Integrable system with peakon, weak kink, and kink-peakon interactional solutions

Zhijun Qiao\textsuperscript{1,*}, Baoqiang Xia\textsuperscript{1,2†}, Jibin Li\textsuperscript{3‡}
\textsuperscript{1}Department of Mathematics, University of Texas-Pan American, Edinburg, Texas 78541, USA
\textsuperscript{2}School of Mathematical Sciences, Jiangsu Normal University, Xuzhou, Jiangsu 221116, P. R. China
\textsuperscript{3}Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, P. R. China

Abstract

In this paper, we study an integrable system with both quadratic and cubic nonlinearity:
\begin{align*}
m_t &= bu_x + \frac{1}{2} k_1 [m(u^2 - u_x^2)]_x + \frac{1}{2} k_2 (2mu_x + m_x u), \quad m = u - u_{xx} , \text{ where } b, k_1 \text{ and } k_2 \\
&\text{are arbitrary constants. This model is kind of a cubic generalization of the Camassa-Holm (CH) equation: } m_t + m_x u + 2mu_x = 0. \text{ The equation is shown integrable with its Lax pair, bi-Hamiltonian structure, and infinitely many conservation laws. In the case of } b = 0, \text{ the peaked soliton (peakon) and multi-peakon solutions are studied. In particular, the two-peakon dynamical system is explicitly presented and their collisions are investigated in details. In the case of } b \neq 0 \text{ and } k_2 = 0, \text{ the weak kink and kink-peakon interactional solutions are found. Significant difference from the CH equation is analyzed through a comparison.}
\end{align*}
In the paper, we also study all possible smooth one-soliton solutions for the system.

Keywords: Generalization of Camassa-Holm equation, Peakon, Weak kink, Lax pair, Integrable system.

PACS: 02.30.Ik, 04.20.Jb.

\textsuperscript{*}E-mail address: qiao@utpa.edu
\textsuperscript{†}E-mail address: xiabaoqiang@126.com
\textsuperscript{‡}E-mail address: lijb@zjnu.cn
1 Introduction

The Camassa-Holm (CH) equation

\[ m_t - bu_x + 2mu_x + m_xu = 0, \quad m = u - u_{xx}, \]  

(1)

was derived by Camassa and Holm [1] as a shallow water wave model. In recent years, this equation has attracted much attention in soliton theory. In the literature, this equation was implicitly implied in the paper of Fuchssteiner and Fokas on hereditary symmetries as a very special case [2]. However, since the work of Camassa and Holm [1], various studies on this equation have been remarkably developed. The CH equation possesses many important integrable properties. For instance, it admits Lax representation, bi-Hamiltonian structures, multi-soliton solutions, and algebro-geometric solutions [3]-[8]. Also, it is integrable by the inverse scattering transformation [11]. The most remarkable feature of the CH equation (1) is to admit peaked soliton (peakon) solutions in the case of \( b = 0 \). A peakon is a weak solution in some Sobolev space with corner at its crest. The stability and interaction of peakons were discussed in several references [15]-[18]. In addition to the CH equation being an integrable model with peakon solutions, other integrable peakon models have been found. Those models include the Degasperis-Procesi equation [19]-[22] and the cubic nonlinear peakon equations [23]-[26].

The present paper focuses on the following equation with both quadratic and cubic nonlinearity:

\[ m_t = bu_x + \frac{1}{2}k_1 \left[ m(u^2 - u_x^2) \right]_x + \frac{1}{2}k_2(2mu_x + m_xu), \quad m = u - u_{xx}, \]  

(2)

where \( b, k_1 \) and \( k_2 \) are three arbitrary constants. It is clear that equation (2) is reduced to the CH equation (1) when we take \( k_1 = 0, k_2 = -2 \). For \( k_1 = -2, k_2 = 0 \), equation (2) is exactly the cubic nonlinear equation:

\[ m_t - bu_x + \left[ m(u^2 - u_x^2) \right]_x = 0, \quad m = u - u_{xx}, \]  

(3)

which was derived independently by Fokas [27], by Fuchssteiner [28], by Olver and Rosenau [4], and by Qiao [23, 29] where the equation was derived from the two-dimension Euler system, and Lax pair, the M/W-shape solitons and cuspon solutions were presented. Recently, Gui, Liu, Olver and Qu worked out the peakon stability for the cubic nonlinear equation (3) in the case of \( b = 0 \) [30].

Equation (2) is actually a linear combination of CH equation (1) and cubic nonlinear equation (3). Therefore, we may view equation (2) as a generalization of the CH equation, or simply call equation (2) a generalized CH equation. This structure is very similar to the one in dealing with the Gardner equation, known as a linear combination of KdV and mKdV equations,
which has important applications in various areas of physics [31, 32]. In fact, the structure of Gardner equation is our mainly starting point to study equation (2). We also notice that by some appropriate rescaling, equation (2) might implicitly be implied in the paper of Fokas and Fuchssteiner in the context of hereditary symmetries [27, 28].

In this paper, we first present the Lax representation, bi-Hamiltonian structure and infinitely many conservation laws for equation (2). This indicates that equation (2) is completely integrable. Then we show that this equation possesses the single peakon of traveling wave type as well as multi-peakon solutions in the case of \( b = 0 \). In particular, for the one-peakon case, we find that the complex peakon (i.e. peakon with complex coefficient) appears by properly choosing the parameters \( k_1 \) and \( k_2 \). For the two-peakon case, the peakon dynamical system is explicitly presented and their collisions are discussed in details. Significant difference between the CH equation (1) and the generalized CH equation (2) (for the case of \( b = 0 \)) is analyzed through a comparison.

Another purpose of this paper is to show that equation (2) with \( k_2 = 0 \) (namely, cubic nonlinear equation (3)) allows the weak kink solution in the case of \( b \neq 0 \). Different from the multi-peakon solutions in the form of linear superpositions of the single-peakon, equation (2) with \( k_2 = 0 \) and \( b \neq 0 \) does not allow the multi-kink solution in the form of the superpositions of single-kink. However, we find that equation (2) with \( k_2 = 0 \) and \( b \neq 0 \) allows the solutions in the form of the superpositions of single-kink and multi-peakon. In particular, the weak kink and kink-peakon interactional solutions are shown and plotted. Within our knowledge, this is probably the first time discussing the weak kink and kink-peakon interactional solutions. In the paper, we also study all possible smooth one-soliton solutions.

2 Lax pair, bi-Hamiltonian structure and conservation laws

Let us consider the following pair of linear spectral problems

\[
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}_x = U \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}, \quad U = \frac{1}{2} \begin{pmatrix}
-\alpha & \lambda m \\
-k_1 \lambda m - k_2 \lambda & \alpha
\end{pmatrix},
\]

(4)

\[
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}_t = V \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}, \quad V = \frac{1}{2} \begin{pmatrix}
A & B \\
C & -A
\end{pmatrix},
\]

(5)
where
\[
\alpha = \sqrt{1 - \lambda^2 b},
\]
\[
A = \lambda^2 \alpha + \frac{\alpha}{2} k_1 (u^2 - u_x^2) + \frac{1}{2} k_2 (\alpha u - u_x),
\]
\[
B = -\lambda^{-1} (u - \alpha u_x) - \frac{1}{2} \lambda m \left[ k_1 (u^2 - u_x^2) + k_2 u \right],
\]
\[
C = \lambda^{-1} \left[ k_1 (u + \alpha u_x) + k_2 \right] + \frac{1}{2} \lambda \left[ k_1^2 m (u^2 - u_x^2) + k_1 k_2 (m u + u^2 - u_x^2) + k_2^2 u \right].
\]

One can easily see that the compatibility condition of (4) and (5) is
\[
U_t - V_x + [U, V] = 0.
\]

Substituting the expressions of \( U \) and \( V \) into (7), we find that (7) is nothing but equation (2), namely, (4) and (5) are the Lax pair of (2).

A direct computation shows that equation (2) has the following bi-Hamiltonian structure
\[
m_t = J \frac{\delta H_1}{\delta m} = K \frac{\delta H_2}{\delta m},
\]

where
\[
J = k_1 \partial m \partial^{-1} m \partial + \frac{1}{2} k_2 (\partial m + m \partial) + b \partial, \quad H_1 = \frac{1}{2} \int_{-\infty}^{+\infty} (u^2 + u_x^2) dx,
\]
\[
K = \partial - \partial^3, \quad H_2 = \frac{1}{8} \int_{-\infty}^{+\infty} \left( k_1 u^4 + 2 k_1 u^2 u_x^2 - \frac{1}{3} k_1 u_x^4 + 2 k_2 u^3 + 2 k_2 u u_x^2 + 4 bu^2 \right) dx.
\]

Let us now construct conservation laws of equation (2). Let \( \omega = \frac{\phi_2}{\phi_1} \), then \( \omega \) satisfies the following Riccati equation
\[
\omega_x = -\frac{1}{2} \lambda (k_1 m + k_2) + \alpha \omega - \frac{1}{2} \lambda m \omega^2.
\]

Based on (4) and (5), we obtain
\[
(\ln \phi_1)_x = -\frac{\alpha}{2} + \frac{1}{2} \lambda m \omega, \quad (\ln \phi_1)_t = -\frac{1}{2} A - \frac{1}{2} B \omega,
\]

which yields the conservation law of equation (2):
\[
\rho_t = F_x,
\]

where
\[
\rho = m \omega,
\]
\[
F = \frac{1}{2} m \left[ k_1 (u^2 - u_x^2) + k_2 u \right] \omega - \frac{1}{2} \left[ \alpha k_1 (u^2 - u_x^2) + k_2 (\alpha u - u_x) \right] \lambda^{-1} + (u - \alpha u_x) \omega \lambda^{-2}.
\]
Usually $\rho$ and $F$ are called a conserved density and an associated flux, respectively. Next, we derive the explicit forms of conservation densities in the case of $b = 0$. In this case, equation (11) becomes

$$\omega_x = -\frac{1}{2}\lambda(k_1m + k_2) + \omega - \frac{1}{2}\lambda m \omega^2.$$  

(15)

We expand $\omega$ in terms of negative powers of $\lambda$ as below:

$$\omega = \sum_{j=0}^{\infty} \omega_j \lambda^{-j}. \quad (16)$$

Substituting (16) into (15) and equating the coefficients of powers of $\lambda$, we arrive at

$$\omega_0 = \frac{1}{m} \sqrt{-k_1 m^2 - k_2 m}, \quad \omega_1 = \frac{2k_1 m^2 + k_2 (2m + m_x)}{2k_1 m^2 + 2k_2 m^2},$$  

$$\omega_{j+1} = \frac{1}{m \omega_0} \left[ \omega_j - \omega_{j,x} - \frac{1}{2} m \sum_{i+k=j+1, \ i,k \geq 1} \omega_i \omega_k \right], \quad j \geq 1. \quad (17)$$

Inserting (16) and (17) into (14), we finally get the following infinitely many conserved densities and the associated fluxes

$$\rho_0 = \sqrt{-k_1 m^2 - k_2 m}, \quad F_0 = \frac{1}{2} \left[ k_1 (u^2 - u_x^2) + k_2 u \right] \sqrt{-k_1 m^2 - k_2 m},$$  

$$\rho_1 = \frac{2k_1 m^2 + k_2 (2m + m_x)}{2k_1 m^2 + 2k_2 m^2}, \quad F_1 = -\frac{1}{2} \left[ k_1 (u^2 - u_x^2) + k_2 (u - u_x) \right] + \frac{1}{2} \left[ k_1 (u^2 - u_x^2) + k_2 u \right] \rho_1,$$  

$$\rho_{j+1} = \frac{1}{\omega_0} \left( \omega_j - \omega_{j,x} - \frac{1}{2} m \sum_{i+k=j+1, \ i,k \geq 1} \omega_i \omega_k \right), \quad j \geq 1,$$  

$$F_{j+1} = (u - u_x) \omega_{j-1} + \frac{1}{2} \left[ k_1 (u^2 - u_x^2) + k_2 u \right] \rho_{j+1}, \quad j \geq 1. \quad (18)$$

We can also consider the expansions of $\omega$ in the positive powers of $\lambda$:

$$\omega = \sum_{j=0}^{\infty} \omega_j \lambda^j. \quad (19)$$

Substituting (19) into equation (15) and comparing powers of $\lambda$ lead to

$$\omega_{2j} = 0, \quad j \geq 0,$$  

$$\omega_1 = \frac{1}{2} [k_1 (u + u_x) + k_2], \quad \omega_{2j+1,x} - \omega_{2j+1} = -\frac{1}{2} m \sum_{i+k=2j, \ i,k \geq 0} \omega_i \omega_k, \quad j \geq 1. \quad (20)$$

Equation (20) shows that one may solve a first-order differential equation to obtain nontrivial $\omega_{2j+1}, j \geq 1$ in this sequence. For brevity, we omit the result since this may involve in nonlocal expressions in $u$. 


3 Peakon solutions in the case of $b = 0$

Applying the operator $(1 - \partial_x^2)^{-1}$ to equation (2), we obtain

$$u_t = \frac{1}{6}k_1(3u^2 u_x - u_x^3) + \frac{1}{2}k_2 uu_x + \frac{1}{6}k_1(1 - \partial_x^2)^{-1}u_x^3$$

$$+ \frac{1}{2}\partial_x(1 - \partial_x^2)^{-1} \left( 2bu + k_1(uu_x^2 + \frac{2}{3}u^3) + k_2(u^2 + \frac{1}{2}u_x^2) \right).$$

Taking the convolution with the Green’s function $G(x) = \frac{1}{2} \exp(-|x|)$ for the Helmholtz operator $(1 - \partial_x^2)$, equation (21) can be rewritten as

$$u_t = \frac{1}{6}k_1(3u^2 u_x - u_x^3) + \frac{1}{2}k_2 uu_x + \frac{1}{6}k_1 G(x) * u_x^3$$

$$+ \frac{1}{2}\partial_x \left( G(x) * [2bu + k_1(uu_x^2 + \frac{2}{3}u^3) + k_2(u^2 + \frac{1}{2}u_x^2)] \right).$$

In this section, we derive the single and multi-peakon solutions of equation (2) with $b = 0$.

3.1 Single-peakon solutions

Let us suppose the single-peakon solution of equation (2) with $b = 0$ in the form of

$$u = Ce^{-|x-ct|},$$

where $C$ is to be determined. The derivatives of expression (23) do not exist at $x = ct$, thus (23) cannot satisfy equation (2) with $b = 0$ in the classical sense. However, in the weak sense, we can write out the expressions of $u_x$, $u_t$ and $m$ with help of distribution:

$$u_x = -C \text{sgn}(x - ct)e^{-|x-ct|}, \quad u_t = ec \text{sgn}(x - ct)e^{-|x-ct|}, \quad m = 2c\delta(x - ct).$$

Substituting (23) and (24) into the weak form (22) with $b = 0$, we are able to find that $C$ should satisfy

$$\frac{1}{3}k_1C^2 + \frac{1}{2}k_2C + c = 0.$$  \hspace{1cm} (25)

For $k_1 = 0, k_2 \neq 0$, we obtain $C = \frac{2c}{k_2}$. In particular, when $k_2 = -2$, (23) is exactly the single-peakon solution $u = ce^{-|x-ct|}$ of the CH equation (1) with $b = 0$. For $k_1 = -2$ and $k_2 = 0$, we recover the single-peakon solution $u = \pm \sqrt{\frac{3c}{2}}e^{-|x-ct|}$ of the cubic nonlinear CH equation (3) with $b = 0$ [30]. In general, for $k_1 \neq 0$, we may obtain

$$C = \frac{-3 \left( \sqrt{3k_2} \pm \sqrt{3k_2^2 - 16k_1c} \right)}{4\sqrt{3k_1}}.$$  \hspace{1cm} (26)
If $3k_2^2 - 16k_1c \geq 0$, then $C$ is a real number. For example, we choose $k_1 = k_2 = -2$ and $c > 0$, then the corresponding single-peakon solution is $u = -\frac{3 \pm \sqrt{9 + 24c}}{4} e^{-|x-ct|}$. If $3k_2^2 - 16k_1c < 0$, then $C$ is a complex number. This means the peakon solution with complex coefficient is obtained. For example, we set $k_2 = -2$ and $k_1 = \frac{1}{c}$, then we have $u = \frac{(3 \pm \sqrt{5})i}{2} e^{-|x-ct|}$.

### 3.2 Two-peakon solutions and their dynamics

Let us assume the two-peakon solution to equation (2) with $b = 0$ has the following form

$$u = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|}. \quad (27)$$

The expression of $u$ has two peaks (or troughs) at positions $x = q_1(t)$ and $x = q_2(t)$. $u_x$, $u_t$ and $m$ have the following weak forms:

$$u_x = -p_1 \text{sgn}(x - q_1)e^{-|x-q_1|} - p_2 \text{sgn}(x - q_2)e^{-|x-q_2|},$$

$$u_t = p_1,t e^{-|x-q_1|} + p_2,t e^{-|x-q_2|} + p_1 q_1,t \text{sgn}(x - q_1)e^{-|x-q_1|} + p_2 q_2,t \text{sgn}(x - q_2)e^{-|x-q_2|}, \quad (28)$$

$$m = 2 p_1 \delta(x - q_1) + 2 p_2 \delta(x - q_2).$$

Substituting (27) and (28) into (22) with $b = 0$, we may obtain the following peakon dynamical system

$$\begin{cases}
    p_{1,t} = -\frac{1}{2} k_2 p_1 p_2 \text{sgn}(q_1 - q_2)e^{-|q_1 - q_2|}, \\
    p_{2,t} = -\frac{1}{2} k_2 p_1 p_2 \text{sgn}(q_2 - q_1)e^{-|q_2 - q_1|}, \\
    q_{1,t} = -k_1 p_1 p_2 e^{-|q_1 - q_2|} - \frac{1}{2} k_1 p_1^2 - \frac{1}{2} k_2 (p_1 + p_2 e^{-|q_1 - q_2|}), \\
    q_{2,t} = -k_1 p_1 p_2 e^{-|q_1 - q_2|} - \frac{1}{2} k_1 p_2^2 - \frac{1}{2} k_2 (p_2 + p_1 e^{-|q_1 - q_2|}).
\end{cases} \quad (29)$$

For $k_1 = 0$, $k_2 = -2$, (2) recovers the CH equation and (29) is reduced to the two-peakon dynamics of CH equation with $b = 0$, which Camassa and Holm studied in [1]. In particular, the peakon-antipeakon was taken as below

$$u = p_1(t) \left( e^{-|x-q_1(t)|} - e^{-|x+q_1(t)|} \right),$$

where

$$p_1(t) = c \coth(ct) = -p_2(t), \quad q_1(t) = -\ln \frac{2}{e^{ct} + e^{-ct}} = -q_2(t). \quad (30)$$

Camassa and Holm [1] also pointed out that the collision of the peakon-antipeakon occurs at the moment $t = 0$. At the moment of collision, the amplitudes $p_1$ and $p_2$ become infinite, but the solution $u$ converges to zero [1] [33].

For $k_2 = 0$ and $b = 0$, (2) is exactly the cubic nonlinear equation [1] [23] [27] [28]

$$m_t = \frac{1}{2} k_1 \left[ m(u^2 - u_x^2) \right]_x, \quad m = u - u_{xx}, \quad (31)$$
and the peakon dynamical system (29) is simplified to

\[
\begin{align*}
    p_{1,t} &= 0, \quad p_{2,t} = 0, \\
    q_{1,t} &= -k_1 p_1 p_2 e^{-|q_1 - q_2|} - \frac{1}{3} k_1 p_1^2, \\
    q_{2,t} &= -k_1 p_1 p_2 e^{-|q_1 - q_2|} - \frac{1}{3} k_1 p_2^2.
\end{align*}
\]  

(32)

From the first two equations of (32), we obtain \( p_1(t) = c_1, \) \( p_2(t) = c_2, \) where \( c_1, \) \( c_2 \) are two arbitrary constants. For the case of \( c_1^2 = c_2^2, \) the solutions of \( q_1(t) \) and \( q_2(t) \) in (32) are given by

\[
\begin{align*}
    q_1(t) &= -k_1 (c_1 c_2 e^{-|A_1 - A_2|} + \frac{1}{3} c_1^2) t + A_1, \\
    q_2(t) &= -k_1 (c_1 c_2 e^{-|A_1 - A_2|} + \frac{1}{3} c_2^2) t + A_2,
\end{align*}
\]  

(33)

where \( A_1, A_2 \) are arbitrary integral constants. If \( A_1 = A_2, \) the two-peakon solution is reduced to zero (when \( c_2 = -c_1 \)) or a single-peakon (when \( c_2 = c_1 \)). If \( A_1 \neq A_2, \) the two-peakon can not collide since \( q_1(t) \neq q_2(t) \) for every \( t \in (-\infty, +\infty). \) Especially, in the peakon-antipeakon case \( (c_1 = -c_2), \) the collision can’t occur for \( A_1 \neq A_2. \) This is very different from the case of the CH equation \( [1]. \) For the case of \( c_1^2 \neq c_2^2, \) we obtain the following solution:

\[
\begin{align*}
    p_1(t) &= c_1, \quad p_2(t) = c_2, \\
    q_1(t) &= \text{sgn}(t) \frac{3 k_1 c_1 c_2}{|k_1 (c_1^2 - c_2^2)|} \left( e^{-\frac{1}{3} k_1 (c_1^2 - c_2^2) t} - 1 \right) - \frac{1}{3} k_1 c_1^2 t, \\
    q_2(t) &= \text{sgn}(t) \frac{3 k_1 c_1 c_2}{|k_1 (c_1^2 - c_2^2)|} \left( e^{-\frac{1}{3} k_1 (c_1^2 - c_2^2) t} - 1 \right) - \frac{1}{3} k_1 c_2^2 t.
\end{align*}
\]  

(34)

The two-peakon collision occurs at the moment \( t = 0, \) since \( q_1(0) = q_2(0) = 0. \) Without loss of generality, let us suppose \( c_1^2 < c_2^2. \) From the formula (34), we know that for \( t < 0, \) the tall and fast peakon (with the amplitude \( |c_2| \) and peak position \( q_2 \)) chases after the short and slow peakon (with the amplitude \( |c_1| \) and peak position \( q_1 \)). At the moment of \( t = 0, \) the two-peakon collides and overlaps into the peakon: \( u(x, 0) = (c_1 + c_2) e^{-|x|}. \) After the collision \( (t > 0), \) the two-peakon departs, and the tall and fast peakon surpasses the short and slow one. See Figures 1 and 2 for the developments of this kind of two-peakon.

**Remark 1.** Our results show that the collision of two-peakon of equation (33) is very different from the case of CH equation. In the peakon-antipeakon case, the collision can occur for the CH equation, but could not happen for equation (31). The collision of two-peakon of equation (33) occurs in the “chase” case as shown above (see Figures 1 and 2).

Next, we want to derive the solutions of (29) for the general case \( k_1 \neq 0 \) and \( k_2 \neq 0. \) To do so, let us introduce the transformations \( p(t) = p_1(t) + p_2(t), \) \( q(t) = q_1(t) + q_2(t), \) \( P(t) = p_1(t) - p_2(t), \)
These yield \( Q(t) = q_1(t) - q_2(t) \). Then (29) is transformed to

\[
\begin{align*}
p_t &= 0, \\
q_t &= -\frac{1}{4}k_1(p^2 - P^2)e^{-|Q|} - \frac{1}{4}k_1(p^2 + P^2) - \frac{1}{2}k_2p(1 + e^{-|Q|}), \\
P_t &= -\frac{1}{2}k_2(p^2 - P^2)\text{sgn}(Q)e^{-|Q|}, \\
Q_t &= \frac{1}{3}k_2P(e^{-|Q|} - 1) - \frac{1}{3}k_1pP.
\end{align*}
\]

From the first equation of (35), we know that \( p(t) = A_1 \), where \( A_1 \) is an arbitrary constant. Let \( \Gamma = 1 + \frac{2k_1}{3k_2}A_1 \), then we have the following results.

**Case 1.** If \( 0 < \Gamma \leq 1 \), then (35) admits the following solution

\[
P(t) = \pm a_2 \frac{1 + \tilde{A}_3e^{Bt}}{1 - \tilde{A}_3e^{Bt}}, \quad Q(t) = \pm \ln \frac{4\Gamma a_2^2\tilde{A}_3e^{Bt}}{a_2^2(1 + \tilde{A}_3e^{Bt})^2 - A_1^2(1 - \tilde{A}_3e^{Bt})^2},
\]

\[
q(t) = -\ln \left| \frac{\tilde{A}_3e^{Bt} - \frac{A_1 - a_2}{A_1 + a_2}}{\tilde{A}_3e^{Bt} - \frac{A_1 - a_2}{A_1 + a_2}} \right| - \frac{2k_1a_2^2(3\Gamma - 1)}{3B(\tilde{A}_3e^{Bt} - 1)} - \frac{1}{2}[k_2A_1 + \frac{1}{3}k_1(A_1^2 + a_2^2)]t + D,
\]

where \( a_2 > |A_1| \), \( \tilde{A}_3 > 0 \), \( B = -\frac{1}{2}a_2k_2\Gamma \), and \( D \) is an arbitrary constant.

**Case 2.** If \( \Gamma > 1 \), then (35) has the following solution

\[
P(t) = \pm a_2 \frac{1 - \tilde{A}_3e^{Bt}}{1 + \tilde{A}_3e^{Bt}}, \quad Q(t) = \pm \ln \frac{-4\Gamma a_2^2\tilde{A}_3e^{Bt}}{a_2^2(1 + \tilde{A}_3e^{Bt})^2 - A_1^2(1 - \tilde{A}_3e^{Bt})^2},
\]

\[
q(t) = -\ln \left| \frac{\tilde{A}_3e^{Bt} + \frac{A_1 + a_2}{A_1 - a_2}}{\tilde{A}_3e^{Bt} + \frac{A_1 + a_2}{A_1 - a_2}} \right| + \frac{2k_1a_2^2(3\Gamma - 1)}{3B(\tilde{A}_3e^{Bt} + 1)} - \frac{1}{2}[k_2A_1 + \frac{1}{3}k_1(A_1^2 + a_2^2)]t + D,
\]

where \( 0 < a_2 < |A_1| \), \( \tilde{A}_3 > 0 \), \( B = -\frac{1}{2}a_2k_2\Gamma \), and \( D \) is an arbitrary constant.

Let us now give special two-peakon solutions based on Case 1 and Case 2.

**For Case 1,** let \( A_1 = 0, a_2 = 2, \tilde{A}_3 = 1, k_1 = k_2 = -2, \) and \( D = 0 \), then we have

\[
p(t) = 0, \quad P(t) = 2\coth(t), \quad q(t) = \frac{\coth(t)}{3(e^{2t} - 1)} + \frac{4}{3}t, \quad Q(t) = -\ln \frac{4e^{2t}}{(1 + e^{2t})^2}.
\]

These yield

\[
p_1(t) = \coth(t), \quad q_1(t) = \frac{8}{3(e^{2t} - 1)} + \ln(e^{2t} + 1) - \frac{1}{3}t - \ln 2, \\
p_2(t) = -\coth(t), \quad q_2(t) = \frac{8}{3(e^{2t} - 1)} - \ln(e^{2t} + 1) + \frac{5}{3}t + \ln 2.
\]

Therefore, we obtain the following peakon-antipeakon solution

\[
u(x, t) = \coth(t) \left( e^{-|x - q_1(t)|} - e^{-|x - q_2(t)|} \right),
\]

where \( x \) is an arbitrary constant.
where $q_1(t)$ and $q_2(t)$ are shown in (39). From (38), one can easily know that the collision occurs at the moment $t = 0$ since $Q(0) = 0$. From (39), we may compute

$$\lim_{t \to 0} p_1(t) = - \lim_{t \to 0} p_2(t) = \infty, \quad \lim_{t \to 0} q_1(t) = \lim_{t \to 0} q_2(t) = \infty.$$  \hspace{1cm} (41)

But from (40), we may infer that

$$\lim_{t \to 0} u(x, t) = 0, \quad \text{for every } x \in \mathbb{R},$$  \hspace{1cm} (42)

which indicates that the peakon and the antipeakon vanish when they overlap. Guided by the above results, we may describe the dynamics of peakon-antipeakon solution (40) as follows. For $t < 0$, the peak is at $q_2(t)$ and the trough is at $q_1(t)$. The peak and the trough approach each other as $t$ goes to 0. At the moment of $t = 0$, the peakon and the antipeakon collide and vanish. After their collision ($t > 0$), they depart and redevelop with the trough at $q_2(t)$ and the peak at $q_1(t)$. Figure 3 shows the peakon-antipeakon dynamics.

Remark 2. The amplitudes $p_1(t)$, $p_2(t)$ in formula (39) are the same as those of the CH equation [1], but the peak positions $q_1(t)$, $q_2(t)$ are different (see formulas (30) and (39)). In the CH equation, only $p_1(t)$ and $p_2(t)$ become infinite at the instant of collision [1, 3]. In the new equation (2) with $b = 0$, not only $p_1(t)$, $p_2(t)$ but also $q_1(t)$ and $q_2(t)$ become infinite at the instant of collision (see formula (41)). However, in both case, the peakon-antipeakon vanishes when the overlap occurs (see formula (42)).

For Case 2, choosing $k_1 = k_2 = -2$, $A_1 = 3$, $A_3 = 1$, $\Gamma = 3$, $a_2 = \sqrt{3}$, $B = 3\sqrt{3}$ forces the first two equations in (37) to

$$P(t) = \sqrt{3} \text{sgn}(t) \frac{1 - e^{3\sqrt{3}t}}{1 + e^{3\sqrt{3}t}}, \quad Q(t) = \text{sgn}(t) \ln \frac{6e^{3\sqrt{3}t}}{e^{6\sqrt{3}t} + 4e^{3\sqrt{3}t} + 1}. \hspace{1cm} (43)$$

Substituting (43) into the second equation of (35), we may obtain the expression of $q(t)$ and finally arrive at the solution

$$p_1(t) = \frac{3}{2} - \frac{\sqrt{3}}{2} \text{sgn}(t) \tanh\left(\frac{3\sqrt{3}t}{2}\right), \quad p_2(t) = \frac{3}{2} + \frac{\sqrt{3}}{2} \text{sgn}(t) \tanh\left(\frac{3\sqrt{3}t}{2}\right),$$

$$q_1(t) = \frac{1}{2} \text{sgn}(t) \ln \frac{6e^{3\sqrt{3}t}}{e^{6\sqrt{3}t} + 4e^{3\sqrt{3}t} + 1} - \frac{1}{2} \ln \frac{e^{3\sqrt{3}t} + 2 + \sqrt{3}}{e^{3\sqrt{3}t} + 2 - \sqrt{3}} - \frac{16\sqrt{3}}{9(e^{3\sqrt{3}t} + 1)} + \frac{7}{2} t,$$

$$q_2(t) = -\frac{1}{2} \text{sgn}(t) \ln \frac{6e^{3\sqrt{3}t}}{e^{6\sqrt{3}t} + 4e^{3\sqrt{3}t} + 1} - \frac{1}{2} \ln \frac{e^{3\sqrt{3}t} + 2 + \sqrt{3}}{e^{3\sqrt{3}t} + 2 - \sqrt{3}} - \frac{16\sqrt{3}}{9(e^{3\sqrt{3}t} + 1)} + \frac{7}{2} t.$$  \hspace{1cm} (44)

The formulas (43) show that the collision happens at the moment $t = 0$ since $Q(0) = 0$. From (44), we know that the two-peakon collides and overlaps into the peakon $u(x, 0) = 3e^{-|x+\frac{1}{2}\ln(2+\sqrt{3})+\frac{a_2}{2}|}$ at the moment $t = 0$. After the collision, the two-peakon departs and redevelops. See Figure 4 for the profile of the two-peakon dynamics.
3.3 N-peakon dynamical systems

In general, we suppose that N-peakon has the following form

$$u(x, t) = \sum_{j=1}^{N} p_j(t)e^{-|x-q_j(t)|}.$$  \hfill (45)

Similar to the cases of one-peakon and two-peakon, but with a lengthy calculation, we are able to obtain the following N-peakon dynamical system

$$p_{j,t} = -\frac{1}{2}k_2 p_j \sum_{k=1}^{N} p_k \text{sgn}(q_j - q_k) e^{-|q_j-q_k|},$$

$$q_{j,t} = -\frac{1}{2}k_2 \sum_{k=1}^{N} p_k e^{-|q_j-q_k|} + \frac{1}{2}k_1 \left( \frac{1}{3}p_j^2 - \sum_{i,k=1}^{N} p_i p_k (1 - \text{sgn}(q_j - q_i)\text{sgn}(q_j - q_k)) e^{-|q_j-q_i|-|q_j-q_k|} \right).$$ \hfill (46)

Different from the N-peakon system of the CH equation, the above system can not directly be rewritten in the standard form of a canonical Hamiltonian system. It is very interesting to see whether the above system is an integrable Hamiltonian system under a Poisson structure. We will investigate this in the near future.

4 Weak kink solution in the case of $b \neq 0$

We have already shown that equation (2) admits peakon solutions in the case of $b = 0$. It is natural to ask what kind of solution one may obtain for the case of $b \neq 0$. Here we will reveal that equation (2) with $k_2 = 0$ and $b \neq 0$ (i.e. cubic nonlinear equation (3) with $b \neq 0$) possesses the weak kink and kink-peakon interacted solutions.

4.1 Single weak kink solution

We seek the solution of equation (2) in the form of

$$u = C \text{sgn}(x - ct) \left( e^{-|x-ct|} - 1 \right),$$ \hfill (47)

where the constant $C$ is to be determined. If $C \neq 0$, (47) actually is a kink solution since

$$\lim_{x \to +\infty} u = -\lim_{x \to -\infty} u = -C.$$ \hfill (48)

One may easily check that the first order partial derivatives of (47) read

$$u_x = -Ce^{-|x-ct|}, \quad u_t = ce^{-|x-ct|}.$$ \hfill (49)
Figure 1: The two-peakon solution determined by (34) with $c_1 = 1$, $c_2 = 2$, $k_1 = -2$. Black line: $t = -4$; Red line: $t = -1$; Brown line: $t = 0$ (collision); Blue line: $t = 1$; Green line: $t = 4$.

Figure 2: The two-peakon solution determined by (34) with $C = -1$, $c_2 = 2$, $k_1 = -2$. Black line: $t = -6$; Red line: $t = -2$; Brown line: $t = 0$ (collision); Blue line: $t = 2$; Green line: $t = 6$.

Figure 3: The peakon-antipeakon solution (40). Pink: peakon (and antipeakon) with the peak (and trough) position $q_2$; Green: antipeakon (and peakon) with the trough (and peak) position $q_1$.

Figure 4: The solution $u(x,t)$ with $p_1$, $p_2$ and $q_1$, $q_2$ shown in (44). Black line: $t = -6$; Red line: $t = -0.8$; Brown line: $t = 0$ (collision); Blue line: $t = 2$; Green line: $t = 4$.

Figure 5: The kink solution (54) for $t = 0$.

Figure 6: The kink-peakon interacted solution. Black line: $t = 2$; Blue line: $t = 1$; Green line: $t = 0$. 
The second order partial derivatives of (47) do not exist at $x = ct$. Therefore, like the case of peakon solutions, the kink solution in the form of (47) should also be understood in the weak sense.

Substituting (47) and (49) into the weak form (22), we arrive at
\[ cCe^{-|x-ct|} = \frac{1}{2}[k_2C^2(x-ct) - bC]e^{-|x-ct|} - \frac{1}{2}[k_1C^3 + bC](x-ct)\text{sgn}(x-ct)e^{-|x-ct|}. \] (50)

If two sides of equation (50) match, we should require
\[
\begin{cases}
  k_2 = 0, \\
  -\frac{1}{2}bC - cC = 0, \\
  k_1C^3 + bC = 0,
\end{cases}
\] (51)

which leads to
\[
\begin{cases}
  k_2 = 0, \\
  c = -\frac{1}{2}b, \\
  C = \pm \sqrt{-\frac{b}{k_1}}.
\end{cases}
\] (52)

**Remark 3.** Formula (52) shows that equation (2) possesses the weak kink solution in the form (47) for the case of $k_2 = 0$ and $b \neq 0$. In formula (52), $k_2 = 0$ implies that the CH equation (11) can not allow a weak kink solution in the form of (47). $c = -\frac{1}{2}b$ means the kink wave speed is exactly $-\frac{1}{2}b$. This is very different from the single-peakon solution (the wave speed in the single-peakon (23) is an arbitrary constant $c$).

In particular, we take $k_2 = 0$, $k_1 = -b = 2$, then equation (2) is cast to
\[ m_t + 2u_x - \left[m(u^2 - u_x^2)\right]_x = 0, \quad m = u - u_{xx}, \] (53)
and the corresponding weak kink solution is
\[ u = \text{sgn}(x - t) \left(e^{-|x-t|} - 1\right). \] (54)

See Figure 5 for the profile of this kink wave solution.

### 4.2 Kink-peakon interacted solutions

Let us first point out that equation (2) with $b \neq 0$ does not allow the two-kink solution in the form of the superposition of two single-kink solutions:
\[ u = p_1(t)\text{sgn}(x - q_1(t)) \left(e^{-|x-q_1(t)|} - 1\right) + p_2(t)\text{sgn}(x - q_2(t)) \left(e^{-|x-q_2(t)|} - 1\right). \] (55)
In fact, substituting (56) into the weak form (22) of equation (2), we find that the solution assumed in the form (55) is reduced to zero or single-kink solution (47).

Instead of considering the solution in the form (55), let us make the following ansatz of solution

\[ u = p_1(t) \text{sgn}(x - q_1(t)) \left( e^{-|x-q_1(t)|} - 1 \right) + p_2(t) e^{-|x-q_2(t)|}, \]  

(56)

which actually describes a new phenomena of kink-peakon interacted dynamics in soliton theory. Apparently, with the help of distribution, \( u_x \) and \( u_t \) can be calculated below:

\[ u_x = -p_1 e^{-|x-q_1|} - p_2 \text{sgn}(x - q_2)e^{-|x-q_2|}, \]
\[ u_t = p_1 t \text{sgn}(x - q_1(t)) \left( e^{-|x-q_1(t)|} - 1 \right) + p_2 t e^{-|x-q_2|} + p_1 q_1, e^{-|x-q_1|} + p_2 q_2, t \text{sgn}(x - q_2)e^{-|x-q_2|}. \]  

(57)

Substituting (56) and (57) into (22) with \( k = 0 \), we arrive at

\[
\begin{cases}
    p_1 = \pm \sqrt{-\frac{1}{k_1}}, \\
p_{2,t} = k_1 p_2^2 p_2 \text{sgn}(q_2 - q_1)e^{-|q_1-q_2|}, \\
q_{1,t} = -\frac{1}{b} + k_1 p_1 p_2 \text{sgn}(q_2 - q_1)e^{-|q_1-q_2|}, \\
q_{2,t} = -\frac{1}{b} k_1 p_2^2 - \frac{1}{b} k_1 p_1^2 + k_1 \left( p_1^2 - p_1 p_2 \text{sgn}(q_2 - q_1) \right) e^{-|q_1-q_2|} + k_1 \text{sgn}(q_2 - q_1)p_1 p_2.
\end{cases}
\]  

(58)

Let us choose \( k_1 = -b = 2 \), then \( p_1 = \pm 1 \). Without loss of generality, taking \( p_1 = 1 \) forces (58) to

\[
\begin{cases}
p_{2,t} = 2p_2 \text{sgn}(q_2 - q_1)e^{-|q_1-q_2|}, \\
q_{1,t} = 1 - 2p_2 \text{sgn}(q_2 - q_1)e^{-|q_1-q_2|}, \\
q_{2,t} = -\frac{2}{b} p_2^2 - 1 + 2 \left( 1 - p_2 \text{sgn}(q_2 - q_1) \right) e^{-|q_1-q_2|} + 2p_2 \text{sgn}(q_2 - q_1).
\end{cases}
\]  

(59)

To solve the above system, let us make an assumption \( q_1 < q_2 \). Therefore, integrating equation (59), we may obtain

\[
\begin{cases}
q_1 = t - p_2 + A_1, \\
q_2 = t - p_2 - \ln\left( \frac{1}{9} p_2^2 - \frac{1}{2} p_2 + 1 + \frac{4A_2}{p_2} \right) + A_1, \\
p_{2,t} = \frac{2}{b} p_2^2 - p_2^2 + 2p_2 + A_2.
\end{cases}
\]  

(60)

where \( A_1 \) and \( A_2 \) are two arbitrary constants. Letting \( A_2 = 0 \), then we may solve the third equation of (60) for \( p_2 \), which has the following implicit form:

\[
\ln |p_2| - \frac{1}{2} \ln \left| p_2^2 - \frac{9}{2} p_2 + 9 \right| + \frac{3 \sqrt{7}}{7} \arctan \left( \frac{4p_2 - 9}{3 \sqrt{7}} \right) = 2t + A_3,
\]  

(61)

See Figure 6 for the profile of the kink-peakon interactional solution with \( A_1 = A_2 = A_3 = 0 \). Here, we do not have more further explicit results about the collisions between kinks and peakons due to the implicit form of the kink-peakon solution.
In general, we may make the following ansatz of the solution for equation (2) with $k_2 = 0$ and $b \neq 0$:

$$u = p_0(t) \text{sgn}(x - q_0(t)) \left( e^{-|x - q_0(t)|} - 1 \right) + \sum_{j=1}^{N} p_j(t) e^{-|x - q_j(t)|},$$

(62)

which can be viewed as the interaction of single-kink and $N$-peakon solution. Through a very lengthy calculation, we are able to obtain the following interactional dynamical system of single-kink and $N$-peakon:

$$\begin{cases}
    p_0 = \pm \sqrt{-\frac{b}{k_1}}, \\
    q_{0,t} = \frac{1}{2} k_1 p_0^2 + k_1 p_0 \sum_{i=1}^{N} p_i \text{sgn}(q_0 - q_i) e^{-|q_0 - q_i|} \\
    + \frac{1}{2} k_1 \sum_{i,k=1}^{N} p_ip_k \text{sgn}(q_i - q_k)(\text{sgn}(q_k - q_0) - \text{sgn}(q_i - q_0)) e^{-|q_i - q_k|}, \\
    p_{j,t} = k_1 p_0^2 p_j \text{sgn}(q_j - q_0) e^{-|q_0 - q_j|} + k_1 p_0 p_j \sum_{i=1}^{N} p_i \text{sgn}(q_j - q_i) \text{sgn}(q_j - q_0) e^{-|q_j - q_i|}, \\
    q_{j,t} = \frac{1}{6} k_1 p_j^2 - \frac{1}{2} k_1 p_0^2 (1 - 2e^{-|q_0 - q_j|}) - \frac{1}{2} k_1 \sum_{i,k=1}^{N} p_ip_k (1 - \text{sgn}(q_j - q_i) \text{sgn}(q_j - q_k)) e^{-|q_j - q_i| - |q_j - q_k|} \\
    - k_1 p_0 \sum_{i=1}^{N} p_i \left( \text{sgn}(q_j - q_0) (e^{-|q_0 - q_j|} - 1) e^{-|q_i - q_j|} - \text{sgn}(q_j - q_i) e^{-|q_0 - q_j| - |q_i - q_j|} \right).
\end{cases}

(63)

The above system is not presented in the canonical Hamiltonian system. We still do not know whether this system is integrable under a Poisson structure.

5 Smooth soliton solutions

In this section, we study all possible single smooth soliton solutions to equation (2). To do so, let $u(x, t) = \phi(x - ct) = \phi(\xi)$, where $c$ is the wave speed. Substituting it into equation (2) and integrating it, we have

$$\phi'' \left( c + \frac{1}{2} k_2 \phi + \frac{1}{2} k_1 \phi^2 - \frac{1}{2} k_1 (\phi')^2 \right) = \left( b + c \right) \phi + \frac{3}{4} k_2 \phi^2 + \frac{1}{2} k_1 \phi^3 \right) - \left( \frac{1}{4} k_2 + \frac{1}{2} k_1 \phi \right) (\phi')^2, \quad (64)$$

where "$\phi'$" stands for the derivative with respect to $\xi$. Equation (64) can be rewritten as the following two-dimensional system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = - \frac{ \left( \frac{3}{4} k_2 + \frac{1}{2} k_1 \phi \right) y^2 + \phi((b + c) + \frac{3}{4} k_2 \phi + \frac{1}{2} k_1 \phi^2)}{c + \frac{1}{2} k_2 \phi + \frac{1}{2} k_1 \phi^2 - \frac{1}{4} k_1 y^2}, \quad (65)$$
which has the following first integral
\[
H(\phi, y) = \frac{1}{2} y^2 \left( c + \frac{1}{2} k_2 \phi + \frac{1}{2} k_1 \phi^2 \right) - \frac{1}{8} k_1 y^4 - \left( b + c \right) \phi^2 + \frac{1}{4} k_2 \phi^3 + \frac{1}{8} k_1 \phi^4 = h. \tag{66}
\]

Without loss of generality, we assume that the wave speed \( c \) is a positive constant. Then, system (65) is a three-parameter planar dynamical system with the triple tuple \( (b, k_1, k_2) \). We only pay attention to the bounded solutions of (65).

We notice that for \( k_1 \neq 0 \), the right hand of the second equation in (65) is not continuous on the hyperbola \( c + \frac{1}{2} k_2 \phi + \frac{1}{2} k_1 \phi^2 - \frac{1}{2} k_1 y^2 = 0 \), i.e., \( \left( \phi + \frac{k_2}{2k_1} \right)^2 - y^2 = \frac{k_2^2 - 8k_1 c}{4k_1^2} \). In other words, \( \phi'' \) is not well-defined on this curve in the phase plane \((\phi, y)\). System (65) is called the second kind of singular travelling wave system [34].

Let us assume \( c > 0, k_1 \neq 0 \). Imposing the transformation \( d\xi = (c + \frac{1}{2} k_2 \phi + \frac{1}{2} k_1 \phi^2 - \frac{1}{2} k_1 y^2) d\zeta \) for \( c + \frac{1}{2} k_2 \phi + \frac{1}{2} k_1 \phi^2 - \frac{1}{2} k_1 y^2 \neq 0 \) on system (65) leads to the following cubic system
\[
\begin{cases}
\frac{d\varphi}{d\xi} = y(\varphi^2 - y^2 - R), \\
\frac{dy}{d\xi} = -\varphi^2 + \varphi^3 + A\varphi + B = -\varphi y^2 + \left( \varphi - \frac{k_2}{2k_1} \right) \left( \varphi^2 + \frac{k_2}{2k_1} \varphi + \frac{4k_1(b+c)-k_2^2}{4k_1} \right),
\end{cases} \tag{68}
\]
where \( \phi = \varphi - \frac{k_2}{2k_1}, \eta = \frac{k_2}{2k_1}, R = \frac{k_2^2 - 8k_1 c}{4k_1^2}, A = \frac{8k_1(b+c) - 3k_2^2}{4k_1^2}, \) and \( B = \frac{k_2(k_2^2 - 4k_1(b+c))}{4k_1^2} \). Apparently, (68) has the following first integral:
\[
H_1(\varphi, y) = -\frac{1}{4}(\varphi^2 - y^2)^2 - \frac{1}{2} A\varphi^2 - \frac{1}{2} R y^2 - B\varphi = h. \tag{69}
\]
Thus, we have
\[
y^2 = (\varphi^2 - R) \pm \sqrt{(R^2 - 4h) - 4B\varphi - 2(A + R)\varphi^2}. \tag{70}
\]

To investigate the exact soliton solutions of equation (2), let us consider system (68) with \( B = 0 \), namely homogeneous system.

1. **M-shape and W-shape soliton solutions**

Assume \( k_1 < 0, k_2^2 = 4k_1(b+c) > 0, b + c < 0 \), and let \( \omega_1 = \frac{1}{2|k_1|} \sqrt{b^2 - c^2} \). Then, system (68) is able to be solved with the following smooth M-shape and W-shape soliton solution for equation (2):
\[
\varphi(\chi) = \pm \frac{(b-c)\sqrt{\frac{b+c}{k_1}\cosh(\chi)}}{(b+c) - 2c\cosh^2(\chi)},
\]
\[
\xi(\chi) = -\frac{1}{2} \sqrt{\frac{b-c}{b+c}} \chi - \ln \left( \frac{1 + \sqrt{\frac{b+c}{k_1} \tanh(\chi)}}{1 - \sqrt{\frac{b+c}{k_1} \tanh(\chi)}} \right). \tag{71}
\]
When $-\infty < b < -3c$, the functions defined by (71) take on the graph of smooth M-shape and W-shape profiles shown in Fig. 7 (7-1). When $-3c \leq b < -c$, the functions defined by (71) give smooth solitons shown in Fig. 7 (7-2).

![Fig. 7](image)

(7-1) $\varphi(\xi)$.

(7-2) $\varphi(\xi)$.

Fig. 7 The profiles of waves with respect to $\xi$

2. Smooth kink and anti-kink wave solutions

Suppose $k_1 > 0, k_2^2 = 4k_1(b + c) > 0, b + c > 0, b < 0$, and let $\alpha_2 = \sqrt{\frac{|b|}{b+c}}$, $\omega_2 = \frac{2}{k_1} \sqrt{c(b + c)}$. Then, system (68) may be solved with the following kink and anti-kink wave solutions for equation (2):

$$\varphi(\chi) = \pm \frac{\sqrt{|b|(b+c)\sinh(\chi)}}{(\sqrt{c} + \sqrt{b+c}\cosh(\chi))},$$

$$\xi(\chi) = 2c\chi - \frac{1}{4k_1} \ln \frac{1+2w+w^2}{1-2w+w^2} + \sqrt{c(b+c)} \ln \frac{(w+1)(\alpha_2^2+w+\alpha_2\sqrt{1+\alpha_2^2\cosh(\chi)})}{(w-1)(\alpha_2^2-w+\alpha_2\sqrt{1+\alpha_2^2\cosh(\chi)})},$$

(72)

where $w = \alpha_2 \sinh(\chi)$.

However in the case of $k_1 > 0, k_2^2 = 4k_1(b + c) > 0, b = 0$, we can solve the system (68) with the following kink and anti-kink wave solutions for system (2) in a simpler form:

$$\varphi(\chi) = \frac{\sqrt{|b|}}{1+e^{-\chi}},$$

$$\xi(\chi) = \pm \frac{1}{2}(\chi + \ln \cosh(\chi)).$$

(73)

3. Smooth soliton solutions

Suppose $k_1 > 0, k_2^2 = 4k_1(b + c) > 0, 0 < b < c$, and let $\alpha_3 = \sqrt{\frac{b}{c-b}}$, and $\omega_3 = \frac{2}{k_1} \sqrt{c(c-b)}$. Then, we can solve the system (68) with the following soliton solutions for equation (2):

$$\varphi(\chi) = \pm \frac{c-b}{\sqrt{c+b}\cosh(\chi)},$$

$$\xi(\chi) = 2c\chi - \frac{1}{4k_1} \ln \frac{1+2w+w^2}{1-2w+w^2} + \sqrt{c(c-b)} \ln \frac{(w+1)(\alpha_3^2+w+\alpha_3\sqrt{1+\alpha_3^2\cosh(\chi)})}{(w-1)(\alpha_3^2-w+\alpha_3\sqrt{1+\alpha_3^2\cosh(\chi)})},$$

(74)

where $w = \alpha_3 \sinh(\chi)$. The functions defined by (74) present two soliton solutions shown in Fig. 8 (8-1).

4. Two-crest soliton solutions
Suppose $k_1 > 0$, $k_2 = 0$, $0 < b < c$ or $b \geq c > 0$. Then, we can solve the system (68) with the following two-crest soliton solutions for equation (2)

$$\varphi(\chi) = \pm \frac{c \sqrt{2(b + c) \sinh(\chi)}}{c + b \cosh^2(\chi)},$$

$$\xi(\chi) = -\frac{1}{2} \sqrt{\frac{c}{b + c}} \chi - \ln \left( \frac{1 + \sqrt{b + c} \tanh(\chi)}{1 - \sqrt{b + c} \tanh(\chi)} \right).$$

(75)

The functions defined by (75) give two-crest soliton solutions shown in Fig. 8 (8-2).

Fig.8 The profiles of soliton waves with respect to $\xi$

Remark 4. If $B \neq 0$ in system (68), namely inhomogeneous system, then we may still obtain some exact smooth one-soliton solutions to equation (2) [35].

6 Conclusions and discussions

In this paper, we have presented the Lax representation, bi-Hamiltonian structure and infinitely many conservation laws for equation (2). The peakon solutions for this equation are derived in the case of $b = 0$. For the one-peakon solution, we found that the complex peakon can be obtained through properly choosing the parameters $k_1$ and $k_2$ in equation (2). For the two-peakon solution, its peakon dynamical system was explicitly solved and their collisions were shown in Figures 1, 2, 3, 4. Moreover, we found that equation (2) with $k_2 = 0$ (namely, cubic nonlinear equation (3)) possesses weak kink and kink-peakon interactional solutions in the case of $b \neq 0$. Compared with the CH equation, equation (2) has some different features, such as the dynamics of the two-peakon solutions and the existence of the weak kink and kink-peakon interactional solutions. These differences are mainly caused by both quadratic and cubic nonlinearity in equation (2). In the paper, we also study all possible smooth one-soliton solutions. Other topics, such as cuspons, Darboux transforms, peakon stability, and algebra-geometric solutions, remain to be developed.
ACKNOWLEDGMENTS

This work was supported by the U. S. Army Research Office (Contract/Grant No. W911NF-08-1-0511) and the Texas Norman Hackerman Advanced Research Program (Grant No. 003599-0001-2009).

References

[1] R. Camassa and D.D. Holm, Phys. Rev. Lett. 71 (1993) 1661.
[2] B. Fuchssteiner and A.S. Fokas, Physica D 4 (1981) 47.
[3] R. Camassa, D.D. Holm and J.M. Hyman, Adv. Appl. Mech. 31 (1994) 1.
[4] P. J. Olver and P. Rosenau, Phys. Rev. E 53 (1996) 1900.
[5] A.S. Fokas and Q.M. Liu, Phys. Rev. Lett. 77 (1996) 2347.
[6] H.R. Dullin, G.A. Gottwald and D.D. Holm, Phys. Rev. Lett. 87 (2001) 194501.
[7] F. Gesztesy and H. Holden, Rev. Mat. Iberoamericana 19 (2003) 73.
[8] Z.J. Qiao, Commun. Math. Phys. 239 (2003) 309.
[9] P. Lorenzoni and M. Pedroni, Int. Math. Res. Not. 75 (2004) 4019.
[10] Y.S. Li and J.E. Zhang, Proc. R. Soc. Lond. Ser. A 460 (2004) 2617.
[11] A. Constantin, V.S. Gerdjikov and R.I. Ivanov, Inverse Problems 22 (2006) 2197.
[12] A. Constantin, J. Funct. Anal. 155 (1998) 352.
[13] R. Beals, D. Sattinger and J. Szmigielski, Adv. Math. 140 (1998) 190.
[14] A. Constantin and W.A. Strauss, Comm. Pure Appl. Math. 53 (2000) 603.
[15] A. Constantin and W.A. Strauss, J. Nonlinear Sci. 12 (2002) 415.
[16] R. Beals, D. Sattinger and J. Szmigielski, Adv. Math. 154 (2000) 229.
[17] M.S. Alber, R. Camassa, Y.N. Fedorov, D.D. Holm and J.E. Marsden, Commun. Math. Phys. 221 (2001) 197.
[18] R.S. Johnson, Proc. R. Soc. Lond. A 459 (2003) 1687.
[19] A. Degasperis and M. Procesi, *Asymptotic Integrability Symmetry and Perturbation Theory* eds A. Degasperis and G. Gaeta (Singapore: World Scientific, 1999) pp. 23-37.
[20] A. Degasperis, D.D. Holm and A.N.W. Hone, Theor. Math. Phys. 133 (2002) 1463.
[21] H. Lundmark and J. Szmigielski, Inverse Problems 19 (2003) 1241.
[22] Z.J. Qiao, Acta. Appl. Math, 83 (2004) 199.
[23] Z.J. Qiao, J. Math. Phys. 48 (2006) 112701.
[24] Z.J. Qiao, J. Math. Phys. 48 (2007) 082701.
[25] A.N.W. Hone and J.P. Wang, J. Phys. A: Math. Theor. 41 (2008) 372002.
[26] V. Novikov, J. Phys. A: Math. Theor. 42 (2009) 342002.
[27] A.S. Fokas, Physica D 87 (1995) 145.
[28] B. Fuchssteiner, Physica D 95 (1996) 229.
[29] Z.J. Qiao and X.Q. Li, Theor. Math. Phys. 167 (2011) 584.
[30] G.L. Gui, Y. Liu, P.J. Olver and C.Z. Qu, to appear in Commun. Math. Phys.
[31] J.A. Desanto, Mathematical and Numerical Aspects of Wave Propagation (SIAM, Philadelphia, 1998).
[32] A.R. Osborne, Int. Geophys. 97 (2010) 857.
[33] A. Bressan and A. Constantin, Arch. Ration. Mech. Anal. 183 (2007) 215.
[34] J.B. Li and H.H. Dai, On the Study of Singular Nonlinear Travelling Wave Equations: Dynamical Approach, Science Press, Beijing, 2007.
[35] J.B. Li, Y. Zhang, and X.H. Zhao, Int. J. Bifurcation Chaos 19 (2009) 1995.