Preference learning along multiple criteria: A game-theoretic perspective

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Abstract

The literature on ranking from ordinal data is vast, and there are several ways to aggregate overall preferences from pairwise comparisons between objects. In particular, it is well known that any Nash equilibrium of the zero-sum game induced by the preference matrix defines a natural solution concept (winning distribution over objects) known as a von Neumann winner. Many real-world problems, however, are inevitably multi-criteria, with different pairwise preferences governing the different criteria. In this work, we generalize the notion of a von Neumann winner to the multi-criteria setting by taking inspiration from Blackwell’s approachability. Our framework allows for non-linear aggregation of preferences across criteria, and generalizes the linearization-based approach from multi-objective optimization.

From a theoretical standpoint, we show that the Blackwell winner of a multi-criteria problem instance can be computed as the solution to a convex optimization problem. Furthermore, given random samples of pairwise comparisons, we show that a simple “plug-in” estimator achieves near-optimal minimax sample complexity. Finally, we showcase the practical utility of our framework in a user study on autonomous driving, where we find that the Blackwell winner outperforms the von Neumann winner for the overall preferences.

1 Introduction

Economists, social scientists, engineers, and computer scientists have long studied models for human preferences, under the broad umbrella of social choice theory [Bla48; Arr51]. Learning from human preferences has found applications in interactive robotics for learning reward functions [SDSS17; PLSS19], in medical domains for personalizing assistive devices [ZFWJ+17; BHKS20], and in recommender systems for optimizing search engines [CJRY12; HWD11]. The recent focus on safety in AI has popularized human-in-the-loop learning methods that use human preferences in order to promote value alignment [CLBM+17; SSSE18; ACKK14].

The most popular form of preference elicitation is to make pairwise comparisons [Thu27; BT52; Luc59]. Eliciting such feedback involves showing users a pair of objects and asking them a query: Do you prefer object A or object B? Depending on the application, an object could correspond to a product in a search query, or a policy or reward function in reinforcement learning. A vast body of classical work dating back to Condorcet and Borda [Con85; Bor84] has focused on defining and producing a “winning” object from the result of a set of pairwise comparisons.
Figure 1. (a) Policy A focuses on optimizing comfort, whereas policy B focuses on speed, and we consider pairwise comparisons of these two policies in different environments. (b) Preference matrices, where entry \((i,j)\) of the matrix contains the proportion of comparisons between the pair \((i,j)\) that are won by object \(i\). (The diagonals are set to half by convention). The overall pairwise comparisons are given by the matrix \(P_{\text{Overall}}\), and preferences along each of the criteria by matrices \(P_{\text{Comfort}}\) and \(P_{\text{Speed}}\). Policy R is a randomized policy \(\frac{1}{2}A + \frac{1}{2}B\). While the preference matrices satisfy the linearity assumption individually along speed and comfort, the assumption is violated overall, wherein R is preferred over both A and B.

Dudik et al. [DHSS+15] proposed the concept of a von Neumann winner, corresponding to a distribution over objects that beats or ties every other object in the collection. They showed that under an expected utility assumption, such a randomized winner always exists and overcomes limitations of existing winning concepts—the Condorcet winner does not always exist, while the Borda winner fails an independence of clones test [Sch11]. However, the assumption of expected utility relies on a strong hypothesis about how humans evaluate distributions over objects: it posits that the probability with which any distribution over objects \(\pi\) beats an object is linear in \(\pi\).

Consequences of assuming linearity: In order to better appreciate these consequences, consider as an example\(^1\) the task of deciding between two policies (say A and B) to deploy in an autonomous vehicle. Suppose that these policies have been obtained by optimizing two different objectives, with policy A optimized for comfort and policy B optimized for speed. Figure 1(a) shows a snapshot of these two policies. When compared overall, 60% of the people preferred Policy A over B—making A the von Neumann winner. The linearity assumption then posits that a randomized policy that mixes between A and B can never be better than both A and B; but we see that the Policy \(R = \frac{1}{2}A + \frac{1}{2}B\) is actually preferred by a majority over both A and B! Why is the linearity assumption violated here?

One possible explanation for such a violation is that the comparison problem is actually multi-criteria in nature. If we look at the preferences for the speed and comfort criteria individually in Figure 1(b), we see that Policy A does quite poorly on the speed axis while B lags behind in comfort. In contrast, Policy R does acceptably well along both the criteria and hence is preferred overall to both Policies A and B. It is indeed impossible to come to this conclusion by only observing the overall comparisons. This observation forms the basis of our main proposal: decompose the single overall comparison and ask humans to provide preferences along simpler criteria. This decomposition of the comparison task allows us to place structural assumptions on comparisons.

\(^1\)Note that while this is an illustrative example, we observe a similar trend in our actual user study in Section 5.
along each criterion. For instance, we may now posit the linearity assumption along each criterion separately rather than on the overall comparison task.

In addition to allowing for simplified assumptions, breaking up the task into such simpler comparisons allows us to obtain richer and more accurate feedback as compared to the single overall comparison. Indeed, such a motivation for eliciting simpler feedback from humans finds its roots in the study of cognitive biases in decision making, which suggests that the human mind resorts to simple heuristics when faced with a complicated questions [TK74].

**Contributions:** In this paper, we formalize these insights and propose a new framework for preference learning when pairwise comparisons are available along multiple, possibly conflicting, criteria. As shown by our example in Figure 1, a single distribution that is the von Neumann winner along every criteria might not exist. In order to address this issue, we formulate the problem of finding the “best” randomized policy by drawing on tools from the literature on vector-valued pay-offs in game theory. Specifically, we take inspiration from Blackwell’s approachability [Bla56] and introduce the notion of a Blackwell winner. This solution concept generalizes the concept of a von Neumann winner, and recovers the latter when there is only a single criterion present. Section 3 describes this framework in detail, and Section 4 collects our statistical and computational guarantees for learning the Blackwell winner from data. Section 5 describes a user study with an autonomous driving environment, in which we ask human subjects to compare self-driving policies along multiple criteria such as safety, aggressiveness, and conservativeness. Our experiments demonstrate that the Blackwell winner is able to better trade off utility along these criteria and produces randomized policies that outperform the von Neumann winner for the overall preferences.

2 Related work

This paper sits at the intersection of multiple fields of study: learning from pairwise comparisons, multi-objective optimization, preference aggregation, and equilibrium concepts in games. Here we discuss those papers from these areas most relevant to our contributions.

**Winners from pairwise comparisons.** Most closely related to our work is the field of computational social choice, which has focused on defining notions of winners from overall pairwise comparisons (see the survey [BCEL+16] for a review). Amongst them, three deterministic notions of a winner—the Condorcet [Con85], Borda [Bor84], and Copeland [Cop51] winners—have been widely studied. In more recent work, Dudik et al. [DHSS+15] introduced the notion of a (randomized) von Neumann winner.

Starting with the work of Yue et al. [YBKJ12], there have been several research papers studying an online version of preference learning, called the Dueling Bandits problem. This is a partial information version of the classic $K$-armed bandit problem, in which feedback takes the forms of pairwise comparisons between arms of the bandit. Many algorithms have been proposed, including versions that compete with Condorcet [ZWMD13; ZWR15; AKJ14], Copeland [ZKWD15; WL16], Borda [JKDN15] and von Neumann [DHSS+15] winners.

**Multi-criteria decision making.** The theoretical foundations of decision making based on multiple criteria have been widely studied within the operations research community. This sub-field—called multiple-criteria decision analysis—has focused largely on scoring, classification, and sorting
based on multiple-criteria feedback. See the surveys \cite{PB12; ZD02} for thorough overviews of existing methods and their associated guarantees. The problem of eliciting the user’s relative weighting of the various criteria has also been considered \cite{DZ07}. However, relatively less attention has been paid to the study of randomized decisions and statistical inference, both of which form the focus of our work. From an applied perspective, the combination of multi-criteria assessments has received attention in disparate fields such as psychometrics \cite{Pap11; MPW14}, healthcare \cite{TN08}, and recidivism prediction \cite{Wal11}. In many of these cases, a variety of approaches—both linear and non-linear—have been empirically evaluated \cite{DM10}. Justification for non-linear aggregation of scores has a long history in psychology and the behavioral sciences \cite{GB91; FC94; TK79}.

**Blackwell’s approachability.** In the game theory literature, Blackwell \cite{Bla56} introduced the notion of approachability as a generalization of a zero-sum game with vector-valued payoffs; see Appendix A for more details. Blackwell’s approachability and its connections with no-regret learning and calibrated forecasting have been extensively studied \cite{ABH11; Per13; MPS14}. These connections have enabled applications of Blackwell’s results to problems ranging from constrained reinforcement learning \cite{MBDD19} to uncertainty estimation for question-answering tasks \cite{KE17}. In contrast with such applications of the repeated vector-valued game, our framework for preference learning along multiple criteria deals with a single shot game and uses the idea of the target set to define the concept of a Blackwell winner.

**Stability of Nash equilibria.** Another related body of literature focuses on Nash equilibria in games with perturbed payoffs, under both robust \cite{AB06; Leh12} and uncertain or Bayesian \cite{FL93} formulations; see the recent survey by Perchet \cite{Per14}. Perturbation theory for Nash equilibria has been derived in these contexts, and it is well-known that the Nash equilibrium is not (in general) stable to perturbations of the payoff matrix. On the other hand, Dudik et al. \cite{DHSS15}, working in the context of dueling bandits, consider Nash equilibria of perturbed, symmetric, zero-sum games, but show that the *payoff* of the perturbed Nash equilibrium is indeed stable. That is, even if the equilibrium itself can change substantially with a small perturbation of the payoff matrix, the corresponding payoff is still close to the payoff of the original equilibrium. Our work provides a similar characterization for the multi-criteria setting.

3 Framework for preference learning along multiple criteria

We now set up our framework for preference learning along multiple criteria. We consider a collection of \(d\) objects over which comparisons can be elicited along \(k\) different criteria. We index the objects by the set \([d] := \{1, \ldots, d\}\) and the criteria by the set \([k]\).

3.1 Probabilistic model for comparisons

Since human responses to comparison queries are typically noisy, we model the pairwise preferences as random variables drawn from an underlying population distribution. In particular, the result of a comparison between a pair of objects \((i_1, i_2)\) along criterion \(j\) is modeled as a draw from a Bernoulli distribution, with \(p(i_1, i_2; j) = P(i_1 \succeq i_2 \text{ along criterion } j)\). By symmetry, we must have

\[
p(i_2, i_1; j) = 1 - p(i_1, i_2; j) \quad \text{for each triple } (i_1, i_2, j) \in [d] \times [d] \times [k].
\]
Letting $\Delta_d$ denote the $d$-dimensional probability simplex, consider two probability distributions $\pi_1, \pi_2 \in \Delta_d$ over the $d$ objects. With a slight abuse of notation, let $p(\pi_1, \pi_2; j)$ denote the probability with which an object drawn from distribution $\pi_1$ beats an object drawn independently from distribution $\pi_2$ along criterion $j$. We assume for each individual criterion $j$ that the probability $p(\pi_1, \pi_2; j)$ is linear in the distributions $\pi_1$ and $\pi_2$, i.e. that it satisfies the relation

$$p(\pi_1, \pi_2; j) := E_{i_1 \sim \pi_1, i_2 \sim \pi_2} [p(i_1, i_2; j)].$$

Equation (2) encodes the per-criterion linearity assumption highlighted in Section 1. We collect the probabilities $\{p(i_1, i_2; j)\}$ into a preference tensor $P \in [0, 1]^{d \times d \times k}$ and denote by $P_{d,k}$ the set of all preference tensors that satisfy the symmetry condition (1). Specifically, we have

$$P_{d,k} = \{P \in [0, 1]^{d \times d \times k} \mid P(i_1, i_2; j) = 1 - P(i_2, i_1; j) \text{ for all } (i_1, i_2, j)\}.$$

Let $P^j$ denote the $d \times d$ matrix corresponding to the comparisons along criterion $j$, so that $p(\pi_1, \pi_2; j) = \pi_1^j \pi_2^j$. Also note that a comparison between a pair of objects $(i_1, i_2)$ induces a score vector containing $k$ such probabilities. Denote this vector by $P^j(i_1, i_2) \in [0, 1]^k$, whose $j$-th entry is given by $p(i_1, i_2; j)$. Denote by $P^j(\pi_1, \pi_2)$ the score vector for a pair of distribution $(\pi_1, \pi_2)$.

In the single criterion case when $k = 1$, each comparison between a pair of objects is along an overall criterion. We let $P_{ov} \in [0, 1]^{d \times d}$ represent such an overall comparison matrix. As mentioned in Section 1, most preference learning problems are multi-objective in nature, and the overall preference matrix $P_{ov}$ is derived as a non-linear combination of per-criterion preference matrices $\{P^j\}_{j=1}^k$. Therefore, even when the linearity assumption (2) holds across each criterion, it might not hold for the overall preference $P_{ov}$.

In contrast, when the matrices $P^j$ are aggregated linearly to obtain the overall matrix $P_{ov}$, we recover the assumptions of Dudik et al. [DHSS+15].

### 3.2 Blackwell winner

Given our probabilistic model for pairwise comparisons, we now describe our notion of a Blackwell winner. When defining a winning distribution for the multi-criteria case, it would be ideal to find a distribution $\pi^*$ that is a von Neumann winner along each of the criteria separately: $p(\pi^*, i; j) \geq 0.5$ for all items $i$ along all criteria $j$. However, as shown in our example from Figure 1, such a distribution need not exist: policy A is preferred along the comfort axis, while policy B along speed. We thus need a generalization of the von Neumann winner that explicitly accounts for conflicts between the criteria.

Blackwell [Bla56] asked a related question for the theory of zero-sum games: how can one generalize von Neumann’s minimax theorem to vector-valued games? He proposed the notion of a target set: a set of acceptable payoff vectors that the first player in a zero-sum game seeks to attain. Within this context, Blackwell proposed the notion of approachability, i.e. how the player might obtain payoffs in a repeated game that are close to the target set on average. We take inspiration from these ideas to define a solution concept for the multi-criteria preference problem.

Our notion of a winner also relies on a target set, which we denote by $S \subset [0, 1]^k$, and which in our setting contains score vectors. This set provides a way to combine different criteria by specifying combinations of preference scores that are acceptable. Figure 2 provides an example of two such sets. Observe that for our preference learning problem, the target set $S$ is by definition monotonic with respect to the orthant ordering, that is, if $z_1 \geq z_2$ coordinate-wise, then $z_2 \in S$. 

In the context of the example introduced in Figure 1, two target sets $S_1$ and $S_2$ that capture trade-offs between comfort and speed. Set $S_1$ requires feasible score vectors to satisfy 40% of the population along both comfort and speed. Set $S_2$ requires both scores to be greater than 0.3 but with a linear trade-off: the combined score must be at least 0.9.

Our goal is to then produce a distribution $\pi^*$ that can achieve a target score vector for any distribution with which it is compared—that is $P(\pi^*, \pi) \in S$ for all $\pi \in \Delta_d$. When such a distribution $\pi^*$ exists, we say that the problem instance $(P, S)$ is achievable.

On the other hand, it is clear that there are problem instances $(P, S)$ that are not achievable. While Blackwell’s workaround was to move to the setting of repeated games, preference aggregation is usually a one-shot problem. Consequently, our relaxation instead introduces the notion of a worst-case distance to the target set. In particular, we measure the distance between any pair of score vectors $u,v \in [0,1]^k$ as $\rho(u, v) = \|u - v\|$ for some norm $\|\cdot\|$. Using the shorthand $\rho(u, S) := \inf_{v \in S} \|u - v\|$, the Blackwell winner for an instance $(P, S, \|\cdot\|)$ is now defined as the one that minimizes the maximum distance to the set $S$, i.e.,

$$\pi(P, S, \|\cdot\|) \in \arg \min_{\pi \in \Delta_d} \max_{\pi' \in \Delta_d} \rho(P(\pi, \pi'), S). \quad (4)$$

Observe that equation (4) has an interpretation as a zero-sum game, where the objective of the minimizing player is to make the score vector $P(\pi, \pi')$ as close as possible to the target set $S$.

We now look at commonly studied frameworks for single criterion preference aggregation and multi-objective optimization and show how these can be naturally derived from our framework.

**Example: Preference learning along a single criterion.** A special case of our framework is when we have a single criterion ($k = 1$) and the preferences are given by a matrix $P_{ov}$. The score $P_{ov}(i_1, i_2)$ is a scalar representing the probability with which object $i_1$ beats object $i_2$ in an overall comparison. As a consequence of the von Neumann minimax theorem, we have

$$\max_{\pi_1 \in \Delta_d} \min_{\pi_2 \in \Delta_d} P_{ov}(\pi_1, \pi_2) = \min_{\pi_2 \in \Delta_d} \max_{\pi_1 \in \Delta_d} P_{ov}(\pi_1, \pi_2) = \frac{1}{2}, \quad (5)$$

with any maximizer above called a von Neumann winner [DHSS+15]. Thus, for any preference matrix $P_{ov}$, a von Neumann winner is preferred to any other object with probability at least $\frac{1}{2}$.

Let us show how this uni-criterion formulation can be derived as a special case of our framework. Consider the target set $S = [\frac{1}{2}, 1]$ and choose the distance function $\rho(a, b) = |a - b|$. By equation (5), the target set $S = [\frac{1}{2}, 1]$ is achievable for all preference matrices $P_{ov}$, and so the von Neumann winner and the Blackwell winner $\pi(P_{ov}, [\frac{1}{2}, 1], \|\cdot\|)$ coincide. ♦
Example: Weighted combinations of a multi-criterion problem. We saw in the previous example that the single criterion preference learning problem is quite special: achievability can be guaranteed by the von Neumann winner for set \( S = [\frac{1}{2}, 1] \) for any preference matrix \( P_{ov} \). One of the common approaches used in multi-objective optimization to reduce a multi-dimensional problem to a uni-dimensional counterpart is by introducing a weighted combinations of objectives.

Formally, consider a weight vector \( w \in \Delta_k \) and the corresponding preference matrix

\[
P(w) := \sum_{j \in [k]} w_j P_j,
\]

obtained by combining the preference matrices along the different criteria. A winning distribution can then be obtained by solving for the von Neumann winner of \( P(w) \) given by \( \pi(P(w), [\frac{1}{2}, 1], | \cdot |) \).

The following proposition establishes that such an approach is a special case of our framework, and conversely, that there are problem instances in our general framework which cannot be solved by a simple linear weighing of the criteria.

**Proposition 1.** (a) For every weight vector \( w \in \Delta_k \), there exists a target set \( S_w \in [0, 1]^k \) such that for any norm \( \| \cdot \| \), we have

\[
\pi(P, S_w, \| \cdot \|) = \pi(P(w), [1/2, 1], | \cdot |) \quad \text{for all} \quad P \in \mathcal{P}_{d,k}.
\]

(b) Conversely, there exists a set \( S \) and a preference tensor \( P \) with a unique Blackwell winner \( \pi^\ast \) such that for all \( w \in \Delta_k \), exactly one of the following is true:

\[
\pi(P(w), [1/2, 1], | \cdot |) \neq \pi^\ast \quad \text{or} \quad \arg \max_{\pi \in \Delta_d} \left\{ \min_{i \in [d]} P(\pi, i) \right\} = \Delta_d.
\]

Thus, while the Blackwell winner is always able to recover any linear combination of criteria, the converse is not true. Specifically, part (b) of the proposition shows that for a choice of preference tensor \( P \) and target set \( S \), either the von Neumann winner for \( P(w) \) is not equal to the Blackwell winner, or it degenerates to the entire simplex \( \Delta_d \) and is thus uninformative. Consequently, our framework is strictly more general than weighting the individual criteria.

4 Statistical guarantees and computational approaches

In this section, we provide theoretical results on computing the Blackwell winner from samples of pairwise comparisons along the various criteria.

4.1 Observation model and evaluation metrics.

We operate in the natural passive observation model, where a sample consists of a comparison between two randomly chosen objects along a randomly chosen criterion. Specifically, we assume access to an oracle that when queried with a tuple \( \eta = (i_1, i_2, j) \) comprising a pair of objects \( (i_1, i_2) \) and a criterion \( j \), returns a comparison \( y(\eta) \sim \text{Ber}(p(i_1, i_2; j)) \). Each query to the oracle constitutes one sample. In the passive sampling model, the tuple of objects and criterion is sampled uniformly, with replacement, that is \( (i_1, i_2) \sim \text{Unif}\{[d] \} \) and \( j \sim \text{Unif}\{[k] \} \) where \( \text{Unif}\{A\} \) denotes the uniform distribution over the elements of a set \( A \).
Given access to samples \( \{y_1(\eta_1), \ldots, y_n(\eta_n)\} \) from this observation model, we define the empirical preference tensor (specifically the upper triangular part)

\[
\tilde{P}_n(i_1, i_2, j) := \frac{1}{1 \vee \sum_i y_i(\eta_i \mid \eta_i = (i_1, i_2, j))} \sum_{i=1}^n y_i(\eta_i \mid \eta_i = (i_1, i_2, j))
\]

for \( i_1 < i_2 \), \( i\in [d] \), where each entry of the upper-triangular tensor is estimated using a sample average and the remaining entries are calculated to ensure the symmetry relations implied by the inclusion \( \tilde{P}_n \in \mathcal{P}_{d,k} \).

As mentioned before, we are interested in computing the solution \( \pi^* := \pi(P, S, \| \cdot \|) \) to the optimization problem (4), but with access only to samples from the passive observation model. For any estimator \( \tilde{\pi} \in \Delta_q \) obtained from these samples, we evaluate its error based on its value with respect to the tensor \( P \), i.e.,

\[
\Delta_P(\tilde{\pi}, \pi) := v(\tilde{\pi}; S, P, \| \cdot \|) - v(\pi^*; S, P, \| \cdot \|). \tag{7}
\]

Note that the error \( \Delta_P \) implicitly also depends on the set \( S \) and the norm \( \| \cdot \| \), but we have chosen our notation to be explicit only in the preference tensor \( P \). For the rest of this section, we restrict our attention to convex target sets \( S \) and refer to them as valid sets. Having established the background, we are now ready to provide sample complexity bounds on the estimation error \( \Delta_P(\tilde{\pi}, \pi^*) \).

### 4.2 Upper bounds on the error of the plug-in estimator

Recall the definition of the function \( v \) from equation (4), and define, for each preference tensor \( \tilde{P} \), an optimizer

\[
\pi(\tilde{P}) \in \arg \min_{\pi \in \Delta_d} v(\pi; S, \tilde{P}, \| \cdot \|). \tag{8}
\]

Also recall the empirical preference tensor \( \tilde{P}_n \) from equation (6). With this notation, the plug-in estimator is given by \( \tilde{\pi}_{\text{plug}} = \pi(\tilde{P}_n) \) and the target (or true) distribution by \( \pi^* = \pi(P) \).

While, our focus in this section is to provide upper bounds on the error of the plug-in estimator \( \tilde{\pi}_{\text{plug}} \), we first state a general perturbation bound which relates the error of the optimizer \( \pi(\tilde{P}) \) to the deviation of the tensor \( \tilde{P} \) from the true tensor \( P \). We use \( P(\cdot, i) \in [0, 1]^{d \times k} \) to denote a matrix formed by viewing the \( i \)-th slice of \( P \) along its second dimension. Finally, recall our definition of the error \( \Delta_P(\tilde{\pi}, \pi^*) \) from equation (7).

**Theorem 1.** Suppose the distance \( \rho \) is induced by the norm \( \| \cdot \|_q \) for some \( q \geq 1 \). Then for each valid target set \( S \) and preference tensor \( \tilde{P} \), we have

\[
\Delta_P(\pi(\tilde{P}), \pi^*) \leq 2 \max_{i \in [d]} \| \tilde{P}(\cdot, i) - P(\cdot, i) \|_{\infty, q}. \tag{9}
\]

Note that this theorem is entirely deterministic: it bounds the deviation in the optimal solution to the problem (4) as a function of perturbations to the tensor \( P \). It also applies uniformly to all valid target sets \( S \). In particular, this result generalizes the perturbation result of Dudik et al. [DHSS+15, Lemma 3] which obtained such a deviation bound for the single criterion problem with \( \pi^* \) as the von Neumann winner. Indeed, one can observe that by setting the distance \( \rho(u, v) = |u - v| \) in Theorem 1 for the uni-criterion setup, we have the error \( \Delta_P(\pi(\tilde{P}), \pi^*) \leq 2\| \tilde{P} - P \|_{\infty, \infty} \), matching the bound of [DHSS+15].

Let us now illustrate a consequence of this theorem by specializing it to the plug-in estimator, and with the distances given by the \( \ell_\infty \) norm.
Corollary 1. Suppose that the distance $\rho$ is induced by the $\ell_\infty$-norm $\| \cdot \|_\infty$. Then there exists a universal constant $c > 0$ such that given a sample size $n > cd^2 k \log \left( \frac{cdk}{\delta} \right)$, we have for each valid target set $S$

$$\Delta_P(\hat{\pi}_{\text{plug}}, \pi^*) \leq c \sqrt{\frac{d^2 k}{n} \log \left( \frac{cdk}{\delta} \right)}, \quad (10)$$

with probability greater than $1 - \delta$.

The bound (10) implies that the plug-in estimator $\hat{\pi}_{\text{plug}}$ is an $\epsilon$-approximate solution whenever the number of samples scales as $n = \tilde{O}(d^2 k)$. Observe that this sample complexity scales quadratically in the number of objects $d$ and linearly in the number of criteria $k$. This scaling represents the effective dimensionality of the problem instance, since the underlying preference tensor $P$ has $O(d^2 k)$ unknown parameters. Notice that the corollary holds for sample size $n = \tilde{\Omega}(d^2 k)$; this should not be thought of as restrictive, since otherwise, the bound (10) is vacuous.

4.3 Information-theoretic lower bounds

While Corollary 1 provides an upper bound on the error of the plug-in estimator that holds for all valid target sets $S$, it is natural to ask if this bound is sharp, i.e., whether there is indeed a target set $S$ for which one can do no better than the plug-in estimator.

In this section, we address this question by providing lower bounds on the minimax risk

$$\mathcal{M}_{n,d,k}(S, \| \cdot \|_\infty) := \inf_{\hat{\pi}} \sup_P \mathbb{E}[\Delta_P(\hat{\pi}, \pi^*)], \quad (11)$$

where the infimum is taken over all estimators that can be computed from $n$ samples from our observation model. It is important to note that the error $\Delta_P$ is computed using the $\ell_\infty$ norm and for the set $S$. Our lower bound will apply to the particular choice of target set $S_0 = \left[ \frac{1}{2}, 1 \right]^k$.

Theorem 2. There are universal constants $c, c'$ such that for all $d \geq 4$, $k \geq 2$, and $n \geq cd^4 k$, we have

$$\mathcal{M}_{n,d,k}(S_0, \| \cdot \|_\infty) \geq c' \sqrt{\frac{d^2 k}{n}}. \quad (12)$$

Comparing equations and (10) and (12), we see that for the $\ell_\infty$-norm and the set $S_0$, we have provided upper and lower bounds that match up to a logarithmic factor in the dimension. Thus, the plug-in estimator is indeed optimal for this pair $(\| \cdot \|_\infty, S_0)$. Further, observe that the above lower bound is non-asymptotic, and holds for all values of $n \gtrsim d^4 k$. This condition on the sample size arises as a consequence of the specific packing set used for establishing the lower bound, and improving it is an interesting open problem.

However, Theorem 2 raises the question of whether the set $S_0$ is special, or alternatively, whether one can obtain an $S$-dependent lower bound. The following proposition shows that at least asymptotically, the sample complexity for any polyhedral set $S$ obeys a similar lower bound.

Proposition 2 (Informal). Suppose that we have a valid polyhedral target set $S$, and that $d \geq 4$. There exists a positive integer $n_0(d, k, S)$ such that for all $n \geq n_0(d, k, S)$ we have

$$\mathcal{M}_{n,d,k}(S, \| \cdot \|_\infty) \gtrsim \sqrt{\frac{d^2 k}{n}}. \quad (13)$$
We defer the formal statement and proof of this proposition to Appendix B. This proposition establishes that the plugin estimator \( \hat{\pi}_{\text{plug}} \) is indeed optimal in the \( \ell_\infty \) norm for a broad class of sets \( S \). Note that the result is asymptotic in nature: in order for the proposition to hold, we require that the number of samples is greater than the value \( n_0 \). This number \( n_0 \) depends on problem dependent parameters, and we provide an exact expression for \( n_0 \) in the appendix.

### 4.4 Instance-specific analysis for the plug-in estimator

In the previous section we established that the error \( \Delta_P(\hat{\pi}_{\text{plug}}, \pi^*) \) of the plug-in estimator scales as \( \tilde{O}\left(\sqrt{\frac{d^2 k}{n}}\right) \) for any choice of preference tensor \( P \) and target set \( S \) when the distance function \( \rho = \| \cdot \|_\infty \). In this section, we study the adaptivity properties of the plug-in estimator \( \hat{\pi}_{\text{plug}} \) and obtain upper bounds on the error \( \Delta_P(\hat{\pi}_{\text{plug}}, \pi^*) \) that depend on the properties of the underlying problem instance.

In the main text, we will restrict our focus to the uni-criterion setup with \( k = 1 \) and the target set \( S = [\frac{1}{2}, 1] \), in which case the Blackwell winner coincides with the von Neumann winner. Furthermore, we will consider the case where the preference matrix \( P \) has a unique von Neumann winner \( \pi^* \). This is formalized in the following assumption.

**Assumption 1** (Unique Nash equilibrium). The matrix \( P \) belongs to the set of preference matrices \( \mathcal{P}_{d,1} \) and has a unique mixed Nash equilibrium \( \pi^* \), that is, \( \pi^*_i > 0 \) for all \( i \in [d] \).

For the more general analysis, we refer the reader to Appendix C. For any preference matrix \( P \in \mathcal{P}_{d,1} \) and the Bernoulli passive sampling model discussed in Section 4 let us represent by \( \Sigma_i \) the diagonal matrix corresponding to the variances along the \( i^{th} \) column of the matrix \( P \) with

\[
\Sigma_i = \text{diag}(P(1, i) \cdot (1 - P(1, i)), \ldots, P(d, i) \cdot (1 - P(d, i))).
\]

Given this notation, we now state an informal corollary (of Theorem 4 in the appendix) which shows that the error \( \Delta_P(\hat{\pi}_{\text{plug}}, \pi^*) \) depends on the worst-case alignment of the Nash equilibrium \( \pi^* \) with the underlying covariance matrices \( \Sigma_i \).

**Corollary 2** (Informal). For any preference matrix \( P \) satisfying Assumption 1, confidence \( \delta > 0 \), and number of samples \( n > n_0(P, \delta) \), we have that the error \( \Delta_P \) of the plug-in estimate \( \hat{\pi}_{\text{plug}} \) satisfies

\[
\Delta_P(\hat{\pi}_{\text{plug}}, \pi^*) \leq c \cdot \sqrt{\frac{\sigma_P^2 d^2}{n} \log \left( \frac{d}{\delta} \right)},
\]

with probability at least \( 1 - \delta \), where the variance \( \sigma_P^2 := \max_{i \in [d]} (\pi^*)\Sigma_i \pi^* \).

We defer the proof of the above to Appendix C. A few comments on the above corollary are in order. Observe that it gives a high probability bound on the error \( \Delta_P \) of the plug-in estimator \( \hat{\pi}_{\text{plug}} \). Compared with the upper bounds of Corollaries 1 and 4, the asymptotic bound on the error above is instance-dependent – the effective variance \( \sigma_P^2 \) depends on the underlying preference matrix \( P \). In particular, this variance measures how well the underlying von Neumann winner \( \pi^* \) aligns with the variance associated with each column of the matrix \( P \). In the worst case, since each entry of \( P \) is bounded above by 1, the variance \( \sigma_P^2 = 1 \) and we recover the upper bounds from Corollaries 1 and 4 for the uni-criterion case. More interestingly, the bound provided by Corollary 2 can be significantly sharper (by a possibly dimension-dependent factor) than its worst-case counterpart. We explore concrete examples of this in Appendix C.
4.5 Computing the plug-in estimator

In the last few sections, we discussed the statistical properties of the plug-in estimator, and showed that its sample complexity was optimal in a minimax sense. We now turn to the algorithmic question: how can the plug-in estimator \( \hat{\pi}_{\text{plug}} \) be computed? Our main result in this direction is the following theorem that characterizes properties of the objective function \( v(\pi; P, S, \| \cdot \|) \).

**Theorem 3.** Suppose that the distance function is given by an \( \ell_q \) norm \( \| \cdot \|_q \) for some \( q \geq 1 \). Then for each valid target set \( S \), the objective function \( v(\pi; P, S, \| \cdot \|_q) \) is convex in \( \pi \), and Lipschitz in the \( \ell_1 \) norm, i.e.,

\[
|v(\pi_1; P, S, \| \cdot \|_q) - v(\pi_2; P, S, \| \cdot \|_q)| \leq k^{\frac{1}{q}} \cdot \| \pi_1 - \pi_2 \|_1 \text{ for each } \pi_1, \pi_2 \in \Delta_d.
\]

Theorem 3 establishes that the plug-in estimator can indeed be computed as the solution to a (constrained) convex optimization problem. In Appendix D, we discuss a few specific algorithms based on zeroth-order and first-order methods for obtaining such a solution and an analysis of the corresponding iteration complexity for these methods; see Propositions 5 and 6 in the appendix. These methods differ in the way they access the target set \( S \): while zeroth-order methods require a distance oracle to the target set, the first-order methods require a stronger projection oracle to this constraint set.

5 Autonomous driving user study

In order to evaluate the proposed framework, we applied it to an autonomous driving environment. The objective is to study properties of the randomized policies obtained by our multi-criteria framework—the Blackwell winner for specific choices of the target set—and compare them with the alternative approaches of linear combinations of criteria and the single-criterion (overall) von Neumann winner. We briefly describe the components of the experiment here; see Appendix E for more details.

**Self-driving Environment.** Figure 1(a) shows a snapshot of one of the worlds in this environment with the autonomous car shown in orange. We construct three different worlds in this environment:

- **W1:** The first world comprises an empty stretch of road with no obstacles (20 steps).
- **W2:** The second world consists of cones placed in a given sequence (80 steps).
- **W3:** The third world has additional cars driving at varying speeds in their fixed lanes (80 steps).

**Policies.** For our base policies, we design five different reward functions encoding different self-driving behaviors. These polices, named Policy A-E, are then set to be the model predictive control based policies based on these reward functions wherein we fix the planning horizon to 6. See Appendix E for a detailed description of these reward functions.

A randomized policy \( \pi \in \Delta_5 \) is given by a distribution over the base policies A-E. Such a randomized policy is implemented in our environment by randomly sampling a base policy from the mixture distribution after every \( H = 18 \) time steps and executing this selected policy for that duration. To account for the randomization, we execute each such policy for 5 independent runs in each of the worlds and record these behaviors.
Subjective Criteria. We selected five subjective criteria with which to compare the policies, with questions asking which of the two policies was C1: Less aggressive, C2: More predictable, C3: More quick, C4: More conservative, and had C5: Less collision risk. Such a framing of question ensures that higher score value along any of C1-C5 is preferred; thus a higher score along C1 would imply less aggressive while along C2 would mean more predictable.

In addition to these base criteria, we also consider an Overall Preference which compares any pair of policies in an aggregate manner. For this criterion, the users were asked to select the policy they would prefer when riding to their destination. Additionally, we also asked the users to rate the importance of each criterion in their overall preference.

Main Hypotheses. Our hypotheses focus on comparing the randomized policies given by the Blackwell winner, the overall von Neumann winner, and those given by weighing the criteria linearly.

MH1 There exists a set \( S \) such that the Blackwell winner with respect to \( S \) and \( \ell_\infty \)-norm produced by our framework outperforms the overall von Neumann winner.

MH2 The Blackwell winner for oblivious score sets \( S \) outperforms both oblivious\(^2\) and data-driven weights for linear combination of criteria.

Independent Variables. The independent variable of our experiment is the choice of algorithms for producing the different randomized winners. These comprise the von Neumann winner based on overall comparisons, Blackwell winners based on two oblivious target sets, and 9 different linear combinations weights (3 data-driven and 6 oblivious).

We begin with the two target sets \( S_1 \) and \( S_2 \) for our evaluation of the Blackwell winner, which were selected in a data-oblivious manner. Set \( S_1 \) is an axis-aligned set promoting the use of safer policies with score vector constrained to have a larger value along the collision risk axis. Similar to Figure 2(b), the set \( S_2 \) adds a linear constraint along aggressiveness and collision risk. This target set thus favors policies that are less aggressive and have lower collision risk. For evaluating hypothesis MH2, we considered several weight vectors, both oblivious and data-dependent, comprising the average of the users’ self-reported weights, that obtained by regressing the overall criterion on C1-C5, and a set of oblivious weights. See Appendix E for details of the sets \( S_1 \) and \( S_2 \), and the weights \( w_{1:9} \).

Data collection. The experiment was conducted in two phases, both of which involved human subjects on Amazon Mechanical Turk (Mturk). See Appendix E for an illustration of the questionnaire.

The first phase of the experiment involved preference elicitation for the five base policies A-E. Each user was asked to provide comparison data for all ten combinations of policies. The cumulative comparison data is given in Appendix E, and the average weight vector elicited from the users was found to be \( w_1 = [0.21, 0.19, 0.20, 0.18, 0.22] \). We ran this study with 50 subjects.

In the overall preference elicitation, we saw an approximate ordering amongst the base policies: \( C \succ E \succsim D \succ B \succ A \). Thus, Policy C was the von Neumann winner along the overall criterion. For each of the linear combination weights \( w_1 \) through \( w_9 \), Policy C was the weighted winner.

\(^2\)We use the term oblivious to denote variables that were fixed before the data collection phase and data-driven to denote those which are based on collected data.
The Blackwell winners R1 and R2 for the sets $S_1$ and $S_2$ with the $\ell_\infty$ distance were found to be $R1 = [0.09, 0.15, 0.30, 0.15, 0.31]$ and $R2 = [0.01, 0.01, 0.31, 0.02, 0.65]$. In the second phase, we obtained preferences from a set of 41 subjects comparing the randomized polices R1 and R2 with the baseline policies A-E. The results are aggregated in Table 1 in Appendix E.

**Analysis for main hypotheses.** Given that the overall von Neumann winner and those corresponding to weights $w_{1.9}$ were all Policy C, hypotheses MH1 and MH2 reduced to whether users prefer at least one of $\{R1, R2\}$ to the deterministic policy C, that is whether $P_{ov}(C, R1) < 0.5$ or $P_{ov}(C, R2) < 0.5$.

Policies C and E were preferred to R1 by 0.71 and 0.61 fraction of the respondents, respectively. On the other hand, R2 was preferred to the von Neumann winner C by 0.66 fraction of the subjects. Using the data, we conducted a hypothesis test with the null and alternative hypotheses given by

$$H_0 : P_{ov}(C, R2) \geq 0.5, \quad \text{and} \quad H_1 : P_{ov}(C, R2) < 0.5.$$ 

Among the hypotheses that make up the (composite) null, our samples have the highest likelihood for the distribution $Ber(0.5)$. We therefore perform a one-sided hypothesis test with the Binomial distribution with number of samples $n = 41$, success probability $p = 0.5$ and number of successes $x = 14$ (indicating number of subjects which preferred Policy C to R1). The p-value for this test was obtained to be 0.0298. This supports both our claimed hypotheses MH1 and MH2.

6 Discussion and future work

In this paper, we considered the problem of eliciting and learning from preferences along multiple criteria, as a way to obtain rich feedback under weaker assumptions. We introduced the notion of a Blackwell winner, which generalizes many known winning solution concepts. We showed that the Blackwell winner was efficiently computable from samples with a simple and optimal procedure, and also that it outperformed the von Neumann winner in a user study on autonomous driving. Our work raises many interesting follow-up questions: How does the sample complexity vary as a function of the preference tensor $P$? Can the process of choosing a good target set be automated? What are the analogs of our results in the setting where pairwise comparisons can be elicited actively?

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Appendices

A  Blackwell’s approachability

Blackwell [Bla56] introduced the concept of approachability as a generalization of the minimax theorem to vector-valued payoffs. Formally, a Blackwell game is an extension of two-player zero-sum games with vector-valued reward functions.

Let $\mathcal{X}, \mathcal{Y}$ denote the action spaces for the two players and $r : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}^k$ be the corresponding vector-valued reward function. Further, let $S \subseteq \mathbb{R}^k$ denote a target set. The objective of player 1 is to ensure that the reward vector $r$ lies in the set $S$ while that of player 2 is ensure that the reward $r$ lies outside this set $S$. Following [ABH11], we introduce the notion of satisfiability and response-satisfiability.

**Definition 1** (Satisfiability). For a Blackwell game parameterized by $(\mathcal{X}, \mathcal{Y}, r, S)$, we say that,

- $S$ is **satisfiable** if there exists $x \in \mathcal{X}$ such that for all $y \in \mathcal{Y}$, we have that $r(x, y) \in S$.
- $S$ is **response-satisfiable** if for every $y \in \mathcal{Y}$, there exists $x \in \mathcal{X}$ such that $r(x, y) \in S$.

In the case of scalar rewards, Von Neumann’s minimax theorem indicates that any set which is satisfiable is also response-satisfiable. In other words, there exists a strategy for Player 1, oblivious of Player 2’s strategy which ensures that the reward belongs to the set $S$ if the set $S$ is response-satisfiable. The existence of such a relation was crucial in obtaining the concept of the Von Neumann winner described in Section 3 for the uni-criterion problem.

However, such a statement fails to hold in the general vector-valued case (see [ABH11] for a counterexample). In order to overcome this limitation, Blackwell [Bla56] defined the notion of approachability as follows.

**Definition 2** (Blackwell’s Approachability). Given a Blackwell game $(\mathcal{X}, \mathcal{Y}, r, S)$, we say that a set $S$ is **approachable** if there exists an algorithm $A$ which selects points in $\mathcal{X}$ such that for any sequence $y_1, \ldots, y_t \in \mathcal{Y}$,

$$
\lim_{T \to \infty} \rho \left( \frac{1}{T} \sum_{t=1}^{T} r(x_t, y_t), S \right) \to 0,
$$

where $x_t = A(y_1, \ldots, y_{t-1})$ is the algorithm’s play at time $t$ for some distance function $\rho$.

Blackwell’s celebrated theorem guarantees that any set $S$ is approachable if and only if it is response-satisfiable. This means that while no single choice of action in the set $\mathcal{X}$ can guarantee a response in the set $S$, there is an algorithm that ensures that in the repeated game, the average reward vector approaches the set $S$ for any choice of opponent play.

Note that our definition of achievability is a stronger requirement than Blackwell’s approachability. While approachability requires the time-averaged payoff in a repeated game to belong to the pre-specified set $S$, achievability requires the same to be true in a single-shot play of the game. Indeed, as the following lemma shows, one can construct examples of multi-criteria preference problems which are approachable but not achievable.

**Proposition 3** (Approachability does not imply achievability). There exists a preference tensor $P \in \mathcal{P}_{d,k}$ and a target set $S \subset [0,1]^k$ such that
(a) For the Blackwell game given by $(\Delta_d, \Delta_d, P, S)$, the set $S$ is approachable, and

(b) The set $S$ is not achievable with respect to $P$.

Proof. We will consider an example in a 2-dimensional action space with 2 criteria. Consider the preference matrix given by:

\[
P^1 = \begin{bmatrix}
\frac{1}{2} & 1 \\
0 & \frac{1}{2}
\end{bmatrix}
\quad \text{and} \quad
P^2 = \begin{bmatrix}
\frac{1}{2} & 0 \\
1 & \frac{1}{2}
\end{bmatrix},
\]

along with the convex set $S = \left[ \frac{1}{2}, 1 \right]^2$. The tensor $P$ represents the strongest possible trade-off between the two objects: Object 1 is preferred over 2 along the first criterion while the reverse is true for the second criterion.

The Blackwell game given by $(\Delta_d, \Delta_d, P, S)$ can indeed be shown to be approachable. The set $S$ is response-satisfiable since for every strategy $y \in \Delta_d$ chosen by the column player, the choice of $x = y$ would yield a reward vector $P(x, y) = \left[ \frac{1}{2}, 1 \right] \in S$. Then, by Blackwell’s theorem [Bla56], the set $S$ is approachable.

In contrast, consider any choice of distribution $\pi_1 = [p, 1-p]$ for the multi-criteria preference problem. The corresponding score vectors for responses $i_2 = 1, 2$ are given by:

\[
r_1 = P(\pi_1, i_2 = 1) = \left[ \frac{p}{2}, 1 - \frac{p}{2} \right] \quad \text{and} \quad
r_2 = P(\pi_1, i_2 = 2) = \left[ \frac{1}{2} + \frac{p}{2}, \frac{1}{2} - \frac{p}{2} \right].
\]

For any choice of the parameter $p \in [0, 1]$, one cannot have both $r_1$ and $r_2$ simultaneously belong to the set $S$. Hence, we have that the set $S$ is not achievable with respect to $P$.

This example can be extended to any arbitrary dimension $k$ by extending the tensor to have $P^j$ equal to the all-half matrix for any criterion $j > 2$ and the target set to be $S = \left[ \frac{1}{2}, 1 \right]^k$. Similarly, in order to extend the example to any dimension, consider the preference tensor (for $k = 2$)

\[
P^1_d = \begin{bmatrix}
P^1 & P^{1/2} & \cdots & P^{1/2} \\
P^{1/2} & P^1 & \cdots & P^{1/2} \\
\vdots & \vdots & \ddots & \vdots \\
P^{1/2} & P^{1/2} & \cdots & P^1
\end{bmatrix}
\quad \text{and} \quad
P^2_d = \begin{bmatrix}
P^2 & P^{1/2} & \cdots & P^{1/2} \\
P^{1/2} & P^2 & \cdots & P^{1/2} \\
\vdots & \vdots & \ddots & \vdots \\
P^{1/2} & P^{1/2} & \cdots & P^2
\end{bmatrix},
\]

with the smaller matrices $P^1$ and $P^2$ from equation (15) at the diagonal and $P^{1/2}$ denoting the all-half tensor of the appropriate dimension. A similar calculation as for the $d = 2$ case yields that the set $S$ is not achievable. This establishes the required claim. \hfill \Box

B Proof of main results

In this section, we provide proofs of all the results stated in the main paper. Appendix D to follow collects some additional results and their proofs.

B.1 Proof of Proposition 1

We establish both parts of the proposition separately.
B.1.1 Proof of part (a)
For any weight vector \( w \in \Delta_k \), consider the set
\[
S_w = \left\{ r \in [0, 1]^k \mid \langle w, r \rangle \geq 1/2 \right\}.
\]
The set \( S_w \) is clearly convex. Indeed, for any two vectors \( r_1, r_2 \in S_w \) and any scalar \( \alpha \in [0, 1] \), we have
\[
\langle w, \alpha r_1 + (1 - \alpha) r_2 \rangle = \alpha \langle w, r_1 \rangle + (1 - \alpha) \langle w, r_2 \rangle \in \left[ \frac{1}{2}, 1 \right].
\]
It is straightforward to verify that the set \( S_w \) is also monotonic with respect to the orthant ordering.

We now show that a von Neumann winner \( \pi^* \) of the (single-criterion) preference matrix \( P_w := P(w) \) can be written as \( \pi(P, S_w, \| \cdot \|) \) for an arbitrary choice of norm \( \| \cdot \| \). For each \( \tilde{\pi} \in \Delta_2 \), we have
\[
\langle w, P(\pi^*, \tilde{\pi}) \rangle = \sum_{j \in [k]} w_j P^j(\pi^*, \tilde{\pi}) = P_w(\pi^*, \tilde{\pi}) \geq \frac{1}{2},
\]
where the inequality (i) follows since \( \pi^* \) is a von Neumann winner for the matrix \( P_w \). Thus, we have the inclusion \( P(\pi^*, \tilde{\pi}) \in S_w \) for all \( \tilde{\pi} \in \Delta_k \), so that \( \max_{\tilde{\pi} \in \Delta_2} \rho(P(\pi^*, \tilde{\pi}), S_w) = 0 \) for any distance metric \( \rho \). Consequently, we have
\[
\pi^* \in \arg \min_{\pi \in \Delta_k} \max_{\tilde{\pi} \in \Delta_2} \rho(P(\pi, \tilde{\pi}), S_w),
\]
which establishes the claim for part (a).

B.1.2 Proof of part (b)

Consider the multi-criteria preference instance given by target set \( S = [\frac{1}{2}, 1]^k \), the \( \ell_\infty \) distance function and the preference tensor \( P \)
\[
P^1 = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad P^2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad P^j = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}
\]
The unique Blackwell winner for this instance \((P, S, \| \cdot \|_\infty)\) is given by
\[
\pi^*(P, S, \| \cdot \|_\infty) = [1/2, 1/2]. \tag{16}
\]
For any weight \( w \in [0, 1]^k \), consider the von Neumann winners corresponding to the weighted matrices \( P_w \)
\[
\pi(P_w, [1/2, 1], \| \cdot \|) = \begin{cases}
[1,0] & \text{for } w \text{ s.t. } P_w(1, 2) > 0.5 \\
[0,1] & \text{for } w \text{ s.t. } P_w(1, 2) < 0.5 \\
\pi \in \Delta_2 & \text{otherwise}
\end{cases}
\tag{17}
\]
Comparing equations (16) and (17) establishes the required claim.
B.2 Proof of Theorem 1

Let us use the shorthand $\tilde{\pi} := \pi(\tilde{P})$. We begin by decomposing the desired error term as

$$\Delta_P(\tilde{\pi}, \pi^*) = v(\tilde{\pi}; S, \tilde{P}, \| \cdot \|) - v(\tilde{\pi}; S, \tilde{P}, \| \cdot \|) + v(\tilde{\pi}; S, \tilde{P}, \| \cdot \|) - v(\pi^*; S, \tilde{P}, \| \cdot \|) + v(\pi^*; S, \tilde{P}, \| \cdot \|) - v(\pi^*; S, \tilde{P}, \| \cdot \|)$$

In order to obtain a bound on the perturbation errors, note that for any distribution $\pi$, we have

$$v(\pi; S, P, \| \cdot \|) = \max_i \rho(P(\pi, i_1, S)) - \max_i \rho(\tilde{P}(\pi, i_2, S))$$

where step (i) follows by setting the $i_2$ equal to $i_1$. Noting that the distance is given by the $\ell_q$ norm, we have

$$v(\pi; S, P, \| \cdot \|) = \max_i \min \| P(\pi, i) - z_1 \|_q - \min \| \tilde{P}(\pi, i) - z_2 \|_q$$

where the inequality (i) follows by setting $z_2$ equal to $z_1$. Taking a supremum over all distributions $\pi$ completes the proof.

B.3 Proof of Corollary 1

By Theorem 1, it suffices to provide a bound on the quantity $\max_i \| P(\cdot, i) - \tilde{P}(\cdot, i) \|_{\infty, \infty}$ for the plug-in preference tensor $\tilde{P}$. Now by definition, we have

$$\max_i \| P(\cdot, i) - \tilde{P}(\cdot, i) \|_{\infty, \infty} = \max_{i_1, i_2, j} \| P^j(i_1, i_2) - \tilde{P}^j(i_1, i_2) \| .$$

For each $i = (i_1, i_2, j)$ representing some index of the tensor, let $N_i := \#\{\ell \mid \eta_\ell = i \}$ denote the number of samples observed at that index. Since $N_i$ can be written as a sum of i.i.d. Bernoulli random variables, applying the Hoeffding bound yields

$$\Pr \left\{ \left| N_i - \frac{n}{d^2 k} \right| \geq c \sqrt{\frac{n \log(c/\delta)}{d^2 k}} \right\} \leq \delta \text{ for each } \delta \in (0, 1).$$

Note that we also have $n \geq c_0 d^2 k \log(c_1 d/\delta)$ by assumption. For a large enough choice of the constants $(c_0, c_1)$, applying the union bound yields the sequence of sandwich relations

$$\frac{n}{2d^2 k} \leq N_i \leq \frac{3n}{2d^2 k} \text{ for all indices } i \text{ with probability greater than } 1 - \delta. \quad (19)$$

Furthermore, conditioned on $N_i$ (for $i = (i_1, i_2, j)$), the Hoeffding bound yields the relation

$$\Pr \left\{ \left| P^j(i_1, i_2) - \tilde{P}^j(i_1, i_2) \right| \geq c \sqrt{\frac{\log(c/\delta)}{N_i}} \right\} \leq \delta \text{ for each } \delta \in (0, 1).$$
Putting this together with a union bound, we have

$$\Pr \left\{ \max_{i_1, i_2, j} |P^j(i_1, i_2) - \hat{P}^j(i_1, i_2)| \geq c \sqrt{\frac{\log(cd^2k/\delta)}{\min_i N_i}} \right\} \leq \delta. \quad (20)$$

Combining inequalities (19) and (20) with a final union bound completes the proof.

**B.4 Proof of Theorem 2**

Suppose throughout that $k \geq 2$, and recall the axis-aligned convex target set $S_0 = [\frac{1}{2}, 1]^k$. We split our proof into two cases depending on whether $d$ is even or odd.

**Case 1: $d$ even.** We use Le Cam’s method and construct two problem instances with preference tensors given by $P_0$ and $P_1$. Two key elements in the construction are the following $2 \times 2 \times 2$ tensors, which we denote by $P_{cr}$ and $\tilde{P}_{cr}$, respectively. Their entries are given by

- $P_{cr}^1 = \left[ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} + \gamma & \frac{1}{2} \\ \frac{1}{2} - \gamma & \frac{1}{2} & \frac{1}{2} \end{array} \right]$
- $P_{cr}^2 = \left[ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} - \gamma & \frac{1}{2} \\ \frac{1}{2} + \gamma & \frac{1}{2} & \frac{1}{2} \end{array} \right]$
- $\tilde{P}_{cr}^1 = \left[ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} + \gamma & \frac{1}{2} \\ \frac{1}{2} - \gamma & \frac{1}{2} & \frac{1}{2} \end{array} \right]$
- $\tilde{P}_{cr}^2 = \left[ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} - \gamma & \frac{1}{2} \\ \frac{1}{2} + \gamma & \frac{1}{2} & \frac{1}{2} \end{array} \right]$

Note that these tensors are parameterized by a scalar $\gamma \in [0, 1/2]$, whose exact value we specify shortly. Also denote by $P_{1/2}$ the $2 \times 2 \times 2$ all-half tensor. We are now ready to construct the pair of $d \times d \times k$ preference tensors $(P_0, P_1)$.

In order to construct tensor $P_0$, we specify its entries on the first two criteria according to

$$P_{0}^{1/2} = \left[ \begin{array}{cccc} P_{1/2} & P_{1/2} & \cdots & P_{1/2} \\ P_{1/2} & P_{cr} & \cdots & P_{1/2} \\ \vdots & \ddots & \ddots & \vdots \\ P_{1/2} & P_{1/2} & \cdots & P_{cr} \end{array} \right], \quad (21)$$

and set the entries on the remaining $k - 2$ criteria to $1/2$.

On the other hand, the first two criteria of the tensor $P_1$ are given by

$$P_{1}^{1/2} = \left[ \begin{array}{cccc} \tilde{P}_{cr} & P_{1/2} & \cdots & P_{1/2} \\ P_{1/2} & P_{cr} & \cdots & P_{1/2} \\ \vdots & \ddots & \ddots & \vdots \\ P_{1/2} & P_{1/2} & \cdots & P_{cr} \end{array} \right], \quad (22)$$

with the entries on the remaining $k - 2$ criteria once again set identically to $1/2$.

Note that the tensors $P_0$ and $P_1$ only differ on the first $2 \times 2 \times 2$ block. Furthermore, the following lemma provides an exact calculation of the values $\min_{\pi} v(\pi; P_0, S_0, \| \cdot \|_\infty)$ and $\min_{\pi} v(\pi; P_1, S_0, \| \cdot \|_\infty)$.  

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Lemma 1. We have

\[ V_0 := \min_{\pi} v(\pi; P_0, S_0, \| \cdot \|_\infty) = 0 \quad \text{and} \quad V_1 := \min_{\pi} v(\pi; P_0, S_0, \| \cdot \|_\infty) = \frac{\gamma}{3d - 2}. \]

Given samples from these two instances, we now use Le Cam’s lemma [see Tsy08, Chap 2] to lower bound the minimax risk as

\[ \mathcal{M}_{n,d,k}(S_0, \| \cdot \|_\infty) \geq \frac{|V_0 - V_1|}{2} \left( 1 - \| P_0^n - P_1^n \|_{TV} \right) = \frac{\gamma}{2(3d - 2)} \left( 1 - \| P_0^n - P_1^n \|_{TV} \right), \quad (23) \]

where \( P_0^n \) and \( P_1^n \) are the probability distributions induced on sample space by the passive sampling strategy applied to the tensor \( P_0 \) and \( P_1 \), respectively.

Using Pinsker’s inequality, the decoupling property for KL divergence and the fact that \( \text{KL}(P||Q) \leq \chi^2(P||Q) \), we have

\[ \| P_0^n - P_1^n \|_{TV} \leq \sqrt{\frac{n}{2} \chi^2(P_1||P_0)} \leq \sqrt{\frac{n}{2} \chi^2(P_1||P_0)}. \quad (24) \]

The chi-squared distance between the two distributions \( P_0 \) and \( P_1 \) is given by

\[ \chi^2(P_1||P_0) = \frac{1}{d^2 k} \sum_{(i_1,i_2,j)} \left( \frac{P_1^j(i_1,i_2)}{P_2^j(i_1,i_2)} - 1 \right)^2 \overset{(i)}{=} \frac{2}{d^2 k} \left( \left( \frac{2\gamma}{d} \right)^2 + \left( -\frac{2\gamma}{d} \right)^2 \right) = \frac{16\gamma^2}{d^4 k}, \]

where step (i) follows from the fact that \( P_1 \) and \( P_2 \) differ only in 4 entries and that the passive sampling strategy samples each index uniformly at random. Putting together the pieces, we have:

\[ \mathcal{M}_{n,d,k}(S_0, \| \cdot \|_\infty) \geq \frac{\gamma}{2(3d - 2)} \left( 1 - \sqrt{\frac{n}{2} \frac{16\gamma^2}{d^4 k}} \right) \overset{(ii)}{=} \frac{1}{48\sqrt{2}} \sqrt{\frac{d^2 k}{n}}, \]

where step (ii) follows by setting \( \gamma^2 = \frac{d^4 k}{32n} \) and using the fact that \( 3d - 2 \leq 3d \). Note that since we require \( \gamma^2 \leq \frac{1}{d} \), the above bound is valid only for \( n \geq d^4 k \). This concludes the proof for even \( d \).

**Case 2: \( d \) odd.** By assumption, we have \( d \geq 5 \). In this case, we construct \( P_0 \) and \( P_1 \) exactly as before, but replace \( P_{cr} \) in the last two rows of both \( P_0 \) and \( P_1 \) with the following modified \( 3 \times 3 \times 2 \) tensor:

\[ P_{1,cr}^3 = \begin{bmatrix} 1 - \gamma & 1 - \gamma & 1 - \gamma \\ \frac{1}{2} - \gamma & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad P_{2,cr}^3 = \begin{bmatrix} 1 + \gamma & 1 + \gamma & 1 + \gamma \\ \frac{1}{2} + \gamma & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \]

By mimicking its proof, it can be verified that this modification ensures that the corresponding values \( V_0 \) and \( V_1 \) still satisfy Lemma 1. Thus, the lower bound remains unchanged up to constant factors.

\[ \square \]

**B.4.1 Proof of Lemma 1**

Let us compute the two values separately.
Computing $V_0$. The choice of distribution $\pi^* = [1, 0, \ldots, 0]$ yields the score vector $[1/2, 1/2, \ldots, 1/2]$, which is in the set $S_0$. Thus, we have $V_0 = 0$.

Computing $V_1$. Note that the optimal distribution $\pi^*$ achieving the value $V_1$ will be of the form

$$\pi^* = [p/2, p/2, (1-p)/(d-2), \ldots, (1-p)/(d-2)]$$

for some $p \in [0,1]$. This follows from the symmetry in the preference tensor for row objects ranging from 3 to $d$. Given such a distribution $\pi^*$, the distance of the reward vector from the set $S_0$ is given by

$$\inf_{z \in S} \|P(\pi^*, i_2) - z\|_\infty = \begin{cases} \frac{2p}{d} & i_2 = 1, 2 \\ \frac{\gamma(1-p)}{d-2} & \text{o.w.} \end{cases}$$

Thus, for any value of $p > 2d/(3d-2)$, the distance is maximized for $i_2 \in \{1, 2\}$, and yields a value $\gamma p/(2d)$. On the other hand, for $p < 2d/(3d-2)$, the maximizing index is $i_2 \geq 3$, and the maximizing value is $\gamma(1-p)/(d-2)$. Optimizing these values for $p$ yields the claim.

**B.5 Instance-specific lower bounds**

In this section, we give a formal statement of Proposition 2 along with its proof. We begin by defining some notation. For any $\alpha, \beta \in [-\frac{1}{2}, \frac{1}{2}]$ and choice of criteria $j_1, j_2 \in [k]$, we define the tensor $P_{\alpha,\beta}^{(j_1,j_2)} \in [0,1]^{2 \times 2 \times k}$ as

$$P_{\alpha,\beta}^{j_1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} + \alpha \\ \frac{1}{2} - \alpha & \frac{1}{2} \end{bmatrix}, \quad P_{\alpha,\beta}^{j_2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} + \beta \\ \frac{1}{2} - \beta & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad P_{\alpha,\beta}^j = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \text{ for } j \neq \{j_1, j_2\}.$$

Further, we denote by $P_{1/2}$ the all-half tensor whose dimensions may vary depending on the context. Any distribution $\pi$ over the two objects can be parameterized by a value $q \in [0,1]$ with $q$ being the probability placed on the first object and $1-q$ the probability on the second object. We will consider the distance function given by the $\ell_\infty$ norm. Given this distance function, we overload our notation for the value

$$v(q; P_{\alpha,\beta}^{(j_1,j_2)}; S) = \max_i [\rho(P_{\alpha,\beta}^{(j_1,j_2)}(q, i), S)] \quad \text{and} \quad V(P_{\alpha,\beta}^{(j_1,j_2)}; S) = \min_q v(q; P_{\alpha,\beta}^{(j_1,j_2)}; S).$$

We now state our main assumption for the score set $S$ which allows us to formulate our lower bound.

**Assumption 2.** There exists a pair of criteria $(j_1, j_2)$, values $\alpha_0 \in (0, \frac{1}{2}]$ and $\beta_0 \in [-\frac{1}{2}, 0]$, and a gap parameter $\gamma > 0$ such that

$$V(P_{1/2}; S) + \gamma \leq V(P_{\alpha_0,\beta_0}^{(j_1,j_2)}; S)$$

for the all-half tensor $P_{1/2} \in [0,1]^{2 \times 2 \times k}$.

The assumption above indicates that there exists a pair of criteria along which one can observe some sort of trade-off when they interact with the underlying score set $S$. The preference tensor $P_{\alpha_0,\beta_0}^{(j_1,j_2)}$ captures this trade-off and the gap parameter $\gamma$ quantifies it. Going forward, we assume without loss of generality that $(j_1, j_2) = (1, 2)$ and drop the dependence of the tensor on these indices, writing $P_{\alpha_0,\beta_0} \equiv P_{\alpha_0}^{(1,2)}$. The following lemma indicates the importance of the special values of $(\alpha, \beta) = (0, 0)$ for which $P_{0,0} = P_{1/2}$.
Lemma 2. For any \( \alpha, \beta \in [\frac{-1}{2}, \frac{1}{2}] \), we have \( V(P_{0,0}; S) \leq V(P_{\alpha,\beta}; S) \).

The above lemma establishes that for any set, the value attained by setting \( (\alpha_0, \beta) = (0,0) \) will be lower than any other setting of the same parameters. For any parameter \( \delta \in [0,1] \), denote by \( P_{\text{wt},\delta} \) the weighted tensor

\[
P_{\text{wt},\delta} = (1 - \delta)P_{0,0} + \delta P_{\alpha_0,\beta_0}.
\]

In order to understand the value \( V(P_{\text{wt},\delta}; S) \), we establish the following structural lemma which gives us insight into how this value varies as a function of the parameter \( \delta \in [0,1] \).

Lemma 3. Consider a target set \( S \) that is given by an intersection of \( h \) half-spaces. Then, the value function \( V(P_{\text{wt},\delta}; S) \) is a piece-wise linear and continuous function of \( \delta \in [0,1] \) with at most \( 4h \) pieces.

The above lemma states that the value \( V(P_{\text{wt},\delta}; S) \) is a piece-wise linear function of \( \delta \). Consider the first such piece which has a non-zero slope. Such a line has to exist since \( V(P_{\text{wt},\delta}) \) is continuous in \( \delta \) and we have \( V(P_{\text{wt},0}) < V(P_{\text{wt},1}) \). Also, this slope has to be positive since we know from Lemma 2 that \( V(P_{\text{wt},0}) \leq V(P_{\text{wt},\delta}) \) for any \( \delta \in [0,1] \). Denote the starting point of this line by \( \delta_0 \) and the corresponding slope by \( m_0 \), and observe that the value \( V(P_{\text{wt},\delta_0}) = V(P_{\text{wt},0}) \). With this notation, we now proceed to prove the lower bound on sample complexity for any polyhedral target score set \( S \).

Proposition 4 (Formal). Suppose that we have a valid polyhedral target set \( S \) satisfying Assumption 2 with parameters \( (\alpha_0, \beta_0) \). Then, there exists a universal constant \( c \) such that for all \( d \geq 4 \), \( k \geq 2 \) and \( n \geq \frac{d^2k(1-\delta_0\alpha_0)^2}{\alpha_0^2 + \beta_0^2} \), we have

\[
\mathcal{M}_{n,d,k}(S, \| \cdot \|_\infty) \geq c \frac{m_0(\frac{1}{2} - \delta_0\alpha_0)}{\sqrt{\alpha_0^2 + \beta_0^2}} \sqrt{\frac{d^2k}{n}}.
\]

Proof. For this proof, we focus on the case when the number of criteria \( k \) is even. The proof for the case when \( k \) is odd can be obtained similar to the proof of Theorem 2.

We use Le Cam’s method for obtaining a lower bound on the minimax value and construct the lower bound instances using the tensor given by \( P_{\text{wt},\delta} \). For some \( \delta \in [0,1] \) (to be fixed later), consider the parameter \( \delta_1 = \delta_0 + \delta \). Using these values of \( \delta_0 \) and \( \delta_1 \), we create the following two instances \( P_0 \) and \( P_1 \):

\[
P_0 = \begin{bmatrix} P_{\text{wt},\delta_0} & P_{1/2} & \cdots & P_{1/2} \\ P_{1/2} & P_{\alpha_0,\beta_0} & \cdots & P_{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{1/2} & P_{1/2} & \cdots & P_{\alpha_0,\beta_0} \end{bmatrix} \quad \text{and} \quad P_1 = \begin{bmatrix} P_{\text{wt},\delta_1} & P_{1/2} & \cdots & P_{1/2} \\ P_{1/2} & P_{\alpha_0,\beta_0} & \cdots & P_{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{1/2} & P_{1/2} & \cdots & P_{\alpha_0,\beta_0} \end{bmatrix},
\]

where \( P_{\alpha_0,\beta_0} \) is as given by Assumption 2. The following lemma now shows that that there exists a small enough \( \delta \) such that the value function \( V(P_{\text{wt},\delta}; S) \) is linear in the range \( \delta \in [\delta_0, \delta_1] \).

Lemma 4. There exists a \( \delta \in (0,1) \) such that for all \( \delta \in [0, \delta] \) and \( \delta_1 = \delta_0 + \delta \), we have

(a) The value \( V(P_{\text{wt},\delta_1}; S) = V(P_{\text{wt},\delta}; S) + \delta m_0 \).
(b) The minimizer \( \pi^*_1 \) for \( P^*_1 \) is given by \( \pi^*_1 = [q_0, 1 - q_0, 0, \ldots, 0] \).

We defer the proof of this lemma to the end of the section. Thus, for a small enough value of \( \delta \in [0, \bar{\delta}] \), we have \( |\mathcal{V}(P_0) - \mathcal{V}(P_1)| = \delta m_0 \). As was shown in the proof of Theorem 2, the minimax rate is lower bounded as

\[
\mathcal{M}_{n,d,k}(S, \| \cdot \|_\infty) \geq \frac{\mathcal{V}(P_0) - \mathcal{V}(P_1)}{2} \left( 1 - \| P^n_0 - P^n_1 \|_{TV} \right) \geq \frac{\delta m_0}{2} \left( 1 - \sqrt{\frac{n}{2}} \chi^2(P_1\|P_0) \right),
\]

where \( P^n_0 \) and \( P^n_1 \) are the probability distributions induced on sample space by the passive sampling strategy and the preference tensor \( P_0 \) and \( P_1 \) respectively. In order to obtain the requisite lower bound, we proceed to compute an upper bound on the chi-squared distance between the two distributions \( P_0 \) and \( P_1 \) as

\[
\chi^2(P_1\|P_0) = \frac{1}{d^2 k} \sum_{(i_1,i_2,j)} \left( \frac{P^j_1(i_1,i_2)}{P^j_0(i_1,i_2)} - 1 \right)^2
\]

\[
\leq \frac{2}{d^2 k} \left( \left( \frac{\alpha_0^2 \delta^2}{\frac{1}{2} - \delta_0 \alpha_0} \right) + \left( \frac{\beta_0^2 \delta^2}{\frac{1}{2} + \delta_0 \beta_0} \right) \right)
\]

\[
\leq \frac{2 \delta^2}{d^2 k} \left( \frac{\alpha_0^2 + \beta_0^2}{\frac{1}{2} - \delta_0 \alpha_0} \right),
\]

where (i) follows from the fact that the instances \( P_0 \) and \( P_1 \) differ only in 4 entries and (ii) follows from the assumption that \( |\alpha_0| \geq |\beta_0| \). Now, substituting the value of \( \delta^2 = \frac{d^2 k}{4n} \cdot \left( \frac{1}{2} - \delta_0 \alpha_0 \right)^2 \) and using the above bound with equation (27), we have

\[
\mathcal{M}_{n,d,k}(S, \| \cdot \|_\infty) \geq \frac{m_0 \left( \frac{1}{2} - \delta_0 \alpha_0 \right)}{8 \sqrt{\alpha_0^2 + \beta_0^2}} \sqrt{\frac{d^2 k}{n}},
\]

which holds whenever we have \( \delta \in [0, \bar{\delta}] \) or equivalently \( n \geq \frac{d^2 k \left( \frac{1}{2} - \delta_0 \alpha_0 \right)^2}{4 \delta^2 \alpha_0^2 + \beta_0^2} \). This establishes the desired claim.

\[\square\]

**B.5.1 Proof of Lemma 2**

For any \( \alpha, \beta \in [-\frac{1}{2}, \frac{1}{2}] \), consider the value

\[
\mathcal{V}(P_{\alpha,\beta}; S) = \min_{q \in [0,1]} \max_{i \in [0,1]} [\rho(P_{\alpha,\beta}(q, i), S)]
\]

\[
= \min_{q \in [0,1]} \max_{\tau \in [0,1]} [\rho(P_{\alpha,\beta}(q, \tau), S)]
\]

\[
\geq \rho \left( \left[ \frac{1}{2} \right]^k, S \right) = \mathcal{V}(P_{1/2}; S),
\]

where (i) follows by setting \( \tau = q \) and \( \left[ \frac{1}{2} \right]^k \) denotes the vector with each entry set to half. This establishes the claim. \[\square\]
B.5.2 Proof of Lemma 3

Let us denote by $q_0$ any minimizer of the value $v(q; \mathbf{P}_{\alpha_0,\beta_0}, S)$ and the two score vectors corresponding to the choices for $i$ in equation (25) by $z_{1,i} := \mathbf{P}_{\alpha_0,\beta_0}(q_0, i)$. Observe that for $\mathbf{P}_{\mathbf{w},\delta}$, the distribution given by $q_0$ is still a minimizer of its value. Further, the score vectors for the two column choices are given by:

$$z_{\delta,i} = (1 - \delta) \left[ \frac{1}{2} \right]^k + \delta z_{1,i} \quad \text{for } i = \{1, 2\}.$$

Recall that the distance function is given by $\rho(z_{\delta,i}, S) = \min_{z \in S} \|z_{\delta,i} - z\|_\infty$. Now, the minimizer $z$ will lie on the closest hyperplane(s) to the point $z_{\delta,i}$. In order to establish the claim, it suffices to show that for any fixed hyperplane $H$, the distance function given by $\rho(z_{\delta,i}, H)$ is a piece-wise linear function for $\delta \in [0, 1]$.

Let us consider a point $z_{\delta,i}$ which does not belong to the half-space given by $H$, since otherwise, the distance to the half-space is 0. If we have $\rho(z_{\delta,i}, H) = \zeta$, then the vector $z_{\delta,i} + \zeta 1_k$ must lie on the hyperplane $H$. This follows from the monotonicity property of the hyperplane $H$.

For any $\delta = \frac{1}{2} \delta_1 + \frac{1}{2} \delta_2$ such that $z_{\delta_1,i}$ and $z_{\delta_2,i}$ do not belong to the half-space given by $H$, we have

$$\rho(z_{\delta,i}) = \frac{1}{2} \rho(z_{\delta_1,i}) + \frac{1}{2} \rho(z_{\delta_2,i}),$$

where the above equality follows since $z_{\delta_1,i} + \zeta 1_k$ and $z_{\delta_2,i} + \zeta 2 1_k$ both lie on the hyperplane $H$ and therefore $z_{\delta,i} + \zeta \frac{1}{2} 1_k$ also lies on the hyperplane. Combined with the fact that for any point $z_{\delta,i}$ which lies in the half-space given by $H$, the distance $\rho(z_{\delta,i}, H) = 0$, we have that the function $\rho(z_{\delta,i}, H)$ is a piece-wise linear function with at most 2 linear pieces for $\delta \in [0, 1]$.

Since $\rho(z_{\delta,i}, S)$ is a minimum over $h$ hyperplanes, this function is itself a piece-wise linear function with at most $2h$ pieces. The desired claim now follows from noting that the value function $\mathcal{V}(\mathbf{P}_{\mathbf{w},\delta}, S)$ is a maximum over two piece-wise linear functions each with at most $2h$ pieces. \hfill \square

B.5.3 Proof of Lemma 4

Consider $\delta_1 = \delta_0 + \delta$ such that $\delta_0$ and $\delta_1$ share the same linear piece. This can be guaranteed to hold true for all $\delta \leq \tilde{\delta}_1$ by the piecewise linear nature of the value $\mathcal{V}(\mathbf{P}_{\mathbf{w},\delta})$.

For part (b) of the claim, let us consider the tensor $\tilde{\mathbf{P}} = \mathbf{P}_1(\mathbf{3} :, 3 :)$ formed by removing the first two rows and columns from the tensor $\mathbf{P}_1$. Then, from Assumption 2, we have that $\mathcal{V}(\tilde{\mathbf{P}}; S) \geq \mathcal{V}(\mathbf{P}_{1/2}; S) + \tilde{\gamma}$ for some $\tilde{\gamma} > 0$. Selecting a value of $\tilde{\delta}_2$ such that $\tilde{\delta}_2 m_0 \leq \tilde{\gamma}$, we can ensure that condition (b.) is satisfied.

Finally, setting $\delta = \min(\delta_1, \tilde{\delta}_2)$ completes the proof. \hfill \square

B.6 Proof of Theorem 3

Let us prove the two claims of the theorem separately. We use the shorthand $v(\pi) := v(\pi; \mathbf{P}, S, \|\cdot\|)$ for convenience.

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We use $H$ to denote the hyperplane and the half-space induced by it interchangeably.
**Establishing convexity.** Consider any two distributions \( \pi_1, \pi_2 \in \Delta_k \) and a scalar \( \alpha \in [0, 1] \). Since the set \( S \) is closed and convex, we have

\[
v(\alpha \pi_1 + (1-\alpha) \pi_2) = \max_{i \in [d]} \min_{z \in S} \rho(\alpha \pi_1 + (1-\alpha) \pi_2, i, z)
\]

\[
\overset{(i)}{=} \max_{i \in [d]} \min_{z_1, z_2 \in S} \left[ \rho(\alpha \pi_1, i, (1-\alpha) \pi_2, i, z_1 + (1-\alpha) z_2) \right]
\]

\[
\overset{(ii)}{\leq} \max_{i \in [d]} \left( \alpha \cdot \min_{z_1 \in S} \left[ \rho(\pi_1, i, z_1) \right] + (1-\alpha) \cdot \min_{z_2 \in S} \left[ \rho(\pi_2, i, z_2) \right] \right)
\]

\[
\leq \alpha v(\pi_1) + (1-\alpha) v(\pi_2)
\]

where (i) follows from the convexity of \( S \) and linearity of the preference evaluation (Eq. (2)), (ii) follows from the convexity of the distance function given by \( \ell_q \) norm and (iii) follows from distributing the max over the two terms. This establishes the first part of the theorem.

**Establishing the Lipschitz bound.** Consider any two distributions \( \pi_1, \pi_2 \in \Delta_d \). The difference in their value function can then be upper bounded as

\[
|v(\pi_1) - v(\pi_2)| = |\max_{i_1 \in [d]} \rho(\pi_1, i_1, S) - \max_{i_2 \in [d]} \rho(\pi_2, i_2, S)|
\]

\[
\overset{(i)}{\leq} \max_{i \in [d]} \left| \rho(\pi_1, i, S) - \rho(\pi_2, i, S) \right|
\]

\[
= \max_{i \in [d]} \left| \min_{z_1 \in S} \rho(\pi_1, i, z_1) - \min_{z_2 \in S} \rho(\pi_2, i, z_2) \right|
\]

\[
\overset{(ii)}{\leq} \max_{i \in [d]} \max_{z \in S} \left| \rho(\pi_1, i, z) - \rho(\pi_2, i, z) \right|
\]

where (i) follows from using the inequality \( \max_x f(x) - \max_y g(y) \leq \max_x |f(x) - g(x)| \) and (ii) follows through a similar inequality \( \min_x f(x) - \min_y g(y) \leq \max_x |f(x) - g(x)| \). Since the distance function \( \rho \) is specified by the \( \ell_q \) norm \( \| \cdot \|_q \), we have

\[
|v(\pi_1) - v(\pi_2)| \leq \max_{i \in [d]} \| \pi_1 - \pi_2 \|_q
\]

\[
= \left[ \sum_{j=1}^{k} \left( \langle \pi_1 - \pi_2, P^j(\cdot, i) \rangle \right)^q \right]^{\frac{1}{q}}
\]

\[
\overset{(i)}{\leq} k^{\frac{1}{q}} \cdot \| \pi_1 - \pi_2 \|_1
\]

where (i) follows from an application of Hölder’s inequality (\( \ell_1 - \ell_\infty \)) to the inner product \( \langle \pi_1 - \pi_2, P^j(\cdot, i) \rangle \) and the fact that \( P^j(i_1, i_2) \in [0, 1] \) for any \( (i_1, i_2, j) \). This establishes the Lipschitz bound and concludes the proof of the theorem. \( \square \)
C Local asymptotic analysis for plug-in estimator

In this section, we study the adaptivity properties of the plug-in estimator $\hat{\pi}_{\text{plug}}$ and derive upper bounds on the error $\Delta_P(\hat{\pi}_{\text{plug}}, \pi^*)$ which depend on the properties of the underlying problem instance $(P, S, \rho)$. Contrast this analysis with the upper bounds obtained in Corollary 1 and the perturbation result of Dudik et al. [DHSS+15, Lemma 3] which provides a worst-case upper bound on the error $\Delta_P$ independent of the underlying preference tensor $P$.

Our focus in this section will be on the uni-criterion setup with $k = 1$ with the target set $S = [\frac{1}{2}, 1]$ in which case the Blackwell winner coincides with the von Neumann winner. Recall from Section 3.2 that for the uni-criterion setup, the von Neumann winner for a preference matrix $P \in [0, 1]^{d \times d}$ is defined to be the distribution $\pi^*$ satisfying

$$\pi^* \in \arg \max_{\pi \in \Delta_d} \min_{i \in [d]} \pi^\top P e_i,$$

(28)

where $e_i$ denotes the basis vector in the $i^{th}$ direction. Observe that the vector $\pi^*$ corresponds to the mixed Nash equilibrium (NE) strategy of the zero-sum game with pay-off matrix $P$ for the row player (maximizing player). Given this equivalence, we focus on the more general problem of estimating the Nash distribution of a zero-sum game with pay-offs $A \in [0, 1]^{d \times d}$ given sampled access to the matrix $A$.

We consider a slightly modified passive sampling regime introduced in Section 4 wherein each sample consists of an observation $y \sim \mathcal{N}(A_{i_1,i_2}, \sigma^2_{i_1,i_2})$, where the indices $i_1, i_2 \sim \text{Unif}([d])$ are sampled independently. We term this the Gaussian passive sampling model in contrast to the Bernoulli sampling model considered in the main text. Note that in the asymptotic regime (and with suitable rescaling), the Bernoulli sampling model is equivalent to the Gaussian sampling model with variance $\sigma^2_{i_1,i_2} = A_{i_1,i_2} \cdot (1 - A_{i_1,i_2})$. We further assume that the variances satisfy $\max_{i_1,i_2} \sigma^2_{i_1,i_2} \leq 1$. Given access to $n$ samples from this model, we are interested in understanding the performance of the plug-in estimator

$$\hat{\pi}_{\text{plug}} \in \arg \max_{\pi \in \Delta_d} \min_{i \in [d]} \pi^\top \hat{A}_n e_i,$$

where $\hat{A}_n$ is the empirical estimate of the matrix, defined analogous to the estimate $\hat{P}$ in equation (6). In particular, we will be interested in obtaining a bound on the error

$$\Delta_A(\hat{\pi}_{\text{plug}}, \pi^*) := \min_{i \in [d]} (\pi^*)^\top A e_i - \min_{i \in [d]} (\hat{\pi}_{\text{plug}})^\top \hat{A} e_i,$$

which measures the gap in the value obtained when distribution $\hat{\pi}_{\text{plug}}$ is played compared with the value obtained by the Nash distribution $\pi^*$. Observe that the optimization problem for obtaining Nash equilibrium in equation (28) can be written as the following linear program with decision variables $(\pi, t)$

$$\max \ t$$

such that $\pi^\top A e_i \geq t$ for all $i \in [d], \sum_i \pi_i = 1$ and $\pi_i \geq 0$ for all $i \in [d]$. (Nash)

\footnote{For this section we use the notation $\hat{\pi}_{\text{plug}}$ and $\hat{\pi}$ to interchangeably to denote the plug-in estimator.}
The above linear program has \(d + 1\) variables \((\pi, t)\) and \(2d + 1\) constraints including one equality constraint. Similarly, the one can rewrite the objective for the plug-in estimator \(\hat{\pi}_{\text{plug}}\) as the solution to a perturbed version of the above linear program

\[
\max t \\
\text{such that } \pi^T \hat{A}_n e_i \geq t \text{ for all } i \in [d], \\
\sum_i \pi_i = 1 \text{ and } \pi_i \geq 0 \text{ for all } i \in [d].
\]

(Pert)

Before stating our main result concerning the asymptotic distribution of the error \(\Delta_A(\hat{\pi}_{\text{plug}}, \pi^*)\), we introduce some notation first. Let us denote by \(x = (\pi, t)\) the variables and by matrix \(C\) and vector \(c_{\text{sim}}\) the set of constraints in the linear program \((\text{Nash})\), that is,

\[
C := \begin{bmatrix} A^T & -1_d \\ I_d & 0 \end{bmatrix} \text{ and } c_{\text{sim}} := [1_d, 0],
\]

where we have denoted by \(1_d\) the all-ones column vector in \(d\) dimension and by \(I_d\) the \(d \times d\) identity matrix. Using this notation, we can rewrite this LP as

\[
\max t \\
\text{such that } Cx \geq 0, \quad c_{\text{sim}}^T x = 1
\]

(30)

It will also be convenient to define the extended matrix \(C_{\text{ext}} := [C; c_{\text{sim}}^T]\) which contains both the equalit and inequality constraints. For the perturbed version of the linear program \((\text{Pert})\) we denote the analagous matrices respectively by \(\hat{C}\) and \(\hat{C}_{\text{ext}}\). Observe that the simplex constraint encoded by the vector \(c_{\text{sim}}\) is deterministic and hence remains the same for both the original and perturbed linear programs.

Observe that the constraint polytope for the LP \((\text{Nash})\) is a closed convex set since the Nash distribution \(\pi\) belongs to the simplex \(\Delta_d\) and the variable \(t \in [0, 1]\). Therefore, the optimal solution \(x^* = (\pi^*, t^*)\) will lie on one of faces whose dimension \(0 \leq k_f \leq d\). In the special case when \(k_f = 0\), we say that the LP admits a unique solution which is a vertex of the constraint polytope. Let us denote by subsets \(J_1 \subseteq \{1, \ldots, d\}\) and \(J_2 \subseteq \{d + 1, \ldots, 2d\}\) the subset of constraints (rows of the constraint matrix \(C\)) which are tight for the set of optimal solutions and let us represent their union by \(J = J_1 \cup J_2\). Observe that in addition to the equality constraint \(c_{\text{sim}}^T x = 1\), there can be at most \(d\) constraints tight, that is, \(|J| \leq d\). Further, we denote by \(\hat{J}_1, \hat{J}_2\) and \(\hat{J}\) the corresponding subsets for the perturbed linear program \((\text{Pert})\).

We first establish a technical lemma which establishes that given enough samples, the active constraints for the original LP \((\text{Nash})\) given by \(J\) will be contained in the active constraints \(\hat{J}\) for the solution of the perturbed LP \((\text{Pert})\).

**Lemma 5.** Consider the perturbed LP \((\text{Pert})\) for any payoff matrix \(A \in [0, 1]^{d \times d}\) with noise distribution following the Gaussian passive sampling model. Then, for all \(n > n_0(A, \delta)\), we have that the active constraint sets \(J\) for the original LP and \(\hat{J}\) for the perturbed LP satisfy \(\hat{J} \subseteq J\) with probability at least \(1 - \delta\).

We defer the proof of the lemma to the end of the section. Observe that depending on the sampling of the noise variables, the subset \(\hat{J}\) can vary with the noise variables. Each of these
different subset can be seen as adding additional constraints on top of the \(|J|\) constraints which characterize the set of Nash equilibria for the original LP. Thus, when we look at the constraint matrix \(C_{\text{ext}, j}\), any \(x = (\pi, t)\) satisfying \(C_{\text{ext}, j} \cdot x = [0_d, 1]^\top\) will necessarily have \(\pi\) as a Nash equilibrium.

Before stating our main result, we introduce some notation which is essential for the statement. Let use represent by \(\Phi := A_{J_1, J_2}^\top\) the rank \(r\) matrix of constraints which are tight in the perturbed LP and its singular value decomposition by \(\Phi = U \Sigma V^\top\) and the corresponding noisy matrix

\[
\hat{A}_{J_1, J_2}^\top = [U_1 \quad U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix} + \begin{bmatrix} Z_{11} \\ Z_{21} \\ Z_{22} \end{bmatrix},
\]

where the matrix \(Z\) represents average zero-mean gaussian noise with \(n\) samples obtained from the passive sampling model. Further, we let \(\tilde{Z}_n := U^\top Z_n V\) denote the noise matrix rotated by the directions given in \(U\) and \(V\). With this, we state our result which characterizes the error \(\Delta_A\) for the plug-in estimator \(\hat{\pi}_{\text{plug}}\) in terms of properties of the underlying matrix \(A\) and the noise matrix \(\tilde{Z}\).

**Theorem 4.** For any payoff matrix \(A \in [0, 1]^{t \times d}\) and confidence \(\delta \in (0, 1)\), there exists a constant \(n_0(A, \delta)\) such that for samples \(n > n_0(A, \delta)\) obtained via the Gaussian passive sampling model, the error \(\Delta_A\) of the plug-in estimate \(\hat{\pi}_{\text{plug}}\) satisfies

\[
\Delta_A(\hat{\pi}_{\text{plug}}, \pi^*) \leq t^* \max_{i \in J_1} e_i^\top U_1 \left( \tilde{Z}_{11} - \tilde{Z}_{12} \tilde{Z}_{22}^{-1} \tilde{Z}_{21} \right) \Sigma_1^{-1} U_1^\top 1_{j_i} + c \sqrt{d} \cdot \| \Sigma_1^{-1} \|_2 \cdot \| \tilde{Z}_{12} \tilde{Z}_{22}^{-1} \tilde{Z}_{21} - \tilde{Z}_{11} \|_2^2 \\
+ c \sqrt{d} (t - t^*) (1 + \| \left( \tilde{Z}_{11} - \tilde{Z}_{12} \tilde{Z}_{22}^{-1} \tilde{Z}_{21} \right) \Sigma_1^{-1} \|_2)
\]

with probability at least \(1 - \delta\) for some universal constant \(c > 0\).

A few comments on the theorem are in order. Observe that the upper bound on the error is a stochastic quantity where the randomness is not only in the entries of the matrix \(\tilde{Z}\) but also in the matrices \(U\) and \(\Sigma\) which depend on the (possibly) random subsets \(J_1\) and \(J_2\). The upper bound depends primarily on two terms, up to lower order error factors, one measures the alignment of the Schur complement \(\tilde{Z}_{11} - \tilde{Z}_{12} \tilde{Z}_{22}^{-1} \tilde{Z}_{21}\) with the rotated and renormalized ones vector \(1_{j_i}\), and the second which measures the convergence of the empirical value \(\hat{t}\) to the true value \(t^*\). Going forward, we first provide a complete proof this result and then specialize it to the special case when the true Nash \(\pi^*\) is unique and lies in the interior of the simplex \(\Delta_d\) – this greatly simplifies the above expression and allows us to study the problem dependent adaptivity properties of the plug-in estimator \(\hat{\pi}_{\text{plug}}\). Additionally, the probability of error \(\delta\) and the corresponding restriction on sample size in the above statement come from conditioning on the event \(\{J \subseteq \hat{J}\}\) (from Lemma 5).

**Remark 1.** The above analysis can be extended to the multi-criteria preference learning setup \(k > 1\) whenever the distance function \(\rho = \| \cdot \|_\infty\) and the target set \(S\) is a polytope by extending the linear program to handle the additional constraints. Similar to the result above, the final upper bound on the error will then depend only on the constraints which are tight in the original and perturbed programs. It remains an interesting problem to study the asymptotic error for general convex target sets for which the optimization problem can be written as a convex program.
Proof of Theorem 4. We will establish the claim by analyzing the structure of the solution \( \hat{x} = (\hat{\pi}, \hat{t}) \) output by solving the perturbed linear program (Pert). Recall from our notation that given access to \( n \) noisy samples of the matrix \( \mathbf{A} \), the set \( \hat{J} = \hat{J}_1 \cup \hat{J}_2 \) represents the set of constraints which are tight for the perturbed LP with the empirical matrix \( \hat{\mathbf{A}} \) where \( |\hat{J}| = d \). Also, observe that since these samples come from the Gaussian passive sampling model, we will have that the solution \( \hat{x} \) will be unique\(^5\) with probability 1.

Given this uniqueness, we can express the solution \( \hat{x} = (\hat{\pi}, \hat{t}) \) as the solution to the linear system

\[
\begin{pmatrix}
\hat{\mathbf{A}}^\top_{\hat{J}_1} & -1_{|\hat{J}_1|} \\
I_{\hat{J}_2} & 0_{|\hat{J}_2|} \\
1_d & \cdot
\end{pmatrix}
\begin{pmatrix}
\hat{\pi}_{|\hat{J}_1|} \\
\hat{\pi}_{|\hat{J}_2|} \\
\hat{t}
\end{pmatrix}
= \begin{pmatrix}
0_{|\hat{J}_1|} \\
0_{|\hat{J}_2|} \\
1
\end{pmatrix}.
\]

Let us denote by the vector \( b_j := [-1_{|\hat{J}_1|}, 0_{|\hat{J}_2|}]^\top \) and by the matrix \( \hat{C}_j := [\hat{\mathbf{A}}^\top_{\hat{J}_1}, I_{\hat{J}_2}] \). Using a standard block matrix inversion formula, we have that the output solution

\[
\hat{\pi} = \hat{t} \hat{C}_j^{-1} b_j \quad \text{and} \quad \hat{t} = \frac{1}{1_d \hat{C}_j^{-1} b_j}.
\]

In order to further simplify the above expression, let us denote by \( \hat{\mathbf{A}}_{\hat{J}_1, \hat{J}_2} \) the matrix formed by selecting the rows \( \hat{J}_1 \) and the columns \( \hat{J}_2 := [d] \setminus \hat{J}_2 \) from the matrix \( \hat{\mathbf{A}} \). The estimate \( \hat{\pi} \) is then given by

\[
\hat{\pi}_{\hat{J}_2} = -\hat{t} \hat{\mathbf{A}}_{\hat{J}_1, \hat{J}_2}^{-T} \cdot 1_{|\hat{J}_1|} \quad \text{and} \quad \hat{\pi}_{\hat{J}_2} = 0.
\]

Plugging in the above value of the estimate \( \hat{\pi} \) into the error term \( \Delta_{\mathbf{A}}(\hat{\pi}, \pi^*) \), we obtain

\[
\Delta_{\mathbf{A}}(\hat{\pi}, \pi^*) = \min_{i \in [d]} e_i^\top \mathbf{A}^\top \pi^* - \min_{i' \in [d]} e_i^\top \mathbf{A}^\top \hat{\pi}
\]

\[
\begin{align*}
&\overset{(i)}{=} \max_{i' \in [d]} \min_{i \in [d]} e_i^\top \mathbf{A}^\top_{1, i' \hat{J}_2} \pi_{i' \hat{J}_2} - e_i^\top \mathbf{A}^\top_{1, i' \hat{J}_2} \hat{\pi}_{i' \hat{J}_2} \\
&\overset{(ii)}{\leq} \max_{i \in \hat{J}_1} e_i^\top \mathbf{A}^\top_{1, \hat{J}_1 \hat{J}_2} \left( \pi_{\hat{J}_2} - \hat{\pi}_{\hat{J}_2} \right) \\
&\overset{(iii)}{=} \max_{i \in \hat{J}_1} e_i^\top \mathbf{A}_{1, \hat{J}_1 \hat{J}_2} \left( \hat{t} \mathbf{A}^{-T}_{\hat{J}_1, \hat{J}_2} - t^* (\mathbf{A}^\top_{\hat{J}_1, \hat{J}_2})^\top \right) 1_{|\hat{J}_1|}
\end{align*}
\]

where equality (i) follows from noting that one of the Nash equilibria will have the components \( \pi^*_{\hat{J}_2} = 0 \) from Lemma 5, (ii) follows from upper bounding the min and noting that the only columns of \( \mathbf{A} \) that can be minimizers are those in \( \hat{J}_1 \) for large enough samples \( n \), and (iii) follows by substituting the values of \( \hat{\pi} \) and the nash distribution \( \pi^* \). We can further split the error term into two components, one which looks at the error in value \( \hat{t} \), and the other corresponding to the error in matrix \( \hat{\mathbf{A}} \).

\[
\Delta_{\mathbf{A}}(\hat{\pi}, \pi^*) \leq t^* \max_{i \in \hat{J}_1} e_i^\top \mathbf{A}_{1, \hat{J}_1 \hat{J}_2} \left( \hat{\mathbf{A}}^{-T}_{\hat{J}_1, \hat{J}_2} - (\mathbf{A}^\top_{\hat{J}_1, \hat{J}_2})^\top \right) 1_{|\hat{J}_1|} + (\hat{t} - t^*) \max_{i \in \hat{J}_1} e_i^\top \mathbf{A}_{1, \hat{J}_1 \hat{J}_2} \hat{\mathbf{A}}^{-T}_{\hat{J}_1, \hat{J}_2} 1_{|\hat{J}_1|} \quad (32)
\]

\(^5\)For the case when an entire column of matrix \( \mathbf{A} \) is determinisitc, those constraints (if tight) can be combined with the other deterministic constraints and the analysis can proceed from there.
Let us denote by $\Phi := A_{\hat{J}_1, \hat{J}_2}^\top$. Then, we can rewrite the matrix $\hat{A}_{\hat{J}_1, \hat{J}_2} = \Phi + Z_n$ where the matrix $Z_n$ represents the zero-mean noise from the Gaussian passive sampling model. Further, let $\Phi = U\Sigma V^\top$ denote the SVD of the matrix $\Phi$. With this, the first term in the above decomposition for any fixed value of $i$ is given by

$$e_i^\top \Phi((\Phi + Z_n)^{-1} - \Phi^i)1_{|\hat{J}_1|} = e_i^\top U \Sigma \left((\Sigma + U^\top Z_n V)^{-1} - \Sigma^i\right) U^\top 1_{|\hat{J}_1|}.$$  

Let us denote by $\hat{Z}_n := U^\top Z_n V$ the effective noise matrix. Using the block matrix inversion formula, the above expression can be written as

$$e_i^\top \Phi((\Phi + Z_n)^{-1} - \Phi^i)1_{|\hat{J}_1|} = e_i^\top \left[U_1 \quad U_2\right] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_1 + \hat{Z}_{11} & \hat{Z}_{12} \\ \hat{Z}_{21} & \hat{Z}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^\top \\ U_2^\top \end{bmatrix} 1_{|\hat{J}_1|},$$

where $\Sigma_1$ is the diagonal matrix with non-zero singular value of $\Phi$ and equality (i) follows from the fact that $U_2^\top 1_{|\hat{J}_1|} = 0$. To see this, recall that $U$ represents the column space of the matrix $\Phi = A_{\hat{J}_1, \hat{J}_2}^\top$, that is the row space of the matrix $A_{\hat{J}_1, \hat{J}_2}$ with $U_2$ representing the null space of this matrix. Since we know that $(\pi^*)^\top A_{\hat{J}_1, \hat{J}_2}^\top = t^* 1_{|\hat{J}_1|}$, all vectors in the null space will necessarily have to be orthogonal to the vector $1_{|\hat{J}_1|}$. Combining the above error bound with equation (32), we find that

$$\Delta_{\mathbf{A}}(\hat{\pi}, \pi^*) \leq t^* \max_{i \in J_1} e_i^\top U_1 \Sigma_1 \left((\Sigma_1 + \hat{Z}_{11} - \hat{Z}_{12} \hat{Z}_{22}^{-1} \hat{Z}_{21})^{-1} - \Sigma_1^{-1}\right) U_1^\top 1_{|\hat{J}_1|}$$

$$+ (t - t^*) \max_{i \in J_1} e_i^\top U_1 \Sigma_1 \left((\Sigma_1 + \hat{Z}_{11} - \hat{Z}_{12} \hat{Z}_{22}^{-1} \hat{Z}_{21})^{-1}\right) U_1^\top 1_{|\hat{J}_1|},$$

where inequality (i) follows from the Taylor series expansion $(I - X)^{-1} = \sum_{s=0}^\infty X^s$ and holds whenever $\|X\|_2 \leq 1$, which can be established for $n$ large enough, and inequality (ii) follows from another such Taylor series expansion followed by a Cauchy–Schwarz inequality. This establishes the desired claim.
Asymptotic error under uniqueness assumption. Having established an upper bound on the error for the general setup in Theorem 4, we now consider the specific scenario where the payoff matrix $A$ is a preference matrix and has a unique von Neumann winner $\pi^\ast$. This is formalized in the following assumption.

**Assumption 3** (Unique Nash equilibrium). The payoff matrix $A$ belongs to the set of preference matrices $P_{d,1}$ and has a unique mixed Nash equilibrium $\pi^\ast$, that is, $\pi^i > 0$ for all $i \in [d]$.

For any preference matrix $A \in P_{d,1}$ and the Bernoulli passive sampling model discussed in Section 4, the asymptotic variance for the Gaussian passive sampling model is $\sigma^2_{i,j} = A_{i,j} \cdot (1 - A_{i,j})$. Let us represent by $\Sigma_i$ the diagonal matrix corresponding to the variances along the $i$th column of the matrix $A$ with $$\Sigma_i = \text{diag}(A_{1,i} \cdot (1 - A_{1,i}), \ldots, A_{d,i} \cdot (1 - A_{d,i})).$$

Given this notation, we now state a corollary which specializes the result of Theorem 4 to payoff matrices $A$ that satisfy the above assumption.

**Corollary 3.** For any payoff matrix $A$ satisfying Assumption 3 and confidence $\delta \in (0, 1)$, there exists a constant $n_0(A, \delta)$ such that for all samples $n > n_0(A, \delta)$, we have that the error $\Delta_A$ of the plug-in estimate $\hat{\pi}_\text{plug}$ satisfies

$$\Delta_A(\hat{\pi}_\text{plug}, \pi^\ast) \leq \|Z_n \pi^\ast\|_\infty + O_d(\|Z_n\|_2^2)$$

$$\leq c \cdot \sqrt{\sigma^2_A d^2 \log \left(\frac{d}{\delta}\right)} + O_d(\|Z_n\|_2^2),$$

where $c$ is constant, and therefore contributes as a lower order term.

We make a few remarks on the above corollary. Observe that the above is a high probability bound on the error $\Delta_A$ of the plug-in estimator $\hat{\pi}_\text{plug}$. Compared with the upper bounds of Corollaries 1 and 4, the asymptotic bound on the error above is instance dependent – the effective variance $\sigma^2_A$ depends on the underlying preference matrix $A$. In particular, this variance measures how well does the underlying von Neumann winner $\pi^\ast$ align with each variance associated with each column of the matrix $A$. In the worst case, since each entry of $A$ is bounded above by 1, the variance $\sigma^2_A = 1$ and we recover back the upper bounds from Corollaries 1 and 4 for the uni-criterion case. The second term in the upper bound comprising the operator norm of the sampling noise, $\|Z_n\|_2^2$, can be shown to be $O_d(\frac{1}{n})$ with high probability\(^6\), and therefore contributes as a lower order term.

**Proof of Corollary 3.** Observe that Assumption 3 implies that the set of tight constraints for the LP (Nash) are the ones corresponding to payoff matrix $A$. That is, the set $J_1 = [d]$ and $J_2 = \phi$. Following Lemma 5, we have, for $n$ large enough, the subset of tight constraints for the perturbed LP (Pert) satisfy $J_1 = J_1$ and $J_2 = J_2$. Further, the uniqueness assumption guarantees that the matrix $A$ is full rank and hence, invertible.

Since the matrix $\hat{A}$ is itself a preference matrix (by construction), the value $\hat{t} = t^\ast = \frac{1}{2}$ and therefore, using the upper bound on the error from Theorem 4, we have,

$$\Delta_A(\hat{\pi}, \pi^\ast) \leq t^\ast \max_{i \in [d]} \left[e_i^T Z_n A^{-T} 1_d \right] + O_d(\|Z_n\|_2^2)$$

$$\leq \|Z_n \pi^\ast\|_\infty + O_d(\|Z_n\|_2^2)$$

\(^6\) $O_d$ notation hides the dependence on the dimensionality $d$. 

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where the final inequality follows by noting that \( \pi^* = t^* A^{-T} 1_d \) and recall that \( Z_n \) denotes the zero mean noise-matrix obtained by the passive Gaussian sampling model with (asymptotic) variance \( \sigma^2_{i,j} = A_{i,j} \cdot (1 - A_{i,j}) \). Let us denote by \( \Sigma_A \) the diagonal matrix measuring the alignment of the Nash equilibrium \( \pi^* \) with the variance of the \( i^{th} \) column of the underlying matrix \( A \), that is,

\[
\Sigma_A(i,j) = \begin{cases} 
(\pi^*)^T \Sigma_i \pi^* & \text{for } i = j \\
0 & \text{otherwise} 
\end{cases}.
\]

Following a similar calculation as in the proof of Corollary 4, we have that each entry of the matrix \( Z_{i,j} \) will have samples \( N_{i,j} = \Theta\left(\frac{d^2}{n}\right) \). Combined with a standard bound for the maximum of sub-Gaussian random variables [Wai19], we have for \( n > n_0(A, \delta) \), with probability at least \( 1 - \delta \),

\[
\Delta_A(\hat{\pi}, \pi^*) \leq c \cdot \sqrt{\frac{\sigma^2_A d^2}{n} \log \left( \frac{d}{\delta} \right)} + O_d(||Z_n||^2_2),
\]

for some universal constant \( c > 0 \) and where the variance \( \sigma^2_A = \max_{i \in [d]} \Sigma_A(i,i) \).

**Example: Generalized Rock-Papers-Scissor.** While in the worst-case, the variance determining the sample complexity of learning Nash from samples is \( \sigma^2_A = \Theta(1) \), we will now construct a family of preference matrices \( A^{(d)} \), for different values of dimension \( d \), and show that \( \sigma^2_A = O\left(\frac{1}{d}\right) \). This exhibits that the plug-in estimator \( \hat{\pi}_{\text{plug}} \) can indeed adapt to the problem complexity and has a sample complexity of \( \tilde{O}\left(\frac{d^2}{\epsilon^2}\right) \) for these class of easier problems compared to the worst-case complexity of \( \tilde{O}\left(\frac{d^6}{\epsilon^2}\right) \).

Our example is a high-dimensional generalization of the classical Rock-Papers-Scissors (RPS) game. Recall, that the pay-off matrix for the RPS game is

\[
A^{\text{RPS}} = \begin{pmatrix}
R & 0.5 & 0 & 1 \\
0.5 & 1 & 0.5 & 0 \\
0 & 1 & 0.5 \\
S & 0 & 1 & 0.5
\end{pmatrix}.
\]

Observe that the above payoff matrix encodes a deterministic game: Rock beats Scissor, Scissor beats Paper, and Paper beats Rock. Similar to this, we define a randomized version of the above RPS game with payoffs where we allow a small probability 0.25 with which the lesser preferred item in a match-up can defeat the other, for example, Scissor against Rock. Explicitly, such a payoff matrix \( A^{(3)} \) is given by

\[
A^{(3)} = \begin{pmatrix}
R & 0.50 & 0.25 & 0.75 \\
0.50 & 0.75 & 0.50 & 0.25 \\
0.25 & 0.75 & 0.50
\end{pmatrix}.
\]

Similar to the deterministic RPS game, the above randomized game can be seen to have a unique Nash equilibrium with \( \pi^* = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}] \).

We now describe a \( d \)-dimensional generalization of the above payoff matrix, for any odd value of \( d = 2d' + 1 \). For the element \( e_1 \), the first entry will be set to 0.50, the next \( d' \) entries to be 0.25 and the final \( d' \) entries to be 0.75 – game-theoretically, this means that the element \( e_1 \) loses to elements
\[ e_2 - e_{d+1} + 1 \] and is preferred over elements \( e_{d+2} - e_{2d+1} \), both with probability 0.75. Similarly, for the \( i^{th} \) element, the row \( A_i^{(d)} \) is given by

\[
A^{(d)}(i, j) = \begin{cases} 
0.50 & \text{for } j = i \\
0.25 & \text{for } j \in [i + 1 \mod d, i + d' \mod d] \\
0.75 & \text{for } j \in [i + d' + 1 \mod d, i + 2d' \mod d] 
\end{cases}
\]

It is easy to see from the form of the pay-off matrix that each element \( e_i \) is preferred over \( d' \) elements and has a lower preference than \( d' \) elements. By the symmetry of the pay-off matrix, the unique Nash equilibrium is given by the distribution \( \pi^* = \frac{1}{d}1_d \) which lies in the interior of the simplex \( \Delta_d \) and hence satisfies Assumption 3. Further, we can compute the variance \( \sigma^2_{A^{(d)}} \) as

\[
\sigma^2_{A^{(d)}} = \max_{i \in [d]} (\pi^*)^\top \Sigma_i \pi^* = \max_i \left( \frac{1}{4d^2} + \sum_{j \neq i} \frac{3}{16d^2} \right) \leq \frac{1}{d^2} + \frac{3}{16d^2}.
\]

Plugging this variance in the upper bound obtained in Corollary 3, for \( n > n_0(A, \delta) \), we have

\[
\Delta_{A^{(d)}}(\hat{\pi}_{\text{plug}}, \pi^*) \leq c \sqrt{\frac{d}{n} \log \left( \frac{d}{\delta} \right)}
\]

with probability greater than \( 1 - \delta \). Thus, to obtain an \( \epsilon \)-accurate solution for the payoff matrix \( A^{(d)} \), the plug-in estimator \( \hat{\pi}_{\text{plug}} \) requires \( \tilde{O}(d^2) \) samples, a factor \( d \) less than the worst-case sample complexity of \( \tilde{O}(d^2) \).

**Proof of Lemma 5**

Recall from our discussion above that the the constraint set for the LP (Nash), the constraint polytope is a closed convex set and has a finite number \( |V| = O((2d + 2)^{\frac{d+1}{2}}) \), which follows from McMullen’s theorem [McM70]. Let us denote each vertex of the polytope by \( x_v = (\pi_v, t_v) \) and the corresponding set of \( d \) constraints which define the vertex by \( J_v \). Further, let \( V^* \) denote the set of optimal vertices.

Because of the random Gaussian noise, for \( n = \Omega(d^2) \), we have that the solution \( \hat{x} = (\hat{\pi}_{\text{plug}}, \hat{t}) \) will be unique with probability 1. In order to establish the claim of the lemma, we can equivalently show that for any vertex \( x_v \notin V^* \), the corresponding vertex \( \hat{x}_v \) will not be output by the perturbed LP. This follows from the observation that for \( n > O(d^2 \log(1/\delta)) \) and for any vertices \( x_v \), we have

\[
P \left( |\hat{t}_v - t_v| \geq c \sqrt{\frac{d^2}{n} \log \left( \frac{1}{\delta} \right)} \right) \leq \delta,
\]

for some universal constant\(^7\) \( c > 0 \). Taking a union bound over all \( |V| \) vertices, we have

\[
P \left( \exists x_v \text{ s.t. } |\hat{t}_v - t_v| \geq c \sqrt{\frac{d^2}{n} \log \left( \frac{|V|}{\delta} \right)} \right) \leq \delta.
\]

\(^7\)To be clear, the value of the constant \( c \) can change values across lines, but will always remain a universal constant independent of problem parameters.
Therefore, whenever \( \max_v t_v \leq t^* - \gamma \) for some \( \gamma > 0 \), we have that after \( n > c \frac{d^2}{\gamma} \log \left( \frac{|V|}{\delta} \right) \), with probability at least \( 1 - \delta \), we have,

\[
\max_{v \notin V^*} \hat{t}_v < \min_{v \in V^*} \hat{t}_v ,
\]

and therefore the perturbed LP will have active constraint set satisfying \( J \subseteq \hat{J} \). This proves the desired claim.

\[\text{D Additional results and their proofs}\]

This section covers additional sample complexity results as well as optimization algorithms for finding the Blackwell winner of a multi-criteria preference learning instance.

\[\text{D.1 Sample complexity bounds for } \ell_1\text{-norm}\]

**Corollary 4.** Suppose that the distance \( \rho \) is induced by the \( \ell_1 \)-norm \( \| \cdot \|_1 \). Then there are universal constants such that given a sample size \( n > c_1 d^2 k \log(c dk) \), then for each valid target set \( S \), we have

\[
\Delta_{\mathbf{P}}(\hat{\pi}_{\text{plug}}, \pi^*) \leq c_3 k \sqrt{\frac{d^2 k n \log\left( c_3 dk / \delta \right)}{\min_{i_1 \in [d]} N_{(i_1, i_2, j)}}}
\]

with probability exceeding \( 1 - \delta \).

**Proof.** Being somewhat more explicit with our notation, let \( N_{(i_1, i_2, j)} \) denote the number of samples observed under the passive sampling model at index \((i_1, i_2, j)\) of the tensor. Proceeding as in equation (20), we have

\[
\Pr \left\{ \| P_j(\cdot, i_2) - \hat{P}_j(\cdot, i_2) \|_\infty \geq c_1 \sqrt{\frac{\log(c_1 d / \delta)}{\min_{i_1 \in [d]} N_{(i_1, i_2, j)}}} \right\} \leq \delta.
\]

Summing over all criteria \( j \in [k] \) and then applying the union bound yields

\[
\Pr \left\{ \| P(\cdot, i_2) - \hat{P}(\cdot, i_2) \|_{\infty, 1} \geq c k \sqrt{\frac{\log(c k d / \delta)}{\min_{i_1, j} N_{(i_1, i_2, j)}}} \right\} \leq \delta.
\]

Finally, in order to obtain a bound on the maximum deviation in the \((\infty, 1)\)-norm, we take a union bound over all \( d \) choices of the index \( i_2 \), and apply inequality (19) to obtain

\[
\max_{i_2} \| P(\cdot, i_2) - \hat{P}(\cdot, i_2) \|_{\infty, 1} \leq c k \sqrt{\frac{d^2 k n \log\left( c dk / \delta \right)}{\min_{i_1, j} N_{(i_1, i_2, j)}}}
\]

with probability exceeding \( 1 - \delta \). \(\square\)

A few comments regarding the corollary are in order. The above corollary suggests that the sample complexity required for obtaining an \( \epsilon \)-accurate solution with respect to the \( \ell_1 \) norm is \( n = \tilde{O}(d^k k^4) \). Observe that this bound is a factor of \( k^2 \) worse than the corresponding one for \( \ell_\infty \) norm established in Corollary 1. This additional sample complexity occurs since for any vector \( v \in \mathbb{R}^k \), we have \( \| v \|_1 \leq k \| v \|_\infty \). This implies that the error when measured with respect to \( \ell_1 \) can be up to \( k \) times larger; since the sample complexity scales as \( \frac{1}{\epsilon^2} \), the corresponding increase with respect to the number of criteria \( k \) is quadratic.
D.2 Optimization algorithms

Recall that Theorem 3 established that the objective function $v(\pi; P, S, \| \cdot \|_q)$ is convex in $\pi$ and Lipschitz with respect to the $\ell_1$ norm. This implies that one could compute the plug-in solution $\hat{\pi}_{\text{plug}}$ as a solution to a constrained optimization problem. In this section, we discuss a few specific algorithms based on zeroth-order and first-order methods for obtaining such a solution.

D.2.1 Zeroth-order optimization

Zeroth-order methods for minimizing a function $f(x)$ over $x \in X$ work with a function query oracle. That is, at each time step, the algorithm has access to an oracle which returns the value $f(x)$ for any point $x \in X$. In our setup, since we are interested in minimizing the value function $v(\pi; P, S, \rho)$ over $\pi \in \Delta_d$, such a function query requires access to the target set $S$ via an oracle $O_S^0$ such that

$$O_S^0(z) \rightarrow \min_{z_1 \in S} \rho(z, z_1),$$

for the underlying distance function $\rho(\cdot)$. The oracle $O_S^0$ essentially takes as input a score vector $z \in [0, 1]^k$ and outputs the distance of this point to the target set $S$. Given this oracle, it is easy to see that for any $\pi$, one can compute the corresponding value function $v(\pi; P, S, \rho)$.

There have been several algorithms proposed for optimization with such oracles when the underlying function $f$ is convex [FKM05; ADX10; Sha13; DJWW15; NS17; Sha17] or non-convex, smooth [GL13]. The key idea in the proposed algorithms is to utilize the zeroth-order oracle to construct estimates of the (sub-)gradient of the function $f$ using a class of techniques called randomized smoothing. The algorithms then differ in the construction of these estimates depending on the underlying randomness as well as on the number of oracle calls during each time step.

Given the results of Theorem 3, we can restrict our focus on algorithms for the class of convex Lipschitz function $f$. To this end, Shamir [Sha17] proposed an algorithm for optimizing such functions which required two function evaluations at each time. The algorithm, adapted to the multi-criteria preference learning problem, is detailed in Algorithm 1. For our setup, we select the negative entropy regularization, $r(\pi) = \sum_i \pi_i \log(\pi_i)$ to suit the geometry of our domain $X = \Delta_d$.

At each time step $t$, the proposed algorithm maintains an estimate of the distribution $\pi_t$, and queries the function value $v(\cdot; P, S, \rho)$ at two points—namely, $\pi_t + \delta u_t$ and $\pi_t - \delta u_t$—where the

---

Algorithm 1: Zeroth-order method for multi-criteria preference learning

| **Input**: Time steps $T$, step size $\eta$, smoothing radius $\delta$ |
| **Initialize**: $\theta_1 = 0$ |
| for $t = 1, \ldots, T$ do |
| $\pi_t = \arg \max_{\pi \in \Delta_d} (\theta_t, \pi) - r(\pi)$ where $r(\pi) = \sum_i \pi_i \log(\pi_i)$ |
| Sample $u_t$ uniformly from the Euclidean unit sphere $\{u \mid \|u\|_2 = 1\}$ |
| For every $i \in [d]$, query points $z_{1,i} = P(\pi_t + \delta u_t, i)$ and $z_{2,i} = P(\pi_t + \delta u_t, i)$ |
| Set $v(\pi_t + \delta u_t; P, S, \rho) = \max_i \rho(z_{1,i}, S)$ and $v(\pi_t - \delta u_t; P, S, \rho) = \max_i \rho(z_{2,i}, S)$ |
| Set sub-gradient estimate $\hat{g}_t = \frac{d}{\pi_t} (v(\pi_t + \delta u_t; P, S, \rho) - v(\pi_t - \delta u_t; P, S, \rho)) u_t$ |
| Update $\theta_{t+1} = \theta_t - \eta \hat{g}_t$ |
| **Output**: $\bar{\pi}_T = \frac{1}{T} \sum_{t=1}^T \pi_t$ |
random vector $u$ is sampled uniformly from the Euclidean unit sphere and $\delta > 0$ represents the smoothing radius. Given these queries, we compute the sub-gradient estimate

$$\hat{g}_t := \frac{d}{2\delta} (v(\pi_t + \delta u_t; \mathbf{P}, S, \rho) - v(\pi_t - \delta u_t; \mathbf{P}, S, \rho)) u_t.$$  

This sub-gradient estimate is then used to update the parameter estimate $\pi_{t+1}$ using the mirror descent algorithm with the specified regularization function. The zeroth-order method in Algorithm 1 does not require the underlying function to be smooth and hence works for our problem setup with arbitrary non-differentiable distance functions. We can now obtain the following convergence result, based on Theorem 1 from the work of Shamir [Sha17].

**Proposition 5.** Under the conditions of Theorem 3, suppose that we run Algorithm 1 for $T$ iterations with step-size $\eta_t = \frac{c}{k\sqrt{d}\sqrt{T}}$ and smoothing radius $\delta = \frac{c\log d}{\sqrt{T}}$. Then the resulting sequence $\tilde{v}(\pi_T; \mathbf{P}, S, \| \cdot \|_q) \leq \min_{\pi \in \Delta_d} v(\pi; \mathbf{P}, S, \| \cdot \|_q) + c\tilde{\gamma} \cdot \sqrt{\frac{d\log^2 d}{T}}$

where $\tilde{v}_T = \frac{1}{T} \sum_{t=1}^T \pi_t$.

**Proof.** By Theorem 3, the value function $v(\pi; \mathbf{P}, S, \| \cdot \|_q)$ is convex and $L_v = k^{\frac{1}{2}}$-Lipschitz with respect to $\| \cdot \|_1$. Also, the choice of the regularizer $r(\pi) = \sum_i \pi_i \log(\pi_i)$ is 1-strongly convex with respect to the $\| \cdot \|_1$. Plugging in the above values in Theorem 1 from [Sha17] establishes the above convergence rate.

Thus, in order to obtain a distribution $\hat{\pi}$ that is $\varepsilon$-close to $\pi^*$ in function value, we need to run Algorithm 1 for $T = O\left(\frac{k^{\frac{3}{2}} d \log^2 d}{\varepsilon^2}\right)$ iterations. Also, note that each iteration of the algorithm requires $d$ calls to the oracle $O^0_S$. Therefore the total oracle complexity of the procedure is $O\left(\frac{k^{\frac{3}{2}} d \log^2 d}{\varepsilon^2}\right)$.

### D.3 First-order optimization

In this section, consider some first-order methods to compute the plug-in estimator. Let us denote by $\partial v(\pi)$ the set of sub-differentials of the function $v(\cdot; \mathbf{P}, S, \| \cdot \|_q)$ evaluated at $\pi$. Further, let the set $\Gamma(\pi)$ denote the set of maximizers for a policy $\pi$, that is,

$$\Gamma(\pi) = \left\{ \tilde{\pi} \in \Delta_d \mid \tilde{\pi} \in \arg \max_{\pi_2 \in \Delta_d} \min_{z \in S} \| \mathbf{P}(\pi, \pi_2) - z \| \right\}. \quad (38)$$

Note that both of these quantities depend implicitly on the tuple $(S, \mathbf{P}, \| \cdot \|_q)$, but we have dropped this dependence in the notation. Given the setup above, Lemma 6 below characterizes this set $\partial v(\pi)$ for any smooth $\ell_q$ norm (with $1 < q < \infty$).

**Lemma 6.** Suppose that the distance is induced by a smooth $\ell_q$ norm for $1 < q < \infty$. Then the set of sub-differentials of $v$ at $\pi$ is given by:

$$\partial v(\pi) = \text{conv} \left\{ \mathbf{P}(\cdot, \pi_2) [\mathbf{P}(\pi, \pi_2) - \Pi_S(\mathbf{P}(\pi, \pi_2))] \mid \pi_2 \in \Gamma(\pi) \right\},$$

where $\Pi_S(z)$ denotes the unique projection of the point $z$ onto set $S$ along $\| \cdot \|_q$.  

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Algorithm 2: First-order method for multi-criteria preference learning

**Input:** Time steps $T$, step size $\eta$

**Initialize:** $\theta_1 = 1_k$

for $t = 1, \ldots, T$

- Set the distribution $\pi_t = \frac{\theta_t}{\|\theta_t\|_1}$
- Obtain $g_t \in \text{conv} \left\{ \frac{P(\pi_1, \pi_2) - \Pi_S(P(\pi_1, \pi_2))}{\|P(\pi_t, \pi_2) - \Pi_S(P(\pi_1, \pi_2))\|_q} \mid \pi_2 \in \Gamma(\pi_t) \right\}$ [See eq.(38) for $\Gamma(\pi_t)$]
- Update $\theta_{t+1} = \pi_t \exp(-\eta g_{t,i})$

**Output:** $\bar{\pi}_T = \frac{1}{T} \sum_{t=1}^{T} \pi_t$

We defer the proof of the above lemma to later in the section. Note that in order to access such a sub-gradient, we need access to an oracle $O^1_S$ that provides projection queries of the form

$$O^1_S(z) \rightarrow \arg \min_{z_1 \in S} \rho(z, z_1).$$

The oracle $O^1_S$ takes in a point $z$ and outputs the closest point in the set $S$ to this point. Given such an oracle, we can compute the sub-gradient of the function $v(\pi; P, S, \rho)$ using Lemma 6 by evaluating it at the point given by $P(\pi, \pi_2)$ for some $\pi_2 \in \Gamma(\pi)$.

Given access to such a projection oracle $O^1_S$, we detail out a procedure based on a standard implementation of mirror descent with entropic regularization (or Exponentiated gradient method) in Algorithm 2 to minimize the objective $v(\pi; P, S)$. Note that we select the negative entropy function, $r(\pi) = \sum_i \pi_i \log(\pi_i)$, as the regularization function for the mirror descent procedure since our parameter space is given by the simplex $\Delta_k$ and the negative entropy function is known to be 1-strongly convex with respect to $\| \cdot \|_1$ over this space.

The algorithm works by maintaining at each time instance a distribution $\pi_t$ over the set of objects and updates it via an exponentiated gradient update. That is, the sub-gradient $g_t$ is evaluated at the current point $\pi_t$ using access to both $O^1_S$ and $O^0_S$, and is used to update each coordinate of the variable $\theta_t$. The updated distribution $\pi_{t+1}$ is obtained via a KL-projection of $\theta_t$ onto the simplex $\Delta_k$, which can be shown to be equivalent to the normalization $\theta/\|\theta\|_1$. We now proceed to prove a convergence result for this gradient-based Algorithm 2, based on a standard analysis of the mirror descent procedure (for example, see [Bub15, Theorem 4.2]).

**Proposition 6.** Suppose the conditions of Theorem 3 hold and consider any $\ell_q$-norm for $1 < q < \infty$. Suppose that running Algorithm 1 for $T$ iterations with step-size $\eta_t = \frac{1}{k^q} \sqrt{\frac{2 \log d}{T}}$ produces a sequence $\pi_1, \pi_2, \ldots, \pi_T$. Then we have

$$v(\bar{\pi}_T; P, S, \| \cdot \|_q) \leq \min_{\pi \in \Delta_d} v(\pi; P, S, \| \cdot \|_q) + k^\frac{1}{q} \sqrt{\frac{2 \log d}{T}}$$

where $\bar{\pi}_T = \frac{1}{T} \sum_{t=1}^{T} \pi_t$.

**Proof.** Note that the function $v(\pi; P, S, \| \cdot \|_q)$ is convex and $k^{\frac{1}{q}}$-Lipschitz with respect to the $\ell_1$ norm from Theorem 3. Further, the mirror map given by negative entropy function is 1-strongly convex with respect to $\| \cdot \|_1$. Plugging in these values in Theorem 4.2 from [Bub15] establishes the required convergence rate.
In order to obtain an \( \epsilon \)-accurate solution in function value, it suffices to run the above algorithm for
\[
T = O\left( \frac{k^8 \log d}{\epsilon^2} \right)
\]
iterations, with each iteration using 1 call to the oracle \( O^1_S \) and \( d \) calls to the oracle \( O^0_S \) (to obtain the set \( \Gamma \)). Thus, we see that the total oracle complexity changes as
\[
O^1_S: O\left( \frac{k^8 \log d}{\epsilon^2} \right) \text{ calls and } O^0_S: O\left( \frac{k^2 q \log d}{\epsilon^2} \right) \text{ calls – effectively, an } O(d \log d) \text{ decrease in the calls to } O^0_S \text{ is compensated by a corresponding increase of } O\left( \frac{\log d}{\epsilon^2} \right) \text{ calls to the stronger oracle } O^1_S.
\]

**Proof of Lemma 6.** Consider the function \( \phi(\pi_1, \pi_2) = \max_{z \in S} \| P(\pi_1, \pi_2) - z \| \) over the domain \( \pi_2 \in \Delta_d \). For any fixed \( \pi_2 \), we have that the function \( \phi(\pi_1, \pi_2) \) is convex in \( \pi_1 \). Thus, by Danskin’s theorem, we have that the subdifferential set is given by:
\[
\partial v(\pi) = \text{conv} \left\{ \frac{\partial \phi(\pi, \pi_2)}{\partial \pi} \mid \pi_2 \in \Gamma(\pi) \right\},
\]
where \( \text{conv} \) represents the convex hull of the set. Let us now focus on the partial derivative \( \frac{\partial \phi(\pi, \pi_2)}{\partial \pi} \) for any \( \pi_2 \) which is a maximizer. This partial derivative involves differentiation of a metric projection onto a convex set, which has been studied extensively in the literature of convex analysis [PR70; Zaj84; AT14]. Recently, Balestro et al. [BMT19] established that for distance functions given by smooth norms, the derivative of metric projection for any \( z \notin S \) is given by:
\[
\nabla \rho(z, S) = \nabla \min_{z_2 \in S} \| z - z_2 \| = \frac{z - \Pi_S(z)}{\| z - \Pi_S(z) \|},
\]
where \( \Pi_S(z) \) denotes the unique projection of the point \( z \) onto set \( S \). Combining this with the chain rule of differentiation, we have that:
\[
\frac{\partial \phi(\pi, \pi_2)}{\partial \pi} = \frac{P(\cdot, \pi_2) \left[ P(\pi, \pi_2) - \Pi_S(P(\pi, \pi_2)) \right]}{\| P(\pi, \pi_2) - \Pi_S(P(\pi, \pi_2)) \|_q}.
\]
The above, in conjunction with equation (39) establishes the desired claim.

**E Details of user study**

In this section, we provide the deferred details of the user study from Section 5.

**Self-driving environment.** The self-driving environment consists of an autonomous car which can be controlled by providing real-valued inputs acceleration and angular acceleration at every time step. We allow the policies to have access to the dynamics of this environment. Observe that there is no explicit reward function in the environment and each policy differs in the way it optimizes a chosen reward function to drive the car forward in a safe manner.

**Policies.** The MPC based Policies A-E were constructed by optimizing linear rewards comprising features F1-F9 as

- **F1** Distance from the starting point along y-axis.
- **F2** Velocity of the autonomous car.
F3 Distance from the center of each lane.
F4 Gaussian collision detector for nearby objects.
F5 Collision detector which works at smaller radii than F4.
F6 Over-speeding feature which penalizes higher speeds.
F7 Reward for over-taking vehicles in the front.
F8 Gaussian off-road detector.
F9 Reward to promote speeding up near obstacles.

For each of the base policy, we set the weights of the features to encode different driving behaviors.

Pol A programmed to prefer the right-most lane and progress forward at a slow speed.
Pol B programmed to prefer the left-most lane and move forward as fast as possible.
Pol C programmed to be conservative, avoids collision and proceeds forward.
Pol D programmed to get attracted towards other cars and obstacles.
Pol E programmed to prefer center lane and exhibit opportunistic behavior by moving ahead of other cars.

Details of target set and linear weights. We selected the two data-oblivious sets to trade-off between the criteria C1-C5 as

\begin{equation}
S_1 = \{ z \mid z \in [0,1]^5, z_1 \geq 0.3, z_2 \geq 0.3, z_3 \geq 0.2, z_4 \geq 0.3, z_5 \geq 0.4 \},
\end{equation}

\begin{equation}
S_2 = \{ z \mid z \in [0,1]^5, z_1 \geq 0.25, z_2 \geq 0.25, z_3 \geq 0.25, z_4 \geq 0.25, z_5 \geq 0.25, z_1 + z_5 \geq 0.9 \}. (40)
\end{equation}

In addition, we selected 9 set of weights \(w_{1:9}\) for linearly combining the different criteria.

- \(w_1\): Average of the users’ self-reported weights.
- \(w_2\): Weight vector obtained by regressing the overall criterion on C1-C5 with squared loss as

\[
w_2 \in \arg \min_{w \in \Delta^5} \sum_{i_1,i_2} (P_{ov}(i_1,i_2) - \sum_j w(j)P^j(i_1,i_2))^2.
\]

- \(w_3\): Weight obtained by regressing Bradley-Terry-Luce (BTL) scores. The BTL parametric model assumes a real-valued score \(v_i\) for each policy and posits that \(\Pr(\text{Pol } i \geq \text{ Pol } j) = \exp(v_i)/\exp(v_i) + \exp(v_j)\). Denoting the scores obtained from the overall preferences by \(v^{ov}\) and those obtained from the individual criteria by \(v^j\) for \(j \in [5]\), the weight

\[
w_2 \in \arg \min_{w \in \Delta^5} \sum_i (v_i^{ov} - \sum_j w(j)v^j_i)^2.
\]

- \(w_4\): Data-oblivious weight \(w_4 = [0.2,0.2,0.2,0.2,0.2]\).
\( w_5 \): Data-oblivious weight \( w_5 = [0.25, 0.5/3, 0.5/3, 0.5/3, 0.25] \).

\( w_6 \): Data-oblivious weight \( w_6 = [0.30, 0.4/3, 0.4/3, 0.4/3, 0.30] \).

\( w_7 \): Data-oblivious weight \( w_7 = [0.5/3, 0.5/3, 0.25, 0.5/3, 0.25] \).

\( w_8 \): Data-oblivious weight \( w_8 = [0.4/3, 0.4/3, 0.3, 0.4/3, 0.30] \).

\( w_9 \): Data-oblivious weight \( w_9 = [0.3, 0.1/2, 0.3, 0.1/2, 0.3] \).

The set of data oblivious weights were chosen to account for different trade-offs along the criteria C1-C5 including the uniform weight \( w_4 \).

**Data Collection.** Table 1 shows the comparison data collected from the Mturk users in both the phases of the experiment. The entry \( i, j \) of the comparison matrices represents the fraction of users which preferred Policy \( i \) over Policy \( j \). The top 5 rows and columns of each matrix correspond to the baseline policies while the bottom rows correspond to the two randomized policies R1 and R2 obtained as the Blackwell winner corresponding to sets \( S_1 \) and \( S_2 \) respectively.

In addition, we would like to highlight some details from an experiment design perspective. Since the experiment was run in two phases, we could not guarantee the same set of subjects to participate in both parts of the experiment. In order to limit distribution shifts, we restricted the nationality of the subjects to United States and began both the phases on the same time and day of the week. Also, in order to prevent biased evaluations, the ordering of the policy pairs as well as the ordering policies within a comparison was randomized across the users.

Figures 3, 4 and 5 shows the experiment setup we used for obtaining comparison data from Amazon Mechanical Turk users consisting of the instructions, the policy comparison page and the questionnaire that the users were asked to fill out.
Instructions

In this experiment, the objective is to select amongst a given alternatives of self-driving cars based on their performance along different objectives. We will show you self-driving cars, operated by different softwares (or algorithms) which leads them to exhibit different behaviors in different environments. In each part of the experiment, we will show you a pair of self-driving softwares and how they behave in certain environments. The behavior of the driving policies will be shown from a bird’s eye view. We will then ask you comparative questions which will ask you to select one of the driving softwares according to a specified criterion and ask you the reasoning behind your choices.

Instructions

During the experiment, please remember the following:

- It is important that you carefully observe the behavior of the softwares in the provided environments before responding to the following questions based on that.
- You will be allowed to proceed to the next part of the experiment only once you have responded to all the comparison questions and have specified the appropriate justification for your choices.
- Please note that the main car driven by the software will be coloured in Orange while the other companion cars will be shown in Black.
- Each of the softwares has been labelled as Software [G, H]. Across the different experiments, the naming of the software remains consistent. For instance, Software A will remain the same software during each of the individual experiments of the survey. Note that some of these policies make use of randomization and their behavior might differ across experiments.

Figure 3. Instructions provided to the users before the experiment began. The users were asked to compare behavior of policies and were told to expect some policies to exhibit a randomized behavior.
Figure 4. Layout of the experiment where each panel shows a GIF exhibiting a Policy controlling the autonomous vehicle in one of the worlds of the environment. The users were instructed to compare behaviors across each of the columns before proceeding to answer the questions.
Questions

Q1*. Which of the two softwares exhibits a less aggressive behavior? :
  ○ Software H    ○ Software G

Q2*. Which of the two softwares is more predictable in their behavior? That is, for which of the two softwares do you think you will be able to anticipate its performance in a new environment? :
  ○ Software H    ○ Software G

Q3*. Which of the two softwares will get you to your destination the quickest? :
  ○ Software H    ○ Software G

Q4*. Which of the two softwares is more conservative in its driving approach? :
  ○ Software H    ○ Software G

Q5*. Which of the two softwares has a lower risk of collision with another car or an obstacle? :
  ○ Software H    ○ Software G

Q6*. [Overall Preference] Imagine you were to select one of the two softwares to get you to your destination. Which of the two softwares would you prefer? :
  ○ Software H    ○ Software G

* Please provide a brief sentence about how you made your selections. (Press ‘Enter’ after typing the sentence)

* For each of the following characteristic, please indicate their relevance in determining the overall preference between the softwares. Please take into account all the experiments that you completed in this study. (5 = extremely important, 1 = had little importance)

Aggressiveness of the software:
  ○ 1    ○ 2    ○ 3    ○ 4    ○ 5

Predictability of the software:
  ○ 1    ○ 2    ○ 3    ○ 4    ○ 5

Speed or quickness of the software:
  ○ 1    ○ 2    ○ 3    ○ 4    ○ 5

Conservativeness of the software:
  ○ 1    ○ 2    ○ 3    ○ 4    ○ 5

Collision Risk of the software:
  ○ 1    ○ 2    ○ 3    ○ 4    ○ 5

Figure 5. Layout of the questions panel comprising the 6 comparison questions and the form for reporting the relevance of each criterion in the overall evaluation.
Table 1. Each matrix consists of pairwise comparisons between policies elicited from a user study with around 50 participants on Mturk. An entry $i,j$ of the comparison matrices represents the fraction of users which preferred Policy $i$ over Policy $j$. Policies A-E comprise the base set of policies while Policies R1-R2 are the randomized Blackwell winners obtained from the sets in equation (40). While Policy C is the overall von Neumann winner, Policy R2 is preferred over it by 66% of the users.

### References

[AB06] Michele Aghassi and Dimitris Bertsimas. “Robust game theory”. In: *Mathematical Programming* 107 (2006), pp. 231–273 (Cited on page 4).

[ABH11] Jacob Abernethy, Peter L Bartlett, and Elad Hazan. “Blackwell approachability and no-regret learning are equivalent”. In: *Proceedings of the Conference on Learning Theory*. 2011 (Cited on pages 4, 14).

[ACKK14] Saleema Amershi, Maya Cakmak, William Bradley Knox, and Todd Kulesza. “Power to the people: The role of humans in interactive machine learning”. In: *AI Magazine* 35.4 (2014), pp. 105–120 (Cited on page 1).

[ADX10] Alekh Agarwal, Ofer Dekel, and Lin Xiao. “Optimal algorithms for online convex optimization with multi-point bandit feedback.” In: *Proceedings of the Conference on Learning Theory*. 2010 (Cited on page 34).

[AKJ14] Nir Ailon, Zohar Karnin, and Thorsten Joachims. “Reducing dueling bandits to cardinal bandits”. In: *Proceedings of the International Conference on Machine Learning*. 2014 (Cited on page 3).

[Arr51] Kenneth Joseph Arrow. *Social Choice and Individual Values*. Wiley, 1951 (Cited on page 1).

[AT14] Alexey Rostislavovich Alimov and Igor’Germanovich Tsar’kov. “Connectedness and other geometric properties of suns and Chebyshev sets”. In: *Fundamentalnaya i Prikladnaya Matematika* 19.4 (2014), pp. 21–91 (Cited on page 37).
[BCEL+16] Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D Procaccia. *Handbook of Computational Social Choice*. Cambridge University Press, 2016 (Cited on page 3).

[BHIKS20] Erdem Bıyık, Nicolas Huynh, Mykel J Kochenderfer, and Dorsa Sadigh. “Active preference-based gaussian process regression for reward learning”. In: *arXiv preprint arXiv:2005.02575* (2020) (Cited on page 1).

[Bla48] Duncan Black. “On the rationale of group decision-making”. In: *Journal of Political Economy* 56.1 (1948), pp. 23–34 (Cited on page 1).

[Bla56] David Blackwell. “An analog of the minimax theorem for vector payoffs.” In: *Pacific Journal of Mathematics* 6.1 (1956), pp. 1–8 (Cited on pages 3–5, 14, 15).

[BMT19] Vitor Balestro, Horst Martini, and Ralph Teixeira. “Convex analysis in normed spaces and metric projections onto convex bodies”. In: *arXiv preprint arXiv:1908.08742* (2019) (Cited on page 37).

[Bor84] JC de Borda. “Mémoire sur les élections au scrutin”. In: *Histoire de l’Academie Royale des Sciences pour 1781* (1784) (Cited on pages 1, 3).

[BT52] Ralph Allan Bradley and Milton E Terry. “Rank analysis of incomplete block designs: I. The method of paired comparisons”. In: *Biometrika* 39.3/4 (1952), pp. 324–345 (Cited on page 1).

[Bub15] Sébastien Bubeck. “Convex optimization: Algorithms and complexity”. In: *Foundations and Trends® in Machine Learning* 8 (2015) (Cited on page 36).

[CJRY12] Olivier Chapelle, Thorsten Joachims, Filip Radlinski, and Yisong Yue. “Large-scale validation and analysis of interleaved search evaluation”. In: *ACM Transactions on Information Systems (TOIS)* 30.1 (2012), pp. 1–41 (Cited on page 1).

[CLBM+17] Paul F Christiano, Jan Leike, Tom Brown, Miljan Martic, Shane Legg, and Dario Amodei. “Deep reinforcement learning from human preferences”. In: *Advances in Neural Information Processing Systems*. 2017 (Cited on page 1).

[Con85] Marquis de Condorcet. “Essai sur l’application de l’analyse a la probabilité des decisions rendues a la pluralité des voix”. In: (1785) (Cited on pages 1, 3).

[Cop51] Arthur H Copeland. *A reasonable social welfare function*. Tech. rep. mimeo, University of Michigan, 1951 (Cited on page 3).

[DHSS+15] Miroslav Dudík, Katja Hofmann, Robert E Schapire, Aleksandr Slivkins, and Masrour Zoghi. “Contextual Dueling Bandits”. In: *Proceedings of the Conference on Learning Theory*. 2015 (Cited on pages 2–6, 8, 25).

[DJWW15] John C Duchi, Michael I Jordan, Martin J Wainwright, and Andre Wibisono. “Optimal rates for zero-order convex optimization: The power of two function evaluations”. In: *IEEE Transactions on Information Theory* 61.5 (2015) (Cited on page 34).

[DM10] Karen M Douglas and Robert J Mislevy. “Estimating classification accuracy for complex decision rules based on multiple scores”. In: *Journal of Educational and Behavioral Statistics* 35.3 (2010), pp. 280–306 (Cited on page 4).
Michael Doumpos and Constantin Zopounidis. “Regularized estimation for preference disaggregation in multiple criteria decision making”. In: Computational Optimization and Applications 38.1 (2007), pp. 61–80 (Cited on page 4).

Deborah Frisch and Robert T Clemen. “Beyond expected utility: rethinking behavioral decision research.” In: Psychological Bulletin 116.1 (1994), p. 46 (Cited on page 4).

Abraham D Flaxman, Adam Tauman Kalai, and H Brendan McMahan. “Online convex optimization in the bandit setting: gradient descent without a gradient”. In: Proceedings of the ACM-SIAM Symposium on Discrete Algorithms. 2005 (Cited on page 34).

Drew Fudenberg and David K Levine. “Self-confirming equilibrium”. In: Econometrica: Journal of the Econometric Society (1993), pp. 523–545 (Cited on page 4).

William M Goldstein and Jane Beattie. “Judgments of relative importance in decision making: The importance of interpretation and the interpretation of importance”. In: Frontiers of Mathematical Psychology. 1991, pp. 110–137 (Cited on page 4).

Saeed Ghadimi and Guanghui Lan. “Stochastic first-and zeroth-order methods for nonconvex stochastic programming”. In: SIAM Journal on Optimization 23.4 (2013) (Cited on page 34).

Katja Hofmann, Shimon Whiteson, and Maarten De Rijke. “A probabilistic method for inferring preferences from clicks”. In: Proceedings of the International Conference on Information and Knowledge Management. 2011 (Cited on page 1).

Kevin Jamieson, Sumeet Katariya, Atul Deshpande, and Robert Nowak. “Sparse dueling bandits”. In: Proceedings of the International Conference on Artificial Intelligence and Statistics. 2015 (Cited on page 3).

Volodymyr Kuleshov and Stefano Ermon. “Estimating uncertainty online against an adversary”. In: Proceedings of the AAAI Conference on Artificial Intelligence. 2017 (Cited on page 4).

Elhud Lehrer. “Partially specified probabilities: decisions and games”. In: American Economic Journal: Microeconomics 4.1 (2012), pp. 70–100 (Cited on page 4).

R Duncan Luce. Individual choice behavior: A theoretical analysis. John Wiley, 1959 (Cited on page 1).

Sobhan Miryoosefi, Kianté Brantley, Hal Daume III, Miroslav Dudik, and Robert E Schapire. “Reinforcement Learning with Convex Constraints”. In: Advances in Neural Information Processing Systems. 2019 (Cited on page 4).

Peter McMullen. “The maximum numbers of faces of a convex polytope”. In: Mathematika 17.2 (1970), pp. 179–184 (Cited on page 32).

Shie Mannor, Vianney Perchet, and Gilles Stoltz. “Approachability in unknown games: Online learning meets multi-objective optimization”. In: Proceedings of the Conference on Learning Theory. 2014 (Cited on page 4).

Matthew T McBee, Scott J Peters, and Craig Waterman. “Combining scores in multiple-criteria assessment systems: The impact of combination rule”. In: Gifted Child Quarterly 58.1 (2014), pp. 69–89 (Cited on page 4).
[NS17] Yurii Nesterov and Vladimir Spokoiny. “Random gradient-free minimization of convex functions”. In: Foundations of Computational Mathematics 17.2 (2017) (Cited on page 34).

[Pap11] John P Papay. “Difference tests, different answers: The stability of teacher value-added estimates across outcome measures”. In: American Educational Research Journal 48.1 (2011), pp. 163–193 (Cited on page 4).

[PB12] Jean-Charles Pomerol and Sergio Barba-Romero. Multicriterion decision in management: principles and practice. Vol. 25. Springer Science & Business Media, 2012 (Cited on page 4).

[Per13] Vianney Perchet. “Approachability, regret and calibration: implications and equivalences”. In: arXiv preprint arXiv:1301.2663 (2013) (Cited on page 4).

[Per14] Vianney Perchet. “A note on robust Nash equilibria with uncertainties”. In: RAIRO-Operations Research 48.3 (2014), pp. 365–371 (Cited on page 4).

[PLSS19] Malayandi Palan, Nicholas C Landolfi, Gleb Shevchuk, and Dorsa Sadigh. “Learning reward functions by integrating human demonstrations and preferences”. In: arXiv preprint arXiv:1906.08928 (2019) (Cited on page 1).

[PR70] Jean-Paul Penot and Robert Ratsimaharo. “Characterizations of metric projections in Banach spaces and applications”. In: Abstract and Applied Analysis. Vol. 3. 1970 (Cited on page 37).

[Sch11] Markus Schulze. “A new monotonic, clone-independent, reversal symmetric, and condorcet-consistent single-winner election method”. In: Social Choice and Welfare 36.2 (2011), pp. 267–303 (Cited on page 2).

[SDSS17] Dorsa Sadigh, Anca D Dragan, Shankar Sastry, and Sanjit A Seshia. “Active preference-based learning of reward functions.” In: Robotics: Science and Systems. 2017 (Cited on page 1).

[Sha13] Ohad Shamir. “On the complexity of bandit and derivative-free stochastic convex optimization”. In: Proceedings of the Conference on Learning Theory. 2013 (Cited on page 34).

[Sha17] Ohad Shamir. “An optimal algorithm for bandit and zero-order convex optimization with two-point feedback”. In: The Journal of Machine Learning Research 18.1 (2017) (Cited on pages 34, 35).

[SSSE18] William Saunders, Girish Sastry, Andreas Stuhlmueller, and Owain Evans. “Trial without error: towards safe reinforcement learning via human intervention”. In: Proceedings of the International Conference on Autonomous Agents and MultiAgent Systems. 2018 (Cited on page 1).

[Thu27] Louis L Thurstone. “A law of comparative judgment.” In: Psychological review 34.4 (1927), p. 273 (Cited on page 1).

[TK74] Amos Tversky and Daniel Kahneman. “Judgment under uncertainty: Heuristics and biases”. In: Science 185.4157 (1974), pp. 1124–1131 (Cited on page 3).

[TK79] Amos Tversky and Daniel Kahneman. “Prospect theory: An analysis of decision under risk”. In: Econometrica 47.2 (1979), pp. 263–291 (Cited on page 4).
