CONSERVED QUANTITIES, GLOBAL EXISTENCE AND BLOW-UP FOR A GENERALIZED CH EQUATION

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Abstract. In this paper, we study conserved quantities, blow-up criterions, and global existence of solutions for a generalized CH equation. We investigate the classification of self-adjointness, conserved quantities for this equation from the viewpoint of Lie symmetry analysis. Then, based on these conserved quantities, blow-up criterions and global existence of solutions are presented.

1. Introduction. In this paper, we study the following shallow water equation with high-order nonlinearities, called a generalized Camassa Holm (g-CH) equation

\[ u_t - u_{xxt} + (b + 1)u^{m+1}u_x - bu^m u_x u_{xx} - u^{m+1}u_{xxx} = 0, \quad t > 0, \quad x \in \mathbb{R}, \]  
\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}, \]  

where \( b \neq 0 \) is a constant, \( m \) is a nonnegative integer, and \( u = u(t, x) \) is a function of \( t \) and \( x \). Letting \( y := u - u_{xx} \) sends equation (1) to the following g-CH family form

\[ y_t + u^{m+1}y_x + bu^m u_x y = 0, \quad t > 0, \quad x \in \mathbb{R}. \]  

The most famous member of the g-CH family (1) is the Camassa-Holm (CH) equation \((m = 0, b = 2)\)

\[ u_t - u_{xxt} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0. \]  

It is a nonlinear dispersive wave equation that models the propagation of unidirectional irrotational shallow water waves over a flat bed [4, 12, 38, 39], as well as water waves moving over an underlying shear flow [40]. The CH equation also arises in the

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study of a certain non-Newtonian fluids [3] and also models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [14]. This equation was originally derived by Fuchssteiner and Fokas in studying completely integrable generalizations of the KdV equation with bi-Hamiltonian structures (see [21]), and it has been widely studied in recent years. The CH equation has many remarkable properties, such as a bi-Hamiltonian structure, Lax completely integrability, infinitely many conservation laws, peakons, wave breaking, etc. Due to its abundant physical and mathematical properties, many physicists and mathematicians pay great attention to this equation. The Cauchy problem and inverse spectral problem for the CH equation were investigated in [5, 7, 13]. Thereafter, Constantin, Escher, McKeen et al investigated the wave-breaking of the Cauchy problem for the CH equation (see [6, 8, 10, 43]). The CH equation possesses the algebro-geometric solutions and has already been extended to the whole integrable hierarchy-the CH hierarchy [49]. Xin and Zhang considered the weak solutions to the CH equation [54]. Later, the global conservative and dissipative solutions for the CH equation were studied by Bressan, Constantin [1, 2] and Holden, Raynaud [28].

The family of equations (1) also includes other peakon shallow water wave models, such as Degasperis-Procesi (DP) equation [17] \((m = 0, b = 3)\), Novikov equation [47] \((m = 1, b = 3)\) and the following \(b\)-equation \((m = 0)\)

\[
u_t - u_{xx}t + (b + 1)uu_x - uu_{xxx} - bu_xu_{xx} = 0.
\]

The DP equation was proposed by Degasperis and Procesi in 1999 when handling asymptotic analysis, which pretty much looks the same as the case for the CH equation [17]. This equation is integrable [15] with \(3 \times 3\) Lax pair and has already been generalized to the whole integrable hierarchy-the DP hierarchy [50]. In [11], the inverse scattering method for the DP equation is studied based on a \(3 \times 3\) matrix Riemann-Hilbert (RH) problem, where the solution of the DP equation is given in terms of the solution of the RH problem. The DP equation is also shown to possess the shock peakons [42] and algebro-geometric solutions [33]. The Novikov equation was deduced by Novikov as an integrable equation with cubic nonlinearity that possesses similar properties to the CH and DP equations (see [47]). The bi-Hamilton structure, an infinite sequence of conserved quantities, exact peakon solutions, the explicit formulas for multipeakon solutions, well-posedness, illposedness and blow-up property have been investigated by several authors, see [31, 32, 45, 55] and the references therein. Equation (1) could be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any \(b \neq 1\) by an appropriate Kodama transformation (see [18, 19]). The \(b\)-equation also admits peakon solutions for any \(b \in \mathbb{R}\) (see [16, 29]). The necessary conditions for integrability, well-posedness, blow-up phenomena and global solutions for the \(b\)-equation were shown in [44, 20, 25, 40].

Recently, equation (1) (or (3)) has attracted lot of attention. Grayshan and Himonas constructed the peakon solutions for this equation and showed that the solution map for these equations is not uniformly continuous in Sobolev spaces with exponent less than \(\frac{3}{2}\) (see [24]). The well-posedness in the sense of Hadamard and the continuity of data-to-solution map have been investigated by Himonas and Holliman in [26]. The persistence properties, unique continuation, Hölder continuity, well-posedness in Besov spaces, persistence properties in weighted \(L^p\) spaces and blow-up phenomena have been studied in [27, 30, 57, 58, 59]. It is worth pointing out that, in those studies, conserved quantities always play important roles.
In this paper, we study the conserved quantities, blow-up phenomena and global existence of solutions to equation (1). From the viewpoint of Lie symmetry and self-adjointness, we first construct some useful conservation laws of (1). Then by those conserved quantities we can analyze the global existence of solutions. In what follows, let us briefly present the notations, definition of nonlinear self-adjointness and Ibragimov’s theorem on conservation laws. Consider a $s$-th order nonlinear equation

$$E(x, u, u(1), u(2), \ldots, u(s)) = 0$$  \hfill (5)

with $n$ independent variables $x = (x_1, x_2, \ldots, x_n)$ and a dependent variable $u = u(x)$, where $u(s) = \partial^s u$. A symmetry generator of (5) is denoted by

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u}. \hfill (6)$$

Let

$$E^*(x, u, v, u(1), v(1), \ldots, u(s), v(s)) := \frac{\delta L}{\delta u} = 0 \hfill (7)$$

be the adjoint equation of equation (5), where $L = vE$ is called formal Lagrangian, $v = v(x)$ is a new dependent variable and

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{m=1}^{s} (-1)^m D_{i_1} \cdots D_{i_m} \frac{\partial}{\partial u_{i_1 \cdots i_m}}$$

denotes the Euler-Lagrange operator.

Let us now state the definition of nonlinear self-adjointness for a equation, see [35, 37, 22, 23, 36, 51, 52] and references therein.

**Definition 1.** equation (5) is said to be nonlinearly self-adjoint if the equation obtained from the adjoint equation (7) by the substitution $v = \phi(x, u)$ with a certain function $\phi(x, u) \neq 0$ is identical with the original equation (5). In other words, the following equation holds:

$$E^*|_{v=\phi} = \lambda(x, u, u(1), \ldots) E \hfill (8)$$

for some differential function $\lambda = \lambda(x, u, u(1), \ldots)$.

Particularly, if (8) holds for a certain function $\phi$ such that $\phi_u \neq 0$ and $\phi_{x_i} \neq 0$ for some $x_i$, equation (5) is called weak self-adjoint. If (8) holds for a certain function $\phi$ such that $\phi = \phi(u) \neq u$ and $\phi'(u) \neq 0$, then equation (5) is called quasi-self-adjoint. If (8) holds for $\phi = u$, then equation (5) is called (strictly) self-adjoint.

Let us recall the conservation theorem given by Ibragimov in [34].

**Theorem.** (Ibragimov [34]) Any Lie point, Lie-Bäcklund and non-local symmetry generated by

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} \hfill (9)$$

of equation (5) provides a conservation law $D_i(C^i) = 0$ for the system comprising equation (5) and its adjoint equation (7). The conserved vector $C = (C^i)$ is given by

$$C^i = \xi^i L + W \left[ \frac{\partial C}{\partial u_{i_1}} - D_j \left( \frac{\partial C}{\partial u_{ij}} \right) + \frac{\partial C}{\partial u_{ik}} \right] + D_k D_j \left( \frac{\partial C}{\partial u_{ijkl}} \right) + \cdots \hfill (10)$$

$$+ D_j(W) \left[ \frac{\partial C}{\partial u_{ij}} - D_k \left( \frac{\partial C}{\partial u_{jk}} \right) + \cdots \right] + \cdots$$
where \( W = \eta - \xi^i u_i \) is the Lie characteristic function and \( \mathcal{L} = vE \) is the formal Lagrangian.

We investigate the conserved quantities for equation (1) from the viewpoint of nonlinear adjointness, and give some classification of global existence of solution by the parameters \( b \) and \( m \).

**Theorem 1.1.** Let \( m \) be a nonnegative integer. Equation (1) is nonlinear self-adjoint for \( b = m + 2; b = m + 1; \) or \( m = 0 \) and any \( b \). Moreover, equation (1) is strict self-adjoint \( b = m + 2 \).

For the conserved quantities, we have the following classification result.

**Theorem 1.2.** Equation (1) admits conservation laws \( D_t C^t + D_x C^x = 0 \) with following local components. For \( b = m + 2 \),

\[
C_t^1 = u^2 + u_x^2, \quad C_x^1 = 2(u^{m+3} - u^{m+2}u_{xx} - uu_{xt});
\]

for \( b = \frac{m+1}{2} \),

\[
C_t^2 = u^2 + 2u_x^2 + u_{xx}^2, \quad C_x^2 = (u - u_{xx})^2 u^{m+1} - 2uu_{xt} - 2u_t u_x;
\]

for \( b = m + 1; \) or \( m = 0 \), \( b \in \mathbb{R} \),

\[
C_t^3 = u - u_{xx}, \quad C_x^3 = \frac{b+1}{m+2} u^{m+2} - u^{m+1} u_{xx} + \frac{1}{2}(b - m - 1)(3m-1)u^{m}u_x^2;
\]

Especially, for \( m = 0 \) and \( b = 3 \),

\[
C_t^4 = -3e^{\pm 2x}u, \quad C_x^4 = e^{\pm x}[\pm 2(u_t + uu_x) - u_{xt} - u_x^2 - uu_{xx}].
\]

For \( b \neq m + 1 \), the related conservation laws are given by

\[
C_t^5 = y|y|^{\frac{m+1}{b-1}}, \quad C_x^5 = u^{m+1}y|y|^{\frac{m+1}{b-1}};
\]

\[
C^5_{5,1} = |y|^{\frac{m+1}{b}}, \quad C^5_{5,1} = u^{m+1}|y|^{\frac{m+1}{b}};
\]

\[
C^5_{5,2} = y^{\frac{m+1}{b}}, \quad C^5_{5,2} = u^{m+1}y^{\frac{m+1}{b}}
\]

provided \( y^{\frac{m+1}{b}} \) makes sense, where \( y = u - u_{xx} \).

From the above theorem, we derive the following corollary.

**Corollary 1.** Let \( u \) be a solution of (1) and \( y = u - u_{xx} \), then it holds:

For \( b = m + 2 \), the \( H^1 \)-norm of \( u \)

\[
\|u\|_{H^1(\mathbb{R})} = \int_R (u^2 + u_x^2)dx = C_1
\]

is conserved; for \( b = \frac{m+1}{2} \), the weight \( H^2 \)-norm of \( u \) is conserved, that is,

\[
\int_R y^2dx = \int_R (u^2 + 2u_x^2 + u_{xx}^2)dx = C_2;
\]

for \( b = m + 1 \) or \( m = 0 \), \( b \in \mathbb{R} \), the mass for equation (1)

\[
\int_R udx = \int_R ydx = C_3
\]

is a constant; for \( m = 0 \) and \( b = 3 \), the weighted mass is conserved:

\[
\int_R e^{\pm 2x}udx = C_4;
\]
and for \( b \neq m + 1 \),
\[
\int_{\mathbb{R}} |y|^{\frac{m+1}{2}} \, dx = C_5, \quad \int_{\mathbb{R}} |y|^{\frac{m+1}{2}} \, dx = C_6, \quad \int_{\mathbb{R}} y^{\frac{m+1}{2}} \, dx = C_7
\]  
(22)
provided \( y^{\frac{m+1}{2}} \) make sense.

Let us now consider the blow-up and existence results for equation (1). We first give a precise blow-up scenario.

**Theorem 1.3.** Let \( u_0 \in H^s(\mathbb{R}) \) with \( s > \frac{3}{2} \) and \( T \) be the maximal existence time of the solution \( u \) to \((1)\) with the initial data \( u_0 \). If \( b \neq \frac{m+1}{2} \), then the corresponding solution blow up in finite time if and only if
\[
\lim_{t \to T^-} \liminf_{x \to \mathbb{R}} \left( b - \frac{m+1}{2} \right) u^m u_x (x, t) = -\infty.
\]  
(23)

This theorem covers blow-up scenario results to the CH equation in \([9]\), DP equation in \([56]\), \( b \)-equation in \([20]\) and Novikov equation in \([45]\). It also similar to Theorem 1.2 in \([59]\), but we give a different proof. It is very interesting that, for \( s > 5/2 \), we can show another precise blow-up scenario. Indeed, consider the associated Lagrangian scale of \((1)\),
\[
\begin{cases}
\frac{\partial q(t, x)}{\partial t} = u^{m+1}(t, q(t, x)), & 0 < t < T, \quad x \in \mathbb{R}, \\
q(0, x) = x, & x \in \mathbb{R},
\end{cases}
\]
where \( u \in C([0, T]; H^s) \) is the solution of \((1)\) with initial data \( u_0 \in H^s \) with \( s > \frac{3}{2} \) and \( T > 0 \) is the maximal time of existence. We can derive a conserved quantity
\[
y(t, q(t, x)) q_x(t, x)^{\frac{m}{m+1}} = y(t, q(t, x)) q_x(t, x)^{\frac{m}{m+1}}|_{t=0} = y_0(x),
\]
which gives rise to a new precise blow-up scenario.

**Theorem 1.4.** Let \( u_0 \in H^s(\mathbb{R}) \) with \( s > \frac{5}{2} \) and \( T \) be the maximal existence time of the solution \( u \) to \((1)\) with the initial data \( u_0 \). For any \( b \in \mathbb{R} \), then the corresponding solution blow up in finite time if and only if
\[
\lim_{t \to T^-} \liminf_{x \to \mathbb{R}} (bu^m u_x)(x, t) = -\infty.
\]  
(24)

From Theorem 1.3 and 1.4 we easily obtain a global existence result.

**Theorem 1.5.** Let \( b \in [0, \frac{m+1}{2}] \) and \( u_0 \in H^s(\mathbb{R}) \) with \( s > \frac{5}{2} \). Then the corresponding solution to equation \((1)\) is defined globally in time.

For \( m = 0 \), our result generalizes partially the result on global existence for the \( b \)-equation (Theorem 4.3 in \([20]\)). Moreover, we don’t need the assumption \( u_{0,xx} \in L^\frac{1}{2} \).

Finally, based on the conserved quantities obtained in Corollary 1, we can present the following classifications of global existence results for solutions to equation \((1)\), which also can be found in \([59]\).

**Theorem 1.6.** Let \( u_0 \in H^s \) with \( s > \frac{3}{2} \). Then the solution of the problem \((1)\) remains regular globally in time provided that one of the following conditions occurs:

(i) \( b = \frac{m+1}{2} \);

(ii) \( b \in (0, m + 1) \), additionally, \( u_0 \in H^s \cap W^{2, \frac{m+1}{2}} \);

(iii) \( b = m + 2 \) and \( y_0 = (1 - \partial_y^2) u_0 \) doesn’t change sign on \( \mathbb{R} \); or

(iv) \( b = m + 1 \); or \( b \in \mathbb{R} \) and \( m = 0 \), additionally, \( y_0 \) doesn’t change sign on \( \mathbb{R} \).
It is worth to note that Theorem 1.5 and (ii) in Theorem 1.6 don’t cover each other.

The remainder of this paper is organized as follows. In Section 2, we discuss the nonlinear self-adjointness of equation (1) and find some conserved quantities based on this concept. In Section 3, we establish some results on blow-up scenario and global existence of solutions to equation (1) (or (3)).

2. Conserved quantities. Conserved quantities are very important and can be used to show some properties of solutions for some nonlinear equations, such as apriori estimates, global existence and stability of solutions. However, it seems not easy to find some useful conserved quantities for some nonlinear wave equations. Here, we will apply the concepts of self-adjointness and the Ibragimov’s theorem on conservation laws to construct some conserved quantities for equation (1).

First, we rewrite equation (1) directly
\[ E = u_t - u_{xxx} + (b + 1)u^{m+1}u_x - bu^m u_x u_{xx} - u^{m+1}u_{xxx} = 0 \]  \tag{25} and let \( v = v(t, x) \) be a new dependent variable. It’s adjoint equation is given by
\[ E^* = \frac{\delta(L)}{\delta u} = -v_t + v_{xxx} + (b + 1)(m + 1)vu^m u_x - (b + 1)D_x(vu^{m+1}) - (m + 1)vu^m u_{xxx} + D_x^3(vu^{m+1}) - bnu^m u_x u_{xx} + bD_x(vu^m u_x) = 0, \] \tag{26}

where \( L = vE \) is the formal Lagrangian.

Let \( \lambda_i = \lambda_i(t, x, u, u_t, u_x, \cdots) \) \( (i = 0, 1, 2, \cdots) \) be differential functions and, by the concept of nonlinear self-adjointness, we have
\[ E^*|_{v=v(t,x,u,u_t,u_x,\cdots)} = \lambda_0E + \lambda_1D_t(E) + \lambda_2D_x(E) + \cdots. \] \tag{27} Solving equation (27), we can see that:

(i) If \( m \geq 0 \) is an integer, then
\[
\begin{cases}
  v = Cu, & \text{for } b = m + 2; \\
  v = C, & \text{for } b = m + 1; \\
  v = C(u - u_{xx}), & \text{for } b = \frac{m + 1}{2}.
\end{cases}
\]

(ii) If \( m = 0 \), then \( v = C \) for any \( b; v = C + C_1e^{2x} + C_2e^{-2x} \) for \( b = 3 \), where \( C, C_1 \) and \( C_2 \) are arbitrary constants. Hence, we have demonstrated the following result.

**Proposition 1.** Let \( m \) be a nonnegative integer. Equation (1) is nonlinear self-adjoint for

(i) \( b = m + 2 \) with the substitution \( v = Cu; b = m + 1 \) with \( v = C \) and \( b = \frac{m + 1}{2} \) with \( v = C(u - u_{xx}); \) And

(ii) \( m = 0 \), for any \( b \) with \( v = C \) and for \( b = 3 \) with \( v = C + C_1e^{2x} + C_2e^{-2x} \). Further, equation (1) is strict self-adjoint with substitution \( v = u \) for \( b = m + 2 \).

**Proof of Theorem 1.1.** Theorem 1.1 is deduced immediately from above proposition.

Now we investigate the symmetries of equation (1) by the classical Lie symmetry analysis [48]. Consider the vector field
\[ X = \xi^t \partial_t + \xi^x \partial_x + \eta \partial_u, \] \tag{28}
which has the fourth-order prolongation, from (1),
\[ X^{(3)} = X + \eta^{(1)}_t \partial_t + \eta^{(1)}_x \partial_x + \eta^{(2)}_{xx} \partial_{xx} + \eta^{(3)}_{xxx} \partial_{xxx}, \]
the functions \( \eta^{(1)}_t, \eta^{(1)}_x, \eta^{(2)}_{xx}, \eta^{(3)}_{xxx} \) can be expressed via the components of the vector field \( \xi^t, \xi^x \) and \( \eta \). The invariant condition
\[ X^{(3)} E = \lambda(t, x, u) E, \]
leads to the following classification result.

**Proposition 2.** For general \( m \in \mathbb{N} \cup \{0\} \) and \( b \neq 0 \) equation (1) admits the following symmetries
\[ X_1 = \partial_x, \quad X_2 = \partial_t \quad \text{and} \quad X_3 = -(m+1)t \partial_t + u \partial_u. \]
Particularly, for \( m = 1 \) and \( b = 3 \), (1) possesses symmetries
\[ X_4 = e^{2x}(\partial_x + u \partial_u) \quad \text{and} \quad X_5 = e^{-2x}(-\partial_x + u \partial_u) \]
besides \( X_1, X_2 \) and \( X_3 \).

Next, from equation (3) we consider the following system
\[
\begin{align*}
E_1 &= y_t + u^{m+1}y_x + bu^m u_x y = 0, \\
E_2 &= y - u + u_{xx} = 0.
\end{align*}
\] (29)
Assume that the form Lagrangian of this system is \( L_s = vE_1 + zE_2 \), where \( v = v(t, x) \) and \( z = z(t, x) \) are two dependent variables. Then the adjoint system of (29) is
\[
\begin{align*}
E_1^* &= \frac{\delta L_s}{\delta u}, \\
F_1^* &= \frac{\delta L_s}{\delta y}.
\end{align*}
\] (30)
By the definition of nonlinear self-adjointness, we have
\[
\begin{align*}
E_1^*|_{w=h,z=g} &= \mu_1 E_1 + \tau_1 F_1, \\
F_1^*|_{w=h,z=g} &= \mu_2 E_1 + \tau_2 F_1,
\end{align*}
\] (31)
where \( h = h(t, x, u, u_t, u_x, \cdots), g = g(t, x, u, u_t, u_x, \cdots), \mu_i \) and \( \tau_i \) \( (i = 1, 2) \) are differentiable functions.

On the other hand, let
\[ Y = \xi^t \partial_t + \xi^x \partial_x + \eta \partial_u + \rho \partial_y \]
be the symmetry operator of (29). From system (31), we get
\[ \mu_1 = z = \tau_1 = \tau_2 = 0, \quad \mu_2 + w_y = 0, \quad bw_y + (b - m - 1)w = 0. \]
Solving this system and going through the process of classical Lie symmetry analysis, we can obtain the following result.

**Proposition 3.** For \( m \in \mathbb{N} \cup \{0\}, b \neq m + 1 \), system (29) is quasi-selfadjoint with the substitution \( w = |y|^{\frac{m-1}{b}}, z = 0 \). Moreover, the symmetries admitted by (29) are given by
\[
\begin{align*}
Y_1 &= y \partial_y, \\
Y_2 &= t \partial_t + x \partial_x, \\
Y_3 &= -(m+1)f(t) \partial_t + f'(t)u \partial_u,
\end{align*}
\] (33)
where \( f(t) \) is a differentiable function.
Based on Propositions 1–3, we now can apply Ibragimov’s theorem to construct some conservation laws for equation (1).

**Proof of Theorem 1.2** Rewriting the formal Lagrangian \( \mathcal{L} = vE \) in the symmetric form

\[
\mathcal{L} = v\left[u_t - \frac{1}{3}(u_{xxx} + u_{txx}) + au^{m+1}u_x - bu^mu_xu_{xx} - cu^{m+1}u_{xxx}\right],
\]

where \( v \) is given in Proposition 1. From Ibragimov’s theorem, for a general generator \( X \) given by (28), we have that the Lie characteristic function \( W = \eta - \xi^t u_t - \xi^x u_x \), the density is given by

\[
C^t = \xi^t \mathcal{L} + W(\mathcal{L}_{u_t} + D^2_{x} \mathcal{L}_{u_{xxx}}) + D_x W(\mathcal{L}_{u_{xxxx}}) + D^2_x W(\mathcal{L}_{u_{xxxx}})
\]

\[
= \xi^t \mathcal{L} + W\left(v - \frac{1}{3}v_{xx}\right) + \frac{1}{3}v_x D_x W - \frac{1}{3}v D^2_x W;
\]

and the flux is

\[
C^x = \xi^x \mathcal{L} + W(\mathcal{L}_{u_x} - D_x \mathcal{L}_{u_{xxxx}}) + D_x W(\mathcal{L}_{u_{xxxx}}) + D^2_x W(\mathcal{L}_{u_{xxxx}})
\]

\[
+ D_t W(\mathcal{L}_{u_{xxxx}}) + D_t W(\mathcal{L}_{u_{xxxx}}) + D^2_t W(\mathcal{L}_{u_{xxxx}})
\]

\[
= \xi^x \mathcal{L} + W\left[(b+1)u^{m+1}v - bu^mu_{xx}v + D_x(bu^mu_xv) - D^2_x(u^m v)\right]
\]

\[
- \frac{2}{3}v_{xx} + \frac{1}{3}v_x D_t W + D_x W[\frac{1}{3}v_t - bu^mu_x] - \frac{2}{3}v D_x W
\]

\[
- \frac{2}{3}v D^2_x W.
\]

(34)

For the symmetries \( X_1 - X_5 \), from the formula (34) and (35) we obtain readily some conservation laws for equation (1).

For the case \( m = 0, b \in \mathbb{R} \) or \( b = m + 1 \), the substitution function is \( v = 1 \) (we take \( C = 1 \)). Let us construct the conserved vector corresponding to the time translation group with the generator \( X_1 = \partial_t \). For this operator, we have \( W = -u_t \). Therefore, we obtain the following conserved vector

\[
C^t = -\frac{3}{2}u_{xx} + (b+1)u^{m+1}u_x - u^{m+1}u_{xxx} - bu^mu_xu_{xx}
\]

\[
= D_x\left(\frac{m+1-b}{2}u^{m+2} + \frac{b+1}{m+2}u^{m+2} - u^{m+1}u_x - \frac{2}{3}u_{xx}\right) + \frac{m(b-m-1)}{2}u^{m-1}u_x^3,
\]

\[
C^x = m(m+1-b)u^{m-1}u_x^2 - (m+1-b)u^mu_xu_{xx} - (b+1)u^{m+1}u_t
\]

\[
+ (m+1)u^{m+1}u_t + \frac{u^{m+1}u_{xx} + \frac{2}{3}u_{xx}}{u_x}
\]

\[
= -D_t\left(\frac{m+1-b}{2}u_x^2 + \frac{b+1}{m+2}u^{m+2} - u^{m+1}u_x - \frac{2}{3}u_{xx}\right)
\]

\[
+ \frac{3m(b-m-1)}{2}u^{m-1}u_x^2 u_t,
\]

(36)

which can be simplified to (note that \( m = 0, b \in \mathbb{R} \) or \( b = m + 1 \) in this case)

\[
C^t = 0, \quad C^x = 0,
\]

that is, this conservation law is trivial. In this case, for the generator \( X_2 = \partial_x \), we also show that the corresponding conservation law is trivial. However, for \( X_3 = -(m+1)t\partial_t \), from the formula (34) and (35) we get the conservation law as follows
which means that the weighted mass is conserved, which can be reduced to
\[ C^t = u - u_{xx}, \quad C^x = \frac{b+1}{m+2}u^{m+2} - u^{m+1}u_{xx} + \frac{1}{2}(b - m - 1)(3m - 1)u^m u_x^2, \]
that is, \((C^t_4, C^x_4)\), which is given by (13).

The other conservation laws in the theorem can be given by similar processes to above. As for \((C^t_5, C^x_5)\), we need to use Proposition \(3\) By the Theorem of Ibragimov, we obtain that
\[ C^t = y|y|^{\frac{m+1}{2}}, \quad C^x = u^{m+1} |y|^{\frac{m+1}{2}}. \]

Moreover, in view of equation (37) it is easy to check that
\[ \|u\|_{H^1(\mathbb{R})} = \int_\mathbb{R} (u^2 + u_x^2)dx = C_1, \quad \text{for } b = m + 2. \]

Similarly, from \(C^t_2\), we can see that
\[ \int_\mathbb{R} y^2dx = \int_\mathbb{R} (u^2 + 2u_x^2 + u_{xx}^2)dx = C_2, \quad \text{for } b = \frac{m+1}{2}, \]
which means that the weighted \(H^2\) norm of \(u\) is conserved if \(b = \frac{m+1}{2}\). By \(C^t_3\), we can conclude that
\[ \int_\mathbb{R} udx = \int_\mathbb{R} ydx = C_3, \quad \text{for } b = m + 1 \text{ or } m = 0, b \in \mathbb{R}. \]

This shows that the mass is conserved for equation (14). From \(C^t_4\), we see that the weighted mass is conserved,
\[ \int_\mathbb{R} e^{\pm 2x}udx = C_4, \quad \text{for } m = 0 \text{ and } b = 3. \]

Finally, we deduce from \(C^t_5\), \(C^t_{5,1}\) and \(C^t_{5,2}\) that for \(b \neq m + 1\)
\[ \int_\mathbb{R} y|y|^{\frac{m+1}{2}-1}dx = C_5, \quad \int_\mathbb{R} y^{\frac{m+1}{2}}dx = C_6, \quad \int_\mathbb{R} y^{\frac{m+1}{8}}dx = C_7. \]
3. Blow-up scenario and global existence. In this section we derive the precise blow-up scenario and show some global existence results for the strong solutions to equation \(\text{(3)}\).

Denote the operator \((1 - \partial^2_x)^{\frac{1}{2}}\) by \(\Lambda\) and the kernel of \(\Lambda^{-2}\) by \(G := \frac{1}{2}e^{-|x|}\). Then \(\Lambda^{-2}f = G * f\) for all \(f \in L^2(\mathbb{R})\) and \(G * y = u\). Using this identity, equation \(\text{(3)}\) can be reformulated in the following form

\[
\begin{aligned}
&\begin{cases}
  u_t + u^{m+1}u_x + G * g + \partial_x G * h = 0 & t > 0, x \in \mathbb{R}, \\
  u(0, x) = u_0(x),
\end{cases}
\end{aligned}
\tag{38}
\]

or in the equivalent form

\[
\begin{aligned}
&\begin{cases}
  u_t + u^{m+1}u_x + (1 - \partial^2_x)^{-1}g + \partial_x (1 - \partial^2_x)^{-1}h = 0 & t > 0, x \in \mathbb{R}, \\
  u(0, x) = u_0(x),
\end{cases}
\end{aligned}
\tag{39}
\]

where

\[
g = \frac{m(b - m - 1)}{2}u_m u_x^2, \quad h = \frac{b}{m + 2}u_m^3 + \frac{3m + b - 1}{2}u_m u_x^2. \tag{40}
\]

First, we recall the local well-posedness of the Cauchy problem for equation \(\text{(3)}\) in \(H^s(\mathbb{R})\), \(s > \frac{3}{2}\). By Kato’s semigroup theorem or Galerkin-type approximation scheme, the authors of [26, 57] showed the following result.

**Theorem 3.1.** (26, 57) Let \(u_0 \in H^s(\mathbb{R})\) with \(s > \frac{3}{2}\). Then there exist a maximal \(T = T(u_0)\) and a unique solution \(u(t, x)\) to the problem \(\text{(3)}\) such that

\[u = u(., u_0) \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})).\]

Moreover, the solution depends continuously on the initial data, i.e. the mapping \(u_0 \to u(., u_0) : H^s(\mathbb{R}) \to C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))\) is continuous.

We will need the following useful lemmas.

**Lemma 3.2.** (41) If \(s > 0\), then \(H^s \cap L^\infty\) is an algebra. Moreover,

\[\|fg\|_{H^s} \leq c(\|f\|_{L^\infty}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{L^\infty}),\]

where \(c\) is a constant depending only on \(s\).

**Lemma 3.3.** (41) If \(s > 0\), then

\[\|A^s f g\|_{L^2} \leq c(\|\partial_x f\|_{L^\infty}\|A^{s-1}g\|_{L^2} + \|A^s f\|_{L^2}\|g\|_{L^\infty}),\]

where \([A, B]\) denote the commutator of linear operator \(A\) and \(B\), \(c\) is a constant depending only on \(s\).

Now, let us consider the associated Lagrangian scale of \((1)\), that is, the initial value problem

\[
\begin{aligned}
&\begin{cases}
  \frac{\partial q(t, x)}{\partial t} = u^{m+1}(t, q(t, x)), & x \in \mathbb{R}, \quad 0 < t < T, \\
  q(0, x) = x, & x \in \mathbb{R},
\end{cases}
\end{aligned}
\tag{41}
\]

where \(u \in C([0, T]; H^s)\) is the solution of \((1)\) with initial data \(u_0 \in H^s\) with \(s > \frac{3}{2}\) and \(T > 0\) is the maximal time of existence. A direct calculation shows that \(q_x(t, x) = (m + 1)u^m u_q(t, q(t, x))q_x(t, x)\). So, for \(t \in [0, T]\), \(x \in \mathbb{R}\), we have

\[q_x(t, x) = e^{\int_0^t u^{m+1}(\tau, q(\tau, x))d\tau} > 0. \tag{42}\]

In view of equations \(\text{(3)}\) and \(\text{(41)}\), one gets easily that

\[\frac{d}{dt}[g(t, q(t, x))q_x(t, x)]^{\frac{1}{m+1}} = (y_t + u^{m+1}y_x + bu^m u_y)q_x^{\frac{1}{m+1}} = 0,\]
which means that
\[ y(t, q(t, x)) q_x(t, x) = y(t, q(t, x)) q_x(t, x) \big|_{t=0} = y_0(x). \] (43)

We now consider some global existence results for the strong solutions to equation (3). Let us state a global existence result.

**Theorem 3.4.** Let \( u_0 \in H^s(\mathbb{R}) \) with \( s > \frac{3}{2} \) and \( T \) be the maximal existence time of the solution \( u \) to (3) corresponding to the initial data \( u_0 \). If there exists a constant \( M > 0 \) such that
\[ \|u\|_{L^\infty} + \|u_x\|_{L^\infty} \leq M, \quad t \in [0, T), \] (44)
then the \( H^s \)-norm of \( u(\cdot, t) \) does not blow up on \([0, T)\).

**Proof.** The idea of the proof is classical and is similar to those used in [20, 59] and the references therein. However, there seems to be a gap in the proof of Theorem 4.1 in [59]. So we give the details of the proof here.

Let \( u \) be the solution to equation (39) with the initial data \( u_0 \in H^s, s > \frac{3}{2} \). And let \( T \) be the maximal existence time of the corresponding solution \( u \), which is guaranteed by Theorem 3.1. Assume that (44) holds. Applying the operator \( \Lambda^s \) to equation (39), multiplying by \( \Lambda^s u \), and integrating over \( \mathbb{R} \) yield that
\[ \frac{d}{dt} \|u\|_{H^s}^2 = -2(u^{m+1}u_x, u)_s - 2(u, f_{11})_s - 2(u, f_{12})_s, \] (45)
where
\[ f_{11} = \Lambda^{-2}g, \quad f_{12} = \partial_x \Lambda^{-2}h. \]

And \( g, h \) are given by (10).

For the first term of right side in (45), from Lemma 3.3 we have the following estimate:
\[ |(u^{m+1}u_x, u)_s| = |(\Lambda^s u^{m+1}u_x, \Lambda^s u)_0 + (u^{m+1} \Lambda^s u_x, \Lambda^s u)_0| \]
\[ \leq |(\Lambda^s u^{m+1}u_x)_{L^\infty}||\Lambda^s u||_{L^2} + \frac{1}{2}|(m+1)u^m u_x \Lambda^s u, \Lambda^s u)_0| \]
\[ \leq c(\|\partial_x u^{m+1}\|_{L^\infty}||\Lambda^{s-1}u_x||_{L^2} + \|\Lambda^s u^{m+1}\|_{L^1}||u_x||_{L^\infty})||u||_{H^s} \]
\[ + \frac{1}{2}(m+1)\|u^m u_x\|_{L^\infty}||\Lambda^s u, \Lambda^s u)_0| \]
\[ \leq c(\|u^m u_x\|_{L^\infty}||u||_{H^s} + \|u\|_{L^\infty}||u||_{H^s} ||u_x||_{L^\infty})||u||_{H^s} \]
\[ + \frac{1}{2}(m+1)\|u^m u_x\|_{L^\infty}||u||_{H^s}^2, \]
where we have used the estimate
\[ \|\Lambda^s u^{m+1}\|_{L^2} = \|uu^m\|_{H^s} \leq c(||u||_{L^\infty}||u^m||_{H^s} + ||u||_{H^s} ||u^m||_{L^\infty}) \]
\[ \leq c(||u||_{L^\infty}||u^m||_{H^s} + ||u||_{H^s} ||u^m||_{L^\infty}) \] (46)
which can be obtained by applying Lemma 3.2 repetitiously.

As for the second term on the right in (45), when \( m = 0 \) it would be disappear; when \( m = 1 \), it can be estimated, we refer to the proof of Theorem 3.1 in [53]. Now, we assume here that \( m \geq 2 \). It follows from the Hölder inequality that
\[ |(u, f_{11})_s| \leq \frac{|m(m-1)|}{2}||\Lambda^{s-2}(u^{m-1}u^3_x), \Lambda^s u)_0| \]
\[ \leq c\|\Lambda^s u\|_{L^2} ||u^{m-2}(u^{m-1}u^3_x)||_{L^2} \]
\[ \leq c\|u||_{H^s} ||u^{m-1}u^3_x||_{H^{s-1}}. \]
Thanks to Lemma 3.2, we see that
\[ \|u^{m-1}u_x^2\|_{H^{s-1}} \leq c(\|u^{m-1}u_x^2\|_{L^\infty}\|u_x\|_{H^{s-1}} + \|u^{m-1}u_x^2\|_{H^{s-1}}\|u_x\|_{L^\infty}) \leq c\left(\|u^{m-1}u_x^2\|_{L^\infty}\|u_x\|_{H^{s-1}} + \left(\|u^{m-1}u_x\|_{L^\infty}\|u_x\|_{H^{s-1}} + \|u^{m-1}u_x\|_{H^{s-1}}\|u_x\|_{L^\infty}\right)\|u_x\|_{L^\infty}\right) \]
which has been obtained similarly to (44). Thus, we obtain
\[ |(u, f_{11})| \leq c(\|u\|_{L^\infty}\|u_x\|_{L^\infty}^2 + \|u\|_{L^\infty}^{m-2}\|u_x\|_{L^\infty}^3)\|u_x\|_{H^{s-1}}^2. \]

For the last term in (45), we need to estimate two parts \(|(\partial_x \Lambda A u^{m-2}u_x, u)_s|\) and \(|(\partial_x \Lambda A u^{m-2}u^2_x, u)_s|\) respectively. First, from Lemma 3.3 we see that
\[ |(\partial_x \Lambda A u^{m-2}u^2_x, u)_s| \leq c(\|u^{m-1}u_x\|_{L^\infty}\|u_x\|_{L^\infty} + \|u\|_{L^\infty}^{m-2}\|u_x\|_{L^\infty}^3)\|u_x\|_{H^{s-1}}^2. \]

We have also used the estimate similar to (46) here. We next estimate the term \(|(\partial_x \Lambda A u^{m-2}u^2_x, u)_s|\). Note that \(H^s\) and \(H^{s-1}\) are algebraic with \(s > \frac{3}{2}\). So we have
\[ |(\partial_x \Lambda A u^{m-2}u^2_x, u)_s| \leq c\|u^{m-1}u_x\|_{H^{s-1}}\|u\|_{H^s} + c\|u\|_{H^s}\left(\|u^{m-1}u_x\|_{L^\infty}\|u_x\|_{H^{s-1}} + \|u^{m-1}u_x\|_{H^{s-1}}\|u_x\|_{L^\infty}\right) \leq c\|u\|_{H^s}^2\left(\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty} + \|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right). \]

Therefore, we see that from above estimates
\[ \frac{d}{dt}\|u\|_{H^s}^2 \leq c\|u\|_{L^\infty}^{m-2}\|u_x\|_{L^\infty}\left(\|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2\right)\|u\|_{H^s}^2. \]

Then from the Gronwall’s inequality and assumption of the theorem, we obtain
\[ \|u\|_{H^s}^2 \leq c\|u_0\|_{H^s}^2, \quad (49) \]
provided (44) holds. This completes the proof of the theorem. \(\square\)

Now, we prove Theorem 1.3

**Proof of Theorem 1.3** By Theorem 3.4 and a simple density argument, it is needed only to show the desired result is valid when \(s \geq 2\). Note \(y = u - u_{xx}\), it’s easy to see that
\[ \|u\|_{H^2}^2 \leq \|y\|_{L^2}^2 = \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2)dx \leq 2\|u\|_{H^2}^2. \quad (50) \]
For \( b \neq \frac{m+1}{2} \), taking account for (43) we know that
\[
\|y\|_{L^2}^2 = \int_{\mathbb{R}} g^2(t, q(t, s)) q_x(t, s) ds = \int_{\mathbb{R}} y_0^2(s) q_x(t, s)^{1 - \frac{2b}{m+1}} ds.
\]
Due to the equality (42), one has
\[
\|y\|_{L^2} = \|y_0(x) e^{\left(\frac{m+1}{2} - b\right) \int_0^1 (u^m u_x)(\tau, q(\tau, x)) d\tau}\|_{L^2}.
\]
Assume (23) is not valid. Then there is some positive number \( M_1 > 0 \) such that
\[
\|u\|_{H^2}^2 \leq \|y\|_{L^2}^2 \leq e^{M_1 t} \|y_0\|_{L^2}^2,
\]
which combining Theorem 3.4 shows that the solution does not blow up in finite time for \( s \geq 2 \).

Conversely, the Sobolev embedding theorem \( H^s(\mathbb{R}) \hookrightarrow L_\infty(\mathbb{R}) \) (with \( s > \frac{5}{2} \)) implies that if (23) holds, the corresponding solution blows up in finite time, which completes the proof.

**Proof of Theorem 1.4** Let \( u \) be the solution to equation (39) with the initial data \( u_0 \in H^s \), \( s > \frac{5}{2} \) and \( T \) be the maximal existence time of the corresponding solution \( u \).

Thanks to the equalities (42) and (43), we have
\[
\|y\|_{L^\infty} = \|y(t, q(t, \cdot))\|_{L^\infty} = \|y_0(x) q_x(t, x)^{\frac{1}{m+1}}\|_{L^\infty}
= \|y_0(x) e^{b \int_0^1 (u^m u_x)(\tau, q(\tau, x)) d\tau}\|_{L^\infty}. \tag{51}
\]
Suppose that (24) is not valid. Then there exists a positive number \( M_2 > 0 \) such that
\[
\|y\|_{L^\infty} \leq e^{M_2 T} \|y_0\|_{L^\infty}. \tag{52}
\]
Note that \( u = G * y \) and \( u_x = \partial_x G * y \). From the Young inequality, it is easy to see that
\[
\|u\|_{L^\infty} \leq \|G\|_{L^1} \|y\|_{L^\infty} = \|y\|_{L^\infty} \quad \text{and} \quad \|u_x\|_{L^\infty} \leq \|\partial_x G\|_{L^1} \|y\|_{L^\infty} = \|y\|_{L^\infty},
\]
which combining (52) imply
\[
\|u\|_{L^\infty} + \|u_x\|_{L^\infty} \leq ce^{M_2 T} \|y_0\|_{L^\infty}. \tag{53}
\]
On the other hand, since \( u_0 \in H^s \) and \( s > 5/2 \), by the Sobolev embedding theorem we have
\[
\|y_0\|_{L^\infty} = \|u_0 - u_{0,xx}\|_{L^\infty} \leq c\|u_0\|_{H^s}.
\]
This together with (53) and Theorem 3.4 completes the proof.

**Proof of Theorem 1.5** The result is obtained immediately from Theorem 1.3 and 1.4.

**Proof of Theorem 1.6** Applying the conserved quantities in Corollary 1 and some analysis technique, one easily obtains the global existence results, for the details we refer to [59].
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