STOCHASTIC HEAT EQUATIONS DRIVEN BY LÉVY PROCESSES

TONGKEUN CHANG AND MINSUK YANG

Abstract. We study stochastic heat equations driven by a class of Lévy processes:

\[ du = \Delta u \, dt + g \, dX_t \quad \text{in} \quad \mathbb{R}_T^d, \quad u(0, x) = 0 \quad \text{in} \quad x \in \mathbb{R}^d. \]

We prove the corresponding estimate

\[ \|u\|_{H^k(\mathbb{R}_T^d)} \leq c(p, T) \|g\|_{B^{k-2}_p(\mathbb{R}_T^d)} \]

for \(2 \leq p < \infty\) and \(k \in \mathbb{R}\).

2000 Mathematics Subject Classification. Primary: 60H15, Secondary: 35R60.

Keywords and phrases: Stochastic heat equation, Lévy process, Sobolev space, Besov space.

1. Introduction

In this paper, we study the following stochastic heat equation

\[ \begin{cases} 
du = (\Delta u + f) \, dt \\
u|_{t=0} = u_0
\end{cases} \quad \text{in} \quad \mathbb{R}_T^d, \]

where \(\mathbb{R}_T^d := (0, T) \times \mathbb{R}^d\) for \(0 < T < \infty\). We assume that \(X_t\) is a one-dimensional Lévy process satisfying some conditions on a probability space, which is explained in Section 2. We allow \(f\) and \(g\) to be random. To solve the problem (1.1), we may consider the following two problems

(1.2) \[ \begin{cases} 
du = (\Delta u + f) \, dt \\
u|_{t=0} = u_0
\end{cases} \quad \text{in} \quad \mathbb{R}^d, \]

and

(1.3) \[ \begin{cases} 
\Delta u \, dt + g \, dX_t \\
u|_{t=0} = 0
\end{cases} \quad \text{in} \quad \mathbb{R}_T^d. \]

Since the problem (1.2) has been well studied, we shall focus on the problem (1.3). For \(g \in L^p(0, T; S'(\mathbb{R}^d))\) (where \(S'(\mathbb{R}^d)\) is the space of tempered distributions), the solution of (1.3) can be represented by

\[ u(t, x) = \int_0^t T_{t-s} g(s, x) \, dX_s. \]

Here, \(T_{t-s} g(s, x) = \Gamma(t-s, \cdot) * g(s, \cdot)(x)\), where \(\Gamma(t, x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}\) and \(\ast\) is the standard convolution in \(\mathbb{R}^d\).

The first author was supported by the National Research Foundation of Korea(NRF-2010-0016699).
For the Brownian motion case, a theory was developed by N.V. Krylov [8]. Since the Burkholder-Davis-Gundy inequality implies
\[
\mathbb{E} \int_0^T \|\nabla u(s, \cdot)\|_{L^p(\mathbb{R}^d)}^p ds \leq c(p) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left( \int_0^t |\nabla T_{t-s} g(s, x)|^2 ds \right)^{p/2} dx dt,
\]
he showed that for \(2 \leq p < \infty\) there is a positive constant \(c(p)\) independent of \(T\) such that

\[
\mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left( \int_0^t |\nabla T_{t-s} g(s, x)|^2 ds \right)^{p/2} dx dt \leq c(p) \mathbb{E} \int_0^T \|g(s, \cdot)\|^p_{L^p(\mathbb{R}^d)} ds.
\]

(1.5)

He proved this inequality by interpolating \(L^2\) estimates via Plancherel’s theorem and sophisticated BMO estimates. Using the properties of Sobolev spaces, it was generalized for \(k \in \mathbb{R}\)

\[
\mathbb{E} \int_0^T \|u(s, \cdot)\|^p_{H^k_p(\mathbb{R}^d)} ds \leq c(p, T) \mathbb{E} \int_0^T \|g(s, \cdot)\|^p_{H^{k-1}_p(\mathbb{R}^d)} ds.
\]

(1.6)

Here, the function space \(H^k_p(\mathbb{R}^d)\) is the usual Sobolev space (see Section 2).

For the general Lévy process case, a few results are known for these types of Sobolev estimates. In this case, instead of the Burkholder-Davis-Gundy inequality Kunita’s inequality is applicable and it produces

\[
\mathbb{E} \int_0^T \|\nabla u(s, \cdot)\|_{L^p(\mathbb{R}^d)}^p ds \leq c \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left( \int_0^t |\nabla T_{t-s} g(s, x)|^2 ds \right)^{p/2} dx dt
\]

\[
+ c \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_0^t |\nabla T_{t-s} g(s, x)|^p ds dx dt.
\]

(1.7)

The first term on the right-hand side is the same as in (1.5), but the second term on the right-hand side is new. Recently, Z. Chen and K. Kim [3] proved that for \(2 \leq p < \infty\) and \(\epsilon > 0\), there is a constant \(c(\epsilon, p, T) > 0\) such that

\[
\mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_0^t |\nabla T_{t-s} g(s, x)|^p ds dx dt \leq c(\epsilon, p, T) \mathbb{E} \int_0^T \|g(s, \cdot)\|^p_{H^{1-\frac{2}{p}+\epsilon}_p(\mathbb{R}^d)} ds
\]

(1.8)

under some assumptions on Lévy measure.

Now we state our main results.

**Proposition 1.** Let \(0 < T < \infty\) and \(1 < p < \infty\). There are positive constants \(c_1(p, T)\) and \(c_2(p)\) such that

\[
\mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_0^t |T_{t-s} g(s, x)|^p ds dx dt \leq c_1(p, T) \mathbb{E} \int_0^T \|g(s, \cdot)\|^p_{B^{\frac{2}{p}}_p(\mathbb{R}^d)} ds
\]

(1.9)

and

\[
\mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_0^t |T_{t-s} g(s, x)|^p ds dx dt \leq c_2(p) \mathbb{E} \int_0^T \|g(s, \cdot)\|^p_{B^{\frac{2}{p}}_p(\mathbb{R}^d)} ds.
\]

(1.10)

To prove Proposition 1, we shall use the Littlewood-Paley theory and then prove variants of Hardy’s inequality. Using (1.6), (1.7), Proposition 1 and the mapping properties of the pseudo-differential operators \((I - \Delta)^{s/2}\) and \((-\Delta)^{s/2}\) (see (1) and (2) of Remark 1), we can obtain our main theorem.
Theorem 1. Let $0 < T < \infty$ and $2 \leq p < \infty$. If $\beta_2 < \infty$ and $\beta_p < \infty$, then there are positive constants $c_1(p,T)$ and $c_2(p)$ such that

$$
\|u\|_{H_0^\beta(R^d_T)} \leq c_1(p,T)\|\tilde{g}\|_{B_p^{-\frac{k}{2}}(R^d_T)},
$$

$$
\|u\|_{H_0^\beta(R^d_T)} \leq c_2(p)\|\tilde{g}\|_{B_p^{-\frac{k}{2}}(R^d_T)},
$$

where $\beta_p$ is defined in (2.6) and stochastic Banach spaces $H_p^k(R^d_T)$, $B_p^k(R^d_T)$, $\dot{H}_p^k(R^d_T)$ and $\dot{B}_p^k(R^d_T)$ are defined in (2.5).

A direct consequence of Theorem 1 is the following corollary which follows from the fact that $H_p^k(R^d)$ is continuously embedded in $B_p^k(R^d)$ for $k \in \mathbb{R}$ and $2 \leq p < \infty$ and the property of real interpolation; (see [2] Theorem 6.4.4 and Theorem 6.3.1).

Corollary 1. For $0 < T < \infty$ and $2 \leq p < \infty$

$$
\|u\|_{B_p^k(R^d_T)} \leq c(p,T)\|\tilde{g}\|_{B_p^{-\frac{k}{2}}(R^d_T)},
$$

$$
\|u\|_{B_p^k(R^d_T)} \leq c(p)\|\tilde{g}\|_{B_p^{-\frac{k}{2}}(R^d_T)},
$$

$$
\|u\|_{\dot{B}_p^k(R^d_T)} \leq c(p)\|\tilde{g}\|_{\dot{B}_p^{-\frac{k}{2}}(R^d_T)},
$$

The organization of the paper is as follows. In Section 2, we introduce precise definitions of function spaces and conditions concerning Lévy processes. In Section 3, we prepare basic lemmas for the heat kernel. In Section 4, we reduce Proposition 1 to Lemma 3. In Section 5, we prove our main Lemma 3. In Section 6, we prove Theorem 1. In Section 7, we apply our method to SPDE with fractional Laplace operator.

2. Preliminaries

2.1. Sobolev and Besov spaces. Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ denote the class of Schwartz functions on $\mathbb{R}^d$. The space $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^d)$ is the dual space, i.e., the space of continuous linear functionals on $\mathcal{S}$. Given $f \in \mathcal{S}$, we define the Fourier transform and the inverse Fourier transform of $f$ by

$$
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x) dx, \quad \mathcal{F}^{-1}(f)(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} f(\xi) d\xi.
$$

The definition of Fourier transform is naturally extended to a tempered distribution $f$; (see chapter 9 in [5]). We define the operators

$$
(I - \Delta)^{k/2}f = \mathcal{F}^{-1}((1 + 4\pi^2|\xi|^2)^{k/2}\hat{f}),
$$

$$
(-\Delta)^{k/2}f = \mathcal{F}^{-1}((2\pi|\xi|)^k\hat{f})
$$

for $k \in \mathbb{R}$ and for $f \in \mathcal{S}(\mathbb{R}^d)$. Let $k \in \mathbb{R}$ and $1 < p < \infty$. The (nonhomogeneous) Sobolev space $H_p^k(\mathbb{R}^d)$ is defined as

$$
H_p^k(\mathbb{R}^d) = \left\{ f \in \mathcal{S}' \mid \|f\|_{H_p^k} := \| (I - \Delta)^{k/2} f \|_{L^p} < \infty \right\},
$$
and the homogeneous Sobolev space $\dot{H}_p^k(\mathbb{R}^d)$ is defined as

$$\dot{H}_p^k(\mathbb{R}^d) = \left\{ f \in S' / P \mid \|f\|_{\dot{H}_p^k} := \|(-\Delta)^{k/2} f\|_{L^p} < \infty \right\},$$

where $S'/P$ denote the set of all tempered distributions modulo polynomials. Note that to avoid working with equivalence classes of functions we identify two distributions in $\dot{H}_p^k(\mathbb{R}^d)$ whose difference is a polynomial.

Before we give the definition of Besov spaces, we prepare the setup. We fix a function $\psi \in S(\mathbb{R}^d)$ satisfying $\hat{\psi}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\psi}(\xi) = 0$ for $|\xi| \geq 2$ and then define $\hat{\phi}(\xi) = \hat{\psi}(\xi) - \hat{\psi}(2\xi)$. Note also that

$$\supp \hat{\phi}(\xi) \subset \{1/2 \leq |\xi| \leq 2\}.$$  

We define for $j \in \mathbb{Z}$

$$\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi)$$

so that for all $\xi \in \mathbb{R}^d$

$$1 = \hat{\psi}(\xi) + \sum_{j=1}^{\infty} \hat{\phi}_j(\xi)$$

and for all $\xi \neq 0$

$$1 = \sum_{j=-\infty}^{\infty} \hat{\phi}_j(\xi).$$

Let $k \in \mathbb{R}$ and $1 \leq p \leq \infty$. The (nonhomogeneous) Besov space $B_p^k(\mathbb{R}^d)$ is defined as

$$B_p^k(\mathbb{R}^d) = \left\{ f \in S' \mid \|f\|_{B_p^k} := \|\psi * f\|_{L^p} + \left( \sum_{j=1}^{\infty} (2^{kj}\|\phi_j * f\|_{L^p})^p \right)^{1/p} < \infty \right\},$$

and the homogeneous Besov space $\dot{B}_p^k(\mathbb{R}^d)$ is defined as

$$\dot{B}_p^k(\mathbb{R}^d) = \left\{ f \in S' / P \mid \|f\|_{\dot{B}_p^k} := \left( \sum_{j=-\infty}^{\infty} (2^{kj}\|\phi_j * f\|_{L^p})^p \right)^{1/p} < \infty \right\},$$

where $*$ denotes the standard convolution in $\mathbb{R}^d$. We note that whenever $\phi \in S$ and $f \in S'$, $\phi * f$ is a well defined function.

**Remark 1.**  
1. For all $k, s \in \mathbb{R}$, the pseudo-differential operator $(I - \Delta)^{s/2}$ is isomorphism from $H_p^k(\mathbb{R}^d)$ to $H_p^{k-s}(\mathbb{R}^d)$ and from $B_p^k(\mathbb{R}^d)$ to $B_p^{k-s}(\mathbb{R}^d)$.
2. For all $k, s \in \mathbb{R}$, the pseudo-differential operator $(-\Delta)^{s/2}$ is isomorphism from $\dot{H}_p^k(\mathbb{R}^d)$ to $\dot{H}_p^{k-s}(\mathbb{R}^d)$ and from $\dot{B}_p^k(\mathbb{R}^d)$ to $\dot{B}_p^{k-s}(\mathbb{R}^d)$.
3. In particular, if $1 < p < \infty$ and $k$ is a nonnegative integer, then $H_p^k(\mathbb{R}^d)$ is the set of functions satisfying

$$\sum_{0 \leq |\alpha| \leq k} \int_{\mathbb{R}^d} |\partial^\alpha f(x)|^p dx < \infty,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ and $\partial^\alpha f = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d} f$ is a distributional derivative.
2.2. Stochastic Banach spaces. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) be a probability space, where \(\{\mathcal{F}_t : t \geq 0\}\) is a filtration of \(\sigma\)-fields \(\mathcal{F}_t \subset \mathcal{F}\) with \(\mathcal{F}_0\) containing all \(\mathbb{P}\)-null subsets of \(\Omega\). Assume that a one-dimensional \(\{\mathcal{F}_t\}\)-adapted Lévy processes \(X_t\) is defined on \((\Omega, \mathcal{F}, \mathbb{P})\). We denote the expectation of a random variable \(X(\omega)\), \(\omega \in \Omega\) by \(\mathbb{E}[X]\) or simply \(\mathbb{E}X\). We consider \(g\) as a Banach space-valued stochastic process and so \((\Omega \times (0, T), \mathcal{P}, \mathbb{P} \otimes \mathbb{L}([0, T]))\) is a suitable choice for their common domain, where \(\mathcal{P}\) is the predictable \(\sigma\)-field generated by \(\{\mathcal{F}_t : t \geq 0\}\) (see, e.g., pp. 84–85 of [7]) and \(\mathbb{L}([0, T])\) is the Lebesgue measure on \((0, T)\). We define the stochastic function space

\[
\mathbb{H}_p^k(\mathbb{R}^d) = L^p(\Omega \times (0, T), \mathcal{P}, H^k_{\mathbb{P}}(\mathbb{R}^d))
\]

with the norm

\[
\|f\|_{\mathbb{H}_p^k(\mathbb{R}^d)} = \left(\mathbb{E} \int_0^T \|f(s, \cdot)\|_{H^k_{\mathbb{P}}(\mathbb{R}^d)}^p ds\right)^{1/p}.
\]

The stochastic function spaces \(\hat{\mathbb{H}}_p^k(\mathbb{R}^d), \mathbb{H}_p^k(\mathbb{R}^d)\) and \(\hat{\mathbb{H}}_p^k(\mathbb{R}^d)\) are defined similarly.

2.3. Lévy process. A Lévy process \(X_t := X(t)\) is a stochastic process satisfying

(L1) \(X(0) = 0\) a.s.,
(L2) \(X(t)\) has stationary and independent increments,
(L3) \(X(t)\) is stochastically continuous, i.e. for all \(a > 0\) and for all \(s \geq 0\),

\[
\lim_{t \to s} \mathbb{P}(|X(t) - X(s)| > a) = 0.
\]

A process \(X_t\) is càdlàg if \(X_t\) has left limit and is right continuous. Since every Lévy process has a càdlàg modification that is itself a Lévy process (see Theorem 2.1.8 in [1]), we may assume all Lévy processes \(X_t\) are càdlàg. Let \(t \geq 0\) and Borel sets \(A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})\). We denote

\[
N(t, A) = \#\{0 \leq s \leq t : X(s) - X(s-) \in A\},
\]

the intensity measure \(\nu(A) = \mathbb{E}[N(1, A)]\), and the compensated Poisson random measure

\[
\tilde{N}(t, A) = N(t, A) - t\nu(A).
\]

Note that \(\nu(A)\) is the Lévy measure of \(X_t\). By the Lévy-Ito decomposition (see more details in [1]), there exist a constant \(c \in \mathbb{R}^d\) and a positive-definite matrix \(A\) such that

\[
X_t = ct + AB_t + \int_{|z| < 1} z\tilde{N}(t, dz) + \int_{|z| \geq 1} zN(t, dz),
\]

where \(B_t\) is a \(d\)-dimensional Brownian motion. If we denote \(\tilde{c} := c + \nu(\{|z| \geq 1\})\), we may write

\[
X_t = \tilde{c}t + AB_t + \int_{\mathbb{R}^d} z\tilde{N}(t, dz).
\]

Note that \(\tilde{N}(t, z)\) is martingale. Since the result for the Brownian motion is known, we assume that \(\tilde{c} = 0\) and \(A = 0\) for the simplicity. Finally, we denote

\[
\beta_p = \int_{\mathbb{R}^d} |z|^p \nu(dz).
\]
3. Basic Heat Kernel Estimates

We give basic lemmas for the heat kernel that will be useful in the sequel.

**Notation** 1. We denote $f \lesssim g$ if $f \leq cg$ for some positive constant $c$.

**Lemma 1.** There exists a constant $c > 0$ such that for all $j \in \mathbb{Z}$

$$\left\| \mathcal{F}^{-1}(\hat{\phi}_j(x)e^{-t|\xi|^2}) \right\|_{L^1} \lesssim e^{-ct2^j},$$

where $\hat{\phi}_j$ is defined in (2.2). The implicit constant depends only on the dimension $d$.

**Proof.** Let $K_j(t, x) = \mathcal{F}^{-1}(\hat{\phi}_j(x)e^{-t|\xi|^2})(x)$. By a simple scaling

$$K_j(t, x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \hat{\phi}(2^{-j} \xi)e^{-t|\xi|^2} d\xi = 2^{jd} \int_{\mathbb{R}^d} e^{2\pi i 2^j x \cdot \xi} \hat{\phi}(\xi)e^{-2^{2j}t|\xi|^2} d\xi = 2^{jd} K_0(2^j t, 2^j x).$$

Observe that

$$(I - \Delta_\xi) e^{2\pi i x \cdot \xi} = (1 + 4\pi^2 |x|^2) e^{2\pi i x \cdot \xi}.$$ 

Carrying out the repeated integrations by parts gives

$$(1 + 4\pi^2 |x|^2)^N K_0(t, x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} (I - \Delta_\xi)^N (\hat{\phi}(\xi)e^{-t|\xi|^2}) d\xi$$

for all $N \in \mathbb{N}$. Since supp $\hat{\phi}(\xi) \subset \{1/2 \leq |\xi| \leq 2\}$, we have

$$(1 + 4\pi^2 |x|^2)^N |K_0(t, x)| \lesssim \sup_{\xi} |(I - \Delta_\xi)^N (\hat{\phi}(\xi)e^{-t|\xi|^2})|.$$ 

A direct computation shows that for some $c > 0$

$$\sup_{\xi} |(I - \Delta_\xi)^N (\hat{\phi}(\xi)e^{-t|\xi|^2})| \lesssim e^{-ct}.$$ 

We choose $N > d$ so that

$$\int_{\mathbb{R}^d} |K_j(t, x)| dx = \int_{\mathbb{R}^d} |K_0(2^j t, x)| dx \lesssim e^{-c2^{2j} t}.$$ 

This completes the proof. \qed

Given $g \in H^k_p(\mathbb{R}^d)$, we denote

$$T_t g(x) = 1(\cdot) * g(x).$$

In fact, for $k < 0$, it is the convolution of a function with a tempered distribution, that is,

$$\Gamma_t * g(x) = \begin{cases} \int_{\mathbb{R}^d} \Gamma(t, x - y) g(y) dy & k \geq 0 \\
\langle g, \Gamma(t, x - \cdot) \rangle & k < 0, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ means the duality paring between $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$.

**Lemma 2.** There exists a constant $c > 0$ such that for all $j \in \mathbb{Z}$

$$\left\| T_t(\phi_j * g)(s, \cdot) \right\|_{L^p} \lesssim e^{-c2^{2j} t} \left\| \phi_j * g(s, \cdot) \right\|_{L^p},$$

where $\phi * g(s, x) := \phi * g(s, \cdot)(x)$. The implicit constant depends only on the dimension $d$. 
Proof. We have
\[ T_t(\phi_j \ast g)(s, x) = \mathcal{F}^{-1}(e^{-t|\xi|^2} \hat{\phi}_j(\xi)\hat{g}(s, \xi))(x). \]
From the support condition (2.1),
\[ \hat{\phi}_j(\xi) = (\hat{\phi}_{j-1}(\xi) + \hat{\phi}_j(\xi) + \hat{\phi}_{j+1}(\xi))\hat{\phi}_j(\xi). \]
Thus we have
\[ T_t(\phi_j \ast g)(s, x) = \sum_{-1 \leq k \leq 1} \mathcal{F}^{-1}(\hat{\phi}_{j+k}(\xi)e^{-t|\xi|^2})*\mathcal{F}^{-1}(\hat{\phi}_j(\xi)\hat{g}(s, \xi))(x). \]
Young’s convolution inequality gives
\[ \|T_t(\phi_j \ast g)(s, \cdot)\|_{L^p} \leq \sum_{-1 \leq k \leq 1} \|\mathcal{F}^{-1}(\hat{\phi}_{j+k}(\xi)e^{-t|\xi|^2})\|_{L^1}\|\phi_j \ast g(s, \cdot)\|_{L^p} \]
and therefore the result follows from Lemma 1. \(\square\)

4. Proof of Proposition 1

First we consider (1.9). Using (2.3) we decompose
\[ g(s, x) = \psi \ast g(s, x) + \sum_{j=1}^{\infty} \phi_j \ast g(s, x). \]
Using (4.1) and Minkowski’s inequality, we have
\[ \mathbb{E} \int_0^T \int_0^t \|T_{t-s}g(s, \cdot)\|^p_{L^p}dsdt \leq \mathbb{E} \int_0^T \int_0^t \|T_{t-s}(\psi \ast g)(s, \cdot)\|^p_{L^p}dsdt \]
\[ + \mathbb{E} \int_0^T \int_0^t \left( \sum_{j=1}^{\infty} \|T_{t-s}(\phi_j \ast g)(s, \cdot)\|_{L^p} \right)^pdsdt. \]
By Young’s inequality, the first term of right-hand side is dominated by
\[ \mathbb{E} \int_0^T \int_0^t \|\Gamma(t-s, \cdot)\|^p_{L^1}\|\psi \ast g(s, \cdot)\|^p_{L^p}dsdt = \mathbb{E} \int_0^T \int_0^t \|\psi \ast g(s, \cdot)\|^p_{L^p}dsdt. \]
Using lemma 2, the second term of right-hand side is dominated by
\[ \mathbb{E} \int_0^T \int_0^t \left( \sum_{j=1}^{\infty} e^{-c2^j(t-s)}\|\phi_j \ast g(s, \cdot)\|_{L^p} \right)^pdsdt. \]
Hence, we have
\[ \mathbb{E} \int_0^T \int_0^t \|T_{t-s}g(s, \cdot)\|^p_{L^p}dsdt \leq \mathbb{E} \int_0^T \int_0^t \|\psi \ast g(s, \cdot)\|^p_{L^p}dsdt + \left( \sum_{j=1}^{\infty} e^{-c2^j(t-s)}\|\phi_j \ast g(s, \cdot)\|_{L^p} \right)^pdsdt. \]
If we denote
\[ f_j(t-s) := e^{-c2^j(t-s)} \quad \text{and} \quad g_j(s) := \|\phi_j \ast g(s, \cdot)\|_{L^p}, \]
(4.2)
then to prove (1.9), it suffices to show that

\begin{equation}
\int_0^T \int_0^t \left( \sum_{1 \leq j < \infty} f_j(t-s)g_j(s) \right)^p \, ds dt \lesssim \int_0^T \sum_{1 \leq j < \infty} 2^{-2j} g_j(s)^p \, ds.
\end{equation}

Now we consider (1.10). Using (2.4), we decompose

\[ g(s,x) = \sum_{j=-\infty}^{\infty} \phi_j \ast g(s,x). \]

Using similar calculation with the above estimation, we obtain

\begin{align*}
\mathbb{E} \int_0^T \int_0^t \left| \sum_{j=-\infty}^{\infty} f_j(t-s)g_j(s) \right|^p \, ds dt \\
&\leq \mathbb{E} \int_0^T \int_0^t \left( \sum_{j=-\infty}^{\infty} \left| T_{t-s}(\phi_j \ast g)(s,\cdot) \right|_{L^p} \right)^p \, ds dt \\
&\leq \mathbb{E} \int_0^T \int_0^t \left( \sum_{j=-\infty}^{\infty} e^{-c2^{2j}(t-s)} \left| \phi_j \ast g(s,\cdot) \right|_{L^p} \right)^p \, ds dt.
\end{align*}

Hence, to prove (1.10), it suffices to show that

\begin{equation}
\int_0^T \int_0^t \left( \sum_{j=-\infty}^{\infty} f_j(t-s)g_j(s) \right)^p \, ds dt \lesssim \int_0^T \sum_{j=-\infty}^{\infty} 2^{-2j} g_j(s)^p \, ds.
\end{equation}

We shall prove the inequalities (4.3) and (4.4) in Section 5.

5. PROOF OF MAIN LEMMA

Lemma 3. For $0 < T < \infty$ and $1 < p < \infty$

\begin{equation}
\int_0^T \int_0^t \left| \sum_{j=1}^{\infty} f_j(t-s)g_j(s) \right|^p \, ds dt \lesssim \int_0^T \sum_{j=1}^{\infty} 2^{-2j} |g_j(s)|^p \, ds
\end{equation}

and

\begin{equation}
\int_0^T \int_0^t \left| \sum_{j=-\infty}^{\infty} f_j(t-s)g_j(s) \right|^p \, ds dt \lesssim \int_0^T \sum_{j=-\infty}^{\infty} 2^{-2j} |g_j(s)|^p \, ds.
\end{equation}

Proof. We only prove (5.1) since the proof of (5.2) is almost the same. In order to use the decay of the function $f_j(t)$, we separate the indices of the summation as

\begin{equation}
2^{1-p} \int_0^T \int_0^t \left( \sum_{j=1}^{\infty} f_j(t-s)g_j(s) \right)^p \, ds dt \\
\leq \int_0^T \int_0^t \left( \sum_{2^{2j}(t-s) \leq 1} f_j(t-s)g_j(s) \right)^p \, ds dt + \int_0^T \int_0^t \left( \sum_{2^{2j}(t-s) > 1} f_j(t-s)g_j(s) \right)^p \, ds dt \\
:= J_1 + J_2.
\end{equation}
If $2^{2j}(t-s) \leq 1$, then $f_j(t-s) \leq c$ for some positive constant $c$ depending only on $d$. Hence, using H"older’s inequality, we have

$$J_1 \leq \int_0^T \int_0^t \left( \sum_{2^{2j}(t-s) \leq 1} 2^{j/(p-1)} \right)^{p-1} \sum_{2^{2j}(t-s) \leq 1} 2^{-j} g_j(s)^p ds dt. $$

Summing a geometric series, we have

$$\left( \sum_{2^{2j}(t-s) \leq 1} 2^{j/(p-1)} \right)^{p-1} \lesssim (t-s)^{-1/2}. $$

Changing the order of integration and summation, we get

$$J_1 \lesssim \int_0^T \int_0^t (t-s)^{-1/2} \sum_{2^{2j}(t-s) \leq 1} 2^{-j} g_j(s)^p ds dt $$

\[\text{(5.4)}\]

$$= \int_0^T \sum_{j=1}^\infty 2^{-j} g_j(s)^p \int_s^{s+2^{-2j}} (t-s)^{-1/2} dtds $$

$$\lesssim \int_0^T \sum_{j=1}^\infty 2^{-2j} g_j(s)^p ds. $$

Now, we estimate $J_2$. Let us fix $2 < r < 2p$. Using H"older’s inequality, we obtain

$$J_2 = \int_0^T \int_0^t \left( \sum_{2^{2j}(t-s) > 1} 2^{rj/p} f_j(t-s) 2^{-rj/p} g_j(s) \right)^p ds dt $$

$$\lesssim \int_0^T \int_0^t \left( \sum_{2^{2j}(t-s) > 1} 2^{rj/(p-1)} f_j(t-s)^p/(p-1) \right)^{p-1} \sum_{2^{2j}(t-s) > 1} 2^{-rj} g_j(s)^p ds dt. $$

Since $f_j(t-s) \lesssim 2^{-2j}(t-s)^{-1}$ for $2^{2j}(t-s) > 1$, summing a geometric series, we have

$$\left( \sum_{2^{2j}(t-s) > 1} 2^{rj/(p-1)} f_j(t-s)^p/(p-1) \right)^{p-1} \lesssim \left( \sum_{2^{2j}(t-s) > 1} 2^{j(r-2p)/(p-1)} \right)^{p-1} (t-s)^{-p} $$

$$\lesssim (t-s)^{-r/2}. $$

By changing the order of integration and summation, we get

$$J_2 \lesssim \int_0^T \int_0^t (t-s)^{-r/2} \sum_{2^{2j}(t-s) > 1} 2^{-rj} g_j(s)^p ds dt $$

\[\text{(5.5)}\]

$$\leq \int_0^T \sum_{j=1}^\infty 2^{-rj} g_j(s)^p \int_s^{s+2^{-2j}} (t-s)^{-r/2} dtds $$

$$\lesssim \int_0^T \sum_{j=1}^\infty 2^{-2j} g_j(s)^p ds. $$

From (5.3), (5.4) and (5.5), we obtain (5.1). □
6. Proof of Theorem 1

Since the proofs are similar, we only prove the first inequality. Let $u$ be a function defined in (1.4). Since we have

$$(I - \Delta)^{k/2}u(t, x) = \int_0^t \langle \Gamma_t-s, (I - \Delta)^{k/2}g(s, \cdot) \rangle dX_s = \int_0^t T_{t-s}((I - \Delta)^{k/2}g)(s, x) dX_s,$$

it is sufficient to prove the case $k = 0$, that is,

$$\|u\|_{L^p(\mathbb{R}^d_T)} = \left( \mathbb{E} \int_0^t \|u(s, \cdot)\|_{L^p(\mathbb{R}^d)} ds \right)^{1/p} \lesssim \|g\|_{B^{-2/p}_p(\mathbb{R}^d_T)}. \quad (6.1)$$

Actually, if this estimate is proved, then, by (1) of remark 1, we obtain

$$\|(I - \Delta)^{k/2}u\|_{L^p(\mathbb{R}^d_T)} \lesssim \|(I - \Delta)^{k/2}g\|_{B^{-2/p}_p(\mathbb{R}^d_T)}$$

and hence we have

$$\|u\|_{H^{kp}(\mathbb{R}^d_T)} \lesssim \|g\|_{B^{k-2p}_p(\mathbb{R}^d_T)}.$$

Using Kunita’s inequality (see pp. 332-335 in [9], corollary 4.4.24 in [1]), we have

$$\|u\|_{L^p(\mathbb{R}^d_T)}^{p} = \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left| \int_0^t T_{t-s}g(s, \cdot)(x) dX_s \right|^p dxdt$$

$$\lesssim \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |T_{t-s}g(s, x)|^p |z|^p \nu(dz) dsdxdt$$

$$+ \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left( \int_0^t \int_{\mathbb{R}^d} |T_{t-s}g(s, x)|^2 |z|^2 \nu(dz) ds \right)^{p/2} dxdt$$

$$= \beta_p \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_0^t |T_{t-s}g(s, x)|^p dsdxdt$$

$$+ \beta_2^p \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left( \int_0^t |T_{t-s}g(s, x)|^2 ds \right)^{p/2} dxdt. \quad (6.2)$$

From (1.9) and (6.2), we get (6.1).

7. SPDE with Fractional Laplace Operator

In this section, we give an application to the SPDE with fractional Laplace operator.

$$\begin{cases}
du = -(-\Delta)^{\alpha}u dt + gdX_t & \mathbb{R}^d_T \\
u|_{t=0} = u_0 & \mathbb{R}^d,
\end{cases} \quad (7.1)$$

where $(-\Delta)^{\alpha}u$, $0 < \alpha < 1$, is the fractional Laplacian of $u$ defined by

$$(-\Delta)^{\alpha}u(x) := c(d, \alpha) \int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{n+2\alpha}} dy \quad (7.2)$$

with $c(d, \alpha)$ is a normalization constant. The fractional Laplacian of $u$ also can be defined as a pseudo-differential operator

$$(-\Delta)^{\alpha}u(x) = \mathcal{F}^{-1}((2\pi|\xi|)^{2\alpha}\hat{u}(\xi))(x). \quad (7.3)$$
The solution $u$ of (7.1) is represented by

$$u(t, x) = \int_0^t P_{t-s} g(s, x) dX_s,$$

where $P_t g(s, x) = p(t, \cdot) * g(s, \cdot)(x)$ with fundamental solution $p(t, x)$ of the fractional Laplace equation which is given by

$$p(t, x) = \mathcal{F}^{-1}(e^{-\frac{|\xi|^{2\alpha}}{2}})(x).$$

By a slight modification of the proof of Proposition 1, one can prove the following estimate

**Proposition 2.** Let $0 < T < \infty$ and $2 \leq p < \infty$. There is a positive constant $c$ such that

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} |P_{t-s} g(s, x)|^p ds dx dt \leq c \mathbb{E} \int_0^T \|g(t, \cdot)\|_{B_p^{- \frac{2\alpha}{p}}(\mathbb{R}^d)}^p dt.$$

A direct consequence is the following theorem.

**Theorem 2.** For $2 \leq p < \infty$,

$$\|u\|_{H_p^k(\mathbb{R}^d)} \leq c(p, T) \|g\|_{B_p^{- \frac{2\alpha}{p}}(\mathbb{R}^d)}.$$

**Proof.** We sketch the proof of Theorem 2. From the same reasoning in the proof of Theorem 1, we may assume that $k = 0$. Using the Kunita’s inequality, we have for some $c > 0$

$$\mathbb{E} \int_0^T \|u(s, \cdot)\|_{L_p(\mathbb{R}^d)}^p ds \leq c \beta_p \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |P_{t-s} g(s, x)|^p ds dx dt$$

$$+ c \beta_p^2 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left( \int_0^t |P_{t-s} g(s, x)|^2 ds \right)^{p/2} dx dt.$$

H. Kim and I. Kim[6] showed that for $2 \leq p < \infty$

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left( \int_0^t |P_{t-s} g(s, x)|^2 ds \right)^{p/2} dx dt \leq c \mathbb{E} \int_0^T \|g(t, \cdot)\|_{H_p^{- \frac{1}{2}}(\mathbb{R}^d)}^p dt$$

for some $c > 0$ (see also [4]). By the same proof as in Lemma 1 and Lemma 2, one can obtain

$$\|P_t (\phi_j * g)(s, \cdot)\|_{L_p} \lesssim e^{-c2^{2j+1}t} \|\phi_j * g(s, \cdot)\|_{L_p}.$$

Similar to the proof of Theorem 1, we can obtain the result. \qed

**References**

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge University Press, 2009.

[2] J. Bergh and J. Lofström, *Interpolation spaces*, An introduction, Springer-Verlag, Berlin(1976).

[3] Z. Chen, K. Kim, *An $L^p$-theory of non-divergence form SPDEs driven by Levy processes*, arXiv:1007.3295.

[4] T. Chang and K. Lee, *On a stochastic partial differential equation with a fractional Laplacian operator*, Stochastic Process. Appl., to appear.

[5] G. B. Folland, *Real analysis, Modern Techniques and Their Applications*, (1999).

[6] I. Kim and K. Kim, *A generalization of the Littlewood-Paley inequality for the fractional Laplacian $(-\Delta^{\alpha/2})$*, J. Math. Anal. Appl., 388, no. 1, 175-190(2012).

[7] N.V. Krylov, *An analytic approach to SPDEs*, Stochastic partial differential equations: six perspectives, Math. Surveys Monogr., 64, 185-242, Amer.Math.Soc., Providence, RI, 1999.

[8] N.V. Krylov, *A generalization of the Littlewood-Paley inequality and some other results related to stochastic partial differential equations*, Ulam Quart., 2, no. 4, 16-26 (1994).
[9] H. Kunita, *Stochastic differential equations based on \textit{Lévy processes and stochastic flows of diffeomorphisms},* in Real and Stochastic Analysis, New Perspectives, ed. M. M. Rao, Birkhauser Boston Basel Berlin pp. 305-75, 2004.

[10] E. Stein, *Singular integrals and differentiability properties of functions,* Princeton, N.J, 1970.

T. Chang: Department of Mathematics, Yonsei University, Seoul 120-749, Republic of Korea
*E-mail address: chang7357@yonsei.ac.kr*

M. Yang: Department of Mathematics, Yonsei University, Seoul 120-749, Republic of Korea
*E-mail address: kusnim@gmail.com*