AVERAGES OF FOURIER COEFFICIENTS OF SIEGEL MODULAR FORMS AND REPRESENTATION OF BINARY QUADRATIC FORMS BY QUADRATIC FORMS IN FOUR VARIABLES

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To the memory of Hiroshi Saito

Abstract. Let $-d$ be a a negative discriminant and let $T$ vary over a set of representatives of the integral equivalence classes of integral binary quadratic forms of discriminant $-d$. We prove an asymptotic formula for $d \to \infty$ for the average over $T$ of the number of representations of $T$ by an integral positive definite quaternary quadratic form and obtain bounds for averages of Fourier coefficients of linear combinations of Siegel theta series. We also find an asymptotic estimate from below on the number of binary forms of fixed discriminant $-d$ which are represented by a given quaternary form. In particular, we can show that for growing $d$ a positive proportion of the binary quadratic forms of discriminant $-d$ is represented by the given quaternary quadratic form.

1. Introduction.

If $A$ is an integral positive definite symmetric $(m \times m)$ matrix one calls a solution $X \in M(m \times n, \mathbb{Z})$ of the system of quadratic equations of the form

$$Q_A(X) = ^tXAX = T$$

where $T$ is another (half-) integral symmetric matrix of size $n \leq m$ an (integral) representation of $T$ by $S$. It is well known that the local global principle of Minkowski and Hasse is not valid for integral representations, but if $m$ is large enough compared to $n$ one can prove that a positive definite $T$ which is represented by $S$ over all $\mathbb{Z}_p$ and is large enough in a suitable sense is indeed represented by $S$ over the rational integers $\mathbb{Z}$, at least under some mild additional conditions. The bound on the size of $m$ necessary for this has recently been pushed down to $m \geq n + 3$, again under suitable additional conditions, in [10], see also [27] for an attempt to optimize those additional conditions. The case $m = n + 2$ brings some limitations due to the existence of the so called spinor exceptions (see [16] [14]), taking these into account a result of the desired type could be reached in [6] for $n = 1, m = 3$, i.e. for representations of sufficiently large numbers by ternary forms. An analogous result for $n = 2$ asserting that all binary quadratic forms of sufficiently large determinant (and perhaps large minimum) are represented by a given quaternary quadratic form if they are represented locally everywhere appears to be out of reach at the moment. However, Einsiedler, Lindenstrauss, Michel, and Venkatesh obtained recently a result (not yet published) for a certain subset of $T$ by adapting Linnik’s ergodic method, which was invented to deal with the ternary case, to the problem of representation of binary forms by quaternary forms. They also obtained a result on representation of $T$ “on average” under the assumption of the so called Linnik condition.

MSC 2000: Primary 11E12, Secondary 11F27 11F30 11F46 11E45.
In this note we prove similar, but somewhat different average results starting out from work of Böcherer and the author \cite{3,4} on Siegel theta series of quaternary positive definite quadratic forms (or quadratic lattices) attached to ideals in Eichler orders of square free level in definite quaternion algebras. For such a quaternary quadratic lattice \( \Lambda \) we proved there among others that the average \( r_{av}(\Lambda, d) \) over binary symmetric matrices \( T \) of fixed discriminant \(-d\) of the representation numbers \( r(\Lambda, T) \) can be related to the product of the representation numbers of \( d \) by two attached ternary lattices \( L, L' \).

In the present article we show first that the results of \cite{3,4} can be extended to a more general situation than treated there. Whereas in those articles we had to restrict attention to quaternary quadratic forms of square free level \( N \) and determinant \( N^2 \), we can now allow more general levels in the case of square determinant and can also treat forms of prime determinant and certain forms of square free determinant \( \Delta \). In these latter cases the role of the pair of ternary \( \mathbb{Z} \)-lattices in \cite{3,4} is taken over by one ternary lattice over the integers of \( \mathbb{Q}(\sqrt{\Delta}) \). Taken together the results now cover in particular all maximal quaternary quadratic lattices of prime level.

An important role in these generalizations is played by ideas of Hiroshi Saito which he explained to Böcherer and me in a letter in 2001; they relate our work to Hijikata’s theory of optimal embeddings of quadratic orders.

We show then that the formulas for averaged representation numbers imply almost immediately asymptotic formulas and estimates for the \( r_{av}(\Lambda, d) \) and hence also for averages of Fourier coefficients of linear combinations of Siegel theta series of these forms (Yoshida liftings). If such a linear combination is a cusp form we obtain upper bounds for the averaged Fourier coefficients which appear to be new and are sharper than what can be deduced from the known estimates for the individual Fourier coefficients. As another consequence of the asymptotic formula for the average representation number \( r_{av}(\Lambda, d) \) we can derive a lower bound for the number of binary quadratic forms of fixed discriminant which are represented by a given integral quaternary quadratic form, in particular, we can show that a positive proportion of these binary forms is represented.

I heard of the results of Einsiedler, Lindenstrauss, Michel, and Venkatesh in the talk of Michel at the International Colloquium on Automorphic Representations and \( L \)-functions 2012 at the Tata Institute of Fundamental Research, Mumbai, where a first version of this note was also written during a stay following that colloquium. I thank the institute and in particular Dipendra Prasad for their hospitality.

2. Quaternary lattices and quaternion algebras

Following the work of Brandt and Eichler (see \cite{9}) on the connection between quadratic spaces of dimension 4 and their orthogonal groups on one side and quaternion algebras on the other side, Ponomarev has studied in \cite{22} in detail the correspondence between quaternary quadratic lattices and orders and ideals in quaternion algebras. Using these results we will consider the following two situations.

Case A: Square discriminant

Let \( D \) be a quaternion algebra over \( \mathbb{Q} \) with reduced norm \( n \) and reduced trace \( \text{tr} \), view \( D \) as a quadratic space with quadratic form \( n \) and associated symmetric bilinear form \( b(x, y) = \text{tr}(xy) \), where \( y \mapsto \bar{y} \) is the usual involution (quaternionic conjugation) on \( D \). We assume \( D \) to be ramified at \( \infty \) and denote by \( N_0 \) the product of the finite primes \( p \) at which \( D \) is ramified (i.e., \( D \otimes \mathbb{Q}_p \) is a division algebra). The special orthogonal group \( SO(D, n) \) of the quadratic space \( (D, n) \) is then isomorphic to \( \{ (x, y) \in D^* \times D^* \mid n(x) = n(y) \} / Z(D^*) \), where the center \( Z(D^*) = \mathbb{Q}^* \) is embedded diagonally into \( D^* \times D^* \); an isomorphism is given by sending the class of \( (x, y) \) to the map \( \sigma_{x,y} \) given by \( \sigma_{x,y}(d) = xdy^{-1} \). This
isomorphism extends naturally to the $v$-adic completions for a valuation $v = v_p$ of $\mathbb{Q}$ and to the adelicization. Let $R$ be an order (of full rank) in $D$ and consider the decomposition

$$D_h^\times = \bigcup_{j=1}^h D_q^\times y_j R_h^\times$$

of the adelicization $D_h^\times$ into double cosets, where $R_h = D_\infty \times \prod_p R_p$ is the adelicization of the order $R$ and where $R_p = R \otimes \mathbb{Z}_p$. The $y_j$ may be chosen to be of norm 1 if the $R_p$ have $\mathbb{Z}_p^\times$ contained in their norm groups, which will be the case in the sequel. The ideals $g_iRy_j^{-1} =: I_{ij}$ with left order $R_i = y_iRy_j^{-1}$ and right order $R_j = y_jRy_j^{-1}$ represent then all isometry classes of lattices in the genus of $(R, n)$, with some classes possibly occurring more than once.

Let $D$ vary over the quaternion algebras over $\mathbb{Q}$ which are ramified at $\infty$ and let $R \subseteq D$ vary over the orders in $D$ such that $R_p$ contains the maximal order of the unramified quadratic extension of $\mathbb{Q}_p$ if $p$ ramifies in $D$ and is an intersection of at most two maximal orders in $D_h$ if $p$ splits in $D$. We obtain then in this way representatives of all genera of quaternionic lattices $(L, g)$ of square discriminant with $q(L)\mathbb{Z} = \mathbb{Z}$, for which the Jordan splitting at the prime $p$ consists of a binary (even) unimodular and a binary (even) $p^r$-modular component for some $r_p$ for all finite primes $p$. These orders are Eichler orders of level $p^r$ at the split primes $p$ and orders of level $p^r$ with odd $r_p$ in the terminology of [21] for the ramified primes. We write $N_1$ for the product of these $p$-adic levels for the ramified primes, $N_2$ for the product of the $p$-adic levels for the $p$ at which $D$ splits and $N = N_1N_2$ for the global level of the order in question.

We will restrict our attention to these orders and lattices in the sequel. We further denote for $s\|N$ (i.e., $s \mid N$ with gcd$(s, N/s) = 1$) by $I_{ijs}^\star$ the lattice obtained as $I_{ijs}^\star \cap \mathbb{Z}[\frac{1}{s}]I_{ij}$, where $\#$ indicates taking the dual lattice, and by $I_{ijs}^\star$ the same lattice with the quadratic forms scaled by the factor $s$. We notice that this last lattice is in the same genus as the $I_{ij}$ and is hence isometric to some $I_{k'j'}$. We call the $I_{ijs}^\star$ the rescaled partial duals of $I_{ij}$ and write $\text{rd}(I_{ij}, N')$ for the set of all $I_{ijs}^\star$ with $s\|N'$, where $N'\|N$ is some fixed exact divisor of $N$.

For any lattice $\Lambda$ with positive definite quadratic form $q$ and symmetric bilinear form $b$ and a positive semidefinite matrix $T = (t_{kl})_{k,l=1,2} \in M^{\text{sym}}(2, \mathbb{Z})$ we write

$$r(\Lambda, T) := \{|(x, y) \in \Lambda \mid \begin{pmatrix} 2q(x) & b(x, y) \\ b(x, y) & 2q(y) \end{pmatrix} = T\}|$$

for the number of representations of $T$ by the quadratic lattice $(\Lambda, q)$ and $r^*(\Lambda, T)$ for the number of primitive representations, i.e., representations $(x, y)$ where $\mathbb{Z}x + \mathbb{Z}y$ is a direct (but not necessarily orthogonal) summand of $\Lambda$ as a $\mathbb{Z}$-module. This notation will in particular be applied to the lattices $I_{ij}, I_{ij}^\star$ with the norm form $n$ on them. We write

$$L_i := \{x \in \mathbb{Z} \cdot 1 + 2R_i \mid \text{tr}(x) = 0\}$$

and $r(L_i, t) = \{|x \in L_i \mid n(x_i) = t\}$ for a positive integer $t$. The quadratic lattice $(L_i, n)$ has level $4N$ and discriminant $32N^2$. It is easily checked that for a fixed decomposition $N = N_1N_2$ all the $L_i$ are in the same genus of quadratic lattices.

**Case B: Non-square discriminant**

Let $D$ be a quaternion algebra over $\mathbb{Q}$ ramified at $\infty$ and let $K = \mathbb{Q}(\sqrt{\Delta})$ be a real quadratic field of discriminant $\Delta$. The quaternion algebra $D_K = D \otimes K$ over $K$ is then ramified at the two infinite places of $K$ and at all those finite places of $K$ lying over a prime $p \in \mathbb{N}$ which splits in $K/\mathbb{Q}$ and for which $D$ is ramified at $p$. We denote by $x \mapsto \hat{x}$ the extension of the standard involution on $D$ to $D_K$ and
by $\tau$ the extension of the non-trivial Galois automorphism of $K/\mathbb{Q}$ to $D_K$ (with $\tau(d \otimes a) = d \otimes \tau(a)$).

Then

$$V := \{ x \in D_K \mid \tau(x) = x \},$$

equipped with the norm form, is a quaternary quadratic space over $\mathbb{Q}$ of determinant $\Delta$, its special orthogonal group is isomorphic to

$$\{ \alpha \in D_K \mid n(\alpha) \in \mathbb{Q}^\times \}/\mathbb{Q}^\times,$$

where the class of $\alpha$ acts via $x \mapsto \alpha x \tau(\alpha)^{-1}$. Varying $D$ and $\Delta$ one obtains all positive definite quaternary quadratic spaces over $\mathbb{Q}$ of non-square discriminant.

For later use we notice that for a prime $p$ ramified in $K/\mathbb{Q}$ the Witt invariant of $V$ (see e. g. [20, Ch. V, §3]) is 1 if $D$ splits at $p$, $-1$ if $D$ is ramified at $p$.

An order $R$ in $D_K$ is called symmetric if $\tau(R) = R$, in that case $R \cap V =: \Lambda$ is a $\mathbb{Z}$-lattice of rank 4 on $V$, and the genus of $\Lambda$ consists of the isometry classes of the lattices $y\Lambda \tau(y)^{-1} = yR \tau(y)^{-1} \cap V$ for $y \in D_K^1 := \{ y \in D_K^1 \mid n(y) \in \mathbb{Q}^\times \}$, where the extension of $\tau$ to the adeles of $K$ interchanges the components for the two places of $K$ above a rational prime which splits in $D$ and $\Delta$ one obtains all classes in the genus of $\Lambda$ (not necessarily only once). The $y_i$, having norm in $\mathbb{Q}_K^\times$, can be chosen to have norm 1 for the orders considered below.

We notice that $I_i := y_i R \tau(y_i)^{-1}$ is an ideal with left order $R_i = y_i R y_i^{-1}$ and right order $\tau(R_i)$.

The genus of maximal lattices on $V$ is obtained by taking for $R_{(p)}$ a symmetric Eichler order of level $(p)$ if $D$ is ramified at the rational prime $p$ and $p$ is inert in $K/\mathbb{Q}$, and a (symmetric) maximal order of $(D_K)_v$ at all other finite places $v$ of $K$. The determinant of these maximal lattices is then $\Delta N^2$, where $N$ is the product of all finite primes which ramify in $D$ but not in $K/\mathbb{Q}$. We will restrict later to the case where this determinant is square free, odd and congruent to 1 modulo 4.

3. Averages of representation numbers

With notations as above and some positive definite quadratic lattice $\Lambda$ we write for a discriminant $-d$

$$r_{av}(\Lambda, d) = \sum_T r(\Lambda, T)/\varepsilon(T),$$

where the sum is over a set of $SL_2(\mathbb{Z})$-equivalence classes of integral symmetric matrices $T = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ of (signed) discriminant $b^2 - 4ac = -d$ and where $\varepsilon(T)$ denotes the number of proper automorphisms (or units) of $T$, i. e. the number of $U \in SL_2(\mathbb{Z})$ with $^tUTU = T$. The primitive average $r_{av}(\Lambda, d)$ is defined analogously by using the primitive representation numbers $r^*(\Lambda, T)$.

In [3, 4] we related the average representation numbers $r_{av}(I_{ij}, d)$ for the (proper) ideals $I_{ij}$ of Eichler orders of square free level to sums of products $r(L_i, d)r(L_j, d)$ of representation numbers of the associated ternary lattices introduced in case $\Lambda$ in the previous section. Since we want to establish a similar relation in a more general context we recall the setup.

With $D, R \subseteq D$ as in Section 2, case $\Lambda$ let $k_1 \subseteq D$ be an imaginary quadratic field and let $k_2 \subseteq D$ be isomorphic to $k_1$. The set

$$\{ z \in D \mid zk_1z \subseteq k_2 \}$$
is then the orthogonal sum $U_1 \perp U_2$ of two two-dimensional subspaces $U_1, U_2$ of $D$ with $k_1U_\nu = U_\nu = U_\nu k_2$ for $\nu = 1, 2$, and we can associate to the optimally embedded quadratic suborders $\sigma_1 = k_1 \cap R$ and $\sigma_2 = k_2 \cap R$ of $R$ the primitive
binary sublattices $M_\nu = U_\nu \cap R (\nu = 1, 2)$ of $R$, which have a quadratic left order $\mathfrak{o}_1$ and right order $\mathfrak{o}_2$.

More generally, with the $R_i, I_{ij}$ as defined above we have for $1 \leq i, j \leq h$ the binary lattices $M_{\nu} = U_\nu \cap I_{ij}$ with associated quadratic left order $\mathfrak{o}_1 = k_1 \cap R_i$ and quadratic right order $\mathfrak{o}_2 = k_2 \cap R_j$ contained in the left order $R_i$ resp. in the right order $R_j$ of the ideal $I_{ij}$. Conversely, given a primitive binary sublattice $M \subseteq I_{ij}$ we obtain its optimally embedded (into $R_i$ resp. $R_j$) quadratic left and right orders. It is then easy to see that the determinant of $M$ and the determinants of its associated quadratic orders, viewed as binary quadratic lattices, generate the same rational square class. Moreover, considering the ternary lattices $L_i \subseteq \mathcal{R}_i$ defined in Section 2 we showed that one has a bijection between $\{ x \in L_i \text{ primitive } | n(x) = d \} / \{ \pm 1 \}$ and optimally embedded quadratic suborders of discriminant $-d$ in $R_i$, so that we obtain in summary a correspondence between pairs of lines in $L_i, L_j$ on one side and (pairs of) primitive binary sublattices of $I_{ij}$. So far this construction does not depend on the level of the quaternion orders involved.

A local computation then showed in the case of square free level that the discriminant of the completion $(M_\nu)_p$ at a prime $p$ is equal to the lcm of the discriminants of the associated quadratic orders, unless $p$ divides the level $N$ and both discriminants are units at $p$, in which case one of the $(M_\nu)_p$ has the same discriminant as the quadratic orders, while for the other one this discriminant is multiplied by $p^2$.

A partial generalization of this is:

**Proposition 1.** Let $R$ be an order in $D$ such that the completion $R_p$ is

- an order of level $p^\nu$ with $r_p$ odd (see [21]) for the primes $p$ at which $D$ is ramified
- an Eichler order of level $p^\nu$ for the primes $p$ at which $D$ splits,

with $r_p = 0$ for almost all $p$. Let $N_1, N_2, N$ be as in Section 2, case A.

Let primitive vectors $x \in L_i, x' \in L_j$ be given with $d = n(x), d' = n(x')$ in the same rational square class and $p \nmid d \cdot d'$ for all $p \mid N$ with $r_p > 1$, let $M_1, M_2$ be the two primitive binary sublattices of $I_{ij}$ described above.

Then

\[
\begin{align*}
\text{disc}(M_1) &= -\text{lcm}(d, d')s_1^2 \\
\text{disc}(M_2) &= -\text{lcm}(d, d')s_2^2,
\end{align*}
\]

where $s_1, s_2$ are relatively prime, $s_1 \mid N, s_2 \mid N$ and $p \mid N$ divides $s_1 s_2$ if and only if $p \mid d \cdot d'$.

**Proof.** This has been proven for square free $N$ in [3, 4]. It remains to do the local computations for a prime $p$ with $p^2 \mid N$ and $p \nmid d \cdot d'$. As in [3, 4] we may assume $R_i = R_j = R = I_{ij}$.

If $p \mid N_1$ we have $R_p \subseteq R_p^{\text{max}}$, the unique maximal order in the division algebra $D_p$, and $R_p$ as a quadratic lattice is the orthogonal sum of a binary unimodular anisotropic sublattice $K_1$, isometric to the maximal order of the unique unramified quadratic extension of $Q_p$ with its norm form as quadratic form, and a $p^\nu$-scaled copy $K_2$ of this binary lattice. In particular, the quadratic suborders $\mathfrak{o}, \mathfrak{o}'$ of $R_p$ associated to $x, x'$ are isomorphic to this unramified quadratic maximal order.

By Lemma 6 b) of [3], we have $z_1 \in \mathcal{R}_p^{\text{max}}$, with $z_1^{-1} \mathfrak{o} z_1 = \mathfrak{o}'$. The orthogonal complement $K_1$ of $\mathfrak{o}_p$ in $R_p$ is then mapped by conjugation with $z_1$ to a $p^\nu$-modular (hence $p^\nu$-$\mathcal{R}_p$-maximal) lattice on the anisotropic subspace orthogonal to $\mathfrak{o}_p$. Since the $p^\nu$-maximal lattice on an anisotropic space is unique, we have $z_1^{-1} K_1 z_1 = K_1'$, where $K_1'$ is the orthogonal complement in $R_p$ of $\mathfrak{o}'$. In particular, we have $z_1^{-1} R_p z_1 = R_p$, which by Theorem 4.2 of [13] implies $z_1 \in R_p^{\text{max}}$. This gives $(M_1)_p = \mathfrak{o}_p z_1$, so $(M_1)_p$ is unimodular and can be split off orthogonally in $R_p$. 

The other lattice \((M_2)_p\) is then a primitive lattice on the space complementary to \((M_1)_p\), hence isometric to a \(p^n\)-scaled copy of \((M_1)_p\). For \(p \mid N_2\) we can copy the proof of [3] Proposition 2, with the only difference that the index of \(R_p^1\) in the two maximal orders \(R_p^0, R_p^1\) is now not \(p\) but \(p^n\), and similarly the index of the ternary lattices occurring there changes from \(p\) to \(p^n\). One obtains as there that one can find a \(z \in R^\times\) with \(z^{-1}oz = o'\), and as above we see that one of \((M_1)_p, (M_2)_p\) has discriminant \(-n(x)\) and the other one \(-p^{2n} \cdot n(x)\). 

As in [3] Theorem 2.1 we obtain from this:

**Theorem 2.** Let \(-d\) be a discriminant which is not divisible by any \(p \mid N\) with \(r_p > 1\) and write \(N_d := \frac{N}{\gcd(N, d)}\). Then

\[
\sum_{s \mid N_d} \sum_{m^2 \mid d} r_{av}(I_{ij}^{s}, \frac{d}{m^2}) = r(L_i, d) r(L_j, d)
\]

or equivalently

\[
\sum_{s \mid N_d} r_{av}(I_{ij}^{s}, d) = \sum_{m \in \mathbb{N}, m^2 \mid d} \mu(m) r(L_i, \frac{d}{m^2}) r(L_j, \frac{d}{m^2}).
\]

If \(N_d\) is a prime power this simplifies to

\[
\sum_{m^2 \mid d} r_{av}(I_{ij}, \frac{d}{m^2}) = \begin{cases} r(L_i, d) r(L_j, d) & N_d = 1 \\ \frac{1}{2} r(L_i, d) r(L_j, d) & N_d \neq 1 \end{cases}
\]

respectively

\[
r_{av}(I_{ij}, d) = \begin{cases} \sum_{m \in \mathbb{N}, m^2 \mid d} \mu(m) r(L_i, \frac{d}{m^2}) r(L_j, \frac{d}{m^2}) & N_d = 1 \\ \frac{1}{2} \sum_{m \in \mathbb{N}, m^2 \mid d} \mu(m) r(L_i, \frac{d}{m^2}) r(L_j, \frac{d}{m^2}) & N_d \neq 1 \end{cases}.
\]

In particular, if in addition \(-d\) is fundamental the sum over \(m^2 \mid d\) can be omitted in the above formulas.

Before treating the case of non-square discriminant (case B) we rephrase the setting of case A since this rephrasing facilitates the proof in case B:

Firstly, as has been pointed out to us by Hiroshi Saito [24], our results from [3, 4] (and their extension above) can be viewed using Hijikata’s theory of embeddings of quadratic orders into quaternion orders [12]. For this, let \(-d\) as above be a negative discriminant and \(f = X^2 - sX + n \in \mathbb{Z}[X]\) with \(s^2 - 4n = -d\). Then by associating to \(x \in L_i\) with \(n(x) = d\) the linear map \(\varphi : k := \mathbb{Q}[X]/(f) \to D\) given by \(\varphi(1) = 1, \varphi(X + (f)) = \frac{s + X}{2}\), one obtains a one to one correspondence between the \(x \in L_i\) with \(n(x) = d\) and the ring homomorphisms \(\varphi : k_f \to D\) with \(\varphi(X + (f)) \in R_i\). In this correspondence, primitive vectors \(x\) of norm \(d\) correspond to optimal embeddings of the quadratic order of discriminant \(-d\) in \(k_f\) into \(R_i\).

Given two such embeddings \(\varphi_1, \varphi_2\) there exists \(z \in D^\times\) with \(z^{-1} \varphi_1 z = \varphi_2\), and the space

\[
U(\varphi_1, \varphi_2) = \varphi_1(k_f) z = z \varphi_2(k_f)
\]

is independent of the choice of \(z\). Moreover, \(\varphi_2\) given by \(\varphi_2(X + (f)) = \frac{s - X}{2}\) has the same image \(\varphi_2(k_f) = \frac{-s}{2\varphi_2(k_f)}\) as \(\varphi_2\), and the spaces \(U_1, U_2\) described above are (with some numbering) the same as \(U(\varphi_1, \varphi_2), U(\varphi_1, \varphi_2)\). In addition, we have \(U(\varphi_1, \varphi_2) = U(\varphi_1, \varphi_2), U(\varphi_1, \varphi_2) = U(\varphi_1, \varphi_2)\), so that we obtain a two to one correspondence between pairs of embeddings of orders in \(k_f\) into quaternion orders \(R_i, R_j\), at least one of which is optimal, on one side and primitive binary sublattices.
\( M \) of \( I_{ij} \) on the other side. Our arguments from \([3, 4]\) can then be replaced using results from \([12]\).

To combine this setup with the results of Ponomarev \([22]\) for the treatment of case B we notice that in case A the second Clifford algebra \( C^+(D) \) of the quadratic space \((D, n)\) can be identified with \( D \oplus D \cong D \oplus K \) with \( K = \mathbb{Q} \oplus \mathbb{Q} \) in the way described in \([22]\) and that the quadratic space \((D, n)\) is then retrieved as being isometric to

\[
\bar{D} := \{ \alpha \in C^+(D) \mid \bar{\alpha}^r = \alpha \},
\]

where \( \tau \) is the involution on \( C^+(D) = D \otimes K \) induced by the automorphism \((x_1, x_2) \mapsto (x_2, x_1)\) of \( K = \mathbb{Q} \oplus \mathbb{Q} \), and where \( \alpha \mapsto \bar{\alpha} \) is the extension of the quaternionic conjugation on \( D \) to \( D \otimes K \) which is trivial on \( 1 \otimes K \).

We can then view a pair \((\varphi_1, \varphi_2)\) of embeddings of \( k_f \) into \( D \) as giving an embedding \( \varphi : y \mapsto (\varphi_1(y), \varphi_2(y)) \in C^+(D) \) of \( k_f \) into \( C^+(D) \). If \( \varphi_1, \varphi_2 \) map the order \( \sigma_f \) of discriminant \(-d\) into \( R_i, R_j \) respectively, \( \varphi = (\varphi_1, \varphi_2) \) maps \( \sigma_f \) into the order \( R_i \oplus R_j \) of \( D \oplus D \cong C^+(D) \). This embedding is optimal in the sense that \( \varphi(k_f) \cap (R_i \oplus R_j) = \varphi(\sigma_f) \) if and only if at least one of \( \varphi_1, \varphi_2 \) is an optimal embedding of \( \sigma_f \) into \( R_i \) resp. \( R_j \).

Taking, as above, \( z \in D^\times \) with \( z^{-1} \varphi_1 z = \varphi_2 \) and writing \( \alpha = (z, \bar{z}) \in C^+(D) \), we have \( \bar{\alpha}^r = \alpha \),

\[
\alpha^{-1}(\varphi_1, \bar{\varphi}_2)\alpha = (\varphi_2, \bar{\varphi}_1) = (\varphi_1, \varphi_2)^r
\]

and

\[
(\varphi_1, \varphi_2)\alpha = (\varphi_1 z, \varphi_2 \bar{z}) = (\varphi_1 z, \varphi_2 z),
\]

so that

\[
\{(\varphi_1(y), \varphi_2(y))\alpha \mid y \in k_f\} \subseteq C^+(D)
\]

is the image of the two-dimensional space \( U(\varphi_1, \varphi_2) \subseteq D \) from above under the identification of \( D \) with \( \bar{D} \subseteq C^+(D) \).

The results from \([3, 4]\) and their reinterpretation in Saito’s approach can therefore also be viewed as a computation of discriminants in the two to one correspondence between optimal embeddings of quadratic orders in \( k_f \) into the order \( R_i \oplus R_j = (y_i, y_j)(R \oplus R)(y_i, y_j)^{-1} \subseteq C^+(D) \) and primitive binary sublattices of \( I_{ij} = D \cap (y_i, y_j)(R \oplus R)(y_i, y_j)^{-1} \subseteq D \subseteq C^+(D) \).

To generalize this to the non split case we need a variant of the Skolem-Noether theorem:

**Theorem 3.** Let \( B \) be a central simple algebra over the field \( K \) of odd characteristic with an involution \( x \mapsto x^f \) of the second kind, denote by \( K_0 \) the fixed field of \( I \).

Let \( C_1 \) be a commutative \( K_0 \)-algebra with an embedding (of \( K_0 \)-algebras) \( \varphi : C_1 \to B \) and assume that \( \varphi(C_1) \) and \( K \) are linearly disjoint over \( K_0 \), i. e., the \( K \)-algebra \( \varphi(C_1) \) generated by \( \varphi(C_1) \) is isomorphic to \( K \otimes_{K_0} C_1 \). Denote by \( C_2 \) the centralizer (commutant) in \( B \) of \( \varphi(C_1) \).

Then the set of \( \alpha \in B \) such that \( \alpha \varphi^f = \varphi \alpha \) and \( \alpha^f = \alpha \) is a \( K_0 \)-vector space of dimension \( \dim_K(C_2) \).

If \( B \) is a division algebra there exists \( \alpha \in B \) with \( \alpha^f = \alpha \) such that \( \alpha^{-1} \varphi \alpha = \varphi^f \).

**Proof.** By the assumption of linear disjointness we can continue the \( K_0 \)-isomorphism \( \varphi(x) \mapsto \varphi(x)^f \) from \( \varphi(C_1) \) to \( \varphi(C_1)^f \) to a \( K \)-algebra isomorphism \( \rho \) from \( \varphi(C_1) \) to the \( K \)-algebra generated by \( \varphi(C_1) \).

By the theorem of Skolem-Noether there exists \( \alpha_1 \in B \) with \( \alpha_1^{-1} b \alpha_1 = \rho(b) \) for all \( b \in \varphi(C_1) \), in particular we have \( \alpha_1^{-1} \varphi(x) \alpha_1 = \varphi(x)^f \) for all \( x \in C_1 \).

The set of \( \alpha \in B \) such that \( \alpha \varphi(x)^f = \varphi(x) \alpha \) holds for all \( x \in C_1 \) is then a non zero \( K \)-vector space \( W_0 \) of dimension \( \dim_K(C_2) \). Moreover, for \( \alpha \in W_0 \) application of \( I \) gives \( \alpha^f \varphi^f = \varphi \alpha^f \), so that \( I \) operates on \( W_0 \), viewed as a vector space over
\( K_0 \subseteq K \). Multiplication by a \( c \in K \) with \( c^t = -c \) switches the +1 and the -1 eigenspace of \( I \) in \( W_0 \), so both eigenspaces have the same dimension over \( K_0 \), namely \( \dim_K(C_2) \). If \( B \) is a division algebra, all nonzero \( \alpha \in W_0 \) are invertible and satisfy \( \alpha^{-1} \varphi \alpha = \varphi^J \).

**Remark.** One could also use recent results of Villa [28] for the proof of the above theorem.

**Corollary 4.** Let \( D \) be a quaternion algebra over \( \mathbb{Q} \) with standard involution \( b \mapsto b \), let \( K \) be a quadratic extension of \( \mathbb{Q} \) and put \( D_K = D \otimes \mathbb{Q} K \), let \( \tau \) be the nontrivial automorphism of \( K \). Denote by \( I \) the involution \( b \mapsto \tau(b) \) of the second kind of \( D_K \), by \( V \) the (4-dimensional) \( \mathbb{Q} \) vector space of all \( I \)-invariant elements of \( D_K \) and equip \( D_K \) and \( V \) with the quaternionic norm form of \( D_K \) as a quadratic form. Let \( k = \mathbb{Q}(\sqrt{-d}) \) be a quadratic extension of \( \mathbb{Q} \) of discriminant \( -d \) which can be embedded into \( D_K \) and is linearly disjoint from \( K \).

Then there is a bijective correspondence between (unordered) pairs of embeddings \( \varphi_1, \varphi_2 : k \to D_K \) with \( \varphi_2(x) = \varphi_1(x) \) for all \( x \in k \) and binary quadratic subspaces \( W \) of determinant \( d \) of \( V \). This correspondence is given by associating to \( \varphi \in \{ \varphi_1, \varphi_2 \} \) as above the set \( W \) of \( \alpha \in V \) with \( \varphi(x) \alpha = \alpha \varphi(x)^J \) for all \( x \in k \).

**Proof.** Given an embedding \( \varphi \), the set \( W \subseteq V \) is a 2-dimensional \( \mathbb{Q} \) vector space by the Theorem, and if we have \( \alpha \in W \) invertible, one sees that \( W = \{ \varphi(x) \alpha \mid x \in k \} \), so that the quadratic space \( W \) has the same determinant \( d \) as the quadratic space \( k \) (equipped with the norm form). If \( \varphi_1, \varphi_2 \) are as above the associated sets of \( \alpha \in V \) are obviously the same. Conversely, given a binary subspace \( W \) of determinant \( d \) of \( V \), we consider an orthogonal basis \( \{w_1, w_2\} \) and put \( b := w_1 w_2^t \), we have \( \tr(b) = 0 \) and \( n(b) = d \). Associating to \( W \) the embedding \( \varphi : k = \mathbb{Q}(\sqrt{-d}) \to D_K \) with \( \varphi(\sqrt{-d}) = b \) we have constructed the inverse of the map \( \varphi \to W \) defined above. \( \square \)

**Corollary 5.** With notations as above we let (as in the discussion of case \( B \) in Section 2) \( R \) be a symmetric order in \( D_K \) and \( \Lambda_i = y_i R \tau(y_i)^{-1} \cap V \) be a lattice in the genus of \( \Lambda_i := R \cap V \), put \( R_i := y_i R \tau(y_i)^{-1} \). Then we have a bijection between pairs \((\varphi, \psi)\) of optimal embeddings of orders of (not necessarily fundamental) discriminant \(-d\) in \( k = \mathbb{Q}(\sqrt{-d}) \) into \( R_i \) and of primitive binary sublattices of determinant \( dm^2 \) of \( \Lambda_i \) with \( m \in \mathbb{Z} \).

**Proof.** The assertion follows directly from the previous corollary by taking the preimage of \( \varphi(k) \cap R_i \) as order in \( k \) and \( \Lambda_i \cap W \) as binary sublattice in \( \Lambda_i \). \( \square \)

As in case \( A \) it remains to compute the discriminants in the matching of the last corollary. For this we restrict now to the case that \( 2 \) is unramified in \( K/\mathbb{Q} \) and that the primes ramifying in \( D \) are precisely those that are ramified in \( K/\mathbb{Q} \). This implies that the discriminant of \( \Lambda \) is odd and square free (congruent to 1 modulo 4).

A more general situation (as in case \( A \)) can probably be handled as well, but the matching leads then to quite complicated and unpleasant formulas, since we have to handle determinants \( d, ds^2, d\Delta, ds^2\Delta \) simultaneously, where \( s \) runs over divisors of the level.

**Lemma 6.** Let \( D \) be a quaternion algebra over \( \mathbb{Q} \), let \( \Delta \) be an odd square free determinant, \( K = \mathbb{Q}(\sqrt{\Delta}) \), \( D_K = D \otimes \mathbb{Q} K \), assume that the primes ramifying in \( D \) are precisely those that are ramified in \( K/\mathbb{Q} \).

Let \( R \) be a symmetric maximal order in \( D_K \) and \( \Lambda = R \cap V \), where \( V = \{ \alpha \in D_K \mid \alpha^\tau = \alpha \} \).

With a double coset decomposition \( D^1_{K, \Lambda} = \bigcup_{i=1}^h D^1_K y_i R^1_{\Lambda} \) (with \( y_1 = 1 \) and \( n(y_i) = 1 \) for all \( i \)) put \( I_i := y_i R \tau(y_i)^{-1}, \Lambda_i = V \cap I_i \) and \( R_i = y_i R \tau(y_i)^{-1} \).
Let $k$ be a quadratic field different from $K$ which can be embedded into $D_K$ and $\varphi : \mathcal{O} \to R_i$ an optimal embedding of the order $\mathcal{O} \subseteq k$ of discriminant $-d$ of $k$.

Then the primitive binary sublattice $M \subseteq V$ of $\Lambda$, associated to $\varphi$ by Corollary 5, has determinant $d$.

Proof. As in case A we check this locally, writing the index $p$ indicating completion only where necessary for clarification in the sequel. Also, for this it is enough to consider $\Lambda = \Lambda_1$.

If $p$ is a prime that splits in $K/Q$ we are locally in the situation of case A and can apply the result proven there.

If $p$ is inert in $K/Q$ the algebra $D \otimes K_p$ is isomorphic to the matrix ring over $K_p$ and we can, as in [3,4], find an $\alpha_1 \in R_p^\times$ with $\alpha_1^{-1}\varphi(x)\alpha_1 = \tau(\varphi(x))$ for all $x \in k$ (the argument given in [3,4] is for the matrix ring over $\mathbb{Z}_p$ but stays valid if $\mathbb{Z}_p$ is replaced by the integers of any local field). Since $K_p/\mathbb{Q}_p$ is unramified, Hilbert’s Theorem 90 holds for the local units, and similarly as in [17, Section 2.5] we can modify $\alpha$ by a unit $\epsilon \in \mathcal{O}_p^\times$ to obtain $\alpha \in R_p^\times \cap V$ with $\alpha^{-1}\varphi(x)\alpha = \tau(\varphi(x))$ for all $x \in k$. Since $M = \varphi(\alpha)$ by our construction, $M$ has the same determinant $d$ as $\mathcal{O}$ (with the norm form as quadratic form).

If $p \neq 2$ is ramified in $K/Q$, we have by assumption that $p$ is ramified in $D$. We denote by $\tilde{R} = \tilde{R}_p = \{z \in D_p \mid n(z) \in \mathbb{Z}_p\}$ the unique maximal order in $D_p$ and choose $R = R_p = \tilde{R} \otimes \mathcal{O}_K + D^{(p)} \otimes \mathcal{O}_K$ with $\delta = \sqrt{\Delta} \in K$ and $D^{(p)} = \{z \in D_p \mid n(z) \in p\mathbb{Z}_p\}$; it is easily verified that $R$ is indeed a symmetric maximal order in (the completion of) $D_K$ and that $R \cap D = \tilde{R}$ holds (identifying $D$ with $D \otimes 1 \subseteq D_K$).

As above we write $k = k_f = \mathbb{Q}[X]/(f)$ with $f = X^2 - sx + n \in \mathbb{Z}[X]$ and $s^2 - 4n = -d$ and $\mathcal{O} = \mathcal{O}_f = \mathbb{Z}[X]/(f)$. To a vector $w \in L_f = \{z \in \mathbb{Z}_p1+2R_p = R_p \mid tr(z) = 0\}$ of norm $d$ we associate the embedding $\varphi$ of $\mathcal{O}$ into $\tilde{R}$ given by $2\varphi(X + (f)) - s = w$ and vice versa. As in [3,4] by this correspondence between the embedding $\varphi$ and the vector $w$ we obtain a bijection between optimal embeddings of $\mathcal{O}$ into $R$ and $\mathbb{Z}$-primitive vectors $w \in L$ (i.e., $rw \in L$ with $r \in \mathbb{Q}$ implies $r \in \mathbb{Z}$) with $n(w) = d$.

The vector $w$ can be written as $v_1 \otimes 1 + v_2 \otimes \delta$ with $v_1, v_2 \in D$ of trace 0, we have $\tau(w) = -v_1 \otimes 1 + v_2 \otimes \delta$ and $d = n(v_1) + n(v_2)$. The condition $n(w) = d \in \mathbb{Z}$ then implies that $v_1, v_2$ are orthogonal with respect to the trace bilinear form on $D$.

The vector $\alpha := r\varphi(v_1) \otimes \delta + s\varphi(v_2) \otimes \delta + t\varphi(\Delta)$ with $r, t \in \mathbb{Q}$ satisfies then $\tau(\alpha) = \alpha$ and $\tau(\alpha) = w\alpha$ and hence $\alpha^{-1}\varphi(x)\alpha = \tau(\varphi(x))$ for all $x \in k$.

We notice that in the trace zero part $\tilde{R}^0$ of $\tilde{R}$ each vector of norm in $\mathbb{Z}_p^\times$ can be split off orthogonally with $p$-modular orthogonal complement, in particular, all vectors in $\tilde{R}^0$ orthogonal to it have norm divisible by $p$.

We have
\[ n(\alpha) = \Delta n(v_2)(r^2 + t^2n(v_1) + t^2\Delta n(v_2)) = \Delta n(v_2)(r^2 + t^2d). \]

Since $R = R_p$ is symmetric we have $v_1 \otimes 1 \in R, v_2 \otimes \delta \in R$. This implies $v_1 \in \tilde{R}, v_2 \in p^{-1}\tilde{R}$ with $pn(v_2) \in \mathbb{Z}$, and at least one of $v_1, v_2$ is primitive in $\tilde{R}$. In particular, since $\tilde{R}$ consists of all elements of $D$ of integral norm, we have $n(v_1) \in \mathbb{Z}_p^\times \cup p\mathbb{Z}_p^\times$ or $n(v_2) \in p^{-2}\mathbb{Z}_p^\times \cup p^{-1}\mathbb{Z}_p^\times$. In the second case we see that in fact $n(v_2) \in p^{-1}\mathbb{Z}_p^\times$ must hold. We can then choose $r = 1, t = p$ above and obtain $\alpha \in R$ with $n(\alpha) \in \mathbb{Z}_p^\times$, hence $\alpha \in R^\times$, which proves the assertion in this case.

If $p \varphi(v_2)$ with $\nu > 0$, $v_2$ primitive in $\tilde{R}$, we write $t = p^{-\nu}\tilde{t}$ and have $\Delta n(v_2)t^2 = p^{-\nu}n(v_2)^2 + \Delta n(v_2)t = p^{-\nu}n(v_2)^2 + \Delta n(v_2)t$. If $d \in \mathbb{Z}_p^\times$ one has $n(v_1) \in \mathbb{Z}_p^\times$ and therefore $n(v_2) \in p\mathbb{Z}_p^\times, n(v_1) \in p\mathbb{Z}_p^\times$, which implies $p^{-1}v_2 \otimes \delta \in R, p^{-1}v_1 \otimes \delta \in R$ and hence $\alpha \in R$ for $r \in p^{-\nu}\mathbb{Z}_p, \tilde{t} \in \mathbb{Z}_p$. With $\nu = p, r = p^{-\nu}$ we obtain $n(\alpha) \in \mathbb{Z}_p^\times$ since $\Delta n(v_2)t^2 \in \mathbb{Z}_p^\times$ and $\Delta n(v_2)t^2 \in \mathbb{Z}_p^\times$. 

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If \( d \in \mathbb{Z}_p^\times \) we have \( n(v_1) \in \mathbb{Z}_p^\times \) and hence \( p^{-1}v_1\tilde{v}_2 \odot \delta \in R \) if \( n(\tilde{v}_2) \in \mathbb{Z}_p^\times \) and \( p^{-1}v_1\tilde{v}_2 \odot \delta \in R, p^{-2}v_1\tilde{v}_2 \odot \delta \in R \) if \( n(\tilde{v}_2) \in \mathbb{Z}_p^\times \). In the first case we put \( \ell = 1, r = 1 \), in the second case we put \( \ell = 1, r = p^{-\nu} \). In both cases we obtain \( \alpha \in R \) with \( n(\alpha) \in \mathbb{Z}_p^\times \), which proves the assertion.

If finally \( d \in \mathbb{Z}_p^\times \), we must have \( n(v_1) \in \mathbb{Z}_p^\times, n(v_2) \in \mathbb{Z}_p^\times, \tilde{v}_2 = v_2 \). We can then take \( \alpha = v_2 \odot \delta \) with \( n(\alpha) = n(v_2)\Delta \in \mathbb{Z}_p^\times \). This gives the 2-dimensional subspace generated by \( v_2 \odot \delta, -\Delta n(v_2) + v_1v_2 \odot \delta \) of \( V \); its intersection with \( \Lambda \) is the lattice \( \mathbb{Z}v_2 \odot \delta + \mathbb{Z}(-n(v_2) + \Delta^{-1}v_1v_2 \odot \delta) \), with imprimitive Gram matrix (divisible by \( p \) once) of determinant \( n(v_2)^2d \in d(\mathbb{Z}_p^\times)^2 \).

**Remark.** The reader may wonder (at least the author did) why certain types of possible binary sublattices of \( \Lambda \) do not appear in the correspondence. One checks easily (by reducing modulo \( p \)) that indeed at primes \( p \) which are inert in \( K/\mathbb{Q} \) the lattice \( \Lambda_p \) has no primitive binary sublattices with \( p \)-imprimitivity Gram matrix (because its reduction mod \( p \) has Witt index 1).

**Theorem 7.** Let \( \Delta \) be an odd square free discriminant with an odd number of prime factors and \( V \) the (unique up to isometry) positive definite quaternionic space of discriminant \( \Delta \) which has Witt invariant \(-1\) at all the primes dividing \( \Delta \). Realize \( V \) by letting \( D \) be the definite quaternion algebra over \( \mathbb{Q} \) ramified at all primes dividing \( \Delta \) and \( K = \mathbb{Q}(\sqrt{\Delta}), D_K = D \otimes_{\mathbb{Q}} K, V = \{ \alpha \in D_K \mid \alpha^2 = \alpha \} \), with the quaternionic norm form as quadratic form.

Let \( \Lambda = \mathbb{R} \cap V \), where \( \mathbb{R} \) is a symmetric maximal order in \( D_K \), be a maximal lattice (of determinant \( \Delta \)) on \( V \), put \( L = \{ z \in \mathbb{Z}I + 2\mathbb{R} \mid \text{tr}(x) = 0 \} \). Let \( L_i = y_iR\tau(y_i^{-1}) \) with \( y_i \in D_K^\times, n(y_i) \in \mathbb{Q}_p^\times \) be a lattice in the genus of \( \Lambda \), put \( L_i = y_iLy_i^{-1} \).

Then for all \( d \in \mathbb{N} \) one has

\[
r^*(\Lambda, d) = r^*(L_i, d),
\]

where \( r^*(L_i, d) \) denotes the number of \( \mathbb{Z} \)-primitive representations of \( d \) by \( L_i \).

**Proof.** This follows from the previous Lemma.

**Remark.**

a) The assertion in case B looks smoother than the result of Theorem 2. As explained earlier this is due to the fact that we have restricted our attention to the maximal lattices \( \Lambda \) on the particular type of quadratic space \( V \) described above. It appears that the formulas for more general \( \Lambda \) of non-square determinant will become messier than in the split case.

At least for \( \Lambda \) of prime determinant our case is the general one.

b) As in \([3, 4]\) the formula for primitive representations above implies the relation

\[
r(L, d) = \sum_{m^2|d} \mu(m)mr_{av}(\Lambda, \frac{d}{m^2})
\]

for representation numbers without primitivity condition. From this one can obtain a formula relating the Koecher-Maaß series of a type II Yoshida lift (as described in \([29]\)) with the Mellin transform of a corresponding Hilbert modular form of weight \( 3/2 \), replacing Corollary 2.2 of \([3]\) in the non-split situation of case B. Moreover, as in \([4]\), one can then obtain a proof of Böcherer’s conjecture for the Koecher-Maaß series of these Yoshida liftings of type II. Details of this will be worked out separately.

4. Average asymptotic formula and applications

We keep the notations of the preceding section for the cases \( \Lambda, B \) respectively. In particular, in case \( \Lambda \) we consider an order \( R \) in the definite quaternion algebra \( D \) over \( \mathbb{Q} \) such that the completion \( \hat{R}_p \) is
• an order of level \( p^{r_p} \) with \( r_p \) odd (see [21]) for the primes \( p \) at which \( D \) is ramified.

• an Eichler order of level \( p^{r_p} \) for the primes \( p \) at which \( D \) splits, with \( r_p = 0 \) for almost all \( p \) and write \( N_1 \) for the product of the \( p^{r_p} \) for the finite primes \( p \) at which \( D \) ramifies, \( N_2 \) for the product of the local levels \( p^{r_p} \) for the primes \( p \mid N_1 \) and \( N = N_1N_2 \). The lattice \( L \) of determinant \( N^2 \) and level \( N \) considered in case A is then one of the ideals \( I \) introduced in Section 2, with associated ternary \( \mathbb{Z} \)-lattices \( L = L_i, L' = L_j \) as described in Section 2. In case B we consider maximal \( \mathbb{Z} \)-lattices \( L \) of odd square free determinant \( \Delta \equiv 1 \mod 4 \), which are realized in the quadratic space \( V = \{ \alpha \in D_K = D \otimes K \mid \tau(\alpha) = \hat{a} \} \) over \( \mathbb{Q} \), where \( K = \mathbb{Q}(\sqrt{\Delta}) \) and \( D \) is a definite quaternion algebra over \( \mathbb{Q} \) ramified precisely at the primes ramified in \( K/Q \), i.e., at those dividing \( \Delta \). To such a \( L \) we have associated a ternary \( \sigma_K \)-lattice \( L \) in the trace zero part of \( D_K \).

Lemma 8. With notations as above one has

\[
\begin{align*}
\langle L, d \rangle \langle L', d \rangle &= r(\langle \text{gen}(L), d \rangle^2 + O(d^{-\frac{1}{4}}),
\langle L, d \rangle \langle L', d \rangle &= r^*(\langle \text{gen}(L), d \rangle^2 + O(d^{-\frac{1}{4}})) \quad \text{in case } A,
\langle L, d \rangle &= r^*(\langle \text{spn}(L), d \rangle + O(d^{-\frac{1}{4}})) \quad \text{in case } B,
\end{align*}
\]

where \( r(\langle \text{gen}(L), d \rangle \rangle \) respectively \( r^*(\langle \text{gen}(L), d \rangle \rangle \) denotes Siegel’s weighted average of the numbers of representations respectively \( \mathbb{Z} \)-primitive representations of \( d \) by the isometry classes of lattices in the genus of \( L \) and where \( r^*(\langle \text{spn}(L), d \rangle \rangle \) denotes the analogous average over the isometry classes in the spinor genus of \( L \).

Moreover, one has \( \langle \text{gen}(L) = \langle \text{spn}(L) \rangle \) in case A and

\[
\langle \text{spn}(L), d \rangle = \langle \text{gen}(L), d \rangle \cdot \begin{cases} 2 \\ 1 \end{cases},
\]

depending on whether the square class of \( d \) is or is not spinor exceptional in the sense of [19].

Proof. In both cases one sees from [15, 7] that the local spinornorms of the lattice are all square classes of units at all finite places. In case A this implies that the genus of \( L, L' \) consists of only one spinor genus and the assertion follows from [25, 6] and the currently best estimate for Fourier coefficients of cusp forms of weight 3/2 (orthogonal to the one-dimensional theta series) of [11].

In case B the genus of \( L \) consists of more than one spinor genus if the class number of \( K \) is even, with all the lattices \( L_i \) arising from the orders \( R_i = y_i R_i^{-1} \) with \( n(y_i) \in \mathbb{Z} \) in the same spinor genus. From [8] one reads off that \( d \) is a primitive spinor exception if and only if the square class of \( d \) is spinor exceptional in the sense of [10] (this case occurs if and only if \( \mathbb{Q}(\sqrt{-d\Delta}, \sqrt{\Delta}) \) is unramified over \( \mathbb{Q}(\sqrt{\Delta}) \)).

In this case by [10] the genus of \( L \) splits into two halves (“half genera”) consisting of equally many spinor genera and such that \( \langle \text{spn}(L'), d \rangle \rangle \) resp. \( r^*(\langle \text{spn}(L'), d \rangle \rangle \) has the same value for all lattices \( L' \) in the same half, in particular it is zero for the lattices in one of the half genera and equal to \( 2\tau(\langle \text{gen}(L), d \rangle) \) resp. \( 2r^*(\langle \text{gen}(L), d \rangle) \)) for lattices in the other half genera.

It is easily checked that for all \( d \) represented locally everywhere by \( L \) there are binary lattices of determinant \( d \) which are represented locally everywhere primitively by \( \Lambda_i \), hence are represented by some lattice \( y_i \Lambda_i y_i^{-1} \) with \( n(y_i) \in \mathbb{Z} \) in the genus of \( \Lambda_i \). Theorem 7 then implies that \( d \) is represented primitively by the ternary \( \mathbb{Z} \)-lattice \( y_i \Lambda_i y_i^{-1} \), which is in the spinor genus of \( L \). From the argument above we obtain \( 2r^*(\langle \text{gen}(L), d \rangle) = r^*(\langle \text{spn}(L), d \rangle) \) in the case that the square class of \( d \) is spinor exceptional. On the other hand, \( r^*(\langle \text{gen}(L), d \rangle) = r^*(\langle \text{spn}(L), d \rangle) \) follows from [10] if the square class of \( d \) is not exceptional.
By [26] $r^\ast(L, d) - r^\ast(\text{spn}(L), d)$ is the $\mathbb{Z}$-primitive Fourier coefficient
\[ a_f^\ast(d) = \sum_{m \in \mathbb{N}, m^2 | d} \mu(m) a_f(d/m^2) \]
at $d$ of a Hilbert cusp form $f$ of weight $3/2$ in the orthogonal complement of the unary theta series with Fourier coefficients $a_f(d)$.

The estimate of [2][11] for the $a_f(d)$ then proves the assertion. Notice that the restrictive condition on the represented number of [11, Theorem 2] can be dropped here since we have established that $d$ is either not in a spinor exceptional square class or is primitive spinor exceptional represented by the spinor genus of $L$. □

**Theorem 9.** Let $-d$ run through discriminants such that binary lattices of determinant $d$ are represented primitively by the genus of $\Lambda$, and assume in case $A$ that $d$ is not divisible by any prime $p | N$ such that $r_p > 1$, write $N_d := \frac{N}{\gcd(N, d)}$. Let $\epsilon > 0$.

Then there are constants $C_i = C_i(\epsilon, N)$ for $i = 1, 2$ such that:

a) In case $A$ for $N = p^s$ a prime power one has
\[ r_{av}^\ast(\Lambda, d) = r_{av}^\ast(\text{gen}(\Lambda), d) + O(d^{1 - \frac{1}{14} + \epsilon}) \]
and
\[ r_{av}(\Lambda, d) \geq C_2 d^{1 - \epsilon} \text{ for } d > C_1. \]

b) In case $A$ with $N$ having more than one prime factor let $N' \| N$ be a fixed divisor of $N$ dividing $N$ exactly and consider $d$ as above with $N_d := \frac{N}{\gcd(N, d)} = N'$. Then one has
\[ r_{av}^\ast(\text{rp}(\Lambda, N'), d) = \sigma_0(N') r_{av}^\ast(\text{gen}(\Lambda), d) + O(d^{1 - \frac{1}{14} - 1}) \]
and
\[ r_{av}(\text{rp}(\Lambda, N'), d) \geq C_2 d^{1 - \epsilon} \text{ for } d > C_1 \text{ with } N_d = N'. \]

Here we write $r_{av}^\ast(\text{rp}(\Lambda, N'), d) = \sum_{n | N'} r_{av}^\ast(\Lambda^{*, s}, d)$ as in Section 2 and analogously for $r_{av}(\text{rp}(\Lambda, N'), d)$.

c) In case $B$, in the situation of Theorem [4] one has
\[ r_{av}^\ast(\Lambda, d) = r_{av}^\ast(\text{spn}(L), d) + O(d^{1 - \frac{1}{14} + \epsilon}) \]
and
\[ r_{av}(\Lambda, d) \geq C_2 d^{1 - \epsilon} \text{ for } d > C_1. \]

If one restricts here to $d$ for which $-d$ is fundamental the local condition is satisfied in case $A$ if and only if $p$ is not split in $\mathbb{Q}((\sqrt{-d}))$ for all $p | N_1$ and is not inert in this quadratic extension for all $p | N_2$. It is satisfied in case $B$ for all positive $d$ for which $-d$ is a discriminant.

**Proof.** For assertion b) we write $e_i = |R_i^\ast|$ and have
\[
\sigma_0(N') r_{av}^\ast(\text{gen}(\Lambda), d) = \sum_{i, j} \sum_{s | N'} r_{av}^\ast(I_{ij}^{*, s}, d) e_i e_j \\
= \sum_{i, j} \sum_{m \in \mathbb{N}, m^2 | d} \mu(m) r(L_i, \frac{d}{m^2}) r(L_j, \frac{d}{m^2}) e_i e_j \\
= \sum_{m \in \mathbb{N}, m^2 | d} \mu(m) (r(\text{gen}(L), \frac{d}{m^2})^2). \]
Since we have

\[ R_{av}^* (\text{rpd}(\Lambda, N'), d) = \sum_{m \in [h, m^2] \mid d} \mu(m) r(L, \frac{d}{m^2}) r(L', \frac{d}{m^2}) \]

by equation 3.2, the asymptotic formula follows from the previous lemma. The lower bound given for \( R_{av}^*(\Lambda, d) \) then follows from

\[ \sigma_0(N_d) R_{av}^*(\text{gen}(\Lambda), d) \geq \sigma_0(N_d) R_{av}^*(\text{gen}(\Lambda), d) \rightarrow c_d^1 \]

for some constant \( c > 0 \) if \( d \) is large enough and represented locally everywhere. The other cases are proved similarly; in case c) we make use of the fact that \( R^*(\text{spn}(L), d) \) is equal to either \( R^*(\text{gen}(L), d) \) or \( 2 R^*(\text{gen}(L), d) \) by Lemma 8 if \( d \) is as assumed. □

Remark. The appearance of the set of all rescaled partial duals \( \Lambda^{*,s} \) with \( s \mid N' \) of \( \Lambda \) in the case of composite \( N \) is somewhat unsatisfactory. The number of lattices involved is obviously \( 2^{\omega(N') \cdot \sigma(N_d) h(\text{gen}(\Lambda), d)} \), but the set of distinct isometry classes of these lattices may be smaller. The argument from [4] would allow to replace representations of \( T \) by \( \Lambda^{*,s} \) by representations of \( sT \) by \( \Lambda \) itself, if one likes that better.

We restrict attention now to fundamental discriminants \(-d\).

**Proposition 10.** Let \(-d\) be a fundamental discriminant as in Theorem 9 and \( N_d = \frac{N}{\gcd(N, d)} \). Put

\[ w = \begin{cases} 
6 & \text{if } d = 3 \\
4 & \text{if } d = 4 \\
2 & \text{otherwise}
\end{cases} \]

and denote by \( h(-d) \) the number of \( SL_2(\mathbb{Z}) \)-equivalence classes of binary quadratic forms of discriminant \(-d\) (equal to the ideal class number of \( \mathbb{Q}(\sqrt{-d}) \)) and by \( 2^t \) the number of genera of such forms. Then in case A we have:

(4.1) \[ h(-d) \sigma_0(N_d) r(\text{gen}(\Lambda), T) = w r(\text{gen}(L), d)^2, \]

where \( \sigma_0 \) denotes the number of divisors function and \( T \) is an integral binary symmetric matrix of discriminant \(-d\).

In case B we have

(4.2) \[ \frac{h(-d)}{2^t} \sum_{T_i} r(\text{gen}(\Lambda, T_i)) = w r(\text{spn}(L), d), \]

where the \( T_i \) run over a set of representatives of the genera of discriminant \(-d\).

**Proof.** In case A Siegel’s weighted average \( r(\text{gen}(\Lambda), T) \) of the numbers of representations of \( T \) by the classes of quadratic lattices in the genus of \( (\Lambda, n) \) is independent of \( T \) since the local lattices allow similitudes with an arbitrary unit as similitude norm. In particular \( r(\text{gen}(\Lambda), T) \) has the same value for \( T \) in all genera of binary quadratic forms of discriminant \(-d\). The assertion then follows as in the proof of Theorem 9.

In case B such similitudes do not exist and \( r(\text{gen}(\Lambda), T) \) is still the same for \( T \) in the same genus but may be different for \( T \) in different genera of binary quadratic...
forms of discriminant \(-d\), in particular it may be zero for some and nonzero for others. We have then
\[
wr(\text{gen}(L), d) = wr_{av}(\text{gen}(\Lambda, d)) = \frac{h(-d)}{2^t} \sum_{T_i} r(\text{gen}(\Lambda, T_i))
\]
as asserted.

Of course, the assertion could also be established by a direct computation of local densities. □

We can now combine this last result with Theorem 9 and obtain a surprising result about representations of individual forms \(T\) by \(\Lambda\). For this we denote by \(\mu = \mu(\text{gen}(\Lambda))\) the mass of the genus of the quadratic lattice \(\Lambda\) and by \(o_{\text{max}} = o_{\text{max}}(\text{gen}(\Lambda))\) the maximal order of the group of automorphisms of a lattice in the genus of \(\Lambda\). With this notation we have:

**Theorem 11.** Let \(-d\) run through fundamental discriminants satisfying the conditions of Theorem 9. In case A with \(N\) having more than one prime factor let \(N'\) be a fixed exact divisor of \(N\) and restrict further to \(d\) with \(N_d = N'\). Denote by \(\nu_d\) the number of binary quadratic forms \(T\) of discriminant \(-d\) with \(r(\text{rp}(\Lambda, N'), T) \neq 0\) in case A with \(N\) not being a prime power, with \(r(\Lambda, T) \neq 0\) in the other cases. Then for all \(\delta > 0\) there is a constant \(C_3 = C_3(\delta)\) such that for all \(d \geq C_3\) as above one has
\[
\nu_d \geq (1 - \delta) \frac{\sigma_0(N') h(-d)}{\mu o_{\text{max}}},
\]
in case A with \(N\) not being a prime power,
\[
\nu_d \geq (1 - \delta) \frac{h(-d)}{\mu o_{\text{max}}},
\]
in case A with \(N\) being a prime power, and
\[
\nu_d \geq (1 - \delta) \frac{h(-d)}{2^t \mu o_{\text{max}}},
\]
in case B.

In particular, for \(d\) large enough a positive proportion of the classes of binary quadratic forms of discriminant \(-d\) is represented by at least one among the rescaled partial dual lattices of \(\Lambda\) comprising the set \(\text{rp}(\Lambda, N_d)\) in case A with \(N\) not being a prime power, by \(\Lambda\) itself in the other cases.

**Proof.** From Siegel’s mass formula we see that we must have
\[
r(\Lambda, T) \leq o_{\text{max}} \mu r(\text{gen}(\Lambda), T)
\]
in case A with \(N\) a prime power and in case B,
\[
r(\text{rp}(\Lambda, N'), T) \leq o_{\text{max}} \mu r(\text{gen}(\Lambda), T),
\]
in case A with \(N\) not a prime power, since all terms in the weighted sum for \(r(\text{gen}(\Lambda), T)\) are non negative.

On the other hand, as in the proposition we see in case A from the asymptotic formula of Theorem 9 that the sum of these terms over the classes of \(T\) of discriminant \(-d\) is asymptotic to \(h(-d) r(\text{gen}(\Lambda, T))\) respectively to \(\sigma_0(N') h(-d) r(\text{gen}(\Lambda, T))\), so we can estimate it for sufficiently large \(d\) from below by \((1 - \delta)\) times this number. The number \(\nu_d\) of terms contributing to the sum must therefore be at least as large as asserted.

In case B we have
\[
r(\text{gen}(\Lambda, T)) \leq \frac{2^t wr(\text{gen}(L), d)}{h(-d)}
\]
by the proposition, and the same argument as above gives the assertion in this case too.

\[\square\]

**Remark.**

a) The “sufficiently large” in the theorem is not effective since our use of the asymptotic formula relies on the fact that \( h(-d) \) is at least of the order of \( d^{\frac{k_2}{2} - \epsilon} \) for all \( \epsilon > 0 \), which estimate is well known to be ineffective.

b) Our method appears not to be suitable to guarantee the existence of \( T \) which is represented by \( \Lambda \) and lies in a given small (say of size \( d^\delta \) with \( 0 < \delta < \frac{1}{2} \)) subset of the class group; such a result was obtained by Einsiedler, Michel, Lindenstrauss and Venkatesh by the ergodic method.

c) If the number of represented forms \( T \) is as small as it can be the representation number \( r(\Lambda, T) \) (or \( r(\text{rd}(\Lambda, N'), T) \)) has to be rather large, i.e. at least of order of magnitude of the number \( r(\text{gen}(R), T) \) from Siegel’s mass formula (in fact, even larger).

d) In contrast to most other results about representation of quadratic forms of rank \( > 1 \) by quadratic forms our result does not at all involve the minimum of the binary quadratic form \( T \) to be represented, at least not explicitly.

5. Fourier coefficients of Siegel modular forms

The results from [4] quoted above were used there to compute averages of Fourier coefficients of special Siegel modular forms of degree 2, the Yoshida liftings of type I. These averages appear there as the coefficients in the Dirichlet series of Koecher and Maass associated to such a Siegel modular form. An immediate consequence is the following:

**Proposition 12.** Let \( F \in M_k^{(2)}(\Gamma_0(N)) \) be the Yoshida lifting of a pair \( f_1, f_2 \) of primitive cuspidal elliptic Hecke eigenforms of weights \( k_1 = 2, k_2 = 2k - 2 \) and square free level \( N \) and denote by

\[
a(F, d) = \sum_T a(F, T) \epsilon(T)
\]

the \( d \)-th coefficient of its Koecher-Maass series, where \( a(F, T) \) is the \( T \)-th Fourier coefficient of \( F \), the sum is over a set of \( SL_2(\mathbb{Z}) \)-equivalence classes of integral symmetric matrices \( T = \left( \begin{array}{cc} a & b \\ b & -ac \end{array} \right) \) of (signed) discriminant \( b^2 - 4ac = -d \), and where \( \epsilon(T) \) denotes the number of proper automorphisms (or units) of \( T \).

Then for any \( \epsilon > 0 \) there is a constant \( C \) such that one has for all \( d \)

\[
a(F, d) \leq Cd^{\frac{k_2}{2} - \frac{1}{4} + \epsilon}.
\]

If above we replace the cusp form \( f_1 \) by the (appropriate) Eisenstein series of weight 2 and level \( N \) we obtain

\[
a(F, d) \leq Cd^{\frac{k_2}{2} - \frac{1}{4} + \epsilon}.
\]

**Remark.** The exponent at \( d \) in our estimate depends on what is known about Fourier coefficients of modular forms in one variable of half integral weight. We discuss this dependence and possible improvements and compare our result on the average Fourier coefficient \( a(F, d) \) with known results and conjectures about the size of the individual Fourier coefficients \( a(F, T) \):

a) If one assumes the generalized Ramanujan conjecture for cusp forms of half integral weight in the complement of the one-dimensional theta series the exponents above would improve to \( \frac{k_2}{2} - \frac{1}{4} + \epsilon \) in the first case and to \( \frac{k_2}{2} - \frac{1}{4} + \epsilon \) in the second case above.
b) In the second case above the form $F$ is in the space of Maaß or Saito-Kurokawa lifts, so that its Fourier coefficient at $T$ depends only on the determinant of $T$ and the summation over $T$ just multiplies the individual coefficient by the number of classes of $T$ of determinant $d$.

c) Similar estimates can be obtained for vector valued Yoshida liftings associated to a pair of elliptic cusp forms $k_1, k_2$ with $k_1 > 2, k_2 > 2$, using the results of [5]. If such a vector valued lifting has transformation type $\text{Sym}^2 \otimes \det^k$ we obtain an exponent $\frac{k}{2} - 2\delta + \epsilon$ at $d$ with $\delta = \frac{1}{19}$ unconditionally and $\delta = \frac{1}{8}$ under the assumption of the generalized Ramanujan conjecture for cusp forms of half integral weight $\geq \frac{5}{2}$. Similar estimates can also be obtained for Yoshida liftings of type II, using Theorem 9.

d) A well known conjecture of Böcherer relates $a(F, d)$ to the square of the central critical value of the twist by $\chi_{-d}$ of the spin $L$-function $Z_{\text{spin}}(s, F)$ of $F$, if $F$ is a cuspidal Hecke eigenform for the full Siegel modular group $\text{Sp}_2(\mathbb{Z})$, more precisely it says that one should have

$$Z_{\text{spin}}(k-1, F, \chi_{-d}) = c_F d^{1-k} (a(F,d))^2.$$  

It was shown in [19] that the completed zeta function

$$
\left(\frac{2\pi}{d}\right)^{-2s} \Gamma(s)\Gamma(s-k+2)Z_{\text{spin}}(s, F, \chi_{-d})
$$

has analytic continuation and satisfies a functional equation under $s \mapsto 2k - 2 - s$. A subconvexity bound in the conductor aspect for the central value of these twists would therefore give an estimate of the type

$$Z_{\text{spin}}(k-1, F, \chi_{-d}) = O(d^{1-\delta})$$

for some $\delta > 0$. Hence, if we assume Böcherer’s conjecture and such a subconvexity bound for the central value of the twisted spin zeta function (with respect to the conductor $d$ of the character) we get bounds for $a(F,d)$ of the above type (with $\delta$ in place of $1/8$); if we assume Böcherer’s conjecture and the Lindelöf hypothesis (again with respect to $d$) for this central value we get the version given in a) which assumes the generalized Ramanujan conjecture for cusp forms of half integral weight. In the case that the level of our modular forms is not 1 we get the same type of estimate if we assume in addition to a generalized version of Böcherer’s conjecture that the twisted spin zeta function has the same type of functional equation as given in [19] in the case of level 1 and satisfies the subconvexity bound mentioned above. In the special case of the Yoshida liftings both the generalized version of Böcherer’s conjecture and the required subconvexity estimate are known by [4].

e) In [18] the Fourier coefficient at $T$ with $\det(T) = d$ of a cusp form of weight $k$ and degree 2 for the full Siegel modular group is estimated to be $O(d^{\frac{k}{2} - \frac{3}{4} + \epsilon})$. If we multiply this estimate by the number of integral equivalence classes of $T$ of determinant $d$ we obtain an exponent of $\frac{k}{2} + \frac{1}{8} + \epsilon$. The conjecture of Resnikoff and Saldaña [23] would give an exponent of $\frac{k}{2} - \frac{1}{4} + \epsilon$ for the individual Fourier coefficient and hence an exponent of $\frac{k}{2} - \frac{1}{4} + \epsilon$ for the sum $a(F,d)$, which bound should be compared with the conjectural bound $\frac{k}{2} + \frac{1}{4} + \epsilon$ obtained above for Yoshida liftings outside the Maaß space under the assumption of the generalized Ramanujan conjecture for (good) cusp forms of half integral weight, or for general cusp forms orthogonal to the Maaß space under the assumption of Böcherer’s conjecture and the Lindelöf conjecture for the twisted spin zeta function. If we consider only forms in
the complement of the Maaß space we see hence that our result for Yoshida liftings gives results for the averaged coefficient $a(F, d)$ that are significantly better than those obtained by multiplying the estimate for the individual Fourier coefficient by the number of summands.

We might summarize the statements of the remark above in the speculation that an estimate of the type

\begin{equation}
(5.3) \quad a(F, d) \leq C d^{\frac{5}{2} - \delta + \epsilon}
\end{equation}

with some $\delta > 0$ could be true and perhaps provable for a general Siegel modular cusp form of degree 2, including as in part c) of the above remark the vector valued case. We might also speculate that similar estimates may hold for averaged Fourier coefficients of higher degree Siegel cusp forms.

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