A New Supersymmetric Extension of Conformal Mechanics

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Abstract
In this paper a new supersymmetric extension of conformal mechanics is put forward. The beauty of this extension is that all variables have a clear geometrical meaning and the super-Hamiltonian turns out to be the Lie-derivative of the Hamiltonian flow of standard conformal mechanics. In this paper we also provide a supersymmetric extension of the other conformal generators of the theory and find their “square-roots”. The whole superalgebra of these charges is then analyzed in details. We conclude the paper by showing that, using superfields, a constraint can be built which provides the exact solution of the system.

1 Introduction
In 1976 a conformally-invariant quantum mechanical model was proposed and solved in ref. [1]. New interest in the model has been recently generated by the discovery [2] that the dynamics of a particle near the horizon of an extreme Reissner-Nordström black-hole is governed in its radial motion by the Lagrangian of ref. [1]. A supersymmetric extension of conformal mechanics was proposed later by two independent groups [3] and also the supersymmetric version seems to hold some interest for black-hole physics.

In this paper we shall put forward a new supersymmetric extension of conformal mechanics. It is based on a path-integral approach to classical mechanics developed in ref. [4]. The difference between our extension and the one of ref. [3], which was tailored on the supersymmetric quantum mechanics of Witten [5], is that the authors of ref. [3] took the original conformal Hamiltonian and added a Grassmannian part in order to make the whole Hamiltonian supersymmetric. Our procedure and extension is different and more geometrical as will be explained later on in the paper.

The paper is organized in the following manner: In section 2 we give a very brief outline of conformal mechanics [1] and of the supersymmetric extension present in the literature [3]; in section 3 we put forward our supersymmetric extension and explain its geometrical structure. In the same section we build a whole set of charges connected with our extension and study their algebra in
detail. In section 4 we show that, differently from the superconformal algebra of \[3\] where the even part had a spinorial representation on the odd part, ours is a non-simple superalgebra whose even part has a reducible and integer representation on the odd part. In section 5 we give a superspace version for our model and, like for the old conformal mechanics, we provide an exact solution of the model. This is given by solving a constraint in superfield space. Details of the calculations can be found in a longer version of this paper \[3\].

2 The Old Conformal and Supersymmetric Extended Mechanics.

The Lagrangian for conformal mechanics proposed in \[1\] is

\[ L = \frac{1}{2} \left[ \ddot{q}^2 - g \frac{q'}{q^2} \right]. \] (1)

This Lagrangian is invariant under the following transformations:

\[ t' = \frac{\alpha t + \beta}{\gamma t + \delta}; \quad q'(t') = \frac{q(t)}{(\gamma t + \delta)}; \quad \text{with} \quad \alpha \delta - \beta \gamma = 1, \] (2)

which are nothing else than the conformal transformations in 0+1 dimensions. They are made of the combinations of the following three transformations:

\[ t' = \alpha^2 t \quad \text{dilations,} \] (3)
\[ t' = t + \beta \quad \text{time-translations,} \] (4)
\[ t' = \frac{t}{\gamma t + 1} \quad \text{special-conformal transformations.} \] (5)

The associated Noether charges \[1\] are:

\[ H = \frac{1}{2} \left( p^2 + \frac{g}{q^2} \right); \quad D = tH - \frac{1}{4} (qp + pq); \quad K = t^2 H - \frac{1}{2} t (qp + pq) + \frac{1}{2} q^2. \] (6)

The quantum algebra of the three Noether charges above is:

\[ [H, D] = iH; \quad [K, D] = -iK; \quad [H, K] = 2iD. \] (7)

The \( H, D, K \) above are explicitly dependent on \( t \), but they are conserved, i.e.:

\[ \frac{\partial D}{\partial t} \neq 0; \quad \frac{\partial K}{\partial t} \neq 0; \quad \frac{dD}{dt} = \frac{dK}{dt} = 0. \] (8)

As the \( H, D, K \) are conserved, their expressions at \( t = 0 \) which are

\[ H_0 = \frac{1}{2} \left[ p^2 + \frac{g}{q^2} \right]; \quad D_0 = -\frac{1}{4} [qp + pq]; \quad K_0 = \frac{1}{2} q^2 \] (9)

satisfy the same algebra as those at time \( t \).

The supersymmetric extension of this model, proposed in ref. \[3\], has the following Hamiltonian:
\[
H_{\text{SUSY}} = \frac{1}{2} \left( p^2 + \frac{g}{q^2} + \sqrt{g} \psi \psi' \right),
\]  
(10)

where \( \psi, \psi' \) are Grassmannian variables whose anticommutator is \([\psi, \psi']_{+} = 1\). In \( H_{\text{SUSY}} \) there is a first bosonic piece which is the conformal Hamiltonian of eq.(11), plus a Grassmannian part. Note that the equations of motion for \( q \) have an extra piece with respect to the equations of motion of the old conformal mechanics [1].

\( H_{\text{SUSY}} \) can be written as a particular form of supersymmetric quantum mechanics [3]:

\[
H_{\text{SUSY}} = \frac{1}{2} [Q, Q'^{\dagger}] + \frac{1}{2} \left( p^2 + \left( \frac{dW}{dq} \right)^2 - [\psi^{\dagger}, \psi] \frac{d^2W}{dq^2} \right),
\]  
(11)

where the supersymmetry charges are given by:

\[
Q = \psi^{\dagger} \left( -ip + \frac{dW}{dq} \right); \quad Q'^{\dagger} = \psi \left( ip + \frac{dW}{dq} \right);
\]  
(12)

and where \( W \) is the superpotential which, in this case of conformal mechanics, turns out to be:

\[
W(q) = \sqrt{g} \log q.
\]  
(13)

If we perform a supersymmetric transformation combined with a conformal one generated by the \((H, K, D)\), we get what is called a superconformal transformation. In order to understand this better let us list the following eight operators:

\[
H = \frac{1}{2} \left[ p^2 + \frac{g + 2\sqrt{g}B}{q^2} \right]; \quad D = -\frac{[q, p]}{4}; \quad K = \frac{q^2}{2};
\]  
(14)

\[
B = \frac{[\psi^{\dagger}, \psi]}{2}; \quad Q = \psi^{\dagger} \left( -ip + \frac{\sqrt{g}}{q} \right); \quad Q'^{\dagger} = \psi \left( ip + \frac{\sqrt{g}}{q} \right);
\]  
(15)

\[
S = \psi^{\dagger} q; \quad S'^{\dagger} = \psi q.
\]  
(16)

The algebra of these operators is closed and given in the table below:

\[
\begin{array}{llll}
[H, D] = iH; & [K, D] = -iK; & [H, K] = 2iD; \\
[Q, H] = 0; & [Q'^{\dagger}, H] = 0; & [Q, D] = \frac{i}{2}Q; \\
[Q'^{\dagger}, K] = S'^{\dagger}; & [Q, K] = -S; & [Q'^{\dagger}, D] = \frac{1}{2}Q'^{\dagger}; \\
[S, K] = 0; & [S'^{\dagger}, K] = 0; & [S, D] = -\frac{1}{2}S; \\
[S'^{\dagger}, D] = -\frac{i}{2}S'^{\dagger}; & [S, H] = -Q; & [S'^{\dagger}, H] = Q'^{\dagger}; \\
[Q, Q'^{\dagger}] = 2H; & [S, S'^{\dagger}] = 2K; & [B, Q] = Q; \\
[B, S] = S; & [B, S'^{\dagger}] = -S'^{\dagger}; & [B, Q'^{\dagger}] = -Q'^{\dagger}; \\
[Q, S'^{\dagger}] = \sqrt{g} - B + 2iD; & & \\
\end{array}
\]

The square-brackets \([., .] \) in the algebra above are graded-commutators. We notice that the commutators of the supersymmetry generators \((Q, Q'^{\dagger})\) with the three conformal generators \((H, K, D)\) generate a new operator which is \( S \). Including this new one we generate an algebra which is closed provided that we introduce the operator \( B \) of eq.(15) which is the last operator we need.
3 A New Supersymmetric Extension.

In this section we are going to present a new supersymmetric extension of conformal mechanics. This extension is based on a path-integral approach to classical mechanics (CM) developed in ref. [4]. Let us start with a system living on a 2n-dimensional phase space $\mathcal{M}$—whose coordinates we indicate as $\phi^a$ with $a = 1, \ldots, 2n$, i.e.: $\phi^a = (q^1, \ldots, q^n; p^1, \ldots, p^n)$—and having an Hamiltonian $H(\phi)$. The equations of motion are then:

$$\dot{\phi}^a = \omega^{ab}\partial_b H$$

where $\omega^{ab}$ is the usual symplectic matrix. By path integral for CM we mean a functional integral that forces all paths in $\mathcal{M}$ to sit on the classical ones. The classical analog of the quantum generating functional is then:

$$Z_{CM}[J] = N \int D\phi \, \delta[\phi(t) - \phi_{cl}(t)] exp \left[ \int J\phi \, dt \right] ,$$

where $\phi$ are the $\phi^a \in \mathcal{M}$, $\phi_{cl}$ are the solutions of the equations of motion. $J$ is an external current and $\delta[.]$ is a functional Dirac delta which forces every path $\phi(t)$ to sit on a classical ones $\phi_{cl}(t)$. Let us first rewrite the functional Dirac delta in the $Z_{CM}$ above as:

$$\delta[\phi - \phi_{cl}] = \delta[\phi^a - \omega^{ab}\partial_b H] \det[\delta^a_b \partial_t - \omega^{ac}\partial_c \partial_b H].$$

The next step is to write the $\delta[.]$ as a Fourier transform over some new variables $\lambda_a$ and to exponentiate the determinant via Grassmannian variables $\bar{c}_a, c^a$. The final result is

$$Z_{CM}[0] = \int D\phi^a D\lambda_a Dc^a D\bar{c}_a \exp \left[ i \int dt \bar{L} \right] \quad \text{with} \quad \bar{L} = \lambda_a [\dot{\phi}^a - \omega^{ab}\partial_b H] + i\bar{c}_a [\delta^a_b \partial_t - \omega^{ac}\partial_c \partial_b H] c^b .$$

One can derive the equations of motion from this Lagrangian which, for $\phi^a$, are the standard Newton equations while for $c^a$ are the equations of the first variations. This last thing allow us to identify the $c^a$ with the basis of the forms $d\phi^a$.

The equations of motions can be derived [4] also from the Hamiltonian associated to the Lagrangian above which is

$$\bar{H} = \lambda_a \omega^{ab}\partial_b H + i\bar{c}_a \omega^{ac}(\partial_c \partial_b H)c^b . \quad (17)$$

As the $Z_{CM}$ is a path integral we can also define the concept of commutator as Feynman did in the quantum case. The result [4] is: $\langle [\phi^a, \lambda_b] \rangle = i\delta^a_b$; $\langle [\bar{c}_b, c^a] \rangle = \delta^a_b$.

Using this operatorial formulation and the fact that the $c^a$ can be identified with forms, it is easy to prove that $\bar{H}$ is nothing else than the Lie-derivative of the Hamiltonian flow [4] generated by $H$. The reader may remember that the concept of Lie-derivative was mentioned also in the second of refs. [5]. There anyhow the connection between Lie-derivative and Hamiltonian was not as direct as here. Moreover the Lie-derivative was not linked to the flow generated by the conformal potential (1) but with the flow generated by the superpotential (13).

The Hamiltonian $\bar{H}$ has various universal symmetries [4] all of which have been studied geometrically [4] [8]. The associated charges are:

$$Q_{BRS} = ic^a \lambda_a ; \quad \bar{Q}_{BRS} = ic^a \omega^{ab} \lambda_b ;$$

$$Q_g = c^a \bar{c}_a ; \quad C = \frac{\omega^{ab} c^a c^b}{2} ; \quad \bar{C} = \frac{\omega^{ab} \bar{c}_a \bar{c}_b}{2}$$

$$N_H = c^a \partial_a H ; \quad \bar{N}_H = \bar{c}_a \omega^{ab} \partial_b H .$$

4
Using the correspondence between Grassmannian variables and forms, the $Q_{BRS}$ turns out to be nothing else than the exterior derivative on phase-space. The $Q_g$, or ghost charge, is the form-number which is always conserved by the Lie-derivative. Similar geometrical meanings can be found for the other charges that are listed above. Of course linear combinations of them are also conserved and there are two combinations which deserve our attention. They are the following charges:

$$Q_H \equiv Q_{BRS} - \beta N_H; \quad \overline{Q}_H \equiv \overline{Q}_{BRS} + \beta \overline{N}_H;$$  \tag{19}

(\text{where } \beta \text{ is an arbitrary dimensionful parameter}) which are true supersymmetry charges because, besides commuting with $\tilde{H}$, they give: $[Q_H, \overline{Q}_H] = 2i\beta \tilde{H}$. This proves that our $\tilde{H}$ is supersymmetric. To be precise it is an $N = 2$ supersymmetry. One realizes immediately that $H$ acts as a sort of superpotential for the supersymmetric Hamiltonian $\tilde{H}$. All this basically means that we can obtain a supersymmetric Hamiltonian $\tilde{H}$ out of any system with Hamiltonian $H$ and, moreover, our $\tilde{H}$ has a nice geometrical meaning being the Lie-derivative of the Hamiltonian flow generated by $H$.

We will now build the $\tilde{H}$ of the conformal invariant system given by the Hamiltonian of eq.(6), that means we insert the $H$ of eq.(6) into the $\tilde{H}$ of eq.(17). The result is:

$$\tilde{H} = \lambda q_p + \lambda_p \frac{q}{q^T} + i\bar{c}_p c^\theta - 3i\bar{c}_p c^\theta \frac{q}{q^T},$$  \tag{20}

where the indices $(,)^q$ and $(,)^p$ on the variables $(\lambda, c, \bar{c})$ replace the indices $(,)^a$ which appeared in the general formalism because here we have only one degree of freedom. The two supersymmetric charges of eq.(19) are in this case

$$Q_H = Q_{BRS} + \beta (q c^\theta - p c^\theta); \quad \overline{Q}_H = \overline{Q}_{BRS} + \beta \left(\frac{q}{q^T} \bar{c}_p + p \bar{c}_q\right).$$  \tag{21}

It was one of the central points of the original paper on conformal mechanics that the Hamiltonians of the system could be, beside $H_0$ of eq.(6), also $D_0$ or $K_0$ or any linear combination of them. In the same manner as we built the Lie-derivative $\tilde{H}$ associated to $H_0$, we can also build the Lie-derivatives associated to the flow generated by $D_0$ and $K_0$. We just have to insert $D_0$ or $K_0$ in place of $H$ as superpotential in the $\tilde{H}$ of eq.(17). Calling the associated Lie-derivatives as $\tilde{D}_0$ and $\tilde{K}_0$, what we get is:

$$\tilde{D}_0 = \frac{1}{2}[\lambda_p p - \lambda_q q + i(\bar{c}_p c^\theta - \bar{c}_q c^\theta)]; \quad \tilde{K}_0 = -\lambda_p q - i\bar{c}_p c^\theta.$$  \tag{22}

Both $\tilde{D}_0$ and $\tilde{K}_0$ are “supersymmetric” in the sense that there are the “square” of some charges: $[Q_D, \overline{Q}_D] = 4i\gamma \tilde{D}_0; \quad [Q_K, \overline{Q}_K] = 2i\alpha \tilde{K}_0$, given by

$$Q_D = Q_{BRS} + \gamma(q c^\theta + p c^\theta); \quad \overline{Q}_D = \overline{Q}_{BRS} + \gamma(p \bar{c}_p - q \bar{c}_q);$$  \tag{23}

$$Q_K = Q_{BRS} - \alpha q c^\theta; \quad \overline{Q}_K = \overline{Q}_{BRS} - \alpha q \bar{c}_p.$$  \tag{24}

The $\alpha$ and $\gamma$ are arbitrary constant variables like $\beta$ was. Let us list all the operators we have found so far:
\[\tilde{H} = \lambda_q p + \lambda_p g \frac{q}{q^3} + i \bar{c}_q c^p - 3i \bar{c}_p c^q \frac{g}{q^4}; \quad H = \frac{1}{2} \left( p^2 + \frac{q}{g^2} \right); \]
\[\tilde{K}_0 = -\lambda_p q - i \bar{c}_p c^q; \quad K_0 = \frac{1}{2} q^2; \]
\[\tilde{D}_0 = \frac{1}{2} \left( \lambda_p p - \lambda_q q + i (\bar{c}_p c^p - \bar{c}_q c^q) \right); \quad D_0 = -\frac{1}{2} g p; \]
\[Q_{BRS} = i(\lambda_q c^p + \lambda_p c^q); \quad Q_{BRS} = i(\lambda_p \bar{c}_q - \lambda_q \bar{c}_p) ; \]
\[Q_H = Q_{BRS} + \beta \left( \frac{q}{g} c^q - p c^p \right); \quad Q_H = \bar{Q}_{BRS} + \beta \left( \frac{g}{q} \bar{c}_p + p \bar{c}_q \right); \]
\[Q_K = Q_{BRS} - \alpha q c^p; \quad Q_K = \bar{Q}_{BRS} - \alpha q \bar{c}_p; \]
\[Q_D = Q_{BRS} + \gamma (q c^q + p c^p); \quad Q_D = \bar{Q}_{BRS} + \gamma (p c^p + q \bar{c}_q).\]

We find that they are the minimum number in order to make a closed algebra which is written in the TABLE below:

**TABLE 2**

| Table Entry | Expression |
|-------------|------------|
| $[\tilde{H}, \tilde{D}_0]$ | $i\tilde{H}$ |
| $[Q_H, \tilde{H}]$ | $0$ |
| $[Q_H, \tilde{D}_0]$ | $i(Q_H - Q_{BRS})$ |
| $[Q_H, \tilde{K}_0]$ | $i\beta \gamma^{-1}(Q_D - Q_{BRS})$ |
| $[Q_{BRS}, \tilde{H}]$ | $[\bar{Q}_{BRS}, \tilde{H}] = 0$ |
| $[Q_D, \tilde{H}]$ | $-2i\gamma \beta^{-1}(Q_H - Q_{BRS})$ |
| $[Q_D, \tilde{K}_0]$ | $2i\gamma \alpha^{-1}(Q_K - Q_{BRS})$ |
| $[Q_D, \tilde{D}_0]$ | $0$ |
| $[Q_K, \tilde{H}]$ | $-i\alpha \gamma^{-1}(Q_D - Q_{BRS})$ |
| $[Q_K, \tilde{D}_0]$ | $-i(Q_K - Q_{BRS})$ |
| $[Q_K, \tilde{K}_0]$ | $0$ |
| $[Q_H, \tilde{Q}_0]$ | $i\beta \tilde{H} + 2i\gamma \tilde{D}_0 - 2\beta \gamma H$ |
| $[Q_K, \tilde{Q}_0]$ | $i\alpha \tilde{K}_0 + 2i\gamma \tilde{D}_0 + 2\alpha \gamma K$ |
| $[Q_H, \tilde{Q}_{BRS}] = [\tilde{Q}_0, \tilde{Q}_{BRS}] = i\beta \tilde{H}$ |
| $[Q_D, \tilde{Q}_{BRS}] = [\tilde{Q}_0, \tilde{Q}_{BRS}] = 2i\gamma \tilde{D}_0$ |
| $[\tilde{Q}_{(\ldots)}, H] = \beta^{-1}(Q_{BRS} - Q_{H})$ |
| $[\tilde{Q}_{(\ldots)}, D_0] = (2\gamma)^{-1}(Q_{BRS} - Q_{D})$ |
| $[\tilde{Q}_{(\ldots)}, K_0] = \alpha^{-1}(Q_{BRS} - Q_{K})$ |
| $[\tilde{H}, \tilde{K}_0]$ | $[H, \tilde{K}_0] = 2iD$ |
| $[\tilde{H}, \tilde{D}_0]$ | $-i\tilde{K}_0$ |
| $[\tilde{H}, \tilde{K}_0]$ | $= 2i\tilde{D}_0$ |
| $[\tilde{Q}_0, \tilde{Q}_{BRS}] = [\tilde{Q}_0, \tilde{Q}_{BRS}] = 0$ |
| $[\tilde{Q}_0, \tilde{Q}_{(\ldots)}] = 2i\alpha \tilde{K}_0$ |

All other commutators\(\square\) are zero.

We notice that for our supersymmetric extension we need 14 charges (see TABLE 2) in order for the algebra to close, while in the extension of ref. \[\square\] one needs only 8 charges (see TABLE 1).

\(^{1}\)The $Q_{(\ldots)}$ appearing in the table can be any of the following operators: $Q_{BRS}, Q_H, Q_D, Q_K$ and the same holds for $\bar{Q}_{(\ldots)}$. Obviously all commutators are between quantities calculated at the same time.
4 Superconformal Algebras Associated to the Two Extensions.

A Lie superalgebra \( \mathfrak{g} \) is an algebra made of even \( E \) and odd \( O \) generators whose graded commutators look like:

\[
[E_m, E_n] = F_{mn}^p E_p; \quad [E_m, O_\alpha] = G_{m\alpha}^\beta O_\beta; \quad [O_\alpha, O_\beta] = C_{\alpha\beta}^m E_m;
\]

and where the structure constants \( F_{mn}^p, G_{m\alpha}^\beta, C_{\alpha\beta}^m \) satisfy generalized Jacobi identities.

The second relation of eq.(25) is usually interpreted by saying that the even part of the algebra has a representation on the odd part. This is clear if we consider the odd part as a vector space and that the even part acts on this vector space via the graded commutators.

For superconformal algebras the usual folklore says that the even part of the algebra has his conformal subalgebra represented spinorially on the odd part. This is true only in a relativistic setting and it is not always the case in a non-relativistic one. We will now analyze both the case of ref. \[3\] and ours.

Let us start from the superalgebra of ref. \[3\] which is given in eqs.(14)–(16). The even part of this superalgebra \( \mathcal{G}_0 \) can be organized in an \( SO(2,1) \) form as follows:

\[
\mathcal{G}_0 : \begin{cases}
B_1 = \frac{1}{2} \left[ \frac{K}{a} - aH \right] \\
B_2 = \tilde{D} \\
J_3 = \frac{1}{2} \left[ \frac{K}{a} + aH \right]
\end{cases}
\]

where \( a \) is the same parameter introduced in \[3\] with dimension of time.

On the other side the odd part \( \mathcal{G}_1 \) is:

\[
\mathcal{G}_1 : \begin{cases}
Q \\
\tilde{Q} \\
S \\
\tilde{S}
\end{cases}
\]

It is easy to work out, using the results of TABLE 1, the action of the \( \mathcal{G}_0 \) on \( \mathcal{G}_1 \). The result is summarized in table 6 of ref. \[6\].

Considering the odd part as a vector space, let us build the following 4 “vectors”:

\[
|q\rangle \equiv Q + Q^\dagger; \quad |p\rangle \equiv S - S^\dagger; \quad |r\rangle \equiv Q - Q^\dagger; \quad |s\rangle \equiv S + S^\dagger.
\]

Next let us take the Casimir operator of the algebra \( \mathcal{G}_0 \) which is \( \mathcal{C} = B_1^2 + B_2^2 - J_3^2 \) and apply it to the state \( |q\rangle \):

\[
\mathcal{C}|q\rangle = [B_1, [B_1, Q + Q^\dagger]] + [B_2, [B_2, Q + Q^\dagger]] - [J_3, [J_3, Q + Q^\dagger]] = -\frac{3}{4} |q\rangle.
\]

The same happens for the state \( |p\rangle \). The factor \( -\frac{3}{4} = -\frac{1}{2}(\frac{3}{2} + 1) \) above indicates that the \((|q\rangle, |p\rangle)\) space carries a spinorial representation. It is possible to prove the same for the other two vectors.

Let us now turn the same crank for our supersymmetric extension of conformal mechanics. Looking at the TABLE 2 of our operators, we can organize the even part \( \mathcal{G}_0 \), as follows:
The LHS is the usual $SO(2, 1)$ while the RHS is formed by three translations because they commute among themselves. So the overall algebra is the Euclidean group $E(2, 1)$.

The odd part of our superalgebra is made of 8 operators (see TABLE 2). As we did before for the model of [3], we will now evaluate for our model the action of $G_0$ on the odd part. The result is summarized in table 9 of ref. [6]. It is easy [6] to realize from that table that the following three vectors (where for simplicity we have made the choice $a = \sqrt{\beta/\alpha}$ and $\eta \equiv \gamma/\sqrt{\alpha\beta}$):

\[
\begin{align*}
B_1 &= \frac{1}{2} \left( \frac{\vec{K}}{a} - a\vec{H} \right); & P_1 &= 2D; \\
B_2 &= \vec{D}; & P_2 &= aH - \frac{\vec{K}}{a}; \\
J_3 &= \frac{1}{2} \left( \frac{\vec{K}}{a} + a\vec{H} \right); & P_0 &= aH + \frac{\vec{K}}{a}.
\end{align*}
\]

make an irreducible representation of the conformal subalgebra. In fact one easily [6] obtains:

\[
\begin{align*}
B_1|q_H\rangle &= -i\frac{1}{2}|q_D\rangle; & B_2|q_H\rangle &= -i|q_H\rangle; & J_3|q_H\rangle &= -i\frac{1}{2}|q_D\rangle \\
B_1|q_K\rangle &= -i\frac{1}{2}|q_D\rangle; & B_2|q_K\rangle &= i|q_K\rangle; & J_3|q_K\rangle &= i\frac{1}{2}|q_D\rangle \\
B_1|q_D\rangle &= -i(|q_H\rangle + |q_K\rangle); & B_2|q_D\rangle &= 0; & J_3|q_D\rangle &= i(|q_H\rangle - |q_K\rangle).
\end{align*}
\]

The Casimir operator is given, as before, by: $C = B_1^2 + B_2^2 - J_3^2$ but we must remember to use, for $B_1$, $B_2$ and $J_3$, the operators contained in TABLE 4. It is then easy to check that: $C|q_H\rangle = -2|q_H\rangle$.

The same we get for the other two vectors $|q_K\rangle$, $|q_D\rangle$, so the eigenvalue in the equation above is $C = -2 = -1(1 + 1)$ and this indicates that those vectors make a spin-1 representation. It is also easy to prove that there is another spin-1 representation and two spin-0. The details can be found in ref. [6].

We wanted to present this analysis in order to underline a further difference between our supersymmetric extension and the one of [3] whose odd part $G_1$, as we showed before, carries two spin one-half representations.

5 Exact Solution of the Model and Its Superspace Formulation.

We will now present a superspace formulation of our model like the authors of ref. [3] did for theirs. We have to enlarge the “base space” $(t)$ to a superspace $(t, \theta, \bar{\theta})$ where $(\theta, \bar{\theta})$ are Grassmannian
partners of \((t)\). It is then easy to put all the variables \((\phi^a, c^a, \lambda_a, \bar{c}_a)\) in a single superfield \(\Phi\) defined as follows:

\[
\Phi^a(t, \theta, \bar{\theta}) = \phi^a(t) + \theta c^a(t) + \bar{\theta} \omega^{ab} \bar{c}_b(t) + i \bar{\theta} \theta \omega^{ab} \lambda_b(t).
\] (31)

This superfield had already been introduced in ref. [4]. It is a scalar field under the supersymmetry transformations of the system. It is a simple exercise to find the expansion of any function \(F(\Phi^a)\) of the superfields in terms of \(\theta, \bar{\theta}\). For example, choosing as function the Hamiltonian \(H\) of a system, we get:

\[
H(\Phi^a) = H(\phi) + \theta N_H - \bar{\theta} \bar{N}_H + i \bar{\theta} \theta \tilde{H}.
\] (32)

From eq. (32) it is easy to prove that:

\[
i \int H(\Phi) d\theta d\bar{\theta} = \tilde{H}.
\] (33)

Here we immediately notice a crucial difference with the supersymmetric QM model of ref. [3]. In the language of superfields (see the second of ref. [3]) those authors obtained the supersymmetric potential of their Hamiltonian by inserting the superfield into the superpotential (which is given by eq. (13)) and integrating in something like \(\theta, \bar{\theta}\), while we get the potential part of our supersymmetric Hamiltonian by inserting the superfield into the normal potential of the conformal mechanical model given in \([4]\).

The space \((\phi^a, c^a, \lambda_a, \bar{c}_a)\) somehow can be considered as a target space whose base space is the superspace \((t, \theta, \bar{\theta})\). The action of the various charges listed in our TABLE 2 is on the target-space variables but we can consider it as induced by some transformations on the base-space. If we collectively indicate the charges acting on \((\phi^a, c^a, \lambda_a, \bar{c}_a)\) as \(\Omega\), we shall indicate the generators of the corresponding transformations on the base space as \(\hat{\Omega}\). The relation between the two is the following:

\[
\delta \Phi^a = -\varepsilon \hat{\Omega} \Phi^a \quad \text{where} \quad \delta \Phi^a = [\varepsilon \Omega, \Phi^a],
\] (34)

with \(\varepsilon\) the commuting or anticommuting infinitesimal parameter of our transformations and \([\cdot, \cdot]\) the graded commutators of our formalism.

Using the relations above it is easy to work out the superspace representation of the operators of eq. (18). They are:

\[
\hat{Q}_{BRS}^\alpha = -\partial_\alpha; \quad \hat{Q}_{BRS} = \partial_\alpha; \quad \hat{Q}_B = \bar{\theta} \partial_\bar{\alpha} - \theta \partial_\alpha; \quad \hat{Q}_H = \bar{\theta} \partial_\bar{\alpha} + \theta \partial_\alpha.
\] (35)

Via the charges above it is easy to write down also the supersymmetric charges of eq. (19):

\[
\hat{Q}_H = -\partial_\alpha - \beta \bar{\theta} \partial_t; \quad \hat{Q}_H = \partial_\alpha + \beta \theta \partial_t.
\]

Proceeding in the same way, via the relation (34), it is a long but easy procedure to give a superspace representation to the charges \(Q_D, \bar{Q}_D, Q_K, \bar{Q}_K\) of eqs. (28)–(31). The result \([6]\) is:

\[
\hat{H} = i \frac{\partial}{\partial t}; \quad \hat{D} = it \frac{\partial}{\partial t} - \frac{i}{2} \sigma_3; \quad \hat{K} = it^2 \frac{\partial}{\partial t} - i t \sigma_3 - i \sigma_-;
\]

\[
\hat{Q}_D^\dagger = \frac{\partial}{\partial \theta} - 2\gamma \bar{\theta} t \frac{\partial}{\partial t} + \gamma \theta \sigma_3; \quad \hat{Q}_D = \frac{\partial}{\partial \bar{\theta}} + 2\gamma \theta t \frac{\partial}{\partial t} - \gamma \theta \sigma_3;
\] (36)
\[ \hat{Q}_K = -\frac{\partial}{\partial \theta} - \alpha \bar{\theta} t^2 \frac{\partial}{\partial t} + \alpha \bar{\theta} \sigma_3 + \alpha \bar{\theta} \sigma_-; \quad \hat{Q}_K^t = \frac{\partial}{\partial \theta} + \alpha \bar{\theta} t^2 \frac{\partial}{\partial t} - \alpha t \sigma_3 - \alpha \bar{\theta} \sigma_-; \]

\[ \hat{H} = \bar{\theta} \theta \frac{\partial}{\partial t}; \quad \hat{D} = \bar{\theta} \theta \left( t \frac{\partial}{\partial t} - \frac{1}{2} \sigma_3 \right); \quad \hat{K} = \bar{\theta} \theta \left( t^2 \frac{\partial}{\partial t} - t \sigma_3 - \sigma_- \right). \]

In the previous expressions the \( \sigma_3 \) and \( \sigma_- \) are the Pauli matrices while the index \( (\cdot)^t \) indicates an explicit dependence on \( t \) which appears for the following reasons. Let us go back to relation (6):

\[ H = H_0; \quad D = tH + D_0; \quad K = t^2 H + 2tD_0 + K_0; \]

from which we get:

\[ \hat{D} = t\hat{H} + \hat{D}_0; \quad \hat{K} = t^2 \hat{H} + 2t\hat{D}_0 + \hat{K}_0. \]  

(37)

The “square roots” of these operators will depend explicitly on the time \( t \) and they are those listed in eqs.(36). For more details about their derivation we invite the reader to consult reference [6].

Let us now turn to the solution of our model. The original conformal mechanical model was solved exactly in eq.(2.35) of reference [1]. The solution was given by the relation:

\[ q^2(t) = 2t^2 H - 4tD_0 + 2K_0. \]  

(38)

As \((H, D_0, K_0)\) are constants of motion, once their values are assigned we stick them in eq. (38), and we get a relation between “\( q \)” (on the LHS of (38) ) and “\( t \)” on the RHS. This is the solution of the equation of motion with “initial conditions” given by the values we assign to the constants of motion \((H, D_0, K_0)\). The reader may object that we should give only two constant values (corresponding to the initial conditions \((q(0), \dot{q}(0))\)) and not three. Actually the three values assigned to \((H, D_0, K_0)\) are not arbitrary because, as it was proven in eq.(2-36) of ref. [1], these three quantities are linked by a constraint: \((HK_0 - D_0^2) = \frac{g^4}{4}\) where “\( g \)” is the coupling which entered the original Hamiltonian (see eq.(1) of the present paper). Having one constraint among the three constants of motion brings them down to two.

What we want to do here is to see if a relation analogous to (38) exists also for our supersymmetric extension or in general if the supersymmetric system can be solved exactly. The answer is \textit{yes} and it is based on a very simple trick. Let us first remember how \( \hat{H} \) and \( H \) are related: \( i \int H(\Phi) \ d\theta d\bar{\theta} = \hat{H} \).

The same relation holds for \( \hat{D}_0 \) and \( \hat{K}_0 \) with respect to \( D_0 \) and \( K_0 \):

\[ i \int D_0(\Phi) \ d\theta d\bar{\theta} = \hat{D}_0; \quad i \int K_0(\Phi) \ d\theta d\bar{\theta} = \hat{K}_0. \]  

(39)

Of course the same kind of relations holds also for the explicitly time-dependent quantities: \( i \int D(\Phi) \ d\theta d\bar{\theta} = \hat{D}; \quad i \int K(\Phi) \ d\theta d\bar{\theta} = \hat{K} \).

Let us now build the following quantity:

\[ 2t^2 H(\Phi) - 4tD(\Phi) + 2K(\Phi); \]

(40)
this is functionally the RHS of eq.(38) with the superfield $\Phi^a$ replacing the normal phase-space variable $\phi^a$. It is then clear that the following relation holds:

$$(\Phi^q)^2 = 2t^2H(\Phi) - 4tD(\Phi) + 2K(\Phi).$$

The reason it holds is because, in the proof [1] of the analogous one in $q$-space, the only thing the authors used was the functional form of the $(H,D,K)$. So that relation holds irrespective of the arguments, $\phi$ or $\Phi$, which enter our functions provided that the functional form of them remains the same. From the form of the superfields it is then easy to expand in $\theta \bar{\theta}$ the RHS and LHS of the relation (40) and get four relations, one for each of the variables $(q, c^a, \bar{c}_p, \lambda_p)$, entering our formalism. These relations solves the model completely. For more details we refer the reader to ref. [1].

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