Zeta Functions of the Dirac Operator on Quantum Graphs

JM Harrison,¹,a) T Weyand,¹,b) and K Kirsten²,c)

¹)Department of Mathematics, Baylor University, Waco, TX 76798, USA
²)GCAP-CASPER, Department of Mathematics, Baylor University, Waco, TX 76798, USA

(Dated: 28 June 2016)

We construct spectral zeta functions for the Dirac operator on metric graphs. We start with the case of a rose graph, a graph with a single vertex where every edge is a loop. The technique is then developed to cover any finite graph with general energy independent matching conditions at the vertices. The regularized spectral determinant of the Dirac operator is also obtained as the derivative of the zeta function at a special value. In each case the zeta function is formulated using a contour integral method, which extends results obtained for Laplace and Schrödinger operators on graphs.

PACS numbers: 03.65.-w, 02.10.Ox, 03.65.Pm

a)Electronic mail: jon_harrison@baylor.edu
b)Electronic mail: tracy_weyand@baylor.edu
c)Electronic mail: klaus_kirsten@baylor.edu
I. INTRODUCTION

Quantum graphs are well studied objects in mathematical physics where they are typically employed to model properties of quasi-one-dimensional structures like carbon nanotubes and photonic crystals, or to investigate the quantum mechanics of systems with chaotic classical dynamics\cite{2,16}. While most investigation of quantum graphs focuses on the Laplace and Schrödinger operators, there has also been recent interest in properties of the Dirac operator, particularly in relation to thin carbon structures where surprisingly the Dirac equation provides the effective model for non-relativistic electronic properties of the system\cite{4,12,17}.

In this article we construct spectral zeta functions of the Dirac operator on metric graphs. A lot of work has gone into understanding the Ihara zeta function associated with combinatorial graphs. The Ihara zeta function of a combinatorial graph is defined by an Euler product over the set of primitive closed loops which do not involve backtracking (primitive periodic orbits)\cite{13,18,19}. In addition to the adjacency structure of a combinatorial graph, a quantum graph consists of intervals connecting pairs of adjacent vertices equipped with a differential operator. Here, rather than the Ihara zeta function, it is natural to consider the zeta function associated with the point spectrum \{\lambda_j\} of the quantum graph,

\[
\zeta(s) = \sum_j \lambda_j^{-s},
\]

where the prime denotes that any eigenvalues of zero are omitted. If the eigenvalues are the positive integers this is just the Riemann zeta function. The results that we report here for the Dirac operator extend similar results obtained by some of the authors for the zeta functions of the Laplace\cite{9} and Schrödinger operators\cite{10}.

The article is laid out as follows. Section III defines the Dirac operator on a graph. In section III we construct the spectral zeta function of the Dirac operator on a rose graph, a graph with a single vertex where every edge is a loop. This explicit example contains the main features of the contour integral approach we adopt. Developing this technique we formulate the zeta function of a general graph Dirac operator first without mass, section IV and then with mass, section V. As a corollary the zeta-regularized spectral determinant is obtained from the zeta function in each case. The rose graph example we use has a natural analogy with the well studied case of the Laplacian on a star graph with the standard Neumann-like vertex conditions, where the wavefunction is continuous at vertices and the
outgoing derivatives sum to zero. As the Dirac extension of this canonical example is not well known, for completeness, we include the derivation of the secular equation of the rose graph as an appendix.

II. THE DIRAC OPERATOR ON A GRAPH

A graph $G$ consists of a set of vertices $\mathcal{V}$ with some pairs of vertices connected by bonds, so a bond $b = (u, v)$ consists of an unordered pair of vertices $u, v \in \mathcal{V}$; see figure 1. We extend $G$ to a metric graph where each bond $b = (u, v)$ is associated with an interval $[0, L_b]$. The vertices $u$ and $v$ lie at the ends of the interval and the choice of orientation for the interval will turn out to be arbitrary. We refer to $L_b$ as the length of $b$, and the bonds are enumerated so that $b \in \mathcal{B} = \{1, 2, \ldots, B\}$. Here we consider only finite graphs where there are a finite number of bonds $\mathcal{B}$ and the length of every interval is finite.

A quantum graph is a metric graph together with a differential operator that acts on functions defined on the set of intervals associated with the bonds. In this paper, we consider the one-dimensional time-independent Dirac operator on the intervals,

$$D := -i\hbar c \alpha \frac{d}{dx} + mc^2 \beta,$$

where $\alpha$ and $\beta$ are matrices that satisfy $\alpha^2 = \beta^2 = I$ and $\alpha \beta + \beta \alpha = 0$, the Dirac algebra in one dimension. To simplify notation, from now on we assume $\hbar = c = 1$. We may think of a Dirac operator on a metric graph as representing the restriction of the Dirac equation in three dimensions to a one-dimensional network. Hence, it is natural to require $\alpha$ and $\beta$ to be $4 \times 4$ matrices. (One might instead choose $\alpha$ and $\beta$ to be $2 \times 2$ matrices, the simplest irreducible representation of the one-dimensional Dirac algebra. However, in this case the wavefunctions depend on the orientation of the intervals which is unphysical. To resolve
this and allow time-reversal invariance, one must use pairs of edges, one oriented in each direction. The Hilbert space for our operator is the direct sum of Hilbert spaces for each bond

$$\mathcal{H} = \bigoplus_{b=1}^{B} L^2([0, L_b]) \otimes \mathbb{C}^4.$$  \hspace{1cm} (3)

To fix a domain of functions in $\mathcal{H}$ on which $\mathcal{D}$ is self-adjoint, we must introduce appropriate matching conditions at the vertices. General vertex conditions can be specified via a pair of $4B \times 4B$ matrices $A$ and $B$. A function $\psi \in \mathcal{H}$ satisfies the boundary conditions if

$$A \psi^+ + B \psi^- = 0 \hspace{1cm} (4)$$

where

$$\psi^+ = \left( \psi_1^1(0), \psi_2^1(0), \ldots, \psi_1^B(0), \psi_2^B(0), \psi_1^1(L_1), \psi_2^1(L_1), \ldots, \psi_1^B(L_B), \psi_2^B(L_B) \right)^T, \hspace{1cm} (5)$$

$$\psi^- = \left( -\psi_3^1(0), \psi_4^1(0), \ldots, -\psi_3^B(0), \psi_4^B(0), \psi_3^1(L_1), -\psi_4^1(L_1), \ldots, \psi_3^B(L_B), -\psi_4^B(L_B) \right)^T. \hspace{1cm} (6)$$

Here $\psi_j^b(x_b)$ is the $j$-th component of the four component function on the bond $b$. Then $\mathcal{D}$ is self-adjoint if and only if

$$\text{rank}(A, B) = 4B \quad \text{and} \quad AB^\dagger = BA^\dagger.$$  

This classification of self-adjoint vertex matching conditions was obtained by Bolte and Harrison and emulates a typical classification of self-adjoint Laplacians on graphs by Kostrykin and Schrader.

Eigenspinors $\psi_k$ that satisfy $\mathcal{D}\psi_k = E(k)\psi_k$ are comprised of plane waves on each bond. While $\alpha$ and $\beta$ are only required to satisfy the Dirac algebra, in order to make the calculations explicit, we choose

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \hspace{1cm} (7)$$
Then, for positive energy, the eigenspinors have the form

\[
\psi^b(x_b) = \mu^b_{\alpha}(1) e^{ikx_b} + \mu^b_{\beta}(0) e^{i\gamma(k)} + i\hat{\gamma}(k) \
\]

with

\[
\gamma(k) := \frac{\sqrt{k^2 + m^2} - m}{k}, \quad E(k) = \sqrt{k^2 + m^2},
\]

while negative energy eigenspinors that satisfy \( \mathcal{D}\psi_k = -E(k)\psi_k \) have the form,

\[
\psi^b(x_b) = \mu^b_{\alpha}(0) e^{ikx_b} + \mu^b_{\beta}(1) e^{i\gamma(k)} + i\hat{\gamma}(k) \
\]

\[
\begin{pmatrix}
1 \\
0 \\
i\gamma(k) \\
0
\end{pmatrix}
\]

with the same definitions of \( \gamma \) and \( E \).

Taking the positive energy case, the matching conditions (4) take the form,

\[
\begin{pmatrix}
1 \\
i\gamma(k) \\
0 \\
1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
-\gamma(k) \\
0
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix}
\begin{pmatrix}
I_{2B} \\
e^{iKL} \\
e^{-iKL}
\end{pmatrix}
\begin{pmatrix}
\frac{-I_{2B} \quad I_{2B}}{I_{2B} \quad -e^{-iKL}}
\end{pmatrix}
\begin{pmatrix}
\mu \\
\hat{\mu}
\end{pmatrix} = 0,
\]

where \( e^{iKL} \) denotes the diagonal matrix \( \text{diag}\{e^{ikL_1}, e^{ikL_2}, \ldots, e^{ikL_B}, e^{ikL_B}\} \) and furthermore \( L = \text{diag}\{L_1, L_1, L_2, L_2, \ldots, L_B, L_B\} \) is used to define other diagonal matrices similarly. The vector \( \mu = (\mu^1_{\alpha}, \mu^1_{\beta}, \mu^2_{\alpha}, \mu^2_{\beta}, \ldots, \mu^B_{\alpha}, \mu^B_{\beta})^T \) and the vector \( \hat{\mu} \) is defined similarly. We see that if the coefficients \( \mu \) and \( \hat{\mu} \) define an eigenspinor, then,

\[
\det \left( \begin{pmatrix}
I_{2B} \\
e^{iKL} \\
e^{-iKL}
\end{pmatrix} + i\gamma(k) \begin{pmatrix}
-1 \\
e^{-iKL} \\
e^{iKL}
\end{pmatrix} \begin{pmatrix}
\mu \\
\hat{\mu}
\end{pmatrix} \right) = 0.
\]

For \( k \notin \{n\pi/L_b\}_{n \in \mathbb{N}, b \in B} \), multiplying on the right by

\[
\begin{pmatrix}
-I_{2B} \\
I_{2B}
\end{pmatrix}^{-1} \begin{pmatrix}
e^{iKL} \\
e^{-iKL}
\end{pmatrix} = \begin{pmatrix}
e^{-iKL} \\
e^{iKL}
\end{pmatrix} \begin{pmatrix}
-I_{2B} \\
I_{2B}
\end{pmatrix} \begin{pmatrix}
\frac{-I_{2B}}{2i \sin kL} \\
0
\end{pmatrix} = 0.
\]

For \( k \notin \{n\pi/L_b\}_{n \in \mathbb{N}, b \in B} \), multiplying on the right by

\[
\det \left( \begin{pmatrix}
I_{2B} \\
e^{iKL} \\
e^{-iKL}
\end{pmatrix}^{-1} \right) = \det \left( \begin{pmatrix}
e^{-iKL} \\
e^{iKL} \\
-I_{2B}
\end{pmatrix} \begin{pmatrix}
\frac{-I_{2B}}{2i \sin kL} \\
0
\end{pmatrix} \right) = 0.
\]
we obtain instead of (12),

$$\det \begin{pmatrix} A + \gamma(k)B \begin{pmatrix} \cot kL & -\csc kL \\ -\csc kL & \cot kL \end{pmatrix} \end{pmatrix} = 0.$$  \hspace{1cm} (14)

Equation (14) is a secular equation for the Dirac operator. Roots $k_j$ of the left hand side correspond to energy eigenvalues according to (9). Notice that in this form the matching conditions appear explicitly in the secular equation.

Following the same argument for negative energy eigenspinors one obtains the secular equation,

$$\det \begin{pmatrix} \gamma(k)A - B \begin{pmatrix} \cot kL & -\csc kL \\ -\csc kL & \cot kL \end{pmatrix} \end{pmatrix} = 0.$$  \hspace{1cm} (15)

We will denote roots of the negative energy secular equation $\tilde{k}_j$, where each root corresponds to a negative eigenvalue $-\sqrt{\tilde{k}_j^2 + m^2}$.

If $m = 0$ the positive energy secular equation (14) reads,

$$\det \begin{pmatrix} A + B \begin{pmatrix} \cot kL & -\csc kL \\ -\csc kL & \cot kL \end{pmatrix} \end{pmatrix} = 0,$$  \hspace{1cm} (16)

and the negative energy equation (15) reads,

$$\det \begin{pmatrix} A - B \begin{pmatrix} \cot kL & -\csc kL \\ -\csc kL & \cot kL \end{pmatrix} \end{pmatrix} = 0.$$  \hspace{1cm} (17)

Changing $k$ to $-k$ we see that the equations agree. This is not surprising as, with $m = 0$, changing the sign of $k$ changes positive to negative energy eigenspinors and vice versa. Hence for $m = 0$ we consider (16) to be the single secular equation whose positive roots are the positive energy eigenvalues and whose negative roots are the negative energy eigenvalues.

For matrices $A$ and $B$ that define a self-adjoint realization of the Dirac operator, $(A - i\gamma(k)B)$ is invertible and we can write (13) in the form,

$$\bar{\mu} = -(A - i\gamma(k)B)^{-1}(A + i\gamma(k)B) \begin{pmatrix} 0 & e^{ikL} \\ e^{ikL} & 0 \end{pmatrix} \bar{\mu},$$  \hspace{1cm} (18)

where

$$\bar{\mu} = \begin{pmatrix} I & 0 \\ 0 & e^{-ikL} \end{pmatrix} \begin{pmatrix} \mu \\ \hat{\mu} \end{pmatrix}.$$  \hspace{1cm} (19)
Then

\[ T = -(A - i\gamma(k)B)^{-1}(A + i\gamma(k)B) \quad (20) \]

is called the transition matrix. When \( A \) and \( B \) define a self-adjoint realization of the Dirac operator, \( T \) is unitary. Applying \( T \) to a vector of incoming plane wave coefficients at the vertices of the graph yields a vector of outgoing coefficients at the vertices. The matrix

\[
\begin{pmatrix}
0 & e^{ikL} \\
e^{ikL} & 0
\end{pmatrix}
\]

in \([18]\) transforms outgoing coefficients to incoming coefficients at the opposite end of the bond; the phase of the coefficient at the two ends of a bond \( b \) differs by \( e^{ikL_b} \). Alternatively the transition matrix can be obtained by combining scattering matrices at each of the vertices. From \([18]\) we see that an eigenspinor is defined by a vector \( \tilde{\mu} \) of plane wave coefficients invariant under the action of

\[ T \begin{pmatrix} 0 & e^{ikL} \\ e^{ikL} & 0 \end{pmatrix}, \]

which is referred to as the quantum evolution operator. For such an eigenspinor,

\[ \det \left( I - T \begin{pmatrix} 0 & e^{ikL} \\ e^{ikL} & 0 \end{pmatrix} \right) = 0, \quad (21) \]

an alternate form of the secular equation.

A. Effect of time-reversal symmetry

For the Dirac equation the standard representation of the time-reversal operator is,

\[ \mathcal{T} = i \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \mathcal{K}, \quad (22) \]

where \( \mathcal{K} \) is complex conjugation and \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). \( \mathcal{T} \) is an anti-unitary operator which squares to \(-I\). As shown in \([3]\), time-reversal symmetry requires the transition matrix satisfy,

\[ T^T = \begin{pmatrix} J^{-1} & & \\ & \ddots & \\ & & J^{-1} \end{pmatrix} T \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix} \quad (23) \]
If we choose matrices $\mathbb{A}$ and $\mathbb{B}$ of the form
\[
\mathbb{A} = \left( \tilde{\mathbb{A}} \otimes \mathbb{I}_2 \right) U \quad \mathbb{B} = \left( \tilde{\mathbb{B}} \otimes \mathbb{I}_2 \right) U \tag{24}
\]
where $U$ is a block diagonal matrix
\[
U = \text{diag}\{u_1^o, \ldots, u_B^o, u_1^t, \ldots, u_B^t\} \tag{25}
\]
with $u_b^o \in \text{SU}(2)$ and $o, t$ standing for the origin and terminus of the bond $b$ respectively, then the Dirac operator is self-adjoint if $\text{rank}(\tilde{\mathbb{A}}, \tilde{\mathbb{B}}) = 2B$ and $\tilde{\mathbb{A}} \tilde{\mathbb{B}}^\dagger$ is self-adjoint. Furthermore, if
\[
\bar{T} := -\left( \tilde{\mathbb{A}} - i\tilde{\mathbb{B}} \right)^{-1} \left( \tilde{\mathbb{A}} + i\tilde{\mathbb{B}} \right) \tag{26}
\]
is symmetric, then $T$ satisfies $\text{(23)}$ and $\mathbb{A}$ and $\mathbb{B}$ define time-reversal symmetric boundary conditions for the Dirac operator. Time-reversal symmetry for the Laplace operator requires a symmetric transition matrix, so matching conditions $\tilde{\mathbb{A}}$ and $\tilde{\mathbb{B}}$ that define self-adjoint time-reversal symmetric matching conditions for the Laplacian provide a pair of matrices $\mathbb{A}$ and $\mathbb{B}$ defining a self-adjoint time-reversal symmetric realization of the Dirac operator.

If we consider the case of a graph with time-reversal symmetry where the vertex matching conditions have the form $\text{(24)}$, the secular equation $\text{(14)}$ simplifies significantly,
\[
\det \left( \left( \tilde{\mathbb{A}} \otimes \mathbb{I}_2 \right) U + \gamma(k) \left( \tilde{\mathbb{B}} \otimes \mathbb{I}_2 \right) U \left( \begin{array}{cc} \cot kL & -\csc kL \\ -\csc kL & \cot kL \end{array} \right) \right) = 0 . \tag{27}
\]
Post multiplying by $U^{-1}$,
\[
\det \left( \left( \tilde{\mathbb{A}} \otimes \mathbb{I}_2 \right) + \gamma(k) \left( \tilde{\mathbb{B}} \otimes \mathbb{I}_2 \right) \left( \begin{array}{cc} \cot kL & -\csc kLW \\ -\csc kLW^{-1} & \cot kL \end{array} \right) \right) = 0 , \tag{28}
\]
where $W = \text{diag}\{w^1, \ldots, w^B\}$ with $w^b = u_b^o(u_b^t)^{-1} \in \text{SU}(2)$. Each $w^b$ can be diagonalized so
\[
w^b = (w^b)^{-1} \left( \begin{array}{cc} e^{i\theta_b} & 0 \\ 0 & e^{-i\theta_b} \end{array} \right) \omega^b . \tag{29}
\]
Setting $\Omega = \text{diag}\{\omega^1, \ldots, \omega^B\}$ and noting that $\Omega$ commutes with $\csc kL$ and $\cot kL$,
\[
\left( \Omega^{-1} \quad 0 \right) \left( \tilde{\mathbb{A}} \otimes \mathbb{I}_2 \right) + \gamma(k) \left( \tilde{\mathbb{B}} \otimes \mathbb{I}_2 \right) \left( \begin{array}{cc} \cot kL & -\csc kLW \\ -\csc kLW^{-1} & \cot kL \end{array} \right) \left( \begin{array}{cc} \Omega & 0 \\ 0 & \Omega \end{array} \right) , \tag{30}
\]
where $D = \text{diag}\{e^{i\theta_1}, e^{-i\theta_1}, e^{i\theta_2}, \ldots, e^{-i\theta_B}\}$. Since $\det \Omega = 1$,

$$
\det \left( \tilde{A} \otimes I_2 \right) + \gamma(k) \left( \tilde{B} \otimes I_2 \right) \begin{pmatrix} \cot kL & -\csc kLD \\ -\csc kLD^{-1} & \cot kL \end{pmatrix} = 0. \tag{31}
$$

Now, applying row and column permutations to separate the odd and even rows and columns, we see that

$$
\det \begin{pmatrix} \tilde{A} + \gamma(k)\tilde{B} \end{pmatrix} \begin{pmatrix} \cot kL & -\csc kL e^{i\theta} \\ -\csc kL e^{-i\theta} & \cot kL \end{pmatrix} = 0, \tag{32}
$$

where $\tilde{L} = \text{diag}\{L_1, \ldots, L_B\}$. Notice that if the matrices $A$ and $B$ defining the vertex conditions are real this is just,

$$
\left| \det \begin{pmatrix} \tilde{A} + \gamma(k)\tilde{B} \end{pmatrix} \begin{pmatrix} \cot kL & -\csc kL e^{i\theta} \\ -\csc kL e^{-i\theta} & \cot kL \end{pmatrix} \right|^2 = 0. \tag{33}
$$

In either case the reduced form of the secular equation allow the subsequent results for the zeta function to be evaluated more easily in the presence of time-reversal invariance for any particular graph.

**B. Spectral zeta function**

In this paper we construct and analyze the spectral zeta function of the Dirac operator on a quantum graph. The spectral zeta function generalizes the Riemann zeta function where the sum over integers is replaced with a sum over the graph eigenvalues,

$$
\zeta(s) = 2 \sum_{j=1}^{\infty} \left[ E(k_j)^{-s} + (-E(\tilde{k}_j))^{-s} \right] = 2 \sum_{j=1}^{\infty} \left[ (k_j^2 + m^2)^{-s/2} + (-1)^{-s}(k_j^2 + m^2)^{-s/2} \right]. \tag{34}
$$

The factor of two simply incorporates Kramers’ degeneracy and the prime indicates that any zero eigenvalues are ignored. The sum runs over positive roots of both the positive and negative energy secular equations. In the zero mass case this simplifies, and

$$
\zeta(s) = 2 \sum_{j=-\infty}^{\infty} \tilde{k}_j^{-s}, \tag{35}
$$

where $k_j$ is a positive or negative non-zero root of (16).
III. THE DIRAC ROSE GRAPH

We will construct the zeta functions of general graph Dirac operators. However, to
illustrate the technique, we first consider an example which can be analyzed more explicitly,
the rose graph. We will see that this is the Dirac analogue of the star graph with Neumann-
like matching conditions that is frequently studied in the case of the Laplace operator. A
rose graph consists of one vertex and $B$ bonds where each bond is a loop that begins and
ends at the vertex; see figure 2.

![Figure 2. A rose graph with eight bonds.](image)

Let

$$v^b(x_b) = \begin{pmatrix} \psi_1^b(x_b) \\ \psi_2^b(x_b) \end{pmatrix} \quad \text{and} \quad w^b(x_b) = \begin{pmatrix} -\psi_4^b(x_b) \\ \psi_3(x_b) \end{pmatrix}. \quad (36)$$

We consider the following self-adjoint matching conditions at the central vertex. Firstly

$$u^0_b v^b(0) = u^t_b v^b(L_b) = \xi \quad (37)$$

for all bonds $b$ where $\xi$ is not a constant vector but rather a placeholder for the value of the
$2B$ vectors which agree at the vertex. Secondly

$$\sum_{b=1}^{B} u^0_b w^b(0) - \sum_{b=1}^{B} u^t_b w^b(L_b) = 0. \quad (38)$$

Recall, $u^b \in SU(2)$ with $o, t$ standing for the origin and terminus of $b$.

Such matching conditions are analogous to the commonly studied Neumann-like boundary
conditions for the graph Laplacian. In the Dirac case the elements of $SU(2)$ generate spin
rotation at the vertices. These vertex conditions can be encoded by a pair of matrices $A$
\[ \begin{bmatrix}
1 & -1 & 0 & \ldots & 0 \\
0 & 1 & -1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & -1 \\
0 & \ldots & 0 & 0 & 0 \\
\end{bmatrix}, \quad 
\begin{bmatrix}
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 \\
1 & 1 & \ldots & 1 \\
\end{bmatrix}. \] (39)

In the case of the rose graph without mass, the secular equation (16) takes a particularly simple form
\[ \sum_{b=1}^{B} \frac{\cos \theta_b - \cos kL_b}{\sin kL_b} = 0, \quad \text{where} \quad \cos \theta_b = \frac{1}{2} \text{tr}(u^b_0(u^b_t)^{-1}). \] (40)

The argument is given in Appendix A for completeness. The roots of (40) are energy eigenvalues of the Dirac rose. To see the analogy with the Laplace operator on a star graph with Neumann-like matching conditions, let \( \theta_b = 0 \) for all bonds \( b \), which corresponds to switching off the spin rotation at the vertices. Then the secular equation becomes
\[ \sum_{b=1}^{B} \tan \frac{kL_b}{2} = 0, \] (41)

which is the secular equation of the Laplace operator on a star graph with Neumann-like matching conditions and bonds of length \( L_b/2 \).

Let
\[ f(z) = z^{B} \frac{\cos \theta_b - \cos L_b z}{\sin L_b z} \] (42)

where \( z \in \mathbb{C} \). The zeros \( k_j \) of \( f \) are the eigenvalues of the Dirac operator while the factor of \( z \) in the definition of \( f \) removes the pole at the origin. By the argument principle, we can express the zeta function of the Dirac operator with zero mass as
\[ \zeta(s) = \frac{1}{\pi i} \int_C z^{-s} \frac{d}{dz} \log f(z) \, dz, \] (43)

where the contour \( C \) encloses both positive and negative zeros of \( f \) and avoids poles.

Poles of \( f \) lie on the real axis at integer multiples of \( \pi/L_b \) for each bond \( b \). From the structure of \( f \), we can see that on the positive real axis \( f \) is strictly increasing between the poles while on the negative real axis it is strictly decreasing between the poles. Hence, for incommensurate bond lengths, \( f \) has exactly one zero between each pair of adjacent poles.
(treating zero as a pole). We locate the branch cut of the logarithm at an angle $\alpha$ to the positive real axis with $0 < \alpha < \pi$ to avoid the zeros and poles of $f$. Figure 3 (i) shows the contour $C$ used to construct the zeta function.

![Contour C](image)

\[ \text{Contour } C \]

\[ \text{Contour } C' \]

FIG. 3. Contours $C$ and $C'$ are shown in (i) and (ii) respectively. The branch cut of the logarithm is located at an angle $\alpha$ to the positive real axis to avoid the zeros and poles of $f$ which are shown with filled and empty circles respectively.

To analyze $\zeta(s)$, we deform the contour from $C$ to $C'$, see figure 3 (ii). The horizontal part of the contour $C'$ will be sent to $-\infty$ in the imaginary coordinate. Given the form of $C'$, it is natural to break $\zeta$ into three parts,

\[ \zeta(s) = \zeta_p(s) + \zeta_b(s) + \zeta_l(s), \quad (44) \]

where $\zeta_p$ is the contribution from the poles of $f$, $\zeta_b$ is the integral around the branch cut and $\zeta_l$ is the integral along the horizontal line.

To evaluate $\zeta_p(s)$ we must subtract residues $-z^{-s}$ at the poles $z$ of $f$,

\[ \zeta_p(s) = 2 \sum_{b=1}^{B} \left( \sum_{n=-\infty}^{-1} \left( \frac{n\pi}{L_b} \right)^{-s} + \sum_{n=1}^{\infty} \left( \frac{n\pi}{L_b} \right)^{-s} \right) \]

\[ = 2(e^{-i\pi s} + 1) \zeta_R(s) \sum_{b=1}^{B} \left( \frac{\pi}{L_b} \right)^{-s}, \quad (45) \]

where $\zeta_R(s)$ is the Riemann zeta function.

The integral along the horizontal line where $z = k + it$ is

\[ \zeta_l(s) = \frac{1}{\pi i} \int_{-\infty}^{\infty} (k + it)^{-s} \frac{d}{dk} \log f(k + it) \, dk, \quad (46) \]
which we consider in the limit \( t \to -\infty \). We observe that
\[
 f(k + it) = i(k + it) \sum_{b=1}^{B} \left( \frac{2 \cos \theta_b - (e^{i(k+it)L_b} + e^{-i(k+it)L_b})}{e^{i(k+it)L_b} - e^{-i(k+it)L_b}} \right)
\]
\[
 \sim k + it
\] (47)
and therefore \( \frac{d}{dz} \log f(z) \sim 1/z \). Hence \( \zeta(s) \to 0 \) provided \( \text{Re}(s) > 0 \).

Finally, the integral around the branch cut is,
\[
 \zeta_b(s) = \frac{1}{\pi i} \int_{0}^{\infty} (ue^{i\alpha})^{-s} \frac{d}{du} \log f(ue^{i\alpha}) \, du
\] (48)
\[
 = e^{i(\pi-\alpha)s} \frac{2 \sin \pi s}{\pi} \int_{0}^{\infty} u^{-s} \frac{d}{du} \log f(ue^{i\alpha}) \, du .
\] (49)

Repeating the argument used for \( \zeta_b \), we see \( \frac{d}{du} \log f(ue^{i\alpha}) \sim 1/u \) as \( u \to \infty \). Hence the integral (49) converges at infinity provided \( \text{Re}(s) > 0 \). On the other hand,
\[
 f(0) = \sum_{b=1}^{B} \frac{\cos \theta_b - 1}{L_b} ,
\] (50)
and as \( u \to 0 \)
\[
 \frac{d}{du} f(ue^{i\alpha}) = e^{i\alpha} \sum_{b=1}^{B} \frac{\cos \theta_b - \cos uL_b e^{i\alpha}}{\sin uL_b e^{i\alpha}} + ue^{i\alpha} \sum_{b=1}^{B} \frac{L_b e^{i\alpha}(1 - \cos \theta_b \cos uL_b e^{i\alpha})}{\sin^2 uL_b e^{i\alpha}}
\] (51)
\[
 \sim e^{i\alpha} \sum_{b=1}^{B} \left( \frac{\cos \theta_b - (1 - (uL_b e^{i\alpha})^2/2 + \ldots)}{uL_b e^{i\alpha}} + 1 - \cos \theta_b (1 - (uL_b e^{i\alpha})^2/2 + \ldots) \right)
\] (52)
\[
 \sim u .
\] (53)

Therefore the integral (49) converges in the strip \( 0 < \text{Re}(s) < 2 \).

Combining the results,
\[
 \zeta(s) = 2(e^{-i\pi s} + 1)\zeta_R(s) \sum_{b=1}^{B} \left( \frac{\pi}{L_b} \right)^{-s} + e^{i(\pi-\alpha)s} \frac{2 \sin \pi s}{\pi} \int_{0}^{\infty} u^{-s} \frac{d}{du} \log f(ue^{i\alpha}) \, du ,
\] (54)
for \( 0 < \text{Re}(s) < 2 \).

To obtain an analytic continuation of \( \zeta(s) \) also valid for \( \text{Re}(s) \leq 0 \), we can split the integral at \( u = 1 \). The restriction to \( \text{Re}(s) > 0 \) came from the behavior at infinity. Let
\[
 \tilde{f}(u) = \sum_{b=1}^{B} \frac{\cos \theta_b - \cos L_b e^{i\alpha} u}{\sin L_b e^{i\alpha} u} .
\] (55)
Then
\[ \hat{f}(u) = \sum_{b=1}^{B} \frac{2 \cos \theta_b - (e^{i u L_b \cos \alpha} + e^{-i u L_b \cos \alpha})}{e^{i u L_b \cos \alpha} - e^{-i u L_b \cos \alpha}} \sim i B , \tag{56} \]
in the limit \( u \to \infty \). Similarly
\[ \frac{d}{du} \hat{f}(u) = e^{i \alpha} \sum_{b=1}^{B} L_b \left( 1 - \cos \theta_b \cos u L_b e^{i \alpha} \right) \sim e^{-c u} , \tag{57} \]
for some \( c > 0 \). Therefore \( \frac{d}{du} \log \hat{f}(u) \sim e^{-c u} \) as \( u \to \infty \). Then, writing \( \log f(u e^{i \alpha}) = \log u e^{i \alpha} + \log \hat{f}(u) \), one can expand the integral in (54) to obtain the following theorem.

**Theorem 1.** For the Dirac operator with zero mass on a rose graph with vertex conditions given by (37) and (38), the zeta function for \( \text{Re}(s) < 2 \) is given by,
\[ \zeta(s) = e^{i(\pi - \alpha) s} \frac{2 \sin \pi s}{\pi} \left[ \int_0^1 u^{-s} \frac{d}{du} \log u e^{i \alpha} \hat{f}(u) \, du + \int_1^\infty u^{-s} \frac{d}{du} \log \hat{f}(u) \, du + \frac{1}{s} \right] + 2(e^{-i \pi s} + 1) \zeta_R(s) \sum_{b=1}^{B} \left( \frac{\pi}{L_b} \right)^{-s} \tag{58} \]
where
\[ \hat{f}(u) = \sum_{b=1}^{B} \frac{\cos \theta_b - \cos L_b e^{i \alpha} u}{\sin L_b e^{i \alpha} u} . \tag{59} \]

**A. Spectral determinant of the rose graph**

As a corollary, it is straightforward to evaluate the spectral determinant from the zeta function. The spectral determinant is formally the product of the eigenvalues,
\[ \det'(D) = \prod_{j=-\infty}^{\infty} k_j^2 , \tag{60} \]
where each eigenvalue \( k_j \) has been squared in the product due to Kramers’ degeneracy. The representation of the zeta function in theorem 1 allows a regularized spectral determinant to be evaluated directly, \( \det'(D) = \exp(-\zeta'(0)) \).

Differentiating the pole contribution,
\[ \zeta'_R(0) = i \pi B - 2B \log(2\pi) + 2 \sum_{b=1}^{B} \log \left( \frac{\pi}{L_b} \right) , \tag{61} \]
where we used \( \zeta_R(0) = -1/2 \) and \( \zeta'_R(0) = -\log(2\pi)/2 \).
Notice that,
\[
\frac{d}{ds} e^{i(\pi - \alpha)s} \frac{2 \sin \pi s}{\pi s} = 2e^{i(\pi - \alpha)s} \left( i(\pi - \alpha) \frac{\sin \pi s}{\pi s} + \frac{\pi s \cos \pi s - \sin \pi s}{\pi s^2} \right) \tag{62}
\]
\[
\rightarrow 2i(\pi - \alpha) \tag{63}
\]
as \( s \to 0 \). Hence,
\[
\zeta'(0) = \zeta_p'(0) + i2(\pi - \alpha)
\]
\[
+ 2 \left( \log \left( e^{i\alpha \hat{f}(1)} \right) - \log \left( \sum_{b=1}^{B} \frac{\cos \theta_b - 1}{L_b} \right) + \log (iB) - \log \hat{f}(1) \right) \tag{64}
\]
\[
= i\pi B - 2B \log(2\pi) + 2 \sum_{b=1}^{B} \log \left( \frac{\pi}{L_b} \right) + 2\pi i - 2 \log \left( \sum_{b=1}^{B} \frac{\cos \theta_b - 1}{L_b} \right) + 2\log(iB). \tag{65}
\]
Therefore the spectral determinant is given by,
\[
\det'(D) = \left( -1 \right)^{B+1} B^2 \left( \sum_{b=1}^{B} \frac{\cos \theta_b - 1}{L_b} \right)^2 \prod_{b=1}^{B} (2L_b)^2. \tag{66}
\]
Note that the spectral determinant is independent of \( \alpha \), the angle of the branch cut of the logarithm, as expected.

IV. ZETA FUNCTION OF A GENERAL GRAPH WITHOUT MASS

The contour integral technique introduced in the case of a rose graph extends to any graph whose vertex matching conditions define a self-adjoint Dirac operator. Let
\[
f(z) = z^{4B-1} \det \left( A + B \begin{pmatrix} \cot zL & -\csc zL \\ -\csc zL & \cot zL \end{pmatrix} \right) \tag{67}
\]
\[
= \frac{1}{z} \det \left( zA + zB \begin{pmatrix} \cot zL & -\csc zL \\ -\csc zL & \cot zL \end{pmatrix} \right). \tag{68}
\]
Notice that as \( z \to 0 \),
\[
\det \left( zA + zB \begin{pmatrix} \cot zL & -\csc zL \\ -\csc zL & \cot zL \end{pmatrix} \right) \to \det \left( B \begin{pmatrix} L^{-1} & -L^{-1} \\ -L^{-1} & L^{-1} \end{pmatrix} \right) = 0, \tag{69}
\]
and the first non-zero term in its power series expansion about \( z = 0 \) is therefore proportional to \( z \). Without loss of generality, we assume that \( f(0) \neq 0 \), as if \( f(0) \) happens to be zero
it can be made non-zero by a perturbation of the edge lengths. Hence, \( f \) has roots at the non-zero solutions of the secular equation (16) and,

\[
f(z) \sim c_0 + c_M z^M + O(z^{M+1})
\]

as \( z \to 0 \), with \( c_0, c_M \) the first two non-zero coefficients in the power-series expansion which can be computed for any given graph with fixed vertex conditions.

For a general graph with zero mass, the spectral zeta function is then given by

\[
\zeta(s) = \frac{1}{i\pi} \int_C z^{-s} \frac{d}{dz} \log f(z) \, dz
\]

where \( C \) is the same contour that was used in the rose graph case, figure (3). Note that we also assume that the matrices \( A \) and \( B \) are independent of \( k \); if \( A \) and \( B \) were \( k \) dependent this could change the location of the zeros and poles of \( f \). Again, we develop the spectral zeta function by transforming the contour from \( C \) to \( C' \). Then

\[
\zeta(s) = \zeta_p(s) + \zeta_b(s) + \zeta_l(s) ,
\]

where \( \zeta_p \) is the contribution from the poles of \( f \), \( \zeta_b \) is the integral around the branch cut, and \( \zeta_l \) is the integral along the horizontal line (that will be sent to negative infinity in the imaginary coordinate).

The poles of \( f \) are located at \( z = n\pi/L_b, n \neq 0 \), so as for the rose graph,

\[
\zeta_p(s) = 2(e^{-i\pi s} + 1) \zeta_R(s) \sum_{b=1}^{B} \left( \frac{\pi}{L_b} \right)^{-s}.
\]

On the horizontal line let \( z = k + it \). In the limit \( t \to -\infty \), \( \cot(zL) \to i \) and \( \csc(zL) \to 0 \), therefore \( f(z) \sim z^{4B-1} \det(A + iB) \). This means that \( \frac{d}{dz} \log f(z) \sim 1/z \) and consequently \( \zeta_l(s) \to 0 \) for \( \text{Re}(s) > 0 \).

For the integral around the branch cut, set \( z = u e^{i\alpha} \) where the imaginary part of \( z \) is positive, see figure 3. Then

\[
\zeta_b(s) = e^{i(\pi - \alpha)} \frac{2 \sin \pi s}{\pi} \int_0^\infty u^{-s} \frac{d}{du} \log f(u e^{i\alpha}) \, du .
\]

As \( u \to \infty \),

\[
\det \left( A + B \begin{pmatrix}
\cot uLe^{i\alpha} & -\csc uLe^{i\alpha} \\
-\csc uLe^{i\alpha} & \cot uLe^{i\alpha}
\end{pmatrix} \right) \to \det (A - iB) ,
\]
and hence \( f(ue^{i\alpha}) \sim (ue^{i\alpha})^{4B-1} \det(A - iB) \). Consequently \( \frac{d}{du} \log f(ue^{i\alpha}) \sim 1/u \) and (74) converges at infinity provided that \( \text{Re}(s) > 0 \). From (70) \( \frac{d}{du} \log f(ue^{i\alpha}) \sim u^{M-1} \) as \( u \to 0 \). Therefore (74) converges at zero for \( \text{Re}(s) < M \), and \( \zeta_b(s) \) is defined in the strip \( 0 < \text{Re}(s) < M \) where \( M \geq 1 \).

Combining the results,

\[
\zeta(s) = 2(1 + e^{-\pi s})\zeta_R(s) \sum_{b=1}^{B} \left( \frac{\pi}{L_b} \right)^{-s} + e^{i(\pi-\alpha)s}\frac{2\sin\pi s}{\pi} \int_0^{\infty} u^{-s} \frac{d}{du} \log f(ue^{i\alpha}) \, du
\]

for \( 0 < \text{Re}(s) < M \).

To remove the restriction to \( \text{Re}(s) > 0 \) we again split the integral at \( u = 1 \). Let

\[
\hat{f}(u) = \det \left( A + iB \begin{pmatrix} \cot ue^{i\alpha}L & -\csc ue^{i\alpha}L \\ -\csc ue^{i\alpha}L & \cot ue^{i\alpha}L \end{pmatrix} \right).
\]

(77)

Since \( \text{Im} e^{i\alpha} > 0 \),

\[
\hat{f}(u) \sim \det \left( A - iB \begin{pmatrix} e^{2iLe^{i\alpha}u} + 1 & -2e^{iLe^{i\alpha}u} \\ -2e^{iLe^{i\alpha}u} & e^{2iLe^{i\alpha}u} + 1 \end{pmatrix} \right)
\]

(78)

as \( u \to \infty \). Consequently \( \hat{f}'/\hat{f} \) decays exponentially as \( u \to \infty \). Then splitting the integral at \( u = 1 \) and using \( \log f(ue^{i\alpha}) = \log(\hat{f}(u)) + (4B - 1)(i\alpha + \log u) \) to develop the integral from 1 to \( \infty \), we obtain the following theorem.

**Theorem 2.** For the Dirac operator with zero mass on a graph with local vertex matching conditions defined by an energy independent pair of matrices \( A \) and \( B \) with \( AB^\dagger = BA^\dagger \), the zeta function is given by,

\[
\zeta(s) = e^{i(\pi-\alpha)s}\frac{2\sin\pi s}{\pi} \left[ \int_0^1 u^{-s} \frac{d}{du} \log \left( (ue^{i\alpha})^{4B-1} \hat{f}(u) \right) \, du + \int_1^{\infty} u^{-s} \frac{d}{du} \log \hat{f}(u) \, du \right] + e^{i(\pi-\alpha)s}\frac{2(4B - 1)\sin\pi s}{\pi s} + 2(1 + e^{-\pi s})\zeta_R(s) \sum_{b=1}^{B} \left( \frac{\pi}{L_b} \right)^{-s},
\]

where \( \text{Re}(s) < M \) for some \( M > 1 \) and

\[
\hat{f}(u) = \det \left( A + B \begin{pmatrix} \cot ue^{i\alpha}L & -\csc ue^{i\alpha}L \\ -\csc ue^{i\alpha}L & \cot ue^{i\alpha}L \end{pmatrix} \right).
\]

(79)

(80)
A. Spectral determinant with zero mass

The regularized spectral determinant is \( \det'(D) = \exp(-\zeta'(0)) \). Using the analogy with the rose graph case, which has the same pole contribution, we have

\[
\zeta'(0) = \zeta_p'(0) + 2 \left( \log \left( f(e^{i\alpha}) \right) - \log \left( \lim_{u \to 0} f(u e^{i\alpha}) \right) + \log \left( \lim_{u \to \infty} \hat{f}(u) \right) - \log \left( \hat{f}(1) \right) \right) \\
+ 2i(4B - 1)(\pi - \alpha)
\]

\[
= \zeta_p'(0) + 2i(4B - 1)\alpha - 2 \log c_0 + 2 \log(\det(A - iB)) + 2i(4B - 1)(\pi - \alpha)
\]

\[
= i\pi B - 2B \log(2\pi) + 2 \sum_{b=1}^{B} \log \left( \frac{\pi}{L_b} \right) - 2 \log c_0 \\
+ 2 \log(\det(A - iB)) + 2i(4B - 1)\pi
\]

(81)

where we used \( f(e^{i\alpha}) = e^{i(4B - 1)\alpha} \hat{f}(1) \) and (75). The constant \( c_0 \) comes from the power series expansion of \( f \) about zero; see (70). Therefore the spectral determinant is,

\[
\det'(D) = \frac{c_0^2 (-1)^B}{(\det(A - iB))^2} \prod_{b=1}^{B} (2L_b)^2 .
\]

(82)

V. ZETA FUNCTION OF A GENERAL GRAPH WITH MASS

To derive the zeta function with non-zero mass we start from the two secular equations (14) and (15) whose roots correspond to positive and negative energy solutions respectively. Let

\[
f(z) = \det \left( A + \gamma(z)B \begin{pmatrix} \cot zL & - \csc zL \\ - \csc zL & \cot zL \end{pmatrix} \right),
\]

(83)

\[
\hat{f}(t) = \det \left( A + \hat{\gamma}(t)B \begin{pmatrix} \coth tL & - \csch tL \\ - \csch tL & \coth tL \end{pmatrix} \right),
\]

(84)

where

\[
\gamma(z) = \frac{\sqrt{z^2 + m^2} - m}{z},
\]

(85)

\[
\hat{\gamma}(t) = \frac{\sqrt{t^2 - m^2} + im}{t} .
\]

(86)
So \( f(it) = \hat{f}(t) \) and the positive energy secular equation reads \( f(k) = 0 \) for \( k > 0 \). Similarly we set,

\[
g(z) = \text{det} \left( \gamma(z) \mathbb{A} - \mathbb{B} \begin{pmatrix} \cot zL & -\csc zL \\
-\csc zL & \cot zL \end{pmatrix} \right), \tag{87}
\]

\[
\hat{g}(t) = \text{det} \left( \hat{\gamma}(t) \mathbb{A} - \mathbb{B} \begin{pmatrix} \coth tL & -\csch tL \\
-\csch tL & \coth tL \end{pmatrix} \right), \tag{88}
\]

so the negative energy secular equation reads \( g(k) = 0 \) for \( k > 0 \).

The contribution of the positive part of the spectrum to the spectral zeta function is

\[
\zeta^+(s) = \frac{1}{1\pi} \int_C (z^2 + m^2)^{-s/2} \frac{d}{dz} \log f(z) \, dz , \tag{89}
\]

where the contour \( C \), shown in figure 4, is chosen to enclose the zeros of \( f \) while avoiding poles. We locate the branch cut of the logarithm at an angle \( \alpha \) and branch cuts of \((z+im)^{-s/2}\) and \((z-im)^{-s/2}\) on the imaginary axis. Transforming the contour from \( C \) to \( C' \), see figure 4 (ii),

\[
\zeta^+(s) = \zeta_p^+(s) + \zeta_b^+(s) , \tag{90}
\]

where \( \zeta_p^+ \) is the contribution from the poles of \( f \) and \( \zeta_b^+ \) is given by the integral along the imaginary axis.

![Contour C and C'](image)

FIG. 4. Contours \( C \) and \( C' \). We again locate the branch cut of the logarithm at an angle \( \alpha \) and branch cuts of \((z+im)^{-s/2}\) and \((z-im)^{-s/2}\) are located on the imaginary axis. The zeros and poles of \( f \) (or \( g \)) are shown with filled and empty circles respectively.
The poles of \( f \) occur at \( z = n\pi/L_b, n \neq 0 \) so

\[
\zeta^+(s) = 2 \sum_{b=1}^{B} \sum_{n=1}^{\infty} \left( \frac{(n\pi/L_b)^2 + m^2}{2} \right)^{-s/2} \frac{d}{dt} \log \hat{f}(t) dt
\]

\[
= 2 \sum_{b=1}^{B} \left( \frac{\pi}{L_b} \right)^{-s} E \left( \frac{s}{2}, \left( \frac{m L_b}{\pi} \right)^2 \right), \tag{91}
\]

where \( E(\alpha, c) \) is an Epstein type zeta function\(^{5-7,14}\)

\[
E(\alpha, c) = \sum_{n=1}^{\infty} (n^2 + c)^{-\alpha}. \tag{92}
\]

For the imaginary axis integral let \( z = it + \epsilon \) for some sufficiently small \( \epsilon > 0 \).

\[
\zeta^+(s) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \left( (it + \epsilon)^2 + m^2 \right)^{-s/2} \frac{d}{dt} \log \hat{f}(t) dt
\]

\[
= -\frac{1}{i\pi} \int_{0}^{\infty} \left( (it + \epsilon)^2 + m^2 \right)^{-s/2} \frac{d}{dt} \log \hat{f}(t) dt + \frac{1}{i\pi} \int_{0}^{\infty} \left( (it - \epsilon)^2 + m^2 \right)^{-s/2} \frac{d}{dt} \log \hat{f}(t) dt
\]

\[
= \frac{2}{\pi} \sin \left( \frac{\pi s}{2} \right) \int_{m}^{\infty} (t^2 - m^2)^{-s/2} \frac{d}{dt} \log \hat{f}(t) dt, \tag{93}
\]

where we used \( \hat{f}(t) = \hat{f}(-t) \) and we take the limit \( \epsilon \to 0 \) in the last step. We note that

\[
\gamma'(t) = (t^2 - m^2)^{-s/2} - \left( \frac{(t^2 - m^2)^{1/2} + im}{t^2} \right)
\]

\[
= \frac{m^2 - im(t^2 - m^2)^{1/2}}{t^2(t^2 - m^2)^{1/2}}. \tag{94}
\]

Hence as \( t \to m^+ \), the integrand behaves as \( (t^2 - m^2)^{-\frac{(s+1)}{2}} \) and the integral converges at \( t = m \) for \( \text{Re} \, s < 1 \).

As \( t \to \infty \) up to exponentially damped terms,

\[
\hat{f}(t) \sim \det(A + \gamma(t)B)
\]

\[
\sim \det(A + B) + c_1 t^{-1} + \cdots + c_{4B} t^{-4B}. \tag{97}
\]

So \( \hat{f}'/\hat{f} \sim t^{-2} \) and the integral converges at infinity for \( \text{Re} \, s > -1 \).

The contribution of the negative energy eigenvalues is evaluated similarly, namely

\[
\zeta^+(s) = \frac{(-1)^{-s}}{i\pi} \int_{C} (z^2 + m^2)^{-s/2} \frac{d}{dz} \log g(z) dz, \tag{98}
\]

\[
(98)
\]
where \( C \) is now chosen to enclose the zeros and avoid the poles of \( g \). Applying the same contour transformation,

\[
\zeta^-(s) = \zeta_p^-(s) + \zeta_b^-(s)
\]

with the pole contribution,

\[
\zeta_p^-(s) = (-1)^{-s} \zeta_p^+(s)
\]

and

\[
\zeta_b^-(s) = \frac{2(1)^{-s}}{\pi} \sin \left( \frac{\pi s}{2} \right) \int_m^\infty (t^2 - m^2)^{-\frac{s}{2}} \frac{d}{dt} \log \hat{g}(t) \, dt,
\]

which is convergent again in the strip \(-1 < \text{Re} \, s < 1\). Combining the results, we obtain the following theorem.

**Theorem 3.** For the Dirac operator with non-zero mass on a graph with local vertex matching conditions defined by a pair of \( k \) independent matrices \( A \) and \( B \) with \( AB^\dagger = BA^\dagger \), the zeta function in the strip \(-1 < \text{Re} \, s < 1\) is given by

\[
\zeta(s) = \frac{2}{\pi} \sin \left( \frac{\pi s}{2} \right) \left( \int_m^\infty (t^2 - m^2)^{-\frac{s}{2}} \frac{d}{dt} \log \hat{f}(t) \, dt + (-1)^{-s} \int_m^\infty (t^2 - m^2)^{-\frac{s}{2}} \frac{d}{dt} \log \hat{g}(t) \, dt \right) + 2(1 + (-1)^{-s}) \sum_{b=1}^B \left( \frac{\pi}{L_b} \right)^s E \left( \frac{s}{2}, \left( \frac{mL_b}{\pi} \right)^2 \right).
\]

**A. Spectral determinant with mass**

Again the regularized spectral determinant is \( \det'(D) = \exp(-\zeta'(0)) \). Differentiating the pole contribution,

\[
\zeta_p'(0) = \sum_{b=1}^B \left[ 4 \left\{ \frac{1}{2} E' \left( 0, \left( \frac{mL_b}{\pi} \right)^2 \right) + \log \left( \frac{L_b}{\pi} \right) E \left( 0, \left( \frac{mL_b}{\pi} \right)^2 \right) \right\} \\
+ 2\pi i E \left( 0, \left( \frac{mL_b}{\pi} \right)^2 \right) \right].
\]

Using the analytical continuation of the Epstein zeta function\(^5\),

\[
E(\alpha, c) = -\frac{1}{2c^\alpha} + \frac{\sqrt{\pi}}{2c^{\alpha-1/2} \Gamma(\alpha)} \times \left\{ \Gamma \left( \alpha - \frac{1}{2} \right) + 4 \sum_{\ell=1}^\infty \frac{1}{(\pi \ell \sqrt{c})^{1/2-\alpha}} K_{1/2-\alpha}(2\pi \ell \sqrt{c}) \right\},
\]

21
with $K_u(z)$ the Kelvin functions\(^8\), it is easily verified that

$$E(0, c) = -\frac{1}{2}$$

(104)

and

$$E'(0, c) = \frac{1}{2} \log c - \pi \sqrt{c} - \log \left(1 - e^{-2\pi \sqrt{c}}\right).$$

(105)

From here we find

$$\zeta'(0) = 2 \sum_{b=1}^{B} \left[ \log \left(\frac{m}{1 - e^{-2mL_b}}\right) - mL_b\right] - i\pi B.$$  

(106)

Differentiating the integral terms,

$$\zeta'_b(0) = \lim_{t \to \infty} \log \hat{f}(t) - \log \hat{f}(m) + \lim_{t \to \infty} \log \hat{g}(t) - \log \hat{g}(m)$$

$$= \log \det (A + B) + \log \det (A - B) - 2 \log \det \left( A + iB \left( \begin{array}{cc} \coth mL & -\text{csch} mL \\ -\text{csch} mL & \coth mL \end{array} \right) \right).$$

(107)

Hence the spectral determinant is, after elementary simplifications,

$$\det'(\mathcal{D}) = \left( \det \left( A + iB \left( \begin{array}{cc} \coth mL & -\text{csch} mL \\ -\text{csch} mL & \coth mL \end{array} \right) \right) \right)^2 (\det(A + B))^{-1}(\det(A - B))^{-1}$$

$$\times \left(-1\right)^B \prod_{b=1}^{B} \frac{2 \sinh(mL_b)}{m} \left[ \frac{2 \sinh(mL_b)}{m} \right]^2.$$  

(108)

(109)

It is important to note that, the representation of the spectral zeta function in theorem 3 for $m > 0$ is not valid in the limit $m \to 0$ where the integrals diverge. Consequently one cannot take $m \to 0$ in the spectral determinant (109) and the zero mass case must be obtained independently, as was done in the previous section. Despite this, formulas for the spectral determinant with and without mass, equations (109) and (82) respectively, show a number of common features.

VI. CONCLUSIONS

We have constructed the spectral zeta functions of the Dirac operator on finite quantum graphs. The contour integral technique we employ allows one to analyze general graphs
with general, local, energy-independent, vertex conditions in a single calculation. Results for individual graphs are straightforward to extract from the general case given a pair of matrices \( A \) and \( B \) specifying the vertex conditions. The approach combines results for the secular equations of the Dirac operator with the argument principle and allows us to obtain an integral representation of the zeta function valid in a strip of the complex plane, theorems 2 and 3. Analytic continuation to any region of the plane can be obtained from the asymptotic behavior of the secular equation. As a special case, we see that the zeta function of the Dirac rose graph has a particularly simple form, theorem 1. In each case, as a straightforward corollary, we obtained the regularized spectral determinant, equations (66), (82) and (109).

ACKNOWLEDGMENTS

The authors would like to thank Rachel Wilkerson for helpful suggestions. This work was partially supported by a grant from the Simons Foundation (354583 to Jon Harrison).

Appendix A: Derivation of secular equation for a Dirac rose graph

As the test case of the Dirac operator on a rose graph, shown in figure 2, is much less well known than the corresponding case of the Laplace operator on a star graph with Neumann-like matching conditions, we include the derivation of the secular equation\( ^{11} \). Importantly the secular equation can be expressed without using matrices despite the spinor valued nature of the wavefunction on the bonds and the nontrivial spin dynamics at the vertex.

We split the four component wavefunctions on the bonds of the rose into pairs of two component functions,

\[
v^b(x_b) = \begin{pmatrix} \psi^b_1(x_b) \\ \psi^b_2(x_b) \end{pmatrix} \quad \text{and} \quad w^b(x_b) = \begin{pmatrix} -\psi^b_4(x_b) \\ \psi^b_3(x_b) \end{pmatrix}. \tag{A1}
\]

Then the rose graph is given by matching conditions at the vertex such that firstly,

\[
u^b_0 v^b(0) = u^b_L v^b(L_b) = \xi \quad \text{for all } b, \tag{A2}
\]

where \( \xi \) is used not as a fixed vector but rather as a placeholder for the common value at
the vertex. Secondly,

$$\sum_{b=1}^{B} u_o^b w^b(0) = \sum_{b=1}^{B} u_t^b w^b(L_b). \quad (A3)$$

The $u_{o/t}^b$ are SU(2) matrices with $o,t$ standing for the element of SU(2) associated with the origin and terminus of the bond $b$ respectively. Such a vertex condition defines a self-adjoint realization of the Dirac operator on the graph. The definition is analogous to the definition of Neumann-like vertex conditions for the Laplace operator. Applying such vertex conditions to a rose graph rather than a star graph is required in order to produce nontrivial spin dynamics.

Using the plane-wave solution (8) for zero mass, (A2) becomes

$$u_o^b (\mu^b + \hat{\mu}^b) = u_t^b (\mu^b e^{ikL_b} + \hat{\mu}^b e^{-ikL_b}) = \xi, \quad (A4)$$

where

$$\mu^b = \begin{pmatrix} \mu^b_\alpha \\ \mu^b_\beta \end{pmatrix} \quad \text{and} \quad \hat{\mu}^b = \begin{pmatrix} \hat{\mu}^b_\alpha \\ \hat{\mu}^b_\beta \end{pmatrix}. \quad (A5)$$

Then

$$(u_o^b - u_t^b e^{ikL_b}) \mu^b = -(u_o^b - u_t^b e^{-ikL_b}) \hat{\mu}^b, \quad (A6)$$

and

$$\mu^b + \hat{\mu}^b = (u_o^b)^{-1} \xi. \quad (A7)$$

Eliminating $\hat{\mu}^b$

$$\mu^b = \frac{1}{2i \sin kL_b} (u_t^b)^{-1} (u_o^b - u_t^b e^{-ikL_b})(u_o^b)^{-1} \xi, \quad (A8)$$

assuming $\sin kL_b \neq 0$. The second part of the vertex condition (A3) reads

$$i\gamma(k) \sum_{b=1}^{B} u_o^b (-\mu^b + \hat{\mu}^b) = i\gamma(k) \sum_{b=1}^{B} u_t^b (-\mu^b e^{ikL_b} + \hat{\mu}^b e^{-ikL_b}), \quad (A9)$$

which simplifies to

$$\sum_{b=1}^{B} (u_t^b e^{ikL_b} - u_o^b(b)) \mu^b = 0 \quad (A10)$$
using equation \((A6)\). Substituting \((A8)\) into \((A10)\),

\[
\sum_{b=1}^{B} \frac{1}{\sin kL_b} \left( u_t^b e^{ikL_b} - u_o^b \right) \left( u_o^b - u_t^b e^{-ikL_b} \right) (u_o^b)^{-1} \xi = 0
\]

\[
\sum_{b=1}^{B} \frac{1}{\sin kL_b} \left( e^{ikL_b} I_2 - u_o^b(u_t^b)^{-1} \right) \left( I_2 - u_t^b(u_o^b)^{-1} e^{-ikL_b} \right) \xi = 0
\]

\[
\sum_{b=1}^{B} \frac{1}{\sin kL_b} \left( 2 \cos(kL_b) I_2 - u_t^b(u_o^b)^{-1} - u_o^b(u_t^b)^{-1} \right) \xi = 0. \tag{A11}
\]

For each bond we may define an angle \(\theta_b \in [0, \pi]\) via,

\[
u_t^b(u_o^b)^{-1} + u_o^b(u_t^b)^{-1} = \text{tr} \left( u_o^b(u_t^b)^{-1} \right) I = 2 \cos \theta_b I. \tag{A12}
\]

Then \(E(k) = \sqrt{k^2 + m^2}\) is an energy eigenvalue of the rose graph if and only if \(k\) is a solution of

\[
\sum_{b=1}^{B} \frac{\cos \theta_b - \cos kL_b}{\sin kL_b} = 0. \tag{A13}
\]

This is the secular equation from which we derived our results for the rose graph.

REFERENCES

1. G. Berkolaiko, E. B. Bogomolny, and J. P. Keating. Star graphs and Šeba billiards. *J. Phys. A*, 34(3):335–350, 2001.
2. G. Berkolaiko and P. Kuchment. *Introduction to quantum graphs*, volume 186 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.
3. J. Bolte and J. M. Harrison. Spectral statistics for the Dirac operator on graphs. *J. Phys. A: Math. Gen.*, 36:2747, 2003.
4. A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim. The electronic properties of graphene. *Rev. Mod. Phys.*, 81:109, 2009.
5. E. Elizalde and A. Romeo. Expressions for the zeta-function regularized Casimir energy. *J. Math. Phys.*, 30:1133–1139, 1989.
6. P. Epstein. Zur Theorie allgemeiner Zetafunktionen. *Math. Ann.*, 56:615–644, 1903.
7. P. Epstein. Zur Theorie allgemeiner Zetafunktionen II. *Math. Ann.*, 63:205–216, 1907.
8. I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series and Products*. Academic Press, New York, NY, 1965.
9 J. Harrison and K. Kirsten. Zeta functions of quantum graphs. *J. Phys. A: Math. Theor.*, 44:235301, 2011.

10 J. Harrison, K. Kirsten, and C. Texier. Spectral determinants and zeta functions of Schrödinger operators on metric graphs. *J. Phys. A: Math. Theor.*, 45:125206, 2012.

11 J. M. Harrison and B. Winn. Intermediate statistics for a system with symplectic symmetry: the Dirac rose graph. *J. Phys. A: Math. Theor.*, 45:435101, 2012.

12 R. R. Hartmann, N. J. Robinson, and M. E. Portnoi. Smooth electron waveguides in graphene. *Phys. Rev. B*, 81:245431, 2010.

13 K. Hashimoto. Zeta functions of finite graphs and representations of the p-adic groups. *Adv. Studies Pure Math.*, 15:211–280, 1989.

14 K. Kirsten. Generalized multidimensional Epstein zeta functions. *J. Math. Phys.*, 35:459–470, 1994.

15 V. Kostrykin and R. Schrader. Kirchoff’s rule for quantum wires. *J. Phys. A: Math. Gen.*, 32:595–630, 1999.

16 P. Kuchment. Quantum graphs: I. Some basic structures. *Waves Random Media*, 14:S107–S128, 2004.

17 K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov. Two-dimensional gas of massless Dirac fermions in graphene. *Nature*, 438:197–200, 2005.

18 H. M. Stark and A. A. Terras. Zeta functions of finite graphs and coverings. *Adv. Math.*, 121:124–165, 1996.

19 T. Sunada. *L-functions in geometry and some applications*, volume 1201 of *Lecture Notes in Mathematics*, pages 266–284. Springer, Berlin, 1986.