Flavor-dependent U(3) Nambu Jona Lasinio coupling constant

Fabio L. Braghin
Instituto de Física, Federal University of Goias
Av. Esperança, s/n, 74690-900, Goiânia, GO, Brazil

Abstract

A non-perturbative one gluon exchange quark-antiquark interaction is considered to compute flavor dependent U(3) Nambu-Jona-Lasinio (NJL)-type interaction of the form $G_{ij} \Gamma(\psi\lambda\Gamma\psi)(\psi\lambda\Gamma\psi)$ for $i,j = 0...8$ and $\Gamma = I, i\gamma_5$ from one loop polarization process with non-degenerate $u$-$d$-$s$ quark effective masses. The resulting NJL-type coupling constants in all channels are resolved in the long-wavelength limit and numerical results are presented for different choices of an effective gluon propagator. Leading deviations with respect to a flavor symmetric coupling constant are found to be of the order of $(M_{f_2}^2 - M_{f_1}^2)/\sqrt{(M_{f_2}^2 + M_{f_1}^2)}$, for $n = 1, 2$, where $M_{f_i}$ are the effective masses of quarks $f_1, f_2 = u, d$ and $s$. The scalar channel coupling constants $G_{ij,s}$ can be considerably smaller than pseudoscalar ones. The effect of the flavor-dependence of coupling constants for the masses of pions and kaons may be nearly of the same order of magnitude as the effect of the $u$-$d$ $s$ quark mass non-degeneracy. The effect of these coupling constants is also verified for some of the light scalar mesons masses, usually described by quark-antiquark states, and for some observables of the pseudoscalar mesons.

1 Introduction

Theoretical investigations and predictions for low energy strong interacting systems have important support from QCD effective models among which the Nambu-Jona-Lasinio (NJL) type models [1, 2, 3]. They are suitable for describing phenomena related to the dynamical chiral symmetry breaking (DChSB) according to which massive constituent quarks can be defined and are responsible for most part of the hadron masses. The NJL model can be derived in terms of QCD degrees of freedom in different ways [4, 5, 6, 7, 8, 9]. Lately, an effective gluon mass was found to be appropriate to parameterize the deep infrared behavior of a gluon propagator and quark-NJL coupling constant has been identified roughly to $G_{NJL} \propto 1/M_{QCD}^2$. Non-perturbative or effective gluon propagators take into account part of the non-Abelian gluon dynamics and they might be suitable to provide numerical estimates for hadron properties. Eventually they make possible a clear relation of fundamental processes and fundamental degrees of freedom with the NJL model framework and description since they are expected, at least, to produce DChSB. In addition to the explicit chiral and flavor symmetry breaking due to the non-degenerate current quark mass, the couplings to electromagnetic fields also break these symmetries contributing to masses [2, 10, 11, 12, 13, 14] and coupling constants [15]. It might be interesting to verify if, and to what extent, NJL coupling constant receive flavor-dependent contributions. This should be of relevance for a fine tuning description hadron masses and dynamics.

Current quark masses, at the energy scale $\mu = 2$ GeV, are approximately $m_u \simeq 2.1$ MeV, $m_d \simeq 4.7$ MeV and $m_s \simeq 93$ MeV [16] and they get amplified due to the DChSB with the formation of quark-antiquark scalar condensates. In spite of the need of the electromagnetic corrections to describe fully hadron masses there are well-known strong-interaction contributions, for example, for masses of pions and kaons, the quasi-Goldstone bosons, that are, respectively, of the order of $m_{\pi^\pm} = m_{\pi^0} \approx 0.1$ MeV and $m_{K^\pm} = m_{K^0,\bar{K}^0} \approx \pm 5.3$ MeV [17]. This contribution for the pions mass difference is very small due to the small difference between up and down quark masses. The electromagnetic neutral and charged pions mass difference is somewhat larger, of the order of $4$ MeV [11, 12, 13, 14]. The charged and neutral kaons mass difference has the opposite sign of the pion mass difference and is larger due to the larger strange quark mass. Flavor symmetry breaking corrections are small for light hadrons but important for a good description of hadrons. In QCD, the quark current masses are the only parameters that control flavor symmetry breaking. This issue has far consequences in some effective approaches as in Chiral Perturbation Theory, as an Effective Field Theory for the low energy regime [15]. By starting from QCD to understand effective models, one might expect that the flavor symmetry breaking encoded in different quark current masses might spread and have consequences to a variety of parameters and coupling constants in effective models by means of quantum effects. Therefore NJL- coupling constants might also be expected to receive flavor-dependent contributions from quantum effects. Schwinger-Dyson equations (SDE) approach for light and heavy hadrons indicates couplings constants might be flavor dependent [19]. The lightest scalar mesons, that could be expected to be chiral partners of the pseudoscalar ones, do not seem to be compatible with usual quark model due to the apparent impossibility of fitting all the
experimental data with quark-antiquark structures as prescribed by the simplest quark model [20, 21] although it might be partially appropriated [20, 16, 22, 23]. One of the specific problems with the attempt to describe some of the lightest scalar mesons in a U(3) nonet from the standard NJL-model scheme is the inverted mass behavior of the \(a_0(1^+)\) and \(K^*(1(2^+))\) [24]. In the present work this issue appears again and, although no complete solution for this problem is obtained or proposed, we expect to show further insights for this problem. In fact there are several different theoretical calculations with different proposals for their structures such as composed by mixed states with tetraquarks, glueballs, mesons molecules or coupled channels resonances [21, 25, 26, 27, 25, 29, 30, 31, 32]. The full problem of the light scalar mesons structure will not be really addressed in the present work. Nevertheless it becomes interesting to introduce as many different effects as possible to test their individual contributions and the predictive power of the model.

In this work explicit chiral and flavor symmetry breaking contributions to the NJL coupling constant are derived by considering vacuum polarization in a flavor U(3) model in which quark-antiquark interaction is mediated by a (non-perturbative) gluon exchange. These coupling constants are resolved in the local long-wavelength limit in terms of quark and gluon propagators and they will be used to calculate light pseudoscalar and scalar mesons masses. An effective non-perturbative gluon propagator will be considered such as to incorporate to some extent non-Abelian dynamics with the crucial requirement to produce dynamical chiral symmetry breaking and the large constituent quark masses due to the gluon cloud. The method extends previous works for u-d-s or u-d degenerate quarks [5, 33]. Because the logics and steps of the calculation has been shown with details in previous works, in the next section the main steps are only briefly outlined. In Section 3 numerical estimations for the long-wavelength local limit for the resulting four point quark interaction are presented as NJL-type flavor-dependent interactions. This is done for two types of effective gluon propagators and different values of Lagrangian and effective quark masses. In Section 4 the effects of such flavor-dependent NJL-coupling constants are verified on some light mesons masses. Because not all the scalars might seemingly be described by quark-antiquark states, and the pseudoscalar \(\eta - \eta'\) mesons require further interactions [34], the masses of pseudoscalar pions and kaons and of the light scalar \(a_0\) and \(K^*\) (or \(\kappa\)) will be investigated. Being that the flavor-dependent coupling constants \(G_{ij}\) were found to describe the light pseudoscalar mesons masses, further observables will be presented in Section (3.3) to assess the change in their values if \(G_{ij}\) are used to redefine the gap equations.

### 2 Quark determinant and leading current-current interactions

The following low energy quark effective action [37, 38, 39] will be considered:

\[
Z = N \int D[\bar{\psi}, \psi] \exp i \int_x \left[ \bar{\psi} (i \slashed{D} - m_f) \psi - \frac{g^2}{2} \int_y j^a_{\mu}(x) \bar{R}^{ab}_{\mu
u}(x-y) j^b_{\nu}(y) + \bar{\psi} J + J^* \psi \right],
\]

(1)

Where the color quark current is \(j^a_{\mu} = \bar{\psi} \lambda_a \gamma^\mu \psi\), the sums in color, flavor and Dirac indices are implicit, \(\int_x\) stands for \(\int dx, a, b, \ldots = 1, \ldots, N_c^2 - 1\) stand for color in the adjoint representation, and \(m_f\) is the quark current masses matrix [10], with indices of the flavor SU(3) fundamental representation \(f = u, d, s\). The adjoint representation of flavor SU(3) will be used with indices \(i, j, k = 0, 1, \ldots, N_f^2 - 1\) with the additional matrix \(\lambda_0 = \sqrt{2/3}\) to complete the U(3) algebra. In several gauges the gluon kernel is usually argued to be written in terms of the transversal and longitudinal components in momentum space, \(R_T(k)\) and \(R_L(k)\), as: \(\bar{R}^{ab}_{\mu\nu}(k) = \delta_{ab} \left[ (g^{\mu\nu} - k^\mu k^\nu) R_T(k) + \frac{k^\mu k^\nu}{k^2} R_L(k) \right]\). Although this decomposition may not be exact in general because of confinement-related effects [10], it is important to emphasize that numerical results will be calculated by considering effective gluon propagators whose contributions from other components might be parameterized into these two components. Besides that, it can be shown that contributions of other terms, of the form \(\delta(k^2)\) or derivatives of it, in the effective gluon propagator for the results can be expected to be much smaller. Even if other terms arise from the non Abelian structure of the gluon sector, the quark-quark interaction [11] is a leading term of the QCD effective action. The use of a dressed (non-perturbative) gluon propagator already takes into account non-Abelian contributions that guarantees important effects. Among these, it will be assumed and required that this dressed gluon propagator provides enough strength for DChSB as obtained for example in [41, 42, 43, 44, 45]. To some extent the present work will follow previous developments by adopting the background field method to introduce the background quarks that, dressed by gluons, give rise to the constituent quarks. A complete account of the calculations below was presented in Refs. [5, 33, 46, 47].

To explore the flavor structure of the interaction in the action \([11]\), that will be denoted by \(\Omega\), a Fierz transformation is performed for the quark-antiquark channel resulting in \(\mathcal{F}(\Omega) = \Omega_F\). We are only interested in this work in the color-singlet scalar-pseudoscalar states sector and vector and axial currents will be neglected. Color non-singlet terms are suppressed by a factor \(1/N_c\) and they might give rise to higher order colorless contributions [40, 48]. The following bilocal currents are needed to describe the resulting terms: \(j^a_{\mu}(x,y) = \bar{\psi}(x) \lambda_a \Gamma^a \psi(y)\) where \(q = s, p\), for scalar and pseudoscalar currents: \(\Gamma_s = \lambda_i I\) (with the 4x4 Dirac identity), \(\Gamma_p = i\gamma^\nu \lambda_i\), where \(\lambda_i\) are the flavor SU(3) Gell Mann matrices \((i = 1...8)\) and \(\lambda_0 = I\sqrt{2/3}\). The resulting \(s\) and \(p\) non local interactions are the following:

\[
\Omega_F = 4 \alpha g^2 \left\{ [j^s_{\mu}(x,y)j^s_{\nu}(y,x)] + [j^p_{\mu}(x,y)j^p_{\nu}(y,x)] R(x-y) \right\},
\]

(2)
where $\alpha = 2/9$ and $R(x-y) \equiv R = 3R_T(x-y) + R_L(x-y)$.

Next, the quark field will be split into the background field, $\bar{\psi}$ constituent quark, and the sea quark field, $\psi_2$, that might either form light mesons and the chiral condensate. This sort of decomposition is not exclusive to the background field method (BFM) and it is found in other approaches \[49\]. At the one loop BFM level it is enough to perform this splitting for the bilinears $\bar{\psi}\Gamma_4\psi$ \[53\] \[48\], and it can be written that:

$$
\bar{\psi}\Gamma^a\psi \rightarrow (\bar{\psi}\Gamma^a\psi)_2 + (\bar{\psi}\Gamma^a\psi)_1,
$$

(3)

where $(\bar{\psi}\psi)_2$ will be treated in the usual way as sea quark of the NJL model and the full interaction $\Omega_F$ is split accordingly $\Omega_F \rightarrow \Omega_1 + \Omega_2 + \Omega_{12}$ where $\Omega_{12}$ contains the interactions between the two components. This separation preserves chiral symmetry, and it might not be simply a low and high energies mode separation. The auxiliary field method makes possible to introduce the light quark-antiark quark chiral states, the chiral condensate and mesons. Therefore this procedure improves one loop BFM since it allows to incorporate DCSB. Because it is a standard procedure in the field, it will not be described. To make possible a clear evaluation of the effects of the resulting NJL-coupling constants the corresponding gap equations at this level, which can arise for the local limit of a auxiliary scalar field, will be considered to be those of the NJL model, given in eq. \[27\] for the case of coupling constants $G_{ff} = G_0 = 10 \text{ GeV}^{-2}$ as discussed below. This guarantees a clear and fair subsequent comparison of the effects of the flavor dependent coupling constants. Otherwise, the relation between effective masses and NJL-coupling constants would not be clear. The non trivial solutions for these gap equations allow one to define the quark effective masses, $M^*_f = m_f + S_f$ where $S_f (f = u,d,s)$. The quark kernel can be written as:

$$
S_{0}(x-y) = (i\partial - M^*_f)^{-1}\delta(x-y).
$$

(4)

The aim of this work is to present corrections to the NJL-type interaction so that the meson sector in terms of auxiliary fields will be neglected. The quark determinant can then be written as:

$$
S_d = C_0 + \frac{i}{2} Tr \ln \left\{ \left( 1 + S_0 \left( \sum_{q} a_q \Gamma_{qj} \right) \right)^* \left( 1 + S_0 \left( \sum_{q} a_q \Gamma_{qj} \right) \right) \right\},
$$

(5)

where the following quantities were defined:

$$
\sum_{q} a_q \Gamma_{qj} = \sum_{q} a_q \Gamma_{qj}(x,y) = 2K_0 R(x-y) \left[ (\bar{\psi}(y)\lambda_i\psi(x)) + i\gamma_5\lambda_i(\bar{\psi}(y)i\gamma_5\lambda_i\psi(x)) \right],
$$

(6)

where $K_0 = \alpha g^2$ and $C_0 = \frac{i}{2} Tr \ln [S_0^{-1}S_{0}^{-1}]$ that reduces to a constant in the generating functional.

A large quark mass expansion is performed with a zero order derivative expansion \[51\] for the local limit. The first term of the expansion is a non-degenerate mass term $M_{3,f}$ proportional to the masses from gap equations \[47\] that will not be investigated further. The second order terms of the expansion correspond to four-fermion interactions with chiral and flavor symmetries breaking. These terms, in the local limit, can be written as:

$$
\mathcal{L} = M_{3,f} (\bar{\psi}\psi)_f + G_{ij,s} (\bar{\psi}\lambda^i\psi)(\bar{\psi}\lambda^j\psi) + G_{ij} (\bar{\psi}i\gamma_5\lambda^i\psi)(\bar{\psi}i\gamma_5\lambda^j\psi)
$$

$$
= M_{3,f} (\bar{\psi}\psi)_f + G_{ij} \left[ (\bar{\psi}\lambda^i\psi)(\bar{\psi}\lambda^j\psi) + (\bar{\psi}i\gamma_5\lambda^i\psi)(\bar{\psi}i\gamma_5\lambda^j\psi) \right] - G_{ij}^{ab} (\bar{\psi}\lambda^i\psi)(\bar{\psi}\lambda^j\psi),
$$

(7)

where the coefficients were resolved and, after a Wick rotation for the Euclidean momentum space in the (very long-wavelength) zero momentum transfer limit, they are the following:

$$
M_{3,f} = d_1 N_c K_0 Tr_{PD} Tr_{PF} \int \frac{d^4k}{(2\pi)^4} S_0(k)\lambda_i R(k),
$$

(8)

$$
G_{ij,s} = G_{ij} + G_{ij}^{ab} = d_2 N_c K_0^2 Tr_{PD} Tr_{PF} \int \frac{d^4k}{(2\pi)^4} S_0(k)\lambda_i R(k) S_0(k)\lambda_j R(k),
$$

(9)

$$
G_{ij} = d_2 N_c K_0^2 Tr_{PD} Tr_{PF} \int \frac{d^4k}{(2\pi)^4} S_0(k) R(k)i\gamma_5\lambda_i S_0(k) R(k)i\gamma_5\lambda_j,
$$

(10)

where $Tr_{PD}, Tr_{PF}$ are the traces in Dirac and flavor indices, $d_n = \frac{1-(1+n)^n}{2}$. $S_0(k)$ is the Fourier transform of $S_0(x-y)$. If the effective gluon propagator has other terms proportional to $\delta(k^2)$, or derivatives, \[40\] it can be shown that their resulting contribution will be suppressed, if not disappearing, with respect to the finite momenta component encoded in $R(k)$, at least, by factors $1/(16\pi^2 M^*_f M^*_f)$. Eventually these further contributions may also vanish because of explicit dependences on internal loop momenta $k_i$. To calculate these traces in flavor indices the following strategy was adopted. Each of the quark propagators, originally in the fundamental representation, was written as a combination of kernels in the adjoint representation by diagonalizing with the correct diagonal GellMann matrices.
This makes possible an unambiguous and straightforward calculation of the flavor-traces in these equations. Quark mass matrix can be written as:

\[ M = M_0 \sqrt{3/2} \lambda_0 + M_3 \lambda_3 + M_8 \sqrt{3} \lambda_8, \]  

where \( M_0, M_3, M_8 \) are combinations of the up, down and strange quark effective masses. The quark propagator can then be written as:

\[ S_{0\mu}(k) = I \left[ A(i\mathbf{k} + M_0) + 2M_6 C + \frac{2}{3} M_3 B \right] + \lambda_3 \left[ B(i\mathbf{k} + M_0) + M_3 (A + C) + M_8 B \right] + \lambda_8 \sqrt{3} \left[ C(i\mathbf{k} + M_0) + M_8 (A - C) + \frac{M_3 B}{3} \right], \]  

where

\[ A = \frac{1}{3} \left( \frac{1}{R_u} + \frac{1}{R_d} + \frac{1}{R_s} \right), \]

\[ B = \frac{1}{2} \left( \frac{1}{R_u} - \frac{1}{R_d} \right), \]

\[ C = \frac{1}{6} \left( \frac{1}{R_u} + \frac{1}{R_d} - \frac{2}{R_s} \right). \]

In these equations \( R_f = (k^2 - M_f^2 + i\epsilon) \). Flavor traces of up to four Gell Mann matrices were calculated, i.e. \( Tr_F(\lambda_m \lambda_n \lambda_j \lambda_j) \) where \( m, n = 0, 3, 8 \) and \( i, j = 0, \ldots, 8 \), with all combinations. The four-point interactions above \( G_{ij} \), in the limit of degenerate quark masses \( m_u = m_d = m_s \), reduce to those of Ref. \[33\] with a different coefficient due to the \( U(3) \) group. This way of writing Eq. (7) suggests a non-unambiguous definition of a nearly chiral "symmetric part" of NJL interaction as \( G_{ij} \) and the chiral symmetry breaking part \( G_{ij}^{sb} \) that arises only for the scalar sector. \( G_{ij} \) is not really a chiral-symmetric interaction because of the flavor symmetry breakings for all the flavor channels \( i, j \). The integrals for \( G_{ij} \) have two components, one of them strongly dependent on momentum \( k \) and the other strongly dependent on the quark masses, whereas \( G_{ij}^{sb} \) is written only in terms of the second of these integrals. This difference between the integrals \( G_{ij} \) favors the above separation of regular and (strongly) symmetry breaking (sb) coupling constant. An important property of these coupling constants is the following:

\[ G_{ij} = G_{ji}, \quad G_{ij,s} = G_{ji,s}. \]  

All the integrals in Eqs. (9) and (10) are ultraviolet (UV) finite and infrared regular if the gluon propagator contains a parameter such as a gluon effective mass or Gribov type parameter. \( G_{ij}, G_{ij}^{sb} \) and \( G_{ij}^{sb} \) have dimension of mass\(^{-2}\). For some observables, however, it is more appropriate to define the following coupling constants between quark currents in the fundamental representation:

\[ G_{ij}(\bar{\psi}\lambda^i\psi)(\bar{\psi}\lambda^j\psi) = 2G_{f_1f_2}(\bar{\psi}\psi)_{f_1}(\bar{\psi}\psi)_{f_2}, \]  

(17)

being that none of the types of mixing terms, \( G_{i\neq j} \) or \( G_{f_1\neq f_2} \), will be considered in most part of the present work.

### 3 Numerical results

In the following, two types of the effective gluon propagators, that incorporate the quark-gluon running coupling constant \( g \), are written and will be considered for the numerical calculations. The first effective gluon propagator is a transversal one extracted from Schwinger Dyson equations calculations \[43,44\]. It can be written as:

\[ D_{1,2}(k) = g^2 R_T(k) = \frac{8\pi^2}{\omega^2} D e^{-k^2/\omega^2} + \frac{8\pi^2 \gamma_m E(k^2)}{\ln \left[ \tau + (1 + k^2/\Lambda_{QCD}^2)^2 \right]}, \]  

(18)

where \( \gamma_m = 12/(33 - 2N_f) \), \( N_f = 4 \), \( \Lambda_{QCD} = 0.234 \text{GeV} \), \( \tau = e^2 - 1 \), \( E(k^2) = [1 - \exp(-k^2/[4m_t^2]])/k^2 \), \( m_t = 0.5 \text{GeV} \), \( D = 0.55^3/\omega \) (GeV\(^{-2}\)) and \( \omega = 0.5 \text{GeV} \).

The second type of gluon propagator is based in a longitudinal effective confining parameterization \[41\] that can be written as:

\[ D_{II,\alpha=5,6}(k) = g^2 R_L,\alpha(k) = \frac{K_F}{(k^2 + M_5^2)^2}, \]  

where \( K_F = (0.5\sqrt{2\pi})^2/0.6 \text{ GeV}^2 \), as considered in previous works \[47,48\], with, however, either a constant effective gluon mass \( M_5 = 0.8 \text{ GeV} \) or a running effective mass given by:

\[ M_6 = \frac{0.5}{1 + k^2/\omega_6^2} \text{GeV} \]  

for \( \omega_6 = 1 \text{GeV} \).
It can be noted that these effective gluon propagators exhibit different normalizations and resulting numerical values for eqs. (9) and (10) might be quite different. Instead of addressing specific issues on the gluon propagators and their normalizations a more pragmatic approach was adopted so that the relevant issue is not their normalization but the overall momentum dependence that contributes in the momentum integrals that generate the flavor-dependencies of the results. Therefore a normalization procedure, common to all effective gluon propagators, to compare different results is needed. In addition to that, to make easier the comparison of the flavor-dependent effects a reference coupling constant with the value $G_0 = 10$ GeV$^{-2}$ will be considered and the following normalized quantities will be displayed in the Tables below:

$$G_{ij}^n = \frac{G_{ij}}{G_0}, \quad G_{ij,s}^n = \frac{G_{ij,s}}{G_0}$$

(20)

In a first analysis the resulting coupling constants, $G_{ij}$ and $G_{ij,s}$, are expected to be additive corrections to a constant NJL-model coupling constant. Given that the overall absolute values are not determined, this sort of multiplicative normalization was chosen instead of an additive correction. Therefore coupling constants are normalized with respect to the channel $G_{ij=11}^0 = 10$ GeV$^{-2}$ that is close to usual values adopted in the literature for the NJL model. Besides that, this way of normalizing coupling constants becomes more appropriate to investigate consequences for the differences between each of the flavor-channels, in particular for the light mesons mass differences.

3.1 Flavor-dependence of coupling constants

Besides the gluon propagators, that contain implicitly a quark-gluon running coupling constant, the quark constituent masses for the quark propagator are also needed. Sets of values for the parameters that will be considered are shown in Table (1) with the Lagrangian quark masses $m_u$, $m_d$ and $m_s$. These values, together with the ultraviolet (three-dimensional) cutoff $\Lambda$, were chosen such that they satisfy a gap equation with a NJL-coupling constant of reference $G_0$ and this makes possible a comparison of the contribution of the particular flavor-dependent coupling constant on the results. Besides that, it will be required that the resulting neutral pion and kaon masses are in quite good agreement with experimental values for the case of symmetric coupling constant $G_{ij} = G_0 = 10$ GeV$^{-2}$. A combination of up and down quark masses, with the UV cutoff, determine neutral (or charged) pion mass and, with the additional strange quark mass, one obtains the neutral (or charged) kaon mass. Although only the effective quark masses are needed for the estimation of coupling constants, the other parameters of the Table are needed to find the mesons masses. Sets of parameters have two main labels $X$ and $Y$ which correspond to different ways of dealing with the scalar mesons channels below. The subscripts in labels $X$ and $Y$ are just numbers to identify a set of parameters, they have no physical meaning. The numerical method used to solve the gap equations and the BSE required to identify each set of values for the quark masses and cutoff and for each possible solution of the BSE. For that, a name $X$ or $Y$ was chosen with numbers to identify them. $X$ stands for sets of parameters with which scalar mesons masses were found by considering the same coupling constants as for the pseudoscalar mesons channel, i.e. $G_{ij} = G_{ij,s}$. This is equivalent to set $G_{ij}^X = 0$ as discussed above. Besides that, sets of parameters $X$ will produce somewhat better results, as discussed below. $Y$ stands for sets of parameters with which scalar mesons were found with coupling constants obtained from eqs. (9) instead of $G_{ij}$, i.e. $G_{ij,s} \neq G_{ij}$.

| set of parameters | $M_u$ | $M_d$ | $M_s$ | $m_u$ | $m_d$ | $m_s$ | $\Lambda$ |
|------------------|-------|-------|-------|-------|-------|-------|---------|
| $X_{20} = Y_{18}$ | 389   | 399   | 600   | 3     | 7     | 123   | 675     |
| $X_{21}$         | 392   | 396   | 600   | 4     | 6     | 123   | 675     |
| $Y_{14}$         | 362   | 362   | 574   | 5     | 5     | 123   | 665     |
| $Y_{19}$         | 391   | 395   | 600   | 4     | 6     | 163   | 675     |

In Tables (2) and (3) the resulting values of coupling constants $G_{ij}$ are presented for each set of masses and for the gluon effective propagators: $D_{1,2}$, $D_{1,5}$ and $D_{1,6}$. In Table (4) the resulting values of $G_{ij,s}$ are presented for the same sets $Y$ of Table (3). The set with $G_0$ corresponds to a constant and symmetric choice of reference $G_{ij} = G_{ij,s} = 10b_{ij}$ GeV$^2$. This set, $G_0$, is independent of the effective gluon propagator and it was included to make possible a clearer analysis of the effects of the flavor-dependence of the coupling constants on the quark-antiquark mesons masses as discussed above. Note that $G_0$ has the same value of the normalized $G_{ij}^X$ and this is important for understanding the role of the flavor-dependent coupling constants on observables. The set of parameters $Y_{18}$ has the same values of $G_{ij}$ as the set $X_{20}$ in Table (2) and it was included separated in this Table to make simpler the comparison of
results, in particular for the scalar mesons channel in the next section. Some entries were not included because they are equal to those already displayed in the Table being CP-conserving interactions:

\[ G_{22} = G_{11}, \quad G_{55} = G_{44}, \quad G_{77} = G_{66}, \]
\[ G_{22,s} = G_{11,s}, \quad G_{55,s} = G_{44,s}, \quad G_{77,s} = G_{66,s}. \]  

(21)

Table 2: Numerical results for \( G_{ij} \) for the sets of parameters \( X \) and different gluon propagators. The entries for \( G_0 \) are simply defined in this Table and they correspond to fixed values independent of any gluon propagator for the sake of comparisons.

| SET    | \( G^{n}_{11} \) GeV\(^{-2} \) | \( G^{n}_{33} \) GeV\(^{-2} \) | \( G^{n}_{44} \) GeV\(^{-2} \) | \( G^{n}_{66} \) GeV\(^{-2} \) | \( G^{n}_{88} \) GeV\(^{-2} \) | \( G^{n}_{00} \) GeV\(^{-2} \) | \( G^{n}_{03} \) GeV\(^{-2} \) | \( G^{n}_{08} \) GeV\(^{-2} \) | \( G^{n}_{38} \) GeV\(^{-2} \) |
|--------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| \( X_{20} \)-\( D_{1.2} \) | 10.00 | 10.00 | 9.77 | 9.69 | 7.61 | 8.60 | 0.11 | 1.80 | 0.14 |
| \( X_{20} \)-\( D_{11.5} \) | 10.00 | 10.00 | 9.82 | 9.76 | 8.13 | 8.90 | 0.08 | 1.41 | 0.10 |
| \( X_{20} \)-\( D_{11.6} \) | 10.00 | 10.00 | 9.89 | 9.70 | 7.72 | 8.66 | 0.10 | 1.72 | 0.13 |
| \( X_{20} \)-\( G_0 \) | 10.00 | 10.00 | 10.00 | 10.00 | 10.00 | 10.00 | 0 | 0 | 0 |
| \( X_{21} \)-\( D_{1.2} \) | 10.00 | 10.00 | 9.74 | 9.71 | 7.61 | 8.60 | 0.04 | 1.80 | 0.05 |
| \( X_{21} \)-\( D_{11.5} \) | 10.00 | 10.00 | 9.80 | 9.78 | 8.12 | 8.90 | 0.03 | 1.41 | 0.04 |
| \( X_{21} \)-\( D_{11.6} \) | 10.00 | 10.00 | 9.76 | 9.73 | 7.72 | 8.66 | 0.04 | 1.72 | 0.05 |
| \( X_{21} \)-\( G_0 \) | 10.00 | 10.00 | 10.00 | 10.00 | 10.00 | 10.00 | 0 | 0 | 0 |

The interaction channels exclusively of up and down quarks, \( G_{11} \) and \( G_{33} \), are seen to be close to each other, i.e. almost no flavor dependent correction. In fact, in the leading order \( G_{33} - G_{11} \propto (M^*_s - M^*_d)/(M^*_s + M^*_d)^2 \sim 10^{-4} \) that is very small. The channels involving strange quarks have a larger deviation from the symmetric limit because \( G_{ij} = 4, 5, 8, 0 - G_{\text{sym}} \propto (M^*_s - M^*_d)/(M^*_s + M^*_d)^n \) for \( n = 1, 2 \), where \( G_{\text{sym}} \) is obtained by the limit of equal quark masses, the flavor symmetric limit. Moreover, larger strange quark masses induce smaller values of the coupling constants. This is in qualitative agreement with [19]. There are also mixing couplings \( G_{ij \neq j} \) among the neutral channels from the diagonal generators of flavor \( U(3) \) group, i.e. for \( i, j = 0, 3, 8 \). In the pseudoscalar channel they are all proportional to the quark mass differences in the leading order: \( G_{03} \propto (M^*_d - M^*_u)/M^* \), \( G_{08} \propto (M^*_s - M^*_d)/M^* \) and \( G_{38} \propto (M^*_d - M^*_u)(M^*_s - M^*_d)/M^* \). Note that the mixing \( G_{08} \) has the largest values and the difference between its values for the sets of parameters \( X_{20} \) and \( X_{21} \) is too small. Together with \( G_{03} \) and \( G_{38} \) these effective coupling constants can be associated to the \( \eta \), \( \eta' \) - \( \pi^0 \) mixings. These mixings, however, will not be investigated in the present work. For the set of parameters \( Y_{14} \) in Table 3, that contains \( m_u = m_d \), it yields \( G_{03} = G_{38} = 0 \). The set of parameters with \( G_0 \) necessarily implies \( G_{i \neq j} = 0 \).
Table 3: Numerical results for $G_{ij}$ for the sets of parameters $Y$ and different gluon propagators. The entries for $G_0$ are simply defined in this Table and they correspond to fixed values independent of any gluon propagator for the sake of comparisons. The sets of parameters $Y_{18}$ has the same values of $X_{20}$ in Table (2).

| SET   | $G_{11}^d$ GeV$^{-2}$ | $G_{33}^d$ GeV$^{-2}$ | $G_{44}^d$ GeV$^{-2}$ | $G_{66}^d$ GeV$^{-2}$ | $G_{88}^d$ GeV$^{-2}$ | $G_{00}^d$ GeV$^{-2}$ | $G_{03}^d$ GeV$^{-2}$ | $G_{08}^d$ GeV$^{-2}$ | $G_{38}^d$ GeV$^{-2}$ |
|-------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| $Y_{14}$-$D_{1,2}$ | 10.00 | 10.00 | 9.69 | 9.69 | 7.52 | 8.53 | 0 | 2.39 | 0 |
| $Y_{14}$-$D_{11,5}$ | 10.00 | 10.00 | 9.76 | 9.76 | 8.06 | 8.85 | 0 | 1.46 | 0 |
| $Y_{14}$-$D_{11,6}$ | 10.00 | 10.00 | 9.71 | 9.71 | 7.63 | 8.60 | 0 | 1.04 | 0 |
| $Y_{14}$-$G_0$ | 10.00 | 10.00 | 10.00 | 10.00 | 10.00 | 10.00 | 0 | 0 | 0 |
| $Y_{18}$-$D_{1,2}$ | 10.00 | 10.00 | 9.77 | 9.69 | 7.61 | 8.60 | 0.11 | 1.80 | 0.14 |
| $Y_{18}$-$D_{11,5}$ | 10.00 | 10.00 | 9.82 | 9.76 | 8.13 | 8.90 | 0.08 | 1.41 | 0.10 |
| $Y_{18}$-$D_{11,6}$ | 10.00 | 10.00 | 9.78 | 9.70 | 7.72 | 8.67 | 0.10 | 1.72 | 0.13 |
| $Y_{18}$-$G_0$ | 10.00 | 10.00 | 10.00 | 10.00 | 10.00 | 10.00 | 0 | 0 | 0 |
| $Y_{19}$-$D_{1,2}$ | 10.00 | 10.00 | 9.74 | 9.71 | 7.61 | 8.60 | 0.04 | 2.04 | 0.06 |
| $Y_{19}$-$D_{11,5}$ | 10.00 | 10.00 | 9.80 | 9.78 | 8.12 | 8.90 | 0.03 | 1.33 | 0.04 |
| $Y_{19}$-$D_{11,6}$ | 10.00 | 10.00 | 9.76 | 9.73 | 7.72 | 8.66 | 0.04 | 1.95 | 0.05 |
| $Y_{19}$-$G_0$ | 10.00 | 10.00 | 10.00 | 10.00 | 10.00 | 10.00 | 0 | 0 | 0 |

The coupling constants $G_{ij,s}$, eq. (9) of the scalar sector, Table (4), are only shown for the sets of parameters $Y$ because for the sets of parameters $X$ it was considered that $G_{sb} \to 0$. The gluon propagator $D_{1,2}$ yields, in average, considerably lower values of coupling constants being that the normalization (20) is, at least in part, responsible for that. These coupling constants may be even negative (repulsive), and the effective gluon propagator $D_{11,5}$ yields the largest values. The relative values of the $G_{ij,s}$ with respect to the $G_{ij}$ are highly dependent on the quark propagator structure similarly to the problems that emerge in form factors [52]. Nevertheless the relative shift of values of $G_{ij,s}$ (and the corresponding changes in the mesons masses) for each of the channels for a particular set of parameters should be meaningful. In average the scalar channel coupling constants $G_{88,s}$ and $G_{00,s}$ are somewhat lower than the others. Detailed investigations of the mesons mixings and the whole scalars mesons octet/nonet mass problems are outside the scope of this work.
3.2 Effect on some light mesons masses

In this section the effect of the flavor-dependent coupling constants on the masses of light quark-antiquark pseudoscalar pions and kaons is analyzed. Besides that, the effect on some of the scalar quark-antiquark states, usually associated to the scalars $a_0$ and $K^*$, is also analyzed. This will be done according to the following quark structure [23, 24]:

$$ a_0^u \sim (\bar{u}u - \bar{d}d), \quad a_0^d \sim \bar{d}u, \quad K_0^\pm, \bar{K}_0^\mp \sim \bar{s}u, \bar{s}d, \quad K_+^* \sim \bar{s}u, \bar{s}d. $$

It must be kept in mind, however, that the scalars sector should not be expected to be fully worked out and described due to the particularities of their structures quoted in the Introduction. Problems in the description of light scalars are particularly stronger for the $\sigma$ that seemingly cannot be a quark-antiquark state as reminded in the Introduction.

The Bethe-Salpeter equation, or bound state equation, (BSE) for the quark-antiquark meson sector in the NJL model is usually investigated at the Born approximation level, therefore for constant Bethe-Salpeter kernel that can be written as $K = 2G$ [23, 53, 54] for the case of diagonal interaction $G_{ii}$ therefore by neglecting mixing interactions.

It provides the following condition to determine a particular meson mass $P_0^M = M$ of the channel $\Gamma, i \ (\Gamma = I, i\gamma_5$ respectively for S, PS, scalar or pseudoscalar):

$$ 1 - 2G_{ii} I_{\Gamma,i,f_1,f_2}^i(P_0^M = M^2, \vec{P} = 0) = 0, $$

for the corresponding flavor $i=0, \ldots, 8$ of the quark-antiquark state, when written in terms of the $f_1, f_2 = u, d, s$ quark flavors. Note that the dependence of the results on the effective gluon propagator is encoded in the resulting value of $G_{ii}$ as discussed above. The following integral, for the four-momentum $P$ of the meson, was defined:

$$ I_{\Gamma,i,f_1,f_2}^i(P) = itr_{F,C,D} \int d^4k (2\pi)^4 \left[ S_{i,f_1}(k) \Gamma_{\lambda_i} S_{i,f_2}(k + P) \Gamma_{\lambda_i} \right], $$

where $tr_{F,C,D}$ stands for the traces in flavor, color and Dirac indices. Note that the indices $i, \lambda$ of the GellMann matrices of the adjoint representation are tied with the indices $f_1, f_2$ of the fundamental representation of the quark propagators for each particular channel in the integral and in the coupling constants $G_{ii}$. In the pseudoscalar channel the following association appears: for the charged and neutral pions: $(i = 1, 2$ with $f_1, f_2 = u, d/d, u)$ and $(i = 3,$
with \( f_1, f_2 = \bar{u}u + \bar{d}d \) respectively. For the charged and neutral kaons: \((i = 4, 5 \text{ and } f_1, f_2 = \bar{u}, s, \bar{s}, u) \) and \((i = 6, 7 \text{ with } f_1, f_2 = \bar{d}s/\bar{s}d) \) respectively. By the usual reduction of eq. \((23)\) with the GAP equation considered so far that is the one with the coupling constant of reference \( G_0 \), that eliminates the quadratic UV divergence, the following forms respectively for the pseudoscalar and scalar mesons are obtained:

\[
(P^2 - (M_{f_1}^* - M_{f_2}^*)^2)G_{ij}I_{ij}^{P} = \frac{G_{ij}}{2G_0} \left( \frac{m_{f_1}}{M_{f_1}^*} + \frac{m_{f_2}}{M_{f_2}^*} \right) + 1 - \frac{G_{ij}}{G_0},
\]

\[
(P^2 - (M_{f_1}^* + M_{f_2}^*)^2)G_{ij}I_{ij}^{S} = \frac{G_{ij}}{2G_0} \left( \frac{m_{f_1}}{M_{f_1}^*} + \frac{m_{f_2}}{M_{f_2}^*} \right) + 1 - \frac{G_{ij}}{G_0},
\]

When the coupling constants become equal to \( G_0 \) and this equation reduces to the usual BSE with an unique coupling constant \([2,3]\). The Goldstone theorem is straightforwardly verified by considering the usual chiral limit for which the effective quark masses are all equal. The integral \( I_{ij}^{P} \) in eq. \((25)\) is UV logarithmic divergent and it was solved with the same 3-dimensional UV cutoff exhibited in Table \((1)\). Besides that, the pole of the scalar quark-antiquark bound state \(|P_0| = M_S\), where \( M_S \) is the mass of the scalar meson, might be located in the region of external momenta larger than the sum of two quark effective masses \( P_0 > (M_{f_1}^* + M_{f_2}^*) \) such that there might have additional poles in the integrals \( I_{ij}^{S} \) indicating instability of the bound state. An IR cutoff \([55]\), \( \Lambda_{IR} = 120 \text{ MeV} \), was used in this case. Its contribution for the pseudoscalar mesons masses can be neglected as usual is. The value of this cutoff is somewhat smaller than values in the literature because these larger values lead to a too large suppression of the momentum integrations modifying the scalar mesons masses and their mass differences. It is also well known that these integrals might be dependent on the regularization method used \([50]\). However it has been shown that the regularization method used usually does not modify the light mesons properties preserving quite well the predictive power of the model \([57]\). Besides that, if one is interested in the influence of the flavor-dependent coupling constants on the energy/mass of the quark-antiquark meson bound state, i.e. in mesons mass differences, the regularization method should not produce leading order effect. Finally, masslessness of pions and kaons is recovered in the chiral limit as described above and in the literature\([2]\).

In Table \((5)\) results for some of the light mesons masses are presented for the sets of parameters \( X \) and three effective gluon propagators presented above \((D_{I,2}, D_{I,5} \text{ and } D_{I,6})\). The case for value of reference, \( G_0 \), which is independent of the gluon propagator is also considered. There are two types of comparisons to be done for a given set \( X \) or \( Y \) below. Firstly, by reading the lines of the Tables, one can obtain the neutral-charged mesons mass difference for the pseudoscalar and scalar mesons. By reading the columns of the Tables, always within a particular set of parameter \( X \) or \( Y \), it is possible to verify the role of the flavor-dependent coupling constants (for each given effective gluon propagator). Although the masses of neutral mesons were used to choose the particular sets of parameters in Table \((1)\), the charged mesons masses are obtained as a consequence of the choice of the sets of parameters, being therefore predictions. It is important to emphasize again that one must be concerned with the mesons mass differences rather than the absolute values of masses. This is because slightly different absolute values for the masses are easily obtained whereas the mass difference between neutral and charged mesons are consequences of quark masses differences and also specific values of \( G_{ij} \) or \( G_{ij,s} \). Furthermore, and more importantly, the comparison among the results from different effective gluon propagators \((D_I, D_{II} \text{ and } G_0)\), within a particular set of parameter \( X \) or \( Y \), indicates the contribution of varying specifically the flavor dependent coupling constants \( G_{ij} \) or \( G_{ij,s} \). Lower values for the pion masses, of the order of 135 or 136 \text{ MeV}, were chosen for the fixed cutoff, instead of larger values such as 140\text{ MeV} with larger UV cutoffs, because an electromagnetic contribution for the charged pion mass is also expected. Besides that, the mass difference between neutral and charged pions due to strong Interaction is small and it depends, first of all, on the up and down Lagrangian quark mass difference \( \delta_{ud} = (m_d - m_u) \). Consequently it is dependent on \( M_u^* - M_d^* \); or, more specifically, \((m_u^2 - m_d^2) \sim (M_u^* - M_d^*)^2/(M_u^* + M_d^*)^2 \) as expected \([12]\). The smaller \( \delta_{ud} \) from set \( X_{21} \) induces smaller neutral and charged pion mass difference. Besides that, the smallness of \( \delta_{ud} \) also favors smaller charged and neutral kaons mass differences. The neutral-charged pion mass difference for \( X_{20} \) is around 0.3 – 0.4\text{ MeV} slightly larger than the obtained by other sets such as \( X_{21} \). It indicates that the up and down quark masses chosen for this set are slightly larger than needed. The difference between the coupling constants \( G_{11} \) and \( G_{33} \), however, in general is very small (of the order of \( 10^{-3} \) \text{ GeV}, not showed in the Tables above) such that, usually, it does not cause meaningful change in the pion masses. The exception was found for the set \( X_{20} \) (the one with larger u-d mass difference) for which the shifts from the values obtained with \( G_0 \) can be of the order of 0.1\text{ MeV}, that is slightly smaller than the mass difference between neutral and charged pions.

The mass difference between neutral and charged kaons is also considerably in better agreement with expectations for the \( X_{21} \) set of parameters than for the set \( X_{30} \). These mass differences are proportional to \((M_u^* - M_d^*)/(M_u^* + M_d^*) \) (where \( f_1, f_2 = u, d \text{ or } s \)) in agreement with other works \([12]\). However the most interesting comparison to be noted is the fact that different values for \( G_{ii} \), due to different effective gluon propagators for a given set \( X_{20} \) or \( X_{21} \), lead to different shifts in the kaons masses. The coupling constants \( G_{14} \) and \( G_{66} \) are smaller than the constant value \( G_0 \) for all the gluon propagators and therefore the kaons masses are shifted to larger values. These shifts are around
4 – 6 MeV, although the resulting effect of each effective gluon propagator considered might be smaller, of the order of 1 – 5MeV. The shifts in kaons masses due to flavor dependent coupling constants have nearly the same modulus as the mass difference between neutral and charged kaons. The weak decay constant will be calculated in the next section.

Concerning the scalar channel: The leading effect for the scalar mesons masses are the large quark effective masses. The conditions for each of the scalar quark-antiquark (with flavors $f_1 - f_2$) bound state \((25)\) might be written approximately as:

$$M_{S,f_1 f_2}^2 \sim M_{PS,f_1 f_2}^2 + (M_{f_1} + M_{f_2})^2 + O\left(\frac{1}{G_{ij} I_{f_1 f_2,S}}\right),$$ \hspace{1cm} \text{(26)}$$

where $M_S$ and $M_{PS}$ are the corresponding scalar and pseudoscalar mesons masses for that particular channel (with quark-antiquark $f_1 - f_2$), $\tilde{G}_{ij} = G_{ij}$ or $G_{ij,s}$, and $\tilde{I}_{f_1 f_2,S}$ is the UV logarithmic divergent integral that however depends on $M_{S,f_1 f_2}^2$. This approximate equation is in agreement with particular limits of equations from other works \[27\]. The largest contribution for the masses of the scalar mesons masses come from the second term, i.e. the sum of the quark and antiquark effective masses, and also the first term for the $K^*$ with quark structure analogous to the kaons. Besides that, the IR cutoff in the integrals for the scalar mesons masses makes the integrals to be slightly suppressed with respect to the values obtained in the pseudoscalar channel. The contributions of the coupling constants of the last term are usually smaller than the first two terms. The resulting $a_0 - K^*$ mass hierarchy, according to the structure \[22\], is inverted as it occurs in the simplest versions of the NJL model \[24\]. It is possible to fit correctly their masses either by introducing other Lagrangian interactions or by seeking specific values of quark masses and coupling constants $G_{ij}$ to fit the desired values. This second procedure, however, is too much artificial from the physical point of view. Therefore the entries in the Tables allow for a limited comparison that may be useful, mainly, for the dependence of the mesons masses on each gluon propagator, or conversely, on flavor dependent coupling constants. It turns out however that it is also an interesting comparison for the mass differences of charged and neutral mesons. The overall pattern of masses is similar to the one of the pseudoscalar channel: set of parameters $X_{20}$ yields larger neutral-charged mass difference than the set of parameters $X_{21}$. However the mass difference is of the order of 2MeV for the $a_0^\pm - a_0^0$ isotriplet and around 4MeV for the isoduplets $K_0^*, K_1^+$ for the set of parameters $X_{21}$ that is better than the set $X_{20}$ whose mass difference are quite large. The shifts in the $K^*$ masses due to the coupling constants are sizeable only for the set of parameters $X_{20}$. The experimental values for these neutral-charged mesons mass differences are seemingly slightly smaller than for the case of the kaons for example. Maybe the flavor dependent coupling constants for the scalar channels should have the role of compensating the quark-effective mass non degeneracy effect on the neutral-charged scalar mesons mass differences. The effective gluon propagators used in this work did not provide the corresponding needed $G_{ij,s}$ to reduce accordingly the neutral-charged scalar mesons mass differences.
Table 5: Masses of pseudoscalar and scalar mesons states for the sets of parameters $X$ and the corresponding coupling constants given above without the electromagnetic mass corrections. For this set $X$ the bound state equation for scalars and pseudoscalars were solved with the same coupling constant $G_{ij}^0$. In the last line there are experimental values from [10]. 

| SET    | $M_{\pi^0}$ | $M_{\pi^\pm}$ | $M_{K^0}$ | $M_{K^\mp}$ | $M_{K^0}/M_{\pi^0}$ | $M_{K^0}/M_{K^\mp}$ |
|--------|-------------|----------------|-----------|--------------|----------------------|----------------------|
| $X_{20}$-$D_{1.2}$ | 135.8 | 136.2 | 502 | 492 | 781/789 | 1009/999 |
| $X_{20}$-$D_{11.5}$ | 135.9 | 136.2 | 501 | 492 | 780/789 | 1003/995 |
| $X_{20}$-$D_{11.6}$ | 135.8 | 136.2 | 502 | 492 | 781/789 | 1009/999 |
| $X_{20}$-$G_0$ | 135.9 | 136.2 | 496 | 488 | 781/789 | 1008/998 |
| $X_{21}$-$D_{1.2}$ | 136.1 | 136.2 | 499 | 495 | 787/789 | 1006/1002 |
| $X_{21}$-$D_{11.5}$ | 136.1 | 136.2 | 498 | 494 | 787/789 | 1006/1002 |
| $X_{21}$-$D_{11.6}$ | 136.1 | 136.2 | 499 | 495 | 787/789 | 1006/1002 |
| $X_{21}$-$G_0$ | 136.1 | 136.2 | 494 | 490 | 787/789 | 1006/1002 |
| Exp. | 135 | 139.6 | 497.6 | 493.7 | 980++ | 892-896++ |

In Table 5 the masses of pseudoscalar and scalar mesons are presented for the sets of parameters $Y_{14}$, $Y_{18}$ and $Y_{19}$, with the different effective gluon propagators, and the corresponding coupling constants given above in Tables 4 and 5. For these sets $Y$, the bound state equation for scalars and pseudoscalars were solved with different coupling constants, $G_{ij}^0$ and $G_{ij}^p$ respectively.

In the pseudoscalar channel, the trends are very similar to the previous table: results for the set of parameters $Y_{19}$ is the same as those for $X_{20}$ in Table 5 whereas resulting pattern of set of parameters $Y_{19}$ is very similar to the one for $X_{21}$. Therefore similar conclusions apply. The effect of the lowering of the coupling constants $G_{44}$ and $G_{66}$ is to push the kaons masses to slightly larger values than the masses obtained with $G_0$. The effect of the change in the coupling constants on the scalar mesons masses is considerably larger than this effect on the pseudoscalar masses. And this might be interesting for the correct complete description of the scalars structures. In addition to these sets, there is a set of parameters $Y_{14}$ for which $m_u = m_d$ and therefore $M_{K^0} = M_{K^\mp}$ that yield all pions with the same mass, and all kaons with the same mass as expected. Nevertheless, the coupling constants $G_{44}$ and $G_{66}$ are slightly, but sufficiently, different for the different gluon propagators $D_{1.2}, D_{11.5}$ and $D_{11.6}$ as much as in the other sets $Y$.

The main tendency presented in the scalar mesons masses is the larger mesons masses. This is, in one hand, due to contribution of the quark effective masses in eq. 22. In another hand, there is a non leading effect that is the fact that the scalar coupling constants $G_{ij}^0$ are pretty much smaller than $G_{ij}^p$. The largest shifts in the scalar masses appear in the $K^*$ states because the relative changes in the values of $G_{44,s} - G_{66,s}$ are larger than changes in $G_{33,s} - G_{11,s}$ together with strange quark effective mass values. Besides that, there is almost no unique trend for the shifts in the scalar mesons masses. This is due to the very diverse values obtained for the scalar channel coupling constants $G_{ij}^0$ in Table 4. The resulting overall mass difference between the $a_0$ and $K^*$, in average, might be as large as 320MeV for example for set $Y_{14} - D_{1.2}$ or 218MeV for example for $Y_{14} - D_{11.6}$. The neutral-charged scalar mesons mass differences are quite different for each of the set of parameters. This suggests, again, that both the quark effective mass and flavor-dependent coupling constants might contribute for a fine-tuning of hadrons spectra and interactions. The main needed effect of inverting the mass hierarchy of $a_0$ and $K^*$ does not occur.
Table 6: Masses of pseudoscalar and scalar mesons states for the sets of parameters Y and the corresponding coupling constants given above without electromagnetic mass corrections. For this set Y the bound state equation for scalars and pseudoscalars were solved respectively with coupling constants $G_{ij,s}$ and $G_{ij}$. In the last line there are experimental values from [16]. ∗ it has an electromagnetic contribution (∼ 4MeV). ++ comparison valid within the assumption for the scalar mesons quark flavor structure adopted in (22).

| SET     | $M_{\pi^0}$ | $M_{\pi^+}$ | $M_{K^0}$ | $M_{K^+}$ | $M_{a_0}/M_{a_+}$ | $M_{s^0}/M_{s^+}$ |
|---------|-------------|-------------|-----------|-----------|------------------|------------------|
| Y14–D1,2 | 135.1       | 135.1       | 492       | 492       | 758/758          | 970/970          |
| Y14–D1,5 | 135.1       | 135.1       | 491       | 491       | 729/729          | 964/964          |
| Y14–D1,6 | 135.1       | 135.1       | 492       | 492       | 740/740          | 978/978          |
| Y14–G0   | 135.1       | 135.1       | 487       | 487       | 725/725          | 946/946          |
| Y18–D1,2 | 135.8       | 136.2       | 502       | 492       | 824/830          | 1068/1060        |
| Y18–D1,5 | 135.9       | 136.2       | 501       | 492       | 788/793          | 1028/1017        |
| Y18–D1,6 | 135.8       | 136.2       | 502       | 492       | 812/820          | 1067/1045        |
| Y18–G0   | 135.9       | 136.2       | 496       | 488       | 781/789          | 1009/998         |
| Y19–D1,2 | 136.1       | 136.2       | 499       | 495       | 831/835          | 1037/1037        |
| Y19–D1,5 | 136.1       | 136.2       | 498       | 494       | 792/793          | 1024/1020        |
| Y19–D1,6 | 136.1       | 136.2       | 499       | 495       | 817/820          | 1048/1045        |
| Y19–G0   | 136.1       | 136.2       | 494       | 490       | 787/789          | 1006/1002        |
| Exp.     | 135         | 139.6∗      | 498       | 494       | 980++           | 892-896++        |

3.3 Leading effects on gap equations and other observables

The effects of the flavor dependent coupling constants $G_{ij}$ of Table 2, without the mixing couplings, on different observables are presented in this section. To have a more complete idea of these effects, observables were calculated firstly by considering the NJL model with the coupling constant of reference, $G_0 = 10 GeV$, and usual resulting solutions from the gap equations shown in Table 1. Secondly, by recalculating effective quark masses from gap equations with the coupling constants shown in Table 2.

First of all therefore, the new or corrected gap equations in the Euclidean momentum space are written as:

$$M_f^* - m_f = 4N_c G_{ff} \int \frac{d^4k}{(2\pi)^4} \frac{M_f^*}{k^2 + M_f^*},$$

where $G_{ff}$ were extracted from eq. (17) in the absence of both types of mixing interactions: $G_{i\not{j}}$ and $G_{f_1\not{f_2}}$. The resulting values are presented in the first lines of Table 7 for the sets of parameters X and compared to the initial values for $G_0$. The sets of parameters Y yield too small values of coupling constants $G_{ij,s}$ which are not strong enough to allow for the DChSB from eqs. (27). These too low values of these coupling constants might be rather a consequence of the common multiplicative normalization adopted in eq. (20) as discussed above. Therefore, because this normalization have shown to be extremely appropriated for pseudoscalar channel as discussed in the last section and, besides that, the scalar channel is not really completely addressed in this work, this discussion will be restricted to the sets of parameters X. The reduced values, mainly of $G_{ss}$ and $G_{SS}$ which by the way are mostly dependent on the strange quark mass and which provide smaller contributions for $G_{ss}$ than for $G_{uu}$ or $G_{dd}$, lead to reduced value of the strange effective quark mass $M_f^*$. Since the self consistent way of solving the model presents some further complications, including instabilities of the solutions, in the following we present results that indicate the tendency of the observables when calculating them with flavor -dependent coupling constants and corrected effective masses.
Observables predicted by the model are exhibited in Table (7), by considering two different ways of calculating them, compared to experimental or expected values (e.v.).

The up, down and strange chiral scalar quark-antiquark condensates are implicitly calculated in the gap equations and they can be written as:

\[ <(\bar{q}q)_f> \equiv -Tr(\Sigma_{0,f}(k)) , \tag{28} \]

where \( \Sigma_{0,f}(k) \) is the quark propagator. In Table (7) \( <q\bar{q}>_G \) stands for the quark condensates calculated with flavor-dependent coupling constants \( G_{ij} \) from Table (2), but with the original quark effective masses \( M_q^f \). Due to the larger reduction of \( G_{qs} \) (\( f=\bar{u},d \) and \( s \)) the strange quark condensate is increased. Secondly, the condensates calculated with the same flavor dependent coupling constants but with corrected effective masses \( M_q^f \) are written as \( <q\bar{q}>_{M_q^c} \). Their values are improved with respect to all the other values and get closer to lattice calculations. It is worth stressing, lattice results had been calculated at the energy scale of \( \mu \sim 1-2 \) GeV in the works quoted and others cited therein.

The sea-quark couplings to pions and kaons, \( G_{qq,\pi} \) and \( G_{qq,K} \), as the residue of the poles of the BSE at the Born level, eq. (28), [2] [3] are presented as calculated with the initial quark effective masses and with the corrected quark masses by means of the equation:

\[ G_{qqPS} = \left( \frac{\partial \Pi_{ij}(P^2)}{\partial P^2_0} \right)^{-2} \bigg|_{(P_0,P) = 0} , \tag{29} \]

where the flavor indices are tied with the quantum numbers of the meson \( PS \) as shown above. Please note that the sets of parameters \( G_0 \) do not receive correction from the flavor-dependent coupling constants and can be used for comparison. The effect of the corrected quark masses is to reduce the difference between \( G_{qq\pi} \) and \( G_{qqK} \) because the effective masses \( M_q^f \) are closer to each other.

The charged pseudoscalar mesons (pions and kaons) weak decay constant were also calculated from 24 [3]:

\[ F_{ps} = \frac{N_c G_{qqPS}}{4} \int \frac{d^4q}{(2\pi)^4} Tr_{F,D}(\gamma_{i,\mu,\gamma_{\lambda},\lambda}) S_{f_1}(q + P/2) \lambda_j S_{f_2}(q - P/2) , \tag{30} \]

where \( f_1, f_2 \) correspond to the quark/antiquark of the meson and \( i, j \) are the associated flavor indices as discussed for eq. (28). In Table (7) they are presented for the flavor dependent coupling constants and corrected masses \( M_q^f \). The values for the sets of parameters with \( G_0 \) provide the results for the case of flavor-dependent coupling constants. Because the strange quark mass decreases with the use of the flavor-dependent interaction, the kaon decay constant has its value closer to the pion decay constant being the only observable whose behavior is not the expected one.

Finally, the pseudoscalar mesons mixing that is responsible for the eta-eta’ mass difference will be shortly addressed according to the following ansatz. The logics of the auxiliary field method was adopted and the pseudoscalar flavor-quark current interactions can be exchanged by auxiliary fields \((P_i \sim \bar{q}_i\gamma_{\lambda},\lambda,q)\). Within the auxiliary field method the following identification, which yields the correct dimension of each of the fields, can be done by implementing functional delta functions in the generating functional [35] [39]: \( \delta((\bar{q}_i\lambda,q) - \bar{q}_j\lambda,q) \). By considering effective masses for the adjoint representation auxiliary fields, \( M_{ai}^2 P_i^2 \), the following terms with mixings can be written for the neutral mesons whose states are obtained from the diagonal generators of the algebra:

\[ \mathcal{L}_{mix} = \frac{M_{\eta}^2}{2} P_3^2 + \frac{M_{\eta}^2}{2} P_8^2 + \frac{M_{\eta}^2}{2} P_0^2 + \frac{G_{08}}{G_0^2} P_0 P_8 + \frac{G_{03}}{G_0^2} P_0 P_3 + \frac{G_{38}}{G_0^2} P_3 P_8 , \tag{31} \]

where \( M_{ai}^2 \) include the contributions from \( G_{i=j} \) derived above. The mixing terms \( G_{i\neq j} \), however, are exclusively obtained from the one-loop interactions [10]. The neutral pion \( (P_3) \) mixings and mass are now neglected and by performing the usual rotation to mass eigenstates \( \eta, \eta' \) [14] it can be written:

\[
|\eta> = \cos \theta_{ps}|P_8> - \sin \theta_{ps}|P_0>, \\
|\eta'> = \sin \theta_{ps}|P_8> + \cos \theta_{ps}|P_0>.
\]

By calculating it and comparing to the above \( 0-8 \) mixing, the following \( \eta-\eta' \) mixing angle is obtained:

\[ \sin(2\theta_{ps}) = \frac{2G_{08}}{G_0^2(M_{\eta}^2 - M_{\eta'}^2)} , \tag{33} \]

The values for \( \theta_{ps} \) are shown in the Table (7) and they are smaller than the expected values. As remarked above, the coupling constants \( G_{08} \) are basically the same for the two different sets of parameters \( X_{20} \) and \( X_{21} \). This new mechanism for mesons mixings may be not sufficient for describing the full mixing.

In the last lines of the Table the reduced chi-square, \( \chi^2_{red} \) is shown for the sets of parameters \( X \) for which ten observables have been taken into account, two of which are fitted parameters/observables \( (M_{\pi^0} \) and \( M_{K^0} \).
The first $\chi^2_{red}$ was done by using $<\bar{q}q>_G$, the second by using $<\bar{q}q>_M$ and the third without the predictions for the quark scalar condensates. The reason is that the values for the chiral quark condensates have large (and the largest) deviations from the expected values obtained from lattice calculations, therefore their contributions for the $\chi^2_{red}$ are too large. So the analysis of the reduced chi square can be done by considering separately the behavior of $<\bar{q}q>$. Results show the tendency of $\chi^2_{red}$ when comparing for the initial calculation, for $G_0$, with the contribution of the flavor-dependent coupling constants by means of the effective masses $M'$. Besides that, note that the numerical difference between the quark condensates calculated in lattices, LQCD, have a large deviation from the NJL-prediction. Another interesting comparison of the $\chi^2_{red}$, that shows specifically the effects of the flavor -dependent coupling constant in $\chi^2_{red,M'}$ for a specific set of parameters $X$, is between the sets of parameters $G_0$ (with no flavor-dependent coupling constants effects) with the other sets ($I, 2, 3, 5$ and $I, 6$). The same comparison between $G_0$ and the other sets $X$ for the reduced chi-squared, $\chi^2_{red,G}$ and $\chi^2_{red,M'}$ (no $<\bar{q}q>$), might be misleading because the chiral condensates (whose flavor-dependent corrections are larger) either are not corrected by $G_{ij}$ or are not taken into account.

Table 7: Observables for non self consistent calculations by considering the parameters $X_{20}$ and $X_{21}$ discussed above for frozen values of the mesons masses. (e.v.) correspond to the experimental or expected values. In the last lines the reduced chi-square are presented for three different calculations and for each of the sets of parameters, $X_{20}, X_{21}$, by considering two fitted observables $M_{ps}$ and $M_{K_G}$. Masses, decay constants and chiral condensate $<\bar{q}q>$ are written in MeV, coupling constants, $G_{psK_G}, \theta_{ps}$ and $\chi^2$ are dimensionless.

| $X_{20}$ | $X_{21}$ | $X_{20}$ | $X_{21}$ | $X_{20}$ | $X_{21}$ | $X_{20}$ | $X_{21}$ |
|---------|---------|---------|---------|---------|---------|---------|---------|
| $M_u^0(G_0)$ | 389 | 389 | 389 | 389 | 392 | 392 | 392 | 392 |
| $M_u^*(G_{uu})$ | 307 | 325 | 311 | 389 | 310 | 328 | 314 | 392 |
| $M_d^0(G_0)$ | 399 | 399 | 399 | 399 | 396 | 396 | 396 | 396 |
| $M_d^*(G_{dd})$ | 319 | 336 | 325 | 399 | 316 | 333 | 320 | 396 |
| $M_{ss}^*(G_0)$ | 600 | 600 | 600 | 600 | 600 | 600 | 600 | 600 |
| $M_{ss}^*(G_{ss})$ | 349 | 400 | 360 | 600 | 349 | 400 | 360 | 600 |
| $-<\bar{u}u>(G^0)$ | 348 | 346 | 348 | 338 | 349 | 347 | 348 | 338 |
| $-<\bar{u}u>(G_1)$ | 322 | 326 | 322 | 338 | 322 | 326 | 323 | 338 |
| $-<\bar{d}d>(G^0)$ | 350 | 348 | 349 | 340 | 350 | 347 | 349 | 340 |
| $-<\bar{d}d>(G_1)$ | 324 | 328 | 325 | 340 | 324 | 328 | 325 | 340 |
| $-<\bar{s}s>(G^0)$ | 424 | 407 | 420 | 363 | 424 | 407 | 420 | 363 |
| $-<\bar{s}s>(G_1)$ | 331 | 340 | 333 | 363 | 331 | 340 | 333 | 363 |
| $G_{qqr}(M^0)$ | 3.28 | 3.40 | 3.31 | 3.83 | 3.28 | 3.40 | 3.31 | 3.83 |
| $G_{qqr}(M^1)$ | 3.38 | 3.61 | 3.43 | 4.50 | 3.39 | 3.63 | 3.44 | 4.59 |
| $f_{\pi}(M^0)$ | 95.6 | 97.5 | 96.0 | 103.1 | 95.6 | 97.6 | 96.0 | 103.1 |
| $f_{K}(M^0)$ | 97.8 | 100.8 | 98.5 | 107.3 | 97.6 | 100.7 | 98.3 | 107.2 |
| $\theta_{ps}(0)$ | -8.2° | -6.5° | -7.9° | 0.0 | -8.3° | -6.5° | -7.9° | 0.0 |
| $\chi^2_{red,G}$ | 313 | 295 | 309 | 244 | 314 | 295 | 309 | 245 |
| $\chi^2_{red,M'}$ (no $<\bar{q}q>$) | 87 | 92 | 88 | 116 | 86 | 92 | 87 | 117 |

4 Final remarks

In this work, flavor symmetry breaking corrections to the NJL-type quark interactions were derived from a quark-antiquark interaction mediated by dressed gluon exchange. All resulting coupling constants are directly proportional to the quark-gluon running coupling constant and they depend on the quark and (effective) gluon propagators. Whereas the coupling constants $G_{ij}$, defined as almost chiral-symmetric couplings, can be associated to the pseudoscalar channel, the coupling constants $G_{ij,s}$ of the scalar channels can be numerically quite smaller and they present stronger dependence on the gluon propagator. Different sets of coupling constants $G_{ij}$ and $G_{ij,s}$ were obtained from sets of quark masses by employing different effective gluon propagators. Different sets of parameters that were labeled as $X$ or $Y$ correspond to different solutions of a NJL gap equation for a coupling constant $G_0 = 10 GeV^{-2}$ of reference. Although the quark effective masses differences induce the flavor-dependence of coupling constants, $G_{ij}, G_{ij,s}$, the effective gluon propagator also slightly contributes for the determination of their relative strength. The effects of the flavor-dependent coupling constants were identified by comparing results obtained with them with
the results for the fixed reference value $G_0$ in the Tables. To make possible a correct assessment of the effects of the flavor-dependent coupling constants, a normalization for $G_{ij}$ was proposed being defined to the pseudoscalar channel and therefore more reasonable for pseudoscalar interactions. The channels with strangeness develop smaller values of $G_{ij}$, i.e., larger deviations from $G_0$, due to the larger strange quark mass. The mixing type interactions $G_{ij,\pi}$ and $G_{ij,\eta}$ were found to be, in averaged, small, being proportional to the quark effective mass differences: $M^*_u - M^*_d$ and/or $M^*_s - M^*_u$. One set of parameters, Y14, was defined with $m_u = m_d$ and the resulting coupling constants $G_{ij}$ and mesons masses calculated with it, carry this information: $G_{0.3} = G_{38} = 0$ and also $m_{\eta^0} = m_{\eta_1}$ and so on. These mixings, $G_{i\neq j}$, yield light mesons mixings and, although the mixing angle for the $\eta - \eta'$ mixing has been calculated, other consequences will be investigated in another work.

The charged and neutral pion mass difference was found to be very small and of the order of 0.1 MeV and it is basically due to the small up-down quark mass difference, in agreement with expectations. The effect of the coupling constants $G_{i1}, G_{33}$ is however still slightly smaller and it was almost not really identified, except for a particular set of parameters with slightly larger up-down quark mass difference. The resulting part of the pion mass difference comes from electromagnetic effects that were not calculated in this work. The neutral-charged kaons mass difference was obtained to be of the order of $4 - 10$ MeV. Both the quark mass difference and the flavor dependent couplings however yield kaon mass differences of the same order of magnitude. The flavor dependent coupling constants $G_{44}$ and $G_{60}$ induce mass shifts of the order of 2-4 MeV but it could reach 6 MeV for some of the sets of parameters. Both flavor dependences should have to be considered in the NJL: the mass and coupling constant flavor dependence. This goes along the very idea of considering the NJL as an effective model for QCD, being that the initial QCD-flavor dependence, parameterized in the quark Lagrangian masses, would have consequences for all the effective parameters of the resulting effective model. This is analogous to the flavor-breaking dependence of parameters in EFT, such as ChPT.

The effects of the flavor-dependent coupling constants on some of the light scalar mesons, $a_0$ and $K^*$ (or $\kappa$), follow nearly the same patterns of the pseudoscalar mesons. The shifts of the masses due to changes in the coupling constants, however, might not be as large as the changes in the quark effective masses. The largest effects due to varying $G_{i1,\pi}$ were found to be of the order of 30-50 MeV. The usual problem of inverted hierarchy of the scalar mesons $a_0$ and $K^*$ showed up because of the pattern of the values of coupling constants does not correct it. Other contributions for these scalar channel quark-antiquark interactions are expected to correct this inverted mass hierarchy. Nevertheless, the $K^*(890)$ mesons masses might be approximately obtained by the following ad hoc set of parameters: $m_u = 3$ MeV, $m_d = 5$ MeV, $m_s = 143$ MeV, $\Lambda = 840$ MeV, $\Lambda_{IR} = 120$ MeV, $G_{44} = 3.65$ and $G_{60} = 3.60$, which yields values close to the experimental ones: $m_{K^*} = 891$ MeV and $m_{K^*} = 888$ MeV. The same type of fitting is possible for the $a_0(980)$ mesons although the physical meaning or content is not clear. The experimental values for these neutral-charged mesons mass differences, however, might be smaller than for the case of the kaons for example. The effective gluon propagators used in this work however can provide the corresponding needed $G_{i1,\pi}$ or $G_{i1,\pi}$ to reduce accordingly the neutral-charged scalar mesons mass differences. However, one might expect that both non degeneracy of quark masses values and flavor-dependent coupling constants contribute to keep the correct experimental behavior of neutral-charge scalar mass differences.

The resulting coupling constants found above define new gap equations as presented in the last section. The corrected effective mass provide observables, as calculated in Section III.C, in average in better agreement with expected or experimental values. To conclude that, note that the cutoff and current quark masses were kept fixed in such a way to show clearly the effects of the flavor-dependent coupling constants. The main source of shifts of the values is the strange quark effective mass that decreases with the reduction of the coupling constant $G_{ss}$. The consequences in the kaon decay constant and on the strange quark condensate are clear. An interesting issue to note is the new mechanism for the mesons mixings by means of the resulting coupling constants $G_{i\neq j}, G_{i\neq j,\pi}$ or equivalently $G_{f1\neq f2}$. The values found however were not enough to reproduce the complete $\eta - \eta'$ mass difference. Further calculations are needed and they should help to constraint further the corresponding components of the (effective) quark-antiquark interactions. The reduced chi-squared was calculated for ten observables being two of them fitted observables. The sets of parameters with the flavor-dependent coupling constants were shown to provide considerably better results. The kaon decay constant is the only observable that present worse values when receiving corrections due to $G_{i1,\pi}$ in the present calculation. On the other hand, the values of scalar chiral condensates are largely improved. These solutions for the corrected gap equations induce further ambiguities to define either a new cutoff or different values for Lagrangian quark masses. As such, a fully self consistent numerical calculation may be expected for which the gap equations and the flavor-dependent coupling constants are solved at once. In this program non stable results easily appear since shifts in quark effective masses might be reasonably large for fixed cutoff and current quark masses. This problem may be worsen if weaker NJL coupling constants $G_0 < 10$ GeV$^{-2}$ are considered. This problem prevents a direct and immediate self consistent solution for the gap equations, coupling constants and eventually BSE described above. A more complete account of the mixing interactions contributions for the light mesons spectra and other observables will be treated separately in another work.

Acknowledgements
F.L.B. is member of INCT-FNA, Proc. 464898/2014-5 and he acknowledges partial support from CNPq-312072/2018-0 and CNPq-421480/2018-1. The author thanks a short conversation with B. El Bennich another one with G. Krein.

References

[1] Y. Nambu, G. Jona-Lasinio, Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity I, Phys. Rev. 122, 345 (1961).

[2] S. P. Klevansky The Nambu-Jona-Lasinio model of quantum chromodynamics, Rev. Mod. Phys. 64, 649 (1992).

[3] U. Vogl, W. Weise, The Nambu and Jona-Lasinio model: Its implications for Hadrons and Nuclei, Progr. in Part. and Nucl. Phys. 27, 195 (1991).

[4] H. Kleinert, 1978 in Erice Summer Institute 1976, Understanding the Fundamental Constituents of Matter, 289, Plenum Press, New York, ed. by A. Zichichi.

[5] A. Paulo Jr., F.L. Braghin, Vacuum polarization corrections to low energy quark effective couplings, Phys. Rev. D 90, 014049 (2014).

[6] J. L. Cortés, J. Gamboa, L. Velásquez, A Nambu-Jona-Lasinio like model from QCD at low energies, Phys. Lett. B 432, 397 (1998).

[7] K.-I. Kondo, Toward a first-principle derivation of confinement and chiral-symmetry-breaking crossover transitions in QCD, Phys. Rev. D 82, 065024 (2010).

[8] P. Costa, O. Oliveira, P.J.Silva, What does low energy physics tell us about the zero momentum gluon propagator, Phys. Lett. B 695, 454 (2011).

[9] T. Hell, S. Rossner, M. Cristoforetti, W. Weise, Dynamics and thermodynamics of a nonlocal Polyakov–Nambu–Jona-Lasinio model with running coupling, Phys. Rev. D 79, 014022 (2009).

[10] V.A. Miransky I.A. Shovkovy, Quantum field theory in a magnetic field: From quantum chromodynamics to graphene and Dirac semimetals, Phys. Rept. 576, 209 (2015). J. O. Andersen, W. R. Naylor, A. Tranberg, Phase diagram of QCD in a magnetic field Rev. Mod. Phys. 88, 025001 (2016).

[11] S. Basak et al, Lattice computation of the electromagnetic contributions to kaon and pion masses, Phys. Rev. D99, 034503 (2019).

[12] J. F. Donoghue, A.F. Perez, The Electromagnetic Mass Differences of Pions and Kaons, Phys.Rev. D55, 7075 (1997).

[13] J.F. Donoghue, Light quark masses and chiral symmetry, Annu. Rev. Nucl. Part. Sci. 39, 1 (1989).

[14] D. Giusti, et al, Leading isospin-breaking corrections to pion, kaon and charmed-meson masses with Twisted-Mass fermions Phys. Rev. D95, 114505 (2017).

[15] F. L. Braghin, SU(2) low energy quark effective couplings in weak external magnetic field, Phys. Rev. D 94, (2016) 074030.

[16] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, (2018) 030001.

[17] J. Gasser, H. Leutwyler, Quark masses, Phys. Rept. 87, 77 (1982).

[18] D.B. Kaplan, Lectures on effective field theory - ICTP-SAIFR, (2016).

[19] M. Chen, L. Chang, A Pattern for the Flavor Dependent Quark-antiquark Interaction, Chin. Phys C 43, (2019) 114103. M. Chen, Lei Chang, Y.-x. Liu, Be Meson Spectrum Via Dyson-Schwinger Equation and Bethe-Salpeter Equation Approach, Phys. Rev. D 101, (2020) 056002

[20] J.R. Pelaez, Status of light scalar mesons as non-ordinary mesons, Journ. of Phys.: Conference Series 562, 012012 (2014).

[21] J. R. Pelaez, From controversy to precision on the sigma meson: a review on the status of the non-ordinary \( f_0(500) \) resonance, Phys. Rep. 658, 1 (2016).
[22] M.X. Su, L.Y. Xiao, H.Q. Zheng, On the scalar nonet in the extended Nambu–Jona-Lasinio model, Nuclear Physics A 792, 288 (2007).

[23] L. Maiani, F. Piccinini, A.D. Polosa, V. Riquer, New Look at Scalar Mesons, Phys. Rev. Lett. 93, 212002 (2004).

[24] A.A. Osipov, B. Hiller, A.H. Blin, J. da Providência, Effects of eight-quark interactions on the hadronic vacuum and mass spectra of light mesons, Annals of Phys. 322, 2021 (2007).

[25] A. A. Osipov, B. Hiller, A. H. Blin, J. da Providencia, Effects of eight-quark interactions on the hadronic vacuum and mass spectra of light mesons, Annals of Phys. 322, 2021 (2007).

[26] F. Giacosa, A. Koenigstein, R.D. Pisarski, How the axial anomaly controls flavor mixing among mesons, Phys. Rev. D97, 091901(R) (2018).

[27] V. Dmitrasinovic, UA1 breaking and scala mesons in the Nambu and Jona-Lasinio model Phys. Rev. C 53, 1383 (1996).

[28] Some proposals not fully confirmed: M. Albaladejo, J. A. Oller, Identification of a Scalar Glueball, Phys. Rev. Lett. 101, 252002 (2008). C. Amsler, F. E. Close, Evidence for a scalar glueball, Phys. Lett. B 353, 385 (1995).

[29] E. Klempt and A. Zaitsev, Glueballs, Hybrids, Multiquarks. Experimental facts versus QCD inspired concepts, Phys. Rep. 454, 1 (2007).

[30] A. H. Fariborz, R. Jora, J. Schechter, Toy model for two chiral nonets, Phys. Rev. D 72, 034001 (2005).

[31] Long-Cheng Gu et al, Scalar glueball in radiative J/ψ decay on the lattice, Phys. Rev. Lett. 110, 021601 (2013).

[32] T. Wolkanowski, F. Giacosa, D.H. Rischke, $a_0(980)$ revisited, Phys. Rev. D 93, 014002 (2016).

[33] F.L. Braghin, SU(2) Higher-order effective quark interactions from polarization, Phys. Lett. B 761 (2016) 424.

[34] The light pseudoscalar mesons mass hierarchy is fully described by a variety of approaches when introducing degrees of freedom usually related to the $U_A(1)$ symmetry breakdown [35]. This last term is effectively responsible for the so called $\eta - \eta'$ mixing that makes the masses of the full pseudoscalar nonet to pin down experimental values [36]. Interestingly it has been pointed out in [5] that the sixth order quark interaction in flavor $U(3)$ NJL model, usually pointed out as consequence of a $U_A(1)$ symmetry breaking phenomenon, can also be obtained from polarization corrections of NJL-type models.

[35] E. Witten, Nucl. Phys. B 156, 269 (1979); G. Veneziano, Nucl. Phys. B 159, 213 (1979). G. 't Hooft, Phys. Rev. D 14, 3432 (1976); Erratum: ibid D 18, 2199 (1978).

[36] C. Rosenzweig, J. Schechter and C. G. Trahern, Phys. Rev. D 21, 3388 (1980). R. Alkofer and I. Zahed, Mod. Phys. Lett. A 4, 1737 (1989); R. Alkofer and I. Zahed, Phys. Lett. B 238, 149 (1990).

[37] C.D. Roberts, R.T. Cahill, J. Pracshifka, The effective action for the Goldstone modes in a global colour symmetry model of QCD, Ann. of Phys. 188, 20 (1988).

[38] B. Holdom, Approaching low-energy QCD with a gauged, nonlocal, constituent-quark model, Phys. Rev. D 45, 2534 (1992).

[39] D. Ebert, H. Reinhardt, M.K. Volkov, Effective hadron theory of QCD, Pr. Part. Nucl. Phys. 33, 1 (1994).

[40] P. Lowdon, Nonperturbative structure of the photon and gluon propagators, Phys. Rev. D96, 065013 (2017).

[41] J. M. Cornwall, Entropy, confinement, and chiral symmetry breaking, Phys. Rev. D 83, 076001 (2011).

[42] K. Higashijima, Dynamical chiral-symmetry breaking, Phys. Rev. D 29, (1984) 1228. V. A. Miransky, Sov. J. Nucl. Phys. 38, 280 (1983).

[43] K.I. Aoki, et al., Prog. Theor. Phys. 84, 683 (1990).

[44] D. Binosi, L. Chang, J. Papavassiliou, C.D. Roberts, Bridging a gap between continuum-QCD and ab initio predictions of hadron observables, Phys. Lett. B 742, 183 (2015) and references therein.

[45] A. Bashir, et al., Collective Perspective on Advances in Dyson–Schwinger Equation QCD, Commun. Theor. Phys. 58,79 (2012). I.C. Cloet, C.D. Roberts, Explanation and prediction of observables using continuum strong QCD, Prog. Part. Nucl. Phys. 77, 1 (2014).
F.L. Braghin, Quark and pion effective couplings from polarization effects, Eur. Phys. Journ. A 52, 134 (2016).

F.L. Braghin, Pion Constituent Quark Couplings strong form factors: A dynamical approach, Phys. Rev. D 99, 014001 (2019).

F.L. Braghin, Low energy constituent quark and pion effective couplings in a weak external magnetic field, Eur. Phys. Journ. A 54, 45 (2018).

F. L. Braghin, Expanding nonhomogeneous configurations of the $\lambda\phi^4$ model, Phys. Rev. D 64, 125001 (2001).

L. F. Abbott, Introduction to the background field method, Acta Phys. Pol. B 13, 33 (1982).

U. Mosel, Path Integrals in Field Theory, An Introduction, (2004) Springer.

F.L. Braghin, Light vector and axial mesons effective couplings to constituent quarks, Phys. Rev. D97, 054025 (2018); Phys. Rev. D 101, 039901(E) (2020).

S. Klimt et al, Generalized SU(3) Nambu-Jona-Lasinio model (I), Nucl. Phys. A516, 429 (1990).

T. Eguchi, H. Sugawara, Extended model of elementary particles based on an analogy with superconductivity, Phys. Rev. D10, 4257 (1974). T. Eguchi, New approach to collective phenomena in superconductivity models, ibid 14, 2755 (1976).

D. Ebert, T. Feldmann, H. Reinhardt, Extended NJL model for light and heavy mesons without $q\bar{q}$ thresholds, Phys. Lett. B 388, 154 (1996).

F. E. Serna, B. El-Bennich, G. Krein, Charmed mesons with a symmetry-preserving contact interaction, Phys. Rev. D96, 014013 (2017).

H. Kohyama, D. Kimura, T. Inagaki, Parameter fitting in three-flavor Nambu-Jona-Lasinio model with various regularizations, Nucl. Phys. B906, 524 (2016).

H. Reinhardt, R. Alkofer, Instanton-induced flavour mixing in mesons, Phys. Lett. B 207, 482 (1988).

A.A. Osipov, B. Hiller, Eur. Phys. J. C 35, 223 (2004). 3. A.A. Osipov, B. Hiller, J. Moreira, A.H. Blin, Eur. Phys. J. C 46, 225 (2006). 4. A.A. Osipov, B. Hiller, A.H. Blin, Eur. Phys. J. A 49, 14 (2013).

M. Jamin, Flavour-symmetry breaking of the quark condensate and chiral corrections to the Gell-Mann-Oakes-Renner relation Phys.Lett. B538, 71 (2002).

C.T.H. Davies et al, Determination of the quark condensate from heavy-light current-current correlators in full lattice QCD, Phys. Rev. D 100, 034506 (2019).

S. Aoki, FLAG, Review of lattice results concerning low energy particle physics, Eur. Phys. Journ. C 80, 113 (2020).