Debiased and threshold ridge regression for linear model with heteroskedastic and dependent error

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Abstract

Focusing on a high dimensional linear model $y = X\beta + \epsilon$ with dependent, non-stationary, and heteroskedastic errors, this paper applies the debiased and threshold ridge regression method that gives a consistent estimator for linear combinations of $\beta$; and derives a Gaussian approximation theorem for the estimator. Besides, it proposes a dependent wild bootstrap algorithm to construct the estimator’s confidence intervals and perform hypothesis testing. Numerical experiments on the proposed estimator and the bootstrap algorithm show that they have favorable finite sample performance.

Research on a high dimensional linear model with dependent (non-stationary) errors is sparse, and our work should bring some new insights to this field.

1 Introduction

Linear regression with independent and identically distributed (i.i.d.) errors is a fundamental topic in statistical inference. The classical setting assumes the dimension of parameters in a linear model is constant. Under this setting, research has been proposed on estimation, e.g., Zou and Hastie [1] and Zou [2]; confidence intervals construction/hypothesis testing, e.g., Chatterjee and Lahiri [3, 4]; and prediction, e.g., Stine [5] and Zhang and Politis [6]. We also refer Seber and Lee [7] for an extensive introduction.

In reality, however, errors in a linear model can be dependent or have different distributions. For example, as Vogelsang [8] and Petersen [9] suggested, heteroscedasticity,
autocorrelation and spatial correlation present in panel data. In this case confidence intervals developed for i.i.d. errors may fail to capture the correct probability. New tools are developed to adapt to the non-i.i.d. errors. Andrews [10] and Kim and Sun [11] considered estimating the ordinary least square estimator’s covariance matrix; Kelejian and Prucha [12] and Vogelsang [8] proposed test statistics; Sun and Wang [13] and Conley et al. [14] worked on inference and hypothesis testing, etc. Despite great success, their works focus on the classical situation, i.e., the dimension of parameters is considered to be fixed and does not change as the sample size increases.

In the modern era, observations may have a comparable or even larger dimension than the number of samples. So the dimension of parameters cannot be considered as fixed. In order to perform consistent estimation, statisticians need to assume the underlying parameters are sparse (i.e., the parameters contain many zeros), and proceed with statistical inference based on this assumption. Lasso is a suitable algorithm for this setting since it conducts an implicit model selection, i.e., zeroing out parameters that are not significant, see Tibshirani [15]. More recent work includes Zhao and Yu [16], Meinshausen and Bühlmann [17] and Meinshausen and Yu [18] for model selection; Zhang and Zhang [19], Zhang and Cheng [20] and Chatterjee and Lahiri [13] for statistical inference and hypothesis testing; Greenshtein and Ritov [21] for prediction and Zou [2] for algorithm improvement. We refer Bühlmann and van de Geer [22] for a comprehensive overview of the Lasso method on high dimensional data set.

Lasso is not the only choice for fitting a high-dimensional linear model. Fan and Li [23] introduced a new penalty function, called SCAD, that is continuously differentiable and maintains the sparsity of the underlying model. Lee et al. [24], Liu and Yu [25] and Tibshirani et al. [26] considered Post-selection inference, i.e., performing model selection with Lasso, then fitting ordinary least square regression on the selected parameters. Shao and Deng [27] applied thresholding on ridge regression to recover the sparsity of a linear model; Zhang and Politis [28] made a further improvement through debiasing and thresholding. After debiasing and thresholding, the ridge regression estimator had a comparable performance to the complex models like threshold Lasso or post-selection inference. Moreover, the estimator had a closed-form formula, making it easy to derive theoretical guarantees.

This paper focuses on fitting a high dimensional linear model, constructing confidence intervals, and performing hypothesis testing, with the presence of dependent and heteroskedas-
tic errors. Since non-i.i.d. errors may appear in fixed dimensional linear models, they may also appear in high dimensional linear models. Unfortunately, performing statistical inference for a high dimensional linear model with dependent and heteroskedastic errors is challenging. Wu and Wu [29] proposed an oracle inequality; Han and Tsay [30] proved the consistency of the Lasso estimator. However, their works relied on a stationary assumption, e.g., definition 1.3.3 in Brockwell and Davis [31].

Suppose a high dimensional sparse linear model $y = X\beta + \epsilon$ with dependent, non-stationary, heteroskedastic errors $\epsilon = (\epsilon_1, ..., \epsilon_n)^T$. Here $\epsilon_i, i = 1, ..., n$ are not necessary to be linear processes. This paper focuses on estimating linear combinations of parameters $\gamma = M\beta$ with $M$ a given matrix. We adopt the debiased and threshold ridge regression estimator proposed in Zhang and Politis [28]. After selecting a suitable ridge parameter and a threshold, this method has a comparable performance to Lasso and is easily analyzed.

Our work includes proving the model selection consistency as well as the consistency of the estimator; deriving a Gaussian approximation theorem for the estimator and constructing a confidence interval for $\gamma$. We are also interested in testing the statistical hypothesis

$$\text{null: } M\beta = z \text{ versus the alternative: } M\beta \neq z$$

(1)

with $z$ a given vector. To achieve this goal, we adopt a dependent wild bootstrap (Shao [32]) and provide its theoretical guarantee. Since there is little research on statistical inference for a high dimensional linear model with non-i.i.d. errors, this paper should shed some light on this field.

The remainder of this paper is organized as follows: section 2 introduces frequently used notations and assumptions. Section 3 presents the consistency results and the Gaussian approximation theorem for the proposed estimator. Section 4 constructs a confidence interval for $\gamma = M\beta$, and tests the null hypothesis $\gamma = z$ versus the alternative hypothesis $\gamma \neq z$ via dependent wild bootstrap. Section 5 provides numerical experiments, and section 6 makes conclusions. Technical proofs are deferred to the Appendix.
2 Preliminarily

Our work considers the fixed design linear model

\[ y = X\beta + \epsilon \tag{2} \]

where the unknown parameter vector \( \beta = (\beta_1, \ldots, \beta_p)^T \in \mathbb{R}^p \), and the \( n \times p (p < n) \) fixed (nonrandom) design matrix \( X = (x_{ij})_{i=1,\ldots,n; j=1,\ldots,p} \) is assumed to have rank \( p \). Denote \( y = (y_1, \ldots, y_n)^T \) and the errors \( \epsilon = (\epsilon_1, \ldots, \epsilon_n)^T \). From the thin singular value decomposition (theorem 7.3.2 in [33]), \( X = PAQ^T \) with \( P = (p_{ij})_{i=1,\ldots,n; j=1,\ldots,p} \) and \( Q = (q_{ij})_{i,j=1,\ldots,p} \) respectively being the \( n \times p \) and \( p \times p \) orthonormal matrix, i.e., \( P^TP = Q^TQ = QQ^T = I_p \).

Here \( I_p \) denotes the \( p \times p \) identity matrix. And \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p) \), \( \lambda_i, i = 1, 2, \ldots, p \) are singular values of \( X \). Define the set \( \mathcal{N}_b = \{ i = 1, 2, \ldots, p : |\beta_i| > b \} \), \( \forall b > 0 \). We are interested in constructing the confidence region for \( \gamma = M\beta \), and testing the statistical hypothesis \( H_0 \).

After choosing a suitable threshold \( b \), define \( c_{ij}, i = 1, 2, \ldots, p; j = 1, 2, \ldots, p \) as \( c_{ij} = \sum_{k \in \mathcal{N}_b} m_{ik}q_{kj} \), and \( M = \{ i = 1, 2, \ldots, p_1 : \sum_{j=1}^{p_1} c_{ij}^2 > 0 \} \). Define \( \tau_i = \left( \frac{1}{n} + \sum_{j=1}^{p} c_{ij}^2 \right)^{1/2} \left( \frac{\lambda_j}{\lambda_j + \rho n} + \frac{\rho_n \lambda_j}{(\lambda_j + \rho n)^2} \right)^2 \), \( i = 1, 2, \ldots, p_1 \). Define \( \Sigma = (\sigma_{ij})_{i,j=1,\ldots,n} \) with \( \sigma_{ij} = \mathbb{E} \epsilon_i \epsilon_j \) as the covariance matrix of errors.

To quantify the dependency among random variables, Wu [34] (also see Wu and Wu [35]) introduced the concept ‘physical dependence’. However, this concept was designed for stationary random variables. We extend this concept to adapt to non-stationary errors \( \epsilon \): suppose \( \epsilon_i, i = \ldots, -1, 0, 1, \ldots \) are independent (not necessarily identically distributed) random variables, and \( \epsilon_i = g_i(\ldots, \epsilon_{i-1}, \epsilon_i) \), here \( g_i \) are measurable functions, \( i = 1, 2, \ldots, n \). Define \( \mathcal{F}_i \) as the \( \sigma \)-field generated by \( \ldots, \epsilon_{i-1}, \epsilon_i \), so \( \epsilon_i \) is \( \mathcal{F}_i \) measurable. Suppose independent random variables \( \epsilon_i^1, i = \ldots, -1, 0, 1, \ldots \) are independent with \( \epsilon_j, j = \ldots, -1, 0, 1, \ldots \); and \( \epsilon_i^1 \) has the same distribution as \( \epsilon_i \), \( \forall i \). Define \( \epsilon_{i,j} = g_i(\ldots, \epsilon_{i-j-2}, \epsilon_{i-j-1}, \epsilon_{i-j}, \epsilon_{i-j+1}, \ldots, \epsilon_i) \); and the filter \( \mathcal{F}_{i,j} \) as the \( \sigma \)-field generated by \( \epsilon_{i-j}, \epsilon_{i-j+1}, \ldots, \epsilon_i \). Here \( i \in \mathbb{Z}, j \geq 0 \). For a constant \( m \geq 1 \), define the norm \( \| \cdot \|_m = (\mathbb{E} | \cdot |^m)^{1/m} \). Then define \( \delta_{i,j,m} = \| \epsilon_i - \epsilon_{i,j} \|_m \).

This paper applies the standard order notations \( O(\cdot), o(\cdot), O_p(\cdot) \) and \( o_p(\cdot) \): for two numerical sequence \( a_n, b_n \), we say \( a_n = O(b_n) \) if \( \exists \) a constant \( C > 0 \) such that \( |a_n| \leq C|b_n| \) for all \( n \); and \( a_n = o(b_n) \) if \( \lim_{n \to \infty} a_n/b_n = 0 \). For two random variable sequences \( X_n, Y_n \), we say
\(X_n = O_p(Y_n)\) if for any \(0 < \varepsilon < 1\), \(\exists\) a constant \(C_\varepsilon\) such that \(\sup_n \text{Prob}(|X_n| \geq C_\varepsilon |Y_n|) \leq \varepsilon\); and \(X_n = o_p(Y_n)\) if \(X_n/Y_n \to_p 0\). See definition 1.9 and chapter 1.5.1 of Shao [36]. All order notations and convergences in this paper are understood to hold as the sample size \(n \to \infty\).

For a set \(A\), we use \(|A|\) to indicate the number of elements in \(A\). For a vector \(a = (a_1, \ldots, a_n)^T\), define its \(q\) norm as \(\|a\|_q = (\sum_{i=1}^n |a_i|^q)^{1/q}\). For a matrix \(T\), define (and mainly use) the operator norm \(\|T\|_2 = \max_{\|a\|_2 = 1} \|Ta\|_2\). The notation \(\to\) and \(\to_p\) respectively indicates convergence in \(\mathbb{R}\), and convergence in probability. \(\exists\) and \(\forall\) respectively means ‘there exists’ and ‘for all’. Define \(\text{Prob}^*(\cdot) = \text{Prob}(\cdot|y)\) and \(\mathbb{E}^* = \mathbb{E}(\cdot|y)\), i.e., the probability and the expectation in the bootstrap world.

This paper uses the following assumptions:

**Assumptions**

1. The fixed design matrix \(X\) has rank \(p \leq n\). There exists constants \(c_\lambda, C_\lambda > 0, 1/2 \geq \eta > 0\) such that
   \[C_\lambda n^{1/2} \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq c_\lambda n^{\eta}, \forall n\]
   (3)

   Besides, \(\max_{i=1, \ldots, n, j=1, \ldots, p} |x_{ij}| = O(1)\). \(p_1 = O(1)\). Here \(p_1\) is the number of simultaneous linear combinations in (1).

2. \(\varepsilon_i = 0, i = 1, 2, \ldots, n\). There exists constants \(m > \frac{\eta}{\eta} \) and \(\alpha > 1\) such that
   \[\sup_{k=0, 1, \ldots} (k + 1)^m \sum_{j=k}^\infty \max_{i=1, 2, \ldots, n} \delta_{i,j,m} = O(1)\] and
   \[\max_{i=1, 2, \ldots, n} \|\varepsilon_i\|_m = O(1)\]
   (4)

   Besides, \(\exists\) a constant \(c_\Sigma > 0\) such that \(\Sigma\’s\) minimum eigenvalue is greater than \(c_\Sigma\) for \(\forall n\); here \(\Sigma = \mathbb{E}\varepsilon\varepsilon^T\).

3. \(\|\beta\|_2 = O(n^{\alpha \beta})\) with a constant \(\alpha_\beta\) such that \(0 \leq \alpha_\beta < 3\eta\), and \(|N_{bn}| = o(n^{\eta-1/m})\).

4. \(\rho_n = O(n^{\eta-\delta})\) with a constant \(\delta\) such that \(\frac{\eta + \eta \rho_n}{2} < \delta < 2\eta\). Besides, there exists constants \(C_\beta > 0\) and \(0 < \nu_0 < \eta - \frac{3}{m}\) such that \(b_n = C_\beta n^{-\nu_0}\) for \(\forall n\). Assume \(\exists\) a constant \(0 < c_\beta < 1\) such that \(\max_{\beta \in N_{bn}} |\beta_i| \leq c_\beta \times b_n\) and \(\min_{i \in N_{bn}} |\beta_i| \geq b_n/c_\beta\).

5. \(\mathcal{M}\) is not empty. And \(\exists\) constants \(0 < c_\mathcal{M} \leq C_\mathcal{M} < \infty\) such that \(c_\mathcal{M} \leq \sum_{k \in N_{bn}} m_{ik}^2 (= \sum_{j=1}^p c_{ij}^2) \leq C_\mathcal{M}\) for \(\forall i \in \mathcal{M}\) and \(\forall n\). Also assume
   \[\max_{i=1, \ldots, p_1} |\sum_{j \notin N_{bn}} m_{ij} \beta_j| = o(1/\sqrt{n})\] and \(\max_{i=1, \ldots, n} |\sum_{j \notin N_{bn}} x_{ij} \beta_j| = o(n^{1/m-\frac{\eta}{\eta}})\)
   (5)
6. The matrix \((c_{ij})_{i \in M, j=1,2,...,p}\) has rank \(|M|\) and
\[
c^* = \max_{i \in M} \left\{ \frac{1}{n} \sum_{j=1}^{p} c_{ij} p_{ij} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2} \right) \right\} = o(n^{-1/4} \log^{-2}(n)) \quad (6)
\]
Here \(z = \max(\frac{3}{2}, \frac{3n}{2n-2})\).

7. Suppose a function \(K : \mathbb{R} \to [0, \infty)\) is symmetric, continuously differentiable, \(K(0) = 1\), \(\int_{\mathbb{R}} K(x)dx < \infty\), and \(K\) is decreasing on \([0, \infty)\). Define the Fourier transformation of \(K\) as \(\mathcal{F}K(x) = \int_{\mathbb{R}} K(t) \exp(-2\pi itx)dt\). Assume \(\mathcal{F}K(x) \geq 0, \forall x \in \mathbb{R}\) and \(\int_{\mathbb{R}} \mathcal{F}K(x)dx < \infty\).

Define a bandwidth parameter \(k_n\) satisfying \(\lim_{n \to \infty} k_n = \infty\), \(k_n = o\left(\sqrt{n^{2-\nu_b}-4/m}\right)\), and \(k_n = O\left(\sqrt{n^{2-2\nu_b}}\right)\).

In assumption 6, \((c_{ij})_{i \in M, j=1,2,...,p}\) denotes the sub-matrix formed by selecting \(M\) rows from the matrix \((c_{ij})_{i=1,...,n, j=1,...,p}\).

Our work mainly focuses on fixed design, i.e., no randomness involves in the design matrix \(X\). In the case of random design, results in this paper hold true after conditioning on the design matrix \(X\), i.e., replacing \(\text{Prob}(\cdot)\) by \(\text{Prob}(\cdot|X)\); \(\mathbb{E}\cdot\) by \(\mathbb{E}\cdot|X\); \(\text{Prob}^\ast(\cdot)\) by \(\text{Prob}(\cdot|X,y)\) and \(\mathbb{E}^\ast\cdot\) by \(\mathbb{E}\cdot|X,y\).

Example 1 introduces a situation in which assumption 1 is satisfied.

Example 1
Suppose \(n > p\) and \(p/n \to c \in (0,1)\). Choose \(X = (x_{ij})_{i=1,...,n, j=1,...,p}\) such that each \(x_{ij}\) is a realization of i.i.d. random variables with mean 0, variance 1, and finite fourth order moment. According to Bai and Yin [37], the smallest eigenvalue of \(\frac{1}{n}X^TX\) converges to \((1-\sqrt{c})^2\) almost surely as \(n \to \infty\). So the smallest singular value of \(X\), being the square root of \(X^TX\)’s smallest eigenvalue, is greater than \(1-\sqrt{c}\sqrt{n}\) for sufficiently large \(n\) almost surely. On the other hand, the largest eigenvalue of \(\frac{1}{n}X^TX\) converges to \((1+\sqrt{c})^2\) as \(n \to \infty\).

So the largest singular value of \(X\) has order \(O(\sqrt{n})\) almost surely.

Remark 1
1. Under stationary assumptions, e.g., Wu [33]
\[
\delta_{i,j,m} = \|e_0 - e_{0,j}\|_m = \delta_{0,j,m} \Rightarrow \sum_{j=1}^{\infty} \max_{i=1,...,n} \delta_{i,j,m} = \sum_{j=1}^{\infty} \delta_{0,j,m} \quad (7)
\]
and (4) coincides with (2.8) in Wu and Wu [32], i.e., the dependence adjusted norm condition. Therefore, assumption 2 can be recognized as the dependence adjusted norm condition.
for non-stationary random variables.

2. According to Shao [32] and Fourier inversion theorem (Theorem 8.26 in [38]), \( \forall x = (x_1, ..., x_n)^T \in \mathbb{R}^n \),

\[
\sum_{s=1}^{n} \sum_{j=1}^{n} x_s x_j K \left( \frac{x_s - x_j}{k_n} \right) = \int_{\mathbb{R}^n} \sum_{s=1}^{n} \sum_{j=1}^{n} x_s x_j F_K(z) \exp \left( 2\pi i z \frac{x_s - x_j}{k_n} \right) dz
\]

\[
= \int_{\mathbb{R}^n} F_K(z) \left| \sum_{s=1}^{n} x_s \exp \left( \frac{2\pi i z x_s}{k_n} \right) \right|^2 dz \geq 0
\]

(8)

Therefore, the matrix \( \left\{ K \left( \frac{x_s - x_j}{k_n} \right) \right\}_{s,j=1,2,...,n} \) is symmetric positive semi-definite. One example of \( K \) satisfying assumption 7 is \( K(x) = \exp(-x^2/2) \Rightarrow F_K(x) = \sqrt{2\pi} \exp(-2\pi^2 x^2) \).

3. Since

\[
\sum_{l=1}^{p} \left( \sum_{j=1}^{p} c_{ij} p_{lj} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2} \right) \right)^2 \leq \sum_{j=1}^{p} c_{ij}^2 \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2} \right)^2 < \tau_i
\]

(9)

Assumption 6 requires no single element in the array \( \left\{ \sum_{j=1}^{p} c_{ij} p_{lj} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2} \right) \right\}_{i=1,...,n} \) dominates others. We add 1/n in \( \tau_i \) to prevent the normalizing parameters from being 0.

4. Assumption 2 implies a polynomial decay of \( \epsilon \)'s covariance, i.e., \( \max_{1 \leq i < j \leq n} (1 + j - i)^{\alpha} \times |E_\epsilon \epsilon_j| = O(1) \). We will prove this in appendix A. Panel data may require different types of dependency, e.g., clustered dependency or user-defined spatial dependency (Vogelsang [28]). Our work needs to be adjusted if those dependencies show up.

3 Consistency and Gaussian approximation theorem

This paper applies the ridge regression estimator introduced in Zhang and Politis [28]. Use notations in section 2. Suppose the classical ridge regression estimator \( \hat{\beta}^* \) and the de-biased estimator \( \tilde{\beta} \) as

\[
\tilde{\beta} = \beta^* + \rho_n Q(\Lambda^2 + \rho_n I_p)^{-1} Q^T \beta^*
\]

Define \( \hat{\mathcal{N}}_{b_n} = \{ i = 1, 2, ..., p : |\tilde{\beta}_i| > b_n \} \), here \( b_n > 0 \) is a given number (related to the sample size \( n \)). Define \( \tilde{\gamma} = (\tilde{\gamma}_1, ..., \tilde{\gamma}_p)^T \) such that \( \tilde{\gamma}_i = \hat{\gamma}_i \times 1_{i \in \hat{\mathcal{N}}_{b_n}} \); and \( \hat{\gamma} = (\hat{\gamma}_1, ..., \hat{\gamma}_p)^T = M \tilde{\gamma} \). \( 1_{i \in \hat{\mathcal{N}}_{b_n}} = 1 \) if \( i \in \hat{\mathcal{N}}_{b_n} \) and 0 otherwise. This paper will use \( \hat{\gamma} \) to
estimate $\gamma = (\gamma_1, ..., \gamma_p)^T = M\beta$.

**Remark 2**

Remark 2 in Zhang and Politis [28] explained why $\tilde{\beta}$ helps decrease the bias as well as the estimation error. Generally speaking, the bias of a classical ridge regression estimator $\tilde{\beta}^*$ is $-\rho_n Q(\Lambda^2 + \rho_n I_p)^{-1} Q^T \beta$. We can estimate the bias by $-\rho_n Q(\Lambda^2 + \rho_n I_p)^{-1} Q^T \tilde{\beta}^*$, then subtract the estimated bias from $\tilde{\beta}^*$, yielding the new estimator $\tilde{\beta}$. Compared to $\tilde{\beta}^*$, $\tilde{\beta}$ increases the variance but decreases the bias. So with a suitable choice of $\rho_n$, the estimation error will decrease.

The consistency of $\tilde{\beta}$ is not a free lunch. We need $||\beta||_2$ not to be very large (which is achieved if $\beta$ is sparse) and $\rho_n/\lambda^2_p \to 0$ to maintain consistency.

The consistency of $\tilde{\beta}$ consists of two aspects: the model selection consistency, i.e., $\check{N}_{bn} = N_{bn}$ (defined in section 2) with probability tending to 1. And the consistency of $\check{\gamma}$, i.e., $\max_{i=1,2,...,p} |\check{\gamma}_i - \gamma_i| = o_p(1)$. Theorem 1 will prove both consistencies. Another important result in theorem 1 is that the estimated errors $\check{\epsilon} = y - X\check{\beta}$ are close to the underlying errors $\epsilon$. This result is the theoretical foundation for bootstrap algorithm [1].

**Theorem 1**

Suppose assumption 1 to 5. Then

1. $\operatorname{Prob}(\hat{X}_{bn} \neq N_{bn}) = o(1)$ and $\max_{i=1,2,...,p} |\hat{\gamma}_i - \gamma_i| = O_p(n^{-\eta})$ (11)

2. Define $\check{\epsilon} = (\check{\epsilon}_1, ..., \check{\epsilon}_n)^T$ such that $\check{\epsilon}_i = y_i - \sum_{j=1}^p x_{ij} \hat{\beta}_j$, then

$$\max_{i=1,2,...,n} |\check{\epsilon}_i - \epsilon_i| = o_p(n^{1/(2m) - \eta/2})$$ (12)

(12) is not self-evident for a high dimensional linear model. We refer Mammen [39] for a detailed explanation.

If the dimension of a parameter vector changes according to the sample size, then the estimator $\hat{\beta}$ does not have an asymptotic distribution, and the classical central limit theorem fails. Nevertheless, statisticians still can use normal random variables to approximate $\hat{\beta}$’s distribution. In reality, joint normal random variables can be generated easily by a computer. So statisticians can perform Monte-Carlo simulations to derive the asymptotic probability of
an event or make a consistent confidence interval. This idea is common for high dimensional statistics, see e.g., Chernozhukov et al. [40] and Zhang and Wu [41].

Following this idea, theorem 2 presents a Gaussian approximation theorem of $\hat{\gamma}$. Define $\xi = (\xi_1, ..., \xi_n)^T$ as joint normal random variables with $\mathbb{E}\xi = 0, \mathbb{E}\xi\xi^T = \Sigma$(the covariance matrix of errors $\epsilon$), and

$$H(x) = \text{Prob}\left(\max_{i \in M} \frac{1}{\tau_i} \sum_{j=1}^{p} \sum_{l=1}^{n} c_{ij} \left(\frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2}\right) p_{lj} \xi_l \leq x\right), \quad x \in \mathbb{R} \quad (13)$$

See section 2 for the meaning of notations. $\tau_i$ and $M$ are defined by choosing $b = b_n$. According to assumption 5, $M$ is not empty. So $H$ is well-defined.

**Theorem 2**

Suppose assumption 1 to 6. Then

$$\sup_{x \in \mathbb{R}} |\text{Prob}\left(\max_{i = 1, ..., p_1} \frac{1}{\tau_i} \sum_{j=1}^{p} m_{ij} \beta_j \leq x\right) - H(x)| = o(1) \quad (14)$$

Here $\tilde{\tau}_i = \frac{1}{n} + \sum_{j=1}^{p} \tilde{c}_{ij}^2 \left(\frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2}\right)^2$ and $\tilde{c}_{ij} = \sum_{k \in \tilde{N}_b} m_{ik} q_{kj}; i = 1, 2, ..., p_1, j = 1, 2, ..., p$.

Define $c_{1-\alpha}$ as the $1 - \alpha$ quantile of $H(x)$, i.e., $c_{1-\alpha} = \inf \{x \in \mathbb{R} : H(x) \geq 1 - \alpha\}$. Theorem 2 implies that the set

$$\left\{\gamma \in \mathbb{R}^{p_1} : \max_{i = 1, ..., p_1} \frac{|\tilde{\gamma}_i - \gamma_i|}{\tilde{\tau}_i} \leq c_{1-\alpha}\right\} \quad (15)$$

is an asymptotic consistent confidence region for $\gamma$.

### 4 Dependent wild bootstrap

In order to calculate (15), we need a method that derives(or approximates) $c_{1-\alpha}$. Since $M, N_{b_n}$, and $\Sigma$ are unknown a priori and $H$ does not have a closed-form formula, we develop a Monte-Carlo algorithm(algorithm 1) that finds $c_{1-\alpha}$. The advantage of algorithm 1 is that it avoids explicitly estimating $\Sigma$, which is hard for non-stationary random variables $\epsilon$. Our method applies the idea of the dependent wild bootstrap. It was first introduced by Shao [32], and has been used in linear regression (Conley et al. [42]), time series analysis...
Our target is to construct the confidence interval for \( \gamma = M\beta \). And Perform the statistical hypothesis testing for the null hypothesis \( M\beta = z \) with \( z = (z_1, ..., z_p)\) a known vector versus the alternative hypothesis \( M\beta \neq z \). Notably, this test problem receives lots of attentions in econometrics, e.g., Gonçalves and Vogelsang [46], Sun and Kim [47], and Conley et al. [14].

**Algorithm 1** (Bootstrap inference and hypothesis testing)

**Input:** Design matrix \( X \) and dependent variables \( y \) satisfying \( y = X\beta + \varepsilon \); the linear combination matrix \( M \). Ridge parameter \( \rho_n \), threshold \( b_n \), bandwidth \( k_n \), a kernel function \( K : \mathbb{R} \to [0, \infty) \). Nominal coverage probability \( 1 - \alpha \), number of bootstrap replicates \( B \).

**Additional input for testing:** \( z = (z_1, ..., z_p)\).

1. Derive the estimator \( \hat{\beta} \) defined in section 3 and \( \hat{\tau}_i, i = 1, ..., n \) defined in theorem 2

Then calculate \( \hat{\gamma} = (\hat{\gamma}_1, ..., \hat{\gamma}_p) = M\hat{\beta}, \hat{\varepsilon} = (\hat{\varepsilon}_1, ..., \hat{\varepsilon}_n) = y - X\hat{\beta} \).

2. Generate joint normal random variables \( \varepsilon_{1,1}^*, ..., \varepsilon_{n,1}^* \) with mean 0 and covariance matrix \( \{ K \left( \frac{t-j}{k_n} \right) \}_{i,j=1,...,n} \). Then define \( \varepsilon^* = (\varepsilon_{1,1}^*, ..., \varepsilon_{n,1}^*) \) such that \( \varepsilon_i^* = \varepsilon_i \times \varepsilon_{i,1}^*, i = 1, 2, ..., n \).

3. Calculate

\[
\begin{align*}
y^* &= X\hat{\beta} + \varepsilon^* \\
\tilde{\beta}^* &= (X^TX + \rho_n I_p)^{-1}X^Ty^* \\
\tilde{\beta}^* &= (\tilde{\beta}_1^*, ..., \tilde{\beta}_p^*) = \tilde{\beta}^* + \rho_n Q(\Lambda^2 + \rho_n I_p)^{-1}Q^T \tilde{\beta}^* \\
\tilde{N}_{bn}^* &= \{ i = 1, 2, ..., p : |	ilde{\gamma}_i^*| > b_n \} \\
\text{and } \tilde{\beta}^* &= (\tilde{\beta}_1^*, ..., \tilde{\beta}_p^*) \text{ such that } \tilde{\beta}_i^* = \tilde{\beta}_i^* \times 1_{1 \in \tilde{N}_{bn}^*}
\end{align*}
\]

4. Derive \( \tilde{\gamma}^* = (\tilde{\gamma}_1^*, ..., \tilde{\gamma}_p^*) = M\tilde{\beta}^*, \tilde{c}_{ij} = \sum_{k \in \tilde{N}_{bn}^*} m_{ik}q_{kj} \) and

\[
\tilde{c}_{ij}^* = \sqrt{\frac{1}{n} + \sum_{j=1}^p \tilde{c}_{ij}^2} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2} \right). \text{ Then calculate}
\]

\[
\delta_b = \max_{i=1,2,...,p} \left| \frac{\tilde{\gamma}_i^* - \tilde{\gamma}_i}{\tilde{c}_{ij}^*} \right|
\]

5. Repeat step 2 to 4 for \( b = 1, ..., B \), and calculate the \( 1 - \alpha \) sample quantile \( \tilde{\gamma}_{1-\alpha}^* \) of \( \delta_b^*, b = 1, 2, ..., B \).

6. For constructing the confidence region) The \( 1 - \alpha \) confidence region for the parameter
of interest $\gamma = M\beta$ is given by the set
\[
\left\{ \gamma = (\gamma_1, \ldots, \gamma_p) : \max_{i=1,\ldots,p_1} \frac{\hat{\gamma}_i - \gamma_i}{\hat{s}_i} \leq \tilde{c}_{1-\alpha} \right\}
\] (18)

6.b (For hypothesis testing) Reject the null hypothesis if
\[
\max_{i=1,\ldots,p_1} \frac{\hat{\gamma}_i - \gamma_i}{\hat{s}_i} > \hat{c}^*_{1-\alpha}
\] (19)

For a numerical sequence $\delta_1 \leq \delta_2 \leq \ldots \leq \delta_n$, the $1 - \alpha$ sample quantile $C_{1-\alpha}$ is defined as
\[
C_{1-\alpha} = \delta_*, \text{ such that } i_* = \min \left\{ i = 1, \ldots, n : \frac{1}{n} \sum_{j=1}^{n} 1_{\delta_j \leq \delta_i} \geq 1 - \alpha \right\}
\] (20)

Define the conditional quantile of $\delta^*_b, b = 1, \ldots, B$ as
\[
c^*_{1-\alpha} = \inf \left\{ x \in \mathbb{R} : \text{Prob}^* \left( \max_{i=1,\ldots,p_1} \frac{\hat{\gamma}_i^*-\hat{\gamma}_i}{\hat{s}_i^*} \leq x \right) \geq 1 - \alpha \right\}, \quad 0 < \alpha < 1
\] (21)

According to theorem 1.2.1 in Politis et al. [48], the sample quantile $\hat{c}^*_{1-\alpha}$ converges to $c^*_{1-\alpha}$ as $B \to \infty$. So consistency of algorithm 1 is assured by showing
\[
\text{Prob} \left( \max_{i=1,\ldots,p_1} \frac{\hat{\gamma}_i - \gamma_i}{\hat{s}_i} \leq c^*_{1-\alpha} \right) \to 1 - \alpha.
\] We derive this result in theorem 3.

**Theorem 3**

Suppose assumption 1 to 7. Then
\[
\sup_{x \in \mathbb{R}} |\text{Prob}^* \left( \max_{i=1,\ldots,p_1} \frac{\hat{\gamma}_i^*-\hat{\gamma}_i}{\hat{s}_i^*} \leq x \right) - H(x)| = o_p(1)
\]
and
\[
\text{Prob} \left( \max_{i=1,\ldots,p_1} \frac{\hat{\gamma}_i - \gamma_i}{\hat{s}_i} \leq \tilde{c}_{1-\alpha} \right) \to 1 - \alpha
\] (22)

Here $H$ is defined in (13).

5 Numerical simulations

5.1 Selecting hyper-parameters

Statisticians need to fine-tune the ridge regression parameter $\rho_n$, the threshold $b_n$, and the bandwidth $k_n$. $\rho_n$ and $b_n$ can be chosen by (ten-fold) cross-validation, i.e., separate the data-
sign matrix $X$ and the dependent variables $y$ into disjoint training set $(X_{\text{train}}, y_{\text{train}})$ and validation set $(X_{\text{valid}}, y_{\text{valid}})$; for each choice of $\rho_n$ and $b_n$, use $(X_{\text{train}}, y_{\text{train}})$ to fit $\hat{\beta}$; and calculate $\|y_{\text{valid}} - X_{\text{valid}}\hat{\beta}\|_2$. The optimal $\rho_n$ and $b_n$ should minimize $\|y_{\text{valid}} - X_{\text{valid}}\hat{\beta}\|_2$.

This paper implements the ten-fold cross validation. See Arlot and Celisse [49] for a further introduction on the cross validation methods.

Fine-tuning $k_n$ is more challenging. Politis and White [50] introduced an automatic bandwidth selection algorithm; Shao [32] applied this algorithm for the dependent wild bootstrap. Andrews [10] and Kim and Sun [11] considered selecting bandwidth in the HAC estimation setting. Following Shao, this paper applies Politis and White’s algorithm [50] on fitted residuals $\hat{\epsilon} = (\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n)^T = y - X\hat{\beta}$ to select $k_n$. However, $\epsilon$ is not assumed to be stationary. So this algorithm may result in a suboptimal bandwidth.

5.2 Generating data

The numerical experiment applies the linear model $y = X\beta + \epsilon$. $X$ is generated by i.i.d. normal random variables with mean 0 and variance 1, and is fixed in each experiment. $\beta = (\beta_1, \ldots, \beta_p)^T$ is generated by the following strategy

$$\beta_i = 0.1 \times (i + 6) \text{ for } i = 1, 2, \ldots, 10 \text{ and } 0 \text{ otherwise} \quad (23)$$

Define $e_i, i \in \mathbb{Z}$ as i.i.d. normal random variables with mean 0 and variance 1. Choose $a = (a_1, a_2, \ldots, a_n)^T$ such that $a_i = e_i^2 e_{i-1}^2 - 1$. Then define $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T = (1/4) \times Ha$(the factor 1/4 avoids $\epsilon$’s variances being too large). Here $H = (h_{ij})_{i,j=1,\ldots,n}$, $h_{ij} = 0$ for $j > i$, $h_{ii} = 1$, $h_{ij}$ is generated by uniform distribution in $[0.6, 0.9]$ for $i - 10 \leq j < i$ and $h_{ij} = s_{ij}/(i - j)^z$ for $j < i - 10$. $s_{ij}$ is generated by uniform distribution on $[-1, 1]$. $H$ is fixed in each experiment. For $E\epsilon_i = E\epsilon_i^2 = E\epsilon_{i-1}^2 = 1 = 0$, we have $E\epsilon = 0$. Moreover, $a$ is not white noise since $Ea_i a_{i-1} = 2$. $\epsilon$ is not linear because of $a$, and is not stationary because of $H$. Figure 1 plots an observation of the errors $\epsilon$, and the first $10 \times 10$ elements of $\epsilon$’s covariance matrix. In figure 1(a), the errors demonstrate strong dependency, i.e., $\epsilon_{i+1}$ is likely to be large if $\epsilon_i$ is large. In figure 1(b), $\epsilon$’s covariance matrix is not a Toeplitz matrix, so the distribution of $\epsilon$ is not stationary.

The linear combination matrix $M$ is generated by i.i.d. normal random variables with mean 0.5 and variance 0.25, and is fixed in each experiment. The hyper-parameters $\rho_n, b_n, k_n
Figure 1: Figure 1(a) plots an observation of the errors $\epsilon$, and figure 1(b) plots the heatmap for the first $10 \times 10$ elements of $\epsilon$’s covariance matrix. Values in each grid represent the corresponding covariance. The covariance matrix is calculated by simulating 40000 samples of the random vector $\epsilon = (\epsilon_1, ..., \epsilon_n)^T$.

are tuned by methods described in section 5.1. The sample size $n$, dimension $p$ and $z$ vary in each experiment. We store the information about experiments in table 1.

Table 1: Experiment parameters. ‘No.’ abbreviates ‘the experiment number’, $n$ is the sample size. $p$ is the dimension of parameters. $p_1$ is the number of linear combinations. $z$ is defined in section 5.2. $\rho_n$, $b_n$, $k_n$ respectively is the selected ridge parameter, the selected threshold, and the selected bandwidth defined in section 2. $\lambda_p$ is the smallest singular value of the design matrix $X$. We use R-Package ‘np’[51] to facilitate choosing $k_n$.

| No. | $n$ | $p$ | $p_1$ | $z$ | $\rho_n$ | $b_n$ | $k_n$ | $\lambda_p$ |
|-----|-----|-----|-------|-----|----------|-------|-------|-------------|
| 1   | 500 | 250 | 20    | 2.5 | 23.87    | 0.56  | 17.67 | 6.95        |
| 2   | 500 | 400 | 20    | 2.5 | 34.97    | 0.66  | 23.42 | 2.47        |
| 3   | 500 | 250 | 60    | 2.5 | 69.22    | 0.47  | 16.75 | 6.66        |
| 4   | 800 | 640 | 60    | 2.5 | 31.30    | 0.61  | 17.99 | 3.32        |
| 5   | 3000| 1500| 20    | 2.5 | 117.10   | 0.31  | 46.80 | 13.46       |
| 6   | 500 | 250 | 20    | 2.2 | 79.07    | 0.48  | 15.76 | 6.53        |

5.3 Performance of linear regression algorithms

This section compares the performance of the debiased and threshold ridge regression($thsDeb$) to the classical linear regression algorithms including Lasso, the Ridge regression($Ridge$), the threshold Lasso($thsLas$), the threshold ridge regression($thsRid$) and the elastic net.
The evaluation indices consist of the estimation loss \(\| M \hat{\beta} - M \beta \|_2 \) and the prediction loss \(\| X \hat{\beta} - X \beta \|_2 \). The latter one is frequently used in evaluating linear regression algorithms, see e.g. Dalalyan et al. [52].

We are also interested in the model selection performance (i.e., whether an algorithm can recover the sparsity of the underlying linear model or not) of the linear regression algorithms. Following the idea of Fithian et al. [53], we evaluate this through the frequency of model misspecification (i.e., \( \hat{N}_b \neq N_{b_n} \)), the average size of model misspecification \( |\hat{N}_b \Delta N_{b_n}| \) (\( \Delta \) denotes the symmetric difference, i.e. \( A \Delta B = (A - B) \cup (B - A) \)); and the average false discovery rate \( \frac{|\hat{N}_{b_n} - N_{b_n}|}{\max(1, |\hat{N}_{b_n}|)} \). Since Lasso, ridge regression and elastic net do not have thresholds, for these algorithms we consider \( i \in \hat{N}_b \) if \( |\hat{\beta}_i| > 0.01 \).

According to figure 2, figure 3 and table 2, the debiased and threshold ridge regression, the threshold Lasso and the threshold ridge regression have small estimation loss and prediction loss compared to the other methods. Besides, these three methods are able to recover the underlying sparsity of the linear model correctly. Notably, even if Lasso is famous for generating sparse linear regression estimators, further thresholding still significantly improves its performance.

The necessity of debiasing can be illustrated in figure 2 and figure 3. Although the threshold ridge regression has a small estimation loss when \( \rho_n \) is close to its optimal value, the estimation loss surges when \( \rho_n \) deviates from its optimal value. However, after debiasing, even if a cross validation selects a sub-optimal \( \rho_n \), the estimator’s performance does not notably deteriorate. In reality, cross validation algorithms cannot assure selecting the optimal ridge parameter \( \rho_n \). So an algorithm that is not sensitive to the slight fluctuations in hyper-parameters is preferable.

Figure 2 also plots the relation between thresholds and the estimation losses. In the experiments, the cross validation always selects sub-optimal thresholds. However, the sub-optimal thresholds form wide-intervals. Therefore, slight fluctuations in the threshold \( b_n \) should not deteriorate the performance of the regression algorithm as well.

5.4 Performance of bootstrap algorithm

This section evaluates the performance of algorithm 1 as well as the bootstrap algorithms designed for independent errors, including Efron’s bootstrap (Efron [54] and Chatterjee and Lahiri [3]) and the wild bootstrap (Mammen [55]), for the linear model with dependent
Figure 2: Estimation losses of linear regression algorithms for case 1 to 4. ‘thsDeb’ represents the debiased and threshold ridge regression method; ‘thsLas’ represents the threshold Lasso method and ‘thsRid’ represents the threshold ridge regression method. Dots reveal the optimal ridge parameters for each method selected by 10-fold cross validation. y-axis records the error $\| \hat{M} \hat{\beta} - M\beta \|_2$, here $\hat{\beta}$ is the estimator of parameters $\beta$ by different methods. We choose $l_1$ ratio 0.5 in the ElasticNet. The small graphs below each of the graphs plot the error of the debiased and threshold ridge regression method with respect to different thresholds. $\rho_n$ and $b_n$ are chosen by 10-fold cross validation.
Table 2: Model selection performance of various linear regression estimators. Hyper-parameters are chosen by ten-fold cross validation. The overscore represents calculating the sample mean among 1000 simulations. ‘FDR’ abbreviates the false discovery rate. We omit experiment 3 here for it coincides with experiment 1.

| No. | Algorithm | P(\(\hat{N}_{bn} \neq N_{bn}\)) | \(\hat{N}_{bn} \Delta N_{bn}\) | FDR | \(\|X\hat{\beta} - X\beta\|\) |
|-----|-----------|-----------------|-----------------|-----|----------------|
| 1   | thsDeb    | 0.211           | 0.334           | 0.010 | 11.335         |
|     | Lasso     | 0.943           | 8.663           | 0.386 | 14.992         |
|     | thsLas    | 0.190           | 0.261           | 0.005 | 10.212         |
|     | Ridge     | 1.0             | 112.18          | 0.918 | 31.216         |
|     | thsRid    | 0.323           | 0.411           | 0.006 | 12.369         |
|     | Elastic net | 1.0             | 27.71           | 0.706 | 18.992         |
| 2   | thsDeb    | 0.790           | 1.233           | 0.014 | 21.575         |
|     | Lasso     | 1.0             | 86.28           | 0.892 | 25.873         |
|     | thsLas    | 0.142           | 0.323           | 0.015 | 10.797         |
|     | Ridge     | 1.0             | 181.10          | 0.948 | 35.431         |
|     | thsRid    | 0.801           | 1.214           | 0.019 | 27.727         |
|     | Elastic net | 1.0             | 122.05          | 0.923 | 30.638         |
| 4   | thsDeb    | 0.377           | 0.568           | 0.009 | 17.365         |
|     | Lasso     | 1.0             | 125.95          | 0.924 | 31.442         |
|     | thsLas    | 0.079           | 0.259           | 0.011 | 10.667         |
|     | Ridge     | 1.0             | 284.67          | 0.966 | 46.353         |
|     | thsRid    | 0.409           | 0.585           | 0.006 | 18.101         |
|     | Elastic net | 1.0             | 184.87          | 0.948 | 38.673         |
| 5   | thsDeb    | 0.031           | 0.045           | 0.004 | 9.781          |
|     | Lasso     | 1.0             | 32.81           | 0.728 | 18.394         |
|     | thsLas    | 0.093           | 0.248           | 0.017 | 9.703          |
|     | Ridge     | 1.0             | 419.87          | 0.977 | 60.853         |
|     | thsRid    | 0.035           | 0.055           | 0.004 | 9.933          |
|     | Elastic net | 1.0             | 79.88           | 0.879 | 27.446         |
| 6   | thsDeb    | 0.138           | 0.236           | 0.012 | 10.856         |
|     | Lasso     | 0.967           | 9.792           | 0.422 | 14.445         |
|     | thsLas    | 0.079           | 0.157           | 0.010 | 9.229          |
|     | Ridge     | 1.0             | 105.775         | 0.913 | 30.164         |
|     | thsRid    | 0.173           | 0.261           | 0.010 | 10.874         |
|     | Elastic net | 1.0             | 28.313          | 0.711 | 18.617         |
errors. The results are demonstrated in table 3.

In table 3, the bootstrap algorithms have similar performance and are able to capture the correct $1 - \alpha$ confidence intervals. Meanwhile, the dependent wild bootstrap tends to generate a wider confidence interval than Efron’s bootstrap and the wild bootstrap. Notably, the number of simultaneous linear combinations $p_1$ affects the performance of bootstrap algorithms. If $p_1$ is too large, then the bootstrap algorithms may generate an unnecessarily wide confidence interval. Numerical results show that the Efron’s bootstrap and the wild bootstrap are robust to the dependent and heteroskedastic errors. However, the consistency of those algorithms cannot be assured.

6 Conclusion

This paper applies the debiased and threshold ridge regression method to a linear model $y = X\beta + \epsilon$. Then it derives a dependent wild bootstrap algorithm that constructs confidence
Table 3: Performance of various bootstrap algorithms. The number of bootstrap replicates is 1000 and the nominal coverage probability is 90%. We use $P$ to abbreviate the real coverage probability and $C$ to abbreviate the average 90% quantile of $\max_{i=1,\ldots,p} |\hat{\gamma}_i^* - \hat{\gamma}_i|$. $P$ and $C$ are derived through 1000 simulations. ‘Efron’ means ‘Efron’s bootstrap’ and ‘Wild’ means the wild bootstrap.

| No | Algorithm $P$ | Algorithm $C$ | Efron $P$ | Efron $C$ | Wild $P$ | Wild $C$ |
|----|---------------|---------------|-----------|-----------|----------|----------|
| 1  | 90.7%         | 9.32          | 88.0%     | 8.83      | 88.4%    | 8.86     |
| 2  | 89.1%         | 17.14         | 87.7%     | 16.71     | 87.2%    | 16.76    |
| 3  | 95.1%         | 11.77         | 94.4%     | 11.26     | 94.0%    | 11.31    |
| 4  | 89.8%         | 14.38         | 89.8%     | 13.89     | 88.8%    | 13.86    |
| 5  | 90.2%         | 4.72          | 89.1%     | 4.62      | 89.7%    | 4.64     |
| 6  | 92.8%         | 11.38         | 92.2%     | 10.70     | 92.1%    | 10.75    |

...intervals for linear combinations of parameters $\gamma = M\beta$; or tests the statistical hypothesis

$$M\beta = z \text{ versus } M\beta \neq z$$  \hspace{1cm} (24)

Here $M$ is a known matrix, and $z$ is the given expected value of $M\beta$.

Our work suits a high dimensional setting, i.e., the dimension of parameters $\beta$ may grow as the sample size increases. Besides, the adopted regression method and the bootstrap procedure are consistent for dependent, non-stationary, heteroskedastic errors $\epsilon$. $\epsilon$ is not necessary to be a linear process as well.

Numerical simulations indicate that the ridge regression estimator has a comparable model selection and estimation performance to complex methods like the threshold Lasso. Furthermore, it is robust to the non-optimal choice of the ridge parameter $\rho_n$ as well as the threshold $b_n$. Therefore, it can be considered as a practical method to handle real-life problems.

Our work generalizes results in Conley et al. [14] to a more realistic setting. We have not yet seen many works on a high-dimensional linear model with non-stationary errors. So this work should bring new insights to this field.

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A  Some useful results

Notations in the appendix should coincide with section 2 unless we specify their meanings. This section starts with some useful corollaries of assumption 2. From corollary C.9 in [50],

\[ \epsilon_i = \lim_{j \to \infty} \mathbb{E} \epsilon_i | \mathcal{F}_{i,j} \] almost surely; and \( \lim_{j \to \infty} \| \epsilon_i - \mathbb{E} \epsilon_i | \mathcal{F}_{i,j} \| = 0, \forall i = 1, 2, \ldots, n. \) For

\[ \| \mathbb{E} \epsilon_i | \mathcal{F}_{i,j} - \mathbb{E} \epsilon_i | \mathcal{F}_{i,j-1} \| = \| \mathbb{E} (\epsilon_i - \epsilon_{i,j}) | \mathcal{F}_{i,j} \| \leq \| \epsilon_i - \epsilon_{i,j} \| \leq \max_{i=1,2,\ldots,n} \| \delta_{i,j,m} \| \tag{25} \]

Therefore, assumption 2 implies \( \sup_{i=1,2,\ldots,n} k \geq 0 (k+1)^{\alpha} \sum_{j=0}^{\infty} \| \mathbb{E} \epsilon_i | \mathcal{F}_{i,j} - \mathbb{E} \epsilon_i | \mathcal{F}_{i,j-1} \| = O(1). \) For any \( a_i \in \mathbb{R}, i = 1, 2, \ldots, n \) and \( s \in \mathbb{Z}^+, \) define \( M_{i,s} = \sum_{j=n+1-i} a_j (\mathbb{E} \epsilon_j | \mathcal{F}_{j,s} - \mathbb{E} \epsilon_j | \mathcal{F}_{j,s-1}) \), then \( M_{i,s} \) is measurable in the filter \( \mathcal{F}_{n,s+1} \). Besides, \( M_{i+1,s} - M_{i,s} = a_{n-s} \mathbb{E} \epsilon_{n-s} | \mathcal{F}_{n-i,s} - a_{n-s} \mathbb{E} \epsilon_{n-s} | \mathcal{F}_{n-i,s-1} \). Apply \( \pi - \lambda \) theorem to the \( \lambda \)-system \( \{ A \in \mathcal{F}_{n,s+1} : \mathbb{E} (\mathbb{E} \epsilon_{n-s} | \mathcal{F}_{n-i,s} ) \mathbf{1}_A = \mathbb{E} (\mathbb{E} \epsilon_{n-s} | \mathcal{F}_{n-i,s-1} ) \mathbf{1}_A \} \) and the \( \pi \)-system \( \{ A_n \times A_{n-1} \times \ldots \times A_{n-s+1} \} \). Here \( A_i \) is generated by \( \epsilon_i \). Then \( \mathbb{E} (\mathbb{E} \epsilon_{n-s} | \mathcal{F}_{n-i,s} - \mathbb{E} \epsilon_{n-i} | \mathcal{F}_{n-i,s-1} ) | \mathcal{F}_{n,s+1} = 0 \) almost surely. In other words, \( \{ M_{i,s} \}_{i=1,2,\ldots,n} \) is a martingale. From Burkholder’s inequality (theorem 1.1 in [57]),

\[ \| M_{n,s} \| \leq C \left( \sum_{j=1}^{n} a_j^2 (\mathbb{E} \epsilon_j | \mathcal{F}_{j,s} - \mathbb{E} \epsilon_j | \mathcal{F}_{j,s-1})^2 \right)^{m/2} \leq C \left( \sum_{j=1}^{n} a_j^2 \max_{i=1,2,\ldots,n} \delta_{i,s,m} \right) \leq C \sum_{j=1}^{n} a_j^2 \tag{26} \]

Here \( C \) is independent of \( s \). In particular, from theorem 2 in [58] and assumption 2,

\[ \| \sum_{j=1}^{n} a_j \epsilon_j \| \leq \| \sum_{j=1}^{n} a_j \mathbb{E} \epsilon_j | \mathcal{F}_{j,0} \| + \sup_{s=1}^{\infty} \| M_{n,s} \| \leq C \left( \sum_{j=1}^{n} a_j^2 \right) + C \left( \sum_{j=1}^{n} a_j^2 \max_{i=1,2,\ldots,n} \delta_{i,s,m} \right) = O \left( \sum_{j=1}^{n} a_j^2 \right) \tag{27} \]

Here \( C \) is a constant. Since \( m > 2 \), (27) implies \( \Sigma \)'s largest eigenvalue has order \( O(1) \). For \( \forall 1 \leq i < j \leq n \), assumption 2 and (25) imply

\[ | \mathbb{E} \epsilon_i | \mathcal{F}_{j,j-1} | = | \mathbb{E} \epsilon_i (\epsilon_j - \mathbb{E} \epsilon_j | \mathcal{F}_{j,j-1}) | \leq \| \epsilon_i \|_2 \times \| \epsilon_j - \mathbb{E} \epsilon_j | \mathcal{F}_{j,j-1} \|_2 \leq \| \epsilon_i \| \times \sum_{j=i-1}^{\infty} \| \mathbb{E} \epsilon_j | \mathcal{F}_{j,j-1} - \mathbb{E} \epsilon_j | \mathcal{F}_{j,j-1} \| \Rightarrow \max_{1 \leq i < j \leq n} (1+j-i)^{\alpha} \times | \mathbb{E} \epsilon_i | = O(1) \tag{28} \]

In other words, assumption 2 implies a polynomial decay on the error’s covariance.

For \( \forall \tau, \psi > 0, \ z \in \mathbb{R}, \) define \( F_\tau(x_1, \ldots, x_n) = \frac{1}{\tau} \log(\sum_{i=1}^{n} \exp(\tau x_i)); \ G_\tau(x_1, \ldots, x_n) = \frac{1}{\tau} \log(\sum_{i=1}^{n} \exp(-\tau x_i) + \sum_{i=1}^{n} \exp(-\tau x_i)) = F_\tau(x_1, \ldots, x_n, -x_1, \ldots, -x_n); \)

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\[ g_0(x) = (1 - \min(1, \max(x, 0)))^4; \ g_{\psi,z}(x) = g_0(\psi(x - z)); \text{and} \ h_{\psi,z}(x_1, \ldots, x_n) = g_{\psi,z}(G_x(x_1, \ldots, x_n)). \]

From lemma A.2 and (8) in [40] and (S1) to (S5) in [59], \( g_* = \sup_{x \in \mathbb{R}} |g''_0(x)| + |g''_0(x)| < \infty; 1_{x \leq z} \leq g_{\psi,z}(x) \leq 1_{x \leq z + 1/\psi}; \) and \( \sup_{x \in \mathbb{R}} |g''_{\psi,z}(x)| \leq g_* \psi, \sup_{x \in \mathbb{R}} |g''_{\psi,z}(x)| \leq g_* \psi^2. \]

Besides, \( \frac{\partial^2 F}{\partial x_j \partial x_k} \geq 0; \sum_{i=1}^n \frac{\partial^2 F}{\partial x_i \partial x_j} = 1; \sum_{i=1}^n \sum_{j=1}^n \left| \frac{\partial^2 F}{\partial x_i \partial x_j} \right| \leq 2\tau; \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \left| \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k} \right| \leq 6\tau^2; \) and \( F_*(x_1, \ldots, x_n) - \frac{\log(n)}{\tau} \leq \max_{i=1,2,\ldots,n} x_i \leq F_*(x_1, \ldots, x_n). \)

Therefore,

\[ G_*(x_1, \ldots, x_n) - \frac{\log(2n)}{\tau} \leq \max_{i=1,2,\ldots,n} |x_i| \leq G_*(x_1, \ldots, x_n) \tag{29} \]

Since \( \frac{\partial G_*}{\partial x_1} = \frac{\partial F}{\partial x_1} - \frac{\partial^2 F}{\partial x_1 \partial x_n} \) we get \( \sum_{i=1}^n \left| \frac{\partial G_*}{\partial x_i} \right| \leq 1. \) For \( \frac{\partial^2 G_*}{\partial x_j \partial x_k} = \frac{\partial^2 F}{\partial x_j \partial x_k} - \frac{\partial^2 F}{\partial x_j \partial x_n} - \frac{\partial^2 F}{\partial x_k \partial x_n} \), we have \( \sum_{i=1}^n \sum_{j=1}^n \left| \frac{\partial^2 G_*}{\partial x_i \partial x_j} \right| \leq 2\tau. \)

\[ \frac{\partial^3 G_*}{\partial x_i \partial x_j \partial x_k} = \left( \frac{\partial^3 F_1}{\partial x_i \partial x_j \partial x_k} + \frac{\partial^3 F_2}{\partial x_i \partial x_j \partial x_k} + \frac{\partial^3 F_3}{\partial x_i \partial x_j \partial x_k} \right) - \left( \frac{\partial^3 F_1}{\partial x_i \partial x_j \partial x_k} + \frac{\partial^3 F_2}{\partial x_i \partial x_j \partial x_k} + \frac{\partial^3 F_3}{\partial x_i \partial x_j \partial x_k} \right) \tag{30} \]

we have \( \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \left| \frac{\partial^3 G_*}{\partial x_i \partial x_j \partial x_k} \right| \leq 6\tau^2. \) Since \( \frac{\partial G_*(x_1, \ldots, x_n)}{\partial x_1} = g_{\psi,z}(g_{\psi,z}(x_1, \ldots, x_n)) \frac{\partial g_{\psi,z}}{\partial x_1}, \) we get \( \sum_{i=1}^n \left| \frac{\partial h_{\psi,z}}{\partial x_i} \right| \leq g_* \psi. \)

\[ \frac{\partial^2 h_{\psi,z}}{\partial x_i \partial x_j} = g_{\psi,z}(g_{\psi,z}(x_1, \ldots, x_n)) \frac{\partial G_{\psi,z}}{\partial x_i} \frac{\partial G_{\psi,z}}{\partial x_j} + g_{\psi,z}(g_{\psi,z}(x_1, \ldots, x_n)) \frac{\partial^2 G_{\psi,z}}{\partial x_i \partial x_j} \]

\[ \Rightarrow \sum_{i=1}^n \sum_{j=1}^n \left| \frac{\partial h_{\psi,z}}{\partial x_i \partial x_j} \right| \leq g_* \psi^2 + 2g_* \psi \tau \]

\[ \frac{\partial^3 h_{\psi,z}}{\partial x_i \partial x_j \partial x_k} = g_{\psi,z}(g_{\psi,z}(x_1, \ldots, x_n)) \frac{\partial G_{\psi,z}}{\partial x_i} \frac{\partial G_{\psi,z}}{\partial x_j} \frac{\partial G_{\psi,z}}{\partial x_k} + g_{\psi,z}(g_{\psi,z}(x_1, \ldots, x_n)) \frac{\partial^3 G_{\psi,z}}{\partial x_i \partial x_j \partial x_k} \]

\[ + g_{\psi,z}(g_{\psi,z}(x_1, \ldots, x_n)) \left( \frac{\partial^2 G_{\psi,z}}{\partial x_i \partial x_j} \frac{\partial G_{\psi,z}}{\partial x_k} + \frac{\partial^2 G_{\psi,z}}{\partial x_j \partial x_k} \frac{\partial G_{\psi,z}}{\partial x_i} + \frac{\partial^2 G_{\psi,z}}{\partial x_k \partial x_i} \frac{\partial G_{\psi,z}}{\partial x_j} \right) \]

\[ = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \left| \frac{\partial^3 h_{\psi,z}}{\partial x_i \partial x_j \partial x_k} \right| \leq g_* \psi^3 + 6g_* \psi \tau^2 + 6g_* \psi \tau^2 \]

**B Proof of theorem 1 and theorem 2**

We first prove necessary lemmas.

**Lemma 1**

*Suppose random variables \( \epsilon_i, i = 1, 2, \ldots, n \) satisfy assumption 2, and \( \{\gamma_{ij}\}_{i=1,2,\ldots,k,j=1,2,\ldots,n} \in \mathbb{R}^{k \times n} \) satisfy*
max_{i=1,2,...,k} \sum_{j=1}^{n} \gamma_{ij}^2 \leq D^2. \quad \text{Then} \quad \max_{i=1,2,...,k} | \sum_{j=1}^{n} \gamma_{ij} \epsilon_j | = O_p(k^{1/m} D) \quad (32)

Proof. Form \[27], \forall \xi > 0,

\[ \text{Prob} \left( \max_{i=1,2,...,k} | \sum_{j=1}^{n} \gamma_{ij} \epsilon_j | > \xi \right) \leq \frac{1}{\xi^m} \sum_{i=1}^{k} \left\| \sum_{j=1}^{n} \gamma_{ij} \epsilon_j \right\|_m^m = O \left( \frac{k}{\xi^m} \times \left( \max_{i=1,2,...,k} \sum_{j=1}^{n} \gamma_{ij}^2 \right)^{m/2} \right) \] \quad (33)

is proved by choosing \( \xi = Ck^{1/m} D \) with a sufficiently large constant \( C \).

Lemma 1 lays the theoretical foundation for the threshold ridge regression estimator \( \hat{\beta} \)'s (model selection) consistency. Lemma 2 is a corollary of Chernozhukov et al. \[60\]. It introduces some properties of a Gaussian random vector.

**Lemma 2**

1. Suppose \( \epsilon = (\epsilon_1, ..., \epsilon_n)^T \) are joint normal random variables, \( E\epsilon = 0, E\epsilon \epsilon^T = \Sigma = (\sigma_{ij})_{i,j=1,...,n} \) is non-singular; and \( \exists 0 < c_0 \leq C_0 < \infty \) such that \( c_0 \leq \sigma_{ii} \leq C_0, i = 1, 2, ..., n \). Then

\[ \sup_{x \in \mathbb{R}} | \text{Prob} \left( \max_{i=1,2,...,n} | \epsilon_i | \leq x + \delta \right) - \text{Prob} \left( \max_{i=1,2,...,n} | \epsilon_i | \leq x \right) | \leq C \delta (1 + \sqrt{\log(n)} + \sqrt{\log(\delta)}) \] \quad (34)

Here \( C \) only depends on \( c_0 \) and \( C_0 \).

2. In addition suppose \( \xi = (\xi_1, ..., \xi_n)^T \) are joint normal random variables with \( E\xi = 0, E\xi \xi^T = \Sigma^* = (\sigma^*_{ij})_{i,j=1,2,...,n} \) and \( \Delta = \max_{i,j=1,2,...,n} | \sigma_{ij} - \sigma^*_{ij} | < 1 \). Then

\[ \sup_{x \in \mathbb{R}} | \text{Prob} \left( \max_{i=1,2,...,n} | \xi_i | \leq x \right) - \text{Prob} \left( \max_{i=1,2,...,n} | \xi_i | \leq x \right) | \leq C^* \left( \Delta^{1/3} (1 + \log^3(n)) + \frac{\Delta^{1/6}}{1 + \log^{1/4}(n)} \right) \] \quad (35)

for \( n = 1, 2, ... \). Here \( C^* \) only depends on \( c_0 \) and \( C_0 \).

We emphasize that \( \Sigma^* \) can be singular. From theorem 4.1.5 in \[33\] \( \Sigma^* = QAQ^T, A = \text{diag}(\lambda_1, ..., \lambda_r, 0, 0, ..., 0), 0 \leq r < n, \) and \( Q = (q_{ij})_{i,j=1,...,n} \) satisfies \( QQ^T = Q^T Q = I_n \). We define \( \tau = Q^T \xi = (\tau_1, ..., \tau_r, 0, ..., 0)^T \) almost surely. For any continuously differentiable
function $f$ such that $E \sum_{i=1}^{n} |\frac{\partial f}{\partial x_i}| < \infty$, lemma 2 in [40] implies

$$E\xi(f) = \sum_{j=1}^{r} q_j E\tau_j f(Q\tau) = \sum_{j=1}^{r} \sum_{k=1}^{n} q_j q_k E(\tau_j \tau_k) \times E \frac{\partial f}{\partial x_k} (Q\tau) = \sum_{l=1}^{n} E \frac{\partial f}{\partial x_l} (\xi) E\xi \xi_l$$

(36)

This observation assures that (35) works for degenerated $\xi$.

Proof. Since $|\epsilon_i| = \max(\epsilon, -\epsilon) \Rightarrow \max_{i=1,2,\ldots,n} |\epsilon_i| = \max(\max_{i=1,2,\ldots,n} \epsilon_i, \max_{i=1,2,\ldots,n} -\epsilon_i)$, and $-\epsilon$ has the same distribution as $\epsilon$,

$$\sup_{x \in \mathbb{R}} \left( \text{Prob} \left( \max_{i=1,2,\ldots,n} |\epsilon_i| \leq x + \delta \right) - \text{Prob} \left( \max_{i=1,2,\ldots,n} |\epsilon_i| \leq x \right) \right) \leq \sup_{x \in \mathbb{R}} \text{Prob} \left( x < \max_{i=1,2,\ldots,n} \epsilon_i \leq x + \delta \right)$$

$$+ \sup_{x \in \mathbb{R}} \text{Prob} \left( x < \max_{i=1,2,\ldots,n} -\epsilon_i \leq x + \delta \right) \leq 2 \sup_{x \in \mathbb{R}} \text{Prob} \left( \max_{i=1,2,\ldots,n} |\epsilon_i - x| \leq \delta \right)$$

(37)

From theorem 3 and (18), (19) in Chernozhukov et al.[60] (also see lemma 3 in Zhang and Politis [28]), we define $\sigma = \min_{i=1,2,\ldots,n} \sigma_{ii}$ and $\tau = \max_{i=1,2,\ldots,n} \sigma_{ii}$,

$$\sup_{x \in \mathbb{R}} \text{Prob} \left( \left| \max_{i=1,2,\ldots,n} \epsilon_i - x \right| \leq \delta \right) \leq \frac{\sqrt{2} \delta}{\sigma} \left( \sqrt{\log(n)} + \sqrt{\max(1, \log(\sigma) - \log(\delta))} \right)$$

$$+ \frac{4\sqrt{2} \delta}{\sigma} \times \left( \frac{\tau}{\sigma} \sqrt{\log(n)} + 2 + \frac{\tau}{\sigma} \sqrt{\max(0, \log(\sigma) - \log(\delta))} \right)$$

$$\leq \frac{\sqrt{2} \delta}{c_0} \left( \sqrt{\log(n)} + 1 + |\log(c_0)| + |\log(C_0)| + \sqrt{\log(\delta)} \right)$$

$$+ \frac{4\sqrt{2} \delta C_0}{c_0} \left( \sqrt{\log(n)} + 2 + \sqrt{|\log(c_0)| + |\log(C_0)|} + \sqrt{\log(\delta)} \right)$$

(38)

and we prove (34).

Without loss of generality, assume $\epsilon$ is independent of $\xi$. Similar to Chernozhukov et al.[60], for any $0 \leq t \leq 1$, we define random variables $Z_i(t) = \sqrt{t} \epsilon_i + \sqrt{1-t} \xi_i$. According
Suppose $\epsilon$ in (31), (36), and theorem 2.27 in Folland [38],

$$Eh_{\tau,\psi,\epsilon}(\epsilon_1, \ldots, \epsilon_n) - Eh_{\tau,\psi,\epsilon}(\xi_1, \ldots, \xi_n)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \int_{0}^{1} t^{-1/2} E\left( \frac{\partial h_{\tau,\psi,\epsilon}(Z_i(t), \ldots, Z_n(t))}{\partial x_i} \right) dt - \frac{1}{2} \sum_{i=1}^{n} \int_{0}^{1} (1-t)^{-1/2} E\left( \frac{\partial h_{\tau,\psi,\epsilon}(Z_i(t), \ldots, Z_n(t))}{\partial x_i} \right) dt$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \left( \sigma_{ik} - \sigma_{*ik} \right) \int_{0}^{1} E\left( \frac{\partial^2 h_{\tau,\psi,\epsilon}(Z_i(t), \ldots, Z_n(t))}{\partial x_i \partial x_k} \right) dt$$

$$\Rightarrow |Eh_{\tau,\psi,\epsilon}(\epsilon_1, \ldots, \epsilon_n) - Eh_{\tau,\psi,\epsilon}(\xi_1, \ldots, \xi_n)| \leq \Delta \times (g_\epsilon \psi^2 + g_\epsilon \psi \tau)$$

(39)

For any $x \in \mathbb{R}$ and given $\tau, \psi > 0$, define $t = \frac{1}{\psi} + \frac{\log(2n)}{\psi}$, then

$$\text{Prob} \left( \max_{i=1,2,\ldots,n} |\epsilon_i| \leq x \right) - \text{Prob} \left( \max_{i=1,2,\ldots,n} |\xi_i| \leq x \right)$$

$$\leq \text{Prob} \left( \max_{i=1,2,\ldots,n} |\epsilon_i| \leq x - t \right) + C t (\sqrt{\log(n)} + \sqrt{\log(t)} + 1) - \text{Prob} \left( \max_{i=1,2,\ldots,n} |\xi_i| \leq x \right)$$

$$\leq Eh_{\tau,\psi,x-\frac{t}{\psi}}(\epsilon_1, \ldots, \epsilon_n) - Eh_{\tau,\psi,\xi}(\xi_1, \ldots, \xi_n) + C t (\sqrt{\log(n)} + \sqrt{\log(t)} + 1)$$

$$\geq \text{Prob} \left( \max_{i=1,2,\ldots,n} |\epsilon_i| \leq x + t \right) - C t (\sqrt{\log(n)} + \sqrt{\log(t)} + 1) - \text{Prob} \left( \max_{i=1,2,\ldots,n} |\xi_i| \leq x \right)$$

$$\geq Eh_{\tau,\psi,\epsilon+x+\frac{\log(2n)}{\psi}}(\epsilon_1, \ldots, \epsilon_n) - Eh_{\tau,\psi,\xi}(\xi_1, \ldots, \xi_n) - C t (\sqrt{\log(n)} + \sqrt{\log(t)} + 1)$$

$$\Rightarrow \sup_{x \in \mathbb{R}} \left| \text{Prob} \left( \max_{i=1,2,\ldots,n} |\epsilon_i| \leq x \right) - \text{Prob} \left( \max_{i=1,2,\ldots,n} |\xi_i| \leq x \right) \right|$$

$$\leq \Delta \times (g_\epsilon \psi^2 + g_\epsilon \psi \tau) + C t (\sqrt{\log(n)} + \sqrt{\log(t)} + 1)$$

(40)

We choose $\tau = \psi = \left(1 + \log^{3/2}(n)\right)/\Delta^{1/3}$, then $\exists$ a constant $C_1 > 0$ such that $\frac{1}{\epsilon_1} \frac{\Delta^{1/3}}{1+\log^{3/2}(n)} \leq t = \Delta^{1/3} \left( \frac{1+\log(2)}{1+\log^{3/2}(n)} \right) + \frac{\Delta^{1/3}}{1+\log^{3/2}(n)} \leq C_1 \Delta^{1/3}$ for $n = 1, 2, \ldots$ and we prove [35].

Lemma 2 approximates the distribution of the linear combinations of errors $\epsilon$ by a multivariate normal random vector. This lemma can be used to derive the Gaussian approximation theorem for the estimator $\hat{\beta}$.

**Lemma 3** (Gaussian approximation theorem)

**Suppose** $\epsilon_i, i = 1, 2, \ldots, n$ **satisfy assumption 2**, and $(\gamma_{ij})_{i=1,\ldots,p_1,j=1,\ldots,n} \in \mathbb{R}^{p_1 \times n}$ **has** **rank** $p_1$, $p_1 = O(1)$, $p_1 \leq n$. **Besides**, suppose $\exists$ **constants** $0 < c_\gamma \leq C_\gamma \leq C_{\gamma}$ such that $c_\gamma \leq \sum_{j=1}^{n} \gamma_{ij}^2 \leq C_\gamma$ for $i = 1, 2, \ldots, p_1$ and $n = 1, 2, \ldots$. **Define** $\gamma = \max_{i=1,\ldots,p_1,j=1,\ldots,n} |\gamma_{ij}|$
and assume $\gamma^* = o(n^{-1/4} \log^{-1}(n))$, here $z = \max(q, \frac{3\log n}{\log q})$. Then

$$\sup_{x \in \mathbb{R}} |\text{Prob} \left( \max_{i=1,2,\ldots,p_1} \left| \sum_{j=1}^{n} \gamma_{ij} \varepsilon_j \right| \leq x \right) - \text{Prob} \left( \max_{i=1,2,\ldots,p_1} \left| \sum_{j=1}^{n} \gamma_{ij} \xi_j \right| \leq x \right)| = o(1) \quad (41)$$

Here $\xi = (\xi_1, \ldots, \xi_n)^T$ has multivariate normal distribution, $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi\xi^T = \Sigma$, the covariance matrix of $\varepsilon$.

Proof. For any given $\psi, \tau > 0$, define $t = \frac{1}{\psi} + \frac{\log(2s)}{\tau}$. Assumption 2, (27), and (40) imply

$$\sup_{x \in \mathbb{R}} |\text{Prob} \left( \max_{i=1,2,\ldots,p_1} \left| \sum_{j=1}^{n} \gamma_{ij} \varepsilon_j \right| \leq x \right) - \text{Prob} \left( \max_{i=1,2,\ldots,p_1} \left| \sum_{j=1}^{n} \gamma_{ij} \xi_j \right| \leq x \right)|$$

$$\leq \sup_{x \in \mathbb{R}} |\text{E}h_{r,\psi,x}(\sum_{j=1}^{n} \gamma_{ij} \varepsilon_j, \ldots, \sum_{j=1}^{n} \gamma_{p_1,j} \varepsilon_j) - \text{E}h_{r,\psi,x}(\sum_{j=1}^{n} \gamma_{ij} \xi_j, \ldots, \sum_{j=1}^{n} \gamma_{p_1,j} \xi_j)|$$

$$+ O \left( t(\sqrt{\log(p_1)} + \sqrt{|\log(t)|} + 1) \right) \quad (42)$$

For any integer $s > 0$, (26) implies

$$\| \sum_{j=1}^{n} \gamma_{ij} (\varepsilon_j - \mathbb{E}\varepsilon_j|F_{j,s}) \|_m \leq \sum_{k=s}^{r} \| \sum_{j=1}^{n} \gamma_{ij} (\mathbb{E}\varepsilon_j|F_{j,k+1} - \mathbb{E}\varepsilon_j|F_{j,k}) \|_m$$

$$\leq C \sum_{j=1}^{n} \gamma_{ij} \sum_{k=s}^{r} \max_{j,k,m} \delta_{j,k,m} \quad (43)$$

$$\Rightarrow \max_{i=1,2,\ldots,p_1} \| \sum_{j=1}^{n} \gamma_{ij} (\varepsilon_j - \mathbb{E}\varepsilon_j|F_{j,s}) \|_m = O \left( \frac{1}{(s+1)^{\alpha r}} \right)$$

Here $C$ is a constant. Therefore,

$$\sup_{x \in \mathbb{R}} |\text{E}h_{r,\psi,x}(\sum_{j=1}^{n} \gamma_{ij} \varepsilon_j, \ldots, \sum_{j=1}^{n} \gamma_{p_1,j} \varepsilon_j) - \text{E}h_{r,\psi,x}(\sum_{j=1}^{n} \gamma_{ij} \xi_j, \ldots, \sum_{j=1}^{n} \gamma_{p_1,j} \xi_j)|$$

$$\leq g_{\psi} \mathbb{E} \max_{i=1,2,\ldots,p_1} \| \sum_{j=1}^{n} \gamma_{ij} (\varepsilon_j - \mathbb{E}\varepsilon_j|F_{j,s}) \|_m = O \left( \frac{\psi}{(s+1)^{\alpha r}} \right) \quad (44)$$

For any integer $k > s$, define the big block $S_{it} = \sum_{j=(l-1)\times(k+s)+k}^{(l-1)\times(k+s)+k+1} \gamma_{ij} \mathbb{E}\varepsilon_j|F_{j,s}$ and the small block $s_{it} = \sum_{j=(l-1)\times(k+s)+k+1}^{(l-1)\times(k+s)+k+2} \gamma_{ij} \mathbb{E}\varepsilon_j|F_{j,s}$ for $l = 1, 2, \ldots, r = \lfloor \frac{n}{k+s} \rfloor$. Here $\lfloor x \rfloor$ denotes the largest integer that is smaller than or equal to $x$. If $(k+s)r + k < n$, we define $S_{i(r+1)} = \sum_{j=(k+s)r+1}^{n} \gamma_{ij} \mathbb{E}\varepsilon_j|F_{j,s}$ and $s_{i(r+1)} = 0$. If $(k+s)r + k < n$, we define $S_{i(r+1)} = \sum_{j=(k+s)r+1}^{n} \gamma_{ij} \mathbb{E}\varepsilon_j|F_{j,s}$ and $s_{i(r+1)} = \sum_{j=(k+s)r+1}^{n} \gamma_{ij} \mathbb{E}\varepsilon_j|F_{j,s}$.
1, 2, ..., \(r + 1\) are independent with each other; \((s_{11}, ..., s_{p1})^T, l = 1, 2, ..., r + 1\) are independent with each other; and \(\sum_{i=1}^n \gamma_{ij} E_{ij} |F_{j,s} = \sum_{j=1}^{i+1} (S_{ij} + s_{ij})\). For \(E_{ij} |F_{j,s} = E_{ij} |F_{j,0} + \sum_{l=1}^i E_{ij} |F_{j,l} - E_{ij} |F_{j,l-1}\). (26) and (27) imply

\[
|E_{h_{s,\psi,s}}(\sum_{j=1}^{i+1} S_{ij} + s_{ij}, \ldots, \sum_{j=1}^{i+1} S_{p1j} + s_{p1j}) - E_{h_{s,\psi,s}}(\sum_{j=1}^{i+1} S_{ij}, \ldots, \sum_{j=1}^{i+1} S_{p1j})| \leq g_s \sum_{i=1}^{p1} ||S_{ij}|| \leq O \left( \frac{n \gamma^s}{\kappa} \right)
\]

Here we define \(\gamma_{ij} = 0\) for \(i = 1, 2, ..., p1, j > n\).

Define \(\xi_{j,s}, j = 1, 2, ..., n\) as the joint normal random variables with mean 0 and the same covariance matrix as \(E_{ij} |F_{j,s}, j = 1, 2, ..., n\); and are independent of \(e_i, i = ..., -1, 0, 1, \ldots\).

Define \(S_{il}^* = \sum_{j=1}^{(l-1)(k+s)+k+i} \gamma_{ij} \xi_{j,s}\) for \(l = 1, ..., r\) and \(S_{r+1}^* = \sum_{j=r(k+s)+1}^\infty \gamma_{ij} \xi_{j,s}\).

Define \(H_{it} = \sum_{j=1}^{i-1} S_{ij} + \sum_{j=i+1}^{i+r} S_{ij}\), then \(H_{it} + S_{it} = H_{it+1} + S_{it+1}^* \cdot S_{il}, i = 1, \ldots, p1, l = 1, \ldots, r + 1\) has joint normal distribution (we treat a constant as a degenerated normal random variable) and \(E S_{il}^* S_{jk}^* = E S_{il} S_{jk}\) for \(l \neq k, (S_{il}^*, S_{jk}^*)^T\) are independent with each other. From Taylor’s theorem,

\[
|E (h_{r,\psi,s} (H_{1j} + S_{1j}, \ldots, H_{p1j} + S_{p1j}) - h_{r,\psi,s} (H_{1j} + S_{1j}^*, \ldots, H_{p1j} + S_{p1j}^*)) | H_{1j}, ..., H_{p1j} | \leq \left| \sum_{l=1}^{p1} \frac{\partial h_{r,\psi,s} (H_{1j}, \ldots, H_{p1j})}{\partial x_{1j}} E (S_{lj} - S_{lj}^*) \right| + \frac{1}{2} \sum_{l=1}^{p1} \sum_{l=1}^{p1} \frac{\partial^2 h_{r,\psi,s} (H_{1j}, \ldots, H_{p1j})}{\partial x_{1j} \partial x_{1j}} E (S_{lj} S_{lj} - S_{lj}^* S_{lj}^*) \right| + g_s (\psi^3 + \tau \psi^2 + \psi^2) E (\max_{i=1,2, \ldots, p1} |S_{ij}|^3 + \max_{i=1,2, \ldots, p1} |S_{ij}|^3 ) \leq g_s (\psi^3 + \tau \psi^2 + \psi^2) \sum_{i=1}^{p1} (||S_{ij}|| + ||S_{ij}^*||) \leq C \left( \frac{\psi^3 + \tau \psi^2 + \psi^2}{\max_{i=1,2, \ldots, p1} \left( \sum_{j=(l-1)(k+s)+1}^{(j-1)(k+s)+k} \gamma_{il}^2 \right)^{3/2} } \right)^{3/2}
\]
Here $C$ is a constant. Since \( \max_{i=1, \ldots, p_1} \sum_{j=1}^n \gamma_{ij}^2 \leq C \gamma \),

\[
\sup_{x \in \mathbb{R}} |E h_{\tau, \psi, x}(\sum_{j=1}^{r+1} S_{ij}, \ldots, S_{p_1j}) - E h_{\tau, \psi, x}(\sum_{j=1}^{r+1} S_{ij}', \ldots, S_{p_1j}')| 
\leq \sum_{j=1}^{r+1} \sup_{x \in \mathbb{R}} |E h_{\tau, \psi, x}(H_{ij} + S_{ij} + P_{ij}) - E h_{\tau, \psi, x}(H_{ij} + S_{ij}', + P_{ij}')| 
\leq C(\psi^3 + \psi^2 + \psi \tau)^3 \sum_{j=1}^{r+1} \sum_{i=1}^{p_1} \left( \sum_{l=(j-1)(k+s)+1}^{(j-1)(k+s)+k} \gamma_{ij}^2 \right)^{3/2} 
\leq O \left( (\psi^3 + \psi^2 + \psi \tau)^3 \times \max_{i=1, \ldots, p_1} \left( \sum_{j=1}^{r+1} \sum_{l=(j-1)(k+s)+k}^{(j-1)(k+s)+k} \gamma_{ij}^2 \right)^{3/2} \right)
\]

which has order \( O \left( (\psi^3 + \psi^2 + \psi \tau)^3 \times \gamma^* \sqrt{\tau} \right) \). We define \( s_{ij}^l = \sum_{j=(l-1)(k+s)+k}^{l} \gamma_{ij} \xi_{j,s} \) for \( l = 1, 2, \ldots, r \), and \( s_{ij}^{r+1} = \sum_{j=(r+1)(k+s)+1}^{n} \gamma_{ij} \xi_{j,s} \). For \( s_{ij}^l \) has normal distribution,

\[
||E h_{\tau, \psi, x}(\sum_{j=1}^{r+1} S_{ij}', \ldots, S_{p_1j}') - E h_{\tau, \psi, x}(\sum_{j=1}^{r+1} S_{ij}', \ldots, S_{p_1j}')||_m = O \left( \psi \max_{i=1, \ldots, p_1} \left( \sum_{j=1}^{r+1} s_{ij}^l \right)^2 \right)
\]

According to (45), this has order \( O \left( \psi \times \gamma^* \sqrt{\frac{\tau^3}{\pi}} \right) \). From (34), (28), and 0.9.7 in [33],

\[
||E h_{\tau, \psi, x}(\sum_{j=1}^{n} \gamma_{ij} \xi_{j,s}, \ldots, \gamma_{p_1j} \xi_{j,s}) - E h_{\tau, \psi, x}(\sum_{j=1}^{n} \gamma_{ij} \xi_{j,s}, \ldots, \gamma_{p_1j} \xi_{j,s})||_m 
\leq (g_x \psi^2 + g_x \psi \tau) \times \max_{i=1, \ldots, p_1} \left( \sum_{l=1}^{n} \sum_{k_1=1}^{n} \gamma_{ij} \gamma_{j,k_2} \left( \|E \epsilon_{k_1} | F_{k_1,s} \times E \epsilon_{k_2} | F_{k_2,s} \| - \sigma_{k_1 k_2} \right) \right) 
\leq (g_x \psi^2 + g_x \psi \tau) \times \max_{i=1, \ldots, p_1} \left( \sum_{l=1}^{n} \sum_{k_1=1}^{n} \gamma_{ij} \gamma_{j,k_2} \left( \|E \epsilon_{k_1} | F_{k_1,s} \times E \epsilon_{k_2} | F_{k_2,s} \| - \epsilon_{k_2} \right) \right) + C(\psi^3 + \psi^2 + \psi \tau)^3 \times \max_{i=1, \ldots, p_1} \left( \sum_{l=1}^{n} \sum_{k_1=1}^{n} \gamma_{ij}^2 \right) \left( \frac{1}{1 + \frac{|k_1 - k_2|^2}{s}} \right)
\]

Here \( \sigma_{ij} = E \epsilon_{ij} \).

Define \( V \) such that \( \frac{1}{V} = \gamma^* \times n^{1/4} \log^3(n) \to 0 \). Choose \( k = \lceil \sqrt{n} \rceil, \psi = \tau = \psi^3, \text{ and } s = \psi^3(n+1) \log^3(n+1)(n) \). Then we prove (41).
proof of theorem \[ 7 \]

\[
\max_{i=1,2,\ldots,p} \sum_{j=1}^{n} \left( \sum_{l=1}^{p} q_{lj} \left( \frac{\lambda_{j}}{\lambda_{j}^{2} + \rho_{n}} + \frac{\rho_{n} \lambda_{j}}{(\lambda_{j}^{2} + \rho_{n})^{2}} \right) p_{lj} \right)^{2} = \max_{i=1,2,\ldots,p} \sum_{j=1}^{n} q_{ij}^{2} \left( \frac{\lambda_{j}}{\lambda_{j}^{2} + \rho_{n}} + \frac{\rho_{n} \lambda_{j}}{(\lambda_{j}^{2} + \rho_{n})^{2}} \right)^{2} \leq 4 \frac{1}{\lambda_{p}^{2}} \tag{50}
\]

Define \( s = (s_1, \ldots, s_p)^T = Q^T \beta \), from lemma \[ 1 \]

\[
\tilde{\beta}_{i} - \beta_{i} = -\rho_{n}^{2} \sum_{j=1}^{p} \frac{q_{ij} \beta_{j}}{\lambda_{p}^{2}} + \max_{i=1,2,\ldots,p} \sum_{j=1}^{n} q_{ij} \left( \frac{\lambda_{j}}{\lambda_{j}^{2} + \rho_{n}} + \frac{\rho_{n} \lambda_{j}}{(\lambda_{j}^{2} + \rho_{n})^{2}} \right) p_{lj} \epsilon_{i} = O_{p} \left( n^{1/m-\eta} \right) \Rightarrow \max_{i=1,2,\ldots,p} \left| \tilde{\beta}_{i} - \beta_{i} \right| \tag{51}
\]

Therefore,

\[
\text{Prob} \left( \tilde{N}_{bn} \neq N_{bn} \right) \leq \text{Prob} \left( \min_{i \in N_{bn}} |\tilde{\beta}_{i}| \leq b_{n} \right) + \text{Prob} \left( \max_{i \in N_{bn}} |\tilde{\beta}_{i}| > b_{n} \right) \leq \text{Prob} \left( \min_{i \in N_{bn}} |\beta_{i}| - \max_{i \in N_{bn}} |\tilde{\beta}_{i} - \beta_{i}| \leq b_{n} \right) + \text{Prob} \left( \max_{i \in N_{bn}} |\beta_{i}| + \max_{i \in N_{bn}} |\tilde{\beta}_{i} - \beta_{i}| > b_{n} \right) \tag{52}
\]

From assumption \( 4 \), \( \min_{i \in N_{bn}} |\beta_{i}| - b_{n} \geq (1/c_{b} - 1)b_{n} \) and \( b_{n} - \max_{i \in N_{bn}} |\beta_{i}| \geq (1 - c_{b})b_{n} \).

So \( \text{Prob} \left( \tilde{N}_{bn} \neq N_{bn} \right) = o(1) \) is proved by \[ 51 \].

If \( \tilde{N}_{bn} = N_{bn} \), we define \( \{c_{ij}\}_{i=1,\ldots,p, j=1,\ldots,p} \) as in section \[ 2 \] with \( b = b_{n} \) and \( \gamma = (\gamma_{1}, \ldots, \gamma_{p})^{T} = M \beta \). Then

\[
\tilde{\gamma}_{i} - \gamma_{i} = \sum_{j \in N_{bn}} m_{ij}(\tilde{\beta}_{j} - \beta_{j}) = \sum_{j \in N_{bn}} m_{ij} \beta_{j} = -\rho_{n}^{2} \sum_{j=1}^{p} \frac{c_{ij} \beta_{j}}{\lambda_{j}^{2} + \rho_{n}}^{2} + \sum_{j=1}^{p} \sum_{l=1}^{n} c_{ij} \left( \frac{\lambda_{j}}{\lambda_{j}^{2} + \rho_{n}} + \frac{\rho_{n} \lambda_{j}}{(\lambda_{j}^{2} + \rho_{n})^{2}} \right) p_{lj} \epsilon_{i} - \sum_{j \in N_{bn}} m_{ij} \beta_{j} \tag{53}
\]
For $\max_{i=1,\ldots,n_1} |\rho_n^2 \sum_{j=1}^n \frac{c_{ij} \lambda_j^3}{(\lambda_j^2 + \rho_n)^2} | \leq \frac{\rho_n^2 \sqrt{\sum_{i=1}^n |\beta_i|}}{\lambda_p} \times \|\beta\|_2 = O(n^{\alpha \beta - 2\delta})$, and
\[
\max_{i=1,2,\ldots,n_1} \sum_{j=1}^n \left( \sum_{j=1}^p c_{ij} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{2}{\lambda_j^2 + \rho_n} \right) p_{ij} \right)^2 = \max_{i=1,2,\ldots,n_1} \sum_{j=1}^p c_{ij} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{2}{\lambda_j^2 + \rho_n} \right)^2 \leq \frac{4C_M}{\lambda_p^2}
\] (54)lemma 1 implies $\max_{i=1,2,\ldots,n_1} |\tilde{\gamma}_i - \gamma_i| = O_p(n^{-\eta})$.

If $\tilde{N}_{\alpha_1} = N_{\alpha_1}$, we define $u_{ij} = \sum_{k \in \tilde{N}_{\alpha_1}} x_{ik} q_{kj}, i = 1, \ldots, n, j = 1, \ldots, p$,
\[
\tilde{e}_i - \epsilon_i = - \sum_{j \in \tilde{N}_{\alpha_1}} x_{ij}(\tilde{\beta}_j - \beta_j) + \sum_{j \notin \tilde{N}_{\alpha_1}} x_{ij}\beta_j
\] (55)
\[
\tilde{e}_i - \epsilon_i = - \sum_{j \in \tilde{N}_{\alpha_1}} x_{ij}(\tilde{\beta}_j - \beta_j) + \sum_{j \notin \tilde{N}_{\alpha_1}} x_{ij}\beta_j
\]
For $\max_{i=1,2,\ldots,n_1, n} |\sum_{j=1}^p u_{ij} s_j| \leq \frac{\rho_n^2 \sqrt{\sum_{i=1}^n u_{ij}^2}}{\lambda_p} \times \max_{i=1,2,\ldots,n} \sqrt{\sum_{j=1}^p u_{ij}^2} = O(n^{\alpha \beta - 2\delta} \sqrt{|\tilde{N}_{\alpha_1}|})$, and
\[
\max_{i=1,2,\ldots,n} \sum_{j=1}^n \left( \sum_{j=1}^p c_{ij} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{2}{\lambda_j^2 + \rho_n} \right) p_{ij} \right)^2 \leq \frac{4}{\lambda_p^2} \max_{i=1,2,\ldots,n} \sum_{j \in \tilde{N}_{\alpha_1}} x_{ij}^2 = O\left(|\tilde{N}_{\alpha_1}| \times n^{-2\eta}\right)
\] (56)from lemma 1
\[
\max_{i=1,2,\ldots,n} |\tilde{e}_i - \epsilon_i| = O_p\left(n^{(1/m-\eta)/2}\right) + O_p\left(n^{\alpha \beta - 2\delta} \sqrt{|\tilde{N}_{\alpha_1}|}\right) + O_p\left(n^{1/m-\eta} \times \sqrt{|\tilde{N}_{\alpha_1}|}\right)
\] (57)
and we prove 12.

proof of theorem 2 Define $v_{it} = \frac{1}{\tau_i} \left( \sum_{j=1}^p c_{ij} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{2}{\lambda_j^2 + \rho_n} \right) p_{ij} \right)$, then $v_{it} = 0$ if $i \notin M$.
If $\tilde{N}_{\alpha_2} = N_{\alpha_2}$, then $\tilde{\gamma}_i = \gamma_i$. From 55) and Cauchy-Schwarz inequality,
\[
\max_{i=1,\ldots,n} \left| \tilde{e}_i - \epsilon_i \right| \leq \max_{i=1,\ldots,p_1} \frac{\rho_n^2}{\tau_i} \|c_{ij}\|_1 \|\beta_i\|_1 + \max_{i=1,\ldots,p_1} \frac{1}{\tau_i} \sum_{j \notin \tilde{N}_{\alpha_1}} m_{ij} \beta_j
\] (58)
which has order \(O(n^{a_\beta + n^{-2\delta}} + o(1))\). Besides, if \(i \in \mathcal{M}\), then for sufficiently large \(n\)

\[
\sum_{l=1}^{n} v_l^2 = \frac{1}{\sum_{j=1}^{m} \nu_j^2} \sum_{j=1}^{p} \gamma_j^2 \left( \frac{\lambda_j}{\nu_j^2 + \lambda_j} + \frac{\rho_n \lambda_j}{(\nu_j^2 + \lambda_j)^2} \right)^2 \leq 1
\]

\[
\sum_{l=1}^{n} v_l^2 \geq \frac{1}{1 + \frac{1}{\sum_{j=1}^{m} \nu_j^2} \left( \frac{\lambda_j}{\nu_j^2 + \lambda_j} + \frac{\rho_n \lambda_j}{(\nu_j^2 + \lambda_j)^2} \right) \geq 1 + \frac{1}{1 + \frac{4C^2}{c_M}} > 0
\] (59)

Assumption 6 implies that the matrix \(\{v_{ij}\}_{i \in \mathcal{M}, j=1,...,n}\) has rank \(|\mathcal{M}|\). Define \(t = \frac{2\|\beta\|_2}{\chi_2} + \max_{i=1,...,p} \sqrt{n} \|\sum_{j \not\in N_{ih}} \beta_j\|_1\), lemma 6 and lemma 7 imply

\[
\text{Prob} \left( \max_{i=1,...,p} \left| \frac{\hat{\gamma}_i - \sum_{j=1}^{m} \nu_j \beta_j }{\tau_i} \right| \leq x \right) - \text{Prob} \left( \max_{i \in \mathcal{M}} | \sum_{l=1}^{n} v_l \xi_l | \leq x \right) 
\]

\[
\leq \text{Prob} \left( \hat{N}_{ih} \neq N_{ih} \right) + | \text{Prob} \left( \max_{i \in \mathcal{M}} | \sum_{l=1}^{n} v_l \epsilon_l | \leq x + t \right) - \text{Prob} \left( \max_{i \in \mathcal{M}} | \sum_{l=1}^{n} v_l \xi_l | \leq x + t \right) |
\]

\[
+ C t (1 + \sqrt{\log(p_1) + \sqrt{\log(t)}})
\]

\[
\text{Prob} \left( \max_{i=1,...,p} \left| \frac{\hat{\gamma}_i - \sum_{j=1}^{m} \nu_j \beta_j }{\tau_i} \right| \leq x \right) - \text{Prob} \left( \max_{i \in \mathcal{M}} | \sum_{l=1}^{n} v_l \xi_l | \leq x \right) 
\]

\[
\geq - \text{Prob} \left( \hat{N}_{ih} \neq N_{ih} \right) - | \text{Prob} \left( \max_{i \in \mathcal{M}} | \sum_{l=1}^{n} v_l \epsilon_l | \leq x - t \right) - \text{Prob} \left( \max_{i \in \mathcal{M}} | \sum_{l=1}^{n} v_l \xi_l | \leq x - t \right) |
\]

\[
- C t (1 + \sqrt{\log(p_1) + \sqrt{\log(t)}})
\]

\[
\Rightarrow \sup_{x \in \mathbb{R}} | \text{Prob} \left( \max_{i=1,...,p} \left| \frac{\hat{\gamma}_i - \sum_{j=1}^{m} \nu_j \beta_j }{\tau_i} \right| \leq x \right) - \text{Prob} \left( \max_{i \in \mathcal{M}} | \sum_{l=1}^{n} v_l \xi_l | \leq x \right) |
\]

\[
\leq \text{Prob} \left( \hat{N}_{ih} \neq N_{ih} \right) + \sup_{x \in \mathbb{R}} | \text{Prob} \left( \max_{i \in \mathcal{M}} | \sum_{l=1}^{n} v_l \epsilon_l | \leq x \right) - \text{Prob} \left( \max_{i \in \mathcal{M}} | \sum_{l=1}^{n} v_l \xi_l | \leq x \right) |
\]

\[
+ C t (1 + \sqrt{\log(p_1) + \sqrt{\log(t)}})
\] (60)

and we prove (14). Here \(C\) is a constant. \(\square\)

C Proof of theorem 3

Errors in the linear model 2 can be dependent, non-stationary and heteroskedastic. So it is hopeless to estimate the errors’ variance and covariance. However, it is still possible to make a consistent estimation of the estimator \(\hat{\gamma} = M \hat{\beta}\)'s variances. Lemma 4 justifies this idea.

Lemma 4
Suppose random variables \( \epsilon_i, i = 1, 2, ..., n \) satisfy assumption 2; \( (\gamma_{ij})_{i=1,...,p_1, j=1,...,n} \in \mathbb{R}^{p_1 \times n} \) satisfy conditions in lemma 3, the kernel function \( K \) satisfies assumption 7; and \( k_n > 0 \) is a chosen bandwidth. Then

\[
\max_{i,j=1,\ldots,p_1} \left| \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \gamma_{i s_1} \gamma_{j s_2} K \left( \frac{s_1 - s_2}{k_n} \right) \right| \leq C \left( \sum_{s=1}^{n} \gamma_{i}^{2} \right) \left( \sum_{s=1}^{n} \gamma_{j}^{2} \right) \left( \sum_{s=0}^{\infty} \frac{1 - K(s/k_n)}{1 + s^{2\alpha}} \right) \leq 2C \sum_{s=1}^{n} \gamma_{s}^{2} \sum_{s=0}^{\infty} \frac{1 - K(s/k_n)}{1 + s^{2\alpha}}.
\]

\[
\sum_{s=0}^{\infty} \frac{1 - K(s/k_n)}{1 + s^{2\alpha}} \leq \max_{x \in [0,1]} |K'(x)| \sum_{s=0}^{k_n} \frac{s}{1 + s^{2\alpha}} + \sum_{s=k_n+1}^{\infty} \frac{1}{1 + s^{2\alpha}} = O \left( \frac{1}{k_n} \left( 1 + \int_{[1,k_n]} x^{1-\alpha} dx \right) + \int_{[k_n,\infty]} x^{-\alpha} dx \right).
\]

Here \( \sigma_{s_1 s_2}, s_1, s_2 = 1, ..., n \) are defined in section 2 and \( z = \max\left( \frac{2}{\pi}, \frac{3n}{2\alpha\sigma_{\epsilon}} \right) \).

Proof. From (28) and section 0.9.7 in [33], there exists a constant \( C > 0 \) such that

\[
\left| \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \gamma_{i s_1} \gamma_{j s_2} \right| \leq C \left( \sum_{s=1}^{n} \gamma_{i}^{2} \right) \left( \sum_{s=1}^{n} \gamma_{j}^{2} \right) \left( \sum_{s=0}^{\infty} \frac{1 - K(s/k_n)}{1 + s^{2\alpha}} \right) \leq 2C \sum_{s=1}^{n} \gamma_{s}^{2} \sum_{s=0}^{\infty} \frac{1 - K(s/k_n)}{1 + s^{2\alpha}}.
\]

\[
\sum_{s=0}^{\infty} \frac{1 - K(s/k_n)}{1 + s^{2\alpha}} \leq \max_{x \in [0,1]} |K'(x)| \sum_{s=0}^{k_n} \frac{s}{1 + s^{2\alpha}} + \sum_{s=k_n+1}^{\infty} \frac{1}{1 + s^{2\alpha}} = O \left( \frac{1}{k_n} \left( 1 + \int_{[1,k_n]} x^{1-\alpha} dx \right) + \int_{[k_n,\infty]} x^{-\alpha} dx \right)
\]

we have \( \max_{i,j=1,\ldots,p_1} \left| \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \left( 1 - K \left( \frac{s_1 - s_2}{k_n} \right) \right) \gamma_{s_1} \gamma_{s_2} \sigma_{s_1 s_2} \right| = O(v_n).

On the other hand, define \( \zeta_{i,k} = \epsilon_i \epsilon_{i+k} - \sigma_{i+k} = h_{i,i+k}(..., \epsilon_{i+k-1}, \epsilon_{i+k}) \) (i.e., \( \zeta_{i,k} \) is \( \mathcal{F}_{i+k} \) measurable), and define \( \zeta_{i,k,t} = h_{i,i+k}(..., \epsilon_{i+k-t-1}, \epsilon_{i+k-t}, \epsilon_{i+k-t+1}, ..., \epsilon_{i+k}) \). Here...
\[
\psi_{i,k,t,m/2} = \|\xi_{i,k} - \xi_{i,k,t}\|_{m/2} = \|\epsilon_i \epsilon_{i+k} - \epsilon_{i,t-k} \epsilon_{i+k,t}\|_{m/2}
\]
\[
\leq \|\epsilon_i - \epsilon_{i,t-k}\| \epsilon_{i+k}\|_{m} + \|\epsilon_{i,t-k}\| \|\epsilon_{i+k} - \epsilon_{i+k,t}\|_{m}
\]
\[
\Rightarrow \max_{i=1,\ldots,n-k} \psi_{i,k,t,m/2} \leq C \max_{i=1,\ldots,n} \delta_{i,t-k,m} + C' \max_{i=1,\ldots,n} \delta_{i,t,m} \text{ if } t-k \geq 0
\]
\[
\text{and } C' \max_{i=1,\ldots,n} \delta_{i,t,m} \text{ if } t-k < 0 \tag{65}
\]

Here \(\epsilon_{i,t} = \epsilon_i \) if \( t < 0 \) and \( C \) is a constant. For fixed \( i, j \), define
\[
N_{s,k,t} = \sum_{t=n-k+1-s}^{n-k} \gamma_{il} \gamma_{jl+k} (E_{\xi_{i,k,t}} | F_{t+k,t} - E_{\xi_{i,k,t}} | F_{t+k,t-1})
\]
then \( N_{s,k,t} \) is \( F_{n,s+t-1} \) measurable. Apply \( \pi - \lambda \) theorem to the \( \lambda \)-system
\[
\{ A \in F_{n,s+t-1} \mid E (E_{\xi_{n-k-s,t}} | F_{n-s,t}) \times I_A = E (E_{\xi_{n-k-s,t}} | F_{n-s,t-1}) \times I_A \} \tag{66}
\]
and the \( \pi \)-system \( \{ A_n \times A_{n-1} \times \ldots \times A_{n-s-t+1} \} \), \( A_i \) is generated by \( \epsilon_i \). Then we know that \( N_{s,k,t}, s = 1, 2, \ldots, n-k \) form a martingale for any given \( k, t \). From \( 25, 26 \) and \( 27 \)
\[
\| \sum_{t=1}^{n-k} \sum_{l=1}^{n-k} \gamma_{il} \gamma_{jl+k} \|_{m/2} \leq \sum_{t=1}^{n-k} \| \sum_{l=1}^{n-k} \gamma_{il} \gamma_{jl+k} \|_{m/2} + \sum_{t=1}^{n-k} \| N_{n-k,t} \|_{m/2}
\]
\[
\leq C \sum_{t=1}^{n-k} \sum_{l=1}^{n-k} \gamma_{il} \gamma_{jl+k} \| \xi_{t,k} \|_{m/2} + C \sum_{t=1}^{n-k} \sum_{l=1}^{n-k} \| \xi_{t,k} \|_{m/2}
\]
\[
\text{Here } C \text{ is a constant independent with } i, j. \text{ Since } \| \xi_{t,k} \|_{m/2} \leq \| \epsilon_i \| \times \| \epsilon_{i+k} \| + |\sigma_{i+k}| \text{ and}
\]
\[
\sum_{t=1}^{\infty} \max_{i=1,\ldots,n-k} \psi_{i,t,k,m/2} \leq 2C \sum_{t=0}^{\infty} \max_{i=1,\ldots,n} \delta_{i,t,m} \text{ with } C \text{ a constant} \tag{68}
\]
we have
\[
\max_{i,j=1,\ldots,p_t} \frac{\sum_{t=1}^{n-k} \gamma_{il} \gamma_{jl+k} \| \xi_{t,k} \|_{m/2}}{\sqrt{\sum_{t=1}^{n-k} \gamma_{il}^2 \| \xi_{t,k} \|_{m/2}}} = O(1) \tag{69}
\]
Therefore
\[
\sum_{t=1}^{n} \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \gamma_{s_1} \gamma_{s_2} K \left( \frac{s_1 - s_2}{k_n} \right) (\epsilon_{s_1} \epsilon_{s_2} - \sigma_{s_1} \sigma_{s_2}) \| \xi_{t,k} \|_{m/2}
\]
\[
\leq 2 \sum_{l=0}^{n-k} \sum_{s=1}^{n-l} \gamma_{s} \gamma_{s+l} \| \xi_{t,k} \|_{m/2} \tag{70}
\]
\[
\leq C \sum_{l=0}^{\infty} K \left( \frac{l}{k_n} \right) \times \max_{i=1,\ldots,p_t,j=1,\ldots,n} |\gamma_{ij}| \text{ with } C \text{ a constant}
\]
For $K$ is decreasing on $[0, \infty)$,
\[
\sum_{l=1}^{\infty} K(l/k_n) \leq \sum_{l=1}^{\infty} \int_{[l-1,l]} K(x/k_n) dx = k_n \int_{[0,\infty)} K(x) dx = O(k_n) \quad (71)
\]

From (64) and (70), we prove (61).

**proof of theorem 3**

Recall $E^* = E \cdot |y|$, $Prob^*(\cdot) = Prob(\cdot|y)$, and define $\epsilon_i^*, i = 1, \ldots, n$ as in algorithm 1, $\forall a = (a_1, ..., a_n)^T \in \mathbb{R}^n$, $\sum_{i=1}^n a_i \epsilon_i^*$ has normal distribution. Section 0.9.7 in [33] and (71) implies

\[
E^* |\sum_{i=1}^{n} a_i \epsilon_i^*|^m = C \left( \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j K \left( \frac{|i-j|}{k_n} \right) \right)^{m/2} = O \left( k_n^{m/2} \times \|a\|_2^m \right) \quad (72)
\]

Here $C = E|Y|^m$, $Y$ has normal distribution with mean 0 and variance 1. Therefore, $\forall \xi > 0$,

\[
Prob^* \left( \max_{i=1,\ldots,p} \left| \frac{1}{\xi^m} \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} p_{ij} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2} \right) \epsilon_i^* \right| > \xi \right)
\]

\[
\leq \frac{1}{\xi^m} \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} p_{ij} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2} \right) \epsilon_i^* \epsilon_j^* \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2} \right)^2 \leq \frac{\alpha^m}{\xi^m} \times \max_{i=1,\ldots,n} |\epsilon_i|^m \times \frac{\alpha^m p}{\lambda_n^2} \quad (73)
\]

Here $C$ is a constant. From theorem 4, $\max_{i=1,\ldots,n} |\epsilon_i|^m \leq 2^m \max_{i=1,\ldots,n} |\epsilon_i|^m + 2^m \max_{i=1,\ldots,n} |\epsilon_i - \epsilon_i|^m = O_p(n)$. If $\tilde{\beta}_k = N_{b_n}$, (51) implies $\|\tilde{\beta} - \tilde{\beta}\|_2 \leq 2 \sqrt{\sum_{i \in N_{b_n}} (\tilde{\beta}_i - \beta_i)^2} + 2 \|\tilde{\beta}\|_2 = O_p(n^{\alpha})$. Define $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_p)^T = Q^T \tilde{\beta}$. If $\tilde{X}_{b_n} = X_{b_n}$, then for any given $0 < a < 1$, there exists a constant $C > 0$ such that $\|\tilde{s}\|_2 \leq C n^{\alpha}$ and $\max_{i=1,\ldots,n} |\epsilon_i|^m \leq C n$ with probability at least $1 - a$. From (73), $\forall \epsilon > 0$,

\[
\tilde{s}_i - \hat{s}_i = -\rho_n^2 \sum_{j=1}^{p} q_{ij} \tilde{s}_j + \rho_n \lambda_j p_{ij} \left( \frac{\lambda_j}{\lambda_j^2 + \rho_n} \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2} \right) \epsilon_i^* 
\]

\[
\Rightarrow \quad \text{Prob}^* \left( \max_{i=1,\ldots,p} |\tilde{s}_i - \hat{s}_i| > 2 \epsilon m^{-\alpha} \right)
\]

\[
\leq \text{Prob}^* \left( \frac{n \epsilon^2 \rho_n^2 \|\tilde{s}\|_2^2}{\lambda_n^4} > \epsilon \right) + C' \frac{\epsilon^2}{\epsilon m} \times \left( \sqrt{k_n} \times n^{2/m - \eta + \nu} \right)^m
\]

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Here $C'$ depends on $C$. If $\tilde{N}_b = N_b$,

$$
Prob^*(\tilde{N}_b \neq N_b) \leq Prob^\ast\left(\max_{i \in N_{b\ast}} |\tilde{\beta}_i| > b_n \right) + Prob^\ast\left(\min_{i \in N_{b\ast}} |\tilde{\beta}_i| \leq b_n \right)
$$

$$
\leq Prob^\ast\left(\max_{i \in N_{b\ast}} |\tilde{\beta}_i - \tilde{\beta}_i| > b_n - \max_{i \in N_{b\ast}} |\tilde{\beta}_i| \right) + Prob^\ast\left(\max_{i \in N_{b\ast}} |\tilde{\beta}_i| \geq \min_{i \in N_{b\ast}} |\tilde{\beta}_i| - b_n \right)
$$

(75)

From assumption 4 and (51), with probability tending to 1, $\max_{i \in N_{b\ast}} |\tilde{\beta}_i| \leq \frac{2k_n}{\tau_i}$. (74) implies $Prob^\ast\left(\tilde{N}_b \neq N_b \right) = o_p(1)$.

If $\tilde{N}_b = N_b = N_b\ast$, define $c_{ij}, i = 1, \ldots, p, j = 1, \ldots, p$ as in section 2.

$$
T_i = \sum_{j \in N_{b\ast}} m_{ij} (\tilde{\beta}_j - \tilde{\beta}_i)
$$

(76)

Besides, $\tilde{T}_i = \tau_i$, and $c_{ij} = c_{ij}, i = 1, \ldots, p, j = 1, \ldots, p$. Define $v_{il} = \frac{1}{\tau_i} \sum_{j=1}^p c_{ij} p_{ij} (\frac{\lambda_j}{\lambda_j^2 + \rho_n} + \frac{\rho_n \lambda_j}{(\lambda_j^2 + \rho_n)^2})$, $i = 1, \ldots, p, l = 1, \ldots, n$; and $z = (z_1, \ldots, z_n)^T$ such that $z_i = \sum_{l=1}^n v_{il} t_l$. We have $E z_i z_j = \sum_{l=1}^n \sum_{l=1}^n v_{il} v_{lj} \tilde{\epsilon}_l \tilde{\epsilon}_j K\left(\frac{l_1 - l_2}{K_n}\right)$. Form section 0.9.7 in [53], lemma 4 and theorem 1.

$$
|E z_i z_j - \sum_{l_1=1}^n \sum_{l_2=1}^n v_{il_1} v_{lj_2} \sigma_{l_1 l_2}| \leq \sum_{l_1=1}^n \sum_{l_2=1}^n v_{il_1} v_{lj_2} K\left(\frac{l_1 - l_2}{K_n}\right) \left(\tilde{\epsilon}_l \tilde{\epsilon}_j - \epsilon_{l_1} \epsilon_{l_2}\right)
$$

(77)

$$
\leq 2 \sum_{l=0}^\infty K\left(\frac{l}{K_n}\right) \times \left(\sum_{l=1}^n v_{i l_1}^2 \tilde{\epsilon}_l^2 + \sum_{l=1}^n v_{i l_2}^2 \tilde{\epsilon}_j^2 + \sum_{l=1}^n v_{i l_1}^2 \tilde{\epsilon}_l^2 + \sum_{l=1}^n v_{i l_2}^2 \tilde{\epsilon}_j^2\right) + \sum_{l_1=1}^n \sum_{l_2=1}^n v_{il_1} v_{lj_2} \epsilon_{l_1} \epsilon_{l_2} K\left(\frac{l_1 - l_2}{K_n}\right) - \sum_{l_1=1}^n \sum_{l_2=1}^n v_{il_1} v_{lj_2} \sigma_{l_1 l_2}
$$

$$
= o_p(k_n x n^{-1/2}) + o_p(k_n x n^{-1/4} \log^{-\varepsilon}(n)) + o_p(v_n) = o_p(1)
$$

$v_n$ is defined in lemma 4 and $z$ is defined in assumption 6. From lemma 2, assumption 6, (27), and (59), $\sup_{x \in \mathbb{R}} |Prob^\ast(\max_{i \in \mathbb{M}} |z_i| \leq x) - H(x)| = o_p(1)$. Since

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Define $t = \frac{\|\beta\|_2^2}{\lambda_p^2}$ and assume $\hat{N}_{bn} = N_{bn}$,

\begin{align*}
\text{Prob}^* \left( \max_{i=1,\ldots,p_1} \frac{|\hat{\gamma}_i^* - \hat{\gamma}_i|}{\hat{\gamma}_i^*} \leq x \right) - H(x) &\leq \text{Prob}^* \left( \hat{N}_{bn} \neq N_{bn} \right) \\
+ \left( \text{Prob}^* \left( \max_{i=1,\ldots,p_1} |\hat{\gamma}_i| \leq x + t \right) - H(x + t) \right) + Ct(1 + \sqrt{\log(p_1) + \sqrt{\log(t)}}) &\leq \text{Prob}^* \left( \hat{N}_{bn} \neq N_{bn} \right) \\
\text{and} \quad \text{Prob}^* \left( \max_{i=1,\ldots,p_1} \frac{|\hat{\gamma}_i^* - \hat{\gamma}_i|}{\hat{\gamma}_i^*} \leq x \right) - H(x) &\geq -\text{Prob}^* \left( \hat{N}_{bn} \neq N_{bn} \right) \\
+ \left( \text{Prob}^* \left( \max_{i=1,\ldots,p_1} |\hat{\gamma}_i| \leq x - t \right) - H(x - t) \right) - Ct(1 + \sqrt{\log(p_1) + \sqrt{\log(t)}}) &\leq \text{Prob}^* \left( \hat{N}_{bn} \neq N_{bn} \right) \\
\Rightarrow \sup_{x \in \mathbb{R}} \left| \text{Prob}^* \left( \max_{i=1,\ldots,p_1} \frac{|\hat{\gamma}_i^* - \hat{\gamma}_i|}{\hat{\gamma}_i^*} \leq x \right) - H(x) \right| &\leq \text{Prob}^* \left( \hat{N}_{bn} \neq N_{bn} \right) + \sup_{x \in \mathbb{R}} \left| \text{Prob}^* \left( \max_{i=1,\ldots,p_1} |\hat{\gamma}_i| \leq x \right) - H(x) \right| + Ct(1 + \sqrt{\log(p_1) + \sqrt{\log(t)}}) = o_p(1)
\end{align*}

(79)

For the non-degenerated joint normal distribution is absolutely continuous with respect to Lebesgue measure, $\forall 0 < \alpha < 1$, $\exists c_{1-\alpha} \in \mathbb{R}$ such that $H(c_{1-\alpha}) = 1 - \alpha$. $\forall 0 < \tau < \min(\alpha/2, (1 - \alpha)/2)$, assign $x = c_{1-\alpha + \tau}, c_{1-\alpha - \tau}$ in (79), then $c_{1-\alpha - \tau} \leq \hat{\gamma}_i \leq c_{1-\alpha + \tau}$ with probability tending to 1. Therefore, we have $\text{Prob} \left( \max_{i=1,\ldots,p_1} \frac{1}{\hat{\gamma}_i} |\hat{\gamma}_i - \gamma_i| \leq c_{1-\alpha} \right) \leq \text{Prob} (c_{1-\alpha} > c_{1-\alpha + \tau}) + \text{Prob} \left( \max_{i=1,\ldots,p_1} \frac{1}{\hat{\gamma}_i} |\hat{\gamma}_i - \gamma_i| \leq c_{1-\alpha + \tau} \right)$; and $\text{Prob} \left( \max_{i=1,\ldots,p_1} \frac{1}{\hat{\gamma}_i} |\hat{\gamma}_i - \gamma_i| \leq c_{1-\alpha} \right) \geq -\text{Prob}(c_{1-\alpha} < c_{1-\alpha - \tau}) + \text{Prob}(\max_{i=1,\ldots,p_1} \frac{1}{\hat{\gamma}_i} |\hat{\gamma}_i - \gamma_i| \leq c_{1-\alpha - \tau})$. By setting $\tau \to 0$, theorem 2 implies 22.

\[ \square \]