A duality principle and its application to a model in superconductivity

Fabio Silva Botelho
Department of Mathematics
Federal University of Santa Catarina - UFSC
Florianópolis, SC - Brazil

Abstract

This article develops a duality principle applicable to the Ginzburg-Landau system in superconductivity. The main results are obtained through standard tools of convex analysis, functional analysis, calculus of variations and duality theory. In the last section, we present the general result for the case including a magnetic field and the respective magnetic potential in a local extremal context.

1 Introduction

In this work we present a theorem which represents a duality principle suitable for a large class of non-convex variational problems.

At this point we refer to the exceptionally important article "A contribution to contact problems for a class of solids and structures" by W.R. Bielski and J.J. Telega, [4], published in 1985, as the first one to successfully apply and generalize the convex analysis approach to a model in non-convex and non-linear mechanics.

The present work is, in some sense, a kind of extension of this previous work [4] combined with a D.C. approach presented in [10] and others such as [3], which greatly influenced and inspired my work and recent book [6].

First, we recall that about the year 1950 Ginzburg and Landau introduced a theory to model the superconducting behavior of some types of materials below a critical temperature $T_c$, which depends on the material in question. They postulated the free density energy may be written close to $T_c$ as

$$F_s(T) = F_n(T) + \frac{\hbar}{4m} \int_\Omega |\nabla \phi|^2 \, dx + \frac{\alpha(T)}{4} \int_\Omega |\phi|^4 \, dx - \frac{\beta(T)}{2} \int_\Omega |\phi|^2 \, dx,$$

where $\phi$ is a complex parameter, $F_n(T)$ and $F_s(T)$ are the normal and superconducting free energy densities, respectively (see [2] for details). Here $\Omega \subset \mathbb{R}^3$ denotes the superconducting sample with a boundary denoted by $\partial \Omega = \Gamma$. The complex function $\phi \in W^{1,2}(\Omega; \mathbb{C})$ is intended to minimize $F_s(T)$ for a fixed temperature $T$. 
Denoting $\alpha(T)$ and $\beta(T)$ simply by $\alpha$ and $\beta$, the corresponding Euler-Lagrange equations are given by:

$$
\begin{cases}
-\frac{\hbar}{2m}\nabla^2\psi + \alpha|\phi|^2\phi - \beta\phi = 0, & \text{in } \Omega \\
\frac{\partial\phi}{\partial n} = 0, & \text{on } \partial\Omega.
\end{cases}
$$

This last system of equations is well known as the Ginzburg-Landau (G-L) one. In the physics literature is also well known the G-L energy in which a magnetic potential here denoted by $A$ is included. The functional in question is given by:

$$
J(\psi, A) = \frac{1}{8\pi} \int_{\mathbb{R}^3} |\text{curl} A - B_0|^2 \, dx + \frac{\hbar^2}{4m} \int_{\Omega} \nabla\phi - \frac{2ie}{\hbar c} A\phi \, \nabla^2 \phi \, dx + \frac{\alpha}{4} \int_{\Omega} |\phi|^4 \, dx - \frac{\beta}{2} \int_{\Omega} |\phi|^2 \, dx
$$

Considering its minimization on the space $U$, where

$$
U = W^{1,2}(\Omega; \mathbb{C}) \times W^{1,2}(\mathbb{R}^3; \mathbb{R}^3),
$$

through the physics notation the corresponding Euler-Lagrange equations are:

$$
\begin{cases}
\frac{1}{2m} \left(-i\hbar\nabla - \frac{2e}{c} A\right)^2 \phi + \alpha|\phi|^2\phi - \beta\phi = 0, & \text{in } \Omega \\
(i\hbar\nabla\phi + \frac{2e}{c} A\phi) \cdot n = 0, & \text{on } \partial\Omega,
\end{cases}
$$

and

$$
\begin{cases}
\text{curl} (\text{curl} A) = \text{curl} B_0 + \frac{4\pi}{c} \tilde{J}, & \text{in } \Omega \\
\text{curl} (\text{curl} A) = \text{curl} B_0, & \text{in } \mathbb{R}^3 \setminus \overline{\Omega},
\end{cases}
$$

where

$$
\tilde{J} = \frac{ie\hbar}{2m} \left(\phi^*\nabla\phi - \phi\nabla\phi^*\right) - \frac{2e^2}{mc} |\phi|^2 A.
$$

and

$$
B_0 \in L^2(\mathbb{R}^3; \mathbb{R}^3)
$$
is a known applied magnetic field.

At this point, we emphasize to denote generically

$$
\langle g, h \rangle_{L^2} = \int_{\Omega} \text{Re}[g]\text{Re}[h] \, dx - \int_{\Omega} \text{Im}[g]\text{Im}[h] \, dx,
$$

$\forall g, h \in L^2(\Omega; \mathbb{C})$, where $\text{Re}[a], \text{Im}[a]$ denote the real and imaginary parts of $a$, $\forall a \in \mathbb{C}$, respectively.

Moreover, existence of a global solution for a similar problem has been proved in [7].

Finally, for the subsequent theoretical results we assume a simplified atomic units context.

**Remark 1.1.** At this point of our analysis and on, we consider a finite dimensional model version in a finite differences or finite elements context, even though the spaces and operators have not been relabeled. So, also in such a context, the expression

$$
\int_{\Omega} \frac{(v_i^*)^2}{2v_0^* - K} \, dx,
$$

2
indeed means

$$(v_1^*)^T(2v_0^* + K I_d)^{-1}v_1^*$$

where $I_d$ denotes the identity matrix $n \times n$ and

$$2v_0^* + K I_d$$

denotes the diagonal matrix with the vector

$$\{2v_0^*(i) + K\}_{n \times 1}$$
as diagonal, for some appropriate $n \in \mathbb{N}$ defined in the discretization process.

2 A brief initial description of our proposal

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a regular boundary denoted by $\partial \Omega$. Let $U = W_0^{1,2}(\Omega)$ and let $J : U \to \mathbb{R}$ be a functional defined by

$$J(u) = G_1(u) + F_1(u) - \langle u, f \rangle_{L^2},$$

where

$$G_1(u) = \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx,$$

and

$$F_1(u) = \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx,$$

where $\alpha, \beta$ and $\gamma$ are positive real constants and $f \in L^2(\Omega)$.

Observe that there exists $\eta \in \mathbb{R}$ such that

$$\eta = \inf_{u \in U} J(u) = J(u_0),$$

for some $u_0 \in U$. (We recall that the existence of a global minimizer may be proven by the direct method of the calculus of variations).

In our approach, we combine the ideas of J.J. Telega [3, 4] generalizing the approach in Ekeland and Temam [8] for establishing the dual functionals through the Legendre transform definition, with a D.C. approach for non-convex optimization developed by J.F. Toland, [10].

At this point we would define the functionals $F : U \to \mathbb{R}$ and $G : U \to \mathbb{R}$, where

$$F(u) = \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{K}{2} \int_\Omega u^2 \, dx,$$

and

$$G(u, v) = -\frac{\alpha}{2} \int_\Omega (u^2 - \beta + v)^2 \, dx + \frac{K}{2} \int_\Omega u^2 \, dx + \langle u, f \rangle_{L^2},$$
so that

\[ J(u) = F(u) - G(u,0) \]
\[ = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}, \quad (5) \]

where such a functional, for a large \( K > 0 \), is represented as a difference of two convex functionals in a large domain region proportional to \( K > 0 \).

The second step is to define the corresponding dual functionals \( F^*, G^* \), where

\[ F^*(v_1^*) = \sup_{u \in U} \{ \langle u, v_1^* \rangle_{L^2} - F(u) \} \]
\[ = \frac{1}{2} \int_{\Omega} \frac{(v_1^*)^2}{K - \gamma \nabla^2} \, dx \quad (6) \]

and

\[ G^*(v_1^*, v_0^*) = \sup_{w \in L^2} \inf_{v \in L^2} \{ \langle u, v_1^* \rangle_{L^2} - \langle v, v_0^* \rangle_{L^2} - G(u, v) \} \]
\[ = \sup_{w \in L^2} \inf_{v \in L^2} \{ \langle u, v_1^* \rangle_{L^2} - \langle w - u^2 + \beta, v_0^* \rangle_{L^2} + \frac{\alpha}{2} \int_{\Omega} w^2 \, dx - \langle u, f \rangle_{L^2} \} \]
\[ = \sup_{w \in L^2} \inf_{v \in L^2} \{ \langle u, v_1^* \rangle_{L^2} - \langle w - u^2 + \beta, v_0^* \rangle_{L^2} + \frac{\alpha}{2} \int_{\Omega} w^2 \, dx - \langle u, f \rangle_{L^2} \} \]
\[ = \frac{1}{2} \int_{\Omega} \frac{(v_1^* - f)^2}{2v_0^* + K} \, dx - \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx \]
\[ - \beta \int_{\Omega} v_0^* \, dx \quad (7) \]

if \( 2v_0^* + K > 0 \) in \( \overline{\Omega} \).

Defining

\[ E = \{ v_0^* \in C(\overline{\Omega}) \text{ such that } 2v_0^* + K > K/2, \text{ in } \overline{\Omega} \}, \]

where \( K > 0 \) is such that

\[ \frac{1}{\alpha} > \frac{2}{K}, \]

it may be proven that for \( K > 0 \) sufficiently large,
\[
\inf_{v^*_1 \in L^2} \sup_{v^*_0 \in E} \{ F^*(v^*_1) - G^*(v^*_1, v^*_0) \} \geq \inf_{u \in U} J(u).
\]

Equality concerning this last result may be obtained in a local extremal context and, under appropriate optimality conditions to be specified, also for global optimization.

At this point we highlight the maximization in \( v^*_0 \) with the restriction \( v^*_0 \in E \) does not demand a Lagrange multiplier, since for the value of \( K > 0 \) specified the restriction is not active.

We emphasize this approach is original and substantially different from all those, of other authors, so far known.

Finally, for a more general model, in the next section we formally prove that a critical point for the primal formulation necessarily corresponds to a critical point of the dual formulation. The reciprocal may be also proven.

3 The duality principle for a local extremal context

In this section we state and prove the concerning duality principle. We recall the existence of a global minimizer for the related functional has been proven in [7]. At this point it is also worth mentioning an extensive study on duality theory and applications for such and similar models is developed in [6].

**Theorem 3.1.** Let \( \Omega, \Omega_1 \subset \mathbb{R}^3 \) be open, bounded and connected sets with regular (Lipschitzian) boundaries denoted by \( \partial \Omega \) and \( \partial \Omega_1 \) respectively.

Assume \( \Omega_1 \) is convex and \( \Omega \subset \Omega_1 \). Consider the functional \( J : U \to \mathbb{R} \) where

\[
J(\phi, A) = \frac{\gamma}{2} \int_{\Omega} |\nabla \phi - i \rho A \phi|^2 \, dx
+ \frac{\alpha}{2} \int_{\Omega} (|\phi|^2 - \beta)^2 \, dx + \frac{1}{8\pi} \| \text{curl} \, A - B_0 \|^2_{0, \Omega_1}
\]

where \( \alpha, \beta, \gamma, \rho \) are positive real constants, \( i \) is the imaginary unit and

\[
U = U_1 \times U_2,
\]

\[
U_1 = C^1(\overline{\Omega}; \mathbb{C}), \quad U_2 = C^1(\overline{\Omega_1}; \mathbb{R}^3),
\]

both with the norm \( \| \cdot \|_{1, \infty} \).

Moreover,

\[
\phi : \Omega \to \mathbb{C}
\]

is the order parameter,

\[
A : \Omega_1 \to \mathbb{R}^3
\]

is the magnetic potential and \( B_0 \in C^1(\overline{\Omega_1}; \mathbb{R}^3) \), is an external magnetic field.

Defining,

\[
B_2 = \{ A \in C^1(\overline{\Omega_1}; \mathbb{R}^3) : \text{div} \, A = 0 \text{ in } \Omega_1, A \cdot n = 0, \text{ on } \partial \Omega_1 \},
\]
where \( n \) denotes the outward normal to \( \partial \Omega_1 \), suppose \((\phi_0, A_0) \in C^1(\overline{\Omega}; \mathbb{C}) \times B_2\) is such that

\[
\delta J(\phi_0, A_0) = 0.
\]

and

\[
\delta^2 J(\phi_0, A_0) > 0.
\]

Denoting also generically

\[
(\nabla - i\rho A)^*(\nabla - i\rho A) = |\nabla - i\rho A|^2,
\]

define \( F : U \to \mathbb{R} \) by

\[
F(\phi, A) = \frac{\gamma}{2} \int_{\Omega} |\nabla \phi - i\rho A \phi|^2 \, dx + \frac{K}{2} \int_{\Omega} |\phi|^2 \, dx,
\]

\[G : U \times C(\Omega) \to \mathbb{R} \]

defined by

\[
G(\phi, A, v) = -\frac{\alpha}{2} \int_{\Omega} (|v|^2 - \beta + v)^2 \, dx - \frac{1}{8\pi} \|\text{curl} \, A - B_0\|^2_{0, \Omega_1} + \frac{K}{2} \int_{\Omega} |\phi|^2 \, dx + \langle \phi, f \rangle_{L^2},
\]

\[
F^*(v_1^*, A) = \sup_{\phi \in U_1} \{ \langle \phi, v_1^* \rangle_{L^2} - F(\phi, A)\}
\]

\[
= \frac{1}{2} \int_{\Omega} \frac{(v_1^*)^2}{(\gamma|\nabla - i\rho A|^2 + K)} \, dx,
\]

\[
\hat{G}^*(v_1^*, v_0^*, A) = \sup_{\phi \in U_1} \inf_{v \in C(\Omega)} \{ \langle \phi, v_1^* \rangle_{L^2} - \langle v, v_0^* \rangle_{L^2} - G(\phi, A, v)\}
\]

\[
= -\frac{1}{2} \int_{\Omega} \frac{(v_1^* - f)^2}{2v_0^* - K} \, dx
- \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx - \beta \int_{\Omega} v_0^* \, dx
+ \frac{1}{8\pi} \|\text{curl} \, A - B_0\|^2_{0, \Omega_1},
\]

if

\[-2v_0^* + K > 0 \text{ in } \overline{\Omega},\]

and

\[
J^*(v_1^*, v_0^*, A) = -F^*(v_1^*, A) + \hat{G}^*(v_1^*, v_0^*, A)
\]

\[
= -\frac{1}{2} \int_{\Omega} \frac{(v_1^*)^2}{(\gamma|\nabla - i\rho A|^2 + K)} \, dx
- \frac{1}{2} \int_{\Omega} \frac{(v_1^*)^2}{2v_0^* - K} \, dx
- \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx - \beta \int_{\Omega} v_0^* \, dx
+ \frac{1}{8\pi} \|\text{curl} \, A - B_0\|^2_{0, \Omega_1}.
\]
Furthermore, define
\[ \hat{v}_0^* = \alpha(|\phi_0|^2 - \beta), \]
\[ \hat{v}_1^* = (2\hat{v}_0^* - K)\phi_0, \]
and
\[ E = \{ v_0^* \in C(\bar{\Omega}) : -2v_0^* + K > K/2, \text{ in } \bar{\Omega} \}. \]

Under such hypotheses and assuming also \( \hat{v}_0^* \in E \), we have
\[ \delta J^*(\hat{v}_1^*, \hat{v}_0^*, A_0) = 0. \]
\[ J(\phi_0, A_0) = J^*(\hat{v}_1^*, \hat{v}_0^*, A_0). \]

Moreover, defining
\[ J_1^*(v_1^*, A) = \sup_{v_0^* \in E} J^*(v_1^*, v_0^*, A), \]
for \( K > 0 \) such that
\[ \frac{1}{\alpha} > \frac{2}{K} \]
and sufficiently large, we have
\[ \delta J_1^*(\hat{v}_1^*, A_0) = 0, \]
\[ \delta^2 J_1^*(\hat{v}_1^*, A_0) > 0 \]
so that there exist \( r, r_1 > 0 \) such that

\[ J(\phi_0, A_0) = \min_{(\phi, A) \in B_r(\phi_0, A_0)} J(\phi, A) \]
\[ = \inf_{(v_1^*, A) \in B_{r_1}(\hat{v}_1^*, A_0)} J_1^*(v_1^*, A) \]
\[ = J_1^*(\hat{v}_1^*, A_0) \]
\[ = \inf_{(v_1^*, A) \in B_{r_1}(\hat{v}_1^*, A_0)} \left\{ \sup_{v_0^* \in E} J^*(v_1^*, v_0^*, A) \right\} \]
\[ = J^*(\hat{v}_1^*, \hat{v}_0^*, A_0). \] (13)

**Proof.** We start by proving that
\[ \delta J^*(\hat{v}_1^*, \hat{v}_0^*, A_0) = 0. \]

Observe that from
\[ \frac{\partial J(\phi_0, A_0)}{\partial \phi} = 0, \]
and
\[ \frac{\partial J(\phi_0, A_0)}{\partial A} = 0, \]
we have
\[
\begin{align*}
\begin{cases}
\gamma |\nabla - i\rho A_0|^2 \phi_0 + 2\alpha (|\phi_0|^2 - \beta)\phi_0 = 0, & \text{in } \Omega \\
(\nabla \phi_0 - i\rho A_0\phi_0) \cdot n = 0, & \text{on } \partial \Omega,
\end{cases}
\end{align*}
\tag{14}
\]
and
\[
\begin{align*}
\begin{cases}
\text{curl } (\text{curl } A_0) = \text{curl } B_0 + 4\pi \tilde{J}_0, & \text{in } \Omega \\
\text{curl } (\text{curl } A_0) = \text{curl } B_0, & \text{in } \Omega_1 \setminus \overline{\Omega},
\end{cases}
\end{align*}
\tag{15}
\]
where
\[
\tilde{J} = -2i\gamma \text{Im} [(\phi^*_0 \nabla \phi_0)] - \rho^2 |\phi_0|^2 A_0.
\]
Observe also that
\[
\frac{\partial J^* (\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial v_0^*} = \frac{\hat{v}_0^*}{(2\hat{v}_0^* - K)^2} - \frac{\hat{v}_0^*}{\alpha} = 0.
\tag{16}
\]
Summarizing, we have got
\[
\frac{\partial J^* (\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial v_0^*} = 0.
\]
Moreover, from the first line in equation (14), we obtain
\[
\gamma |\nabla - i\rho A_0|^2 \phi_0 + K \phi_0 + 2\alpha (|\phi_0|^2 - \beta)\phi_0 - K \phi_0 = 0, \text{ in } \Omega,
\]
so that
\[
\hat{v}_1^* = \frac{(\hat{v}_1^*)^2}{2\hat{v}_0^* - K} = -\frac{\hat{v}_0^*}{\gamma |\nabla - i\rho A_0|^2 + K},
\tag{17}
\]
Hence,
\[
\phi_0 = \frac{(\hat{v}_1^*)^2}{2\hat{v}_0^* - K} = -\frac{\hat{v}_0^*}{\gamma |\nabla - i\rho A_0|^2 + K},
\]
and thus,
\[
\frac{\partial J^* (\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial v_1^*} = -\frac{\hat{v}_0^*}{\gamma |\nabla - i\rho A_0|^2 + K} - \hat{v}_1^* = -\phi_0 + \phi_0 = 0.
\tag{18}
\]
Also, denoting
\[
H_1 = \frac{\partial [\gamma |\nabla - i\rho A_0|^2]}{\partial A} \left[ \frac{1}{2} (\hat{v}_1^*)^2 (\gamma |\nabla - i\rho A_0|^2 + K)^2 \right] + \frac{1}{4\pi} \{ \text{curl } (\text{curl } A_0) - \text{curl } B_0 \}
\]
\[
= \frac{\partial [\gamma |\nabla - i\rho A_0|^2]}{\partial A} \left[ \frac{|\phi_0|^2}{2} \right] + \frac{1}{4\pi} \{ \text{curl } (\text{curl } A_0) - \text{curl } B_0 \}
\]
\[
= \frac{1}{4\pi} \{ \text{curl } (\text{curl } A_0) - \text{curl } B_0 \} - \tilde{J}_0,
\tag{19}
\]
and
\[ H_2 = \frac{1}{4\pi} \{ \text{curl (curl } A_0) - \text{curl } B_0 \}, \]
we get
\[ \frac{\partial J^*(\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial A} = \begin{cases} H_1, & \text{in } \Omega, \\ H_2, & \text{in } \Omega_1 \setminus \overline{\Omega}. \end{cases} \quad (20) \]
Summarizing,
\[ \frac{\partial J^*(\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial A} = 0. \]
Such last results may be denoted by
\[ \delta J^*(\hat{v}_1^*, \hat{v}_0^*, A_0) = 0. \]
Also
\[ \frac{\partial J^*_1(\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial v_1^*} = \frac{\partial J^*_1(\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial v_1^*} + \frac{\partial J^*_1(\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial \hat{v}_0^*} \frac{\partial \hat{v}_0^*}{\partial v_1^*} = 0. \quad (21) \]
and
\[ \frac{\partial J^*_1(\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial A} = \frac{\partial J^*_1(\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial A} = 0, \]
so that we may denote
\[ \delta J^*_1(\hat{v}_1^*, A_0) = 0. \]
Furthermore, we may easily compute,
\[ J^*(\hat{v}_1^*, \hat{v}_0^*, A_0) = -F^*(\hat{v}_1) + \hat{G}^*(\hat{v}_1^*, \hat{v}_0^*, A_0) \]
\[ = -\langle \phi_0, \hat{v}_1^* \rangle_{L^2} + F(\phi_0, A_0) + \langle \phi_0, \hat{v}_1^* \rangle_{L^2} - \langle 0, \hat{v}_0^* \rangle_{L^2} - G(\phi_0, A_0) \]
\[ = J(\phi_0, A_0). \quad (22) \]
Define now
\[ J_1(u_1^*, A_0) = \sup_{v_0^* \in E} J^*(u_1^*, v_0^*, A_0). \]
Observe that, in particular, we have
\[ J^*_1(\hat{v}_1^*) = J^*(\hat{v}_1^*, \hat{v}_0^*, A_0), \]
where the concerning supremum is attained through the equation
\[ \frac{\partial J^*(\hat{v}_1^*, \hat{v}_0^*, A_0)}{\partial v_0^*} = 0, \]
that is
\[
\frac{(\dot{v}_1^*)^2}{(2\dot{v}_0^* - K)^2} - \frac{\dot{v}_0^*}{\alpha} - \beta = |\phi_0|^2 - \frac{\dot{v}_0^*}{\alpha} - \beta = 0. \quad (23)
\]
Taking the variation in \(v_1^*\) in such an equation, we get
\[
\frac{2\ddot{v}_1^*}{(2\dot{v}_0^* - K)^2} - \frac{4(\ddot{v}_1^*)^2}{(2\dot{v}_0^* - K)^3} \frac{\partial \dot{v}_0^*}{\partial v_1^*} - \frac{1}{\alpha} \frac{\partial \dot{v}_0^*}{\partial v_1^*} = 0,
\]
so that
\[
\frac{\partial \dot{v}_0^*}{\partial v_1^*} = \frac{\frac{2\phi_0}{(2\phi_0 - K)}}{\frac{1}{\alpha} + \frac{4|\phi_0|^2}{2\phi_0 - K}},
\]
where, as previously indicated,
\[
\phi_0 = \frac{\ddot{v}_1^*}{2\dot{v}_0^* - K}.
\]
At this point we observe that
\[
\frac{\partial^2 J^*(\ddot{v}_1^*, A_0)}{\partial (v_1^*)^2} = \frac{\partial^2 J^*(\ddot{v}_1^*, \ddot{v}_0^*, A_0)}{\partial (v_1^*)^2} + \frac{\partial^2 J^*(\ddot{v}_1^*, \ddot{v}_0^*, A_0)}{\partial v_1^* \partial v_0^*} \frac{\partial \dot{v}_0^*}{\partial v_1^*} = -\frac{1}{\gamma |\nabla - i\rho A_0|^2 + K} - \frac{1}{2\dot{v}_0^* - K} + \frac{4\alpha |\phi_0|^2}{[(2\dot{v}_0^* - K)^2]^{1 + \frac{4\alpha |\phi_0|^2}{2\phi_0 - K}}} = \frac{-2\dot{v}_0^* - 4\alpha |\phi_0|^2 + K - \gamma |\nabla - i\rho A_0|^2 - K}{(K + \gamma |\nabla - i\rho A_0|^2)(2\dot{v}_0^* + 4\alpha |\phi_0|^2 - K)} - \frac{\delta_{\phi_0}^2 J(\phi_0, A_0)}{\gamma |\nabla - i\rho A_0|^2(2\dot{v}_0^* + 4\alpha |\phi_0|^2 - K)} > 0.
\]
Summarizing,
\[
\frac{\partial^2 J^*(\ddot{v}_1^*, A_0)}{\partial (v_1^*)^2} > 0.
\]
For \(K > 0\) sufficiently big we may easily obtain
\[
\frac{\partial^2 J^*(\ddot{v}_1^*, A_0)}{\partial (A_0)^2} > 0
\]
and
\[ \frac{\partial^2 J^*(\hat{v}_1^*, A_0)}{\partial v_1^* \partial A} \approx O(1/K), \]
so that
\[ \delta^2 J^*(\hat{v}_1^*, A_0) > 0. \]

From these last results, there exists \( r, r_1 > 0 \) such that
\[
J(\phi_0, A_0) = \min_{(\phi, A) \in B_r(\phi_0, A_0)} J(\phi, A) \\
= \inf_{(v_1^*, A) \in B_{r_1}(\hat{v}_1^*, A_0)} J^*(v_1^*, A) \\
= J^*(\hat{v}_1^*, A_0) \\
= \inf_{(v_1^*, A) \in B_{r_1}(\hat{v}_1^*, A_0)} \left\{ \sup_{v_0^* \in C} J^*(v_1^*, v_0^*, A, v) \right\} \\
= J^*(\hat{v}_1^*, \hat{v}_0^*, A_0). \tag{25}
\]

The proof is complete.

References

[1] R.A. Adams and J.F. Fournier, Sobolev Spaces, 2nd edn. (Elsevier, New York, 2003).
[2] J.F. Annet, Superconductivity, Superfluids and Condensates, 2nd edn. (Oxford Master Series in Condensed Matter Physics, Oxford University Press, Reprint, 2010)
[3] W.R. Bielski, A. Galka, J.J. Telega, The Complementary Energy Principle and Duality for Geometrically Nonlinear Elastic Shells. I. Simple case of moderate rotations around a tangent to the middle surface. Bulletin of the Polish Academy of Sciences, Technical Sciences, Vol. 38, No. 7-9, 1988.
[4] W.R. Bielski and J.J. Telega, A Contribution to Contact Problems for a Class of Solids and Structures, Arch. Mech., 37, 4-5, pp. 303-320, Warszawa 1985.
[5] D. Bohm, Quantum Theory (Dover Publications INC., New York, 1989).
[6] F. Botelho, Functional Analysis and Applied Optimization in Banach Spaces, (Springer Switzerland, 2014).
[7] F. Botelho, A Classical Description of Variational Quantum Mechanics and Related Models, Nova Science Publishers, New York, 2017.
[8] I.Ekeland and R.Temam, Convex Analysis and Variational Problems. North Holland (1976).
[9] L.D. Landau and E.M. Lifschits, Course of Theoretical Physics, Vol. 5- Statistical Physics, part 1. (Butterworth-Heinemann, Elsevier, reprint 2008).
[10] J.F. Toland, A duality principle for non-convex optimisation and the calculus of variations, Arch. Rath. Mech. Anal., 71, No. 1 (1979), 41-61.