Zeta-polynomials, Hilbert polynomials, and the Eichler–Shimura identities

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Abstract
Recently, Ono et al. answered problems of Manin by defining zeta-polynomials $Z_f(s)$ for even weight newforms $f \in S_k(\Gamma_0(N))$; these polynomials can be defined by applying the “Rodriguez–Villegas transform” to the period polynomial of $f$. It is known that these zeta-polynomials satisfy a functional equation $Z_f(s) = \pm Z_f(1 - s)$ and they have a conjectural arithmetic-geometric interpretation. Here, we give analogous results for a slightly larger class of polynomials which are also defined using the Rodriguez–Villegas transform.

Keywords: Period polynomials, Modular forms, Zeta-polynomials, Eichler–Shimura relations, Hilbert polynomials

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1 Introduction and statement of results
Let $f \in S_k(\Gamma_0(N))$ be a newform of even weight $k$ and level $N$, and let $L(f,s)$ be the $L$-function associated with $f$. Manin [8] speculated that the critical $L$-values $L(f,1), L(f,2), \ldots, L(f,k-1)$ can be assembled in a natural way to build a zeta-polynomial. This polynomial $Z_f(s)$ should

(i) satisfy a functional equation $Z_f(s) = \pm Z_f(1 - s)$,

(ii) obey the "Riemann hypothesis:" if $Z_f(\rho) = 0$ then $\text{Re}(\rho) = 1/2$, and

(iii) have an arithmetic-geometric interpretation.

Recently, Ono et al. [9] defined a zeta-polynomial $Z_f(s)$ which satisfies properties (i) and (ii) above. Assuming the Bloch–Kato Tamagawa Number Conjecture, it also satisfies property (iii) by encoding the arithmetic of a combinatorial arithmetic-geometric object called “Bloch–Kato complex” for $f$.

Although $Z_f(s)$ can be defined as a sum involving weighted moments of critical $L$-values and signed Stirling numbers of the first kind, it is more convenient here to instead express it in terms of (a slightly normalized version of) the period polynomial of $f$, which is given by

$$R_f(X) := \left(\frac{\sqrt{N}}{2\pi}\right)^{k-1} \frac{(k-2)!}{(2\pi)^{k-1}} \sum_{n=0}^{k-2} \frac{(2\pi X)^n}{n!(\sqrt{N})^n} L(f,k - n - 1).$$
Period polynomials are well-studied objects which are known to have many beautiful properties. For example, it is known that $R_f(X)$ satisfies its own Riemann hypothesis: all of its roots occur on the unit circle $|X| = 1$, as proved in [4,6]. See Sect. 2.1 for additional background information about $R_f(X)$. In [9], $Z_f(s)$ is described as the unique polynomial which satisfies

$$
\frac{R_f(X)}{(1 - X)^{w+1}} = \sum_{n \geq 0} Z_f(-n)X^n,
$$

where $w := k - 2$. This relationship between $R_f(X)$ and $Z_f(s)$ is known as the “Rodriguez-Villegas transform,” and a key theorem of Rodriguez–Villegas [10] allows Ono, Rolen, and Sprung to translate the Riemann hypothesis for $R_f(X)$ into statements (i) and (ii) about $Z_f(s)$.

Zeta-polynomials are relatively new objects, and little else is currently known about their properties. However, the results described thus far give evidence that known properties of a newform $f$ and its period polynomial $R_f(X)$ could be translated into the realm of the zeta-polynomial $Z_f(s)$. This could give us more insight into the behavior of zeta-polynomials.

The goal of this article is to offer additional evidence in this direction.

Note, however, that the results here are general enough that they do not require $R(X)$ to be the period polynomial of a newform (and for example, our results will apply to even/odd parts of period polynomials). Thus, we fix the following notation: let $w \geq 2$ be even, let $R(X) \in \mathbb{C}[X]$ be any polynomial of degree at most $w$, and let $Z(s)$ be the unique polynomial which satisfies

$$
\frac{R(X)}{(1 - X)^{w+1}} = \sum_{n \geq 0} Z(-n)X^n,
$$

(1)

Our first result assumes the identity

$$
R(X) + \epsilon i^w R(1/X) = 0
$$

(2)

(where $\epsilon = \pm 1$ is a constant) and interprets its meaning in terms of $Z(s)$. Note here that Eq. (2) is important because it is known to be true for any period polynomial (as well as its even and odd parts) associated with a newform $f \in S_k(\Gamma_0(N))$, where $\epsilon = \pm 1$ is the eigenvalue of $f$ under the Fricke involution.

**Theorem 1** Let $w \geq 2$ be even, let $R(X) \in \mathbb{C}[X]$ be any polynomial of degree at most $w$, and let $Z(s)$ be the polynomial satisfying (1). If $R(X)$ satisfies (2), then we have that

$$
Z(s) + \epsilon i^w Z(1 - s) = 0.
$$

**Remark 1** Since the period polynomial $R_f(X)$ satisfies Eq. (2) and the conclusion of the theorem gives property (i) above, one can view Theorem 1 as a generalization of property (i) proved by Ono et al. [9] which uses completely different techniques (and does not depend on the Riemann hypothesis for period polynomials).

One may also consider what the “Eichler–Shimura relations” for period polynomials associated with cusp forms of level 1 (as described in Sect. 2) tell us about zeta-polynomials. Thus, we suppose that
\[ R(X) + (-iX)^w R(1/X) = 0, \quad (3) \]

\[ R(X) + (-iX)^w R \left( \frac{X - i}{iX} \right) + (-iX - 1)^w R \left( \frac{-i}{-iX - 1} \right) = 0, \quad (4) \]

and obtain the following result.

**Theorem 2** Let \( w \geq 2 \) be even, let \( R(X) \in \mathbb{C}[X] \) be any polynomial of degree at most \( w \), and let \( Z(s) \) be the polynomial satisfying (1). If \( R(X) \) satisfies (3) and (4), then

\[ Z(s) + i^w Z(1 - s) = 0 \]

and for any positive integer \( n \) we have that

\[
Z(-n) + (-i)^w \sum_{m=1}^{n+1} a_m Z(1 - m) \\
+ \sum_{k \geq 0} \sum_{m=0}^{k+n} \sum_{j=0}^{k+n-m} \binom{k+n}{m+w} \binom{w+1}{j} (-1)^{j+1} (-i)^k (1-i)^{m+w+1} Z(m+j-k-n) = 0,
\]

where

\[
\frac{(1-x)^{w+1} (x+i)^n}{(x+i-ix)^{w+1} (ix)^{n+1}} = \sum_{m=-n-1}^{\infty} a_m x^m.
\]

**Example 1** For instance, consider the unique newform \( \Delta \in S_{12}(\Gamma_0(1)) \). Here (using [11], Section 4.1 of [9], or Section 1.1 of [7]), we have that

\[
R_{\Delta}(X) \approx 0.114379 \cdot \left( \frac{36}{691} X^{10} + X^8 + 3X^6 + 3X^4 + X^2 + \frac{36}{691} \right) \\
+ 0.00926927 \cdot \left( 4X^9 + 25X^7 + 42X^5 + 25X^3 + 4X \right)
\]

and

\[
Z_{\Delta}(s) \approx (5.11 \times 10^{-7}) s^{10} - (2.554 \times 10^{-6}) s^9 + (6.01 \times 10^{-5}) s^8 - (2.25 \times 10^{-4}) s^7 \\
+ 0.00180s^6 - 0.00463s^5 + 0.0155s^4 - 0.0235s^3 + 0.0310s^2 \\
- 0.0199s + 0.00596.
\]

Since \( \Delta \) is the unique normalized cusp form of weight 12 and level 1, we have that \( R_{\Delta}(X) \) satisfies Eqs. (2), (3), and (4), so Theorems 1 and 2 apply, although of course the first statement of Theorem 2 that

\[ Z_{\Delta}(s) - Z_{\Delta}(1 - s) = 0 \]

is the same as Theorem 1, and it is already known by [9]. However, one can now consider

\[ R_{\Delta}'(X) := 4X^9 + 25X^7 + 42X^5 + 25X^3 + 4X \]

(although we have abused notation a bit here by scaling to omit the constant 0.00926927). Although the roots of this polynomial can be understood using work of Conrey et al. in [3], this polynomial does not satisfy the Riemann hypothesis that \( R_{\Delta}(X) \) does, so the work
in [9] does not apply here. Since \( R_\Delta(X) \) still satisfies Eqs. (2), (3), and (4), Theorems 1 and 2 still apply to the zeta-polynomial

\[
Z_\Delta(s) = \frac{10}{36288} - \frac{5}{36288}s^9 + \frac{7}{2160}s^8 - \frac{367}{30240}s^7 + \frac{833}{8640}s^6 - \frac{2137}{8640}s^5 \\
+ \frac{70841}{90720}s^4 - \frac{13193}{11340}s^3 + \frac{403}{360}s^2 - \frac{727}{1260}s^1.
\]

In particular, we have that

\[
Z_\Delta(s) - Z_\Delta(1 - s) = 0.
\]

Analogous statements hold for \( Z_\Delta^+(s) \) as well.

**Example 2** Similarly, we now consider the unique newform \( f = q - 8q^2 + 12q^3 + 64q^4 + \cdots \in S_8(\Gamma_0(2)) \). Here (using [11] or Section 2 of [2]), we have that

\[
R_f(X) = \omega^+ R_f^+(X) + \omega^- R_f^-(X),
\]

where \( \omega^+ \approx 0.00995207, \omega^- \approx 0.0419648 \), and

\[
R_f^+(X) = 2X^6 + 17X^4 + 17X^2 + 2, \\
R_f^-(X) = 2X^5 + 5X^3 + 2X.
\]

Applying the Rodriguez–Villegas transform (1), we have

\[
Z_f(s) = \omega^+ Z_f^+(s) + \omega^- Z_f^-(s),
\]

where

\[
Z_f^+(s) = \frac{19}{360}s^6 - \frac{19}{120}s^5 + \frac{43}{36}s^4 - \frac{17}{8}s^3 + \frac{171}{360}s^2 - \frac{223}{60}s + 2, \\
Z_f^-(s) = \frac{1}{80}s^6 - \frac{13}{80}s^5 + \frac{13}{48}s^4 - \frac{23}{48}s^3 + \frac{43}{60}s^2 - \frac{29}{60}s.
\]

Note that \( R_f(X), R_f^+(X), \) and \( R_f^-(X) \) each satisfy Eq. (2) with \( \varepsilon = 1 \), so Theorem 1 guarantees that

\[
Z_f(s) - Z_f(1 - s) = 0
\]

(which is also already known by [9]) and also that

\[
Z_f^+(s) - Z_f^-(1 - s) = 0.
\]

Finally, we have an alternative way of describing how \( R(X) \) satisfies property (iii) above.

**Theorem 3** Let \( w \geq 2 \) be even, let \( R(X) \in \mathbb{C}[X] \) be any polynomial of degree at most \( w \), and let \( Z(s) \) be the polynomial satisfying (1). If \( R(X) \) satisfies (2) and \( Z(s) \) has integer coefficients with positive leading term, then \( Z(s) \) is a Hilbert polynomial.

**Example 3** For example, consider \( w := 2, R(X) := 2X^2 - 2, \) and

\[
Z(s) := 4s - 2.
\]
By Theorem 1, we have that \( Z(s) + Z(1 - s) = 0 \) since \( R(X) \) and \( Z(s) \) satisfy (1) and (2) with \( \varepsilon = -1 \). By Theorem 3, we also have that \( Z(s) \) is a Hilbert polynomial; for example, one can compute (using, for example, [5]) that \( Z(s) = 4s - 2 \) is the Hilbert polynomial of \( R/I \), where \( R := k[x, y, z] \) and \( I := \langle x^4 + y^4 \rangle \).

This paper is organized as follows. In Sect. 2, we will review the relevant background related to period polynomials, zeta-polynomials, and Hilbert polynomials. In Sects. 3, 4, and 5, we will prove Theorems 1, 2, and 3, respectively.

2 Preliminaries

2.1 Period polynomials of modular forms

First, we must define our notation and review the required background related to modular forms and their period polynomials; period polynomials give a context for Theorems 1, 2, and 3 by providing natural applications of these results. For additional information, see, for example, the discussions in [7] and [2].

Here we follow the standard notation: let \( \mathbb{H} \) denote the upper half plane. For an even integer \( k \) and \( \gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}_2^+(\mathbb{R}) \), we define the slash operator \( |_k \) for holomorphic functions \( f : \mathbb{H} \to \mathbb{C} \) by

\[
(f |_k \gamma)(\tau) := (ad - bc)^{k/2}(c\tau + d)^{-k} f \left( \frac{a\tau + b}{c\tau + d} \right).
\]

If \( N \) is a positive integer, we let \( S_k(\Gamma_0(N)) \) denote the space of cusp forms of weight \( k \) on \( \Gamma_0(N) \).

Now, we summarize the theory of period polynomials. Let \( f \in S_k(\Gamma_0(N)) \) be a cusp form of even weight \( k \) and level \( N \), and set \( w := k - 2 \). The period polynomial associated with \( f \) is given by

\[
r_f(X) := \int_0^\infty f(\tau)(X - \tau)^w \, d\tau,
\]

which is a polynomial in the space

\[
\mathbb{V}_w := \{ P \in \mathbb{C}[X] : \text{deg}(P) \leq w \}.
\]

We also define \( r_f^+(X) \) and \( r_f^-(X) \) to be the even and odd parts of \( r_f(X) \), respectively, and note that \( r_f^\pm(X) \in \mathbb{V}_w^\pm \) (where of course \( \mathbb{V}_w^+ \) and \( \mathbb{V}_w^- \) are defined to be the set of even and odd polynomials of degree at most \( w \), respectively). There is an action of \( \text{PGL}_2^+(\mathbb{R}) \) on \( \mathbb{V}_w \) via the slash operator \( |_{-w} \).

Note that if the cusp form \( f \) is an eigenfunction of the Fricke involution \( W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \), i.e., \( f |_k W_N = \varepsilon f \) for \( \varepsilon \in \{ \pm 1 \} \), then it follows that \( r_f \) (as well as \( r_f^\pm \)) satisfies

\[
r_f |_{-w} (1 + \varepsilon W_N) = 0. \tag{5}
\]

(This fact can also be obtained using the functional equation of the \( L \)-function associated with \( f \).)

On the other hand, if the modular form \( f \) has level \( N = 1 \), then one can show that \( r_f \) (as well as \( r_f^\pm \)) satisfies the Eichler–Shimura relations

\[
r_f |(1 + S) = 0 \tag{6}
\]

\[
r_f |(1 + U + U^2) = 0. \tag{7}
\]
where 
\[
S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Thus, we define 
\[
\mathbb{W}_w := \{ P \in \mathbb{V}_w : P | (1 + S) = P | (1 + U + U^2) = 0 \}
\]
and note that \( r_f \in \mathbb{W}_w \). The following result of Eichler–Shimura illustrates the importance of the period polynomial \( r_f(X) \).

**Theorem 4**  The map

\[
S_k(\text{SL}_2(\mathbb{Z})) \rightarrow \mathbb{W}_w^- \\
f \mapsto r_f^-
\]
is an isomorphism. The map

\[
S_k(\text{SL}_2(\mathbb{Z})) \rightarrow \mathbb{W}_w^+ \\
f \mapsto r_f^+
\]
is an injection whose image is a subspace of \( \mathbb{W}_w^+ \) of codimension 1.

### 2.2 Zeta-polynomials for modular form periods

Let \( f \in S_k(\Gamma_0(N)) \) be a newform of even weight \( k \geq 4 \). As discussed in the introduction, Ono et al. considered in [9] a reformulated version of the period polynomial

\[
R_f(X) := (\sqrt{N}/i)^{k-1}r_f(X/i\sqrt{N}).
\]

These polynomials serve as the inspiration for this work, so we note here that equation (5) gives

\[
R_f(X) + \varepsilon i^w X^w R_f(1/X) = 0,
\]
i.e., \( R_f(X) \) satisfies Eq. (2). Also, in the special case where \( N = 1 \), the Eichler–Shimura relations (6) and (7) above tell us that 

\[
R_f(X) + (-iX)^w R_f(1/X) = 0,
\]
\[
R_f(X) + (-iX)^w R_f \left( \frac{X - i}{-iX} \right) + (-iX - 1)^w R_f \left( \frac{-i}{-iX - 1} \right) = 0,
\]
i.e., \( R_f(x) \) satisfies Eqs. (3) and (4) when \( N = 1 \) (with \( \varepsilon = 1 \)). As discussed in the introduction, the zeta-polynomials for modular form periods \( Z_f(s) \) are given by

\[
\frac{R_f(X)}{(1 - X)^{w+1}} = \sum_{n \geq 0} Z_f(-n)X^n.
\]
2.3 Hilbert polynomials

Here, we give the necessary background related to Hilbert polynomials; for more information, see [1]. Fix a field \( k \), let \( R = \bigoplus_{j \geq 0} R_j \) be a graded \( k \)-algebra, and suppose that \( R \) is standard (i.e., that it can be finitely generated by elements of \( R_1 \)). The Hilbert series of \( R \) is the formal power series

\[
\sum_{j \geq 0} \dim_k (R_j) X^j.
\]

It is known that the Hilbert series can be written as

\[
\frac{U(X)}{(1 - X)^r} = \sum_{j \geq 0} \dim_k (R_j) X^j
\]

for some positive integer \( r \) and some polynomial \( U(X) \), and it is also known that there exists a polynomial \( P_R(X) \in \mathbb{Q}[X] \) such that

\[
P_R(j) = \dim_k (R_j)
\]

for all sufficiently large \( j \).

Thus, we make the following definition: a polynomial \( H(X) \in \mathbb{Q}[X] \) is called a Hilbert polynomial if there exists a standard graded \( k \)-algebra \( R \) such that \( H(X) = P_R(X) \). Work of Brenti [1] investigates which polynomials are Hilbert polynomials, and how to measure “how far” a polynomial is from being Hilbert. Along the way, Brenti proves the following useful results.

**Theorem 5** (Theorems 3.5 and 3.14 of [1]) Let \( H(X) \in \mathbb{Z}[X] \) be a polynomial with positive leading term.

- There exists \( M \in \mathbb{N} \) such that \( H(X + j) \) is a Hilbert polynomial for any \( j \geq M \).
- If \( H(X) \) is a Hilbert polynomial then \( H(X + 1) \) is a Hilbert polynomial.

3 Proof of Theorem 1

Let \( w \geq 2 \) be even, let \( R(X) = \sum_{j=0}^{w} a_j X^j \in \mathbb{C}[X] \), and let \( Z(s) \) be the polynomial satisfying

\[
\frac{R(X)}{(1 - X)^{w+1}} = \sum_{n \geq 0} Z(-n) X^n.
\]

In order to better understand the relationship between \( R(X) \) and \( Z(s) \), we use Newton’s Binomial Theorem, which says that

\[
\frac{1}{(1 - X)^{w+1}} = \sum_{n \geq 0} \binom{w+n}{n} X^n.
\]

Thus, we have

\[
\sum_{n \geq 0} Z(-n) X^n = \frac{R(X)}{(1 - X)^{w+1}}
\]

\[
= \left( \sum_{j=0}^{w} a_j X^j \right) \left( \sum_{n \geq 0} \binom{w+n}{n} X^n \right)
\]

\[
= \sum_{n \geq 0} \left( \sum_{j=0}^{w} a_j \binom{w+n-j}{w} \right) X^n
\]
so we can now express $Z(s)$ explicitly by

$$Z(s) = \sum_{j=0}^{w} a_j \binom{w-s-j}{w}.$$  

Now, to prove Theorem 1, we suppose Eq. (2), i.e., that

$$a_j + \varepsilon^i a_{w-j} = 0$$

for $0 \leq j \leq w$. Thus,

$$Z(1-s) = \sum_{j=0}^{w} a_j \binom{w-1+s-j}{w} = \sum_{j=0}^{w} a_{w-j} \binom{w-1+s-(w-j)}{w}$$

$$= \sum_{j=0}^{w} -\varepsilon^i a_j \binom{w-1+s+j}{w} = \sum_{j=0}^{w} -\varepsilon^i a_j \binom{w-s-j}{w}$$

$$= -\varepsilon^i Z(s),$$

as desired. □

4 Proof of Theorem 2

First, note that the first statement of Theorem 2 follows from Theorem 1 by letting $\varepsilon = 1$.

To complete the proof of Theorem 2, we note that Eq. (4) tells us that for any positive integer $n$, we have that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{R(z)}{(1-z)^{w+1}} z^{-(n+1)} \, dz + \frac{1}{2\pi i} \int_{\gamma} \frac{(-iz)^w R \left( \frac{z-i}{z-i} \right)}{(1-z)^{w+1}} z^{-(n+1)} \, dz$$

$$+ \frac{1}{2\pi i} \int_{\gamma} \frac{(-iz-1)^w R \left( \frac{-i}{iz} \right)}{(1-z)^{w+1}} z^{-(n+1)} \, dz = 0,$$

where $\gamma$ is a small circle with center 0 (oriented counter-clockwise). Our proof will follow by interpreting each integral of Eq. (8); note that by Cauchy's integral formula, the first integral is

$$\frac{1}{2\pi i} \int_{\gamma} \frac{R(z)}{(1-z)^{w+1}} z^{-(n+1)} \, dz = Z(-n).$$

Next, we note that the second integral is (by applying Eq. (3) and then setting $x = -iz/(z-i)$)

$$\frac{1}{2\pi i} \int_{\gamma} \frac{(-iz)^w R \left( \frac{z-i}{z} \right)}{(1-z)^{w+1}} z^{-(n+1)} \, dz$$

$$= \frac{-1}{2\pi i} \int_{\gamma} \frac{(-iz)^w R \left( \frac{z-i}{z} \right) \chi_{w+1}^{w+1} z^{-(n+1)} \, dz}{(1-z)^{w+1}}$$

$$= \frac{-(-i)^w}{2\pi i} \int_{\gamma} \frac{R \left( \frac{z-i}{z} \right) (z-i)^{w+2}}{(1-z)^{w+1}} z^{-(n+1)} \, dz$$

$$= \frac{(-i)^w}{2\pi i} \int_{\gamma_0} \frac{R(x)}{(1-x)^{w+1}} (x+i)^{w+1} (x+i)^{w+1} \, dx$$
Thus, we define a function

\[ g(z) = \sum_{m=1}^{\infty} a_{-m} Z(1 - m), \]

where \( \gamma_0 \) is a small circle with center 0 and

\[ \frac{(1 - x)^{w+1}(x + i)^n}{(x + i - ix)^{w+1}(ix)^{n+1}} = \sum_{m=-n-1}^{\infty} a_{m} x^m. \]

Finally, by Cauchy’s integral formula and Eq. (3), the third integral is the coefficient of \( z^n \) in the expansion for

\[ (-iz - 1) w R \left( \frac{-i}{-iz - 1} \right) \]

about 0. Thus, we define a function \( g \) by

\[ g(z - i) = \frac{-R(z - i)}{(1 - z)^{w+1}} = \frac{-R(z - i)}{(1 - z + i)^{w+1}} \cdot \frac{(1 - z + i)^{w+1}}{(1 - z)^{w+1}}. \]

Then

\[ g(z) = \frac{-R(z)}{(1 - z)^{w+1}} \cdot \frac{(1 - z)^{w+1}}{(1 - i - z)^{w+1}} \]

\[ = \left( - \sum_{n=0}^{\infty} Z(-n) z^n \right) (1 - z)^{w+1} \left( \sum_{m=0}^{\infty} \left( \frac{m + w}{w} \right) \frac{z^m}{(1 - i)^{m+w+1}} \right) \]

\[ = \sum_{k \geq 0} \left[ \sum_{m=0}^{k} \left( \frac{m + w}{w} \right) \frac{1}{(1 - i)^{m+w+1}} \sum_{j=0}^{k-m} \left( \frac{w + 1}{j} \right) (-1)^{j+1} Z(m + j - k) \right] z^k. \]

Now, set \( b_k \) to be the expression inside the brackets above, so that \( g(z) = \sum_{k \geq 0} b_k z^k \).

Since the radius of convergence of this series is \( \sqrt{2} \), we may substitute to find

\[ g(z - i) = \sum_{k \geq 0} b_k (z - i)^k = \sum_{n \geq 0} \sum_{k \geq 0} b_{k+n} \binom{k+n}{n} (i)^k z^n \]

\[ = \sum_{n \geq 0} \sum_{k \geq 0} \sum_{m=0}^{k+n} \sum_{j=0}^{k+n-m} \binom{k+n}{n} \binom{m+w}{w} \binom{w+1}{j} \times \frac{(-1)^{j+1} (i)^k}{(1-i)^{m+w+1}} Z(m + j - k - n) \]

\[ \cdot Z(m + j - k - n) \]

Thus, the third integral is

\[ \sum_{k \geq 0} \sum_{m=0}^{k+n} \sum_{j=0}^{k+n-m} \binom{k+n}{n} \binom{m+w}{w} \binom{w+1}{j} \frac{(-1)^{j+1} (i)^k}{(1-i)^{m+w+1}} Z(m + j - k - n), \]

completing the proof. \( \square \)
5 Proof of Theorem 3
Suppose for the sake of contradiction that $Z(s)$ is not a Hilbert polynomial. By the first part of Theorem 5, there exists some $M \in \mathbb{N}$ such that $Z(s+j)$ is Hilbert for any $j \geq M$ (and we may suppose without loss of generality that $M$ is minimal, i.e., that $Z(s+M)$ is Hilbert and $Z(s+M-1)$ is not Hilbert).

Note that by Theorem 1

$$Z(s+M) = -\epsilon i^w Z(1 - M - s)$$

is Hilbert, so the second part of Theorem 5 tells us that

$$-\epsilon i^w Z(2 - M - s) = Z(M + s - 1)$$

is also Hilbert. This is a contradiction. $\square$

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