Modified Wilcoxon-Mann-Whitney tests of stochastic dominance

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Abstract

Given independent samples from two univariate distributions, one-sided Wilcoxon-Mann-Whitney statistics may be used to conduct rank-based tests of stochastic dominance. We broaden the scope of applicability of such tests by showing that the bootstrap may be used to conduct valid inference in a matched pairs sampling framework permitting dependence between the two samples. Further, we show that a modified bootstrap incorporating an implicit estimate of a contact set may be used to improve power. Numerical simulations indicate that our test using the modified bootstrap effectively controls the null rejection rates and can deliver more or less power than that of the Donald-Hsu test. In the course of establishing our results we obtain a weak approximation to the empirical ordainance dominance curve permitting its population density to diverge to infinity at zero or one at arbitrary rates.

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1 Introduction

Stochastic dominance is a central concept in a wide range of research areas including economics and finance. It captures the notion that one random variable is stochastically larger than another. Given two cumulative distribution functions (CDFs) $F_1$ and $F_2$ on the real line, which are throughout this article always assumed to be continuous with common support (see Assumption 2.1 below), we say that $F_1$ stochastically dominates $F_2$ if $F_1(x) \leq F_2(x)$ for all real $x$. While higher-order forms of stochastic dominance such as second-order stochastic dominance are also of much interest, we are concerned in this article only with stochastic dominance defined in the first-order sense just given.

It is common to base tests of stochastic dominance on the one-sided Kolmogorov-Smirnov (KS) statistic $\sup_{x \in \mathbb{R}}(\hat{F}_1(x) - \hat{F}_2(x))$, where $\hat{F}_1$ and $\hat{F}_2$ are the empirical counterparts to $F_1$ and $F_2$. Depending on the application, $\hat{F}_1$ and $\hat{F}_2$ may be computed from two samples drawn independently from $F_1$ and $F_2$, or from a single bivariate sample drawn from a joint distribution with margins $F_1$ and $F_2$. We refer to the former sampling framework as independent sampling and to the latter as matched pairs. In the independent sampling framework, the asymptotic distribution of the one-sided KS statistic when $F_1 = F_2$ (the so-called least favorable case) was obtained in Smirnov (1939), while the exact sampling distribution was studied in numerous subsequent articles surveyed in Durbin (1973); see also McFadden (1989) for an entry-point to this literature aimed at economists, and an extension to tests of second-order stochastic dominance. Simulation-based and bootstrap procedures for constructing asymptotically valid critical values for the one-sided KS statistic were proposed in Barrett and Donald (2003). Modifications to these procedures developed in Donald and Hsu (2016) can be applied in both the independent and matched pairs sampling frameworks, and can deliver substantial improvements to power given the suitable selection of a tuning parameter. Further contributions developing tests of first- or higher-order stochastic dominance using a range of statistics include Anderson (1996), Schmid and Trede (1996), Davidson and Duclos (2000), Hall and Yatchew (2005), Linton, Maasoumi and Whang (2005), Bennett (2008), Linton, Song and Whang (2010) and Lok and Tabri (2021), among others.
**Figure 1.1:** An empirical ODC for two samples of ten observations. The one-sided KS statistic is determined by the length of the vertical double-headed arrow. The one-sided WMW statistic is determined by the shaded area.

To motivate the statistic studied in this article – the one-sided WMW statistic, defined in Section 2.3 – it is useful to observe that both it and the one-sided KS statistic can be expressed as functionals of the empirical ordinal dominance curve (ODC) defined by the composition $\hat{R} = \hat{F}_1 \circ \hat{Q}_2$, where $\hat{Q}_2$ is the empirical counterpart to $Q_2$, the quantile function corresponding to $F_2$. In Figure 1.1 we display an empirical ODC $\hat{R}$ computed from two samples of size ten. The one-sided KS statistic is the maximum degree by which $\hat{R}$ exceeds the 45-degree line, providing a measure of the degree to which stochastic dominance is empirically violated. The one-sided WMW statistic provides another measurement of the degree to which stochastic dominance is empirically violated: it is equal to the shaded area in Figure 1.1, which differs negligibly (see Lemma 2.1 below) from the area of the set of points lying below the graph of $\hat{R}$ and above the 45-degree line. Numerical simulations reported in Schmid and Trede (1996) for the independent sampling framework indicate that a test of stochastic dominance based on the one-sided WMW statistic may deliver more or less power than one based on the one-sided KS statistic depending on the way in which stochastic dominance is violated. It may therefore be useful to compare the results of both tests in empirical applications.
The asymptotic distribution of the one-sided WMW statistic when $F_1 = F_2$ was obtained in Schmid and Trede (1996) for the independent sampling framework. In this article we extend this result by obtaining the asymptotic distribution of the one-sided WMW statistic for both the independent and matched pairs sampling frameworks, and at all pairs of CDFs $F_1$ and $F_2$ belonging to the boundary of the null, by which we mean all pairs satisfying the null hypothesis and also satisfying $F_1(x) = F_2(x)$ for all $x$ in a set of positive measure within the common support of $F_1$ and $F_2$. This extension is contained in Proposition 2.2 below. The asymptotic distribution in the matched pairs sampling framework when $F_1 = F_2$ depends on the unknown copula $C$ for the matched pairs, meaning that the asymptotic critical values obtained in Schmid and Trede (1996) for the independent sampling framework may not be applied. We show in Proposition 3.3 that the bootstrap can be used to construct critical values for the one-sided WMW statistic which deliver a limiting rejection rate no greater than the nominal level when the null hypothesis is satisfied, equal to the nominal level when $F_1 = F_2$, and equal to one when the null hypothesis is violated.

It was shown in Donald and Hsu (2016) that the bootstrap critical values for the one-sided KS statistic proposed in Barrett and Donald (2003) may be modified in such a way as to deliver substantially improved power while retaining control of null rejection rates. Related modifications were used in Linton, Song and Whang (2010) and Lok and Tabri (2021) to improve the power of bootstrap tests of stochastic dominance based on a different statistic resembling a one-sided reweighted Cramér-von Mises statistic. In all three cases, the bootstrap critical value is modified in such a way as to raise the limiting rejection rate to the nominal level on the boundary of the null, thereby improving power. The modifications depend on a tuning parameter which must be chosen to decay to zero or diverge to infinity at a specified rate as the sample sizes increase. We show in Proposition 3.4 that a similar outcome may be achieved by suitably modifying our bootstrap critical value for the one-sided WMW statistic. The modification involves adjusting the resampled bootstrap statistics to incorporate an implicit estimate of the contact set, by which we mean the set of values in the unit interval on which the graph of the ODC $R$ coincides with the 45-degree line. The bootstrap modifications in Linton, Song and Whang (2010) and Lok and Tabri (2021) also
rely on estimation of a contact set, whereas the modification in Donald and Hsu (2016) is based on a distinct recentering method adapted from Hansen (2005). We report numerical simulations in Section 4 providing evidence that, with a suitable choice of tuning parameter, our modification to the bootstrap critical value can lead to a large increase in power while maintaining control of null rejection rates.

Our demonstration of the asymptotic validity of our bootstrap and modified bootstrap critical values for the one-sided WMW statistic relies on the fact that the empirical ODC \( \hat{R} \), suitably normalized, converges weakly to a centered Gaussian process. Existing results establishing such weak convergence – see the discussion at the end of Section 2.4.1 – have done so with respect to the uniform norm, and have typically required \( R \) to have bounded density. This requirement is very restrictive, ruling out even the canonical case where \( F_1 \) and \( F_2 \) are Gaussian distributions with common scale and different locations, as in this case \( R \) has density diverging to infinity at zero or one. In Aly, Csörgő and Horváth (1987), weak convergence of the normalized empirical ODC in the uniform norm is established under conditions permitting the density of \( R \) to diverge to infinity at zero and one, but only at a rate slow enough to satisfy a Chibisov-O’Reilly condition. Further, it is shown that weak convergence in the uniform norm need not be satisfied if the Chibisov-O’Reilly condition is dropped. Making use of important recent work on the weak convergence of empirical quantile processes in Kaji (2018), we establish in Proposition 2.1, without imposing any condition on the rate of divergence of the density of \( R \) at zero and one, that weak convergence of the normalized empirical ODC holds with respect to the \( L^1 \) norm. We also establish a bootstrap analogue to this weak convergence in Proposition 3.2. Since the one-sided WMW statistic differs negligibly from the area below the graph of \( \hat{R} \) and above the 45-degree line, weak convergence in the \( L^1 \) norm is sufficient for us to characterize the asymptotic behavior of the one-sided WMW statistic and its bootstrap counterpart. Propositions 2.1 and 3.2 may be of independent interest in other contexts where the asymptotic behavior of the empirical ODC is a relevant concern.

The remainder of this article is structured as follows. We define our null and alternative hypotheses in Section 2.1, our sampling frameworks in Section 2.2, and the one-sided WMW
statistic in Section 2.3. We discuss the asymptotic behavior of the empirical ODC and one-sided WMW statistic in Section 2.4, the construction of our standard and modified bootstrap critical values in Section 3.1, and the asymptotic properties of tests based on these critical values in Section 3.2. Numerical simulations are reported in Section 4, and final remarks provided in Section 5. Proofs of numbered mathematical statements are collected together in Section 6.

Throughout this article, we let $\ell^\infty[0, 1]$ denote the space of bounded real functions on $[0, 1]$ equipped with the uniform norm. We let $C[0, 1]$ and $C[0, 1]^2$ denote the spaces of continuous real functions on $[0, 1]$ and $[0, 1]^2$ respectively, equipped with the uniform norm. We let $L^1(0, 1)$ denote the space of integrable real functions on $(0, 1)$ equipped with the $L^1$ norm, identifying functions which coincide on a set of measure one. We let $\Rightarrow$ denote Hoffman-Jørgensen weak convergence in a normed space or product of normed spaces.

## 2 The one-sided WMW statistic

### 2.1 Hypothesis formulation

Let $F_1 : \mathbb{R} \to \mathbb{R}$ and $F_2 : \mathbb{R} \to \mathbb{R}$ be CDFs. Define the quantile function $Q_2 : (0, 1) \to \mathbb{R}$ by $Q_2(u) = \inf \{ x \in \mathbb{R} : F_2(x) \geq u \}$, and the ODC $R : (0, 1) \to (0, 1)$ by $R(u) = F_1(Q_2(u))$. The ODC is also commonly called a PP plot, and is closely related to the receiver operating characteristic curve. We impose the following regularity conditions on $F_1$, $F_2$ and $R$.

**Assumption 2.1.** (a) $F_1$ and $F_2$ have common support $I$, a closed and convex subset of $\mathbb{R}$. (b) $F_1$ and $F_2$ are continuous on $\mathbb{R}$ and strictly increasing on $I$. (c) $R$ is continuously differentiable on $(0, 1)$, with strictly positive derivative $r : (0, 1) \to \mathbb{R}$.

Under Assumption 2.1, $R$ approaches zero (one) at the left (right) endpoint of the unit interval. If $F_1$ and $F_2$ admit densities $f_1$ and $f_2$ then $r$ is given by

$$r(u) = \frac{f_1(Q_2(u))}{f_2(Q_2(u))}. \quad (2.1)$$
Note that \( r \) need not be bounded under Assumption 2.1. For instance, if \( F_1 \) and \( F_2 \) are Gaussian CDFs with variance one and means one and zero respectively, then \( r \) is given by

\[
    r(u) = \exp \left( Q_2(u) - \frac{1}{2} \right),
\]

which diverges to infinity as \( u \to 1 \). If we interchange \( F_1 \) and \( F_2 \) then \( r(u) \) instead diverges to infinity as \( u \to 0 \). The case where \( r \) diverges to infinity at zero or one may thus be regarded as being of canonical importance.

Stochastic dominance may be reformulated as a property of the ODC. Specifically, \( F_1 \) stochastically dominates \( F_2 \) if and only if \( R(u) \leq u \) for all \( u \in (0, 1) \). The hypotheses we seek to discriminate between are thus

\[
    H_0 : R(u) \leq u \text{ for all } u \in (0, 1), \\
    H_1 : R(u) > u \text{ for some } u \in (0, 1).
\]

The null hypothesis \( H_0 \) is satisfied when \( F_1 \) stochastically dominates \( F_2 \), while the alternative hypothesis \( H_1 \) is satisfied when such dominance does not occur.

### 2.2 Sampling frameworks

We consider two sampling frameworks: independent samples and matched pairs. In both frameworks we observe two independent and identically distributed (iid) samples \( \{X^1_i\}_{i=1}^{n_1} \) and \( \{X^2_i\}_{i=1}^{n_2} \) drawn from \( F_1 \) and \( F_2 \) respectively. The sample sizes \( n_1 \) and \( n_2 \) are viewed as functions of an underlying index \( n \in \mathbb{N} \) satisfying

\[
    \frac{n_1 n_2}{n_1 + n_2} \to \infty \quad \text{and} \quad \frac{n_2}{n_1 + n_2} \to \lambda \in (0, 1) \quad \text{as} \quad n \to \infty. \tag{2.3}
\]

In the independent sampling framework the two samples are assumed to be independent of one another, and we define \( C \) to be the product copula \( C(u, v) = uv \). In the matched pairs sampling framework we require that \( n_1 = n_2 = n \), so that (2.3) is satisfied with \( \lambda = 1/2 \), and require the pairs \( \{(X^1_i, X^2_i)\}_{i=1}^n \) to be iid. We let \( C \) denote the copula for each pair \( (X^1_i, X^2_i) \),
uniquely determined by Sklar's theorem since $F_1$ and $F_2$ are continuous under Assumption 2.1.

**Assumption 2.2.** The iid samples $\{X_1^i\}_{i=1}^{n_1}$ and $\{X_2^i\}_{i=1}^{n_2}$ drawn from $F_1$ and $F_2$ satisfy one of the following conditions.

(i) *(Independent sampling.)* $\{X_1^i\}_{i=1}^{n_1}$ and $\{X_2^i\}_{i=1}^{n_2}$ are mutually independent, and the sample sizes $n_1$ and $n_2$ satisfy (2.3).

(ii) *(Matched pairs.)* The sample sizes $n_1$ and $n_2$ satisfy $n_1 = n_2 = n$, the pairs $\{(X_1^i, X_2^i)\}_{i=1}^n$ are iid, and the bivariate copula $C$ for those pairs has maximal correlation strictly less than one.

Assumption 2.2 is identical to the same numbered assumption in Sun and Beare (2021), which concerns the related context of Lorenz dominance testing.

### 2.3 Construction of test statistic

Define the empirical CDFs $\hat{F}_1 : \mathbb{R} \to \mathbb{R}$ and $\hat{F}_2 : \mathbb{R} \to \mathbb{R}$ by

$$\hat{F}_j(x) = \frac{1}{n_j} \sum_{i=1}^{n_j} 1(X_j^i \leq x), \quad j \in \{1, 2\},$$

the empirical quantile function $\hat{Q}_2 : (0, 1) \to \mathbb{R}$ by $\hat{Q}_2(u) = \inf\{x \in \mathbb{R} : \hat{F}_2(x) \geq u\}$, and the empirical ODC $\hat{R} : (0, 1) \to [0, 1]$ by $\hat{R}(u) = \hat{F}_1(\hat{Q}_2(u))$. All “hatted” quantities in this article are estimated from data and implicitly indexed by $n$.

The test statistic we consider is the one-sided WMW statistic studied in Schmid and Trede (1996). It provides an estimate of the area that lies below the graph of the ODC $R$ and above the 45-degree line. Define the functional $\mathcal{F} : L^1(0, 1) \to \mathbb{R}$ by

$$\mathcal{F}(h) = \int_0^1 \max\{h(u) - u, 0\} \, du.$$
The area that lies below the graph of $R$ and above the 45-degree line is then $\mathcal{F}(R)$, which satisfies

$$\mathcal{F}(R) = \int_{-\infty}^{\infty} \max\{F_1(x) - F_2(x), 0\} dF_2(x). \quad (2.6)$$

We have $\mathcal{F}(R) = 0$ under $H_0$, and $\mathcal{F}(R) > 0$ under $H_1$. The one-sided WMW statistic is obtained by replacing $F_1$ and $F_2$ on the right-hand side of (2.6) with their empirical counterparts $\hat{F}_1$ and $\hat{F}_2$, and then scaling by the square root of $T_n = n_1n_2/(n_1 + n_2)$ to obtain a statistic with nondegenerate limit distribution on the boundary of the null. It is observed in Schmid and Trede (1996) that

$$\int_{-\infty}^{\infty} \max\{\hat{F}_1(x) - \hat{F}_2(x), 0\} d\hat{F}_2(x) = \frac{1}{n_2} \sum_{i=1}^{n_2} \max\{\hat{F}_1(X^2_{(i)}) - \frac{i}{n_2}, 0\}, \quad (2.7)$$

where $X^2_{(1)}, \ldots, X^2_{(n_2)}$ are the sample observations $X^2_1, \ldots, X^2_{n_2}$ ranked from smallest to largest. The one-sided WMW statistic is thus

$$\hat{S} = \frac{T_{n}^{1/2}}{n_2} \sum_{i=1}^{n_2} \max\{\hat{F}_1(X^2_{(i)}) - \frac{i}{n_2}, 0\}, \quad (2.8)$$

which is $T_{n}^{1/2}$ times the shaded area in Figure 1.1. Note that $\hat{S}$ is determined by the empirical ODC $\hat{R}$, which is determined by the ranks of the $n_1 + n_2$ observations $X^j_i$. This means that both $\hat{S}$ and $\hat{R}$ are unaffected if all observations $X^j_i$ are replaced with $g(X^j_i)$ using some strictly increasing transformation $g : \mathbb{R} \rightarrow \mathbb{R}$. For this reason $\hat{S}$ and $\hat{R}$ are said to be rank-based.

**Lemma 2.1.** The statistic $\hat{S}$ satisfies $\hat{S} \leq T_{n}^{1/2} \mathcal{F}(\hat{R}) \leq \hat{S} + T_{n}^{1/2}/(2n_2)$.

Lemma 2.1 indicates that $\hat{S}$ may be regarded as a computationally convenient approximation to $T_{n}^{1/2} \mathcal{F}(\hat{R})$. The maximum approximation error $T_{n}^{1/2}/(2n_2)$ vanishes asymptotically under Assumption 2.2.
2.4 Asymptotic properties

2.4.1 Weak convergence of the empirical ordinal dominance process

Let $B$ be a centered Gaussian random element of $C[0, 1]^2$ with covariance kernel

$$
\text{Cov}(B(u, v), B(u', v')) = C(u \wedge u', v \wedge v') - C(u, v)C(u', v'),
$$

(2.9)

and let $B_1$ and $B_2$ be the random elements of $C[0, 1]$ given by $B_1(u) = B(u, 1)$ and $B_2(u) = B(1, u)$. Note that $B_1$ and $B_2$ are Brownian bridges, and that these Brownian bridges are independent if $C$ is the product copula but may be dependent in the matched pairs sampling framework. A central ingredient to our study of the asymptotic behavior of the one-sided WMW statistic is the following result establishing weak convergence of the empirical ordinal dominance process $T_n^{1/2}(\hat{R} - R)$ in $L^1(0, 1)$.

**Proposition 2.1.** Under Assumptions 2.1 and 2.2, we have

$$
T_n^{1/2}(\hat{R} - R) \Rightarrow \mathcal{R} := \lambda^{1/2}B_1 \circ R - (1 - \lambda)^{1/2}r \cdot B_2 \quad \text{in } L^1(0, 1).
$$

(2.10)

To the best of our knowledge, Proposition 2.1 provides the first weak approximation to the empirical ODC which permits arbitrary rates of divergence of $r$ at zero and one (subject to integrability, which is automatic since $r$ is the derivative of $R$). Weak and strong approximations in the uniform norm used in Hsieh and Turnbull (1996), Beare and Moon (2015), Tang, Wang and Tebbs (2017), Beare and Shi (2019) and Wang and Tang (2021) require $r$ to be bounded. Theorem 3.1 in Aly, Csörgő and Horváth (1987) establishes a weak approximation in the uniform norm under the requirement that $r$ satisfies a Chibisov-O’Reilly condition, effectively bounding the rate of divergence of $r$ at zero and one. It is shown there that weak convergence in the uniform norm need not hold if the Chibisov-O’Reilly condition is dropped. Proposition 2.1 shows that the Chibisov-O’Reilly condition can be dropped if we require only weak convergence in $L^1(0, 1)$. All of the references just cited deal only with the case of independent sampling, with the exception of Wang and Tang (2021), which concerns a matched pairs sampling framework.
Our proof of Proposition 2.1 proceeds broadly as follows. We first observe that, since \( \hat{R} \) is rank-based and \( F_1 \) is strictly increasing on the common support of \( F_1 \) and \( F_2 \), we may replace our \( n_1 + n_2 \) observations \( X_i^j \) with \( F_1(X_i^j) \) without affecting \( \hat{R} \). The ODC for the transformed observations is the same as the ODC for the original observations. Since \( F_1 \) is continuous, the transformed observations \( F_1(X_i^j) \) are uniformly distributed on \((0, 1)\). It is therefore without loss of generality to normalize \( F_1 \) to be the CDF of the uniform distribution on \((0, 1)\), so that \( Q_2 = R \). By applying results on the weak convergence of empirical quantile processes in \( L^1(0, 1) \) obtained in Kaji (2018) – see Section 6 for details – we arrive at the weak convergence

\[
T_n^{1/2} \begin{pmatrix} \hat{F}_1 - F_1 \\ \hat{Q}_2 - Q_2 \end{pmatrix} \overset{\text{d}}{\rightarrow} \begin{pmatrix} \lambda^{1/2} B_1 \\ -(1 - \lambda)^{1/2} r \cdot B_2 \end{pmatrix} \quad \text{in } \ell^\infty[0, 1] \times L^1(0, 1).
\]

Proposition 2.1 follows from (2.11) by applying the functional delta method – see e.g. Theorem 2.8 in Kosorok (2008) – with the composition map \((F, Q) \mapsto F \circ Q\). Suitable Hadamard differentiability of this map is obtained in Lemma 6.2.

2.4.2 Asymptotic distribution of test statistic

In view of Lemma 2.1, the difference between our test statistic \( \hat{S} \) and \( T_n^{1/2} F(\hat{R}) \) is asymptotically negligible. It will be more convenient for us to study the behavior of the latter quantity. The null hypothesis of stochastic dominance is satisfied if and only if \( F(R) = 0 \). In this case we have

\[
T_n^{1/2} F(\hat{R}) = T_n^{1/2}(F(\hat{R}) - F(R)).
\]

In view of the weak convergence of \( T_n^{1/2}(\hat{R} - R) \) established in Proposition 2.1, we can obtain the limit distribution of \( T_n^{1/2}(F(\hat{R}) - F(R)) \) in (2.12) by applying the functional delta
method. Define the map \( \mathcal{F}'_R : L^1(0, 1) \to \mathbb{R} \) by
\[
\mathcal{F}'_R(h) = \int_{B_+(R)} h(u) \, du + \int_{B_0(R)} \max\{h(u), 0\} \, du,
\]
where \( B_+(R) \) and \( B_0(R) \) are the sets
\[
B_+(R) = \{ u \in (0, 1) : R(u) > u \} \quad \text{and} \quad B_0(R) = \{ u \in (0, 1) : R(u) = u \}.
\]

We refer to the set \( B_0(R) \) as the contact set. It is established in Lemma 6.3 that \( \mathcal{F}'_R \) is the Hadamard directional derivative of \( \mathcal{F} \) at \( R \). Note that if the contact set has positive measure, which must be the case when the pair of CDFs \( F_1 \) and \( F_2 \) belongs to the boundary of the null, then \( \mathcal{F}'_R \) is not linear and so \( \mathcal{F} \) is not Hadamard differentiable at \( R \), but only Hadamard directionally differentiable. See Fang and Santos (2019) for definitions of these notions of differentiability and a discussion of their importance in statistics and econometrics. Other recent contributions to econometrics in which the distinction between Hadamard differentiability and Hadamard directional differentiability plays an important role include Beare and Moon (2015), Kaido (2016), Beare and Fang (2017), Seo (2018), Beare and Shi (2019), Chen and Fang (2019a,b), Fang (2019), Sun and Beare (2021) and Beare (2021).

While standard accounts of the functional delta method require the map in question to be Hadamard differentiable, it was shown independently in Shapiro (1991) and Dümbgen (1993) that the weaker condition of Hadamard directional differentiability is sufficient. An application of the functional delta method with the Hadamard directionally differentiable map \( \mathcal{F} \) allows us to deduce the following result from Proposition 2.1.

**Proposition 2.2.** Under Assumptions 2.1 and 2.2, we have
\[
T_n^{1/2}(\mathcal{F}(\hat{R}) - \mathcal{F}(R)) \rightsquigarrow \mathcal{F}'_R(R) \quad \text{in } \mathbb{R}.
\]
If \( H_0 \) is satisfied then we have \( \hat{S} \rightsquigarrow \mathcal{F}'_R(R) \) in \( \mathbb{R} \), whereas if \( H_1 \) is satisfied then \( \hat{S} \) diverges in probability to infinity.

Proposition 2.2 establishes the limit distribution of \( T_n^{1/2}(\mathcal{F}(\hat{R})) \), and therefore of \( \hat{S} \), at all
null configurations. The limit is nondegenerate when the pair of CDFs \( F_1 \) and \( F_2 \) belongs to the boundary of the null, and is degenerate (zero) at other null configurations. Schmid and Trede (1996) observed, under independent sampling, that at null configurations such that \( F_1 = F_2 \) (i.e., the least favorable case) the test statistic \( \hat{S} \) is asymptotically distributed as \( \int_0^1 \max \{ \mathcal{B}(u), 0 \} du \), with \( \mathcal{B} \) a Brownian bridge. This follows from Proposition 2.2 by noting that, under independent sampling, if \( F_1 = F_2 \) then \( \mathcal{R} \) is a Brownian bridge. We may therefore construct a test of \( H_0 \) with limiting rejection rate no greater than \( \alpha \) at all null configurations, and equal to \( \alpha \) at null configurations with \( F_1 = F_2 \), by rejecting \( H_0 \) when \( \hat{S} \) exceeds the \((1 - \alpha)\)-quantile of \( \int_0^1 \max \{ \mathcal{B}(u), 0 \} du \). Schmid and Trede (1996) calculate the 0.9, 0.95 and 0.99 quantiles to be 0.39, 0.48 and 0.68 respectively.

In the matched pairs sampling framework \( \mathcal{R} \) is no longer a Brownian bridge at all null configurations such that \( F_1 = F_2 \), and depends on the unknown copula \( C \). The critical values computed by Schmid and Trede (1996) therefore no longer apply. In the following section we propose a bootstrap scheme to produce critical values for \( \hat{S} \) which apply under both independent sampling and matched pairs. Further, we show how power may be improved by incorporating an implicit estimate of the contact set into our bootstrap scheme.

### 3 Bootstrap procedures

#### 3.1 Construction of bootstrap critical values

##### 3.1.1 Standard bootstrap critical values

Our baseline procedure to obtain a bootstrap critical value for the test statistic \( \hat{S} \) is as follows. We first construct bootstrap CDFs \( \hat{F}_1^* : \mathbb{R} \rightarrow \mathbb{R} \) and \( \hat{F}_2^* : \mathbb{R} \rightarrow \mathbb{R} \) by setting

\[
\hat{F}_j^*(x) = \frac{1}{n_j} \sum_{i=1}^{n_j} W_{i,n_j}^j \mathbb{1}(X_i^j \leq x), \quad j \in \{1, 2\},
\]

where \( W_{n_1}^1 = (W_{1,n_1}^1, \ldots, W_{n_1,n_1}^1) \) and \( W_{n_2}^2 = (W_{1,n_2}^2, \ldots, W_{n_2,n_2}^2) \) are random weights generated independently of the data. The way in which the weights are generated depends
on the sampling framework. With independent samples we draw \( W_{n_1}^1 \) and \( W_{n_2}^1 \) independently of one another from the multinomial distribution with equal probabilities over the categories 1, \ldots, n_1 and 1, \ldots, n_2 \) respectively. With matched pairs we draw \( W_{n_1}^1 \) from the multinomial distribution with equal probabilities over the categories 1, \ldots, n, and then set \( W_{n_2}^2 = W_{n_1}^1 \). In either sampling framework, we then construct the bootstrap quantile function \( \hat{Q}^*_2 : (0,1) \to \mathbb{R} \) by setting \( \hat{Q}^*_2(u) = \inf\{x \in \mathbb{R} : \hat{F}^*_2(x) \geq u\} \), and the bootstrap ODC \( \hat{R}^* : (0,1) \to (0,1) \) by setting \( \hat{R}^*(u) = \hat{F}^*_1(\hat{Q}^*_2(u)) \). We then compute the bootstrap test statistic

\[
\hat{S}^* = \frac{T_n^{1/2}}{n_2} \sum_{i=1}^{n_2} \max\{\hat{R}^*\left(\frac{i}{n_2}\right) - \hat{R}\left(\frac{i}{n_2}\right), 0\}.
\]  

(3.1)

To obtain a test with nominal level \( \alpha \) we independently generate a large number \( N \) of bootstrap test statistics and choose as our critical value the \( [N(1 - \alpha)] \)-th largest of these. We reject \( H_0 \) when \( \hat{S} \) exceeds this critical value.

\subsection{3.1.2 Modified bootstrap critical values}

Our second bootstrap procedure involves modifying the bootstrap statistics defined in (3.1) so as to incorporate an implicit estimate of \( B_0(R) \), the contact set. The first step in this procedure is to compute, for \( i \in \{1, \ldots, n_2\} \), an estimate \( \hat{V}_i \) of the variance of \( T_n^{1/2} \hat{R}(i/n_2) \). This is discussed in more detail below. Next, to generate a single bootstrap test statistic, we generate \( \hat{R}^* \) in the same way as in the standard bootstrap procedure described in Section 3.1.1, and then compute the modified bootstrap test statistic

\[
\tilde{S}^* = \frac{T_n^{1/2}}{n_2} \sum_{i=1}^{n_2} \max\{\hat{R}^*\left(\frac{i}{n_2}\right) - \hat{R}\left(\frac{i}{n_2}\right), 0\} \mathbb{1}\left(T_n^{1/2} \left(\hat{R}\left(\frac{i}{n_2}\right) - \frac{i}{n_2}\right) \geq -\tau_n \hat{V}_i^{1/2}\right),
\]  

(3.2)

where \( \tau_n \in (0,\infty) \) is a tuning parameter. Note that if we set \( \tau_n = \infty \) in (3.2) then we recover the standard bootstrap test statistic defined in (3.1). To obtain a test with nominal level \( \alpha \) we independently generate a large number \( N \) of modified bootstrap test statistics and choose as our critical value the \( [N(1 - \alpha)] \)-th largest of these. We reject \( H_0 \) when \( \tilde{S} \)
exceeds this critical value.

The role of the indicator function in (3.2) is to exclude summands for which \( \hat{R}(i/n_2) \) falls below \( i/n_2 \) by more than \( \tau_n \) estimated standard deviations. It provides an implicit estimate of the contact set. This will be made more clear in Section 3.2.3. In the development of asymptotics to follow we will assume that \( \tau_n \) diverges to infinity at a controlled rate as \( n \to \infty \). The numerical simulations reported in Section 4 may be used to guide the choice of \( \tau_n \) in practice.

The variance estimators \( \hat{V}_i \) may be chosen to approximate the pointwise variances of the weak limit \( \mathcal{R} \) of \( T_n^{1/2}(\hat{R} - R) \) given in Proposition 2.1.

**Proposition 3.1.** Under Assumptions 2.1 and 2.2, \( \mathcal{R}(u) \) has variance equal to

\[
\lambda(R(u) - R(u)^2) + (1 - \lambda)r(u)^2(u - u^2) - 2\lambda^{1/2}(1 - \lambda)^{1/2}r(u)(C(R(u), u) - R(u)u)
\]

for each \( u \in (0, 1) \). In particular, for each \( u \in B_0(R) \), we have \( \text{Var}(\mathcal{R}(u)) = u - C(u, u) \).

There are a range of possibilities for estimating the pointwise variances in Proposition 3.1. It is sufficient for our purposes to focus on estimators of \( \text{Var}(\mathcal{R}(u)) \) that work well for \( u \in B_0(R) \). We therefore propose setting

\[
\hat{V}_i = \frac{i}{n_2} - \frac{i^2}{n_2^2}
\]

under independent sampling, or

\[
\hat{V}_i = \frac{i}{n} - \hat{C}(\frac{i}{n}, \frac{i}{n})
\]

with matched pairs, where \( \hat{C} : (0, 1)^2 \to \mathbb{R} \) is the empirical copula

\[
\hat{C}(u, v) = \frac{1}{n} \sum_{i=1}^{n} 1(\hat{F}_1(X_i^1) \leq u, \hat{F}_2(X_i^2) \leq v).
\]

Note that in the latter case \( \hat{C} \) and thus \( \hat{V}_i \) are rank-based, meaning that they depend on the data only through the ranks of the observations.
3.2 Asymptotic properties

3.2.1 Weak convergence of the bootstrap ordinal dominance process

To study the asymptotic behavior of our bootstrap procedures we require a bootstrap analogue to the weak convergence of the empirical ordinal dominance process established in Proposition 2.1. We provide such an analogue in Proposition 3.2. In the statement of this result we write $P \xrightarrow{W} W$ to denote weak convergence conditional on the data in probability in the sense of Kosorok (2008, pp. 19–20). Such convergence should be understood to concern the distribution of an object depending on both our data $\{X_1^1\}_{i=1}^{n_1}$ and $\{X_2^1\}_{i=1}^{n_2}$ and our bootstrap weights $W_1^n$ and $W_2^n$ when we regard the data as being held fixed.

Proposition 3.2. Under Assumptions 2.1 and 2.2, we have

$$T_n^{1/2}(\hat{R}^* - \hat{R}) \xrightarrow{P} W \text{ in } L^1(0, 1).$$

(3.5)

3.2.2 Standard bootstrap critical values

Our standard bootstrap test statistic defined in (3.1) may also be written as

$$\hat{S}^* = \int_0^1 \max\{T_n^{1/2}(\hat{R}^*(u) - \hat{R}(u)), 0\} du.$$  

(3.6)

Let $R_0$ be the ODC which arises when $F_1 = F_2$; i.e., $R_0(u) = u$. Then, recalling the form of the directional derivative $F'_R$ in (2.13), we deduce from (3.6) that $\hat{S}^* = F'_{R_0}(T_n^{1/2}(\hat{R}^* - \hat{R}))$. Noting that $F'_{R_0}$ is a continuous map from $L^1(0, 1)$ to $\mathbb{R}$, we may apply the continuous mapping theorem in conjunction with Proposition 3.2 to obtain

$$\hat{S}^* \xrightarrow{P} W F'_{R_0}(\mathcal{R}) = \int_0^1 \max\{\mathcal{R}(u), 0\} du \text{ in } \mathbb{R}.$$  

(3.7)
We know from Proposition 2.2 that the asymptotic distribution of $\hat{S}$ under $H_0$ is $F'_R(\mathcal{R})$, which satisfies

$$F'_R(\mathcal{R}) = \int_{B_0(R)} \max\{\mathcal{R}(u), 0\} du \leq \int_{0}^{1} \max\{\mathcal{R}(u), 0\} du$$

under $H_0$. It is therefore straightforward to deduce the following result characterizing the asymptotic rejection probabilities of our test using the standard bootstrap critical value.

**Proposition 3.3.** Suppose that Assumptions 2.1 and 2.2 hold. Fix $\alpha \in (0, 1/2)$ and let $\hat{c}_{1-\alpha}$ denote the $(1-\alpha)$-quantile of the distribution of $\hat{S}^*$ conditional on the data; that is,

$$\hat{c}_{1-\alpha} = \inf \left\{ c \in \mathbb{R} : P\left( \hat{S}^* \leq c \right| \{X_i^1\}_{i=1}^{n_1}, \{X_i^2\}_{i=1}^{n_2} \right) \geq 1 - \alpha \right\}.$$  \hspace{1cm} (3.9)

(i) If $H_0$ is true, then $P(\hat{S} > \hat{c}_{1-\alpha})$ converges to a limit no greater than $\alpha$, and equal to $\alpha$ if $F_1 = F_2$.

(ii) If $H_0$ is false, then $P(\hat{S} > \hat{c}_{1-\alpha}) \rightarrow 1$.

**3.2.3 Modified bootstrap critical values**

To study our modified bootstrap procedure we adopt an asymptotic framework in which the tuning parameter $\tau_n$ is assumed to diverge to infinity at a controlled rate as $n \to \infty$.

**Assumption 3.1.** As $n \to \infty$ we have $\tau_n \to \infty$ and $T_n^{-1/2}\tau_n \to 0$.

We claimed in Section 3.1.2 that the indicator function in (3.2) has the effect of providing an implicit estimate of the contact set $B_0(R)$. The implicit estimate we were referring to is the set

$$\hat{B}_0(R) = \left\{ u \in (0, 1) : T_n^{1/2}(\hat{R}(u) - u) \geq -\tau_n \hat{\nu}_{[n2u]}^{1/2} \right\}.$$  \hspace{1cm} (3.10)
Using \( \widehat{B}_0(R) \) we may define the data dependent map \( \widehat{F}_R' : L^1(0, 1) \rightarrow \mathbb{R} \) by

\[
\widehat{F}_R'(h) = \int_{\widehat{B}_0(R)} \max\{h(u), 0\} du. \tag{3.11}
\]

The map \( \widehat{F}_R' \) may be viewed as an implicit estimate of the directional derivative \( F'_R \) defined in (2.13). It does not incorporate an estimate of the set \( B_+(R) \) because this set is empty under \( H_0 \). It is easy to see that our modified bootstrap test statistic \( \tilde{S}^* \) defined in (3.2) may also be written as

\[
\tilde{S}^* = \widehat{F}_R'(T_{n,1/2}^1(\hat{R}^* - \hat{R})) , \tag{3.12}
\]

revealing the connection between \( \tilde{S}^* \) and the implicitly estimated contact set and directional derivative.

The representation of our modified bootstrap test statistic in (3.12) is useful because it facilitates the application of results in Fang and Santos (2019) providing conditions sufficient for the validity of modified bootstrap procedures. By showing that \( \widehat{F}_R' \) suitably approximates \( F'_R \) under \( H_0 \), we are able to deduce the following result from Proposition 3.2 above and Theorem 3.2 in Fang and Santos (2019).

**Lemma 3.1.** Under Assumptions 2.1, 2.2 and 3.1, if \( H_0 \) is true we have

\[
\widehat{F}_R'(T_{n,1/2}^1(\hat{R}^* - \hat{R})) \overset{P}{\Rightarrow} \mathcal{F}_R'(\mathcal{R}) \quad \text{in} \ \mathbb{R}. \tag{3.13}
\]

Using Lemma 3.1 we establish the following result characterizing the asymptotic rejection probabilities of our test using the modified bootstrap critical value.

**Proposition 3.4.** Suppose that Assumptions 2.1, 2.2 and 3.1 are satisfied. Fix \( \alpha \in (0, 1/2) \) and let \( \hat{c}_{1-\alpha} \) denote the \( (1 - \alpha) \)-quantile of the distribution of \( \tilde{S}^* \) conditional on the data; that is,

\[
\hat{c}_{1-\alpha} = \inf \left\{ c \in \mathbb{R} : P \left( \tilde{S}^* \leq c \bigg| \{X_1^1\}_{i=1}^{n_1}, \{X_2^2\}_{i=1}^{n_2} \right) \geq 1 - \alpha \right\} . \tag{3.14}
\]
(i) If $H_0$ is true, and $R(u) = u$ on a set of positive measure, then
\[
\lim_{\eta \downarrow 0} \lim_{n \to \infty} P(\hat{S} > \max\{\tilde{c}_{1-\alpha}, \eta\}) = \lim_{n \to \infty} P(\hat{S} > \tilde{c}_{1-\alpha}) = \alpha.
\]

(ii) If $H_0$ is true, and $R(u) < u$ almost everywhere, then $P(\hat{S} > \max\{\tilde{c}_{1-\alpha}, \eta\}) \to 0$ for each \(\eta > 0\).

(iii) If $H_0$ is false, then $P(\hat{S} > \max\{\tilde{c}_{1-\alpha}, \eta\}) \to 1$ for each \(\eta \geq 0\).

The constant $\eta$ appearing in the statement of Proposition 3.4 resembles what is referred to in Andrews and Shi (2013, p. 625) as an infinitesimal uniformity factor. Its role is to control the limiting rejection rate at null configurations for which $R(u) < u$ almost everywhere. At such configurations, both the test statistic $\hat{S}$ and modified bootstrap critical value $\tilde{c}_{1-\alpha}$ converge in probability to zero, making it difficult to characterize the rejection rate within our first-order asymptotic framework. See Donald and Hsu (2016, pp. 559–560), Beare and Shi (2019, pp. 15–16) and Sun and Beare (2021, p. 195) for further discussion of infinitesimal uniformity factors in closely related contexts. Also see Linton, Song and Whang (2010), where the regularity condition introduced in Definition 3 plays a similar role to an infinitesimal uniformity factor by excluding null configurations at which the test statistic and critical value converge in probability to zero. Donald and Hsu (2016) recommend setting $\eta$ equal to a very small value such as $10^{-6}$ in practice, and using $\max\{\tilde{c}_{1-\alpha}, \eta\}$ as a critical value rather than $\tilde{c}_{1-\alpha}$, but they report that in numerical simulations there is no difference between setting $\eta = 10^{-6}$ and $\eta = 0$. We have found the same in the numerical simulations reported in Section 4, and on this basis recommend ignoring the infinitesimal uniformity factor and using $\tilde{c}_{1-\alpha}$ as a critical value. We conjecture that Proposition 3.4(ii) remains true with $\eta = 0$, but have not been able to rigorously establish this claim.
4 Numerical simulations

4.1 Simulation design

To investigate the small sample properties of our bootstrap tests of stochastic dominance we ran a number of Monte Carlo simulations. In each simulation we used $10^5$ Monte Carlo repetitions to compute rejection rates. In each of these repetitions we randomly generated data $\{X^1_i\}_{i=1}^{n_1}$ and $\{X^2_i\}_{i=1}^{n_2}$ using either the independent or matched pairs sampling framework, and using this data randomly generated $10^3$ bootstrap samples to compute bootstrap critical values. In each simulation for the independent sampling framework we set $n_1 = n_2 = n$ and set $F_1$ equal to the CDF of the uniform distribution on $(0, 1)$ and $Q_2$ equal to a given ODC $R$. In each simulation for the matched pairs sampling framework we specified $F_1$ and $Q_2$ in the same way as in the independent sampling framework, and used a given copula $C$ to generate dependence between the pairs. Note that it is without loss of generality to confine attention to the case where $F_1$ is uniform because our tests are rank-based.

To provide a point of comparison, we report rejection rates obtained using the bootstrap test of stochastic dominance proposed in Donald and Hsu (2016) alongside those obtained using our modified WMW test. The Donald-Hsu test is similar in spirit to our test, with asymptotic properties comparable to those established in Proposition 3.4, but is based on the one-sided KS statistic rather than the one-sided WMW statistic, and uses a recentering method to modify bootstrap critical values rather than a contact set estimator.

We do not report comparisons with the bootstrap tests of stochastic dominance proposed in Linton, Song and Whang (2010) and Lok and Tabri (2021). While these tests are similar in spirit to our test and to the Donald-Hsu test, they have an unfortunate property: the statistic employed, while resembling a one-sided Cramér-von Mises statistic, is in fact not rank-based and is therefore not invariant to strictly increasing transformations of the data. The property of (first-order) stochastic dominance is invariant under such transformations: a random variable $X^1$ stochastically dominates another random variable $X^2$ if and only if $g(X^1)$ stochastically dominates $g(X^2)$ for every strictly increasing $g$. Lehmann’s principle of invariance (Lehmann, 1959, p. 215) therefore dictates that the outcome of a statistical
test of first-order stochastic dominance should be invariant to strictly increasing transformations of the data. This is the case with the modified WMW test developed here and with the Donald-Hsu test, but is not true of the tests proposed in Linton, Song and Whang (2010) and Lok and Tabri (2021). We found in unreported simulations that it was easy to make the rejection rates obtained using the latter tests either much greater or much less than those for the modified WMW and Donald-Hsu tests simply by taking different increasing transformations of the data. Meaningful comparison is therefore impossible. Our observation that the tests of first-order stochastic dominance in Linton, Song and Whang (2010) and Lok and Tabri (2021) do not satisfy Lehmann’s principle of invariance does not apply to the tests of higher-order stochastic dominance developed in these articles. In Section 5 we discuss the potential adaptation of the results in this article to tests of stochastic dominance based on a one-sided Cramér-von Mises statistic depending only on the ranks of the data.

4.2 Null rejection frequencies

4.2.1 Independent sampling framework

In Table 4.1 we report rejection rates at the least favorable case $R(u) = u$ for our modified WMW test and for the Donald-Hsu test with independent samples of size $n$. We use nominal significance levels $\alpha = .05, .01$ and sample sizes $n = 25, 50, 100, 200, 500, 1000$. The modified bootstrap critical values for our test were computed using the tuning parameter values $\tau_n = .5, .75, 1, 1.25, 1.5, \infty$, with $\tau_n = \infty$ corresponding to our standard bootstrap critical value described in Section 3.1.1. The tuning parameter for the Donald-Hsu test ($a_N$ in their notation) was set equal to the values $-.025, -.05, -.1, -.15, -.2, -\infty$, with the value $-\infty$ corresponding to one of the standard bootstrap procedures described in Barrett and Donald (2003). Note that the numerical simulations reported in Donald and Hsu (2016) use a tuning parameter value of $-.1\sqrt{\log \log(n_1 + n_2)}$, which decreases from $-.117$ to $-.142$ as the two equal sample sizes increase from 25 to 1000, thus falling within the range of tuning parameter values considered here. We see in Table 4.1 that our modified WMW test delivers a rejection rate which is generally close to, but slightly less than, the nominal rejection rate.
Some over-rejection is observed with the smallest sample sizes and tuning parameters. On the other hand, the Donald-Hsu test delivers a rejection rate which is generally close to, but slightly greater than, the nominal rejection rate. As expected, the rejection rates rise as the tuning parameters decrease in magnitude.

In Figure 4.1 we report rejection rates obtained with independent samples of size $n = 500$ and nominal level $\alpha = .05$ at two parametric families of ODCs satisfying the null hypothesis of stochastic dominance, graphing the rejection rates as a function of the ODC parameter $\gamma$. The top-left and bottom-left panels of Figure 4.1 display the two families of ODCs. The family in the top-left is parametrized as $R_\gamma(u) = u^{1+\gamma}$ with $\gamma \geq 0$, and the family in the bottom-left is parametrized as

$$R_\gamma(u) = \begin{cases} 
\Phi(\Phi^{-1}(u)) & \text{for } u \in (0, .5) \\
u & \text{for } u \in [.5, 1) 
\end{cases}$$

with $\gamma \geq 0$, where $\Phi$ is the standard normal CDF and $\Phi^{-1}$ the corresponding quantile function. Note that for the bottom-left family the contact set is always $[.5, 1)$ when $\gamma > 0$.

Table 4.1: Rejection rates at the least favorable case with independent samples.
Figure 4.1: Rejection rates for a range of null configurations using independent samples of size $n = 500$ and nominal level $\alpha = .05$. ODC families are displayed in top-left and bottom-left panels. Rejection rates for the modified WMW test are displayed in top-center and bottom-center panels, with $\tau_n = .5, .75, 1, 1.25, 1.5, \infty$, and triangles superimposed on the curves for $\tau_n = \infty$. Top-right and bottom-right panels show rejection rates for the modified WMW test with $\tau_n = .75$ and the Donald-Hsu test with tuning parameter $-1.135$, with asterisks superimposed on the curve for the Donald-Hsu test.

So that we are on the boundary of the null, whereas for the top-left family the contact set is empty when $\gamma > 0$, so that we are not on the boundary of the null. In both families we obtain the least favorable case $R(u) = u$ when $\gamma = 0$.

In the top-center and bottom-center panels of Figure 4.1 we plot the rejection rates obtained using our modified WMW test as a function of the parameter $\gamma$ for the ODC families in the top-left and bottom-left panels. Separate curves are plotted for each of the tuning parameter values $\tau_n = .5, .75, 1, 1.25, 1.5, \infty$, with the curves for smaller tuning parameter values lying above those for larger values. We superimpose triangles on the curve for $\tau_n = \infty$ to improve visibility. An interesting pattern is apparent wherein the rejection rates with $\tau_n < \infty$ initially decrease as we raise $\gamma$ above zero – essentially decreasing to zero in the top-center panel – before rising back toward the nominal level of .05 as $\gamma$ becomes larger. The
rejection rates with $\tau_n = \infty$ decrease smoothly to zero in both the top-center and bottom-center panels. The results in the top-center panel are particularly encouraging because these null configurations do not belong to the boundary of the null and therefore, as discussed at the end of Section 3.2.3, our modified bootstrap critical values are not guaranteed by Proposition 3.4 to lead to limiting rejection rates no greater than the nominal level unless they are adjusted using an infinitesimal uniformity factor, which we have not done here.

In the top-right and bottom-right panels of Figure 4.1 we plot the rejection rates obtained using the modified WMW test with $\tau_n = .75$ and the Donald-Hsu test with tuning parameter $-.139$ (the latter superimposed with asterisks) as a function of the parameter $\gamma$ for the ODC families in the top-left and bottom-left panels. In the top-right panel we see that the rejection rate for the Donald-Hsu test does not share the interesting non-monotone behavior exhibited by the modified WMW test, and instead drops quickly to zero as we raise $\gamma$ above zero. On the other hand, we see in the bottom-left panel that the Donald-Hsu test is more successful than the WMW test in maintaining a rejection rate close to the nominal level on the boundary of the null, at least for the family of ODCs considered here.

### 4.2.2 Matched pairs sampling framework

In Table 4.2 we report rejection rates at the least favorable case using the matched pairs sampling framework. These results may be compared directly to those reported in Table 4.1 for the independent sampling framework. The only difference in the design of the simulations is that in Table 4.2 the data are generated such that the copula $C$ for each matched pair is Gaussian with correlation parameter $\rho$. We report results for $\rho = .25, .5, .75$. The results are broadly similar to those in Table 4.1, with the rejection rates using the modified WMW test tending to fall below the nominal level, and the rejection rates using the Donald-Hsu test tending to fall modestly above the nominal level. The degree to which the rejection rates for the modified WMW test fall below the nominal level increases as the dependence between matched pairs increases.

In Figure 4.2 we report rejection rates with matched pairs ($n = 500, \alpha = .05$) at the two parametric families of ODCs used to produce the results displayed in Figure 4.1. The
| $\rho$ | $\alpha$ | n | Mod. Wilcoxon-Mann-Whitney | Donald-Hsu |
|------|------|---|----------------|-----------|
|      |      |   | Tun. par. | .5 | .75 | 1 | 1.25 | 1.5 | $\infty$ | -.025 | -.05 | -.1 | -.15 | -.2 | $\infty$ |
| 25   | .05  | 4.6 | 3.8 | 2.4 | 2.4 | 7.2 | 7.2 | 7.2 | 7.0 | 6.6 |
| 50   | .05  | 5.4 | 5.1 | 4.4 | 4.4 | 4.4 | 5.9 | 5.9 | 5.9 | 5.7 |
| 100  | .05  | 5.9 | 5.9 | 4.4 | 4.4 | 4.4 | 5.9 | 5.9 | 5.9 | 5.7 |
| 200  | .05  | 6.3 | 6.3 | 5.7 | 5.7 | 5.7 | 6.0 | 6.0 | 6.0 | 5.8 |
| 500  | .05  | 6.7 | 6.7 | 6.0 | 6.0 | 6.0 | 6.0 | 6.0 | 6.0 | 5.8 |
| 1000 | .05  | 7.1 | 7.1 | 6.1 | 6.1 | 6.1 | 6.1 | 6.1 | 6.1 | 5.8 |
| 25   | .05  | 6.1 | 6.1 | 5.7 | 5.7 | 5.7 | 5.9 | 5.9 | 5.9 | 5.7 |
| 50   | .05  | 6.5 | 6.5 | 6.0 | 6.0 | 6.0 | 6.0 | 6.0 | 6.0 | 5.8 |
| 100  | .05  | 6.9 | 6.9 | 6.2 | 6.2 | 6.2 | 6.2 | 6.2 | 6.2 | 5.8 |
| 200  | .05  | 7.3 | 7.3 | 6.4 | 6.4 | 6.4 | 6.4 | 6.4 | 6.4 | 5.8 |
| 500  | .05  | 7.7 | 7.7 | 6.7 | 6.7 | 6.7 | 6.7 | 6.7 | 6.7 | 5.8 |
| 1000 | .05  | 8.1 | 8.1 | 7.0 | 7.0 | 7.0 | 7.0 | 7.0 | 7.0 | 5.8 |
| 25   | .05  | 5.1 | 5.1 | 5.7 | 5.7 | 5.7 | 5.9 | 5.9 | 5.9 | 5.7 |
| 50   | .05  | 5.5 | 5.5 | 6.0 | 6.0 | 6.0 | 6.0 | 6.0 | 6.0 | 5.8 |
| 100  | .05  | 5.9 | 5.9 | 6.2 | 6.2 | 6.2 | 6.2 | 6.2 | 6.2 | 5.8 |
| 200  | .05  | 6.3 | 6.3 | 6.4 | 6.4 | 6.4 | 6.4 | 6.4 | 6.4 | 5.8 |
| 500  | .05  | 6.7 | 6.7 | 6.7 | 6.7 | 6.7 | 6.7 | 6.7 | 6.7 | 5.8 |
| 1000 | .05  | 7.1 | 7.1 | 7.0 | 7.0 | 7.0 | 7.0 | 7.0 | 7.0 | 5.8 |

Table 4.2: Rejection rates at the least favorable case with matched pairs.
Figure 4.2: Rejection rates for a range of null configurations using matched pairs ($n = 500$, $\alpha = .05$). ODC families are displayed in top-left and bottom-left panels of Figure 4.1. The left, center and right columns of panels correspond to Gaussian copulae with $\rho = .25, .5, .75$. Rejection rates are plotted for the modified WMW test with $\tau_n = .75$ (superimposed with squares) and with $\tau_n = \infty$ (superimposed with triangles), and the Donald-Hsu test with tuning parameter $-1.139$ (superimposed with asterisks).

left, center and right columns of panels in Figure 4.2 correspond to Gaussian copulae with correlation parameter $\rho = .25, .5, .75$ respectively. The top (bottom) row of panels in Figure 4.2 corresponds to the family of ODCs displayed in the top-left (bottom-left) panel in Figure 4.1. Each panel of Figure 4.2 displays curves plotting the rejection rate for three tests: the modified WMW test with $\tau_n = .75$ (superimposed with squares) and with $\tau_n = \infty$ (superimposed with triangles) and the Donald-Hsu test with tuning parameter $-1.139$ (superimposed with asterisks). The results are very similar overall to those reported in Figure 4.1 for the independent sampling framework. The two curves for the modified WMW test with $\tau_n = \infty$ and the Donald-Hsu test are difficult to distinguish in the top row of panels.
4.3 Alternative rejection frequencies

4.3.1 Independent sampling framework

In Figure 4.3 we report rejection rates obtained with independent samples of size $n = 500$ and nominal level $\alpha = .05$ at two parametric families of ODCs not satisfying the null hypothesis of stochastic dominance in general. The two families are displayed in the top-left and bottom-left panels of Figure 4.3. The family in the top-left is parametrized as $R_\gamma(u) = u^{1-\gamma}$ with $\gamma \geq 0$, and the family in the bottom-left is parametrized as $R_\gamma(u) = \Phi(e^{\gamma}\Phi^{-1}(u))$ with $\gamma \in \mathbb{R}$. In both families we obtain the least favorable case $R(u) = u$ when $\gamma = 0$, while for other values of $\gamma$ the null hypothesis is not satisfied. The crucial difference between the two families is that, when the null hypothesis is not satisfied, the graph of $R_\gamma$ is everywhere above the 45-degree line with the top-left family but is partially below the 45-degree line with the bottom-left family.

The top-center and bottom-center panels in Figure 4.3 display the rejection frequencies of our modified WMW test with tuning parameter values $\tau_n = .5, .75, 1.25, 1.5, \infty$. The curve for $\tau_n = \infty$, which corresponds to our standard bootstrap critical values, has triangles superimposed. In both panels we see the rejection frequencies increase from approximately $\alpha = .05$ to one as $\gamma$ moves away from zero, reflecting the consistency of the tests. In the top-center panel the curves plotted for different tuning parameter values are indistinguishable. In the bottom-center panel there is a clear separation between the curves, with the rejection frequencies using the standard bootstrap critical values well below the rejection frequencies using the modified bootstrap critical values. The very different behavior displayed in the two panels can be understood by observing that the modified bootstrap statistic $\tilde{S}^*$ differs from the standard bootstrap statistic $\hat{S}^*$ only when the graph of the empirical ODC $\hat{R}$ falls below the 45-degree line by more than some threshold depending on the tuning parameter. This happens very infrequently in the simulations generating the rejection frequencies in the top-center panel because here we have $R_\gamma(u) > u$ for all $u$ when $\gamma < 0$; but frequently in those generating the rejection frequencies in the bottom-center panel because here we have $R_\gamma(u) < u$ for $u \in (.5, 1)$ when $\gamma < 0$, and $R_\gamma(u) < u$ for $u \in (0, .5)$ when $\gamma > 0$. 

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Figure 4.3: Rejection rates for a range of alternative configurations using independent samples of size $n = 500$ and nominal level $\alpha = .05$. ODC families are displayed in top-left and bottom-left panels. Rejection rates for the modified WMW test are displayed in top-center and bottom-center panels, with $\tau_n = .5, .75, 1, 1.25, 1.5, \infty$, and triangles superimposed on the curves for $\tau_n = \infty$. Top-right and bottom-right panels show rejection rates for the modified WMW test with $\tau_n = .75$ and the Donald-Hsu test with tuning parameter $- .139$, with asterisks superimposed on the curve for the Donald-Hsu test.

In the top-right and bottom-right panels of Figure 4.3 we plot the rejection frequencies for the modified WMW test with $\tau_n = .75$ and the Donald-Hsu test with tuning parameter $- .139$ against one another. Rejection frequencies for the Donald-Hsu tests are superimposed with asterisks. We see that the rejection frequencies for the two tests are similar. A slight power advantage for the modified WMW test is observed in the top-right panel, and a slight power advantage for the Donald-Hsu test in the bottom-right panel.

4.3.2 Matched pairs sampling framework

Figure 4.4 shows how the rejection frequencies plotted in Figure 4.3 are affected when we adopt a matched pairs sampling framework, with Gaussian copulae characterizing the
Figure 4.4: Rejection rates for a range of alternative configurations using matched pairs ($n = 500$, $\alpha = .05$). ODC families are displayed in top-left and bottom-left panels of Figure 4.3. The left, center and right columns of panels correspond to Gaussian copulae with $\rho = .25, .5, .75$. Rejection rates are plotted for the modified WMW test with $\tau_n = .75$ (superimposed with squares) and with $\tau_n = \infty$ (superimposed with triangles), and the Donald-Hsu test with tuning parameter $-.139$ (superimposed with asterisks).

dependence between pairs. The left, center and right columns of panels in Figure 4.4 correspond to correlation parameters $\rho = .25, .5, .75$ respectively, while the top (bottom) row of panels corresponds to the family of ODCs displayed in the top-left (bottom-left) panel in Figure 4.3. Rejection frequencies are plotted for the modified WMW test with $\tau_n = .75$ (superimposed with squares) and with $\tau_n = \infty$ (superimposed with triangles), and the Donald-Hsu test with tuning parameter $-.139$ (superimposed with asterisks). In both rows of panels we see that the power for all three tests improves as $\rho$ increases. In the top row of panels we see that the rejection rates for the modified WMW test with $\tau_n = .75$ and with $\tau_n = \infty$ are indistinguishable, as they were with independent samples in Figure 4.3. The slight power advantage of these tests over the Donald-Hsu test widens as $\rho$ increases, becoming quite substantial with $\rho = .75$. In the bottom row of panels we see that the slight power advantage of the Donald-Hsu test observed with independent samples erodes as the
correlation parameter increases. The rejection frequencies for the modified WMW test with \( \tau_n = .75 \) and the Donald-Hsu test are nearly indistinguishable when \( \rho = .5 \) or \( \rho = .75 \).

5 Further discussion

In this article we have extended the one-sided WMW test of stochastic dominance introduced in Schmid and Tröde (1996) by showing how to construct bootstrap critical values which may be validly applied in independent and matched pairs sampling frameworks, and showing how these bootstrap critical values may be modified in such a way as to raise the limiting rejection rate to the nominal level on the boundary of the null, thereby improving power. Our asymptotic results and numerical simulations indicate that our modified WMW test behaves similarly to the test of stochastic dominance proposed in Donald and Hsu (2016), which involves comparing a one-sided KS statistic with bootstrap critical values modified using a recentering procedure. In applications it may be useful to implement both tests and compare the results.

An obvious direction for future research is to adapt the procedures developed here so that they may be applied to the one-sided Cramér-von Mises statistic

\[
\hat{S}_{cvm} = \frac{T_n}{n_2} \sum_{i=1}^{n_2} \max \left\{ \hat{R} \left( \frac{i}{n_2} \right) - \frac{i}{n_2}, 0 \right\}^2,
\]

obtaining by squaring each summand in the definition of the one-sided WMW statistic in (2.8) and scaling by \( T_n^{1/2} \). The primary obstacle to obtaining such an adaptation is that Proposition 2.1 establishes weak convergence of the empirical ordinal dominance process only with respect to the \( L^1 \) norm. Weak convergence with respect to the stronger \( L^2 \) norm is needed if we wish to apply the functional delta method to deduce the null asymptotic distribution of \( \hat{S}_{cvm} \). Such a strengthening of Proposition 2.1 may be possible. The key input to the proof of Proposition 2.1 is the demonstration in Kaji (2018) that the empirical
quantile function $\hat{Q}_2$ satisfies

$$n_2^{1/2} (\hat{Q}_2 - Q_2) \rightsquigarrow -Q'_2 \cdot B_2 \quad \text{in } L^1(0, 1),$$

(5.1)

provided that $F_2$ is continuously differentiable with strictly positive derivative on its support, and has finite absolute moment of order $2 + \epsilon$ for some $\epsilon > 0$. For this weak convergence to be strengthened to hold in the $L^2$ norm, a necessary condition is that $Q'_2 \cdot B_2$ is square integrable (with probability one). In the proof of Proposition 2.1 a normalization is applied under which $Q_2 = R$. A relevant question is thus whether the potential divergence of $r(u)$ to infinity as $u$ decreases to zero or increases to one can be fast enough to violate square integrability of $r \cdot B_2$. Focusing on the behavior at zero, if we suppose that $r(u)$ is regularly varying at zero then, since it is integrable, it must satisfy $r(u) = u^{-1+\epsilon} \ell(u)$ for some $\epsilon > 0$ and some function $\ell(u)$ slowly varying at zero. The Lévy modulus of continuity theorem implies that $B_2(u)^2 = O_{a.s.}(u \log^2(1/u))$ for small $u$. We therefore obtain $r(u)^2 B_2(u)^2 = O_{a.s.}(u^{-1+2\epsilon} \log^2(1/u) \ell(u))$ for small $u$, consistent with square integrability of $r \cdot B_2$. A similar heuristic argument applies when $u$ is close to one. It thus seems plausible to us that it may be possible to strengthen the weak convergence in (5.1) to hold in the $L^2$ norm under a bounded support condition on $F_2$ and perhaps also a regular variation condition on $Q'_2$. Such a strengthening would facilitate the study of one-sided Cramér-von Mises statistics using the approach taken in this article, and also advance our understanding of empirical quantile and ordinal dominance processes.

6 Proofs

Here we provide proofs of all numbered lemmas and propositions in Sections 2 and 3, as well as those of supporting lemmas to be stated.

Proof of Lemma 2.1. It should be clear from Figure 1.1 that $T_n^{-1/2} \hat{S}$, represented by the red shaded area, falls short of $\mathcal{F}(\hat{R})$ by an amount equal to the area contained in the small white triangles comprising the area which lies above the 45-degree line and below the graph of $\hat{R}$.
but is not shaded in red. There can be at most \( n^2 \) such triangles, each with area \( \frac{1}{(2n^2)} \), so the total shortfall is at most \( \frac{1}{(2n^2)} \).

Let \( \mathbb{L} \) be the space of Borel functions \( h : \mathbb{R} \to \mathbb{R} \) that are bounded, have left- and right-limits \( h(-\infty) := \lim_{x \to -\infty} h(x) \) and \( h(\infty) := \lim_{x \to \infty} h(x) \), and satisfy

\[
\int_{-\infty}^{0} |h(x) - h(-\infty)| \, dx < \infty \quad \text{and} \quad \int_{0}^{\infty} |h(x) - h(\infty)| \, dx < \infty.
\]

Equip \( \mathbb{L} \) with the norm

\[
\|h\|_{\mathbb{L}} := \left( \sup_{x \in \mathbb{R}} |h(x)| \right) \vee \left( \int_{-\infty}^{0} |h(x) - h(-\infty)| \, dx + \int_{0}^{\infty} |h(x) - h(\infty)| \, dx \right).
\]

The following lemma is a minor extension of Proposition 2.1 in Kaji (2018), which deals with the univariate case. It is used in the proof of Proposition 2.1.

**Lemma 6.1.** Under Assumptions 2.1 and 2.2, if \( F_1 \) and \( F_2 \) have finite \((2 + \epsilon)\)-th absolute moment for some \( \epsilon > 0 \), then

\[
\begin{pmatrix}
\frac{n_1^{1/2}}{2}(\hat{F}_1 - F_1) \\
\frac{n_2^{1/2}}{2}(\hat{F}_2 - F_2)
\end{pmatrix}
\overset{\mathcal{D}}{\to}
\begin{pmatrix}
\mathcal{B}_1 \circ F_1 \\
\mathcal{B}_2 \circ F_2
\end{pmatrix}
\quad \text{in } \mathbb{L} \times \mathbb{L}.
\] (6.1)

**Proof of Lemma 6.1.** A version of the result for nonnegative random variables, proved using Proposition 2.1 in Kaji (2018), is stated as Lemma 5.1 in Sun and Beare (2021). The proof for real random variables is the same.

A second ingredient to the proof of Proposition 2.1 is the following lemma concerning Hadamard differentiability of the composition map. Note that it is not implied by Lemma 3.9.27 in van der Vaart and Wellner (1996) due to the fact that the space inhabited by one of the functions to be composed and its perturbation is endowed only with the \( L^1 \) norm rather than the stronger uniform norm.

**Lemma 6.2.** Let \( A \) be the set of all Borel measurable maps \( Q : (0, 1) \to [0, 1] \). The map \( \phi : \ell^{\infty}[0,1] \times A \subset \ell^{\infty}[0,1] \times L^1(0,1) \to L^1(0,1) \) given by \( \phi(F,Q) = F \circ Q \) is Hadamard
differentiable tangentially to \( C[0, 1] \times L^1(0, 1) \) at any \((F, Q)\) such that \( F \) is the identity. Its derivative \( \phi'_{F,Q} : C[0, 1] \times L^1(0, 1) \to L^1(0, 1) \) is given by \( \phi'_{F,Q}(g, h) = g \circ Q + h \).

**Proof of Lemma 6.2.** Our goal is to show that, for all converging sequences \( t_n \downarrow 0, g_n \to g \in C[0, 1] \) and \( h_n \to h \in L^1(0, 1) \) with \( Q + t_n h_n \in A \) for all \( n \), we have

\[
\lim_{n \to \infty} \left\| \frac{(F + t_n g_n) \circ (Q + t_n h_n) - F \circ Q}{t_n} - g \circ Q - h \right\|_1 = 0. \tag{6.2}
\]

Since \( F \) is the identity, the \( L^1 \) norm in (6.2) is easily seen to be bounded by

\[
\| (g_n - g) \circ (Q + t_n h_n) \|_1 + \| h_n - h \|_1 + \| g \circ (Q + t_n h_n) - g \circ Q \|_1. \tag{6.3}
\]

We seek to show that the three terms in (6.3) converge to zero as \( n \to \infty \). For the first two terms, this is obvious. Thus we need only show that

\[
\lim_{n \to \infty} \int_0^1 \left| g(Q(u) + t_n h_n(u)) - g(Q(u)) \right| du = 0. \tag{6.4}
\]

Since \( h_n \to h \in L^1(0, 1) \), we have \( t_n h_n(u) \to 0 \) for a.e. \( u \in (0, 1) \). Since \( g \) is continuous, it follows that the integrand in (6.4) is converging pointwise a.e. to zero. Since \( g \) is bounded, we deduce from the dominated convergence theorem that (6.4) is satisfied.

**Proof of Proposition 2.1.** As discussed in Section 2.4.1, the empirical ODC \( \hat{R} \) is invariant to strictly increasing transformations of the data, and so for the purposes of proving Proposition 2.1 we may assume without loss of generality that \( F_1 \) is the CDF of the uniform distribution on \((0, 1)\), so that \( Q_2 = R \) and thus \( Q_2 \) satisfies the regularity conditions placed on \( R \) in Assumption 2.1. Since \( T_n/n_1 \to \lambda \) and \( T_n/n_2 \to 1 - \lambda \) under Assumption 2.2, we may rewrite the joint weak convergence guaranteed by Lemma 6.1 as

\[
T_n^{1/2} \begin{pmatrix} \hat{F}_1 - F_1 \\ \hat{F}_2 - F_2 \end{pmatrix} \sim \begin{pmatrix} \lambda^{1/2} B_1 \circ F_1 \\ (1 - \lambda)^{1/2} B_2 \circ F_2 \end{pmatrix} \text{ in } \mathbb{L} \times \mathbb{L}. \tag{6.5}
\]
From (6.5) we obtain the weak convergence
\[ T_n^{1/2} \left( \hat{F}_1 - F_1 \right) \Rightarrow \left( \lambda^{1/2} B_1 \circ F_1 - (1 - \lambda)^{1/2} Q_2 \cdot B_2 \right) \quad \text{in } L_2 \times L_1(0, 1). \] (6.6)
by applying the functional delta method, using Theorem 1.3 in Kaji (2018) to obtain suitable Hadamard differentiability of the map from distribution functions to quantile functions. Since we have normalized \( F_1 \) to be the CDF of the uniform distribution, (6.6) implies the weak convergence in (2.11). Another application of the functional delta method, using Lemma 6.2 to obtain suitable Hadamard differentiability of the composition map, allows us to deduce from (2.11) that
\[ T_n^{1/2} (\hat{F}_1 \circ \hat{Q}_2 - F_1 \circ Q_2) \Rightarrow \lambda^{1/2} B_1 \circ Q_2 - (1 - \lambda)^{1/2} R \cdot B_2 \quad \text{in } L_1(0, 1), \]
which may be rewritten as in (2.10). \( \square \)

In the following lemma establishing Hadamard directional differentiability of \( \mathcal{F} \) we allow \( R \) to be an arbitrary element of \( L_1(0, 1) \), but when applying the lemma \( R \) will be our ODC.

**Lemma 6.3.** The functional \( \mathcal{F} : L_1(0, 1) \to \mathbb{R} \) is Hadamard directionally differentiable at any \( R \in L_1(0, 1) \), with directional derivative \( \mathcal{F}_R' : L_1(0, 1) \to \mathbb{R} \) given by
\[ \mathcal{F}_R'(h) = \int_{B_+(R)} h(u) du + \int_{B_0(R)} \max\{h(u), 0\} du, \] (6.7)
where \( B_+(R) \) and \( B_0(R) \) are the sets
\[ B_+(R) = \{ u \in (0, 1) : R(u) > u \} \quad \text{and} \quad B_0(R) = \{ u \in (0, 1) : R(u) = u \}. \] (6.8)

**Proof of Lemma 6.3.** Let \( \{t_n\} \subset \mathbb{R}_+ \) and \( \{h_n\} \subset L^1(0, 1) \) be sequences such that \( t_n \downarrow 0 \) and \( h_n \to h \in L^1(0, 1) \). Since the absolute difference between two functions is at least as great
as the absolute difference between their nonnegative parts, we have
\[ |\max\{R(u) + t_nh_n(u) - u, 0\} - \max\{R(u) + t_nh(u) - u, 0\}| \leq t_n|h_n(u) - h(u)| \]
for all \( u \in (0, 1) \). Therefore, \( t_n^{-1}(F(R + t_nh_n) - F(R)) \) is equal to
\[
\int_0^1 (\max\{h(u) + t_n^{-1}(R(u) - u), 0\} - \max\{t_n^{-1}(R(u) - u), 0\})du + o(1). \tag{6.9}
\]
The integrand in (6.9) is equal to \( \max\{h(u), 0\} \) on \( B_0(R) \), converges pointwise to \( h(u) \) on \( B_+(R) \), and converges pointwise to zero elsewhere. Applying the dominated convergence theorem with \( |h| \) as a dominator, we deduce that \( t_n^{-1}(F(R + t_nh_n) - F(R)) \to F'_R(h) \). \( \square \)

**Proof of Proposition 2.2.** Immediate from Proposition 2.1 and Lemma 6.3 by applying the functional delta method using the directionally differentiable functional \( F \) (see e.g. Fang and Santos, 2019, Thm. 2.1).

**Proof of Proposition 3.1.** From the definition of \( R \) given in Proposition 2.1 we have
\[
\text{Var}(R(u)) = \lambda \text{Var}(B_1(R(u))) + (1 - \lambda)r(u)^2\text{Var}(B_2(u))
- 2\lambda^{1/2}(1 - \lambda)^{1/2}r(u)\text{Cov}(B_1(R(u)), B_2(u)). \tag{6.10}
\]
Using the covariance kernel of \( B \) in (2.9), we obtain \( \text{Var}(B_1(R(u))) = R(u) - R(u)^2 \), \( \text{Var}(B_2(u)) = u - u^2 \), and \( \text{Cov}(B_1(R(u)), B_2(u)) = C(R(u), u) - R(u)u \). This establishes the first assertion of Proposition 3.1. The second assertion follows by setting \( R(u) = u \) and \( r(u) = 1 \). \( \square \)

Let \( \overset{\text{s.a.}}{\rightharpoonup} \) denote weak convergence conditional on the data almost surely in the sense of Kosorok (2008, pp. 19–20). The next lemma is used to prove Proposition 3.2.

**Lemma 6.4.** Under Assumptions 2.1 and 2.2, if \( F_1 \) and \( F_2 \) have finite \((2 + \epsilon)\)-th absolute moment for some \( \epsilon > 0 \), then
\[
\left( \begin{array}{c}
\frac{n_1}{2}(\hat{F}_1 - F_1) \\
\frac{n_2}{2}(\hat{F}_2 - F_2)
\end{array} \right) \overset{\text{s.a.}}{\rightharpoonup} \left( \begin{array}{c}
B_1 \circ F_1 \\
B_2 \circ F_2
\end{array} \right) \text{ in } \mathbb{L} \times \mathbb{L}. \tag{6.11}
\]
**Proof of Lemma 6.4.** A version of the result for nonnegative random variables, proved using results of Kaji (2018), is stated as Lemma 5.2 in Sun and Beare (2021). The proof for real random variables is the same.

**Proof of Proposition 3.2.** As in the proof of Proposition 2.1, we may assume without loss of generality that $F_1$ is the CDF of the uniform distribution on $(0, 1)$, so that $Q_2 = R$ and thus $Q_2$ satisfies the regularity conditions placed on $R$ in Assumption 2.1. Since $T_{n_1} / n_1 \to \lambda$ and $T_{n_2} / n_2 \to 1 - \lambda$ under Assumption 2.2, we may rewrite the joint conditional weak convergence guaranteed by Lemma 6.4 as

$$T_{n_1}^{1/2} \left( \frac{\hat{F}_1^* - \hat{F}_1}{\hat{F}_2^* - \hat{F}_2} \right) \xrightarrow{\text{ass}^*} W \left( \frac{\lambda^{1/2}B_1 \circ F_1}{(1 - \lambda)^{1/2}B_2 \circ F_2} \right) \quad \text{in } \mathbb{L} \times \mathbb{L}. \quad (6.12)$$

From (6.12) we obtain the conditional weak convergence

$$T_{n_1}^{1/2} \left( \frac{\hat{F}_1^* - \hat{F}_1}{\hat{Q}_2^* - \hat{Q}_2} \right) \xrightarrow{\text{p}^*} W \left( \frac{\lambda^{1/2}B_1 \circ F_1}{(1 - \lambda)^{1/2}Q_2' \cdot B_2} \right) \quad \text{in } \mathbb{L} \times L^1(0, 1) \quad (6.13)$$

by applying the functional delta method for the bootstrap (Kosorok, 2008, Theorem 2.9), using Theorem 1.3 of Kaji (2018) to obtain suitable Hadamard differentiability of the map from distribution functions to quantile functions. Since we have normalized $F_1$ to be the CDF of the uniform distribution, (6.13) implies that

$$T_{n_1}^{1/2} \left( \frac{\hat{F}_1^* - \hat{F}_1}{\hat{Q}_2^* - \hat{Q}_2} \right) \xrightarrow{\text{p}^*} W \left( \frac{\lambda^{1/2}B_1}{(1 - \lambda)^{1/2}r \cdot B_2} \right) \quad \text{in } \ell^{\infty}[0, 1] \times L^1(0, 1). \quad (6.14)$$

Another application of the functional delta method for the bootstrap, using Lemma 6.2 to obtain suitable Hadamard differentiability of the composition map $(F, Q) \mapsto F \circ Q$, yields the desired result. □

The next lemma, which establishes a simple fact about bivariate copulae, is used in the proofs of Lemmas 3.1 and 6.6.
Lemma 6.5. A bivariate copula $C$ with maximal correlation less than one satisfies $C(u, u) < u$ for all $u \in (0, 1)$.

Proof of Lemma 6.5. Let $(U, V)$ be a pair of random variables with joint CDF $C$ and suppose that $C(u, u) = u$ for some $u \in (0, 1)$. Let $f : (0, 1) \to \{0, 1\}$ be the indicator function of the interval $(0, u)$. Then

$$
\text{Cor}(f(U), f(V)) = \frac{\text{Cov}(f(U), f(V))}{\sqrt{\text{Var}(f(U))\text{Var}(f(V))}} = \frac{C(u, u) - u^2}{u - u^2} = 1,
$$

implying that $C$ has maximal correlation one. □

The next lemma is used in the proofs of Propositions 3.3 and 3.4.

Lemma 6.6. Suppose that $\alpha \in (0, 1/2)$, that $H_0$ is satisfied, and that $B_0(R)$ has positive measure. Then the CDF of $F'_R(R)$ is continuous and strictly increasing at its $(1 - \alpha)$-quantile.

Proof of Lemma 6.6. We first observe that since $R$ is Gaussian and the directional derivative $F'_R$ is continuous and convex, Theorem 11.1 of Davydov, Lifshits and Smorodina (1998) implies that the CDF of $F'_R(R)$ is continuous everywhere except perhaps at zero, and that if it assigns probability less than one to zero then it is strictly increasing on $(0, \infty)$. Thus if the CDF of $F'_R(R)$ is not continuous and strictly increasing at its $1 - \alpha$ quantile, then it must assign probability of at least $1 - \alpha$ to zero. But since $B_+ (R)$ is empty under $H_0$, we have

$$
P(F'_R(R) > 0) \geq P \left( \int_{B_0(R)} R(u) \, du > 0 \right).
$$

The latter probability is equal to $1/2$ since $B_0(R)$ has positive measure and $R$ is centered and Gaussian, with positive variance on $B_0(R)$ by Proposition 3.1 and Lemma 6.5. Since $\alpha \in (0, 1/2)$, we deduce that $P(F'_R(R) = 0) < 1 - \alpha$, and thus conclude that the CDF of $F'_R(R)$ is continuous and strictly increasing at its $1 - \alpha$ quantile. □

Proof of Proposition 3.3. Let $A$ denote the set of continuity points of the CDF of $F'_{R_0}(R)$. The
conditional weak convergence in (3.7) implies, via Lemma 10.11(i) of Kosorok (2008), that

\[
P\left( \hat{S}^* \leq c \Big| \{X_i^{1}\}_{i=1}^{n_1}, \{X_i^{2}\}_{i=1}^{n_2} \right) \rightarrow P(\mathcal{F}_{R_0}'(\mathcal{R}) \leq c) \tag{6.15}
\]

in probability for each \( c \in A \). Fix \( \epsilon > 0 \). Lemma 6.6 establishes that the CDF of \( \mathcal{F}_{R_0}'(\mathcal{R}) \) is strictly increasing at its \((1 - \alpha)\)-quantile \( c_{1-\alpha} \), so we may choose \( c_1, c_2 \in A \) such that \( c_{1 - \alpha} - \epsilon < c_1 < c_{1 - \alpha} < c_2 < c_{1 - \alpha} + \epsilon \) and

\[
P(\mathcal{F}_{R_0}'(\mathcal{R}) \leq c_1) < 1 - \alpha < P(\mathcal{F}_{R_0}'(\mathcal{R}) \leq c_2). \tag{6.16}
\]

If \( |\hat{c}_{1-\alpha} - c_{1-\alpha}| > \epsilon \) then either \( \hat{c}_{1-\alpha} < c_1 \) or \( \hat{c}_{1-\alpha} > c_2 \); that is, either

\[
P\left( \hat{S}^* \leq c_1 \Big| \{X_i^{1}\}_{i=1}^{n_1}, \{X_i^{2}\}_{i=1}^{n_2} \right) \geq 1 - \alpha, \quad \text{or} \quad P\left( \hat{S}^* \leq c_2 \Big| \{X_i^{1}\}_{i=1}^{n_1}, \{X_i^{2}\}_{i=1}^{n_2} \right) < 1 - \alpha. \tag{6.17, 6.18}
\]

Since \( c_1, c_2 \in A \), the convergence in (6.15) implies that the left-hand sides of (6.17) and (6.18) converge in probability to \( P(\mathcal{F}_{R_0}'(\mathcal{R}) \leq c_1) \) and \( P(\mathcal{F}_{R_0}'(\mathcal{R}) \leq c_1) \) respectively. Thus, due to (6.16), the probability of either (6.17) or (6.18) being satisfied converges to zero. We conclude that \( P(|\hat{c}_{1-\alpha} - c_{1-\alpha}| > \epsilon) \rightarrow 0 \). Since \( \epsilon \) was arbitrary, it follows that \( \hat{c}_{1-\alpha} \rightarrow c_{1-\alpha} \) in probability.

If \( H_0 \) is satisfied then Proposition 2.2 implies that \( \hat{S} \) converges in distribution to \( \mathcal{F}_R'(\mathcal{R}) \). Therefore, since (i) \( \hat{c}_{1-\alpha} \rightarrow c_{1-\alpha} \) in probability, and (ii) the CDF of \( \mathcal{F}_R'(\mathcal{R}) \) is continuous at \( c_{1-\alpha} \) by Lemma 6.6, we have

\[
P(\hat{S} > \hat{c}_{1-\alpha}) \rightarrow P(\mathcal{F}_R'(\mathcal{R}) > c_{1-\alpha}) \leq P(\mathcal{F}_{R_0}'(\mathcal{R}) > c_{1-\alpha}) = 1 - \alpha,
\]

where the inequality follows from the fact that \( \mathcal{F}_R'(\mathcal{R}) \leq \mathcal{F}_{R_0}'(\mathcal{R}) \). If \( F_1 = F_2 \) then \( R = R_0 \) and the inequality holds with equality.

If \( H_1 \) is satisfied then Proposition 2.2 implies that \( \hat{S} \) diverges in probability to infinity. Since \( \hat{c}_{1-\alpha} \rightarrow c_{1-\alpha} \) in probability, we deduce that \( P(\hat{S} > \hat{c}_{1-\alpha}) \rightarrow 1 \). \( \square \)
Our final two lemmas are used in the proof of Lemma 3.1.

**Lemma 6.7.** Under Assumptions 2.1 and 2.2, $T_n^{1/2} (\hat{R}(u) - R(u)) = O_p(1)$ for each $u \in (0, 1)$.

**Proof of Lemma 6.7.** As in the proof of Proposition 2.1, we may assume without loss of generality that $F_1$ is the CDF of the uniform distribution on $(0, 1)$, so that $Q_2 = R$ and thus $Q_2$ satisfies the regularity conditions placed on $R$ in Assumption 2.1. Begin by writing

$$T_n^{1/2} (\hat{R}(u) - R(u)) = \sqrt{\frac{n_2}{n_1 + n_2}} \cdot n_1^{1/2} (\hat{F}_1(\hat{Q}_2(u)) - F_1(\hat{Q}_2(u))) + \sqrt{1 - \frac{n_2}{n_1 + n_2}} \cdot n_2^{1/2} (\hat{Q}_2(u) - Q_2(u)).$$  \(6.19\)

Since $n_2/(n_1 + n_2) \to \lambda$ under Assumption 2.2, the first term on the right-hand side of (6.19) is $O_p(1)$ by Donsker’s theorem, and the second term is $O_p(1)$ by Example 3.9.21 in van der Vaart and Wellner (1996).

**Lemma 6.8.** Under Assumptions 2.1 and 2.2, $\hat{V}_{[n_2u]} \to u - C(u, u)$ in probability for each $u \in (0, 1)$.

**Proof of Lemma 6.8.** Under independent sampling the result is true since

$$\hat{V}_{[n_2u]} = \frac{[n_2u]}{n_2} - \left( \frac{[n_2u]}{n_2} \right)^2 \to u - u^2 = u - C(u, u).$$

Under matched pairs the result follows from the pointwise consistency of the empirical copula (see e.g. van der Vaart and Wellner, 1996, Example 3.9.29) since

$$\hat{V}_{[nu]} = \frac{[nu]}{n} - \hat{C}(u, u) \to u - C(u, u)$$

in probability.

**Proof of Lemma 3.1.** It suffices to verify Assumptions 1–4 of Theorem 3.2 of Fang and Santos (2019). Assumption 1 is implied by Lemma 6.3. Assumption 2 is implied by Proposition 2.1. Assumption 3 is implied by Proposition 3.2. Note that the measurability conditions in
Assumption 3 are trivially satisfied in our context because $\hat{R}$ and $\hat{R}^*$ are measurable functions of the data and bootstrap weights. To see why, observe that $\hat{R}$ is uniquely determined by the $n_2$ random variables $\hat{F}_1(X_{(1)}^2), \ldots \hat{F}_1(X_{(n_2)}^2)$, each of which can only take the values $0, 1/n_1, \ldots, 1$. Thus $\hat{R}$ is a simple map into $L^1(0,1)$, hence measurable. Similarly, $\hat{R}^*$ is a simple map into $L^1(0,1)$, hence measurable.

For Assumption 4 of Theorem 3.2 of Fang and Santos (2019) to be satisfied, it suffices to show (see their Remark 3.4) that (i) $|\hat{F}_R'(h_2) - \hat{F}_R'(h_1)| \leq \|h_2 - h_1\|_1$ for any $h_1, h_2 \in L^1(0,1)$, and that (ii) $\hat{F}_R'(h) \to F_R'(h)$ in probability for any $h \in L^1(0,1)$. Condition (i) is satisfied because

$$|\hat{F}_R'(h_2) - \hat{F}_R'(h_1)| \leq \int_0^1 |\max\{h_2(u),0\} - \max\{h_1(u),0\}| du \leq \|h_2 - h_1\|_1.$$ 

To verify condition (ii), observe that

$$|\hat{F}_R'(h) - F_R'(h)| \leq \int_{B_0(R) \triangle B_0(R)} \max\{h(u),0\} du,$$

where $B_0(R) \triangle B_0(R)$ is the symmetric difference of $B_0(R)$ and $B_0(R)$. It thus follows from Markov's inequality and Tonelli’s theorem that, for any $\epsilon > 0$,

$$P \left( \frac{|\hat{F}_R'(h) - F_R'(h)|}{\epsilon} > 1 \right) \leq \epsilon^{-1} E \left| \hat{F}_R'(h) - F_R'(h) \right| \leq \epsilon^{-1} \int_0^1 P \left( u \in \overline{B_0(R) \triangle B_0(R)} \right) \max\{h(u),0\} du.$$

Condition (ii) will therefore follow from the dominated convergence theorem if we can show that, for each $u \in (0,1)$,

$$P \left( u \in \overline{B_0(R) \triangle B_0(R)} \right) \to 0. \quad (6.20)$$

For each $u \in B_0(R)$ we have

$$P \left( u \in \overline{B_0(R) \triangle B_0(R)} \right) = P \left( T_n^{1/2}(\hat{R}(u) - R(u)) < -\tau_n \hat{V}_{[n_2u]}^{1/2} \right). \quad (6.21)$$
Lemma 6.7 establishes that $T_n^{1/2}(\hat{R}(u) - R(u)) = O_P(1)$ for all $u \in (0, 1)$, while Lemmas 6.5 and 6.8 establish that $\hat{V}_n^{1/2}$ converges in probability to a positive constant for all $u \in (0, 1)$. Thus, since $\tau_n \to \infty$ under Assumption 3.1, the probability on the right-hand side of (6.21) converges to zero. On the other hand, for each $u \in (0, 1) \setminus B_0(R)$ we have

$$P \left( u \in B_0(R) \triangle B_0(R) \right) = P \left( T_n^{1/2}(\hat{R}(u) - R(u)) + T_n^{1/2}(R(u) - u) \geq -\tau_n \hat{V}_n^{1/2} \right).$$

The latter probability converges to zero by Lemmas 6.5, 6.7 and 6.8 since $R(u) < u$ outside of $B_0(R)$ under $H_0$ and $T_n^{1/2} \tau_n \to 0$ under Assumption 3.1. Thus (6.20) is satisfied. This completes our verification of Assumptions 1-4 of Theorem 3.2 of Fang and Santos (2019). Our claimed result follows from theirs. \hfill \Box

**Proof of Proposition 3.4.** We first prove part (i). The proof is similar to that of Proposition 3.3(i). Let $A$ denote the set of continuity points of the CDF of $\mathcal{F}_R' (\mathcal{R})$. The conditional weak convergence established in Lemma 3.1 implies, via Lemma 10.11(i) of Kosorok (2008), that

$$P \left( \tilde{S}^* \leq c \mid \{X_1^1\}_{i=1}^{n_1}, \{X_1^2\}_{i=1}^{n_2} \right) \to P(\mathcal{F}_R' (\mathcal{R}) \leq c)$$

in probability for each $c \in A$. Fix $\epsilon > 0$. When $R(u) = u$ on a set of positive measure Lemma 6.6 establishes that the CDF of $\mathcal{F}_R' (\mathcal{R})$ is strictly increasing at its $(1 - \alpha)$-quantile $c_{1-\alpha}$, so we may choose $c_1, c_2 \in A$ such that $c_{1-\alpha} - \epsilon < c_1 < c_{1-\alpha} < c_2 < c_{1-\alpha} + \epsilon$ and

$$P(\mathcal{F}_R' (\mathcal{R}) \leq c_1) < 1 - \alpha < P(\mathcal{F}_R' (\mathcal{R}) \leq c_2).$$

(6.23)

If $|\tilde{c}_{1-\alpha} - c_{1-\alpha}| > \epsilon$ then either $\tilde{c}_{1-\alpha} < c_1$ or $\tilde{c}_{1-\alpha} > c_2$; that is, either

$$P \left( \tilde{S}^* \leq c_1 \mid \{X_1^1\}_{i=1}^{n_1}, \{X_1^2\}_{i=1}^{n_2} \right) \geq 1 - \alpha, \quad \text{or}$$

$$P \left( \tilde{S}^* \leq c_2 \mid \{X_1^1\}_{i=1}^{n_1}, \{X_1^2\}_{i=1}^{n_2} \right) < 1 - \alpha.$$

(6.24)

(6.25)

Since $c_1, c_2 \in A$, the convergence in (6.22) implies that the left-hand sides of (6.24) and (6.25) converge in probability to $P(\mathcal{F}_R' (\mathcal{R}) \leq c_1)$ and $P(\mathcal{F}_R' (\mathcal{R}) \leq c_2)$ respectively. Thus, due
to (6.23), the probability of either (6.24) or (6.25) being satisfied converges to zero. We conclude that $P(|\tilde{c}_{1-\alpha} - c_{1-\alpha}| > \epsilon) \to 0$. Since $\epsilon$ was arbitrary, it follows that $\tilde{c}_{1-\alpha} \to c_{1-\alpha}$ in probability.

If $H_0$ is satisfied then Proposition 2.2 implies that $\hat{S}$ converges in distribution to $\mathcal{F}'_R(\mathcal{R})$. Therefore, since (i) $\tilde{c}_{1-\alpha} \to c_{1-\alpha}$ in probability, and (ii) the CDF of $\mathcal{F}'_R(\mathcal{R})$ is continuous at $c_{1-\alpha}$ by Lemma 6.6 when $R(u) = u$ on a set of positive measure, we have

$$P(\hat{S} > \max\{\tilde{c}_{1-\alpha}, \eta\}) \to P(\mathcal{F}'_R(\mathcal{R}) > c_{1-\alpha}) = 1 - \alpha$$

whenever $\eta < c_{1-\alpha}$. This proves part (i).

Part (ii) is true because, when $R(u) < u$ almost everywhere, we have $\hat{S} \to \mathcal{F}'_R(\mathcal{R}) = 0$ in probability by Proposition 2.2. To see why part (iii) is true, observe first that in view of Lemma 2.1 and Proposition 2.2 we have $T^{-1/2}\hat{S} \to \mathcal{F}(R) > 0$ in probability under $H_1$, so that $P(\hat{S} > \eta) \to 1$ for each $\eta > 0$. Moreover, since $\tilde{c}_{1-\alpha} \leq \hat{c}_{1-\alpha}$, we have $P(\hat{S} > \tilde{c}_{1-\alpha}) \to 1$ by Proposition 3.3(ii).

\begin{flushright}$\square$\end{flushright}

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