On linear groups over weakly locally finite division rings

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Abstract

In this paper, we give the definition of \textit{weakly locally finite} division rings and we show that the class of these rings strictly contains the class of locally finite division rings. Further, we study multiplicative subgroups in these rings. Some skew linear groups are also considered. Our new obtained results generalize previous results for centrally finite case.

\textbf{Key words:} division ring; centrally finite; locally finite; weakly locally finite; linear groups.

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1 Introduction

Let $D$ be a division ring with center $F$. Recall that $D$ is centrally finite if $D$ is a finite dimensional vector space over $F$; $D$ is locally finite if for every finite subset $S$ of $D$, the division subring $F(S)$ of $D$ generated by $S \cup F$ is a finite dimensional vector space over $F$. If $a$ is an element from $D$, then we have the field extension $F \subseteq F(a)$. Obviously, $a$ is algebraic over $F$ if and only if this extension is finite. We say that a non-empty subset $S$ of $D$ is algebraic over $F$ if every element of $S$ is algebraic over $F$. A division ring $D$ is algebraic over center $F$ (briefly, $D$ is algebraic), if every element of $D$ is algebraic over $F$. Clearly, a locally finite division ring is algebraic. It was conjectured that any algebraic division ring is locally finite (this is known as the Kurosch’s Problem for division rings [5]). However, this problem remains still open in general. There exist locally finite division rings which are not centrally finite. In this paper, we define a weakly locally finite division ring as a division ring in which for every finite subset $S$, the division subring generated by $S$ is centrally finite. It can be shown that every locally finite division ring is weakly locally finite. The converse is not true. Moreover, in the text, we give one example of weakly locally finite division ring which is not even algebraic. Next, we study subgroups in weakly locally finite division rings. In particular, we give the affirmative answer to one of Herstein’s conjectures [4] for these rings. Some linear groups are also investigated. Our new obtained results generalize previous results for centrally finite case. The symbols and notation we use in this paper are standard and they should be found in the literature on subgroups in division rings and on skew linear groups. In particular, for a division ring $D$ we denote by $D^*$ and $D'$ the multiplicative group and the derived group of $D$ respectively.

2 Definitions and examples

Definition 2.1. We say that a division ring $D$ is weakly locally finite if for every finite subset $S$ of $D$, the division subring generated by $S$ in $D$ is centrally finite.

It can be shown that every division subring of a centrally finite division ring is itself centrally finite. Using this fact, it is easy to see that every locally finite division ring is weakly locally finite.

Our purpose in this section is to construct an example showing the difference between the class of locally finite division rings and the class of weakly locally finite division rings. In order to do so, following the general Mal’cev-Neumann construction of Laurent series rings, we construct a Laurent series ring with a base ring which is an extension of the field $\mathbb{Q}$ of rational numbers. The ring we construct in the following proposition is weakly locally finite but it is not even algebraic.

Proposition 2.2. There exists a weakly locally finite division ring which is not algebraic.
Proof. Denote by \( G = \bigoplus_{i=1}^{\infty} \mathbb{Z} \) the direct sum of infinitely many copies of the additive group \( \mathbb{Z} \). For any positive integer \( i \), denote by \( x_i = (0, \ldots, 0, 1, 0, \ldots) \) the element of \( G \) with 1 in the \( i \)-th position and 0 elsewhere. Then \( G \) is a free abelian group generated by all \( x_i \) and every element \( x \in G \) is written uniquely in the form

\[
x = \sum_{i \in I} n_i x_i,
\]

with \( n_i \in \mathbb{Z} \) and some finite set \( I \).

Now, we define an order in \( G \) as follows:

For elements \( x = (n_1, n_2, n_3, \ldots) \) and \( y = (m_1, m_2, m_3, \ldots) \) in \( G \), define \( x < y \) if either \( n_1 < m_1 \) or there exists \( k \in \mathbb{N} \) such that \( n_1 = m_1, \ldots, n_k = m_k \) and \( n_{k+1} < m_{k+1} \). Clearly, with this order \( G \) is a totally ordered set.

Suppose that \( p_1 < p_2 < \ldots < p_n < \ldots \) is a sequence of prime numbers and \( K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots) \) is the subfield of the field \( \mathbb{R} \) of real numbers generated by \( \mathbb{Q} \) and \( \sqrt{p_1}, \sqrt{p_2}, \ldots \), where \( \mathbb{Q} \) is the field of rational numbers. For any \( i \in \mathbb{N} \), suppose that \( f_i : K \to K \) is \( \mathbb{Q} \)-homomorphism satisfying the following condition:

\[
f_i(\sqrt{p_i}) = -\sqrt{p_i}; \quad \text{and} \quad f_i(\sqrt{p_j}) = \sqrt{p_j} \quad \text{for any} \ j \neq i.
\]

It is easy to verify that \( f_i f_j = f_j f_i \) for any \( i, j \in \mathbb{N} \).

- **Step 1. Proving that, for \( x \in K, f_i(x) = x \) for any \( i \in \mathbb{N} \) if and only if \( x \in \mathbb{Q} \):**

  The converse is obvious. Now, suppose that \( x \in K \) such that \( f_i(x) = x \) for any \( i \in \mathbb{N} \). By setting \( K_0 = \mathbb{Q} \) and \( K_i = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_i}) \) for \( i \geq 1 \), we have the following ascending series:

\[
K_0 \subset K_1 \subset \ldots \subset K_i \subset \ldots
\]

If \( x \notin \mathbb{Q} \), then there exists \( i \geq 1 \) such that \( x \in K_i \setminus K_{i-1} \). So, we have \( x = a + b\sqrt{p_i} \), with \( a, b \in K_{i-1} \) and \( b \neq 0 \). Since \( f_i(x) = x, 0 = x - f_i(x) = 2b\sqrt{p_i} \), a contradiction.

- **Step 2. Constructing a Laurent series ring:**

  For any \( x = (n_1, n_2, \ldots) = \sum_{i \in I} n_i x_i \in G \), define \( \Phi_x := \prod_{i \in I} f_i^{n_i} \). Clearly \( \Phi_x \in \text{Gal}(K/\mathbb{Q}) \) and the map \( \Phi : G \to \text{Gal}(K/\mathbb{Q}) \), defined by \( \Phi(x) = \Phi_x \) is a group homomorphism. The following conditions hold.

  i) \( \Phi(x_i) = f_i \) for any \( i \in \mathbb{N} \).

  ii) If \( x = (n_1, n_2, \ldots) \in G \), then \( \Phi_x(\sqrt{p_i}) = (-1)^{n_i} \sqrt{p_i} \).

  For the convenience, from now on we write the operation in \( G \) multiplicatively. For \( G \) and \( K \) as above, consider formal sums of the form

\[
\alpha = \sum_{x \in G} a_x x, \quad a_x \in K.
\]
For such an $\alpha$, define the support of $\alpha$ by $\text{supp}(\alpha) = \{x \in G : a_x \neq 0\}$. Put

$$D = K((G, \Phi)) := \left\{\alpha = \sum_{x \in G} a_x x, a_x \in K : \text{supp}(\alpha) \text{ is well-ordered}\right\}.$$  

For $\alpha = \sum a_x x$ and $\beta = \sum b_x x$ from $D$, define

$$\alpha + \beta = \sum_{x \in G} (a_x + b_x) x; \quad \text{and} \quad \alpha \beta = \sum_{x \in G} \left(\sum_{y \in z} a_x \Phi_x(b_y)\right) z.$$

With operations defined as above, $D = K((G, \Phi))$ is a division ring (we refer to [6, pp. 243-244]). Moreover, the following conditions hold.

iii) For any $x \in G, a \in K, xa = \Phi_x(a)x$.

iv) For any $i \neq j$, $x_i \sqrt{p_i} = -\sqrt{p_i} x_i$ and $x_j \sqrt{p_i} = \sqrt{p_i} x_j$.

v) For any $i \neq j$ and $n \in \mathbb{N}, x_i^n \sqrt{p_i} = (-1)^n \sqrt{p_i} x_i^n$ and $x_j^n \sqrt{p_i} = \sqrt{p_i} x_j^n$.

**Step 3. Finding the center of $D$:**

Put $H = \{x^2 : x \in G\}$ and $\mathbb{Q}((H)) = \left\{\alpha = \sum_{x \in H} a_x x, a_x \in \mathbb{Q} : \text{supp}(\alpha) \text{ is well-ordered}\right\}$. It is easy to check that $H$ is a subgroup of $G$ and for every $x \in H, \Phi_x = \text{Id}_K$.

Denote by $F$ the center of $D$. We claim that $F = \mathbb{Q}((H))$. Suppose that $\alpha = \sum_{x \in H} a_x x \in \mathbb{Q}((H))$. Then, for every $\beta = \sum_{y \in G} b_y y \in D$, we have $\Phi_x(b_y) = b_y$ and $\Phi_y(a_x) = a_x$. Hence

$$\alpha \beta = \sum_{z \in G} \left(\sum_{xy = z} a_x \Phi_x(b_y)\right) z = \sum_{z \in G} \left(\sum_{xy = z} a_x b_y\right) z,$$

$$\beta \alpha = \sum_{z \in G} \left(\sum_{xy = z} b_y \Phi_y(a_x)\right) z = \sum_{z \in G} \left(\sum_{xy = z} a_x b_y\right) z.$$

Thus, $\alpha \beta = \beta \alpha$ for every $\beta \in D$, so $\alpha \in F$.

Conversely, suppose that $\alpha = \sum a_x x \in F$. Denote by $S$ the set of all elements $x$ appeared in the expression of $\alpha$. Then, it suffices to prove that $x \in H$ and $a_x \in \mathbb{Q}$ for any $x \in S$. In fact, since $\alpha \in F$, we have $\sqrt{p_i} \alpha = \alpha \sqrt{p_i}$ and $\alpha x_i = x_i \alpha$ for any $i \geq 1$, i.e. $\sum_{x \in S} \sqrt{p_i} a_x x = \sum_{x \in S} \Phi_x(\sqrt{p_i}) a_x x$ and $\sum_{x \in S} a_x (x_i x) = \sum_{x \in S} \Phi_{x_i}(a_x)(x_i x)$. Therefore, by conditions mentioned in the beginning of Step 2, we have $\sqrt{p_i} a_x = \Phi_x(\sqrt{p_i}) a_x = (-1)^n \sqrt{p_i} a_x$ and $a_x = \Phi_{x_i}(a_x) = f_i(a_x)$ for any $x = (n_1, n_2, \ldots) \in S$. From the first equality it follows that $n_i$ is even for any $i \geq 1$. Therefore $x \in H$. From the second equality it follows that $a_x = f_i(a_x)$ for any $i \geq 1$. So by Step 1, we have $a_x \in \mathbb{Q}$. Therefore $\alpha \in \mathbb{Q}((H))$. Thus, $F = \mathbb{Q}((H))$.

**Step 4. Proving that $D$ is not algebraic over $F$:**
Suppose that $\gamma = x_1^{-1} + x_2^{-1} + \ldots$ is an infinite formal sum. Since $x_1^{-1} < x_2^{-1} < \ldots$, $\text{supp}(\gamma)$ is well-ordered. Hence $\gamma \in D$. Consider the equality

$$a_0 + a_1 \gamma + a_2 \gamma^2 + \ldots + a_n \gamma^n = 0, \quad a_i \in F. \quad (2)$$

Note that $X = x_1^{-1} x_2^{-1} \ldots x_n^{-1}$ does not appear in the expressions of $\gamma, \gamma^2, \ldots, \gamma^{n-1}$ and the coefficient of $X$ in the expression of $\gamma^n$ is not algebraic over $F$. Therefore, the coefficient of $X$ in the expression on left side of the equality (2) is $a_n n!$. It follows that $a_n = 0$. By induction, it is easy to see that $a_0 = a_1 = \ldots = a_n = 0$. Hence, for any $n \in \mathbb{N}$, the set $\{1, \gamma, \gamma^2, \ldots, \gamma^n\}$ is independent over $F$. Consequently, $\gamma$ is not algebraic over $F$.

**Step 5. Constructing a division subring of $D$ which is a weakly locally finite:**

Consider the element $\gamma$ from Step 4. For any $n \geq 1$, put

$$R_n = F(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}, x_1, x_2, \ldots, x_n, \gamma),$$

and $R_\infty = \bigcup_{n=1}^{\infty} R_n$. First, we prove that $R_n$ is centrally finite for each positive integer $n$.

Consider the element

$$\gamma_n = x_{n+1}^{-1} + x_{n+2}^{-1} + \ldots \quad (\text{infinite formal sum}).$$

Since $\gamma_n = \gamma - (x_1^{-1} + x_2^{-1} + \ldots + x_n^{-1})$, we conclude that $\gamma_n \in R_n$ and

$$F(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}, x_1, x_2, \ldots, x_n, \gamma) = F(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}, x_1, x_2, \ldots, x_n, \gamma_n).$$

Note that $\gamma_n$ commutes with all $\sqrt{p_i}$ and all $x_i$ (for $i = 1, 2, \ldots, n$). Therefore

$$R_n = F(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}, x_1, x_2, \ldots, x_n, \gamma_n) = F(\gamma_n)(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}, x_1, x_2, \ldots, x_n).$$

In combination with the equalities $(\sqrt{p_i})^2 = p_i, x_i^2 \in F, \sqrt{p_i} x_j = x_j \sqrt{p_i}, i \neq j, \sqrt{p_i} x_i = -x_i \sqrt{p_i}$, it follows that every element $\beta$ from $R_n$ can be written in the form

$$\beta = \sum_{0 \leq \varepsilon_i, \mu_i \leq 1} a_{(\varepsilon_1, \ldots, \varepsilon_n, \mu_1, \ldots, \mu_n)}(\sqrt{p_1})^{\varepsilon_1} \ldots (\sqrt{p_n})^{\varepsilon_n} x_1^{\mu_1} \ldots x_n^{\mu_n}, \text{ where } a_{(\varepsilon_1, \ldots, \varepsilon_n, \mu_1, \ldots, \mu_n)} \in F(\gamma_n).$$

Hence $R_n$ is a vector space over $F(\gamma_n)$ having the finite set $B_n$ which consists of the products

$$(\sqrt{p_1})^{\varepsilon_1} \ldots (\sqrt{p_n})^{\varepsilon_n} x_1^{\mu_1} \ldots x_n^{\mu_n}, 0 \leq \varepsilon_i, \mu_i \leq 1$$

as a base. Thus, $R_n$ is a finite dimensional vector space over $F(\gamma_n)$. Since $\gamma_n$ commutes with all $\sqrt{p_i}$ and all $x_i$, $F(\gamma_n) \subseteq Z(R_n)$. It follows that $\dim_{F(\gamma_n)} R_n \leq \dim_{F(\gamma_n)} R_n < \infty$ and consequently, $R_n$ is centrally finite.
For any finite subset \( S \subseteq R_\infty \), there exists \( n \) such that \( S \subseteq R_n \). Therefore, the division subring of \( R_\infty \), generated by \( S \) over \( F \) is contained in \( R_n \), which is centrally finite. Thus, \( R_\infty \) is weakly locally finite.

- **Step 6. Finding the center of \( R_\infty \):**

  We claim that \( Z(R_\infty) = F \). Put \( S_n = \{ \sqrt{p_1}, \ldots, \sqrt{p_n}, x_1, \ldots, x_n \} \). Since for any \( i \neq j \), \( x_i^2, (\sqrt{p_i})^2 \in F \), \( x_ix_j = x_jx_i \), \( \sqrt{p_i}\sqrt{p_j} = \sqrt{p_j}\sqrt{p_i} \), \( x_i\sqrt{p_j} = \sqrt{p_j}x_i \), every element from \( F[S_n] \) can be expressed in the form

\[
\alpha = \sum_{0 \leq \varepsilon_i, \mu_i \leq 1} a_{(\varepsilon_1, \ldots, \varepsilon_n, \mu_1, \ldots, \mu_n)} (\sqrt{p_1})^{\varepsilon_1} \ldots (\sqrt{p_n})^{\varepsilon_n} x_1^{\mu_1} \ldots x_n^{\mu_n}, \quad a_{(\varepsilon_1, \ldots, \varepsilon_n, \mu_1, \ldots, \mu_n)} \in F. \tag{3}
\]

Moreover, the set \( B_n \) consists of products \( (\sqrt{p_1})^{\varepsilon_1} \ldots (\sqrt{p_n})^{\varepsilon_n} x_1^{\mu_1} \ldots x_n^{\mu_n}, 0 \leq \varepsilon_i, \mu_i \leq 1 \) is finite of \( 2^{2n} \) elements. Hence, \( F[S_n] \) is a finite dimensional vector space over \( F \). So, it follows that \( F[S_n] = F(S_n) \). Therefore, every element from \( F(S_n) \) can be expressed in the form (3).

In the first, we show that \( Z(F(S_1)) = F \). Thus, suppose that \( \alpha \in Z(F(S_1)) \). Since \( x_1^2, (\sqrt{p_1})^2 = p_1 \in F \) and \( x_1\sqrt{p_1} = -\sqrt{p_1}x_1 \), every element \( \alpha \in F(S_1) = F(\sqrt{p_1}, x_1) \) can be expressed in the following form:

\[
\alpha = a + b\sqrt{p_1} + cx_1 + d\sqrt{p_1}x_1, \quad a, b, c, d \in F.
\]

Since \( \alpha \) commutes with \( x_1 \) and \( \sqrt{p_1} \), we have

\[
ax_1 + b\sqrt{p_1}x_1 + cx_1^2 + d\sqrt{p_1}x_1^2 = ax_1 - b\sqrt{p_1}x_1 + cx_1^2 - d\sqrt{p_1}x_1^2,
\]

and

\[
a\sqrt{p_1} + bp_1 - c\sqrt{p_1}x_1 - dp_1x_1 = a\sqrt{p_1} + bp_1 + c\sqrt{p_1}x_1 + dp_1x_1.
\]

From the first equality it follows that \( b = d = 0 \), while from the second equality we obtain \( c = 0 \). Hence, \( \alpha = a \in F \) and consequently, \( Z(F(S_1)) = F \).

Suppose that \( n \geq 1 \) and \( \alpha \in Z(F(S_n)) \). By (3), \( \alpha \) can be expressed in the form

\[
\alpha = a_1 + a_2\sqrt{p_n} + a_3x_n + a_4\sqrt{p_n}x_n, \quad \text{with } a_1, a_2, a_3, a_4 \in F(S_{n-1}).
\]

From the equality \( \alpha x_n = x_n\alpha \), it follows that

\[
a_1x_n + a_2\sqrt{p_n}x_n + a_3x_n^2 + a_4\sqrt{p_n}x_n^2 = a_1x_n - a_2\sqrt{p_n}x_n + a_3x_n^2 - a_4\sqrt{p_n}x_n^2.
\]

Therefore, \( a_2 + a_4x_n = 0 \) and consequently we have \( a_2 = a_4 = 0 \). Now, from the equality \( \alpha\sqrt{p_n} = \sqrt{p_n}\alpha \), we have \( a_1\sqrt{p_n} - a_3\sqrt{p_n}x_n = a_1\sqrt{p_n} + a_3\sqrt{p_n}x_n \) and it follows that \( a_3 = 0 \). Therefore, \( \alpha = a_1 \in F(S_{n-1}) \) and this means that \( \alpha \in Z(F(S_{n-1})) \).

Thus, we have proved that \( Z(F(S_n)) \subseteq Z(F(S_{n-1})) \). By induction we can conclude that
Step 7. Proving that $R_\infty$ is not algebraic over $F$:

It was shown in Step 4 that $\gamma \in R_\infty$ is not algebraic over $F$. \qed

3 Herstein’s Conjecture for weakly locally finite division rings

Let $K \not\subseteq D$ be division rings. Recall that an element $x \in D$ is radical over $K$ if there exists some positive integer $n(x)$ depending on $x$ such that $x^{n(x)} \in K$. A subset $S$ of $D$ is radical over $K$ if every element from $S$ is radical over $K$. In 1978, I.N. Herstein (cf. [1]) conjectured that given a subnormal subgroup $N$ of $D^*$, if $N$ is radical over center $F$ of $D$, then $N$ is central, i.e., $N$ is contained in $F$. Herstein, himself in the cited above paper proved this fact for the special case, when $N$ is torsion group. However, the problem remains still open in general. In [3], it was proved that this conjecture is true in the finite dimensional case. In this section, we shall prove that this conjecture is also true for weakly locally finite division rings. First, we note the following two lemmas we need for our further purpose.

**Lemma 3.1.** Let $D$ be a division ring with center $F$. If $N$ is a subnormal subgroup of $D^*$, then $Z(N) = N \cap F$.

**Proof.** If $N$ is contained in $F$, then there is nothing to prove. Thus, suppose that $N$ is non-central. By [9, 14.4.2, p. 439], $C_D(N) = F$. Hence $Z(N) \subseteq N \cap F$. Since the inclusion $N \cap F \subseteq Z(N)$ is obvious, $Z(N) = N \cap F$. \qed

**Lemma 3.2.** If $D$ is a weakly locally finite division ring, then $Z(D')$ is a torsion group.

**Proof.** By Lemma 3.1, $Z(D') = D' \cap F$. For any $x \in Z(D')$, there exists some positive integer $n$ and some $a_i, b_i \in D^*, 1 \leq i \leq n$, such that

$$x = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1}.$$ 

Set $S := \{a_i, b_i : 1 \leq i \leq n\}$. Since $D$ is weakly locally finite, the division subring $L$ of $D$ generated by $S$ is centrally finite. Put $n = [L : Z(L)]$. Since $x \in F$, $x$ commutes with every element of $S$. Therefore, $x$ commutes with every element of $L$, and consequently, $x \in Z(L)$. So,

$$x^n = N_{L/Z(L)}(x) = N_{L/Z(L)}(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1}) = 1.$$ 

Thus, $x$ is torsion. \qed
In [4, Theorem 1], Herstein proved that, if in a division ring $D$ every multiplicative commutator $aba^{-1}b^{-1}$ is torsion, then $D$ is commutative. Further, with the assumption that $D$ is a finite dimensional vector space over its center $F$, he proved [4, Theorem 2] that, if every multiplicative commutator in $D$ is radical over $F$, then $D$ is commutative. Now, using Lemma 3.2 we can carry over the last fact for weakly locally finite division rings.

**Theorem 3.3.** Let $D$ be a weakly locally finite division ring with center $F$. If every multiplicative commutator in $D$ is radical over $F$, then $D$ is commutative.

**Proof.** For any $a, b \in D^*$, there exists a positive integer $n = n_{ab}$ depending on $a$ and $b$ such that $(aba^{-1}b^{-1})^n \in F$. Hence, by Lemma 3.2 it follows that $aba^{-1}b^{-1}$ is torsion. Now, by [4, Theorem 1], $D$ is commutative. \qed

The following theorem gives the affirmative answer to Conjecture 3 in [4] for weakly locally finite division rings.

**Theorem 3.4.** Let $D$ be a weakly locally finite division ring with center $F$ and $N$ be a subnormal subgroup of $D^*$. If $N$ is radical over $F$, then $N$ is central, i.e. $N$ is contained in $F$.

**Proof.** Consider the subgroup $N' = [N, N] \subseteq D'$ and suppose that $x \in N'$. Since $N$ is radical over $F$, there exists some positive integer $n$ such that $x^n \in F$. Hence $x^n \in F \cap D' = Z(D')$. By Lemma 3.2, $x^n$ is torsion, and consequently, $x$ is torsion too. Moreover, since $N$ is subnormal in $D^*$, so is $N'$. Hence, by [4, Theorem 8], $N' \subseteq F$. Thus, $N$ is solvable, and by [9, 14.4.4, p. 440], $N \subseteq F$. \qed

In Herstein’s Conjecture a subgroup $N$ is required to be radical over center $F$ of $D$. What happen if $N$ is required to be radical over some proper division subring of $D$ (which not necessarily coincides with $F$)? In the other words, the following question should be interesting: “Let $D$ be a division ring and $K$ be a proper division subring of $D$ and given a subnormal subgroup $N$ of $D^*$. If $N$ is radical over $K$, then is it contained in center $F$ of $D$?” In the following we give the affirmative answer to this question for a weakly locally finite ring $D$ and a normal subgroup $N$.

**Lemma 3.5.** Let $D$ be a weakly locally finite division ring with center $F$ and $N$ be a subnormal subgroup of $D^*$. If for every elements $x, y \in N$, there exists some positive integer $n_{xy}$ such that $x^{n_{xy}}y = yx^{n_{xy}}$, then $N \subseteq F$.

**Proof.** Since $N$ is subnormal in $D^*$, there exists the following series of subgroups

$$N = N_1 \lhd N_2 \lhd \ldots \lhd N_r = D^*.$$
Suppose that \( x, y \in N \). Let \( K \) be the division subring of \( D \) generated by \( x \) and \( y \). Then, \( K \) is centrally finite. By putting \( M_i = K \cap N_i, \forall i \in \{1, \ldots, r\} \) we obtain the following series of subgroups
\[
M_1 \lhd M_2 \lhd \ldots \lhd M_r = K^*.
\]

For any \( a \in M_1 \subseteq N_1 = N \), suppose that \( n_{ax} \) and \( n_{ay} \) are positive integers such that 
\[
a^{n_{ax}}x = xa^{n_{az}} \text{ and } a^{n_{ay}}y = ya^{n_{ay}}. \text{ Then, for } n := n_{ax}n_{ay} \text{ we have } a^n = (a^{n_{ax}})^{n_{ay}} = (xa^{n_{ax}x^{-1}})^{n_{ay}} = xa^{n_{ax}n_{ay}x^{-1}} = xa^n, \text{ and } a^n = (a^{n_{ay}})^{n_{ax}} = (ya^{n_{ay}y^{-1}})^{n_{ax}} = ya^{n_{ax}n_{ay}y^{-1}} = ya^n. \text{ Therefore } a^n \in Z(K). \text{ Hence } M_1 \text{ is radical over } Z(K). \text{ By Theorem 3.4, } M_1 \subseteq Z(K). \text{ In particular, } x \text{ and } y \text{ commute with each other. Consequently, } N \text{ is abelian group. By [9, 14.4.4, p. 440], } N \subseteq F. \]

**Theorem 3.6.** Let \( D \) be a weakly locally finite division ring with center \( F \), \( K \) be a proper division subring of \( D \) and suppose that \( N \) is a normal subgroup of \( D^* \). If \( N \) is radical over \( K \), then \( N \subseteq F \).

**Proof.** Suppose that \( N \) is not contained in the center \( F \). If \( N \setminus K = \emptyset \), then \( N \subseteq K \).

By [9, p. 433], either \( K \subseteq F \) or \( K = D \). Since \( K \neq D \) by the assertion, it follows that \( K \subseteq F \). Hence \( N \subseteq F \), that contradicts to the assertion. Thus, we have \( N \setminus K \neq \emptyset \).

Now, to complete the proof of our theorem we shall show that the elements of \( N \) satisfy the requirements of Lemma 3.5. Thus, suppose that \( a, b \in N \). We examine the following cases:

1) Case 1: \( a \in K \).

- Subcase 1.1: \( b \notin K \).

We shall prove that there exists some positive integer \( n \) such that \( a^n b = ba^n \). Thus, suppose that \( a^n b \neq ba^n \) for any positive integer \( n \). Then, \( a + b \neq 0, a \neq \pm 1 \) and \( b \neq \pm 1 \). So we have 
\[
x = (a + b)a(a + b)^{-1}, \ y = (b + 1)a(b + 1)^{-1} \in N.
\]

Since \( N \) is radical over \( K \), we can find some positive integers \( m_x \) and \( m_y \) such that 
\[
x^{m_x} = (a + b)a^{m_x}(a + b)^{-1}, \ y^{m_y} = (b + 1)a^{m_y}(b + 1)^{-1} \in K.
\]

Putting \( m = m_xm_y \), we have 
\[
x^m = (a + b)a^{m}(a + b)^{-1}, \ y^m = (b + 1)a^{m}(b + 1)^{-1} \in K.
\]

Direct calculations give the equalities 
\[
x^m b - y^m b + x^m a - y^m = x^m(a + b) - y^m(b + 1) = (a + b)a^m - (b + 1)a^m = a^m(a - 1),
\]
from that we get the following equality 
\[
(x^m - y^m)b = a^m(a - 1) + y^m - x^m a.
\]
If \((x^m - y^m) \neq 0\), then \(b = (x^m - y^m)^{-1}[a(a^m - 1) + y^m - x^m a] \in K\), that is a contradiction to the choice of \(b\). Therefore \((x^m - y^m) = 0\) and consequently, \(a^m(a - 1) = y^m(a - 1)\). Since \(a \neq 1\), \(a^m = y^m = (b+1)a^m(b+1)^{-1}\) and it follows that \(a^mb = ba^m\), a contradiction.

- Subcase 1.2: \(b \in K\).

Consider an element \(x \in N \setminus K\). Since \(xb \notin K\), by Subcase 1.1, there exist some positive integers \(r, s\) such that \(a^rxb = xba^r\) and \(a^sx = xa^s\). From these equalities it follows that \(a^{rs} = (xb)^{-1}a^{rs}(xb) = b^{-1}(x^{-1}a^{rs}x)b = b^{-1}a^{rs}b\), and consequently, \(a^{rs}b = ba^{rs}\).

2°) Case 2: \(a \notin K\).

Since \(N\) is radical over \(K\), there exists some positive integer \(m\) such that \(a^m \in K\). By Case 1, there exists some positive integer \(n\) such that \(a^{mn}b = ba^{mn}\). □

4 Some skew linear groups

Let \(D\) be a division ring with center \(F\). In the following we identify \(F^*\) with \(F^*I := \{\alpha I | \alpha \in F^*\}\), where \(I\) denotes the identity matrix in \(GL_n(D)\).

In [7, Theorem 1], it was proved that if \(D\) is centrally finite, then any finitely generated subnormal subgroup of \(D^*\) is central. This result can be carried over for weakly locally finite division rings as the following.

**Theorem 4.1.** Let \(D\) be a weakly locally finite division ring. Then, every finitely generated subnormal subgroup of \(D^*\) is central.

**Proof.** Since \(N\) is finitely generated and \(D\) is weakly locally finite, the division subring generated by \(N\), namely \(L\), is centrally finite. By [7, Theorem 1], \(N \subseteq Z(L)\). Consequently, \(N\) is abelian. Now, by [9, 14.4.4, p. 440], \(N \subseteq Z(D)\). □

The following theorem is a generalization of Theorem 5 in [1].

**Theorem 4.2.** Let \(D\) be a weakly locally finite division ring with center \(F\) and \(N\) be an infinite subnormal subgroup of \(GL_n(D)\), \(n \geq 2\). If \(N\) is finitely generated, then \(N \subseteq F\).

**Proof.** Suppose that \(N\) is non-central. Then, by [8, Theorem 11], \(SL_n(D) \subseteq N\). So, \(N\) is normal in \(GL_n(D)\). Suppose that \(N\) is generated by matrices \(A_1, A_2, ..., A_k\) in \(GL_n(D)\) and \(T\) is the set of all coefficients of all \(A_j\). Since \(D\) is weakly locally finite, the division subring \(L\) generated by \(T\) is centrally finite. It follows that \(N\) is a normal finitely generated subgroup of \(GL_n(L)\). By [1, Theorem 5], \(N \subseteq Z(GL_n(L))\). In particular, \(N\) is abelian and consequently, \(SL_n(D)\) is abelian, a contradiction. □

**Lemma 4.3.** Let \(D\) be a division ring and \(n \geq 1\). Then, \(Z(SL_n(D))\) is a torsion group if and only if \(Z(D')\) is a torsion group.
Proof. The case \( n = 1 \) is clear. So, we can assume that \( n \geq 2 \). Denote by \( F \) the center of \( D \). By [2, §21, Theorem 1, p.140],

\[
Z(SL_n(D)) = \{ dI | d \in F^* \text{ and } d^n \in D' \}.
\]

If \( Z(SL_n(D)) \) is a torsion group, then, for any \( d \in Z(D') = D' \cap F, dI \in Z(SL_n(D)) \). It follows that \( d \) is torsion. Conversely, if \( Z(D') \) is a torsion group, then, for any \( A \in Z(SL_n(D)) \), \( A = dI \) for some \( d \in F^* \) such that \( d^n \in D' \). It follows that \( d^n \) is torsion. Therefore, \( A \) is torsion.

**Theorem 4.4.** Let \( D \) be a non-commutative algebraic, weakly locally finite division ring with center \( F \) and \( N \) be a subgroup of \( GL_n(D) \) containing \( F^*, n \geq 1 \). Then \( N \) is not finitely generated.

Proof. Recall that if a division ring \( D \) is weakly locally finite, then \( Z(D') \) is a torsion group (see Lemma 3.2). Therefore, by Lemma 4.3 \( Z(SL_n(D)) \) is a torsion group.

Suppose that there is a finitely generated subgroup \( N \) of \( GL_n(D) \) containing \( F^* \). Clearly \( N/N' \) is a finitely generated abelian group, where \( N' \) denotes the derived subgroup of \( N \). Then, in virtue of [9, 5.5.8, p. 113], \( F^*N'/N' \) is a finitely generated abelian group.

**Case 1:** \( \text{char}(D) = 0 \).

Then, \( F \) contains the field \( \mathbb{Q} \) of rational numbers and it follows that \( \mathbb{Q}^*I/(\mathbb{Q}^*I \cap N') \cong \mathbb{Q}^*N'/N' \). Since \( F^*N'/N' \) is finitely generated abelian subgroup, \( \mathbb{Q}^*N'/N' \) is finitely generated too, and consequently \( \mathbb{Q}^*I/(\mathbb{Q}^*I \cap N') \) is finitely generated. Consider an arbitrary \( A \in \mathbb{Q}^*I \cap N' \). Then \( A \in F^*I \cap SL_n(D) \subseteq Z(SL_n(D)) \). Therefore \( A \) is torsion. Since \( A \in \mathbb{Q}^*I \), we have \( A = dI \) for some \( d \in \mathbb{Q}^* \). It follows that \( d = \pm 1 \). Thus, \( \mathbb{Q}^*I \cap N' \) is finite. Since \( \mathbb{Q}^*I/(\mathbb{Q}^*I \cap N') \) is finitely generated, \( \mathbb{Q}^*I \) is finitely generated. Therefore \( \mathbb{Q}^* \) is finitely generated, that is impossible.

**Case 2:** \( \text{char}(D) = p > 0 \).

Denote by \( \mathbb{F}_p \) the prime subfield of \( F \), we shall prove that \( F \) is algebraic over \( \mathbb{F}_p \). In fact, suppose that \( u \in F \) and \( u \) is transcendental over \( \mathbb{F}_p \). Put \( K := \mathbb{F}_p(u) \), then the group \( K^*/(K^* \cap N') \) considered as a subgroup of \( F^*N'/N' \) is finitely generated. Considering an arbitrary \( A \in K^*I \cap N' \), we have \( A = (f(u)/g(u))I \) for some \( f(X), g(X) \in \mathbb{F}_p[X], ((f(X), g(X)) = 1 \) and \( g(u) \neq 0 \). As mentioned above, we have \( f(u)^s/g(u)^s = 1 \) for some positive integer \( s \). Since \( u \) is transcendental over \( \mathbb{F}_p \), \( f(u)/g(u) \in \mathbb{F}_p \). Therefore, \( K^*I \cap N' \) is finite and consequently, \( K^*I \) is finitely generated. It follows that \( K^* \) is finitely generated, hence \( K \) is finite. Hence \( F \) is algebraic over \( \mathbb{F}_p \) and it follows that \( D \) is algebraic over \( \mathbb{F}_p \). Now, in virtue of Jacobson’s Theorem [6, (13.11), p. 219], \( D \) is commutative, a contradiction.

\[\square\]
Corollary 4.5. Let $D$ be an algebraic, weakly locally finite division ring. If the group $GL_n(D), n \geq 1,$ is finitely generated, then $D$ is commutative.

If $M$ is a maximal finitely generated subgroup of $GL_n(D)$, then $GL_n(D)$ is finitely generated. So, the next result follows immediately from Corollary 4.5.

Corollary 4.6. Let $D$ be an algebraic, weakly locally finite division ring. If the group $GL_n(D), n \geq 1,$ has a maximal finitely generated subgroup, then $D$ is commutative.

By the same way as in the proof of Theorem 4.4, we obtain the following corollary.

Corollary 4.7. Let $D$ be a non-commutative algebraic, weakly locally finite division ring with center $F$ and $S$ is a subgroup of $GL_n(D)$. If $N = F^* S$, then $N/N'$ is not finitely generated.

Proof. Suppose that $N/N'$ is finitely generated. Since $N' = S'$ and $F^* I/(F^* I \cap S') \simeq F^* S'/S', F^* I/(F^* I \cap S')$ is a finitely generated abelian group. Now, by the same arguments as in the proof of Theorem 4.4, we conclude that $D$ is commutative. 

Corollary 4.8. Let $D$ be a non-commutative algebraic, weakly locally finite division ring. Then, $D^*$ is not finitely generated.

Proof. Take $N = S = GL_n(D)$ in Corollary 4.7 and have in mind that $[GL_n(D), GL_n(D)] = SL_n(D)$, we have $D^* \simeq GL_n(D)/SL_n(D)$ is not finitely generated.

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