Affine-Detection Loophole in Quantum Data Processing

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Here is considered a specific detection loophole, that is relevant not only to testing of quantum nonlocality, but also to some other applications of quantum computations and communications. It is described by a simple affine relation between different quantum “data structures” like pure and mixed state, separable and inseparable one. It is shown also, that due to such relations imperfect device for a classical model may mimic measurements of quantum correlations on ideal equipment.

I. INTRODUCTION

Let us consider a few standard formulae. A density matrix for one qubit can be represented as:

\[ \rho = \frac{1}{2} (I + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z), \] (1)

where \( I \) is the unit matrix, \( \sigma_\mu \) are three Pauli matrices and \( a = (a_1, a_2, a_3) \) is 3D vector of real parameters \( |a| \leq 1 \). All pure states due to \( \rho^2 = \rho \) satisfy to simple condition

\[ |a| = 1. \] (2)

It is clear, that any mixed state for such a simple model is related with pure state via affine transformation

\[ A_\alpha: \rho' = \alpha \rho + (1 - \alpha) I / N \] (3)

where \( \alpha > 1 \) is a real parameter, \( N = 2 \) is dimension of Hilbert space for example under consideration (cf. 2 \& 3). An inverse transformation may be expressed using same formula with \( 1/\alpha \),

\[ A_\alpha^{-1} = A_{1/\alpha} \] (4)

For models with higher dimensional systems or more than one qubit the transformation Eq. (3) maps to pure states only some class of mixed ones, denoted as pseudo-pure \[ 5 \]. The relation between pure and pseudo-pure states discussed above devotes special consideration and generalizations.

II. PSEUDO-PURE STATES AND ENSEMBLE COMPUTATIONS

Let us consider an unitary evolution (quantum gate) \( U \):

\[ \rho \rightarrow U \rho U^\dagger \] (5)

Such evolution does not change unit matrix \( I \) and so we have a set of similar models linked by the transformations \( A_\alpha \) Eq. (3). Really the models are equivalent in more rigor sense, that can be described using a commutative diagram

\[
\begin{align*}
\rho & \rightarrow U \rho U^\dagger \\
\downarrow A & \quad \downarrow A \\
\rho' & \rightarrow U \rho' U^\dagger
\end{align*}
\] (6)

Here vertical arrows represent the affine transformation \( A \) Eq. (3) and commutativity means simply, that a path ‘right – down’ is equivalent to ‘down – right’, i.e., Eq. (6) does not change a structure of the model.

In our case the affine equal models may be physically very different. It is clear from an example with the Bloch sphere for one qubit. It is enough to consider a sphere of pure states Eq. 2 with radius one and thin layer with radius \( 1/\alpha \) representing some family of mixed states inside of this unit ball. It is clear, that unitary transformations does not change this smaller sphere, but it can be mapped to space of pure states by Eq. (3).

Unlike of approach with vectors in Hilbert spaces, there only direction does matter and length is fixed, here scale, “length” Eq. (2) of traceless part of density matrix is important parameter of the model. Similar, but more difficult picture can be suggested for higher dimensional and multi-qubit systems, where already not all mixed states could be used for such a model. But for one qubit any mixed state is “pseudo-pure.”

Let us consider now measurements in such a models. If to choose a measurement basis with \( N \) states \( |\mu_k\rangle \), then probabilities are described as:

\[ p_k = \text{Tr}(\rho |\mu_k\rangle \langle \mu_k|). \] (7)

The probabilities of corresponding outcomes for affine equal models are related as

\[ p'_k = \alpha p_k - (a - 1)/N. \] (8)

Due to Eq. (6) the measurements let us distinguish these models, but similarity is such a strong, that in NMR quantum computation where \( a \) is very big the mixed (pseudo-pure) states anyway are used for simulations of quantum gates for pure states \[ 5 \]. The NMR is one of the

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most developed branch of quantum computations and it could be suggested, that in other areas such “affine loopholes” also may have importance, but still did not investigated with proper attention. It should be mentioned yet, that problem discussed here is related also with quantum communications and formally it is some kind of detection loophole (see below and also 1, 2).

III. GENERAL MEASUREMENT PROBLEMS

The main problem here, that such loophole is rather hidden due to common principles of any physical measurement. A measured value $X$ usually contains two main kinds of systematic errors: rescaling $s$ and background noise $b$, and so “true” value $X_0$ is calculated usually as

$$X_0 = s (X - b).$$

Both $X$ and $X_0$ here are classical values, there $X_0$ may be considered as result of some imaginary ideal experiment, contrasting with real raw outcome $X$.

The formula is appropriate for any measurements, but it is discussed here in quantum mechanical framework, because it is similar with Eq. 5 and overestimation of noise may produce specific problems, like treatment mixed state as pure (discussed above) and separable state as inseparable. Last case is discussed below and maybe even more important: it is known, that specific quantum phenomena is related with nonlocality and inseparable states. Local quantum system like qubit may be simulated by classical model 13 and so testing quantum mechanics is related nowadays with different kinds of non-local experiments. Overestimation of noise in Eq. 9 can produce a possibility to treat a classical process as a quantum one. It is discussed below and clear already from well known notion, that classical description of a quantum nonlocal process has a problem of “negative probabilities” 3.

IV. SEPARABILITY AND QUANTUM NONLOCALITY

It was described above, that the affine transformation Eq. 3 may maps any state of qubit (or a pseudo-pure state of higher dimensional systems) to pure one. But Eq. 3 may also describe transformations between other classes of states.

In two-party quantum communications there is important notion of separable system 1 with density matrix represented as

$$\rho = \sum_\mu \lambda_\mu \rho_\mu^I \otimes \rho_\mu^I, \quad \lambda_\mu > 0.$$  

For a multi-party system the theory is more complicated due to absence of an unique decomposition, but using the main idea, that a separable state can be considered as a composition of classically correlated systems, it is possible to define it also as the similar sum with positive coefficients. If it is not possible to represent density matrix in such a way, i.e., always appear some negative coefficients, the state is called inseparable (or entangled, but the first term seems more appropriate in present context for mixed states).

It is reasonable in present paper to consider question about different classes of states related by the affine transformation Eq. 3 and an answer is rather trivial:

For arbitrary number of qubits any state can be produced from a separable state by affine transformation Eq. 3 with some parameter $a$.

i.e., inseparable and even pure states can be produced from “classically correlated states” for some $a > 1$ 2, 3, 4.

• Proof. Let us consider an inseparable state with $n$ qubits. It is possible to use the unique decomposition with up to $4^n$ tensor products of $(I, \sigma_i)$ and rewrite it as some sum of up to $6^n$ tensor products of pure states

$$\rho_\mu^\pm \equiv \frac{1}{2}(I \pm \sigma_\mu).$$

It is enough to use expressions $\sigma_\mu = \rho_\mu^+ - \rho_\mu^-$ and $I = \rho_\mu^+ + \rho_\mu^-$ (let it be $I = \rho_\mu^+ + \rho_\mu^-$ for certainty). Let us consider now some negative term

$$- \alpha \rho_\mu^+ \otimes \cdots \otimes \rho_\mu^+,$$

and rewrite it as

$$\alpha (I - \rho_\mu^+ \otimes \cdots \otimes \rho_\mu^+) - \alpha I,$$

where $I \equiv I \otimes \cdots \otimes I$ can be represented as

$$I = (\rho_\mu^+ + \rho_\mu^-) \otimes \cdots \otimes (\rho_\mu^+ + \rho_\mu^-).$$

The Eq. 13 with $2^n$ terms includes Eq. 12 and so instead of the negative term it is possible to write $2^n - 1$ positive terms and $-\alpha I$. Using such transformations it is possible to get rid of all negative coefficients except of one for $I \equiv I \otimes \cdots \otimes I$, but this last coefficient may be deleted using the affine transformation Eq. 3. It is enough to rewrite Eq. 3 as

$$\rho' = a(\rho + \frac{1-a}{N^a} I),$$

where $N = 2^n$. To delete negative term $-xI$, it is necessary to use the affine transformation with

$$\frac{1-a}{N^a} = x \implies a = \frac{1}{1 + N x}$$

So the inseparable state can be transformed to a separable one using Eq. 3 with $0 < a < 1$ and so an inverse transformation Eq. 4 with $a = N x + 1$ is the necessary map for the separable state.

Let us consider an example with the Bell state $|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$.
errors can be considered of a complimentary kind, but here it has a specific inter-
reduction of sensitivity to one kind of events due to events proportional lost of events

associated with intensities of different sources and the bound of inseparability must be based on the minimal possible value. The parameter values necessary, but it is appropriate for our purposes. Due to

It should be mentioned, that responsible for such “con-

Due to such relation some “classically correlated states” can be used as a model of “pure quantum” system after appropriate distortion Eq. [8] of measurement statistic. Is it possible to observe such pseudo-quantum effects due to errors of experimental equipment or mis-interpreting of data? Formally, it could be really so, but here the model is more difficult than for one system. Say, for measurement of intensity in one beam, the values \(X_0, X, \text{ background } b\) in Eq. [9] may be simply associated with intensities of different sources and the error discussed above may be related for example with overestimation of the background intensity \(b\).

It should be mentioned that responsible for such “con-
stant error” \(b\), together with an unknown background noise and fixed leak of events per unit time, could be also more difficult schemes. Let us consider two complimentary events with probabilities \(p_0\) and \(p_1 = 1 - p_0\). A proportional lost of events \(s\) is simply corrected due to condition \(p_0 + p_1 = 1\) and as an example of other kind of errors can be considered symmetric misclassification

\[
p_i' = p_i + \frac{s}{2}(p_i - p_1) = p_i + \frac{s}{2}(2p_i - 1) = (1 + \varepsilon)p_i - \frac{s}{2},
\]

\[i \equiv 1 - i, \quad p_0' + p_1' = 1.\]

Such misclassification may be related for example with reduction of sensitivity to one kind of events due to events of a complimentary kind, but here it has a specific inter-
est due to the precise correspondence with Eq. [8] for \(N = 2\) and \(a = 1 + \varepsilon\).

For a two-beam setup errors related with any separate beam may not produce the discussed effect of “pseudo-quantum nonlocality,” but it can appears in a count of a coincidence rate. Here is again the rescaling \(s\) may not be relevant for such kind of error and it is related with the background or threshold coefficient \(b\) in Eq. [9].

Really any good experimentalist always feels such kind of problems even without using of commutative diagrams like Eq. [4] and produces many test and calibrations of experimental equipment to get rid or estimate \(s\) and \(b\) in Eq. [9] together with more complicated sources of errors [10]. But this affine detection loophole discussed here, because it belongs to some general class of problems related with any observation and experiment, when very different physical models may produce mathematically almost equal behavior.

Let us compare it with well known Bell ideas [11] about necessity of taking into account classical theories reproducing “quantitative” result of quantum mechanics for few fixed angles and with an absolute different shape of a curve for the angular dependence (“saw” instead of “co-
sine”, see [11] and also [6, 7]). It was one of reason to use four different angles in such kind of experiments [11]. So it was interesting here to consider a classical model with “qualitative” same behavior (e.g., it would be same “cosine” shape for the angular dependence, but with different absolute values, cf [7]). Such apparent similarity may provokes wrong interpretations.

**VI. CONCLUSION**

In the paper was discussed problems related with specific kind of affinity Eq. [3] between different systems and models.

\[
|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) - |1\rangle(|0\rangle \otimes |0\rangle - |0\rangle(|1\rangle \otimes |1\rangle),
\]

\[
\frac{1}{4}(\rho_x^0 \otimes \rho_z^0 + \rho_z^0 \otimes \rho_x^+ + \rho_x^+ \otimes \rho_z^0 + \rho_z^0 \otimes \rho_x^+ + \rho_y^+ \otimes \rho_y^- + \rho_y^- \otimes \rho_y^+).
\]
Say, using a “broken” device with a threshold:

\[
X = \begin{cases} 
X_0 - \theta; & X_0 \geq \theta \\
0; & X_0 < \theta 
\end{cases}
\]  

(16)

and an ensemble of separable, i.e., “classically correlated” states, we may produce same result as with an ideal measurement device for inseparable states. For example state Eq. (15) is separable and may be modeled by classical local model with hidden variables, but due to specific errors Eq. (3) the same system would mimic any statistical property of inseparable Bell state, including Bell inequalities.

It is really disappointing fact, that an experimenter should use more and more complicated equipment and measurement techniques to test quantum correlations, that may be equivalent with a result of measurement of classical correlations on a non-ideal device. It may be taken into account in relation with questions about necessity of subtlest experiments. Maybe it really still exists some place for doubts and loopholes. Not in quantum mechanics, but in absolutely perfect tests using the contemporary kind of classical equipment.

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[15] It should be mentioned, that although the qubit can be modeled by some classical program with a generator of random numbers for simulation of a measurement, to treat it as “a classical statistical model” is not a very good idea. A difference is even more transparent in case of a qutrit (a quantum system with three states) — despite of possibility of a computer simulation with a random number generator and few states (e.g., $2N^2 - N$ for an $N$-dimensional system [2]) it may not be treated as a classical statistical model due to the Kochen-Specker theorem. The resolution of this apparent paradox is a fact, that not any formal manipulation with the random numbers in an abstract and complicated algorithm corresponds to some classical stochastic process (certainly, it may also does not correspond to any quantum process, e.g., it can describe cloning, evolution of some other universes or simply contains program errors).
[16] It is slightly refined version of proof suggested in [3]. Here is necessary to emphasize two properties of the (over)complete set of projectors $\rho_k^\mu$ used in present proof: (1) they linear span is whole space of $N \times N$ density matrices and (2) any element of set belongs to some complete family with $N_k$ orthogonal projectors. This proof may be applied to any number of “quNit” ($N$-dimensional systems) even with different $N_k$, if to use higher-dimensional generalizations of the set $\rho_k^\mu$ with same properties (1, 2) [4]. As an example may be used “complete set of projectors” with $2N_k^2 - N_k$ elements for each $N_k$-dimensional (sub)system described in [1]. An “existence” proof of equivalent theorem for an arbitrary number of quNits also can be found in [2].