Confining solutions of $SU(3)$ Yang - Mills theory

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Abstract

Spherically and cylindrically symmetric solutions of $SU(3)$ Yang - Mills theory are found, whose gauge potentials have confining properties. The spherically symmetric solutions give field distributions which have a spherical surface on which the gauge fields become infinite (which is similar to bag models of confinement), and the other solution has a potential which increases at large distances. The cylindrically symmetric solution describes a classical field “string” (flux tube) of the kind which is expected to form between quarks in the dual superconductor picture of confinement. These solutions with classical confining behaviour appear to be typical solutions for the classical $SU(3)$ Yang - Mills equations. This implies that the confining properties of the classical $SU(3)$ Yang - Mills theory are general properties of this theory.
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I. INTRODUCTION

The strong, nuclear interaction (quantum chromodynamics or QCD) is thought to be described by a quantized SU(3) gauge theory. In this paper we will examine solutions to the classical field equations of an SU(3) gauge theory. The reason for investigating these classical field configurations is to see if they might shed some light on the confinement mechanism which is hypothesized to occur in the strong interaction. Although a full explanation of the confinement mechanism may require that one consider the fully quantized theory, the solutions presented in this paper have properties which mimic the behaviour of various phenomenological explanations of confinement. In particular the various solutions exhibit a bag-like structure similar to the bag models \([1]\) of confinement, an almost linearly increasing potential such as those used in the study of heavy quark bound states \([2]\), and a string like structure as found in the dual superconducting picture of confinement.

The drawback of the classical configurations presented here is that they all have infinite field energy when their energy densities are integrated over. This can be compared with the finite energy monopole and dyon solutions of Yang-Mills field theory \([3]\) \([4]\). At the classical level one might expect that only solutions which have fields that become infinite (and thus have an infinite field energy) are capable of giving a confining behaviour. In the context of SU(2) Yang-Mills theory it has been shown \([5]\) that, at the classical level, finite energy solutions, like monopoles, do not lead to confinement, while infinite energy solutions do lead to confinement. Quantum effects may modify these classical solutions to soften the infinite field strengths and energies in the same way that quantum effects soften the singularity of the Coulomb solution in E&M.

II. SPHERICALLY SYMMETRIC ANSATZ

The ansatz for the SU(3) gauge field we take as in \([6]\) \([7]\):

\[
A_0 = \frac{2 \varphi(r)}{ir^2} \left( \lambda^5 x - \lambda^5 y + \lambda^7 z \right) + \frac{1}{2} \lambda^a \left( \lambda^a_{ij} + \lambda^a_{ji} \right) \frac{x^i x^j}{r^2} w(r),
\]  

(1)
\[ A_i^a = \left( \lambda_i^a - \lambda_j^a \right) \frac{x^j}{r^2} (f(r) - 1) + \lambda_{jk}^a \left( \epsilon_{ilj} x^k + \epsilon_{ilk} x^j \right) \frac{x^l}{r^3} v(r), \]  

(2)

here \( \lambda^a \) are the Gell-Mann matrices; \( a = 1, 2, \ldots, 8 \) is a color index; the Latin indices \( i, j, k, l = 1, 2, 3 \) are the space indices; \( i^2 = -1 \); \( r, \theta, \varphi \) are the usual spherically coordinates.

Substituting Eqs. (1) - (2) into the Yang-Mills equations

\[ \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} F^a_{\mu \nu} \right) + f^{abc} F^b_{\mu \nu} A^c_\mu = 0, \]  

(3)

gives the following system of equations for \( f(r), v(r), w(r) \) and \( \varphi(r) \)

\[ r^2 f'' = f^3 - f + 7f v^2 + 2vw\varphi - f \left( w^2 + \varphi^2 \right), \]  

(4)

\[ r^2 v'' = v^3 - v + 7vf^2 + 2fw\varphi - v \left( w^2 + \varphi^2 \right), \]  

(5)

\[ r^2 w'' = 6w \left( f^2 + v^2 \right) - 12fv\varphi, \]  

(6)

\[ r^2 \varphi'' = 2\varphi \left( f^2 + v^2 \right) - 4fvw. \]  

(7)

This set of equations is difficult to solve even numerically thus we will investigate various simplified cases when only two of the functions are nonzero. Under this assumption there are three cases. In the first case \((f, w) = 0\) or \((v, w) = 0\) Eqs. (4) - (7) reduce to a form similar to the system of equations studied in [8] which yield the well-known dyon solutions.

We will examine the cases where \( w = \varphi = 0, \) and \( f = \varphi = 0 \) (or \( v = \varphi = 0 \)).

**A. The SU(3) bag**

In this case we set \( w = \varphi = 0 \) so that Eqs. (4) - (7) reduce to the following form

\[ r^2 f'' = f^3 - f + 7f v^2, \]  

(8)

\[ r^2 v'' = v^3 - v + 7vf^2. \]  

(9)

To simplify the equations further we take \( f(r) = v(r) = q(r)/\sqrt{8}. \) This reduces Eqs. (8) - (10) to

\[ r^2 q'' = q(q^2 - 1) \]  

(10)
This is the Wu-Yang equation. In addition to the monopole solutions to this equation it is also known that this equation possesses a solution which becomes infinite on a spherical surface. If one lets this spherical surface be at \( r = r_0 \) then in the limit \( r \to r_0 \) the form of the solution approaches

\[
q(r) \approx \frac{\sqrt{2}r_0}{r_0 - r}
\]

(11)

Using Eq. (11) to find \( f(r) \), \( v(r) \) and inserting these back into Eq. (2) shows that the \( A_i^a \) gauge field develops a singularity on the sphere of radius \( r = r_0 \). It is easy to solve Eq. (10) numerically (for this work we used the Mathematica numerical differential solver routine). In solving Eq. (10) we considered that near \( r = 0 \) the function \( q(r) \) had a series expansion like

\[
q(r) = 1 + q_2 r^2 + ...
\]

(12)

where \( q_2 \) is some constant. Chosing a specific \( q_2 \) at some radius \( r = r_i \) sets the initial conditions on \( q(r_i), q'(r_i) \) for the numerical solution. The choice of these initial conditions determined the radius on which \( q(r) \) became singular. In Fig. 1 we show a typical example of \( q(r) \). This type of field configuration is somewhat similiar to a bag-like structure, and it has been shown that such structures lead to the confinement of a test particle placed in the field of this solution. If complex gauge fields are allowed or if scalar fields are introduced into the field equations it is possible to find analytical solutions which possesses gauge fields which are singular on some spherical surface of radius \( r = r_0 \). Several authors have remarked on the mathematical similiarity between the above solution and the Schwarzschild solution of general relativity, which leads to a gravitational type of confinement.

**B. The SU(3) bunker**

Here we examine the \( f = \varphi = 0 \) case. The case \( v = \varphi = 0 \) is entirely analogous. From Eqs. (1) - (7) the equations for the ansatz functions become
\[ r^2 v'' = v^3 - v - vw^2, \quad (13) \]
\[ r^2 w'' = 6wv^2. \quad (14) \]

Near \( r = 0 \) we took the series expansion form for \( v \) and \( w \) as

\[ v = 1 + v_2 \frac{r^2}{2!} + ..., \quad (15) \]
\[ w = w_3 \frac{r^3}{3!} + ... \quad (16) \]

where \( v_2, w_3 \) were constants which determined the initial conditions on \( v \) and \( w \) as in the last section. In the asymptotic limit \( r \to \infty \) the form of the solutions to Eqs. (13) - (14) approaches the form

\[ v \approx A \sin (x^\alpha + \phi_0), \quad (17) \]
\[ w \approx \pm \left[ \alpha x^\alpha + \frac{\alpha - 1}{4} \cos \left(2x^\alpha + 2\phi_0\right) \right], \quad (18) \]
\[ 3A^2 = \alpha (\alpha - 1). \quad (19) \]

where \( x = r/r_0 \) is a dimensionless radius and \( r_0, \phi_0, \) and \( A \) are constants. The second, strongly oscillating term in \( w(r) \) is kept since it contributes to the asymptotic behaviour of \( w'' \). As in the previous case we did not find an analytical solution for Eqs. (13) - (14) but it is straightforward to solve these equations numerically. A typical solution is shown in Fig. 2. The strongly oscillating behaviour of \( v(r) \) resulted in the space part of the gauge field of Eq. (2) being strongly oscillating. The ansatz function \( w(r) \) increases as some power of \( x \) as \( x \to \infty \), and would lead to the confinement of a test particle placed in the background field of this solution. (For the bunker solution there is some subtlety associated with the confinement of the test particle due to pair creation when the test particle scatters off the potential. This is essentially related to the Klein paradox and is discussed in Refs. [20] [21]).

The type of confinement given by this bunker solution is different from the bag-like solution of the previous sub-section: First the confining behaviour of the bag-like solution came from the “magnetic” part of the gauge field \( A^a_i \) through the ansatz functions \( v(r), f(r) \), while in the present case it is the “electric” part of the gauge field \( A_0^a \) which gives confinement.
through the ansatz function \( w(r) \). Second, the bag-like solution confines a test particle by the field strength becoming infinitely large at some finite value of \( r \), while the present solution confines a test particle by the field strength increasing without bound as \( r \to \infty \).

The power law with which \( w(r) \) increases changes as \( r \) increases. In Fig. 3 we show a plot of \( \log(w) - \log(x) \) for the solution of Fig. 2. At around \( \log(x) \approx 0.7 \) the slope of the line (and therefore the power law increase of \( w(r) \)) changes from \( \alpha \approx 2.8 \) to \( \alpha \approx 1.3 \). Depending on the initial conditions we found that for \( x \) near the origin \( \alpha \) was in the range \( \approx 2 - 3 \) while as \( x \) became large \( \alpha \) decreased to the range of \( \approx 1.2 - 1.8 \). In studies of heavy quark bound states [2] a potential which increases as \( r \to \infty \) is often used to successfully model the excited states of these systems. In these studies the increase is usually linear in \( r \).

The “magnetic” and “electric” fields associated with this solution can be found from \( A'^a_\mu \), and have the following behaviour

\[
H^a_r \propto \frac{v^2 - 1}{r^2}, \quad H^a_\varphi \propto v', \quad H^a_\theta \propto v', \quad E^a_r \propto \frac{r w' - w}{r^2}, \quad E^a_\varphi \propto \frac{vw}{r}, \quad E^a_\theta \propto \frac{vw}{r},
\]

(20)

(21)

here for \( E^a_r, H^a_\theta \), and \( H^a_\varphi \) the color index \( a = 1, 3, 4, 6, 8 \) and for \( H^a_r, E^a_\theta \) and \( E^a_\varphi \) \( a = 2, 5, 7 \).

The asymptotic behaviour of \( H^a_\varphi, H^a_\theta \) and \( E^a_\varphi, E^a_\theta \) is dominated by the strongly oscillating function \( v(r) \). If quantum corrections were applied to this solution it is expected that these strongly oscillating fields would be smoothed out and not play a significant role in the large \( r \) limit. From Eqs. (20) - (21) and the asymptotic form of \( v(r), w(r) \) the radial components of the “magnetic” and “electric” have the following asymptotic behaviour

\[
H^a_r \propto \frac{1}{r^2}, \quad E^a_r \propto \frac{1}{r^{2-\alpha}}.
\]

(22)

where the strongly oscillating portion of \( H^a_r \) is assumed not to contribute in the limit of large \( r \) due to smoothing by quantum corrections. The radial “electric” field falls off slower than \( 1/r^2 \) (since \( \alpha > 1 \)) indicating the presence of a confining potential. The \( 1/r^2 \) fall off of \( H^a_r \) indicates that this solution carries a “magnetic” charge. This was also true for the simple solutions discussed in Refs. [6] [11]. It can also be shown in the same way that the bag-like
solution of the previous section also carries a “magnetic” charge. This leads to the result that if a test particle is placed in the background field of either the bag or bunker solution, this composite system will have unusual spin properties \[22\] (i.e. if the test particle is a boson the system will behave as a fermion, and if the test particle is a fermion the system will behave as a boson).

Just as for the bag solution, the biggest drawback of the present solution is its infinite field energy. The bunker solution has an asymptotic energy density proportional to

\[
E \propto 4v^2 r^2 + \frac{2}{3} \left( \frac{w' - w}{r^2} \right)^2 + 4 \frac{v^2 w^2}{r^4} + \frac{2}{r^4} \left( v^2 - 1 \right)^2 \approx \frac{2}{3} \frac{\alpha^2 (\alpha - 1)(3\alpha - 1)}{x^{4-2\alpha}}
\]  

(23)

Since we found \(\alpha > 1\) this energy density will yield an infinite field energy when integrated over all space. This can be compared with the finite field energy monopole and dyon solution \[4\]. However, as remarked previously, it has been demonstrated \[5\] that the finite energy monopole solutions do not trap a test particle while the infinite energy solutions do.

What is the physical meaning of this solution? As in the case of the bag-like solution one can examine the motion of a test particle in the background field of the bunker solution, and find in this way that the test particle will tend to remain confined due to the increasing gauge potential. Another possible interpretation is that this solution is the Yang-Mills analog to the Coulomb potential in electrostatics. An electron can exist as an asymptotic state while a quark can not. Therefore, the bunker solution can be thought of as the far field of a color charge - “quark”. The fact that the bunker solution possesses an infinite field energy then indicates that an isolated quark is not allowed as an observable free state.

The Coulomb solution of electrostatics also possesses an infinite field energy, but the manner in which the field energy becomes infinite is different than for the bunker solution. Any point electric charge such as the electron has a singularity at \(r = 0\), but the “quark” field of the bunker solution has a singularity at \(r = \infty\). To follow through on this interpretation of the bunker solution as an isolated “quark”, one should investigate what happens when two bunker solutions are placed in the vicinity of one another. In this way one might hope that the combination of two bunker solutions would lead to a localized, finite energy field.
configuration. Then if one tried to separate the two “quarks” the field energy would become infinite. However the nonlinear character of the classical SU(3) field equations make this a difficult problem beyond the scope of the present work. Finally it can be noted that this solution is in a sense asymptotical free since at \( r = 0 \) the gauge potential \( A^a_\mu \to 0 \).

**III. THE GAUGE “STRING”**

Let us write down the following ansatz

\[
\begin{align*}
A^2_t &= f(\rho), \\
A^5_z &= v(\rho), \\
A^7_\varphi &= \rho w(\rho),
\end{align*}
\]

(24)-(26)

Here we use the cylindrical coordinate system \( z, \rho, \varphi \). The color index \( a = 2, 5, 7 \) corresponds to an embedding of \( SU(2) \) in \( SU(3) \). Using Eqs. (24) - (26) the Yang - Mills equations become

\[
\begin{align*}
f'' + \frac{f'}{\rho} &= f \left( v^2 + w^2 \right), \\
v'' + \frac{v'}{\rho} &= v \left( -f^2 + w^2 \right), \\
w'' + \frac{w'}{\rho} - \frac{w}{\rho^2} &= w \left( -f^2 + v^2 \right),
\end{align*}
\]

(27)-(29)

Let us examine the simple case \( w = 0 \) which reduces Eqs. (27) - (29) to

\[
\begin{align*}
f'' + \frac{f'}{\rho} &= f v^2, \\
v'' + \frac{v'}{\rho} &= -v f^2.
\end{align*}
\]

(30)-(31)

At origin \( \rho = 0 \) the solution has the following form

\[
\begin{align*}
f &= f_0 + f_2 \frac{\rho^2}{2} + \ldots, \\
v &= v_0 + v_2 \frac{\rho^2}{2} + \ldots.
\end{align*}
\]

(32)-(33)
Substituting Eqs. (32) - (33) into (30) - (31) we find that

\[ f_2 = \frac{1}{2}f_0v_0^2, \] (34)

\[ v_2 = -\frac{1}{2}v_0f_0^2. \] (35)

The asymptotic behaviour of the ansatz functions \( f, v \) and the energy density \( E \) can be given as

\[ f \approx 2 \left[ x + \frac{\cos(2x^2 + 2\phi_1)}{16x^3} \right], \] (36)

\[ v \approx \sqrt{2} \sin \left( \frac{x^2 + \phi_1}{x} \right), \] (37)

\[ E \propto f'^2 + v'^2 + f^2v^2 \approx \text{const}, \] (38)

where \( x = \rho/\rho_0 \) is the dimensionless radius, and \( \rho_0, \phi_1 \) are constants. To solve the system in Eqs. (30) - (31) for all \( r \) we again used numerical methods. A typical solution for \( f \) and \( v \) is shown in Fig. 4. As in the solution of the previous section we have a confining potential \( A_i^2 = f(\rho) \) and a strongly oscillating potential \( A_5^a = v(\rho) \). Depending on the relationship between \( v_0 \) and \( f_0 \) the energy density near \( \rho = 0 \) will be either a hollow (\( i.e. \) an energy density less than the asymptotic value) or a hump (\( i.e. \) an energy density greater than the asymptotic value). On account of this and the cylindrical symmetry of this solution we call this the “string” solution. The quotation marks indicate that this is a string from an energetic point of view, not from the potential (\( A_i^a \)) or field strength (\( F_{\mu\nu}^a \)) point of view. After quantization the oscillating functions will most likely vanish and only the confining potential and constant energy density will remain.

This “string”-like solution can be thought of as describing the classical gauge field between two “quarks”. Similar string-like configurations are thought to occur in the dual superconductor picture of confinement, and lattice calculations (nonperturbative quantization) also may give evidence for such structures.
IV. DISCUSSION

In this paper we have examined several non-trivial classical solutions of the $SU(3)$ Yang-Mills theory. Each of these solutions demonstrated some type of confining behaviour, indicating that this may be a general property of the classical $SU(3)$ Yang-Mills theory, and also that some form of this behaviour may carry over to the quantized theory. These infinite energy solutions to Eqs. (4) - (7) represent typical solutions to the classical field equations in the sense that they arise for a wide range of initial conditions. In contrast to this the simple $SU(3)$ monopole and dyon solutions investigated in Refs. [6] [10] are unique solutions in the sense that they arise for only certain initial conditions. In addition the infinite energy solutions investigated here give rise to a classical type of confining behaviour which neither the $SU(3)$ solutions of Refs. [6] [10] or the finite energy [3] [4] solutions possess.

The physical significance of the spherically symmetric cases is motivated by noting the similiarities between these solutions and various phenomenological models of confinement. The first solution will confine a quantum test particle via the spherical singularity in the “magnetic” part of the gauge field in a manner similiar to some bag models. Studies of such bag-like field configurations with scalar [14] and spinor [16] [17] test particles have been carried out. In both cases it was found that the test particles were confined inside $r = r_0$, and in Ref. [17] a somewhat realistic spectrum of hadron masses was obtained in this way. The second solution has the “electric” part of the gauge field increasing like $r^\alpha$ for large $r$ with $\alpha > 1$. If this field configuration is taken as representing the far field of an isolated “quark” then the infinitely increasing field strength can be taken to indicate the impossibility of isolating an individual “quark”. In contrast isolated electrons exist in nature since they generate electric fields which decrease at infinity. The third solution has a string-like structure from an energetic point of view. Similiar string-like structures are found in the dual superconductor picture of confinement. Just as two interacted electrons generate an electric field which is essentially the superposition of the electric fields of the individual electrons, so two interacting quarks are thought to generate a string-like flux tube...
which runs from one quark to the other. The “string” solution obtained above is a classical model of such a field distribution. It appears as a string-like structure on the background of the field with constant energy density. The strongly oscillating components of this and the bunker solution will most likely be smoothed out once quantum effects are taken into account.

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Fig.1. The $q(r)$ function for the $SU(3)$ bag. The initial conditions for this solution were $q_2 = 0.1$ and $r_i = 0.001$.

Fig.2. The $w(x)$ confining function, and the $v(x)$ oscillating function of the $SU(3)$ bunker solution. The initial conditions for this particular solution were $v_2 = 0.1$, $w_3 = 2.0$, and $x_i = 0.001$.

Fig.3. A plot of $\log(w) - \log(x)$ of the solution from Fig. 2 showing the different power law behaviour in the small $x$ and large $x$ regions.

Fig.4. The $SU(3)$ “string” solution with the linearly confining function $f(x)$ and the strongly oscillating function $v(x)$. The initial conditions for this solution were $f_0 = 0.75$, $v_0 = 0.75$ and $x_i = 0.001$. 
V. Dzhunushaliev and D. Singleton Fig. 1

\[ q(x) \]
Fig. 2
