THE KEMPF-NESS THEOREM AND INVARIANT THEORY

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ABSTRACT. We give new proofs of some well-known results from Invariant Theory using the Kempf-Ness theorem.

1. INTRODUCTION

This article does not contain any new results. Its goal is to deduce some well-known results of Invariant Theory from the Kempf-Ness theorem.

In the sequel $G$ denotes a complex reductive algebraic group. By a small fraktur letter we denote the Lie algebra of a Lie group denoted by the corresponding capital Latin letter.

Let us state the results we want to prove.

Theorem 1 (The Matsushima criterion). Let $H$ be an algebraic subgroup of $G$. The homogeneous space $G/H$ is affine iff $H$ is reductive.

The theorem was proved independently by Matsushima, \[M\], and Onischik \[O\]. It has many different proofs, see \[A\] for references and one more proof. One part of the theorem (the "if" part) is easy. We give a new proof of the other part.

Theorem 2 (The Luna criterion). Let $H$ be a reductive subgroup of $G$, $X$ an affine $G$-variety and $x$ an $H$-stable point in $X$. Then $N_G(H)x$ is closed in $X$ iff $Gx$ is.

The Luna criterion for orbit’s closedness was originally proved in \[L\]. The proof is quite involved. An alternative (and easier) proof was obtained by Kempf, \[Ke\]. Again, the "if" part of the theorem is easy, and we give a new proof of the difficult part.

Theorem 3. Let $H_1, H_2$ be reductive subgroups of $G$. Then the action $H_1 : G/H_2$ is stable, i.e. an orbit in general position is closed.

This result also has different proofs, see \[V\] for details.

The proofs of all three theorems are based on the Kempf-Ness criterion for the orbit closedness which we state now.

Let $V$ be a $G$-module and $K$ a compact form of $G$. Choose a $K$-invariant hermitian scalar product $(\cdot, \cdot)$ on $V$. Define a map $\mu : V \rightarrow \mathfrak{k}^*$ by the formula

\begin{equation}
(\mu(v), \xi) = \frac{1}{2i}(\xi v, v)
\end{equation}

$\mu(v)$ lies in $\mathfrak{k}^*$ because the image of $\mathfrak{k}$ in $\text{gl}(V)$ consists of skew-hermitian operators. The map $\mu$ is the moment map for the action $K : V$.

Theorem 4 (The Kempf-Ness criterion). For $v \in V$ the orbit $Gv$ is closed iff $Gv \cap \mu^{-1}(0) \neq \emptyset$.

A stronger result was proved in \[KN\].

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2. Proofs

At first, for convenience of a reader we give

Proof of the Kempf-Ness criterion. Let us note that, again, one implication in the theorem is easy. Denote by \(|v|||\) the length of a vector \(v \in V\) with respect to \((\cdot, \cdot)\). If an orbit \(Gv\) is closed, then there is a point \(v_0 \in Gv\) such that \(|v_0||| = \min_{u \in Gv} ||u||\). Thus for any \(\xi \in \mathfrak{k}\) there is the equality \(\frac{d}{dt}(\exp(it\xi)v, \exp(it\xi)v)|_{t=0} = 0\), or, equivalently, \(0 = (i\xi v, v) + (v, i\xi v) = 2i(\xi v, v) = -4\langle \mu(v), \xi \rangle\).

Conversely, let \(v \in V\) be such that \(\mu(v) = 0\). Assume that the orbit \(Gv\) is not closed. It follows from the Hilbert-Mumford criterion that there exists a one-parameter subgroup \(\tau : \mathbb{C}^* \rightarrow G\) such that \(\lim_{t \rightarrow 0} \tau(t)v\) exists and is not equal to \(v\) and \(\tau\) is compatible with \(K\), i.e. \(\tau(z) \in K\) if \(|z| = 1\). The last statement follows easily from a proof of the criterion given, for example, in [Kr].

Let \(v = \sum_{i \in \mathbb{Z}} v_i\) be the weight decomposition with respect to \(\tau\). It can be easily seen that \((v_i, v_j) = 0\) provided \(i \neq j\). The limit \(\lim_{t \rightarrow 0} \tau(t)v\) exists if \(v_i = 0\) for all \(i < 0\) and is equal to \(v_0\). The equality \(\langle \mu(v), \xi \rangle = 0\) for \(\xi = \frac{d}{dt} \tau(t)\) can be rewritten as \(\sum_{i \in \mathbb{Z}} i(v_i, v_i) = 0\). Hence \(v_i = 0\) for \(i > 0\) and \(v = v_0 = \lim_{t \rightarrow 0} \tau(t)v\).

Proof of Theorem 4. First suppose that \(H\) is a reductive subgroup of \(G\). Then \(G/H\) is a categorical quotient in the sense of Geometric Invariant Theory, see, for example, [PV]. In particular, \(G/H\) is affine.

Suppose now that \(G/H\) is affine. Then there are a \(G\)-module \(V\) and a closed \(G\)-equivariant embedding \(G/H \rightarrow V\) (see [PV], §1). Choose a compact form \(K \subset G\) and a \(K\)-invariant hermitian scalar product \((\cdot, \cdot)\) on \(V\). By (the easy part of) Theorem 3 one can find a point \(v \in G/H \cap \mu^{-1}(0)\).

The real 2-form \(\omega(u, v) = \Im(u, v)\) on \(V\) is symplectic. Moreover, for any complex submanifold \(X \subset V\) the restriction of \(\omega\) to \(X\) is again symplectic. Clearly, \(\omega\) is \(K\)-invariant. In particular,

\[
\omega(\xi v_1, v_2) + \omega(v_1, \xi v_2) = 0, \forall \xi \in \mathfrak{k}, u, v \in V
\]

The equality \(\omega\) applied for \(v_1 = v, v_2 = \eta v, \eta \in \mathfrak{k}\), implies

\[
\langle \mu(v), [\xi, \eta] \rangle = \omega(\xi v, \eta v)
\]

By (3), the orbit \(Kv\) is an isotropic submanifold of \(Gv\). Therefore \(2 \dim_{\mathbb{R}} Kv \leq \dim_{\mathbb{R}} Gv\) or, equivalently,

\[
\dim_{\mathbb{R}} Kv \leq \dim_{\mathbb{C}} Gv.
\]

But \(\dim_{\mathbb{R}} Kv = \dim_{\mathbb{R}} K - \dim_{\mathbb{R}} \mathfrak{k}, \dim_{\mathbb{C}} G = \dim_{\mathbb{C}} G - \dim_{\mathbb{R}} (\mathfrak{g}_v \cap \mathfrak{k})\). Thus (4) implies \(\dim_{\mathbb{R}} (\mathfrak{g}_v \cap \mathfrak{k}) \geq \dim_{\mathbb{C}} \mathfrak{g}_v\). It follows that \(\mathfrak{g}_v \cap \mathfrak{k}\) is a real form of \(\mathfrak{g}_v\). Therefore \(\mathfrak{g}_v\) is reductive.

Proof of Theorem 2. For convenience of a reader we give a proof of an easy part of the theorem due to Luna. That is, we prove that if \(Gx\) is closed, then \(N_G(H)x\) is. Let \(x \in Gx\). We have \(T_y((Gx)^H) = (\mathfrak{g}/\mathfrak{g}_y)^H = \mathfrak{g}^H/\mathfrak{g}_y^H = T_y(G^H y) = T_y(N_G(H)y)\). It follows that \((Gx)^H\) is a smooth variety and its components are \(N_G(H)^o\)-orbits. In particular, for any \(y \in X^H\) the orbit \(N_G(H)y\) is closed.

Now we prove that if \(N_G(H)x\) is closed, then so is \(Gx\). Let us embed \(X\) into a \(G\)-module \(V\).

One can choose compact forms \(K, K_1, K_2\) of \(G\), \(N_G(H), H\), respectively, such that \(K_2 \subset K_1 \subset K\). We choose a \(K\)-invariant hermitian scalar product \((\cdot, \cdot)\) on \(V\) and define the moment map \(\mu\) by (1).

Let \(v \in V^H\) be such that \(N_G(H)v\) is closed. Denote by \(\mu_1\) the moment map for the action \(K_1 : V^H\) (in (1) we take the restriction of \((\cdot, \cdot)\) to \(V^H\) for the scalar product). Choose an invariant scalar product \((\cdot, \cdot)\) on \(\mathfrak{k}\) and identify \(\mathfrak{k}, \mathfrak{k}_1\) with their duals via \((\cdot, \cdot)\). Let us prove
that \( \mu_1 = \mu \vert_{V^H} \). It can be seen directly, that the map \( \mu : V \to \mathfrak{k} \) is \( K \)-equivariant. It follows that \( \mu(V^H) \subset \mathfrak{k}^K \subset \mathfrak{k}_1 \). The equality \( \mu_1 = \mu \vert_{V^H} \) follows now directly from the definitions of \( \mu, \mu_1 \).

By Theorem 4 applied to the action \( N_G(H) : V^H \), one can choose a vector \( v_1 \in N_G(H)v \cap \mu_1^{-1}(0) \). Since \( \mu_1 = \mu \vert_{V^H}, v_1 \in \mu^{-1}(0) \). Applying Theorem 4 again, we see that the orbit \( Gv \) is closed.

\[ \square \]

**Proof of Theorem 3.** Embed \( G/H_2 \) into a \( G \)-module \( V \), fix a compact form \( K \subset G \), a \( K \)-invariant hermitian scalar product \((\cdot,\cdot)\) on \( V \) and a compact form \( K_1 \) of \( H \) such that \( K_1 \subset K \).

Let \( \pi \) denote the natural projection \( \mathfrak{k}^* \to \mathfrak{k}_1^* \). The map \( \pi \circ \mu \) is the moment map for the action \( K_1 : V \). There exists \( v \in G/H_2 \) such that \( \mu(v) = 0 \). Note that \( \mu(kv) = 0 \) for all \( k \in K \). In particular, \( \pi \circ \mu(Kv) = 0 \). By Theorem 4 the orbit \( H_1 kv \) is closed for any \( k \in K \). It remains to check that the subset \( Kv \) is dense in \( Gv \). Assume the converse: there exists a proper closed subvariety \( Y \subset G/H_2 \) containing \( Kv \). Replacing \( Y \) by \( \bigcap_{k \in K} kY \) we may assume that \( Y \) is \( K \)-invariant. Since \( K \) is Zariski-dense in \( G, Y \) is \( G \)-invariant. Contradiction. \[ \square \]

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