Dynamics of Predator-Prey Model Interaction with Intraspecific Competition

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Abstract. In this paper, we have formulated and studied a dynamic predator-prey model. We have formulated intraspecific competition of predators. Holling type I and Holling type II response function have been used for the consumption of prey by two predators respectively. We have analyzed the positivity of solutions, existence of equilibria, and stability of the proposed system around these equilibrium points. Condition for local stability was obtained by eigenvalue approach and Routh-Hurwitz Criterion. Finally, some numerical simulation have been presented to investigate the dynamic of the system.

1. Introduction
The interaction between organisms is imperative to carry on life. The interactions that take place will have an impact on the quality of life of these creatures. The main reason for organisms is to survive.

Observing the interactions between organisms is not easy. One of the alternative disciplines that make it possible to observe this is mathematical modeling. Mathematical modeling in the field of ecology is very interesting to study considering the many factors that affect the growth and life of living populations and the balance of organisms. The process of dynamics of organisms can be modeled mathematically by using differential equations involving continuous-time or discrete-time. One of the mathematical models used to explain this natural phenomenon is the prey-predator population model. Competition between predators and harvesting factors in populations is very important in the discipline of ecology. Many researchers can evoke interesting things from behavioral dynamics in population ecosystems. By combining the two aspects above, namely the aspects of competition between predators and harvesting, population dynamics can be expressed in a model.

There are many researchers who model prey-predator interactions. Alebraheem and Abu Hasan [1] examined the resistance of predators in the prey-predator model system with non-periodic solutions. Das [2] discusses the dynamics of the prey-predator with diseased predators. Li et al., [3] discussed global dynamics of a prey-predator model with antipredator behavior and two predators. Agarwal and Fathak [4] analyzed persistence and optimal harvesting of prey-predator model with hollig type III functional response. Gupta and Chandra [5] discuss the dynamics of the prey-predator model by quadratic harvesting. Kar and Matsuda [6] discusses global dynamics and control of predator prey models with Holling type III response functions. Kar et al., [7] proposed and analysed a dynamic reaction model of a prey-predator system with stage-structure for predator. Mukhopadhyay and Bhattacharyya [8], examined the effect of harvesting and competition between predators in the prey-predator model. Ndam et al., [9] discuss a model of interaction of three species in one habitat.
Upadhyay and Raw [10] discuss the complex dynamics of a three-species food chain model with Holling type III response functions.

2. Assumptions and model

This research has studied model from the other researchers. In this model, there is an interference between predators as modeled by other researchers [1,8,10,11]. There are researchers who studied the intraspecific competition coefficient [11]. The response function is used by researchers in their models [12-14]. Researchers frequently use the Holling-type I response function [3,5,8] and Holling-type II [1,5,8,11,15,14].

The one prey-two predator model studied involves interaction between predators and interactions between individuals in the predator population. Following are the assumptions used in the model:

a. The prey growth rate uses the logistical growth rate,
b. Predators compete with each other for prey,
c. Predator I uses the Holling type II response function for predation,
d. Predator II uses the Holling type I response function for predation,
e. There is an intraspecific competition for each predator.

The model is formulated as follows:

\[
\frac{dP}{dt} = rP - \frac{rP^2}{K} - \frac{\alpha_1PH_1}{1 + \theta P} - \alpha_2PH_2
\]
\[
\frac{dH_1}{dt} = e_1\alpha_1PH_1 - g_1H_1^2 - \beta_1H_1H_2 - d_1H_1
\]
\[
\frac{dH_2}{dt} = e_2\alpha_2PH_2 - g_2H_2^2 - \beta_2H_1H_2 - d_2H_2
\]

with initial condition

\[
P(0) \geq 0, H_1(0) \geq 0, H_2(0) \geq 0.
\]

The two predatory species (H_1 and H_2) are assumed to have direct access to prey (P). They exert the same influence on prey and they compete. The effect of disturbance in the growth rate of competitors is assumed to be proportional to the density of the predator population with \( \beta_1 \) and \( \beta_2 \) respectively given disturbance rates. The parameter \( \alpha_1 \) and \( \alpha_2 \) represent predator rates for predator species \( H_1 \) and \( H_2 \) respectively. The parameter and represent coefficient intraspecific competition for two predators and respectively, here \( d_1 \) and \( d_2 \) are their mortality rate. The parameter \( e_1 \) and \( e_2 \) are the predator’s conversion efficiency. The parameter \( \theta \) represents the half-saturation constant. However, the predation functions of the two predators were made different - one following a Holling type I response and the other following a Holling type II response. Besides experiencing a reduction due to the predation function, the prey population grew logistically with r as the intrinsic growth rate and K as the holding capacity.

3. Equilibrium points and stability analysis

3.1. Equilibrium points

Equilibrium points of the system (1) are given below:

- The trivial equilibrium point \( E_0 = (0, 0, 0) \),
- The predators free equilibrium point \( E_1 = (K, 0, 0) \),
- The \( H_1 \)-free boundary equilibrium state \( E_2 = \left( \frac{K(d_1a_2 + rg_2)}{Ke_2a_2^2 + rg_2}, 0, \left( \frac{Ke_2a_2 \cdot d_2}{Ke_2a_2^2 + rg_2} \right) \right) \).
• The $H_2$-free boundary equilibrium state $E_3 = (P^*, H_1^*, 0)$, where
$H_1^* = \frac{r}{K\alpha_1} \left[ KP^* \theta - P^* \theta + K - P^* \right]$.
and $P^*$ the only positive root of the cubic equation
$[r\theta^3 - r\theta + Kr\theta^2]s^3 + [K\alpha_1^2 - r\theta - Kr\theta \theta_1 - 2Kr\theta \theta_1]s - d_1\alpha_1 - K\theta_1 = 0$

• The interior equilibrium point $E_4 = (P^{**}, H_1^{**}, H_2^{**})$, where
$H_1^{**} = K(P^{**})^2 e_2 \alpha_2^2 - KP^* d_2 \alpha_2 + KP^* e_2 \alpha_2^2 - KP^* r\theta g_2 + (P^{**})^2 r\theta g_2$
$- Kd_2 \alpha_2 - K\theta g_2 + \frac{P^* r\theta_2}{K\alpha_2 \beta_2 + \alpha_2 \beta_2 - \alpha_1 \beta_2}$
$H_2^{**} = P^* e_2 \alpha_2 K\alpha_1 + KP^* r\theta_2 - (P^{**})^2 r\theta_2 + d_2 \alpha_1 + K\theta_2 - \frac{P^* r\theta_2}{K\alpha_2 \beta_2 + \alpha_2 \beta_2 - \alpha_1 \beta_2}$
and $P^{**}$ the only positive root of the cubic equation
\[
[KE_2 \theta^2 \alpha_2^2 g_1 + r\theta^2 \theta_1 g_1 - r\theta^2 \beta_2] \theta^3 + [Kd_1 \theta^2 \alpha_2 \beta_2 + 2KE_2 \theta^2 \alpha_2^2 g_1 + K\theta^2 \beta_1 \beta_2 + 2r\theta_1 g_1 - Kd_2 \theta^2 \alpha_2 g_1 - Kd_2 \theta^2 \alpha_2 \beta_2 - Ke_1 \theta^2 \alpha_2 \beta_1 - KKe_2 \theta^2 \alpha_2 \beta_1 - K\theta^2 \theta_1 g_1 - 2r\theta_1 \beta_2] \theta^2 + [2Kd_1 \theta^2 \beta_2 + Kd_2 \theta_1 \beta_1 + Ke_1 \alpha_2 \beta_2 + Ke_2 \alpha_2 \beta_1 + 2K\theta^2 \beta_1 \beta_2 + r\theta_1 g_2 - Kd_1 \theta_1 g_1 - 2Kd_2 \theta_1 g_1 - Ke_1 \alpha_2 \beta_2 - Ke_1 \alpha_2 \beta_1 - 2K\theta_1 g_1 - K\theta_1 \beta_2 + Kd_2 \alpha_2 g_1 - Kd_1 \alpha_2 g_1 - Kd_2 \alpha_2 g_1 - K\theta_1 g_1] = 0
\]

3.2. Stability analysis
The stability analysis equilibrium point of the system (1) is studied and determined. The point $E_0$ is
trivial equilibrium point. The predators free equilibrium point $E_1 = (K, 0, 0)$ is stable if
$Kd_1 \theta + d_1 > K\alpha_1 \theta$ (3)
and
$Ke_2 \alpha_2 > d_2$. (4)

Jacobian matrix of the model system (1) is
\[
J = \begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{bmatrix}
\]
where,
\[
J_{11} = \frac{2r}{P^{**}} - a_1 H_1 + a_1 P^{**} ; J_{12} = -a_2 P ; J_{13} = -a_2 P ; J_{21} = \frac{e_1 a_1 H_1}{P^{**}} + \frac{e_1 a_1 P^{**}}{P^{**}} ; J_{22} = -2d_2 H_1 - d_1 ; J_{23} = -H_1 ; J_{31} = e_2 a_2 H_2 ; J_{32} = -\beta_2 H_2 ; J_{33} = e_2 a_2 \beta_2 H_1 - 2g_2 H_2 - d_2.
\]

**Theorem 1** The solutions of the system (1) are non-negative. Equilibrium point
$E_2 = (K(d_2 \alpha_2 + rg_2), 0, r(E_2 \alpha_2 - d_2))$ local stable if
$\frac{e_2 a_1 K(d_2 \alpha_2 + rg_2)}{K \alpha_2^{2+rg_2}} < \frac{\beta_2 K(\alpha_2 \alpha_2 - d_2)}{K \alpha_2^{2+rg_2}} + d_1$
$\frac{2(d_2 \alpha_2 + rg_2)}{K \alpha_2^{2+rg_2}} \frac{e_2 a_2 K(d_2 \alpha_2 + rg_2)}{K \alpha_2^{2+rg_2}} < \frac{2g_2 d(\alpha_2 \alpha_2 - d_2)}{K \alpha_2^{2+rg_2}} + d_2$ and $J_{11}J_{33} > J_{13}J_{31}$.

**Proof**. The result of substitution equilibrium point $E_2$ to Jacobian Matrix (5)
\[
J(E_2) = \begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{bmatrix}
\]
where
\[ J_{11}^0 = \frac{2\alpha K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2}, \quad J_{12}^0 = \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2}, \quad J_{13}^0 = \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2}, \quad J_{21}^0 = 0, \]
\[ J_{22}^0 = \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2} - d_1, \quad J_{23}^0 = 0, \quad J_{31}^0 = \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2} - \beta, \quad J_{32}^0 = \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2} - \beta, \quad J_{33}^0 = \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2} - \beta.
\]

Characteristic equation matrix \( J(E_2) \) is

\[
\text{det}(J(E_2) - \lambda I) = 0. \tag{7}
\]

One of the roots equation (7) is \( \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2} - \beta, \) Eigenvalue is negative if \( \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2} < \beta \). The other eigenvalues are roots of the quadratic equation

\[
\lambda^2 \left( J_{11}^0 + J_{33}^0 \right) \lambda + J_{11}^0 J_{33}^0 - J_{13}^0 J_{31}^0 = 0. \tag{8}
\]

The roots of the equation (8) is negative if \( J_{11}^0 + J_{33}^0 < 0, J_{11}^0 < 0, J_{33}^0 < 0 \) dan \( J_{41}^0 J_{43}^0 > J_{13}^0 J_{31}^0 \).

**Theorem 2** Equilibrium point \( E_3 = (P^*, H_1^*, 0) \) local stable if \( J_{11}^0 + J_{33}^0 < 0, J_{11}^0, J_{33}^0 > J_{12}^0, J_{21}^0, J_{12}^0, J_{21}^0, J_{33}^0 > J_{11}^0, J_{33}^0, \) and \( A_1 A_2 > A_3 \).

**Proof.** The result of substitution equilibrium point \( E_3 \) to Jacobian Matrix (5)

\[
J(E_3) = \begin{bmatrix}
J_{11}^0 & J_{12}^0 & J_{13}^0 \\
J_{21}^0 & J_{22}^0 & J_{23}^0 \\
J_{31}^0 & J_{32}^0 & J_{33}^0
\end{bmatrix}
\tag{9}
\]

where

\[
J_{11}^0 = r_1 - \frac{2p^*}{K} \cdot a_1 H_1^* + \frac{a_2 P^* H_1^*}{p^*} + \frac{a_2 p^*}{p^*} + \frac{a_2 P^*}{p^*}, \quad J_{12}^0 = \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2}, \quad J_{13}^0 = -a_2 P^*, \quad J_{21}^0 = \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2}, \quad J_{22}^0 = \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2} - 3, \quad J_{23}^0 = \frac{\alpha \lambda K(e_2 x_2 + r_2)}{Ke_2 x_2 + r_2} - 3.
\]

Characteristics equation matrix \( J(E_3) \) is

\[
\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \tag{10}
\]

where,

\[
A_1 = -(J_{11}^0 + J_{22}^0 + J_{33}^0)
\]

\[
A_2 = J_{11}^0 J_{22}^0 J_{33}^0 + J_{12}^0 J_{13}^0 - J_{12}^0 J_{21}^0
\]

\[
A_3 = J_{13}^0 J_{23}^0 J_{31}^0 - J_{13}^0 J_{23}^0 J_{31}^0.
\]

To ensure the stability of model system with equilibrium point \( E_3 \), the point must qualify of the Routh-Hurtwiz criteria. The equation (10) have negative roots if \( J_{11}^0 + J_{22}^0 + J_{33}^0 < 0, J_{11}^0, J_{22}^0, J_{33}^0 > J_{12}^0, J_{21}^0, J_{12}^0, J_{21}^0, J_{33}^0 > J_{11}^0, J_{33}^0, \) and \( A_1 A_2 > A_3 \).

**Theorem 3** Equilibrium point \( E_4 = (P^{**}, H_1^{**}, H_2^{**}) \) local stable if \( J_{11}^0 + J_{22}^0 + J_{33}^0 < 0, J_{11}^0, J_{22}^0, J_{33}^0 > J_{12}^0, J_{21}^0, J_{12}^0, J_{21}^0, J_{33}^0 > J_{11}^0, J_{33}^0, \) and \( A_1 A_2 > A_3 \).

**Proof.** The result of substitution equilibrium point \( E_3 \) to Jacobian Matrix (5)
\[ J(E_4) = \begin{bmatrix} J_{11}^4 & J_{12}^4 & J_{13}^4 \\ J_{21}^4 & J_{22}^4 & J_{23}^4 \\ J_{31}^4 & J_{32}^4 & J_{33}^4 \end{bmatrix} \]  
\[ (11) \]

where

\[ J_{11}^4 = r - \frac{2\rho_{\text{m}}}{K} - \frac{a_1 H_{\text{m}}}{p - \rho_{\text{m}} + 1} - \frac{2 a_2 H_{\text{m}}}{(p - \rho_{\text{m}} + 1)^2} - \alpha_2 H_{\text{m}}, \]
\[ J_{12}^4 = -\frac{a_1}{p - \rho_{\text{m}}} J_{13}^4 = -\alpha_2 P_{\text{m}}, \]
\[ J_{21}^4 = \frac{e_1 a_1 H_{\text{m}}}{p - \rho_{\text{m}}} - \frac{e_1 a_1 H_{\text{m}}}{(p - \rho_{\text{m}} + 1)^2}, \]
\[ J_{22}^4 = \frac{e_2 a_2 P_{\text{m}}}{p - \rho_{\text{m}}}, \]
\[ J_{23}^4 = \beta_1 H_{\text{m}}, \]
\[ J_{32}^4 = -\beta_2 H_{\text{m}}. \]

Characteristics equation matrix \( J(E_4) \) is

\[ \lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0 \]  
\[ (12) \]

where,

\[ B_1 = -(J_{11}^4 + J_{22}^4 + J_{33}^4) \]
\[ B_2 = J_{11}^4 J_{22}^4 + J_{11}^4 J_{33}^4 + J_{22}^4 J_{33}^4 - J_{12}^4 J_{21}^4 - J_{13}^4 J_{31}^4 - J_{23}^4 J_{32}^4 \]
\[ B_3 = J_{11}^4 J_{22}^4 J_{33}^4 + J_{11}^4 J_{32}^4 J_{33}^4 + J_{13}^4 J_{31}^4 J_{32}^4 - J_{12}^4 J_{31}^4 J_{32}^4 - J_{13}^4 J_{23}^4 J_{31}^4 - J_{23}^4 J_{31}^4 J_{32}^4 - J_{13}^4 J_{23}^4 J_{32}^4 - J_{13}^4 J_{32}^4 J_{33}^4 - J_{23}^4 J_{31}^4 J_{33}^4. \]

To ensure the stability of model system with equilibrium point \( E_4 \), the point must qualify of the Routh-Hurwitz criteria, \( J_{11}^4 + J_{22}^4 + J_{33}^4 < 0, J_{11}^4 J_{22}^4 + J_{11}^4 J_{33}^4 + J_{22}^4 J_{33}^4 > J_{12}^4 J_{21}^4 + J_{13}^4 J_{31}^4 + J_{23}^4 J_{32}^4, \)
\[ J_{11}^4 J_{22}^4 J_{33}^4 + J_{11}^4 J_{32}^4 J_{33}^4 + J_{13}^4 J_{31}^4 J_{32}^4 > J_{12}^4 J_{31}^4 J_{32}^4 + J_{13}^4 J_{23}^4 J_{31}^4 + J_{23}^4 J_{31}^4 J_{32}^4, \]
and \( B_1 B_2 > B_3 \).

4. Numerical Simulation

This simulation aims to show the stability of the system visually. The parameter values was adopted from literature (3,7,8,10,11,13). We try simulated the model with some condition. The first condition with parameter with \( g_1 = 0.1 \) dan \( g_2 = 0.2 \). The second condition with parameter \( g_1 = 0.1 \) dan \( g_2 = 0.1 \). The third condition with \( g_1 = 0.2 \) dan \( g_2 = 0.15 \).

To see the system is in a stable state, a numerical simulation is performed with parameter estimates according to the following table 1.

| Table 1. Parameter values |
|---------------------------|
| **Parameter** | **Values** |
| \( r \) | 1 |
| K | 100 |
| \( a_1 \) | 0.21 |
| \( a_2 \) | 0.274 |
| \( c_1 \) | 1 |
| \( c_2 \) | 1 |
| \( g_1 \) | 0.1 |
| \( g_2 \) | 0.2 |
| \( \beta_1 \) | 0.05 |
| \( \beta_2 \) | 0.06 |
| \( \theta \) | 0.01 |
| \( d_1 \) | 0.05 |
| \( d_2 \) | 0.06 |
With the parameter values in Table 1, the simulation results are given nonnegative equilibrium points

\[ E_0 = (0, 0, 0), \quad E_1 = (100, 0, 0), \quad E_2 = (2.808137423, 0, 3.547148269), \]

\[ E_3 = (2.568492476, 4.75876326, 0), \quad E_4 = (1.934788444, 2.718360251, 1.535152093). \]

![Figure 1](image1.png)

**Figure 1.** Numerical simulation model. (a) prey population density (b) first predator population density (c) second predator population density

![Figure 2](image2.png)

**Figure 2.** Time series of the model with \( g_1 = 0.1, \ g_2 = 0.2 \) and initial value

\[ P(0) = 5, \ H_1(0) = 5, \ H_2(0) = 5. \]
The first simulation shown in Figure 1 shows that the dynamics between populations occur due to several factors including intraspecific competition. Intraspecific competition coefficients can interfere with the growth and activity of predators. There is a difference in the plot results in Figure 2, a situation when the intraspecific coefficient of the second predator is greater than that of the first predator. Figure 2 and Figure 4 look very different. Figure 4 with a larger intraspecific predator coefficient shows an image that is inversely proportional to Figure 2. Figure 3 shows the same dynamics as Figure 4 with the intraspecific coefficient of both predators being equal.

5. Conclusion
In this section, we will make conclusions of this research. This study focuses on the dynamics of prey-predator populations and the intraspecific competition of the predators. The intraspecific competition coefficient has an impact on the system. The system will be stable and exist if the intraspecific competition of the first predator and the second predator is balanced.
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