Ergodicity variation in a long range interacting one-dimensional Ising spin system subject to a time-varying magnetic field

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Abstract. We consider one dimensional Ising spin system in a transverse uniform time-dependent magnetic field. The asymptotic behavior of the bipartite entanglements between the terminal spin and each one of the other spins along the chain is investigated and compared at different spin-spin interaction ranges, from nearest neighbor to infinite long range, under the separate action of two different magnetic fields, constant and time-varying. We find that each of the nearest neighbor and next to nearest neighbor bipartite entanglements reach an asymptotic final state that is independent of the initial condition or the variation in the interaction range showing perfect ergodic behavior at quite short interaction ranges. However, the nearest neighbor entanglement maintains this behavior at a slightly longer ranges. The other bipartite entanglements assume a zero value within these interaction ranges. At intermediate short and long interaction ranges, the asymptotic states of all entanglements become strongly dependent on the initial state and the interaction range, deviating from the ergodic behavior observed before. The maximum asymptotic entanglement attainable between a pair of spins takes place at a long interaction range value that increases with the distance between the spins. At the infinite long range interaction, the dynamics of all bipartite entanglements coincide. Great care should be taken in constructing both.

1. Introduction
Investigating the dynamics of quantum many body system is crucial for understanding plenty of physical systems of interest. The short range interaction between the constituents of many body systems has been in the focus of interest for decades. This is due to the fact that traditional systems were mainly interacting through coulomb potentials that are screened by the existence of charges, which substantially limits the coupling range. The recent groundbreaking in engineering new many body systems such as the ultra-cold atoms, ions and polar molecules trapped in optical lattices that are tunable using electromagnetic radiation sparked great interest in studying long range interactions, as they all enjoy \([1, 2, 3, 4, 5]\). Particularly, a great progress was achieved in isolating and controlling trapped ions and ultracold Rydberg atoms, where the individual particles are addressable and the inter-particle interactions are tunable all the way from the short to infinite range \([6, 7, 8]\). These systems were customized to yield effective one and three dimensional Heisenberg spin 1/2 chains and lattices. The speed with which information and quantum
The correlation between the different parts of a short range interacting many body system is propagating is limited by the Lieb-Robinson bounds that lead to an effective light cone [9]. Nevertheless, in systems with long range interaction these bounds are not valid anymore and were found to be exceeded [6]. One of the most important dynamical properties of many body systems that significantly varies with the interaction range is ergodicity. When the value of a physical observable is calculated, according to statistical physics consideration, the average of that observable over an infinite number of copies (Gibbs ensemble) of the same system is evaluated to yield an ensemble average of the observable. If this ensemble average coincides with the long-time average of a single copy the system, the system is said to be ergodic [10]. The main idea is that as the physical system evolves for a long time it has to visit all accessible states in the allowed phase space, which assures the two averages to be equal. Systems with short range interaction enjoy convexity, due to their additivity property, in the allowed space of their extensive thermodynamic variables and therefore any of these variables values are accessible within this space. In contrary, systems with long range interaction lack this additivity, as a result the extensive observables of such system may not be able to access the whole spectrum of values. In other words, some subspaces of the phase space become inaccessible for the system because of an opening energy gap due to disconnectivity in the allowed phase space.

These characteristics may lead to a broken ergodicity in these long range interacting systems. Studying ergodicity in many body systems with long range interaction was brought to the focus of interest due to the inspiring experimental developments in this type of systems [11, 12, 13, 14]. In a previous work, we have showed how the entanglement dynamics and ergodicity can be tuned in a short range interacting spin system using impurities and coupling anisotropy [15]. In this paper, we study a one-dimensional long-range interacting spin system under the effect of a time-dependent external magnetic field. We show how the non-equilibrium dynamics and its relaxation behavior, especially ergodicity, are significantly affected by the interaction range. Particularly, we demonstrate how the bipartite entanglements between the terminals spin and its nearest neighbor and next to nearest neighbor behave ergodically at quite short interaction ranges, whereas all the other bipartite entanglements vanish at these ranges at all times. Furthermore, the effect of long range interaction on all bipartite entanglements in the system is explored to highlight how it forces the system to a nonergodic behavior. Also, the amount of entanglement contained in the asymptotic states of the system and its dependence on the distance between the spins and the interaction range is discussed. This paper is organized as follows. In the next section, we introduce the model and calculations. In Sec. 3, we present our results and discuss them. We conclude in Sec. 4.

2. The Model

We consider an open boundary one-dimensional system of 11 localized spin-1/2 particles coupled through Ising exchange interaction and subject to an external transverse time-dependent magnetic field of strength $h(t)$. The Hamiltonian for such a system is given by

$$H = - \sum_{<i,j>} J_{ij} \sigma_i^x \sigma_j^x - h(t) \sum_i \sigma_i^z$$

(1)

where $\sigma_i^\delta (\delta = x, y$ or $z)$ are the Pauli operators and $<i,j>$ is a pair of sites on the chain such that

$$J_{ij} = J / (r_i - r_j)^\alpha$$

(2)

for all pairs of spins, where we assume that the distance between any two nearest neighbor spins is $r$. For convenience, we set $J = r = 1$. For a system of 11 spins, the Hilbert space is huge with $2^{11}$ dimensions, yet it is exactly diagonalizable using the standard computational techniques. Exactly solving Schrodinger
equation of the Hamiltonian (1), yields the system energy eigenvalues \( \{E_i\} \) and eigenfunctions \( \{\psi_i\} \). The density matrix of the system is defined by

\[
\rho = |\psi_0\rangle \langle \psi_0|
\]

(3)

Where \( |\psi_0\rangle \) is the ground state energy of the entire spin system. We confine our interest to the entanglement between two spins, at any sites \( i \) and \( j \). All the information needed in this case, at any moment \( t \), is contained in the reduced density matrix \( \rho_{ij}(t) \) which can be obtained from the entire system density matrix by integrating out all the spins states except \( i \) and \( j \). We adopt the entanglement of formation, as a well known measure of entanglement where Wootters [16] has shown that, for a pair of binary qubits, the concurrence \( C \), which goes from 0 to 1, can be taken as a measure of entanglement. The concurrence between two sites \( i \) and \( j \) is defined as

\[
C(\rho) = \max\{0, \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4\}
\]

(4)

where the \( \epsilon_i 's \) are the eigenvalues of the Hermitian matrix \( R \equiv \sqrt{\rho \tilde{\rho} \sqrt{\rho}} \) with \( \tilde{\rho} = (\sigma^y \otimes \sigma^y) \rho^* (\sigma^y \otimes \sigma^y) \) and \( \sigma^y \) is the Pauli matrix of the spin in y direction. The dynamics of entanglement is evaluated using the step-by-step time-evolution projection technique [17]. In this technique we assume that our system is initially, at \( t_0 \), in the ground state at zero temperature \( |\phi\rangle \) with energy, say, \( \varepsilon \) in an external magnetic field with strength \( a \). The magnetic field is turned to a new value \( b \) and the system Hamiltonian becomes \( H \) with \( N \) eigenpairs \( E_i \) and \( |\psi_i\rangle \). The original state \( |\phi\rangle \) can be expanded in the basis \( \{|\psi_i\rangle\} \):

\[
|\phi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + \ldots + c_N|\psi_N\rangle
\]

(5)

where \( c_i = \langle \psi_i |\phi\rangle \).

When \( H \) is independent of time between \( t \) and \( t_0 \) then we can write

\[
U(t, t_0)|\psi_{i(t_0)}\rangle = e^{-iH(t-t_0)}|\psi_{i(t_0)}\rangle = e^{-iE_i(t-t_0)/\hbar}|\psi_{i(t_0)}\rangle
\]

(6)

where \( U(t, t_0) \) is the time evolution operator. The ground state will evolve with time as

\[
|\phi(t)\rangle = c_1|\psi_1\rangle e^{-iE_1(t-t_0)} + c_2|\psi_2\rangle e^{-iE_2(t-t_0)} + \ldots + c_N|\psi_N\rangle e^{-iE_N(t-t_0)}
\]

\[
= \sum_{i=1}^{N} c_i |\psi_i\rangle e^{-iE_i(t-t_0)}
\]

(7)

and the pure state density matrix becomes

\[
\rho(t) = |\phi(t)\rangle \langle \phi(t)|
\]

(8)
Figure 1. The time varying magnetic field described by
\[ h(t) = (b - a)/2 \times [\tanh(\omega(t - t_0) + 1) + a] \] (in units of J), where \( a = 1, b = 3.5, \omega = 0.1 \) and \( t_0 = 50 \) (solid blue line) versus time (in units of \( J^{-1} \)), and the constant magnetic field \( h = b = 3.5 \) (dash dotted red line).

Simply any complicated function can be treated as a collection of step functions. When the state evolves to the next step just repeat the procedure to get the next step results. Of course the lack of smoothness in the magnetic field function represents a challenging obstacle in the calculations but this can be overcome by choosing a proper small enough time step. Because the size of our 11-site system is still manageable although hard computationally, in our actual calculations, we included all the \( 2^{11} \) states in every step, without any truncation of the higher energy eigenstates, which avoids any approximation in this step. But the method itself can be implemented at larger size systems, where cutting off higher energy eigenstates might be a necessary action in that case.

3. Results and discussion
We investigate the dynamical reaction of the one-dimensional Ising system to an applied time-dependent magnetic field with a form
\[ h(t) = (b - a)/2 \times [\tanh(\omega(t - t_0) + 1) + a] \],
where it starts initially at \( h = \omega = 0.1 \) and \( t_0 = 50 \) but smoothly turns to \( h = b \) for \( t \gg t_0 \) as shown in Fig. 1. The Ising spin system with short range, nearest neighbor, interaction encounters a phase transition at a critical magnetic field value of \( h_c = 1 \) (in units of J). The system exists in a ferromagnetic phase for \( h < h_c \) and in paramagnetic for \( h > h_c \). In our calculations we choose \( a = 1, b = 3.5 \) and \( \omega = 0.1 \), which guarantees the short range interacting system to be in the paramagnetic phase with no possible phase transition. We set \( t_0 = 50 \) (in units of \( J^{-1} \)).
Figure 2. Time evolution of the entanglement $C_{1,2}$ in (a) for short range interactions and (b) for long range interactions and $C_{1,3}$ in (c) for short range interactions and (d) for long range interactions. The legend for the short range ($\alpha = 100$ (solid brown line), $\alpha = 10$ (dashed green line), $\alpha = 3$ (dotted blue line) and $\alpha = 2$ (dash dotted red line)) and long range ($\alpha = 1.5$ (solid brown line), $\alpha = 1$ (dashed green line), $\alpha = 0.5$ (dotted blue line) and $\alpha = 0$ (dash dotted red line)) interactions are as shown in panels (a) and (b) respectively. The curves correspond to the time varying magnetic field whereas the straight lines correspond to the constant magnetic field cases.

We compare the dynamic behavior of the system under the time varying magnetic field with that under a constant magnetic field $h = b$, where according to the ergodic assumption the system should relax after a long time to the same state. The bipartite entanglement between any pair of spins remains constant under a constant magnetic field which is represented by straight lines in the different graphs. We discuss the dynamics of the entanglement between the terminal spin 1 and its nearest neighbor spin 2, $C_{1,2}$, in Fig. 2(a) and (b) and also between 1 and its next to nearest neighbor 3, $C_{1,3}$, in Fig. 3(a) and (b). In Fig. 2(a) the time evaluation of entanglement $C_{1,2}$ is depicted. For different values of the exponent $\alpha = 100, 10, 3, 2$ which all are within the interaction short range. As can be noticed there is a very slight difference between the effect of $\alpha = 100$ and 10 on the behavior of $C_{1,2}$, which starts with a constant value around 0.34 but then decreases monotonically under the effect of the time varying magnetic field until coinciding with the constant magnetic field state with steady state value around 0.139. As the value of $\alpha$ decreases, to 3 and then to 2, the initial state of the entanglement deviates from the previous cases significantly where it starts at smaller values, 0.23 and 0.11 respectively, and increases reaching a peak before relaxing.
again to asymptotic steady state values that are different from each other and slightly higher than the previous case of higher exponent values as shown in the inner panel of Fig. 2(a). It is important to note that the dynamic behavior of $C_{1,2}$ shows perfect ergodicity for the short range exponents $\alpha = 100$ and 10, where the initial states are different but the final steady states coincide. Nevertheless, for smaller exponent values 3 and 2, which are still considered in the short range (greater than 1), their final states deviate from the previous ones although they coincide with their own constant magnetic field states, which can be considered as attaining partial ergodicity. On the other hand, the monotonic increase in the value of $C_{1,2}$ corresponding to $\alpha = 100$ and 10 indicates that increasing the magnetic field with time in the interaction range doesn't cause any phase transition in the system as expected and was explained earlier.

![Figure 3](image)

**Figure 3.** Time evolution of the entanglement $C_{1,3}$ in (a) for short range interactions and (b) for long range interactions. The legend for the short range and long range interactions are as shown in Fig. 2(a) and (b) respectively.

But the non-monotonic variation in $C_{1,2}$ as the magnetic field increases with time for $\alpha = 3$ and 2 is a sign of tendency to critical behavior of the entanglement and its derivatives in this interaction range, which indicates a possible phase transition at the thermodynamic limit. The effect of a smaller interaction exponent is explored in Fig. 2(b), where we set $\alpha = 1.5, 1, 0.5$ and 0. The final states values are close for $\alpha = 1.5$ and 1 but clearly are different from that of $\alpha = 0.5$ and 0, where they themselves are different from each other.
When we reach $\alpha = 0.5$ and 0, which are in the long and infinite ranges respectively, the variation in $C_{1,2}$ is monotonic again. Comparing Fig. 2(b) with (a), one can notice that the final steady state value decreases as we go toward longer ranges, lower $\alpha$ values. This is consistent with the entanglement sharing rule, where as expected the entanglement between spin 1 and its nearest neighbor spin should decrease as the entanglement between 1 and the other spins beyond the nearest neighbors increases as the interaction range increases.

The dynamics of the next to nearest neighbor bipartite entanglement $C_{1,3}$ shows similarities and differences from that of $C_{1,2}$. While the asymptotic behavior of $C_{1,3}$ shows complete independent of the initial state of the system for nearest neighbor interaction (very large exponent), which stays the same for lower interaction exponents and even for as low as $\alpha = 15$, which we have tested, the asymptotic value
corresponding to $\alpha = 10$ is slightly higher than that of $\alpha = 100$, as illustrated in Fig. 3(a) and its inner panel. The asymptotic value of $C_{1,3}$ increases even considerably further as $\alpha$ decreases, which is expected, since increasing the interaction range should enhance the beyond nearest neighbor entanglements. But as the exponent goes down from 1.0 to 0.5 and then to 0, the corresponding asymptotic value of $C_{1,3}$ decreases, as a result of more entanglement sharing among different spins, as shown in Fig. 3(b). Again the entanglement varies non-monotonically for $\alpha = 3, 2, 1.5$ and 1 but otherwise for higher and lower exponents values in a similar fashion to what was found for $C_{1,2}$. This means that the entanglement between the next to nearest neighbors shows ergodic behavior at larger exponents compared with the nearest neighbor case. The bipartite entanglements between spin 1 and spins beyond 3, where found to be zero at the nearest neighbor interaction range and longer. The time evolution of the entanglement between the terminal spin 1 and the central spin 6, $C_{1,6}$, is plotted in Fig. 4(a) and (b) for the same set of exponent values as in figure 2. The entanglement corresponding to the short range exponents $\alpha = 100, 10$ and 3 maintains a zero value at all times, whereas for $\alpha = 2$, the entanglement revives for a short period of time before vanishing again as can be seen in figure 4(a). Further increment of the interaction range causes the entanglement to sustain a small but non-zero value which varies with time as illustrated figure 4(b). Interestingly, the value of the asymptotic state varies with the exponents in a non-conventional way. It starts with a small value, around 0.008, corresponding to $\alpha = 1.5$ but then increases to about 0.035 for

![Figure 5. Time evolution of the entanglement $C_{1,11}$ in (a) for short range interactions and (b) for long range interactions. The legend for the short range and long range interactions are as shown in Fig. 2(a) and (b) respectively.](image)

Finally, we monitor the bipartite entanglement between the two terminal spins 1 and 11, $C_{1,11}$. The entanglement is zero in the short interaction range corresponding to $\alpha = 100, 10, 3$ and 2 as can be seen in Fig. 5(a). As the interaction range increases further, the entanglement assumes a non-zero value. At $\alpha = 1.5$ the entanglement starts at a finite value before vanishing under the effect of the higher magnetic field strength, as illustrated in Fig. 5(b). As the interaction range is increased ($\alpha = 1$), the asymptotic value becomes 0.0151, which increases even further at the exponent $\alpha$ value 0.5, where it becomes 0.029. However, when the system reaches the infinite long range, $\alpha = 0$, the asymptotic value drops abruptly to about 0.008. In fact, comparing the dynamics of all bipartite entanglements, $C_{1,2}$, $C_{1,3}$, $C_{1,6}$ and $C_{1,11}$, at the infinite interaction range, one finds that they all exactly coincide starting from the same initial state ending
up to the same final state. On the other hand, comparing their asymptotic behavior at the different exponent values one can notice that each one of them reaches its maximum asymptotic value at a different interaction range exponent, from the values we consider here, except C₁₂ and C₁₃, where both have the maximum asymptotic value (0.1422 and 0.0435 respectively) at the same exponent value 1.5. The maximum asymptotic values for C₁₅ and C₁₁ (0.0357 and 0.0296 respectively) at the exponent values 1 and 0.5 respectively. Now clearly, examining the ergodicity of the system by comparing all entanglements at different interaction ranges indicates that perfect ergodicity is observed at quite short interaction ranges, although the exact range varies from one bipartite entanglement to the other, whereas all other cases can be considered as showing partial ergodicity at the other interaction ranges.

4. Conclusion
We considered a one dimensional open boundary chain made up of 11 interacting spin-1/2 particles. The interaction between the spins is of the Ising type and the chain is exposed to a transverse uniform time-dependent magnetic field. We studied the dynamics and examined the ergodicity of the spin system at different interaction ranges, all the way from nearest neighbor to infinite long range. We compared the reaction of the system bipartite entanglements to a constant magnetic field with that of a time-varying smoothly increasing magnetic field, which has a final strength equals to that of the constant magnetic field. For the system to be ergodic it has to asymptotically in time reach the same final equilibrium state regardless of its initial state. The dynamics of the bipartite entanglements between the terminal spin and each one of the other spins along the chain were studied and compared with each other at different spin-spin interaction ranges. The nearest neighbor and the next to nearest neighbor entanglements showed perfect ergodicity behavior at quite short range interactions but the former maintains this behavior even at longer interaction ranges. The other bipartite entanglements beyond the next to the nearest neighbors remains zero with time at short range interactions. At intermediate short range and long range interactions, each one of the bipartite entanglements reaches a final asymptotic state that is the same under either the constant or the time varying magnetic field. Nevertheless, the final state varies considerably depending on the interaction range and the distance between the two spins, which means the perfect ergodicity is broken.

Furthermore, the interaction range at which the value of the bipartite entanglement in the final state is maximum varies depending on the two-spin distance, but in general the range increases as the distance increases and lies within the long range for all pairs of spins. At the infinite long range interaction, the time evolution of all bipartite entanglements coincide starting from the same initial state until reaching the same exact very small final state value, which means quantum correlation propagation between all pairs of spins takes the same time and has the same strength regardless of the distance. It is interesting in the future to investigate how the degree of anisotropy in the spin-spin interaction, deviating from the complete anisotropy considered here, would affect the dynamical behavior and ergodicity variation of the spin system compared with what have been observed in the current case.

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