Quantum Hall phases of two-component bosons

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The recent production of synthetic magnetic fields acting on electroneutral particles, like atoms or photons, has boosted the interest in the quantum Hall physics of bosons. Adding pseudospin-1/2 to the bosons greatly enriches the scenario, as it allows them to form an interacting integer quantum Hall (IQH) phase with no fermionic counterpart. Here we show that, for a small two-component Bose gas on a disk, the complete strongly correlated regime, extending from the integer phase at filling factor \( \nu = 2 \) to the Halperin phase at filling factor \( \nu = 2/3 \), is well described by composite fermionization of the bosons. Moreover we study the edge excitations of the IQH state, which, in agreement with expectations from topological field theory, are found to consist of forward-moving charge excitations and backward-moving spin excitations. Finally, we demonstrate how pair-correlation functions allow one to experimentally distinguish the IQH state from competing states, like non-Abelian spin singlet (NASS) states.

\[ \int \text{Introduction.} \] Recent progress in producing strong synthetic gauge fields in neutral systems like atomic quantum gases \([1, 2]\) or photonic fluids \([3]\) has catalyzed the research in bosonic quantum Hall states. While in the fractional quantum Hall (FQH) regime the bosonic states are often simply the counterparts of fermionic states, a significant difference occurs for non-interacting particles: Instead of forming an IQH liquid as fermions do, the bosons’ fate is to condense. However, as has been strikingly predicted by effective field theory \([4, 5]\), this does not exclude the possibility of an IQH phase of bosons. This phase can be obtained in a two-component system at filling factor \( \nu = 2 \). As has been confirmed by numerical studies \([6, 7]\), repulsive two-body contact interaction favors this phase against competing FQH phases. In contrast to the fermionic case, interactions are a crucial ingredient for the IQH physics of bosons.

Different from FQH states, IQH states have no anyonic excitations, nor do they exhibit topological degeneracies in non-trivial geometries (e.g. tori). Nevertheless, they possess topologically protected edge states which due to Wen’s edge-bulk correspondence \([8]\) make them distinct to conventional bulk insulators. A particularly appealing property of the edge in spin-singlet systems is the fact that it can be excited in twofold ways: by spinless charge excitations (“holons”) or by charge-neutral spin excitations (“spinons”) \([9]\). For the edge of the bosonic IQH phase, a \( K \)-matrix description predicts opposite velocities for these two types of excitations \([8]\), as a consequence of one positive and one negative eigenvalue of the \( K \)-matrix. This interesting property has been discussed before for a FQH state of spin-1/2 fermions at \( \nu = 2/3 \) in a singlet \([11, 12]\).

In the context of FQH physics, two-component Bose gases have been considered in a torus geometry \([13, 14]\), where ground state degeneracies suggest them as a candidate for realizing NASS phases \([15]\). Quantum many-body states with non-Abelian excitations are particularly relevant, as their use for topological quantum computations has been proposed \([16]\). A recent study of two-component bosons in a spherical geometry \([7]\), however, gave rise to some controversy: Competitors of the NASS states are the composite fermion states which have Abelian topological order.

In this Letter we shed further light on the quantum Hall physics of two-component bosons by performing a systematic numerical study in a disk geometry. After briefly introducing different trial wave functions, we study for \( N = 6 \) particles all incompressible states on the Yrast line, starting with the IQH state at \( L_z = 9 \) (in units of \( h \)) and ending with the Halperin state at \( L_z = 21 \), where the system is able to fully avoid contact interaction. We find all the incompressible states to be well described by the composite fermion (CF) approach \([17]\). We then study (for \( N = 8 \)) the edge excitations of the IQH phase. Apart from some exceptions in the forward-moving branch, we find number and spin of the edge excitations to precisely agree with the predictions from effective theory. A model of the edge states based on CF theory is found to accurately describe the wave functions of the backmoving states. It is shown that the forward moving states can be modeled by multiplying the ground state wave function with symmetric polynomials. Finally, we demonstrate how pair-correlation functions distinguish the IQH state from competing states in an experiment.

\[ \int \text{System and trial wave functions.} \] We study a system of \( N \) two-component bosons described by the Hamiltonian \( H = \sum_i^N \left[ \frac{\hat{p}_i^2 - A(z_i)}{2m} \right]^2 + \frac{\hbar^2}{m} \omega^2 |z_i|^2 + V_0 \sum_{i<j} \delta(z_i - z_j) \), where \( z_i = x_i + iy_i \) represents the position of the boson, \( A(z) = \frac{B}{2}(x, -y) \) is a gauge potential, \( m, V_0 \), and \( \omega \) are positive parameters specifying the mass, the two-body interaction strength, and the frequency of a harmonic
confined. The single-particle part of $H$ has a Landau level (LL) structure, and is solved by Fock-Darwin (FD) states $\varphi_{n\ell}\ell$, which in the lowest Landau level (LLL) read $\varphi_{n\ell}(z) \propto z^{\ell} \exp(-|z|^2/4)$, in units of length given by $\lambda = \sqrt{\hbar/(M\omega)}$, and $\omega_0 \equiv \sqrt{\omega^2 + \lambda^2}$.

One way to account for interactions is the CF theory developed by Jain [17]. It provides a compelling picture to understand both IQH and FQH phases on an equal footing: By attaching magnetic fluxes to each particle, one obtains CFs, which are assumed to behave like non-interacting particles, that is, they fill one or several LLs. Originally, this theory has been formulated for fermions, but it has been extended to bosonic quantum Hall phases triggered by the experimental progress in producing synthetic gauge fields acting on ultracold atoms [18].

Recently, CF states for two-component Bose systems have been introduced and studied in a spherical geometry [2].

Here we consider a two-component Bose system on a disk, for which CF states can be constructed in a similar way. Omitting the omnipresent Gaussian factor, we write [2],

$$\Psi_{L_z}^{n_a,n_b} = \mathcal{P}_{\text{LLL}}[\Phi_{n_a}(\{z_a\}) \Phi_{n_b}(\{z_b\})].$$  \hspace{1cm} (1)

The last term is a Jastrow factor $J(\{z\}) = \prod_{i<j}(z_i - z_j)$, which attaches one magnetic flux to each particle, turning the bosons into CFs. The wave function of the composite particles is given by the Slater determinants $\Phi_{n_a}$ and $\Phi_{n_b}$, for particles of type $a$ and $b$, respectively. The indices $n_a(b)$ yield the number of LLs occupied by the CFs. If $n_a = n_b$, the total spin is zero, $S = 0$. Importantly, negative $n_a$ and $n_b$ shall refer to flux-reversed LLs: $\Phi_{-n} = \Phi_{n}^*$. Finally, $\mathcal{P}_{\text{LLL}}$ projects back into the LLL of the bosonic system. We perform this projection in the standard way by replacing conjugate variables $z^*$ by derivatives $\partial_z$.

The only difference between Eq. (1) and the corresponding definition on a sphere is the fact that in closed geometries the number of states in each LL is finite. This gives rise to the notion of “completely filled” LLs, and the state $\Psi_{[n_a,n_b]}$ is uniquely defined. Depending on the sign of $n = n_a + n_b$, its filling factor is $\nu = n/(n \pm 1)$. Contrarily, on a disk, there is more than one way to distribute $N_a$ ($N_b$) particles in $n_a$ ($n_b$) LLs. Typically each choice leads to a different total angular momentum $L_z$, such that wave functions at different angular momentum $L_z$ correspond to the same filling factor $\nu$ in the thermodynamic limit. Note that, for $|n_a| = |n_b| = 1$, however, the wave functions are unique also on a disk. In particular, $\Psi_{[-1,1]}$ has $L_z = N^2/4$ and corresponds to an integer filling factor, $\nu = 2$. In contrast to all other CF wave functions with fractional filling, this wave function might describe an IQH liquid.

Another important trial wave function, obtained within the CF theory by putting all composite particles to the LLL ($\Psi_{[1,1]}$), is the Halperin state [19], explicitly given by:

$$\Psi_{\text{H}} \sim \prod_{i<j}(z_{ia} - z_{jb})^2 \prod_{i<j}(z_{ib} - z_{jb})^2 \prod_{i,j}(z_{ia} - z_{jb}).$$  \hspace{1cm} (2)

It is a spin singlet wave function at filling $\nu = 2/3$, with zero energy in a two-body contact potential, and describes an Abelian FQH phase. A series of non-Abelian quantum Hall states can be constructed from it by forming $k$ clusters, putting each cluster into a Halperin state, and symmetrizing over all possible clusterizations [15]. In this way, one obtains the NASS states at filling factor $\nu = 2k/3$ as the zero-energy eigenstates of $(k + 1)$-body contact interaction.

Yrast line. We have studied $N = 6$ two-component bosons in the LLL on a disk by exactly diagonalizing the SU(2)-symmetric two-body contact interaction. The presence of an additional harmonic trapping in $H$ which is invariant under spatial rotations along the $z$-axis and under spin rotations will not modify the eigenstates of the system, but simply increases the energy eigenvalues by a value proportional to $L_z$. Properly choosing the trapping frequency, one can tune the ground state of the system to different $L_z$.

The system’s Yrast line, that is the spectrum of the interaction energy at fixed $L_z$, is shown in Fig. 1. Different $L_z = 9, 12, 15, 18, 21$ correspond to incompressible states, where an increase of angular momentum does not directly lead to a decrease in energy. Notably, for all these $L_z$ it is possible to construct CF states. Moreover, exact ground states and CF states agree in spin, and have very good overlap (> 0.97). At $L_z = 21$, the overlap equals 1, as the Halperin state of Eq. (2) becomes the exact ground state. At $L_z = 18$, two CF states with $S_z = (N_a - N_b)/2 = 0$ can be constructed: $\Psi_{[1,2]}$ and $\Psi_{[2,1]}$. Accordingly, the ground state is a triplet, but notably, also the antisymmetric combination of the two states gives rise to a quasi-degenerate singlet state. For $L_z = 15$, the CF construction yields a unique singlet phase, $\Psi_{[-2,-2]}$, with overlap
0.9878 and large gap. For \( L_z = 12 \), the situation is similar to \( L_z = 18 \), with a triplet ground state and a quasi-degenerate singlet state obtained from two possible CF states, \( \Psi^{-1, -2} \) and \( \Psi^{-2, -1} \). The incompressible phase with smallest \( L_z \) is found for \( L_z = 9 \): the clearly gapped ground state is a singlet and has large overlap (0.985) with \( \Psi^{-1, -1} \).

**Edge physics of the IQH phase.** We now focus on this lowest-\( L_z \), state on the Yrust line, for which we can extend our numerical study to \( N = 8 \) particles and, accordingly, \( L_z = 16 \). Compared to \( N = 6 \), we find an only slightly smaller overlap, \( |\langle \text{GS} | \Psi^{-1, -1} \rangle| = 0.9709 \). As \( \Psi^{-1, -1} \) describes a spin singlet with integer filling \( \nu = 2 \), and the phase turns out to be strongly gapped and incompressible, all prerequisites for an IQH phase are fulfilled. Previous studies provided evidence of the integer topological character of this phase by analyzing the properties of the entanglement entropy on a sphere \([6,7]\), and the uniqueness of the ground state on a torus \([8]\). In the present paper, we consider the equivalent system in a plane, and focus on the physics at the edge to characterize its topology \([8]\).

An effective theory of the edge physics in fermionic singlet states \([11]\) is applicable also to the bosonic IQH state. It allows for a straightforward counting of the edge excitations. This theory is based on the observation that edge excitations of a spin singlet state might either be excitations which change angular momentum of the spin-up (down) particles, or be excitations which flip the spin of some particles. Thus, the effective edge Hamiltonian has the form \([11]\) \( H_{\text{edge}} \propto v_s \langle S_z^2 + \sum_{l} (b_l^\dagger b_l) + v_c \sum_{l} (c_l^\dagger c_l) \rangle \). Here, the first term denotes the spinon excitations with velocity \( v_s \), and the second term the holon excitations with velocity \( v_c \). The operators \( b_l \) and \( c_l \) annihilate bosonic modes at angular momentum \( l \).

An edge excitation at \( |\Delta L_z| = 1 \) can thus be achieved either by \( \langle S_z^2 \rangle = 1 \) and \( \langle b_l^\dagger b_l \rangle = \langle c_l^\dagger c_l \rangle = 0 \), or by \( \langle S_z^2 \rangle = 0 \) and \( \langle b_l^\dagger b_l \rangle = \delta_{l1} \) and \( \langle c_l^\dagger c_l \rangle = 0 \), giving rise to three states forming a spin triplet excitation, or by a spin singlet charge excitation with \( \langle S_z^2 \rangle = 0 \) and \( \langle b_l^\dagger b_l \rangle = 0 \) and \( \langle c_l^\dagger c_l \rangle = \delta_{l1} \). In the case where \( v_s < 0 \) and \( v_c > 0 \), the spin triplet excitation corresponds to \( \Delta L_z = -1 \), and the spin singlet to \( \Delta L_z = +1 \).

Extending this counting to excitations with \( |\Delta L_z| > 1 \), we find that the multiplicities of the \( c \) modes are given by the same counting which also applies to the Laughlin state, namely the number of positive-integer sums which add up to \( |\Delta L_z| \). From the effective theory, all these lowest excitations are expected to be spin-singlets. Mixed charge-spin excitations would have higher energies. For \( v_c > 0 \), these modes are located at \( \Delta L_z > 0 \). For the spin branch, the counting is trivial: At \( |\Delta L_z| = 2 \), four possible choices are possible, two with \( \langle S_z^2 \rangle = 0 \) and two with \( \langle S_z^2 \rangle = 1 \), thus giving rise to a triplet and a singlet. For \( |\Delta L_z| = 3 \), two triplets and one singlet are expected, and for \( |\Delta L_z| = 4 \), we expect two singlets, two triplets, and one SU(2) multiplet with total spin \( S = 2 \). Again, mixed charge/spin excitations are expected at higher energies, and for \( v_s < 0 \) the spinon modes must have \( \Delta L_z < 0 \).

**Backward moving edge states.** In the spectrum shown in Fig. 2 we find one gapped triplet ground state at \( \Delta L_z = -1 \), and two quasi-degenerate gapped ground states, one singlet and one triplet, at \( \Delta L_z = -2 \). This perfectly matches with the counting expected from effective theory. Also at \( \Delta L_z = -3 \) and \( \Delta L_z = -4 \), the spin of the lowest states agrees with the spin predicted by effective theory, but the degeneracy lifting within the ground state manifold becomes larger than the gap to the excited states. A particularly striking confirmation of the effective theory is the fact that at \( \Delta L_z = -4 \) a \( S = 2 \) multiplet becomes member of the ground state manifold.

A simple intuitive explanation for the presence of back-moving state, which directly leads to a scheme for constructing trial wave functions, can be given in terms of the CF approach: Since the ground state, \( \Psi^{-1, -1} \), describes an IQH phase of CFs which are subjected to a flux-reversed magnetic field, a forward-directed edge excitation of the CFs constitutes a backward-directed edge excitation of the bosons. More formally, as a consequence of the complex conjugation of the Slater determinants in \( \Psi^{-1, -1} \), the edge excitation of the CFs (that is the shift of one or several CFs to higher angular momentum) will correspond to a reduced angular momentum of the bosons.

Following this reasoning, we have constructed trial wave functions for edge states with \( -4 \leq \Delta L_z \leq -1 \). For example, consider the state with \( S_z = 0 \) at \( \Delta L_z = -1 \): The ground state \( \Psi^{-1, -1} \) consists of four spin-up and four spin-down CFs, each filling the FD states with \( \ell = 0, \ldots, 3 \) in the flux-reversed LLL. An edge state can
then obtained in two ways: for either the spin-up or the spin-down CFs, we replace the FD state with \( \ell = 3 \) by a FD state with \( \ell = 4 \), which after complex conjugation leads to \( \Delta L_z = -1 \). Strikingly, after projecting these wave functions into the LLL, both choices lead to exactly the same wave function, and we recover a single state at \( S_z = 0 \), as demanded by both the effective theory and the numerical results. This becomes more remarkable for \( \Delta L_z < -1 \): At \( \Delta L_z = -2 \), we find five ways to construct \( S_z = 0 \) edge states, but they reduce to two linearly independent states. At \( \Delta L_z = -3 \), ten different constructions lead to three states, and at \( \Delta L_z = -4 \), twenty constructions yield precisely five different states. Thus, the CF construction is in perfect agreement with the counting of modes. Apart from the counting, also the overlaps of the trial states with the exact states are remarkably high. They are explicitly given within Fig. 2, and for any of the eleven edge states in the interval \(-4 \leq \Delta L_z \leq -1\) they are larger than 0.82, demonstrating the power of the CF description.

Forward moving edge states. For \( \Delta L_z > 0 \), the effective theory predicts spin singlet ground states, with degeneracy 1, 2, 3, 5, . . . for \( \Delta L_z = 1, 2, 3, 4, . . . \). Indeed we find a single singlet ground state at \( \Delta L_z = 1 \), though it is not separated by a large gap from a second, low-lying triplet state, see Fig. 2. Also at \( \Delta L_z = 2 \), there is a singlet ground state, but a nearby second state in the spectrum is a triplet state, instead of a second spin singlet. At \( \Delta L_z = 3 \), even the ground state is a triplet. It has been argued that forwardmoving edge states have a large velocity and thus merge with bulk excitations, spoiling the spectral structure expected from effective theory \cite{11, 12}. Moreover, we note that the state \( \Psi^{-1,-1} \) is the first incompressible state on the Yrast line. Therefore, while backmoving modes of this state do not interfere with forward moving edge modes of other incompressible states, the forwardmoving excitations of \( \Psi^{-1,-1} \) are expected to mix with backmoving modes of an incompressible triplet phase at \( L_z = 20 \) (for \( N = 8 \)).

Nevertheless, it is possible to identify some states in the spectrum of Fig. 2 as forward moving edge states of \( \Psi^{-1,-1} \). We construct them by multiplying the ground state by homogeneous polynomials which are symmetric in all variables. Such excitation might either act on the bosons, that is on the wave function \( \Psi^{-1,-1} \), or on the composite fermions, that is on the CF wave function before LLL projection. Remarkably, the latter approach yields slightly better results.

For \( \Delta L_z = 1 \), the construction yields one singlet, having overlap 0.9709 with the exact state. Note that this is precisely the overlap of the exact ground state at \( \Delta L_z = 0 \) with \( \Psi^{-1,-1} \), suggesting that the construction of the edge itself is exact, and the slight deviation of the overlap from unity is caused by a discrepancy between the ground state at \( L_z = 16 \) and the CF state. Also, the ground states at both \( \Delta L_z = 0 \) and \( \Delta L_z = 1 \) have exactly the same energy.

At \( \Delta L_z = 2 \), the energy of only the sixth state in the spectrum, a singlet, matches with the ground state energy at \( \Delta L_z = 0 \). This state is well reproduced (again overlap 0.9709) by our construction of edge states which now yields two singlet states. At lower energies, we find two singlet states, two triplet states, and one \( S = 2 \) multiplet. Each of the two singlet states has an overlap around 0.63 with our edge state construction, suggesting that a linear combination of the two states would reasonably well agree. In that way, we can, out of the three low-energy singlet states, recognize two as the edge states predicted by effective theory.

Experimental realization. The Hamiltonian studied here can be created in the laboratory by subjecting two-component bosonic atoms to artificial magnetic fields. Notably, such systems are flexible in size, and could be tuned from the microscopic regime (accessible by exact diagonalization) to the macroscopic regime (beyond exact diagonalization). This has interesting applications: While a competition between the NASS state and the CF state at filling factor \( \nu = 4/3 \) takes place in the thermodynamic limit \cite{3}, or for \( N \geq 16 \) on a disk, in smaller systems, accessible to numerics, the two states occur at different \( L_z \). The favored phase could be determined, however, by an experiment. To illustrate this, let us refer to a different competition which takes place for \( N = 8 \) at \( L_z = 16 \): We have already seen that the CF picture with the \( \nu = 2 \) state describes well the ground state (overlap 0.97), but an alternative trial wave function is the \( \nu = 4/3 \) NASS state (overlap 0.52). Note that the CF state and the NASS state themselves have overlap 0.41, despite their different topological order. The overlaps certainly give a clear picture in favor of the CF state.
but they are not accessible to experiment. What can be measured instead, are pair-correlation functions, that is, the probability distribution of finding one particle somewhere in space, after another particle has been fixed at a given point. As shown in Fig. [3] the pair-correlation functions well distinguish between CF and NASS state, and a measurement of them would be able to identify the ground state.

**Conclusions.** We have studied quantum Hall phases of two-component bosons on a disk. All incompressible phases are understood in the CF picture. The edge states identify the IQH phase of bosons. This phase could be realized in experiments with cold atoms, and detected by measuring pair-correlation functions.

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