THE FORMULA FOR THE QUASICENTRAL MODULUS IN THE CASE OF SPECTRAL MEASURES ON FRACTALS

DAN-VIRGIL VOICULESCU*

Abstract. We prove a general ampliation homogeneity result for the quasicentral modulus of an $n$-tuple of operators with respect to the $(p, 1)$ Lorentz normed ideal. We use this to prove a formula involving Hausdorff measure for the quasicentral modulus of $n$-tuples of commuting Hermitian operators the spectrum of which is contained in certain Cantor-like self-similar fractals.

1. Introduction

The quasicentral modulus $k_n(\tau)$ is a number associated with an $n$-tuple $\tau$ of Hermitian operators relative to a normed ideal $(\mathcal{J}, |\cdot|)$ of compact operators. It underlies many questions on normed ideal perturbations of $n$-tuples of operators (see the recent survey [11]), and it also had applications in non-commutative geometry in work on the spectral characterization of manifolds [1].

We proved in [7] that in the case of $\tau$ an $n$-tuple of commuting Hermitian operators and if the normed ideal is the $(n, 1)$-Lorentz ideal, which we denote by $\mathcal{C}_n$, the corresponding quasicentral modulus $k_n^{-1}(\tau)$ has the property that $(k_n^{-1}(\tau))^{1/n}$ is proportional to the integral w.r.t. $n$-dimensional Lebesgue measure of the multiplicity function of $\tau$.

Here we prove a similar result in fractional dimension. More precisely instead of a cube in $\mathbb{R}^n$ which contains the spectrum of $\tau$ we assume there is a fixed self-similar fractal in $\mathbb{R}^n$ of Hausdorff dimension $p > 1$ containing the spectrum $\sigma(\tau)$. The analogous formula we prove has the exponent $n$ replaced by $p$ and the integral of the multiplicity function is with respect to $p$-Hausdorff measure. For technical reasons the class of fractals is rather restricted, only certain totally disconnected sets, that is Cantor-like fractals are considered. One should certainly expect this can be extended to a larger class of fractals, the present paper being only a first step.

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What made the extension of the formula to fractional dimension possible is a completely general ampliation homogeneity result $k_{-p}(\tau \otimes I_m) = m^{1/p}k_{-p}(\tau)$. Such a result was previously known only for $p = \infty$ (9) and for $p = 1$ (7). Note that in (7) we obtained easily an ampliation homogeneity result for $k_p(\tau)$, that is with $C_{-p}$ replaced by the Schatten–v. Neumann class $C_p$. For $p = 1$ we have $C_1 = C_{-1}$, but for $p > 1$ the result for $k_p(\tau)$ turned out to be trivial after we showed in (8) that in this case $k_p(\tau) \in \{0, \infty\}$. The interesting quantity which replaces $k_p(\tau)$ is $k_{-p}(\tau)$.

The paper has six sections including this introduction. Section 2 contains preliminaries about the quasicentral modulus. Section 3 is devoted to the ampliation homogeneity theorem. In section 4 we collected preliminaries concerning the class of Cantor-type fractals we consider. The formula for $k_{-p}(\tau)$ in the fractal setting is obtained in section 5. Section 6 deals with concluding remarks.

2. Operator preliminaries

By $\mathcal{H}$ we denote a separable complex Hilbert space of infinite dimension and by $\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H}), \mathcal{R}(\mathcal{H})$ the bounded operators, the compact operators and the finite rank operators. When no confusion will arise we will simply write $\mathcal{K}, \mathcal{R}$ and we will denote by $\mathcal{R}^+_1(\mathcal{H})$ the finite rank positive contractions $0 \leq A \leq I$ on $\mathcal{H}$. The $(p, 1)$ Lorentz normed ideal of compact operators will be denoted by $(C_{-p}, | \cdot |_p)$. We recall that the norm is $|T|_p = \sum_{j \in \mathbb{N}} s_j j^{-1+1/p}$ where $s_1 \geq s_2 \geq \ldots$ are the eigenvalues of $|T| = (T^*T)^{1/2}$ in decreasing order. If $(C_p, | \cdot |_p)$ is the Schatten–von Neumann $p$-class, then $C_1 = C_1^-$. More on normed ideals can be found in (5) and (6).

We shall also use the following notation for operations on $n$-tuples of operators, in line with (7). If $\tau = (T_i)_{1 \leq i \leq n} \in (\mathcal{B}(\mathcal{H}))^n$ and $X, Y \in \mathcal{B}(\mathcal{H})$ then we use

\[
X \tau Y = (XT_iY)_{1 \leq i \leq n} \quad [X, \tau] = ([X, T_i])_{1 \leq i \leq n} \quad \tau^* = (T_i^*)_{1 \leq i \leq n}.
\]

If also $\sigma = (S_i)_{1 \leq i \leq n} \in (\mathcal{B}(\mathcal{H}))^n$ then we write

\[
\sigma + \tau = (S_i + T_i)_{1 \leq i \leq n} \quad \sigma \oplus \tau = (S_i \oplus T_i)_{1 \leq i \leq n} \quad \tau \otimes I_m = (T_i \otimes I_m)_{1 \leq i \leq n}
\]

where $I_m$ is the identity operator on $\mathbb{C}^m$. When we identify $\mathcal{H} \otimes \mathbb{C}^m$ and $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ we also have

$$\tau \otimes I_m \cong \underbrace{\tau \oplus \cdots \oplus \tau}_{m\text{-times}}.$$
Further, we consider norms

\[ \|\tau\| = \max_{1 \leq i \leq n} \|T_i\| \]

\[ |\tau|_{\mathcal{J}} = \max_{1 \leq i \leq n} |T_i|_{\mathcal{J}}. \]

The quasicentral modulus of an \( n \)-tuple \( \tau = (T_i)_{1 \leq i \leq n} \) with respect to a normed ideal \((\mathcal{J}, |\cdot|_{\mathcal{J}})\) (see [7], [9]) is the number

\[ k_{\mathcal{J}}(\tau) = \lim \inf_{A \in \mathbb{R}^*_+} |[\tau, A]|_{\mathcal{J}} \]

where the lim inf is w.r.t. the natural order on \( \mathbb{R}^*_+ \). This definition is also equivalent to

\[ k_{\mathcal{J}}(\tau) = \inf \{ \alpha \in [0, \infty] | \alpha = \lim_{k \to \infty} |[\tau_k, A]|_{\mathcal{J}}, A_k \uparrow I \} \]

or the same with \( w - \lim_{k \to \infty} A_k = I \) instead of \( A_k \uparrow I \). If \( J = C_p^\infty \) we denote \( k_{\mathcal{J}}(\tau) \) by \( k_{-p}^- (\tau) \).

We should also record as the next proposition the results in [7] Prop. 1.4 and Prop. 1.6.

**Proposition 2.1.** If \( \tau^{(j)} \in \mathcal{B}(\mathcal{H})^n, j \in \mathbb{N} \) then we have

\[ \max_{j=1,2} k_{\mathcal{J}}(\tau^{(j)}) \leq k_{\mathcal{J}}(\tau^{(1)} \oplus \tau^{(2)}) \leq k_{\mathcal{J}}(\tau^{(1)}) + k_{\mathcal{J}}(\tau^{(2)}) \]

and

\[ k_{\mathcal{J}}\left( \bigoplus_{j \in \mathbb{N}} \tau^{(j)} \right) = \lim_{m \to \infty} k_{\mathcal{J}}\left( \bigoplus_{1 \leq j \leq m} \tau^{(j)} \right). \]

If \( \lambda^{(j)} \in \mathbb{C}^n \) and \( \lambda^{(j)} \otimes I_{3\mathcal{H}} \in \mathcal{B}(\mathcal{H})^n \) then we have

\[ k_{\mathcal{J}}(\tau^{(1)} \oplus \cdots \oplus \tau^{(m)}) = k_{\mathcal{J}}\left( (\tau^{(1)} - \lambda^{(1)} \otimes I_{3\mathcal{H}}) \oplus \cdots \oplus (\tau^{(m)} - \lambda^{(m)} \otimes I_{3\mathcal{H}}) \right). \]

Finally, if \( \tau \) is a \( n \)-tuple of commuting Hermitian operators we denote by \( \sigma(\tau) \subset \mathbb{R}^n \) the joint spectrum and by \( E(\tau; \omega) \) the spectral projection of \( \tau \) for the Borel set \( \omega \subset \mathbb{R}^n \).

### 3. Ampliation Homogeneity

**Theorem 3.1.** If \( \tau \) is a \( n \)-tuple of bounded operators and \( 1 \leq p \leq \infty \) then

\[ k_p^- (\tau \otimes I_m) = m^{1/p} k_p^- (\tau). \]

The cases \( p = 1 \) and \( p = \infty \) have already been proved ([7] Prop. 1.5 and [9] Prop. 3.9).

We begin the proof with a couple of lemmas.
Lemma 3.1. Let $X_j \in C_p$, $j \in \mathbb{N}$, $p \in [1, \infty]$ be so that $|X_j|_p^- \leq C$. If
\[
\lim_{j \to \infty} \|X_j\| = 0
\]
then we have
\[
\lim_{j \to \infty} (|X_j \otimes I_m|_p^- - m^{1/p}|X_j|_p^-) = 0.
\]

Proof. Let $s_1^{(j)} \geq s_2^{(j)} \geq \ldots$ be the eigenvalues of $(X_j^*X_j)^{1/2}$. Then
\[
|X_j \otimes I_m|_p^- = \sum_{k \in \mathbb{N}} s_k^{(j)}((m(k - 1) + 1)^{-1+1/p} + \ldots + (mk)^{-1+1/p})
\]
\[
\geq m^{1/p} \sum_{k \in \mathbb{N}} s_k^{(j)}k^{-1+1/p} = m^{1/p}|X_j|_p^-.
\]
On the other hand, given $\epsilon > 0$ there is $N$ so that $k \geq N \Rightarrow (m(k - 1) + 1)^{-1+1/p} + \ldots + (mk)^{-1+1/p} \leq (1 + \epsilon)m^{1/p}k^{-1+1/p}$.
This gives
\[
|X_j \otimes I_m|_p^- \leq (1 + \epsilon)m^{1/p} \sum_{k \geq N} s_k^{(j)}k^{-1+1/p} + N\|X_j\|
\]
\[
\leq (1 + \epsilon)m^{1/p}|X_j|_p^- + N\|X_j\|.
\]
Thus we have
\[
0 \leq |X_j \otimes I_m|_p^- - m^{1/p}|X_j|_p^- \leq \epsilon m^{1/p}|X_j|_p^- + N\|X_j\|.
\]
Since $\epsilon > 0$ is arbitrary and $\|X_j\| \to 0$ we get the desired result when $j \to \infty$. \qed

Corollary 3.1. Let $X_j = (X_{ji})_{1 \leq i \leq n}$ be $n$-tuples of operators so that $|X_j|_p^- \leq C$ where $p \in [1, \infty]$ and $\lim_{j \to \infty} \|X_j\| = 0$. Then we have
\[
\lim_{j \in \infty} (m^{1/p}|X_j|_p^- - |X_j \otimes I_m|_p^-) = 0.
\]

Lemma 3.2. If $\tau \in (\mathcal{B}(\mathcal{H}))^n$ and $(\mathcal{J}, | \cdot \rangle)$ is a normed ideal so that $k_{\mathcal{J}}(\tau) < \infty$, then there are $B_j \in \mathcal{R}_{\mathcal{J}}^+$ so that $B_j \uparrow I$ and
\[
\lim_{j \to \infty} ||[\tau, B_j]|_{\mathcal{J}} = k_{\mathcal{J}}(\tau)
\]
\[
\lim_{j \to \infty} \|\tau, B_j\| = 0.
\]

Proof. It suffices to show that given $\epsilon > 0$ and $P$ a finite rank Hermitian projector we can find $B \in \mathcal{R}_{\mathcal{J}}^+$ so that $B \geq P$ and
\[
|[B, \tau]|_{\mathcal{J}} \leq k_{\mathcal{J}}(\tau) + \epsilon
\]
\[
\|B, \tau\| \leq \epsilon.
\]
Such that $B$ can be constructed as follows. We find recursively $P = P_1 \leq P_2 \leq P_3 \leq \ldots$ finite rank Hermitian projectors and $A_j \in \mathbb{R}_+^+$ so that

$$A_j \geq P_j, \quad ||[A_j, \tau]|_3| \leq k_3(\tau) + \epsilon$$

$$\tau P_j = P_{j+1} \tau P_j, \quad \tau^* P_j = P_{j+1} \tau^* P_j$$

$$P_{j+1} \geq A_j.$$ If we put $Q_j = P_{j+1} - P_j$ if $j \geq 1$ and $Q_0 = P_1 = P$ we have

$$Q_e \tau Q_s \neq 0 \Rightarrow |r - s| \leq 1$$

and

$$A_j = (Q_0 + \cdots + Q_{j-1}) + Q_j A_j Q_j.$$ This gives

$$Q_e [\tau, A_j] Q_s \neq 0$$

$\Rightarrow |r - s| \leq 1$ and $j - 1 \leq r, s \leq j + 1$. It follows that if $|k - j| \geq 4$ ($[\tau, A_j])^* [\tau, A_k] = 0$ so that

$$||[\tau, A_1 + A_2 + \cdots + A_{4N}]|| \leq 2||\tau||.$$ Thus if $B_N = N^{-1}(A_1 + A_2 + \cdots + A_{4N})$ we have $B_N \geq P, B_N \in \mathbb{R}_+^+$,

$$||[B_N, \tau]|_3| < k_3(\tau) + \epsilon$$

and

$$||[B_N, \tau]|_3| \leq 2N^{-1}||\tau||.$$ Thus if $2N^{-1}||\tau|| < \epsilon$ we may take $B = B_N$. $\square$

**Lemma 3.3.** If $m \in \mathbb{N}$ and $(\mathcal{J}, |\cdot|)$ is a normed ideal and $\tau \in (B(\mathcal{H}))^m$ is so that $k_3(\tau) < \infty$, then there are $A_j \in \mathbb{R}_+^+$, $A_j \uparrow I$ so that

$$k_3 (\tau \otimes I_m) = \lim_{j \to \infty} ||[\tau \otimes I_m, A_j \otimes I_m]|_3|$$

and

$$\lim_{j \to \infty} ||[\tau, A_j]|_3| = 0.$$ 

**Proof.** Let $G$ be the group $S_m \times \mathbb{Z}_2^m$ of permutation matrices with $\pm 1$ entries and $g \to U_g$ its representation on $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ which is $\otimes I_3$ the representation on $\mathbb{C}^m$. Then the commutant $\{U_g : g \in G\}'$ is $B(\mathcal{H}) \otimes I_m$ and the map $\Phi : B(\mathcal{H}) \to B(\mathcal{H}) \otimes I_m$ given by $\Phi(X) = |G|^{-1} \sum_{g \in G} U_g X U_g^*$ is the projection of norm one which preserves the trace. If $B \in R_1^m(\mathcal{H})$ we have

$$||[B, \tau \otimes I_m]|_3| = |U_g[B, \tau \otimes I_m]U_g^*|_3 = |[U_g B U_g^*, \tau \otimes I_m]|_3|$$

which gives by taking the mean over $G$

$$||[\Phi(B), \tau \otimes I_m]|_3| \leq ||[B, \tau \otimes I_m]|_3|$$

and clearly $B_j \uparrow I \otimes I_m$ implies $\Phi(B_j) \uparrow I \otimes I_m$. Thus if $A_j \otimes I_m = \Phi(B_j)$ and $B_j \uparrow I \otimes I_m$ are so that

$$\lim_{j \to \infty} ||[B_j, \tau \otimes I_m]|_3| = k_3 (\tau \otimes I_m)$$

and

$$\lim_{j \to \infty} ||[B_j, \tau \otimes I_m]|_3| = 0$$
then we have
\[ \lim_{j \to \infty} \sup_{\| \|} \| [A_j \otimes I_m, \tau \otimes I_m] \|_{\mathcal{A}} \leq k_3(\tau \otimes I_m) \]
and
\[ \lim_{j \to \infty} \| [A_j \otimes I, \tau \otimes I_m] \| = 0. \]
Since on the other hand
\[ \liminf_{j \to \infty} \| [A_j \otimes I_m, \tau \otimes I_m] \|_{\mathcal{A}} \geq k_3(\tau \otimes I_m) \]
we conclude that
\[ \lim_{j \to \infty} \| [A_j \otimes I_m, \tau \otimes I_m] \|_{\mathcal{A}} = k_3(\tau). \]
\[ \square \]

**Proof of Theorem 3.1.** Using Lemma 3.3 we can find \( A_j \in \mathcal{R}_1^+ \), \( A_j \uparrow I \) when \( j \to \infty \) so that
\[ \lim_{j \to \infty} \| [A_j, \tau] \otimes I_m \|_p^- = k_p^-(\tau \otimes I_m) \]
and
\[ \lim_{j \to \infty} \| [A_j, \tau] \| = 0. \]
Using Corollary 3.1 we infer that
\[ \lim_{j \to \infty} m^{1/p} [A_j; \tau]_p^- = k_p^-(\tau \otimes I_m) \]
which implies that
\[ m^{1/p} k_p^-(\tau) \leq k_p^-(\tau \otimes I_m). \]
On the other hand, Lemma 3.2 shows that there are \( A_j \uparrow I, A_j \in \mathcal{R}_1^+ \) so that
\[ \lim_{j \to \infty} \| [\tau, A_j] \|_p^- = k_p^-(\tau) \]
and
\[ \lim_{j \to \infty} \| [\tau, A_j] \| = 0. \]
Then by Corollary 3.1 we get that
\[ m^{1/p} k_p^-(\tau) = \lim_{j \to \infty} \| [\tau, A_j] \otimes I_m \|_p^- \]
\[ = \lim_{j \to \infty} \| [\tau \otimes I_m, A_j \otimes I_m] \|_p^- \]
\[ \geq k_p^-(\tau \otimes I_m) \]
which concludes the proof. \( \square \)
4. Fractal preliminaries

To keep things simple the fractal context will be certain totally disconnected Cantor-like self-similar sets and on which a certain Hausdorff measure is a Radon measure on Borel sets.

We consider a non-empty compact set $K \subset \mathbb{R}^n$ and a $N$-tuple of maps
$$F_i(x) = \lambda(x - b(i)) + b(i), \quad 1 \leq i \leq N$$
where $0 < \lambda < 1$ and $b(i) \in \mathbb{R}^n$, so that
$$K = \bigcup_{1 \leq i \leq N} F_iK$$
and we assume that
$$i_1 \neq i_2 \Rightarrow F_{i_1}K \cap F_{i_2}K = \emptyset.$$
Note that the open set condition for the contractions $F_i$ (see [4], page 121) in this case can be satisfied with an open neighborhood of $K$. The Hausdorff and box-dimension of $K$ are equal to
$$p = \log N / \log(1/\lambda)$$
by [4] Thm. 8.6, and the $p$-Hausdorff measure of $K$ is finite and non-zero. This Hausdorff measure is often referred to as the Hutchinson measure of $K$.

Note also the uniqueness of $K$ given the maps $F_i, 1 \leq i \leq N$ ([4] Thm. 8.3).

If $w \in \{1, \ldots, N\}^m$ we put $|w| = m$ and define $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ and $K_w = F_wK$. In particular if $|w| = |w'|$ then $K_w$ and $K_{w'}$ are congruent and in particular have the same $p$-Hausdorff measure and diameter. Moreover
$$K = \bigcup_{|w|=L} K_w$$
and for any $L \in \mathbb{N}$ the union is disjoint. On $K$ there is a unique Radon measure $\mu$ so that
$$\mu(K_w) = N^{-|w|} = \lambda^{|w|p}$$
and on Borel sets $\mu = cH_p$ where $H_p$ denotes the $p$-Hausdorff measure and $c$ is a constant. We use the $l^\infty$-norm $|(x_1, \ldots, x_n)| = \max_{1 \leq i \leq n} |x_i|$ on $\mathbb{R}^n$. Note that
$$\text{diam}(K_w) = c \cdot \lambda^{|w|}$$
for some constant $c$.

We shall also assume that the Hausdorff dimension $p \geq 1$.

If $\tau$ is a $n$-tuple of commuting Hermitian operators on $\mathcal{H}$ with spectrum $\sigma(\tau) \subset K$, like with Lebesgue measure here on $K$ with $\mu$ that is with $H_p$ the Hilbert space splits
$$\mathcal{H} = \mathcal{H}_{\text{psing}} \oplus \mathcal{H}_{\text{pac}}$$
where $\mathcal{H}_{\text{psing}}, \mathcal{H}_{\text{pac}}$ are reducing subspaces for $\tau$ and consist of vectors $\xi$ so that
$$\langle E(\tau; \cdot)\xi, \xi \rangle$$
is singular w.r.t. \( \mu \) and respectively absolutely continuous w.r.t. \( \mu \), that is w.r.t. \( H_p \). (Use for instance 1.6.3 in \[3\].) For more on Hausdorff-measure and on fractals see \[3\] and \[4\].

5. \( k_p^- \) in the fractal setting

In this section we study \( k_p^-(\tau) \) where \( \tau \) is a \( n \)-tuple of commuting Hermitian operators with \( \sigma(\tau) \subset K \), in the context of Section 4. We assume \( p \geq 1 \) (and for certain results we will require \( p > 1 \)).

**Lemma 5.1.** Assume \( \tau \) is a \( n \)-tuple of commuting Hermitian operators with \( \sigma(\tau) \subset K \) and with a cyclic vector \( \xi \). Then for some constant \( C \) depending only on \( K \), we have

\[
k_p^-(\tau) \leq C(H_p(\sigma(\tau)))^{1/p}.
\]

**Proof.** Let \( \Omega(L) = \{ w \mid |w| = L, \ K_w \cap \sigma(\tau) \neq \emptyset \} \) and \( G(L) = \bigcup_{w \in \Omega(L)} K_w \). Then \( G(L) \) is open in \( K \) and there is \( L_0 \) so that for a given \( \epsilon > 0 \) we have \( L \geq L_0 \Rightarrow H_p(G(L)) \leq H_p(\sigma(\tau)) + \epsilon \). Let further \( E_w = E(\tau; K_w) \) and observe that since \( G(L) \supset \sigma(\tau) \) we have

\[
\sum_{w \in \Omega(L)} E_w = I.
\]

We also have

\[
H_p(G_L) = |\Omega(L)| H_p(K) \lambda^{L^p}.
\]

Let \( P_L \) be the orthogonal projection onto \( \sum_{w \in \Omega(L)} \mathbb{C} E_w \xi \), and \( P_w \) the orthogonal projection onto \( \mathbb{C} E_w \xi \). Remark that the non-zero \( E_w \xi \) are an orthogonal basis on \( P_L \mathcal{H} \) and rank \( P_L \leq |\Omega_L| \). We have \([P_L, E_w] = 0\) if \( |w| = L \) and \( L_1 \geq L_2 \Rightarrow P_{L_1} \geq P_{L_2} \) since each \( E_w \) with \( |w| = L \) is the sum of \( E_{w'} \) with \( |w'| = L_1 \). Remark also that

\[
\|[P_L, \tau]\| = \max_{|w| = L} \|[P_L, \tau]E_w]\|
\]

\[
= \max_{|w| = L} \|[P_w, E_w \tau]\|
\]

\[
\leq \max_{|w| = L} 2diam(\sigma(E_w \tau | E_w \mathcal{H}))
\]

\[
\leq \max_{|w| = L} 2diam(K_2) \leq 2 \lambda^L \text{diam}(K).
\]
Lemma 5.2. Assume \( \nu \) is the spectral measure \( E(\tau; \cdot) \) is singular w.r.t. \( H_p \). Then we have

\[ k_p^- (\tau) = 0. \]

Proof. Since \( \tau \) is the orthogonal sum of \( n \)-tuples with cyclic vector, it suffices to prove the lemma when \( \tau \) has a cyclic unit vector \( \xi \). The absolute continuity class on \( K \) of \( E(\tau; \cdot) \) is then the same as the absolute continuity class of the scalar measure \( \nu = \langle E(\tau; \cdot)\xi, \xi \rangle \).

Given \( \epsilon > 0 \) we can find a compact set \( C_m \) which is a finite union of \( K_w \) so that \( H_p(C_m) \leq \epsilon \) and \( \nu(K \setminus C_m) < 2^{-m} \). Since \( \nu(K \setminus C_m) \subset C_m \) the preceding lemma gives that

\[ k_p^- (\tau \mid E(\tau; C_m)\xi) \leq \epsilon \cdot c \cdot H_p(C_m) \leq c \cdot \epsilon. \]

On the other hand \( \nu(K \setminus C_m) \leq 2^{-m} \) gives

\[ \|\xi - E(\tau; C_m)\xi\|^2 \leq 2^{-m}. \]

Since \( (\tau)'' \) is a maximal abelian von Neumann algebra in \( B(\mathcal{H}) \), \( \xi \) being cyclic is also separating and we infer \( E(\tau; C_m) \rightarrow^w I \) and hence \( E(\tau; C_m)^* \rightarrow I \) as \( m \rightarrow \infty \). This gives that we can find \( A_m \in \mathfrak{M}_1^+ \), \( A_m \leq E(\tau; C_m) \), \( \|A_m, \tau\|_p \leq k_p^- (\tau \mid E(\tau; C_m)) \) and \( A_m \uparrow I \). It follows that

\[ k_p^- (\tau) \leq c \cdot \epsilon \]

and \( \epsilon \) being arbitrary \( k_p^-(\tau) = 0. \)
Note that on $K$ the $p$-Hausdorff measure satisfies an Ahlfors regularity condition

$$C^{-1}r^p \leq H_p(B(x, r)) \leq Cr^p$$

if $r \leq 1$ for some $C > 0$. The right half of this $H_p(B(x, r)) \leq Cr^p$ if $r \leq 1$ is the sub-regularity condition where $p > 1$ required in [2] Cor. 4.7 to show that $k^-_p(\tau_K) > 0$ where $\tau_K$ is the $n$-tuple of multiplication operators by the coordinate functions in $L^2(K, H_p | K)$. Thus we have

**Lemma 5.3.** ([2]) Assume $p > 1$ then $k^-_p(\tau_K) > 0$.

More generally if $\omega \subset K$ is a Borel set let $\tau_\omega$ be the $n$-tuple of multiplication operators by the coordinate functions in $L^2(\omega, H_p | \omega)$ (this is the same as $\tau_K | L^2(\omega, H_p | \omega)$ since $L^2(\omega, H_p | \omega) \subset L^2(K, H_p | K)$). A key part of the proof of the main theorem will be to evaluate $k^-_p(\tau_\omega)$ for increasingly general $\omega$, along lines similar of Lebesgue measure on $\mathbb{R}^n$ considered in [7].

We also define a constant $\gamma_K = \frac{(k^-_p(\tau_K))^p}{H_p(K)}$ where $p$ is the Hausdorff dimension of $K$. Lemma 5.1 and Lemma 5.3 imply that $0 < k^-_p(\tau_K) < \infty$ so that $0 < \gamma_K < \infty$.

**Theorem 5.1.** Let $\tau$ be a $n$-tuple of commuting Hermitian operators with $\sigma(\tau) \subset K$ and assume $p > 1$. Then we have

$$(k^-_p(\tau))^p = \gamma_K \int_K m(x)dH_p(x)$$

where $m$ is the multiplicity function of $\tau$.

**Proof.** Using Lemma 5.2 and the decomposition $\mathcal{H} = \mathcal{H}_{psing} \oplus \mathcal{H}_{pac}$ the proof reduces to the case when the spectral measure of $\tau$ is absolutely continuous w.r.t. $H_p$, that is when $\mathcal{H} = \mathcal{H}_{pac}$. In view of Proposition 2.1 a further reduction is possible to the case when $\tau$ has finite cyclicity, that is when the multiplicity function is bounded. Since when $\tau$ has a cyclic vector and $H_p$-absolutely continuous spectral measure it is unitarily equivalent to a $\tau_\omega$, it means that the proof reduces to the case when $\tau = \tau_{w_1} \oplus \cdots \oplus \tau_{w_m}$ for some Borel sets $\omega_j \subset K, 1 \leq j \leq m$. In view of the last assertion in Proposition 2.1 the theorem holds for $\tau_{w_1} \oplus \cdots \oplus \tau_{w_m}$ iff it holds for

$$\tau_{F_{w_1}(\omega_1)} \oplus \cdots \oplus \tau_{F_{w_m}(\omega_m)}$$

where $|w_1| = \cdots = |w_m|$ because

$$\tau_{F_{w_j}(\omega_j)} \simeq F_{w_j}(\tau_{\omega_j}).$$

We may then choose $|w_j|$ sufficiently large and so that the $F_{w_j}(\omega_j), 1 \leq j \leq m$ are disjoint, which implies that

$$\tau_{F_{w_1}(\omega_1)} \oplus \cdots \oplus \tau_{F_{w_m}(\omega_m)} \simeq \tau_\omega$$

where

$$\omega = F_{w_1}(\omega_1) \cup \cdots \cup F_{w_m}(\omega_m).$$
Thus the proof has been reduced to showing that

\[
(k_p^-(\tau_\omega))^p = \gamma_K H_p(\omega).
\]

First, assume \(\omega\) is a finite union of \(K_w\). Since \(K_w\) is a disjoint union of \(K_w\), with \(|w'| \geq |w|\) we may assume

\[
\omega = K_{w_1} \cup \cdots \cup K_{w_m}
\]

where \(|w_1| = \cdots = |w_m|\) and \(w_1, \ldots, w_m\) are distinct. These \(K_{w_j}\) are congruent and using again the last assertion in Prop. 2.1 the proof of this case reduces to proving the theorem for \(\tau = \tau_{K_w} \otimes I_m\). The multiplicity function is \(m\) times the indicator function of \(K_w\) so that the right-hand side in the formula we want to prove is

\[
\gamma_K m H_p(K_w)
\]

\[
= \frac{(k^+_p(\tau_K))^p}{H_p(K)} \cdot m \cdot \lambda^{|w|} \cdot H_p(K)
\]

\[
= m \cdot (\lambda^{|w|} k_{p}^-(\tau_K))^p = m(k_p^-(\tau_{K_w}))^p.
\]

On the left-hand side we have

\[
(k_p^-(\tau_{K_w} \otimes I_m))^p = (m^{1/p} k_p^-(\tau_{K_w}))^p
\]

by Thm. 3.1 which equals the right-hand side.

Next we prove the theorem for \(\tau_\omega\) when \(\omega \subset K\) is a general open subset. Let \(\omega^{(l)}\) be the union of the \(K_w \subset \omega\) with \(|w| \leq L\). The \(\omega^{(l)}\) are clopen subsets of \(K\) and are finite unions of \(K_w\) so that the theorem holds for \(\omega^{(l)}\) and the theorem for \(\tau_\omega\) is obtained using Prop. 2.1, which gives \(k_p^-(\tau_{\omega^{(l)}}) \uparrow k_p^-(\tau_\omega)\).

Finally let \(\omega \subset K\) be a Borel set and let \(C\) be compact and \(G\) in \(K\) be open so that \(C \subset \omega \subset G\) and \(H_p(G \setminus C) < \epsilon\) for a given \(\epsilon > 0\). We have \(|k_p^-(\tau_\omega) - k_p^-(\tau_G)| \leq |k_p^-(\tau_{G \setminus C})| = (\gamma_K \epsilon)^{1/p}\) using the fact that \(G \setminus C\) is open in \(K\) and Prop. 2.1. Thus

\[
|k_p^-(\tau_\omega) - (\gamma_K H_p(\omega))^{1/p}|
\]

\[
\leq |k_p^-(\tau_\omega) - k_p^-(\tau_G)| + |k_p^-(\tau_G) - (\gamma_K H_p(\omega))^{1/p}|
\]

\[
\leq (\gamma_K \epsilon)^{1/p} + |(\gamma_K H_p(G))^{1/p} - (\gamma_K H_p(\omega))^{1/p}|
\]

\[
\leq (\gamma_K \epsilon)^{1/p} + |(\gamma_K (H_p(\omega) + \epsilon))^{1/p} - (\gamma_K H_p(\omega))^{1/p}|
\]

Since \(\epsilon > 0\) was arbitrary, we get \(k_p^-(\tau_\omega) = (\gamma_K H_p(\omega))^{1/p}\). \(\square\)

**Corollary 5.1.** Assume \(\sigma(\tau) \subset K\) and \(p > 1\). Then \(k_p^-(\tau) = 0\) iff the spectral measure of \(\tau\) is singular w.r.t. \(H_p\).

**Remark 5.1.** In [8] we showed for a \(n\)-tuple \(\tau\) and a normed ideal \(\mathcal{J}\) that there is a largest reducing subspace for \(\tau\) on which \(k_\mathcal{J}\) vanishes. In the case of commuting \(n\)-tuples of Hermitian operators and \(\mathcal{J} = \mathcal{C}_n^-\) this subspace is the subspace where the spectral measure is singular w.r.t. Lebesgue measure.
The theorem we proved in this section shows that if $\sigma(\tau) \subset K$ and $p > 1$, then the largest reducing subspace on which $k_p^-$ vanishes for the restriction of $\tau$ is precisely $\mathcal{H}_{psing}$.

6. Concluding remarks

Remark 6.1. It is natural to wonder whether in general

$$(k_p^-(\tau_1 \oplus \tau_2))^p = (k_p^-(\tau_1))^p + (k_p^-(\tau_2))^p$$

which would be much more than the ampliation homogeneity we proved. If $p = 1$ this is known to be true \[7\]. For $1 < p \leq \infty$ this is an open problem. While a negative answer would not be surprising, it is certainly desirable to clarify this issue.

Remark 6.2. To get results for more general self-similar fractals than the Cantor-like $K$ we considered it may be useful to replace $k_p^-$ by $\tilde{k}_p^-$ the variant of $k_p^-$ considered in \[7\] pages 13–16. This amounts to extending the norms of normed ideals to $n$-tuples, not by the max of norm on the components but by the norm of $(T_1^*T_1 + \cdots + T_n^*T_n)^{1/2}$ that is the modulus in the polar decomposition of the column

$$\begin{pmatrix} T_1 \\ \vdots \\ T_m \end{pmatrix}.$$  

This $|\tau|_\tilde{p}$ has the advantage over $|\tau|_p$ of being invariant under rotations, that is if $(u_{ij})_{1 \leq i,j \leq n}$ is a unitary matrix then the $n$-tuple $(\sum_j u_{ij}T_j)_{1 \leq i \leq n}$ has the same $\sim$-norm as $\tau = (T_i)_{1 \leq i \leq n}$.

In particular $\tilde{k}_p^-$ may be better suited to handle self-similar sets $K$ when we use more general $F_i(x) = \lambda U_i(X - b(i)) + b(i)$ where $U_i \in O(n)$. In particular it is quite straightforward to use $\sim$-norms in \S 3 and to see that ampliation homogeneity still holds for $\tilde{k}_p^-$ which we record as the next theorem.

Theorem 6.3. If $\tau$ is an $n$-tuple of bounded operators and $1 \leq p \leq \infty$ then

$$\tilde{k}_p^-(\tau \otimes I_m) = m^{1/p}\tilde{k}_p^-(\tau).$$

Remark 6.4. In \[10\] we give an extension in another direction to the formula for $k_n^-(\tau)$ in \[7\] to hybrid perturbations.

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Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720-3840

E-mail address: dvv@math.berkeley.edu