CUTOFF FOR THE BIDIRECTIONAL EAST PROCESS

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Abstract. This paper will examine the cutoff for a random process on the hypercube, \( \{0,1\}^L \), closely related to the East Process. In this process, every coordinate has two \( \frac{1}{2} \)-Poisson clocks at each coordinate which add the coordinate to the previous or next one when they ring. We show that the cutoff is \( \frac{L}{v} \) with a window of order \( \sqrt{L} \), where \( v \) is the speed of the front. We compare these results to the cutoff for the East Process and the cutoff for a non-local version of this same process [5], [2].

1. Introduction

The East Process is derived from the Ising model and was first studied by Jäckle and Eisinger in 1991 [6]. A version of the process, defined over the hypercube \( \{0,1\}^L \), involves a Poisson clock with rate 1 at every pair of consecutive coordinates, and adding the left to the right one modulo 2 when the corresponding clock rings. In 2002, Aldous and Diaconis studied the spectral gap to show that the mixing time of the model has order \( L \) [1]. More recently, in 2015, Ganguly, Lubetzky and Martinelli proved cutoff, showing that the cutoff was in fact \( \frac{L}{v} \) with a window of order \( \sqrt{L} \), where \( v \) is the speed of the front of the process [5].

In this paper we study a similar process on the hypercube, where the added coordinate can be added in either direction. Here, we have a Poisson clocks with rate \( \frac{1}{2} \) at each ordered pair of consecutive coordinates, where the first is added to the second modulo 2 when the corresponding clock rings. We define the mixing time \( T_{\text{mix}}(L, \epsilon) \) as

\[
T_{\text{mix}}(L, \epsilon) = \min_{\tau \in \mathbb{R}} \left( \sup_{\omega \in \{0,1\}^L} ||\mu_\omega^\tau - \pi||_L < \epsilon \right),
\]

where \( || \cdot ||_L \) denotes the total variation distance over \( \{0,1\}^L \) and we arrive at the result

Theorem 1. For any fixed \( 0 < \epsilon < 1 \) and large enough \( L \),

\[
T_{\text{mix}}(L, \epsilon) = v^{-1}L \pm C_\epsilon \sqrt{L},
\]

for some constant \( C_\epsilon \) depending on \( \epsilon \).

In other words, our process is a version of the East process which has an added feature of bidirectionality. There are many interesting recent developments over the hypercube. A remarkable result for a slightly different process is Ben-Hamou and Peres’s study of another stratified random walk on the hypercube, where every ordered pair (not necessarily consecutive) of coordinates has a Poisson rate of \( \frac{1}{2L} \).

Key words and phrases. East process, mixing times, cutoff, hypercube, Markov chains, coupling.

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(such that rings occur equally often), and when the corresponding clock rings, the first is added to the second modulo 2. In this case, they proved a cutoff of $\frac{3}{2} \log L$ \cite{2}. Our process lies in some sense between the two previously studied models, and can be thought of either as a bidirectional East process or a version the random walk studied by Ben-Hamou and Peres where we only choose consecutive coordinates.

In 2017, Nestoridi showed that all non-local Markov chains on the hypercube have cutoff of $O(\log L)$ (assuming an expected number of $L$ updates every step) \cite{7}. Other recent relevant results include those Nestoridi and Nguyen \cite{8} as well as by Collevecchio and Griffiths \cite{4}, where the spectral technique was used to achieve cutoff.

Such processes can be considered as the column processes of random walks on $GL_n(\mathbb{F}_2)$, over a matrix where rows are added to each other. This has already been done for the East model, for which the matrix walk is on an upper triangular matrix by Peres and Sly \cite{9} who showed that the mixing time is $O(n)$ (assuming continuous time as introduced), and Ganguly and Martinelli \cite{5} who showed that the cutoff is in fact the same as for the single column. For the stratified random walk, however, the mixing time for finitely many columns is still an open question. An interesting next step would be to see how this model behaves in finitely many columns, as it will not be an upper triangular matrix walk.

We first formally describe our process over all integers as follows. For a configuration $\omega \in \{0, 1\}^\mathbb{Z}$, we associate two rate-\(\frac{1}{2}\) Poisson processes with every point $x \in \mathbb{Z}$, denoted by $\{t_{x,k} : k \in \mathbb{N}\}$ and $\{t_{x,k}^+ : k \in \mathbb{N}\}$. Now, at each time $t_{x,n}$ we query $\omega_x$ and add it to $\omega_{x-1}$. Similarly, at each time $t_{x,n}^+$ we query $\omega_x$ and add it to $\omega_{x+1}$. We will in particular be examining the behavior of this process with regards to its "front". To do so, we consider the subset $\Omega^* \subset \Omega$ defined by

$$\Omega^* = \{\omega \in \Omega : \sup_{x \in \mathbb{Z}} (|x| : \omega_x = 1) < \infty\}.$$ 

In other words, $\Omega^*$ is the set of configurations for which there is a rightmost 1, and we write $X(\omega) = \sup_{x \in \mathbb{Z}} (|x| : \omega_x = 1)$. Furthermore, let $\Omega_F$ be the set of configurations $\omega \in \Omega^*$ such that $X(\omega) = 0$. We denote the process by $\omega(t)$ and the law by $\mu^t_\omega$ for time $t$. Moreover, let $v$ be the "speed" of the front, we can write

$$\lim_{t \to \infty} \frac{X(\omega(t))}{t} \overset{p}{\to} v.$$ 

In this paper we begin by outlining properties of this random process which we will then use form a coupling argument that shows the Markov Chain mixes well in an interval behind the front. We then proceed to examine the behavior at the front which will help us find the cutoff for the overall process.

**Acknowledgments.** This is the result of an undergraduate summer research project advised by Evita Nestoridi. The author is very grateful for her valuable guidance and many fruitful discussions.

2. Setting the scene

The key idea needed to obtain our result involves examining the behavior of the process behind the front, and showing that it mixes well, which will prove Theorem 2. To do so, we introduce a coupling which will depend on some spacing conditions (which we will later justify in Theorems 3.2 and 3.3). However, we begin by establishing some useful properties of our process.
2.1. **Useful results.** The first lemma introduces a maximum speed, above which it is unlikely that the front will travel. This will be helpful as it will allow us to keep track of a likely upper bound for the front of the process.

**Lemma 2.1.** (similar to Lemma 2.6 in [5]) For \( x < y \in \mathbb{Z} \) and \( 0 \leq s < t \), let \( F(x, y; s, t) \) be the event that one of the two following ordered sequences occurs

- \( s \leq t^x < t^{x+1} < \cdots < t^y < t \),
- \( s \leq t_y < t_{y-1} < \cdots < t_x < t \),

where \( t_i \) or \( t^i \) represents a time in which the corresponding Poisson clock at site \( i \) rings. Then there exists a constant \( v_{\text{max}} \) such that, for all \( |y - x| \geq v_{\text{max}}(t - s) \),

\[
P(F(x, y; s, t)) \leq e^{-|x-y|}.
\]

We call this event the **linking event** between \( x \) and \( y \) in time interval \([s, t]\).

*Proof.* To prove this, we simply observe that the event \( F(x, y; s, t) \) is equal to the probability that a Poisson process with rate \( \frac{1}{2} \) has at least \( |x - y| \) instances within time \( t - s \). \(\square\)

We will use the concept of a linking event by thinking about it as dependence of information. Given a configuration \( \omega \), for \( a \leq b \leq c \), if two configurations match on \([a, c]\) at time \( s \), but do not match on \([b, c]\) at time \( t > s \), the event \( F(a, b; s, t) \) must occur.

The following proposition will assert that we can use the maximum number of consecutive zeroes to bound the total variation distance from the stationary distribution.

**Proposition 2.2.** (equivalent to Proposition 2.3) Let \( \Lambda = [1, 2, \ldots, \ell] \), \( \omega \in \Omega \) and \( \omega(0) = 1 \). Let \( \Delta(\omega) \) denote the maximum number of consecutive zeroes in the interval \( \Lambda \). Then, there exist positive constants \( c \) and \( m \) such that

\[
||\mu_{\omega}^t - \pi||_\Lambda \leq \ell c^{\Delta(\omega)} e^{-tm},
\]

where \( \pi \) is the stationary distribution, \( \mu \) is the law of the process and we take the total variation distance on the interval \( \Lambda \).

This proposition follows from Proposition 4.3, [3], which is defined for the East Process, and where the proof considers a "distinguished zero" which moves to the
right at specific clock rings. We extended the result to our process by considering two moving distinguished zeroes. We call this a "pair of distinguished zeroes", they start at the same point but one moves left on rings at times $t_k$, and the other moves right at times $t^k$. The proof of Proposition 4.3 [3] is divided into two steps:

- Conditioning on the right of a distinguished zero.
- Relaxation on the left of a distinguished zero.

We instead reformulate the proof to the following two steps:

- Conditioning on either side a pair of distinguished zeroes.
- Relaxation in between a pair of distinguished zeroes.

We can then use the same reasoning as in the proof of Proposition 2.3, [5] to arrive at the result.

2.2. Discussing speed. We introduce another result which will be useful and can be replicated from [5].

**Lemma 2.3.** (Lemma 2.8, [5]) Taking $v_{\max}$ as defined in Lemma 2.1, there exist constants $v_{\min} > 0$ and $\gamma > 0$ such that

$$\sup_{\omega \in \Omega} \mathbb{P}_\omega(X(\omega(t)) \in [X(\omega) + v_{\min}t, X(\omega) + v_{\max}]) \geq 1 - e^{-\gamma t}.$$ 

Note that we have now introduced constants $v_{\min}$ and $v_{\max}$ and by the lemma above we can think of these as likely bounds for the speed of the front.

We can in fact compute that $v = \frac{1}{2} (1 - \mathbb{E}_\pi(\omega_{-1}))$, where $\omega_{-1}$ is the value in place $-1$ for some configuration $\omega$ distributed with the stationary distribution $\pi$. This is simply because as $\mu_\omega$ approaches $\pi$, the front will move right with probability $\frac{1}{2}$, and left with probability $\frac{1}{2}$ conditioned on the value in place $-1$ being a 1. Using the invariance of the measure $\pi$, we can compute that $\frac{1}{2} < \mathbb{E}_\pi(x_{-1}) < \frac{2}{3}$, meaning that we have $\frac{1}{6} < v < \frac{1}{4}$. Interestingly enough, using the same calculations for the East process, we only get the upper bound of $v_{\text{East}} < \frac{1}{4}$. The speed for the East Process was first studied by Blondel in 2013, where he showed that the process does in fact converge to a speed [3].

2.3. Defining spacing conditions. Finally, before introducing Theorem 2, we introduce two spacing conditions which will outline that it is unlikely to have many consecutive zeroes behind the front.

**Definition 2.4.** Given $\delta, \epsilon \in (0, \frac{1}{2})$ and an interval $I$, we say $\omega \in \Omega$ satisfies the $\left(\delta, \epsilon\right)$-Weak Spacing Condition (WSC) in $I$ if the largest sub-interval of $I$ where $\omega$ is identically equal to zero has length at most $\delta |I|^\epsilon$. We denote by $W_{\ell,t}$ the set of configurations which fail to satisfy WSC in the interval $[-v_{\min}t, -\ell) \cap \mathbb{Z}$, where $\ell = t^\epsilon$ and $v_{\min}$ represents the minimum speed of the front introduced in Lemma 2.3.

**Definition 2.5.** We say that given a configuration $\omega \in \Omega$ and an interval $I$, $\omega$ satisfies the Strong Spacing Condition (SSC) in $I$ if the largest sub-interval of $I$ where $\omega$ is identically zero has length at most $10 \log |I|$. We denote by $S_{\ell}$ the set of configurations which fail to satisfy SSC in the interval $[-3(v_{\max}/v_{\min})\kappa \ell, -\kappa \log \ell)$, where $\kappa$ is some fixed constant.
3. Mixing behind the front

We are now ready to introduce our main theorem, which describes the behavior of the process behind the front.

**Theorem 2.** *(similar to Thm. 3.1 [5]*) There exists constants $\alpha \in (0, 1)$ and $v^* > 0$ such that

$$\sup_{\omega \in \Omega_F} \| \mu^t_\omega - \pi \|_{[-v^* t, 0]} = O(e^{-\alpha t}),$$

where $\pi$ is the stationary measure on $[-v^* t, 0]$.

To prove this theorem we will introduce a recursive coupling $M^t_{\omega, \omega'}$ for starting configurations $\omega, \omega' \in \Omega_F$ on an interval $I_n$ behind the front. The coupling will consist of $N$ rounds, each round $n$ ending at time $t_n$. Thus, $t_n$ is the "remainder" of time left, and is less than the next round would have been. Let $\Delta_n = t_n - (1-\epsilon)t_n$. We define $I_n = [a_n, 0]$, with $a_n = -v_{\min}t_n + 2v_{\max}\Delta_n$. We want our coupling to satisfy

$$M^{t_N}(\omega_{t_N} \neq \omega'_{t_N} \text{ in the interval } I_N) = O(e^{-\alpha t}) \tag{1}$$

Note that if we choose $v^* = v_{\min} - 3\epsilon v_{\max}$, then the interval $I_N$ will in fact include $[-v^* t, 0]$. We can check that this is true.

$$-v_{\min}t + 2v_{\max}\Delta N \leq -v_{\min}t + 3\epsilon v_{\max}t$$

$$\iff v_{\min}(t - t_N) \leq v_{\max}(et + 2(t - t_N)).$$

For the two configurations to be coupled at time $t$, it is enough to ensure that they are coupled on $I_N$ at time $t_N$ and that the event $\mathcal{F}(a_N, -v^* t; t_N, t)$ does not occur.

To address the linking event, we note that

$$| - v^* t - a_N | = (t - t_N)(2v_{\max} - v_{\min}) + v_{\max}t$$

$$\geq v_{\max}t,$$

and use Lemma 2.1 above to bound the probability of $\mathcal{F}(a_N, -v^* t; t_N, t)$ by $e^{v_{\max}t}$. Combining this with (1) we get

$$M^t_{\omega, \omega'}(\exists x \in [-v^* t, 0] : \omega_x \neq \omega'_x) \leq M^{t_N}(\omega \neq \omega' \text{ in the interval } I_N)$$

$$+ \mathbb{P}(\mathcal{F}(a_N, -v^* t; t_N, t)),$$

$$\leq O(e^{-\alpha t}) + e^{-v_{\max}t}.$$

Thus, proving (1) will be enough to prove Theorem 2.

3.1. Coupling Argument. We now take a closer look at the coupling introduced in [5], where we attempt to iteratively couple two configurations in an interval behind the front, as the fronts of the configurations move. The success of this coupling will depend on Theorems 3.2 and 3.3 which address the likelihood of meeting the spacial conditions defined in Definitions 2.4 and 2.5 and which we will prove later. In every round $n$ we have the following considerations.

- Time: For some fixed constant $\kappa$, let $\Delta_1 = (\kappa/v_{\min})t'$, $\Delta_2 = \kappa \log t$ and $\Delta = \Delta_1 + \Delta_2$. Here, each round $n$ will take time $\Delta$, where $\Delta_1$ will be the burn-in part of the round and $\Delta_2$ is the mixing part of the round. Let $s_n = t_n + \Delta_1$, and let $t_{n+1} = s_n + \Delta_2$. The first round starts at time $t_0 = (1-\epsilon)t$.
\[ \Lambda_{n+1} \quad \Lambda_n \quad 1 \quad 0 \quad 0 \quad \ldots \]

**Figure 2.** A visualisation of the coupling

- \( I_n = [-v_{\text{min}} t_n + 2v_{\text{max}} \Delta n, 0] \), this is the interval on which we are aiming to couple \( \omega \) and \( \omega' \). Note that, as we see in Figure 2, \( I_{n+1} \subseteq I_n \) as the difference between their end points is \( (2v_{\text{max}} - v_{\text{min}})\Delta \).

- \( \Lambda_n = [-v_{\text{min}} s_n, -3(v_{\text{max}}/v_{\text{min}})\kappa t_n] \). This is an interval overlapping but to the left of \( I_n \) (see Figure 2). Theorem 3.2 will show that on this interval, Markov Chains are relatively well mixed, and this will thus be the motivation for extending a coupling from \( \Lambda_n \) to \( I_n \). Note here, again as seen in Figure 2, that \( \Lambda_n \subseteq \Lambda_{n+1} \) and that \( \Lambda_n \) overlaps with \( I_n \), but each of its endpoints are to the left of the corresponding endpoint of \( I_n \).

- Front: we will redefine the space after each round so that the 0 is the front \( X(w(t_n)) \).

We also consider the following two couplings

- Basic coupling - we choose the same \( x \) in \( \omega \) and \( \omega' \) and we update the same coordinate to the extent that is possible.

- Maximal coupling - the optimal coupling from the standard definition of total variation distance.

The intuition behind this is that we will first let the two configurations evolve by the maximal coupling for the *burn-in* part of the round, expecting them (by Theorem 3.2) to couple on the interval \( \Lambda_n \). If they do, we will utilize either the basic or maximal coupling in the *mixing* part of the round to push the matching so that the two configurations are coupled closer to the front, on the interval \( I_n \). Formally, \([5]\) define the process as follows.

We define the family of couplings \( \{M^{(n)}\} \) for \( \{(\mu^{t_n}_{\omega}, \mu^{t_n}_{\omega'})\}_{n=0}^{N} \), such that \( M^{(0)} \) is the trivial product coupling and we define \( M^{(n+1)} \) from \( \omega(t_n) \) as follows.

1. If \( \omega \) and \( \omega' \) match on \( I_n \) at time \( t_n \), we call this event \( E_n \). We then follow the basic coupling for the whole round, and they will likely match on \( I_{n+1} \) unless the linking event \( F(a_n, a_{n+1}; t_n, t_{n+1}) \) occurs to "bring in" discrepancies.

2. If \( E_n \) does not hold, we use the maximal coupling on interval \( \Lambda_n \) for the burn-in part of the round.
   a) If after this, the two chains agree on \( \Lambda_n \), call this event \( G_n \). We attempt to "push" their agreement right towards the front of the process. To do so, we find the rightmost common zero of \( \omega(s_n) \) and \( \omega'(s_n) \) in \( \Lambda_n \) (or the right boundary of \( \Lambda_n \) if no such zero exists) and call this point \( x^* \). Now we consider the probability that event \( A_n \) occurs: one of the Poisson clocks at \( x^* \) or 0 will ring in \([s_n, t_{n+1}]\). We consider this event before actually deciding which coupling to use as it is independent of the coupling.

   i) If \( A_n \) holds, there is an update, we simply continue by the basic coupling.

   This is the most likely outcome.
ii) If $A_n$ does not hold, we complete the mixing part of the round following the maximal coupling on the interval between $x_*$ and the origin, since we know that this is an isolated interval which will mix independently from the rest.

![Figure 3](image)

**Figure 3.** We use the maximal coupling on the purple interval

b) If $G_n$ does not hold, on the other hand, the chains did not agree on $\Lambda_n$, we let them evolve according to the basic coupling for the rest of the round.

Now, to show (1), we prove the following recursive formula

**Theorem 3.1** (Coupling process, Claim 3.5, [5]). If $p_n = M^{t_n}(\omega \neq \omega'$ in the interval $I_n$),

$$p_{n+1} \leq C e^{-\epsilon t'} + (1 - e^{-2\Delta \epsilon}/2)p_n.$$ 

**Proof.** We follow the outline of the coupling and consider every possible outcome of each event to show that if $E_n$ holds, or the linking event $F(a_n, a_{n+1}; t_n, t_{n+1})$ between the left endpoint of the interval $I_n$, $a_n$, and the left endpoint of the interval $I_{n+1}$, $a_{n+1}$, occurred. Note that

$$|a_n - a_{n+1}| = 2v_{\max} \Delta - v_{\min}(t_{n+1} - t_n) \geq v_{\max}(t_{n+1} - t_n)$$

and thus by Lemma 2.1,

$$\mathbb{P}(E_{n+1}^c | E_n) = O(e^{-|a_n - a_{n+1}|}) = O(e^{-v_{\max} t'}).$$

Note that $|a_n - a_{n+1}| \geq v_{\max} \Delta$ and that $\Delta$ behaves as $O(t')$ as $t \to \infty$.

2. Otherwise, we assume $E_n^c$ and follow the maximal coupling on $\Lambda_n$.

a) If $G_n$ holds, so $\omega$ and $\omega'$ match on $\Lambda_n$:

i) If there is an update either at the point $x_*$ or in the origin $(A_n)$, we continue with the basic coupling, and we will repeat until the next iteration.

ii) If $A_n$ does not hold, we use the maximal coupling on the closed interval between $x_*$ and the origin. We want to show that if neither $x_*$ nor the origin are updated, and we use the maximal coupling between the two, then the probability of $E_{n+1}$ is large. Now, consider the event $B$ that

- $E_n^c, G_n, A_n^c$ are fulfilled,
- $x_*$ is within a distance of $\epsilon \log t$ from the right boundary of $\Lambda_n$, and
- $\omega(\Delta_1)$ and $\omega'(\Delta_1)$ satisfy SSC in $[-3(v_{\max}/v_{\min})\kappa t', -\kappa \epsilon \log t]$.

We will show that $\mathbb{P}(B | E_n^c, G_n, A_n^c)$ is large. First note that for the second condition not to be fulfilled, we must have $\epsilon \log t$ ones in a row, the probability of this is close to the probability of this event for $\pi$ by Theorem 3.2, which is $2^{-\epsilon \log t} = t^{-\epsilon \epsilon}$. 

For the third point, we will show in Theorem 3.3 that the probability of \( \omega, \omega' \) not satisfying the weak spacing condition in the desired interval is \( O(t^{-2\epsilon}) \), and thus we have
\[
P(\mathcal{B}|\mathcal{E}^c_n, \mathcal{G}_n, \mathcal{A}^c_n) = 1 - O(t^{-\epsilon\epsilon})
\]
for some constant \( \epsilon \).

Now, we will show that given \( \mathcal{B} \), there is a high likelihood of \( \mathcal{E}_{n+1} \) occurring. In fact we can bound \( P(\mathcal{E}_{n+1}^c|\mathcal{B}) \) from above by using Proposition 2.2 and linking events. Recall that Proposition 2.2 uses an upper bound on the number of consecutive zeroes to bound the total variation distance from the stationary distribution. More specifically, we have
\[
P(\mathcal{E}_{n+1}^c|\mathcal{B}) \leq \mathcal{F}(a_n, a_{n+1}; t_n, t_{n+1}) + ||\mu_{\Delta}^{2\Delta_2} - \pi||\|\cdot\|=3(\nu_{\text{max}}/\nu_{\text{min}})\kappa t_n, 0] \leq e^{-|a_n-a_{n+1}|} + (3\kappa t)\epsilon \log t e^{-\Delta_2 m} = O(t^{-\epsilon})
\]
if we choose \( \kappa \) large enough. (Note that \( \Delta_2 \) depends on \( \kappa \).

Combining (2) and (3), we have
\[
P(\mathcal{E}_{n+1}^c|\mathcal{E}^c_n, \mathcal{G}_n, \mathcal{A}^c_n) \geq \frac{1}{2}.
\]

Furthermore, note that \( P(\mathcal{A}^c_n|\mathcal{E}^c_n; \mathcal{G}_n) = e^{-2\Delta_2} \) and since \( \mathcal{E}^c_n \) and \( \mathcal{G}_n \) will occur with probability \( O(e^{-\epsilon}) \) (as we will show below), we can assume that \( P(\mathcal{A}^c_n) \approx e^{-2\Delta_2} \).

b) To account for the probability that \( \omega \) and \( \omega' \) may not agree on \( \Lambda_n \), the event \( \mathcal{G}^c_n \), we will use Theorem 3.2 which will show that \( P(\mathcal{E}_{n+1}^c) = O(e^{-1/\epsilon}) \).

Now, since \( \mathcal{E}^c_n \cap \mathcal{G}_n \) will occur with probability \( (1 - O(e^{-\epsilon})) \) and \( P(\mathcal{A}^c_n, \mathcal{E}^c_n, \mathcal{G}_n) \geq \frac{1}{2} e^{-2\Delta_2} \), we have completed the proof. \( \square \)

3.2. Spacing conditions. We will now provide upper bounds on probabilities of not satisfying the spacing conditions defined in Definitions 2.4 and 2.5. Let \( \ell = t \) as in Definition 2.4, and let \( t_{\ell} = t - \kappa \ell / \nu_{\text{min}} \). Recall that we denote the law for the process at time \( t \) starting at configuration \( \omega \) as \( \mu_{\omega}^t \).

**Theorem 3.2.** (Equivalent to Thm 3.3, (1) and (3), [5]) There exists \( \delta \) small enough and \( \kappa \) large enough such that for all \( t \) large enough
\[
\sup_{\omega \in \Omega} \mu_{\omega}^t(\mathcal{W}_{t, t}) = O(e^{-\epsilon/2}),
\]
\[
\sup_{\omega \in \Omega} ||\mu_{\omega}^t(\cdot|\mathcal{F}_{t}) - \pi||\|\cdot\|_{[\nu_{\text{min}} t - 3(\nu_{\text{max}}/\nu_{\text{min}})\kappa \ell]} = O\left(e^{-t/\epsilon}\right) + 1_{\mathcal{W}_{t, t}}(\omega(t_{\ell})).
\]

This theorem asserts that the process mixes well far enough from the front, with low probabilities of breaking the weak spacing condition.

**Proof.** We simply sketch the proof of this theorem as it is proved in [5]. First, to prove (4), we note that we can bound the probability of a configuration not satisfying the WSC from Definition 2.4 from above. As before, we shift our definition so that instead of having the origin be the front at time \( t \), we let the origin be the front at time 0. Then, we define \( a \) to be the front at time \( t \), and we consider \( \mathcal{W}_{t, t} \) on the interval \([0, a - \ell]\) rather than \([-\nu_{\text{min}} t, -\ell]\) as per Definition 2.4. Recall that this is the set of configurations which fail to satisfy WSC on the given interval. We do this because we are interested in fixing the front at time 0 rather than at time \( t \) as this allows us to sum over possible fronts at time \( t \).
Now, under this new set of definitions, we observe that any configuration $\omega \in \mathcal{W}_{\ell,t}$, must have a point $x$, at which the spacing condition breaks, for which

- $0 \leq x \leq a - \ell$,
- $\omega$ is identically zero in the interval $[x, x + \delta(v_{\min}t)/2]$,
- the hitting time of $x$, $\tau_x < t$, and
- the linking event $F(x, a; \tau_x, t)$ occurred.

Let $V$ be the occurrence $\omega \in \mathcal{W}_{\ell,t}$, meaning that $\omega$ does not satisfy the Weak Spacing Condition.

Now, if we consider any value for $a$ within the interval $[v_{\min}t, v_{\max}t]$, we can bound the probability of event $V$ by a double sum over all values of $a$ and all values of $x$. By Lemma 2.3,

$$P(a \in [v_{\min}t, v_{\max}t]) \geq 1 - e^{-\gamma t}.$$ 

From here we will split our sum into two cases, in the case where $|x - a| \geq v_{\max}(t - \tau_x)$, we can use Lemma 2.1 to bound the probability of the linking event. However, if $|x - a| < v_{\max}(t - \tau_x)$ we can note that the probability that the weak spacing condition is not satisfied is less than $||P_{\omega(\tau_x)} - \pi||_{L^\infty} + 2^{-\frac{t}{2}(v_{\min}t)}$, simply by calculating the distance from configurations. Conditioning on possible values of $|x - a|$ and combining these together, we obtain the first result.

Now we sketch the proof of (5), which allows us to expect our coupling to mix well on the intervals $\Lambda_n$. The main two ideas behind this proof is that one, the process far enough from the front does not depend on the location of the front (Proposition 2.11, [5]) and two, there is a universal bound on total variation distance for our process using gaps (proposition 2.3) which assumes that the weak spacing conditions hold.

We define $\mathcal{F}_{t_\ell}$ to be the state of the chain at time $t_\ell$. Now we introduce the following theorem to address configurations which do not satisfy the strong spacing condition (SSC), defined in Definition 2.5.

**Theorem 3.3.** [Equivalent to 3.3, line 2] There exists $\delta$ small enough and $\kappa$ large enough such that for all $t$ large enough

$$\sup_{\omega \in \Omega_F} \mu_\ell^t(S_\ell | \mathcal{F}_{t_\ell}) = O(t^{-7\epsilon}) + 1_{W_{\ell,t}(\omega(t_\ell))}. \quad (6)$$

The proof will be omitted but can be reproduced from [5], Theorem 3.3, (3.8).

4. MIXING ON WHOLE INTERVAL

To arrive at our result we use Theorem 2, to examine the increments of movement of the front, which will then help us examine the behavior at the front, which will help imply our result. We first recall Theorem 2:
Theorem 2. (similar to Thm. 3.1 [5]) There exists constants $\alpha \in (0,1)$ and $v^* > 0$ such that
\[
\sup_{\omega \in \Omega_F} \| \mu^F_\omega - \pi \|_{[-v^*t,0]} = O(e^{-\alpha t}),
\]
where $\pi$ is the stationary measure on $[-v^*t,0]$.

Now, we take $\omega \in \Omega_F$ (such that $X(\omega) = 0$), and we define the increments in the position of the front $\xi_n = X(\omega(t_n)) - X(\omega(t_{n-1}))$. Let $\Delta > 0$, $t_n = n\Delta$, and let $N = \lfloor t/\Delta \rfloor$. Note that
\[
X(\omega(t)) = \sum_{n=1}^{N} \xi_n + [X(\omega(t)) - X(\omega(t_N))].
\] (7)

4.1. Increments at the front. We will now prove several properties of the increments $\{\xi_n\}_{n=1}^N$ and how they behave as random variables depending on some starting configuration $\omega$.

Lemma 4.1. (similar to Corollary 3.2 in [5]) Let $f : \mathbb{R} \mapsto [0,\infty)$ be such that $e^{-|x|}f^2(x) \in L^1(\mathbb{R})$. Then
\[
C_f = \sup_{\omega \in \Omega_F} \mathbb{E}_\omega [f(\xi_1)^2] < \infty.
\] (8)
Moreover, there exists a constant $\gamma > 0$ such that
\[
\sup_{\omega \in \Omega_F} \mathbb{E}_\omega [f(\xi_n)] - \mathbb{E}_\omega [f(\xi_1)] = O(e^{-\gamma n}) \forall n \geq 1,
\] (9)
and, $\forall j < n$
\[
\sup_{\omega \in \Omega_F} |\text{Cov}_\omega(\xi_j,\xi_n) - \text{Cov}_\omega(\xi_1,\xi_{n-j})| = O(e^{-\gamma j}),
\] (10)
\[
\sup_{\omega \in \Omega_F} |\text{Cov}_\omega(\xi_j,\xi_n)| = O(e^{-\gamma(n-j)}) .
\] (11)

To prove this result we must perform some computations which we walk through below. The key idea is that Theorem 2 tells us that any configuration $\omega$ will mix "well" on the interval $[-v^*t,0]$, and since we are only looking at $\xi_j$ which depends only on the front part of the configuration, it is not very likely that we have to worry about the configuration to the left of $-v^*t$.

Proof: To prove (8) and (9), we simply follow the reasoning in the proof of Corollary 3.2, [5]. We prove (10). Here, the idea is that for a large $j$ relative to $n$, $\text{Cov}_\omega(\xi_j,\xi_n)$ is similar to $\text{Cov}_\pi(\xi_1,\xi_{n-j})$ because there has been "time for mixing" before we reach $\xi_j$.

We rewrite $\text{Cov}_\omega(\xi_j,\xi_n) = \mathbb{E}_\omega(\xi_j\xi_n) - \mathbb{E}_\omega(\xi_j)\mathbb{E}_\omega(\xi_n)$.

Reformatting $\text{Cov}_\pi(\xi_1,\xi_{n-j+1})$ in the same way, our goal is to show that
\[
\sup_{\omega \in \Omega_F} |\mathbb{E}_\omega(\xi_j\xi_n) - \mathbb{E}_\pi(\xi_j\xi_{n-j+1})| + |\mathbb{E}_\pi(\xi_1)\mathbb{E}_\pi(\xi_{n-j+1}) - \mathbb{E}_\omega(\xi_j)\mathbb{E}_\omega(\xi_n)| = O(e^{-\gamma j}).
\]
First note that by (9)
\[
\sup_{\omega \in \Omega_F} |\mathbb{E}_\pi(\xi_1) - \mathbb{E}_\omega(\xi_j)| = O(e^{-\gamma j}),
\]
\[
\sup_{\omega \in \Omega_F} |\mathbb{E}_\pi(\xi_{n-j+1}) - \mathbb{E}_\omega(\xi_n)| = O(e^{-\gamma j}),
\]
which implies that $\sup_{\omega \in \Omega} |\mathbb{E}_\pi(\xi_1)\mathbb{E}_\pi(\xi_{n-j+1}) - \mathbb{E}_\omega(\xi_j)\mathbb{E}_\omega(\xi_n)| = O(e^{-\gamma \alpha})$. So it is enough to prove that
\[
\sup_{\omega \in \Omega} |\mathbb{E}_\omega(\xi_j\xi_n) - \mathbb{E}_\pi(\xi_1\xi_{n-j+1})| = O(e^{-\gamma \alpha}).
\]
Now we use the same linking event reasoning again as earlier as well as Theorem 2, note that by the same reasoning as above we have
\[
\text{Finally we prove (11). Here the intuition is that if } \sup_{\omega \in \Omega} |E_\pi(\xi_j, \xi_n) - E_\omega(\xi_j, \xi_n)| \leq O(e^{-\gamma \alpha}), \text{so we can in fact write this all as } O(e^{-\gamma \alpha}).
\]
Finally we prove (11). Here the intuition is that if $j$ is small relative to $n$, then $Cov_\omega(\xi_j, \xi_n)$ is small because there has been "time for mixing" between $\xi_j$ and $\xi_n$. Note that by the same reasoning as above we have
\[
E_\omega(\xi_j)O(e^{-(n-j)\alpha}) = O(e^{-(n-j)\alpha}).
\]
Now, we will prove that each of the covariance terms is small, rewriting
\[
Cov_\omega(\xi_j, \xi_n) = E_\omega(\xi_j\xi_n) - E_\omega(\xi_j)E_\omega(\xi_n).
\]
Then, writing
\[
E_\omega(\xi_j) = \int d\mu_{\omega}^{j-1}(\omega')E_\omega'(\xi_{n-j+1}).
\]
Combining this with several uses of (12) gives us
\[
E_\omega(\xi_j\xi_n) = \int E_\omega'(\xi_1\xi_{n-j+1})d\mu_{\omega}^{j-1}(\omega')
= \int \int E_\omega''(\xi_{n-j})[X(\omega'') - X(\omega')]d\mu_{\omega}^{\Delta}(\omega'')d\mu_{\omega}^{j-1}(\omega')
\]
Now, using (9),
\[
= \int \int (E_\pi(\xi_1) + O(e^{-(n-j)\alpha}))[X(\omega'') - X(\omega')]d\mu_{\omega}^{\Delta}(\omega'')d\mu_{\omega}^{j-1}(\omega')
= (E_\pi(\xi_1) + O(e^{-(n-j)\alpha}))\int \int [X(\omega'') - X(\omega')]d\mu_{\omega}^{\Delta}(\omega'')d\mu_{\omega}^{j-1}(\omega')
\]
Note that the double integral is in fact equal to $E_\omega(\xi_j)$,
\[
= (E_\pi(\xi_1) + O(e^{-(n-j)\alpha}))E_\omega(\xi_j)
\]
We do the same for $Cov_\omega(\xi_1, \xi_{n-j+1})$ to guarantee that there difference is also $O(e^{-\gamma(n-j)\alpha})$. Now, note that by (9) we have
\[
\sup_{\omega \in \Omega} |\int d\mu_{\omega}^{j-1}(\omega')E_\omega'(\xi_{n-j+1}) - E_\pi(\xi_1)| \leq O(e^{-(n-j+1)\alpha})
\]
which allows us to write
\[
Cov_\omega(\xi_j, \xi_n) = Cov_\omega(\xi_j, E_\pi(\xi_1)) + O(e^{-(n-j)\alpha})E_\omega(\xi_j)
\]
by the linearity of covariance. It remains to show that
\[ \text{Cov}_\omega(\xi_j, \mathbb{E}_\pi(\xi_1)) = 0, \]
however this is true because \( \mathbb{E}_\pi(\xi_1) \) is independent of \( \omega \).

4.2. Behavior of the front. Now, using Lemma 4.1 we can show the following theorem about the front.

**Theorem 4.2.** There exists a non-negative constant \( \sigma_* \) such that for all \( \omega \in \Omega_F \)
\[ \mathbb{E}_\omega[X(\omega(t))] = vt + O(1), \]  \tag{13}
\[ \lim_{t \to \infty} \frac{1}{t} X(\omega(t)) = v, \]  \tag{14}
\[ \lim_{t \to \infty} \frac{1}{t} \text{Var}_\omega(X(\omega(t))) = \sigma_*^2. \]  \tag{15}

This theorem will almost directly prove our desired result.

**Proof.** We begin by noting that
\[ \frac{d}{dt} \mathbb{E}_\omega[X(\omega(t))] = \frac{1}{2}(1 - \mathbb{E}_\pi(x_{-1})) \]
which by the definition of \( v \) as discussed in the introduction. This immediately proves (13).

Now we move to prove (14). By Chebyshev’s inequality, we have
\[ \mathbb{P}_\omega\left( \left( \frac{X(\omega(t)) - vt}{t} \right)^2 > \epsilon \right) \leq \frac{1}{\epsilon} \mathbb{E}_\omega\left( \left( \frac{X(\omega(t)) - vt}{t} \right)^2 \right) \leq \frac{c}{t\epsilon} \]  \tag{16}

To prove the last inequality, assume that \( t = t_N \) for some \( N \in \mathbb{N} \),
\[ \mathbb{E}_\omega \left( (X(\omega(t)) - vt)^2 \right) = \mathbb{E}_\omega \left( \left( \sum_{j=1}^{N} \xi_j - v\Delta N \right)^2 \right) \]
\[ = \sum_{j=1}^{N} \mathbb{E}_\omega(\xi_j - v\Delta)^2 + \sum_{j \neq k} \mathbb{E}_\omega((\xi_j - v\Delta)(\xi_k - v\Delta)) \]

Now, note that since \( \mathbb{E}_\omega(X(\omega(t))) = vt + O(1) \) by (13), for any \( i \leq n \)
\[ = \sum_{j=1}^{N} \mathbb{E}_\omega(\xi_j - \mathbb{E}_\omega(\xi_j))^2 + \sum_{j \neq k} \mathbb{E}_\omega((\xi_j - \mathbb{E}_\omega(\xi_j))(\xi_k - \mathbb{E}_\omega(\xi_k))) + O(n) \]
\[ = \sum_{j=1}^{N} \text{Var}(\xi_j) + \sum_{j \neq k} \text{Cov}(\xi_j, \xi_k) + O(n) \]

Now we use Lemma (4.1) to note that \( \text{Var}(\xi_j) \) is bounded so we can write
\[ = \sum_{j \neq k} \text{Cov}(\xi_j, \xi_k) + O(n) \]
Finally, using (11) from Lemma 4.1,
\[ = 2 \sum_{j<k} O(e^{-\gamma(k-j)^2}) + O(n) \]
\[ \leq 2n \sum_{i=1}^{n} O(e^{-\gamma i^2}) + O(n) = O(n), \]
because the series \( \sum_{i=1}^{\infty} O(e^{-\gamma i^2}) \) converges. Thus, we have that \( E_{\omega} \left( (X(\omega(t)) - vt)^2 \right) = O(n) = O(t) \) which is enough to prove the inequality (16). Finally, we prove (15).

First we note that if we split the movement of the front \( X(\omega(t)) \) into the movements over time intervals of \( \Delta \), as we write in (7), we can consider the sum of the variances.

\[ \sum_{n=1}^{N} \text{Var}_{\omega}(\xi_n) + \text{Var}(X(\omega(t)) - X(\omega(t_N))) \]

(17)

Now, we write
\[ |\text{Var}_{\omega}(\xi_n) - \text{Var}_{\pi}(\xi_1)| = |E_{\omega}(\xi_n^2) - E_{\pi}(\xi_1^2) + E_{\pi}(\xi_1)^2 - E_{\omega}(\xi_n)^2| \]
\[ \leq |E_{\omega}(\xi_n^2) - E_{\pi}(\xi_1^2)| + |E_{\pi}(\xi_1)^2 - E_{\omega}(\xi_n)^2| \]

Now, using Lemma 4.1, (9) with \( f(x) = x^2 \), we get that for any \( \omega \)
\[ |E_{\omega}(\xi_n^2) - E_{\pi}(\xi_1^2)| = O(e^{-\gamma n^2}). \]

To deal with the second term, we write
\[ |E_{\pi}(\xi_1)^2 - E_{\omega}(\xi_n)^2| = |E_{\pi}(\xi_1) - E_{\omega}(\xi_n)||E_{\pi}(\xi_1) + E_{\omega}(\xi_n)| \]

Note that we know that the first factor behaves as \( O(e^{-\gamma n^2}) \). Moreover, note that the second factor is bounded with high probability by \( 2\alpha_{\max}\Delta \), thus the product will also behave as \( O(e^{-\gamma n^2}) \). With this, we can conclude
\[ \text{Var}_{\omega}(\xi_n) = \text{Var}_{\pi}(\xi_1) + O(e^{-\gamma n^2}) \]

Now, taking (17) and with the limit as \( t \to \infty \), we can write
\[ \lim_{t \to \infty} \frac{1}{t} \left( \sum_{n=1}^{N} \text{Var}_{\omega}(\xi_n) + \text{Var}(X(\omega(t)) - X(\omega(t_N))) \right) = \lim_{t \to \infty} \frac{N}{t} \text{Var}_{\pi}(\xi_1) = \frac{1}{\Delta} \text{Var}_{\pi}(\xi_1) \]

(18)

However, note that to express \( \text{Var}_{\omega}(X(\omega(t))) \) using this equation, we must also consider the covariances. We can show that
\[ \lim_{t \to \infty} \frac{2}{t} \left[ \sum_{j<n}^{N} \text{Cov}_{\omega}(\xi_j, \xi_n) + \sum_{n=1}^{N} \text{Cov}_{\omega}(\xi_n, X(\omega(t)) - X(\omega(t_N))) \right] = \frac{2}{\Delta} \sum_{n \geq 2} \text{Cov}_{\pi}(\xi_1, \xi_n) \]

(19)

This follows from Lemma 4.1, (10), by first considering \( t \) such that \( t = N\Delta \), and then realizing that the remainder is negligible as \( t \to \infty \). Now, combining (18) and (19), we have
\[ \lim_{t \to \infty} \frac{1}{t} \text{Var}_{\omega}(X(\omega(t))) = \frac{1}{\Delta} \left[ \text{Var}_{\pi}(\xi_1) + 2 \sum_{n \geq 2} \text{Cov}_{\pi}(\xi_1, \xi_n) \right] \]
Now, to finish the proof, it remains to show that the right side is nonnegative. To do this, first note (10) implies that
\[
\sup_{\Delta} \left| \sum_{n \geq 2} \Cov_{\pi}(\xi_1, \xi_n) \right| < \infty
\]
and that
\[
\lim_{\Delta \to \infty} \Var_{\pi}(\xi_1) = \infty.
\]
Thus, we can choose a \( \Delta \) such that
\[
|\Var_{\pi}(\xi_1)| > 2 \left| \sum_{n \geq 2} \Cov_{\pi}(\xi_1, \xi_n) \right|
\]
which concludes the proof.

4.3. Proving the main result. Now we can finally prove our main result. We recall Theorem 1:

**Theorem 1.** For any fixed \( 0 < \epsilon < 1 \) and large enough \( L \),
\[
T_{\text{mix}}(L, \epsilon) = v^{-1}L \pm C_\epsilon \sqrt{L},
\]
for some constant \( C_\epsilon \) depending on \( \epsilon \).

**Proof.** Recall that \( T_{\text{mix}}(L, \epsilon) \) is the minimum time \( t \) such that \( d(t) \leq \epsilon \), where \( d(t) = \sup_{\omega \in \Omega} \| \mu_{\omega} - \pi_L \|_{TV} \), where both measures are considered on the interval \( L \). We consider \( t = \frac{L}{v} + s\sqrt{L} \),
\[
\mathbb{P}_\omega(X(\omega(t)) < L) = \mathbb{P}_\omega(X(\omega(t)) - vt < -s\sqrt{L})
\]
\[
\leq \mathbb{P}_\omega((X(\omega(t)) - vt)^2 > s^2v^2L) \quad (20)
\]
\[
\leq s^{-2}v^{-2}L^{-1}\mathbb{E}_\omega((X(\omega(t)) - vt)^2) \quad (21)
\]
\[
\leq s^{-2}v^{-2}L^{-1}\Var_\omega(X(\omega(t))) \quad (22)
\]

Note that we use Chebyshev’s inequality in (22). Now, using this we can write
\[
d_{TV}(t) \leq \lim_{L \to \infty} \mathbb{P}(X(\omega(t)) < L)
\]
\[
\leq s^{-2}v^{-2} \lim_{L \to \infty} L^{-1}\Var_\omega(X(\omega(t)))
\]
\[
\leq s^{-2}v^{-2} \lim_{t \to \infty} \frac{\Var_\omega(X(\omega(t)))}{t} \cdot \frac{t}{L}
\]
\[
\leq s^{-2}v^{-3}\sigma_\epsilon^2
\]
Thus, if we choose \( s_\epsilon \) such that \( s_\epsilon^{-2}v^{-3}\sigma_\epsilon^2 = \epsilon \), so \( s_\epsilon = \frac{1}{\sigma_\epsilon \sqrt{\epsilon}} \), we have
\[
d_{TV}(t) \leq \epsilon.
\]
Thus, we have shown that
\[
|T_{\text{mix}}(L, \epsilon) - v^{-1}L| \leq s_\epsilon \sqrt{L}
\]
and this concludes the proof.

**Remark.** This is not an optimal dependence on \( \epsilon \). While, we have shown that \( |T_{\text{mix}}(L, \epsilon) - v^{-1}L| \leq C_\epsilon \sqrt{\frac{L}{\epsilon}} \) for some constant \( C_\epsilon \), one can in fact prove a stronger cutoff, namely
\[
|T_{\text{mix}}(L, \epsilon) - v^{-1}L| \leq C\Phi^{-1}(1-\epsilon)\sqrt{L},
\]
where \( \phi \) is the c.d.f. of \( N(0, 1) \). This is done by proving a central limit theorem on the behavior of \( X(\omega(t)) \). The proof is included in [5] and will not be repeated here.
CUTOFF FOR THE BIDIRECTIONAL EAST PROCESS

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