Angular Momentum and Energy Structure of the Coherent State of a 2D Isotropic Harmonic Oscillator

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Abstract The angular momentum structure and energy structure of the coherent state of a 2D isotropic harmonic oscillator were investigated. Calculations showed that the average values of angular momentum and energy (except the zero-point energy) of this nonspreading 2D wavepacket are identical to those of the corresponding classical oscillator moving along a circular or an elliptic orbit.

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I. Introduction

The coherent state of a harmonic oscillator was first constructed by Schrödinger[1,2] and was widely used in the description of coherent light sources and in communication theory at optical frequency.[3,4] The main motivation of Schrödinger was to investigate the relation between quantum mechanics and classical mechanics.[5] His aim was to find a special kind of quantum state — a nonspreading wavepacket whose center follows the corresponding classical motion. He believed that it is only a question of computational skill to accomplish the same thing for electron in the hydrogen atom. However, wavepackets describing the Kepler orbits in a hydrogen atom were yet to be discovered, which is usually considered to be connected with the nonuniformity of the hydrogen spectrum.[6] Therefore, someone tried to find the wavepackets constructed by the superposition of Rydberg’s states.[7] Nieto and Simmons have constructed approximate (not exact) coherent states for particle in general one-dimensional (1D) potentials.[8,9]

In classical mechanics a 2D isotropic harmonic oscillator follows, in general, an elliptic orbit, which is reduced to a circular orbit or a straight line in special cases. It is expected that the coherent states of a 2D isotropic harmonic oscillator are nondispersing wavepackets with centers moving along elliptic orbits. However, as we know, maybe due to computational difficulties, the classical correspondence (angular momentum structure, energy structure) of such coherent states has not been investigated in detail. In this letter the angular momentum and energy constituents of such 2D nonspreading wavepackets are calculated and it is shown that the centers of the wavepackets follow the identical elliptic orbits as the corresponding 2D classical oscillator.

II. Angular Momentum and Energy Constituents of the Coherent State of a 2D Isotropic Harmonic Oscillator

The Schrödinger’s coherent state of a 1D harmonic oscillator is well known[1,2]

$$\psi_{\xi_0}(\xi, t) = \frac{e^{1/2}}{\pi^{1/4}} \exp \left[ -\frac{i\omega t}{2} - \frac{1}{2} \xi^2 - \frac{1}{4} \xi_0^2(1 + e^{2i\omega t}) + \xi_0 \xi e^{-i\omega t} \right],$$

(1)

where $$\xi = \alpha x$$, $$\xi_0 = \alpha x_0$$, $$\alpha = \sqrt{M \omega / \hbar}$$, and $$|\psi|^2 = (\alpha / \sqrt{\pi}) \exp[-(\xi - \xi_0 \cos \omega t)^2]$$. The shape of this wavepacket remains unchanged as time progresses and the position of its center is located at $$\xi = \xi_0 \cos \omega t$$, which is the same as the motion of a classical oscillator with amplitude $$x_0 = \xi_0 / \alpha$$ and natural angular frequency $$\omega$$. 
Assuming that the phase of the coherent state along the $y$ direction is $\pi/2$ retarded with respect to that along the $x$ direction,

$$
\psi_{0\eta}(\eta, t) = \frac{\alpha^{1/2}}{\pi^{1/4}} \exp\left[ -\frac{t(\omega t - \pi/2)}{2} - \frac{1}{2} \eta^2 - \frac{1}{4} \eta_0^2 (1 - e^{2\omega t}) + i \eta_0 \eta e^{-i\omega t} \right],
$$

where $\eta = \alpha y$, $\eta_0 = \alpha y_0$, whose center is located at $\eta = \eta_0 \cos(\omega t - \pi/2)$. Thus the coherent state of a 2D isotropic harmonic oscillator is

$$
\psi_{0\eta_0}(\xi, \eta, t) = \frac{\alpha}{\pi^{1/2}} \exp\left[ -i\omega t + \frac{\pi}{4} - \frac{1}{2} (\xi^2 + \eta^2) - \frac{1}{4} \xi_0^2 (1 + e^{2\omega t}) - \frac{1}{4} \eta_0^2 (1 - e^{2\omega t}) \right] e^{-i\xi_0 \xi + i\eta_0 \eta}.
$$

The wavefunction at initial time ($t = 0$) is (the trivial constant phase factor $e^{i\pi/4}$ being neglected)

$$
\psi_c(\xi, \eta) = \frac{\alpha}{\pi^{1/2}} \exp\left[ -\frac{1}{2} (\xi^2 + \eta^2) - \frac{1}{2} \xi_0^2 + (\xi_0 \xi + i\eta_0 \eta) \right].
$$

This 2D coherent state is a nonstationary state which is a coherent superposition of infinite stationary states. To investigate its angular momentum structure and energy structure, we may expand Eq. (4) in terms of the simultaneous eigenstates of the complete set of conserved quantities ($\hat{H}, \hat{l}_z$), and the moduli of expansion coefficients are time-independent. The normalized simultaneous eigenstates of ($\hat{H}, \hat{l}_z$) for a 2D isotropic oscillator may be expressed as

$$
\psi_{mn_\varphi}(\tilde{\rho}, \varphi) = \left[ \frac{n_r! \alpha^2}{\pi^{1/4}} \right]^{1/2} e^{im\varphi} \tilde{\rho}^{|m|} e^{-\tilde{\rho}^2/2} L^{(m)}_{n_r} (\tilde{\rho}^2)
$$

with $n_r$, $|m| = 0, 1, 2, \cdots$, $\tilde{\rho} = \alpha \rho = \alpha \sqrt{x^2 + y^2} = \sqrt{\xi^2 + \eta^2}$, where $L$ is the generalized Laguerre polynomial$^{[9]}$ and the corresponding eigenvalue is

$$
E = E_N = (N + 1) \hbar \omega, \quad N = 2n_r + |m| = 0, 1, 2, \cdots.
$$

The expansion coefficients of $\psi_c$ in terms of $\psi_{mn_\varphi}$ are

$$
C_{mn_\varphi} = \int_0^{2\pi} d\varphi \int_0^\infty \rho d\rho \psi_c(\xi, \eta) \psi_{mn_\varphi}^*(\tilde{\rho}, \varphi),
$$

which can be calculated in the following two cases:

1) $\xi_0 = \eta_0$ (circular orbit)

Substituting Eqs (4) and (5) into Eq. (7), it is shown (see Appendix) that

$$
C_{mn_\varphi} = \begin{cases}
\xi_0^m e^{-\xi_0^2/2} \left( \frac{1}{m!} \right)^{1/2} \delta_{m, 0}, & (m \geq 0), \\
0, & (m < 0).
\end{cases}
$$

This is expected because the quantum state corresponding to a classical circular orbit must have $n_r = 0$ (radial wavefunction without node). $m \geq 0$ in Eq. (8) means that the oscillator moves counterclockwise along a circular orbit. If the phase of the coherent state along the $y$ direction is $\pi/2$ advanced with respect to that along the $x$ direction, $C_{mn_\varphi}$ does not vanish only for $m < 0$, which means that the circular motion is clockwise.

Using Eq. (8) we may investigate the angular momentum structure and energy structure of the 2D coherent state (3). First, the average value of $m$ is

$$
\bar{m} = \sum_{m=0}^{\infty} m \xi_0^m e^{-\xi_0^2/2} \frac{1}{m!} = \xi_0^2,
$$

hence, the average value of angular momentum $l_z$ is

$$
\bar{l}_z = \bar{m} \hbar = \xi_0 \hbar = M \omega x_0^2 = MR^2 \omega,
$$

where $R = x_0$ is the radius of circular orbit. It is seen that $\bar{l}_z$ is the same as the angular momentum of the corresponding classical 2D oscillator moving along a circular orbit with
radius \( R \) and angular frequency \( \omega \). Second, we may calculate the average value of energy using Eq. (8) (note \( n_r = 0, N = |m|) \)

\[
\bar{H} = (\bar{m} + 1)\hbar \omega = MR^2\omega + \hbar \omega .
\]

It is seen that \( \bar{H} \) (except the zero-point energy \( \hbar \omega \)) is just the energy of the corresponding classical oscillator moving along a circular orbit with radius \( R \) and angular frequency \( \omega \).

2) \( \xi_0 \neq \eta_0 \) (elliptic orbit)

Let

\[
A = (\xi_0 - \eta_0)/2, \quad B = (\xi_0 + \eta_0)/2,
\]

we obtain (see Appendix)

\[
C_{mn_r} = \begin{cases} \frac{1}{2^{1/2} n_r! n_r!} e^{-\xi_0^2/2} e^{\lambda n} A^{n_r} B^{m+n_r}, & (m \geq 0), \\ \frac{1}{2^{1/2} n_r! (n + m)!} e^{-\xi_0^2/2} e^{\lambda n} A^{-m+n_r}, & (m < 0). \end{cases}
\]

Using Eq. (13) we may calculate

\[
\bar{n}_r(m \geq 0) = \sum_{m \geq 0, n_r} |C_{mn_r}|^2 n, \quad \bar{n}_r(m < 0) = \sum_{m < 0, n_r} |C_{mn_r}|^2 n,
\]

\[
\bar{n}_r(m \geq 0) = \sum_{m \geq 0, n_r} |C_{mn_r}|^2 n, \quad \bar{n}_r(m < 0) = \sum_{m < 0, n_r} |C_{mn_r}|^2 n.
\]

For example,

\[
\bar{n}_r(m \geq 0) = e^{-\xi_0^2} e^{2AB} \left[ \frac{A^2}{1!} (e^{\lambda^2} - 1) + 2 \frac{A^4}{2!} \left( e^{\lambda^2} - 1 - \frac{B^4}{2!} \right) + \ldots \right] \\
+ 3 \frac{A^6}{3!} \left( e^{\lambda^2} - 1 - \frac{B^2}{1!} - \frac{B^4}{2!} \right) + \ldots
\]

Similarly, it can be shown that

\[
\bar{n}_r(m < 0) = e^{-\xi_0^2} e^{2AB} \left[ \frac{B^2}{2!} \left( \frac{A^6}{4!} + 4 \frac{A^8}{6!} + \ldots \right) + \ldots \right].
\]

Hence, we get

\[
\bar{n}_r(m \geq 0) + (\bar{n}_r + n)(m < 0) = A^2 e^{-\xi_0^2} + 2AB + B^2 = A^2.
\]

Similarly,

\[
\bar{n}_r(m < 0) + (\bar{n}_r + n)(m \geq 0) = B^2.
\]

Equation (19) plus and minus equation (18) result in, respectively,

\[
\bar{n}_r + |\bar{m}| = A^2 + B^2,
\]

\[
\bar{m} = B^2 - A^2.
\]

Therefore, we get

\[
\bar{l}_z = \bar{m} \hbar = (B^2 - A^2)\hbar = \xi_0 \eta_0 \hbar = x_0 y_0 M \omega,
\]

which is just the angular momentum of a classical oscillator moving along an elliptic orbit with semi-major and semi-minor axes of \( x_0 \) and \( y_0 \). The average value of energy is

\[
\tilde{H} = (2n_r + |m| + 1)\hbar \omega = (A^2 + B^2)\hbar \omega + \hbar \omega = \frac{1}{2} (\xi_0^2 + \eta_0^2) \hbar \omega + \lambda \hbar \omega = \frac{1}{2} (x_0^2 + y_0^2) M \omega^2 + \hbar \omega,
\]

which is also the same as that of a classical oscillator moving along an elliptic orbit (except the zero-point energy \( \hbar \omega \)).
Appendix A

1) $\xi_0 = \eta_0$ (circular orbit)

$$C_{mn_\rho} = \int \rho d \rho d \varphi \frac{\alpha}{\pi^{1/2}} \exp \left[ -\frac{1}{2} \xi_0^2 - \frac{1}{2} (\xi^2 + \eta^2) + \xi_0 (\xi + i \eta) \right] \times \left( \frac{n_\rho! \alpha^2}{\pi [m_\rho + n_\rho]!} \right)^{1/2} e^{-i m \varphi} \rho |m_\rho| e^{-\frac{1}{4} \rho^2 L_{m_\rho}^m (\rho^2)}.$$  \hspace{1cm} (A1)

Using $\xi^2 + \eta^2 = \rho^2$, $\xi + i \eta = \rho e^{i \varphi}$, and

$$\int_0^{2\pi} \exp[\xi_0 \rho e^{i \varphi}] e^{i m \varphi} d \varphi = \left\{ \begin{array}{ll} 2\pi (\xi_0 \rho)^m / m!, & (m \geq 0), \\ 0, & (m < 0), \end{array} \right.$$  \hspace{1cm} (A2)

we get

$$C_{mn_\rho} = \left[ \frac{n_\rho!}{(m_\rho + n_\rho)!} \right]^{1/2} e^{-\xi_0^2 / 2} \frac{2 \xi_0 m_\rho}{m!} \int_0^\infty d \rho \rho e^{-|m_\rho| + 1} \rho^2 L_{m_\rho}^m (\rho^2).$$  \hspace{1cm} (A3)

Using

$$2 \int_0^\infty x^{2\lambda+1} e^{-x^2} L_n^\mu (x^2) dx = (-\pi)^{\lambda + \frac{1}{2}} \Gamma (\lambda + 1) \left( \frac{\lambda - \mu}{\pi} \right),$$  \hspace{1cm} (A4)

it is seen that the integral in Eq. (A3) does not vanish only for $n_\rho = 0$,

$$C_{mn_\rho} = e^{-\xi_0^2 / 2} \left( \frac{1}{m!} \right)^{3/2} \frac{2 \xi_0 m_\rho}{m!} \delta_{n_\rho,0}, \quad (m \geq 0)$$

$$= \left\{ \begin{array}{ll} \xi_0^m e^{-\xi_0^2 / 2} \left( \frac{1}{m!} \right)^{1/2} \delta_{n_\rho,0}, & (m \geq 0), \\ 0, & (m < 0). \end{array} \right.$$  \hspace{1cm} (A5)

2) $\xi_0 \neq \eta_0$ (elliptic orbit)

$$C_{mn_\rho} = \left[ \frac{n_\rho!}{\pi^2 (m_\rho + n_\rho)!} \right]^{1/2} e^{-\xi_0^2 / 2} \int e^{i \eta_0 \eta} e^{-i m \varphi} \rho |m_\rho| + 1 e^{-\rho^2 L_{m_\rho}^m (\rho^2)} d \rho d \varphi.$$  \hspace{1cm} (A6)

Using

$$\int_0^{2\pi} d \varphi \exp[\xi_0 \xi + i \eta_0 \eta] e^{-i m \varphi} = \left\{ \begin{array}{ll} 2\pi \sum_{k=0}^\infty \frac{(A \rho)^{k+m}}{k!(k+m)!}, & (m \geq 0), \\ 2\pi \sum_{k=0}^\infty \frac{(A \rho)^{k-m}}{k!(k-m)!}, & (m < 0), \end{array} \right.$$  \hspace{1cm} (A7)

equation (A6) is reduced to Eq. (13).

References

[1] E. Schrödinger, Naturwissenschaften 14 (1926) 166.
[2] L.I. Schiff, Quantum Mechanics, 3rd ed., McGraw-Hill, NY (1968) p. 74.
[3] R.J. Glauder, Phys. Rev. Lett. 10 (1963) 84; Phys. Rev. 130 (1963) 2529; ibid. 131 (1963) 2766.
[4] e.g., J.R. Klauder and E.C.G. Sudershan, Fundamentals of Quantum Optics, Benjamin, NY (1968); J.R. Klauder and B. Skagertam, Coherent States, World Scientific, Singapore (1985).
[5] A letter to Max Planck, Letters on Wave Mechanics, ed. K. Pizibram, Vision, London (1967).
[6] e.g., G. Alber and P. Zoller, Phys. Rep. 199 (1991) 231; M. Nauenberg, C. Stroud and J. Yeazell, Scientific American (1994) p. 24.
[7] M.M. Nieto and L.M. Simmons Jr., Phys. Rev. Lett. 41 (1978) 207; Phys. Rev. D20 (1979) 1321, 1332, 1342.
[8] S.K. Roy and V. Singh, Phys. Rev. D35 (1982) 3413.
[9] WANG ZhuXi and GUO DunRen, Theory of Special Functions, Science Press, Beijing (1979); P.M. Morse and H. Feshbach, Method of Theoretical Physics, McGraw-Hill, NY (1953).