VERTEX PARTITIONS OF CHORDAL GRAPHS

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Abstract. A $k$-tree is a chordal graph with no $(k + 2)$-clique. An $\ell$-tree-partition of a graph $G$ is a vertex partition of $G$ into ‘bags’, such that contracting each bag to a single vertex gives an $\ell$-tree (after deleting loops and replacing parallel edges by a single edge). We prove that for all $k \geq \ell \geq 0$, every $k$-tree has an $\ell$-tree-partition in which every bag induces a connected $\lfloor k/(\ell + 1) \rfloor$-tree. An analogous result is proved for oriented $k$-trees.

1. Introduction

Let $G$ be an (undirected, simple, finite) graph with vertex set $V(G)$ and edge set $E(G)$. The neighbourhood of a vertex $v$ of $G$ is denoted by $N(v) = \{w \in V(G) : vw \in E(G)\}$. A chord of a cycle $C$ is an edge not in $C$ whose endpoints are both in $C$. $G$ is chordal if every cycle on at least four vertices has a chord. A $k$-clique ($k \geq 0$) is a set of $k$ pairwise adjacent vertices. A $k$-tree is a chordal graph with no $(k + 2)$-clique. The tree-width of $G$, denoted by $\text{tw}(G)$, is the minimum $k$ such that $G$ is a subgraph of a $k$-tree. It is well known that $G$ is a $k$-tree if and only if $V(G) = \emptyset$, or $G$ has a vertex $v$ such that $G \setminus v$ is a $k$-tree, and $N(v)$ is a $k'$-clique for some $k' \leq k$.

Let $G$ and $H$ be graphs. The elements of $V(H)$ are called nodes. Let $\{H_x \subseteq V(G) : x \in V(H)\}$ be a set of subsets of $V(G)$ indexed by the nodes of $H$. Each set $H_x$ is called a bag. The pair $(H, \{H_x \subseteq V(G) : x \in V(H)\})$ is an $H$-partition of $G$ if:

- $\forall$ vertices $v$ of $G$, $\exists$ node $x$ of $H$ with $v \in H_x$, and
- $\forall$ distinct nodes $x$ and $y$ of $H$, $H_x \cap H_y = \emptyset$, and
- $\forall$ edge $vw$ of $G$, either
  - $\exists$ node $x$ of $H$ with $v \in H_x$ and $w \in H_x$, or
  - $\exists$ edge $xy$ of $H$ with $v \in H_x$ and $w \in H_y$.

For brevity we say $H$ is a partition of $G$. A $k$-tree-partition is an $H$-partition for some $k$-tree $H$. A tree-partition is a 1-tree-partition. Tree-partitions were independently introduced by Seese [13] and Halin [12], and have since been investigated

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by a number of authors \cite{2, 3, 6, 7, 12, 13}. The main property of tree-partitions that has been studied is the maximum cardinality of a bag, called the *width* of the tree-partition. The minimum width over all tree-partitions of a graph $G$ is the *tree-partition-width*\footnote{Tree-partition-width has also been called *strong tree-width* \cite{2, 3}.} of $G$, denoted by $\text{tpw}(G)$. A graph with bounded degree has bounded tree-partition-width if and only if it has bounded tree-width \cite[12]{3}. In particular, for every graph $G$, Seese \cite{13} proved that $\text{tw}(G) \leq 2 \text{tpw}(G) - 1$, and Ding and Oporowski \cite{6} proved that $\text{tpw}(G) \leq 24 \text{tw}(G) \max\{\Delta(G), 1\}$, where $\Delta(G)$ is the maximum degree of $G$. See \cite{1, 5, 8, 9} for other results related to tree-width and vertex partitions.

Tree-partition-width is not bounded above by any function solely of tree-width. For example, wheel graphs have bounded tree-width and unbounded tree-partition-width, as observed by Bodlaender and Engelfriet \cite{3}. Thus, it seems unavoidable that the maximum degree appears in an upper bound on the tree-partition-width. This fact, along with other applications, motivated Dujmović et al. \cite{10, 11} to study the structure of the bags in a tree-partition. In this paper we continue this approach, and prove the following result (in Section 2).

**Theorem 1.** Let $k$ and $\ell$ be integers with $k \geq \ell \geq 0$. Let $t = \left\lfloor \frac{k}{\ell + 1} \right\rfloor$. Every $k$-tree $G$ has an $\ell$-tree-partition in which each bag induces a connected $t$-tree in $G$.

It is easily seen that Theorem 1 is tight for $G = K_{k+1}$ and for all $\ell$. Note that Theorem 1 can be interpreted as a statement about chromomorphisms (see \cite{15, 16}).

Dujmović et al. \cite{10, 11} proved that every $k$-tree has a tree-partition in which each bag induces a $(k - 1)$-tree. Thus Theorem 1 with $\ell = 1$ improves this result. That said, the tree-partition of Dujmović et al. \cite{10, 11} has a number of additional properties that were important for the intended application. We generalise these additional properties in Section 3. The price paid is that each bag may now induce a $(k - \ell)$-tree, thus matching the result of Dujmović et al. \cite{10, 11} for $\ell = 1$. Note that the proof of Dujmović et al. \cite{10, 11} uses a different construction to the one given here.

### 2. Proof of Theorem 1

We proceed by induction on $|V(G)|$. If $V(G) = \emptyset$, then the result holds with $V(H) = \emptyset$ regardless of $k$ and $\ell$. Now suppose that $|V(G)| \geq 1$. Thus $G$ has a vertex $v$ such that $G \setminus v$ is a $k$-tree, and $N(v)$ is a $k'$-clique for some $k' \leq k$. By induction, $G \setminus v$ has an $\ell$-tree-partition $H$ in which each bag induces a connected $t$-tree. Let $C = \{x \in V(H) : N(v) \cap H_x \neq \emptyset\}$. Since $N(v)$ is a clique, $C$ is a clique of $H$ (by the definition of $H$-partition). Since $H$ is an $\ell$-tree, $|C| \leq \ell + 1$.

**Case 1.** $|C| \leq \ell$: Add one new node $y$ to $H$ adjacent to each node $x \in C$. Since $C$ is a clique of $H$ and $|C| \leq \ell$, $H$ remains an $\ell$-tree. Let $H_y = \{v\}$. The other
bags remain unchanged. Since \( t \geq 0 \), \( H_y \) induces a connected \( t \)-tree (\( = K_1 \)) in \( G \). Thus \( H \) is now a partition of \( G \) in which each bag induces a connected \( t \)-tree in \( G \).

Case 2. \( |C| = \ell + 1 \): There is a node \( y \in C \) such that \( |N(v) \cap H_y| \leq t \), as otherwise \( |N(v)| \geq (t+1)|C| = ((k/\ell+1)) + 1) \geq k + 1 \). Add \( v \) to the bag \( H_y \). Let \( u \in N(v) \cap H_y \). Every neighbour of \( v \) not in \( H_y \) is adjacent to \( u \) (in \( G \setminus v \)). Thus \( H \) is a partition of \( G \). \( H_y \) induces a connected \( t \)-tree in \( G \), since \( H_y \setminus \{v\} \) induces a connected \( t \)-tree in \( G \setminus v \), and the neighbourhood of \( v \) in \( H_y \) is a clique of at least one and at most \( t \) vertices. The other bags do not change. Thus each bag of \( H \) induces a connected \( t \)-tree in \( G \). \( \square \)

3. Oriented Partitions

Let \( G \) be an oriented graph with arc set \( A(G) \). Let \( \hat{G} \) be the underlying undirected graph of \( G \). The in- and out-neighbourhoods of a vertex \( v \) of \( G \) are respectively denoted by \( N^-(v) = \{u \in V(G) : uv \in A(G)\} \) and \( N^+(v) = \{w \in V(G) : vw \in A(G)\} \). It is easily seen that an (undirected) graph \( G \) is a \( k \)-tree if and only if there is an acyclic orientation of \( G \) such that for every vertex \( v \) of \( G \), \( N^-(v) \) is a \( k' \)-clique for some \( k' \leq k \). An oriented graph with this property is called an oriented \( k \)-tree. Let \( G \) and \( H \) be oriented graphs. An oriented \( H \)-partition of \( G \) is an \( \hat{H} \)-partition of \( \hat{G} \) such that for every arc \( xy \) of \( H \), and for every edge \( vw \) of \( \hat{G} \) with \( v \in H_x \) and \( w \in H_y \), \( vw \) is oriented from \( v \) to \( w \). This concept is similar to an oriented homomorphism (see [1,4] for example).

**Theorem 2.** Let \( k \) and \( \ell \) be integers with \( k \geq \ell \geq 0 \). Let \( t = k - \ell \). Every oriented \( k \)-tree \( G \) has an oriented \( \ell \)-tree partition \( H \) in which each bag induces a weakly connected oriented \( t \)-tree in \( G \). Moreover, for every node \( x \) of \( H \), the set of vertices \( Q(x) = \bigcup_{v \in H_x} (N^-(v) \setminus H_x) \) is a \( k' \)-clique of \( G \) for some \( k' \leq k \).

The construction in the proof of Theorem 2 only differs from that of Theorem 1 in the choice of the node \( y \) in Case 2.

**Proof.** We proceed by induction on \( |V(G)| \). If \( V(G) = \emptyset \), then the result holds with \( V(H) = \emptyset \) regardless of \( k \) and \( \ell \). Now suppose that \( |V(G)| \geq 1 \). Since \( G \) is acyclic, there is a vertex \( v \) of \( G \) such that \( N^+(v) = \emptyset \). \( N^-(v) \) is a \( k' \)-clique for some \( k' \leq k \), and \( G \setminus v \) is an oriented \( k \)-tree. By induction, there is an oriented \( \ell \)-tree partition \( H \) of \( G \setminus v \) in which each bag induces a weakly connected oriented \( t \)-tree in \( G \setminus v \). Moreover, for every node \( x \) of \( H \), \( Q(x) \) is a \( k' \)-clique for some \( k' \leq k \). Let \( C = \{x \in V(H) : N^-(v) \cap H_x \neq \emptyset \} \). Since \( N^-(v) \) is a clique, \( C \) is a clique of \( H \). Since \( H \) is an oriented \( \ell \)-tree, \( |C| \leq \ell + 1 \).

Case 1. \( |C| \leq \ell \): Add one new node \( y \) to \( H \) adjacent to each node \( x \in C \). Orient each new edge from \( x \) to \( y \). Obviously \( H \) remains acyclic. Since \( C \) is a clique of \( H \) and \( |C| \leq \ell \), \( H \) remains an oriented \( \ell \)-tree. Let \( H_y = \{v\} \). The other bags are unchanged. Since \( t \geq 0 \), \( H_y \) induces a weakly connected oriented \( t \)-tree (\( = K_1 \)) in \( G \). All edges of \( G \) that are incident to a vertex in \( H_y \) are oriented into the vertex in
$H_y$. Thus $H$ is now an oriented partition of $G$ in which each bag induces a weakly connected oriented $t$-tree in $G$. Now $Q(y) = N^-(v)$, which is a $k'$-clique for some $k' \leq k$. $Q(x)$ is unchanged for nodes $x \neq y$. Hence the theorem is satisfied.

**Case 2.** $|C| = \ell + 1$: The clique $C$ induces an acyclic tournament in $H$. Let $y$ be the sink of this tournament. Since $|N^-(v) \cap H_x| \geq 1$ for every node $x \in C \setminus \{y\}$, $|N^-(v) \cap H_y| \leq k' - (|C| - 1) \leq k - \ell = t$. Add $v$ to the bag $H_y$.

Consider a neighbour $u$ of $v$. Since $N^+(v) = \emptyset$, $uv$ is oriented from $u$ to $v$. Say $u \in H_z$ with $z \neq y$. Then $z$ is in the clique $C$. Thus $zy$ is an edge of $H$. Since $y$ is a sink of $C$, $zy$ is oriented from $z$ to $y$. Thus $H$ is now an oriented partition of $G$. $H_y$ induces a weakly connected oriented $t$-tree in $G$, since $H_y \setminus \{v\}$ induces an oriented $t$-tree in $G \setminus v$, and the in-neighbourhood of $v$ in $H_y$ is a clique of at least one and at most $t$ vertices. The other bags do not change. Thus each bag of $H$ induces a weakly connected oriented $t$-tree in $G$.

$Q(y)$ is not changed by the addition of $v$ to $H_y$, as there is at least one vertex $u \in N^-(v) \cap H_y$, and any vertex in $N^-(v) \setminus H_y$ is also in $N^-(u) \setminus H_y$. For nodes $x \neq y$, $Q(x)$ is unchanged by the addition of $v$ to $H_y$, since $v$ is not in the in-neighbourhood of any vertex. Hence the theorem is satisfied. \[\square\]

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