LOWER CENTRAL SERIES
OF BAUMSLAG–SOLITAR GROUPS

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We look at Baumslag–Solitar groups. Lower central series are given for residually nilpotent Baumslag–Solitar groups. For some Baumslag–Solitar groups that are not residually nilpotent, we find the intersection of all terms of the lower central series. Also we point out non-Abelian Baumslag–Solitar groups having lower central series of length 2. For some Baumslag–Solitar groups, a connection is found between an intersection of subgroups of finite index and an intersection of terms of the lower central series.

INTRODUCTION

The Baumslag–Solitar groups

$$BS(m, n) = \langle a, t \mid t^{-1}a^mt = a^n \rangle, \quad m, n \in \mathbb{Z} \setminus \{0\},$$

were introduced in [1] as examples of groups among which there are non-Hopfian ones (i.e., have a proper quotient isomorphic to the group itself), for instance, $BS(2, 3)$. Finitely generated non-Hopfian groups are not residually finite, so these groups also exemplify nonresidually finite one-relator groups. A review of residual properties of Baumslag–Solitar groups can be found in [2, 3].

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In the present paper, we look at lower central series of Baumslag–Solitar groups. In particular, we describe lower central series of residually nilpotent Baumslag–Solitar groups. For groups that are not residually nilpotent, it seems interesting to find a lower central series to understand the reason why residual nilpotency is missing. In fact, there are two causes: either the lower central series stabilizes at some finite step, or all of its terms are different but their intersection is nontrivial. In the latter case, it will be of interest to examine a transfinite lower series. We will point out unsolvable Baumslag–Solitar groups having lower central series of length 2.

Moldavanskii in [4] proved that the Baumslag–Solitar group $BS(m, n)$, $0 < m \leq |n|$, is residual nilpotent iff either $m = 1$ and $n \neq 2$ or $n = \varepsilon m$ and $m > 1$ is a power of a prime, with $\varepsilon = \pm 1$. For every group in this class, we describe a lower central series. We also prove that the Baumslag–Solitar group is residual torsion-free nilpotent iff $n = 1$, i.e., $BS(m, n)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Obviously, the quotient of an arbitrary group with respect to the intersection of all terms of the lower central series is a maximal residually nilpotent quotient. For some Baumslag–Solitar groups, we describe the intersection of all terms of the lower central series, and also find a connection between an intersection of terms of the lower central series and an intersection of subgroups of finite index.

1. PRELIMINARIES

The following definitions are due to Mal’tsev [5]. Let $\mathfrak{C}$ be a class of groups. We say that a group $G$ is residually $\mathfrak{C}$ (or simply $\mathfrak{C}$-residual) if for every nonidentity element $g \in G$ there is a homomorphism $\varphi$ of the group $G$ onto a group in $\mathfrak{C}$ such that $\varphi(g) \neq 1$. If $\mathfrak{C}$ is a class of finite groups, then $G$ is said to be residually finite (briefly, RF). If $\mathfrak{C}$ is a class of finite $p$-groups, then $G$ is said to be residually $p$-finite (briefly, RF$_p$). If $\mathfrak{C}$ is a class of nilpotent groups, then $G$ is called a residually nilpotent (briefly, RN) group. Denote by $\mathfrak{rF}$ the class of RF groups, by $\mathfrak{rF}_p$ the class of RF$_p$ groups, by $\{\mathfrak{rF}_p\}_p$ groups that are RF$_p$ for some prime $p$, and by $\mathfrak{rN}$ the class of RN groups. Obviously, for finitely generated groups, the following hold:

$$\mathfrak{rF}_p \subset \{\mathfrak{rF}_p\}_p \subset \mathfrak{rN} \subset \mathfrak{rF}.$$ 

It is not hard to see that all the inclusions are strict. Notice that for arbitrary groups, the inclusion $\mathfrak{rN} \subseteq \mathfrak{rF}$ does not hold. Indeed, a quasicyclic group $C_{p^\infty}$ is not residually finite, but, being Abelian, is residually nilpotent.

For subsets $A, B \subseteq G$, by $[A, B]$ we denote a subgroup generated by all possible commutators $[a, b] = a^{-1}b^{-1}ab, a \in A, b \in B$. A lower central series of a group $G$ is the sequence of groups

$$G = \gamma_1 G \geq \gamma_2 G \geq \ldots,$$

where $\gamma_{i+1} G = [\gamma_i G, G], \ i = 1, 2, \ldots.$

In particular, $G' = \gamma_2 G = [G, G]$ is the derived subgroup of $G$. A group $G$ is residually nilpotent iff the intersection of all terms of the lower central series is trivial, i.e.,

$$\gamma_\omega G = \bigcap_{i=1}^{\infty} \gamma_i G = 1.$$
The length of the lower central series of a group $G$ is the least $N$ for which $\gamma_N G = \gamma_{N+1} G$. The length equals either a natural number or a limit ordinal $\omega$.

The Baumslag–Solitar groups $BS(m, n)$, $BS(n, m)$, and $BS(-m, -n)$ are pairwise isomorphic, and there is no loss of generality in assuming that the parameters satisfy the inequalities $0 < m \leq |n|$. Throughout the paper, we assume that these inequalities hold.

The criterion of being residually finite for $BS(m, n)$ was found in [1] and [6]; a simple proof was expounded in [2].

**PROPOSITION A** [1, 6]. The group $BS(m, n)$ is RF if and only if either $m = 1$ or $|n| = m > 1$.

The criterion of being residually $p$-finite, $p$ is a prime, for $BS(m, n)$ was obtained in [7] (see also [2]).

**PROPOSITION B** [2, 7]. The group $BS(m, n)$ is RF if and only if either $m = 1$ and $n \equiv 1 \pmod{p}$, or $m = n = p^r$ for some $r \geq 0$, or $m = -n = 2^r$ for some $r \geq 0$ and $p = 2$.

Since every RF group is RN, Proposition B implies that the following groups are residually nilpotent:

$BS(1, 1 + pk), k \in \mathbb{Z}, BS(p^r, p^r), r \geq 0$.

We see that compared to the class of RF groups, the above list does not contain groups $BS(m, m)$, where $m$ is greater than 1 and is not a power of a prime.

2. LOWER CENTRAL SERIES

**Solvable Baumslag–Solitar groups.** Such groups have the form

$BS(1, n) = \langle a, t \mid t^{-1} at = a^n \rangle$.

In view of [8], it is not hard to show that every element $g$ of the group $BS(1, n)$ can be uniquely represented as a product

$g = t^k a^l t^{-r}, k, r \in \mathbb{N} \cup \{0\}, l \in \mathbb{Z}$,

where $n \not| l$ for $r > 0$. For instance,

$tat^{-2} = t^2(t^{-1} at)t^{-2} = t^2a^{n+1}t^{-2}$.

Obviously, for $n = 1$, the group $BS(1, 1)$ is a free Abelian group of rank 2 and is therefore torsion-free residually nilpotent. If $n = -1$, then $BS(1, -1)$ is the fundamental group of the Klein bottle, for which the following proposition is well known and readily verifiable.

**PROPOSITION 1.** The group

$G = BS(1, -1) = \langle a, t \mid t^{-1} at = a^{-1} \rangle$
is residually nilpotent and terms of its lower central series have the form
\[ \gamma_{s+1}G = \langle a^{2^s} \rangle, \ s \geq 1, \]
\[ \gamma_\omega G = \bigcap_{s \geq 1} \gamma_s G = 1. \]

The theorem below describes how the lower central series of solvable Baumslag–Solitar groups are structured and generalizes the previous proposition.

**THEOREM 1.** Let \( G = BS(1,n) \), \( n \neq 1 \). Then:

1. \( \gamma_i G, \ i > 1 \), consists of elements
   \[ t^l a^{n-1} t^{-l}, \ l \in \mathbb{N} \cup \{0\}, \ \alpha \in \mathbb{Z}; \]
2. the group \( \gamma_i G, \ i > 1 \), is isomorphic to the additive group \( (n-1)^{i-2}\mathbb{Z}[1/n] \); in particular, \( G \) is metabelian;
3. \( G = G' \times \mathbb{Z} \), where \( \mathbb{Z} \) is generated by an element \( t \); the quotient \( \gamma_i G/\gamma_{i+1} G \) is isomorphic to \( \mathbb{Z}_{n-1} \) for all \( i > 1 \).

**Proof.** (1) Since
\[ a^{n-1} = a^{-1} t^{-1} at = [a, t], \]
the subgroup \( H \) generated by elements \( t^l a^{n-1} t^{-l} \) is contained in the derived subgroup \( G' \). We claim that every element of \( H \) has the form
\[ t^l a^{n-1} t^{-l}, \ l \in \mathbb{N} \cup \{0\}, \ \alpha \in \mathbb{Z}. \]
Indeed, from \( H \) we take two elements
\[ h_1 = t^{l_1} a^{\alpha_1(n-1)} t^{-l_1}, \ h_2 = t^{l_2} a^{\alpha_2(n-1)} t^{-l_2} \]
and find their product. If \( l_1 < l_2 \), then
\[ h_1 \cdot h_2 = t^{l_1} a^{\alpha_1(n-1)} t^{-l_1} \cdot t^{l_2} a^{\alpha_2(n-1)} t^{-l_2} \]
\[ = t^{l_1} a^{\alpha_1(n-1)} t^{(l_2-l_1)} a^{\alpha_2(n-1)} t^{-l_2} \]
\[ = t^{l_2} a^{\alpha_1(n-1) n^{l_2-l_1} + \alpha_2(n-1)} t^{-l_2} \]
\[ = t^{l_2} a^{(n-1)(\alpha_1 n^{l_2-l_1} + \alpha_2)} t^{-l_2}. \]
If \( l_1 > l_2 \), then
\[ h_1 \cdot h_2 = t^{l_1} a^{\alpha_1(n-1)} t^{-l_1} \cdot t^{l_2} a^{\alpha_2(n-1)} t^{-l_2} \]
\[ = t^{l_1} a^{\alpha_1(n-1)} t^{-(l_1-l_2)} a^{\alpha_2(n-1)} t^{-l_2} \]
\[ = t^{l_1} a^{\alpha_1(n-1) + \alpha_2(n-1) n^{l_1-l_2}} t^{-l_1} \]
\[ = t^{l_1} a^{(n-1)(\alpha_1 + \alpha_2 n^{l_1-l_2})} t^{-l_1}. \]
If \( l_1 = l_2 \), then the result of the product is evident.

We claim that \( H \) is normal in \( G \). In fact,

\[
\begin{align*}
-1 \left( t^l a^{n-1} t^{-l} \right) t &= t^l a^{n(n-1)} t^{-l} = (t^l a^{n-1} t^{-l})^n, \\
-1 \left( t^l a^{n-1} t^{-l} \right) t^{-1} &= t^{l+1} a^{n(n-1)} t^{-l-1}, \\
-1 \left( t^l a^{n-1} t^{-l} \right) a &= t^l (t^{-l} a^{-1} t^l) a^{-1} (t^{-l} a t^l) t^{-l} \\
&= t^l a^{-n^l} a^{-n^l} t^{-l} \\
&= t^l a^{n-1} t^{-l}.
\end{align*}
\]

Note also that the last relation \([a, t^l a^{n-1} t^{-l}] = 1\) implies that \( H \) is a commutative subgroup. Since

\[
\left\{ t, a \mid t^{-1} a t = a^n, \; a^{n-1} = 1 \right\} = \left\{ t, a \mid t^{-1} a t = a, \; a^{n-1} = 1 \right\},
\]

the quotient \( G/H \) is Abelian, and hence \( G' \leq H \).

We proceed by induction on \( i \geq 2 \). Suppose that we have already proved that

\[
\gamma_i G = \left\{ t^l a^{\alpha(n-1)^i} t^{-l} \mid l \in \mathbb{N} \cup \{0\}, \; \alpha \in \mathbb{Z} \right\}.
\]

Since \([G', a] = 1\) and \( G'' = 1\), it follows that \( \gamma_{i+1} G \subseteq [\gamma_i G, t] \). We have

\[
[t^l a^{\alpha(n-1)^i} t^{-l}, t] = t^l a^{-\alpha(n-1)^i} t^{-l} t^{-1} t^l a^{\alpha(n-1)^i} t^{-l} t = t^l a^{\alpha(n-1)^i} t^{-l}.
\]

As above, we verify that the product of elements of the form \( t^l a^{\alpha(n-1)^i} t^{-l}, \; l \in \mathbb{N} \cup \{0\}, \; \alpha \in \mathbb{Z} \), has the same form. Thus

\[
\gamma_{i+1} G = \left\{ t^l a^{\alpha(n-1)^i} t^{-l} \mid l \in \mathbb{N} \cup \{0\}, \; \alpha \in \mathbb{Z} \right\}.
\]

(2) We show that \( G' \cong \mathbb{Z}[1/n] \). To do this, we define \( b = a^{n-1} \) and construct a mapping

\[
\varphi : G' \rightarrow \mathbb{Z}[1/n]
\]

using the rule

\[
t^l b^\alpha t^{-l} \mapsto \frac{\alpha}{n^l}, \; l \in \mathbb{N} \cup \{0\}.
\]

The mapping is onto. Taking into account that every element in the group \( G' \) is uniquely represented as \( t^l b^\alpha t^{-l}, \; l \in \mathbb{N} \cup \{0\}, \; \alpha \in \mathbb{Z} \), where \( n \not| \alpha \) for \( l > 0 \), we conclude that the mapping \( \varphi \) is an embedding. We verify that \( \varphi \) preserves the operations. As above, in \( G' \) we choose two elements

\[
h_1 = t^{l_1} a^{\alpha_1(n-1)} t^{-l_1}, \; h_2 = t^{l_2} a^{\alpha_2(n-1)} t^{-l_2},
\]

and, using the above computations, define

\[
\varphi(h_1 \cdot h_2) = \frac{\alpha_1 n^{l_2 - l_1} + \alpha_2}{n^{l_2}} \text{ for } l_1 < l_2,
\]
\[
\varphi(h_1 \cdot h_2) = \frac{\alpha_1 + \alpha_2 n^{l_1-l_2}}{n^{l_1}} \quad \text{for } l_1 > l_2.
\]

On the other hand,
\[
\varphi(h_1) \cdot \varphi(h_2) = \frac{\alpha_1}{n^{l_1}} + \frac{\alpha_2}{n^{l_2}} = \frac{\alpha_1 n^{l_2} + \alpha_2 n^{l_1}}{n^{l_1}+l_2},
\]

which equals
\[
\frac{\alpha_1 n^{l_2-l_1} + \alpha_2}{n^{l_2}} \quad \text{for } l_1 < l_2,
\]
\[
\frac{\alpha_1 + \alpha_2 n^{l_1-l_2}}{n^{l_1}} \quad \text{for } l_1 > l_2.
\]

If we compare the expressions obtained we see that \(\varphi\) is a homomorphism. Thus \(\varphi\) is an isomorphism.

Furthermore, if we apply \(\varphi\) to a subgroup \(\gamma_i G\) we obtain the induced isomorphism \(\gamma_i G \cong (n-1)^i \mathbb{Z}[1/n]\).

(3) It remains to prove that \(\gamma_i G/\gamma_{i+1} G \cong \mathbb{Z}_{n-1}\) holds for all \(i > 1\).

Let \(x_i, i \in \mathbb{N} \cup \{0\}\), be free generators for a free Abelian group. Then the group \(\mathbb{Z}[1/n]\) can be defined by the following genetic code:
\[
\mathbb{Z}[1/n] \cong \langle x_i, i \in \mathbb{N} \cup \{0\} \mid n x_i = x_{i-1}, i \in \mathbb{N}, [x_k, x_l] = 1 \text{ for all } k, l \rangle,
\]

where the isomorphism is defined by a mapping \(n^{-i} \to x_i, i \in \mathbb{N} \cup \{0\}\). Consequently,
\[
\mathbb{Z}[1/n]/(n-1)\mathbb{Z}[1/n] \cong \langle x_i, i \in \mathbb{N} \cup \{0\} \mid (n-1)x_0 = 0, nx_i = x_{i-1}, (n-1)x_i = 0, i \in \mathbb{N}, [x_k, x_l] = 1 \text{ for all } k, l \rangle
\]
\[
\cong \langle x_i, i \in \mathbb{N} \cup \{0\} \mid (n-1)x_0 = 0, nx_i = x_{i-1}, n x_i = x_i, i \in \mathbb{N}, [x_k, x_l] = 1 \text{ for all } k, l \rangle
\]
\[
\cong \langle x_0 \mid (n-1)x_0 = 0 \rangle \cong \mathbb{Z}_{n-1}.
\]

In particular,
\[
\mathbb{Z}[1/n] = \langle x_0, (n-1)\mathbb{Z}[1/n] \rangle.
\]

After multiplying the above equality by \((n-1)^i\), we obtain
\[
(n-1)^i \mathbb{Z}[1/n] = \langle (n-1)^i x_0, (n-1)^{i+1} \mathbb{Z}[1/n] \rangle.
\]

If, for some \(1 \leq k \leq n-1\),
\[
k(n-1)^i x_0 \in (n-1)^{i+1}\mathbb{Z}[1/n],
\]

then
\[
kx_0 \in (n-1)\mathbb{Z}[1/n],
\]
i.e., \((n - 1)\) divides \(k\), a contradiction. Therefore, there exists an isomorphism

\[(n - 1)^i\mathbb{Z}[1/n]/(n - 1)^{i+1}\mathbb{Z}[1/n] \cong \mathbb{Z}_{n-1}.
\]

The theorem is proved.

Theorem 1 gives rise to the following previously established result.

**COROLLARY** [4]. (1) The group \(BS(1,n)\) is residually nilpotent if and only if \(n \neq 2\).

(2) The group \(BS(1,2)\) is not residually nilpotent, and the length of its lower central series is equal to 2, i.e.,

\[\gamma_2 BS(1,2) = \gamma_3 BS(1,2).\]

**Proof.** Let \(n \neq 2\). It suffices to show that \(\bigcap_{i \geq 2} (n - 1)^{i-2}\mathbb{Z}[1/n] = 0\). Let \(\alpha \in \bigcap_{i \geq 2} (n - 1)^{i-2}\mathbb{Z}[1/n]\). Then \(\frac{\alpha}{(n-1)^{i-2}} \in \mathbb{Z}[1/n]\) for all \(i \geq 2\), which is impossible since the denominator of the fraction \(\frac{\alpha}{(n-1)^{i-2}}\) increases indefinitely with \(i\).

Let \(n = 2\). Then

\[G = BS(1,2) = \langle t, a \mid t^{-1}at = a^2 \rangle.\]

Since \(a = [a, t]\), we have \(a \in \gamma_s G, s \geq 2\). Hence \(a \in \gamma_\omega G\). Furthermore,

\[\langle t, a \mid t^{-1}at = a^2, a = 1 \rangle \cong \mathbb{Z}.
\]

Therefore, \(\gamma_2 G \leq \gamma_\omega G\). Consequently,

\[\gamma_2 G = \gamma_\omega G.
\]

Since \(\gamma_2 G \neq 1\), the group \(BS(1,2)\) is not residually nilpotent. The corollary is proved.

**Unsolvable Baumslag–Solitar groups.** Unsolvable RF Baumslag–Solitar groups are of the form \(BS(m, \epsilon m)\), \(m > 1, \epsilon = \pm 1\). In [4], it was proved that the group \(BS(m, \epsilon m), m > 1, \epsilon = \pm 1\), is residually nilpotent iff \(m\) is a power of a prime.

The proposition below describes the structure of the lower central series for Baumslag–Solitar groups of the form \(BS(m, \epsilon m), m > 1, \epsilon = \pm 1\).

**PROPOSITION 2.** Let \(m > 1\) be an arbitrary natural number. The following isomorphisms hold:

(1) \(\gamma_i (BS(m, m)) \cong \gamma_i (\mathbb{Z} \ast \mathbb{Z}_m), i > 1;\)

(2) \(\gamma_i (BS(m, -m)) \cong \langle a^{2i-1} \rangle \times \gamma_i (\mathbb{Z} \ast \mathbb{Z}_m), i > 1.\)

**Proof.** First consider a group \(G = BS(m, m)\). There are natural homomorphisms

\[\varphi_i : \gamma_i G \longrightarrow \gamma_i (\mathbb{Z} \ast \mathbb{Z}_m), i \geq 1,
\]

induced by the homomorphism

\[\varphi : G \longrightarrow (\mathbb{Z} \ast \mathbb{Z}_m),\]
taking an element $a^m$ to unity. The element $a^m$ is central, so $\text{Ker } \varphi_2$ coincides with the subgroup generated by $a^m$, and we obtain the following short exact sequence:

$$1 \rightarrow \langle a^m \rangle \rightarrow G \rightarrow \mathbb{Z} \ast \mathbb{Z}_m \rightarrow 1.$$ 

In order to verify that $\varphi_i, i \geq 2$, are isomorphisms, we need only show that $\text{Ker } \varphi_2 = 1$, i.e., $\gamma_2 G \cap \langle a^m \rangle = 1$.

The derived subgroup $\gamma_2 G$ of a group $G$ is generated by commutators of the form

$$[t^k, a^l], \ k, l \in \mathbb{Z} \setminus \{0\},$$

as the image of the derived subgroup of a free group of rank 2. Keeping in mind that $a^m$ is a central element of the group $G$, we may assume that $1 \leq l \leq m - 1$.

The derived subgroup $\gamma_2 (\mathbb{Z} \ast \mathbb{Z}_m)$ of the free product

$$\mathbb{Z} \ast \mathbb{Z}_m = \langle t, a \mid a^m = 1 \rangle$$

is a free group on free generators

$$[t^k, a^l], \ k \in \mathbb{Z} \setminus \{0\}, \ 1 \leq l \leq m - 1.$$ 

Therefore, there is a natural homomorphism

$$\psi : \gamma_2 (\mathbb{Z} \ast \mathbb{Z}_m) \rightarrow \gamma_2 G,$$

which is inverse to $\varphi_2$. Thus $\varphi_2$ is an isomorphism, and so $\varphi_i, i \geq 2$, are also isomorphisms.

Consider a group $G = BS(m, -m)$. First we show that there is an isomorphism

$$\gamma_2 G \cong \langle a^{2m} \rangle \times \gamma_2 (\mathbb{Z} \ast \mathbb{Z}_m).$$

The derived subgroup of a group $G$ is generated by commutators of the form

$$[t^k, a^l], \ k, l \in \mathbb{Z} \setminus \{0\},$$

as the image of the derived subgroup of a free group of rank 2. It is straightforward to verify that for any $k, l \in \mathbb{Z},$

$$[[t^k, a^l], a^m] = 1.$$

By virtue of the commutator identity

$$[z, xy] = [z, y][z, x]^y,$$

we conclude that the derived subgroup of $G$ is generated by commutators of the form

$$[t^k, a^l], \ k \in \mathbb{Z} \setminus \{0\}, \ 1 \leq l \leq m - 1,$$
and by degrees of the word $a^{2m}$. As above, the specified commutators generate a free group isomorphic to the derived subgroup of the free product $\mathbb{Z} * \mathbb{Z}_m$. Consequently, the homomorphism

$$\varphi_2 : \gamma_2 G \longrightarrow \gamma_2(\mathbb{Z} * \mathbb{Z}_m)$$

has an inverse, i.e., it splits. In virtue of the fact that $a^{2m}$ is a central element in $\gamma_2 G$, we obtain an isomorphism

$$\gamma_2 G \cong \langle a^{2m} \rangle \times \gamma_2(\mathbb{Z} * \mathbb{Z}_m).$$

Note also that the group $\gamma_2(\mathbb{Z} * \mathbb{Z}_m)$ is naturally embedded in the group $\gamma_2 G$; i.e., an isomorphism sign can be replaced by an equal sign.

Suppose that for some $i \geq 2$, there is an isomorphism

$$\gamma_i G \cong \langle a^{2i-1m} \rangle \times \gamma_i(\mathbb{Z} * \mathbb{Z}_m),$$

and the group $\gamma_i(\mathbb{Z} * \mathbb{Z}_m)$ is naturally embedded in $\gamma_i G$. Then

$$\gamma_{i+1} G = [\gamma_i G, G]$$

$$= \langle \langle a^{2i-1m} \rangle \times \gamma_i(\mathbb{Z} * \mathbb{Z}_m), G \rangle \rangle$$

$$= \langle \langle a^{2i-1m} \rangle, G \rangle \times [\gamma_i(\mathbb{Z} * \mathbb{Z}_m), G] \rangle$$

$$= \langle a^{2i-1m} \rangle \times \gamma_i(\mathbb{Z} * \mathbb{Z}_m), \mathbb{Z} * \mathbb{Z}_m \rangle$$

$$= \langle a^{2i-1m} \rangle \times \gamma_{i+1}(\mathbb{Z} * \mathbb{Z}_m).$$

The proposition is proved.

Thus Theorem 1 and Proposition 2 yield the criterion of being residually nilpotent for Baumslag–Solitar groups which was established in [4].

**THEOREM 2.** The group $BS(m, n)$, $0 < m \leq |n|$, is RN if and only if either $m = 1$ and $n \neq 2$ or $|n| = m > 1$ and $m$ is a power of a prime.

Denote by $rFBS$ the set of RF Baumslag–Solitar groups, by $rNBS$ the set of RN Baumslag–Solitar groups, by $rF_pBS$ the set of $p$-Baumslag–Solitar groups, and by $\{rF_pBS\}_p$ the union of sets $rF_pBS$ over all prime $p$. The following inclusions hold:

$$rF_pBS \subset \{rF_pBS\}_p \subset rNBS \subset rFBS.$$

In this case the difference $rFBS \setminus rNBS$ contains a group $BS(1, 2)$ and groups $BS(m, \pm m)$, where $m$ is not a power of a prime. The difference $rNBS \setminus \{rF_pBS\}_p$ contains a set of groups $BS(p^r, -p^r)$, $p > 2$ is a prime.
3. INTERSECTION OF TERMS OF THE LOWER CENTRAL SERIES

We know that if a group \( G \) is residually nilpotent, then \( \gamma_\omega G = 1 \). Otherwise, the quotient \( G/\gamma_\omega \) is the maximal residually nilpotent quotient of a group \( G \). In this section we describe an intersection of terms of the lower central series of some Baumslag–Solitar groups.

Obviously, we have

**Lemma 1.** An Abelianization \( BS(m, n)^{ab} \) is isomorphic to \( \langle a, t \mid a^{n-m} = [a, t] = 1 \rangle \). In particular, the following hold:
- if \( n = m + 1 \) then \( BS(m, n)^{ab} \cong \mathbb{Z} \);
- if \( n = m \) then \( BS(m, n)^{ab} \cong \mathbb{Z} \times \mathbb{Z} \);
- if \( n \neq m, m + 1 \) then \( BS(m, n)^{ab} \cong \mathbb{Z} \times \mathbb{Z}_{m-m} \).

We also need

**Lemma 2.** In the group \( BS(m, n) \), the inclusion \( a^{(n-m)^i} \in \gamma_{i+1}BS(m, n) \) holds for all \( i \geq 0 \).

**Proof.** From the defining relation for \( BS(m, n) \), we derive
\[
[a^m, t] = a^{n-m},
\]
i.e., \( a^{n-m} \in \gamma_2 BS(m, n) \). Raising both parts of the defining relation for \( BS(m, n) \) to the power \( n - m \), we obtain
\[
t^{-1} a^{m(n-m)} t = a^{n(n-m)} \iff a^{m(m-n)} t^{-1} a^{m(n-m)} t = a^{(n-m)^2}.
\]
Consequently,
\[
[[a^m, t]^n, t] = a^{(n-m)^2}.
\]
It remains to use induction on \( i \). The lemma is proved.

Below we will consider Baumslag–Solitar groups that are not residually nilpotent.

Among the non-Abelian Baumslag–Solitar groups, \( BS(m, m+1), m > 1 \), has the shortest lower central series. We have

**Proposition 3.** The length of the lower central series of a group
\[
G = BS(m, m+1) = \langle a, t \mid t^{-1} a^m t = a^{m+1} \rangle, \ m > 1,
\]
is equal to 2.

**Proof.** Since \( a = [a^m, t] \), we have \( a \in \gamma_s G \) with \( s \geq 2 \). Consequently, \( a \in \gamma_\omega G \). Furthermore, there is an isomorphism
\[
\langle a, t \mid t^{-1} a^m t = a^{m+1}, \ a = 1 \rangle \cong \mathbb{Z}.
\]
Therefore, \( \gamma_2 G \leq \gamma_\omega G \). Hence \( \gamma_2 G = \gamma_\omega G \). The proposition is proved.

By \( d = \text{GCD}(m, n) \) we denote the greatest common divisor of numbers \( m \) and \( n \). Consider a group \( G = BS(m, n) \) for \( m = kd \) and \( n = kd + d \), where \( k, d \in \mathbb{Z} \). In \( G \), we have
\[
a^d = [a^{kd}, t] \in \gamma_2 G.
\]
Hence \(a^d \in \gamma \omega G\). A generalization of Proposition 3 is

**THEOREM 3.** For a group \(G = BS(kd, kd + d), k, d \in \mathbb{Z}\), the following statements hold:

1. if \(d\) is a power of a prime or \(d = 1\) then \(\gamma \omega G = \langle a^d \rangle^G\);
2. if \(d\) is not a power of a prime then \(\gamma \omega G > \langle a^d \rangle^G\).

**Proof.** There is an isomorphism
\[
G/\langle a^d \rangle^G = \langle a, t \mid a^d = 1 \rangle \cong \mathbb{Z} \ast \mathbb{Z}_d.
\]

If \(d\) is a power of a prime or \(d = 1\), then the free product \(\mathbb{Z} \ast \mathbb{Z}_d\) is residually nilpotent, so \(\gamma \omega (G/\langle a^d \rangle^G) = 1\). Hence
\[
\gamma \omega G \subseteq \langle a^d \rangle^G.
\]
The inverse inclusion has been pointed out above.

If \(d\) is not a power of a prime, then \(\gamma \omega (\mathbb{Z} \ast \mathbb{Z}_d) \neq 1\), and so \(\gamma \omega G > \langle a^d \rangle^G\). The theorem is proved.

We fix a Baumslag–Solitar group \(G = BS(m, n), d = \text{GCD}(m, n)\), which is not RF. Let \(R\) be an intersection of all subgroups of finite index in \(G\). As shown in [9], \(R\) is the normal closure of the set of commutators
\[
[t^k a^d t^{-k}, a], \; k \in \mathbb{Z},
\]
in \(G\). Let \(\overline{G} = G/R\) be the greatest RF quotient of \(G\). Then
\[
\overline{G} = \langle a, t \mid t^{-1} a^m t = a^n, [t^k a^d t^{-k}, a] = 1, \; k \in \mathbb{Z} \rangle.
\]
Define in \(\overline{G}\) the subgroup
\[
\overline{A} = \langle t^k a^d t^{-k}, \; k \in \mathbb{Z} \rangle.
\]
Then \(\overline{A}\) is an Abelian subgroup normal in \(\overline{G}\), and the following isomorphism holds (see [9]):
\[
\overline{G}/\overline{A} = \langle a, t \mid a^d = 1 \rangle = \mathbb{Z}_d \ast \mathbb{Z}.
\]

It would be interesting to find a connection between the subgroups \(R\) and \(\gamma \omega G\).

Notice that if \(d \geq 2\) then \(BS(m, n)\) contains a free non-Abelian group. More exactly, the following holds:

**PROPOSITION 4.** If \(d \geq 2\), then \(G = BS(m, n)\) contains a free non-Abelian subgroup of infinite rank.

**Proof.** Let \(D\) be the Cartesian subgroup of the group \(\mathbb{Z}_d \ast \mathbb{Z}\), i.e., the kernel of the natural homomorphism \(\mathbb{Z}_d \ast \mathbb{Z} \rightarrow \mathbb{Z}_d \times \mathbb{Z}\). Clearly, \(D = (\overline{G}/\overline{A})'\). It is well known that \(D\) is a free group on free generators
\[
[t^k, a^s], \; k \in \mathbb{Z}, \; 1 \leq s \leq d - 1.
\]
Consequently, if \(d \geq 2\), then the groups \(\overline{G}\) and \(G\) contain a free subgroup of infinite rank. The proposition is proved.
Now we note that \( R \subseteq G' \), and, as stated in Lemma 1,

\[
G/G' \cong \mathbb{Z} \times \mathbb{Z}_{n-m},
\]

while

\[
\overline{G}/\overline{G'}A \cong \mathbb{Z} \times \mathbb{Z}_d.
\]

If \( n - m > d \), then \( \overline{G} \) is a proper subgroup of \( \overline{G'}A \). If \( n - m = d \), then \( \overline{G} = \overline{G'}A \), i.e., \( \overline{A} \subseteq \overline{G'} \).

The quotient \( G/\gamma \omega G \) is residually nilpotent, so the intersection of its subgroups of finite index is trivial. Consequently, the subgroup \( R \) is contained in \( \gamma \omega G \), and the following inclusion holds:

\[
R = \left\langle [t^ka^d-t^{-k}a], \ k \in \mathbb{Z} \right\rangle \subseteq \gamma \omega G.
\]

Put \( \gamma \omega G/R = \gamma \omega \overline{G}. \) Note that \( \gamma \omega \overline{G} < \gamma \omega G \) in general.

Let \( A \leq G \) be such that \( A/R = \overline{A} \). There are inclusions

\[
G \geq G'A \geq A \geq R, \ G \geq G' \geq \gamma \omega G \geq R
\]

with the following quotients:

\[
G/G'A \cong \mathbb{Z} \times \mathbb{Z}_d, \ G/A \cong \mathbb{Z} \ast \mathbb{Z}_d, \ G/G' \cong \mathbb{Z} \times \mathbb{Z}_{n-m},
\]

\[
G'/A \cong \begin{cases} F_{\infty} & \text{if } d \geq 2, \\ \mathbb{Z} & \text{if } d = 1. \end{cases}
\]

We also observe that the quotient \( \gamma \omega G/R \) is Abelian.

If \( n = m + d \), i.e., \( m = k \) and \( n = kd + d \), then, as noted, \( A \subseteq G' \), and the following inclusions hold:

\[
G \geq G' \geq A \geq R.
\]

In particular, \( \gamma_2 G/\gamma \omega G \) contains a free group of infinite rank for \( d \geq 2 \).

If \( d \) is a power of a prime, then the quotient \( \overline{G}/\overline{A} \cong \mathbb{Z} \ast \mathbb{Z}_d \) is residually nilpotent, \( \gamma \omega \overline{G} \subseteq \overline{A} \), and the following inclusions hold:

\[
G \geq G'A \geq A \geq \gamma \omega G \geq R.
\]

If \( n = m + d \) and \( d \) is a power of a prime, then

\[
G \geq G' \geq A \geq \gamma \omega G \geq R.
\]

If \( n = m + 1 \), then, as stated in Proposition 3, \( G' = A = \gamma \omega G \). Thus we have in fact proved

**PROPOSITION 5.** There are inclusions

\[
G \geq G'A \geq A \geq R, \ G \geq G' \geq \gamma \omega G \geq R,
\]

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where the quotient $A/R$ is Abelian. In particular, the following hold:

1. if $n - m \geq d > 1$, then
   \[ G/G' A \cong \mathbb{Z} \times \mathbb{Z}_d, \quad G/A \cong \mathbb{Z} \ast \mathbb{Z}_d, \quad G/G' \cong \mathbb{Z} \times \mathbb{Z}_{n-m}, \quad G'/A \cong F_{\infty}; \]

2. if $n - m > d = 1$, then $A$ contains $G'$ as a proper subgroup, and
   \[ G/A \cong \mathbb{Z}, \quad A/G' \cong \mathbb{Z}_{n-m}; \]

3. if $n - m = d = 1$, then $G' = A = \gamma_\omega G$;

4. if $d$ is a power of a prime, then
   \[ G \geq G'A \geq A \geq \gamma_\omega G \geq R; \]

5. if $n = m + d$ and $d$ is a power of a prime, then
   \[ G \geq G' \geq A \geq \gamma_\omega G \geq R. \]

**Remark.** If $n - m = d = 1$, then $\gamma_\omega G/R \cong \mathbb{Z}^\infty$, and the following isomorphism holds:
\[ \overline{G} \cong \mathbb{Z} \wr \mathbb{Z}, \]
where $\mathbb{Z} \wr \mathbb{Z}$ is a direct wreath product of two infinite cyclic groups, which is isomorphic to a group generated by matrices
\[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix}, \]
where $\xi$ is a transcendental number.

In connection with Proposition 5, the following problems arise.

**Question 1.** Find the structure of a quotient $\gamma_\omega G/R$ for $n - m \geq d > 1$ and $n - m > d = 1$.

**Question 2.** Find relations between the subgroups $A$ and $\gamma_\omega G$ if $d$ is not a power of prime.

Also of interest are the following:

**Question 3.** Is it true that if $G$ is a Baumslag–Solitar group then $\gamma_\omega G = \gamma_{\omega+1} G$ where $\gamma_{\omega+1} G = [\gamma_\omega G, G]$?

Recall [10] that a finitely generated group acting on a tree with all vertex and edge stabilizers infinite cyclic is called a *generalized Baumslag–Solitar group* (a GBS group).

**Question 4.** Which of the GBS groups are residually nilpotent?

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