A solvable model of the breakdown of the adiabatic approximation

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Abstract

Let \( L \geq 0 \) and \( 0 < \varepsilon \ll 1 \). Consider the following time-dependent family of 1D Schrödinger equations with scaled and translated harmonic oscillator potentials

\[ i\varepsilon \partial_t u_\varepsilon = -\frac{1}{2} \partial_x^2 u_\varepsilon + V(t, x) u_\varepsilon, \quad u_\varepsilon(-L-1, x) = \frac{\pi}{4} \exp(-x^2/2), \]

where

- \( V(t, x) = (t+L)^2 x^2/2 \), \( t < -L \),
- \( V(t, x) = 0 \), \( -L \leq t \leq L \), and
- \( V(t, x) = (t-L)^2 x^2/2 \), \( t > L \).

The initial value problem is explicitly solvable in terms of Bessel functions. Using the explicit solutions we show that the adiabatic theorem breaks down as \( \varepsilon \to 0 \).

For the case \( L = 0 \) complete results are obtained. The survival probability of the ground state \( \frac{\pi}{4} \exp(-x^2/2) \) at microscopic time \( t = 1/\varepsilon \) is \( 1/\sqrt{2} + O(\varepsilon) \). For \( L > 0 \) the framework for further computations and preliminary results are given.

1 Introduction

Let \( \mathcal{H} \) be a Hilbert space and \( \{H(t): -a < t < a\} \) a family of selfadjoint operators in \( \mathcal{H} \). Suppose that the time dependent Schrödinger equation

\[ i\varepsilon \partial_t u(t) = H(t) u(t) \]
with a small parameter $0 < \varepsilon \ll 1$ generates a unique unitary propagator $U_\varepsilon(t,s)$ and that $t \mapsto (H(t) - i)^{-1} \in B(H)$ is of class $PC^2(\mathbb{R})$, i.e. piecewise $C^2$. Suppose further that $H(t)$ has an isolated simple eigenvalue $\lambda(t)$ with an associated normalized eigenfunction $\varphi(t)$, both of class $PC^1$ for $t \in (-a,a)$:

$$H(t)\varphi(t) = \lambda(t)\varphi(t), \quad \|\varphi(t)\|^2 = 1, \quad -a < t < a.$$ 

Then, the classical theorem of adiabatic approximation due to Born-Fock ([2]) and Kato ([8]) implies that the solution $u_\varepsilon(t) = U_\varepsilon(t,0)\varphi(0)$ of the initial value problem:

$$i\varepsilon \partial_t u_\varepsilon(t) = H(t)u_\varepsilon(t), \quad u_\varepsilon(0) = \varphi(0), \quad (1.2)$$

follows the eigenstate of $H(t)$ as $0 < \varepsilon \to 0$ for a time interval $-\delta < t < \delta$ for $0 < \delta < a$, at least modulo a term of order $\varepsilon$:

$$\|u_\varepsilon(t) - e^{-i\varepsilon \int_0^t \lambda(s)ds} \varphi(t)\| \leq C_\delta \varepsilon \quad (1.3)$$

where $\|\cdot\|$ is the norm of $L^2(\mathbb{R})$. This theorem has been substantially elaborated and extended to more general situations and it has been widely applied in various fields of mathematical physics, see e.g. Teufel’s monograph [10] and the references therein.

**The eigenvalue dives into the continuum.** Next we consider the situation that eigenvalue $\lambda(t)$ dives into the continuous spectrum of $H(t)$ at, say, $t = -L > -a$, stays in the continuum of $H(t)$ for $-L \leq t \leq L$ and comes out again for $t > L$ as an isolated eigenvalue of $H(t)$. Under the assumption that $\lambda(t)$ remains as an (embedded) eigenvalue of $H(t)$ for $-L \leq t \leq L$, then a general argument has been established and a result similar to (1.3) is obtained ([10]). Moreover, the result has been applied by Dürr-Pickl [5] to the Dirac equation to explain the adiabatic pair creation and by Cornean-Jensen-Knörr-Nenciu [3] to specific finite rank perturbations of Schrödinger equations. However, if $H(t)$ has no embedded eigenvalues for $-L \leq t \leq L$ and the eigenvalue $\lambda(t)$ “melts away into the continuum”, then there is no general theory to deal with the problem; it is even not clear what is meant by the adiabatic approximation. We should mention that embedded eigenvalues in the continuum are very unstable under a perturbation and for genuinely time dependent Hamiltonians embedded eigenvalues would hardly persist for any finite time interval.
Harmonic oscillators which become the free Hamiltonian. To understand these phenomena we study an explicitly solvable model. More precisely, we study the solution of the Schrödinger equation which can be written in terms of the macroscopic time variable as

\[ i\varepsilon \partial_t u_{\varepsilon} = -\frac{1}{2} \partial_x^2 u_{\varepsilon} + V(t, x) u_{\varepsilon}, \quad u_{\varepsilon}(-L - 1, x) = \varphi_0(x), \quad (1.4) \]

which is a scaled harmonic oscillator for \( t < -L \) and \( t > L \) and \( V(t, x) = 0 \) for \(-L \leq t \leq L\):

\[ V(t, x) = \begin{cases} (t + L)^2 x^2 / 2, & t < -L, \\ 0, & -L \leq t \leq L, \\ (t - L)^2 x^2 / 2, & L > t, \end{cases} \quad (1.5) \]

and the initial state \( \varphi_0(x) = (\pi^{\frac{1}{4}})^{-1} e^{-x^2/2} \) is the normalized ground state of the initial Hamiltonian \( H(-L - 1) = -(1/2)\Delta + (1/2)x^2 \). We are particularly interested in the asymptotic behavior as \( \varepsilon \to 0 \) of \( u_{\varepsilon}(t, x) \) at \( t = L + 1 \) when \( H(t) \) again becomes \( -(1/2)\Delta + (1/2)x^2 \).

It is well known that the equation (1.4) generates a unique unitary propagator \( \{ U_{\varepsilon}(t, s) : -\infty < t, s < \infty \} \) which is simultaneously an isomorphism of \( \mathcal{S}(\mathbb{R}) \) and of \( \Sigma(2n), n = 0, 1, \ldots, \) the domain of \( -(1/2)\Delta + (1/2)x^2 \)^n. For \( \varphi \in \Sigma(2), \mathbb{R} \times \mathbb{R} \ni (t, s) \mapsto U_{\varepsilon}(t, s)\varphi \in L^2(\mathbb{R}) \) is \( C^1 \) in \((t, s)\) and \( u_{\varepsilon}(t) = U_{\varepsilon}(t, -L - 1)\varphi \) (see [6]). We should emphasize, however, that \( H(t) \) fails to satisfy the assumptions of the theory of adiabatic approximation in two ways: (1) all eigenvalues dive into continuum simultaneously; (2) the domain of \( H(t) \) has a sharp transition at time \( t = -L \) and \( t = L \) and the resolvent \( (H(t) + c^2)^{-1} \) is not of class \( C^1 \) at these points.

We shall study (1.4) in the microscopic time variable, viz. we change the time variable to \( s = t/\varepsilon \) and study \( v_{\varepsilon}(s, x) = u_{\varepsilon}(\varepsilon s, x) \). \( v_{\varepsilon}(s, x) \) satisfies

\[ i\partial_s v_{\varepsilon} = -\frac{1}{2} \partial_x^2 v_{\varepsilon} + V(\varepsilon s, x) v_{\varepsilon}, \quad v_{\varepsilon}(-\varepsilon^{-1}(L + 1), x) = \varphi_0(x) \quad (1.6) \]

and, as we only consider (1.6) in what follows we denote the microscopic time variable again by \( t \) instead of \( s \). Our result will be rather complete in the case \( L = 0 \), however, when \( L > 0 \), the situation becomes exceedingly complicated and we have to be satisfied with partial results which should be considered as the starting point for further study.
The case \( L = 0 \)

We first consider the case \( L = 0 \), viz. the case that eigenvalues of \( H(t) \) touch upon the continuum only at time \( t = 0 \) but all simultaneously. We should mention that the problem for this case has been studied by Bachmann et al. ([1]) by a method very different from ours and the results slightly overlap.

We record a few lemmas which we shall use in what follows. The first one can be found in [11].

**Lemma 2.1.** Let \( l_\varepsilon(t) \) be the solution of the Riccati equation

\[
l_\varepsilon'(t) + il_\varepsilon(t)^2 = i\varepsilon^2 l^2 \tag{2.1}
\]

with initial condition

\[
l_\varepsilon(-1/\varepsilon) = 1. \tag{2.2}
\]

Suppose that \( m_\varepsilon(t) \) solves

\[
im_\varepsilon'(t) = \frac{1}{2} m_\varepsilon(t) l_\varepsilon(t), \quad m_\varepsilon(-1/\varepsilon) = 1/\pi^{1/4}. \tag{2.3}
\]

Then, \(|m_\varepsilon(t)|^4 = \pi^{-1} \Re l_\varepsilon(t)\) and

\[
v_\varepsilon(t, x) = m_\varepsilon(t)e^{-l_\varepsilon(t)x^2/2} \tag{2.4}
\]

is the solution of the initial value problem for the Schrödinger equation

\[
i\partial_t v_\varepsilon = -(1/2)\partial_x^2 v_\varepsilon + (t^2 \varepsilon^2 x^2/2)v_\varepsilon, \quad v_\varepsilon(-1/\varepsilon) = \pi^{-1/4}e^{-x^2/4}. \tag{2.5}
\]

**General solutions of the Riccati equation.** Bessel functions of the first kind \( J_\nu(z) \) and the second kind \( Y_\nu(z) \) are defined by

\[
J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{(z^2/4)^k}{k! \Gamma(\nu + k + 1)}, \tag{2.6}
\]

\[
Y_\nu(z) = \frac{J_\nu(z) \cos \nu \pi - J_{-\nu}(z)}{\sin \nu \pi}. \tag{2.7}
\]

They are linearly independent solutions of Bessel’s equation

\[
z^2 J_\nu(z) + z J'_\nu(z) + (z^2 - \nu^2) J_\nu(z) = 0
\]

and their positive zeros are interlaced (see [9] 10.21.3).
Lemma 2.2. Let $\varepsilon > 0$ and $\kappa \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{R}$ (the projective space). Define $w(s, \kappa)$ for $s \geq 0$ by

$$ w(s, \kappa) = s^{1/8} \left( - J_{\frac{1}{4}}(2\sqrt{s}) + \kappa J_{\frac{3}{4}}(2\sqrt{s}) \right) \tag{2.8} $$

where the principal branches are assumed for the Bessel functions. Define

$$ \tilde{w}_\varepsilon(t, \kappa) = w\left(\frac{\varepsilon^2 t^4}{16}, \kappa\right), \quad t > 0. \tag{2.9} $$

Then, $\tilde{w}_\varepsilon(t, \kappa)$ may be analytically continued to an entire function of $t \in \mathbb{C}$ and it does not vanish on the real line.

**Proof.** From the definition of Bessel functions (2.6), we have

$$ J_\nu(2\sqrt{s}) = s^{\nu/2} M_\nu(s), \quad M_\nu(s) = \sum_{k=0}^{\infty} (-1)^k \frac{s^k}{k! \Gamma(\nu + k + 1)}, \tag{2.10} $$

and $M_\nu(s)$ is evidently an entire function of $s \in \mathbb{C}$. It follows that

$$ w(s, \kappa) = - M_{-\frac{1}{4}}(s) + \kappa s^{\frac{1}{4}} M_{\frac{1}{4}}(s) \tag{2.11} $$

and

$$ \tilde{w}_\varepsilon(t, \kappa) = - M_{-\frac{1}{4}}(\varepsilon^2 t^4/16) + (\kappa \sqrt{\varepsilon} t/2) M_{\frac{1}{4}}(\varepsilon^2 t^4/16) \tag{2.12} $$

is an entire function of $t \in \mathbb{C}$. As $w(s, \kappa)$ is a linear combination of $J_{\frac{1}{4}}(2\sqrt{s})$ and $J_{-\frac{1}{4}}(2\sqrt{s})$ with non $\mathbb{R}$-related coefficients, $\tilde{w}_\varepsilon(t, \kappa) \neq 0$ for $t > 0$. But (2.12) shows $\tilde{w}_\varepsilon(-t, \kappa) = \tilde{w}_\varepsilon(t, -\kappa)$ and the same is true for $t < 0$ and, $\tilde{w}_\varepsilon(0, \kappa) = - M_{-\frac{1}{4}}(0) = - \Gamma(3/4)^{-1} \neq 0$. This completes the proof. \hfill \Box

Lemma 2.3. Let $\varepsilon \neq 0$, $\kappa \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{R}$ and $s = \varepsilon^2 t^4/16$. Let $w(s, \kappa)$ and $\tilde{w}_\varepsilon(t, \kappa)$ be as in Lemma 2.2. Then:

(1) With $\kappa$ being an arbitrary constant, the general solution of the Riccati equation (2.1) is given by

$$ l_\varepsilon(t, \kappa) = \frac{-4i sw'(s, \kappa)}{tw(s, \kappa)} = -i \frac{\tilde{w}_\varepsilon'(t, \kappa)}{\tilde{w}_\varepsilon(t, \kappa)} \tag{2.13} $$

It is a holomorphic function of $t$ in a complex neighborhood of the real line.
We may express \( l_\varepsilon(t, \kappa) \) without using derivatives:

\[
\begin{align*}
\frac{4is^{\frac{1}{2}}(\kappa J_{\frac{1}{4}}(2\sqrt{s}) + J_{\frac{3}{4}}(2\sqrt{s}))}{t(-J_{\frac{3}{4}}(2\sqrt{s}) + \kappa J_{\frac{1}{4}}(2\sqrt{s}))} \\
= \frac{-i(8\kappa \varepsilon^{\frac{1}{2}}M_{\frac{3}{4}}(s) + \varepsilon^2 t^3 M_{\frac{1}{4}}(s))}{2(-2M_{\frac{1}{4}}(s) + \kappa \varepsilon^{\frac{1}{2}} t M_{\frac{1}{4}}(s))}.
\end{align*}
\]  

(2.14)

(2.15)

For the solution \( l_\varepsilon(t, \kappa) \) we have as \( t \to 0 \)

\[
l_\varepsilon(t, \kappa) = \frac{8\kappa \varepsilon^{\frac{1}{2}} M_{\frac{1}{4}}(s) - \varepsilon^2 t^3 M_{\frac{3}{4}}(s)}{2(-2M_{\frac{1}{4}}(s) + \kappa \varepsilon^{\frac{1}{2}} t M_{\frac{1}{4}}(s))}, \quad a = 2\kappa \Gamma(3/4)/\Gamma(1/4).
\]  

(2.16)

Proof. Define \( w_1(s) = s^{1/8}(aJ_{\frac{1}{4}}(2\sqrt{s}) + bY_{\frac{1}{4}}(2\sqrt{s})) \) for \( s \geq 0 \). Davies([4]), pages 67-78 shows that general solution of (2.1) is given by

\[
l_\varepsilon(t) = -\frac{4isw'_1(s)}{tw_1(s)} \tag{2.17}
\]

with arbitrary constants \( a \) and \( b \) which are not \( \mathbb{R} \)-related. If we use \( Y_{\frac{1}{4}} = J_{\frac{1}{4}} - \sqrt{2}J_{-\frac{1}{4}} \) and set \( \kappa = (a + b)/(\sqrt{2}b) \in P^1(\mathbb{C}) \), the right of (2.17) becomes \( l_\varepsilon(t, \kappa) \) of (2.13). Lemma 2.2 implies (1).

For proving (2) we use the recurrence formula of Bessel functions (see (10.6.5) of [9]):

**Lemma 2.4.** Let \( C_\nu(z) \) be any of \( J_\nu(z), Y_\nu(z), H^{(1)}_\nu(z), H^{(2)}_\nu(z) \) or any non-trivial linear combination of these functions, the coefficients in which are independent of \( z \) and \( \nu \). Define \( f_\nu(z) = z^\nu C_\nu(\lambda z^q) \), where \( p, q, \) and \( \lambda \neq 0 \) are real or complex constants, then

\[
z f'_\nu(z) = \lambda q z^q f_{\nu-1}(z) + (p - \nu q) f_\nu(z)
\]

as long as the principal branch is considered for \( C_\nu(z) \),

Conside \( f_\nu(z) \) for \( C_\nu(z) = aJ_\nu(z) + bY_\nu(z) \) with

\[
p = \frac{1}{8}, \quad q = \frac{1}{2}, \quad \nu = \frac{1}{4}, \quad \lambda = 2.
\]

Then \( \lambda q = 1, \ p - \nu q = 0 \) and (2.18) implies that for \( w_1(s) \) of (2.17) we have

\[
l_\varepsilon(t) = \frac{-4is^{\frac{1}{2}}(s^{\frac{1}{8}} C_{\frac{1}{4}}(2\sqrt{s}))'}{s^{\frac{1}{8}} C_{\frac{1}{4}}(2\sqrt{s})} = \frac{-4is^{\frac{1}{2}} C_{\frac{1}{4}}(2\sqrt{s})}{s^{\frac{1}{8}} C_{\frac{1}{4}}(2\sqrt{s})}.
\]  

(2.19)
In the right side of (2.19) substitute $C_{\frac{1}{4}}(z) = aJ_{\frac{1}{4}}(z) + bY_{\frac{1}{4}}(z) = (a + b)J_{\frac{1}{4}}(z) - \sqrt{2b}J_{\frac{1}{4}}(z)$,
\[ C_{-\frac{3}{4}}(z) = aJ_{-\frac{3}{4}}(z) + bY_{-\frac{3}{4}}(z) = (a + b)J_{-\frac{3}{4}}(z) + \sqrt{2b}J_{\frac{3}{4}}(z) \]
with $z = 2\sqrt{s}$ and reduce by the common factor $\sqrt{2b}$. In the denominator we have
\[ \frac{s^{\frac{1}{2}}}{\sqrt{2b}} C_{\frac{1}{4}}(2\sqrt{s}) = s^{1/8} (-J_{-\frac{1}{4}}(2\sqrt{s}) + \kappa J_{\frac{1}{4}}(2\sqrt{s})) = w(s, \kappa) \] (2.20)
and in the numerator
\[ \frac{s^{\frac{3}{2}}}{\sqrt{2b}} C_{-\frac{3}{4}}(2\sqrt{s}) = s^{\frac{3}{2}} (\kappa J_{-\frac{3}{4}}(z) + J_{\frac{3}{4}}(z)) = \frac{\sqrt{\varepsilon \kappa t}}{2} M_{-\frac{3}{4}}(s) + s M_{\frac{3}{4}}(s). \] (2.21)
Plugging these in (2.19) we obtain (2.14). If we use (2.11) and the last expression in (2.21) we obtain (2.15) which manifests that $l_\varepsilon(t, \kappa)$ is a meromorphic function of $t$.

For proving (3), we use (2.15). The numerator has an asymptotic expansion $-i(8\kappa \varepsilon^{\frac{1}{2}} \Gamma(1/4)^{-1} + O(\varepsilon^2 t^3))$ and the denominator $2(-2\Gamma(3/4)^{-1} + \kappa \varepsilon^{\frac{1}{2}} t \Gamma(5/4)^{-1} + O(\varepsilon^2 t^4))$ as $t \to 0$, hence,
\[ l_\varepsilon(t, \kappa) = \frac{-i8\kappa \varepsilon^{\frac{1}{2}} \Gamma(1/4)^{-1} + O(\varepsilon^2 t^3)}{2(-2\Gamma(3/4)^{-1} + \kappa \varepsilon^{\frac{1}{2}} t \Gamma(5/4)^{-1} + O(\varepsilon^2 t^4))} = \frac{(2i\kappa \varepsilon^{\frac{1}{2}} \Gamma(3/4)^{-1} + O(\varepsilon^2 t^3) \Gamma(1/4) + O(\varepsilon^2 t^4))^{-1}}{1 - a \varepsilon^{\frac{1}{2}} t + O(\varepsilon^2 t^4)}, \quad a = \frac{2\kappa \Gamma(3/4)^{-1} \Gamma(1/4)}{\Gamma(1/4)}. \]
Statement (3) follows. \[ \square \]

**The initial condition.** Having obtained the general solution $l_\varepsilon(t, \kappa)$ of (2.1), we need to determine $\kappa = \kappa_\varepsilon$ such that the initial condition $l_\varepsilon(-1/\varepsilon, \kappa_\varepsilon) = 1$ is satisfied. We define $l_\varepsilon'(t) = l_\varepsilon(t, \kappa_\varepsilon)$ and $\tilde{l}_\varepsilon(t) = -l_\varepsilon'(-t)$,
We have introduced $\tilde{l}_\varepsilon(t)$ as we want to deal with a positive variable. Then (2.15) implies for $t > 0$ that

$$
\tilde{l}_\varepsilon(t) = \frac{-i(8(-\kappa_\varepsilon)\varepsilon^{\frac{3}{2}}M_{-\frac{3}{4}}(s) + \varepsilon^2 t^2 M_\frac{1}{4}(s))}{2(-2M_{-\frac{3}{4}}(s) + (-\kappa_\varepsilon)\varepsilon^{\frac{3}{2}} t M_\frac{1}{4}(s))}
$$

$$
= \frac{-4i s^\frac{3}{4}(-\kappa_\varepsilon J_{-\frac{3}{4}}(2\sqrt{s}) + J_{\frac{1}{4}}(2\sqrt{s}))}{t(-J_{-\frac{1}{4}}(2\sqrt{s}) - \kappa_\varepsilon J_{\frac{1}{4}}(2\sqrt{s}))}
$$

$$
= l_\varepsilon(t, -\kappa_\varepsilon) = -\frac{i\tilde{w}'(t, -\kappa_\varepsilon)}{\tilde{w}(t, -\kappa_\varepsilon)},
$$

(2.22)

Thus, $\tilde{l}_\varepsilon(1/\varepsilon) = -1$ is satisfied if (and only if)

$$
-1 = \left. \frac{-4i s^\frac{3}{4}(-\kappa_\varepsilon J_{-\frac{3}{4}}(2\sqrt{s}) + J_{\frac{1}{4}}(2\sqrt{s}))}{t(-J_{-\frac{1}{4}}(2\sqrt{s}) - \kappa_\varepsilon J_{\frac{1}{4}}(2\sqrt{s}))} \right|_{t=1/\varepsilon} = \frac{i(-\kappa_\varepsilon J_{-\frac{3}{4}}(1/2\varepsilon) + J_{\frac{1}{4}}(1/2\varepsilon))}{J_{-\frac{1}{4}}(1/2\varepsilon) + \kappa_\varepsilon J_{\frac{1}{4}}(1/2\varepsilon)}
$$

where we used $2\sqrt{s} = 1/(2\varepsilon)$ when $t = 1/\varepsilon$ in the last expression. Solving this equation for $\kappa_\varepsilon$ leads to

$$
\kappa_\varepsilon = -\frac{J_{-\frac{1}{4}} + iJ_{\frac{1}{4}}}{J_{-\frac{1}{4}} - iJ_{\frac{1}{4}}} \left| \frac{1}{\varepsilon} \right.
$$

(2.23)

We recall the following special case of [9, 10.17.3].

**Lemma 2.5.** For real $x$, we have with $\omega = x - \frac{1}{2} \nu \pi - \frac{\pi}{4}$ that as $x \to \infty$

$$
J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left( \cos \omega - \frac{4\nu^2 - 1}{8x} \sin \omega + O\left( \frac{1}{x^2} \right) \right).
$$

(2.24)

Application of this result to the right hand side of (2.23) yields

$$
\kappa_\varepsilon = -e^{\frac{i\pi}{4}} + (2\sqrt{2})^{-1} e^{-i\left(\frac{1}{4} + \frac{3}{4}\right)} + O(\varepsilon^2) = -e^{\frac{i\pi}{4}} + O(\varepsilon), \quad \varepsilon \to 0.
$$

(2.25)

We omit the details.

**Lemma 2.6.** The solution $l_\varepsilon^*(t)$ of the initial value problem for the Riccati equation

$$
l_\varepsilon^*(t) + il_\varepsilon^*(t)^2 = i\varepsilon^2 t^2, \quad l_\varepsilon^*(-1/\varepsilon) = -1
$$

is given by (2.14) with $\kappa$ given by $\kappa_\varepsilon$ of (2.23) or (2.25):

$$
l_\varepsilon^*(t) = \frac{-4i s^\frac{3}{4}((\kappa_\varepsilon J_{-\frac{3}{4}}(2\sqrt{s}) + J_{\frac{1}{4}}(2\sqrt{s}))}{t(-J_{-\frac{1}{4}}(2\sqrt{s}) + \kappa_\varepsilon J_{\frac{1}{4}}(2\sqrt{s}))}, \quad s = \varepsilon^2 t^4 / 16
$$

(2.27)

where the principal branch is assumed for Bessel functions.
Asymptotic behavior of $l^*_\varepsilon(1/\varepsilon)$ as $\varepsilon \to 0$.

**Lemma 2.7.** As $\varepsilon \to 0$, we have

$$l^*_\varepsilon(1/\varepsilon) = \frac{1 - 2\sqrt{2}i \cos(1/\varepsilon)}{3 + 2\sqrt{2} \sin(1/\varepsilon)} + O(\varepsilon) \quad (2.28)$$

and $\Re l^*_\varepsilon(1/\varepsilon)$ oscillates between $(3 + 2\sqrt{2})^{-1}$ and $(3 - 2\sqrt{2})^{-1}$ as $\varepsilon \to 0$.

**Proof.** From (2.27) we have

$$l^*_\varepsilon(1/\varepsilon) = \frac{-i(\kappa_\varepsilon J_{-\frac{1}{4}} + J_{\frac{3}{4}})}{-J_{-\frac{1}{4}} + \kappa_\varepsilon J_{\frac{1}{4}}} \quad , \quad \kappa_\varepsilon = -\frac{J_{-\frac{1}{4}} + iJ_{\frac{3}{4}}}{J_{\frac{1}{4}} - iJ_{-\frac{1}{4}}} \quad (2.29)$$

Thus,

$$l^*_\varepsilon(1/\varepsilon) = \frac{2J_{\frac{3}{4}} J_{-\frac{1}{4}} + i(J_{\frac{3}{4}} J_{\frac{1}{4}} - J_{-\frac{1}{4}} J_{-\frac{3}{4}})}{2J_{-\frac{1}{4}} J_{\frac{1}{4}} + i(J_{\frac{1}{4}} J_{\frac{3}{4}} - J_{-\frac{3}{4}} J_{-\frac{1}{4}})} \quad (2.30)$$

and we may compute the asymptotic value of (2.30) as $\varepsilon \to 0$ by applying once more (2.24). This yields (2.28) and the lemma follows. \qed

**The amplitude function** $m_\varepsilon(t)$. We next solve initial value problem (2.3) associated with $l^*_\varepsilon(t)$ which reads

$$\frac{m'_\varepsilon(t)}{m_\varepsilon(t)} = \frac{l^*_\varepsilon(t)}{2i}, \quad m_\varepsilon(-1/\varepsilon) = \pi^{-1/4}.$$

For the same reason as before, we consider $\tilde{m}_\varepsilon(t) = m_\varepsilon(-t)$. The expression (2.22) for $\tilde{l}_\varepsilon(t)$ implies

$$\frac{\tilde{m}'_\varepsilon(t)}{\tilde{m}_\varepsilon(t)} = \frac{m'_\varepsilon(-t)}{m_\varepsilon(-t)} = -\frac{l^*_\varepsilon(-t)}{2i} = \frac{\tilde{l}_\varepsilon(t)}{2i} = -\frac{\tilde{w}'_\varepsilon(t, -\kappa_\varepsilon)}{2\tilde{w}_\varepsilon(t, -\kappa_\varepsilon)} \quad (2.31)$$

Recall (2.8) and (2.12) for the definition of $\tilde{w}_\varepsilon(t, \kappa)$. Integrating (2.31) yields $\tilde{m}_\varepsilon(t) = A_\varepsilon \tilde{w}_\varepsilon(t, -\kappa_\varepsilon)^{-1/2}$ for a constant $A_\varepsilon$ for $t > 0$, viz.

$$m_\varepsilon(-t) = A_\varepsilon(-s^{1/8} J_{-\frac{1}{4}}(2\sqrt{s}) - \kappa_\varepsilon s^{1/8} J_{\frac{1}{4}}(2\sqrt{s}))^{-1/2} \quad (2.32)$$

$$= A_\varepsilon(-M_{-\frac{1}{4}}(s) - \kappa_\varepsilon t M_{\frac{1}{4}}(s))^{-1/2}. \quad (2.33)$$
Thus the initial condition $m_\epsilon(-1/\epsilon) = \pi^{-\frac{1}{4}}$ is satisfied if

$$
\pi^{-\frac{1}{4}} = A_\epsilon \left( - s^{1/8} J_{-\frac{1}{4}}(2\sqrt{s}) - \kappa_\epsilon s^{1/8} J_{\frac{1}{4}}(2\sqrt{s}) \right)^{-1/2} \bigg|_{t=1/\epsilon}.
$$

(2.34)

By virtue of (2.24) and (2.25), \((\cdots)\) on the right hand side is equal to (with $\alpha = \frac{1}{2\epsilon} - \frac{\pi}{4}$)

$$
\frac{2^{\frac{1}{2}} \epsilon^{\frac{1}{4}}}{\pi^{\frac{1}{4}}} \left( - \cos \left( \alpha + \frac{\pi}{8} \right) + e^{\frac{i\pi}{4}} \cos \left( \alpha - \frac{\pi}{8} \right) + O(\epsilon) \right) = \frac{\epsilon^{\frac{1}{4}}}{\pi^{\frac{1}{4}}} e^{-i(\alpha' - \pi/8)} + O(\epsilon^{\frac{1}{4}}),
$$

and we have

$$
A_\epsilon = \frac{\epsilon^{\frac{1}{4}}}{\pi^{\frac{1}{4}}} e^{-i(\alpha' - \pi/8)}(1 + O(\epsilon)).
$$

(2.35)

(2.33) implies that $m_\epsilon(t)$ is given by changing $\kappa_\epsilon$ to $-\kappa_\epsilon$ in the right hand side of (2.32) or (2.33). This proves the first statement of the following lemma.

**Lemma 2.8.** (1) The solution of the initial value problem (2.3) associated with $l^*_\epsilon(t)$ is given by

$$
\left. m_\epsilon(t) \right|_{t=1/\epsilon} = A_\epsilon \left( - s^{1/8} J_{-\frac{1}{4}}(2\sqrt{s}) - \kappa_\epsilon s^{1/8} J_{\frac{1}{4}}(2\sqrt{s}) \right)^{-1/2}.
$$

(2.36)

where $\kappa_\epsilon$ and $A_\epsilon$ are asymptotically given by (2.25) and (2.35) respectively and the branch of the square root should be chosen such that $m_\epsilon(-1/\epsilon) = \pi^{-\frac{1}{4}}$.

(2) As $\epsilon \to 0$,

$$
m_\epsilon(1/\epsilon) = \pi^{-1/4} \left( \sqrt{2e^{i/\epsilon} + i} \right)^{1/2} + O(\epsilon).
$$

(2.37)

where the branch of the square root should be chosen by the continuity.

**Proof.** By virtue of (2.34) and (2.36),

$$
m_\epsilon(1/\epsilon)^2 = \pi^{-\frac{1}{4}} \left| - J_{-\frac{1}{4}} - \kappa_\epsilon J_{\frac{1}{4}} \right|_{1/\epsilon}.
$$

and we compute the asymptotic value of the right side by using (2.24). We obtain (2.37).
Asymptotic behavior at $t = 0$. In the following section we need $l^*_ε(0)$ and $m_ε(0)$. We already computed $l^*_ε(0) = iaε^{1/2}$ in (2.16) where $κ$ in the expression for $a$ should be taken as $κ = κ_ε$ (see (2.25)). The following lemma immediately follows from (2.33) or (2.36).

Lemma 2.9. As $t → 0$, $m_ε(t)$ has the following asymptotic expansion

$$m_ε(t) = -iA_εΓ(3/4)^{1/2}(1 - 2Γ(3/4)/Γ(1/4)κ_ε t + O(t^2)), \quad (2.38)$$

where $A_ε$ and $κ_ε$ are as in (2.35) and (2.25) respectively.

Lemma 2.9 shows how the adiabatic approximation breaks down as $t → 0$: The adiabatic approximation would yield $l_ε(t) = εt/2$ for (minus) the exponent of the Gaussian as $ε → 0$ whereas the leading term in (2.16) is $iaε^{1/2}$ which does not go to zero as $t → 0$. The corresponding term of order $εt$ appears only as the second term $ia^2εt ∼ C^2εt/2$, $C = 2Γ(3/4)/Γ(1/4) ≈ 0.676$. The state at time $t = 0$, $v_ε(0, x)$, is a Gaussian, which is a result of general theorems (see [7]), but the speed of spreading is $Cε^{1/4}√t$ times slower than the one given by the adiabatic approximation and, at time zero, it remains as a finite Gaussian of size $Cε^{-1/4}$ whereas the adiabatic approximation gives a completely flat Gaussian.

Behavior of $v_ε(1/ε)$ as $ε → 0$ and the survival probability. The following theorem states the main result of this section for the case $L = 0$. The theorem explicitly exhibits that the state at the microscopic time $1/ε$, when the Hamiltonian returns to the initial $-(1/2)d^2/dx^2 + (1/2)x^2$, is highly oscillating as $ε → 0$ and the adiabatic approximation is completely broken down.

We have proven the following theorem:

Theorem 2.10. (1) Let $l^*_ε(1/ε)$ and $m_ε(1/ε)$ be given by (2.28) and (2.37). Then the solution $v_ε(t, x)$ of the initial value problem (1.6) satisfies as $ε → 0$,

$$∥v_ε(1/ε, x) − m_ε(1/ε)e^{-l^*_ε(1/ε)x^2/2}∥ ≤ C_ε. \quad (2.39)$$

We have modulo a term of order $O(ε)$

$$|m_ε(1/ε)|^4 = π^{-1}(3 + 2√2 sin(1/ε)) = π^{-1}Re^*_ε(1/ε) \quad (2.40)$$

(2) The survival probability of ground state $φ_0(x) = π^{-1/4}e^{-x^2/2}$ at time $1/ε$ is equal to $1/√2 + O(ε)$. 

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Remak 2.11. The survival probability in part (2) was also computed in [1, Theorem 1].

Proof. We only prove (2). Using (2.39) and explicitly computing the Gaussian integral, we obtain

\[ \langle v_{\varepsilon}(1/\varepsilon), \varphi_0 \rangle = \int_{\mathbb{R}} \pi^{-\frac{1}{4}} e^{-x^2/2} m_{\varepsilon}(1/\varepsilon) e^{-t_{\varepsilon}^2(1/\varepsilon)x^2/2} dx + O(\varepsilon) \]  
\[ = \frac{\sqrt{2}\pi^{1/4} m_{\varepsilon}(1/\varepsilon)}{(1 + t_{\varepsilon}^2(1/\varepsilon))^{1/2}} + O(\varepsilon). \]  

(2.41) Insert (2.37) for \( m_{\varepsilon}(1/\varepsilon) \) and (2.28) for \( t_{\varepsilon}^*(1/\varepsilon) \). Since

\[ (\sqrt{2}e^{i/\varepsilon} + i) \left( 1 + \frac{1 - 2\sqrt{2}i \cos(1/\varepsilon)}{3 + 2\sqrt{2} \sin(1/\varepsilon)} \right) = e^{i/\varepsilon} \frac{6\sqrt{2} + 8 \sin(1/\varepsilon)}{3 + 2\sqrt{2} \sin(1/\varepsilon)} = e^{i/\varepsilon} 2\sqrt{2}, \]

we conclude that

\[ |\langle v_{\varepsilon}(1/\varepsilon), \varphi_0 \rangle|^2 = \frac{2(3 + 2\sqrt{2} \sin(1/\varepsilon))}{6\sqrt{2} + 8 \sin(1/\varepsilon)} + O(\varepsilon) = \frac{1}{\sqrt{2}} + O(\varepsilon). \]

\[ \Box \]

3 The case \( L > 0 \).

We next study the case \( L > 0 \) and examine how the asymptotic behavior as \( \varepsilon \to 0 \) of the solution depends on the macroscopic length \( L \) of time which the particle has spent in the continuum of \(- (1/2) \Delta \). We let \( v_{\varepsilon}(t, x) \) be the solution of the initial value problem (1.6) with \( L > 0 \). Then, by translating in time the result for the case \( L = 0 \) by \(- L/\varepsilon \), we see from (2.16) with \( \kappa = \kappa_{\varepsilon} \) and (2.38) that

\[ v_{\varepsilon}(-L/\varepsilon, x) = -iA_x \Gamma(3/4)^{1/2} e^{-ia_x x^2/2}, \quad a_x = \frac{2\Gamma(3/4)}{\Gamma(1/4)}^{1/2} \]

Solution at the time exiting the continuum. We may explicitly compute

\[ v_{\varepsilon}(L/\varepsilon, x) = (e^{-2iLH_0/\varepsilon} v_{\varepsilon}(-L/\varepsilon))(x) \]
\[
\int_{\mathbb{R}} e^{i\varepsilon(x-y)^2} dy = 0
\]

\[
\int_{\mathbb{R}} e^{i\varepsilon(x-y)^2} dy = \frac{-iA_\varepsilon}{(-2\alpha_\varepsilon L + \varepsilon)^{1/2}} e^{2(-2\alpha_\varepsilon L + \sqrt{\varepsilon})x^2}. \quad (3.1)
\]

Solution after the particle exits the continuum. We want to evaluate at time \( t = \frac{L+1}{\varepsilon} \) the solution of

\[
i\partial_t v_\varepsilon(t, x) = -\frac{1}{2} \Delta v_\varepsilon + \frac{(t - L/\varepsilon)^2\varepsilon^2x^2}{2} v_\varepsilon
\]

when \( v_\varepsilon(L/\varepsilon, x) \) is given by (3.1). Then, translation of \( t \) by \( L/\varepsilon \) once again shows that \( v_\varepsilon((L+1)/\varepsilon, x) = z_\varepsilon(1/\varepsilon, x) \), where \( z_\varepsilon(t, x) \) is the solution of

\[
i\partial_t z_\varepsilon(t, x) = -\frac{1}{2} \Delta z_\varepsilon + \frac{t^2\varepsilon^2x^2}{2} z_\varepsilon,
\]

\[
z_\varepsilon(0, x) = -iA_\varepsilon \Gamma(3/4) \frac{1}{2} e^{-\frac{-i\varepsilon a_\varepsilon}{2(-2\alpha_\varepsilon L + \sqrt{\varepsilon})}} x^2. \quad (3.3)
\]

We know from (2.14) that \( z_\varepsilon(t, x) \) is of the form

\[
z_\varepsilon(t, x) = m^*_\varepsilon(t)e^{-l_\varepsilon(t, \gamma)x^2/2}
\]

where \( l_\varepsilon(t, \gamma) \) and \( m^*_\varepsilon(t) \) are given by (2.14) and (2.36) respectively, with \( \gamma \) in place of \( \kappa \) and \( B_\varepsilon \) in place of \( A_\varepsilon \), in particular,

\[
l_\varepsilon(t, \gamma) = \frac{-4is^{3/4}(\gamma J_{-3/4}(2\sqrt{s}) + J_{3/4}(2\sqrt{s}))}{t(-J_{-3/4}(2\sqrt{s}) + \gamma J_{3/4}(2\sqrt{s})^2) - 1/2}. \quad (3.4)
\]

\[
m^*_\varepsilon(t) = B_\varepsilon \left( -s^{1/8}J_{-3/4}(2\sqrt{s}) + \gamma s^{1/8}J_{3/4}(2\sqrt{s}) \right)^{-1/2}. \quad (3.5)
\]

We will choose \( \gamma \) and \( B_\varepsilon \) such that the initial condition (3.3) is met, viz.

\[
l_\varepsilon(0, \gamma) = \frac{i\varepsilon a_\varepsilon}{-2\alpha_\varepsilon L + \sqrt{\varepsilon}}, \quad (3.6)
\]

\[
m^*_\varepsilon(0) = \frac{-iA_\varepsilon \Gamma(3/4) \frac{1}{2} \varepsilon^{1/4}}{(-2\alpha_\varepsilon L + \sqrt{\varepsilon})^{1/2}}. \quad (3.7)
\]

By virtue of (2.10) we may evaluate \( l_\varepsilon(0, \gamma) \) of (3.4) and \( m^*_\varepsilon(0) \) of (3.5):

\[
l_\varepsilon(0, \gamma) = \frac{2i\gamma \varepsilon^{1/4} \Gamma(3/4)}{\Gamma(1/4)}, \quad m^*_\varepsilon(0) = -iB_\varepsilon \Gamma(3/4)^{1/2}. \quad (3.8)
\]
Equating the right hand sides of (3.6) and (3.7) with those of (3.8), we have
\[ \gamma = \frac{\varepsilon^{\frac{3}{4}} \kappa_\varepsilon}{-2a_\varepsilon L + \sqrt{\varepsilon}}, \quad B_\varepsilon = \frac{A_\varepsilon \varepsilon^{\frac{1}{4}}}{(-2a_\varepsilon L + \sqrt{\varepsilon})^{1/2}}. \] (3.9)

Hereafter we write \( C_1 = \Gamma(3/4)/\Gamma(1/4) \) so that \( a_\varepsilon = 2\kappa_\varepsilon C_1 \). Note that \( \gamma = \kappa_\varepsilon \) and \( B_\varepsilon = A_\varepsilon \) when \( L = 0 \) as they should be.

**Solution when the Hamiltonian returns to** \(-(1/2)d^2/dx^2 + x^2/2\). We study the behavior as \( \varepsilon \to 0 \) of \( l_\varepsilon(1/\varepsilon, \gamma) \) and \( m_\varepsilon(1/\varepsilon) \). They are given by
\[ l_\varepsilon(1/\varepsilon, \gamma) = \frac{-i(\gamma J_{-\frac{3}{4}}(z) + J_{\frac{3}{4}}(z))}{(-J_{-\frac{3}{4}}(z) + \gamma J_{\frac{3}{4}}(z))} \bigg|_{z=\frac{\varepsilon}{2\pi}}, \] (3.10)
\[ m_\varepsilon(1/\varepsilon) = B_\varepsilon ((z/2)^{1/4}J_{-\frac{3}{4}}(z) + \gamma(z/2)^{1/4}J_{\frac{3}{4}}(z))^{-1/2} \bigg|_{z=\frac{\varepsilon}{2\pi}}. \] (3.11)

We substitute the first of (3.9) for \( \gamma \), which yields
\[ l_\varepsilon(1/\varepsilon, \gamma) = i\frac{-4\kappa_\varepsilon C_1 L J_{-\frac{3}{4}}(z) + \varepsilon^{1/2}(\kappa_\varepsilon J_{-\frac{3}{4}}(z) + J_{\frac{3}{4}}(z))}{-4\kappa_\varepsilon C_1 L J_{-\frac{3}{4}}(z) - \varepsilon^{1/2}(\kappa_\varepsilon J_{\frac{3}{4}}(z) - J_{-\frac{3}{4}}(z))} \bigg|_{z=\frac{\varepsilon}{2\pi}}. \] (3.12)

We substitute \( \kappa_\varepsilon = -e^{\frac{i\pi}{2}} + O(\varepsilon) \) in (3.12) and use (2.24). Denote \( \alpha = (2\varepsilon)^{-1} - 4^{-1}\pi \).

Then, as \( \varepsilon \to 0 \), \( l_\varepsilon(1/\varepsilon) \) (omitting \( \gamma \) in the notation) is asymptotically equal to
\[ i \frac{L_1 e^{\frac{i\pi}{4}} \cos(\alpha - \frac{3\pi}{8}) + \varepsilon^{1/2} (-e^{\frac{i\pi}{4}} \cos(\alpha - \frac{3\pi}{8}) + \cos(\alpha + \frac{3\pi}{8}))) + O(\varepsilon)}{L_1 e^{\frac{i\pi}{4}} \cos(\alpha + \frac{3\pi}{8}) + \varepsilon^{1/2} (\cos(\alpha + \frac{3\pi}{8}) + e^{\frac{i\pi}{4}} \cos(\alpha - \frac{3\pi}{8}))} \bigg|_{z=\frac{\varepsilon}{2\pi}}. \]

After a simple but tedious computation we simplify the equation above and obtain the following lemma.

**Lemma 3.1.** Define \( \rho = \frac{1}{2\varepsilon} - \frac{\pi}{8} \) and \( B = L_1 + \sqrt{2\varepsilon} \). As \( \varepsilon \to 0 \), we have
\[ l_\varepsilon(1/\varepsilon) = \frac{\varepsilon + i \left( B^2 \sin 2\rho - \sqrt{2\varepsilon} B \cos 2\rho \right) + O(\varepsilon)}{B^2 + \varepsilon + B^2 \cos 2\rho + \sqrt{2\varepsilon} B \sin 2\rho + O(\varepsilon)} \] (3.13)
We note that in the denominator we have

\[ A(\varepsilon) \equiv B^2 + \varepsilon + B^2 \cos 2 \rho + \sqrt{2 \varepsilon} B \sin 2 \rho \]
\[ = B^2 \left( 1 + \frac{\varepsilon}{B^2} + \left( 1 + \frac{2 \varepsilon}{B^2} \right)^{\frac{1}{2}} \cos(2 \rho - \beta) \right) \geq C \varepsilon^2 \quad (3.14) \]

where

\[ \sin \beta = \frac{\sqrt{2 \varepsilon}}{(B^2 + 2 \varepsilon)^{\frac{1}{2}}}. \]

However, \( A(\varepsilon) + O(\varepsilon) \) can be controlled only when \( A(\varepsilon) \geq C \varepsilon^{1-\delta} \) for a \( \delta > 0 \) and this requires

\[ \cos(2 \rho - \beta) > -1 + C \varepsilon^{1-\delta}, \quad \delta > 0, \quad (3.15) \]

in which case we have indeed

\[
\frac{A(\varepsilon)}{B^2} = 1 + \frac{\varepsilon}{B^2} + \left( 1 + \frac{2 \varepsilon}{B^2} \right)^{\frac{1}{2}} \cos(2 \rho - \beta) \\
> 1 + \frac{\varepsilon}{B^2} + \left( 1 + \frac{\varepsilon}{B^2} - O(\varepsilon^2) \right)(-1 + C \varepsilon^{1-\delta}) \geq C \varepsilon^{1-\delta}. \quad (3.16)
\]

Let \( \Omega \subset (0, 1) \) be the set of \( \varepsilon \) which does not satisfy (3.15). Then, Taylor’s formula implies for some \( C > 0 \) that

\[
\Omega \subset \bigcup_{n=0}^{\infty} \{ \varepsilon > 0 : \left| \frac{1}{\varepsilon} - \pi - \beta - (2n + 1)\pi \right| < C \varepsilon^{(1-\delta)/2} \}. \quad (3.17)
\]

The definition of \( \beta \) and Taylor’s formula imply

\[
\sin \beta = \beta - O(\beta^3) = \frac{2 \varepsilon}{L_1} \left( 1 + \frac{2 \sqrt{2 \varepsilon}}{L_1} + \frac{4 \varepsilon}{L_1^2} \right)^{-\frac{1}{2}} = \frac{\sqrt{2 \varepsilon}}{L_1} (1 - \frac{\sqrt{2 \varepsilon}}{L_1} + O(\varepsilon))
\]

and as \( \varepsilon \to 0 \)

\[ \beta = \frac{\sqrt{2 \varepsilon}}{L_1} + O(\varepsilon). \quad (3.18) \]

(3.18) implies that \( \varepsilon > 0 \) which satisfies (3.17) must satisfy

\[
\left| \frac{1}{\varepsilon} - \frac{\pi}{4} - \frac{\sqrt{2 \varepsilon}}{L_1} - (2n + 1)\pi \right| < C \varepsilon^{(1-\delta)/2}, \quad (3.19)
\]
for some $n$ and (another) constant $C > 0$. We want to solve (3.19) for $\varepsilon$ in terms of $n$. (3.19) is equivalent to

$$\left| \varepsilon - \frac{1}{\frac{\pi}{4} + \frac{1}{L_1} + (2n + 1)\pi} \right| < \frac{C\varepsilon^{\frac{n-\delta}{2}}}{\frac{\pi}{4} + \frac{1}{L_1} + (2n + 1)\pi}. \quad (3.20)$$

For small $\varepsilon > 0$ or for large $n$, this implies $(4n\pi)^{-1} \leq |\varepsilon| \leq C(n\pi)^{-1}$ and

$$\left| \varepsilon - \frac{1}{\frac{\pi}{4} + \frac{1}{L_1} + (2n + 1)\pi} \right| < Cn^{-\frac{n-\delta}{2}}.$$

Define

$$\varepsilon(n) = \left( \frac{\pi}{4} + (2n + 1)\pi \right)^{\frac{1}{2}}. \quad (3.21)$$

Then,

$$|\sqrt{\varepsilon} - \varepsilon(n)| = \left| \frac{\varepsilon - \varepsilon(n)^2}{\sqrt{\varepsilon} + \varepsilon(n)} \right| \leq Cn^{-\frac{n-\delta}{2}}$$

and

$$\left| \frac{1}{\frac{\pi}{4} + \frac{2\varepsilon}{L_1} + (2n + 1)\pi} - \frac{1}{\frac{\pi}{4} + \frac{2\varepsilon(n)}{L_1} + (2n + 1)\pi} \right| \leq Cn^{-\frac{n-\delta}{2}}.$$

In this way, we have shown that for some $C > 0$

$$\Omega \subset \tilde{\Omega} = \bigcup_{n=0}^{\infty} \left\{ \varepsilon : \left| \varepsilon - \frac{1}{\frac{\pi}{4} + \frac{2\varepsilon(n)}{L_1} + (2n + 1)\pi} \right| < Cn^{-(5-\delta)/2} \right\}. \quad (3.22)$$

Thus, we have obtained the following theorem.

**Theorem 3.2.** Let $0 < \delta < 1$, $B = L_1 + \sqrt{2}\varepsilon$, $\varepsilon(n)$ be define by (3.21), and $\tilde{\Omega}$ by (3.22) with a suitable constant $C > 0$. Denote $\rho = \frac{B}{2\varepsilon} - \frac{\pi}{8}$. Then, for $\varepsilon \notin \tilde{\Omega}$, $B^2 + \varepsilon + B^2\cos 2\rho + \sqrt{2}\varepsilon B \sin 2\rho \geq C\varepsilon^{1-\delta}$ and as $\varepsilon \to 0$

$$l_{\varepsilon}(1/\varepsilon) = \varepsilon + i \left( \frac{B^2 \sin 2\rho - \sqrt{2}\varepsilon B \cos 2\rho}{B^2 + \varepsilon + B^2 \cos 2\rho + \sqrt{2}\varepsilon B \sin 2\rho} \right) \left( 1 + O(\varepsilon^{\delta}) \right), \quad (3.23)$$

We notice that $\Re l_{\varepsilon}(1/\varepsilon) \leq C\varepsilon^{\delta}$ and $|v_{\varepsilon}(1/\varepsilon, x)| \leq C_{\varepsilon}\exp(-C\varepsilon^{\delta}x^2/L)$ for $\varepsilon \notin \tilde{\Omega}$. Recall that the free Schrödinger operator $-\Delta$ has a zero resonance with resonant function 1. It follows that, as $\varepsilon \notin \tilde{\Omega}$ approaches 0, $v_{\varepsilon}(1/\varepsilon, x)$ approaches an oscillating function of the magnitude of the resonant function.
of $-\Delta$ on every compact interval of $\mathbb{R}$, and it does so faster, when the length $L$ becomes longer, $2L$ being the time the particle stays as a free particle. Here the behavior as $\varepsilon \to 0$ of the imaginary part of $l_\varepsilon(1/\varepsilon)$ heavily depends on how $\varepsilon$ approaches 0, however, we shall not pursue this point any further here.

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