Stochastic differential equation model for spontaneous emission and carrier noise in semiconductor lasers

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Abstract

We present a new stochastic differential equation model for the spontaneous emission noise and carrier noise in semiconductor lasers. The correlations between these two types of noise have often been neglected in recent studies of the effects of the noise on the laser dynamics. However, the classic results of Henry show that the intensity noise and the carrier noise are strongly negatively correlated. Our model demonstrates how to properly account for these correlations since the corresponding diffusion coefficients agree exactly with those derived by Henry. We show that in fact in the correct model the spontaneous emission noise and the carrier noise are driven by the same Wiener processes. Furthermore, we demonstrate that the nonzero correlation time of the physical noise affects the mean dynamics of both the electric field amplitude and the carrier number. We show that these are systematic corrections that can be described by additional drift terms in the model.

1 Introduction

The inherent noise in semiconductor laser systems is a complex topic [1–9] that has received renewed attention due to recent interest in the dynamics of semiconductor lasers with optical feedback [10–18]. Spontaneous recombination within the gain medium leads to two types of intrinsic noise: spontaneous emission noise and carrier noise. The former, also referred to as field noise, results from emitted photons in the lasing mode, and the latter, also referred to as shot noise, is due to the discrete and instantaneous nature of these events [16, 19]. Spontaneous emission noise affects the linewidth of a semiconductor laser and can give rise to the phenomenon of mode hopping in which the laser transitions from, e.g., multi-mode to single-mode operation [20]. Enhanced spontaneous emission and linewidth are distinguishing features of semiconductor lasers

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compared to other types of lasers and, in particular, contribute to the extreme sensitivity of semiconductor lasers to optical feedback [21].

In this paper we consider the problem of modeling the intrinsic noise in semiconductor lasers within the framework of stochastic differential equations (SDEs). Spontaneous emission noise is often accounted for in models of the semiconductor laser system by including a Langevin noise term in the field equation [10,12,14,17]. Such a system that contains Langevin noise terms is rigorously interpreted by defining Wiener processes that drive the noise. The advantage of introducing this additional mathematical framework is that the rich theory of stochastic differential equations (see, e.g., [22,23]) can then be employed to analyze the physical system. In Ref. [8] a Fokker-Planck equation for the field amplitude was derived, and from this the corresponding SDE accounting for the spontaneous emission noise was obtained. In addition to spontaneous emission noise, carrier noise has also been considered within the dynamical model by the inclusion of another Langevin noise term in the equations (see, e.g., [16,18]). However, an SDE model that precisely describes the relationship between the Wiener processes that drive both types of noise has not been derived or presented. We show in this paper that the SDE framework is necessary in order to correctly account for the correlations between the two types of noise within the dynamical model of the semiconductor laser system.

In numerical studies of the effects of the intrinsic noise on the laser dynamics, typically the correlations between the spontaneous emission noise and the carrier noise either have not been considered or have been explicitly neglected under the presumption that they are negligible. However, the classic work of Henry shows that these two types of noise are closely related. In Refs. [5,6], Henry derived the steady-state diffusion coefficients associated with the intensity and phase of the field and the carrier number. These results show that while the phase noise and the carrier noise are uncorrelated, the intensity noise and the carrier noise are strongly negatively correlated. While these diffusion coefficients characterize the statistical distribution of the noise, they do not describe the effects of the noise on the laser dynamics, for which an SDE model is necessary. In this paper we present a new SDE model for which the corresponding diffusion coefficients agree exactly with those derived by Henry. We therefore show how to correctly account for, within the dynamical model, the strong negative correlation between the intensity noise and the carrier noise. In particular, in this correct model the same Wiener processes that drive the spontaneous emission noise also drive the carrier noise.

We begin with the rate equations for the complex amplitude \( E \) of the electric field inside the cavity and the carrier number \( N \), in which we include terms representing the spontaneous emission noise and the carrier noise. We consider the more general model that includes a delayed feedback term that represents optical feedback. Our starting point is therefore the system (see, e.g., [10,12,14])
\[
\frac{dE}{dt}(t) = \frac{1 + i\alpha}{2} \left( G(E(t), N(t)) - \gamma \right) E(t) + \kappa e^{-i\omega \delta} E(t - \delta) + \zeta(t) \\
\frac{dN}{dt}(t) = \gamma_e \left( C N^\text{th} - N(t) \right) - G(E(t), N(t)) |E(t)|^2 + \xi(t)
\]

where \( G \) is the material gain function given by

\[
G(E(t), N(t)) = g \left[ N(t) - N^0 \right] \frac{1}{1 + s|E(t)|^2}.
\]
We will discuss below the precise meaning of the diffusion coefficients given by (3). First, we introduce an SDE model for which the corresponding diffusion coefficients agree exactly with (3). Let $E^R$ and $E^I$ denote the real and imaginary parts, respectively, of $E$. In order to state our model and the results of the next section as generally as possible we let the electron-hole recombination rate $S$ be an arbitrary function of the carrier number $N$. We will show that the diffusion coefficients given by (3) correspond to the noise terms in the following SDE system:

\[ dE_t = \frac{1 + i\alpha}{2} \left( G(E_t, N_t) - \gamma \right) E_t dt + \kappa e^{-i\omega\delta} E_{t-\delta} dt \]

\[ + \sqrt{2\beta N_t} \left( dW_t^A + i dW_t^B \right) \]  

\[ dN_t = \gamma e \left( C N_{th}^* - N_t \right) dt - G(E_t, N_t) |E_t|^2 dt \]

\[ - 2\sqrt{2\beta N_t} \left( E_t^R dW_t^A + E_t^I dW_t^B \right) \]

\[ + \sqrt{2S(N_t)} dW_t^C \]  

(4a)

where the Wiener processes $W^A, W^B, W^C$ are mutually independent (here and in the rest of the paper we use standard SDE notation). Note that the noise terms in both equations are multiplicative since the spontaneous emission noise depends on the carrier number through the spontaneous emission rate and the carrier noise depends on both the field and the carrier number. We also note that in the presence of optical feedback ($\kappa \neq 0$), (4) is a stochastic differential delay equation (SDDE) system. Therefore, in this case, its solution is non-Markovian.

An SDE model for the intensity and the phase can be found from (4a). The equation for the intensity is found by using the Itô product formula:

\[ dI_t = d \left( E_t E_t^* \right) = E_t dE_t^* + E_t^* dE_t + \left( dE_t \right) \left( dE_t^* \right) \]

where the last term is computed using the Itô calculus. The equation for the phase can then be obtained by using the Itô formula. The resulting equations for the intensity and the phase are

\[ dI_t = \left( G(I_t, N_t) - \gamma \right) I_t dt + 2\kappa \text{Re} \left( e^{i\omega\delta} E_t \overline{E_{t-\delta}} \right) dt \]

\[ + 4\beta N_t dt + 2\sqrt{2\beta N_t} \left( E_t^R dW_t^A + E_t^I dW_t^B \right) \]  

(5a)

\[ d\theta_t = \frac{\alpha}{2} \left( G(I_t, N_t) - \gamma \right) dt - \kappa \text{Re} \left( i e^{-i\omega\delta} \frac{E_t \overline{E_{t-\delta}}}{E_t} \right) dt \]

\[ + \sqrt{2\beta N_t} \left( E_t^R dW_t^B - E_t^I dW_t^A \right) \]  

(5b)

where $\text{Re}(z)$ denotes the real part of the complex number $z$, and where we have explicitly denoted the fact that the material gain function $G$ depends on $E_t$ only through $I_t$. 

4
We discuss here the case $\kappa = 0$ in which the solution of (4) is a Markov process so that there is an associated Fokker-Planck equation that describes the evolution of the probability density function of the system's state \[2\] (there is no Fokker-Planck equation for the non-Markovian system corresponding to the case $\kappa \neq 0$). In the white-noise driven model (4) the noise terms give rise to diffusion terms in the corresponding Fokker-Planck equation. The coefficients of these diffusion terms can be found from the stochastic differential equation system as follows. In the Fokker-Planck equation, $E$ and $N$ are independent variables, and so we let $\langle \cdot \rangle$ denote the expectation operator that treats these variables as such, i.e., as deterministic quantities. Let $F_I$, $F_\theta$, and $F_N$ denote the intensity, phase, and carrier noises, respectively, that is,

\[
F_I(t)dt = 2\sqrt{2\beta N_t} \left( E_t^R dW_t^A + E_t^I dW_t^B \right), \quad (6a)
\]

\[
F_\theta(t)dt = \frac{\sqrt{2\beta N_t}}{I_t} \left( E_t^R dW_t^B - E_t^I dW_t^A \right), \quad (6b)
\]

and

\[
F_N(t)dt = -2\sqrt{2\beta N_t} \left( E_t^R dW_t^A + E_t^I dW_t^B \right)
+ \sqrt{2S(N_t)}dW_t^C. \quad (6c)
\]

For $j, k = I, \theta, N$, the diffusion coefficients $D_{jk}$ are then given by

\[
\langle F_j(t)F_k(s) \rangle = 2D_{jk}\delta(t-s). \quad (7)
\]

The diffusion coefficients corresponding to the SDE system can be found by using (6), (7), and the independence of the Wiener processes $W_t^A$, $W_t^B$, and $W_t^C$. The diffusion coefficients obtained this way agree exactly with those derived by Henry that are given in \[3\].

3 Systematic corrections due to nonzero correlation time

In the system \[4\] the spontaneous emission noise and the carrier noise are represented by white noises, i.e., their correlation functions are taken to be delta functions so that their correlation times are equal to zero. Physically, the Langevin reservoir forces are not, however, delta-correlated; their correlation time is nonzero. Nonetheless, since their correlation time is small compared to all of the typical relaxation times of the system \[3\] they are normally modeled by white noise. However, when the noise in a system is multiplicative (state-dependent), the nonzero correlation time of the physical noise can affect the mean dynamics of the system. Furthermore, the magnitude of these effects does not depend on the size of the small correlation time, that is, these effects do not vanish for exceedingly small correlation times. Mathematically, these effects can be seen by using a colored noise process (i.e., a process having a
nonzero correlation time) to model the physical noise and subsequently taking
the white-noise limit of the system. This procedure gives rise to additional drift
terms in the system called the Stratonovich corrections that capture the effects
of the nonzero correlation time on the mean dynamics. This fact is sometimes
referred to as the Wong-Zakai theorem [25] and is discussed in, e.g., Gardiner’s
text [22].

Because the spontaneous emission noise and the carrier noise are multiplicative,
such effects are present in the semiconductor laser system. The purpose
of this section is to derive these corrections. To do this we replace the white
noises in (4) with noise processes having small but nonzero correlation times.
As a model for the noise we use the stationary Ornstein-Uhlenbeck colored noise
process. This choice is a good model of the physical noise since it is a Gaus-
sian process with a rapidly (exponentially) decaying correlation function. For

\[ i = A, B, C, \]

we define the process \( \eta^i \) as the stationary solution of the SDE

\[ d\eta^i_t = -\frac{1}{\tau}\eta^i_t dt + \frac{1}{\tau}dW^i_t \]  

where \( \tau > 0 \). The stationary solution of (8) is the solution corresponding to
the initial condition that has a mean-zero Gaussian distribution with var-
iance \((2\tau)^{-1}\). Defined this way, each \( \eta^i \) is a mean-zero Gaussian process with autoco-
variance function

\[ E[\eta^i_t \eta^j_s] = K(t-s) = \frac{1}{2\tau}e^{-|t-s|/\tau} \]  

(we let \( E[\cdot] \) denote expectation, in contrast to the electric field amplitude \( E \) which is never followed by brackets in this paper). For \( i \neq j \), the processes \( \eta^i \) and \( \eta^j \) are independent by the independence of \( W^i \) and \( W^j \). In view of (8) each
process \( \eta^i \) has correlation time \( \tau \), and \( K(t-s) \to \delta(t-s) \) as \( \tau \to 0 \). More
precisely, as \( \tau \to 0 \),

\[ \int_0^t \eta^i_s ds \to W^i_t \]

(see, e.g., [26]). This justifies replacing in (4) each \( dW^i_t \) with \( \eta^i_t dt \) to get

\[ dE_t = \frac{1 + i\alpha}{2} \left( G(E_t, N_t) - \gamma \right) E_t dt + \kappa e^{-i\omega \delta} \int_0^t E_{t-s} ds dt + \sqrt{2\beta N_t} \left( \eta^A_t + i\eta^B_t \right) dt \]

\[ dN_t = \gamma e \left( CN^{\text{th}} - N_t \right) dt - G(E_t, N_t) |E_t|^2 dt - 2\sqrt{2\beta N_t} \left( E_t^A \eta^A_t + E_t^B \eta^B_t \right) dt \]

\[ + \sqrt{2S(N_t)} \eta^C_t dt \]  

(10a)

(10b)

We now derive the limit of the system (10) as the correlation time \( \tau \) of the
noises goes to zero. We show that the resulting white-noise driven system is

\[
dE_t = \frac{1 + i\alpha}{2} \left( G(E_t, N_t) - \gamma \right) E_t dt + \kappa e^{-i\omega \delta} E_{t-\delta} dt - \beta E_t dt + \sqrt{2\beta N_t} \left( dW_t^A + idW_t^B \right)
\]

\[
dN_t = \gamma e \left( C N^{\text{th}} - N_t \right) dt - G(E_t, N_t) |E_t|^2 dt + 2\beta |E_t|^2 dt - 4\beta N_t dt + \frac{1}{2} S'(N_t) dt
\]

\[
- 2\sqrt{2\beta N_t} \left( E_t^R dW_t^A + E_t^l dW_t^B \right) + \sqrt{2S(N_t) dW_t^C}
\]

(11a)

\[
dN_t = \gamma e \left( C N^{\text{th}} - N_t \right) dt - G(E_t, N_t) |E_t|^2 dt + 2\beta |E_t|^2 dt - 4\beta N_t dt + \frac{1}{2} S'(N_t) dt
\]

\[
- 2\sqrt{2\beta N_t} \left( E_t^R dW_t^A + E_t^l dW_t^B \right) - \frac{\sqrt{2S(N_t)} dW_t^C}{2}
\]

(11b)

where \( S' \) is the derivative of the electron-hole recombination rate. Note that the system (11) is not equal to (4); there are additional drift terms in (11) that are due to the dependence of the strengths of the noises on the carrier number and the field. These terms are \( -\beta E_t dt \) in (11a) and \( 2\beta |E_t|^2 dt, -4\beta N_t dt, \) and \( \frac{1}{2} S'(N_t) dt \) in (11b). We now turn to the derivation of (11).

### 3.1 Derivation

For convenience we let \( f(E_t, E_{t-\delta}, N_t) dt \) denote the first two terms on the right-hand side of (10a) and \( h(E_t, N_t) dt \) denote the first two terms on the right-hand side of (10b). We solve (8) for \( \eta_t dt \) and substitute the resulting expression into (10) to get

\[
dE_t = f(E_t, E_{t-\delta}, N_t) dt - \sqrt{2\beta N_t} \left( \tau d\eta^A_t + i\tau d\eta^B_t \right)
\]

\[
+ \sqrt{2\beta N_t} \left( dW_t^A + idW_t^B \right)
\]

\[
dN_t = h(E_t, N_t) dt + 2\sqrt{2\beta N_t} \left( E_t^R \tau d\eta^A_t + E_t^l \tau d\eta^B_t \right)
\]

\[
- 2\sqrt{2\beta N_t} \left( E_t^R dW_t^A + E_t^l dW_t^B \right)
\]

\[
- \sqrt{2S(N_t)} \tau d\eta^C_t + \sqrt{2S(N_t)} dW_t^C .
\]
It is useful to work with the integral form of this system:

\[ E_t = E_0 + \int_0^t f(E_s, E_{s-\delta}, N_s) \, ds \]

\[ - \int_0^t \sqrt{2\beta N_s} \left( \tau \eta^A_s + i \tau \eta^B_s \right) \, ds + \int_0^t \sqrt{2\beta N_s} \left( dW^A_s + i dW^B_s \right) \]  

(12a)

\[ N_t = N_0 + \int_0^t h(E_s, N_s) \, ds \]

\[ + \int_0^t \sqrt{2\beta N_s} \left( E^R_s \tau \eta^A_s + E^I_s \tau \eta^B_s \right) \, ds \]

\[ - \int_0^t \sqrt{2\beta N_s} \left( E^R_s dW^A_s + E^I_s dW^B_s \right) \]

\[ - \int_0^t \sqrt{2\beta N_s} \left( dW^C_s + \int_0^t \sqrt{2\beta N_s} dW^C_s \right) \]

We integrate by parts the second integral in (12a) and the second and fourth integrals in (12b) to get

\[ E_t = E_0 - \sqrt{2\beta N_t} \left( \tau \eta^A_t + i \tau \eta^B_t \right) \]

\[ + \sqrt{2\beta N_0} \left( \tau \eta^A_0 + i \tau \eta^B_0 \right) + \int_0^t f(E_s, E_{s-\delta}, N_s) \, ds \]

\[ + \int_0^t \sqrt{\frac{\beta}{2 N_s}} \left( \tau \eta^A_s + i \tau \eta^B_s \right) \, dN_s \]

\[ + \int_0^t \sqrt{2\beta N_s} \left( dW^A_s + i dW^B_s \right) \]  

(13a)

\[ N_t = N_0 + 2\sqrt{2\beta N_t} \left( E^R_t \tau \eta^A_t + E^I_t \tau \eta^B_t \right) \]

\[ - 2\sqrt{2\beta N_0} \left( E^R_0 \tau \eta^A_0 + E^I_0 \tau \eta^B_0 \right) - \sqrt{2S(N_t)} \tau \eta^C_t \]

\[ + \sqrt{2S(N_0)} \tau \eta^C_0 + \int_0^t h(E_s, N_s) \, ds \]

\[ - \int_0^t \sqrt{\frac{2\beta}{N_s}} \left( E^R_s \tau \eta^A_s + E^I_s \tau \eta^B_s \right) \, dN_s \]

\[ - \int_0^t \sqrt{2\beta N_s} \left( \tau \eta^A_s dE^R_s + \tau \eta^B_s dE^I_s \right) \]

\[ - \int_0^t \sqrt{2\beta N_s} \left( E^R_s dW^A_s + E^I_s dW^B_s \right) \]

\[ + \int_0^t \frac{S'(N_s)}{\sqrt{2S(N_s)}} \tau \eta^C_s \, dN_s + \int_0^t \sqrt{2S(N_s)} dW^C_s \]  

(13b)
Note that since the solution \((E, N)\) of \((10)\) is differentiable the Itô terms arising from the stochastic integration by parts formula are zero. At this point we need the equations for \(E_R\) and \(E_I\) which follow from \((10a)\):

\[
dE_R^t = \text{Re}\left( f(E_t, E_{t-\delta}, N_t) \right) dt + \sqrt{2\beta N_t} \eta^A_t dt \tag{14a}
\]
\[
dE_I^t = \text{Im}\left( f(E_t, E_{t-\delta}, N_t) \right) dt + \sqrt{2\beta N_t} \eta^B_t dt . \tag{14b}
\]

In \((13)\) we substitute \((10b)\), \((14a)\), and \((14b)\) for \(dN_s\), \(dE_R^t\), and \(dE_I^t\) to get

\[
E_t = E_0 + U^{E}_t + \int_0^t f(E_s, E_{s-\delta}, N_s) ds \\
+ H^{E}_t + \int_0^t \sqrt{2\beta N_s} \left( dW^A_s + idW^B_s \right)
\]
\[
N_t = N_0 + U^{N}_t + \int_0^t h(E_s, N_s) ds + H^{N}_t \\
- \int_0^t 2\sqrt{2\beta N_s} \left( E^{R}_s dW^A_s + E^{I}_s dW^B_s \right) \\
+ \int_0^t \sqrt{2S(N_s)} dW^C_t
\]

where

\[
U^{E}_t = -\sqrt{2\beta N_t} \left( \tau \eta^A_t + i\tau \eta^B_t \right) + \sqrt{2\beta N_0} \left( \tau \eta^A_0 + i\tau \eta^B_0 \right) \\
+ \int_0^t \sqrt{\frac{\beta}{2N_s}} h(E_s, N_s) \left( \tau \eta^A_s + i\tau \eta^B_s \right) ds ,
\]
\[
H^{E}_t = -\int_0^t 2\beta \left( \tau \eta^A_s + i\tau \eta^B_s \right) \left( E^{R}_s \eta^A_s + E^{I}_s \eta^B_s \right) ds \\
+ \int_0^t \sqrt{\frac{\beta S(N_s)}{N_s}} \left( \tau \eta^A_s + i\tau \eta^B_s \right) \eta^C_s ds ,
\]
\[
U_i^N = 2\sqrt{\beta N_i} \left( E_i^R \tau \eta_i^A + E_i^I \tau \eta_i^B \right) \\
- 2\sqrt{\beta N_0} \left( E_0^R \tau \eta_0^A + E_0^I \tau \eta_0^B \right) \\
- \sqrt{2S(N_i) \tau \eta_i^C} + \sqrt{2S(N_0) \tau \eta_0^C} \\
- \int_0^t \sqrt{\frac{2\beta}{N_s}} h(E_s, N_s) \left( E_s^R \tau \eta_s^A + E_s^I \tau \eta_s^B \right) ds \\
- \int_0^t 2\sqrt{2\beta N_s} \text{Re} \left( f(E_s, E_s-\delta, N_s) \right) \tau \eta_s^A ds \\
- \int_0^t 2\sqrt{2\beta N_s} \text{Im} \left( f(E_s, E_s-\delta, N_s) \right) \tau \eta_s^B ds \\
+ \int_0^t \frac{S'(N_s)}{\sqrt{2S(N_s)}} h(E_s, N_s) \tau \eta_s^C ds ,
\]

and

\[
H_i^N = \int_0^t 4\beta \tau \left( E_s^R \eta_s^A + E_s^I \eta_s^B \right)^2 ds \\
- \int_0^t 4\sqrt{\beta S(N_s) \eta_s^C} \left( E_s^R \tau \eta_s^A + E_s^I \tau \eta_s^B \right) \eta_s^C ds \\
- \int_0^t 4\beta N_s \left( \tau (\eta_s^A)^2 + \tau (\eta_s^B)^2 \right) ds \\
- \int_0^t 2\sqrt{\frac{\beta N_s}{S(N_s)}} S'(N_s) \tau \eta_s^C \left( E_s^R \eta_s^A + E_s^I \beta \eta_s^B \right) ds \\
+ \int_0^t S'(N_s) \tau (\eta_s^C)^2 ds .
\]

Now, in view of (9), for all \( i \) and all \( s \), \( \tau \eta_s^i \) has mean zero and variance \( \tau/2 \). Hence, \( \tau \eta_s^i \to 0 \) as the correlation time \( \tau \to 0 \). Therefore, \( U_i^E \) and \( U_i^N \) converge to zero as \( \tau \to 0 \). The terms \( H_i^E \) and \( H_i^N \) converge to the integrals of the additional drift terms in (11a) and (11b), respectively. More precisely, as the
correlation time $\tau \to 0$, 
\[- \int_0^t 2\beta \left( \tau \eta^A_s + i \tau \eta^B_s \right) (E^R_s \eta^A_s + E^I_s \eta^B_s) ds \quad \rightarrow \quad - \int_0^t \beta E_s ds , \quad (15a)\]
\[
\int_0^t 4\beta \tau \left( E^R_s \eta^A_s + E^I_s \eta^B_s \right)^2 ds \quad \rightarrow \quad \int_0^t 2\beta |E_s|^2 ds , \quad (15b)\]
\[- \int_0^t 4\beta N_s \left( \tau (\eta^A_s)^2 + \tau (\eta^B_s)^2 \right) ds \quad \rightarrow \quad - \int_0^t 4\beta N_s ds , \quad (15c)\]
\[
\int_0^t S'(N_s) \tau (\eta^C_s)^2 ds \quad \rightarrow \quad \int_0^t \frac{1}{2} S'(N_s) ds , \quad (15d)\]
and
\[
\int_0^t \sqrt{\frac{\beta S(N_s)}{N_s}} \left( \tau \eta^A_s + i \tau \eta^B_s \right) \eta^C_s ds , \]
\[- \int_0^t 2 \sqrt{\frac{\beta S(N_s)}{N_s}} \left( E^R_s \tau \eta^A_s + E^I_s \tau \eta^B_s \right) \eta^C_s ds , \quad (15e)\]
\[- \int_0^t 2 \sqrt{\frac{\beta N_s}{S(N_s)}} \frac{S'(N_s) \tau \eta^C_s}{S(N_s)} \left( E^R_s \eta^A_s + E^I_s \eta^B_s \right) ds \quad \rightarrow \quad 0 . \]
We will show (15a). The limits (15b) – (15e) follow by the same general argument. Let
\[
I \equiv \int_0^t \left( 2 \left( \tau \eta^A_s + i \tau \eta^B_s \right) (E^R_s \eta^A_s + E^I_s \eta^B_s) - E_s \right) ds . \]
The limit (15a) is equivalent to $I \to 0$, which is a consequence of
\[
I_1 \equiv \int_0^t \left( 2 \tau (\eta^A_s)^2 E^R_s - E^R_s \right) ds \quad \rightarrow \quad 0 , \quad (16a)\]
\[
I_2 \equiv \int_0^t \left( 2i \tau (\eta^B_s)^2 E^I_s - i E^I_s \right) ds \quad \rightarrow \quad 0 , \quad (16b)\]
and
\[
I_3 \equiv \int_0^t 2 \tau \eta^A_s \eta^B_s (E^I_s + i E^R_s) ds \quad \rightarrow \quad 0 . \quad (16c)\]
We show (16a); the limits (16b) and (16c) follow similarly. We first define the new process $\tilde{\eta}^A$ by $\tilde{\eta}^A_s = \sqrt{\tau} \eta^A_s$. Then $\tilde{\eta}^A$ solves the equation
\[
d\tilde{\eta}^A_s = -\tilde{\eta}^A_s ds + d\tilde{W}^A_s \quad (17)\]
where \( \tilde{W}^A \) is the Wiener process defined by \( \tilde{W}^A_s = \tau^{-1/2} W^{\tau} \). We express the integral \( I_1 \) in terms of this process:

\[
I_1 = \int_0^t E^R_s \left( 2 \left( \tilde{\eta}^A_{s/t} \right)^2 - 1 \right) ds .
\]

Let \( \{ t_j : 0 \leq j \leq n \} \) be a partition of \([0, t]\) such that \( t_{j+1} - t_j = t/n = O(\sqrt{\tau}) \). We can then write

\[
I_1 = \tau \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}/\tau} E^R_{t_j} \left( 2 \left( \tilde{\eta}^A_{r/t} \right)^2 - 1 \right) dr .
\]

In each of the integrals in this sum the length of the interval of integration is \( O(\tau^{-1/2}) \). Thus, since the slower variable \( E^R \) evolves on a timescale of order one (in \( \tau \)), for \( \tau \) sufficiently small we can approximate \( I_1 \) by

\[
\tau \sum_{j=0}^{n-1} E^R_{t_j} \int_{t_j}^{t_{j+1}/\tau} \left( 2 \left( \tilde{\eta}^A_{r/t} \right)^2 - 1 \right) dr .
\]

We now estimate

\[
\left| \int_{t_j}^{t_{j+1}/\tau} \left( 2 \left( \tilde{\eta}^A_{r/t} \right)^2 - 1 \right) dr \right| .
\]

In view of (19), we have \( E[|\tilde{\eta}^A_{r/t}|^2] = 1/2 \). Therefore, for fixed \( r \), the integrand in (19) is a mean-zero random variable. Thus, the integral in (19) can be thought of as a sum of \( O(\tau^{-1/2}) \) identically distributed (by the stationarity of \( \eta^A \)) mean-zero random variables. Furthermore, equations (17) and (9) show that these random variables are weakly correlated, since the covariance function of the process \( \tilde{\eta}^A \) decays exponentially with an exponential decay constant equal to one. Thus, in view of the law of large numbers, (19) is on the order of \( \tau^{-1/4} \) (this can also be shown by explicitly calculating the second moment of (19)). Therefore, since \( n = O(\tau^{-1/2}) \), an upper bound for the order of (15) is \( \tau^{1/4} \) (this is, in fact, an overestimate since it does not take into account cancellations between positive and negative terms in the sum). This estimate gives (16a). The limits (16b), (16c), and (15b) – (15e) follow from the same general argument, using the fact that, for \( i \neq j \), \( E[\eta^A_i \eta^A_j] = 0 \) by the independence of \( \eta^A_i \) and \( \eta^A_j \). This completes the derivation of (11).

4 Conclusion

We have considered the problem of modeling spontaneous emission noise and carrier noise within the framework of stochastic differential equations. The
stochastic differential equation model that we presented has corresponding diffusion coefficients that agree exactly with those derived in the classic works of Henry [5,6]. While recent studies of the effects of the noise on the laser dynamics have neglected the correlations between the spontaneous emission noise and the carrier noise, our model describes the strong negative correlation between the intensity noise and the carrier noise. More precisely, we have shown that the same Wiener processes that drive the spontaneous emission noise also drive the carrier noise. This result could have important consequences for the dynamics. For example, over any time interval the contribution to the carrier number of the part of the carrier noise that does not involve the electron-hole recombination rate $S$ is exactly the negative of the contribution of the spontaneous emission noise to the intensity of the electric field. Clearly then, the inherent noise cannot be adequately described by previously used dynamical models in which the spontaneous emission noise and the carrier noise are uncorrelated. In the presence of optical feedback in particular, there has been observed a disagreement between the experimentally measured and the numerically calculated values of certain properties of the semiconductor laser system. While the inclusion of noise in the dynamical model gives better agreement, previously used models do not satisfactorily reproduce for example features of the intensity distribution such as the narrow peak at large intensities [17].

We have also shown that the nonzero correlation time of the physical noise affects the semiconductor laser dynamics in a systematic way. The magnitude of these effects is independent of the size of the small correlation time of the inherent noise and thus does not vanish for exceedingly small correlation times. The effects of the nonzero correlation time of the physical noise are captured by the additional drift terms in (11). To be precise, these effects are the combined result of the dependence of the strength of the spontaneous emission noise on the carrier number, the dependence of the strength of the carrier noise on both the carrier number and the field, the correlations between the spontaneous emission noise and the carrier noise, and the nonzero correlation time of the noise. The system (11) reveals that the spontaneous emission noise and carrier noise affect the mean values of both the complex amplitude of the field and the carrier number. Comparing (11) with (1), we see that simply ignoring the noise terms in (1) does not yield the correct deterministic part of the system because there is a contribution from the stochastic fluctuations. This contribution is the result of correlations in the physical noise, which influence the mean dynamics of the semiconductor laser system.

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