DIFFERENTIATION OF THE WRIGHT FUNCTIONS WITH RESPECT TO PARAMETERS AND OTHER RESULTS

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Abstract: In this work we discuss the derivatives of the Wright functions (of the first and the second kind) with respect to parameters. Differentiation of these functions leads to infinite power series with coefficient being quotients of the digamma (psi) and gamma functions. Only in few cases it is possible to obtain the sums of these series in a closed form. Functional form of the power series resembles those derived for the Mittag-Leffler functions. If the Wright functions are treated as the generalized Bessel functions, differentiation operations can be expressed in terms of the Bessel functions and their derivatives with respect to the order. It is demonstrated that in many cases it is possible to derive the explicit form of the Mittag-Leffler functions by performing simple operations with the Laplace transforms of the Wright functions. The Laplace transform pairs of the both kinds of the Wright are discussed for particular values of the parameters. Some transform pairs serve to obtain functional limits by applying the shifted Dirac delta function. We expect that the present analysis will find several applications in physics and more generally in applied sciences. Indeed these special functions of the Mittag-Leffler and Wright type have already found application in rheology and in stochastic processes where fractional calculus is relevant. Then careful readers can take benefit from our new results presented in this paper for novel applications.

Keywords: Derivatives with respect to parameters; Wright functions; Mittag-Leffler functions; Laplace transforms; functional limits.

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1 Introduction

Partial differential equations of fractional order are successively applied for modeling time and space diffusion, stochastic processes, probability distributions and other diverse natural phenomena. They are extremely important in physical processes that can be described by using the fractional calculus. In the mathematical literature, when solution of these fractional differential equations is desired, we frequently encounter introduced and named after him, the Wright functions. At beginning, at 1933 [1] and at 1940 [2], these functions were considered as a some kind of generalization of the Bessel functions, but today they play an significant independent role in the theory of special functions. There are many investigations devoted to analytical properties and applications of the Wright functions, but here are mentioned only two survey papers where essential material on the subject is included [3,4]. These functions turn out to be particular cases of higher transcendental functions as recently shown in interesting surveys by Kiryakova [5], Srivastava [6].

In this paper we discuss three quite different subjects which are associated with the Wright functions. In the first part, the Wright function $W_{\alpha,\beta}(t)$ where $t$ is the argument and $\alpha$ and $\beta$ are the parameters, is differentiated with respect to parameters and derived expressions are compared with similar formulas for the Mittag-Leffler functions. In a continuous effort, after investigating differentiation of the Bessel functions and the Mittag-Leffler functions with respect to their parameters [7-9], this mathematical operation is extended here to the Wright functions. Special attention is devoted to the cases when the Wright functions can be reduced to the Bessel functions and expressed in a closed form. The auxiliary functions, $F_{\alpha}(t)$ and $M_{\alpha}(t)$, which were introduced for the first time in 1990's by Mainardi, see [4], and now are called the Mainardi functions, are as well discussed in this section.

Functional behaviour of derivatives with respect to the order is also presented in graphical form. The presented plots were prepared by evaluation of sums of infinite series by using MATHEMATICA program.

The second part of this paper is dedicated to the Laplace transform pairs of the Wright functions. It is demonstrated how the Laplace transforms of the Wright functions are useful for obtaining explicit expressions for the Mittag-Leffler functions. Finally, we discuss the functional limits which are associated with the Wright and the Mittag-Leffler functions. These limits can be derived by applying the delta sequence in the form of the shifted Dirac function. This delta sequence is directly related to the order of Bessel function and was introduced by Lamborn in 1969, see [8-11] and [18-20].
Throughout this paper all mathematical operations or manipulations with functions, series, integrals, integral representations and transforms are formal and it is assumed that arguments and parameters are real numbers. There will be no proofs of validity of derived results, though they are presumed to be correct considering that in a part they were previously obtained independently by other methods.

2 The Wright functions of the first and of the second kind

The Wright functions $W_{\alpha,\beta}(z)$ are defined by the series representation as a function of complex argument $z$ and parameters $\alpha$ and $\beta$.

$$W_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\alpha k + \beta)}.$$ (1)

They are entire functions of $z \in \mathbb{C}$ for $\alpha > -1$ and for any complex $\beta$ (here always $\beta \geq 0$). According to Mainardi see i.e. the appendix F of [4], we distinguish the Wright function of the first kind for $\alpha \geq 0$, and of the second kind for $-1 < \alpha < 0$. This distinctions is justified for the difference in the asymptotics representations in the complex domain and in the Laplace transforms for the real positive argument. For our purposes we recall their Laplace transforms for positive argument $t$, We have, by using the symbol $\div$ to denote the juxtaposition of a function $f(t)$ with its Laplace transform $\tilde{f}(s)$, for the first kind, when $\alpha \geq 0$

$$W_{\alpha,\beta}(\pm t) \div \frac{1}{s} E_{\lambda,\mu}(\pm \frac{1}{s}).$$ (2)

for the second kind, when $-1 < \alpha < 0$ and putting for convenience $\nu = -\alpha$ so $0 < \nu < 1$

$$W_{-\nu,\beta}(-t) \div E_{\nu,\nu+\nu}(-s).$$ (3)

Above we have introduced the Mittag-Leffler function in two parameters $\alpha > 0$, $\beta \in \mathbb{C}$ defined as its convergent series for all $z \in \mathbb{C}$:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$ (4)

For more details on the special functions of the Mittag-Leffler type we refer the interested readers to the treatise by Gorenflo et al [7] where in the recent 2-nd edition also the Wright functions are treated in some detail.
3 Differentiation of the Wright functions of the first kind with respect to parameters

We first compare the Wright functions of the first kind with the two-parameter Mittag-Leffler functions for \( \alpha > 0 \) and \( \beta \geq 0 \) from which they differ only by the absence of factorials.

Direct differentiation of series with respect to the parameter \( \alpha \) gives

\[
\frac{\partial W_{\alpha,\beta}(t)}{\partial \alpha} = -\sum_{k=1}^{\infty} \left( \frac{\psi(\alpha k+\beta)}{\Gamma(\alpha k+\beta)} \right) k \, t^k = \sum_{k=1}^{\infty} \left( \frac{\psi(\alpha k+\beta)}{(k-1)!\Gamma(\alpha k+\beta)} \right) k \, t^k
\]

and with respect to the parameter \( \beta \)

\[
\frac{\partial W_{\alpha,\beta}(t)}{\partial \beta} = -\sum_{k=1}^{\infty} \left( \frac{\psi(\alpha k+\beta)}{\Gamma(\alpha k+\beta)} \right) k \, t^k
\]

The second derivatives are

\[
\frac{\partial^2 W_{\alpha,\beta}(t)}{\partial \alpha^2} = \sum_{k=1}^{\infty} \left( \frac{\psi(\alpha k+\beta)}{\Gamma(\alpha k+\beta)} \right) k \, t^k
\]

and

\[
\frac{\partial^2 W_{\alpha,\beta}(t)}{\partial \beta^2} = \sum_{k=1}^{\infty} \left( \frac{\psi(\alpha k+\beta)}{\Gamma(\alpha k+\beta)} \right) k \, t^k
\]

We note that for the Mittag-Leffler and the Wright functions we have the same functional expressions, but in the case of the Wright functions factorials appear. Contrary to the Mittag-Leffler functions \([9-10]\), summation of these series by using MATHEMATICA gives only few results in a closed form in terms of the generalized hypergeometric functions

\[
\left. \frac{\partial W_{\alpha,\beta}(t)}{\partial \alpha} \right|_{\alpha=1,\beta=0} = -\sum_{k=1}^{\infty} \left( \frac{\psi(k)}{(k-1)!k^2} \right) t^k = t_0 F_1 (; 1; t)
\]

and

\[
\left. \frac{\partial W_{\alpha,\beta}(t)}{\partial \beta} \right|_{\alpha=1,\beta=0} = -\sum_{k=0}^{\infty} \left( \frac{\psi(k)}{(k-1)!k} \right) t^k = \frac{1}{2} t_0 F_1 (; 1; t) - t_0 F_1 (; 2; t) \ln t + \sqrt{t} K_1 (2 \sqrt{t})
\]

and

\[
\left. \frac{\partial W_{\alpha,\beta}(t)}{\partial \alpha} \right|_{\alpha=1,\beta=0} = -\sum_{k=0}^{\infty} \left( \frac{\psi(k+1)}{(k)!k^2} \right) t^k = 0 F_1 (; 1; t)
\]

and

\[
\left. \frac{\partial W_{\alpha,\beta}(t)}{\partial \beta} \right|_{\alpha=1,\beta=0} = -\sum_{k=0}^{\infty} \left( \frac{\psi(k+1)}{(k)!} \right) t^k = 0 F_1 (; 1; t)
\]
In the last case, \( \alpha = \beta = 1 \), in the Brychkov compilation of infinite series [12], the sum is expressed in terms of the modified Bessel functions

\[
\frac{\partial W_{\alpha,\beta}(t)}{\partial \beta} |_{\alpha=\beta=1} = -\sum_{k=0}^{\infty} \left( \psi(k+1) \frac{1}{(k!)^2} \right) t^k = -\frac{1}{2} \ln t I_0(2\sqrt{t}) - K_0(2\sqrt{t})
\]

(11)

Using MATHEMATICA program, values of derivatives with respect to parameters \( \alpha \) and \( \beta \) of the Wright functions of the first kind were calculated for the argument \( 0.25 \leq t \leq 4.0 \) and for parameters \( 0 \leq \alpha \leq 5.0 \) and \( 0 \leq \beta \leq 2.0 \).

In Figure 1 is illustrated behaviour of derivatives with respect to parameter \( \alpha \) at different values if the argument \( t \).

As can be observed, in the \( 0 < \alpha < 1 \) region, exists minimum, with increasing \( \alpha \) all curves tend to zero value. The absolute value of the minimum increases with increasing argument.

Derivatives with respect to the parameters \( \alpha \) and \( \beta \) when the argument \( t \) is constant, are presented in Figure 2. The functional form of curves with the change of \( \beta \) values is similar to that observed previously in Figure 1.

In order to compare the behaviour of derivatives with respect to \( \alpha \) with those with respect to \( \beta \), the same conditions were imposed on \( t \) and \( \beta \) in Figures 3 and 4 as are in Figures 1 and 2.

As can be observed, the similarity of corresponding curves is evident, the only difference is that the absolute values of the minima are lower for derivatives with respect to parameter \( \beta \) than to \( \alpha \).

4 Differentiation of the Wright functions of the second kind with respect to parameters

We then consider among the Wright functions of the second kind the functions, \( F_\alpha(t) \) and \( M_\alpha(t) \), introduced by Mainardi,

\[
F_\alpha(t) = W_{-\alpha,0}(t) ; \quad 0 < \alpha < 1 \\
M_\alpha(t) = W_{-\alpha,1-\alpha}(t) ; \quad 0 < \alpha < 1 \\
F_\alpha(t) = \alpha t M_\alpha(t)
\]

(12)

Their series expansions explicitly read

\[
F_\alpha(t) = \sum_{k=1}^{\infty} \frac{(-t)^k}{k! \Gamma(-\alpha k)} = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} \Gamma(\alpha k + 1) \sin(\pi \alpha k)
\]

(13)

and

\[
M_\alpha(t) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k! \Gamma(-\alpha (k+1) + 1)} = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-t)^{k-1}}{k!} \Gamma(\alpha k) \sin(\pi \alpha k)
\]

(14)
Figure 1: Derivatives of the Wright functions of the first kind with respect to parameter $\alpha$ as a function of $\alpha$ for $\beta = 1$: 1: $t = 0.5$; 2: $t = 1.0$; 3: $t = 1.5$; 4: $t = 1.75$; 5: $t = 2.0$.

Direct differentiation of (13) and (14) gives

\[
\frac{\partial F_\alpha(t)}{\partial \alpha} = \frac{1}{\pi} \sum_{k=1}^{\infty} k^{-1} (k-1)! \Gamma(\alpha k + 1) \left[ \psi'(\alpha k + 1) + \psi^2(\alpha k + 1) \right] \sin(\pi \alpha k) + \frac{2}{\pi} \cos(\pi \alpha k) \psi(\alpha k)
\]

Using the last equation in (12) we have

\[
\frac{\partial F_\alpha(t)}{\partial \alpha} = t M_\alpha(t) + \alpha t \frac{\partial M_\alpha(t)}{\partial \alpha}
\]

The second derivatives of these functions are

\[
\frac{\partial^2 F_\alpha(t)}{\partial \alpha^2} = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{k^2}{(k-1)!} \Gamma(\alpha k + 1) \left[ \psi'(\alpha k + 1) + \psi^2(\alpha k + 1) \right] \sin(\pi \alpha k) + 2 \frac{\pi}{\pi} \cos(\pi \alpha k) \psi(\alpha k) - \pi^2 \sin(\pi \alpha k)
\]

and

\[
\frac{\partial^2 M_\alpha(t)}{\partial \alpha^2} = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{k^2}{(k-1)!} \Gamma(\alpha k + 1) \left[ \psi'(\alpha k) + \psi^2(\alpha k) \right] \sin(\pi \alpha k) + 2 \frac{\pi}{\pi} \cos(\pi \alpha k) \psi(\alpha k) - \pi^2 \sin(\pi \alpha k)
\]

They are interrelated by

\[
\frac{\partial^2 F_\alpha(t)}{\partial \alpha^2} = 2 t \frac{\partial M_\alpha(t)}{\partial \alpha} + \alpha t \frac{\partial^2 M_\alpha(t)}{\partial \alpha^2}
\]
Figure 2: Derivatives of the Wright functions of the first kind with respect to parameter \( \alpha \) as a function of \( \alpha \) and \( \beta = 1 \) for \( t = 2.0 \).

5 The Laplace transforms and the Wright functions of the first kind

The Laplace transforms of the Wright functions are expressed in terms of two-parameter the Mittag-Leffler functions [3,4]

\[
L\{W_{\alpha,\beta}(\pm \lambda t)\} = \frac{1}{s} E_{\alpha,\beta}\left(\pm \frac{\lambda}{s}\right) ; \quad \alpha > 0 ; \quad \lambda > 0 \tag{20}
\]

Applying operational rules of the Laplace transformation we have, see [13], [15-16],

\[
L\{e^{\pm \rho W_{\alpha,\beta}(\lambda t)}\} = \frac{1}{s \mp \rho} E_{\alpha,\beta}\left(\frac{\lambda}{s \mp \rho}\right) ; \quad \lambda, \rho > 0 \tag{21}
\]

and this permits to obtain

\[
L\{\sinh(\rho t) W_{\alpha,\beta}(\lambda t)\} = \frac{1}{2} \left\{ \frac{1}{s \mp \rho} E_{\alpha,\beta}\left(\frac{\lambda}{s \mp \rho}\right) - \frac{1}{s \mp \rho} E_{\alpha,\beta}\left(\frac{1}{s \mp \rho}\right) \right\}
\]
\[
L\{\cosh(\rho t) W_{\alpha,\beta}(\lambda t)\} = \frac{1}{2} \left\{ \frac{1}{s \mp \rho} E_{\alpha,\beta}\left(\frac{1}{s \mp \rho}\right) + \frac{1}{s \mp \rho} E_{\alpha,\beta}\left(\frac{1}{s \mp \rho}\right) \right\} \tag{22}
\]

Multiplication \[20\] by \( t \) gives

\[
L\{t W_{\alpha,\beta}(\lambda t)\} = -\frac{d}{ds} \left\{ \frac{1}{s} E_{\alpha,\beta}\left(\frac{\lambda}{s}\right) \right\} = -\left\{ -\frac{1}{s^2} E_{\alpha,\beta}\left(\frac{\lambda}{s}\right) + \frac{1}{s} \frac{d}{ds} E_{\alpha,\beta}\left(\frac{\lambda}{s}\right) \right\} \tag{23}
\]
Figure 3: Derivatives of the Wright functions with respect to parameter $\beta$ as a function of $\alpha$ for $\beta = 1$ and 1: $t = 0.5$; 2 $t = 1.0$; 3 $t = 1.5$; 4 $t = 1.75$; 5 $t = 2.0$.

Derivative of the Mittag-Leffler function is

$$\frac{d}{ds}E_{\alpha,\beta}\left(\frac{1}{s}\right) = -\frac{\lambda}{s^2}\left\{E_{\alpha,\beta-1}\left(\frac{1}{s}\right) - (\beta - 1)E_{\alpha,\beta}\left(\frac{1}{s}\right)\right\}$$  \hspace{1cm} (24)

and finally we have

$$L\{t W_{\alpha,\beta}(\lambda t)\} = \frac{1}{s^2}\left\{\left(\frac{\alpha}{\lambda} - 1\right)E_{\alpha,\beta}\left(\frac{1}{s}\right) + E_{\alpha,\beta-1}\left(\frac{1}{s}\right)\right\}$$  \hspace{1cm} (25)

In case that the Wright functions are expressed as the Bessel functions (see (2.18)), the Laplace transforms are known for $\beta = 0, 1, 2$ [12]

$$\int_0^\infty e^{-st}W_{1,1}\left(-\frac{\lambda^2 t^2}{4}\right)dt = \int_0^\infty e^{-st}f_0(\lambda t)dt = \frac{1}{\sqrt{s^2 + \lambda^2}}$$

$$\int_0^\infty e^{-st}W_{1,2}\left(-\frac{\lambda^2 t^2}{4}\right)dt = \frac{1}{2} \int_0^\infty e^{-st}f_1(\lambda t)dt = \frac{1}{\sqrt{s^2 + \lambda^2}}$$

$$\int_0^\infty e^{-st}W_{1,3}\left(-\frac{\lambda^2 t^2}{4}\right)dt = \frac{1}{\lambda^2} \int_0^\infty e^{-st}f_2(\lambda t)dt = \frac{1}{s^2 + \lambda^2}$$

$$\frac{1}{\lambda} \left\{\frac{1}{\sqrt{s^2 + \lambda^2}} + \frac{1}{3} \left[\frac{1}{s + \sqrt{s^2 + \lambda^2}}\right]^3\right\}$$  \hspace{1cm} (26)
Figure 4: Derivatives of the Wright functions with respect to parameter $\beta$ as a function of $\alpha$ and $\beta = 1$ for $t = 2.0$.

From (91) it follows that

$$
\int_0^\infty e^{-st}W_{1,\beta+1}(-\lambda t)dt = \frac{1}{\lambda^{\beta/2}} \int_0^\infty e^{-st}t^{-\beta/2}J_\beta(2\sqrt{\lambda t})dt
$$

(27)

and this integral equality is useful to derive explicit forms of the Mittag-Leffler functions. Starting with $\beta = 0$ we have [14]

$$
\int_0^\infty e^{-st}W_{1,1}(-\lambda t)dt = \int_0^\infty e^{-st}J_0(2\sqrt{\lambda t})dt = \frac{1}{s} e^{-\lambda/s} ; \quad \text{Res} > 0
$$

(28)

but from [20]

$$
L\{W_{1,1}(-\lambda t)\} = \frac{1}{s} E_{1,1}(-\frac{\lambda}{s})
$$

(29)

and therefore by comparing the expected result is reached

$$
E_{1,1}(-\frac{\lambda}{s}) = e^{-\lambda/s}
$$

$$
\tau = \frac{\lambda}{s}
$$

$$
E_{1,1}(-\tau) = e^{-\tau}
$$

(30)
Introducing \( \beta = 1 \) into \([27]\) we have \([14]\)

\[
\int_0^\infty e^{-st}W_{1,2}(-\lambda t)\,dt = \frac{1}{\sqrt{\lambda}} \int_0^\infty e^{-st}I_{1/2}(2\sqrt{\lambda t})\,dt =
\]

\[
\sqrt{\frac{\pi}{\lambda}} e^{-\lambda/2}I_{1/2}(\frac{1}{\sqrt{\lambda}}) = \frac{2}{\lambda} e^{-\lambda/2} \sinh(\frac{1}{2\sqrt{\lambda}}) = \frac{1}{\lambda} (1 - e^{-\lambda/\lambda})
\]

\[
\int_0^\infty e^{-st}W_{1,2}(-\lambda t)\,dt = \frac{1}{s} E_{1,2}(-\frac{1}{s})
\]

\[
\tau = \frac{1}{s} \rightarrow E_{1,2}(-\tau) = \frac{1}{\tau} (1 - e^{-\tau})
\]

In general case, the Laplace transform can be expressed in terms of the incomplete gamma function \( \gamma(a,z) \) \([12]\)

\[
\int_0^\infty e^{-st}W_{1,\beta+1}(-\lambda t)\,dt = \frac{1}{\sqrt{s}} \int_0^\infty e^{-st}t^{-\beta/2}J_{\beta}(2\sqrt{\lambda t})\,dt =
\]

\[
e^{\pi \beta \beta^{-1} - \lambda t} e^{-\lambda/\lambda} \gamma(\beta, \frac{1}{\lambda} e^{-\pi \beta}) ; \quad \text{Res} > 0
\]

and therefore

\[
L \{ W_{1,\beta+1}(-\lambda t) \} = \frac{1}{\sqrt{s}} \frac{e^{\pi \beta \beta^{-1}}}{\lambda \Gamma(\beta)} e^{-\lambda/\lambda} \gamma(\beta, \frac{1}{\lambda} e^{-\pi \beta})
\]

\[
E_{1,\beta+1}(-z) = \frac{e^{\pi \beta}}{\Gamma(\beta)^{2}} e^{-z} \gamma(\beta, z e^{-\pi \beta})
\]

For \( \beta = 2 \), we have \( \exp(\pm 2i\pi) \) and

\[
E_{1,3}(-z) = \frac{1}{2} e^{-z} \gamma(2, z) = \frac{1}{2} e^{-z} \int_0^z e^{-t} t \,dt
\]

If \( n \) is positive integer, then

\[
\gamma(n,z) = \Gamma(n) P(n,z)
\]

\[
P(n,z) = 1 - \left( 1 + z + \frac{z^2}{2!} + ... + \frac{z^{n-1}}{(n-1)!} \right) e^{-z}
\]

\[
\gamma(2,z) = 1 - (1 + z) e^{-z}
\]

There are some equivalent expressions in the form given in \([12]\)

\[
E_{1,\beta+1}(-z) = \frac{e^{\pi \beta}}{\Gamma(\beta)^{2}} e^{-z} \gamma(\beta, z e^{-\pi \beta})
\]

\[
E_{1,\beta+1}(-z) = \frac{e^{\pi \beta \beta^{-1} - \lambda t}}{\Gamma(\beta + 1)^{2}(\beta + 1)^{1/2}} M(1 - \beta)/2, \beta/2(z)
\]

\[
E_{1,\beta+1}(-z) = \frac{1}{\Gamma(\beta + 1)^{2}} _1F_1(1; \beta + 1; -z) = \frac{1}{\Gamma(\beta)} \int_0^1 e^{zt}(1 - t)^{\beta-1} \,dt
\]

For \( \beta \) being positive integer \( n \), the last equation links the Mittag-Leffler functions with the Kummer functions (see also Appendix A in \([14]\) for other results).

For positive values of argument \( t \) we have

\[
W_{1,\beta+1}(t) = t^{\beta/2} I_{\beta}(2 \sqrt{t})
\]
and therefore

\[ \int_0^\infty e^{-st}W_{1,\beta+1}(\lambda t)\,dt = \frac{1}{\lambda^{\beta/2}} \int_0^\infty e^{-st}t^{-\beta/2}I_\beta(2\sqrt{\lambda t})\,dt \]  

(38)

For \( \beta = 0 \), this gives [14]

\[ \int_0^\infty e^{-st}W_{1,1}(\lambda t)\,dt = \int_0^\infty e^{-st}I_0(2\sqrt{\lambda t})\,dt = \frac{1}{\lambda} e^{\lambda/s} \]

(39)

but

\[ L\{W_{1,1}(\lambda t)\} = \frac{1}{s} E_{1,1}\left(\frac{1}{s}\right) \]

(40)

and by comparing the expected result is reached

\[ E_{1,1}\left(\frac{1}{s}\right) = e^{\lambda/s} \]

(41)

If \( \beta = 1 \), then [12]

\[ \int_0^\infty e^{-st}W_{1,2}(\lambda t)\,dt = \frac{1}{\sqrt{\lambda}} \int_0^\infty e^{-st}t^{-1/2}I_1(2\sqrt{\lambda t})\,dt = \frac{1}{s}(e^{\lambda/s} - 1) \]

\[ \int_0^\infty e^{-st}W_{1,2}(-\lambda t)\,dt = \frac{1}{s} E_{1,2}\left(\frac{1}{s}\right) \]

(42)

\[ z = \frac{1}{s} \]

\[ E_{1,2}(z) = \frac{1}{2}(e^{z} - 1) \]

In general case [12]

\[ \int_0^\infty e^{-st}W_{1,\beta+1}(\lambda t)\,dt = \frac{1}{\lambda^{\beta/2}} \int_0^\infty e^{-st}t^{-\beta/2}I_\beta(2\sqrt{\lambda t})\,dt = e^{z\beta/2} \gamma(\beta, \frac{1}{s}) \]

\[ L\{W_{1,\beta+1}(\lambda t)\} = \frac{1}{s} E_{1,\beta+1}\left(\frac{1}{s}\right) \]

(43)

\[ z = \frac{1}{s} \]

\[ E_{1,\beta+1}(z) = \frac{e^z}{z\beta} \gamma(\beta, z) \]

where the incomplete gamma function can be expressed in terms of the Kummer function

\[ \gamma(\beta, z) = \frac{z^\beta}{\beta} e^{-z} _1F_1(1; \beta + 1; z) = \frac{z^\beta}{\beta} _1F_1(1; \beta + 1; -z) \]

(44)

From (97) and (98) it follows that

\[ E_{1,\beta+1}(z) = \frac{e^z}{z\beta} \gamma(\beta, z) = \frac{1}{\beta} _1F_1(1; \beta + 1; z) = \frac{e^z}{\beta} _1F_1(1; \beta + 1; -z) \]

(45)

Particular values of the incomplete gamma function of interest are

\[ \gamma(1, z) = (1 - e^{-z}) \]

\[ E_{1,2}(z) = \frac{1}{2}(e^{z} - 1) \]

(46)
In derivation explicit expressions for the Mittag-Leffler functions the recurrence relation of the incomplete gamma function
\[ \gamma(a+1, z) = a \gamma(a, z) - z^a e^{-z} \] (48)
is very useful. For example, for \( n = 1, 2, 3 \) we have
\begin{align*}
\gamma(1, z) &= \frac{1}{2} (1 - e^{-z}) \\
\gamma(2, z) &= \gamma(1, z) - z e^{-z} = \frac{1}{2} (1 - e^{-z}) - z e^{-z} \\
\gamma(3, z) &= \gamma(2, z) - z^2 e^{-z} = \frac{1}{2} (1 - e^{-z}) - 2z e^{-z} \\
\gamma(n+1, z) &= n \gamma(n, z) - z^e z^{-e} ; \quad n = 1, 2, 3, \ldots
\end{align*}
and previously derived formulas in (41) and in (42) are reached. For \( n + 1/2 \), from (48) it follows that
\begin{align*}
\gamma(1/2, z) &= \sqrt{\pi} e^e \text{erf}(\sqrt{z}) \\
\gamma(3/2, z) &= \sqrt{\pi} e^e \text{erf}(\sqrt{z}) - z e^{-z} \\
\gamma(5/2, z) &= 2 \left[ \sqrt{\pi} e^e \text{erf}(\sqrt{z}) - z e^{-z} \right] - z^e e^{-z}
\end{align*}
and immediately this gives by using (43)
\begin{align*}
E_{1,3/2}(z) &= \sqrt{\pi} e^e \text{erf}(\sqrt{z}) \\
E_{1,5/2}(z) &= \sqrt{\pi} e^e \text{erf}(\sqrt{z}) - z e^{-z} \\
E_{1,7/2}(z) &= \sqrt{\pi} e^e \text{erf}(\sqrt{z}) - z e^{-z} - z^e e^{-z}
\end{align*}
The Laplace transforms of the Mainardi functions are
\[ L \left\{ \frac{1}{\alpha} F_\alpha (\frac{1}{z}) \right\} = L \left\{ \frac{1}{\alpha} \gamma(\frac{1}{z^{\alpha}}) \right\} = e^{-\lambda s} \] (52)
\begin{align*}
0 < \alpha < 1 ; \quad \lambda > 0
\end{align*}
and
\[ L \left\{ \frac{1}{\alpha} F_\alpha (\frac{1}{z}) \right\} = L \left\{ \frac{1}{\alpha} \gamma(\frac{1}{z^{\alpha}}) \right\} = \frac{1}{\alpha s^{\alpha}} e^{-\lambda s} \] (53)
\begin{align*}
0 < \alpha < 1 ; \quad \lambda > 0
\end{align*}
or the term of the Wright function
\[ L \left\{ \frac{1}{\alpha} \gamma(\frac{1}{z}) \right\} = L \left\{ \frac{1}{\alpha} \gamma(\frac{1}{z^{\alpha}}) \right\} = \frac{1}{\alpha s^{\alpha}} e^{-\lambda s} \] (54)
The inverse Laplace transforms are known only for \( \alpha = 1/2 \) and \( \alpha = 1/3 \).
\begin{align*}
L^{-1} \left\{ e^{-\lambda s^{1/2}} \right\} &= \frac{\lambda e^{-\lambda^{2}/4}}{2 \sqrt{\pi} \lambda^{3/2}} \\
L^{-1} \left\{ e^{-\lambda s^{1/3}} \right\} &= \frac{\lambda e^{-\lambda^{2}/2}}{2 \sqrt{\pi} \lambda^{3/2}} \\
\frac{1}{\alpha} F_{1/2}(\frac{1}{z}) &= \frac{1}{\alpha} F_{1/2}(\frac{1}{z^{1/2}}) = \frac{1}{\alpha} F_{1/2}(\frac{1}{z^{1/2}}) = \frac{1}{\alpha} F_{1/2}(\frac{1}{z^{1/2}}) = \frac{1}{\alpha} F_{1/2}(\frac{1}{z^{1/2}}) = \frac{1}{\alpha} F_{1/2}(\frac{1}{z^{1/2}}) = \frac{1}{\alpha} F_{1/2}(\frac{1}{z^{1/2}}) = \frac{1}{\alpha} F_{1/2}(\frac{1}{z^{1/2}})
\end{align*}
\[ F_{1/2}(\lambda \tau) = \frac{\lambda^{2}}{2} M_{1/2}(\lambda \tau) = \frac{\lambda^{2} e^{-\lambda^{2}/2}}{2 \sqrt{\pi}} \] (55)
Multiplication by $t$ of the Mainardi function in [55] is equivalent to

$$L\left\{ \frac{1}{t^{\lambda/2}} \right\} = L\left\{ \frac{1}{t^{\lambda/2}} M_{1/2} \left( \frac{1}{t^{\lambda/2}} \right) \right\} = -\frac{d}{dt} \left( e^{-\lambda s^{1/2}} \right) = \frac{1}{2\sqrt{\pi t}} e^{-\lambda s^{1/2}}$$

$$L^{-1} \left\{ \frac{1}{t^{\lambda/2}} e^{-\lambda s^{1/2}} \right\} = \frac{\lambda e^{-\lambda^2/4t}}{2 \sqrt{\pi t}}$$

$$F_{1/2} \left( \frac{1}{t^{\lambda/2}} \right) = \frac{\lambda}{21^{1/2}} M_{1/2} \left( \frac{1}{t^{\lambda/2}} \right) = \frac{\lambda e^{-\lambda^2/4t}}{2 \sqrt{\pi t}}$$

$$F_{1/2} (\lambda \tau) = \frac{\lambda^2}{2 \sqrt{\pi}} M_{1/2} (\lambda \tau) = \frac{\lambda^2 e^{-\lambda^2/4}}{2 \sqrt{\pi}}$$

The same results, but in terms of the Wright functions can be written as

$$L\left\{ 2 W_{-1/2,0} \left( -\frac{1}{t^{\lambda/2}} \right) \right\} = L\left\{ \frac{1}{t^{\lambda/2}} W_{-1/2,1/2} \left( -\frac{1}{t^{\lambda/2}} \right) \right\} = \frac{\lambda}{3^{1/2}} e^{-\lambda s^{1/2}}$$

$$L^{-1} \left\{ \frac{1}{t^{\lambda/2}} e^{-\lambda s^{1/2}} \right\} = \frac{1}{\sqrt{\pi t}} e^{-\lambda^2/4t}$$

$$2 W_{-1/2,0} \left( -\frac{1}{t^{\lambda/2}} \right) = \frac{\lambda}{t^{\lambda/2}} W_{-1/2,1/2} \left( -\frac{1}{t^{\lambda/2}} \right) = \frac{\lambda}{\sqrt{\pi t}} e^{-\lambda^2/4t}$$

$$2 W_{-1/2,0} \left( -\lambda \tau \right) = \lambda \tau W_{-1/2,1/2} \left( -\lambda \tau \right) = \frac{\lambda^3}{\sqrt{\pi}} e^{-\lambda^2 \tau^2/4}$$

In general case of the multiplication by $t^n$, the differentiation of exponential functions can be expressed in terms of the Bessel functions [12]

$$L\left\{ t^n F_{1/2} \left( \frac{1}{t^{\lambda/2}} \right) \right\} = L\left\{ \frac{1}{t^{\lambda/2}} M_{1/2} \left( \frac{1}{t^{\lambda/2}} \right) \right\} = (-1)^n \frac{d^n}{ds^n} \left( e^{-\lambda s^{1/2}} \right) = \left( \frac{\lambda}{2n+1/2} \right) K_{n+1/2} (\lambda s^{1/2})$$

and therefore from (58) we have

$$L\left\{ 2 t^n W_{-1/2,0} \left( -\frac{1}{t^{\lambda/2}} \right) \right\} = L\left\{ \lambda t^{n-1/2} W_{-1/2,1/2} \left( -\frac{1}{t^{\lambda/2}} \right) \right\} = (-1)^n \lambda \frac{d^n}{ds^n} \left( e^{-\lambda s^{1/2}} \right) = \left( \frac{\lambda^{n+1/2} e^{-2n s^{1/2}}}{2n-1/2} \right) K_{n+1/2} (\lambda s^{1/2})$$

If $\alpha = 1/3$, then [3,4]

$$L\left\{ \frac{1}{t} F_{1/3} \left( \frac{1}{t^{1/3}} \right) \right\} = L\left\{ \frac{\lambda}{3^{1/4} 3} M_{1/3} \left( \frac{1}{t^{1/3}} \right) \right\} = e^{-\lambda s^{1/3}}$$

but using [15]

$$L\left\{ \frac{\lambda^{3/2}}{3 \pi^{3/2}} K_{1/3} \left( \frac{2 \lambda^{3/2}}{ \sqrt{27} t} \right) \right\} = e^{-\lambda s^{1/3}}$$

we have

$$3 F_{1/3} \left( \frac{1}{t^{1/3}} \right) = \frac{\lambda}{t^{1/3}} M_{1/3} \left( \frac{1}{t^{1/3}} \right) = \frac{\lambda^{3/2}}{\pi^{1/2} t^{1/2}} K_{1/3} \left( \frac{2 \lambda^{3/2}}{ \sqrt{27} t} \right)$$

The same result is available from [3,4]

$$L\left\{ 3 F_{1/3} \left( \frac{1}{t^{1/3}} \right) \right\} = L\left\{ \frac{\lambda}{t^{1/3}} M_{1/3} \left( \frac{1}{t^{1/3}} \right) \right\} = \frac{\lambda}{2^{2/3}} e^{-\lambda s^{1/3}}$$

and [15]

$$L\left\{ \frac{\lambda^{3/2}}{\pi^{1/2} t^{1/2}} K_{1/3} \left( \frac{2 \lambda^{3/2}}{ \sqrt{27} t} \right) \right\} = \frac{\lambda e^{-\lambda s^{1/3}}}{2^{2/3}}$$
In terms of the Wright functions it can be expressed by
\[
3 W_{-1/3,0}(-\frac{\lambda}{t^{1/3}}) = \frac{\lambda}{t^{1/3}} W_{-1/3,2/3}(-\frac{\lambda}{t^{1/3}}) = \frac{\lambda^{3/2}}{\pi t^{1/2}} K_{1/3}(\frac{2 \lambda^{3/2}}{\sqrt{27 t}}) \tag{65}
\]

6 Functional limits associated with the Wright functions.

In 1969 Lamborn, see see [8-11] and [18-20] proposed the following delta sequence for representation of the shifted Dirac delta function
\[
\delta(x - 1) = \lim_{\nu \to \infty} [\nu J_{\nu}(\nu x)] \tag{66}
\]
As it was demonstrated over the 2000-2008 period by Apelblat [16-18], this delta sequence is useful for evaluation of the asymptotic relations, limits of series, integrals and integral representations of elementary and special functions.

If the Lamborn expression is multiplied by a function \(f(tx)\) and integrated from zero to infinity with respect to variable \(x\) we have
\[
f(t) = \int_0^\infty f(tx) \delta(x - 1) \, dx = \lim_{\nu \to \infty} \left[ \nu \int_0^\infty f(tx) J_{\nu}(\nu x) \, dx \right] \tag{67}
\]
In such way, the function \(f(t)\) is represented by the asymptotic limit of the infinite integral of product of \(f(tx)\) and the Bessel function \(J_{\nu}(\nu x)\). If the right-hand integral in (67) can be evaluated in the closed form, then the limit can be regarded as the generalization of the l’Hospital’s rule.
\[
f(t) = \lim_{\nu \to \infty} \nu \Phi(t, \nu) \tag{68}
\]
For the Wright function treated as the generalized the Bessel function
\[
f(t) = W_{1,\beta+1}(-\frac{t^2}{4}) = \left(\frac{2}{t}\right)^\beta I_\beta(t) \tag{69}
\]
It follows from (4.1) and (69) that
\[
f(t) = \lim_{\nu \to \infty} \nu \int_0^\infty J_{\nu}(\nu x) \left(\frac{2}{t}\right)^x I_\beta(t) \, dx = W_{1,\beta+1}(-\frac{t^2}{4}) \tag{70}
\]
\[
\Phi(t, \nu, \beta) = \int_0^\infty x^{-\beta} J_{\nu}(\nu x) I_\beta(t) \, dx
\]
However, the infinite integral in (70) is known [19]
\[
\Phi(t, \nu, \mu) = \int_0^\infty x^{-\beta} J_{\nu}(\nu x) I_\beta(t) \, dx = \left(\frac{t}{2}\right)^\beta \frac{1}{\nu^2} F_1 \left(\frac{\nu + 1}{2}, \frac{1 - \nu}{2}; \beta + 1; \frac{t^2}{\nu^2}\right) \tag{71}
\]
and therefore the Wright function is represented by the following limit
\[
W_{1,\beta+1}\left(-\frac{t^2}{4}\right) = \lim_{y \to \infty} \left\{ 2F_1\left(\frac{\nu+1}{2}, \frac{1-v}{2}; \beta+1; \frac{y^2}{4}\right) \right\}
\]  \quad (72)

or in the equivalent form
\[
W_{1,\beta+1}(-x) = \lim_{y \to \infty} \left\{ 2F_1\left(\frac{\nu+1}{2}, \frac{1-v}{2}; \beta+1; \frac{x^2}{4}\right) \right\}
\]  \quad (73)

For \( \beta = 0 \), we have
\[
W_{1,1}\left(-\frac{t^2}{4}\right) = J_0(t) = \lim_{y \to \infty} \left\{ 2F_1\left(\frac{\nu+1}{2}, \frac{1-v}{2}; 1; \frac{y^2}{4}\right) \right\}
\]  \quad (74)

and for \( \beta = \pm 1/2 \),
\[
W_{1,3/2}\left(-\frac{t^2}{4}\right) = \sqrt{2} f_{1/2}(t) = \lim_{y \to \infty} \left\{ 2F_1\left(\frac{\nu+1}{2}, \frac{1-v}{2}; \frac{1}{2}; \frac{y^2}{4}\right) \right\} = \frac{2 \sin t}{\sqrt{2}}
\]  \quad (75)

Hypergeometric functions (75) are known in different form [20]
\[
\begin{align*}
2F_1\left(a, 1-a; \frac{3}{2}; (\sin z)^2\right) &= \frac{\sin[(2a-1)z]}{(2a-1)\sin z} \\
2F_1\left(a, 1-a; \frac{1}{2}; (\sin z)^2\right) &= \frac{\cos[(2a-1)z]}{\cos z} \\
\end{align*}
\]  \quad (76)

If the delta sequence in (66) is used together with integral transforms having different kernels \( T \), the we have [17,18]
\[
\begin{align*}
\lim_{v \to \infty} [v \int_0^\infty f(\xi, \lambda) T[f(x), \xi] d\xi] &= 0 \\
\lim_{v \to \infty} [v \int_0^\infty f(x) T[f(\xi, \lambda), x] d\xi] &= T(1, \lambda)
\end{align*}
\]  \quad (77)

In the case of the Laplace transformation, (77) can be written in the following way
\[
\begin{align*}
\int_0^\infty e^{-\xi x} f(x) d\xi &= \frac{v^\nu}{\sqrt{v^2 + \xi^2} [\xi + \sqrt{v^2 + \xi^2}]^v} \\
\lim_{v \to \infty} \left\{ v^{\nu+1} \int_0^\infty \frac{f(\xi, \lambda)}{\sqrt{v^2 + \xi^2} [\xi + \sqrt{v^2 + \xi^2}]^v} d\xi \right\} &= L(1, \lambda)
\end{align*}
\]  \quad (78)

Introducing (20) into (78) we have
\[
\lim_{v \to \infty} \left\{ v^{\nu+1} \int_0^\infty \frac{W_{a,\beta}(\lambda, \xi)}{\sqrt{v^2 + \xi^2} [\xi + \sqrt{v^2 + \xi^2}]^v} d\xi \right\} = \frac{1}{s} E_{a,\beta}(\frac{\lambda}{s}) \bigg|_{s=1} = E_{a,\beta}(\lambda) \quad (79)
\]

The same operation performed with (21) gives
\[
\begin{align*}
\lim_{v \to \infty} \left\{ v^{\nu+1} \int_0^\infty \frac{e^{\rho \xi} W_{a,\beta}(\lambda, \xi)}{\sqrt{v^2 + \xi^2} [\xi + \sqrt{v^2 + \xi^2}]^v} d\xi \right\} &= \frac{1}{1+\rho} E_{a,\beta}(\frac{\lambda}{1+\rho}) \\
\lim_{v \to \infty} \left\{ v^{\nu+1} \int_0^\infty \frac{e^{-\rho \xi} W_{a,\beta}(\lambda, \xi)}{\sqrt{v^2 + \xi^2} [\xi + \sqrt{v^2 + \xi^2}]^v} d\xi \right\} &= \frac{1}{1+\rho} E_{a,\beta}(\frac{\lambda}{1+\rho})
\end{align*}
\]  \quad (80)
and using (25)
\[
\lim_{v \to \infty} \left\{ v^{v+1} \int_0^\infty \frac{\xi W_{\alpha,\beta}(\lambda \xi)}{\sqrt{\xi^2 + \sqrt{\xi^2 + \xi^2}^v}} \, d\xi \right\} = \frac{1}{\pi \lambda} [(\alpha - \beta + 1)E_{\alpha,\beta}(\lambda) + E_{\alpha,\beta-1}(\lambda)]
\]

(81)

The Laplace transform in (43) leads to
\[
\lim_{v \to \infty} \left\{ v^{v+1} \int_0^\infty \frac{W_{1,\beta+1}(\lambda \xi)}{\sqrt{\xi^2 + \xi^2 [\xi + \sqrt{\xi^2 + \xi^2}^v]}} \, d\xi \right\} = e^{-\lambda} \lambda \Gamma(\beta) \gamma(\beta, \lambda) = E_{1,\beta+1}(\lambda)
\]

(82)

For integer values of parameters \( \beta \) in (82), the limits of the Wright functions can be represented by simple expressions. For \( \beta = 1 \), we have
\[
\lim_{v \to \infty} \left\{ v^{v+1} \int_0^\infty \frac{W_{1,1}(\lambda \xi)}{\sqrt{\xi^2 + \xi^2 [\xi + \sqrt{\xi^2 + \xi^2}^v]}} \, d\xi \right\} = e^\lambda
\]

(83)

and
\[
\lim_{v \to \infty} \left\{ v^{v+1} \int_0^\infty \frac{W_{1,1}(-\lambda \xi)}{\sqrt{\xi^2 + \xi^2 [\xi + \sqrt{\xi^2 + \xi^2}^v]}} \, d\xi \right\} = e^{-\lambda}
\]

(84)

For \( \beta = 1 \), the corresponding limits are
\[
\lim_{v \to \infty} \left\{ v^{v+1} \int_0^\infty \frac{W_{1,2}(\lambda \xi)}{\sqrt{\xi^2 + \xi^2 [\xi + \sqrt{\xi^2 + \xi^2}^v]}} \, d\xi \right\} = \frac{2}{\lambda} e^{-\lambda} \sinh(\lambda/2)
\]

(85)

\[
\lim_{v \to \infty} \left\{ v^{v+1} \int_0^\infty \frac{W_{1,2}(-\lambda \xi)}{\sqrt{\xi^2 + \xi^2 [\xi + \sqrt{\xi^2 + \xi^2}^v]}} \, d\xi \right\} = \frac{1}{\lambda} (e^\lambda - 1)
\]

(86)

\[
\lim_{v \to \infty} \left\{ v^{v+1} \int_0^\infty \frac{W_{1,2}(-\lambda^2 \xi^2)}{\sqrt{\xi^2 + \xi^2 [\xi + \sqrt{\xi^2 + \xi^2}^v]}} \, d\xi \right\} = \frac{2}{1 + \sqrt{1 + \lambda^2}}
\]

(87)

Similarly, for \( \beta = 3 \), the functional limit is
\[
\lim_{v \to \infty} \left\{ v^{v+1} \int_0^\infty \frac{W_{1,3}(-\lambda^2 \xi^2)}{\sqrt{\xi^2 + \xi^2 [\xi + \sqrt{\xi^2 + \xi^2}^v]}} \, d\xi \right\} = \frac{1}{\lambda} \left( \frac{\lambda}{1 + \sqrt{1 + \lambda^2}} + \frac{1}{3} \left( \frac{\lambda}{1 + \sqrt{1 + \lambda^2}} \right)^3 \right)
\]

(88)

The Laplace transforms of the Mainardi functions from (58) and (59) are
\[
\lim_{v \to \infty} \left\{ v^{v+1} \int_0^\infty \frac{\xi^n F_{\alpha/2}(\sqrt{\xi^2})}{\sqrt{\xi^2 + \xi^2 [\xi + \sqrt{\xi^2 + \xi^2}^v]}} \, d\xi \right\} = \frac{\lambda^{n+1}}{2^{n-1/2} \sqrt{n}} K_{n-1/2}(\lambda)
\]

(89)
and from (63) we have

\[
\lim_{\nu \to \infty} \left\{ \nu^{\nu + 1} \int_0^\infty \frac{F_{1/3}(\frac{\lambda}{\nu^2})}{\sqrt{\nu^2 + \xi^2} [\xi + \sqrt{\nu^2 + \xi^2}]^\nu} d\xi \right\} = \frac{\lambda}{3} e^{-\lambda}
\]  

(90)

7 Conclusions

Parameters of the Wright functions were treated as variables and derivatives with respect to them were derived and discussed. These derivatives are expressible in terms of infinite power series with quotients of digamma and gamma functions in their coefficients. The functional form of these series resembles those which were derived for the Mittag-Leffler functions. Only in few cases, it was possible to obtain the sums of these series in a closed form. The differentiation operation when the Wright functions are treated as the generalized Bessel functions leads to the Bessel functions and their derivatives with respect to the order. Simple operations with the Laplace transforms of the Wright functions of the first kind give explicit forms of the Mittag-Leffler functions. Applying the shifted Dirac delta function, permits to derive functional limits by using the Laplace transforms of the Wright functions.

Finally we would like to draw attention of the interested readers to the recent papers [23], [24], [25] where some noteworthy applications of the Wright functions of the first and of the second kind are discussed. Relevant applications can be expected in the field of special functions in fractional calculus for which we address the interested readers to its extensive literature, see e.g. the papers [26]-[32].

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Appendix: Differentiation of the Wright functions with respect to parameters versus Bessel functions

Initially, the Wright functions (of the first kind) were treated as the generalized Bessel functions because for parameters $\alpha = 1$ and $\beta + 1$ they become

\[
W_{1,\beta+1}(-\frac{t^2}{4}) = \left( \frac{t}{2} \right)^\beta J_\beta(t)
\]

\[
W_{1,\beta+1}(\frac{t^2}{4}) = \left( \frac{t}{2} \right)^\beta I_\beta(t)
\]  

(91)
Differentiation of the Wright functions in \[91\] with respect to parameter $\beta$ gives

\[
\frac{\partial (W_{1,\beta+1}(t^2 \frac{\pi}{\beta}))}{\partial \beta} = \left( \frac{2}{\beta} \right)^\beta \ln \left( \frac{2}{\beta} \right) I_\beta(t) + \frac{\partial I_\beta(t)}{\partial \beta} \tag{92}
\]

However, differentiation of the Bessel functions with respect to the order can be expressed by \[13\]

\[
\frac{\partial J_\beta(t)}{\partial \beta} = \pi \beta \int_0^{\pi/2} \tan \theta Y_0 \left( t \left[ \sin \theta \right]^2 \right) J_\beta \left( t \left[ \cos \theta \right]^2 \right) d\theta
\]

\[
\frac{\partial J_\beta(t)}{\partial \beta} = -2 \beta \int_0^{\pi/2} \tan \theta K_0 \left( t \left[ \sin \theta \right]^2 \right) J_\beta \left( t \left[ \cos \theta \right]^2 \right) d\theta
\]

\[\Re \beta > 0\]

In particular cases, differentiation with respect to the parameter $\beta$ can be explicitly expressed \[8\]

\[
\left( \frac{\partial J_\beta(t)}{\partial \beta} \right)_{\beta=0} = \frac{\pi}{2} Y_0(t)
\]

\[
\left( \frac{\partial J_\beta(t)}{\partial \beta} \right)_{\beta=0} = -K_0(t)
\]

and therefore

\[
\left( \frac{\partial W_{1,\beta+1}(t^2 \frac{\pi}{\beta})}{\partial \beta} \right)_{\beta=0} = -\ln \left( \frac{2}{\beta} \right) J_0(t) + \frac{\pi}{2} Y_0(t)
\]

\[
\left( \frac{\partial W_{1,\beta+1}(t^2 \frac{\pi}{\beta})}{\partial \beta} \right)_{\beta=0} = -\ln \left( \frac{2}{\beta} \right) J_0(t) - K_0(t)
\]

For $\beta = 1$ we have

\[
\left( \frac{\partial J_\beta(t)}{\partial \beta} \right)_{\beta=1} = \frac{J_0(t)}{t} + \frac{\pi}{2} Y_1(t)
\]

\[
\left( \frac{\partial J_\beta(t)}{\partial \beta} \right)_{\beta=1} = K_1(t) - \frac{J_0(t)}{t}
\]

which gives

\[
\left( \frac{\partial W_{1,\beta+1}(t^2 \frac{\pi}{\beta})}{\partial \beta} \right)_{\beta=1} = \left( \frac{2}{\beta} \right)^\beta \left[ -\ln \left( \frac{2}{\beta} \right) J_1(t) - \frac{J_0(t)}{t} + \frac{\pi}{2} Y_1(t) \right]
\]

\[
\left( \frac{\partial W_{1,\beta+1}(t^2 \frac{\pi}{\beta})}{\partial \beta} \right)_{\beta=1} = \left( \frac{2}{\beta} \right)^\beta \left[ -\ln \left( \frac{2}{\beta} \right) J_1(t) + K_1(t) - \frac{J_0(t)}{t} \right]
\]

Derivatives for $\text{beta} = 1/2$ are

\[
\left( \frac{\partial J_\beta(t)}{\partial \beta} \right)_{\beta=1/2} = \sqrt{\frac{2}{\pi t}} \left[ \sin t \text{Ci}(2t) - \cos t \text{Si}(2t) \right]
\]

\[
\left( \frac{\partial J_\beta(t)}{\partial \beta} \right)_{\beta=1/2} = \sqrt{\frac{1}{2\pi t}} \left[ e^t Ei(-2t) - e^{-t} Ei(2t) \right]
\]

\[
J_{1/2}(t) = \sqrt{\frac{2}{\pi t}} \sin t \quad ; \quad I_{1/2}(t) = \sqrt{\frac{2}{\pi t}} \sinh t
\]

18
which leads to

\[
\left( \frac{\partial W_{1,\beta+1}(-\frac{x^2}{4})}{\partial \beta} \right)_{\beta=1/2} = \frac{2}{\sqrt{\pi t}} \left[ -\ln \left( \frac{1}{2} \right) \sin t + \sin t 
Ci(2t) - \cos t Si(2t) \right]
\]

\[
\left( \frac{\partial W_{1,\beta+1}(\frac{t^2}{4})}{\partial \beta} \right)_{\beta=1/2} = \frac{2}{\sqrt{\pi t}} \left[ -\ln \left( \frac{1}{2} \right) \sinh t + \frac{1}{2} \left( e^t Ei(-2t) - e^{-t} Ei(2t) \right) \right]
\]

(99)

If variable is changed to \( t = 2x^{1/2} \), these results can be equivalently written in different form, for example (95) is

\[
\left( \frac{\partial W_{1,\beta+1}(-x)}{\partial \beta} \right)_{\beta=0} = -\ln \sqrt{x} I_0(2 \sqrt{x}) + \frac{\eta}{2} Y_0(2 \sqrt{x})
\]

\[
\left( \frac{\partial W_{1,\beta+1}(x)}{\partial \beta} \right)_{\beta=0} = -\ln \sqrt{x} I_0(2 \sqrt{x}) - K_0(2 \sqrt{x})
\]

(100)

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