Mathematical modeling of thermal processes in building enclosures

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Abstract. In the development of automatic control systems (ACS) for heat supply of buildings, mathematical models of their control objects play a significant role, one of the most important elements of which being the enclosing structures. The existing models of thermal processes indoors either do not take into account the dynamic effect of the enclosing structures, or are complex analytical expressions, the use of which in the engineering practice of the synthesis of ACS for the indoor thermal regime is difficult. The purpose of the study was to create a linear mathematical model of the thermal interaction of the external and internal air environment through the enclosing structures. To achieve this goal, a functional diagram of a heated room as a control object was developed, which describes the mechanism of thermal interaction of its elements. On its basis, design schemes of thermal processes for external and internal enclosing structures were developed. They were used to create systems of equations linking temperatures on the surfaces of layers of external enclosing structures and air temperatures outside and inside the premises. At the same time, the assumption was introduced that the dynamic temperature distribution over the thickness of the enclosing structure is linear. To find the temperatures at the boundaries of the layers, the average temperature value over the thickness of the structure was used. As a result of the study, equations for the dynamics of heat transfer processes were obtained, and a generalized structure of a model of multilayer external enclosing structures of a room as an element of a control object was developed. The resulting model can be used to develop a generalized mathematical model of the thermal regime of both individual rooms and the building as a whole.

1. Introduction
The process of optimization of heat energy consumption, provided that comfortable microclimate conditions are ensured, is one of the most important tasks in the operation of residential and public buildings. Its solution is closely related to the use of automation tools that control and regulate heating, air conditioning and ventilation systems [1, 2]. The most important role in ensuring comfortable temperature regime in rooms is played by heating systems.

There is a wide variety of heating systems in buildings. They differ in the types of heat supply sources, methods of delivery and distribution of the heating medium between consumers, etc. One of the common options for organizing heat supply to buildings is a centralized system in which the source of heat energy is a boiler plant (thermal station), and the heating medium is water.

The development of any control system requires knowledge of the mathematical model of its control object. In our case, we consider the processes of heat exchange between the external
environment, building structural elements, the heating system and the air of the building as such control object. The constituent elements of the control object have spatial distribution, therefore their dynamics is described by differential equations in partial derivatives [3].

Currently, there is a number of mathematical models of heated rooms and enclosing structures, in particular [3, 4]. As a rule, they are aimed at solving computational problems, i.e. determination of thermal regime parameters for various combinations of external and internal conditions in which the premises of the buildings are located. The high complexity of these models complicates their adaptation and application in the engineering practice of the synthesis of heat supply control systems. Therefore, the purpose of this study is to develop a linear mathematical model of thermal processes in enclosing structures that is convenient for use in problems of synthesis involving systems for automatic control of thermal conditions of buildings.

2. Mathematical Modeling of Enclosing Structures as a Control Object Element

2.1. Definition of the control object

The internal energy of the room air is determined by the dynamic balance of heat fluxes arising from its interaction with the heating device, enclosing structures and the outdoor environment. Therefore, we take the total of the heating device HD, external (ExS) and internal (InS) enclosing structures and room air (RA) as the control object of the room air temperature control system (Figure 1).

The output coordinate of the control object is the room air temperature $T_{ra}$, the control action is the heat flux rate of the heat carrier of the heating device $G_1$, and the main disturbance is the temperature of the outside air $T_{ext}$.

![Figure 1. Functional diagram of the control object](image)

2.2. Assumptions made when developing a mathematical model

Indoor thermal processes are characterized by a complex dynamic interaction of spatially distributed elements. Moreover, some of their thermophysical parameters are nonlinearly dependent on temperature, which significantly complicates the mathematical model. Therefore, when developing a linear mathematical model focused on the use of synthesis of automatic control systems in analytical methods, it is necessary to introduce the following assumptions:

1. All spatially distributed parameters of the control object are replaced by their average values. This allows us to proceed to the mathematical model of the control object with lumped parameters.
2. We assume that the densities and heat capacities of air and water are constant and independent of temperature.
3. The modeled room is located in the middle of the building and has only one outside wall.
4. The thermal regimes of the modeled room and those adjacent to it coincide.
5. The temperature distribution over the thickness of structures is linear, both in statics and dynamics.
6. There are no infiltration, insolation or household heat fluxes in the room.

2.3. Mathematical model of external and internal enclosing structures

External enclosing structures can include both transparent elements (windows, stained-glass windows) and opaque elements (walls). Walls play the main role in thermal protection of premises from the external environment and, as a rule, are multilayer. We will assume that external enclosing structures consist of four layers: textured facade layer 1, insulation 2, cellular concrete 3 and cement mortar 4.

In order to compile a system of equations describing the dynamics of external enclosing structures, a design scheme has been developed (Figure 2), showing the mechanism of heat exchange between the inside air environment and the outside air through a multi-layer enclosing structure.

The upper part of the design scheme shows heat fluxes in different layers of the enclosing structure. The heat flux of each layer includes two components: static (transmission) and dynamic, which appears only when the temperature head of the layer changes. In stationary mode, $Q_0 = Q_{1c} = Q_{2c} = Q_{3c} = Q_{4c}$, i.e. the same transmission heat flux passes through all layers.

In the design scheme, the dashed line describes the stationary temperature distribution in different layers of the building enclosure. The horizontal solid lines in the diagram correspond to the approximation of the spatial temperature distribution within the layer by its average value.

![Figure 2. Calculation scheme of thermal processes in a four-layer enclosing structure](image)

Outside air with a temperature $T_{ext}$ affects the outer surface of the wall, which has a temperature $T_{1b}$, by convection. If $T_{1b} \neq T_{ext}$, then under the influence of the temperature head $(T_{ext} - T_{1b})$ there occurs a heat flux $Q_{0c}$ [5]

$$Q_{0c} = (T_{ext} - T_{1b}) \cdot \alpha_i \cdot F_c$$  \hspace{1cm} (1)

Where $F_c$ is the wall area, $\alpha_i$ is the coefficient of heat transfer by convection at the outer surface of the wall, which is usually determined by the Frank formula [5]

$$\alpha_i = 7,34 \cdot e^{0.656 \nu} + 3,78 \cdot e^{-1.91 \nu},$$  \hspace{1cm} (2)
Here \( v \) is the air velocity. In order to preserve the linearity of the equation (1), we will assume that \( v = \text{const.} \).

Heat flux \( Q_{0c} \) causes a change in the internal energy of layer 1, which can be described by the expression for the average temperature

\[
T_{1,av} = \frac{1}{C_1 \cdot m_1} \int Q_{1d} \, dt,
\]

Where \( C_i \) and \( m_i \) – the specific heat of the material and the mass of layer 1 of the enclosing structure, \( Q_{1d} \) – dynamic heat flux

\[
Q_{1d} = Q_{0c} - Q_{1c}.
\]

Transmission heat flux \( Q_{1c} \) can be found by equation [5]

\[
Q_{1c} = (T_{1,av} - T_{2b}) \cdot \frac{\frac{1}{R_i}}{F_c},
\]

where \( T_{2b} \) is the temperature on the inner surface between layers 1 and 2, \( R_i \) is the thermal resistance of layer 1

\[
R_i = \frac{\delta_i}{\lambda_i},
\]

here \( \delta_i \) and \( \lambda_i \) – thickness and thermal conductivity of layer 1 material.

If we assume that both in statics and dynamics, the temperature distribution over the layer thickness is linear, then the average value of the temperature of layer 1 \( T_{1,av} \) can be defined as the arithmetic mean of temperature values \( T_{1b} \) and \( T_{2b} \) on its borders

\[
T_{1,av} = \frac{T_{1b} + T_{2b}}{2}.
\]

Thermal processes in layers 2 and 3 are similar to those considered above, therefore, based on expressions (3) - (7) one can have the following system of equations

\[
\begin{cases}
Q_{1c} = Q_{2c} + Q_{2d}, \\
Q_{2c} = (T_{2,av} - T_{3b}) \cdot \frac{\frac{1}{R_2}}{F_c}, \\
T_{2,av} = \frac{T_{2b} + T_{3b}}{2}, \\
Q_{2c} = Q_{3c} + Q_{3d}, \\
Q_{3c} = (T_{3,av} - T_{4b}) \cdot \frac{\frac{1}{R_3}}{F_c}, \\
T_{3,av} = \frac{T_{3b} + T_{4b}}{2}.
\end{cases}
\]

Where \( Q_{2c}, Q_{3c} \) – transmission heat fluxes of layers 2 and 3, \( Q_{2d}, Q_{3d} \) – dynamic heat fluxes, \( T_{2,av}, T_{3,av} \) – average values of temperatures, \( C_2, C_3 \) and \( m_2, m_3 \) – specific heat capacities of materials and masses of layers 2 and 3, \( T_{3b}, T_{4b} \) – temperatures at the boundaries of layers 2-3 and 3-4, \( R_2, R_3 \) - thermal resistance of layers 2 and 3.

Thermal processes in the last 4th layer of the building enclosure can be described by the equations


\[ \begin{align*}
Q_{\text{tc}} &= Q_{\text{tc}} + Q_{\text{td}}, \\
Q_{\text{tc}} &= (T_{\text{ave}} - T_{\text{sb}}) \cdot \frac{2}{R_{\text{d}}} \cdot \frac{1}{F_{\text{c}}}, \\
T_{\text{ave}} &= \frac{1}{C_{\text{d}} \cdot m_{\text{d}}} \int Q_{\text{td}} \, dt, \\
T_{\text{ave}} &= \frac{T_{\text{sb}} + T_{\text{sb}}}{2}.
\end{align*} \]  

(9)

Where \( Q_{\text{tc}} \) – transmission heat flux of layer 4, \( Q_{\text{td}} \) – dynamic heat flux, \( T_{\text{ave}} \) – average temperature value, \( C_{\text{d}} \) and \( m_{\text{d}} \) – specific heat capacity of the material and mass of layer 4, \( T_{\text{sb}} \) is the temperature on the inner surface of the enclosing structure, \( R_{\text{d}} \) is the thermal resistance of layer 4.

If \( T_{\text{sb}} \neq T_{\text{ra}} \), where \( T_{\text{ra}} \) is the average room air temperature, then a convective heat flux will appear between the wall surface and the room air.

\[ Q_{\text{tc}} = Q_{\text{ad}} = (T_{\text{sb}} - T_{\text{ra}}) \cdot \alpha_{2} \cdot F_{\text{c}}, \]

(10)

here \( \alpha_{2} \) is the convection heat transfer coefficient at the inner wall surface \([5]\)

\[ \alpha_{2} = 1.66 \cdot \sqrt{(T_{\text{ra}} - T_{\text{sb}})}. \]

(11)

In order to preserve the linearity of the equation (10), we will assume that \( (T_{\text{ra}} - T_{\text{sb}}) = \Delta T = \text{const} \), then

\[ \alpha_{2} = 1.66 \cdot \sqrt{\Delta T}. \]

(12)

If we generalize the above equations and represent them in operator form, we can obtain a mathematical model of thermal processes for an n-layer enclosing structure

\[ \begin{align*}
Q_{\text{bc}}(p) &= (T_{\text{ext}} - T_{\text{ib}}(p)) \cdot \alpha_{1} \cdot F_{\text{c}}, \\
Q_{\text{ai-1}b} (p) &= Q_{\text{bc}}(p) + Q_{ad}(p), \\
Q_{\text{ai}}(p) &= (T_{\text{ave}}(p) - T_{\text{ai+1b}}(p)) \cdot \frac{2}{R_{i}} \cdot \frac{1}{F_{\text{c}}}, \\
T_{\text{ave}}(p) &= \frac{1}{C_{i} \cdot m_{i} \cdot p} Q_{\text{id}}(p), \\
T_{\text{ave}}(p) &= \frac{2}{2} \\
Q_{\text{nc}} &= Q_{ad}(p) = (T_{\text{ai+1b}}(p) - T_{\text{ra}}(p)) \cdot \alpha_{2} \cdot F_{\text{c}},
\end{align*} \]

(13)

Where \( i \in I \ldots n \) is the ordinal number of the layer, \( p \) is the Laplace operator.

The system of equations (13) allows synthesizing the generalized structure of the mathematical model of thermal processes in a multilayer enclosing structure (Figure 3). It is a sequential connection of operators \( S \), each of which describes thermal processes at the internal and external boundaries of the enclosing structure. Operators \( S_{i} \) (Figure 4) and \( S_{n} \) (Figure 5) describe convective heat transfer processes on the outer surfaces of the enclosing structure. Operators \( S_{2}, \ldots, S_{i}, \ldots, S_{n-1} \) (Figure 6) are a mathematical description of conductive thermal processes at the internal boundaries between the layers of the enclosing structure.
The transparent part of the enclosing structure (window, stained-glass window) belongs to non-heat-consuming structures, therefore the thermal processes in it can be described by the expression for transmission heat losses.
\[ Q_{es}(p) = (T_{ra}(p) - T_{w}) \cdot \frac{1}{R_w} \cdot \frac{1}{F_w}, \]

where \( R_w \) and \( F_w \) – thermal resistance and area of the window structure.

Then the total heat flux of the external enclosing structure

\[ Q_{es}(p) = Q_{es}(p) + Q_{en}(p). \]

To describe the thermal processes of internal enclosing structures (internal partitions, floors), a design scheme has been developed (Figure 7). According to it, the volume of each internal structure is divided into two equal zones of influence from the air inside the modeled room and the air from rooms adjacent to it. In accordance with the assumption that the thermal regimes of neighboring rooms are identical, there are no transmission heat fluxes through the internal enclosing structures.

![Figure 7. Calculation scheme of thermal processes in an internal enclosing structure](image)

When the heating system is turned on (off), the room air temperature \( T_r \) begins to change. As a result thermal pressure and dynamic heat flow arise between the air and the surface of the internal structures \( T_{ra} - T_{w,av} \)

\[ Q_{ia} = (T_{ra} - T_{w,av}) \cdot \alpha_a \cdot F_{ins}, \]

where \( F_{ins} \) is the area of internal enclosing structures, \( T_{w,av} \) – average temperature of the internal enclosing structure

\[ T_{w,av} = \frac{2}{C_w \cdot m_w} \int Q_{ia} \, dt, \]

where \( C_w, m_w \) – specific heat capacity of the material and the mass of the structure.

Let us represent the equations (16) and (17) in operator form

\[
\begin{align*}
Q_{ia}(p) &= (T_{ra}(p) - T_{w,av}(p)) \cdot \alpha_a \cdot F_{ins}, \\
T_{w,av}(p) &= \frac{2}{C_w \cdot m_w \cdot p} Q_{ia}(p).
\end{align*}
\]

2.4. Mathematical model of thermal processes for room air

Room air is an accumulator of heat flows passing through external and internal enclosing structures, as well as of heat gains from internal heat sources (people, household and heating appliances). The air temperature in the heated room has a certain spatial distribution. However, supposing that natural air circulation leads to its even distribution, we assume that the room air temperature is a lumped parameter and is equal to the average value. We should also assume the absence of infiltration, insolation, and household heat flows. Then the average room temperature is
\[ T_{ra} = \frac{1}{C_a \cdot m_a} \int (Q_{ra} + Q_{wa} + Q_{hd}) \, dt, \]
or in operator form
\[ T_{ra}(p) = \frac{1}{C_a \cdot m_a \cdot p} (Q_{ra}(p) + Q_{wa}(p) + Q_{hd}(p)), \quad (19) \]
where \( C_a, m_a \) - specific heat and mass of room air.

3. Results and discussion

Let us combine the previously developed models in order to obtain a structural scheme of a generalized control object. In this case, we assume that the opaque part of the external enclosing structure has four layers and contains an opening filled with a transparent window structure. As a result, the structure of a generalized control object is obtained (shown in Figure 8). In it, the model of the opaque part of the enclosing structure is represented by blocks \( S_1 - S_4 \), the model of the opening transparent part (window) – by the \( S_w \) block, and the model of the internal enclosing structures – by the \( S_{ins} \) block. The operation of the heater is taken into account with the heat flow \( Q_{hd} \), while we assume that its value is constant and does not depend on the air temperature in the room.

Figure 8. The structure of the mathematical model of a multilayer enclosing structure

The advantage of the resulting structure is its linearity, which allows us to obtain transfer functions connecting the variable values of the control object by the method of structural transformations. In the future, they can be used in the synthesis of automatic control systems for the thermal regimes of buildings.

4. Conclusion

Multilayer building enclosures have a complex dynamic mechanism of thermal interaction with the external and internal air environments of the premises. The developed mathematical model of these processes can be specified for any number of layers. It is presented in the form of three types of blocks, combining which you can easily synthesize the setup of the model of an n-layer structure. In this case, blocks can simulate individual physical layers of a structure or correspond to groups of intermediate layers into which physical layers can be divided to improve the accuracy of modeling.
the latter case, it is necessary to additionally solve the problem of minimizing the number of intermediate layers for the purpose of ensuring acceptable calculation accuracy.

The important feature of the obtained model is its orientation towards application in the field of synthesis of systems for automatic control of air temperature in a room, where the apparatus of transfer functions is widely used. The linearity of the created model makes it possible to obtain transfer functions connecting the variables of the generalized control object, either by solving a system of operator equations or by the method of structural transformations. In the future, these can be used in the synthesis of automatic control systems for the thermal regime of a building.

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