The influence of the Hawking-Page phase transition on the black hole information

Dao-Quan Sun

School of Physics, Sun Yat-sen University,
Guangzhou 510275, P. R. China

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Abstract

We study the holographic entanglement entropy at the Hawking-Page phase transition point for the BTZ black hole in the gauge/gravity setup and derive a first law of entanglement thermodynamics for the boundary field theory. We find that when the Hawking-Page phase transition takes place, the Page curve will be modified and then all the information of the black hole will come out at the critical temperature. Furthermore, the influence of the first-order phase transition of black holes on the Page curve is universal. These results mean that black holes information does not have to escape by the radiation, and the first-order phase transition of black holes can also cause information to escape.

*lygsdq@outlook.com

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I. INTRODUCTION

In 1975, Hawking [1] presented that a black hole radiates as a black body with a temperature. This discovery solves the contradiction of the temperature in the thermodynamics of black holes and implies the internal relation among gravity, thermodynamics and quantum mechanics. However, this produced the black hole information paradox. The discovery of the AdS/CFT correspondence means that the question of whether information can come out of a black hole is answered by Maldacena [2]. By the AdS/CFT correspondence, Ryu-Takayanagi [3] proposed a holographic description of the entanglement entropy. It was subsequently generalized and applied to the black hole information problem by researchers.

The paper [4] addressed a relation between the entanglement entropy of an excited state and its energy, which is analogous to the first law of thermodynamics like relation for a sufficiently small subsystem. More recently, some authors [5, 6] generalize the entanglement temperature using the first law of black hole thermodynamics. The entropy of a thermodynamical system is related to the internal energy of the concerned system via the first law of thermodynamics. On the other hand, the entanglement entropy is a good measure of quantum information for a pure state. Now, the holographic entanglement entropy is also related to the internal energy of the concerned system via the first law of thermodynamics like relation. This shows that the energy of the black hole connects the quantum information to the thermodynamic information of the black hole. When the Hawking-Page phase transition takes place, the energy of black holes suffers a sudden change accompanied. Thus this means that there may be a link between the Hawking-Page phase transition and the black hole information. While the Page curve as a function of time [7, 8] which expected the entanglement entropy between the outgoing radiation and the quantum state associated to the remaining black hole, if black hole evaporation is a unitary
process. Therefore, it may pose a question whether there is an influence of the Hawking-Page phase transition on the black hole information. The Hawking-Page phase transition began with the paper of Hawking and Page [9] who demonstrated the existence of a certain phase transition between a thermal radiation phase and a stable black hole phase in an AdS spacetime. Since then the black hole phase transition and critical phenomena have been extended to a variety of more complicated backgrounds [10, 12]. One of the ideas is that the cosmological constant $\Lambda$ [10, 13] in an asymptotically AdS black hole space time can be interpreted as a pressure [14]

$$P = -\frac{\Lambda}{8\pi G_N}$$

in a thermodynamic sense.

In this paper, we study the holographic entanglement entropy at the Hawking-Page transition point for the BTZ black hole in the gauge/gravity setup and derive a first law of entanglement thermodynamics for the boundary field theory. We use the cosmological constant interpreted as pressure to explore the relation between the Hawking-Page phase transition, the Page curve and the black hole information in an AdS spacetime. Furthermore, the influence of the first-order phase transition of black hole on the Page curve is also studied.

The paper is organized as follows. In Section II, we employ the AdS/CFT correspondence holographically to compute the entanglement entropy at the Hawking-Page transition point for the BTZ black hole. In Section III, we study the relation between the Hawking-Page phase transition and the Page curve. Section IV is reserved for conclusions and discussions.

II. THE ENTANGLEMENT ENTROPY AT THE HAWKING-PAGE TRANSITION POINT

In this section, we calculate the holographic entanglement entropy at the Hawking-Page phase transition point for a spherical region in this geometry. For a three-
dimensional Schwarzschild AdS black hole corresponding to the BTZ black hole, the metric can be written as

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\theta^2 \]

\[ f(r) = -8G_N M + \frac{r^2}{R^2}, \] (1)

where the cosmological constant \( \Lambda = -\frac{2}{R^2} \). This geometry is then dual to a CFT at the boundary by the AdS/CFT correspondence. Here the lapse function \( f(r) \) contains the information about the excitation properties of the boundary CFT. For a pure AdS spacetime the three-dimensional lapse function is

\[ f(r) = 1 + \frac{r^2}{R^2} \] (2)

This expression corresponds to the ground state in the two-dimensional boundary CFT. In the boundary, we divide the total system into \( A \) and \( B \). The entanglement entropy \( S_A \) for a subsystem \( A \) is defined by \( S_A = -\text{Tr}\rho_A \log \rho_A \), where \( \rho_A = \text{Tr}_B \rho_{\text{total}} \) is the reduced density matrix of the subsystem \( A \) and defined by tracing out with respect to \( B \) from the total density matrix of the system. In the gravity dual, the formula for the holographic entanglement entropy is given by [3]

\[ S_A = \frac{\text{Area}(\gamma_A)}{4G_N} \] (3)

where \( \gamma_A \) is a minimal area surface extending into the bulk.

We assume the system is on a fixed time slice and then we choose the entangling surface of subsystem \( A \) as \( \theta = \theta_c \) and \( 0 < \theta_c < \frac{\pi}{2} \). The entangling surface in the bulk can be written as \( r(\theta) \) and we have the minimum radius at \( \theta = 0 \) and \( r \to \infty \) at \( \theta = \theta_c \). Therefore its area is computed as

\[ \text{Area} = 2\pi \int_0^{\theta_c} d\theta \sqrt{r^2(\theta) + \frac{r^2(\theta)}{f(r(\theta))}}. \] (4)
This leads to subsystem size $\theta = \theta_c$ for the pure AdS spacetime, the solution to this variational problem is
\[ r^2(\theta) = \frac{R^2 \cos^2 \theta_c}{\sin^2 \theta_c - \sin^2 \theta}, \] (5)
and the holographic entanglement entropy $S_A^{(0)}$ is [15][16]
\[ S_A^{(0)} = \frac{c}{3} \log \left( \frac{2R}{\varepsilon \sin \theta_c} \right), \] (6)
where $\varepsilon$ is a short distance cut-off in the boundary, $c = \frac{3R}{2G_N}$ is the central charge of the two-dimensional boundary CFT. For a three-dimensional Schwarzschild AdS black hole background [1], the holographic entanglement entropy is [16][17]
\[ S_A = \frac{c}{3} \log \left( \frac{\sinh(2\pi T_H R \theta_c)}{\varepsilon \pi T_H} \right), \] (7)
where $T_H$ is the Hawking temperature of the black hole. The paper [18] shows that the Hawking-Page phase transition takes place between a thermal radiation phase and a stable black hole phase in an AdS spacetime when the temperature of black hole $T_{HP} = \hbar \sqrt{\frac{-\Lambda}{4\pi^2}}$ and $M = \frac{1}{8G_N}$. Now, we consider that the Hawking-Page phase transition takes place for the BTZ black hole, and then the holographic entanglement entropy is written as
\[ S_A^{(c)} = \frac{c}{3} \log \left( \frac{\sinh(\hbar \sqrt{-\Lambda} R \theta_c)}{\frac{1}{2} \varepsilon \hbar \sqrt{-\Lambda}} \right), \] (8)
where $\Lambda$ is cosmological constant. The reduced amount of holographic entanglement entropy from the exited state to the ground state of CFT in the subsystem $A$ is given by
\[ \Delta S_A^{(c)} = \frac{c}{3} \log \left( \frac{\sinh(\hbar \sqrt{-\Lambda} R \theta_c)}{R \hbar \sqrt{-\Lambda} \sin \theta_c} \right). \] (9)

The Hawking-Page phase transition takes place and the reduced amount of energy in the subsystem $A$ can be written as
\[ \Delta E_A^{(c)} = \frac{\theta_c}{8\pi G_N}. \] (10)
The first law of thermodynamics like relation for a sufficiently small subsystem is as follows

$$\Delta E^c_A = T_{ent}\Delta S^{(c)}_A,$$

(11)

where entanglement temperature is given by

$$T_{ent} = k \cdot \theta^{-1}_c,$$

(12)

where $k$ is an order one constant, and the result of (12) agrees with [4]. From the above results, we can see that when the Hawking-Page phase transition takes place, the system of the boundary CFT undergoes a transition from the excited state to the ground state. Therefore, there is a change of energy $\Delta E^c_A$ related to the amount of quantum information accompanied.

III. THE INFLUENCE OF A PHASE TRANSITION ON THE PAGE CURVE

In this section, we first discuss the influence of the Hawking-Page phase transition on the Page curve. According to the paper [18], when the Hawking-Page phase transition takes place, a stable black hole phase becomes a thermal radiation phase in the bulk. Then clearly there is a jump in entropy of thermodynamics

$$\Delta S^{(c)}_{BH} = \frac{\pi R}{2\hbar G_N},$$

(13)

and the latent heat is

$$\Delta E^c = T_{HP}\Delta S^{(c)}_{BH} = \frac{1}{4G_N}.$$

(14)

From the above results, we can see that when the Hawking-Page phase transition takes place, there is a change of energy of latent heat $\Delta E^c$. Clearly there is a jump in entropy of thermodynamics and corresponds to a jump in entanglement entropy.
Namely, this energy $\Delta E^c$ connects the information of thermodynamics of a black hole and the quantum information of the boundary CFT.

Now, let us consider the influence of the Hawking-Page phase transition on the Page curve. If the Hawking-Page phase transition does not take place, it is well known that the Page curve is illustrated in Fig. 1.

![Figure 1. The Page curve without a Hawking-Page phase transition.](image)

In this model, we assume that the modes may escape the bulk with the absorbing boundary conditions, and the further hypothesis that the black hole evaporation is regarded as a quasi-static approximation process. The relation between the black hole temperature and mass is given as

$$T_{BH} = \frac{\hbar \sqrt{8 G_N M}}{2 \pi R}. \quad (15)$$

When the temperature of the black hole reaches the critical temperature $T_{HP} = \hbar \sqrt{-\frac{\Lambda}{4 \pi^2}}$ and $M = \frac{1}{8 G_N}$, then the Hawking-Page phase transition takes place. If the Hawking-Page phase transition takes place after the Page time, the entanglement
entropy is limited by the black hole entropy and becomes a decreasing function of time, and then the Page curve is described in Fig. 2 which shows that the Hawking-Page phase transition takes place at the time $t_{c_2}$ and when the thermal entropy of black hole $S_{BH}^{(c)} = \frac{\pi R}{2\hbar G_N}$.  

Figure 2. If the Hawking-Page phase transition take place after page time, that the Page curve(blue) is described in this picture.

If the Hawking-Page phase transition takes place before the Page time, the entanglement entropy is a monotonically increasing function of time, which closely tracks the coarse grained thermal entropy of the radiation, and then the Page curve is described in Fig. 3 which show that the Hawking-Page phase transition take place at the time $t_{c_1}$, when the thermal entropy of black hole $S_{BH}^{(c)} = \frac{\pi R}{2\hbar G_N}$.

These results mean that the black hole information does not have to escape by the radiation, and the phase transition of the black hole can also cause information to escape. According to the paper [19], a latent heat of thermodynamical phase transition of black holes can be interpreted as a transition of the quantized black
hole. If we interpret Hawking-Page phase transition as a jump between the two levels, we can compute the black hole entropy jump between the excited state level $n$ and the ground state level 0 is

$$S_{BH,n}^{(c)} - S_{BH,0}^{(c)} = S_{BH,n}^{(c)} - S_{BH,0}^{(c)}.$$  \hspace{1cm} (16)

Then, we may think that the information of black holes can also escape by the transition.

Figure 3. If the Hawking-Page phase transition take place before page time, that the Page curve(blue) is described in this picture.

From the above results, we can infer that if there is a first-order phase transition of black holes, the influence of the first-order phase transition of black holes on the Page curve will be universal. Namely, it also applies to higher dimensional and different kinds of black holes. For example, for a charged AdS black hole, the paper [20] shows that small-large black hole thermodynamic phase transitions which is identified with liquid-gas phase transitions of the van der Waals fluid. The fig [4] shows that the
Figure 4. If a first order phase transition take place after page time, that the Page curve(blue) is described in this picture.

Figure 5. If a first order phase transition take place before page time, that the Page curve(blue) is described in this picture.
first order phase transition takes place for a charged AdS black hole at the time $t_{c2}$ which is after page time $t_p$ then the page curve is described by this picture. The fig 5 shows that the first order phase transition takes place for a charged AdS black hole at the time $t_{c1}$ which is before Page time $t_p$ then the Page curve is described by this picture. The Page curve may be more complex for the first-order phase transition of a charged AdS black hole, here we only point out one possible situation. It is an interesting future problem to study the influence of the first-order phase transition of black holes on the Page curve.

According to the paper [19], a charged AdS black hole, the latent heat $L$ of each black hole molecule transiting from one phase to another phase can be calculated as [19]

$$L = \frac{T\Delta S}{N} = \frac{T\gamma l_P^3}{3} \sqrt{\frac{\alpha}{4\pi}} (\sqrt{n_{LBH}} - \sqrt{n_{SBH}}) dP/dT, \tag{17}$$

the latent heat of thermodynamical phase transition of black holes be interpreted as a transition between the number of quanta of area of LBH and the number of quanta of area of SBH. In the same way, we can think that the information of black holes can also escape by the transition when the first-order phase transition takes place.

IV. CONCLUSION

The holographic entanglement entropy at the Hawking-Page transition point for the BTZ black hole in the gauge/gravity setup has been studied and a first law of entanglement thermodynamics for the boundary field theory has been derived. We use the cosmological constant interpreted as pressure to explore the relation between the Hawking-Page phase transition, the Page curve and the black hole information in an AdS spacetime. The results show that if there is the Hawking-Page phase transition, the Page curve will be modified and then all the information of the black hole will escape at the critical temperature. We further infer that the influence of
the first-order phase transition of black holes on the Page curve is universal. Namely, it also applies to higher dimensional and different kinds of black holes. These results mean that black holes information does not have to escape by the radiation, and the first-order phase transition of black holes can also cause information to escape. The information of black holes can escape by the transition when the first-order phase transition takes place.

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