Force Equation That Describes the Quantum States of a Relativistic Spinless Particle

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Newton’s second law may be used to obtain a wave equation, which reduces to Schrodinger’s equation in the nonrelativistic limit and for a conservative force.

I. INTRODUCTION

A quantum mechanical version of Newton’s second law has existed for some time [1, 2]. By substituting $\psi = \sqrt{f}e^{iS/\hbar}$ into Schrodinger’s equation, it is possible to arrive at a continuity equation,

$$\frac{\partial f}{\partial t} + \nabla \cdot (fv) = 0,$$  \hspace{1cm} (1)

and the quantum mechanical version of Newton’s second law,

$$m \frac{d^2 r}{dt^2} = -\nabla (V + Q),$$  \hspace{1cm} (2)

where $v = \nabla S/m$ is velocity, and

$$Q = \frac{\hbar^2}{2m} \left[ \left( \frac{\nabla f}{f} \right) \cdot \left( \frac{\nabla f}{f} \right) - \left( \frac{\nabla^2 f}{2f} \right) \right]$$  \hspace{1cm} (3)

is a quantum mechanical potential energy [1, 2]. Here, $\psi$ is the wavefunction, $f$ is the probability density, $V$ is the classical potential energy, $r$ is the configuration space coordinate vector, $t$ is time, $m$ is the particle’s mass, $i = \sqrt{-1}$, $\hbar = h/(2\pi)$, and $h$ is Planck’s constant. This type of formulation has been used as the basis of a significant number of recent studies [3, 4]. Although Newton’s second law describes the trajectory of a point particle, Newton’s second law may also be used to obtain a wave equation that describes the quantum states of a relativistic spinless particle.

II. PRELIMINARIES

As a starting point, it is assumed that a time-dependent probability density in configuration space, $f(r, t)$, and a time-dependent probability density in momentum space, $g_p(p, t)$, describe the state of a spinless particle. A wavevector $k$ is defined in relation to momentum $p$ by $p = \hbar k$. Wavefunctions in configuration space, momentum space, and wavevector space, denoted $\chi_p(p, t)$, $\chi_k(k, t)$, and $\chi_{kp}(k, t)$, respectively, are defined in relation to probability densities in configuration space, momentum space, and wavevector space, denoted $f(r, t)$, $g_p(p, t)$, and $g_k(k, t)$, respectively, such that $f(r, t) = \psi^*(r, t)\psi(r, t)$, $g_p(p, t) = \chi^*_{p}(p, t)\chi_p(p, t)$,

and $g_k(k, t) = \chi^*_{k}(k, t)\chi_k(k, t)$. Here, a complex conjugate is indicated with a superscript star ($^*$). As a postulate, $\chi_k(k, t)$ is taken to be the three-dimensional Fourier transform of $\psi(r, t)$,

$$\chi_k(k, t) = \frac{1}{(2\pi)^{3/2}} \int \psi(r, t)e^{-ik \cdot r} d^3r,$$   \hspace{1cm} (4)

where $\int d^3r$ is an integration over configuration space. The inverse transform is

$$\psi(r, t) = \frac{1}{(2\pi)^{3/2}} \int \chi_k(k, t)e^{ik \cdot r} d^3k,$$   \hspace{1cm} (5)

where $\int d^3k$ is an integration over wavevector space.

The expectation value of an arbitrary expression $A(p)$ that is written in terms of momentum is evaluated in momentum space as

$$\langle A(p) \rangle = \int A(p)g_p(p, t)d^3p = \int A(p)\chi^*_p(p, t)\chi_p(p, t)d^3p,$$   \hspace{1cm} (6)

where $\int d^3p$ is an integration over momentum space. The same expectation value can be evaluated as an integration over wavevector space using

$$\langle A(p) \rangle = \int A(\hbar k)\chi^*_k(k, t)\chi_k(k, t)d^3k.$$   \hspace{1cm} (7)

The expectation value can be evaluated as an integration over configuration space using

$$\langle A(p) \rangle = \int \psi^*(r, t)O[\psi(r, t)]d^3r,$$   \hspace{1cm} (8)

where $O$ is a differential operator in configuration space, and $\langle \psi | O | \psi \rangle$ is understood to indicate that $O$ operates on $\psi$. Substituting $\psi$ given by Eq. (5) into Eq. (8) gives

$$\langle A(p) \rangle = \frac{1}{(2\pi)^3} \int \chi^*_k(k_1, t)\chi_k(k, t)e^{-i k_1 \cdot r}O(e^{ik \cdot r})d^3r d^3k_1,$$   \hspace{1cm} (9)

where the subscript on $k_1$ is used to distinguish two different sets of wavevector space integration variables. An expression for $O$ that allows the expectation value to be evaluated using Eq. (8) is one that satisfies

$$O(e^{ik \cdot r}) = A(\hbar k)e^{ik \cdot r}.$$   \hspace{1cm} (10)

Substituting Eq. (10) into Eq. (9) yields

$$\langle A(p) \rangle = \int A(\hbar k)\chi^*_k(k_1, t)\chi_k(k, t)\delta(k - k_1)d^3k_1d^3k,$$   \hspace{1cm} (11)
where \( \delta(k - k_1) \) is the Dirac delta function, as represented by

\[
\delta(k - k_1) = \frac{1}{(2\pi)^3} \int e^{i(k-k_1)\cdot r} d^3 r, \tag{12}
\]

Carrying out the integration \( \int d^3 k_1 \) in Eq. (11) yields Eq. (7).

### III. DIFFERENTIAL EXPRESSIONS

Note that Eq. (8) can be written as

\[
\langle A(p) \rangle = \int A(p)f(r,t)d^3r, \tag{13}
\]

where \( A(p) \) in configuration space is understood to represent a differential expression defined by

\[
A(p) = \frac{O[\psi(r,t)]}{\psi(r,t)}. \tag{14}
\]

For example, the expectation value of \( p \) would be written as \( \langle p \rangle = \int pf(r,t)d^3r \), where \( p \) represents a differential expression in configuration space. Two differential expressions that are defined according to Eq. (14) from operators that satisfy Eq. (10) are

\[
p = -\frac{i\hbar}{\psi(r,t)} \nabla \psi(r,t), \tag{15}
\]

and

\[
p^2 = -\hbar^2 \frac{\nabla^2 \psi(r,t)}{\psi(r,t)}. \tag{16}
\]

It can be shown that \( p^2 = p \cdot p^* \) when \( \psi = e^{ikr} \).

### IV. RELATIVISTIC EQUATION

The time rate of change of the differential expression, \( p(r,t) \), is

\[
\frac{dp}{dt} = \frac{\partial p}{\partial t} + (v \cdot \nabla) p = \frac{\partial p}{\partial t} + (\frac{p \cdot \nabla}{m}) p, \tag{17}
\]

where, velocity is written in terms of relativistic momentum as \( v = p/m_\gamma \), where

\[
m_\gamma = m\sqrt{1 + \frac{p^2}{(mc)^2}}, \tag{18}
\]

\( m \) is the particle’s rest mass, and \( c \) is the speed of light. The expression \( v = p/m_\gamma \) is obtained by inverting \( p = mv/\sqrt{1 - v^2/c^2} \). According to Newton’s second law, \( F = dp/dt \), where \( F \) is the sum of the forces that act on the particle, and \( dp/dt \) is the time rate of change of the particle’s relativistic momentum. With Newton’s second law, and employing the vector identity, \( \nabla(a \cdot b) = a \times (\nabla \times b) + b \times (\nabla \times a) + (a \cdot \nabla)b + (b \cdot \nabla)a \), where \( a \) and \( b \) are vectors, a force equation that describes the quantum states of a relativistic spinless particle is written as

\[
F = \frac{\partial p}{\partial t} + \nabla p^2 - \frac{p \times (\nabla \times p)}{m_\gamma}. \tag{19}
\]

Equation (19) may be considered a wave equation, because \( p \) and \( p^2 \) represent differential expressions that contain the wavefunction \( \psi \). Equation (19) applies even when the presence of a nonconservative force does not allow a potential energy to be defined.

### V. NONRELATIVISTIC, CONSERVATIVE-FORCE EQUATION

In the nonrelativistic \( (m_\gamma \to m) \) limit, and for a conservative force \( F = -\nabla V \), where \( V \) is potential energy, Eq. (19) is written as

\[
-\nabla V = \frac{\partial p}{\partial t} + \nabla \left( \frac{p^2}{2m} \right) - \frac{p \times (\nabla \times p)}{m}. \tag{20}
\]

The first term on the right is written as

\[
\frac{\partial p}{\partial t} = \frac{\partial}{\partial t} \left( -i\hbar \frac{\nabla \psi}{\psi} \right) = \nabla \left( \frac{\hbar}{p} \frac{\nabla \psi}{\psi} \right), \tag{21}
\]

where the second equality is arrived at by direct substitution of \( \psi(r,t) \). The second term on the right in Eq. (20) represents the gradient of the classical kinetic energy. The associated differential expression is

\[
\nabla \left( \frac{p^2}{2m} \right) = \nabla \left( \frac{-\hbar^2}{2m} \frac{\nabla^2 \psi}{\psi} \right). \tag{22}
\]

The third term on the right in Eq. (20) is zero,

\[
p \times (\nabla \times p) = 0. \tag{23}
\]

With Eqs. (21) - (23), Eq. (20) is written as

\[
\nabla \left( \frac{-\hbar^2}{2m} \frac{\nabla^2 \psi}{\psi} + V \psi - i\hbar (\partial \psi/\partial t) \right) = 0. \tag{24}
\]

For \( \psi = \psi(r,t) \), Eq. (24) is satisfied if the numerator of the quotient equals zero. Setting the numerator equal to zero, the resulting equation can be written as Schrödinger’s equation,

\[
-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = i\hbar \frac{\partial \psi}{\partial t}. \tag{25}
\]

For a stationary state, Eq. (24) is written as \( \nabla [p^2/(2m) + V] = 0 \), where \( \partial p/\partial t = 0 \) and Eq. (23) are used. The expression \( \nabla [p^2/(2m) + V] = 0 \) is satisfied by a spatially constant classical energy \( E = p^2/(2m) + V \). Upon substituting \( \psi \) into \( E = p^2/(2m) + V \), the resulting equation can be written as the time-independent version of Schrödinger’s equation,

\[
-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = E \psi. \tag{26}
\]
VI. RELATIVISTIC, CONSERVATIVE-FORCE EQUATION

Relativistic versions of Eqs. (25) and (26) are readily obtained, provided the second term on the right in Eq. (19) is rewritten as the gradient of the relativistic energy:

\[
\nabla p^2 = \nabla \left[ \sqrt{(mc^2)^2 + c^2 p^2} \right].
\]

(27)

The associated differential expression is

\[
\nabla p^2 = \nabla \left[ \sqrt{(mc^2)^2 - (\hbar c)^2} \frac{\nabla^2 \psi}{\psi} \right].
\]

(28)

With \( F = -\nabla V \), and with Eqs. (21), (23), and (28), Eq. (19) is written as

\[
\nabla \left( \sqrt{(mc^2)^2 - (\hbar c)^2} \frac{\nabla^2 \psi}{\psi} \right) \psi + V \psi = \imath \hbar \frac{\partial \psi}{\partial t}.
\]

(29)

With \( \psi = \psi(r,t) \), the numerator of the quotient is set equal to zero, and the resulting equation is written as

\[
\sqrt{(mc^2)^2 - (\hbar c)^2} \frac{\nabla^2 \psi}{\psi} \psi + V \psi = \imath \hbar \frac{\partial \psi}{\partial t}.
\]

(30)

The stationary state version of Eq. (30) is

\[
\sqrt{(mc^2)^2 - (\hbar c)^2} \frac{\nabla^2 \psi}{\psi} \psi + V \psi = E \psi,
\]

(31)

where \( E \) now represents the sum of the relativistic energy and the potential energy.

VII. CONCLUSION

In summary, a way to arrive at Schrödinger’s equation from Newton’s second law was developed by writing momentum expressions as differential expressions. Newton’s second law was used to write a force equation, Eq. (19), which represents a wave equation that describes the quantum states of a relativistic spinless particle. Equation (19) was used to obtain Schrödinger’s equation (both time-dependent and time-independent versions) in the nonrelativistic limit and for a conservative force. Relativistic versions of both versions of Schrödinger’s equation were obtained.

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