The Cornell confining potential from spontaneous breaking of scale symmetry

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Abstract

We show that one can obtain naturally the Cornell confining potential from the spontaneous symmetry breaking of scale invariance in gauge theory. At the classical level a confining force is obtained and at the quantum level, using a gauge invariant but path-dependent variables formalism, the Cornell confining potential is explicitly obtained.

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I. INTRODUCTION

The question of confinement in gauge theories has been approached with the use of many different techniques and ideas, like lattice gauge theory techniques \cite{1} and non-perturbative solutions of Schwinger-Dyson’s equations \cite{2}. All these approaches have the goal of proving the existence of a linear potential between static quark sources.

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The study of the spectrum of heavy quark-antiquark systems is very well understood. However, as is well known, the binding energy of an infinitely heavy quark-antiquark pair represents a fundamental concept in QCD which is expected to play an important role in the understanding of quark confinement. In this respect we recall that the famous "Cornell potential" [3] was postulated in order to simulate the features of QCD, that is,

$$V = - \frac{\kappa}{r} + \frac{r}{a^2},$$  \hspace{1cm} (1)

here $a$ is a constant with the dimensions of length.

It is worthwhile remarking at this point that the appearance of the scale $a$ in the Cornell potential (1) is very important. One should take notice that the original gauge field theory does not have any scales. Furthermore gauge theories with no scale have a symmetry which is associated to this, scale invariance. Thus it follows that the confinement phenomena breaks the scale invariance as the Cornell potential (1) explicitly shows by introducing the scale $a$.

In this paper we will investigate the connection between scale symmetry breaking and confinement. In particular we will show the appearance of the Cornell potential (1) after spontaneous breaking of scale invariance in a specific model [4]. The quark-antiquark potential is then calculated using the gauge invariant variables formalism [5].

We also draw attention to the fact that the scale invariant model studied [4] introduces, in addition to the standard gauge fields also maximal rank gauge field strengths of four indices in four dimensions, $F_{\mu\nu\alpha\beta} = \partial_{[\mu} A_{\nu\alpha\beta]}$ where $A_{\nu\alpha\beta}$ is a three index potential. The integration of the equations of motion of the $A_{\nu\alpha\beta}$ field introduces a constant of integration $M$ which breaks the scale invariance. As we will see, the linear term in the Cornell potential arises from the constant of integration $M$. When $M = 0$ the equations of motion reduce to those of the standard gauge field theory.

A short note on the history of these kind of models and this way of breaking scale invariance is in order here. This technique for breaking scale invariance was used first in generally covariant theories containing a dilaton field in Refs. [6,7], in the context of a general
type of models which were studied (in non scale invariant form) before [8]. This approach
has also been used to dynamically generate the tension of strings and branes [9]. In Refs.
[6–9] the maximal rank gauge field strength derives from a potential which is composite out
of $D$-scalars.

In order to calculate the potential energy between a quark-antiquark pair we will use the
gauge-invariant but path-dependent variables formalism [5]. Here the quark-antiquark state
is made gauge invariant by the introduction of a gauge field cloud which is basically the
path-ordered exponential of the gauge field potential along the path where the two charges
are located. This methodology has been used previously in many examples for studying
features of screening and confinement in gauge theories [10–12].

II. SCALE INVARIANCE BREAKING AND GENERATION OF CONFINEMENT

We will study the scale symmetry breaking in the context of an Abelian theory. The
non-Abelian generalization presents no problems [5].

Our starting point is the well known action

$$S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right),$$

(2)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This theory is invariant under the scale symmetry

$$A_\mu (x) \mapsto A'_\mu (x) = \lambda A_\mu (\lambda x),$$

(3)

here $\lambda$ is a constant.

Let us now rewrite (2) with the use of an auxiliary field $\omega$

$$S = \int d^4x \left( -\frac{1}{4} \omega^2 + \frac{1}{2} \omega \sqrt{-F_{\mu\nu} F^{\mu\nu}} \right).$$

(4)

From the equation of the $\omega$ field we get

$$\omega = \sqrt{-F_{\mu\nu} F^{\mu\nu}},$$

(5)
and replacing (5) back into (4) we get then (2). Substituting (5) in (4) is a valid operation because (5) is a constraint equation. Under a scale transformation $\omega$ transforms as

$$\omega \mapsto \lambda^2 \omega (\lambda x).$$  \hspace{1cm} (6)

Let us now introduce a charge in the theory (4): we will now keep the form (4) but now $\omega$ will not be an elementary field, rather $\omega$ will be given by

$$\omega = \varepsilon^{\mu\nu\alpha\beta} \partial_{[\mu} A_{\nu\alpha\beta]}. \hspace{1cm} (7)$$

Notice that we have introduced a new degree of freedom, the three index potential and it generates the 4-index field strength $F_{\mu\nu\alpha\beta} \equiv \partial_{[\mu} A_{\nu\alpha\beta]}$, a ”maximal rank” (of 4-indices in 4-dimensions) field strength. In that case the equation of motion of $A_{\nu\alpha\beta}$ is

$$\varepsilon^{\gamma\delta\alpha\beta} \partial_\beta \left( \omega - \sqrt{-F_{\mu\nu} F_{\mu\nu}} \right) = 0, \hspace{1cm} (8)$$

which is integrated to give

$$\omega = \sqrt{-F_{\mu\nu} F_{\mu\nu}} + M. \hspace{1cm} (9)$$

The integration constant $M$ spontaneously breaks the scale invariance, since both $\omega$ and $\sqrt{-F_{\mu\nu} F_{\mu\nu}}$ transform as in Eq.(6) but $M$ does not transform. Notice that $M$ has the same dimensions as the field strength $F_{\mu\nu}$, that is, dimensions of $(length)^{-2}$. We further observe that the variation of the $A_\mu$ field produces the following equation

$$\frac{\partial}{\partial x^\mu} \left( \omega \frac{F_{\mu\nu}}{\sqrt{-F_{\alpha\beta} F_{\alpha\beta}}} \right) = \frac{\partial}{\partial x^\mu} \left[ \left( \sqrt{-F_{\alpha\beta} F_{\alpha\beta}} + M \right) \frac{F_{\mu\nu}}{\sqrt{-F_{\alpha\beta} F_{\alpha\beta}}} \right] = 0, \hspace{1cm} (10)$$

as we will see in the next section, the introduction of the unusual $M$ term leads to the generation of confinement. One may suspect this because the consideration of the $M$ term alone is known to lead to such behavior. In that case the equations of motion are obtained from an action of the form

$$S = k \int d^4 x \sqrt{-F_{\mu\nu} F_{\mu\nu}}, \hspace{1cm} (11)$$
where $k$ is a constant. Such model leads to confinement, as shown in Refs. [13,14], and to string solutions. Among other properties it is known that electric monopoles do not exist [14]. We will see however that the consideration of the two terms in (10) leads to a richer structure in particular to solutions containing Coulomb and linear parts, as in the Cornell potential.

III. CLASSICAL SOLUTIONS AND EFFECTIVE ACTIONS

In order to illustrate the discussion, we now study the equation (10) for the case of a spherically symmetric electric field $F_{0i} = -E_i$ and $F_{ij} = 0$, where $E = E(r)\hat{r}$. Then (10) gives

$$\nabla \cdot \left( E + \frac{M}{\sqrt{2}} \hat{r} \right) = 0,$$

which is solved by

$$E = -\frac{M}{\sqrt{2}} \hat{r} + \frac{q}{r^2} \hat{r}.$$

The scalar potential $V$ that gives rise to such electric field is

$$V = -\frac{M}{\sqrt{2}} r + \frac{q}{r},$$

which is indeed resembles very much the Cornell potential (1). Notice that so far (14) refers to the field of one charge and not yet to the interaction energy between two charges. We will see that such interaction energy also has the Cornell form, even at the quantum level. Since Abelian solutions are solutions of the non-Abelian theory, these solutions are also relevant for the non-Abelian generalization.

Before approaching the quantum theory (which will be treated in some approximations) we want to define effective actions that give the equations of motion (10). Indeed one can easily see that

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{M}{4} \sqrt{-F_{\mu\nu} F^{\mu\nu}},$$

(15)
reproduces Eqs. (10).

Since the full treatment of the quantum theory is rather difficult, instead of using (15) we restrict ourselves to a "truncated" phase space model where we consider spherical coordinates \((r, \theta, \varphi)\) in addition to time, but where we set \(F_{ij} = 0 = F_{0r} = F_{0\theta}\) and consider only \((t, r)\) dependence of \(F_{0r}\). Then instead of (15), we consider

\[
S = 4\pi \int drr^2 \mathcal{L}_{\text{eff}},
\]

where

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} (F_{0r})^2 - \frac{M \sqrt{2}}{4} F_{0r}.
\]

Similar kind of "reduced phase space" which take into account only the spherical degrees of freedom have been used elsewhere in other examples, see for example Ref. [15].

IV. INTERACTION ENERGY

As already mentioned, our aim now is to calculate the interaction energy between external probe sources in the model (16). To do this, we will compute the expectation value of the energy operator \(H\) in the physical state \(\vert \Phi \rangle\), which we will denote by \(\langle H \rangle_\Phi\). The starting point is the two-dimensional space-time Lagrangian (16):

\[
\mathcal{L} = 4\pi r^2 \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{M \sqrt{2}}{8} \varepsilon_{\mu\nu} F^{\mu\nu} \right\} - A_0 J^0,
\]

where \(J^0\) is the external current. A notation remark, in (18), \(\mu, \nu = 0, 1\), also, \(x^1 \equiv r \equiv x\) and \(\varepsilon^{01} = 1\).

We now proceed to obtain the Hamiltonian. For this we restrict our attention to the Hamiltonian framework of this theory. The canonical momenta read \(\Pi^\mu = -4\pi x^2 \left(F^0{}^\mu + \frac{M \sqrt{2}}{8} \varepsilon^{0\mu}\right)\), which results in the usual primary constraint \(\Pi^0 = 0\), and \(\Pi^i = -4\pi x^2 \left(F^0{}^i + \frac{M \sqrt{2}}{8} \varepsilon^{0i}\right)\). The canonical Hamiltonian following from the above Lagrangian is:

\[
H_C = \int dx \left( \Pi_1 \partial^1 A^0 - \frac{1}{8\pi x^2} \Pi_1 \Pi^1 - \frac{M \sqrt{2}}{4} \varepsilon^{01} \Pi_1 + A_0 J^0 \right).
\]
The consistency condition $\dot{\Pi}_0 = 0$ leads to the secondary constraint $\Gamma_1(x) \equiv \partial_1 \Pi^1 - J^0 = 0$. It is straightforward to check that there are no further constraints in the theory, and that the above constraints are first class. The extended Hamiltonian that generates translations in time then reads $H = H_C + \int dx \left( c_0(x)\Pi_0(x) + c_1(x)\Gamma_1(x) \right)$, where $c_0(x)$ and $c_1(x)$ are the Lagrange multipliers. Moreover, it follows from this Hamiltonian that $A_0(x) = [A_0(x), H] = c_0(x)$, which is an arbitrary function. Since $\Pi_0 = 0$, neither $A^0$ nor $\Pi^0$ are of interest in describing the system and may be discarded from the theory. The Hamiltonian then takes the form

$$H = \int dx \left( -\frac{1}{8\pi x^2} \Pi_1 \Pi^1 - \frac{M\sqrt{2}}{4} \varepsilon^{01} \Pi_1 + c'(\partial_1 \Pi^1 - J^0) \right),$$

where $c'(x) = c_1(x) - A_0(x)$.

According to the usual procedure we introduce a supplementary condition on the vector potential such that the full set of constraints becomes second class. A convenient choice is found to be [5,10–12]

$$\Gamma_2(x) \equiv \int_{C_{\xi x}} dz' A_{\nu}(z) \equiv \int_0^1 d\lambda x^1 A_1(\lambda x) = 0,$$

where $\lambda (0 \leq \lambda \leq 1)$ is the parameter describing the spacelike straight path $x^1 = \xi^1 + \lambda (x - \xi)^1$, and $\xi$ is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to $\xi^1 = 0$. In this case, the only nontrivial Dirac bracket is

$$\left\{ A_1(x), \Pi^1(y) \right\}^* = \delta^{(1)}(x - y) - \partial_1 \int_0^1 d\lambda x^1 \delta^{(1)}(\lambda x - y).$$

We are now equipped to compute the interaction energy between pointlike sources in the model (16), where a fermion is localized at the origin $0$ and an antifermion at $y$. As we have already mentioned, we will calculate the expectation value of the energy operator $H$ in the physical state $|\Phi\rangle$. From our above discussion, we see that $\langle H \rangle_\Phi$ reads

$$\langle H \rangle_\Phi = \langle \Phi \left| \int dx \left( -\frac{1}{8\pi x^2} \Pi_1 \Pi^1 - \frac{M\sqrt{2}}{4} \varepsilon^{01} \Pi_1 \right) \right| \Phi \rangle.$$
\[ |\Phi\rangle \equiv \overline{\Psi}(y)\,\Psi(0) = \overline{\psi}(y)\exp\left(\frac{i e}{\hbar} \int_0^y dz_1 A_i(z)\right)\psi(0)|0\rangle, \tag{24} \]

where \(|0\rangle\) is the physical vacuum state. As we have already indicated, the line integral appearing in the above expression is along a spacelike path starting at \(0\) and ending \(y\), on a fixed time slice.

Taking into account the above Hamiltonian structure, we observe that

\[ \Pi_1(x)|\overline{\Psi}(y)\,\Psi(0)\rangle = \overline{\Psi}(y)\,\Psi(0)\,\Pi_1(x)|0\rangle - e\int_0^y dz_1 \delta^{(1)}(z_1 - x)|\Phi\rangle. \tag{25} \]

Inserting this back into (23), we get

\[ \langle H \rangle_{\Phi} = \langle H \rangle_0 + \frac{e^2}{8\pi} \int dx \frac{1}{x^2} \left(\int_0^y dz_1 \delta^{(1)}(z_1 - x)\right)^2 + \frac{M\sqrt{2}e}{4} \int dx \left(\int_0^y dz_1 \delta^{(1)}(z_1 - x)\right), \tag{26} \]

where \(\langle H \rangle_0 = \langle 0|\,H\,|0\rangle\). We further note that

\[ \frac{e^2}{2} \int dx \left(\int_0^y dz \delta(z_1 - x)\right)^2 = \frac{e^2}{2} L, \tag{27} \]

with \(|y| \equiv L\). Inserting this into Eq.(26), the interaction energy in the presence of the static charges will be given by

\[ V = -\frac{e^2}{8\pi L} + \frac{M\sqrt{2}e}{4} L, \tag{28} \]

which has the Cornell form. In this way the static interaction between fermions arises only because of the requirement that the \(|\overline{\Psi}\Psi\rangle\) states be gauge invariant.

**V. CONCLUSIONS**

We have found that in the context of a model where scale invariance is spontaneously broken, the Cornell confining potential between quark-antiquark naturally appears. The solutions appear also relevant to the non-Abelian generalizations of the model. Once again, the gauge-invariant formalism has been very economical in order to obtain the interaction
energy, this time showing a confining effect in $(3 + 1)$ dimensions. Other aspects of $QCD$ concern gluon confinement, in addition to the quark-antiquark confinement we have studied so far. Indeed, preliminary studies indicate that Eq.(10), do not support plane wave solutions, which is a clear hint of gluon confinement. We will report on these issues in a future publication. Finally, it would also be interesting to see if this model can describe other confined states, like baryons, glueballs, etc.

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