Waves on shearing current for an arbitrary inertial viewer

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Abstract

Real world water waves often propagate on current. And, the measurement of waves and current is an important task for coastal and marine engineers. Modern marine measurement technologies (i.e. unmanned autonomous vehicles, drones) often propagate with different velocities relative to the current and waves. It has been shown that, due to a loss of Galilean invariance, viewer velocity has a non-trivial effect on the mathematical form of the wave and uniform current problem. It is demonstrated herein that similar complexities arise for shearing currents. The work provides a generalized formulation of the wave and shearing current boundary value problem for an arbitrary inertial viewer. The moving viewer dispersion relation for the special case of a constant current shear is also derived.

1 Introduction

Real world water waves often propagate on current. The measurement of waves and current is an important task for coastal and marine engineers. Such measurements are also relevant to general geophysical research, e.g. understanding sea surface hydrodynamics and climate. Modern marine measurement technologies (e.g., unmanned autonomous vehicles, drones) often propagate with arbitrary velocities relative to the current and waves. It has been shown, for waves on uniform current, invariance between descriptions given by different inertial viewers is not generally preserved by Galilean transformations [4]. This is, in part, due to the non-Galilean invariance of Bernoulli’s equation (see also [5]). In real world environments, it is common to have significant variation of current with depth. In such cases, uniform current cannot reasonably be assumed.

Water waves have modal structures which are modified by horizontal and vertical changes in current. Amplitude changes of waves on non-homogeneous moving media and slowly varying wave guides have been discussed [1, 2]. Effects of horizontal variations on internal waves have been investigated using the geometric optics approximation [3, 12]. The geometric optics approximation has also been used to derive an adiabatic invariant for waves on horizontally and vertically changing flows [11]. The lowest order approximation yields a governing equation for the vertical structure of the fluid velocity which is equivalent to Rayleigh’s equation or the inviscid Orr-Sommerfeld equation [8, 9]. There is no general

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solution for the linear theory of waves on arbitrary shearing current. However, approximate dispersion relations have been found and demonstrated for certain profiles [3, 10]. This approach was extended to provide an approximated form of the Wave Action Equation for horizontally inhomogeneous currents on vertical shear [3]. Although the effects of vertically inhomogeneous current on a linear monochromatic wavetrain are reasonably well understood, there is no discussion regarding how viewer motion may modify the mathematical form of the problem. This issue has been overlooked because it is generally assumed that the problem is Galilean invariant. Since a breaking of Galilean invariance was shown for the uniform current case [3], it is of important to investigate the effects of viewer velocity on the shearing current problem. This is of particular interest since it is becoming more common to measure wave-current fields using instrumentation on moving platforms, both above and below the water surface.

In the following, the surface water wave on shearing current problem is rewritten for an arbitrary inertial viewer. The boundary value problem is transformed using Galilean transformations for both coordinates and derivatives. In a most general case, viewer velocity may be assumed to have nonzero components in all three spatial dimensions. Equations of motion governing the vertical wave velocity profile are derived. With regards to the governing equation, the effects of the viewer velocity appear in the usual coefficients and as additional derivatives of the independent variable, increasing the order of the differential equation. The classical Rayleigh equation is recovered for a fixed viewer. The kinematic free surface boundary condition is also modified by the viewer velocity. The effect of horizontal velocity components appear in the usual coefficients. Similar to the governing equation, a vertical velocity component increases the order of the differential equation. There is no general analytical solution for waves on shearing current. Nonetheless, analytical solutions can be found for certain special cases. As an example, the case of a constant current shear is considered. An analytical form of the dispersion relation is derived. It is shown to reduce to known forms in deep water for uniform and zero current.

2 Equations of motion

Consider incompressible, inviscid and rotational flow and a viewer denoted by $S$. $\mathbf{u} = \{u, v, w\}$ is the flow velocity components, $p$ is the pressure and $\eta$ is the free surface. Momentum conservation is given by Euler’s equations,

$$
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p = 0.
$$

(1)

Wherein, $\nabla \equiv \{\partial_x, \partial_y, \partial_z\}$. Equation (1) follows from the full Navier-Stokes equations in the absence of viscous effects. Mass conservation is given by the continuity equation,

$$
\nabla \cdot \mathbf{u} = 0.
$$

(2)

The free surface boundary conditions are

$$
\frac{\partial \eta}{\partial t} + \mathbf{u} \cdot \nabla \eta - w = 0, \quad z = 0, \quad (3)
$$

$$
p - \rho g \eta = 0, \quad z = 0, \quad (4)
$$

$$
w = 0, \quad z = -h, \quad (5)
$$

$$
w = 1, \quad z = 0. \quad (6)
$$
Equation (3) is the kinematic free surface boundary condition, (4) is the dynamic free surface boundary condition, (5) is the kinematic condition for an impermeable flat bottom and (6) normalizes the amplitude of the harmonic.

Decompose the flow into temporal mean (current) and oscillatory (wave) components,

\[
\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(z) + \epsilon \tilde{\mathbf{u}}(\mathbf{x}, t),
\]

\[
\eta(\mathbf{x}, t) = \epsilon \tilde{\eta}(\mathbf{x}, t),
\]

\[
p(\mathbf{x}, t) = \text{constant} + \epsilon \tilde{p}(\mathbf{x}, t).
\]

Wherein, \(\mathbf{U} = \{U_x(z), U_y(z), 0\}\) is the horizontal current. Tildes denote small oscillatory perturbations corresponding to Fourier components of the wave motion. Substitute (7)-(9) into (1)-(6). The linearized wave motion is governed by terms \(O(\epsilon)\),

\[
\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla\right) \tilde{\mathbf{u}} + w \frac{d\mathbf{U}}{dz} = -\nabla \tilde{p},
\]

\[
\nabla \cdot \tilde{\mathbf{u}} = 0.
\]

The boundary conditions become

\[
\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla\right) \tilde{\eta} - w = 0, \quad z = 0,
\]

\[
p - \rho g \tilde{\eta} = 0, \quad z = 0,
\]

\[
w = 0, \quad z = -h,
\]

\[
w = 1, \quad z = 0.
\]

The mathematical form of equations (10)-(15) were derived for a fixed viewer who sees the fluid (in the absence of wave motion) move with the velocity \(\mathbf{U}\). In a real world context, this viewer would correspond to a measurement instrumentation attached to a fixed platform relative to the earth (e.g. wave gauge attached to sea floor in shallow water). However, modern measurement technologies may also move with arbitrary velocities and directions relative to the earth (and waves and current). For example, they may be ship-towed, attached to unmanned autonomous vehicles or deployed on a moving drone. It is a common assumption that (10)-(15) will have the same form for all inertial viewers. However, it will be shown that this is not generally the case within a Galilean framework since the system exhibits broken Galilean boost invariance.

### 3 System in an inertial reference frame

Assume an inertial reference frame \(S'\) moving with a constant velocity \(\mathbf{V} = \{V_x, V_y, V_z\}\) relative to \(S\). In the Galilean framework, invariance between inertial reference frames is given by the following transformations,

\[
\mathbf{x} = \mathbf{x}' + \mathbf{V} t', \quad t = t',
\]

\[
\nabla = \nabla', \quad \partial_t = \partial_t' - \mathbf{V} \cdot \nabla'.
\]
These are the Galilean transformations. In $S'$, Euler's equation and the continuity equation become

\[
\left( \frac{\partial}{\partial t'} + (U - V) \cdot \nabla' \right) \hat{u}' + \hat{w}' \frac{dU}{dz'} = -\nabla' \hat{p}',
\]

\[
\nabla' \cdot \hat{u}' = 0.
\]

The boundary conditions take the form,

\[
\left( \frac{\partial}{\partial t'} + (U - V) \cdot \nabla' \right) \hat{\eta}' - \hat{w}' = 0, \quad z' = 0,
\]

\[
\hat{p}' - \rho g \hat{\eta}' = 0, \quad z' = 0,
\]

\[
\hat{w}' = 0, \quad z' = -h,
\]

\[
\hat{w}' = 1, \quad z' = 0.
\]

Primes denote quantities in $S'$. For clarity in writing, the primes will now be dropped. Introducing temporal and spatial periodicity into the oscillatory terms yields,

\[
\hat{u}(x, t) = \hat{u}(z) \exp \left[ i (k_x x + k_y y - (-1)^n \omega_d t) \right],
\]

\[
\hat{\eta}(x, t) = \hat{\eta} \exp \left[ i (k_x x + k_y y - (-1)^n \omega_d t) \right],
\]

\[
\hat{p}(x, t) = \hat{p}(z) \exp \left[ i (k_x x + k_y y - (-1)^n \omega_d t) \right].
\]

Wherein, $n = 1, 2$ differentiate between solution branches (see appendix C), $\hat{u} = \{\hat{u}, \hat{v}, \hat{w}\}$, and $\omega_d = \omega - k \cdot V$ is the Doppler shifted frequency ($\omega$ is the frequency measured in $S$).

Combining (17) and (18), using the standard derivation procedure for a fixed viewer (see, for example, [11]), yields a governing equation in terms of $\hat{w}$ in $S'$,

\[
\frac{\partial^2 \hat{w}}{\partial z^2} - \left( \frac{|k|^2}{(-1)^{n+1} C - \mathcal{U} + \mathcal{V}} \right) \hat{w} + \frac{i V_z}{|k|} \frac{\partial^2 \hat{U}}{\partial z^2} \left( \frac{|k|^2}{(-1)^{n+1} C - \mathcal{U} + \mathcal{V}} \right) \frac{\partial \hat{w}}{\partial z} = 0.
\]

Wherein, the following quantities have been introduced,

\[
\mathcal{U} = \frac{k \cdot U}{|k|}, \quad \mathcal{V} = \frac{k \cdot \{V_x, V_y\}}{|k|}, \quad C = \frac{\omega_d}{|k|}.
\]

$\mathcal{U}$ and $\mathcal{V}$ are the projections of the current and viewer velocities on the wave. $C$ is the wave celerity. As expected, for $\{V_x, V_y\} = V_z = 0$ (a fixed viewer), Rayleigh’s equation is recovered. It is of interest to inspect the influence of different viewer velocity components on (26). When $V_x \neq 0$ and $V_y \neq 0$ and $V_z = 0$, the general functional form of Rayleigh’s equation is preserved; however, additional secularities appear in the coefficients. Perhaps more problematic is the case of a viewer velocity with a vertical component, i.e. $V_z \neq 0$. This introduces an imaginary term in (26) with no analogy in $S$. Moreover, it introduces a third order derivative, increasing the order of the differential equation governing the independent variable $w$. This renders the boundary value problem under-determined. An additional (possibly symmetry) condition will be required in order to uniquely determine the solution.
Combining (20) and (17) and eliminating \( \hat{u} \) *vis a vis* (18) yields a combined free surface boundary condition in \( S' \),

\[
\frac{\partial \hat{w}}{\partial z} + \left( \frac{1}{((-1)^{n+1} C - \hat{U} + \hat{V})} \frac{\partial \hat{U}}{\partial z} - \frac{g}{((-1)^{n+1} C - \hat{U} + \hat{V})^2} \right) \hat{w} - \frac{iV_z}{k((-1)^{n+1} C - \hat{U} + \hat{V})} \frac{\partial^2 \hat{w}}{\partial z^2} = 0. \tag{28}
\]

As expected, setting \( \{V_x, V_y\} = V_z = 0 \) recovers a known form of the boundary condition for a fixed viewer. Similar to (26), when \( V_z \neq 0 \) an imaginary term without analogy in \( S' \) appears. Therefore, the system exhibits a loss of Galilean boost invariance for a viewer moving with a vertical velocity component (e.g. an underwater autonomous vehicle changing depth).

4 Approximate dispersion relation: constant shear

There is no general analytical solution to (26)-(28). However, for certain current profiles and viewers, one may find analytical forms of dispersion. Consider here the simple case of a constant shear and viewer moving in the horizontal plane \( (V_x \neq 0, V_y \neq 0, \text{and} \ V_z = 0) \). The current profile will have the form,

\[
\hat{U}(z) = A + Bz. \tag{29}
\]

The dispersion relation becomes,

\[
\mathcal{C} = \frac{(-1)^{-3n} \left( -2k (A - \mathcal{V}) + \left( \mathcal{B} + (-1)^m \sqrt{\mathcal{B}^2 + 4gk \coth(kh)} \right) \tanh(kh) \right)}{2k}. \tag{30}
\]

Wherein, \( m = 1, 2 \) and \( n = 1, 2 \). In deep water, \( kh \to \infty \), (30) becomes

\[
\mathcal{C} = \frac{(-1)^{-3n} \left( -2k (A - \mathcal{V}) + \left( \mathcal{B} + (-1)^m \sqrt{\mathcal{B}^2 + 4gk} \right) \right)}{2k}. \tag{31}
\]

Known forms can be recovered for a fixed viewer. Consider two special cases: uniform and zero current in deep water.

**Case 1:** Fixed viewer of wave propagation on uniform current in deep water \( (\mathcal{B} \to 0, kh \to \infty) \),

\[
\mathcal{C} = \frac{(-1)^{-3n} \left( -kA + (-1)^m \sqrt{gk} \right)}{k}. \tag{32}
\]

**Case 2:** Fixed viewer of wave propagation on still and deep water \( (A \to 0, \mathcal{B} \to 0, kh \to \infty) \),

\[
\mathcal{C} = \frac{(-1)^{-3n+m} \sqrt{gk}}{k}. \tag{33}
\]

Thus, the dispersion relation, (30), is consistent with known approximations in the limits of deep water with uniform and zero current.
5 Discussion

The linear gravity wave on shearing current problem was generalized for an arbitrary inertial viewer using Galilean transformations. Results demonstrate that the functional form of the boundary value problem is not the same for all inertial viewers. Thus, Galilean invariance applies only in a restricted sense. The effect of a vertical viewer velocity component is of particular interest. In this case, imaginary terms appear in the governing equation and free surface condition which have no analogy for a fixed viewer. It should be noted that the effect remains in the limit of zero current. If the viewer velocity is strictly in the horizontal plane, there is no imaginary term; however, new secularities appear in the usual coefficients.

There is no general analytical solution for wave propagation on an arbitrary shearing current profile. Nonetheless, for some simple cases analytical solutions may be found. As an example, the dispersion relation for a moving viewer of waves on current with constant shear was derived. For simplicity, it was assumed the viewer moves only along a horizontal plane. This avoids the imaginary terms which are generated by a vertical component in the viewer velocity. The new generalized dispersion relation was shown to recover known forms in the limits of uniform and zero current in deep water.

Wave measurement instrumentations often propagate with different directions and velocities relative to the waves and current. And, in most real world environments, there will be vertical variation in a current profile. Using a Galilean framework, the preceding results demonstrate the combined effect of a moving viewer and current shear on the functional form of dispersion. The results are of high importance to the interpretation of wave and current measurements by moving platforms.

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