Abstract. At the core of the quest for a logic for PTIME is a mismatch between algorithms making arbitrary choices and isomorphism-invariant logics. One approach to overcome this problem is witnessed symmetric choice. It allows for choices from definable orbits which are certified by definable witnessing automorphisms.

We consider the extension of fixed-point logic with counting (IFPC) with witnessed symmetric choice (IFPC+WSC) and a further extension with an interpretation operator (IFPC+WSC+I). The latter operator evaluates a subformula in the structure defined by an interpretation. This structure possibly has other automorphisms exploitable by the WSC-operator. For similar extensions of pure fixed-point logic (IFP) it is known that IFP+WSC+I simulates counting which IFP+WSC fails to do. For IFPC+WSC it is unknown whether the interpretation operator increases expressiveness and thus allows studying the relation between WSC and interpretations beyond counting.

We separate IFPC+WSC from IFPC+WSC+I by showing that IFPC+WSC is not closed under FO-interpretations. By the same argument, we answer an open question of Dawar and Richerby regarding non-witnessed symmetric choice in IFP. Additionally, we prove that nesting WSC-operators increases the expressiveness using the so-called CFI graphs. We show that if IFPC+WSC+I canonizes a particular class of base graphs, then it also canonizes the corresponding CFI graphs. This differs from various other logics, where CFI graphs provide difficult instances.

1. Introduction

The quest for a logic for PTIME is one of the prominent open questions in finite model theory [CH82, Gro08]. It asks whether there is a logic defining exactly all polynomial-time decidable properties of finite structures. While Fagin’s theorem [Fag74] initiated descriptive complexity theory by showing that there is a logic capturing NPTIME, the question for PTIME is still open. One problem at the core of the question is a mismatch between logics and algorithms. For algorithms, it is common to make arbitrary choices as long as the
output is still isomorphism-invariant. In general, it is undecidable whether an algorithm is isomorphism-invariant. Showing this is usually part of the proof that the algorithm is correct. On the other hand, every reasonable logic is required to be isomorphism-invariant by design [Gur88], so in contrast to algorithms it is not possible to define something non-isomorphism-invariant. That is, a logic has to enforce isomorphism-invariance syntactically and it is generally not clear how algorithms making choices can be implemented in a logic.

For totally ordered structures, inflationary fixed-point logic (IFP) captures $P_{\text{time}}$ due to the Immerman-Vardi Theorem [Imm87b]. On ordered structures, no arbitrary choices are needed and the total order is used to “choose” the unique minimal element. Thus, the lack of making choices is crucial on unordered structures. We therefore would like to support choices in a logic while still guaranteeing isomorphism-invariance. There are logics in which arbitrary choices can be made (e.g. [AB87, DR03b]), but for these it is undecidable whether a formula is isomorphism-invariant [AB87]. In particular, such a logic fails to be a logic capturing $P_{\text{time}}$ in the sense of Gurevich [Gur88]. Similarly, when extending structures by an arbitrary order it is undecidable whether a formula is order-invariant, i.e., it evaluates equally for all such orders (see [GKL+07]).

One approach to overcome the lack of choices in logics is to support a restricted form of choice. If only choices from definable orbits are allowed, that is from sets of definable objects related by an automorphism of the input structure, the output is still guaranteed to be isomorphism-invariant. This form of choice is called symmetric choice (SC). However, it is unknown whether orbits can be computed in $P_{\text{time}}$. So it is unknown whether a logic with symmetric choice can be evaluated in $P_{\text{time}}$ because during the evaluation it has to verified that the choice-sets are indeed orbits. This is solved by handing over the obligation to check whether the choice-sets are orbits from the evaluation to the formulas themselves. To make a choice, not only a choice-set but also a set of witnessing automorphisms has to be defined. These automorphisms certify that the choice-set is indeed an orbit in the following way: For every pair of elements $a$ and $b$ in the choice-set, an automorphism mapping $a$ to $b$ has to be provided. This condition guarantees evaluation in $P_{\text{time}}$. We call this restricted form of choice witnessed symmetric choice (WSC).

Besides witnessed symmetric choice, other operators were proposed to extend the expressiveness of logics not capturing $P_{\text{time}}$ including a counting operator (see [Ott97]) and an operator based on logical interpretations [GH98]. It was shown that witnessed symmetric choice increases the expressiveness of IFP [GH98]. It was also shown that counting operators increase the expressiveness of IFP and of the logic of Choiceless Polynomial Time (CPT) [BGS02]. However, for the combination of counting and choices not much is known. In this article, we investigate the relation of counting, witnessed symmetric choice, and interpretations to better understand their expressive power.

Extending IFP with symmetric and witnessed symmetric choice was first studied by Gire and Hoang [GH98]. They extend IFP with (witnessed) symmetric choice (IFP+SC and IFP+WSC). They show that IFP+WSC distinguishes CFI graphs over ordered base graphs, which IFP (and even fixed-point logic with counting IFPC) fails to do [CFI92]. Afterwards IFP+SC was studied by Dawar and Richerby [DR03a]. They allowed for nested symmetric choice operators, proved that parameters of choice operators increase the expressiveness, showed that nested symmetric choice operators are more expressive than a single one, and conjectured that with additional nesting depth the expressiveness increases. Recently, the extension of CPT with witnessed symmetric choice (CPT+WSC) was studied by Schweitzer and Lichter [LS22]. CPT+WSC has the interesting property that a CPT+WSC-definable
isomorphism test on a class of structures implies a CPT+WSC-definable canonization for this class. Canonization is the task of defining an isomorphic but totally ordered copy of the input structure. The only requirement is that the class of structures is closed under individualization, so under assigning unique colors to vertices. This is unproblematic in most cases [KSS15, Mat79]. Individualization is natural in the context of choices because a choice is, in some sense, an individualization. The concept of canonization is essential in the quest for a logic for \( \text{Ptime} \). It provides the routinely employed approach to capture \( \text{Ptime} \) on a class of structures: define canonization, obtain isomorphic and ordered structures, and apply the Immerman-Vardi Theorem (e.g. [ZGGP14, Gro17, GN19, LS21]). While CPT+WSC has the nice property that defining isomorphism implies canonization, we do not know whether the same holds for CPT or whether CPT+WSC is more expressive than CPT. Proving this requires separating CPT from \( \text{Ptime} \), which has been open for a long time.

(Witnessed) symmetric choice has the drawback that it can only choose from orbits of the input structure. This structure might have complicated orbits that we cannot define or witness in the logic. However, there could be a reduction to a different structure with easier orbits exploitable by witnessed symmetric choice. For logics, the natural concept of a reduction is an interpretation, i.e., defining a structure in terms of another one. Interpretations are in some sense incompatible with (witnessed) symmetric choice because we always have to choose from orbits of the input structure. To exploit a combination of choices and interpretations, Gire and Hoang proposed an interpretation operator [GH98]. It evaluates a formula in the image of an interpretation. For logics closed under interpretations (e.g. IFP, IFPC, and CPT) such an interpretation operator does not increase expressiveness. However, for the extension with witnessed symmetric choice this is different: IFP+WSC is less expressive than the extension of IFP+WSC with the interpretation operator. The interpretation operator together with witnessed symmetric choice simulates counting. However, it is indicated in [GH98] that which witnessed symmetric choice alone fails to simulate counting.

We are interested in the relation between witnessed symmetric choice and the interpretation operator not specifically for IFP but more generally. Most of the existing results in [GH98, DR03a] showing that (witnessed) symmetric choice or the interpretation operator increases in some way the expressiveness of IFP are based on counting. However, counting is not the reason for using witnessed symmetric choice. Counting can be achieved naturally in IFPC. Thus, it is unknown whether the interpretation operator increases expressiveness of IFPC+WSC. In CPT, it is not possible to show that witnessed symmetric choice or the interpretation operator increase expressiveness without separating CPT from \( \text{Ptime} \) [LS22].

Overall, a natural base logic for studying the interplay of witnessed symmetric choice and the interpretation operator is IFPC. On the one hand, separation results based on counting are not applicable in IFPC. On the other hand, there are known IFPC-undefinable \( \text{Ptime} \) properties, namely the already mentioned CFI query, which can be used to separate extensions of IFPC. The CFI construction assigns to a connected graph, the so-called base graph, two non-isomorphic CFI graphs: one is called even and the other is called odd. The CFI query is to define whether a given CFI graph is even.

**Results.** We define the logics IFPC+WSC and IFPC+WSC+I. They extend IFPC with a fixed-point operator featuring witnessed symmetric choice and the latter additionally with an interpretation operator. We show that the interpretation operator increases expressiveness of the logic:

**Theorem 1.1.** IFPC+WSC < IFPC+WSC+I ≤ \( \text{Ptime} \).
In particular, this separates IFPC+WSC from PTIME. Such a result does not follow from existing techniques because separating IFP+WSC from PTIME is based on counting in [GH98]. Moreover, we show that IFPC+WSC is not even closed under FO-interpretations. The same construction shows that IFP+SC is not closed under FO-interpretations, too. We thereby answer an open question of Dawar and Richerby in [DR03a]. Proving Theorem 1.1 relies on the CFI construction. Similarly to [LS22], we show that if IFPC+WSC+I distinguishes orbits, then IFPC+WSC+I defines a canonization. We apply this to CFI graphs:

**Theorem 1.2.** If IFPC+WSC+I canonizes a class of colored base graphs $K$ of minimal degree 3 (closed under individualization), then IFPC+WSC+I canonizes the class of CFI graphs $\text{CFI}(K)$ over $K$.

The conclusion is that for IFPC+WSC+I a class of CFI graphs is not more difficult than the corresponding class of base graphs, which is different in many other logics [CF92, GP19, Lic21, DGL23]. However, to canonize the CFI graphs in our proof, the nesting depth of WSC-fixed-point operators and interpretation operators increases. We show that this increase is unavoidable.

**Theorem 1.3.** There is a class of base graphs $K$ such that

1. WSCI(IFPC) defines a canonization for $K$,
2. WSCI(IFPC) does not define the CFI query for $\text{CFI}(K)$, and
3. WSCI(WSCI(IFPC)) defines a canonization for $\text{CFI}(K)$.

Here WSCI($L$) is the fragment of IFPC+WSC+I using IFPC-formula-formation-rules to compose $L$-formulas and an additional interpretation operator nested inside a WSC-fixed-point operator. Theorem 1.3 can be seen as a first step towards an operator nesting hierarchy for IFPC+WSC+I.

**Our Techniques.** We adapt the techniques of [LS22] from CPT to IFPC to define a WSC-fixed-point operator. It has some small but essential differences to [GH98, DR03a]. Similar to [LS22] for CPT, Gurevich’s canonization algorithm [Gur97] is expressible in IFPC+WSC: It suffices to distinguish orbits of a class of individualization-closed structures to define a canonization.

To prove Theorem 1.2, we use the interpretation operator to show that if IFPC+WSC+I distinguishes orbits of the base graphs, then IFPC+WSC+I distinguishes also orbits of the CFI graphs and thus canonizes the CFI graphs. The CFI-graph-canonizing formula nests one WSC-fixed-point operator (for Gurevich’s algorithm) and one interpretation operator (to distinguish orbits) more than the orbit-distinguishing formula of the base graphs. To show that this increase in nesting depth is necessary, we construct double CFI graphs. We start with a class of CFI graphs $\text{CFI}(K')$ canonized in WSCI(IFPC). We create a class of new base graphs $K$ from the $\text{CFI}(K')$-graphs. Applying the CFI construction once more, $\text{CFI}(K)$ is canonized in WSCI(WSCI(IFPC)) but not in WSCI(IFPC): To define orbits of $\text{CFI}(K)$, we have to define orbits of the base graph, for which we need to distinguish the CFI graphs $\text{CFI}(K')$.

To prove IFPC+WSC $<$ IFPC+WSC+I, we construct a class of asymmetric structures, i.e., structures without non-trivial automorphisms, for which isomorphism is not IFPC-definable. Because asymmetric structures have only singleton orbits, witnessed symmetric choice is not beneficial, thus IFPC+WSC = IFPC, and isomorphism is not IFPC+WSC-definable. These structures combine CFI graphs and the so-called multipedes [GS96], which
are asymmetric and for which IFPC fails to distinguish orbits. An interpretation removes
the multipedes and reduces the isomorphism problem to the ones of CFI graphs. Thus,
isomorphism of this class of structures is IFPC+WSC+I-definable.

**Related Work.** The logic IFPC was separated from \textsc{Ptime} by the CFI query [CFI92]. CFI
graphs not only turned out to be difficult for IFPC but variants of them were also used to
separate rank logic [Lic21] and the more general linear-algebraic logic [DGL23] from \textsc{Ptime}.
CPT was shown to define the CFI query for ordered base graphs [DRR08] and base graphs
of maximal logarithmic color class size [PSS16]. Defining the CFI query for these graphs in
CPT turned out to be comparatively more complicated than in IFP+WSC for ordered base
graphs in [GH98]. It is still open whether CPT defines the CFI query for all base graphs.

The definitions of the (witnessed) symmetric choice operator in [GH98, DR03a] differ
at crucial points form the one in [LS22] and in this article: The formula defining the wit-
nessing automorphism has access to the obtained fixed-point. This is essential to implement
Gurevich’s canonization algorithm but has the drawback to impose (possibly) stronger or-
bit conditions. CPT+WSC in [LS22] is actually a three-valued logic using, beside true and
false, an error marker for non-witnessed choices. This is needed for CPT because fixed-
point computations in CPT do not necessarily terminate in a polynomial number of steps.
Instead, computation is aborted (and orbits cannot be witnessed). For IFPC this problem
does not occur because fixed-points are always reached within polynomially many steps.

There are more approaches to integrate choices in first-order logic: Choice-operators
independent of a fixed-point operator were studied in [BG00, Ott00]. They are no candidates
to capture \textsc{Ptime} because of nondeterminism, undecidable syntax, or too high complexity.
A similar statement holds for the nondeterministic version of the fixed-point operator with
choice, where choices can be made from arbitrary choice sets and not only of orbits [DR03b].
For a more detailed overview, we refer to [Ric04].

Multipedes [GS96] are asymmetric structures, which are not characterized up to isomor-
phism in \(k\)-variable counting logic for every fixed number of variables \(k\). Asymmetry turns
multipedes to hard instances for graph isomorphism algorithms in the individualization-
refinement framework [NS18, AS21]. The size of a multipede not identifiable in \(k\)-variable
counting logic is large compared to \(k\). There are asymmetric graphs with similar properties,
but whose order is linear in \(k\) [DK19]. Both constructions are based on the CFI construction.

There is another remarkable but not directly-connected coincidence to lengths of res-
olution proofs. Resolution proofs for non-isomorphism of CFI-graphs have exponential
size [Tor13]. When adding a global symmetry rule (\textsc{SRC-I}), which exploits automorphisms
of the formula (so akin to symmetric choice), the length becomes polynomial [SS21]. For
asymmetric multipedes the length in the \textsc{SRC-I} system is still exponential [TW22]. But
when considering the local symmetry rule (\textsc{SRC-II}) exploiting local automorphisms (so
somewhat akin to symmetric choice after restricting to a substructure with an interpreta-
tion) the length becomes polynomial again [SS21].

**Structure of this Article.** We review preliminaries including the logic IFPC and logical
interpretations in Section 2. Next, we introduce in Section 3 the logics IFPC+WSC and
IFPC+WSC+I. We review the CFI construction in Section 4 and consider their canoniza-
tion (Theorem 1.2) in Section 5. We prove that the increase in the operator nesting depth is
unavoidable to canonize CFI graphs (Theorem 1.3) in Section 6. In Section 7, we separate
IFPC+WSC from IFPC+WSC+I (Theorem 1.1). We end with a discussion in Section 8.
2. Preliminaries

We set $[k] := \{1, \ldots, k\}$. For some set $N$, the $i$-th entry of a $k$-tuple $\bar{t} \in N^k$ is denoted by $t_i$ and its length by $|\bar{t}| = k$. The set of all tuples of length at most $k$ is $N^{\leq k}$ and the set of all tuples of finite length is $N^*$.

A relational signature is a set of relation symbols \(\{R_1, \ldots, R_\ell\}\) with associated arities \(\text{ar}(R_i)\). We use letters $\tau$ and $\sigma$ for signatures. Let $\tau = \{R_1, \ldots, R_\ell\}$ be a signature. A $\tau$-structure is a tuple $\mathfrak{A} = (A, R_1^A, \ldots, R_\ell^A)$ where $R_i^A \subseteq A^{\text{ar}(R_i)}$ for every $i \in [\ell]$. The set $A$ is called the universe of $\mathfrak{A}$ and its elements vertices. We use fraktur letters $\mathfrak{A}$ and $\mathfrak{B}$ for relational structures and denote their universe always by $A$ and $B$. For vertices, we use the letters $u$, $v$, and $w$. For $\sigma \subseteq \tau$, the reduct $\mathfrak{A} \mid \sigma$ is the restriction of $\mathfrak{A}$ to the relations contained in $\sigma$. For a subset $A' \subseteq A$, we denote by $\mathfrak{A}[A']$ the substructure of $\mathfrak{A}$ induced by $A'$. We sometimes also view a tuple $\bar{u} \in A^*$ as a set and write $\mathfrak{A}[\bar{u}]$ for $\mathfrak{A}\{u_i \mid i \in |\bar{u}|\}$.

In this article we consider finite structures.

A colored graph is an $\{E, \preceq\}$-structure $G = (V, E^G, \preceq^G)$. The binary relation $E$ is the edge relation and the binary relation $\preceq$ is a total preorder. Its equivalence classes are the color classes or just colors. We usually just write $G = (V, E, \preceq)$ for a colored graph. The neighborhood of a vertex $u \in V$ in $G$ is $N^G(u)$. For a subset $W \subseteq V$ of vertices of $G$, the subgraph of $G$ induced by $W$ is $G[W]$. The graph $G$ is $k$-connected if $|V| > k$ and, for every $V' \subseteq V$ of size at most $k - 1$, the graph $G \setminus V'$ obtained from $G$ by deleting all vertices in $V'$ is connected. The treewidth of a graph measures how close a graph is to being a tree (see e.g. [DR07]). We omit a formal definition here and only use that if a graph $G$ is a minor of $H$, so $G$ can be obtained from $H$ by deleting vertices, deleting edges, and contracting edges, then the treewidth of $G$ is at most the treewidth of $H$.

For two $\tau$-structures $\mathfrak{A}$ and $\mathfrak{B}$, an isomorphism $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a bijection $A \rightarrow B$ such that $\bar{u} \in R_i^A$ if and only if $\varphi(\bar{u}) = (\varphi(u_1), \ldots, \varphi(u_{\text{ar}(R_i)}) \in R_i^B$ for all $R_i \in \tau$ and all $\bar{u} \in A^{\text{ar}(R_i)}$. For $\bar{u} \in A^k$ and $\bar{v} \in B^k$, the structures $(\mathfrak{A}, \bar{u})$ and $(\mathfrak{B}, \bar{v})$ are isomorphic, denoted $(\mathfrak{A}, \bar{u}) \cong (\mathfrak{B}, \bar{v})$, if there is an isomorphism $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ satisfying $\varphi(\bar{u}) = \bar{v}$. An automorphism $\varphi$ of $(\mathfrak{A}, \bar{u})$ is an isomorphism $(\mathfrak{A}, \bar{u}) \rightarrow (\mathfrak{A}, \bar{u})$. We say that $\varphi$ fixes $\bar{u}$ and write $\text{Aut}((\mathfrak{A}, \bar{u}))$ for the set of all automorphisms fixing $\bar{u}$. We will use the same notation also for other objects than tuples, e.g., for automorphisms fixing relations. A $k$-orbit of $(\mathfrak{A}, \bar{u})$ is a maximal set of $k$-tuples $O \subseteq A^k$ such that for every $\bar{v}, \bar{w} \in O$ there is an automorphism $\varphi \in \text{Aut}((\mathfrak{A}, \bar{u}))$ satisfying $\varphi(\bar{v}) = (\varphi(v_1), \ldots, \varphi(v_k)) = \bar{w}$.

Fixed-Point Logic with Counting. We recall fixed-point logic with counting IFPC (proposed in [Imm87a], also see [Ott97]). Let $\tau$ be a signature and $\mathfrak{A} = (A, R_1^A, \ldots, R_\ell^A)$ be a $\tau$-structure. We extend $\tau$ and $\mathfrak{A}$ with counting. Define $\tau^\# := \tau \uplus \{\cdot, +, 0, 1\}$ and $\mathfrak{A}^\# := (A, R_1^A, \ldots, R_\ell^A, N, \cdot, +, 0, 1)$ to be the two-sorted $\tau^\#$-structure that is the disjoint union of $\mathfrak{A}$ and $N$.

$\text{IFPC}[\tau]$ is a two-sorted logic using the signature $\tau^\#$. Element variables range over the vertices and numeric variables range over the natural numbers. For element variables we use the letters $x$, $y$, and $z$, for numeric variables the Greek letters $\nu$ and $\mu$, and for numeric terms the letters $s$ and $t$. IFPC-formulas are built from first-order formulas, a fixed-point operator, and counting terms. When quantifying over numeric variables, their range needs to be bounded to ensure PTIME-evaluation: For an IFPC-formula $\Phi$, a closed numeric IFPC-term $s$, a numeric variable $\nu$ possibly free in $\Phi$, and a quantifier $Q \in \{\forall, \exists\}$,
the formula
\[ Qv \leq s. \Phi \]
is an IFPC-formula. An inflationary fixed-point operator defines a relation \( R \). We allow \( R \) to mix vertices with numbers. For an IFPC\([\tau, R]\)-formula \( \Phi \) and variables \( \bar{x}\bar{\mu} \) possibly free in \( \Phi \), the fixed-point operator
\[
[\text{if} \, R\bar{x}\bar{\mu} \leq s. \Phi](\bar{x}\bar{\mu})
\]
is an IFPC\([\tau]\)-formula. Here, \( s \) is a \(|\bar{\mu}|\)-tuple of closed numeric terms which bounds the values of \( \bar{\mu} \) similar to the case of a quantifier. The crucial element of IFPC are counting terms. They count the number of tuples satisfying a formula. Let \( \Phi \) be an IFPC-formula with possibly free variables \( \bar{x} \) and \( \bar{\nu} \) and let \( \bar{s} \) be a \(|\bar{\nu}|\)-tuple of closed numeric IFPC-terms. Then
\[
\#\bar{x}\bar{\nu} \leq \bar{s}. \Phi
\]
is a numeric IFPC-term.

IFPC-formulas (or terms) are evaluated over \( \mathfrak{A}^\# \). For a numeric term \( s(\bar{x}\bar{\nu}) \), we denote by \( s^\mathfrak{A} : A^{[\bar{x}] \times [\bar{\nu}]} \to \mathbb{N} \) the function mapping the values for the free variables of \( s \) to the value that \( s \) takes in \( \mathfrak{A}^\# \). Likewise, for a formula \( \Phi(\bar{x}\bar{\nu}) \), we write \( \Phi^\mathfrak{A} \subseteq A^{[\bar{x}] \times [\bar{\nu}]} \) for the set of values for the free variables satisfying \( \Phi \). Evaluating a counting term for a formula \( \Phi(\bar{y}\bar{x}\bar{\mu}\bar{\nu}) \) and a \(|\bar{\nu}|\)-tuple \( \bar{s} \) of closed numeric terms is defined as follows:
\[
(\#\bar{x}\bar{\nu} \leq \bar{s}. \Phi)^\mathfrak{A}((\bar{u}\bar{\nu})) := \left\{ \bar{w} \bar{n} \in A^{[\bar{x}] \times [\bar{\nu}]} \mid n_i \leq s_i^\mathfrak{A} \text{ for all } i \in [|\bar{\nu}|], \bar{u}\bar{w}\bar{n} \bar{m} \in \Phi^\mathfrak{A} \right\}.
\]

**Finite Variable Counting Logic.** The \( k \)-variable first-order logic with counting \( \mathcal{C}_k \) extends the \( k \)-variable fragment of first-order logic (FO) with counting quantifiers \( \exists^2 \forall x. \Phi \) stating that at least \( j \) distinct vertices satisfy \( \Phi \) (see [Ott97]). Bounded variable logics with counting are a useful tool to prove IFPC-undefinability. For every \( n \in \mathbb{N} \), every IFPC-formula using \( k \) variables is equivalent on structures of order up to \( n \) to a \( \mathcal{C}_{O(k)} \)-formula.

Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be two \( \tau \)-structures and \( \bar{u} \in A^\ell \) and \( \bar{v} \in B^\ell \). We say that a logic \( L \) distinguishes \( (\mathfrak{A}, \bar{u}) \) from \( (\mathfrak{B}, \bar{v}) \) if there is an \( L \)-formula \( \Phi \) with \( \ell \) free variables such that \( \bar{u} \in \Phi^\mathfrak{A} \) and \( \bar{v} \notin \Phi^\mathfrak{B} \). Otherwise, the structures are called \( L \)-equivalent. We write \( (\mathfrak{A}, \bar{u}) \simeq^k \mathcal{C}_k \) \( (\mathfrak{B}, \bar{v}) \) if \( (\mathfrak{A}, \bar{u}) \) and \( (\mathfrak{B}, \bar{v}) \) are \( \mathcal{C}_k \)-equivalent. The logics \( \mathcal{C}_k \) are used to prove IFPC-undefinability as follows: Let \( (\mathfrak{A}_k, \mathfrak{B}_k) \) be a sequence of finite structures for every \( k \in \mathbb{N} \) such that \( \mathfrak{A}_k \) has a property \( P \) but \( \mathfrak{B}_k \) does not. If \( \mathfrak{A}_k \simeq^k \mathcal{C}_k \mathfrak{B}_k \) for every \( k \), then IFPC does not define \( P \).

The logic \( \mathcal{C}_k \) can be characterized by an Ehrenfeucht-Fraïssé-like pebble game – the bijective \( k \)-pebble game [Hel96]. The game is played on two structures \( \mathfrak{A} \) and \( \mathfrak{B} \) by two players called Spoiler and Duplicator. There are \( k \) pebble pairs \( (p_i, q_i) \) for every \( i \in [k] \). Positions in the game are tuples \( (A, \bar{u} \bar{v}; B, \bar{v}) \) for tuples \( \bar{u} \in A^{\leq k} \) and \( \bar{v} \in B^{\leq k} \) of the same length. For every \( i \in [|\bar{u}|] \), a pebble \( p_j \) is placed on the atom \( u_i \) and the pebble \( q_j \) is placed on \( v_i \) from some \( j \in [k] \). It will not matter which pebble pair \( (p_j, q_j) \) is used for the \( i \)-th entry of \( \bar{u} \) and \( \bar{v} \). The game proceeds as follows. If \(|A| \neq |B|\), then Spoiler wins. Otherwise, Spoiler picks up a pair of pebbles \( (p_i, q_i) \) (may it be already placed on the structures are not). Duplicator answers with a bijection \( \lambda : A \to B \). Spoiler places the pebble \( p_i \) on a vertex \( w \in A \) and \( q_i \) on \( \lambda(w) \in B \). If in the resulting position \( (\mathfrak{A}, \bar{u} ; \mathfrak{B}, \bar{v}) \) there is no pebble-respecting local isomorphism, that is, the map defined via \( u_i \mapsto \nu_i \) is not an isomorphism \( (\mathfrak{A}(\bar{u}), \bar{u}) \to (\mathfrak{B}(\bar{v}), \bar{v}) \), then Spoiler wins. Otherwise, the game continues with the next round. Duplicator wins if Spoiler never wins. We say that Spoiler respectively Duplicator has a
winning strategy in position \((A, \bar{u}; B, \bar{v})\) if Spoiler respectively Duplicator can always win the game regardless of the moves of the other player.

For all finite \(\tau\)-structures \(A\) and \(B\) and all tuples \(\bar{u} \in A^{\leq k}\) and \(\bar{v} \in B^{\leq k}\), Spoiler has a winning strategy in the bijective \(k\)-pebble game in position \((A, \bar{u}; B, \bar{v})\) if and only if \((A, \bar{u}) \not\in_k^L (B, \bar{v})\) [Hel96].

**Logical Interpretations.** A logical interpretation is the logical correspondence to an algorithmic reduction. It transforms a relational structure to another one. An IFPC\(\tau, \sigma\)-interpretation defines a partial map from \(\tau\)-structures to \(\sigma\)-structures, where the map is defined in terms of IFPC-formulas operating on tuples of the input \(\tau\)-structure. In the case of IFPC, these tuples not only contain vertices but also numbers. For sake of readability, we use in the following \(\bar{x}, \bar{y}\), and \(\bar{z}\) for a tuple of both element and numeric variables and \(\bar{u}\) and \(\bar{v}\) for a tuple of both vertices and numbers.

Let \(\tau\) and \(\sigma = \{R_1, \ldots, R_{\ell}\}\) be relational signatures. A \(d\)-dimensional IFPC\(\tau, \sigma\)-interpretation \(\Theta(\bar{z})\) with parameters \(\bar{z}\) is a tuple

\[
\Theta(\bar{z}) = (\Phi_{\text{dom}}(\bar{z}x), \Phi_{\exists}(\bar{z}x\bar{y}), \Phi_{R_1}(\bar{z}x_1 \ldots \bar{x}_{ar(R_1)}), \ldots, \Phi_{R_{\ell}}(\bar{z}x_1 \ldots \bar{x}_{ar(R_\ell)}), \bar{s})
\]

of IFPC\(\tau\)-formulas and a \(j\)-tuple \(\bar{s}\) of closed numeric IFPC\(\tau\)-terms, where \(j\) is the number of numeric variables in \(\bar{x}\). The tuples of variables \(\bar{x}, \bar{y}\), and all the \(\bar{x}_i\) are of length \(d\) and agree on whether the \(k\)-th variable is an element or numeric variable. Let \(A\) be a \(\tau\)-structure and \(\bar{u} \in (A \cup \mathbb{N})^{\leq d}\) match the types of the parameter variables \(\bar{z}\) (element or numeric). We now define \(\Theta(A, \bar{u})\). Assume that up to reordering the first \(j\) variables in \(\bar{x}\) are numeric variables and set \(D := \{0, \ldots, s_1^{\alpha_1}\} \times \cdots \times \{0, \ldots, s_{\ell}^{\alpha_{\ell}}\} \times A^{d-j}\). We define a \(\sigma\)-structure \(B = (B, R_1^{\alpha_1}, \ldots, R_{\ell}^{\alpha_{\ell}})\) via

\[
B := \left\{ \bar{v} \in D \mid \bar{u}\bar{v} \in \Phi_{\text{dom}}^{\alpha_1} \right\},
\]

\[
R_i^{\alpha_i} := \left\{ (\bar{v}_1, \ldots, \bar{v}_{ar(R_i)}) \in B^{ar(R_i)} \mid \bar{u}\bar{v}_1 \ldots \bar{v}_{ar(R_i)} \in \Phi_{R_i}^{\alpha_i} \right\}
\]

for all \(i \in [\ell]\).

Finally, using the relation \(E\), we define the image of the interpretation:

\[
\Theta(A, \bar{u}) := \left\{ \frac{B/E}{\mathcal{B}} \right\} \quad \text{if } E \text{ is a congruence relation on } \mathcal{B},
\]

\[
\text{undefined otherwise.}
\]

An interpretation is called *equivalence-free* if \(\Phi_{\exists}(\bar{z}x\bar{y})\) is the formula \(\bar{x} = \bar{y}\).

When we consider a logic \(L\) that is an extension of IFPC, then the notion of an \(L[\tau, \sigma]\)-interpretation is defined exactly in the same way by replacing IFPC-formulas or terms with \(L\)-formulas or terms. For logics, which do not possess numeric variables like FO or IFP, the notion of an interpretation is similar and just omits the numeric part, i.e., there is no numeric term \(s\) bounding the range of numeric variables.

A property \(P\) of \(\tau\)-structures is \(L\)-reducible to a property \(Q\) of \(\sigma\)-structures if there is an \(L[\tau, \sigma]\)-interpretation \(\Theta\) such that, for every \(\tau\)-structure \(A\), it holds that \(A \in P\) if and only if \(\Theta(A) \in Q\). A logic \(L'\) is *closed under \(L\)-interpretations* (or \(L\)-reductions) if for every property \(P\) that is \(L\)-reducible to an \(L'\)-definable property \(Q\), the property \(P\) itself is \(L'\)-definable (cf. [Ebb85, Ott97]). We say that \(L'\) is closed under interpretations if \(L'\) is closed under \(L'\)-interpretations.
3. Witnessed Symmetric Choice

We extend IFPC with an inflationary fixed-point operator with witnessed symmetric choice. Let \( \tau \) be a relational signature and \( R, R^* \), and \( S \) be new relation symbols not contained in \( \tau \) of arities \( \text{ar}(R) = \text{ar}(R^*) \) and \( \text{ar}(S) \). The letter \( p \) is used for an element parameter in this section. We define the WSC-fixed-point operator with parameters \( \bar{p} \bar{v} \). If

- \( \Phi_{\text{step}}(\bar{p}\bar{x}\bar{v}) \) is an IFPC+WSC[\( \tau, R, S \)]-formula such that \( |\bar{x}| = \text{ar}(R) \),
- \( \Phi_{\text{choice}}(\bar{p}\bar{y}\bar{v}) \) is an IFPC+WSC[\( \tau, R \)]-formula such that \( |\bar{y}| = \text{ar}(S) \),
- \( \Phi_{\text{wit}}(\bar{p}\bar{y}\bar{y}'z_1z_2\bar{v}) \) is an IFPC+WSC[\( \tau, R, R^* \)]-formula where \( |\bar{y}| = |\bar{y}'| = \text{ar}(S) \), and
- \( \Phi_{\text{out}}(\bar{p}\bar{v}) \) is an IFPC+WSC[\( \tau, R^* \)]-formula,

then

\[
\Phi(\bar{p}\bar{v}) = \text{ifp-wsc}_{R,A,R^*,S,Y,Y';z_1z_2}.(\Phi_{\text{step}}(\bar{p}\bar{x}\bar{v}), \Phi_{\text{choice}}(\bar{p}\bar{y}\bar{v}), \Phi_{\text{wit}}(\bar{p}\bar{y}\bar{y}'z_1z_2\bar{v}), \Phi_{\text{out}}(\bar{p}\bar{v}))
\]

is an IFPC+WSC[\( \tau \)]-formula. The formulas \( \Phi_{\text{step}} \), \( \Phi_{\text{choice}} \), \( \Phi_{\text{wit}} \), and \( \Phi_{\text{out}} \) are called step formula, choice formula, witnessing formula, and output formula, respectively. The free variables of \( \Phi \) are the ones of \( \Phi_{\text{step}} \) apart from \( \bar{x} \), the ones of \( \Phi_{\text{choice}} \) apart from \( \bar{y} \), the ones of \( \Phi_{\text{wit}} \) apart from \( \bar{y}, \bar{y}', z_1 \) and \( z_2 \), and the ones of \( \Phi_{\text{out}} \). That is, the variables \( \bar{x} \) are bound in \( \Phi_{\text{step}} \), the variables \( \bar{y} \) are bound in \( \Phi_{\text{choice}} \) and \( \Phi_{\text{wit}} \), and the variables \( \bar{y}', z_1 \), and \( z_2 \) are bound in \( \Phi_{\text{wit}} \). Note that only element variables are used for defining the fixed-point in the WSC-fixed-point operator. This suffices for our purpose in this article and increases readability. We expect that our arguments also work with numeric variables in the fixed-point. For the sake of readability, we will now omit the free numeric variables \( \bar{v} \) when defining the semantics of the WSC-fixed-point operator. Fixing numeric parameters does not change orbits or automorphisms.

**Evaluation with Choices.** Intuitively, the WSC-fixed-point operator \( \Phi \) is evaluated as follows: Let \( \mathfrak{A} \) be a \( \tau \)-structure and \( \bar{u} \in A^{[\bar{p}]} \) be a tuple of parameters. We define a sequence of relations called stages \( \emptyset =: R^\mathfrak{A}_1 \subset \cdots \subset R^\mathfrak{A}_t = R^\mathfrak{A}_{t+1} =: (R^*)^\mathfrak{A} \). Given the relation \( R^\mathfrak{A}_t \), the choice formula defines the choice-set \( T^\mathfrak{A}_{i+1} \) of the tuples \( \bar{v} \) satisfying \( \Phi_{\text{choice}} \), i.e.,

\[
T^\mathfrak{A}_{i+1} := \{ \bar{v} \mid \bar{u}\bar{v} \in \Phi^\mathfrak{A}_{\text{choice}} \}
\]

We pick an arbitrary tuple \( \bar{v} \in T^\mathfrak{A}_{i+1} \) and set \( S^\mathfrak{A}_{i+1} := \{ \bar{v} \} \) or \( S^\mathfrak{A}_{i+1} := \emptyset \) if no such \( \bar{v} \) exists. The step formula is used on the structure \( (\mathfrak{A}, R^\mathfrak{A}_t, S^\mathfrak{A}_{i+1}) \) to define the next stage in the fixed-point iteration:

\[
R^\mathfrak{A}_{i+1} := R^\mathfrak{A}_t \cup \{ \bar{w} \mid \bar{u}\bar{w} \in \Phi^\mathfrak{A}_{\text{step}} \}
\]

We proceed in that way until a fixed-point \( (R^*)^\mathfrak{A} \) is reached. Because we define an inflationary fixed-point, it is guaranteed to exist. This fixed-point is in general not isomorphism-invariant, i.e., not invariant under applying automorphisms of \( (\mathfrak{A}, \bar{u}) \).

We ensure that \( \Phi \) is still isomorphism-invariant as follows: First, we only allow choices from orbits, which the witnessing formula has to certify. A set \( N \subseteq \text{Aut}((\mathfrak{A}, \bar{u})) \) witnesses a relation \( R \subset A^k \) as \( (\mathfrak{A}, \bar{u}) \)-orbit, if for every \( \bar{v}, \bar{v}' \in R \), there is a \( \varphi \in N \) satisfying \( \bar{v} = \varphi(\bar{v}') \). Because we need the notion of witnessing orbits only for isomorphism-invariant sets, we do not need to check whether \( R \) is a proper subset of an orbit. We require that \( \Phi_{\text{wit}} \) defines a set of automorphisms. Intuitively, for \( \bar{v}, \bar{v}' \in T^\mathfrak{A}_{i+1} \), a map \( \varphi_{\bar{v},\bar{v}'} \) is defined via

\[
w \mapsto w' \quad \text{whenever} \quad \bar{u}\bar{v}\bar{v}'ww' \in \Phi_{\text{wit}}^{(\mathfrak{A}, R^\mathfrak{A}_t, (R^*)^\mathfrak{A})}.
\]
The set of all these maps for all $v, v' \in T_{i+1}$ has to witness $T_{i+1}$ as $(\mathcal{A}, \bar{u}, R_{i+1}, \ldots, R_{\ell})$-orbit. Note here, that the witnessing formula always has access to the fixed-point. Actually, we do not require that $\varphi_{v,v'}$ maps $v$ to $v'$ but only the set of all $\varphi_{v,v'}$ has to witness the orbit. If some choice is not witnessed, $\Phi$ is not satisfied by $\bar{u}$. Otherwise, the output formula is evaluated on the defined fixed-point:

$$\Phi^\mathfrak{A} := \Phi^\mathfrak{B}(R^\mathfrak{A})$$

Because all choices are witnessed, all possible fixed-points (for different choices) are related by an automorphism of $(\mathcal{A}, \bar{u})$ and thus either all of them or none of them satisfy the output formula.

**An Example.** We give an illustrating example (by adapting an example for CPT+WSC in [LS22]). We show that the class of threshold graphs (i.e., graphs that can be reduced to the empty graph by iteratively deleting universal or isolated vertices) is IFPC+WSC-definable (it is actually IFP-definable, but illustrates the WSC-fixed-point operator). The set of all isolated or universal vertices of a graph forms a 1-orbit (note that there cannot be an isolated and a universal vertex at the same time). We choose one vertex of this orbit, collect the chosen vertex in a unary relation $S$, and repeat pretending that the vertices in $R$ are deleted. If all vertices are contained in the obtained fixed-point $R^*$, then the graph is a threshold graph. The choice formula $\Phi_{\text{choice}}$ defines the set of all isolated or universal vertices not in $R$. The step formula $\Phi_{\text{step}}$ adds the chosen vertex, which is the only vertex in the relation $S$, to $R$. The output formula $\Phi_{\text{out}}$ checks whether $R^*$ contains all vertices and so defines whether it was possible to delete all vertices:

$$\Phi_{\text{choice}}(y) := \neg R(y) \land \left( (\forall z. \neg R(y) \Rightarrow E(y,z)) \lor (\forall z. \neg R(y) \Rightarrow \neg E(y,z)) \right),$$

$$\Phi_{\text{step}}(x) := R(x) \lor S(x),$$

$$\Phi_{\text{out}} := \forall x. R^*(x).$$

Witnessing orbits is easy: To show that two isolated (or universal, respectively) vertices $y$ and $y'$ are related by an automorphism, it suffices to define their transposition as follows:

$$\Phi_{\text{wit}}(y, y', z_1, z_2) := (z_1 = y \land z_2 = y') \lor (z_2 = y \land z_1 = y') \lor (y \neq z_1 = z_2 \neq y').$$

To the end, the formula ifp-wsc$_{R,x;R^*;S,y,y';z_1,z_2}$, $(\Phi_{\text{step}}, \Phi_{\text{choice}}, \Phi_{\text{wit}}, \Phi_{\text{out}})$ defines the class of threshold graphs.

**Formal Semantics.** To define the semantics of the WSC-fixed-point operator formally, we use the WSC*-operator that is defined in [LS22]. The WSC*-operator captures the idea of fixed-point iterations with choices from orbits for arbitrary isomorphism-invariant functions. Let $\mathcal{A}$ be a $\tau$-structure and $\bar{u} \in A^k$. We denote by $\text{HF}(A)$ the set of all hereditary finite sets over $A$. The WSC*-operator defines for isomorphism-invariant functions $f_{\text{step}}, f_{\text{wit}}, f_{\text{choice}} : \text{HF}(A) \times \text{HF}(A) \rightarrow \text{HF}(A)$ and $f_{\text{choice}} : \text{HF}(A) \rightarrow \text{HF}(A)$, the set

$$W = \text{WSC}^*(f_{\text{step}}, f_{\text{wit}}, f_{\text{choice}})$$

of all $\text{HF}(A)$-sets obtained in the following way. Starting with $b_0 := \emptyset$, define a sequence of sets as follows: Given $b_\ell$, define the choice-set $c_{\ell} := f_{\text{choice}}(b_\ell)$, pick an arbitrary $d_{\ell} \in c_{\ell}$ (or $d = \emptyset$ if $c_{\ell} = \emptyset$), and set $b_{\ell+1} := f_{\text{step}}(b_\ell, d_{\ell})$. Let $b^* := b_\ell$ for the smallest $\ell$ satisfying $b_\ell = b_{\ell+1}$ (if it exists, which in our case of inflationary fixed-points is always the case).
Then we include \( b^* \) in \( W \) if, for every \( i \in [\ell] \), the set \( f_{\text{wit}}^{\mathfrak{A}, \bar{u}}(b_i, b^*) \) is a set of automorphisms witnessing \( c_i \) as \((\mathfrak{A}, \bar{u}, b_1, \ldots, b_i)\)-orbit. Of course \( b^* \) is not unique and depends on the choices of the \( d_i \) and thus \( W \) is not necessarily a singleton set. It is proven in [LS22] that \( f_{\text{wit}}^{\mathfrak{A}, \bar{u}} \) witnesses the choices of all \( d_i \) for either every possible \( b^* \) obtained in the former way or for none of them. In particular, \( W \) is an \((\mathfrak{A}, \bar{u})\)-orbit. Formally, the set \( W \) is defined using trees capturing all possible choices. For more details, we refer to [LS22]. We now define the semantics of the WSC-fixed-point operator

\[
\Phi(\bar{p}) = \text{ifp-wsc}_{R, \bar{x}; R^*; S, \bar{y}, \bar{y}'; z_1 z_2} \left( \Phi_{\text{step}}(\bar{p} \bar{x}), \Phi_{\text{choice}}(\bar{p} \bar{y}), \Phi_{\text{wit}}(\bar{p} \bar{y} \bar{y}' z_1 z_2), \Phi_{\text{out}}(\bar{p}) \right)
\]

using tuples (implicitly encoded as \( HF(A) \)-sets). Let \( \bar{u} \in A^{[\bar{p}]} \). We set

\[
\begin{align*}
\Phi_{\text{step}}^{\mathfrak{A}, \bar{u}}(R^\mathfrak{A}, S^\mathfrak{A}) &:= R^\mathfrak{A} \cup \left\{ \bar{w} \mid \bar{u}\bar{w} \in \Phi_{\text{step}}^{\mathfrak{A}, R^\mathfrak{A}, S^\mathfrak{A}} \right\}, \\
\Phi_{\text{choice}}^{\mathfrak{A}, \bar{u}}(R^\mathfrak{A}) &:= \left\{ \bar{v} \mid \bar{u}\bar{v} \in \Phi_{\text{choice}}^{\mathfrak{A}, R^\mathfrak{A}} \right\}, \text{ and} \\
\Phi_{\text{wit}}^{\mathfrak{A}, \bar{u}}(R^\mathfrak{A}, (R^*)^\mathfrak{A}) &:= \left\{ \varphi\bar{w}\bar{v}' \mid \bar{v}, \bar{v}' \in \Phi_{\text{choice}}^{\mathfrak{A}, \bar{u}}(R^\mathfrak{A}) \right\}
\end{align*}
\]

where \( \varphi \bar{w}\bar{v}' = \left\{ (w, w') \in A^2 \mid \bar{u}\bar{w}w' \bar{v} \in \Phi_{\text{wit}}^{\mathfrak{A}, R^\mathfrak{A}, (R^*)^\mathfrak{A}} \right\} \).

The function \( \Phi_{\text{step}}^{\mathfrak{A}, \bar{u}}(R^\mathfrak{A}, S^\mathfrak{A}) \) evaluates the step formula \( \Phi_{\text{step}} \) and adds its output to \( R^\mathfrak{A} \), which defines the inflationary fixed-point. The function \( \Phi_{\text{choice}}^{\mathfrak{A}, \bar{u}}(R^\mathfrak{A}) \) defines the choice-set by evaluating the choice formula \( \Phi_{\text{choice}} \). Finally, the function \( \Phi_{\text{wit}}^{\mathfrak{A}, \bar{u}}(R^\mathfrak{A}, (R^*)^\mathfrak{A}) \) defines a set (possibly) of automorphisms by evaluating the witnessing formula \( \Phi_{\text{wit}} \) for all tuples in the current relation. Now set \( W_\Phi^{(\mathfrak{A}, \bar{u})} := \text{WSC}^{\mathfrak{A}, \bar{u}}(\Phi_{\text{step}}^{\mathfrak{A}, \bar{u}}, \Phi_{\text{choice}}^{\mathfrak{A}, \bar{u}}, \Phi_{\text{wit}}^{\mathfrak{A}, \bar{u}}) \).

We define the semantics of the WSC-fixed-point operator \( \Phi \) as follows. Let \( \mathfrak{A} \) be a \( \tau \)-structure and let \( \text{sig}(\Phi) \subseteq \tau \) be the relation symbols mentioned in \( \Phi \) (excluding those bound by fixed-point operators or WSC-fixed-point operators). We define

\[
\Phi^\mathfrak{A} := \left\{ \bar{u} \mid \bar{u} \in \Phi_{\text{out}}^{\mathfrak{A}, (R^*)^\mathfrak{A}}(\Phi^\mathfrak{A}, \varphi) \text{ for some } (R^*)^\mathfrak{A} \in W_\Phi^{(\mathfrak{A}, \text{sig}(\Phi), \bar{u})} \right\}.
\]

Note that if \( W_\Phi^{(\mathfrak{A}, \text{sig}(\Phi), \bar{u})} = \emptyset \), that is, not all choices could be witnessed, then we have \( \bar{u} \notin \Phi^\mathfrak{A} \). Also note that because \( W_\Phi^{(\mathfrak{A}, \text{sig}(\Phi), \bar{u})} \) is an \((\mathfrak{A} \upharpoonright \text{sig}(\Phi), \bar{u})\)-orbit, \( \bar{u} \in \Phi_{\text{out}}^{(\mathfrak{A}, \text{sig}(\Phi), (R^*)^\mathfrak{A})} \) holds for either every \((R^*)^\mathfrak{A} \in W_\Phi^{(\mathfrak{A}, \text{sig}(\Phi), \bar{u})} \) or for no \((R^*)^\mathfrak{A} \in W_\Phi^{(\mathfrak{A}, \text{sig}(\Phi), \bar{u})} \). Finally, note that the WSC-fixed-point operator is evaluated on the reduct \( \mathfrak{A} \upharpoonright \text{sig}(\Phi) \). In that way, adding more relations to \( \mathfrak{A} \) which are not mentioned in the formula does not change the orbit structure of \( \mathfrak{A} \upharpoonright \text{sig}(\Phi) \) and so does not change whether \( \Phi \) is satisfied or not. This is a desirable property of a logic [Ebb85]. This reduce semantics of a choice operator can also be found in [DR03a]. Finally, we conclude that IFPC+WSC is isomorphism-invariant:

**Lemma 3.1.** For every IFPC+WSC[\( \tau \) ]-formula \( \Phi \) and every \( \tau \)-structure \( \mathfrak{A} \), the set \( \Phi^\mathfrak{A} \) is a union of \((\mathfrak{A} \upharpoonright \text{sig}(\Phi))\)-orbits.

**Proof.** The proof is straight-forward by induction on the formula using that \( W_\Phi^{(\mathfrak{A}, \bar{u})} \) is an \((\mathfrak{A}, \bar{u})\)-orbit. \( \square \)
3.1. Extension with an Operator for Logical Interpretations. We extend the logic IFPC+WSC with another operator using interpretations. First, every IFPC+WSC-formula is an IFPC+WSC+I-formula. Second, if \( \Theta(p\nu) \) is an IFPC+WSC+I[\( \tau, \sigma \)]-interpretation with parameters \( p\nu \) and \( \Phi \) is an IFPC+WSC+I[\( \sigma \)]-sentence, then the interpretation operator

\[
\Psi(p\nu) = I(\Theta(p\nu); \Phi)
\]

is an IFPC+WSC+I[\( \tau \)]-formula with free variables \( p\nu \). The semantics is defined as follows:

\[
I(\Theta(p\nu); \Phi)^A := \left\{ \bar{u}n \in A^{p|\pi|} \times \mathbb{N}^{p|\pi|} \mid \Phi^{\Theta(\bar{A}, \bar{u}n)}(\bar{u}n) \neq \emptyset \right\}.
\]

Note that \( \Phi^{\Theta(\bar{A}, \bar{u}n)}(\bar{u}n) \neq \emptyset \) if and only if \( \Phi \) is satisfied. The interpretation operator allows to evaluate a subformula in the image of an interpretation. Thus, by definition, IFPC+WSC+I is closed under interpretations. For IFPC, such an operator does not increase the expressive power because IFPC is already closed under IFPC-interpretations (see [Ott97]). For IFPC+WSC, this is not clear: Because \( \Theta(\bar{A}, \bar{u}n) \) may have a different automorphism structure, \( \Phi \) may exploit the WSC-fixed-point operator in a way which is not possible in \( \bar{A} \).

Indeed, we will later see that IFPC+WSC is not even closed under FO-interpretations. We now study the properties of IFPC+WSC+I and its relation to IFPC+WSC. To do so, we first review a construction providing a non-IFPC-definable \( \text{Ptime} \)-property.

4. The CFI Construction

In this section we recall the CFI construction that was introduced by Cai, F¨urier, and Immerman [CFI92]. At the heart of the construction are the so-called CFI gadgets. These gadgets are used to obtain from a base graph two non-isomorphic CFI graphs.

The CFI Gadget. The degree-\( d \) CFI gadget consists of \( d \) pairs of edge vertices \( \{a_{i0}, a_{i1}\} \) for every \( i \in [d] \) and the set of gadget vertices \( \{b \in \mathbb{F}_2^d \mid b_1 + \cdots + b_d = 0\} \). There is an edge between \( a_{ij} \) and \( b \) if and only if \( b_i = j \). Edge vertices and gadget vertices receive different colors. If, using \( d \) additional colors, every edge-vertex-pair \( \{a_{i0}, a_{i1}\} \) receives its own color for every \( i \in [d] \), then the CFI gadget realizes precisely the automorphisms exchanging the vertices \( \{a_{i0}, a_{i1}\} \) for an even number of \( i \in [d] \).

We also need a variant of the CFI gadgets that does not use the gadget vertices but a \( d \)-ary relation instead. Every gadget vertex \( b \) is replaced by the \( d \)-tuple \( (a_{i1_b}, \ldots, a_{id_b}) \) in said relation. This gadget has the same automorphisms (with respect to the edge vertices). It has the benefit that not the whole gadget can be fixed by fixing a single gadget vertex but has the drawback that the arity of the relations depends on the degree. For an overview of the different variants of CFI gadgets we refer to [Lic21].

CFI Graphs. A base graph is a connected, simple, and possibly colored graph. Let \( G = (V, E, \preceq) \) be a colored base graph. In the context of CFI graphs, we call the vertices \( V \) of \( G \) base vertices, call its edges \( E \) base edges, and use fraktur letters for base vertices or edges. For a function \( f : E \rightarrow \mathbb{F}_2 \), we construct the CFI graph \( \text{CFI}(G, f) \) as follows: First, replace every vertex of \( G \) by a CFI gadget of the same degree. In that way, we obtain for every base edge \( \{u, v\} \in E \) two edge-vertex-pairs \( \{a_{i0}, a_{i1}\} \) and \( \{a_{j0}', a_{j1}'\} \). The first one is given by the gadget of \( u \) and the second one by the gadget of \( v \). Second, add edges such that \( \{a_{ik}, a_{j\ell}'\} \) is an edge if and only if \( k + \ell = f(\{u, v\}) \). We say that the edge vertices \( \{a_{i0}, a_{i1}\} \) originate from \( (u, v) \), the edge vertices \( \{a_{j0}', a_{j1}'\} \) originate from \( (v, u) \), respectively, and that
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Figure 1. Construction of a CFI graph from a base graph: Figure (a) shows a part of a colored base graph $G$. Figure (b) shows the two gadgets for the red base vertex $u$ and the blue base vertex $v$ in the graph $CFI(G, f)$. The figure assumes that $f(\{u, v\}) = 0$. Gadget vertices inherit the color from their base vertex. An edge vertex of the gadget of $u$ inherits the color from the order pair $(u, \mathbf{w})$ of the corresponding neighbor $\mathbf{w}$ of $u$. This color is indicated by asymmetrically coloring a vertex in the figure with two colors.

the gadget vertices of the gadget for a base vertex $u$ originate from $u$. The color of edge and gadget vertices is obtained from the color of its origin given by $\preceq$.

It is well-known [CFI92] that, for every $f, g : E \to \mathbb{F}_2$, we have $CFI(G, g) \cong CFI(G, f)$ if and only if $\sum g := \sum_{e \in E} g(e) = \sum f$. Hence, if we are only interested in the graph up to isomorphism, we also write $CFI(G, 0)$ and $CFI(G, 1)$. A CFI graph $CFI(G, f)$ is even if $\sum f = 0$ and odd otherwise. A base edge $e \in E$ is called twisted by $f$ and $g$ if $g(e) \neq f(e)$. Twisted edges can be “moved around” using path isomorphisms (see e.g. [Lic21]): If $u_1, \ldots, u_\ell$ is a path in $G$, then there is an isomorphism $\varphi : CFI(G, g) \to CFI(G, g')$, where $g'(e) = g(e)$ apart from $e_1 := \{u_1, u_2\}$ and $e_2 := \{u_{\ell-1}, u_{\ell}\}$ which satisfy $g'(e_1) = g(e_1) + 1$. The automorphism $\varphi$ is the identity on all vertices apart from the ones whose origin is contained in the path $u_1, \ldots, u_{\ell}$. If the path is actually a cycle, we obtain an automorphism. If $G$ is totally ordered, then every isomorphism is composed of path-isomorphisms and every automorphism of $CFI(G, g)$ is the composition of such cycle-automorphisms. For a class of base graphs $\mathcal{K}$, set

$$CFI(\mathcal{K}) := \{ CFI(G, g) \mid G = (V, E) \in \mathcal{K}, g : E \to \mathbb{F}_2 \}$$

to be the class of CFI graphs over $\mathcal{K}$. The CFI query for $CFI(\mathcal{K})$ is to decide whether a given CFI graph in $CFI(\mathcal{K})$ is even. We collect some facts on CFI graphs:
Lemma 4.1 [DR07, Theorem 3]. If $G$ is of minimum degree 2 and has treewidth at least $k$, in particular if $G$ is $k$-connected, then $\text{CFI}(G, 0) \preceq_C \text{CFI}(G, 1)$.

In particular, the CFI query for a class of base graphs of unbounded treewidth or connectivity is not IFPC-definable. We now consider orbits of CFI graphs. The following lemma is well-known:

Lemma 4.2. Let $G = (V, E, \preceq)$ be a colored base graph, let $\{u, v\} \in E$, let $\mathfrak{A} = \text{CFI}(G, f)$ for some $f : E \to \mathbb{F}_2$, let $\bar{w} \in A^*$ be an $\ell$-tuple of edge vertices, and let the origin of $w_i$ be $(\bar{w}_i, \bar{w}_i')$ for every $i \in [\ell]$. Then both edge vertices with origin $(u, v)$ are in the same 1-orbit of $(\mathfrak{A}, \bar{w})$ if and only if $\{u, v\}$ is part of a cycle in $G - \{\{w_i, w_i'\} | i \in [||\bar{w}||]\}$.

Proof. Set $E' := \{\{w_i, w_i'\} | i \in [\ell]\}$. Assume that there is a cycle in $G - E'$ containing $\{u, v\}$. Then we can use a cycle-automorphism for that cycle to exchange the two edge vertices pairs with origin $(u, v)$. Because this automorphism is the identity on all vertices apart from the ones whose origin is contained in the cycle, it in particular fixes $\bar{w}$.

For the other direction, assume that there is an automorphism $\varphi$ mapping one edge vertex with origin $(u, v)$ to the other one. Then in particular $\varphi$ has to exchange both. We can assume that $\varphi$ is base-vertex-respecting, that is, $\varphi$ maps vertices to vertices of the same origin. If $\varphi$ was not base-vertex-respecting, $\varphi$ induces a non-trivial automorphism of the base graph, whose inverse can be combined with $\varphi$ to a base-vertex-respecting automorphism (see [Pag23]). But a base-vertex-respecting automorphism of $(\mathfrak{A}, \bar{w})$ is composed out of cycle-automorphism not using the edges $E'$. So there is in particular a single cycle in $G - E'$ containing $\{u, v\}$. \qed

We sketch the proof of the following lemma to illustrate the requirement of high connectivity on the base graph.

Lemma 4.3 [GP19, Lemma 3.14]. Let $G = (V, E, \preceq)$ be an ordered and $(k + 2)$-connected base graph and let $\mathfrak{A} = \text{CFI}(G, f)$ for some $f : E \to \mathbb{F}_2$. Let $\bar{w} \in A^{\leq k}$ and $\{u, v\} \in E$ be a base edge such that no vertex in $\bar{w}$ has origin $u$, $v$, $(u, v)$, or $(v, u)$. Then the two edge vertices with origin $(u, v)$ are contained in the same orbit of $(\mathfrak{A}, \bar{w})$.

Proof. We first assume that $\bar{w}$ only consists of gadget vertices. Let $V_{\bar{w}} \subseteq V$ be the set of origins of all vertices in $\bar{w}$. Because every vertex in $G$ has degree at least $k + 2$ (since $G$ is $(k + 2)$-connected), the vertices $u$ and $v$ have degree at least 2 in $G \setminus V_{\bar{w}}$. Because $G$ is $(k + 2)$-connected, there is a $u$-$v$-path in $G \setminus V_{\bar{w}}$ not using the edge $\{u, v\}$ (removing $v$ from $G \setminus V_{\bar{w}}$ removes at most $k + 1$ vertices from $G$). So there is a cycle in $G \setminus V_{\bar{w}}$ using the edge $\{u, v\}$ and thus there is an automorphism exchanging the two edge vertices with origin $(u, v)$.

Now assume that there is an edge vertex with origin $(u, v)$ in $\bar{w}$. Let $\bar{w}'$ be obtained from $\bar{w}$ by replacing this vertex with a gadget vertex with origin $u$. Every automorphism fixing $\bar{w}'$ also fixes $\bar{w}$. \qed

5. Canonization of CFI Graphs in IFPC+WSC+I

In this section we show that with respect to canonization the CFI construction “loses its power” in IFPC+WSC+I in the sense that canonizing CFI graphs is not harder than canonizing the base graphs. In the following, we work with a class of base graphs closed under individualization. Intuitively, this means that the class is closed under assigning some
vertices unique new colors. We adapt a result of [LS22] from CPT+WSC to IFPC+WSC and show that it suffices to define a single orbit for every individualization of a graph to obtain a definable canonization. The canonization approach adapts Gurevich’s canonization algorithm [Gur97] and requires the WSC-fixed-point operator. We then show that once we define orbits of (the closure under individualization of the) base graphs, we can define orbits of CFI graphs and hence canonize them. Here we need the interpretation operator to reduce a CFI graph to its base graph to define its orbits.

Intuitively, individualizing vertices is just a tuple of parameters. However, the number of vertices to individualize is not bounded and we need to encode them via relations in the first-order setting.

**Definition 5.1 (Individualization of Vertices).** Let $\mathfrak{A}$ be a relational structure. A binary relation $\leq^A \subseteq A^2$ is an individualization of a set of vertices $V \subseteq A$ if $\leq^A$ is a total order on $V$ and $\leq^A \subseteq V^2$. We say that $\leq^A$ is an individualization if it is an individualization of some $V \subseteq A$ and that the vertices in $V$ are individualized by $\leq^A$.

The relation $\leq^A$ defines a total order on the individualized vertices, so intuitively it assigns unique colors to these vertices. Instead of $(\mathfrak{A}, \leq^A)$, one can intuitively think of $(\mathfrak{A}, \bar{u})$ where $\bar{u}_1 \leq^A \cdots \leq^A \bar{u}_{|\bar{u}|}$ are the vertices individualized by $\leq^A$.

**Definition 5.2 (Closure under Individualization).** Let $\mathcal{K}$ be a class of $\tau$-structures. The closure under individualization $\mathcal{K}^\leq$ of $\mathcal{K}$ is the class of $(\tau \cup \{\leq\})$-structures such that $(\mathfrak{A}, \leq^A) \in \mathcal{K}^\leq$ for every $\mathfrak{A} \in \mathcal{K}$ and every individualization $\leq^A$.

In the following, let $L$ be one of the logics IFPC, IFPC+WSC, or IFPC+WSC+I. We adapt some notions related to canonization from [LS22] to the first-order setting. Note that all following definitions implicitly include the closure under individualization.

**Definition 5.3 (Canonization).** Let $\mathcal{K}$ be a class of $\tau$-structures. An $L$-canonization for $\mathcal{K}$ is an $L[\tau \cup \{\leq\}, \tau \cup \{\leq, \leq\}]$-interpretation $\Theta$ satisfying the following:

1. $\leq^{\Theta(\mathfrak{A})}$ is a total order on $\Theta(\mathfrak{A})$ for every $\mathfrak{A} \in \mathcal{K}^\leq$.
2. $\mathfrak{A} \cong \Theta(\mathfrak{A}) \upharpoonright (\tau \cup \{\leq\})$ for every $\mathfrak{A} \in \mathcal{K}^\leq$, and
3. $\Theta(\mathfrak{A}) \cong \Theta(\mathfrak{B})$ if and only if $\mathfrak{A} \cong \mathfrak{B}$ for every $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}^\leq$.

The structure $\Theta(\mathfrak{A})$ is called the $\Theta$-canon (or just the canon if unambiguous) of $\mathfrak{A}$. We say that $L$ canonizes $\mathcal{K}$ if there is an $L$-canonization for $\mathcal{K}$.

We make same remarks to the former definition. Condition 3 requires isomorphism of ordered structures. For a logic $L$ possessing numbers (as in our case), Condition 3 can equivalently be stated with equality. While Condition 3 is essential for algorithmic canonizations, it is implied by Condition 2 for $L$-definable canonizations: Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}^\leq$. If $\mathfrak{A} \cong \mathfrak{B}$, then $\Theta(\mathfrak{A}) \cong \Theta(\mathfrak{B})$ because $L$ is isomorphism-invariant. If $\Theta(\mathfrak{A}) \cong \Theta(\mathfrak{B})$, then by Condition 2, $\mathfrak{A} \cong \Theta(\mathfrak{A}) \upharpoonright (\tau \cup \{\leq\}) \cong \Theta(\mathfrak{B}) \upharpoonright (\tau \cup \{\leq\}) \cong \mathfrak{B}$.

**Definition 5.4 (Distinguishable $k$-Orbits).** The logic $L$ distinguishes the $k$-orbits of a class of $\tau$-structures $\mathcal{K}$, if there is an $L[\tau \cup \{\leq\}]$-formula $\Phi(x, y)$ that has $2k$ free variables $x$ and $y$ such that $|x| = |y| = k$ and that defines, for every $\mathfrak{A} \in \mathcal{K}^\leq$, a total preorder on $A^k$ whose equivalence classes coincide with the $k$-orbit partition of $\mathfrak{A}$.

Note that because $\Phi$ defines a total preorder, $L$ does not only define the $k$-orbit partition but also orders the $k$-orbits.
Definition 5.5 (Ready for Individualization). A class of \(\tau\)-structures \(\mathcal{K}\) is ready for individualization in \(L\) if there is an \(L[\tau \uplus \{\leq\}]\)-sentence \(\Phi\) defining, for every \(\mathfrak{A} \in \mathcal{K}^{\leq}\), a set of vertices \(O = \Phi^\mathfrak{A}\) such that

- \(O\) is a 1-orbit of \(\mathfrak{A}\),
- \(|O| > 1\) if \(\mathfrak{A}\) has a non-trivial 1-orbit, and
- if \(O = \{u\}\) is a singleton set, then \(u\) is not individualized by \(\mathfrak{A}\) unless \(\mathfrak{A}\) individualizes \(A\), i.e., all vertices.

The following is a similar statement as for CPT+WSC in [LS22] and the proof is analogous (note that [LS22] includes definable isomorphism, which we do not do here):

Lemma 5.6. Let \(L\) be one of the logics IFPC+WSC and IFPC+WSC+I and let \(\mathcal{K}\) be a class of \(\tau\)-structures. The following are equivalent:

1. \(L\) defines a canonization for \(\mathcal{K}\).
2. \(L\) distinguishes the \(k\)-orbits of \(\mathcal{K}\) for every positive \(k \in \mathbb{N}\).
3. \(\mathcal{K}\) is ready for individualization in \(L\).

Proof. We show \((1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)\). To show \((1) \Rightarrow (2)\), two \(k\)-tuples are ordered according to the lexicographical order on the canons when individualizing these tuples. For \((2) \Rightarrow (3)\), one orders the 1-orbits and picks the minimal (according to that order) non-trivial orbit. If such an orbit does not exist, the minimal singleton 1-orbit whose vertex is not individualized is picked. Finally, to show \((3) \Rightarrow (1)\), one defines a variant of Gurevich’s canonization algorithm [Gur97] using a WSC-fixed-point operator. This is done exactly as in [LS22] for CPT+WSC. It is easy to see there that once the formula defining the orbit is an \(L\)-formula, the algorithm can easily be expressed in \(L\) (essentially collecting individualized vertices in a relation). 

While stating Lemma 5.6 requires some technical definitions, it simplifies defining canonization using WSC-fixed-point operators. Their use is hidden in Gurevich’s canonization algorithm: Witnessing automorphisms do not have to be defined explicitly. Also note that counting is useful for canonizing in IFPC because numbers serve as vertices in the canon.

Lemma 5.7. Let \(\mathcal{K}\) be a class of colored base graphs of minimal degree 3. If IFPC+WSC+I distinguishes 2-orbits of \(\mathcal{K}\), then \(\text{CFI}(\mathcal{K})\) is ready for individualization in IFPC+WSC+I.

Proof. Let \(G = (V, E, \leq) \in \mathcal{K}\), let \(g: E \to \mathbb{F}_2\), and let \(\mathfrak{A} = (\text{CFI}(G, g), \leq^\mathfrak{A})\), where \(\leq^\mathfrak{A}\) individualizes some vertices of \(\text{CFI}(G, g)\). We assume that only edge vertices are individualized. Instead of individualizing a gadget vertex with origin \(u\), one can equivalently individualize an edge vertex with origin \((u, v)\) for every \(v \in N_G(u)\). This step is IFPC-definable.

We denote by \(\text{orig}(\leq^\mathfrak{A})\) the set of all (directed) base edges \((u, v)\) that are the of some edge vertex individualized by \(\leq^\mathfrak{A}\). We denote by \(G - \text{orig}(\leq^\mathfrak{A})\) the graph obtained from \(G\) by deleting the edges in \(\text{orig}(\leq^\mathfrak{A})\) viewed as undirected edges. Note that individualizing an edge vertex implies individualizing a directed base edge, which is equivalent to individualizing two base vertices. In that way, we denote by \((G, \text{orig}(\leq^\mathfrak{A}))\) the graph \(G\) when individualizing base vertices such that exactly the base edges in \(\text{orig}(\leq^\mathfrak{A})\) are implicitly individualized.

We analyze the cases in which there are non-trivial 1-orbits of \((\mathfrak{A}, \leq^\mathfrak{A})\). If there is a 2-orbit of \((G, \text{orig}(\leq^\mathfrak{A}))\) containing base edges part of a cycle in \(G - \text{orig}(\leq^\mathfrak{A})\), we obtain a non-trivial 1-orbit of \((\mathfrak{A}, \leq^\mathfrak{A})\) as follows:
Claim 5.8. Let \( O \) be a 2-orbit of \((G, \text{orig}(\leq^3))\). If every (directed) edge in \( O \) is part of a cycle in \( G - \text{orig}(\leq^3) \), then the set of edge-vertex-pairs \( \{u \mid \text{the origin of } u \text{ is in } O\} \) is a 1-orbit of \((A, \leq^3)\).

Proof. Because \( O \) is a 2-orbit, either all or none of the directed base edges in \( O \) are part of a cycle in \( G - \text{orig}(\leq^3) \). By Lemma 4.2, the edge-vertex-pairs with origin in \( O \) are in the same 1-orbit. Using an automorphism of the base graph, the edge vertices with origin \((u, v) \in O\) can be mapped to the edge vertices with origin \((u', v') \in O\) for every \((u, v), (u', v') \in O\). That is, \( \{u \mid \text{the origin of } u \text{ is in } O\} \) is a subset of a 1-orbit of \((A, \leq^3)\). It cannot be a strict subset because then an edge vertex with origin in \( O \) has to be mapped to an edge vertex with origin not in \( O \), which contradicts that \( O \) is an orbit. \( \square \)

In case that no such 2-orbit exists, there are possibly other non-trivial 1-orbits. They arise from automorphisms of the base graph. An edge-vertex-pair-order of a set of base edges \( E' \subseteq E \) is a set of edge vertices \( R \) such that, for every edge-vertex-pair \( \{u_1, u_2\} \) with origin \((u, v) \) such that \( \{u, v\} \in E' \), exactly one of \( u_1 \) and \( u_2 \) is contained in \( R \). Intuitively, \( R \) defines an order per edge-vertex-pair with origin in \( E' \), but does not order edge-vertex-pairs of different origins.

Claim 5.9. There is an IFPC-formula (uniformly in \( G \)) that defines an edge-vertex-pair-order on all base edges \( E \) if \( G - \text{orig}(\leq^3) \) has no cycles.

Proof. We first show the following: Whenever for a base vertex there is an edge-vertex-pair-order \( R \) of all incident edges apart from one, then \( R \) can be extended to the remaining incident edge. This is done as follows (cf. Figure 2). Let \( u \in V \) be a base vertex and let \( N_G(u) = \{v_1, \ldots, v_d\} \). Assume w.l.o.g. that \( R \) orders the edge-vertex-pairs of the base edges \( \{u, v_i\} \in E \) for every \( i \in [d - 1] \) and let \( v_i \in R \) be the edge vertex with origin \((u, v_i)\) for every \( i \in [d - 1] \). By construction of the CFI graphs, there is exactly one gadget vertex \( u \) of the gadget of \( v_i \) adjacent to \( v_i \) for all \( i \in [d - 1] \). Because every gadget vertex is adjacent to exactly one edge vertex per incident edge-vertex-pair, we can add this unique edge vertex \( v_d \) with origin \((u, v_d)\) and the unique edge vertex \( v'_d \) with origin \((v_d, u)\) adjacent to \( v_d \) to \( R \). These two vertices can clearly be defined in IFPC (without an order on the \( v_i \)).
We propagate this approach through gadgets. Assume there is an edge-vertex-pair-order $R$ of $E'$ such that $G - E'$ has no cycles. Then $G - E'$ is a forest and there are vertices of degree one in $G - E'$ (unless $E' = E$). For all degree-one vertices of $G - E'$, we extend $R$ to the remaining incident edge as shown before. So unless $E' = E$, we added more edges to $E'$. Surely, $G - E'$ has still no cycles. So we can repeat this process using a fixed-point operator to define an edge-vertex-pair-order of $E$.

It is clear that we can turn $\leq^3$ into an edge-vertex-pair-order $R$ of $\text{orig}(\leq^3)$ (seen as undirected edges): For every $(u, v) \in \text{orig}(\leq^3)$, at least for one of the edge-vertex-pairs with origin $(u, v)$ is individualized by $\leq^3$. We put the $\leq^3$-minimal such vertex $u$ into $R$. For the edge-vertex-pair with origin $(v, u)$, we add the unique edge vertex adjacent to $u$ to $R$ (if both $(u, v), (v, u) \in \text{orig}(\leq^3)$, we start with the directed edge containing the $\leq^3$-minimal vertex). One easily sees that $R$ is IFPC-definable. By assumption, $G - \text{orig}(\leq^3)$ has no cycles and thus we can define an edge-vertex-pair-order of $E$.

**Claim 5.10.** Suppose $R$ is an isomorphism-invariant edge-vertex-pair-order of $E$, that is, $\text{Aut}((\mathfrak{A}, \leq^3)) = \text{Aut}((\mathfrak{A}, \leq^3, R))$, and $O$ is a 2-orbit of $(G, \text{orig}(\leq^3))$. Then the set of edge vertices $\{u \in R \mid \text{the origin of } u \text{ is in } O\}$ is a 1-orbit of $(\mathfrak{A}, \leq^3)$.

**Proof.** Let $(u, v), (u', v') \in O$, i.e., there is an automorphism $\varphi \in \text{Aut}((G, \text{orig}(\leq^3)))$ such that $\varphi((u, v)) = (u', v')$. Every automorphism of $(G, \text{orig}(\leq^3))$ induces and automorphism of $(\mathfrak{A}, \leq^3)$. So there is an automorphism $\psi \in \text{Aut}((\mathfrak{A}, \leq^3))$ mapping the edge-vertex-pair with origin $(u, v)$ to the one with origin $(u', v')$. Because $R$ is isomorphism-invariant, $\psi$ has to map edge vertices in $R$ to edge vertices in $R$. Hence, $\{u \in R \mid \text{the origin of } u \text{ is in } O\}$ is a subset of a 1-orbit of $(\mathfrak{A}, \leq^3)$. This set cannot be a proper subset of an orbit because $R$ is isomorphism-invariant.

Let $\Phi_{2\text{-orb}}(\bar{x}, \bar{y})$ be an IFPC+WSC+I-formula distinguishing 2-orbits of $K$ (that is, by definition, of $K_{\leq^2}$). We cannot evaluate $\Phi_{2\text{-orb}}$ on $\mathfrak{A}$ to define 2-orbits of $(G, \text{orig}(\leq^3))$ because $\mathfrak{A}$ has a more complicated automorphism structure than $G$ and it is not clear how to witness orbits. Here we use the interpretation operator. We define an IFPC-interpretation defining the base graph $(G, \text{orig}(\leq^3))$. Intuitively, we contract all gadgets to a single vertex, remove all edge vertices, and instead directly connect the contracted gadgets. To do so, we need the following claim:
Claim 5.11. For every $u \in A$, it holds that $u$ is a gadget vertex if and only if, for every $v \in N_3(u)$, there are two distinct $w, w' \in N_3(v) \setminus \{u\}$ of different color. Every two distinct gadget vertices $u, v \in A$ with origins $u$ and $v$

(a) have distance 2 or 4 if and only if $u = v$ and
(b) have distance 3 or 5 if and only if $\{u, v\} \in E$.

Proof. We show the first claim (cf. Figure 3). Let $u \in A$. Assume that $u$ is a gadget vertex with origin $u$ and let $v \in N_3(u)$. Then $v$ is an edge vertex with origin $(u, v)$. Because the minimal degree of $G$ is 3, there is another gadget vertex $u' \neq u$ with origin $u$ adjacent to $v$. The edge vertex with origin $(v, u)$ adjacent to $v$ has a different color than $u'$. To show the other direction, assume that for every neighbor $v \in N_3(u)$ there are two distinct $w, w' \in N_3(v) \setminus \{u\}$ of different color. For a sake of contradiction, suppose that $u$ is an edge vertex with origin $(u, v)$ adjacent to $v$. Consider the unique edge $v$ vertex with origin $(v, u)$ adjacent to $u$. Every neighbor of $v$ is either $u$ or a gadget vertex with origin $v$, but which all have the same color. Hence, $u$ is a gadget vertex.

To show the second claim (cf. Figure 3 again), let $u = \bar{a}$ and $v = \bar{b}$ be gadget vertices with origins $u$ respectively $v$, i.e., $\bar{a} \in \mathbb{F}_2^{\lceil N_G(u) \rceil}$ and $\bar{b} \in \mathbb{F}_2^{\lceil N_G(v) \rceil}$ such that $0 = a_1 + \cdots + a_{\lceil N_G(u) \rceil} = b_1 + \cdots + b_{\lceil N_G(v) \rceil}$. For part a), assume $u = v$. By construction of the CFI gadget, $u$ and $v$ are not adjacent and have a common neighbor if and only if $a_i \neq b_i$ for some $i \in [\lceil N_G(u) \rceil]$. If $a_i \neq b_i$ for all $i \in [\lceil N_G(u) \rceil]$, then there is another gadget vertex to which both $u$ and $v$ have distance 2 because $u$ has degree at least 3. For the other direction, assume $a_i \neq b_i$ have distance 2 or 4. For every neighbor $w$ of $u$, there is an $\{u, w\} \in E$ such that $w$ is an edge vertex with origin $(u, w)$ and every neighbor of $w$ is a gadget vertex with origin $u$ or an edge vertex with origin $(w, u)$. Hence, if $u$ and $v$ have distance 2, then $v = u$. The case of distance 4 is similar, here all distance 4 vertices are either vertices of the same gadget or edge vertices.

Proving Part b) is similar: Assume $\{u, v\} \in E$. If $a_i = b_j$ where the $i$-th edge-vertex-pair of the gadget of $u$ is connected to the $j$-th one of the gadget of $v$, then $u$ and $v$ have distance 3 (via two edge vertices with origin $(u, v)$ and $(v, u)$), otherwise they have distance 5. For the other direction, all distance 3 vertices of $u$ are either gadget vertices of gadgets whose origin $w$ satisfies $\{u, w\} \in E$ or edge vertices.

So the interpretation $\Theta = (\Phi_{\text{dom}}, \Phi_{\preceq}, \Phi_E, \Phi_{\preceq}, \Phi_{\preceq})$ defines the base graph:

- $\Phi_{\text{dom}}(x) := \forall y. \ E(x, y) \implies \exists z_1, z_2. \ z_1 \neq x \land z_2 \neq x \land E(y, z_1) \land E(y, z_2) \land z_1 < z_2$,
- $\Phi_{\preceq}(x, y) := x = y \lor \Phi_{\text{dist}}^2(x, y) \lor \Phi_{\text{dist}}^4(x, y)$,
- $\Phi_E(x, y) := \Phi_{\text{dist}}^3(x, y) \lor \Phi_{\text{dist}}^5(x, y)$,
- $\Phi_{\preceq}(x, y) := x \preceq y$.

We used formulas $\Phi_{\text{dist}}(x, y)\ell$ defining that $x$ and $y$ have distance $\ell$. It remains to define $\Phi_{\preceq}$, for which we omit a formal definition: We use $\preceq$ to define an order on the base edges in $\text{orig}(\preceq^3)$ and individualize the corresponding base vertices using this order.

Now we are able to evaluate $\Phi_{\text{2-orb}}$ on the base graph. We extend $\Theta$ by two parameters $z_1$ and $z_2$ for two edge vertices, for which we want to know whether their origins are in the same 2-orbit of $G$. We add two fresh binary relation symbols $S_1$ and $S_2$ and let $\Theta$ define $S_i$ to contain the origin of the edge vertex of $z_i$ for every $i \in [2]$. 


We lift the total preorder of the base vertices to the edge vertices using the interpretation operator:

$$\Phi_{\text{base-orb}}(z_1, z_2) := I(\Theta(z_1, z_2); \forall \bar{x}\bar{y}. S_1(\bar{x}) \Rightarrow S_2(\bar{y}) \Rightarrow \Phi_{\text{2-orb}}(\bar{x}, \bar{y})).$$

The formula $\Phi_{\text{base-orb}}$ defines whether the origins of two edge vertices are in the same orbit of $(G, \text{orig}(\leq^3))$. Note that $\Phi_{\text{2-orb}}$ does not mention the additional relations $S_1$ and $S_2$ and thus every WSC-fixed-point operator in $\Phi_{\text{2-orb}}$ is evaluated on a structure not containing $S_1$ and $S_2$ (by the reduct semantics) and so indeed on a $K^{\leq}$-graph.

Finally, we can indeed check whether there is an orbit as required by Claim 5.8: Consider the equivalence classes induced by $\Phi_{\text{base-orb}}$ on the edge vertices and check the existence of the required cycle. We define the minimal such orbit if such one exists.

If no such orbit exists, then $G - \text{orig}(\leq^3)$ contains no cycles and we can define an edge-vertex-pair-order $R$ of $E$ by Claim 5.9 in IFPC. Because $R$ is IFPC-definable, $R$ is in particular isomorphism-invariant. If there is a non-trivial 2-orbit of $G - \text{orig}(\leq^3)$, we pick the minimal one to define a non-trivial 1-orbit of $(\mathfrak{A}, \leq^3)$ using $R$ and Claim 5.10.

If that is also not the case, then there is no non-trivial 2-orbit of $G - \text{orig}(\leq^3)$, that is, there is a definable total order on the (directed) base edges. Together with the edge-vertex-pair-order $R$, we define a total order on all edge-vertex-pairs, which can be extended to a total order on $\mathfrak{A}$. We define the minimal vertex which is not individualized in $\leq^3$ if it exists. Otherwise, all vertices are individualized. We pick the $\leq$-minimal one. This concludes the proof of Lemma 5.7.

We now can prove Theorem 1.2, stating that if IFPC+WSC+I canonizes $\mathcal{K}$, then IFPC+WSC+I canonizes $\text{CFI}(\mathcal{K})$, too.

**Proof of Theorem 1.2.** Assume that IFPC+WSC+I defines a canonization for $\mathcal{K}$. Then IFPC+WSC+I distinguishes 2-orbits of $\mathcal{K}$ by Lemma 5.6. Hence, $\text{CFI}(\mathcal{K})$ is ready for individualization in IFPC+WSC+I by Lemma 5.7 and so IFPC+WSC+I defines a canonization for $\text{CFI}(\mathcal{K})$ by Lemma 5.6.

The restriction to base graphs of minimal degree 3 simplifies the proof of Lemma 5.7, namely defining the base graph of a given CFI graph. However, this restriction is not necessary and Theorem 1.2 can be extended to all base graphs.

**Corollary 5.12.** If $\mathcal{K}$ is a class of base graphs of bounded degree, then IFPC+WSC+I defines canonization for $\mathcal{K}$ if and only if it defines canonization for $\text{CFI}(\mathcal{K})$.

**Proof.** One direction is by Theorem 1.2. For the other direction, let $\mathcal{K}$ be graph class of maximal degree $d$. Then there is a $d$-dimensional IFPC-interpretation $\Theta$ such that $\Theta(G)$ is the even CFI graph over $G$ for every base graph $G \in \mathcal{K}^{\leq}$. Individualized vertices are translated into the coloring and thus into gadgets of unique color. Together with the canonization-defining interpretation and the base-graph-defining one in the proof of Theorem 1.2, we obtain a canonization for $\mathcal{K}^{\leq}$. Note that this approach of defining a CFI graph to canonize the base graph cannot work for arbitrary base graphs because for large degrees the CFI graphs get exponentially large (which cannot be defined by an IFPC-interpretation).

We make the following observation. First, Theorem 1.2 can be applied iteratively: If IFPC+WSC+I canonizes $\mathcal{K}$, then IFPC+WSC+I canonizes $\text{CFI}(\mathcal{K})$, and so IFPC+WSC+I canonizes $\text{CFI}(\text{CFI}(\mathcal{K}))$. Second, every application of Theorem 1.2 adds one WSC-fixed-point
operator (to define Gurevich’s algorithm in Lemma 5.6) and one interpretation operator (for the base-graph-defining interpretation in Lemma 5.7). More precisely, the nesting depth of these operators increases. We will show that this is in some sense necessary: The CFI query on a variation of $\text{CFI}(\text{CFI}(\mathcal{K}))$ cannot be defined without nesting two WSC-fixed-point and two interpretation operators.

6. The CFI Query and Nesting of Operators

In this section we show that the increased nesting depth of WSC-fixed-point and interpretation operators used to canonize CFI graphs in Theorem 1.2 is unavoidable. Because IFPC does not define the CFI query, it is clear that even if the class of base graph has IFPC-distinguishable orbits, the CFI query cannot be defined in IFPC (e.g., on a class of all ordered graphs). So, the nesting depth of WSC-fixed-point operators has to increase. However, for orbits distinguishable in IFPC+WSC+I but not in IFPC it is not clear whether the nesting depth has to increase. We now show that this is indeed the case.

Intuitively, we want to combine non-isomorphic CFI graphs into a new base graph and then apply the CFI construction again. To define orbits of these double CFI graphs, one has to define the CFI query for the base CFI graphs, which cannot be done without a WSC-fixed-point operator. However, parameters of WSC-fixed-point operators complicate matters. If parameter of a WSC-fixed-point operator is used to fix a vertex contained in one of the base CFI graphs, then this base CFI graph can be distinguished from all the others. Hence, orbits of this base CFI graph can be defined without defining the CFI query. To overcome this problem, we use multiple copies of the base CFI graphs such that their number exceeds the number of parameters.

If we want to make choices from the base CFI graphs not containing a parameter, we have to define the CFI query for them, which increases the nesting depth. If we do not do so, we can essentially only make choices in the base CFI graphs containing a parameter vertex. That is, we can move the twist to the other base CFI graphs. Proving that in this case the CFI query cannot be defined without more operators requires formal effort: We introduce a logic which allows for quantifying over all individualizations of the base CFI graphs containing parameters. This is (potentially) more powerful than making choices, but we still can prove non-definability of the CFI query. This logic has the benefit that we can characterize it via a pebble game.

First, we introduce a formalism for nesting WSC-fixed-point and interpretation operators in Section 6.1. Second, we provide a graph construction combining multiple base CFI graphs in Section 6.2. Third, we define a logic quantifying over certain individualizations in Section 6.3 and prove that it cannot distinguish CFI graphs from the prior graph construction. Last, we make the argument sketched above formal in Section 6.4 and show that these CFI graphs require nested WSC-fixed-point operators.

6.1. Nested WSC-Fixed-Point and Interpretation Operators. We next consider IFPC+WSC+I-formulas with restricted nesting depth of WSC-fixed-point and interpretation operators. Let $\text{IFPC} \subseteq L \subseteq \text{IFPC}+\text{WSC}+\text{I}$ be a subset of IFPC+WSC+I. We write $\text{WSC}(L)$ for the set of formulas obtained from $L$ using IFPC-formula-formation rules and WSC-fixed-point operators, for which the step, choice, witnessing, and output formulas are $L$-formulas. Likewise, we define $I(L)$: One can use an additional interpretation operator $I(\Theta, \Psi)$, where the interpretation $\Theta$ is an $L$-interpretation and $\Psi$ is an $L$-formula. Note
that $L \subseteq \text{WSC}(L)$, $L \subseteq \text{I}(L)$, and that $\text{I}(\text{IFPC}) = \text{IFPC}$ because IFPC is closed under IFPC-interpretations.

We abbreviate $\text{WSCI}(L) := \text{WSC}(\text{I}(L))$ and $\text{WSCI}^{k+1}(L) := \text{WSC}(\text{WSCI}^k(L))$. Our goal is to provide a class of CFI graphs proving $\text{WSCI}(\text{IFPC}) < \text{WSCI}^2(\text{IFPC})$. Recall the construction in Lemmas 5.6 and 5.7:

**Corollary 6.1.** Let $K$ be a class of base graphs.

1. If $L$ distinguishes 2-orbits of $K$, then $\text{CFI}(K)$ is ready for individualization in $\text{I}(L)$.
2. If $\text{CFI}(K)$ is ready for individualization in $L$, then $\text{WSC}(L)$ defines a canonization for $\text{CFI}(K)$.
3. If $L$ distinguishes 2-orbits of $K$, then $\text{WSCI}(L)$ defines a canonization for $\text{CFI}(K)$.

**Proof.** The first claim follows from the proof of Lemma 5.7. The second claim follows from the proof of Lemma 5.6 and the fact that implementing Gurevich’s canonization algorithm requires one WSC-fixed-point operator [LS22]. Combining the first and second claim yields the last one.

---

**6.2. Color Class Joins and CFI Graphs.** We define a composition operation of graphs: Let $G_1, \ldots, G_\ell$ be connected colored graphs, such that all $G_i$ have the same number of color classes $c$. The color class join $J_{cc}(G_1, \ldots, G_\ell)$ is defined as follows: Start with the disjoint union of the $G_i$ and add $c$ additional vertices $u_1, \ldots, u_c$. Then add for every $i \in [c]$ edges between $u_i$ and every vertex $v$ in the $i$-th color class of every $G_j$ (cf. Figure 4). The resulting colored graph $J_{cc}(G_1, \ldots, G_\ell)$ has $2c$ color classes: One color class for each $u_i$ and the union of the color classes of the $G_j$. We call the $G_j$ the parts of $J_{cc}(G_1, \ldots, G_\ell)$. The vertices $u_i$ are called the join vertices and the other ones the part vertices. The part of a part vertex $v$ is the graph $G_j$ containing $v$. The color class join has two crucial properties: First, defining orbits of $J_{cc}(G_1, \ldots, G_\ell)$ is at least as hard as defining isomorphism of the $G_j$.

**Lemma 6.2.** If two part vertices $v$ and $v'$ are in the same orbit of $J_{cc}(G_1, \ldots, G_\ell)$, then the part of $v$ is isomorphic to the one of $v'$. 

---

**Figure 4.** Visualization of color graphs joins: For three graphs $G_1$, $G_2$, and $G_3$, each with three color classes, the figure shows the color graph join $J_{cc}(G_1, G_2, G_3)$. Edges inside the graphs $G_1$ to $G_3$ are not drawn. For each color class, one new vertex is added and connected to all existing vertices of that color class. The new receives a new unique color.
Proof. Because every \( G_i \) is connected and the part and join vertices have different colors, an automorphism of \( J_{cc}(G_1, \ldots, G_\ell) \) mapping \( v \) to \( v' \) has to map the part of \( v \) to the part of \( v' \). In particular, these parts are isomorphic.

Second, the automorphism structure of a part \( G_j \) is independent of individualizing vertices in other parts (or join vertices).

Lemma 6.3. Let \( \bar{w} \) be a tuple of vertices of \( J_{cc}(G_1, \ldots, G_\ell) \), let \( j \in [\ell] \), and let \( v \) and \( v' \) be vertices of \( G_j \). If there is no \( i \in [\bar{w}] \) such that \( G_j \) is the part of \( w_i \) and if \( v \) and \( v' \) are in the same orbit of \( G_j \), then \( v \) and \( v' \) are in the same orbit of \( (J_{cc}(G_1, \ldots, G_\ell), \bar{w}) \).

Proof. Because \( v \) and \( v' \) are in the same orbit of \( G_j \), there is a \( \varphi \in \text{Aut}(G_j) \) such that \( \varphi(v) = v' \). We extend \( \varphi \) to \( J_{cc}(G_1, \ldots, G_\ell) \) by the identity on all join vertices and all parts apart from \( G_j \). Because \( \varphi \) respects the colors classes of \( G_j \) and the join vertex \( u_i \) is adjacent to all vertices in the \( i \)-th color class of \( G_j \) for every \( i \in [\ell] \), adjacency of part vertices of \( G_j \) and join vertices is preserved by \( \varphi \). Thus, \( \varphi \in \text{Aut}((J_{cc}(G_1, \ldots, G_\ell), \bar{w})) \) because the part of \( w_i \) is not \( G_j \) for every \( i \in [\bar{w}] \) and \( \varphi \) is the identity on all other parts. \( \square \)

Color Class Joins of CFI Graphs. Now we apply color class joins to CFI graphs: For connected colored graphs \( G, H, \) and \( K \) with the same number of color classes, we define

\[
J^k_{cc}(G, H, K) := J_{cc}(G, \ldots, G, H, \ldots, H, K, \ldots, K).
\]

Let \( \mathcal{K} \) be a class of colored base graphs. For \( G \in \mathcal{K} \) and \( g \in \mathbb{F}_2 \), we define

\[
\text{CFI}^k(G, g) := J^k_{cc}(\text{CFI}(G, 0), \text{CFI}(G, g), \text{CFI}(G, 1)),
\]

\[
\text{CFI}^k(\mathcal{K}) := \left\{ \text{CFI}^k(G, g) \mid G \in \mathcal{K}, g \in \mathbb{F}_2 \right\},
\]

\[
\text{CFI}^\omega(\mathcal{K}) := \bigcup_{k \in \mathbb{N}} \text{CFI}^k(\mathcal{K}).
\]

Lemma 6.4. Let \( \text{IFPC} \subseteq L \subseteq \text{IFPC}+\text{WSC}+\text{I} \) be a subset of \( \text{IFPC}+\text{WSC}+\text{I} \) closed under \( \text{IFPC} \)-formula-formation rules. If \( L \) canonizes \( \text{CFI}(\mathcal{K}) \), then \( L \) canonizes \( \text{CFI}^\omega(\mathcal{K}) \).

Proof. First, we can easily distinguish join vertices from part vertices in IFPC for graphs in \( \text{CFI}^\omega(\mathcal{K}) \) because the join vertices are in singleton color classes and the part vertices are not. So for a given part vertex \( u \) of \( G \), we can define the set of all part vertices contained in the same part as \( u \) (namely the ones reachable without using a join vertex), that is, we can define the CFI graph in \( \text{CFI}(\mathcal{K}) \) containing \( u \).

Let \( \Theta_{\text{can}} \) be an \( L \)-canonization of \( \text{CFI}(\mathcal{K}) \) (that is, by definition, a canonization of \( \text{CFI}^\omega(\mathcal{K}) \)). Then we obtain an \( L \)-interpretation \( \Theta_{\text{part-can}}(x) \) that given a \( \text{CFI}^\omega(\mathcal{K}) \)-graph canonizes the part of \( x \). It essentially is \( \Theta_{\text{can}} \) but only considers the (definable) set of part vertices in the same part as \( x \). So every choice-set in the evaluation of \( \Theta_{\text{part-can}}(x) \) will be a set of part vertices in the same part as \( x \). Thus, a choice-set is an orbit if and only if the corresponding choice-set in the evaluation of \( \Theta_{\text{can}} \) is an orbit. The witnessing automorphisms are obtained by extending the witnessing automorphisms defined in \( \Theta_{\text{can}} \) with the identity on all vertices not in the part of \( x \). In that way, exactly the same choices are successfully witnessed by \( \Theta_{\text{part-can}} \) as by \( \Theta_{\text{can}} \). Note that \( \Theta_{\text{part-can}} \) is an \( L \)-interpretation because \( L \) is closed under \( \text{IFPC} \)-formula-formation rules.
We use $\Theta_{\text{part-can}}(x)$ to define the canon of every part containing an $\preceq$-individualized vertex. These canons can be ordered according to the order of the individualized vertices. For the remaining parts not containing individualized vertices, we use $\Theta_{\text{part-can}}(x)$ to determine how many of them are even respectively odd CFI graphs. We obtain a canon for the even and odd graph (if they occur) and define as many copies as needed (using numeric variables). The copies can be ordered. We finally take the disjoint union of all these canons, and lastly add the join vertices, which all is IFPC-definable.

**CFI Graphs of Color Class Joins.** Now, we use color class joins as base graphs. We introduce terminology for CFI graphs over color class joins. Let $G_1, \ldots, G_\ell$ be colored base graphs with the same number of colors and let $h \in \mathbb{F}_2$. We transfer the notion of part and join vertices from $H := J_{cc}(G_1, \ldots, G_\ell)$ to $\mathcal{A} := \text{CFI}(H, h)$. The $G_i$-part of $\mathcal{A}$ is the set of vertices originating from a vertex or edge of $G_i$ in $H$. These vertices are called *part vertices* of $G_i$. A vertex is just a part vertex, if it is a part vertex of some $G_i$. The remaining vertices are the *join vertices*.

We consider a special class of individualizations of $\mathcal{A}$. Let $\bar{u} \in A^\ell$. A part of $\mathcal{A}$ is $\bar{u}$-pebbled if $u_i$ is a part vertex of that part for some $i \in [k]$. Otherwise, the part is $\bar{u}$-unpebbled. The set of $\bar{u}$-pebbled-part vertices $V_\bar{u}(\mathcal{A})$ is the set of all join vertices and all part vertices of all $\bar{u}$-pebbled parts. The set of $\bar{u}$-pebbled-part individualizations $P_\bar{u}(\mathcal{A})$ is the set of all individualizations of $V_\bar{u}(\mathcal{A})$.

**Definition 6.5** (Unpebbled-Part-Distinguishing). For a tuple $\bar{u} \in A^\ell$, a relation $R \subseteq A^k$ is $\bar{u}$-unpebbled-part-distinguishing if there are $m \in [k]$ and $i \neq j \in [\ell]$ such that

1. both the $G_i$-part and the $G_j$-part of $\mathcal{A}$ are $\bar{u}$-unpebbled,
2. there is a $\bar{v} \in R$ such that $v_m$ is a part vertex of $G_i$, and
3. for every $\bar{w} \in R$, $w_m$ is not a part vertex of $G_j$.

**Lemma 6.6.** Let $\bar{u} \in A^\ell$ and $R \subseteq A^k$. Assume that $G_i \nsubseteq G_j$, that $G_i$ and $G_j$ are $\bar{u}$-unpebbled, and some $\bar{v} \in R$ contains a vertex of a $\bar{u}$-unpebbled part. If $R$ is an orbit of $(\mathcal{A}, \bar{u})$, then $R$ is $\bar{u}$-unpebbled-part-distinguishing.

**Proof.** Let the part of $v_m$ be $\bar{u}$-unpebbled and let this part be the $G_n$-part. Assume that $R$ is an orbit of $(\mathcal{A}, \bar{u})$. If the part of $w_m$ is neither $G_i$ nor $G_j$ for all $\bar{w} \in R$, then $n \notin \{i, j\}$. Thus, $R$ is $\bar{u}$-unpebbled-part-distinguishing because the $G_i$-part and the $G_n$-part are $\bar{u}$-unpebbled, $v_m$ is a part vertex of $G_n$, and for every $\bar{w} \in R$, $w_m$ is not a part vertex of $G_i$.

Otherwise, there is a $\bar{w} \in R$ such that the part of $w_m$ is w.l.o.g. $G_i$. Then no automorphism can map $w_m$ to a vertex whose part is $G_j$ because $G_i \nsubseteq G_j$ (Lemma 6.2). Thus, $w_m$ is not a part vertex of $G_j$ for every $\bar{w} \in R$ because $R$ is an orbit. It follows that $R$ is $\bar{u}$-unpebbled-part-distinguishing.

### 6.3. Quantifying over Pebbled-Part Individualizations

We now define an extension of $\mathcal{C}_k$ which allows for quantifying over pebbled-part individualizations. Our (unnatural) extension $\mathcal{P}_k$ can only be evaluated on CFI graphs over color class joins and will be a tool to show $\text{WSC(IFPC)}$-undefinability. The benefit of this logic is that we can characterize it via a pebble game. Whenever $\Phi(\bar{x})$ is a $\mathcal{C}_k[\tau, \preceq_P]$-formula (for a binary relation symbol $\preceq_P \notin \tau$), then

$$\exists^P \preceq_P. \Phi(\bar{x})$$
is a $P_k[\tau]$-formula. $P_k[\tau]$-formulas can be combined as usual in $C_k$ with Boolean operators and counting quantifiers. Note that $\exists^F$-quantifiers cannot be nested. Let $G_1, \ldots, G_t$ be colored base graphs with the same number of colors, $g \in F_2$, and $A = CFI(J_{cc}(G_1, \ldots, G_n), g)$. The $\exists^F$-quantifier has the following semantics:
\[
(\exists^F \preceq P. \Phi)^A := \left\{ \bar{u} \in A^{[k]} \mid \bar{u} \in \Phi^{A, \preceq P} \text{ for some } \preceq P \in P_k(A) \right\}.
\]
That is, the $\exists^F$-quantifier quantifies over a pebbled-part individualization for the free variables of $\Phi$. Note that the quantifier does not bind first-order variables.

We now characterize $P_k$ by an Ehrenfeucht-Fraïssé-like pebble game, which is an extension of the bijective $k$-pebble game. The $P_k$-game is a game between two players called Spoiler and Duplicator. There are two types of pebbles. First, there a $k$ pebble pairs $(p_i, q_i)$ for every $i \in [k]$. Second, there are pebble pairs $(a_i, b_i)$ for every $i \in \mathbb{N}$. The game is played on CFI graphs over color class joins $A$ and $B$ satisfying $|A| = |B|$ (otherwise Spoiler wins immediately). A position in the game is a tuple $(A, \preceq P, \bar{u}, B, \preceq P, \bar{v})$, where $\bar{u} \in A^{\leq k}$ and $\bar{v} \in B^{\leq k}$ are of the same length, and $\preceq P$ and $\preceq P$ individualize vertices (possibly none) originating from up to $k$ part or join vertices in $A$ and $B$ respectively. A pebble $p_i$ is placed on $u_i$ and the corresponding pebble $q_i$ is placed on $v_i$ for some $j \in [k]$ (the exact pebble pair $(p_j, q_j)$ placed on the $i$-th entries will not matter). The pebble $a_i$ is placed on the $i$-th vertex individualized by $\preceq P$ and $b_i$ on the $i$-th vertex individualized by $\preceq P$. The initial position is $(A, \emptyset, (); B, \emptyset, ())$, where $(\cdot)$ denotes the empty tuple.

Spoiler can perform two kinds of moves. A regular move proceeds as in the bijective $k$-pebble game: Spoiler picks up a pair of pebbles $(p_i, q_i)$. Then Duplicator provides a bijection $\lambda: A \to B$. Spoiler places $p_i$ on $u \in A$ and $q_i$ on $\lambda(u)$.

A $P$-move proceeds as follows and can only be performed once by Spoiler (that is, if $\preceq P = \preceq P = \emptyset$): Spoiler places $a_i$-pebbles (respectively $b_i$-pebbles) exactly on the $\bar{u}$-pebbled-part vertices of $A$ (respectively on the $\bar{v}$-pebbled-part vertices of $B$). Duplicator responds by placing the $b_i$-pebbles (respectively the $a_i$-pebbles) on exactly the $\bar{v}$-pebbled-part vertices of $B$ (respectively the $\bar{u}$-pebbled-part vertices of $A$).

If after a round there is no pebble-respecting local isomorphism of the pebble-induced substructures of $A$ and $B$, then Spoiler wins. Duplicator wins if Spoiler never wins.

**Lemma 6.7.** For every $k \geq 3$, Spoiler has a winning strategy in the $P_k$-game at position $(A, \emptyset, \bar{u}; B, \emptyset, \bar{v})$ if and only if the logic $P_k$ distinguishes $(A, \bar{u})$ and $(B, \bar{v})$.

**Proof.** Let $k \geq 3$. We first consider positions $(A, \preceq P, \bar{u}; B, \preceq P, \bar{v})$ where Spoiler already has performed the $P$-move, i.e., $\preceq P, \preceq P \neq \emptyset$. Then the remaining game is essentially just the bijective $k$-pebble game, where for each $i \in \mathbb{N}$ the vertices pebbled by $a_i$ and $b_i$ are put in a unique singleton relation. Spoiler has a winning strategy at the position $(A, \preceq P, \bar{u}; B, \preceq P, \bar{v})$ if and only if $C_k$ distinguishes $(A, \preceq P, \bar{u})$ and $(B, \preceq P, \bar{v})$ because, for $k \geq 3$, we can define the $i$-th vertex individualized by $\preceq P$ or $\preceq P$.

We now prove by induction on the number of rounds that if Spoiler has a winning strategy in the $P_k$-game at position $(A, \emptyset, \bar{u}; B, \emptyset, \bar{v})$, then $P_k$ distinguishes $(A, \bar{u})$ and $(B, \bar{v})$. If Spoiler wins without performing a $P$-move, then Spoiler wins the bijective pebble game and $C_k$ and thus also $P_k$ distinguishes $(A, \bar{u})$ and $(B, \bar{v})$. So assume Spoiler eventually performs a $P$-move.

Assume that Spoiler performs a regular move and picks up the $i$-th pebble pair $(p_i, q_i)$. The argument is essentially the same as for the bijective $k$-pebble game: For every bijection
\(\lambda : A \rightarrow B\), there is a \(w \in A\) such that Spoiler wins the \(P_k\)-game when placing \(p_i\) on \(w\) and \(q_i\) on \(\lambda(w)\). For all these positions, there is a \(P_k\)-formula distinguishing them by induction hypothesis. So some Boolean combination of these distinguishing formulas is satisfied by a different number of vertices in \((\mathfrak{A}, \bar{u})\) and \((\mathfrak{B}, \bar{v})\) and we can distinguish them using a counting quantifier.

Now assume that Spoiler performs a \(P\)-move: By symmetry, assume Spoiler places \(a_i\)-pebbles on all \(\bar{u}\)-pebbled-part vertices of \(\mathfrak{A}\) (inducing the individualization \(\leq_p^\mathfrak{A}\)). Then, for every placement of the \(b_j\)-pebbles by Duplicator on the \(\bar{v}\)-pebbled-part vertices of \(\mathfrak{B}\) (inducing \(\leq_p^\mathfrak{B}\)), Spoiler has a winning strategy in the bijective \(k\)-pebble game at position \((\mathfrak{A}, \leq_p^\mathfrak{A}, \bar{u}; \mathfrak{B}, \leq_p^\mathfrak{B}, \bar{v})\) (no \(P\)-move is allowed anymore). Then, as argued before, there is a \(C_i\)-formula \(\Phi\) distinguishing \((\mathfrak{A}, \leq_p^\mathfrak{A}, \bar{u})\) and \((\mathfrak{B}, \leq_p^\mathfrak{B}, \bar{v})\). So the \(P_k\)-formula \(\exists \leq_P^p \cdot \Phi\) distinguishes \((\mathfrak{A}, \bar{u})\) and \((\mathfrak{B}, \bar{v})\).

To show the other direction, we prove by induction on the quantifier depth that if a \(P_k\)-formula \(\Phi\) distinguishes \((\mathfrak{A}, \bar{u})\) and \((\mathfrak{B}, \bar{v})\), then Spoiler has a winning strategy in the \(P_k\)-game at position \((\mathfrak{A}, \emptyset, \bar{u}; \mathfrak{B}, \emptyset, \bar{v})\). If \(\Phi\) is actually a \(C_i\)-formula, then Spoiler wins the bijective \(k\)-pebble game so in particular the \(P_k\)-game. If \(\Phi = \Psi \land \Psi'\), \(\Psi \lor \Psi'\), or \(\neg \Psi\), then one of \(\Psi\) and \(\Psi'\) distinguishes \((\mathfrak{A}, \bar{u})\) and \((\mathfrak{B}, \bar{v})\). If \(\Phi\) is a counting quantifier \(\exists \forall x. \Psi\), then Spoiler performs a regular move. Because \(\Phi\) has at most \(k - 1\) free variables, Spoiler can pick up a pair of pebbles \((p_i, q_i)\). Whatever bijection \(\lambda\) Duplicator chooses, there is a vertex \(w\) such that, by symmetry, \(w\) satisfies \(\Psi\) in \(\mathfrak{A}\) but \(\lambda(w)\) does not satisfy \(\Psi\) in \(\mathfrak{B}\). That is, Spoiler places \(p_i\) on \(w\) and \(q_i\) on \(\lambda(w)\) and wins by the induction hypothesis.

To the end, assume that \(\Phi\) is the \(P_k\)-formula \(\exists \leq_P^p \cdot \Psi\). By symmetry, we assume that \((\mathfrak{A}, \bar{u})\) satisfies \(\Phi\) but \((\mathfrak{B}, \bar{v})\) does not. So there is a \(\leq_p^\mathfrak{A} \in P_b(\mathfrak{A})\) satisfying \(\bar{u} \in \Psi(\mathfrak{A}, \leq_p^\mathfrak{A})\) such that, for every \(\leq_p^\mathfrak{B} \in P_b(\mathfrak{B})\), it holds that \(\bar{v} \notin \Psi(\mathfrak{B}, \leq_p^\mathfrak{B})\). Because \(\Psi\) is a \(C_i\) formula, Spoiler has a winning strategy in the \(k\)-bijective pebble game at position \((\mathfrak{A}, \leq_p^\mathfrak{A}, \bar{u}; \mathfrak{B}, \leq_p^\mathfrak{B}, \bar{v})\). By performing a \(P\)-move and placing the \(a_i\) pebbles according to \(\leq_p^\mathfrak{A}\), Spoiler obtains a winning strategy in the \(P_k\) game at position \((\mathfrak{A}, \emptyset, \bar{u}; \mathfrak{B}, \emptyset, \bar{v})\).

Note that the former lemma only holds for \(k \geq 3\) because with fewer variables we cannot define the \(i\)-th individualized vertex in \(C_k\) and thus cannot check local isomorphisms. Extending the logic to make the lemma hold for every \(k\) only complicates matters and is not needed in the following.

**Lemma 6.8.** Let \(G_1, \ldots, G_{k+1}\) be colored base graphs, each with \(c > k\) color classes, such that \(\text{CFI}(G_i, 0) \preceq_C^k \text{CFI}(G_i, 1)\) for every \(i \in [k+1]\). Then Duplicator has a winning strategy in the \(P_k\)-game played on \(\text{CFI}(J_{cc}(G_1, \ldots, G_{k+1}), 0)\) and \(\text{CFI}(J_{cc}(G_1, \ldots, G_{k+1}), 1)\).

**Proof.** Let \(\mathfrak{A} = \text{CFI}(J_{cc}(G_1, \ldots, G_{k+1}), 0)\) and \(\mathfrak{B} = \text{CFI}(J_{cc}(G_1, \ldots, G_{k+1}), 1)\). In this proof, we call the \(G_j\)-part in \(\mathfrak{A}\) and \(\mathfrak{B}\) just the \(j\)-th part. Let \(V_j \subseteq A = B\) be the set of join vertices and \(V_j \subseteq A = B\) be the set of part vertices of the \(j\)-th part for every \(j \in [k+1]\) (recall that CFI graphs are defined over the same vertex set).

We show that Duplicator is able to maintain the following invariant. At every position \((\mathfrak{A}, \leq_p^\mathfrak{A}, \bar{u}; \mathfrak{B}, \leq_p^\mathfrak{B}, \bar{v})\) during the game there is an isomorphism \(\psi : (\mathfrak{B}, \bar{v}, \leq_p^\mathfrak{B}) \rightarrow (\mathfrak{B}', \bar{v}, \leq_p^\mathfrak{B})\) for some \(\mathfrak{B}'\), that moves the twisted edge, so \(B' = B\), and there is a \(j \in [k+1]\) satisfying the following:

1. If Spoiler has not performed the \(P\)-move (i.e., the \(a_i\) and \(b_i\) pebbles are not placed), then the part of every \(u_i\) and every \(v_i\) is not the \(j\)-th one.
(2) If Spoiler has performed the P-move, then no vertex of the \( j \)-th part is individualized by \( \leq_P^{\Lambda} \) or \( \leq_P^{B} \).

(3) There is an isomorphism \( \varphi: (\Lambda, \leq^{\Lambda}, \bar{u})[A \setminus V_j] \to (\psi(\mathcal{B}), \leq^B, \bar{v})[B \setminus V_j] \) respecting the parts, that is, \( \varphi \) maps the \( i \)-th part of \( \Lambda \) to the \( i \)-th part of \( \psi(\mathcal{B}) \).

(4) \( (\Lambda, \leq, \bar{u})[V_j \cup V_j] \simeq^A \psi(\mathcal{B}), \psi(\leq), \bar{v})[V_j \cup V_j] \) for some individualization \( \leq \) of \( V_j \).

Note that Property 4 is satisfied either by none or all such individualizations. Clearly, the invariant is satisfied initially. Assume that it is Spoiler’s turn and Spoiler performs a regular move. Spoiler picks up a pebble pair \( (p, q) \) in \( V_j \). Let this be the \( \ell \)-th part along that path. Thus, we can extend the restriction \( \varphi \) to \( V_j \) as follows:

\[
\lambda(w) := \begin{cases} 
\varphi(w) & \text{if } w \notin V_j, \\
\lambda'(w) & \text{otherwise.}
\end{cases}
\]

Because in this game the individualization \( \leq \) is used for \( \Lambda \) and the image \( \psi(\leq) \) is used in for \( \mathcal{B} \), the bijection \( \lambda' \) necessarily has to map the \( i \)-th \( \leq \)-individualized vertex to the \( i \)-th \( \psi(\leq) \)-individualized vertex. That is, \( \lambda \) and \( \lambda' \) necessarily agree on the join vertices, that is, \( \lambda(w) = \lambda'(w) \) for all \( w \in V_j \). Spoiler places \( p_i \) on \( w \) and \( q_i \) on \( \lambda(w) \). The pebbles still induce a local isomorphism (because \( \varphi \) is an isomorphism and \( \lambda \) is given by a winning strategy). Properties 1 and 2 are obviously satisfied. If \( w \notin V_j \), then \( \varphi(w) = \lambda(w) \) and the isomorphism \( \varphi \) still satisfies Property 3. If in particular \( w \notin V_j \), then Property 4 is satisfied because the new pebble is not placed on \( (\Lambda, \leq, \bar{u})[V_j \cup V_j] \) respectively on \( \psi(\mathcal{B}), \psi(\leq), \bar{v})[V_j \cup V_j] \). If \( w \in V_j \), then the pebbles are placed according to a winning strategy of Duplicator and thus Property 4 is satisfied, too. If otherwise \( w \in V_j \), then \( w \) is not in the domain of \( \varphi \) and thus \( \varphi \) satisfies Property 3. Property 4 is satisfied because \( \lambda(w) = \lambda'(w) \) and \( \lambda' \) was obtained by a winning strategy of Duplicator.

If Spoiler has not performed the P-move, then Duplicator extends \( \varphi \) as follows. There is another \( \bar{u} \)-unpebbled part different from the \( j \)-th one because there are at most \( k - 1 \) pebbles placed (one pebble pair is picked up). Let this be the \( \ell \)-th part for some \( \ell \neq j \) and let \( \{u, v\} \) be the twisted edge between \( \Lambda \) and \( \psi(\mathcal{B}) \), which is contained in the \( j \)-th part (by Property 3). There is a path from one of \( u \) or \( v \) into the \( \ell \)-th part in \( J_{\alpha}(G_1, \ldots, G_{k+1}) \) only using vertices of the \( j \)-th and \( \ell \)-th part and one join vertex \( \mathfrak{w} \) such that it does not use the origin of the pebbled vertices: Both \( G_j \) and \( G_{\ell} \) are connected, do not contain any pebbles, and there are \( c > k \) color classes, so one join vertex is not pebbled. Hence, there is a path-isomorphism \( \psi': (\mathcal{B}', \bar{v}) \to (\mathcal{B}'', \bar{v}) \) moving the twist from the \( j \)-th into the \( \ell \)-th part along that path. Thus, we can extend the restriction

\[
\psi'|_{A \setminus V_{j} \setminus V_{\ell}} \circ \varphi|_{A \setminus V_{j} \setminus V_{\ell}}: (\Lambda, \bar{u})[A \setminus V_{j} \setminus V_{\ell}] \to (\psi'(\psi(\mathcal{B})), \bar{v})[B \setminus V_{j} \setminus V_{\ell}]
\]

to \( V_j \) because the twist is now in the \( \ell \)-th part. That is, we obtain an isomorphism \( \varphi': (\Lambda, \bar{u})[A \setminus V_{j}] \to (\psi' \circ \psi(\mathcal{B}), \bar{v})[B \setminus V_{j}] \) which agrees with \( \varphi \) on \( A \setminus V_{j} \setminus V_{\ell} \) apart from gadget vertices originating from \( \mathfrak{w} \) and the edge vertices originating from the edge incident to \( \mathfrak{w} \) into the \( j \)-th and \( \ell \)-th part. These are the only vertices in \( A \setminus V_{j} \setminus V_{\ell} \) for which the
isomorphism $\psi'$ is not the identity. Duplicator extends $\varphi$ on $V_j$ using $\varphi'$ to the bijection $\lambda$.

$$\lambda(w) := \begin{cases} 
\varphi(w) & \text{if } w \notin V_j, \\
\varphi'(w) & \text{otherwise.}
\end{cases}$$

Spoiler places $p_i$ on $w$ and $q_i$ on $\lambda(w)$. If $w \notin V_j$ (and so $\lambda(w) = \varphi(w) \notin V_j$), then Properties 1 to 4 are clearly satisfied (now for $\ell$ instead for $j$) because $\varphi$ is an isomorphism. For the same reason, the pebbles induce a local isomorphism.

So assume $w \in V_j$. By Property 1, the first pebble is placed on the $j$-th part and the $\ell$-th part does not contain a pebble. Then the restriction of $\lambda$ to $A \setminus V_\ell$ can be turned into an isomorphism $(A, \emptyset, \tilde{u}w)[A \setminus V_\ell] \to (B'', \emptyset, \tilde{p}_\lambda(w))[B \setminus V_\ell]$ by applying a local automorphism of the gadget of $w$ (which is not pebbled) that moves the twist from the $j$-th into the $\ell$-th part according to $\psi'$. In particular, the pebbles induce a local isomorphism and Property 3 holds. Property 4 is satisfied because the $\ell$-th part does not contain a pebble: If $\text{CFI}(G_\ell, 0) \simeq^k \text{CFI}(G_\ell, 1)$, then $(A, \preceq_J)[V_J \cup V_\ell] \simeq^k (B'', \psi(\preceq_J))[V_J \cup V_\ell]$, too, because $(A, \preceq_J)[V_J \cup V_\ell]$ just extends $\text{CFI}(G_\ell, 0)$ by gadgets for the join vertices, which are all fixed by $\preceq_J$ (and likewise for $(B'', \psi(\preceq_J))[V_J \cup V_2]$ and $\text{CFI}(G_\ell, 1)$).

Finally, let Spoiler perform the $P$-move. Assume by symmetry that Spoiler places the $a_i$ pebbles on the $\tilde{u}$-pebbled parts. Duplicator places the pebble $b_i$ on $\varphi(a_i)$ for all $i$. Because the pebbles are placed according to the isomorphism $\varphi$, there is a pebble-respecting local isomorphism. Properties 1 and 2 are clearly satisfied. Property 3 is satisfied because no pebble is placed in the $j$-th part (similar to the $\ell$-th part in the former case).

6.4. **Nesting Operators to Define the CFI Query is Necessary.** We use CFI graphs over color class joins of CFI graphs to construct a new class of base graphs, for which WSCI$^2$(IFPC) defines the CFI query but WSCI(IFPC) does not. Fix a class of totally ordered and 3-regular base graphs $K := \{G_i \mid i \in \mathbb{N}\}$ such that $G_i$ has treewidth at least $i$ for every $i \in \mathbb{N}$. Such a class exists because we can obtain from some graph $G'_i$ of treewidth $i$ (e.g., a clique of size $i + 1$) a 3-regular graph of treewidth at least $i$ as follows: If $G'_i$ has a vertex $u$ of degree greater than 3, then we obtain a new graph $G''_i$ by splitting $u$ off into two vertices (onto which we equally distribute the edges incident to $u$) and connecting them via an edge. Contracting this edge yields back $G'_i$. Thus, $G'_i$ is a minor of $G''_i$ and thus the treewidth of $G''_i$ is at least the treewidth of $G'_i$. We repeat this procedure until every vertex has degree 3.

**Lemma 6.9.** $\text{CFI}(\text{CFI}(G_k, g), 0) \simeq^k \text{CFI}(\text{CFI}(G_k, g), 1)$ for every $k \in \mathbb{N}$ and $g \in F_2$.

**Proof.** Let $k \in \mathbb{N}$ and $g \in F_2$. The graph $G_k$ has treewidth at least $k$. The CFI construction does not decrease the treewidth because $G_k$ is a minor of $\text{CFI}(G_k, g)$ (cf. [DR07]). Hence, $\text{CFI}(G_k, g)$ has treewidth at least $k$ and $\text{CFI}(\text{CFI}(G_k, g), 0) \simeq^k \text{CFI}(\text{CFI}(G_k, g), 1)$ by Lemma 4.1. \hfill \square

**Lemma 6.10.** WSCI$^2$(IFPC) defines the CFI query for $\text{CFI}(\text{CFI}^\omega(K))$.

**Proof.** IFPC distinguishes 2-orbits of $K$ because $K$-graphs are totally ordered. By Corollary 6.1, WSCI(IFPC) canonizes CFI($K$). From Lemma 6.4 it follows that WSCI(IFPC) canonizes $\text{CFI}^\omega(G)$ and so also distinguishes 2-orbits of $\text{CFI}^\omega(G)$. Again due to Corollary 6.1, WSCI$^2$(IFPC) canonizes CFI($\text{CFI}^\omega(K)$). In particular, WSCI$^2$(IFPC) defines the CFI query for $\text{CFI}(\text{CFI}^\omega(K))$. \hfill \square
To show that WSCI(IFPC) does not define the CFI query for CFI(CFI(K)), we will use the following idea: Suppose that WSCI(IFPC)-formula Φ defines the CFI query for CFI(CFI(K)) and we evaluate Φ on CFI(CFI(G)) for some ℓ > |p|. If Φ always defines choice-sets containing only tuples of vertices in parameter-pebbled parts, then the twist can be moved in the parameter-unpebbled parts. Because all choices are made in parameter-pebbled parts, the output formula of Φ, which is an IFPC-formula, essentially has to define the CFI query for CFI(CFI(K)), which is not possible (Lemma 6.9). Otherwise, Φ makes a choice in parameter-unpebbled parts. But for that, Φ has to distinguish CFI(G, 0) from CFI(G, 1) to define orbits of CFI(CFI(G)). So the choice IFPC-formula has to define the CFI query for CFI(K), which is also not possible. Making this idea formal turns out to be tedious.

**Lemma 6.11.** WSCI(IFPC) does not define the CFI query for CFI(CFI(K)).

**Proof.** For a sake of contradiction, suppose that Φ is a WSCI(IFPC)-formula defining the CFI query for CFI(CFI(K)). W.l.o.g., we assume that Φ binds no variable twice. Because I(IFPC) = IFPC, we can assume that Φ is a WSC(IFPC)-formula.

Let Ψ₁(¯x₁), . . . , Ψₘ(¯xₘ) be all WSC-fixed-point operators that are subformulas of Φ. For the moment assume that all free variables ¯xₖ are element variables. Let the number of distinct variables of Φ be k and let ℓ := ℓ(k) ≥ max{k, 3} for some function ℓ(k) to be defined later. We consider the subclass CFI(CFI(CFI(K)) ⊆ CFI(CFI(K)). We partition K as follows: First, let K_Π be the set of all G ∈ K such that, for every g ∈ F₂, there are h ∈ F₂, j ∈ [m], and a |x_j|-tuple u of vertices of CFI(CFI(CFI(G, g), h)) such that

1. all choice-sets during the evaluation of Ψ_j(¯u) on CFI(CFI(G, g), h) are indeed orbits and

2. one of these choice-sets is ¯u-unpebbled-part-distinguishing.

Second, set K_cfi := K \ K_Π. Clearly, at least one of K_Π and K_cfi is infinite, which is a contradiction as shown in the two following claims.

**Claim 6.12.** K_Π is finite.

**Proof.** We claim that there is an IFPC-formula defining the CFI query for CFI(K_Π). First, we show that there is an IFPC-interpretation that, for G ∈ K_Π (and even in K), maps a CFI graph CFI(G, g) and an h ∈ F₂ to the graph (CFI(CFI(G, g), h), ≤) such that ≤ individualizes the vertices of ℓ + 1 many CFI(G, 0)-parts, ℓ + 1 many CFI(G, 1)-parts, and all join vertices of CFI(CFI(G, g), h). The following mappings are definable by IFPC-interpretations:

(a) Map CFI(G, g) to the base graph G (which is an ordered graph).
(b) Map G and g′ ∈ F₂ to (CFI(G, g′), ≤), such that ≤ is a total order on CFI(G, g′). This map is IFPC-definable because G is 3-regular.
(c) Map CFI(G, g), (CFI(G, 0), ≤₀), and (CFI(G, 1), ≤₁) to (CFI(CFI(G, g), ≤′), where ≤₀ and ≤₁ are total orders and ≤′ individualizes all vertices of the ℓ + 1 parts of CFI(G, 0) and CFI(G, 1) as well as the join vertices.
(d) Finally, map (CFI(CFI(G, g), ≤′) and h ∈ F₂ to (CFI(CFI(G, g), h), ≤) such that ≤ individualizes the required vertices. This map is IFPC-definable since CFI(CFI(G, g)) is of bounded degree: The graph G has color class size 1 and is 3-regular, so CFI(G, h) has color class size 4 and degree at most 3. That is, CFI(G, h) has color class size 4ℓ + 4 and degree at most 4ℓ + 4 (the join vertices), which is a constant.
By composing these IFPC-interpretations, we obtain the required one. We now show that we can simulate each $\Psi_j$ on $(\text{CFI}(\text{CFI}^{l+1}(G,g), h), \leq)$ in IFPC such that we determine the parity of $\text{CFI}(G,g)$.

For every WSC-fixed-point operator $\Psi_j(\bar{x}_j)$, we consider every $h \in \mathbb{F}_2$ and every possible $|\bar{x}_j|$-tuple $\bar{u}$ of $\leq$-individualized vertices for the parameters $\bar{x}_j$. We simulate the evaluation of the WSC-fixed-point operator $\Psi_j$ in IFPC as follows: Because $\Phi$ is a WSC(IFPC) formula, the step, choice, witnessing, and output formula of $\Psi_j$ are IFPC-formulas. We evaluate the choice formula and check whether all tuples in the defined relation are composed of $\bar{u}$-pebbled-part vertices. If that is the case, we resolve the choice deterministically using the lexicographical order of $\leq$ on the tuples (recall that $\leq$ individualizes all $\bar{u}$-pebbled-part vertices by construction). Next, we evaluate the step formula. The simulation is continued until there is a choice-set not solely composed of vertices of $\bar{u}$-pebbled-part parts. Because the choice-set is by definition of $\mathcal{K}_\text{orb}$ an orbit, it is $\bar{u}$-unpebbled distinguishing by Lemma 6.6. So the choice-set contains (at some index) vertices of the $\text{CFI}(G,g)$-parts and either of the $\text{CFI}(G,0)$-parts or of $\text{CFI}(G,1)$-parts: Because isomorphic parts are in the same orbit, either vertices of all isomorphic parts or none of them occur because they cannot be distinguished.

At least one of the $\text{CFI}(G,0)$-parts and the $\text{CFI}(G,1)$-parts each is not $\bar{u}$-pebbled because $|\bar{u}| \leq k < \ell + 1$. So one of these is isomorphic to the $\text{CFI}(G,g)$-parts. The graphs $\text{CFI}(G,0)$ and $\text{CFI}(G,1)$ were added by the interpretation, so we can actually remember their parity and thus defined the parity of $\text{CFI}(G,g)$.

It remains to prove that such a combination of $j$, $h$, and $\bar{u}$ always exists. By construction of $\mathcal{K}_\text{orb}$, they exist when testing all possible $|\bar{x}_j|$-tuples for $\bar{u}$ (and not only those of $\leq$-individualized vertices). But, because $k < \ell + 1$, there is always an automorphism $\varphi$ (ignoring the individualization) mapping $\bar{u}$ to the vertices of the $\text{CFI}(G,0)$-parts and the $\text{CFI}(G,1)$-parts (because $\text{CFI}(G,g)$ is isomorphic to one of them). Because $\Psi_j$ has no access to the individualization, $\Psi_j$ is satisfied by $\bar{u}$ if and only if $\Psi_j$ is satisfied by $\varphi(\bar{u})$. So indeed IFPC defines the CFI query for $\text{CFI}(\mathcal{K}_\text{orb})$.

Now, for sake of contradiction, assume that $\mathcal{K}_\text{orb}$ is infinite. So for every $k$, there is a $j \geq k$ such that $G_j \in \mathcal{K}_\text{orb}$ and $\text{CFI}(G_j,0) \simeq^2_\ell \text{CFI}(G_j,1)$ by Lemma 4.1. This contradicts that IFPC defines the CFI query for $\text{CFI}(\mathcal{K}_\text{orb})$.

Claim 6.13. $\mathcal{K}_\text{cfi}$ is finite.

Proof. Assume that $\mathcal{K}_\text{cfi}$ is infinite. So there is an $\ell' > \ell$ such that $G = G_{\ell'} \in \mathcal{K}_\text{cfi}$. By definition of $\mathcal{K}_\text{cfi}$, there is a $g \in \mathbb{F}_2$ such that for all $h \in \mathbb{F}_2$, all $j \in [m]$, and all $|\bar{x}_j|$-tuples $\bar{u}$ of $\text{CFI}(\text{CFI}^{l+1}(G,g), h))$

(1) some choice-set during the evaluation of $\Psi_j(\bar{u})$ on $\text{CFI}(\text{CFI}^{l+1}(G,g), h)$ is not an orbit or

(2) all choice-sets are not $\bar{u}$-unpebbled-part-distinguishing.

We claim that there is a $\mathcal{P}_c$-formula equivalent to the CFI-query-defining formula $\Phi$ on $\text{CFI}(\text{CFI}^{l+1}(G,g), 0)$ and $\text{CFI}(\text{CFI}^{l+1}(G,g), 1)$. We first translate every WSC-fixed-point operator $\Psi_i(\bar{x}_i)$ (for $i \in [m]$) into an equivalent IFPC-formula which uses a fresh relation symbol $\leq_P$. The relation $\leq_P$ is intended to be interpreted as an $\bar{u}$-pebbled-part individualization when using $\bar{u}$ for the parameters $\bar{x}_i$. Let $i \in [m]$ be arbitrary. Again, the step, choice, witnessing, and output formulas of $\Psi_i$ are IFPC-formulas because $\Psi_i$ is a WSC(IFPC)-formula. We simulate $\Psi_i$ by an IFPC-formula using the relation $\leq_P$. If all choice-sets during the evaluation for $\bar{u}$ are not $\bar{u}$-unpebbled-part-distinguishing, then all
choice-sets contain solely tuples composed out of the vertices individualized by $\leq P$ (otherwise a choice-set would be $\bar{u}$-unpebbled-part-distinguishing by Lemma 6.6). So if all tuples in a choice-set are composed of the individualized vertices, we can resolve all choice deterministically using the lexicographical order on tuples given by $\leq P$. Otherwise, some choice-set during the evaluation will not be an orbit by definition of $K_{\text{cfi}}$ and we immediately evaluate to false because we make (or will make) a choice out of a non-orbit. If this was never the case, we check in the end whether all choices were indeed witnessed. Let $\Psi_i(\bar{x}_i)$ be an IFPC-formula, which implements exactly this approach to simulate $\Psi_i$. The formula $\Psi_i(\bar{x}_i)$ can be constructed to use not more than $\ell(k)$ distinct variables such that no variable is bound twice.

For every number $n$, every $k$-variable IFPC-formula not binding variables twice can be unwound into a $C_k$-formula equivalent on structures of size up to $n$ (see [Ott97]). So, for $\ell = \ell(k)$ and $n = \lceil \text{CFI}(\text{CFI}^{\ell+1}(G, g), 0) \rceil$, we can unwind $\Psi_i(\bar{x}_i)$ to a $C_{\ell}$-formula $\Psi_i(\bar{x}_i)$. Then the $P_\ell$-formula

$$\Phi_i(\bar{x}_i) := \exists^n P \leq_P. \Psi_i(\bar{x}_i)$$

is equivalent to $\Psi_i$ on $\text{CFI}(\text{CFI}^{\ell+1}(G, g), 0)$ and $\text{CFI}(\text{CFI}^{\ell+1}(G, g), 1)$. To see this, note that $\Psi_i$ evaluates equally for every pebbled-part individualization $\leq_P$: The individualization $\leq_P$ is only used to resolve choices. If all choice-sets were witnessed as orbits (not fixing $\leq_P$), then indeed $\Psi_i$ evaluates equally for all $\leq_P$ (neither the step, the choice, the witnessing, nor the output formula use $\leq_P$). If a choice-set is not witnessed as orbit, this is surely also true for all $\leq_P$.

We replace every WSC-fixed-point operator $\Psi_i$ by $\Phi_i$ in $\Phi$ and continue to unwind the remaining IFPC-part of $\Phi$ yielding a $P_\ell$-formula equivalent to $\Phi$ on $\text{CFI}(\text{CFI}^{\ell+1}(G, g), 0)$ and $\text{CFI}(\text{CFI}^{\ell+1}(G, g), 1)$, which by assumption distinguishes the two graphs. This contradicts Lemma 6.8: The ordered graph $G$ has more than $\ell$ vertices (and thus color classes), so $\text{CFI}(G, g')$ has also more than $\ell$ color classes for every $g' \in \mathbb{F}_2$. Thus, we have $\text{CFI}(\text{CFI}(G, g'), 0) \not\cong^\delta \text{CFI}(\text{CFI}(G, g'), 1)$ by Lemma 6.9. So, by Lemma 6.8, Duplicator has a winning strategy in the $P_\ell$-game played on $\text{CFI}(\text{CFI}^{k+1}(G, g), 0)$ and $\text{CFI}(\text{CFI}^{k+1}(G, g), 1)$. Hence, $P_\ell$ does not distinguish the graphs by Lemma 6.7, which is a contradiction.

Finally, we have to consider the case of free numeric variables. Let the numeric variables in $\Phi$ be $\bar{v}$. For each numeric variable, there is a closed numeric WSC(IFPC)-term bounding its value. Let these terms be $\bar{s}$. Because we cannot evaluate the WSC(IFPC)-terms, we construct upper-bound-defining IFPC-terms $\bar{t}$ only depending on the size of the input structure. To obtain these, we construct upper-bound-defining terms recursively: For $0, 1, \cdot \cdot \cdot , +$ this is obvious. For a counting quantifier $\# \bar{x} \bar{y} \leq s'$. $\Phi$, the upper bound is defined by the IFPC-term $(\# \bar{x} \bar{x} = \bar{x}) \cdot t'_1 \cdot \cdot \cdot t'_{|\bar{v}|}$, where $t'_i$ is the upper-bound-defining IFPC-term recursively obtained for $s'_i$ for every $i \in [|\bar{v}]$. Note that we do not recurse on $\Phi$ and in particular not on a WSC-fixed-point operator. For $G \in K$, set $N(G) := \{0, \ldots, t^3_1\} \times \cdot \cdot \cdot \times \{0, \ldots, t^3_{|\bar{v}|}\}$ to be the possible values for the numeric variables for $\mathfrak{A} = \text{CFI}(\text{CFI}^{\ell+1}(G, g), h)$ (which only depends on $|\mathcal{A}|$).

To partition $K$ into $K_{\text{orb}}$ and $K_{\text{cfi}}$, we not only consider $|\bar{x}_j|$-tuples $\bar{u}$ of vertices of $\text{CFI}(\text{CFI}^{\ell+1}(G, g), h)$ but tuples $\bar{u} \bar{a}$ for $\bar{a} \in N(G)$. To extend Claim 6.12 to numeric variables, we have to test all possible values for the free numeric variables according to the upper-bound-defining term $\bar{t}$ and find the unpebbled-part-distinguishing choice-set. To adapt the proof of Claim 6.13, we obtain for every $i \in [m]$ and every tuple of values $\bar{a} \in N(G)$ for the
free numeric variables, a $P_\ell$-formula $\Phi_i^\alpha(x_i) := \exists P \vdash P. \tilde{\Psi}_i^\alpha(x_i)$ satisfying $\tilde{u} \in (\Phi_i^\alpha)^\mathbb{A}$ if and only if $\tilde{u} \in \Psi_i^\alpha$ for every $\mathbb{A} \in \{\mathrm{CFI}(\mathrm{CFI}^{\ell+1}(G, g), h) \mid h \in F_2\}$. In the same way, free numeric variables of IFPC-formulas are eliminated and we can use these formulas to construct the $P_\ell$-formula equivalent to $\Phi$.

Now we can show that we cannot avoid the additional operators to canonize CFI graphs (which implies defining the CFI query) as shown in Corollary 6.1.

**Proof of Theorem 1.3.** We consider the class of base graphs $\mathrm{CFI}^\omega(\mathcal{K})$. $\mathrm{WSCI}^2(\mathrm{IFPC})$ defines the CFI query for $\mathrm{CFI}(\mathrm{CFI}^\omega(\mathcal{K}))$ by Lemma 6.10. But $\mathrm{WSCI}(\mathrm{IFPC})$ does not define the CFI query for $\mathrm{CFI}(\mathrm{CFI}^\omega(\mathcal{K}))$ by Lemma 6.11.

Note that our proofs only use IFPC-interpretations in interpretation operators.

**Corollary 6.14.** $\mathrm{IFPC} < \mathrm{WSCI}(\mathrm{IFPC}) < \mathrm{WSCI}^2(\mathrm{IFPC})$.

It seems natural that $\mathrm{WSCI}^n(\mathrm{IFPC}) < \mathrm{WSCI}^{n+1}(\mathrm{IFPC})$ for every $n \in \mathbb{N}$. Possibly, this hierarchy can be shown by iterating our construction (e.g. $\mathrm{CFI}((\mathrm{CFI}^\omega)^n(\mathcal{K}))$, where $(\mathrm{CFI}^\omega)^n$ denotes $n$ applications of the CFI$^\omega$-operator).

We have seen that nesting WSC-fixed-point and interpretation operators increases the expressiveness of IFPC+WSC+I. However, we have not shown that the interpretation operator is necessary for that or whether WSC-fixed-point operators suffice. We will show in the next section that the interpretation operator indeed increases the expressiveness.

### 7. Separating IFPC+WSC from IFPC+WSC+I

In this section we separate IFPC+WSC from IFPC+WSC+I, that is, we show that the interpretation operator strictly increases expressiveness. To do so, we will define a class of structures $\mathcal{K}$ without non-trivial automorphisms. Thus, there are only singleton orbits and the WSC-fixed-point operator can be expressed by a (non-WSC) fixed-point operator: Either all choice-sets are singletons or the WSC-fixed-point operator evaluates to false. We will show that isomorphism of $\mathcal{K}$-structures is not definable in IFPC and thus not in IFPC+WSC. However, we will show that there is an IFPC-interpretation reducing $\mathcal{K}$-isomorphism to isomorphism of CFI graphs (over ordered base graphs) and thus $\mathcal{K}$-isomorphism is IFPC+WSC+I-definable.

We will combine two known constructions. We start with CFI graphs. In the next step, we will modify the CFI graphs to remove all automorphisms. This will be achieved by gluing a CFI graph to a so-called multipede [GS96]. The multipedes are structures without non-trivial automorphisms, for which IFPC fails to define the orbit partition. To prove that isomorphism of this gluing is not IFPC-definable, either, we will combine winning strategies of Duplicator in the bijective $k$-pebble game of CFI graphs and multipedes. In order to successfully combine the strategies, we will require that the base graphs of the CFI graphs have large connectivity. For the multipedes, we will show the existence of sets of vertices with pairwise large distance. The multipede can be removed by an FO-interpretation reducing the isomorphism problem of the gluing to the CFI query and hence the isomorphism problem is IFPC+WSC+I-definable.
7.1. Multipedes. We now review the multipede structures [GS96]. These structures are also based on the CFI gadgets. Most importantly, these structures are asymmetric, i.e., their only automorphism is the trivial one.

The base graph of a multipede is a bipartite graph \( G = (V, W, E, \leq) \), where \( \leq \) is some total order on \( V \cup W \) and every vertex in \( V \) has degree 3. We obtain the multipede structure \( \text{MP}(G) \) as follows. For every base vertex \( u \in W \), there is a vertex pair \( F(u) = \{u_0, u_1\} \) called a segment. We also call \( u \in W \) itself a segment. A single vertex \( u_i \) is called a foot. Vertices \( v \in V \) are called constraint vertices.

For every constraint vertex \( v \in V \), a degree-3 CFI gadget with three edge-vertex-pairs \( \{a^j_0, a^j_1\} \) (for \( j \in [3] \)) is added. Let \( N_G(v) = \{u^1_i, u^2_i, u^3_i\} \) (the order of the \( u^i \) is given by \( \leq \)). Then \( a^j_i \) is identified with the foot \( u^j_i \) for every \( j \in [3] \) and \( i \in \mathbb{F}_2 \). To construct multipedes, we use the relation-based CFI gadgets, that is we do not add further vertices to the feet but a ternary relation \( R \) containing all triples \( (u^1_i, u^2_i, u^3_i) \) with \( i_1 + i_2 + i_3 = 0 \). Because all constraint vertices in \( V \) have degree 3, the construction yields ternary structures of the fixed signature \( \{R, \leq\} \). The coloring \( \leq \) is again obtained from \( \leq \) (the feet of a segment have the color of the segment). We collect properties of multipedes.

**Definition 7.1 (Odd Graph).** A bipartite graph \( G = (V, W, E, \leq) \) is odd if, for every \( \emptyset \neq X \subseteq W \), there exists a \( v \in V \) such that \( |X \cap N_G(v)| \) is odd.

**Lemma 7.2 [GS96].** If \( G \) is an odd and ordered bipartite graph, then \( \text{MP}(G) \) is asymmetric.

**Definition 7.3 (k-Meager).** A bipartite graph \( G = (V, W, E, \leq) \) is called k-meager, if every set \( X \subseteq W \) of size \( |X| \leq 2k \) satisfies \( |\{v \in V \mid N_G(v) \subseteq X\}| \leq 2|X| \).

Intuitively, if \( G \) is a k-meager bipartite graph, then \( C_k \) cannot distinguish between the two feet of a segment in the structure \( \text{MP}(G) \). For a set \( X \subseteq W \), we define the feet-induced subgraph \( G[[X]] := G[X \cup \{v \in V \mid N_G(v) \subseteq X\}] \) to be the subgraph induced by the feet in \( X \) and all constraint vertices only adjacent to feet in \( X \). We extend the notation to the multipede: \( \text{MP}(G)[[X]] \) is the substructure induced by all feet whose segment is contained in \( X \). For a tuple \( \bar{u} \) of feet of \( \text{MP}(G) \), we define \( S(\bar{u}) := \{u \in W \mid u_i \in F(u) \text{ for some } i \leq |\bar{u}|\} \) to be the set of the segments of all feet in \( \bar{u} \).

**Lemma 7.4 [GS96].** Let \( G \) be a k-meager bipartite graph, \( A = \text{MP}(G) \), and \( \bar{u}, \bar{v} \in A^k \). If there is a local automorphism \( \varphi \in \text{Aut}(A[[S(\bar{u})]]) \) with \( \varphi(\bar{u}) = \bar{v} \), then \( (A, \bar{u}) \simeq_k (A, \bar{v}) \).

Gurevich and Shelah [GS96] showed that odd and k-meager bipartite graphs exist. However, we need a more detailed understanding of these graphs for our construction. In particular, we are interested in sets of segments which have pairwise large distance.

**Definition 7.5 (k-Scattered).** For a bipartite graph \( G = (V, W, E) \), a set \( X \subseteq W \) is called k-scattered if every distinct \( u, v \in X \) have distance at least \( 2k \) in \( G \).

We require pairwise distance \( 2k \) for a k-scattered set \( X \) because we are actually only interested in segments (which alternate with constraint vertices in paths in \( G \)).

**Lemma 7.6.** For every \( k \in \mathbb{N} \), there is an odd and k-meager bipartite graph \( G = (V, W, E) \) and a k-scattered set \( X \subseteq W \) of size \( |X| \geq k^2 \).
Proof. The multipedes constructed in [GS96] are sparse graphs generated by a random process. We that these graphs contain a k-scattered set of size at least \( k^2 \). Fix an arbitrary \( k \in \mathbb{N} \). We start with a vertex set \( U \) of size \( n \). An event \( E(n) \) is called almost sure if the probability that \( E \) occurs tends to 1 when \( n \) grows to infinity. Pick \( \epsilon < (2k+3)^{-1} \) and add independently with probability \( p = n^{-2+\epsilon} \) for every 3-element subset \( \{u_1, u_2, u_3\} \subseteq U \) a vertex \( v \) to \( V \) adjacent to the \( u_i \). Now, it is almost surely possible to remove at most \( n/4 \) vertices from \( U \) forming subgraphs that are exceedingly dense (and all constraint vertices in \( v \) incident to them), which results in an odd and \( k \)-meager bipartite graph [GS96]. We show that in this graph a \( k \)-scattered set \( X \) of size \( k^2 \) exists almost surely.

Claim 7.7. Almost surely, every vertex \( u \in W \) has degree at most \( n^{1.5k^{-1}} \).

Proof. Let \( m \) be the least integer such that \( m > n^{(1.5k)^{-1}} \). The probability that a given vertex has degree larger than \( n^{(1.5k)^{-1}} \) is \( \binom{n}{m} \cdot p^m \). So the probability that some vertex has degree larger than \( n^{(1.5k)^{-1}} \) is at most the following:

\[
n \cdot \left( \frac{n^2}{m} \right) \cdot p^m = n \cdot \left( \frac{n^2}{m} \right) \cdot (n^{-2+\epsilon})^m \\
\leq \left( \frac{n^2}{m} \right) \cdot n^{(-2+2(2k+3)^{-1})m+1} \\
\leq \left( \frac{en^2}{m} \right)^m \cdot n^{(-2+2(2k+3)^{-1})m+1} \\
\leq \left( \frac{en^2 - (1.5k)^{-1}}{m} \right)^m \cdot n^{(-2+2(2k+3)^{-1})m+1} \\
= e^m \cdot n^{(1.5k^{-1})m} \cdot n^{(-2+2(2k+3)^{-1})m+1} \\
= e^m \cdot n^{(2(2k+3)^{-1}-(1.5k)^{-1})m+1} \\
= n^{(ln n)^{-1} \cdot m} \cdot n^{((2k+3)^{-1}-(1.5k)^{-1})m+1} \\
= e^m \cdot n^{((2k+3)^{-1}-(1.5k)^{-1})m+1} \\
\leq n^{((2k+3)^{-1}-(1.5k)^{-1}+(ln n)^{-1})((1.5k^{-1})^1+1)+1} = o(1).
\]

The last step holds because \( (2k+3)^{-1} - (1.5k)^{-1} < 0 \) and thus for sufficiently large \( n \) it holds that \( \ell := (2k+3)^{-1} - (1.5k)^{-1} + (ln n)^{-1} < 0 \) and \( \ell(n^{(1.5k)^{-1}} + 1) + 1 \) tends to \(-\infty\).

Claim 7.8. If every vertex \( u \in W \) has degree at most \( n^{(1.5k)^{-1}} \), then for every vertex \( u \in W \) at most \( 3^k n^{\frac{2}{3}} \) vertices \( u' \in W \) have distance at most \( 2k \) to \( u \).

Proof. By construction, every vertex in \( V \) has degree 3. So at most \( 3n^{(1.5k)^{-1}} \) vertices have distance 2 to \( u \). Repeating the argument, at most \( (3n^{(1.5k)^{-1}})^k = 3^k n^{\frac{2}{3}} \) vertices in \( W \) have distance 2\( k \) to \( u \).

Almost surely, every vertex \( u \in W \) has degree at most \( n^{(1.5k)^{-1}} \) (Claim 7.7). We show that in this case a \( k \)-scattered set \( X \) of size at least \( k^2 \) exists. Note that we determined the probability before removing the \( \frac{4}{7} \) “bad” vertices. So we first remove some set of size at most \( \frac{4}{7} \) from \( U \) (which only decreases the degree of the remaining vertices). We now repeatedly apply Claim 7.8. If \( X \) is a \( k \)-scattered set, then at most \( |X| \cdot 3^k n^{\frac{2}{3}} \) vertices have distance at most \( 2k \) to a vertex in \( X \). Then pick one of the other vertices, add
it to $X$, so $X$ is still $k$-scattered, and repeat. In that way we find a $k$-scattered set $X$ of size $|X| \geq \frac{2}{3}n \cdot (3^k n^k)^{-1} = \frac{2}{3}3^{-k} n^3$. Finally, for sufficiently large $n$, we have that $k^2 \leq \frac{2}{3}3^{-k} n^3$.

We want to use a $k$-scattered set $X$ to ensure that in the bijective $k$-pebble game played on multipedes placing a pebble on one foot of a segment in $X$ has no effect on the other segments in $X$. To make this argument formal, we need to consider how information of pebbles is spread through the multipedes. For now, fix an arbitrary ordered bipartite graph $G = (V, W, E, \leq)$.

**Definition 7.9** (Closure). Let $X \subseteq W$. The attractor of $X$ is

$$\text{attr}(X) := X \cup \bigcup_{u \in V, \ |N_G(u) \setminus X| \leq 1} N_G(u).$$

The set $X$ is closed if $X = \text{attr}(X)$. The closure $\text{cl}(X)$ of $X$ is the inclusion-wise minimal closed superset of $X$.

**Lemma 7.10** [GS96]. If $G$ is $k$-meager and $X \subseteq W$ of size at most $k$, then $|\text{cl}(X)| \leq 2|X|$.

Let $X$ be closed. A set $Y \subseteq X$ is a component of $X$ if $G[[Y]]$ is a connected component of $G[[X]]$. That is, every constraint vertex contained in $G[[X]]$ is contained in $G[[Y]]$ or in $G[[X \setminus Y]]$.

**Lemma 7.11.** Let $X \subseteq W$. If $\text{cl}(X) = Y_1 \cup Y_2$ is the disjoint union of two components $Y_1$ and $Y_2$, then $X$ can be partitioned into $X_1 \cup X_2$ such that $\text{cl}(X_i) = Y_i$ for every $i \in [2]$.

**Proof.** Define $X_i := Y_i \cap X$ for every $i \in [2]$. Let $i \in [2]$ be arbitrary. We show that $\text{cl}(X_i) = Y_i$. It is clear that $\text{cl}(X_i) \subseteq Y_i$. For the other direction, suppose that $u \in Y_i \setminus \text{cl}(X_i)$. Then there must be a constraint vertex in $G[[Y]]$ which has a neighbor in $Y_1$ and another one in $Y_2$. This contradicts that the $Y_i$ are components of $\text{cl}(X)$. □

**Lemma 7.12.** Let $Y \subseteq W$ be closed and $u \in W$ have distance at least 4 to $Y$. Then $\text{cl}(Y \cup \{u\}) = Y \cup \{u\}$ and $u$ forms a singleton component of $Y \cup \{u\}$.

**Proof.** For a sake of contradiction, assume that there exists a $v \in \text{cl}(Y \cup \{u\}) \setminus (Y \cup \{u\})$. Then there is a constraint vertex $w$, of which one is $v$ and the other two are contained in $\text{cl}(Y \cup \{u\})$. If the two other neighbors are contained in $\text{cl}(Y)$, then $v$ is already contained in $\text{cl}(Y)$, which is a contradiction. So one of the two neighbors is $u$. But then $u$ has distance 2 to $Y$, which contradicts our assumption. □

**Lemma 7.13.** Let $k \geq 2$, $G$ be $2k$-meager, $Y \subseteq W$ be of size at most $k$, and $X \subseteq W$ be $6k$-scattered. Then at most $|Y|$ vertices of $X$ do not form singleton components in $\text{cl}(X \cup Y)$.

**Proof.** Because $X \subseteq W$ is $6k$-scattered, all distinct $u, v \in X$ have distance at least $12k$ in $G$. Hence, for every $w \in Y$, there is at most one $u \in X$ such that $w$ has distance less than $6k$ to $u$. Let $X' \subseteq X$ be the set of vertices with distance at most $6k$ to $Y$. Thus, $|X'| \leq |Y|$. So, $|X' \cup Y| \leq 2k$ and thus $|\text{cl}(X' \cup Y)| \leq 4k$ by Lemma 7.10 because $G$ is $2k$-meager. It follows that $|\text{cl}(X' \cup Y) \setminus X' \setminus Y| \leq 2k$. Because every component of $\text{cl}(X' \cup Y)$ contains a vertex of $X' \cup Y$, every vertex in $\text{cl}(X' \cup Y)$ has distance at most $4k$ to every vertex in $X' \cup Y$.

Since $|X'| \leq |Y|$, it suffices to show that all vertices in $X \setminus X'$ form singleton components of $\text{cl}(X \cup Y)$. Let $X \setminus X' = \{w_1, \ldots, w_{\ell}\}$ and set $X_i := \{w_1, \ldots, w_i\}$ for every
0 ≤ i ≤ ℓ. We show by induction on i ≤ ℓ that all vertices in Xi form singleton components in \( \text{cl}(X_i ∪ X′ \cup Y) \). For i = 0, the claim trivially holds. So assume i > 0. As seen before, every vertex in \( \text{cl}(X′ \cup Y) \) has distance at most 4k to X′ ∪ Y. By construction of X′, the vertex \( w_i \) has distance at least 6k + 1 to Y. Hence, \( w_i \) has distance at least 2k ≥ 4 to \( \text{cl}(X′ \cup Y) \). Because X is 6k-scattered, \( w_i \) has distance at least 12k to all other \( w_j \). Because, by the inductive hypothesis, all vertices in \( X_{i−1} \) form singletons components, \( w_i \) has distance at least 4 to \( \text{cl}(X_i ∪ X′ \cup Y) \). Hence, by Lemma 7.12, all vertices in \( X_i \setminus X′ \) form singleton components of \( \text{cl}(X \cup Y) \).

**Lemma 7.14.** Let \( k ≥ 2 \), \( G \) be 2k-meager, \( X ⊆ W \) be 6k-scattered, and let \( Y ⊆ W \) be of size at most \( k \). Then \( X \cup Y \) can be partitioned into \( Z_1, \ldots , Z_ℓ \) such that \( |\text{cl}(Z_i) \cap X| ≤ 1 \) for every \( i ∈ [ℓ]\) and \( \text{cl}(Z_1), \ldots , \text{cl}(Z_ℓ) \) are the components of \( \text{cl}(X \cup Y) \).

**Proof.** We partition \( X \cup Y \) using Lemma 7.11 into \( Z_1, \ldots , Z_ℓ \) such that \( \text{cl}(Z_i), \ldots , \text{cl}(Z_ℓ) \) are precisely the components of \( \text{cl}(X \cup Y) \). Let \( i ∈ [ℓ] \) be arbitrary but fixed. We prove that \( |\text{cl}(Z_i) \cap X| ≤ 1 \). By Lemma 7.13, all apart from \( |Y| \) vertices in \( X \) are contained in a singleton component. If \( \text{cl}(Z_i) \) is such a singleton component, we are done. Otherwise, \( |Z_i| ≤ |Z_i \cap X| \cup |Y| ≤ 2|Y| \leq 2k \) and \( |\text{cl}(Z_i)| ≤ 4k \) by Lemma 7.10 and because \( G \) is 2k-meager. Because \( \text{cl}(Z_i) \) is a component, all vertices in \( \text{cl}(Z_i) \) have pairwise distance at most 8k. So \( |\text{cl}(Z_i) \cap X| \leq 1 \), because \( X \) is 6k-scattered (and thus vertices in \( X \) have distance at least 12k).

**Lemma 7.15** [GS96]. Let \( X ⊆ W \), \( G \) be k-meager, \( ϕ ∈ \text{Aut}(\text{MP}(G)[[X]]) \), and \( |X| ≤ k \). Then \( ϕ \) extends to an automorphism of \( \text{MP}(G)[[\text{cl}(X)]] \).

**Lemma 7.16.** Let \( Y ⊆ W \), \( G \) be k-meager, \( ϕ ∈ \text{Aut}(\text{MP}(G)[[\text{cl}(Y)]]), \) and \( |Y| < k \). Then for every \( u ∈ W \setminus \text{cl}(Y) \) and all feet \( v, v′ ∈ F(u) \) (possibly \( v = v′ \)), there is an extension \( ψ \) of \( ϕ \) to an automorphism of \( \text{MP}(G)[[\text{cl}(Y \cup \{u\})]] \) satisfying \( ψ(v) = v′ \).

**Proof.** The condition \( ψ(v) = v′ \) defines \( ψ \) uniquely on the feet of \( u \), that is, given \( ϕ \), \( v \), and \( v′ \) the map \( ψ \) is determined. If \( v = v′ \), then the \( ψ \) maps the feet of \( u \) to itself and otherwise it exchanges them. Using Lemma 7.15 and noting that \( |Y \cup \{u\}| ≤ k \), it suffices to show that \( ψ \) is an automorphism of \( \text{MP}(G)[[Y \cup \{u\}]] \). Because \( u \notin \text{cl}(Y) \), there is no CFI gadget of which one edge-vertex-pair form the feet of \( u \) and the other edge-vertex-pairs are contained in \( \text{cl}(Y) \) (otherwise \( u \) would be in the closure). So indeed, \( ψ \) is a local automorphism.

**Lemma 7.17.** Let \( k ≥ 2 \), \( G \) be 2k-meager, \( X = \{u_1, \ldots , u_ℓ\} ⊆ W \) be 6k-scattered, \( Y ⊆ W \) be such that \( X \cap \text{cl}(Y) = \emptyset \) and \( |Y| < k \), and \( ϕ ∈ \text{Aut}(\text{MP}(G)[[Y]]) \). Then for all tuples \( \bar{u}, \bar{u}′ ∈ F(u_1) × \cdots × F(u_ℓ) \), there exists an extension \( ψ \) of \( ϕ \) to an automorphism of \( \text{MP}(G)[[X \cup Y]] \) satisfying \( ψ(\bar{u}) = \bar{u}′ \).

**Proof.** Let \( \bar{u}, \bar{u}′ ∈ F(u_1) × \cdots × F(u_ℓ) \). For every \( i ∈ [ℓ] \), there is by Lemma 7.16 an extension \( ψ_i \) of \( ϕ \) to \( \text{MP}(G)[[\text{cl}(Y \cup \{u_i\})]] \) satisfying \( ψ_i(u_i) = u′_i \).

By Lemma 7.14, all \( u_i \) are in different components of \( \text{cl}(Y \cup X) \) because \( G \) is 2k-meager and \( X \) is 6k-scattered. So we can extend every \( ψ_i \) on all components of \( \text{cl}(Y \cup X) \), on which \( ψ_i \) is not defined (in particular the ones containing all other \( u_j \) for \( j ≠ i \)), by the identity map and obtain an automorphism of \( \text{MP}(G)[[\text{cl}(Y \cup X)]] \) (two components are never connected by a CFI gadget). Hence, composing all the extended \( ψ_i \) yields the desired automorphism.

The previous lemma will be extremely useful in the bijective k-pebble game: If the pebbles are placed on the feet in \( Y \), we can simultaneously for all feet in \( X \setminus \text{cl}(Y) \) place
arbitrary pebbles and still maintain a local automorphism. Such sets $X$ will allow us to glue another graph to the multipede at the feet in $X$. Whatever restrictions on placing pebbles are imposed by the other graph, we still can maintain local automorphisms in the multipede.

**Lemma 7.18.** Let $k \geq 2$, $G$ be 2k-meager, $X \subseteq W$ be 6k-scattered, and $Y \subseteq W$ be of size at most $k$. Then $|cl(Y) \cap (X \setminus Y)| \leq |Y \setminus X|$.

**Proof.** We partition $X \cup Y$ using Lemma 7.14 into $Z_1, \ldots, Z_j$ such that the $cl(Z_i)$ are the components of $cl(X \cup Y)$ and at most one vertex of $X$ is contained in one $Z_i$. Up to reordering, assume that for some $\ell \leq j$ the components $Z_1, \ldots, Z_\ell$ are all components $Z_i$ such that $Z_i \cap Y \neq \emptyset$ and, for some $m \leq \ell$, the $Z_1, \ldots, Z_m$ are all $Z_i$ such that additionally $cl(Z_i) \cap (X \setminus Y) \neq \emptyset$.

Then clearly $cl(Y) \subseteq \bigcup_{i=\ell}^j cl(Z_i)$ and $\ell \leq |Y|$. Because $G$ is 2k-meager, $X$ is 6k-scattered, and $|Y| \leq k$, it follows from Lemma 7.14 that $|cl(Z_i) \cap X| \leq 1$ for every $i \in [\ell]$. Thus, $|cl(Z_i) \cap (X \setminus Y)| = 1$ for every $i \in [m]$ and $|cl(Z_i) \cap (X \setminus Y)| = 0$ for every $m < i \leq \ell$. It follows that $cl(Y) \cap (X \setminus Y) \subseteq \bigcup_{i=1}^m cl(Z_i) \cap (X \setminus Y)$. Hence, $|cl(Y) \cap (X \setminus Y)| \leq m$.

Every vertex in $Y \cap X$ has to be contained in some $Z_i$ such that $m < i \leq \ell$ because, for every $i \leq m$, we have $|cl(Z_i) \cap X| \leq 1$ and so $Z_i \cap (Y \cap X) = \emptyset$ since $Z_i$ contains a vertex of $X \setminus Y$. But for $i \leq \ell$, every $Z_i$ contains at least one vertex of $Y$. That is, $m \leq |Y \setminus X|$. \qed

### 7.2. Gluing Multipedes to CFI Graphs

We now glue CFI graphs to multipedes. First, we alter the used CFI-construction. Instead of two edge-vertex-pairs for the same base edge $\{u,v\}$ (one with origin $(u,v)$ and one with origin $(v,u)$), we contract the edges between these two edge-vertex-pairs (which form a matching) and obtain a single edge-vertex-pair with origin $\{u,v\}$. This preserves all relevant properties of CFI graphs. In particular, there are parity-preserving IFPC-interpretations mapping a CFI graph with two edge-vertex-pair per base edge to the corresponding CFI graphs with only one edge-vertex-pair per base edge and vice-versa. In this section, we write $\text{CFI}(H,f)$ for CFI graphs of this modified construction. Using only one edge-vertex-pair per base edge removes technical details from the following.

Let $G = (V^G, W^G, E^G, \leq^G)$ be an ordered bipartite graph, let $H = (V^H, E^H, \leq^H)$ be an ordered base graph, let $f : E^H \to \mathbb{F}_2$, and let $X \subseteq W^G$ have size $|X| = |E^H|$. We define the ternary structure $\text{MP}(G) \cup_X \text{CFI}(H,f)$ called the *gluing* of the multipede $\text{MP}(G) = (A, R^\text{MP}(G), \preceq^\text{MP}(G))$ and the CFI graph $\text{CFI}(H,f) = (B, E^\text{CFI}(H,f), \preceq^\text{CFI}(H,f))$ at $X$ as follows (see Figure 5 for an illustration): The $i$-th edge-vertex-pair of $\text{CFI}(H,f)$ is the edge-vertex-pair such that its origin $\{u,v\}$ is the $i$-th edge in $H$ according to $\leq^H$. We start with the disjoint union of $\text{MP}(G)$ and $\text{CFI}(H,f)$ and identify the $i$-th edge-vertex-pair of $\text{CFI}(H,f)$ with the $i$-th segment in $X$ (according to $\leq^G$). We finally turn the edges $E^\text{CFI}(H,f)$ into a ternary relation by extending every edge $(u,v) \in E^\text{CFI}(H,f)$ to the triple $(u,v,v)$. In this way, we obtain the $(R, \preceq)$-structure $\text{MP}(G) \cup_X \text{CFI}(H,f)$, where $R$ is the union of $R^\text{MP}(G)$ with the triples $(u,v,v)$ defined before and $\preceq$ is the total preorder obtained from joining $\preceq^\text{MP}(G)$ and $\preceq^\text{CFI}(H,f)$ by moving all gadget vertices of $\text{CFI}(H,f)$ to the end.

**Lemma 7.19.** If $\text{MP}(G)$ is asymmetric, then $\text{MP}(G) \cup_X \text{CFI}(H,f)$ is asymmetric.

**Proof.** Every automorphism of $\text{MP}(G) \cup_X \text{CFI}(H,f)$ is in particular an automorphism of $\text{CFI}(H,f)$. Every non-trivial automorphism of $\text{CFI}(H,f)$ exchanges the two vertices of some
Figure 5. Gluing multipedes to CFI graphs: The figure shows the gluing $\text{MP}(G) \cup_X \text{CFI}(H,f)$ of the multipede $\text{MP}(G)$ and the CFI graph $\text{CFI}(H,f)$ at the set of segments $X$. The multipede is drawn in blue, the segments in $X$ are drawn in red, and the gadget vertices of the CFI graph are drawn in green. Only some segments and CFI gadgets are shown. Two relational CFI gadgets of the multipede are shown, where different colors are used for each gadget. The edge-vertex-pairs of the CFI graphs are identified with the vertex-pairs of the segments in $X$. Formally, every vertex pair of each segment has a unique color and there is only a single ternary relation.

edge-vertex-pairs because $H$ is totally ordered. Because every edge-vertex-pair is identified with a segment of $\text{MP}(G)$ and $\text{MP}(G)$ is asymmetric, $\text{MP}(G) \cup_X \text{CFI}(H,f)$ is asymmetric, too.

We now show that $\text{MP}(G) \cup_X \text{CFI}(H,0) \simeq_k^r \text{MP}(G) \cup_X \text{CFI}(H,1)$ if $G$ and $H$ satisfy certain conditions. Let $\bar{u}$ be a vertex-tuple of $\text{MP}(G) \cup_X \text{CFI}(H,f)$, i.e., $\bar{u}$ contains either gadget vertices of the gadgets in $\text{CFI}(H,f)$ or feet of $\text{MP}(G)$. We also write $S(\bar{u})$ for the set of segments of all feet in $\bar{u}$.

1. The segments $S(\bar{u})$ are directly-fixed by $\bar{u}$.
2. The segments $\text{cl}(S(\bar{u})) \setminus S(\bar{u})$ are closure-fixed by $\bar{u}$.
3. A segment $u \in X$ is gadget-fixed by $\bar{u}$ if the feet of $u$ are identified with some edge-vertex-pair with origin $\{v, w\}$ in $\text{CFI}(H,f)$ such that there is a gadget vertex with origin $v$ or $w$ in $\bar{u}$.
4. A segment is fixed by $\bar{u}$ if it is directly fixed, closure-fixed, or gadget-fixed by $\bar{u}$.

Lemma 7.20. Let $r \geq k \geq 2$ and $\bar{u}$ be a vertex-tuple of $\text{MP}(G) \cup_X \text{CFI}(H,f)$ of length at most $k$. If $H$ is $r$-regular, $G$ is $2k$-meager, and $X$ is $6k$-scattered, then at most $r \cdot |\bar{u}|$ segments are fixed by $\bar{u}$. If $\bar{u}$ contains $i$ gadget vertices and $\ell$ segments in $X$ are directly-fixed by $\bar{u}$, then at most $|\bar{u}| - i - \ell$ segments in $X$ are closure-fixed by $\bar{u}$.
Proof. Assume that $\bar{u}$ contains $j$ feet and $i$ gadget vertices. So, $i + j = |\bar{u}| \leq k$. Then at most $ri$ segments are gadget-fixed by $\bar{u}$ because $G$ is $r$-regular. From Lemma 7.10 it follows that $|\text{cl}(S(\bar{u}))| \leq 2j$ because $G$ is $k$-meager. So at most $2j$ segments are directly-fixed or closure-fixed. Then $ri + 2j \leq ri + rj = r(i + j) = r|\bar{u}|$ because $i + j = |\bar{u}|$ and $r \geq k \geq 2$.

By Lemma 7.18, it holds that $|\text{cl}(S(\bar{u})) \cap (X \setminus S(\bar{u}))| \leq |S(\bar{u}) \setminus X|$ because $G$ is $k$-meager, $X$ is $6k$-scattered, and $|S(\bar{u})| \leq k$. That is, the number of closure-fixed segments in $X$ is bounded by the number of directly-fixed segments not in $X$. The number of directly-fixed segments not in $X$ is $|\bar{u}| - i - \ell$. \hfill \qed

We now combine winning strategies of Duplicator in the bijective pebble game on multipedes and CFI graphs:

Lemma 7.21. Let $r \geq k \geq 3$, $G$ be $2rk$-meager, $H$ be $r$-regular and at least $(k+2)$-connected, and $X$ be $6rk$-scattered. Then $\mathbb{MP}(G) \cup_X \text{CFI}(H,0) \simeq \mathbb{C} \mathbb{I} \mathbb{F}(G) \cup_X \text{CFI}(H,1)$.

Proof. Let $\mathfrak{A} = \mathbb{MP}(G)$, $\mathfrak{B} = \text{CFI}(H,0)$, and $\mathfrak{B}' = \text{CFI}(H,1)$. We show that Duplicator has a winning strategy in the bijective $k$-pebble game played on $\mathfrak{A} \cup_X \mathfrak{B}$ and $\mathfrak{A} \cup_X \mathfrak{B}'$.

For a set of segments $Y$ and a tuple $\bar{u}$, we denote by $\bar{u}Y$ the restriction of $\bar{u}$ to all feet whose segment is contained in $Y$, by $\bar{u}_G$ the restriction of $\bar{u}$ to all gadget vertices, and by $\bar{u}_F$ the restriction of $\bar{u}$ to all feet.

Duplicator maintains the following invariant. At every position $(\mathfrak{A} \cup_X \mathfrak{B}, \bar{u}; \mathfrak{A} \cup_X \mathfrak{B}', \bar{u}')$ in the game there exist tuples of feet $\bar{v}_g, \bar{v}_c$ of $\mathfrak{A} \cup_X \mathfrak{B}$ and $\bar{v}_g', \bar{v}_c'$ of $\mathfrak{A} \cup_X \mathfrak{B}'$ satisfying the following:

1. For every segment gadget-fixed by $\bar{u}$ (and so by $\bar{u}'$), there is exactly one foot of this segment contained in $\bar{v}_g$ respectively in $\bar{v}'_g$. No foot of another segment is contained in $\bar{v}_g$ respectively in $\bar{v}'_g$.
2. For every segment contained in $X$ and closure-fixed by $\bar{u}$ (and so by $\bar{u}'$), there is exactly one foot of this segment contained in $\bar{v}_c$ respectively in $\bar{v}'_c$. No foot of another segment is contained in $\bar{v}_c$ respectively in $\bar{v}'_c$.
3. There is a $\varphi \in \text{Aut}(\mathfrak{A}[[S(\bar{u}_F \bar{v}_g \bar{v}_c)])$ satisfying $\varphi(\bar{u}_F \bar{v}_g \bar{v}_c) = \bar{u}'_F \bar{v}'_g \bar{v}'_c$.
4. $(\mathfrak{B}, \bar{u}_X \bar{u}_G \bar{v}_c) \simeq \mathfrak{B}'(\bar{u}_X \bar{v}'_G \bar{v}'_c)$.
5. For every base vertex $u \in V^H$, it holds that $(\mathfrak{B}', \bar{u}_G \bar{v}_g)[V_u] \cong (\mathfrak{B}, \bar{u}_G \bar{v}_g)[V_u]$, where $V_u$ is the set of all gadget vertices with origin $u$ and all edge vertices with origin $\{u,v\}$ for some $v \in N_G(u)$.

Regarding Property 3, we have $S(\bar{u}_F \bar{v}_g \bar{v}_c) = S(\bar{u}'_F \bar{v}'_g \bar{v}'_c)$ and $|\bar{u}_F \bar{v}_g \bar{v}_c| = |\bar{u}_F \bar{v}'_g \bar{v}'_c|$ by Lemma 7.20, so the local automorphism $\varphi$ extends to the closure of $S(\bar{u}_F \bar{v}_g)$ by Lemma 7.15 because $G$ is $rk$-meager. Regarding Property 4, note that $|\bar{u}_X \bar{u}_G \bar{v}_c| = |\bar{u}'_X \bar{u}'_G \bar{v}'_c| \leq k$: By Lemma 7.18, the number of closure-fixed segments in $X$ is at most $|\bar{v}_c| \leq k - |\bar{u}_G| - |\bar{u}_X|$. Hence, $|\bar{u}_X \bar{u}_G \bar{v}_c| = |\bar{u}_X| + |\bar{u}_G| + |\bar{v}_c| \leq k$. (Note that $|\bar{u}_X| = |\bar{u}'_X|$, $|\bar{u}_G| = |\bar{u}'_G|$, etc., because otherwise Duplicator has already lost the game). Property 5 is needed because $|\bar{v}_g| \leq rk$ (possibly with equality), which exceeds $k$. Thus, Property 5 cannot be implied by Property 4. Property 5 guarantees that we pick the vertices $\bar{v}_g$ and $\bar{v}_c$ consistently.

Intuitively, we want to play two games. Game I is played with $rk$ pebbles on the multipede at position $(\mathfrak{A}, \bar{u}_F \bar{v}_g \bar{v}_c; \mathfrak{A}, \bar{u}_F \bar{v}'_g \bar{v}'_c)$. Game II is played with $k$ pebbles on the CFI graphs at position $(\mathfrak{B}, \bar{u}_X \bar{u}_G \bar{v}_c; \mathfrak{B}', \bar{u}_X \bar{u}'_G \bar{v}'_c)$. We use the winning strategy of Duplicator in both games (Lemmas 4.1 and 7.4) to construct a winning strategy in the bijective $k$-pebble game played at position $(\mathfrak{A} \cup_X \mathfrak{B}, \bar{u}; \mathfrak{A} \cup_X \mathfrak{B}', \bar{u}')$. Intuitively, we can do so because in Game I
we artificially fixed all gadget-fixed segments and in Game II we artificially fixed all closure-fixed segments in $X$ (and only the segments in $X$ are identified with edge-vertex-pairs of the CFI graphs).

Now assume that it is Spoiler’s turn. When Spoiler picks up a pair of pebbles $(p_i, q_i)$ from the structures, we first update the tuples $\bar{v}_{gf}$, $\bar{v}_{gf}'$, $\bar{v}_{cf}$, and $\bar{v}_{cf}'$: If a segment is no longer gadget-fixed or closure-fixed, then we remove the corresponding entries in the corresponding tuples. Clearly the invariant is maintained.

We describe how Duplicator defines a bijection $\lambda$ between $\mathcal{A} \cup_X \mathcal{B}$ and $\mathcal{A} \cup_X \mathcal{B}'$ by defining $\lambda(w)$ using the following case distinction:

(a) Assume the vertex $w$ is a foot whose segment is contained in $\text{cl}(S(\bar{u}_F \bar{v}_{gf} \bar{v}_{cf}))$. The automorphism $\varphi$ from Property 3 extends to an automorphism of $\mathcal{A}[[\text{cl}(S(\bar{u}_F \bar{v}_{gf} \bar{v}_{cf}))]]$ by Lemma 7.15. We set $\lambda(w) := \varphi(w)$. (This is actually Duplicator’s winning strategy in Game I [GS96], but the exact strategy is needed later).

(b) Assume $w$ is a foot not covered by the previous case. The bijection given by Duplicator’s winning strategy in Game I defines $\lambda(w)$ (actually, $\lambda(w)$ is an arbitrary foot of the same segment as $w$).

(c) Finally, assume $w$ is a gadget vertex. We use the bijection given by Duplicator’s winning strategy in Game II to define $\lambda(w)$.

Now Spoiler places the pebble pair $(p_i, q_i)$ on the vertices $w$ and $w' := \lambda(w)$. We update the tuples $\bar{v}_{gf}$, $\bar{v}_{gf}'$, $\bar{v}_{cf}$, and $\bar{v}_{cf}'$ as follows:

(a) Assume $w$ (and thus also $w'$) is a gadget vertex. Property 4 clearly holds because we followed Duplicator’s winning strategy in Game II. No new segments in $X$ get closure-fixed by $\bar{uw}$ respectively $\bar{uw}'$, so we just do not change $\bar{v}_{cf}$ respectively $\bar{v}_{cf}'$ and Property 2 is satisfied. We satisfy Properties 1, 3, and 5 by picking feet from the new gadget-fixed segments as follows:

- Assume that a segment becomes gadget-fixed by $\bar{uw}$ respectively $\bar{uw}'$ (and so the segment is in $X$) that is already closure-fixed by $\bar{u}$ respectively $\bar{u}'$. We pick the same feet as in $\bar{v}_{cf}$ and $\bar{v}_{cf}'$ and append them to $\bar{v}_{gf}$ and $\bar{v}_{gf}'$, respectively. Thus, Property 1 holds. Because the local automorphism from Property 3 already maps $\bar{v}_{cf}$ to $\bar{v}_{cf}'$, appending the corresponding entries to $\bar{v}_{gf}$ and $\bar{v}_{gf}'$ satisfies Property 3. Because the closure-fixed segments are part of the pebbled vertices in Game II (Property 4), appending these feet to $\bar{v}_{gf}$ and $\bar{v}_{gf}'$ satisfies Property 5.

- Assume that a segment $u$ becomes gadget-fixed by $\bar{uw}$ respectively $\bar{uw}'$ that is not closure-fixed by $\bar{u}$ respectively $\bar{u}'$. For both $\mathcal{B}$ and $\mathcal{B}'$, we pick the unique feet $v$ respectively $v'$ of the segment $u$ adjacent to the newly pebbled gadget vertex and append them to $\bar{v}_{gf}$ and $\bar{v}_{gf}'$, respectively. Hence, Property 1 is satisfied. Property 3 is satisfied by Lemma 7.17: We can pick for non-closure-fixed segments arbitrary feet and still find a local automorphism mapping them onto each other. For a sake of contradiction, assume that Property 5 is not satisfied by appending $v$ and $v'$. Then there is a base vertex $v \in V^H$ such that $(\mathcal{B}', \bar{u}_G \bar{v}_{gf} v)[V_6] \not\equiv (\mathcal{B}, \bar{u}_G' \bar{v}_{gf}' v')[V_6]$. There must be a pebble placed on a gadget vertex with origin $v$, i.e., both $\bar{u}_G$ and $\bar{u}_G'$ are nonempty, because otherwise $|\bar{v}_{gf}| \leq k$ and $|\bar{v}_{gf}'| \leq k$ and thus the two edge vertices of the segment $u$ form an orbit by Lemma 4.3 since $H$ is $(k+2)$-connected (the lemma also holds in the setting of a single edge-vertex-pair per base edge). In particular, $(\mathcal{B}', \bar{u}_G v)[V_6] \not\equiv (\mathcal{B}, \bar{u}_G' v')[V_6]$. Note that $v$ respectively $v'$ are $C_3$-definable because they are the unique vertices in the segment $u$ adjacent to $w$ respectively $w'$. So
Spoiler can win Game II by picking up a pebble pair (which is neither placed on \( w \) nor the gadget of \( v \)) and placing it on \( v \). Because \( k \geq 3 \), such a pebble pair actually exists. But that contradicts Property 4 and hence Property 5 is satisfied.

(b) Assume \( w \) (and thus also \( w' \)) is a foot. Thus, no segments get gadget-fixed by \( \bar{u}w \) respectively \( \bar{u}'w' \) that were not already gadget-fixed by \( \bar{u} \) respectively \( \bar{u}' \). So without picking further feet, \( \bar{v}_g \) and \( \bar{v}'_g \) satisfy Properties 1 and 5. Possibly a new segment \( u \in X \) becomes closure-fixed. By Lemma 7.20, there can be at most one such segment.

We satisfy Properties 2 to 4 as follows:

- Assume that \( u \) is already gadget-fixed by \( \bar{u} \) respectively \( \bar{u}' \). We append the vertices \( v \) and \( v' \) whose segment is \( u \) in \( \bar{v}_g \) respectively \( \bar{v}'_g \) to \( \bar{v} \) and \( \bar{v}' \), respectively. So Property 2 holds. Property 3 is satisfied because the local automorphism \( \varphi \) already maps \( \bar{v}_g \) to \( \bar{v}'_g \) and hence appending \( v \) respectively \( v' \) satisfies Property 2. Property 4 is satisfied because of Property 5: Fixing a single gadget vertex fixes all edge vertices adjacent to the gadget (which is \( C_3 \)-definable) and by Property 5 we have chosen \( \bar{v}_g \) respectively \( \bar{v}'_g \) consistently with the pebbles on the gadgets. So we can actually place a pebble pair \( v \) and \( v' \) and Property 4 holds.

- Otherwise, \( u \) is not gadget-fixed by \( \bar{u} \) respectively \( \bar{u}' \). So the two feet of \( u \) form an orbit in \((\mathcal{B}, \bar{u}_X \bar{u}_G \bar{v}_c)\) and likewise in \((\mathcal{B}, \bar{u}'_X \bar{u}'_G \bar{v}'_c)\) by Lemma 4.3 because \( H \) is \((k+2)\)-connected. So for every choice of feet in \( u \), Property 4 is satisfied. We make an arbitrary choice of a foot \( v \) of the segment \( u \) in \( \mathcal{B} \), which we append to \( \bar{v}_G \). Because \( u \) is closure-fixed by \( \bar{u}w \), we can extend \( \varphi \) also to the segment \( u \). We pick \( v' = \varphi(u) \) and append it to \( \bar{v}'_c \). So Property 3 is satisfied.

Hence, Duplicator is able to maintain the invariant. We update \( \bar{u} \) and \( \bar{u}' \) to include \( w \) respectively \( w' \). We show that the pebbles induce a local isomorphism: By Property 4, the pebbles induce a local isomorphism \( \bar{u}_X \bar{u}_G \bar{v}_c \mapsto \bar{u}'_X \bar{u}'_G \bar{v}'_c \) on the CFI graphs. This local isomorphism extends to all gadget-fixed segments by Property 5, that is, the map \( \bar{u}_X \bar{u}_G \bar{v}_c \bar{v}_g \mapsto \bar{u}'_X \bar{u}'_G \bar{v}'_c \bar{v}'_g \) is a local isomorphism of the CFI graphs. By Property 3, the map \( \bar{u}_F \bar{v}_g \bar{v}_c \bar{v}_g \mapsto \bar{u}'_F \bar{v}'_g \bar{v}'_c \bar{v}'_g \) is a local automorphism on the multipede. Because the maps agree on \( \bar{v}_g \bar{v}_c \bar{v}_g \), the combined map \( \bar{u}_X \bar{u}_G \bar{v}_c \bar{v}_g \bar{u}_F \mapsto \bar{u}'_X \bar{u}'_G \bar{v}'_c \bar{v}'_g \bar{u}'_F \) is a local isomorphism of the gluings. In particular, the map \( \bar{u}_X \bar{u}_G \bar{u}_F \mapsto \bar{u}'_X \bar{u}'_G \bar{u}'_F \) is a local isomorphism of the gluings, but that is just the map \( \bar{u} \mapsto \bar{u}' \). So Duplicator does not lose in this round and by induction wins the bijection \( k \)-pebble game.

Theorem 7.22. There is an FO-interpretation \( \Theta \) and, for every \( k \in \mathbb{N} \), a pair of ternary \( \{(R, \preceq)\text{-structures} (\mathfrak{A}_k, \mathfrak{B}_k)\text{ such that} \)

(a) \( \preceq \) is a total preorder on \( \mathfrak{A} \) and \( \mathfrak{B} \),

(b) \( \mathfrak{A}_k \) and \( \mathfrak{B}_k \) are asymmetric,

(c) \( \mathfrak{A}_k \simeq \mathfrak{B}_k \),

(d) \( \mathfrak{A}_k \not\simeq \mathfrak{B}_k \), and

(e) \( \Theta(\mathfrak{A}_k) \) and \( \Theta(\mathfrak{B}_k) \) are non-isomorphic CFI graphs over the same ordered base graph.

The interpretation \( \Theta \) is 1-dimensional and equivalence-free.

Proof. Let \( k \geq 2 \) be arbitrary and \( H \) be a clique of size \( k+4 \). Thus, \( H \) is \( r := (k+3)\)-regular, \((k+2)\)-connected, and has \( m := \frac{1}{2}(k+4)(k+3) \leq r(k+2) \) edges. There exists an odd and \((6r(k+2))\)-meager bipartite graph \( G = (V, W, E) \) that contains a \( 6r(k+2) \)-scattered set \( X \subseteq W \) of size \( m \leq (6r(k+2)) \) by Lemma 7.6. Equip the graphs \( G \) and \( H \) with arbitrary total orders.
We now prove Theorem 1.1 and separate IFPC+WSC from IFPC+WSC+I.

Set $\mathfrak{A}_k := \text{MP}(G) \cup_X \text{CFI}(H, 0)$ and $\mathfrak{B}_k := \text{MP}(G) \cup_X \text{CFI}(H, 1)$. Clearly, $\mathfrak{A}_k \not\cong \mathfrak{B}_k$ and $\mathfrak{A}_k \cong^k \mathfrak{B}_k$ by Lemma 7.21 because $H$ is $(k + 2)$-connected and $r$-regular, $G$ is $(6r(k + 2))$-meager and so in particular $2rk$-meager, and $X$ is $6r(k + 2)$-scattered and so in particular $6rk$-scattered. By Lemma 7.2, the multipede $\text{MP}(G)$ is asymmetric because $G$ is odd. Thus, $\mathfrak{A}_k$ and $\mathfrak{B}_k$ are asymmetric by Lemma 7.19.

It remains to define the interpretation $\Theta$, which maps $\mathfrak{A}_k$ to $\text{CFI}(H, 0)$ respectively $\mathfrak{B}_k$ to $\text{CFI}(H, 1)$. Recall that the gluing extends edges of the CFI graphs to triples by repeating the last entry and multipedes do not contain such triples. So we can easily define the vertices contained in the “CFI triples”. By taking the induced graph on these vertices and by shortening the triples back to pairs, one defines the CFI graphs again. This is done by the following 1-dimensional and equivalence-free FO-$\{R, \leq\}$-interpretation $\Theta = (\Phi_{\text{dom}}(x), \Psi_E(x, y), \Psi_\leq(x, y))$:

\[
\Phi_{\text{dom}}(x) := \exists y. R(x, y, y) \lor R(y, x, x),
\]
\[
\Psi_E(x, y) := R(x, y, y),
\]
\[
\Psi_\leq(x, y) := x \leq y.
\]

Clearly $\Theta(\mathfrak{A}_k) = \text{CFI}(H, 0)$ and $\Theta(\mathfrak{B}_k) = \text{CFI}(H, 1)$. □

We now prove Theorem 1.1 and separate IFPC+WSC from IFPC+WSC+I.

**Proof of Theorem 1.1.** Consider the class of $\{R, \leq\}$-structures $\mathcal{K}$ given by Theorem 7.22. To ensure that the reduct semantics does not add automorphisms when not using $\leq$ in a formula, we additionally encode $\leq$ into $R$. We add a directed path of length $i$ and another one of length $i + 1$ to every vertex in the $i$-th color class. In that way, we obtain the class of $\{R, \leq\}$-structures $\mathcal{K}'$, for which also the $\{R\}$-reducts are asymmetric.

Clearly, there is a 1-dimensional and equivalence-free FO-interpretation $\Theta_\mathcal{K}$ mapping a $\mathcal{K}'$-structure back to the corresponding $\mathcal{K}$-structure (remove all vertices of out-degree at most 1 because we attached to all original vertices two directed paths).

We argue that IFPC+WSC = IFPC on $\mathcal{K}'$. If a WSC-fixed-point operator mentions $R$, then the structure is asymmetric, i.e., there are only singleton orbits. Hence, choosing becomes useless and can be simulated by (non-WSC)-fixed-point operators. If a WSC-fixed-point operator does not mention $R$, then the $\{\leq\}$-reduct is fully determined by the number and sizes of the color classes (which must be equal for $\mathfrak{A}_k$ and $\mathfrak{B}_k$ for every $k \geq 3$ because otherwise $C_3$ distinguishes $\mathfrak{A}_k$ and $\mathfrak{B}_k$). The number of color classes and their sizes are clearly IFPC-definable.

For every $k \in \mathbb{N}$, there are two non-isomorphic structures $\mathfrak{A}_k \cong^k \mathfrak{B}_k$ in $\mathcal{K}'$ and thus IFPC does not define the isomorphism problem of $\mathcal{K}'$. It remains to show that IFPC+WSC+I defines the isomorphism problem. The CFI-query on ordered base graphs is definable in IFPC+WSC+I by Theorem 1.2. By Corollary 6.1, there is actually a WSCI(IFPC) = WSC(IFPC)-formula $\Phi_{\text{CFI}}$ defining the CFI query on ordered base graphs. Let $\Theta_{\text{CFI}}$ be the FO-interpretation extracting the CFI graphs from $\mathcal{K}$-structures given by Theorem 7.22. Then the I(WSC(IFPC))-formula

\[
\text{I}(\Theta_{\text{CFI}} \circ \Theta_\mathcal{K}; \Phi_{\text{CFI}})
\]

defines the isomorphism problem of $\mathcal{K}'$-structures: $\Theta_{\text{CFI}}$ reduces by Theorem 7.22 the isomorphism problem of $\mathcal{K}$ to the isomorphism problem of CFI graph over ordered base graphs, which is defined by $\Phi_{\text{CFI}}$. Note that because both $\Theta_{\text{CFI}}$ and $\Theta_\mathcal{K}$ are 1-dimensional and equivalence-free FO-interpretations, so is $\Theta_{\text{CFI}} \circ \Theta_\mathcal{K}$, too. □
**Corollary 7.23.** IFPC+WSC \( < \) Ptime.

**Corollary 7.24.** WSC(IFPC) \( < \) I(WSC(IFPC)).

Note that the prior corollary refines Corollary 6.14, which states that

\[
WSC(IFPC) = WSCI(IFPC) < WSCI^2(IFPC) = WSC(I(WSC(IFPC))).
\]

We actually expect that WSC(IFPC) \( < \) I(WSC(IFPC)) \( < \) WSC(I(WSC(IFPC))) because it seems unlikely that I(WSC(IFPC)) defines the CFI query of the base graphs of Theorem 1.3.

**Corollary 7.25.** IFPC+WSC is not closed under IFPC-interpretations and not even under 1-dimensional and equivalence-free FO-interpretations.

*Proof.* The interpretation \( \Theta_{CFI} \circ \Theta_K \) in the proof of Theorem 1.1 is 1-dimensional and equivalence-free. The interpretation actually only removes vertices. The claim follows.

Similarly, we can answer the question of Dawar and Richerby in [DR03a] whether IFP+SC is closed under FO-interpretations in the negative:

**Corollary 7.26.** IFP+SC is not closed under 1-dimensional and equivalence-free FO-interpretations.

*Proof.* We consider the same structures: IFP = IFP+SC for the constructed structures because they only have trivial orbits and it does not make a difference whether we need to witness choices. Because IFPC does not define isomorphism, surely IFP does neither. Because IFP+WSC defines the CFI query for ordered base graphs [GH98], so does IFP+SC. We conclude that IFP+SC is not closed under FO-interpretations, too.

**8. Discussion**

We defined the logics IFPC+WSC and IFPC+WSC+I to study the combination of witnessed symmetric choice and interpretations beyond simulating counting. Instead, we provided graph constructions to prove lower bounds. IFPC+WSC+I canonizes CFI graphs if it canonizes the base graphs, but operators have to be nested. We proved that this increase in nesting depth is unavoidable using double CFI graphs obtained by essentially applying the CFI construction twice. Does iterating our construction further show an operator nesting hierarchy in IFPC+WSC+I? We have seen that also in the presence of counting the interpretation operator strictly increases the expressiveness. So indeed both, witnessed symmetric choice and interpretations are needed to possibly capture Ptime. This answers the question of the relation between witnessed symmetric choice and interpretations for IFPC. But it remains open whether IFPC+WSC+I captures Ptime. Here, iterating our CFI construction is of interest again: If one shows an operator nesting hierarchy using this construction, then one in particular will separate IFPC+WSC+I from Ptime because our construction does not change the signature of the structures. Studying this remains for future work.
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