Renormalization scheme with diminished ultraviolet divergences

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We present new renormalization scheme, which being based on strict ordering of mathematical operations, avoids the need of infinite counterterms in evaluation of one loop diagrams in the Standard Model. The UV divergences call for renormalization only when overlapping divergences occur in multi-loops diagrams, in which case they can be subtracted. Further suppression of singularities due to the multiloop tadpoles is shown. In all studied cases yet, suggested procedure provides the finite non-polynomial structure, which is equivalent to results already known form other scheme. Unphysical polynomial structure of diagrams, including anomalies, differs from conventional schemes. Consequences and conceptual differences are discussed for a set of several school diagrams calculated explicitly for purpose of presented paper. It includes e.g. set of vacuum polarization diagrams, ABJ triangle and scalar sunset diagram.

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I. INTRODUCTION

Renormalization is the procedure, which should be done in order to get a meaningful comparison between quantum field theory predictions and experiments. An excellent agreement between theory and and various experiments on muon magnetic moment, 1 2, electroweak and QCD precise fits 4 5 with the interrelation of both 8, is recently based on multi-loop Feynman diagram evaluation and thus represents the test of Standard Model within the renormalization program as its intrinsic and unavoidable part.

Almost half century ago, after publication of the paper 9, the most economic way to address the issue of renormalization is to deal with divergent momentum space integrals by method of dimensional renormalization/regularization (DR). The old fashionable approaches 10–14 turned to be cumbersome when dealing with overlapping divergences in multiloop calculations, however the guiding role of Pauli-Villars regularization method in development of DR scheme was unsubstitutable in the case of chiral anomalies 15, 16. Albeit the Standard Model was proven to be renormalizable algebraically 17, i.e. without the use of any regularization, the DR scheme persist as most useful way to renormalize quantum corrections and the essence of DR in the Standard Model at one loop level becomes a part of textbooks 18.

Mass hierarchy problem due to the quadratically divergent diagrams was one of the most cited reasons for theoretical introduction and longstanding experimental search of hypothetical super-symmetric partners. Few years ago, a Large Hadron Collider achieved 14 TeV - the highest energy in human driven particle collision on the Earth. After the experimental confirmation of relatively light Higgs boson 19, 20 and as there is no good experimental evidence for a new interaction beyond the Standard model at presently accessible energies, this is still the selfenergy in the Higgs boson propagator, which (seemingly) leads to a suspicious friction between bare and renormalized parameters of the Standard Model. It follows that the corresponding problems of naturalness can be ill-defined and might not be problems at all 21.

The study of hadrons at low energy is beyond the reach of perturbation theory, for which purpose a various nonperturbative methods are available at these days. To name one, the formalism of Dyson-Schwinger equations (DSEs) 22–26 is particularly suited for this purpose. Irrespective of softening of QCD coupling at large momenta only after a careful removal of ultraviolet(UV) singularities from the equations, one can get meaningful predictions for hadron properties. Thus for instance, without regularizing DSEs one would not get correct pion properties, which are tightly related with the solution of the quark gap equation. Recall, nonperturbative phenomena like dynamical quark mass generation and related chiral symmetry breaking are almost entirely governed by quantum loops of strongly coupled QCD. Within conventional method of regularization, there is basically no conceptual difference between renormalization of perturbation series and renormalization of nonperturbative set of DSEs. As consequence, there would be always a huge difference between the bare and the renormalized values of parameters in QCD corner of the Standard Model.

In presented paper we offer the renormalization scheme, which avoids the need of infinite counterterms at least at one loop level in sort of renormalizable quantum field theories. As one of the consequence in this approximation, there are small differences between the bare and renormalized values of masses and couplings in the Standard model. We do not extend Standard Model for this purpose, neither we introduce super-symmetric partners or new interactions with higher number of differentiations, nevertheless found the solution. Softened UV singularities reappear only in case of diagrams with overlapping character of divergences. Such diagrams can be calculated by presented method as
well, providing ordinary subtractions are required in this case. Hence we call presented RS as the one with reduced ultraviolet singularities or as in the title of the paper.

A key point to achieve our goal is to ensure that all momentum integrations in given diagram are finite before performing integration explicitly. We dot introduce a regulator function for this purpose, instead of we proposed a new calculation scheme based on exploitation of auxiliary Feynman parameter integral, into which the original momentum space divergence is transferred. As a rule, we do not perform any momentum integration before momentum integral is made finite by proposed method. We allow and require conventional shifting of the integration variable and define the order of operations. This definition is such that for finite diagrams it provides the conventional result. This technical step is described in th Sec.2.

In Sec. 3 we exhibit the calculation on simple example of scalar diagrams. Vacuum polarization diagrams with fermions and heavy vectors bosons are considered in the next Sec. 4 and 5 respectively. Up to constant terms, the obtained unrenormalized results correspond to the renormalized results in DR evaluated at specific t’Hooft scale $\mu$. The main conceptual difference is the absence of infinite constants as well as the structure of anomalies. For this purpose the Abelian ABJ chiral anomalous triangle diagram is rederived in proposed RS in this paper. We also show how to disentangle with overlapping divergences and derive the result for (not only) two loop sunset diagram in details. The role of scalar multiloop tadpole diagrams is emphasized as they can be important in further diminishing of ultraviolet divergences.

The proposed RS represent meaningful calculation scheme, which has not been yet studied in the literature. Although the appropriate calculations are not hard to understand and quite simple to follow, the whole procedure is rather involved and technical, thus we decided as much as possible not to refer to large achievements made in other renormalization schemes. Proposed scheme is not yet designed for studies of Conformal Field Theories, but rather for models with massive particles. We do not discuss the infrared divergences and also avoid discussion of renormgroup equations, which are not essential for understanding of the topic at this stage.

II. DEFINITIONS AND MAIN THEOREM

For the Standard Model quantized in renormalizable gauges, the highest degree of divergence that occur for individual Feynman diagram is two and according to the behavior of hard cut off we use to say such a diagram is quadratically divergent. Similarly, for zero degree of divergence we adopt the standard convention and call such diagram logarithmically divergent.

Singularities, which are presented in Green’s functions need to be removed without disrupting predicative power of theory. Feynman parameterization represents primary step in this respect and most of known RS including, t’Hooft-Veltman DR scheme as well as algebraic BPHZ scheme, deeply rely on its use. It is well known, that divergences in Quantum Field Theory can be sent into the Feynman parameter integrals, which when happen accidentally during the calculation is usually regarded as inconvenience, since if unnoticed it can complicate the track of the renormalization program, especially when dealing with multiloop diagrams.

Here we change completely the philosophy and our priority is to make all the momentum space integrals finite before the integrations are actually performed and for this purpose we transfer all divergences into the Feynman parametric integral in advance. We will always work in 3+1D Minkowski space, or equally in the 4D Euclidean space and do not continue to non-integer dimensions. This naturally avoids the problems with intrinsically integer dimensional object like LeviCivita pseudotensor $\epsilon_{\alpha\beta\gamma\delta}$ or $\gamma_5$ matrix.

Let us illustrate the method at one loop. Using the Feynman parameterization one can transform an arbitrary tensor integral into the following scalar integral:

$$i \int \frac{d^4k}{(2\pi)^4} \frac{k^{\mu_1}..k^{\mu_j}}{[k^2 + k \cdot p - M^2 + i\epsilon]^\alpha} = \prod_{m=0}^{j-1} \frac{1}{(j - m - \alpha)} \partial \frac{d^4k}{(2\pi)^4} \left( \frac{-\lambda_F^2}{-\lambda_F^2} \right)^{f} \frac{i}{[k^2 + k \cdot p - M^2 + i\epsilon]^\alpha - j} \ , \tag{2.1}$$

where we have inserted the unit with arbitrary dummy parameter $\lambda_F$ for which we choose the dimension of mass for convenience. In what follows, the Feynman parametric integral in the form

$$\frac{1}{\alpha^{\gamma_1} b^{\gamma_2}} = \frac{\Gamma(\gamma_1 + \gamma_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \lim_{\epsilon_F \to 0} \int_{\epsilon_F} dz \left( \frac{1 - z}{az + b(1 - z)} \right)^{\gamma_1 + \gamma_2} \ , \tag{2.2}$$

is applied once again, such that the Eq. (2.1) is equivalently written like

$$\prod_{m=0}^{j-1} \frac{1}{(j - m - \alpha)} \partial \lim_{\epsilon_F \to 0} \int_{\epsilon_F} dz \int \frac{d^4k}{(2\pi)^4} \frac{i \delta(f, \alpha - 1) z^{(f-1)}(1 - z)^{\alpha - j - 1}(-\lambda_F^2)^f}{[(k^2 + k \cdot p - M^2 + i\epsilon)z - \lambda_F^2(1 - z)]^{\alpha - j + f}} \ , \tag{2.3}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^{\mu_1}..k^{\mu_j}}{[k^2 + k \cdot p - M^2 + i\epsilon]^\alpha} = \prod_{m=0}^{j-1} \frac{1}{(j - m - \alpha)} \partial \frac{d^4k}{(2\pi)^4} \left( \frac{-\lambda_F^2}{-\lambda_F^2} \right)^{f} \frac{i}{[k^2 + k \cdot p - M^2 + i\epsilon]^\alpha - j} \ , \tag{2.1}$$

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where beta functions $\beta(\gamma_1, \gamma_2)$ is shorthand notation for the three gamma functions prefactor which appears in front of the limit at rhs. of Eq. (2.2) and we choose $f$ such that $\alpha - j + f > 2$ which makes the momentum integral convergent. A practical choice $\alpha - j + f = 3$ can be made, since integer values are always appreciated in multiloop calculations.

Performing the momentum integration gives

$$
\prod_{m=0}^{j-1} \frac{1}{(j - m - \alpha)} \frac{\partial}{\partial p_{\mu_m}} \lim_{\epsilon_F \rightarrow 0} \frac{\Gamma(a - j + f - 2)}{(4\pi)^2 \Gamma(\varphi) \Gamma(a - j)} \int_{\epsilon_F}^{1} dz \frac{(-1)^{(j+1)} \Lambda^{(2j)}(1 - z)^{(a-j-1)}}{z(a-j+1)p^2 - M^2 - \Lambda^2 \left(1 - \frac{1}{z}\right) + i\epsilon^{a-j+f-2}},
$$

where all gamma functions are finite and the all potential UV divergences, if presented, has been transported into the single Feynman integral.

As it is ease to show, for the finite parameter $\lambda_F$ one gets at least a single pole of the form $\epsilon_F^{-1}$ for quadratically divergent diagrams, while the term $\ln \epsilon_F$ appears for log divergent diagrams. The results would be quite complicated and certainly less useful when compared to traditional DRS, where all divergences are presented by harmless pole in the variable $(4 - d)$. To get rid of computational complications and in order to get fruitful results, one could recognize that more meaningful mass scale is not the variable $\lambda_F$ alone, but rather the ratio

$$
\mu^2 = \frac{\lambda^2}{\epsilon_F},
$$

Thus to achieve simple and useful prescription, the simultaneous limit $\epsilon_F \rightarrow 0$, $\lambda_F \rightarrow 0$ should be taken in (2.4), such that the ratio (2.5) is kept finite.

For this and future purpose we introduce shorthand notation

$$
\hat{L} I(p, \epsilon_F, \lambda_F) := \lim_{\epsilon_F \rightarrow 0} \left[ \lim_{\lambda_F \rightarrow 0} I(p, \epsilon_F, \lambda_F) \right],
$$

where $I$ is some single loop integral, i.e. the Eq. (2.4).

After making the limit (2.6), the all one loop integrals which appears in the Standard Model calculations turn to be finite. This is because the integration over the variable $z$ provides the regulator $\epsilon_F$ in a way it always meets its own $\lambda_F$ and thus the double limit (2.6) render one loop Feynman diagrams finite in $3+1$ dimensional space. Those $\Lambda_F$’s which left unpaired gives zero in the limit (2.6) and there is no divergent piece left, which would call for subtraction. Case by case, this will be explicitly shown in sections which follow, where the renormalization prescription is completed by fixing the scale $\mu$ according to physical observable and symmetry constrains.

Procedure described above we will call $L$-operation and it works well for quantum models showing at most quadratic divergences. Obviously it cannot be used directly in presence of quartic or higher divergences. As consequence, the Standard Model bare parameters stay finite at one loop approximation in the presented RS. Actually, to that order, instead of usual renormalization program stay with usual subtractions of infinities, this is just the right ordering of mathematical operations, which makes all one loop Feynman diagram finite.

In case of more multiloop Feynman diagrams the situation is slightly more complicated, due to the diagrams with overlapping divergences. Nevertheless, as we will explain, they can be calculated loop by loop iteratively and theory can be renormalized in a quite standard way afterward.

More concretely, in case of multiple loops integral, one can finish the $L$-operation for a particular loop and then to use per-partes integration with respect to the old Feynman variable in order to prepare next, not yet integrated loop, into the form (2.1) suited for the next integration again. The exception are massless fields where a pure $\ln k^2$ term can occur, which then overlaps with next loop in the following way:

$$
i \int \frac{d^4 k_2}{(2\pi)^4} \frac{\ln \frac{(k_2 + q)^2}{\mu^2}}{[k_2^2 - m^2 + i\epsilon][((k_2 - q)^2 - m^2 + i\epsilon]},
$$

where we took the two propagators in (already second) loop only for illustration (so we are dealing with two log overlapping subdivergences in our exemplary case). For such a log term one can use the following paramaterization

$$
\ln \frac{p^2 + i\epsilon}{-\mu^2} = \int_0^1 \frac{du(p^2 + \mu^2)}{(p^2 u - \mu^2 (1 - u) + i\epsilon)},
$$

which after additional matching by means of Feynman formula (2.2) with the rest of (2.7) leads again into the form (2.1) and the $\hat{L}$-operation is applicable again.
A repeatably use of $L$-operation does not necessary lead to a finite final result, but “remaining singularity” is transferred into the Feynman variable integrals in an easy way to understand. Infinities in multiple loops integrals can be subtracted algebraically and removed by introduction of subtraction polynomial with finite number of terms. The renormalization program is then finished by identifying the polynomial coefficients with counter-term part of the Lagrangian. We assume it can be done in symmetry preserving way in multiloop case, however could be proved from order by order in perturbation theory. This last step then completes a formal proof of renormalizability at given studied order.

We finish this Section by note about (perturbative) Unitarity and causality, which is established in similar way as for instance in DRS: the proposed prescription does not change spacetime structure of the loop integral.

### III. BUBBLES AND TADPOLES RENDERED FINITE IN 3+1 DIMENSIONS

In this Section we begin with simple exercise and calculate the scattering amplitude and correction to propagator at simple scalar theory with the interacting Lagrangian

$$L_{\text{int}} = -\frac{h}{4!}\phi^4(x), \quad (3.1)$$

where $\phi$ is the real scalar field. The interaction is a part of the Standard Model, however it is often considered alone for pedagogical purposes 27, 28, 29, 30, 18. This simple theory contains several types of diagrams we are interested in: the log divergent bubble and the quadratic divergent tadpole. Let us begin with former, which when written in Minkowski momentum space reads

$$b(q^2, m^2) = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2 + i\epsilon][(k - q)^2 - m^2 + i\epsilon]}. \quad (3.2)$$

This diagram builds the one loop approximated scattering amplitude of four $\phi$ particles

$$M(s, t, u) = h + \sum_{q^2=s,t,u} \frac{h^2}{2} b(q^2, m^2), \quad (3.3)$$

where $s, t, u$ are Mandelstam variable composed from the individual incoming and outgoing particle momenta as usually $s = (p_1 + p_2)^2; \ldots$.

In order to apply $L$-operation to the bubble diagram (3.2) one observes that $\alpha = 2$ in the Eq. (2.2) and hence it is enough to take $f = 1$ for making the momentum integration finite. Explicitly written:

$$b(q^2, m^2) = i \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{\lambda_F^2}{[-\lambda_F^2][k^2x + (k-q)^2(1-x) - m^2 + i\epsilon]^2}$$

$$= \hat{L} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \int_{\epsilon_F}^1 dz \frac{\lambda_F^2}{[(k^2 + q^2x(1-x) - m^2) z - \lambda_F^2(1-z) + i\epsilon]^3},$$

$$= \hat{L} \int \frac{1}{4\pi^2} \int_0^1 dx \int_{\epsilon_F}^1 dz \frac{\lambda_F^2}{z(\frac{q^2x(1-x) - m^2 + \lambda_F^2}{-\lambda_F^2}) z - \lambda_F^2 + i\epsilon]}$$

$$= \hat{L} \int \frac{1}{4\pi^2} \int_0^1 dx \left[ \ln \epsilon_F + \ln \left( \frac{q^2x(1-x) - m^2 + i\epsilon}{-\lambda_F^2} \right) \right]. \quad (3.4)$$

Thus for a finite parameter $\lambda_F$ the formula (3.4) involves logarithmic divergence $\ln \epsilon_F$, which after performing the limit (2.7) gives us familiar result:

$$b(q^2; m^2) = \frac{1}{(4\pi)^2} \ln \left( \frac{-q^2x(1-x) + m^2 - i\epsilon}{\mu^2} \right). \quad (3.5)$$

Comparing the Eq. (3.5) to the expression for renormalized bubble in MS DR scheme one finds they are formally identical. However a certain care is needed, since the renormalization constant, which relates the bare coupling $h_o$ with renormalized value $h$ is simply $Z_h = 1$ in our scheme and only trivial RGE arises at one loop

$$\frac{dh}{d\mu} = \frac{dh_o}{d\mu} = 0. \quad (3.6)$$
In other words, the renormalized constant $h(\mu) = h$ as well as the “renormalized” mass $m(\mu) = m$ do not run at one loop approximation in proposed RS. As a consequence, the scale $\mu$ should not be identified with t’Hooft renormalization scale of DR scheme. As usually, it can be eliminated by the requirement that the physical quantities should be independent on $\mu$, which forces us to choose the scale $\mu$ uniquely in our case. It can be done by giving a concrete meaning of what the coupling $h$ could exactly means for (virtual) experimentalists. One can use the cross section at zero participants momentum to define the coupling
\[ \sigma(s, t, u = 0) = h^2, \] (3.7)
in which case one needs to take $\mu^2 = m^2$, ensuring thus $b(0, m^2) = 0$.

Of course, one can perform finite renormalization in order to define renormalized $h(\mu)$, which would give the original meaning to the parameter $\mu$ as a renormalization scale, however we do not follow this quite artificial possibility here.

In what follows we will make the inspection of quadratic divergent loop integral which appears in the inverse of the $\phi$ propagator. It reads:
\[ G^{-1} = p^2 - m^2 - \frac{h}{2} a(m^2) - \ldots, \]
\[ a(m^2) = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon}, \] (3.8)
where dots represent higher loops contributions considered in the Section VII.

In order to evaluate the tadpole function $a(q)$ one can take $f = 2$ in the Eq. (2.1), implying the second and the first power in the Feynman formula (2.2). Actual derivation reads:
\[ a(m^2) = i \int \frac{d^4k}{(2\pi)^4} \frac{\lambda_F^4}{-4\lambda^2_F[k^2 - m^2 + i\epsilon]} = -\tilde{L} \int \frac{d^4k_E}{(2\pi)^4} \int_\epsilon^1 dz \frac{2\lambda^2_F(1 - z)}{([-(k_E^2 - m^2)z + \lambda^2_F(1 - z) + i\epsilon]^3 \lambda^2_F(1 - z)} \]
\[ = \frac{\tilde{L}}{(4\pi)^4} \int_\epsilon^1 dz \frac{\lambda^2_F(1 - z)}{z^2[(m^2 - \lambda^2_F)z + \lambda^2_F]} = \frac{\tilde{L}}{(4\pi)^4} \left[ -\frac{\lambda^2_F}{\epsilon z} + \frac{\lambda^2_F}{\epsilon z} + m^2 \ln \frac{\epsilon z m^2}{\lambda^2_F} \right] \]
\[ = \frac{1}{(4\pi)^4} \left[ \mu^2 + m^2 \ln \frac{m^2}{\mu^2} \right]. \] (3.9)

The result is finite in the limit (2.6), however if one needs for any reason, the original singularities can be easily traced back, noting that for a finite $\lambda_F$ the scale $\mu$ represents linearly divergent variable.

The proof for the third possible case which appears in $\phi^4$ theory and in the Standard Model at all is, when one deals with finite momentum integral. It is quite trivial to show that in this case the direct momentum integration and $L$–operation are equal.

A. Physical consequence

The last line in (3.9) express the finite, albeit quite arbitrary, quantum correction to a classical -bare- Lagrangian mass $m$. In $\Phi^4$ theory the physical on shell mass is given by
\[ m^2_{\text{OS}} = m^2 + \frac{h}{32\pi^2} \left[ \mu^2 + m^2 \ln \frac{m^2}{\mu^2} \right], \] (3.10)
thus the Eq. (3.10) just tells how the scale $\mu$ relates physical mass with the “bare” Lagrangian mass. Using the condition (3.7) one gets unique expression
\[ m^2_{\text{OS}} = m^2 + \frac{h}{32\pi^2} m^2. \] (3.11)

In prescription above, we have sketched what should be done in order to keep the theory selfconsistent in the proposed RS applied to simple $\phi^4$ theory.

In case of the Standard Model, the scalar higgs doublet enters the classical potential, which is responsible for the spontaneous symmetry breaking of $SU(2) \times U(1)$ gauge symmetry. The triple higgs vertex are generated already at classical level, generating thus other contributing diagrams beyond tree level, which together with further quantum
corrections stemming from Yukawa and gauge interaction determine the the value of Higgs condensate \( v = \langle h(x) \rangle = 246 \text{GeV} \). Its value is determined from the minimum of higgs potential - in fact the first DSE- for the higgs field

\[
\frac{\delta \Gamma}{\delta h(x)} = 0 .
\]

This is just this condition which must fix the unphysical scale \( \mu \) according to observable masses of \( W^\pm, Z^0 \) bosons. Other subtlety is that the generating functional \( \Gamma \) turns to be gauge fixing dependent as the irreducible Green’s functions are. Here, as for any other RS, we assume additional constrains could be imposed on the renormalized Green’s functions in a way that constructed observable are gauge fixing independent \[31\]. How much proposed RS naturally respects such constraints and how much naively pre-renormalized results need to be modified when one considers one or two loop corrections can be determined in future studies.

IV. LEPTONIC VHP

Perturbative contribution to the fermion vacuum polarization has the following familiar form

\[
\Pi^{\mu\nu}(q) = -ie^2 T_r \int \frac{d^4l}{(2\pi)^4} \gamma^\mu \frac{J + m}{l^2 - m^2 + i\epsilon} \gamma^\nu \frac{J + \not{q} + m}{(l + q)^2 - m^2 + i\epsilon} \frac{\lambda_F^2}{[\lambda_F^2 - \lambda_{F,5}^2]^2}, \tag{4.1}
\]

where we have performed the shift of the integration variable for a convenience of the reader, who can easily compare with the calculation performed in the DR scheme (see for instance in \[18\]).

Following the receipt described in the Section I we will use the formula \[22\] (with powers \( \gamma_1 = \gamma_2 = 2 \)) in order to match denominators in the Eq. \[4.1\]. It gives us

\[
\Pi^{\mu\nu}(q) = -4ie^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \int_{\epsilon_F}^1 dz \lambda_F^2 z(1-z) \frac{2g^\mu\nu k^2 - g^\mu\nu k^2 + [q^2 g^\mu\nu - 2q^\mu q^\nu]x(1-x) + m^2 g^\mu\nu}{z^4[k^2 + q^2 x(1-x) - m^2 + i\epsilon]^2} - \lambda_F^2 (1-z)/z + i\epsilon]^4 . \tag{4.2}
\]

Performing the integration over the momentum we get

\[
\Pi^{\mu\nu}(q) = \int \frac{4e^2}{(4\pi)^2} \int_0^1 dx \int_{\epsilon_F}^1 dz \left[ -g^\mu\nu(1-z)\lambda_F^2 \right] z^3 [J(x, q, m) - F(x, q, m, \lambda_F^2)]
\]

\[
+ \int \frac{4e^2}{(4\pi)^2} \int_0^1 dx \int_{\epsilon_F}^1 dz \left[ g^\mu\nu q^2 x(1-x) - 2q^\mu q^\nu x(1-x) + m^2 g^\mu\nu \right] z^3 [J(x, q, m) - F(x, q, m, \lambda_F^2)]^2 ;
\]

\[
J(x, q, m) \equiv q^2 x(1-x) - m^2 + i\epsilon . \tag{4.3}
\]

Integrating over the variable \( z \) and performing the limit \[2.6\] explicitly gives us the following result

\[
\Pi^{\mu\nu}(q) = \frac{e^2}{4\pi^2} \left[ g^\mu\nu (\mu^2 - m^2) - g^\mu\nu \frac{q^2}{6} + \frac{1}{3} q^\mu q^\nu \right] + \frac{e^2}{2\pi^2} \int_0^1 dx [-g^\mu\nu q^2 + q^\mu q^\nu] x(1-x) \ln \left( \frac{J(x, q, m)}{\mu^2} \right) , \tag{4.4}
\]

where we have also integrated over the variable \( x \) in the first line by using the following relation

\[
\int_0^1 dx x(1-x) = \frac{1}{6} . \tag{4.6}
\]

It is clear now, that the scale \( \mu \) could not be confused with the renormalization scale, since now we must take \( \mu = m \) in order to get massless photon. The second line automatically provides the result identical with \( MS \) DR scheme for \( \mu = m \), i.e. it corresponds to on-mass shell renormalization scheme since the transverse part vanishes for \( q^2 = 0 \).

To prevent redefinition of the fine structure constant \( \alpha \) one can sent the constant of transverse part in \[4.4\] into the photon renormalization function

\[
(Z_3 - 1) = \frac{\alpha}{6\pi} , \tag{4.7}
\]
when considering VHP contribution from single charged lepton.

The first line in the Eq. (4.3) involves the pseudo-anomalous term, it violates gauge invariance by a constant term. Such term has no physical relevance and it does not lead to the violation of the unitarity or renormalizability. Its contribution to S-matrix vanishes when applying on the on-shell spinor and it can be fully absorbed by a gauge fixing parameter.

Actually the one loop corrected photon propagator written in a class of covariant gauges meets it’s standard form:

\[ G^{\mu\nu}(q) = \frac{-g^{\mu\nu} + \frac{2q^\mu q^\nu}{q^2}}{q^2 (1 - \Pi_{os}(q^2))} + \xi_F \frac{q^\mu q^\nu}{q^2}. \]  

(4.8)

with

\[ \Pi_{os}(q^2) = \sum_f c_f^2 \frac{2\alpha}{\pi} \int_0^1 dx x (1-x) \ln \frac{J_f(x,q)}{-m_f^2}; \]

\[ \xi_F = \frac{1}{\xi^{-1} + \sum_f c_f^2 \frac{\alpha}{\pi}} = \xi \left(1 - \frac{\xi \sum_f c_f^2 \frac{\alpha}{\pi}}{1 + \xi \sum_f c_f^2 \frac{\alpha}{\pi}} \right), \]  

(4.9)

where the sum runs over all charged fermions (color counts as well) and \(e_f\) is the the fermion charge in units of electron charge.

The quantization in axial or in the class of covariant gauges seems to be preferable choice for RS presented here. The variable \(\xi_F\) is then Fadeev-Popov gauge fixing parameter nonlinearly shifted by a constant, noting that the extreme case of Landau gauge \(\xi = \xi_F\) since the shift is finite.

Remarkable fact is that in the presented RS this is the interaction itself, which allows to write down the photon propagator without performing gauge fixing at all, recalling in Abelian theory the limit \(\xi \to \infty\) eliminates the gauge fixing from the action (it is not the case for other the Standard Model gauge interactions due to their non-Abelian character, where at least ghosts should be introduced due to the perturbative unitarity [32, 33]).

A more standard situation is for contribution to photon vacuum polarization due to the charged scalars, which remains exactly massless in proposed RS. As we will see in the next section the same is true for contributions due to loop containing \(W^\pm\) vector fields.

V. MASSIVE YANG-MILLS THEORY- QED OF SM W BOSONS

Application of proposed RS to massless Yang-Mills theory and hence to perturbative QCD requires a special care due to the presence of infrared singularities. The author believe that the main potential of presented RS is in its application in nonperturbative context of Dyson-Schwinger equations, in which case the mass generation and non-Abelian Schwinger mechanism prevents the appearance of infrared singularities. The intended study lies beyond the scope of presented paper and herein as the first instance, instead of study of QCD diagrammatic we concern massive vector QED contribution to vacuum polarization, considering the vector \(W\) bosons.

Massive \(W\) bosons gain their masses nondynamically through the breaking of \(SU(2)/U_Y(1)\) symmetry in the Standard Model. For this purpose, we will work in the \(R_\xi\) gauges, which albeit primarily designed for DR scheme, are equally suited for presented RS as well. We are not going to high orders of theory and concern one loop corrections. We will show that photon remains massless and the proper photon self-energy is transverse in QED with \(W^+\) vector bosons. Both conditions can be mutually satisfied, if there is no correction to longitudinal part of photon polarization function \(\Pi^L(q)\), what will be shown explicitly in the following. Obviously then \(\mu\)-parameter dependent terms for each subset of gauge invariant diagrams can be freely traded to physical masses. In case of the photon it simply means \(\Pi_R(0) = 0\).

The general form of photon self-energy of in \(R_\xi\) gauges consist from seven distinct one loop Feynman diagrams, the one with fermion loop was discussed separately in the previous Section due to its own peculiarity. Each diagram can be decomposed to two terms containing the transverse and longitudinal projector with its own scalar form factor function. These functions can be written in terms of linear combinations of two functions \(a(M_i)\) and \(b(q^2, M_i, M_j)\), the later with the prefactor \(c_1 + c_2 q^2\). All diagrams are individually finite for a finite gauge parameter \(\xi\).

Actually, it is quite obvious that we do not need to proceed Feynman parameterization again and again and the result can be derived by a simple algebraic manipulation, e.g. by using the identity \(2(k.q) = [(l+q)^2 - m^2] - [l^2 - m^2] - q^2\) and by shifts of integral variables. Thus for instance one can immediately write for the following integral:

\[ iL \int \frac{d^4l}{(2\pi)^4} \frac{(l,q)^2}{[l^2 - m^2 + i\epsilon][(l+q)^2 - m^2 + i\epsilon]} = iL \int \frac{d^4l}{(2\pi)^4} \left[ \frac{l.q}{2[l^2 - m^2 + i\epsilon]} - \frac{l.q - q^2}{2[(l+q)^2 - m^2 + i\epsilon]} \right]. \]
\[ + i \mathcal{L} \int \frac{d^4 l}{(2\pi)^4} \frac{q^2}{2[(l - m^2 + i\epsilon)(l + q)^2 - m^2 + i\epsilon]} = \frac{q^2}{2} a(m^2) + \frac{q^2}{4} b(q^2, m^2). \]  

(5.1)

which will be used in the following derivation.

In t’Hooft-Feynman gauge, which we use for simplicity here, the most dominant expression is given by diagram with two trilinear \( WW\gamma \) vertices. The vertices have usual Yang-Mills structure and after some algebra the familiar contribution reads

\[ \Pi_{\mu\nu}(1, q) = \frac{e^2}{2} i \mathcal{L} \int \frac{d^4 l}{(2\pi)^4} U_\lambda \frac{N_{\mu\nu}(q, l)}{\left[(l - M_W^2 + i\epsilon)[(l + q)^2 - M_W^2 + i\epsilon]\right]}, \]

(5.2)

where

\[ N^{\mu\nu}(q, l) = 10 \delta^{\mu\nu} + 4 (l^\mu q^\nu + l^\nu q^\mu - 2 q^\mu q^\nu + g^{\mu\nu}(2l^2 + 2k.q + 5q^2)), \]

(5.3)

where the unit \( U_\lambda = 1 = (\lambda_\rho^2/\lambda_\rho^2) \) is a formal remainder that we should perform \( L \)-operation instead of direct momentum integration.

Projecting (5.2) with \( P_L(q) = q^\mu q^\nu/q^2 \) and \( P_T(q) = g^{\mu\nu} - q^\mu q^\nu/q^2 \) one gets the contribution to the longitudinal and to the transverse part of selfenergy. The former reads

\[ \Pi_L(1, q^2) = \frac{e^2}{2} i \mathcal{L} \int \frac{d^4 l}{(2\pi)^4} U_\lambda \frac{2l^2 + 12l.q + 3q^2 + 10(l.q)^2}{\left[(l - M_W^2 + i\epsilon)[(l + q)^2 - M_W^2 + i\epsilon]\right]}, \]

\[ = \frac{7e^2}{2} a(M_W^2) + e^2(M_W^2 - \frac{q^2}{4} b(q^2, M_W^2)). \]

(5.4)

The tadpole diagram with \( WWAA \) quartic vertex gives the term proportional solely to metric tensor. In T-L decomposition of polarization function it thus gives

\[ \Pi_L(2, q^2) = -3e^2 a(M_W^2). \]

(5.5)

Two diagrams with two lines of charged scalar Goldstone are purely transverse , giving thus trivial contribution to the function \( \Pi_L \). A single one loop diagram, which involves one scalar and one vector propagator provides the following nontrivial contribution

\[ \Pi_L(3, q^2) = -e^2 m_W^2 b(q^2, M_W^2). \]

(5.6)

The forth and the last contribution follows from the diagram with two ghost propagators, which reads

\[ \Pi_L(4, q^2) = -\frac{e^2}{2} a(M_W^2) + e^2 \frac{q^2}{4} b(q^2, M_W^2). \]

(5.7)

Summing all one loop diagrams one finally gets:

\[ \Pi_L(q^2) = \sum_{i=1}^{4} \Pi_L(i, q^2) = 0, \]

(5.8)

which means the WTI \( \sigma^\mu \Pi_{\mu\nu} = 0 \) is satisfied in the gauge sector automatically and one does not need to use the variable \( \mu \) to ensure the gauge invariance.

The scale \( \mu \) is fixed by imposing \( \Pi_T(0) = 0 \), noting its value must be kept the same in every gauge invariant subset of diagrams (while, it can be, and in fact, sometimes it must be taken different in different invariant subsets). Since one can use the Feynman rules for \( R_L \) gauge, the calculation of transverse correction to the SM gauge boson propagators is straightforward. For observable quantities involving one loop corrections we expect results are identical to those derived in the DR scheme.

VI. ABJ TRAPEZOID DIAGRAM

"True" anomalies in quantum field theory are common name for quantum corrections that do not respect Ward identities in any scheme and when one can not avoid anomalous terms completely. Among them a chiral anomaly \cite{15, 16} played the pivotal historical role.
Considering single quark (or lepton), the chiral anomaly would appear in the well known sum of triangle diagrams which would violate at least one of the Ward identity listed here:

\[-(p + q)\mu \Gamma^5_{\mu\nu\delta}(p, q, m) = 2m\Gamma^5_{\nu\rho}(p, q, m) ; \quad p^\delta\Gamma^5_{\mu\nu\delta}(p, q, m) = p^\nu\Gamma^5_{\mu\nu\delta}(p, q, m) = 0; \quad (6.1)\]

where \(m\) is the quark(lepton) mass, which appears in all three quark propagators inside the triangle loop and we will adopt standard conventions used in the textbook \[18\], i.e. \(p\) and \(q\) are outgoing photons, implying that \((p+q)\) is the four-momentum associated with external line of axial-vector vertex (i.e. \(Z^0\) boson corresponds with the external in case of the Standard Model). For calculation using the Pauli-Villars regularization we refer the standard textbook \[18\]. Recall also here, that the DRS rule for dealing with \(\gamma_5\) matrix were adjusted such that the original chiral anomaly is reproduced and for a subtleties associated with the use of DRS in case of dealing with intrinsically 4-dimensional object like \(\gamma_5\) matrix or Levi-Civita tensor \(\epsilon_{\alpha\beta\gamma\mu}\) see for instance \[34\].

Standard Model is anomaly free theory, anomaly cancel between leptonic and quarks diagram for each family individually and thus the existence of the RS scheme which preserves both Ward identities (6.1) is not merely of

To show that anomaly does not occur in presented RS for diagrams with single kind of the fermion is at least instructive and we begin with the proof of electromagnetic WTI

\[p^\delta\Gamma^5_{\mu\nu\delta}(p, q; m) = 2i\text{Tr} \int \frac{d^4k}{(2\pi)^4} U_\lambda \frac{k+\hat{q}+m}{(k+q)^2-m^2+ie} \gamma^\nu \frac{k-\hat{p}+m}{(k-p)^2-m^2+ie} \gamma^\mu \gamma^5 \]

\[-2i\text{Tr} \int \frac{d^4k}{(2\pi)^4} U_\lambda \frac{k+\hat{q}+m}{(k+q)^2-m^2+ie} \gamma^\nu \frac{k+m}{k^2-m^2+ie} \gamma^\mu \gamma^5 \]

\[-i\hat{L} \int \frac{d^4k}{(2\pi)^4} U_\lambda \frac{8i\epsilon_{\mu\nu\rho} (q^\rho p^\sigma + k^\rho p^\sigma - q^\rho k^\sigma)}{[k^2-m^2+ie][(k+q)^2-m^2+ie]} \] \( (6.2) \)

where again the unit \(U_\lambda = 1 = \lambda^2_F/\lambda^2_F\) is a remainder that we should perform \(L\)-operation. Note the second line in Eq. (6.2) vanishes exactly, since being proportional to \(q^\rho q^\sigma \epsilon_{\alpha\beta\gamma\mu}\) where \(I(q, m)\) is the finite function.

In the result (6.2) we will use the Feynman variable \(x\) in order to match the denominators and the variable \(z\) for purpose of \(L\)-operation. It gives us

\[-i\hat{L} \int \frac{d^4k}{(2\pi)^4} \int_{\epsilon_F}^1 dz \int_0^1 dx i8\Gamma(3) [\lambda^2_F \epsilon_{\mu\nu\rho} (q^\rho p^\sigma + xq^\rho p^\sigma - q^\rho p^\sigma (1-x))] z^3 [k^2-m^2+ie][(p+q)^2(1-x)-m^2+ie] \] \( (6.3) \)

After changing the ordering of integrations one gets finite momentum integral and we perform the shift of the momenta by making the substitution \(k = k_{\text{new}} - qx + p(1-x)\) and integrate over the momentum \(k_{\text{new}}\). The result can be written as follows

\[-i\hat{L} \frac{8i}{(4\pi)^2} \int_{\epsilon_F}^1 dz \int_0^1 dx \frac{\lambda^2_F \epsilon_{\mu\nu\rho} (q^\rho p^\sigma + xq^\rho p^\sigma - q^\rho p^\sigma (1-x))}{z^3[(p+q)^2(1-x)-m^2+ie]} \]

\( (6.4) \)

which, without any ambiguity gives zero of the form:

\[-p^\delta\Gamma^5_{\mu\nu\delta}(p, q; m) = -\frac{8i}{(4\pi)^2} \epsilon_{\mu\nu\rho} (q^\rho p^\sigma - q^\rho p^\sigma) \frac{1}{\sqrt{z^3}} \int_0^1 dx (1-x) \ln \frac{(p+q)^2x(1-x)-m^2+ie}{-\mu^2} = 0 \]

which proves the validity of the electromagnetic Ward identity.

The proof of the chiral Ward identity follows very similar steps. Here we refer to the page 462 Eq. (13.15) in the textbook \[18\], which we start with, however without presence of Pauli-Villars regulators, but rather with the symbol of \(L\)-operation instead. It reads

\[-(p + q)\mu \Gamma^5_{\mu\nu\delta}(p, q, m) = 2i\text{Tr} \int \frac{d^4k}{(2\pi)^4} U_\lambda \frac{k+\hat{q}+m}{(k+q)^2-m^2+ie} \gamma^\nu \frac{k+m}{k^2-m^2+ie} \gamma^\delta \gamma^5 \]

\[-2i\text{Tr} \int \frac{d^4k}{(2\pi)^4} U_\lambda \gamma^\nu \frac{k+m}{k^2-m^2+ie} \gamma^\delta \frac{k-\hat{p}+m}{(k-p)^2-m^2+ie} \gamma^5 \]

\[+ 2m\Gamma^5_{\alpha\beta}(p, q) \]

\( (6.6) \)
Performing $L$–operations one gets zero separately for each of the first two lines in the following form
\[
\epsilon_{\nu\delta\alpha\beta}p^\alpha p^\beta b(p, m) = 0,
\epsilon_{\nu\delta\alpha\beta}q^\alpha q^\beta b(q, m) = 0,
\tag{6.7}
\]
where again $b(q, m)$ stands for the finite function $\langle 3.3 \rangle$.

The third line in the Eq. (6.6) is proportional to the pseudoscalar-vector-vector triangle i.e. to the rhs. of axial WTI and finishes our proof. Notably, it reads
\[
\Gamma^5_{\alpha\beta}(p, q) = 2iTr \int \frac{d^4k}{(2\pi)^4} \frac{k + \xi + m}{(k + q)^2 - m^2 + i\epsilon} \gamma^\nu \frac{k + m}{k^2 - m^2 + i\epsilon} \gamma^5 \frac{k - \xi + m}{(k - p)^2 - m^2 + i\epsilon} \gamma^5
\tag{6.8}
\]
and since being finite, the both sides of Axial Ward Identity are independent on the scale $\mu$.

A. Physical consequence

Appearance of the anomalous term $8i\epsilon_{\nu\delta\alpha\beta}p^\alpha q^\beta$ in (6.1) in some RSs can be regarded as consequence of scheme dependent operations required there, since this term is surely absent in the proposed RS. Contrary to assertions frequently made in the literature, there should not be a deep physical consequence due to the anomaly. Actually, we can even set $m_\pi = 0$ in ideal case and pretend the pion is perfect Goldstone boson of broken $SU_L \times SU_R$ symmetry, nevertheless the decay amplitude of the process $\pi \rightarrow \gamma\gamma$ will not vanish.

In what follows, we will use the historical triangle diagram expression (6.8)
\[
\Gamma^5_{\nu\mu}(p, q) = 8m_qi\epsilon_{\nu\mu\alpha\beta}p^\alpha q^\beta F(p^2, q^2, m_\pi^2)
\]
\[
F(p^2, q^2, m_\pi^2) = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k + q)^2 - m_q^2 + i\epsilon} \frac{1}{k^2 - m_\pi^2 + i\epsilon} \frac{1}{(k - p)^2 - m_\pi^2 + i\epsilon}
\tag{6.9}
\]
as the approximation for the amplitude for the pionic decay in the limit of the constant quark mass and the pionic point-like vertex. Using this, we will simply show that the decay amplitude does not vanishes in the chiral limit, irrespective of how chiral symmetry breaking is broken in QCD. Actually, we can even set $m_\pi = 0$ in ideal case and pretend the pion is perfect Goldstone boson of broken $SU_L \times SU_R$ symmetry, nevertheless the decay amplitude of the process $\pi \rightarrow \gamma\gamma$ will not vanish.

For on-shell pion and physical photons we can get for the amplitude (6.9)
\[
F(0, 0, m_\pi^2) = \frac{-1}{(4\pi)^2} \int_0^1 dx \frac{1}{m_\pi^2 x} \ln \frac{m_\pi^2 (1 - x) - m_q^2}{-m_q^2},
\tag{6.10}
\]
which does not vanish for exactly massless pion, since in this limit we get:
\[
F(0, 0, 0) = \frac{1}{2m_q^2 (4\pi)^2}.
\tag{6.11}
\]

Recall that in equations above, $m_q$ stands for the $u, d$ constituent quark mass $m_q = M_q(0) \approx 250 - 300 MeV$, rather then for current quark mass $M_q(\zeta = 2 GeV) \approx 5MeV$ in which case the decay would be badly faster (here we use usual notation for RS invariant dynamical quark mass $M(p^2)$).

In derivation above, we have ignored the normalization of the pion vertex, which can trade the inverse of mass into the decay constant of the charge pion $f_\pi$. Before doing this explicitly, recall that the modern nonperturbative calculations [35, 41] replace the naive constituent quark propagators and electromagnetic vertices by their fully dressed versions as well as the pion pointlike vertex $\gamma_5$ must be replaced by the momentum dependent Bethe-Salpeter pion vertex function in accordance to the solution of QCD Dyson-Schwinger/Bethe-Salpeter equations system. In mentioned framework, there is no doubt, the pion decays in the chiral limit whilst this is the phenomena of QCD dynamical chiral symmetry breaking and associated quark mass generation in QCD, which prevents the pionic decay is not too fast in the Nature!

Now we can go back into the relations (6.11) and correct the proper normalization by inserting factorized properly normalized Bethe-Salpeter pion vertex into the graph. Then using known identity for the Bethe-Salpeter vertex function $\Gamma(x, 0) \approx M(x)/f_\pi$, it gives the familiar result for relevant decay amplitude:
\[
8\Gamma(0, 0)M(0)F(0, 0, m_\pi^2) \approx 8\Gamma(0, 0)M(0)F(0, 0, 0) = \frac{1}{4f_\pi^2},
\tag{6.12}
\]
which is independent on the quark mass, however where one should keep in mind the fact that it is infrared mass $M(0)$ rather then the current one, which leads to this result. (The result is derived within the use of factorization, noting the vertex function could be included into the loop integral in more precise treatment. However such factorization is well justified here, since the form factor $F$, in fact the relation (6.9), is perfectly finite number.)

To conclude our short non-perturbative digression, let us state clearly that in order to get the correct value of the neutral pion decay width, the chiral anomaly is not required to be there. There are further open questions relating to anomalies. For instance we speculate here, that the absence of anomalies in proposed RS does not represent a good hint for any statement about renormalizability of quantum field models, simply due to fact that the divergences can reappear in diagrams with overlapping divergences (which are sources of non-renormalizability at higher orders of perturbation theory [42]). However, if it would turn that this the only one loop anomaly (i.e. neither of WTIs is violated by higher orders), which vanishes in proposed RS and which does not in other schemes, then we can move some oldfashioned and known nonrenormalizable quantum models into the list of renormalizable ones.

VII. OVERLAPPING DIVERGENCES-SUNSET DIAGRAM

In proposed RS, multiloop diagrams with no overlapping divergences are all finite. These diagrams just involves already known finite subdiagrams, or more trivially, the Euclidean momentum integrations are finite without application of $L$-operation. In order to illustrate one such a case we complete two loops expansion in $\Phi^4$ (3.1) theory and state the result for the double tadpole diagram [43]. After effortless calculation the unrenormalized result reads:

$$DTad(q^2, m^2) = \frac{h^2}{4} a(m^2) b(0, m^2),$$  \hspace{1cm} (7.1)

which for a reasonable choice of the scale $\mu$ represents a small contribution being proportional to the constant $h^2/[4(4\pi)^4]$, even vanishing exactly for $\mu^2 = m^2$. Not surprisingly, similar statement is valid for vacuum diagrams (those without external legs): all vacuum diagrams without overlapping divergences turn to be finite after multiply applied L-operation.

Those diagrams where several sub-loops share the same propagator line(s) and where at least two separate momentum integrations are divergent are called diagrams with overlapping divergences. A meaningful RS could provide a result for multiloop Feynman diagram with overlapping divergences as well. Here we will show the appropriate treatment for the case of sunset diagram.

The sunset diagram is the only single two loop irreducible diagram in $\Phi^4$ theory and there are certainly more possibilities how it can be renormalized in proposed RS. Here we present probably the simplest way, which is particularly suited for very generic case of multiloops diagrams in the Standard Model. It consists from two steps. As the first one the integrations over the momenta are performed by repeated use of $L$-operation, and the subtractions of infinities will be made algebraically at the second step.

Two limits (2.6) are required, which is now what the symbol of $L$-operation stands in case of the following expression for the sunset diagram

$$Sun(q^2, m^2) = iL \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{1}{[k^2 - m^2 + i\epsilon][l^2 - m^2 + i\epsilon]},$$  \hspace{1cm} (7.2)

where we have skipped a constant symmetry prefactor (which is $h^2/6$ if $\Phi^4$ theory is considered).

Note also, we put the symbol of $L$ operation in front of the integral in order to stress a strictly defined ordering of mathematical operations. Performing the first $L$-operation for purpose of the integration over the momentum $k$ we should get the following expression

$$Sun(q^2, m^2) = iL \int \frac{d^4l}{(2\pi)^4} b((q + l)^2, m^2),$$  \hspace{1cm} (7.3)

where $B$ stand for the bubble integral (5.5).

Let us perform the substitution $x = (1 - z)/2$ in the integral expression for $b((q + l)^2)$, which after a short algebra gives

$$(4\pi)^2 Sun(q^2, m^2) = \ln(1/4)a(m^2) + iL \int \frac{d^4l}{(2\pi)^4} \int_0^1 dz \ln \frac{(q + l)^2(1 - z^2) - 4m^2 + i\epsilon}{-\mu^2} \frac{1}{l^2 - m^2 + i\epsilon}.$$  \hspace{1cm} (7.4)

After per partes integration we can get

$$(4\pi)^2 Sun(q^2, m^2) = \ln\frac{m^2}{\mu^2}a(m^2) + iL \int \frac{d^4l}{(2\pi)^4} \int_0^1 dz \frac{2z^2(q + l)^2}{(1 - z^2)((q + l)^2 - 4m^2 + i\epsilon)[l^2 - m^2 + i\epsilon]}.$$

(7.5)
Let us perform substitution $z \to \omega$ such that

$$\omega = \frac{4m^2}{1 - z^2},$$

(7.6)

which allows us write the expression for sunset in terms of bubble and tadpole

$$(4\pi)^2 Sun(q^2, m^2) = \ln \left( \frac{m^2}{\mu^2} \right) a(m^2) + \hat{L} \int \frac{d^4l}{(2\pi)^4} \int_0^\infty d\omega \frac{\sqrt{1 - 4m^2/\omega}}{\omega (q + l)^2 - \omega + i\epsilon}.$$ 

(7.7)

where the function $b(p^2; \omega, m^2)$ is the expression for the bubble integral with different masses: $\sqrt{\omega}$ and $m$. It is worthwhile to calculate this integral separately here.

$$b(p^2; \omega, m^2) = \frac{i L}{(4\pi)^4} \int_0^1 dz \int_0^1 dy \frac{\lambda_F^2 - \lambda_F^2}{z^2 (1 - y) - m^2 y - \omega (1 - y) - \lambda_F^2 \zeta} - 2 \lambda_F^2 z.$$

(7.8)

Obviously the L-operation does not kill divergences in the sunset diagram completely, they are just moved into the spectral (Feynman) integral. The second line in the Eq. (7.7) contains the term which is log divergent in the spectral variable $\omega$, while the term in the third line turns to be linearly divergent in this variable.

In what follows we show they can be subtracted algebraically by adding the appropriate counterterms into the Lagrangian. For this purpose we substitute (7.8) into the expression (7.7) and integrate per-parts with respect to the variable $y$:

$$(4\pi)^2 Sun(q^2, m^2) = \ln \left( \frac{m^2}{\mu^2} \right) a(m^2) + a(m^2) \int_0^\infty d\omega \frac{\sqrt{1 - 4m^2/\omega}}{\omega} + \frac{1}{(4\pi)^2} \int_0^\infty d\omega \frac{\sqrt{1 - 4m^2/\omega}}{\omega} \ln \frac{\omega}{\mu^2}$$

(7.9)

where we have labeled

$$\Omega = \frac{m^2 y + \omega (1 - y)}{y (1 - y)}.$$

(7.10)

Now it is obvious that the Eq. (7.9) takes the form of un-subtracted dispersion relation, which means that the infinities can be absorbed into the mass counter-term

$$\delta m = Sun(p^2; \zeta, m^2) + c_1,$$

(7.11)

and by the field redefinition

$$Z = 1 + \delta \phi, \quad \delta \Phi = \frac{d Sun(x, m^2)}{dx} |_{x=\zeta} + c_2$$

(7.12)

where $c_1, c_2$ are arbitrary constants and $\zeta$ is some suitably chosen scale.

Actually, owing unrenormalized result (7.9) is enough to make an explicit comparison with calculation performed in other RSs. Making the substitution (7.12) and subtracting divergent term as suggested, we can get the familiar result [26], i.e. the expression obtained via dimensional regularization followed by subtractions (or equivalently by R-operation) establishing thus BPHZ momentum scheme for which $c_1 = c_2 = 0$. Notably, both results are equivalent to the Cutkosky rules method applied to the sunset diagram earlier [14], [15].
A. Multiloops in the SM, further hint for the reduction of UV infinities

Obviously, in case of multiloop diagrams repeated use of $L$ operation with simultaneous per-partes integrations over the Feynman (or spectral) variables is possible. All logarithm and dilogarithm functions, which appear after the integration can be turned to integrals with rational kernel, exactly in the same way as has been shown in case of the sunset diagram. For example, one can imagine three loop vacuum diagram with sunset involved in as a subdiagram (There is exactly one such diagram in the Standard Model - the Higgs vacuum three loop diagram). In this case one can use the formula (7.9) and continue the calculation by applying Feynman paramaterization and $L$-operation again. After this repetition, one can arrive into renormalized finite result plus polynomial part, the later can be subtracted away algebraically.

Since the Standard Model is the theory with broken symmetries by specific Higgs mechanism, another interesting (and not yet announced in the literature) feature appears at two loop level. It turns out that the UV divergences are further suppressed due to the associated generation of tadpoles diagrams (generic name “tadpole” is used for all diagrams, which contain a subdiagram with no any possibility to insert external momentum into its loop, noting that all untouched tadpoles with single external leg must vanish in the Standard Model [31, 46, 47], since there is no linear term in the Higgs field at all orders of perturbation theory). These tadpoles, albeit not too much important in RSs with subtractions, can kill the dominant singularity in the multiloop diagrams with overlapping divergences. Especially in our case, the quadratic divergence vanishes when the sunset diagram is summed with the tadpole sunset diagram.

Recall, the sunset tadpole has the symmetry factor $S = 48$, i.e. 1/2times smaller then the sunset diagram ($S = 96$). This diagram is coupled to the Higgs propagator line with the second power of Higgs cubic interaction, which has coupling strength $g_3 = -\sqrt{2\lambda m_H^2}$.

Taking account all factors, the sunset tadpole contributes to the inverse Higgs propagator with the following negative amount

$$\frac{48\lambda g_3^2}{-m_H^2} = -96\lambda^2$$ (7.13)

i.e. the prefactor is just minus of the Standard Model Higgs sunset diagram. Recall, that in the Eq. (7.13) $m_H$ is the physical Higgs mass, $\lambda$ is the Higgs Standard Model quartic coupling, thus in usual convention the Higgs vev. is $\langle v \rangle = \sqrt{\mu_H^2/2\lambda}$ and Higgs mass $m_H^2 = 4\mu_H^2$.

Both diagrams contribute to the Higgs selfenergy by their sum

$$96\lambda^2(Sun(q^2) + SunTad(q^2)) = 96\lambda^2(Sun(q^2) - Sun(0)),$$ (7.14)

which in fact supplies the first algebraic subtraction and the only soft logarithmic divergence survives in the sum. In words: In the pure Higgs sector, presented RS provides two loops result with the UV divergent structure radically suppressed when compared to other schemes.

Of course, more complete evaluation of radiative corrections to the Higgs boson in the Standard Model are rather involved already at one loop level [48] and its extension to two loops will require more comprehensive study then presented here. It is worthwhile to mention that looking at the two loop gauge-higgs sector, one can conclude that the self-regulating mechanism we have sketched above is not a generic feature of quantum field theory and is not own to the Standard Model at least as known today. There is only partial subtraction of quadratic divergences, since those proportional to $e^2$ survive at two loop.

VIII. CONCLUSION

Presented method provides finite quantum loop corrections at one loop level as well as it provides a unique prescription to regularize and renormalize multiloop diagrams. In later case higgs sunset contribution to the higgs boson mass turns to be log divergent only due to the self-regulating two loop tadpole mechanism. In all studied cases the non-polynomial part is equivalent to those calculated in other known renormalization schemes, e.g. in DRS.

Calculation of one loop diagrams containing odd number of $\gamma$ matrices does not require a special care and usual definition of $\gamma_5$ matrix anticommuting with all other dirac gamma matrices is enough for purpose of calculation. Curiosity of presented renormalization scheme is the absence of the axial anomaly in the ABJ triangle graph. The occurrence of pseudoanomaly in QED fermionic part of vacuum polarization appears already at one loop in the presented scheme, however it can be absorbed to unphysical gauge term. It does not spoil renormalizability, but instead of it allows us to write down the gauge propagator in interacting theory without explicit presence of gauge fixing term (at least at one loop level).
In perturbation theory, the absence of one loop UV infinities may be regarded as a mathematical curiosity or as a consequence of introduced mathematical trick and we do not expect some extra novelties at higher orders of perturbation theory, that are not known already from conventional approach. However the scheme can be promising in two respects: one is nonperturbative framework of Dyson-Schwinger equations, especially when the equations for Green's functions are solved by the use of the integral representation \[23, 26, 41, 49, 50\]. The second possible utilization is the physics beyond the Standard Model, where a further diminishing of UV divergences can take a place. Actually, further observation of UV singularity reduction due to the presence of tadpole, represents interesting hint for self-regulating mechanism in non-supersymmetric quantum models. In this respect the gauge-higgs unified models \[51, 53\] are particularly interesting in this respect, not necessarily being defined within the use extradimensions \[54\]. A significance of presented scheme for naively non-renormalizable theories, e.g. for CHPT used in low energy hadron physics or for Little Higgs models \[55-62\] is an open question. A finitne of the loop diagrams have nontrivial issues for stability of the Higgs effective potential and has certainly impact on new physics of simple extensions of the Standard Model \[63-66\]. Possible relaxing on “naturalness” then offer much wider parameter space for a models with scalar sector extensions \[67-72\].

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