Main purpose.
Initial value problem / geometric evolution under weak regularity
Coupling with compressible matter (shock waves)
Geometry with weak regularity (impulsive gravitational waves)
Global geometric structure

Outline.

▶ Elements from mathematical general relativity
▶ Manifolds with distributional curvature
▶ Global foliation for Einstein-Euler spacetimes with Gowdy-symmetry on $\mathbb{T}^3$
▶ Characteristic initial value problem for plane symmetric spacetimes with weak regularity
▶ Uniqueness of Schwarzschild-de Sitter spacetime
Elements from mathematical general relativity

\((M, g)\): time-oriented Lorentzian \((n + 1)\)-manifold with signature \((- , + , \ldots , +)\)

- Lorentzian metric: \(X \in T_p M\) is time-like, null, or space-like if \(g(X, X)\) is negative, zero, or positive.

- Time-like curve (observer), null curve (photon).

- Time-orientation: distinguish between future and past directions.
Causality.

- **Chronological future** \( \mathcal{I}^+(p) \)
  Set of points \( q \in M \) attainable from \( p \) by a future-oriented, time-like continuous curve (a trip) \( \gamma : (a, b) \to M \).

- **Causal future** \( \mathcal{J}^+(p) \)

- **Future domain of dependence** \( \mathcal{D}^+(S) \)
  Set of points \( p \in M \) such that every past-endless trip containing \( p \) meets the given set \( S \).

- **Future Cauchy hypersurface in** \( M \): \( \mathcal{D}^+(S) = M \)
Matter spacetime.

\((M, g)\) satisfying Einstein’s field equations

\[ G = \kappa \, T, \quad G_{\alpha\beta} = \kappa \, T_{\alpha\beta}. \]

- Einstein tensor
- Ricci tensor
- Scalar curvature

\[ G = Rc - \frac{R}{2} \, g \]

\[ Rc \quad R \]
The energy-momentum tensor satisfies the (contracted, second) Bianchi identity ($\delta$: formal adjoint of $d$)

$$\delta T = 0,$$

that is,

$$\nabla_\beta T^{\alpha\beta} = 0$$

with, for a perfect fluid,

\[
T = (\mu + p) \, u \otimes u + p \, g
\]

- Unit time-like velocity vector $u$
- Proper mass-energy density $\mu \geq 0$
- Pressure $p = p(\mu)$, satisfying the dominant energy condition: for every time-like vector $X$, the time-like energy flux $T(X, \cdot)$ is causal, with $T(X, X) \geq 0$. 
Formulation of the initial value problem.

Initial data set.

- Riemannian 3-manifold \((\mathcal{M}, \bar{g})\)
- Symmetric 2-covariant tensor field [2nd fund. form] \(\bar{k}\)
- Matter fields (energy density, current) [measured orthog.] \(\bar{\rho}, \bar{J}\)
- Einstein’s constraint equations [Gauss-Codazzi eq.]

\[
\bar{R} + (\text{tr}\bar{k})^2 - |\bar{k}|^2 = 16\pi \bar{\rho}, \quad \delta \bar{k} - d(\text{tr}\bar{k}) = 8 \pi \bar{J}^b.
\]
Initial value problem. Future globally hyperbolic development of the initial data set

- Lorentzian manifold satisfying Einstein equations $(M, g)$
- Foliation with normal 1-form $n$
- Embedding $\psi: \overline{M} \to \mathcal{H}_0 \subset M$
- Induced metric $\overline{g}$
- Second fundamental form $\overline{k}$
- Matter fields $\rho, J$

$$\rho := T(n, n) = T^{\alpha\beta} n_\alpha n_\beta, \quad J^b := T(n, \cdot) + T(n, n) n,$$

$\overline{\rho}, \overline{J}$ being their restrictions to $\mathcal{H}_0$ ($\overline{J}$ being tangent).
Huge literature on solving the Einstein’s field equations.

- Einstein constraint equations
  Bartnik, Choquet-Bruhat, Isenberg, etc.

- Vacuum or matter models / large data / small times:
  *Maximal globally hyperbolic* development of the initial data set.
  Choquet-Bruhat, Geroch, etc

- Vacuum / small data / geodesically complete
  Stability of the Minkowski space established by Christodoulou and Klainerman. Improved by Bieri.
  Friedrich, Andersson-Moncrief, Klainerman-Rodnianski,
  Choquet-Bruhat–Moncrief, Lindblad–Rodnianski.
- Models with symmetries / large data / Penrose conjecture
  - Vacuum \hspace{1cm} Moncrief, Isenberg, etc
  - Scalar field \hspace{1cm} Christodoulou, Dafermos, etc
  - Vlasov’s kinetic model \hspace{1cm} Andreasson, Rendall, etc
  - Euler equations of compressible fluids (weak solutions, shock / gravitational waves) \hspace{1cm} PLF, Rendall, Stewart

**Issues raised by physics:**
- Boundary of the maximal hyperbolic development
- Nature of “singularities” (coordinate or geometric)
- Penrose’s strong censorship conjecture (generic inextendibility)

**NEED:** optimal control of the geometry in terms of the curvature
Distributions on manifolds. \( M \): differential \( m \)-manifold.

- Canonical embedding \( f \in L^1_{\text{loc}}(M) \mapsto f \in \mathcal{D}'(M) \), via
  \[
  \langle f, \omega \rangle_{\mathcal{D}', \mathcal{D}} := \int_M f \omega, \quad \omega \in \mathcal{D}\Lambda^m(M).
  \]

- Space of scalar distributions \( \mathcal{D}'(M) \): dual of the space \( \mathcal{D}\Lambda^m(M) \) of all (compactly supported) \( m \)-form fields, or densities.

- Space of distribution densities \( \mathcal{D}'\Lambda^m(M) \): dual of the space \( \mathcal{D}(M) \) of compactly supported functions.
From Stokes formula, for all smooth vector fields $X$, functions $f$, and all $\omega \in \mathcal{D}\Lambda^m(M)$ one gets

$$\int_M (Xf) \omega = -\int_M f \mathcal{L}_X \omega.$$  

The derivative $XA$ by a smooth vector field of a scalar distribution $A$:

$$\left\langle XA, \omega \right\rangle_{\mathcal{D}' , \mathcal{D}} := -\left\langle A, \mathcal{L}_X \omega \right\rangle_{\mathcal{D}' , \mathcal{D}}, \quad \omega \in \mathcal{D}\Lambda^m(M).$$

Notation. $\mathcal{T}_q^p(M) := C^\infty T^p_q(M)$.

**Tensor distributions.**

Space of all tensor distributions $\mathcal{D}' T^p_q(M)$: all $C^\infty(M)$-multi-linear maps

$$A : \mathcal{T}_0^1(M) \times \ldots \times \mathcal{T}_0^1(M) \times \mathcal{T}_1^0(M) \times \ldots \times \mathcal{T}_1^0(M) \to \mathcal{D}'(M).$$

Embedding $A \in L^1_{loc} T^p_q(M) \subset \mathcal{D}' T^p_q(M)$:

$$\left\langle A(X_1), \ldots , X_q, \theta^{(1)}, \ldots , \theta^{(p)}) , \omega \right\rangle_{\mathcal{D}' , \mathcal{D}} := \int_M A(X_1), \ldots , X_q, \theta^{(1)}, \ldots , \theta^{(p)}) \omega.$$
Distributional connection.

- A distributional connection: \( \nabla : \mathfrak{X}_0(M) \times \mathfrak{X}_0(M) \to \mathcal{D}' T^1_0(M) \)
satisfies the linearity and Leibnitz properties for all smooth fields.

Connection with \( L^2 \) regularity.

- An \( L^2_{\text{loc}} \) connection (square-integrable):

\[
\nabla_X Y \in L^2_{\text{loc}} T^1_0(M) \quad \text{for all } X, Y \in \mathfrak{X}_0(M).
\]

- Its extension: \( \nabla : \mathfrak{X}_0(M) \times L^2_{\text{loc}} T^1_0(M) \to \mathcal{D}' T^1_0(M) \)

\[
\langle \nabla_X Y, \theta \rangle = X(\langle Y, \theta \rangle) - \langle Y, \nabla_X \theta \rangle
\]

\[
X \in \mathfrak{X}_0(M), \quad Y \in L^2_{\text{loc}} T^1_0(M), \quad \theta \in \mathcal{D} T^0_1(M).
\]
Sobolev space: \( H^1 \): functions with square-integrable derivatives
\( H^{-1}_{\text{loc}} \): dual space (distributions)

**Definition (Notion of distributional curvature).**

Let \( \nabla \) be an \( L^2_{\text{loc}} \) connection on a differentiable manifold \( M \).

- The Riemann tensor

\[
\mathbf{Rm} : \mathcal{T}^1_0(M) \times \mathcal{T}^1_0(M) \times \mathcal{T}^1_0(M) \rightarrow D' T^1_0(M),
\]

\[
\langle \mathbf{Rm}(X, Y)Z, \theta \rangle = X \langle \nabla_Y Z, \theta \rangle - Y \langle \nabla_X Z, \theta \rangle
- \langle \nabla_Y Z, \nabla_X \theta \rangle + \langle \nabla_X Z, \nabla_Y \theta \rangle - \langle [X, Y]Z, \theta \rangle
\]

is well-defined as a distribution in \( H^{-1}_{\text{loc}} \).

- The Ricci tensor \( \mathbf{Ric} \) is similarly well-defined as a distribution in \( H^{-1}_{\text{loc}} \).
Proposition (Stability under $L^2_{\text{loc}}$ convergence).

Let $\nabla^{(n)}$ be a sequence of $L^2_{\text{loc}}$ connections on $M$, converging in the sense

$$\nabla_X^{(n)} Y \to \nabla_X^{(\infty)} Y \quad \text{in } L^2_{\text{loc}}$$

for all $X, Y \in \mathfrak{X}^1_0(M)$. Then, the distributional Riemann and Ricci tensors $\mathbf{Rm}^{(n)}$ and $\mathbf{Ric}^{(n)}$ of the connections converge in $\mathcal{D}'$ to the distributional curvature tensors of the limiting connection

$$\mathbf{Rm}^{(n)} \to \mathbf{Rm}^{(\infty)}, \quad \mathbf{Ric}^{(n)} \to \mathbf{Ric}^{(\infty)}.$$
**Distributional metrics.** The equation $\nabla g = 0$ is meaningless since non-smooth connections do not act on non-smooth tensors.

**Definition**

The *distributional Levi-Cevita connection* of a distributional metric $g$ is the operator

$$\nabla^b : (X, Y) \in \mathfrak{X}_0^1(M) \times \mathfrak{X}_0^1(M) \mapsto \nabla^b_X Y \in D' T^0_1(M),$$

defined by the "dual version" of Koszul formula

$$\langle \nabla^b_X Y, Z \rangle := \frac{1}{2} \left( X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\
- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \right).$$

In the weak sense and for all $X, Y, Z \in \mathfrak{X}_0^1(M)$

$$\nabla^b_X Y - \nabla^b_Y X - [X, Y]^b = 0$$

$$X(g(Y, Z)) - \langle \nabla^b_X Y, Z \rangle - \langle Y, \nabla^b_X Z \rangle = 0.$$
Proposition (Stability under distributional convergence).

Let $g^{(n)}$ be a sequence of distributional metrics converging in the sense

$$g^{(n)} \to g^{(\infty)}$$

in the distribution sense $\mathcal{D}'$.

Then the distributional Levi-Cevita connection $\nabla^b_{-}^{(n)}$ associated with $g^{(n)}$ converges in $\mathcal{D}'$ to the connection $\nabla^b_{-}^{(\infty)}$ associated with $g^{(\infty)}$, i.e.

$$\langle \nabla^b_{-}^{(n)} Y, Z \rangle \to \langle \nabla^b_{-}^{(\infty)} Y, Z \rangle$$

in $\mathcal{D}'(\mathcal{M})$

for all $X, Y, Z \in \mathcal{K}^1_0(\mathcal{M})$. 

Metrics with $H^1$ regularity.

**Theorem**

If $\nabla$ is the Levi-Cevita connection of a metric $g$ of class $H^1_{\text{loc}} \cap L^\infty_{\text{loc}}$ which is also uniformly non-degenerate

$$|\det(g)| \geq c > 0,$$

then it is of class $L^2_{\text{loc}}$ and, moreover, its scalar curvature ($E^{(\alpha)}$ being a local frame)

$$R = g^{\alpha\beta} \text{Ric}(E^{(\alpha)}, E^{(\beta)})$$

is well-defined in a weak sense as a distribution in $H^{-1}_{\text{loc}}$. 
Proof. Since $H^1_{\text{loc}} \cap L^\infty_{\text{loc}}$ is an algebra and $g$ is uniformly non-degenerate, it follows that $g^{\alpha\beta} \in H^1_{\text{loc}} \cap L^\infty_{\text{loc}}(M)$. Moreover, one has

$$\text{Ric}(E(\alpha), E(\beta))$$

$$= \langle E^{(\sigma)}, Rm(E(\alpha), E(\sigma))E(\beta) \rangle$$

$$= A_{\alpha\beta} - B_{\alpha\beta}$$

$$- \langle \nabla E(\alpha) E^{(\sigma)}, \nabla E^{(\sigma)} E(\beta) \rangle + \langle \nabla E^{(\sigma)} E(\alpha), \nabla E(\beta) \rangle - \langle E^{(\sigma)}, \nabla [E(\alpha), E(\sigma)] E(\beta) \rangle.$$  

Since the last three terms belong to $L^1_{\text{loc}}(M)$, we only need to define the product of $g^{\alpha\beta}$ with the distributions

$$A_{\alpha\beta} := E(\alpha)(\langle E^{(\sigma)}, \nabla E^{(\sigma)} E(\beta) \rangle), \quad B_{\alpha\beta} := E(\sigma)(\langle E^{(\sigma)}, \nabla E(\alpha) E(\beta) \rangle).$$

This is done by letting

$$g^{\alpha\beta} A_{\alpha\beta}$$

$$:= E(\alpha)(g^{\alpha\beta} \langle E^{(\sigma)}, \nabla E^{(\sigma)} E(\beta) \rangle) - (E(\alpha)g^{\alpha\beta}) \langle E^{(\sigma)}, \nabla E^{(\sigma)} E(\beta) \rangle,$$

which clearly is a distribution, and similarly for $g^{\alpha\beta} B_{\alpha\beta}$. 


Proposition (Stability under $H^1_{\text{loc}}$ convergence).

Let $g^{(n)}$ be a sequence of $H^1_{\text{loc}}$ metric tensors converging in the sense

$$g^{(n)} \to g^{(\infty)} \quad \text{in } H^1_{\text{loc}}$$

and such that the inverses $g^{-1}_{(n)}$ converge locally in $L^\infty_{\text{loc}}$ to $g^{-1}_{(\infty)}$.

- The Levi-Cevita connections $\nabla^{(n)}$ associated with $g^{(n)}$ are of class $L^2_{\text{loc}}$ and converge in $L^2_{\text{loc}}$ to the connection $\nabla^{(\infty)}$ of the limit $g^{(\infty)}$.

- The distributional Riemann, Ricci, and scalar curvature tensors

$$Rm^{(n)}, \quad Ric^{(n)}, \quad R^{(n)}$$

of the connections $\nabla^{(n)}$ converge in $\mathcal{D}'$ to the limiting curvature tensors of $\nabla^{(\infty)}$

$$Rm^{(\infty)}, \quad Ric^{(\infty)}, \quad R^{(\infty)}.$$
Consider a connection $\nabla$ of class $L^2_{\text{loc}}(M) \cap W^{1,p}_{\text{loc}}(M^\pm)$ having a jump discontinuity along a smooth hypersurface $\mathcal{H} \subset M$

$$M = M^- \cup M^+, \quad M^- \cap M^+ = \mathcal{H}.$$ 

The corresponding operators $\nabla^\pm$ have well-defined (distributional) Riemann and Ricci curvatures $Rm^\pm$ and $\text{Ric}^\pm$ defined in $M^\pm$.

Belong to $L^1_{\text{loc}}(M^\pm)$, at least, and to $L^p_{\text{loc}}(M^\pm)$ if $p \geq m/2$.

**Question.** Determine the distributional curvature of $\nabla$. 


Dirac measure.

- Dirac measure supported by $\mathcal{H} \subset M$: the 1-form distribution $\delta_\mathcal{H} \in \mathcal{D}'_1(M)$

\[ X \in \mathfrak{X}_0^1(M) \mapsto \langle \delta_\mathcal{H}, X \rangle \in \mathcal{D}'(M), \]

\[ \langle \delta_\mathcal{H}, X \rangle, \omega \rightharpoonup \mathcal{D}', \mathcal{D} = \int_\mathcal{H} i_X \omega, \quad \omega \in \mathcal{D} \Lambda^m(M). \]

- Write $[A]_\mathcal{H} := A^+ - A^-$ for the jump of a tensor field $A$ across $\mathcal{H}$, and write the “regular part” as

\[ A^{\text{reg}} := \begin{cases} A^+ & \text{in } M^+, \\ A^- & \text{in } M^- \end{cases}. \]

Remarks. 1. The distributional derivative $\nabla V$ of a vector field is $\nabla_X V := (\nabla_X V)^{\text{reg}} + [V]_\mathcal{H} \langle \delta_\mathcal{H}, X \rangle$.

2. $\langle \delta_\mathcal{H}, X \rangle$ depends on $X$ only via its restriction to $\mathcal{H}$, so $\delta_\mathcal{H}$ can be applied to fields defined on $\mathcal{H}$.

3. If $X$ is tangent to the hypersurface ($X \in \mathfrak{X}_0^1(\mathcal{H})$), then $\langle \delta_\mathcal{H}, X \rangle = 0$. 
Jump relations associated with a singular connection

Choose an adapted local frame, as follows:
\( E_{(i)}, \ i = 1, \ldots, m - 1 \): frame on \( \mathcal{H} \).
\( E_{(\alpha)}, \ \alpha = 1, \ldots, m \): frame on the manifold \( M \).

\( E^{(\alpha)} \): dual frame of \( 1 \)-form fields.

---

**Proposition (Curvature of an \( L^2_{\text{loc}} \) connection).**

- The distributional Riemann curvature:
  \[
  Rm(X, Y)Z = (Rm(X, Y)Z)^{\text{reg}} + [\nabla_Y Z]_\mathcal{H} \langle \delta_\mathcal{H}, X \rangle \\
  \quad - [\nabla_X Z]_\mathcal{H} \langle \delta_\mathcal{H}, Y \rangle.
  \]

- The distributional Ricci curvature:
  \[
  Ric(X, Y) = (Ric(X, Y))^{\text{reg}} + [\langle E^{(\alpha)}, \nabla_{E^{(\alpha)}} Y \rangle]_\mathcal{H} \langle \delta_\mathcal{H}, X \rangle \\
  \quad - [\langle E^{(m)}, \nabla_X Y \rangle]_\mathcal{H} \langle \delta_\mathcal{H}, E^{(m)} \rangle.
  \]
Properties of singular parts of curvature tensors.

- The singular part of the Riemann tensor vanishes if and only if the connection $\nabla$ is continuous across $\mathcal{H}$.

- The singular part of the Ricci tensor vanishes if and only if the components $\langle E^{(m)}, \nabla_{E^i} X \rangle$ and $\langle E^{(j)}, \nabla_{E^{(j)}} X \rangle$ are continuous across $\mathcal{H}$ for all fields $X \in \mathcal{T}^1_0(M)$.

- Suppose $\nabla$ is the $L^2$ Levi-Cevita connection of a uniformly non-degenerate metric $g$ and suppose that $g$ is $W^{2,p}_{\text{loc}}$ on each side of some hypersurface $\mathcal{H}$, then

$$R := R^{\text{reg}} + \left[ \langle g^{m\beta} E^{(j)} - g^{j\beta} E^{(m)}, \nabla_{E^{(j)}} E^{(\beta)} \rangle \right]_{\mathcal{H}} \langle \delta_{\mathcal{H}}, E^{(m)} \rangle.$$

From this, one recovers the classical junction conditions.
Global foliation of Einstein-Euler spacetimes with Gowdy-symmetry on $T^3$

P.G. LeFloch and A.D. Rendall, Preprint ArXiv 1004.0427.

Einstein spacetimes with matter $(M,g)$:

$$G^{\alpha\beta} := R^{\alpha\beta} - \frac{R}{2} g^{\alpha\beta} = T^{\alpha\beta}$$

- Compressible perfect fluids: $T^{\alpha\beta} = (\mu + p) u^{\alpha} u^{\beta} + p g^{\alpha\beta}$

- Global foliation and geometric structure.

- Gowdy symmetry assumption.
  Inhomogeneous cosmology with a “big bang” or “big crunch”.
  Gravitational waves. Relevant to develop quantum gravity models.

**Weak regularity.**

- Shock waves propagating at (about) the speed of sound.

- Impulsive gravitational waves propagating at the speed of light.
Gowdy symmetry on $T^3$.

- $(3 + 1)$-dim. Lorentzian manifold $(M, g)$ with topology $I \times T^3$.

- Admits the Lie group $T^2$ as an isometry group acting on $T^3$: generated by two (linearly independent) vector fields $X, Y$

  \[ \mathcal{L}_X g = \mathcal{L}_Y g = 0 \]

  Commuting: \[ [X, Y] = 0 \]

  Spacelike: \[ g(X, X) > 0, \quad g(Y, Y) > 0 \]

  and, in addition,

  **Orthogonally transitive:**

  the distribution of 2-planes \( \left( \text{Vect}(X, Y) \right)^\perp \) is Frobenius integrable.

In the vacuum, the Gowdy spacetimes (1974).
The orthogonality condition is made explicit as follows:

- There exist two vectors $Z, T$ orthogonal to $X, Y$ so that
  \[ \varepsilon(X, Y, \cdot, \cdot)^\# = Z \otimes T - T \otimes Z, \]
  where $\varepsilon$ is the canonical volume form.

- The distribution of covectors $g(X, \cdot), g(Y, \cdot)$ is Frobenius integrable if and only if $Z, T, [Z, T]$ are linearly dependent, that is
  \[ \varepsilon(Z, T, [Z, T], \cdot) = 0. \]

- After some calculations, the condition is found to be equivalent to have vanishing “twist constants” (identified by Chrusciel)
  \[ \langle \varepsilon(X, Y, \cdot, \cdot), \nabla X \rangle_g = \langle \varepsilon(X, Y, \cdot, \cdot), \nabla Y \rangle_g = 0, \]
  \[ \varepsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta = \varepsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma Y^\delta = 0. \]

The vacuum Einstein equations imply that the above (twist) quantities are constants.
Gowdy symmetric initial data set on $T^3$.

- **Geometry:**
  A Riemannian 3-manifold $(T^3, \bar{g}, \bar{k})$ together with a symmetric tensor field $\bar{k}$ (second fundamental form).

- **Matter content:**
  $\bar{\rho}$ (energy density, a scalar field, mass-energy measured by an observer moving orthogonally to the foliation slices) and $\bar{J}$ (momentum, a vector field) on $T^3$. (Equivalent to prescribing $\bar{\mu}, \bar{u}$.)

- **Einstein constraint equations** (Gauss-Codazzi like equations).

- **Invariance under Gowdy symmetry on $T^3$.**
Weakly regular, Gowdy symmetric $T^3$-spacetimes

- Metric in $H^1(\Sigma)$ (squared-integrable, first-order derivatives) on every spacelike slice $\Sigma$.

- Fluid variables $\rho \geq 0$ (scalar field) and $J$ (vector field) in $L^1(\Sigma)$.

- Weak solution to the Einstein equations in the distribution sense:

$$G^{\alpha\beta} := R^{\alpha\beta} - \frac{R}{2}g^{\alpha\beta} = T^{\alpha\beta}$$

  curvature well-defined as a distribution in $H^{-1}(\Sigma)$

- Entropy condition associated with the Euler equations.

Remark. Even lower regularity sufficient. (see below).
Theorem (Global geometry of Gowdy-symmetric matter spacetimes. LeFloch & Rendall, 2010).

Given any weakly regular, Gowdy-symmetric $T^3$-initial data set $(\bar{g}, \bar{k}, \bar{\rho}, \bar{J})$:

- **Existence result:**
  A weakly regular, Gowdy-symmetric $T^3$-spacetime $(\mathcal{M}, g, \rho, J)$ which is a future Cauchy development of $(\bar{g}, \bar{k}, \bar{\rho}, \bar{J})$ globally covered by a single chart in areal coordinates:
  Foliation by spacelike hypersurfaces such that the time variable $t$ coincides with (plus or minus) the area $R$ of the (two-dimensional spacelike) orbits of the $T^2$ isometry group.

- **Global structure:**
  Distinguish whether $R$ is increasing or decreasing toward the future

  $\text{expanding spacetime: } t = R \in [c_0, +\infty), \quad c_0 > 0,$
  $\text{contracting spacetime: } t = -R \in [c_0, c_1), \quad c_0 < c_1 \leq 0.$
Earlier works on Gowdy symmetry.

- Vacuum case: Berger, Chrusciel, Eardley, Isenberg, Moncrief, Ringstrom

- Kinetic matter models: Rendall, Andreasson, Dafermos

- Fluid models:
  - Christodoulou (general relativistic fluids)
  - LeFloch & Stewart (local-in-time result)
  - Rendall & Stahl (shock formation)

Present result / new difficulties.

- Metric coefficients need not be smooth:
  - functional space $H^1$
  - impulsive gravitational waves (i.e. curvature singularities)

- Fluid variables with arbitrary large amplitude:
  - shock waves and curvature discontinuities
  - solutions contain vacuum states ($\rho \geq 0$)

- Compactness framework:
  - weak convergence techniques, Young measures
  - mathematical entropies for the Euler equations
Other issues.

– On the boundary of the future Cauchy development:
  
  ▶ $c_1 = 0$ ? Explicit counter-examples for the Einstein-Euler equations (probably unstable).
  
  ▶ Penrose’s strong cosmic censorship (in-extendibility of the future development) ?
    
    ▶ Expanding case: OK.
    ▶ Contracting case: known in the vacuum case only, and for generic initial data (Ringström, 2008).

– Generalizations:
  
  ▶ Cosmological constant $\Lambda$
  
  ▶ Existence of a CMC foliation
Choice of coordinates (Geroch, Gowdy).

- Quotient manifold \( \tilde{M} := M / T^2 \).
  - Since \((X, Y) =: (X_1, X_2)\) are linearly independent and spacelike
    \[
    R^2 := \det(\lambda_{ab}) > 0, \quad \lambda_{ab} := g(X_a, X_b).
    \]
  - Killing fields: The area \( R > 0 \) is a constant on each orbit.
  - Lorentzian metric on \( \tilde{M} \):
    \[
    h := g - \lambda^{ab} X_a \otimes X_b.
    \]

- Choice of coordinates \((t, \theta, x, y)\).
  - \(x, y\) span the orbits of symmetry
  - The metric coefficients depend on \((t, \theta)\), only, and are periodic in \(\theta\).
  - Key property: the gradient \( \nabla R \) is timelike.
Conformal coordinates \((\tau, \theta, x, y)\):

\[
  g = e^{2(\eta-U)} (-d\tau^2 + d\theta^2) + e^{2U} (dx + A\,dy)^2 + e^{-2U} R^2\,dy^2,
\]

where \(\eta, U, A, R\) with \(R > 0\) depend on \(\tau, \theta\), and \(R\) represents the area of the orbits of the \(T^2\) symmetry group.

Coupled system of 8 PDE’s:

- Evolution equations: 4 second-order, semi-linear wave equations for \(U, A, \eta, R\).

- Constraint equations: 2 nonlinear differential equations for \(\nabla R\).

- Euler equations: 2 nonlinear hyperbolic equations for \(\rho, J^1\).
Areal coordinates \((t, \theta, x, y)\):

\[ g = e^{2(\nu-U)}(-dt^2 + \alpha^{-1}d\theta^2) + e^{2U}(dx + A\,dy)^2 + e^{-2U}t^2\,dy^2, \]

in which \(U, A, \nu, \alpha\) depend on \(t, \theta\), and \(t = R\) coincides with the area of the orbits of the \(T^2\) symmetry group.

Coupled system of 6 PDE’s:

- Evolution: second-order nonlinear wave equations for \(U, A, \nu\).
- Constraint: nonlinear differential equation for \(\alpha\).
- Euler equations: 2 nonlinear hyperbolic equations for \(\rho, J^1\), while \(J^2 = J^3 = 0\).
  Equivalently, a (normalized) component \(V\) of the velocity and the energy density \(\mu\).
Evolution equations for $U, A, \nu$

$$(t \alpha^{-1/2} U_t)_t - (t \alpha^{1/2} U_\theta)_\theta = \frac{e^{4U}}{2t\alpha^{1/2}} (A_t^2 - \alpha A_\theta^2) + t \alpha^{1/2} \Pi_1$$

$$(t^{-1} \alpha^{-1/2} A_t)_t - (t^{-1} \alpha^{1/2} A_\theta)_\theta = -\frac{4}{t \alpha^{1/2}} (U_t A_t - \alpha U_\theta A_\theta) + \alpha^{1/2} \Pi_2$$

and similarly for $\nu$, where $\Pi_1, \Pi_2$ depend on the geometric and fluid variables.

Constraint equation for $\alpha$

$$\alpha(t, \theta) = \overline{\alpha}(\theta) \exp \left( -2(1 - k^2) \int_0^t t' \left( e^{2(\nu - U) \mu} \right)(t', \theta) \, dt' \right)$$

with prescribed initial value $\overline{\alpha} > 0$. 
Euler equations in general geometry.

\[
\begin{align*}
\left( a_1 (\mu + (\mu + p(\mu)) \frac{V^2}{1 - V^2}) \right)_t + \left( a_2 (\mu + p(\mu)) \frac{V}{1 - V^2} \right)_\theta &= \Sigma_1 \\
\left( a_3 (\mu + p(\mu)) \frac{V}{1 - V^2} \right)_t + \left( a_4 ((\mu + p(\mu)) \frac{V^2}{1 - V^2} + p(\mu)) \right)_\theta &= \Sigma_2
\end{align*}
\]

- Fluid unknowns $\mu \geq 0$ and $V \in (-1, 1)$
- $a_1, a_2, \ldots$ depend on the geometric variables
- $\Sigma_1, \Sigma_2$ depend on both the geometric and the fluid variables
A weakly regular solution to the Einstein-Euler equations: measurable functions $U, A, \nu, \alpha, M, V$ defined on $I \times S^1 := [c_0, \infty) \times S^1$:

- **Regularity:**
  \[
  U_t, A_t, U_\theta, A_\theta \in L^\infty_{\text{loc}}(I, L^2(S^1)), \\
  \nu_t, \nu_\theta \in L^\infty_{\text{loc}}(I, L^1(S^1)), \\
  \alpha > 0 \text{ with } \alpha, \alpha^{-1} \in L^\infty_{\text{loc}}(I, L^\infty(S^1)), \\
  M \in L^\infty_{\text{loc}}(I, L^1(S^1)), \quad M \geq 0, \quad |V| \leq 1.
  \]

- **Solutions in the distribution sense:** evolution equations, constraint equation, and Euler equations.

- **Entropy inequalities:**
  \[
  \nabla_\alpha \mathcal{F}^\alpha(M, V) \leq \mathcal{G}(M, V)
  \]
  for all convex weak entropy flux $\mathcal{F}^\alpha$ defining conservation laws to the relativistic Euler equations.
Area of the orbits of the symmetry group.

Evolution and constraint equations for the area function \( R \)

\[
\Box R = Re^{2(\tau-U)}(\rho - P_1),
\]

\[
\frac{R_{\theta\theta}}{R} = \frac{1}{R} (\eta_\tau R_\tau + \eta_\theta R_\theta) - (U_\tau^2 + U_\theta^2) - \frac{e^{4U}}{4R^2} (A_\tau^2 + A_\theta^2) - e^{2(\eta-U)} \rho,
\]

\[
\frac{R_{\tau\theta}}{R} = \frac{1}{R} (\eta_\tau R_\theta + \eta_\theta R_\tau) - 2 U_\tau U_\theta - \frac{e^{4U}}{2R^2} A_\tau A_\theta + e^{2(\eta-U)} J^1,
\]

where \( P_1 \) is a nonlinear expression in the fluid (and metric) variables.
Definition

Weak solutions to the Einstein-Euler equations in the class

\[ U_\tau, A_\tau \in L^\infty_{\text{loc}}(L^2(S^1)), \quad U, A \in L^\infty_{\text{loc}}(W^{1,2}(S^1)), \]
\[ \eta_\tau \in L^\infty_{\text{loc}}(L^1(S^1)), \quad \eta \in L^\infty_{\text{loc}}(W^{1,1}(S^1)), \]
\[ R_\tau \in L^\infty_{\text{loc}}(L^\infty(S^1)), \quad R \in L^\infty_{\text{loc}}(W^{1,\infty}(S^1)), \]
\[ \rho, J^1 \in L^\infty_{\text{loc}}(L^1(S^1)). \]
Proposition

- The functions $R_\tau, R_\theta$ are continuous in both variables and

  Contracting case (a): $R_\tau < -|R_\theta|$,  
  Expanding case (b): $R_\tau > |R_\theta|$,  
  Case (c): $R$ is constant and the spacetime is flat and vacuum.

- In Cases (a) and (b) the gradient of the function $R$ is timelike.

- In Case (a) the components of the gradient $\nabla R$ are uniformly controled in $L^\infty$:

  $$ |R_\theta| \leq -R_\tau \leq 2 \sup_{S^1} |R_\tau|. $$

- Second-order derivatives $R_{\theta t}$ and $R_{\theta\theta}$ uniformly controled

  $$ \sup_{\tau \geq \tau_0} \int_{S^1} (|R_{\tau\theta}| + |R_{\theta\theta}|)(\tau, \cdot) \, d\theta \lesssim \int_{S^1} (|R_\tau| + |R_\theta|)(\tau_0, \cdot) \, d\theta + \int_{S^1} \rho(\tau_0, \cdot) \, d\theta. $$
Characteristic initial value problem for plane symmetric spacetimes with weak regularity

P.G. LeFloch and J.M. Stewart, Preprint ArXiv 1004.2343v1.

Class of spacetimes

- “Polarized” Gowdy spacetimes (hypersurface orthogonal)
- Compressible perfect fluid

Objective

- Initial data set: characteristic value problem with data prescribed on two null hypersurfaces intersecting along a 2-plane.
- Spacetimes with weak regularity.
- Penrose’s strong censorship conjecture for weakly regular spacetimes.
Characteristic value problems. For general spacetimes Friedrich, Stewart, Dossa, Cagnac, Rendall, Christodoulou, Choquet-Bruhat, Chrusciel.

Recent works on vacuum, general but more regular spacetimes: Christodoulou (generic formation of trapped surfaces), Klainerman, Rodnianski.

Matter model

- Perfect fluid with pressure equal to its mass-energy density (null fluid), \( p = \mu \), with energy-momentum tensor

\[
T^{\alpha\beta} = 2\mu u^\alpha u^\beta + \mu g^{\alpha\beta}.
\]

- The sound speed coincides with the light speed.

- The (second) contracted Bianchi identity yields the Euler equations

\[
(u^\alpha \nabla_\alpha \mu) u^\beta + \mu (\nabla_\alpha u^\alpha) u^\beta + \mu u^\alpha \nabla_\alpha u^\beta - \frac{1}{2} \nabla^\beta \mu = 0.
\]
**Irrotational fluids.** (Taub, Rodnianski-Speck)

- Assume the existence of a (scalar) potential $\psi$ with timelike gradient (using the signature $(+,−,−,−)$ for the metric)

\[
\nabla_\beta \psi \nabla^\beta \psi > 0, \quad u^\alpha = \frac{\nabla^\alpha \psi}{\sqrt{\nabla_\beta \psi \nabla^\beta}}.
\]

- After normalization (by replacing $\psi$ with $F(\psi)$), the Euler equations imply the so-called Bernouilli’s equation

\[
\mu = \nabla^\alpha \psi \nabla_\alpha \psi = 4e^{-2a} \psi_u \psi_v
\]

(in characteristic variables, defined below), which yields the energy density in terms of the potential.

- A single matter equation for the scalar field $\psi$. 
Characteristic coordinates.
Each Killing field is assumed to be hypersurface orthogonal ("plane symmetry")

\[
g = e^{2a} (-dt^2 + dx^2) + e^{2b} (e^{2c} dy^2 + e^{-2c} dz^2) \\
= -e^{2a} dudv + e^{2b} (e^{2c} dy^2 + e^{-2c} dz^2)
\]

\(a, b, c\) depend upon the characteristic variables

\[u = t - x, \quad v = t + x.
\]

Components of the Einstein tensor:

\[G_{00} = 2 \left(-2 a_u b_u + b_{uu} + b_u^2 + c_u^2\right)
\]
\[G_{01} = 2 \left(-b_{uv} - 2 b_u b_v\right)
\]
\[G_{11} = 2 \left(-2 a_v b_v + b_{vv} + b_v^2 + c_v^2\right)
\]
\[G_{22} = 4e^{-2a+2b+2c} \left(a_{uv} + b_{uv} + b_u b_v - b_u c_v - b_v c_u - c_{uv} + c_u c_v\right)
\]
\[G_{33} = 4e^{-2a+2b-2c} \left(a_{uv} + b_{uv} + b_u b_v + b_u c_v + b_v c_u + c_{uv} + c_u c_v\right).
\]
Essential field equations in adapted coordinates.

- Gauge freedom: solving a decoupled equation for $b$ allows us to choose coordinates so that

$$e^{2b} = \frac{1}{2}|u + v|,$$

and the region of interest is:

$$\{u + v < 0\}$$

the hypersurface $u + v = 0$ being a (physical or coordinate) singularity.

- Evolution equations: two singular wave equations of Euler-Poisson-Darboux type for $\psi, c$

- Constraint equations: two ODE’s for $a$
Characteristic initial value problem.

- To each \((u_0, v_0)\) with \(u_0 + v_0 < 0\), one associates a 2–plane \(P_0 = P_0(u_0, v_0)\) and one considers the spacetime region \(\mathcal{I}_0 = \mathcal{I}_0(u_0, v_0)\) limited by the null hypersurfaces

\[
\mathcal{N}_0 = \mathcal{N}(v_0) := \{v = v_0\}, \quad \overline{\mathcal{N}}_0 = \overline{\mathcal{N}}(u_0) := \{u = u_0\},
\]

and the (spacelike, coordinate singularity) hypersurface

\[
\mathcal{B}_0 := \{u + v = 0\}.
\]

- Double-null foliation:

\[
\mathcal{N}(v) := \{(u', v') / v' = v\}, \quad \overline{\mathcal{N}}(u) := \{(u', v') / u' = u\}.
\]

- Prescribe initial data for the geometry and the matter on \(\mathcal{N}_0 \cup \overline{\mathcal{N}}_0\), which may have a jump discontinuity on the 2-plane \(P_0\) and have weak regularity.
Analysis in adapted coordinates. Notation for functions $f = f(u, v)$

$$
\bar{M}_{u, v}^{u_0, v_0}[f] := \sup_{v_0 < v' < v} \left( \int_{u_0}^{u} |f(\cdot, v')|^2 \, du' \right)^{1/2}
$$

$$
\bar{M}_{u_0, v_0}^{u_0, v_0}[f] := \sup_{u_0 < u' < u} \left( \int_{v_0}^{v} |f(u', \cdot)|^2 \, dv' \right)^{1/2}.
$$

**Notion of $H^1$ regular spacetime**

A $H^1$ regular spacetime is determined in characteristic coordinates $(u, v)$ by the continuous metric coefficients $a, b, c$ and fluid potential $\psi$:

- $a$ satisfies the normalization $a(u_0, v_0) = 0$ on the two-plane $P_0$.

- $b$ is given by $e^{2b(u, v)} = \frac{1}{2} |u + v|$.

- The (semi-)norms $\bar{M}_{u_0, v_0}^{u, v}[a_u^{1/2}, c_u, \psi_u]$ and $\bar{M}_{u, v}^{u_0, v_0}[a_v^{1/2}, c_v, \psi_v]$ are finite for every $(u, v)$ satisfying $u + v < 0$ and $u_0 < u, v_0 < v$.

- The Einstein field equations hold in the sense of distributions.
Proposition (Existence theory in a characteristic rectangle).

\( \mathcal{D} = \mathcal{D}(u_0, v_0; u, v) \): a characteristic rectangle not intersecting the singularity hypersurface. Let \( \psi, c \) and \( \overline{\psi}, \overline{c} \) be continuous and defined on

\[ \mathcal{N}_0 = \{ u_0 < u' < u; \quad v' = v_0 \}, \quad \overline{\mathcal{N}}_0 = \{ u' = u_0; \quad v_0 < u' < u \}, \]

with finite semi-norms \( M^{u_0, v_0}_{u, \psi}[c, \psi] \) and \( M^{u_0, v_0}_{u, \psi}[\overline{c}, \overline{\psi}] \).

Then, there exists an \( H^1 \) regular spacetime determined by functions \( a, b, c, \psi : \mathcal{D} \to \mathbb{R} \) satisfying the Einstein field equations and

\[
\begin{align*}
c(\cdot, v_0) &= c, & \psi(\cdot, v_0) &= \psi & \text{on } \mathcal{N}_0, \\
c(u_0, \cdot) &= \overline{c}, & \psi(u_0, \cdot) &= \overline{\psi} & \text{on } \overline{\mathcal{N}}_0.
\end{align*}
\]

Second-order derivatives of \( \psi, c \) defined in the sense of distributions, only.
Final statement in geometric form.

**Definition**

An **initial data with weak regularity**: Let \((\mathcal{N}_0, e^a dUdydz)\) and \((\bar{\mathcal{N}}_0, e^{\bar{a}} dVdydz)\) be two plane symmetric 3-manifolds with boundaries identified along a two-plane \(P_0 := \{U = U_0, V = V_0\}\):

\[\mathcal{N}_0 := \{U > U_0\}, \quad \bar{\mathcal{N}}_0 := \{V > V_0\}\]

- \(a, \bar{a}\) belong to the Sobolev space \(W^{1,1}\) and are normalized:
  \[a|_{P_0} = \bar{a}|_{P_0} = 0\]

- \(\psi_0, \Phi_{00}\) and \(\overline{\psi}_4, \overline{\Phi}_{22}\) are (plane-symmetric) functions defined on \(\mathcal{N}\) and \(\bar{\mathcal{N}}\), respectively, with
  \[0 \leq \Phi_{00} \in L^1(\mathcal{N}_0), \quad 0 \leq \overline{\Phi}_{22} \in L^1(\bar{\mathcal{N}}_0)\]
  \[\psi_0 \in H^{-1}(\mathcal{N}_0), \quad \overline{\psi}_4 \in H^{-1}(\bar{\mathcal{N}}_0)\]

- The connection NP scalars \(\rho_0, \sigma_0, \lambda_0, \mu_0\) on \(P_0\).
Theorem (Global causal structure of plane-symmetric matter spacetimes, LeFloch & Stewart, 2010).

- There exists an $H^1$ regular spacetime $(\mathcal{M}, g)$ satisfying the plane-symmetric, Einstein equations describing self-gravitating, irrotational fluids with prescribed data

\[
\begin{align*}
(\rho, \sigma, \lambda, \mu) &= (\rho_0, \sigma_0, \lambda_0, \mu_0) \quad \text{on } \mathcal{P}_0 \\
(a, \psi_0, \Phi_{00}) &= (a, \psi_0, \Phi_{00}) \quad \text{on } \mathcal{N}_0 \\
(a, \psi_4, \Phi_{22}) &= (\bar{a}, \bar{\psi}_4, \bar{\Phi}_{22}) \quad \text{on } \overline{\mathcal{N}}_0.
\end{align*}
\]

- The future Cauchy development has past boundary

\[
\{ U_0 > U > U_0; V = V_0 \} \cup \{ U = U_0; \overline{V}_0 > V > V_0 \} \subset \overline{\mathcal{N}}_0 \cup \overline{\mathcal{N}}_0.
\]

- For generic initial data, the curvature blows up and no longer makes sense as a distribution as one approaches its ($H^1$ regular) future boundary

\[
\mathcal{B}_0 := \{ F(U) + G(V) = 0 \}
\]

for some functions $F, G$, so that the spacetime is inextendible beyond $\mathcal{B}_0$ within the class of $H^1$ regular spacetimes.

http://philippelefloch.wordpress.com
Uniqueness of Schwarzschild-de Sitter spacetime

P.G. LeFloch and L. Rozoy, ArXiv 1009.0936.

- Black hole uniqueness theorem for static vacuum spacetimes
- Final state of the evolution of matter under self-gravitating forces
- Very limited number of such spacetimes. Case $\Lambda \leq 0$: Israel, Hawking, Beig, Simon, Chrusciel, etc.
- Case $\Lambda > 0$
Static spacetime with maximal compact spacelike slices

- Time-oriented, $(3 + 1)$-dimensional Lorentzian manifold $N$

- Topology $N \cong \mathbb{R} \times M$ and Lorentzian metric $g = -f^2 \, dt^2 + g$

  $t$ is a coordinate on $\mathbb{R}$ increasing toward the future

- $M$ connected, orientable, smooth topological 3-manifold with boundary $\partial M$, endowed with a Riemannian metric $g$ of class $W^{2,2}(M)$, such that $M \cup \partial M =: \overline{M}$ compact

- Lapse function $f : M \to (0, +\infty)$ in the Sobolev space $W^{2,2}(M)$ and vanishes at the boundary. Regularity condition: the level set where the lapse function achieves its maximum is a regular surface.

- Einstein’s vacuum equations with positive cosmological constant $\Lambda > 0$

  \[ R_{\mu\nu} = \Lambda \, g_{\mu\nu}. \]
By definition:

- \( \mathbf{T} := \partial / \partial t \) future-oriented, timelike Killing field:

\[
\mathcal{L}_T g = 0, \quad g(T, T) < 0.
\]

- Hypersurfaces \( t = \text{const.} \) are orthogonal to \( \mathbf{T} \).
- The lapse function is positive in \( \mathcal{M} \) and vanishes on \( \partial \mathcal{M} \), so that the so-called horizon

\[
\mathcal{H} := \{ f = 0 \},
\]

coincides with the boundary of the slices \( \mathcal{H} = \partial \mathcal{M} \), which need not be connected.
**Schwarzschild-de Sitter spacetime** discovered by Kottler in 1918.

Fix $m, \Lambda > 0$ satisfying $(3m)^2 \Lambda \in (0, 1)$.

The *interior domain of the Kottler spacetime* $\mathcal{N}_{K, m, \Lambda}$ with metric $g_{K, m, \Lambda}$, is the static spacetime:

- Compact spacelike slices $\mathcal{M}_{K, m, \Lambda} \simeq (r_K^-, r_K^+) \times S^2$

- Lapse function $(f_{K, m, \Lambda}(r))^2 := 1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2$

- Riemannian metric $g_{K, m, \Lambda} := \frac{dr^2}{(f_{K, m, \Lambda}(r))^2} + r^2 g_{S^2}$, with $r \in [r_K^-, r_K^+]$

- $g_{S^2}$ : canonical metric on $S^2$

- $r_K^\pm = r_{K, \pm, m, \Lambda}$ : positive roots of the cubic $r \mapsto r(f_{K, m, \Lambda}(r))^2$. 
Observe that:

- The horizon of a Kottler spacetime $\mathcal{H}_{K,m,\Lambda}$ has two connected components

$$\mathcal{H}_{K,m,\Lambda}^\pm := \{ r = r_{K,m,\Lambda}^\pm \}.$$ 

- $N_{K,m,\Lambda}$ may be extended beyond its horizon:
  - an “inner horizon” connecting to an interior black hole region
  - a cosmological horizon connecting to a non-compact exterior domain of communication (asymptotic to de Sitter).

- The interior domain is, both mathematically and physically, the region of interest.
  As $\Lambda \to 0$, converges to the outer communication domain of the Schwarzschild spacetime.
De Sitter spacetimes.

- Parametrized by the cosmological constant $\Lambda > 0$.
- $N_{dS,\Lambda}$: one domain of communication of the de Sitter spacetime,
  - whose spacelike slices have the topology of a half-sphere $S^3_+$
  - whose horizon $H_{dS,\Lambda}$ admits a single component diffeomorphic to $S^2$.

Theorem (Uniqueness theorem for Kottler spacetimes)

- The interior domain of the Kottler spacetimes $N_{K,m,\Lambda}$ parameterized by their mass $m > 0$ and cosmological constant $\Lambda > 0$
- together with the domain of communication $N_{dS,\Lambda}$ of the de Sitter spacetimes

are, up to global isometries, the unique static spacetimes with maximal compact spacelike slices and regular maximal level set, satisfying Einstein’s field equations with positive cosmological constant.
Remarks:

- No restriction assumed a priori on the topology of the spacelike slices.

- Topology identified in the conclusion of the theorem, together with the metric.

- Low regularity:
  
  - Possibly singular boundary behavior: when $f \to 0$
  
  - Possibly singular foliation: critical points $\nabla f = 0$. 
Sketch of the proof.

- Einstein equations on the 4-dim. spacetime equivalent to a problem on the 3-manifold $\mathcal{M}$

- Partial differential equations for the lapse $f$ and metric $g$

\[ \nabla df - (\Delta f) g - f \, Rc = 0, \]
\[ R = 2\Lambda > 0. \]

- Taking the trace of the Einstein equations:

\[ \Delta f = -\frac{R}{2} f. \]

so $f$ is an eigenfunction of the Laplace operator on the (unknown) manifold $(\mathcal{M}, g)$. 
Determine all triplets of solutions \((\mathcal{M}, g, f)\) and, in particular, determine the topology of \(\mathcal{M}\).

From the lapse function of the \((3 + 1)\)-foliation, define a (possibly) degenerate \((2 + 1)\)-foliation.

Investigate the topology and geometry of its leaves.
Hawking mass density,
defined from the Gauss curvature and mean-curvature of the 2-slices
\[ m_H := R^{(2)} - \frac{H^2}{2}. \]

Pointwise version of Penrose inequality on the horizon
– Low regularity when \( f \to 0 \).

This allows us to identify a topological 2-sphere \( S_* \) within the
connected components of the horizon, together with a nearby
foliation \( S_{*,s} \) with \( s \to 0 \):
– Lower bound on Hawking’s mass density:
\[ m_{H,S_*} := \liminf_{s \to 0} \left( \min_{S_{*,s}} m_H \right) \geq \frac{R}{6}. \]
An “optimal” Kottler model with well-chosen ADM mass is introduced.

Covers the region limited by well-chosen level sets of the lapse function.

Suitable maximum principle arguments are developed for Einstein’s field equations of static spacetimes.

– Possibly degenerate \((2 + 1)\)-foliation \(f = \text{Const.}\).

– Singularities in \(1/|\nabla f|\).