On some results of Agélas concerning the GRH and of Vassilev-Missana concerning the prime zeta function*†

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Abstract

A recent paper by Agélas [Generalized Riemann Hypothesis, 2019, hal-00747680v3] claims to prove the Generalized Riemann Hypothesis (GRH) and, as a special case, the Riemann Hypothesis (RH). We show that the proof given by Agélas contains an error. In particular, Lemma 2.3 of Agélas is false. This Lemma 2.3 is a generalisation of Theorem 1 of Vassilev-Missana [A note on prime zeta function and Riemann zeta function, Notes on Number Theory and Discrete Mathematics, 22, 4 (2016), 12–15]. We show by several independent methods that Theorem 1 of Vassilev-Missana is false. We also show that Theorem 2 of Vassilev-Missana is false.

This note has two aims. The first aim is to alert other researchers to these errors so they do not rely on faulty results in their own work. The second aim is pedagogical — we hope to show how these errors could have been detected earlier, which may suggest how similar errors can be avoided, or at least detected at an early stage.

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1 Introduction

In [1], Agélas states

Claim 1 (Agélas, Theorem 2.1). For any Dirichlet character \( \chi \mod k \), the Dirichlet L-function \( L(\chi, s) \) has all its non-trivial zeros on the critical line \( \Re(s) = \frac{1}{2} \).

This is the Generalized Riemann Hypothesis, probably formulated by Adolf Piltz in 1884 (see Davenport [3, p. 124]). A special case, which corresponds to the principal character \( \chi_0(n) = 1 \) and the Riemann zeta-function \( \zeta(s) \), is the well-known Riemann Hypothesis [10].

Agélas defines the half-plane \( \mathcal{A} := \{ s \in \mathbb{C} : \Re(s) > 1 \} \), and two Dirichlet series (convergent for \( s \in \mathcal{A} \))

\[
P(\chi, s) := \sum_{p \in \mathcal{P}} \chi(p) p^{-s}
\]

and

\[
P_2(\chi, s) := \sum_{p \in \mathcal{P}} \chi(p)^2 p^{-s},
\]

where \( \mathcal{P} \) is the set of primes \( \{2, 3, 5, \ldots\} \).

When trying to understand the proof of Claim 1 by Agélas, we considered the case of the Riemann zeta-function. Since this was sufficient to find an error in the proof, we only need to consider this case. Thus we can take \( \chi(p) = 1 \), so \( P(\chi, s) \) and \( P_2(\chi, s) \) both reduce to the usual prime zeta function [4]

\[
P(s) := \sum_{p \in \mathcal{P}} p^{-s}.
\]

This function is also considered by Vassilev-Missana [11]. It is well-known (and a proof may be found in [4, p. 188]) that, for \( \Re(s) > 1 \),

\[
P(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log \zeta(k s).
\]  

Vassilev-Missana states

Claim 2 (Vassilev-Missana, Theorem 1). For integer \( s > 1 \), the relation

\[
(1 - P(s))^2 = \frac{2}{\zeta(s)} - 1 + P(2s)
\]

holds.

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1 We use the word “Claim” for a statement which we may later prove to be false.

2 It is not clear why Vassilev-Missana imposes such a strong restriction on \( s \); we might expect the relation to hold for all \( s \in \mathcal{A} \) or perhaps (using analytic continuation) for almost all \( s \in \{z \in \mathbb{C} : \Re(z) > 0\} \).
Agélas states

“Lemma 2.3 appears as an extension of Theorem 1 of Vassilev-Missana (2016), we give here the details of the proof as it is at the heart of the Theorem obtained in this paper. For this, we borrow the arguments used in Vassilev-Missana (2016).”

He then states

Claim 3 (Agélas, Lemma 2.3). For $s \in \mathcal{A}$, we have

$$(1 - P(\chi, s))^2 L(\chi, s) - (P_2(\chi, 2s) - 1)L(\chi, s) = 2.$$ 

In the case that we consider, namely $L(\chi, s) = \zeta(s)$, both Claim 2 and Claim 3 amount to the same relation, which we can write in an equivalent form as

$$\frac{2}{\zeta(s)} = 2 - 2P(s) + (P(s))^2 - P(2s). \quad (2)$$

In §2 we show that (2) is false. This implies that Lemma 2.3 of Agélas is false, as is Theorem 1 of Vassilev-Missana. Theorem 2.1 of Agélas (the GRH) may be true, but has not been proved. Lemmas 2.4 and 2.5 of Agélas depend on Lemma 2.3, so are most likely false. Theorem 2 of Vassilev-Missana is also false, as discussed in §3.

2 Disproving (2)

We give several methods to disprove (2).

Method 1. Expand each side of (2) as a Dirichlet series $\sum a_n n^{-s}$. On the right-hand side (RHS), the only terms with nonzero coefficients $a_n$ are for integers $n$ of the form $p^\alpha q^\beta$, where $p$ and $q$ are primes, $\alpha \geq 0$, and $\beta \geq 0$. However, on the left-hand side (LHS), we find $a_{30} = -2 \neq 0$, since $30 = 2 \times 3 \times 5$ has three distinct prime factors, implying that $\mu(30) = -1$. By the uniqueness of Dirichlet series that converge absolutely for all sufficiently large values of $\Re(s)$ [6, Thm. 4.8], we have a contradiction, so (2) is false. □

Remark 1. Instead of 30 we could take any squarefree positive integer with greater than two prime factors. A somewhat analogous situation arises when bounding the coefficients of the cyclotomic polynomial $\Phi_n(x)$ (see for example [7, 9]). Migotti [8] showed that if $n$ has at most two distinct odd prime factors, then the coefficients of $\Phi_n(x)$ are all in the set $\{0, -1, 1\}$. However, this is not necessarily true if $n$ has greater than two distinct odd prime factors. For example, $\Phi_{105}(x)$ contains terms $-2x^7$ and $-2x^{41}$. □
Method 2. We can evaluate both sides of (2) numerically for one or more convenient values of \( s \). If we take \( s = 2k \) for some positive integer \( k \), then the LHS of (2) can easily be evaluated using Euler’s formula
\[
\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}}{2 \cdot (2k)!} B_{2k},
\]
where \( B_{2k} \) is a Bernoulli number. The RHS can be evaluated by using (1). Taking \( k = 1 \), i.e. \( s = 2 \), the LHS is \( 12/\pi^2 = 1.2158542 \) and the RHS is \( 1.2230397 \) (both values correct to 7 decimals). Thus, \( |\text{LHS} - \text{RHS}| > 0.007 \). This is a contradiction, so (2) is false.

Remark 2. It is always a good idea to verify identities numerically whenever it is convenient to do so. A surprising number of typos and more serious errors can be found in this manner. Early mathematicians such as Euler, Gauss, and Riemann were well aware of the value of numerical computation, even though they lacked the electronic tools and mathematical software that we have today.

If we had followed the philosophy of “experimental mathematics” [2], we would have attempted method 2 first. It is only because we were familiar with the computation of cyclotomic polynomials that we thought of using method 1 before trying method 2.

Method 3. We consider the behaviour of each side of (2) near \( s = 1 \). On the LHS there is a simple zero at \( s = 1 \), since the denominator \( \zeta(s) \) has a simple pole. On the RHS there is a logarithmic singularity of the form \( a(\log(s-1))^2 + b\log(s-1) + O(1) \). This is a contradiction, so (2) is false.

Method 4. For this method we need to assume that both sides of (2) have been extended by analytic continuation into the critical strip. The LHS of (2) has singularities precisely where \( \zeta(s) \) has zeros. Using (1), \( P(s) \) has singularities at these zeros (say \( \rho \)), and also at \( \rho/2, \rho/3, \ldots \). Thus, on the line \( \Re(s) = 1/2 \), the LHS has infinitely many simple poles and no logarithmic singularities, whereas the RHS has infinitely many logarithmic singularities. This is a contradiction, so (2) is false.

Method 5. The LHS of (2) has a meromorphic continuation to the whole of the complex plane, with poles wherever \( \zeta(s) \) has zeros. On the other hand, it is known that \( P(s) \) has a natural boundary at the line \( \Re(s) = 0 \), and by extending the argument of Method 4 we can show that the RHS of (2) also has a natural boundary at this line. Again, this is a contradiction.

3This is because \( \zeta(s) \) has infinitely many simple zeros on the critical line [5].
3 Theorem 2 of Vassilev-Missana is false

Vassilev-Missana \[11\] Theorem 2] makes the following claim.

Claim 4. For integer \( s > 1 \),

\[
P(s) = 1 - \sqrt{2/\zeta(s)} - \sqrt{2/\zeta(2s)} - \sqrt{2/\zeta(4s)} - \sqrt{2/\zeta(8s)} - \cdots \quad (3)
\]

Proof that Claim 4 is incorrect. Assume that Claim 4 is correct. Replacing \( s \) by \( 2s \) and using the result to simplify (3), we obtain

\[
1 - P(s) = \sqrt{2/\zeta(s)} - (1 - P(2s)).
\]

Squaring both sides of (4) and simplifying gives (2), but we showed in §2 that (2) is incorrect. This contradiction shows that Claim 4 is incorrect. \( \square \)

Remark 3. An alternative is to resort to a variation on method 2 above. With \( s = 2 \) we find numerically that \( P(s) \approx 0.4522 \) and

\[
1 - \sqrt{2/\zeta(s)} - \sqrt{2/\zeta(2s)} - \sqrt{2/\zeta(4s)} - \cdots \approx 0.4588 \neq P(s),
\]

where the numerical values are correct to 4 decimal places. Thus, (3) is incorrect.

Remark 4. It may not be clear what the infinite expression on the RHS of (3) means. We state Claim 4 more precisely as

\[
P(s) = 1 - \lim_{n \to \infty} \sqrt{2/\zeta(s)} - \sqrt{2/\zeta(2s)} - \sqrt{2/\zeta(4s)} - \cdots \sqrt{2/\zeta(2^n s)}.
\]

(5)

The limit exists and is real if \( s \) is real, positive, and sufficiently large.

To evaluate (5) numerically, we start with a sufficiently large value of \( n \), then evaluate the nested square roots in (5) by working from right to left, using the values of \( \zeta(2^n s), \zeta(2^{n-1} s), \ldots, \zeta(2 s), \zeta(s) \).

\( \square \)

\( ^4 \)As before, it is not clear why the integer restriction needs to be imposed.
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Postscript (21 March 2021)

Kannan Soundararajan kindly pointed out that there has been some dis-
cussion of the paper by Vassilev-Missana [11] on the MathOverflow website. For this, see [https://mathoverflow.net/questions/288847/](https://mathoverflow.net/questions/288847/).

In particular, the anonymous user Lucia pointed out the Dirichlet series argument (see Method 1 above) for disproving (2). Despite this, we have not found any erratum or retraction by the author of [11].

Since the comments by user Klangen on MathOverflow regarding the numerical investigation of Claim 4 are inconclusive, we mention that, for the accurate numerical evaluation of (5), it is desirable to use

\[
P(s) = 1 - \lim_{n \to \infty} \sqrt{2/\zeta(s) - \sqrt{2/\zeta(2s) - \sqrt{2/\zeta(4s) - \cdots}} - 1},
\]

(6)
since this has the same limit, but converges faster as \(n \to \infty\). An explanation is that, when \(n\) is large, we have \(\zeta(2^n s) = 1 + O(2^{-2^n s}) \approx 1\), so the “tail” of the expression \(3\) is approximately

\[X := \sqrt{2 - \sqrt{2 - \sqrt{2 - \cdots}}}.
\]

By squaring we see that \(X\) satisfies the quadratic equation \(X^2 = 2 - X\), whose only positive real root is \(X = 1\). Thus, it is better to approximate the tail by 1, as in (6), than by 0, as in (5).