BINDING CONDITION FOR A GENERAL CLASS OF QUANTUM FIELD HAMILTONIANS

C. GÉRARD AND I. SASAKI

Abstract. We consider a system of a quantum particle interacting with a quantum field and an external potential \( V(x) \). The Hamiltonian is defined by a quadratic form \( H^V = H^0 + V(x) \), where \( H^0 \) is a quadratic form which preserves the total momentum. \( H^0 \) and \( H^V \) are assumed to be bounded from below. We give a criterion for the positivity of the binding energy \( E_{\text{bin}} = E^0 - E^V \), where \( E^0 \) and \( E^V \) are the ground state energies of \( H^0 \) and \( H^V \). As examples of the result, the positivity of the binding energy of the semi-relativistic Pauli-Fierz model and Nelson type Hamiltonian is proved.

1. Introduction

We consider a Hamiltonian of the form

\[
H^V = H^0 + V \otimes I,
\]

acting on the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^d; dx) \otimes \mathcal{K} \), where \( \mathcal{K} \) is a Hilbert space, \( H^0 \) is a semi-bounded quadratic form on \( \mathcal{H} \) and \( V \) is the operator of multiplication by a real function \( V(x) \) in \( L^2(\mathbb{R}^d; dx) \). We are interested in the ground state energy \( E^V \) of \( H^V \). The binding energy of the system is defined by

\[
E_{\text{bin}} = E^0 - E^V.
\]

In this paper, we give a criterion for \( E_{\text{bin}} \) to be strictly positive.

Hamiltonians of the form (1) appear in models of a quantum particle interacting with a quantum field. One of the important examples is the Pauli-Fierz Hamiltonian, for which \( d = 3 \), \( \mathcal{K} \) is the bosonic Fock space over \( L^2(\mathbb{R}^3 \times \{1, 2\}) \) and

\[
H^0 = H_{\text{PF}}^0 := \frac{1}{2m}(p \otimes I + \sqrt{\alpha}A(x))^2 + I \otimes H_f
\]

where \( H_f \) is the free photon energy, \( \alpha \) is the fine structure constant, \( A(x) \) is the quantized vector potential and \( V(x) \) is the nuclear potential (see [3]). The positivity of the binding energy is established under certain conditions on the potential \( V(x) \).
energy is used as a hypothesis to establish the existence of a ground state of the Pauli-Fierz model in [3]. In [3] the positivity of the binding energy is obtained by assuming that

\[
\frac{p^2}{2m} + V(x)
\]

has a negative energy ground state. In this paper, we generalize the method developed in [3] and apply it to several types of quantum field Hamiltonians such that the semi-relativistic Pauli-Fierz Hamiltonian, the Pauli-Fierz Hamiltonian with dipole approximation and Nelson type Hamiltonians.

2. Definitions and Main Results

If \( \mathcal{H} \) is a Hilbert space we denote by \( ( \cdot | \cdot )_\mathcal{H} \) the scalar product on \( \mathcal{H} \). If \( A \) is a quadratic form on \( \mathcal{H} \), we denote by \( Q(A) \) its form domain and the value of \( A \) will be denoted by \( (\Psi | A\Phi)_\mathcal{H} \) for \( \Psi, \Phi \in Q(A) \). We use the same notation for the quadratic form associated to a self-adjoint operator \( A \), with domain \( Q(A) = \text{Dom}(|A|^\frac{1}{2}) \).

We now formulate the hypotheses of Thm. 2.1 below.

Let \( L^2(\mathbb{R}^d; dx) \) be the space of square integrable functions on \( \mathbb{R}^d \) with variable \( x = (x_1, \ldots, x_d) \), and \( \mathcal{K} \) be a separable complex Hilbert space. We denote by \( p = (p_1, \ldots, p_d) = -i\nabla_x \) the momentum operator on \( L^2(\mathbb{R}^d; dx) \) The Hilbert space of the total system is:

\[
\mathcal{H} := L^2(\mathbb{R}^d; dx) \otimes \mathcal{K}
\]

We fix a quadratic form \( H^0 \) on \( \mathcal{H} \) and an external potential \( V : \mathbb{R}^d \to \mathbb{R} \) which is a real Borel measurable function. The multiplication by \( V(x) \) is denoted by the same symbol.

The Hamiltonian of the system is obtained from the quadratic form on \( \mathcal{H} \) defined by

\[
H^V := H^0 + V.
\]

We assume the following conditions:

(H.1) There exists a dense domain \( D_0 \) such that

\[
D_0 \subseteq Q(H^0) \cap Q(V)
\]

and \( H^V \) and \( H^0 \) are closable and bounded from below on \( D_0 \).

(H.2) There exist a vector of commuting self-adjoint operators \( P_f = (P_{f,1}, \ldots, P_{f,d}) \) on \( \mathcal{K} \) such that \( H^0 \) commutes with

\[
P := (P_1, \ldots, P_d),
\]

\[
P_j = p_j \otimes I + I \otimes P_{f,j},
\]

namely, for all \( k \in \mathbb{R}^d, e^{ik \cdot P}D_0 = D_0 \) and it holds that

\[
(e^{ik \cdot P}\Psi | H^0 e^{ik \cdot P}\Phi) = (\Psi | H^0 \Phi)
\]

for all \( \Psi, \Phi \in D_0 \) and \( k \in \mathbb{R}^d \).
From (H.1), $H^V$ and $H^0$ are closable on $\mathcal{D}_0$, and we denote by $\bar{H}^V$, $\bar{H}^0$ the self-adjoint operators associated to the closure of $H^V$, $H^0$. Let

$$E^V := \inf \sigma(\bar{H}^V) = \inf_{\Psi \in \mathcal{D}_0, \|\Psi\| = 1} (\Psi|H^V\Psi)_\mathcal{H},$$

$$E^0 := \inf \sigma(\bar{H}^0) = \inf_{\Psi \in \mathcal{D}_0, \|\Psi\| = 1} (\Psi|H^0\Psi)_\mathcal{H},$$

be the ground state energies. The key assumption of the main theorem is the following:

(H.3) There exist a measurable real function $K(k)$ such that

$$\frac{1}{2}\{\Omega(k) + \Omega(-k) - 2\Omega(0)\} \leq K(k) \text{ on } \mathcal{D}_0, \forall k \in \mathbb{R}^d,$$

where $\Omega(k) := e^{-ik \cdot x} H^0 e^{ik \cdot x}$.

We set

$$h := K(p) + V$$

which is a quadratic form on $L^2(\mathbb{R}^d; dx)$. We assume that

(H.4) There exists a non-trivial subspace $\mathcal{D}_1$ of $L^2(\mathbb{R}^d; dx)$ with $\mathcal{D}_1 \subset Q(K(p)) \cap Q(V)$ such that for all $f \in \mathcal{D}_1$ and $\Psi \in \mathcal{D}_0$, $f(x)\Psi \in \mathcal{D}_0$. Moreover $\mathcal{D}_1$ is invariant under the complex conjugation, i.e. $\bar{f} \in \mathcal{D}_1$ for all $f \in \mathcal{D}_1$.

We define

$$e_0 := \inf_{f \in \mathcal{D}_1, \|f\| = 1} (f|hf)_{L^2}.$$ 

The main theorem in this paper is the following.

**Theorem 2.1.** Assume the hypotheses (H.1)–(H.4). Then the inequality

$$E^V \leq E^0 + e_0$$

holds. In particular, if $e_0 < 0$, then $E_{\text{bin}} \geq -e_0 > 0$.

3. **Proof of Theorem 2.1**

For arbitrary small $\epsilon$, we choose normalized vectors $F \in \mathcal{D}_0$, $f \in \mathcal{D}_1$ such that

$$(F|H^0 F)_\mathcal{H} \leq E^0 + \epsilon,$$

$$(f|hf)_{L^2} \leq e_0 + \epsilon.$$ 

Since by (H.4) $h$ commutes with the complex conjugation, the function $f$ can be chosen to be real. We consider the following extended Hilbert space

$$\mathcal{H}_\text{ex} := L^2(\mathbb{R}^d; dy) \otimes \mathcal{H},$$

which naturally identified with the sets of $\mathcal{H}$-valued square integrable functions $L^2((\mathbb{R}^d; dy); \mathcal{H})$. For $y \in \mathbb{R}^d$, we set $F_y := e^{iy \cdot P} F$ and consider the $\mathcal{H}$-valued function:

$$\Phi : \mathbb{R}^d \ni y \mapsto \Phi_y := f(x)F_y \in \mathcal{H}.$$
The theorem will follow easily from the following three claims:

(5) \( \Phi \in \mathcal{H}_{\text{ex}}, \|\Phi\| = 1, \)

(6) \( \Phi \in Q(I \otimes H^0), \quad (\Phi | f \otimes H^0 \Phi)_{\mathcal{H}_{\text{ex}}} \leq (f | H^0 F)_\mathcal{H} + (f | K(p)f)_{L^2}, \)

(7) \( \Phi \in Q(I \otimes V), \quad (\Phi | I \otimes V \Phi)_{\mathcal{H}_{\text{ex}}} = (f | Vf)_{L^2}. \)

Let us first prove (5), (6) and (7). We have:

\[
\int_{\mathbb{R}^d} \|\Phi_y\|^2_{\mathcal{H}} dy = \int_{\mathbb{R}^d} \|f(x)e^{iy \cdot P} F\|^2_{\mathcal{H}} dy = \int_{\mathbb{R}^d} \|e^{-iy \cdot P} f(x)e^{iy \cdot P} F\|^2_{\mathcal{H}} dy
\]

\[
= \int_{\mathbb{R}^d} |f(x - y)|^2 dy = \|f\|^2_{L^2(\mathbb{R}^d)} = \|f\|_{\mathcal{H}}^2 = 1,
\]

which proves (5). Since \( H^0 \) is bounded below, (6) will follow from

(8) \( (\Phi | I \otimes H^0 \Phi)_{\mathcal{H}_{\text{ex}}} = \int_{\mathbb{R}^d} (\Phi_y | H^0 \Phi_y)_{\mathcal{H}} dy \leq (f | H^0 F)_\mathcal{H} + (f | K(p)f)_{L^2}, \)

using that \( F \in Q(H^0) \) and \( f \in Q(K(p)) \).

Denoting by \( \mathcal{F} : L^2(\mathbb{R}^d; dy) \ni f \mapsto \hat{f} \in L^2(\mathbb{R}^d; dk) \) the unitary Fourier transform, we have:

\[
\int_{\mathbb{R}^d} (\Phi_y | H^0 \Phi_y)_{\mathcal{H}} dy = \int_{\mathbb{R}^d} (f(x)F_y | H^0 f(x)F_y)_{\mathcal{H}} dy
\]

\[
= \int_{\mathbb{R}^d} (e^{-iy \cdot P} f(x)e^{iy \cdot P} F | e^{-iy \cdot P} H^0 e^{iy \cdot P} f(x)e^{iy \cdot P} F)_{\mathcal{H}} dy
\]

\[
= \int_{\mathbb{R}^d} (f(x - y) | H^0 f(x - y)F)_{\mathcal{H}} dy
\]

\[
= \int_{\mathbb{R}^d} (e^{ik \cdot x} \hat{f}(k) | H^0 e^{ik \cdot x} \hat{f}(k)F)_{\mathcal{H}} dk
\]

\[
= \int_{\mathbb{R}^d} |\hat{f}(k)|^2 (F | \Omega(k)F)_{\mathcal{H}} dk.
\]

Since \( f \) is real valued, we have:

\[
\int_{\mathbb{R}^d} |\hat{f}(k)|^2 (F | \Omega(k)F)_{\mathcal{H}} dk = \frac{1}{2} \int_{\mathbb{R}^d} |\hat{f}(k)|^2 (F | (\Omega(k) + \Omega(-k) - 2\Omega(0))F)_{\mathcal{H}} dk + \|f\|^2_{L^2} (F | H^0 F)_\mathcal{H}
\]

\[
\leq \|f\|^2 \int_{\mathbb{R}^d} |\hat{f}(k)|^2 K(k) dk + \|f\|^2_{L^2} (F | H^0 F)_\mathcal{H}
\]

\[
= (F | H^0 F)_\mathcal{H} + (f | K(p)f)_{L^2},
\]

which proves (6).

Similarly we have

\[
(\Phi | I \otimes V \Phi)_{\mathcal{H}_{\text{ex}}} = \int_{\mathbb{R}^d} (f(x)F_y | V(x) f(x)F_y)_{\mathcal{H}} dy
\]

\[
= \int_{\mathbb{R}^d} (e^{-iy \cdot P} f(x)F_y | e^{-iy \cdot P} V(x) f(x)F_y)_{\mathcal{H}} dy
\]

\[
= \int_{\mathbb{R}^d} (f(x - y) | V(x - y) f(x - y)F)_{\mathcal{H}} dy
\]

\[
= (f | Vf)_{L^2} \|f\|^2 = (f | Vf)_{L^2},
\]

which proves (7). From (5), (6) and (7) we obtain

\[
E^V \leq (\Phi | I \otimes V \Phi)_{\mathcal{H}_{\text{ex}}} \leq (f | H^0 F)_\mathcal{H} + (f | (K(p) + V)f)_{L^2} \leq E^0 + E_0 + 2\varepsilon.
\]
Since \( \epsilon \) is arbitrary we obtain the theorem.

4. Examples

In this section we give some examples to which Thm. 2.1 can be applied. If \( \mathfrak{h} \) is a Hilbert space, we denote by

\[
\Gamma_s(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \otimes_n^s \mathfrak{h}
\]

the bosonic Fock space over \( \mathfrak{h} \). The vacuum vector in \( \Gamma_s(\mathfrak{h}) \) will be denoted by \( \Omega, a^*(t), a(t) \) for \( t \in \mathfrak{h} \) denote the creation/annihilation operators.

4.1. Semi-relativistic Pauli-Fierz Hamiltonians. The semi-relativistic Pauli-Fierz Hamiltonian is defined as follows: we take \( d = 3 \) and

\[
\mathcal{K} = \Gamma_s\left( L^2(\mathbb{R}^3 \times \{1, 2\}) \right),
\]

\[
H^V = H_{\text{SRPF}}^V := \sqrt{(p \otimes I + \sqrt{\alpha} A(x))^2 + m^2} - m + I \otimes H_f + V \otimes I,
\]

where \( \alpha \in \mathbb{R} \) is a coupling constant and \( m > 0 \) is the mass of the electron (see \cite{1}). The quantized vector potential \( A(x) \) is defined by

\[
A(x) = \frac{1}{\sqrt{2}} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \sqrt{k} \frac{\Lambda(k)}{|k|^{1/2}} e^{(\lambda)}(k) (e^{i \mathbf{k} \cdot \mathbf{x}} a_\lambda(k) + e^{-i \mathbf{k} \cdot \mathbf{x}} a^*_\lambda(k)),
\]

where \( a^*_\lambda(k), a_\lambda(k) \) are creation and annihilation operators on \( \mathcal{K}, \Lambda \) is a real-function such that \( \Lambda, |k|^{-1/2} \Lambda \in L^2(\mathbb{R}^3) \) and the polarization vectors \( e^{(\lambda)} : \mathbb{R}^3 \to \mathbb{R}^3 \) satisfy

\[
e^{(\lambda)}(k) \cdot e^{(\lambda')} (k) = \delta_{\lambda,\lambda'}, \quad e^{(\lambda)}(k) \cdot k = 0.
\]

The free photon energy \( H_f \) is defined by

\[
H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| a^*_\lambda(k) a_\lambda(k) dk
\]

Let

\[
\mathcal{F}_{\text{fin}} := \mathcal{L}[\{a^*(f_1) \cdots a^*(f_n) \Omega_{\text{photon}}, \Omega_{\text{photon}} | f_j \in C^\infty_0 (\mathbb{R}^3 \times \{1, 2\}), j = 1, 2, \ldots, n, n \in \mathbb{N}\}]
\]

be a finite photon subspace where \( \Omega_{\text{photon}} = (1, 0, 0, \ldots) \in \mathcal{K} \). We set

\[
\mathcal{D}_0 = C^\infty_0(\mathbb{R}^3) \otimes \mathcal{F}_{\text{fin}},
\]

\[
\mathcal{P}_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} k a^*_\lambda(k) a_\lambda(k) dk
\]

where \( \otimes \) indicates the algebraic tensor product. Then the above operator satisfy the condition (H.1) and (H.2). Moreover, it is proved that (H.3) holds with \( K(k) = \sqrt{k^2 + m^2} - m \) (see
C. GÉRARD AND I. SASAKI

We assume that $V \in L^1_{\text{loc}}(\mathbb{R}^3; dx)$ and set $\mathcal{D}_1 = C_0^\infty(\mathbb{R}^3)$. Then (H.4) holds. Therefore $E^{V}_{\text{SRPF}} \leq E^0_{\text{SRPF}} + e_0$ holds with

$$E^V_{\text{SRPF}} := \inf_{\Psi \in \mathcal{D}_0 : \|\Psi\| = 1} (\Psi | H^V_{\text{SRPF}} \Psi)_H, \quad \Psi = V, 0$$

$$e_0 = \inf_{f \in C_0^\infty : \|f\| = 1} (f | (\sqrt{p^2 + m^2} - m + V)f)_{L^2}.$$  

4.2. Pauli-Fierz Hamiltonian with dipole approximation. The Pauli-Fierz Hamiltonian with dipole approximation is defined by

$$H^V_{\text{DP}} = \frac{1}{2m} (p \otimes I + \sqrt{\alpha} A(0))^2 + I \otimes H_f + V \otimes I,$$

where $A(0)$ is defined in [9] with $x = 0$. $H^V_{\text{DP}}$ is defined on $\mathcal{D}_0 = C_0^\infty \hat{\otimes} \mathcal{F}_{\text{fin}}$. Clearly (H.1) holds. The operator $H^0_{\text{DP}}$ is not translation invariant, but it preserves the particle momentum $p$. Hence we set

$$P_f = 0, \quad P = p.$$  

Then (H.2) holds. For this Hamiltonian, we have

$$\frac{1}{2} (\Omega(k) + \Omega(-k) - 2\Omega(0)) = \frac{k^2}{2m},$$

which implies that (H.3) holds with $K(k) = k^2/2m$. We assume that $V \in L^1_{\text{loc}}(\mathbb{R}^3)$ and set $\mathcal{D}_1 = C_0^\infty(\mathbb{R}^3)$. Then (H.4) holds. Therefore the inequality $E^V_{\text{DP}} \leq E^0_{\text{DP}} + e_0$ holds with

$$E^V_{\text{DP}} := \inf_{\Psi \in \mathcal{D}_0 : \|\Psi\| = 1} (\Psi | H^V_{\text{DP}} \Psi)_H, \quad \Psi = V, 0$$

$$e_0 = \inf_{f \in C_0^\infty : \|f\| = 1} (f | (\frac{p^2}{2m} + V)f)_{L^2}.$$  

4.3. Nelson type Hamiltonians. We define the Nelson type Hamiltonian by

$$\mathcal{K} = \Gamma_s(L^2(\mathbb{R}^d)),$$

$$H^V = H^V_{\text{Nel}} := B(p^2) \otimes I + I \otimes H_f + P(\phi(x)),$$

where $B : \mathbb{R}_+ \to \mathbb{R}_+$ is a Bernstein function, i.e.,

$$B(u) \geq 0, \quad B(0) = 0, \quad (-1)^n \frac{d^n B(u)}{du^n} \geq 0, \quad n = 1, 2, \ldots.$$  

The field operator $\phi(x)$ is defined by

$$\phi(x) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} (g(k)e^{-ik \cdot x} a^*(k) + \overline{g(k)} e^{ik \cdot x} a(k)) dk$$

with $g \in L^2(\mathbb{R}^d), a^*, a$ are creation and annihilation operators on $\mathcal{K}$ and $P$ is a real, bounded below polynomial.

The free boson Hamiltonian $H_f$ is defined by

$$H_f = \int_{\mathbb{R}^d} \omega(k) a^*(k)a(k) dk,$$
where $\omega$ is a non-negative function. We refer the reader to [2] for a recent study of the Nelson-type Hamiltonians with Bernstein function type kinetic energy. We set

$$\mathcal{F}_{\text{fin}} := \mathcal{L}\{a^*(f_1) \cdots a^*(f_n)\Omega_b, \Omega_b| f_j \in C_0^\infty(\mathbb{R}^d), j = 1, 2, \ldots, n, n \in \mathbb{N}\},$$

$$\mathcal{D}_0 = C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}_{\text{fin}},$$

where $\Omega_b = (1, 0, 0, \ldots) \in \mathcal{K}$. Assume that $V \in L^1_{\text{loc}}(\mathbb{R}^d)$. By (13), we have

$$B(u) \leq \frac{u^3}{6} + B''(0) \frac{u^2}{2} + B'(0)u.$$ 

Hence, $C_0^\infty(\mathbb{R}^d) \subset \text{Dom}(B(p^2))$. Then $H_{\text{Nel}}^V$ and $H_{\text{Nel}}^0$ are well-defined on $\mathcal{D}_0$ and (H.1) holds. We set

$$P_f = \int_{\mathbb{R}^d} ka^*(k)a(k)dk.$$ 

Then, $H_{\text{Nel}}^0$ commutes with $P_j = p_j \otimes I + I \otimes P_{f,j}, j = 1, \ldots, d$ and (H.2) holds. Next we check (H.3). We note that

$$\Omega(k) + \Omega(-k) - 2\Omega(0) = B((p + k)^2) + B((p - k)^2) - 2B(p^2).$$

We have the following lemma:

**Lemma 4.1.** For all $p, k \in \mathbb{R}^d$, the inequality

$$\frac{1}{2} \left(B((p + k)^2) + B((p - k)^2) - 2B(p^2)\right) \leq B(k^2).$$

holds.

**Proof.** It is known that any Bernstein function can be written in the form

$$B(u) = a + bu + \int_{\mathbb{R}^+} (1 - e^{-tu}) \mu(dt), \quad (u \geq 0)$$

where $a, b$ are non-negative constants and $\mu$ is a non-negative measure on $\mathbb{R}^+$ such that $\int_{\mathbb{R}^+} \min\{t, 1\} \mu(dt) < \infty$ (see [2]). Hence it is sufficient to prove the inequality

$$e^{-(p+k)^2t} - e^{-(p-k)^2t} + 2e^{-p^2t} \leq 2(1 - e^{-k^2t}),$$

for all $t \geq 0$ and $p, k \in \mathbb{R}^d$. If $t = 0$, (15) is trivial. Without loss of generality, one can set $t = 1$. Moreover we can assume that $k = (\kappa, 0, 0), \kappa \geq 0$ by the spherical symmetry of (15). Then (15) will follow from

$$b_\kappa(p) := e^{-(p_1 + \kappa)^2} - e^{-(p_1 - \kappa)^2} + 2e^{-p_1^2} \leq 2(1 - e^{-\kappa^2}),$$

where $p_1$ is the first component of $p$. This completes the proof.
where \( p = (p_1, p_2, p_3) \). It is enough to show that \( b_\kappa(p_1) \leq 2(1 - e^{-\kappa^2}) \) for \( \kappa > 0 \) and \( p_1 > 0 \).

We set \( p_1 = a\kappa \) with \( a > 0 \). Then

\[
b_\kappa(p_1) = e^{-a^2\kappa^2} \left[-e^{-\kappa^2}(e^{-2a\kappa^2} + e^{2a\kappa^2}) + 2\right] \\
\leq e^{-a^2\kappa^2} \left[-2e^{-\kappa^2} + 2\right] \\
\leq 2(1 - e^{-\kappa^2}),
\]

where we used the inequality \( e^{-2a\kappa^2} + e^{2a\kappa^2} \geq 2 \) and \( e^{-a^2\kappa^2} \leq 1 \).

Lemma 4.1 implies that (H.3) holds with \( K(k) = B(k^2) \). By setting \( \mathcal{D}_1 = C_0^\infty(\mathbb{R}^d) \), (H.4) holds. Therefore, by Theorem 2.1, \( E_{\text{Nel}}^V \leq E_{\text{Nel}}^0 + \epsilon_0 \) holds with

\[
E_{\text{Nel}}^2 := \inf_{\Psi \in \mathcal{D}_0, \|\Psi\|=1} (\Psi^\dagger H_{\text{Nel}}^V \Psi)_{\mathcal{H}}, \quad \sharp = V, 0 \\
\epsilon_0 := \inf_{f \in \mathcal{D}_1, \|f\|=1} (f^\dagger (B(p^2) + V) f)_{L^2}.
\]

**References**

[1] F. Hiroshima and I. Sasaki, *On the ionization energy of semi-relativistic Pauli-Fierz model for a single particle*, Kokyuroku Bessatsu B21 (2010), 25–34.

[2] J. Lőrinczi, F. Hiroshima, and V. Betz, *Feynman-Kac-type theorems and Gibbs measures on path space*, vol. 34, Walter De Gruyter, 2011, Seminar on Probability, Studies in Mathematics.

[3] E. H. Lieb M. Griesemer and M. Loss, *Ground states in non-relativistic quantum electrodynamics*, Invent Math 145 (2001), no. 1, 557–595.