Euclidean Scalar Green Function in a Higher Dimensional Global Monopole Spacetime

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Abstract

We construct the explicit Euclidean scalar Green function associated with a massless field in a higher dimensional global monopole spacetime, i.e., a (1 + d)-spacetime with $d \geq 3$ which presents a solid angle deficit. Our result is expressed in terms of an infinite sum of products of Legendre functions with Gegenbauer polynomials. Although this Green function cannot be expressed in a closed form, for the specific case where the solid angle deficit is very small, it is possible to develop the sum and obtain the Green function in a more workable expression. Having this expression it is possible to calculate the vacuum expectation value of some relevant operators. As an application of this formalism, we calculate the renormalized vacuum expectation value of the square of the scalar field, $\langle \Phi^2(x) \rangle_{\text{Ren.}}$, and the energy-momentum tensor, $\langle T_{\mu\nu}(x) \rangle_{\text{Ren.}}$, for the global monopole spacetime with spatial dimensions $d = 4$ and $d = 5$.

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1. Introduction

In this paper we consider the Euclidean scalar Green function associated with a massless field in the higher dimensional global monopole spacetime. We define this spacetime as a generalization of the previous one given in Ref. [1] (the metric of cosmic string spacetime which is spatially flat was considered in Ref. [2]). The generalization of the Euclidean line element of [1] for higher dimensional case is given by

\[ ds^2 = d\tau^2 + \frac{dr^2}{\alpha^2} + r^2 d\Omega_{d-1}^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \]

(1)

where \( \mu, \nu = 0, 1, 2...d \), with \( d \geq 3 \) and \( x^\mu = (\tau, r, \theta_1, \theta_2, ..., \theta_{d-2}, \phi) \). The coordinates are defined in the intervals \( \tau \in (-\infty, \infty) \), \( \theta_i \in [0, \pi] \) for \( i = 1, 2...d-2 \), \( \phi \in [0, 2\pi] \) and \( r \geq 0 \). The parameter \( \alpha \) which codify the presence of the global monopole is smaller than unity. In this coordinate system the metric tensor is explicitly defined as shown below:

\[
\begin{align*}
g_{00} &= 1 \\
g_{11} &= 1/\alpha^2 \\
g_{22} &= r^2 \\
g_{jj} &= r^2 \sin^2 \theta_1 \sin^2 \theta_2...\sin^2 \theta_{j-2},
\end{align*}
\]

(2)

for \( 3 \leq j \leq d \), and \( g_{\mu\nu} = 0 \) for \( \mu \neq \nu \).

This spacetime corresponds to a pointlike global monopole. It is not flat: the scalar curvature \( R = (d-1)(d-2)(1-\alpha^2)/r^2 \), and the solid angle associated with a hypersphere with unity radius is \( \Omega = 2\pi^{d/2}\alpha^2/\Gamma(d/2) \), so smaller than ordinary one. The energy-momentum tensor associated with this object has a diagonal form and its non-vanishing components read \( T_0^0 = T_1^1 = (\alpha^2 - 1)(d-2)/r^2 \).

The Euclidean scalar Green function associated with a massless field in the geometry defined by (2) should obey the non-homogeneous second order differential equation

\[
(\Box - \xi R) G_E(x, x') = -\delta^n(x, x') = -\frac{\delta^n(x - x')}{\sqrt{g}},
\]

(3)

where \( n = d + 1 \), \( \Box \) denotes the covariant d’Alembertian in the spacetime defined by (2), \( \xi \) is an arbitrary coupling constant, \( \delta^d(x, x') \) is the bidensity Dirac distribution and \( R \) is the scalar curvature.
The vacuum expectation value of the square of the scalar field, \( \langle \Phi^2(x) \rangle \), is given by the evaluation of the Green function at the same point which provides a divergent result. So, in order to obtain a finite and well defined one, we should apply some renormalization procedure. The method which we shall adopt here is the point-splitting renormalization procedure. The basic idea of this method consists of subtracting from the Green function all the divergences which appears in the coincidence limit. (As it is well known this procedure is ambiguous in even dimensions and the ambiguity is a consequence of mass scale parameter \( \mu \) which is introduced after the renormalization procedure. In this way the final result contains a finite part which depends on the scale \( \mu \).)

In [3] Wald observed that the singular behavior of the Green function in the coincidence limit has the same structure as the Hadamard function, so his proposal was to subtract from the Green function the Hadamard one. In fact the use of the Hadamard function to renormalize the vacuum expectation value of the energy-momentum tensor in a curved spacetime has been first introduced by Adler et. al. [4]. In Ref. [3] Wald added a modification to this technique in order to provide the correct result for the trace anomaly. In this way the renormalized vacuum expectation value of the square of the field operator is given by:

\[
\langle \Phi^2(x) \rangle_{\text{Ren.}} = \lim_{x' \to x} \left[ G_E(x, x') - G_H(x, x') \right]. \tag{4}
\]

The renormalized vacuum expectation value of the energy-momentum tensor can also be obtained in a similar way:

\[
\langle T_{\mu\nu}(x) \rangle_{\text{Ren.}} = \lim_{x' \to x} D_{\mu\nu}(x, x') \left[ G_E(x, x') - G_H(x, x') \right], \tag{5}
\]

where \( D_{\mu\nu}(x, x') \) is a bivector differential operator. Moreover this vacuum expectation value should be conserved, i.e.,

\[
\nabla_\mu \langle T^\mu_\nu(x) \rangle_{\text{Ren.}} = 0, \tag{6}
\]

and gives us the correct conformal trace anomaly,

\[
\langle T^\mu_\mu(x) \rangle_{\text{Ren.}} = \frac{1}{(4\pi)^{n/2}} a_{n/2}(x), \tag{7}
\]

for \( n = 1 + d \) even and

\[
\langle T^\mu_\mu(x) \rangle_{\text{Ren.}} = 0, \tag{8}
\]

for \( n = 1 + d \) odd.
for \( n = 1 + d \) odd \([3]\).

In this paper we study the quantum field theory associated with a massless scalar field in the background spacetime defined by (2). More specifically we shall specialize in the cases where the spatial section of our spacetime has dimensions \( d = 4 \) and \( d = 5 \). In section 2, we construct explicitly the Euclidean Green function for a massless scalar field in a higher dimensional global monopole spacetime. We show that this Green function is expressed in terms of an infinite sum of product of Legendre function with Gegenbauer polynomials. In section 3 we calculate explicitly the renormalized vacuum expectation value of the square of the scalar field operator for \( d = 4 \) and \( d = 5 \) dimensions in the case where \( \eta^2 \ll 1 \), being \( \eta^2 = 1 - \alpha^2 \). We show that for six dimensions the expression obtained presents an ambiguity given by the mass scale parameter and that this ambiguity vanishes if we assume for the non-minimal coupling constant its conformal value, \( \xi = 1/5 \). In section 4 we present the formal expressions for the vacuum expectation values of the energy-momentum tensor for the cases \( d = 4 \) and \( d = 5 \). For the six dimensional case we present, after a long calculation, an explicit expression for the scale dependent term up to the first order in \( \eta^2 \). We explicitly show that this term is traceless for the conformal coupling. In section 5 we present our conclusions and some important remarks about this paper. Although the formalism developed here is applied only when the dimension of the space section is \( d \geq 3 \), in the appendix we present an expression for the Green function in a limiting case \( d \to 2 \).

2. Green Function

In this section we calculate the Euclidean scalar Green function associated with a massless field in the spacetime described by (2). This Green function must obey the non-homogeneous second order differential equation

\[
(\Box - \xi R) G_E(x, x') = -\delta^d(x, x') = -\frac{\delta^2(x - x')}{\sqrt{g}}, \tag{9}
\]

\(^1\)The parameter \( \alpha \) is related with the energy scale where the global symmetry of the system is spontaneously broken, see Ref. \([1]\). For a typical grand unified theory in a physical \((1 + 3)\)-dimensional spacetime, this scale is of order \(10^{16}\)Gev. So \(1 - \alpha^2 = \eta^2 \sim 10^{-5}\).
where we have introduced the non-minimal coupling of the scalar field with the geometry. As we have already said, the scalar curvature for this spacetime is \( R = (d - 1)(d - 2)(1 - \alpha^2)/r^2 \).

The Euclidean Green function can also be obtained by the Schwinger-DeWitt formalism as follows:

\[
G_E(x, x') = \int_0^\infty ds K(x, x'; s) ,
\]

where the heat kernel, \( K(x, x'; s) \), can be expressed in terms of eigenfunctions of the operator \( \Box - \xi R \) as follows:

\[
K(x, x'; s) = \sum_\sigma \Phi_\sigma(x)\Phi_\sigma^*(x') \exp(-s\sigma^2) ,
\]

\( \sigma^2 \) being the corresponding positively defined eigenvalue. Writing

\[
(\Box - \xi R) \Phi_\sigma(x) = \sigma^2 \Phi_\sigma(x) ,
\]

we obtain the complete set of normalized solutions of the above equation:

\[
\Phi_\sigma(x) = \sqrt{\frac{\alpha p}{2\pi r^{d/2-1}}} e^{-i\omega \tau} J_{\nu_n}(pr) Y(n, m_j; \phi, \theta_j) ,
\]

with

\[
\sigma^2 = \omega^2 + \alpha^2 p^2 ,
\]

\( Y(n, m_j; \phi, \theta_j) \) being the hyperspherical harmonics of degree \( n \) \[6\], \( J_\nu \) the Bessel function of order

\[
\nu_n = \alpha^{-1} \sqrt{(n + (d - 2)/2)^2 + (d - 4)(d - 2)(1 - \alpha^2)(\xi - \bar{\xi})} ,
\]

with \( \bar{\xi} = \frac{d - 2}{4(d - 1)} \). So according to (11) our heat kernel is given by

\[
K(x, x'; s) = \int_{-\infty}^{\infty} d\omega \int_0^\infty dp \sum_{n, m_j} \Phi_\sigma(x)\Phi_\sigma^*(x') e^{-s\sigma^2}
\]

\[
= \frac{1}{8\alpha^2 \Gamma(d/2)} \frac{1}{s^{d/2-1}} \frac{\Gamma(d/2)}{d - 2} e^{-\Delta r^2 \alpha^2 \gamma^2 s} \times
\]

\[
\sum_{n=0}^\infty [2(n - 1) + d] I_{\nu_n} \left( \frac{rr'}{2\alpha^2 s} \right) C_n^m \left( \cos \gamma \right) ,
\]

\[16\]
\( I_\nu \) being the modified Bessel function, \( C_n^\mu(x) \) the Gegenbauer polynomial of degree \( n \) and order \( \mu \) and \( \gamma \) is the angle between two arbitrary directions. Our final expression \((10)\) was obtained using the addition theorem for the hyperspherical harmonics \((\text{III})\) and the integral table \((\text{IV})\).

Now we are in position to obtain the Euclidean Green function substituting \((16)\) into \((10)\). Our final result is:

\[
G_E^{(d)}(x, x') = \frac{1}{4\pi^{d/2+1}} \frac{1}{(rr')^{d/2}} \frac{\Gamma(d/2)}{d-2} \times \frac{1}{d-2} \times \sum_{n=0}^{\infty} [2(n-1) + d] Q_{\nu_n-1/2}(u) C_n^{d-2}(\cos \gamma),
\]

where

\[
u = \frac{\alpha^2 \Delta r^2 + r^2 + r'^2}{2rr'}
\]

and \( Q_\mu \) is the Legendre function. Unfortunately because the dependence of the order of the Legendre functions on the parameter \( \alpha \) is not a simple one, it is not possible to develop the summation and to obtain a closed expression for the Green function even in the simpler case \( d = 3 \) \((\text{V})\). However for \( \xi = \xi', \nu_n \) becomes equal to \((n + (d-2)/2)/\alpha \) and for \( \gamma = 0 \) it is possible to represent this Green function in an integral form using the integral representation for the Legendre function

\[
Q_{\nu-1/2}(\cosh \rho) = \frac{1}{\sqrt{2}} \int_{\rho}^{\infty} dt \frac{e^{-\nu t}}{\sqrt{\cosh t - \cosh \rho}}
\]

and

\[
C_n^{d-2}(1) = \frac{(n + d - 3)!}{n!(d - 3)!}.
\]

Now it is possible to develop the summation and after some intermediate steps we get

\[
G_E^{(d)}(r, \tau; r', \tau') = \frac{1}{4\pi^{d/2+1}} \frac{1}{(rr')^{d-1}} \frac{\Gamma(d/2)}{2^{d-3/2}} \times \int_{\rho}^{\infty} dt \frac{1}{\sqrt{\cosh t - \cosh \rho}} \frac{\cosh(t/2\alpha)}{(\sinh(t/2\alpha))^{d-1}}.
\]
For the case where $\alpha = 1/2$, i.e., a large solid angle deficit, the above expression can be written in a simpler form by performing the integral. Making an appropriate transformation of variable, $t := \arccosh \left( \frac{u - 1}{q} + 1 \right)$, we get:

$$G_E^{(d)}(r, \tau; r', \tau') = \frac{1}{4\sqrt{2\pi^{d/2+1}}} \frac{\Gamma(d/2)}{d-2} \frac{(rr')^{d-2}}{[(r - r')^2 + \Delta \tau^2/4]^{d/2-1}} \times \frac{1}{[(r + r')^2 + \Delta \tau^2/4]^{d/2-1}}. \quad (22)$$

The next section is devoted to the obtainment of the renormalized vacuum expectation value of the square of the field operator. As an explicit application we shall develop these quantities for the cases where $d = 4$ and $d = 5$.

**3. Calculation of $\langle \Phi^2(x) \rangle_{\text{Ren.}}$.**

The calculation of the vacuum expectation value of the square of the field operator is obtained computing the Green function in the coincidence limit:

$$\langle \Phi^2(x) \rangle = \lim_{x' \to x} G_E(x, x'). \quad (23)$$

However this procedure provides a divergent result. In order to obtain a finite and well defined result we must apply in this calculation some renormalization procedure. Here we shall adopt the point-splitting renormalization. This procedure is based upon a divergence subtraction scheme in the coincidence limit of the Green function. In [3], Wald examined the behavior of the Green function in this limit. He observed that its divergences have the same structure as given by the Hadamard function, which on the other hand can be explicitly written in terms of the square of the geodesic distance between the two points. So, we shall adopt the following prescription: we subtract from the Green function the Hadamard one before applying the coincidence limit as shown below:

$$\langle \Phi^2(x) \rangle_{\text{Ren.}} = \lim_{x' \to x} [G_E(x, x') - G_H(x, x')]. \quad (24)$$

Now let us develop this calculation explicitly. As we have mentioned above, it is not possible to proceed exactly the summation which appears
in the Green function. By this reason the best that we can do is to obtain an approximate expression for it, developing a series expansion in powers of the parameter $\eta^2 = 1 - \alpha^2$, which is much smaller than unity for the physical four-dimensional spacetime. Moreover, because we want to take the coincidence limit in the Green function, let us take first $\gamma = 0$ and $\Delta \tau = 0$ in (L7). The approximate expression for the order of the Legendre function, up to the first power in $\eta^2$ is

$$\nu_n \approx \left(n + \frac{d - 2}{2}\right)(1 + \eta^2/2) + \frac{(d - 1)(d - 2)(\xi - \tilde{\xi})}{2n + d - 2}\eta^2 + O(\eta^4).$$

(25)

We also need to develop the summation

$$S = \sum_{n=0}^{\infty} \frac{[2n + d - 2](n + d - 3)!}{n!} e^{-\nu_n t}.$$  

(26)

Substituting the approximate expression for $\nu_n$ into the summation above, we get after some intermediate steps,

$$S = \frac{(d - 2)!}{2^{d-2}} \frac{\cosh(t/2)}{\sinh^{d-1}(t/2)} \left[1 - \frac{t\eta^2(d - 1)}{2\sinh(t)} \left(1 + 4\xi \sinh^2(t/2)\right)\right].$$

(27)

So the approximate Green function is given by

$$G_E^{(d)}(r, r') = \frac{1}{2^{d+1/2} \pi^{d/2 + 1}} \frac{\Gamma(d/2)}{(rr')^{d-2}/2} \int_\rho^{\infty} dt \frac{1}{\sqrt{\cosh t - \cosh \rho}} \times \frac{\cosh(t/2)}{\sinh^{d-1}(t/2)} \left[1 - \frac{t\eta^2(d - 1)}{2\sinh(t)} \left(1 + 4\sinh^2(t/2)\right)\right].$$

(28)

In his beautiful paper Christensen [5] has given a general expression for the Hadamard function for any dimensional spacetime, which is expressed in terms of the square of the geodesic distance $2\sigma(x, x')$. Moreover, there he has called attention for the different behavior of this function when the dimension of the spacetime is an even or odd number. In latter case there is no logarithmic terms in the expansion of the Hadamard. So because of this fact we shall develop, separately, the calculation of the renormalized vacuum expectation value of the square of the field operator, for $n = 1 + d$ odd and even.
Following [5], below we write down the Hadamard function for the massless case when the dimension of the spacetime be an odd number. This function is given by

\[
G_H(x, x') = \frac{\Delta^{1/2}(x, x')}{\sigma^{n/2-1}(x, x')} \frac{1}{2(2\pi)^{n/2}} \sum_{k=0}^{\infty} a_k(x, x') \sigma^k(x, x') \frac{\Gamma(n/2 - k - 1)}{2^k},
\]

where \(\Delta(x, x')\), the Van Vleck-Moretti determinant and the factor \(a_k(x, x')\), for \(k = 0, 1, 2\), have been computed by many authors. See Refs. [9] and [10].

Now let us apply this formalism for the case \(n = 5\), i.e., \(d = 4\). The Euclidean Green function in this case is

\[
G_E^{(4)}(r, r') = \frac{1}{16\sqrt{2\pi}^3} \frac{1}{(rr')^{3/2}} \int_0^\infty dt \frac{1}{\sqrt{\cosh t - \cos \rho}} \times \frac{\cosh(t/2)}{\sinh^{3/2}(t/2)} \left[ 1 - \frac{3t\eta^2}{2\sinh(t)}(1 + 4\xi \sinh^2(t/2)) \right].
\]

Because we need to evaluate the Hadamard function in the coincidence limit we can write this function exhibiting only its divergent contributions as shown below:

\[
G_H(x, x') = \frac{1}{16\sqrt{2\pi}^2} \frac{1}{\sigma^{3/2}(x, x')} [1 + (1/6 - \xi) R(x) \sigma(x, x')] ,
\]

where we have substituted the explicit expression for \(\Delta, a_0\) and \(a_1\) in the coincidence limit. The scalar curvature in this five-dimensional spacetime is \(R = 6\eta^2/r^2\). The radial one-half of the geodesic distance, \(\sigma(x, x') = (1/2\alpha^2)(r-r')^2\), in our approximation is equal to \(\sigma \approx (1/2)(r-r')^2(1+\eta^2+...).\) Now substituting (30) and (31) into (24) we get after a long calculation

\[
\langle \Phi^2(x) \rangle_{\text{Ren.}} = \frac{3\eta^2}{64\pi r^3} (\xi - 3/16) .
\]

We can see that for the conformal coupling in five dimensional spacetime, \(\xi = 3/16\), the above expectation vanishes, i.e., the renormalized vacuum expectation value of the operator \(\Phi^2(x)\) is zero up to the first order in \(\eta^2\).

The six dimensional case will be analysed now. This case, together with the four dimensional one studied in [8], exhibit explicitly the ambiguity in
the renormalization procedure given by a mass subtraction point \( \mu \). (Moretti in \cite{11}, has used the local \( \zeta \)-function renormalization technique to show, by a general argument, that the scale ambiguity is present in the calculation of vacuum expectation value of the square of scalar field operator in a curved spacetime of dimension even.)

The Hadamard function in even dimensions is explicitly written in the paper by Christensen \cite{5}, so we shall not reproduce it here. The singular behavior of the Hadamard function in the six dimensional spacetime is

\[
G_H(x, x') = \frac{\Delta^{1/2}(x, x')}{16\pi^3} \left[ \frac{a_0(x, x')}{\sigma^2(x, x')} + \frac{a_1(x, x')}{2\sigma(x, x')} - \frac{a_2(x, x')}{4} \ln \left( \frac{\mu^2 \sigma(x, x')}{2} \right) \right].
\]

(33)

Substituting the expressions for the factors \( a_k \), we get, up to the first order in the parameter \( \eta^2 \), the following result:

\[
G_H(r, r') = \frac{1}{2\pi^3} \left[ \frac{(1 - 2\eta^2)}{(r - r')^4} + \frac{(1 - 6\xi)\eta^2}{2r^2(r - r')^2} - \frac{\eta^2}{4r^4} \ln \left( \frac{\mu^2 (r - r')^2}{4} \right) \right].
\]

(34)

Now taking \( d = 5 \) in (28) and substituting the result, together with the above equation, into (24) we get after some calculation the following expression:

\[
\langle \Phi^2(x) \rangle_{\text{Ren.}} = -\frac{\eta^2}{96\pi^3 r^4} \left( \frac{47}{25} - 10\xi \right) + \frac{\eta^2}{8\pi^3 r^4} \ln(\mu r).
\]

(35)

We can see that for the conformal coupling in six dimension, \( \xi = 1/5 \), there is no ambiguity, the logarithmic contribution disappears and we get \( \langle \Phi^2(x) \rangle_{\text{Ren.}} = \eta^2/800\pi^3 r^4 \). (In order to obtain the above result we expressed the logarithmic term which is present in (33) in terms of the \( Q_0(\pi) \), \( \pi \) being \( \frac{r^2 + r'^2}{2rr'} \) and write this Legendre function in its integral representation (19)).

The next section is devoted to the analysis of the vacuum expectation value of the energy-momentum tensor in five and six dimensional spacetime.

### 4. Vacuum Expectation Value of the Energy-Momentum Tensor

In this paper we are working with a massless scalar quantum field theory in the metric spacetime defined by (4), which does not present any dimensional
parameter. Moreover we are adopting the natural system units where \( \hbar = c = 1 \), so because of these reasons the physical quantities calculated in this model can only depend on the radial coordinate \( r \) or on the renormalization mass scale \( \mu \). By dimensional point we could expect that \( \langle \Phi^2(x) \rangle_{\text{Ren.}} \) be proportional to \( 1/r^{n-2} \) and \( \langle T_{\mu,\nu}(x) \rangle_{\text{Ren.}} \) proportional to \( 1/r^n \). The factor of proportionality should be given in terms of the parameter \( \eta^2 \) and the non-minimal coupling \( \xi \). As to the square of the scalar field, this calculation has been done in this paper for the cases where the dimension of the spacetime is 5 and 6 up to the first order in \( \eta^2 \). In this section we want to analyze the vacuum expectation value of the energy-momentum tensor. We start considering the five dimensional case.

The renormalized vacuum expectation value of the energy-momentum tensor in five dimensions does not depend on the subtraction mass parameter; so there is no logarithmic term and it can be written in a general form by

\[
\langle T^\mu_\nu(x) \rangle_{\text{Ren.}} = \frac{A^\mu_\nu(\xi, \eta^2)}{r^5}.
\]

(36)

Because there is no trace anomaly in odd dimension, for \( \xi = 3/16 \), we can write

\[
\langle T^\mu_\mu(x) \rangle_{\text{Ren.}} = 0,
\]

(37)

so \( A^\mu_\mu = 0 \). On the other hand, by symmetry of this spacetime, the above vacuum expectation value should be diagonal. Moreover the conservation condition

\[
\nabla_\nu \langle T^\nu_\mu(x) \rangle_{\text{Ren.}} = 0,
\]

(38)

imposes additional restrictions on the components of the tensor \( A^\nu_\mu \). So under these conditions we have:

\[
A^0_0 = A^1_1, \ A^2_2 = A^3_3 = A^4_4.
\]

(39)

Using these relations and the traceless condition we can express all nonzero components of \( A^\nu_\mu \) in terms of one of them, let us choose \( A^0_0 \), so we can write

\[
A^\nu_\mu = A^0_0 \ diag ( 1, 1, -2/3, -2/3, -2/3 ) .
\]

(40)

The explicit calculation of the component \( A^0_0 \) involves an extensive calculation which we shall not do here.
The vacuum expectation value of the energy-momentum tensor in six dimensions requires more details. By the expression obtained for the vacuum expectation value of the square of the field, it is possible to infer that there exists a logarithmic contribution to this tensor. Moreover by the trace anomaly [5] we have

\[ \langle T^\mu_{\mu}(x) \rangle_{\text{Ren.}} = \frac{1}{64\pi^3 r^6} a_3(x) \cdot \text{(41)} \]

So we can conclude that the general expression for this object is

\[ \langle T^\nu_{\mu}(x) \rangle_{\text{Ren.}} = \frac{1}{64\pi^3 r^6} \left[ A^\nu_\mu(\eta^2, \xi) + B^\nu_\mu(\eta^2, \xi) \ln(\mu r) \right] , \text{(42)} \]

where \( A^\nu_\mu \) in principal are arbitrary numbers. Because the cutoff factor \( \mu \) is completely arbitrary, there is an ambiguity in the definition of this renormalized vacuum expectation value. Moreover the change in this quantity under the change of the renormalization scale is given in terms of the tensor \( B^\nu_\mu \) as shown below:

\[ \langle T^\nu_{\mu}(x) \rangle_{\text{Ren.}}(\mu) - \langle T^\nu_{\mu}(x) \rangle_{\text{Ren.}}(\mu') = \frac{1}{64\pi^3 r^6} B^\nu_\mu(\eta^2, \xi) \ln(\mu/\mu') \cdot \text{(43)} \]

In Ref. [5], Christensen pointed out that the difference between them is given in terms of the effective action which depends on the logarithmic terms whose final expression, in arbitrary even dimension, is

\[ \langle T^\nu_{\mu}(x) \rangle_{\text{Ren.}}(\mu) - \langle T^\nu_{\mu}(x) \rangle_{\text{Ren.}}(\mu') = \frac{1}{(4\pi)^{\gamma/2}} \frac{\delta}{\sqrt{g} \delta g^\mu\nu} \int d^n x \sqrt{g} a_{n/2}(x) \ln(\mu/\mu') \cdot \text{(44)} \]

In our six dimensional case we need the factor \( a_3(x) \). The explicit expression for this factor can be found in the paper by Gilkey [12] and in a more systematic form in the paper by Jack and Parker [13], for a scalar second order differential operator \( D^2 + X \), \( D_\mu \) being the covariant derivative including gauge field and \( X \) an arbitrary scalar function. This expression involves 46 terms and we shall not repeat it here in a complete form. The reason is that our calculation has been developed up to the first order in the parameter \( \eta^2 \) and only the quadratic terms in Riemann and Ricci tensors, and in the scalar curvature are relevant for us. This reduces to 12 the number of terms which will be considered. Discarding the gauge fields and taking \( X = -\xi R \) we get:

\[ a_3(x) = \frac{1}{6} \left( \frac{1}{6} - \xi \right) \left( \frac{1}{5} - \xi \right) R^\square R + \frac{\xi^2}{12} R^\mu R_{,\mu} + \frac{\xi}{90} R^\mu R_{\mu} + \frac{\xi}{36} R^\mu R_{,\mu} \]
\[-\frac{1}{7!} \left[ 28\Box R + 17 R_{\mu\nu} R^{\mu\nu} - 2 R_{\mu\nu;\rho} R^{\mu\nu;\rho} - 4 R_{\mu\nu;\rho} R^{\mu\nu;\rho;\sigma} + 9 R_{\mu\nu\rho\sigma;\gamma} R^{\mu\nu\rho\sigma;\gamma} - 8 R_{\mu\nu} \Box R^{\mu\nu} + 24 R_{\mu\nu} R^{\mu\nu;\rho} + 12 R_{\mu\nu;\rho} \Box R^{\mu\nu;\rho} \right] + O(R^3) \text{.} \tag{45} \]

This expression is of sixth order derivative on the metric tensor. Our next step is to take the functional derivative of \( \bar{\sigma}_x(x) \). Using the expressions for the functional derivative of the Riemann and Ricci tensor, together with the scalar curvature \([4]\), we obtain after a long calculation the following expression for the tensor \( B^\nu_\mu \): \[
B^\nu_\mu(\eta^2, \xi) = \frac{r^6}{6} \left[ -\delta^\nu R^{\Box^2} \left( \xi^2 - \frac{\xi}{3} + \frac{23}{840} \right) + \frac{1}{40} \Box^2 R^\nu_\mu + \nabla^\nu \nabla_\mu R \left( \xi^2 - \frac{\xi}{3} + \frac{1}{42} \right) \right] + O(R^2) \text{.} \tag{46} \]

Moreover, developing all the terms which appear in the above equation we obtain after some calculations \[
B^\nu_\mu(\eta^2, \xi) = \frac{\eta^2}{225} \text{ diag } (\, 6, 6, -3, -3, -3, -3 \, ) + 16\eta^2(\xi - 1/5)(\xi - 2/15) \text{ diag } (\, 1, -4, 2, 2, 2, 2 \, ) . \tag{47} \]

As in the last case it is possible to make some restrictions on the tensor \( A^\nu_\mu \). Again because of the spherical symmetry of the problem we can infer that this tensor should be diagonal. Moreover the renormalized vacuum expectation value of the energy-momentum tensor must be conserved, i.e.: \[
\langle T^\nu_\mu(x) \rangle_{\text{Ren.}} = 0 . \tag{48} \]

From these six equations and defining the variable \( T = 64\pi^3 r^6 \langle T^\mu_\mu(x) \rangle_{\text{Ren.}} \), we obtain \[
A^0_0 = T + A^1_1 - B^1_1 + (B^1_1 - B^0_0) \ln(\mu r) \tag{49} \]
and \[
A^2_2 = A^3_3 = A^4_4 = A^5_5 = \frac{B^1_1}{4} - \frac{A^1_1}{2} - \left( B^2_2 + \frac{B^1_1}{2} \right) \ln(\mu r) . \tag{50} \]

When the non-minimal coupling \( \xi \) coincides with the conformal one, \( 1/5 \), we get \[
A^0_0 = T + A^1_1 - B^1_1 , \tag{51} \]
\[ A_2^2 = A_3^3 = A_4^4 = A_5^5 = \frac{B_1^1}{4} - \frac{A_1^1}{2}, \]  
\text{(52)}

with
\[ T = r^6 a_3(x) = -\frac{1}{4200} \Box^2 R + O(R^2) = \frac{8}{350} \eta^2 + O(\eta^4). \]
\text{(53)}

Again the complete evaluation of \( \langle T_{\mu\nu}(x) \rangle_{\text{Ren.}} \) requires the knowledge of at least one component of the tensor \( A_{\mu\nu} \), say \( A_{11}^1 \). However we do not attempt to do this straightforward and long calculation here.

5. Concluding Remarks

In this paper we have found, in a formal expression for the Euclidean scalar Green function associated with a massless field in higher dimensional global monopole spacetime defined by (2), i.e., spacetime where the dimension is bigger than four and presents a solid angle deficit. The expression for this function is given in terms of an infinite sum of products of Legendre functions with Gegenbauer polynomials or hyperspherical harmonics. Having this Green function in our hands we can use it to calculate the vacuum expectation value of some physically relevant operators, as the square of the field and the energy-momentum tensor. We have applied this formalism to calculate \( \langle \Phi^2(x) \rangle \) and \( \langle T_{\mu\nu}(x) \rangle \) in five and six dimensions. However these calculations become effective only for the case when the parameter \( \alpha \), associated with the solid angle deficit, is close to unity. In this case we can expand the Green function in powers of the parameter \( \eta^2 = 1 - \alpha^2 \), and obtain closed results for these two quantities.

As it was mentioned the vacuum expectation values for these quantities are divergent and these divergences are consequence of the evaluation of the two points Green function in the coincidence limit. In order to obtain finite and well defined results we adopted the point-splitting renormalization procedure and eliminate all divergences subtracting from the Green function the Hadamard one. An interesting result of our calculation was that the renormalized vacuum expectation value of the square of the field in the five dimensional global monopole spacetime vanishes when we take for the non-minimal coupling constant \( \xi \) the conformal value \( 3/16 \). A similar calculation in the six dimensional case shows that a finite contribution remains for \( \xi = 1/5 \) which is independent of the mass cutoff parameter.
Moreover, in six dimensional case, there appears in the renormalized vacuum expectation value of the energy-momentum tensor, the function $a_3(x)$. This function, according to [13] on pp. 159, is associated with a purely geometric (divergent) Lagrangian that should renormalize the modified classical Einstein one. When similar terms are inserted into the gravitational action, the field equation is modified by the presence of order six terms proportional to:

$$c_1 g_{\mu\nu} \Box^2 R + c_2 \Box^2 R_{\mu\nu} + c_3 \nabla_\mu \nabla_\nu \Box R + O(R^2) .$$

(54)
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6. Appendix

In this appendix we present a non-trivial extension of our Green function (17) for the case where \( d = 2 \). Although this Green function constructed for \( d \geq 3 \), presents a pole at \( d = 2 \), the Gegenbauer polynomial is also not defined when its order is zero. However it is possible to get such expression if we admit that the \( (17) \) is a function of a continuous parameter \( d \). According to \[16\], it is possible to obtain a well defined limit for the ratio of the Gegenbauer polynomial by its order when it goes to zero. This limit is given by

\[
\lim_{d \rightarrow 2} \frac{1}{d - 2} C_{\frac{d-2}{2}}(x) = \frac{T_n(x)}{n},
\]

where \( T_n(x) \) is the Chebychev polynomials type I. Moreover in the limit \( d = 2 \), \( \nu_n = |n|/\alpha \). However we have to be careful when we try to substitute the above limit into \( (17) \). The definition of the Chebychev polynomials type I by its generating function reproduces explicitely the polynomials with order \( n \geq 1 \) in a recurrence equation separated from the \( T_0(x) \), see \[16\]. Taking into account this fact we have to consider a multiplicity factor in the summation. Finally we arrive at the following Green function is:

\[
G^{(2)}(x, x') = \frac{1}{2\pi^2} \frac{1}{\sqrt{rr'}} \sum_{n=0}^{\infty} Q_{\nu_n-1/2} T_n(\cos \gamma) \epsilon(n),
\]

(56)

with \( \epsilon(0) = 1 \) and \( \epsilon(n > 0) = 2 \). However \( T_n(\cos \gamma) = \cos(n\gamma) \), see again \[16\] on pp. 631. Finally substituting the integral representation for the Legendre function \( (19) \), we obtain

\[
G^{(2)}(x, x') = \frac{1}{2\pi^2} \frac{1}{\sqrt{2rr'}} \int_{\rho}^{\infty} dt \frac{1}{\sqrt{\cosh t - \cosh \rho \cosh(t/\alpha) - \cos(\gamma)}} \sinh(t/\alpha).
\]

(57)

This equation is equivalent with the Euclidean Green function for the three dimensional conical space given in Eq. (2.19) of \[17\]. We can see that changing in a compatible way the coordinates \( r \) by \( r/\alpha \), \( \gamma \) by \( \gamma/\alpha \) and \( \tau \) by \( z \).
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