A WEAK APPROACH TO THE STOCHASTIC DEFORMATION OF CLASSICAL MECHANICS

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Abstract. We establish a transfer principle, providing a canonical form of dynamics to stochastic models, inherited from their classical counterparts. The stochastic deformation of Euler–Lagrange conditions, and the associated Hamiltonian formulations, are given as conditions on laws of processes. This framework is shown to encompass classical models, and the so-called Schrödinger bridges. Other applications and perspectives are provided.

1. Introduction. A relevant set of paths satisfying classical Euler–Lagrange conditions (see [1], [2] or [9]) is canonically embedded, in a set of probability measures (or laws) of continuous semi-martingales. Thus, these conditions become conditions on laws, involving local characteristics of the evaluation process, a trivial semi-martingale under the probability measures concentrated on the associated paths.

This motivates the following transfer principle: starting from a classical condition on the dynamics, in order to eliminate second order time derivatives, one first integrates it, so that constants appear. Then, one substitutes for configuration variables the canonical process and its corresponding local characteristics; one substitutes martingales of the canonical filtration for the constants. The originality of this formulation, with respect to other stochastic deformations of classical mechanics (for instance see [3] or [10] and the references therein), is to provide canonical conditions on laws of stochastic processes. In particular, we embed the stochastic deformation of mechanics in the realm of probabilities on Polish spaces, where the related problems can be conveniently tackled, using tools of functional analysis.

The description of dynamical systems may involve randomness at two different levels: either its kinematics, or its dynamics, can be perturbed by some exogenous

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factor, whose effect is modeled by stochastic processes. In this paper, both cases are considered.

For the sake of clarity, we focus on Euler–Lagrange conditions. Both solutions to its classical counterpart, and the so-called Schrödinger bridges, meet this condition. Schrödinger bridges (whose precise definition will be stated in subsection 2.2.1.), achieve the \textit{primal attainment} to specific dynamical Schrödinger problems (see [11]). They were first introduced in [13], by the celebrated physicist E. Schrödinger, to provide a new insight into the probabilistic content of Quantum mechanics. Their construction as mathematical objects, the study of their properties, and the related applications, have been extensively developed by many authors ever since (see [11] for a synthetic survey on the domain, and [18]). Incidentally, although these processes were first introduced heuristically, to share common features with quantum mechanics, one of the present authors showed that these Schrödinger bridges have many properties analogous to those of classical mechanics. For instance, an analogue of the Noether theorem has been provided (see [14]) for Schrödinger’s bridges, where calculus can be performed explicitly, and it has been noticed, that those satisfy some conditions analogous to classical Euler–Lagrange conditions. Moreover in [14] it was observed that, in a Noether theorem for Schrödinger bridges, \textit{martingales were analogues to classical constants}. These analogies strongly suggests to perform some stochastic deformation of classical mechanics by a kind of \textit{embedding} (see [4] for another attempt toward this direction). This paper provides a straightforward mathematical consistency for this intuition, and reduces the analogies to common features.

For the sake of clarity the present paper, which is meant to be a short pedagogical presentation of this topic is rather informal. As far as Schrödinger bridges are concerned (for this paper to be short enough, and self contained), we work on hypotheses which are not optimal; we refer to [11] and to the references therein for further extensions and references. The definitions extend to separable Hilbert spaces valued path spaces \( H \), using the \( H \)-\textit{differentiation} (see [8]). Finally, for a similar approach in a different context see [5].

The structure of this paper is the following. In Section 2, we set the general framework, the notation, and we state the weak stochastic Euler-Lagrange conditions. We also show in which acceptation it extends the usual one. We particularly focus on probabilities which meet this condition for the kinematic energy, called here \textit{stochastic geodesics}. Then, we investigate several connections to PDE (partial differential equations), and we provide the associated Hamiltonian formulation. In Section 3, we investigate several related models. Schrödinger bridges are briefly introduced, and we mention further extensions and perspectives.

2. Stochastic deformation of classical mechanics.

2.1. General framework and notations. To model trajectories, we first consider the separable Banach space \( W := C([0,1],\mathbb{R}) \), of continuous real valued maps, on the interval \([0,1], \) endowed with the norm \( |.|_\infty \) of uniform convergence : \( |f|_\infty = \sup_{t \in [0,1]} |f(t)| \), for \( f \in W \). Denoting by \( \mathcal{P}_W \) the set of Borel probabilities on \( W \), which is endowed with the topology of weak convergence in measure, we have the topological embedding

\[
W \hookrightarrow \mathcal{P}_W,
\]
δ denoting the map \( \delta : \omega \in W \to \delta_\omega \in \mathcal{P}_W \), where, for \( \omega \in W \), \( \delta_\omega \) denotes the Dirac measure concentrated on the path \( \omega \). That is for \( A \in \mathcal{B}(W) \), the Borelian sigma-algebra of \( W \), \( \delta_\omega(A) = 1_A(\omega) = 0 \) if \( \omega \notin A \) and 1 if \( \omega \in A \). Indeed for these topologies, \( \delta \) is an homeomorphism of \( W \) onto its image. Sometimes we will regard a continuous process \((u_t)_{t \in [0,1]}\), defined on a complete probability space \((\Omega, \mathcal{A}, \mathcal{P})\), as a usual stochastic process (a family of random variables, indexed by the time), and sometimes we will regard it as a random continuous path or trajectory; that is, as a continuous process \((\mathcal{F}_t)\)–martingale \((M_t)\) is said to be càdlàg, if the sample trajectory \( t \in [0,1] \to M_t(\omega) \in \mathbb{R} \) is a right-continuous map, with left limits, for any \( \omega \in W \), up to a \( \nu \)–negligible set; this regularity assumption avoids measurability issues, in subsequent applications. To ensure the existence of such modifications, entitled by the so-called usual conditions (see [7]), denoting the natural filtration of the evaluation process by \( (\mathcal{B}_t^0(W))_{t \in [0,1]} \), i.e. \( \mathcal{B}_t^0(W) := \sigma(W_s, s \leq t) \), for \( t \in [0,1] \), we rather consider its \( \nu \)–usual augmentation \((\mathcal{F}_t^\nu)_{t \in [0,1]} \). It is given by \( \mathcal{F}_t^\nu := \mathcal{B}_t^\nu(W)^\nu \), for all \( t \in [0,1] \); if \( \mathcal{G} \) denotes a sub-sigma field of \( \mathcal{B}(W) \), then \( \mathcal{G}^\nu \) denotes its \( \nu \)–completion. In particular, if \( \nu := \delta_\omega \), then \( \mathcal{F}_t^\nu \) is constant and coincide with \( \mathcal{B}(W)^\nu \), for all \( t \in [0,1] \); in this particular case, the latter is the set of all subsets of \( W \). Actually, in the classical case, we require much regularity on trajectories, than being continuous. Therefore, we consider \( H \subset W \), the linear subspace of absolutely continuous \( h \in W \), with a square integrable derivative (i.e., \( \dot{h} \in H \) if there exists a map \( s \in [0,1] \to \dot{h}_s \), such that \( \int_0^1 |\dot{h}_s|^2 ds < \infty \)). The Cameron-Martin space \( H \) is turned into an Hilbert space by the inner product

\[
<k, h>_{\mathcal{H}} = \int_0^1 \dot{h}_s \dot{k}_s ds,
\]

for \( h = \int_0^1 \dot{h}_s ds \) and \( k = \int_0^1 \dot{k}_s ds \), both elements of \( H \); \( ||.||_{\mathcal{H}} = \sqrt{<.>_{\mathcal{H}}} \) denotes the associated norm. Similarly, within the stochastic framework, the integrability of suitable processes will be handled by means of the Hilbert space \( L^2(\nu, H) \), of equivalence classes of \( \mathcal{B}(W)^\nu/\mathcal{B}(W) \)–measurable maps \( u : W \to W \), such that \( u \in H \) \( \nu \)–a.s. and \( t \to u_t \) is \((\mathcal{F}_t^\nu)\)–adapted, which are identified when they are \( \nu \)–a.s. equal, and satisfy \( E_{\nu}[|u|^2_H] < \infty \).

2.2. The set of probabilities describing kinematics. In this subsection, we define a subset \( S \) of \( \mathcal{P}_W \) whose elements model some admissible kinematics. We denote by \( S \) the set of \( \nu \in \mathcal{P}_W \), such that there exists a continuous martingale \((M_t^\nu)\), with \( M_0^\nu = 0 \), and an absolutely continuous, \((\mathcal{F}_t^\nu)\)–adapted, process \( b^\nu := \int_0^t u_t^\nu dt \)
on \((W, \mathcal{B}(W)^\nu, \nu)\), such that
\[
W_t - W_0 = M_t^\nu + \int_0^t v_s^\nu ds, \nu - a.s.,
\]
for all \(t \in [0, 1]\), and such that the \textit{predicable covariation process} \((< M^\nu >_t)\) is absolutely continuous, with \(< M^\nu > = \int_0^t \alpha_s^\nu ds\), for some \((\mathcal{F}_t^\nu)\)-predicable (see [7]) process \((\alpha_t^\nu)\). We recall that \((W_t)\) denotes the canonical (evaluation) process \((2)\).

Modulo further assumptions on the \textit{covariation process} of \(M^\nu\), which will always be satisfied in our examples, the martingale representation theorem implies that \(\nu\) is the law of a solution of a, generally non Markovian, stochastic differential equation of the form
\[
dX_t = \sigma_t(X)dB_t + v_t^\nu(X)dt.
\]
We call \(M^\nu\) (resp. \(b^\nu\)) \textit{the martingale} (resp. \textit{the finite variation part}) of \(\nu\), we call \((\alpha_t^\nu)\) (resp. \((v_t^\nu)\)) \textit{the dispersion process} (resp. \textit{the velocity process}) of \(\nu\), and \((3)\) will be referred to as the \textit{kinematic equation} of \(\nu\). The velocity and dispersion processes are the \textit{local characteristics} of \(\nu\) mentioned in the introduction. We also indicate technical subtilities which may be skipped by readers not familiar with stochastic analysis. \(M^\nu\) and \(b^\nu\) are only defined canonically as \(\nu\)-equivalence classes of \(\mathcal{B}(W)^\nu/\mathcal{B}(W)\)-measurable maps, \(\mathcal{B}(W)^\nu\) denoting the \(\nu\)-completion of \(\mathcal{B}(W)\). On the other hand, to be totally rigorous, \((v_t^\nu)\) is only defined uniquely as a \(\lambda \otimes \nu\)-a.s. (\(\lambda\) the Lebesgue measure) equivalence class of optional processes (see [7]). Under our assumptions, predicable modifications always exist in this class, and it is what we pick, except if we explicitly mention that we take a modification which is merely optional (see [7]).

\textbf{Example 1.} \(\Omega^2_{[0,1]}\) will denote the set of absolutely continuous paths on \([0, 1]\) with a square integrable derivative. For \(x \in \mathbb{R}\), let \(j_x : h \in H \rightarrow j_x(h) \in W\) where \(j_x(h)_t = x + h_t\), \(\Omega^2_{[0,1]} := \{j_x(h), x \in \mathbb{R}, h \in H\}\). For \(h \in \Omega^2_{[0,1]}(\subset W)\), let \(\nu := \delta_h\) be the Dirac measure concentrated on \(h\) (see section 1.1). Then \(\nu \in \mathcal{S}\) with \(M^\nu = 0\) and \(b^\nu = h\ \nu - a.s..\) In particular, if we endow \(\Omega^2_{[0,1]}\) (resp. \(\mathcal{S}\)) with the topologies induced on these subspaces of \(W\) (resp. of \(\mathcal{P}_W\)) by those given in section 1.1, the set \(\Omega^2_{[0,1]}\) is actually embedded into \(\mathcal{S}\) by
\[
\Omega^2_{[0,1]} \rightarrow \delta \mathcal{S}.
\]
This suggests that physically, if one thinks of elements of \(\Omega^2_{[0,1]}\) (or of one of its subsets) as the set of configurations describing admissible kinematics, then one should think of elements of \(\mathcal{S}\) as being the set of configurations describing admissible kinematics in this generally random framework. We denote by \(\mathcal{S}_{cl}\) the image of \(\Omega^2_{[0,1]}\) by \(\delta\) i.e.
\[
\mathcal{S}_{cl} := \{\delta_k, k \in \Omega^2_{[0,1]}\} \subset \mathcal{S}
\]
In particular, for \(\nu \in \mathcal{S}_{cl}\), denoting the velocity of \(\nu\) by \((v_t^\nu)\) \((3)\) reads
\[
W_t = W_0 + \int_0^t v_s^\nu ds.
\]
Using the terminology of stochastic calculus this means that \((W_t)\) solves the particular stochastic differential equation
\[
dX_t = v_t^\nu(X)dt
\]
on the probability space \((W, \nu)\) with the filtration \((\mathcal{F}_t^\nu)\). To be concrete, let \(g_{x,y} \in \Omega^2_{[0,1]}\) be the straight line \(g_{x,y}(t) = x + t(y-x)\). Then

\[ v_t^\nu = \frac{W_t - W_0}{t} = \frac{W_1 - W_t}{1-t} = \frac{y - W_t}{1-t} \lambda \otimes \nu - a.e. \]

and the coordinate process \((W_t)\) solves the stochastic differential equation

\[ dX_t = \frac{y - X_t}{1-t} dt ; \quad X_0 = x, \]

which reduces here to an ODE.

**Example 2.**

- We denote by \(\mathbb{S}_B\) the set of the \(\nu \in \mathbb{S}\) whose martingale part \(M^\nu\) is a Brownian motion. By Levy’s criterion \(<M^\nu>_t = \nu - a.s.\), for all \(t \in [0,1]\).
- (Probabilities absolutely continuous with respect to the Wiener measure.) For \(x \in \mathbb{R}\), let \(\mu^x\) be the Wiener measure starting from \(x\). It is the unique element of \(\mathcal{P}_W\), such that the coordinate process \((t, \omega) \rightarrow W_t(\omega)\) is a Brownian motion with \(W_0 = x\) under \(\mu^x\). For \(\nu \in \mathcal{P}_W\), we set

\[ \mu_{W_0, \nu} := \int \nu(dx) \mu^x \]

the Wiener measure with initial distribution \(W_0 \nu\) (i.e. the law of \(W_0\) under the probability \(\nu\)). The relative entropy of a probability \(\nu\) with respect to \(\mu_{W_0, \nu}\) is defined by \(\mathcal{H}(\nu|\mu_{W_0, \nu}) := E_\nu \left[ \ln \frac{d\nu}{d\mu_{W_0, \nu}} \right]\); if \(\nu \ll \mu_{W_0, \nu}\) (i.e. absolutely continuous), and by \(\infty\) otherwise. By a classical application of Girsanov’s theorem for \(\nu \ll \mu_{W_0, \nu}\), we obtain \(\nu \in \mathbb{S}_B\). Moreover, by [6], we have

\[ \mathcal{H}(\nu|\mu_{W_0, \nu}) = E_\nu \left[ \int_0^1 (v_t^\nu)^2 dt \right], \]

and we define

\[ \mathbb{S}_B^f := \{ \nu \in \mathcal{P}_W : \mathcal{H}(\nu|\mu_{W_0, \nu}) < \infty \}. \]

- Time reversal of \(\mathbb{S}_B^f\). The time reversal map on \(W\), is the Borel measurable map \(R : \omega \in W \rightarrow R(\omega) \in W\) given by \(R(\omega)(t) = \omega(1-t)\) for \((t, \omega) \in [0,1] \times W\). If \(\nu \in \mathbb{S}_B^f\) then \(R_* \nu \in \mathbb{S}_B^f\); \(R_* \nu\) denotes the image of the probability \(\nu\), by \(R\), which is also the law of the continuous process \(t \rightarrow W_{1-t}\), under the probability \(\nu\).

2.3. Constraints on the dynamics: The weak stochastic Euler-Lagrange conditions. Since we interpret (3) as extending admissible kinematics, it is natural to consider that the dispersion process should be treated as a state variable. For this reason we allow an extended Lagrangian, which depends on the dispersion term, of the form

\[ \mathcal{L} : (t, q, v, a) \in [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{L}(q, v, a) \in \mathbb{R}, \]

where \(\mathcal{L}\) is \(C^1\). By taking a particular \(\mathcal{L}\), which does not depend on \(a\), we recover the Lagrangian of classical mechanics, which we will particularly examine below. Thus, for a smooth map \(V : \mathbb{R} \rightarrow \mathbb{R}\) we define

\[ \mathcal{L}^V(q, v, a) := \frac{v^2}{2} - V(q); \quad (5) \]
we call $V$ the potential associated with $\mathcal{L}^V$, and $\mathcal{L}^V$ the classical Lagrangian associated with $V$. Further hypothesis on $\mathcal{L}$ and $V$ will be assumed when necessary.

**Definition 2.1.** We say that a probability $\nu \in \mathcal{S}$ satisfies the weak stochastic Euler-Lagrange condition for $\mathcal{L} : (t, x, v, a) \to \mathcal{L}_t(x, v, a)$, if there exists a $(\mathcal{F}_t) - \text{càdlàg}$ martingale $(N_t^\nu)_{t \in [0,1]}$, on the complete probability space $(\mathcal{W}, \mathcal{B}(\mathcal{W})^\nu, \nu)$, such that

$$
\frac{\partial \mathcal{L}}{\partial v}(W_t, v_t^\nu, \alpha_t^\nu) - \int_0^t \frac{\partial \mathcal{L}}{\partial q}(W_s, v_s^\nu, \alpha_s^\nu) ds = N_t^\nu \lambda \otimes \nu - a.e. \quad (6)
$$

($v_t^\nu$) (resp. ($\alpha_t^\nu$)) denoting the velocity (resp. dispersion) process of $\nu$. A probability $\nu \in \mathcal{S}$ which satisfies (6) for $\mathcal{L}^0(q, v) = \frac{1}{2} |v|^2$ will be called a stochastic geodesic (on the flat space).

**Remark 1.** For technical reasons, $(N_t)_{t \in [0,1]}$ is only assumed to be defined on $[0,1]$, and it is not assumed to be closable, as a martingale, on $[0,1]$. We also mention here that this definition extends by localization by requiring that $(N_t^\nu)_{t \in [0,1]}$ is a local martingale instead of a martingale. However, in all the examples and applications found until now, the definition stated above was sufficient.

**Example 3.** For $\nu \in \mathcal{S}_{cl}$ (see Example 1), since $\nu = \delta_h$ for some $h \in \Omega_{[0,1]}$ (see Example 1), $(\mathcal{F}_t)$ is constant and equal to $\mathcal{B}(\mathcal{W})^\nu$. Thus the càdlàg $(\mathcal{F}_t^\nu)$—martingales can be identified with constants and we have $b^\nu = h \nu - a.s. \ (b^\nu$ denotes the finite variation part of $\nu$ see above). Therefore, the weak stochastic Euler-Lagrange condition for a smooth $\mathcal{L} : (x, v, a) \to \mathcal{L}(x, v) \in \mathcal{R}$ (which does not depend on the dispersion term $a$) holds if and only if there exists $c \in \mathcal{R}$, such that

$$
\frac{\partial \mathcal{L}}{\partial v}(h_t, \dot{h}_t) - \int_0^t \frac{\partial \mathcal{L}}{\partial q}(h_s, \dot{h}_s) ds = c \lambda \otimes \nu - a.e..
$$

Equivalently, $\frac{\partial \mathcal{L}}{\partial v}(h_t, \dot{h}_t)$ is a.e. differentiable, and

$$
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial v}(h_t, \dot{h}_t) \right) = \frac{\partial \mathcal{L}}{\partial q}(h_t, \dot{h}_t) \ a.e..
$$

When $h$ is $C^1$, this equation reduces to the classical Euler–Lagrange condition.

**Example 4.** (Statistical mixture of geodesics are stochastic geodesics.) For $x, y \in \mathcal{R}$ and $g_{x,y}$ as in Example 1, $\delta_{g_{x,y}}$ is a stochastic geodesic since $v_t^\nu = y - x \lambda \otimes \nu - a.e.$.

**Example 5.** For $x, y \in \mathcal{R}$ let $\mu_{x,y}$ be the element of $\mathcal{P}_\mathcal{W}$ which is the law of the pinned Brownian motion, i.e. the conditioned Wiener measure so that $W_0 = x$ and $W_1 = y$ (in particular when $x = y$ the so-called Brownian bridge from $x$). Then
\( \mu_{x,y} \) is a stochastic geodesic. Indeed (see [7]) we have \( \mu_{x,y} \in \mathcal{S}_B \) (see Example 2) and

\[
v_t^\nu = \frac{y - W_t}{1 - t} \lambda \otimes \nu - a.e..
\]

From basic properties of the pinned Brownian motion, the result follows.

**Remark 2.** If \( \nu \in \mathcal{S} \) satisfies the weak stochastic Euler-Lagrange condition, for a classical Lagrangian \( \mathcal{L}^V \), associated with some potential \( V \) (see (5)), then the coordinate process satisfies

\[
dW_t = dZ_t - \nabla V(t, W_t)dt
\]

where \( Z_t = M_t^\nu + \int_0^t N_t^\nu ds \). We can also interpret the related condition on processes, as a perturbations of the classical fundamental principle of dynamics, by a noise \((Z_t)\) which is the sum of a martingale part \( M_t^\nu \), and a finite variation part \( \int_0^t N_t^\nu dt \), the latter being the primitive of another martingale.

**Remark 3.** Given a Lagrangian \( \mathcal{L} : (t, x, v, a) \to \mathcal{L}_\mathcal{E}(x, v) \) not depending on the dispersion term \( a \), denote by \( \Omega^{E.L.}[0,1](\mathcal{L}) \) the set of the \( C^2 \) continuous paths on \([0,1]\) satisfying the classical Euler-Lagrange condition, and denote by \( \mathcal{S}^{E.L.}(\mathcal{L}) \) the set of \( \nu \in \mathcal{S} \) satisfying the weak stochastic Euler-Lagrange condition (Definition 2.1) for the same Lagrangian \( \mathcal{L} \). If \( \Omega^{E.L.}[0,1](\mathcal{L}) \) and \( \mathcal{S}^{E.L.}(\mathcal{L}) \) are endowed respectively with the topologies induced on these subspaces of \( \mathcal{W} \) (resp. of \( \mathcal{P}_\mathcal{W} \)) by those considered in section 1.1, then using (1) we have the embedding

\[
\Omega^{E.L.}[0,1](\mathcal{L}) \hookrightarrow \mathcal{S}^{E.L.}(\mathcal{L}).
\]

This provides a rigorous meaning to the assertion that classical mechanics is actually embedded in the weak stochastic extension (or deformation) of mechanics we advocate here.

2.4. **Example from partial differential equations.** The next Example 6 formulates in probabilistic terms what happens in a not random context. We formulate it using the weak stochastic Euler-Lagrange conditions, so that example 7 will sounds natural. For a more systematic approach to the relation with PDE in the case of Schrödinger Bridges, see [15].

**Example 6.** Assume that \( u : (t, x) \in [0,1] \times \mathbb{R} \to u(t, x) \) solves

\[
\partial_t u + (u, \nabla) u = - \nabla V
\]

for \( V : (t, x) \in [0,1] \times \mathbb{R} \to V(t, x) \in \mathbb{R} \) a smooth map and that, for \( x \in \mathbb{R} \) \( dh_t^x = u(t, h_t^x)dt \) has a solution on \([0,1]\) with \( h_0^x = x \). Setting \( \nu := \delta_{h^x} \), we have \( \nu \in \mathcal{S}_{cl} \subset \mathcal{S} \) and \( v_t^\nu = u(t, W_t) \lambda \otimes \nu - a.e.. \) That is,

\[
W_t = W_0 + \int_0^t u(s, W_s) ds \text{ for all } t \nu - a.s..
\]

Moreover

\[
dv_t^\nu = (\partial_t u + u, \nabla) u)(t, W_t)dt = - \nabla V(t, W_t)dt,
\]

on the complete probability space \((\mathcal{W}, \mathcal{B}(\mathcal{W})', \nu)\). Whence, \( \nu \) satisfies the weak stochastic Euler-Lagrange condition, for the classical Lagrangian \( \mathcal{L}^V \) associated with \( V \) (see (5)), for \( N_t^\nu \) the constant process equal to some \( c \in \mathbb{R} \)
Example 7. Assume that for some \( \nu \in \mathcal{S} \) whose finite variation part (resp. dispersion process) is denoted by \( b^\nu := \int_0^t v_t^\nu dt \) (resp. by \( (\alpha_t^\nu) \)), there exist smooth maps \( u : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \), and \( A : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \), solving

\[
\partial_t u + (u, \nabla)u + A \frac{\Delta u}{2} = -\nabla V \lambda \otimes \nu - a.e.,
\]

and such that

\[
E_\nu \left[ \int_0^T (\nabla u)^2(t, W_t)A(t, W_t)dt \right] < \infty,
\]

for all \( T < 1 \). Moreover, we assume that \( v_t^\nu = u(t, W_t) \lambda \otimes \nu - a.e., \) and that \( \alpha_t^\nu = A(t, W_t) \lambda \otimes \nu - a.e. \) hold. Applying Itô’s formula, with standard notations, we obtain

\[
dv_t^\nu = (\partial_t u + u, \nabla u + A \frac{\Delta u}{2})(t, W_t) + (\nabla u)(t, W_t)\,dM_t^\nu = -\nabla V(t, W_t)dt + dN_t^\nu,
\]

on the probability space \( (W, \mathcal{B}(W)^\nu, \nu) \), where \( N_t^\nu := \int_0^t (\nabla u)(t, W_t)\,dM_t^\nu \). Whence, \( \nu \) satisfies the weak stochastic Euler-Lagrange condition for \( \mathcal{L}^V \) (see (5)).

2.5. Hamiltonian formulation. In this subsection, we use Bismut’s approach (see [3] and the references therein), to obtain the Hamiltonian formulation of the weak stochastic Euler-Lagrange condition of Definition 2.1. We further assume that the map \( \mathcal{L} : (t, q, v, a) \rightarrow \mathcal{L}_t(q, v, a) \) is smooth, and both superlinear, and strictly convex in \( v \). Thus, for any \( t \in [0, 1] \), the map

\[
(q, v, a) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow (q, p(q, v, a, t), a) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}
\]

is an homeomorphism, where \( p(q, v, a, t) := \partial_a \mathcal{L}_t(q, v, a) \). Let \( (q, p, a) \in \mathbb{R} \rightarrow (q, v(q, p, a, t), a) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) be its inverse at \( t \). The Hamiltonian associated with \( \mathcal{L} \) is defined by

\[
H(x, p, a, t) := \sup_{v \in \mathbb{R}} (pv - \mathcal{L}_t(x, v, a)).
\]

By our hypothesis it satisfies

\[
H(x, p, a, t) = p v(x, p, a, t) - \mathcal{L}(x, v(q, p, a, t)).
\]

For \( \nu \in \mathcal{S} \) whose velocity (resp. dispersion) process is \( (v_t^\nu) \) (resp. \( (\alpha_t^\nu) \)) the momentum process \( (p_t^\nu) \) for \( \mathcal{L} \) is the process on the probability space \( (W, \mathcal{B}(W)^\nu, \nu) \) defined by

\[
(t, \omega) \in [0, 1] \times W \rightarrow p_t^\nu := \partial_p \mathcal{L}_t(W_t, v_t^\nu, \alpha_t^\nu) \in \mathbb{R}
\]

By definition, it satisfies

\[
\partial_q H(x, p_t^\nu, \alpha_t^\nu) = -\partial_q \mathcal{L}_t(W_t, v_t^\nu, \alpha_t^\nu) \lambda \otimes \nu - a.e.
\]

and

\[
v_t^\nu = \partial_p H(W_t, p_t^\nu, \alpha_t^\nu, t) \lambda \otimes \nu - a.e.
\]

Denoting the martingale part of \( \nu \) by \( M^\nu \), equation (10) reads

\[
dW_t = dM_t^\nu + \partial_p H(W_t, p_t^\nu, \alpha_t^\nu, t)dt.
\]

By (9) the weak stochastic Euler-Lagrange condition is clearly equivalent to the existence of a càdlàg \( (\mathcal{F}_t^\nu) \)—martingale \( (N_t^\nu) \) such that \( (p_t^\nu) \) becomes a semi—martingale of the form

\[
dp_t^\nu = dN_t^\nu - \partial_q H(W_t, p_t^\nu, \alpha_t^\nu, t)dt.
\]

Given \( \nu \in \mathcal{S} \), it satisfies the weak stochastic Euler—Lagrange condition if and only if there exists a martingale \( (N_t^\nu) \) such that (11) and (12) hold. We call (11) and (12)
3. Description of random mechanics for classical Lagrangians. Given a smooth map $V$, and the associated Lagrangian $L^V$, we examine several examples. Informally, we still interpret $S$, whose elements satisfy a kinematic condition of the form (3), as describing admissible kinematics.

3.1. Deterministic kinematics and random dynamics. In this subsection we focus on stochastic systems of the form

$$\begin{cases}
  dX_t = Y_t dt \\
  dY_t = dZ_t - (\nabla V)(t, X_t) dt
\end{cases}$$

for some martingale $(Z_t)$. Assume that $(X_t)_{t \in [0,1]}$ and $(Y_t)_{t \in [0,1]}$ are two adapted processes defined on the same probability space $(\Omega, \mathcal{A}, P)$, with a filtration $(\mathcal{A}_t)_{t \in [0,1]}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{P}$–complete). $(X_t)$ describes the position, whereas $(Y_t)$ describes the velocity of the particle. Describe the kinematics by

$$dX_t = Y_t dt,$$

where $(Y_t)$ meets the integrability condition $E_P \left[ \int_0^1 Y_t^2 dt \right] < \infty$. While $(Y_t)$ can be random, we say that (14) is a deterministic kinematics, in the sense that no noise is involved explicitly in the equation. Define the map $X: \omega \in \Omega \to X(\omega) := \int_0^t Y_s dt \in W$, and the probability measure $\nu := X_* P \in \mathcal{P}_W$, the image measure of $\mathcal{P}$ by $X$. Then, $\nu \in S$, with $M^\nu = 0$ (its martingale part). Moreover, if we see its finite variation part $b^\nu := \int_0^t v^\nu_s dt$ as a map $W \to H$, then the composition $b^\nu \circ X$ defines a process adapted to the filtration generated by $X$ on $(\Omega, \mathcal{A}, P)$, and for all $h = \int_0^t \dot{h}_s ds \in L^2_2(\nu, H)$

$$E_P \left[ <b^\nu \circ X, h \circ X>_H \right] = E_X, P \left[ \int_0^1 \dot{h}_s dW_s \right] = E_P \left[ \int_0^1 \dot{h}_s \circ XY_s ds \right].$$

Therefore,

$$b^\nu \circ X := \int_0^t E_P \left[ Y_t | \sigma(X_s, s \leq t) \right] dt \quad \nu - a.s.\ (15)$$

In particular, by standard results on optional projections, this implies that if $t \to Y_t$ is right continuous (resp. càdlàg), then $(v^\nu_t)$ can be chosen to be right continuous (resp. càdlàg). All the elements of $\nu \in S$ such that $M^\nu = 0$ and $b^\nu \in L^2_2(\nu, H)$, i.e. whose kinematic equations are of the form

$$dW_t(\omega) = v^\nu_t(\omega) dt,$$

can be obtained in this way, that is, starting from an equation such as (14). Denoting by $S^2_{kdet}$, this set of probabilities, we now assume that the dynamics is random, and satisfies an equation of the form

$$dY_t = dZ_t - (\nabla V)(t, X_t) dt,$$
where \((Z_t)_{t \in [0,1]}\) is an \((\mathcal{A}_t)\)-càdlàg martingale such that \(E \rho \left[ \int_0^1 Z_t^2 dt \right] < \infty\). Setting \(N_t^\nu := \nu_t^\nu + \int_0^t (\nabla V)(W_s)ds\) \(\nu - a.s.,\) for all \(t \in [0,1]\), (15) implies

\[
E_\nu [(N_t^\nu - N_s^\nu)\theta] = E_\nu [(N_t^\nu - N_s^\nu) \circ X]\n\]

\[
= E_\nu \left[ \left( E_\nu [Y_t|\sigma(X_u, u \leq t)] - E_\nu [Y_s|\sigma(X_u, u \leq s)] + \int_s^t \nabla V(X_\sigma)d\sigma \right) \theta \circ X \right]\n\]

\[
= E_\nu \left[ \left( E_\nu \left[ Y_t - Y_s + \int_s^t \nabla V(X_\sigma)d\sigma \right] \sigma(X_u, u \leq s) \right) \theta \circ X \right]\n\]

\[
= E_\nu \left[ (E_\nu |Z_t - Z_s|\sigma(X_u, u \leq s)) \theta \circ X \right],
\]

for any \(s \leq t\), and \(\theta \in L^2(\nu, \mathbb{R})\) which is \((\mathcal{F}_s^\nu)\)-measurable. Since \(\sigma(X_s, s \leq s) \subset \mathcal{A}_s\) and \((Z_t)\) is an \((\mathcal{A}_t)\)-martingale, the last term of the above equation vanishes. Thus, for all \(s \leq t\) we obtain \(E_\nu [N_t^\nu|\mathcal{F}_s^\nu] = N_s^\nu \nu - a.s.\). Whence, by definition of \((N_t^\nu)\), \(\nu\) satisfies the weak stochastic Euler–Lagrange condition for \(\mathcal{L}^\nu\). Any \(\nu \in \mathbb{S}_B^2\) satisfying the weak stochastic Euler Lagrange condition for \(\mathcal{L}^\nu\), can be obtained in this way.

3.2. Brownian kinematics with random dynamics. We recall that \(\mathbb{S}_B\) and \(\mathbb{S}_B^f\) are defined in Example 2. When \(\nu \in \mathbb{S}_B\), a representation similar to the one of the previous subsection can be obtained by systems of the form

\[
dX_t = dB_t + Y_t dt.
\]

Moreover, for those probabilities satisfying the weak stochastic Euler-Lagrange condition, a representation of the dynamical equation for the pair \((X_t, Y_t)\) is still given by (16). Here we focus on some particular cases, which do not encompass all processes satisfying the weak stochastic Euler-Lagrange conditions for Brownian kinematics, but naturally arise from specific variational problems of stochastic control. Finally, we present some other extensions of the stochastic case.

3.2.1. Schrödinger bridges. For \(\nu_0 \in \mathcal{P}_\mathbb{R}\) (i.e. a Borel probability on \(\mathbb{R}\)), we denote by \(\mu_{\nu_0}\) the Wiener measure with initial distribution \(\nu_0\) (see Example 2). From now on, we assume that \(V : \mathbb{R} \rightarrow \mathbb{R}\) is a smooth map, such that

\[
\int_0^1 |V(W_s)| ds < \infty \quad \mu_{\nu_0} - a.s.,
\]

and

\[
E_{\mu_{\nu_0}} \left[ \exp \left( \int_0^1 V(W_s) ds \right) \right] < \infty.
\]

To apply Feynman–Kac’s formula (see [7] p. 383), we assume as well, that \(V\) satisfies \(\limsup \frac{V(x)}{x^2} < \frac{1}{4}\), and that its derivative of order \(\alpha \leq 2m\) are of polynomial growth order, for some \(m \geq 2\). There is a probability \(\mu_V\) naturally related to \(V\) and \(\nu_0\). It is defined as the absolutely continuous probability with respect to \(\mu_{\nu_0}\) whose density is given by

\[
\frac{d\mu_V}{d\mu_{\nu_0}} = \frac{\exp \left( \int_0^1 V(W_s) ds \right)}{Z},
\]

\(Z\) denoting the normalization constant, such that \(\mu_V(W) = 1\). Physically, one may think of \(\mu_V\) as describing the theoretical distribution of some particles affected by both a deterministic potential \(V\) and by the perturbation of many little impacts,
whose effect is described by a Brownian motion. It will be shown below that $\mu_V \in S$ satisfies the weak stochastic Euler-Lagrange condition for $\mathcal{L}^V$. Following the terminology of [6] (inherited from [16]) the Schrödinger Bridge for $V$ with initial distribution $\nu_0$ is a continuous process with law $\nu \in S_B^f$ (with $W_{0*}\nu = \nu_0$), of finite entropy with respect to $\mu_V$ ($\mathcal{H}(\nu|\mu_V) := E_\nu[\ln \frac{d\nu}{d\mu_V}] < \infty$), and whose density is of the precise product form

$$\frac{d\nu}{d\mu_V} = f(W_0)g(W_1),$$

(17)

for some positive measurable $f, g : \mathbb{R} \to \mathbb{R}$, which we assume to be smooth for the sake of simplicity. For $\nu_1 \in \mathcal{P}_\mathbb{R}$ we denote by $Z_{\nu_0,\nu_1}$ the set of Schrödinger Bridges for $V$ such that $W_{0*}\nu = \nu_0$ and $W_{1*}\nu = \nu_1$. $Z_{\nu_0,\nu_1}$ can be empty for some choices of $\nu_0$ and $\nu_1$. Henceforth, we assume that $\nu_0$ and $\nu_1$ are such that this is not the case. Using the definition of the relative entropy, Jensen’s inequality yields that if $\nu^* \in Z_{\nu_0,\nu_1}$, then it attains

$$\inf \{ \mathcal{H}(\nu|\mu_V)|\nu \in \mathcal{P}_\mathbb{R}, W_{0*}\nu = \nu_0, W_{1*}\nu = \nu_1 \}$$

so that, by Sanov’s theorem (see [6] for details), Schrödinger Bridges correspond to some interpolation processes between $\nu_0$ and $\nu_1$. Moreover, since the entropy is also strictly convex, $\nu^*$ is unique, given $\nu_0$ and $\nu_1$. Using the entropy representation formula of [6], $\nu^* \in Z_{\nu_0,\nu_1}$ also attains the infimum of the following variational problem

$$\inf \left\{ \mathcal{E}_\nu \left[ \int_0^1 \mathcal{L}^V(W_s, v_s)ds \right] \nu \in S_B^f, W_{0*}\nu = \nu_0, W_{1*}\nu = \nu_1 \right\},$$

(18)

where $\mathcal{L}^V$ is the map given by (5). Those Bridges are also known as Bernstein (or variational, or reciprocal) processes. Proposition 1 below recalls some basic properties of Schrödinger Bridges stemming from Itô’s formula, Girsanov’s theorem, and the Feynman–Kac’s formulas (see [7]) :

**Proposition 1.** Under the above hypothesis, for $\nu \in Z_{\nu_0,\nu_1}$, whose density is of the specific form (17), the following assertions hold :

(i) $\nu \in S_B^f$, and its finite variation part $b^\nu := \int_0^1 v_s^\nu dt$, is such that

$$v_s^\nu = \nabla \ln \theta(t, W_t) \lambda \otimes \nu - a.e.,$$

(19)

$\theta$ denoting the non-negative map on $(0,1) \times \mathbb{R}$, which is defined by

$$\theta(t,x) := E_{\nu_{\nu_0}} \left[ g(W_1) \exp \left( \int_0^t V(W_s)ds \right) \right] \frac{1}{Z} \bigg| W_t = x \bigg].$$

In particular, it satisfies

$$\partial_t \theta = H \theta,$$

(20)

where $H\theta := -\frac{\Delta \theta}{2} - V\theta$.

(ii) The time reversal $R_{\nu} \nu$ of $\nu$ (see Example 2) is in $S_B$, and its finite variation part $b_{R\nu} := \int_0^1 v_t^{R\nu} dt$, is such that

$$v_t^{R\nu} = \nabla \ln \theta^*(1-t, W_t) \lambda \otimes R_{\nu} - a.e.,$$

(21)

where $\theta^*$ denotes the non-negative map on $(0,1) \times \mathbb{R}$, which is defined by

$$\theta^*(t,x) := p_{\nu_0}(t,x)E_{\mu_{\nu_0}} \left[ f(W_0) \exp \left( \int_0^t V(W_s)ds \right) \right] W_t = x ,$$

$$\theta^*(t,x) := p_{\nu_0}(t,x)E_{\nu_{\nu_0}} \left[ g(W_1) \exp \left( \int_0^t V(W_s)ds \right) \right] W_t = x \bigg].$$

(22)
and where for \( t \in (0, 1] , x \in \mathbb{R} , p_{\nu_0}(t, x) \) denotes the density of \( W_{t, \nu_0} \) w.r.t. the Lebesgue measure. In particular, it satisfies

\[
- \partial_t \theta^* = H \theta^* ,
\]

where \( H \theta^* := - \Delta \theta^* - V \theta^* . \)

(iii) For \( t \in (0, 1] , W_{t, \nu} \) is absolutely continuous w.r.t. the Lebesgue measure, with density

\[
\rho_t(x) := \theta(t, x) \theta^*(t, x) .
\]

Remark 4. • The terminology “Euclidean quantum mechanics” is sometimes used in relation with the particular form of the density of these processes. Indeed, if we define \( \bar{V} = -V , \) so that

\[
H := - \frac{\Delta}{2} + \bar{V} ,
\]

and if we substitute “formally” \( t \to it \) (the operation frequently achieved by physicists under the name “Wick’s rotation”) in the above PDE, then we obtain formally a density of the form \( \rho_t(x) = \Psi \Psi^* , \) where \( \Psi \) (resp. \( \Psi^* \)) solves Schrödinger’s equation (resp. its conjugate), for the potential \( \bar{V} \).

• Using Example 2, Proposition 1 implies that \( \nu \) is the law of a solution, to a stochastic differential equation (see [7]), of the form

\[
dX_t = dB_t + \nabla \ln \theta(t, X_t) dt ; \text{ Law}(X_0) = \nu_0 ,
\]

whereas its time reversal \( R_{\nu} \nu \) is the law of a solution to another stochastic differential equation of the form

\[
d\bar{X}_t = dB_t + \nabla \ln \theta^*(1 - t, \bar{X}_t) dt ; \text{ Law}(\bar{X}_0) = \nu_1 .
\]

The following proposition shows that both Schrödinger Bridges, and their time reversal, satisfy the same weak stochastic Euler-Lagrange condition:

Proposition 2. If \( \nu \in \mathcal{Z}_{\nu_0, \nu_1} \) then both \( \nu \) and its time reversal \( R_{\nu} \nu \) satisfy the weak stochastic Euler–Lagrange condition for \( \mathcal{L}^V \).

Sketch of the proof. With the notation of Proposition 1, we set \( \phi = \ln \theta \) (resp. \( \phi^* = \ln \theta^*(1 - t, x) \)) so that both \( \phi \) and \( \phi^* \) satisfy

\[
\partial_t f + \frac{\Delta f}{2} + \frac{\nabla f^2}{2} + V = 0 .
\]

Hence \( \nabla \phi \) and \( \nabla \phi^* \) satisfy (7) with \( A = 1 \). Therefore, by (19) and (21) of Proposition 1, the result follows as a particular case of Example 7.

Remark 5. Taking \( \nu_1 = W_{1, \mu_V} \) and \( f = g = 1 \) (constant equal to 1), \( \mu_V \) satisfies the weak stochastic Euler-Lagrange condition for \( \mathcal{L}^V \).

3.3. Related variational problems, and further perspectives. Least action principles, more general than (18), have been investigated recently, whose optimum will be shown to also satisfy the weak stochastic Euler–Lagrange conditions. Given \( \nu \in \mathcal{P}_W \), denote by \( (W_0 \times W_1)_\nu \) the joint law of \( W_0 \) and \( W_1 \) (the coordinate process) under \( \nu \). For \( \gamma \), a Borel probability on \( \mathbb{R} \times \mathbb{R} \), consider

\[
\inf \left( \left\{ E_{\nu} \left[ \int_0^1 \mathcal{L}^V(W_s, v_{s}^e) ds \right] \nu \in \mathcal{S}_{B_1}(W_0 \times W_1)_\nu = \gamma \right\} \right) .
\]

These problems extend those associated to Schrödinger bridges (see [12]). Indeed, if \( \nu^S \in \mathcal{Z}_{\nu_0, \nu_1} \) for some \( \nu_1 \), then it obviously attains the infimum of (25), for \( \gamma = \)}
Under mild conditions, when the optimum of (25) is attained, it can also be shown to satisfy some stochastic Euler–Lagrange condition. As a matter of fact, this deeply relies on the possibility to extend the classical calculus of variations to this weak stochastic framework. Such construction entails difficulties, specific to stochastic frameworks. Essentially, perturbations by variations may yield information loss. Within the weak framework provided here, this difficulty can be efficiently overcome, so that to obtain a construction in full consistency with the classical one. This task, which justifies the framework \textit{a posteriori}, will be achieved by one of the authors, in a forthcoming work.

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