Representation of Entangled States

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Abstract

Identifying a pair of correlated qubits with the pure entangled state \( \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \) is an idealization unless the pair is so prepared using an appropriate quantum gate operating on a known state. Questions related to the reference frame for measurement of the entangled state are considered.

Introduction

The only way to certify a specific pure entangled state is to start with a known pure state, say \( |00\rangle \), and use an appropriate 2-qubit transformation to drive it to the desired state.

This note presents some observations concerning entanglement states. Specifically, the question of the use of an enlarged reference frame to test entangled particles is discussed.

Imperfect entanglement

In the decay of a spinless system into a pair of spin \( \frac{1}{2} \) particles, say electrons, if the spin of one electrons is found in a particular orientation to be \( \frac{1}{2} \), then the spin of the other electron is \( -\frac{1}{2} \). It is customary to represent the entangled state as \( |\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \). Likewise, in atomic SPS cascade, if the two emitted photons are detected in opposite directions, they appear to have the same polarization. The state of the photons is usually represented by: \( |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \), where 0 and 1 are horizontal and vertical polarization.
In general, one seeks states such as \( |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + k|11\rangle) \), where \(|k| = 1\) in many applications. Entanglement, where the probability amplitudes of the terms in the superposition are \(1, -1, i, -i\) (i.e. \( k = \pm 1, \pm i \)) may be called maximal or perfect.

In experiments on entangled photons created using spontaneous parametric down-conversion\[5, 7, 8, 9\], about one out of 10,000 trials produces an entangled pair and the probability of getting a double-pair is even lower\[6\]. Some sorting procedure is used to post-select entangled pairs out of the large number created by the source.

When there is maximal correlation and 0 and 1 are obtained with equal probability on qubits tested from the source, the principle of least information requires that we represent the state by a density matrix and not as a pure state.

The degree of entanglement at the point of emission depends on the physical process. Some processes may not be perfectly symmetric with respect to the entangled variable. The mixedness of an entangled state may require a large ensemble to establish.

**Generating entangled photons**

We may, in principle, generate the pure entangled state \( \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \) by operating on the \(|00\rangle\) of a pure state by many operators, such as \( U \):

\[
U = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1
\end{bmatrix}
\]

However, no deliberately engineered implementation of \( U \) can be absolutely precise. Due to the inevitable imprecision in implementing the components of \( U \), the actual entanglement would not be perfect.

For example, if the implemented \( U \) is:

\[
U' = \frac{1}{\sqrt{2}} \begin{bmatrix}
e^{i\theta_1} & 0 & 0 & 1 \\
0 & e^{i\theta_2} & 1 & 0 \\
0 & 1 & -e^{i\theta_2} & 0 \\
1 & 0 & 0 & -e^{i\theta_1}
\end{bmatrix}
\]
where $\theta_i$s are small random errors, an application of the $4 \times 4$ Hadamard operator

$$H = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$
on $U'|00\rangle$ will reveal that the state is not correctly rotated.

**The reference frame**

To test entanglement it is assumed that the experimenters, if they are at different locations, share the same reference frame which, in general, will be three-dimensional. For photons, the polarization state is determined by the oscillating electrical or magnetic vectors that are mutually orthogonal and perpendicular to the direction of propagation. The propagation direction is assumed to fix one of the three axes of the reference frame and, therefore, the problem is reduced to one of two dimensions only.

In the general case, one must view a qubit $a|0\rangle + b|1\rangle$ as a point on a unit sphere (Figure 1). Here it is assumed that the measurements are with respect to the XY-plane; the third axis represents the phase variable $\theta$.

The most general rotation operation on the qubit $|0\rangle$ may be represented by

$$\begin{bmatrix} \alpha e^{i\theta_1} & \beta e^{-i\theta_2} \\ \beta e^{i\theta_2} & -\alpha e^{-i\theta_1} \end{bmatrix}$$

where $\theta_1$ and $\theta_2$ are phase angles. This leads to the superposition state $\alpha e^{i\theta_1}|0\rangle + \beta e^{i\theta_2}|1\rangle$.

If no reference axis is available then one can use the full complement of three axes. The qubit may then be written as: $a|0\rangle + b|1\rangle + c|2\rangle$.

Thus in the BB84 quantum cryptography protocol[1], one could represent polarization along 9 different directions rather than just two. These 9 directions would be:

$$|0\rangle, |1\rangle, |2\rangle, |0\rangle + |1\rangle, |0\rangle - |1\rangle, |0\rangle + |2\rangle, |0\rangle - |2\rangle, |1\rangle + |2\rangle, |1\rangle - |2\rangle$$

These 9 directions belong to 6 different frames in the X, Y, and Z planes. Generalized Bell states for such a case can be easily defined[3].

3
Figure 1: The qubit sphere \((\alpha, \beta, \theta)\). The vertical circles represent \(|1\rangle\) and its phase shifts. The circle on the right represents \(1/2^{1/2}(|0\rangle + e^{i\theta}|1\rangle)\), which are various combinations of \(|0\rangle\) with phase shifted \(|1\rangle\) (i.e. 45° polarized photons, for example). The point A is \(e^{i\pi/2}|1\rangle\); B is \(1/2^{1/2}(|0\rangle + i|1\rangle)\); C is \(1/2^{1/2}(|0\rangle + |1\rangle)\).
Enlarged basis set

In the Bell basis, the entanglements are for the mutually orthogonal states:

\[ |00\rangle + |11\rangle, |00\rangle - |11\rangle, |01\rangle + |10\rangle, |01\rangle - |10\rangle \]

This set may be enlarged by considering weights \( \pm i \) and in applications such as dense coding one may use the basis states containing \( \pm i \) rather than the usual Bell basis. The enlarged set of basis states is:

\[ |00\rangle + |11\rangle, |00\rangle - |11\rangle, |01\rangle + |10\rangle, |01\rangle - |10\rangle, \\
|00\rangle + i|11\rangle, |00\rangle - i|11\rangle, |01\rangle + i|10\rangle, |01\rangle - i|10\rangle \]

The operators given below represent relevant transformations that form a group:

\[
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\
D = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 \\ -i & 0 \end{bmatrix}
\]

The multiplications products for the elements of this group are shown in Table 1.

**Table 1:** A group of elementary quantum operators where the item in the left column comes first in the multiplication

| mult | I | A | B | C | D | E | F | G |
|------|---|---|---|---|---|---|---|---|
| I    | I | A | B | C | D | E | F | G |
| A    | A | I | C | B | G | F | E | D |
| B    | B | C | I | A | F | G | D | E |
| C    | C | B | A | I | E | D | G | F |
| D    | D | E | F | G | B | C | I | A |
| E    | E | D | G | F | A | I | C | B |
| F    | F | G | D | E | I | A | B | C |
| G    | G | F | E | D | C | B | A | I |
If one did not wish to use the operators containing $\pm i$, then the subgroup consisting of the operators $I, A, B, C$ will suffice.

Corresponding to the use of the Hadamard operator $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ to distinguish between $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$, one may distinguish between $|0\rangle + i|1\rangle$ and $|0\rangle - i|1\rangle$ using the operator $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}$.

The idea of the enlarged basis may be used in cryptography[1, 2]. The four states of the quantum cryptographic protocol may be viewed as polarizations at $-45^\circ, 0^\circ, +45^\circ, +90^\circ$ and these polarizations are recovered by the use of the $0/90^\circ$ and $-45/+45^\circ$ bases. Since the shared reference frame is $0/90^\circ$, it makes the two pairs of states asymmetric in the sense that one pair has no superposition whereas the other does. The states $|0\rangle + |1\rangle, |0\rangle - |1\rangle, |0\rangle + i|1\rangle, |0\rangle - i|1\rangle$ constitutes a set where each state is a superposition and it may be used in place of the usual set.

Conclusion

Many applications in quantum information science require entangled pure states. In some of those, like dense coding, a small deviation from maximal entanglement appears to make only a correspondingly small difference in the final results. On the other hand, in teleportation one would like to have a very precisely defined maximally entangled pure state. We could create this entangled state using an appropriate transformation on a state such as $|00\rangle$.

This indicates the importance of hardware implementation of basic quantum gates. This is also significant because quantum lithography[4] using entanglement of groups of photons could increase etching resolution beyond the diffraction limit.

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