Sumudu Transform with Modified Homotopy Perturbation Method to Solve Two Point Singular Boundary Value Problems

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Abstract. In this paper, a class of two point singular boundary value problems (SBVPs) has been solved by a new powerful technique. This technique relies on the Sumudu transform (ST) and modified homotopy perturbation method (MHPM). The proposed scheme is based on the freedom of the homotopy perturbation method (HPM) by introducing a suitable initial approximation; in addition, the residual error will be canceled in several points of the interest interval (RECP). Furthermore, only a first order approximation of the modified Sumudu transform homotopy perturbation method (MSTHPM) is required. The proposed method has been applied to solve several examples and the results have been compared with the exact solutions, it is found that the proposed solutions are of high accuracy and, therefore, MSTHPM is extremely efficient, simple and can be applied to other nonlinear problems.

1. Introduction
In the last two decades, many analytical approximate methods have been presented to solve two point boundary value problems. Most of these problems generally occur commonly in many areas of Physics, Chemistry and engineering. Recently, many researchers have introduced various methods to obtain approximate solutions for linear and nonlinear singular differential equations (SDEs), such as variational iteration method [1-3], adomian decomposition method [4-9], differential transform method [10, 11], and homotopy perturbation method [12-14], Amongst all these methods, the homotopy perturbation method (HPM) has been considered as one of the most popular one, due to its simplicity and its wide range of applications.

The HPM was suggested by He in [15, 16] and had been proven by many authors [17-20] to be a powerful mathematical tool for solving various types of nonlinear problems, which represent a large number of modern science branches. In some applications when series solution is searched for the HPM method has some drawbacks which reduce the efficiency of the method due to repeated calculations and calculations of massive unneeded terms. Hence, many authors had improved this scheme by integrating it with other methods to avoid these drawbacks; one of them is the Sumudu transform homotopy perturbation method STHPM which is a powerful combination of the sumudu transformation and the HPM to obtain a highly accurate technique for solving boundary value problems BVPs. The advantage of this method is that it is simple, efficient and reliable which also reduces the volume computational work involved. Moreover, the proposed method gives approximate solutions without any limitations. Singh and Devendra [21]
coupled STHPM with He’s polynomials to find the solution of nonlinear partial equations with the initial conditions, and it has been presented by many authors to be a powerful mathematical tool for solving a wide range of nonlinear operator equations [22-24]. However, the work which will be introduced in this paper is quite different from the techniques used to solve these types of problems; we will modify STHPM to improve the accuracy of the solution, particularly at unknown endpoints of the interval. The methodology used in the proposed method is based primarily on the exploitation of the freedom of the HPM in the choice of an arbitrary linear function as a suitable initial approximation; furthermore, the residual error will be canceled in some points of the interval. Implementing these steps will further accelerate the convergence of the approximate solutions, as it will be shown, the MSTHPM present more accurate results in a first order approximation when compared with the exact solution.

In this present study, the main goal is to employ the modified Sumudu transform homotopy perturbation method (MSTHPM) in solving two point singular boundary value problems. The proposed method provides the exact solution, where the advantage lies in its applicability and effectiveness for obtaining approximate solutions for other types of nonlinear equations.

In this paper the Homotopy perturbation method, Sumudu transformations and the combination of ST and HPM are presented in section 2,3 and 4. In section 5, Numerical application of the method is illustrated by three test examples to demonstrate the efficiency of the method. Conclusion is given in section 6.

2. Homotopy Perturbation Method (HPM)

To explain the fundamental idea of the HPM [15, 16], consider the following nonlinear differential equation:

$$L(u) + N(u) = f(r) \quad (1)$$

$$\beta(u, \partial u/\partial t), \quad r \in \Gamma \quad (2)$$

where L and N is a linear and nonlinear operators respectively, f(r) is a known analytical function, \(\beta\) is a boundary operator and \(\Gamma\) is the domain boundary for \(\Omega\).

We construct the HPM as \(\nu(\rho, r) \times [0, 1] \rightarrow \mathbb{R}\) which satisfies:

$$H(\nu, \rho) = L(\nu) = L(u_0) + \rho[-L(u_0) - N(\nu) + f(r)], \quad \rho \in [0, 1], \quad r \in \Omega \quad (3)$$

where \(\rho\) is an embedding parameter, its values are varied from 0 to 1, \(u_0\) is the initial approximate solution for Eq(1) which satisfies the boundary conditions.

Suppose the solution for Eq(3) can be expressed as a power series of \(\rho\) as

$$\nu = \nu_0 + \rho \nu_1 + \rho^2 \nu_2 + \rho^3 \nu_3 + \ldots, \quad (4)$$

The values for the sequence \(\nu_0, \nu_1, \nu_2, \ldots\) can be found by substituting Eq(4) into Eq(3) and equating coefficients of \(\rho\) with the same power. When \(\rho \rightarrow 1\), it gives the approximate solution for Eq(1) as

$$u = \lim_{\rho \rightarrow 1} \nu = \nu_0 + \nu_1 + \nu_2 + \nu_3 + \ldots, \quad (5)$$

the series (6) has been proved its convergence in [15].
3. Sumudu Transform (ST):
Watugula [25] introduced Sumudu transform as a new integral and is defined as:

\[ F(u) = S\{ f(t) \} = \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} f(t) dt. \] (6)

In this work we used the following properties of ST:

(i) \( S\{ t^n \} = n! u^n \) \hspace{1cm} (7)

(ii) \( S\{ f^{(n)}(t) \} = \frac{1}{u^n} F(u) - \frac{1}{u^n} \sum_{k=0}^{n-1} u^k f^{(k)}(0) \) \hspace{1cm} (8)

where \( f^{(0)}(0) = f(0), f^{(k)}(t), k=1, 2, 3,...,n-1 \) are the k-th derivatives of the function \( f(t) \), and \( S\{ f^{(n)}(t) \} = F(u) \). If \( F(u) \) is the Sumudu transform of \( f(t) \), then \( f(t) \) is called the inverse Sumudu transform of \( F(u) \) and is expressed by \( f(t) = S^{-1}\{ F(u) \} \), where the inverse Sumudu transform operator is \( S^{-1} \).

4. Modified Sumudu Transform Homotopy Perturbation Method (MSTHPM)
In order to illustrate the basic idea of this method, we employ MSTPM to give analytical and approximate solutions for a nonlinear BVPs, as Eq(1). Thus, the same steps of HPM follows until Step 3, after which \( L(u_0) \) was substituted by an arbitrary function \( Z(r) \), where the freedom of HPM was exploited. To solve the problems in this work, it is sufficient to select a polynomial trial function with unknown parameters, A, B, C to be determined.

Taking the Sumudu transform of both sides of Eq(3), we get:

\[ S\{ L(V) \} = S\{ Z(r) + \rho[-Z(r) - N(V) + f(r)] \}. \] (9)

Employing the differential property of ST, that gives:

\[ V = S^{-1}(u^n) \left\{ \frac{1}{u^n} V(0) + \frac{1}{u^{n-1}} V'(0) + ... + \frac{1}{u} V^{(n-1)}(0) \right\} + \]

\[ S^{-1}\left\{ (u^n) S\{ Z(r) + \rho[-Z(r) - N(V) + f(r)] \} \right\} \] (10)

obtained upon solving for \( V \) and applying the inverse Sumudu transform \( S^{-1} \).

Suppose that:

\[ V = \sum_{n=0}^\infty \rho^n \nu_n \] (11)

is a power series solution of Eq(1).

Next, substituting Eq(11) into Eq(10), we obtain

\[ \sum_{n=0}^\infty \rho^n \nu_n = S^{-1}\left\{ (u^n) \left\{ \frac{1}{u^n} V(0) + \frac{1}{u^{n-1}} V'(0) + ... + \frac{1}{u} V^{(n-1)}(0) \right\} \right\} + \]

\[ S^{-1}\left\{ (u^n) S\{ Z(r) + \rho[-Z(r) - N(\sum_{n=0}^\infty \rho^n \nu_n) + f(r)] \} \right\} \] (12)
Equating the identical power terms of $\rho$, we obtain:

$$\rho^0 : \nu_0 = S^{-1} \left\{ (u^n) \left[ \frac{1}{u^n} V(0) + \frac{1}{u^{n-1}} V'(0) + \ldots + \frac{1}{u} V^{(n-1)}(0) \right] + S \left\{ Z(r) \right\} \right\}$$  \hspace{1cm} (13)

$$\rho^1 : \nu_1 = S^{-1} \left\{ (u^n) S \left\{ -N(\nu_0) - Z(r) + f(r) \right\} \right\}$$  \hspace{1cm} (14)

$$\rho^2 : \nu_2 = S^{-1} \left\{ (u^n) S \left\{ -N(\nu_0, \nu_1) \right\} \right\}$$  \hspace{1cm} (15)

$$\rho^3 : \nu_3 = S^{-1} \left\{ (u^n) S \left\{ -N(\nu_0, \nu_1, \nu_2) \right\} \right\}$$  \hspace{1cm} (16)

$$\ldots$$

$$\rho^j : \nu_j = S^{-1} \left\{ (u^n) S \left\{ -N(\nu_0, \nu_1, \ldots, \nu_j) \right\} \right\}$$  \hspace{1cm} (17)

Suppose that the initial approximation is $V(0) = u_0 = \alpha_0, V'(0) = \alpha_1, \ldots, V^{n-1}(0) = \alpha_{n-1}$; Thus an approximate solution will be:

$$u = \lim_{\rho \to 1} \nu_0 + \nu_1 + \nu_2 + \nu_3 + \ldots$$  \hspace{1cm} (18)

The values of $A, B, C, \ldots, \alpha_i$ are adequately calculated by solving the algebraic system, which is derived as follows:

1. Equation (18) should satisfy the boundary conditions at the end points of the interval.
2. In order to determine the values of all the parameters, we need to solve more algebraic equations by adding to those mentioned in Eq(1), until we get the same number of equations and parameters to be determine. If the number of additional equations is $j$-equations, then to obtain $R(r, A, B, C, \ldots, \alpha_i) = L(u(r, A, B, C, \ldots, \alpha_i)) + N(u(r, A, B, C, \ldots, \alpha_i)) - f(r)$, we should combine the following $j$-equations $R(r_1, A, B, C, \ldots, \alpha_i) = R(r_2, A, B, C, \ldots, \alpha_i) = \ldots = R(r_j, A, B, C, \ldots, \alpha_i) = 0$, where the residual is defined, by substituting Eq(18) into Eq(1). Suppose that $u(r, A, B, C, \ldots, \alpha_i)$ is the approximate solution of Eq(1) given by Eq(18), then the points $r_1, r_2, r_3, \ldots, r_j$ belong to the interest interval.

5. CASE STUDIES

In this section, we will solve the examples by MSTHPM to get the exact solution, to illustrate the solution procedures of the proposed method in solving singular two point BVPs:

**Example I:** Consider the following singular bvp:

$$y'' + \frac{1}{t} y' + y = \frac{t^2}{16} + \frac{5}{4}, \hspace{1cm} 0 \leq t \leq 1$$  \hspace{1cm} (19)

with the boundary conditions

$$y'(0) = 0, \hspace{1cm} y(1) = \frac{17}{16}$$

The exact solution of the problem is $y(t) = 1 + \frac{t^2}{16}$

By applying the MSTHPM method to find a solution for Eq(19), we define the operators as:

$$L(y) = y''(t)$$  \hspace{1cm} (20)
\[ N(y) = \frac{1}{t} y' + y - \frac{t^2}{16} - \frac{5}{4} \]  

(21)

Next, we build a homotopy in accordance with Eq(3), thus we get an approximate analytical solution:

\[ y'' = Z(t) + \rho(-Z(t) - \frac{1}{t} y' - y + \frac{t^2}{16} + \frac{5}{4}) \]  

(22)

where we have substituted \( L(u_0) \) for a function of \( Z(t) \), which will be defined later.

Applying ST, we obtain

\[ S\{y''\} = S \left\{ Z(t) + \rho(-Z(t) - \frac{1}{t} y' - y + \frac{t^2}{16} + \frac{5}{4}) \right\} \]  

(23)

employing the differential property of ST for \( n = 2 \), we have:

\[ \frac{1}{u^2} Y(u) - \frac{1}{u^2} y(0) - \frac{1}{u} y'(0) = S \left\{ Z(t) + \rho(-Z(t) - \frac{1}{t} y' - y + \frac{t^2}{16} + \frac{5}{4}) \right\} \]  

(24)

that gives

\[ y(t) = S^{-1} \left\{ u^2 \left( A + \frac{B}{u^2} + S \left\{ Ct + B + \rho(-Ct - B - \frac{1}{t} (\sum_{n=0}^{\infty} \rho^n \nu_n) - \frac{t^2}{16} + \frac{5}{4}) \right\} \right) \right\} \]  

(25)

obtained upon solving for \( Y(u) \) and applying the inverse Sumudu transform \( S^{-1} \), where we define \( A = y(0) \), and using \( y'(0) = 0 \).

Then, we suppose that the series solution for \( y(t) \) is given by

\[ y(t) = \sum_{n=0}^{\infty} \rho^n \nu_n \]  

(26)

Likewise, to get a highly accurate approximation, it is sufficient to choose \( Z(t) \), as a linear function

\[ Z(t) = Ct + B \]  

(27)

By substituting equations (26) and (27) into (25), we have:

\[ \sum_{n=0}^{\infty} \rho^n \nu_n = S^{-1} \left\{ u^2 \left( A + B u^2 + C u^3 \right) \right\} \]  

(28)

By comparing the coefficients of like powers of \( \rho \), we obtain:

\[ \rho^0 : \nu_0(t) = S^{-1} \left\{ A + B u^2 + C u^3 \right\} \]  

(29)

\[ \rho^1 : \nu_1(t) = S^{-1} \left\{ (u^2) S \left\{ -B - Ct - \frac{1}{t} \nu_0' - \nu_0 + \frac{t^2}{16} + \frac{5}{4} \right\} \right\} \]  

(30)

By solving equations (29) - (30) for \( \nu_0(t) \) and \( \nu_1(t) \), we obtain:
\[ \rho^0 : \nu_0(t) = A + \frac{B}{2} t^2 + \frac{C}{6} t^3 \]  

(31)

\[ \rho^1 : \nu_1(t) = \frac{5 - 4A - 8B}{8} t^2 - \frac{C}{4} t^3 + \frac{1 - 8B}{192} t^4 - \frac{C}{120} t^5 \]  

(32)

By substituting (31) and (32) into (26) and evaluating the limit when \( \rho \to 1 \), the first order approximate solution is given by:

\[ y(t) = A + \frac{5 - 4A - 4B}{8} t^2 - \frac{C}{12} t^3 + \frac{1 - 8B}{192} t^4 - \frac{C}{120} t^5. \]  

(33)

Next, we apply the boundary condition \( y(1) = \frac{17}{16} \) on Eq(33), to calculate the values of \( A, B \) and \( C \). In addition, we follow the MSTHPM algorithm, by substituting (33) into (19) and evaluate the resultant expression for the values \( t = 0.30 \) and \( t = 0.70 \), which lies in \([0,1]\). Upon following the above procedure, we have a system of equations for \( A, B \) and \( C \), to obtain the values:

\[ A = 1 \quad B = 0.125 \quad C = 0 \]  

(34)

after rounding the values to fourth decimal places, and substituting (34) into (33), The obtained results were found to be very close to the exact solution with some truncation error:

\[ y(t) = 1 + \frac{t^2}{16} \]  

(35)

**Example II:** Consider the inhomogeneous Bessel equation:

\[ y'' + \frac{1}{t} y' + y = 4 - 9t + t^2 - t^3, \quad 0 \leq t \leq 1, \quad y'(0) = 0, \quad y(1) = 0 \]  

(36)

The exact solution of the problem is \( y(t) = t^2 - t^3 \)

By applying the MSTHPM method to find a solution for (36),

\[ y'' = Z(t) + \rho(-Z(t) - \frac{1}{t} y' - y + 4 - 9t + t^2 - t^3) \]  

(37)

Applying the Sumudu transform (ST) to Eq(37), we obtain:

\[ S\{y''\} = S\left\{Z(t) + \rho(-Z(t) - \frac{1}{t} y' - y + 4 - 9t + t^2 - t^3)\right\} \]  

(38)

Next, employing the differential property of ST for \( n = 2 \), gives:

\[ y(t) = S^{-1}\left\{u^2 A + S\left\{Z(t) + \rho(-Z(t) - \frac{1}{t} y' - y + 4 - 9t + t^2 - t^3)\right\}\right\} \]  

(39)

which was obtained upon solving for \( Y(u) \) and applying the inverse of the Sumudu transforms \( S^{-1} \), where we define \( A = y(0) \), and using \( y'(0) = 0 \)

Then, we suppose that the series solution for \( y(t) \) is given by

\[ y(t) = \sum_{n=0}^{\infty} \rho^n \nu_n \]  

(40)
After substituting (27) and (40) into (39), we have:

\[
\sum_{n=0}^{\infty} \rho^n \nu_n = S^{-1} \left\{ u^2 \left( \frac{A}{u^2} + (Ct + B) \right) \right\} + \nonumber \]

\[
S^{-1} \left\{ u^2 \left( S \left\{ \rho(- (Ct + B) - \frac{1}{t} \left( \sum_{n=0}^{\infty} \rho^n \nu_n \right)' - \left( \sum_{n=0}^{\infty} \rho^n \nu_n \right) + 4 - 9t + t^2 + t^3 \right) \right\} \right\} \quad (41) \nonumber 
\]

the identical coefficients of \( \rho \) can be readily identified as:

\[
\rho^0 : \nu_0(t) = S^{-1} \left\{ A + Bu^2 + Cu^3 \right\} \nonumber \quad (42) \nonumber 
\]

\[
\rho^1 : \nu_1(t) = S^{-1} \left\{ (u^2)S \left\{ -B - Ct - \frac{1}{t} \nu_0' - \nu_0 + 4 - 9t + t^2 + t^3 \right\} \right\} \nonumber \quad (43) \nonumber 
\]

By solving the equations (42) - (43) for \( \nu_0(t) \) and \( \nu_1(t) \), we obtain:

\[
\rho^0 : \nu_0(t) = A + \frac{B}{2} t^2 + \frac{C}{6} t^3 \nonumber \quad (44) \nonumber 
\]

\[
\rho^1 : \nu_1(t) = \frac{4 - A - 2B}{2} t^2 - \frac{6 - C}{4} t^3 + \frac{2 - B}{24} t^4 - \frac{6 - C}{120} t^5 \nonumber \quad (45) \nonumber 
\]

By substituting (44) and (45) into (40) and evaluate the lim when \( \rho \to 1 \), the first order approximates solution is given by:

\[
y(t) = A + \frac{4 - A - B}{2} t^2 - \frac{18 + C}{12} t^3 + \frac{2 - B}{24} t^4 - \frac{6 + C}{120} t^5. \nonumber \quad (46) \nonumber 
\]

Next, we apply the boundary condition \( y(1) = 0 \) on Eq(46), to calculate the values of A, B and C. In addition, we follow the MSTHPM algorithm, by substituting (46) into (36) and evaluate the resultant expression for the values \( t = 0.30 \) and \( t = 0.75 \), which lies in \([0,1]\). Upon following the above procedure, we have a system of equations for A, B and C, to obtain the values:

\[
A = 0 \quad B = 2 \quad C = -6 \nonumber \quad (47) \nonumber 
\]

after rounding the values to fourth decimal places, and substituting (47) into (46), The obtained results were found to be very close to the exact solution with some truncation error:

\[
y(t) = t^2 - t^3 \nonumber \quad (48) \nonumber 
\]

Example III: Consider the following singular bvp:

\[
y'' + \frac{2}{t} y' - (1 - t^2)y = -t^4 + 2t^2 - 7, \quad 0 \leq t \leq 1, \quad y'(0) = 0, \quad y(1) = 0 \nonumber \quad (49) \nonumber 
\]

The exact solution is \( y(t) = 1 - t^2 \)

Applying the MSTHPM method for (49) to obtain an accurate solution,

By building a homotopy, we obtain:

\[
y'' = Z(t) + \rho(-Z(t) - \frac{2}{t} y' + (1 - t^2)y - t^4 + 2l^2 - 7) \nonumber \quad (50) \nonumber 
\]
Applying the Sumudu transform to (50), we get:
\[ S \{ y'' \} = S \left\{ Z(t) + \rho(-Z(t) - \frac{2}{t} y' + (1 - t^2) y - t^4 + 2t^2 - 7) \right\} \] (51)

Next, employing the differential property of ST for \( n = 2 \), gives:
\[ y(t) = S^{-1} \left\{ u^2 \left( \frac{A}{u^2} + (Ct + B) \right) + \rho(-Ct - B - \frac{2}{t} \sum_{n=0}^{\infty} \rho^n \nu_n)' + (1 - t^2)(\sum_{n=0}^{\infty} \rho^n \nu_n) - t^4 + 2t^2 - 7 \right\} \] (52)

which was obtained upon solving for \( Y(u) \) and applying the inverse of the Sumudu transforms \( S^{-1} \), where we define \( A = y(0) \), and using \( y'(0) = 0 \).

We suppose that the series solution for \( y(t) \) is given by
\[ y(t) = \sum_{n=0}^{\infty} \rho^n \nu_n \] (53)

After substituting (27) and (53) into (52) we get:
\[ \sum_{n=0}^{\infty} \rho^n \nu_n = S^{-1} \left\{ u^2 \left( \frac{A}{u^2} + (Ct + B) \right) + \rho(-Ct - B - \frac{2}{t} \sum_{n=0}^{\infty} \rho^n \nu_n)' + (1 - t^2)(\sum_{n=0}^{\infty} \rho^n \nu_n) - t^4 + 2t^2 - 7 \right\} \] (54)

the identical coefficients of \( \rho \) can be readily identified as:
\[ \rho^0 : \nu_0(t) = S^{-1} \left\{ A + Bu^2 + Cu^3 \right\} \] (55)
\[ \rho^1 : \nu_1(t) = S^{-1} \left\{ (u^2)S \left( -B - Ct - \frac{2}{t} \nu_0 + (1 - t^2)\nu_0 - t^4 + 2t^2 - 7 \right) \right\} \] (56)

By solving the equations (55) - (56) for \( \nu_0(t) \) and \( \nu_1(t) \), we obtain:
\[ \rho^0 : \nu_0(t) = A + \frac{B}{2} t^2 + \frac{C}{6} t^3 \] (57)
\[ \rho^1 : \nu_1(t) = \frac{(A - 3B - 7)}{2} t^2 - \frac{(C)}{3} t^3 + \frac{(4 - 2A + B)}{24} t^4 + \frac{C}{120} t^5 - \frac{(2 + B)}{60} t^6 - \frac{C}{252} t^7 \] (58)

By substituting (57) and (58) into (53) and evaluating the lim when \( \rho \rightarrow 1 \), results in a first order approximation will be
\[ y(t) = A + \frac{A - 2B - 7}{2} t^2 - \frac{C}{6} t^5 + \frac{4 - 2A + B}{24} t^4 + \frac{C}{120} t^5 - \frac{(2 - B)}{60} t^6 - \frac{C}{252} t^7 \] (59)

Next, we apply the boundary condition \( y(1) = 0 \) on Eq(59), to calculate the values of \( A, B \) and \( C \). In addition, we follow the MSTHPM algorithm, by substituting (59) into (49) and evaluate the resultant expression for the values \( t = 0.30 \) and \( t = 0.80 \), which lies in \([0,1]\). Upon following the above procedure, we have a system of equations for \( A, B, \) and \( C \), to obtain the values:
\begin{equation}
A = 1 \quad B = -2 \quad C = 0
\end{equation}

(60)

after rounding the values to fourth decimal places, and substituting (60) into (59), The obtained results were found to be very close to the exact solution with some truncation error:

\begin{equation}
y(t) = 1 + t^2
\end{equation}

(61)

6. Conclusions

We have demonstrated the reliability of the MSTHPM method by solving singular two point boundary value problems to obtain precise approximate solutions. The suggested method is simple and easy and produces highly accurate solutions. We have performed this method on three examples of nonhomogeneous singular BVPs. Furthermore, the ordinary differential equations have been solved by the algebraic system to obtain the unknown conditions. The results showed that there is a possibility to accelerate the convergence of the solution by using the MSTHPM for a given singular BVP. Other advantages of this method is that the first order approximation is employed where it can be used to solve other types of nonlinear problems.

Acknowledgment

The authors gratefully acknowledge the financial assistance provided by Universiti Sains Malaysia under the Research University grant scheme 1001/PMATHS/8011041.

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