CONVEXITY PROPERTIES OF GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Abstract. We study the power mean inequality for the generalized trigonometric and hyperbolic functions with two parameters. The generalized $p$-trigonometric and $(p, q)$-trigonometric functions were introduced by Lindqvist and Takeuchi, respectively.

1. Introduction and Main Results

The generalized trigonometric functions were introduced by Lindqvist [17] two decades ago. These $p$-trigonometric functions, $p > 1$, coincide with the usual trigonometric functions for $p = 2$. Recently, the $p$-trigonometric functions have been studied extensively, see for example [5, 8, 11, 16] and their references. Drábek and Manásevich [11] considered a certain $(p, q)$-eigenvalue problem with the Dirichlet boundary condition and found the complete solution to the problem. The solution of a special case is the function $\sin_{p,q}$, which is the first example of so called $(p, q)$-trigonometric function. Motivated by the $(p, q)$-eigenvalue problem Takeuchi [19] introduced the $(p, q)$-trigonometric functions, $p, q > 1$. These $(p, q)$-trigonometric functions have recently been studied also in [6, 7, 9, 10, 12], and the functions agree with the $p$-trigonometric functions for $p = q$.

The Gaussian hypergeometric function is the analytic continuation to the slit plane $\mathbb{C}\setminus[1, \infty)$ of the series

$$F(a, b; c; z) = \sum_{n \geq 0} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \quad |z| < 1,$$

for given complex numbers $a, b$ and $c$ with $c \neq 0, -1, -2, \ldots$. Here $(a, 0) = 1$ for $a \neq 0$, and $(a, n)$ is the shifted factorial function or the Appell symbol

$$(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1)$$

for $n = 1, 2, \ldots$. The hypergeometric function is used to define the $(p, q)$-trigonometric functions and it has numerous special functions as its special or limiting cases [1].

For $p, q > 1$ and $x \in (0, 1)$ the function $\arcsin_{p,q}$ is defined by

$$\arcsin_{p,q}(x) = \int_0^x (1 - t^q)^{-1/p} dt = x F\left(\frac{1}{p}, \frac{1}{q}; 1 + \frac{1}{q}; x^q\right),$$

We also define for $x \in (0, 1)$ $\arccos_{p,q}(x) = \arcsin_{p,q}((1 - x^p)^{1/q})$ (see [12] Prop. 3.1) and for $x \in (0, \infty)$

$$\arcsinh_{p,q}(x) = \int_0^x (1 + t^p)^{-1/p} dt = x F\left(\frac{1}{p}, \frac{1}{q}; 1 + \frac{1}{q}; -x^q\right) = \left(\frac{x^p}{1 + x^q}\right)^{1/p} F\left(1, \frac{1}{p}, \frac{1}{q}; 1 + \frac{x^q}{1 + x^q}\right).$$

Their inverse functions are given by

$$\sin_{p,q}, \cos_{p,q} : (0, \pi_{p,q}/2) \rightarrow (0, 1), \quad \sinh_{p,q} : (0, \infty) \rightarrow (0, \infty),$$

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where $\pi_{p,q}/2 = \arcsin_{p,q}(1)$. Now, we define $\tan_{p,q} : (0, \pi_{p,q}/2) \to (0, \infty)$ by
\[
\tan_{p,q}(x) = \frac{\sin_{p,q}(x)}{\cos_{p,q}(x)}, \quad \cos_{p,q}(x) \neq 0,
\]
and we denote its inverse by $\arctan_{p,q}$.

We introduce next a geometric definition for the functions $\sin_{p,q}$ and $\cos_{p,q}$. The definition is based on the formula [12, eq. (2.7)]
\[
|\sin_{p,q}(x)|^q + |\cos_{p,q}(x)|^p = 1.
\]
The usual trigonometric functions are geometrically defined by the unit circle. In the same manner we can define the $(p, q)$-trigonometric functions. Instead of the unit circle we use the Lamé-curve (or generalized superellipse) defined by
\[
C = \{(x, y) \in \mathbb{R}^2 : x = \cos^{2/p} t, y = \sin^{2/q} t, t \in [0, \pi/2]\}.
\]
Because $(\sin^{2/q} t)^q + (\cos^{2/p} t)^p = 1$ it is clear by (1.1) that $C = D$ for
\[
D = \{(x, y) \in \mathbb{R}^2 : x = \sin_{p,q}(t), y = \cos_{p,q}(t), t \in [0, \pi_{p,q}/2]\}.
\]
The curve $C$ is defined only in the first quadrant, but by reflections we can easily extend $C$ to form a circular curve, see Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Left: Curve $C$ for $(p, q) = (4, 3)$ and points $(0, \sin_{4,3} x), (\cos_{4,3} x, 0)$ and $(\cos_{4,3} x, \sin_{4,3} x)$. Right: Curve $C$ for $(p, q) = (4, 3)$ extended to all four quadrants (black) and the unit circle (gray).}
\end{figure}

Now, we recall the definition of convex functions with respect to power means. For $a \in \mathbb{R}$ and $x, y > 0$, the power mean $M_a$ of order $a$ is defined by
\[
M_a(x, y) = \begin{cases} 
\left(\frac{x^a + y^a}{2}\right)^{1/a}, & a \neq 0, \\
\sqrt[2]{xy}, & a = 0.
\end{cases}
\]
We consider the function $f : I \subset (0, \infty) \to (0, \infty)$, which is continuous, and let $M_a(x, y)$ and $M_b(x, y)$ be the power means of order $a$ and $b$ of $x > 0$ and $y > 0$. For $a, b \in \mathbb{R}$ we say that $f$ is $M_a M_b$-convex (concave) or simply $(a, b)$-convex (concave), if for those $a$ and $b$ we have
\[
f(M_a(x, y)) \leq (\geq) M_b(f(x), f(y)) \text{ for all } x, y \in I.
\]
It is important to mention here that $(1, 1)$-convexity means the usual convexity, $(1, 0)$ means the log-convexity and $(0, 0)$-convexity is the so-called geometrical (multiplicative) convexity. Moreover, it is known that if the function $f$ is differentiable, then it is $(a, b)$-convex (concave) if and only if $x \mapsto x^{1-a} f'(x)/(f(x))^{b-1}$ is increasing (decreasing). See [3] for more details.

In the next results we can see that the above mentioned $(p, q)$ generalized trigonometric functions are $(a, a)$-convex/concave. These results refine the earlier results in [6].
Theorem 1. If \( p, q > 1 \) and \( a \geq 1 \), then \( \arcsin_{p,q} \) is \((a, a)\)-convex on \((0, 1)\), \( \arctan_{p,q} \) is \((a, a)\)-concave on \((0, 1)\), while \( \arsinh_{p,q} \) is \((a, a)\)-concave on \((0, \infty)\). In other words, if \( p, q > 1 \) and \( a \geq 1 \), then we have

\[
\arcsin_{p,q}(M_a(r, s)) \leq M_a(\arcsin_{p,q}(r), \arcsin_{p,q}(s)), \quad r, s \in (0, 1),
\]

\[
\arctan_{p,q}(M_a(r, s)) \geq M_a(\arctan_{p,q}(r), \arctan_{p,q}(s)), \quad r, s \in (0, 1),
\]

\[
\arsinh_{p,q}(M_a(r, s)) \geq M_a(\arsinh_{p,q}(r), \arsinh_{p,q}(s)), \quad r, s > 0.
\]

Theorem 2. If \( p, q > 1 \) and \( a \geq 1 \), then \( \sin_{p,q} \) is \((a, a)\)-concave, and \( \cos_{p,q} \), \( \tan_{p,q} \), \( \sinh_{p,q} \) are \((a, a)\)-convex on \((0, 1)\). In other words, if \( p, q > 1 \), \( a \geq 1 \) and \( r, s \in (0, 1) \), then the next inequalities are valid

\[
\sin_{p,q}(M_a(r, s)) \geq M_a(\sin_{p,q}(r), \sin_{p,q}(s)),
\]

\[
\cos_{p,q}(M_a(r, s)) \leq M_a(\cos_{p,q}(r), \cos_{p,q}(s)),
\]

\[
\tan_{p,q}(M_a(r, s)) \leq M_a(\tan_{p,q}(r), \tan_{p,q}(s)),
\]

\[
\sinh_{p,q}(M_a(r, s)) \leq M_a(\sinh_{p,q}(r), \sinh_{p,q}(s)).
\]

The next theorems improve some of the above results.

Theorem 3. If \( p, q > 1 \), \( a \leq 0 \) and \( b \in \mathbb{R} \) or \( 0 < a \leq b \) and \( b \leq 1 \), then \( \arcsin_{p,q} \) is \((a, b)\)-convex on \((0, 1)\), and in particular if \( p = q \), then the function \( \arcsin_p = \arcsin_{p,p} \) is \((a, b)\)-convex on \((0, 1)\). In other words, if \( p, q > 1 \), \( a \leq 0 \) and \( b \in \mathbb{R} \) or \( 0 < a \leq b \) and \( b \leq 1 \), then for all \( r, s \in (0, 1) \) we have

\[
\arcsin_{p,q}(M_a(r, s)) \leq M_b(\arcsin_{p,q}(r), \arcsin_{p,q}(s)).
\]

Theorem 4. If \( p, q > 1 \), \( a \leq 0 \geq b \) or \( 0 < b \leq a \) and \( a \leq 1 \), then \( \arsinh_{p,q} \) is \((a, b)\)-concave on \((0, \infty)\), and in particular if \( p = q \), then the function \( \arsinh_p = \arsinh_{p,p} \) is \((a, b)\)-concave on \((0, \infty)\). In other words, if \( p, q > 1 \), \( a \leq 0 \geq b \) or \( 0 < b \leq a \) and \( a \leq 1 \), then for all \( r, s \in (0, \infty) \) we have

\[
\arsinh_{p,q}(M_a(r, s)) \geq M_b(\arsinh_{p,q}(r), \arsinh_{p,q}(s)).
\]

It is worth to mention that Chu et al. \([10]\) very recently proved implicitly the \((p_1, 0)\)-convexity of \( \arcsin_{p,q} \) for \( p_1 \leq 0 \), \( p, q > 1 \), and the \((p_2, 0)\)-concavity of \( \arsinh_{p,q} \) for \( p_2 \geq 0 \), \( p, q > 1 \). Our approach is similar but somewhat easier, since we applied in the above cases directly the results from \([3]\). We also mention that recently Jiang and Qi \([13]\) proved that the \( \arcsin_{p,q} \) is \((0, 0)\)-convex, by using the same idea as we used in this paper. See also \([4, 5]\) for more details.

Now, observe that by using the change of variable \( u = (1 - t^p)^{1/p} \)

\[
\arccos_{p,q}(x) = \arcsin_{p,q}(y) = \int_0^y (1 - t^p)^{-1/p} dt, \quad \text{where} \quad y = (1 - x^p)^{1/q},
\]

can be written as

\[
\arccos_{p,q}(x) = \frac{p}{q} \int_0^1 f(u) du, \quad \text{where} \quad f(u) = (1 - u^p)^{1/q - 1} u^{p - 2}.
\]

Consequently, we have

\[
\int_0^x f(u) du = \frac{q}{p} \arccos_{p,q}(0) - \frac{q}{p} \arccos_{p,q}(x) = \frac{q}{p} \arcsin_{p,q}(1) - \frac{q}{p} \arcsin_{p,q}(x).
\]

The next result is about this integral.

Theorem 5. The function \( x \mapsto \pi_{p,q}/2 - \arccos_{p,q}(x) \) is \((a, b)\)-convex on \((0, 1)\) if \( p \in (1, 2] \), \( q > 1 \), \( a < 0 \) and \( b \in \mathbb{R} \) or if \( p, q > 1 \), \( a \leq 0 \) and \( b \geq 0 \). In other words, if \( p \in (1, 2] \), \( q > 1 \), \( a < 0 \) and \( b \in \mathbb{R} \) or if \( p, q > 1 \), \( a \leq 0 \) and \( b \geq 0 \), then for all \( r, s \in (0, 1) \) we have

\[
\pi_{p,q}/2 - \arccos_{p,q}(M_a(r, s)) \leq M_b(\pi_{p,q}/2 - \arccos_{p,q}(r), \pi_{p,q}/2 - \arccos_{p,q}(s)).
\]
It is worth to mention that if in the above theorem we take \( a = b = 0 \), then we have that the function \( x \mapsto \arcsin_{p,q}(1) - \arccos_{p,q}(x) \) is geometrically convex on \((0,1)\) for \( p, q > 1 \), and in particular if \( p = q \), then the function \( x \mapsto \arcsin_{p,p}(1) - \arccos_{p,p}(x) \) is geometrically convex on \((0,1)\) for \( p > 1 \). In other words, for \( p, q > 1 \) and \( r, s \in (0,1) \) we have
\[
\pi_{p,q}/2 - \arccos_{p,q}(\sqrt{rs}) \leq \sqrt{(\pi_{p,q}/2 - \arccos_{p,q}(r))(\pi_{p,q}/2 - \arccos_{p,q}(s))}. 
\]

2. Preliminary results

In this section we present some preliminary results which will be used in the proof of the main theorems. The first known result considers the \((p, q)\)-trigonometric functions whereas the other results consider properties of real functions.

Lemma 1. [12] For all \( p, q > 1 \) and \( x \in (0, \pi_{p,q}/2) \), we have
\[
(sin_{p,q}(x))' = cos_{p,q}(x), \quad (cos_{p,q}(x))' = -\frac{p}{q}(cos_{p,q}(x))^{2-p}(sin_{p,q}(x))^{q-1}, \quad (tan_{p,q}(x))' = 1 + \frac{p}{q}(sin_{p,q}(x))^{q}(cos_{p,q}(x))^{-p}.
\]

The following result is known as the monotone form of l’Hospital’s rule.

Lemma 2. [2] Theorem 1.25] For \(-\infty < a < b < \infty\), let \( f, g : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\), and be differentiable on \((a, b)\). Let \( g'(x) \neq 0 \) on \((a, b)\). If \( f'/g' \) is increasing (decreasing) on \((a, b)\), then so are
\[
x \mapsto \frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad x \mapsto \frac{f(x) - f(b)}{g(x) - g(b)}.
\]

If \( f'/g' \) is strictly monotone, then the monotonicity in the conclusion is also strict.

The following two results consider the convexity of the inverse function.

Lemma 3. [15] Theorem 2] Let \( J \subset \mathbb{R} \) be an open interval, and let \( f : J \to \mathbb{R} \) be strictly monotonic function. Let \( f^{-1} : f(J) \to J \) be the inverse to \( f \). If \( f \) is convex and increasing, then \( f^{-1} \) is concave.

Lemma 4. [18] Proposition 2] Let \( f : (a, b) \to (c, d) = f(a, b) \subset \mathbb{R} \) be a convex function and let \( f^{-1} : (c, d) \to (a, b) \) be its inverse. If \( f \) is increasing (decreasing), then \( f^{-1} \) is increasing (decreasing) and concave (convex).

The following two results consider the \((p, q)\)-convexity of the function.

Lemma 5. [3] Lemma 3] Let \( a, b \in \mathbb{R} \) and let \( f : [\alpha, \beta] \to (0, \infty) \) be a differentiable function for \( \alpha, \beta \in (0, \infty) \). The function \( f \) is \((a, b)\)-convex \((a, b)\)-concave) if and only if \( x \mapsto x^{1-a}f'(x)(f(x))^{b-1} \) is increasing (decreasing).

Lemma 6. [3] Theorem 7] Let \( \alpha, \beta \in (0, \infty) \) and \( f : [\alpha, \beta] \to (0, \infty) \) be a differentiable function. Denote \( g(x) = \int_x^\alpha f(t) \, dt \) and \( h(x) = \int_x^\beta f(t) \, dt \). Then
(a) If for all \( a \in [0,1] \) the function \( f \) is \((a,0)\)-concave, then the function \( g \) is \((a, b)\)-concave for all \( a \in [0,1] \) and \( b \leq 0 \). In addition the function \( x \mapsto x^{1-a}f(x) \) is increasing for all \( a \in [0,1] \) and \( b \leq 0 \). Moreover, if for all \( a \in \mathbb{R} \) the function \( x \mapsto x^{1-a}f(x) \) is increasing, then \( g \) is \((a, b)\)-convex for all \( a \in \mathbb{R} \) and \( b \geq 1 \).
(b) If for all \( a \in [0,1] \) the function \( f \) is \((a,0)\)-concave, then the function \( g \) is \((a, b)\)-concave for all \( a \in [0,1] \) and \( b \leq 0 \). In addition the function \( x \mapsto x^{1-a}f(x) \) is decreasing for all \( a \in [0,1] \) and \( b \geq 1 \). Moreover, if for all \( a \in \mathbb{R} \) the function \( x \mapsto x^{1-a}f(x) \) is decreasing, then \( g \) is \((a, b)\)-convex for all \( a \in \mathbb{R} \) and \( b \geq 1 \).
(c) If for all \( a \notin (0,1) \) we have \( \alpha^{1-a} f(\alpha) = 0 \) and the function \( f \) is \((a,0)\)-convex, then \( g \) is \((a,b)\)-convex for all \( a \notin (0,1) \) and \( b \geq 0 \). If, in addition the function \( x \mapsto x^{1-a} f(x) \) is increasing for all \( a \notin (0,1) \), then \( g \) is \((a,b)\)-convex for all \( a \notin (0,1) \) and \( b < 0 \).

(d) If for all \( a \notin (0,1) \) we have \( \beta^{1-a} f(\beta) = 0 \) and the function \( f \) is \((a,0)\)-convex, then \( g \) is \((a,b)\)-convex for all \( a \notin (0,1) \) and \( b \geq 0 \). If, in addition the function \( x \mapsto x^{1-a} f(x) \) is decreasing for all \( a \notin (0,1) \), then \( g \) is \((a,b)\)-convex for all \( a \notin (0,1) \) and \( b < 0 \).

Next we introduce a convexity result for the functions \( \tan_{p,q} \) and \( \arctan_{p,q} \).

**Lemma 7.** If \( p, q > 1 \), then the function \( \tan_{p,q} \) is increasing and convex on \((0, \pi_{p,q}/2)\), while the function \( \arctan_{p,q} \) is increasing and concave on \((0, 1)\).

**Proof.** By Lemma 1, we get
\[
[\tan_{p,q}(x)]'' = 1 + \frac{p}{q^2} \left[ \frac{pq (\sin_{p,q}(x))^{q-1} (\cos_{p,q}(x))^{1+p} + p^2 \cos_{p,q}(x) (\sin_{p,q}(x))^{2q-1}}{(\cos_{p,q}(x))^{2p}} \right],
\]
which is positive since both \( \sin_{p,q} \) and \( \cos_{p,q} \) are positive. Hence \( \tan_{p,q} \) is convex. By Lemma 1, we observe that the derivative of \( \tan_{p,q} \) is positive and thus \( \tan_{p,q} \) is increasing. It follows from Lemma 7 that \( \arctan_{p,q} \) is increasing and concave. \( \square \)

Now, we focus on some monotonicity results.

**Lemma 8.** Let \( a \geq 0 \), \( p, q > 1 \) and consider the functions \( f, h : (0, 1) \to \mathbb{R} \) and \( g : (0, \infty) \to \mathbb{R} \), defined by
\[
\begin{align*}
  f(x) &= \left( \frac{\arcsin_{p,q}(x)}{x} \right)^{a} (\arcsin_{p,q}(x))', \\
  g(x) &= \left( \frac{\arcsinh_{p,q}(x)}{x} \right)^{a} (\arcsinh_{p,q}(x))', \\
  h(x) &= \left( \frac{\arctan_{p,q}(x)}{x} \right)^{a} (\arctan_{p,q}(x))'.
\end{align*}
\]

Then the function \( f \) is increasing and the functions \( g \) and \( h \) are decreasing.

**Proof.** It follows from the definition that
\[
  x \mapsto (\arcsin_{p,q}(x))' = (1 - x^q)^{-1/p},
\]
is increasing since
\[
(\arcsin_{p,q}(x))'' = \frac{q x^{q-1}(1 - x^q)^{-1-1/p}}{p} > 0.
\]
By Lemma 2, the function \( x \mapsto (\arcsin_{p,q} x)/x \) is increasing \((0,1)\) and consequently the function \( x \mapsto ((\arcsin_{p,q} x)/x)^a \) is increasing too on \((0,1)\). Since the product of increasing functions is increasing, the function \( f \) is increasing. Now, the function \( \arctan_{p,q} \) is increasing and concave \((0,1)\) by Lemma 7, which implies that \( x \mapsto (\arctan_{p,q}(x))' \) is decreasing on \((0,1)\). Using again Lemma 2, it follows that \( x \mapsto (\arctan_{p,q}(x))/x \) is decreasing on \((0,1)\), and hence is the function \( h \). Finally, the proof for function \( g \) follows similarly as the proof for \( f \), because
\[
(\arcsinh_{p,q}(x))'' = \frac{-q x^{q-1}(1 + x^q)^{-1-1/p}}{p} < 0. \quad \square
\]

**Lemma 9.** If \( a \geq 0 \) and \( p, q > 1 \), then the function
\[
  x \mapsto \left( \frac{\sin_{p,q}(x)}{x} \right)^{a} (\sin_{p,q}(x))'
\]
is decreasing on \((0,1)\), while the functions
\[
  x \mapsto \left( \frac{\cos_{p,q}(x)}{x} \right)^{a} (\cos_{p,q}(x))', \quad x \mapsto \left( \frac{\tan_{p,q}(x)}{x} \right)^{a} (\tan_{p,q}(x))'
\]
and

\[ x \mapsto \left( \frac{\sinh_{p,q}(x)}{x} \right)^a (\sinh_{p,q}(x))' \]

are increasing on \((0,1)\).

**Proof.** Let \(f(x) = \arcsin_{p,q}(x)\), where \(x \in (0,1)\). We get

\[ f'(x) = \frac{1}{(1-x^p)^{1/p}}, \]

which is positive and increasing, hence \(f\) is convex. Clearly \(\sin_{p,q}\) is increasing, and by Lemma 2 is concave. This implies that \(x \mapsto (\sin_{p,q}(x))'\) is decreasing, and \(x \mapsto (\sin_{p,q}(x))/x\) is also decreasing by Lemma 2.

Similarly we get that \(x \mapsto (\cos_{p,q}(x))'\), \(x \mapsto (\tan_{p,q}(x))'\) and \(x \mapsto (\sin_{p,q}(x))'\) are increasing, and the assertion follows from Lemma 2.

3. Proofs of the main results

**Proof of the Theorem 1** Let \(0 < x < y < 1\), and \(u = ((x^a + y^a)/2)^{1/a} > x\). Let us denote \(h_1(x) = \arcsin_{p,q}(x), h_2(x) = \arctan_{p,q}(x), h_3(x) = \arcsinh_{p,q}(x)\) and for \(i \in \{1,2,3\}\) define

\[ g_i(x) = h_i(u) - \frac{h_i(x)^a + h_i(y)^a}{2}. \]

Differentiating with respect to \(x\), we get \(du/dx = (1/2)(x/u)^{a-1}\) and

\[ g_i'(x) = \frac{1}{2} a h_i(x)^{a-1} \frac{d}{dx}(h_i(u)) \left( \frac{x}{u} \right)^{a-1} - \frac{1}{2} a h_i(x)^{a-1} \frac{d}{dx}(h_i(x)) = \frac{a}{2} x^{a-1}(f_i(u) - f_i(x)) \]

where

\[ f_i(x) = \left( \frac{h_i(x)}{x} \right)^{a-1} \frac{d}{dx}(h_i(x)). \]

By Lemma 8 \(g_i'\) is positive and \(g_2', g_3'\) are negative. Hence \(g_1\) is increasing and \(g_2, g_3\) are decreasing. This implies that \(g_1(x) < g_1(y) = 0\), \(g_2(x) > g_2(y) = 0\), \(g_3(x) > g_3(y) = 0\), and the assertion follows.

**Proof of the Theorem 2** The proof is similar to the proof of Theorem 1 and follows from Lemma 8.

**Proof of Theorem 3** Let us consider the function \(f : (0,1) \rightarrow (0, \infty), f(t) = (1 - t^q)^{-1/p}.\) If \(a \leq 0\), then the function

\[ t \mapsto \frac{t^{1-a} f'(t)}{f(t)} = \frac{q}{p} t^{q-a} \]

is increasing on \((0,1)\), that is \(f\) is \((a,0)\)-convex on \((0,1)\), according to Lemma 3. Since \(t^{1-a}(1 - t^q)^{-1/p} \rightarrow 0\), as \(t \rightarrow 0\), and \(t^{1-a}(1 - t^q)^{-1/p}\) is increasing for \(a \leq 0\) as a product of two increasing and positive functions from Lemma 3 we deduce that \(x \mapsto \arcsin_{p,q}(x)\) is \((a,b)\)-convex for \(a \leq 0\) and \(b \geq 0\) or \(a \leq 0\) and \(b < 0\), that is, for \(a \leq 0\) and \(b \in \mathbb{R}\).

According to Theorem 1 the function \(x \mapsto \arcsin_{p,q}(x)\) is \((a,a)\)-convex for \(a \geq 1\) and \(p,q > 1\). Taking into account that the power mean is increasing with respect to its order it follows that \(x \mapsto \arcsin_{p,q}(x)\) is \((a,b)\)-convex on \((0,1)\) for \(1 \leq a \leq b\) and \(p,q > 1\).

Let \(u,v : (0,1) \rightarrow (0, \infty)\) be the functions defined by \(u(x) = x^{1-a}\) and \(v(x) = \varphi(x)(\varphi(x))^{b-1}\) where \(\varphi(x) = \arcsin_{p,q}(x)\). Then we have that \(u'(x) = (1-a)x^{-a} \geq 0\) if \(a \leq 1\), \(x \in (0,1)\), and

\[ v'(x) = (\varphi(x))^{b-2}(\varphi''(x))\varphi(x) + (b-1)(\varphi'(x))^2 \geq 0 \]
if $b \geq 1$ and $x \in (0,1)$, since $\varphi$ is convex. Thus we obtain $x \mapsto x^{1-a} \varphi'(x)(\varphi(x))^{b-1}$ is decreasing on $(0,1)$ if $a \leq 1$ and $b \geq 1$. According to Lemma \[\Box\] it follows that $x \mapsto \arcsin_{p,q}(x)$ is $(a,b)$-convex on $(0,1)$ for $a \leq 1$ and $b \geq 1$.

Finally, we would like to show that the $(a,b)$-convexity of $x \mapsto \arcsin_{p,q}(x)$ is the case when $b \geq a$ and $b \geq 1$ can be proved by a somewhat different approach. For this, consider the function $\varphi : (0,1) \to (0,\infty)$, defined by $\varphi(x) = x^{1-a} \varphi'(x)(\varphi(x))^{b-1}$. Then we have

$$
\Delta(x) = \frac{x\varphi'(x)}{\varphi(x)} = 1 - a + \frac{q}{p} \frac{x^q}{1-x^q} + (b-1) \frac{x\varphi'(x)}{\varphi(x)},
$$

and

$$
\Delta'(x) = \frac{q}{p} \frac{q x^{q-1}}{(1-x^q)^2} + (b-1) \left( \frac{x\varphi'(x)}{\varphi(x)} \right)' \geq 0
$$

if $x \in (0,1)$ and $b \geq 1$, since $\varphi$ is $(0,0)$-concave, as we proved before. Consequently we have $\Delta(x) \geq \Delta(0) = b-a \geq 0$ if $x \in (0,1)$ and $b \geq a$. In other words, if $b \geq a$ and $b \geq 1$, then $x \mapsto \arcsin_{p,q}(x)$ is $(a,b)$-convex on $(0,1)$. \[\Box\]

**Proof of Theorem 4.** Observe that for $f : (0, \infty) \to (0,\infty)$, $f(t) = (1+tv)^{-1/p}$, we have

$$
\left( \frac{tf'(t)}{f(t)} \right)' = -\frac{q^2}{p} \frac{t^{q-1}}{(1+tv)^{q-2}} < 0 \quad \text{for all } t > 0 \text{ and } p, q > 1.
$$

Consequently, $f$ is geometrically concave on $(0, \infty)$ and according to Lemma \[\Box\] the function

$$
x \mapsto \arcsinh_{p,q}(x) = \int_0^x (1+tv)^{-1/p} dt
$$

is also geometrically concave (or $(0,0)$-concave) on $(0,\infty)$.

Now, consider the functions $u, v, w : (0, \infty) \to (0,\infty)$, defined by

$$
u(x) = \frac{x \varphi'(x)}{\varphi(x)}, \quad v(x) = x^{-a}, \quad w(x) = (\varphi(x))^b,
$$

where $\varphi(x) = \arcsin_{p,q}(x)$. Observe that for $a \geq 0$ and $b \leq 0$, the function $v$ and $w$ are decreasing, and hence in this case, by using the fact that $\varphi$ is $(0,0)$-concave, we get that

$$
x \mapsto \frac{x \varphi'(x)}{\varphi(x)} x^{-a} (\varphi(x))^b
$$

is also decreasing on $(0, \infty)$ as a product of three positive decreasing functions. In other words, $\varphi$ is $(a,b)$-concave on $(0,\infty)$ for $a \geq 0$ and $b \leq 0$.

By Theorem \[\Box\] the function $\varphi$ is $(a,a)$-concave for $a \geq 1$. It follows from the monotonicity of the power mean that $\varphi$ is $(a,b)$-concave for $b \leq a$ and $a \geq 1$. \[\Box\]

**Proof of Theorem 5.** Consider the function $f : (0,1) \to (0,\infty)$, defined by

$$
f(t) = (1-tp)^{1/(a-1)} t^{p-2}.
$$

Then we have

$$
\left( \frac{t^{1-a} f'(t)}{f(t)} \right)' = \frac{p}{q} \frac{(q-1)(p-a)t^{p-a-1} + at^{2p-a-1}}{(1-tp)^2} + a(p-2)t^{-a-1}.
$$

If $a = 0$, then we have that $f$ is geometrically convex on $(0,1)$ for $p, q > 1$. This in turn implies that the function

$$
x \mapsto \lambda(x) = \frac{q}{p} \arcsin_{p,q}(1) - \frac{q}{p} \arccos_{p,q}(x) = \int_0^x f(t) dt
$$

is also geometrically convex on $(0,1)$. Moreover, if $a < 0$, then taking into account that

$$
\left( \frac{t^{1-a} f'(t)}{f(t)} \right)' = \frac{p}{q} \frac{(q-1)t^{p-a-1}(p+a(t^p-1))}{(1-tp)^2} + a(p-2)t^{-a-1},
$$
we obtain that for $p \in (1, 2]$, $q > 1$ the function $f$ is $(a, 0)$-convex. Since $t^{1-a}(1-t^p)^{1/q-1}t^{p-2} \to 0$ as $t \to 0$, according to Lemma $\mathbb{B}$ the function $\lambda$ is $(a, b)$-convex for $a < 0$, $b \geq 0$ and $p \in (1, 2]$, $q > 1$. On the other hand, the function

$$t \mapsto t^{1-a}(1-t^p)^{1/q-1}t^{p-2} = t^{p-a-1}(1-t^p)^{1/q-1}$$

is increasing on $(0, 1)$ for $p, q > 1$ and $a < 0$. Appealing again to Lemma $\mathbb{B}$ the function $\lambda$ is $(a, b)$-convex for $a < 0$, $b < 0$ and $p \in (1, 2]$, $q > 1$.

Finally, if we consider the functions $u, v, w : (0, 1) \to (0, \infty)$, defined by

$$u(x) = \frac{x\lambda'(x)}{\lambda(x)}, \quad v(x) = x^{-a}, \quad w(x) = (\lambda(x))^b,$$

then we get the $x \mapsto x^{1-a}\lambda'(x)/(\lambda(x))^{b-1}$ is increasing on $(0, 1)$ as a product of three positive increasing functions $u, v$ and $w$ when $a \leq 0$ and $b \geq 0$. Here we used that $\lambda$ is geometrically convex on $(0, 1)$.

\[\square\]

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