SPHERICAL BLACK HOLES CANNOT SUPPORT SCALAR HAIR

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ABSTRACT

The static spherically symmetric “black hole solution” of the Einstein - conformally invariant massless scalar field equations presented in [1] and [2], is critically examined. It is shown that the stress energy tensor is ill-defined at the horizon, and that its evaluation through suitable regularization yields ambiguous results. Consequently, the configuration fails to represent a genuine black hole solution. With the removal of this solution as a counterexample to the no hair conjecture, we argue that the following appears to be true:

Spherical black holes cannot carry any kind of classical scalar hair.
The theory of a scalar field conformally coupled to Einstein’s gravity in four spacetime dimensions is described by the action:

$$S[g, \Phi] = \int \sqrt{g} d^4x \left[ -\frac{1}{16\pi G} R - \frac{1}{12} \Phi^2 R + \frac{1}{2} \nabla_a \Phi \nabla^a \Phi \right]$$

implying the following equations of motion:

$$(1 - \alpha \Phi^2) R_{mn} = \alpha [4 \nabla_m \Phi \nabla_n \Phi - 2 \Phi \nabla_m \nabla_n \Phi - g_{mn} \nabla^l \Phi \nabla_l \Phi] \quad (1a)$$

and

$$\nabla^m \nabla_m \Phi = 0 \quad (1b)$$

where all indices are four dimensional, $\alpha = \frac{k}{6}$ with $k = 8\pi G$, and $G$ stands for the gravitational coupling constant.

Using a suitable generation technique, Bekenstein [1] (see also [2]), constructed an exact asymptotically flat, static, spherical solution of the above equations. Specifically, the solution is described by:

$$ds^2 = -(1 - \frac{r_o}{r})^2 dt^2 + (1 - \frac{r_o}{r})^{-2} dr^2 + r^2 d\Omega \quad (2a)$$

$$\Phi = -\frac{C}{r - r_o} \quad (2b)$$

where $r_o > 0$ is an arbitrary constant, and $C = C(r_0, \alpha) = r_o \alpha^{-\frac{1}{2}}$. The coordinate chart is supposed to cover the domain of outer communications of a black hole spacetime with the horizon located at $r = r_o$.

The solution is interesting in several respects. First, it explicitly demonstrates that a given spacetime geometry may locally support two entirely different stress tensor as sources of the Einstein equations. In the present case, the geometry is that of the extreme Reissner-Nordstrom spacetime and corresponds to a solution of the Einstein’s equations with a source given, either by the stress tensor of a $U(1)$ field, or the stress
tensor of a conformal massless scalar field. Furthermore, the solution (2-ab) is unique in the following sense: All asymptotically flat, static, spherically symmetric solutions of (1a,b) with nontrivial \( \Phi \), other than the Bekenstein solution (2a,b), do not possess a regular horizon [3].

Finally, the scalar field \( \Phi \) given in (2b) is diverging on the horizon. Initially, this divergence was considered as a pathology of the solution, and a black hole interpretation of the solution was not advanced. However, further analysis in [4], suggested that the divergence of \( \Phi \) on the horizon may be innocuous. The regularity of the geometry at the horizon, ensures that tidal gravitational forces are bounded there. Test particles following geodesics, “feel” nothing peculiar as they cross the horizon, since, as we have already mentioned, the spacetime geometry is that of the extreme Reissner-Nordstrom black hole which possess a regular horizon. It was, moreover, argued that, even when considering particles coupled to the scalar field itself, physical pathologies due to the divergence were not expected, as these particles would require an infinite amount of proper time to reach the horizon [4]. Consequently, Bekenstein, with the encouragement of DeWitt [4], reconsidered previous thoughts [1] and interpreted (2a,b) as genuine family of black hole solutions of Einstein’s- conformally invariant massless scalar field equations. Partially, however, due to the fact that the solution appears to be unstable [5], and partially due to the lack of evidence of long range forces mediated by scalar field, astrophysically, the solution never enjoyed the same status as the well known family of Kerr-Newman black holes. It does, however, raise the following question: One of the basic requirements used in the establishment of black hole uniqueness theorems is that of a regular event horizon. Whenever scalar fields are involved, regularity of the horizon is normally associated with the requirement that the scalar field is bounded at the horizon. If, on the other hand, the Bekenstein solution is a genuine black hole, then one would need to reexamine the established uniqueness theorems by relaxing the boundedness behavior of the scalar fields on the horizon. It was this issue that motivated us to take
a closer look at the Bekenstein solution.

Another interesting aspect of (2a-b) is its relation to the no hair conjecture. Let us note that the new family is characterized by the $ADM$ mass $\frac{r_o}{2}$ and scalar charge $Q = r_o \alpha^{\frac{1}{2}}$. On the other hand, only the $ADM$ mass defined by a two-surface integral at infinity, is associated with an asymptotic conservation law. The same parameter characterizes also the familiar Schwarzschild black holes; therefore, the solution (2-ab) carries “hair” and indeed represents a counterexample to Wheeler’s graphical statement “black holes have no hair” a precursor of the ”No Hair Conjecture”. The “No Hair Conjecture” has often been interpreted in a different way by different authors, so we shall be more precise. Following Bizon [6], we would say that a certain theory allows a hairy black hole if there is a need to specify quantities other than conserved charges defined at asymptotic infinity, in order to characterize completely a stationary black hole solution within that theory. With this definition, the well known dilatonic black hole solution [7, 8] does not constitute a hairy solution, by virtue of the fact that it is the unique static solution [9] of that theory, and it is specified completely by the values of the electric charge and of its ADM mass (the Reissner-Nordstrom configuration is not a solution of the theory).

Having clarified the meaning of “hair”, we shall argue in the present paper against a black hole interpretation of the solution (2-ab). In fact, we shall show that the divergence of the field $\Phi$ at the horizon has rather severe consequences. Specifically, we shall show that the extension of (2-ab) to a chart that includes the horizon, fails to satisfy Einstein’s equation at the horizon, and, therefore, it cannot be considered as a genuine black hole solution.

To do so, let us first rewrite (2) in advanced Eddington-Finkelstein coordinates. Such coordinate chart is advantageous since it can be extended through the horizon (in fact a portion of it). One then has:
\[ ds^2 = -V^2 du^2 + 2dudr + r^2 d\Omega \]  

where \( V^2 = -\xi^a \xi_a = (1 - \frac{r_0}{r})^2 \) is the square of the Killing field that is timelike at infinity, and \( r \) is a null coordinate varying in the range \((0, \infty)\). The extension of the metric is given by the same expression in all the chart, but the extension of the scalar field through the horizon has a sign ambiguity. This is easily resolved by noting that, in these coordinates, the field equation for \( \Phi = \Phi(r) \) admits a first integral:

\[ V^2 r^2 \frac{\partial \Phi}{\partial r} = \text{const} \]  

thus, the expression for the scalar field is given by eq.(2.b) throughout the chart (in fact, we will see that we have to give a distributional meaning to this configuration, and that only this extension can be considered as a distributional solution).

In view of the fact that the metric is regular at \( r = r_o \), we shall explicitly check whether or not Einstein’s equations hold at the horizon. The fact that \( l^a = \left( \frac{\partial}{\partial r} \right)^a \) is a smooth null vector field (even across \( r = r_o \)), implies that the quantity \( R_{mn}l^m l^n \) is finite everywhere, and, in particular, at the horizon. Moreover, direct calculation shows that it actually vanishes. The next step would be to compute \( T_{mn}l^m l^n|_{r_o} \) and compare its value with that of \( R_{mn}l^m l^n|_{r_o} \). However, due to the unboundedness of \( \Phi \), special care must be taken. One may naively compute \( T_{mn}l^m l^n|_{r_o} \) by identifying it with

\[ \lim_{r \to r_0} (T_{mn}l^m l^n|_r) \]  

Computing explicitly the right-hand side of (5) for \( r \neq r_0 \) leads to

\[ T_{mn}l^m l^n|_r = \frac{\alpha}{(1 - \alpha \Phi^2)} [4(\Phi')^2 - 2\Phi \Phi''] = O \]  

where the prime indicates derivative with respect to \( r \). We thus find that the limit is well-defined and is zero. This identification would suggest that Einstein’s equations hold at
the horizon. However, that is not correct. The problem with the above procedure is that this result follows from the fact that Einstein’s equations hold everywhere outside the horizon, and thus they obviously hold in the limit as we approach the horizon. However, this limiting procedure, still does not tell us whether or not Einstein’s equations hold at the horizon. In other words, we still do not know what the value of $T_{mn}l^m l^n|_{r_o}$ actually is. If indeed $T_{mn}l^m l^n|_{r}$ was continuous in a neighborhood of $r_o$, one could identify the value of $T_{mn}l^m l^n|_{r_o}$ with the limit obtained above, but that is precisely the question we are addressing. In fact, the right-hand side of (6) at $r_0$ is of the form $0 \times (\infty - \infty)$, and thus is ill-defined.

The problem, of course, has to do with the fact that the scalar field is not really well-defined throughout the spacetime, and, in particular, at the horizon. In order to try to make sense of this field configuration, we might consider the generalized solutions, i.e., solutions in the distributional sense, and in particular we will take $\Phi$ as the principal value of $\frac{-C}{r-r_0}$ distribution. This means considering the solution as a functional on the space of test functions $X = C_0^\infty(M)$ of infinitely differentiable functions of compact support in the manifold $M$, and to define for all $f \in X$

$$< \phi, f > = \lim_{\epsilon \to 0} \left\{ \int_{D^1(\epsilon)} \frac{-C}{r-r_0} f \sqrt{|g| d^4x} + \int_{D^2(\epsilon)} \frac{-C}{r-r_0} f \sqrt{|g| d^4x} \right\}. \quad (7)$$

where $D^1(\epsilon) = \{ x \in M/r(x) > r_0 + \epsilon \}$ and $D^2(\epsilon) = \{ x \in M/r(x) > r_0 + \epsilon \}$. We then ask whether or not this $\phi$ is a distributional solution of the scalar field equation (1b). This question is then whether, for all $f \in X$,

$$< \nabla^m \nabla_m \Phi, f > = < \Phi, \nabla^m \nabla_m f > = 0 \quad (8)$$

The answer is in the affirmative, has been seen from the following evaluation:

$$< \Phi, \nabla^m \nabla_m f > = \lim_{\epsilon \to 0} \left\{ \int_{D^1(\epsilon)} \frac{-C}{r-r_0} \nabla^m \nabla_m f \sqrt{|g| d^4x} + \int_{D^2(\epsilon)} \frac{-C}{r-r_0} \nabla^m \nabla_m f \sqrt{|g| d^4x} \right\}. \quad (9)$$

integrating by parts, and recalling that $f$ has compact support, we find
\[ <\Phi, \nabla^m \nabla_m f > = -C \times \lim_{\epsilon \to 0} \left\{ \int_{\partial D^1(\epsilon)} \left[ \frac{1}{r-r_0} \nabla_m f - f \nabla_m \frac{1}{r-r_0} \right] n^m \sqrt{|h|} d^3x \right. \]

\[ + \int_{\partial D^2(\epsilon)} \left[ \frac{1}{r-r_0} \nabla_m f - f \nabla_m \frac{1}{r-r_0} \right] n^m \sqrt{|h|} d^3x \right\}. \tag{10} \]

where \( \partial D \) indicates the boundary of the region \( D \), \( h \) is the determinant of the induced metric on the boundary, and \( n^m \) is the outward pointing unit normal to it. Thus

\[ <\Phi, \nabla^m \nabla_m f > = -C \times \lim_{\epsilon \to 0} \left\{ \int_{r=r_0+\epsilon} \left[ \frac{1}{\epsilon} \frac{\partial f}{\partial u} + \frac{\epsilon}{r^2} \frac{\partial f}{\partial r} + \frac{1}{r^2} f \right] r^2 \sin(\theta) d\theta d\phi du \right. \]

\[ - \int_{r=r_0-\epsilon} \left[ \frac{-1}{\epsilon} \frac{\partial f}{\partial u} + \frac{\epsilon}{r^2} \frac{\partial f}{\partial r} + \frac{1}{r^2} f \right] r^2 \sin(\theta) d\theta d\phi du \right\} \tag{11} \]

where we have used \( n^m = (1-r_0/r)^{-1} (\partial/\partial u)^m + (1-r_0/r)(\partial/\partial r)^m \) and \( \sqrt{|h|} = |(1-r_0/r)| r^2 \sin(\theta) \). Noting that the first term in the integrals can be integrated out to yield zero, because \( f \) has compact support, and that the remaining integrals cancel out, due to the continuity of \( f \) and its derivatives, we see that, in effect, we have a distributional solution to the scalar field equation. That this is not a trivial result is evidenced by the well known fact that \( 1/r \) is a solution of sourceless Laplace’s equation in flat \( R^3 - 0 \) but that when the origin is included we have a distributional solution of Laplace’s equation with a \( \delta \) “function” source. Moreover the above calculation shows that we had to take the current extension of the scalar field through the horizon in order to have a distributional solution.

Next, we turn to Einstein’s equations. We note that the spacetime metric, being the same as that in the Reissner Nordstrom solution, is regular everywhere, so the right-hand side of Einstein’s equations presents no problem. The issue is then the left-hand side of the equations, namely the energy momentum tensor. The first issue is whether we can give any meaning to it. The problem is similar to the one often found in quantum field theory, and that is the origin of the infinities that plague the theory, i.e., the fact that one is forced to deal with expressions that contain products of
distributions at “the same point”. The latter is an operation that is not mathematically well-defined. In quantum field theory, this problem is dealt with through the process of regularization and the subsequent renormalization of the expressions through the subtraction of formally divergent terms in a well-defined fashion. In the particular case of the energy momentum operator in quantum field theory in curved spacetime, the renormalization consists in the subtraction of divergent terms corresponding to the vacuum expectation value of the tensor operator in Minkowski spacetime. More precisely, the Hadamard anzats for the bi-distribution used in the substraction scheme that renormalizes the energy-momentum tensor is motivated by the vacuum two point funton in Minkowski spacetime [10]. In our case, since we are dealing with classical solutions, the analogous terms would correspond to the energy momentum tensor for the configuration \( \Phi = 0 \) in Minkowski spacetime, which is \( T_{mn} = 0 \); thus, we do not have any canonical expression that can be subtracted in order to renormalize the ill-defined quantities we are dealing with.

We could consider whether it is possible to give meaning to the expression in question by means of a regularization without the subsequent renormalization. The fact that the product of distributions is not well-defined suggests the answer is negative, but let’s examine specifically the problems we encounter in the attempts to do so.

We start by noting that the theory of distributions provides the means to regularize expressions through the following theorem [11]:

*Every distribution is the limit in the distributional sense of functions of class \( C^\infty_0 \).*

The idea is, then, to try to assign a value, to \( T_{mn}l^ml^n|_r \) through a regularization procedure. In fact it is more convenient to look at both sides of eq. (1a) contracted with \( l^ml^n \) and to consider their regularized values. To this effect we consider \( \Phi \) as the (distributional) limit when \( a \to 0 \) of a class of \( C^\infty_0 \) functions \( \Phi_a \). We must chose \( \Phi_a \) so that it tends to \( \Phi \) pointwise everywhere, except at the horizon (because, when consider-
ing test functions with support away from the horizon, the distribution $\Phi$ corresponds to a regular function).

Next, we note that the RHS. of eq (1a) vanishes identically (because the metric (3) has $R^m_{\ m l n} = 0$) and finally we turn to the LHS of eq (1a) contracted with $l^m l^n$ which is given by $A(\Phi) \equiv 2\alpha[2(\Phi')^2 - \Phi\Phi'']$ To actually compute it we replace $\Phi$ by its regularized version $\Phi_a$, and, at the end, remove the regulator (i.e., take the limit $a \to 0$). In that manner, if the R.H.S. is well-defined, the procedure would yield a well-defined distribution. Moreover, if the energy momentum tensor is to be considered as well-behaved, the procedure should yield a finite expression; and if, furthermore, Einstein’s equations are to be said to hold, the resulting value should be the zero distribution.

We can employ as $\Phi_{1,a}$ the function given by [12]:

$$\Phi_{1,a} = \frac{C(r - r_o)}{a^2 + (r - r_o)^2} \tag{12}$$

One may easily verify that as $a \to 0$, $\Phi_{1,a}$ converges pointwise to $\Phi$ everywhere in the Eddington- Finkelstein chart minus the horizon.

Carrying through the above procedure, we obtain:

$$A(\Phi_{1,a}) = \frac{4r_o^2}{a^4} \frac{1}{(1 + y^2)^3} \tag{13}$$

where $y = \frac{r - r_o}{a}$. Therefore, at the horizon, one finds $A = \frac{4r_o^2}{a^4}$. This seems to indicate that the R.H.S of eq. (1,a) is divergent at the horizon with a singularity of the type $(\delta)^4$ (The height of $A$ is order $a^{-4}$ and its width is of order $a$).

However, we can take a different class of regularizing functions [13], namely:

$$\Phi_{2,a} = \frac{C(r - r_o)^3}{(a^2 + (r - r_o)^2)^3} \tag{14}$$

or more generally
\[ \Phi_{n,a} = \frac{C(r - r_o)^{2n-1}}{(a^2 + (r - r_o)^2)^{2n}} \] (15)

These classes of functions also converge pointwise to \( \Phi \) as \( a \to 0 \) everywhere in the Eddington-Finkelstein chart minus the horizon.

Now, if we use the functions \( \Phi_{2,a} \), and repeat the procedure we find:

\[
A(\Phi_{2,a}) = \frac{8r_o^2 y^4(3 + y^2)}{a^4 (1 + y^2)^6}
\] (16)

where \( y = \frac{r - r_0}{a} \). Evaluating at \( r - r_0 \), we find \( A(\Phi_{2,a})(y = 0) = 0 \), in contrast with what was found using the first regularization. In this case, we actually have a divergence, but of a more complicated nature, in fact the form of eq. (16) suggests a singularity of the type \( (\delta'' - \delta')^4 \). The main point is, however, that the result depends on the form of the functions employed in the regularization, which indicates the type of problems one encounters in trying to give meaning to the product of singular distributions. In other words, we see, in a clear way, that there is no canonical procedure to give meaning to the R.H.S of eq. (1.a). This would be, however, an obvious prerequisite that must be satisfied in order to be able to state that one has a solution of Einstein’s equations.

The above discussion shows that Bekenstein’s configuration cannot be considered to satisfy the Einstein’s equations on the whole manifold including the horizon. Consequently, it cannot represent a regular black hole spacetime solution. On the other hand, the scalar field equation can be considered as satisfied in the distributional sense. And, of course, for \( r > r_0 \), we have a perfectly valid solution of the coupled Einstein-scalar field equations.

The removal of the Bekenstein solution as a possible black hole changes the scenario describing black holes admitting classical scalar hair. In view of the fact that spherical black holes cannot carry hair of massive - massless [14], or even arbitrarily self interacting scalar field, as long as it is minimally coupled to gravity [15, 16], combined with the
results of [17] for the conformal coupling and those of [18] for arbitrary coupling, appears to indicate strongly the validity of the following conjecture:

**Spherical black holes cannot carry any kind of classical hair associated to a scalar field.**

On physical grounds, one may attribute the absence of any kind of scalar hair as due to the fact that scalar mediated forces are attractive in nature, and, thus, there is nothing to counterbalance the equally attractive gravitational force. In contrast, vector mediated forces can be repulsive, and, thus, help to balance the gravitational attraction [19]. Needles to say that all those issues need a closer examination.

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