AN EXAMPLE OF A MEASURE ASSOCIATED WITH A PATH
ON $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

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Abstract. With every path on $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ there is associated a measure on $\mathbb{Z}_p$. The group $\mathbb{Z}_p^\times$ acts on measures. We consider two measures. One measure is associated to a path from $0\bar{1}$ to a root of unity $\xi$ of order prime to $p$. Another measure is associated to a path from $0\bar{1}$ to $\xi^{-1}$ and next it is acted by $-1 \in \mathbb{Z}_p^\times$. We show that the sum of these measures can be defined in a very elementary way. Integrating against this sum of measures we get $p$-adic Hurwitz zeta functions constructed previously by Shiratani.

0. Introduction

Let $K$ be a number field, let $z \in \mathbb{P}^1(K) \setminus \{0, 1, \infty\}$ and let $\gamma$ be a path on $\mathbb{P}^1_K \setminus \{0, 1, \infty\}$ from $0\bar{1}$ to $z$, i.e. an isomorphism of the corresponding fiber functors. Let $p$ be a fixed prime number. The Galois group $G_K$ acts on

$$\pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, 0\bar{1})$$

- the pro-$p$ étale fundamental group. Let $\mathbb{Q}_p\{\{X, Y\}\}$ be the $\mathbb{Q}_p$-algebra of non-commutative formal power series in two non-commuting variables $X$ and $Y$. Let

$$E: \pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, 0\bar{1}) \rightarrow \mathbb{Q}_p\{\{X, Y\}\}$$

be the continuous multiplicative embedding given by $E(x) = \exp X$ and $E(y) = \exp Y$, where $x$ and $y$ are standard generators of $\pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, 0\bar{1})$. For any $\sigma \in G_K$ we define

$$f_{\gamma}(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, 0\bar{1})$$

and

$$\Lambda_{\gamma}(\sigma) := E(f_{\gamma}(\sigma)) \in \mathbb{Q}_p\{\{X, Y\}\}.$$
Grothendieck. The coefficients of the power series \( \Lambda_\gamma(\sigma) \) are analogues of the multizeta numbers studied already by Euler. For an arbitrary path \( \gamma \) the coefficients of the power series \( \Lambda_\gamma(\sigma) \) are analogues of values of iterated integrals evaluated at \( z \).

Observe that
\[
\Lambda_\gamma(\sigma) = 1 + l_\gamma(z)(\sigma)X \mod I^2 + (Y)
\]
for a certain \( l_\gamma(z)(\sigma) \in \mathbb{Z}_p \), where \( I \) is the augmentation ideal of \( \mathbb{Q}_p \{ \{X,Y\} \} \) and \((Y)\) is the principal ideal generated by \( Y \). Let us set
\[
\Delta_\gamma(\sigma) := \exp(-l_\gamma(z)(\sigma)X) \cdot \Lambda_\gamma(\sigma) .
\]

One possible way to calculate (some) coefficients of the power series \( \Lambda_\gamma(\sigma) \) and some other power series \( \Lambda_\chi(\sigma) \) is to use symmetries of \( \mathbb{F}_1 \{ \{0,1,\infty\} \} \), i.e. the so called Drinfeld-Ihara relations (see [3] and [5]). For example in [10], we have calculated even polylogarithmic coefficients of the power series \( \Lambda_\gamma(\sigma) \) using the symmetries of \( \mathbb{F}_1 \{ \{0,1,\infty\} \} \).

In [7] the authors have constructed a measure on \( \mathbb{Z}_p \) for any path \( \gamma \) and expressed the \( k \)-th polylogarithmic coefficient of the power series \( \log \Delta_\gamma(\sigma) \) as integrals of the polynomial \( x^{k-1} \) against this measure recovering the old result of O. Gabber (see [2]). Let us denote this measure by \( K(z)_\gamma \).

Now we shall describe the main result of this note. Let \( m \) be a positive integer not divisible by \( p \). Let us set
\[
\xi_m = \exp\left(\frac{2\pi \sqrt{-1}}{m}\right).
\]

Let \( 0 < i < m \). Further we chose paths \( \beta_i \) (resp. \( \beta_{m-i} \)) on \( \mathbb{P}_1 \{ \{0,1,\infty\} \} \) from 0 to \( \xi^i_m \) (resp. \( \xi^{m-i}_m \)) such that \( l_\beta_i(\xi_m^i) = 0 \) and \( l_{\beta_{m-i}}(\xi_m^{m-i}) = 0 \).

In [11] using the symmetry \( \bar{z} \mapsto 1/\bar{z} \) of \( \mathbb{F}_1 \{ \{0,1,\infty\} \} \) we have shown that the polylogarithmic coefficient in degree \( k \) of the formal power series
\[
\log \Lambda_{\beta_{m-i}}(\sigma) + (-1)^k \log \Lambda_{\beta_i}(\sigma)
\]
is equal \( B_k(\chi^i)(1 - \chi^i(k)) \), where \( B_k(X) \) is the \( k \)-th Bernoulli polynomial and \( \chi : G_{\mathbb{Q}(\mu_m)} \rightarrow \mathbb{Z}_p^\times \) is the cyclotomic character (see [11, Theorem 10.2.]). In this paper we shall calculate the same polylogarithmic coefficients using the measure
\[
K(\chi_m^{m-i})_{\beta_{m-i}} + \iota(K(\chi_m^i)_{\beta_i}),
\]
where \( \iota \) is the complex conjugation acting on measures. To calculate these measures we use the symmetry \( \bar{z} \mapsto 1/\bar{z} \) of the tower of coverings
\[
\mathbb{P}_1^1 \{ \{0,\infty\} \cup \mu_{p^n} \} \rightarrow \mathbb{P}_1^1 \{ \{0,1,\infty\} \}, \bar{z} \mapsto \psi^\sigma
\]
of \( \mathbb{P}_1^1 \{ \{0,1,\infty\} \} \). However in contrast with the calculations in [11] we need to work only with terms in degree 1. We show that the measure \( K(\chi_m^{m-i})_{\beta_{m-i}} + \iota(K(\chi_m^i)_{\beta_i}) \) is the sum of the Bernoulli measure \( E_{1,\chi} \) (see [6, the formula E.1 on page 38]) and the measure we denote by \( \mu_{\chi}(\iota^{-1}) \). The definition of the measure \( \mu_{\chi}(\iota^{-1}) \) is very elementary and perhaps it is well known. From this it follows immediately the formula for the \( k \)-th polylogarithmic coefficient of the power series (1). The measure we get, allows to get the \( p \)-adic Hurwitz zeta functions as Mellin transform in the same way as the \( p \)-adic \( L \)-functions are the Mellin transforms of the measure \( \psi E_{1,\psi} \), where \( \psi \) is a character on \( \mathbb{Z}_p^\times \) (see [6, Chapter 4]).
1. An example of a measure on $\mathbb{Z}_p$

This section can be seen as an attempt to construct a measure on $\mathbb{Z}_p$ which to a subset $a + p^n\mathbb{Z}_p$ associates $1/p^n$. We found the measure in question studying Galois actions on torsors of paths (see section 3). The measure is elementary and we think that it should be known.

If $a \in \mathbb{Z}_p$ and $a = \sum_{i=0}^{\infty} \alpha_ip^i$ with $0 \leq \alpha_i \leq p - 1$ then we set

$$v_n(a) := \sum_{i=0}^{n} \alpha_ip^i \quad \text{and} \quad t_{n+1}(a) := \frac{a - v_n(a)}{p^{n+1}}.$$ 

Let us fix a positive integer $m > 1$. For $k \in \mathbb{Q}^\times$, $k = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $(b, m) = 1$ we define

$$[k]_m \in \mathbb{N}$$

by the following two conditions

$$0 \leq [k]_m < m \quad \text{and} \quad b[k]_m \equiv a \mod m.$$ 

Let us assume that $p$ does not divide $m$. Let $i$ be such that $0 < i < m$. Observe that

$$[p^{-r}[ip^{-n}]]_m = [ip^{-(n+r)}]_m.$$ 

We define a sequence of integers

$$(k_r(i))_{r \in \mathbb{N}}$$

by the equalities

$$p[ip^{-r}] = [ip^{-(r-1)}] + k_{r-1}(i)m.$$ 

Observe that

$$0 < \frac{[ip^{-(r-1)}]}{m} < 1 \quad \text{and} \quad 0 < \frac{p[ip^{-r}]}{m} < p.$$ 

Hence it follows that

$$0 \leq k_r(i) \leq p - 1$$

for all $r \geq 0$. Applying successively the formula (3) we get

$$p^n[ip^{-n}] = i + (\sum_{\alpha=0}^{n-1} k_\alpha(i)p^\alpha)m.$$ 

It follows from (4) that

$$\frac{-i}{m} = \sum_{\alpha=0}^{\infty} k_\alpha(i)p^\alpha$$

and

$$\frac{i}{m} = 1 + \sum_{\alpha=0}^{\infty} (p - 1 - k_\alpha(i))p^\alpha.$$ 

Another consequence of (4) is the equality

$$t_{n}\left(\frac{-i}{m}\right) = \frac{-[ip^{-n}]_m}{m}.$$
For any positive integer \(a\) such that \(0 \leq a < p^n\) we set
\[
\delta_n(a) := \begin{cases} 
-1 & \text{if } a \geq 1 + \sum_{\alpha=0}^{n-1}(p - 1 - \kappa_\alpha(i))p^\alpha, \\
0 & \text{if } a < 1 + \sum_{\alpha=0}^{n-1}(p - 1 - \kappa_\alpha(i))p^\alpha.
\end{cases}
\]

**Definition-Proposition 1.1.** The function from the open-closed subsets of \(Z_p\) to \(Z_p\) defined by the formula
\[
\mu\left(\frac{i}{m}\right)(a + p^nZ_p) := \frac{[ip^{-n}]_m}{m} + \delta_n(a)
\]
for \(0 \leq a < p^n\) is a measure.

**Proof.** Let \(0 \leq a < p^n\). We have
\[
\sum_{b=0}^{p-1} \mu\left(\frac{i}{m}\right)(a + bp^n + p^{n+1}Z_p) = \sum_{b=0}^{p-1} \left(\frac{[ip-(n+1)]_m}{m} + \delta_{n+1}(a + bp^n)\right) = \frac{p[ip^{-n+1}]_m}{m} + \sum_{b=0}^{p-1} \delta_{n+1}(a + bp^n) = \frac{[ip^{-n}]_m}{m} + k_n(i) + \sum_{b=0}^{p-1} \delta_{n+1}(a + bp^n)
\]
by the equality (3). Observe that
\[
\sum_{b=0}^{p-1} \delta_{n+1}(a + bp^n) := \begin{cases} 
-k_n(i) - 1 & \text{if } a \geq 1 + \sum_{\alpha=0}^{n-1}(p - 1 - \kappa_\alpha(i))p^\alpha, \\
-k_n(i) & \text{if } a < 1 + \sum_{\alpha=0}^{n-1}(p - 1 - \kappa_\alpha(i))p^\alpha.
\end{cases}
\]
Hence finally we get
\[
\sum_{b=0}^{p-1} \mu\left(\frac{i}{m}\right)(a + bp^n + p^{n+1}Z_p) = \frac{[ip^{-n}]_m}{m} + \delta_n(a) = \mu\left(\frac{i}{m}\right)(a + p^nZ_p).
\]

**Proposition 1.2.** For \(k \geq 1\) we have
\[i) \int_{Z_p} x^{k-1}d\mu\left(\frac{i}{m}\right)(x) = \frac{1}{k}(B_k\left(\frac{i}{m}\right) - B_k),\]
\[ii) \int_{Z_p} x^{k-1}d\mu\left(\frac{i}{m}\right)(x) = \frac{1}{k}(B_k\left(\frac{i}{m}\right) - B_k) - \frac{p^{k-1}}{k}(B_k\left(\frac{[ip^{-n}]_m}{m}\right) - B_k).
\]

**Proof.** First we shall prove the formula i). Let us calculate the Riemann sum
\[
\sum_{\alpha=0}^{p^n-1} \alpha^{k-1} \mu\left(\frac{i}{m}\right)(\alpha + p^nZ_p) = \sum_{\alpha=0}^{p^n-1} \alpha^{k-1} \left(\frac{[ip^{-n}]_m}{m} + \delta_n(\alpha)\right) = \frac{[ip^{-n}]_m}{m} \sum_{\alpha=0}^{p^n-1} \alpha^{k-1} - \sum_{\alpha=0}^{p^n-1} \alpha^{k-1} + \sum_{\alpha=0}^{p^n-1} \alpha^{k-1}.
\]
Observe that
\[
\sum_{\alpha=0}^{p^n-1} \alpha^{k-1} = \frac{1}{k}(B_k\left(\frac{[ip^{-n}]_m}{m}\right) - B_k)
\]
and it tends to \(\frac{1}{k}(B_k\left(\frac{[ip^{-n}]_m}{m}\right) - B_k)\) if \(n\) tends to \(\infty\). Hence the formula i) of the proposition follows because \(\sum_{\alpha=0}^{p^n-1} \alpha^{k-1}\) tends to 0 if \(n\) tends to \(\infty\).
Observe that
\[
\int_{\mathbb{Z}_p^\times} x^{-1} d\mu\left(\frac{i}{m}\right)(x) = \int_{\mathbb{Z}_p^\times} x^{-1} d\mu\left(\frac{i}{m}\right)(x) - \int_{\mathbb{Z}_p^\times} x^{-1} d\mu\left(\frac{i}{m}\right)(x).
\]

We shall calculate Riemann sums for the integral \( \int_{\mathbb{Z}_p^\times} x^{-1} d\mu\left(\frac{i}{m}\right)(x) \). We have
\[
\sum_{\alpha=0}^{p^n-1} (p\alpha)^{k-1} \mu(p\alpha + p^{n+1}\mathbb{Z}_p) = \sum_{\alpha=0}^{p^n-1} p^k \alpha^{k-1} \left[ \frac{ip^{n-1}}{m} \right] + \sum_{\alpha=0}^{p^n-1} p^k \alpha^{k-1} \delta_{n+1}(p\alpha).
\]

The first sum tends to 0 if \( n \) tends to \( \infty \). Observe that
\[
\sum_{\alpha=0}^{p^n-1} p^k \alpha^{k-1} \delta_{n+1}(p\alpha) = \sum_{0<\alpha<p^n, \alpha \geq v_n\left(\frac{i}{m}\right)} p^k \alpha^{k-1} (-1) = -\sum_{\alpha=0}^{p^n-1} p^k \alpha^{k-1} + \sum_{0<\alpha<p^n, \alpha < v_n\left(\frac{i}{m}\right)} p^k \alpha^{k-1}.
\]

Let \( 0 \leq \beta_0 < p \) be such that \( v_n\left(\frac{i}{m}\right) \equiv \beta_0 \) modulo \( p \). Then
\[
v^{-1}_n\left(\left[\frac{ip^{n-1}}{m}\right]\right) = \begin{cases} 1 + \frac{1}{p} (v_n\left(\frac{i}{m}\right) - \beta_0) & \text{if } \beta_0 \neq 0, \\ \frac{1}{p} v_n\left(\frac{i}{m}\right) & \text{if } \beta_0 = 0. \end{cases}
\]

Hence it follows that
\[
\sum_{0<\alpha<p^n, \alpha < v_n\left(\frac{i}{m}\right)} p^k \alpha^{k-1} = p^{k-1} \sum_{\alpha=0}^{v_{n-1}\left(\left[\frac{ip^{n-1}}{m}\right]\right)-1} \alpha^{k-1}.
\]

If \( n \) tends to \( \infty \) the last sum tends to \( p^{k-1} \frac{1}{k} (B_k(\frac{ip^{n-1}}{m}) - B_k) \). Hence the proof of the formula ii) is finished. \( \square \)

If \( c \in \mathbb{Z}_p^\times \setminus \mu_{p-1} \) we define
\[
\mu_c\left(\frac{i}{m}\right) := \mu\left(\frac{i}{m}\right) - c\mu\left(\frac{i}{m}\right) \circ c^{-1}.
\]

Then we have
\[
\frac{1}{1-c^k} \int_{\mathbb{Z}_p^\times} x^{-1} d\mu_c\left(\frac{i}{m}\right)(x) = \frac{1}{k} (B_k\left(\frac{i}{m}\right) - B_k).
\]

\textbf{Corollary 1.3.} Let \( P : \mathbb{Z}_p[[\mathbb{Z}_p]] \rightarrow \mathbb{Z}_p[[T]] \) be the Iwasawa isomorphism given by \( P(1) = 1 + T \). Then
\[
P(\mu\left(\frac{i}{m}\right))(T) = \frac{(1+T)^{\frac{1}{c}} - 1}{T}
\]

and
\[
P(\mu_c\left(\frac{i}{m}\right)) = \frac{(1+T)^{\frac{1}{c}} - 1}{T} - \frac{c((1+T)^{\frac{1}{c}} - 1)}{(1+T)^c - 1}.
\]

\textbf{Proof.} The power series \( P(\mu\left(\frac{i}{m}\right))(\exp X - 1) \) is equal to \( \sum_{k=0}^{\infty} \left( \int_{\mathbb{Z}_p} x^k d\mu\left(\frac{i}{m}\right)(x) \right) X^k \). Hence by the point i) of Proposition 1.2 it is equal
\[
\sum_{k=0}^{\infty} \frac{1}{(k+1)!} (B_{k+1}\left(\frac{i}{m}\right) - B_{k+1}) X^k.
\]
It follows from the definition of the Bernoulli numbers and the Bernoulli polynomials that this power series is equal \( \exp \frac{X}{\exp 1 - X} \). Replacing \( \exp X \) by \( 1 + T \) we get the power series \( P(\mu(\frac{i}{m}))(T) \).

We denote by \( \omega : \mathbb{Z}_p^\times \to \mu_{p-1} \subset \mathbb{Z}_p^\times \) the Teichmüller character. For \( x \in \mathbb{Z}_p^\times \) we set \( [x] := x\omega(x)^{-1} \).

Let us define

\[
\tilde{H}_p(1 - s, \omega^b, \frac{i}{m}) := \int_{\mathbb{Z}_p^\times} [x]^s x^{-1} \omega(x)^b d\mu(\frac{i}{m})(x).
\]

**Proposition 1.4.** Let \( k \equiv b \pmod{p-1} \). Then

\[
\tilde{H}_p(1 - k, \omega^b, \frac{i}{m}) = \frac{1}{k} (B_k(\frac{i}{m}) - B_k) - \frac{p^{k-1}}{k} (B_k(\frac{[ip-1]m}{m}) - B_k).
\]

**Proof.** We have

\[
\tilde{H}_p(1 - k, \omega^b, \frac{i}{m}) = \int_{\mathbb{Z}_p^\times} [x]^k x^{-1} \omega(x)^b d\mu(\frac{i}{m})(x) = \int_{\mathbb{Z}_p^\times} x^{k-1} d\mu(\frac{i}{m})(x).
\]

Hence the proposition follows from the formula ii) of Proposition 1.2. \( \square \)

**Remark 1.5.** A function closely related to our function \( \tilde{H}_p(1 - s, \omega^b, \frac{i}{m}) \) appears in a paper of Shiratani (see [9, Theorem 1, case \( p \nmid f \)]).

2. **Action of the complex conjugation on measures**

We define an action of \( \mathbb{Z}_p^\times \) on the group ring \( \mathbb{Z}_p[\mathbb{Z}_p] \) by the formula

\[
\alpha(\sum_{i=1}^n a_i(x_i)) = \alpha \sum_{i=1}^n a_i(\alpha^{-1} x_i)
\]

and we extend by continuity to the action of \( \mathbb{Z}_p^\times \) on \( \mathbb{Z}_p[[\mathbb{Z}_p]] \). The action of \(-1 \in \mathbb{Z}_p^\times\) we denote by \( \iota \). Then

\[
\mathbb{Z}_p[[\mathbb{Z}_p]] = \mathbb{Z}_p[[\mathbb{Z}_p]]^+ \oplus \mathbb{Z}_p[[\mathbb{Z}_p]]^-,
\]

where \( \iota \) acts on \( \mathbb{Z}_p[[\mathbb{Z}_p]]^+ \) (resp. on \( \mathbb{Z}_p[[\mathbb{Z}_p]]^- \)) as the identity (resp. as the multiplication by \(-1\)). For any \( \mu \in \mathbb{Z}_p[[\mathbb{Z}_p]] \) we have the decomposition

\[
\mu = \mu^+ + \mu^-,
\]

where \( \mu^+ = \frac{1}{2}(\mu + \iota(\mu)) \in \mathbb{Z}_p[[\mathbb{Z}_p]]^+ \) and \( \mu^- = \frac{1}{2}(\mu - \iota(\mu)) \in \mathbb{Z}_p[[\mathbb{Z}_p]]^- \). Observe that

\[
\int_{\mathbb{Z}_p} x^{k-1} d\mu = (-1)^k \int_{\mathbb{Z}_p} x^{k-1} d\mu.
\]

Hence it follows

\[
\int_{\mathbb{Z}_p} x^{k-1} d\mu^+ := \begin{cases} 0 & \text{for } k \text{ odd}, \vspace{0.5em} \\
\int_{\mathbb{Z}_p} x^{k-1} d\mu & \text{for } k \text{ even} \end{cases}
\]
and
\[ (10) \quad \int_{\mathbb{Z}_p} x^{k-1} d\mu^- := \begin{cases} \int_{\mathbb{Z}_p} x^{k-1} d\mu & \text{for } k \text{ odd,} \\ 0 & \text{for } k \text{ even.} \end{cases} \]

In [11, Proposition 10.5] we have shown that
\[ (11) \quad \int_{\mathbb{Z}_p} x^{k-1} d(K(\xi_{m}^{-i}) + K(\xi_{m}^{i})) = \frac{1}{k} B_k\left(\frac{i}{m}\right)(1 - \chi^k) \quad \text{for } k \text{ even} \]
and
\[ (12) \quad \int_{\mathbb{Z}_p} x^{k-1} d(K(\xi_{m}^{-i}) - K(\xi_{m}^{i})) = \frac{1}{k} B_k\left(\frac{i}{m}\right)(1 - \chi^k) \quad \text{for } k \text{ odd.} \]

Hence it follows from (9) and (10) that
\[ (13) \quad \int_{\mathbb{Z}_p} x^{k-1} d\left( (K(\xi_{m}^{i}) + K(\xi_{m}^{i}))^+ + (K(\xi_{m}^{i}) - K(\xi_{m}^{i}))^- \right) = \frac{1}{k} B_k\left(\frac{i}{m}\right)(1 - \chi^k) \quad \text{for } k \geq 1. \]

Observe that
\[ (K(\xi_{m}^{-i}) + K(\xi_{m}^{i}))^+ + (K(\xi_{m}^{i}) - K(\xi_{m}^{i}))^- = K(\xi_{m}^{i}) + \iota(K(\xi_{m}^{i})). \]

Hence we get
\[ (14) \quad \int_{\mathbb{Z}_p} x^{k-1} d(K(\xi_{m}^{i}) + \iota(K(\xi_{m}^{i}))) = \frac{1}{k} B_k\left(\frac{i}{m}\right)(1 - \chi^k) \quad \text{for } k \geq 1. \]

The proof of the formulas (11) and (12) given in [11] is based on the symmetry \( z \mapsto 1/z \) of \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) and the study of the polylogarithmic coefficients (at \( YX^{k-1} \)) of the power series \( \Lambda_\beta(\sigma) \) and \( \Lambda_{\beta - i}(\sigma) \).

In this paper we calculate explicitly the measure \( K(\xi_{m}^{i}) + \iota(K(\xi_{m}^{i})) \). We use also the symmetry \( z \mapsto 1/z \) of the tower of coverings
\[ \mathbb{P}^1 \setminus ((0, \infty) \cup \mu_{p^n}) \rightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}, \quad z \mapsto z^{p^n} \]
but only in degree 1.

The third possible method to calculate the measure \( K(\xi_{m}^{i}) + \iota(K(\xi_{m}^{i})) \) is to use the explicit formula for measures \( K(z) \) (see [7, Proposition 3]). Compare the three different proofs of Proposition 5.13 in [8]. Two proofs are given in [8] and the third one in [11] (the second proof of Lemma 4.1.)

3. MEASURES ASSOCIATED WITH ROOTS OF UNITY

We set
\[ \xi_r := \exp\left(\frac{2\pi\sqrt{-1}}{r}\right) \]
for a natural number \( r \). Let us set
\[ V_n := \mathbb{P}_Q^1 \setminus ((0, \infty) \cup \mu_{p^n}). \]
We recall that \( \pi_1(V_n, 0) \) - pro- \( p \) étalé fundamental group - is free on generators \( x_n \) - loop around 0 - and \( y_{n,i} \) - loops around \( \xi_{p^n}^i \) for \( 0 \leq i < p^n \).
For each $0 < i < m$, let $\alpha_i$ be a path on $V_0 = \mathbb{P}_\mathbb{Q}^1 \setminus \{0, 1, \infty\}$ from $01$ to $\xi^i_m$ which is the composition of an arc from $01$ to $0\xi^i_m$ in an infinitesimal neighbourhood of $0$ followed by the canonical path (straight line) from $0\xi^i_m$ to $\xi^i_m$.

Let us set

$$\beta_i := \alpha_i \cdot x^{-\frac{p}{m}}.$$  

Observe that $l(\xi^i_m)\beta_i = 0$. If we regard the path $\alpha_i$ as the path on $V$ then we denote it by $n\alpha_i$. Then

$$n\beta_i := n\alpha_i \cdot x^{-\frac{p}{m}}$$

is also a path on $V_n$. Let $\tilde{\beta}^n_i$ (resp. $\tilde{\alpha}^n_i$) be the lifting of $\beta_i$ (resp. $\alpha_i$) to $V_n$ starting from $01$. Let $0 \leq j < p^n$. We denote by $s^n_j$ a lifting of $x^n_j$ to $V_n$ starting from $01$. Observe that $s^n_j$ is a path on $V_n$ from $01$ to $0\xi^n_{p^n}$.

**Lemma 3.1.** We have

$$\tilde{\beta}^n_i = n\beta_{[ip^{-n}]} = n\alpha_{[ip^{-n}]} \cdot x^{-\frac{[ip^{-n}]}{m}}.$$  

**Proof.** Observe that the lifting of $x^{-\frac{p}{m}}$ to $V_n$ is equal $s^{v_{n-1}(\frac{-1}{m}) \cdot x^{n_t}_(\frac{-1}{m})}$. The lifting of $\alpha_i$ to $V_n$ is a path (an arc) from $01$ to $\xi^n_{p^n}$ in the positive sense composed with the canonical path from $\xi^n_{p^n}$ to $\xi^n_{p^n \cdot p^n}$ in the positive sense. Hence the lifting of $\beta_i$ is the composition of $s^{v_{n-1}(\frac{-1}{m}) \cdot x^{n_t}_(\frac{-1}{m})}$ with the lifting of $\alpha_i$ multiplied by $\xi^{v_{n-1}(\frac{-1}{m})}$. We have

$$\xi^{v_{n-1}(\frac{-1}{m})} \xi^n_{p^n \cdot p^n} = \xi^{m \cdot v_{n-1}(\frac{-1}{m}) + i}.$$  

Observe that $0 \leq v_{n-1}(\frac{-1}{m}) \cdot m + i < p^n m$ and that $p^n$ divides $v_{n-1}(\frac{-1}{m}) \cdot m + i$. Moreover we have $\frac{v_{n-1}(\frac{-1}{m}) \cdot m + i}{p^n} \cdot p^n \equiv i$ modulo $m$. Hence it follows that

$$\frac{v_{n-1}(\frac{-1}{m}) \cdot m + i}{p^n} = [ip^{-n}]_m.$$  

Therefore we get

$$-\frac{[ip^{-n}]}{m} = -\frac{1}{p^n}(v_{n-1}(\frac{-1}{m}) + \frac{i}{m}) = t_n(\frac{-1}{m}).$$  

Hence it follows that the lifting of $\beta_i$ is $n\alpha_{[ip^{-n}]} \cdot x^{-\frac{[ip^{-n}]}{m}}$.  

To simplify the notation we set

$$r_n = [ip^{-n}]_m$$

and $v_{n-1} = v_{n-1}(\frac{-1}{m})$. Then we have

$$\tilde{\beta}^n_i = n\alpha_{r_n} \cdot x^{\frac{r_n}{m}}$$

and $\tilde{\beta}^n_{m-i} = n\alpha_{m-r_n} \cdot x^{\frac{r_n}{m}-1}$.
Let $h : V_n \to V_0$ be given $\frac{1}{2} \to 1/3$. Let $p_n$ be the canonical path from $\vec{1}$ to $\vec{10}$ on $V_n$, $t_n$ a path from $10$ to $1\infty$ (half circle in the positive sense in an infinitesimal neighbourhood of $1$) and $q_n = h(p_n)$. We set
\[
\Gamma_n := q_n \cdot t_n \cdot p_n .
\]

**Lemma 3.2.** We have
\[
\hat{\beta}_{m-i}^n = h(\beta_{m-i}^n) \cdot \Gamma_n \cdot z_n^{-1} \cdot x_n \cdot y_{n-1} \cdot \ldots \cdot y_{n-v_{n-1}} \cdot x_n^{-1}
\]
in $\pi_1(V_n, 01)$.

**Proof.** One checks that $n \alpha_{m-r_n} = h(n \alpha_{r_n}) \cdot \Gamma_n \cdot x_n \cdot y_{n-1} \cdot \ldots \cdot y_{n-v_{n-1}}$. The formula of the lemma follows from Lemma 3.1.

**Lemma 3.3.** Let $\sigma \in G_{\mathbb{Q}(\mu_m)}$. Then writing additively we have
\[
\int_{\gamma_n} (\sigma) = \sum_{k=0}^{p^n-1} E_{1, \chi(\sigma)}^{(n)}(k)y_{n,k} \mod \left(\pi_1(V_n, 01), \pi_1(V_n, 01)\right) .
\]

**Proof.** See the proof of Lemma 4.1 in [11] or the second proof of Proposition 5.13 in [8].

It follows from Lemma 3.2 that
\[
\int_{\beta_{m-i}^n} (\sigma) = \Gamma_n^{-1} h(\int_{\beta_{m-i}^n}(\sigma)) \cdot \Gamma_n \cdot \int_{\gamma_n}(\sigma) .
\]

\[
(z_n^{-1} \cdot x_n \cdot y_{n-1} \cdot \ldots \cdot y_{n-v_{n-1}} \cdot x_n^{-1})^{-1} \cdot \sigma(z_n^{-1} \cdot x_n \cdot y_{n-1} \cdot \ldots \cdot y_{n-v_{n-1}} \cdot x_n^{-1})
\]
modulo $(\pi_1(V_n, 01), \pi_1(V_n, 01))$. Hence writing the result additively we get
\[
\sum_{k=0}^{p^n-1} K^{(n)}(\xi_{m}^{-1})(\sigma)(k)y_{n,k} = \sum_{k=0}^{p^n-1} K^{(n)}(\xi_{m}^{i})(\sigma)(k)y_{n,-k} + \sum_{k=0}^{p^n-1} E_{1, \chi(\sigma)}^{(n)}(k)y_{n,k} +
\]
\[
\sum_{k=0}^{p^n-1} (1 - \chi(\sigma)) \frac{[ip^{-n}]m}{m} y_{n,k} - \sum_{j=1}^{v_{n-1}(-1)} y_{n,-j} + \sum_{j=1}^{v_{n-1}(\frac{j\chi(\sigma)}{m})} y_{n,[j\chi(\sigma)]_m}
\]
modulo $(\pi_1(V_n, 01), \pi_1(V_n, 01))$. Observe that $v_{n-1}(\frac{1}{m}) = p^n - v_{n-1}(\frac{1}{m})$, Hence the last two sums we can rewrite in the form
\[
\sum_{j=v_{n-1}(\frac{1}{m})}^{p^n-1} y_{n,j} + \chi(\sigma) \sum_{j=v_{n-1}(\frac{1}{m})}^{p^n-1} y_{n,[j\chi(\sigma)]_m} .
\]

Comparing coefficients at $y_{n,k}$ we get for $0 \leq k < p^n$
\[
K^{(n)}(\xi_{m}^{-1})(\sigma)(k) - K^{(n)}(\xi_{m}^{i})(\sigma)(-k) =
\]
\[
E_{1, \chi(\sigma)}^{(n)}(k) + \frac{[ip^{-n}]m}{m} + \delta(\sigma) \frac{[ip^{-n}]m}{m} + \chi(\sigma) \delta(\sigma)[\chi(\sigma)^{-1}k]_{p^n} =
\]
\[
E_{1, \chi(\sigma)}^{(n)}(k) + \mu_{\chi(\sigma)}(\frac{i}{m})(k)
\]
by the definition of the measure $\mu_{\chi(\sigma)}(\frac{i}{m})$. 
Theorem 3.5. Let $m$ be a positive integer not divisible by $p$ and let $0 < i < m$. Then we have
\[ K(\xi_m^{-i})(\sigma) + \iota(K(\xi_m^i)(\sigma)) = E_{1,\chi(\sigma)} + \mu_{\chi(\sigma)}\left(\frac{i}{m}\right). \]

Proof. The theorem follows from the formula (15). \qed

Corollary 3.6. Let $\sigma \in G_{Q(\mu_m)}$ be such that $\chi(\sigma)^{p-1} \neq 1$. Then we have

i) \[ \frac{1}{1 - \chi(\sigma)^{k}} \int_{Z_p} x^{k-1} d(K(\xi_m^{-i})(\sigma) + \iota(K(\xi_m^i)(\sigma))) = \frac{B_k(i/m)}{k}. \]

ii) \[ P(K(\xi_m^{-i})(\sigma) + \iota(K(\xi_m^i)(\sigma)))(T) = \frac{(1 + T)^{\chi(\sigma)^{k}}}{1 + T} \frac{\chi(\sigma)^{k}(1 + T)^{\chi(\sigma)^{k}}}{1 + T} = 1. \]

Proof. The point i) of the corollary follows from Theorem 3.5 and the formula (7). The point ii) follows immediately from Corollary 1.8 and the equality $P(E_{1,\chi(\sigma)}(T)) = \frac{1}{T} - \frac{\chi(\sigma)^{k}}{(1 + T)^{\chi(\sigma)^{k}} - 1}$. \qed

Now we define
\[ L^\beta(1 - s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) := \frac{1}{1 - \omega(\chi(\sigma)^{\beta})} \int_{Z_p} [x]^s x^{-1} \omega(x)^{\beta} d((K(\xi_m^{-i}) + \iota(K(\xi_m^i)))(\sigma)). \]

Theorem 3.7. Let $\sigma \in G_{Q(\mu_m)}$ be such that $\chi(\sigma)^{p-1} \neq 1$. i) Let $k \equiv \beta$ modulo $(p - 1)$. Then
\[ L^\beta(1 - k, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) = \frac{1}{k} B_k(i/m) - p^{k-1} \frac{1}{k} B_k([p^{-1}m]/m). \]

ii) Let $\sigma, \sigma_1 \in G_{Q(\mu_m)}$ be such that $\chi(\sigma)^{p-1} \neq 1$ and $\chi(\sigma_1)^{p-1} \neq 1$. Then
\[ L^\beta(1 - s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) = L^\beta(1 - s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma_1), \]

i.e. the function $L^\beta(1 - s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma)$ does not depend on $\sigma$.

Proof. For $k \equiv \beta$ modulo $p - 1$ we have
\[ L^\beta(1 - k, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) = \frac{1}{1 - \chi(\sigma)^{k}} \int_{Z_p} x^{k-1} d(\mu_{\chi(\sigma)}(i/m) + E_{1,\chi(\sigma)}), \]

by Theorem 3.5. It follows from [6, Theorem 2.3] that
\[ \int_{Z_p} x^{k-1} dE_{1,\chi(\sigma)}(i) = E_{1,\chi(\sigma)}^{(n)}(\mu(i)) \]

of the measure $E_{1,\chi(\sigma)}$ implies that
\[ \frac{1}{1 - \chi(\sigma)^{k}} \int_{Z_p} x^{k-1} dE_{1,\chi(\sigma)} = (1 - p^{k-1}) \frac{1}{k} B_k. \]

Integrating the function $x^{k-1}$ against the measure $\mu_{\chi(\sigma)}(i/m)$ we get
\[ \frac{1}{\chi(\sigma)^{k}} \int_{Z_p} x^{k-1} d\mu_{\chi(\sigma)}(i/m)(x) = \]
\[
\frac{1}{\chi(\sigma)^k - 1} \left( \int_{\mathbb{Z}_p^x} x^{k-1}d\mu\left(\frac{i}{m}\right)(x) - \int_{\mathbb{Z}_p^x} x^{k-1}d(\chi(\sigma)\mu(\frac{i}{m}) \circ \chi(\sigma)^{-1})(x) \right).
\]

Observe that \( \int_{\mathbb{Z}_p^x} x^{k-1}d(\chi(\sigma)\mu(\frac{i}{m}) \circ \chi(\sigma)^{-1})(x) = \chi(\sigma)^k \int_{\mathbb{Z}_p^y} y^{k-1}d\mu(\frac{i}{m})(y) \) if we set \( \chi(\sigma)y = x \). It follows from Proposition 1.9 that

\[
\frac{1}{\chi(\sigma)^k - 1} \int_{\mathbb{Z}_p^x} x^{k-1}d\mu(\chi(\sigma))\left(\frac{i}{m}\right) = \frac{1}{k} \left( B_k\left(\frac{i}{m}\right) - B_k \right) - p^{k-1} \frac{1}{k} \left( B_k\left(\frac{[ip-1]m}{m}\right) - B_k \right).
\]

After the addition of (16) and (17) we get the point i) of the theorem.

Concerning the point ii) observe that the functions \( L^\beta(1-s, (\xi_m^{-i}) + i(\xi_m^i); \sigma) \) and \( L^\beta(1-s, (\xi_m^{-i}) + i(\xi_m^i); \sigma_1) \) coincide for \( k \equiv \beta \) modulo \( (p-1) \). Hence these functions are equal because they are equal on a dense subset of \( \mathbb{Z}_p \). \( \square \)

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