Computing Crisp Simulations and Crisp Directed Simulations for Fuzzy Graph-Based Structures

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Abstract

Like bisimulations, simulations and directed simulations are used for analyzing graph-based structures such as automata, labeled transition systems, linked data networks, Kripke models and interpretations in description logic. Simulations characterize the class of existential modal formulas, whereas directed simulations characterize the class of positive modal formulas. These notions are worth studying. For example, one may be interested in checking whether a given finite automaton simulates another or whether an object in a linked data network has all positive properties that another object has. To deal with vagueness and uncertainty, fuzzy graph-based structures are used instead of crisp ones. In this article, we design efficient algorithms with the complexity $O((m+n)n)$ for computing the largest crisp simulation and the largest crisp directed simulation between two finite fuzzy labeled graphs, where $n$ is the number of vertices and $m$ is the number of nonzero edges of the input fuzzy graphs. We also adapt them to computing the largest crisp simulation and the largest crisp directed simulation between two finite fuzzy automata.

1 Introduction

Bisimulations and simulations are used for characterizing similarity between states in automata, labeled transition systems and Kripke models. When applied to domains such as linked data networks or description logics, they can be used for characterizing similarity between objects or individuals. Phrases like “two states are bisimilar” or “[something] simulates another” are familiar in computer science. Searching Google Scholar with both the keywords “bisimulation” and “simulation” returns more than 13000 results. This shows their popularity and usefulness in computer science (see, e.g., [26, 23, 27, 28, 11, 1, 24]).

Bisimulations have the following logical characterization in modal logic: modal formulas are invariant under bisimulations and, in image-finite Kripke models, two states are bisimilar (i.e., in the largest bisimulation relation) iff they cannot be distinguished by any modal formula (see, e.g., [1]). There are different variants of modal logic and different corresponding notions of bisimulation. Besides, the second assertion (called the Hennessy-Milner property) can be made stronger by replacing “image-finite” with “modal saturated”. The largest auto-bisimulation of a Kripke model is an equivalence relation.

Simulations require fewer conditions than bisimulations and characterize only the class of existential modal formulas, which are formulas in negation normal form without universal modal operators. Namely, they do not require the “backward” condition(s) of bisimulations, existential modal formulas are preserved under simulations and, in image-finite Kripke models, a state $x'$ simulates a state $x$ iff $x'$ satisfies all existential modal formulas that $x$ satisfies (see, e.g., [1]). The largest auto-simulation of a Kripke model is a pre-order.
The notion of directed simulation was introduced by Kurtonina and de Rijke [14] for modal logic. It was later formulated and studied by Divroodi and Nguyen [8] for description logics. Directed simulations are similar to bisimulations in that they use both the “forward” and “backward” conditions. The difference is that, if a state \( x \) is directedly simulated by a state \( x' \), then the label of \( x \) is only required to be a subset of the label of \( x' \), whereas two states in a bisimulation relation must have the same label. Directed simulations characterize the class of positive modal formulas [13]. That is, positive modal formulas are preserved under directed simulations and, in image-finite Kripke models, a state \( x' \) directedly simulates a state \( x \) iff \( x' \) satisfies all positive modal formulas that \( x \) satisfies. The largest directed auto-simulation of a Kripke model is a preorder.

The domains to which bisimulations, simulations and directed simulations are applied, such as automata, labeled transition systems, Kripke models, linked data networks and interpretations in description logic, are graph-based structures. To deal with vagueness and uncertainty, fuzzy graph-based structures are used instead, including fuzzy automata, fuzzy transition systems and weighted social networks. In such structures, both labels of vertices (states or individuals) and labels of edges (transitions or connections) can be fuzzified. Extending the notions of bisimulation, simulation and directed simulation for fuzzy graph-based structures, we can consider crisp or fuzzy relations for them. Crisp bisimulations/simulations have been studied for fuzzy transition systems [3] [30], weighted automata [25], fuzzy modal logics [9] and fuzzy description logics [16, 19]. Fuzzy bisimulations/simulations have been studied for fuzzy automata [9] [18], weighted/fuzzy social networks [10] [13], fuzzy modal logics [9] and fuzzy description logics [19] [20] [21]. As shown in [9] [19], the logical characterization of crisp bisimulations differs from the logical characterization of fuzzy bisimulations (in fuzzy modal/description logics under the Gödel semantics) in that it uses a logical language extended with involutive negation or the Baaz projection operator. A similar claim can be stated for the difference between crisp simulations (respectively, crisp directed simulation) and fuzzy simulations (respectively, fuzzy directed simulation) [17].

The objective of this article is to design efficient algorithms for computing crisp simulations and crisp directed simulations between fuzzy graph-based structures. As related works, the closest ones are discussed below.

- In [22] Paige and Tarjan gave an efficient algorithm with the complexity \( O((m + n) \log n) \) for computing the coarsest partition of a finite crisp graph, where \( n \) is the number of vertices and \( m \) is the number of edges of the graph. This problem is of the same nature as the problem of computing the largest auto-bisimulation of a finite crisp graph. Bloom and Paige [2] and Henzinger et al. [12] gave algorithms with the complexity \( O((m + n)n) \) for computing the largest auto-simulation of a crisp labeled transition system or a crisp graph, respectively.

- Adapting the idea of the above mentioned algorithm of Henzinger et al. [12], in [15] we gave an algorithm with the complexity \( O((m + n)n) \) for computing the largest directed auto-simulation of a finite crisp graph. Furthermore, in that paper we also gave algorithms with the complexity \( O((m + n)^2 n^2) \) for computing the largest auto-simulation and the largest directed auto-simulation of a finite crisp graph for the setting with counting successors.

- In [25] Stanimirović et al. gave algorithms with the complexity \( O(n^3) \) for computing the greatest right/left invariant Boolean equivalence matrix of a weighted automaton over an additively idempotent semiring. Such matrices are closely related to crisp bisimulations. They also gave algorithms with the complexity \( O(n^5) \) for computing the greatest right/left invariant Boolean quasi-order matrix of a weighted automaton over an additively idempotent semiring. Such matrices are closely related to crisp simulations.

- In [21] Nguyen and Tran gave algorithms with the complexity \( O((m + n)n) \) for computing the greatest fuzzy bisimulation/simulation between two finite fuzzy interpretations in the fuzzy description logic \( \mathcal{fALC} \) under the Gödel semantics, where \( n \) is the number of individuals and \( m \) is the number of nonzero instances of roles in the given fuzzy interpretations. They also adapted the algorithms to computing the greatest fuzzy bisimulation/simulation between fuzzy finite automata, as well as for dealing with other fuzzy description logics.
In this article, we design efficient algorithms with the complexity $O((m+n)n)$ for computing the largest crisp simulation and the largest crisp directed simulation between two finite fuzzy labeled graphs, where $n$ is the number of vertices and $m$ is the number of nonzero edges of the input fuzzy graphs. The motivations of this work are as follows:

- The research problem is worth studying. Given two finite fuzzy automata, we may be interested in checking whether one crisply simulates the other. This is related to computing the largest crisp simulation between the fuzzy automata. As another potential application, given two objects in a fuzzy linked data network, we may wonder whether one has all positive properties with a greater or equal degree than the other has. This is related to computing the largest crisp directed auto-simulation of the fuzzy network.

- As far as we know, there were no algorithms directly formulated for computing the largest crisp simulation and the largest crisp directed simulation between two finite fuzzy graph-based structures (such as fuzzy automata, fuzzy labeled transition systems, fuzzy Kripke models, and fuzzy interpretations in description logics). One can try to adapt the earlier mentioned algorithm with the complexity $O(n^5)$ by Stanimirović et al. [25] to computing the largest crisp simulations and the largest crisp directed simulations, but the complexity order $O(n^5)$ is too high. One can also crisp a given fuzzy labeled graph by treating a fuzzy $r$-labeled edge $(x, y)$ with a degree $d \in (0, 1]$ as the set of all crisp $(r, d_i)$-labeled edges $(x, y)$, where $d_i \in (0, d]$ is any fuzzy value occurring in the specification of the input graph, and then apply one of the algorithms given in [2] [12] [15] to the obtained crisp graph. The complexity of the resulting algorithm is of order $O(l(m+n)n)$, where $l$ is the number of fuzzy values occurring in the specification of the input fuzzy graph. In the worst case, $l$ can be $m+n$ and $O((m+n)n)$ is the same as $O((m+n)^2n)$, which is still too high.

- We choose to formulate our algorithms for fuzzy labeled graphs because the notion of fuzzy labeled graphs is universal. It covers fuzzy labeled transition systems, fuzzy Kripke models and fuzzy interpretations in description logic. Fuzzy automata are also a special case of fuzzy labeled graphs (see Section 5).

The rest of this article is structured as follows. In Section 2, we define fuzzy labeled graphs, define crisp simulations and crisp directed simulations between such graphs, and present some properties. In Section 3, we present our algorithm of computing the largest crisp simulation between two finite fuzzy labeled graphs, prove it correctness and analyze its complexity. In Section 4, we extend and adapt that algorithm to computing the largest crisp directed simulation between two finite fuzzy labeled graphs. In Section 5, we adapt our algorithms to computing the largest crisp simulation and the largest crisp directed simulation between two finite fuzzy automata. We conclude the article in Section 6.

2 Preliminaries

Fix a finite set $\Sigma_v$ of vertex labels and a finite set $\Sigma_e$ of edge labels. A fuzzy labeled graph, hereafter called a fuzzy graph for short, is a triple $G = \langle V, E, L \rangle$, where $V$ is a set of vertices, $E : V \times \Sigma_e \times V \to [0, 1]$ is called the fuzzy set of labeled edges, and $L : V \to (\Sigma_v \to [0, 1])$ is called the labeling function of vertices. It is finite if $V$ is finite. Given vertices $x, y \in V$, a vertex label $p \in \Sigma_v$, and an edge label $r \in \Sigma_e$, $L(x)(p)$ means the degree of that $p$ is a member of the label of $x$, and $E(x, r, y)$ means the degree of that there is an edge labeled by $r$ from $x$ to $y$.

Given $f$ and $g$ of type $\Sigma_v \to [0, 1]$, we write $f \leq g$ to denote that $f(p) \leq g(p)$ for all $p \in \Sigma_v$.

Let $G = \langle V, E, L \rangle$ and $G' = \langle V', E', L' \rangle$ be fuzzy graphs. A binary relation $Z \subseteq V \times V'$ is called a (crisp) simulation between $G$ and $G'$ if the following two conditions hold (for all possible values of the free variables), where $\rightarrow$ and $\land$ denote the usual crisp logical connectives:

\[
Z(x, x') \rightarrow L(x) \leq L'(x') \quad (1)
\]

\[
Z(x, x') \land E(x, r, y) > 0 \rightarrow \exists y' \in V' (Z(y, y') \land E(x, r, y) \leq E(x', r, y')). \quad (2)
\]
To make it clear, as the connectives $\rightarrow$ and $\land$ are crisp, the conditions mean that:

1. if $Z(x, x')$ holds, then $L(x) \leq L'(x')$;
2. if $Z(x, x')$ holds and $E(x, r, y) > 0$, then there exists $y' \in V'$ such that $Z(y, y')$ holds and $E(x, r, y) \leq E(x', r, y')$.

A binary relation $Z \subseteq V \times V'$ is called a (crisp) directed simulation between $G$ and $G'$ if Conditions (1) and (2) and the following condition hold (for all possible values of the free variables):

$$Z(x, x') \land E(x', r, y') > 0 \rightarrow \exists y \in V (Z(y, y') \land E(x', r, y) \leq E(x, r, y)).$$

**Remark 2.1**

1. We allow the empty binary relation to be a simulation (respectively, directed simulation) between $G$ and $G'$. This is for increasing simplicity in talking about the largest simulation (respectively, directed simulation) between $G$ and $G'$.
2. Some works in the literature on simulations use the condition $Z(x, x') \rightarrow L(x) = L'(x')$ instead of Condition (1). Our choice of using (1) does not reduce the generality, because one can choose $\Sigma_0, L_2$ and $L'_2$ so that $L(x) = L'(x')$ over $\Sigma_0$ if $L_2(x) \leq L'_2(x')$ over $\Sigma_0$ for all $(x, x') \in V \times V'$.
3. The definition of (crisp) directed simulation differs from the definition of (crisp) bisimulation in that it uses Condition (1) instead of the condition $Z(x, x') \rightarrow L(x) = L'(x')$.

**Proposition 2.2** Let $G$, $G'$ and $G''$ be fuzzy graphs and let $G = \langle V, E, L \rangle$.

1. The relation $Z = \{ (x, x) \mid x \in V \}$ is a simulation (respectively, directed simulation) between $G$ and itself.
2. If $Z_1$ is a simulation (respectively, directed simulation) between $G$ and $G'$, and $Z_2$ is a simulation (respectively, directed simulation) between $G'$ and $G''$, then $Z_1 \circ Z_2$ is a simulation (respectively, directed simulation) between $G$ and $G''$.
3. If $Z$ is a set of simulations (respectively, directed simulations) between $G$ and $G'$, then $\bigcup Z$ is also a simulation (respectively, directed simulation) between $G$ and $G'$.
4. Every directed simulation between $G$ and $G'$ is also a simulation between $G$ and $G'$.

The proof of this proposition is straightforward.

The following corollary follows from the third assertion of Proposition 2.2.

**Corollary 2.3** The largest simulation (respectively, directed simulation) between arbitrary fuzzy graphs exists.

A (crisp) auto-simulation (respectively, directed auto-simulation) of $G$ is a simulation (respectively, directed simulation) between $G$ and itself. The following corollary immediately follows from Proposition 2.2.

**Corollary 2.4** The largest auto-simulation (respectively, directed auto-simulation) of a fuzzy graph is a pre-order.

**Example 2.5** Let $\Sigma_v = \{ p \}$, $\Sigma_e = \{ r \}$ and let $G$ and $G'$ be the fuzzy graphs specified and illustrated below:

- $G = \langle V, E, L \rangle$, where $V = \{ a, b, c, d \}$ and $L(a)(p) = 0.8$, $L(b)(p) = 0.8$, $L(c)(p) = 0.7$, $L(d)(p) = 0.9$, $E(a, r, b) = 0.6$, $E(b, r, c) = 0.5$, $E(c, r, d) = 0.4$, $E(d, r, a) = 0.3$.

- $G' = \langle V', E', L' \rangle$, where $V' = \{ a', b', c', d' \}$ and $L'(a')(p) = 0.8$, $L'(b')(p) = 0.8$, $L'(c')(p) = 0.7$, $L'(d')(p) = 0.9$, $E'(a', r', b') = 0.6$, $E'(b', r', c') = 0.5$, $E'(c', r', d') = 0.4$, $E'(d', r', a') = 0.3$.

- $Z = \{ (x, x) \mid x \in V \}$ is a simulation (respectively, directed simulation) between $G$ and $G'$.
Therefore, $E(a, r, b) = 0.7$, $E(b, r, c) = 0.6$, $E(b, r, d) = 0.7$, $E(c, r, d) = 0.5$, $E(d, r, b) = 0.6$, $E(x, r, y) = 0$ for the other triples $\langle x, y \rangle$ with $x, y \in V$;

- $G' = \langle V', E', L' \rangle$, where $V' = \{e, f\}$ and
- $L'(e)(p) = 0.8$, $L'(f)(p) = 0.9$,
- $E'(e, r, e) = 0.6$, $E'(e, r, f) = 0.7$, $E'(f, r, e) = 0.6$,
- $E'(x, r, y) = 0$ for the other triples $\langle x, y \rangle$ with $x, y \in V'$.

It can be checked that $Z_0 = \{\langle b, e \rangle, \langle c, e \rangle, \langle d, f \rangle\}$ is a simulation between $G$ and $G'$. Let $Z$ be the largest simulation between $G$ and $G'$. We show that $Z = Z_0$ by justifying that

$$\{\langle a, e \rangle, \langle a, f \rangle, \langle b, f \rangle, \langle c, f \rangle, \langle d, e \rangle\} \cap Z = \emptyset.$$ 

Observe the following.

- $\langle d, e \rangle \notin Z$ because $L(d) \not\leq L'(e)$.
- $\langle c, f \rangle \notin Z$ because (2) cannot hold for $\langle x, x', y \rangle = \langle c, f, d \rangle$ (since $\langle d, e \rangle \notin Z$).
- $\langle b, f \rangle \notin Z$ because (2) cannot hold for $\langle x, x', y \rangle = \langle b, f, d \rangle$.
- $\langle a, e \rangle \notin Z$ because (2) cannot hold for $\langle x, x', y \rangle = \langle a, e, b \rangle$ (since $\langle b, f \rangle \notin Z$).
- $\langle a, f \rangle \notin Z$ because (2) cannot hold for $\langle x, x', y \rangle = \langle a, f, b \rangle$.

Therefore, $\{\langle b, e \rangle, \langle c, e \rangle, \langle d, f \rangle\}$ is the largest simulation between $G$ and $G'$.

**Example 2.6** Let $\Sigma_v$, $\Sigma_e$, $G$ and $G'$ be as in Example 2.5. Now, let $Z$ be the largest directed simulation between $G$ and $G'$. We show that $Z = \emptyset$. Since every directed simulation between $G$ and $G'$ is also a simulation between $G$ and $G'$, by the claim of Example 2.5 $Z \subseteq \{\langle b, e \rangle, \langle c, e \rangle, \langle d, f \rangle\}$. Hence, it suffices to show that

$$\{\langle b, e \rangle, \langle c, e \rangle, \langle d, f \rangle\} \cap Z = \emptyset.$$ 

Observe the following.

- $\langle c, e \rangle \notin Z$ because (3) cannot hold for $\langle x, x', y \rangle = \langle c, e, c \rangle$ (since $\langle d, e \rangle \notin Z$).
- $\langle b, e \rangle \notin Z$ because (3) cannot hold for $\langle x, x', y \rangle = \langle b, e, e \rangle$ (since $\{\langle c, e \rangle, \langle d, e \rangle\} \cap Z = \emptyset$).
- $\langle d, f \rangle \notin Z$ because (3) cannot hold for $\langle x, x', y \rangle = \langle d, f, e \rangle$ (since $\langle b, e \rangle \notin Z$).

Therefore, $\emptyset$ is the largest directed simulation between $G$ and $G'$.
Example 2.7 Let $\Sigma_v = \{p\}$ and $\Sigma_e = \{r\}$. Let $G_2$ and $G'_2$ be the fuzzy graphs illustrated below and specified in a similar way as done for $G$ and $G'$ in Example 2.5.

\[ G_2 \]

\[ G'_2 \]

It is straightforward to show that $Z = \{b, c, d\} \times \{e, f\}$ is the largest simulation (respectively, directed simulation) between $G_2$ and $G'_2$. ■

3 Computing Simulations

In this section, we design an algorithm for computing the largest simulation between two given finite fuzzy graphs $G = (V, E, L)$ and $G' = (V', E', L')$.

For $r \in \Sigma_e$, $x, y \in V$ and $x', y' \in V'$, we denote

\[ \text{Next}_r(x) = \{ y \in V \mid E(x, r, y) > 0 \} \quad \text{Prev}_r(y) = \{ x \in V \mid E(x, r, y) > 0 \} \]

\[ \text{Next}'_r(x') = \{ y' \in V' \mid E'(x', r, y') > 0 \} \quad \text{Prev}'_r(y') = \{ x' \in V' \mid E'(x', r, y') > 0 \} \]

3.1 The Skeleton of the Algorithm

We first formulate our algorithm on an abstract level without implementation details. The aim is to facilitate understanding the skeleton of the algorithm and its correctness. Implementation details and complexity analysis will be presented in the next subsection.

It can be checked that Condition (2) holds for all $r \in \Sigma_e$, $x, y \in V$ and $x' \in V'$ iff the following condition holds for all $r \in \Sigma_e$, $y \in V$ and $x' \in V'$, where the suprema are taken in the complete lattice $[0,1]$.

\[ \sup\{E(x, r, y) \mid x \in \text{Prev}_r(y) \land (x, x') \in Z\} \leq \sup\{E'(x', r, y') \mid y' \in \text{Next}'_r(x') \land (y, y') \in Z\} \]  (4)

Figure 1 is helpful for illustrating Condition (4). It is worth emphasizing that this condition is involved with $\langle r, y, x' \rangle$. Our algorithm constructs the largest relation $Z \subseteq V \times V'$ that satisfies Conditions (4) and (1) for all possible values of the free variables. It can be informally expressed as follows:

1. Initialize $Z$ by setting it to the largest subset of $V \times V'$ that satisfies Condition (1).

2. Initialize $\text{removed}_Z$ by setting it to the empty set. This variable stands for the set of pairs removed from $Z$ and remained to be processed. So, $Z \cup \text{removed}_Z$ plays the role of a subset of a previous value of $Z$.

3. For each $\langle r, y, x' \rangle \in \Sigma_e \times V \times V'$, move all the pairs $\langle x, x' \rangle \in Z$ that falsify Condition (4) to the set $\text{removed}_Z$. This can be restated as follows: for each $\langle r, y, x' \rangle \in \Sigma_e \times V \times V'$ and $x \in \text{Prev}_r(y)$, if $\langle x, x' \rangle \in Z$ but the following condition does not hold, then move $\langle x, x' \rangle$ from $Z$ to $\text{removed}_Z$.

\[ E(x, r, y) \leq \sup\{E'(x', r, y') \mid y' \in \text{Next}'_r(x') \land (y, y') \in Z\} \]  (5)
4. While $\text{removed}_{Z_{tp}} \neq \emptyset$:
   - extract $\langle y, y' \rangle$ from $\text{removed}_{Z_{tp}}$;
   - for each $r \in \Sigma_e$, $x' \in \text{Prev}_{r}^{'}(y')$ and $x \in \text{Prev}_{r}(y)$, if $\langle x, x' \rangle \in Z$ but the following condition does not hold, then move $\langle x, x' \rangle$ from $Z$ to $\text{removed}_{Z_{tp}}$.

$$E(x, r, y) \leq \sup\{E'(x', r, z') \mid z' \in \text{Next}_{r}^{'}(x') \land \langle y, z' \rangle \in Z \cup \text{removed}_{Z_{tp}}\} \quad (6)$$

The steps 1–3 together form the initialization, whereas the step 4 is the main loop of the algorithm. The pseudocode is formulated as Algorithm 1 (on page 7).

![Diagram of the algorithm](image)

**Figure 1:** An illustration for the algorithms given in Section 3.

**Example 3.1** Let $\Sigma_v = \{p\}$, $\Sigma_e = \{r\}$ and let $G$ and $G'$ be the fuzzy graphs specified in Example 2.5. Consider the execution of Algorithm 1 for $G$ and $G'$.

- After executing the statement 2, we have that $Z = \{(a, b, c) \times \{e, f\}\} \cup \{(d, f)\}$ and $\text{removed}_{Z_{tp}} = \emptyset$.

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**Algorithm 1:** ComputeSimulation

**Input:** finite fuzzy graphs $G = (V, E, L)$ and $G' = (V', E', L')$.

**Output:** the largest simulation between $G$ and $G'$.

1. $Z := \{(x, x') \in V \times V' \mid L(x) \leq L'(x')\}$;
2. $\text{removed}_{Z_{tp}} := \emptyset$;
3. foreach $\langle r, y, x' \rangle \in \Sigma_e \times V \times V'$ do
   4. ProcessPrev$(r, y, x')$;
4. while $\text{removed}_{Z_{tp}} \neq \emptyset$ do
   5. extract $\langle y, y' \rangle$ from $\text{removed}_{Z_{tp}}$;
   6. foreach $r \in \Sigma_e$ and $x' \in \text{Prev}_{r}^{'}(y')$ do
      7. ProcessPrev$(r, y, x')$;
8. return $Z$;

---

**Procedure** ProcessPrev$(r, y, x')$ // process $\text{Prev}_{r}(y)$ with respect to $\langle r, y, x' \rangle$

1. $d := \sup\{E'(x', r, y') \mid y' \in \text{Next}_{r}^{'}(x') \land \langle y, y' \rangle \in Z \cup \text{removed}_{Z_{tp}}\}$;
2. foreach $x \in \text{Prev}_{r}(y)$ do
3. if $E(x, r, y) > d$ and $\langle x, x' \rangle \in Z$ then
   4. move $\langle x, x' \rangle$ from $Z$ to $\text{removed}_{Z_{tp}}$. 

---

Example 3.1 Let $\Sigma_v = \{p\}$, $\Sigma_e = \{r\}$ and let $G$ and $G'$ be the fuzzy graphs specified in Example 2.5. Consider the execution of Algorithm 1 for $G$ and $G'$.
• Executing the “foreach” loop specified by the statements 3 and 4 for the iteration involved with \( \langle r, y, x' \rangle \),
  - if \( \langle r, y, x' \rangle \in \{ \langle r, a, e \rangle, \langle r, a, f \rangle, \langle r, b, e \rangle \} \), then no changes are made,
  - if \( \langle r, y, x' \rangle = \langle r, b, f \rangle \), then \( \langle a, f \rangle \) is moved from \( Z \) to \( \text{removed}_{Z,tp} \),
  - if \( \langle r, y, x' \rangle \in \{ \langle r, c, e \rangle, \langle r, c, f \rangle, \langle r, d, e \rangle \} \), then no changes are made,
  - if \( \langle r, y, x' \rangle = \langle r, d, f \rangle \), then \( \langle b, f \rangle \) and \( \langle c, f \rangle \) are moved from \( Z \) to \( \text{removed}_{Z,tp} \).

• Thus, before executing the statement 5, we have that
  \[
  Z = (\{a, b, c\} \times \{e\}) \cup \{(d, f)\}
  \]
  \[
  \text{removed}_{Z,tp} = \{(a, f), (b, f), (c, f)\}.
  \]

• Executing the “while” loop, for the iteration involved with \( \langle y, y' \rangle \) extracted from \( \text{removed}_{Z,tp} \),
  - if \( \langle y, y' \rangle = \langle a, f \rangle \), then no additional changes are made,
  - if \( \langle y, y' \rangle = \langle b, f \rangle \), then \( \langle a, e \rangle \) is moved from \( Z \) to \( \text{removed}_{Z,tp} \),
  - if \( \langle y, y' \rangle \in \{ \langle c, f \rangle, \langle a, e \rangle \} \), then no additional changes are made.

• The algorithm returns \( Z = \{(b, e), (c, e), (d, f)\} \). This coincides with the claim of Example 2.5 that this relation is the largest simulation between \( G \) and \( G' \).

\[
\text{Lemma 3.2} \quad \text{The following assertions are invariants of the “while” loop of Algorithm 1:}
\]

1. The largest simulation between \( G \) and \( G' \) is a subset of \( Z \).
2. For all \( \langle r, y, x' \rangle \in \Sigma_e \times V \times V' \), if \( x \in \text{Prev}_r(y) \) and \( Z[x, x'] \) holds, then (6) holds.

\[
\text{Proof.} \quad \text{Let} \ Z_0 \text{ be the largest simulation between} \ G \text{ and} \ G'. \text{ It must satisfy Condition (4), i.e.,}
\]
\[
\sup\{E(x, r, y) \mid x \in \text{Prev}_r(y) \land (x, x') \in Z_0\} \leq \sup\{E'(x', r, y') \mid y' \in \text{Next}^r(y') \land (y, y') \in Z_0\}. \quad (7)
\]

The first assertion is an invariant because it holds after the initialization (i.e., after executing the statement 1) and the removal of any pair from \( Z \) at a later step is justifiable by using (7) and the induction assumption (which means that after such a removal it still holds that \( Z_0 \subseteq Z \)). It is straightforward to check that the second assertion is also an invariant of the “while” loop of Algorithm 1.

\[
\text{Theorem 3.3} \quad \text{Algorithm 1 always terminates and returns the largest simulation between the input fuzzy graphs.}
\]

\[
\text{Proof.} \quad \text{After the initialization at the statement 1, no pairs are added to} \ Z. \text{ The pairs added to} \ \text{removed}_{Z,tp} \text{ are the ones extracted from} \ Z. \text{ Hence, the total number of pairs added to} \ \text{removed}_{Z,tp} \text{ is bounded. Each iteration of the “while” loop of Algorithm 1 extracts one pair from} \ \text{removed}_{Z,tp}. \text{ Hence, the loop and the algorithm itself always terminate. At the end, we have that} \ \text{removed}_{Z,tp} = \emptyset. \text{ Hence, by the second assertion of Lemma 3.2, the resulting} \ Z \text{ satisfies Condition (4) (for all possible values of the free variables). Due to the initialization, it also satisfies Condition (1). Thus, it is a simulation between} \ G \text{ and} \ G'. \text{ By the first assertion of Lemma 3.2 it follows that the returned} \ Z \text{ is the largest simulation between} \ G \text{ and} \ G'.
\]
3.2 Implementation Details and Complexity Analysis

By $|E|$ we denote $|\{(x, y) \in V \times V : E(x, y) > 0\}|$, and similarly for $|E'|$. Let $n = |V| + |E'|$ and $m = |E| + |E'|$. Assume that $|\Sigma_e|$ and $|\Sigma_e|$ are constants. In this subsection, we give details on how to implement Algorithm 1 so that its complexity is of order $O((m + n)n)$. In particular, we provide an improved algorithm together with a complexity analysis.

For simplicity, without loss of generality we assume that $V = 0..(|V| - 1)$, $V' = 0..(|V' - 1)$ and $\Sigma_e = 0..(\Sigma_e - 1)$. In practice, the input data for $G$ and $G'$ can be given in a friendly format using names for vertices and edge labels, but the conversion (using maps) for the input as well as for the output can be done in time $O((m + n)\log n)$, which is within $O((m + n)n)$.

From the input data we construct arrays $\text{Next'}$, $\text{Prev'}$, $E$ and $E'$ such that, for $r \in \Sigma_e$, $x, y \in V$ and $x', y' \in V'$, $\text{Next}'[r, x]$ is a vector representing $\text{Next}'[x', x, r, y]$, $E[r, x, y] = E(x, y, r)$, and similarly for $\text{Prev}[r, y]$, $\text{Prev}'[r, y']$ and $E'[x', r, y']$. The construction can be done in time $O(n^2)$.

We use an array $\text{EdgeID}' : V' \times \Sigma_e \times V' \rightarrow 0..(|E' - 1)$ for identifying nonzero edges of $G'$ so that, if $\langle x'_1, r_1, y'_1 \rangle$ and $\langle x'_2, r_2, y'_2 \rangle$ are different triples from $V' \times \Sigma_e \times V'$ such that $E'[x'_1, r_1, y'_1] > 0$ and $E'[x'_2, r_2, y'_2] > 0$, then $\text{EdgeID}'[x'_1, r_1, y'_1] \neq \text{EdgeID}'[x'_2, r_2, y'_2]$. Such an array can be constructed in time $O(n^2 + m \log m)$. After having been constructed, $\text{EdgeID}'$ will never change.

For the improved algorithm, we implement $Z$ as an array of type $V \times V' \rightarrow \text{bool}$ and $\text{removed}_Z_{tp}$ as a queue. We also use the variables $\text{rcNext}'$, $\text{rcNextElem}'$ and $\text{rcPrev}$ for representing arrays described below.

- $\text{rcNext}'$ is an array such that, for $\langle r, x, y \rangle \in \Sigma_e \times V \times V'$, $\text{rcNext}'[r, x, y]$ is a doubly linked list consisting of the vertices $y' \in \text{Next}'[x', x, r, y]$ such that $Z[y, y'] \lor (y, y') \in \text{removed}_Z_{tp}$ holds. The list is sorted in ascending order with respect to $E'[x', r, y]$. The prefix “rc” stands for “remaining for consideration”. The vertex contained in an element of $\text{rcNext}'[r, x, y]$ is called the key of that element.
- $\text{rcNextElem}'$ is an array with indices from $V \times (0..(|E' - 1))$ such that, if $Z[y, y'] \lor (y, y') \in \text{removed}_Z_{tp}$, $r \in \Sigma_e$, $x' \in \text{Prev}'[y']$ and $e' = \text{EdgeID}'[x', r, y']$, then $\text{rcNextElem}'[y, e']$ is (a reference to the element of the doubly linked list $\text{rcNext}'[r, x, y]$ whose key is $y'$.
- $\text{rcPrev}$ is an array such that, for $\langle r, x, y \rangle \in \Sigma_e \times V \times V'$, $\text{rcPrev}[r, y, x']$ is a vector consisting of the vertices $x \in \text{Prev}_r(y)$ such that the following condition (which is a reformulation of (3)) holds

$$E[x, r, y] \leq \sup\{E'[x', r, y'] \mid y' \in \text{Next}'[x', x, r, y] \land (Z[y, y'] \lor (y, y') \in \text{removed}_Z_{tp})\}$$  (8)

and the vector is sorted in ascending order with respect to $E[x, r, y]$.

Algorithm 2 (on page 10) is a reformulation of Algorithm 1 using the above described data structures. It is given together with its subroutines $\text{UpdateRecPrev}$ (for updating $\text{rcPrev}$) and $\text{Initialize}$ on page 10. The procedure $\text{UpdateRecPrev}$ corresponds to the procedure $\text{ProcessPrev}$ used in Algorithm 1. Roughly speaking, the reformulation relies on the following two points.

- Computing $d$ specified by the statement 1 of the procedure $\text{ProcessPrev}(r, x, y')$ used in Algorithm 1 is implemented in Algorithm 2 by the statements 14 of the procedure $\text{UpdateRecPrev}(r, y, x')$ so that its cost is bounded by a constant. It is done by using the sorted list $\text{rcNext}'[r, x, y']$. This list is doubly linked so that deleting any element from the list can be done in constant time. Such a deletion is triggered when a pair $(y, y')$ is extracted from $\text{removed}_Z_{tp}$ by the statement 2 of Algorithm 2. In such a situation, to guarantee that $\text{rcNext}'$ satisfies its specification, we need to delete the element $u$ whose key is $y'$ from the list $\text{rcNext}'[r, y, x']$. A reference to $u$ is kept in $\text{rcNextElem}'[y, e']$, where $e' = \text{EdgeID}'[x', r, y']$. The deletion is done by the statements 35 of Algorithm 2. This shows the key role of the arrays $\text{rcNext}'$, $\text{rcNextElem}'$ and $\text{EdgeID}'$. 

9
**Procedure** UpdateRcPrev(r, y, x') // for updating rcPrev

1. if rcNext'[r, y, x'] is empty then d := 0;
2. else
3. let y' be the key of the last element of rcNext'[r, y, x'];
4. d := E'[x', r, y'];
5. while rcPrev[r, y, x'] is not empty do
6. let x be the last element of rcPrev[r, y, x'];
7. if E[x, r, y] ≤ d then break;
8. delete from rcPrev[r, y, x'] the last element;
9. if Z[x, x'] then
10. Z[x, x'] := false;
11. add ⟨x, x'⟩ to removed_Ztp;

**Procedure** Initialize

1. construct arrays Next', Prev', Prev', E', and EdgeID' according to their specifications;
2. foreach (x, x') ∈ V × V' do Z[x, x'] := L(x) ≤ L'(x');
3. set removed_Ztp to the empty queue;
4. foreach (r, x') ∈ Σ_e × V' do
5. sort Next'[r, x'] in ascending order w.r.t. E'[x', r, y'] for y' ∈ Next'[r, x'];
6. foreach (r, y) ∈ Σ_e × V do
7. sort Prev[r, y] in ascending order w.r.t. E[x, r, y] for x ∈ Prev[r, y];
8. foreach y ∈ V and e' ∈ 0..(|E' | − 1) do rcNextElem'[y, e'] := null;
9. foreach (r, y, x') ∈ Σ_e × V × V' do
10. construct a vector rcPrev[r, y, x'] of all x ∈ Prev[r, y] such that Z[x, x'] holds, preserving the order;
11. construct a doubly linked list rcNext'[r, y, x'] whose elements’ keys are all
12. y' ∈ Next'[r, x'] such that Z[y, y'] holds, preserving the order;
13. foreach element u of rcNext'[r, y, x'] do
14. let y' be the key of u and let e' = EdgeID'[x', r, y'];
15. rcNextElem'[y, e'] := u;
16. UpdateRcPrev(r, y, x');

**Algorithm 2: ComputeSimulationEfficiently**

**Input**: finite fuzzy graphs G = ⟨V, E, L⟩ and G' = ⟨V', E', L'⟩.

**Output**: the largest simulation between G and G'.

1. Initialize();
2. while removed_Ztp is not empty do
3. extract ⟨y, y'⟩ from removed_Ztp;
4. foreach r ∈ Σ_e and x' ∈ Prev'[r, y'] do
5. e' := EdgeID'[x', r, y'];
6. u := rcNextElem'[y, e'];
7. delete u from rcNext'[r, y, x'];
8. rcNextElem'[y, e'] := null;
9. UpdateRcPrev(r, y, x');
10. return the relation corresponding to Z;
Lemma 3.4 The following assertions are invariants of the “while” loop of Algorithm $2$

1. The data structures rcNext, rcNextElem and rcPrev satisfy their specifications.

2. The largest simulation between $G$ and $G'$ is a subset of $\{\langle x, x' \rangle \in V \times V' \mid Z[x, x']\}$.

3. For all $\langle r, y, x' \rangle \in \Sigma \times V \times V'$, if $x \in \text{Prev}_r(y)$ and $Z[x, x']$ holds, then $x \in \text{rcPrev}[r, y, x']$.

Proof. It is easy to check that the first assertion is an invariant of the “while” loop of Algorithm $2$.

Consider the second assertion and let $Z_0$ be the largest simulation between $G$ and $G'$.

Recall that $Z_0$ satisfies (7).

We prove that $Z_0 \subseteq \{\langle x, x' \rangle \in V \times V' \mid Z[x, x']\}$ by induction on the step number during the execution of Algorithm $2$.

The base case occurs after the initialization of $Z$ by the statement 2 of the procedure Initialize.

The induction hypothesis clearly holds for the base case.

For the induction step, assume that the induction hypothesis holds before calling the procedure UpdateRcPrev($r, y, x'$).

We only need to prove that it still holds after executing that procedure.

Let $d$ be the value computed by the statements 1-4 of UpdateRcPrev($r, y, x'$).

By the first invariant, we have that

$$d = \sup \{E'(x', r, y') \mid y' \in \text{Next}'(x') \land (Z[y, y'] \lor \langle r, y', y' \rangle \in \text{removed}_Z\}.$$

By the induction assumption, $Z_0 \subseteq \{\langle x, x' \rangle \in V \times V' \mid Z[x, x']\}$. Hence,

$$d \geq \sup \{E'(x', r, y') \mid y' \in \text{Next}'(x') \land \langle r, y', y' \rangle \in Z_0\}.$$

By (7), it follows that

$$d \geq \sup \{E(x, r, y) \mid x \in \text{Prev}_r(y) \land \langle x, x' \rangle \in Z_0\}.$$

Therefore, after executing the statements of UpdateRcPrev($r, y, x'$), it still holds that $Z_0 \subseteq \{\langle x, x' \rangle \in V \times V' \mid Z[x, x']\}$. This completes the proof that the second assertion is an invariant of the “while” loop of Algorithm $2$.

The third assertion is also an invariant of the loop because it holds after executing the statements of Initialize and, whenever $x$ is deleted from $\text{rcPrev}[r, y, x']$, $Z[x, x']$ is set to false.

We give below the main theorem of this section.

Theorem 3.5 Algorithm $2$ runs in time $O((m + n)n)$ and returns the largest simulation between given finite fuzzy graphs $G = \langle V, E, L \rangle$ and $G' = \langle V', E', L' \rangle$, where $n = |V| + |V'|$ and $m = |E| + |E'|$.

Proof. First, consider the complexity of the procedure Initialize.

- As mentioned earlier, constructing the arrays Next', Prev, Previous', E and $E'$ can be done in time $O(n^2)$, whereas constructing the array EdgeID' can be done in time $O(n^2 + m \log m)$.

- Hence, the statement 1 (of the procedure) runs in time $O(n^2 + m \log m)$.

- The statement 2 runs in time $O(n^2)$. The statement 8 runs in constant time.
• The loops specified by the statements 4-7 run in time $O(n + m \log m)$.

• The loop specified by the statement 8 runs in time $O(nm)$.

• The loop specified by the statements 9-14 runs in time $O(n(n + m))$.

• The loop specified by the statements 15 and 16 runs in time $O(n(n + m))$.

Summing up, the procedure Initialize runs in time $O((m + n)n)$.

Now consider the complexity of the “while” loop of Algorithm 2.

• The statements 3 and 5-8 of Algorithm 2 as well as the statements 1-4 and 6-11 of the procedure UpdateRcPrev$(r, y, x')$ run in constant time.

• Each iteration of the “foreach” loop of Algorithm 2 is involved with a pair $\langle y, y' \rangle$ extracted from removed$_tZ_{tp}$ and a vertex $x' \in \text{Prev}_r(y')$.

• Each iteration of the “while” loop of the procedure UpdateRcPrev$(r, y, x')$ is involved with the triple $\langle r, x, y \rangle \in \Sigma_e \times V \times V'$ and a vertex $x \in \text{Prev}_r(y)$. If $x$ is deleted from rcPrev$(r, y, x')$, then we ascribe the cost of the involved iteration to the edge $\langle x, r, y \rangle$ and $x' \in \text{Prev}_r(y')$ which together identify the iteration of the “foreach” loop of Algorithm 2 that calls UpdateRcPrev$(r, y, x')$.

Thus, the “while” loop of Algorithm 2 runs in time $O((m + n)n)$.

The “return” statement of Algorithm 2 runs in time $O(n^2)$.

Summing up, Algorithm 2 runs in time $O((m + n)n)$.

At the end of the execution of Algorithm 2, removed$_tZ_{tp}$ is empty. By the first and third assertions of Lemma 3.4 (see, among others, (8)), it follows that $Z$ satisfies Condition (4). By the initialization, $Z$ also satisfies Condition (1). Hence, $Z$ is a simulation between $G$ and $G'$. Together with the second assertion of Lemma 3.4, this implies that the relation returned by Algorithm 2 is the largest simulation between $G$ and $G'$.

4 Computing Directed Simulations

In this section, we extend and adapt the algorithms given in the previous section to obtain algorithms for computing the largest directed simulation between two given finite fuzzy graphs $G = \langle V, E, L \rangle$ and $G' = \langle V', E', L' \rangle$. We use all notions and data structures introduced in the previous section.

It can be checked that Condition (3) holds for all $r \in \Sigma_e, x \in V$ and $x', y' \in V'$ iff the following condition holds for all $\langle r, x, y \rangle \in \Sigma_e \times V \times V'$.

$$\sup\{E(x', r, y') \mid x' \in \text{Prev}_r(y') \land \langle x, x' \rangle \in Z\} \leq \sup\{E(x, r, y) \mid y \in \text{Next}_r(x) \land \langle y, y' \rangle \in Z\}. \quad (9)$$

In some sense, Conditions (3) and (9) are dual to Conditions (2) and (4), respectively. Algorithm 3 (ComputeDirectedSimulation) given on page 13 is our adapted extension of Algorithm 1 for computing the largest directed simulation between $G$ and $G'$. It takes into account the mentioned duality. In particular,

• the procedure ProcessPrev$(r, x, y')$ is dual to the procedure ProcessPrev$(r, y, x')$,

• the statements 11 and 710 of Algorithm 3 are the same as the statements 114 and 558 of Algorithm 1, respectively,

• the loops specified by the statements 114 and 710 of Algorithm 3 are dual to the loops specified by its statements 314 and 910, respectively.
Example 4.1 Let $\Sigma_v = \{p\}$, $\Sigma_e = \{r\}$ and let $G$ and $G'$ be the fuzzy graphs specified in Example 2.5. Consider the execution of Algorithm 3 for $G$ and $G'$.

- As stated in Example 3.1 for Algorithm 1, before executing the statement $5$, we have that
  \[
  Z = \{(a, b, c) \times \{e\}\} \cup \{(d, f)\}
  \]
  \[
  removed_Z \uparrow P = \{(a, f), (b, f), (c, f)\}.
  \]

- Executing the “foreach” loop specified by the statements $5$ and $6$, for the iteration involved with $\langle r, x', y' \rangle$,
  - if $\langle r, x, y' \rangle \in \{(r, a, e), (r, a, f), (r, b, e), (r, b, f)\}$, then no changes are made,
  - if $\langle r, x, y' \rangle = \langle r, c, e \rangle$, then $\langle c, e \rangle$ is moved from $Z$ to $removed_Z \uparrow P$,
  - if $\langle r, x, y' \rangle \in \{(r, c, f), (r, d, e), (r, d, f)\}$, then no changes are made.

- Thus, before executing the statement $7$, we have that
  \[
  Z = \{(a, e), (b, e), (d, f)\}
  \]
  \[
  removed_Z \uparrow P = \{(a, f), (b, f), (c, f), (c, e)\}.
  \]

- Executing the “while” loop, for the iteration involved with $\langle y, y' \rangle$ extracted from $removed_Z \uparrow P$,
  - if $\langle y, y' \rangle = \langle a, f \rangle$, then no additional changes are made,
  - if $\langle y, y' \rangle = \langle b, f \rangle$, then $\langle a, e \rangle$ is moved from $Z$ to $removed_Z \uparrow P$,
  - if $\langle y, y' \rangle = \langle c, f \rangle$, then no additional changes are made,
  - if $\langle y, y' \rangle = \langle c, e \rangle$, then $\langle b, e \rangle$ is moved from $Z$ to $removed_Z \uparrow P$. 


### Algorithm 3: ComputeDirectedSimulation

**Input**: finite fuzzy graphs $G = (V, E, L)$ and $G' = (V', E', L')$.

**Output**: the largest directed simulation between $G$ and $G'$.

```plaintext
1 $Z := \{(x, x') \in V \times V' \mid L(x) \leq L'(x')\};$
2 $removed_Z \uparrow P := \emptyset;$
3 foreach $\langle r, y, x' \rangle \in \Sigma_e \times V \times V'$ do
   ProcessPrev$(r, y, x');$
4 foreach $\langle r, x, y' \rangle \in \Sigma_e \times V \times V'$ do
   ProcessPrev$(r, x, y');$
5 while $removed_Z \uparrow P \neq \emptyset$ do
   extract $\langle y, y' \rangle$ from $removed_Z \uparrow P;$
   foreach $r \in \Sigma_e$ and $x' \in prev_r(y')$ do
      ProcessPrev$(r, y, x');$
   foreach $r \in \Sigma_e$ and $x \in prev_r(y)$ do
      ProcessPrev$(r, x, y');$
6 return $Z;$
```


Lemma 4.2 The following assertions are invariants of the “while” loop of Algorithm 3:

1. If \( (y, y') = (a, c) \), then no additional changes are made.
2. If \( (y, y') = (b, c) \), then \( (d, f) \) is moved from \( Z \) to \( \text{removed}_Ztp \).
3. If \( (y, y') = (d, f) \), then no additional changes are made.

- The algorithm returns \( Z = \emptyset \). This coincides with the claim of Example 2.6 that \( \emptyset \) is the largest directed simulation between \( G \) and \( G' \).

The following lemma is a counterpart of Lemma 3.2. It can be proved analogously.

Lemma 4.2 The following assertions are invariants of the “while” loop of Algorithm 3:

1. The largest directed simulation between \( G \) and \( G' \) is a subset of \( Z \).
2. For all \( (r, y, x') \in \Sigma_e \times V \times V' \), if \( x \in \text{Prev}_r(y) \) and \( Z[x, x'] \) holds, then \( (6) \) holds.
3. For all \( (r, x, y') \in \Sigma_e \times V \times V' \), if \( x' \in \text{Prev}_r'(y') \) and \( Z[x, x'] \) holds, then the following counterpart of \( (6) \) holds.

\[
E'(x', r, y') \leq \sup\{E(x, r, y) \mid y \in \text{Next}_r(x) \land (y, y') \in Z \cup \text{removed}_Ztp\}
\] (10)

The following theorem is a counterpart of Theorem 3.3

Theorem 4.3 Algorithm 3 always terminates and returns the largest directed simulation between the input fuzzy graphs.

This theorem can be proved analogously as done for Theorem 3.3. In particular, in addition to the second assertion of Lemma 4.2 and Condition 4, the proof also exploits the third assertion of Lemma 4.2 and Condition 9.

Like its counterpart, Algorithm 3 has been formulated on an abstract level without implementation details in order to increase simplicity and facilitate understanding. We now refine this algorithm by giving implementation details so that the resulting algorithm has a complexity of order \( O((m + n)n) \). In short, the new algorithm is an adapted extension of Algorithm 2.

Apart from the data structures described in Section 3.2, we also use arrays \( \text{Next}, \text{EdgeID}, \text{rcNext}, \text{rcNextElem} \) and \( \text{rcPrev}' \), which are dual to \( \text{Next}' \), \( \text{EdgeID}' \), \( \text{rcNext}' \), \( \text{rcNextElem}' \) and \( \text{rcPrev} \), respectively. For clarity, they are explicitly specified below.

- For \( r \in \Sigma_e \) and \( x \in V \), \( \text{Next}[r, x] \) is a vector representing \( \text{Next}_r(x) \).
- \( \text{EdgeID} : V \times \Sigma_e \times V \rightarrow (\mathbb{0},(|E| - 1)) \) is an array used for identifying nonzero edges of \( G \) with the following properties: if \( (x_1, r_1, y_1) \) and \( (x_2, r_2, y_2) \) are different triples from \( V \times \Sigma_e \times V \) such that \( E[x_1, r_1, y_1] > 0 \) and \( E[x_2, r_2, y_2] > 0 \), then \( \text{EdgeID}[x_1, r_1, y_1] \neq \text{EdgeID}[x_2, r_2, y_2] \).
- \( \text{rcNext} \) is an array such that, for \( (r, x, y') \in \Sigma_e \times V \times V' \), \( \text{rcNext}[r, x, y'] \) is a doubly linked list consisting of the vertices \( y \in \text{Next}_r(x) \) such that \( Z[y, y'] \cup ((y, y') \in \text{removed}_Ztp) \) holds. The list is sorted in ascending order with respect to \( E[x, r, y] \). The vertex contained in an element of \( \text{rcNext}[r, x, y'] \) is called the key of that element.
- \( \text{rcNextElem} \) is an array with indices from \( V' \times (\mathbb{0},(|E| - 1)) \) such that, if \( Z[y, y'] \cup ((y, y') \in \text{removed}_Ztp) \), \( r \in \Sigma_e \), \( x \in \text{Prev}_r(y) \) and \( e = \text{EdgeID}[x, r, y] \), then \( \text{rcNextElem} [y', e] \) is (a reference to) the element of the doubly linked list \( \text{rcNext}[r, x, y'] \) whose key is \( y \).
- \( \text{rcPrev}' \) is an array such that, for \( (r, x, y') \in \Sigma_e \times V \times V' \), \( \text{rcPrev}'[r, x, y'] \) is a vector consisting of the vertices \( x' \in \text{Prev}_r'(y') \) such that the following condition holds

\[
E'[x', r, y'] \leq \sup\{E[x, r, y] \mid y \in \text{Next}_r(x) \land (Z[y, y'] \cup ((y, y') \in \text{removed}_Ztp))\}
\] (11)

and the vector is sorted in ascending order with respect to \( E[x', r, y'] \).
procedure updateRcPrev'(r, x, y') // for updating rcPrev'
1 if rcNext[r, x, y'] is empty then d := 0;
2 else
3 let y be the key of the last element of rcNext[r, x, y'];
4 d := E[x, r, y];
5 while rcPrev'[r, x, y'] is not empty do
6 let x' be the last element of rcPrev'[r, x, y'];
7 if E'[x', r, y'] ≤ d then break;
8 delete from rcPrev'[r, x, y'] the last element;
9 if Z[x, x'] then
10 Z[x, x'] := false;
11 add ⟨x, x'⟩ to removed_Ztp;

procedure InitializeDS // a counterpart of Initialize for directed simulations
1 construct arrays Next, Next', Prev, Prev', E, E', EdgeID, EdgeID' according to their
2 specifications;
3 foreach ⟨x, x'⟩ ∈ V × V' do Z[x, x'] := L(x) ≤ L'(x');
4 set removed_Ztp to the empty queue;
5 foreach ⟨r, x⟩ ∈ Σe × V do
6 sort Next[r, x] in ascending order w.r.t. E[x, r, y] for y ∈ Next[r, x];
7 foreach ⟨r, x'⟩ ∈ Σe × V' do
8 sort Next'[r, x'] in ascending order w.r.t. E'[x', r, y'] for y' ∈ Next'[r, x'];
9 foreach ⟨r, y⟩ ∈ Σe × V do
10 sort Prev[r, y] in ascending order w.r.t. E[x, r, y] for x ∈ Prev[r, y];
11 foreach ⟨r, y'⟩ ∈ Σe × V' do
12 sort Prev'[r, y'] in ascending order w.r.t. E'[x', r, y'] for x' ∈ Prev'[r, y'];
13 foreach y ∈ V and e' ∈ 0..(|E'| − 1) do rcNextElem'[y, e'] := null;
14 foreach y' ∈ V' and e ∈ 0..(|E| − 1) do rcNextElem'[y', e] := null;
15 foreach ⟨r, y, x'⟩ ∈ Σe × V × V' do
16 construct a vector rcPrev[r, y, x'] of all x ∈ Prev[r, y] such that Z[x, x'] holds, preserving
17 the order;
18 construct a doubly linked list rcNext'[r, y, x'] whose elements' keys are all
19 y' ∈ Next'[r, x'] such that Z[y, y'] holds, preserving the order;
20 foreach element u of rcNext'[r, y, x'] do
21 let y' be the key of u and let e' = EdgeID'[x', r, y'];
22 rcNextElem'[y', e'] := u;
23 foreach ⟨r, x, y⟩ ∈ Σe × V × V' do
24 construct a vector rcPrev'[r, x, y] of all x' ∈ Prev'[r, y] such that Z[x, x'] holds, preserving
25 the order;
26 construct a doubly linked list rcNext[r, x, y'] whose elements' keys are all y ∈ Next[r, x]
27 such that Z[y, y'] holds, preserving the order;
28 foreach element u of rcNext[r, x, y'] do
29 let y be the key of u and let e = EdgeID[x, r, y];
30 rcNextElem[y', e] := u;
31 foreach ⟨r, y, x⟩ ∈ Σe × V × V' do UpdateRcPrev(r, y, x');
32 foreach ⟨r, x, y⟩ ∈ Σe × V × V' do UpdateRcPrev'(r, x, y');
Algorithm 4: ComputeDirectedSimulationEfficiently

Input: finite fuzzy graphs $G = (V, E, L)$ and $G' = (V', E', L')$.

Output: the largest directed simulation between $G$ and $G'$.

1. InitializeDS(): // defined on page 15
2. while removed.Z.tp is not empty do
3.     extract $(y, y')$ from removed.Z.tp;
4.     foreach $r \in \Sigma_e$ and $x' \in \text{Prev}'[r, y']$ do
5.         $e' := \text{EdgeID}'[x', r, y']$;
6.         $u := \text{rcNextElem}'[y, e']$;
7.         delete $u$ from $\text{rcNext}'[r, y, x']$;
8.         $\text{rcNextElem}'[y, e'] := \text{null}$;
9.         UpdateRcPrev(r, y, x'); // defined on page 10
10.    foreach $r \in \Sigma_e$ and $x \in \text{Prev}[r, y]$ do
11.        $e := \text{EdgeID}[x, r, y]$;
12.        $u := \text{rcNextElem}[y', e]$;
13.        delete $u$ from $\text{rcNext}[r, x, y]$;
14.        $\text{rcNextElem}[y', e] := \text{null}$;
15.        UpdateRcPrev(r, x, y'); // defined on page 15
16. return the relation corresponding to $Z$;

Algorithm 4 (ComputeDirectedSimulationEfficiently) given on page 16 is a reformulation of Algorithm 3 using the above mentioned data structures. It is an adapted extension of Algorithm 2 (ComputeSimulationEfficiently). In particular, observe the following.

- The procedure $\text{UpdateRcPrev}'(r, x, y')$ (on page 15) is dual to the procedure $\text{UpdateRcPrev}(r, y, x')$.
- The procedure InitializeDS (on page 15) is a counterpart of the procedure Initialize. It initializes not only $\text{EdgeID}'$, $\text{rcNext}'$, $\text{rcNextElem}'$ and $\text{rcPrev}$ but also $\text{EdgeID}$, $\text{rcNext}$, $\text{rcNextElem}$ and $\text{rcPrev}$ dually. The postfix “DS” relates to “directed simulation”.
- The statements 2-9 of Algorithm 4 are the same as the statements 2-9 of Algorithm 2.
- The statements 10-15 of Algorithm 4 are dual to the statements 8-13 of Algorithm 2.

We have implemented Algorithm 4 in C++ and shared the code [18] to allow experiments with it. The following lemma is a counterpart of Lemma 3.4. It can be proved analogously.

**Lemma 4.4** The following assertions are invariants of the “while” loop of Algorithm 4.

1. The data structures $\text{rcNext}$, $\text{rcNext}'$, $\text{rcNextElem}$, $\text{rcNextElem}'$, $\text{rcPrev}$ and $\text{rcPrev}'$ satisfy their specifications.
2. The largest direct simulation between $G$ and $G'$ is a subset of $\{ (x, x') \in V \times V' \mid Z[x, x'] \}$.
3. For all $(r, y, x') \in \Sigma_e \times V \times V'$, if $x \in \text{Prev}_1(y)$ and $Z[x, x']$ holds, then $x \in \text{rcPrev}[r, y, x']$.
4. For all $(r, x, y') \in \Sigma_e \times V \times V'$, if $x' \in \text{Prev}_1'(y')$ and $Z[x, x']$ holds, then $x' \in \text{rcPrev}'[r, x, y']$.

The following theorem is a counterpart of Theorem 3.5. It can be proved analogously, using Lemma 4.4 instead of Lemma 3.3.

**Theorem 4.5** Algorithm 4 runs in time $O((m + n)n)$ and returns the largest directed simulation between given finite fuzzy graphs $G = (V, E, L)$ and $G' = (V', E', L')$, where $n = |V| + |V'|$ and $m = |E| + |E'|$. 

16
5 Adapting to Fuzzy Automata

In this section, we adapt Algorithms 2 and 4 to computing the largest crisp simulation and the largest crisp directed simulation between two finite fuzzy automata.

A fuzzy automaton over a (finite) alphabet $\Sigma$ is a tuple $A = \langle Q, \delta, \sigma, \tau \rangle$, where $Q$ is a non-empty set of states, $\delta : Q \times \Sigma \times Q \to [0, 1]$ is called the fuzzy transition function, $\sigma : Q \to [0, 1]$ the fuzzy set of initial states, and $\tau : Q \to [0, 1]$ the fuzzy set of terminal states (cf. \[17\]). It is finite if $Q$ is finite.

Let $A = \langle Q, \delta, \sigma, \tau \rangle$ and $A' = \langle Q', \delta', \sigma', \tau' \rangle$ be fuzzy automata over an alphabet $\Sigma$. A relation $Z \subseteq Q \times Q'$ is a (crisp) simulation between $A$ and $A'$ if the following conditions hold (for all possible values of the free variables), where $\rightarrow$ and $\wedge$ denote the usual crisp logical connectives:

\[
\sigma(x) > 0 \rightarrow \exists x' \in Q' (Z(x, x') \wedge \sigma(x) < \sigma'(x')) \tag{12}
\]

\[
Z(x, x') \wedge \delta(x, r, y) > 0 \rightarrow \exists y' \in Q' (Z(y, y') \wedge \delta(x, r, y) < \delta'(x', r, y')) \tag{13}
\]

\[
Z(x, x') \wedge \tau(x) > 0 \rightarrow \tau(x) < \tau'(x'). \tag{14}
\]

The above definition of crisp simulations between fuzzy automata is an adaptation of the definition of “forward (fuzzy) simulation” between fuzzy automata introduced by Čirić et al. in \[3\]. A relation $Z \subseteq Q \times Q'$ is a (crisp) directed simulation between $A$ and $A'$ if Conditions (12)-(14) and the following one hold (for all possible values of the free variables):

\[
Z(x, x') \wedge \delta'(x', r, y') > 0 \rightarrow \exists y \in Q (Z(y, y') \wedge \delta'(x', r, y') < \delta(x, r, y)) \tag{15}
\]

Given a fuzzy automaton $A = \langle Q, \delta, \sigma, \tau \rangle$ over an alphabet $\Sigma$, we define the fuzzy graph corresponding to $A$ to be the fuzzy graph $G = \langle V, E, L \rangle$ using $\Sigma_e = \Sigma$ and $\Sigma_v = \{i, f\}$ such that:

- $V = Q \cup \{v_i, v_f\}$, where $v_i$ and $v_f$ are new vertices, with $v_i$ standing for the new unique initial state, and $v_f$ the new unique final state,
- $L(v_i)(i) = L(v_f)(f) = 1$, $L(v_i)(f) = L(v_f)(i) = 0$, and $L(x)(i) = L(x)(f) = 0$ for $x \in Q$ (thus, $i \in \Sigma_e$ is used to identify $v_i$ and $f \in \Sigma_e$ is used to identify $v_f$),
- for every $r \in \Sigma_e$, $x, y \in Q$ and $z \in V$:
  - $E(x, r, y) = \delta(x, r, y)$,
  - $E(v_i, r, x) = \sigma(x)$ and $E(x, r, v_f) = \tau(x)$,
  - $E(z, r, v_i) = E(v_f, r, z) = E(v_i, r, v_f) = 0$.

The above definition is a counterpart of the definition of the fuzzy interpretation (in description logic) corresponding to a fuzzy automaton \[21\]. The following lemma can be proved in a straightforward way.

Lemma 5.1 Let $A = \langle Q, \delta, \sigma, \tau \rangle$ and $A' = \langle Q', \delta', \sigma', \tau' \rangle$ be fuzzy automata over the same alphabet. Let $G = \langle V, E, L \rangle$ and $G' = \langle V', E', L' \rangle$ be the fuzzy graphs corresponding to $A$ and $A'$, respectively. Let $v_i, v_f \in V$ and $v'_i, v'_f \in V'$ be the vertices such that $L(v_i)(i) = L(v_f)(f) = L'(v'_i)(i) = L'(v'_f)(f) = 1$. If $Z \subseteq Q \times Q'$, then $Z$ is a simulation (respectively, directed simulation) between $A$ and $A'$ iff $Z \cup \{(v_i, v'_i), (v_f, v'_f)\}$ is a simulation (respectively, directed simulation) between $G$ and $G'$.

Let Algorithm 2 (respectively, Algorithm 3) be the algorithm that, given finite fuzzy automata $A = \langle Q, \delta, \sigma, \tau \rangle$ and $A' = \langle Q', \delta', \sigma', \tau' \rangle$ over the same alphabet $\Sigma$, computes the largest simulation (respectively, directed simulation) between $A$ and $A'$ as follows:

1. construct the fuzzy graph $G = \langle V, E, L \rangle$ that corresponds to $A$ and let $v_i, v_f \in V$ be the added vertices (with $L(v_i)(i) = L(v_f)(f) = 1$);
2. construct the fuzzy graph $G' = \langle V', E', L' \rangle$ that corresponds to $A'$ and let $v'_i, v'_f \in V'$ be the added vertices (with $L'(v'_i)(i) = L'(v'_f)(f) = 1$);
3. run Algorithm 2 (respectively, Algorithm 4) to compute the largest simulation (respectively, directed simulation) $Z$ between $G$ and $G'$;

4. return $Z - \{\langle v_i, v'_i \rangle, \langle v_f, v'_f \rangle \}$.

**Theorem 5.2** Algorithm 2' (respectively, Algorithm 4') returns the largest simulation (respectively, directed simulation) between the given finite fuzzy automata $A = (Q, \delta, \sigma, \tau)$ and $A' = (Q', \delta', \sigma', \tau')$, which are over the same alphabet $\Sigma$. Its complexity is of order $O((m + n)n)$, where $n = |Q| + |Q'|$ and $m = |\delta| + |\delta'|$, with

$$|\delta| = |\{(x, r, y) \in Q \times \Sigma \times Q : \delta(x, r, y) > 0\}|$$

$$|\delta'| = |\{(x', r, y') \in Q' \times \Sigma \times Q' : \delta(x', r, y') > 0\}|.$$

This theorem follows immediately from Lemma 5.1 and Theorem 3.5 (respectively, Theorem 4.5) and the fact that the first two steps of the algorithm (for constructing $G$ and $G'$) can be done in time $O(m + n)$.

### 6 Conclusions

As far as we know, before the current work there were no algorithms directly formulated for computing the largest crisp simulation and the largest crisp directed simulation between two finite fuzzy graph-based structures. One can try to adapt the algorithm of computing the greatest right/left invariant Boolean quasi-order matrix of a weighted automaton over an additively idempotent semiring, given by Stanimirović et al. in [25], to obtain algorithms for those tasks. However, the complexity order $O(n^5)$ is too high. Crisping a given finite fuzzy graph and then applying one of the algorithms given in [2] [12] [15] to the obtained crisp graph also results in an algorithm with a high complexity order, $O(l(m + n)n)$, where $l$ is the number of fuzzy values occurring in the specification of the input graph and, in the worst case, can be $m + n$.

In this article, we have given efficient algorithms with the complexity $O((m+n)n)$ for computing the largest crisp simulation and the largest crisp directed simulation between two finite fuzzy labeled graphs, where $n$ is the number of vertices and $m$ is the number of nonzero edges of the input fuzzy graphs. We have also adapted them to computing the largest crisp simulation and the largest crisp directed simulation between two finite fuzzy automata.

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