RESTRICTION OF THE OSCILLATOR REPRESENTATION TO DUAL PAIRS: SOME PROJECTIVE CASES

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Abstract. We study here the restriction of the oscillator representation of the symplectic group \( Sp(2p(m+n), \mathbb{R}) \) to two different subgroups, namely \( O(m,n; \mathbb{R}) \) and \( Sp(2p, \mathbb{R}) \). We use the duality correspondence introduced by Howe to analyze these restrictions, and determine sufficient conditions on \( m, n \) and \( p \) so that the modules obtained are projective. The duality correspondence gives a description of the restriction in terms of lowest and highest modules, and we conclude by using gradings and filtrations to identify the modules.

1. Introduction

A very classical problem in representation theory is the understanding of the restriction of a representation \( \Pi \) of a group \( G \) to one of its subgroups \( H \). In that setting, it is often useful to analyze \( \text{Hom}_H(\Pi, \pi) \), where \( \pi \) is a representation of \( H \). For this purpose, one may use the derived functors \( \text{Ext}^n_H(\Pi, \pi) \) to understand \( \text{Hom}_H(\Pi, \pi) \) itself. Calculating \( \text{Ext}^n_H(\Pi, \pi) \) is not necessarily easier than \( \text{Hom}_H(\Pi, \pi) \), but their Euler characteristic might be. This difficult part can become much simpler when we have a projective representation of \( G \). In this case, \( \text{Ext}^n_H(\Pi, \pi) \) vanishes for every \( n > 0 \). This is one of the basic motivations here: the projectivity of a representation is an extremely powerful property. The link between Euler characteristic and projectivity is emphasized in [2], for example.

We focus on dual pairs, an approach introduced in the framework of the duality correspondence for the oscillator representation. A dual pair is a pair \( (G, G') \) of subgroups of a symplectic group \( Sp(V) \), such that \( G \) is the centralizer of \( G' \) in \( Sp(V) \). Two dual pairs \( (G, G') \) and \( (H, H') \) of \( Sp(V) \) together are called a seesaw dual pair if \( (G \supset H, H' \supset G') \).

More precisely, we consider the oscillator representation \( \omega \) of \( Sp(2p(m+n), \mathbb{R}) \) with the seesaw dual pair \( \left( (U(m,n), U(p)), (O(m,n; \mathbb{R}), Sp(2p, \mathbb{R})) \right) \). We restrict \( \omega \) to \( O(m,n; \mathbb{R}) \), and to \( Sp(2p, \mathbb{R}) \) respectively, and analyze the cases when these restrictions are projective. Because \( U(p) \) is a compact group, the restriction of \( \omega \) to \( U(m,n) \) is discrete. It is therefore enough to analyze each \( U(m,n) \)-summand. We show in theorems 1 and 2 that imposing a relation between the variables \( m, n \) and \( p \) that determine the size of these groups is sufficient to force the projectivity of
these restrictions. The restriction to $O(m, n; \mathbb{R})$ becomes projective under a slightly stronger condition than being in the stable range:

**Theorem.** The restriction of the oscillator representation of $Sp(2p(m + n), \mathbb{R})$ to $O(m, n; \mathbb{R})$ is projective if $p > m + n$.

Using $(O(m, n; \mathbb{R}), Sp(2p, \mathbb{R}))$, we obtain a result related to the semistable range, namely:

**Theorem.** The restriction of the oscillator representation of $Sp(2p(m + n), \mathbb{R})$ to $Sp(2p, \mathbb{R})$ is projective if $\min(n, m) > 2p$.

It might seem unusual to focus on only one representation of one chosen group. Due to the importance of the oscillator representation in many different topics, this is however not surprising. This representation also appears in some books and papers as (Segal-Shale)-Weil representation, harmonic or metaplectic representation, among many other names. As mentioned in [10], for example, it is a fundamental object for the study of the minimal representations of classical groups, not only the symplectic group. Many different models of the oscillator representation can be found in the literature. Lecture 2 of [10], and Adams' notes from [1] present several realizations, and provide explanations of which model is the most appropriate depending on the context.

Seesaw dual pairs appear in the work of Kudla for the first time, in [9], and have been extensively used since that. Howe gives many results about dual pairs and their use together with the oscillator representation, for example in [4] and [5]. We focus here on compact dual pairs, i.e., dual pairs with one compact group, since it allows us to decompose representations under the action of the compact member.

The theory of duality correspondence, first introduced by Howe and also called theta correspondence, describes explicitly the subrepresentations that appear in the decomposition of the oscillator representation after restriction to a dual pair. For our case of interest, namely irreducible dual pairs of real reductive groups, the duality correspondence can be found in Adams’ notes [1]. The description of the restriction is made in terms of highest (and lowest) weight modules, whose theory is used in the technical part of our result.

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2. **Generalities**

In this section, we introduce the mathematical objects that we use, and we recall some well-known results. The main goal is to introduce most of the notations, and to make a list of the different tools that are used here.
The basic set-up will be the following: let $G$ be a Lie group with complexified Lie algebra $\mathfrak{g}$, and let $K \subset G$ be a maximal compact Lie subgroup. We denote by $\mathfrak{k}$ the complexified Lie algebra of $K$, and we choose a Cartan subalgebra $\mathfrak{t}$ of both $\mathfrak{g}$ and $\mathfrak{k}$.

2.1. **Graded algebras.** Let $A$ be a ring with filtration $A_n$, and corresponding graded algebra $\text{Gr}(A) = \bigoplus A_n/A_{n-1}$. Let $M, N$ be two $A$-modules with filtrations denoted by $M_n$ and $N_n$. Assume that $A_n M_m \subset M_{m+n}$ for all $m, n$, and similarly for $N_m$. We write $\text{Gr}(M) = \bigoplus M_n/M_{n-1}$ and $\text{Gr}(N) = \bigoplus N_n/N_{n-1}$ for the corresponding graded $A$-modules.

**Proposition 1.** Let $T : M \to N$ be a morphism of $A$-modules preserving the filtrations $M_n$ and $N_n$ and such that the corresponding graded morphism of $\text{Gr}(A)$-modules $T \text{Gr} : \text{Gr}(M) \to \text{Gr}(N)$ is surjective. If $\dim(M_n) = \dim(N_n)$ for all $n$, then $T$ is an isomorphism of $A$-modules.

This basic result is extremely useful to conclude the proof of our results. Indeed, the graded pieces of our modules are easier to describe than the whole modules themselves, so we use filtrations to obtain isomorphisms.

2.2. **Frobenius reciprocity.** Let $A, B$ be two rings with $A \subset B$. We recall the definition of a projective module, which is one of the central notions:

**Definition.** We say that a $B$-module $P$ is projective if for any $B$-modules $M, N$ and homomorphisms $f : N \rightarrow M$, $g : P \rightarrow M$ with $f$ onto, there exists a homomorphism $h : P \rightarrow N$ such that $f \circ h = g$.

When we work with tensor products of $(\mathfrak{g}, K)$-modules, projectivity can be directly deduced from a corollary of the Frobenius reciprocity, recalled here.

**Proposition 2** (Frobenius reciprocity). Let $M$ be an $A$-module and $N$ be a $B$-module. We have a vector space isomorphism

$$\text{Hom}_B(B \otimes_A M, N) \cong \text{Hom}_A(M, N).$$

**Corollary 1.** Let $Q$ be an $A$-module, and let $P = B \otimes_A Q$. If $Q$ is a projective $A$-module, then $P$ is a projective $B$-module.

2.3. **Highest weight modules.** We do not give details about the theory of highest weight modules, as it can be found in many textbooks, as [7] for example. We mainly introduce our notations for these objects here.

We fixed a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, and we can therefore define the root system $\Delta$ of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Fixing a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ determines the positive and negative roots in $\Delta$, we write $\Delta^+$ for the positive roots. We denote by $\mathfrak{q}$ the parabolic subalgebra $\mathfrak{q} = \mathfrak{k} + \mathfrak{b}$ obtained by summing the Lie algebra of $K$ and the fixed Borel subalgebra.
For a weight \( \lambda \) of \( \mathfrak{g} \), we write \( F_\lambda \) for the irreducible \( \mathfrak{t} \)-module with highest weight \( \lambda \), and \( E_\lambda \) for the irreducible \( \mathfrak{g} \)-module with highest weight \( \lambda \). We use \( N(\lambda) \) to denote the \( \mathfrak{U}(\mathfrak{g}) \)-module \( \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(q^\mathfrak{g}) \) \( F_\lambda \), where \( \mathfrak{U}(\mathfrak{a}) \) is the universal enveloping algebra of \( \mathfrak{a} \) for any Lie algebra \( \mathfrak{a} \).

### 2.4. Irreducibility criterion

We state here an irreducibility criterion for \( \mathfrak{U}(\mathfrak{g}) \)-module of the form \( N(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(q^\mathfrak{g}) \) \( F_\lambda \). This allows us to make a crucial identification that leads to the main result. We write \( \Delta_\alpha \) for the compact roots, namely the roots of \( \mathfrak{k} \) with respect to \( \mathfrak{t} \), and \( \Delta_n = \Delta \setminus \Delta_\alpha \) denotes the non-compact roots. We also define \( \Delta_\alpha^+ = \Delta_\alpha \cap \Delta^+ \), \( \Delta_n^+ = \Delta_n \cap \Delta^+ \). The product

\[
\frac{2 < \lambda, \alpha >}{< \alpha, \alpha >}, \quad \text{for any } \alpha \in \Delta \text{ and } \lambda \in \mathfrak{t}^* \]

is denoted by \( (\lambda)_\alpha \). As usual, \( \rho \) is half of the sum of the positive roots and we write \( s_\alpha \) for the reflection through the hyperplane determined by the root \( \alpha \).

**Proposition 3.** Assume for any \( \alpha \in \Delta_n^+ \) with \( (\lambda + \rho)_\alpha \in \mathbb{Z}_{>0} \), there is \( \gamma \in \Delta_n \) with \( (\lambda + \rho)_\gamma = 0 \) and \( s_\alpha(\gamma) \in \Delta_\alpha \). Then \( N(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(q^\mathfrak{g}) \) \( F_\lambda \) is irreducible. Moreover, if \( \mathfrak{g} \) is of type \( \mathfrak{A}_n \), it is a necessary and sufficient condition.

This appears as Corollary 6.3 and Theorem 6.4 in [3].

### 2.5. \( (\mathfrak{g}, \mathfrak{K}) \)-modules

The modules that we use in this paper are \( (\mathfrak{g}, \mathfrak{K}) \)-modules. We give here the definition and state a basic but fundamental result. More details can be found in [8].

**Definition.** A \( (\mathfrak{g}, \mathfrak{K}) \)-module is a complex vector space \( V \) with an action of \( \mathfrak{g} \) and an action of \( \mathfrak{K} \) such that

1. for all \( v \in V, k \in \mathfrak{K}, X \in \mathfrak{g} \), we have \( k \cdot (X \cdot v) = (\text{Ad}(k)X) \cdot (k \cdot v) \),
2. \( V \) is \( \mathfrak{K} \)-finite, i.e., for every \( v \in V \), \( K \cdot v \) is a finite dimensional vector space,
3. for all \( v \in V, Y \in \mathfrak{t} \), we have \( \left( \frac{d}{dt} \exp(tY) \cdot v \right)_{|t=0} = Y \cdot v \).

In this definition, (1) is a compatibility condition between the action of \( \mathfrak{K} \) on \( V \), the action of \( \mathfrak{g} \) on \( V \) and the action of \( \mathfrak{K} \) on \( \mathfrak{g} \). Part (3) forces the compatibility between the action of \( \mathfrak{t} \) on \( V \) as a Lie subalgebra of \( \mathfrak{g} \) and the action of \( \mathfrak{t} \) on \( V \) as the complexified Lie algebra of \( \mathfrak{K} \).

As a consequence of Frobenius reciprocity, we have the following result:

**Proposition 4.** Let \( V \) be a \( (\mathfrak{g}, \mathfrak{K}) \)-module. Then \( \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{t}) \) \( V \) is a projective \( \mathfrak{U}(\mathfrak{g}) \)-module.

**Proof.** By \( \mathfrak{K} \)-finiteness, every \( (\mathfrak{g}, \mathfrak{K}) \)-module is \( \mathfrak{U}(\mathfrak{t}) \)-projective. Now the result is a direct application of corollary 1. \( \square \)
2.6. Oscillator representation. We are interested in a particular representation of the symplectic group \(Sp(2p, \mathbb{R})\) on \(L^2(\mathbb{R}^p)\), called the oscillator representation. We follow here the construction presented in [6], where more details can be found. Note that we never use explicitly this construction, but only the duality correspondence, which specifically applies to this representation.

The oscillator representation for \(Sp(2p, \mathbb{R})\) will be constructed from a representation of the Lie algebra \(sp(2p, \mathbb{R})\) on the space of Schwarz functions on \(\mathbb{R}^p\).

**Definition.** The space of Schwarz functions on \(\mathbb{R}^n\), denoted \(S(\mathbb{R}^n)\), is the set of functions \(f \in C^\infty(\mathbb{R}^n)\) such that

\[
\sup_{(x_1, \ldots, x_n) \in \mathbb{R}^n} |x_1^{\alpha_1} \ldots x_n^{\alpha_n} \frac{\partial^{\beta_1 + \cdots + \beta_n} f(x_1, \ldots, x_n)}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}| < \infty
\]

for all \((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_{>0}^n\).

When \(p = 1\), the group \(Sp(2p, \mathbb{R})\) is isomorphic to \(SL(2, \mathbb{R})\), and it is easy to describe the explicit construction of the oscillator representation. If we pick a standard basis \((h, e, f)\) of \(sl(2, \mathbb{R})\), we can define a representation \(\omega\) on \(S(\mathbb{R})\) by:

\[
\omega(h) = x \frac{d}{dx} + \frac{1}{2}, \quad \omega(e) = \frac{i}{2} x^2, \quad \omega(f) = \frac{i}{2} \frac{d^2}{dx^2}.
\]

Exponentiating this \(sl(2, \mathbb{R})\)-module, we obtain a unitary representation of the double cover of \(SL(2, \mathbb{R})\) on the space \(L^2(\mathbb{R})\).

A similar construction can be done for any \(p\), but we do not explain the details here. We therefore obtain structure of \(sp(2p, \mathbb{R})\)-module on \(S(\mathbb{R}^p)\), which is a derived representation of the double cover of \(Sp(2p, \mathbb{R})\) on \(L^2(\mathbb{R}^p)\). Several constructions and different models for \(p > 1\) can be found in Lecture 2 in [10] or in [1]. We denote this representation by \(\omega\) and call it the oscillator representation of the symplectic group \(Sp(2p, \mathbb{R})\). Note that \(\omega\) is a \((g, k)\)-module for \(G = Sp(2p, \mathbb{R})\) and \(K\) its maximal compact subgroup.

2.7. Reductive dual pairs. When we restrict the oscillator representation to a subgroup of the symplectic group, it is useful to use another group to decompose this restriction. This can be done using pairs of groups called dual pairs.

**Definition.** A pair \((G, G')\) of subgroups in a symplectic group \(Sp(2q, \mathbb{R})\) is a reductive dual pair if

1. \(G\) and \(G'\) act reductively on \(\mathbb{R}^{2q}\),
2. \(G\) and \(G'\) are centralizers of each other inside \(Sp(2q, \mathbb{R})\).

Moreover, if \(G\) is compact, we say that \((G, G')\) is a compact dual pair.

It is also useful to consider two dual pairs with a particular relation, as introduced by Kudla in [9]:
Definition. Two dual pairs \((G, G')\) and \((H, H')\) form a seesaw dual pair if we have the inclusions \(H \subset G\) and \(G' \subset H'\). We denote it by \(\left((G, G'), (H, H')\right)\).

2.8. Duality correspondence. This section introduces briefly the idea of duality correspondence, which can be used to calculate the restriction of the oscillator representation to some subgroups.

The duality correspondence is a decomposition of the oscillator representation \(\omega\) of a symplectic group \(Sp(q, \mathbb{R})\), under the action of a subgroup. We assume that we have a compact dual pair \((G, G')\), so that we can decompose \(\omega\) under the action of \(G\).

Recall that if \(G\) is a compact group with finite dimensional representation \(\pi\), we can decompose \(\pi\) as

\[ \pi = \bigoplus_{\sigma} (\text{Hom}_G(\sigma, \pi) \otimes \sigma), \]

where the sum is taken over all the irreducible representations \(\sigma\) of \(G\). Indeed, if \(T \in \text{Hom}_G(\sigma, \pi)\) and \(v \in \sigma\), then \(T(v) \in \pi\) and we have a map

\[ \text{Hom}_G(\sigma, \pi) \times \sigma \to \pi, \quad (T, v) \mapsto T(v). \]

This map can be extended to a map \(\text{Hom}_G(\sigma, \pi) \otimes \sigma \to \pi\), and it is injective when \(\sigma\) is irreducible. Since \(G\) is compact, \(\pi\) is completely reducible, hence \(\pi = \bigoplus_{\sigma} (\text{Hom}_G(\sigma, \pi) \otimes \sigma)\). Note that here, \(G\) does not act on \(\text{Hom}_G(\sigma, \pi)\); this only denotes the multiplicity of \(\sigma\) in \(\pi\).

We apply the same method here, in the sense that we use the action of a compact group \(G\) on \(\omega\), and obtain a decomposition

\[ \omega = \bigoplus_{\sigma} (\text{Hom}_G(\sigma, \omega) \otimes \sigma), \]

where the sum is taken over all the irreducible representations \(\sigma\) of \(G\). We denote \(\text{Hom}_G(\sigma, \omega)\) by \(\theta(\sigma)\) in this case. The Lie algebra \(g'\) of \(G'\) acts naturally on \(\theta(\sigma) = \text{Hom}_G(\sigma, \omega)\) as follows: for \(X \in g', T \in \text{Hom}_G(\sigma, \omega)\) and \(v \in \sigma\), we have \((X \cdot T)(v) = X \cdot (T(v))\), where \(X \cdot (T(v))\) comes from the action of \(g'\) on \(\omega\).

The duality correspondence gives an explicit description of \(\theta(\sigma)\) in some specific cases. In general, we know that \(\theta(\sigma)\) is a highest weight module, and we denote its highest weight by \(\tau\). We use, as before, \(E_\tau\) to denote the irreducible \(g'\)-module with highest weight \(\tau\). Note that \(\tau\) is also a dominant weight for \(\mathfrak{k}'\), the Lie algebra of a maximal compact subgroup \(K'\) of \(G'\), so \(\tau\) is also the highest weight of a finite dimensional representation of \(\mathfrak{k}'\): we write \(F_\tau\) for the irreducible \(\mathfrak{k}'\)-module with highest weight \(\tau\). In most cases, the duality correspondence does not give us the highest weight \(\tau\) directly from \(\sigma\), but it produces a lowest weight \(\tau'\) related to \(\tau\).

We give here the correspondence for the two cases that we use. This correspondence can be found with more details in [1].
Since $\mathfrak{gl}_{m+n}(\mathbb{C})$ is the complexified Lie algebra of $U(m, n)$, the duality correspondence for $(U(p), U(m, n))$ can be expressed in term of $U(p)$-modules and $\mathfrak{gl}_{m+n}(\mathbb{C})$-modules:

**Proposition 5.** The duality correspondence for the pair $(U(p), U(m, n))$ is given by

$$\sigma = (a_1 + \frac{m-n}{2}, \ldots, a_k + \frac{m-n}{2}, b_1 + \frac{m-n}{2}, \ldots, b_l + \frac{m-n}{2})$$

$$\mapsto\tau' = (a_1 + \frac{p}{2}, \ldots, a_k + \frac{p}{2}, \ldots, \frac{p}{2} \oplus (-\frac{p}{2}, \ldots, -\frac{p}{2}, b_1 - \frac{p}{2}, \ldots, b_l - \frac{p}{2}),$$

where $\sigma$ defines an irreducible highest weight $U(p)$-module and $\tau'$ defines an irreducible lowest weight $\mathfrak{gl}_{m+n}(\mathbb{C})$-module. All such weights occur, with the constraints $k + l \leq p$, $k \leq m$, $l \leq n$.

For $O(n, \mathbb{R})$, it is not obvious how we can describe a representation using highest weights, since it is disconnected. However, we can use the embedding $O(n, \mathbb{R}) = U(n) \cap GL(n, \mathbb{R})$ of $O(n, \mathbb{R})$ into $U(n)$ and the highest weights of $U(n)$. Given a highest weight $\lambda$ of $U(n)$ and some parameter $\epsilon = \pm 1$, we say that the representation of $O(n, \mathbb{R})$ with highest weight $(\lambda, \epsilon)$ is the irreducible summand of the representation of $U(n)$ with highest weight $\lambda$ that contains the highest weight vector, tensored with the $\text{sgn}$ representation of $O(n, \mathbb{R})$ if $\epsilon = -1$.

**Proposition 6.** The duality correspondence for the pair $(O(n, \mathbb{R}), Sp(2p, \mathbb{R}))$ is given by

$$\sigma = (a_1, \ldots, a_k, 0, \ldots, 0; \epsilon)$$

$$\mapsto\tau' = (a_1 + \frac{n}{2}, \ldots, a_k + \frac{n}{2}, \ldots, \frac{1-\epsilon}{2}(n-2k), \ldots, \frac{n}{2}, \ldots, \frac{n}{2}, \frac{n}{2}, \ldots, \frac{n}{2})$$

where $\sigma$ defines an irreducible highest weight $O(n, \mathbb{R})$-module in the sense explained previously and $\tau'$ defines an irreducible lowest weight $\mathfrak{sp}(2p, \mathbb{C})$-module. All such weights occur, with the constraints $k \leq \lfloor \frac{n}{2} \rfloor$, and $k + \frac{1-\epsilon}{2}(n-2k) \leq p$.

3. Restrictions

3.1. **Set-up.** We consider the seesaw dual pair

$$(U(m, n), U(p)), (O(m, n; \mathbb{R}), Sp(2p, \mathbb{R}))$$

inside $Sp(2p(m + n), \mathbb{R})$, with oscillator representation $\omega$. We want to understand the restriction of $\omega$ to $O(m, n; \mathbb{R})$ and to $Sp(2p, \mathbb{R})$, and analyze the cases when these restrictions are projective. This will be done by first restricting to $U(m, n)$ using the
dual pair \((U(p), U(m, n))\), and then restricting further to \(O(m, n; \mathbb{R})\). In the second case, we will use the action of \(O(m, \mathbb{R}) \times O(n, \mathbb{R})\) to decompose \(\omega\) and then focus on a pair of the form \((O(n, \mathbb{R}), Sp(2p, \mathbb{R}))\).

We recall that our notations are as follows:

- \((G, G')\) a dual pair with \(G\) compact,
- \(G, G'\) real Lie groups, with complexified Lie algebras \(\mathfrak{g}, \mathfrak{g}'\),
- \(K'\) a maximal compact subgroup of \(G'\), with complexified Lie algebra \(\mathfrak{k}'\),
- \(\mathfrak{t}'\) a Cartan subalgebra of both \(\mathfrak{g}'\) and \(\mathfrak{k}'\),
- \(\mathfrak{b}'\) a Borel subalgebra of \(\mathfrak{g}'\),
- \(\mathfrak{q}' = \mathfrak{t}' + \mathfrak{b}'\) a parabolic subalgebra of \(\mathfrak{g}'\).

In order to understand the restriction of \(\omega\) to \(G'\), we let \(G\) act and we obtain a decomposition of the form \(\omega = \bigoplus (\sigma \otimes \theta(\sigma)) = \bigoplus (\sigma \otimes E_{\tau})\), where \(\sigma\) is an irreducible representation of \(G\) with highest weight \(\sigma\), \(E_{\tau}\) is an irreducible representation of \(G'\) with highest weight \(\tau\) and the sum is taken over all the possible \(\sigma\).

We know that \(E_{\tau}\) is irreducible and it is a quotient of \(N(\tau) = \mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{q}')} F_{\tau}\). So if \(N(\tau)\) is also irreducible, this forces \(N(\tau) = E_{\tau}\). In this case, we have an explicit description of the restriction of \(\omega\), which makes it easier to analyze. The goal is now to determine which \(N(\tau)\) are irreducible, and which \(\sigma\) they correspond to. We will do this in two different settings, depending which restriction of \(\omega\) we want to understand.

3.2. **Restriction to** \(O(m, n; \mathbb{R})\). Our very first step uses the dual pair \((G, G') = (U(p), U(m, n))\) to restrict \(\omega\) to \(U(m, n)\). Once we understand the restriction to \(U(m, n)\), we will be able to restrict further to \(O(m, n; \mathbb{R})\).

We first decompose \(\omega\) under the action of the compact group \(U(p)\). We therefore obtain a decomposition of the form \(\omega = \bigoplus_{\sigma} (\sigma \otimes \theta(\sigma)) = \bigoplus_{\sigma} (\sigma \otimes E_{\tau})\), where \(\sigma\) is an irreducible representation of \(U(p)\) and \(E_{\tau}\) is a representation of \(U(m, n)\). Explicitly, the correspondence is given between \(\sigma\) and the lowest weight \(\tau'\) of \(E_{\tau}\) as follows:

\[
\sigma = \left(\frac{m-n}{2}, \ldots, \frac{m-n}{2}, \frac{m-n}{2}, \ldots, \frac{m-n}{2}, b_1 + \frac{m-n}{2}, \ldots, b_l + \frac{m-n}{2}\right)
\]

\[
\tau' = \left(\frac{p}{2}, \ldots, \frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}\right) \oplus \left(-\frac{p}{2}, \ldots, -\frac{p}{2}, b_1 - \frac{p}{2}, \ldots, b_l - \frac{p}{2}\right),
\]

with the constraints \(k + l \leq p\), \(k \leq m\), \(l \leq n\).

The issue is that we have a correspondence between the highest weight \(\sigma\) for \(U(p)\) and the lowest weight \(\tau'\) for \(U(m, n)\), but our irreducibility criterion works for a highest weight. We can solve this problem using a conjugation by the longest element of the Weyl group of \(G'\), denoted by \(w_0\). This conjugation will send \(U(m, n)\) to \(U(n, m)\), but it will also switch positive and negative roots, so we will be able to
work with a highest weight. To avoid unnecessary confusion of notation, we will still denote our group by $G'$ after conjugation by $w_0$.

Now, instead of working with the lowest weight
\[ \tau' = (a_1 + \frac{p}{2}, \ldots, a_k + \frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}) \oplus (-\frac{p}{2}, \ldots, -\frac{p}{2}, b_1 - \frac{p}{2}, \ldots, b_l - \frac{p}{2}) \]
on $U(m, n)$, we can equivalently work with the highest weight
\[ \tau = (-\frac{p}{2}, \ldots, -\frac{p}{2}, b_1 - \frac{p}{2}, \ldots, b_l - \frac{p}{2}) \oplus (a_1 + \frac{p}{2}, \ldots, a_k + \frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}) \]
on $U(n, m)$. Since we start with a highest weight $\sigma$ for $U(p)$ expressed as
\[ \sigma = (a_1 + \frac{m-n}{2}, \ldots, a_k + \frac{m-n}{2}, \frac{m-n}{2}, \ldots, \frac{m-n}{2}, b_1 + \frac{m-n}{2}, \ldots, b_l + \frac{m-n}{2}) \]
we have the conditions $a_1 \geq \cdots \geq a_k \geq 0$ and $0 \geq b_1 \geq \cdots \geq b_l$.

We can now apply the irreducibility criterion given by proposition 3 to determine in which cases $N(\tau)$ is irreducible. We are working with $G' = U(n, m)$ and $K' = U(n) \times U(m)$. The complexified Lie algebra of $G'$ is $\mathfrak{g}_{m+n}(\mathbb{C})$, so we have a root system of type $A_n$, which implies that this criterion is both necessary and sufficient for the irreducibility of $N(\tau)$.

We apply the criterion to
\[ \tau = (-\frac{p}{2}, \ldots, -\frac{p}{2}, b_1 - \frac{p}{2}, \ldots, b_l - \frac{p}{2}) \oplus (a_1 + \frac{p}{2}, \ldots, a_k + \frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}). \]
Therefore, we need to carefully analyze the root systems occurring here:

- The roots for $G'$ have the form $e_i - e_j$ with $1 \leq i, j \leq n + m$ and $i \neq j$. If $i \leq j$, we have a positive root.
- The non-compact positive roots for $G'$ are the roots not coming from $K'$, i.e., roots of the form $e_i - e_j$ with $1 \leq i \leq n$ and $n + 1 \leq j \leq n + m$. We will write $\alpha_{ij} = e_i - e_j$ for the corresponding non-compact root.
- We can calculate $\rho$ as half the sum of the positive roots, and we obtain:
\[ \rho = \left( \frac{m+n-1}{2}, \ldots, \frac{m+n-2i+1}{2}, \ldots, \frac{-m-n+1}{2} \right). \]

We look at $\tau = (-\frac{p}{2}, \ldots, -\frac{p}{2}, b_1 - \frac{p}{2}, \ldots, b_l - \frac{p}{2}) \oplus (a_1 + \frac{p}{2}, \ldots, a_k + \frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$, with $0 > b_1 \geq \cdots \geq b_l$ and $a_1 \geq \cdots \geq a_k > 0$. For simplicity of notations, we will assume that $a_i$ and $b_j$ can be equal to zero, and we will rewrite $\tau$ as
\[ \tau = (b_1 - \frac{p}{2}, \ldots, b_n - \frac{p}{2}) \oplus (a_{n+1} + \frac{p}{2}, \ldots, a_{n+m} + \frac{p}{2}) \]
with $b_n \leq \cdots \leq b_1 \leq 0$ and $0 \leq a_{n+m} \leq \cdots \leq a_{n+1}$. 

\[ \cdots \]
Here we obtain $(\tau + \rho)_{\alpha_{ij}} = b_i - a_j + j - i - p$ with $1 \leq i \leq n$ and $n+1 \leq j \leq n+m$. Since we know that $b_i - a_j \leq 0$ for all $i, j$, we conclude that if $p \geq m + n - 1$, then $(\tau + \rho)_{\alpha_{ij}}$ is non-positive for all $i, j$, so there is nothing to check using this criterion. So for $p \geq m + n - 1$, we have the irreducibility of $N(\tau)$.

However, we cannot improve this condition on $p$ without being specific about the values of $a_i$ and $b_j$, as we can see by working out some small examples. So we will only use the case where $p \geq m + n - 1$ for further work.

Indeed, if we look at the case $m = n = p = 2$, where $p \geq m + n - 1$, we can play with values of $a_i$ and $b_j$ to get an irreducible $N(\tau)$ or a reducible $N(\tau)$:

1. $b_2 = -2 \leq b_1 = -1 \leq 0 \leq a_4 = 1 \leq a_3 = 2$ gives $\tau = (-2, -3) \oplus (3, 2)$, and $(\tau + \rho)_{\alpha_{ij}} < 0$ for all $i, j$, meaning that $N(\tau)$ is irreducible.
2. $b_2 = -1 \leq b_1 = 0 \leq 0 \leq a_4 = 0 \leq a_3 = 1$ gives $\tau = (-1, -2) \oplus (2, 1)$, and $(\tau + \rho)_{\alpha_{ij}} < 0$ except for $i = 1, j = 4$. However, there is no non-compact root $\alpha_{ij}$ such that $(\tau + \rho)_{\alpha_{ij}} = 0$. By the irreducibility criterion, and since we have a root system of type $A_n$, $N(\tau)$ is reducible.

The first example shows that $p \geq m + n - 1$ is a sufficient but not necessary condition for the irreducibility of $N(\tau)$ and therefore for the projectivity of the restriction of the oscillator representation to $O(m, n; \mathbb{R})$. However, the second example does not mean that it is not possible to improve this criterion: in this case, $N(\tau)$ is reducible and our method cannot be used. But this does not mean that the restriction will not necessarily be projective.

3.2.1. Conclusion for $O(m, n; \mathbb{R})$. We just saw that if $p \geq m + n - 1$, then $N(\tau)$ is irreducible for any $\tau$. In particular, we have

$$E_\tau = N(\tau) = \mathfrak{U}(\mathfrak{gl}_{m+n}(\mathbb{C})) \otimes_{\mathfrak{U}(\mathfrak{q}')} F_\tau,$$

since the complexified Lie algebra of $U(m, n)$ is equal to $\mathfrak{gl}_{m+n}(\mathbb{C})$. However, this is a restriction to $U(m, n)$, and we would like to restrict further to $O(m, n; \mathbb{R})$. Now we obtain

$$E_\tau \mid_{\mathfrak{o}(m, n, \mathbb{C})} = \left( \mathfrak{U}(\mathfrak{gl}_{m+n}(\mathbb{C})) \otimes_{\mathfrak{U}(\mathfrak{q}')} F_\tau \right) \mid_{\mathfrak{o}(m, n, \mathbb{C})}$$

$$= \mathfrak{U}(\mathfrak{o}(m, n, \mathbb{C})) \otimes_{\mathfrak{U}(\mathfrak{o}(m, \mathbb{C}) \times \mathfrak{o}(n, \mathbb{C}))} (F_\tau \mid_{\mathfrak{o}(m, \mathbb{C}) \times \mathfrak{o}(n, \mathbb{C}))}.$$

This identification of the restriction is not obvious, but will be proved in the next section, namely in theorem 3. Now, $F_\tau \mid_{\mathfrak{o}(m, \mathbb{C}) \times \mathfrak{o}(n, \mathbb{C})}$ is an $(\mathfrak{o}(m, n, \mathbb{C}), O(m, \mathbb{R}) \times O(n, \mathbb{R}))$-module, so proposition 4 implies that $E_\tau \mid_{\mathfrak{o}(m, n, \mathbb{C})}$ is a projective $\mathfrak{o}(m, n, \mathbb{C})$-module. This is enough to prove:

**Theorem 1.** If $p > m + n$, the restriction of the oscillator representation $\omega$ of $Sp(2p(m + n), \mathbb{R})$ to $O(m, n; \mathbb{R})$ is projective.
Proof. Under the action of $U(p)$, we had the decomposition $\omega = \bigoplus_{\sigma} (\sigma \otimes E_\tau)$. We saw previously that if $p \geq m + n - 1$, then $E_\tau = N(\tau)$ is a projective $\mathfrak{o}(m, n, \mathbb{C})$-module. Since $\mathfrak{o}(m, n, \mathbb{C})$ does not act on $\sigma$, which is a finite dimensional space, we obtain that $\sigma \otimes E_\tau$ is projective as an $\mathfrak{o}(m, n, \mathbb{C})$-module. We decomposed $\omega$ as a direct sum of such spaces, and the direct sum of projective modules is also projective, so we conclude that the restriction of $\omega$ is a projective $\mathfrak{o}(m, n, \mathbb{C})$-module. \qed

3.3. Restriction to $Sp(2p, \mathbb{R})$. We want to apply a similar method to understand the restriction of the oscillator representation to $Sp(2p, \mathbb{R})$. To do so, we will first decompose $\omega$ under the action of $O(m, \mathbb{R}) \times O(n, \mathbb{R})$. We can then write $\omega$ as $\omega = \omega_m \otimes \omega_n^*$, where $\omega_m$ is a highest weight module for $O(m, \mathbb{R})$ and $\omega_n^*$ is a lowest weight module for $O(n, \mathbb{R})$. Since $O(m, \mathbb{R})$ and $O(n, \mathbb{R})$ are compact, we can decompose each piece as before. So we have $\omega_m = \bigoplus_{\sigma} (\sigma \otimes \theta(\sigma)) = \bigoplus_{\sigma} (\sigma \otimes E_\tau)$, summing over all the irreducible representations $\sigma$ of $O(m, \mathbb{R})$. Similarly, $\omega_n^* = \bigoplus_{\bar{\sigma}} (\bar{\sigma} \otimes \theta(\bar{\sigma})) = \bigoplus_{\bar{\sigma}} (\bar{\sigma} \otimes E_{\bar{\tau}})$, summing over all the irreducible representations $\bar{\sigma}$ of $O(n, \mathbb{R})$. Here $E_\tau$, $E_{\bar{\tau}}$ denote representations of $Sp(2p, \mathbb{R})$, since $(O(m, \mathbb{R}) \times O(n, \mathbb{R}), Sp(2p, \mathbb{R}) \times Sp(2p, \mathbb{R}))$ is a dual pair in $Sp(2p(m + n), \mathbb{R})$. We can therefore express $\omega$ as

$$\omega = \omega_m \otimes \omega_n^* = \left( \bigoplus_{\sigma} (\sigma \otimes E_\tau) \right) \otimes \left( \bigoplus_{\bar{\sigma}} (\bar{\sigma} \otimes E_{\bar{\tau}}) \right) = \bigoplus (\sigma \otimes \bar{\sigma} \otimes (E_\tau \times E_{\bar{\tau}})).$$

We can now look closer at one of the dual pairs, say $(O(n, \mathbb{R}), Sp(2p, \mathbb{R}))$. Here, the correspondence between $\sigma$ and the lowest weight $\tau'$ of $E_\tau$ is given by:

$$\sigma = (a_1, \ldots, a_k, 0, \ldots, 0; \epsilon) \quad \mapsto \quad \tau' = (a_1 + \frac{n}{2}, \ldots, a_k + \frac{n}{2}, \frac{n}{2} + 1, \ldots, \frac{n}{2} + 1, \frac{n}{2}, \ldots, \frac{n}{2}),$$

with the constraints $k \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $k + \frac{1 - \epsilon}{2}(n - 2k) \leq p$.

So we use $G' = Sp(2p, \mathbb{R})$ and $K' = U(p)$. Here we do not have a root system of type $A_n$, so the criterion is sufficient (but not necessary) for the irreducibility of $N(\tau)$. We have a correspondence between the highest weight $\sigma$ for $O(n, \mathbb{R})$ and the lowest weight $\tau'$ for $Sp(2p, \mathbb{R})$, but, as before, we need to write the corresponding highest weight $\tau$. We can again conjugate by the longest element of the Weyl group of $G$, which will switch positive and negative roots but not change $G$.

So instead of working with the lowest weight

$$\tau' = (a_1 + \frac{n}{2}, \ldots, a_k + \frac{n}{2}, \frac{n}{2} + 1, \ldots, \frac{n}{2} + 1, \frac{n}{2}, \ldots, \frac{n}{2}),$$
we can work with the highest weight
\[
\tau = \left( -\frac{n}{2}, \ldots, -\frac{n}{2}, -\frac{n-1}{2}, \ldots, -\frac{n}{2}, -a_k - \frac{n}{2}, \ldots, -a_1 - \frac{n}{2} \right).
\]

Since we start with a highest weight \(\sigma\) for \(O(n, \mathbb{R})\), we have \(a_1 \geq \cdots \geq a_k \geq 0\) and \(\epsilon = \pm 1\).

We will now apply proposition 3 to \(\tau\) on \(G' = Sp(2p, \mathbb{R})\). The root system occurring here has the following properties:

- The roots for \(G'\) have the form
  - \(e_i - e_j\) with \(1 \leq i, j \leq p\) and \(i \neq j\). If \(i \leq j\), it is a positive root.
  - \(\pm(e_i + e_j)\) with \(1 \leq i, j \leq p\) and \(i \neq j\), and \(e_i + e_j\) is a positive root.
  - \(\pm2e_i\) with \(1 \leq i \leq p\), and \(2e_i\) is a positive root.
- The non-compact positive roots for \(G'\) are the roots not coming from \(K'\), i.e., roots of the form \(e_i + e_j\) with \(1 \leq i, j \leq p\), \(i \neq j\) and roots of the form \(2e_i\) with \(1 \leq i \leq p\).
- We can calculate \(\rho\) as half the sum of the positive roots, and we obtain:
  \[
  \rho = (p, \ldots, \underbrace{p + 1 - i, \ldots, 1}_\text{\(i\)-th coordinate}).
  \]

Depending on the value of the parameter \(\epsilon\), we have different values of \(\tau\). We will therefore look at the two cases separately.

3.3.1. Case \(\epsilon = 1\). If \(\epsilon = 1\), we have \(\frac{1-\epsilon}{2}(n-2k) = 0\), so the highest weight \(\tau\) can be written as
\[
\tau = \left( -\frac{n}{2}, \ldots, -\frac{n}{2}, -a_k - \frac{n}{2}, \ldots, -a_1 - \frac{n}{2} \right).
\]

Therefore, the different products between \(\tau + \rho\) and a non-compact positive root are as follows:

\[
(\tau + \rho)_{2e_i} = \begin{cases} 
  p + 1 - i - \frac{n}{2} & \text{if } 1 \leq i \leq p - k \\
  p + 1 - i - a_{p+1-i} & \text{if } p - k < i \leq p 
\end{cases},
\]

\[
(\tau + \rho)_{e_i+e_j} = \begin{cases} 
  2p + 2 - i - j - n & \text{if } 1 \leq i, j \leq p - k \\
  2p + 2 - i - j - a_{p+1-i} & \text{if } 1 \leq i \leq p - k < j \leq p \\
  2p + 2 - i - j - a_{p+1-i} - a_{p+1-j} & \text{if } p - k < i, j \leq p
\end{cases}.
\]

If we take \(n \geq 2p\), all these products are non-positive, and by proposition 3, there is nothing to check: \(N(\tau)\) is irreducible. But we can do slightly better: the condition \(n \geq 2p - 1\) is also sufficient. Indeed, for \(n = 2p - 1\), we have \(p + 1 - i - \frac{n}{2} = \frac{3}{2} - i\), which is not an integer. All the other products are still non-positive, so the criterion
can be apply without any further checking, and \( N(\tau) \) is irreducible. And as before, we cannot improve this condition without being specific about the values \( a_k \).

3.3.2. Case \( \epsilon = -1 \). If \( \epsilon = -1 \), we have \( \frac{1 - \epsilon}{2}(n-2k) = n-2k \), so the highest weight \( \tau \) is more complicated. It can be written as

\[
\tau = \left(-\frac{n}{2}, \ldots, -\frac{n}{2}, -\frac{n}{2} - 1, \ldots, -\frac{n}{2} - 1, -a_k - \frac{n}{2}, \ldots, -a_1 - \frac{n}{2}\right).
\]

The different products between \( \tau + \rho \) and a non-compact positive root are:

\[
(\tau + \rho)_{2e_i} = \begin{cases} 
p + 1 - i - \frac{n}{2} & \text{if } 1 \leq i \leq p + k - n \\
p - i - \frac{n}{2} & \text{if } p + k - n < i \leq p - k, \\
p + 1 - i - \frac{n}{2} - a_{p+1-i} & \text{if } p - k < i \leq p
\end{cases}
\]

\[
(\tau + \rho)_{e_i + e_j} = \begin{cases} 
2p + 2 - i - j - n & \text{if } 1 \leq i, j \leq p + k - n \\
2p - i - j - n & \text{if } p + k - n < i, j \leq p - k \\
2p + 2 - i - j - n - a_{p+1-i} - a_{p+1-j} & \text{if } p - k < i, j \leq p \\
2p + 1 - i - j - n & \text{if } 1 \leq i \leq p + k - n \\
2p + 1 - i - j - n - a_{p+1-j} & \text{if } p + k - n < i \leq p - k \\
2p + 2 - i - j - n - a_{p+1-j} & \text{if } 1 \leq i \leq p + k - n \\
\end{cases}
\]

As before, if we take \( n \geq 2p \), all these products are non-positive, and by proposition 3, \( N(\tau) \) is irreducible. We can again refine this, since for \( n = 2p - 1 \), we have \( p + 1 - i - \frac{n}{2} = \frac{3}{2} - i \) which is not an integer, and all the other products are non-positive. So \( N(\tau) \) is irreducible for all \( n \geq 2p - 1 \), and this is the best that we can do to stay in a general case.

3.3.3. Conclusion for \( Sp(2p, \mathbb{R}) \). As we saw by writing the oscillator representation as

\[
\omega = \omega_m \otimes \omega_n^* = \left( \bigoplus_{\sigma} (\sigma \otimes E_\sigma) \right) \otimes \left( \bigoplus_{\bar{\sigma}} (\bar{\sigma} \otimes E_{\bar{\sigma}}) \right),
\]

we can analyze the situation using the pairs \( (O(m, \mathbb{R}), Sp(p, \mathbb{R})) \) and \( (O(n, \mathbb{R}), Sp(p, \mathbb{R})) \). We now need to put our results together. For this purpose, we write \( q^+ = \mathfrak{t}' + \mathfrak{b}' + \mathfrak{b}' \), where \( \mathfrak{b}' + \mathfrak{b}' \) are opposite choices of Borel subalgebras in \( \mathfrak{sp}(2p, \mathbb{C}) \) and \( \mathfrak{t}' \) is the complexified Lie algebra of \( K' \). The fact that \( \omega_m \) is a highest weight
module means that $E_\tau$ is a quotient of $N^+(\tau) = \mathcal{U}(\mathfrak{sp}(2p, \mathbb{C})) \otimes_{\mathcal{U}(\mathfrak{q}^+)} F_\tau$ but $E_\tau$ is a quotient of $N^-(\tau) = \mathcal{U}(\mathfrak{sp}(2p, \mathbb{C})) \otimes_{\mathcal{U}(\mathfrak{q}^-)} F_\tau$ since $\omega_n^+$ is a lowest weight module. However, our irreducibility criterion from proposition 3 can be applied to any choice of Borel subalgebra, so we can use our previous calculation for both cases.

If $n, m \geq 2p-1$, we have the identifications $E_\tau = N^+(\tau) = \mathcal{U}(\mathfrak{sp}(2p, \mathbb{C})) \otimes_{\mathcal{U}(\mathfrak{q}^+)} F_\tau$ and $E_\tau = N^-(\tau) = \mathcal{U}(\mathfrak{sp}(2p, \mathbb{C})) \otimes_{\mathcal{U}(\mathfrak{q}^-)} F_\tau$. Therefore, we can write $\omega$ as

$$\omega = \bigoplus_{\sigma, \bar{\sigma}} ((\sigma \otimes \bar{\sigma}) \otimes (E_\tau \otimes E_\tau)) = \bigoplus_{\sigma, \bar{\sigma}} (\omega \otimes \big(\mathcal{U}(\mathfrak{sp}(2p, \mathbb{C})) \otimes_{\mathcal{U}(\mathfrak{q}^+)} F_\tau\big) \otimes \big(\mathcal{U}(\mathfrak{sp}(2p, \mathbb{C})) \otimes_{\mathcal{U}(\mathfrak{q}^-)} F_\tau\big)).$$

Again, as in the first case, $\mathcal{U}(\mathfrak{sp}(2p, \mathbb{R}))$ does not act on the finite dimensional space $\sigma \otimes \bar{\sigma}$. In the next section, we will show that the tensor product of $E_\tau$ and $E_\tau$ is a projective $\mathfrak{sp}(2p, \mathbb{C})$-module. So we can write $\omega$ as a direct sum of such objects, and consequently we proved:

**Theorem 2.** If $\min(n, m) > 2p$, the restriction of the oscillator representation $\omega$ of $\mathcal{U}(\mathfrak{sp}(2p(m + n), \mathbb{R}))$ to $\mathcal{U}(\mathfrak{sp}(2p, \mathbb{R}))$ is projective.

4. Identifications and Projectivity

This technical section is here to complete the proof of both theorems 1 and 2, through identifications of some tensor products.

4.1. **Restriction to $O(m, n; \mathbb{R})$.** The goal here is to prove that we have an identification

$$\left(\mathcal{U}G_{m+n}(\mathbb{C})\right) \otimes_{\mathcal{U}(\mathfrak{q}')} F_{\tau} \big|_{\mathcal{U}(\mathfrak{o}(m, n, \mathbb{C}))} = \mathcal{U}(\mathfrak{o}(m, n, \mathbb{C})) \otimes_{\mathcal{U}(\mathfrak{o}_m(\mathbb{C}) \times \mathfrak{o}_n(\mathbb{C}))} \left(\mathcal{U}G_{m+n}(\mathbb{C}) \otimes_{\mathcal{U}(\mathfrak{q}')} F_{\tau} \big|_{\mathcal{U}(\mathfrak{o}_m(\mathbb{C}) \times \mathfrak{o}_n(\mathbb{C}))}\right),$$

with notations as in the previous section. We will analyze this restriction for a more general case, i.e., for a module of the form $\mathcal{U}G_{m+n}(\mathbb{C}) \otimes_{\mathcal{U}(\mathfrak{q}')} E$.

4.1.1. **Definitions and notations.** We keep the notations used in the previous section. So we have $G' = U(m, n)$, with corresponding complex Lie algebra $\mathfrak{g}'$. We consider the Cartan decomposition $\mathfrak{g}' = \mathfrak{t}' + \mathfrak{p}'$. We can identify $\mathfrak{g}' = \mathfrak{gl}_{m+n}(\mathbb{C})$,

$$\mathfrak{t}' = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \mid X \in \mathfrak{gl}_m(\mathbb{C}), Y \in \mathfrak{gl}_n(\mathbb{C}) \right\} \cong \mathfrak{gl}_m(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$$

and $\mathfrak{p}' = \mathfrak{p}^+ + \mathfrak{p}^-$ where

$$\mathfrak{p}' = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mid A \in M_{m,n}(\mathbb{C}), B \in M_{n,m}(\mathbb{C}) \right\},$$
\[ p^+ = \{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \mid A \in M_{m,n}(\mathbb{C}) \} \text{ and } p^- = \{ \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \mid B \in M_{n,m}(\mathbb{C}) \}. \]

We will also write \( q' = \mathfrak{t}' + p^- \). Note that using \( q' \) for this sum is not misleading, since we can choose the Borel subalgebra \( \mathfrak{b}' \) so that this new definition agrees with the first one as \( q' = \mathfrak{t}' + \mathfrak{b}' \).

We will also consider the subgroup of \( G' \) given by \( H = O(m, n, \mathbb{R}) \), with complexified Lie algebra \( \mathfrak{h} = \mathfrak{o}(m, n, \mathbb{C}) \) and Cartan decomposition \( \mathfrak{h} = \mathfrak{t} + \mathfrak{p} \). Here we have

\[ \mathfrak{t} = \{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \mid X \in \mathfrak{o}_m(\mathbb{C}), Y \in \mathfrak{o}_n(\mathbb{C}) \} \cong \mathfrak{o}_m(\mathbb{C}) \times \mathfrak{o}_n(\mathbb{C}) \]

and

\[ p = \{ \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \mid A \in M_{m,n}(\mathbb{C}) \}. \]

Note that here we do not have a decomposition of \( p \) as a sum \( p^+ + p^- \).

**Lemma 1.** The map \( p \xrightarrow{\iota} p' \xrightarrow{\pi} p^+ \) is a bijection, where \( \iota : p \to p' \) is the inclusion and \( \pi : p' \to p^+ \) is the projection.

We consider a finite dimensional \( \mathfrak{t}' \)-module \( E \). By letting \( p^- \) act trivially on \( E \), this will become a \( q' \)-module and we can form \( V_E^+ = \bigoplus_{n \geq 0} \mathfrak{sl}(q') \otimes_{\mathfrak{u}(q')} E \), which is a \( \mathfrak{sl}(q') \)-module. Note that as vector spaces, we have the isomorphism \( V_E^+ \cong S(p^+) \otimes E \), with \( S(p^+) \) the symmetric algebra on \( p^+ \).

We can also use \( E \) to form a \( \mathfrak{u}(\mathfrak{h}) \)-module. By restriction, we can see \( E \) as a \( \mathfrak{t} \)-module, denoted \( E |_{\mathfrak{t}} \), and form the tensor product \( V_E = \bigoplus_{n \geq 0} \mathfrak{sl}(\mathfrak{h}) \otimes_{\mathfrak{u}(\mathfrak{t})} (E |_{\mathfrak{t}}) \). Similarly, there is an isomorphism of vector spaces \( V_E \cong S(p) \otimes (E |_{\mathfrak{t}}) \).

**4.1.2. Gradings and filtrations.** We can define a filtration \( V_E = \bigoplus_n (V_E)_n/(V_E)_{n-1} \) by setting

\[ (V_E)_n = \sum_{r \leq n} S(p)[r] \mathfrak{u}(\mathfrak{t}) \otimes_{\mathfrak{u}(\mathfrak{t})} (E |_{\mathfrak{t}}). \]

We therefore have \( (V_E)_0 = 1 \otimes (E |_{\mathfrak{t}}) \) and

\[ (V_E)_n/(V_E)_{n-1} \cong S(p)[n] \mathfrak{u}(\mathfrak{t}) \otimes_{\mathfrak{u}(\mathfrak{t})} (E |_{\mathfrak{t}}). \]

By a similar construction on \( V_E^+ \), we have the filtration \( V_E^+ = \bigoplus_n (V_E^+)_n/(V_E^+)_{n-1} \), where

\[ (V_E^+)_n = \sum_{r \leq n} S(p^+)[r] \mathfrak{u}(\mathfrak{t}') \otimes_{\mathfrak{u}(\mathfrak{t}')} E \]

And we obtain \( (V_E^+)_0 = 1 \otimes E \), and

\[ (V_E^+)_n/(V_E^+)_{n-1} \cong S(p^+)[n] \mathfrak{u}(\mathfrak{t}') \otimes_{\mathfrak{u}(\mathfrak{t}')} E. \]
4.1.3. Identification of the restriction $V_E^+ \mid \mathfrak{h}$. By Frobenius reciprocity, we have a map

$$T : V_E \to V_E^+, \ 1 \otimes e \mapsto 1 \otimes e$$

for any $e \in E$. This map is extended to the whole $V_E$ by looking at the action of an element of $S(p)$ on $1 \otimes e$. It is enough to look at the action of $S(p)[n]$ and extend by linearity.

Recall that using lemma 1, if we start with an element $x \in p$, we can use $i$ to see $x$ as an element of $p_h$ and then we can decompose $x = y + z$ where $y \in p^+_h$ and $z \in p^-_h$.

We will write $\{x_1, \ldots, x_r\}$ for a basis of $p$. So an element of $S(p)[n]$ can be written $x_1 \ldots x_n$, with possible repetitions in the indices. Recall that we have an inclusion of $p$ in $p'$, and $p$ is in bijection with $p^+$ (we could do the same with $p^-$) so we can choose a basis $\{y_1, \ldots, y_r\}$ of $p^+$ and a basis of $\{z_1, \ldots, z_r\}$ of $p^-$ such that $x_i = y_i + z_i$ in $p'$.

Consequently, we extend the map $T$ so that

$$T(x_1 \ldots x_n \otimes e) = (y_1 + z_1) \ldots (y_n + z_n) \otimes e$$

for any $e \in E$.

Using this definition of $T$, we want to show the following result:

**Proposition 7.** The map $T : V_E \to V_E^+$ defined previously preserves the filtrations.

This is just a consequence of the following lemma, whose proof consists only of technical calculations and is therefore omitted here:

**Lemma 2.** The action of $(y_1 + z_1) \ldots (y_n + z_n)$ on $1 \otimes e$ is given by

$$(y_1 + z_1) \ldots (y_n + z_n) \otimes e = y_1 \ldots y_n \otimes e + \text{ elements of } (V_E^+)^{n-1}.$$

We can now prove the main result, namely:

**Theorem 3.** The map $T : V_E \to V_E^+$ is an isomorphism of $\mathfrak{u}(\mathfrak{h})$-modules, and it is induced by an isomorphism of $S(p)$-modules on the graded spaces $T_{Gr} : Gr(V_E) \to Gr(V_E^+)$ by the identification $p \cong p^+$.

**Proof.** By proposition 7, we know that $T((V_E)_n) \subset (V_E^+)_n$. We will now show that the restriction $T \mid (V_E)_n$ is surjective onto $(V_E^+)_n$. Indeed, a basis of $(V_E)_n$ is given by elements of the form $x_1 \ldots x_r \otimes e$ with $r \leq n$, $x_i \in p$ and $e \in E$, and a basis of $(V_E^+)$ is given by elements of the form $y_1 \ldots y_s \otimes e$ with $s \leq n$, $y_i \in p^+$ and $e \in E$. So the description by lemma 2 of the image $x_1 \ldots x_n \otimes e$ as

$$T(x_1 \ldots x_n \otimes e) = y_1 \ldots y_n \otimes e + \text{ elements of } (V_E^+)^{n-1}$$

is enough to show the surjectivity of $T : (V_E)_n \to (V_E^+)_n$, by induction on $n$ and using the linearity of $T$. 


We also need to compare the dimensions of \((V_E)_n\) and \((V_E^+)_n\), as complex vector spaces. We have the vector space identifications
\[
(V_E)_n = \sum_{r \leq n} S(p)[r] \mathfrak{U}(\mathfrak{t}) \otimes \mathfrak{U}(\mathfrak{t}) E |_t \cong \sum_{r \leq n} S(p)[r] \otimes E
\]
and
\[
(V_E^+)_n = \sum_{r \leq n} S(p^+)[r] \mathfrak{U}(\mathfrak{t}') \otimes \mathfrak{U}(\mathfrak{t}') E \cong \sum_{r \leq n} S(p^+)[r] \otimes E.
\]
Using these descriptions and the bijection between \(p\) and \(p^+\), we deduce that the dimensions of \((V_E)_n\) and \((V_E^+)_n\) have to be equal.

By proposition 1, we conclude that \(T\) is an isomorphism. This shows that
\[
\left( \mathfrak{U}(\mathfrak{g}') \otimes \mathfrak{U}(\mathfrak{q}') \right) E |_h \cong \mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{t}) (E |_t),
\]
and concludes the proof of theorem 1. \(\square\)

4.2. Tensor product on \(Sp(2p, \mathbb{R})\). Our goal is now to show that the tensor product
\[
\left( \mathfrak{U}(\mathfrak{sp}(2p, \mathbb{C})) \otimes \mathfrak{U}(\mathfrak{q}^+ \mathfrak{F}_T) \otimes \mathfrak{U}(\mathfrak{q}^- \mathfrak{F}_T') \right)
\]
is a projective \(\mathfrak{U}(\mathfrak{sp}(2p, \mathbb{C}))\)-module. We will show this in a more general case, with
\[
\left( \mathfrak{U}(\mathfrak{sp}(2p, \mathbb{C})) \otimes \mathfrak{U}(\mathfrak{q}^+ \mathfrak{F}) \otimes \mathfrak{U}(\mathfrak{q}^- \mathfrak{F}^\prime) \right),
\]
for some modules \(E\) and \(F\).

4.2.1. Definitions and notations. We fix now, as in the previous section, \(g' = \mathfrak{sp}_{2p}(\mathbb{C})\), and we have the corresponding Cartan decomposition of \(g'\) as \(g' = \mathfrak{t}' + \mathfrak{p}'\). By choice of \(g' = \mathfrak{sp}_{2p}(\mathbb{C})\), we therefore have
\[
\mathfrak{t}' = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^T \end{pmatrix} \mid X \in \mathfrak{gl}_p(\mathbb{C}) \right\} \cong \mathfrak{gl}_p(\mathbb{C}) \quad \text{and} \quad \mathfrak{p}' = \left\{ \begin{pmatrix} Y \\ Z \end{pmatrix} \mid Y = Y^T, Z = Z^T \right\},
\]
that we can decompose further as
\[
\mathfrak{p}^+ = \left\{ \begin{pmatrix} Y \\ Z \end{pmatrix} \mid Y = Y^T \right\} \quad \text{and} \quad \mathfrak{p}^- = \left\{ \begin{pmatrix} Y \\ Z \end{pmatrix} \mid Z = Z^T \right\}.
\]

We note that \(\mathfrak{p}^+\) and \(\mathfrak{p}^-\) are both commutative Lie algebras, but they do not commute with each other. Indeed, we have \(0 \neq [\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{t}\). We will let \(\{\alpha_i\}\) denote an ordered basis of \(\mathfrak{p}^-\) and \(\{\beta_j\}\) denote an ordered basis of \(\mathfrak{p}^+\).

We fix a Cartan subalgebra \(\mathfrak{t}'\) of \(g'\) that is also a Cartan subalgebra for \(\mathfrak{t}'\). We define two more subalgebras of \(g'\) as follows: \(\mathfrak{q}^+ = \mathfrak{t}' + \mathfrak{p}^+\) and \(\mathfrak{q}^- = \mathfrak{t}' + \mathfrak{p}^-\).
We will now consider two finite dimensional $\mathfrak{t}'$-modules $E$ and $F$. We can let $\mathfrak{p}^+$ act on $E$ by zero, so that $E$ becomes a $\mathfrak{q}^+$-module. Similarly, we let $\mathfrak{p}^-$ act on $F$ by zero and obtain a $\mathfrak{q}^-$-module. We can therefore define

$$V_E = \mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{b}^+)} E$$

and

$$V_F = \mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{b}^-)} F,$$

that are both $(\mathfrak{g}', K')$-modules by construction. We also define

$$V = \mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{b}')} (E \otimes F).$$

4.2.2. Gradings and filtrations. By Poincaré-Birkhoff-Witt theorem, we have a grading on both $\mathfrak{U}(\mathfrak{p}^+)$ and $\mathfrak{U}(\mathfrak{p}^-)$, defined using a basis of $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$). Since $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are commutative, we have $\mathfrak{U}(\mathfrak{p}^+) = S(\mathfrak{p}^+)$ and $\mathfrak{U}(\mathfrak{p}^-) = S(\mathfrak{p}^-)$, i.e., the universal enveloping algebra is the same as the symmetric algebra. We can therefore identify the graded piece of a degree $n$, denoted $\mathfrak{U}(\mathfrak{p}^+)[n]$ with the space of homogeneous polynomials of degree $n$, written as $S(\mathfrak{p}^+)[n]$ (and similarly for $\mathfrak{p}^-$).

We write $M_n$ for the subspace of elements of degree less or equal to $n$ in $\mathfrak{U}(\mathfrak{p}')$, i.e., we have

$$M_n = \sum_{r+s \leq n} (S(\mathfrak{p}^-)[r] \otimes S(\mathfrak{p}^+)[s]) \cong \oplus_{i \leq n} S(\mathfrak{p}')[i].$$

This allows us to define a filtration on $V$: we can write $V = \oplus_n V_n / V_{n-1}$ where

$$V_n = M_n \mathfrak{U}(\mathfrak{t}') \otimes_{\mathfrak{U}(\mathfrak{b}')} (E \otimes F).$$

Note that $V_0 = E \otimes F$. By the description of $M_n$ as $\oplus_{i \leq n} S(\mathfrak{p}')[i]$, we observe that the quotient $M_n / M_{n-1}$ can be identified with $S(\mathfrak{p}')[n]$. Therefore we obtain

$$V_n / V_{n+1} = S(\mathfrak{p}')[n] \otimes_{\mathfrak{U}(\mathfrak{b}')} (E \otimes F).$$

We can define a similar filtration on $V_E \otimes V_F$: 

$$(V_E \otimes V_F)_n = \sum_{r+s \leq n} \left( (M_r \mathfrak{U}(\mathfrak{t}) \otimes_{\mathfrak{U}(\mathfrak{b}^+)} E) \otimes (M_s \mathfrak{U}(\mathfrak{t}) \otimes_{\mathfrak{U}(\mathfrak{b}^-)} F) \right).$$

We observe that, as vector spaces, this is equivalent to

$$(V_E \otimes V_F)_n = \sum_{r+s \leq n} \left( S(\mathfrak{p}^-)[r] \otimes_{\mathfrak{U}(\mathfrak{b}^+)} E \right) \otimes \left( S(\mathfrak{p}^+)[s] \otimes_{\mathfrak{U}(\mathfrak{b}^-)} F \right).$$

We obtain

$$(V_E \otimes V_F)_n / (V_E \otimes V_F)_{n-1} = \sum_{r+s = n} \left( S(\mathfrak{p}^-)[r] \otimes_{\mathfrak{U}(\mathfrak{b}^+)} E \right) \otimes \left( S(\mathfrak{p}^+)[s] \otimes_{\mathfrak{U}(\mathfrak{b}^-)} F \right).$$
4.2.3. Identification of the tensor product $V_E \otimes V_F$. Since $E \otimes F = V_0$ is naturally a subset of $V$, we can use Frobenius reciprocity to extend this inclusion to a map 

$$T : V \rightarrow V_E \otimes V_F,$$

for all $e \in E$ and $f \in F$. We extend this map so that it is is compatible with the module structure and we want to show that it preserves the filtrations defined previously. Note that we will sometimes write $e \otimes f$ instead of $1 \otimes (e \otimes f)$ to simplify notations.

Since $\{\alpha_i\}$ is a basis of $p^-$ and $\{\beta_j\}$ a basis of $p^+$, we can write any basis element of $S(p)$ in the form $\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l$ with possible repetitions in the indices. In particular, the action of the element $\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l$ on $e \otimes f$ is given by $\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l \otimes (e \otimes f)$, which is an element in $V_k \otimes V_l$ in the filtration described above. This element should be mapped by $T$ to

$$(\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l) \cdot T(e \otimes f) = (\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l) \cdot ((1 \otimes e) \otimes (1 \otimes f)).$$

The next lemma implies that the filtrations are preserved.

**Lemma 3.** With the notations above, we have

$$T((\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l)(e \otimes f)) = (\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l) \cdot T(e \otimes f) = \alpha_1 \ldots \alpha_k e \otimes \beta_1 \ldots \beta_l f + v,$$

with $v \in (V_E \otimes V_F)_{k+l-1}$.

**Proof.** This can be checked by direct computation and induction on $l$ and $k$. \qed

Our next step will be given by the following theorem:

**Theorem 4.** The map $T : V \rightarrow V_E \otimes V_F$, induced by an isomorphism of $S(p)$-modules on the graded spaces $T_G : Gr(V) \rightarrow Gr(V_E \otimes V_F)$, is an isomorphism of $\mathfrak{U}(g')$-modules.

**Proof.** We saw that $T(V_n) \subset (V_E \otimes V_F)_n$. We will show that $T_G : Gr(V) \rightarrow Gr(V_E \otimes V_F)$ is in fact surjective and that both $V_n$ and $(V_E \otimes V_F)_n$ have the same dimension, for every $n$.

Recall that we can write the graded piece of $V_E \otimes V_F$ of degree $n$ as

$$(V_E \otimes V_F)_n/(V_E \otimes V_F)_{n-1} = \sum_{k+l=n} \left( S(p^-)[k] \otimes_{\mathfrak{U}(b')} E \right) \otimes \left( S(p^+)[l] \otimes_{\mathfrak{U}(b-)} F \right).$$

So any element in $(V_E \otimes V_F)_n/(V_E \otimes V_F)_{n-1}$ is of the form $(\alpha_1 \ldots \alpha_k \otimes e) \otimes (\beta_1 \ldots \beta_l \otimes f)$ with $k + l = n$. The surjectivity of $T : V_n/V_{n-1} \rightarrow (V_E \otimes V_F)_n/(V_E \otimes V_F)_{n-1}$ is then clear from the work done before: if $k + l = n$, then we saw that

$$T(\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l \otimes (e \otimes f)) = (\alpha_1 \ldots \alpha_k \otimes e) \otimes (\beta_1 \ldots \beta_l \otimes f) + v,$$
with \( v \in (V_E \otimes V_F)_{n-1} \), so
\[
T(\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l \otimes (e \otimes f)) = (\alpha_1 \ldots \alpha_k \otimes e) \otimes (\beta_1 \ldots \beta_l \otimes f) \pmod{(V_E \otimes V_F)_{n-1}}
\]
when \( T \) is considered as a map on the graded pieces of degree \( n \). By definition, we
know that \( \alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l \otimes (e \otimes f) \) is an element of degree \( n = k + l \), so it is an
element of \( V_n/V_{n-1} \). This shows that
\[
T : V_n/V_{n-1} \to (V_E \otimes V_F)_{n}/(V_E \otimes V_F)_{n-1}
\]
is a surjective map for every \( n \).

Looking at the dimensions, we recall that we had
\[
(V_E \otimes V_F)_n = \sum_{r+s \leq n} \left( S(p^-)[r] \otimes \mathfrak{u}(b^+)^E \right) \otimes \left( S(p^+)[s] \otimes \mathfrak{u}(b^-)^F \right),
\]
and
\[
V_n = \sum_{i \leq n} S(p^i)[i] \otimes \mathfrak{u}(p') E \otimes F = \sum_{r+s \leq n} \left( S(p^-)[r] \otimes S(p^+)[s] \right) \otimes \mathfrak{u}(p') E \otimes F.
\]
Considered as \( \mathbb{C} \)-vector spaces, these two spaces have the same dimension, namely
\[
\dim(V_n) = \dim(E) \dim(F) \left( \sum_{r+s \leq n} \dim(S(p^-)[r]) \dim(S(p^+)[s]) \right) = \dim((V_E \otimes V_F)_n).
\]
A direct application of proposition 1 concludes the proof.

**Corollary 2.** The tensor product \( V_E \otimes V_F \) is a projective \( \mathfrak{U}(\mathfrak{g}') \)-module.

**Proof.** The previous theorem implies that \( V_E \otimes V_F \cong V \cong \mathfrak{U}(\mathfrak{g}') \otimes \mathfrak{u}(p') (E \otimes F) \).
We can then apply proposition 4, since \( E \otimes F \) is a \( (\mathfrak{g}', K') \)-module.

This concludes the proof of theorem 2.

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