Supersymmetry in the Dirac equation for generalized Coulomb potential

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Abstract
We propose a symmetry of the Dirac equation under the interchange of signs of eigenvalues of the Dirac’s $K$ operator. We show that the only potential which obeys this requirement is the Coulomb one for both vector and scalar cases. Spectrum of the Dirac Equation is obtained algebraically for arbitrary combination of Lorentz-scalar and Lorentz-vector Coulomb potentials using the Witten’s Superalgebra approach. The results coincides with that, known from the explicit solution of the Dirac equation.

1 Introduction

The Kepler problem has an additional symmetry, more precisely, additional conserved quantity, so-called Laplace-Runge-Lenz (LRL) vector (for relevant references see, for example, [1]). Because of this fact there appears extended algebra together with orbital momentum, which is isomorphic to $SO(4)$ (for negative total energies). But components of LRL vector as generators of this algebra do not participate in any geometric transformations. In 1926 W.Pauli [2] considered this algebra in quantum mechanics and obtained Hydrogen atom spectrum by only algebraic methods. In 30-ies V.Fock [3] considered the Schroedinger equation for the Kepler problem in momentum representation and had shown that the Coulomb spectrum has an $SO(4)$ symmetry in energy-momentum space. Afterwards algebraic methods attract more wide applications and it was cleared up in the last decades that the hidden symmetry of the Kepler problem is closely related to the supersymmetry of Hydrogen atom [4, 5].

The aim of our paper is to determine what the Dirac equation tells us about this problem. We can see that very interesting physical picture emerges.

For the Kepler problem in the Dirac equation Johnson and Lippmann [6] published very brief abstract in Physical Review at 1950. They had written that there is an additional conserved quantity

$$A = \hat{\sigma} \cdot \hat{r} r^{-1} - i \left( \frac{\hbar c}{e^2} \right) (mc^2)^{-1} j \rho_1 (H - mc^2 \rho_3), \quad (1)$$

which plays the same role in Dirac equation, as the LRL vector in Schroedinger equation.

As regards to the more detailed derivation, to the best our knowledge, is not published in scientific literature (one of the curious fact in the history of physics of 20th century). Moreover as far as commutativity of the Johnson-Lippmann(JL) operator with the Dirac Hamiltonian is concerned, it is usually mentioned that it can be proved by "rather tedious calculations." [7].
In papers [8, 9], we developed rather simple and transparent way for deriving the JL operator. We obtained this operator and at the same time proved its commutativity with the Hamiltonian.

After we consider necessity to be convinced, that the Coulomb problem is distinguishable in this point of view. We considered the Dirac equation in arbitrary central potential, $V(r)$ and shown that the symmetry, which will be defined more precisely below, takes place only for Coulomb potential.

Moreover it is remarkable that this property remains valid for arbitrary linear combination of Lorentz-vector and Lorentz-scalar potentials [11]. Therefore we expect that the spectrum for this general case can be obtained by pure algebraic methods, i.e. without solving of corresponding Dirac equation.

Our paper is organized in the following manner: First of all we define a symmetry, which in fact leads to Witten’s superalgebra. Then we determine the general class of so-called $K$-odd operators and apply this result to pure vector and then to mixed cases. At the end we derive the spectrum by algebraic methods on the ground of supersymmetric approach. We show that the final expression for ground state energy of Hydrogen atom coincides with that obtained by solving of equations of motion.

2 Survey Of Symmetry

Let consider the Dirac Hamiltonian in arbitrary central field, $\hat{V}(r)$, which in general may be Lorentz-scalar or 4th component of the Lorentz vector (or their combination)

$$H = \vec{\alpha} \cdot \vec{p} + \beta m + \hat{V}(r)$$

(2)

The so-called Dirac’s $K$-operator, defined as

$$K = \beta(\vec{\Sigma} \cdot \vec{l} + 1)$$

(3)

commutes with this Hamiltonian for arbitrary $\hat{V}(r)$, $[K, H] = 0$. Here $\vec{\Sigma}$ is the spin matrix

$$\vec{\Sigma} = \begin{pmatrix}
\vec{\sigma} & 0 \\
0 & \vec{\sigma}
\end{pmatrix}$$

and $\vec{l}$ - orbital momentum vector-operator.

It is evident that the eigenvalues of Hamiltonian also are labelled by eigenvalues of $K$. For example, the well-known Sommerfeld formula for the Coulombic spectrum looks like [12]

$$E_{n,|\kappa|} = m \left[ 1 + \frac{(Z\alpha)^2}{n - |\kappa| + \sqrt{\kappa^2 - (Z\alpha)^2}} \right]^{-1/2}$$

It manifests explicit dependence on $\kappa$, more precisely on $|\kappa| = j + 1/2$. It is surprising that for other solvable potentials the degeneracy with respect to signs of $\kappa$ does not take place. Physically this degeneracy leads to the forbidden of the Lamb shift. Indeed, positive $\kappa = j + 1/2$ corresponds to aligned spin $j = l + 1/2$, i.e. to states ($s_{1/2}, p_{3/2}, etc$), while negative $\kappa = -(j + 1/2)$ corresponds to unaligned spin $j = l - 1/2$, i.e. to states ($p_{1/2}, d_{3/2}, etc$). So the absence of the Lamb shift ($s_{1/2} - p_{1/2}$) is a consequence of above mentioned degeneracy $\kappa \to -\kappa$. Naturally for description of this degeneracy one has to find an operator that mixes these two signs. It is clear that such an operator, say $A$, must be anticommuting with $K$:

$$\{A, K\} = AK + KA = 0$$

(4)
If at the same time this operator should commute with the Hamiltonian, then it'll generate the symmetry of the Dirac equation. Therefore, we need an operator $A$ with the following properties:

$$\{A, K\} = 0 \text{ and } [A, H] = 0$$  \hfill (5)

**It is our definition of symmetry we are looked for.**

It is worth to mention that after this operator is constructed, we will be able to define relativistic supercharges as follows:

$$Q_1 = A, \quad Q_2 = i\frac{AK}{|K|}$$

It is obvious that

$$\{Q_1, Q_2\} = 0, \quad Q_1^2 = Q_2^2$$

and we can find Witten’s superalgebra $[13]$, where $Q_1^2 \equiv h$ is a so-called, Witten’s Hamiltonian.

### 3 Symmetry Operators

Now our goal is a construction of the operator $A$. We know, that there is a Dirac’s $\gamma^5$ matrix, that anticommutes with $K$. **What else?** There is a **simple theorem** $[9]$, according to which arbitrary $(\vec{\Sigma} \cdot \vec{V})$ type operator, where $\vec{V}$ is a vector with respect of $\vec{l}$ and is perpendicular to it, $(\vec{l} \cdot \vec{V}) = (\vec{V} \cdot \vec{l})$, anticommutes with $K$:

$$\{(\vec{\Sigma} \cdot \vec{V}), K\} = 0$$  \hfill (6)

It is evident that the class of operators anticommuting with $K$ (so-called $K$-odd operators) is much wider. Any operator of the form $\hat{O}(\vec{\Sigma} \cdot \vec{V})$, where $\hat{O}$ commutes with $K$, but is otherwise arbitrary, also is a $K$-odd. This fact will be used below.

#### 3.1 Pure Vector Potential

Let first establish the form of symmetry operator for vector potential only $\hat{V}(r) = V(r)$. By using the above mentioned theorem we wish to construct the $K$-odd operator $A$, that commutes with $H$. It is clear that there remains large freedom according to the above mentioned remark about $\hat{O}$ operators - one can take $\hat{O}$ into account or ignore it.

We have the following physically interesting vectors at hand which obey the requirements of our theorem. They are

$$\hat{r} - \text{unit radius-vector and } \vec{p} - \text{linear momentum vector}$$  \hfill (7)

Both of them are perpendicular to $\vec{l}$. Constraints of this theorem are also satisfied by LRL vector $\vec{A} = \hat{r} - \frac{i}{2ma}[\vec{p} \times \vec{l} - \vec{l} \times \vec{p}]$, but this vector is associate to the Coulomb potential. Hence we abstain from its consideration for now. Thus, we choose the following $K$-odd terms

$$(\vec{\Sigma} \cdot \hat{r}), \quad K(\vec{\Sigma} \cdot \vec{p}) \text{ and } K\gamma^5$$  \hfill (8)

and let probe the combination

$$A = x_1(\vec{\Sigma} \cdot \hat{r}) + ux_2K(\vec{\Sigma} \cdot \vec{p}) + ux_3K\gamma^5f(r)$$  \hfill (9)
Here the coefficients $x_i (i = 1, 2, 3)$ are chosen in such a way that $A$ operator is Hermitian
for arbitrary real numbers and $f(r)$ is an arbitrary scalar function to be determined later from the symmetry requirements. Let’s calculate

$$0 = [A, H] = (\bar{\Sigma} \cdot \hat{r})\{x_2 V'(r) - x_3 f'(r)\} +$$
$$+ 2i\beta K \gamma^5 \left\{ \frac{x_1}{r} - mx_3 f(r) \right\}$$

We have a diagonal matrix in the first term, while the antidiagonal matrix in the second one. Therefore two equations follow:

$$x_2 V'(r) = x_3 f'(r), \quad x_3 m f(r) = \frac{x_1}{r}$$

One can find from these equations:

$$V(r) = \frac{x_1}{x_2 mr}$$

Therefore only the Coulomb potential corresponds to the above required $\pm \kappa$ degeneracy. The final form of obtained $A$ operator is the following:

$$A = x_1\{ (\bar{\Sigma} \cdot \hat{r}) - \frac{i}{ma} K (\bar{\Sigma} \cdot \vec{p}) + \frac{i}{mr} K \gamma^5 \}$$

where unessential common factor $x_1$ may be omitted and after using known relations for
Dirac matrices, this expression may be reduced to the form

$$A = \gamma^5 \left\{ \bar{\alpha} \cdot \vec{r} - \frac{i}{ma} K \gamma^5 (H - \beta m) \right\}$$

Above and here $a$ is a strengh of Coulomb potential, $a = Z\alpha$. Precisely this operator is
given in Johnson’s and Lippmann’s abstract [6].

What the real physical picture is standing behind this? Taking into account the relation

$$K (\bar{\Sigma} \cdot \vec{p}) = -i\beta \left( \bar{\Sigma} \cdot \frac{1}{2} [\vec{p} \times \vec{l} - \vec{l} \times \vec{p}] \right)$$

one can recast our operator in the following form

$$A = \bar{\Sigma} \cdot \left( \vec{r} - \frac{i}{2ma} \beta [\vec{p} \times \vec{l} - \vec{l} \times \vec{p}] \right) + \frac{i}{mr} K \gamma^5$$

One can see that in the non-relativistic limit, i.e. $\beta \to 1$ and $\gamma^5 \to 0$ our operator reduces to

$$A \to A_{NR} = \bar{\sigma} \cdot \left( \vec{r} - \frac{i}{2ma} [\vec{p} \times \vec{l} - \vec{l} \times \vec{p}] \right)$$

Note the LRL vector in the parenthesis of this equation. Therefore relativistic supercharge reduces to the projection of the LRL vector on the electron spin direction. Precisely this operator was used in the case of Pauli electron [14].

Because the Witten’s Hamiltonian is

$$A^2 = 1 + K^2 \left( \frac{H^2}{m^2} - 1 \right)$$

and it consists only mutually commuting operators, it is possible their simultaneous diagonalization and replacement by corresponding eigenvalues. For instance, the energy of ground state is

$$E_0 = \left( 1 - \frac{(Z\alpha)^2}{\kappa^2} \right)^{1/2}$$
By using the ladder procedure, familiar for SUSY quantum mechanics, the Sommerfeld
formula can be easily derived [7].

3.2 Inclusion of Lorentz-scalar Potential

Let’s remark that if we include the Lorentz-scalar potential as well

\[ H = \vec{\alpha} \cdot \vec{p} + \beta m + V(r) + \beta S(r) \] (19)

this last Hamiltonian also commutes with \( K \)-operator, but does not commute with the above
JL operator.

On the other hand, non-relativistic quantum mechanics is indifferent with regard of the
Lorentz variance properties of potential. Therefore it is expected that in case of scalar
potential the description of hidden symmetry should also be possible. In other words, the
JL operator must be generalized to this case.

For this purpose it is necessary to increase number of \( K \)-odd structures. One has to use
our theorem in the part of additional \( \hat{O} \) factors.

Now let us probe the following operator

\[ X = x_1 \left( \hat{\Sigma} \cdot \hat{r} \right) + x'_1 \left( \hat{\Sigma} \cdot \hat{r} \right) + 2 i x_2 K \left( \hat{\Sigma} \cdot \hat{p} \right) + 2 i x'_2 \gamma^5 \beta S(r) \] (20)

We included factor \( \hat{O} = H \) in the first structure and at the same time the matrix \( \hat{O} = \beta \)
in the third structure. Both of them commute with \( K \) and are permissible by our theorem.
The form (20) is a minimal extension of the previous picture, because only the first order
structures in \( \hat{r} \) and \( \hat{p} \) participate. For turning to the previous case one must take \( x'_1 = x'_3 = 0 \)
and \( S(r) = 0 \). Calculation of relevant commutators gives:

\[ [X, H] = \gamma^5 \beta K \left\{ \frac{2 i x_1}{r} - 2 i x_3 (m + S) f_1(r) + \frac{2 i x'_1}{r} V(r) \right\} + \]
\[ + K \left( \hat{\Sigma} \cdot \hat{r} \right) \left\{ x_2 V'(r) - x_3 f'_1(r) \right\} + \]
\[ + K \beta \left( \hat{\Sigma} \cdot \hat{r} \right) \left\{ x_2 S'(r) - x'_3 f'_2(r) \right\} + \]
\[ + \gamma^5 K \left\{ \frac{2 i x'_1 (m + S)}{r} - 2 i x'_3 (m + S) f_2(r) \right\} + \]
\[ + \beta K \left\{ \frac{2 i x'_1}{r} - 2 i x'_3 f_2(r) \right\} \left( \hat{\Sigma} \cdot \hat{p} \right) \]

Equating this expression to zero, we derive matrix equation, then passing to \( 2 \times 2 \) representa-
tion we must equate to zero the coefficients standing in front of diagonal and antidiagonal
elements. In this way it follows equations:

(i) From antidiagonal structures (\( \gamma^5, \gamma^5 \beta K \))

\[ \frac{x_1}{r} - x_3 (m + S) f_1(r) + \frac{x'_1}{r} V(r) = 0 \] (22)
\[ \frac{x'_1}{r} (m + S) - (m + S) f_2(r) = 0 \]

(ii) From diagonal structures (\( K \left( \hat{\Sigma} \cdot \hat{r} \right), \beta \left( \hat{\Sigma} \cdot \hat{r} \right), \beta K \left( \hat{\Sigma} \cdot \hat{p} \right) \)):

\[ x_2 V'(r) - x_3 f'_1(r) = 0 \] (23)
\[ \frac{x_1}{r} - x_2 (m + S) V(r) + \frac{x'_1}{r} V(r) = 0 \]
\[ x_2 S'(r) - x'_3 f_2'(r) = 0 \]
\[ \frac{x'_1}{r} - x'_3 f_2(r) = 0 \]

Integrating the first and third equations in (23) for vanishing boundary conditions at infinity, we obtain

\[ f_1(r) = \frac{x_2}{x_3} V(r), \quad f_2(r) = \frac{x_2}{x'_3} S(r) \tag{24} \]

and taking into account the last equation from (23), we have

\[ f_2(r) = \frac{x'_1}{x'_3 r} \tag{25} \]

Therefore, according to (24), we obtain finally

\[ S(r) = \frac{x'_1}{x_2 r} \tag{26} \]

**So, the scalar potential must be Coulombic.**

Inserting Eq.(24) into the first equation of (22) and solving for \( V(r) \), one derives

\[ V(r) = \frac{x_1}{r} \frac{1}{x_2 (m + S) - \frac{x'_1}{r}} \tag{27} \]

At last, using here the expression (26), we find

\[ V(r) = \frac{x_1}{x_2 m r} \tag{28} \]

Therefore we have asserted that the \( \pm \kappa \) degeneracy is a symmetry of the Dirac equation only for Coulomb potential (for any general combination of Lorentz scalar and 4th component of a Lorentz vector).

It seems that the conservation of LRL vector is a macroscopic manifestation of symmetry, which is present in microworld.

Further, if we take into account above obtained solutions, one can reduce the \( X \) operator to more compact form

\[ X = (\vec{\Sigma} \cdot \hat{r})(m a_V + a_S H) - i K \gamma^5 (H - \beta m) \tag{29} \]

where the following notations are used

\[ a_V = -\frac{x_1}{x_2 m}, \quad a_S = -\frac{x'_1}{x_2} \]

Here \( a_i \)-s are the constants of corresponding Coulomb potentials

\[ V(r) = -\frac{a_V}{r}, \quad S(r) = -\frac{a_S}{r} \]

In conclusion we want to remark, that this expression for \( X \) was obtained earlier by Leviatan [11], who used the radial decomposition and separation of spherical angles in the Dirac equation.

Our approach is 3-dimensional, without any referring to radial equation and, therefore is more systematic and rather easy.
4 Algebraic Derivation of Spectrum

Now we want to obtain spectrum of the Dirac Hamiltonian pure algebraically, without any referring on equations of motion. Our method is based on Witten’s superalgebra, established in Sec.II above.

Now we explore this algebra in a manner as in paper [7]. One defines a SUSY ground state $|0\rangle$:

$$h|0\rangle = X^2|0\rangle = 0 \quad \rightarrow \quad X|0\rangle = 0 \quad (30)$$

Because $X^2$ is a square of Hermitian operator, it has a positive definite spectrum and one is competent to take zero this operator itself in ground state. By this requirement we’ll obtain Hamiltonian in this ground state and, correspondingly, ground state energy. After that by well known ladder procedure we can construct the energies of all excited levels. We believe that this method requires more strong justification, but nevertheless we are convinced, that it is true.

Let equate $X = 0$ and solve $H$ from Eq. (29):

$$H = m[(\bar{\alpha} \cdot \hat{r})a_S + iK]^{-1}[iK\beta - a_V(\bar{\alpha} \cdot \hat{r})] = \frac{m}{\kappa^2 + a_S^2}N \quad (31)$$

where

$$N \equiv [(\bar{\alpha} \cdot \hat{r})a_S - iK][iK\beta - a_V(\bar{\alpha} \cdot \hat{r})] =$$

$$= -a_Sa_V + K[K\beta + a_V(\bar{\alpha} \cdot \hat{r})] - ia_SK\beta(\bar{\alpha} \cdot \hat{r}) \quad (32)$$

Now we try to diagonalize this operator using Foldy-Wouthuysen [15] like transformation. Because the second and third terms do not commute with each others we need several (at least two) such transformations.

We choose the first transformation in the following manner

$$\exp(iS_1) = \exp\left(-\frac{1}{2}\beta(\bar{\alpha} \cdot \hat{r})w_1\right) \quad (33)$$

It is evident that

$$\exp(iS_1)(\bar{\alpha} \cdot \hat{r}) \exp(-iS_1) = \exp(2iS_1)(\bar{\alpha} \cdot \hat{r}) \quad (34)$$

$$\exp(iS_1)\beta \exp(-iS_1) = \exp(2iS_1)\beta$$

Moreover

$$\exp(iS_1)K \exp(-iS_1) = K, \quad \exp(iS_1)\beta K \exp(-iS_1) = \exp(2iS_1)\beta K \quad (35)$$

and

$$\exp(iS_1)\beta(\bar{\alpha} \cdot \hat{r}) \exp(-iS_1) = \beta(\bar{\alpha} \cdot \hat{r}) \quad (36)$$

Therefore the first transformation acts as

$$N' \equiv \exp(iS_1)N \exp(-iS_1) =$$

$$= -a_Sa_V + K \exp(2iS_1)[K\beta + a_V(\bar{\alpha} \cdot \hat{r})] - ia_SK\beta(\bar{\alpha} \cdot \hat{r}) \quad (37)$$
But \( \exp(2iS_1) = \cosh w_1 + i\beta(\alpha \cdot \hat{r}) \sinh w_1 \). Make use of this relation, we have

\[
\exp(2iS_1)[K\beta + ia_V(\alpha \cdot \hat{r})] = \\
\beta[K \cosh w_1 + a_V \sinh w_1] + K(\alpha \cdot \hat{r})[ia_V \cosh w_1 + iK \sinh w_1]
\]

Now in order to get rid of non-diagonal \((\alpha \cdot \hat{r})\) terms, we must choose

\[
\tanh w_1 = -\frac{a_V}{K}
\]

Using simple trigonometric relations we arrive at

\[
\exp(2iS_1)[K + ia_V(\alpha \cdot \hat{r})] = K^{-1} \beta \sqrt{\kappa^2 - a_V^2}
\]

Let us perform the second F.-W. transformation

\[
N'' = \exp(iS_2)N' \exp(-iS_2), \quad \text{where} \quad S_2 = -\frac{1}{2}(\alpha \cdot \hat{r})w_2
\]

Now

\[
\exp(iS_2)K\beta \exp(-iS_2) = \exp(2iS_2)K\beta
\]

\[
\exp(iS_2)K\beta(\alpha \cdot \hat{r}) \exp(-iS_2) = \exp(2iS_2)K\beta(\alpha \cdot \hat{r})
\]

\[
\exp(2iS_2) = \cos w_2 - i(\alpha \cdot \hat{r}) \sin w_2
\]

Therefore

\[
N'' = -aSa_V + K\sqrt{\kappa^2 - a_V^2} \exp(2iS_2)\beta - \\
-ia_s \exp(2iS_2)K\beta(\alpha \cdot \hat{r})
\]

\[
= -aSa_V + K\sqrt{\kappa^2 - a_V^2} \beta \cos w_2 + iK\sqrt{\kappa^2 - a_V^2} \beta(\alpha \cdot \hat{r}) \sin w_2 - \\
-ia_s K\beta(\alpha \cdot \hat{r}) \cos w_2 + a_s K\beta \sin w_2
\]

Requiring absence of \((\alpha \cdot \hat{r})\) terms we have

\[
\tan w_2 = \frac{a_s}{\sqrt{\kappa^2 - a_V^2}}
\]

Therefore

\[
N'' = -aSa_V + K\beta \sqrt{\kappa^2 - a_V^2 + a_s^2}
\]

and finally

\[
H = \frac{m}{\kappa^2 + a_s^2} \{-aSa_V + K\beta \sqrt{\kappa^2 - a_V^2 + a_s^2}\}
\]

For eigenvalues in ground state we have

\[
E_0 = \frac{m}{\kappa^2 + a_s^2} \left[-aSa_V \pm \kappa \sqrt{\kappa^2 - a_V^2 + a_s^2}\right]
\]

Now let us remember the result obtained by explicit solution of the Dirac equation for this case [16].
\[ E = \frac{-a_S a_V}{a_V^2 + (n - |\kappa| + \gamma)^2} \]

\[ \pm \sqrt{\left(\frac{a_S a_V}{a_V^2 + (n - |\kappa| + \gamma)^2}\right)^2 + \frac{(n - |\kappa| + \gamma)^2 - a_S^2}{a_V^2 + (n - |\kappa| + \gamma)^2}}, \]  

(48)

where

\[ \gamma^2 = \kappa^2 - a_S^2 \]

(49)

In the ground state \( n = 1, j = 1/2 \rightarrow |\kappa| = j + 1/2 = 1 \), there remains

\[ E_0 = m \left[ \frac{-a_S a_V}{a_V^2 + \gamma^2} \pm \sqrt{\left(\frac{a_S a_V}{a_V^2 + \gamma^2}\right)^2 + \gamma^2 - a_S^2} \right], \]

(50)

which after obvious manipulations reduces to our above derived expression (47). Therefore by only algebraic methods we have obtained the correct expression for ground state energy. For obtaining of total spectrum it is sufficient now to use the Witten’s algebra. Following the paper [7], the ordinary ladder procedure consists in change (for our case):

\[ \gamma \rightarrow \gamma + n - |\kappa| \]

Making this, it follows from our lowest energy formula (47) the correct expression for total energy spectrum, Eq. (48).

5 Conclusions

In conclusion, by using of pure algebraic manipulations we obtained the spectrum of generalized Coulomb problem of the Dirac equation for arbitrary combination of Lorentz-scalar and Lorentz-vector potentials. This fact demonstrates a power of symmetry considerations. It is interesting by itself that the requirement of validity of Witten’s algebra, or equivalently, \( S(2) \) supersymmetry of the Dirac Hamiltonian for arbitrary central potentials (scalar or vector) leads to \textit{unique role of the Coulomb potential}.

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References

[1] R.C.O’Connel and K. Jagannathan. Am.J.Phys.71, 243, (2003)
[2] W.Pauli. Z. Phys. 36,336 (1926)
[3] V,Fock. Z.Phys. 98,145 (1935).
[4] A.A.Stahlhofen. Helv.Phys.Acta. 70,372 (1997).
[5] J.P. Dahl, T. Jorgensen. Int. J. Quantum Chemistry. 53, 161 (1995)
[6] M.H. Johnson and B.A. Lippmann. Phys. Rev. 78, 329(A), (1950).
[7] H. Katsura and H. Aoki. J. Math. Phys. 47, 032302 (2006).
[8] T.T. Khachidze and A.A. Khelashvili. Mod. Phys. Lett. A20, 2277 (2005).
[9] T.T. Khachidze and A.A. Khelashvili. Am. J. Phys. 74, 628 (2006).
[10] T.T. Khachidze and A.A. Khelashvili. In Proc. of Cairo Int. Conf. on High Energy Phys. (CICHEP II) p. 279 (2007); ArXiv hep-th/0602181.
[11] A. Leviatan. Phys. Rev. Lett. 92, 202501 (2004).
[12] A. Sommerfeld. Atombau und Spectralinien (Fridr, Braunschweig, 1951), Vol. II
[13] E. Witten. Nucl. Phys. B188, 513 (1981).
[14] R.D. Tangeman and J.A. Tjon. Phys. Rev. A48 1089 (1993).
[15] L. Foldy and S. Wouthuysen. Phys. Rev. 72, 29 (1950).
[16] W. Greiner, B. Muller and J. Rafelski. Quantum Electrodynamics of Strong Fields, Springer-Verlag. Berlin, Heidelberg, New York, Tokyo, 1985, p. 83.