Characterization of C-symmetric Toeplitz operators for a class of conjugations in Hardy spaces

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\textbf{ABSTRACT}

In this article, we introduce a new class of conjugations in the scalar-valued Hardy space $H^2_{\mathbb{C}}(\mathbb{D})$ and provide a characterization of a complex symmetric Toeplitz operator $T_\phi$ with respect to these newly introduced conjugations in various cases. Moreover, we obtain a characterization of a complex symmetric block Toeplitz operator $T_{\Phi_1}$ on the vector-valued Hardy space $H^2_{\mathbb{C}^2}(\mathbb{D})$ with respect to certain conjugations introduced in [Câmara MC, Kliś-Garlicka K, Ptak M. Complex symmetric completions of partial operator matrices. Linear and Multilinear Algebra. 2019; DOI: 10.1080/03081087.2019.1631246 ], [Kang D, Ko E, Lee JE. Remarks on complex symmetric Toeplitz operators. Linear Multilinear Algebra. 2020; DOI: 10.1080/03081087.2020.1842847 ], [Ko E, Lee JE. Remark on complex symmetric operator matrices. Linear Multilinear Algebra. 2019;67(6):1198–1216].

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1. Introduction and preliminaries

Complex symmetric operators on Hilbert spaces are natural generalizations of complex symmetric matrices, and the study of complex symmetric (in short C-symmetric) operators was initiated by Garcia, Putinar and Wogen in [1–4]. A well-known class of operators, namely all normal operators, Hankel operators and truncated Toeplitz operators are included in the class of complex symmetric operators. For more materials on complex symmetric operators and related topics including historical comments we refer the reader to [1–10] and the references cited therein.

The following concept is a straightforward generalization of the conjugate-linear map $z \rightarrow \bar{z}$ on the one-dimensional Hilbert space $\mathbb{C}$.

\textbf{Definition 1.1 ([2])}: A conjugation on a complex Hilbert space $\mathcal{H}$ is a function $C: \mathcal{H} \rightarrow \mathcal{H}$ which satisfies the following three properties:

(i) conjugate linear: $C(\alpha x + \beta y) = \bar{\alpha}Cx + \bar{\beta}Cy$, for all $x, y \in \mathcal{H}$ and $\forall \alpha, \beta \in \mathbb{C}$,
(ii) involutive: $C^2 = I$, 

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In this connection it is worth mentioning that Garcia and Putinar have shown in [1] that for any given conjugation $C$ on a separable Hilbert space $\mathcal{H}$ there exists an orthonormal basis $\{e_n : n \in \mathbb{N}_0\}$ such that $Ce_n = e_n$, where $\mathbb{N}_0$ denotes the set of all non-negative integers. Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$.

**Definition 1.2:** An operator $T \in \mathcal{B}(\mathcal{H})$ is called $C$-symmetric if there exists a conjugation $C$ on $\mathcal{H}$ such that $CTC = T^*$. If $T$ is $C$-symmetric for some conjugation $C$, then $T$ is called complex symmetric.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc in the complex plane and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Let $L^2(\mathbb{T})$ be the Lebesgue (Hilbert) space on $\mathbb{T}$ and let $L^\infty(\mathbb{T})$ be the Banach space of all essentially bounded functions on $\mathbb{T}$. Now it is well known that $\{e_n(z) = z^n : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$, where $\mathbb{Z}$ is the set of all integers. Therefore, if $f \in L^2(\mathbb{T})$, then the function $f$ can be expressed as

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n,$$

where $\hat{f}(n)$ denotes the $n$th Fourier coefficient of $f$ and $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty$. Recall that $\mathcal{H}$-valued Hardy space over the unit disc $\mathbb{D}$ in $\mathbb{C}$ is denoted by $H^2_\mathcal{H}(\mathbb{D})$ and defined by

$$H^2_\mathcal{H}(\mathbb{D}) := \left\{ f(z) = \sum_{n=0}^{\infty} a_nz^n : \|f\|^2_{H^2_\mathcal{H}(\mathbb{D})} := \sum_{n=0}^{\infty} \|a_n\|^2_{\mathcal{H}}, \; z \in \mathbb{D}, \; a_n \in \mathcal{H} \right\}.$$

In this article, we mainly focus on two particular vector-valued Hardy spaces, namely the classical Hardy spaces $H^2_\mathbb{C}(\mathbb{D})$ and $H^2_\mathbb{C}^2(\mathbb{D})$ corresponding to $\mathcal{H} = \mathbb{C}$ and $\mathcal{H} = \mathbb{C}^2$, respectively. For any $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator $T_\phi : H^2_\mathbb{C}(\mathbb{D}) \rightarrow H^2_\mathbb{C}(\mathbb{D})$ is defined by the formula

$$T_\phi(f) = P(\phi f), \; f \in H^2_\mathbb{C}(\mathbb{D}),$$

where $P$ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2_\mathbb{C}(\mathbb{D})$. It is well known that $T_\phi$ is bounded if and only if $\phi \in L^\infty(\mathbb{T})$, and moreover, $\|T_\phi\| = \|\phi\|_{\infty}$. Note that we can also identify the Hardy space $H^2_\mathcal{H}(\mathbb{D})$ as a Hilbert space tensor product between $H^2_\mathbb{C}(\mathbb{D})$ and $\mathcal{H}$, that is $H^2_\mathcal{H}(\mathbb{D}) = H^2_\mathbb{C}(\mathbb{D}) \otimes \mathcal{H}$.

Let $L^2_\mathbb{C}^2(\mathbb{T}) = L^2(\mathbb{T}) \otimes \mathbb{C}^2$, and let $L^\infty_{M_2}(\mathbb{T}) = L^\infty(\mathbb{T}) \otimes M_2$, where $M_2$ is the set of all $2 \times 2$ complex matrices. Now for $\Phi \in L^\infty_{M_2}(\mathbb{T})$, the block Toeplitz operator with symbol $\Phi$ is the operator $T_\Phi$ on the vector-valued Hardy space $H^2_\mathbb{C}^2(\mathbb{D})$ defined by

$$T_\Phi(f) = \tilde{P}(\Phi f), \; f \in H^2_\mathbb{C}^2(\mathbb{D}),$$

where $\tilde{P}$ is the orthogonal projection of $L^2_\mathbb{C}^2(\mathbb{T})$ onto $H^2_\mathbb{C}^2(\mathbb{D})$. In particular, if $\Phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix}$ where $\phi_i \in L^\infty(\mathbb{T})$ for $1 \leq i \leq 4$, then the block Toeplitz operator has the following
representation:

\[ T_\Phi = \begin{bmatrix} T_{\Phi_1} & T_{\Phi_2} \\ T_{\Phi_3} & T_{\Phi_4} \end{bmatrix}. \]

For more materials on block Toeplitz operator and related topics we refer the reader to [11].

The study of complex symmetric operators, Toeplitz operators and block Toeplitz operators provides important connections with various problems in the field of physics and most importantly in the field of mechanics [7, 12–14]. Normal operators are examples of complex symmetric operators and the characterization of normal Toeplitz operators was given by Brown and Halmos in [15]. In other words, they proved that \( T_\phi \) is normal if and only if \( \phi = \alpha + \beta \rho \) for some real-valued function \( \rho \in L^\infty(\mathbb{T}) \) and \( \alpha, \beta \in \mathbb{C} \). Note that, if \( \phi \in L^\infty(\mathbb{T}) \), then \( T_\phi \) may not be a complex symmetric operator. Therefore, in general, it is a difficult problem to describe when a Toeplitz operator is complex symmetric. In this direction, recently, Guo and Zhu [16] have raised the following interesting question:

Characterize a complex symmetric Toeplitz operator on the Hardy space \( H^2_C(\mathbb{D}) \). This question has motivated researchers to identify special classes of conjugations on Hardy spaces. More precisely, for certain conjugations \( C \) with explicit forms, it is an interesting question to characterize \( C \)-symmetric Toeplitz operators. Recently, Ko and Lee in [9] gave a characterization of a complex symmetric Toeplitz operator \( T_\phi \) on \( H^2_C(\mathbb{D}) \) with respect to some special conjugations. More precisely, they considered the family of conjugations \( C_{\mu, \lambda} \) on \( H^2_C(\mathbb{D}) \) defined by

\[ C_{\mu, \lambda}f(z) = \mu \overline{f}(\lambda z) \]

for \( \mu, \lambda \in \mathbb{T} \), and proved the following theorem:

**Theorem 1.3:** If \( \phi \in L^\infty(\mathbb{T}) \), then \( T_\phi \) is \( C_{\mu, \lambda} \)-symmetric if and only if \( \hat{\phi}(-n) = \hat{\phi}(n)\lambda^n \) for all \( n \in \mathbb{Z} \), where \( \hat{\phi}(n) \) is the \( n \)th Fourier coefficient of \( \phi \).

In this context, a recent result by Waleed Noor [17] proves that: if \( T_\phi \) is complex symmetric on \( H^2_C(\mathbb{D}) \) with continuous symbol \( \phi \) on \( \mathbb{T} \), then \( \phi(\mathbb{T}) \) is a nowhere winding curve. We refer the reader to articles [18, 19] for important results in the study of complex symmetric Toeplitz operators on Bergman spaces and Dirichlet spaces. Moreover, very recently, Kang, Ko and Lee have provided a characterization of complex symmetric block Toeplitz operator \( T_\Phi \) with respect to some special conjugations on the vector-valued Hardy space \( H^2_{C^2}(\mathbb{D}) \).

Motivated by all these works (most importantly, [8–10]), our principle aim in this article is to obtain characterizations of complex symmetric Toeplitz operators \( T_\phi \) and \( T_\Phi \) on the Hardy spaces \( H^2_C(\mathbb{D}) \) and \( H^2_{C^2}(\mathbb{D}) \), respectively, with respect to certain new conjugations defined as follows. Let \( p \in \mathbb{N} \) and let \( S_p \) denote the symmetric group defined over a finite set of \( p \) symbols. Thus, \( S_p \) consists of the permutations that can be performed on the \( p \) symbols. For \( \sigma \in S_p \), we denote \( O(\sigma) \) as the order of the permutation \( \sigma \). Now for any \( \sigma \in S_p \) with \( O(\sigma) = 2 \), let \( C_\sigma : H^2_C(\mathbb{D}) \mapsto H^2_C(\mathbb{D}) \) be defined by

\[
C_\sigma \left( \sum_{k=0}^{\infty} \sum_{m=0}^{p-1} a_{m+pk} z^{m+pk} \right) = \sum_{k=0}^{\infty} \sum_{m=0}^{p-1} \sigma(a_{m+pk}) z^{m+pk},
\]

(1)
where for fixed $p$ and $k$, $\sigma$ is a permutation on the set $\{a_{pk}, a_{1+p+k}, \ldots, a_{(p-1)+pk}\}$. Then, it is easy to verify from the definition that $C_\sigma$ is a conjugation on $H^2_C(\mathbb{D})$.

This paper is organized as follows. In Section 2, we provide a characterization of complex symmetric Toeplitz operators $T_\phi$ with respect to a special case of (1), that is, the conjugation $C_{i,j}^p$ for some fixed $p \in \mathbb{N}$ and $i, j \in \mathbb{N}$ such that $i \neq j$ (see Theorem 2.2). Section 3 deals with the characterization of Toeplitz operators $T_\phi$ with respect to a conjugation $C_n$ on $H^2_C(\mathbb{D})$ which is again a special case of (1). In Section 4, we give a characterization of block Toeplitz operators $T_\phi$ with respect to the conjugations $C$ (see (50)) and $\overline{C}$ (see (58)) on $H^2_C(\mathbb{D})$, respectively (see Theorems 4.1, 4.2 and 4.4) that were introduced earlier in [5, 8, 10].

2. Transpositions type of conjugations

In this section, we study the complex symmetry of Toeplitz operators $T_\phi$ on $H^2_C(\mathbb{D})$ with respect to a class of conjugations, which are special cases of (1). For a fixed integer $p \in \mathbb{N}$, let us choose $i, j \in \mathbb{N}$ such that $0 \leq i < j < p$ and define the map $C_{i,j}^p : H^2_C(\mathbb{D}) \mapsto H^2_C(\mathbb{D})$,

$$C_{i,j}^p \left( \sum_{k=0}^{\infty} a_k z^k \right) \mapsto \sum_{k=0}^{\infty} \tilde{a}_{j+p} z^{i+p+k} + \sum_{k=0}^{\infty} \tilde{a}_{i+p} z^{j+p+k} + \sum_{k=0}^{p-1} \sum_{m=0}^{\infty} \tilde{a}_{m+p} z^{m+p+k}. \quad (2)$$

It follows from the definition above that $C_{i,j}^p$ is a conjugation on $H^2_C(\mathbb{D})$. The following lemma shows that this class of conjugations are unitarily equivalent to each other.

**Lemma 2.1:** The conjugation $C_{i,j}^p$ is unitarily equivalent to $C_{i',j'}^p$.

**Proof:** For $k \in \mathbb{N}$, let $U : H^2_C(\mathbb{D}) \rightarrow H^2_C(\mathbb{D})$ be a unitary defined by

$$U(z^{i+p}) = z^{i'} + p; \quad U(z^{j+p}) = z^{j'} + p; \quad U(z^{i'+p}) = z^{i'+p}; \quad U(z^{j'+p}) = z^{j'+p},$$

and for $m \neq i, j, i', j'$, $U(z^{m+p}) = z^{m+p}$. Let us consider the case where $\{i, j\} \cap \{i', j'\} = \emptyset$. Then

$$C_{i,j}^p UC_{i',j'}^p (z^{i+p}) = C_{i,j}^p U(z^{i+p}) = C_{i,j}^p (z^{i'+p}) = z^{i'+p} = U(z^{i+p}).$$

Similarly,

$$C_{i,j}^p UC_{i',j'}^p (z^{j+p}) = C_{i,j}^p U(z^{j+p}) = C_{i,j}^p (z^{i'+p}) = z^{i'+p} = U(z^{j+p}),$$

$$C_{i,j}^p UC_{i',j'}^p (z^{i'+p}) = C_{i,j}^p U(z^{i'+p}) = C_{i,j}^p (z^{i'+p}) = z^{i'+p} = U(z^{i'+p})$$

and

$$C_{i,j}^p UC_{i',j'}^p (z^{j'+p}) = C_{i,j}^p U(z^{j'+p}) = C_{i,j}^p (z^{j'+p}) = z^{j'+p} = U(z^{j'+p}).$$

On the other hand for $m \neq i, j, i', j'$, we have

$$C_{i,j}^p UC_{i',j'}^p (z^{m+p}) = C_{i,j}^p U(z^{m+p}) = C_{i,j}^p (z^{m+p}) = z^{m+p} = U(z^{m+p}).$$
Combining the above relations we get $U = C_{i}^{j} U C_{p}^{j} C_{i}^{j}$, which further implies $U^{*} C_{p}^{j} U = C_{p}^{j}$. The remaining cases $\{i = i'; j \neq j\}$ and $\{i \neq i'; j = j\}$ can be proved in a similar manner.

Our next aim is to identify necessary and sufficient conditions on the symbol $\phi \in L^{\infty}(\mathbb{T})$, for which the Toeplitz operator $T_{\phi}$ is complex symmetric with respect to the conjugation $C_{p}^{i} C_{p}^{j}$. Let $\phi(z) = \sum_{k=-\infty}^{\infty} \hat{\phi}(k) z^{k} \in L^{\infty}(\mathbb{T})$ be the Fourier expansion of $\phi(z)$. Let us begin by assuming that $T_{\phi}$ is complex symmetric with respect to the conjugation $C_{p}^{i} C_{p}^{j}$, that is,

$$T_{\phi} C_{p}^{i} C_{p}^{j} = C_{p}^{i} C_{p}^{j} T_{\phi}.$$  

(3)

Now, it is well known that the set $\{z^{r} p^{k} : k \geq 0, 0 \leq r \leq p - 1\}$ forms an orthonormal basis of $H_{o}^{2}(\mathbb{D})$ and therefore, corresponding to $r = j$ we have

$$T_{\phi} C_{p}^{i} j (z^{r} p^{k}) = T_{\phi} (z^{i} p^{k}) = \sum_{m=0}^{\infty} \hat{\phi} (m - i p - k) z^{m}$$

and

$$C_{p}^{i} j (z^{r} p^{k}) = \sum_{m=0}^{\infty} \hat{\phi} (-(m - j - p k)) z^{m}$$

$$= \sum_{m=0}^{\infty} \hat{\phi} (-i m - j p - k) z^{m} + \sum_{m=0}^{\infty} \hat{\phi} (-j m - k) z^{j m} + \sum_{m=0}^{\infty} \sum_{t=0}^{p-1} \hat{\phi} (-t m - j p - k) z^{j t m}.$$  

(5)

Thus, by substituting $m - k = l \in \mathbb{Z}$ in (4) and (5), and using (3) we get

$$\hat{\phi}(pm) = \hat{\phi}(-pm), \quad \hat{\phi}(j i - p) = \hat{\phi}(j - i - p), \quad \hat{\phi}(t - i p) = \hat{\phi}(j - i - p).$$  

(6)

Moreover, for $r = i$, by repeating the above calculations and using the relation $T_{\phi} C_{p}^{i} (z^{r} p^{k}) = C_{p}^{i} j (z^{i} p^{k})$ we conclude for $l \in \mathbb{Z}$ that

$$\hat{\phi}(i - j + p l) = \hat{\phi}(i - j - p l), \quad \hat{\phi}(p l) = \hat{\phi}(-p l), \quad \hat{\phi}(j - t + p l) = \hat{\phi}(j - t - p l).$$

(7)

Now for $0 \leq r \leq p - 1$ such that $r \neq i, j$ we have

$$T_{\phi} C_{p}^{i} j (z^{r} p^{k}) = \sum_{m=0}^{\infty} \hat{\phi} (m - r p - k) z^{m}$$

(8)
and

\[
\begin{align*}
C_{lp} T_\phi(z^{r+pk}) &= C_{lp} \left( \sum_{m=0}^{\infty} \hat{\phi} \left( -(m - r - pk) \right) z^m \right) \\
&= \sum_{m=0}^{\infty} \hat{\phi} \left( -(j + pm - r - pk) \right) z^{j+pm} \\
&+ \sum_{m=0}^{\infty} \hat{\phi} \left( -(i + pm - r - pk) \right) z^{i+pm} \\
&+ \sum_{m=0}^{\infty} \hat{\phi} \left( -(r + pm - r - pk) \right) z^{r+pm} \\
&+ \sum_{m=0}^{p-1} \sum_{s=0}^{p-1} \hat{\phi} \left( -(s + pm - r - pk) \right) z^{s+pm}.
\end{align*}
\]

Finally, by using (3) we obtain the following conditions:

\[
\begin{align*}
\hat{\phi}(i - r + pl) &= \hat{\phi}(r - j - pl), \\
\hat{\phi}(j - r + pl) &= \hat{\phi}(r - i - pl), \\
\hat{\phi}(s - r + pl) &= \hat{\phi}(r - s - pl), \\
\hat{\phi}(pl) &= \hat{\phi}(-pl),
\end{align*}
\]

(8)

where \(m - k = l \in \mathbb{Z}\) and \(0 \leq s \leq p - 1\) such that \(s \neq i, j, r\). All the conditions obtained in (8) are enlisted in the following table:

| Condition | Description |
|-----------|-------------|
| \(\phi(pl) = \phi(-pl)\) | \(0 \leq a \leq p - 1\) such that \(a \neq i, j\), and \(0 \leq b \leq p - 1\) such that \(b \neq a, i, j\), then |
| \(\phi(j - i + pl) = \phi(j - i - pl), \quad \phi(i - j + pl) = \phi(i - j - pl),\) | |
| \(\phi(a - i + pl) = \phi(a - i - pl), \quad \phi(a - j + pl) = \phi(a - j - pl),\) | |
| \(\phi(i - a + pl) = \phi(a - j - pl), \quad \phi(j - a + pl) = \phi(a - i - pl),\) | |
| \(\phi(b - a + pl) = \phi(a - b - pl).\) | |

(9)

Our next aim is to simplify the above relations between the Fourier coefficients of \(\phi\) obtained in (9) by assuming some more restrictions on \(i, j\) and \(p\).

**Case I:** First we assume that \(p\) is even and \(|i - j| = \frac{p}{2}\). That is, \(j - i = \frac{p}{2}\) and \(i - j = -\frac{p}{2}\). Therefore, \(j = \frac{p}{2} + i\) and hence the above conditions mentioned in (9) become

| Condition | Description |
|-----------|-------------|
| \(\hat{\phi}(\frac{p}{2} + pl) = \hat{\phi}(\frac{p}{2} - pl), \quad \hat{\phi}(-\frac{p}{2} + pl) = \hat{\phi}(-\frac{p}{2} - pl),\) | \(0 \leq a \leq p - 1\) such that \(a \neq i, j\), and \(0 \leq b \leq p - 1\) such that \(b \neq a, i, j\), then |
| \(\hat{\phi}(a - i + pl) = \hat{\phi}(\frac{p}{2} + i - a - pl), \quad \hat{\phi}(a - \frac{p}{2} - i + pl) = \hat{\phi}(i - a - pl),\) | |
| \(\hat{\phi}(i - a + pl) = \hat{\phi}(a - \frac{p}{2} - i - pl), \quad \hat{\phi}(\frac{p}{2} + i - a + pl) = \hat{\phi}(a - i - pl),\) | |
| \(\hat{\phi}(b - a + pl) = \hat{\phi}(a - b - pl).\) | |

(10)
The following two tables consist of different values of $a$ that are essential in the sequel:

| $a$    | 0     | 1     | $\cdots$ | $i-1$ | $i+1$ | $i+2$ | $\cdots$ | $\frac{p}{2} + i - 1$ |
|--------|-------|-------|----------|------|------|------|----------|----------------------|
| $a-i$  | $-i$  | $1-i$ | $\cdots$ | $-1$ | 1    | 2    | $\cdots$ | $\frac{p}{2} - 1$    |
| $i-a$  | $i$   | $i-1$ | $\cdots$ | 1    | $-1$ | $-2$ | $\cdots$ | $1 - \frac{p}{2}$    |
| $\frac{p}{2} + i - a$ | $\frac{p}{2} + i$ | $\frac{p}{2} + i - 1$ | $\cdots$ | $\frac{p}{2} + 1$ | $\frac{p}{2} - 1$ | $\frac{p}{2} - 2$ | $\cdots$ | 1                     |
| $a - \frac{p}{2} - i$ | $-\frac{p}{2} - i$ | $1 - \frac{p}{2} - i$ | $\cdots$ | $-\frac{p}{2} - 1$ | $1 - \frac{p}{2}$ | $2 - \frac{p}{2}$ | $\cdots$ | $-1$                  |

(11)

Sub-case I: Suppose $\frac{p}{2}$ is even. Then, we have the following $\frac{p}{4}$ many pairs:

$$(1, \frac{p}{2} - 1), (2, \frac{p}{2} - 2), \ldots, (\frac{p}{4} - 1, \frac{p}{4} + 1), (\frac{p}{4}, \frac{p}{4}).$$

(13)

For proceeding further, we need the following conditions as mentioned in the fourth row of the table (10), that is for $l \in \mathbb{Z}$, and for $0 \leq a \leq p - 1$ such that $a \neq i, \frac{p}{2} + i$, we have

$$\hat{\phi}(a - i + pl) = \hat{\phi}\left(\frac{p}{2} + i - a - pl\right), \quad \hat{\phi}(a - \frac{p}{2} - i + pl) = \hat{\phi}(i - a - pl).$$

(14)

Therefore, using the column corresponding to $a = i + 1$ in (11) and using (14) we get

$$\hat{\phi}(1 + pl) = \hat{\phi}\left(\frac{p}{2} - 1 - pl\right) = \hat{\phi}\left(-\frac{p}{2} + 1 - p(l - 1)\right) = \hat{\phi}(1 + p(l - 1)) \quad \forall \ l \in \mathbb{Z},$$

where, in the last equality, we have used the column corresponding to $a = \frac{p}{2} + i + 1$ in (12). Again, using (14) and using the columns corresponding to $a = i + 2$ and $a = \frac{p}{2} + i + 2$ in (11) and (12), respectively, we obtain

$$\hat{\phi}(2 + pl) = \hat{\phi}\left(\frac{p}{2} - 2 - pl\right) = \hat{\phi}\left(-\frac{p}{2} + 2 - p(l - 1)\right) = \hat{\phi}(2 + p(l - 1)) \quad \forall \ l \in \mathbb{Z}.$$  

Therefore, by repeating the same argument as above and using (11), (12) and (14) we conclude

$$\hat{\phi}(r + pl) = \hat{\phi}(r + p(l - 1)) \quad \forall \ l \in \mathbb{Z} \quad \text{and} \quad 1 \leq r \leq \frac{p}{4}.$$  

(15)

Since $\phi \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$ and hence $\sum_{k=-\infty}^{\infty} |\hat{\phi}(k)|^2 < \infty$, then Equation (15) yields

$$\hat{\phi}(r + pl) = 0 \quad \forall \ l \in \mathbb{Z} \quad \text{and} \quad 1 \leq r \leq \frac{p}{4}.$$  

(16)

By observing the symmetricity of the pair in (18) and using Equations (14) and (16) we conclude

$$\hat{\phi}\left(\frac{p}{4} + s + pl\right) = \hat{\phi}\left(\frac{p}{4} - s - pl\right) = 0 \quad \forall \ l \in \mathbb{Z} \quad \text{and} \quad 1 \leq s \leq \frac{p}{4} - 1.$$  

(17)
Furthermore, using the third row of the table (10) we get
\[ \hat{\phi} \left( \frac{P}{2} + pl \right) = \hat{\phi} \left( \frac{P}{2} - pl \right) = \hat{\phi} \left( \frac{P}{2} - p - p(l - 1) \right) \]
\[ = \hat{\phi} \left( -\frac{P}{2} - p(l - 1) \right) = \hat{\phi} \left( -\frac{P}{2} + p(l - 1) \right) = \hat{\phi} \left( \frac{P}{2} + p(l - 2) \right) \quad \forall \ l \in \mathbb{Z}, \]
and hence by the similar argument as in (16) we conclude
\[ \hat{\phi} \left( \frac{P}{2} + pl \right) = 0 \quad \forall \ l \in \mathbb{Z}. \quad (18) \]

Next, we need the following conditions as mentioned in the fifth row of the table (10), that is for \( l \in \mathbb{Z} \), and for \( 0 \leq a \leq p - 1 \) such that \( a \neq i, \frac{p}{2} + i \), we have
\[ \hat{\phi} \left( i - a + pl \right) = \hat{\phi} \left( a - \frac{P}{2} + i - pl \right), \quad \hat{\phi} \left( \frac{P}{2} + i - a + pl \right) = \hat{\phi} \left( a - i - pl \right). \quad (19) \]

Again, by using the columns corresponding to \( a = i + 1, i + 2 \) in (11) and using (19) we get
\[ \hat{\phi} \left( -1 + pl \right) = \hat{\phi} \left( -\frac{P}{2} + 1 - pl \right) = \hat{\phi} \left( \frac{P}{2} + 1 - p(l + 1) \right) = \hat{\phi} \left( -1 + p(l + 1) \right) \quad \forall \ l \in \mathbb{Z}, \]
\[ \hat{\phi} \left( -2 + pl \right) = \hat{\phi} \left( -\frac{P}{2} + 2 - pl \right) = \hat{\phi} \left( \frac{P}{2} + 2 - p(l + 1) \right) = \hat{\phi} \left( -2 + p(l + 1) \right) \quad \forall \ l \in \mathbb{Z}. \]

Therefore, by repeating the same argument as above and using (11), (12) and (19) we conclude
\[ \hat{\phi} \left( -r + pl \right) = \hat{\phi} \left( -r + p(l + 1) \right) \quad \forall \ l \in \mathbb{Z} \text{ and } 1 \leq r \leq \frac{P}{4}. \]

Thus, by the similar argument as in (16) we conclude
\[ \hat{\phi} \left( -r + pl \right) = 0 \quad \forall \ l \in \mathbb{Z} \text{ and } 1 \leq r \leq \frac{P}{4}. \quad (20) \]

Consequently, by using Equations (19) and (20) we get
\[ \hat{\phi} \left( \frac{P}{2} + r + pl \right) = \hat{\phi} \left( -r - pl \right) = 0 \quad \forall \ l \in \mathbb{Z} \text{ and } 1 \leq r \leq \frac{P}{4}. \quad (21) \]

Furthermore, using (20) we also conclude
\[ \hat{\phi} \left( \frac{P}{2} + \frac{P}{4} + s + pl \right) = \hat{\phi} \left( -(\frac{P}{4} - s) + p(l + 1) \right) = 0 \quad \forall \ l \in \mathbb{Z} \text{ and } 1 \leq s \leq \frac{P}{4} - 1. \quad (22) \]

Thus, by combining all the conditions obtained in (16), (17), (18), (21) and (22) we get
\[ \hat{\phi}(r + pl) = 0 \quad \forall \ l \in \mathbb{Z} \text{ and } 1 \leq r \leq p - 1. \]

Sub-case II: Suppose \( \frac{p}{2} \) is odd. Then, we have the following \( \frac{p/2-1}{2} \) many pairs:
\[ \left( 1, \frac{p}{2} - 1 \right), \left( 2, \frac{p}{2} - 2 \right), \ldots, \left( \frac{p/2-1}{2}, \frac{p/2-1}{2} + 1 \right). \]
Therefore, by proceeding with the similar arguments as in Sub-case I we conclude,

\[ \hat{\phi}(r + pl) = 0 \quad \forall l \in \mathbb{Z} \quad \text{and} \quad 1 \leq r \leq p - 1. \]

**Case II:** Here we assume that \( p = mq + 1 \) for some natural number \( m \geq 2, i = q - 1, \) and \( j = p - 1. \) Now by rewriting the relations obtained in the table (9), we have the following.

- **Interchange Rule:** (Intc)
  \[
  \hat{\phi}(c + pl) = \hat{\phi}(d - pl) \quad \text{for} \quad \begin{cases} |c|, |d| \in \{1, \ldots, p - 1\} \setminus \{p - q\} \quad \text{such that} |c + d| = p - q, \\ c = d = \pm (p - q). \end{cases}
  \]
  (23)

- **Sign Rule:** (Sgn)
  \[
  \hat{\phi}(c + pl) = \hat{\phi}(-c - pl) \quad \text{for} \quad \begin{cases} c \in \{1, \ldots, p - 3\} \quad \text{if} \quad q = 1, \\ c \in \{1, \ldots, p - 2\} \quad \text{if} \quad q \geq 2. \end{cases}
  \]
  (24)

Our next aim is to show that \( \hat{\phi}(k + pl) = 0 \) for all \( k \in \{1, 2, \ldots, p - 1\} \) and for all \( l \in \mathbb{Z}. \) Suppose \( q = 1 \) and \( p = m + 1 \) is odd (that is, \( m \) is even). Now for \( m = 2, \) we have

\[
\hat{\phi}(1 + pl) = \hat{\phi}(1 + 3l) \overset{(\text{Intc})}{=} \hat{\phi}(1 - 3l) \overset{(\text{Adj})}{=} \hat{\phi}(-2 - 3(l - 1)) \overset{(\text{Intc})}{=} \hat{\phi}(3 + 3(l - 1)) \quad \forall l \in \mathbb{Z},
\]
(25)

where we have used Equation (23) and the symbol (Adj) stands for the adjustment of the Fourier coefficients. Furthermore, by using Equation (23) we have for \( m > 2 \) (\( m \) is even) that

\[
\hat{\phi}(1 + pl) \overset{(\text{Intc})}{=} \hat{\phi}((p - 2) - pl) \overset{(\text{Adj})}{=} \hat{\phi}(-2 - p(l - 1)) \overset{(\text{Intc})}{=} \hat{\phi}(3 - p)
\]

\[+ p(l - 1) \overset{(\text{Adj})}{=} \hat{\phi}(3 + p(l - 2))
\]

\[= \cdots \overset{(\text{Adj})}{=} \hat{\phi}((p - 2) + p(l - (p - 3))) \overset{(\text{Intc})}{=} \hat{\phi}(1 - p(l - (p - 3)))
\]

\[\overset{(\text{Adj})}{=} \hat{\phi}((-1 - p) - p(l - (p - 2)))
\]

\[\overset{(\text{Intc})}{=} \hat{\phi}((1 - p) + p(l - (p - 2))) \overset{(\text{Adj})}{=} \hat{\phi}(1 + p(l - (p - 1))) \quad \forall l \in \mathbb{Z}. \quad (26)
\]

Now if \( q = 1 \) and \( p = m + 1 \) is even (that is, \( m \) is odd), then again by using Equations (23) and (24) we get

\[
\hat{\phi}(1 + pl) \overset{(\text{Intc})}{=} \hat{\phi}((p - 2) - pl) \overset{(\text{Adj})}{=} \hat{\phi}(-2 - p(l - 1))
\]

\[= \cdots \overset{(\text{Adj})}{=} \hat{\phi}((-p - 2) - p(l - (p - 3)))
\]

\[\overset{(\text{Intc})}{=} \hat{\phi}(-1 + p(l - (p - 3))) \overset{(\text{Adj})}{=} \hat{\phi}((p - 1) + p(l - (p - 2)))
\]
\[
\begin{align*}
\text{(Intc)} & \quad \hat{\phi}\left((p - 1) - p(l - (p - 2))\right) \quad \text{(Adj)} \quad \hat{\phi}\left(-1 - p(l - (p - 1))\right) \\
\text{(Sgn)} & \quad \hat{\phi}(1 + p(l - (p - 1))) \quad \forall \quad l \in \mathbb{Z}.
\end{align*}
\]

Therefore, by the similar argument as in (16), Equations (25), (26) and (27) yield that

\[
\hat{\phi}(k + pl) = 0 \quad \forall \quad l \in \mathbb{Z} \quad \text{and} \quad k \in \{1, 2, \ldots, p - 1\}.
\]  

Next we consider \( q \geq 2 \), then by using (23) we get

\[
\hat{\phi}(1 + pl) \overset{\text{(Intc)}}{=} \hat{\phi}\left((p - q) - 1 - pl\right) \overset{\text{(Adj)}}{=} \hat{\phi}\left(-(q + 1) - p(l - 1)\right) \\
\overset{\text{(Intc)}}{=} \hat{\phi}\left(-p + (2q + 1) + p(l - 1)\right) \overset{\text{(Adj)}}{=} \hat{\phi}\left((2q + 1) + p(l - 2)\right) \\
\vdots \\
\overset{\text{(Adj)}}{=} \hat{\phi}\left((-1)^{(m - 1)}((m - 1)q + 1) + (-1)^{(m - 1)}p(1 - (m - 1))\right) \quad \forall \quad l \in \mathbb{Z}.
\]

To proceed further, let us denote \( l^{(k)} = l - k(m - 1), \quad k \in \mathbb{N}, \quad l \in \mathbb{Z}, \quad m \geq 2.\) 

Sub-case I: Suppose \( p = mq + 1 \) such that \( m \) is odd. Then, Equation (29) yields

\[
\hat{\phi}(1 + pl) = \hat{\phi}\left(((m - 1)q + 1) + p(1 - (m - 1))\right) = \hat{\phi}\left((p - q) + pl^{(1)}\right) \overset{\text{(Intc)}}{=} \hat{\phi}\left((p - q) - pl^{(1)}\right) \\
\overset{\text{(Adj)}}{=} \hat{\phi}\left(-q - pl^{(1)} - 1\right) \overset{\text{(Intc)}}{=} \hat{\phi}\left((2q - p) + p(l^{(1)} - 1)\right) \overset{\text{(Adj)}}{=} \hat{\phi}\left(2q + p(l^{(1)} - 2)\right) \\
\overset{\text{(Intc)} \ldots \text{(Adj)}}{=} \hat{\phi}\left(-mq - p(l^{(1)} - m)\right) = \hat{\phi}\left(-(p - 1) - p(l^{(1)} - m)\right) \\
\overset{\text{(Adj)}}{=} \hat{\phi}\left(1 - p \left(l^{(1)} - (m - 1)\right)\right) = \hat{\phi}\left(1 - pl^{(2)}\right) \quad \forall \quad l \in \mathbb{Z},
\]

where we have used Equation (23). Moreover, again by using Equation (23) we get

\[
\hat{\phi}\left(1 - pl^{(2)}\right) \overset{\text{(Intc)}}{=} \hat{\phi}\left((q - p) - 1 + pl^{(2)}\right) \overset{\text{(Adj)}}{=} \hat{\phi}\left((q - 1) + p(l^{(2)} - 1)\right) \\
\overset{\text{(Intc)}}{=} \hat{\phi}\left(p - (2q - 1) - p(l^{(2)} - 1)\right) \overset{\text{(Adj)}}{=} \hat{\phi}\left(-(2q - 1) - p(l^{(2)} - 2)\right) \\
\overset{\text{(Intc)} \ldots \text{(Adj)}}{=} \hat{\phi}\left((-1)^{(m - 1)}(mq - 1) + (-1)^{m - 1}p(l^{(2)} - m)\right) = \hat{\phi}\left(p - 2 + p(l^{(2)} - m)\right) \\
\overset{\text{(Adj)}}{=} \hat{\phi}\left(-2 + p \left(l^{(2)} - (m - 1)\right)\right) = \hat{\phi}(-2 + pl^{(3)}) \quad \forall \quad l \in \mathbb{Z}.
\]
Therefore, by repeating the similar arguments as in (30) and (31) we conclude

\[
\hat{\phi}(1 + pl) = \hat{\phi}(1 - pl^{(2)}) = \hat{\phi}(-2 + pl^{(3)}) = \cdots = \begin{cases} 
\hat{\phi}(-(q - 1) + pl^{(q)}) & \text{if } q \text{ is odd,} \\
\hat{\phi}((q - 1) - pl^{(q)}) & \text{if } q \text{ is even,}
\end{cases} 
\]

(Adj) \begin{align*}
\hat{\phi}((p - q) + 1 + p (l^{(q)} - 1)) & \text{if } q \text{ is odd,} \\
\hat{\phi}((q - p) - 1 - p (l^{(q)} - 1)) & \text{if } q \text{ is even,}
\end{align*}

(Intc) \begin{align*}
\hat{\phi}(-1 - p(l^{(q)} - 1)) & \text{if } q \text{ is odd,} \\
\hat{\phi}(1 + p(l^{(q)} - 1)) & \text{if } q \text{ is even,}
\end{align*}

(Sgn) \phi \left(1 + p(l^{(q)} - 1)\right) \forall l \in \mathbb{Z}. \quad (32)

Therefore, by combining all the conditions obtained in (30), (31) and (32) we have the following chain of relations:

\[
\hat{\phi}(1 + pl) = \hat{\phi}(-(q + 1) - p(l - 1)) = \hat{\phi}((2q + 1) + p(l - 2)) = \cdots = \hat{\phi} ((p - q) + pl^{(1)}) \\
= \hat{\phi}(-q + p(l^{(1)} - 1)) = \hat{\phi}(2q + p(l^{(1)} - 2)) = \cdots = \hat{\phi}(-mq - p(l^{(1)} - m)) \\
= \hat{\phi} (1 - pl^{(2)}) = \hat{\phi}(-2 + pl^{(3)}) = \cdots = \begin{cases} 
\hat{\phi}(-(q - 1) + pl^{(q)}) & \text{if } q \text{ is odd,} \\
\hat{\phi}((q - 1) - pl^{(q)}) & \text{if } q \text{ is even,}
\end{cases} 
\]

\[= \hat{\phi} (1 + p(l^{(q)} - 1)) \forall l \in \mathbb{Z}. \quad (33)\]

**Sub-case II:** Suppose \( p = mq + 1 \) such that \( m \) is even. Then, by applying (23), Equation (29) becomes

\[
\hat{\phi}(1 + pl) = \hat{\phi}(-(m - 1)q + 1) - p(l - (m - 1))) = \hat{\phi}(-(p - q) - pl^{(1)}) \\
\overset{\text{(Intc)}}{=} \hat{\phi}(-(p - q) + pl^{(1)}) \\
\overset{\text{(Adj)}}{=} \hat{\phi}(q + p(l^{(1)} - 1)) \overset{\text{(Intc)}}{=} \hat{\phi}((p - 2q) - p(l^{(1)} - 1)) \overset{\text{(Adj)}}{=} \hat{\phi}(-2q - p(l^{(1)} - 2)) \\
\overset{\text{(Intc)}}{=} \cdots \overset{\text{(Adj)}}{=} \hat{\phi}(-mq - p(l^{(1)} - m)) \overset{\text{(Intc)}}{=} \hat{\phi}(-(p - 1) - p(l^{(1)} - m) \\
\overset{\text{(Adj)}}{=} \hat{\phi} (1 - p(l^{(1)} - (m - 1))) \\
= \hat{\phi} (1 - pl^{(2)}) \overset{\text{(Intc)}}{=} \hat{\phi} ((q - p) - 1 + pl^{(2)}) \overset{\text{(Adj)}}{=} \hat{\phi} (q - 1 + p(l^{(2)} - 1)) \\
\overset{\text{(Intc)}}{=} \hat{\phi} (p - (2q - 1) - p(l^{(2)} - 1)) \overset{\text{(Adj)}}{=} \hat{\phi} (-2q - 1 - p(l^{(2)} - 2)) \\
\overset{\text{(Intc)}}{=} \cdots \overset{\text{(Adj)}}{=} \hat{\phi} ((-1)^{m-1}(mq - 1) + (-1)^{m-1}p(l^{(2)} - m)) \\
= \hat{\phi} (2 - p - p(l^{(2)} - m)) \\
\overset{\text{(Adj)}}{=} \hat{\phi} (2 - p(l^{(2)} - (m - 1))) = \hat{\phi}(2 - pl^{(3)}) = \cdots = \hat{\phi} (q - 1 - pl^{(q)}) \\
\overset{\text{(Adj)}}{=} \hat{\phi} ((q - p) - 1 - p(l^{(q)} - 1)) \overset{\text{(Intc)}}{=} \hat{\phi} (1 + p(l^{(q)} - 1)) \forall l \in \mathbb{Z}. \quad (34)
Therefore, by the similar argument as in (16) and by applying Sign Rule, Equations (33) and (34) yield that

\[
\forall l \in \mathbb{Z}, \quad \begin{cases}
\hat{\phi}(\pm 1 + pl) = \hat{\phi}(\pm 2 + pl) = \cdots = \hat{\phi}(\pm (q - 1) + pl) = 0, \\
\hat{\phi}(\pm q + pl) = \hat{\phi}(\pm 2q + pl) = \cdots = \hat{\phi}(\pm mq + pl) = 0, \\
\hat{\phi}(\pm (q + 1) + pl) = \hat{\phi}(\pm (2q + 1) + pl) = \cdots = \hat{\phi}(\pm ((m - 1)q + 1) + pl) = 0.
\end{cases}
\] (35)

Now for the remaining terms, by using (23) and (35) we have

\[
\forall l \in \mathbb{Z}, \quad \begin{cases}
\hat{\phi}(q + 2 + pl) & \overset{(Adj)}{=} \hat{\phi}((q - p + 2) + p(l + 1)) & \overset{(Intc)}{=} \hat{\phi}(-2 - p(l + 1)) = 0, \\
\vdots \\
\hat{\phi}(q + (q - 1) + pl) & \overset{(Adj)}{=} \hat{\phi}((q - p + (q - 1)) + p(l + 1)) \\
& \overset{(Intc)}{=} \hat{\phi}(-(q - 1) - p(l + 1)) = 0.
\end{cases}
\] (36)

Similarly, by employing the similar argument as in (36) we conclude

\[
\forall l \in \mathbb{Z}, \quad \begin{cases}
\hat{\phi}(2q + 2 + pl) = \hat{\phi}(2q + 3 + pl) = \cdots = \hat{\phi}(3q - 1 + pl) = 0, \\
\vdots \\
\hat{\phi}((m - 2)q + 2 + pl) = \hat{\phi}((m - 2)q + 3 + pl) = \cdots = \hat{\phi}((m - 1)q - 1 + pl).
\end{cases}
\] (37)

Moreover, again by applying (23) and (35) we conclude

\[
\forall l \in \mathbb{Z}, \quad \begin{cases}
\hat{\phi}((m - 1)q + 1 + 1 + pl) = \hat{\phi}((p - q) + 1 + pl) & \overset{(Intc)}{=} \hat{\phi}(-1 - pl) = 0, \\
\vdots \\
\hat{\phi}((m - 1)q + 1 + (q - 1) + pl) = \hat{\phi}((p - q) + (q - 1) + pl) \\
& \overset{(Intc)}{=} \hat{\phi}(-(q - 1) - pl) = 0.
\end{cases}
\] (38)

Finally, combining all the conditions obtained in (35), (36), (37) and (38) we get

\[\hat{\phi}(r + pl) = 0 \quad \forall l \in \mathbb{Z} \quad \text{and} \quad 1 \leq r \leq p - 1.\]

Summing up we have the following result.

**Theorem 2.2:** Let \( \phi(z) = \sum_{k=0}^{\infty} \hat{\phi}(z)z^k \in L^\infty(\mathbb{T}) \), and let \( T_\phi \) be the Toeplitz operator corresponding to the symbol \( \phi \). Let \( p, i, j \in \mathbb{N} \) be such that \( 0 \leq i < j < p \), and let \( C_p^{ij} \) be the corresponding conjugation defined as in (2) on \( H^2_{\mathbb{C}}(\mathbb{D}) \). If either

(i) \( p \) is even and \( |i - j| = \frac{p}{2} \), or

(ii) \( p = mq + 1 \) for some natural number \( m \geq 2 \) such that \( i = q - 1 \) and \( j = p - 1 \),

then \( T_\phi \) is \( C_p^{ij} \)-symmetric if and only if

\[\hat{\phi}(pl) = \hat{\phi}(-pl), \quad \text{and} \quad \hat{\phi}(r + pl) = 0 \quad \forall l \in \mathbb{Z} \quad \text{and} \quad 1 \leq r \leq p - 1.\]
Remark 2.3: We expect that Theorem 2.2 is also valid for any $0 \leq i < j < p$ and we leave the general case for future investigation.

3. Conjugation related to model spaces

In this section, we consider a special type of conjugation on $H^2_\mathbb{C}(\mathbb{D})$ different from those discussed in the previous section which essentially arose from the study of natural conjugation in model spaces. Let $p = n \in \mathbb{N}$ and consider the special permutation

$$
\sigma : \begin{pmatrix}
    a_{nk} & a_{nk+1} & \cdots & a_{nk+m} & \cdots & a_{nk+(n-1)} \\
    a_{nk+(n-1)} & a_{nk+(n-2)} & \cdots & a_{nk+(n-m-1)} & \cdots & a_{nk}
\end{pmatrix}
$$

on the set \{a_{nk}, a_{nk+1}, \ldots, a_{nk+(n-1)}\} for $k \in \mathbb{N} \cup \{0\}$, and $0 \leq m \leq n - 1$. Then, from (1) it follows that

$$
C_n = C_\sigma \left( \sum_{k=0}^{\infty} \sum_{m=0}^{n-1} a_{nk+m}z^{nk+m} \right) = \sum_{k=0}^{\infty} \sum_{m=0}^{n-1} \bar{\hat{a}}_{nk+(n-m-1)}z^{nk+m}, \tag{39}
$$

where $\sum_{k=0}^{\infty} \sum_{m=0}^{n-1} a_{nk+m}z^{nk+m} \in H^2_\mathbb{C}(\mathbb{D})$. As earlier, it is easy to verify that $C_n$ is a conjugation on $H^2_\mathbb{C}(\mathbb{D})$. Our main aim in this section is to provide a necessary and sufficient conditions on the symbol $\phi \in L^\infty(\mathbb{T})$ whenever the Toeplitz operator $T_\phi$ is complex symmetric with respect to the conjugation $C_n$. Let $\phi \in L^\infty(\mathbb{T})$ and let $\hat{\phi}(z) = \sum_{k=-\infty}^{\infty} \hat{\phi}(k)z^k$. Now we assume that the Toeplitz operator $T_\phi$ is complex symmetric with respect to this conjugation $C_n$. Therefore,

$$
C_n T_\phi C_n = T_\phi^* \quad \text{that is} \quad T_\phi C_n = C_n T_\phi. \tag{40}
$$

It is well known that \{z^{nk+a} : 0 \leq a \leq n-1, k \geq 0\} is an orthonormal basis of the Hardy space $H^2_\mathbb{C}(\mathbb{D})$. Now applying the definition of $C_n$ (see (39)) it follows that

$$
C_n T_\phi (z^{nk+a}) = C_n \left( \sum_{j=0}^{\infty} \hat{\phi}(nk + a - j)z^j \right) = \sum_{j=0}^{\infty} \sum_{m=0}^{n-1} \hat{\phi} \left( nk - nj - ((n-1) - a) + m \right) z^{nj+m} \tag{41}
$$

and

$$
T_\phi C_n (z^{nk+a}) = T_\phi \left( z^{nk+(n-1)-a} \right) = \sum_{j=0}^{\infty} \sum_{m=0}^{n-1} \hat{\phi} \left( nj - nk - ((n-1) - a) + m \right) z^{nj+m}. \tag{42}
$$

Therefore, by substituting $k - j = l \in \mathbb{Z}$ in (41) and (42), and using (40) we obtain

$$
\hat{\phi} \left( nl - ((n-1) - a) + m \right) = \hat{\phi} \left( -nl - ((n-1) - a) + m \right), \tag{43}
$$
where \(0 \leq a \leq n-1\) and \(0 \leq m \leq n-1\). In particular if \(a = n-1\) and \(m = 0\), then Equation (43) yields
\[
\hat{\phi}(nl) = \hat{\phi}(-nl) \quad \forall l \in \mathbb{Z}.
\] (44)
The following table is essential in the sequel consisting of two different values of \(a\), namely \(a = 0\) and \(a = n-1\):

| \(a = 0\) | \(a = n-1\) |
|---|---|
| \(\phi(nl - (n-1)) = \phi(-nl - (n-1))\) | \(\phi(nl) = \phi(-nl)\) |
| \(\hat{\phi}(nl - (n-2)) = \hat{\phi}(-nl - (n-2))\) | \(\hat{\phi}(nl + 1) = \hat{\phi}(-nl + 1)\) |
| \(\vdots\) | \(\vdots\) |
| \(\hat{\phi}(nl - (n-r)) = \hat{\phi}(-nl - (n-r))\) | \(\hat{\phi}(nl + r) = \hat{\phi}(-nl + r)\) |
| \(\vdots\) | \(\vdots\) |
| \(\hat{\phi}(nl - 1) = \hat{\phi}(-nl - 1)\) | \(\hat{\phi}(nl + (n-2)) = \hat{\phi}(-nl + (n-2))\) |
| \(\phi(nl) = \phi(-nl)\) | \(\hat{\phi}(nl + (n-1)) = \hat{\phi}(-nl + (n-1))\),

where \(0 \leq r \leq n-1\) and \(l \in \mathbb{Z}\). Therefore, by using the above table (45) we conclude
\[
\hat{\phi}(r + nl) = \hat{\phi}(r - nl) = \hat{\phi}(r - n - n(l - 1)) = \hat{\phi}(r - n + n(l - 1)) = \hat{\phi}(r + n(l - 2)) \quad \forall l \in \mathbb{Z},
\] (46)
where \(1 \leq r \leq n-1\). On the other hand, note that \(\phi \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})\) and hence \(\sum_{k=\infty}^{\infty} |\hat{\phi}(k)|^2 < \infty\). As a result Equation (46) yields
\[
\hat{\phi}(r + nl) = 0 \quad \forall l \in \mathbb{Z} \quad \text{and} \quad 1 \leq r \leq n-1.
\]
So, if we assume that \(T_\phi\) is \(C_n\) symmetric, then as a necessary condition of this fact we obtain
\[
\hat{\phi}(nl) = \hat{\phi}(-nl) \quad \text{and} \quad \hat{\phi}(r + nl) = 0 \quad \text{for any} \ l \in \mathbb{Z} \quad \text{and} \quad 1 \leq r \leq n-1.
\]
Now conversely, if we assume \(\phi \in L^\infty(\mathbb{T})\) be such that \(\hat{\phi}(nl) = \hat{\phi}(-nl)\) and \(\hat{\phi}(r + nl) = 0\) for any \(l \in \mathbb{Z}\) and \(1 \leq r \leq n-1\). Then, using the definition of the conjugation \(C_n\) (see (39)) one can easily check that
\[
\left( C_n T_\phi - T_\phi C_n \right) (z^{nk+a})
= \sum_{j=0}^{\infty} \sum_{m=0}^{n-1} \left( \hat{\phi}(nk - nj - (n-1) - a) + m \right)
- \hat{\phi}(nj - nk - (n-1) - a) + m \right) z^{nj+m} = 0,
\]
for any \(0 \leq a \leq n-1\) and \(k \geq 0\). Combining all the above observations, we have the following main result in this section.

**Theorem 3.1:** Let \(\phi(z) = \sum_{k=\infty}^{\infty} \hat{\phi}(k)z^k \in L^\infty(\mathbb{T})\), and let \(T_\phi\) be the Toeplitz operator on \(H_2^\infty(\mathbb{T})\) corresponding to the symbol \(\phi\). Then, \(T_\phi\) is complex symmetric with respect to the
conjugation $C_n$ if and only if $\hat{\phi}(nl) = \hat{\phi}(-nl)$ and $\hat{\phi}(r + nl) = 0$, for any $l \in \mathbb{Z}$ and $1 \leq r \leq n - 1$.

The following two corollaries provide a characterization on the symbol $\phi$ whenever the Toeplitz operator $T_\phi$ is normal and unitary.

**Corollary 3.2:** Let $\phi \in L^\infty$. If $T_\phi$ is complex symmetric with respect to the conjugation $C_n$, then $T_\phi$ is normal if and only if $\phi(z) = \hat{\phi}(0) + 2e^{-i\theta/2}\mathfrak{R}(\sum_{l=1}^{\infty} e^{i\theta/2} \hat{\phi}(nl) \mathfrak{R}(zn^l))$, where $\mathfrak{R}(z)$ denotes the real part of $z$.

**Proof:** Let $T_\phi$ be complex symmetric with respect to the conjugation $C_n$, then by Theorem 3.1, we have

$\hat{\phi}(nl) = \hat{\phi}(-nl)$ and $\hat{\phi}(r + nl) = 0$ for any $l \in \mathbb{Z}$ and $1 \leq r \leq n - 1$.

It is well known that $T_\phi$ is a normal operator if and only if there exists a unit modular constant $\alpha$ such that $\hat{\phi}(n) = \alpha \hat{\phi}(-n)$ (for more details see [9, 15, 20, 21]). So if $T_\phi$ is a normal, $C_n$-symmetric Toeplitz operator then

$\hat{\phi}(nl) = \alpha \hat{\phi}(nl)$ and $\hat{\phi}(r + nl) = 0$ for any $l \in \mathbb{Z}$ and $1 \leq r \leq n - 1$.

Therefore, the Fourier series representation of the symbol $\phi$ is given by

$\phi(z) = \hat{\phi}(0) + 2\sum_{l=1}^{\infty} \hat{\phi}(nl) \mathfrak{R}\left(z^{nl}\right) = \hat{\phi}(0) + \sum_{l=1}^{\infty} \hat{\phi}(nl) \mathfrak{R}\left(z^{nl}\right) + \sum_{l=1}^{\infty} \hat{\phi}(nl) \mathfrak{R}\left(z^{nl}\right)$

$= \hat{\phi}(0) + \sum_{l=1}^{\infty} \hat{\phi}(nl) \mathfrak{R}\left(z^{nl}\right) + \sum_{l=1}^{\infty} e^{i\theta} \hat{\phi}(nl) \mathfrak{R}\left(z^{nl}\right)$ [taking $\alpha = e^{-i\theta}$]

$= \hat{\phi}(0) + 2e^{-i\theta/2}\mathfrak{R}\left(\sum_{l=1}^{\infty} e^{i\theta/2} \hat{\phi}(nl) \mathfrak{R}(zn^l)\right)$. ■

**Corollary 3.3:** Let $\phi \in L^\infty$ and let $T_\phi$ be complex symmetric with respect to the conjugation $C_n$, then $T_\phi$ is unitary if and only if $\phi = \alpha$, where $\alpha$ is a complex number such that $|\alpha| = 1$.

**Proof:** Let $\phi \in L^\infty$ and let $T_\phi$ be complex symmetric with respect to the conjugation $C_n$. It is trivially true that if $\phi = \alpha$ with $|\alpha| = 1$, then $T_\phi$ is unitary. For the converse part, note that if $T_\phi$ is an unitary operator then for every non-negative integer $k$, we have

$1 = \langle T_\phi z^k, T_\phi z^k \rangle = \langle \hat{T_\phi} z^k, T_\phi z^k \rangle$

$= \left( \sum_{m=-k}^{\infty} \hat{\phi}(m) z^{m+k}, \sum_{q=-k}^{\infty} \hat{\phi}(q) z^{q+k} \right) = \sum_{m=-k}^{\infty} |\hat{\phi}(m)|^2$, (47)

which furthermore gives that

$\hat{\phi}(k) = 0$ for $k < 0$. (48)
Moreover, since \( T_\phi \) is complex symmetric with respect to the conjugation \( C_n \), so by Theorem 3.1, we have

\[
\dot{\phi}(nl) = \dot{\phi}(-nl) \quad \text{and} \quad \dot{\phi}(r + nl) = 0 \quad \forall \ l \in \mathbb{Z} \quad \text{and} \quad 1 \leq r \leq n - 1.
\]

(49)

Therefore, from the above relations (49) together with (48), we conclude that \( \dot{\phi}(k) = 0 \) for all non-zero integer \( k \), and from (47), we have \( |\dot{\phi}(0)| = 1 \). Thus, \( \phi = \dot{\phi}(0) \), with \( |\dot{\phi}(0)| = 1 \).

4. Conjugations in \( H^2_{\mathbb{C}^2} (\mathbb{D}) \)

In this section, we study complex symmetric block Toeplitz operators on \( H^2_{\mathbb{C}^2} (\mathbb{D}) \) with respect to some special conjugations on \( H^2_{\mathbb{C}^2} (\mathbb{D}) \) introduced earlier in [5, 8, 10]. Let \( C_2 \) be a conjugation on \( H^2_{\mathbb{C}^2} (\mathbb{D}) \) defined as in (39) corresponding to \( n = 2 \). Next we define a map \( C : H^2_{\mathbb{C}^2} (\mathbb{D}) \rightarrow H^2_{\mathbb{C}^2} (\mathbb{D}) \) whose block matrix representation is the following:

\[
C = \frac{1}{\sqrt{2}} \begin{bmatrix} C_2 & C_2 \\ C_2 & -C_2 \end{bmatrix}.
\]

(50)

Then, it is important to observe that \( C \) is a conjugation on \( H^2_{\mathbb{C}^2} (\mathbb{D}) \) (see [10, Corollary 2.8]). For more on \( 2 \times 2 \) conjugation matrices on \( \mathcal{H} \oplus \mathcal{H} \) we refer to [10], where \( \mathcal{H} \) is any complex Hilbert space. Let \( \Phi \in L^\infty_{\mathcal{M}_2} (\mathbb{T}) \) be such that \( \Phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix} \), where \( \phi_i \in L^\infty (\mathbb{T}) \) for \( 1 \leq i \leq 4 \), and let \( T_\Phi = \begin{bmatrix} T_{\phi_1} & T_{\phi_2} \\ T_{\phi_3} & T_{\phi_4} \end{bmatrix} \) be corresponding block Toeplitz operator on \( H^2_{\mathbb{C}^2} (\mathbb{D}) \).

First we assume that the Toeplitz operator \( T_\Phi \) is complex symmetric with respect to the conjugation \( C \), that is

\[
T_\Phi C = CT_\Phi^*.
\]

which implies

\[
C_2 (T_{\phi_1} + T_{\phi_2}) C_2 = T_{\bar{\phi}_1} + T_{\bar{\phi}_2}, \quad (T_{\phi_3} - T_{\phi_4}) C_2 = C_2 (T_{\bar{\phi}_3} - T_{\bar{\phi}_4})
\]

(51)

and

\[
C_2 (T_{\phi_3} + T_{\phi_4}) C_2 = C_2 (T_{\bar{\phi}_1} - T_{\bar{\phi}_2}), \quad C_2 (T_{\phi_3} - T_{\phi_4}) C_2 = T_{\bar{\phi}_3} - T_{\bar{\phi}_4}.
\]

(52)

Thus, by applying Theorem 3.1 for \( n = 2 \) and using Equations (51) and (52) we conclude

(i) \( T_{\phi_1 + \phi_2} \) is \( C_2 \) symmetric, that is \( \phi_1 + \phi_2(2l) = \phi_1 + \phi_2(-2l) \) and \( \phi_1 + \phi_2(2l + 1) = 0 \), for all \( l \in \mathbb{Z} \),

(ii) \( T_{\phi_3 - \phi_4} \) is \( C_2 \) symmetric, that is \( \phi_3 - \phi_4(2l) = \phi_3 - \phi_4(-2l) \) and \( \phi_3 - \phi_4(2l + 1) = 0 \), for all \( l \in \mathbb{Z} \),

(iii) \( C_2 (T_{\phi_3} - T_{\phi_4}) C_2 = T_{\bar{\phi}_3} + T_{\bar{\phi}_4} \).
Suppose $\phi_1 - \phi_2 = \psi_1$ and $\phi_3 + \phi_4 = \psi_2$, and let $\psi_1(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}_1(n)z^n$ and $\psi_2(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}_2(n)z^n$. Then, the above condition (iii) becomes

$$T_{\psi_1} C_2 = C_2 T_{\psi_2}. \quad (53)$$

Next by applying the definition of $C_2$ and using (53) we get for any $m \geq 0$ that

$$T_{\psi_1} C_2 (z^{2m}) = C_2 T_{\psi_2} (z^{2m})$$

$$\Rightarrow T_{\psi_1} (z^{2m+1}) = C_2 \left( \sum_{n=0}^{\infty} \hat{\psi}_2(-(n-2m))z^n \right)$$

$$\Rightarrow \sum_{n=0}^{\infty} \hat{\psi}_1(n-2m-1)z^n = \sum_{k=0}^{\infty} \hat{\psi}_2(-(2k-2m))z^{2k+1}$$

$$+ \sum_{k=0}^{\infty} \hat{\psi}_2(-(2k+1-2m))z^{2k},$$

which by equating the Fourier coefficient yields the following conditions:

$$\forall k, m \geq 0, \quad \left\{ \begin{array}{l} \hat{\psi}_1(2k-2m-1) = \hat{\psi}_2(-(2k+1-2m)), \\ \hat{\psi}_1(2k+2m) = \hat{\psi}_2(-(2k-2m)). \end{array} \right. \quad (54)$$

Similarly, for any $m \geq 0$ we also get

$$T_{\psi_1} C_2 (z^{2m+1}) = C_2 T_{\psi_2} (z^{2m+1})$$

$$\Rightarrow T_{\psi_1} (z^{2m}) = C_2 \left( \sum_{n=0}^{\infty} \hat{\psi}_2(-(n-2m-1))z^n \right)$$

$$\Rightarrow \sum_{n=0}^{\infty} \hat{\psi}_1(n-2m)z^n = \sum_{k=0}^{\infty} \hat{\psi}_2(-(2k-2m-1))z^{2k+1}$$

$$+ \sum_{k=0}^{\infty} \hat{\psi}_2(-(2k+1-2m-1))z^{2k},$$

which leads to the following conditions:

$$\forall k, m \geq 0, \quad \left\{ \begin{array}{l} \hat{\psi}_1(2k-2m+1) = \hat{\psi}_2(-(2k-2m-1)), \\ \hat{\psi}_1(2k-2m) = \hat{\psi}_2(-(2k-2m)). \end{array} \right. \quad (55)$$

Substituting $k - m = l \in \mathbb{Z}$ in (54) and (55) we have the following set of conditions:

$$\forall l \in \mathbb{Z}, \quad \left\{ \begin{array}{l} \phi_1 - \phi_2(2l) = \phi_3 + \phi_4(-2l), \\ \phi_1 - \phi_2(2l-1) = \phi_3 + \phi_4(-2l-1), \\ \phi_1 - \phi_2(2l+1) = \phi_3 + \phi_4(-2l+1). \end{array} \right. \quad (56)$$
Moreover, using the above mentioned conditions in (56) we conclude
\[
\hat{\phi}_1 - \phi_2(2l + 1) = \phi_3 + \phi_4(-2l + 1) = \phi_3 + \phi_4(-2(l - 1) - 1) \\
= \hat{\phi}_1 - \phi_2(2(l - 1) - 1) = \hat{\phi}_1 - \phi_2(2(l - 2) + 1) \quad \forall \ l \in \mathbb{Z},
\]
and hence \(\hat{\phi}_1 - \phi_2(2l + 1) = 0\), for all \(l \in \mathbb{Z}\) since \(\phi_1 - \phi_2 \in L^\infty(\mathbb{T}) \subseteq L^2(\mathbb{T})\). Similarly, we also conclude \(\phi_3 + \phi_4(2l + 1) = 0\), for all \(l \in \mathbb{Z}\). Consequently, combining all the above obtained conditions we get
\[
\forall \ l \in \mathbb{Z}, \ \left\{ \begin{array}{l}
\hat{\phi}_1 + \phi_2(2l) = \phi_1 + \phi_2(-2l), \quad \hat{\phi}_1 + \phi_2(2l + 1) = 0, \\
\phi_3 - \phi_4(2l) = \phi_3 - \phi_4(-2l), \quad \phi_3 - \phi_4(2l + 1) = 0, \\
\hat{\phi}_1 - \phi_2(2l) = \phi_3 + \phi_4(-2l), \quad \hat{\phi}_1 - \phi_2(2l + 1) = 0, \quad \phi_3 + \phi_4(2l + 1) = 0,
\end{array} \right.
\]
which after slight modifications reduces to
\[
\forall \ l \in \mathbb{Z}, \ \left\{ \begin{array}{l}
\hat{\phi}_1 + \phi_2(2l) = \phi_1 + \phi_2(-2l), \quad \phi_3 - \phi_4(2l) = \phi_3 - \phi_4(-2l), \\
\hat{\phi}_1 - \phi_2(2l) = \phi_3 + \phi_4(-2l), \quad \hat{\phi}_1(2l + 1) = 0 \quad \text{for } 1 \leq i \leq 4.
\end{array} \right. \quad (57)
\]
Summing up, we have the following theorem.

**Theorem 4.1:** Let \(\Phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix} \in L^\infty_{M_2}(\mathbb{T})\), and let \(C : H^2_{C_2}(\mathbb{D}) \rightarrow H^2_{C_2}(\mathbb{D})\) be a conjugation on \(H^2_{C_2}(\mathbb{D})\) whose block matrix representation is \(\frac{1}{\sqrt{2}} \begin{bmatrix} C_2 & C_2 \\ C_2 & -C_2 \end{bmatrix}\), where \(C_2\) is conjugation on \(H^2_{C_2}(\mathbb{D})\) defined as in (39). Then, the Toeplitz operator \(T_\Phi\) is complex symmetric with respect to the conjugation \(C\) if and only if
\[
\hat{\phi}_1 + \phi_2(2l) = \phi_1 + \phi_2(-2l), \quad \phi_3 - \phi_4(2l) = \phi_3 - \phi_4(-2l), \\
\hat{\phi}_1 - \phi_2(2l) = \phi_3 + \phi_4(-2l), \quad \hat{\phi}_1(2l + 1) = 0, \quad \text{for all } l \in \mathbb{Z} \quad \text{and} \quad 1 \leq i \leq 4.
\]

Alternatively, by adding and subtracting the conditions obtained in (57) we have the following theorem.

**Theorem 4.2:** Let \(\Phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix} \in L^\infty_{M_2}(\mathbb{T})\), and let \(C : H^2_{C_2}(\mathbb{D}) \rightarrow H^2_{C_2}(\mathbb{D})\) be a conjugation on \(H^2_{C_2}(\mathbb{D})\) whose block matrix representation is \(\frac{1}{\sqrt{2}} \begin{bmatrix} C_2 & C_2 \\ C_2 & -C_2 \end{bmatrix}\), where \(C_2\) is conjugation on \(H^2_{C_2}(\mathbb{D})\) defined as in (39). Then, the Toeplitz operator \(T_\Phi\) is complex symmetric with respect to the conjugation \(C\) if and only if
\[
\hat{\phi}_1 + \phi_2(2l) = \phi_1 + \phi_2(-2l), \quad \phi_1 + \phi_3(2l) = \phi_1 + \phi_3(-2l), \\
\phi_1 + \phi_4(2l) = \phi_1 + \phi_4(-2l), \quad \hat{\phi}_i(2l + 1) = 0 \quad \text{for all } l \in \mathbb{Z} \quad \text{and} \quad 1 \leq i \leq 4.
\]
Remark 4.3: Let \( \Phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix} \in L_{\infty}^2(\mathbb{T}) \), and let \( T\Phi = \begin{bmatrix} T_{\phi_1} & T_{\phi_2} \\ T_{\phi_3} & T_{\phi_4} \end{bmatrix} \) be the corresponding block Toeplitz operator on \( H_{C_2}^2(\mathbb{D}) \). Then, using Theorems 4.2 and 3.1 we conclude the following: If \( T_{\phi_i} \) is complex symmetric with respect to the conjugation \( C_2 \) on \( H_{C_2}^2(\mathbb{D}) \) for all \( 1 \leq i \leq 4 \), then \( T\Phi \) is complex symmetric with respect to the conjugation \( C \) on \( H_{C_2}^2(\mathbb{D}) \). Conversely, if \( T\Phi \) is complex symmetric with respect to the conjugation \( C \) on \( H_{C_2}^2(\mathbb{D}) \) and if one of \( T_{\phi_i} \) is complex symmetric with respect to the conjugation \( C_2 \) on \( H_{C_2}^2(\mathbb{D}) \), then rest of \( T_{\phi_i} \) is also complex symmetric with respect to the conjugation \( C_2 \) on \( H_{C_2}^2(\mathbb{D}) \).

Next, let \( C_1 \) and \( C_2 \) be two conjugations on \( H_{C_2}^2(\mathbb{D}) \) defined as in (39) corresponding to \( n = 1 \) and \( n = 2 \) respectively. Then, it is easy to verify that \( C_1 \) commutes with \( C_2 \), that is \( C_1 C_2 = C_2 C_1 \). Let \( \tilde{C} : H_{C_2}^2(\mathbb{D}) \rightarrow H_{C_2}^2(\mathbb{D}) \) be a conjugation on \( H_{C_2}^2(\mathbb{D}) \) defined in [10, Corollary 2.8] whose block matrix representation is the following:

\[
\tilde{C} = \frac{1}{\sqrt{2}} \begin{bmatrix} C_2 & C_1 \\ C_1 & -C_2 \end{bmatrix}.
\]

Our next aim is to investigate the complex symmetry of the Toeplitz operator \( T\Phi : H_{C_2}^2(\mathbb{D}) \rightarrow H_{C_2}^2(\mathbb{D}) \) having symbol \( \Phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix} \in L_{\infty}^2(\mathbb{T}) \) with respect to the conjugation \( \tilde{C} \). As earlier, we first assume \( T\Phi \) is complex symmetric with respect to \( \tilde{C} \), that is

\[
T\Phi \tilde{C} = \tilde{C} T^* \Phi
\]

\[
\iff \begin{bmatrix} T_{\phi_1} & T_{\phi_2} \\ T_{\phi_3} & T_{\phi_4} \end{bmatrix} \begin{bmatrix} C_2 & C_1 \\ C_1 & -C_2 \end{bmatrix} = \begin{bmatrix} C_2 & C_1 \\ C_1 & -C_2 \end{bmatrix} \begin{bmatrix} T_{\phi_1} & T_{\phi_3} \\ T_{\phi_2} & T_{\phi_4} \end{bmatrix},
\]

which yields the following set of conditions:

\[
\begin{align*}
T_{\phi_1} C_2 + T_{\phi_2} C_1 &= C_2 T_{\phi_1} + C_1 T_{\phi_2}, \\
T_{\phi_1} C_1 - T_{\phi_2} C_2 &= C_2 T_{\phi_3} + C_1 T_{\phi_4},
\end{align*}
\]

\[
T_{\phi_3} C_2 + T_{\phi_4} C_1 = C_1 T_{\phi_3} - C_2 T_{\phi_4},
\]

\[
T_{\phi_3} C_1 - T_{\phi_4} C_2 = C_1 T_{\phi_3} - C_2 T_{\phi_4}.
\]

(59)

Therefore, as earlier using the definition of \( C_1, C_2 \) and using (59), we get for any \( m \geq 0 \) that

\[
(T_{\phi_1} C_2 + T_{\phi_2} C_1) (z^{2m}) = (C_2 T_{\phi_1} + C_1 T_{\phi_2}) (z^{2m})
\]

\[
\Longrightarrow \sum_{n=0}^{\infty} \hat{\phi}_1 (n-2m-1) z^n + \sum_{n=0}^{\infty} \hat{\phi}_2 (n-2m) z^n
\]

\[
= C_2 \left( \sum_{n=0}^{\infty} \hat{\phi}_1 ((n-2m)) z^n \right)
\]
which by substituting the index $k - m = l \in \mathbb{Z}$ and equating the Fourier coefficient leads to the following conditions:

\[
\begin{align*}
\phi_1(2l) + \phi_2(2l + 1) &= \phi_1(-2l) + \phi_2(-2l - 1), \\
\phi_1(2l - 1) + \phi_2(2l) &= \phi_1(-2l - 1) + \phi_2(-2l).
\end{align*}
\]

Similarly, by repeating the above similar calculations and using the equation \((T_{\phi_1} C_2 + T_{\phi_2} C_1)(z^{2m+1}) = (C_2 T_{\phi_1} + C_1 T_{\phi_2})(z^{2m+1})\) for any \(m \geq 0\) we get the following conditions:

\[
\forall l \in \mathbb{Z}, \quad \begin{align*}
\phi_1(2l) + \phi_2(2l - 1) &= \phi_1(-2l) + \phi_2(-2l + 1), \\
\phi_1(2l + 1) + \phi_2(2l) &= \phi_1(-2l + 1) + \phi_2(-2l).
\end{align*}
\]

Therefore, continuing the above process, the equation \(T_{\phi_1} C_1 - T_{\phi_2} C_2 = C_2 T_{\phi_3} - C_1 T_{\phi_4}\) yields

\[
\forall l \in \mathbb{Z}, \quad \begin{align*}
\phi_1(2l) - \phi_2(2l - 1) &= \phi_4(-2l) + \phi_3(-2l - 1), \\
\phi_1(2l + 1) - \phi_2(2l) &= \phi_4(-2l - 1) + \phi_3(-2l), \\
\phi_1(2l - 1) - \phi_2(2l) &= \phi_4(-2l + 1) + \phi_3(-2l), \\
\phi_1(2l) - \phi_2(2l + 1) &= \phi_4(-2l) + \phi_3(-2l + 1).
\end{align*}
\]  \hfill (60)

Similarly, the equation \(T_{\phi_3} C_2 + T_{\phi_4} C_1 = C_1 T_{\phi_1} - C_2 T_{\phi_2}\) leads to the following set of conditions:

\[
\forall l \in \mathbb{Z}, \quad \begin{align*}
\phi_4(2l) + \phi_3(2l - 1) &= \phi_1(-2l) - \phi_2(-2l - 1), \\
\phi_4(2l + 1) + \phi_3(2l) &= \phi_1(-2l - 1) - \phi_2(-2l), \\
\phi_4(2l - 1) + \phi_3(2l) &= \phi_1(-2l + 1) - \phi_2(-2l), \\
\phi_4(2l) + \phi_3(2l + 1) &= \phi_1(-2l) - \phi_2(-2l + 1).
\end{align*}
\]  \hfill (61)
Furthermore, the equation $T_{\phi_3}C_1 - T_{\phi_4}C_2 = C_1 T_{\phi_3} - C_2 T_{\phi_4}$ gives the following set of conditions:

$$
\forall l \in \mathbb{Z},
\begin{align*}
\hat{\phi}_3(2l) - \hat{\phi}_4(2l - 1) &= \hat{\phi}_3(-2l) - \hat{\phi}_4(-2l - 1), \\
\hat{\phi}_3(2l + 1) - \hat{\phi}_4(2l) &= \hat{\phi}_3(-2l - 1) - \hat{\phi}_4(-2l), \\
\hat{\phi}_3(2l - 1) - \hat{\phi}_4(2l) &= \hat{\phi}_3(-2l + 1) - \hat{\phi}_4(-2l), \\
\hat{\phi}_3(2l) - \hat{\phi}_4(2l + 1) &= \hat{\phi}_3(-2l) - \hat{\phi}_4(-2l + 1).
\end{align*}
$$

It is important to observe that some repetition is there in the set of conditions obtained in (60) and (61). Thus, by removing those repetition we get the following complete list of conditions:

$$
\forall l \in \mathbb{Z},
\begin{align*}
\hat{\phi}_1(2l) + \hat{\phi}_2(2l + 1) &= \hat{\phi}_1(-2l) + \hat{\phi}_2(-2l - 1), \\
\hat{\phi}_1(2l - 1) + \hat{\phi}_2(2l) &= \hat{\phi}_1(-2l - 1) + \hat{\phi}_2(-2l), \\
\hat{\phi}_1(2l) + \hat{\phi}_2(2l - 1) &= \hat{\phi}_1(-2l - 1) + \hat{\phi}_2(-2l + 1), \\
\hat{\phi}_1(2l + 1) + \hat{\phi}_2(2l) &= \hat{\phi}_1(-2l + 1) + \hat{\phi}_2(-2l), \\
\hat{\phi}_1(2l) - \hat{\phi}_2(2l - 1) &= \hat{\phi}_4(-2l - 1) - \hat{\phi}_3(-2l - 1), \\
\hat{\phi}_1(2l - 1) - \hat{\phi}_2(2l) &= \hat{\phi}_4(-2l + 1) - \hat{\phi}_3(-2l), \\
\hat{\phi}_1(2l) - \hat{\phi}_2(2l + 1) &= \hat{\phi}_4(-2l - 1) - \hat{\phi}_3(-2l), \\
\hat{\phi}_1(2l + 1) - \hat{\phi}_2(2l) &= \hat{\phi}_3(-2l - 1) - \hat{\phi}_4(-2l), \\
\hat{\phi}_3(2l - 1) - \hat{\phi}_4(2l) &= \hat{\phi}_3(-2l + 1) - \hat{\phi}_4(-2l), \\
\hat{\phi}_3(2l) - \hat{\phi}_4(2l + 1) &= \hat{\phi}_3(-2l) - \hat{\phi}_4(-2l + 1),
\end{align*}
$$

which after simplifying again we obtain the following set of minimal conditions:

$$
\forall l \in \mathbb{Z},
\begin{align*}
\hat{\phi}_1 + \hat{\phi}_4(2l + 1) &= 0, \\
\hat{\phi}_2 - \hat{\phi}_3(2l + 1) &= 0, \\
\hat{\phi}_1(2l) - \hat{\phi}_1(2l + 2) &= \hat{\phi}_1(-2l) - \hat{\phi}_1(-2l - 2), \\
\hat{\phi}_2(2l) - \hat{\phi}_1(2l - 2) &= \hat{\phi}_2(-2l) - \hat{\phi}_2(-2l + 2), \\
\hat{\phi}_3(2l) - \hat{\phi}_1(2l - 2) &= \hat{\phi}_3(-2l) - \hat{\phi}_3(-2l + 2), \\
\hat{\phi}_4(2l) - \hat{\phi}_1(2l + 2) &= \hat{\phi}_4(-2l) - \hat{\phi}_4(-2l - 2), \\
\hat{\phi}_2 + \phi_3(2l) + \hat{\phi}_1(2l - 1) &= \phi_2 + \phi_3(-2l) + \phi_1(2l + 1), \\
\phi_1 + \phi_4(2l) + \phi_2 + \phi_3(2l + 1) &= \phi_1 - \phi_4(-2l) + \phi_2 + \phi_3(-2l - 1).
\end{align*}
$$

Summing up, we have the following theorem in this section.

**Theorem 4.4:** Let $\Phi = \left[ \begin{array}{c} \phi_1 \\ \phi_3 \\ \phi_4 \end{array} \right] \in L^\infty_{M_2}(\mathbb{T})$, and let $\tilde{\mathcal{C}} : H^2_\mathcal{C}(\mathbb{D}) \rightarrow H^2_\mathcal{C}(\mathbb{D})$ be a conjugation on $H^2_\mathcal{C}(\mathbb{D})$ whose block matrix representation is $\frac{1}{\sqrt{2}} \begin{bmatrix} C_2 & C_1 \\ C_1 & -C_2 \end{bmatrix}$, where $C_1$ and $C_2$ are conjugations on $H^2_\mathcal{C}(\mathbb{D})$ defined as in (39). Now if the Toeplitz operator $T_\Phi$ is complex
symmetric with respect to the conjugation $\tilde{C}$, then for any $l \in \mathbb{Z}$ we get

\[
\begin{align*}
\hat{\phi}_1 + \phi_4 (2l + 1) &= 0, \quad \hat{\phi}_2 - \phi_3 (2l + 1) = 0, \\
\hat{\phi}_1 (2l) - \hat{\phi}_1 (2l + 2) &= \phi_1 (-2l) - \phi_1 (-2l - 2), \\
\hat{\phi}_2 (2l) - \hat{\phi}_1 (2l - 2) &= \phi_2 (-2l) - \phi_2 (-2l + 2), \\
\hat{\phi}_3 (2l) - \hat{\phi}_1 (2l - 2) &= \phi_3 (-2l) - \phi_3 (-2l + 2), \\
\hat{\phi}_4 (2l) - \hat{\phi}_1 (2l + 2) &= \phi_4 (-2l) - \phi_4 (-2l - 2),
\end{align*}
\]

\[
\begin{align*}
\phi_2 + \phi_3 (2l) + \phi_1 - \phi_4 (2l - 1) &= \phi_2 + \phi_3 (-2l) + \phi_1 - \phi_4 (-2l - 1), \\
\phi_1 - \phi_4 (2l) + \phi_2 + \phi_3 (2l + 1) &= \phi_1 - \phi_4 (-2l) + \phi_2 + \phi_3 (-2l - 1).
\end{align*}
\]

5. Concluding remarks

It is important to observe that in Sections 2 and 3, we gave a characterization of complex symmetric Toeplitz operators $T_\phi$ on $H^2_C(D)$ with respect to conjugations that are special cases of $C_\sigma$ (as defined in (1)). Hence, it is natural to ask the following question:

**Question:** Characterize complex symmetric Toeplitz operators $T_\phi$ on $H^2_C(D)$ with respect to the conjugation $C_\sigma$ defined in (1).

We expect to have similar type of characterizations as obtained in Theorems 2.2 and 3.1 and leave this as a subject for future investigation.

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