Qubit decoherence by low-frequency noise

K. Rabenstein, V.A. Sverdlov, and D.V. Averin

Department of Physics and Astronomy, Stony Brook University, SUNY, Stony Brook, NY 11794-3800

(Dated: June 16, 2018)

We have derived explicit non-perturbative expression for decoherence of quantum oscillations in a qubit by low-frequency noise. Decoherence strength is controlled by the noise spectral density at zero frequency while the noise correlation time \( \tau \) determines the time \( t \) of crossover from the \( 1/\sqrt{\tau} \) to the exponential suppression of coherence. We also performed Monte Carlo simulations of qubit dynamics with noise which agree with the analytical results and show that most of the conclusions are valid for both Gaussian and non-Gaussian noise.

PACS numbers: 03.65.Yz, 03.67.Lx, 72.70.+m
have the value $v$, where $v_0 \ll \Delta$, one can determine the rate of accumulation of this phase by expanding the energies in noise amplitude $v(t)$. Also, in this case the dephasing rate is larger than the transition rate and can be calculated disregarding the transitions. The factor $F(t)$ describing suppression in time of coherence between the two states (i.e., suppression of the off-diagonal element $\rho_{12}$ of the qubit density matrix in the energy basis; $\rho_{12}(t) = F(t)\rho_{12}(0)e^{-i\delta t}$) can be written then as follows:

$$
F(t) = \langle \exp\{-i \int_0^t \frac{v(t')}{\Omega} + \frac{\Delta^2 v^2(t')}{2\Omega^3}dt'\} \rangle. \tag{3}
$$

For Gaussian noise, the correlation function $\overline{v}$ determines the noise statistics completely. In this case, it is convenient to take the average in Eq. (3) by writing it as a functional integral over noise. For this purpose, and also for use in the numerical simulations, we start with the “transition” probability $p(v_1, v_2, \delta t)$ for the noise to have the value $v_2$ a time $\delta t$ after it had the value $v_1$:

$$
p(v_1, v_2, t) = \langle \exp\{-i \int_0^t \frac{\varepsilon v(t')}{\Omega} + \frac{\Delta^2 v^2(t')}{2\Omega^3}dt'\} \rangle. \tag{4}
$$

Using this expression we introduce the probability of specific noise realization as $p_0(v_1)\cdot p(v_1, v_2, \delta t_1)\cdot p(v_2, v_3, \delta t_2)\cdot \ldots$, where $p_0(v) = (2\pi v_0^2)^{-1/2}\exp\{-v^2/2v_0^2\}$ is the stationary Gaussian probability distribution of $v$. Taking the limit $\delta t_i \to 0$ we see that the average over the noise can be written as the following function integral:

$$
\langle \ldots \rangle = \int dv_0 dv(t) Dv(t')\ldots \times \exp\left\{-\frac{v(0)^2 + v(t)^2}{4v_0^2} - \frac{1}{4v_0^2}\int_0^t dt'\left(\tau^2 v^2 + v'^2\right)\right\}. \tag{5}
$$

Since the average in Eq. (3) with the weight (5) is now given by the Gaussian integral, it can be calculated straightforwardly:

$$
F(t) = F_0(t) \exp\left\{-\alpha^2\left(\frac{\nu t}{\tau} - 2\text{coth}\frac{\nu t}{2\tau} + \nu^{-1}\right)\right\}, \tag{6}
$$

$$
F_0(t) = \varepsilon^{\nu/2\tau}[\cosh(\nu t/\tau)]^{1/2} + \frac{1}{2\nu}\sinh(\nu t/\tau)^{-1/2},
$$

where $\nu = \sqrt{1 + 2v_0^2\Delta^2/\Omega^3}$ and $\alpha \equiv \varepsilon\tau v_0/\Omega\nu^{3/2}$.

Equation (6) is our main analytical result for dephasing by the Gaussian noise. To analyze its implications, we start with the case $\varepsilon = 0$, where pure qubit dephasing vanishes in the standard perturbation theory. Dephasing (6) is still non-vanishing and its strength depends on the noise spectral density at zero frequency $S_v(0) = 2v_0^2\tau$ through $\nu = \sqrt{1 + i\epsilon\delta, \tau \equiv S_v(0)/\Delta}$. For small and large times $t$ Eq. (6) simplifies to:

$$
F(t) = \left\{ \begin{array}{ll}
\frac{1 + t/\tau}{1 + t/\tau + ist/2\tau}, & t \ll \tau, \\
\frac{\sqrt{e^{-(\gamma+\delta)t}(1 + \nu)} - 1}{2\sqrt{\nu}} + 1, & t \gg \tau,
\end{array} \right.
$$

where

$$
\gamma = \frac{1}{2\tau} \left[ \frac{(1 + s^2)^{1/2} + 1}{2} \right]^{1/2} - 1. \tag{7}
$$

Besides suppressing coherence, the noise also shifts the frequency of qubit oscillations. The corresponding frequency renormalization is well defined for $t \gg \tau$:

$$
\delta = \frac{1}{2\tau} \left[ \frac{(1 + s^2)^{1/2} - 1}{2} \right]^{1/2}. \tag{8}
$$
Suppression of coherence \( t \ll \tau \) for Gaussian noise can be qualitatively understood as the result of averaging over the static distribution of noise \( v \). In contrast to this, at large times \( t \gg \tau \), the noise appears to be \( \delta \)-correlated, the fact that naturally leads to the exponential decay \( 7 \).

This interpretation means that the two regimes of decay should be generic to different models of the low-frequency noise. Indeed, they exist for the non-Gaussian noise considered below, and are also found for Gaussian noise with \( 1/f \) spectrum \[ 15 \]. Crossover between the two regimes takes place at \( t \simeq \tau \), and the absolute value of \( F(t) \) in the crossover region can be estimated as \( (1 + s^2)^{-1/4} \), i.e. \( s \) determines the amount of coherence left to decay exponentially. The rate \( 8 \) of exponential decay shows a transition from the quadratic to square-root behavior as a function of \( s \).

The rate of exponential decay at large times \( t \gg \tau \) is given by the same Eq. (2).

Non-zero qubit bias \( \varepsilon \) leads to additional dephasing \( F(t)/F_0(t) \) described by the last exponential factor in Eq. (6). One can see that similar to \( F_0(t) \), additional dephasing exhibits the crossover at \( t \simeq \tau \) from “inhomogeneous broadening” (averaging over the static distribution of the noise \( v \)) to exponential decay at \( t \gg \tau \). In contrast to \( F_0(t) \), the short-time decay is now Gaussian:

\[
\ln \left( \frac{F(t)}{F_0(t)} \right) = \frac{\varepsilon^2}{\Omega^2} \int_0^t dt' v(t') / (1 + is(\Delta/\Omega)^3), \quad t \ll \tau.
\]

We see that, again, the rate of exponential decay depends non-trivially on the noise spectral density \( S_v(0) \), changing from direct to inverse proportionality to \( S_v(0) \) at small and large \( s \), respectively.

Our approach can be used to calculate the rate of exponential decay at large times \( t \) for Gaussian noise with arbitrary spectral density \( S_v(\omega) \). Such a noise can be represented as a sum of noises \( 2 \) and appropriate transformation of variables in this sum enables one to write the average over the noise as a functional integral similar to \( 9 \). For calculation of the relaxation rate at large \( t \), the boundary terms in the integral \( 9 \) can be neglected and it is dominated by the contribution from the “bulk” which can be conveniently written in terms of the Fourier components

\[
v_n = (2/t)^{1/2} \int_0^t dt' v(t') \sin \omega_n t', \quad \omega_n = \pi n / t.
\]

Then, \( \langle ... \rangle = \int Dv... \exp\left\{ -(1/2) \sum_n v_n^2 / S_v(\omega_n) \right\} \). Combining this equation and Eq. (4) we get at large \( t \):

\[
F(t) = \exp \left\{ -\frac{t}{2} \left[ \frac{\varepsilon^2 \Omega S_v(0)}{\Omega^3 + iS_v(0) \Delta^2} + \frac{1}{\pi} \int_0^\infty d\omega \ln(1 + iS_v(\omega) \Delta^2 / \Omega^2) \right] \right\}.
\]

For unbiased qubit, \( \varepsilon = 0 \), this equation coincides with the one obtained by more involved diagrammatic perturbation theory in quadratic coupling \([13]\).

To check how well the analytic theory described above works for finite noise amplitude \( v_0 \), and to see how sensitive the results are to the assumption of the Gaussian noise, we performed Monte Carlo simulations of the qubit oscillations under the influence of Gaussian and non-Gaussian noise. We looked specifically at the coherent oscillations of a qubit with Hamiltonian \( 11 \) that start in one of the eigenstates of the \( \sigma_z \) operator, focusing on the case \( \varepsilon = 0 \). The qubit density matrix was averaged over up to \( 10^7 \) realizations of noise. In the case of Gaussian noise, realizations were built using the transition probability \( 4 \). For non-Gaussian noise we used the model of Ref. \([16]\), which should provide an appropriate description of the situation when a qubit is coupled to several fluctuators with similar characteristic time scale \( \tau \) of the fluctuations. In this model the fluctuators create random qubit bias \( v \) which remains constant for some (random) time interval after which it is updated and the new value remains constant during the next time interval, etc. The time intervals between bias updates are taken to be distributed according to the Poisson distribution with characteristic time \( \tau \). For more direct comparison with the Gaussian noise, we assumed Gaussian distribution \( p_0(v) \) of \( v \). The correlation function of \( v(t) \) defined in this way is given by the same Eq. (4).

Example of the oscillations dephased by such a noise is given in Fig. 2. It shows real part of the off-diagonal

![FIG. 2: The profile of coherent quantum oscillations in an unbiased qubit dephased by the non-Gaussian noise with characteristic amplitude \( v_0 = 0.15\Delta \) and correlation time \( \tau = 300\Delta^{-1} \) obtained by direct simulation of qubit dynamics with noise. Solid line is the exponential fit of the oscillation amplitude at large times. Dashed line is the initial \( 1/\sqrt{\tau} \) decay caused by effectively static distribution of \( v \).](image)
element $\rho_{12}(t)$ of the qubit density matrix in the energy eigenstates basis. For oscillations starting in one of the $\sigma_z$ eigenstates, $\rho_{12}(0) = 1/2$. Similarly to the case of Gaussian noise, we consider only weak noise, $v_0 \ll \Delta$.

In this case, there is a crossover at $t \approx \tau$ in the oscillation amplitude from the initial $1/\sqrt{t}$ suppression of coherence due to averaging over static potential distribution: $\rho_{12}(t) = \rho_{12}(0)/(1 + iv_0^2 t/\Delta)^{1/2}$ (neglecting all terms of order $t/\tau$), to exponential suppression at $t \gg \tau$.

The rate of the exponential decay can be found analytically as follows. Expansion of the average qubit density matrix $\rho(t)$ in the number of “jumps” of $v(t)$ leads to the Dyson-like equation for its evolution [10]:

$$\rho(t) = e^{-t/\tau} \langle S(t,0) \rho(0) S(t,0) \rangle + \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} \langle S(t,t') \rho(t') S(t,t') \rangle,$$  \hspace{1cm} (11)

where $\langle \rangle$ denotes the average over the distribution of $v$. For weak noise, introducing slowly-varying amplitude $r$ of $\rho_{12}(t) = r(t) e^{-\Delta t}$, one can reduce Eq. (11) to the equation for $r(t)$ neglecting rapidly oscillating terms:

$$r(t) = e^{-t/\tau} r(0)(1 + iv_0^2 t/\Delta)^{-1/2} + \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} r(t')(1 + iv_0^2 (t - t')/\Delta)^{-1/2}.$$  \hspace{1cm} (12)

With the exponential ansatz for $r(t)$: $r(t) \propto e^{-(1-\lambda)t/\tau}$ Eq. (12) gives then equation for the parameter $\lambda$:

$$\lambda = \int_0^\infty dx e^{-x} \left[ 1 + \frac{ixs}{2\lambda} \right]^{-1/2}. \hspace{1cm} (13)$$

(Omission of the first term in Eq. (12) is justified by the final result for the oscillation decay rate $\gamma = (1 - \text{Re}[\lambda])/\tau$.) Asymptotics of $\gamma$ found from Eq. (13) are:

$$\gamma = \frac{1}{\tau} \times \left\{ \begin{array}{l} s^2/8, \quad s \ll 1, \\ 1 - 16\pi/s^2, \quad s \gg 1. \end{array} \right. \hspace{1cm} (14)$$

The rate $\gamma(s)$ evaluated from Eq. (13) is shown in Fig. 1b, together with the pure dephasing rates found numerically by fitting the oscillation amplitude (similar to that shown in Fig. 2) at $t \gg \tau$ and subtracting the contribution $S_\gamma(\Delta)$ to dephasing from real transitions. One can see that Eq. (13) indeed gives an accurate description of pure dephasing rates. Similarly to the case of Gaussian noise, $\gamma$ depends only on $S_\gamma(0)$. In both situations, $\gamma \propto v_0^4 \tau/\Delta^2$ for small $v_0^2 \tau/\Delta$, the fact that can be explained by the lowest-order perturbation theory in qubit energy fluctuations. In the non-perturbative regime, however, the behavior of $\gamma$ as function of $S_\gamma(0)$ is model-dependent and varies from saturation [14] to $\sqrt{S_\gamma(0)}$-growth [5].

In summary, we developed non-perturbative theory of qubit dephasing within two models of Gaussian and non-Gaussian low-frequency noise and performed Monte Carlo simulations of qubit dynamics within these models. The theory agrees well with simulations and shows that the decoherence strength is controlled by the noise spectral density at zero frequency. It allows for generalizations in several experimentally-relevant directions and should be useful for analysis of observed shapes of quantum qubit oscillations.

This work was supported in part by ARDA and DOD under the DURINT grant # F49620-01-1-0439 and by the NSF under grant # 032551. The authors would like to thank T. Duty, K. Likharev, J. Lukens, Yu. Makhlin, Y. Nakamura, Yu. Pashkin, and A. Schnirman for useful discussions.

References:

[1] Y. Nakamura, Yu.A. Pashkin, and J.S. Tsai, Nature 398, 786 (1999).
[2] J.R. Friedman, V. Patel, W. Chen, S.K. Tolpygo, and J.E. Lukens, Nature 406, 43 (2000).
[3] C.H. van der Wal, A.C.J. ter Haar, F.K. Wilhelm, R.N. Schouten, C. Harms, T.P. Orlando, S. Lloyd, and J.E. Mooij, Science 290, 773 (2000); I. Chiorescu, Y. Nakamura, C.J.P.M. Harmans, and J.E. Mooij, Science 299, 1869 (2003).
[4] D. Vion, A. Aassime, A. Cottet, P. Joyez, H. Pothier, C. Urbina, D. Esteve, and M.H. Devoret, Science 296, 886 (2002).
[5] J.M. Martinis, S. Nam, J. Aumentado, and C. Urbina, Phys. Rev. Lett. 89, 117901 (2002).
[6] E. Il'ichev, N. Oukhanski, A. Izmalkov, Th. Wagner, M. Grjancar, H.-G. Meyer, A. Yu. Smirnov, A.M. van den Brink, M.H.S. Amin, and A.M. Zagosskin, Phys. Rev. Lett. 91, 097906 (2003).
[7] T. Duty, D. Gunnarsson, K. Bladh, R.J. Schoelkopf, P. Delsing, cond-mat/0305433.
[8] J. Claudon, F. Balestro, F.W.J. Hekking, and O. Buisson, “Coherent oscillations in a current-biased dc SQUID”, (2004).
[9] Yu. A. Pashkin, T. Yamamoto, O. Astafiev, Y. Nakamura, D.V. Averin, and J.S. Tsai, Nature 421, 823 (2003).
[10] K. Blum, Density Matrix Theory and Applications, (Plenum, NY, 1991).
[11] Y. Nakamura, Yu.A. Pashkin, T. Yamamoto, and J.S. Tsai, Phys. Rev. Lett. 88, 047901 (2002).
[12] K. Rabenstein, and D. V. Averin, Turk. J. Phys. 27, 313 (2003); cond-mat/0310193.
[13] E. Paladino, L. Faoro, G. Falci, and R. Fazio, Phys. Rev. Lett. 88, 228304 (2002); E. Paladino, L. Faoro, G. Falci, Adv. Sol. State Phys. 43, 747 (2003); T. Ikutaka and Y. Tokura, Phys. Rev. B 67, 195320 (2003); Y.M. Galperin, B.L. Altshuler, D.V. Shantsev, cond-mat/0312490.
[14] N.V. Prokof'ev and P.C.E. Stamp, Rep. Prog. Phys. 63, 669 (2000).
[15] Yu. Makhlin and A. Schnirman, JETP Lett. 78, 497 (2003); cond-mat/0308297.
[16] A.I. Burshtein, JETP 21, 567 (1965).