Surface Rayleigh-Lamb waves in stratified substrate

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Abstract. Propagation of Rayleigh - Lamb waves has shortage of theoretical studies and demands in deeper analyses. Understanding the process of propagation of Rayleigh-Lamb waves depending on the geometrical and acoustical properties of the individual layers within the Earth's crust is of great importance for the NDT of Earth exploration and development of seismic protection of buildings and structures. Propagation of these waves in the Earth's crust upper layers is studied by analytical methods. A mathematical model for analyzing Rayleigh - Lamb waves propagating in stratified media is worked out. The analytical solution is constructed using Cauchy complex formalism and the method called Modified Transfer Matrix (MTM). As an example a multilayer system modeling the Earth's crust, consisting of sedimentary rocks, a layer of granite, basalt and mantle, is considered. Equation for dispersion relations is derived. Fundamental modes of the dispersion curves for a stratified media are obtained.

1. Introduction
Surface waves vertically polarized in a stratified half-space are usually called Rayleigh-Lamb waves [1]. The physical and geometric properties of the Earth's crust layers affect to acoustic parameters of Rayleigh-Lamb waves. These waves make a great contribution to the transfer of seismic energy during earthquakes [2,3] therefore it is very important to research this relationship, which can be used to predict the characteristics of the waves. Therefore, the analytical solution for the Rayleigh-Lamb wave’s propagation in the Earth's crust is important for both geological exploration and the development of seismic protection systems for buildings and structures. However, some unresolved problems demands in deeper analyses of some peculiarities associated with propagation of these waves.

2. Basic Notations
The phase velocity is one of the main acoustic parameters of Rayleigh-Lamb waves. Knowing the phase velocity of the Rayleigh-Lamb waves, we can determine the wave number.

The frequency, phase velocity and wave number interconnected by the following relation

\[ c_f = \frac{\omega}{r}, \]

where \( c_f \) is the phase velocity; \( \omega \) is the frequency of the waves; \( r \) is the wave number.
The dispersion equation of Rayleigh-Lamb wave can be written in explicit form as \( \omega = \omega(r) \) or in implicit form as \( f(\omega, r) = 0 \). Graphical representation of the roots of dispersion equation in the plane \((r, \omega)\) called a dispersion curve.

The equation of motion for elastic homogeneous multilayer anisotropic media can be written in vector form

\[
\text{div}_x \mathbf{C} \cdot \nabla \mathbf{u} - \rho \ddot{\mathbf{u}} = 0,
\]

where \( \mathbf{u} \) is the displacement field, \( \rho \) is the material density, \( \mathbf{C} \) is elasticity tensor, which is expected to be positive definite.

Probably, the first analysis of wave propagation in an isotropic plate was based on an approximate equation of motion associated with hypothesis Bernoulli-Euler [4]. The study in [5] is based on the classical theory of plates. In addition to approximate methods [6], a more general approach to solving the equations of motion is based on the use of potentials. Dispersion relations for different types of waves without using the theory of plate bending can be obtained [7].

The velocities of longitudinal and transverse waves \( c_p \) and \( c_s \) in isotropic media, respectively:

\[
c_p^2 = \frac{\lambda + 2\mu}{\rho} \quad \text{and} \quad c_s^2 = \frac{\mu}{\rho},
\]

where \( \lambda \) and \( \mu \) are Lamé constants.

Research [8] showed that Poisson was the first who classified two types of wave displacement: longitudinal and transverse. However, the general solution of Poisson didn’t include the vector potential. First scalar and vector potentials were presented in [9]. Thus the general solution as the sum of the scalar and the vector potential gradient can be represented in [8, 9] for the determination of the displacement field.

Following [10-13] generally solution of equation (2.2) for Rayleigh-Lamb wave can be represented in exponential form

\[
\mathbf{u}(\mathbf{x}, t) = \mathbf{f}(\zeta) e^{i(\mathbf{n} \cdot \mathbf{x} - \zeta t)},
\]

where \( r \) is the wave number; \( \mathbf{f} \) is unknown vector function of the imaginary coordinates \( \zeta \), determined by the change in the amplitude of the wave front; \( \zeta = i r (\mathbf{x} \cdot \mathbf{v}) \) is imaginary variable; \( \mathbf{x} \) is the coordinate of the point; \( \mathbf{v} \) is the unit normal vector (a vector perpendicular to the middle surface); \( \mathbf{n} \) is the direction of wave propagation; \( c \) is the phase velocity.

For analysis waves of Rayleigh-Lamb propagating in the layer of anisotropic medium it is possible to using different variants of six-dimensional formalism when the second-order vector equations of motion transforms to first order equations.

The Cauchy formalism is considered in detail in [14].

For this formalism the harmonic Lamb waves representation is determined in the form (2.4). New vector function introduced:

\[
\mathbf{V}(\zeta) = \partial_\zeta \mathbf{f}(\zeta).
\]

The equation of motion (2.2) is converted to Cauchy normal form (if the tensor \( \mathbf{C} \) is positive certainty), which is a matrix equation of the first order

\[
\partial_\zeta \begin{pmatrix} \mathbf{f} \\ \mathbf{V} \end{pmatrix} = \mathbf{G} \begin{pmatrix} \mathbf{f} \\ \mathbf{V} \end{pmatrix}.
\]

(2.5) (2.6)
where \( \begin{bmatrix} f \\ V \end{bmatrix} \) is three-dimensional vector function; \( G \) is fundamental matrix.

\[
G = \begin{bmatrix} O & I \\ -A_1^{-1} \cdot A_3 & -A_2^{-1} \cdot A_3 \end{bmatrix},
\]

where \( A_1 = v \cdot C \cdot v; \ A_2 = v \cdot C \cdot n + n \cdot C \cdot v; \ A_3 = n \cdot C \cdot n - \rho c^2 I \).

With (2.7), the general solution (2.6) can be represented in form

\[
\begin{bmatrix} f(\zeta) \\ V(\zeta) \end{bmatrix} = e^{\xi G} \cdot X_6.
\]

Vector of boundary conditions is determined by

\[
t_v(\zeta) = (A_4 \delta \zeta + A_4) \cdot f(\zeta),
\]

where \( A_4 = v \cdot C \cdot n \).

Combining (2.8) and (2.9), on both sides of the layer we obtain the displacements and surface tractions in the form

\[
\begin{bmatrix} f(\pm irh) \\ t_v(\pm irh) \end{bmatrix} = Z \cdot \exp(\pm irh G) \cdot X_6,
\]

where

\[
Z = \begin{bmatrix} I & O \\ A_4 & A_4 \end{bmatrix}.
\]

If we express the displacements and stresses on one layer side, through the appropriate vectors on the other layer side then it can be excluded \( X_6 \) from the equation (2.11):

\[
\begin{bmatrix} f(-irh) \\ t_v(-irh) \end{bmatrix} = T \cdot \begin{bmatrix} f(+irh) \\ t_v(+irh) \end{bmatrix},
\]

where \( r = \frac{\omega}{c} \) is wave number, \( 2h \) is plate thickness, matrix \( T \) has the form

\[
T = Z \cdot e^{-2irh G} \cdot Z^{-1}.
\]

The representation (2.12) is valid if the matrix \( G \) is degenerate. The condition for the degeneracy of the matrix is the following expression

\[
\det \begin{bmatrix} O & I \\ I & O \end{bmatrix} \cdot T \cdot \begin{bmatrix} I \\ 0 \end{bmatrix} = 0,
\]

where \( \begin{bmatrix} O & I \end{bmatrix} \) is move from zero effort on the surface and \( \begin{bmatrix} I \\ 0 \end{bmatrix} \) is zero displacement of effort.

3. **Formulation of the problem**

This study examines four layer stratified media as shown in figure 1.

\[
3
\]
The crust is modeled as a four-layer stratified substrate with characteristic properties. The upper layers are layers of sedimentary layer, granite and basalt. The lower layer of the substrate modulates the mantle.

The thickness of each layer of which varies from \(-h_k + \Delta h_k, k = 1, n\) - layer number.

First layer properties: \(E_1 = 3.2 \times 10^{10} \text{Pa}, \nu_1 = 0.35, \rho_1 = 2700, h_1 = 5\text{km}\) (sedimentary layer).

Second layer properties: \(E_2 = 5.0 \times 10^{10} \text{Pa}, \nu_2 = 0.25, \rho_2 = 2600, h_2 = 5\text{km}\) (granite).

Third layer properties: \(E_3 = 6.5 \times 10^{10} \text{Pa}, \nu_3 = 0.22, \rho_3 = 2750, h_3 = 5\text{km}\) (basalt).

Substrate: \(E_s = 1 \times 10^{11} \text{Pa}, \nu_s = 0.2, \rho_s = 3300, h_s = 15\text{km}\) (mantle).

The outer surfaces of the layers are assumed to be free: Traction-free boundary conditions are written as

\[
\begin{align*}
    t_x \big|_{x = -h_k} &= 0 \\
    t_x \big|_{x = h_k} &= 0,
\end{align*}
\]

where \(t_x\) and \(t_y\) are stresses on the surfaces of the first and last layers, respectively.

For a stratified media, the interface boundary conditions are conditions of the ideal mechanical contact:

\[
\begin{align*}
    u_x \big|_{x = -h_k} &= u_x \big|_{x = h_k}, \\
    t_y \big|_{x = -h_k} &= -t_y \big|_{x = h_k},
\end{align*}
\]

where index \(k\) is referred to the corresponding layer. Equations (3.2) are written in local coordinate systems. The origin of coordinate system is center of the corresponding layers.

The movement of the particles of the media is described by equation (2.2).

In this problem it is necessary to solve equations (2.2) with conditions of boundary (3.1).

The solution is constructed using Cauchy complex formalism and the method called Modified Transfer Matrix (MTM) [15].

4. Results

The solution was built for a two-layer system and then projected for a multi-layer system.

The six-dimensional field for each layer can be written as a fundamental matrix \(e^{iG_x x} \)

\[
\begin{pmatrix}
    u_k(x,t) \\
    v_k(x,t)
\end{pmatrix} = e^{iG_x x} \int X_{6k} e^{i\beta(x-x')} dx',
\]

(3.3)
where \( \mathbf{v}_k(x,t) = V_k e^{i(n_x x - c t)} \), \( V_k \) determined by (2.5); \( x \) – is the coordinate of the point; \( X_{6k} \) – a six-dimensional complex vector.

Substitution of (3.3) into condition (3.2) gives (3.4)

\[
\begin{pmatrix}
    \mathbf{I} & \mathbf{O} \\
    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}
\end{pmatrix}
(\mathbf{e}^{i\mathbf{G}_k x} \cdot \mathbf{X}_0)
= 
\begin{pmatrix}
    \mathbf{I} & \mathbf{O} \\
    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}
\end{pmatrix}
(\mathbf{e}^{i\mathbf{G}_k x} \cdot \mathbf{X}_0)
= 
\begin{pmatrix}
    \mathbf{I} & \mathbf{O} \\
    \mathbf{X}_{62}
\end{pmatrix}
(3.4)
\]

If the tensor \( \mathbf{C}_i \), \( k = 1, 2 \), is positive definite, then all matrices of size 6 \( \times \) 6 in (3.4), non-degenerate, and a six-dimensional complex vector \( X_{62} \) can be represented in terms \( X_{6l} \) –

\[
X_{62} = (\mathbf{e}^{i\mathbf{G}_k x}) \cdot 
\begin{pmatrix}
    \mathbf{I} & \mathbf{O} \\
    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}
\end{pmatrix}
^{-1}
\begin{pmatrix}
    \mathbf{I} & \mathbf{O} \\
    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}
\end{pmatrix}
(\mathbf{e}^{i\mathbf{G}_k x} \cdot \mathbf{X}_0) = \mathbf{X}_6.
(3.5)
\]

Expression (3.5) forms the basis of the transfer matrix method, while the matrices appearing on the right side are known as transfer matrices. Considering (3.5), the conditions (3.1) can be written as

\[
\mathbf{M} \cdot \mathbf{X}_6 = 0,
(3.6)
\]

The matrix \( \mathbf{M} \) can be written as

\[
\mathbf{M} = 
\begin{pmatrix}
    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}\n    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}
\end{pmatrix}
(\mathbf{e}^{i\mathbf{G}_k x})
\begin{pmatrix}
    \mathbf{I} & \mathbf{O} \\
    \mathbf{O} & \mathbf{O}
\end{pmatrix}
\begin{pmatrix}
    \mathbf{I} & \mathbf{O} \\
    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}
\end{pmatrix}
(\mathbf{e}^{i\mathbf{G}_k x})
\begin{pmatrix}
    \mathbf{I} & \mathbf{O} \\
    \mathbf{X}_{6l}
\end{pmatrix}
(3.7)
\]

In (3.7) \( \mathbf{(A_i)_j} \), \( \mathbf{(A_i)_l} \), \( k = 1, 2 \) – layer number.

Existence of a nontrivial solution for the equation (2.2) is equivalent to satisfying the condition is equivalent to satisfying the condition

\[
\det(\mathbf{M}) = 0,
(3.8)
\]

For multi-layer system:

\[
\mathbf{M} = 
\begin{pmatrix}
    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}\n    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}
\end{pmatrix}
(\mathbf{e}^{i\mathbf{G}_k x})
\prod_{k=1}^{m}
\begin{pmatrix}
    \mathbf{I} & \mathbf{O} \\
    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}
\end{pmatrix}
(\mathbf{e}^{i\mathbf{G}_k x})
\begin{pmatrix}
    \mathbf{I} & \mathbf{O} \\
    \mathbf{(A_i)_j} & \mathbf{(A_i)_l}
\end{pmatrix}
(3.9)
\]

The results of analytical solutions as shown in figure 2.

**Figure 2.** Fundamental modes of Rayleigh - Lamb waves dispersion curves.
5. Conclusions
A model of a stratified media was developed. Based on the method combining the Cauchy six-dimensional complex formalism and the method called Modified Transfer Matrix, the fundamental modes of the Rayleigh-Lamb waves dispersion curves were studied for waves propagating in a stratified media.

The propagation of Rayleigh-Lamb waves in the Earth's crust upper layers was investigated. The study of the dispersion curves of Rayleigh-Lamb waves allowed us studying both geometrical and acoustical properties of the individual layers within the Earth's crust. Understanding the process of propagation of Rayleigh-Lamb waves depending on the properties of the Earth's crust is of great importance for the NDT of Earth exploration and development of seismic protection of buildings and structures.

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