A slow-to-start traffic model related to a M/M/1 queue

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Abstract. We consider a system of ordered cars moving in \( \mathbb{R} \) from right to left. Each car is represented by a point in \( \mathbb{R} \); two or more cars can occupy the same point but cannot overpass. Cars have two possible velocities: either 0 or 1. An unblocked car needs an exponential random time of mean 1 to pass from speed 0 to speed 1 (slow-to-start). Car \( i \), say, travels at speed 1 until it (possibly) hits the stopped car \( i-1 \) to its left. After the departure of car \( i-1 \), car \( i \) waits an exponential random time to change its speed to 1, travels at this speed until it hits again stopped car \( i-1 \) and so on. Initially cars are distributed in \( \mathbb{R} \) according to a Poisson process of parameter \( \lambda < 1 \). We show that every car will be stopped only a finite number of times and that the final relative car positions is again a Poisson process with parameter \( \lambda \). To do that, we relate the trajectories of the cars to a \( M/M/1 \) stationary queue as follows. Space in the traffic model is time for the queue. The initial positions of the cars coincide with the arrival process of the queue and the final relative car positions match the departure process of the queue.

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1. Introduction

Interest in traffic models is old. In 1935 Greenshield (see Helbing [14] and Chowdhury et al [9] for a review on traffic models) introduced the first traffic study. In 1959 Greenberg [13] called the attention to the importance of the area. In 1992 Nagel-Schreckenberg [17] study traffic using probabilistic cellular automata, computer simulations and mean field models. Slow-to-start models intend to capture the behavior of cars that come out of a traffic jam: a driver needs a moment to speed the car. Nagel and Schreckenberg [17], Schadschneider and Schreckenberg [19], Gray and Griffeath [12], Chowdhury, Santen and Schadschneider [9] and Yaguchi [21] studied these type of models. Other studies of traffic have been done by Krug and Spohn [15], Barlovic, Santen, Schadschneider and Schreckenberg [2], Fuks [10], Boccara and Fuks [8], Belitsky and Ferrari [4], Belitsky, Krug, Neves and Schütz [5], Blank [6,7], Helbing [14], Wolfram [22] among many others.

We introduce a slow-to-start traffic model which is continuous in space and time. Initially cars are distributed in the straight line $\mathbb{R}$ following a Poisson process of parameter $\lambda$; cars may have speed 1 or 0. All cars start at zero speed and each car waits a random (delay) time exponentially distributed with mean 1 to change its speed from 0 to 1. Delay times of different cars are independent. After the delay time, each car moves at speed 1 until it collides with a stopped car to its left or forever if it does not collide. When car $i$ collides with car $i-1$, its speed drops immediately to zero and remains blocked until car $i-1$ leaves. Then car $i$ waits a further random time with exponential distribution before moving. And so on. At each time there are cars at speed 1 and cars at speed 0. We prove that if $\lambda < 1$ every car will eventually be unblocked forever and the final relative positions of cars are distributed as a Poisson process of rate $\lambda$.

The main tool is a relation between the traffic model and a $M/M/1$ stationary queuing system. Customers arrive at rate $\lambda$ according to a Poisson process to a single server whose service times are independent and exponentially distributed with mean 1. There exists a stationary version for the queue when $\lambda < 1$. The stationary process is reversible so that its distribution is invariant by reversing the arrow of time. As a consequence the customer departure process is a Poisson process as the arrival process. This is the famous Burke’s theorem.

In Section 2 we define the process and give the main results. In Section 3 we construct a semi-infinite traffic model and define traffic cycles. In Section 4 we define the queue process and construct workload cycles. In Section 5 we relate traffic final relative car-positions and customer exit times in the queue. In Section 6 we use an approach of Thorisson to construct the stationary traffic process using cycles and in Section 7 we do it for the queue and conclude the proofs of the results.
2. Definition of model and main results

We consider a system of cars moving in \( \mathbb{R} \) from right to left. Cars have two possible velocities: either 0 or 1. Cars are represented by points and cannot overpass. A car needs an exponential random time of mean 1 to pass from speed 0 to speed 1. A typical car starts at speed 0, waits an exponentially distributed random time to change its speed to 1, travels at this velocity until it is blocked by another stopped car, waits the stopped car to leave, waits another exponential time to get again velocity 1 and so on.

We construct a sequence \((\Pi(t), V(t)); t \geq 0\) of car trajectories, \(\Pi(t) = (\pi_i(t), i \in \mathbb{Z})\) and \(V(t) = (v_i(t), i \in \mathbb{Z})\). For each \(i\), \(\pi_i : \mathbb{R}_+ \to \mathbb{R}\) is a piecewise linear function almost everywhere differentiable; \(\pi_i(t)\) represents the position of car \(i\) at time \(t\) and \(v_i(t) \in \{0, 1\}\) its velocity. The initial car positions are given by \((y_i, i \in \mathbb{Z})\), with \(y_i \in \mathbb{R}\) and \(y_i < y_{i+1}\) for all \(i\): set \(\pi_i(0) = y_i\). The initial velocities are all null: \(v_i \equiv 0\). The trajectories must satisfy the following properties:

(i) \(\pi_i(0) = y_i\)
(ii) \(\dot{\pi}_i(t) = -v_i(t)\) if \(v_i\) is continuous at \(t\)
(iii) \(v_i(t)\) jumps from 0 to 1 at rate 1 if \(\pi_i(t) > \pi_{i-1}(t)\).
(iv) \(v_i(t) = 0\) if \(\pi_i(t) = \pi_{i-1}(t)\)

If instead of taking \(i\) in \(\mathbb{Z}\) we consider a semi-infinite configuration of cars with initial positions \(y_0 \leq y_1 \leq \ldots\), the trajectories can be constructed inductively. Car 0 waits an exponential time of rate 1 and then assumes velocity 1; since it will not be blocked by any other car, it will continue at speed 1 for ever. Car 1 does the same, but if it collides with car 0 (at \(y_0\)), then it stops and after car 0 leaves, it waits an extra exponential time and then goes, and so on. Let \(\pi_i := (\pi_i(t), t \geq 0)\). If we think \(i\) as discrete time, the process of trajectories is Markovian in the sense that the law of the trajectory \(\pi_i\) given \(\pi_{i-1}, \ldots, \pi_0\) depends only on \(\pi_{i-1}\). Our first result says that it is possible to construct a spatially translation invariant traffic process if the initial positions of the cars are given by a Poisson process of rate \(\lambda\).

Proposition 1. If \((y_i; i \in \mathbb{Z})\) is a stationary Poisson process in \(\mathbb{R}\) with rate \(\lambda\) and \(\lambda < 1\) then there exists a spatially stationary version of the process \(\Pi = ((\pi_i(t); t \geq 0), i \in \mathbb{Z})\) with initial positions \(\pi_i(0) = y_i\) for all \(i \in \mathbb{Z}\).

In the stationary Poisson process by convention \(y_0 < 0 < y_1\), so that \(-y_0, y_1\) and \(y_{i+1} - y_i\) for \(i \neq 0\) are iid random variables exponentially distributed. Let

\[ t_i := \sup\{s : v_i(s) = 0\} \]

(possibly equal to infinity) be the last time car \(i\) has velocity 0. From \(t_i\) on, car \(i\) goes freely at speed 1. We say that car \(i\) is free at times \(t > t_i\).

Proposition 2. Assume \(\lambda < 1\) and consider the traffic process \(\Pi\) of Proposition 1. For each \(i \in \mathbb{Z}\), \(t_i\) is finite almost surely.
Let $D_i$ be the total delay of car $i$ defined by

$$D_i := \left| -t_i - (\pi_i(t_i) - y_i) \right|$$

(3)

Here $-t_i$ is the displacement car $i$ would have at time $t_i$ if it started at speed 1 and had never been blocked and $\pi_i(t_i) - y_i$ is the actual displacement by that time. The absolute value of the difference is the delay of car $i$. Call

$$s_i := y_i + D_i$$

(4)

To understand the meaning of $s_i$, observe that if a car starts at position $s_i$ at speed 1 and it is never blocked, then its position at time $t$ coincides with $\pi_i(t)$ for all times $t \geq t_i$.

**Proposition 5.** If $\lambda < 1$ then $\{s_i; i \in \mathbb{Z}\}$ is a Poisson process of parameter $\lambda$. Furthermore, $\pi_i(t) - \pi_{i-1}(t) = s_i - s_{i-1}$ for $t > \max\{t_i, t_{i-1}\}$.

The statement is that $\{s_i; i \in \mathbb{Z}\}$ as a subset of $\mathbb{R}$ is a Poisson process; the specification of the indexes may corrupt the Poisson property. These results say that if the initial position of cars are distributed according to a Poisson process and all with zero speed, then every car will eventually have velocity 1 and the relative car positions will be distributed according to a Poisson process. For this reason we call $s_i$ the **final relative position** of car $i$.

Let $y \in \mathbb{R}$ and for initial position $y_i > y$ let $r_i$ be the random time defined by

$$r_i(y) := \sup((\pi_i)^{-1}(y))$$

this is last time the time car $i$ is at position $y$. Let $T(y)$ be the first time all cars crossing $y$ after $T(y)$ are free:

$$T(y) := \inf\{r_i(y); t_i \leq r_m(y) \text{ for all } m \geq i\}$$

**Proposition 6.** If $\lambda < 1$ then $T(y) < \infty$ almost surely for all $y \in \mathbb{R}$.

### 3. Construction of trajectories for a semi-infinite initial car configuration

We construct car trajectories for initial car positions $0 = y_0 < y_1 < \ldots$ and relate them to a $M/M/1$ queue starting with an empty system at arrival of a customer.

The trajectories will be defined as a function of a marked Poisson process

$$(y, \xi) = ((y_i, \xi_{i,m}); i \geq 0, 0 \leq m \leq i)$$

(7)

where $y = (y_i, i \geq 0)$ is a Poisson process on $[0, \infty)$ with rate $\lambda$ with a car added at the origin (that is, $y_0 = 0$ and $(y_{i+1} - y_i; i \geq 0)$ are iid exponential random variables with mean $1/\lambda$) and the delay times $\xi = ((\xi_{i,m}; 0 \leq m \leq i), i \geq 0)$ are iid exponential random variables with mean 1. The sequences $\xi$ and $y$ are independent. These times are used as follows. Car $i$ may collide with car $i - 1$ at sites $y_0, \ldots, y_{i-1}$; if the collision occurs at $y_m$, then after car $i - 1$ leaves $y_m$, car $i$ waits $\xi_{i,m}$ units of time before taking again speed 1. More rigorously, car 0 starts at the origin and waits $\xi_{0,0}$ units of time.
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to start moving at speed 1. Since there are no cars to the left of car 0, it will never be blocked and we define

\[
\pi_0(t) := \begin{cases} 
0 & \text{if } 0 < t \leq \xi_{0,0} \\
-t + \xi_{0,0} & \text{if } t > \xi_{0,0}
\end{cases}
\]  

(8)

The trajectory of car \(i\) is then defined as a function of the trajectory of car \(i - 1\) and the waiting times \(\xi_{i,m}; 0 \leq m \leq i\) as follows. We shall define \(A_{i,m}\) as the time car \(i\) arrives to \(y_m\) and \(B_{i,m}\) as the time car \(i\) departs from \(y_m\). Clearly \(A_{i,m} \leq B_{i,m}\) and if car \(i\) is not blocked at \(y_m\), then \(A_{i,m} = B_{i,m}\). Let

\[
A_{i,i} := 0, \quad B_{i,i} := \xi_{i,i}
\]

and then inductively assume \(A_{i-1,m}, B_{i-1,m}\) are defined for all \(0 \leq m \leq i - 1\), as well as \(A_{i,m}\) and \(B_{i,m}\). Then set

\[
A_{i,m-1} := B_{i,m} + y_m - y_{m-1}
\]

(9)

\[
B_{i,m-1} := \begin{cases} 
A_{i,m-1} & \text{if } A_{i,m-1} > B_{i-1,m-1} \\
B_{i-1,m-1} + \xi_{i,m-1} & \text{if } A_{i,m-1} < B_{i-1,m-1}
\end{cases}
\]  

(10)

In words: the arrival of car \(i\) to site \(m - 1\) occurs \((y_m - y_{m-1})\) time units after its departure from site \(m\). The departure of car \(i\) from site \(y_{m-1}\) occurs immediately (at arrival time) if car \(i - 1\) has already left or, \(\xi_{i,m-1}\) units of time after \(B_{i-1,m-1}\), the departure of car \(i - 1\) from site \(y_{m-1}\).

The vector \(((A_{i,m}; B_{i,m}), 0 \leq m \leq i)\) determines the trajectory \((\pi_i(t), t \geq 0)\):

\[
\pi_i(t) := \sum_{k=0}^{i} y_k 1\{t \in (A_{i,k}, B_{i,k})\} + (y_k + B_{i,k} - t) 1\{t \in (B_{i,k}, A_{i,k-1})\}
\]

(11)

where \(1\{\cdot\}\) is the indicator function of the set \(\{\cdot\}\). The total delay of car \(i\) defined in (3) satisfies

\[
D_i = \sum_{k=0}^{i} (B_{i,k} - A_{i,k})
\]

(12)

To stress the dependence on \((y, \xi)\) we write \(A_{i,k}(y, \xi), B_{i,k}(y, \xi), \text{ etc.}\). Let \(s(y, \xi) = (s_0, s_1, \ldots)\), where the final relative position \(s_i\) is defined as a function of \(y_i\) and \(D_i\) as in (4). Let \(\sigma(y, \xi) = (\sigma_0, \sigma_1, \ldots)\) be the sequence defined by \(\sigma_0 := \xi_{0,0}\) and for \(i \geq 1\)

\[
\sigma_i := \begin{cases} 
B_{i,0} - B_{i-1,0} & \text{if } s_{i-1} > y_i, \\
\xi_{i,i} & \text{otherwise}
\end{cases}
\]

(13)

\(\sigma_i\) is called the final delay of car \(i\).

Lemma 14. \(\sigma = (\sigma_0, \sigma_1, \ldots)\) is a sequence of iid random variables with exponential law of mean 1. Furthermore \(\sigma\) is independent of \((y_i; i \geq 0)\).
**Proof** Fix the trajectory of car $i - 1$. There are two cases: either car $i$ is blocked at 0 by car $i - 1$ or not. In the first case $B_{i,0} = B_{i-1,0} + \xi_{i,0}$, so that $\sigma_i = \xi_{i,0}$ which is independent of the trajectories $(\pi_m; 0 \leq m \leq i - 1)$ and in particular of $(\sigma_m; 0 \leq m \leq i - 1)$. In the second case the label of the leftmost blocking position of car $i$ is given by

$$K := \min\{k \leq i : B_{i-1,k} + \xi_{i,k} > B_{i-1,0} - (y_k - y_0)\}$$

here by convention $B_{i-1,i} = 0$. $K$ is a stopping time for $(\xi_{i,i-m}; m \geq 0)$; that is, the event $\{K = k\}$ is a function of $(\xi_{i,i-m}; 0 \leq m \leq k)$. But the dependence on $\xi_{i,k}$ is only on the event $\{\xi_{i,k} > B_{i-1,0} - (y_k - y_0) - B_{i-1,k}\}$. Then, given this event,

$$\sigma_i = \xi_{i,K} - (B_{i-1,0} - (y_K - y_0) - B_{i-1,K})$$

is exponentially distributed with mean one and independent of $(\pi_m; 0 \leq m \leq i - 1)$.

**Traffic Cycle** Call

$$X := \min\{y_i > 0 : y_i > s_{i-1}\} \quad (15)$$

$$N := \min\{i > 0 : y_i > s_{i-1}\} \quad (16)$$

$$C := (\{(\pi_i(t); t \geq 0), i \in \{0, \ldots, N - 1\}\}, N, X) \quad (17)$$

We say that $C$ is a cycle with length $X$ and $N$ cars involved. The cycle $C$ consists of a space interval $X$ and $N$ car trajectories in the time interval $[0, \infty)$ with starting positions in $[0, X)$; however, since car $i$ is free after time $B_{i,0}$, the trajectories are determined by the set $(\{(\pi_i(t); t \in [0, B_{i,0}) ; i \in \{0, \ldots, N - 1\}\}$ or, alternatively by the arrival/departure times of the $N$ cars to/from sites $y_0, \ldots, y_{N-1}$ given by $(A_{i,m}, B_{i,m}); 0 \leq i \leq N - 1, 0 \leq m \leq i)$.

The cycle $C$ induces a stochastic process $Z(C) = (Z(y,C), y \in [0, X])$ given by the interval-valued vector (with dimension depending on $y$)

$$Z(y,C) := (\pi_{i}^{-1}(y); y \leq y_i < X) \quad (18)$$

The $k$th coordinate of $Z(y,C)$ contains the time interval spent at $y$ by the $k$th car to the right of $y$. Assuming this car has label $i$ there are two cases: (a) if $y = y_m$ for some $m$, the interval is $\pi_{i}^{-1}(y) = [A_{i,m}, B_{i,m}]$ and (b) if $y$ is not an initial car position the interval is a point, because car $k$ will not be delayed at $y$. $Z(y,C)$ is a vector of zero length if $y \in [y_{N-1}, X)$.

The cycle $C$, its length $X$ and its number $N$ cars are functions of $y = (y_0, y_1, \ldots)$ and $\xi = (\xi_{i,m}; i \geq 0, 0 \leq m \leq i)$:

$$C = C(y, \xi); \quad X = X(y, \xi); \quad N = N(y, \xi)$$

Given a cycle $C$ we can recover the length of the cycle, the number of cars involved, the initial car positions, the final delay times and the final relative car positions which are denoted

$$N(C), X(C), y_i(C), \sigma_i(C), s_i(C), \quad i = 0, \ldots, N(C) - 1. \quad (19)$$

The final relative car positions in the cycle coincide with the last passage through the origin:

$$s_i = B_{i,0}, \quad \text{for } i = 0, \ldots, N - 1 \quad (20)$$
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Figure 1. The first cycle has 4 cars. Car 1 collides with car 0 and car 2 collides with car 1 at \( y_0 \) (the trajectories are drawn slightly separated at \( y_0 \); the actual trajectories partially intersect at \( y_0 \)). Car 3 does not collide with previous cars, but since its starting position \( y_3 \) is to the right of \( s_2 \), it still belongs to the cycle. Car 4 initial position \( y_4 \) is to the right of \( s_3 \) so that the new cycle starts at \( S_1 = y_4 \).

4. FIFO M/M/1 queue

The FIFO M/M/1 queue is a Markov process in \( \mathbb{N} = \{0, 1, \ldots \} \). At rate \( \lambda > 0 \) customers arrive into the system, stay in line and one customer at a time is served at rate 1. This means that the service times are exponentially distributed with mean 1. The customers respect the time of arrival: first in, first out in the jargon of queuing theory. We use the following notation

- \( \tilde{y}_i \) arrival time of customer \( i \)
- \( \tilde{\sigma}_i \) service time of customer \( i \).
- \( \tilde{s}_i \) exit time of customer \( i \).
The queue size at time \( y \) can be constructed as a function of a marked Poisson process \( \left((\tilde{y}_i, \tilde{\sigma}_i); i \in \mathbb{Z}\right) \). The arrival process \( (\tilde{y}_i; i \in \mathbb{Z}) \) is a Poisson process of parameter \( \lambda \). The service times \( (\tilde{\sigma}_i; i \in \mathbb{Z}) \) are iid random variables with exponential law of mean 1. The sequences are independent.

The workload \( W_y, y \in \mathbb{R} \) is the amount of service due by the server at time \( y \). This is the time a customer arriving at time \( y \) has to wait to start to be served. This process is continuous to the right with limits to the left and piecewise linear. \( W_y \) jumps at times \( \tilde{y}_i \), the arrival time of customer \( i \) by an amount \( \tilde{\sigma}_i \), the service time of this customer and it decreases continuously with derivative \(-1\) until hitting 0, where stays until next arrival. The workload process must satisfy the following evolution equations:

\[
\frac{dW_y}{dy} = -1\{W_y > 0\} \quad \text{for } y \neq \tilde{y}_i, \quad i \in \mathbb{Z} \quad (21)
\]

\[
W_{\tilde{y}_i} = W_{\tilde{y}_i} + \tilde{\sigma}_i \quad (22)
\]

Construction of the workload process with an arrival to an empty system

Let \( \tilde{y} = (\tilde{y}_0, \tilde{y}_1, \ldots) \) be a Poisson process of rate \( \lambda \) as \( y \), with \( y_0 = 0 \) and let \( \tilde{\sigma} = (\tilde{\sigma}_0, \tilde{\sigma}_1, \ldots) \) be a sequence of iid exponential random variables with mean 1 as \( \sigma \). Define

\[
W_0 := \tilde{\sigma}_0 \quad (23)
\]

and then, recursively, for \( i \geq 1 \):

\[
W_{\tilde{y}_i} := [W_{\tilde{y}_{i-1}} - (\tilde{y}_i - \tilde{y}_{i-1})]^+ + \tilde{\sigma}_i \quad (24)
\]

\[
W_y := [W_{\tilde{y}_{i-1}} - (y - \tilde{y}_{i-1})]^+, \quad y \in (\tilde{y}_{i-1}, \tilde{y}_i) \quad (25)
\]

To stress the dependence of \( (\tilde{y}, \tilde{\sigma}) \) we denote \( W(\tilde{y}, \tilde{\sigma}) = (W_y(\tilde{y}, \tilde{\sigma}); y \geq 0) \) the process defined by (23), (24) and (25).

The exit time of customer \( i \) is defined by

\[
\tilde{s}_i := \tilde{y}_i + W_{\tilde{y}_i} \quad (26)
\]

for \( i \geq 0 \). That is, the \( i \)th arrival time plus the workload of the server at arrival (included the \( i \)th service time). Define

\[
\tilde{s}(\tilde{y}, \tilde{\sigma}) := (\tilde{s}_0, \tilde{s}_1, \ldots) \quad (27)
\]

Workload cycle

Let \( \tilde{X} \) be the first time after time zero the workload jumps from 0 to a positive value:

\[
\tilde{X} := \inf\{y > 0: W_y = 0, W_y > 0\}
\]

Define

\[
\tilde{C} := (W_y; y \in [0, \tilde{X}))
\]

and let \( \tilde{N} \) be the number of arrivals during the cycle:

\[
\tilde{N} := \max\{i : \tilde{y}_i \in [0, \tilde{X})\} + 1
\]

(we add 1 to take account of the arrival at time 0).
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Clearly \( \tilde{C}, \tilde{X} \) and \( \tilde{N} \) are function of \((\tilde{y}, \tilde{\sigma})\):

\[
\tilde{C} = \tilde{C}(\tilde{y}, \tilde{\sigma}), \quad \tilde{X} = \tilde{X}(\tilde{y}, \tilde{\sigma}), \quad \tilde{N} = \tilde{N}(\tilde{y}, \tilde{\sigma})
\]

**Lemma 28.** If \( \lambda < 1 \) then \( \tilde{X} \) has finite expectation and there exists a stationary version of the workload process denoted \( W = (W_y; y \in \mathbb{R}) \).

For a proof see Loynes [16]; we prove this later using cycles, following [20]. The proof of the following theorem can be found in Prabhu [18], theorem 8 (page 98) or Baccelli and Brémaud [1].

**Theorem 1** (Burke’s Theorem). The exit times \((s_i; i \in \mathbb{Z})\) of the stationary process \( W \) is a Poisson process with parameter \( \lambda \).

![Figure 2. Workload evolution related with the traffic example of Figure 1. The cycle has 4 customers. The exit time \( s_3 \) of customer 3 occurs before the time arrival \( y_4 \) of customer 4 so that a new cycle starts at \( S_4 = y_4 \). The service times \( \tilde{\sigma}_i \) coincide with the final delays \( \sigma_i \) and the customer exit times \( \tilde{s}_i \) coincide with the final relative car positions \( s_i \) of Figure 1.](image)

5. Queue associated to traffic model

The car trajectories generated by semi-infinite car positions \( y \) and car-delays \( \xi \) defined in (7) generate final car-delays \( \sigma = \sigma(y, \xi) \) and last passages through the origin \( B_{i,0}(y, \xi) \). The queue generated by arrivals \( y \) and waiting times \( \sigma \) produce a workload process \( W_y(y, \sigma) \). The relation between these processes is given by

\[
W_y = (B_{i,0} - y)^+ \quad \text{for } y \in [y_i, y_{i+1})
\]

(29)
This follows from the definitions (23)-(25) and (13). In the corresponding cycles this relation reads

$$W_y = \begin{cases} B_{i,0} - y & \text{if } y_i \leq y < y_{i+1} \leq y_{N-1}, \\ 0 & \text{if } y_{N-1} \leq y \leq y_N \end{cases}$$

(30)

As a consequence of (29), the queue exit times $\tilde{s} = \tilde{s}(\mathbf{y}, \mathbf{\sigma})$ coincide with the final relative car-positions $\mathbf{s}$. The length and number of elements in the respective cycles agree:

**Lemma 31.** Let $(\mathbf{y}, \mathbf{\xi})$ be semi infinite car initial positions and delay times as in (7). Then

$$\tilde{s}(\mathbf{y}, \mathbf{\sigma}(\mathbf{y}, \mathbf{\xi})) = s(\mathbf{y}, \mathbf{\xi})$$

(32)

$$\tilde{X}(\mathbf{y}, \mathbf{\sigma}(\mathbf{y}, \mathbf{\xi})) = X(\mathbf{y}, \mathbf{\xi}), \quad \tilde{N}(\mathbf{y}, \mathbf{\sigma}(\mathbf{y}, \mathbf{\xi})) = N(\mathbf{y}, \mathbf{\xi})$$

(33)

**Proof** (33) is a consequence of (32) and the definitions. It suffices to prove (32) for each cycle. $\sigma_0 = \xi_{0,0} = D_0$, so that by (1), $s_0 = 0 + \sigma_0 = \tilde{s}_0$ by (20) because $W_0 = \sigma_0$. For $1 < i < N$,

$$\tilde{s}_i = y_i + W_{y_i} = B_{i,0} = s_i$$

by (26), (30) and (20).

6. **Stationary traffic process**

Let $((\mathbf{y}_n, \mathbf{\xi}_n); n \in \mathbb{Z})$ be a iid sequence of Poisson processes and delay times with the same law as $(\mathbf{y}, \mathbf{\xi})$. Let $C_n = C(\mathbf{y}_n, \mathbf{\xi}_n)$ be the cycles generated by these variables. Then $(C_n; n \in \mathbb{Z})$ is a sequence of iid cycles with the same distribution as $C$. Let $X_n$ be the length of cycle $C_n$, $N_n$ the number of cars involved with this cycle. By (33) and Lemma 28 $X_n$ has finite expectation. Let

$$S_n^o := \begin{cases} 0 & \text{if } n = 0 \\ \sum_{k=1}^n X_k & \text{if } n > 0 \\ -\sum_{k=n}^0 X_k & \text{if } n \leq 0 \end{cases} \quad M_n := \begin{cases} 0 & \text{if } n = 0 \\ \sum_{k=1}^n N_k & \text{if } n > 0 \\ -\sum_{k=n}^0 N_k & \text{if } n \leq 0 \end{cases}$$

(34)

Define the traffic process $Z^o = (Z^o_y; y \in \mathbb{R})$ by

$$Z^o_y := Z_{y-S_n}(C_n) \quad \text{for } y \in [S_n^o, S_{n+1}^o), \quad n \in \mathbb{Z}$$

(35)

as the process obtained by juxtaposing the cycles one after the other and putting the beginning of cycle 0 at the origin; recall the definition of $Z_y(C)$ in (18). In the process $Z^o$ the cars of a cycle are not blocked by cars of the previous cycle; in particular, all cars starting at positions $S_n$, $n \in \mathbb{Z}$, will be free after waiting an exponential time of mean 1.

Let $U$ be a uniform random variable in $[0, 1]$ independent of $Z^o$. Let $\mathbb{P}^o$ be the law of $((\mathbf{y}_n, \mathbf{\xi}_n); n \in \mathbb{Z}), U$) and $\mathbb{E}^o$ the corresponding expectation. As $Z^o$ is a function of $((\mathbf{y}_n, \mathbf{\xi}_n); n \in \mathbb{Z})$, we can also think that $\mathbb{P}^o$ is the law of $(Z^o, U)$. Let $\theta_r$ be the
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translation operator defined by \((\theta_rZ)_y = Z_{y-r}\). Define \(Z\) as a function of \(Z^o\) and \(U\) as the process \(Z^o\) “as seen from \(-(1-U)X_0\)”:

\[
Z := \theta_{-(1-U)X_0}Z^o \quad y \in \mathbb{R}
\]

and define the law \(\mathbb{P}\) by size biasing cycle zero:

\[
d\mathbb{P} := \frac{X_0}{\mathbb{E}^oX_0}d\mathbb{P}^o
\]

The following result is in Theorem 4.1 of Chapter 8 of Thorisson [20]; see also Figure 2.1 in that chapter for a better understanding of the relation between \(Z\) and \(Z^o\).

**Lemma 38.** The law of \(Z\) under \(\mathbb{P}\) is stationary.

For test functions \(f\) (from the space where \(Z\) is defined to \(\mathbb{R}\)) the law of \(Z\) under \(\mathbb{P}\) satisfies

\[
\mathbb{E}_f(Z) = \frac{\mathbb{E}^o[X_0f(\theta_{-X_0(1-U)}Z^o)]]}{\mathbb{E}^oX_0}
\]

where \(X_0\) is the length of cycle 0 of \(Z^o\). To obtain a sample of \(Z\) under \(\mathbb{P}\), first sample a process with a cycle starting at the origin with the size biased law \(\mathbb{P}\), then translate the origin to a point uniformly distributed in cycle 0.

Since the traffic process \(Z\) is constructed as juxtaposition of cycles, the initial car positions, the final delays and the final relative car positions can be recovered as follows using the notation of (19):

\[
S_0 = \sup\{y \leq 0 : Z_{y-} = 0, Z_y > 0\},
\]

\[
S_n = \inf\{y > S_{n-1} : Z_{y-} = 0, Z_y > 0\}, n > 0,
\]

\[
S_n = \sup\{y < S_{n+1} : Z_{y-} = 0, Z_y > 0\}, n < 0;
\]

\[
X_n = S_{n+1} - S_n, \quad C_n = (Z_{y-S_n}; y \in [S_n, S_n + X_n)), \quad N_n = N(C_n); (42)
\]

\[
y_k = S_n + y_k - M_n(C_n), \quad \sigma_k = \sigma_k - M_n(C_n), \quad s_k = X_n + s_k - M_n(C_n),
\]

for \(M_n \leq k < M_{n+1}\).

Denote \(y(Z)\), \(\sigma(Z)\) and \(s(Z)\) the initial car-positions, the final delays and final relative car-positions of the process \(Z\).

**Lemma 45.** The law of \(y(Z)\) and \(\sigma(Z)\) under \(\mathbb{P}\) are stationary and \(y(Z)\) and \(\sigma(Z)\) are independent. In particular \(y(Z)\) is a Poisson process of parameter \(\lambda\) and \(\sigma(Z)\) is a sequence of iid exponential random variables of mean 1. Furthermore \(s(Z)\) is a Poisson process of parameter \(\lambda\).

Stationarity of \(y(Z)\) and \(\sigma(Z)\) is immediate consequence of stationarity of \(Z\). The other properties follow from the stationary construction of the queue and relation (32) as shown in the next section.
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Trajectory extension. The trajectory of the car starting at \( y_i \in [S_n, S_{n+1}] \) is defined only for \( t \in [0, B_{i,0}(C_n)] \); see (18). We extend the trajectory by just continuing at speed 1 from this point on:

\[
\pi_i(t) = \begin{cases} 
S_n + \pi_{i-M_n}(t) & \text{if } y_i \in [S_n, S_{n+1}], \ t \in [0, B_{i,0}(C_n)] \\
S_n - t + B_{i-M_n,0}(C_n) & \text{if } y_i \in [S_n, S_{n+1}], \ t > B_{i,0}(C_n)
\end{cases}
\]

(46)

where \( \pi_{i-M_n}(t) \) is the trajectory of the \( i \)th particle of cycle \( C_n \). Using definitions (46) and (18) the corresponding process \( Z \) has a countable number of coordinates at each position \( y \); the \( k \)th coordinate indicates the interval of time spent at \( y \) by the \( k \)th car initially to the right of \( y \). We abuse notation and continue calling \( Z \) this process. Notice that \( y(Z) \), \( \sigma(Z) \) and \( s(Z) \) remain unchanged and Lemma 45 holds for this extension.

7. Stationary workload process

Assume \( \lambda < 1 \) and let \( (y_n, \xi_n) \) be the sequence introduced in the beginning of previous section. Let \( \sigma_n = \sigma(y_n, \xi_n) \) as defined in (13). By (33) the workload cycles \( \hat{C}_n = \hat{C}(y_n, \sigma_n) \) have the same length as the traffic cycles \( C_n \): \( \hat{X}_n = X_n \) and the number of cars in cycle \( C_n \) is the same as the number of customers in cycle \( \hat{C}_n \): \( \hat{N}_n = N_n \). Let \( S_n \) as in (34) and define \( W^\circ = (W^\circ_y; \ y \in \mathbb{R}) \) by

\[
W^\circ_y = W_{y-S_n}((\hat{C}_n)^n) \quad \text{for } y \in [S_n^o, S_{n+1}^o), \ n \in \mathbb{Z}
\]

(47)

and the process \( W = (W_y; \ y \in \mathbb{R}) \) by

\[
W_y = W_{y+(1-U)X^o_0} \quad y \in \mathbb{R}
\]

(48)

where \( U \) is the same variable used in (36). As before, for \( \mathbb{P} \) given by (37),

Lemma 49. Under \( \mathbb{P} \) the law of \( W \) is stationary.

\( W \) is a stationary M/M/1 queue with arrivals \( y(W) \) and departures \( \hat{s}(W) \). Hence the arrival process \( y(W) \) is a stationary Poisson process of rate \( \lambda \) in \( \mathbb{R} \). By Burke’s Theorem \( \mathbb{I} \) the same is true for the departure process \( \hat{s}(W) \).

The stationary workload process \( W \) and the stationary traffic process \( Z \) are constructed in the same space (as function of \( ((y_n, \xi_n); \ n \in \mathbb{Z}), U \) so that they have exactly the same cycles and the initial and relative final car positions of \( Z \) coincide respectively with the arrival and departure process of \( W \):

\[
y(Z) = y(W), \quad s(Z) = \hat{s}(W)
\]

(50)

This finishes the proof of Lemma \( \mathbb{I} \) and Proposition \( \mathbb{I} \).

8. Final remarks

When the initial density of cars is smaller than the inverse of the delay time, \( \lambda < 1 \) the process is called subcritical. We have described how a stationary configuration of initial car positions organize the departure from speed zero to speed 1 under the
rule slow-to-start in the subcritical case. The method relates the space-stationary one-dimensional slow-to-start traffic model with the workload process of a $M/M/1$ time-stationary queuing system. If the initial position of cars is Poisson then the final relative position of free cars is also Poisson. The same method shows that if the initial position of cars $y$ is a stationary ergodic process with density $\lambda < 1$ (density is the mean number of cars per unit length) then the final relative car position process coincides with the departure process of the queue with arrival process $y$.

The supercritical case merits to be investigated. When $\lambda > 1$ cars form “traffic jams” on a subset of the initial positions $y$, depending on time. As time grows, the density of this set goes to zero and the number of cars per traffic jam increases. We agree with an anonymous referee who conjectures that as $t$ goes to infinity the set of traffic jams suitably rescaled converges to a Poisson process of rate 1. The critical case $\lambda = 1$ should also show traffic jams.

Another model to research might include spontaneous stops at some rate $\nu$ keeping the slow-to-start rule. These kind of models will be closer to those studied by Nagel and Schreckenberg and Gray and Griffeath.

We are investigating the same phenomena for cellular automata in $\mathbb{Z}$. The approach can be applied but there are complications coming from the hard core interaction. The results are similar.

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