Obtaining optical soliton solutions of the cubic–quartic Fokas–Lenells equation via three different analytical methods

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Received: 8 July 2022 / Accepted: 16 August 2022 / Published online: 29 September 2022
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Abstract
In this study, we have focused on finding soliton solutions of the cubic–quartic Fokas–Lenells equation, which models the nonlinear pulse transmission through optical fiber, is a pretty new and updated model. The main motivation of this study is to produce new solutions with previously unused methods for data transmission models in fiber optic cables together with the developing technology, which has been used very frequently today. We have used three different efficient analytical methods, namely, the Sinh–Gordon expansion method, enhanced modified extended tanh expansion method and unified Riccati equation expansion method. By modifying our proposed method and using it effectively, we have obtained much more optical solutions. We have supported the results with the 3D surface, 2D and contour plots of the soliton solutions, such as singular, periodic, periodic-singular, dark and M-shaped solitons. The originality of the method we propose is that it has never been used before. With the innovation of the method, we have obtained M-shaped solitons, which are very rarely obtainable in soliton waves. On the other hand, enhanced modified extended tanh expansion method have provided us more extra solutions. As the last and most important part, in order to examine physical behaviors, we have investigated the effect of the coefficient of the fourth order dispersion parameter on wave propagation by presenting the graphical representation.

Keywords Cubic–quartic Fokas–Lenells equation · Optical soliton solutions · Enhanced modified extended tanh expansion method · Sinh–Gordon expansion method · Unified Riccati equation expansion method

This article is part of the Topical Collection on Quantum technology, Quantum walks and quantum image processing: Emerging Trends, Issues and Challenges.

Guest edited by Ahmed Abd El-Latif and Salvador Venegas-Andraca.

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1 Introduction

In recent years, the concept of optics has become popular, especially in data transmission and other technologies with developing technology. These developments in technology have caused the necessity of the modeling of many optical phenomena such as energy transmission, refraction of the light beam, and scattering in optical fibers. When one reviews the current literature, one can see that there is an incomparable increase in optics. Some important topics we have chosen from the literature studies that we think will be useful in this sense are as follows: models for pulse propagation on fibers (Kudryashov 2019), analysis of edge filtering for fiber Bragg gratings sensors (Dey et al. 2021), investigation of the effects of nonlinearity laws on optical pulse (Topkara et al. 2010), classification of plastics using lasers (Yan et al. 2021), phase difference of electromagnetic waves (Muhibullah 2021), polarization preserving fibres and birefringent fibres (Biswas 2002) and optical solitons of Schrödinger equation in case of potential and nonlinearity (Akinyemi et al. 2021).

Generally, nonlinear Schrödinger equations (NLSEs), a type of nonlinear partial differential equation (NLPDE), are used for modeling optical media. Such as, optical solitons of NLSE in magneto-optic waveguide (Mathanaranjan et al. 2021), influence of fractional time order on W-shaped NLSE (Houwe et al. 2021), coupled NLSE in multicomponent inhomogenous optical fiber (Gao et al. 2021), optical solitons of NLSE with time-dependent coefficients (Saha et al. 2009), chiral NLSE in optics (Esen et al. 2021a), Darboux-dressing transformation for NLSE system (Yang et al. 2021), interacting airy beams in fractional NLSE (Chen et al. 2021), optical dromions for perturbed fractional NLSE (Rizvi et al. 2021) and broken-unbroken solutions for NLSE (Yu et al. 2021). Besides, there are some optical phenomena equations which models transmission on fibers or optical media. Namely, Kundu–Mukherjee–Naskar equation (Onder et al. 2022), Lakshmanan–Porsezian–Daniel equation (Biswas et al. 2018), Ginzburg–Landau equation (Mirzazadeh et al. 2016), Gerdjikov–Ivanov equation (Kaur and Wazwaz 2018), Biswas–Milovic equation (Cinar et al. 2021), Chen–Lee–Liu equation (Esen et al. 2021b) and Fokas–Lenells equation (Triki and Wazwaz 2017).

Now back to our own work, we have reviewed all the details of the cubic–quartic Fokas–Lenells equation (CQFLE) which is a type of Fokas–Lenells equation (FLE). FLE models the femtosecond pulse propagation through optical silica fibers, and it arises in an optical media that have nonlinear pulse propagation (Triki and Wazwaz 2017). FLE is a generalized type of NLSE. Also known as the fully integrable equation derived from NLSE with the usage of two Hamiltonian operators (Triki and Wazwaz 2017; Mahak and Akram 2020a). Since the light has different colors and speeds, chromatic dispersion (CD) is observed during signal transmission in optic fiber. This causes the transmitted signal waves to expand even further and even overlap. Self-phase modulation (SPM) is one of the way to regularize this unbalanced transmission. However, where CD is low, the cubic and quartic nonlinearity terms are added to model and control this balance (Yıldırım et al. 2021; Biswas and Arshed 2018). When the CD term is omitted and the third-order dispersion (3OD) and fourth-order dispersion (4OD) terms are added, the resulting model is the cubic–quartic Fokas–Lenells equation (CQFLE) Zayed et al. (2021):

\[
\begin{align*}
  i\phi_t + i\alpha \phi_{xxx} + b \phi_{xxxx} + |\phi|^2 (c \phi + i \alpha \phi_x) - i [\alpha \phi_x + \alpha (|\phi|^{2m}) \phi_x + \mu (|\phi|^{2m})_x \phi] &= 0,
\end{align*}
\]  

(1)

where \( \phi = \phi(x, t) \) is soliton profile, \( a, b, c, d, \alpha, \mu \) are the coefficients of 3OD, 4OD, Kerr law nonlinearity, nonlinear dispersion, inter-modal dispersion (IMD), self-steepening and higher order dispersion terms, respectively. In addition, \( m \) is the degree of full nonlinearity term. CQFLE is a quite new model thus there exist a few studies in the literature. Obtaining optical
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The organization of the paper is as follows: Sect. 2 contains obtaining the nonlinear ordinary differential equation (NODE) form of CQFLE and the application of the proposed methods to the considered equation. Section 3 includes 3D surface figures and 2D and contour plots of the obtained soliton solutions. Lastly, Sect. 4 is the conclusion part.

2 Mathematical analysis

2.1 Obtaining nonlinear ordinary differential form of CQFL equation

Let us consider the following wave transformation:

\[ \Phi = \Phi(\eta), \quad \Phi' = \Phi'(\eta), \quad \Phi'' = \Phi''(\eta), \quad \Phi''' = \Phi'''(\eta), \quad (i = 1, 2, \ldots, N) \]

where \( \eta = x - vt \), \( \theta = -kx + wt + \psi_0 \), \( \psi_0 \) is phase constant. Substituting the Eq. (2) into the Eq. (1) and decomposing the obtained equation into real and imaginary parts, the following equations can be obtained, respectively.

\[ b\Phi^{(IV)} + (6bk^2)\Phi'' + (c + kd - k\lambda)\Phi^3 - (3bk^4 + k\alpha + w)\Phi = 0, \]  \hspace{1cm} (3)

\[-(v + a + 8bk^3)\Phi' + (d - 3\lambda + 2\mu)\Phi^2\Phi' + (a - 4bk)\Phi''' = 0, \]  \hspace{1cm} (4)

where \( \Phi = \Phi(\eta) \), \( \Phi' \) denotes \( \frac{d}{d\eta} \) and Eq. (3) is the NODE form of Eq. (1). Assuming that \( m = 1 \), we obtain the following constraint conditions from Eq. (4).

\[ a = 4bk, \quad d = 3\lambda - 2\mu \quad \text{and} \quad v = -\alpha - 8bk^3. \]  \hspace{1cm} (5)

In Eq. (3), balancing the terms \( \Phi^{(IV)}, \Phi^3 \), one can compute \( N = 2 \), which is the balancing constant.

2.2 Application of Sinh–Gordon expansion method to the CQFL equation

Let us suppose that Eq. (3) has a solution in general form as follows:

\[ \Phi(H) = A_0 + \sum_{i=1}^{N} \cosh^{-1}(H)[A_i \cosh(H) + B_i \sinh(H)], \]  \hspace{1cm} (6)

where \( H = H(\eta) \), \( N \) is balancing constant, \( A_0, A_i \) and \( B_i \) \((i = 1, 2, \ldots, N)\) are real values to be determined later and \( H(\eta) \) satisfies the following equation:

\[ \frac{dH}{d\eta} = \sqrt{\sinh^2(H) + C}, \]  \hspace{1cm} (7)

where \( C \) is integration constant and Eq. (7) gives the following solutions:

\[ \sinh(H) = \mathrm{i} \text{sech}(\eta), \quad \cosh(H) = \tanh(\eta), \]  \hspace{1cm} (8)
\[ \sinh(H) = \text{csch}(\eta), \quad \cosh(H) = \text{coth}(\eta). \] (9)

where \( C = 0 \). Taking into account that \( N = 2 \), the Eq. (6) turns into following form:

\[ \Phi(H) = A_0 + A_1 \cosh(H) + B_1 \sinh(H) + \cosh(H) \left[ A_2 \cosh(H) + B_2 \sinh(H) \right]. \] (10)

Substituting the Eq. (10) and its related derivatives to Eq. (3) by considering Eq. (7), we have a polynomial. Then, equating the coefficients of \( \cosh^i(H) \sinh^j(H), i = (0...6), j = (0, 1) \) to zero.

\[
cosh^0(H) : (c + k(d - \lambda))A_0^0 + (-3b k^4 + ((3\lambda - 3d)B_1^2 - a)k - 3B_1^2 c - w)A_0 \\
+ 12 \left( k^2 - \frac{4}{3} \right) b A_2 = 0,
\]

\[
cosh^0(H)\sinh(H) : -3 \left( b k^4 + 2b k^2 + \left( -\frac{2}{3} + \frac{d}{3} \right) B_1^2 + (\lambda - d) A_0^2 + \frac{B_1^2 c}{3} \right) \\
- c A_0^2 - \frac{5b}{3} + \frac{w}{3} B_1 = 0,
\]

\[
cosh^1(H) : (-3b k^4 - 12b k^2 + ((-3\lambda + 3d)A_0^0 + (3\lambda - 3d)B_1^2 - a)k + 3c A_0^2 \\
- 3B_1^2 c + 16b - w)A_1 + 6B_1 B_0 A_0 ((\lambda - d)k - c) = 0,
\]

\[
cosh^1(H)\sinh(H) : (3b k^4 - 30b k^2 + ((-3\lambda + 3d)A_0^0 + (3\lambda - 3d)B_1^2 - a)k + 3c A_0^2 \\
- 3B_1^2 c + 61b - w)B_2 - 6B_1 A_0 A_1 ((\lambda - d)k - c) = 0,
\]

\[
cosh^2(H) : -3A_1 b k^4 + 48A_2 b k^2 + (-3A_1 (\lambda - d) A_0^0 + 3(A_1^2 - B_1^2) \\
(\lambda - d)A_2 + ((3\lambda - 3d)B_1^2 - a)A_2 + 6B_2 A_1 ((\lambda - d)k - c)k + 3A_0 A_2 c \\
- 3c(-A_1^2 - B_1^2 + B_2^2)A_0 + (-3B_1^2 c + 136b - w)A_2 - 6A_1 B_2 c = 0,
\]

\[
cosh^2(H)\sinh(H) : (c + k(d - \lambda))B_1^3 + \left( 12b k^2 - 6(\lambda - d) \left( A_0 A_2 + \frac{A_1^2}{2} - \frac{B_2^2}{2} \right) k \\
+ 6c A_0 A_2 + 3c A_1^2 - 3B_1^2 c - 28b)B_1 - 6B_1 A_1 ((\lambda - d)k - c) A_0 = 0,
\]

\[
cosh^3(H) : (c + k(d - \lambda))A_1^1 + \left( 12b k^2 - 6(\lambda - d) \left( A_0 A_2 + \frac{B_1^2}{2} - \frac{B_2^2}{2} \right) k + 6c A_0 A_2 \\
+ 3B_1^2 c - 3B_2^2 c - 40b)A_1 - 6B_2 B_1 ((\lambda - d)k - c) (A_0 - A_2) = 0,
\]

\[
cosh^3(H)\sinh(H) : ((\lambda - d)k - c)B_2^3 + \left( 36b k^2 - 6(\lambda - d) \left( A_0 A_2 + \frac{A_1^2}{2} + \frac{B_2^2}{2} \right) k \\
+ 6c A_0 A_2 + 3c A_1^2 + 3B_1^2 c - 180b)B_2 - 6B_1 A_1 ((\lambda - d)k - c) A_2 = 0,
\]

\[
cosh^4(H) : -3((\lambda - d)k - c)A_0 A_2^2 + \left( 36b k^2 - 3(A_1^2 + B_1^2 - B_2^2) (\lambda - d)k + 3c A_1^2 \\
+ 3B_1^2 c - 3B_2^2 c - 240b)A_2 - 3B_2 (A_0 B_2 + 2A_1 B_1) ((\lambda - d)k - c) = 0,
\]

\[
cosh^4(H)\sinh(H) : \left( (-3\lambda + 3d)k + 3c)A_2^2 + (-3\lambda + 3d)k + 3c)B_2^2 + 24b \right) A_1 \\
+ 6(c + k(d - \lambda))B_2 A_2 A_1 = 0,
\]

\[
cosh^5(H) : \left( (-3\lambda + 3d)k + 3c)A_2^2 + (-3\lambda + 3d)k + 3c)B_2^2 + 24b \right) A_1 \\
+ 6(c + k(d - \lambda))B_2 A_2 A_1 = 0,
\]

\[
cosh^5(H)\sinh(H) : \left( 3(d - \lambda) \left( A_0^2 + \frac{B_1^2}{3} \right) k + 3c A_2^2 + B_2^2 c + 120b \right) B_2 = 0,
\]

\[
cosh^6(H) : \left( 3(d - \lambda) \left( A_0^2 + \frac{B_1^2}{3} \right) k + 3c A_2^2 + B_2^2 c + 120b \right) A_0 = 0.
\]
The above system produces a many solution sets but we present some of them. One can easily get the others.

Substituting the SET\textsubscript{1,1} and SET\textsubscript{1,2} into Eq. (10) by considering the Eq. (2), (5), (8), (9), we reach the following solutions:

\begin{align}
\phi_{1,1}(x, t) &= e^{i\theta} B_2 \coth(x - vt) \csch(x - vt), \\
\phi_{1,2}(x, t) &= e^{i\theta} B_2 \tanh(x - vt) \sech(x - vt), \\
\phi_{1,3}(x, t) &= e^{i\theta} \frac{6\sqrt{-10\delta b}}{\delta} \coth(x - vt) \csch(x - vt), \\
\phi_{1,4}(x, t) &= e^{i\theta} \frac{6\sqrt{-10\delta b}}{\delta} \tanh(x - vt) \sech(x - vt),
\end{align}

In Eqs. (13)–(16), $\nu = -\alpha - 8bk^3$, $\theta = \left(-\frac{\sqrt{15}}{3} + \left(\frac{8b - a\sqrt{15}}{3}\right)t + \psi_0\right)$ and $\delta b < 0$.

### 2.3 Application of eMETEM to the CQFL equation

According to eMETEM, we seek the solution of Eq. (3) in the following form:

\[
\Phi(\eta) = A_0 + \sum_{i=1}^{N} A_i R_i(\eta) + \sum_{i=1}^{N} \frac{B_i}{R(\eta)},
\]

where $N$ is the balancing constant which has been computed as $N = 2$, $A_0, A_i, B_i, (i = 1, 2, \ldots, N)$ are real numbers to be determined later and $A_N, B_N$ should not be zero, simultaneously. Moreover, $R(\eta)$ satisfies the following Riccati differential equation:

\[
\frac{dR}{d\eta} - \omega - R^2(\eta) = 0,
\]

where $R(\eta)$ admits the solutions given in Table 1 which covers the more solutions of Eq. (18) Ozisik et al. (2022).

where $\eta_0, a_\epsilon, \epsilon$, and $\omega$ are free real parameters and $\epsilon = \pm 1$.

Considering that $N = 2$, the proposed solution in Eq. (17) converts into the following form:
\[ \Phi(\eta) = A_0 + A_1 R(\eta) + A_2 R^2(\eta) + \frac{B_1}{R(\eta)} + \frac{B_2}{R^2(\eta)}. \] (19)

Substituting Eqs. (19) and (18) into Eq. (3), we get a polynomial in power of \( R(\eta) \). Then equating the coefficients of \( R(\eta)^j \), \( j = {-6}, .., 0, ..6 \) to zero, we construct the following algebraic system:

\[
R^{-6}(\eta) = 120b B_2 \omega^4 + B_1^2 k d - B_2^2 k \lambda + B_2^3 c = 0,
\]

\[
R^{-5}(\eta) = B_1 ((d - \lambda) k + c) B_2^2 + 8 b \omega^4 = 0,
\]

\[
R^{-4}(\eta) = B_2 (12 b \omega^2 k^2 + (A_1 B_2 + B_2) (d - \lambda) k + 80 b \omega^3 + B_2 c A_0 + B_1^2 c) = 0,
\]

\[
R^{-3}(\eta) = (c + k (d - \lambda)) B_1^2 + (12 b \omega^2 k^2 - 6 A_0 B_2 (\lambda - d) k + 40 b \omega^3 + 6 B_2 c A_0) B_1 - 3 B_2^2 A_1 ((d - \lambda) k - c) = 0,
\]

\[
R^{-2}(\eta) = -3 A_2 ((\lambda - d) k - c) B_2^2 + (3 b k^4 + 48 b \omega k^2 + (-6 b_1 (\lambda - d) A_1 + (-3 \lambda + 3 d) A_0^2 - 128 k^3 + 12 b A_0 k^2 + ((\lambda - d) A_0) 0 - c) k + 16 b \omega^2 + 6 b c A_2) = 0,
\]

\[
R^{-1}(\eta) = -3 A_1 ((\lambda - d) k - c) B_1^2 + (3 b k^4 + 6 b \omega k^2 + ((-3 \lambda + 3 d) A_0^2 - 6 A_2 (\lambda - d) B_2 - \alpha) k + 16 b \omega^2 + 6 b c A_2 + 3 c A_0^2 - w) B_1 - 6 A_1 B_0 ((\lambda - d) k - c) = 0,
\]

\[
R^0(\eta) = -3 A_0 b k^4 + 12 b (A_0^2 k^2 + B_2) k^2 + ((d - \lambda) A_0 + (-6 A_2 (\lambda - d) B_2 - B_1 (\lambda - d) A_1 - \alpha) A_0 - 3 (\lambda - d) (A_2 B_2 + A_2 B_1) k + A_0^2 c + (6 b_1 c A_2 + 6 b_2 c A_1 - w) A_0 + (16 b \omega^2 + 3 b_2 c) A_1 + (3 c A_0^2 + 16 b \omega) B_2 = 0,
\]

\[
R^1(\eta) = -3 B_1 ((\lambda - d) k - c) A_1^2 + (3 b k^4 + 12 b \omega k^2 + ((-3 \lambda + 3 d) A_0^2 - 6 A_2 (\lambda - d) B_2 - \alpha) k + 16 b \omega^2 + 6 b c A_2 + 3 c A_0^2 - w) A_1 - 6 A_2 B_1 ((\lambda - d) k - c) = 0,
\]

\[
R^2(\eta) = -3 B_2 ((\lambda - d) k - c) A_2^2 + (3 b k^4 + 48 b \omega k^2 + ((-6 b_1 (\lambda - d) A_1 + (-3 \lambda + 3 d) A_0^2 - 128 k^3 + 12 b A_0 k^2 + ((\lambda - d) A_0) 0 - c) k + 16 b \omega^2 + 6 b c A_2) = 0,
\]

\[
R^3(\eta) = (c + k (d - \lambda)) A_1^2 + (12 b \omega^2 k^2 - 6 A_0 A_2 (\lambda - d) k + 40 b A_1 + 3 c A_1^2 B_2) + (\lambda - d) k - c) = 0,
\]

\[
R^4(\eta) = (A_0 A_2 + 2 A_1^2) (d - \lambda) k + c A_0 A_2 + c A_1^2 + 80 b A_1) A_2 = 0,
\]

\[
R^5(\eta) = A_1 ((d - \lambda) k + c) A_0^2 + 8 b = 0,
\]

\[
R^6(\eta) = 120 b A_2 + (c + k (d - \lambda)) A_2^2 = 0.
\]

Solution of the above system, gives the following sets:

\[
\text{SET}_{2,1} : \begin{cases}
A_0 = 0, A_1 = 0, A_2 = \frac{6 \sqrt{-10 b}}{\sqrt{2 \sqrt{15} (d - \lambda) \sqrt{-\omega + 3 c}}}, B_1 = 0, \\
B_2 = -A_2 \omega^2, k = \frac{\sqrt{30 \omega}}{3}, w = \frac{128 b \omega^2}{3} - \frac{2 \sqrt{-15 \omega a}}{3}.
\end{cases}
\] (20)
Table 1 The generalized and enhanced solutions of Eq. (18)

| For $\omega < 0$ | For $\omega > 0$ |
|------------------|------------------|
| $R_1(\eta) = -\sqrt{-\omega \tanh(\sqrt{-\omega}(\eta + \eta_0))}$ | $R_8(\eta) = \sqrt{\omega \tanh(\sqrt{\omega}(\eta + \eta_0))}$ |
| $R_2(\eta) = -\sqrt{-\omega \coth(\sqrt{-\omega}(\eta + \eta_0))}$ | $R_9(\eta) = \sqrt{\omega \coth(\sqrt{\omega}(\eta + \eta_0))}$ |
| $R_3(\eta) = -\sqrt{-\omega \tanh(2\sqrt{-\omega}(\eta + \eta_0))} + \text{csch}(2\sqrt{-\omega}(\eta + \eta_0))$ | $R_{10}(\eta) = \sqrt{\omega \tan(2\sqrt{\omega}(\eta + \eta_0))} + \csc(2\sqrt{\omega}(\eta + \eta_0))$ |
| $R_4(\eta) = \left( \omega - \sqrt{-\omega \tanh(\sqrt{-\omega}(\eta + \eta_0))} \right) \left( \frac{1}{1 + \sqrt{-\omega \tanh(\sqrt{-\omega}(\eta + \eta_0))}} \right)$ | $R_{11}(\eta) = \sqrt{\omega (1 - \sec(\sqrt{\omega}(\eta + \eta_0)))}$ |
| $R_5(\eta) = \sqrt{-\omega (5 - 4\cosh(2\sqrt{-\omega}(\eta + \eta_0)))}$ | $R_{12}(\eta) = \sqrt{\omega (4 - 5\cos(2\sqrt{\omega}(\eta + \eta_0)))}$ |
| $R_6(\eta) = \frac{\sqrt{-\omega (\omega^2 + b^2)}}{\omega \sinh(2\sqrt{-\omega}(\eta + \eta_0))} + b$ | $R_{13}(\eta) = \frac{\sqrt{\omega (\omega^2 - b^2)}}{\omega \sin(2\sqrt{\omega}(\eta + \eta_0))} + b$ |
| $R_7(\eta) = \sqrt{-\omega} \left[ 1 - \frac{2\alpha}{\omega \cos(2\sqrt{-\omega}(\eta + \eta_0)) - \cosh(2\sqrt{-\omega}(\eta + \eta_0))} \right]$ | $R_{14}(\eta) = i\sqrt{\omega} \left[ 1 - \frac{2\alpha}{\omega \sin(2\sqrt{\omega}(\eta + \eta_0)) - \sin(2\sqrt{\omega}(\eta + \eta_0))} \right]$ |
| $R_{15}(\eta) = -\frac{1}{\pi \eta_0}$ for $\omega = 0$ |

Remark 1 For simplicity, the following abbreviations are used in Eqs. (24)–(38).
Substituting $R(\eta)$ functions in Table 1 into Eq. (19) and considering the wave transformation in Eq. (2), we can get the solutions of Eq. (1). One can easily get the any solution functions by inserting the sets in Eqs. (20)–(23) into resultant function. But in order not to take up much volume, we give the solutions of Eq. (1) in general form as in the following forms:

For $\omega < 0$;

\[
\phi_{2,1}(x, t) = e^{i\omega} \left( A_0 + A_1 \tanh P_1 + A_2 \tanh^2 P_1 + \frac{B_1}{\tanh P_1} + \frac{B_2}{\tanh^2 P_1} \right),
\]

\[
\phi_{2,2}(x, t) = e^{i\omega} \left( A_0 + A_1 \coth P_1 + A_2 \coth^2 P_1 + \frac{B_1}{\coth P_1} + \frac{B_2}{\coth^2 P_1} \right),
\]

\[
\phi_{2,3}(x, t) = e^{i\omega} \left( A_0 + A_1 \Psi_1 + A_2 \Psi^2_1 + \frac{B_1}{\Psi_1} + \frac{B_2}{\Psi^2_1} \right),
\]

\[
\phi_{2,4}(x, t) = e^{i\omega} \left( A_0 + A_1 \left( \frac{\omega + \tanh P_1}{1 - \tanh P_1} \right) + A_2 \left( \frac{\omega + \tanh P_1}{1 - \tanh P_1} \right)^2 + \frac{B_1}{(\omega + \tanh P_1)} \left( \frac{1}{1 - \tanh P_1} \right)^2 + \frac{B_2}{(\omega + \tanh P_1)} \left( \frac{1}{1 - \tanh P_1} \right)^2 \right),
\]

\[
\phi_{2,5}(x, t) = e^{i\omega} \left( A_0 + A_1 \left( \frac{\cosh P_1}{\sinh P_1} \right) + A_2 \left( \frac{\cosh P_1}{\sinh P_1} \right)^2 + \frac{B_1}{\cosh P_1} \left( \frac{1}{\sinh P_1} \right)^2 + \frac{B_2}{\cosh P_1} \left( \frac{1}{\sinh P_1} \right)^2 \right),
\]

\[
\phi_{2,6}(x, t) = e^{i\omega} \left( A_0 + A_1 \left( \frac{\gamma_1 - \cosh P_2}{\sinh P_2} \right) + A_2 \left( \frac{\gamma_1 - \cosh P_2}{\sinh P_2} \right)^2 + \frac{B_1}{\gamma_1 - \cosh P_2} \left( \frac{1}{\sinh P_2} \right)^2 + \frac{B_2}{\gamma_1 - \cosh P_2} \left( \frac{1}{\sinh P_2} \right)^2 \right),
\]
\[ \phi_{2,7}(x, t) = e^{i\theta} \left( A_0 + A_1 \left( \psi_2 \right) + A_2 \left( \psi_2 \right)^2 + \frac{B_1}{\psi_2} + \frac{B_2}{\left( \psi_2 \right)^2} \right). \] (30)

For \( \omega > 0 \):

\[ \phi_{2,8}(x, t) = e^{i\theta} \left( A_0 + A_1 \tan P_1 + A_2 \tan P_1^2 + \frac{B_1}{\tan P_1} + \frac{B_2}{\tan P_1^2} \right). \] (31)

\[ \phi_{2,9}(x, t) = e^{i\theta} \left( A_0 + A_1 \cot P_1 + A_2 \cot P_1^2 + \frac{B_1}{\cot P_1} + \frac{B_2}{\cot P_1^2} \right), \] (32)

\[ \phi_{2,10}(x, t) = e^{i\theta} \left( A_0 + A_1 \Omega_1 + A_2 \Omega_1^2 + \frac{B_1}{\Omega_1} + \frac{B_2}{\Omega_1^2} \right), \] (33)

\[ \phi_{2,11}(x, t) = e^{i\theta} \left[ A_0 + A_1 \left( \sqrt{\omega} \left( 1 - \tan \left( \Delta_2 \right) \right) \right) + A_2 \left( \sqrt{\omega} \left( 1 - \tan \left( \Delta_2 \right) \right) \right)^2 + \frac{B_1}{\sqrt{\omega} \left( 1 + \tan \left( \Delta_2 \right) \right)} + \frac{B_2}{\sqrt{\omega} \left( 1 + \tan \left( \Delta_2 \right) \right)^2} \right]. \] (34)

\[ \phi_{2,12}(x, t) = e^{i\theta} \left( A_0 + A_1 \left( \frac{\csc P_1}{\sin P_1} \right) + A_2 \left( \frac{\csc P_1}{\sin P_1} \right)^2 + \frac{B_1}{\csc P_1} + \frac{B_2}{\csc P_1^2} \right), \] (35)

\[ \phi_{2,13}(x, t) = e^{i\theta} \left( A_0 + A_1 \left( \frac{\gamma_2 - \csc P_2}{\sin P_2} \right) + A_2 \left( \frac{\gamma_2 - \csc P_2}{\sin P_2} \right)^2 + \frac{B_1}{\gamma_2 - \csc P_2} + \frac{B_2}{\gamma_2 - \csc P_2^2} \right). \] (36)

\[ \phi_{2,14}(x, t) = e^{i\theta} \left( A_0 + A_1 \Omega_2 + A_2 \Omega_2^2 + \frac{B_1}{\Omega_2} + \frac{B_2}{\Omega_2^2} \right). \] (37)

For \( \omega = 0 \):

\[ \phi_{2,15}(x, t) = e^{i\theta} \left( A_0 - A_1 \left( \frac{1}{x - vt} \right) + A_2 \left( \frac{1}{x - vt} \right)^2 - \frac{B_1}{\frac{1}{x - vt}} + \frac{B_2}{\frac{1}{x - vt}^2} \right), \] (38)

where \( a_c, b_c \) are real constants and \( v = -\alpha - 8bk^3 \).

Here, we present the some solutions which are plotted, in Figs. 2, 3, 4, 5. Considering the Eqs. (26), (27) with \( SET_{2,1} \) in Eq. (20), together, we get:
Combination of Eq. (33) and $SET_{2,3}$ in Eq. (22), gives,

$$
\phi_{2,3}(x, t) = e^{i\theta} \left( \frac{6\sqrt{-10b} \Psi_2^2}{\sqrt{2\sqrt{15} (d - \lambda) \sqrt{-\omega + 3c}}} - \frac{6\omega^2 \sqrt{-10b}}{\sqrt{2\sqrt{15} (d - \lambda) \sqrt{-\omega + 3c}} \Psi_2^2} \right),
$$

$$
\phi_{2,4}(x, t) = e^{i\theta} \left( \frac{6\sqrt{-10b}}{\sqrt{2\sqrt{15} (d - \lambda) \sqrt{-\omega + 3c}}} \left( \frac{(\omega + \tanh P_1)}{(1 - \tanh P_1)} \right)^2 - \frac{6\omega^2 \sqrt{-10b}}{\sqrt{2\sqrt{15} (d - \lambda) \sqrt{-\omega + 3c}}} \left( \frac{1 - \tanh P_1}{\omega + \tanh P_1} \right)^2 \right),
$$

where $B_2$ is an arbitrary real parameter.

Considering the Eq. (36) with $SET_{2,2}$ in Eq. (21), together, we get:

$$
\phi_{2,10}(x, t) = e^{i\theta} \left( \frac{B_2 (\omega + \Omega_2^2)}{\omega \Omega_2^2} \right).
$$

where $\Omega_2$ is an arbitrary real parameter.

2.4 Application of UREEM to the CQFL equation

According to the UREEM (Sirendaoreji 2017), we seek a solution of Eq. (3) in the following form:

$$
\Phi(\eta) = \sum_{i=0}^{N} \sigma_i F_i(\eta), \quad \sigma_N \neq 0,
$$

where $N$ is the balancing term which has been computed $N = 2$, $\sigma_i$, $(i = 1, 2 \ldots N)$ are real values to be determinated later. Besides, $F(\eta)$ satisfies the following equation:

$$
\frac{dF}{d\eta} - c_0 - c_1 F(\eta) - c_2 F^2(\eta) = 0,
$$

in which $c_0, c_1, c_2$ are real constants and $F(\eta)$ admits the solutions given in Table 2.

where $\delta = c_1^2 - 4c_0c_2$. Considering $N = 2$, the Eq. (43) turns into the following equation:

$$
\Phi(\eta) = \sigma_0 + \sigma_1 F(\eta) + \sigma_2 F^2(\eta),
$$

Substituting Eqs. (45) and (44) into Eq. (3), we get a polynomial in power of $F(\eta)$. Then equating the coefficients of $F(\eta)^j, (j = 0, \ldots 6)$ to zero, the following algebraic system is obtained:
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\[ F^0(\eta) : b(12k^2 + 16c_0c_2 + 14c_1^2)c_0^2\sigma_2 + (c_1c_0(6k^2 + 8c_0c_2 + c_1^2)\sigma_1 - 3\sigma_0k^4)b \]
\[ - (\lambda - d)\sigma_0^2 + a)k - \sigma_0^2c + w)\sigma_0 = 0, \]

\[ F^1(\eta) : 6bc_1(6k^2 + 20c_0c_2 + 5c_1^2)c_0\sigma_2 - \sigma_1\left( 3k^4 + 3(-4c_0c_2 - 2c_1^2)k^2 - 16c_0^2c_2^2 - 22c_0c_1^2c_2 \right) - c_1^2b + 3(\lambda - d)\sigma_0^2 + a)k - 3\sigma_0^2c + w) \]
\[ = 0, \]

\[ F^2(\eta) : \left( -3k^4 + (48c_0c_2 + 24c_1^2)k^2 + 136c_0^2c_2^2 + 232c_0c_1^2c_2 + 16c_1^4 \right) + b + (\lambda - 3\lambda + 3d)\sigma_0^2 - a)k \]
\[ + 3\sigma_0^2c - w)\sigma_2 + 3(c_1c_2(6k^2 + 20c_0c_2 + 5c_1^2)b - \sigma_0\sigma_1((\lambda - d)k - c)\sigma_1 = 0, \]

\[ F^3(\eta) : (10c_1c_2(6k^2 + 44c_0c_2 + 13c_1^2)b - \sigma_0\sigma_1((\lambda - d)k - c)\sigma_2 + \left( c_2^2\left( 12k^2 + 40c_0c_2 \right) + 50c_1^2b - \sigma_1^2((\lambda - d)k - c) \right)\sigma_1 = 0, \]

\[ F^4(\eta) : -3\sigma_0((\lambda - d)k - c)\sigma_2^2 + (240c_0c_2^3 + 330c_1^2c_2^2 + 36c_1^2k^2) - 3\sigma_1^2((\lambda - d)k - c)\sigma_2 \]
\[ + 60b\sigma_1c_1c_2^2 = 0, \]

\[ F^5(\eta) : -3\sigma_1((\lambda - d)k - c)\sigma_2^2 + 336b\sigma_4c_1c_2^3 + 24b\sigma_1c_2^4 = 0. \]

\[ F^6(\eta) : (c + k(d - \lambda))\sigma_2^2 + 120b\sigma_2c_1^4 = 0. \]

Solving this system gives the following set:

\[
\text{SET}_{3,1} : \left\{ \begin{array}{l}
\sigma_0 = \frac{\sigma_2(5c_1^2 + 6k^2)}{20c_2^2}, \quad \sigma_1 = \frac{c_1\sigma_2}{c_2}, \quad \sigma_2 = \frac{\sigma_1}{2c_2}, \quad c = - \frac{120b c_1^2 + dk \sigma_2^2 - k\lambda\sigma_1^2}{\sigma_1^2}, \\
c_0 = \frac{5c_1^2 + 6k^2}{20c_2^2}, \quad w = - \frac{k(219b k^3 + 25a)}{25}.
\end{array} \right. \tag{46}
\]

Combination of the SET_{3,1}, Eq. (2), Eq. (45) and Table 2, gives,

\[ \phi_{3,1}(x, t) = e^{i\eta} \left( \frac{\sigma_2(5c_1^2 + 6k^2)}{20c_2^2} + \frac{c_1\sigma_2}{c_2} - \frac{c_1}{2c_2} \right) - \sqrt{\delta} \tanh \left( \frac{\sqrt{\delta}(x-\nu t)}{2c_2} \right) \]
\[ + \sigma_2 \left( \frac{c_1}{2c_2} - \frac{\sqrt{\delta} \tanh \left( \frac{\sqrt{\delta}(x-\nu t)}{2c_2} \right)}{2c_2} \right)^2 \] \tag{47}

where \( \delta = c_1^2 - 4c_0c_2 < 0, \nu = -a - 8bk^3. \)

\[ \phi_{3,4}(x, t) = e^{i\eta} \left( \frac{\sigma_2(5c_1^2 + 6k^2)}{20c_2^2} + \frac{c_1\sigma_2}{c_2} - \frac{c_1}{2c_2} - \sqrt{-\delta} \cot \left( \frac{\sqrt{-\delta}(x-\nu t)}{2c_2} \right) \right) \]
\[ + \sigma_2 \left( \frac{c_1}{2c_2} - \frac{\sqrt{-\delta} \cot \left( \frac{\sqrt{-\delta}(x-\nu t)}{2c_2} \right)}{2c_2} \right)^2 \] \tag{48}
where $\delta = c_1^2 - 4c_0c_2 > 0$.

## 3 Results and discussion

Figure 1 shows 3D, contour and 2D plots of $\phi_{1,4}(x, t)$ in Eq. (16) with $\alpha = 0.1, b = 1, d = 0.1, c = -1, \lambda = -1$ and $\psi_0 = 1$. In Fig. 1a, we depict 3D view of $|\phi_{1,4}(x, t)|$ which represents M-shaped soliton solution. Figure 1b, c show the 3D plots of real and imaginary parts of $\phi_{1,4}(x, t)$ with aforementioned parameters. Figure 1d–f show the contour plots of modulus, real and imaginary parts of $\phi_{1,4}(x, t)$.

Figure 1a, g show the M-shaped soliton character as double-bright soliton which is not a common type of soliton for $\phi_{1,4}(x, t)$. In Fig. 1g, the shape of soliton is conserved for different $t$ values and the soliton is traveling towards the left. But, Fig. 1h, i show behavior of $\text{Re}(\phi_{1,4}(x, t))$ and $\text{Im}(\phi_{1,4}(x, t))$ soliton, in addition, both $\text{Re}(\phi_{1,4}(x, t))$ and $\text{Im}(\phi_{1,4}(x, t))$ are traveling towards left with different $t$ values also behaviors of solitons are changing according to the time. For example, in Fig. 1h, there are two dark solitons with different amplitudes for $t = 0$ value (red line), but there are two bright solitons with different amplitudes for $t = 0.2$ value (blue line).

In Fig. 1g–i we illustrate modulus, real and imaginary part of $\phi_{1,4}(x, t)$ in 2D, $t = 0$ (red colour) and $t = 0.2$ (blue colour), respectively, for $\alpha = 0.1, b = 1, d = 0.1, c = -1, \lambda = -1$ and $\psi_0 = 1$. These figures, represent the traveling wave properties of the solution such that soliton moves to left along the x-axis.

Figure 2 shows some plots of $\phi_{2,4}(x, t)$ in Eq. (40) with $\alpha = 30, b = -1, B_2 = 1, d = 0.1, c = 7, \lambda = -1, \omega = -0.4, a_c = -2, b_c = 1$ and $\psi_0 = 1$. In Fig. 2a, 3D view of $|\phi_{2,4}(x, t)|$ is plotted and this soliton type is known as singular soliton. Figure 2b shows the contour plot of $|\phi_{2,4}(x, t)|$ in Eq. (40) with mentioned parameters. In Fig. 2c, we depicted the 2D plots of $|\phi_{2,4}(x, t)|$ in Eq. (40) while the wave is traveling through to the right along the x-axis.

Figure 3 shows various plots of $\phi_{2,10}(x, t)$ in Eq. (41) with $\alpha = 3, b = -1, \omega = 4$ and $\psi_0 = 0$. In Fig. 3a, 3D view of $|\phi_{2,10}(x, t)|$ is plotted and this soliton type is known as periodic-singular soliton. Figure 3b shows the contour plot of $|\phi_{2,10}(x, t)|$ in Eq. (41) with mentioned parameters. Figure 3c shows the 2D plots of $|\phi_{2,10}(x, t)|$ in Eq. (41) at $t = 0$ and $t = 3$ while the wave is traveling through to the right along the x-axis.

Figure 4 shows some portraits of $\phi_{2,13}(x, t)$ in Eq. (42) with $\alpha = 1.25, b = 1, \omega = 0.4, \alpha = 1.25, b = 1, \omega = 0.4, A_2 = 0.1, b_c = -2, a_c = 1$ and $\psi_0 = 1$. In Fig. 4a, 3D view of $|\phi_{2,13}(x, t)|$ is plotted and this soliton type is known as singular soliton. Figure 4b shows the contour plot of $|\phi_{2,13}(x, t)|$ in Eq. (42) with mentioned parameters. Figure 4c shows the 2D plots of $|\phi_{2,13}(x, t)|$ in Eq. (42) at $t = 0$ and $t = 3$ while the wave is traveling through to the left along the x-axis.

Figure 5 shows some graphical representations of $\phi_{2,3}(x, t)$ in Eq. (39) with $\alpha = 30, b = -1, \omega = -0.4, B_2 = 0.1, \lambda = -1, d = 0.1, c = 7$ and $\psi_0 = 1$. In Fig. 5a, 3D
view of $|\phi_{2,3}(x,t)|$ is plotted and this soliton type is known as M-shaped soliton. Figure 5b shows the contour plot of $|\phi_{2,3}(x,t)|$ in Eq. (39) with mentioned parameters. Figure 5c shows the 2D plots of $|\phi_{2,3}(x,t)|$ in Eq. (39) at $t = 0$ and $t = 0.4$ while the wave is traveling through to the right along the x-axis.

In Fig. 6a, 2D plots of $|\phi_{1,4}(x,t)|$ in Eq. (16) are placed. Graphs are obtained with $\alpha = 0.1, \psi_0 = 1, B_2 = 1, d = 0.1, c = -1$ and $\lambda = -1$. There are four different styled plots in Fig. 6. Red stripe is obtained selecting $t = 0$ and $b = 1$, blue stripe is obtained selecting $t = 0$ and $b = 3$, orange dashed stripe is obtained selecting $t = 0.15$ and $b = 3$, lastly purple dashed stripe is obtained selecting $t = 0.15$ and $b = 3$. We can obtain two different results from the Fig. 6. Firstly increasing $b$, coefficient of 4OD, is resulted as an increase in the height of the soliton in Eq. (16). This result is arised from the parameter $b$ is placed in the numerator part of $B_2$ in Eq. (16). Another result of the Fig. 6 is that increasing parameter $b$ is results as an increase in the velocity of the soliton in Eq. (16). This result is reasoned from the wave transform in Eq. (2) includes a negative $v$ parameter. Thus decreasing in $v$ affects the velocity in a positive way. Also, constraint conditions in Eq. (5) are considered, velocity $v$ is decreased with increasing in $b$ such as $v = -a - 8bk^3$.

In Fig. 6b, periodic-singular soliton $|\phi_{2,10}(x,t)|$ in Eq. (41) is plotted with $\alpha = 3, b = 1, B_2 = -1$ and $\psi_0 = 0$. The figure has two different styled plots, red wave is obtained with $\omega = 0.4$ and the blue one is obtained with $\omega = 0.6$. When the wave
transformation in Eq. (2) is considered, \(w\) is the wave number and the wave number is inversely proportional to the wave length. So increasing wave number affects the wave length in a negative way. Considering the \(\text{SET}_{2,3}\), parameter \(w\) is affected by parameter \(b\) in a negative way so when \(b\) is increased, the wave number \(w\) is decreased, and consequently wave length increases and frequency decreases naturally. These consecutive results are shown in Fig. 6b.

Figure 7 shows some plots of \(|\phi_{3,1}(x, t)|\) in Eq. (48) with \(\alpha = 1, \beta = 1, k = 1, \sigma_2 = -1, c_0 = 1, c_1 = 3, c_2 = 1\) and \(\psi_0 = 1\). In Fig. 7a, 3D view of \(|\phi_{3,1}(x, t)|\) is plotted and this soliton type is known as dark soliton. Figure 7b shows the contour plot of \(|\phi_{3,1}(x, t)|\) in Eq. (48) with mentioned parameters. Figure 7c shows the 2D plots of \(|\phi_{3,1}(x, t)|\) in Eq. (48) at \(t = 0\) and \(t = 0.4\) while the wave is traveling through to the left along the x-axis.

Figure 8 shows some views of \(|\phi_{3,4}(x, t)|\) in Eq. (48) with \(\alpha = 1, \beta = 1, k = 1, \sigma_2 = -1, c_0 = 1, c_1 = 1, c_2 = 3\) and \(\psi_0 = 1\). In Fig. 8a, 3D view of \(|\phi_{3,4}(x, t)|\) is plotted and this soliton type is known as periodic-singular soliton. Figure 8b shows the contour plot of \(|\phi_{3,4}(x, t)|\) in Eq. (48) with mentioned parameters. Figure 8c shows the 2D plots of \(|\phi_{3,4}(x, t)|\) in Eq. (48) at \(t = 0, t = 0.1\) and \(t = 0.13\) while the wave is traveling through to the left along the x-axis.

If we make a general comparison over the results obtained, the soliton types obtained in this study; They are M-shaped, singular, singular-periodic, periodic and dark solitons. When other CQFLE studies in the literature are examined, it is seen that they obtain M-shaped, singular and bright soliton solutions (Yıldırım et al. 2021; Zayed et al. 2021). Although we see that the difference is due to the methods used, when we examine the other studies in which the mentioned methods are used, we can see that different styles of soliton solutions are also obtained such as kink (Neirameh and Parvaneh 2021), dark
and bright (Zayed et al. 2020). In addition to all this, we find it useful to examine some other methods that the authors can use to find optical solitons such as extended rational sine-cosine and sinh–cosh method (Mahak and Akram 2019a), $G'/G$ method (Akram and Mahak 2018), uniform algebraic method (Mahak and Akram 2019b) and auxiliary equation method (Mahak and Akram 2020b).

![Fig. 4](image1.png)

**Fig. 4** The various plots of $|\phi_{2,13}(x,t)|$ in Eq. (41)

![Fig. 5](image2.png)

**Fig. 5** The various plots of $|\phi_{2,3}(x,t)|$ in Eq. (39)

![Fig. 6](image3.png)

**Fig. 6** 2D plots of $|\phi_{1,4}(x,t)|$ for Eq. (16) in Fig. 6a, $|\phi_{2,10}(x,t)|$ for Eq. (41) in Fig. 6b
4 Conclusion

In this paper, we handled the CQFL equation which models the nonlinear pulse transmission through optical fiber. This new model is obtained by adding the cubic and quartic nonlinearities when the chromatic dispersion is low due to different colors of light having different velocities. The equation models nonlinear pulse transmission on the optical silica fibers. We obtained the NODE form of the model by using wave transformation in Eq. (2). Then we applied three different analytical methods to the NODE. Thus various analytical soliton solutions of the model are received. The novelty of the paper is to obtain many solutions of considered equation, which have different kinds of solitons. Especially, we obtained singular, periodic, periodic-singular, dark and M-shaped soliton solutions such that M-shaped is a rare type of soliton. In addition, we used eMETEM therefore the obtained solutions are pretty new. We plotted the 3D surface and contour graphs of the obtained solutions. We stated the direction of the traveling wave via 2D plots and then we interpreted velocity, the height of wave envelope, wave number and wave length in case of changing parameters. As a result, we hope that this study will help future researches in the optic field of physics.

Author contributions Ismail Onder and Muslum Ozisik conceived of the presented idea. Aydin Secer developed the theory and performed the computations. Mustafa Bayram verified the analytical methods. Mustafa Bayram supervised the findings of this work. All authors discussed the results and contributed to the final manuscript. Aydin Secer and Mustafa Bayram prepared all figures. Aydin Secer and Muslum Ozisik planned and carried out the simulations. All authors discussed the results and commented on the manuscript and
Mustafa Bayram contributed to the design and implementation of the research, to the analysis of the supervised work. All authors reviewed the manuscript.

**Funding** No funding was received to assist with the preparation of this manuscript.

**Availability of data and materials** All data generated or analyzed during this study are included in this article.

**Declarations**

**Conflict of interest** The authors declare that there is no conflict of interest whatsoever.

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