Flat top solitons on linear gaussian potential

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Abstract. The study of Nonlinear Schrodinger Equation has been wide focus from many researchers especially analysing the result of collision as it describes the soliton propagation. This paper considers the soliton scattering of cubic-quintic Nonlinear Schrodinger Equation on localized Gaussian potential. By applying Super-Gaussian ansatz as the trial function for variational approximation (VA) method, the soliton interaction may acquire flat-top shape with appropriate parameters. The result of VA will be compared to numerical analysis to check the accuracy of analytical predictions.

1. Introduction

Physicists and mathematicians all around the globe have always been interested in studying the solution of nonlinear Schrodinger equation (NLSE) as it has some major contributions in the optics, fluid mechanics, atomic physics and more. Soliton solution of NLSE is particularly one of the most investigated matters in mathematics and theoretical physics. Soliton is caused by the balance between weak nonlinear and dispersive effects in the medium that can be explained through NLSE \cite{1}. It is a particle-like solitary wave that can propagate in nonlinear media without changing its shape and velocity.

Different kind of matter-wave solitons such as dark, bright and gap soliton have been observed experimentally by many researchers. In Bose-Einstein condensates (BECs) the dark solitons emerge with repulsive interactions between atoms where nonlinearity is defocusing, while nonlinearity is focusing for bright solitons where the interatomic interaction should be attractive \cite{2}. Meanwhile, gap solitons emerge between nonlinearity and periodicity of the medium.

The aim of this research is to investigate the scattering of single soliton on the Gaussian potential in CQNLS in form of flat-top soliton and understand the conditions when the solitons interacts with the external potential. Flat-top solitons in BECs arise as the atoms repel with each other, originating from three-atom collisions which prevents the attraction resulting from two-atom interaction \cite{2}. The scattering of flat-top soliton on delta potential was considered on the paper \cite{11}. Also it is worth to note that the experimental observation of soliton scattering on delta potential was reported in the paper \cite{10}.

The NLSE is the partial differential equation and must be simplified to ordinary differential equation (ODE) to be solved. Anderson \cite{4} was the first one discussing the interaction of soliton within NLSE by variational approximation (VA) method. The main objective of VA method is to
provide approximate expressions for pulse width, pulse amplitude and nonlinear frequency chirp [4]. This method is applied into this work to derive approximate terms of the two-coupled equation for soliton’s width and center-of-mass position which will describe the scattering of the soliton after meeting the external potential \(V(x)\). Then the result is compared numerically to confirm a good agreement with the analytical result.

The organization of this paper is follows. The model and governing equations, and problem statement were introduced in Section 2. We discuss the main method of VA while analysing its result in the Section 3, and the discussion of the numerical simulations of the dynamics of single soliton interacting to the external potential is presented in Section 4. Finally, in Section 5 we conclude the paper.

2. The governing equations and variational analysis

Before implementing the two methods to solve the problem, it is necessary to understand the model and main equations studied in this work. The main equation for this scattering of flat-top solitons over Gaussian potential is based on the generalized NLSE which is Cubic-Quintic NLSE.

\[
i\psi_t + \frac{1}{2}i\psi_{xx} + V(x)\psi + \alpha|\psi|^2\psi - \beta|\psi|^4\psi = 0
\]  

(1)

where \(V(x)\) is external potential, \(\psi(x,t)\) is the macroscopic wave function of the condensate, \(\alpha\) is the self-focusing cubic nonlinearity and \(\beta\) is the defocusing quantic nonlinearity. The coefficient of nonlinearity is considered positive (\(\alpha > 0\) and \(\beta > 0\)) corresponding to attraction between atoms in the condensate, or nonlinearity is focused in optics applications in order for the system to support bright matter-wave solitons [5]. However, equation (1) is not integrable in this case and does not have exact soliton solution. Hence, the interaction of localized soliton like solution of CQNLSE is considered with external potentials.

\[
V(x) = U_0 e^{-x^2}
\]  

(2)

For this paper, we study the scattering of a flat-top soliton by Gaussian potential barrier (2) where \(U_0\) stands for the strength. Soliton may preserve the identity after interaction with the potential, but its parameters may become time dependent because of perturbations. Therefore, variational approximation method is applied to study the evolution of soliton parameters. The trial function must be chosen wisely as it determines the success of variational approximation. For NLSE with cubic nonlinearity, remarkable progress has been done by implementing Gaussian and hyperbolic secant trial functions with the application of variational approximation. On the other hand, as the higher order nonlinear terms are included in the NLSE, the shapes of localized states may differ greatly from the previous function, and the options must be considered [2].

By applying variational approximation method in this paper, we consider the transmission of a soliton interacts with external Gaussian potential. As for flat-top soliton, the super-Gaussian trial function (3) is chosen in order to achieved the objective.

\[
\psi(x,t) = Ae^{-\left(\frac{x-\xi}{a}\right)^m} + ib(x-\xi) + iv(x-\xi) + i\phi
\]  

(3)

\(A, a, \xi, b, v, \) and \(\phi\) are the amplitude, width, center of mass position, chirp parameter, velocity and initial phase of the soliton respectively. The parameter \(m\) is the super Gaussian index of soliton and we consider that it is constant in further calculation [2]. The norm of the wave function for Lagrangian density above given by \(N = \int_{-\infty}^{\infty} |\psi|^2 dx = 2A^2a\Gamma(1+M),\) where \(2A^2a\Gamma(1+M) = 1\) and \(M = 1/2m\).
The $\Gamma(x)$ here is the gamma function and given by definition $\nu = \xi$. The Lagrangian density (4) is used to verify the governing equation (1). Then by trial function (3) and equation (4), we calculate the effective Lagrangian with spatial integration of the Lagrangian density $L = \int_{-\infty}^{\infty} \mathcal{L} \, dx$ which is shown by equation (5).

$$\mathcal{L} = \frac{i}{2} \left( \psi \psi^* - \psi^* \psi \right) + \frac{1}{2} |\psi_x|^2 - V(x)|\psi|^2 - \frac{\alpha}{2} |\psi|^4 + \frac{\beta}{3} |\psi|^6 \tag{4}$$

$$L = \psi_t - \frac{1}{2} \xi \psi^2 + \left( a^2 b_t + 2 b^2 a^2 \right) \frac{\Gamma(1+3M)}{3\Gamma(1+M)} + \frac{\Gamma(2-M)}{8M\Gamma(1+M)} a^2 - \frac{\alpha}{2M+2\Gamma(1+M)} a$$

$$+ \frac{\beta}{4M+2\Gamma(1+M)} a^2 - \frac{U_0}{2\Gamma(1+M)} \int_{-\infty}^{\infty} e^{-x^2-(x-\xi/a)^2} \, dx \tag{5}$$

By using Euler-Lagrange equations $d/dt(\partial \mathcal{L}/\partial q_t) - \partial \mathcal{L}/\partial q_i = 0$, where $q_i$ is the time dependent collective coordinates for $a$, $\xi$, and $b$, we calculate the variational parameters collective coordinate equations, the two coupled equations for width (6) and center-of-mass position (7). These equations are the main result of this work and it describe the interaction of soliton by Gaussian potential $V(x)$.

$$a_{tt} = \frac{3\Gamma(2-M)}{4M\Gamma(1+3M)a^3} - \frac{3\alpha}{2M+2\Gamma(1+3M)a^2} + \frac{\beta}{2.3M\Gamma(1+3M)\Gamma(1+M)a^3}$$

$$+ \frac{3U_0}{2\Gamma(1+M)} \frac{\partial}{\partial a} \left( \int_{-\infty}^{\infty} e^{-x^2-(x-\xi/a)^2} \, dx \right) \tag{6}$$

$$\xi_{tt} = -\frac{U_0}{2a\Gamma(1+M)} \frac{\partial}{\partial \xi} \left( \int_{-\infty}^{\infty} e^{-x^2-(x-\xi/a)^2} \, dx \right) \tag{7}$$

When the flat top soliton is located far from the potential $|\xi| \gg a$, it is clear from the equations (6) - (7) are decoupled and one has a soliton moving freely with initial velocity and fixed at initial moment equilibrium width $a_0$.

$$a_{tt} = \frac{3\Gamma(2-M)}{4M\Gamma(1+3M)a^3} - \frac{3\alpha}{2M+2\Gamma(1+3M)a^2} + \frac{\beta}{2.3M\Gamma(1+3M)\Gamma(1+M)a^3} \tag{8}$$

Equation (8) is analogous to the motion equation of a unit mass particle in anharmonic potential $a_{tt} = -\partial U(a)/\partial a$ and we can also determine $a_0$ from this equation, which is the width corresponding to the point where potential $U(a)$ reaches its minimum.

In case when the potential is absence, i.e. $U_0 = 0$ and the equations (6)-(7) are decoupled, from equation (6), it is possible to determine the approximate stationary width ($a_{tt} = 0$) soliton solution of the main equation CQNLSE (1) which is showed by equation (9) below.
The system becomes coupled when the soliton is reaching the potential barrier, and we can consider full system of equations (6)-(7). At this time, we can understand that the oscillations of soliton shape can be excited around this stationary point, also the flat top soliton is either reflected or transmitted. Variational analysis also predicts that as a result of interaction with localized potentials, the oscillations of the soliton’s width \( a(t) \) can be excited. The velocity \( \xi \) in this case is a constant free parameter.

### 3. Numerical simulations of flat top soliton scattering

The result of direct numerical simulations of flat top soliton governed by CQNLSE (1) are explained and presented in this section. This work aims to study the significant quantum behaviours which are the quantum reflection and the quantum transmission of matter-wave soliton due to the interaction with external potential well. We consider the flat top scattering with two different variables, which are velocity \( v \) and the potential strength \( U_0 \). The initial parameter values of stationary flat top soliton are fixed as follow, \( a_0 = 2.78481, M = 0.11146 \), and the coefficients are \( \alpha = 100, \beta = 400 \) [2]. We state the initial position as \( \xi(0) = -10 \).

### 3.1 Scattering of flat-top soliton on Gaussian potential well, velocity \( v \) as the variable.

The corresponding simulations of soliton’s interaction by the potential well with varying soliton velocity \( v \) at three different periods are shown in the Figure 1 and Figure 2. The first period represents the initial time before the flat-top soliton move, second period as the time incident between the flat-top soliton and the potential well, and last period as the time after the interaction. In figure 1, the soliton is reflected from the potential well with the slower velocity. In the meantime, the soliton is transmitted through the potential when the velocity is faster and can be clearly observed from figure 2. The critical velocity here is \( v_c = 0.61 \) and the quantum reflection occurs when \( v < 0.61 \) where some excitations of the potential appear in the well thus reflect the soliton back from the well.

![Figure 1](image-url)

**Figure 1.** Scattering of a flat top soliton by Gaussian potential well with \( U_0 = 0.3, a = 1, x_0 = -5 \) at intial velocity \( v = 0.1 \) according to variational equations (6)-(7). (a) flat top soliton at \( t = 0 \). (b) propagation of soliton close to potential well at \( t = 26s \). (c) flat top soliton after meeting the incident at \( t = 47.6s \).
Figure 2. Scattering of a flat top soliton by Gaussian potential well with $U_0 = 0.3$, $a = 1$, $x_0 = -5$ at initial velocity $v = 1.0$ according to variational equations (6)-(7). (a) flat top soliton at $t = 0$. (b) propagation of soliton meet the potential well at $t = 5.4s$. (c) flat top soliton after meeting the incident at $t = 14.0s$.

3.2 Scattering of flat-top soliton on Gaussian potential well, potential strength $U_0$ as the variable.

The corresponding simulations of flat top soliton’s interaction by the potential well with varying potential strength $U_0$ at three different periods are shown in the Figure 3 and Figure 4. Based on figure 3, the flat-top soliton is transmitted through potential well at $U_0 = 0.15$ due to the lack of potential strength to block the soliton from propagating through it. But then the soliton if reflected from the potential when the potential strength is higher, and it is proved in figure 4. The critical value for potential strength here is $U_{0c} = 0.25$ as the potential well is much stronger than the force produced by the soliton propagation, hence preventing the soliton to pass through it.

Figure 3. Flat top soliton scattering by Gaussian potential well with $U_0 = 0.15$, $a = 1$, $x_0 = -5$ at initial velocity $v = 0.5$ according to variational equations (6)-(7). (a) flat top soliton at $t = 0$. (b) propagation of soliton meet the potential well at $t = 15.0s$. (c) flat top soliton after meeting the incident at $t = 25.0s$. 
Figure 4. Flat top soliton scattering by Gaussian potential well with $U_0 = 0.30, a = 1, x_0 = -5$ at initial velocity $v = 0.5$ according to variational equations (6)-(7). (a) flat top soliton at $t = 0$. (b) propagation of soliton meet the potential well at $t = 15.0\text{s}$. (c) flat top soliton after meeting the incident at $t = 20.0\text{s}$.

4. Conclusion
The flat-top soliton of CQNLSE scattering on the external Gaussian potential have been studied. We identified the quantum reflection and quantum transmission of the soliton by the potential well. Besides, the critical value separating the scenario is determined successfully. The result of variational equations is defined by variational approximation and is compared with the direct numerical solution for more accurate solutions.

We expect that in the future the interaction between flat-top soliton with less complicated potential form to improve the experimental output and the accuracy. We also propose a new and unexpected result of scattering process with two-soliton molecule of flat-top shape.

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