CURVATURE-ADAPTED SUBMANIFOLDS
OF SEMI-RIEMANNIAN GROUPS

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Abstract. We study semi-Riemannian submanifolds of arbitrary codimension in a Lie group $G$ equipped with a bi-invariant metric. In particular, we show that, if the normal bundle of $M \subset G$ is closed under the Lie bracket, then any normal Jacobi operator $K$ of $M$ equals the square of the associated invariant shape operator $\alpha$. This permits to understand curvature adaptedness to $G$ geometrically, in terms of left translations. For example, in the case where $M$ is a Riemannian hypersurface, our main result states that the normal Jacobi operator commutes with the ordinary shape operator precisely when the left-invariant extension of each of its eigenspaces has first-order tangency with $M$ along all the others. As a further consequence of the equality $K = \alpha^2$, we obtain a new case-independent proof of a well-known fact: every three-dimensional Lie group equipped with a bi-invariant semi-Riemannian metric has constant curvature.

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1. INTRODUCTION AND MAIN RESULT

Given a Riemannian manifold, it is natural to study submanifolds whose geometry is somehow adapted to that of the ambient space. This idea led to the concept of curvature-adapted submanifold [9, 1, 2]; a concept that, since its introduction, has attracted the interest of many geometers.

Here we define curvature adaptedness in a slightly more general setting, that of a semi-Riemannian manifold $(Q, g) \equiv Q.$

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Let $M$ be an $m$-dimensional semi-Riemannian submanifold of $Q$, let $N^1M$ be its unit normal bundle, and let $R$ be the ambient curvature tensor. For $(p, \eta) \equiv \eta \in N^1_pM$, the normal Jacobi operator

$$K_\eta \equiv K: T_pM \to T_pQ$$

$$v \mapsto R(\eta, v)\eta$$

of $M$ (with respect to $\eta$) measures the curvature of the ambient manifold along $\eta$. On the other hand, denoting by $N$ a unit normal local extension of $\eta$ along $M$, and by $\nabla$ the Levi-Civita connection of $Q$, the shape operator

$$A_\eta \equiv A: T_pM \to T_pM$$

$$v \mapsto \pi^\top \nabla_v N$$

of $M$ (with respect to $\eta$) describes the curvature of $M$ as a submanifold of $Q$; here $\pi^\top$ denotes orthogonal projection onto $T_pM$.

**Definition 1.1.** One says that $M$ is curvature adapted (to $Q$) at a point $p \in M$ if, for every $\eta \in N^1_pM$,

1. The normal Jacobi operator leaves $T_pM$ invariant, i.e., $K(T_pM) \subset T_pM$;
2. The operators $A$ and $K$ commute, i.e., $K \circ A = A \circ K$.

Consequently, one calls $M$ curvature adapted if it is curvature adapted at $p$ for all $p \in M$.

**Remark 1.2.** Condition (1) in Definition 1.1 is always satisfied for hypersurfaces.

**Remark 1.3.** Both $A$ and $K$ are self-adjoint with respect to the induced semi-Riemannian metric. Thus, if these operators are also diagonalizable, then $M$ is curvature adapted at $p$ precisely when they share a common orthonormal basis of eigenvectors (Lemma 2.1). Recall that diagonalizability is automatic if the metric is positive definite.

**Remark 1.4.** In general, it could also be interesting to consider a weaker notion of curvature adaptedness, obtained by replacing the operator $K$ with $\pi^\top K$ in the definition. This modification would make any totally umbilic submanifold necessarily curvature adapted.

It is easy to see (e.g., using [12, Lemma 2]) that every semi-Riemannian submanifold of a real semi-Riemannian space form is curvature adapted. However, for other ambient spaces, the definition is restrictive. For example, if $Q$ is an $(m+1)$-dimensional nonflat complex space form with complex structure $J$, then $A$ and $K$ commute precisely when $-J\eta$ is an eigenvector of $A$. Also, if $Q$ is an $(m+1)$-dimensional nonflat quaternionic space form with quaternionic structure $I$, then $A$ and $K$ commute precisely when the maximal subspace of $T_pM$ invariant under $I$ is also invariant under $A$; see [1] and [6, sec. 9.8].

In a symmetric space of nonconstant curvature, the situation is more involved, yet many interesting results have been obtained. Among others (see for example [14, 19, 15, 16]), the most important is arguably Gray’s theorem [13, Theorem 6.14], which states that any tubular hypersurface around a curvature-adapted submanifold is itself curvature adapted.
Gray’s theorem has been further generalized to the family of Riemannian manifolds such that, for every geodesic $\gamma$, the Jacobi operator $R(\dot{\gamma}, \cdot)\dot{\gamma}$ is diagonalizable by a parallel orthonormal frame field along $\gamma$. It turns out that, in such spaces, the classification of the curvature-adapted submanifolds is fully determined by that of the curvature-adapted hypersurfaces [3, 2].

In this article, we shall examine the case where $Q$ is a semi-Riemannian group, i.e., a Lie group $G \equiv (G, \langle \cdot, \cdot \rangle)$ equipped with a bi-invariant semi-Riemannian metric $\langle \cdot, \cdot \rangle$. In particular, we will focus our attention on the class of (semi-Riemannian) submanifolds of $G$ having closed normal bundle; here by closed we mean that each normal space of $M$ corresponds, under the group’s left action, to a subspace of $\mathfrak{g}$ that is closed under the Lie bracket, i.e., to a Lie subalgebra (Definition 2.5).

Examples of Lie groups $G$ abounds [22]: every semisimple Lie group can be furnished with a bi-invariant metric, and every compact Lie group admits one that is Riemannian. Note that, if $G$ is absolutely simple, then any bi-invariant metric on $G$ is a scalar multiple of the Killing form of $\mathfrak{g}$.

In order to explain our main result, we first set up some notation. Provided that $K$ is diagonalizable, let $(e_1, \ldots, e_m)$ be an orthonormal basis of eigenvectors of $K$, that is, a basis of $T_pM$ such that $|\langle e_j, e_h \rangle| = \delta_{jh}$ and $K(e_j) = \lambda_j e_j$ for all $j, h = 1, \ldots, m$; for each $j$, let $e^L_j$ denote the left-invariant extension of $e_j$.

**Theorem 1.5.** Let $M$ be a semi-Riemannian submanifolds of a semi-Riemannian group $G$. Assume that $K$ is diagonalizable and that the normal space of $M$ at $p$ is closed under the Lie bracket. Then the following statements are equivalent:

(i) $A$ and $K$ commute.

(ii) If $\lambda_j$ and $\lambda_h$ are distinct eigenvalues, then $e_j(\langle N, e^L_h \rangle) = 0$.

Moreover, if the induced metric is positive definite at $p$, then $e_j(\langle N, e^L_h \rangle) = e_h(\langle N, e^L_j \rangle)$ whenever $e_j$ and $e_h$ are in distinct eigenspaces.

**Remark 1.6.** For all $j = 1, \ldots, m$, we have $\lambda_j = -\sec(e_j, \eta) \leq 0$; see section 2.

Roughly speaking, Theorem 1.5(ii) expresses the condition that the left-invariant extension of each eigenspace of $K$ is “orthogonal to first order” to $N$ (at $p$) along all other eigenspaces; in other words, along any curve that starts at $p$ and is tangent to a different eigenspace.

In particular, in the case of hypersurfaces, the condition on the normal space is automatically satisfied. Specializing the theorem to that case, we obtain the result below.

**Corollary 1.7.** If $M$ is a hypersurface and $K$ is diagonalizable, then the following statements are equivalent:

(i) $A$ and $K$ commute.

(ii) The left-invariant extension of each eigenspace of $K$ has first-order tangency with $M$ along all other eigenspaces.

The main significance of Theorem 1.5 lies in the fact that it permits to understand curvature adaptedness to $G$ geometrically, in terms of left translations. More precisely, it reveals that, in order for a generic submanifold (with closed normal bundle) to be curvature adapted, its tangent bundle needs to behave
reasonably well under left translations. Note that the tangent bundle of a Lie subgroup is fully left-invariant; conversely, if $M$ is a closed, connected submanifold that contains the identity and has left-invariant tangent bundle, then it is a Lie subgroup.

The basic fact that allows us to prove Theorem 1.5 is that the shape operator of $M \subset H$ with respect to $\eta$, being $H$ any Lie group with a left-invariant semi-Riemannian metric, decomposes as the sum of two terms [24]: an invariant shape operator, which depends only on $\eta, T_p M$ and $H$; plus a second term, here denoted by $\mathcal{W}$, that is closely related to the Gauss map of $M$; see section 3 for details. In particular, if the metric is bi-invariant and the normal bundle is closed, then the invariant shape operator commutes with $K$ (Proposition 3.5), and so, by linearity, commutativity of $A$ and $K$ reduces to that of $\mathcal{W}$ and $K$.

In fact, if $\dim M = 2$ or $M$ is Riemannian, then the nonzero eigenvalues of $K$ have even multiplicities (Corollary 3.8), which leads us to the following conclusions.

**Proposition 1.8.** Every two-dimensional surface with closed normal bundle is curvature adapted to $G$.

**Proposition 1.9.** If $M$ is a three-dimensional Riemannian submanifold of $G$ with closed normal bundle, and if $K \neq 0$ for all $\eta \in N^1_p M$, then the following statements are equivalent:

(i) $M$ is curvature adapted in a neighborhood $U$ of $p$.

(ii) For all $\eta \in N^1 U$, the 0-eigenvector of $K$ is an eigenvector of $A$.

Proposition 1.8 implies that, if $\dim G = 3$, then every Riemannian surface in $G$ is curvature adapted. This is by no means surprising, because every three-dimensional semi-Riemannian group has constant curvature. Indeed, it is well known that the Lie algebra of such a Lie group is either abelian or isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$ [22]; in each of the latter two cases, a direct computation would reveal that the curvature of the Killing form (which, up to scaling, coincides with the metric) vanishes. Alternatively, since the Ricci curvature of $G$ is proportional to the Killing form, the statement follows from the fact that every three-dimensional Einstein manifold has constant curvature [4, p. 49]; see Remark 1.11.

On the other hand, applying classical results of Cartan and Dajczer–Nomizu (which, for the reader’s convenience, are included in section 2), we can prove the following statement.

**Lemma 1.10.** Suppose that $Q$ is a Riemannian (resp., Lorentzian) manifold of dimension at least three. If, for some $p \in Q$ and each $x \in T_p Q$ such that $\langle x, x \rangle = 1$ (resp., $-1$), the map from $x^\perp$ to $x^\perp$ defined by $y \mapsto R(x, y)x$ is a multiple of the identity, then $Q$ has constant curvature.

**Proof.** Assume the hypothesis of the lemma. If $x, y, z$ are orthonormal vectors in $T_p Q$ (resp., if $x, y, z$ are orthonormal vectors in $T_p Q$ and $\langle x, x \rangle = -1$), then $\langle R(x, y)z, x \rangle = -\langle R(x, y)x, z \rangle = 0$. Applying [8, Lemma 1.17] (resp., [7, Theorem 1a]) and Schur’s lemma [21, p. 96, Exercise 21(b)], the claim follows.

Corollary 3.8 thus yields a case-independent proof that, in dimension three, every metric Lie group has constant curvature.
Remark 1.11. An alternative case-independent proof of the same fact may be sketched as follows. First, by Levi decomposition and the absence of simple Lie algebras of dimension one and two, observe that a three-dimensional Lie algebra is either solvable or simple. However, if a nonabelian solvable Lie algebra of dimension three admits an ad-invariant bilinear form, then such form is necessarily degenerate [10, Proposition 2.3]; in other words, a three-dimensional nonabelian Lie algebra admitting an ad-invariant metric is (absolutely) simple.

Suppose, thus, that $G$ is absolutely simple. Then any semi-Riemannian metric on $G$ is a scalar multiple of the Killing form of $g$, implying that $G$ is an Einstein manifold. Hence, in dimension three, it has constant curvature [4, p. 49].

The remainder of the paper is organized as follows. In the next section we briefly review some background material. In section 3 we introduce the invariant shape operator and examine its properties. In section 4 we then prove Theorem 1.5 and Proposition 1.9. We conclude with Appendix A, where, for the sake of illustration, we give a direct proof that condition (ii) in Theorem 1.5 holds whenever $p$ is an umbilical point of $M$.

Some final remarks about notation:

1. The indices $j, h, i$ satisfy $j, h \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, m + n\}$; note that we always use Einstein summation convention.

2. For $x \in T_p H$, we denote the left-invariant extension of $x$ by $x^L$.

2. Preliminaries

Here we recall some basic results that are used throughout the paper; see e.g. [18, 17, 21] for further details about semi-Riemannian geometry and metric Lie groups.

2.1. Semi-Euclidean vector spaces. Let $V$ be a real vector space, of finite dimension $d$, equipped with a nondegenerate symmetric bilinear form $f$, and let $L$ be an endomorphism of $V$ that is self-adjoint with respect to $f$, i.e., such that $f(L(x), y) = f(x, L(y))$ for all $x, y \in V$. Recall that a basis $v_1, \ldots, v_d$ of $V$ is called orthonormal if $|f(v_j, v_h)| = \delta_{jh}$ for all $j, h = 1, \ldots, d$.

It is well known that, if $f$ is positive definite, then $L$ is diagonalizable by an orthonormal basis of eigenvectors of $L$. Moreover, two self-adjoint endomorphisms of $V$ commute if and only if they share a common orthonormal basis of eigenvectors.

While it is not possible to fully extend these classic results beyond the positive definite case, something interesting can still be said.

Lemma 2.1. If $L$ is diagonalizable, then there exists an orthonormal basis of eigenvectors. Moreover, any two diagonalizable self-adjoint endomorphisms of $V$ commute if and only if they share a common orthonormal basis of eigenvectors.

Proof. Assume that $L$ is diagonalizable. Since $V$ is the direct sum of mutually orthogonal eigenspaces of $L$, each eigenspace must be nondegenerate, and so it has an orthonormal basis; by concatenating these bases, we obtain an orthonormal basis of $V$.

As for the second statement, one direction is obvious; for the other, it suffices to note that any two commuting linear maps on $V$ preserve each other’s eigenspaces.
2.2. Semi-Riemannian geometry. Let \((Q, g)\) be a semi-Riemannian manifold, and let \(\nabla\) be its Levi-Civita connection. The curvature endomorphism \(R: \mathfrak{X}(Q)^3 \to \mathfrak{X}(Q)\) of \((Q, g)\) is the \((1, 3)\)-tensor field on \(Q\) defined by
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.
\]

(Caution: some authors define the curvature endomorphism as the negative of ours.)

Being \(R\) a tensor, if \(x, y, z\) are vectors in \(T_p Q\), then the value \(R(x, y) z\) is independent of the extension of \(x, y, z\) and thus well-defined.

Suppose that \(x \in T_p Q\) has unit length, i.e., that \(|g(x, x)| = 1\). Then the Jacobi operator of \((Q, g)\) with respect to \(x\) is the linear map
\[
K_x: x^\perp \to x^\perp \quad \text{y} \mapsto R(x, y)x.
\]

Clearly, by the symmetry by pairs of the \((0, 4)\)-curvature tensor (obtained by lowering the last index of \(R\)), the operator \(K_x\) is self-adjoint with respect to \(g\).

Next, suppose that \(x, y \in T_p Q\) are orthonormal. Then the sectional curvature \(\text{sec}(x, y)\) of the nondegenerate plane spanned by \(x\) and \(y\) is given by the formula
\[
\text{sec}(x, y) = g(R(x, y)y, x) = -g(K_x(y), y),
\]
where the last equality follows from the skew-symmetry of the \((0, 4)\)-curvature tensor.

In the Riemannian setting, an important criterion for discerning whether a manifold has constant sectional curvature is provided by the following lemma.

**Lemma 2.2 ([5], [8, Lemma 1.17]).** Suppose that \(Q\) is a Riemannian manifold, and that \(\dim Q \geq 3\). If, at some point \(p \in Q\), the curvature tensor satisfies \(g(R(x, y)z, x) = 0\) whenever \(x, y, z\) are orthonormal, then all sectional curvatures of \(Q\) at \(p\) are equal.

Several generalizations of Lemma 2.2 to the semi-Riemannian setting have appeared [12, 7, 20]. Below we recall the one that is most relevant to our discussion.

**Definition 2.3.** Let \(x, y \in T_p Q\). We say that the pair \((x, y)\) is orthonormal of signature \((-+, +)\) if \(g(x, x) = -1\), \(g(y, y) = 1\), and \(g(x, y) = 0\).

**Theorem 2.4 ([7, Theorem 1a]).** Suppose that \(\dim Q \geq 3\). If, at some point \(p \in Q\), the curvature tensor satisfies \(g(R(x, y)z, x) = 0\) whenever \((x, y)\) is orthonormal of signature \((-+, +)\) and \(g(x, z) = g(y, z) = 0\), then all nondegenerate two-planes have the same sectional curvature.

2.3. Semi-Riemannian groups. Let \(G\) be a Lie group equipped with a left- and right-invariant (i.e., bi-invariant) semi-Riemannian metric \(\langle \cdot, \cdot \rangle\), and let \(\mathfrak{g}\) be its Lie algebra, that is, the Lie algebra of left-invariant vector fields on \(G\). As customary, we identify \(\mathfrak{g}\) with the tangent space \(T_e G\) of \(G\) at the identity \(e\).

Suppose that \(X, Y, Z \in \mathfrak{X}(G)\) are left-invariant, i.e., that \(X, Y, Z \in \mathfrak{g}\). Then the Levi-Civita connection is given by
\[
(1) \quad \nabla_X Y = -\nabla_Y X = \frac{1}{2} [X, Y]
\]
and the curvature endomorphism by
\[ R(X,Y)Z = \frac{1}{4}[Z,[X,Y]]. \]

In addition, the following equality holds:
\[ \langle [X,Y], Z \rangle = \langle X, [Y,Z] \rangle. \]

Suppose that \( x, y \in T_pG \) are orthonormal; let \( x^L, y^L \) be their left-invariant extensions. The sectional curvature of the two-plane spanned by \( x \) and \( y \) may be computed by
\[ \sec(x,y) = \frac{1}{4} \langle [x^L, y^L], [x^L, y^L] \rangle. \]

Note that \( \sec(x,y) \geq 0 \), with equality if and only if \( [x^L, y^L] = 0 \).

Let \( M \) be a semi-Riemannian submanifold of \( G \).

**Definition 2.5.** The normal space \( N_pM \) is said to be closed (under the Lie bracket) if \( dL_{p^{-1}}(N_pM) \) is a Lie subalgebra of \( g \). Consequently, one calls the normal bundle of \( M \) closed (under the Lie bracket) if every normal space is closed.

It is clear that \( N_pM \) is closed exactly when \( \exp(N_pM) \) is contained in a Lie subgroup of \( G \).

**Remark 2.6.** Following [26], the normal space \( N_pM \) is called abelian if \( \exp(N_pM) \) is contained in a totally geodesic, flat submanifold of \( G \). It is easy to see that \( N_pM \) is abelian if and only if \( dL_{p^{-1}}(N_pM) \) is an abelian subalgebra of \( g \).

### 3. The invariant shape operator

In this section we consider the general case of an orientable semi-Riemannian submanifold \( M \) of a Lie group \( H \) equipped with a left-invariant metric \( \langle \cdot, \cdot \rangle \). Given a unit normal vector \( \eta \) of \( M \) at \( p \), the invariant shape operator of \( M \) (with respect to \( \eta \)) is the map
\[ \alpha : T_pM \rightarrow T_pM \]
\[ v \mapsto \pi^\top \nabla_v \eta^L, \]
where, as usual, \( \pi^\top \) is the orthogonal projection onto \( T_pM \) and \( \eta^L \) the left-invariant extension of \( \eta \).

The significance of the invariant shape operator lies in the fact that it represents the deviation of the ordinary shape operator from the differential of the Gauss map of \( M \), as we now explain.

Let \( N^1M \) be the unit normal bundle of \( M \), and let \( S^{m+n-1}_e \) be the unit sphere inside the Lie algebra \( \frakh \) of \( H \). The Gauss map of \( M \) is the map
\[ G : N^1M \rightarrow S^{m+n-1}_e \]
\[ (p, \eta) \mapsto d(L_{p^{-1}})(\eta); \]
here \( L_{p^{-1}} : T_pH \rightarrow \frakh \) denotes left translation by \( p^{-1} \).

Let \( N \) be a unit normal vector field along \( M \) such that \( N_p = \eta \), and consider the map \( \bar{G} = G \circ N \). Its differential at \( p \) is a linear map \( T_pM \rightarrow G(p, \eta)^\perp \). Thus, since \( d(L_{p^{-1}}) \) takes \( \eta \) to \( G(p, \eta) \) and is an isometry, it follows that
\[ W = \pi^\top \circ d(L_p) \circ d\bar{G} \]
is an endomorphism of $T_pM$.

**Remark 3.1.** The Gauss map of a hypersurface in a metric Lie group was first defined by Ripoll in [24]. It is worth pointing out that our definition can be extended to the case where the ambient manifold is parallelizable [25], or even just Killing-parallelizable [11].

Clearly, if $H = \mathbb{R}^{m+1}$, then $\mathcal{G}$ is the classical Gauss map of $M$, whereas $\mathcal{W}$ its shape operator. In our setting, the following result holds.

**Proposition 3.2** (cf. [25, p. 769]). For any $v \in T_pM$,

$$
(4) \quad A(v) = \alpha(v) + \mathcal{W}(v).
$$

**Proof.** Let $(b_1, \ldots, b_{m+n})$ be an orthonormal basis of $T_pH$ such that $b_1, \ldots, b_m \in T_pM$ and $b_{m+n} = \eta$. For each $i$, let $b^L_i$ be the left-invariant extension of $b_i$, so that $(b^L_1, \ldots, b^L_{m+n} = \eta^L)$ is an orthonormal frame for $H$ (and a basis of $T_eH$).

If $q \in M$—writing $N^i$ as a shorthand for $\langle N, b^L_i \rangle$—then

$$
\bar{\mathcal{G}}(q) = d(L_{q^{-1}})(N_q) = d(L_{q^{-1}})(N^i(q)b^L_i|_q) = N^i(q)b^L_i|_e.
$$

Thus, if $v \in T_pM$, then

$$
\begin{align*}
\bar{\mathcal{G}}(v) &= dN^i(v)b^L_i|_e = v(N^i)b^L_i|_e. \\
\end{align*}
$$

Since

$$
\begin{align*}
d(L_p)(d\bar{\mathcal{G}}(v)) &= v(N^i)b_i, \\
\end{align*}
$$

it follows that

$$
\begin{align*}
(5) \quad \pi^\top d(L_p)(d\bar{\mathcal{G}}(v)) &= v(N^i)b_j. \\
\end{align*}
$$

On the other hand,

$$
\begin{align*}
A(v) &= \pi^\top \nabla_v N^i b^L_i \\
&= \pi^\top (N^i(p)\nabla_v b^L_i + v(N^i)b_i). \\
\end{align*}
$$

Since, by construction, $N^1(p) = \cdots = N^{m+n-1}(p) = 0$ and $N^{m+n}(p) = 1$, we have

$$
(6) \quad A(v) = \alpha(v) + v(N^j)b_j,
$$

which, together with (5), gives (4). □

**Remark 3.3.** Proposition 3.2 shows that $\mathcal{W}$ does not depend on the particular choice of normal vector field $N$ but only on its value at $p$.

**Remark 3.4.** Using equation (6), it is not difficult to see that statement (ii) in Theorem 1.5 is nothing but the coordinate expression, with respect to the frame $(e^L_1, \ldots, e^L_m)$, of the condition

$$
(7) \quad \pi_j \text{Im } \mathcal{W}|_{\Lambda_h} = 0 \quad \text{for all } j, h = 1, \ldots, m \text{ such that } \lambda_j \neq \lambda_h,
$$

where $\Lambda_h$ is the eigenspace of $K$ corresponding to the eigenvalue $\lambda_h$, and where $\pi_j$ is the orthogonal projection onto $\Lambda_j$. Note that (7) holds if and only if $\mathcal{W}$ leaves the eigenspaces of $K$ invariant.

A useful property of the invariant shape operator, which is crucial in proving Theorem 1.5, is contained in the following proposition.
Proposition 3.5. If $\langle \cdot, \cdot \rangle$ is bi-invariant and the normal space $N_pM$ is closed, then

1. $K = \alpha \circ \alpha$, and so $\alpha$ and $K$ commute;
2. $K$ leaves $T_pM$ invariant.

The proof will be based on a lemma.

Lemma 3.6. Under the hypotheses of Proposition 3.5, $\alpha(v) = \nabla_v \eta^L$.

Proof. Let $\xi$ be a unit normal vector at $p$. Since $N_pM$ is closed,

$$[\eta^L, \xi^L] \in dL_p^{-1}(N_pM),$$

under the usual identification of $\mathfrak{g}$ with $T_eG$. By the bi-invariance of the metric, it follows that

$$\langle [\eta^L, \xi^L], \xi^L \rangle = \langle [\eta^L, \xi^L], \eta^L \rangle = 0,$$

which implies $1/2 [v^L, \eta^L]_p = \nabla_v \eta^L \in T_pM$, and so $\alpha(v) = \pi^\top \nabla_v \eta^L = \nabla_v \eta^L$. □

Remark 3.7. The converse of Lemma 3.6 holds: if the metric is bi-invariant and $\alpha(v) = \nabla_v \eta^L$ for all $(v, \eta) \in T_pM \times N_pM$, then $N_pM$ is closed. In other words, given a nondegenerate subspace $S$ of $\mathfrak{g}$, the orthogonal complement $S^\perp$ of $S$ is closed under the Lie bracket if and only if $[S, S^\perp] \subset S$.

Proof of Proposition 3.5. Clearly, being the second assertion in the proposition a direct consequence of the first, we only need to prove the latter.

Let $v \in T_pM$. Since $K$ is tensorial, the value $K(v)$ may be computed in terms of the left-invariant extensions $v^L$ and $\eta^L$ of $v$ and $\eta$:

$$K(v) = R(\eta^L, v^L)\eta^L.$$

Assume that the metric is bi-invariant. Then, using (1) and (2), we have

$$K(v) = \frac{1}{4} [\eta^L, [\eta^L, v^L]]$$

$$= \nabla_{\nabla_v \eta^L} \eta^L.$$

From here the statement follows directly from Lemma 3.6. □

Corollary 3.8. Suppose that the induced semi-Riemannian metric on $M$ is positive definite at $p \in M$. Then, under the hypotheses of Proposition 3.5, the nonzero eigenvalues of $K$ are negative and have even multiplicities.

Proof. We deduce from equations (1) and (3) that the invariant shape operator $\alpha$ of $G$ is skew-adjoint with respect to the semi-Riemannian metric. On the other hand, $K$ is self-adjoint. Thus, if the induced metric on $M$ is positive definite at $p$ and $K(T_pM) \subset T_pM$, then the matrices of $K$ and $\alpha$ with respect to any orthonormal basis of $T_pM$ are symmetric and skew-symmetric, respectively. Since $K = \alpha \circ \alpha$ when the hypotheses of Proposition 3.5 are fulfilled, the statement follows from [23, Theorem 2]. □
Remark 3.9. Dropping the assumption that the metric is positive definite, the matrix of $\alpha$ in any orthonormal basis $(b_1, \ldots, b_m)$ of $T_p M$ becomes

$$
\begin{pmatrix}
0 & -\langle \alpha(b_1), b_2 \rangle & \cdots & -\langle \alpha(b_1), b_m \rangle \\
\frac{\langle \alpha(b_1), b_2 \rangle}{\langle b_2, b_2 \rangle} & 0 & \cdots & -\langle \alpha(b_2), b_m \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\langle \alpha(b_1), b_m \rangle}{\langle b_m, b_m \rangle} & \frac{\langle \alpha(b_2), b_m \rangle}{\langle b_m, b_m \rangle} & \cdots & 0
\end{pmatrix}
$$

Hence, when $m = 2$, the operator $\alpha \circ \alpha$ is a multiple of the identity regardless of the signature of the metric.

4. Proof of the main result

We are now ready to prove Theorem 1.5 and Proposition 1.9.

Proof of Theorem 1.5. It follows from equation (6) that

$$A(e_j) = \alpha(e_j) + \sum_{h=1}^{m} e_j(\langle N, e_h^L \rangle)e_h.$$ 

Hence, by linearity of $K$, we have

$$K(A(e_j)) = K(\alpha(e_j)) + \sum_{h=1}^{m} e_j(\langle N, e_h^L \rangle)K(e_h),$$

whereas

$$A(K(e_j)) = \alpha(K(e_j)) + \sum_{h=1}^{m} K(e_j)(\langle N, e_h^L \rangle)e_h.$$ 

Assume that $N_p M$ is closed; this way, since $G$ is equipped with a bi-invariant metric, $K$ and $\alpha$ commute by Proposition 3.5. It follows that $K(A(e_j)) = A(K(e_j))$ exactly when

$$\sum_{h=1}^{m} \lambda_h e_j(\langle N, e_h^L \rangle)e_h = \sum_{h=1}^{m} \lambda_j e_j(\langle N, e_h^L \rangle)e_h.$$ 

Being $(e_1, \ldots, e_m)$ a basis of $T_p M$, we conclude that $A$ and $K$ commute if and only if $e_j(\langle N, e_h^L \rangle) = 0$ for all $j$ and $h$ such that $\lambda_j \neq \lambda_h$.

It remains to show that $e_j(\langle N, e_h^L \rangle) = e_h(\langle N, e_j^L \rangle)$ when $\lambda_j \neq \lambda_h$ and the induced metric on $M$ is positive definite at $p$. To this end, identify $\alpha$, $A$, and $K$ with their matrices in the basis $(e_j)_{j=1}^{m}$. The first is a skew-symmetric matrix, by equation (3), whereas $A$ is symmetric and $K$ diagonal. Since $\alpha$ and $K$ commute, the $(j, h)$-entries of $\alpha K$ and $K \alpha$ are equal, and so we must have $\alpha_{jh} \lambda_h = \lambda_j \alpha_{jh}$.

Assume $\lambda_j \neq \lambda_h$. Then $\alpha_{jh} = -\alpha_{hj} = 0$ and so, by equation (4),

$$A_{jh} = \langle W(e_j), e_h \rangle,$$

$$A_{hk} = \langle W(e_h), e_j \rangle,$$

implying $\langle W(e_j), e_h \rangle = \langle W(e_h), e_j \rangle$ by symmetry of $A$. □
Proof of Proposition 1.9. Suppose that $M$ is a three-dimensional Riemannian submanifold with closed normal bundle, and suppose that $\alpha \neq 0$. It follows by Corollary 3.8 that $K$ has one zero eigenvalue, while the remaining two are equal. Without loss of generality, we may assume that $\lambda_3 = 0$. Since $K \neq 0$, it is clear that $\lambda_1 = \lambda_2 \neq 0$.

Extend $\eta$ to a unit normal vector field $N$ along $M$. Then, by continuity, the multiplicity of $\lambda_3$ is locally constant, i.e., there exists a neighborhood $U = U(N)$ of $p$ in $M$ such that the extension of $K$ has two negative definite eigenvalues in $U$.

Assume that $A$ and $K$ commute, i.e., they share a common basis of eigenvectors. Since the 0-eigenspace of $K$ is one-dimensional, it follows that $e_3$ is an eigenvector of $A$. Conversely, if $e_3$ is an eigenvector of $A$, then its other two eigenvectors lie in the $\lambda_1$-eigenspace of $K$, from which we infer that $A$ and $K$ commute. \hfill \Box

Appendix A.

Here we present a direct proof of the following obvious corollary of Theorem 1.5.

Corollary A.1. Suppose that $N_p M$ is closed. If the shape operator of $M$ with respect to $\eta$ is a multiple of the identity, then $e_j(\langle N, e^L_h \rangle) = 0$ for all $e_j, e_h$ in different eigenspaces.

Proof. Assume the hypotheses of the corollary. Since $\alpha$ commutes with $K$, each eigenspace of $K$ is invariant under $\alpha$. Moreover, being $A$ a multiple of the identity, $\langle A(e_j), e_h \rangle = 0$ for $j \neq h$, and so equation (6) implies

$$\langle \alpha(e_j), e_h \rangle = \pm e_j(\langle N, e^L_H \rangle) \text{ for } j \neq h.$$ 

Clearly, if $\lambda_j \neq \lambda_h$, then $e_j(\langle N, e^L_H \rangle) = 0$, because $\alpha(e_j)$ and $e_h$ are in different (orthogonal) eigenspaces of $K$. \hfill \Box

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