Lattice Gauge Field Interpolation for Chiral Gauge Theories

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The importance of lattice gauge field interpolation for our recent non-perturbative formulation of chiral gauge theory is emphasized. We illustrate how the requisite properties are satisfied by our recent four-dimensional non-abelian interpolation scheme, by going through the simpler case of $U(1)$ gauge fields in two dimensions.

1. INTRODUCTION

In every major scenario for physics above the $\text{TeV}$ scale non-perturbative chiral gauge theory dynamics is expected to play an important role, yet our understanding of this dynamics is very limited. Recently we proposed a non-perturbative formulation of chiral gauge theory on the lattice, in the hope that the important features of these theories will be calculable in future computer simulations [1,2]. We briefly review it here, focussing on a lattice gauge field interpolation procedure which is the crucial feature of our construction.

The fermions are taken to live on a euclidean lattice with spacing $f$. Fermion doublers are eliminated by the Rome Group method of using a gauge non-invariant Wilson term [3]. (The Nielsen-Ninomiya Theorem tells us that we must break chiral gauge invariance to eliminate the fermions doublers.) The central problem is then to recover gauge invariance in the continuum limit. Before summing over gauge fields this can be arranged in a fairly simple way for any anomaly-free theory [4]. However, once gauge fields are integrated over, new divergences can lead to uncontrollable violations of gauge invariance. A solution to this problem is to cut off the gauge field momenta by a scale $\Lambda_b \ll 1/f$ ($f\Lambda_b, f, 1/\Lambda_b \to 0$ in the continuum limit.) Ref. [4] gives detailed lattice power-counting arguments to show that this can be achieved by obtaining the $f$-lattice gauge fields as an interpolation of gauge fields living on a lattice of spacing $b \sim 1/\Lambda_b \gg f$, which are summed over using the standard Wilson action. In practice it is possible that $b/f$ may not have to be too large for computing the properties of low-lying states [4]. (For another possible way of cutting off gauge field momenta see ref. [5].)

It is sufficient for the interpolation procedure to satisfy the following properties [4], most easily stated by imagining interpolating the $b$-lattice link variables, $U$, all the way to the continuum to give gauge fields, $a_\mu[U]$. (i) Transverse continuity: the interpolation describes a differentiable continuum gauge field inside each $b$-lattice hypercube, whose transverse components are continuous across hypercube boundaries (the longitudinal components can jump). (ii) Lattice spacetime symmetries should be respected. (iii) Gauge covariance: A $b$-lattice gauge transformation changes the interpolation only by a continuous gauge transformation. (iv) Locality: The gauge-invariant behavior of the interpolation should depend locally on $U$, in particular it is sufficient if the trace of any continuum Wilson loop depends only on the $U$ lying on $b$-hypercubes through which the Wilson loop passes.

We have detailed such an interpolation procedure for non-abelian gauge fields in four dimensions in ref. [2]. Below, we describe the more transparent case of interpolating $U(1)$ gauge fields in two dimensions, following a procedure which readily generalizes to the non-abelian case. For simplicity we deal with the case $f = 0$ (the continuum) and work in units where $b = 1$.

2. $U(1)$ 2-D Interpolation

In order to maintain transverse continuity it is helpful to build the interpolation from the lowest
dimensional sublattices up. We therefore begin by interpolating the link variables,
\[ U_\mu(s) = e^{iA_\mu(s)}, \quad |A_\mu(s)| < \pi, \] (1)
along the points of each plaquette edge. (We are neglecting the measure-zero set of lattice fields where at least one of the link variables equals exactly \(-1\).) The simplest such interpolation is
\[ a_\mu(s + t\hat{\mu}) \equiv A_\mu(s), \quad 0 \leq t < 1. \] (2)
Note that parallel transport along the links agrees between the lattice and the continuum fields.

We now attempt to interpolate the lattice field into a plaquette interior in such a way as to agree with the above interpolation on the plaquette edges. In order to satisfy locality we try to do this interpolation for each plaquette, using as input only its bounding link variables. In order to satisfy gauge covariance the strategy is to do a lattice gauge transformation on the bounding links of the plaquette which put the link variables into a complete axial gauge. Thus all gauge equivalent lattice fields on the plaquette edges are taken to the same gauge-fixed field (‘almost’ the same in the non-abelian case), which we denote by \( \bar{U} \). This lattice configuration will then be smoothly interpolated to the plaquette interior to give a continuum field \( \bar{\pi}_\mu \). We will then try to find a smooth gauge transformation inside the plaquette which makes the result agree with the one-dimensional edge interpolation, \eqref{eq:2}. At this last stage we will fail, but in a way which we can understand and then correct.

In detail, let us fix some plaquette and use local coordinates \((z_1, z_2), z_\mu = 0, 1\) for the vertices of the plaquette. The lattice gauge transformation,
\[ \Omega[U](z) = U_1(s)^{z_1}U_2(s + z_1\hat{1})^{z_2}, \] (3)
takes \( U_\mu \) to
\[ \bar{U}_1(0, 1) = U_2(s)U_1(s + \hat{2})U_2^{-1}(s + 1)U_1^{-1}(s), \]
\[ \bar{U}_1(0, 0) = \bar{U}_2(0, 0) = \bar{U}_2(1, 0) = 1. \] (4)
This lattice field is easily interpolated into the plaquette interior,
\[ \bar{\pi}_1(t_1, t_2) = t_2\bar{T}_1(0, 1) \]
\[ \bar{\pi}_2(t_1, t_2) = 0, \] (5)
where \((t_1, t_2) : 0 \leq t_{1,2} \leq 1\) are local continuum coordinates for the plaquette interior.

The problem is now to find a continuum gauge transformation \( \omega \) which takes \( \bar{\pi}_\mu \) to a gauge field agreeing with eq. \( \eqref{eq:2} \), so that we can be assured of transverse continuity across plaquette boundaries. In fact it is not hard to see that this demand essentially fixes \( \omega \) on the plaquette boundary to be
\[ \omega(t_1, 0) = e^{i\theta_1A_1(s)}, \]
\[ \omega(1, t_2) = U_1(s)e^{\omega_2A_2(s+1)}, \]
\[ \omega(0, t_2) = e^{i\theta_2A_2(s)}, \]
\[ \omega(t_1, 1) = U_2(s)e^{i\theta_1(A_1(s+2)-A_1(0,1))}, \] (6)
thereby specifying a map from the plaquette boundary (topologically a circle) to \( U(1) \) (topologically also a circle). We can therefore associate a topological winding number (integer) to this map for each plaquette. Unless this winding is zero, \( \omega \) cannot be continuously extended from the plaquette boundary to the interior, and we are stuck. It is simple to show that the winding number associated to the plaquette at \( s, N(s) \), is equal to
\[ I[A_2(s) + A_1(s + \hat{2}) - A_2(s + 1) - A_1(s)] = N(s), \] (7)
\( I[y] \equiv \text{nearest integer to } y \) and is generically non-zero.

The way out of this impasse is to generalize the edge interpolation to
\[ a_\mu(s + t\hat{\mu}) = A_\mu(s) + 2\pi n_\mu(s), \quad 0 \leq t < 1, \] (8)
where the \( n_\mu(s) \) are integer-valued and do not affect agreement of parallel transport between the link variables and the interpolation. (In the notation of ref. \footnote{3}, \( n_\mu = -\epsilon_{\mu\nu}N_\nu \)) If these integers are chosen to satisfy
\[ \sum_{\mu\nu} \epsilon_{\mu\nu}(n_\nu(s + \hat{\mu}) - n_\nu(s)) = N(s), \] (9)
it is easy to show that this new definition differs from the original by a gauge transformation defined on the plaquette boundary with winding number \(-N(s)\). Therefore the gauge transformation which makes \( \bar{\pi}_\mu \) agree with eq. \( \eqref{eq:9} \), \( \tilde{\omega} \), has
winding number zero and can be smoothly extended to the plaquette interior, allowing us to get an interpolation $a_\mu = \tilde{\omega}_\mu$. A simple choice for this extension of $\tilde{\omega}$ yields for $a_\mu(s_1 + t_1, s_2 + t_2)$,

\begin{equation}
\begin{aligned}
a_1 &= (1 - t_2) \left( A_1(s) + 2\pi n_1(s) \right) \\
&\quad + t_2 \left( A_1(s + 2) + 2\pi n_1(s + 2) \right) \\
a_2 &= (1 - t_1) \left( A_2(s) + 2\pi n_2(s) \right) \\
&\quad + t_1 \left( A_2(s + 1) + 2\pi n_2(s + 1) \right).
\end{aligned}
\end{equation}

(10)

If $\sum N(s) \neq 0$ then in fact there is no consistent solution to eq. (9). The reason is that the boundary conditions for the interpolation then correspond to a continuum configuration with topological charge $\sum N(s) \neq 0$, which cannot be represented by a single smooth periodic gauge field. While configurations with non-zero topological charge are physically important, we do not need them in our proposal for lattice chiral gauge theory, because their effects can be inferred from the sector with zero topological charge using cluster decomposition for the full theory. For $\sum N(s) = 0$ eq. (9) has many solutions and it is simple to pick one [2].

Let us check the four central requirements for a successful interpolation. (i) It is straightforward to see that eq. (10) defines a transversely continuous gauge field. (ii) Even though we had to pick the axes of our complete gauge fixing somehow, the gauge-invariant behavior of our interpolation is covariant under lattice translations and rotations, though the gauge dependent form is not. To see this in the $U(1)$ case is easy, since from eqs. (10, 9) the continuum field strength is a constant in each plaquette and is just the logarithm of the plaquette field strength (with absolute value less than $\pi$). For the same reason (iii) and (iv) are also obvious, the non-locality in choosing the $n_\mu$ does not infect the gauge invariant part of the interpolation (ie. the field strength for the $U(1)$ case).

2.1. Non-abelian 4-D Interpolation

While non-abelian interpolation in four dimensions is technically more complicated, the basic steps are the same. The topological obstruction to making higher dimensional interpolations agree with lower dimensional ones now occurs in four dimensions, because the smallest non-abelian group, $SU(2)$, is topologically the 3-sphere as is a hypercube boundary. The resolution of the problem generalizes the 2-D $U(1)$ case. One extra complication not seen in two dimensions is that the choice of axes for the complete gauge fixing (important for maintaining gauge covariance) does lead to a breaking of lattice rotational covariance in the gauge-invariant behavior of the interpolation. To repair this one needs to allow the orientation of the axes in each hypercube to be different and to determine this orientation by the gauge-invariant behavior of the link field $U$ itself. Then if $U$ is rotated, so do the axes. See ref. [2].

The interpolation in ref. [6] differs in that the authors directly interpolate $\omega$ instead of $\tilde{\omega}$, thus getting singular gauge fields whenever any $N(s)$ is non-zero. Such singular gauge fields are unsuitable for our formulation of chiral gauge theories. Their interpolation also breaks lattice rotational covariance in the non-abelian case.

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