On the Existence of New Conservation Laws for the Spaces of Different Curvatures

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Abstract

It is known that corresponding to each isometry there exist a conserved quantity. It is also known that the Lagrangian of the line element of a space is conserved. Here we investigate the possibility of the existence of “new” conserved quantities, i.e. other than the Lagrangian and associated with the isometries, for spaces of different curvatures. It is found that there exist new conserved quantities only for the spaces of zero curvature or having a section of zero curvature.

Introduction

Sophus Lie investigated the role of transformation theory in classical integration methods. He studied continuous transformations that depend on parameter(s)

\[ s^* = s^* (s, x^i; \varepsilon), \quad x^{i*} = x^{i*} (s, x^i; \varepsilon), \]

(1)
and satisfy the axioms of a group and, hence, form a group of point transformations. Then he, with the help of these transformations (called the symmetry transformations or symmetries), provided different integration strategies for differential equations. An important class of these symmetries which leave the action principle invariant is that of the Noether symmetries, i.e. $X$ is defined as a Noether symmetry if

$$X \int L(s, x^i, \dot{x}^i) \, ds = 0; \ i = 1, 2, \ldots, n$$

(2)

where $L$ is the Lagrangian, $x^i = (x^1, x^2, x^3, \ldots, x^n)$ is a point in the $n$-dimensional underlying manifold, $\dot{x}^i$ is the derivative of $x^i$ with respect to $s$ and

$$X = \xi \left(s, x^i, \dot{x}^i\right) \frac{\partial}{\partial s} + \eta^a \left(s, x^i, \dot{x}^i\right) \frac{\partial}{\partial x^a}, \ a, i = 1, 2, \ldots, n$$

(3)

where

$$\xi \left(s, x^i\right) = \frac{\partial s^a}{\partial \varepsilon} \bigg|_{\varepsilon = 0}, \ \eta^0 \left(s, x^i\right) = \frac{\partial x^a}{\partial \varepsilon} \bigg|_{\varepsilon = 0}, \ \dot{\eta}^a \left(s, x^i, \dot{x}^i\right) = \frac{d\eta^a}{ds} - \dot{x}^a \frac{d\xi}{ds}.$$  

For some gauge function $A(s, x^i)$ eq.(2) can equivalently be given as

$$XL + L \frac{d}{ds} \xi = \frac{d}{ds} A,$$

(4)

where

$$\frac{d}{ds} = \frac{\partial}{\partial s} + \dot{x}^a \frac{\partial}{\partial x^a}.$$  

For each Noether symmetry one can reduce the equation of motion (Euler-Lagrange equation) in order by two [1]. Also, corresponding to each symmetry a conservation law is associated [2], and a conserved quantity, $T$, is obtained as [1]

$$T = \xi (\dot{x}^a L_{\dot{x}^a} - L) - \eta^a L_{\dot{x}^a} + A(s, x^i).$$

(5)

In Differential Geometry the usual Lagrangian for the line element

$$ds^2 = g_{ab} \left(x^c\right) dx^a dx^b, \ a, b, c = 1, 2, \ldots, n$$

(6)
One may find isometries (functions that preserve the distance between two points on a manifold) for the metric given by eq. (6). In the context of isometries a lot of work has been done [3,4]. One can associate geometrically conserved quantities to isometries. One may also find the Noether symmetries for the action of the line element. Recently, Bokhari et al find the Noether symmetries associated with the usual Lagrangian of different line elements [5]. These Noether symmetries may provide new conserved quantities. The aim of this paper is to generalize the idea and obtain general result for the existence of new conserved quantities.

Existence of New Conservation Laws

Before discussing the existence of conservation laws we first prove the following

**Theorem 1:**

\[ \mathbf{X} = s \frac{\partial}{\partial s} \]  

is not a Noether symmetry generator associated with the Lagrangian for the line element.

**Proof:** For \( \mathbf{X} \) to be a Noether symmetry it must satisfy eq. (4). But then

\[ \frac{d}{ds} A = -L (s, x^a, \dot{x}^a), \]  

or

\[ \left( \frac{\partial}{\partial s} + \dot{x}^i \frac{\partial}{\partial x^i} \right) A (s, x^c) = -g_{ab} (x^c) \dot{x}^a \dot{x}^b. \]
which is not possible as the left hand side can not be quadratic in velocities. So it is not a Noether symmetry associated with the Lagrangian for the line element.

Now, a result on the existence of new conservation law(s) is given in the form of the following

**Theorem 2:** (i) There exist new conservation laws for the spaces of zero curvature or for the spaces having a section of zero curvature; (ii) There does not exist new conservation laws for a space of non-zero curvature and not having a section of zero curvature.

Proof: For the proof of this theorem two already known results are required. The first result is that the algebra of the Noether symmetries form a subalgebra of the symmetries of the differential equations (geodesic equations or equations of motion) [1]. The second result is that for a space of non-zero curvature with isometry algebra \( h \), the symmetry algebra of the geodesic equations is \( h \oplus d_2 \) provided that there is no section of zero curvature; if there is an m-dimensional section of zero curvature and the isometry algebra for the orthogonal space is \( h_1 \), then the symmetry algebra of the geodesic equations is \( h_1 \oplus sl(m+2) \) [6], where \( d_2 \) is a 2-dimensional dilation algebra with basis

\[
X_0 = \frac{\partial}{\partial s}, \quad X_1 = s \frac{\partial}{\partial s}. \quad (11)
\]

(i) For the n-dimensional spaces of zero curvature the isometry algebra is \( so(n) \otimes_s \mathbb{R}^n \), where \( \otimes_s \) is the semi direct product, and the symmetries of the geodesic equations form \( sl(n+2, \mathbb{R}) \). Let two sets of symmetries forming the isometry algebra and algebra of the symmetries of the geodesic equations be S and I. Since the Noether symmetries form a
subalgebra of the symmetries of the geodesic equations, therefore, for the spaces with zero curvature the new conservation laws are associated with the Noether symmetries spanned by the space $S - I$. The conserved quantity associated with $X_0$, which is also an element of $sl(n + 2, \mathbb{R})$, is the Lagrangian. So, finally, the new conservation laws can be obtained from the space $S - I - X_0$.

Similarly, for the spaces of non-zero curvature with an $m$-dimensional section of zero curvature admitting the isometry algebra $h_1 \oplus [so(n) \otimes \mathbb{R}^m]$ and the symmetry algebra of the geodesic equations $h_1 \oplus sl(m + 2)$, the space spanned by $S - I - X_0$ may give new conservation laws.

(ii) For the spaces of non-zero curvature there are only two additional symmetries of the geodesic equations given in eq. (11), other than isometries.

Since the Noether symmetries form a subalgebra of the symmetries of the geodesic equations, therefore, the Noether symmetries can not be other than $h \oplus d_2$. As $h$ are the isometries, $X_0$ is associated with the Lagrangian, $L$, and $X_1$ is not a Noether symmetry by theorem 1. Therefore, there does not exist new conservation law for the spaces of non-zero curvature with no section of zero curvature. This completes the proof of theorem 2.

**Example 1:** Consider Bertotti - Robinson like metric [4] (having 2-dimensional section of zero curvature)

$$ds^2 = \cosh^2 \left( \frac{x}{a} \right) dt^2 - dx^2 - (dy^2 + dz^2),$$  \hspace{1cm} (12)
and to determine Noether symmetries use

\[ L = \cosh^2 \left( \frac{x}{a} \right) \dot{t}^2 - \dot{x}^2 - (y^2 + z^2). \]

This metric admits 9 Noether symmetries which are

\[ \begin{align*}
X_0 &= \partial / \partial t, & X_1 &= z \partial / \partial y - y \partial / \partial z, \\
X_2 &= \partial / \partial z, & X_3 &= \partial / \partial y, \\
X_4 &= -\frac{1}{a} \tanh \frac{x}{a} \sin \frac{t}{a} \partial / \partial t + \cos \frac{t}{a} \partial / \partial x, \\
X_5 &= \tanh \frac{x}{a} \cos \frac{t}{a} \partial / \partial t + \sin \frac{t}{a} \partial / \partial x, \\
X_6 &= \partial / \partial s, \\
X_7 &= s \partial / \partial y \quad \text{with gauge term } A = 2y \\
X_8 &= s \partial / \partial z \quad \text{with gauge term } A = 2z
\end{align*} \]

The first six symmetries are the isometries of the spacetime and \( X_6 \) corresponds to the Lagrangian. The new conserved quantities corresponding to \( X_7 \) and \( X_8 \) are \( T = s \dot{y} - y \) and \( T = s \dot{z} - z \) respectively.

**Example 2:** The Lagrangian for the line element (having zero curvature everywhere)

\[ ds^2 = dx^2 + dy^2 + dz^2, \]

is

\[ L = \dot{x}^2 + \dot{y}^2 + \dot{z}^2. \]

The Noether symmetries are as follows

\[ \begin{align*}
X_0 &= \partial / \partial x, & X_1 &= \partial / \partial y, & X_2 &= \partial / \partial z,
\end{align*} \]
\[ X_3 = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}, \]
\[ X_4 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \]
\[ X_5 = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \]
\[ X_6 = \frac{\partial}{\partial s}, \]
\[ X_7 = s^2 \frac{\partial}{\partial s} + sx \frac{\partial}{\partial x} + sy \frac{\partial}{\partial y} + sz \frac{\partial}{\partial z}, \text{ with gauge term } A = (x^2 + y^2 + z^2) \]
\[ X_8 = 2s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}, \text{ with gauge term } A = 0 \]
\[ X_9 = s \frac{\partial}{\partial x}, \text{ with gauge term } A = 2x \]
\[ X_{10} = s \frac{\partial}{\partial y}, \text{ with gauge term } A = 2y \]
\[ X_{11} = s \frac{\partial}{\partial z}, \text{ with gauge term } A = 2z \]

The first six symmetries are the isometries of the space and \( X_6 \) corresponds to the Lagrangian. The new conserved quantities corresponding to \( X_7, X_8, X_9, X_{10} \) and \( X_{11} \) are \( T = s^2L - 2s(x\ddot{x} + y\ddot{y} + z\ddot{z}) + (x^2 + y^2 + z^2), \)
\[ T = sL - (x\ddot{x} + y\ddot{y} + z\ddot{z}), \]
\[ T = s\dot{x} - x, \]
\[ T = s\dot{y} - y \text{ and } T = s\dot{z} - z \] respectively.

**Conclusion**

In this paper the new conserved quantities for the spaces of different curvatures are discussed with the help of method introduced by Lie. It is shown that for the spaces of non-zero curvature there does not exist any conservation law other than the Lagrangian and those associated with isometries. However, new conserved quantities exist for the spaces of zero curvature or the spaces having a section of zero curvature. It is worth mentioning here that an n-dimensional space of zero curvature admits a total number of \( \frac{n(n-1)}{2} + 2n + 3 = \frac{n^2 + 3n + 6}{2} \) symmetries. Among which \( \frac{n(n-1)}{2} + n \) (rotation
and translations) are the isometries and one corresponds to the Lagrangian. And the remaining \( n + 2 \) correspond to the new conserved quantities. Also, for the spaces with an \( m \)-dimensional section of zero curvature we have the following

**Conjecture:** *The spaces with an \( m \)-dimensional section of zero curvature admit \( m \) new conserved quantities and the corresponding Noether symmetries have the form*

\( s \frac{\partial}{\partial x^i}, \; i = 1, 2, \ldots, m. \)

**Acknowledgments**

The author is grateful to Professor Azad A. Siddiqui and Dr. Khalid Saifullah for their valuable suggestions.

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