Induced nilpotent orbits and birational geometry

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To the memory of Professor Masayoshi Nagata

Introduction

Let $G$ be a complex simple algebraic group and let $\mathfrak{g}$ be its Lie algebra. A nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is an orbit of a nilpotent element of $\mathfrak{g}$ by the adjoint action of $G$ on $\mathfrak{g}$. Then $\mathcal{O}$ admits a natural symplectic 2-form $\omega$ and the nilpotent orbit closure $\overline{\mathcal{O}}$ has symplectic singularities in the sense of [Be] and [Na3] (cf. [Pa], [Hi]). In [Ri], Richardson introduced the notion of so-called the Richardson orbit. A nilpotent orbit $\mathcal{O}$ is called Richardson if there is a parabolic subgroup $Q$ of $G$ such that $\mathcal{O} \cap n(q)$ is an open dense subset of $n(q)$, where $n(q)$ is the nil-radical of $q$. Later, Lusztig and Spaltenstein [L-S] generalized this notion to the induced orbit. A nilpotent orbit $\mathcal{O}$ is an induced orbit if there are a parabolic subgroup $Q$ of $G$ and a nilpotent orbit $\mathcal{O}'$ in the Levi subalgebra $\mathfrak{l}(q)$ of $q := \text{Lie}(Q)$ such that $\mathcal{O}$ meets $n(q) + \mathcal{O}'$ in an open dense subset. If $\mathcal{O}$ is an induced orbit, one has a natural map (cf. (1.2))

$$\nu : G \times Q (n(q) + \overline{\mathcal{O}'}) \to \overline{\mathcal{O}}.$$ 

The map $\nu$ is a generically finite, projective, surjective map. This map is called the generalized Springer map. In this paper, we shall study the induced orbits from the viewpoint of birational geometry. For a Richardson orbit $\mathcal{O}$, the Springer map $\nu$ is a map from the cotangent bundle $T^*(G/Q)$ of the flag variety $G/Q$ to $\overline{\mathcal{O}}$. In [Fu], Fu proved that, if $\overline{\mathcal{O}}$ has a crepant (projective) resolution, it is a Springer map. Note that $Q$ is not unique (even up to the conjugate) for a Richardson orbit $\mathcal{O}$. This means that $\overline{\mathcal{O}}$ has many different crepant resolutions. In [Na], the author has given a description of all crepant resolutions of $\overline{\mathcal{O}}$ and proved that any two different crepant resolutions are connected by Mukai flops. The purpose of this paper is to generalize
these to all nilpotent orbits $O$. If $O$ is not Richardson, $\bar{O}$ has no crepant resolution. The substitute of a crepant resolution, is a $\mathbb{Q}$-factorial terminalization. Let $X$ be a complex algebraic variety with rational Gorenstein singularities. A partial resolution $f : Y \to X$ of $X$ is said to be a $\mathbb{Q}$-factorial terminalization of $X$ if $Y$ has only $\mathbb{Q}$-factorial terminal singularities and $f$ is a birational projective morphism such that $K_Y = f^*K_X$. A $\mathbb{Q}$-factorial terminalization is a crepant resolution exactly when $Y$ is smooth. Recently, Birkar-Cascini-Hacon-McKernan [B-C-H-M] have established the existence of minimal models of complex algebraic varieties of general type. As a corollary of this, we know that $X$ always has a $\mathbb{Q}$-factorial terminalization. In particular, $\bar{O}$ should have a $\mathbb{Q}$-factorial terminalization. The author would like to pose the following conjecture.

**Conjecture.** Let $O$ be a nilpotent orbit of a complex simple Lie algebra $\mathfrak{g}$. Let $\tilde{O}$ be the normalization of $\bar{O}$. Then one of the following holds:

1. $\tilde{O}$ has $\mathbb{Q}$-factorial terminal singularities.
2. There are a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ and a nilpotent orbit $O'$ of $\mathfrak{l}$ such that (a): $O = \text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(O')$ and (b): the normalization of $G \times ^Q (\mathfrak{n}(\mathfrak{q}) + O')$ is a $\mathbb{Q}$-factorial terminalization of $\tilde{O}$ via the generalized Springer map.

Moreover, if $\tilde{O}$ does not have $\mathbb{Q}$-factorial terminal singularities, then every $\mathbb{Q}$-factorial terminalization of $\tilde{O}$ is of the form (2). Two $\mathbb{Q}$-factorial terminalizations are connected by Mukai flops (cf. [Na], p.91).

In this paper, we shall prove that Conjecture is true when $\mathfrak{g}$ is classical.

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**Notations and convention.** Let $X$ be a normal algebraic variety. Then, $X$ is called $\mathbb{Q}$-factorial if, for any Weil divisor $D$ on $X$, its suitable multiple $mD$ ($m > 0$) is a Cartier divisor. Assume that the canonical divisor $K_X$ is Cartier. We say that $X$ has only terminal singularities if, for a resolution $\pi : Y \to X$, $K_Y = \pi^*K_X + \Sigma a_i E_i$ with $a_i > 0$ for all $i$. Here $E_i$ run through all $\pi$-exceptional prime divisors. For details in birational geometry, see the first part of [Ka] (or one can find a quick guide in [Na 4], Preliminaries). On the other hand, for details on the basic properties on nilpotent orbits, see [C-M] and [K-P].
§1. Preliminaries

(1.1) Nilpotent orbits and resolutions: Let $G$ be a complex simple algebraic group and let $\mathfrak{g}$ be its Lie algebra. $G$ has the adjoint action on $\mathfrak{g}$. The orbit $\mathcal{O}_x$ of a nilpotent element $x \in \mathfrak{g}$ for this action is called a nilpotent orbit. By the Jacobson-Morozov theorem, one can find a semi-simple element $h \in \mathfrak{g}$, and a nilpotent element $y \in \mathfrak{g}$ in such a way that $[h, x] = 2x$, $[h, y] = -2y$ and $[x, y] = h$. For $i \in \mathbb{Z}$, let

$$\mathfrak{g}_i := \{ z \in \mathfrak{g} \mid [h, z] = iz \}.$$ 

Then one can write

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i.$$ 

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with $h \in \mathfrak{h}$. Let $\Phi$ be the corresponding root system and let $\Delta$ be a base of simple roots such that $h$ is $\Delta$-dominant, i.e. $\alpha(h) \geq 0$ for all $\alpha \in \Delta$. In this situation,

$$\alpha(h) \in \{0, 1, 2\}.$$ 

The weighted Dynkin diagram of $\mathcal{O}_x$ is the Dynkin diagram of $\mathfrak{g}$ where each vertex $\alpha$ is labeled with $\alpha(h)$. A nilpotent orbit $\mathcal{O}_x$ is completely determined by its weighted Dynkin diagram. A Jacobson-Morozov parabolic subalgebra for $x$ is the parabolic subalgebra $p$ defined by

$$p := \bigoplus_{i \geq 0} \mathfrak{g}_i.$$ 

Let $P$ be the parabolic subgroup of $G$ determined by $p$. We put

$$n_2 := \bigoplus_{i \geq 2} \mathfrak{g}_i.$$ 

Then $n_2$ is an ideal of $p$; hence, $P$ has the adjoint action on $n_2$. Let us consider the vector bundle $G \times^P n_2$ over $G/P$ and the map

$$\mu : G \times^P n_2 \to \mathfrak{g}$$

defined by $\mu([g, z]) := Ad_g(z)$. Then the image of $\mu$ coincides with the closure $\bar{\mathcal{O}_x}$ of $\mathcal{O}_x$ and $\mu$ gives a resolution of $\bar{\mathcal{O}_x}$ (cf. [K-P], Proposition 7.4). We call $\mu$ the Jacobson-Morozov resolution of $\bar{\mathcal{O}_x}$. The orbit $\mathcal{O}_x$ has a natural closed non-degenerate 2-form $\omega$ (cf. [C-G], Prop. 1.1.5., [C-M], 1.3). By $\mu$, $\omega$ is regarded as a 2-form on a Zariski open subset of $G \times^P n_2$. By [Pa], [Hi], it
extends to a 2-form on $G \times P \mathfrak{n}_2$. In other words, $\tilde{O}_x$ has symplectic singularity. Let $\tilde{O}_x$ be the normalization of $O_x$. In many cases, one can check the $\mathbb{Q}$-factoriality of $\tilde{O}_x$ by applying the following lemma to the Jacobson-Morozov resolution:

**Lemma (1.1.1).** Let $\pi : Y \to X$ be a projective resolution of an affine variety $X$ with rational singularities. Let $\rho$ be the relative Picard number for $\pi$. If $\text{Exc}(\pi)$ contains $\rho$ different prime divisors, then $X$ is $\mathbb{Q}$-factorial.

**Proof.** Recall that two line bundles $L, L'$ on $Y$ are called $\pi$-numerically equivalent if $(L.C) = (L'.C)$ for every proper curve $C$ on $Y$ such that $\pi(C)$ is a point. Let $N^1(\pi)$ be the group of $\pi$-numerical classes of line bundles on $Y$, and put $N^1(\pi)_\mathbb{Q} := N^1(\pi) \otimes \mathbb{Q}$. By definition, the Picard number $\rho$ is the dimension of the $\mathbb{Q}$-vector space $N^1(\pi)_\mathbb{Q}$. Let $E_i$, $(1 \leq i \leq \rho)$ be the prime divisors contained in $\text{Exc}(\pi)$. We shall prove that $N^1(\pi)_\mathbb{Q} = \oplus \mathbb{Q}[E_i]$. It suffices to show that $[E_i]$’s are linearly independent. Put $d := \dim Y$. Assume $\Sigma a_i[E_i] = 0$. Put $m$ to be the largest number among $\{\dim \pi(E_i)\}$. We take $m$ very ample divisors $H_1, \ldots, H_m$ on $X$ in such a way that $\cap H_j$ intersects with $\pi(E_i)$, in finite points, for each $i$ with $\dim \pi(E_i) = m$, but $\cap H_j$ does not intersect with $\pi(E_i)$ for each $i$ with $\dim \pi(E_i) < m$. By the Bertini theorem, we may assume that $\cap \pi^*(H_j)$ is non-singular. We cut out $\cap \pi^*(H_j)$ further by $d - m - 2$ very ample divisors $L_1, \ldots, L_{d-m-2}$ on $Y$. Then the resulting variety is a smooth surface $S$. The restriction of $\pi$ to $S$ gives a birational contraction map $\pi|_S : S \to \tilde{S}$. For $i$ with $\dim \pi(E_i) = m$, we put $C_i := \pi^*(H_1) \cap \ldots \cap \pi^*(H_m) \cap L_1 \cap \ldots \cap L_{d-m-2} \cap E_i$. Then $C_i$ is a (possibly reducible) curve contained in $\text{Exc}(\pi|_S)$. Moreover, two different such $C_i$’s do not have no common irreducible components. By a theorem of Grauert, the intersection matrix of $\pi|_S$-exceptional curves, is negative definite, which implies that $[C_i]$ are linearly independent in $N^1(\pi|_S)_\mathbb{Q}$. Let us consider the restriction map

$$\iota : N^1(\pi)_\mathbb{Q} \to N^1(\pi|_S)_\mathbb{Q}.$$  

Note that $\iota([E_i]) = 0$ for $i$ with $\dim \pi(E_i) < m$, and $\iota([E_i]) = [C_i]$ for $i$ with $\dim \pi(E_i) = m$. Since $\Sigma a_i[E_i] = 0$ in $N^1(\pi)_\mathbb{Q}$, we have $\Sigma a_i[C_i] = 0$ in $N^1(\pi|_S)_\mathbb{Q}$. This implies that $a_i = 0$ for all $i$ with $\dim \pi(E_i) = m$. Next, replace $m$ by the second largest number among $\{\dim \pi(E_i)\}$ and repeat the same procedure; then we finally conclude that $a_i = 0$ for all $i$. Now let us prove that $X$ is $\mathbb{Q}$-factorial. Let $D$ be a prime Weil divisor of $X$ and let $D'$ be the prime divisor of $Y$ obtained as the proper transform of $D$. There are rational numbers $b_i$ such that $[D'] + \Sigma b_i[E_i] = 0$ in $N^1(\pi)_\mathbb{Q}$. Since $X$ has only
rational singularities, \( l(D' + \Sigma b_i[E_i]) \) is the pull-back of a Cartier divisor \( M \) on \( X \) for some integer \( l > 0 \). This implies that \( lD \) is linearly equivalent to the Cartier divisor \( M \).

(1.2) Induced orbits

(1.2.1). Let \( G \) and \( g \) be the same as in (1.1). Let \( Q \) be a parabolic subgroup of \( G \) and let \( q \) be its Lie algebra with Levi decomposition \( q = l \oplus n \). Here \( n \) is the nil-radical of \( q \) and \( l \) is a Levi-part of \( q \). Fix a nilpotent orbit \( \mathcal{O}' \) in \( l \). Then there is a unique nilpotent orbit \( \mathcal{O} \) in \( g \) meeting \( n + \mathcal{O}' \) in an open dense subset ([L-S]). Such an orbit \( \mathcal{O} \) is called the nilpotent orbit induced from \( \mathcal{O}' \) and we write

\[
\mathcal{O} = \text{Ind}_l^g(\mathcal{O}').
\]

Note that when \( \mathcal{O}' = 0 \), \( \mathcal{O} \) is the Richardson orbit for \( Q \). Since the adjoint action of \( Q \) on \( q \) stabilizes \( n + \mathcal{O}' \), one can consider the variety \( G \times^Q (n + \mathcal{O}') \). There is a map

\[
\nu : G \times^Q (n + \mathcal{O}') \to \bar{\mathcal{O}}
\]

defined by \( \nu([g, z]) := \text{Ad}_g(z) \). Since \( \text{Codim}_q(\mathcal{O}') = \text{Codim}_q(\mathcal{O}) \) (cf. [C-M], Prop. 7.1.4), \( \nu \) is a generically finite dominating map. Moreover, \( \nu \) is factorized as

\[
G \times^Q (n + \mathcal{O}') \to G/Q \times \bar{\mathcal{O}} \to \bar{\mathcal{O}}
\]

where the first map is a closed embedding and the second map is the 2-nd projection; this implies that \( \nu \) is a projective map. In the remainder, we call \( \nu \) the generalized Springer map for \( (Q, \mathcal{O}') \).

(1.2.2). Assume that \( Q \) is contained in another parabolic subgroup \( \bar{Q} \) of \( G \). Let \( \bar{L} \) be the Levi part of \( \bar{Q} \) which contains the Levi part \( L \) of \( Q \). Let \( \bar{q} = \bar{l} \oplus \bar{n} \) be the Levi decomposition. Note that \( \bar{L} \cap Q \) is a parabolic subgroup of \( \bar{L} \) and \( l(\bar{L} \cap Q) = l \). Let \( \mathcal{O} \subset \bar{l} \) be the nilpotent orbit induced from \( (\bar{L} \cap Q, \mathcal{O}') \). Then there is a natural map

\[
\pi : G \times^Q (n + \mathcal{O}') \to G \times^\bar{Q} (\bar{n} + \bar{\mathcal{O}}_1)
\]

which factorizes \( \nu \) as \( \bar{\nu} \circ \pi = \nu \). Here \( \bar{\nu} \) is the generalized Springer map for \( (\bar{Q}, \mathcal{O}_1) \).

(1.2.3). Assume that there are a parabolic subgroup \( Q_L \) of \( L \) and a nilpotent orbit \( \mathcal{O}_2 \) in the Levi subalgebra \( l(Q_L) \) such that \( \mathcal{O}' \) is the nilpotent orbit induced from \( (Q_L, \mathcal{O}_2) \). Then there is a parabolic subgroup \( \bar{Q}' \) of \( G \)
such that $Q' \subset Q$, $I(Q') = I(Q_L)$ and $O$ is the nilpotent orbit induced from $(Q', O_2)$. The generalized Springer map $\nu'$ for $(Q', O_2)$ is factorized as

$$G \times Q' (n' + \bar{O}_2) \to G \times Q (n + \bar{O}') \to \bar{O}.$$

**Lemma (1.2.4).** Let

$$\nu : G \times Q (n + \bar{O}') \to \bar{O}$$

be a generalized Springer map defined in (1.2.1). Then the normalization of $G \times Q (n + \bar{O}')$ is a symplectic variety.

**Proof.** We shall prove that the pull-back of the Kostant-Kirillov form $\omega$ on $O$ gives a non-degenerate 2-form on $G \times Q (n + \bar{O}')$. This is enough for proving (1.2.4). In fact, $G \times Q (n + \bar{O}')$ is locally a product of $\bar{O}'$ and a non-singular variety; hence the normalization of $G \times Q (n + \bar{O}')$ has only rational Gorenstein singularities. The Kostant-Kirillov form extends to a regular 2-form on any resolution of $G \times Q (n + \bar{O}')$ as explained in (1.1).

Therefore, the normalization of $G \times Q (n + \bar{O}')$ is a symplectic variety. The following argument is analogous to [Pa]. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset I$. There is an involution $\phi_\mathfrak{g}$ of $\mathfrak{g}$ which stabilizes $\mathfrak{h}$ and which acts on the root system $\Phi$ via $-1$. Put $n_- := \phi_\mathfrak{g}(n)$. Take a point $[1, y + y'] \in G \times Q (n + \bar{O}')$ so that $y \in n$, $y' \in \bar{O}'$ and $y + y' \in O$. The tangent space of $G \times Q (n + \bar{O}')$ at $[1, y + y']$ is decomposed as

$$T_{[1, y + y']} = n_- \oplus T_{y + y'}(n + \bar{O}').$$

Since $Q \cdot (y + y')$ coincides with the Zariski open dense subset $O \cap (n + \bar{O}') \subset n + \bar{O}'$, an element $v \in T_{[1, y + y']}$ can be written as

$$v = v_1 + [v_2, y + y'], \quad v_1 \in n_-, \quad v_2 \in q.$$

Let $d\nu_* : T_{[1, y + y']} \to T_{\nu([1, y + y'])} O$ be the tangential map for $\nu$. Then

$$d\nu_*(v) = [v_1 + v_2, y + y'].$$

Take one more element $w \in T_{[1, y + y']}$ in such a way that

$$w = w_1 + [w_2, y + y'], \quad w_1 \in n_-, \quad w_2 \in q.$$

Denote by $\langle , \rangle$ the Killing form of $\mathfrak{g}$. By the definition of the Kostant-Kirillov form, one has

$$\omega(\nu_*(v), \nu_*(w)) := \langle y + y', [v_1 + v_2, w_1 + w_2] \rangle.$$
Note that \( \langle y + y', [v_1, w_1] \rangle = \langle y, [v_1, w_1] \rangle \), and \( \langle y + y', [v_2, w_2] \rangle = \langle y', [v_2, w_2] \rangle \). Therefore,

\[
\omega(d\nu(v), d\nu(w)) = \\
\langle y, [v_1, w_1] \rangle + \langle y + y', [v_1, w_2] \rangle + \langle y + y', [v_2, w_1] \rangle + \langle y', [v_2, w_2] \rangle = \\
\langle y, [v_1, w_1] \rangle + \langle v_1, [w_2, y + y']_n \rangle - \langle [v_2, y + y']_n, w_1 \rangle + \omega([v_2, y'], [w_2, y']),
\]

where \( [w_2, y + y']_n \) (resp. \( [v_2, y + y']_n \)) is the nil-radical part of \( [w_2, y + y'] \) (resp. \( [v_2, y + y'] \)) in the decomposition \( T_{y + y'}(n + O') = n + T_y O' \). Let \( O_r \subset g \) be the Richardson orbit for \( Q \), and let \( \pi : T^*(G/Q) \to \bar{O}_r \) be the Springer map. The first part \( \langle y, [v_1, w_1] \rangle + \langle v_1, [w_2, y + y']_n \rangle - \langle [v_2, y + y']_n, w_1 \rangle \) corresponds to the 2-form on \( T^*(G/Q) \) obtained by the pull-back of the Kostant-Kirillov 2-form on \( O_r \) by \( \pi \) (cf. [Pa]), which is non-degenerate on \( T^*(G/Q) \). Let us consider the second part \( \omega([v_2, y'], [w_2, y']) \). Denote by \( [v_2, y + y']_l \) (resp. \( [w_2, y + y']_l \)) the \( T_y O' \)-part of \( [v_2, y + y'] \) (resp. \( [w_2, y + y'] \)) in the decomposition \( T_{y + y'}(n + O') = n + T_y O' \). Then \( [v_2, y'] = [v_2, y + y']_l \); and \( [w_2, y'] = [w_2, y + y']_l \); hence, the second part is the Kostant-Kirillov form on \( O' \). The arguments above show that \( \nu^* \omega \) is non-degenerate at \( [1, y + y'] \) for an arbitrary \( y \in n \) and for an arbitrary \( y' \in O' \). By the \( G \)-equivariance of \( \nu \), we have the lemma.

(1.3) Nilpotent orbits in classical Lie algebras: When \( g \) is a classical Lie algebra, \( g \) is naturally a Lie subalgebra of \( \text{End}(V) \) for a complex vector space \( V \). Then we can attach a partition \( d \) of \( n := \dim V \) to each orbit as the Jordan type of an element contained in the orbit. Here a partition \( d := [d_1, d_2, ..., d_k] \) of \( n \) is a set of positive integers with \( \Sigma d_i = n \) and \( d_1 \geq d_2 \geq ... \geq d_k \). Another way of writing \( d \) is \( [d_1^{s_1}, ..., d_k^{s_k}] \) with \( d_1 > d_2 > ... > d_k > 0 \). Here \( d_i^{s_i} \) is an \( s_i \) times \( d_i \): \( d_i, d_i, ..., d_i \). The partition \( d \) corresponds to a Young diagram. For example, \( [5, 4^2, 1] \) corresponds to

```
  +---+---+---+---+---+
  |   |   |   |   |   |
  +---+---+---+---+---+
  |   |   |   |   |   |
  +---+---+---+---+---+
  |   |   |   |   |   |
  +---+---+---+---+---+
  |   |   |   |   |   |
  +---+---+---+---+---+
  |   |   |   |   |   |
  +---+---+---+---+---+
  |   |   |   |   |   |
  +---+---+---+---+---+
  |   |   |   |   |   |
  +---+---+---+---+---+
  |   |   |   |   |   |
  +---+---+---+---+---+
```

When an integer \( e \) appears in the partition \( d \), we say that \( e \) is a member of \( d \). We call \( d \) very even when \( d \) consists with only even members, each having even multiplicity.
Let us denote by $\epsilon$ the number 1 or $-1$. Then a partition $d$ is $\epsilon$-admissible if all even (resp. odd) members of $d$ have even multiplicities when $\epsilon = 1$ (resp. $\epsilon = -1$). The following result can be found, for example, in [C-M, §5].

**Proposition (1.3.1)** Let $\mathcal{N}o(g)$ be the set of nilpotent orbits of $g$.

1. $(A_{n-1})$: When $g = sl(n)$, there is a bijection between $\mathcal{N}o(g)$ and the set of partitions $d$ of $n$.

2. $(B_n)$: When $g = so(2n + 1)$, there is a bijection between $\mathcal{N}o(g)$ and the set of $\epsilon$-admissible partitions $d$ of $2n + 1$ with $\epsilon = 1$.

3. $(C_n)$: When $g = sp(2n)$, there is a bijection between $\mathcal{N}o(g)$ and the set of $\epsilon$-admissible partitions $d$ of $2n$ with $\epsilon = -1$.

4. $(D_n)$: When $g = so(2n)$, there is a surjection $f$ from $\mathcal{N}o(g)$ to the set of $\epsilon$-admissible partitions $d$ of $2n$ with $\epsilon = 1$. For a partition $d$ which is not very even, $f^{-1}(d)$ consists of exactly one orbit, but, for very even $d$, $f^{-1}(d)$ consists of exactly two different orbits.

Take an $\epsilon$-admissible partition $d$ of a positive integer $m$. If $\epsilon = 1$, we put $g = so(m)$ and if $\epsilon = -1$, we put $g = sp(m)$. We denote by $\mathcal{O}_d$ a nilpotent orbit in $g$ with Jordan type $d$. Note that, except when $\epsilon = 1$ and $d$ is very even, $\mathcal{O}_d$ is uniquely determined. When $\epsilon = 1$ and $d$ is very even, there are two possibilities for $\mathcal{O}_d$. If necessary, we distinguish the two orbits by the labelling: $\mathcal{O}_d'$ and $\mathcal{O}_d''$. Let us fix a classical Lie algebra $g$ and study the relationship among nilpotent orbits in $g$. When $g$ is of type $B$ or $D$ (resp. $C$), we only consider the $\epsilon$-admissible partitions with $\epsilon = 1$ (resp. $\epsilon = -1$). We introduce a partial order in the set of the partitions of (the same number): for two partitions $d$ and $f$, $d \geq f$ if $\Sigma_{i \leq k}d_i \geq \Sigma_{i \leq k}f_i$ for all $k \geq 1$. On the other hand, for two nilpotent orbits $\mathcal{O}$ and $\mathcal{O}'$ in $g$, we write $\mathcal{O} \geq \mathcal{O}'$ if $\mathcal{O}' \subset \mathcal{O}$. Then, $\mathcal{O}_d \geq \mathcal{O}_f$ if and only if $d \geq f$. When $d$ and $f$ are $\epsilon$-admissible partitions with $f \geq g$, we call this pair an $\epsilon$-degeneration or simply a degeneration.

Now let us consider the case $g$ is of type $B$, $C$ or $D$.

Assume that an $\epsilon$-degeneration $d \geq f$ is minimal in the sense that there is no $\epsilon$-admissible partition $d'$ (except $d$ and $f$) such that $d \geq d' \geq f$. Kraft and Procesi [K-P] have studied the normal slice $N_{d,f}$ of $\mathcal{O}_f \subset \mathcal{O}_d$ in such cases. If, for two integers $r$ and $s$, the first $r$ rows and the first $s$ columns of $d$ and $f$ coincide and the partition $(d_1, ..., d_r)$ is $\epsilon$-admissible, then one can erase these rows and columns from $d$ and $f$ respectively to get new partitions $d'$ and $f'$ with $d' \geq f'$. If we put $\epsilon' := (-1)^*\epsilon$, then $d'$ and $f'$ are both
\( \epsilon' \)-admissible. The pair \((d', f')\) is also minimal. Repeating such process, one can reach a degeneration \( d_{irr} \geq f_{irr} \) which is irreducible in the sense that there are no rows and columns to be erased. By [K-P], Theorem 2, \( N_{d,f} \) is analytically isomorphic to \( N_{d_{irr}, f_{irr}} \) around the origin. According to [K-P], a minimal and irreducible degeneration \( d \geq f \) is one of the following:

- **a:** \( g = sp(2), \ d = (2), \) and \( f = (1^2) \).
- **b:** \( g = sp(2n) \) \((n > 1)\), \( d = (2n) \), and \( f = (2n - 2, 2) \).
- **c:** \( g = so(2n + 1) \) \((n > 0)\), \( d = (2n + 1) \), and \( f = (2n - 1, 1^2) \).
- **d:** \( g = sp(4n + 2) \) \((n > 0)\), \( d = (2n + 1, 2n + 1) \), and \( f = (2n, 2n, 2) \).
- **e:** \( g = so(4n) \) \((n > 0)\), \( d = (2n, 2n) \), and \( f = (2n - 1, 2n - 1, 1^2) \).
- **f:** \( g = so(2n + 1) \) \((n > 1)\), \( d = (2^2, 1^{2n-3}) \), and \( f = (1^{2n+1}) \).
- **g:** \( g = sp(2n) \) \((n > 1)\), \( d = (2, 1^{2n-2}) \), and \( f = (1^{2n}) \).
- **h:** \( g = so(2n) \) \((n > 2)\), \( d = (2^2, 1^{2n-4}) \), and \( f = (1^{2n}) \).

In the first 4 cases \((a, b, c, d, e)\), \( \mathcal{O}_f \) have codimension 2 in \( \mathcal{O}_d \). In the last 3 cases \((f, g, h)\), \( \mathcal{O}_f \) have codimension \( \geq 4 \) in \( \mathcal{O}_d \).

**Proposition (1.3.2)** Let \( \mathcal{O} \) be a nilpotent orbit in a classical Lie algebra \( g \) of type \( B, C \) or \( D \) with Jordan type \( d := [(d_1)^{s_1}, \ldots, (d_k)^{s_k}] \) \((d_1 > d_2 > \ldots > d_k)\). Let \( \Sigma \) be the singular locus of \( \mathcal{O} \). Then \( \text{Codim}_\mathcal{O}(\Sigma) \geq 4 \) if and only if the partition \( d \) has full members, that is, any integer \( j \) with \( 1 \leq j \leq d_1 \) is a member of \( d \). Otherwise, \( \text{Codim}_\mathcal{O}(\Sigma) = 2 \).

**Proof.** Assume that \( \text{Codim}_\mathcal{O}(\Sigma) \geq 4 \). We shall prove that \( d \) has full members. Suppose, to the contrary, that there is some \( i \) with \( d_i \geq d_{i+1} + 2 \). Then one can find a minimal degeneration \( d \geq f \) where \( f \) is one of the following:

\[
[\ldots, d_i^{s_i-1}, (d_{i+1} + 1)^2, d_{i+1}^{s_{i+1}-1}, \ldots], \quad d_i = d_{i+1} + 2
\]

\[
[\ldots, d_i^{s_i-1}, d_i - 2, d_{i+1} + 2, d_{i+1}^{s_{i+1}-1}, \ldots]
\]

\[
[\ldots, d_i^{s_i-1}, d_i - 2, (d_{i+1} + 1)^2, d_{i+1}^{s_{i+1}-2}, \ldots]
\]

\[
[\ldots, d_i^{s_i-2}, (d_i - 1)^2, d_{i+1} + 2, d_{i+1}^{s_{i+1}-1}, \ldots]
\]

\[
[\ldots, d_i^{s_i-2}, (d_i - 1)^2, (d_{i+1} + 1)^2, d_{i+1}^{s_{i+1}-2}, \ldots]
\]

By the row-column erasing one gets an irreducible, minimal degeneration \( d_{irr} \geq f_{irr} \) of type \( a, b, c, d \) or \( e \). This is a contradiction; hence \( d \) has full members. Conversely, assume that \( d \) has full members, i.e. \( d_j = k - j + 1 \).
for all $1 \leq j \leq k$. Then, for every minimal degeneration $d \geq f$, $f$ is one of the following:

$$[..., (k - i + 1)^{s_i-2}, (k - i)^{s_i-1+4}, (k - i - 1)^{s_i-2-2}, ...]$$

$$[..., (k - i + 1)^{s_i-1}, (k - i)^{s_i-1+2}, (k - i - 1)^{s_i-2-1}, ...].$$

By the row-column erasing one gets an irreducible, minimal degeneration $d_{irr} \geq f_{irr}$ of type $f, g$ or $h$. Therefore, $\text{Codim}_\Delta(\Sigma) \geq 4$.

(1.4) Jacobson-Morozov resolutions in the case of classical Lie algebras (cf. [CM], 5.3): Let $V$ be a complex vector space of dimension $m$ with a non-degenerate symmetric (or skew-symmetric) form $<, >$. In the symmetric case, we take a basis $\{e_i\}_{1 \leq i \leq m}$ of $V$ in such a way that $< e_j, e_k > = 1$ if $j + k = m + 1$ and otherwise $< e_j, e_k > = 0$. In the skew-symmetric case, we take a basis $\{e_i\}_{1 \leq i \leq m}$ of $V$ in such a way that $< e_j, e_k > = 1$ if $j < k$ and $j + k = m + 1$, and $< e_j, e_k > = 0$ if $j + k \neq m + 1$. When $(V, <, >)$ is a symmetric vector space, $\mathfrak{g} := \text{so}(V)$ is the Lie algebra of type $B_{(m-1)/2}$ (resp. $D_{m/2}$) if $m$ is odd (resp. even). When $(V, <, >)$ is a skew-symmetric vector space, $\mathfrak{g} := \text{sp}(V)$ is the Lie algebra of type $C_{m/2}$. In the remainder of this paragraph, $\mathfrak{g}$ is one of these Lie algebras contained in $\text{End}(V)$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra consisting of all diagonal matrices, and let $\Delta$ be the standard base of simple roots. Let $x \in \mathfrak{g}$ be a nilpotent element. As in (1.1), one can choose $h, y \in \mathfrak{g}$ in such a way that $\{x, y, h\}$ is a $\text{sl}(2)$-triple.

If necessary, by replacing $x$ by its conjugate element, one may assume that $h \in \mathfrak{h}$ and $h$ is $\Delta$-dominant. Assume that $x$ has Jordan type $d = [d_1, ..., d_k]$. The diagonal matrix $h$ is described as follows. Let us consider the sequence of integers of length $m$:

$$d_1 - 1, d_1 - 3, ..., -d_1 + 3, -d_1 + 1, d_2 - 1, d_2 - 3, ..., -d_2 + 3, -d_2 + 1, ..., d_k - 1, d_k - 3, ..., -d_k + 3, -d_k + 1.$$

Rearrange this sequence in the non-increasing order and get a new sequence $p_1^{t_1}, ..., p_l^{t_l}$ with $p_1 > p_2 > ... > p_l$ and $\Sigma t_i = m$. Then

$$h = \text{diag}(p_1^{t_1}, ..., p_l^{t_l}).$$

Here $p_i^{t_i}$ means the $t_i$ times of $p_i$'s: $p_i, p_i, ..., p_i$. It is then easy to describe explicitly the Jacobson-Morozov parabolic subalgebra $\mathfrak{p}$ of $x$ and its ideal $\mathfrak{n}_2$ (cf. (1.1)). The Jacobson-Morozov parabolic subgroup $P$ is the stabilizer group of certain isotropic flag $\{F_i\}_{1 \leq i \leq r}$ of $V$. Here, an isotropic flag of $V$
(of length $r$) is a increasing filtration $0 \subset F_1 \subset F_2 \subset \ldots \subset F_r \subset V$ such that $F_{r+1-i} = F_i^\perp$ for all $i$. The flag type of $P$ is $(t_1, \ldots, t_l)$. The nilradical $n := \oplus_{i>0} \mathfrak{g}_i$ of $\mathfrak{p}$ consists of the elements $z$ of $\mathfrak{g}$ such that $z(F_i) \subset F_{i-1}$ for all $i$. On the other hand, it depends on the weighted Dynkin diagram for $x$ how $n_2$ takes its place in $n$.

**Example (1.4.1).** Assume that $d$ has full members (cf. (1.3.2)). Then, it can be checked that

$$n_2 = \{ z \in \mathfrak{g}; z(F_i) \subset F_{i-2} \text{ for all } i \}.$$  

Let us consider the Jacobson-Morozov resolution

$$\mu : G \times^P n_2 \to \mathcal{O}_d.$$  

Take $z \in \mathcal{O}_d$. One can look at the fiber $\mu^{-1}(z)$ by using the characterization of $n_2$ above. Assume that $G = Sp(V)$. We prepare two kinds of skew-symmetric vector space $V_d(d: \text{even})$, and $W_{2d}(d: \text{odd})$ as follows. The $V_d$ is a $d$ dimensional vector space with a basis $e_{d-1}, e_{d-3}, \ldots, e_{-d+3}, e_{-d+1}$. The skew-symmetric form $\langle \cdot, \cdot \rangle$ is defined in such a way that $\langle e_i, e_j \rangle = 1$ (resp. $\langle e_i, e_j \rangle = -1$) when $i+j = 0$ and $i > j$ (resp. $i+j = 0$ and $i < j$), and $\langle e_i, e_j \rangle = 0$ when $i+j \neq 0$. Let $z_d$ be the endomorphism of $V_d$ defined by the $d \times d$ matrix $J_d$ with $J_d(i, i+1) = 1$ ($1 \leq i \leq d/2)$, $J_d(i, i+1) = -1$ ($d/2+1 \leq i \leq d-1$) and otherwise $J_d(i,j) = 0$. The $W_{2d}$ is a $2d$ dimensional vector space with a basis $f_{d-1}, f_{d-3}, \ldots, f_{-d+3}, f'_{d-1}, f'_d, f'_d, \ldots, f'_{-d+3}, f'_{-d+1}$. The skew-symmetric form $\langle \cdot, \cdot \rangle$ is defined in such a way that $\langle f_i, f'_j \rangle = 1$ ($\langle f_i, f_j \rangle = -1$) when $i+j = 0$, and otherwise $\langle f_i, f'_j \rangle = 0$, $\langle f'_i, f'_j \rangle = 0$. Let $z_{2d}$ be the endomorphism defined by the matrix $J'_{2d}$ with $J'_{2d}(i, i+1) = 1$ ($1 \leq i \leq d-1$), $J'_{2d}(d, d+1) = 0$, $J'_{2d}(i, i+1) = -1$ ($d+1 \leq i \leq 2d-1$) and otherwise $J'_{2d}(i,j) = 0$. Write $d$ as $[k^{s_k}, (k-1)^{s_{k-1}}, \ldots, 2^{s_2}, 1^{s_1}]$ with $s_i > 0$. Note that when $i$ is odd, $s_i$ is even. In the notation above, $d_1 = \ldots = d_{s_{k-1}} = k_1$, $d_{s_{k-1}} = \ldots = d_{s_{k-1} + s_{k-2}} = k - 1$ and so on. Then, $t_1 = s_k, t_2 = s_{k-1}, t_3 = s_k + s_{k-1}, t_4 = s_{k-1} + s_{k-3}, \ldots$ We may assume that

$$V = \bigoplus_{d: \text{even}} (V_d)^{\oplus s_d} \oplus \bigoplus_{d: \text{odd}} (W_{2d})^{\oplus s_d/2}$$

and $z|_{V_d} = z_d, z|_{W_{2d}} = z_{2d}$. An element of $\mu^{-1}(z)$ is the isotropic flag $\{F_i\}$ of $V$ with flag type $(t_1, \ldots, t_l)$ which satisfies $z(F_i) \subset F_{i-2}$ for all $i$. One can find such a flag as follows. Put $F_1 := z^{k-1}(V) (= \text{Im}(z^{k-1}))$ and define $F_{2k-2} :=$
We next put $F_2 := z^{k-2}(F_{2k-2})$ and $F_{2k-3} := F_2^\perp$. The subsequent step is similar; we put inductively $F_i := z^{k-i}(F_{2k-i})$ and $F_{2k-i-1} := F_i^\perp$.

When $G = SO(V)$, $d$ can be written as $[k^{s_k}, (k-1)^{s_{k-1}}, \ldots, 2^{s_2}, 1^{s_1}]$ where $s_i$ is even when $i$ is even. We prepare two kinds of symmetric vector spaces $V_d$ ($d$: odd) and $W_{2d}$ ($d$: even). Then

$$V = \left( \bigoplus_{d: \text{odd}} (V_d)^{\oplus s_d} \right) \oplus \left( \bigoplus_{d: \text{even}} (W_{2d})^{\oplus s_d/2} \right).$$

The description of the flag corresponding to $\mu^{-1}(z)$ is quite similar.

**Lemma (1.4.2)** Assume that $d$ has full members. For each minimal $\epsilon$-degeneration $d \geq f$, the fiber $\mu^{-1}(O_f)$ has codimension 1 in $G \times^P n_2$.

**Proof.** (1): By the proof of (1.3.2), if $d$ has full members, then, for every minimal degeneration $d \geq f$, its reduction $d_{\text{irr}} \geq f_{\text{irr}}$ is of type $f$, $g$ or $h$. If it is of type $f$, the normal slice $N_{d,f}$ of $O_f \subset \tilde{O}_d$ is isomorphic to the germ of $\tilde{O}_{[2,12n-3]}(\subset so(2n + 1))$ ($n > 1$) at 0. Similarly, if it is of type $g$ (resp. $h$), $N_{d,f}$ is isomorphic to the germ of $\tilde{O}_{[2,12n-3]}(\subset sp(2n))$ ($n > 1$) (resp. $\tilde{O}_{[2,12n-4]}(\subset so(2n))$ ($n > 2$)) at 0. Note that they are all isolated singularities. Except when $d_{\text{irr}} \geq f_{\text{irr}}$ is of type $h$ with $n = 3$, these germs have $Q$-factorial terminal singularities. Indeed, they are isolated symplectic singularities of dim $\geq 4$; hence they have only terminal singularities. The $Q$-factoriality of them is checked by using the Jacobson-Morozov resolutions of them. The exceptional locus of each Jacobson-Morozov resolution consists of a flag variety with $b_2 = 1$; this implies the $Q$-factoriality. If $\text{Codim } \mu^{-1}(O_f) \geq 2$, then $N_{d,f}$ is not $Q$-factorial. Therefore, the lemma follows in these cases.

(2): The only exception is when $d_{\text{irr}} \geq f_{\text{irr}}$ is of type $h$ with $n = 3$. In this case, the germ of $\tilde{O}_{[2,12]}(\subset so(6))$ at 0 has only terminal singularity because it is an isolated symplectic singularity with dim 6. But its Jacobson-Morozov resolution has $\text{Gr}_{\text{iso}}(2,6)$ as its exceptional divisor and $\text{b}_2(\text{Gr}_{\text{iso}}(2,6)) = 2$. By this observation, we know that $\tilde{O}_{[2,12]}$ is not $Q$-factorial. Let $d = [k^{s_k}, (k-1)^{s_{k-1}}, \ldots, 2^{s_2}, 1^{s_1}]$ be an $\epsilon$-admissible partition of $m$. Our exceptional case only occurs when $s_j = 2$ for some $j$ with $(-1)^{j+1} = \epsilon$. Then

$$f = \ldots, (j + 1)^{s_{j+1}-2}, j^6, (j - 1)^{s_{j-1}-2}, \ldots].$$

Let

$$\mu : G \times^P n_2 \to \tilde{O}_d$$

be the Jacobson-Morozov resolution. Take $z \in O_f$. We shall prove that the isotropic Grassmann variety $G_{\text{iso}}(2,6)$ is contained in $\mu^{-1}(z)$ by a similar
argument to (1.4.1). We only discuss here the case when $G = Sp(V)$, but the case of $G = SO(V)$ is quite similar. As in (1.4.1), we take two skew-symmetric spaces $V_d$ ($d$: even) and $W_{2d}$ ($d$: odd). We may assume that $V$ is isomorphic to

$$
\bigoplus_{d: \text{even}, d \not= j-1, j, j+1} (V_d)^{\oplus s_d} \oplus W_{2j-2}^{\oplus s_{j-1}/2-1} \oplus V_j^{\oplus 6} \oplus W_{2j+2}^{\oplus s_{j+1}/2-1} \oplus \bigoplus_{d: \text{odd}, d \not= j-1, j, j+1} (W_{2d})^{\oplus s_d}/2
$$

and $z|_{V_d} = z_d$, $z|_{W_{2d}} = z_{2d}$. We must find isotropic flags $\{F_i\}$ of $V$ with flag type $(t_1, \ldots, t_l)$ which satisfies $z(F_i) \subset F_{i-2}$ for all $i$. Here $(t_1, \ldots, t_l)$ is the same one as in (1.4.1). As in (1.4.1), we define $F_i$ with $i \leq k - j - 1$ by $F_i := z^{k-i}(F_{2k-i})$ and $F_{2k-i-1} := F_i^\perp$. But, the situation is different from (1.4.1) when $i = k - j$. We cannot put $F_{k-j} := z^j(F_{k+j})$. In fact, dim $z^j(F_{k+j})$ is exactly 2 less than the dimension $F_{k-j}$ should have because the exponent $s_{j+1} - 2$ of $j + 1$ in $f$ is exactly 2 less than that of $j + 1$ in $d$. Let us consider the direct summand $V_j^{\oplus 6}$ of $V$. The kernel of the endomorphism $z_j : V_j \to V_j$ has dimension 1 and is spanned by $e_{j-1}$. Take 6 copies $e_{j-1}^{(1)}, \ldots, e_{j-1}^{(6)}$ of $e_{j-1}$. Then, $\text{Ker}(z_j^{\oplus 6})$ is a 6 dimensional vector space spanned by $e_{j-1}^{(1)}, \ldots, e_{j-1}^{(6)}$. We want to choose two dimensional subspace $L \subset \text{Ker}(z_j^{\oplus 6})$ and want to define $F_{k-j}$ as $z^j(F_{k+j}) + L$. Since $F_{k-j}$ should be contained in $z^{j-1}(F_{k-j})^{\perp}$, we must choose $L$ in such a way that

$$
z^j(F_{k+j}) + L \subset z^{j-1}(z^j(F_{k+j}) + L)^\perp.
$$

Let $v = \Sigma a_i e_{j-1}^{(i)}$ and $w = \Sigma b_i e_{j-1}^{(i)}$ be a basis of $L$. Then the condition above is equivalent to

$$
\Sigma a_i^2 = \Sigma b_i^2 = \Sigma a_i b_i = 0.
$$

This means that $[L] \in Gr_{iso}(2, 6)$. For such an $L$, we put $F_{k-j} := z^j(F_{k+j}) + L$. Once $F_{k-j}$ is fixed, we define, for $i \geq k - j + 1$, $F_i := z^{k-i}(F_{2k-i})$ and $F_{2k-i-1} := F_i^\perp$. One can check that $\{F_i\}$ is a desired flag. Therefore, $Gr_{iso}(2, 6) \subset \mu^{-1}(z)$. Since dim $Gr_{iso}(2, 6) = 5$ and dim $N_{d,f} = 6$, this implies that $\mu^{-1}(O_f)$ has codimension 1 in $G \times \mathbb{P} n_2$.

**Corollary (1.4.3)** Assume that $d$ is an $\epsilon$-admissible partition and it has full members. Let $\bar{O}_d$ be the normalization of $O_d$. Then, $\bar{O}_d$ has only $Q$-factorial terminal singularities except when $g = so(4n + 2)$, $n \geq 1$ and $d = [2^{2n}, 1^2]$.

**Proof.** Let $k$ be the maximal member of $d$. Then there are $k - 1$ minimal degenerations $d \geq f$. By Lemma (1.4.2), $\text{Exc}(\mu)$ contains at least $k - 1$
irreducible divisors. When \( \epsilon = 1 \) (i.e. \( g = so(V) \)) and there is a minimal degeneration \( d \geq f \) with \( f \) very even, there are two nilpotent orbits with Jordan type \( f \). Thus, in this case, \( \text{Exc}(\mu) \) contains at least \( k \) irreducible divisors. On the other hand, for the Jacobson-Morozov parabolic subgroup \( P \), \( b_2(G/P) = k - 1 \) when \( g = sp(V) \), or \( g = so(V) \) with \( \dim V \) odd. When \( g = so(V) \) and \( \dim V \) is even, we must be careful; if the flag type of \( P \) is of the form \( (p_1, ..., p_{k-1}; 2; p_{k-1}, ..., p_1) \), \( b_2(G/P) = k \). This happens when \( \dim V = 4n + 2 \) and \( d = [2^2n, 1^2] \) or when \( \dim V = 8m + 4n + 4 \) and \( d = [4^{2m}, 3, 2^{2n}, 1] \). In the latter case, \( d \) has a minimal degeneration \( d \geq f \) with \( f = [4^{2m}, 2^{2n} + 2] \), which is very even. Note that \( b_2(G/P) \) coincides with the relative Picard number \( \rho \) of the Jacobson-Morozov resolution. By these observations, we know that \( \mu \) has at least \( \rho \) exceptional divisors except when \( g = so(4n + 2), n \geq 1 \) and \( d = [2^{2n}, 1^2] \). Therefore, \( \mathcal{O}_d \) are \( \mathbb{Q} \)-factorial in these cases. By (1.3.2) they have terminal singularities. When \( g = so(4n + 2), n \geq 1 \) and \( d = [2^{2n}, 1^2] \), \( \mathcal{O}_d \) is a Richardson orbit and the Springer map gives a small resolution of \( \bar{O}_d \). Therefore, \( \bar{O}_d \) has non-\( \mathbb{Q} \)-factorial terminal singularities.

(1.5) Induced orbits in classical Lie algebras: Let \( d = [d_1^{s_1}, ..., d_k^{s_k}] \) be an \( \epsilon \)-admissible partition of \( m \). According as \( \epsilon = 1 \) or \( \epsilon = -1 \), we put \( G = SO(m) \) or \( G = Sp(m) \) respectively. Assume that \( d \) does not have full members. In other words, for some \( p \), \( d_p \geq d_{p+1} + 2 \) or \( d_k \geq 2 \). We put \( r = \sum_{1 \leq j \leq p} s_j \). Then \( \mathcal{O}_d \) is an induced orbit (cf. [C-M], 7.3). More explicitly, there are a parabolic subgroup \( Q \) of \( G \) with (isotropic) flag type \( (r, m - 2r, r) \) with Levi decomposition \( q = l \oplus n \), and a nilpotent orbit \( \mathcal{O}' \) of \( l \) such that \( \mathcal{O}_d = \text{Ind}^l_q(\mathcal{O}') \). Here, \( l \) has a direct sum decomposition \( l = gl(r) \oplus g' \), where \( g' \) is a simple Lie algebra of type \( B_{(m-2r-1)/2} \) (resp. \( D_{(m-2r)/2} \), resp. \( C_{(m-2r)/2} \)) when \( \epsilon = 1 \) and \( m \) is odd (resp. \( \epsilon = 1 \) and \( m \) is even, resp. \( \epsilon = -1 \)). Moreover, \( \mathcal{O}' \) is a nilpotent orbit of \( g' \) with Jordan type \( [(d_1 - 2)^{s_1}, ..., (d_p - 2)^{s_p}, d_{p+1}^{s_{p+1}}, ..., d_k^{s_k}] \).

Let us consider the generalized Springer map

\[
\nu : G \times^Q (n(q) + \mathcal{O}') \to \bar{O}_d
\]

(cf. (1.2)).

**Lemma (1.5.1).** The map \( \nu \) is birational. In other words, \( \deg(\nu) = 1 \).

**Proof.** We only discuss the case \( G = Sp(m) \). It is enough to prove that \( \nu^{-1}(z) \) is a point for \( z \in \mathcal{O}_d \). We prepare two kinds of skew-symmetric vector space \( V_d \) (\( d \): even), and \( W_{2d} \) (\( d \): odd) as follows. The \( V_d \) is a \( d \)}
dimensional vector space with a basis $e_1, e_2, ..., e_d$. The skew-symmetric form $\langle \ , \ \rangle$ is defined in such a way that $\langle e_i, e_j \rangle = 1$ (resp. $\langle e_i, e_j \rangle = -1$) when $i + j = d + 1$ and $i > j$ (resp. $i + j = d + 1$ and $i < j$), and $\langle e_i, e_j \rangle = 0$ when $i + j \neq d + 1$. Let $z_d$ be the endomorphism of $V_d$ defined by the $d \times d$ matrix $J_d$ with $J_d(i, i + 1) = 1$ $(1 \leq i \leq d/2), J_d(i, i + 1) = -1$ $(d/2 + 1 \leq i \leq d - 1)$ and otherwise $J_d(i, j) = 0$. The $W_{2d}$ is a $2d$ dimensional vector space with a basis $f_1, ..., f_d, f_1', ..., f_d'$. The skew-symmetric form $\langle \ , \ \rangle$ is defined in such a way that $\langle f_i, f_j' \rangle = 1$ ($\langle f_i', f_j \rangle = -1$) when $i + j = d + 1$, and otherwise $\langle f_i, f_j' \rangle = 0, \langle f_i', f_j \rangle = 0$. Let $z_{2d}$ be the endomorphism defined by the matrix $J_{2d}'$ with $J_{2d}'(i, i + 1) = 1$ $(1 \leq i \leq d - 1), J_{2d}'(d, d + 1) = 0, J_{2d}'(i, i + 1) = -1$ $(d + 1 \leq i \leq 2d - 1)$ and otherwise $J_{2d}'(i, j) = 0$. Note that, in the partition $d, s_i$ is even if $d_i$ is odd. When $d_i$ is even, we put $U_i := V_{d_i}^{\oplus s_i}$ and define $z_i \in \text{End}(U_i)$ by $z_i = z_{d_i}^{\oplus s_i}$. When $d_i$ is odd, we put $U_i := W_{2d_i}^{\oplus s_i/2}$ and define $z_i \in \text{End}(U_i)$ by $z_i = z_{2d_i}^{\oplus s_i/2}$. Let us consider the skew-symmetric vector space $V$ defined by

$$V := \oplus_{1 \leq i \leq k} U_i.$$ 

Then we may assume that $z$ is an endomorphism of $V$ defined by $z = \oplus z_i$. Each $U_i$ has a filtration $0 \subset U_{i, 1} \subset U_{i, 2} \subset ... \subset U_{i, d_i} = U_i$ defined by $U_{i, j} := \text{Im}(z_i^{d_i-j})$. By definition, $U_{i, d_i-1} = (U_i, 1)^\perp$. The problem is to find an $r$ dimensional isotropic subspace $F$ of $V$ in such a way that $z(F) = 0$ and $z|_{(F^\perp/F)}$ has Jordan type $[(d_1-2)^{s_1}, ..., (d_p-2)^{s_p}, d_{p+1}^{s_{p+1}}, ..., d_k^{s_k}]$. We shall prove that $F = \oplus_{1 \leq i \leq p} U_{i, 1}$. First note that $F \subset \oplus_{1 \leq i \leq k} U_{i, 1}$ since $z(F) = 0$. Next, one can check that $F^\perp \subset \oplus_{1 \leq i \leq k} U_{i, 1}$. In fact, if not, then one can find some $v \in F^\perp/F$ with $z^{d_i-2}(v) \neq 0$, which contradicts that $z|_{F^\perp/F}$ has Jordan type $[(d_1-2)^{s_1}, ..., (d_p-2)^{s_p}, d_{p+1}^{s_{p+1}}, ..., d_k^{s_k}]$. By taking the dual with respect to the skew-symmetric form, one has $U_{1, 1} \subset F$. We put $F_2 := F \cap \oplus_{2 \leq i \leq k} U_{i, 1}$. Let $(F_2)^\perp$ be the orthogonal complement of $F_2$ in $\oplus_{2 \leq i \leq k} U_i$ (not in $V$). Then

$$F^\perp/F = (\bigoplus_{2 \leq j \leq d_i-1} U_{1, j}) \oplus (F_2)^\perp/F_2.$$ 

Then $z|_{(F_2)^\perp/F_2}$ has Jordan type $[(d_2-2)^{s_1}, ..., (d_p-2)^{s_p}, d_{p+1}^{s_{p+1}}, ..., d_k^{s_k}]$. We apply the same argument to $F_2$ to conclude that $U_{2, 1} \subset F_2$. In particular, $U_{2, 1} \subset F$. In this way, we can prove inductively that $U_{i, 1} \subset F$ for $i \leq p$. Since $\dim(\oplus_{1 \leq i \leq p} U_{i, 1}) = r$, we have $F = \oplus_{1 \leq i \leq p} U_{i, 1}$.

**Remark (1.5.2).** A nilpotent orbit is called *rigid* if it is not induced from any other nilpotent orbit in a proper Levi subalgebra of $\mathfrak{g}$. In a simple Lie
algebra \(g\) of type B, C or D, \(O_d\) is rigid if and only if \(d\) has full members and any odd (resp. even) member \(d_i\) does not have multiplicity \(s_i = 2\) when \(\epsilon = 1\) (resp. \(\epsilon = -1\)) (cf. [CM]). By Corollary (1.4.3), for a rigid orbit \(O_d\), \(O_d\) has only \(Q\)-factorial terminal singularities. But, even if \(O_d\) has only \(Q\)-factorial terminal singularities, \(O_d\) is not necessarily rigid. Assume that an odd (resp. even) member \(d_p\) has multiplicity 2 when \(\epsilon = 1\) (resp. \(\epsilon = -1\)). In this case, we have an induction of another type. Namely, put \(r = \sum_{1 \leq j \leq p-1} s_j + 1\). Then, there are a parabolic subgroup \(Q\) of \(G\) with (isotropic) flag type \((r, m-2r, r)\) with Levi decomposition \(q = l \oplus n\), and a nilpotent orbit \(O'\) in \(l\) such that \(O_d = \text{Ind}^g_l(O')\). The orbit \(O'\) is contained in \(g'\), and its Jordan type is \([(d_1 - 2)^{s_1}, \ldots, (d_{p-1} - 2)^{s_{p-1}}, (d_p - 1)^2, d_{p+1}^{s_{p+1}}, \ldots, d_k^{s_k}]\). As explained above, if \(d\) has full members, but \(O_d\) is not rigid, then \(O_d\) has an induction of this type. But, for such an induction, the generalized Springer map \(\nu\) is not birational.

§2. Main Results

(2.1) Let \(X\) be a complex algebraic variety with rational Gorenstein singularities. A partial resolution \(f : Y \rightarrow X\) of \(X\) is said to be a \(Q\)-factorial terminalization of \(X\) if \(Y\) has only \(Q\)-factorial terminal singularities and \(f\) is a birational projective morphism such that \(K_Y = f^*K_X\). In particular, when \(Y\) is smooth, \(f\) is called a crepant resolution. In general, \(X\) has no crepant resolution; however, by [B-C-H-M], \(X\) always has a \(Q\)-factorial terminalization. But, in our case, the \(Q\)-factorial terminalization can be constructed very explicitly without using the general theory in [B-C-H-M].

**Proposition (2.1.1).** Let \(O\) be a nilpotent orbit of a classical simple Lie algebra \(g\). Let \(\bar{O}\) be the normalization of \(O\). Then one of the following holds:

(1) \(\bar{O}\) has \(Q\)-factorial terminal singularities.

(2) There are a parabolic subalgebra \(q\) of \(g\) with Levi decomposition \(q = l \oplus n\) and a nilpotent orbit \(O'\) of \(l\) such that (a): \(O = \text{Ind}^g_l(O')\) and (b): the normalization of \(G \times \mathbb{Q} \ (n(q) + O')\) is a \(Q\)-factorial terminalization of \(\bar{O}\) via the generalized Springer map.

**Proof.** When \(g\) is of type A, every \(\bar{O}\) has a Springer resolution; hence (2) always holds. Let us consider the case \(g\) is of B, C or D. Assume that (1) does not hold. Then, by (1.4.3), the Jordan type \(d\) of \(O\) does not have full members except when \(g = \text{so}(4n + 2), n \geq 1\) and \(d = [2^{2n}, 1^2]\). In the exceptional case, \(O\) is a Richardson orbit and the Springer map gives a crepant resolution of \(\bar{O}\); hence (2) holds. Now assume that \(d\) does not have
full members. Then, by (1.5), $\mathcal{O}$ is an induced nilpotent orbit and there is a generalized Springer map

$$\nu : G \times Q (n(q) + \bar{O}') \to \bar{O}.$$ 

This map is birational by (1.5.1). Let us consider the orbit $\mathcal{O}'$ instead of $\mathcal{O}$. If (1) holds for $\mathcal{O}'$, then $\nu$ induces a $Q$-factorial terminalization of $\bar{O}$. If (1) does not hold for $\mathcal{O}'$, then $\mathcal{O}'$ is an induced orbit. By (1.2.3), one can replace $Q$ with a smaller parabolic subgroup $Q'$ in such a way that $\mathcal{O}$ is induced from $(Q', \mathcal{O}_2)$ for some nilpotent orbit $\mathcal{O}_2 \subset l(Q')$. The generalized Springer map $\nu'$ for $(Q', \mathcal{O}_2)$ is factorized as

$$G \times Q' (n' + \bar{O}_2) \to G \times Q (n + \bar{O}') \to \bar{O}.$$ 

The second map is birational as explained above. The first map is locally obtained by a base change of the generalized Springer map

$$L(Q) \times L(Q) \cap Q' (n(L(Q) \cap Q') + \bar{O}_2) \to \bar{O}'.$$

This map is birational by (1.5.1). Therefore, the first map is also birational, and $\nu'$ is birational. This induction step terminates and (2) finally holds.

(2.2) We shall next show that every $Q$-factorial terminalization of $\bar{O}$ is of the form in Proposition (2.1.1) except when $\bar{O}$ itself has $Q$-factorial terminal singularities. In order to do that, we need the following Proposition.

**Proposition (2.2.1).** Let $\mathcal{O}$ be a nilpotent orbit of a classical simple Lie algebra $\mathfrak{g}$. Assume that a $Q$-factorial terminalization of $\bar{O}$ is given by the normalization of $G \times Q (n(q) + \bar{O}')$ for some $(Q, \mathcal{O}')$ as in (2.1.1). Assume that $Q$ is a maximal parabolic subgroup of $G$ (i.e. $b_2(G/Q) = 1$), and this $Q$-factorial terminalization is small. Then $Q$ is a parabolic subgroup corresponding to one of the following marked Dynkin diagrams and $\mathcal{O}' = 0$: 

- $A_{n-1}$ ($k < n/2$)
  $$\begin{array}{ccccccc}
  \circ & \cdots & k & \cdots & \circ \\
  \circ & \cdots & n-k & \cdots & \circ \\
  \end{array}$$ 
- $D_n$ ($n : \text{odd} \geq 5$)
  $$\begin{array}{ccccccc}
  \times \circ & \cdots & \circ \\
  \end{array}$$
Proof. Assume that \( g \) is of type \( A \). Then every nilpotent orbit closure has a crepant resolution via the Springer map. Once \( \tilde{O} \) has a crepant resolution, every \( \mathbb{Q} \)-factorial terminalization is a crepant resolution (cf. [Na 2]). Then the claim follows from [Na], Proposition 5.1. Assume that \( g \) is of type \( B, C \) or \( D \). Since the \( \mathbb{Q} \)-factorial terminalization is small, \( \tilde{O} \) has terminal singularities. By (1.3.2) and (1.4.3), \( O \) is the nilpotent orbit in \( \text{so}(4n + 2) \), \( n \geq 1 \) with Jordan type \( [2^n, 1^2] \). In this case, \( O \) is a Richardson orbit and \( \tilde{O} \) has exactly two crepant resolutions via the Springer maps \( G \times \mathbb{Q} n(\mathfrak{q}) \to \tilde{O} \) corresponding to two marked Dynkin diagrams of type \( D \) listed above (cf. [Na], p.92). Therefore, \( \tilde{O} \) has no other \( \mathbb{Q} \)-factorial terminalizations.

The following is the main theorem:

**Theorem (2.2.2).** Let \( O \) be a nilpotent orbit of a classical simple Lie algebra \( g \). Then \( \tilde{O} \) always has a \( \mathbb{Q} \)-factorial terminalization. If \( \tilde{O} \) itself does not have \( \mathbb{Q} \)-factorial terminal singularities, then every \( \mathbb{Q} \)-factorial terminalization is given by the normalization of \( G \times \mathbb{Q} (n(\mathfrak{q}) + \bar{O}') \) in (2.1.1). Moreover, any two such \( \mathbb{Q} \)-factorial terminalizations are connected by a sequence of Mukai flops of type \( A \) or \( D \) defined in [Na], pp. 91, 92.

Proof. The first statement is nothing but (2.1.1). The proof of the second statement is quite similar to that of [Na], Theorem 6.1. Assume that \( \tilde{O} \) does not have \( \mathbb{Q} \)-factorial terminal singularities. Then, by (2.1.1), one can find a generalized Springer (birational) map

\[
\nu : G \times \mathbb{Q} (n(\mathfrak{q}) + \bar{O}') \to \tilde{O}.
\]

Let \( X_Q \) be the normalization of \( G \times \mathbb{Q} (n(\mathfrak{q}) + \bar{O}') \). Then \( \nu \) induces a \( \mathbb{Q} \)-factorial terminalization \( f : X_Q \to \tilde{O} \). The relative nef cone \( \overline{\text{Amp}}(f) \) is a rational, simplicial, polyhedral cone of dimension \( b_2(G/Q) \) (cf. (1.2.2) and [Na], Lemma 6.3). Each codimension one face \( F \) of \( \overline{\text{Amp}}(f) \) corresponds to a birational contraction map \( \phi_F : X_Q \to Y_Q \). The construction of \( \phi_F \) is as follows. The parabolic subgroup \( Q \) corresponds to a marked Dynkin diagram \( D \). In this diagram, there are exactly \( b_2(G/Q) \) marked vertexes. Choose a marked vertex \( v \) from \( D \). The choice of \( v \) determines a codimension one face \( F \) of \( \overline{\text{Amp}}(f) \). Let \( D_v \) be the maximal, connected, single marked Dynkin subdiagram of \( D \) which contains \( v \). Let \( \bar{D} \) be the marked Dynkin diagram obtained from \( D \) by erasing the marking of \( v \). Let \( \bar{Q} \) be the parabolic
subgroup of $G$ corresponding to $\bar{D}$. Then, as in (1.2.2), we have a map

$$\pi : G \times^Q (n + \bar{O}') \to G \times^{\bar{Q}} (\bar{n} + \bar{O}_1).$$

Let $Y_Q$ be the normalization of $G \times^{\bar{Q}} (\bar{n} + \bar{O}_1)$. Then $\pi$ induces a birational map $X_Q \to Y_Q$. This is the map $\phi_F$. Note that $\pi$ is locally obtained by a base change of the generalized Springer map

$$L(\bar{Q}) \times^{L(Q) \cap Q} (n(L(\bar{Q}) \cap Q) + \bar{O}') \to \bar{O}_1.$$

Let $Z(l(q))$ (resp. $Z(l(\bar{q}))$) be the center of $l(q)$ (resp. $l(\bar{q})$). By the definition of $\bar{Q}$, the simple factors of $l(\bar{q})/Z(l(\bar{q}))$ are common to those of $l(q)/Z(l(q))$ except one factor, say $m$. Put $O'' := O' \cap m$. By (2.2.1), $\pi$ (or $\phi_F$) is a small birational map if and only if $O'' = 0$ and $D_v$ is one of the single Dynkin diagrams listed in (2.2.1). In this case, one can make a new marked Dynkin diagram $D'$ from $D$ by replacing $D_v$ by its dual $D^*_v$ (cf. [Na], Definition 1).

Let $Q'$ be the parabolic subgroup of $G$ corresponding to $D'$. We may assume that $Q$ and $Q'$ are both contained in $\bar{Q}$. The Levi part of $Q'$ is conjugate to that of $Q$; hence there is a nilpotent orbit in $l(q')$ corresponding to $O'$. We denote this orbit by the same $O'$. Then $O$ is induced from $(Q', O')$. As above, let $X_{Q'}$ be the normalization of $G \times^Q (n(q') + \bar{O}')$. Then we have a birational map $\phi_F : X_{Q'} \to Y_Q$. The diagram

$$X_Q \to Y_Q \leftarrow X_{Q'}$$

is a flop. Assume that $g : X \to \bar{O}$ is a $Q$-factorial terminalization. Then, the natural birational map $X \leftarrow \to X_Q$ is an isomorphism in codimension one. Let $L$ be a $g$-ample line bundle on $X$ and let $L_0 \in \text{Pic}(X_Q)$ be its proper transform of $L$ by this birational map. If $L_0$ is $f$-nef, then $X = X_Q$ and $f = g$. Assume that $L_0$ is not $f$-nef. Then one can find a codimension one face $F$ of $\overline{\text{Amp}}(f)$ which is negative with respect to $L_0$. Since $L_0$ is $f$-movable, the birational map $\phi_F : X_Q \to Y_Q$ is small. Then, as seen above, there is a new (small) birational map $\phi'_F : X_{Q'} \to Y_Q$. Let $f' : X_{Q'} \to \bar{O}$ be the composition of $\phi'_F$ with the map $Y_Q \to \bar{O}$. Then $f'$ is a $Q$-factorial terminalization of $\bar{O}$. Replace $f$ by this $f'$ and repeat the same procedure; but this procedure ends in finite times (cf. [Na], Proof of Theorem 6.1 on pp. 104, 105). More explicitly, there is a finite sequence of $Q$-factorial terminalizations of $\bar{O}$:

$$X_0(= X_Q) \leftarrow \to X_1(= X_{Q'}) \leftarrow \to \ldots \leftarrow \to X_k(= X_{Q_k})$$
such that $L_k \in \text{Pic}(X_k)$ is $f_k$-nef. This means that $X = X_{Q_k}$.

**Example (2.3).** We put $G = SP(12)$. Let $\mathcal{O}$ be the nilpotent orbit in $sp(12)$ with Jordan type $[6, 3^2]$. Let $Q_1 \subset G$ be a parabolic subgroup with flag type $(3, 6, 3)$. The Levi part $l_1$ of $q_1$ has a direct sum decomposition

$$l_1 = \mathfrak{g}l(3) \oplus C_3,$$

where $C_3$ is isomorphic to $sp(6)$ up to center. Let $\mathcal{O}'$ be the nilpotent orbit in $sp(6)$ with Jordan type $[4, 1^2]$. Then $\mathcal{O} = \text{Ind}_{l_1}^{sp(12)}(\mathcal{O}')$. Next consider the parabolic subgroup $Q_2 \subset SP(6)$ with flag type $(1, 4, 1)$. The Levi part $l_2$ of $q_2$ has a direct sum decomposition

$$l_2 = \mathfrak{g}l(1) \oplus C_2,$$

where $C_2$ is isomorphic to $sp(4)$ up to center. Let $\mathcal{O}''$ be the nilpotent orbit in $sp(4)$ with Jordan type $[2, 1^2]$. Then $\mathcal{O}' = \text{Ind}_{l_2}^{sp(6)}(\mathcal{O}'')$. One can take a parabolic subgroup $Q \subset SP(12)$ with flag type $(3, 1, 4, 1, 3)$ in such a way that the Levi part $l$ of $q$ contains the nilpotent orbit $\mathcal{O}''$. Then $\mathcal{O}$ is the nilpotent orbit induced from $\mathcal{O}''$. We shall illustrate the induction step above by

$$([2, 1^2], sp(4)) \rightarrow ([4, 1^2], sp(6)) \rightarrow ([6, 3^2], sp(12)).$$

Since $\tilde{O}''$ has only $Q$-factorial terminal singularities, the $Q$-factorial terminalization of $\tilde{O}$ is given by the generalized Springer map

$$\nu : G \times^Q (n(q) + \tilde{O}'') \rightarrow \tilde{O}.$$

The induction step is not unique; we have another induction step

$$([2, 1^2], sp(4)) \rightarrow ([4, 3^2], sp(10)) \rightarrow ([6, 3^2], sp(12)).$$

By these inductions, we get another generalized Springer map

$$\nu' : G \times^{Q'} (n(q') + \tilde{O}'') \rightarrow \tilde{O},$$

where $Q'$ is a parabolic subgroup of $G$ with flag type $(1, 3, 4, 3, 1)$. This gives another $Q$-factorial terminalization of $\tilde{O}$. The two $Q$-factorial terminalizations of $\tilde{O}$ are connected by a Mukai flop of type $A_3$. 
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