Very ampleness for Theta on the compactified Jacobian

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1 Introduction

If $X$ is a smooth, complete, connected curve over an algebraically closed field, then the Jacobian $J_0$, parametrizing invertible sheaves on $X$ of Euler characteristic 0, is projective and admits a canonical ample divisor $\Theta$, the Theta divisor. If $g$ denotes the genus of $X$, then $\Theta$ is the scheme-theoretic image of the Abel-Jacobi morphism $X^{g-1} \rightarrow J_0$, given by

$$(p_1, \ldots, p_{g-1}) \mapsto \mathcal{O}_X(p_1 + \cdots + p_{g-1}).$$

It follows from [14, Section 17, p. 163] that $3\Theta$ is very ample.

In the singular case, D’Souza has constructed a natural compactification $\bar{J}_0$ for the Jacobian $J_0$ of a complete, integral curve over an algebraically closed field [5]. The scheme $\bar{J}_0$ parametrizes torsion-free, rank 1 sheaves of Euler characteristic 0 on $X$. A natural question in this context is whether there is a canonical Cartier divisor on $\bar{J}_0$ extending the notion of the classical Theta divisor.

The above question was partially and independently answered in [6] and [19]. In these two works the same canonical line bundle $L$ on $\bar{J}_0$ and the same global section $\theta$ of $L$ are defined. For smooth curves, the zero scheme of $\theta$ is the classical Theta divisor $\Theta$. In [19] Soucaris shows that the zero scheme of the restriction of $\theta$ to the maximum reduced subscheme of $\bar{J}_0$ is a Cartier divisor. Both [6] and [19] show that $L$ is ample. It remains to determine whether the zero scheme of $\theta$ on $\bar{J}_0$ is a Cartier divisor in general, and what is the minimum $n$ such that $L^\otimes n$ is very ample.

In this article our main concern is with the latter question. We will show that $L^\otimes n$ is very ample for $n$ at least equal to a specified lower bound (Theorem 7.) If $X$ has at most ordinary nodes or cusps as singularities, then our lower bound is 3. Our main tool is to use theta sections $\theta_E$ associated to vector bundles $E$ on $X$. The theta sections were used by Faltings [9] to construct the moduli of semistable vector bundles on a smooth, complete curve without using Geometric Invariant Theory (see also [18].) In a forthcoming work [7], [8] we will apply such method to construct the compactified Jacobian for families of reduced curves.

The importance of Theorem 7 is that we obtain a canonical projective embedding of $\bar{J}_0$ in $\mathbf{P}(H^0(\bar{J}_0, L^\otimes n))$, for $n$ minimum such that $L^\otimes n$ is very ample. By studying the structure of the homogeneous coordinate ring of $\bar{J}_0$ in $\mathbf{P}(H^0(\bar{J}_0, L^\otimes n))$, maybe in a way analogous to Mumford’s in [15] and [16], we might be able to understand better the algebraic structure of $\bar{J}_0$.

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Notation. We will often deal with parameter spaces, that is, spaces whose points are classes representing certain objects. In such context, we will employ the usual bracket notation $[F]$ for the point representing the object $F$. If $E$ is a vector bundle on a scheme $Y$, we denote by $P_Y(E)$ the corresponding projective bundle over $Y$. By a point we mean a closed point.

2 The compactified Jacobian

Let $X$ be a complete, integral curve over an algebraically closed field $k$. Denote by $g$ the arithmetic genus of $X$, and by $\omega$ the dualizing sheaf on $X$. A coherent sheaf $I$ on $X$ is torsion-free if $I_x$ is a torsion-free $\mathcal{O}_x$-module for every $x \in X$. A coherent sheaf $I$ on $X$ is rank 1 if $I$ is generically invertible. By [4, p. 96], the sheaf $\omega$ is torsion-free, rank 1. Fix an ample line bundle $\mathcal{O}_X(1)$ on $X$. For every coherent sheaf $F$ on $X$, let $\text{Quot}_X^p(F)$ denote Grothendieck’s Quot-scheme [10], parametrizing quotients of $F$ with Hilbert polynomial $p(t)$ with respect to $\mathcal{O}_X(1)$. We will drop the superscript $p(t)$ whenever it is not important.

For every integer $d$, let $\bar{J}_d$ denote the compactified Jacobian functor. For each $k$-scheme $S$, the set $\bar{J}_d(S)$ consists of equivalence classes of $S$-flat coherent sheaves $\mathcal{I}$ on $X \times S$ such that $\mathcal{I}(s)$ is torsion-free, rank 1 of Euler characteristic $d$ for every $s \in S$. (Two sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ are called equivalent if there is an invertible sheaf $N$ on $S$ such that $\mathcal{I}_1 \cong \mathcal{I}_2 \otimes N$.) D’Souza [5] and Altman and Kleiman [3], [4] have shown that $\bar{J}_d$ is represented by a (projective) scheme $\bar{J}_d$, the compactified Jacobian. Here we present yet another proof of the representability of $\bar{J}_d$ by a scheme, a proof more suitable for treating the question of very ampleness in Section 4.

For every torsion-free, rank 1 sheaf $I$ on $X$, let $e(I) := \max_{x \in X} \dim_k I(x)$.

Since $I$ is generically invertible, then $1 \leq e(I) < \infty$.

**Proposition 1** Let $I$ be a torsion-free, rank 1 sheaf on $X$ of Euler characteristic $d$. Then, for every integer $r \geq \max(e(\text{Hom}_X(I, \omega)), 2)$, there is a vector bundle $E$ on $X$ of rank $r$ and degree $-rd - 1$ such that:

(i) $h^0(X, I \otimes E) = 0$ and $h^1(X, I \otimes E) = 1$;

(ii) the unique (modulo $k^*$) non-zero homomorphism $I \to E^* \otimes \omega$ is an embedding with torsion-free cokernel.

**Proof.** Let $m >> 0$ be an integer such that $H^0(X, I(-m)) = 0$ and $\text{Hom}_X(I, \omega)(m)$ is generated by global sections. Since $r \geq \max(e(\text{Hom}_X(I, \omega)), 2)$ and $k$ is infinite, then there is a surjection $p: \mathcal{O}_X^{r \oplus r} \twoheadrightarrow \text{Hom}_X(I, \omega)(m)$. Applying $\text{Hom}_X(\cdot, \omega)$ to $p$, we obtain an embedding $I(-m) \hookrightarrow \omega^{\oplus r}$, whose cokernel is torsion-free since $\text{Ext}_X^1(F, \omega) = 0$ for every torsion-free sheaf $F$ on $X$ [4, p. 96]. Twisting by $\mathcal{O}_X(m)$ and letting $E := \mathcal{O}_X(-m)^{\oplus r}$, we get that $H^0(X, I \otimes E) = 0$ and there is a short exact sequence on $X$ of the form

$$0 \to I \overset{\mu}{\to} E^* \otimes \omega \overset{\omega}{\to} C \to 0,$$

where $C$ is torsion-free.
Let \( h := h^1(X, I \otimes E) \). If \( h = 1 \), then the proposition is proved. We will show by descending induction on \( h \) that we can choose \( E \) as in the above paragraph with \( h = 1 \). Suppose \( h > 1 \). Let \( \lambda: I \to E^* \otimes \omega \) be a homomorphism that is not a multiple of \( \mu \). Since \( I \) is simple by [4, Lemma 5.4, p. 83], then the composition \( \rho := q \circ \lambda \) is not zero. Since \( C \) is torsion-free, then there is a regular point \( x \in X \) such that \( \rho(x) \neq 0 \). Let \( \sigma: C(x) \to I(x) \) be a splitting for \( \rho(x) \). Let

\[ F := (\ker(E^* \to E^*(x) \xrightarrow{q(x)} C(x) \xrightarrow{\sigma} I(x)))^*. \]

(We implicitly chose a trivialization of \( \omega \) at \( x \). Any other choice of trivialization yields the same subsheaf \( F \).) Then \( F \) is a vector bundle of \( \deg F = \deg E + 1 \) and rank \( r \). By definition of \( F \), we have that \( \mu \) factors through an embedding \( \mu': I \to F^* \otimes \omega \), but \( \lambda \) does not. Thus \( h^1(X, I \otimes F) < h^1(X, I \otimes E) \). Since \( \deg F = \deg E + 1 \), then \( H^0(X, I \otimes F) = 0 \). It is clear that the cokernel of \( \mu' \) is torsion-free. The induction proof is complete. \( \square \)

**Corollary 2** The functor \( \check{J}_d \) is representable by a scheme.

**Proof.** First note that properties (i) and (ii) in the statement of Proposition 1 are open on \( I \). More precisely, given a vector bundle \( E \) on \( X \) of rank \( r \) and degree \( -rd - 1 \), the subfunctor \( U_E \subseteq \check{J}_d \), parametrizing sheaves \( I \) satisfying properties (i) and (ii) in the statement of Proposition 1, is open. By Proposition 1, the subfunctors \( U_E \) cover \( \check{J}_d \). Thus to show that \( \check{J}_d \) is representable we need only show that each \( U_E \) is representable.

Fix a vector bundle \( E \) on \( X \) of rank \( r \) and degree \( -rd - 1 \). Let

\[ V \subseteq \text{Quot}_X(E^* \otimes \omega) \]

be the open subscheme parametrizing those quotients \( q: E^* \otimes \omega \to G \) such that both \( G \) and \( \ker(q) \) are torsion-free, \( \ker(q) \) has rank 1,

\[ h^0(X, \ker(q) \otimes E) = 0 \quad \text{and} \quad h^1(X, \ker(q) \otimes E) = 1. \]

(2.1) There is a morphism of functors \( V \to U_E \) sending a quotient \([q] \in V\) to its kernel, \([\ker(q)] \in U_E \). It follows from (2.1) that the latter morphism is an isomorphism. The proof is complete. \( \square \)

We will say that a vector bundle \( E \) on \( X \) of rank \( r \) and degree \( -rd - 1 \) satisfying properties (i) and (ii) in the statement of Proposition 1 represents \( I \). We remark that the property of representing \( I \) is open.

### 3 The Theta divisor

Assume from now on that \( g > 0 \). Let \( \mathcal{I} \) be a universal relatively torsion-free, rank 1 sheaf on \( X \times \check{J}_0 \) over \( \check{J}_0 \). Denote by \( p: X \times \check{J}_0 \to \check{J}_0 \) the projection map. Define

\[ \mathcal{L} := (\det Rp_*(\mathcal{I}))^{-1}, \]

where \( \det Rp_* \) denotes the determinant of cohomology associated with the projection \( p \). (For a brief description of \( Rp_* \), see [6] or [19]; for a more in-depth development of the theory of determinants, see [12].) Since the sheaf \( \mathcal{I} \) has relative Euler
characteristic 0 over $J_0$, then $\mathcal{L}$ is independent on the choice of a universal sheaf $\mathcal{I}$, and there is a canonical global section $\theta$ of $\mathcal{L}$ whose zero scheme $\Theta$ parametrizes torsion-free, rank 1 sheaves $I$ of Euler characteristic 0 on $X$ such that

$$h^0(X, I) = h^1(X, I) \neq 0.$$  

Equivalently, by Serre’s duality, $\Theta$ consists of the torsion-free, rank 1 sheaves of Euler characteristic 0 that can be embedded into the dualizing sheaf $\omega$. In other words, $\Theta$ is (set-theoretically) the image of the $(g - 1)$-th component of the Abel-Jacobi map:

$$\mathcal{A}^{g-1}: \text{Quot}_{X}^{g-1}(\omega) \to J_0,$$

where $\mathcal{A}^{g-1}$ sends a quotient $[q] \in \text{Quot}_{X}^{g-1}(\omega)$ to its kernel, $[\ker(q)] \in J_0$ (cf. [4, p. 87].) We say that $\mathcal{L}$ is the Theta line bundle, and $\Theta$ is the Theta divisor (even though it is not known whether $\Theta$ is actually a Cartier divisor in general.)

If $X$ is smooth, then

$$\text{Quot}^{g-1}_{X}(\omega) \cong \text{Hilb}^{g-1}_{X} = \text{Symm}^{g-1}(X),$$

where $\text{Hilb}^{g-1}_{X} := \text{Quot}^{g-1}_{X}(\mathcal{O}_X)$ is the Hilbert scheme, parametrizing $(g - 1)$-uples of points in $X$, and $\text{Symm}^{g-1}(X)$ is the symmetric product of $(g - 1)$ copies of $X$. Hence $\Theta$ corresponds to the classical Theta divisor (cf. Section 1.)

Assume that $X$ is locally planar, that is, that the embedding dimension of each point of $X$ is at most 2. Equivalently, assume that $X$ can be embedded into a quasi-projective smooth surface [2]. Then $\text{Quot}^{g-1}_{X}(\omega)$ and $J_0$ are integral, local complete intersections of dimensions $g - 1$ and $g$, respectively (Since locally planar curves are Gorenstein, then $\text{Quot}^{g-1}_{X}(\omega) \cong \text{Hilb}^{g-1}_{X}$, and thus our statement follows from [1, Cor. 7 and Thm. 9].) In this case, $\Theta$ is an irreducible, local complete intersection, effective Cartier divisor on $J_0$. Moreover, it is clear that $\mathcal{A}^{g-1}$ is an isomorphism over the open subscheme of $\Theta$ parametrizing torsion-free, rank 1 sheaves $I$ with $h^1(X, I) = 1$. From [4, Prop. 3.5.ii, p. 76], this open subscheme is dense. Since $\Theta$ is Cohen-Macaulay and irreducible, and $\text{Quot}^{g-1}_{X}(\omega)$ is integral, then $\Theta$ is also integral. We observe that the assumption that $X$ is locally planar is essential in the above argument. If $X$ is not locally planar, then $J_0$ is not irreducible (cf. [11] or [17, Thm. A]), and may have dimension greater than $g$ (cf. [1, Ex. 13, p. 10].)

We observe that the above notions and arguments can be extended to families of integral, complete curves without difficulty [6]. Moreover, the formation of $\theta$ and $\mathcal{L}$ commutes with base change, since so does the determinant of cohomology. From this observation it follows that Poincaré’s formula holds for locally planar curves. Namely, we claim that, if $X$ is locally planar, then the self-intersection $\Theta^g$ is equal to $g!$. In fact, the claim is known for smooth curves [13, §2]. Since every locally planar curve is part of a family whose general member is a smooth curve, then we may apply the principle of conservation of intersection number to prove our claim.

As we have already remarked, it is not known whether $\Theta$ is always a Cartier divisor. Nevertheless, Soucaris showed that the zero scheme of the restriction of the canonical section $\theta$ to the maximum reduced subscheme of $J_0$ is a Cartier divisor [19, Thm. 8, p. 236].
4 Very ampleness

Recall the notations of Section 3. If \( E \) is a vector bundle on \( X \) with \( \deg E = 0 \), then \( \mathcal{I} \otimes E \) has relative Euler characteristic 0 over \( \bar{J}_0 \). Therefore, the invertible sheaf

\[
\mathcal{L}_E := (\det R^p (\mathcal{I} \otimes E))^{-1}
\]

on \( \bar{J}_0 \) has a canonical global section \( \theta_E \), whose zero scheme \( \Theta_E \) parametrizes torsion-free, rank 1 sheaves \( I \) of Euler characteristic 0 on \( X \) such that

\[
h^0(X, I \otimes E) = h^1(X, I \otimes E) = 0.
\]

As before, \( \mathcal{L}_E \) and \( \theta_E \) are independent on the choice of a universal sheaf \( \mathcal{I} \).

**Lemma 3** Let \( E \) and \( F \) be vector bundles on \( X \) of same rank and degree 0. If \( \det E \cong \det F \), then \( \mathcal{L}_E \cong \mathcal{L}_F \).

**Proof.** By Seshadri in [18, Lemma 2.5, p. 165]. \( \square \)

By Lemma 3, if \( E \) is a vector bundle on \( X \) of rank \( n \) and \( \det E \cong \mathcal{O}_X \), then \( \mathcal{L}_E \cong \mathcal{L}^\otimes n \). Thus we may consider \( \theta_E \) as a global section of \( \mathcal{L} \).

For every integer \( d \), let \( J^d \) be the Jacobian of \( X \), parametrizing invertible sheaves of degree \( d \) on \( X \). Recall that \( J^d \) is connected, quasi-projective and smooth.

**Lemma 4** Let \( n \geq 2 \). For each \( i = 1, \ldots, n \), let \( d_i \) be an integer and \( U_i \subseteq J^{d_i} \) be a non-empty, open subset. Let \( L \) be an invertible sheaf of degree \( d_1 + \cdots + d_n \). Then there are points \( [L_i] \in U_i \) for every \( i = 1, \ldots, n \) such that

\[
L \cong \bigotimes_{i=1}^n L_i.
\]

**Proof.** Consider the morphism \( \phi: U_1 \times \cdots \times U_{n-1} \to J^{d_n} \), given by

\[
([M_1], \ldots, [M_{n-1}]) \mapsto [L \otimes M_1^{-1} \otimes \cdots \otimes M_{n-1}^{-1}].
\]

It is clear that the image \( V \) of \( \phi \) is open in \( J^{d_n} \). Since \( J^{d_n} \) is irreducible, then \( V \cap U_n \neq \emptyset \). Thus there is a point \( [L_i] \in U_i \) for each \( i = 1, \ldots, n \) such that

\[
L_n \cong L \otimes L_1^{-1} \otimes \cdots \otimes L_{n-1}^{-1}.
\]

The proof is complete. \( \square \)

**Theorem 5** The sheaf \( \mathcal{L}^\otimes n \) is generated by global sections if \( n \geq 2 \).

**Proof.** Fix \( n \geq 2 \). Let \( I \) be a torsion-free, rank 1 sheaf on \( X \) of Euler characteristic 0. We will show that there is a vector bundle \( E \) on \( X \) of rank \( n \) and \( \det E \cong \mathcal{O}_X \) such that

\[
h^0(X, I \otimes E) = h^1(X, I \otimes E) = 0.
\]
In this case, the section $\theta_E$ generates $\mathcal{L}^\otimes n$ at $[I]$, thereby proving the theorem.

By the proof of [19, Prop. 7, p. 235], there is an invertible sheaf $L$ on $X$ of degree 0 such that

$$h^0(X, I \otimes L) = h^1(X, I \otimes L) = 0.$$ 

By semicontinuity, there is an open, dense subset $U \subseteq J^0$, containing $[L]$, such that if $[M] \in U$, then

$$h^0(X, I \otimes M) = h^1(X, I \otimes M) = 0.$$ 

From Lemma 4, with $U_i := U$ for every $i = 1, \ldots, n$, there are invertible sheaves $M_1, \ldots, M_n$ of degree 0 on $X$ such that

$$h^0(X, I \otimes M_i) = h^1(X, I \otimes M_i) = 0$$

for every $i = 1, \ldots, n$, and

$$M_1 \otimes \cdots \otimes M_n \cong \mathcal{O}_X.$$ 

If we now let $E := M_1 \oplus \cdots \oplus M_n$, then $E$ satisfies (5.1) and $\det E \cong \mathcal{O}_X$. The proof is complete. \hfill \Box

Soucaris had used [19, Prop. 7, p. 235] to show that the pullback of $\mathcal{L}^\otimes 2$ to the normalization of $\bar{J}_0$ is generated by global sections [19, Prop. 9, p. 236].

If $S$ is a $k$-scheme and $\mathcal{F}$ is a vector bundle on $X \times S$ of relative degree $d$ over $S$, then we denote by $\pi_S : S \to J^d$ the determinant morphism, mapping $s \in S$ to $[\det \mathcal{F}(s)] \in J^d$.

**Lemma 6** Let $F_1, \ldots, F_n$ be vector bundles on $X$ of same rank $r$ and same degree $d$. Then there are a connected, smooth $k$-scheme $S$ and a vector bundle $\mathcal{F}$ on $X \times S$ such that $\pi_S$ is smooth, and $F_i \cong \mathcal{F}(s_i)$ for some $s_i \in S$, for each $i = 1, \ldots, n$.

**Proof.** Let $m >> 0$ be such that $F_i(m)$ is generated by global sections for every $i = 1, \ldots, n$. Since $k$ is infinite, then there is an exact sequence of the form

$$0 \to \mathcal{O}_X(-m)^{\oplus (r-1)} \to F_i \to (\det F_i)((r-1)m) \to 0$$

for each $i = 1, \ldots, n$. Let $\mathcal{P}$ be a universal sheaf on $X \times J^d$. Let $p : X \times J^d \to J^d$ denote the projection map, and let $\mathcal{V} := R^1 p_*(\mathcal{P}(-rm))^{\oplus (r-1)}$. Choose $m >> 0$ such that $\mathcal{V}$ is locally free, and let $T := \mathcal{P}_J^d(\mathcal{V}^r)$. Since $\mathcal{V}$ is locally free, then $T$ is smooth over $J^d$. Since $J^d$ is connected, smooth and quasi-projective, then so is $T$. The scheme $T$ parametrizes $\mathcal{O}_X$-module extensions of $L((r-1)m)$ by $\mathcal{O}_X(-m)^{\oplus (r-1)}$ for invertible sheaves $L$ on $X$ of degree $d$. Thus there is $s_i \in T$ corresponding to (6.1) for each $i = 1, \ldots, n$. Since $T$ is quasi-projective, then there is an affine open subscheme $S \subseteq T$ containing $s_1, \ldots, s_n$. Since $S$ is affine, then

$$\mathcal{V}(S) = \text{Ext}_{X \times S}^1(\mathcal{P}|_{X \times S}, (r-1)m), \mathcal{O}_{X \times S}(-m)^{\oplus (r-1)}.$$ 

Let $q : \mathcal{V}_T^r \to \mathcal{Q}$ be the universal quotient on $T$ over $J^d$. Then $q$ induces an extension of the form

$$0 \to \mathcal{O}_{X \times S}(-m)^{\oplus (r-1)} \otimes \mathcal{Q} \to \mathcal{F} \to \mathcal{P}|_{X \times S}((r-1)m) \to 0$$

on $X \times S$ that specializes to (6.1) over $s_i$, for each $i = 1, \ldots, n$. By construction, $\pi_S$ is equal to the restriction to $S$ of the structure morphism $T \to J^d$. Thus $\pi_S$ is smooth. The proof is complete. \hfill \Box

Let $e_X := \max_I e(I)$, where the maximum runs over all torsion-free, rank 1 sheaves on $X$. If $S$ is a $k$-scheme, we say that an $S$-flat coherent sheaf $\mathcal{C}$ on $X \times S$ is relatively torsion-free if $\mathcal{C}(s)$ is torsion-free for every $s \in S$. 

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Theorem 7 The sheaf $L^\otimes n$ is very ample for every $n \geq \max(e_X, 2) + 1$.

Proof. Fix $n \geq \max(e_X, 2) + 1$. By Theorem 5, the sheaf $L^\otimes n$ is generated by global sections. We need only show that $L^\otimes n$ separates points and tangent vectors on $J_0$. The former is Step 1, while the latter is Step 2 below.

Step 1: Let $I_1$ and $I_2$ be non-isomorphic torsion-free, rank 1 sheaves on $X$ of Euler characteristic 0. Then there is a vector bundle $E$ on $X$ of rank $n$ and $\det E \cong O_X$ such that $\theta_E([I_1]) \neq 0$, but $\theta_E([I_2]) = 0$.

Proof of Step 1: By Proposition 1, since $n \geq \max(e_X, 2) + 1$, there is a vector bundle $F_i$ on $X$ of rank $n - 1$ and degree $-1$ representing $I_i$ for each $i = 1, 2$. From Lemma 6, since the property of representing a torsion-free, rank 1 sheaf is open, we may assume that there are a non-empty, connected, smooth $k$-scheme $S$, and a vector bundle $F$ on $X \times S$ of rank $n - 1$ and relative degree $-1$ over $S$ such that the determinant morphism $\pi_F$ is smooth, and $F(s)$ represents both $I_1$ and $I_2$ for every $s \in S$.

By replacing $S$ with an open, dense subscheme if necessary, we may assume that for each $i = 1, 2$ there is an exact sequence

$$0 \to I_i \otimes O_S \xrightarrow{\lambda_i} F^* \otimes \omega \xrightarrow{q_i} C_i \to 0$$

on $X \times S$, where $C_i$ is a relatively torsion-free sheaf over $S$. If the composition $\rho := q_2 \circ \lambda_1$ were zero over a certain $s \in S$, then $\lambda_1(s)$ would factor through $I_2$, and since $\chi(I_1) = \chi(I_2)$ we would have that $I_1 \cong I_2$. Thus $\rho: I_1 \otimes O_S \to C_2$ is an embedding with $S$-flat cokernel. Since $C_2$ is relatively torsion-free, by replacing $S$ with an open, dense subscheme if necessary, we may assume that there is a regular point $x \in X$ such that $\rho(x): I_1(x) \otimes O_S \to C_2(x)$ is an embedding with free cokernel. Let $\sigma: C_2(x) \to I_1(x) \otimes O_S$ be a splitting for $\rho(x)$. Let

$$G := (\ker(F^* \to F^*(x) \xrightarrow{q_2(x)} C_2(x) \xrightarrow{\sigma} I_1(x) \otimes O_S))^*.$$

(As in the proof of Proposition 1, we implicitly chose a trivialization of $\omega$ at $x$.) Then $G$ is a vector bundle on $X \times S$ of rank $n - 1$ and relative degree 0 over $S$.

Moreover, $\det G(s) \cong \det F(s) \otimes O_X(x)$ for every $s \in S$. Thus the determinant morphism $\pi_G$ is also smooth. In addition, $\lambda_2$ factors through $G^* \otimes \omega$, but $\lambda_1(s)$ does not factor through $G^*(s) \otimes \omega(s)$ for any $s \in S$. Thus

$$h^0(X, I_1 \otimes G(s)) = 0, \quad \text{but} \quad h^0(X, I_2 \otimes G(s)) \neq 0$$

for every $s \in S$.

By the proof of [19, Prop. 7, p. 235] (see the proof of Theorem 5), there is an open dense subset $U \subseteq J^0$ such that

$$h^0(X, I_1 \otimes L) = h^0(X, I_2 \otimes L) = 0$$

for every $[L] \in U$. By Lemma 4 applied to $U_1 := \pi_G(S)$ and $U_2 := U$, there are $s \in S$ and $[L] \in U$ such that

$$(\det G(s)) \otimes L \cong O_X.$$

It is clear that $E := G(s) \oplus L$ satisfies $\det E \cong O_X$ and

$$h^0(X, I_1 \otimes E) = 0, \quad \text{but} \quad h^0(X, I_2 \otimes E) \neq 0.$$

Thus $\theta_E([I_1]) \neq 0$, but $\theta_E([I_2]) = 0$. The proof of Step 1 is complete. \qed
Step 2: Let $I$ be a torsion-free, rank 1 sheaf on $X$ with $\chi(I) = 0$. Let $v \in T_{\overline{J}_0}([I])$ be a non-zero tangent vector on $\overline{J}_0$ at $[I]$. Then there is a vector bundle $E$ on $X$ of rank $n$ and $\det E \cong \mathcal{O}_X$ such that $\Theta_E$ contains $[I]$ but not $v$.

Proof of Step 2: As in Step 1, we may assume that there are a non-empty, connected, smooth $k$-scheme $S$, and a vector bundle $\mathcal{F}$ on $X \times S$ of rank $n - 1$ and relative degree $-1$ over $S$, such that the determinant morphism $\pi_\mathcal{F}$ is smooth, and $\mathcal{F}(s)$ represents $I$ for every $s \in S$.

By replacing $S$ with an open, dense subscheme if necessary, we may assume that there is an exact sequence

$$0 \to I \otimes \mathcal{O}_S \xrightarrow{\lambda} \mathcal{F}^* \otimes \omega \xrightarrow{\delta} \mathcal{C} \to 0$$

on $X \times S$, where $\mathcal{C}$ is a relatively torsion-free sheaf over $S$. By the proof of Corollary 2, we have natural identifications

$$(7.1) \quad T_{\overline{J}_0}([I]) = T_{Q(s),[q(s)]} = \text{Hom}_X(I, \mathcal{C}(s))$$

for every $s \in S$, where $Q(s) := \text{Quot}_X(\mathcal{F}^*(s) \otimes \omega)$. So there is a homomorphism $\nu: I \otimes \mathcal{O}_S \to \mathcal{C}$ such that $\nu(s) = v$ under the identification (7.1) for every $s \in S$. Since $v \neq 0$, then $\nu$ is an embedding with $S$-flat cokernel. Since $\mathcal{C}$ is relatively torsion-free, by replacing $S$ with an open, dense subscheme if necessary, there is a regular point $x \in X$ such that $\nu(x)$ is an embedding with free cokernel. Let $\sigma: \mathcal{C}(x) \to I(x) \otimes \mathcal{O}_S$ be a splitting for $\nu(x)$. Let

$$\mathcal{G} := (\ker(\mathcal{F}^* \to \mathcal{F}^*(x) \xrightarrow{q(x)} \mathcal{C}(x) \xrightarrow{\sigma} I(x) \otimes \mathcal{O}_S))^*.$$ 

(As in the proof of Step 1, we implicitly chose a trivialization of $\omega$ at $x$.) Then $\mathcal{G}$ is a vector bundle on $X \times S$ of rank $n - 1$ and relative degree 0 over $S$. Moreover, $\det \mathcal{G}(s) \cong \det \mathcal{F}(s) \otimes \mathcal{O}_X(x)$ for every $s \in S$. Thus the determinant morphism $\pi_{\mathcal{G}}$ is smooth. In addition, $\lambda$ factors through $\mathcal{G}^* \otimes \omega$. Thus $[I] \in \Theta_{\mathcal{G}(s)}$ for every $s \in S$. On the other hand, since $\nu(x)$ is an embedding, then $\nu$ does not belong to $\Theta_{\mathcal{G}(s)}$ for any $s \in S$.

The reader is invited to repeat the argument in the last paragraph of the proof of Step 1 to finish the proof of Step 2. The proof of Theorem 7 is complete. \[\square\]

Remark 8. Let $x \in X$. Let $\overline{\mathcal{O}}_x$ denote the normalization of $\mathcal{O}_x$. Let $\delta_x$ denote the length of $\overline{\mathcal{O}}_x/\mathcal{O}_x$. If $I$ is a torsion-free, rank 1 module over $\overline{\mathcal{O}}_x$, then it is easy to show that $I$ is isomorphic to a submodule of $\overline{\mathcal{O}}_x$ containing $\mathcal{O}_x$. Thus

$$(8.1) \quad \dim_k I(x) \leq \delta_x + 1.$$ 

If the conductor, $\mathcal{C}_x := (\mathcal{O}_x : \overline{\mathcal{O}}_x)$, is the maximal ideal $m_x$ of $\mathcal{O}_x$, then equality in (8.1) is achieved for $I = \overline{\mathcal{O}}_x$ only; otherwise the inequality (8.1) is always strict. Let

$$\delta_X := \max_{x \in X} \delta_x.$$ 

Since $X$ is generically non-singular, then $\delta_X < \infty$. It follows from (8.1) that $e_X \leq \delta_X + 1$.

Theorem 7 states that $\mathcal{L}^\otimes 3$ is very ample if $e_X \leq 2$. This is the case for $X$ non-singular, or with at most ordinary nodes or cusps as singularities, as $\delta_X \leq 1$. It is clear that if $e_X \leq 2$ then $X$ is locally planar. If $\delta_X = 2$, then $e_X < 2$ if and only if $X$ is locally planar. If $\delta_X = 3$, then $e_X \leq 2$ if and only if $X$ is locally planar and $m^2_x \neq \mathcal{C}_x$ for every $x \in X$. Note that the planar curve $X \subseteq \mathbb{P}_k^2$, given as the zero scheme of $\widetilde{\psi}_x \cdots \tilde{\psi}_1$, has $e_X = 3$. 
Question 9. It follows from the proof of the theorem in [14, Section 17, p. 163] that, if $X$ is smooth, then $3\Theta$ is very ample, and the sections $\theta_E$ associated to completely decomposable vector bundles $E$ (that is: vector bundles $E$ of the form $L_1 \oplus L_2 \oplus L_3$, where $L_i$ is an invertible sheaf of degree 0 for $i = 1, 2, 3$, and $L_1 \otimes L_2 \otimes L_3 \cong \mathcal{O}_X$), are enough to embed $J_0$ into a projective space. We might ask: for which integral curves $X$ are such sections enough to embed $\bar{J}_0$ into a projective space? The proof of Theorem 7 shows that the sections $\theta_E$ associated to vector bundles $E$ of the form $F \oplus L$, where $F$ is a vector bundle of rank $\max(e_X, 2)$ and degree 0, the sheaf $L$ is invertible of degree 0 and $(\det F) \otimes L \cong \mathcal{O}_X$, are enough to embed $\bar{J}_0$ into a projective space.

Example 10. Let $X$ be a complete, integral curve of arithmetic genus $g = 1$. As a subset, $\Theta$ is the locus of torsion-free, rank 1 sheaves $I$ with Euler characteristic 0 such that $h^0(X, I) > 0$. Since $\chi(\mathcal{O}_X) = 0$, then any non-zero section $\mathcal{O}_X \to I$ must be an isomorphism. Since $\Theta$ is integral by Subsection 3, then $\Theta = [\mathcal{O}_X]$, as Cartier divisors of $\bar{J}_0$.

By [3, Ex. 8.9.iii, p. 109], the first component of the Abel-Jacobi map,

$$A^1: X \to \bar{J}_{-1}$$

$$x \mapsto [m_x],$$

where $m_x$ denotes the maximal ideal sheaf of $x$, is an isomorphism. Fix a regular point $x \in X$. Then we have an isomorphism $\phi_x: \bar{J}_{-1} \to \bar{J}_0$, by sending $[I] \in \bar{J}_{-1}$ to $[I(x)] \in \bar{J}_0$. Under the composition $\psi := \phi_x \circ A^1$, the Cartier divisor $\Theta$ corresponds to the Cartier divisor $[x]$ in $X$.

Let $n \geq 3$ be an integer. The complete linear system associated to $\mathcal{O}_X(nx)$ gives rise to an embedding $X \hookrightarrow \mathbb{P}^{n-1}$. If $H \subseteq \mathbb{P}^{n-1}$ is a hyperplane intersecting $X$ at regular points $y_1, \ldots, y_n$, then $[y_1] + \cdots + [y_n]$ is a Cartier divisor on $X$ whose associated invertible sheaf is $\mathcal{O}_X(nx)$. Under $\psi$, the divisor $[y_1] + \cdots + [y_n]$ corresponds to $\Theta_E$, where

$$E = (\mathcal{O}_X(y_1) \oplus \cdots \oplus \mathcal{O}_X(y_n)) \otimes \mathcal{O}_X(-x).$$

It follows now from Bertini’s theorem that the theta sections of degree $n$ associated to completely decomposable vector bundles generate $H^0(\bar{J}_0, \mathcal{L}^\otimes n)$ for every $n \geq 0$. (In case $n \leq 2$ it is easy to check the latter statement directly.) Thus, for the case of curves of arithmetic genus 1, Question 9 is answered in the affirmative.

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References

1. Altman, A., Iarrobino, A., Kleiman, S.: Irreducibility of the compactified Jacobian. In: Real and complex singularities, Oslo 1976 (Proc. 9th nordic summer school, pp. 1–12) Sijthoff and Noordhoff 1977.
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2. Altman, A., Kleiman, S.: Bertini theorems for hypersurface sections containing a subscheme. Commun. Algebra 7 (8), 775–790 (1979).
3. Altman, A., Kleiman, S.: Compactifying the Picard scheme II. Am. J. Math. 101, 10–41 (1979).
4. Altman, A., Kleiman, S.: Compactifying the Picard scheme. Adv. Math. 35, 50–112 (1980).
5. D’Souza, C.: Compactification of generalized Jacobian. Proc. Indian Acad. Sci. Sect. A, Math. Sci. 88 no. 5, 419–457 (1979).
6. Esteves, E.: The presentation functor and Weierstrass theory for families of local complete intersection curves. M.I.T. Ph.D. thesis 1994.
7. Esteves, E.: Separation properties of theta functions. Available from the e-print service at alg-geom@eprints.math.duke.edu, September 1997.
8. Esteves, E.: Compactifying the relative Jacobian over families of reduced curves. Available from the e-print service at alg-geom@eprints.math.duke.edu, September 1997.
9. Faltings, G.: Stable G-bundles and projective connections. J. Algebr. Geom. 2, 507–568 (1993).
10. Grothendieck, A.: Techniques de construction et théorèmes d’existence en géométrie algébrique IV: les schemas de Hilbert (Séminaire Bourbaki, vol. 221) 1961.
11. Kleiman, S., Kleppe, H.: Reducibility of the compactified Jacobian. Compos. Math. 43, 277–280 (1981).
12. Knudsen, F., Mumford, D.: The projectivity of the moduli space of stable curves I. Math. Scand. 39, 19–55 (1976).
13. Mattuck, A.: On symmetric products of curves. Proc. Am. Math. Soc. 13, 82–87 (1962).
14. Mumford, D.: Abelian varieties, Oxford University Press 1970.
15. Mumford, D.: On the equations defining Abelian varieties. I. Invent. Math. 1, 287–354 (1966).
16. Mumford, D.: On the equations defining Abelian varieties. II and III. Invent. Math. 3, 75–135 and 215–244 (1967).
17. Rego, C.: The compactified Jacobian. Ann. Sci. École Norm. Sup. 13, 211–223 (1980).
18. Seshadri, C.S.: Vector bundles on curves. In: R.S. Elman et al.: Linear Algebraic Groups and Their Representations, Los Angeles, California 1992 (Contemp. Math., vol. 153, pp. 163–200) Providence, RI: American Mathematical Society 1993.
19. Soucaris, A.: The ampleness of the theta divisor on the compactified Jacobian of a proper and integral curve. Compos. Math. 93, 231–242 (1994).