One-Loop Spectroscopy of Scalar Three-Point Functions
in planar $\mathcal{N} = 4$ super Yang-Mills Theory

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Abstract

We report on a systematic study of scalar field three-point functions in planar $SU(N) \mathcal{N} = 4$ super Yang-Mills theory at the one-loop level. For this we have computed a sample of 70 structure constants at one-loop order involving primary operators of up to and including length five built entirely from scalar fields. We observe in all 17 cases occurring in our sample that the one-loop structure constant of two protected chiral primary operators and one unprotected operator is given by a simple linear function involving the anomalous scaling dimension of the latter. Moreover, a similar simple one-loop formula is proven for the three-point structure constants of the Konishi operator and two arbitrary protected or un-protected operators. It is again determined by the anomalous scaling dimensions of the operators involved.
1 Introduction and Conclusions

Following the discovery of integrable structures [1–3] in the AdS/CFT correspondence [4], our understanding of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory [5] and the dual $AdS_5 \times S^5$ superstring theory has greatly advanced. To a large extent this progress occurred in the problem of finding the exact all-loop form of the anomalous scaling dimensions of local gauge invariant operators of the gauge theory alias the spectrum of string excitations in the string model. The key was a mapping of the problem to an integrable spin chain which emerged from a one-loop perturbative study of the diagrams involved by Minahan and Zarembo [1]. Moving on to higher loops the spectral problem was mapped to the diagonalization of a long-range spin chain model, whose precise microscopic form remains unknown [3, 6]. Nevertheless assuming integrability the spin-chain S-matrix could be algebraically constructed and the spectral problem was rephrased for asymptotically long operators to the solution of a set of nested Bethe equation [7] (for reviews see [8, 9]). The central remaining problem is now the understanding of wrapping interactions, which affect short operators at lower loop orders [10], [11]. From the algebraic viewpoint important progress was made by thermodynamic Bethe ansatz techniques [12] which also lead to a conjecture for the exact numerical scaling dimensions of the Konishi operator, the shortest unprotected operator in the theory [13].

Next to the scaling dimensions there also exist remarkable all-order results in planar $\mathcal{N} = 4$ SYM for supersymmetric Wilson-loops of special geometries [14] as well as for scattering ampli-
tudes of four and five external particles \[15\], being closely related to light-like Wilson lines \[16\], see \[17\] for reviews.

Given these advances in finding exact results it is natural to ask if one can make similar statements for three-point functions of local gauge invariant operators. Due to conformal symmetry the new data appearing are the structure constants which have a nontrivial coupling constant \(\lambda = g^2N\) dependence and also appear in the associated operator product expansion. In detail we have for renormalized operators

\[
\langle \tilde{O}_\alpha(x_1) \tilde{O}_\beta(x_2) \tilde{O}_\gamma(x_3) \rangle = \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha + \Delta_\beta - \Delta_\gamma}|x_{23}|^{\Delta_\beta + \Delta_\gamma - \Delta_\alpha}|x_{13}|^{\Delta_\alpha + \Delta_\gamma - \Delta_\beta}|\mu|^{\gamma_\alpha + \gamma_\beta + \gamma_\gamma},
\]

where \(\Delta_\alpha = \Delta_\alpha^{(0)} + \lambda \gamma_\alpha\) denotes the scaling dimensions of the operators involved with \(\Delta^{(0)}\) the engineering and \(\gamma\) the anomalous scaling dimensions, \(\mu\) the renormalization scale and

\[
C_{\alpha\beta\gamma} = C_{\alpha\beta\gamma}^{(0)} + \lambda C_{\alpha\beta\gamma}^{(1)} + O(\lambda^2)
\]

is the scheme independent structure constant representing the new observable arising in three-point functions one would like to find. Similar to the case of two-point functions there are non-renormalization theorems for three-point correlation functions of chiral primary (or 1/2 BPS) operators, whose structure constants do not receive radiative corrections \[18\].

The study of three-point functions involving non-protected operators allowing for a non-trivial coupling constant dependence of the structure constants is still largely in its infancy. Direct computations of three-point functions are \[19\]–\[25\] while \[26\] analyzed the problem indirectly through an OPE decomposition of four-point functions of chiral primaries. The works \[20, 25\] focused on non-extremal correlators involving scalar two-impurity operators which are particularly relevant in the BMN limit. The mixing problem of these operators with fermion and derivative impurities was analyzed in \[27\]. \[21\] considered extremal correlators of a very special class of operators allowing an interesting map to spin-chain correlation functions, while \[28\] addresses similar questions from the perspective of the non-planar contribution of the dilatation operator.

The two works \[22, 23\] considered the general problem of finding the structure constants of scalar field primary operators discussing important aspects of scheme independence for the determination of \(C_{\alpha\beta\gamma}^{(1)}\). In this paper we shall continue this work and report on a systematic one-loop study of short single trace conformal primary operators built from the six real scalar fields of the theory in the planar limit. For this we developed a combinatorial dressing technique to promote tree-level non-extremal three-point correlation functions to the one-loop level which is similar to the results reported in \[23\]. This is then used to compute a total of 70 structure constant at the one-loop level involving 11 different scalar field conformal primary operators up to and including length five. The restriction to this particular set of operators arose from the necessity to lift the operator degeneracy in the scalar sector by diagonalising the two-point functions at one-loop. However, the mixing problem in the sector with fermionic and derivative insertions was not resolved, which in general contributes structure constants at the \(O(\lambda)\) level. The main motivation for this spectroscopic study is to provide data to test and develop future conjectures on the form of the three-point structure constants potentially making use of integrability.
Next to providing this one-loop data two general observations could be made. Firstly, in all cases that we computed the structure constants involving two protected (1/2 BPS or chiral primary) operators with an unprotected operator follow a simple linear expression in the anomalous scaling dimensions. In the normalization conventions of (1) and (8) for renormalized operators we find the relation

\[ \frac{C_{\alpha\beta\gamma, \text{non-extremal}}^{(1)}}{C_{\alpha\beta\gamma, \text{non-extremal}}^{(0)}} = -\frac{1}{2} \gamma_\gamma \quad \text{if} \quad \gamma_\alpha = \gamma_\beta = 0, \]

in all 17 cases that occurred in our study. It should be stressed, however, that possible additional contributions to \( C_{\alpha\beta\gamma, \text{non-extremal}}^{(1)} \) arise from operator mixing at the \( \mathcal{O}(\sqrt{\lambda}) \) order with fermionic insertions or covariant derivative insertions at the \( \mathcal{O}(\lambda) \) level respectively, as was studied in \cite{27,25} \cite{27}. These additional contributions due to operator-mixings beyond the \( SO(6) \) sector have not been taken into account here and might change the above result.

Secondly, for the non-extremal three-point correlator of the Konishi operator \( \mathcal{K} = \text{Tr}(\phi^i \phi^i) \) with two arbitrary scalar field primary operators we prove the relation

\[ \frac{C_{\alpha\beta \mathcal{K}, \text{non-extremal}}^{(1)}}{C_{\alpha\beta \mathcal{K}, \text{non-extremal}}^{(0)}} = -\delta_{\alpha\beta} \left( 2 \frac{\gamma_\alpha}{\Delta_\alpha^{(0)}} + \frac{\gamma_\mathcal{K}}{\Delta_\mathcal{K}^{(0)}} \right), \]

Note that this result is in accordance with (3) for \( \gamma_\alpha = 0 \) as \( \Delta_\mathcal{K}^{(0)} = 2 \). It is important to stress that both results only apply for non-extremal correlation functions. Extremal correlation functions are such that \( \Delta_\gamma^{(0)} = \Delta_\alpha^{(0)} + \Delta_\beta^{(0)} \) i.e. the length of the longest operator is equal to the sum of the two shorter ones. Here there also exists a compact one-loop formula due to Okuyama and Tseng \cite{22} see equation (28).

It would be very interesting to see whether these simple structures are stable at higher loop-order and also for non-purely scalar field primary operators such as the twist \( J \) operators for example. Even more interesting would be a computation of three-point functions involving non-protected operators at strong coupling via classical string theory. Here very interesting first steps were done by constructing suitable spinning string solution in \cite{29} approaching the boundary of \( AdS_5 \) and in the construction of classical string vertex operators \cite{30}.

2 General structure and scheme dependence of two and three-point functions

We want to compute planar two- and three-point functions of local scalar operators at the one-loop order. For this it is important to identify the regularization scheme independent information.

To begin with a scalar two-point function of bare local operators \( \mathcal{O}_\alpha^B(x) \) in a random basis can be brought into diagonal form under a suitable linear transformation \( \mathcal{O}_\alpha = M_{\alpha\beta} \mathcal{O}_\beta^B \) with a coupling constant \( \lambda = \alpha'^2 N \) independent mixing matrix \( M_{\alpha\beta} \) as we are working at the one-loop order.

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\footnote{We thank the authors of these papers for pointing this out to us.}
\[ \langle O_\alpha(x_1) O_\beta(x_2) \rangle = \frac{\delta_{\alpha\beta}}{x_{12}^{2\Delta_\alpha(0)}} (1 + \lambda g_\alpha - \lambda \gamma_\alpha \ln |x_{12}\epsilon^{-1}|^2) , \quad x_{12}^2 := (x_1 - x_2)^2 , \quad (5) \]

where \( \epsilon \) represents a space-time UV-cutoff and \( \Delta_\alpha^{(0)} \) the engineering scaling dimension of \( O_\alpha \). Clearly the finite contribution to the one-loop normalization \( g_\alpha \) is scheme dependent \[22,23\] as a shift in the cutoff parameter \( \epsilon \to \epsilon e^c \) changes

\[ g_\alpha \to g_\alpha + 2 c \gamma_\alpha . \quad (6) \]

One may now define the renormalized operators via

\[ \tilde{O}_\alpha = O_\alpha \left( 1 - \frac{\lambda}{2} g_\alpha - \lambda \gamma_\alpha \ln |\mu \epsilon| + O(\lambda^2) \right) \quad (7) \]

with a renormalization momentum scale \( \mu \) to obtain finite canonical two-point correlation functions

\[ \langle \tilde{O}_\alpha(x_1) \tilde{O}_\beta(x_2) \rangle = \frac{\delta_{\alpha\beta}}{|x_{12}|^{2\Delta_\alpha^{(0)}}} (1 - \lambda \gamma_\alpha \ln |x_{12}\mu|^2 + O(\lambda^2)) = \frac{\delta_{\alpha\beta}}{|x_{12}|^{2\Delta_\alpha^{(0)}} |x_{12}\mu|^{2\lambda \gamma_\alpha}} , \quad (8) \]

allowing one to extract the scheme independent one-loop scaling dimensions \( \Delta_\alpha = \Delta_\alpha^{(0)} + \lambda \gamma_\alpha \).

Moving on to three-point functions of the un-renormalized diagonal operators \( O_\alpha \) one obtains to the one-loop order in \( \lambda \)

\[ \langle O_\alpha(x_1) O_\beta(x_2) O_\gamma(x_3) \rangle = \frac{1}{|x_{12}|^{\Delta_\alpha^{(0)} + \Delta_\beta^{(0)} - \Delta_\gamma^{(0)}} |x_{23}|^{\Delta_\alpha^{(0)} + \Delta_\gamma^{(0)} - \Delta_\beta^{(0)}} |x_{31}|^{\Delta_\beta^{(0)} + \Delta_\gamma^{(0)} - \Delta_\alpha^{(0)}}} \times \left[ C_{\alpha\beta\gamma}^{(0)} \left( 1 + \frac{1}{2} \lambda \left\{ \gamma_\alpha \ln \frac{\epsilon^2 x_{12}^2 x_{23}}{x_{12} x_{23}^2} + \gamma_\beta \ln \frac{\epsilon^2 x_{23}^2 x_{12}}{x_{23} x_{12}^2} + \gamma_\gamma \ln \frac{\epsilon^2 x_{12}^2 x_{23}}{x_{12} x_{23}^2} \right\} \right) \right] + \lambda \tilde{C}_{\alpha\beta\gamma}^{(1)} . \quad (9) \]

Now again the finite one-loop contribution to the structure constant \( \tilde{C}_{\alpha\beta\gamma}^{(1)} \) is scheme dependent \[22,23\] as it changes under \( \epsilon \to \epsilon e^c \) as

\[ \tilde{C}_{\alpha\beta\gamma}^{(1)} \to \tilde{C}_{\alpha\beta\gamma}^{(1)} + c (\gamma_\alpha + \gamma_\beta + \gamma_\gamma) C_{\alpha\beta\gamma}^{(0)} , \quad (no \sums \onindices) . \quad (10) \]

However, the following combination of the unrenormalized three-point function structure constant and the normalization is scheme independent

\[ C_{\alpha\beta\gamma}^{(1)} := \tilde{C}_{\alpha\beta\gamma}^{(1)} - \frac{1}{2} (g_\alpha C_{\alpha\beta\gamma}^{(0)} + g_\beta C_{\alpha\beta\gamma}^{(0)} + g_\gamma C_{\alpha\beta\gamma}^{(0)}) . \quad (11) \]

This is the only datum to be extracted from three-point functions. It also directly arises as the structure constant in the three-point function of the renormalized operators \( \tilde{O}_\alpha \)

\[ \langle \tilde{O}_\alpha(x_1) \tilde{O}_\beta(x_2) \tilde{O}_\gamma(x_3) \rangle = \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha + \Delta_\beta - \Delta_\gamma} |x_{23}|^{\Delta_\beta + \Delta_\gamma - \Delta_\alpha} |x_{31}|^{\Delta_\alpha + \Delta_\gamma - \Delta_\beta} |\mu|^\lambda (\gamma_\alpha + \gamma_\beta + \gamma_\gamma) , \quad (12) \]

\[ ^2 \text{Note that the two-loop diagonalization will involve a mixing matrix proportional to } \lambda. \]
where $C_{\alpha\beta\gamma} = C_{\alpha\beta\gamma}^{(0)} + \lambda C_{\alpha\beta\gamma}^{(1)} + O(\lambda^2)$ is the scheme independent structure constant of (11).

An important point is the following. If one wishes to compute the one-loop piece $C_{\alpha\beta\gamma}^{(1)}$ starting from a generic basis of operators one has to resolve the mixing problem at the two-loop order. This is so as the resulting mixing matrix $M_{\alpha\beta}$ will then receive $O(\lambda^2)$ terms which will contribute to the final $C_{\alpha\beta\gamma}^{(1)}$ through tree-level contractions. If, however, the degeneracy for a given set of states in a representation of $psu(2, 2|4)$ has been completely lifted already at the one-loop order, then this two-loop mixing effect will be absent as the mixing matrix $M_{\alpha\beta}$ cannot receive further corrections. Looking at short primary operators this indeed turns out to be the case up for a large class of short operators, as we will discuss later.

3 The one-loop planar dressing formulae

3.1 Derivation

In this section we derive an efficient set of combinatorial dressing formulae to dress up tree-level graphs to one-loop. Similar formulae appeared in [31].

Following [20] we introduce the 4d propagator and the relevant one-loop integrals in configuration space

\begin{align}
I_{12} &= \frac{1}{(2\pi)^2 x_{12}^2}, \\
Y_{123} &= \int d^4w I_{1w} I_{2w} I_{3w}, \\
X_{1234} &= \int d^4w I_{1w} I_{2w} I_{3w} I_{4w}, \\
H_{12,34} &= \int d^4v d^4w I_{1v} I_{2w} I_{3w} I_{4w}, \\
F_{12,34} &= \frac{(\partial_1 - \partial_2) \cdot (\partial_3 - \partial_4) H_{12,34}}{I_{12} I_{34}}.
\end{align}

We have put the space-time points as indices to the function to make the expressions more compact. These functions are all finite except in certain limits. For example $Y_{123}, X_{1234}$ and $H_{12,34}$ diverge logarithmically when $x_1 \to x_2$. In point splitting regularization one has the limiting formulae ($\lim_{i\to j} x_{ij}^2 = \epsilon^2$)

\begin{align}
X_{1123} &= -\frac{1}{16\pi^2} I_{12} I_{13} \left( \ln \frac{x_{23}^2 \epsilon^2}{x_{12}^2 x_{13}^2} - 2 \right), \\
Y_{112} &= -\frac{1}{16\pi^2} I_{12} \left( \ln \frac{\epsilon^2}{x_{12}^2} - 2 \right) = Y_{122}, \\
F_{12,13} &= -\frac{1}{16\pi^2} \left( \ln \frac{\epsilon^2}{x_{23}^2} - 2 \right) + Y_{123} \left( \frac{1}{I_{12}} + \frac{1}{I_{13}} - \frac{2}{I_{23}} \right).
\end{align}
Figure 1: The generic tree-level three-point function.

\[ X_{1122} = -\frac{1}{8\pi^2} r_{12}^2 \left( \ln \frac{\varepsilon^2}{x_{12}^2} - 1 \right), \]  
\[ F_{12,12} = -\frac{1}{8\pi^2} \left( \ln \frac{\varepsilon^2}{x_{12}^2} - 3 \right). \]  

We introduce a graphical symbol for the scalar propagators and work in a normalization where

\[ \langle \phi^I(x_1)\phi^J(x_2) \rangle_{\text{tree}} u^I_1 u^J_2 = \frac{u_1}{u_2} = (u_1 \cdot u_2) I_{12}, \]  

here the $SO(6)$-indices of the scalar fields are contracted with dummy six-vectors $u^I_1$ and $u^J_2$ for bookmarking purposes.

The one-loop corrections are then built of the following three components

\[ u_1 \overline{\bullet} \bullet u_2 = -\lambda (u_1 \cdot u_2) I_{12} \frac{Y_{112} + Y_{122}}{I_{12}} \quad \text{(self-energy)}, \]  
\[ \frac{u_1}{u_2} \overline{\bullet} \bullet \frac{u_2}{u_3} = \frac{\lambda}{2} (u_1 \cdot u_2)(u_3 \cdot u_4) I_{12} I_{34} F_{12,34} \quad \text{(gluon)}, \]  
\[ \frac{u_1}{u_2} \overline{\bullet} \bullet \frac{u_2}{u_3} = \frac{\lambda}{2} \left[ 2(u_2 \cdot u_3)(u_1 \cdot u_4) - (u_2 \cdot u_4)(u_1 \cdot u_3) \right. \]  
\[ \left. - (u_1 \cdot u_2)(u_3 \cdot u_4) \right] X_{1234} \quad \text{(vertex)}. \]  

With these basic interactions we can now diagrammatically dress up the tree-level two- and three-point correlation functions to the one-loop level. To do so we note that a generic planar three-point function will be made of two-gon and three-gon sub-graphs which need to be dressed, see figure 1.

For the two-gon dressing one finds the basic dressing formula

\[
\begin{array}{c}
\langle u_1 \overline{\bullet} \bullet u_2 \rangle_{\text{1-loop}} = \langle \cdot \overline{\bullet} \bullet \cdot \rangle + \frac{1}{2} \langle \cdot \overline{\bullet} \bullet \cdot \rangle + \frac{1}{2} \langle \cdot \overline{\bullet} \bullet \cdot \rangle
\end{array}
\]
\[ I_{12}^2 \frac{\lambda}{8\pi^2} \left( \ln \frac{\epsilon^2}{x_{12}^2} - 1 \right) \left( u_1 \cdot v_2 v_1 \cdot u_2 - u_1 \cdot u_2 v_1 \cdot v_2 - \frac{1}{2} u_1 \cdot v_1 u_2 \cdot v_2 \right) \]

\[ = I_{12}^2 \frac{\lambda}{8\pi^2} \left( \ln \frac{\epsilon^2}{x_{12}^2} - 1 \right) \left( \begin{array}{cc} \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{array} \right) + \frac{1}{2} \left( \begin{array}{cc} \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{array} \right) , \]  
\[ \text{(23)} \]

where the diagrams in the last line only stand for the index contractions not for propagators. This contraction structure is of course that of an integrable nearest neighbor $SO(6)$ vector spin-chain Hamiltonian as was first noted in [1].

Analogously, for the three-gon we find

\[ \langle u_2 v_1 v_2 u_1 u_2 w_1 w_2 \rangle_{1\text{-loop}} = \frac{I_{12} I_{13} I_{23}}{16\pi^2} \times \left[ \left( \ln \frac{\epsilon^2}{x_{12}^2} - 2 \right) \left( \begin{array}{cc} \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{array} \right) + \frac{1}{2} \left( \begin{array}{cc} \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{array} \right) \right] + \left( \ln \frac{\epsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2 \right) \left( \begin{array}{cc} \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{array} \right) + \frac{1}{2} \left( \begin{array}{cc} \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{array} \right) \right] + \left( \ln \frac{\epsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2 \right) \left( \begin{array}{cc} \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{array} \right) + \frac{1}{2} \left( \begin{array}{cc} \cdot \cdot \cdot & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot \end{array} \right) \right] . \]  
\[ \text{(24)} \]

Again the graphs in the last three lines only represent the index contractions. Interestingly a similar structure to the integrable spin-chain Hamiltonian of (23) emerges also for the one-loop three-gon interactions.

3.2 Gauge invariance and Wilson line contributions

There is one important point we have not addressed so far. The point splitting regularization method that we employed violates gauge invariance as the space-time locations of the two neighboring operators in the trace are no longer coincident. The natural way to recover gauge invariance is to connect the two split points through a straight Wilson line. This, however, gives rise to new diagrams not yet accounted for in which a gluon is radiated off the Wilson line. Luckily we are able to show that this contribution vanishes entirely at the one-loop level for $|\epsilon| \to 0$.

Setting $\epsilon^\mu = x^\mu_{13}$ the Wilson line is parametrized by

\[ x^\mu(\tau) = x_{13}^\mu + \epsilon^\mu \tau , \quad \tau \in [0,1] . \]  
\[ \text{(25)} \]
Figure 2: Additional Feynman-Graphs for extremal three-point functions.

We then have the contribution

\[ \tau_1 \omega_2 = \lambda (u_1 \cdot u_2) (u_3 \cdot u_4) \int_0^1 d\tau \epsilon \cdot (\partial_1 - \partial_2) Y_{12\tau} \]

\[ = -2\lambda (u_1 \cdot u_2) (u_3 \cdot u_4) \int_0^1 d\tau \int d^4 \omega \frac{\epsilon \cdot x_{1\omega}}{(x_{1\omega}^2)^2 x_{2\omega}^2 x_{\tau\omega}^2} . \]  

This five dimensional integral is by power-counting logarithmically divergent for coincident points \( x_3, x(\tau) \rightarrow x_1 \) i.e. \(|\epsilon| \rightarrow 0\) and one has

\[ \lim_{|\epsilon| \rightarrow 0} \int_0^1 d\tau \int d^4 \omega \frac{\epsilon \cdot x_{1\omega}}{(x_{1\omega}^2)^2 x_{2\omega}^2 x_{\tau\omega}^2} \sim \lim_{|\epsilon| \rightarrow 0} \epsilon \cdot x_{12} \left( \ln \frac{\epsilon^2}{x_{12}^2} + \text{finite} + O(\epsilon) \right) \rightarrow 0 . \]

There is also a novel ladder-diagram in which a gluon is exchanged between two Wilson lines extending from \( x_1 \) to \( x_3 \) and from \( x_2 \) to \( x_4 \). This ladder-graph is manifestly finite and vanishes as \( \epsilon^2 \). Therefore all the Wilson line contributions to the point splitting regularization vanish at this order of perturbation theory.

3.3 Extremal three-point functions

Three-point functions of operators with lengths \( \Delta_\alpha^{(0)} \), \( \Delta_\beta^{(0)} \) and \( \Delta_\gamma^{(0)} \) where \( \Delta_\alpha^{(0)} + \Delta_\beta^{(0)} = \Delta_\gamma^{(0)} \) are called extremal. For these extremal functions the dressing formulae above do not hold any longer for two reasons: First, there appear additional diagrams with a gluon exchange or a vertex between non-neighboring propagators as the one in figure 2. These non-nearest neighbor interactions lead to additional terms in the dressing formulae. Second, unlike non-extremal ones extremal three-point functions with double-trace operators contain the same factor of \( N \) as those with single-trace operators. This results in an operator mixing of single-trace with double-trace operators already at tree-level. This is described in detail in [32,22].

We will refrain from studying these extremal three-point correlators in the following. In any case the one-loop structure constants follow a simple pattern: They are a given by a linear function of the anomalous scaling dimensions of the operators involved [22]

\[ C^{(1)}_{\alpha \beta \gamma, \text{extremal}} = \frac{1}{2} C^{(0)}_{\alpha \beta \gamma, \text{extremal}} (\gamma_\alpha + \gamma_\beta - \gamma_\gamma) , \]
hence the three-point problem has been reduced to the two-point one. In particular structure constants of protected operators are free of radiative corrections.

### 3.4 Two convenient regularization schemes

We have seen in (11) how to extract the regularization scheme independent structure constant from a combination of the bare structure constant and the one-loop finite normalization shifts. As the latter arises from the finite contribution to the two-gon dressing (23) one may pick a regularization to simply cancel these contributions. I.e. making the transformation on the point-splitting parameter

$$\epsilon \rightarrow \sqrt{\epsilon} \epsilon$$

transforms

$$\ln \frac{\epsilon^2}{x_{ij}^2} - 1 \rightarrow \ln \frac{\epsilon^2}{x_{ij}^2}, \quad \text{and} \quad \ln \frac{\epsilon^2 x_{ij}^2}{x_{ik} x_{jk}^2} - 2 \rightarrow \ln \frac{\epsilon^2 x_{ij}^2}{x_{ik} x_{jk}^2} - 1.$$  

Hence in this scheme the finite part of the two-gon dressing vanishes resulting in a vanishing finite correction to the two-point functions

$$g_\alpha = 0,$$  

which in turn implies that the bare and the renormalized structure functions coincide in this scheme

$$\tilde{C}^{(1)}_{\alpha\beta\gamma} = C^{(1)}_{\alpha\beta\gamma}.$$  

This implies that the structure function may be read off solely from the three-gon dressings of the non-extremal correlator, which may be graphically represented by

$$C^{(1)}_{\alpha\beta\gamma} = -\frac{1}{16 \pi^2} \sum_{\text{cyclic perm.}} \left[ 3 \times \left( \begin{array}{c} \epsilon_1 \epsilon_2 \epsilon_3 \\ \epsilon_4 \epsilon_5 \epsilon_6 \\ \epsilon_7 \epsilon_8 \epsilon_9 \\ \epsilon_{10} \epsilon_{11} \epsilon_{12} \end{array} \right) + \frac{1}{2} \times \left( \begin{array}{c} \epsilon_1 \epsilon_2 \epsilon_3 \\ \epsilon_4 \epsilon_5 \epsilon_6 \\ \epsilon_7 \epsilon_8 \epsilon_9 \\ \epsilon_{10} \epsilon_{11} \epsilon_{12} \end{array} \right) \right].$$  

Alternatively one may apply the transformation

$$\epsilon \rightarrow \sqrt{\epsilon} \epsilon$$

yielding

$$\ln \frac{\epsilon^2}{x_{ij}^2} - 1 \rightarrow \ln \frac{\epsilon^2}{x_{ij}^2} + 1, \quad \text{and} \quad \ln \frac{\epsilon^2 x_{ij}^2}{x_{ik} x_{jk}^2} - 2 \rightarrow \ln \frac{\epsilon^2 x_{ij}^2}{x_{ik} x_{jk}^2}.$$  

Now the finite contributions to the three-gon dressings vanish and the bare structure constant may be computed from only dressing the two-gons in the tree-level correlator

$$\tilde{C}^{(1)}_{\alpha\beta\gamma} = \frac{1}{8 \pi^2} \sum_{\text{cyclic perm.}} \sum_{\text{all 2-gons}} \left( \begin{array}{c} \epsilon_1 \epsilon_2 \\ \epsilon_3 \epsilon_4 \\ \epsilon_5 \epsilon_6 \\ \epsilon_7 \epsilon_8 \\ \epsilon_9 \epsilon_{10} \\ \epsilon_{11} \epsilon_{12} \end{array} \right) - \frac{1}{2} \times \left( \begin{array}{c} \epsilon_1 \epsilon_2 \\ \epsilon_3 \epsilon_4 \\ \epsilon_5 \epsilon_6 \\ \epsilon_7 \epsilon_8 \\ \epsilon_9 \epsilon_{10} \\ \epsilon_{11} \epsilon_{12} \end{array} \right).$$  

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The scheme independent structure constants can then be calculated using \((11)\) with \(g_\alpha = \gamma_\alpha\) by virtue of \((35)\), i.e.

\[
C^{(1)}_{\alpha\beta\gamma} = \tilde{C}^{(1)}_{\alpha\beta\gamma} - \frac{1}{2} C^{(0)}_{\alpha\beta\gamma} (\gamma_\alpha + \gamma_\beta + \gamma_\gamma).
\]

(37)

In our actual computations we have used both schemes depending on the problem at hand.

| Length | Class | \(SU(4)\)\(_{\text{length}}\) Rep. | Dim. | \(8\pi^2\gamma\) | Operator |
|--------|-------|-------------------------------|------|----------------|----------|
| 2      | 2A    | \([0, 0, 0]_2^+\)             | 1    | 6              | \(\mathcal{K}\) |
|        | 2B    | \([0, 2, 0]_2^+\)             | 20   | 0              | \text{CPO} |
| 3      | 3B    | \([0, 1, 0]_3^-\)             | 6    | 4              | \(\mathcal{O}_{n=1}^{J=1}\) |
|        | 3C    | \([0, 3, 0]_3^-\)             | 50   | 0              | \text{CPO} |
| 4      | 4A    | \([0, 0, 0]_4^+\)             | 1    | \(\frac{1}{2} (13 + \sqrt{41})\) * | |
|        | 4E    | \([0, 0, 0]_4^+\)             | 1    | \(\frac{1}{2} (13 - \sqrt{41})\) * | |
|        | 4B    | \([0, 2, 0]_4^+\)             | 20   | 5 + \(\sqrt{5}\) | \(\mathcal{O}_{n=1}^{J=2}\) |
|        | 4F    | \([0, 2, 0]_4^+\)             | 20   | 5 - \(\sqrt{5}\) | \(\mathcal{O}_{n=1}^{J=2}\) |
|        | 4C    | \([2, 0, 2]_4^+ + [1, 0, 1]_4^-\) | 84 + 15 | 6       | |
|        | 4G    | \([0, 4, 0]_4^+\)             | 105  | 0              | \text{CPO} |
| 5      | 5A    | \([0, 0, 2]_5^+ + [2, 0, 0]_5^+\) | 10 + 10 | 7 + \(\sqrt{13}\) | |
|        | 5H    | \([0, 0, 2]_5^+ + [2, 0, 0]_5^+\) | 10 + 10 | 7 - \(\sqrt{13}\) | |
|        | 5D    | \([0, 1, 0]_5^- + \text{desc}\) | 6 + 252 | 5 + \(\sqrt{5}\) | |
|        | 5I    | \([0, 1, 0]_5^- + \text{desc}\) | 6 + 252 | 5 - \(\sqrt{5}\) | |
|        | 5F    | \([1, 1, 1]_5^+ + [1, 1, 1]_5^-\) | 64 + 64 | 5       | |
|        | 5J    | \([0, 3, 0]_5^-\)             | 50   | 2              | |
|        | 5E    | \([0, 3, 0]_5^- + \text{desc}\) | 50 + 140 | 6       | \(\mathcal{O}_{n=1}^{J=3}\) |
|        | 5K    | \([0, 5, 0]_5^-\)             | 196  | 0              | \text{CPO} |
|        | 5B    | \([0, 1, 0]_5^-\)             | \(6+6\) | 10       | |

**Figure 3:** List of all scalar conformal primary operator up to length 5 with their one-loop anomalous dimensions. Degenerate classes of operators are printed in bold-face. \(\mathcal{K}\) denotes the Konishi and \text{CPO} chiral primary operators. The \(\mathcal{O}_{n=1}^{J}\) refer to the BMN singlet operators in the nomenclature of \([33,25]\) where the quantum mixing with fermion and derivative insertions is resolved. The asterix refers to not resolved fermion and derivative mixings.
4 Results

Using the dressing formulae of section 3 one can in principle straightforwardly compute arbitrary three-point functions by combinatorial means. Clearly, due to the need to sum over all permutations in these dressing formulae the complexity in the computations grows fast and needs to be done on a computer. This has been implemented in a two step procedure. Starting with an arbitrarily chosen basis of operators all two-point functions are computed and then diagonalized. Similarly all three-point functions are computed in the original basis and then projected to the diagonal basis where the structure constants can be extracted. For operators up to length three this was done algebraically with a Mathematica program. Starting with length four the mixing matrix diagonalization could not be performed algebraically any longer and we had to resort to numerics using Matlab. Once the diagonal basis was constructed the numerically obtained structure constants could in most cases be again fitted to algebraic expressions derived by the algebraic form of the one-loop scaling dimensions. This could be done in 62 out of 70 cases.

4.1 Short primary scalar operators up to length 5

We first list all the scalar conformal primary operators up to and including length five. We have independently constructed this list by an explicit diagonalization of the corresponding two-point functions finding complete agreement with the previous analysis of Beisert [34,9] see figure 3. Note that there remain degeneracies in the anomalous scaling dimensions $\gamma$ which we indicate in the table through bold face letters.

The operators up to length three and the length four singlets can be explicitly given and read

$$O_{2A} = \sum_{i=1}^{6} \text{Tr} \left( \phi^i \phi^i \right) = K$$

$$O_{2B,(ij)} = \text{Tr} \left( \phi^i \phi^j \right) \quad (i < j)$$

$$O_{2B,i} = \text{Tr} \left( \phi^i \phi^i \right) - \frac{1}{\sqrt{3}} K \quad (i = 2 \ldots 6)$$

$$O_{3B,i} = \sum_{j=1}^{6} \text{Tr} \left( \phi^i \phi^j \phi^j \right)$$

$$O_{3C,i(jk)} = \text{Tr} \left( \phi^i \phi^j \phi^k \right) \quad (i < j < k)$$

$$O_{3C,ij} = 8 \text{Tr} \left( \phi^i \phi^j \phi^j \right) - \sum_{k=1}^{6} \text{Tr} \left( \phi^i \phi^k \phi^k \right) \quad (i \neq j, j = 2 \ldots 6)$$

$$O_{3C,ii} = 8 \text{Tr} \left( \phi^i \phi^i \phi^i \right) - 3 \sum_{j=1}^{6} \text{Tr} \left( \phi^i \phi^j \phi^j \right) \quad (i = 2 \ldots 6)$$

$$O_{4A} = \sum_{i=1}^{6} \sum_{j=1}^{6} \left[ 4 \text{Tr} \left( \phi^i \phi^j \phi^j \phi^j \right) + \left( 5 - \sqrt{41} \right) \text{Tr} \left( \phi^i \phi^j \phi^j \phi^j \right) \right] + \ldots$$
\[ O_{4E} = \sum_{i=1}^{6} \sum_{j=1}^{6} \left[ 4 \text{ Tr} \left( \phi^i \phi^j \phi^i \phi^j \right) + \left( 5 + \sqrt{41} \right) \text{ Tr} \left( \phi^i \phi^i \phi^j \phi^j \right) \right] + \ldots, \quad (46) \]

The dots in the last two operators indicate possible operator mixings with fermion and derivative insertions which have not been resolved so far. Similarly the operators 4B, 4F and 5E mix with such terms and have been displayed in \[27,25].

Below we list our main results. We computed almost all one-loop structure constants for the non-degenerate operators of up to length five of figure 3. Note that only three-point functions which do not vanish at tree-level are listed. We also stress that the majority of results for the fractions \( C_{\alpha\beta\gamma}^{(1)} / C_{\alpha\beta\gamma}^{(0)} \) have been obtained numerically and the quoted analytical results represents a biases fit allowing as non-rational factors only the square root term appearing in the anomalous scaling dimensions of the operators involved in the particular three-point function. The numerical precision in these fits is typically of order \( 10^{-5} \) or better, for the raw data see the appendix A.2 of \[35\]. Finally, the analytically obtained results are highlighted in bold-face letters.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\mathcal{O}_\alpha & \mathcal{O}_\beta & \mathcal{O}_\gamma & 8\pi^2\gamma_\alpha & 8\pi^2\gamma_\beta & 8\pi^2\gamma_\gamma & -16\pi^2 C_{\alpha\beta\gamma}^{(1)} / C_{\alpha\beta\gamma}^{(0)} \\
\hline
2B & 3B & 3B & 0 & 4 & 4 & \frac{8}{3} \\
2B & 3B & 3C & 0 & 4 & 0 & 4 \\
2B & 3C & 3C & 0 & 0 & 0 & 0 \\
2B & 4A & 4B & 0 & \frac{1}{2} (13 + \sqrt{41}) & 5 + \sqrt{5} & 5 + \sqrt{5} \\
2B & 4A & 4F & 0 & \frac{1}{2} (13 + \sqrt{41}) & 5 - \sqrt{5} & 5 - \sqrt{5} \\
2B & 4B & 4B & 0 & 5 + \sqrt{5} & 5 + \sqrt{5} & \frac{2}{\sqrt{5}} (115 + 14 \sqrt{5}) \\
2B & 4B & 4E & 0 & 5 + \sqrt{5} & \frac{1}{2} (13 - \sqrt{41}) & 5 + \sqrt{5} \\
2B & 4B & 4F & 0 & 5 + \sqrt{5} & 5 - \sqrt{5} & 0 \\
2B & 4B & 4G & 0 & 5 + \sqrt{5} & 0 & 5 + \sqrt{5} \\
2B & 4E & 4F & 0 & \frac{1}{2} (13 - \sqrt{41}) & 5 - \sqrt{5} & 5 - \sqrt{5} \\
2B & 4F & 4F & 0 & 5 - \sqrt{5} & 5 - \sqrt{5} & \frac{2}{\sqrt{5}} (115 - 14 \sqrt{5}) \\
2B & 4F & 4G & 0 & 5 - \sqrt{5} & 0 & 5 - \sqrt{5} \\
2B & 4G & 4G & 0 & 0 & 0 & 0 \\
3B & 3B & 4A & 4 & 4 & \frac{1}{2} (13 + \sqrt{41}) & \frac{1}{50} (261 + 9 \sqrt{41}) \\
3B & 3B & 4B & 4 & 4 & 5 + \sqrt{5} & \frac{1}{11} (87 + 3 \sqrt{5}) \\
\hline
\end{array}
\]

\(^3\)We thank the authors of this work for important discussions on this point.
| $\mathcal{O}_\alpha$ | $\mathcal{O}_\beta$ | $\mathcal{O}_\gamma$ | $8\pi^2\gamma_\alpha$ | $8\pi^2\gamma_\beta$ | $8\pi^2\gamma_\gamma$ | $-16\pi^2C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 3B              | 3B              | 4E              | 4               | 4               | $\frac{1}{2}(13 - \sqrt{41})$ | $\frac{1}{50}(261 - 9\sqrt{41})$ |
| 3B              | 3B              | 4F              | 4               | 4               | $5 - \sqrt{5}$   | $\frac{1}{11}(87 - 3\sqrt{5})$ |
| 3B              | 3C              | 4B              | 4               | 0               | $5 + \sqrt{5}$   | $\frac{1}{11}(39 + 7\sqrt{5})$ |
| 3B              | 3C              | 4F              | 4               | 0               | $5 - \sqrt{5}$   | $\frac{1}{11}(39 - 7\sqrt{5})$ |
| 3B              | 3C              | 4G              | 4               | 0               | 0               | 4               |
| 3C              | 3C              | 4A              | 0               | 0               | $\frac{1}{2}(13 + \sqrt{41})$ | $\frac{1}{2}(13 + \sqrt{41})$ |
| 3C              | 3C              | 4B              | 0               | 0               | $5 + \sqrt{5}$   | $5 + \sqrt{5}$ |
| 3C              | 3C              | 4E              | 0               | 0               | $\frac{1}{2}(13 - \sqrt{41})$ | $\frac{1}{2}(13 - \sqrt{41})$ |
| 3C              | 3C              | 4F              | 0               | 0               | $5 - \sqrt{5}$   | $5 - \sqrt{5}$ |
| 3C              | 3C              | 4G              | 0               | 0               | 0               | 0               |
| 4A              | 4A              | 4A              | $\frac{1}{2}(13 + \sqrt{41})$ | $\frac{1}{2}(13 + \sqrt{41})$ | $\frac{1}{2}(13 + \sqrt{41})$ | $\frac{1}{733}(7185 + 309\sqrt{41})$ |
| 4A              | 4A              | 4E              | $\frac{1}{2}(13 + \sqrt{41})$ | $\frac{1}{2}(13 + \sqrt{41})$ | $\frac{1}{2}(13 - \sqrt{41})$ | $\frac{1}{10}(21 - \sqrt{41})$ |
| 4A              | 4A              | 4G              | $\frac{1}{2}(13 + \sqrt{41})$ | $\frac{1}{2}(13 + \sqrt{41})$ | 0               | $\frac{1}{2}(13 + \sqrt{41})$ |
| 4A              | 4B              | 4B              | $\frac{1}{2}(13 + \sqrt{41})$ | 5 + $\sqrt{5}$   | 5 + $\sqrt{5}$   | 12.3279656 |
| 4A              | 4B              | 4F              | $\frac{1}{2}(13 + \sqrt{41})$ | 5 + $\sqrt{5}$   | 5 - $\sqrt{5}$   | $\frac{1}{2}(9 + \sqrt{41})$ |
| 4A              | 4E              | 4E              | $\frac{1}{2}(13 + \sqrt{41})$ | $\frac{1}{2}(13 - \sqrt{41})$ | $\frac{1}{2}(13 - \sqrt{41})$ | $\frac{1}{10}(21 + \sqrt{41})$ |
| 4A              | 4F              | 4F              | $\frac{1}{2}(13 + \sqrt{41})$ | 5 - $\sqrt{5}$   | 5 - $\sqrt{5}$   | 4.865786 |
| 4A              | 4G              | 4G              | $\frac{1}{2}(13 + \sqrt{41})$ | 0               | 0               | $\frac{1}{2}(13 + \sqrt{41})$ |
| 4B              | 4B              | 4B              | 5 + $\sqrt{5}$   | 5 + $\sqrt{5}$   | 5 + $\sqrt{5}$   | 6.772955 |
| 4B              | 4B              | 4E              | 5 + $\sqrt{5}$   | 5 + $\sqrt{5}$   | $\frac{1}{2}(13 - \sqrt{41})$ | 38.020253 |
| 4B              | 4B              | 4F              | 5 + $\sqrt{5}$   | 5 + $\sqrt{5}$   | 5 - $\sqrt{5}$   | 26.076638 |
| 4B              | 4B              | 4G              | 5 + $\sqrt{5}$   | 5 + $\sqrt{5}$   | 0               | $\frac{4}{19}(25 + 7\sqrt{5})$ |
| 4B              | 4E              | 4F              | 5 + $\sqrt{5}$   | $\frac{1}{2}(13 - \sqrt{41})$ | 5 - $\sqrt{5}$   | $\frac{1}{2}(9 - \sqrt{41})$ |
| 4B              | 4F              | 4F              | 5 + $\sqrt{5}$   | 5 - $\sqrt{5}$   | 5 - $\sqrt{5}$   | 5.374976 |
| 4B              | 4F              | 4G              | 5 + $\sqrt{5}$   | 5 - $\sqrt{5}$   | 0               | $\frac{10}{3}$ |
| 4B              | 4G              | 4G              | 5 + $\sqrt{5}$   | 0               | 0               | 5 + $\sqrt{5}$ |
| \(O_\alpha\) | \(O_\beta\) | \(O_\gamma\) | \(8\pi^2\gamma_\alpha\) | \(8\pi^2\gamma_\beta\) | \(8\pi^2\gamma_\gamma\) | \(-16\pi^2C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}\) |
|---|---|---|---|---|---|---|
| 4E | 4E | 4E | \(\frac{1}{2}(13 - \sqrt{41})\) | \(\frac{1}{2}(13 - \sqrt{41})\) | \(\frac{1}{2}(13 - \sqrt{41})\) | \(\frac{1}{733}(7185 - 309\sqrt{41})\) |
| 4E | 4E | 4G | \(\frac{1}{2}(13 - \sqrt{41})\) | \(\frac{1}{2}(13 - \sqrt{41})\) | 0 | \(\frac{1}{2}(13 - \sqrt{41})\) |
| 4E | 4F | 4F | \(\frac{1}{2}(13 - \sqrt{41})\) | 5 - \(\sqrt{5}\) | 5 - \(\sqrt{5}\) | 4.785995 |
| 4E | 4G | 4G | \(\frac{1}{2}(13 - \sqrt{41})\) | 0 | 0 | \(\frac{1}{2}(13 - \sqrt{41})\) |
| 4F | 4F | 4F | 5 - \(\sqrt{5}\) | 5 - \(\sqrt{5}\) | 5 - \(\sqrt{5}\) | 4.464987 |
| 4F | 4F | 4G | 5 - \(\sqrt{5}\) | 5 - \(\sqrt{5}\) | 0 | \(\frac{4}{19}(25 - 7\sqrt{5})\) |
| 4F | 4G | 4G | 5 - \(\sqrt{5}\) | 0 | 0 | 5 - \(\sqrt{5}\) |
| 4G | 4G | 4G | 0 | 0 | 0 | 0 |
| 2B | 5J | 5J | 0 | 2 | 2 | \(\frac{10}{7}\) |
| 2B | 5J | 5K | 0 | 2 | 0 | 2 |
| 2B | 5K | 5K | 0 | 0 | 0 | 0 |
| 3B | 4A | 5J | 4 | \(\frac{1}{2}(13 + \sqrt{41})\) | 2 | 2 |
| 3B | 4A | 5K | 4 | \(\frac{1}{2}(13 + \sqrt{41})\) | 0 | 4 |
| 3B | 4B | 5J | 4 | 5 + \(\sqrt{5}\) | 2 | \(\frac{8}{3}(13 + 2\sqrt{5})\) |
| 3B | 4B | 5K | 4 | 5 + \(\sqrt{5}\) | 0 | 4 |
| 3B | 4E | 5J | 4 | \(\frac{1}{2}(13 - \sqrt{41})\) | 2 | 2 |
| 3B | 4E | 5K | 4 | \(\frac{1}{2}(13 - \sqrt{41})\) | 0 | 4 |
| 3B | 4F | 5J | 4 | 5 - \(\sqrt{5}\) | 2 | \(\frac{8}{3}(13 - 2\sqrt{5})\) |
| 3B | 4F | 5K | 4 | 5 - \(\sqrt{5}\) | 0 | 4 |
| 3B | 4G | 5J | 4 | 0 | 2 | 2 |
| 3B | 4G | 5K | 4 | 0 | 0 | 4 |
| 3C | 4A | 5J | 0 | \(\frac{1}{2}(13 + \sqrt{41})\) | 2 | 2 |
| 3C | 4B | 5J | 0 | 5 + \(\sqrt{5}\) | 2 | 2 |
| 3C | 4B | 5K | 0 | 5 + \(\sqrt{5}\) | 0 | 5 + \(\sqrt{5}\) |
| 3C | 4E | 5J | 0 | \(\frac{1}{2}(13 - \sqrt{41})\) | 2 | 2 |
| 3C | 4F | 5J | 0 | 5 - \(\sqrt{5}\) | 2 | 2 |
| $\mathcal{O}_\alpha$ | $\mathcal{O}_\beta$ | $\mathcal{O}_\gamma$ | $8\pi^2 \gamma_\alpha$ | $8\pi^2 \gamma_\beta$ | $8\pi^2 \gamma_\gamma$ | $-16\pi^2 C_{\alpha\beta\gamma}^{(1)}/C_{\alpha\beta\gamma}^{(0)}$ |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 3C             | 4F             | 5K             | 0              | 5 $-\sqrt{5}$ | 0              | 5 $-\sqrt{5}$ |
| 3C             | 4G             | 5J             | 0              | 0              | 2              | 2              |
| 3C             | 4G             | 5K             | 0              | 0              | 0              | 0              |

As reported in the introduction we make the general observation, that for a three-point function of two protected operators with one unprotected operator the structure constants follow the simple pattern:

$$C_{\alpha\beta\gamma}^{(1)} = -\frac{1}{2} \gamma_\gamma \text{ if } \gamma_\alpha = \gamma_\beta = 0.$$  \hfill (47)

This occurred in all applicable 17 cases in the above. We again stress that, except for the cases of $K$ and 3B, this result will generically receive corrections from subleading operator-mixing terms with two fermion and two derivative insertions.

### 4.2 Konishi operator with two primary scalar operators of arbitrary lengths

We calculated the three-point function of a Konishi operator with two arbitrary operators of same length from a diagonal basis. The three-point function then takes the general form

$$C_{\alpha\beta\gamma}^{(1)} = -\left(\frac{\gamma_\alpha}{\Delta_\alpha^{(0)}} + \frac{\gamma_\beta}{\Delta_\beta^{(0)}} + \frac{\gamma_\kappa}{\Delta_\kappa^{(0)}}\right) C_{\alpha\beta\gamma}^{(0)} = -\frac{\delta_{\alpha\beta}}{4\pi^2 \sqrt{3}} \left(2\gamma_\alpha + \frac{3}{8\pi^2} \Delta_\alpha^{(0)}\right),$$ \hfill (48)

as already mentioned in the introduction.

This may be shown as follows. Let $\mathcal{K}$ be the length two Konishi operator and the set $\{\mathcal{O}_\alpha\}$ an arbitrary non-diagonal basis for the operators of length $\Delta^{(0)}$ that can be written in terms of attached vectors, namely

$$\mathcal{K} = \frac{1}{\sqrt{12}} \sum_i \text{Tr} \left(\phi^i \phi^i\right)$$ \hfill (49)

$$\mathcal{O}_\alpha = \text{Tr} \left(u_1^\alpha \cdot \phi \cdots u_\Delta^{(0)} \cdot \phi\right) \quad (\Delta^{(0)} > 2).$$ \hfill (50)

Let $Z_k \subset S_k$ denote the set of cyclic permutations of $(1,2,\ldots,k)$.

We choose the renormalization scheme $\varepsilon \to e\varepsilon$ in which only the 2-gons hold finite contributions

$$\langle u_1 \cdots u_2 \rangle_{1\text{-loop}} = t_{12}^2 \frac{\lambda}{8\pi^2} \left(\ln \frac{\varepsilon^2}{x_{12}} + 1\right) \left(\frac{1}{2} \cdots \cdots + 1\right)$$ \hfill (51)
while the 3-gons only contribute to the logarithmic terms. For the two-point functions we get

\[
\langle \mathcal{O}_\alpha(x_1) \mathcal{O}_\beta(x_2) \rangle = I_{12}^{(0)} \sum_{\sigma \in \mathcal{Z}_\Delta^{(0)}} \prod_{i=1}^{\Delta^{(0)}} u^\alpha_{\sigma(i)} u^\beta_{\sigma(i)} + \frac{\lambda}{8\pi^2} \left( \ln \frac{\varepsilon^2}{x_{12}^2} + 1 \right) 
\times \sum_{\tau \in \mathcal{Z}_\Delta^{(0)}} u^\alpha_{\tau(1)} u^\beta_{\tau(2)} u^\alpha_{\tau(2)} u^\beta_{\tau(1)} - u^\alpha_{\tau(1)} u^\beta_{\tau(2)}
\times u^\alpha_{\tau(2)} u^\beta_{\tau(1)} + \frac{1}{2} u^\alpha_{\tau(1)} u^\alpha_{\tau(2)} u^\beta_{\tau(1)} u^\beta_{\tau(2)}
\times \prod_{i=3}^{\Delta^{(0)}} u^\alpha_{\tau(i)} u^\beta_{\tau(i)}.
\] (52)

Now let \( \mathcal{D}_\alpha = M_{\alpha\beta} \mathcal{O}_\beta \) denote a diagonal basis of the length \( \Delta^{(0)} \) subspace. Then

\[
\langle \mathcal{D}_\alpha(x_1) \mathcal{D}_\beta(x_2) \rangle = \frac{1}{x_{12}^{2\Delta^{(0)}}} \left( \delta_{\alpha\beta} + \lambda g_{\alpha\beta} + \lambda \gamma_{\alpha} \delta_{\alpha\beta} \ln \frac{\varepsilon^2}{x_{12}^2} \right) = M_{\alpha\gamma} M_{\beta\delta} \langle \mathcal{O}_\gamma(x_1) \mathcal{O}_\delta(x_2) \rangle
\] (53)

from which we immediately get the condition for tree-level diagonality

\[
\sum_{\sigma \in \mathcal{Z}_\Delta^{(0)}} M_{\alpha\gamma} M_{\beta\delta} \prod_{i=1}^{\Delta^{(0)}} u^\gamma_{\sigma(i)} u^\delta_{\sigma(i)} = (2\pi)^{2\Delta^{(0)}} \delta_{\alpha\beta}.
\] (54)

Using this result we obtain

\[
\langle \mathcal{D}_\alpha(x_1) \mathcal{D}_\beta(x_2) \rangle = \frac{1}{x_{12}^{2\Delta^{(0)}}} \left( \delta_{\alpha\beta} + \frac{\lambda}{8\pi^2} \left( \ln \frac{\varepsilon^2}{x_{12}^2} + 1 \right) \right) \left[ \Delta^{(0)} \delta_{\alpha\beta} - \frac{1}{(2\pi)^{2\Delta^{(0)}}} \right]
\times \sum_{\sigma \in \mathcal{Z}_\Delta^{(0)}} \sum_{\tau \in \mathcal{Z}_\Delta^{(0)}} M_{\alpha\gamma} M_{\beta\delta} \left( u^\gamma_{\tau(1)} u^\delta_{\tau(2)} u^\gamma_{\tau(2)} u^\delta_{\tau(1)} - \frac{1}{2} u^\gamma_{\tau(1)} u^\gamma_{\tau(2)} u^\delta_{\tau(1)} u^\delta_{\tau(2)} \right) \prod_{i=3}^{\Delta^{(0)}} u^\gamma_{\tau(i)} u^\delta_{\tau(i)}
\] (55)

and thus the condition for one-loop diagonality

\[
(2\pi)^{2\Delta^{(0)}} \delta_{\alpha\beta} \left( \Delta^{(0)} - 8\pi^2 \gamma_{\alpha} \right) = \sum_{\sigma \in \mathcal{Z}_\Delta^{(0)}} \sum_{\tau \in \mathcal{Z}_\Delta^{(0)}} M_{\alpha\gamma} M_{\beta\delta} \left( u^\gamma_{\tau(1)} u^\delta_{\tau(2)} u^\gamma_{\tau(2)} u^\delta_{\tau(1)} - \frac{1}{2} u^\gamma_{\tau(1)} u^\gamma_{\tau(2)} u^\delta_{\tau(1)} u^\delta_{\tau(2)} \right) \prod_{i=3}^{\Delta^{(0)}} u^\gamma_{\tau(i)} u^\delta_{\tau(i)}
\] (56)
and

\[ g_α = γ_α. \]  

The three-point functions are

\[ \langle D_α(x_1) Dβ(x_2) K(x_3) \rangle = M_{αγ} M_{βδ} \langle O_α(x_1) O_β(x_2) K(x_3) \rangle \]

\[ = \frac{1}{(2π)^{2\Delta(0)+2} \sqrt{3} x_{12}^{2\Delta(0)+2} x_{13}^{2\Delta(0)-2} x_{23}^{2\Delta(0)-2}} \sum_{γ∈Z_Δ^{(0)}} \sum_{τ∈Z_Δ^{(0)}} M_{αγ} M_{βδ} \prod_{i=1}^{Δ(0)} u_γ^{τ(1)} \cdot u_τ^{δ(1)} \]

\[ + \frac{λ}{8π^{2}} \sum_{ρ∈Z_Δ^{(0)}-2} \left( u_γ^{τ(1)} \cdot u_τ^{δ(1)} u^{γ(2)}_{σρ(1)} \cdot u^{δ(2)}_{τρ(2)} - u^{γ(2)}_{σρ(1)} \cdot u^{δ(2)}_{τρ(2)} u^{γ(2)}_{σρ(2)} \cdot u^{δ(2)}_{τρ(1)} \right) \]

\[ \times u^{γ(0)}_{τ(Δ(0)-1)} \cdot u^{δ(0)}_{τ(Δ(0)-1)} + λ \times \text{logs} \]

\[ = \frac{1}{x_{12}^{2\Delta(0)-2} x_{13}^{2\Delta(0)-2} x_{23}^{2\Delta(0)-2}} \left( C_{αβK}^{(0)} + λ \tilde{C}_{αβK}^{(1)} + λ \times \text{logs} \right) \]  

and we obtain the tree-level structure constant

\[ C_{αβK}^{(0)} = \frac{1}{(2π)^{2\Delta(0)+2} \sqrt{3}} \sum_{γ∈Z_Δ^{(0)}} \sum_{τ∈Z_Δ^{(0)}} M_{αγ} M_{βδ} \prod_{i=1}^{Δ(0)} u_γ^{τ(1)} \cdot u_τ^{δ(1)} \]

\[ = \frac{Δ(0)}{(2π)^{2\Delta(0)+2} \sqrt{3}} \sum_{τ∈Z_Δ^{(0)}} M_{αγ} M_{βδ} \prod_{i=1}^{Δ(0)} u_γ^{τ(1)} \cdot u_τ^{δ(1)}, \]

where we omitted one sum over all permutations in the second line because the first sum already delivers all possible contractions.

Using equation (54) we get

\[ C_{αβK}^{(0)} = \frac{Δ(0)}{4π^2 \sqrt{3}} δ_{αβ}. \]  

The one-loop structure constant is

\[ \tilde{C}_{αβK}^{(1)} = \frac{1}{(2π)^{2\Delta(0)+4} \sqrt{12}} \sum_{γ∈Z_Δ^{(0)}} \sum_{τ∈Z_Δ^{(0)}} \sum_{ρ∈Z_Δ^{(0)}-2} M_{αγ} M_{βδ} \]

\[ × \prod_{i=1}^{Δ(0)-2} \left( u_γ^{τ(1)} \cdot u_τ^{δ(1)} \right) \times u^{γ(0)}_{τ(Δ(0)-1)} \cdot u^{δ(0)}_{τ(Δ(0)-1)} u^{γ(0)}_{τ(Δ(0))} \cdot u^{δ(0)}_{τ(Δ(0))} \]

18
\[-\left( u_{\sigma\rho(1)}^{\gamma} \cdot u_{\tau\rho(2)}^{\delta} u_{\sigma\rho(2)}^{\gamma} u_{\tau\rho(1)}^{\delta} - \frac{1}{2} u_{\sigma\rho(1)}^{\gamma} \cdot u_{\sigma\rho(2)}^{\gamma} u_{\tau\rho(1)}^{\delta} u_{\tau\rho(2)}^{\delta}\right) \]
\times \prod_{i=3}^{\Delta(0)-2} \left( u_{\sigma\rho(i)}^{\gamma} \cdot u_{\tau\rho(i)}^{\delta} \right) \times u_{\sigma(\Delta(0)-1)}^{\gamma} \cdot u_{\tau(\Delta(0)-1)}^{\delta} u_{\sigma(\Delta(0))}^{\gamma} \cdot u_{\tau(\Delta(0))}^{\delta} \right] \]
\begin{align*}
&= \frac{\delta_{\alpha\beta}}{(2\pi)^4 \sqrt{12}} \left[ (\Delta(0) - 2) \Delta(0) - (\Delta(0) - 2) (\Delta(0) - 8\pi^2 \gamma_{\alpha}) \right] \\
&= \frac{(\Delta(0) - 2) \gamma_{\alpha}}{4\pi^2 \sqrt{3}} \delta_{\alpha\beta}, \tag{61}
\end{align*}

where the sum over the $\rho$-permutations gives only a factor of $(\Delta(0) - 2)$ and we made use of equations (54) and (56) in the second step.

The renormalization scheme independent structure constants

\[ C^{(1)}_{\alpha\beta\gamma} = \tilde{C}^{(1)}_{\alpha\beta\gamma} - \frac{1}{2} C^{(0)}_{\alpha\beta\gamma} (g_{\alpha} + g_{\beta} + g_{\gamma}) \tag{62} \]

may now be written down using (60), (61) and (57) to find

\[ C^{(1)}_{\alpha\beta\kappa} = \tilde{C}^{(1)}_{\alpha\beta\kappa} - \frac{1}{2} C^{(0)}_{\alpha\beta\kappa} \left( \gamma_{\alpha} + \gamma_{\beta} + \frac{3}{4\pi^2} \right) = - \left( \frac{\gamma_{\alpha}}{\Delta_{\alpha}} + \frac{\gamma_{\beta}}{\Delta_{\beta}} + \frac{\gamma_{\kappa}}{\Delta_{\kappa}} \right) C^{(0)}_{\alpha\beta\kappa}. \tag{63} \]

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