ON GROUP ANALYSIS OF OPTIMAL CONTROL PROBLEMS IN ECONOMIC GROWTH MODELS

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Abstract. The optimal control problems in economic growth theory are analyzed by considering the Pontryagin’s maximum principle for both current and present value Hamiltonian functions based on the theory of Lie groups. As a result of these necessary conditions, two coupled first-order differential equations are obtained for two different economic growth models. The first integrals and the analytical solutions (closed-form solutions) of two different economic growth models are analyzed via the group theory including Lie point symmetries, Jacobi last multiplier, Prelle-Singer method, $\lambda$-symmetry and the mathematical relations among them.

1. Introduction. This research aims to represent a Lie group-based approach for the analysis of optimal control problems in economic growth models. In the literature, the dynamic optimization is generally used to investigate such models with some constraints. On the other hand, the dynamic optimization in continuous time is more common since it emerges interesting mathematical aspects in the analysis. Furthermore, it has a direct connection among the dynamic programming, variation of calculus, and optimal control theory [5, 31]. In addition, it is well known that Lagrangian, Hamiltonian, and Euler-Lagrange equations play a vital role in the calculus of variations [16, 21, 13, 22, 14]. Most economic models are defined with the integrand function including a discount factor. But in the analysis of these models, the current value Hamiltonian, which is independent of the discount factor, can be preferred since it makes the analysis simpler. Additionally, the present value Hamiltonian, which is dependent on the discount factor, can also be used for analyzing the economic growth models. One can say that this approach is not common in the literature since the discount factor poses more complicated derivatives based on the maximum principle differentiation rules [1].

In economic growth model analysis, an optimal control problem is defined with a state variable, a control variable, and constraint function, besides necessary conditions for the Pontryagin maximum principle by generating the costate variable.

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Necessary conditions of the Pontryagin maximum principle are defined by the first-order conditions (FOCs) for optimality. In the mathematical framework, if these conditions can be written by eliminating the control variable in the system, then the system of coupled first-order ordinary differential equations (ODEs), which depend on state and costate variables, can be determined.

The principal part of this study is based on the fact that two coupled first-order ODEs corresponding to the optimal control problems are investigated by analyzing the Lie point symmetries of the systems. In the literature, the use of the partial Hamiltonian approach in the analysis of the optimal control problems is presented in the studies [23, 24, 25]. It is known that the first-order ODEs have infinite-dimensional Lie algebra [30, 26]. In addition, it is not possible to determine to Lie symmetries of these systems with the package program MathLie or GEM, etc. [4, 8]. It is also well known that Lie group analysis of the ODEs and PDEs is one of the most efficient methods to obtain an analytical solution of differential equations [30] in the literature. In this study, we concentrate on the Lie symmetries of the system to corresponding optimal control problem, then determine its analytical solutions by utilizing some methods related to Lie theory, Jacobi last multiplier, λ-symmetry, and Prelle-Singer. Although Jacobi’s last multiplier (JLM) method was developed by Jacobi, after long years Nucci drew our attention by demonstrating the connection between Lie theory and JLM method [27, 28, 29]. The method of λ-symmetry is developed by Muriel and Romero for ODEs, which have no Lie point symmetries [18, 19, 20]. It is clear that not only the JLM method but also the λ-symmetry method are very applicable approaches to determine the first integrals of ODEs, also they have simple algorithms related to Lie point symmetries of differential equations. In this work, the coupled first-order system of ODEs defining the optimal control problems, which are derived by both current and present value Hamiltonians is analyzed. In addition, the Prelle-Singer (PS) method is considered to analyze the nonlinear economic growth models [17] by pointing out the relationships between Lie group related methods such as Lie point symmetry, JLM, λ-symmetry, and adjoint symmetry.

In the following section, some preliminaries about the definition of the present and the current value Hamiltonian functions are presented. Also, the same section consists of some preliminaries about JLM and λ-symmetry methods. Section 3 consists of the analysis for the present and the current value Hamiltonians of an optimal control problem defined in economics. Section 4 represents the analysis of the economic growth model with logarithmic utility function as a second model via the Prelle-Singer method. Section 5 indicates some conclusions and discussions.
be regarded as two types, namely the present value Hamiltonian and the current value Hamiltonian.

**Definition 2.1.** The present value Hamiltonian is defined as
\[
\bar{H} = F(t, \bar{q}, \bar{c}) e^{-\rho t} + \bar{p}_i f^i(t, \bar{q}, \bar{c}),
\]
where \(\bar{H}\) notation is used for the present value Hamiltonian. In addition, the current value Hamiltonian is defined by
\[
H = F(t, q, c) + p_i f^i(t, q, c).
\]

**Remark 1.** The relationship between the current value Hamiltonian and present value Hamiltonian can be given by
\[
q = \bar{q}, \quad u = \bar{u}, \quad p = \bar{p} e^{\rho t} \quad \text{and} \quad H = \bar{H} e^{\rho t} \quad [9, 10].
\]

**Remark 2.** The necessary conditions for optimal control based on the Pontryagin maximum principle for the present value Hamiltonian \(\bar{H}\) are given by
\[
\frac{\partial \bar{H}}{\partial \bar{c}_i} = 0, \quad \frac{\partial^2 \bar{H}}{\partial \bar{c}_i^2} < 0, \quad (5)
\]
\[
\bar{q}^i = \frac{\partial \bar{H}}{\partial \bar{p}_i}, \quad (6)
\]
\[
\bar{p}^i = -\frac{\partial \bar{H}}{\partial \bar{q}^i}, \quad i = 1, \ldots, n, \quad (7)
\]
and the necessary conditions in terms of current value Hamiltonian \(H\) are
\[
\frac{\partial H}{\partial c_i} = 0, \quad (8)
\]
\[
\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad (9)
\]
\[
\dot{p}^i = -\frac{\partial H}{\partial q^i} + \Gamma_i, \quad i = 1, \ldots, n, \quad (10)
\]
in which \(\Gamma_i\) is generally taken as a nonzero function of \(t, p_i, q^i\) and it is generally formulated as a linear function of costate variable \(p_i\), and the time parameter \(t\).

The equations (5)-(7) and (8)-(10) determine optimality with all variables starting out from any given initial position. Notice that, in this study, the analysis of the optimal control problem is considered by taking account of both the current value Hamiltonian and the present value Hamiltonian functions.

2.1. **The method of Jacobi last multiplier.** It is known that one of the most efficient methods for the investigation of the solutions of differential equations is based on the study of Lie group of transformations. Besides, in the literature, there are some methods related to Lie symmetries such as \(\lambda\)-symmetry and Jacobi last multiplier that is highly effective to determine reduced forms of a given differential equation to study its first integrals. Additionally, the Jacobi last multiplier method has important relations related with first integrals and multiplier for ODEs and multidimensional systems whose Lie symmetries are known [27, 28, 29]. To introduce the approach, let us consider an \(n^{th}\)-order ODE
\[
q^{(n)} = f(t, q, q', q'', \ldots, q^{(n-1)}), \quad (11)
\]
in which an apex is indicated the order of differentiation. If above equation (11) is transformed into an equivalent system of first-order equations, i.e.
\[
w'_i = W_i(t, w_1, \ldots, w_{n-1}),
\]
then its Jacobi last multiplier \( M \) can be determined by solving the following differential equation

\[
\frac{d(\log M)}{dt} + \sum_{i=1}^{n} \frac{\partial W}{\partial w_i} = 0,
\]

i.e.

\[
M = \exp \left( - \int \sum_{i=1}^{n} \frac{\partial W}{\partial w_i} dt \right). \tag{14}
\]

Every different integral gives another multiplier such as \( \tilde{M} \). Therefore, any ratio of two different multipliers \( M/\tilde{M} \) represents the first integral of (11). If we know \( n - 1 \) symmetries of (11) such that

\[
X_i = \sum_{j=1}^{n} \xi_{ij}(x_1, \ldots, x_n) \partial_{x_j}, \quad i = 1, \ldots, n - 1,
\]

then Jacobi last multiplier is represented by \( M = \Delta^{-1} \), verified that \( \Delta \neq 0 \), where

\[
\Delta = \det \begin{bmatrix}
    a_1 & \ldots & a_n \\
    \xi_{1,1} & \ldots & \xi_{1,n} \\
    \vdots & \ddots & \vdots \\
    \xi_{n-1,1} & \ldots & \xi_{n-1,n}
\end{bmatrix}. \tag{16}
\]

2.2. The method of \( \lambda \)-symmetry. We now introduce the procedure emphasizing the \( \lambda \)-symmetry concept and its relation with the Lie symmetries [18, 19, 20]. For this purpose, let assume that a second-order differential equation given by

\[
\ddot{q} = \phi(t, q, \dot{q}), \tag{17}
\]

and then corresponding vector field of (17)

\[
A = \frac{\partial}{\partial t} + \dot{q} \frac{\partial}{\partial q} + \phi(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}}, \tag{18}
\]

is called the total derivative operator and its original notation is \( D_t \) [30]. It can be said that the first integral of the differential equation (17) is any function such as \( I(t, q, \dot{q}) \), which satisfies the relation \( A(I) = D_t I = 0 \). And Lie symmetry generator of the equation (17) is defined as

\[
v = \xi(t, q) \frac{\partial}{\partial t} + \eta(t, q) \frac{\partial}{\partial q}, \tag{19}
\]

and then the characteristic function is given by the formula [30]

\[
Q = \eta - \dot{q} \xi. \tag{20}
\]

**Theorem 2.2.** The vector field \( \partial_q \) is a \( \lambda \)-symmetry of the equation (17) and then \( \lambda \)-function can be derived from the form

\[
\lambda = \frac{A(Q)}{Q}. \tag{21}
\]

**Remark 3.** If \( \partial_q \) is a \( \lambda \)-symmetry of the given equation, then the first integral and integrating factor can be derived by the following steps:

1. Consider \( \psi^{([\lambda](1))] \) as the \( \lambda \)-prolongation of the vector field \( v \), and then a particular solution of

\[
\omega_q + \lambda \cdot \omega_q = 0, \tag{22}
\]
is a first integral of $v^{\lambda(t)}$.

2. Determine $A(\omega)$ and state $A(\omega)$ in terms of $(t, \omega)$ as $A(\omega) = F(t, \omega)$.

3. Find a first integral $G$ of $\partial_t + F(t, \omega)\partial_\omega$.

4. Express $I(t, q, \dot{q}) = G(t, \omega(t, q, \dot{q}))$.

3. An application to an optimal control problem. In this section, the application of the Lie point symmetry group and the related methods to an optimal control problem given in the theory of economics [25] is considered by considering two different Hamiltonian functions. The problem is defined in the form

$$\text{Max } \int_0^\infty \left[ \alpha q - \beta q^2 - \alpha u^2 - \gamma u\right] e^{-rt} dt,$$  \hspace{1cm} (23)

subject to the constraint

$$\dot{q} = u,$$ \hspace{1cm} (24)

in which $\alpha, \beta, \gamma$ are all positive constants, $r$ is a discount factor, $q(t)$ is the state variable, and $u(t)$ is the control variable.

3.1. Current value Hamiltonian systems for the equation (23). In this subsection, we deal with the current value Hamiltonian function, which is convenient for infinite horizon exponentially discounted problems. The Hamiltonian function is formulated for the problem (23) and (24) by using (4) of the form

$$H(t, q, p, u) = \alpha q - \beta q^2 - \alpha u^2 - \gamma u + pu,$$ \hspace{1cm} (25)

where $p(t)$ is the costate variable. If we take the partial derivatives of Hamiltonian function (25) with respect to the related variables according to the equations (8)-(10) and then the associated algebraic and differential relations, which, in fact, are not Hamiltonian equations, for an optimum are written in accordance with (23) and (24) as

$$p = 2\alpha u + \gamma,$$ \hspace{1cm} (26)

$$\dot{q} = u,$$ \hspace{1cm} (27)

$$\dot{p} = 2\beta q - \alpha + pr,$$ \hspace{1cm} (28)

where $\Gamma$ function is defined by $pr$ and the “overdot” represents the derivatives of first-order with respect to time parameter $t$. By utilizing the mathematical methods based on symmetry groups to determine the closed-forms solutions for the related problem, we deal with the system of first-order ODEs. For this purpose, we rewrite the system of equations (26)-(28) in a coupled-form by solving the control variable $u(t)$ from equation (26) by using the application of the implicit function theorem based on the fact that the second derivative of the Hamiltonian function with respect to the control variable $u(t)$ is an invertible map [12]. Thus, one can obtain a system of first-order ODEs in terms of only $p(t)$ and $q(t)$ functions and their first-order time derivatives such that

$$\dot{p} = 2\beta q - \alpha + pr,$$

$$\dot{q} = \frac{p - \gamma}{2\alpha},$$ \hspace{1cm} (29)

Proposition 1. It is possible to derive five-dimensional Lie algebra $L_5$ for the system of first-order coupled differential equations (29).
The determining equations of the system (29) are given by

\[ -2p\alpha\phi - 4\alpha\beta\eta - 2p^2r^2\alpha\xi_p + 4pr\alpha^2\xi_p = 2\alpha^3\xi_p - 8pq\alpha\beta\xi_p + 8q\alpha^2\xi_p - 8q^2\alpha\beta^2\xi_p - p^2r\xi_q + p\alpha\xi_q - 2pq\beta\xi_q + pr\gamma\xi_q - \alpha\gamma\xi_q - 2q\beta\gamma\xi_q - 2pr\alpha\xi_t + 2\alpha^2\xi_t - 4q\alpha\beta\xi_t + 2p\alpha\phi_p - 2\alpha\beta\phi_p + 4q\alpha\beta\phi_p + p\phi_q - \gamma\phi_q + 2\alpha\phi_t = 0, \]

\[ -2\alpha\phi - 2p^2\alpha\xi_p + 2p\alpha^2\xi_p - 4pq\alpha\beta\xi_p + 2pr\alpha\gamma\xi_p - 2\alpha^2\gamma\xi_p + 4q\alpha\beta\gamma\xi_p = 0, \]

\[ + 2p\gamma\xi_q - \gamma^2\xi_q - 2p\alpha\xi_t + 2\alpha\gamma\xi_t + 4pr\alpha^2\eta_p - 4\alpha^3\eta_p + 8q\alpha^2\beta\eta_p + 2pq\eta_q + 4\alpha^2\eta_t = 0. \]

(30)

In fact, to obtain a solution to determining equations may be a much more difficult problem than to obtain a solution of the first-order ordinary differential equation since the Lie point symmetries cannot be determined by using a standard package program. However, it can be shown that the infinitesimal functions \( \xi(t, p, q), \phi(t, p, q), \) and \( \eta(t, p, q) \) are determined by using the analytical integration of the corresponding determining equations (30). We here only represent Lie algebra \( L_5 \) with basis operators of the form

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2\alpha(p\gamma - \alpha + 2q\beta) \frac{\partial}{\partial p} + (p - \gamma) \frac{\partial}{\partial q}, \]

\[ X_3 = (p - \gamma) \frac{\partial}{\partial p} + \frac{2q\beta + r\gamma - \alpha}{2\beta} \frac{\partial}{\partial q}, \]

\[ X_4 = e^{\frac{1}{4}t} \left( -\frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}} \right) \frac{\partial}{\partial p} + e^{\frac{1}{4}t} \left( -\frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}} \right) \frac{\partial}{\partial q}, \]

\[ X_5 = e^{\frac{1}{4}t} \left( -\frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}} \right) \frac{\partial}{\partial p} + e^{\frac{1}{4}t} \left( -\frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}} \right) \frac{\partial}{\partial q}. \]

(31)

Proposition 2. It can be shown that numerous first integrals concerning the five-dimensional Lie algebra \( L_5 \) via the JLM method can be determined, but six different types of these first integrals are presented here.

Proof. The Jacobi last multipliers of (29) are determined via the determinant constructed by Lie point symmetries above. The mathematical connection between the Jacobi last multipliers and Lie point symmetries is based on the construction of a determinant according to the formula (16). For our purpose, let us consider, for example, \( JLM_{13} \). Here the superscript refers to the symmetries, namely \( X_1 \) and
analytical solution to the system (29). For example, the first integral

\[
\Delta_{13} = \det \begin{bmatrix}
1 & 2\beta q - \alpha + pr & \frac{e^{-\gamma}}{2\alpha} \\
1 & 0 & 0 \\
0 & (p - \gamma) & \frac{2q\beta + r\alpha - \alpha}{2\beta}
\end{bmatrix}
\]

\[
= \frac{2\beta(p - \gamma)^2 + 2\alpha(-pr + \alpha - 2q\beta)(-\alpha + 2q\beta + r\gamma)}{4\alpha\beta},
\]

and then Jacobi last multiplier becomes \(JLM_{13} = \frac{1}{\Delta_{13}}\). The other Jacobi last multipliers can be determined by using the other symmetries of the system, namely, \(X_2, X_4, X_5\). Hence, any ratio of two multipliers gives the first integrals (conservation forms) of the system (29) in the form

\[
I_1 = \frac{JLM_{14}}{JLM_{13}} = 2e^{\frac{-1,i(\gamma - \sqrt{\beta q + 2\alpha})}{\beta}} \sqrt{\alpha}
\]

\[
\times (2\beta(p - \gamma)^2 + 2\alpha(-pr + \alpha - 2q\beta)(-\alpha + 2q\beta + r\gamma)),
\]

\[
I_2 = \frac{JLM_{14}}{JLM_{23}} = ae^{\frac{-1,i(\gamma - \sqrt{\beta q + 2\alpha})}{\beta}} \left( -\alpha^{3/2}\sqrt{\alpha r^2 + 4\beta} + 2\sqrt{\alpha}\sqrt{\alpha q}\sqrt{\alpha r^2 + 4\beta} + r\alpha(\alpha - 2q\beta) + p(-r^2\alpha - 2\beta + r\sqrt{\alpha}\sqrt{\alpha r^2 + 4\beta} + 2\beta\gamma) \right),
\]

\[
I_3 = \frac{JLM_{15}}{JLM_{13}} = \frac{ae^{\frac{-1,i(\gamma - \sqrt{\beta q + 2\alpha})}{\beta}}}{2\beta} \left( -\alpha^{3/2}\sqrt{\alpha r^2 + 4\beta} + 2\sqrt{\alpha}\sqrt{\alpha q}\sqrt{\alpha r^2 + 4\beta} + r\alpha(\alpha - 2q\beta) + p(-r^2\alpha - 2\beta + r\sqrt{\alpha}\sqrt{\alpha r^2 + 4\beta} + 2\beta\gamma) \right),
\]

\[
I_4 = \frac{JLM_{34}}{JLM_{23}} = \frac{e^{\frac{-1,i(\gamma - \sqrt{\beta q + 2\alpha})}{\beta}}}{2\beta} \left( \sqrt{\alpha}p\sqrt{\alpha r^2 + 4\beta} - 2(\alpha - 2q\beta) \right),
\]

\[
I_5 = \frac{JLM_{34}}{JLM_{13}} = \frac{e^{\frac{-1,i(\gamma - \sqrt{\beta q + 2\alpha})}{\beta}}}{2\sqrt{\alpha}} \left( -p\sqrt{\alpha r^2 + 4\beta} - 2\sqrt{\alpha}(\alpha - 2q\beta) \right),
\]

\[
I_6 = \frac{JLM_{24}}{JLM_{35}} = \frac{e^{i(\sqrt{\alpha r^2 + 4\beta})/\beta}}{2\alpha pr - \alpha + 2q\beta)(r\sqrt{\alpha} + \sqrt{\alpha r^2 + 4\beta}) + 4\beta\sqrt{\alpha}(p - \gamma)} \times (p\sqrt{\alpha r^2 + 4\beta} + 2\sqrt{\alpha}(\alpha - 2q\beta) - \gamma\sqrt{\alpha r^2 + 4\beta} - r\sqrt{\alpha}(p + \gamma)).
\]

\[
\]

Remark 4. The above first integrals of the coupled system of first-order differential equations (29) are algebraic equations as expected, not differential, which means they have no first-order derivatives of the dependent variables in the equation since the system has only first-order derivatives itself.

Analytical solutions: Using any first integral, it can be possible to obtain an analytical solution to the system (29). For example, the first integral \(I_5\) having the relation \(I_5 = c_1\), where \(c_1\) is a constant, gives the solution for the function \(p(t)\) and then from the first equation of (29), \(q(t)\) is determined. It can be checked that the second equation is automatically satisfied for the solutions of \(p(t)\) and \(q(t)\) as verification of these solutions. This means that the functions \(p(t)\) and \(q(t)\) are the
exact solutions (solution surfaces) of the first-order system (29). Then, the solution for function \( u(t) \) is derived from the equation (26). Finally, the analytical solutions for the optimal control problem including control, state, and costate variables for the equations (23) and (24), which are related to the first integral \( I_5 \), have the form

\[
\begin{align*}
    u(t) &= \frac{1}{8\beta\sqrt{\alpha}\sqrt{\alpha r^2 + 4\beta}} \left( e^{\frac{1}{2}t \left( \frac{r - \frac{\sqrt{\alpha r^2 + 4\beta}}{\sqrt{\pi}}}{\sqrt{\alpha r^2 + 4\beta}} \right)} \right) \left( r\sqrt{\alpha} + \sqrt{\alpha r^2 + 4\beta} \right) \left( c_1 r\sqrt{\alpha} - c_1 \sqrt{\alpha r^2 + 4\beta} + 4\beta e^{\frac{1}{2}t \left( \frac{r - \frac{\sqrt{\alpha r^2 + 4\beta}}{\sqrt{\pi}}}{\sqrt{\alpha r^2 + 4\beta}} \right)} \right), \\
    q(t) &= \frac{1}{4\beta\sqrt{\alpha r^2 + 4\beta}} \left( c_1 e^{\frac{1}{2}t \left( \frac{r - \frac{\sqrt{\alpha r^2 + 4\beta}}{\sqrt{\pi}}}{\sqrt{\alpha r^2 + 4\beta}} \right)} \right) \left( r\sqrt{\alpha} + \sqrt{\alpha r^2 + 4\beta} \right) \\
        &\quad + 2\sqrt{\alpha r^2 + 4\beta} (\alpha - \gamma + 2e^{\frac{1}{2}t \left( \frac{r - \frac{\sqrt{\alpha r^2 + 4\beta}}{\sqrt{\pi}}}{\sqrt{\alpha r^2 + 4\beta}} \right)} \beta), \\
    p(t) &= \frac{1}{r^2\alpha + 4\beta - r\sqrt{\alpha}\sqrt{\alpha r^2 + 4\beta}} \left( 4\beta e^{\frac{1}{2}t \left( \frac{r - \frac{\sqrt{\alpha r^2 + 4\beta}}{\sqrt{\pi}}}{\sqrt{\alpha r^2 + 4\beta}} \right)} \sqrt{\alpha} \sqrt{\alpha r^2 + 4\beta} \right) \\
        &\quad + c_1 e^{\frac{1}{2}t \left( \frac{r - \frac{\sqrt{\alpha r^2 + 4\beta}}{\sqrt{\pi}}}{\sqrt{\alpha r^2 + 4\beta}} \right)} \sqrt{\alpha} \left( r\sqrt{\alpha} - \sqrt{\alpha r^2 + 4\beta} \right) \left( c_1 \sqrt{\alpha} - c_1 \sqrt{\alpha r^2 + 4\beta} + 4\beta e^{\frac{1}{2}t \left( \frac{r - \frac{\sqrt{\alpha r^2 + 4\beta}}{\sqrt{\pi}}}{\sqrt{\alpha r^2 + 4\beta}} \right)} \right),
\end{align*}
\]

Remark 5. The other types of analytical solutions to the system (25) can be found by using the other first integrals, namely, \( I_1, I_2, I_3, I_4, \) and \( I_6 \).

3.2. Present value Hamiltonian systems for the equation (23). In this analysis, it is also possible to consider the present value Hamiltonian function. To construct the present value Hamiltonian form, one takes into account the complete integrand inside the integral (1). Hence, the Hamiltonian function is given in the form

\[
\dot{H}(t, \bar{q}, \bar{p}, \bar{u}) = (\alpha \bar{q} - \beta \bar{q}^2 - 2\alpha^2 - \gamma \bar{u}) e^{-rt} + \bar{p} \bar{u}. \tag{34}
\]

If the first-order conditions (5)-(7), which are now called Hamiltonian equations, for an optimum, are applied to (34) then one can have the following system of equations

\[
\begin{align*}
    \ddot{\bar{p}} &= (2\alpha \bar{u} + \gamma) e^{-rt}, \tag{35} \\
    \ddot{\bar{q}} &= \ddot{\bar{u}}, \tag{36} \\
    \ddot{\bar{q}} &= - (\alpha - 2\beta \bar{q}) e^{-rt}. \tag{37}
\end{align*}
\]

Remark 6. It is clear that both systems are given by (26)-(28) and (35)-(37) are same by using transformation \( q = \bar{q}, u = \bar{u}, p = \bar{p} e^{rt} \) and \( H = \bar{H} e^{rt} \). Similarly, the systems of equations (29) and (38) are same by using the relationship between variables of the present and the current value Hamiltonian systems i.e. \( q = \bar{q}, u = \bar{u}, p = \bar{p} e^{rt}, H = \bar{H} e^{rt} \).

Proposition 3. It is possible to derive four-dimensional Lie algebra \( L_4 \) for the system of first-order coupled differential equations (38).
Proof. It can also be shown that the determining equations can be written in terms of \( t, \, \dot{p}, \) and \( \dot{q} \) variables for the system (38) similar to the current Hamiltonian case as

\[
-2e^{\alpha t}r\alpha q\beta^2 \xi + 4e^{\alpha t}q\alpha \beta \xi - 4e^{\alpha t}\alpha \beta \eta - 2\alpha^3 \xi_\phi + 8q\alpha^2 \beta \xi_\theta - 8q^2 \alpha \beta^2 \xi_\phi + e^{2\alpha t}p\alpha \xi_\eta \\
-2e^{2\alpha t}p\alpha \beta \xi + e^{2\alpha t}\alpha \beta \eta - 4e^{2\alpha t}q\alpha \beta \xi_\phi - 4e^{2\alpha t}q\alpha^2 \xi_\phi - 2e^{2\alpha t}p\alpha \xi_\eta \\
+ 4e^{\alpha t}q\alpha \beta \phi_\eta + e^{3\alpha t}p\phi_\eta - e^{2\alpha t}\gamma \phi_\eta + 2e^{2\alpha t}\alpha \phi_\eta = 0, \\
-2e^{2\alpha t}p\alpha \beta \xi - 2e^{2\alpha t}\alpha \phi + 4e^{2\alpha t}p\alpha \beta \xi_\theta - 2e^{2\alpha t}\alpha \phi \xi_\theta - 2e^{2\alpha t}\alpha \gamma \xi_\phi + 4q\alpha \beta \xi_\phi \\
- e^{3\alpha t}p\alpha^2 \xi_\phi - 2e^{2\alpha t}p\gamma \xi_\phi - e^{2\alpha t}t \phi_\eta \xi_\theta - 2e^{2\alpha t}p\alpha \gamma \xi_\phi + 2e^{2\alpha t}t \phi_\eta \xi_\theta - e^{3\alpha t} \eta_\phi \\
+ 8q^2 \beta \eta_\phi + 2e^{2\alpha t}p\alpha \eta_\phi - 2e^{2\alpha t}t \phi_\eta \eta_\phi + 4e^{2\alpha t} \alpha^2 \eta_\phi = 0.
\]

(39)

The direct integration of the above determining equations leads four-dimensional Lie algebra \( L_4 \) of the system (38)

\[
\begin{align*}
\dot{X}_1 &= -\frac{1}{r} \frac{\partial}{\partial t} + \frac{\bar{p}}{\bar{p}} \frac{\partial}{\partial \bar{p}}, \\
\dot{X}_2 &= e^{\frac{1}{t}(-r - \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})} \frac{\partial}{\partial \bar{p}} - e^{\frac{1}{t}(r - \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})} \frac{\partial}{\partial \bar{q}}, \\
\dot{X}_3 &= e^{\frac{1}{t}(-r + \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})} \frac{\partial}{\partial \bar{p}} + e^{\frac{1}{t}(r + \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})} \frac{\partial}{\partial \bar{q}}, \\
\dot{X}_4 &= \frac{\partial}{\partial t} - 2\beta \eta_\phi - e^{\alpha t} \frac{\partial}{\partial \bar{p}} + \frac{2\beta \bar{r} - 2\bar{q} \beta - \alpha + \eta_\phi}{2\beta} \frac{\partial}{\partial \bar{q}}.
\end{align*}
\]

Remark 7. It can be said that there is no direct relationship between the current Hamiltonian and the present Hamiltonian systems in terms of Lie point symmetries.

**Proposition 4.** *JLM method enables us to determine many first integrals of the system of first-order coupled differential equations (38) with respect to the four-dimensional Lie algebra \( L_4 \) since the ratio of two multipliers yields to the first integral. However, we consider only six different types of first integrals to illustrate related results.*

Proof. To determine the first integrals of the system (38), we consider the Jacobi method by using a similar approach considered in the current Hamiltonian case. As a result, the corresponding first integrals related to the Lie point symmetries (40) can be determined as

\[
I_1 = \frac{JLM_{12}}{JLM_{13}} = e^{\frac{1}{t}(-r + \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})} \frac{\alpha^2 \eta_\phi + \alpha + 2\bar{q} \beta \eta_\phi + \alpha + \eta_\phi}{\sqrt{\alpha^2 \alpha + \sqrt{\alpha^2 + 4\beta}}(\bar{q} \beta + \gamma)} \times (\sqrt{\alpha^2 \alpha + \sqrt{\alpha^2 + 4\beta}}(\bar{q} \beta + \gamma) + 2\beta(-e^{rt} \bar{p} + \gamma)) \\
\times (\sqrt{\alpha^2 \alpha + \sqrt{\alpha^2 + 4\beta}}(-e^{rt} \bar{p} + \gamma) + 2\beta(-e^{rt} \bar{p} + \gamma)).
\]

\[
I_2 = \frac{JLM_{12}}{JLM_{14}} = e^{\frac{1}{t}(-r + \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})} \frac{\alpha^2 \eta_\phi + \alpha + 2\bar{q} \beta \eta_\phi + \alpha + \eta_\phi}{\sqrt{\alpha^2 \alpha + \sqrt{\alpha^2 + 4\beta}}(\bar{q} \beta + \gamma)} \times (-2r \beta(-e^{rt} \bar{p} + \gamma)^2 + 2\alpha(e^{rt} \bar{p} + \alpha + \bar{q} \beta)(-\alpha + 2\bar{q} \beta + \eta_\phi)).
\]
If one considers the present value Hamiltonian as given in (34) and then determined from the Hamiltonian function by the relation
\[ I_3 = \frac{JLM_{12}}{JLM_{34}} = \frac{e^{\frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}}}}{\sqrt{\alpha}(e^{-rt}\bar{p}r + \alpha - 2\bar{q}\beta)}(r\sqrt{\alpha} + \sqrt{\alpha^2 + 4\beta}) + 2\beta(e^{-rt}\bar{p} + \gamma) \]
\[ \times (-\alpha^{3/2}\sqrt{\alpha^2 + 4\beta} + r^2\alpha\gamma + 2\beta(-e^{rt}\bar{p} + \bar{q}\sqrt{\alpha}\sqrt{\alpha^2 + 4\beta} + \gamma) + r(-\alpha^2 + 2\bar{q}\alpha\beta + \gamma\sqrt{\alpha}\sqrt{\alpha^2 + 4\beta}), \]
\[ I_4 = \frac{JLM_{23}}{JLM_{13}} = \frac{e^{-\frac{1}{2}t}(r - \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})}{2r\sqrt{\alpha}\sqrt{\alpha^2 + 4\beta}} \]
\[ \times (\sqrt{\alpha}(e^{rt}\bar{p}r - \alpha + 2\bar{q}\beta)(r\sqrt{\alpha} + \sqrt{\alpha^2 + 4\beta}) + 2\beta(e^{rt}\bar{p} + \gamma)), \]
\[ I_5 = \frac{JLM_{23}}{JLM_{14}} = \frac{e^{-rt}(-2\beta(-e^{rt}\bar{p} + \gamma) + 2\beta\alpha(e^{rt}\bar{p}r - \alpha + 2\bar{q}\beta)(-\alpha + 2\bar{q}\beta + r\gamma))}{2r\sqrt{\alpha}\sqrt{\alpha^2 + 4\beta}} \]
\[ I_6 = \frac{JLM_{34}}{JLM_{14}} = \frac{e^{\frac{1}{2}t(-r - \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})}(2\beta(e^{rt}\bar{p}r - \alpha + 2\bar{q}\beta)(-\alpha + 2\bar{q}\beta + r\gamma) - r^2\alpha\gamma + r(\alpha^2 - 2\bar{q}\alpha\beta + \gamma\sqrt{\alpha}\sqrt{\alpha^2 + 4\beta}))}{2\beta}. \]

(41)

Analytical solutions: To obtain an analytical solution to the system (38), the relation \( I = c \) constant can be used. Let us consider the first integral \( I_4 \). From this relation, \( \bar{q}(t) \) is obtained and then \( \bar{p}(t) \) is determined from the first equation in (38) and the second equation in (38) is automatically satisfied, which represents the verification of the solutions, which is similar to the current Hamiltonian function case. The solution for variable \( \bar{u}(t) \) is determined from the equation (35). Finally, the solutions of the optimal control problem as the control, state, and costate variables are of the form
\[
\bar{u}(t) = -\frac{1}{4\sqrt{\alpha}\beta}(e^{\frac{1}{2}t(r - \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})}(c_1 - 2c_2e^{\frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}}})(r\sqrt{\alpha} + \sqrt{\alpha^2 + 4\beta}),
\]
\[
\bar{q}(t) = \frac{c_1 e^{\frac{1}{2}t(r - \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})}}{r^2\alpha + 4\beta - r\sqrt{\alpha}\sqrt{\alpha^2 + 4\beta}} + \frac{\alpha - r\gamma + 2\beta c_2e^{\frac{1}{2}t(r - \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})}r\sqrt{\alpha}(r\sqrt{\alpha} + \sqrt{\alpha^2 + 4\beta})}{2\beta}, \]
\[
\bar{p}(t) = \frac{e^{-rt}}{2\beta}(-c_1 e^{\frac{1}{2}t(r - \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})}r\sqrt{\alpha}(r\sqrt{\alpha} + \sqrt{\alpha^2 + 4\beta}) + 2\beta(c_2 e^{\frac{1}{2}t(r + \frac{\sqrt{\alpha^2 + 4\beta}}{\sqrt{\alpha}})}\sqrt{\alpha}(r\sqrt{\alpha} + \sqrt{\alpha^2 + 4\beta} + \gamma)),
\]
\[
\text{where } c_1 \text{ and } c_2 \text{ are constants.}
\]

3.3. Lagrangian approach. In addition to the Hamiltonian function, the Lagrangian function can also be used to analyze the optimal control problems. It is known that the Lagrangian function of an optimal control problem can be determined from the Hamiltonian function by the relation \( L = pq - H \). For this purpose if one considers the present value Hamiltonian as given in (34) and then the corresponding Lagrangian function
\[ \tilde{L} = -(\alpha \bar{q} - \beta \bar{q}^2 - \alpha \bar{q}^2 - \gamma \bar{q})e^{-rt}, \]

(43)
is derived. Moreover, the Euler-Lagrange equations for the Lagrangian function $\mathcal{L}$ are written in the form

$$\frac{\delta \mathcal{L}}{\delta q^i} = \frac{\partial \mathcal{L}}{\partial q^i} + \sum_{s \geq 1} (-D_s) \frac{\partial \mathcal{L}}{\partial q_s^i}, \quad \alpha = 1, 2, \ldots, m,$$

(44)

which give the following the second-order ODE in terms of the state variable $\dot{q}(t)$

$$2\alpha \ddot{q} - 2\beta \dot{q} - 2\dot{r} \dot{q} + \alpha - r \gamma = 0.$$

(45)

Remark 8. If $p(t)$ and $u(t)$ functions are eliminated by using the FOCs (26)-(28) for the current value Hamiltonian (25), then the same second-order ODE with the differential equation (45) in terms of $q(t)$ is obtained. For this reason, it is clear that the relation $q = \ddot{q}$ is satisfied and the equations both (26)-(28) and (35)-(37) have the relations $\dot{q} = u$ and $\ddot{q} = \ddot{u}$, then the relation $u = \ddot{u}$ is provided [9, 10].

Remark 9. Based on the above conditions and the definitions given by (25) and (34) one can prove that if and only if the relations $q = \ddot{q}$ and $u = \ddot{u}$ exist then the corresponding properties $p = \dddot{p} e^{rt}$ and $H = \dddot{H} e^{rt}$ are satisfied [9, 10].

Proposition 5. It is clear that the second-order linear ordinary differential equation (45) admits eigth-dimensional Lie algebra $L_8$ [4, 8].

Proof. To examine an analytical solution for the optimal control problem (23) and (24), we deal with the equivalent differential equation (45). The direct integration of determining equations of (45) gives the associated Lie point symmetries as

$$X_1 = \frac{\partial}{\partial t},$$

$$X_2 = - \frac{2q \beta + r \gamma - \alpha}{2 \beta} \frac{\partial}{\partial q},$$

$$X_3 = e^{\frac{\alpha}{2 \beta} t} \left( r - \frac{\sqrt{\alpha (\alpha r^2 + 4 \beta)}}{\sqrt{4 \beta}} \right) \frac{\partial}{\partial q},$$

$$X_4 = e^{\frac{\alpha}{2 \beta} t} \left( r + \frac{\sqrt{\alpha (\alpha r^2 + 4 \beta)}}{\sqrt{4 \beta}} \right) \frac{\partial}{\partial q},$$

$$X_5 = e^{\frac{\alpha}{2 \beta} t} \left( r - \frac{\sqrt{\alpha (\alpha r^2 + 4 \beta)}}{\sqrt{4 \beta}} \right) \frac{\partial}{\partial t} + e^{-\frac{\alpha}{2 \beta} t} \left( \frac{r - \sqrt{\alpha (\alpha r^2 + 4 \beta)}}{\sqrt{4 \beta}} \right) \frac{\partial}{\partial q},$$

$$X_6 = e^{\frac{\alpha}{2 \beta} t} \left( r + \frac{\sqrt{\alpha (\alpha r^2 + 4 \beta)}}{\sqrt{4 \beta}} \right) \frac{\partial}{\partial t} + e^{-\frac{\alpha}{2 \beta} t} \left( \frac{r + \sqrt{\alpha (\alpha r^2 + 4 \beta)}}{\sqrt{4 \beta}} \right) \frac{\partial}{\partial q},$$

$$X_7 = (\alpha^{3/2} \sqrt{\alpha r^2 + 4 \beta} ) \left( 4 e^{\frac{\sqrt{\alpha r^2 + 4 \beta}}{\sqrt{4 \beta}}} q \beta \alpha (\alpha r^2 + 4 \beta) (2 r^2 \alpha + 9 \beta) - 4 \beta \sqrt{\alpha} \sqrt{\alpha r^2 + 4 \beta} \right) \times (2 r^2 \alpha + 9 \beta) (r^3 (\alpha^2 + 4 \beta) \alpha r^2 + 4 \beta) - 4 \beta (4 \alpha^3 + \alpha (\alpha r^2 + 4 \beta)) + 4 \alpha \beta + 4 \alpha^3 \beta^2 \sqrt{\alpha r^2 + 4 \beta} + \alpha^2 \sqrt{\alpha (\alpha r^2 + 4 \beta)} - r^2 \alpha (4 \alpha^3 + \sqrt{\alpha (\alpha r^2 + 4 \beta)})$$
Six different types of first integrals of the second-order ordinary differential equation (45) can be determined concerning the eight-dimensional Lie algebra $L_8$ via the JLM method.

**Proposition 6.** Six different types of first integrals of the second-order ordinary differential equation (45) can be determined concerning the eight-dimensional Lie algebra $L_8$ via the JLM method.

**Proof.** The use of the same procedure based on the Jacobi method discussed in the previous section leads to the associated six-different types of the first integrals to the equation (45) of the form

\begin{align*}
\dot{I}_1 &= \frac{JLM_{12}}{JLM_{33}} = e^{\frac{\sqrt{\alpha r^2 + 4\beta}}{2\alpha}} \left( \frac{(\dot{r}^2 - 1)\alpha + 2\beta q - \dot{q}\sqrt{\alpha (r^2 + 4\beta) + r\gamma}}{\sqrt{\alpha (r^2 + 4\beta) + r\gamma}} \right), \\
\dot{I}_2 &= \frac{JLM_{23}}{JLM_{34}} = e^{\frac{\sqrt{\alpha r^2 + 4\beta}}{2\alpha}} \left( \frac{(\dot{r}^2 - 1)\alpha + 2\beta q + \dot{q}\sqrt{\alpha (r^2 + 4\beta) + r\gamma}}{\sqrt{\alpha (r^2 + 4\beta) + r\gamma}} \right), \\
\dot{I}_3 &= \frac{JLM_{44}}{JLM_{24}} = e^{\frac{\sqrt{\alpha r^2 + 4\beta}}{2\alpha}} \left( 4\alpha \beta \dot{q} - (-\alpha + \sqrt{\alpha (r^2 + 4\beta)}) (\alpha - 2q\beta - r\gamma) \right), \\
\dot{I}_4 &= \frac{JLM_{23}}{JLM_{24}} = e^{\frac{\sqrt{\alpha r^2 + 4\beta}}{2\alpha}} \left( 4\alpha \beta \dot{q} + (\alpha + \sqrt{\alpha (r^2 + 4\beta)}) (\alpha - 2q\beta - r\gamma) \right), \\
\dot{I}_5 &= \frac{JLM_{14}}{JLM_{12}} = 2e^{\frac{\sqrt{\alpha r^2 + 4\beta}}{2\alpha}} \left( \frac{\beta(\dot{r}^2 - 1)\alpha + 2\beta q + \dot{q}\sqrt{\alpha (r^2 + 4\beta) + r\gamma}}{\sqrt{\alpha (r^2 + 4\beta) + r\gamma}} \right), \\
\dot{I}_6 &= \frac{JLM_{13}}{JLM_{14}} = 2e^{\frac{\sqrt{\alpha r^2 + 4\beta}}{2\alpha}} \left( \frac{\beta(\dot{r}^2 - 1)\alpha + 2\beta q + \dot{q}\sqrt{\alpha (r^2 + 4\beta) + r\gamma}}{2\beta} \right).
\end{align*}
where the notation \( \hat{I} \) is used for the first integral representation for this case. 

Remark 10. The first integrals established for the Hamiltonian form do not involve the derivatives of unknown functions. Thus, the first integral is an algebraic equation (not differential) involving unknown functions. But in the case of the Lagrangian form, the first integrals, which are not algebraic equations, involve first-order derivatives of the unknown function. This is the most important mathematical result between Hamiltonian and Lagrangian forms for the model considered.

Remark 11. The first integrals are called the reduced forms (or reductions) of the second-order ODE (45).

Analytical solutions: If we take into account the first integral, for example, \( \hat{I}_6 \) and then it gives

\[
\bar{q}(t) = \frac{1}{2\beta} \left( 4e^{\frac{1}{2\beta} (\alpha \sqrt{\alpha \beta^2 + 4 \beta^2})} c_2 \beta^2 + \alpha - r\gamma \right) + c_1 e^{-\frac{2\beta}{\alpha \sqrt{\alpha \beta^2 + 4 \beta^2}}},
\]

(48)

From the equations (35)-(37), the solutions for other variables of the problem

\[
\bar{u}(t) = -2c_1 \beta e^{-\frac{2\beta}{\alpha \sqrt{\alpha \beta^2 + 4 \beta^2}}} \frac{e^{\frac{1}{2\beta} (\alpha \sqrt{\alpha \beta^2 + 4 \beta^2})}}{\alpha (4\beta + r(\alpha + \sqrt{\alpha\beta^2 + 4\beta^2}))},
\]

\[
\bar{p}(t) = e^{-rt} \left( 2\alpha \left( -2c_1 \beta e^{-\frac{2\beta}{\alpha \sqrt{\alpha \beta^2 + 4 \beta^2}}} \frac{e^{\frac{1}{2\beta} (\alpha \sqrt{\alpha \beta^2 + 4 \beta^2})}}{\alpha (4\beta + r(\alpha + \sqrt{\alpha\beta^2 + 4\beta^2}))} + \gamma \right) \right),
\]

(49)

are obtained, where \( c_1 \) and \( c_2 \) are constants.

3.4. \( \lambda \)-symmetry approach. \( \lambda \)-symmetries can also be considered to analyze the closed-form solutions of the equation (45) based on the fact that there is a notable relation between Lie symmetries and \( \lambda \)-symmetries as mentioned in Section 2.2. Here, we deal with \( \lambda \)-symmetry derived from Lie symmetry \( X_2 \) in (46). For the vector field, for example, \( X_2 \), the infinitesimal functions are \( \xi = 0 \) and \( \eta = \frac{2q\beta + r\gamma - \alpha}{2q} \) and then via the equations (18) and (20), the characteristic function \( Q \) and the vector field \( A \) are written by

\[
Q = \frac{2q\beta + r\gamma - \alpha}{2\beta}, \quad A = \dot{q},
\]

(50)

respectively. Then, \( \lambda \)-symmetry is derived with respect to the Theorem (2.2) such that

\[
\lambda_1(t, q, \dot{q}) = \frac{2\beta \dot{q}}{2\beta q + r\gamma - \alpha}.
\]

(51)

If the steps expressed in Remark 3 are carried out for the \( \lambda \)-symmetry (51) and then the related integration factor and the first integral

\[
\mu_2 = -2\beta q - r\gamma + \alpha
\]

\[
I_2 = -\frac{t}{\alpha} + \frac{2}{\alpha \sqrt{-\alpha^2 - 4\beta}} \left( c_1 \pi - \arctan \frac{\sqrt{\alpha (-r + \frac{4\beta q + r\gamma - \alpha}{r})}}{\sqrt{-\alpha^2 - 4\beta}} \right), \quad c_1 \in \text{Integers}
\]

(52)
are determined, respectively. For the other λ-symmetries related to the further Lie point symmetries in (46), the corresponding integration factors and first integrals can be determined. Secondly, we represent the results for Lie point symmetry \( X_3 \) as

\[
\lambda_3 = \frac{1}{2\alpha}(r\alpha - \sqrt{\alpha(\alpha r^2 + 4\beta)}), \quad \mu_2 = e^{-\frac{1}{2\alpha}t(r\alpha + \sqrt{\alpha(\alpha r^2 + 4\beta)})},
\]

\[
I_3 = \frac{2\beta}{\alpha + \sqrt{\alpha(\alpha r^2 + 4\beta)}} \bar{I}_6.
\]

Lastly, λ-symmetry, integration factor and first integral corresponds to \( X_4 \) are given below

\[
\lambda_4 = \frac{1}{2\alpha}\left(r\alpha + \sqrt{\alpha(\alpha r^2 + 4\beta)}\right), \quad \mu_3 = e^{-\frac{1}{2\alpha}t(r\alpha - \sqrt{\alpha(\alpha r^2 + 4\beta)})},
\]

\[
I_4 = e^{\frac{t(r\alpha + \sqrt{\alpha(\alpha r^2 + 4\beta)})}{2\alpha}} \left(\alpha - r\alpha \dot{q} - 2\beta \dot{q} + \dot{q} \sqrt{\alpha(\alpha r^2 + 4\beta)} - r \gamma\right) - r\alpha + \sqrt{\alpha(\alpha r^2 + 4\beta)}.
\]

**Analytical solutions:** The first integral \( I_1 \) leads to the solution of state variable \( \bar{q}(t) \) and then via the equations (35)-(37), the control \( \bar{u}(t) \) and the costate variables \( \bar{p}(t) \) are determined. As a result, the solution of the system (35)-(37) is of the form

\[
\ddot{q}(t) = \frac{\alpha - \gamma}{2\beta} + c_3 e^{\gamma t/2} \cos[c_1 \pi - \frac{(t + c_2 \alpha)\sqrt{-\alpha r^2 - 4\beta}}{2\sqrt{\alpha}}],
\]

\[
\ddot{u}(t) = \frac{1}{2\alpha^r} c_3 e^{\gamma t/2}(r \sqrt{\alpha} \cos[c_1 \pi - \frac{(t + c_2 \alpha)\sqrt{-\alpha r^2 - 4\beta}}{2\sqrt{\alpha}}] + \sqrt{-\alpha r^2 - 4\beta} \sin[c_1 \pi - \frac{(t + c_2 \alpha)\sqrt{-\alpha r^2 - 4\beta}}{2\sqrt{\alpha}}]),
\]

\[
\ddot{p}(t) = e^{-\gamma t}(c_3 e^{\gamma t/2}(r \sqrt{\alpha} \cos[c_1 \pi - \frac{(t + c_2 \alpha)\sqrt{-\alpha r^2 - 4\beta}}{2\sqrt{\alpha}}] + \sqrt{-\alpha r^2 - 4\beta} \sin[c_1 \pi - \frac{(t + c_2 \alpha)\sqrt{-\alpha r^2 - 4\beta}}{2\sqrt{\alpha}}]),
\]

where \( c_1, c_2, \) and \( c_3 \) are arbitrary constants.

3.5. **Extended PS procedure for the coupled first-order ODEs.** In this section, we introduce the extended PS method for the investigation of the solutions of coupled nonlinear ODEs. For this purpose, let us suppose a system of the coupled first-order ODE of the form

\[
\dot{p} = \phi_1(t, p, q), \quad \dot{q} = \phi_2(t, p, q),
\]

in which \( \phi_i, \quad i = 1, 2 \) are analytic functions. For the equation (56) having a first integral \( I(t, p, q) = C \), with \( C \) constant on the solutions, the total differential is expressed as

\[
dI = I_1 dt + I_p dp + I_q dq = 0,
\]

and the equation (56) can be rewritten in the form

\[
\phi_1 dt - dp = 0, \quad \phi_2 dt - dq = 0.
\]

It is clear that the one-forms (57) and (58) must be proportional to the solutions. Let assume that \( R(t, p, q) \) and \( K(t, p, q) \) are the integrating factors for the first and second equations in (58), respectively. If the first equation in (58) is multiplied by
the factor $R$ and the second equation in (58) is multiplied by the factor $K$, then the following expression

$$dI = (R\phi_1 + K\phi_2)dt - Rdp - Kdq = 0 \quad (59)$$

can be written. By comparing the equations (59) and (57), it yields to

$$I_t = (R\phi_1 + K\phi_2), \quad I_p = -R, \quad I_q = -K. \quad (60)$$

Based on the compatibility conditions, $I_{tp} = I_{pt}, \quad I_{tq} = I_{qt}, \quad I_{pq} = I_{qp}$, the equations for the functions $R$ and $K$ from (60)

$$R_t + \phi_1 R_p + \phi_2 R_q = -(R\phi_1_p + K\phi_2_p),$$
$$K_t + \phi_1 K_p + \phi_2 K_q = -(R\phi_1_q + K\phi_2_q), \quad (61)$$

are obtained. If the equations in (60) are integrated, then the integral of the motion (first integral)

$$I = r_1 + r_2 - \int \left[ K + \frac{d}{dq}(r_1 + r_2) \right], \quad (62)$$

can be obtained, where

$$r_1 = \int (R\phi_1 + K\phi_2)dt \quad \text{and} \quad r_2 = -\int (R + \frac{d}{dp}(r_1))dp. \quad (63)$$

However, it is a fact that the determination of the solution for the equations (61) is not straightforward and then the following transformation [7] can be proposed

$$R = SK, \quad (64)$$

where $S$ is called a null form, which is assumed to be a function of variables $t,p,$ and $q$. Thus, the system of equations (61) can be rewritten in terms of functions $S$ and $K$ such that

$$S_t + \phi_1 S_p + \phi_2 S_q = -\phi_2 q + (\phi_2_q - \phi_1_p)S + \phi_1 q S^2, \quad (65)$$
$$K_t + \phi_1 K_p + \phi_2 K_q = -K(S\phi_1_q + \phi_2 q), \quad (66)$$
$$K_p = SK_q + KS_q, \quad (67)$$

where the last equation (67) is called the compatibility equation.

3.6. **Interconnections between PS method and the other approaches.** This section represents how the PS method can be applied to the coupled nonlinear first-order ODEs. Based on the algorithm given above, if the functions $S$ and $K$ are known and then the first integral of (56) can be derived. On the other hand, the determining equations related to the functions $S$ and $K$ (65) and (66) cannot be solved easily. However, it can be shown that the mathematical relations among PS method, Lie point symmetry, Jacobi last multiplier, adjoint symmetry, $\lambda$-symmetry, and Darboux polynomial [17] can provide an advantage for the determination of functions $S$ and $K$. 
3.6.1. The connection between Lie point symmetries and the PS method. Let us assume that \(\xi(t,p,q), \eta_1(t,p,q),\) and \(\eta_2(t,p,q)\) are the Lie point symmetries of (56). The characteristic notation \(Q\) can be written as
\[
Q_1 = \eta_1 - \dot{\xi} = \eta_1 - \phi_1 \xi, \quad Q_2 = \eta_2 - \dot{\eta} = \eta_2 - \phi_2 \xi,
\]
that are related with \(\phi_1\) and \(\phi_2\), respectively. Hence, the relation between Lie symmetries and PS method can be given by the definition
\[
S = -g = \frac{Q_1}{Q_2},
\]
in which \(S\) null form is the solution of (65).

**Remark 12.** If the Lie symmetries of the equation (56) are known, then by using the relation (69), it is clear that the equation (65) is satisfied. Thus, via the equations (66) and (67), the process of the PS method can be applied.

3.6.2. The connection between Lie point symmetries and JLM method. The method of JLM enables us to find the multipliers of the equation (56) if the Lie symmetries of this equation are known [15, 27, 28]. If \((\xi_1, \eta_{11}, \eta_{12})\) and \((\xi_2, \eta_{21}, \eta_{22})\) are two independent Lie symmetries of (56), then the inverse of the following determinant correspond to Jacobi last multiplier
\[
\Delta = \det \begin{vmatrix} 1 & \dot{p} & \dot{q} \\ \xi_1 & \eta_{11} & \eta_{12} \\ \xi_2 & \eta_{21} & \eta_{22} \end{vmatrix},
\]
namely \(JLM = \frac{1}{\Delta}\).

**Remark 13.** Based on the JLM method, any ratio of two multipliers gives to the first integral of (56), namely, \(I = \frac{JLM_1}{JLM_2}\). It is clear that by using the equation (60), the integrating factors \(R\) and \(K\) can be determined via the equations (65-67).

3.6.3. The connection between adjoint symmetries and PS method. In the literature, the integrating factors and first integrals for systems of ODEs are investigated and it is demonstrated that each integrating factor must be an adjoint symmetry, and the adjoint-invariance condition for an adjoint symmetry corresponds to an integrating factor [6, 2].

First, let us suppose that \(\Lambda_1\) and \(\Lambda_2\) are the adjoint symmetries of (56), then they must satisfy the adjoint equation for the linearized symmetry conditions
\[
D[\Lambda_1] = -(\Lambda_1 \phi_1 + \Lambda_2 \phi_2), \quad D[\Lambda_2] = -(\Lambda_1 \phi_1 + \Lambda_2 \phi_2). \quad (70)
\]
Notice that the first two statements of the equation (61) overlap the above equation (70).

**Remark 14.** The connection between adjoint symmetries and PS method can be given by the integrating factors \(R\) and \(K\) such that
\[
\Lambda_1 = R, \quad \Lambda_2 = K, \quad \Lambda_{1q} = \Lambda_{2p}.
\]
3.6.4. The connection between $\lambda$-symmetries and PS method. Let $\xi_1(t,p,q)$, $\eta_{11}(t,p,q)$, and $\eta_{12}(t,p,q)$ are $\lambda$-symmetries of (56). With some specific choice of such that $\xi_1 = 0$, $\eta_{11} = 1$ and $\eta_{12} = \gamma$, via the connection $g = \gamma$ that is introduced in [17], then $\lambda$-function can be determined by using the below formula

$$\lambda = \phi_1 p + \phi_1 q \gamma.$$  

(73)

Remark 15. The connection between $\lambda$-symmetry and PS method emerges in the form

$$\gamma = -S,$$  

(74)

and the associated $\lambda$-symmetry for the system (56) is given by

$$V = \frac{\partial}{\partial p} - S \frac{\partial}{\partial q}.$$  

(75)

3.6.5. The connection between the Darboux polynomial and JLM method. There are no standard techniques to find first integrals of ODEs, however, there are several special methods widely discussed in the literature. One of them is the Darboux polynomial method. There are simple relations between Darboux polynomials and Jacobi’s last multipliers. The determining equation for the $F$ Darboux polynomial [11] of the equation (56) is defined by

$$D[F] = \alpha F,$$  

(76)

in which $\alpha = (\phi_{2q} + \phi_{1p})$ represents the corresponding cofactor.

Remark 16. The relation between Jacobi last multiplier and Darboux polynomial is given by [17]

$$F = JLM^{-1}.$$  

(77)

4. Economic growth model with logarithmic utility function. This section is devoted to analysis of growth model, which is introduced in [3], where represents the consumer’s utility maximization problem stated by

$$\max_c \int_0^\infty e^{-rt} \ln(c)dt,$$  

(78)

subject to the capital accumulation equation

$$\dot{k}(t) = k^\beta - (\delta - A)k - c, \quad 0 < \beta < 1,$$  

(79)

in which $c(t)$ is the consumption per person, $k(t)$ the capital-labor ratio, $A > 0$ the marginal product of capital, $\beta$ the capital share, $\delta$ the depreciation rate, and $r$ is the rate of time preferences. The model (78) has been investigated via the method of partial Lagrangian by Naz et al. in [23], by utilizing the current value Hamiltonian (78). Because of its nonlinearity, this model cannot be evaluated without using some parametric conditions. In the literature, only one first integral has been determined under the condition $\beta = \frac{r + \delta - A}{2\delta - A}$ [23].

4.1. The economic growth model involving current value Hamiltonian. If similar calculations in Section 3.1 are done for the economic growth model with logarithmic utility function (78), its current value Hamiltonian can be given by the expression

$$H(t, c, s, p) = \ln(c) + p(k^\beta - (\delta - A) - c)$$  

(80)
in which \( p(t) \) is the costate variable. FOCs for the optimal control problem (78) via the equations (8)-(10) can be derived

\[
c = \frac{1}{p},
\]

\[
\dot{k} = k^\beta - (\delta - A)k - c,
\]

\[
\dot{p} = -p(A + \beta k^{\beta - 1} - \delta) + pr.
\]

If first-order necessary conditions above are written by eliminating \( c(t) \) control variable, then the corresponding coupled nonlinear first-order system of ODE

\[
\dot{k} = k^\beta - (\delta - A)k - \frac{1}{p},
\]

\[
\dot{p} = -p(A + \beta k^{\beta - 1} - \delta) + pr,
\]

is obtained. It is clear that the system (82) is highly nonlinear, therefore, the analysis of its Lie point symmetries is very complicated. After lengthy calculations, it can be shown that only two different Lie point symmetries of the system (82) can be determined under the condition \( \beta = \frac{r + \delta - A}{\delta - A} \) in the following form

\[
X_{11} = \frac{e^{rt}}{r} \frac{\partial}{\partial t} + \frac{e^{rt}p}{r} \frac{(\delta - A + r)}{\partial p} + \frac{e^{rt}k}{r} \frac{(A - \delta)}{\partial k},
\]

\[
X_{12} = \frac{\partial}{\partial t}.
\]

For the system (82), there exist two different parametric conditions to determine Lie point symmetries of the system. Without any conditions, it can be shown that the general case of the system (82) has only trivial symmetry \( \frac{\partial}{\partial t} \). These specific conditions on the parameters are represented in the following cases.

Case 1. \( \beta = \frac{r + \delta - A}{\delta - A} \)

The system (82) has only two Lie point symmetries, and then the first integral of the system by using the JLM method cannot be determined since only one multiplier can be obtained in this way. Therefore, in this case, we focus on the Prelle-Singer procedure to analyze the economic growth model. If we consider the Lie point symmetry \( X_{11} \) to derive null form \( S \) from the equation (69) via Remark 12, then \( S \) function is derived as

\[
S_{11} = \frac{(pk - k \frac{\pi}{\pi - 2})(A - \delta)}{p^2(A - r - \delta)},
\]

which is a particular solution of (65). To solve the determining equation for \( K \) (66), the ansatz in the form

\[
K = \frac{S_d}{(A(t, p) + B(t, p)s)^z},
\]

is considered, where \( S_d \) is the denominator of \( S \) and \( z \) is a constant. Since \( K \) is in a rational form while taking differentiation or integration the form of the denominator remains the same but the power of the denominator decreases or increases by a unit order from that of the initial one [7]. If the calculations for the determining equations (65)-(67) are performed, and then the corresponding \( K \) function in the form (85) can be given by

\[
K_{11} = \Lambda_2 = \frac{pk}{r} \frac{\pi}{\pi - 2} c_1 \frac{\pi}{\pi - 2} (A - \delta),
\]
where $c_1$ is a constant. The other integrating factor $R$ is

$$R_{11} = \Lambda_1 = \frac{(pk^{1+\frac{1}{\gamma}} - 1)c_1^{\frac{1}{\alpha}}}{pr(r+\delta - A)} (A - \delta)^2,$$

where $\Lambda_1$ and $\Lambda_2$ are the adjoint symmetries of (82) satisfying the relation $\Lambda_{1k} = \Lambda_{2r}$. By applying the equations (62) and (63), the first integral of (82) is derived as

$$I_{11} = \frac{(pk^{1+\frac{1}{\gamma}} + t(r + \delta - A) - \ln(p))c_1^{\frac{1}{\alpha}} (A - \delta)^2}{r(r+\delta - A)}$$

which satisfies the relation $D_1 I = 0$ and thus the solution of this first integral yields

$$q(t) = \left(\frac{c_1 + t(A - r - \delta) + \ln(p(t))}{p(t)}\right)^{\frac{A - \delta}{\alpha - \frac{1}{\gamma}}},$$

$$p(t) = \ln(1 - c_2) = \frac{e^{(\beta - 1)(t + \frac{c_1}{\alpha - \frac{1}{\gamma}})(A - \delta)}(\beta - 1)(A - \delta)(\frac{A - \delta}{A - r - \delta})}{(r - A + \delta)^2}$$

where $c_1$ and $c_2$ are constants and $\ln(y) = \int \frac{dy}{\ln y}$, which is the logarithmic integral function.

For the system (82), its $\lambda$-function and $\lambda$-symmetry are given via Remark 15

$$\lambda_{11} = \frac{r - pk(A - \delta)(A + k^{\frac{1}{\gamma}} - r - \delta)}{pk(A - \delta)} - \frac{\partial}{\partial p} - \left(\frac{(pk - k^{\frac{1}{\gamma}})(A - \delta)}{p^2(A - r - \delta)}\right) \frac{\partial}{\partial k},$$

and the Darboux polynomial, which verify the condition (76), is given by

$$F_1 = \frac{r + \delta - A}{re^{-\frac{r}{\alpha}}}. $$

By using $X_{12}$ in (83), it is clear that $S$ null form can be derived easily. But the determination for the solution of (66) is not possible, so integrating factor $K$ cannot be obtained. Here the corresponding $S_{12}$, $\lambda_{12}$-function, and $\lambda_{12}$-symmetry are only given

$$S_{12} = \frac{(pk(A + k^{\frac{1}{\gamma}} - \delta) - 1)(A - \delta)}{p^2(A + k^{\frac{1}{\gamma}} - \delta)(r + \delta - A)},$$

$$\lambda_{12} = -A - k^{\frac{1}{\gamma}} + r - \frac{r}{pk(A + k^{\frac{1}{\gamma}} - \delta)} + \delta + \frac{r}{Apk - pk\delta},$$

and the Darboux polynomial, which verify the condition (76), is given by

$$V_{12} = \frac{\partial}{\partial p} - \left(\frac{(pk(A + k^{\frac{1}{\gamma}} - \delta) - 1)(A - \delta)}{p^2(A + k^{\frac{1}{\gamma}} - \delta)(r + \delta - A)}\right) \frac{\partial}{\partial k},$$

Case 2. $A = 2r + \delta$, $\beta = \frac{1}{2}$

The restriction on parameters $A$ and $\beta$ gives three dimensional Lie algebra for the system (82)
\[
X_{21} = \frac{e^{rt}}{r} \frac{\partial}{\partial t} + (-e^{rt}) \frac{\partial}{\partial p} + (2e^{rt}k) \frac{\partial}{\partial k}, \\
X_{22} = \frac{\partial}{\partial t}, \\
X_{23} = -\frac{1}{pr} \frac{\partial}{\partial t} + \frac{\partial}{\partial p} + \left(-\sqrt{k} + 2kr\right) \frac{\partial}{\partial k}.
\]
\[
F_2 = \frac{1}{2e^{-rt}r^2}.
\]

\[H(t, \bar{c}, \bar{k}, \bar{p}) = \ln(\bar{k}) + p(\bar{k}^\delta - (\delta - A) - \bar{c}),\]

and if FOCs are applied to (95) then we obtain

\[
\bar{c} = \frac{e^{-rt}}{\bar{p}}, \\
\dot{\bar{k}} = \bar{k}^\delta - (\delta - A)\bar{k} - c, \\
\dot{\bar{p}} = -\bar{p}(A + \beta\bar{k}^{\delta-1} - \delta).
\]

If one eliminates the control variable \(\bar{c}\) from (96) and then the coupled nonlinear first-order system of ODE

\[
\dot{\bar{k}} = \bar{k}^\delta - (\delta - A)\bar{k} - \frac{e^{-rt}}{\bar{p}}, \\
\dot{\bar{p}} = -\bar{p}(A + \beta\bar{k}^{\delta-1} - \delta),
\]

For this case, we give only corresponding null forms, \(\lambda\)-functions and \(\lambda\)-symmetries of the system (82) related to the Lie algebras \(X_{21}, X_{22}, \) and \(X_{23}\), respectively in the following Table 1.

### Table 1. Null forms, \(\lambda\)-functions and \(\lambda\)-symmetries of the system (82)

| Null Forms | \(\lambda\)-functions and \(\lambda\)-symmetries |
|------------|---------------------------------------------|
| \(S_{21} = \frac{2(-\sqrt{k} + pk)}{p^2}\) | \(\lambda_{21} = \frac{1}{2pk} - \frac{1}{\sqrt{k}} - r, \) \(V_{21} = \frac{\partial}{\partial p} - \left(\frac{2(-\sqrt{k} + pk)}{p^2}\right) \frac{\partial}{\partial k},\) |
| \(S_{22} = \frac{2\sqrt{k}(-1 + p(\sqrt{k}^{2} + 2kr))}{p^2(1 + 2\sqrt{k}r)}\) | \(\lambda_{22} = -\frac{1}{\sqrt{k}} + \frac{1}{2pk + 4prk^{\sqrt{k}/2} - r}, \) \(V_{22} = \frac{\partial}{\partial p} - \left(\frac{2\sqrt{k}(-1 + p(\sqrt{k}^{2} + 2kr))}{p^2(1 + 2\sqrt{k}r)}\right) \frac{\partial}{\partial k},\) |
| \(S_{23} = -\frac{2\sqrt{k}}{p^2}\) | \(\lambda_{23} = \frac{1}{2} \left(\frac{1}{pk} - \frac{1}{\sqrt{k}} - 2r\right), \) \(V_{23} = \frac{\partial}{\partial p} + \left(\frac{2\sqrt{k}}{p^2}\right) \frac{\partial}{\partial k}.\) |
is obtained. It can be seen that there are two different cases for the determination of Lie point symmetries of the system (97). The parametric constraints are considered the same as in Section 4.1.

**Case 1.** $\beta = \frac{r + \delta - A}{\delta + A}$

With the same procedure given in the previous parts, Lie symmetries of the system (97)

$$\tilde{X}_{11} = \frac{e^{rt}}{r} \frac{\partial}{\partial t} + \frac{e^{rt} p (\delta - A)}{r} \frac{\partial}{\partial p} + \frac{e^{rt} k(A - \delta)}{r} \frac{\partial}{\partial k},$$

$$\tilde{X}_{12} = \frac{\partial}{\partial t} + (-\bar{\rho} r) \frac{\partial}{\partial \bar{p}},$$

(98)

can be obtained. In addition, $\tilde{S}_{11}$ can be obtained via Lie symmetry $\tilde{X}_{11}$ in the following form

$$\tilde{S}_{11} = \frac{e^{-rt}(e^{rt} pk - k \bar{p}^2 \pi)(\delta - A)}{p^2(r - A + \delta)},$$

(99)

by deriving the integrating factor $K$, the ansatz form $K = \frac{\bar{S}_{11}}{(A_{t,p} + B_{t,p} \bar{p})^{\frac{r}{2}}}$ is used, where $\bar{S}_{11}$ is the denominator of $\tilde{S}_{11}$. If the determining equations (65)-(67) are solved and then $K_{11}$ is found as below

$$K_{11} = \frac{e^{rt} p(kc_1)^{\frac{\bar{p}^2 \pi}{\delta}} (A - \delta)}{r}.$$  

(100)

The other integrating factor can be obtained via (64) in the following form

$$\tilde{R}_{11} = \frac{(1 - e^{rt} pk^{1+\frac{\bar{p}^2 \pi}{\delta}})c_1^{\frac{\bar{p}^2 \pi}{\delta}} (A - \delta)^2}{pr(r - A + \delta)},$$

(101)

and the first integral of the system (97) is given by utilizing the equations (62)-(63)

$$\bar{I}_{11} = \frac{k^{\frac{\bar{p}^2 \pi}{\delta}} c_1^{\frac{\bar{p}^2 \pi}{\delta}} (A - \delta)^2 (e^{rt} pk^{\frac{\bar{p}^2 \pi}{\delta}} - k^{\frac{\bar{p}^2 \pi}{\delta}}(t(A - \delta) - \ln[p]))}{r(r - A + \delta)},$$

(102)

The corresponding $\lambda$-function and $\lambda$-symmetry, which are derived via the equation (73) and Remark 15, are

$$\bar{\lambda}_{11} = \frac{e^{-rt}(r - e^{-rt} \bar{p}k(A - \delta))(A + \bar{k}^{\frac{\bar{p}^2 \pi}{\delta}} - \delta)}{\bar{p}k(A - \delta)},$$

(103)

$$\bar{V}_{11} = \frac{\partial}{\partial \bar{p}} + \left(\frac{e^{-rt}(e^{rt} \bar{p}k - \bar{k}^{\frac{\bar{p}^2 \pi}{\delta}})(A - \delta)}{\bar{p}^2(\bar{p}^2 + r - A)}\right) \frac{\partial}{\partial \bar{k}}.$$  

The null form, $\lambda$-function and $\lambda$-symmetry corresponding to the Lie symmetry $\tilde{X}_{12}$ can be obtained easily. For this case, the related results are represented as below

$$\tilde{S}_{12} = -\frac{e^{-rt} + \bar{k}(A + \bar{k}^{\frac{\bar{p}^2 \pi}{\delta}} - \delta)(A - \delta)}{\bar{p}(A + \bar{k}^{\frac{\bar{p}^2 \pi}{\delta}} - \delta)(A - \delta)},$$

$$\tilde{\lambda}_{12} = -A - \bar{k}^{\frac{\bar{p}^2 \pi}{\delta}} + \frac{re^{-rt}}{\bar{p}k(1 + \bar{k}^{\frac{\bar{p}^2 \pi}{\delta}}(A - \delta))(A - \delta)} + \delta,$$

(104)

$$\tilde{V}_{12} = \frac{\partial}{\partial \bar{p}} + \left(\frac{e^{-rt} + \bar{k}(A + \bar{k}^{\frac{\bar{p}^2 \pi}{\delta}} - \delta)(A - \delta)}{\bar{p}(A + \bar{k}^{\frac{\bar{p}^2 \pi}{\delta}} - \delta)(A - \delta)}\right) \frac{\partial}{\partial \bar{k}}.$$
Case 2. $A = 2r + \delta$, $\beta = \frac{1}{2}$

For this case, the Lie point symmetries of the system (97) are

$$
\begin{align*}
\dot{X}_{21} &= \frac{e^{rt}}{p^2} \frac{\partial}{\partial t} + (-2e^{rt}) \frac{\partial}{\partial p} + (2e^{rt}k) \frac{\partial}{\partial k}, \\
\dot{X}_{22} &= \frac{\partial}{\partial t} + (-\bar{p}^r) \frac{\partial}{\partial \bar{p}}, \\
\dot{X}_{23} &= -\frac{e^{-rt}}{\bar{p}} \frac{\partial}{\partial t} + (2re^{-rt}) \frac{\partial}{\partial \bar{p}} + \left(\frac{e^{-rt}(\sqrt{k} + 2kr)}{\bar{p}r}\right) \frac{\partial}{\partial \bar{k}}.
\end{align*}
$$

The corresponding null forms $S$-functions, $\lambda$-functions and $\lambda$-symmetries, which relate to above Lie symmetries, are given in the following Table 2.

| Null Forms | $\lambda$-functions and $\lambda$-symmetries |
|------------|-----------------------------------------------|
| $\bar{S}_{21} = -\frac{2(e^{-rt} \sqrt{k-k\bar{p}})}{\bar{p}^2}$ | $\bar{\lambda}_{21} = \frac{e^{-rt}}{2\bar{p}} - \frac{1}{\sqrt{k}} - 2r,$ |
| $\bar{V}_{21} = \frac{\partial}{\partial \bar{p}} + \left(\frac{2(e^{-rt} \sqrt{k-k\bar{p}})}{\bar{p}^2}\right) \frac{\partial}{\bar{p}}$ | |
| $\bar{S}_{22} = -\frac{2(e^{-rt} \sqrt{k-2k\bar{p}(1+2\sqrt{k})})}{\bar{p}^2(1+2\sqrt{k})}$ | $\bar{\lambda}_{22} = -\frac{1}{\sqrt{k}} - 2r + \frac{e^{-rt}}{2\bar{p}^\beta+4\bar{p}k^{\beta+\gamma}},$ |
| $\bar{V}_{22} = \frac{\partial}{\partial \bar{p}} + \left(\frac{2(e^{-rt} \sqrt{k-2k\bar{p}(1+2\sqrt{k})})}{\bar{p}^2(1+2\sqrt{k})}\right) \frac{\partial}{\bar{p}}$ | |
| $\bar{S}_{23} = -\frac{2e^{-rt} \sqrt{k}}{\bar{p}^2}$ | $\bar{\lambda}_{23} = \frac{1}{2} \left(\frac{e^{-rt}}{\bar{p}} - \frac{1}{\sqrt{k}} - 4r\right),$ |
| $\bar{V}_{23} = \frac{\partial}{\partial \bar{p}} + \left(\frac{2e^{-rt} \sqrt{k}}{\bar{p}^2}\right) \frac{\partial}{\bar{p}}$ | |

5. Conclusion. In this study, we investigate the first integrals and the analytical solutions of optimal control problems of economic growth models by using JLM, $\lambda$-symmetry, and Prelle-Singer approaches, having a significant connection with Lie group theory. We both perform the current and the present value Hamiltonians to verify the necessary conditions for optimality. In this way, two different coupled first-order ODEs related to the economic models are taken into account in the study. The calculations show that the current Hamiltonian and the present Hamiltonian functions have different Lie point symmetries. In the case of the current Hamiltonian function, the first model has five-dimensional Lie algebra $L_5$. However, in the case of present value Hamiltonian, it has four-dimensional Lie algebra $L_4$. This can be pointed out as an important property for the analysis of the optimal control problems. Furthermore, after determining the corresponding Lagrangian and applying the Euler-Lagrange equations for the optimal control problem (23), we show that the same equivalent second-order ordinary differential equation in terms of the state variable function $q(t)$ for each Hamiltonian function can be obtained. As a result, we prove that closed-form solutions of optimal control problems can be analyzed by JLM and the $\lambda$-symmetry methods effectively. Furthermore, by the relation $q = \dot{q}$, we indicate that the properties $\dot{q} = u$ and $\ddot{q} = \ddot{u}$ are valid and then
the relation $u = \bar{u}$ is satisfied. Additionally, for the relations $q = \bar{q}$ and $u = \bar{u}$, we show that the properties $p = \bar{p}e^r$ and $H = \bar{H}e^r$ are satisfied.

Furthermore, in this study, we represent the analytical solution methods such as JLM, $\lambda$-symmetry, and Prelle-Singer for solving optimal control problems in economic growth theory. For the second nonlinear model, which is called the economic growth model with a logarithmic utility function, the effectiveness of the Prelle-Singer method for two coupled first-order ODEs has been illustrated since the JLM method is not appropriate to obtain the first integral of the aforementioned system. It is known that the analytical solution procedure of these problems is not very common, but mostly numerical and steady-state solutions have been reported in the literature. Thus, the current study can be considered to represent a novel approach for the analytical analysis of economic growth models, especially for the optimal control problems.

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