Differentially private depth functions and their associated medians

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Abstract

In this paper, we investigate the differentially private estimation of data depth functions and their associated medians. We start with several methods for privatizing depth values at a fixed point, and show that for some depth functions, when the depth is computed at an out of sample point, privacy can be gained for free when \( n \to \infty \). We also present a method for privately estimating the vector of sample depth values, and show that privacy is not gained for free asymptotically. We also introduce estimation methods for depth-based medians for both depth functions with low global sensitivity and depth functions with only highly probably, low local sensitivity. We provide a general Theorem (Lemma 1) which can be used to prove consistency of an estimator produced by the exponential mechanism, provided the asymptotic cost function is uniquely minimized and is sufficiently smooth. We introduce a general algorithm to privately estimate minimizers of a cost function which has low local sensitivity, but high global sensitivity. An application of this algorithm to generate consistent estimates of the projection depth-based median is presented. For these private depth-based medians, we show that it is possible for privacy to be free when \( n \to \infty \).

Keywords — Differential Privacy, Depth function, Multivariate Median, Propose-test-release

1 Introduction

There is a large body of literature that shows simply removing the identifying information about subjects from a database is not enough to ensure data privacy [see Dwork et al., 2017, and the references therein]. Even if only certain summary statistics are released, an adversary can still learn a surprising amount about individuals in a database [Dwork et al., 2017]. This phenomena is largely due to auxiliary information that is known by the adversary. Given the large amount of information about individuals that is publicly available, it is not infeasible to assume that an adversary already knows some information about the individual they wish to learn about. On the contrary, if a statistic is differentially private an adversary cannot learn about the attributes of specific individuals in the original database, regardless of the amount of initial information the adversary possesses. This property, coupled with the lack of assumptions on the data itself needed to ensure privacy, accounts for the volume of recent literature on differentially private statistics.

One part of this literature represents a growing interest in the statistical community in differentially private inference [Wasserman and Zhou, 2010, Awan et al., 2019, Cai et al., 2019, Brunel and Avella-Medina, 2020, see, e.g.,]. One burgeoning area is the connection between robust statistics and differentially private statistics, first discussed by Dwork and Lei [2009]. Private M-estimators were studied by several authors [Lei, 2011, Avella-Medina, 2019]. A connection between private estimators and gross error sensitivity was formalized by Chaudhuri and Hsu [2012], who present upper and lower bounds on the convergence of differentially private estimators in relation to their gross error sensitivity. The connection between private estimators and gross error sensitivity has been further exploited in order to construct differentially private statistics [Avella-Medina, 2019]. Brunel and Avella-Medina [2020] greatly expanded the propose-test-release paradigm of Dwork and Lei [2009] using the concept of the finite sample breakdown point. The same authors use this idea to construct private median estimators with sub-Gaussian errors [Avella-Medina and Brunel, 2019]. Our present work is inspired by these recent papers, where we explore the privatization of depth functions, a robust and nonparametric data analysis tool; given the recent success of robust procedures in the private setting, it is worthwhile to develop and study privatized depth functions and associated medians.

Depth functions facilitate the extension of, among other things, medians and rank statistics to the multivariate setting. The robustness properties of depth functions, including breakdown and gross error sensitivities, are well studied and favourable [Romanazzi, 2001, Chen and Tyler, 2002, Zuo, 2004, Dang et al., 2009], making them a
promising direction of study for use in the private setting. We take the some of the first steps in privatizing depth based inference. The contributions are as follows

- We present several approaches for the privatization of sample depth functions, including a discussion of advantages and disadvantages of each approach.
- We present algorithms for the private release of sample depth values of several popular depth functions. These include halfspace depth [Tukey, 1974], simplicial depth [Liu, 1990], IRW depth [Ramsay et al., 2019] and projection depth [Zuo, 2003]. Our algorithms and analysis can also be applied to depth functions with similar characteristics. We present asymptotic results concerning these private, depth value estimates, showing that pointwise, private depth values can be consistently estimated.
- We present algorithms for generating consistent, private depth-based medians, using the exponential mechanism and the propose-test-release framework of [Dwork and Lei, 2009, Brunel and Avella-Medina, 2020].
- We extend the propose-test-release algorithm of Brunel and Avella-Medina [2020] to be used with the exponential mechanism. We present a general algorithm for releasing a private maximizer of an objective function (or minimizer of a cost function) which may have infinite global sensitivity. The objective function should be such that observing high local sensitivity is unlikely.
- We present a lemma that can be used to prove weak consistency of private estimators generated from the exponential mechanism, even when the cost function is not necessarily differentiable.

Some work has been done surrounding the private computation of halfspace depth regions and the halfspace median [Beimel et al., 2019, Gao and Sheffet, 2020] from a computational geometry point of view. Though Beimel et al. [2019] mentions that the Tukey depth function can be used with the exponential mechanism, they do not study the estimator’s properties from a statistical point of view; it is used as a method of finding a point in the convex hull of a set of points. To the best of our knowledge, no one has attempted to privatize other depth functions.

2 Differential Privacy

In this section we introduce the fundamentals of differential privacy. Two essential concepts are that of a mechanism and that of adjacent databases. In order for a statistic (or database) to be differentially private, it must be stochastically computed [Dwork and Roth, 2014]. This differs from typical data analysis in that all differentially private statistics are generated from a distribution rather than being deterministically computed. In other words, all differentially private statistics $\tilde{T}(X_n)$ admit measures $Q_{\tilde{X}_n}$ given the data $X_n$. We assume here that the data is a random sample of size $n$ such that each observation is in $\mathbb{R}^d$, we will denote the sample by $X_n = \{X_1, \ldots, X_n\}$. We call the procedure that determines $Q_{\tilde{X}_n}$ and then outputs $\tilde{T}(X_n) \sim Q_{\tilde{X}_n}$ a mechanism. We may also refer to the mechanism by $\tilde{T}$ with an abuse of notation.

Along with mechanisms, we also must define adjacent databases. We say that $X_n$ and $Y_n$ (another random sample of size $n$) are adjacent if they differ by one observation. In other words, $X_n$ and $Y_n$ are adjacent if they have symmetric difference equal to one. Equipped with these concepts, we can define differential privacy:

**Definition 1.** A mechanism $\tilde{T}$ is $\epsilon$-differentially private for $\epsilon > 0$ if

$$\frac{Q_{\tilde{X}_n}(B)}{Q_{\tilde{Y}_n}(B)} \leq e^\epsilon$$

holds for all measurable sets $B$ and adjacent $X_n$ and $Y_n$.

The parameter $\epsilon$ should be small, implying that

$$\frac{Q_{\tilde{X}_n}(B)}{Q_{\tilde{Y}_n}(B)} \approx 1,$$

which gives the interpretation that the two measures $Q_{\tilde{X}_n}$ and $Q_{\tilde{Y}_n}$ are almost equivalent. To understand this definition, it helps to think of the problem from the adversary’s point of view. Suppose that we are the adversary and that we have access to all the entries in the database except for one, call it $\theta$, which we are trying to learn about. If $\tilde{T}$ is released, how can we use it to conduct inference about $\theta$? In order to test $H_0: \theta = \theta_0 \text{ vs. } H_1: \theta = \theta_1$ we, as statisticians, would then ask two questions:

*How likely was it to observe $\tilde{T}$ under $H_0$? and How likely was it to observe $\tilde{T}$ under $H_1$?*
Differential privacy stipulates that both of these questions have practically the same answer, making it impossible to infer anything about $\theta$ from $\tilde{T}$. Definition 1 implies that if someone in the dataset was replaced, we are just as likely to have seen $\tilde{T}$ (or some value very close to $\tilde{T}$ if $Q_{X_n}$ is continuous). Another way to interpret the definition is to observe that differential privacy implies that $\text{KL}(Q_{X_n}, Q_{Y_n}) < \epsilon$, where $\text{KL}$ is the Kullback–Leibler divergence; implying that the distributions are necessarily close. Differential privacy is a worst case restriction, in that the inequality covers all databases, and all possible outcomes of the mechanism.

Definition 1 can be difficult to satisfy because the umbrella of 'all databases and mechanism outputs' can include both some extreme databases and extreme mechanism outputs. One may wish to relax this definition over unlikely mechanism outputs; one way to do this is if $B$ is such that $Q_{X_n}(B)$ is very small, then the bound could be allowed to fail. This is called approximate differential privacy or $(\epsilon, \delta)$-differential privacy, in which we have

$$Q_{X_n}(B) \leq e^{\epsilon} Q_{Y_n}(B) + \delta$$

in place of the condition (1). Typically, $\delta < \epsilon$, and $\delta$ can be interpreted as the probability under which the bound is allowed to fail. To see this, observe that for $B$ such that $Q_{X_n}(B) < \delta$, (2) holds regardless of $\epsilon$. We mention that for remainder of the paper, $\epsilon$ and $\delta$ are always assumed to be positive and that sometimes we may have that the privacy parameters are a function of the sample size, and we indicate this with a subscript $n$; $\epsilon_n$, $\delta_n$.

Central to many private algorithms is the concept of sensitivity. Consider some function $T: (\mathbb{R}^d)^n \rightarrow \mathbb{R}^k$ where $(\mathbb{R}^d)^n$ denotes the sample space. Usually $T$ represents a statistic or a data driven objective function. Sensitivity measures how sensitive $T$ is to exchanging one sample point for another. Two important types of sensitivity are local sensitivity and global sensitivity, which are defined as

$$
\text{LS}(T; X_n) = \sup_{Y_n} \| T(X_n) - T(Y_n) \| \quad \text{and} \quad \text{GS}(T) = \sup_{X_n, Y_n} \| T(X_n) - T(Y_n) \| .
$$

In some cases, it is necessary to use different norms and so we add the subscript $G_p$ to indicate global sensitivity computed with respect to the $p$-norm.

We can now introduce some important building blocks of differentially private algorithms. Let $W_1, \ldots, W_k, \ldots$ and $Z_1, \ldots, Z_k, \ldots$ represent a sequence of independent, standard Laplace random variables and a sequence of independent, standard Gaussian random variables respectively. The Laplace and Gaussian mechanisms are essential differentially private mechanisms; they define how much an estimator must be perturbed in order for it to be differentially private.

**Mechanism 1** (Dwork et al. [2006]). Given a statistic $T: (\mathbb{R}^d)^n \rightarrow \mathbb{R}^k$, the mechanism that outputs

$$
\tilde{T}(X_n) = T(X_n) + (W_1, \ldots, W_k)\text{GS}_1(T)/\epsilon
$$

is $\epsilon$-differentially private.

**Mechanism 2** (Dwork et al. [2006], Dwork and Roth [2014]). Given a statistic $T: (\mathbb{R}^d)^n \rightarrow \mathbb{R}^k$, the mechanism that outputs

$$
\tilde{T}(X_n) = T(X_n) + (Z_1, \ldots, Z_k) \sqrt{2\log(1.25/\delta)}\text{GS}_2(T)/\epsilon
$$

is $(\epsilon, \delta)$-differentially private.

This can be improved in strict privacy scenarios [Balle and Wang, 2018]. We can also add noise based on smooth sensitivity [Nissim et al., 2007]. Using smooth sensitivity allows the user to leverage improbable, worst case local sensitivities. Often in practice, statistics are computed by maximizing a data driven objective function $\phi_{X_n}(\cdot)$. We can privatize such a procedure via the exponential mechanism. The exponential mechanism can be defined as follows:

**Mechanism 3** (McSherry and Talwar [2007]). Given the data, consider a function $\phi_{X_n}: \mathbb{R}^d \rightarrow \mathbb{R}$ and define the global sensitivity of such a function as $\text{GS}(\phi) = \sup_{X_n, Y_n} \| \phi_{X_n} - \phi_{Y_n} \|_{\infty}$. Then a random draw from the density $f(v; \phi_{X_n}, \epsilon)$ that satisfies

$$
f(x; \phi_{X_n}, \epsilon) \propto \exp \left( \frac{\epsilon \phi_{X_n}(x)}{2\text{GS}(\phi)} \right),
$$

is an $\epsilon$-differentially private mechanism. It is assumed that

$$
\int_{\mathbb{R}^d} \exp \left( \frac{\epsilon \phi_{X_n}(x)}{2\text{GS}(\phi)} \right) dx < \infty.
$$
The factor of 2 can be removed if the normalizing term is independent of the sample. All of the mechanisms discussed so far require that the statistic has finite global sensitivity. This is a somewhat strict requirement; under the normal model neither the sample mean nor sample median have finite global sensitivity. The sample median does, however, have low local sensitivity, viz.

\[ \text{LS}(\text{Med}(X_n)) \leq |F_n^{-1}(1/2 - 1/n) - F_n^{-1}(1/2 + 1/n)|, \]

where \( F_n^{-1} \) refers to the left continuous quantile function for a distribution \( F \) and \( F_n \) is the empirical distribution of a univariate sample \( X_n \). Since \( 1/n \to 0 \), we expect this value to be small (assuming the sample comes from a distribution which is continuous at its median). Throughout the paper we define the median of a continuous distribution by \( \text{Med}(F) = F^{-1}(1/2) \). Both \( \text{Med}(X_n) \) and \( \text{Med}(F_n) \) are taken to be the usual sample median.

The propose-test-release mechanism, or PTR, can be used to generate private versions of statistics with infinite global sensitivity but highly probable low local sensitivity. The propose-test-release idea was introduced by Dwork and Lei [2009] but was greatly expanded in the recent paper by Brunel and Avella-Medina [2020]. The PTR algorithm of Brunel and Avella-Medina [2020] relies on the truncated breakdown point \( A_\eta \), which is the minimum number of points that must be changed in order to move an estimator by \( \eta \).

\[ A_\eta(T; X_n) = \min \left\{ k : \sup_{Y_n \in D(X_n, k)} |T(X_n) - T(Y_n)| > \eta \right\}, \tag{3} \]

where \( D(X_n, k) \) is the set of all samples that differ from \( X_n \) by \( k \) observations. Unlike the traditional breakdown point, the dependence of \( A_\eta(T; X_n) \) on \( X_n \) is important. PTR works by proposing a statistic, testing if it is insensitive and then releasing it if it is, in fact, insensitive. A private version of \( A_\eta(T; X_n) \) is used to check the sensitivity.

**Mechanism 4.** Given a statistic \( T : (\mathbb{R}^d)^n \to \mathbb{R}^k \), the mechanism that outputs

\[ \tilde{T}(X_n) = \begin{cases} \bot & \text{if } A_\eta(T; X_n) + \frac{1}{\epsilon} W_1 \leq 1 + \frac{\log(2/\delta)}{\delta} \text{ o.w.} \end{cases} \]

is \((2\epsilon, \delta)\) differentially private and the statistic

\[ \tilde{T}(X_n) = \begin{cases} \bot & \text{if } A_\eta(T; X_n) + \frac{\sqrt{2 \log(1.25/\delta)}}{\epsilon} Z_2 \leq 1 + \frac{\log(1.25/\delta)}{\epsilon} \text{ o.w.} \end{cases} \]

is \((2\epsilon, 2\epsilon^2 + \delta^2)\) differentially private.

The release of \( \bot \) means that the dataset was too sensitive for the statistic to be released. The goal is to choose a \( T \) such that releasing \( \bot \) is incredibly unlikely; \( A_\eta(T; X_n) \) should be large with high probability.

All of the mechanisms discussed thus far can be combined to produce more sophisticated algorithms, combining two or more mechanisms is called composition. One type of composition is computing a function of a differentially private statistic, where the function is defined independently of the data. Such statistics are also differentially private. It is also true that sums and products of \( k \) differentially private procedures each with privacy budget \( \epsilon_i \) are \( \sum_{i=1}^k \epsilon_i \) differentially private [Dwork et al., 2006]. This can be improved with advanced composition [Dwork and Roth, 2014].

**Theorem 1 (Dwork and Roth [2014]).** For given \( 0 < \epsilon < 1 \) and \( \delta' > 0 \), the composition of \( k \) mechanisms which are each \( \left( \frac{\epsilon}{2 \sqrt{2k \log(1/\delta')}} \delta \right) \)-differentially private is \((\epsilon, k\delta + \delta')\)-differentially private.

### 3 Data Depth

A data depth function is a robust, nonparametric tool used for a variety of inference procedures in multivariate spaces, as well as in more general spaces. A data depth function gives meaning to centrality, order and outlyingness in spaces beyond \( \mathbb{R} \). Data depth functions do this by giving all points in the working space a rating based on how central the point is in the sample. Precisely, we can write multivariate depth functions as \( D : \mathbb{R}^d \times F_n \to \mathbb{R}^+ \); given the empirical distribution of a sample \( F_n \) and a point in the domain, the depth function assigns a real valued depth to that point. Figure 1(a) shows a sample of 20 points labelled by their depth values, we can see that the points in the centre of the data cloud have larger values. Note that it is not necessary to restrict the domain of the depth function to points in the sample; we can compute depth values for each point in the sample space. The heatmap in Figure 1(a) gives the depth value for each point in the plot. Writing depth functions as functions of the empirical distribution \( \hat{D}(\cdot; F_n) \) rather than functions of the sample provides a natural definition for the population depth function \( D(\cdot; F) \). Figure 1(b) shows the population depth values when \( F \) is the two dimensional standard normal distribution.
Figure 1: (a) Sample halfspace depth values, i.e., $D(X_i; F_n)$, are displayed in white text. The heatmap of the sample depth function, i.e., $D(\cdot; F_n)$, is also displayed. This sample is drawn from a standard, two dimensional normal distribution. (b) Theoretical halfspace depth contours for the standard, two dimensional normal distribution.

Depth functions provide an immediate definition of order statistics; observations can be ordered by their depth values. However, since the ordering of the sample is center outward, the depth-based order statistics have a different interpretation than univariate order statistics. Nevertheless, data depth-based order can be used to define multivariate analogues of many univariate, nonparametric inference procedures. For example, the definition of the depth-based median is

$$\text{Med}(F; D) = \arg\max_{x \in \mathbb{R}^d} D(x; F).$$

Depth-based medians are generally robust, in the sense that they are not affected by outliers. Many depth-based medians have high a breakdown point and favourable properties related to the influence function [Chen and Tyler, 2002, Zuo, 2004]. Furthermore, depth-based medians inherent any transformation invariance properties possessed by the depth function. We can subsequently define sample depth ranks as

$$R_i = \# \{ X_j : D(X_j; F_n) \leq D(X_i; F_n) \},$$

which are the building block of various multivariate depth based rank tests [Liu and Singh, 1993, Serfling, 2002, Chenouri et al., 2011], as well as providing a method to construct trimmed means [Zuo, 2002]. Depth values can also be used directly in testing procedures [Li and Liu, 2004]. Depth functions have also been used for visualization, including the bivariate extension of the boxplot (bagplots) and dd-plots, which allow the analysts to visually compare two samples of any dimension [Liu et al., 1999, Li and Liu, 2004]. In the same vein of data exploration, we can visualise multivariate distributions through one dimensional curves based on depth values [Liu et al., 1999]. In the past decade this depth-based inference framework has expanded to include solutions to clustering [Jörnsten, 2004, Baidari and Patil, 2019], classification [Jörnsten, 2004, Lange et al., 2014], outlier detection [Chen et al., 2009, Cárdenas-Montes, 2014], process monitoring [Liu, 1995], change-point problems [Chenouri et al., 2019] and discriminant analysis [Chakraborti and Graham, 2019]. In summary, depth functions facilitate a framework for robust, nonparametric inference in $\mathbb{R}^d$. A major motivating factor for this work is that by privatizing depth functions, we consequentially privatize many of the procedures in this framework. This means that private depth values imply access to private procedures for nonparametrically estimating location, scale, rank tests, building classifiers and more.

In their seminal paper Zuo and Serfling [2000] give a concrete set of mathematical properties which a multivariate depth function should satisfy in order to be considered a statistical depth function. These properties include

1. **Affine invariance**: This implies any depth based analysis is independent of the coordinate system, particularly the scales used to measure the data.
2. **Maximality at centres of symmetry**: If a distribution is symmetric about a point, then surely this point should be regarded as the most central point.

3. **Decreasing along rays**: This property ensures that as one moves away from the deepest point, the depth decreases.

4. **Vanishing at infinity**: As a point moves toward infinity along some ray, the depth vanishes.

A depth function which satisfies these four properties is known as a statistical depth function. The last three properties are all related to centrality, where the first is to ensure there is no dependence on the measurement system. Not all popular depth functions satisfy all four of these properties, but they typically satisfy most of them. Affine invariance, as discussed previously, ensures that the function is not dependent on the coordinate system which, from a practical point of view, means that the measurement scales can be adjusted freely. Maximality at centre means that if a distribution is symmetric about some point \( \theta \), the depth is maximal at that point. Think of the median coinciding with the mean in the univariate case. Decreasing along rays means that as one moves away from the deepest point along some ray, i.e., moves away from the centre, the depth decreases. This property can be replaced with upper semi-continuity. Vanishing at infinity means that as the point moves along a ray to infinity, its depth approaches 0.

Note that if all four of these properties are not satisfied, it does not necessarily mean that a depth function is invalid or not useful in data analysis; it is merely a limitation to consider.

Aside from coordinate invariance and centrality, there are other properties that are desirable for a depth function to satisfy. We shall list the main ones here

- **Robustness**: A robust depth function implies subsequent inference will be robust, and may make it more amenable to privatization.
- **Consistency/Limiting Distribution**: Consistency for a population depth value and existence of a limiting distribution is useful for developing inference procedures.
- **Continuity**: It can be a building block for consistency and for optimizing the depth function.
- **Computation**: In order to apply depth-based inference, it is necessary that the depth values are computed quickly. Specifically, being able to compute or approximate the depth values in polynomial time with respect to both \( d \) and \( n \) is useful.

On top of having these properties, a depth function that is to be used in the private setting should be insensitive. In other words, the depth function has low global sensitivity and or highly probable, low local sensitivity.

We now introduce several depth functions and evaluate their sensitivities. The first depth function we will discuss is halfspace depth [Tukey, 1974].

**Definition 2 (Halfspace depth).** Let \( S^{d-1} = \{ x \in \mathbb{R}^d : \| x \| = 1 \} \) be the set of unit vectors in \( \mathbb{R}^d \). Define the halfspace depth \( \text{HD} \) of a point \( x \in \mathbb{R}^d \) with respect to some distribution \( X \sim F \) as

\[
\text{HD}(x; F) = \inf_{u \in S^{d-1}} \Pr \left( X^\top u \leq x^\top u \right).
\]

Halfspace depth is the minimum of the projected mass above and below the projection of \( x \), over all univariate projections. We can interpret the sample depth of some point \( x \) as the minimum normalised, univariate, centre-outward rank of \( x \)'s projections amongst the samples’ projections, over all univariate directions. Therefore, if a point is exchanged, all the ranks are shifted by at most one, and the global sensitivity of the unnormalised halfspace depth is 1. We get \( \text{GS} \left( \text{HD} \right) = 1/n \), which leads us to conclude that this depth function is relatively insensitive. In terms of known properties, halfspace depth is a statistical depth function. Its sample depth function is also uniformly consistent [Massé, 2004]. Halfspace depth is frequently cited as being computationally complex Serfling [2006], however, recently an algorithm for computing half-space depth in high dimensions has been proposed [Zuo, 2019].

We can replace the minimum in Definition 2 with an average [Ramsay et al., 2019].

**Definition 3 (Integrated Rank-Weighted Depth).** Define integrated rank-weighted depth as

\[
\text{IRW}(x; F) = \int_{S^{d-1}} \min \left( \Pr \left( X^\top u \leq x^\top u \right), 1 - \Pr \left( X^\top u < x^\top u \right) \right) d\nu(u),
\]

where \( \nu \) is the uniform measure on \( S^{d-1} \).
It immediately follows from the discussion on the sensitivity of halfspace depth that \( \text{GS(IRW)} = 1/n \); this depth function has the interpretation of the average normalised univariate centre-outward rank over all projections. Therefore, IRW depth is also insensitive. Aside from being insensitive, IRW depth also vanishes at infinity and is maximal at points of symmetry. It is invariant under similarity transformations, which is weaker than affine invariance. It is conjectured that this function also has the decreasing along rays property. This depth function is also continuous, and can be approximately computed very quickly [Ramsay et al., 2019]. This depth function’s sample depths are also uniformly consistent and asymptotically normal under mild assumptions.

Another asymptotically normal depth function is simplicial depth, which was introduced by Liu [1988].

Definition 4 (Simplicial Depth). Suppose that \( Y_1, \ldots, Y_{d+1} \) are i.i.d. from \( F \). Define simplicial depth as

\[
\text{SMD}(x; F) = \Pr(x \in \Delta(Y_1, \ldots, Y_{d+1})),
\]

where \( \Delta(Y_1, \ldots, Y_{d+1}) \) is the simplex with vertices \( Y_1, \ldots, Y_{d+1} \).

We can show that sample simplicial depth has finite global sensitivity. Note that

\[
\text{SMD}(x; F_n) = \frac{1}{\binom{n}{d+1}} \sum_{1 \leq i_1 < \ldots < i_{d+1} \leq n} \mathbb{1} \{ X \in \Delta(Y_{i_1}, \ldots, Y_{d+1}) \}.
\]

Changing one observation can influence a maximum of \( \binom{d+1}{d-1} \) terms, and each term has a sensitivity of 1. It follows that \( \text{GS(SMD)} = (d+1)/n \). Although it is insensitive, this depth function can be difficult to compute in even moderate dimensions \( d > 3 \). Simplicial depth is a statistical depth function if \( F \) is angularly symmetric, but fails to satisfy the maximality at centre and decreasing along rays for some discrete distributions.

The investigation by Zuo and Serfling [2000] lead to the study of a general and powerful statistical depth function based on outlyingness functions. Outlyingness functions \( O(\cdot; F) : \mathbb{R}^d \to \mathbb{R}^+ \) measure the degree of outlyingness of a point. A particular version of depth based on outlyingness is projection depth.

Definition 5 (Projection Depth). Given a univariate translation and scale equivariant location measure \( \mu \) and a univariate measure of scale \( \varsigma \) which is equivariant and translation invariant, we can define projected outlyingness as

\[
O(x; F; \mu, \varsigma) = \sup_{u \in S^{d-1}} \left| \frac{u^\top x - \mu(F_u)}{\varsigma(F_u)} \right|
\]

and thus projection depth as,

\[
\text{PD}(x; F; \mu, \varsigma) = \frac{1}{1 + O(x; F; \mu, \varsigma)}.
\]

Typically, \( \mu \) and \( \varsigma \) refer to the median and median absolute deviation, but properties have been investigated for general \( \mu \) and \( \varsigma \). One idea is to design \( \mu \) and \( \varsigma \) such that \( O(x; F_n) \) has low global sensitivity, but that is left to later work. Here, we will use either

\[
O_1(x; F_n) := O(x; F_n; \text{Med, MAD}) = \sup_{\|u\|-1} \left| \frac{u^\top x - \text{Med}(X_n^\top u)}{\text{MAD}(X_n^\top u)} \right|
\]

or

\[
O_2(x; F_n) := O(x; F_n; \text{Med, IQR}) = \sup_{\|u\|-1} \left| \frac{u^\top x - \text{Med}(X_n^\top u)}{\text{IQR}(X_n^\top u)} \right|
\]

The global sensitivities of \( O_1, O_2 \) are unbounded, implying that the global sensitivity of PD is equal to 1. Seeing as the range of projection depth is \([0, 1]\), a global sensitivity of 1 is rather high. Both \( O_1, O_2 \) have bounded local sensitivities, making projection depth a good candidate for the propose-test-release procedure. Note that we use a slight abuse of notation, where \( X_n^\top u \) refers to the sample \( \{X_1^\top u, \ldots, X_n^\top u\} \). We may also refer to the empirical distribution implied by this sample as \( F_n,u \). A thorough investigation of the properties of projection depth was done in the successive papers [Zuo, 2003, 2004]. As a result of these papers, it has been shown that projection depth is a statistical depth function, it also has a limiting distribution and is quite robust against outliers.
4 Private Data Depth

There are several ways in which we could approach privatizing depth functions. A natural and easy way to do this is to start with a differentially private estimate of the distribution of the data $\tilde{F}_n$ and use $\tilde{D}(x, \tilde{F}_n)$, which is differentially private. Computing $\tilde{F}_n$ relies on existing methods for generating private multidimensional empirical distribution functions. This method fails to take advantage of any robustness properties of depth functions; it does not leverage the low sensitivities of the depth function itself. This method also does not give a method for computing distribution functions. This method fails to take advantage of any robustness properties of depth functions; it does not leverage the low sensitivities of the depth function itself. This method also does not give a method for computing distribution functions. This method fails to take advantage of any robustness properties of depth functions; it does not leverage the low sensitivities of the depth function itself. This method also does not give a method for computing distribution functions.

The fact that these mechanisms are differentially private follow from the differential privacy of Mechanisms 1 and 2.

If the global sensitivity of $D$ is finite, then an obvious private estimate is

$$\tilde{D}(x; F_n) = D(x; F_n) + V_{\delta, \epsilon} \cdot \text{GS}(D),$$

where $V_{\delta, \epsilon}$ is independent noise, from the Laplace or Gaussian distribution with scale calibrated to ensure privacy. If a depth function has infinite or large global sensitivity, then since it is likely robust and thus, it makes sense to apply a propose-test-release algorithm.

We can also produce a more direct privatized estimate of $D(\cdot, F)$ based on the sample, such as has been done with histogram bins [Wasserman and Zhou, 2010]. For example, many depth functions are defined based on functions of projections: $h(\cdot; X_n^u, u), u \in \mathcal{U}_n$ where $\mathcal{U}_n$ is some set of directions, i.e., $\mathcal{U}_n \subset S^{d-1}$. We could then produce private versions of $X_n^u$ or private versions of $h(\cdot; X_n^u, u)$, if $h$ is insensitive. The advantage of this approach would be that the entire depth function could be privatized at once, including the sample depth values. In the same vein, recalling that $\nu$ is the uniform measure on $S^{d-1}$, there exists an image measure, $\mu_{x,F_n}(A) = \nu(h^{-1}(A; x, F_n))$ on the Borel sets of the range of the depth, i.e., $A \in \mathcal{B}(I_d)$. If $\mu_{x,F_n}$ is insensitive, then we can construct a differentially private estimator based on random draws from $\mu_{x,F_n}$. This approach is somewhat complicated, and could be tedious since we must setup a sampler for each $x$ at which we want to compute a depth value. We leave these projection type approaches for future research.

From the discussion above, it is clear that a key question is at which points would we like to estimate depth values? Algorithms which estimate the depth of single point are of course of interest; they can be composed to compute depth values at several points privately. Additionally, simple algorithms to compute the depth of a single point can be used as building blocks for private versions of depth based inference procedures. As mention previously, it is also of interest to compute the depth values of the sample points:

$$\tilde{D}(F_n) := (D(X_1; F_n), D(X_2; F_n), \ldots, D(X_n; F_n)).$$

Since $X_i$ appears in both arguments, the sensitivity of $D(X_i; F_n)$ is larger than that of $D(x; F_n)$ $x \notin X_n$. We investigate private methods of estimating the vector of sample depth values. A further question is whether or not we can estimate several depth values from different samples simultaneously, e.g., for use in depth-based clustering. To elaborate, if $X_n$ contains the samples for $J$ groups, then $X_n = \bigcup_{j=1}^J X_n^j$. For example, if we privatize the one dimensional projections of the entire sample $X_n^u$ we can then compute the depth of each point in $X_n$ with respect to each group $X_n^u$.

An important question is how well do the privatized inference procedures perform when compared to their non-private counterparts? Do the privatized depth values converge to their non-private counterparts? If so, what is the rate of convergence? Does this private estimate have a limiting distribution? If so, is the limiting distribution different from that of the non-private limiting distribution? We investigate some of these questions in the next section.

5 Algorithms for Private Depth Values

As mentioned previously, for depth functions with finite global sensitivity, we can make use of the Gaussian and Laplace mechanisms.

**Mechanism 5.** For $x$ given independently of the data, the following estimators

$$\tilde{D}_1(x; F_n) = D(x; F_n) + WGS(D)/\epsilon \quad \text{and} \quad \tilde{D}_2(x; F_n) = D(x; F_n) + ZGS(D)\sqrt{2\log(1.25/\delta)/\epsilon}$$

are $\epsilon$-differentially private and $(\epsilon, \delta)$-differentially private respectively.

The fact that these mechanisms are differentially private follow from the differential privacy of Mechanisms 1 and 2. The following results are immediate.
Theorem 2. For a given depth function, suppose that \(\sqrt{n}(\tilde{D}(x; F_n) - D(x; F)) \xrightarrow{d} \mathcal{Y}_D(x)\), where \(\xrightarrow{d}\) denotes convergence in distribution. Suppose we can write \(GS(D) = C(D)/n\) where \(C(D)\) does not depend on \(n\). Let \(r > 0\) and \(\ell = 1\) or \(\ell = 2\). For depth values generated under Mechanism 5, the following holds

1. For \(\delta_n = o(n^{-k})\) and \(\epsilon_n = O(n^{-1+r})\), \(\tilde{D}_\ell(x; F_n) \xrightarrow{d} D(x; F)\), where \(\xrightarrow{d}\) denotes convergence in probability.

2. For \(\delta_n = o(n^{-k})\) and \(\epsilon_n = O(n^{-1/2+r})\), \(\sqrt{n}(\tilde{D}_\ell(x; F_n) - D(x; F)) \xrightarrow{d} \mathcal{Y}_D(x)\).

It should be noted that choosing \(\delta = o(n^{-k})\) and \(\epsilon = O(n^{-1/2+r})\) maintains a reasonable level of privacy. For example, choosing \(\epsilon \in O(1)\) and \(\delta < 1/n\) is “the most-permissive setting under which \((\epsilon, \delta)\)-differential privacy is a nontrivial guarantee” [Cai et al., 2019]. From Theorem 2 we can conclude that for large samples and small privacy parameters, depth value estimates generated via Mechanism 5 are minimally affected by privatization.

What if we want to calculate depth at a sample point? How we can estimate the vector of depth values at the sample points

\[\tilde{D}(F_n) = (D(X_1; F_n), D(X_2; F_n), \ldots, D(X_n; F_n)).\]

privately? The sample values now appear in both arguments of \(D\) so we must do a bit more work to compute sensitivities. First we look at halfspace and IRW depth. Consider one set of projections \(X^\top_u\) and their corresponding empirical distribution \(F_{n,u}\). We want to compute the sensitivity of the vector \(R_{n,u} = (R_1, \ldots, R_n)\), with

\[R_i = \min\{F_{n,u}(X_i^\top u), 1 - F_{n,u}(X_i^\top u^-)\},\]

and \(F(x-) = P(X < x)\). If we change one observation, that will change at most \(n - 1\) values of \(R_{n,u}\) in the odd case and at most \(n - 2\) values of \(R_{n,u}\) in the even case. Alternatively, we can change one depth value by \([n+1]/n - 1/n\) and \([n+1]/n - 1\) values by \(1/n\). This gives that

\[GS_1(R_{n,u}) = \frac{2}{n} \left(\left\lfloor\frac{n+1}{2}\right\rfloor - 1\right) \approx 1,\]

and that

\[GS_2(R_{n,u}) = \frac{1}{n} \sqrt{\left(\left\lfloor\frac{n+1}{2}\right\rfloor - 1\right)^2 + \left\lfloor\frac{n+1}{2}\right\rfloor} - 1 \approx 1.\]

Since averaging or taking the supremum over such \(R_{n,u}\) does not affect these sensitivities, it follows that for halfspace depth and IRW depth \(GS_1(D) = GS_2(R_{n,u})\). Concerning simplicial depth, with respect to some adjacent dataset, the depth values of the unchanged points can each change by at most \((d + 1)/n\). For the point that is different, we can bound the sensitivity above by \(1 - (d + 1)/n\). It follows that \(GS_1(\text{SMD}) \leq 1\) and \(GS_2(\text{SMD}) \leq \sqrt{1 - (d + 1)/n)^2 + (d + 1)/n)^2} \approx 1\), where \(\text{SMD}\) is the vector \(D\) with \(D = \text{SMD}\). In summary, the global sensitivities of the vector of sample depth values for halfspace, IRW and simplicial depth are all close to 1. Considering we pay \(\epsilon/n\) for each depth value in Mechanism 5, we pay the same privacy budget for \(n\) depth values at \(n\) arbitrary points as we do for the depth values at the \(n\) sample points. In other words, we do not use any extra privacy budget for the fact that we are computing the depth at the sample values. We can then use the following mechanism to estimate the vector of depth values:

Mechanism 6. The following estimators for the vector of depth values of the sample points

\[\tilde{D}_1(F_n) = D(F_n) + (W_1, \ldots, W_n)/\epsilon \quad \text{and} \quad \tilde{D}_2(F_n) = D(F_n) + (Z_1, \ldots, Z_n)/\epsilon\]

are \(\epsilon\)-differentially private and \((\epsilon, \delta)\)-differentially private respectively.

The fact that these mechanisms are differentially private follow from the differential privacy of Mechanisms 1 and 2. For the full vector of sample depth values we do not get privacy for free in the limit. Observe that for

\[\|\tilde{D}_1(F_n) - D(F)\| \leq \|\tilde{D}_1(F_n) - D(F_n)\| + \|D(F_n) - D(F)\| \leq \|(W_1, \ldots, W_n)/\epsilon\| + O_p(n^{1/2}) = O_p(n^{1/2}). \quad (4)\]

The level of noise is the same order as the sampling error, and therefore must be accounted for in inference procedures. For \(\tilde{D}_2\) with \(\delta \propto n^{-k}\), we have that

\[\|\tilde{D}_2(F_n) - D(F)\| \leq O_p(n^{1/2} \log^{1/2} n);\]

the level of noise introduced by the privatization is larger than that of the sampling error.

We now turn our attention to a depth function with high global sensitivity: projection depth. For projection depth, we would like to generate private outlyingness values, which have unbounded sensitivity. Note that Med, MAD, IQR are all robust statistics, in the sense that they are not perturbed by extreme data points. This implies that \(O_1\) and
$O_2$ have an unlikely chance of worst case sensitivity, which would make projection depth a good candidate for the propose-test-release framework [Dwork and Lei, 2009, Brunel and Avella-Medina, 2020]. Suppose that IQR $\approx 1 \forall u$. If

$$\eta \geq \max(F_{n,u}^{-1}(1/2) - F_{n,u}^{-1}(1/2 - 1/n), F_{n,u}^{-1}(1/2 + 1/n) - F_{n,u}^{-1}(1/2))$$

for all $u$ then $A_\eta \approx [n/2] - 1$, which means it is very unlikely that Mechanism 4 will return $\perp$.

**Mechanism 7.** For $\ell = 1$ or $\ell = 2$, define privatized projection depth as

$$\tilde{\text{PD}}_\ell(x; F_n) = \frac{1}{1 + O_{\ell}(x; F_n)},$$

where

$$\tilde{O}_{\ell}(x; F_n) = \begin{cases} \perp & \text{if } A_\eta(\tilde{O}_{\ell}(x; F_n); \mathcal{X}_n) + \frac{\eta \epsilon}{\epsilon} V_1 \leq 1 + \frac{b_\ell}{\epsilon} \text{ a.w.} \\
\tilde{O}_{\ell}(x; F_n) + \frac{\eta \epsilon}{\epsilon} V_2 & \text{if } A_\eta(\tilde{O}_{\ell}(x; F_n); \mathcal{X}_n) + \frac{\eta \epsilon}{\epsilon} V_1 > 1 + \frac{b_\ell}{\epsilon} \end{cases}$$

where $a_\ell$, $b_\ell$, the level of privacy and $V_j$ are according to Mechanism 4.

We can actually show that this algorithm is consistent for the population depth values when using $O_2$ as the outlyingness measure.

**Theorem 3.** Let $\xi_{p,u}$ be the $p^{th}$ quantile of $F_u$. Suppose that for all $h > 0$, $u$

$$|F_u(\xi_{p,u} + h) - F_u(\xi_{p,u})| = M|h|^q(1 + O|h|^{q/2})$$

with $M > 0$, $q > 0$, for $p = 1/4$, $1/2$, $3/4$. Suppose that $\sup_u \xi_{p,u} < \infty$ for $p = 1/4$, $3/4$. For $\eta \propto \frac{\log n}{n^{3/4}}$ with $r > 0$, $\delta_\eta = O(n^{-k})$ and $\frac{n^{1/4} (\log \log n)^{3/4}}{\log \delta_\eta} \epsilon_n \to \infty$ we have that

$$|\tilde{\text{PD}}_2(x; F_n) - PD(x; F; \text{Med}, \text{IQR})| \xrightarrow{p} 0.$$
with \( m_1(u) \) being the median of a dataset the same as \( X_n^\top u \), except that the smallest \( k^* \) observations of \( X_n^\top u \) are replaced with \( x^\top u \) and \( m_2(u) \) being the same as \( m_1(u) \), except instead the largest \( k^* \) observations of \( X_n^\top u \) are replaced. Define
\[
B = \{ F_{n,u}^{-1}(3/4 + k_1/n) - F_{n,u}^{-1}(1/4 + k_2/n) : \ 1 \leq k_1, k_2 \leq k^*, \ |k_1| + |k_2| = k^* \}.
\]
We can then write
\[
\text{lo}(\text{IQR}, u) := \min B \leq \text{IQR}(\hat{y}_n^\top u) \leq \max B := \text{up}(\text{IQR}, u).
\]
Therefore, it holds that
\[
\hat{O}_t^\top(x) \in \left[ \frac{\text{lo}(\text{Med}, u)}{\text{up}(\text{IQR}, u)} \right] \left[ \frac{\text{up}(\text{Med}, u)}{\text{lo}(\text{IQR}, u)} \right] = \left[ \text{lo}(\hat{O}_t^\top(x)), \text{up}(\hat{O}_t^\top(x)) \right]
\]
and we can check if
\[
\max(\hat{O}_t^\top(x) - \text{lo}(\hat{O}_t^\top(x)), \text{up}(\hat{O}_t^\top(x)) - \hat{O}_t^\top(x)) < \eta.
\]
Then if this holds for all \( u \) we must have that \( A_\eta(\hat{O}_2(x; F_0), X_n) \geq k^* \), which gives a lower bound on the truncated breakdown point. This lower bound can be used when implementing Mechanism 7, in the ‘test’ portion of the algorithm.

The methods used to construct private depth values discussed in this section can be used to privatize inference procedures based solely on functions of depth values. For example, a common way to compare scale between two multivariate samples, say \( X_{n1} \) and \( X_{n2} \), is to compute the sample depth values with respect to the empirical distribution of the pooled sample \( X_{n1} \cup X_{n2} \)[Li and Liu, 2004, Chenouri et al., 2011]. We can denote this empirical distribution by \( G_{n1+n2} \). Private depth-based ranks could then be defined as
\[
\bar{R}_{ji} = \{ \#X_{k1} : \bar{D}(X_{k1}; G_{n1+n2}) \leq \hat{D}(X_{ji}; G_{n1+n2}) \},
\]
where \( X_{ji} \) is the \( i^{th} \) observation from sample \( j \). We can use these ranks to privately test for a difference in scale between the two groups with the rank sum test statistic, viz.
\[
\hat{T}(X_{n1} \cup X_{n2}) = \sum_{i=1}^{n1} \bar{R}_{ji}.
\]
The distribution of such a statistic remains the same under the null hypothesis, and (4) can be used to assess its performance under the alternative hypothesis. It is clear that the power will be lowered, as the noise biases the statistic toward failing to reject the null hypothesis. We can also take a similar approach in multivariate, covariance change-point models [Chenouri et al., 2019, Ramsay and Chenouri, 2020]. The algorithms of this section cannot be used to compute private depth-based medians, i.e., private maximizers of the depth functions, and so we investigate algorithms to compute depth-based medians in the next section.

### 6 Private Multivariate Medians

For depth functions with finite global sensitivity, it is natural to estimate the depth-based median using the exponential mechanism. As such, we could generate an observation from
\[
f(v; F_n) \propto \exp \left( -\frac{\epsilon}{2\text{GS(D)}} \hat{D}(v; F_n) \right),
\]
to be used as a private estimate of the D-based median. One issue is that this density is not necessarily valid. For example,
\[
f(v; F_n) = \frac{\exp \left( -\frac{\epsilon}{2\text{GS(HD)}} \hat{D}(v; F_n) \right)}{\int_{\mathbb{R}^d} \exp \left( -\frac{\epsilon}{2\text{GS(HD)}} \hat{D}(v; F_n) \right)}
\]
is not a valid density, since \( \int_{\mathbb{R}^d} \exp \left( -\frac{\epsilon}{2\text{GS(HD)}} \hat{D}(v; F_n) \right) = \infty \). To see this, note that
\[
1 < \exp \left( -\frac{\epsilon}{2\text{GS(HD)}} \hat{D}(v; F_n) \right) < \infty
\]
and so even if we transform this to
\[
\exp \left( -\frac{\epsilon}{2\text{GS(HD)}} (\alpha - \hat{D}(v; F_n)) \right),
\]
it is still bounded below for any \(\alpha\). This implies that

\[
\int_{\mathbb{R}^d} \exp \left( \frac{-\epsilon}{2\text{GS}(D)} \text{HD}(v; F_n) \right) dv = \infty.
\]

Similar results follow for the remaining depth functions, since they all have a range that lies in an interval. If the data for which we would like to estimate the median is within some compact set \(B\), then we can easily reduce the range of the estimator to \(B\) and the density

\[
f(v; F_n) = \frac{\exp \left( \frac{\epsilon}{2\text{GS}(D)} \text{HD}(v; F_n) \right) \mathbb{1} \{ v \in B \}}{\int_B \exp \left( \frac{\epsilon}{2\text{GS}(D)} \text{HD}(v; F_n) \right) dv},
\]

is valid. If there is no clear set \(B\) in which the median will lie then we propose a Bayesian approach, and recommend using a prior \(\pi(v)\) on the median such that

\[
f(v; F_n) = \frac{\exp \left( \frac{\epsilon}{2\text{GS}(D)} \text{HD}(v; F_n) \right) \pi(v)}{\int_{\mathbb{R}^d} \exp \left( \frac{\epsilon}{2\text{GS}(D)} \text{HD}(v; F_n) \right) \pi(v) dv},
\]

is a valid density. Seeing as \(\mathbb{1} \{ v \in B \}\) normalized by \(\int_B dv\) is a special case of a prior, we can summarise this procedure as follows:

**Mechanism 8.** Suppose that \(\text{GS}(D) = C(D)/n\). Suppose also that \(\pi(v)\) is a density chosen independently of the data. Provided

\[
f(v; F_n) = \frac{\exp \left( \frac{\epsilon n}{2C(D)} D(v; F_n) \pi(v) \right)}{\int_{\mathbb{R}^d} \exp \left( \frac{\epsilon n}{2C(D)} D(v; F_n) \right) \pi(v) dv},
\]

is a valid Lebesgue density, a random draw from \(f(v; F_n)\) is an \(\epsilon\)-differentially private estimate of the depth-based median of \(X_n\).

It is imperative that this prior is chosen independently of the data or the privacy of the procedure will be violated. For any depth function bounded above and below it is easy to see that this is a valid density. Suppose that the range of \(D\) is \([0,1]\), then the following inequality holds

\[
1 = \int_{\mathbb{R}^d} \pi(v) dv \leq \int_{\mathbb{R}^d} \exp \left( \frac{\epsilon n}{2C(D)} D(v; F_n) \right) \pi(v) dv \leq \int_{\mathbb{R}^d} \exp \left( \frac{\epsilon n}{2C(D)} \right) \pi(v) dv = \exp \left( \frac{\epsilon n}{2C(D)} \right).
\]

Some asymptotics of the exponential mechanism have been investigated by Awan et al. [2019], but their result requires that the cost function is twice differentiable and convex. Depth functions do not typically satisfy these requirements. The following lemma is useful for proving some asymptotic results related to the exponential mechanism, when the cost function is not necessarily differentiable, but smooth at the limiting minimizer

**Lemma 1.** Let \(0\) be the zero vector in \(\mathbb{R}^d\) and \(\pi(v)\) be a density on \(\mathbb{R}^d\). Suppose that \(\phi_n(\omega, v) : \Omega \times \mathbb{R}^d \to \mathbb{R}^+\) are a sequence of random functions on the probability space \((\Omega, \mathcal{A}, P)\). Assume that

1. \(\lambda_n = Cn^r\) for some \(C > 0\).
2. For \(s > r\), \(\|\phi_n(\omega, \cdot) - \phi(\omega, \cdot)\|_{\infty} = o_P(n^{-s})\).
3. For some \(\alpha > 0\), \(\phi(\omega, v)\) is \(\alpha\)-Hölder continuous in a neighborhood around \(0\) - almost surely. This means that
   \(\|\phi(\omega, v) - \phi(\omega, 0)\| \leq C_1 \|v\|^{\alpha}\) for some constant \(C_1\).
4. \(\phi(\omega, v) = 0\) if and only if \(v = 0\); \(\phi\) is uniquely minimized at \(v = 0\).
5. \(\pi(v)\) is a bounded Lebesgue density which is positive in some neighborhood around \(0\).

Let \(V_n\) be a sequence of random variables whose measure on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) is given by

\[
Q_n(A) = \int_{\Omega} \int_{\mathbb{R}^d} e^{-\lambda_n \phi_n(\omega, v)} \pi(v) dv dP,
\]

for \(A \in \mathcal{B}(\mathbb{R}^d)\). Then \(V_n \xrightarrow{P} 0\).
Lemma 1 may be applied outside the context of depth functions, and can be used to prove weak consistency of an estimator based on the exponential mechanism. When applying Lemma 1, the sequence $\lambda_n$ in Lemma 1 should be replaced by the ratio of the privacy parameter and the global sensitivity of the cost function. This lemma shows that smoother, insensitive cost functions will allow the estimator to be consistent for smaller privacy budgets, provided that the prior is positive in a region around the maximizer. Additionally, if $e^{-\lambda_n \phi_n(w; \cdot)}$ is integrable, then we can let $\pi(v) = 1$ and the result still holds. We can apply Lemma 1 to data depth functions, which results in the following theorem.

**Theorem 4.** Suppose that $\sup_{v} |D(v; F_n) - D(v; F)| = o_p(n^{-s})$ where $s \geq 0$, $\text{GS}(D) = C(D)/n$, the maximum of $D(x; F)$ occurs uniquely at $\theta$ and $D$ is $\alpha$-Hölder continuous at $\theta$, for some $\alpha > 0$. Additionally, suppose that $\pi(v)$ is a bounded Lebesgue density which is positive in a neighborhood around $\theta$. Let $\rho = D(\theta; F) < \infty$, then for $\bar{T}(X_n)$ drawn from the density

$$f(v; F_n) = \frac{\exp \left( - \frac{\epsilon_n}{2\text{GS}(D)}(\rho - D(v; F)) \right) \pi(v)}{\int_{\mathbb{R}^d} \exp \left( - \frac{\epsilon_n}{2\text{GS}(D)}(\rho - D(v; F_n)) \right) \pi(v) dv},$$

it holds that

1. $\bar{T}(X_n) \overset{p}{\to} \theta$ when $\epsilon_n = O(n^{r'}), r' < s - 1$.
2. $\bar{T}(X_n) \overset{d}{\to} T$, when $n\epsilon_n \to K < \infty$, with the density of $T$ proportional to $\exp \left( - \frac{K \rho - D(v; F))}{2\text{GS}(D)} \right)$.

**Remark 1.** The continuity condition is weak, in the sense that $\alpha$ can be very small. Halfspace depth is $\alpha$-Hölder continuous if $F_n$ are $\alpha$-Hölder continuous and $F$ is continuous. IRW depth is $\alpha$-Hölder continuous if $F_n$ are $\alpha$-Hölder continuous. For simplicial depth, we only need $F$ to be $\alpha$-Hölder continuous.

**Remark 2.** For many depth functions, we can choose $s$ arbitrarily close to $1/2$ and the convergence requirement is still satisfied. Therefore choices of $r'$ such that $r'$ is close to $-1/2$ give the fastest rates at which the privacy parameter can decrease to 0 while maintaining consistency of the estimator.

Theorem 4 can then be applied to the three depths of [Tukey, 1974, Liu, 1988, Ramsay et al., 2019]. They are all satisfy the uniform consistency requirement for $s < 1/2$ [Dümbgen, 1992, Massé, 2004, Ramsay et al., 2019] and continuity was discussed in Remark 1. The assumption on uniqueness of the median is trickier, in the sense that these depth-based medians are not necessarily unique. We only need the population depth-based median to be unique, which is satisfied for distributions which are symmetric about a unique point. This holds because these depth functions satisfy the maximality at center property, see, e.g., [Zuo and Serfling, 2000]. Algorithms that implement Mechanism 8 are an interesting line of new research; we cannot directly use, say Markov Chain Monte Carlo methods, without first ensuring that they maintain the privacy of the estimators.

For projection depth, we cannot use the exponential mechanism without injecting a significant level of noise into the estimator and so we instead extend the propose-test-release framework [Brunel and Avella-Medina, 2020] to be used with the exponential mechanism. Suppose $\phi_{X_n} : \mathbb{R}^d \to \mathbb{R}^d$ is some cost function which we would like to minimize. Then, define

$$A_{\eta}(\phi_{X_n}; X_n) = \min \left\{ k \in \mathbb{N} : \sup_{\mathbb{R}^d} \sup_{x \in \mathcal{D}(X_n, k)} |\phi_{X_n}(x) - \phi_{X_n'}(x)| > \eta \right\},$$

as the truncated breakdown point of the cost function. This is a direct extension of the truncated breakdown point of [Brunel and Avella-Medina, 2020] to the functional context; we are essentially using the infinity norm and writing $||\phi_{X_n}(x) - \phi_{X_n'}(x)||_{\infty} > \eta$. We can write the estimator as follows

**Mechanism 9.** Suppose that

$$Q_{X_n}(A) = \int_{A} \frac{\exp(-\phi_{X_n}(v) \frac{dv}{\rho}) dv}{\int_{\mathbb{R}^d} \exp(-\phi_{X_n}(v) \frac{dv}{\rho}) dv}$$

is a valid measure on the $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then the estimator

$$\bar{T}(X_n) = \begin{cases} \bot & \text{if } A_{\eta}(\phi_{X_n}; X_n) + \frac{a_5}{\epsilon} V \leq 1 + \frac{b_5}{\epsilon} \text{ o.w.} \end{cases},$$

with $a_5, b_5$ and $V$ as in Mechanism 4 is a differentially private estimate of $\argmin \phi_{X_n}(v)$. Under the Laplace version, the estimator is $(2\epsilon, \delta)$-differentially private and under the Gaussian version, the estimator is $(2\epsilon, 2\delta)$-differentially private.
This theorem shows that we can still use PTR with the exponential mechanism. The level of privacy under the Gaussian version is slightly higher than that of the original PTR mechanism, which is due to the pure differential privacy of the exponential mechanism.

**Theorem 5.** Suppose that the sequence of random functions $\phi_{X_n}(v) : \mathbb{R}^d \rightarrow \mathbb{R}^+$ satisfy the conditions of Lemma 1 and that

$$\int_{\mathbb{R}^d} \exp \left( - \frac{K \phi(v)}{2} \right) dv < \infty,$$

for any $K > 0$. Suppose that $\tilde{T}(X_n)$ is generated according to Mechanism 9. If the sequences $\epsilon_n, \delta_n, \eta_n$ imply that

$$\Pr \left( A_{n}(\phi_{X_n}; X_n) + V^{\frac{\alpha_n}{\epsilon_n}} \leq \frac{b_n}{\epsilon_n} + 1 \right) \rightarrow 0$$

then it holds that

1. $\tilde{T}(X_n) \overset{d}{=} \arg\min_{v \in \mathbb{R}^d} \phi(v)$ when $\epsilon_n/\eta_n = O(n^r)$, $r < 1/2$
2. $\tilde{T}(X_n) \overset{d}{=} T$, when $\epsilon_n/\eta_n \rightarrow K < \infty$ with the density of $T$ proportional to $\exp \left( - \frac{K \phi(v)}{2} \right)$.

We can now substitute in the outlyingness function and see how this algorithm works for the purposes of privately estimating the projection depth median. A first question is whether or not this density exists. If $\sup_u \text{Med}^d(X_n) < \infty$ and $\inf_u \varsigma(X_n^T u) > 0$ then

$$\int_{\mathbb{R}^d} \exp \left( - \frac{\epsilon \sup_u \|v^T u - \text{Med}(X_n^T u)/\varsigma(X_n^T u)\|}{2\eta} \right) dv \leq C_1 \int_{\|v\| > 1} \exp \left( - \frac{\epsilon \|v\|^2}{2\eta} \right) dv + C_2 \int_{\|v\| \leq 1} \exp \left( - \frac{\epsilon}{2\eta} \right) dv < \infty.$$

Unfortunately, immediately using PTR with the exponential mechanism gives no gains in estimating the projection median over using the global sensitivity of projection depth (which is 1). If the points in $X_n$ are distinct, we have that $A_n(O(\cdot); X_n; X_n) = 1$ for any $\eta$. To see this, suppose that $\gamma_n$ is a neighboring dataset, with $X_1$ changed to be some observation such that $\varsigma(\gamma_n^T u) \neq \varsigma(X_n^T u)$. It follows that for any $u$

$$\sup_x |O^1(x; X_n) - O^1(x; \gamma_n)| \approx \sup_x \left| v^T u \varsigma(X_n^T u) - \varsigma(\gamma_n^T u) \varsigma(X_n^T u)/\varsigma(\gamma_n^T u) \right| = \infty.$$

In order to estimate the projection depth-based median privately, we can truncate the outlyingness function $O$; for $\|x\| > M_n$, set $O(x) = \infty$. Define this function to be

$$O_t(x; X_n; M_n) = \{ O_t(x; X_n) \|x\| < M_n \\infty \|x\| \geq M_n \}.$$

We can now apply Mechanism 9 and Theorem 5 to $O$ in order to privately estimate the projection depth-based median. The following theorem gives reasonable choices of $\eta$ and $\epsilon$ that maintain consistency of the estimate.

**Theorem 6.** Suppose that $O(1) \geq M_n = o(n^{1/2})$, $\sup_u |\text{Med}(F_u)| < \infty$, $\sup_u \text{IQR}(F_u) < \infty$, $\inf_u \text{IQR}(F_u) > 0$ and the conditions of Theorem 3 hold. Further, suppose that $\text{PD}(x; F)$ is uniquely maximized at $\theta$. Let $\tilde{T}(X_n)$ be the estimator of Mechanism 9 with cost function $\phi_{X_n}(v) = O_2(v; X_n; M_n)$, then

1. $\tilde{T}(X_n) \overset{d}{=} \theta$ when $\epsilon_n/\eta_n = O(n^r)$, $r < 1/2$.
2. $\tilde{T}(X_n)$ converges in distribution to a random variable with measure

$$Q(A) = \int_A \int_{\mathbb{R}^d} \exp \left( - (O_2(v; F) - O_2(\theta; F)) \frac{\epsilon_n}{2\eta} \right)dv$$

when $\epsilon_n/\eta_n = K$, $K < \infty$.

The obvious issue is choosing $M_n$ in practice, which can be partially informed by the above theorem. Clearly, if the data is known to be bounded it is easy to choose $M_n$. If the data is not bounded one can choose $M_n$ independent of the data, given domain knowledge. The choice of $M_n$ should not depend on the data, which would violate the consistency theorem; if $M_n$ is chosen based on the data, then $M_n$ could differ between two datasets, implying that $O^1_t(x; X_n) - O^1_t(x; \gamma_n) = \infty$ for some $x$ and consequently the truncated breakdown point of the outlyingness function is 1. Computationally, again the difficulty lies in computing $A_\eta(\phi_{X_n}; X_n)$, for which we can use similar methods as the previous section.
7 Concluding Remarks

We have introduced several mechanisms for private, depth-based inference. These mechanisms include private estimates of point-wise depth values for population depth functions, such as halfspace and projection depth. Such mechanisms have been shown to output consistent estimators, even for cases where the privacy budget is small; \( \epsilon \to 0 \). Notable is that we have shown that one can get consistent estimates of projection depth, even though it has high global sensitivity. We have also introduced algorithms for estimating popular depth-based medians, including the simplicial, halfspace, IRW and projection medians. These algorithms all provide differentially private, consistent estimators of the population median under some very mild conditions. These conditions include the existence of a unique depth-based population median and that the population depth function is smooth at its median. The privacy budget is allowed to decrease to 0, provided it is not too fast, e.g., \( \epsilon_n \propto n^{-r}, \ r < 1/2 \) for the halfspace median. We also provide some general tools for constructing and studying differentially private estimators. We provide a lemma for showing weak consistency of differentially private estimators based on the exponential mechanism, even if the objective function is not differentiable. We also extend the propose-test-release algorithm of Brunel and Avella-Medina [2020] to be used with the exponential mechanism, which allows one to privately estimate maximizers of objective functions which have infinite global sensitivity. We also provide tools to show weak consistency of an estimator based on this algorithm. We apply this algorithm and the related consistency result to the projection depth-based median of Zuo [2003]. In this paper, we have demonstrated that depth functions are very amenable to privatization. This has provided options for different types of private inference via the depth-based inference framework. One benefit is that these inference procedures retain their robustness, which may even be more helpful in the private setting, where the analyst may have only limited access to the database. This work has shown another meaningful connection between robust statistics and differential privacy. As such, it has opened up many avenues for further research, which the authors are exploring. Some of these include how these algorithms perform in different inferential contexts, such as depth-based hypothesis testing and clustering. Another area of interest is the computation of the medians presented in Section 6. Particularly the computation of the truncated breakdown point, which is more difficult than in the one-dimensional setting.

8 Proofs

Proof of Theorem 2. The first property follows directly from consistency of the sample depths and the fact that \( \text{GS}(D) \to 0 \). The second case is true for the same reasons.

Proof of Theorem 3. We show the result for the Laplace case, but the Gaussian case follows the same path. First, we want to show that

\[
\Pr \left( A_\eta \left( O_2 \left( x; F_n \right); X_n \right) \leq 1 + \frac{\log(2/\delta_n) - W_1}{\epsilon_n} \right) \to 0.
\]

Note that

\[
\Pr(|W_1| > \log(2/\delta_n)) = e^{-\log(2/\delta_n)} = (\delta_n/2) = O(n^{-k}),
\]

from the properties of the Laplace distribution and the rate of convergence of \( \delta_n \). We can then write

\[
\Pr \left( A_\eta \left( O_2 \left( x; F_n \right); X_n \right) \leq 1 + \frac{\log(2/\delta_n) - W_1}{\epsilon_n} \right) \leq \Pr \left( A_\eta \left( O_2 \left( x; F_n \right); X_n \right) \leq 1 + \frac{\log(2/\delta_n) - W_1}{\epsilon_n}, W_1 > -\log(2/\delta_n) \right)
\]

\[
+ \Pr \left( A_\eta \left( O_2 \left( x; F_n \right); X_n \right) \leq 1 + \frac{\log(2/\delta_n) - W_1}{\epsilon_n}, W_1 < -\log(2/\delta_n) \right)
\]

\[
\leq \Pr \left( A_\eta \left( O_2 \left( x; F_n \right); X_n \right) \leq 1 + 2\frac{\log(2/\delta_n)}{\epsilon_n} \right) + O(n^{-k}).
\]

Now, let \( \rho_n = 2\frac{\log(2/\delta_n)}{\epsilon_n} \) and we want to show that

\[
\Pr \left( A_\eta \left( O_2 \left( x; F_n \right); X_n \right) \leq 1 + \rho_n \right) \to 0.
\]

It holds that

\[
\Pr \left( A_\eta \left( O_2 \left( x; F_n \right); X_n \right) \leq 1 + \rho_n \right) = \Pr \left( 1 + \rho_n \sup_{j=1}^{1+\rho_n} \sup_{Y \in \mathcal{D}(X_n,j)} |O_2(x; Y_n) - O_2(x; Y_n)| \geq \eta \right).
\]
Consider the Taylor series expansion of \( f(x, y) = x/y \) about \( |x - \text{Med}(F_n)| / \text{IQR}(F_n) \):

\[
O^y_n(x; X_n) = \frac{|x - \text{Med}(F_n)|}{\text{IQR}(F_n)} + \frac{|x - \text{Med}(F_n)| - |x - \text{Med}(F_n)|}{\text{IQR}(F_n)}
\]

\[
= \frac{|x - \text{Med}(X_n^u)|}{\text{IQR}(F_n)} - \frac{|x - \text{Med}(F_n)|}{\text{IQR}(F_n)} (\text{IQR}(X_n^u) - \text{IQR}(F_n)) + R_{n,u}
\]

It is easy to see that \( R_{n,u} = O_p(n^{-1}) \), since \( (\text{IQR}(X_n^u) - \text{IQR}(F_n))^2 = O_p(n^{-1}) \). We can then write

\[
|O^y_n(x; X_n) - O^y_n(x; Y_n)| = \frac{|x - \text{Med}(u^T X_n)|}{\text{IQR}(X_n^u)} - \frac{|x - \text{Med}(u^T Y_n)|}{\text{IQR}(Y_n^u)}
\]

\[
= \frac{|x - \text{Med}(X_n^u)| - |x - \text{Med}(Y_n^u)|}{\text{IQR}(F_n)} - \frac{|x - \text{Med}(F_n)|}{\text{IQR}(F_n)} (\text{IQR}(X_n^u) - \text{IQR}(Y_n^u)) + O_p(n^{-1})
\]

\[
\leq \frac{\text{Med}(Y_n^u) - \text{Med}(X_n^u)}{\text{IQR}(F_n)} + \frac{|x - \text{Med}(F_n)|}{\text{IQR}(F_n)} (\text{IQR}(X_n^u) - \text{IQR}(Y_n^u)) + O_p(n^{-1})
\]

where the last line follows from the reverse triangle inequality. Now, recall that \( Y_n \) differs from \( X_n \) by at most \( \rho_n + 1 \) points. If \( F_{n,u} \) is the empirical distribution corresponding to \( X_n^u \) and \( G_{n,u}(x) = 1 - F_{n,u}(x) \), it holds that

\[
|\text{Med}(Y_n^u) - \text{Med}(X_n^u)| \leq \max \{|F_{n,u}^{-1}(1/2) - F_{n,u}^{-1}(1/2 + (\rho_n + 1)/n)|, |F_{n,u}^{-1}(1/2) - F_{n,u}^{-1}(1/2 - (\rho_n + 1)/n)|\}
\]

\[
= |F_{n,u}^{-1}(1/2) - F_{n,u}^{-1}(1/2 + (\rho_n + 1)/n)|
\]

\[
= |F_{n,u}^{-1}(1/2) - F_{n,u}^{-1}(1/2 + (\rho_n + 1)/n)|
\]

\[
= |G_{n,u}(\text{Med}(F_u)) + R'_{n,u} - 1/2 - G_{n,u}(\text{Med}(F_u)) - R'_{n,u} + 1/2|
\]

\[
= O(n^{-3/4} \log n) \ a.s. .
\]

where the second last line and the last line follow from a Bahadur type representation of quantiles, as long as

\[
\frac{1}{n} \left(1 + 2 \log(2/\delta_n)\right) = O((\log \log n/n)^{3/4}) \] [see Theorem 2 on page 2 of de Haan and Taqqu-Mauro, 1979]. However, we know that \( \frac{1}{n} \left(1 + 2 \log(2/\delta_n)\right) = O((\log \log n/n)^{3/4}) \) holds from the assumptions on \( \epsilon_n \) and \( \delta_n \). We can show something similar for the inter-quartile range by simply replacing 1/2 with 1/4 and 3/4. Now, we must show that

\[
\sup_u O^y_n(x; X_n) - \sup_u O^y_n(x; Y_n) \rightarrow 0 \ a.s. .
\]

We see that

\[
\sup_u O^y_n(x; X_n) - \sup_u O^y_n(x; Y_n) \leq 2 \sup_u |O^y_n(x; X_n) - O^y_n(x; Y_n)| = O(n^{-3/4} \log n) \ a.s. .
\]

where the last line follows from the fact that \( \xi_{1/4,u} - \xi_{1/4,u} \) are bounded as functions of \( u \), implying that \( R'_{n,u}, R'_{n,u} \) are also bounded in \( u \). See the proof of Theorem 2' of de Haan and Taqqu-Mauro, 1979 for the exact expression of \( R'_{n,u}, R'_{n,u} \). This implies that for \( \eta \geq \frac{\log \frac{u}{\epsilon_n}}{n^{1/4}} \) it holds that \( \Pr[A_n(O_2(x; X_n); X_n) \leq 1 + \rho_n] \rightarrow 0 \). Now, since

\[
\frac{\epsilon_n}{\epsilon_n} W_2 \rightarrow 0 \text{ and } \text{PD}(x; F_n;\text{Med}, \text{IQR}) \underset{\text{p}}{\rightarrow} \text{PD}(x; F;\text{Med}, \text{IQR}) \text{ we have that } \text{PD}_2(x; F_n) \underset{\text{p}}{\rightarrow} \text{PD}(x; F; \text{Med}, \text{IQR}).
\]

For the Gaussian case, we have that

\[
\Pr(|Z_1| \sqrt{2 \log(1.25/\delta_n)} > 2 \log(1.25/\delta_n)) \leq 2e^{-\log(1.25/\delta_n)} = 2(\delta_n/1.25) = O(n^{-k}),
\]

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from the properties of the normal distribution and the rate of convergence of \( \delta_n \). We can then write, using the same argument as above,

\[
\Pr \left( A_\eta \left( O_2 (x; F_n) ; x_n \right) \right) \leq 1 + \frac{2 \log(1.25/\delta_n) - Z_1 \sqrt{2 \log(1.25/\delta_n)}}{\epsilon_n} \leq \Pr \left( A_\eta \left( O_2 (x; F_n) ; x_n \right) \right) \leq 1 + \frac{4 \log(1.25/\delta_n)}{\epsilon_n} + O(n^{-k}).
\]

Now, the probability is of the same form of that of the Laplace case, and the same argument applies.

\[\Box\]

**Proof of Lemma 1.** It is clear that

\[
e^{-\lambda_n \phi_n(\omega, v)} \pi(v) dv
\]

is a valid density, since \( e^{-x} \) is bounded for all \( x \in \mathbb{R}^+ \). The goal is to show that \( Q_n \) converges weakly to \( \mathbb{E}_\theta(\cdot) \), since this is equivalent to \( V_n \xrightarrow{P} 0 \). Note that we use \( \mathbb{E}_\theta(B) \) as shorthand for \( \mathbb{E}\{ B \} \). We use the Portmanteau Theorem and show that for all \( \mathbb{E}_\theta() \)-continuity sets \( A, Q_n(A) \to \mathbb{E}_\theta(A) \). Let \( B_n = \{ \omega : \| \phi_0(\omega, \cdot) - \phi(\omega, \cdot) \| < n^{-k} \} \) and let \( Q_n^v(\cdot) \) be the measure corresponding to \( V_n \), conditional on \( \omega \), viz.

\[
Q_n^v(A) = \int_A e^{-\lambda_n \phi_n(\omega, v)} \pi(v) dv.
\]

We can now write

\[
\lim_{n \to \infty} Q_n(A) = \lim_{n \to \infty} \int_{B_n} Q_n^v(A) dP = \lim_{n \to \infty} \int_{B_n} \int_{\Omega} e^{-\lambda_n \phi_n(\omega, v)} \pi(v) dv dP = \lim_{n \to \infty} \int_{B_n} \int_{\Omega} e^{-\lambda_n \phi_n(\omega, v)} \pi(v) dv + \lim_{n \to \infty} \int_{B_n} Q_n^v(A) dP.
\]

It is easy to see that

\[
0 \leq \lim_{n \to \infty} \int_{B_n} Q_n^v(A) dP \leq \lim_{n \to \infty} \int_{B_n} dP = 0,
\]

where the last equality comes from assumption 2. Using this, we can write

\[
\lim_{n \to \infty} Q_n(A) = \lim_{n \to \infty} \int_{B_n} Q_n^v(A) dP + \lim_{n \to \infty} \int_{B_n^c} Q_n^v(A) dP = \lim_{n \to \infty} \int_{B_n} Q_n^v(A) dP = \lim_{n \to \infty} \mathbb{E}_\theta(A).
\]

where the last equality follows from dominated convergence theorem, noting that \( Q_n^v(\cdot) < 1 \). We now consider \( \lim_{n \to \infty} Q_n^v(A) \) for fixed \( \omega \in B_n \). Note that for any \( \mathbb{E}_\theta(\cdot) \)-continuity set, \( \emptyset \) is either an interior point or not in the set. Keeping this in mind, let \( A_1 \) be a \( \mathbb{E}_\theta(\cdot) \)-continuity set such that \( \emptyset \) is interior in \( A_1 \). We can write

\[
\lim_{n \to \infty} Q_n^v(A_1) = \lim_{n \to \infty} \int_{A_1} e^{-\lambda_n \phi_n(\omega, v)} \pi(v) dv = \lim_{n \to \infty} \int_{A_1} e^{-\lambda_n \phi_n(\omega, v) - \phi(\omega, v)} \pi(v) dv + \int_{A_1} e^{-\lambda_n \phi_n(\omega, v) \pi(v)} dv.
\]

Observe that for fixed \( \omega \in B_n \), \( \lambda_n(\phi_n(\omega, v) - \phi(\omega, v)) = O(n^\beta) = o(n^{-k}) = o(n^{-k}) \), where \( \beta > 0 \); we then have that \( \lambda_n(\phi_n(\omega, v) - \phi(\omega, v)) = o(n^{-k}) \). Thus, assumption 1 and 2 imply that \( \lambda_n(\phi_n(\omega, v) - \phi(\omega, v)) \leq Cn^{-\beta} \) for some \( \beta > 0 \) independent of \( v \). Therefore, we can write

\[
\int_{A_1} e^{-\lambda_n \phi_n(\omega, v) \pi(v)} dv = \int_{A_1} e^{-\lambda_n(\phi_n(\omega, v) - \phi(\omega, v) + \phi(\omega, v) - \phi(\omega, v))} \pi(v) dv \\
= \int_{A_1} e^{-\lambda_n(\phi_n(\omega, v) - \phi(\omega, v))} e^{-\lambda_n(\phi(\omega, v) - \phi(\omega, v))} \pi(v) dv \\
\geq \int_{A_1} e^{-Cn^{-\beta}} \pi(v) dv.
\]

For large \( n \) and fixed \( \xi > 0 \), the neighborhood \( N_n^c(0) = \{ x \in \mathbb{R}^d : \| x \| < n^{-\xi} \} \) is in \( A_1 \), since \( \emptyset \) is interior in \( A_1 \). From assumption 3 (Hölder continuity) we have that

\[
\sup_{v \in N_n^c(0)} | \phi(\omega, v) - \phi(\omega, v) | < C_n d^c(n^\xi).
\]

Choose \( \xi \) such that \( \alpha' = \xi \alpha > r \), and write

\[
\int_{A_1} e^{-\lambda_n(\phi(\omega, v) - \phi(\omega, v))} \pi(v) dv \geq \int_{N_n^c(0)} e^{-\lambda_n(\phi(\omega, v) - \phi(\omega, v))} \pi(v) dv \\
\geq C_3 n^{-d} e^{-C_2 n^{-\alpha'}} \lambda_n e^{-Cn^{-\beta}} \\
\geq C_4 n^{-d} e^{-C_2 n^{-\alpha'}} \pi(v) \\
= O(n^{-\delta \xi}).
\]
Note that assumption 5 implies that there exists some \( N \), such that for all \( n > N \), \( \pi(v) \) is bounded below on \( \mathcal{N}_{n} - \Omega \) and so \( \pi \) is absorbed into the constant \( C_3 \). Now, consider \( A_i^j \), a \( \pi(\cdot) \)-continuity set such that \( 0 \) is not interior in \( A_i^j \). There then exists a neighborhood around \( 0 \), call it \( \mathcal{N}_k(0) \), such that \( \mathcal{N}_k(0) \notin A_i^j \). By assumption 3 and 4, we also have that \( \phi(\omega, v) > k' > 0 \) on \( A_i^j \), for some \( k' \) independent of \( n \). It follows easily that

\[
\int_{A_i^j} e^{-\lambda_n \phi(\omega, v)} \pi(v) dv = \int_{A_i^j} e^{-\lambda_n \phi(\omega, v)} e^{-\lambda_n (\phi(\omega, v) - \phi(\omega, v))} \pi(v) dv \\
= O(1) \int_{A_i^j} e^{-\lambda_n \phi(\omega, v)} \pi(v) dv \\
\leq O(1) \int_{A_i^j} e^{-\lambda_n k'} \pi(v) dv \\
= O(e^{-\lambda_n k'}),
\]

where the second equality comes from the fact that \( \lambda_n (\phi(\omega, v) - \phi(\omega, v)) = o(1) \) uniformly in \( v \). The last equality comes from the fact that \( \pi(v) \) is a density with respect to the Lebesgue measure. It then follows that

\[
\lim_{n \to \infty} Q^n_n(\Omega) = \lim_{n \to \infty} \int_{A_i^j} e^{-\lambda_n \phi(\omega, v)} \pi(v) dv + \int_{A_i^j} e^{-\lambda_n \phi(\omega, v)} \pi(v) dv = \lim_{n \to \infty} \frac{O(n^{-d \varepsilon})}{O(n^{-d \varepsilon}) + O(e^{-\lambda_n k'})} = 1,
\]

which immediately gives that \( \lim_{n \to \infty} Q^n_n(A) = \mathbb{I}_A(A) \) for \( \mathbb{I}_A(\cdot) \)-continuity sets \( A \). Then, since \( \lim_{n \to \infty} \mathbb{I} \{ \omega \in B_n \} = 1 \) \( P \)-almost surely by assumption 2, we have that

\[
\lim_{n \to \infty} Q_n(A) = \int_{\Omega} \lim_{n \to \infty} \mathbb{I} \{ \omega \in B_n \} Q^n_n(A) dP = \mathbb{I}_A(A),
\]

which implies that

\[
Q_n \xrightarrow{d} \mathbb{I}_A(\cdot).
\]

**Lemma 2.** Suppose the conditions of Lemma 1 hold, except that \( \lim_{n \to \infty} \lambda_n = K \). Let \( V_n \) be a sequence of random variables whose measure on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) is given by

\[
Q_n(A) = \int_{\Omega} \int_A e^{-\lambda_n \phi(\omega, v)} \pi(v) dv dP,\]

for \( A \in \mathcal{B}(\mathbb{R}^d) \). Then \( V_n \xrightarrow{d} Q(A) \), where

\[
Q(A) = \int_A e^{-K \phi(\omega, v)} \pi(v) dv.
\]

**Proof.** From the proof of Lemma 1, it suffices to look at

\[
\lim_{n \to \infty} \int_A e^{-\lambda_n \phi(\omega, v)} \pi(v) dv,
\]
on \( B_n \). On \( B_n \), \( \| \phi(\omega, v) - \phi(\omega, v) \|_\infty \to 0 \), which implies that \( a_n = e^{-\lambda_n (\phi(\omega, v) - \phi(\omega, v))} \to 1 \), uniformly in \( v \). We can then say that

\[
\lim_{n \to \infty} \int_A e^{-\lambda_n \phi(\omega, v)} \pi(v) dv = \lim_{n \to \infty} \int_A e^{-\lambda_n \phi(\omega, v)} a_n \pi(v) dv = \int_A \lim_{n \to \infty} e^{-\lambda_n \phi(\omega, v)} \pi(v) dv a_n dv = \int_A e^{-K \phi(\omega, v)} \pi(v) dv,
\]

where the last line follows from dominated convergence theorem and the fact that \( \pi \) is a density function. The result follows from the proof of Lemma 1.

**Proof of Theorem 4.** We can write this problem as an application of Lemmas 1 and 2. For the first part, set \( \phi_n = \rho - D(v; F_n) \). Then, we have that \( \frac{1}{2 \pi (\rho_0)} \mathbb{E} e_n = n^{r+1} \frac{1}{2 \pi (\rho_0)} \) with \( r+1 < s \). Assumption 1 of Lemma 1 is then satisfied and the result follows. The second result is a direct application of Lemma 2.
Proof of Remark 1. Note that for halfspace depth consider two points \( x, y \in \mathbb{R}^d \). If \( F_u \) are \( \alpha \)-Hölder continuous, then
\[
|F_u(x^\top u) - F_u(y^\top u)| \leq C|(x - y)^\top u|^\alpha = C||x - y||^\alpha \left(\frac{x - y}{\|x - y\|}\right)^\top u \leq C||x - y||^\alpha. \tag{5}
\]
Now, without loss of generality, suppose that \( \text{HD}(x, F) > \text{HD}(y, F) \). Suppose further that \( u^* \) is such that \( F_{u^*}(y^\top u) = \inf_u F_u(y^\top u) \). There exists such a \( u^* \) because \( F \) is continuous, implying that \( F_u \) is continuous in \( u \), thus, \( F_u \) is continuous function on a compact set. It follows that
\[
|\text{HD}(x, F) - \text{HD}(y, F)| = \inf_u F_u(x^\top u) - F_{u^*}(y^\top u) \leq F_{u^*}(x^\top u) - F_u(y^\top u) \leq C||x - y||^\alpha.
\]
For IRW depth, it holds that
\[
|\text{IRW}(x, F) - \text{IRW}(y, F)| = \int_{S^{d-1}} \min(F_u(x^\top u), 1 - F_u(x^\top u)) - \min(F_u(y^\top u), 1 - F_u(y^\top u))d\nu(u)
\]
\[
\leq \int_{S^{d-1}} 2|F_u(x^\top u) - F_u(y^\top u)| + 2|1 - F_u(x^\top u) - F_u(y^\top u)|d\nu(u)
\]
\[
\leq 4C||x - y||^\alpha \int_{S^{d-1}} 2d\nu(u),
\]
which is a result of (5) and the fact that \( |1 - F_u(x^\top u) - F_u(y^\top u)| \leq 1 \).

For simplicial depth, if \( F \) is \( \alpha \)-Hölder continuous, then we must show that \( \Pr(x \in \Delta(X_1, \ldots, X_{d+1}) \) is also \( \alpha \)-Hölder continuous. It is easy to begin with two dimensions. Consider \( \Pr(x \in \Delta(X_1, X_2, X_3)) \) - \( \Pr(y \in \Delta(X_1, X_2, X_3)) \), as per [Liu, 1990], we need to show that \( \Pr(X_1 \cap X_2 \text{ intersects } \mathcal{F}_y) \leq C||x - y||^\alpha. \) In order for this event to occur, we must have that \( X_1 \) is above \( \mathcal{F}_y \) and \( X_2 \) is below \( \mathcal{F}_y \), but both are projected onto the line segment \( \mathcal{F}_y \) when projected onto the line running through \( \mathcal{F}_y \). The affine invariance of simplicial depth implies we can assume, without loss of generality, that \( x \) and \( y \) lie on the axis of the first coordinate. Let \( x_1 \) and \( y_1 \) be the first coordinates of \( x \) and \( y \). Suppose that \( X_{11} \) is the first coordinate of \( X_1 \). It then follows from \( \alpha \)-Hölder continuity of \( F \) that
\[
\Pr(X_1 \cap X_2 \text{ intersects } \mathcal{F}_y) \leq \Pr(x_1 < X_{11} < y_1) \leq C|x_1 - y_1|^\alpha \leq C||x - y||^\alpha.
\]
In dimensions greater than two, a similar line of reasoning can be used. We can again assume, without loss of generality, that \( x \) and \( y \) lie on the axis of the first coordinate. It holds that
\[
\Pr(x \in \Delta(X_1, X_2, X_3)) - \Pr(y \in \Delta(X_1, X_2, X_3)) \leq \left(\frac{d + 1}{d}\right) \Pr(A_d),
\]
where \( A_d \) is the event that the \( d - 1 \)-dimensional face of the random simplex, formed by \( d \) points randomly drawn from \( F \), intersects the line segment \( \mathcal{F}_y \). It is easy to see that
\[
\Pr(A_d) \leq \Pr(x_1 < X_{11} < y_1) \leq C|x_1 - y_1|^\alpha \leq C||x - y||^\alpha.
\]

Proof of Differential Privacy of Mechanism 9. The proof has the same outline as that of [Brunel and Avella-Medina, 2020], as well as the proof that the exponential mechanism is differentially private, which can be found in [McSherry and Talwar, 2007, Dwork and Roth, 2014]. First, assume that it holds \( |\phi_n(x) - \phi_n(x)| \leq \eta \forall x \), then
\[
\frac{f_{\mathcal{X}_n}(v)}{f_{\mathcal{Y}_n}(v)} = \frac{\exp(-\phi_n(v)\frac{\epsilon^2}{2})}{\exp(\phi_n(v)\frac{\epsilon^2}{2})} \int \exp(-\phi_n(v)\frac{\epsilon^2}{2})dv
\]
\[
\leq e^{\epsilon^2/2} \int \exp(-\phi_n(v)\frac{\epsilon^2}{2})dv
\]
\[
\leq e^{\epsilon^2/2} \epsilon^{\epsilon^2/2} \int \exp(\phi_n(v)\frac{\epsilon^2}{2})dv
\]
\[
= e^\epsilon.
\]
Note that, for \( B \in \mathcal{B}(\mathbb{R}^d) \) (the Borel sets with respect to \( \mathbb{R}^d \)) this implies that
\[
\Pr(\tilde{T}(\mathcal{X}_n) \in B) \leq e^\epsilon \Pr(\tilde{T}(\mathcal{Y}_n) \in B). \tag{6}
\]
It follows from Brunel and Avella-Medina [2020] that \( A_\eta(\phi_{X_n};X_n) \) has global sensitivity equal to 1, since changing one point can at most change the breakdown by 1. Then

\[
\Pr\left(\hat{T}(X_n) \in B\right) = \Pr\left(A_\eta(\phi_{X_n};X_n) + \frac{1}{\epsilon} V \geq 1 + \frac{\log(2/\delta)}{\epsilon}, \hat{T}(X_n) \in B\right)
\]

\[
\leq e^\epsilon \Pr\left(A_\eta(\phi_{Y_n};Y_n) + \frac{1}{\epsilon} V \geq 1 + \frac{\log(2/\delta)}{\epsilon}\right) \Pr(\hat{T}(X_n) \in B)
\]

\[
\leq e^{2\epsilon} \Pr\left(A_\eta(\phi_{Y_n};Y_n) + \frac{1}{\epsilon} V \geq 1 + \frac{\log(2/\delta)}{\epsilon}\right) \Pr(\hat{T}(Y_n) \in B)
\]

\[
= e^{2\epsilon} \Pr\left(\hat{T}(Y_n) \in B\right).
\]

The first inequality is from independence and the fact that \( A_\eta(\phi_{X_n};X_n) + \frac{1}{\epsilon} V \) is an \( \epsilon \)-differentially private estimator. The second inequality is from (6). Now what if there exists an \( x \) such that \( |\phi_{X_n}(x) - \phi_{Y_n}(x)| \geq \eta \)? This implies that \( A_\eta(\phi_{X_n};X_n) = 1 \) and

\[
\Pr\left(\hat{T}(X_n) \in B\right) \leq \Pr\left(A_\eta(\phi_{X_n};X_n) + \frac{1}{\epsilon} V \geq 1 + \frac{\log(2/\delta)}{\epsilon}\right) = \Pr(V \geq \log(2/\delta)) = \delta \leq \delta + e^{2\epsilon} \Pr\left(\hat{T}(Y_n) \in B\right).
\]

This implies that we get \((2\epsilon, \delta)\) differential privacy if \( B \) is restricted to \( \mathcal{B}(\mathbb{R}^d) \). For completeness, we need to include sets of the form \( B = B' \cup \{(\bot, \bot)\} \), where \( B' \in \mathcal{B}(\mathbb{R}^d) \). Consider

\[
\Pr\left(\hat{T}(X_n) \in B\right) = \Pr\left(\hat{T}(X_n) \in B', A_\eta(\phi_{X_n};X_n) + \frac{1}{\epsilon} V \leq 1 + \frac{\log(2/\delta)}{\epsilon}\right) + \Pr\left(A_\eta(\phi_{X_n};X_n) + \frac{1}{\epsilon} V > 1 + \frac{\log(2/\delta)}{\epsilon}\right)
\]

\[
\leq e^{2\epsilon} \left(\Pr\left(\hat{T}(Y_n) \in B'\right) + \Pr\left(A_\eta(\phi_{Y_n};Y_n) + \frac{1}{\epsilon} V > 1 + \frac{\log(2/\delta)}{\epsilon}\right)\right) + \delta
\]

\[
= e^{2\epsilon} \Pr\left(\hat{T}(Y_n) \in B\right) + \delta.
\]

The first inequality comes from the fact that we get \((2\epsilon, \delta)\) differential privacy if \( B \) is restricted to \( \mathcal{B}(\mathbb{R}^d) \) and the fact that \( A_\eta(\phi_{Y_n};Y_n) + \frac{1}{\epsilon} V \) is \( \epsilon \)-differentially private.

Now, suppose that \( V, a_\eta \) and \( b_\delta \) correspond to the Gaussian version of PTR. Then, following the same steps as for the Laplace version gives, for \( B \in \mathcal{B}(\mathbb{R}^d) \),

\[
\Pr\left(\hat{T}(X_n) \in B\right) \leq e^{2\epsilon} \Pr\left(\hat{T}(Y_n) \in B\right) + \delta,
\]

when \( \|\phi_{X_n} - \phi_{Y_n}\|_\infty < \eta \). When \( \|\phi_{X_n} - \phi_{Y_n}\|_\infty \geq \eta \),

\[
\Pr\left(\hat{T}(X_n) \in B\right) \leq \Pr\left(Z \geq \sqrt{2\log(1.25/\delta)}\right) \leq \delta.
\]

We then have that

\[
\Pr\left(\hat{T}(X_n) \in B\right) \leq e^{2\epsilon} \Pr\left(\hat{T}(Y_n) \in B\right) + \delta.
\]

Again, we need to include sets of the form \( B = B' \cup \{(\bot, \bot)\} \), where \( B' \in \mathcal{B}(\mathbb{R}^d) \). Consider

\[
\Pr\left(\hat{T}(X_n) \in B\right) = \Pr\left(\hat{T}(X_n) \in B', A_\eta(\phi_{X_n};X_n) + \frac{\sqrt{2\log(1.25/\delta)}}{\epsilon} Z > 1 + \frac{2\log(1.25/\delta)}{\epsilon}\right)
\]

\[
\leq e^{2\epsilon} \left(\Pr\left(\hat{T}(Y_n) \in B'\right) + \Pr\left(A_\eta(\phi_{Y_n};Y_n) + \frac{\sqrt{2\log(1.25/\delta)}}{\epsilon} Z > 1 + \frac{2\log(1.25/\delta)}{\epsilon}\right)\right) + 2\delta
\]

\[
= e^{2\epsilon} \Pr\left(\hat{T}(Y_n) \in B\right) + 2\delta.
\]

\[\square\]

**Proof of Theorem 5.** The only difference, from the previous proofs of results in this section is that we do not have a prior \( \pi \). The assumed existence of the measures \( Q_{X_n} \) and

\[
Q(A) = \int_A \frac{\exp(-K^{\phi(x)})}{\int_{\mathcal{B}(\mathbb{R}^d)} \exp(-K^{\phi(x)})} dv
\]

remedy this. The proofs of Lemma 1 and Lemma 2 would then imply that \( \hat{T}(X_n) \) satisfies the above convergence results if \( P(\hat{T}(X_n) = \bot) \to 0 \) as \( n \to \infty \). This statement is, however, assumed by the theorem.\[\square\]
Proof of Theorem 6. First, note that because sup_u |Med(F_u)| < ∞, sup_u IQR(F_u) < ∞ and inf_u IQR(F_u) > 0, we have that

\[ O_2(x, F) \geq \|x\| - \sup_{u \in [1]} \mu (F_u) \frac{\|x\|}{\sup_{|u| = 1} \sigma (F_u)}, \]

see page 1477 of [Zuo, 2003]. It immediately follows that

\[ \int_{\mathbb{R}^d} \exp \left( -\frac{\epsilon_n O_2(v; F)}{2\eta_n} \right) dv \leq C_1 \int_{\mathbb{R}^d} \exp \left( -\frac{\epsilon_n \|v\|}{2\eta_n} \right) dv < \infty. \]

For the density

\[ \int_{\mathbb{R}^d} \exp \left( -\frac{\epsilon_n O_2(v, X_n; M_n)}{2\eta_n} \right) dv = \int_{\|v\| < M_n} \exp \left( -\frac{\epsilon_n O_2(v, X_n; M_n)}{2\eta_n} \right) dv < \infty. \]

We then have both

\[ \frac{\exp \left( -\frac{\epsilon_n O_2(v, X_n; M_n)}{2\eta_n} \right)}{\int_{\mathbb{R}^d} \exp \left( -\frac{\epsilon_n O_2(v, X_n; M_n)}{2\eta_n} \right) dv} \quad \text{and} \quad \frac{\exp \left( -\frac{\epsilon_n O_2(v; F)}{2\eta_n} \right)}{\int_{\mathbb{R}^d} \exp \left( -\frac{\epsilon_n O_2(v; F)}{2\eta_n} \right) dv} \]

are valid Lebesgue density functions. We need to show that O_2(v, X_n; M_n) satisfies assumptions 2-4 of Lemma 1. Assumptions 2 and 3 hold on account of sup_u |Med(F_u)| < ∞, sup_u IQR(F_u) < ∞ and inf_u IQR(F_u) > 0, [see Remark 2.5 of Zuo, 2003]. Assumption 4 is implied by the theorem assumptions. We need to show that

\[ \Pr \left( A_{\eta_n} (\phi_{X_n}; X_n) + V_{\alpha_n, \eta_n} \leq \frac{b_{\delta_n}}{\epsilon_n} + 1 \right) \rightarrow 0. \]

First, suppose that \|x\| < M_n. From proof of Theorem 3, we have directly that

\[ \lim_{n \to \infty} \Pr \left( A_{\eta_n} (\phi_{X_n}; X_n) + V_{\alpha_n, \eta_n} \leq 1 + \frac{\log(2/\delta_n)}{\epsilon_n} \right) = 0. \]

The fact that \|x\| < M_n allows us to bounded the remainder in the Taylor series expansion used in the proof of Theorem 3. Now, suppose that \|x\| > M_n, which implies that O_2^+(x; X_n) - O_2^+(x; Y_n) = 0, which implies that

\[ \Pr \left( A_{\eta_n} (\phi_{X_n}; X_n) + V_{\alpha_n, \eta_n} \leq 1 + \frac{\log(2/\delta_n)}{\epsilon_n} \right) = 0. \]

The conditions of Theorem 5 are satisfied and the result follows.

\[ \square \]

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