Characterization of Inner Product Spaces by Strongly Schur-Convex Functions

Miroslaw Adamek

Abstract. Involving the notion of strongly Schur-convex functions we give a new characterization of inner product spaces among norm spaces. We also present a representation theorem for functions which generate strongly Schur-convex sums.

Mathematics Subject Classification. Primary 46C15; Secondary 26B25, 39B62.

Keywords. Inner product space, Schur-convex, strongly Schur-convex, Wright-convex, strongly Wright-convex.

1. Introduction

A rich collection of characterizations of inner product spaces among norm spaces can be found in Amir’s book [8] (see also [1, Chpt. 11], [5–7,9,21]). The best known characterization was given in the classic paper of Jordan and von Neumann [12]. Namely, a norm space $(X, \|\cdot\|)$ is an inner product space if and only if the following equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

holds true for all $x, y \in X$. We call this equality the Jordan–von Neumann identity or the parallelogram law.

In the paper [20] the authors show a characterization of functions which generate strongly Schur-convex sums, obtaining equivalently a representation of such functions. In particular, they prove that the following three sentences are equivalent.

(1) Function $f$ generates strongly Schur-convex sums.
(2) Function $f$ is strongly Wright-convex.
There exists a convex function $g$ and an additive function $a$ such that 
\[ f = g + a + c \| \cdot \|^2. \]
For details, see Theorem 7, p. 180 [20]; it should be mentioned, though, that the domains of investigated functions are subsets of an inner product space; i.e. the unitarity of a space is a sufficient condition for such a result. We will prove that it is also a necessary condition, which, together with the previously mentioned theorem from [20] will give a new characterization of inner product spaces among norm spaces. Moreover, we will obtain a representation theorem for functions generating strongly Schur-convex sums, and it will be a counterpart of the classical Hardy–Littlewood–Pólya majorization theorem (see [11]), works of Schur [22], Karamata [13], Ng [18] and Kominek [14]. The construction of presented theorems in this work was inspired by the works [3,19].

It is worth noting that beside theoretical applications Schur-convex functions have also practical applications in data transmission in cellular networks (see [15]). Thus it seems that research on these functions can be very useful.

2. Main Result

In this paper $X$ will be a space and $X^n$, where $n \geq 2$ is a natural number, will be a Cartesian product of $n$-copies of the space $X$ (i.e. $X^n := X \times \cdots \times X$).

We will start with recalling some definitions presented, for example, in the literature given in brackets and also discussed there, respectively.

**Definition 2.1.** [16,20,22] Let $X$ be a real vector space. For $x, y \in X^n$ we say that $x$ is majorized by $y$, written $x \preceq y$, if
\[ x = y \cdot P \]
for a doubly stochastic matrix $P$ (i.e. a matrix of degree $n$ containing non-negative elements with all rows and columns summing up to 1).

**Definition 2.2.** [20] Let $(X, \| \cdot \|)$ be a real normed space and $D$ be a convex subset of $X$. We say that a function $F: D^n \to \mathbb{R}$ is strongly Schur-convex with modulus $c > 0$ if
\[ F(x) \leq F(y) - c (\| y \|^2 - \| x \|^2) \]
for all $x, y \in D$ such that $x \preceq y$ and $\| \cdot \|$ is the Cartesian norm in the space $X^n$ (i.e. $\| x \| = \sqrt{\| x_1 \|^2 + \cdots + \| x_n \|^2}$ for $n$-tuples $x = (x_1, \ldots, x_n) \in X^n$).

**Definition 2.3.** [17] Let $(X, \| \cdot \|)$ be a real normed space and $D$ be a convex subset of $X$. We say that a function $f: D \to \mathbb{R}$ is strongly Wright-convex with modulus $c > 0$ if
\[ f(t x + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y) - 2ct(1-t)\| x - y \|^2 \]
for all $x, y \in D$ and $t \in [0,1]$. 
**Definition 2.4.** Let \((X, \|\cdot\|)\) be a real normed space and \(D\) be a convex subset of \(X\). We say that a function \(f: D \to \mathbb{R}\) generates strongly Schur-convex sums with modulus \(c > 0\) if for all natural numbers \(n \geq 2\) the function \(F: D^n \to \mathbb{R}\) defined by formula \(F(x_1, \ldots, x_n) := f(x_1) + \cdots + f(x_n)\), is strongly Schur-convex with modulus \(c\).

The following theorem gives a new characterization of inner product spaces. To be clear, the phrase “For all functions \(f: D \to \mathbb{R}\)” is understood as “For all subsets \(D\) of \(X\) and all functions \(f\) defined on \(D\)”.

**Theorem 2.5.** Let \((X, \|\cdot\|)\) be a real normed space. The following conditions are equivalent:

1. For all \(c > 0\) and all functions \(f: D \to \mathbb{R}\), \(f\) generates strongly Schur-convex sums with modulus \(c\) if and only if \(f\) is strongly Wright-convex with modulus \(c\);
2. \((X, \|\cdot\|)\) is an inner product space.

**Proof.** The implication (2) \(\Rightarrow\) (1) is proved in [20] (Theorem 7, p. 180). Assume that (1) holds true. Immediately from the definition of functions \(f\) generating strongly Schur-convex sums with modulus \(c\) for each \(x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in D^n\) \((n \geq 2)\) such that \(x \preceq y\), we have the following two equivalent inequalities

\[
F(x) \leq F(y) - c \left( \|y\|^2 - \|x\|^2 \right),
\]
\[
f(x_1) - c|x_1|^2 + \cdots + f(x_n) - c|x_n|^2 \leq f(y_1) - c|y_1|^2 + \cdots + f(y_n) - c|y_n|^2.
\]

The function \(f(\cdot) := c\|\cdot\|^2\) satisfies the above inequality, and it means that it generates strongly Schur-convex sums with modulus \(c\). Thus, from the adopted assumption, the function \(c\|\cdot\|^2: X \to \mathbb{R}\) is strongly Wright-convex with modulus \(c\) and, consequently, it satisfies the following inequality

\[
c|tx + (1-t)y|^2 + c||(1-t)x + ty||^2 \leq c|x|^2 + c|y|^2 - 2ct(1-t)||x - y||^2,
\]

for all \(x, y \in X\) and \(t \in (0, 1)\). Dividing this inequality by \(c\) and taking \(t = \frac{1}{2}\) we obtain the inequality

\[
2 \left\| \frac{x + y}{2} \right\|^2 \leq \|x\|^2 + \|y\|^2 - \frac{1}{2}||x - y||^2, \quad x, y \in X.
\]

Multiplying the above inequality by 2 and using the homogeneous axiom of norm, we can write the result in the following form

\[
\|x + y\|^2 + \|x - y\|^2 \leq 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X.
\]

Now, putting the standard substitution (i.e. \(u = x + y\) and \(v = x - y\)) in the last inequality, we get the reverse inequality and it means that the norm \(\|\cdot\|\) satisfies the parallelogram law, which, together with the paper of Jordan-von Neumann [12], implies that \((X, \|\cdot\|)\) is an inner product space. The proof is finished. \(\square\)
Remark 2.6. We can replace the first condition in Theorem 2.5 with the following:

(1') For all $c > 0$, a function $f : X \rightarrow \mathbb{R}$ generates strongly Schur-convex sums with modulus $c$ if and only if $f$ is strongly Wright-convex with modulus $c$.

A representation theorem for functions generating strongly Schur-convex sums looks as follows.

Theorem 2.7. Let $(X, \| \cdot \|)$ be a real normed space. The following conditions are equivalent:

1. For a $c > 0$ and a function $f : D \rightarrow \mathbb{R}$, $f$ generates strongly Schur-convex sums with modulus $c$;
2. There exists a convex function $g : D \rightarrow \mathbb{R}$ and an additive function $a : X \rightarrow \mathbb{R}$ such that $f(x) = g(x) + a(x) + c \| x \|^2$, $x \in D$.

Proof. Assuming (1) and using the definition of functions $f$ generating strongly Schur-convex sums with modulus $c$ (Definition 2.4), we conclude that for each $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in D^n \ (n \geq 2)$ such that $x \preceq y$ we receive the following inequality

$$F(x) \leq F(y) - c \left( |||y|||^2 - |||x|||^2 \right),$$

which is equivalent to the inequality

$$f(x_1) - c ||x_1||^2 + \cdots + f(x_n) - c ||x_n||^2 \leq f(y_1) - c ||y_1||^2 + \cdots + f(y_n) - c ||y_n||^2.$$  

It means that the function $h(\cdot) := f(\cdot) - c \cdot || \cdot ||^2$ generates Schur-convex sums. Thus, from the theorem of Ng characterizing functions generating Schur-convex sums (see [14,18]), there exists a convex function $g : D \rightarrow \mathbb{R}$ and an additive function $a : X \rightarrow \mathbb{R}$ such that

$$h(x) = g(x) + a(x), \ x \in D,$$

and consequently, $f$ is of the form

$$f(x) = g(x) + a(x) + c \| x \|^2, \ x \in D.$$

To prove (2) $\Rightarrow$ (1), observe that the function $h$ defined as before (i.e. $h(\cdot) := f(\cdot) - c \cdot || \cdot ||^2$) is a sum of convex and additive functions. And using once more the aforementioned Ng’s theorem, it generates Schur-convex sums, i.e.

$$h(x_1) + \cdots + h(x_n) \leq h(y_1) + \cdots + h(y_n), \ x \preceq y.$$  

Which is equivalent to the following inequalities

$$f(x_1) - c ||x_1||^2 + \cdots + f(x_n) - c ||x_n||^2 \leq f(y_1) - c ||y_1||^2 + \cdots + f(y_n) - c ||y_n||^2,$$

$$f(x_1) + \cdots + f(x_n) \leq f(y_1) + \cdots + f(y_n) - c \left( ||y_1||^2 + \cdots + ||y_n||^2 - ||x_1||^2 - \cdots - ||x_n||^2 \right),$$
for all $x \preceq y$. Finally, taking the function

$$F(x_1, \ldots, x_n) := f(x_1) + \cdots + f(x_n)$$

and the Cartesian norm $||| \cdot |||$ in the space $X^n$, we get the inequality

$$F(x) \leq F(y) - c \left(|||y|||^2 - |||x|||^2\right), \quad x \preceq y.$$ 

It proves that $f$ generates strongly Schur-convex sums with modulus $c$. This finishes the proof. \hfill \square

Combining Theorems 2.5 and 2.7 we immediately obtain the following theorem. Notice that the implication (2) $\Rightarrow$ (1) in Theorem 2.8 is also proved in [17].

**Theorem 2.8.** Let $(X, ||\cdot||)$ be a real normed space. The following conditions are equivalent:

1. For all $c > 0$ and all functions $f : D \to \mathbb{R}$, $f$ is strongly Wright-convex with modulus $c$ if and only if there exists a convex function $g : D \to \mathbb{R}$ and an additive function $a : X \to \mathbb{R}$ such that

   $$f(x) = g(x) + a(x) + c \|x\|^2, \quad x \in D;$$

2. $(X, ||\cdot||)$ is an inner product space.

**Remark 2.9.** We can replace the first condition in Theorem 2.8 with the following:

(1') For all $c > 0$, a function $f : X \to \mathbb{R}$ is strongly Wright-convex with modulus $c$ if and only if there exists a convex function $g : X \to \mathbb{R}$ and an additive function $a : X \to \mathbb{R}$ such that

$$f(x) = g(x) + a(x) + c \|x\|^2, \quad x \in X.$$

**Remark 2.10.** In virtue of Remarks 2.6 and 2.9, taking (1') instead of (1) in Theorems 2.5 and 2.8, respectively, we get stronger implications (1') $\Rightarrow$ (2) than (1) $\Rightarrow$ (2), but the implications (2) $\Rightarrow$ (1') become weaker than (2) $\Rightarrow$ (1).

We end this work with two examples which show that in Theorem 2.5 neither the assumption that $f$ generates strongly Schur-convex sums implies that $f$ is strongly Wright-convex, nor conversely. The examples' construction is based on the ideas of the examples from [10, 19].

**Example 2.11.** Let $X = \mathbb{R}^2$, $\|x\| = |x_1| + |x_2|$, for $x = (x_1, x_2)$ and $f(x) = \|x\|^2$. Observe the following equivalent inequalities.

$$f(x_1) + \cdots + f(x_n) \leq f(y_1) + \cdots + f(y_n) - \left(|||y|||^2 - |||x|||^2\right);$$

$$f(x_1) + \cdots + f(x_n) \leq f(y_1) + \cdots + f(y_n) - \left(||y_1||^2 + \cdots + ||y_n||^2 - \|x_1\|^2 - \cdots - \|x_n\|^2\right);$$

$$||x_1||^2 + \cdots + ||x_n||^2 \leq ||y_1||^2 + \cdots + ||y_n||^2.$$
which means that \( f \) generates Schur-convex sums with modulus \( c = 1 \). However, \( f \) is not strongly Wright-convex with modulus \( c = 1 \). Indeed, for \( x = (1, 0) \), \( y = (0, 1) \) and \( t = \frac{1}{2} \) we have
\[
f \left( \frac{x + y}{2} \right) = 1 \geq 0 = \frac{f(x) + f(y)}{2} - \frac{1}{4} \|x - y\|^2.
\]

**Example 2.12.** Let \( X = \mathbb{R}^2 \), \( \|x\| = |x_1| + |x_2| \), for \( x = (x_1, x_2) \). We show that the function \( f(x) = x_1^2 + x_2^2 \) is strongly Wright-convex with modulus \( c = \frac{1}{2} \). First of all, observe that \( f \) is strongly midconvex with modulus \( c = \frac{1}{2} \) (i.e. \( f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2} - \frac{c}{2} \|x - y\|^2 \), with \( c = \frac{1}{2} \)). Indeed, for arbitrary \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) we have
\[
f \left( \frac{x + y}{2} \right) = \frac{1}{4} \left( x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2 \right)
\]
and
\[
\frac{f(x) + f(y)}{2} - \frac{1}{8} \|x - y\|^2 = \frac{3}{8} \left( x_1^2 + y_1^2 + x_2^2 + y_2^2 \right)
+ \frac{1}{4} \left( x_1y_1 + x_2y_2 - |x_1 - y_1||x_2 - y_2| \right).
\]

Thus
\[
\frac{f(x) + f(y)}{2} - \frac{1}{8} \|x - y\|^2 - f \left( \frac{x + y}{2} \right) = \frac{1}{8} \left( |x_1 - y_1| - |x_2 - y_2| \right)^2 \geq 0,
\]
which means that \( f \) is strongly midconvex with modulus \( c = \frac{1}{2} \). And consequently, \( f \) satisfies the inequality
\[
f \left( tx + (1 - t)y \right) \leq tf(x) + (1 - t)f(y) - \frac{1}{2} t(1 - t) \|x - y\|^2,
\]
for all dyadic numbers \( t \in (0, 1) \) (see [2] and also a generalization in [4]). Moreover, \( f \) is continuous, thus it is a strongly convex function with modulus \( c = \frac{1}{2} \) (i.e. \( f \) satisfies (2.1) for all \( t \in (0, 1) \)). Now, interchanging \( t \) with \( 1 - t \) in (2.1) and adding the obtained inequality side by side to the inequality (2.1), we conclude that \( f \) is strongly Wright-convex with modulus \( c = \frac{1}{2} \).

Suppose that \( f \) generates strongly Schur-convex sums with modulus \( c = \frac{1}{2} \). Then, from Theorem 2.7, there exists a convex function \( g \) and an additive function \( a \) such that
\[
f(x) = g(x) + a(x) + \frac{1}{2} \|x\|^2.
\]
Hence the function
\[
h(x) := f(x) - \frac{1}{2} \|x\|^2 = g(x) + a(x)
\]
would be midconvex, but it is not. Indeed, for \( x = (-1, 1) \) and \( y = (1, 1) \) we get

\[
h \left( \frac{x + y}{2} \right) = \frac{1}{2} > 0 = \frac{h(x) + h(y)}{2}.
\]

**Open Access.** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit [http://creativecommons.org/licenses/by/4.0/](http://creativecommons.org/licenses/by/4.0/).

**References**

[1] Aczél, J., Dhombres, J.: Functional Equations in Several Variables. Encyclopaedia of Mathematics and Its Applications. Cambridge University Press, Cambridge (1989)

[2] Azócar, A., Gimenez, J., Nikodem, K., Sánchez, J.L.: On strongly midconvex functions. Opusc. Math. **31**(1), 15–26 (2011)

[3] Adamek, M.: On a problem connected with strongly convex functions. Math. Inequal. Appl. **19**(4), 1287–1293 (2016)

[4] Adamek, M.: On Jensen-type inequality for \( F \)-convex functions. Math. Inequal. Appl. **22**(4), 1355–1364 (2019)

[5] Alfonso, J.: Some remarks on inequalities that characterize inner product spaces. Int. J. Math. Math. Sci. **15**(1), 31–34 (1992)

[6] Alsina, C., Cruells, P., Tomas, M.S.: Characterizations of inner product spaces related to an isosceles trapezoid property. Arch. Math. (Brno) **35**, 21–27 (1992)

[7] Alsina, C., Sikorska, J., Tomas, M.S.: Norm Derivatives and Characterizations of Inner Product Spaces. World Scientific, Hackensack (2010)

[8] Amir, D.: Characterizations of Inner Product Spaces, Operator Theory: Advances and Applications. Birkhäuser, Basel (1986)

[9] Dadipour, F., Moslehian, M.S.: A characterization of inner product spaces related to the \( p \)-angular distance. J. Math. Anal. Appl. **371**(2), 677–681 (2010)

[10] Dragomir, S.S., Nikodem, K.: Jensen’s and Hermite–Hadamard’s type inequalities for lower and strongly convex functions on normed spaces. Bull. Iran. Math. Soc. **44**(5), 1337–1349 (2018)

[11] Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1952)
[12] Jordan, P., Neumann, J.V.: On inner products in linear metric spaces. Ann. Math. 36, 719–723 (1935)
[13] Karamata, J.: Sur une inégalité relative aux fonctions convexes. Publ. Math. Univ. Belgrade 1, 145–148 (1932)
[14] Kominek, Z.: On additive and convex functionals. Rad. Mat. 36, 267–279 (1987)
[15] Lagen, S., Agustín, A., Vidal, J.: On the superiority of improper Gaussian signaling in wireless interference MIMO scenarios. IEEE Trans. Commun. 64(8), 3350–3368 (2016)
[16] Marshall, A.W., Olkin, I.: Inequalities: Theory of Majorization and Its Applications, Mathematics in Science and Engineering, vol. 143. Academic Press, New York (1979)
[17] Merentes, N., Nikodem, K., Rivas, S.: Remarks on strongly Wright-convex functions. Ann. Pol. Math. 102(3), 271–278 (2011)
[18] Ng, C.T.: Functions generating Schur-convex sums. In: General Inequalities, 5, Oberwolfach, 1986, Internat. Schriftenreihe Numer. Math., vol. 80, pp. 433–438. Birkhäuser, Basel (1987)
[19] Nikodem, K., Páles, Z.S.: Characterizations of inner product spaces by strongly convex functions. Banach J. Math. Anal. 5(1), 83–87 (2011)
[20] Nikodem, K., Rajba, T., Wasowicz, S.Z.: Functions generating strongly Schur-convex sums. In: Bandle, C. (ed.) Inequalities and Applications 2010: Dedicated to the Memory of Wolfgang Walter, pp. 175–182. Springer, Berlin (2012)
[21] Rätz, J.: Characterizations of inner product spaces by means of orthogonally additive mappings. Aequat. Math. 58, 111–117 (1999)
[22] Schur, I.: Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie. Sitzungsber. Berl. Math. Ges. 22, 9–20 (1923)

Mirosław Adamek
Department of Mathematics
University of Bielsko-Biala
ul. Willowa 2
43-309 Bielsko-Biala
Poland
e-mail: madamek@ath.bielsko.pl

Received: February 5, 2020.
Accepted: April 7, 2020.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.