Fermion masses on a warped 6D world with the extra 2D sphere

Akira Kokado
Kobe International University, Kobe 658-0032, Japan

Takesi Saito
Department of Physics, Kwansei Gakuin University, Sanda 669-1337, Japan

(Dated: February 5, 2014)

In a warped 6-dimensional world with an extra 2-dimensional surface of a sphere, we find a 4-dimensional fermion mass formula with a zero mode. The warp factor is given by

$$\phi(\theta, \varphi) = \sin \theta \cos \varphi,$$

which is a solution to the 6-dimensional Einstein equation with the bulk cosmological constant $\Lambda$ and the energy-momentum tensor of the bulk matter fields.

PACS numbers: 11.10.Kk, 04.50.-h, 11.25.Mj

I. INTRODUCTION

The 6-dimensional space is particularly interested in unified theories in higher-dimensions. The extra 2-dimensional compact space generates some useful gauge symmetries for various fields[1]-[4].

In this article we specially confine ourselves to the problem of fermionic masses when the extra 2-dimensional surface is a sphere. In this case we encounter a serious theorem that there is no zero mode in a 4-dimensional fermionic field [5][6]. This theorem can be generalized to any internal space with a positive curvature. Since it is desirable that we have a zero mode in unified theories at least at the first symmetric stage, this theorem is unwelcome. To overcome this difficulty it has been considered to introduce a gauge field. When the gauge field has Dirac’s monopole, we get the zero mode of fermionic fields [7].

As another possibility of obtaining fermion zero mode, we consider here the warped 6-dimensional world model with the extra 2-dimensional surface of a sphere. The line element of this model is

$$ds^2 = g_{AB}dx^A dx^B$$

$$= \phi^2(\theta, \varphi)\eta_{\mu\nu}dx^\mu dx^\nu - a^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

where the 4-dimensional metric, $\eta_{\mu\nu}$, has the signature $(+,-,-,-)$, and the extra 2-dimensional surface is a sphere with a constant radius $a$ and the two spherical angles $x^5$, $x^6 = \theta$ and $x^i = \varphi$ ($0 \leq \theta \leq \pi, -\pi/2 \leq \varphi \leq \pi/2$). In the following we assume $a << 1$, in order to make KK modes negligible. The warp factor is given by

$$\phi(\theta, \varphi) = \sin \theta \cos \varphi,$$

which is a solution to Einstein’s equation with the bulk cosmological constant $\Lambda$ and the energy-momentum tensor of the bulk matter fields. We solve the 6D Dirac equation with a 6D zero-mass in this 6D warped background (1.1). We then find a 4-dimensional fermion mass formula with a zero mode.

In Sec II the warp factor (1.2) is derived. In Sec III the Dirac equation in 6D warped background is summarized. In Sec IV the Dirac equation is solved in the method of separable variables to obtain the fermion mass formula. The final section is devoted to concluding remarks. The Appendix is prepared for derivation of boundary parameters.

II. THE WARP FACTOR

The action of the gravitational system in six dimensions can be written as

$$I = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g}(R + 2\Lambda),$$

where $\kappa_6^2 = 8\pi G_N$ is the six dimensional Newton constant and $\Lambda$ is the bulk cosmological constant. When we have a stress-energy tensor $T_{AB}$
in the bulk, Einstein equations become

\[ R_{AB} - \frac{1}{2} g_{AB} R = \kappa_6^2 (\Lambda g_{AB} + T_{AB}) \]  \hspace{1cm} (2.2) \]

Capital Latin indices run over \( A, B, \ldots = 0, 1, 2, 3, 5, 6 \).

We look for solutions of Eq. (2.2) with the ansatz (1.1). The equation (2.1) is a solution to Eq. (2.2) with stress-energy tensor of the bulk matter fields \[1\] \[8\],

\[ T_{\mu\nu} = -\mu_{\nu} E(\theta, \varphi), \quad T_{ij} = -\epsilon_{ij} P(\theta, \varphi), \quad T_{i\mu} = 0 \] \hspace{1cm} (2.3)

\[ E(\theta, \varphi) = \frac{3}{\kappa_6 a^2 \varphi^2(\theta, \varphi)}, \quad P(\theta, \varphi) = \frac{6}{\kappa_6 a^2 \varphi^2(\theta, \varphi)} \] \hspace{1cm} (2.4)

The results are obtained in the following: From Eq. (2.2) we have four equations as

\[ \frac{3}{\varphi^2} \left( \frac{\partial \phi}{\partial \theta} \right)^2 + \frac{3}{\varphi^2} \phi \left( \frac{\partial \varphi}{\partial \theta} \right)^2 + \frac{3 \cos^2 \theta}{\varphi^2} \phi \frac{\partial \phi}{\partial \theta} - \frac{3 \partial^2 \phi}{\varphi^2} + 1 = \kappa_6 a^2 (E - \Lambda), \] \hspace{1cm} (2.5)

\[ \frac{6}{\varphi^2} \sin^2 \theta \left( \frac{\partial \phi}{\partial \varphi} \right)^2 + \frac{4 \phi}{\varphi} \sin^2 \theta \frac{\partial^2 \phi}{\varphi \partial \varphi^2} + \frac{4 \cos \theta \partial \phi}{\varphi \sin \theta \partial \theta} + \frac{6}{\varphi^2} \left( \frac{\partial \phi}{\partial \theta} \right)^2 = \kappa_6 a^2 (P - \Lambda), \] \hspace{1cm} (2.6)

\[ \cos \theta \frac{\partial \phi}{\varphi \sin \theta \partial \varphi} - \frac{1}{\phi \partial \theta} \partial^2 \phi = 0. \] \hspace{1cm} (2.7)

For such a solution that \( \phi(\theta, \varphi) = \Theta(\theta) \Phi(\varphi) \) takes a maximum value 1 at \( \theta = \pi/2 \) and \( \varphi = 0 \), the last equation is immediately solved as

\[ \Theta(\theta) = C \sin \theta. \] \hspace{1cm} (2.8)

Substituting the result into the other three equations above, we get

\[ \kappa_6^2 a^2 = \frac{10}{\Lambda}, \quad \Phi(\varphi) = \cos \varphi, \quad C = 1, \] \hspace{1cm} (2.9)

and Eqs. (2.3).

To sum up we have

\[ \phi(\theta, \varphi) = \sin \theta \cos \varphi, \] \hspace{1cm} (2.10)

\[ T_{AB} = \frac{3 \Lambda}{10} \delta_{ab} \left( -1, 1, 1, \frac{2 a^2}{\varphi^2}, \frac{2 a^2 \sin^2 \theta}{\varphi^2} \right). \] \hspace{1cm} (2.11)

\section{III. THE 6D DIRAC EQUATION}

We now consider the 6-dimensional massless Dirac equation with the metric (1.1):

\[ i b_A^\dagger \Gamma^A D_A \Psi(x^A) = 0, \] \hspace{1cm} (3.1)

where \( D_A \) denote covariant derivatives, \( \Gamma^A \) the 6-dimensional flat gamma matrices and \( b_A^\dagger \) the sechsein through the definition

\[ g_{AB} = \eta_{AB} b_A^\dagger b_B . \] \hspace{1cm} (3.2)

\( \bar{A}, \bar{B}, \ldots \) are local Lorentz indices.

In six dimensions a spinor

\[ \Psi(x^A) = \left( \begin{array}{c} \psi \\ \xi \end{array} \right), \] \hspace{1cm} (3.3)

has eight components and is equivalent to a pair of 4-dimensional Dirac spinors, \( \psi \) and \( \xi \). We use the following representation of the flat (8 \( \times \) 8) gamma-matrices

\[ \Gamma^\mu = \gamma^\mu \otimes 1, \quad \Gamma^\theta = i \gamma_5 \otimes \tau_1, \quad \Gamma^\phi = i \gamma_5 \otimes \tau_2, \] \hspace{1cm} (3.4)

where \( \tau_i \)'s are Pauli matrices. They satisfy

\[ \{ \Gamma^A, \Gamma^B \} = 2 \eta^{AB}. \] \hspace{1cm} (3.5)

The sechsein for our background metric (1.1) is given by

\[ b_A^\dagger = \left( \frac{1}{a} \delta^A_\mu, \frac{1}{a} \delta^A_\theta, \frac{1}{a \sin \theta} \delta^A_\phi \right). \] \hspace{1cm} (3.6)

From the definition of standard spin-connections the non-vanishing components for them can be found

\[ \omega^\theta_\mu = \frac{1}{a} \cos \theta \cos \varphi, \quad \omega^\phi_\mu = -\frac{1}{a} \sin \varphi, \] \hspace{1cm} (3.7)

\[ \omega^\phi_\theta = \cos \theta. \]
The Dirac equation (3.1) then reduces to
\[
\left[ \frac{1}{\phi} \Gamma^\mu \partial_\mu + \frac{1}{a} \Gamma^\theta (\partial_\theta + \frac{1}{2} \cot \theta - \sin \theta \frac{\phi}{\phi'}) \right. \\
+ \frac{1}{a \sin \theta} \Gamma^\varphi \partial_\varphi \right] \Psi(x^A) = 0 .
\] (3.8)

If we put
\[
\Psi(x^A) = \frac{1}{\phi^2 \sin \theta} \tilde{\Psi}(x^\mu, \theta, \varphi) ,
\] (3.9)
it follows that
\[
\left( i a \phi^{-1} \gamma^\mu \otimes 1 \partial_\mu - \gamma_5 \otimes \hat{\nabla} \right) \tilde{\Psi}(x^\mu, \theta, \varphi) = 0 ,
\] (3.10)
where
\[
\hat{\nabla} \equiv (\tau_1 \partial_\theta + \tau_2 \frac{1}{\sin \theta} \partial_\varphi) .
\] (3.11)

Let us expand \( \tilde{\Psi}(x^\mu, \theta, \varphi) \) into eigenfunctions of the chiral operator \( \gamma_5 \) as follows:
\[
\tilde{\Psi}(x^\mu, \theta, \varphi) = \psi_R(x^\mu)f_+(\theta, \varphi) + \psi_L(x^\mu)f_-(\theta, \varphi) ,
\] (3.12)
where \( \gamma_5 \psi_R(x^\mu) = \psi_R(x^\mu) \) and \( \gamma_5 \psi_L(x^\mu) = -\psi_L(x^\mu) \). If we use the Dirac equation with mass \( m \) in 4-dimension,
\[
iv^\mu \partial_\mu \psi = m\psi ,
\] (3.13)
or equivalently
\[
iv^\mu \partial_\mu \psi_L = m\psi_R ,
\] (3.14)
\[
iv^\mu \partial_\mu \psi_R = m\psi_L ,
\]
it follows that
\[
\hat{\nabla} f_\pm(\theta, \varphi) = \pm \frac{ma}{\phi} f_\pm(\theta, \varphi) .
\] (3.15)
These equations reduce to
\[
- i \hat{\nabla} g_\pm(\theta, \varphi) = \pm \frac{ma}{\phi} g_\pm(\theta, \varphi) ,
\] (3.16)
\[
g_\pm = \frac{f_+ \mp if_-}{2} .
\]

Here it should be noted that there is no zero mode \((m = 0)\) for the 4D Dirac field \( \psi(x^\mu) \), when \( \phi \equiv 1 \). This is because the Dirac operator \( \hat{\nabla} \) is just the Dirac operator on the sphere, its eigenvalues are known to be non-zero
\[
- i \hat{\nabla} g_\pm(\theta, \varphi) = \lambda g_\pm(\theta, \varphi) ,
\] (3.17)
with \( \lambda = \pm 1, \pm 2, \cdots \). Here \( g_\pm(\theta, \varphi) \) is the spinor on the sphere. However, this is not always clear in the case of the 6D warped space. In the next section we would like to see how to fix the fermion mass.

IV. FERMION ON THE EXTRA 2D SPHERE

We would like to solve Eqs. (3.16), i.e.,
\[
- i \left( \tau_1 \partial_\theta + \tau_2 \frac{1}{\sin \theta} \partial_\varphi \right) g_\pm(\theta, \varphi)
\]
\[
= \pm \frac{ma}{\sin \theta \cos \varphi} g_\pm(\theta, \varphi) .
\] (4.1)

This is a separable type of variables, so that we put
\[
g_\pm(\theta, \varphi) = \alpha_\pm(\theta) \left( \begin{array}{c} u_\pm(\varphi) \\ v_\pm(\varphi) \end{array} \right) ,
\] (4.2)
to yield
\[
\partial_\theta \alpha_\pm(\theta) = \frac{C_\pm}{\sin \theta} \alpha_\pm(\theta) ,
\] (4.3)
and
\[
(\partial_\varphi - i C_\pm) u_\pm(\varphi) = \pm \frac{ma}{\cos \theta} v_\pm(\varphi) ,
\] (4.4)
\[
(\partial_\varphi + i C_\pm) v_\pm(\varphi) = \mp \frac{ma}{\cos \theta} u_\pm(\varphi) ,
\] (4.5)
where \( C_\pm \) are arbitrary constants.

The first equation (4.3) can be solved with a constant \( \alpha_\pm \) as
\[
\alpha_\pm(\theta) = \alpha_\pm \tan C_\pm(\theta/a, 2) .
\] (4.6)

From coupled equations (4.4) and (4.5) we have equations of the Schrödinger type,
\[
\left[ \partial_\varphi^2 - \tan \varphi \partial_\varphi + i C_\pm \tan \varphi + C_\pm^2 + \frac{ma^2}{\cos^2 \varphi} \right] u_\pm(\varphi)
\]
\[
= 0 .
\] (4.7)
\( u_\pm(\varphi) \) are given by solutions \( u_\pm(\varphi) \) with use of Eqs. (4.4).
By using a variable \( \zeta = (1 + i \tan \varphi)/2 \), the general solution is given by

\[
u(z) = e^{ic\varphi} \left[ AF(i\alpha, -i\alpha, C + \frac{1}{2}; \zeta) + B\zeta^{1-c} \times F(i\alpha + \frac{1}{2} - C, -i\alpha + \frac{1}{2} - C; \zeta) \right]
\]

where \( F \) is the Gauss series and \( A \) and \( B \) are arbitrary constants. Here we have dropped suffixes \( \pm \).

Let us now consider the behavior of the solution near \( \varphi = \pm \pi/2 \). If we put \( \varphi = \pm \pi/2 \mp \varepsilon \) with a small positive \( \varepsilon \), we have \( \tan \varphi \sim \pm (1/\varepsilon) \) and \( \cos \varphi \sim \varepsilon \). In this region near \( \varphi = \pm \pi/2 \), Eq. (4.9) reduces to

\[
\left[ \partial_{2}^{2} - \tan \varphi \partial_{\varphi} + \frac{m^{2}a^{2}}{\cos^{2} \varphi} \right] u_{\pm}(\varphi) = 0 .
\]

Near \( \varphi = \pm \pi/2 \) it may be simpler to consider solutions of these equations rather than the original exact solutions. General solutions of Eq. (4.9) are given by

\[
u_{\pm} = A_{\pm} \exp[\pm ikw(\varphi)] + B_{\pm} \exp[-\pm ikw(\varphi)],
\]

where

\[
w(\varphi) = \frac{1}{2} \ln \frac{1 + \sin \varphi}{1 - \sin \varphi} = -w(-\varphi),
\]

\[
k = ma .
\]

Note

\[
w(\varphi) \simeq \begin{cases} 
- \ln \varepsilon > 0 & \text{for } \varphi = \frac{\pi}{2} - \varepsilon , \\
\ln \varepsilon < 0 & \text{for } \varphi = -\frac{\pi}{2} + \varepsilon .
\end{cases}
\]

For \( v_{\pm} \) we get

\[
\partial_{\varphi} v_{\pm} = \pm kv_{\pm} .
\]

According to the formula of the warp factor \( \phi(\theta, \varphi) = \sin \theta \cos \varphi \), the 4D metric becomes zero and some components of the energy-momentum tensors (2.11) are divergent at boundaries \( \theta = 0, \pi \) and \( \varphi = \pm \pi/2 \).

We now consider the conserved current of fermion,

\[
J^{A} = \bar{\psi} \Gamma^{A} \psi ,
\]

which satisfies the continuity equation

\[
D_{A} J^{A} = 0 ,
\]

This equation reduces to

\[
\phi^{-1} \partial_{\mu} (\bar{\psi} \Gamma^{\mu} \psi) + a^{-1} \partial_{\theta} (\bar{\psi} \Gamma^{\theta} \psi)
\]

\[
+ a^{-1} \sin^{-1} \theta \partial_{\varphi} (\bar{\psi} \Gamma^{\varphi} \psi) = 0 .
\]

The first 4-dimensional part vanishes because of the 4D field equation. As boundary conditions near \( \theta = 0, \pi \) and \( \varphi = \pm \pi/2 \), we impose

\[
\partial_{\theta} (\bar{\psi} \Gamma^{\theta} \psi) = 0 , \quad \text{at } \theta = \eta, \pi - \eta \quad (4.17)
\]

\[
\partial_{\varphi} (\bar{\psi} \Gamma^{\varphi} \psi) = 0 , \quad \text{at } \varphi = \pm (\pi/2 - \varepsilon) \quad (4.18)
\]

where small quantities \( \varepsilon \) and \( \eta \) will be fixed later.

For the first boundary condition (4.17), we have

\[
\partial_{\theta} \left[ \bar{\psi} (x^{\mu}, \theta, \varphi) \Gamma^{\theta} \psi (x^{\mu}, \theta, \varphi) \right]
\]

\[
eq \partial_{\theta} \left[ - \bar{\psi}_{R} \psi_{L} \cdot \hat{f} \Gamma_{1} f_{-} + h.c. \right] = 0 . \quad (4.19)
\]

Since

\[
f_{+} = g_{+} + g_{-} = \left( \alpha_{+} u_{+} + \alpha_{-} u_{-} / \alpha_{+} v_{+} + \alpha_{-} v_{-} \right) ,
\]

\[
f_{-} = i(g_{+} - g_{-}) = i \left( \alpha_{+} u_{+} / \alpha_{+} v_{+} - \alpha_{-} u_{-} / \alpha_{-} v_{-} \right) ,
\]

we get

\[
\hat{f} \Gamma_{1} f_{-} =
\]

\[
- 2|\alpha_{+}(\theta)|^{2} \left[ A_{+} B_{+} e^{2ikw - c.c.} \right] + 2|\alpha_{-}(\theta)|^{2} \left[ A_{-} B_{-} e^{2ikw - c.c.} \right] - 2 \left[ \alpha_{+}(\theta) \alpha_{+}(\theta) (A_{+} A_{+} - B_{+} B_{+}) - c.c. \right] ,
\]

(4.22)

where

\[
|\alpha_{+}(\theta)|^{2} = |\alpha_{\pm}|^{2} \tan^{2}(\tilde{C}_{\pm}^{+} + \tilde{C}_{\mp}^{+})(\theta/2) ,
\]

\[
\alpha_{\pm}(\theta) = \alpha_{\mp}(\theta) \tan^{2}(\tilde{C}_{\pm}^{+} + \tilde{C}_{\mp}^{+})(\theta/2) .
\]

Hence we see that Eq. (4.22) is independent of \( \theta \) if \( \tilde{C}_{\pm}^{+} + \tilde{C}_{\mp}^{+} = \tilde{C}_{\pm}^{+} + \tilde{C}_{\mp}^{+} = 0 \), that is, \( C_{\pm} = C_{-} \equiv C = -C^{*} \). This means that \( |\alpha_{\pm}(\theta)|^{2} \) and \( \alpha_{\pm}(\theta) \alpha_{\mp}(\theta) \) are all constants, even at \( \theta = \eta, \pi - \eta \).
For the second boundary condition, similarly we have

\[ \partial_\varphi (\bar{\psi} \Gamma^\varphi \psi) = \left( \bar{\psi}_R \psi_L - \bar{\psi}_L \psi_R \right) \frac{4ik}{\cos \varphi} \left[ Ke^{2ikw} - \text{c.c.} \right] = 0 , \]

(4.23)

where

\[ K \equiv \alpha_+ \alpha^*_+ A_+ B^*_+ + \alpha^*_+ \alpha_+ A_+ B^*_+ . \]

(4.24)

From Eqs. (4.23) and (4.12) we have

\[ (K - K^*) k \cos (2k \ln \varepsilon) = 0 , \]

(4.25)

\[ (K + K^*) k \sin (2k \ln \varepsilon) = 0 . \]

(4.26)

The solutions to these equations are

\[ k = ma = \frac{n\pi}{2 |\ln \varepsilon|} , \quad \text{when } K = K^* \]

(4.27)

or

\[ k = ma = \frac{(n + \frac{1}{2})\pi}{2 |\ln \varepsilon|} , \quad \text{and } k = 0 , \]

(4.28)

when \( K = -K^* \)

where \( n = 0, 1, 2, \cdots \). When \( K \) is a general complex number, the possible solution is only \( m = 0 \) mode.

Here, the boundary parameters \( \varepsilon \) and \( \eta \) are given by, in the Appendix,

\[ |\varphi| \leq \frac{\pi}{2} - \varepsilon , \quad \eta \leq \theta \leq \pi - \eta , \]

(4.29)

\[ \varepsilon = a^\frac{3}{2} , \quad \eta = a^\frac{3}{2} , \quad a^3 |\ln a| >> \kappa_6^2/6 . \]

Substituting \( \varepsilon = a^{2/3} \) into Eqs. (4.27) and (4.28) we have the 4D fermion mass formulas

\[ m = \frac{3}{4a |\ln a|} n\pi , \quad (n = 0, 1, 2, \cdots) \]

(4.30)

for \( K \) a real number,

\[ m = \frac{3}{4a |\ln a|} (n + \frac{1}{2})\pi , \quad m = 0 \]

(4.31)

for \( K \) a pure imaginary number, and

\[ m = 0 , \]

(4.32)

for \( K \) a general complex number.

The above inequalities come from the inequality

\[ |T^{(b)}_{AB}| >> |T^{(f)}_{AB}| , \]

(4.33)

where \( T^{(b)}_{AB} \) is the bulk energy-momentum tensor given by Eq. (4.111), while \( T^{(f)}_{AB} \) is the fermion energy-momentum tensor. If the above inequality holds, the back reaction from the fermion may be neglected. The detail will be discussed in the Appendix.

V. CONCLUDING REMARKS

In the 6D warped world model with the extra 2D surface of a sphere, we have derived mass formulas for 4D fermion, and have shown that they include the zero mode.

The mass formulas have come from the boundary conditions and. Here the boundaries of \( \theta \) and \( \varphi \) are so defined by Eqs. (4.29) from the condition that the back reaction from the fermion should be neglected. The \( \varepsilon \) and \( \eta \) are given by \( \varepsilon = a^{2/3} \) and \( \eta = a^{1/2} \), where \( a \) is the radius of the extra dimension with constraint \( a^3 |\ln a| >> \kappa_6^2/6 \). Though the \( a \) is assumed to be very small \( a << 1 \), but it can not tend completely to zero because of this constraint.

As the boundary condition we adopted we make use of the continuity equation of the current \( J^A \), which should hold even at boundaries. Hence the most simple boundary conditions are to put as Eqs. (4.17) and (4.18), separately. These conditions seem to be very weak forms, comparing with other forms of boundary conditions for each wave function of \( f_{\pm} \)’s.

The warp factor is given by \( \phi(\theta, \varphi) = \sin \theta \cos \varphi \), which is a solution to Einstein’s equation with the bulk cosmological constant \( \Lambda \) and the energy-momentum tensor of the bulk matter fields.

Our model provides another possibility of obtaining fermion zero mode, rather than traditional model based on Dirac’s monopole.
Acknowledgments

We would like to express our deep gratitude to T. Okamura for many valuable discussions.

Appendix A: Boundary parameters

The fermion energy-momentum tensor is given by

\[ T_{AB}^{(f)} = i\bar{\psi}(\Gamma_A D_B + \Gamma_B D_A)\psi - g_{AB}i\bar{\psi}\Gamma^M D_M \psi, \]  
(A1)

where the second term becomes zero for the 6D massless fermion. Substituting \( \psi = (\phi^2 \sin^{1/2} \theta)^{-1}\psi \) into Eq.(A.1) we have for the \( \mu\nu \) component

\[ T_{\mu\nu}^{(f)} = \frac{i}{\phi^3 \sin \theta} [\bar{\psi}(\Gamma_{\mu} \partial_{\nu} + \Gamma_{\nu} \partial_{\mu})\psi] \]  
(A2)

\[ = \frac{i}{\phi^3 \sin \theta} [\bar{\psi}_R(\gamma_{\mu} \partial_{\nu} + \gamma_{\nu} \partial_{\mu})\psi_R \cdot f_{\mu}^R f_+ + \bar{\psi}_L(\gamma_{\mu} \partial_{\nu} + \gamma_{\nu} \partial_{\mu})\psi_L \cdot f_{\mu}^L f_-] \]  
(A3, A4)

where

\[ t_{\mu\nu}^R = i\bar{\psi}_R(\gamma_{\mu} \partial_{\nu} + \gamma_{\nu} \partial_{\mu})\psi_R, \]

and

\[ t_{\mu\nu}^L = i\bar{\psi}_L(\gamma_{\mu} \partial_{\nu} + \gamma_{\nu} \partial_{\mu})\psi_L. \]

Let us put

\[ C_{\mu\nu}^{(f)} = t_{\mu\nu}^R f_{\mu}^R f_+ + t_{\mu\nu}^L f_{\mu}^L f_- , \]  
(A7)

then from the inequality \( 10 \) we are enough to check only for diagonal parts

\[ \frac{3\Lambda}{10} \gg \frac{1}{|\phi^3 \sin \theta|} |C_{\mu\nu}^{(f)}|, \]  
(A8)

which reduces to

\[ |\cos^3 \varphi| \geq |\sin^4 \theta \cos^3 \varphi| \gg \frac{10 |C_{\mu\nu}^{(f)}|}{3\Lambda}. \]  
(A9)

Since the 4D energy-momentum tensor element \( C_{\mu\nu}^{(f)} \) may be extremely smaller than the 6D Planck mass, i.e.

\[ |C_{\mu\nu}^{(f)}| < \frac{1}{\kappa_6^2}, \]  
(A10)

we have

\[ \frac{10 |C_{\mu\nu}^{(f)}|}{3\Lambda} < \frac{10}{3\Lambda\kappa_6^2} \equiv \varepsilon^3, \]  
(A11)

by using Eq.(2.9). Hence we get inequalities

\[ |\cos^3 \varphi| \geq \varepsilon^3 >> \frac{10 |C_{\mu\nu}^{(f)}|}{3\Lambda}, \]  
(A12)

\[ |\sin^4 \theta| \geq \varepsilon^3 >> \frac{10 |C_{\mu\nu}^{(f)}|}{3\Lambda}, \]

hence

\[ |\varphi| \leq \frac{\pi}{2} - \varepsilon, \quad \eta \leq \theta \leq \pi - \eta, \]  
(A13)

\[ \varepsilon = a^{1/3}, \quad \eta = a^{1/2}. \]

Here we have dropped the factor 1/3 in \( \varepsilon^3 \), since such a factor is negligible in |\ln \varepsilon| in the mass formulas \( \{A2\} \) and \( \{A28\} \). We have defined the boundaries of \( \varphi \) and \( \theta \) by \( \varepsilon \) and \( \eta \).

For another component we have

\[ T_{55}^{(f)} = \frac{a}{\phi^3 \sin \theta} [\bar{\psi}(i\gamma_5 \otimes \tau_1 \partial_0)\psi] \]  
(A14)

\[ = \frac{ia}{\phi^4 \sin \theta} \left[ \bar{\psi}_R \psi_L f_{\mu}^R f_+ + \bar{\psi}_L \psi_R f_{\mu}^L f_- \right] \]

\[ = \frac{i a C}{\phi^4 \sin \theta} D, \]

where \( D \equiv (N_4 f_{\mu}^L f_- - N_4 f_{\mu}^R f_+) \) and \( N_4 \equiv \bar{\psi}_R \psi_L \cdot |CD| \) may be \( O(1) \) from the normalization condition of 4D fermion and finiteness of \( f^R f \) terms. Hence the inequality becomes

\[ \frac{6a^2 \Lambda}{10\phi^2} >> \frac{a|CD|}{\phi^4 \sin^2 \theta} \approx \frac{a}{\phi^4 \sin^2 \theta}, \]  
(A15)

then

\[ \sin^4 \theta \cos^2 \varphi >> \frac{10}{6a\Lambda} = \frac{a\kappa_6^2}{6}. \]  
(A16)

Since the lower bounds of \( \cos^2 \varphi \) and \( \sin^4 \theta \) are given by \( a^{1/3} \) and \( a^2 \), respectively, we get new inequalities for \( a \),

\[ a^{1/3} >> \kappa_6^2. \]  
(A17)
For the 66 component we get

$$T_{66}^{(f)} = \frac{ia}{\phi^2} \left[ - (N_4 f_+^\dagger \tau_1 f_- - N_4 f_-^\dagger \tau_1 f_+) \right]$$

$$+ \frac{ma}{\cos \varphi} (N_4 |f_+|^2 - N_4^\dagger |f_-|^2) \right]. \quad (A18)$$

Since the second term dominates near $\varphi = \pi/2$, the inequality becomes

$$6a^2 \Lambda \sin^2 \theta \cos^2 \phi > \frac{ma^2 |E|}{\phi^2} \theta \cos \phi, \quad (A19)$$

where $E \equiv N_4 |f_+|^2 - N_4^\dagger |f_-|^2 \approx O(1)$. This reduces to

$$\phi^2 \sin^2 \theta \cos \varphi >> \frac{10m|E|}{6\Lambda}. \quad (A20)$$

Hence we have

$$\sin^2 \theta \cos \varphi >> \frac{10m|E|}{6\Lambda} = \frac{\kappa_6^2 a^2 3n|E|}{6a \ln \frac{\ln a}{a}} \approx \frac{a\kappa_6^2 / 6}{\ln \frac{\ln a}{a}} \quad (A21)$$

that is,

$$a^3 \ln a >> \frac{\kappa_6^2}{6}, \quad (A22)$$

Since $a^{7/3} > a^3 \ln a$, Eq. (A17) is covered by Eq. (A22).

To sum up, regions of $\varphi$ and $\theta$ are given by

$$|\varphi| \leq \frac{\pi}{2} - \varepsilon, \quad \eta \leq \theta \leq \pi - \eta, \quad (A23)$$

$$\varepsilon = a^2, \quad \eta = a^2, \quad \frac{\ln a}{a}$$

together with the constraint (A22).