Abstract versus Concrete Computation on Metric Partial Algebras

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Abstract

Data types containing infinite data, such as the real numbers, functions, bit streams and waveforms, are modelled by topological many-sorted algebras. In the theory of computation on topological algebras there is a considerable gap between so-called abstract and concrete models of computation. We prove theorems that bridge the gap in the case of metric algebras with partial operations.

With an abstract model of computation on an algebra, the computations are invariant under isomorphisms and do not depend on any representation of the algebra. Examples of such models are the ‘while’ programming language and the BCSS model. With a concrete model of computation, the computations depend on the choice of a representation of the algebra and are not invariant under isomorphisms. Usually, the representations are made from the set \( \mathbb{N} \) of natural numbers, and computability is reduced to classical computability on \( \mathbb{N} \). Examples of such models are computability via effective metric spaces, effective domain representations, and type two enumerability.

The theory of abstract models is stable: there are many models of computation, and conditions under which they are equivalent are largely known. The theory of concrete models is not yet stable, though it seems to be converging: several interesting models are known to be equivalent over special types of topological algebra. We investigate the problem of comparing the two types of models and, hence, establishing a unified and stable theory of computation for topological algebras.

First, we show that to compute functions on topological algebras using an abstract model, it is necessary that one must use algebras with partial operations and computable functions that are continuous and multivalued. This multivaluedness is needed even to compute single-valued functions, and so abstract models must be nondeterministic even to compute

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deterministic problems. Then we choose the ‘while’-array programming language as an abstract model for computing on any data type, and extend it with a nondeterministic assignment of “countable choice”. This is the new WhileCC* model. Finally, we introduce the notion of approximable multivalued computation on metric algebras. As a concrete model, we choose effective metric spaces. Among a number of results we prove the following.

For any metric algebra $A$ with an effective representation, any function WhileCC* approximable over $A$ is computable in the effective representation of the metric algebra $A$. Conversely, we show that, under certain reasonable conditions on the effective metric algebra $A$, any function that is effective is also WhileCC* approximable. We give an equivalence theorem, and examples of algebras where equivalence holds.

Keywords: data types, abstract models of computation, concrete models of computation, partial algebra, ‘while’ language, countable choice, nondeterminism, multivalued functions, metric algebras, topological algebras, approximation by ‘while’ programs, effective metric spaces, effective Banach spaces

0 Introduction

The theory of data in computer science is based on many sorted algebras and homomorphisms. The theory originates in the 1960s, and has developed a wealth of theoretical concepts, methods and techniques for the specification, construction, and verification of software and hardware systems. It is a significant achievement in computer science and has exerted a profound influence on programming [Wir91, GTW78, MG85]. However, given the absolutely fundamental nature of its subject matter — data — there are many fascinating and significant open problems. An important general problem is:

To develop a comprehensive theory of specification, computation and reasoning with infinite data.

By infinite data we mean real numbers, spaces of functions, streams of bits or reals, waveforms, multidimensional graphics objects, video, and analogue and digital interfaces. The application areas are obvious: scientific modelling and simulation, embedded systems, graphics and multimedia communications.

Data types containing infinite data are modelled by topological many-sorted algebras. In this paper we consider computability theory on topological algebras and investigate the problem

To compare and integrate high-level, representation independent, abstract models of computation with low-level, representation dependent, concrete models of computation in topological algebras.

Computability theory lies at the technical heart of theories of both specification and reasoning about such systems. There are many disparate ways of defining computable functions on topological algebras and some have (different) significant mathematical theories. In the case of real numbers one can contrast the approaches in books such as [Abe80, Abe01, PER89, Wei00, BCSS97].
Generally speaking, the models of computation for an algebra can be divided into two kinds: the abstract and concrete.

With an abstract model of computation for an algebra the programs do not depend on any representation of the algebra and are invariant under isomorphisms. Abstract models originated in the late 1950s in formalising flowcharts, and include program schemes and many general models of recursion. Examples of such models are the While programming language over any algebra and the Blum-Cucker-Shub-Smale model [BSS89, BCSS97] over the rings of real or complex numbers. The theory of abstract models is stable: there are many models of computation and the conditions under which they are equivalent are largely known [TZ88, TZ00]. For example, ‘while’ programs, flow charts, register machines, Kleene schemes, etc., are equivalent on any algebra; the BCSS models are simply instances obtained by choosing the algebra appropriate to the ring or ordered ring.

With a concrete model of computation for an algebra the programs and computations are not invariant under isomorphisms, but depend on the choice of a representation of the algebra. Usually, the representations are made from the set $\mathbb{N}$ of natural numbers, and computability on an algebra is reduced to classical computability on $\mathbb{N}$. Concrete models originated in the 1940s, in formalising the computable functions on real numbers. Examples of general models are computability via

- effective metric spaces [Mos64],
- computable sequence structures [PER89],
- domain representations [SHT88, SHT95, Eda95, Eda97], and
- type two enumerability [Wei00].

The theory of concrete models is not stable though it seems to be converging: several basic models are known to be equivalent in special cases (see, e.g., [SHT99] where the four general approaches above are shown to be equivalent).

In the theory of computation on algebras, abstract models are implemented by concrete models. Thus, the gap between the models is the gap between high level programming abstractions and low level implementations, and can be explored in terms of the following concepts:

- **Soundness of abstract model**: The functions computable in the abstract model are also computable in the concrete model.

- **Adequacy of abstract model**: The functions computable in the concrete model are computable in the abstract model.

- **Completeness of abstract model**: Functions are computable in the abstract model if, and only if, they are computable in the concrete model.

However, there is a considerable gap between abstract and concrete models of computation, especially over topological data types. For example, the popular abstract model in [BCSS97] is not sound for the main concrete models because of its assumptions about the total computability of relations such as equality. Equality on the real numbers is not everywhere continuous, but in all the concrete models computable functions are continuous (cf. Ceitin’s Theorem [Mos64]). The connection between abstract and concrete models of
computation on the real numbers is examined in [TZ99] where approximation by ‘while’ programs over a particular algebra was shown to be equivalent to the standard concrete model of GL computability over the unit interval.

First attempts at bridging the gap for all topological algebras in general have been made in [Bra96, Bra99], using a generalisation of recursion schemes (abstract computability) and Weihrauch’s type two enumerability (concrete computability). Here we investigate further the problems in comparing the two classes of models and in establishing a unified and stable theory of computation on topological algebras. We prove new theorems that bridge the gap in the case of computations on metric algebras with partial operations.

By reflecting on a series of examples, we show that to compute functions on topological algebras, it is necessary that one must consider

(i) algebras with partial operations,
(ii) computable functions that are both continuous and multivalued, and
(iii) approximations by abstract programs.

In particular, multivalued functions are needed, even to compute single-valued functions. Thus, to prove an equivalence between abstract and concrete models we must include a nondeterministic construct to define multivalued functions, and in this way use nondeterministic abstract models even to compute deterministic problems. We find that

imperative and other abstract programming models must be nondeterministic to express even simple programs on topological data types.

We choose the While programming language as an abstract model for computing on any data type, and extend it with the nondeterministic assignment of countable choice having the form:

\[ x ::= \text{choose } z : b(z, x, y) \]

where \( z \) is a natural number variable and \( b \) is a Boolean-valued operation. This new model is called WhileCC* computability (‘CC’ for “countable choice”, ‘*’ for array variables.) In particular, we introduce a notion of approximable multivalued computation, and formulate and prove the continuity of their semantics. We thus have the partial multivalued functions approximable by a WhileCC* program on \( A \).

As a concrete model, we choose effective metric spaces; this is known to be equivalent with several other concrete models. In computation with effective metric spaces \( A \) we pick an enumeration \( \alpha \) of a subspace \( X \) of \( A \), and construct the subspace \( C_\alpha(X) \) of \( \alpha \)-computable elements of \( A \), enumerated by \( \bar{\alpha} \). We thus have the partial functions computable on \( C_\alpha(X) \) in the representation \( \bar{\alpha} \).

We then prove two theorems that can be summarised (a little loosely) as follows.

**Soundness Theorem:** Let \( A \) be any metric partial algebra with an effective representation \( \alpha \). Suppose \( C_\alpha(X) \) is a subalgebra of \( A \), effective under \( \bar{\alpha} \). Then any function \( F \) on \( A \) that is WhileCC* approximable over \( A \) is computable on \( C_\alpha(X) \) in \( \bar{\alpha} \).

The soundness theorem is technically involved but quite general, and gives new insight into the semantics of imperative programs applied to topological data types. The converse
Adequacy Theorem: Let $A$ be any metric partial algebra $A$ with an effective representation $\alpha$. Suppose the representation $\alpha$ is $\text{WhileCC}^*$ computable and dense. Then any function $F: A \to A$ that is computable on $C_\alpha(X)$ in $\alpha$ and effectively locally uniformly continuous in $\alpha$ is $\text{WhileCC}^*$ approximable over $A$.

These are combined into a Completeness Theorem.

The proper statements of these theorems are given as Theorems A, B and C (in Sections 6, 7 and 8). Some interesting applications to algebras of real numbers and to Banach spaces are studied.

Here is the structure of the paper. We begin, in Section 1, by explaining the role of partiality, continuity and multivaluedness in computation, using simple examples on the real numbers. In Section 2 we describe topological and metric partial algebras and their extensions. In Section 3 we introduce the $\text{WhileCC}^*$ language, give it an algebraic semantics, and define approximable $\text{WhileCC}^*$ computability. We will see that the $\text{WhileCC}^*$ language has a complex semantics. However on total algebras it defines precisely the $\text{While}^*$ computable functions. Section 4 is devoted to examples. In Section 5 we prove the continuity of these $\text{WhileCC}^*$ computable multivalued functions. In Section 6 we introduce our concrete model, effective metric spaces, and prove a Soundness Theorem (Theorem $A_0$) for the special case of surjective enumerations of countable (not necessarily metric) algebras. In Section 7 we define the subspace (or subalgebra) of elements computable in a metric algebra, and then prove the more general Soundness Theorem (Theorem $A$) and, in Section 8, the Adequacy Theorem (Theorem $B$). These are combined into a Completeness Theorem (Theorem $C$) in Section 9. Concluding remarks are made in Section 10.

This work is part of a research programme — starting in [TZ88] and most recently surveyed in [TZ00] — on the theory of computability on algebras, and its application to specifiability and verifiability in different areas of computer science and mathematics. Specifically, it has developed from our studies of real and complex number computation in [TZ92a, TZ99, TZ00], stream algebras in [TZ92b, TZ94] and metric algebras in [TZ01].

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1 Nondeterminism, many-valuedness, non-extensionality, continuity and partiality: Some real number examples

When one considers the relation between abstract and concrete models, a number of intriguing problems appear. We explain them by considering a series of examples. Then we formulate our strategy for solving these problems.

Our chosen abstract and concrete models are introduced later (in Sections 3 and 5, respectively), so we must explain the problems of computing on the real number data type in rather general terms. First, we sketch the abstract and concrete forms of the real number data type. The picture for topological algebras in general will be clear from the examples.
1.1 Abstract versus concrete data types of real numbers; Continuity; Partiality

1.1.1 Abstract and concrete data types of reals. To compute on the set $\mathbb{R}$ of real numbers with an abstract model of computation, we have only to select an algebra $A$ in which $\mathbb{R}$ is a carrier set. Abstract computability on an algebra $A$ is a computability relative to $A$: a function is computable over $A$ if it can be programmed from the operations of $A$ using the programming constructs of the abstract model. Clearly, there are infinitely many choices of operations with which to make an algebra $A$, and hence there are infinitely many choices of classes of abstractly computable functions. All the classes of abstractly computable functions on $\mathbb{R}$ have decent mathematical theories, resembling the theory of the computable functions on the natural numbers — thanks to the general theory of computable functions on many sorted algebras [TZ00].

In contrast, to compute on $\mathbb{R}$ with a concrete model of computation, we choose an appropriate concrete representation $R$, and map

$$\alpha: R \rightarrow \mathbb{R}$$

where $R$ is an algebra made from the set $\mathbb{N}$ of natural numbers. For example, the map will be based on the fact that the reals can be built from the rationals, and hence the naturals, in a variety of equivalent ways (such as Cauchy sequences, decimal expansions, etc.). The computability of functions on the reals is investigated using the theory of computable functions on $\mathbb{N}$, applied to $\mathbb{R}$ via $\alpha$.

To compare this computation theory with abstract models, we choose an algebra $A$ in which $\mathbb{R}$ is a carrier set and, in particular, the operations of $A$ are computable with respect to the representation $\alpha$. For example, multiplication by 3 is not computable in the decimal representation, but the field operations on $\mathbb{R}$ are computable in the Cauchy sequence representation.

We assume that our concrete model is the subspace $\text{CS}$ of Baire space $\mathbb{N}^\mathbb{N}$ consisting of codings of fast Cauchy sequences of rationals, i.e., sequences $(k_n)$ of naturals such that for all $n$ and all $m > n$, $|r_{k_m} - r_{k_n}| < 2^{-n}$, where $r_0, r_1, r_2, \ldots$ is some standard enumeration of the rationals. The representing function

$$\alpha: \text{CS} \rightarrow \mathbb{R}$$

is continuous and onto.

1.1.2 Continuity. Computations with real numbers involve infinite data. The topology of $\mathbb{R}$ defines a process of approximation for infinite data; the functions on the data that are continuous in the topology are exactly the functions that can be approximated to any desired degree of precision.

For abstract models we assume the algebra $A$ that contains $\mathbb{R}$ is a topological algebra, i.e., one in which the basic operations are continuous in its topologies. We expect further that all the computable functions will be continuous. The class of functions that can be
abstractly computed exactly can be quite limited! With abstract models, approximate computations also turn out to be necessary [TZ99].

In the concrete models, moreover, it follows from Ceitin’s Theorem [Mos64] that if a function is computable then it is continuous.

Thus, in both abstract and concrete approaches, an analysis of basic concepts shows that computability implies continuity.

1.1.3 Partiality. In computing with an abstract model on $A$ we assume $A$ has some boolean-valued functions to test data. For example, in computing on $\mathbb{R}$ we need to use the functions

$$=_{R} : \mathbb{R}^{2} \to \mathbb{B} \quad \text{and} \quad <_{R} : \mathbb{R}^{2} \to \mathbb{B}$$

where $\mathbb{B} = \{\text{true}, \text{false}\}$ is the set of booleans.

Use of these functions presents a problem, since total continuous boolean-valued functions on the reals must be constant. This is because the only continuous functions from a connected space to a discrete space are the constant functions. Furthermore, in [TZ99] it was shown that on connected total topological algebras, the ‘while’ and ‘while’-array computable functions are precisely the functions explicitly definable by terms over the algebra.

To study the full range of real number computations, we must therefore redefine these tests as partial boolean-valued functions. Computation with partial algebras has interesting effects on the theory of computable functions, as indicated in [TZ99].

On the basis of these preliminary remarks on the data type of reals, we turn to the examples.

1.2 Examples of nondeterminism and many-valuedness

We now look at three examples of computing functions on $\mathbb{R}$.

Example 1.2.1: Pivot function. Define the function

$$\text{piv} : \mathbb{R}^{n} \rightarrow \{1, \ldots, n\}$$

by

$$\text{piv}(x_{1}, \ldots, x_{n}) = \begin{cases} \text{some } i : x_{i} \neq 0 \text{ if such an } i \text{ exists} \\ \uparrow \quad \text{otherwise} \end{cases} \quad (1)$$

Computation of this pivot is a vital step in the Gaussian elimination algorithm for inverting matrices.

Note that (depending on the precise semantics for the phrase “some $i$” in (1)) piv is nondeterministic or (alternatively) many-valued on $\text{dom}(\text{piv}) = \mathbb{R}^{n}\setminus\{0\}$. Further:

(a) There is no single-valued function which satisfies the definition (1) and is continuous on $\mathbb{R}^{n}$. For such a function, being continuous and integer-valued, would have to be constant on its domain $\mathbb{R}^{n}\setminus\{0\}$, with constant value (say) $j \in \{1, \ldots, n\}$. But its value on the $x_{j}$-axis would have to be different from $j$, leading to a contradiction.
(b) However there is a computable (and hence continuous!) single-valued function
\[
piv_0: \mathbf{CS}^n \to \{1, \ldots, n\}\quad (2)
\]
with a simple algorithm. Note however that \(\text{piv}_0\) is not extensional on \(\mathbf{CS}^n\) (i.e., not well defined on \(\mathbb{R}^n\)), or (equivalently) the map (2) cannot be factored through \(\mathbb{R}^n\):

\[
\begin{array}{c}
\mathbf{CS}^n \\
\alpha \\
\mathbb{R}^n \\
\Longleftarrow \\
\Longleftarrow \\
\downarrow \text{piv}_0 \\
\uparrow \text{?} \\
\{1, \ldots, n\}
\end{array}
\]

In effect, we can regain continuity (for a single-valued function), by foregoing extensionality.

(c) Alternatively, we can maintain continuity and extensionality by giving up single-valuedness. For the many-valued function
\[
piv_\omega: \mathbb{R}^n \to \mathcal{P}_\omega(\{1, \ldots, n\})
\]
(where \(\mathcal{P}_\omega(\ldots)\) denotes the set of countable subsets of \(\ldots\) ) defined by: for all \(k \in \{1, \ldots, n\}\)
\[
k \in \text{piv}_\omega(x_1, \ldots, x_n) \iff x_k \neq 0,
\]
is extensional and continuous, where a function
\[
f: A \to \mathcal{P}_\omega(B)
\]
is defined to be continuous iff for all open \(Y \subseteq B\),
\[
f^{-1}[Y] := \{x \in A \mid f(x) \cap Y \neq \emptyset\}
\]
is open in \(A\). (We will consider continuity of many-valued functions systematically in Section 5.)

Remarks 1.2.2. (i) The many-valued function \(\text{piv}_\omega\) is “tracked” (in a sense to be elucidated in Section 6) by (any implementation of) \(\text{piv}_0\).

(ii) We could only recover continuity of the \(\text{piv}\) function by giving up either extensionality (as in (b)) or single-valuedness (as in (c)).

(ii) Note however that the complete algorithm for inverting matrices is continuous and deterministic (hence single-valued) and extensional, even though it contains \(\text{piv}_0\) as an essential component!

Example 1.2.3: “Choose” a rational arbitrarily near a real. Define a function
\[
F: \mathbb{R} \times \mathbb{N} \to \mathbb{N}
\]
by
\[ F(x, n) = \text{“some” } k : d(x, r_k) < 2^{-n} \] (3)
where (as before) \( r_0, r_1, r_2, \ldots \) is some standard enumeration of the rationals. Note again (as in Example 1.1):

(a) There is no single-valued, continuous function \( F \) satisfying (3). This is because such a function, being continuous with discrete range space, would have to be constant in the first argument.

(b) But there is a single-valued computable (and continuous) function

\[ F_0 : \mathbb{CS} \times \mathbb{N} \to \mathbb{N} \]

trivially – just define

\[ F_0(\xi, n) = \xi_n. \]

This is, again, non-extensional on \( \mathbb{R} \).

(c) Further, there is a many-valued, continuous, extensional function satisfying (1):

\[ F_\omega : \mathbb{R} \times \mathbb{N} \to \mathcal{P}_\omega(\mathbb{N}) \]

where

\[ F_\omega(x, n) = \{ k \mid d(x, r_k) < 2^{-n} \}. \]

Example 1.2.4: Finding the root of a function. This example is adapted from [Wei00]. Consider the function \( f_a \) shown in Figure 1, where \( a \) is a parameter which can assume any real value.
It is defined by

\[ f_a(x) = \begin{cases} 
  x + a + 2 & \text{if } x \leq -1 \\
  a - x & \text{if } -1 \leq x \leq 1 \\
  x + a - 2 & \text{if } 1 \leq x.
\end{cases} \]

This function has either 1 or 3 roots, depending on the size of \( a \). For \( a < -1 \), \( f_a \) has a single (large positive) root; for \( a > 1 \), \( f_a \) has a single (large negative) root; and for \(-1 < a < 1\), \( f_a \) has three roots, two of which become equal when \( a = \pm 1 \).

Let \( g \) be the (many-valued) function, such that \( g(a) \) gives all the non-repeated roots of \( f_a \). This is shown in Figure 2.

![Figure 2](image)

Again, we have the same situation as in the previous examples:

(a) We cannot choose a (single) root of \( f_a \) continuously as a function of \( a \).

(b) However, one can easily choose and compute a root of \( f_a \) continuously as a function of a Cauchy sequence representation of \( a \), i.e., non-extensionally in \( a \).

(c) Finally, \( g(a) \), as a many-valued function of \( a \), is continuous. (Note that in order to have continuity, we must exclude the repeated roots of \( f_a \), at \( a = \pm 1 \).)

**Remark 1.2.5.** Other examples of a similar nature abound, and can be treated similarly; for example, the problem of finding, for a given real number \( x \), an integer \( n > x \).
1.3 Solutions for the abstract model

In the above three examples we have given:

(i) a number of single-valued functions \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) that we want to compute;

(ii) arguments that they are not continuous;

(iii) a prima facie case that they cannot be abstractly computed on the abstract data type \( A \) containing \( \mathbb{R} \) because they are not continuous;

(iv) a prima facie case that they can be computed in the concrete data type \( \mathbf{CS} \);

(v) arguments that they are selection functions for many-valued functions on \( \mathbb{R} \) that are continuous.

At the level of concrete models of computation, there is not really a problem with the issues raised by the above examples, since concrete models work only by computations on representations of the reals (say by Cauchy sequences), as described fully in Sections 5 and 7.

The real problem arises with the construction of abstract models of computation on the reals which should model the phenomena illustrated by these examples, and should, moreover, correspond, in some sense, to the concrete models. Thus we have the question:

*Can such continuous many-valued functions be computed on the abstract data type \( A \) containing \( \mathbb{R} \) using new abstract models of computation? If they can, are the concrete and abstract models then equivalent?*

The rest of this paper deals with these issues. We answer the above question more generally, over many-sorted partial metric algebras \( A \).

The solution presented in this paper is to extend the \textbf{While}\textsuperscript{*} programming language over \( A \) \cite{TZ00} with a nondeterministic “countable choice” programming construct, so that in the rules of program term formation,

\[
\text{choose } z : b
\]

is a new term of type \texttt{nat}, where \( z \) is a variable of type \texttt{nat} and \( b \) is a term of type \texttt{bool}. We will revisit the examples after giving the language definition in Section 3.

Alternatively, one could use other abstract models; for example, one can modify the \( \mu\text{PR}\textsuperscript{*} \) function schemes \cite{TZ00, §8.1} by replacing the constructive least number (\( \mu \)) operator

\[
f(x) \simeq \mu z \in \mathbb{N} [g(x, z) = \texttt{tt}],
\]

where \( g \) is a boolean-valued function, by a nondeterministic choice operator:

\[
f(x) \simeq \text{choose } z \in \mathbb{N} [g(x, z) = \texttt{tt}].
\]

Given suitable semantics, these two approaches turn out to be equivalent.

In \cite{Bra99} a more elaborate set of recursive schemes over many-sorted algebras, with many-valued operations, was presented.
2 Topological partial algebras and continuity

We define some basic notions concerning topological and metric many-sorted partial algebras. We begin with some basic ideas and examples.

2.1 Basic algebraic definitions

A signature Σ (for a many-sorted partial algebra) is a pair consisting of (i) a finite set Sort(Σ) of sorts, and (ii) a finite set Func(Σ) of (basic) function symbols, each symbol $F$ having a type $s_1 \times \cdots \times s_m \rightarrow s$, where $s_1, \ldots, s_m, s \in \text{Sort}(Σ)$; in that case we write $F : s_1 \times \cdots \times s_m \rightarrow s$. (The case $m = 0$ corresponds to constant symbols.)

A Σ-product type has the form $u = s_1 \times \cdots \times s_m$ ($m \geq 0$), where $s_1, \ldots, s_m$ are Σ-sorts. We use the notation $u, v, w, \ldots$ for Σ-product types.

A partial Σ-algebra $A$ has, for each sort $s$ of Σ, a non-empty carrier set $A_s$ of sort $s$, and for each Σ-function symbol $F : u \rightarrow s$, a partial function $F^A : A^u \rightarrow A_s$, where, for the Σ-product type $u = s_1 \times \cdots \times s_m$, we write $A^u =_{df} A_{s_1} \times \cdots \times A_{s_m}$. (The notation $f : X \rightarrow Y$ refers in general to a partial function from $X$ to $Y$.)

The algebra $A$ is total if $F^A$ is total for each Σ-function symbol $F$. Without such a totality assumption, $A$ is called partial.

In this paper we deal mainly with partial algebras. The default assumption is that “algebra” refers to partial algebra. We will, nevertheless, for the sake of emphasis, often speak explicitly of “partial algebras”.

Given an algebra $A$, we write $Σ(A)$ for its signature.

Examples 2.1.1. The following algebras will be used repeatedly as examples in this paper. All but one are total.

(a) The algebra of booleans has the carrier $\mathbb{B} = \{\text{tt, ff}\}$ of sort bool. The signature $Σ(\mathbb{B})$ and algebra $\mathbb{B}$ respectively can be displayed as follows:

| signature $Σ(\mathbb{B})$ | algebra $\mathbb{B}$ |
|---------------------------|-----------------------|
| sorts bool                | carriers $\mathbb{B}$ |
| functions true, false : $→$ bool, and, or : bool$^2 \rightarrow$ bool | functions $\text{tt, ff} : \rightarrow \mathbb{B}$, and$^B, \text{or}^B : \mathbb{B}^2 \rightarrow \mathbb{B}$ |
| end                       | end                   |

Usually the signature can essentially be inferred from the algebra; indeed we will not define the signature where no confusion will arise. Further, for notational simplicity, we will not always distinguish between function names in the signature (true, etc.) and their intended interpretations ($\text{true}^B = \text{tt}$, etc.)

(b) The algebra $\mathcal{N}_0$ of naturals has a carrier $\mathbb{N}$ of sort nat, together with the zero constant
and successor function:

| algebra   | \( \mathcal{N}_0 \) |
| carriers  | \( \mathbb{N} \) |
| functions | \( 0: \mathbb{N} \to \mathbb{N}, \quad S: \mathbb{N} \to \mathbb{N} \) |
| end       |               |

(c) The ring \( \mathcal{R}_0 \) of reals has a carrier \( \mathbb{R} \) of sort real:

| algebra   | \( \mathcal{R}_0 \) |
| carriers  | \( \mathbb{R} \) |
| functions | \( 0, 1: \mathbb{R} \to \mathbb{R}, \quad +, \times: \mathbb{R}^2 \to \mathbb{R}, \quad -: \mathbb{R} \to \mathbb{R} \) |
| end       |               |

where

\[
\text{inv}^\mathcal{R}(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ \uparrow & \text{otherwise} \end{cases}.
\]

This is an example of a partial algebra. More examples of partial algebras will be given later.

Throughout this work we make the following assumption about the signatures \( \Sigma \).

**Assumption 2.1.2 (Instantiation Assumption).** For every sort \( s \) of \( \Sigma \), there is a closed term of that sort, called the default term \( \delta^s \) of that sort.

This guarantees the presence of default values \( \delta^s_A \) in a \( \Sigma \)-algebra \( A \) at all sorts \( s \), and default tuples \( \delta^u_A \) at all product types \( u \).

**Definition 2.1.3 (Expansions and reducts).** Let \( \Sigma \) and \( \Sigma' \) be signatures with \( \Sigma \subset \Sigma' \).

(a) If \( A' \) is a \( \Sigma' \)-algebra, then the \( \Sigma \)-reduct of \( A' \), \( A'|\_\Sigma \), is the algebra of signature \( \Sigma \), consisting of the carriers of \( A' \) named by the sorts of \( \Sigma \) and equipped with the functions of \( A' \) named by the function symbols of \( \Sigma \).

(b) If \( A \) is a \( \Sigma \)-algebra and \( A' \) is a \( \Sigma' \)-algebra, then \( A' \) is a \( \Sigma' \)-expansion of \( A \) iff \( A \) is the \( \Sigma \)-reduct of \( A' \).
2.2 Adding booleans: Standard signatures and algebras

The algebra $B$ of booleans (Example 2.1.1(a)) plays an essential role in computation, as we will see. This motivates the following definition.

**Definition 2.2.1 (Standard signature).** A signature $\Sigma$ is standard if 

(i) it contains the signature of booleans, i.e., $\Sigma(B) \subseteq \Sigma$, and 

(ii) The function symbols of $\Sigma$ include a conditional

$$
\text{if}_s : \text{bool} \times s^2 \rightarrow s
$$

for all sorts $s$ of $\Sigma$ other than $\text{bool}$.

Now given a standard signature $\Sigma$, a sort of $\Sigma$ is called an equality sort if $\Sigma$ includes an equality operator

$$
\text{eq}_s : s^2 \rightarrow \text{bool}.
$$

**Definition 2.2.2 (Standard algebra).** Given a standard signature $\Sigma$, a $\Sigma$-algebra $A$ is a standard if 

(i) it is an expansion of $B$; 

(ii) the conditional operator on each sort $s$ has its standard interpretation in $A$; i.e., for $b \in B$ and $x, y \in A_s$,

$$
\text{if}^A_s(b, x, y) = \begin{cases} 
  x & \text{if } b = \text{tt} \\
  y & \text{if } b = \text{ff};
\end{cases}
$$

(iii) the equality operator $\text{eq}_s$ is interpreted as a partial identity on each equality sort $s$, i.e., for any two elements of $A_s$, if they are identical, then the operator at these arguments returns $\text{tt}$ if it returns anything; and if they are not identical, it returns $\text{ff}$ if anything. More specifically, there are three possible cases. First, the case

$$
\text{eq}^A_s(x, y) = \begin{cases} 
  \text{tt} & \text{if } x = y \\
  \uparrow & \text{otherwise},
\end{cases}
$$

i.e., total equality, represents the situation that equality is “decidable” or “computable” at sort $s$, for example, when $s = \text{nat}$. Second, the case

$$
\text{eq}^A_s(x, y) = \begin{cases} 
  \uparrow & \text{if } x = y \\
  \text{ff} & \text{otherwise}
\end{cases}
$$

represents typically the situation that that equality is “semidecidable”. An example is given by the initial term algebra of an r.e. equational theory. Third, the case

$$
\text{eq}^A_s(x, y) = \begin{cases} 
  \uparrow & \text{if } x = y \\
  \text{ff} & \text{otherwise},
\end{cases}
$$
represents typically the situation that that equality is “co-semidecidable”. Examples are given by the data types of streams and real numbers, as mentioned in 1.1.3; see Example 2.2.4(c) below.

Note that any many-sorted signature \( \Sigma \) can be standardised to a signature \( \Sigma^B \) by adjoining the sort bool together with the standard boolean operations; and, correspondingly, any algebra \( A \) can be standardised to an algebra \( A^B \) by adjoining the algebra \( B \) as well as the conditional and equality operators.

**Examples 2.2.4 (Standard algebras).**

(a) The simplest standard algebra is the algebra \( B \) of the booleans (Example 2.1.1(a)).

(b) A standard algebra of naturals \( N \) is formed by standardising the algebra \( N_0 \) (Example 2.1.1(b)), with (total) equality and order operations on \( \mathbb{N} \):

```plaintext
algebra N
import N_0, B
functions
     if^{N}_{nat} : \mathbb{B} \times \mathbb{N}^2 \to \mathbb{N},
     eq^{N}_{nat}, less^{N}_{nat} : \mathbb{N}^2 \to \mathbb{B}
end
```

(c) A standard partial algebra \( R \) on the reals is formed similarly by standardising the field \( R_1 \) (Example 2.1.1(d)), with partial equality and order operations on \( \mathbb{R} \):

```plaintext
algebra R
import R_1, B
functions
     if^{R}_{real} : \mathbb{B} \times \mathbb{R}^2 \to \mathbb{R},
     eq^{R}_{real}, less^{R}_{real} : \mathbb{R}^2 \to \mathbb{B}
end
```

where

\[
eq^{R}_{real}(x, y) = \begin{cases} 
\uparrow & \text{if } x = y \\
\mathsf{ff} & \text{if } x \neq y.
\end{cases}
\]

and

\[
less^{R}_{real}(x, y) = \begin{cases} 
\mathsf{tt} & \text{if } x < y \\
\mathsf{ff} & \text{if } x > y \\
\uparrow & \text{if } x = y,
\end{cases}
\]

**Discussion 2.2.5 (Semicomputability and co-semicomputability).** The significance of the partial equality and order operations in Example (c) above, in connection with computability and continuity, has been touched on in 1.1.3. The continuity of partial functions will be discussed in §2.5 (and see in particular Example 2.5.3(b)). Regarding computability, these definitions are intended to reflect, or capture the intuition of, the
“semicomputability” of order and the “co-semicomputability” of equality on (a concrete model of) the reals. For given two reals \( x \) and \( y \), represented (say) by their infinite decimal expansions, suppose their decimal digits are being read systematically, the \( n \)-th digit of both at step \( n \). Then if \( x \neq y \) or \( x < y \), this will become apparent after finitely many steps, but no finite number of steps can confirm that \( x = y \).

Throughout this paper, we will assume the following, unless specifically noted to the contrary.

**Assumption 2.2.6 (Standardness Assumption).** The signature \( \Sigma \) and \( \Sigma \)-algebra \( A \) are standard.

### 2.3 Adding counters: \( \mathbb{N} \)-standard signatures and algebras

The standard algebra \( \mathcal{N} \) of naturals (Example 2.2.4(b)) plays, like \( \mathcal{B} \), an essential role in computation. This motivates the following definition.

**Definition 2.3.1 (\( \mathbb{N} \)-standard signature).** A signature \( \Sigma \) is \( \mathbb{N} \)-standard if

(i) it is standard, and

(ii) it contains the standard signature of naturals (Example 2.2.4(b)), i.e., \( \Sigma(\mathcal{N}) \subseteq \Sigma \).

**Definition 2.3.2 (\( \mathbb{N} \)-standard algebra).** Given an \( \mathbb{N} \)-standard signature \( \Sigma \), a corresponding \( \Sigma \)-algebra \( A \) is \( \mathbb{N} \)-standard if it is an expansion of \( \mathcal{N} \).

Note that any standard signature \( \Sigma \) can be \( \mathbb{N} \)-standardised to a signature \( \Sigma^N \) by adjoining the sort \( \text{nat} \) and the operations \( 0, S, \text{eq}_{\text{nat}}, \text{less}_{\text{nat}}, \text{if}_{\text{nat}} \). Correspondingly, any standard \( \Sigma \)-algebra \( A \) can be \( \mathbb{N} \)-standardised to an algebra \( A^N \) by adjoining the carrier \( \mathbb{N} \) together with the corresponding standard functions.

**Examples 2.3.3 (\( \mathbb{N} \)-standard algebras).**

(a) The simplest \( \mathbb{N} \)-standard algebra is the algebra \( \mathcal{N} \) (Example 2.2.4(b)).

(b) We can \( \mathbb{N} \)-standardise the standard real algebra \( \mathcal{R} \) (Example 2.2.4(c)) to form the algebra \( \mathcal{R}^N \).

### 2.4 Adding arrays: Algebras \( A^* \) of signature \( \Sigma^* \)

The significance of arrays for computation is that they provide *finite but unbounded memory*.

Given a standard signature \( \Sigma \), and standard \( \Sigma \)-algebra \( A \), we expand \( \Sigma \) and \( A \) in two stages:

(1°) \( \mathbb{N} \)-standardise these to form \( \Sigma^N \) and \( A^N \), as in §2.3.

(2°) Define, for each sort \( s \) of \( \Sigma \), the carrier \( A^*_s \) to be the set of *finite sequences* or *arrays* \( a^* \) over \( A_s \), of “starred sort” \( s^* \).
The resulting algebras $A^*$ have signature $\Sigma^*$, which extends $\Sigma^N$ by including, for each sort $s$ of $\Sigma$, the new starred sorts $s^*$, and certain new function symbols. Details are given in [TZ00, §2.7] and (an equivalent but simpler version) in [TZ99, §2.4].

The reason for introducing starred sorts is the lack of effective coding of finite sequences within abstract algebras in general.

2.5 Topological partial algebras

We now add topologies to our partial algebras, with the requirement of continuity for the basic partial functions. Background information on topology can be obtained from any standard text, e.g., [Kel55, Dug66, Eng89].

Definition 2.5.1. Given two topological spaces $X$ and $Y$, a partial function $f : X \rightarrow Y$ is continuous if for every open $V \subseteq Y$,

$$f^{-1}[V] = \{ x \in X \mid x \in \text{dom}(f) \text{ and } f(x) \in Y \}$$

is open in $X$.

Definition 2.5.2. (a) A topological partial $\Sigma$-algebra is a partial $\Sigma$-algebra with topologies on the carriers such that each of the basic $\Sigma$-functions is continuous.

(b) An (N-)standard topological partial algebra is a topological partial algebra which is also an (N-)standard partial algebra, such that the carriers $\mathbb{B}$ (and $\mathbb{N}$) have the discrete topology.

Examples 2.5.3. (a) (Discrete algebras.) The standard algebras $\mathcal{B}$ and $\mathcal{N}$ of booleans and naturals respectively (§§2.1, 2.2) are topological (total) algebras under the discrete topology. All functions on them are trivially continuous, since the carriers are discrete.

(b) (Partial real algebra.) An important standard topological partial algebra for our purpose is the real algebra $\mathcal{R}$ (Example 2.2.4(c)), or its N-standardised version $\mathcal{R}^N$ (Example 2.3.3(b)), in which $\mathbb{R}$ has its usual topology, and $\mathbb{B}$ and $\mathbb{N}$ the discrete topology. Recall our earlier discussion (1.1.3) of partiality of tests in connection with continuity, and note that the partial operations $\text{eq}_{\mathbb{R}}^\mathbb{R}$ and $\text{les}_{\mathbb{R}}^\mathbb{R}$ are continuous, in the sense of Definition 2.5.1.

(c) (Partial interval algebras.) Another useful class of standard topological partial algebras are of the form

| algebra | $\mathcal{I}$ |
| import | $\mathcal{R}$ |
| carriers | $I$ |
| functions | $i_f : I \rightarrow \mathbb{R}$, $F_1 : I^{m_1} \rightarrow I$, $\ldots$, $F_k : I^{m_k} \rightarrow I$ |

end
where $I$ is the closed interval $[0, 1]$ (with its usual topology), $i_I$ is the embedding of $I$ into $\mathbb{R}$, and $F_i : I^{m_i} \to I$ are continuous partial functions. These are called \textit{(partial) interval algebras on $I$}. There are also N-standard versions:

| algebra     | $I^N$               |
|-------------|---------------------|
| import      | $\mathcal{R}_N$     |
| carriers    | $I$                 |
| functions   | $i_I : I \to \mathbb{R}$, |
|             | $\ldots$           |

\textit{(d) ($N$-standard total real algebra.)} The algebra $\mathcal{R}_t^N$ is ("t" for "total topological"), defined by

| algebra     | $\mathcal{R}_t^N$ |
|-------------|-------------------|
| import      | $\mathcal{R}_0, \mathcal{N}, \mathcal{B}$ |
| functions   | $i_{\text{real}}^\mathcal{R} : \mathbb{B} \times \mathbb{R}^2 \to \mathbb{R}$, |
|             | $\text{div}_{\text{nat}}^\mathcal{R} : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$, |

Here $\mathcal{R}_0$ is the ring of reals (§2.1.1(c)), $\mathcal{N}$ is the standard algebra of naturals (2.2.4(b)), and $\text{div}_{\text{nat}}$ is division of reals by naturals.

Note that $\mathcal{R}_t^N$ does not contain (total) boolean-valued functions $<$ or $=$ on the reals, since they are not continuous (cf. the partial functions $\text{eq}_{\text{real}}$ and $\text{less}_{\text{real}}$ of $\mathcal{R}$). It is therefore not an expansion of $\mathcal{R}$.

\textbf{Definition 2.5.4 (Extensions of topology to $A^N$ and $A^*$).} Corresponding to the various algebraic expansions of $A$ detailed in §§2.3 and 2.4, there are induced topological expansions.

\begin{itemize}
  \item[(a)] The topological partial N-standard algebra $A^N$, of signature $\Sigma^N$, is constructed from $A$ by giving the new carrier $\mathbb{N}$ the discrete topology.
  
  \item[(b)] The topological partial array algebra $A^*$, of signature $\Sigma^*$, is constructed from $A^N$ as follows. Viewing the elements of $A^*_s$ as (essentially) arrays of elements of $A_s$ of finite length, we can give $A^*_s$ the \textit{disjoint union} topology of the sets $(A_s)^n$ of arrays of length $n$, for all $n \geq 0$, where each set $(A_s)^n$ is given the \textit{product topology} of the sets $A_s$.

  The topology on $A^*$ can also be described as follows. The \textit{basic open sets} in $A^*_s$ are of the form

  $$\{ a^* \in A^*_s \mid \text{Lgth}(a^*) > i_n \text{ and } a^*[i_1] \in U_1, \ldots, a^*[i_n] \in U_n \}$$

  for some $n > 0$, $i_1 < \cdots < i_n$ and open sets $U_1, \ldots, U_n \subseteq A_s$.

  It is easy to check that $A^*$ is indeed a topological algebra, \textit{i.e.}, all the new functions of $A^*$ are continuous.
2.6 Metric algebra

A particular type of topological algebra is a *metric partial algebra*. This is a many-sorted standard partial algebra with an associated metric:

| algebra | $A$ |
| import | $B$, $\mathcal{R}$ |
| carriers | $A_1, \ldots, A_r$ |
| functions | $F^A_1 : A^{u_1} \to A_{s_1}$, $\ldots$, $F^A_k : A^{u_k} \to A_{s_k}$, $d^A_1 : A^2_1 \to \mathbb{R}$, $\ldots$, $d^A_r : A^2_r \to \mathbb{R}$ |

where $B$ and $\mathcal{R}$ are respectively the algebras of booleans and reals (Examples 2.1.1(a), 2.2.4(c)), the carriers $A_1, \ldots, A_r$ are metric spaces with metrics $d^A_1, \ldots, d^A_r$ respectively, $F_1, \ldots, F_k$ are the $\Sigma$-function symbols other than $d_1, \ldots, d_k$, and the (partial) functions $F^A_i$ are all continuous with respect to these metrics, where continuity of a partial function is understood as in Definition 2.5.1.

Clearly, metric algebras can be viewed as special cases of *topological partial algebras*.

Note that the carrier $\mathbb{B}$ (as well as $\mathbb{N}$, if present) has the *discrete metric*, defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

which induces the discrete topology.

We will often speak of a “metric algebra $A$”, without stating the metric explicitly.

**Example 2.6.1.** The partial and total real algebras $\mathcal{R}$, $\mathcal{R}^N$ and $\mathcal{R}^N_t$ (Examples 2.5.3) can be recast as metric algebras in an obvious way.

**Remark 2.6.2 (Extension of metric to $A^*$).** A metric algebra $A$ can be expanded to a metric algebra $A^*$ of arrays over $A$. Namely, given a metric $d_s$ on $A_s$, we define a (bounded) metric $d^*_s$ on $A_s^*$ as follows: for $a^* = (a_1, \ldots, a_k), b^* = (b_1, \ldots, b_l) \in A_s^*$:

$$d^*_s(a^*, b^*) = \begin{cases} 1 & \text{if } k \neq l \\ \min (1, \max_{i=0}^{k-1} d_s(a^*[i], b^*[i])) & \text{otherwise} \end{cases}$$

This gives the same topology on $A^*$ as that induced by the topology on $A$ (Definition 2.5.4) [Eng89].

**Remark 2.6.3 (Product metric on $A$).** If $A$ is a $\Sigma$-metric algebra, then for each $\Sigma$-product sort $u = s_1 \times \cdots \times s_m$, we can define a metric $d_u$ on $A^u$ by

$$d_u((x_1, \ldots, x_m), (y_1, \ldots, y_m)) = \max_{i=1}^m (d_s(x_i, y_i))$$
or more generally, by the $\ell_p$ metric

$$d_u((x_1, \ldots, x_m), (y_1, \ldots, y_m)) = \left( \sum_{i=1}^{m} (d_s(x_i, y_i))^p \right)^{1/p} \quad \left(1 \leq p \leq \infty\right)$$

where $p = \infty$ corresponds to the “max” metric. This induces the product topology on $A^u$.

### 2.7 W-continuity: Another notion of continuity of partial functions

Recall our definition (2.5.1) of continuity of partial functions: $f : X \rightarrow Y$ is continuous if for every open $V \subseteq Y$, $f^{-1}[V]$ is open in $X$.

This is not the only reasonable definition. Another definition, used in [Wei00] and [Bra96, Bra99] (henceforth “W-continuity”), amounts to saying that $f$ is continuous iff its restriction to its domain

$$f \upharpoonright \text{dom}(f) : \text{dom}(f) \rightarrow Y$$

is continuous (as a total function), where $\text{dom}(f)$ has the topology as a subspace of $A$; or, equivalently, iff for every open $V \subseteq Y$, $f^{-1}[V]$ is open in $\text{dom}(f)$.

The following is easily checked:

**Proposition 2.7.1.** $f$ is continuous $\iff$ $f$ is W-continuous and $\text{dom}(f)$ is open.

**Remark 2.7.2.** It is instructive to express these two notions of continuity in terms of metric spaces. Suppose $f : X \rightarrow Y$ where $X$ and $Y$ are metric spaces. Then

(a) $f$ is continuous iff

$$\forall a \in \text{dom}(f) \forall \epsilon > 0 \exists \delta > 0 \forall x \in B(a, \delta) \left( x \in \text{dom}(f) \land f(x) \in B(f(a), \epsilon) \right).$$

(b) $f$ is W-continuous iff

$$\forall a \in \text{dom}(f) \forall \epsilon > 0 \exists \delta > 0 \forall x \in B(a, \delta) \left( x \in \text{dom}(f) \rightarrow f(x) \in B(f(a), \epsilon) \right).$$

Here $B(a, \delta)$ is the open ball with centre $a$ and radius $\delta$.

**Example 2.7.3.** Consider the partial function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $f$ is W-continuous, but not continuous.
3 ‘While’ programming with countable choice

The programming language WhileCC = WhileCC(Σ) is an extension of While(Σ) [TZ00, §2.1, 2.13] with an extra ‘choose’ rule of term formation. We give the complete definition of its syntax and semantics, using the algebraic operational semantics of [TZ00].

Assume Σ is an N-standard signature, and A is an N-standard Σ-algebra.

3.1 Syntax of WhileCC(Σ)

We define four syntactic classes: variables, terms, statements and procedures.

(a) \( Var = Var(Σ) \) is the class of Σ-program variables, and for each Σ-sort \( s \), \( Var_s \) is the class of program variables of sort \( s \): \( a^s, b^s, \ldots, x^s, y^s \ldots \).

(b) \( PTerm = PTerm(Σ) \) is the class of Σ-program terms \( t, \ldots \), and for each Σ-sort \( s \), \( PTerm_s \) is the class of program terms of sort \( s \). These are generated by the rules

\[
t ::= x^s \mid F(t_1, \ldots, t_n) \mid \text{choose } z^{\text{nat}} : b
\]

where \( s, s_1, \ldots, s_n \) are Σsorts, \( F : s_1 \times \cdots \times s_n \to s \) is a Σ-function symbol, \( t_i \in PTerm_{s_i} \) for \( i = 1, \ldots, n \) (\( n \geq 0 \)), and \( b \) is a boolean term, i.e., a term of sort \( \text{bool} \).

Think of ‘choose’ as a generalisation of the constructive least number operator \( \text{least } z : b \) which has the value \( k \) in case \( b[z/k] \) is true and \( b[z/i] \) is defined and false for all \( i < k \), and is undefined in case no such \( k \) exists.

Here ‘choose \( z : b \)’ selects some value \( k \) such that \( b[z/k] \) is true, if any such \( k \) exists (and is undefined otherwise). Which value is selected depends, in general, on the implementation of the algebra \( A \). In our abstract semantics, we will give the meaning as the set of all possible \( k \)’s (hence “countable choice”). Any concrete model will select a particular \( k \), according to the implementation.

Note that the program terms extend the algebraic terms (i.e., the terms over the signature \( Σ \)) by including in their construction the ‘choose’ operator, which is not an operation of \( Σ \). An alternative formulation would be to have ‘choose’ not as part of the term construction, but rather as a new atomic program statement: ‘choose \( z : b \)’. We prefer the present treatment, as it leads to the construction of many-valued term semantics (as we will see), which is interesting in itself, and which we would get anyway if we were to extend our syntax to include many-valued function procedure calls in our term construction.

We write \( t : s \) to indicate that \( t \in PTerm_s \), and for \( u = s_1 \times \cdots \times s_m \), we write \( t : u \) to indicate that \( t \) is a \( u \)-tuple of program terms, i.e., a tuple of program terms of sorts \( s_1, \ldots, s_m \).

We also use the notation \( b, \ldots \) for boolean terms.

(c) \( AtSt = AtSt(Σ) \) is the class of atomic statements \( S_{\text{at}}, \ldots \) defined by

\[
S_{\text{at}} ::= \text{skip} \mid \text{div} \mid x := t
\]

where ‘div’ stands for “divergence” (non-termination), and \( x := t \) is a concurrent assignment, where for some product type \( u \), \( t : u \) and \( x \) is a \( u \)-tuple of distinct variables.
(d) \( Stmt = Stmt(\Sigma) \) is the class of statements \( S, \ldots \), generated by the rules
\[
S ::= S_{\text{at}} \mid S_1; S_2 \mid \text{if } b \text{ then } S_1 \text{ else } S_2 \text{ fi } \mid \text{while } b \text{ do } S \text{ od}
\]

(e) \( Proc = Proc(\Sigma) \) is the class of function procedures \( P, Q, \ldots \). These have the form
\[
P \equiv \text{func in } a \text{ out } b \text{ aux } c \text{ begin } S \text{ end}
\]
where \( a, b \) and \( c \) are lists of input variables, output variables and auxiliary (or local) variables respectively, and \( S \) is the body. Further, we stipulate:

- \( a, b \) and \( c \) each consist of distinct variables, and they are pairwise disjoint,
- all variables occurring in \( S \) must be among \( a, b \) or \( c \),
- the input variables \( a \) must not occur on the lhs of assignments in \( S \),
- initialisation condition: \( S \) has the form \( S_{\text{init}}; S' \), where \( S_{\text{init}} \) is a concurrent assignment which initialises all the output and auxiliary variables, i.e., assigns to each variable in \( b \) and \( c \) the default term (2.1.2) of the same sort.

If \( a : u \) and \( b : v \), then \( P \) is said to have type \( u \to v \), written \( P : u \to v \). Its input type is \( u \).

### 3.2 Algebraic operational semantics of WhileCC

We will interpret programs as countably-many-valued state transformations, and function procedures as countably-many-valued functions on \( A \). Our approach follows the algebraic operational semantics of [TZ00, §§3.4]. First we need some definitions and notation for many-valued functions.

**Notation 3.2.1.**

(a) \( \mathcal{P}_\omega(X) \) is the set of all countable subsets of a set \( X \), including the empty set.
(b) \( \mathcal{P}^+_\omega(X) \) is the set of all countable non-empty subsets of \( X \).
(c) We write \( Y^\uparrow \) for \( Y \cup \{ \uparrow \} \), where \( \uparrow \) denotes divergence.
(d) We write \( f : X \Rightarrow Y \) for \( f : X \to \mathcal{P}_\omega(Y) \).
(e) We write \( f : X \Rightarrow^+ Y \) for \( f : X \to \mathcal{P}^+_\omega(Y) \).

We will interpret a WhileCC procedure
\[
P : u \to s
\]
as a countably-many-valued function \( P^A \) from \( A^u \) to \( A^s_\uparrow \), i.e., as a function
\[
P^A : A^u \to \mathcal{P}_\omega(A^s_\uparrow)
\]
or, in the above notation:
\[
P^A : A^u \Rightarrow^+ A^s_\uparrow.
\]
Remark 3.2.2 (Significance of ↑). Notice that an output of, say, \{2, 5, ↑\} is different from \{2, 5\}, since the former indicates the possibility of divergence. So a semantic function will have, for inputs not in its domain, ‘↑’ as a possible output value.

Definition 3.2.3 (States). (a) For each \(\Sigma\)-algebra \(A\), a state on \(A\) is a family \(\langle \sigma_s \mid s \in \text{Sort}(\Sigma) \rangle\) of functions \(\sigma_s : \text{Var}_s \rightarrow A_s\).

Let \(\text{State}(A)\) be the set of states on \(A\), with elements \(\sigma, \ldots\).

(b) Let \(\sigma\) be a state over \(A\), \(x \equiv (x_1, \ldots, x_n) : u\) and \(a = (a_1, \ldots, a_n) \in A^u\) (for \(n \geq 1\)). The variant \(\sigma\{x/a\}\) of \(\sigma\) is the state over \(A\) formed from \(\sigma\) by replacing its value at \(x_i\) by \(a_i\) for \(i = 1, \ldots, n\).

We give a brief overview of algebraic operational semantics. This was used in [TZ88] for deterministic imperative languages with ‘while’ and recursion (see [TZ00] for the case of \(\text{While}(\Sigma)\)), but it can be applied to a wide variety of imperative languages. It has also been used to analyse compiler correctness [Ste96]. It can also be adapted, as we will see, to a nondeterministic language such as \(\text{WhileCC}^*\).

Assume (i) we have a meaning function for atomic statements

\[\langle S_{\mathsf{at}} \rangle : \text{State}(A) \rightrightarrows \text{State}(A)^\dagger,\]

and (ii) we have defined a pair of functions

\[\text{First} : \text{Stmt} \rightarrow \text{AtSt}\]

\[\text{Rest}^A : \text{Stmt} \times \text{State}(A) \rightarrow \text{Stmt},\]

where, for a statement \(S\) and state \(\sigma\),

\[\text{First}(S)\] is an atomic statement which gives the first step in the execution of \(S\) (in any state), and \(\text{Rest}^A(S, \sigma)\) is a statement (or, in the present context, a finite set of statements) which gives the rest of the execution in state \(\sigma\).

From these we define the computation step function

\[\text{CompStep}^A : \text{Stmt} \times \text{State}(A) \rightrightarrows \text{State}(A)^\dagger\]

by

\[\text{CompStep}^A(S, \sigma) = \langle \text{First}(S) \rangle\sigma^A.\]

from which, in turn, we can define (for the deterministic language of [TZ00]) a computation sequence or (for the present language) a computation tree. The aim is to define a computation tree stage function

\[\text{CompTreeStage}^A : \text{Stmt} \times \text{State}(A) \times \mathbb{N} \rightrightarrows (\text{State}(A)^\dagger)^{<\omega}\]

where \(\text{CompTreeStage}^A(S, \sigma, n)\) represents the first \(n\) stages of \(\text{CompTree}^A(S, \sigma)\). Here \((\text{State}(A)^\dagger)^{<\omega}\) denotes the set of finite sequences from \(\text{State}(A)^\dagger\), interpreted as
finite initial segments of the paths through the computation tree. From this, in turn, are
defined the semantics of statements and procedures.

The intuition behind these semantics is that

for any input \( x \in A^a \), \( P^A(x) \) is the set of all possible outcomes (including di-
vergence), for all possible implementations of the ‘choose’ construct, including
non-constructive implementations!

For if (for a given input \( x \)) the only infinite paths through the semantic computation tr
ee are non-constructive, then \( P^A(a) \) will still include ‘↑’.

We now turn to the details of these definitions.

(a) Semantics of program terms. The meaning of \( t \in PTerm_s \) is a function

\[
[t]^A : State(A) \Rightarrow^+ A^s.
\]

The definition is by structural induction on \( t \):

\[
x]^A \sigma = \{ \sigma(x) \}
\]
\[
e]^A \sigma = \{ e^A \}
\]
\[
[F(t_1, \ldots, t_m)]^A \sigma = \{ y \mid \exists x_1 \in A \cap [t_1] \sigma \ldots \exists x_m \in A \cap [t_m] \sigma : F^A(x_1, \ldots, x_m) \downarrow y \}
\]
\[\cup \{ \uparrow \mid \exists x_1 \in A \cap [t_1] \sigma \ldots \exists x_m \in A \cap [t_m] \sigma : F^A(x_1, \ldots, x_m) \uparrow \}
\]
\[\cup \{ \uparrow \mid \uparrow \in [t_i]^A \sigma \text{ for some } i, 1 \leq i \leq m \}
\]
\[
[if(b, t_1, t_2)]^A \sigma = \{ y \mid (tt \in [b]^A \sigma \land y \in [t_1]^A \sigma) \lor (ff \in [b]^A \sigma \land y \in [t_2]^A \sigma) \}
\]
\[\cup \{ \uparrow \mid \uparrow \in [b]^A \sigma \}
\]
\[
[choose z : b]^A \sigma = \{ n \in \mathbb{N} \mid tt \in [b]^A \sigma \{ z/n \} \}
\]
\[\cup \{ \uparrow \mid \forall n \in \mathbb{N} (ff \in [b]^A \sigma \{ z/n \} \lor \uparrow \in [b]^A \sigma \{ z/n \}) \} \}
\]

Notice that \( [choose z : b]^A \sigma \) could include both natural numbers and ‘↑’, since for any \( n \),
\( [b]^A \sigma \{ z/n \} \) could include both \( tt \) and \( ff \).

(b) Semantics of atomic statements. The meaning of an atomic statement \( S_{at} \in AtSt \) is a function

\[
\langle S_{at} \rangle : State(A) \Rightarrow^+ State(A)^\dagger
\]

defined by:

\[
\langle skip \rangle^A \sigma = \{ \sigma \}
\]
\[
\langle div \rangle^A \sigma = \{ \uparrow \}
\]
\[
\langle x := t \rangle^A \sigma = \{ \sigma[x/a] \mid a \in A \cap [t]^A \sigma \}
\]
\[\cup \{ \uparrow \mid \uparrow \in [t]^A \sigma \}
\]
(c) The First and Rest operations. The operation

\[ \text{First} : \text{Stmt} \to \text{AtSt} \]

is defined exactly as in [TZ00, §3.5], namely:

\[
\text{First}(S) = \begin{cases} 
S & \text{if } S \text{ is atomic} \\
\text{First}(S_1) & \text{if } S \equiv S_1; S_2 \\
\text{skip} & \text{otherwise.}
\end{cases}
\]

The operation

\[ \text{Rest}^A : \text{Stmt} \times \text{State}(A) \rightarrow^+ \text{Stmt}, \]

is defined as follows (cf. [TZ00, §3.5]):

Case 1. \( S \) is atomic. Then

\[ \text{Rest}^A(S, \sigma) = \{ \text{skip} \}. \]

Case 2. \( S \equiv S_1; S_2 \).

Case 2a. \( S_1 \) is atomic. Then

\[ \text{Rest}^A(S, \sigma) = \{ S_2 \}. \]

Case 2b. \( S_1 \) is not atomic. Then

\[ \text{Rest}^A(S, \sigma) = \{ S' \mid S' \in \text{Rest}^A(S_1, \sigma) \} \cup \{ \text{div} \mid \text{div} \in \text{Rest}^A(S_1, \sigma) \}. \]

Case 3. \( S \equiv \text{if } b \text{ then } S_1 \text{ else } S_2 \text{ fi} \). Then \( \text{Rest}^A(S, \sigma) \) contains all of:

\[
\begin{cases} 
S_1 & \text{if } t \in [b]^A\sigma, \\
S_2 & \text{if } f \in [b]^A\sigma, \\
\text{div} & \text{if } \uparrow \in [b]^A\sigma.
\end{cases}
\]

Note that more than one condition may hold.

Case 4. \( S \equiv \text{while } b \text{ do } S_0 \text{ od} \). Then \( \text{Rest}^A(S, \sigma) \) contains all of:

\[
\begin{cases} 
S_0; S & \text{if } t \in [b]^A\sigma, \\
\text{skip} & \text{if } f \in [b]^A\sigma, \\
\text{div} & \text{if } \uparrow \in [b]^A\sigma.
\end{cases}
\]

Note again that more than one condition may hold.

(d) Computation step. From the First function we can define the computation step function

\[ \text{CompStep}^A : \text{Stmt} \times \text{State}(A) \rightarrow^+ \text{State}(A)^\uparrow \]
which is like the one-step computation function $\text{Comp}_A^1$ of [TZ00, §3.4], except for being multi-valued:

$$\text{CompStep}_A^1(S, \sigma) = \langle \text{First}(S) \rangle^A \sigma.$$ 

(e) The computation tree. The computation sequence, which is basic to the semantics of While computations in [TZ00], is replaced here by a computation tree

$$\text{CompTree}_A(S, \sigma)$$

of a statement $S$ at a state $\sigma$. This is an $\omega$-branching tree, branching according to all possible outcomes (i.e., “output states”) of the one-step computation function $\text{CompStep}_A^1$. Each node of this tree is labelled by either a state or ‘↑’.

Any actual (“concrete”) computation of statement $S$ at state $\sigma$ corresponds to one of the paths through this tree. The possibilities for any such path are:

(i) it is finite, ending in a leaf containing a state: the final state of the computation;
(ii) it is finite, ending in a leaf containing ‘↑’ (local divergence);
(iii) it is infinite (global divergence).

Correspondingly, the function $\text{Comp}_A^1$ of [TZ00, §3.4] is replaced by a computation tree stage function

$$\text{CompTreeStage}_A^1 : \text{Stmt} \times \text{State}(A) \times \mathbb{N} \rightarrow^+ (\text{State}(A)^\uparrow)^{<\omega}$$

where $\text{CompTreeStage}_A(S, \sigma, n)$ represents the first $n$ stages of $\text{CompTree}_A(S, \sigma)$. This is defined (like $\text{Comp}_A^1$) by a simple recursion (“tail recursion”) on $n$:

**Basis:** $\text{CompTreeStage}_A(S, \sigma, 0) = \{\sigma\}$, i.e., just the root labelled by $\sigma$.

**Induction step:** $\text{CompTreeStage}_A(S, \sigma, n)$ is formed by attaching to the root $\{\sigma\}$ the following:

(i) for $S$ atomic: the leaf $\{\sigma'\}$, for each $\sigma' \in \langle S \rangle^A \sigma$ (where $\sigma'$ may be a state or $\uparrow$);
(ii) for $S$ not atomic:
   the subtree $\text{CompTreeStage}_A(S', \sigma', n)$, for each $\sigma' \in \text{CompStep}_A^1(S, \sigma)$ ($\sigma' \neq \uparrow$) and $S' \in \text{Rest}_A(S, \sigma)$, as well as the leaf $\{\uparrow\}$ if ‘$\uparrow$’ $\in \text{CompStep}_A^1(S, \sigma)$.

Then $\text{CompTree}_A^1(S, \sigma)$ is defined as the “limit” over $n$ of $\text{CompTreeStage}_A^1(S, \sigma, n)$.

Note that only the leaves of $\text{CompTree}_A^1(S, \sigma)$ may contain ‘$\uparrow$’, indicating “local divergence”.

(f) Semantics of statements. From the semantic computation tree we can easily define the i/o semantics of statements

$$[S]^A : \text{State}(A) \rightarrow^+ \text{State}(A)^\uparrow.$$ 

Namely,
$[S]^A \sigma$ is the set of states and/or ‘↑’ at all leaves in $\text{CompTree}^A(S, \sigma)$, together with ‘↑’ if $\text{CompTree}^A(S, \sigma)$ has an infinite path.

(g) Semantics of procedures. Finally, if

$$P \equiv \text{func in a out b aux c begin } S \text{ end}$$

is a procedure of type $u \rightarrow v$, then its meaning in $A$ is a function

$$P^A : A^u \rightrightarrows A^u \uparrow$$

defined as follows (cf. [TZ00, §3.6]). For $x \in A^u$,

$$P^A(x) = \{ \sigma'(b) \mid \sigma' \in [S]^A \sigma \} \cup \{ \uparrow \mid \uparrow \in [S]^A \sigma \}$$

where $\sigma$ is any state on $A$ such that $\sigma[a] = x$.

Remark 3.2.4. From the initialisation condition (§3.1(e)) it follows by a “functionality lemma” (cf. [TZ00, 3.6.1]) that $P^A$ is well defined.

Definition 3.2.5. A $\text{WhileCC}$ procedure $P : u \rightarrow v$ is deterministic on $A$ if for all $x \in A^u$, $P^A(x)$ is a singleton.

Remark 3.2.6 (Two concepts of deterministic computation). One can distinguish between two notions of deterministic computation: (i) strong deterministic computation, the common concept, in which each step of the computation is determinate; and (ii) weak deterministic computation, in which the output (or divergence) is uniquely determined by (i.e., a unique function of) the input, but the steps in the computation are not determinate. A good example of (ii) is the Gaussian elimination algorithm (§1.2.1, §4.1) which, although defining a unique function (the inverse of a matrix), incorporates the (nondeterministic!) pivot function as a subroutine. In Definition 3.2.5 and elsewhere in this paper, we are concerned with the weak sense of deterministic computation.

Definition 3.2.7. (a) A many-valued function $F : A^u \rightrightarrows A^s \uparrow$ is $\text{WhileCC}$ computable on $A$ if there is a $\text{WhileCC}$ procedure $P$ such that $F = P^A$.

(b) A partial function $F : A^u \rightarrow A^s$ is $\text{WhileCC}$ computable on $A$ if there is a deterministic $\text{WhileCC}$ procedure $P : u \rightarrow s$ such that for all $x \in A^u$,

(i) $F(x) \downarrow y \implies P^A(x) = \{y\}$,

(ii) $F(x) \uparrow \implies P^A(x) = \{\uparrow\}$.

Remark 3.2.8 (Many-valued algebras). As we have seen, the semantics for $\text{WhileCC}$ procedures is given by countably many-valued functions. If we were to start with algebras with many-valued basic operations, as in [Bra96, Bra99], the algebraic operational semantics could handle this just as easily, by adapting the clause for the basic $\Sigma$-function $F$ in part (a) (“Semantics of program terms”) of the semantic definition above.
3.3 The language $\text{WhileCC}^*(\Sigma)$

In [TZ99, TZ00] we worked with the language $\text{While}^*$ rather than $\text{While}$, which can be viewed as $\text{While}$ augmented by auxiliary array and $\text{nat}$ variables [TZ00, §3.13]. The importance of $\text{While}^*$ computability lies in the fact that it forms the basis for a generalised Church-Turing Thesis for computability on abstract many-sorted algebras [TZ00, §8].

Here, similarly, we will work with the language $\text{WhileCC}^* = \text{WhileCC}^*(\Sigma)$, which may be thought of as $\text{WhileCC}(\Sigma)$ augmented by auxiliary array and $\text{nat}$ variables (or as $\text{While}^*(\Sigma)$ augmented by the ‘choose’ construct). More precisely:

Definition 3.3.1 (The $\text{WhileCC}^*(\Sigma)$ language). A $\text{WhileCC}^*(\Sigma)$ procedure is a $\text{WhileCC}(\Sigma^*)$ procedure in which the input and output variables have sorts in $\Sigma$ only. (However the auxiliary variables may have starred sorts or sort $\text{nat}$.)

Thus a $\text{WhileCC}^*(\Sigma)$ procedure defines a countably-many-valued function on any standard $\Sigma$-algebra.

3.4 Some computability issues in the semantics of $\text{WhileCC}^*$ procedures

Some interesting issues in the semantics of $\text{WhileCC}^*$ arise already in the case of computation over the algebra $\mathcal{N}$ of naturals (Example 2.2.4(b)).

(a) Elimination of ‘choose’ from deterministic $\text{WhileCC}^*$ programs over total algebras

The ‘choose’ operator can be eliminated from deterministic $\text{WhileCC}^*$ procedures (cf. Definition 3.2.5 and Remark 3.2.6) over total algebras.

Proposition 3.4.1. For any total $\Sigma$-algebra $A$ and $f : A^u \rightarrow A$, $f$ is $\text{WhileCC}^*$ computable over $A$ $\iff$ $f$ is $\text{While}^*$ computable over $A$.

Proof: ($\Rightarrow$) Let $P$ be a deterministic $\text{WhileCC}^*$ procedure over $A$ which computes $f$. Since $A$ is total, evaluation of any boolean term $b$ over $A$ (relative to a state) converges to $\mathbf{t}$ or $\mathbf{f}$ in $A$. Further, since $P$ is deterministic, its output for a given input is independent of the implementation. Hence every ‘choose’ term in $P$ of the form $\text{choose } z : b[z]$ can be replaced by a ‘while’ loop which tests $b[0]$, $b[1]$, $b[2]$, . . . in turn, i.e., finds the least $k$ for which $b[k]$ is true, if it exists, and diverges otherwise. $\square$

Applying this to the total algebra $\mathcal{N}$, and recalling that $\text{While}^*$ computability over $\mathcal{N}$ is equivalent to partial recursiveness (i.e., classical computability) over $\mathbb{N}$ [TZ00], we have:

Corollary 3.4.2. For any $f : \mathbb{N}^m \rightarrow \mathbb{N}$,

$f$ is $\text{WhileCC}^*$ computable over $\mathcal{N}$ $\iff$ $f$ is partial recursive over $\mathbb{N}$.

(b) Recursive and non-recursive implementations

The semantics $P^A$ of a procedure $P$ is given, for an input $x$, by all paths of the computation tree $T = \text{CompTree}^A(S, \sigma)$ (where $S$ is the body of $P$) representing all possible
computation sequences for $S$ starting at state $\sigma$, where $\sigma[a] = x$, i.e., all possible implementations of instances of the ‘choose’ construct occurring in the execution of $S$ starting at $\sigma$. This gives rise to interesting computation-theoretic issues even in the simple case that $A = N$. In this case we can assume that $T$ is coded as a subset of $\mathbb{N}$ in a standard way. Now any path of $T$ ending in a leaf is finite, and therefore (trivially) recursive. An infinite path or computation sequence (leading to divergence), however, may or may not be recursive.

**Proposition 3.4.3.** There is a $\text{WhileCC}^*(N)$ procedure $P$ such that its computation tree has infinite paths, but no recursive infinite paths.

**Proof:** Our construction of $P$ is based on the construction of a recursive tree with infinite paths, but no recursive infinite paths [Odi99, V.5.25]. Let $A$ and $B$ be two disjoint r.e., recursively inseparable sets, and suppose $A = \text{ran}(f)$ and $B = \text{ran}(g)$ where $f$ and $g$ are total recursive functions. The procedure $P$ can be written in pseudo-code as:

```plaintext
func aux n, k : nat,
    choices*: nat*, { array recording all choices up to present stage n }
    halt : bool
begin
    n := 0;
    choices* := Null;
    halt := false;
    while not halt do
        n := n + 1;
        choices* := Newlength(choices*, n + 1);
        choices*[n] := choose z: (z = 0 or z = 1);
        for k := 0 to n - 1 do
            if (choices*[k] = 0 and k ∈ { f(0), ..., f(n - 1) }) or
                (choices*[k] = 1 and k ∈ { g(0), ..., g(n - 1) })
                then halt := true
        od
    od
end.
```

Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be the successive values (0 or 1) given by the ‘choose’ operator in some given implementation of $P$. Note that at stage $n$,

$$\text{choices}*[k] = \alpha_k \quad \text{for } k = 0, \ldots, n - 1.$$ 

Further, the execution diverges if, and only if, the set $C =_{df} \{ k \mid \alpha_k = 1 \}$ separates $A$ and $B$ (i.e., $A \subseteq C$ and $C \cap B = \emptyset$), in which case $C$, and hence its characteristic function $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$, are non-recursive.
Note finally that for any given sequence $\alpha$ of choices, $\alpha$ is effectively obtainable from the corresponding computation sequence or path, i.e., $\alpha$ is recursive in that path (with a standard coding of the computation tree). Hence, since any infinite sequence $\alpha$ is non-recursive, so is the corresponding infinite path. \qed

Remarks 3.4.4. (1) Clearly, $P$ as defined above is not semantically equivalent to a While$^\ast$$(\mathcal{N})$ procedure. This does not contradict Proposition 3.4.1, since $P$ is not deterministic.

(2) According to our semantics above ($\S$3.2), for $P$ as defined above, $\uparrow \in P^A()$, i.e., divergence is possible. However, if we were to restrict all computation sequences to be recursive, then divergence would not be a possible outcome for $P^A()$. The semantics, as we give it (i.e., all possible computation sequences are included, whether recursive or not) is simpler than this alternative. In any case, as we will see, this choice will not affect continuity considerations (cf. Lemmas 5.1.7 and 5.2.1).

3.5 Approximable WhileCC$^\ast$ computability

The basic notion of computability that we will be using in working with metric algebras is not so much computability, as rather computable approximability on metric algebras, as discussed in [TZ99, $\S$9]. We have to adapt the definition given there to the nondeterministic case with countable choice.

Let $A$ be a metric $\Sigma$-algebra, $u$ a $\Sigma$-product type and $s$ a $\Sigma$-type. Let $P : \text{nat} \times u \to s$ be a WhileCC$^\ast$$(\Sigma^N)$ procedure. Put

$$P_n^A = \text{df } P^A(n, \cdot) : A^u \rightharpoonup^+ A_s^\uparrow.$$

Note that that for all $x \in A^u$, $P_n^A(x) \neq \emptyset$.

Definition 3.5.1 (WhileCC$^\ast$ approximability to a single-valued function).
Let $F : A^u \rightarrow A_s$ be a single-valued partial function on $A$.

(a) $F$ is WhileCC$^\ast$ approximable by $P$ on $A$ if for all $n \in \mathbb{N}$ and all $x \in A^u$:

$$x \in \text{dom}(F) \implies \uparrow \notin P_n^A(x) \subseteq B(F(x), 2^{-n}). \quad (1)$$

(b) $F$ is strictly WhileCC$^\ast$ approximable by $P$ on $A$ if in addition to (1),

$$x \notin \text{dom}(F) \implies P_n^A(x) = \{ \uparrow \}. \quad (2)$$

Remark 3.5.2. If $F$ is strictly approximable by $P$, then (from (1) and (2)) for all $x \in A^u$ and all $n$:

$$F(x) \uparrow \iff \uparrow \in P_n^A(x) \iff P_n^A(x) = \{ \uparrow \}.$$

Clearly, WhileCC$^\ast$ computability is a special case of WhileCC$^\ast$ approximability.
Definition 3.5.3 (WhileCC∗ approximability to a many-valued function).
Let $F : A^u \rightarrow A_s$ be a countably-many-valued function on $A$.

(a) $F$ is **WhileCC∗ approximable** by $P$ on $A$ if for all $n \in \mathbb{N}$ and all $x \in A^u$:

$$F(x) \neq \emptyset \implies \uparrow \notin P^A_n(x) \subseteq \bigcup_{y \in F(x)} B(y, 2^{-n})$$

and

$$F(x) \subseteq \bigcup_{y \in P^A_n(x)} B(y, 2^{-n}).$$

Note that (assuming $\uparrow \notin P^A_n(x)$) the r.h.s. of (3) implies

$$d_H(\overline{F(x)}, \overline{P^A_n(x)}) \leq 2^{-n},$$

and is implied by

$$d_H(\overline{F(x)}, \overline{P^A_n(x)}) < 2^{-n},$$

where $\overline{X}$ denotes the closure of $X$, and $d_H$ is the Hausdorff metric on the set of closed, bounded non-empty subsets of $A_s$ [Eng89, 4.5.23]. (Actually, the Hausdorff metric applies only to the space of closed bounded subsets of a given metric space, so (4) and (5) should be taken as heuristic statements.)

In other words (assuming $F(x) \neq 0$), for all $x \in A^u$ and all $n$, each output of $F(x)$ lies within $2^{-n}$ of some output of $P^A_n(x)$, and vice versa.

(b) $F$ is **strictly WhileCC∗ approximable** by $P$ on $A$ if in addition,

$$F(x) = \emptyset \implies P^A_n(x) = \{ \uparrow \}.$$

Remark 3.5.4. (Cf. Remark 3.5.2.) If $F$ is strictly approximable by $P$, then for all $x \in A^u$ and all $n$:

$$F(x) = \emptyset \iff \uparrow \in P^A_n(x) \iff P^A_n(x) = \{ \uparrow \}.$$

4 Examples of WhileCC∗ computations and approximating computations

4.1 Discussion: Use of ‘choose’ for searching and dovetailing

Following the examples in Section 1, the ‘choose’ construct was introduced to compute many-valued functions. Technically, the ‘choose’ construct strengthens the power of the While language in performing searches. In a partial algebra, simple searches (e.g., “find some $x_k$ in an effectively enumerated set $X = \{ x_0, x_1, x_2, \ldots \}$ satisfying $b(x_k)$”) will obviously fail in general if the search simply follows the given enumeration of $X$ (i.e.,
testing in turn whether \( b(x_0), b(x_1), b(x_2), \ldots \) holds, since the computation of the boolean predicate \( b(x) \) may not terminate for some \( x \).

This problem is overcome, at the concrete model level, by the use of scheduling techniques such as *interleaving* or “dovetailing”: at stage \( n \), do \( n \) steps in testing whether \( b(x_i) \) holds, for \( i = 0, \ldots, n \).

An important function of the ‘choose’ construct, which will recur in our examples, is to simulate such scheduling techniques at the abstract model level. This allows searches over any countable subset \( X \) of an algebra \( A \) that has a computable enumeration

\[
\text{enum}_X : \mathbb{N} \to X,
\]

since we can search \( X \) in \( A \) by assignments such as

\[
x := \text{enum}_X(\text{choose } z : b(\text{enum}_X(z))).
\]

### 4.2 Examples

We now illustrate the use of the WhileCC* language in topological partial algebras with examples, which involve computations which are either many-valued, or approximating, or both. The examples given in §1.2 to motivate many-valued abstract computation are a good place to start. They can be displayed in the table:

|                      | Exact computation                                      | Approximating computation               |
|----------------------|-------------------------------------------------------|----------------------------------------|
| Single-valued        | Gaussian elimination                                   | \( e^x, \sin(x), \text{etc.} \)         |
| Many-valued          | Approx. points in metric algebra                       | All simple roots of polynomial          |

Examples 4.2.1, 4.2.2 and 4.2.4 below are all based on the metric algebra derived from \( \mathcal{R}^N \) (Example 2.3.3(b)).

**Example 4.2.1 (Gaussian elimination).** This is a single-valued exact computation. The algorithm can be found in any standard text of numerical computation, e.g., [Hea97]. It is deterministic, but only in the weak sense (cf. Remark 3.2.6), since it contains, as an essential component, the computation of the *pivot* function (§1.2), which is many-valued, and can be formalised simply with the ‘choose’ construct:

```plaintext
func in x_1, \ldots, x_n : real
    out i : nat
    aux k : nat
begin
    i := choose k : (k = 1 \text{ and } x_1 \neq 0) \text{ or }
         (k = 2 \text{ and } x_2 \neq 0) \text{ or }
         \ldots
         (k = n \text{ and } x_n \neq 0)
end.
```
Example 4.2.2 (Approximations to $e^x$). On the N-standard interval algebra $\mathcal{I}^N$ (Example 2.5.3(c)) we give a While procedure to approximate the function $e^x$ on $I$.

```
func in n: nat, { degree of approximation }
    x: intvl
out s: real { partial sum of power series }
aux y: real, { current term of series }
    k: nat { counter }
begin
    k := 0;
    y := 1;
    s := 1;
    while k < 2^{n+1} do
        k := k + 1;
        y := y × i_I(x)/i_N(k); { y = x^k/k! }
        s := s + y { s = \sum_{i=0}^{k} x^i/i! }
    od
end
```

where $i_I : I \rightarrow \mathbb{R}$ is the embedding of $I$ in $\mathbb{R}$, which is primitive in $\Sigma(\mathcal{I}^N)$, and $i_N : \mathbb{N} \rightarrow \mathbb{R}$ is the embedding of $\mathbb{N}$ in $\mathbb{R}$, which is easily definable in While($\mathcal{R}^N$).

Denoting the above function procedure by $P$, and $\mathcal{I}^N$ by $A$, we have the semantics

$$P^A_n : I \rightarrow \mathbb{R}$$

with

$$P^A_n(x) = \sum_{i=0}^{2^{n+1}} \frac{x^i}{i!}$$

and so for all $x \in I$,

$$d(P^A_n(x), e^x) < 2^{-n},$$

i.e., $e^x$ is While approximable on $\mathcal{I}^N$ by $P$.

This computation of $e^x$ is single-valued, but approximating.

Example 4.2.3 ("Choosing" a member of an enumerated subspace close to an arbitrary element of a metric algebra). Given a metric algebra $A$ with a countable dense subspace $C$, and an enumeration of $C$

$$\text{enum}_C : \mathbb{N} \rightarrow C$$
in the signature, we want to compute a function

\[ F : A \times \mathbb{N} \rightarrow C \]

such that

\[ F(a, n) = \text{"some" } x \in C \text{ such that } d(a, x) < 2^{-n}. \]

This is a generalised version of the problem of approximating reals by rationals (Example 1.2.3).

Here is a WhileCC* procedure (in pseudo-code) for an exact computation of this function. (Note that the real-valued function \( 2^{-n} \) is While computable on \( \mathcal{R}^N \), and hence on \( A \).)

```
func in  a : space,
        n : nat
out  x : space
aux  k : nat
begin
    x := \text{enum}C(\text{choose } k : d(a, \text{enum}C(k)) < 2^{-n})
end
```

This computation is many-valued, but exact.

**Example 4.2.4 (Finding simple roots of a polynomial).** We construct a WhileCC procedure to approximate “some” simple root of a polynomial \( p(X) \) with real coefficients, using the method of bisection. By a simple root of \( p(X) \) we mean a real root at which \( p(X) \) changes sign. (See [Hea97]. In practice, a hybrid method is generally used, involving bisection, Newton’s method, etc.)

Fundamental to the bisection method is the concept of a bracket for \( p(X) \), which means an interval \([a, b]\) such that \( p(a) \) and \( p(b) \) have opposite signs. By rational bracket, we mean a bracket with rational endpoints.

We note the following:

1. Any bracket for \( p \) contains a root of \( p \) (by the Intermediate Value Theorem), in fact a simple root of \( p \).
2. Conversely, any simple root of \( p \) is contained in a rational bracket for \( p \) of arbitrarily small width.
3. If \( x \) is a simple root of \( p \), then any bracket for \( p \) of sufficiently small width which contains \( x \), contains no other simple root of \( p \).
4. If \([a, b]\) is a bracket for \( p \), then, putting \( m = (a + b)/2 \), exactly one of the following holds:
   1. \( p(m) = 0 \); then \( m \) is a root of \( p \) (not necessarily simple);
(ii) \( p(m) \) has the same sign as \( p(a) \); then \([m, b]\) is a bracket for \( p \);

(iii) \( p(m) \) has the same sign as \( p(b) \); then \([a, m]\) is a bracket for \( p \).

It follows from the above that starting with any rational bracket \( J \) for \( p \), we can, by repeated bisection, find a nested sequence of rational brackets

\[ J = J_0, J_1, J_2, \ldots \quad \text{where} \quad \bigcap_{n=0}^{\infty} J_n = \{x\} \]

for some simple root \( x \) of \( p \). Then, letting \( r_n \) be the left-hand endpoint of \( J_n \), we have a fast Cauchy sequence \( \langle r_n \rangle_n \) with limit \( x \).

One complication with our algorithm is the occurrence of case (i) in (4) above, i.e., the case that the midpoint \( m \) of the bracket is itself a root of \( p \), since by the co-semicomputability of equality (Discussion 2.2.5) on \( \mathbb{R} \) we can only verify when \( f(m) \neq 0 \), not when \( f(m) = 0 \). We therefore proceed as follows. By means of the ‘choose’ construct, we search in the middle third (say) of the bracket \([a, b]\) for a “division point”, i.e., a rational point \( d \) such that \( f(d) \neq 0 \), producing either \([a, d]\) or \([d, b]\) as a sub-bracket.

This new bracket may not halve the width of \([a, b]\); in the worst case its width is \( 2/3 \) the width of \([a, b]\). However a second iteration of this procedure leads to a bracket of width at most \((2/3)^2 < 1/2\) the width of \([a, b]\), and so \(2n\) iterations lead to a bracket of width less than \( 2^{-n} \times \text{the width of } [a, b] \).

This new bracket may not halve the width of \([a, b]\); in the worst case its width is \( 2/3(b-a) \). However a second iteration of this procedure leads to a bracket of width at most \((2/3)^2 < 1/2\) the width of \([a, b]\), and so \(2n\) iterations lead to a bracket of width less than \( 2^{-n}(b-a) \).

For convenience, we will use the following two conservative extensions to our “official” programming notation:

(a) Simultaneously choosing two naturals with a single condition:

\[ k_1, k_2 := \text{choose } z_1, z_2 : b[z_1, z_2] \]

which is easily expressible in \textbf{WhileCC} by the use of a primitive recursive pairing function \texttt{pair} on \( \mathbb{N} \) and its inverses \texttt{proj}1, \texttt{proj}2:

\[ k := \text{choose } z : b[\texttt{proj}_1(z), \texttt{proj}_2(z)]; \]

\[ k_1, k_2 := \texttt{proj}_1(k), \texttt{proj}_2(k) \]

(b) Choosing a rational (of type \texttt{real}) satisfying a boolean condition:

\[ q := \text{choose } r^{\text{real}} : ("r is rational" \text{ and } b[r]) \]

Let \( \texttt{rat} : \mathbb{N} \to \mathbb{R} \) be a \textbf{While}-computable enumeration of the rationals in \( \mathbb{R} \). Then this can be interpreted as:

\[ q := \texttt{rat}(\text{choose } k : b[\texttt{rat}(k)]) \]
Finally, a polynomial \( p(X) \) over \( \mathbb{R} \) will be represented by an element \( p^* \) of \( \mathbb{R}^* \):

\[
p^* = (a_0, \ldots, a_{n-1}) = \sum_{i=0}^{n-1} a_i X^{n-i}
\]

Its evaluation at a point \( c \), denoted by \( p^*(c) \), is easily seen to be \textbf{While}(\mathbb{R}) computable in \( p^* \) and \( c \).

Now we give a \textbf{WhileCC}^* procedure for approximably computing some simple root of an input polynomial, in the signature of \( \mathbb{R} \).

\[
\begin{align*}
\text{func} & \quad \text{in} \quad n : \text{nat}, \quad \{ \text{degree of approximation} \} \\
& \quad \text{p}^* : \text{real}^* \quad \{ \text{input polynomial, given by list of coefficients} \} \\
& \quad \text{out} \quad x : \text{real} \quad \{ \text{approximation to root} \} \\
& \quad \text{aux} \quad a, b : \text{real}, \quad \{ \text{endpoints of bracket} \} \\
& \quad \quad d : \text{real}, \quad \{ \text{division point of bracket} \} \\
& \quad \quad k : \text{nat} \quad \{ \text{counter} \} \\
\text{begin} \\
& \quad k := 0; \\
& \quad a, b := \text{choose} \quad a, b : ("a and b are rational" \quad \text{and} \quad a \text{ < } b \text{ < } a + 1 \quad \text{and} \\
& \quad \quad (p^*(a) > 0 \quad \text{and} \quad p^*(b) < 0) \\
& \quad \quad \quad \text{or} \quad (p^*(a) < 0 \quad \text{and} \quad p^*(b) > 0)); \\
\text{while} \quad k < 2n \text{ do} \\
& \quad k := k + 1; \\
& \quad d := \text{choose} \quad d : ("d is rational" \quad \text{and} \quad (2a + b)/3 \text{ < } d \text{ < } (a + 2b)/3 \\
& \quad \quad \text{and} \quad p^*(d) \neq 0); \\
& \quad \text{if} \quad (f(d) > 0 \quad \text{and} \quad f(a) > 0) \quad \text{or} \quad (f(d) < 0 \quad \text{and} \quad f(a) < 0) \\
& \quad \quad \text{then} \quad a, b := d, b \quad \{ \text{new bracket on right part of old} \} \\
& \quad \quad \text{else} \quad a, b := a, d \quad \{ \text{new bracket on left part of old} \} \\
& \quad \text{fi} \\
& \quad \text{od}; \\
& \quad x := a \quad \{ x := b \text{ would also work here} \} \\
\text{end.}
\]

For input natural \( n \) and polynomial \( p \), the output is within \( 2^{-n} \) of some simple root of \( p \). Further, for \textit{any} simple root \( e \) of \( p \), there is \textit{some} implementation of the `choose' operator which will give an output within \( 2^{-n} \) of \( e \). Finally, the computation will diverge if, and only if, \( p \) has no simple roots.

This computation is both many-valued and approximating.
5 Countably-many-valued functions; Continuity of WhileCC* computable functions

In this section we discuss the continuity of countably-many-valued functions, and then prove that the countably-many-valued functions computed by WhileCC* programs are continuous.

5.1 Topology and continuity with countably many values and ‘↑’

Recall Notation 3.2.1.

Definition 5.1.1 (Totality). The function \( f : X \rightrightarrows Y \) is said to be total if for all \( x \in X \), \( f(x) \) is a non-empty subset of \( Y \), i.e., if \( f : X \rightrightarrows^{+} Y \).

Our semantic functions (in Section 6) will typically be of the form

\[
\Phi : A^u \rightrightarrows^{+} A^v. \tag{1}
\]

Remark 5.1.2. We think of the “deterministic version” of (1) as being a total function \( \Phi \), where for each \( x \in X \), \( \Phi(x) \) is a singleton, containing either an element of \( A^v \) (to indicate convergence) or ‘↑’ (to indicate divergence). (Recall Remark 3.2.2.)

We must now consider what it means for such a function (1) to be continuous.

Definition 5.1.3 (Continuity). Let \( f : X \rightrightarrows Y \), where \( X \) and \( Y \) are topological spaces.

(a) For any \( V \subseteq Y \),

\[ f^{-1}[V] = \{ x \in X | f(x) \cap V \neq \emptyset \}, \]

i.e., \( x \in f^{-1}[V] \) iff at least one of the elements of \( f(x) \) lies in \( V \).

(b) \( f \) is continuous (w.r.t. \( X \) and \( Y \)) iff for all open \( V \subseteq Y \), \( f^{-1}[V] \) is open in \( X \).

Remarks 5.1.4. (a) For metric spaces \( X \) and \( Y \), Definition 5.1.3(b) becomes:

\( f : X \rightrightarrows Y \) is continuous iff

\[ \forall a \in X \forall b \in f(a) \forall \epsilon > 0 \exists \delta > 0 \forall x \in B(a, \delta) \ (f(x) \cap B(b, \epsilon) \neq \emptyset). \]

(b) Definition 5.1.3(b) reduces to the standard definition of continuity for total single-valued functions from \( X \) to \( Y \).

(c) It also reduces to the definition of continuity for partial single-valued functions (Definition 2.5.1 and Remark 2.7.2(a)), as we will see below (Remark 5.1.9). We must first see how to extend the topology on \( Y \) to that on \( Y^\uparrow \) (Definition 5.1.6 below).

Definition 5.1.5. For two functions \( f : X \rightrightarrows Y \), \( g : X \rightrightarrows Y \), we define

\[ f \sqsubseteq g \iff \text{for all } x \in X, f(x) \subseteq g(x). \]
Definition 5.1.6 (Topology on $Y^\uparrow$). We extend the topology on $Y$ to $Y^\uparrow (= Y \cup \{ \uparrow \})$ by specifying that the only open set containing $\{ \uparrow \}$ is $Y^\uparrow$. (So $Y^\uparrow$ is a “one-point compactification” of $Y$.)

Now, given a function $f : X \rightrightarrows Y$, we define functions

\[
\begin{align*}
    f^\uparrow : X & \rightrightarrows Y^\uparrow \\
    f^- : X & \rightrightarrows Y
\end{align*}
\]

by

\[
\begin{align*}
    f^\uparrow(x) &= f(x) \cup \{ \uparrow \} \\
    f^-(x) &= f(x) \setminus \{ \uparrow \}.
\end{align*}
\]

In other words, $f^\uparrow$ adds ‘$\uparrow$’ to the set $f(x)$ for each $x \in X$ and $f^-$ removes ‘$\uparrow$’ from every such set. This changes the semantics of $f$ (see Remark 3.2.2), but not its continuity properties, as will be seen from the following technical lemma, which will be used in the proof of continuity of computable functions below ($\S 5.2$).

Lemma 5.1.7. Let $f : X \rightrightarrows Y$ and $g : X \rightrightarrows Y^\uparrow$ be any two functions such that

\[
f \subseteq g \subseteq f^\uparrow,
\]

i.e., for all $x \in X$, $g(x) \neq \emptyset$, and either $g(x) = f(x)$ or $g(x) = f(x) \cup \{ \uparrow \}$. Then

\[
f \text{ is continuous} \iff g \text{ is continuous}.
\]

Proof: ($\Rightarrow$) Suppose $f$ is continuous. We must show $g$ is continuous. Let $V$ be an open subset of $Y^\uparrow$. We must show $g^{-1}[V]$ is open in $X$. There are two cases, according as $\uparrow$ is in $V$ or not.

Case 1: $\uparrow \notin V$, i.e., $V \subseteq Y$. Then $V$ is also open in $Y$ (by definition of the topology on $Y^\uparrow$). Hence $f^{-1}[V]$ is open in $X$, and hence

\[
g^{-1}[V] = \{ x \in X \mid g(x) \cap V \neq \emptyset \} = \{ x \in X \mid f(x) \cap V \neq \emptyset \} \quad \text{since } \uparrow \notin V
\]

is open in $X$.

Case 2: $\uparrow \in V$. Then $V = Y^\uparrow$ (by definition of the topology on $Y^\uparrow$). Hence

\[
g^{-1}[V] = g^{-1}[Y^\uparrow] = X \quad \text{(since $g$ is total)},
\]

which is open in $X$. 

Suppose $g$ is continuous. We must show $f$ is continuous. Let $V$ be an open subset of $Y$. We must show $f^{-1}[V]$ is open in $X$. Since $V$ is also open in $Y^\uparrow$ (by definition of the topology on $Y^\uparrow$), $g^{-1}[V]$ is open in $X$. Hence

$$f^{-1}[V] = \{ x \in X \mid f(x) \cap V \neq \emptyset \}$$
$$= \{ x \in X \mid g(x) \cap V \neq \emptyset \} \quad \text{since } \uparrow \notin V$$
$$= g^{-1}[V]$$

is open in $X$. □

**Corollary 5.1.8.** Suppose $f : X \Rightarrow Y^\uparrow$ (i.e., $f$ is total). Then

$$f \text{ is continuous} \iff f^- \text{ is continuous} \iff f^\uparrow \text{ is continuous}.$$  

**Proof:** Apply Lemma 5.1.7 twice: once with $f^-$ and $f$, and once with $f^-$ and $f^\uparrow$. □

**Remark 5.1.9 (Justification of Remark 5.1.4(c)).** Let $f : X \rightarrow Y$ be a single-valued partial function. Define

(a) $\check{f} : X \Rightarrow Y$ by

$$\check{f}(x) = \left\{ \begin{array}{ll} \{ f(x) \} & \text{if } x \in \text{dom}(f) \\ \emptyset & \text{otherwise} \end{array} \right.$$  

(b) $\hat{f} : X \Rightarrow Y^\uparrow$ by

$$\hat{f}(x) = \left\{ \begin{array}{ll} \{ f(x) \} & \text{if } x \in \text{dom}(f) \\ \{ \uparrow \} & \text{otherwise}. \end{array} \right.$$  

(We can view either $\check{f}$ or $\hat{f}$ as “representing” $f$ in the present context, cf. Remark 5.1.2.) Then

$$f \text{ is continuous (according to Def. 2.5.1)} \iff \check{f} \text{ is continuous (according to Def. 5.1.3)} \iff \hat{f} \text{ is continuous (according to Def. 5.1.3)}$$

The equivalence of the continuity of $f$ and $\check{f}$ follows immediately from the definitions. The equivalence of the continuity of $\check{f}$ and $\hat{f}$ follows from Lemma 5.1.7.

**Remark 5.1.10 (Comparison with W-continuity).** As in §2.7, we can consider another notion of continuity for functions $f : X \Rightarrow Y$ by modifying Definition 5.1.3(b); namely, $f$ is W-continuous iff for all open $V \subseteq Y$, $f^{-1}[V]$ is open in $\text{dom}(f)$. Note that Lemma 5.1.7, and the equivalences given in Remark 5.1.9, also hold for W-continuity.

**Lemma 5.1.11.** Given $f : X \Rightarrow Y^\uparrow$, extend it to $\tilde{f} : X^\uparrow \Rightarrow Y^\uparrow$ by stipulating that $\tilde{f}(\uparrow) = \uparrow$. If $f$ is continuous and total, then $\tilde{f}$ is continuous.
Proof: Let $V$ be an open subset of $Y^\uparrow$. We must show $\tilde{f}^{-1}[V]$ is open in $X^\uparrow$. There are two cases:

Case 1: $\uparrow \notin V$, i.e., $V \subseteq Y$. Then $\tilde{f}^{-1}[V] = f^{-1}[V]$, which is open in $X$, and hence in $X^\uparrow$.

Case 2: $\uparrow \in V$. Then $V = Y^\uparrow$ (by definition of the topology on $Y^\uparrow$). Hence

$$\tilde{f}^{-1}[V] = \tilde{f}^{-1}[Y^\uparrow] = \text{dom}(f) \cup \{ \uparrow \} = X \cup \{ \uparrow \} \quad \text{(since } f \text{ is total)}$$

which is open in $X^\uparrow$. □

Definition 5.1.12 (Composition).
(a) Suppose $f : X \Rightarrow Y$ and $g : Y \Rightarrow Z$. We define $g \circ f : X \Rightarrow Z$ by

$$(g \circ f)(x) = \bigcup \{ g(y) \mid y \in f(x) \}.$$ 

(b) Suppose $f : X \Rightarrow Y^\uparrow$ and $g : Y \Rightarrow Z^\uparrow$. We define $g \circ f : X \Rightarrow^+ Z^\uparrow$ by

$$(g \circ f)(x) = \bigcup \{ g(y) \mid y \in f(x) \cap Y \} \cup \{ \uparrow \mid \uparrow \in f(x) \}.$$ 

Proposition 5.1.13 (Continuity of composition).
(a) If $f : X \Rightarrow Y$ and $g : Y \Rightarrow Z$ are continuous, then so is $g \circ f : X \Rightarrow Z$.

(b) If $f : X \Rightarrow^+ Y^\uparrow$ and $g : Y \Rightarrow^+ Z^\uparrow$ are continuous, then so is $g \circ f : X \Rightarrow^+ Z^\uparrow$.

Proof: (a) Just note that for $W \subseteq Z$,

$$(g \circ f)^{-1}[W] = f^{-1}[g^{-1}[W]].$$

(b) We give two proofs: (i) Note that

$$(g \circ f)^- = g^- \circ f^- : X \Rightarrow Z$$

and use part (a) and Corollary 5.1.8.

(ii) Note that for $W \subseteq Z^\uparrow$,

$$(g \circ f)^{-1}[W] = f^{-1}[g^{-1}[W]]$$

(in the notation of Lemma 5.1.11), and apply Lemma 5.1.11. □
Definition 5.1.14 (Union of functions). Let \( f_i : X \rightarrow Y \uparrow \) be a family of functions for \( i \in I \). Suppose for all \( x \in X \), \( \bigcup_{i \in I} f_i(x) \) is countable. Then we define
\[
\bigcup_{i \in I} f_i : X \rightarrow Y \uparrow
\]
by
\[
\left( \bigcup_{i \in I} f_i \right)(x) = \bigcup_{i \in I} f_i(x).
\]

Lemma 5.1.15. If \( f_i : X \rightarrow Y \uparrow \) is continuous for all \( i \in I \), then so is \( \bigcup_{i \in I} f_i \).

Proof: This follows from the fact that for \( V \subseteq Y \uparrow \),
\[
\left( \bigcup_{i \in I} f_i \right)^{-1}[V] = \bigcup_{i \in I} f_i^{-1}[V].
\]
\( \square \)

Remark 5.1.16. Note that all the results of this subsection (5.1) hold for arbitrary multivalued functions \( f : X \rightarrow \mathcal{P}(Y) \), not necessarily countably-many-valued.

5.2 Continuity of WhileCC computable functions

Let \( A \) be an \( N \)-standard topological \( \Sigma \)-algebra.

In order to prove that \textit{WhileCC} procedures on \( A \) are continuous, we first state and prove a lemma which says that such procedures are (almost) equivalent to \textit{While} procedures (without ‘choose’) in an extended signature, which includes a symbol \( f \) for an “oracle function”. Then we apply Lemma 5.1.7.

Lemma 5.2.1 (Oracle equivalence lemma). Given a \textit{WhileCC}(\( \Sigma \)) statement \( S \), and procedure
\[
P \equiv \text{func in a out b aux c begin } S \text{ end},
\]
we can effectively construct a \textit{While}(\( \Sigma_f \)) statement \( S_f \) and procedure
\[
P_f \equiv \text{func in a out b aux c begin } S_f \text{ end}
\]
in a signature \( \Sigma_f \) which extends \( \Sigma \) by a function symbol \( f : \mathbb{N} \rightarrow \mathbb{N} \), such that, putting
\[
P_A^A \equiv_{df} \bigcup_{f \in \mathcal{F}} P_f^A,
\]
where \( \mathcal{F} = \mathbb{N}^\mathbb{N} \) is the set of all functions \( f : \mathbb{N} \rightarrow \mathbb{N} \) and \( P_f^A \) is the interpretation of \( P_f \) in \( A \) formed by interpreting \( f \) as \( f \), we have
\[
P_A \subseteq P_A^A \subseteq (P_A)^\uparrow.
\]
(1)
(Recall Definitions 5.1.14 and 5.1.5, and the definition of \( P^A : A^u \rightrightarrows A^v \uparrow \) in §3.2(g.).)

**Proof:** Intuitively, \( f \) represents a possible implementation of the ‘choose’ operator: \( f(n) \) is a possible value for the \( n \)th call of this operator in any particular implementation of \( P \). We will then take the union of the interpretations over all such possible implementations.

In more detail: the construction of \( S_f \) from \( S \) is as follows. Let \( c \) be a new “counter”, \( i.e., \) an auxiliary variable of sort \( \text{nat} \) which is not in \( S \). First, it is clear that by “splitting up” assignments in \( S \), and introducing more auxiliary \( \text{nat} \) variables, we can re-write \( S \) in such a way that every occurrence of the ‘choose’ construct is in the context of an assignment of the form

\[
z' := \text{choose } z : b. \tag{2}
\]

where the boolean term \( b \) does not contain the ‘choose’ construct. Now replace each assignment of the form (2) by the pair of assignments

\[
c := c + 1;
\]

\[
\text{if } b(z/f(c)) \text{ then } z' := f(c) \text{ else div}
\]

and initialise the value of \( c \) (at the beginning of the statement) to 0. The result is a \( \text{While}^*(\Sigma_f) \) procedure \( P_f \) with a body \( S_f \) which, for a given interpretation \( f \) of \( f \), “interprets” successive executions of ‘choose’ by successive values of \( f \), when this is possible (\( i.e., \) \( b(z/f(c)) \) has \( \text{tt} \) as one of its values), and otherwise, causes the execution to diverge.

For those \( f \) which (for a given input) always give “good” values for all the successive executions of ‘choose’ assignments (2) in \( S \), \( P_f^A \) will give a possible implementation of \( P \). For all other \( f \), \( P_f^A \) will diverge. Since (for a given input) each \( P_f^A \) either simulates one possible implementation of successive executions of ‘choose’ in \( S \) or diverges, their “union” \( P^A_{\cup} \) gives the result of all possible implementations of ‘choose’, plus divergence; hence the conclusion (1). \( \square \)

**Theorem 5.2.2.** Let

\[
P \equiv \text{func in a out b aux c begin S end} \tag{3}
\]

be a \( \text{WhileCC} \) procedure, where \( a : u \) and \( b : v \). Then the interpretation

\[
P^A : A^u \rightrightarrows A^v\uparrow
\]

is continuous.

**Proof:** In the notation of the Oracle Equivalence Lemma (5.2.1): \( P_f^A \) is continuous for all \( f \in \mathcal{F} \), by the continuity theorem for \( \text{While} \) [TZ00, §6.5]. Hence \( P^A_{\cup} \) is continuous, by Lemma 5.1.15. Hence, by (1) and Lemma 5.1.7, so is \( P^A \). \( \square \)

**Remark 5.2.3.** In the special case that \( P^A \) is deterministic, \( i.e., \) single-valued:

\[
P^A : A^u \rightarrow A^v,
\]

it follows by Remark 5.1.9 that \( P^A \) is continuous according to our definition (2.5.1) of continuity for single-valued partial functions.
Corollary 5.2.4. A WhileCC* computable function on $A$ is continuous.

Proof: Such a function is WhileCC computable on $A^*$, hence (by Theorem 5.2.2) continuous on $A^*$, and hence on $A$. □

5.3 Continuity of WhileCC* approximable functions

Recall Definition 3.5.1 and § 2.7.

Theorem 5.3.1. Let $A$ be a metric $\Sigma$-algebra, and let $F: A^u \to A^v$.

(a) If $F$ is WhileCC* approximable then $F$ is W-continuous.

(b) If also $\text{dom}(F)$ is open in $A^u$ then $F$ is continuous.

Proof: Suppose $F$ is approximable on $A$ by the WhileCC* procedure $P: \text{nat} \times u \to v$. We will show that $F$ is W-continuous, using Remark 2.7.2(b). Given $a \in \text{dom}(F)$ and $\epsilon > 0$, choose $N$ such that

$$2^{-N} < \epsilon/3. \quad (1)$$

Then by Definition 3.5.1,

$$\emptyset \neq P^A_N(a) \subseteq B(F(a), 2^{-N}). \quad (2)$$

Choose $b \in P^A_N(a)$. By (2),

$$d(F(a), b) < 2^{-N}. \quad (3)$$

By Corollary 5.2.3, $P^A_N$ is continuous on $A$, and so by Remark 5.1.4(a), there exists $\delta > 0$ such that

$$\forall x \in B(a, \delta), \ P^A_N(x) \cap B(b, \epsilon/3) \neq \emptyset. \quad (4)$$

Take any $x \in B(a, \delta) \cap \text{dom}(F)$. By Definition 3.5.1 again,

$$P^A_N(x) \subseteq B(F(x), 2^{-N}). \quad (5)$$

By (4), choose $y \in P^A_N(x) \cap B(b, \epsilon/3)$. So

$$d(y, b) < \epsilon/3 \quad (6)$$

and by (5)

$$d(F(x), y) < 2^{-N}. \quad (7)$$

Hence

$$d(F(x), F(a)) \leq d(F(x), y) + d(y, b) + d(b, F(a)) \leq \epsilon$$

by (7), (6), (3) and (1). Part (a) follows by Remark 2.7.2(b).

Part (b) then follows by Proposition 2.7.1. □
Concrete computability and the soundness of WhileCC\textsuperscript{*} computation on countable algebras

To compute on a metric algebra $A$ using a concrete model of computation, we choose a countable subspace $X$ of $A$ and an enumeration

$$\alpha : \mathbb{N} \to X.$$ 

From this we build the space $C_\alpha(X)$ of $\alpha$-computable elements of $A$, and enumerate it with

$$\pi : \mathbb{N} \to C_\alpha(X).$$

In this section we step back from topological algebras and consider computability on arbitrary countable algebras. We show that if an algebra $A$ is enumerated and its basic functions are effective, then functions that are WhileCC\textsuperscript{*} computable on $A$ are also effective. This result is a key lemma in the soundness theorem for WhileCC\textsuperscript{*} approximation in the next section.

6.1 Enumerations and tracking functions for partial functions

Let 

$$X = \langle X_s \mid s \in \text{Sort}(\Sigma) \rangle$$

be a $\text{Sort}(\Sigma)$-indexed family of non-empty sets.

**Definition 6.1.1.** An enumeration of $X$ is a family

$$\alpha = \langle \alpha_s : \Omega_s \to X_s \mid s \in \text{Sort}(\Sigma) \rangle$$

of surjective maps $\alpha_s : \Omega_s \to X_s$, for some family

$$\Omega = \langle \Omega_s \mid s \in \text{Sort}(\Sigma) \rangle$$

of sets $\Omega_s \subseteq \mathbb{N}$. The family $X$ is said to be enumerated by $\alpha$. We say that $\alpha : \Omega \to X$ is an enumeration of $X$, and call the pair $(X,\alpha)$ an enumerated family of sets. (The notation \(\to\) denotes surjections, or onto mappings.)

We also write $\Omega_s = \Omega_{\alpha,s}$ to make explicit the fact that $\Omega_s = \text{dom}(\alpha_s)$.

**Definition 6.1.2 (Tracking and strict tracking functions).** We use the notation $X^u = X_{s_1} \times \cdots \times X_{s_m}$ and $\Omega^u_{\alpha} = \Omega_{\alpha,s_1} \times \cdots \times \Omega_{\alpha,s_m}$, where $u = s_1 \times \cdots \times s_m$.

Let $F : X^u \to X_s$ and $f : \Omega^u_{\alpha} \to \Omega_{\alpha,s}$,

(a) $f$ is a tracking function with respect to $\alpha$, or $\alpha$-tracking function, for $F$, if the following diagram commutes:

$$\begin{array}{c}
X^u \xrightarrow{F} X_s \\
\alpha^u \downarrow \quad \downarrow \alpha_s \\
\mathbb{N}^m \xrightarrow{f} \mathbb{N}
\end{array}$$
in the sense that for all \( k \in \Omega^u_\alpha \)
\[
F(\alpha^u(k)) \downarrow \quad \iff \quad f(k) \downarrow \land f(k) \in \Omega_{\alpha,s} \land F(\alpha^u(k)) = \alpha_s(f(k)).
\]

(b) \( f \) is a strict \( \alpha \)-tracking function for \( F \) if in addition, for all \( k \in \Omega^u_\alpha \)
\[
f(k) \downarrow \quad \implies \quad F(\alpha^u(k)) \downarrow.
\]

Here we use the notation \( \alpha^u(k) = (\alpha_1(k_1), \ldots, \alpha_m(k_m)) \), where \( k = (k_1, \ldots, k_m) \).
(We will sometimes drop the type super- and subscripts.)

**Definition 6.1.3** (\( \alpha \)-computability). (a) Suppose \( A \) is a \Sort(\Sigma)\-family, and \( (X, \alpha) \) an enumerated subfamily of \( A \), i.e., \( X_s \subseteq A_s \) for all \( \Sigma \)-sorts \( s \). Suppose \( F : A^u \to A_s \) and \( f : \mathbb{N}^m \to \mathbb{N} \), such that
\[
F \upharpoonright X^u : X^u \to X_s,
\]
\[
f \upharpoonright \Omega^u_\alpha : \Omega^u_\alpha \to \Omega_{\alpha,s},
\]
and \( f \upharpoonright \Omega^u_\alpha \) is a (strict) \( \alpha \)-tracking function for \( F \upharpoonright X \). We then say that \( f \) is a (strict) \( \alpha \)-tracking function for \( F \).

(b) Suppose now further that \( f \) is a computable (i.e., recursive) partial function. Then \( F \) is said to be (strictly) \( \alpha \)-computable.

**Remarks 6.1.4.** (a) In the situation of Definition 6.1.3, we are not concerned with the behaviour of \( F \) off \( X^u \), or the behaviour of \( f \) off \( \Omega^u_\alpha \).

(b) For convenience, we will always assume:
\[
\Omega_{\alpha,\text{bool}} = \{0, 1\}, \quad \alpha_{\text{bool}}(0) = \text{f}, \quad \alpha_{\text{bool}}(1) = \text{t}
\]
and also (when \( \Sigma \) is \( \mathbb{N} \)-standard):
\[
\Omega_{\alpha,\text{nat}} = \mathbb{N} \quad \text{and} \quad \alpha_{\text{nat}} \quad \text{is the identity on} \ \mathbb{N}.
\]

Assume now that \( A \) is a \( \Sigma \)-algebra and \( (X, \alpha) \) is a \Sort(\Sigma)\-family of subsets of \( A \), enumerated by \( \alpha \).

**Definition 6.1.5** (Enumerated \( \Sigma \)-subalgebra). \( (X, \alpha) \) is said to be an enumerated \( \Sigma \)-subalgebra of \( A \) if \( X \) is a \( \Sigma \)-subalgebra of \( A \).

**Definition 6.1.6** (\( \Sigma \)-effective subalgebra). Suppose \( A \) is a \( \Sigma \)-algebra and \( (X, \alpha) \) is an enumerated \( \Sigma \)-subalgebra. Then \( \alpha \) is said to be

(a) \( \Sigma \)-effective if all the basic \( \Sigma \)-functions on \( A \) are \( \alpha \)-computable; and

(b) strictly \( \Sigma \)-effective if all the basic \( \Sigma \)-functions on \( A \) are strictly \( \alpha \)-computable.
6.2 Soundness Theorem for surjective enumerations

For the rest of this section we will be considering the special case of §6.1 in which the enumerated subalgebra $X$ is $A$ itself, i.e., we assume the enumeration is onto $A$. To emphasise this special situation, we will denote the enumeration by

$$
\beta : \Omega_\beta \to A,
$$

so that $(A, \beta)$ is our enumerated $\Sigma$-algebra.

Given such an enumerated algebra $(A, \beta)$ and a function

$$
F : A^u \longrightarrow A^s,
$$

we have two notions of computability for $F$:

(i) abstract, i.e., WhileCC* computability, as described in Section 3; and

(ii) concrete, i.e., $\beta$-computability, as in Definition 6.1.3, in the special case that $X = A$.

We will prove a soundness theorem (Theorem A0), for these notions of abstract and concrete computability, i.e., (i)$\implies$(ii), assuming strict effectiveness of $\beta$.

A more general soundness theorem (Theorem A), with more general notions of abstract computability (WhileCC* approximability) and concrete computability (computability w.r.t. the computable closure of an enumeration), will be proved in Section 7.

**Theorem A0 (Soundness for countable algebras).** Let $(A, \beta)$ be an enumerated $N$-standard $\Sigma$-algebra such that $\beta$ is strictly $\Sigma$-effective. If $F : A^u \longrightarrow A^s$ is WhileCC* computable on $A$, then $F$ is strictly $\beta$-computable on $A$.

6.3 Proof of Soundness Theorem A0

Assume, then, that $(A, \beta)$ is an enumerated $N$-standard $\Sigma$-algebra and $\beta$ is strictly $\Sigma$-effective.

We will show that each of the semantic functions listed in §3.2(a)–(g) has a computable tracking function. More precisely, we will work, not with the semantic functions themselves, but “localised” functions representing them (cf. [TZ00, §4]).

First we will prove a series of results of the form:

**Lemma Scheme 6.3.1.** For each semantic representing function

$$
\Phi : A^u \Rightarrow A^v
$$

representing one of the semantic functions listed in §3.2(a)–(g), there is a computable tracking function w.r.t. $\beta$, i.e., a function

$$
\varphi : \Omega^u_\beta \longrightarrow \Omega^v_\beta
$$

which commutes the diagram
in the sense that for all $k, l \in \Omega^u_\beta$:
\[
\begin{align*}
\varphi(k) \downarrow l & \implies \beta^v(l) \in \Phi(\beta^u(k)), \\
\varphi(k) \uparrow & \implies \uparrow \in \Phi(\beta^u(k)).
\end{align*}
\]

**Remarks 6.3.2.** (a) Here $\varphi$ is a combination “strict tracking function” and “selection function”. We can think of $\varphi$ as giving one possible implementation of $\Phi$. (Compare the representative functions for various semantic functions in [TZ00, §4].)

(b) We are not concerned with the behaviour of $\varphi$ on $\mathbb{N}^m \setminus \Omega^u_\beta$ (where $m = \text{arity}(u)$). (Cf. Remark 6.1.4(a).)

Theorem A_0 then follows easily (§6.5) from this lemma scheme.

**Proof of Lemma Scheme 6.3.1:** We proceed to prove this lemma scheme by constructing concrete strict tracking functions for the semantic functions in §3.2.

Let $x$ be a $u$-tuple of variables, where $u = s_1 \times \cdots \times s_m$. Let $\mathit{PTerm}_x = \mathit{PTerm}_x(\Sigma)$ be the class of all $\Sigma$-terms with variables among $x$ only, and for all sorts $s$ of $\Sigma$, let $\mathit{PTerm}_{x,s} = \mathit{PTerm}_{x,s}(\Sigma)$ be the class of such terms of sort $s$.

We consider in turn the semantic functions in §3.2, or rather versions of these localised to $x$, i.e., defined only in terms of the state values on $x$ (cf. [TZ00, §4]). For example, we localise the set $\textit{State}(A)$ of states on $A$ to the set
\[
\mathit{State}_x(A) =_{df} A^u
\]
of $u$-tuples of elements of $A$, where a tuple $a \in A^u$ represents a state $\sigma$ (relative to $x$) if $\sigma[x] = a$. The set $A^u$ is, in turn, represented (relative to $\beta$) by the set $\Omega^u_\beta$.

We assume an effective coding, or Gödel numbering, of the syntax of $\Sigma$. We use the notation
\[
\Gamma \mathit{PTerm}_s \sqsupset =_{df} \{ \Gamma t \mid t \in \mathit{PTerm}_s \},
\]
etc., for sets of Gödel numbers of syntactic expressions.

(a) Tracking of term evaluation.

The function
\[
\mathit{PTE}_{x,s}^A : \mathit{PTerm}_{x,s} \times \mathit{State}_x(A) \rightarrow^+ A^s_\uparrow
\]
defined by

\[ PTE_{x,s}^A(t, a) = \lceil t \rceil^A \sigma \]

for any state \( \sigma \) on \( A \) such that \( \sigma[x] = a \), is strictly tracked by a computable function

\[ pte_{x,s}^{A,\beta} : \Gamma PTerm_{x,s} \times \Omega_{\beta}^u \rightarrow \Omega_{\beta,s} \]

so that the following diagram commutes:

\[
\begin{array}{ccc}
PTerm_{x,s} \times \text{State}_x(A) & \xrightarrow{\beta_s} & A_s^\uparrow \\
\downarrow \langle \text{enum}, \beta^u \rangle & & \downarrow \beta_s \\
\Gamma PTerm_{x,s} \times \Omega_{\beta}^u & \xrightarrow{pte_{x,s}^{A,\beta}} & \Omega_{\beta,s}
\end{array}
\]

(where \( \text{enum} \) is the inverse of the Gödel numbering function), in the sense that

\[
\begin{align*}
pte_{x,s}^{A,\beta}(\Gamma t, k) \downarrow & \quad \Rightarrow \quad \beta_s(l) \in PTE_{x,s}^A(t, \beta^u(k)), \\
pte_{x,s}^{A,\beta}(\Gamma t, k) \uparrow & \quad \Rightarrow \quad \uparrow \in PTE_{x,s}^A(t, \beta^u(k)).
\end{align*}
\]

In order to construct such a representing function, we first define the state variant representing function, i.e., a (primitive) recursive function

\[ vart_{x}^{\beta} : \Omega_{\beta}^u \times \Gamma Var_{x} \times \Omega_{\beta,s} \rightarrow \Omega_{\beta,s} \]

such that

\[ \beta^u(vart_{x}^{\beta}(k, \Gamma y, k_0)) = \beta^u(e) \{ y/\beta_s(k_0) \} \]

for \( k \in \Omega_{\beta}^u \), \( y \in \text{Var}_x \) and \( k_0 \in \Omega_{\beta,s} \) (cf. Definition 3.2.3(b)).

We turn to the definition of \( pte_{x,s}^{A,\beta}(\Gamma t, k) \). This is by induction on \( \Gamma t \), or structural induction on \( t \in PTerm_x \), over all \( \Sigma \)-sorts \( s \). The cases are:

- \( t \equiv c \), a primitive constant. Then define
  \[ pte_{x,s}^{A,\beta}(\Gamma t, k) = k_0 \quad \text{where} \quad \beta(k_0) = c^A. \]
  (Such a \( k_0 \) exists by the strict \( \Sigma \)-effectivity of \( \beta \)).

- \( t \equiv x_i \) for some \( i = 1, \ldots, m \), where \( x \equiv x_1, \ldots, x_m \). Note that \( k = (k_1, \ldots, k_m) \in \Omega_{\beta}^u \). So define
  \[ pte_{x,s}^{A,\beta}(\Gamma t, k) = k_i. \]

- \( t \equiv F(t_1, \ldots, t_m) \). Let \( f \) be a computable strict tracking function for \( F \), which exists by the strict \( \Sigma \)-effectivity of \( \beta \). Then define
  \[ pte_{x,s}^{A,\beta}(\Gamma t, k) \simeq f(ppe_{x,s}^{A,\beta}(\Gamma t_1, k), \ldots, ppe_{x,s}^{A,\beta}(\Gamma t_m, k)). \]
From the induction hypothesis applied to $t_1, \ldots, t_m$, the definition of $\text{PTE}$ (§3.2(a)) and the fact that $f$ strictly tracks $F$, we can infer (1) for $t$.

- $t \equiv \text{if}(b, t_1, t_2)$. Define

$$
\text{pte}_{x, s}^{A, \beta}(t, k) \simeq \begin{cases} 
\text{pte}_{x, s}^{A, \beta}(t_1, k) & \text{if } \text{pte}_{x, \text{bool}}^{A, \beta}(b, k) \downarrow 1 \\
\text{pte}_{x, s}^{A, \beta}(t_2, k) & \text{if } \text{pte}_{x, \text{bool}}^{A, \beta}(b, k) \downarrow 0 \\
\uparrow & \text{if } \text{pte}_{x, \text{bool}}^{A, \beta}(b, k) \uparrow.
\end{cases}
$$

From the induction hypothesis applied to $b$, $t_0$ and $t_1$, and the definition of $\text{PTE}$, we can infer (1) for $t$.

- $t \equiv \text{choose } z : t_0$. We define $\text{pte}_{x, s}^{A, \beta}(\lceil t \rceil, k)$ by specifying its computation: find, by dovetailing (recall the discussion in §4.1!) some $n$ such that

$$
\text{pte}_{x, s}^{A, \beta}(\lceil t_0 \rceil, \text{var}^{A}(k, \lceil z \rceil, n)) \downarrow 1
$$

(remember, $\beta(1) = \texttt{t}$, by Remark 6.1.4(b)), so that $\text{pte}_{x, s}^{A, \beta}(\lceil t \rceil, k) = \text{some such } n$, if it exists, and $\uparrow$ otherwise. From the induction hypothesis applied to $t_0$, and the definition of $\text{PTE}$, we can infer (1) for $t$.

(b) Tracking of atomic statement evaluation.

Let $AtSt_x$ be the class of atomic statements with variables among $x$ only. The atomic statement evaluation function on $A$ localised to $x$,

$$
AE_x^A : AtSt_x \times State_x(A) \rightharpoonup State_x(A)^\uparrow,
$$

defined by

$$
AE_x^A(S, a) = \langle \text{eval} \rangle^A \sigma
$$

for any state $\sigma$ such that $\sigma[x] = a$, is strictly tracked by a computable function

$$
ae_x^{A, \beta} : \lceil AtSt_x \rceil \times \Omega^{\text{u}}_{\beta} \longrightarrow \Omega^{\text{u}}_{\beta}
$$

so that the following diagram commutes:

\[ 
\begin{array}{ccc}
AtSt_x \times State_x(A) & \xrightarrow{AE_x^A} & State_x(A)^\uparrow \\
\langle \text{enum}, \beta^{\text{u}} \rangle & \downarrow & \downarrow \beta^{\text{u}} \\
\lceil AtSt_x \rceil \times \Omega^{\text{u}}_{\beta} & \xrightarrow{ae_x^{A, \beta}} & \Omega^{\text{u}}_{\beta}
\end{array}
\]

in the sense that

$$
\begin{align*}
\text{ae}_x^{A, \beta}(\lceil S \rceil, k) \downarrow l & \implies \beta(l) \in AE_x^A(S, \beta(k)), \\
\text{ae}_x^{A, \beta}(\lceil S \rceil, k) \uparrow & \implies \uparrow \in AE_x^A(S, \beta(k)).
\end{align*}
$$

(2)
The definition of \( ae_{\mathcal{A}, \beta}^x(\Gamma S^\gamma, k) \) is given by:

\[
\begin{align*}
    ae_{\mathcal{A}, \beta}^x(\Gamma \text{skip}^\gamma, k) & \downarrow k \\
    ae_{\mathcal{A}, \beta}^x(\Gamma \text{div}^\gamma, k) & \uparrow \\
    ae_{\mathcal{A}, \beta}^x(\Gamma y := t^\gamma, k) & \simeq \begin{cases} 
        \text{vart}_{\mathcal{A}, \beta}^x(k, y, l) & \text{if } pte_{\mathcal{A}, \beta}^{x,s}(s t^\gamma, k) \downarrow l \\
        \uparrow & \text{if } pte_{\mathcal{A}, \beta}^{x,s}(\Gamma t^\gamma, k) \uparrow.
    \end{cases}
\end{align*}
\]

Using (1) and the definition of \( \mathcal{A}E_{\mathcal{A}}^x \) (§3.2(b)), we can infer (2).

(c) Tracking of First and Rest operations.

Let \( \text{Stmt}_x \) be the class of statements with variables among \( x \) only. Consider the functions \( \text{First} \) and \( \text{Rest}^A \) (§3.2(c)). Then \( \text{First} \) is strictly tracked by a computable function

\[
\text{first} : \Gamma \text{Stmt}^\gamma \rightarrow \Gamma \text{AtSt}^\gamma
\]

defined on Gödel numbers in the obvious way, so that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Stmt}^\gamma & \xrightarrow{\text{first}} & \Gamma \text{AtSt}^\gamma \\
\Gamma \text{Stmt}^\gamma & \xrightarrow{\text{First}} & \text{AtSt} \\
\end{array}
\]

(Note that \( \text{first} \), unlike most of the other representing functions here, does not depend on \( \text{State}_x(A) \), or, indeed, on \( A \) or \( x \).) Next, the localised version of \( \text{Rest}^A \):

\[
\text{Rest}^A_x : \text{Stmt}_x \times \text{State}_x(A) \Rightarrow \text{Stmt}_x
\]

defined by

\[
\text{Rest}^A_x(S, a) = \text{Rest}^A(S, \sigma)
\]

for any state \( \sigma \) such that \( \sigma[x] = a \), is strictly tracked by a computable function

\[
\text{rest}^{A, \beta}_x : \Gamma \text{Stmt}_x^\gamma \times \Omega^u_{\beta} \rightarrow \Gamma \text{Stmt}_x^\gamma
\]

so that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Stmt}_x \times \text{State}_x(A) & \xrightarrow{\text{Rest}_x^A} & \text{Stmt}_x \\
\Gamma \text{Stmt}_x^\gamma \times \Omega^u_{\beta} & \xrightarrow{\text{rest}_x^{A, \beta}} & \Gamma \text{Stmt}_x^\gamma \\
\end{array}
\]
in the sense that
\[
\begin{align*}
\text{rest}_x^{A,\beta}(\Gamma S^\gamma, k) \downarrow \Gamma S'^\gamma & \implies S' \in \text{Rest}_x^A(S, \beta(k)), \\
\text{rest}_x^{A,\beta}(\Gamma S^\gamma, k) \uparrow & \implies \text{div} \in \text{Rest}_x^A(S, \beta(k))
\end{align*}
\] (3)

The definition of \( \text{rest}_x^{A,\beta}(\Gamma S^\gamma, k) \), as well as the proof of (3), are by induction on \( \Gamma S^\gamma \), or structural induction on \( S \).

- \( S \) is atomic. Then
  \[ \text{rest}_x^{A,\beta}(\Gamma S^\gamma, k) = \Gamma \text{skip}^\gamma. \]

- \( S \equiv S_1; S_2 \). Then
  \[ \text{rest}_x^{A,\beta}(\Gamma S^\gamma, k) = \begin{cases} 
  \Gamma S_2^\gamma & \text{if } S_1 \text{ is atomic} \\
  \text{rest}_x^{A,\beta}(S_1, k); S_2^\gamma & \text{otherwise}
  \end{cases} \]

- \( S \equiv \text{if } b \text{ then } S_1 \text{ else } S_2 \text{ fi} \). Then
  \[ \text{rest}_x^{A,\beta}(\Gamma S^\gamma, k) \equiv \begin{cases} 
  \Gamma S_1^\gamma & \text{if } \text{pte}_{A,\beta}^\gamma(b, k) \downarrow 1 \\
  \Gamma S_2^\gamma & \text{if } \text{pte}_{A,\beta}^\gamma(b, k) \downarrow 0 \\
  \uparrow & \text{if } \text{pte}_{A,\beta}^\gamma(b, k) \uparrow.
  \end{cases} \]

- \( S \equiv \text{while } b \text{ do } S_0 \text{ od} \). Then
  \[ \text{rest}_x^{A,\beta}(S, k) \equiv \begin{cases} 
  S_0; S & \text{if } \text{pte}_{A,\beta}^\gamma(b, k) \downarrow 1, \\
  \text{skip} & \text{if } \text{pte}_{A,\beta}^\gamma(b, k) \downarrow 0, \\
  \uparrow & \text{if } \text{pte}_{A,\beta}^\gamma(b, k) \uparrow.
  \end{cases} \]

(d) Tracking of a computation step.

The computation step function (§3.2(d)) localised to \( x \):

\[ \text{CompStep}_x^A : \text{Stmt}_x \times \text{State}_x(A) \rightarrow^+ \text{State}_x(A)^\uparrow \]

defined by

\[ \text{CompStep}_x^A(S, a) = \text{CompStep}_x^A(S, \sigma) \]

for any state \( \sigma \) such that \( \sigma[x] = a \), is represented by the computable function

\[ \text{compstep}_x^{A,\beta} : \Gamma \text{Stmt}_x^\gamma \times \Omega_\beta^u \rightarrow \Omega_\beta^u \]

defined by

\[ \text{compstep}_x^{A,\beta}(\Gamma S^\gamma, k) \equiv \text{ae}_x^{A,\beta}((\text{first}(\Gamma S^\gamma), k)). \]
This makes the following diagram commute:

\[
\begin{array}{ccc}
\text{Stmt}_x \times \text{State}_x(A) & \xrightarrow{\text{CompStep}^A_x} & \text{State}_x(A)^\uparrow \\
\langle \text{enum}, \beta^u \rangle & \mapsto & \beta^u \\
\Gamma \text{Stmt}_x \downarrow \times \Omega^u_\beta & \xrightarrow{\text{compstep}^A_{x,\beta}} & \Omega^u_\beta
\end{array}
\]

in the sense that

\[\text{compstep}^A_{x,\beta}(\Gamma S^\gamma, k) \downarrow \Rightarrow \beta(l) \in \text{CompStep}^A_x(S, \beta(k)),\]

\[\text{compstep}^A_{x,\beta}(\Gamma S^\gamma, k) \uparrow \Rightarrow \uparrow \in \text{CompStep}^A_x(S, \beta(k)).\]  

(4)

This is proved easily from the definitions and (2).

(e) Tracking of a computation sequence.

Now consider localised versions of the computation tree stage and computation tree of §3.2(e):

\[
\text{CompTreeStage}^A_x : \text{Stmt}_x \times \text{State}_x(A) \times \mathbb{N} \rightarrow \mathcal{P}((\text{State}_x(A)^\uparrow)^{<\omega})
\]

\[
\text{CompTree}_x^A : \text{Stmt}_x \times \text{State}_x(A) \rightarrow \mathcal{P}((\text{State}_x(A)^\uparrow)^{\leq\omega})
\]

We will define a function which selects a path through the computation tree:

\[
\text{compseq}^A_{x,\beta} : \Gamma \text{Stmt}_x \downarrow \times \Omega^u_\beta \times \mathbb{N} \rightarrow \Omega^u_\beta \cup \{\Gamma^*\gamma\}
\]

(where ‘*’ is a symbol meaning “already terminated”) by recursion on \(n\):

\[
\text{compseq}^A_{x,\beta}(\Gamma S^\gamma, k, 0) = e
\]

\[
\text{compseq}^A_{x,\beta}(\Gamma S^\gamma, k, n + 1) \simeq
\begin{cases}
\Gamma^*\gamma & \text{if } S \text{ is atomic and } n > 0 \text{ and } \text{compseq}^A_{x,\beta}(\Gamma S^\gamma, k, n) \downarrow \\
\Gamma S^\gamma & \text{if } S \text{ is atomic and } n > 0 \text{ and } \text{compseq}^A_{x,\beta}(\Gamma S^\gamma, k, n) \uparrow \\
\text{compseq}^A_{x,\beta}(\text{rest}^A_{x,\beta}(\Gamma S^\gamma, k), \text{compstep}^A_{x,\beta}(\Gamma S^\gamma, k), n) & \text{otherwise.}
\end{cases}
\]

(This is a “tail recursion”: compare definition of \(\text{Comp}_1^A\) in [TZ00, §3.4].)

Writing \(k_n = \text{compseq}^A_{x,\beta}(\Gamma S^\gamma, k, n)\), this defines a (concrete) computation sequence

\[\bar{k} = k_0, k_1, k_2, \ldots\]

for \(S\) from the initial state \(k = k_0\). (Our notation here includes the possibility that some of the \(k_i\) may be \(\Gamma^*\gamma\) or \(\uparrow\).) As can easily be checked, there are three possibilities for \(\bar{e}\) (compare the discussion in §3.2(e)):
(i) For some \( n \), \( k_i \in \Omega^u_\delta \) for all \( i \leq n \) and \( k_i = * \) for all \( i > n \). This represents a computation which terminates at stage \( n \), with final state \( k_n \).

(ii) For some \( n \), \( k_i \in \Omega^u_\delta \) for all \( i < n \) and \( k_i = \uparrow \) for all \( i \geq n \). This represents a non-terminating computation, with local divergence at stage \( n \).

(iii) For all \( i \), \( k_i \in \Omega^u_\delta \). This represents non-terminating computation, with global divergence.

We write \( \bar{k}[n] \) = the initial segment \( k_0, k_1, \ldots, k_n \), with length \( \text{lgth}(\bar{k}[n]) = n + 1 \). We put \( \text{lgth}(\bar{k}) = \infty \). The \( k_i \) are called components of \( \bar{k} \), and of \( \bar{k}[n] \), for all \( i \leq n \).

The computation sequence \( \bar{k} \) then has the following connection with the computation tree \( \text{CompTree}_x^A \). Extend (for now) the definition of \( \beta \) by \( \beta(\tau^\infty) = * \), \( \beta(\uparrow) = \uparrow \), and

\[
\beta(\bar{k}) = \text{df} \beta(k_0), \beta(k_1), \beta(k_2), \ldots \\
\beta(\bar{k}[n]) = \text{df} \beta(k_0), \beta(k_1), \beta(k_2), \ldots, \beta(k_n).
\]

Let \( \tau = \text{CompTree}_x^A(S, \beta(k)) \). Then

(i) If the computation sequence \( \bar{k} \) terminates at stage \( n \), then \( \beta(\bar{k}[n]) \) is a path through \( \tau \) from the root to a leaf (\( = \beta(k_0) \), the final state).

(ii) If for some (smallest) \( n \), \( k_n = \uparrow \), then \( \beta(\bar{k}[n]) \) is a path through \( \tau \) from the root to a leaf (\( = \uparrow \), local divergence).

(iii) If for all \( n \), \( k_n \in \Omega^u_\delta \), then \( \beta(\bar{k}) \) is an infinite path through \( \tau \) (global divergence).

To prove this, we first define an initial segment of \( \bar{k} \) (including \( \bar{k} \) itself) to be acceptable if (i) no component is equal to ‘\(*’\), and (ii) no component, except possibly the last, is equal to \( \uparrow \). Further, an acceptable initial segment of \( \bar{k} \) is maximal (acceptable) if it has no acceptable extension. Thus if \( \bar{k} \) is acceptable, it is automatically maximal. If \( \bar{k}[n] \) is acceptable, it is maximal acceptable provided either \( k_{n+1} = * \) or \( k_n = \uparrow \). We then show:

**Lemma 6.3.3.** Given a computation sequence \( \bar{k} = k_0, k_1, \ldots \) for \( \tau^S \) from \( k \), where \( k_n = \text{compseq}_x^{A,\beta}(\tau^S, k \), \( n \)), let \( \tau = \text{CompTree}_x^A(S, \beta(k)) \). Then with every acceptable initial segment \( \bar{k}[n] \) of \( k \), \( \beta(\bar{k}[n]) \) is a path through \( \tau \) from the root. If \( \bar{k}[n] \) is maximal, then \( \beta(k[n]) \) is a leaf.

**Proof:** Put \( \tau[n] = \text{CompTreeStage}_x^A(S, \beta(k_0), n) \). The proof is by induction on \( n \), comparing the inductive definitions of \( k_n \) and \( \tau[n] \).

**Basis:** \( n = 0 \). This is immediate from the definitions: \( k_0 = k \), and \( \tau[0] = \{ \beta(k_0) \} \).

**Induction step:** Assume the induction hypothesis holds for the initial segment of length \( n \) of the computation sequence for \( \tau^S \) from \( k_1 \), where

\[
S' = \text{rest}_x^{A,\beta}(\tau^S, \beta(k)),
\]

\[
e_1 = \text{compseq}_x^{A,\beta}(\tau^S, k, 1)
\]

\[
\simeq \text{compseq}_x^{A,\beta}(\text{rest}_x^{A,\beta}(\tau^S, k), \text{compstep}_x^{A,\beta}(\tau^S, k), 0)
\]

\[
\simeq \text{compstep}_x^{A,\beta}(\tau^S, e)
\]
i.e., assume the induction hypothesis for the segment \( T \) of length \( n \):

\[
l_0, \ l_1, \ l_2, \ldots, l_n
\]

where \( l_i = e_{i+1} \ (i = 1, \ldots, n) \). Now apply the inductive definitions for \( \text{compseq}^A_\beta(S, \ k, \ n+1) \) (above) and \( \text{CompTreeStage}^A_\beta(S, \ \beta(k), \ n+1) \) (§3.2(e)), and use (3) and (4).

(f) Tracking of statement evaluation.

First we need a constructive computation length function

\[
\text{complength}^A_\beta : \Gamma \text{Stmt}_\beta \times \Omega^u_\beta \rightarrow \mathbb{N}
\]

by (cf. [TZ00, §3.4])

\[
\text{complength}^A_\beta(\Gamma S^\gamma, \ k) \simeq \mu n[\text{compseq}^A_\beta(\Gamma S^\gamma, \ k, \ n+1) \downarrow *]
\]

i.e., the least \( n \) (if it exists) such that for all \( i \leq n \), \( \text{compseq}^A_\beta(\Gamma S^\gamma, \ k, \ i) \downarrow \neq * \) and

\( \text{compseq}^A_\beta(\Gamma S^\gamma, \ k, \ n+1) \downarrow * \).

Thus \( \text{complength}^A_\beta(\Gamma S^\gamma, \ k) \) is undefined (↑) in the case of local or global divergence of the computation sequence for \( \Gamma S^\gamma \) from \( k \).

Now the statement evaluation function (§3.2(f)) localised to \( x \):

\[
\text{SE}^A_x : \text{Stmt}_x \times \text{State}_x(A) \rightrightarrows \text{State}_x(A)^\uparrow
\]

defined by

\[
\text{SE}^A_x(S, a) = [S]^A(\sigma)
\]

for any state \( \sigma \) such that \( \sigma[x] = a \), is strictly tracked by the computable function

\[
\text{se}^A_\beta : \Gamma \text{Stmt}_x \times \Omega^u_\beta \rightarrow \Omega^u_\beta
\]

defined by

\[
\text{se}^A_\beta(\Gamma S^\gamma, \ k) \simeq \text{compseq}^A_\beta(\Gamma S^\gamma, \ k, \ \text{complength}^A_\beta(\Gamma S^\gamma, \ k)).
\]

This makes the following diagram commute:

\[
\begin{array}{ccc}
\text{Stmt}_x \times \text{State}_x(A) & \xrightarrow{SE^A_x} & \text{State}_x(A)^\uparrow \\
\text{\langle enum, } \beta^u \rangle & \downarrow & \se^A_\beta \downarrow \\
\Gamma \text{Stmt}_x \times \Omega^u_\beta & \xrightarrow{.} & \Omega^u_\beta
\end{array}
\]

in the sense that

\[
\text{se}^A_\beta(\Gamma S^\gamma, \ k) \downarrow l \quad \Rightarrow \quad \beta(l) \in \text{SE}^A_x(S, \ \beta(k)),
\]

\[
\text{se}^A_\beta(\Gamma S^\gamma, \ k) \uparrow \quad \Rightarrow \quad \uparrow \in \text{SE}^A_x(S, \ \beta(k)).
\]

This result is clear from the definition of \( \text{complength} \) and Lemma 6.3.1.
(g) Tracking of procedure evaluation.

For a specific triple of lists of variables $a : u$, $b : v$, $c : w$, let $\text{Proc}_{a,b,c}$ be the class of all $\text{WhileCC}^*$ procedures of type $u \rightarrow v$, with declaration `in $a$ out $b$ aux $c$'. The procedure evaluation function ($\S 3.2(g)$) localised to this declaration:

$$\text{PE}^A_{a,b,c} : \text{Proc}_{a,b,c} \times A^u \rightarrow A^v$$

defined by

$$\text{PE}^A_{a,b,c}(P, a) = P^A(a),$$

is strictly tracked by the computable function

$$\text{pe}^A_{a,b,c} : \text{Proc}_{a,b,c} \times \Omega^u_{\beta} \rightarrow \Omega^v_{\beta}$$

defined by the following algorithm. Let $P \in \text{Proc}_{a,b,c}$; say

$$P \equiv \text{proc in } a \text{ out } b \text{ aux } c \text{ begin } S \text{ end}$$

and let $k_0 \in \Omega^u_{\beta}$. Take any $k_1 \in \Omega^v_{\beta}$ and $k_2 \in \Omega^w_{\beta}$. (The choice of $k_1$ and $k_2$ is irrelevant, by Remark 3.2.4.) Put $k \equiv k_0, k_1, k_2$ and put $x \equiv a, b, c$. Compute $s e^A_{x, \beta}(S, k)$. Suppose this converges to $l \equiv l_0, l_1, l_2$, where $l_0 \in \Omega^u_{\beta}$, $l_1 \in \Omega^v_{\beta}$ and $l_2 \in \Omega^w_{\beta}$. Then we define $\text{pe}^A_{a,b,c}(P, k_0) \downarrow l_1$. The following diagram then commutes:

$$\begin{array}{ccc}
\text{Proc}_{a,b,c} \times A^u & \xrightarrow{\text{PE}^A_{a,b,c}} & A^v \\
\downarrow{\text{enum}, \beta^v} & & \downarrow{\beta^v} \\
\Gamma \text{Proc}_{a,b,c} \times \Omega^u_{\beta} & \xrightarrow{\text{pe}^A_{a,b,c}} & \Omega^v_{\beta}
\end{array}$$

in the sense that

$$\text{pe}^A_{a,b,c}(P, k_0) \downarrow l \implies \beta(l) \in \text{PE}^A_{a,b,c}(P, \beta(k)),$$

$$\text{pe}^A_{a,b,c}(P, k) \uparrow \implies \uparrow \in \text{PE}^A_{a,b,c}(P, \beta(k)). \quad (6)$$

This is proved from (5) and the definitions of $\text{PE}$ and $\text{pe}$.

This concludes the proof of Lemma Scheme 6.3.1. □

**Proof of Theorem A₀ (conclusion):** Suppose $F : A^u \rightarrow A_s$ is $\text{WhileCC}^*$ computable on $A$. Then there is a deterministic $\text{WhileCC}^*$ procedure (Definitions 3.2.5/6)

$$P : u \rightarrow s$$

such that for all $a \in A^u$,

$$F(x) \downarrow y \implies P^A(x) = \{y\},$$

$$F(x) \uparrow \implies P^A(x) = \{\uparrow\}.$$

Hence by (g) (above) there is a computable (partial) function

$$f : \Omega^u_{\beta} \rightarrow \Omega_{\beta,s}$$

which strictly tracks $F$, as required. □
7 Soundness of $\text{WhileCC}^*$ approximation

In this section we address the general situation introduced in §6.2, of a partial metric $\Sigma$-algebra $A$ with an enumerated subalgebra $(X, \alpha)$, and prove a more general soundness theorem (Theorem A) for $\text{WhileCC}^*$ approximation.

7.1 Enumerated subspace of metric algebra; Computational closure

Let $A$ be an N-standard metric $\Sigma$-algebra, and $(X, \alpha)$ an enumerated $\text{Sort}(\Sigma)$-family $\langle (X_s, \alpha) | s \in \text{Sort}(\Sigma) \rangle$ of subsets $X_s \subseteq A_s$ ($s \in \text{Sort}(\Sigma)$). Each $X_s$ can be viewed as a metric subspace of the metric space $A_s$. We call $(X, \alpha)$ a $\text{Sort}(\Sigma)$-enumerated (metric) subspace of $A$.

We define from $(X, \alpha)$ a family

$$C_\alpha(X) = \langle C_\alpha(X)_s | s \in \text{Sort}(\Sigma) \rangle$$

of sets $C_\alpha(X)_s$ of $\alpha$-computable elements of $A_s$, i.e., limits in $A_s$ of effectively convergent Cauchy sequences (to be defined below) of elements of $X_s$, so that

$$X_s \subseteq C_\alpha(X)_s \subseteq A_s,$$

with corresponding enumerations

$$\overline{\alpha}_s : \Omega_{\overline{\alpha},s} \rightarrow C_\alpha(X)_s.$$

Writing $\overline{\alpha} = \langle \overline{\alpha}_s | s \in \text{Sort}(\Sigma) \rangle$, we call the enumerated subspace $(C_\alpha(X), \overline{\alpha})$ the computable closure of $(X, \alpha)$ in $A$.

We will generally be interested in (strictly) $\overline{\alpha}$-computable (rather than $\alpha$-computable) functions on $A$ (cf. Definition 6.1.3), as our more general model of concrete computability on $A$.

The sets $\Omega_{\overline{\alpha},s} \subseteq \mathbb{N}$ consist of codes for $C_\alpha(X)_s$ (w.r.t. $\alpha$), i.e., pairs of numbers $c = \langle e, m \rangle$ where

(i) $e$ is an index for a total recursive function defining a sequence $\alpha \circ \{e\}$ in $X_s$, i.e., the sequence

$$\alpha_s(\{e\}(0)), \alpha_s(\{e\}(1)), \alpha_s(\{e\}(2)), \ldots,$$

of elements of $X_s$,

(ii) $m$ is an index for a modulus of convergence for this sequence:

$$\forall k, l \geq \{m\}(n) : d_i(\alpha(\{e\}(k)), \alpha(\{e\}(l))) < 2^{-n}.$$

For any such code $c = \langle e, m \rangle \in \Omega_{\overline{\alpha},s}$, $\overline{\alpha}_s(c)$ is defined as the limit in $A_s$ of the Cauchy sequence (1), and $C_\alpha(X)_s$ is the range of $\overline{\alpha}_s$:
Remark 7.1.1. We may assume, when convenient, that the modulus of convergence for a given code is the identity, i.e., replace (2) by the simpler condition
\[ \forall k, l \geq n : \ d_i(\alpha(\{e\}(k)), \alpha(\{e\}(l))) < 2^{-n}. \]

or, equivalently,
\[ \forall k > n : \ d_i(\alpha(\{e\}(k)), \alpha(\{e\}(n))) < 2^{-n}. \]

This is because any code \( c = \langle e, m \rangle \) satisfying (2) may be effectively replaced by a code for the same element of \( C_\alpha(X)_s \) satisfying (3), namely \( c' = \langle e', m_1 \rangle \), where \( m_1 \) is a standard code for the identity function on \( \mathbb{N} \), and \( e' = \text{comp}(e, m) \), where \( \text{comp}(x, y) \) is a primitive recursive function for “composition” of (indices of) computable functions, i.e.,
\[ \{ \text{comp}(e, m) \}(x) = e(\{ m \}(x)). \]

In case of a code \( c = \langle e, m_1 \rangle \) satisfying (3), the sequence (1) is called a fast (\( \alpha \)-effective) Cauchy sequence. In such a case we will often, for simplicity, refer to \( e \) itself as the “code”, and the argument of \( \alpha \). In this way we will shift between “c-codes” and “\( e \)-codes” as convenient.

Remark 7.1.2. In the case \( s = \text{nat} \), we can simply take \( \Omega_{\pi, \text{nat}} = \Omega_{\alpha, \text{nat}} = \mathbb{N} \), and \( \overline{\alpha}_{\text{nat}} \) as the identity mappings on \( \mathbb{N} \). Similarly, in the case \( s = \text{bool} \), we can take \( \Omega_{\pi, \text{bool}} = \Omega_{\alpha, \text{bool}} = \{0, 1\} \), with \( \overline{\alpha}(0) = \alpha(0) = \text{f} \) and \( \overline{\alpha}(1) = \alpha(1) = \text{t} \). (Cf. Remark 6.1.3(b).)

Remark 7.1.3 (Closure of \( \alpha \)-computability operation). The subspace \( (C_\alpha(X), \overline{\alpha}) \) is “computationally closed in \( A \)”, in the sense that the limit of a (fast) \( \pi \)-effective Cauchy sequence of elements of \( C_\alpha(X) \) is again in \( C_\alpha(X) \), i.e., \( C_\pi(C_\alpha(X)) = C_\alpha(X) \). (Easy exercise.)

Remark 7.1.4. We will usually assume that \( \Omega_{\alpha, s} \) is decidable, in fact, that \( \Omega_{\alpha, s} = \mathbb{N} \) for all sorts \( s \), which is typical in practice, unlike the case for \( \Omega_\pi \). (See the following Example.)

Example 7.1.5 (Constructible reals). The best known nontrivial example of an enumerated subspace \( (X, \alpha) \), and its extension to a subspace of \( \alpha \)-computable elements, is the following. Let \( A \) be the metric algebra \( \mathcal{R} \) of reals (Example 2.6.1), with signature \( \Sigma \). Let \( X_{\text{real}} \) be the set of rationals \( \mathbb{Q} \subset \mathbb{R} \), let \( \Omega_{\alpha, \text{real}} = \mathbb{N} \) and let
\[ \alpha_{\text{real}} : \mathbb{N} \to \mathbb{Q} \]
be a canonical enumeration of \( \mathbb{Q} \). Then \( C_\alpha(\mathbb{Q}) = _{df} C_\alpha(X)_{\text{real}} \subset \mathbb{R} \) is the subspace of recursive or constructible reals. Note that it is a subfield of \( \mathbb{R} \), and hence \( C_\alpha(X) \) is a subalgebra of \( \mathcal{R} \). Further, it is easily verified that \( \overline{\alpha} \) is strictly \( \Sigma(\mathcal{R}) \)-effective. (Cf. Definition 6.1.6.) Note that \( \Omega_{\alpha, \text{real}} = \mathbb{N} \), unlike \( \Omega_{\pi, \text{real}} \), which, by contrast, is non-recursive. (See the previous Remark.)
Remark 7.1.6 (Extension of enumeration to $A^*$). Given an enumeration $\alpha$ of a $\Sigma$-subspace $X$ of $A$, we can extend this canonically to an enumeration $\alpha^*$ of a $\Sigma^*$-subspace $X^*$ of $A^*$. (Easy exercise.) This in turn generates an enumeration $\overline{\alpha}^*$ of a $\Sigma^*$-subspace $C_\alpha(X)^*$ of $\alpha^*$-computable elements of $A^*$. It is easy to see that

(i) if $C_\alpha(X)$ is an $\Sigma$-subalgebra of $A$, then $C_\alpha(X)^*$ is a $\Sigma^*$-subalgebra of $A^*$;

(ii) if $\overline{\alpha}$ is (strictly) $\Sigma$-effective, then $\overline{\alpha}^*$ is (strictly) $\Sigma^*$-effective.

We will usually use this extension (of $(X, \alpha)$ and $(C_\alpha(X), \overline{\alpha})$) to $A^*$ implicitly, i.e., writing ‘$\alpha$’ instead of ‘$\alpha^*$’ etc.

7.2 Soundness Theorem for effective numberings

We now prove the first main theorem mentioned in the Introduction.

Theorem A (Soundness). Let $A$ be an $N$-standard metric $\Sigma$-algebra, and $(X, \alpha)$ an enumerated $\text{Sort}(\Sigma)$-subspace. Suppose the enumerated $\text{Sort}(\Sigma)$-space $(C_\alpha(X), \overline{\alpha})$ of $\alpha$-computable elements of $A$ is a $\Sigma$-subalgebra of $A$, and $\overline{\alpha}$ is strictly $\Sigma$-effective. If $F : A^u \rightarrow A_s$ is $\text{WhileCC}^*$-approximable on $A$, then $F$ is $\overline{\alpha}$-computable on $A$.

Proof: The proof uses the Soundness Theorem $A_0$ (Section 6), or rather the Lemma Scheme 6.4.1 (specifically, part (g) of the proof) applied to the enumerated subalgebra $(C_\alpha(X), \overline{\alpha})$ in place of $(A, \beta)$.

So suppose $F : A^u \rightarrow A_s$ is effectively uniformly $\text{WhileCC}^*$ approximable on $A$. Then there is a $\text{WhileCC}^*(\Sigma)$ procedure

$$P : \text{nat} \times u \rightarrow s$$

such that for all $n \in \mathbb{N}$ and all $x \in \text{dom}(F)$:

$$\uparrow \notin P_n^A(x) \subseteq B(F(x), 2^{-n}). \quad (1)$$

(see Definition 3.5.1). By §6.4(g) (applied to $(C_\alpha(X), \overline{\alpha})$ in place of $(A, \beta)$) there is a computable function

$$f : \mathbb{N} \times \Omega_\alpha^u \rightarrow \Omega_{\overline{\alpha},s}$$

which tracks $P^A$ strictly, in the sense that for all $n \in \mathbb{N}$, $e \in \Omega_\alpha^u$ and $e' \in \Omega_{\overline{\alpha},s}$ (and writing $f_n = f(n, \cdot)$):

$$f_n(e) \downarrow e' \implies \overline{\alpha}(e') \in P_n^A(\overline{\alpha}(e)),$$

$$f_n(e) \uparrow \implies \uparrow \in P_n^A(\overline{\alpha}(e)). \quad (2)$$

We will show how to define a partial recursive $\overline{\alpha}$-tracking function

$$g : \Omega_\alpha^u \rightarrow \Omega_{\overline{\alpha},s}$$

for $F$ as follows.
Given any \( e \in \Omega_{\overline{\alpha}} \), suppose \( \overline{\alpha}(e) \in \text{dom}(F) \), i.e.,

\[
F(\overline{\alpha}(e)) \downarrow \in A_s.
\] (3)

We must show how to define an \( \overline{\alpha} \)-tracking function \( g \) for \( F \), i.e., such that

\[
g(e) \in \Omega_{\overline{\alpha},s} \quad \text{and} \quad \overline{\alpha}(g(e)) = F(\overline{\alpha}(e)).
\] (4)

By (1), for all \( n \)

\[\uparrow \notin P_n^A(\overline{\alpha}(e)) \subseteq B(F(\overline{\alpha}(e)), 2^{-n}).\] (5)

Hence by (2), for all \( n \)

\[
f_n(e) \downarrow \in \Omega_{\overline{\alpha},s}
\] (6a)

and

\[
\overline{\alpha}(f_n(e)) \in P_n^A(\overline{\alpha}(e)).
\] (6b)

and so by (6a) we may assume (by definition of \( \Omega_{\overline{\alpha}} \)) that for all \( n \)

\[
\alpha \circ \{f_n(e)\} \text{ is a fast Cauchy sequence, with limit } \overline{\alpha}(f_n(e)).
\] (7)

Also by (6b) and (5),

\[
d(\overline{\alpha}(f_n(e)), F(\overline{\alpha}(e))) < 2^{-n}.
\] (8)

Now let \( e' \) be a “canonical” index for the (partial) function

\[
\{e'\}: n \mapsto \{f_n(e)\}(n)
\] (9)

obtained uniformly effectively in \( e \). So \( \{e'\} \) is the “diagonal” function formed from the sequence of functions with indices \( f_n(e) \). Consider the sequence \( \alpha_s \circ \{e'\} \), i.e.,

\[
\alpha_s(\{e'\}(0)), \alpha_s(\{e'\}(1)), \alpha_s(\{e'\}(2)), \ldots,
\] (10)

**Claim:** (10) is a Cauchy sequence in \( A_s \), with modulus of convergence \( \lambda n(n+2) \).

**Proof of claim:** For any \( n \) and \( k > n \):

\[
d(\alpha(\{e'\}(k)), \alpha(\{e'\}(n)))
\]

\[
= d(\alpha(\{f_k(e)\}(k)), \alpha(\{f_n(e)\}(n))) \quad \text{by def. (9) of } e'
\]

\[
\leq d(\alpha(\{f_k(e)\}(k)), \overline{\alpha}(f_k(e))) + d(\overline{\alpha}(f_k(e)), \overline{\alpha}(f_n(e))) + d(\overline{\alpha}(f_n(e)), \alpha(\{f_n(e)\}(n)))
\]

\[
= d_1 + d_2 + d_3 \quad \text{(say)}
\]

where

\[
d_1 \leq 2^{-k},
\]

\[
d_3 \leq 2^{-n},
\]
by (7), and
\[
d_2 \leq d(\alpha(f_k(e)), F(\alpha(e))) + d(F(\alpha(e)), \alpha(f_n(e))) \\
< 2^{-k} + 2^{-n}
\]
by (8). Therefore
\[
d(\alpha(e'(k)), \alpha(e'(n))) \leq d_1 + d_2 + d_3 \\
< 2 \cdot 2^{-k} + 2 \cdot 2^{-n} \\
< 2^{-n+2}.
\]
This proves the claim. □

Further, by the method of Remark 7.1.1 (composing \{e'\} with the modulus of convergence), we can replace the index \(e'\) by an \(e\)-code \(e''\) for a fast Cauchy sequence:
\[
\{e''(n) \simeq \{e'(n + 2)\}.
\]
Then we define
\[
g(e) = e''.
\]
We show that \(g\) is an \(\alpha\)-tracking function for \(F\), i.e., (assuming (3)) we show (4). Since \(\alpha \circ \{e''\}\) is a fast Cauchy sequence, with the same limit in \(A\) (if it exists) as \(\alpha \circ \{e'\}\) (by its definition (11)), to prove (4) it is enough to show (by (12)) that
\[
\alpha(\{e'(n)\}) \to F(\alpha(e)) \quad \text{as} \quad n \to \infty.
\]
This follows since
\[
d(\alpha(\{e'(n)\}, F(\alpha(e))) = d(\alpha(\{f_n(e)\}(n)), F(\alpha(e))) \quad \text{by def. (9) of } e' \\
\leq d(\alpha(\{f_n(e)\}(n)), \alpha(f_n(e))) + d(\alpha(f_n(e)), F(\alpha(e))) \\
< 2^{-n} + 2^{-n} \quad \text{by (7) and (8)} \\
= 2^{-n+1}
\]
proving (13). □

Remark 7.2.1. A deterministic version of Theorem A (i.e., without ‘choose’) was proved in [Ste98].
8 Interpretation of concrete in abstract model: Adequacy of WhileCC* approximation

8.1 Adequacy Theorem

In this section we will prove Theorem B, a converse to the result of the previous section. Assume that $A$ is an $N$-standard metric $\Sigma$-algebra, and $(X, \alpha)$ an enumerated $\Sigma$-subspace, with $\alpha$-computable closure $(C_\alpha(X), \overline{\alpha})$.

Note that we are not assuming in this section that $C_\alpha(X)$ is a subalgebra of $A$, or even that $\overline{\alpha}$ is $\Sigma$-effective.

Before stating the theorem, we need a definition.

Definition 8.1.1 ($\alpha$-effective local uniform continuity). A partial function $F : A^u \rightarrow A_s$ is effectively locally uniformly continuous (with respect to $\alpha$) if there is a recursive sequence

$$(k_0, l_0), (k_1, l_1), (k_2, l_2), \ldots$$

of pairs of naturals such that

$$\text{dom}(F) \subseteq \bigcup_{i=0}^{\infty} B_u(\alpha(k_i), 2^{-l_i})$$

and there is a total recursive function $\text{LU}_F : \mathbb{N}^2 \rightarrow \mathbb{N}$ (a modulus of local uniform continuity for $F$) such that for all $i$, all $x, y \in B_u(\alpha(k_i), 2^{-l_i}) \cap \text{dom}(F)$, and all $n$:

$$d_u(x, y) < 2^{-\text{LU}_F(i, n)} \implies d_s(F(x), F(y)) < 2^{-n}.$$ 

Here $B_u(a, \delta)$ is the open ball in $A^u$ with centre $a$ and radius $\delta$. (Recall the definition (2.6.3) of the product metric $d_u$ on $A^u$.)

Example 8.1.2. This phenomenon typically occurs in the situation where $A$ is a countable union of neighbourhoods with compact closure; for example, in the algebra $\mathcal{R}_p$ of reals, $\mathbb{R}$ is the union of the neighbourhoods $(-k, k)$ for $k = 1, 2, \ldots$. Then a continuous function $F$ on $A$ will be uniformly continuous on each of these neighbourhoods.

We are now ready for the theorem.

Theorem B (Adequacy). Let $A$ be an $N$-standard metric $\Sigma$-algebra, $(X, \alpha)$ an enumerated $\text{Sort}(\Sigma)$-subspace, and $(C_\alpha(X), \overline{\alpha})$ the $\text{Sort}(\Sigma)$-subspace of $\alpha$-computable elements of $A$. Suppose that for all $\Sigma$-sorts $s$:

(i) $X_s$ is dense in $A_s$, and

(ii) $\alpha_s : \mathbb{N} \rightarrow A_s$ is WhileCC* -computable on $A$.

Let $F : A^u \rightarrow A_s$ be a function on $A$ such that
(iii) $F$ is effectively locally uniformly continuous w.r.t. $\alpha$, and

(iv) $\text{dom}(F)$ is open.

If $F$ is strictly $\overline{\alpha}$-computable on $A$, then $F$ is $\textbf{WhileCC}^*$ approximable on $A$.

Note the extra condition in Theorem B (apart from assumptions (i)–(iv)), that $F$ be strictly $\overline{\alpha}$-computable.

Remark 8.1.3. From the proof of the theorem, it will be apparent that only sorts $s$ in the domain of $F$ have to satisfy condition (i), and only sorts $s$ in the domain or range of $F$ have to satisfy condition (ii).

The proof uses the following notation.

Notation 8.1.4. For any $k \in \mathbb{N}$, let $e_{\text{con}[k]}$ be a canonical index for the constant function on $\mathbb{N}$ with constant value $k$, i.e., for all $n \in \mathbb{N},$

$$\{e_{\text{con}[k]}\}(n) = k.$$ 

Note that $e_{\text{con}[k]} \in \Omega_{\overline{\alpha}}$ and $\overline{\alpha}(e_{\text{con}[k]}) = \alpha(k)$.

8.2 Proof of Theorem B: Overview

As an aid to the reader, we first give an informal overview of the proof of Theorem B. (See Figure 3.)
Given the assumptions (i) → (iv) of Theorem B, suppose \( F : A^u \rightarrow A_\alpha \) is strictly \( \bar{\alpha} \)-computable by \( f : \Omega^u_{\bar{\alpha}} \rightarrow \Omega^u_{\bar{\alpha},s} \). (In Figure 3, we represent \( f \) as mapping \( \Omega^u_{\alpha} \) to \( \Omega^u_{\alpha,s} \), rather than mapping \( \Omega^u_{\alpha} \) to \( \Omega^u_{\pi,s} \), as a useful approximation, as we will see.) We must describe a \( \text{WhileCC}^* \) procedure which approximates \( F \) on \( A \).

Let \( x \in A^u \). Suppose \( F(x) \downarrow y \). By the density of \( X = \text{ran}(\alpha^u) \) in \( A^u \), and by the openness of \( \text{dom}(F) \), for each \( n \) we can find (using the ‘choose’ operator, as well as the \( \text{WhileCC}^* \) computability of \( \alpha \)) an element \( k_n \) of \( \Omega^u_{\alpha} \) such that \( x_n =_{df} \alpha^u(k_n) \in \text{dom}(F) \), and also \( d(x_n, x) < 2^{-n} \).

Now compute an element \( l_n \) of \( \Omega^u_{\alpha} \) which is a close approximation to \( f(k_n) \), or rather to \( f(e_{\text{con}[k_n]}) \). More precisely, let \( e_n' =_{df} f(e_{\text{con}[k_n]}) \), and let \( l_n =_{df} \{e_n'(n) \}. \) Then \( d(\alpha(l_n), \bar{\alpha}(e_n')) < 2^{-n} \). Put \( y_n = \alpha(l_n) \).

We must now check that the mapping \( (x, n) \mapsto y_n \) defined above is \( \text{WhileCC}^* \) computable, and approximates \( F \). By effective local uniform continuity of \( F \), since \((x_n)_n \) is a fast Cauchy sequence with limit \( x \), \((y_n)_n \) is a Cauchy sequence with computable modulus of convergence and limit \( y \). Note also that \( \text{WhileCC}^* \) computability of \( y_n \) (as a function of \( x \) and \( n \)) uses the \( \text{WhileCC}^* \) computability of \( \alpha \). Hence we can define a \( \text{WhileCC}^* \) procedure \( P : \text{nat} \times u \rightarrow s \) with \( P^A(n, x) \) equal to the set of all such \( y_n \), obtainable in this way from all possible implementations of the ‘choose’ operator. Hence \( F \) is computably approximable by \( P \).

We turn to a precise proof of the theorem.

### 8.3 Proof of Theorem B

First we show, from assumption (iii), that \( F \) has a \( \text{WhileCC}^* \) modulus of continuity, i.e., a function

\[
\text{MC}_F : A^u \times \text{nat} \rightarrow \text{nat}
\]

such that \( \text{dom}(F) \subseteq \text{dom}(\text{MC}_F) \), and for all \( x, y \in \text{dom}(F) \) and for all \( n \),

\[
d(x, y) < 2^{-\text{MC}_F(x, n)} \implies d(F(x), F(y)) < 2^{-n}.
\]

A \( \text{WhileCC}^* \) algorithm for this is easily constructed as follows (using the notation of Definition 8.1.1). With input \( x \in A^u \) and \( n \): first find \( i \) such that

\[
x \in \text{B}(\alpha(k_i), 2^{-l_i}).
\]

(If \( x \notin \text{dom}(F) \), there may be no such \( i \), and the algorithm for \( \text{MC}_F(x, n) \) would then diverge, which is fine, from our viewpoint.) Note that the sequences \((k_i) \) and \((l_i) \) are computable, and also (by assumption (ii)) \( \alpha \) is \( \text{WhileCC}^* \) computable. We also use the primitive operations \( d \) and ‘<’ (partial!) on \( \mathbb{R} \), as well as the ‘choose’ construct, in “finding” a suitable \( i \).

Next (by (2)) find a natural number \( d_0 \) such that

\[
d(x, \alpha(k_i)) + 2^{-d_0} < 2^{-l_i}.
\]

(3)
Here again we use the **WhileCC***- computability of $\alpha$, and the primitive operations $d$, ‘$+$’ and ‘$<$ on $\mathbb{R}$, as well as the ‘choose’ construct, to find a suitable $d_0$.

From (2) and (3),

$$B(x, 2^{-d_0}) \subseteq B(\alpha(k_i), 2^{-l_i}).$$

So define

$$MC_F(x, n) := \max(d_0, LU_F(i, n))$$

which is **WhileCC***- computable, by the above remarks.

Now we will describe (in pseudo-**WhileCC***- code) an algorithm for a **WhileCC***-computable function

$$G : \mathbb{N} \times A \implies A_s \uparrow$$

which approximates $F$, in the sense that for all $n$ and all $x \in \text{dom}(F)$,

$$G_n(x) \subseteq B(F(x), 2^{-n}) \subseteq A_s. \quad (4)$$

With input $n, x$:

(1°) Compute

$$M := MC_F(x, n + 1). \quad (5)$$

(2°) We want to find some $k$ such that both

$$d(\alpha(k), x) < 2^{-M} \quad (6)$$

and

$$\overline{\alpha}(e_{\text{con}[k]}(\{e\}')(n + 1)) \subseteq \text{dom}(F). \quad (7)$$

Assume $x \in \text{dom}(F)$. By the density assumption (i) and openness assumption (iv), such a $k$ exists. Further, by assumption, $F$ has a computable strict $\overline{\alpha}$-tracking function $f$. Then (7) is equivalent to

$$f(e_{\text{con}[k]}) \downarrow. \quad (8)$$

So using the ‘choose’ construct again, search for some $k$ satisfying both (6) and (8). (Note that in practice this ‘choose’ operation would be implemented by dovetailing — recall the discussion in §4.1.)

(3°) Compute $f(e_{\text{con}[k]}) \downarrow e'$. By (7), $e' \in \Omega_\alpha$ and

$$F(\alpha(k)) = F(\overline{\alpha}(e_{\text{con}[k]})) = \overline{\alpha}(f(e_{\text{con}[k]})) = \overline{\alpha}(e').$$

Hence by (1), (5) and (6),

$$d(F(x), \overline{\alpha}(e')) = d(F(x), F(\alpha(k))) < 2^{-n - 1}. \quad (9)$$

(4°) Finally compute

$$y := \alpha(\{e\}'(n + 1)) \quad (10)$$
This is possible by assumption \((ii)\) again. Then, since \(\alpha \circ \{e'\}\) is a fast Cauchy sequence,

\[
d(y, \bar{\alpha}(e')) = d(\alpha(\{e'\}(n + 1)), \bar{\alpha}(e')) \leq 2^{-n-1}. \tag{11}
\]

Hence by (11) and (9),

\[
d(y, F(x)) \leq d(y, \bar{\alpha}(e')) + d(\bar{\pi}(e'), F(x)) < 2^{-n-1} + 2^{-n-1} = 2^{-n}.
\]

Define \(G_n(x)\) to be the set of all possible \(y\) computed as in (10), by all possible implementations of the ‘choose’ construct as used in the above algorithm. Then \(G\) satisfies (4), and is \textbf{WhileCC}* computable, by the above discussion. \(\square\)

\section{Completeness of \textbf{WhileCC}* approximation}

Under certain assumptions, we can combine Theorems A and B into a single equivalence, Theorem C below. We will then look at several examples of metric algebras where our abstract and concrete models are equivalent according to this Theorem.

\subsection{Effective openness}

Note first the following problem: Theorem A concludes with \(\bar{\alpha}\)-computability of \(F\), whereas Theorem B assumes \textit{strong} \(\bar{\pi}\)-computability. To deal with this, we must make an assumption of “effective openness” of \(\text{dom}(F)\). This is handled by strengthening the “effective local uniform continuity” assumption, as follows.

Assume, as before, that \(A\) is an N-standard metric \(\Sigma\)-algebra, \((X, \alpha)\) is an enumerated \(\Sigma\)-subspace of \(A\), and \((C_{\alpha}(X), \bar{\pi})\) is its computable closure in \(A\).

\begin{definition}[(\(\alpha\)-effective openness)]
A subset \(U\) of \(A^u\) (\(u\) a \(\Sigma\)-product type) is \textit{effectively open} (with respect to \(\alpha\)) if there is a recursive sequence

\[
(k_0, l_0), (k_1, l_1), (k_2, l_2), \ldots
\]

of pairs of naturals such that

\[
U = \bigcup_{i=0}^{\infty} B_u(\alpha(k_i), 2^{-l_i}).
\]
\end{definition}

\begin{definition}[(Strong \(\alpha\)-effective local uniform continuity)]
A partial function \(F : A^u \longrightarrow A_s\) is \textit{strongly effectively locally uniformly continuous} (with respect to \(\alpha\)) if there is a recursive sequence

\[
(k_0, l_0), (k_1, l_1), (k_2, l_2), \ldots
\]


of pairs of naturals such that
\[
\text{dom}(F) = \bigcup_{i=0}^{\infty} B_u(\alpha(k_i), 2^{-l_i})
\] (1)
and there is a total recursive function \(LU_F : \mathbb{N}^2 \rightarrow \mathbb{N}\) (a modulus of local uniform continuity for \(F\)) such that for all \(i\), all \(x, y \in B_u(\alpha(k_i), 2^{-l_i})\), and all \(n\):
\[
d(x, y) < 2^{-LU_F(i,n)} \implies d(F(x), F(y)) < 2^{-n}.
\]

**Remark 9.1.3.** The only difference between effective local uniform continuity (Definition 8.1.1) and the “strong” version above is the equality in equation (1).

Let \(F : A^u \rightarrow A_s\) be a function on \(A\). Then clearly:

**Lemma 9.1.4.** Strong \(\alpha\)-effective local uniform continuity of \(F\) implies \(\alpha\)-effective openness of \(\text{dom}(F)\).

**Lemma 9.1.5.** Suppose \(\text{dom}(F)\) is \(\alpha\)-effectively open, and \(\overline{\alpha}\) is strictly \(\Sigma\)-effective. Then
\[
F \text{ is } \overline{\alpha}\text{-computable } \iff F \text{ is strictly } \overline{\alpha}\text{-computable.}
\]

**Proof:** (\(\Rightarrow\)) Note first that the assumptions imply that
\[
\text{dom}(F) = \{ \overline{\alpha}(e) \in \text{dom}(F) \text{ s.t. } d(\overline{\alpha}(e), \alpha(k_i)) < 2^{-l_i} \}
\]
is an r.e. (recursively or computably enumerable) subset of \(\mathbb{N}\), since for all \(e \in \mathbb{N}\)
\[
e \in \text{dom}(F) \iff \exists i \left[ d(\overline{\alpha}(e), \alpha(k_i)) < 2^{-l_i} \right]
\]
in the notation of Definition 9.1.2) which is an r.e. condition, by strict \(\overline{\alpha}\)-computability of \(d\) and \text{less}_\text{real} (implied by strict \(\Sigma\)-effectiveness of \(\overline{\alpha}\)). Hence, if \(f\) is a computable \(\overline{\alpha}\)-tracking function for \(F\), it can be replaced by a strict \(\overline{\alpha}\)-tracking function \(f'\), defined by
\[
f'(e) \simeq \left\{ \begin{array}{ll} f(e) & \text{if } e \in \text{dom}(F) \\ \uparrow & \text{otherwise} \end{array} \right.
\]
which is easily seen to be computable. \(\Box\)

**Lemma 9.1.6.** Suppose \(\text{dom}(F)\) is \(\alpha\)-effectively open, and the mappings \(\alpha_s : \mathbb{N} \rightarrow A_s\) are \(\text{WhileCC}^*\) computable. Then
\[
F \text{ is } \text{WhileCC}^*\text{-approximable } \iff F \text{ is strictly } \text{WhileCC}^*\text{-approximable.}
\]
(Recall Definition 3.5.1.) The proof is an easy exercise.

### 9.2 Completeness

We are ready to state the completeness theorem for \(\text{WhileCC}^*\) approximability relative to \(\overline{\alpha}\)-computability.
Theorem C (Completeness). Let $A$ be an $N$-standard metric $\Sigma$-algebra, and $(X, \alpha)$ an enumerated $\text{Sort}(\Sigma)$-subspace. Suppose the enumerated $\text{Sort}(\Sigma)$-space $(C_\alpha(X), \overline{\alpha})$ of $\alpha$-computable elements of $A$ is a $\Sigma$-subalgebra of $A$. Assume also that for all $\Sigma$-sorts $s$,

(i) $\overline{\alpha}$ is strictly $\Sigma$-effective,

(ii) $X_s$ is dense in $A_s$, and

(iii) $\alpha_s : \mathbb{N} \to A_s$ is $\text{WhileCC}^*$-computable on $A$.

Let $F : A^u \to A_s$ be a function on $A$, such that

(iv) $F$ is strongly effectively locally uniformly continuous w.r.t. $\alpha$.

Then $F$ is (strictly) $\text{WhileCC}^*$ approximable on $A \iff F$ is (strictly) $\overline{\alpha}$-computable on $A$.

Note that the word “strictly” in the equivalence may be omitted or inserted in either side at will.

Proof: From Theorems A and B, together with Lemmas 9.1.4, 9.1.5 and 9.1.6.

9.3 Examples of the application of the Completeness Theorem

(a) Canonical enumerations

The purpose of this example is to make plausible condition (iii) of Theorem C (and, of course, condition (ii) of Theorem B in Section 8), i.e., the assumption of $\text{WhileCC}^*$ computability of the enumeration $\alpha$, by describing a commonly occurring situation which implies it.

Suppose $(X, \alpha)$ is an enumerated $\Sigma$-subalgebra of $A$.

Definition 9.3.1. The enumeration $\alpha : \mathbb{N} \to X$ is effectively determined by a system of generators $G = \langle g_0^s, g_1^s, g_2^s, \ldots \rangle_{s \in \text{Sort}(\Sigma)}$ if, and only if,

(i) $G$ generates $X$ as a $\Sigma$-subalgebra of $A$;

(ii) $\alpha$ is defined as the composition of the maps

\[
\mathbb{N} \xrightarrow{\text{enum}_\Sigma} \text{Term}(\Sigma) \xrightarrow{\text{eval}_G} X
\]

where $\text{enum}_\Sigma$ is the inverse of the Gödel numbering of $\text{Term}(\Sigma)$, and $\text{eval}_G$ is the term evaluation induced by $G$, i.e.,

\[
\text{eval}_G(t) = \llbracket t \rrbracket_{\sigma_G},
\]

where $\sigma_G$ is the state defined by

\[
\sigma_G(x^s_i) = g^s_i
\]
for some standard enumeration \( x_0^s, x_1^s, x_2^s, \ldots \) of the \( \Sigma \)-variables of sort \( s \); and

(iii) if, for any \( \Sigma \)-sort \( s \), the sequence \( \langle g_0^s, g_1^s, g_2^s, \ldots \rangle \) is finite, then each \( g_i^s \) is a \( \Sigma \)-constant, whereas if this sequence is infinite, then the map \( i \mapsto g_i^s \) is a \( \Sigma \)-function.

An enumeration constructed in this way is called canonical w.r.t. \( G \).

**Remark 9.3.2 (Totality of \( eval_G \)).** We assume here that \( eval_G \) (and hence \( \alpha \)) is total. This is achieved by assuming that either

(i) \( A \) is total, or

(ii) \( \text{Term}(\Sigma) \) is replaced by some decidable subset \( \text{Term}'(\Sigma) \) on which \( eval_G \) is total (for example, omitting all terms involving division by 0).

Either one of these assumptions holds in each of the following examples; for example, (i) holds in example (b) below, and (ii) in example (c), resulting in the same “canonical” enumeration \( \alpha \) of \( \mathbb{Q} \) in both cases (even though the algebras are different).

**Proposition 9.3.3.** If \( \alpha \) is effectively determined by a system of generators, then the canonical enumerations \( \alpha_s \) are While\(^*\) computable for all \( \Sigma \)-sorts \( s \).

**Proof:** This follows from the While\(^*\) computability of term evaluation [TZ00, Cor. 4.7]. \( \square \)

The significance of the above definition and proposition is this: it is quite common for an enumeration to be effectively determined by a system of generators; and in such a situation, condition (ii) in Theorem B, and (iii) in Theorem C, will be (more than) satisfied. This will be the case in the following examples.

(b) Partial real algebra

Recall the example (7.1.5) of the enumeration \( \alpha \) of \( \mathbb{Q} \) as a subspace of the N-standardised metric algebra \( \mathbb{R}^N \) of reals (Examples 2.5.3(b) and 2.6.1) and the corresponding enumeration \( \overline{\pi} \) of the set \( C_\alpha(Q) \) of recursive reals. Note that \( \alpha \) is canonical, being effectively determined by the generators \( \{0, 1\} \), and is hence While\(^*\) computable over \( \mathbb{R} \). Further, \( \mathbb{Q} \) is dense in \( \mathbb{R} \), \( C_\alpha(Q) \) is a subfield of \( \mathbb{R} \), and \( \overline{\pi} \) is strictly \( \Sigma(\mathbb{R}) \)-effective. We then have, as a corollary to Theorem C:

**Corollary 9.3.4.** Suppose \( F: \mathbb{R}^n \rightarrow \mathbb{R} \) is strongly effectively locally uniformly continuous. Then

\[
F \text{ is (strictly) WhileCC\(^*\)-approximable on } \mathbb{R}^N \iff F \text{ is (strictly) } \overline{\pi}\text{-computable on } \mathbb{R}.
\]

Examples of functions satisfying the assumption (and also the equivalence) are all the common (partial) functions of elementary calculus, such as \( 1/x \), \( \log x \) and \( \tan x \).
(c) Banach spaces with countable bases

Let $X$ be a Banach space over $\mathbb{R}$ with a countable basis $e_0, e_1, e_2, \ldots$, which means that any element $x \in X$ can be represented uniquely as an infinite sum

$$x = \sum_{i=0}^{\infty} r_i e_i$$

with coefficients $r_i \in \mathbb{R}$ (where the infinite sum is understood as denoting convergence of the partial sums in the norm of $X$). (Background on Banach space theory can be found in any of the standard texts, e.g., [Roy63, TL80].) To program with $X$, we construct a many-sorted algebra $\mathcal{X}$ of the form

```
| algebra   | $\mathcal{X}$ |
|---|---|
| import   | $\mathbb{R}^N$ |
| carriers | $X$ |
| functions| $0: \rightarrow X,$ |
|         | $+ : X^2 \rightarrow X,$ |
|         | $- : X \rightarrow X,$ |
|         | $\odot : \mathbb{R} \times X \rightarrow X,$ |
|         | $\| \cdot \| : X \rightarrow \mathbb{R},$ |
|         | $e : \mathbb{N} \rightarrow X,$ |
|         | if$_X : \mathbb{B} \times X^2 \rightarrow X$ |
```

where $\odot$ is scalar multiplication, $\| \cdot \|$ is the norm function and and $e$ is the enumeration of the basis: $e(i) = e_i$. Note that the algebras $\mathcal{B}$ and $\mathcal{N}$ are implicitly imported, as parts of $\mathbb{R}^N$, so that there are four carriers: $X$, $\mathbb{R}$, $\mathbb{B}$ and $\mathbb{N}$, of sorts vector, scalar, bool and nat respectively.

Let $\Sigma = \Sigma(\mathcal{X})$. Let $\Sigma_0$ be $\Sigma$ without the norm function $\| \cdot \|$, and let $\mathcal{X}_0$ be the reduct of $\mathcal{X}$ to $\Sigma_0$. Then $\Sigma_0$ is the signature of an N-standardised vector space over $\mathbb{R}$, with explicit countable basis.

This can be turned into a metric algebra in the standard way, by defining a distance function on $X$ in terms of the norm:

$$d(x, y) = df \| x - y \|.$$ 

Let $L(\mathbb{Q}, e) \subset X$ be the set of all finite linear combinations of basis elements from $e$ with coefficients in $\mathbb{Q}$. The following are easily shown:

- $L(\mathbb{Q}, e)$ is countable; in fact it has a canonical enumeration

  $$\alpha : \mathbb{N} \rightarrow L(\mathbb{Q}, e)$$

  w.r.t. the generators $e$, which (by (a) above) is While* computable;
• \( L(\mathbb{Q}, e) \) is dense in \( X \);

• \( L(\mathbb{Q}, e) \), with scalar field \( \mathbb{Q} \) (together with carriers \( \mathbb{N} \) and \( \mathbb{B} \)) is a \( \Sigma_0 \)-subalgebra of \( \mathcal{X}_0 \).

Now let \( (C_\alpha(L(\mathbb{Q}, e)), \overline{\alpha}) \) be the enumerated subspace of \( \alpha \)-computable vectors. Then we can see that

• \( C_\alpha(L(\mathbb{Q}, e)), \) with scalar field \( C_\alpha(\mathbb{Q}) \) (together with carriers \( \mathbb{N} \) and \( \mathbb{B} \)) is also a \( \Sigma_0 \)-subalgebra of \( \mathcal{X}_0 \); and moreover,

• \( \overline{\alpha} \) is strictly \( \Sigma_0 \)-effective.

However \( C_\alpha(L(\mathbb{Q}, e)) \) is not necessarily a normed subspace of \( \mathcal{X} \), since it may not be closed under \( \| \cdot \| \), i.e., \( \| x \| \) may not be in \( C_\alpha(\mathbb{Q}) \) for all \( x \in C_\alpha(L(\mathbb{Q}, e)) \); for example, if \( \mathcal{X} \) is the space \( \ell^p \) or \( L^p[0,1] \) where \( p \) is a nonrecursive real (see Examples 9.3.8 below). We must therefore make an explicit assumption for the Banach space \( (X, \| \cdot \|) \) with respect to both the closure of \( C_\alpha(L(\mathbb{Q}, e)) \) under \( \| \cdot \| \), and the \( \overline{\alpha} \)-computability of \( \| \cdot \| \).

Assumption 9.3.5 (\( \overline{\alpha} \)-computable norm assumption for \( (X, \| \cdot \|) \)).

For all \( x \in C_\alpha(L(\mathbb{Q}, e)) \), \( \| x \| \in C_\alpha(\mathbb{Q}) \). Furthermore, the norm function \( \| \cdot \| \) is strictly \( \overline{\alpha} \)-computable.

As we will see, many common examples of Banach spaces satisfy this assumption.

Note that assumption 9.3.5 is equivalent to the following (apparently weaker) assumption, which is often easier to prove:

Assumption 9.3.6 (\((\alpha, \overline{\alpha})\)-computable norm assumption for \( (X, \| \cdot \|) \)). For all \( x \in L(\mathbb{Q}, e) \), \( \| x \| \in C_\alpha(\mathbb{Q}) \). Further, \( \| \cdot \| \) has a computable \((\alpha, \overline{\alpha})\)-tracking function, i.e., a computable function \( f : \mathbb{N} \to \mathbb{N} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
L(\mathbb{Q}, e) & \xrightarrow{\| \cdot \|} & C_\alpha(\mathbb{Q}) \\
\vert \quad \alpha \quad \vert & & \vert \quad \overline{\alpha} \quad \vert \\
\mathbb{N} & \xrightarrow{f} & \Omega_{\overline{\alpha}}
\end{array}
\]

Suppose now that \( (X, \| \cdot \|) \) satisfies the \( \overline{\alpha} \)-computable norm assumption. Then the \( \Sigma_0 \)-subalgebra \( C_\alpha(L(\mathbb{Q}, e)) \) of \( \mathcal{X}_0 \) can be expanded to a \( \Sigma \)-subalgebra of \( \mathcal{X} \) (which we will also write as \( C_\alpha(L(\mathbb{Q}, e))) \), enumerated by \( \overline{\alpha} \), which is strictly \( \Sigma \)-effective.

Now let \( F : X \to \mathbb{R} \) be a (total) linear functional on \( X \). \( F \) is said to be bounded if for some real \( M \),

\[
|F(x)| \leq M \| x \| \quad \text{for all } x \in X.
\]

Write \( \| F \| \) for the least \( M \) for which (1) holds. Then if \( F \) is bounded,

\[
|F(x) - F(y)| \leq \| F \| \cdot \| x - y \| \quad \text{for all } x, y \in X,
\]
and so $F$ is uniformly continuous, in fact it is clearly effectively locally uniformly continuous, and strongly so (since it is total). We may therefore apply Theorem C to $F$.

**Corollary 9.3.7 (Completeness for computation on Banach spaces).** Let $X$ be a Banach space over $\mathbb{R}$ with countable basis, and let $C_{\alpha}(L(\mathbb{Q}, e))$ be the enumerated subspace of $\alpha$-computable vectors, where $\alpha$ is a canonical enumeration of the subspace $L(\mathbb{Q}, e)$. Suppose $(X, \| \cdot \|)$ satisfies the $(\alpha, \overline{\alpha})$-computable norm assumption. Then for any bounded linear functional $F$ on $X$,

$F$ is (strictly) WhileCC* approximable on $\mathcal{X} \iff F$ is (strictly) $\overline{\alpha}$-computable on $X$,

where $\mathcal{X}$ is the $N$-standard algebra formed from $X$ as above.

Finally we give examples of Banach spaces which satisfy this $\overline{\alpha}$-computable norm assumption.

**Examples 9.3.8 (Banach spaces with computable norms).**

(i) For $1 \leq p < \infty$, we have the space $\ell^p$ of all sequences $x = \langle x_n \rangle_{n=0}^\infty$ of reals such that $\sum_{n=0}^\infty |x_n|^p < \infty$, with norm defined by

$$\|x\|_p = \left( \sum_{n=0}^\infty |x_n|^p \right)^{1/p},$$

and a countable basis given by $e_i = \langle e_{i,n} \rangle_{n=0}^\infty$, where

$$e_{i,n} = \begin{cases} 1 & \text{if } i = n, \\ 0 & \text{otherwise}. \end{cases}$$

It is not hard to see that

if $p$ is a recursive real, then $\ell^p$ satisfies the computable norm assumption,

and hence Corollary 9.3.7 can be applied to it.

(ii) For $1 \leq p < \infty$, we have the space $L^p[0,1]$ of all Lebesgue measurable functions $f$ on the unit interval $[0,1]$ such that $\int_0^1 |f|^p < \infty$, with norm defined by

$$\|f\|_p = \left( \int_0^1 |f|^p \right)^{1/p},$$

and a countable basis given by (e.g.) some standard enumeration of all step functions on $[0,1]$ with rational values and (finitely many) rational points of discontinuity, or of all polynomial functions on $[0,1]$ with rational coefficients. Again, it is not hard to see that

if $p$ is a recursive real, then $L^p[0,1]$ satisfies the computable norm assumption,
and hence Corollary 9.3.7 can be applied to it.

(iii) The space $C[0, 1]$ of all continuous functions $f$ on $[0, 1]$, with norm defined by

\[ \|f\|_{\text{sup}} = \sup_{t \in I} |f(t)| \]

and a countable basis given by a standard enumeration of all zig-zag functions on $[0, 1]$ with (finitely many) turning points with rational coordinates, or of all polynomial functions on $[0, 1]$ with rational coefficients. Again, we see that $C[0, 1]$ satisfies the computable norm assumption.

10 Conclusion

We have compared two theories of computable functions on topological algebras, one based on an abstract, high level model of programming and another based on a concrete, low-level implementation model. Our examples and results here, combined with our earlier results [TZ99, TZ00] and those of Brattka [Bra96, Bra99], show that the following are surprisingly necessary features of a comprehensive theory of computation on topological algebras:

1. The algebras have partial operations.
2. Functions are both continuous and multivalued.
3. Classical algorithms in analysis require nondeterministic constructs for their proper expression in programming languages.
4. Indeed, multivalued subfunctions are needed to compute even single-valued functions, and abstract models must be nondeterministic even to compute deterministic problems.
5. Abstract models and effective approximations by abstract models are generally sound for concrete models.
6. Abstract models even with approximation or limit operators are adequate to capture concrete models only in special circumstances.
7. Nevertheless there are interesting examples where equivalence holds.
8. The classical computable functions of analysis can be characterised by abstract models of computation.

Specifically, we examined abstract computation by the basic imperative model of ‘while’-array programs. Many algorithms in practical computation are presented in pseudo-code based on the ‘while’ language. To meet the requirement of feature 2 above we added the simplest form of countable choice to the assignments of the language, and we defined the $\text{WhileCC}^*$ approximable computations. We proved a Soundness Theorem (Theorem A) and an Adequacy Theorem (Theorem B), and combined these into a Completeness Theorem (Theorem C), in the case of metric algebras with partial operations. We considered algebras of real numbers and Banach spaces where equivalence theorems hold.

There are, of course, interesting technical questions to answer in working out the details of the computability theory for the $\text{WhileCC}^*$ model (cf. the theory for single-valued
functions on total algebras in [TZ00]). There are several other important abstract models of computation that may be extended with nondeterministic constructs in order to establish equivalence with concrete models. The abstract model of schemes in [Bra99] is quite general in a number of ways. The topological properties of many valued functions are also in need of investigation.

However, returning to the general problem posed in the Introduction, the features 1–8 above suggest that new research directions are needed to develop a comprehensive theory of specification, computation and reasoning with infinite data. What are the appropriate programming constructs for working with topological computations? What specification techniques are appropriate for continuous systems? What logics are needed to support verification of programs that approximate functions? Our work on computation suggests that some advanced semantic features are necessary. It suggests that the nondeterminism that played an important role in programming methodologies of the late 1970s (e.g., [Dij76] seems to be needed in the proper development of topological programming. There are plenty of algorithms in scientific modelling, numerical analysis and graphics to investigate, using such new theoretical tools.

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