QUENCHED LIMIT THEOREMS FOR RANDOM U(1) EXTENSIONS OF EXPANDING MAPS

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Abstract. In this paper we provide quenched central limit theorems, large deviation principles and local central limit theorems for random U(1) extensions of expanding maps on the torus. The results are obtained as special cases of corresponding theorems that we establish for abstract random dynamical systems. We do so by extending a recent spectral approach developed for quenched limit theorems for expanding and hyperbolic maps to be applicable also to partially hyperbolic dynamics.

1. Introduction

This paper concerns quenched limit theorems for random dynamical systems on a compact smooth Riemannian manifold $M$. Given a measure-preserving mapping $\sigma : \Omega \rightarrow \Omega$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random dynamical system (abbreviated RDS henceforth) on $M$ over $\sigma$ is given as a measurable map $T : N_0 \times \Omega \times M \rightarrow M$ satisfying $T(0, \omega, \cdot) = \text{id}_M$ and $T(n + m, \omega, x) = T(n, \sigma^{m} \omega, T(m, \omega, x))$ for each $n, m \in \mathbb{N}_0$, $\omega \in \Omega$ and $x \in M$, with $\mathbb{N}_0 = \{0, 1, \ldots\}$. Here $\sigma \omega$ denotes the value $\sigma(\omega)$, and $\sigma$ is called a driving system. A standard reference for random dynamical systems is the monograph by Arnold [3]. Here we merely recall that if we denote $T(n, \omega, \cdot) = T^{(n)}(\omega)$ and $T(1, \omega, \cdot) = T^{(1)}(\omega)$, respectively, then we have

$$T(n, \omega, \cdot) = T^{(n)}(\omega) = T^{(n-1)}(\omega) \circ T^{(n-2)}(\omega) \circ \cdots \circ T^{(1)}(\omega).$$

Conversely, given a measurable map $T : \Omega \times M \rightarrow M : (\omega, x) \mapsto T_{\omega}(x)$, the map $T : N_0 \times \Omega \times M \rightarrow M$ defined by (1.1) is an RDS over $\sigma$. We call it the RDS induced by $T$ over $\sigma$.

Given a measurable function $g : \Omega \times M \rightarrow \mathbb{R} : (\omega, x) \mapsto g_{\omega}(x)$, we consider the quenched (i.e. noisewise) Birkhoff sum $(S_{n}g)_{\omega} : M \rightarrow \mathbb{C}$ of $g$, given by

$$\frac{1}{n} - 1 \leq \sum_{j=0}^{n-1} g_{\sigma^j \omega} \circ T^{(j)}(\omega) \quad (n \geq 1, \ \omega \in \Omega).$$

Our ultimate interest is the $\mathbb{P}$-almost sure asymptotic behavior of such Birkhoff sums as $n \rightarrow \infty$. Results on such behavior are known as quenched limit theorems (for the random process $\{g_{\sigma^n \omega} \circ T^{(n)}(\omega)\}_{n \geq 0}$ in the random environment $\{\sigma^n \omega\}_{n \geq 0}$). Correspondingly, results on the asymptotic behavior of the expectation $\mathbb{E}_{\sigma}[S_{n}g]$ are referred to as annealed (i.e. averaged) limit theorems. For the history of annealed and quenched limit theorems for random dynamical systems, we refer to [2, 17, 18].

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In [2], Aimino et al. gave a comprehensive study of annealed limit theorems for abundant random dynamical systems (including random piecewise expanding maps) via a Nagaev-Guivarc’h type perturbative spectral approach for random dynamical systems (see Subsection 2.3 for a brief description of the approach). This was extended by Dragičević et al. [17] to quenched schemes by using the newly-developed theory of Lyapunov spectrum of random operators (see e.g. [26, 27] for a presentation of this theory) and the detailed analysis of the leading Lyapunov exponent of the associated transfer operator. This was recently further extended to random piecewise hyperbolic maps [18]. In this paper we extend the technology in [17] to a class of partially expanding maps with a neutral direction, namely, to U(1) extensions of expanding maps on the circle.

U(1) extensions of expanding maps can be seen as toy models of piecewise hyperbolic flows such as geodesic flows on negatively curved manifolds or dispersive billiard flows (via suspension flows of hyperbolic maps, see [31, 38]), and has been intensively studied by several authors, see e.g. [10, 13, 23–25, 36, 37]. In particular, billiard flows are closely related to kinetic theory of gases, and the random small perturbation (of the ceiling function $\tau$ with noise parameter $\omega$ defined below) studied in this paper can be viewed as random small deformations of the particles in the gas (scatters); see the recent paper by Demers and Liverani [12].

The difficulty in the analysis of these dynamics, compared with the expanding or hyperbolic maps studied in the previous works [17, 18], is the existence of a neutral direction. In fact, for limit theorems to hold the dynamics typically needs to be rapid mixing, but partially expanding (hyperbolic) systems with a neutral direction are not even weakly mixing in general. In [37], the second and third author showed that under a generic condition, randomly perturbed U(1) extensions of expanding maps exhibit quenched exponential mixing, and the corresponding transfer operator cocycles have a Lyapunov spectral gap. In this paper we extend the Nagaev-Guivarc’h type perturbative spectral approach of [17, 18] to random U(1) extensions of expanding maps, and estimates from [37] will play an essential role. To the best of our knowledge, the results in this paper are the first quenched limit theorems for random partially expanding (hyperbolic) systems with a neutral direction, see Theorems A–D below. These results are obtained as special cases of corresponding quenched limit theorems for abstract random dynamical systems, and we in fact reproduce the results of [17, 18] as a consequence of the abstract theorems (see Theorems 2.5, 2.6, 2.7 and 2.8). We hope that this analysis subsequently can be helpful in establishing quenched limit theorems for more complicated random dynamical systems.

1.1. Random U(1) extensions of expanding maps on $S^1$. Let $e_0 : S^1 \to S^1$ be a $C^\infty$ orientation-preserving diffeomorphism on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Let $k \geq 2$ be an integer and set $E_0(x) = ke_0(x) \text{ mod } 1$. Let $\tau_0 : S^1 \to \mathbb{R}$ be a $C^\infty$ function. We consider the skew product $T_0 : T^2 \to T^2$ of class $C^\infty$ given by

\begin{equation}
T_0 : \begin{pmatrix} x \\ s \end{pmatrix} \mapsto \begin{pmatrix} E_0(x) \\ s + \tau_0(x) \text{ mod } 1 \end{pmatrix}
\end{equation}

on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The map $E_0$ is assumed to be an expanding map on $S^1$ in the sense that $\min_x E'_0(x) > 1$, and we then say that $T_0$ is the $U(1)$ extension of the expanding map $E_0$ over $\tau_0$. As mentioned above, this dynamical system can
be seen as a toy model of (piecewise) hyperbolic flows, such as geodesic flows on
negatively curved manifolds or dispersive billiard flows.

Remark. We recall that if $\tau_0$ is cohomologous to a constant function (i.e. there is
a smooth function $u$ such that $\tau_0 - u \circ E_0 + u$ is a constant function), then $T_0$
not even weakly mixing ([11]; see also [10]) which implies that the corresponding
transfer operator does not have a spectral gap (and several limit theorems thus fail to
hold). This would prevent us from applying the Nagaev-Guivarc’h spectral
method. Therefore, in this paper we shall always assume that the function $\tau_0$ is not
cohomologous to a constant function, which is known to be a generic condition [11].

We do so by imposing the slightly stronger condition that $T_0$ is partially captive
which is still a generic condition on $\tau_0$ once $E_0$ is fixed [36]. A precise description
of the partial captivity condition is given in Appendix A.

Let $(\Omega, F, P)$ be a probability space. Let $\mathcal{C}^\infty(T^2, T^2)$ be the space of smooth
endomorphisms on $T^2$, endowed with the Borel $\sigma$-algebra. Given a noise level $\epsilon \geq 0$,
we let $T \equiv T(\epsilon) : \Omega \to \mathcal{C}^\infty(T^2, T^2) : \omega \mapsto T_\omega$ be a measurable map such that $T_\omega$ is
for $P$-almost every $\omega \in \Omega$ of the form

$$
T_\omega : \left( \frac{x}{s} \right) \mapsto \left( \frac{E_\omega(x)}{s + \tau_\omega(x)} \ mod \ 1 \right),
$$

and $\omega \mapsto T_\omega(z)$ is a measurable mapping from $\Omega$ to $T^2$ for each $z = (x, s) \in T^2$.
Here $E_\omega : S^1 \to S^1$ is given by $E_\omega(x) = k e_\omega(x) \ mod \ 1$ where $k \geq 2$ is the same
integer as before while $e_\omega : S^1 \to S^1$ is a $\mathcal{C}^\infty$ diffeomorphism and $\tau_\omega : S^1 \to \mathbb{R}$ is a
$\mathcal{C}^\infty$ function, $P$-almost surely. We also assume that

$$
\text{ess sup}_\omega d_{\mathcal{C}^\infty}(T_\omega, T_0) \leq \epsilon,
$$

where $T_0$ is the partially expanding map given by (1.3) and $d_{\mathcal{C}^\infty}$ is some choice
of metric on $\mathcal{C}^\infty(T^2, T^2)$. Note that $E_\omega$ is an expanding map for any sufficiently
small $\epsilon > 0$ and $P$-almost every $\omega \in \Omega$ (see Remark 3.2 for details), and that when
$\epsilon = 0$, $T_\omega = T_0$ for $P$-almost every $\omega \in \Omega$.

Let $\sigma : \Omega \to \Omega$ be a measurably invertible, ergodic, measure-preserving map
on $(\Omega, F, P)$. Then $(\omega, z) \mapsto T_\omega(z)$ is a measurable map from $\Omega \times T^2$ to $T^2$ (see
Lemma 3.14]), so the map $T$ induced by $T$ over $\sigma$ (recall (1.1)) is an RDS on
$T^2$. If $B(T^2)$ denotes the Borel algebra, it follows from [37, Theorem 1.3] that when $\epsilon$
is sufficiently small, there is a unique function $(\omega, A) \mapsto \mu_\omega(A)$ on $\Omega \times B(T^2)$ such that

(i) $\{\mu_\omega\}_{\omega \in \Omega}$ is a measurable family of probability measures (i.e. $\mu_\omega$ is $P$-almost
surely a probability measure on $T^2$, and $\omega \mapsto \mu_\omega(A)$ is measurable for each
$A \in B(T^2)$),

(ii) $\{\mu_\omega\}_{\omega \in \Omega}$ is $T$-invariant (i.e. $\mu_\omega(\sigma^{-1}A) = \mu_\sigma(A)$ for $P$-almost every $\omega \in \Omega$
and all $A \in B(T^2)$),

(iii) $\mu_\omega$ is $P$-almost surely absolutely continuous.

We define a probability measure $\mu$ on $\Omega \times T^2$ by $\mu(\Gamma \times A) = \int_\Gamma \mu_\omega(A) dP(\omega)$ for
each measurable set $\Gamma \subset \Omega$ and $A \subset T^2$.

Let $r$ be a positive constant and let $g$ be in $L^\infty(\Omega, H^r(T^2))$, that is, $g$
is a measurable map from $\Omega$ to the Sobolev space $H^r(T^2)$ such that $\text{ess sup}_\omega \|g_\omega\|_{H^r} < \infty,$
where $g_\omega = g(\omega)$. For simplicity, we will assume that

$$
E_{\mu_\omega}[g_\omega] = 0 \quad \text{for } P\text{-almost every } \omega \in \Omega.
$$
(Note that if $g \in L^\infty(\Omega, H^r(T^2))$, then $\overline{g}_\omega = g_\omega - E_{\mu_\omega}[g_\omega]$, called the centering, satisfies $E_{\mu_\omega}[g_\omega] = 0$.) We define the asymptotic variance $V$ for $g$ by

$$V = E_p[v], \quad v_\omega = E_{\mu_\omega}\left[ g_\omega^2 + 2 \sum_{n=1}^{\infty} g_\omega \cdot g_{\sigma^n \omega} \circ T^{(n)}_\omega \right].$$

We say that $g$ is non-degenerate if $V > 0$, which was shown in [13] to be equivalent to the non-coboundary condition of $g$ (i.e. non-existence of $\phi \in L^2(\Omega \times M)$ such that $g_\omega = \phi_\omega - \phi_{\sigma \omega} \circ T_\omega$) when $T_\omega$ is a piecewise expanding map.

Exponential decay of quenched correlation functions for the RDS $T$ with respect to $\{\mu_\omega\}_{\omega \in \Omega}$ was investigated in [37] via a spectral approach. As a by-product of key estimates in the analysis of this paper we obtain the following improved version.

**Theorem A.** Let $T_0$ as well as $\sigma$, $T$ and $g$ be as above. In particular, assume that $T_0$ is partially captive. Then, for any sufficiently small $\epsilon \geq 0$, quenched correlation functions decay exponentially fast in the following sense: There are constants $\rho \in (0,1)$, $m > 1$ and $C_g > 0$ such that if $r \geq m$ and $u \in H^r(T^2)$ then

$$\left| E_{\mu_\omega}[g_{\sigma^n \omega} \circ T^{(n)}_\omega u] - E_{\mu_\omega}[g_{\sigma^n \omega} \circ T^{(n)}_\omega] E_{\mu_\omega}[u] \right| \leq C_g \rho^n \|u\|_{H^r}$$

for all $n \geq 1$ and $\mathbb{P}$-almost every $\omega \in \Omega$.

We remark that in our previous study, the coefficient $C_g$ above depended on $\omega$ and we were unable to remove this dependency. By a more careful analysis we are now able to overcome this problem by using a different spectral scheme for a global transfer operator instead; a key step is to establish (2.4) in Section 3. Note also that Theorem A implies exponential decay of annealed correlation functions and moreover that the quenched law of large numbers (or ergodicity) holds, see the discussion following Theorem 2.5 below.

The main results of the paper are the following limit theorems:

**Theorem B** (Central limit theorem). Assume that the conditions in Theorem A are satisfied. Assume also that $g$ is non-degenerate. Then, for any sufficiently small $\epsilon \geq 0$ and $\mathbb{P}$-almost every $\omega \in \Omega$, $(S_n g)_{\omega}$ converges in distribution to a normal distribution with mean 0 and variance $V$ as $n \to \infty$, i.e. for any $a \in \mathbb{R}$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \mu_\omega((S_n g)_{\omega} \leq a) = \frac{1}{\sqrt{2\pi V}} \int_{-\infty}^{a} e^{-\frac{v^2}{2V}} dv.$$

**Theorem C** (Large deviation principle). Assume that the conditions in Theorem A are satisfied. Then, for any sufficiently small $\epsilon \geq 0$, $\mathbb{P}$-almost every $\omega \in \Omega$ and any sufficiently small $\delta > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \log \mu_\omega((S_n g)_{\omega} > n\delta) = -c(\delta),$$

where $c$ is nonnegative, continuous, strictly convex on a neighborhood of 0, vanishing only at 0.

**Theorem D** (Local central limit theorem). Assume that the conditions in Theorem A are satisfied. Assume also that condition (L) holds (see Definition 2.7). Then for $\mathbb{P}$-almost every $\omega \in \Omega$ and every bounded interval $[a, b] \subset \mathbb{R}$ we have

$$\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \sqrt{n} \mu_\omega(t + a \leq (S_n g)_{\omega} \leq t + b) - \frac{1}{\sqrt{2\pi V}} e^{-\frac{b-a}{2V}} = 0.$$
We will prove Theorems A–D by establishing corresponding limit results for an abstract class of random dynamical systems. In Section 2 we present the abstract setting and state the general limit theorems (Theorems 2.5, 2.6, 2.7, and 2.8), and also describe how random U(1) extensions of expanding maps on the circle is included within this class (see Theorem 2.10). The verification of this inclusion is made in Section 3. We then prove the general limit theorems in Section 4.

2. LIMIT THEOREMS FOR ABSTRACT RANDOM DYNAMICAL SYSTEMS

2.1. The abstract setting. Let $(M, G)$ be a measurable space, where $G$ is a $\sigma$-algebra on $M$. We consider a topological vector space $\mathcal{D}$ consisting of complex-valued functions on $M$, and a map $T: M \to M$ preserving $\mathcal{D}$ (i.e., $u \circ T \in \mathcal{D}$ for any $u \in \mathcal{D}$). Then, we define the (Perron-Frobenius) transfer operator $L_T$ of $T$ on the (continuous) dual space $\mathcal{D}'$ of $\mathcal{D}$ by

\begin{equation}
\langle L_T \varphi, u \rangle = \langle \varphi, u \circ T \rangle, \quad \varphi \in \mathcal{D}', \quad u \in \mathcal{D},
\end{equation}

where $\langle \ , \ \rangle$ is the dual pairing. When $T$ is a smooth map on a compact smooth Riemannian manifold $M$ equipped with the normalized Lebesgue measure $\text{Leb}_M$, and $\mathcal{D} = L^1(M)$, it follows from the canonical form $L^1(M) \ni u \mapsto \langle \varphi, u \rangle = \int \varphi u \, d\text{Leb}_M$ of $\varphi \in L^\infty(M)$ as an element of $(L^1(M))'$ that (2.1) is equivalent to $\int L_T \varphi \cdot u \, d\text{Leb}_M = \int \varphi \cdot u \circ T \, d\text{Leb}_M$ for each $\varphi \in L^\infty(M)$ and $u \in L^1(M)$, Hence, if $\det DT(x) \neq 0$ Lebesgue almost everywhere, then by a change of variables one can get

\begin{equation}
L_T \varphi(x) = \sum_{T(y) = x} \frac{\varphi(y)}{|\det DT(y)|}
\end{equation}

for each $\varphi \in L^\infty(M)$ and Lebesgue almost every $x \in M$. Due to the relation (2.1), one may expect that several statistical properties of $T$ directly follow from properties of the spectrum of $L$. Standard references for transfer operators are the monographs by Baladi [5, 6].

In some literature, the transfer operator of $T$ is defined as the bounded operator on a space of functions on a smooth manifold $M$ given by (2.2). However, in the last two decades it has been realized that it is important to investigate the spectrum of the transfer operator on Banach spaces of distributions if one hopes to obtain information on the Sinai-Ruelle-Bowen measures of the dynamics, which are the most relevant measures in smooth dynamical systems theory. We thus employ relation (2.1) for the definition of transfer operators and interpret it in the distributional sense whenever appropriate.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $\sigma$ be as in Subsection 1.1. Let $\mathcal{M}$ be a measurable space consisting of maps preserving $\mathcal{D}$, and $T: \Omega \to \mathcal{M}: \omega \mapsto T_\omega$ a measurable map such that $(\omega, x) \mapsto T_\omega(x)$ is a measurable map from $\Omega \times M$ to $M$. We simply write $L_\omega$ for the transfer operator $L_\omega: \mathcal{D}' \to \mathcal{D}'$ of $T_\omega$.

Definition 2.1. We say that a Banach space $\mathcal{B} \subset \mathcal{D}'$ with a norm $\| \cdot \|$ is $T$-admissible if $L_\omega$ is bounded on $\mathcal{B}$ for $\mathbb{P}$-almost every $\omega \in \Omega$, and there exists a constant $C_0 > 0$ such that

(A1) $\|L_\omega \varphi\| \leq C_0\|\varphi\|, \quad \varphi \in \mathcal{B}, \quad \mathbb{P}$-almost every $\omega$.

Let $\mathcal{Z}(\mathcal{D})$ be a linear subspace of $\mathcal{D}$ given by

$\mathcal{Z}(\mathcal{D}) = \{ u \in \mathcal{D} \mid uv \in \mathcal{D} \text{ for any } v \in \mathcal{D} \}$. 
We say that a vector space $\mathcal{E}$ with a seminorm $\| \cdot \|_\mathcal{E}$ is associated with $(\mathcal{B}, \| \cdot \|)$ if $\mathcal{E} \subset \mathcal{Z}(\mathcal{B})$, $1_M \in \mathcal{E}$ and there exists a constant $C_1 > 0$ such that

$$\max \{ \langle \varphi, u \rangle, \| u \varphi \| \} \leq C_1 \| \varphi \|_\mathcal{E} u, \varphi \in \mathcal{B}, u \in \mathcal{E},$$

(A2) $\| u v \|_\mathcal{E} \leq C_1 \| u \|_\mathcal{E} \| v \|_\mathcal{E}, \quad u, v \in \mathcal{E}.$

We call each element in $\mathcal{E}$ an observable.

We now fix a seminormed vector space $(\mathcal{E}, \| \cdot \|_\mathcal{E})$ associated with a $T$-admissible Banach space $(\mathcal{B}, \| \cdot \|)$. We define a cocycle $(n, \omega, \varphi) \mapsto L^{(n)}_\omega \varphi$ by

$$L^{(n)}_\omega = L_{\omega-1} \circ \cdots \circ L_n, \quad n \geq 1,$$

and $L^{(0)}_\omega = \text{id}_\mathcal{B}$ for each $\omega \in \Omega$, which is called the transfer operator cocycle (of the RDS $T$ induced by $T$ over $\sigma$), and denoted by $(L, \sigma)$ for simplicity.

For each topological space $\mathcal{Y}$, we denote by $B_\mathcal{Y}$ the Borel $\sigma$-algebra of $\mathcal{Y}$. In particular, $B_L(\mathcal{B})$ is the Borel $\sigma$-algebra generated by the norm topology on $L(\mathcal{B})$, where $L(\mathcal{B})$ is the space of all bounded operators on $\mathcal{B}$ equipped with the operator norm. When $\mathcal{B}$ is separable, we also consider the $\sigma$-algebra $S_L(\mathcal{B})$ generated by the strong operator topology on $L(\mathcal{B})$, see [27, Appendix A] for its definition and basic properties. Here we merely recall that a map $A : \Omega \rightarrow L(\mathcal{B})$ is $(\mathcal{F}, S_L(\mathcal{B}))$-measurable if and only if $A$ is strongly measurable, that is, for any $\varphi \in \mathcal{B}$, the map $\omega \mapsto A(\omega)(\varphi)$ is $(\mathcal{F}, B_L(\mathcal{B}))$-measurable. Furthermore, as pointed out in [27] (see also [4, 8]), the $(\mathcal{F}, B_L(\mathcal{B}))$-measurability of $L$ is a strong requirement (at least stronger than the $(\mathcal{F}, S_L(\mathcal{B}))$-measurability). Indeed, [17, 13] ensured the $(\mathcal{F}, B_L(\mathcal{B}))$-measurability of $L$ by assuming that $\{T_\omega \mid \omega \in \Omega\}$ is at most countable. The main hypotheses which we shall place on an abstract RDS in this setting are the following two spectral conditions.

**Definition 2.3** (Uniform spectral gap condition). We say that $T$ satisfies the uniform spectral gap condition (UG) with respect to $\mathcal{B}$ and $\mathcal{E}$ if the following holds: There exists a unique $h \in L^\infty(\Omega, \mathcal{B})$ and a measurable family of probability measures $\{\mu_\omega\}_{\omega \in \Omega}$ on $M$ such that

$$L_\omega h_\omega = h_{\sigma \omega}, \quad \langle h_\omega, u \rangle = \int u d\mu_\omega, \quad \text{ess inf}_{\omega \in \Omega} \| h_\omega \| \geq 1$$

for $\mathbb{P}$-almost every $\omega \in \Omega$ and any $u \in \mathcal{E}$. Furthermore, there is a constant $C_2 > 0$ and $\rho \in (0, 1)$ such that if $\varphi \in \mathcal{B}$ satisfies $\langle \varphi, 1_M \rangle = 0$, then

$$\text{ess sup}_{\omega \in \Omega} \| L^{(n)}_\omega \varphi \| \leq C_2 \rho^n \| \varphi \| \quad \text{for all } n \in \mathbb{N}.$$

**Definition 2.4** (Lasota-Yorke inequality condition). We say that $T$ satisfies the Lasota-Yorke inequality condition (LY) with respect to $\mathcal{B} \subset \mathcal{B}_+$ if the following holds: either

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a Polish space, $\sigma$ is a homeomorphism, and $L$ is $(\mathcal{F}, B_L(\mathcal{B}))$-measurable, or

(ii) $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue-Rokhlin probability space, $\mathcal{B}$ is separable, and $L$ is $(\mathcal{F}, S_L(\mathcal{B}))$-measurable.
Furthermore, \((\mathcal{B}_+, \cdot, \cdot)\) is a Banach space with \(| \cdot | \leq \| \cdot \|\) and \(\mathcal{B} \subset \mathcal{B}_+ \subset \mathcal{D}'\), and it is required that there exist a positive integer \(n_0\) and random variables \(\alpha, \beta\) with values in \(\mathbb{R}_+ = \{x > 0\}\) such that the inclusion \((\mathcal{B}, \| \cdot \|) \hookrightarrow (\mathcal{B}_+, \cdot, \cdot)\) is compact, \(\mathcal{L}\) is \(\mathbb{P}\)-almost surely bounded on \(\mathcal{B}_+\), and

\[
\| L^{(n_0)}_\omega \| \leq \alpha_\omega \| \varphi \| + \beta_\omega \| \varphi \|, \quad \varphi \in \mathcal{B}, \quad \mathbb{P}\text{-almost every } \omega \in \Omega,
\]

with \(\mathbb{E}_\omega [\alpha] < 1\) and \(\mathbb{E}_\omega [\beta] < \infty\).

**Remark.** The expectation \(\mathbb{E}_\omega[u]\) of \(u\) with respect to a probability measure \(\mathbb{P}\) is \(\mathbb{E}_\omega[u] = \int u \, d\mathbb{P}\). It follows from (2.7) and (UG) that

\[
\mathbb{E}_{\mu_\omega} [u \circ T_\omega] = \mathbb{E}_{\mu_{\sigma_\omega}} [u]
\]

for any \(u \in \mathcal{S}\). Moreover, for several random dynamics including random U(1) extensions of expanding maps in Subsection 1.1, one can also show that \(\{\mu_\omega\}_{\omega \in \Omega}\) is \(T\)-invariant, i.e.

\[
\mu_\omega(T_\omega^{-1}A) = \mu_{\sigma_\omega}(A)
\]

for \(\mathbb{P}\)-almost every \(\omega \in \Omega\) and any \(A \in \mathcal{G}\) (or equivalently, [2.6] with \(\{1_A : A \in \mathcal{G}\}\) in place of \(\mathcal{S}\)), although in this paper we will not use this property (use (2.6) instead) to prove quenched limit theorems. Another remark on (UG) is that for any positive \(h_\omega \in (\mathcal{C}^0(M))'\) with \(\langle h_\omega, 1_M \rangle = 1\), there exists a unique probability measure \(\mu_\omega\) such that \(\langle h_\omega, u \rangle = \int u \, d\mu_\omega\) by the Riesz-Markov-Kakutani representation theorem, and that \(\text{ess inf}_{\omega \in \Omega} \| h_\omega \| \geq 1\) if \(\| \cdot \| \geq \| \cdot \|_{L^1}\).

The setting of (i) in (LY) is required in order to apply the Froyland-Lloyd-Quas version of multiplicative ergodic theorem ([26]), while the setting of (ii) comes from the González-Tokman-Quas version of multiplicative ergodic theorem ([27]), see Subsection 2.3. We also remark that it follows from [26, Remark 13] that when \((\Omega, \mathcal{F}, \mathbb{P})\) is a Polish space, the \((\mathcal{F}, B_{L^1(M)}')\)-measurability of \(\mathcal{L}\) is equivalent to the \(\mathbb{P}\)-continuity of \(\mathcal{L}\) (i.e. there is a countable collection of Borel sets \(\{A_n\}_{n \in \mathbb{N}}\) such that \(\bigcup_{n \in \mathbb{N}} A_n\) has \(\mathbb{P}\)-full measure and the restriction of \(\mathcal{L}\) on each \(A_n\) is continuous), which is exactly the condition used in [17,18].

**Remark.** The spectral assumption (UG) may seem to be much stronger than (LY), and for some spaces \(\mathcal{D}\), the inequality (2.5) of (LY) does indeed follow directly from (UG). For example, consider the case when \(M\) is a locally compact space and \(\mathcal{D}\) is the space of all compactly supported continuous functions on \(M\) (with the standard locally convex topology), for which any element \(\varphi \in \mathcal{D}'\) is the difference \(\varphi_+ - \varphi_-\) of two Radon measures \(\varphi_+\) and \(\varphi_-\) by Riesz-Markov-Kakutani representation theorem. Then

\[
\mathcal{B}_+ := \{ \varphi \in \mathcal{D}' : \| \varphi \| < \infty \} \quad \text{with} \quad \| \varphi \| = \varphi_+ + \varphi_-
\]

is a Banach space, and \(\| \langle \varphi, 1_M \rangle \| \leq \| \varphi \|\). Let \(\mathcal{B}, h, C_2, \rho\) be as in (UG), and define projections \(\pi_\omega, \tilde{\pi}_\omega\) on \(\mathcal{B}\) by

\[
\pi_\omega(\varphi) = \langle \varphi, 1_M \rangle h_\omega, \quad \tilde{\pi}_\omega(\varphi) = \varphi - \pi_\omega(\varphi).
\]

Then it follows from (A2) and (UG) that, with \(H = \text{ess sup}_\omega \| h_\omega \| < \infty\),

\[
\text{ess sup}_{\omega \in \Omega} \| L^{(n)}_\omega \pi_\omega(\varphi) \| \leq \text{ess sup}_{\omega \in \Omega} \| \langle \varphi, 1_M \rangle \| \| h_\rho \| \leq H \| \varphi \|.
\]
for each $\varphi \in \mathcal{B}$. On the other hand, since $\langle \pi_\omega(\varphi), 1_M \rangle = 0$, by (A2) and (UG) we have
\[
\underset{\omega \in \Omega}{\text{ess sup}} \| L^{(n)}_\omega \pi_\omega(\varphi) \| \leq C_2 \rho^n \| \pi_\omega(\varphi) \| \leq C' \rho^n \| \varphi \|
\]
with $C' = C_2(1 + H\|1_M\|)$. Hence, noticing that $\varphi = \pi_\omega(\varphi) + \tilde{\pi}_\omega(\varphi)$, we get
\[
\underset{\omega \in \Omega}{\text{ess sup}} \| L^{(n)}_\omega \varphi \| \leq C' \rho^n \| \varphi \| + H|\varphi|,
\]
that is, the inequality (2.8).

However, the “absolute value” operation used in the example above does not immediately generalize for general $\mathcal{D}'$, so it is not clear to us if there is always a natural choice of weak norm $\| \cdot \|$ for which inequality (2.4) of (LY) follows by virtue of (UG) alone. Furthermore, the compactness assumption of $(\mathcal{B}, \| \cdot \|) \to (\mathcal{B}_+, \| \cdot \|)$ in (LY) is not true in general for the weak norm $\| \cdot \|$ defined in (2.7) (e.g. the case when $\mathcal{B} = \mathcal{D}' = L^2(M)$, where $\mathcal{B}_+$ in (2.7) automatically coincides with $\mathcal{B}$).

The abstract setting considered here is quite natural. In addition to random $U(1)$ extensions of expanding maps on the circle, dynamical systems satisfying conditions (UG) and (LY) can be found in [17],[18]:

- For random piecewise expanding maps considered in [17], the conditions can be verified with $\| \cdot \|_{\mathcal{E}} = \| \cdot \|_{L^1} + \text{var}(\cdot) + \| \cdot \|_{L^\infty}$, where $\text{var}(\cdot)$ is the total variation.
- For random hyperbolic maps considered in [18], the conditions are verified with $\| \cdot \|_{\mathcal{E}} = \| \cdot \|_{\mathcal{E}^r}$, where $r > 2$, $\| \cdot \|_{\mathcal{E}} = \| \cdot \|_{L,1}$ and $\| \cdot \|_{\mathcal{E}^r}$ is the Gouëzel-Liverani’s scale of norms given in (2.2) of [19].

**Remark.** For the above examples, the most difficult condition to verify is (2.4) in (UG): note that the coefficient $C_2$ in (2.4) needs to be independent of $\omega$. The techniques to show (2.4) in [17],[18] and this paper are quite different. In particular, we will prove (2.4) for random $U(1)$ extensions of expanding maps without using any abstract perturbation lemma, see Section 3 and compare with [18] (10).

We also remark that for the analysis of random $U(1)$ extensions of expanding maps, we employ the setting (ii) in (LY). This is a difference from [17] and the piecewise hyperbolic part of [18], that used the setting (i) and needed in practice the restriction that $\{T_\omega \mid \omega \in \Omega\}$ is countable.

### 2.2. Limit theorems.

Let $\mu_\omega$, $\omega \in \Omega$ be the measurable family of probability measures on $M$ provided by (UG). Define a probability measure $\mu$ on $\Omega \times M$ by $\mu(\Gamma \times A) = \int_{\Gamma} \mu_\omega(A) \mathbb{P}(d\omega)$ for each measurable set $\Gamma \subset \Omega$ and $A \subset M$. Fix also $g \in L^\infty(\Omega, \mathcal{E})$ (i.e. $g$ is a measurable map from $\Omega$ to $\mathcal{E}$) and assume that the centering condition (1.0) holds for $g$. Our first main result concerns exponential decay of correlation functions.

**Theorem 2.5** (Exponential decay of correlations). Let $(\mathcal{E}, \| \cdot \|_{\mathcal{E}})$ be a semi-normed vector space associated with a $T$-admissible Banach space $(\mathcal{B}, \| \cdot \|)$. Let $g \in L^\infty(\Omega, \mathcal{E})$ satisfy (1.0). Assume that (UG) holds. Then, one can find a constant $C_g > 0$ such that for all $u \in \mathcal{E}$, $n \geq 1$ and $\mathbb{P}$-almost every $\omega \in \Omega$,
\[
\mathbb{E}_{\mu_\omega} \left[ g_{\sigma^n\omega} \circ T^{(n)}_\omega u \right] \leq C_g \rho^n \| u \|_{\mathcal{E}}.
\]
Proof. By (UG) we have \( \int u \, d\mu_\omega = \langle h_\omega, u \rangle \) for \( u \in \mathcal{E} \), so
\[
\mathbb{E}_{\mu_\omega} [g_{\sigma^n} \circ T^{(n)}_\omega u] = \int u g_{\sigma^n} \circ T^{(n)}_\omega \, d\mu_\omega = \langle uh_\omega, g_{\sigma^n} \circ T^{(n)}_\omega \rangle,
\]
where the right-hand side equals \( \langle \mathcal{L}^{(n)}_\omega (uh_\omega), g_{\sigma^n} \rangle \) by (2.1). We would like to use (2.4) to estimate \( \mathcal{L}^{(n)}_\omega (uh_\omega) \), but \( \langle uh_\omega, 1_M \rangle \neq 0 \) in general so this requires modification. Thus, we note that (2.6) and the centering assumption (1.6) gives
\[
\langle \mathcal{L}^{(n)}_\omega (h_\omega), g_{\sigma^n} \rangle = \int g_{\sigma^n} \circ T^{(n)}_\omega \, d\mu_\omega = \int g_{\sigma^n} \, d\mu_\omega = \mathbb{E}_{\mu_\omega} [g_{\sigma^n}] = 0,
\]
which means that
\[
\mathbb{E}_{\mu_\omega} [g_{\sigma^n} \circ T^{(n)}_\omega u] = \langle \mathcal{L}^{(n)}_\omega (uh_\omega), g_{\sigma^n} \rangle - \langle \mathcal{L}^{(n)}_\omega (h_\omega), g_{\sigma^n} \rangle \mathbb{E}_{\mu_\omega} [u]
\]
\[
= \langle \mathcal{L}^{(n)}_\omega ((u - \mathbb{E}_{\mu_\omega} [u])h_\omega), g_{\sigma^n} \rangle.
\]
Since \( \langle (u - \mathbb{E}_{\mu_\omega} [u])h_\omega, 1_M \rangle = \langle h_\omega, u \rangle - \mathbb{E}_{\mu_\omega} [u]\langle h_\omega, 1_M \rangle = 0 \), we can now use (A2) and (2.4) to obtain
\[
\left| \mathbb{E}_{\mu_\omega} [g_{\sigma^n} \circ T^{(n)}_\omega u] \right| \leq C_1 \mathcal{L}^{(n)}_\omega ((u - \mathbb{E}_{\mu_\omega} [u])h_\omega) \| g_{\sigma^n} \|_E
\]
\[
\leq C_1 C_2 \rho^\alpha G (1 + H) \| u \|_E
\]
where \( G = \text{ess sup}_\omega \| g_\omega \|_E \) and \( H = \text{ess sup}_\omega \| h_\omega \|_E \). This completes the proof. \( \square \)

Obviously, Theorem 2.5 implies exponential decay of annealed correlation functions:
\[
\text{for every } u \in \mathcal{E} \text{ and } n \geq 1, \text{ where } T : \Omega \times M \to \Omega \times M \text{ is the skew-product map induced by } (T, \sigma), \text{ i.e. } T(\omega, x) = (\sigma(\omega), T_\omega(x)) \text{ for } (\omega, x) \in \Omega \times M. \text{ Furthermore, it easily follows from Theorem 2.5 that for } \mu \text{-almost every } (\omega, x) \in \Omega \times M, \text{ we have}
\]
\[
\lim_{n \to \infty} \frac{(S_n g)_\omega(x)}{n} = 0,
\]
if for each \( A \in \mathcal{G}, \bar{u} \in \mathcal{E} \) and \( \mathbb{P} \)-almost every \( \omega \) there is a sequence \( (u_n)_{n \geq 1} \subset \mathcal{E} \) such that \( \mathbb{E}_{\mu_\omega} [\bar{u}(1_A - u_n)] \to 0 \) as \( n \to \infty \) (the examples of dynamical systems on\[8\] indeed satisfy this condition). That is, the quenched strong law of large numbers (or ergodicity) holds under a mild condition. We refer to [3] for the relation between fast decay of correlation functions and various limit theorems for deterministic systems.

To state our other results, we further assume that \( g \) is \( \mathbb{P} \)-almost surely a real-valued function. As before, we define the asymptotic variance \( V \) for \( g \) by (1.7) (the existence and boundedness of the sum in (1.7) is ensured by (A2) and Theorem 2.5), and say that \( g \) is non-degenerate if \( V > 0 \).

**Theorem 2.6** (Central limit theorem). Assume that the conditions in Theorem 2.5 together with (LY) hold. Assume also that \( g \) is non-degenerate. Then, for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), \( \frac{(S_n g)_\omega}{\sqrt{n}} \) converges in distribution to a normal distribution with mean 0 and variance \( V \) as \( n \to \infty \), i.e. for any \( a \in \mathbb{R} \), we have
\[
\lim_{n \to \infty} \mu_\omega \left( \frac{(S_n g)_\omega}{\sqrt{n}} \leq a \right) = \frac{1}{\sqrt{2\pi V}} \int_{-\infty}^{a} e^{-\frac{z^2}{2V}} \, dz.
\]
Theorem 2.7 (Large deviation principle). Assume that the conditions in Theorem 2.6 hold. Then, one can find a nonnegative, continuous, strictly convex function \( c : (-\delta_0, \delta_0) \rightarrow \mathbb{R} \) with some \( \delta_0 > 0 \) such that \( c \) vanishes at 0 and that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) and any \( \delta \in (0, \delta_0) \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \mu_\omega((S_n g)_\omega > n\delta) = -c(\delta).
\]

Theorem 2.8 (Local central limit theorem). Assume that the conditions in Theorem 2.6 hold. Assume also that the condition (L) (see Definition 2.12) holds. Then, for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) and every bounded interval \( J \subset \mathbb{R} \), we have

\[
\lim_{n \to \infty} \sup_{z \in \mathbb{R}} \left| \sqrt{nV(\mu_\omega(z + (S_n g)_\omega \in J)} - \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}n}|J| \right| = 0,
\]

where \( |J| \) denotes the length of \( J \).

The proofs of Theorems 2.6, 2.7 and 2.8 will be given in Section 3. An excellent survey paper for limit theorems (including central limit theorems, large deviation principles and local central limit theorems) for deterministic dynamical systems is \[28\]. For limit theorems of random dynamical systems, see \[2, 17–19\] and reference therein. These results were established by the Nagaev-Guivarc'h perturbative spectral method explained in \[22, 23\], which indicates that a variety of other limit theorems (such as almost sure invariance principles, moderate deviations principles and Berry-Esseen theorems) also may hold for random dynamical systems in our abstract setting.

Remark. Soon after we uploaded the first version of this paper to arXiv, Dragičević and Sedro \[21\] also extended limit theorems via the Nagaev–Guivarc'h spectral approach of \[17, 18\] to expanding on average maps, by adapting Pesin theory to transfer operator cocycles (on the space of bounded variation functions). Therefore, although our abstract setting in this subsection is not applicable to expanding on average maps in general, an appropriate change of the setting of this section and its proof according to Dragičević–Sedro’s approach may enable us to include expanding on average maps as a new application of our abstract limit theorems.

Furthermore, Dragičević and Hafouta \[20\] developed an abstract framework tailored for almost surely invariant principle, and in their work estimates appear which are similar to some of the estimates in this paper. However, because our setting is different and slightly more abstract we cannot apply the estimates of \[20\] directly.

2.3. The Nagaev-Guivarc’h perturbative spectral method. Here we briefly recall the idea of the Nagaev-Guivarc’h perturbative spectral method and the difficulties that have to be overcome in order for it to be used for quenched schemes. This will heuristically explain why both (UG) and (LY) are needed, and provide preparation for definition of condition (L).

The twisted transfer operator \( \mathcal{L}_{\theta, \omega} : \mathcal{D}' \to \mathcal{D}' \) (of \( T_\omega \) associated with \( g_\omega \)) with \( \theta \in \mathbb{C} \) and \( \omega \in \Omega \) is given by

\[
\mathcal{L}_{\theta, \omega} \varphi = \mathcal{L}_\omega(e^{i\theta g_\omega} \varphi), \quad \varphi \in \mathcal{D}'.
\]

(Recall \[2.3\] for the multiplication of \( e^{i\theta g_\omega} \) and \( \varphi \).) By \[A1\] and \[A2\], \( \mathcal{L}_{\theta, \omega} \) is bounded on \( \mathcal{D} \) for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) and

\[
\operatorname{ess} \sup_{\omega \in \Omega} \| \mathcal{L}_{\theta, \omega} \varphi - \mathcal{L}_\omega \varphi \| \leq C_0|\theta|Ge^{i|\theta|C_1G}||\varphi||, \quad \varphi \in \mathcal{D},
\]
where $G = \text{ess} \sup_{\omega} \|g_\omega\|_\varphi$. In a manner similar to the one in the definition of $(\mathcal{L}, \sigma)$, we define a cocycle $\mathbb{N}_0 \times \Omega \times \mathcal{B} \ni (n, \omega, \varphi) \mapsto \mathcal{L}_{\theta, \omega}^{(n)} \varphi$ over $\sigma$, and denote it by $(\mathcal{L}_\theta, \sigma)$. Furthermore, for any $\theta \in \mathbb{C}$, $n \in \mathbb{N}$, $\varphi \in \mathcal{B}$ and $\mathbb{P}$-almost every $\omega \in \Omega$, we have

\begin{equation}
\mathcal{L}_{\theta, \omega}^{(n)} \varphi = \mathcal{L}^{(n)}(e^{\theta(S_n g)\omega} \varphi).
\end{equation}

Indeed, it is obviously true for $n = 1$, and if (2.12) holds for a fixed $n \geq 1$, then

\[ \langle \mathcal{L}_{\theta, \omega}^{(n+1)} \varphi, u \rangle = \langle \mathcal{L}_{\sigma^n, \omega}^{(n)}(e^{\theta g_{S_n} n} \mathcal{L}_{\theta, \omega}^{(n)} \varphi), u \rangle = \langle \mathcal{L}_{\theta, \omega}^{(n)} \varphi, e^{\theta g_{S_n} n} u \circ T_{\sigma^n, \omega} \rangle = \langle \mathcal{L}_{\omega}^{(n)}(e^{\theta(S_n g)\omega} \varphi), e^{\theta g_{S_n} n} u \circ T_{\sigma^n, \omega} \rangle = \langle e^{\theta(S_{n+1} g)\omega} \varphi, u \circ T_{\omega}^{(n+1)} \rangle = \langle \mathcal{L}^{(n+1)}(e^{\theta(S_{n+1} g)\omega} \varphi), u \rangle \]

for each $u \in \mathcal{B}$, and we get the claim.

For a random process $\{u_n\}_{n \geq 1}$ on a probability space $(\mathcal{M}, \mathcal{G}, \nu)$, the moment generating function $\mathbb{R} \ni t \to \mathbb{E}_\nu[e^{tu_n}]$ and the characteristic function $\mathbb{R} \ni t \to \mathbb{E}_\nu[e^{it u_n}]$ of $u_n$’s (around $t = 0$) play a fundamental role in the study of the asymptotic behavior of $\{u_n\}_{n \geq 1}$ (see e.g. [22]). Hence, to understand the asymptotic behavior of $(S_n g)\omega$ with respect to $\mu_\omega$, it is of great importance to investigate

\[ \mathbb{E}_{\mu_\omega}[e^{\theta(S_n g)\omega}] = \langle h_\omega, e^{\theta(S_n g)\omega} \rangle = \langle \mathcal{L}_{\theta, \omega}^{(n)} h_\omega, 1_M \rangle \]

around $\theta = 0$. (The first identity follows from (UG) and the second from (2.12).) Therefore, it is natural to expect that several limit theorems should follow from $(\mathcal{L}_\theta, \sigma)$ having nice spectral properties around $\theta = 0$. This idea is called Nagaev-Guivarc’h perturbative spectral method, and has been broadly applied to show limit theorems for abundant deterministic dynamical systems (see [2][17][28][32] and the references therein; this method was originally applied to Markov chains [32]).

The difficulty in applying a (Nagaev-Guivarc’h type) spectral method for quenched schemes is that one should consider a spectral analysis for operator cocycles, and the usual notion of spectrum for a single operator is not useful. This was overcome in [17] by using the theory of Lyapunov spectrum. The notion of Lyapunov spectrum was also used in [37] to study decay of correlation functions for quenched schemes.

Let $\mathcal{H}$ be a Banach space and let $\mathcal{A}$ be a measurable mapping from $\Omega$ to the set of bounded operators on $\mathcal{H}$ (equipped with the operator norm) such that $\omega \mapsto \log^+ \|A_\omega\|$ is $\mathbb{P}$-integrable, and consider the cocycle $(n, \omega, \varphi) \mapsto \mathcal{A}_{\omega}^{(n)} \varphi = A_{\omega}^{n-1} \varphi \circ \cdots \circ A_{\omega}$ over $\sigma$ (denoted by $(\mathcal{A}, \sigma)$). Then, it follows from Kingman’s sub-additive ergodic theorem that the following limits (called the maximal Lyapunov exponent and the index of compactness of $\mathcal{A}$) exist and are $\mathbb{P}$-almost surely independent of $\omega \in \Omega$:

\begin{equation}
\Lambda(\mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \log \| \mathcal{A}_{\omega}^{(n)} \|, \quad \kappa(\mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \log \text{ic}(\mathcal{A}_{\omega}^{(n)}),
\end{equation}

where $\text{ic}(\mathcal{A}_0)$ is the index of compactness of a bounded operator $\mathcal{A}_0 : \mathcal{H} \to \mathcal{H}$ (i.e. $\text{ic}(\mathcal{A}_0)$ is the infimum of $r > 0$ such that the image of the unit ball of $\mathcal{H}$ by $\mathcal{A}_0$ can be covered with finitely many balls of radius $r$). The cocycle $(\mathcal{A}, \sigma)$ is called quasi-compact if $\Lambda(\mathcal{A}) > \kappa(\mathcal{A})$. The notion of quasi-compactness for cocycles
of linear operators was introduced by Thieullen [10], and was key for establishing
the multiplicative ergodic theorem on Banach spaces (see [26, 27] and reference
therein). We shall use the fact that quasi-compact cocycles admit an Oseledets
decomposition in the following sense.

**Definition 2.9.** We say that there is an *Oseledets decomposition* of \((A, \sigma)\) on \(\mathcal{H}\)
if there is a real number \(\bar{\Lambda}(A) < \Lambda(A)\), a measurable splitting of \(\mathcal{H}\) into closed
subspaces \(\mathcal{H} = \mathcal{F}_\omega \oplus \mathcal{R}_\omega\) with \(\dim \mathcal{F}_\omega < \infty\), and a \(\sigma\)-invariant subset \(\tilde{\Omega} \subset \Omega\) of
full measure such that for each \(\omega \in \tilde{\Omega}\), the following holds:

1. The splitting is semi-invariant:
   \[ A_\omega \mathcal{F}_\omega = \mathcal{F}_{\sigma \omega} \quad \text{and} \quad A_\omega \mathcal{R}_\omega \subset \mathcal{R}_{\sigma \omega}. \]
2. If \(\varphi \in \mathcal{F}_\omega \setminus \{0\}\), then
   \[ \lim_{n \to \infty} \frac{1}{n} \log \| A^{(n)} \varphi \| = \Lambda(A). \]
3. If \(\varphi \in \mathcal{R}_\omega\), then
   \[ \limsup_{n \to \infty} \frac{1}{n} \log \| A^{(n)} \varphi \| \leq \bar{\Lambda}(A). \]

It can be easily shown that (UG) implies that \((\mathcal{L}, \sigma)\) admits an Oseledets
decomposition \(\mathcal{B} = \mathcal{F}_\omega \oplus \mathcal{R}_\omega\) with \(\mathcal{F}_\omega = \{ \pi_\omega(f) \mid f \in \mathcal{B}\}, \mathcal{R}_\omega = \{ f - \pi_\omega(f) \mid f \in \mathcal{B}\}\),
\(\Lambda(\mathcal{L}) = 0\) and \(\bar{\Lambda}(\mathcal{L}) = \log \rho\), where \(\pi_\omega(f) = \langle h_\omega, f \rangle h_\omega\) and \(\rho\) is the decay rate
in \[(2.3)\]. On the other hand, \(\mathbb{E}_\varnothing \log^+ \| \mathcal{L} \| < \infty\) by \((A1)\), where \(\log^+ \| \mathcal{L} \| (\omega) = \max \{ \log \| L_\omega \|, 0 \}\) for \(\omega \in \Omega\). Hence, using the multiplicative ergodic theorems in
[26, 27], it can also be shown that (LY) implies that \((\mathcal{L}, \sigma)\) admits an Oseledets
decomposition on \(\mathcal{B}\) \([26]\) for the case (i) of (LY) and \([27]\) for the case (ii)). We
consider the following condition (recall that \(\mathcal{L}_\theta\) depends on \(g\)).

**Definition 2.10 (Oseledets decomposition condition).** We say that \((T, g)\) satisfies
the *Oseledets decomposition condition* \((\text{OD})\) if \(\mathcal{L}_\theta\) admits an Oseledets
decomposition on \(\mathcal{B}\) for any \(\theta \in \mathcal{C}\) with sufficiently small absolute value.

We note that the proofs of Theorems 2.7, 2.8, 2.9 work with (OD) instead of
(LY). However, it is unclear to us whether the Oseledets decomposition of \((\mathcal{L}, \sigma)\)
is stable under the twisting perturbation, i.e. whether the Oseledets decomposition
of \((\mathcal{L}, \sigma)\) implies (OD). On the other hand, the stability of Lasota-Yorke inequality
is easily obtained as in the following proposition, which is the reason why (not
only (UG) but also) (LY) is assumed. In particular, if \(T\) satisfies (LY) then \((T, g)\)
satisfies (OD) for arbitrary \(g \in L^\infty(\Omega, \mathcal{B})\).

**Proposition 2.11.** Assume that \(T\) satisfies (LY). Then the Lasota-Yorke inequality
[2.5] holds with \(\mathcal{L}_\theta\) instead of \(\mathcal{L}\) for any \(\theta \in \mathcal{C}\) with sufficiently small absolute value.

**Proof.** Note that for any \(n \geq 1\) and \(\omega \in \Omega\),

\[ L_{\theta, \omega}^{(n)} \varphi - L_{\omega}^{(n)} \varphi = \sum_{j=0}^{n-1} L_{\theta, \sigma^{n-j} \omega}^{(j)} (L_{\theta, \sigma^{n-j} \omega} - L_{\sigma^{n-j} \omega}) L_{\omega}^{(n-j-1)} \varphi. \]

Hence, it follows from \[\text{[A1]}\] and \[2.11\] that there is a constant \(C \equiv C(g, n) > 0\)
such that

\[ \| L_{\theta, \omega}^{(n)} \varphi - L_{\omega}^{(n)} \varphi \| \leq C \| \theta \| e^{C \| \theta \|} \| \varphi \|, \quad \varphi \in \mathcal{B}. \]
This immediately leads to the implication that \( \mathcal{L}_\theta \) satisfies (2.8) (with \( \alpha \) replaced by \( \alpha + C|\theta|e^{C|\theta|} \)) for any \( \theta \) satisfying \( E_\theta[\alpha] + C|\theta|e^{C|\theta|} < 1 \).

We next show that in the case when \((\Omega, \mathcal{F}, \mathbb{P})\) is a Polish space, the \((\mathcal{F}, \mathcal{B}_{\mathbb{L}(\mathcal{B})})\)-measurability of \( \mathcal{L} \) implies that of \( \mathcal{L}_\theta \). Recall that, since \((\Omega, \mathcal{F}, \mathbb{P})\) is a Polish space, for any topological space \( \mathcal{Y} \) and any map \( A : \Omega \to \mathcal{Y} \), the \((\mathcal{F}, \mathcal{B}_\mathcal{Y})\)-measurability of \( A \) is equivalent to the \( \mathbb{P}\)-continuity of \( A \), see [20, Remark 13]. Assume that \( \mathcal{L} : \Omega \to L(\mathcal{B}) \) is \( \mathbb{P}\)-continuous. Then, since \( g : \Omega \to \mathcal{E} \) is also measurable, one can find a countable collection of Borel sets \( \{A_n\}_{n \in \mathbb{N}} \) such that \( \bigcup_{n \in \mathbb{N}} A_n \) has \( \mathbb{P}\)-full measure and both the restriction of \( \mathcal{L} \) and \( g \) on \( A_n \) are continuous for each \( n \in \mathbb{N} \). Notice that for each \( n \in \mathbb{N} \), the restriction of \( \mathcal{L}_\theta \) on \( A_n \) is the composition of a map \( A_n \times \mathcal{E} \to L(\mathcal{B}) : (\omega, g_0) \mapsto \mathcal{L}_\omega(e^{g_0} \cdot) \) and a continuous map \( A_n \to \mathcal{L}_\omega(e^{g_0} \cdot) : \omega \mapsto (\omega, g_0) \). On the other hand, it is easy to see that for each \( \omega \in A_n \) the map \( g_0 \mapsto \mathcal{L}_\omega(e^{g_0} \cdot) \) is a continuous map from \( \mathcal{E} \) to \( L(\mathcal{B}) \) because

\[
\| \mathcal{L}_\omega(e^{g_0} \varphi) - \mathcal{L}_\omega(e^{g_0'} \varphi) \| \leq C_0 \| e^{g_0} \| \| e^\theta \| \| g_0 - g_0' \| \mathcal{E} \times e^{\theta|C_1|} \|
\]

for each \( \varphi \in \mathcal{B} \) with \( \| \varphi \| = 1 \) and \( g_0, g_0' \in \mathcal{E} \) (recall the calculation to obtain (2.11)). Moreover, for each \( g_0 \in \mathcal{E} \) the map \( \omega \mapsto \mathcal{L}_\omega(e^{g_0} \cdot) \) is a continuous map from \( A_n \) to \( L(\mathcal{B}) \) due to the continuity of the restriction of \( \mathcal{L} \) on \( A_n \) and (A2). In conclusion, \( \mathcal{L}_\theta \) is continuous on each \( A_n \), which implies the \((\mathcal{F}, \mathcal{B}_{\mathbb{L}(\mathcal{B})})\)-measurability of \( \mathcal{L}_\theta \).

Finally, we show that in the case when \((\Omega, \mathcal{F}, \mathbb{P})\) is a Lebesgue-Rokhlin probability space and \( \mathcal{B} \) is separable, the \((\mathcal{F}, \mathcal{S}_{\mathbb{L}(\mathcal{B})})\)-measurability of \( \mathcal{L} \) implies that of \( \mathcal{L}_\theta \). Recall that the \((\mathcal{F}, \mathcal{S}_{\mathbb{L}(\mathcal{B})})\)-measurability of a map \( A : \Omega \to L(\mathcal{B}) \) is equivalent to the strong measurability of \( A \) (i.e. for any \( \varphi \in \mathcal{B} \), the map \( \omega \mapsto A(\omega) \varphi \) is \((\mathcal{F}, \mathcal{B}_\mathcal{B})\)-measurable), see [27, Appendix A]. Assume the strong measurability of \( \mathcal{L} \) and fix \( \varphi \in \mathcal{B} \). As in the previous paragraph, \( \Omega \to \mathcal{B} : \omega \mapsto \mathcal{L}_\theta \omega \varphi \) is the composition of \( \Omega \times \mathcal{E} \to \mathcal{B} : (\omega, g_0) \mapsto \mathcal{L}_\omega(e^{g_0} \varphi) \) and a measurable map \( \Omega \to \Omega \times \mathcal{E} : \omega \mapsto (\omega, g_\omega) \). Furthermore, for each \( \omega \in \Omega \) the map \( \mathcal{E} \ni g_0 \mapsto \mathcal{L}_\omega(e^{g_0} \varphi) \) is continuous, and for each \( g_0 \in \mathcal{E} \) the map \( \Omega \ni \omega \mapsto \mathcal{L}_\omega(e^{g_0} \varphi) \) is measurable due to the strong measurability of \( \mathcal{L} \). Hence, by [11, Lemma 3.14], \( (\omega, g_0) \mapsto \mathcal{L}_\omega(e^{g_0} \varphi) \) is measurable, and thus \( \mathcal{L}_\theta \) is \((\mathcal{F}, \mathcal{S}_{\mathbb{L}(\mathcal{B})})\)-measurable.

\[\square\]

### 2.4. Local central limit theorem

After the preparation in Subsection 2.3 we can now give sufficient conditions under which the local central limit theorem (LCLT) in Theorem 2.8 holds.

For recent progress on LCLT for dynamical systems (with a neutral direction), see [11, 15] and reference therein. As usual in the study of LCLT, we start this subsection from considering a dichotomy between periodic (lattice, arithmetic) and aperiodic (non-lattice, non-arithmetic) cases. We define \( \mathcal{U}_g \) by

\[\mathcal{U}_g = \{ t \in \mathbb{R} \mid \Lambda(\mathcal{L}_{t\mathbf{i}t}) = 0 \}\]

(recall that \( \mathcal{L}_{t\mathbf{i}t} \) depends on \( g \)). In the case when \( \mathcal{L}_\omega \) is \( \mathbb{P}\)-almost surely a mixing piecewise expanding map and \( \mathcal{B} \) is the space of BV functions (i.e. real-valued functions whose total variation is bounded), Dragičević et al. showed in [17, Subsection 4.4] that \( \mathcal{U}_g \) is a subgroup of \((\mathbb{R}, +)\). Furthermore, it follows from the remark in the proof of Lemma 4.13 of [17] that, by assuming that \( g \) is non-degenerate (i.e. \( V > 0 \)), \( \mathcal{U}_g \) is one of the following two cases:

- \( \mathcal{U}_g = a\mathbb{Z} \) for some \( a > 0 \) (periodic);
- \( \mathcal{U}_g = \{ 0 \} \) (aperiodic).
This dichotomy is known to hold for wide classes of Markov processes or deterministic dynamical systems [22, 32, 33]. The conclusion in Theorem 2.8 is a standard form of the LCLT in the aperiodic case, while the LCLT in the periodic case always needs some modification in the term $|J|$ (see [17, 22, 30] for a precise description of the LCLT in the periodic case).

Furthermore, Dragićević et al. also showed that (for their random dynamics and functional space) the aperiodicity of $g$ is equivalent to the following useful spectral condition, see [17, Subsection 4.3.2]. To express our respect to [18], we keep using their terminology "condition (L)" for the condition.

Definition 2.12. We say that $T$ satisfies condition (L) for $g$ if for every bounded interval $J \subset \mathbb{R} \setminus \{0\}$, there exist a real number $\kappa \in (0, 1)$ and a random variable $C : \Omega \rightarrow (0, \infty)$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\|L_{t, \omega}^{(n)}\| \leq C_\omega \kappa^n \text{ for } t \in J \text{ and } n \geq 0.$$

The deterministic version of condition (L) has been also considered and shown to be equivalent to the deterministic version of the aperiodicity condition, see e.g. [32].

Although condition (L) is useful for a functional analytic proof of the LCLT (and indeed we prove Theorem 2.8 by assuming condition (L)), as pointed out in [18] it may not be easy to verify this condition directly. Therefore, as in [18], we employ a more tractable sufficient condition for condition (L) by Hafouta and Kifer [30].

Definition 2.13. We say that $T$ satisfies the Hafouta-Kifer condition (HK) for $g$ if the following holds:

(HK1) $\mathcal{F}$ is the Borel $\sigma$-algebra on a metric space $\Omega$, $\mathbb{P}(U) > 0$ for any open set $U$, and $\sigma$ is a homeomorphism with a periodic point $\omega_0$ with period $\ell_0$ (i.e. $\sigma^{\ell_0} \omega_0 = \omega_0$). Moreover for each $j \in \{0, 1, \ldots, \ell_0 - 1\}$, there exists a neighborhood of $\sigma^j \omega_0$ on which the map $\omega \mapsto T_\omega$ is constant.

(HK2) For any compact interval $J \subset \mathbb{R}$, the family of maps $\{\omega \mapsto L_{t, \omega}\}_{t \in J}$ is equicontinuous at any point in the orbit of $\omega_0$. Furthermore, there exists a constant $B \geq 1$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\|L_{t, \omega}^{(n)}\| \leq B \text{ for any } n \in \mathbb{N} \text{ and } t \in J.$$

(HK3) For any compact interval $J \subset \mathbb{R} \setminus \{0\}$, there exist constants $c > 0$ and $b \in (0, 1)$ such that

$$\left\| \left(L_{t, \omega_0}^{(\ell_0)} \right)^n \right\| \leq cb^n$$

for any $n \in \mathbb{N}$ and $t \in J$.

In [30, Lemma 2.10.4], Hafouta and Kifer showed (in a more abstract form) that condition (HK) implies condition (L). Note that (HK1) is rather a restriction on noise, than a hypothesis of dynamics to be verified. See [18, Section 9] where it is explained that the restriction of (HK1) is mild. Furthermore, the first part of (HK2) follows from a condition on the observable $g$ as follows.

Proposition 2.14. Assume that (HK1) holds and $g : \Omega \rightarrow \mathcal{E}$ is continuous at any point in the set $\{\sigma^j \omega_0 \mid 0 \leq j \leq \ell_0 - 1\}$. Assume that $g$ satisfies the centering condition (1.6). Then the first part of (HK2) holds.
3.1 below. We mention that the above regularity index \( m \) belongs to Sobolev space \( r \) space of observables for all \((1.6)\)

centering condition

unperturbed map \( T \).

Let Theorem 2.16. \( g \) the circle given in Theorem 2.15. Assume that condition \((UG)\)

\((LY)\)

condition \( B \) conditions of Theorem 2.5 with \( \epsilon > 0 \). Then for any sufficiently small \( \epsilon > 0 \), we get

by using (A2) again, we get

\[ \|e^{itg_{\omega}} - e^{itg_0}\| \leq \|g_{\omega} - g_0\| \|\varepsilon\|t|e^{t}|G \]

(recall that \( G = \text{ess sup}_{\omega} \|g_{\omega}\| \)). The first part of (HK2) immediately follows from this inequality and (2.14).

Notice that, as (HK1), the condition in Proposition 2.14 is rather a restriction on observables than a condition to be verified. See [18, Remark 9.2] for the argument that the condition is preserved under the centering by virtue of (HK1), so the centering condition on \( g \) is not essential.

2.5. Application to random U(1) extensions of expanding maps. The following result shows that Theorems A–D follow from Theorems 2.15–2.18.

**Theorem 2.15.** Let \( T \) be the random U(1) extension of an expanding map on the circle with a noise level \( \epsilon > 0 \) described in Subsection 1.1. Assume that \((\Omega,F,\mathbb{P})\) is a Lebesgue-Rokhlin probability space and \( T_0 \) satisfies the partial captivity condition. Then for any sufficiently small \( \epsilon > 0 \) and any sufficiently large integer \( m \), the usual Sobolev space \( H^m(T^2) \) is admissible for \( T \equiv T(\epsilon) \), and \( H^r(T^2) \) is an associated space of observables for all \( r \geq m \). Moreover, \( T \) satisfies the uniform spectral gap condition (UG) with respect to \( H^m(T^2) \) and \( H^r(T^2) \) and the Lasota-Yorke inequality condition (LY) with respect to \( H^m(T^2) \subset H^{m-1}(T^2) \). In particular, \( T \) satisfies the conditions of Theorem 2.7 with \( \mathcal{B} = H^m(T^2) \) and \( \mathcal{E} = H^r(T^2) \) for such \( \epsilon, m, r \).

The Sobolev spaces \( H^m(T^2) \) is the usual one – a definition is given in Subsection 3.1 below. We mention that the above regularity index \( m \) only depends on the unperturbed map \( T_0 \), see Remark 3.2.

**Theorem 2.16.** Let \( T \) be the random U(1) extension of an expanding map on the circle given in Theorem 2.15. Assume that \( g \in L^\infty(\Omega,\mathcal{C}^m(T^2)) \) satisfies the centering condition \((1.4)\).

(i) Assume that \( g_{\omega} \) does not depend on the second variable for \( \mathbb{P} \)-almost every \( \omega \). Then the second part of (HK2) holds.

(ii) Assume also that the set \( \{ \sigma^j\omega_0 \mid 0 \leq j \leq \ell_0 - 1 \} \) is included in the full measure set of \((1.5)\) and that \((S_{\ell_0}g)_{\omega_0} \) is not cohomologous to a constant with respect to \( T^{(\ell_0)}_{\omega_0} \), that is, for any \( c \in \mathbb{R} \) and integrable function \( \psi : T^2 \rightarrow \mathbb{R} \), it does not hold that

\[ (S_{\ell_0}g)_{\omega_0} = \psi \circ T^{(\ell_0)}_{\omega_0} - \psi + c, \]

where \( \ell_0 \) is a positive integer and \( \omega_0 \) is a periodic point with period \( \ell_0 \). Then (HK3) holds for these \( \ell_0 \) and \( \omega_0 \).
Remark. A class of examples satisfying the condition for \( g \) in the item (ii) of Theorem 2.16 is \( \ell_0 = 1 \) and \( g_{\omega_0}(x,s) = \tilde{g}(x) - E_{\mu_{\omega_0}}[\tilde{g}] \) with some \( \tilde{g} \in \mathcal{Y}_\ell \), \( 1 < \ell < k \), where \( \mathcal{Y}_\ell \) is the open and dense subset of \( \mathcal{C}^\ell(S^1) \) given in Theorem A.1 in Appendix A. Indeed, if \( (S_{\omega_0}g)_{\omega_0} = \psi \circ T_{\omega_0} - \psi + c \) with some \( c \in \mathbb{R} \) and \( \psi \in L^1(T^2) \) then \( \tilde{g} = \tilde{\psi} \circ E_{\omega_0} - \tilde{\psi} + c + E_{\mu_{\omega_0}}[\tilde{g}] \) with \( \tilde{\psi} = \int \psi(x,s)ds \) due to the translation invariance of \( \text{Leb}_g \). This contradicts the fact that no \( \tilde{g} \in \mathcal{Y}_\ell \) is cohomologous to a constant (see Appendix A). Therefore, this observable satisfies the statement in item (ii) of Theorem 2.16.

As an immediate consequence, \( T \) satisfies condition (L) for \( g \) as long as the following, more easily verifiable, conditions are met:

**Corollary 2.17.** Let \( T \) be the random \( U(1) \) extension of an expanding map on the circle given in Theorem 2.16 Assume (HK1).

(i) Assume that \( g \in L^\infty(\Omega,\mathcal{F}^m(T^2)) \) satisfies the centering condition \( 1.6 \).

(ii) Assume that \( g : \Omega \to \mathbb{R} \) is continuous at any point in the set \( \{ \sigma^j\omega_0 \mid 0 \leq j \leq \ell_0 - 1 \} \).

(iii) Assume that \( g_{\omega} \) does not depend on the second variable for \( \mathbb{P} \)-almost every \( \omega \).

(iv) Assume that the set \( \{ \sigma^j\omega_0 \mid 0 \leq j \leq \ell_0 - 1 \} \) is included in the full measure set of \( 1.5 \) and that \( (S_{\ell_0}g)_{\omega_0} \) is not cohomologous to a constant with respect to \( T_{\omega_0} \).

Then \( T \) satisfies condition (L) for \( g \).

3. Hypothesis verification

In this section we will prove Theorem 2.16 and thus show that the random \( U(1) \) extension of an expanding map in Section 1 satisfies the conditions appearing in the statements of Theorems 2.5–2.8. We also prove Theorem 2.16 Let therefore \( E \), \( T \) be as in (1.3). In particular, \( \tau \) is a random perturbation of a function \( \tau_0 \), where \( \tau_0 \) is guaranteed not to be cohomologous to a constant function by the assumption of partial captivity. Throughout this subsection we assume that \( (\Omega,\mathcal{F},\mathbb{P}) \) is a Lebesgue-Rokhlin probability space.

**Notation.** In Subsection 2.3 we used the double-indexed operator \( \mathcal{L}_{\theta,\omega} \) with \( \theta \in \mathbb{C} \), \( \omega \in \Omega \). However, this notation will not appear in this subsection, so another double-indexed operator \( \mathcal{L}_{\nu,\omega} \) with \( \nu \in \mathbb{Z} \), \( \omega \in \Omega \) given in (3.3) should cause no confusion. In particular, \( \mathcal{L}_{0,\omega} \) in this section always means \( \mathcal{L}_{\nu,\omega} \) at \( \nu = 0 \). Furthermore, we will use \( C \) and \( C_n \) \( (n \geq 1) \) as positive constants which do not depend on \( \omega \in \Omega \), \( \nu \in \mathbb{Z} \) nor \( \epsilon \geq 0 \), and may change on occasion.

3.1. Preliminaries. We first note that \( T_{\omega} \) preserves \( L^2(T^2) \equiv L^2(T^2,\text{Leb}_T) \) \( \mathbb{P} \)-almost surely, so that we can define the transfer operator \( \mathcal{L}_{\nu} \) of \( T_{\omega} \) on \( L^2(T^2) \equiv (L^2(T^2))^\prime \) by (2.11). As in 3.7 (originally in 2.3), we employ the following decomposition in Fourier modes,

\[
L^2(T^2) = \bigoplus_{\nu \in 2\pi \mathbb{Z}} \mathcal{H}_\nu, \quad \mathcal{H}_\nu = \{ (x,s) \mapsto \psi(x)e^{i\nu s} : \psi \in L^2(S^1) \}.
\]

The spaces \( (\mathcal{H}_\nu, \| \cdot \|_{L^2(S^1)}) \) and \( L^2(S^1) \) are isometrically isomorphic, and the decomposition (3.1) is preserved by the pullback operator \( u \mapsto u \circ T_{\omega} \).
For given $\nu \in 2\pi \mathbb{Z}$ and fixed $\omega \in \Omega$, let $\mathcal{M}^{(n)}_{\nu,\omega}$ denote the operator $u \mapsto u \circ T^{(n)}_{\nu}$ restricted to $\mathcal{H}_\nu$. It is straightforward to check that by identifying $\mathcal{H}_\nu$ with $L^2(\mathbb{S}^1)$ we can view $\mathcal{M}^{(n)}_{\nu,\omega}$ for $\mathbb{P}$-almost every $\omega$ as an operator on $L^2(\mathbb{S}^1)$ given by

$$\mathcal{M}^{(n)}_{\nu,\omega}\psi(x) = \psi(E^{(n)}_\omega(x)) e^{i\nu r^{(n)}(x)}, \quad \psi \in L^2(\mathbb{S}^1),$$

Here $E^{(n)}_\omega = E^{n-1}_\omega \circ E^{n-2}_\omega \circ \cdots \circ E_\omega$ and $r^{(n)}_\omega$ is the quenched Birkhoff sum of $\tau$ given by

$$r^{(n)}_\omega = \sum_{j=0}^{n-1} \tau_{\sigma^j \omega} \circ E^{(j)}_\omega \quad (n \geq 1, \ \omega \in \Omega).$$

Let $\mathcal{L}^{(n)}_{\nu,\omega} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be the adjoint operator of $\mathcal{M}^{(n)}_{\nu,\omega}$. It is straightforward to check that

$$\mathcal{L}^{(n)}_{\nu,\omega}\psi(x) = \sum_{E^{(n)}_\omega(y) = x} \frac{e^{-i\nu r^{(n)}(y)}}{dy} \psi(y), \quad \psi \in L^2(\mathbb{S}^1).$$

Note also that for $\varphi \in L^2(\mathbb{T}^2)$ we have

$$\mathcal{L}_\omega \varphi(x, s) = \sum_{\nu \in 2\pi \mathbb{Z}} e^{i\nu s} \mathcal{L}\nu,\omega \varphi(x), \quad \varphi(x, s) = \sum_{\nu \in 2\pi \mathbb{Z}} \varphi_\nu(x) e^{i\nu s},$$

where $\varphi_\nu(x) = \int_{\mathbb{S}^1} e^{-i \nu y} \varphi(x, s) ds$ and thus

$$\mathcal{L}^{(n)}_{\nu,\omega}(x) = \int_{\mathbb{S}^1} e^{-i \nu s} \mathcal{L}^{(n)}_\omega \varphi(x, s) ds = \mathcal{L}^{(n)}_{\nu,\omega} \varphi_\nu(x), \quad n \geq 1.$$

For $m \geq 0$ we let $H^m(\mathbb{S}^1) \subset L^2(\mathbb{S}^1)$ denote the Sobolev space of regularity index $m$, defined as the set of measurable functions $\psi$ satisfying

$$||\psi||_{H^m(\mathbb{S}^1)}^2 = \sum_{\xi \in 2\pi \mathbb{Z}} (1 + |\xi|^2)^m |\hat{\psi}(\xi)|^2 < \infty.$$

The Fourier coefficients of $\psi \in L^2(\mathbb{S}^1)$ are defined by

$$\hat{\psi}(\xi) = \langle \psi, \phi_{-\xi} \rangle, \quad \xi \in 2\pi \mathbb{Z},$$

where the functions $\phi_{\xi} \in C^\infty(\mathbb{S}^1)$ are given by $\phi_{\xi}(x) = e^{ix\xi}$ for $\xi \in 2\pi \mathbb{Z}$ and $x \in \mathbb{S}^1$.

We also introduce an alternative $\nu$-dependent norm $\| \cdot \|_{H^m(\mathbb{S}^1)}$ on $H^m(\mathbb{S}^1)$ defined by

$$\|\psi\|^2_{H^m(\mathbb{S}^1)} = \sum_{\xi \in 2\pi \mathbb{Z}} (1 + (\xi/\nu)^2)^m |\hat{\psi}(\xi)|^2, \quad \psi \in L^2(\mathbb{S}^1), \quad 0 \neq \nu \in \mathbb{Z}.$$

Let $H^m_\nu(\mathbb{S}^1)$ denote the space $H^m(\mathbb{S}^1)$ equipped with the norm $\| \cdot \|_{H^m_\nu}$. Finally, we let $H^r(\mathbb{T}^2)$ denote the Sobolev space on $\mathbb{T}^2$ with regularity $r$, defined similarly as the set of measurable functions $\varphi$ satisfying

$$\|\varphi\|^2_{H^r(\mathbb{T}^2)} = \sum_{\xi, \nu \in 2\pi \mathbb{Z}} (1 + |\xi|^2 + |\nu|^2)^r |\hat{\varphi}(\xi, \nu)|^2 < \infty,$$

where

$$\hat{\varphi}(\xi, \nu) = \int_{\mathbb{T}^2} e^{-i(x\xi + sy)} \varphi(x, s) dx ds, \quad \xi, \nu \in 2\pi \mathbb{Z}.$$
We shall work with the space \( \mathcal{H}^m(T^2) \subset L^2(T^2) \) defined as the set of measurable functions \( \varphi \) satisfying

\[
\|\varphi\|_{\mathcal{H}^m}^2 = \|\varphi_0\|^2_{H^m} + \sum_{\nu \in 2\pi\mathbb{Z}, \nu \neq 0} (1 + \nu^2)^m \|\hat{\varphi}_\nu\|^2_{H^0} < \infty
\]

where \( \varphi_\nu(x) = \int_{T^2} e^{-ix\xi} \varphi(x, s) \, ds \) as before. (This functional space did not appear in [37].) Observe that computing \( \|\varphi_\nu\|_{H^m} \) involves the Fourier coefficients \( \hat{\varphi}_\nu(\xi) \) of \( x \mapsto \varphi_\nu(x) \). Naturally, these are precisely the Fourier coefficients \( \hat{\varphi}(\xi, \nu) \) of \( (x, s) \mapsto \varphi(x, s) \), since \( \hat{\varphi}_\nu(\xi) = \int_{T^2} e^{-ix\xi} \varphi(x) \, dx \). In fact, the somewhat odd-looking norm in (3.3) is equivalent to the usual Sobolev norm:

**Lemma 3.1.** \( \|\cdot\|_{\mathcal{H}^m} \) is equivalent to \( \|\cdot\|_{H^m(T^2)} \).

**Proof.** Note that for any \( \xi, \nu \in 2\pi\mathbb{Z} \) with \( \nu \neq 0 \),

\[
\frac{(1 + \xi^2 + \nu^2)}{(1 + \nu^2)(1 + (\xi/\nu)^2)} = \frac{1 + \xi^2 + \nu^2}{\xi^2 + \nu^2} \frac{\nu^2}{1 + \nu^2} = \frac{1 + \frac{1}{1 + \frac{\xi}{\nu}}}{1 + \frac{1}{\nu}}
\]

is in an interval \((\frac{1}{2}, 1)\). Indeed, the right-hand side is clearly bounded by 1, and by inserting \( \nu = 1 \) and letting \( |\xi| \to \infty \) we see it is bounded from below by \( \frac{1}{2} \). (Note that since \( \nu \in 2\pi\mathbb{Z} \), the choice \( \nu = 1 \) is actually illegal but the estimate is clearly valid.) Hence the lemma immediately follows in view of the definitions of the two norms.

3.2. Conditions [A1] and [A2] We begin by verifying that conditions [A1] and [A2] are satisfied when \( \mathcal{D} = L^2(T^2) \), \( \mathcal{B} = \mathcal{H}^m(T^2) \) and \( \mathcal{E} = H^r(T^2) \) for some sufficiently large \( m \) depending only on the unperturbed system \( T_0 \) given by (1.3), see Remark 3.2. The dual pairing in (2.1) can thus be expressed via the standard inner product \( \langle \cdot, \cdot \rangle_{L^2} \) on \( L^2(T^2) \) as

\[
\langle \varphi, u \rangle = \langle \varphi, \bar{u} \rangle_{L^2}.
\]

**Remark 3.2.** Assume that the map \( E_0 \) in (1.3) is expanding with an expansion rate \( \lambda_0 = \min_x E_0'(x) \). For sufficiently small noise levels \( \epsilon \) we have in view of (1.3) that \( E_\omega \) is an expanding map \( \mathbb{P} \)-almost surely with an expansion rate strictly larger than \( \lambda = \frac{r}{2}(\lambda_0 + 1) > 1 \). Let \( \lambda^{-\frac{1}{2}} < \rho < 1 \). By combining [27] Theorem 1.4 and [37] Theorem 1.5 we can find a positive integer \( m_0 > 1 \) together with numbers \( \epsilon_0 = \epsilon_0(m) \) and \( \nu_0 = \nu_0(m) \) depending also on an integer \( m \geq m_0 \) for which those results are in force. For the rest of this section we assume this to be the case and only consider noise levels \( \epsilon \) such that \( 0 \leq \epsilon \leq \epsilon_0 \). We make sure to pick \( m \) large enough so that

\[
\log(\lambda^{-m-\frac{1}{2}}k^{\frac{1}{2}}) < 0
\]

where \( k \geq 2 \) is the angle multiplication factor of \( E_0 \). Then \( m \) only depends on the unperturbed system \( T_0 \). Fixing \( r \geq m > 1 \), the same is true for the Sobolev regularity index \( r \) appearing in \( \mathcal{E} = H^r(T^2) \).

**Proposition 3.3.** \( \mathcal{H}^m(T^2) \) is \( T \)-admissible.

**Proof.** In view of Definition 2.1 we need to prove [A1], i.e., that \( \|L_\omega \varphi\|_{\mathcal{H}^m} \leq C\|\varphi\|_{\mathcal{H}^m} \) for all \( \varphi \in \mathcal{H}^m \) and some \( C > 0 \). By (3.4) and the definition of the norm
in \( \mathcal{H}^m \) we have
\[
(3.8) \quad \|L_\omega \varphi\|^2_{\mathcal{H}^m} = \|L_{0, \omega} \varphi_0\|^2_{\mathcal{H}^m(\mathbb{S}^1)} + \sum_{\nu \in 2\pi \mathbb{Z}, \nu \neq 0} (1 + \nu^2)^m \|L_{\nu, \omega} \varphi_\nu\|^2_{\mathcal{H}^m(\mathbb{S}^1)}.
\]
It follows from \([37, \text{Theorem 1.5}]\) that there is a \( \rho_0 \in (0, 1) \) as in Remark 3.2 together with a \( \nu_0 \in 2\pi \mathbb{N} \) and a constant \( c_0 \) such that
\[
(3.9) \quad \|L_{\nu, \omega} \varphi_\nu\|_{\mathcal{H}^m} \leq c_0 \rho_0^n \| \varphi_\nu\|_{\mathcal{H}^m}, \quad n \geq 1,
\]
for any \( \nu \) with \( |\nu| > \nu_0 \) and \( \mathbb{P} \)-almost every \( \omega \in \Omega \). On the other hand, for any \( \nu \neq 0 \) with \( |\nu| \leq \nu_0 \), we have
\[
\nu_0^{-2}(1 + \xi^2) \leq \nu^{-2}(\nu^2 + \xi^2) = 1 + (\xi/\nu)^2 \leq 1 + \xi^2,
\]
so that
\[
(3.10) \quad \nu_0^{-m} \| \varphi_\nu\|_{\mathcal{H}^m} \leq \| \varphi_\nu\|_{\mathcal{H}^m} \leq \| \varphi_\nu\|_{\mathcal{H}^m}.
\]
Hence, combining \([37] \text{for } n = 1 \) with \([37, \text{Theorem 1.4}]\) it follows that there is a constant \( C \) such that
\[
\|L_{\nu, \omega} \varphi_\nu\|_{\mathcal{H}^m} \leq C \| \varphi_\nu\|_{\mathcal{H}^m}
\]
for all \( \nu \neq 0 \), and that
\[
\|L_{0, \omega} \varphi_0\|_{\mathcal{H}^m} \leq C \| \varphi_0\|_{\mathcal{H}^m}.
\]
Combining these estimates with (3.8) we conclude that [A1] holds. \( \square \)

We shall need the following well-known structure result for Sobolev spaces. We include a short direct proof for convenience.

**Lemma 3.4.** Let \( r > 1 \). Then there is a constant \( C_r \) such that for all \( u, \varphi \in H^r(\mathbb{T}^2) \) we have \( u \varphi \in H^r(\mathbb{T}^2) \) and
\[
\|u \varphi\|_{H^r(\mathbb{T}^2)} \leq C_r \| u \|_{H^r(\mathbb{T}^2)} \| \varphi\|_{H^r(\mathbb{T}^2)}.
\]

**Proof.** To shorten notation we shall denote \( (\xi, \nu) \) by \( \xi = (\xi_1, \xi_2) \), and write \( \langle x \rangle = (1 + |x|^2)^{1/2} \) for the Japanese bracket. Following Sjögstrand \([39, \text{Proposition 2.1}]\), we pass to the Fourier side and see that the result follows if we show that
\[
(3.11) \quad \sum_{\xi \in (2\pi \mathbb{Z})^2} \langle \xi \rangle^r \langle \xi \rangle \langle \langle \xi \rangle^{-r} \tilde{\varphi} \ast \langle \xi \rangle^{-r} \tilde{u} \rangle \langle \xi \rangle \langle \langle \xi \rangle^{-r} \tilde{\varphi} \rangle \langle \langle \xi \rangle^{-r} \tilde{u} \rangle \leq C_r \| w\|_{\mathcal{H}^r(\mathbb{T}^2)} \| \tilde{\varphi}\|_{\mathcal{H}^r(\mathbb{T}^2)}
\]
for all non-negative \( \tilde{\varphi}, \tilde{u}, w \in \ell^2 \), where \( \ast \) denotes convolution and the norms are the ones in \( \ell^2 \). Indeed,\[
\| u \varphi\|_{H^r(\mathbb{T}^2)} = \langle \langle \langle \xi \rangle^{-r} \tilde{\varphi} \rangle \ast \langle \langle \xi \rangle^{-r} \tilde{u} \rangle \rangle \langle \langle \xi \rangle^{-r} \tilde{\varphi} \rangle \langle \langle \xi \rangle^{-r} \tilde{u} \rangle \leq 1 + II,
\]
where \( I, II \) are the sums over \( \{|\eta| \geq |\xi|/2\} \) and \( \{|\zeta| \geq |\xi|/2\} \), respectively. In \( I \) we have \( \langle \xi \rangle^{-r} \leq 2^{r} \langle \eta \rangle^{-r} \), so
\[
I \leq 2^r \sum_{\zeta} \left( \sum_{\xi} w(\xi) \langle \xi - \zeta \rangle \langle \xi \rangle^{-r} \tilde{u}(\zeta) \right) \langle \xi \rangle^{-r} \tilde{u}(\zeta) \leq 2^r \| w\|_{\mathcal{H}^r(\mathbb{T}^2)} \| \langle \xi \rangle^{-r} \tilde{\varphi}\|_{\ell^2},
\]
where \( \| \langle \cdot \rangle \|_{L^2} \leq C_r \| \bar{u} \| \) by the Cauchy-Schwartz inequality since \( r > 1 \). Hence I is bounded by a constant times \( \| u \| \) \( \| \overline{\langle \cdot \rangle} \| \). In II we have \( \langle \xi \rangle \leq 2^r \langle \xi \rangle \) so by symmetry the same estimate holds for II, and (3.11) follows.

**Proposition 3.5.** \( H^r(T^2) \) is associated with \( \mathcal{H}^m(T^2) \) for all \( r \geq m > 1 \).

**Proof.** By Lemma 3.4 the multiplication map \( H^r(T^2) \times H^r(T^2) \rightarrow H^r(T^2) \) is continuous. Hence, it suffices to prove that the first inequality in (3.12) is satisfied when \( \| \cdot \|_{\mathcal{H}} = \| \cdot \|_{\mathcal{H}^m} \) and \( \| \cdot \|_H = \| \cdot \|_{H^r} \). It is even enough to show that

\[
\| u \varphi \|_{\mathcal{H}^m} \leq C_{m,r} \| \varphi \|_{\mathcal{H}^m} \| u \|_{H^r}.
\]

Indeed, if \( \psi \in H^r_m(S^1) \) and \( \nu \neq 0 \) then \( \| \psi \|_{L^2(S^1)} \leq \| \nu \|_{H^m(S^1)} \) holds, which implies that \( \| \varphi \|_{L^2(T^2)} \leq \| \varphi \|_{\mathcal{H}^m(T^2)} \).

Hence

\[
\langle \varphi, u \rangle = \langle u \varphi, 1_{T^2} \rangle \leq \| u \varphi \|_{L^2(T^2)} \leq \| u \|_{\mathcal{H}^m(T^2)}
\]

so we always have \( \max \{ \langle \varphi, u \rangle, \| u \varphi \|_{\mathcal{H}^m} \} = \| u \varphi \|_{\mathcal{H}^m} \) in this case, which proves the claim.

But by Lemma 3.1 \( \mathcal{H}^m(T^2) \) coincides with \( H^m(T^2) \), so if \( r \geq m > 1 \) then

\[
\| u \varphi \|_{\mathcal{H}^m} \leq C_m \| \varphi \|_{\mathcal{H}^m} \| u \|_{\mathcal{H}^m} \leq C_m \| \varphi \|_{\mathcal{H}^m} \| u \|_{H^r}
\]

by Lemma 3.1 which establishes (3.12) and the proof is complete.

**3.3. Condition (UG).** Having established that \( H^r(T^2) \) is associated with \( \mathcal{H}^m(T^2) \) we proceed to prove that \( T \) satisfies the uniform spectral gap condition with respect to \( \mathcal{H}^m(T^2) \) and observables in \( H^r(T^2) \). We begin by verifying the existence of invariant measures.

**Proposition 3.6.** For any sufficiently small noise level \( \epsilon > 0 \) there exists a unique \( h \in L^\infty(\Omega, \mathcal{H}^m(T^2)) \) and a family of probability measures \( \{ \mu_\omega \}_{\omega \in \Omega} \) such that

\[
\mathcal{L}_\omega h_\omega = h_{\sigma \omega}, \quad \langle h_\omega, u \rangle = \int u \, d\mu_\omega, \quad \text{ess inf}_{\omega \in \Omega} \| h_\omega \|_{\mathcal{H}^m} \geq 1
\]

for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) and any observable \( u \in H^r(T^2) \).

**Proof.** For sufficiently small \( \epsilon > 0 \) there is by [37] Theorem 1.3 a uniquely defined function \( h = h(\epsilon) : \omega \mapsto h_\omega \) such that \( \mathcal{L}_\omega h_\omega = h_{\sigma \omega} \) and \( \langle h_\omega, 1_{T^2} \rangle = 1 \), together with a family of probability measures \( \{ \mu_\omega \}_{\omega \in \Omega} \) such that \( d \mu_\omega = d\mu_{\omega(2)} \). Hence, if \( u \in H^r(T^2) \subset L^2(\mathbb{T}^2) \) we have \( \langle h_\omega, u \rangle = \int u \, d\mu_\omega \). Note that by [37] Theorem 1.3], \( h_\omega(x, s) \) for \( (x, s) \in \mathbb{T}^2 \) has the form of a tensor product \( h_\omega(x, s) = h_\omega(x) \otimes 1_{S^1}(s) \) with \( x \mapsto h_\omega(x) \in C^{\infty}(S^1) \) satisfying \( \int_{S^1} h_\omega(x) \, dx = 1 \) and \( \text{ess sup}_\omega \| h_\omega \|_{H^m(S^1)} \leq C_m \) for any \( m \in \mathbb{N} \). In particular, decomposing \( h_\omega(x, s) \) into Fourier modes we find that the mode corresponding to \( \nu = 0 \) satisfies \( \langle h_\omega \rangle_0(x) = \int h_\omega(x, s) \, ds = h_\omega(x) \), while \( \langle h_\omega \rangle_\nu(x) = 0 \) for all \( \nu \in 2\pi \mathbb{Z} \) such that \( \nu \neq 0 \). Inspecting the definition of the norm \( \| \cdot \|_{\mathcal{H}^m(T^2)} \) this implies that

\[
\| h_\omega \|_{\mathcal{H}^m(T^2)}^2 = \langle (h_\omega)_0 \rangle^2_{H^m(S^1)} \leq C_m^2
\]

\( \mathbb{P} \)-almost surely, which shows that \( h \in L^\infty(\Omega, \mathcal{H}^m(T^2)) \). Moreover, by the Cauchy-Schwartz inequality we have

\[
1 = \int_{S^1} h_\omega(x) \, dx \leq \| h_\omega \|_{L^2(S^1)} = \| (h_\omega)_0 \|_{L^2(S^1)}
\]

and since \( m \geq 0 \) we thus find that

\[
\| h_\omega \|^2_{\mathcal{H}^m(T^2)} = \| (h_\omega)_0 \|^2_{H^m(S^1)} \geq \| (h_\omega)_0 \|^2_{L^2(S^1)} \geq 1,
\]
so \( \text{ess inf}_{\omega \in \Omega} \| h_\omega \|_{\mathcal{H}^m} \geq 1 \). \( \square \)

We next record a perturbation lemma.

**Lemma 3.7.** There is a \( \rho \in (0, 1) \), an arbitrarily large integer \( n_0 \) and a number \( \epsilon(n_0) \) such that for all \( 0 \leq \epsilon < \epsilon(n_0) \) we have

\[
\| L^{(n_0)}_\omega \varphi \|_{\mathcal{H}^m} \leq \rho^{n_0} \| \varphi \|_{\mathcal{H}^m} + C_{n_0} \| \varphi_0 \|_{H^{m-1}(\mathbb{S}^1)}
\]

where \( \varphi_0(x) = \int_{\mathbb{S}^1} \varphi(x, s) \, ds \). If \( \langle \varphi, 1_{\mathbb{S}^1} \rangle = 0 \) we may take \( C_{n_0} = 0 \).

**Proof.** By \([3.4]\) together with \([3.3] - [3.9]\) we have

\[
\| L^{(n)}_\omega \varphi \|_{\mathcal{H}^m} \leq \| L^{(n)}_{\omega,0} \varphi_0 \|_{H^m} + \sum_{0 < |\nu| \leq n_0} (\nu)^{2m} \| L^{(n)}_{\nu,\omega} \varphi_\nu \|_{H^m} + \sum_{|\nu| > n_0} C_0 \rho_0^{2n} (\nu)^{2m} \| \varphi_\nu \|_{H^m}^2
\]

where \( \rho_0 \in (0, 1) \). Let \( L_{T_0} \) denote the transfer operator induced by the unperturbed map \( T_0 \) and \((L_{T_0})_\nu\) its restriction corresponding to \( L_{\nu,\omega} \). By applying \([37]\) Proposition 4.5] to each \( L_{\nu,\omega} \) with \( |\nu| \leq n_0 \) we can find a \( \rho_1 \in (0, 1) \) together with an integer \( n_0 \) (which can be chosen arbitrarily large) and a number \( \epsilon(n_0) \) such that for all \( |\nu| \leq n_0 \) and \( 0 < \epsilon < \epsilon(n_0) \) we have

\[
\text{ess sup}_{\omega} \| (L_{\nu,\omega}^{(n_0)} - (L_{T_0}^{(n_0)})_{\nu}) \varphi_\nu \|_{H^m} \leq \rho_1 \rho_0^{n_0} \| \varphi_\nu \|_{H^m}.
\]

By \([37]\) Theorem 4.4] the spectral radius of each \((L_{T_0})_\nu : H^m \to H^m\) is strictly less than 1 when \( \nu \neq 0 \). Hence, by increasing \( n_0 \) if necessary, we have for some \( \rho_2 \in (0, 1) \) and all \( 0 < |\nu| \leq n_0 \) that

\[
\| (L_{T_0}^{(n_0)})_{\nu} \varphi_\nu \|_{H^m} \leq \rho_2 \rho_0^{n_0} \| \varphi_\nu \|_{H^m}.
\]

When \( \nu = 0 \), the transfer operator \((L_{T_0})_{\nu=0}\) is the standard transfer operator induced by the uniformly expanding map \( E_0 \) on the circle. Thus, by \([4]\) Lemma 15 we can (by taking \( n_0 \) sufficiently large and increasing \( \rho_2 \) if necessary) find a constant \( C_{n_0} \) such that

\[
\| (L_{T_0}^{(n_0)})_{0} \varphi_0 \|_{H^m} \leq \rho_2 \rho_0^{n_0} \| \varphi_0 \|_{H^m} + C_{n_0} \| \varphi_0 \|_{H^{m-1}}
\]

for all \( \varphi_0 \in H^m \). On the other hand, if \( \langle \varphi, 1_{\mathbb{S}^1} \rangle = 0 \) then the Fourier mode \( \varphi_0(x) \) corresponding to \( \nu = 0 \) satisfies

\[
\int_{\mathbb{S}^1} \varphi_0(x) \, dx = \int_{\mathbb{S}^1} \left( \int_{\mathbb{S}^1} \varphi(x, s) \, ds \right) \, dx = \langle \varphi, 1_{\mathbb{S}^1} \rangle = 0.
\]

This implies that \([3.10]\) holds with \( C_{n_0} = 0 \) for such \( \varphi \). Indeed, it is well-known that the only eigenvalue of \((L_{T_0})_{\nu=0}\) on the unit circle is the simple eigenvalue 1, and the Lyapunov subspace associated to the eigenvalues of modulus 1 is one-dimensional and spanned by the function \( h(\epsilon = 0) \) given in the first paragraph of the proof, see \([34]\) Chapter 3) or \([37]\) Theorem 4.4]. Let us write \( h_0 \) for this function, indicating that \( h_0 \) does not depend on \( \omega \) in contrast to \( h_\omega = h_\omega(\epsilon) \) for \( \epsilon > 0 \). If \( \pi_0 : H^m(\mathbb{S}^1) \to H^m(\mathbb{S}^1) \) denotes the projection \( \pi_0(\psi) = h_0(x) \int_{\mathbb{S}^1} \psi(x) \, dx \) then \( \varphi_0 \) is in the kernel of \( \pi_0 \) by \([3.17]\). In the notation of Definition \([2.9]\) we thus have a splitting \( H^m = \mathcal{F}_0 \oplus \mathcal{R}_0 \) with \( \mathcal{F}_0 = C h_0 \) and if \( \psi \in \mathcal{R}_0 \) then

\[
\lim_{n \to \infty} \frac{1}{n} \log \| (L_{T_0}^{(n)})_{0} \psi \|_{H^m} \leq \tilde{\Lambda}(\mathcal{L}_{T_0})_0 < 0.
\]

This proves the claim since \( \tilde{\Lambda}(\mathcal{L}_{T_0})_0 < \log \rho_2 \) when \( \rho_2 \) is sufficiently large.
By combining (3.14) + (3.16) we see that
\[ \|L^{(n_0)}_{\nu,\omega} \varphi_0\|_{H^m} \leq K_1(n_0)\|\varphi_0\|^2_{H^m} + C^2_{n_0} \|\varphi_0\|^2_{H^{m-1}} \]
where \( K_1(n_0) = (\rho_1^{n_0} + \rho_2^{n_0}) (p_1^{n_0} + \rho_2^{n_0} + 2C_{n_0}) \) and by using (3.10) twice we also get
\[ \sum_{0 < |\nu| \leq n_0} \langle \nu \rangle^{2m} \|L^{(n_0)}_{\nu,\omega} \varphi_0\|_{H^m} \leq K_2(n_0) \sum_{0 < |\nu| \leq n_0} \langle \nu \rangle^{2m} \|\varphi_0\|^2_{H^m} \]
where \( K_2(n_0) = \nu_0^{2m} (p_1^{n_0} + \rho_2^{n_0})^2 \). Since \( 0 < \rho_1, \rho_2 < 1 \) we can choose \( n_0 \) so large that \( K_1(n_0), K_2(n_0) < 1 \), which means that there is a \( \rho_3 \in (0, 1) \) such that both \( K_1(n_0) \) and \( K_1(n_0) \) are bounded by \( \rho_3^{2n_0} \). Inserting this into (3.13) we find that
\[ \|L^{(n_0)}_{\nu,\omega} \varphi\|_{H^m}^2 \leq \rho_3^{2n_0} \left( \|\varphi_0\|^2_{H^m} + \sum_{0 < |\nu| \leq n_0} \langle \nu \rangle^{2m} \|\varphi_0\|^2_{H^m} \right) \]
\[ + \sum_{|\nu| > n_0} c_0 \rho_3^{2n_0} \langle \nu \rangle^{2m} \|\varphi_0\|^2_{H^m} + C_0^2 \|\varphi_0\|^2_{H^{m-1}}. \]

Since \( \rho_0 < 1 \) we may assume that \( c_0 \rho_0^{2n_0} = \rho_4^{2n_0} \) for some \( \rho_4 \in (0, 1) \) by increasing \( n_0 \) if necessary. The result now follows by picking \( \rho = \max(\rho_3, \rho_4) \). \( \Box \)

**Proposition 3.8.** There is an \( \epsilon_0 \) such that for all \( 0 \leq \epsilon < \epsilon_0 \), \( T \) satisfies the uniform spectral gap condition with respect to \( H^m \) and \( H^r(\mathbb{T}^2) \).

**Proof.** In view of Proposition 3.6 we only need to verify (2.4), namely, that there are constants \( C > 0 \) and \( \rho \in (0, 1) \) such that if \( \varphi \in H^m(\mathbb{T}^2) \) satisfies \( \langle \varphi, 1_{\mathbb{T}^2} \rangle = 0 \), then
\[ \text{ess sup}_{\omega \in \Omega} \|L^{(n)}_{\omega} \varphi\|_{H^m} \leq C \rho^n \|\varphi\|_{H^m} \quad \text{for all } n \geq 1. \]
By Lemma 3.7 there is a \( \rho \in (0, 1) \) together with an integer \( n_0 \) and a number \( \epsilon(n_0) \) such that for all \( 0 \leq \epsilon < \epsilon(n_0) \) we have
\[ \|L^{(n_0)}_{\omega} \varphi\|_{H^m} \leq \rho^{n_0} \|\varphi\|_{H^m} \]
for all \( \varphi \in H^m \) such that \( \langle \varphi, 1_{\mathbb{T}^2} \rangle = 0 \).

Now let \( n \geq 1 \) be arbitrary and write \( n = kn_0 + r \) where \( 0 \leq r < n_0 \). Using (A1) and (3.19) this immediately gives
\[ \|L^{(n)}_{\omega} \varphi\|_{H^m} \leq C_0 \rho^{kn_0} \|\varphi\|_{H^m} = \left( \frac{C_0}{\rho} \right)^r \rho^{kn_0 + r} \|\varphi\|_{H^m} \leq C \rho^n \|\varphi\|_{H^m} \]
where \( C \) is independent of \( 0 \leq r < n_0 \). This establishes (3.18) and the proof is complete. \( \Box \)

3.4. **Condition (LY).** Here we show that \( T \) satisfies the Lasota-Yorke inequality condition. Recall that \( (\Omega, \mathcal{F}, \mathbb{P}) \) is assumed to be a Lebesgue-Rokhlin probability space. Since \( H^m(\mathbb{T}^2) \) is equivalent to \( H^m(\mathbb{T}^2) \) which is separable, condition (ii) in Definition 2.4 holds as long as \( \mathcal{L} \) is \( (\mathcal{F}, \mathcal{S}_{L(H^m(\mathbb{T}^2))}) \)-measurable, i.e., if \( \mathcal{L} : \Omega \to L(H^m(\mathbb{T}^2)) \) is strongly measurable. The fact that this holds can easily be shown by adapting the proof on page 981 in [37] of strong measurability of the reduced transfer operator \( L_{\nu,\omega} \). Thus it remains to verify that there is a Banach space \( (\mathcal{B}_+, |\cdot|) \) such that \( H^m(\mathbb{T}^2) \) is compactly embedded in \( \mathcal{B}_+ \) and \( \mathcal{B}_+ \) is continuously
embedded in $L^2(\mathbb{T}^2)$, that $\mathcal{L}$ is $\mathbb{P}$-almost surely bounded on $\mathcal{B}_+$ and that (2.25) holds, namely,

$$\|\mathcal{L}^{(n)}\varphi\|_{\mathcal{H}^m} \leq \alpha_m\|\varphi\|_{\mathcal{H}^m} + \beta_m|\varphi|, \quad \varphi \in \mathcal{H}^m,$$

for some positive integer $n_0$, where $\alpha$ and $\beta$ are random variables with values in $\mathbb{R}_+$ with $\mathbb{E}_P[\alpha] < 1$ and $\mathbb{E}_P[\beta] < \infty$. By the following result we can take $\mathcal{B}_+ = H^{m-1}(\mathbb{T}^2)$.

**Proposition 3.9.** There is an $\epsilon_0$ such that for all $0 \leq \epsilon < \epsilon_0$, $T$ satisfies the Lasota-Yorke inequality condition with respect to $\mathcal{H}^m \subset H^{m-1}(\mathbb{T}^2)$.

**Proof.** By Lemma 3.3, we have

$$\|\mathcal{L}^{(n)}\varphi\|_{\mathcal{H}^m} \leq \rho^{n_0}\|\varphi\|_{\mathcal{H}^m} + C_n\|\varphi\|_{H^{m-1}(\mathbb{T}^2)}$$

where $\rho \in (0, 1)$. Let $H^{j,k}(\mathbb{T}^2)$ be the anisotropic Sobolev space with norm

$$\|\varphi\|_{H^{j,k}} = \sum_{\xi, \nu \in 2\pi \mathbb{Z}} \{(1 + \xi^2)^j + (1 + \nu^2)^k\}|\hat{\varphi}(\xi, \nu)|^2.$$

If $\varphi_0$ is the Fourier mode $\varphi_0(x) = \int_{\mathbb{T}} e^{-i\xi \cdot x} \varphi(x, s) \, ds$ then clearly

$$\|\varphi_0\|_{H^{m-1}(\mathbb{T}^2)} \leq \|\varphi\|_{H^{m-1,0}} \leq \|\varphi\|_{H^{m-1,0}}$$

where the right-hand side is equivalent to the usual norm in $H^{m-1}(\mathbb{T}^2)$. Hence (3.21) gives

$$\|\mathcal{L}^{(n)}\varphi\|_{\mathcal{H}^m} \leq \rho^{n_0}\|\varphi\|_{\mathcal{H}^m} + C_n\|\varphi\|_{H^{m-1}(\mathbb{T}^2)}.$$

It is well-known that $H^m(\mathbb{T}^2)$ is compactly embedded in $H^{m-1}(\mathbb{T}^2)$, and since $\|\cdot\|_{\mathcal{H}^m}$ is equivalent to $\|\cdot\|_{H^m(\mathbb{T}^2)}$ we can thus take $\mathcal{B}_+ = H^{m-1}(\mathbb{T}^2)$, (We also need to check that $\mathcal{L}$ is $\mathbb{P}$-almost surely bounded on $H^{m-1}(\mathbb{T}^2)$, but this is immediate from [A1] by the same reasoning.) In particular, (3.20) holds with $\alpha_\omega \equiv \rho^{n_0}$ and $\beta_\omega \equiv C_n$.

**Proof of Theorem 2.15.** The result follows by virtue of Propositions 3.3, 3.5, 3.8 and 3.9.

**Proof of Theorem 2.16.** We first prove item (i). Since $g_\omega(x, s)$ does not depend on $s$, a straightforward calculation for Sobolev norms shows that there exists a constant $B' \geq 1$ such that

$$\|\mathcal{L}^{(n)}\varphi\|_{H^m} \leq \max\{1, |t|^m\} B'\|\varphi\|_{H^m}$$

for any $m \in \mathbb{N}_0$, $\varphi \in H^m(\mathbb{T}^2)$, $t \in \mathbb{R}$, $n \in \mathbb{N}_0$ and $\mathbb{P}$-almost every $\omega \in \Omega$ (see the proof of [37, Lemma 4.3] for the case of $\mathbb{S}^1$ instead of $\mathbb{T}^2$). Hence, the second part of (HK2) immediately follows.

We next prove item (ii). By the assumption in item (i), there exists $\tilde{g} \in L^\infty(\Omega, \mathcal{E}^m(\mathbb{S}^2))$ such that $g_\omega(x, s) = \tilde{g}(\omega)\varphi(x)$ for all $(\omega, x, s) \in \Omega \times \mathbb{T}^2$. As in [37], given $\theta \in \mathbb{C}$, $\nu \in \mathbb{Z}$ and $n \in \mathbb{N}$, we introduce the Fourier-decomposed twisted transfer operator cocycle $\mathcal{L}_{\theta, \nu, \omega} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ by

$$\mathcal{L}^{(n)}_{\theta, \nu, \omega}\psi(x) = \sum_{E^{(n)}_\omega(y) = x} e^{\theta(S_n \omega)x - i\nu t^{(n)} y} \psi(y).$$

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Since $g_ω$ does not depend on $s$, we have for each $φ(x, s) = \sum_{ν ∈ Z} e^{ινs}ψ_ν(x)$ that
\[ L^{(n)}_{θ, ω}φ(x, s) = \sum_{ν ∈ Z} e^{ινs}L^{(n)}_{θ, ν, ω}ψ_ν(x). \]

Fix $t ∈ ℝ$. Since $L^{(f_0)}_{it, ν, ω_0}$ can be considered as the transfer operator of the expanding
map $L^{(f_0)}_{ω_0}$ with weight $e^{ι(t(S_0g_0)ω_0 − νT^{(f_0)}_{ω_0})}$ whose real part is bounded by 1, due
to \[ 6 \] together with the argument in the proof of \[ 37 \] Proposition 4.1, one can conclude that
the spectral radius of $L^{(f_0)}_{it, ω_0}$ is bounded by 1 and the essential
spectral radius of $L^{(f_0)}_{it, ν, ω_0}$ strictly less than 1 on $H^m(𝕊^1)$. Moreover, the uniform
exponential decay in the semiclassical limit $ν → ∞$ in \[ 37 \] Theorem 1.5 that holds
for $(L^{(f_0)}_{it, ν, ω_0})_{ν ≥ 0}$ with $ℙ$-almost every $ω$ also holds for $(L^{(f_0, N)}_{it, ν, ω_0})_{N ≥ 0}$: for any $ρ > λ^{-1/2}$
and sufficiently large $m$, there are $ν_0 = ν_0(t)$ and $c_0 = c_0(t)$ such that (if the noise
level $ε$ is sufficiently small)
\[ \| (L^{(f_0)}_{it, ν, ω_0})^N \| = \| L^{(f_0, N)}_{it, ν, ω_0} \| ≤ c_0ρ^{c_0N}, \ N ≥ 1, \ |ν| ≥ ν_0. \]

This fact is obtained by repeating the proof of \[ 37 \] Theorem 1.5: the only difference
is that the function $(dE^{(n)}_ω/dx)^{-1}$ is replaced by $e^{ι(t(S_0g_0)ω_0)(dE^{(f_0, N)}_ω)/dx}$ which
does not lead to any complications in the proof. Therefore, the spectral radius of $L^{(f_0)}_{it, ω_0}$ is bounded by 1 and the essential spectral radius of $L^{(f_0)}_{it, ν, ω_0}$ is strictly less than 1
on $H^m(T^2)$ (see the proof of Lemma 3.7).

Furthermore, given a compact interval $J$ of $ℝ \setminus \{0\}$, $L^{(f_0)}_{it, ω_0}$ has no eigenvalue
with radius 1 for each $t ∈ J$. Indeed, arguing by contradiction, assume that there exist
$c ∈ ℝ$ and $γ ∈ H^m(T^2)$ such that
\[ \sum_{z = T^{(f_0)}_{ω_0}(ζ)} e^{ι(t(S_0g_0)ω_0)(ζ)}γ(ζ) = e^{ιc}γ(z). \]

In fact, according to the remark on p. 1479 in \[ 23 \] we even have $γ ∈ C^∞(T^2)$. Then,
repeating the argument in \[ 37 \] Theorem 4.4, we obtain
\[ t(S_0g_0)ω_0(ζ) = ψ(ζ) − ψ(T^{(f_0)}_{ω_0}(ζ)) − c + 2πn(ζ) \]
with $ψ(ζ) = Arg(γ(ζ))$ where $Arg$ is the principal value, for some integer valued
function $ζ ↦ n(ζ)$. However, $n(ζ)$ must be a constant function of $ζ$ since it can be
written as a linear combination of smooth functions. So, we conclude that $(S_0g_0)ω_0
is cohomologous to a constant, which is a contradiction to the statement about $g
in item (ii) of Theorem 2.10$. In conclusion, the spectral radius of $L^{(f_0)}_{it, ω_0}$ is strictly
smaller than 1 for each $t ∈ J$. Therefore, since $J$ is compact and $t → L^{(f_0)}_{it, ω_0}$ is a
continuous map from $J$ to the space of bounded operators on $H^m(T^2)$ (see (2.15)),
we get (HK3) by applying \[ 32 \] Corollary III.13. This completes the proof. □

4. Proofs of Theorem 2.6, 2.7 and 2.8

4.1. Regularity of the top Oseledets space. As in \[ 17 \], we start the proof of
Theorems 2.6, 2.7 and 2.8 by establishing the regularity of the top Oseledets space
of the twisted transfer operator cocycles.

Due to the observation in Subsection 2.3, it follows from (LY) that $(T, g)$ satisfies
the Oseledets decomposition condition (OD): there is a real number $λ_0 > K

and a measurable splitting of $B$ into closed subspaces $B = F_{\theta,\omega} \oplus R_{\theta,\omega}$ with $\dim F_{\theta,\omega} < \infty$ such that $P$-almost surely the following holds:

1. $L_{\theta,\omega}F_{\theta,\omega} = F_{\theta,\sigma,\omega}$ and $L_{\theta,\omega}R_{\theta,\omega} \subset R_{\theta,\sigma,\omega}$,
2. If $\varphi \in F_{\theta,\omega}\setminus \{0\}$, then $\lim_{n \to \infty} \frac{1}{n} \log \|L_{\theta,\omega}^{(n)}\varphi\| = \Lambda_{\theta}$,
3. If $\varphi \in R_{\theta,\omega}$, then $\limsup_{n \to \infty} \frac{1}{n} \log \|L_{\theta,\omega}^{(n)}\varphi\| \leq K$.

For $\theta \in \mathbb{C}$ and $\omega \in \Omega$, let $L_{\theta,\omega}^{*} : B' \to \mathbb{B}'$ be the adjoint operator of $L_{\theta,\omega}$ given by $L_{\theta,\omega}^{*}(\varphi)(\psi) = \langle \varphi, \psi \rangle$ for every $\varphi \in B$, $\psi \in \mathbb{B}$.

Recall that given a map $A$ from an open subset $U$ of a normed vector space $X$ to a normed vector space $Y$, the Fréchet derivative $DA(x)$ of $A$ at $x \in U$ is a bounded operator from $X$ to $Y$ such that

$$\lim_{\|y\|_X \to 0} \frac{\|A(x + y) - A(x) - DA(x)(y)\|_Y}{\|y\|_X} = 0.$$  

The Fréchet derivative of $DA : U \to L(X,Y)$ at $x \in U$ is denoted by $D^2A(x)$, where $L(X,Y)$ is the space of bounded linear operators from $X$ to $Y$. $A$ is said to be of class $C^1$ (resp. $C^2$) if the map $DA$ (resp. $D^2A$) is continuous. We also use the notation $A'$, $A''$ for $DA$, $D^2A$.

**Remark 4.1.** In this section we use $B$ as a sufficiently small ball in $\mathbb{C}$ centered at 0 on which the maps given below are well-defined, even if it may change between occurrences. Furthermore, for simple description, we identify each $\varphi \in X$ with a map $t \mapsto t\varphi$ from $\mathbb{C}$ to $X$ for $X = L^\infty(\Omega,\mathcal{B})$ or $L^\infty(\Omega,\mathbb{C})$, if it makes no confusion. Moreover, for a $C^1$ map $\varphi : B \to X$, we permit us to write $\varphi'(\omega) = \psi(\omega)$ with some $\psi \in X$ to mean $\varphi'(t)(\omega) = t\psi(\omega)$ for every $t \in B$. Finally, in this section we hope to use the notation $h$ as the $C^1$ map from $B$ to $L^\infty(\Omega,\mathbb{B})$ (in Theorem 4.2), which conflicts to another map $h : \Omega \to \mathbb{B}$ given in (UG). To avoid notational confusion, we instead use notation $h_0$ for the map $h : \Omega \to \mathbb{B}$ of (UG).

The following theorem is crucial in this section.

**Theorem 4.2.** There are a ball $B$ in $\mathbb{C}$ centered at 0, $C^1$ maps $h : B \to L^\infty(\Omega,\mathbb{B})$, $w : B \to L^\infty(\Omega,\mathbb{B}')$, and a $C^2$ map $\lambda : B \to L^\infty(\Omega,\mathbb{C})$ such that the following hold:

1. For every $\theta \in B$ and $F$-almost every $\omega \in \Omega$,
   $$L_{\theta,\omega}h_{\theta,\omega} = \lambda_{\theta,\omega}h_{\theta,\sigma,\omega}, \quad \text{and} \quad \langle h_{\theta,\omega}, 1_{M} \rangle = 1.$$
   Furthermore, $h(0) = h_0$ and $\lambda(0) = 1$ $F$-almost surely.
2. If we define $A : B \to L^\infty(\Omega,\mathbb{B})$ by $A(\theta)(\omega) = \log |\lambda(\theta)(\omega)|$ for $\theta \in B$ and $\omega \in \Omega$, then
   $$\hat{\lambda}(0) = \hat{\lambda}'(0) = 0 \quad F$-almost surely
   and
   $$\mathbb{E}_{\theta} \left[ \hat{\lambda}''(0) \right] = V$$
   (recall (1.2) for the definition of $V$).
3. $F_{\theta,\omega}$ is spanned by $h_{\theta,\omega}$ for every $\theta \in B$ and $F$-almost every $\omega \in \Omega$, and
   $$\mathbb{E}_{\theta} \left[ \hat{\lambda}(\theta) \right] = \Lambda_{\theta}.$$
4. For every $\theta \in B$ and $F$-almost every $\omega \in \Omega$,
   $$L_{\theta,\omega}^{*}w_{\theta,\sigma,\omega} = \lambda_{\theta,\omega}w_{\theta,\omega}, \quad \text{and} \quad w_{\theta,\omega}(h_{\theta,\omega}) = 1.$$
We used the notation \( h_{\theta, \omega}, w_{\theta, \omega}, \lambda_{\theta, \omega} \) for \( h(\theta)(\omega), w(\theta)(\omega), \lambda(\theta)(\omega) \), respectively.

The proof of the item (1) and the former statement of the item (2) in Theorem 4.2 will be given in Subsection 4.2. Subsections 4.3 and 4.4 are, respectively, dedicated for the proof of the later statement of the items (2), (3) and (4).

**Remark 4.3.** Note that \( \hat{\Lambda}(\theta)(\omega) \) in Theorem 4.2 corresponds to \( Z(\theta, \omega) \) in [17] (see the proof of Lemma 3.9 of [17]), and \( E_{\theta}[\hat{\Lambda}(\theta)] \) in our context is written as \( \hat{\Lambda}(\theta) \) in that of [17].

Before starting the proof of Theorem 4.2, we give necessary definitions as well as a brief explanation for the strategy of the proof of Theorem 4.2. Define \( I : L^\infty(\Omega, \mathcal{B}) \to L^\infty(\Omega, \mathbb{C}) \) and \( L_\theta : L^\infty(\Omega, \mathcal{B}) \to L^\infty(\Omega, \mathcal{B}) \) by

\[
I(\varphi)(\omega) = \langle \varphi(\omega), 1_M \rangle, \quad L_\theta(\varphi)(\omega) = L_{\theta, \sigma^{-1}} \varphi_{\sigma^{-1}}
\]

for \( \varphi \in L^\infty(\Omega, \mathcal{B}) \) and \( \omega \in \Omega \). \( L_\theta \) is well-defined for any \( \theta \in \mathbb{C} \) by virtue of (2.11). We simply write \( L \) for \( L_0 \). Note that \( I(\theta L(h_0)) = 1 \) for \( \mathbb{P} \)-almost every \( \omega \in \Omega \). We will see that \( \theta \mapsto I \circ L_\theta(h_0) \) is a continuous map from \( \mathbb{C} \) to \( L^\infty(\Omega, \mathcal{B}) \) in the proof of Lemma 4.3. Hence, one can find a small neighborhood \( \mathcal{B} \) of 0 in \( \mathbb{C} \) such that \( I \circ L_\theta(\varphi) \neq 0 \) \( \mathbb{P} \)-almost surely and \( F(\theta, \varphi) : \Omega \to \mathcal{B} \) given by

(4.1)

\[
F(\theta, \varphi) = \frac{L_\theta(\varphi)}{I \circ L_\theta(\varphi)} - \varphi
\]

is well-defined for any \( (\theta, \varphi) \in \mathcal{B} \times (h_0 + \text{Ker}(I)) \). Notice that \( F(\theta, \varphi) \in \text{Ker}(I) \) for each \( (\theta, \varphi) \in \mathbb{B} \times (h_0 + \text{Ker}(I)) \). It also holds that

(4.2) \quad \begin{align*}
F(\theta, h_0) = 0 \quad \text{implies} \quad L_{\theta, \omega} h_{\theta, \omega} = \lambda_{\theta, \omega} h_{\theta, \sigma \omega} \quad \mathbb{P}-\text{almost surely},
\end{align*}

where \( \lambda_{\theta, \omega} = (L_{\theta, \omega}, h_{\theta, \omega}, 1_M) \). Therefore, in Subsections 4.2 and 4.3, we will apply the implicit function theorem to a map \( \hat{\mathbf{F}} : \mathbb{B} \times \text{Ker}(I) \to \text{Ker}(I) \) given by

(4.3)

\[
\hat{\mathbf{F}}(\theta, \varphi) = F(\theta, h_0 + \varphi), \quad (\theta, \varphi) \in \mathbb{B} \times \text{Ker}(I),
\]

in order to prove the regularities of \( h : \mathbb{B} \to L^\infty(\Omega, \mathcal{B}) \) and \( \lambda : \mathbb{B} \to L^\infty(\Omega, \mathbb{C}) \) in Theorem 4.2. Furthermore, in Subsection 4.4, we will apply the implicit function theorem to another map \( G \) induced by the adjoint twisted operators \( L_{\theta, \omega}^* \) to obtain the regularity of \( w : \mathbb{B} \to L^\infty(\Omega, \mathcal{B}) \) in Theorem 4.2.

**4.2. First order regularity.** In this subsection, we show the existence of \( h : \mathbb{B} \to L^\infty(\Omega, \mathcal{B}) \) satisfying the item (1) of Theorem 4.2. We first consider Fréchet derivatives of \( M : \mathbb{C} \times L^\infty(\Omega, \mathcal{B}) \to L^\infty(\Omega, \mathcal{B}) \) at \( \varphi \) and \( \theta \). For any \( \theta \in \mathbb{C} \) and \( \varphi \in L^\infty(\Omega, \mathcal{B}) \), since \( M(\theta, \cdot) = L_\theta : L^\infty(\Omega, \mathcal{B}) \to L^\infty(\Omega, \mathcal{B}) \) is a linear operator,

(4.4) \quad the Fréchet derivative of \( M(\theta, \cdot) \) at \( \varphi \), denoted by \( \partial_\varphi M(\theta, \varphi), \) is \( L_\theta \).

Similarly,

(4.5) \quad the Fréchet derivative \( D_I(\varphi) \) of \( I \) at \( \varphi \) is \( I \)

for any \( \varphi \in L^\infty(\Omega, \mathcal{B}) \) because \( I \) is linear.

Given \( (\theta, \varphi) \in \mathbb{C} \times L^\infty(\Omega, \mathcal{B}) \), we denote by \( \partial_\varphi M(\theta, \varphi) \) the Fréchet derivative of \( M(\cdot, \varphi) \) at \( \varphi \in \mathbb{C} \). Define \( L_{j, \theta} : L^\infty(\Omega, \mathcal{B}) \to L^\infty(\Omega, \mathcal{B}) \) with \( j = 1, 2 \) by

\[
(L_{j, \theta}(\varphi))(\omega) = L_{j, \theta, \sigma^{-1}} \varphi_{\sigma^{-1}} \quad \text{for} \quad \varphi \in L^\infty(\Omega, \mathcal{B}) \quad \text{and} \quad \omega \in \Omega,
\]

where \( L_{j, \theta, \omega} : \mathcal{B} \to \mathcal{B} \) is given by the duality

\[
\langle L_{j, \theta, \omega}(\varphi), u \rangle = \langle \varphi, (g_\omega)^{-1} \sigma \circ T_\omega \rangle.
\]
This is well-defined due to (A2). Note that \( \mathcal{L}_{t,\theta,\omega}\varphi = \mathcal{L}_\omega((g_\omega)^t e^{t g_\omega} \varphi) \).

**Lemma 4.4.** For each \( \theta \in \mathbb{C} \) and \( \varphi \in L^\infty(\Omega, \mathcal{B}) \),

\[
(4.7) \quad \partial_1 \mathbf{M}(\theta, \varphi) = \mathbf{L}_{1,\theta}(\varphi)
\]

under the identification in Remark 4.4.

**Proof.** Let \( \tilde{g}_{t,\omega} = (e^{tg_\omega} - 1)/t - g_\omega \) with \( t \in \mathbb{C} \setminus \{0\} \) and \( \tilde{g}_{0,\omega} = 0 \) for each \( \omega \in \Omega \). We first show that there is a positive constant \( c_t \) converging to 0 as \( t \to 0 \) such that

\[
(4.8) \quad \text{ess sup}_\omega \| \tilde{g}_{t,\omega} \varphi \| \leq c_t \| \varphi \| \quad \text{for } \varphi \in \mathcal{B}.
\]

By Taylor expansion of \( e^{tg_\omega} \),

\[
\tilde{g}_{t,\omega} = \frac{1}{t} \left( \sum_{k=0}^\infty \frac{(tg_\omega)^k}{k!} - 1 \right) - g_\omega = t \sum_{k=0}^\infty \frac{t^k g_\omega^{k+2}}{(k+2)!}.
\]

Therefore, it follows from (A2) that

\[
\| \tilde{g}_{t,\omega} \varphi \| \leq C_1 t \| G^2 \| \varphi \sum_{k=0}^\infty \frac{|tg_\omega|^k}{k!} = C_1 t \| G^2 e^{tg_\omega} \| \varphi
\]

(recall that \( G = \text{ess sup}_\omega \| g_\omega \|, \varphi \)), which concludes (4.8).

For each \( \theta \in \mathbb{C} \) and \( \varphi \in L^\infty(\Omega, \mathcal{B}) \), it follows from (A1) that

\[
\frac{\| (\mathbf{M}(\theta + t, \varphi) - \mathbf{M}(\theta, \varphi) - t\mathbf{L}_{1,\theta}(\varphi)) \varphi \|}{|t|} \leq C_0 \| \tilde{g}_{t,\omega} e^{tg_\omega} \varphi \|.
\]

Hence, by (A2) and (4.8) we get that

\[
(4.9) \quad \frac{\| \mathbf{M}(\theta + t, \varphi) - \mathbf{M}(\theta, \varphi) - t\mathbf{L}_{1,\theta}(\varphi) \|_{L^\infty(\Omega, \mathcal{B})}}{|t|} \leq C_0 C_1 |\theta| G c_t \| \varphi \|_{L^\infty(\Omega, \mathcal{B})},
\]

which converges to 0 as \( t \to 0 \). This completes the proof. \( \square \)

We let \( \tilde{\mathbf{L}}_{n,\theta}(\varphi) = \mathbf{L}_{n,\theta}(h_0 + \varphi) \) and \( \tilde{\mathbf{M}}(\theta, \varphi) = \mathbf{M}(\theta, h_0 + \varphi) \), and define Fréchet derivatives \( \partial_1 \tilde{\mathbf{F}}(\theta, \varphi) \) and \( \partial_2 \tilde{\mathbf{M}}(\theta, \varphi) \) \((j = 1, 2)\) in a similar manner to the definition of \( \partial_1 \mathbf{M}(\theta, \varphi) \). By (4.10) and the chain rule for Fréchet derivatives, it is obvious that

\[
(4.10) \quad \partial_1 \tilde{\mathbf{M}}(\theta, \varphi) = \tilde{\mathbf{L}}_{1,\theta}(\varphi) \quad \text{and} \quad \partial_2 \tilde{\mathbf{M}}(\theta, \varphi) = \mathbf{L}_{\theta}.
\]

**Lemma 4.5.** For each \( \theta \in \mathbb{B} \) and \( \varphi \in \text{Ker}(\mathbf{I}) \),

\[
\partial_1 \tilde{\mathbf{F}}(\theta, \varphi) = \frac{\mathbf{I} \circ \tilde{\mathbf{L}}_{\theta}(\varphi) \cdot \tilde{\mathbf{L}}_{1,\theta}(\varphi) - \tilde{\mathbf{L}}_{\theta}(\varphi) \cdot \mathbf{I} \circ \tilde{\mathbf{L}}_{1,\theta}(\varphi)}{(\mathbf{I} \circ \tilde{\mathbf{L}}_{\theta}(\varphi))^2}
\]

and

\[
\partial_2 \tilde{\mathbf{F}}(\theta, \varphi) = \frac{\mathbf{I} \circ \tilde{\mathbf{L}}_{\theta}(\varphi) \cdot \mathbf{L}_{\theta} - \tilde{\mathbf{L}}_{\theta}(\varphi) \cdot \mathbf{I} \circ \mathbf{L}_{\theta}}{(\mathbf{I} \circ \mathbf{L}_{\theta}(\varphi))^2} - \text{id},
\]

under the identification in Remark 4.7. In particular, for each \( t \in \mathbb{B} \) and \( \psi \in \text{Ker}(\mathbf{I}) \),

\[
\partial_1 \tilde{\mathbf{F}}(0, 0)(t) = t\mathbf{L}_{1,0}(h_0) \quad \mathbb{P}\text{-almost surely}
\]

and

\[
\partial_2 \tilde{\mathbf{F}}(0, 0)(\psi) = \mathbf{L}(\psi) - \psi.
\]
Proof. By the chain rule for Fréchet derivatives, the Fréchet derivatives $D \left[ I \circ \tilde{L}_\theta \right] (\varphi)$ of $I \circ \tilde{L}_\theta = I \circ \tilde{M}(\theta, \cdot)$ at $\varphi$ is calculated as
\begin{equation}
D \left[ I \circ \tilde{L}_\theta \right] (\varphi) = DI(\tilde{L}_\theta(\varphi)) \circ \partial_2 \tilde{M}(\theta, \varphi) = I \circ \tilde{L}_\theta.
\end{equation}
In the last equality, we used (4.4), (4.5) and (4.11). Similarly, for each $\varphi \in \text{Ker}(I)$, the Fréchet derivatives $D \left[ I \circ \tilde{L}_1(\varphi) \right] (\theta)$ of $I \circ \tilde{L}_1(\varphi) = I \circ \tilde{M}(\cdot, \varphi)$ at $\theta \in \mathbb{B}$ is
\begin{equation}
D \left[ I \circ \tilde{L}_1(\varphi) \right] (\theta) = DI(\tilde{L}_1(\varphi)) \circ \partial_1 \tilde{M}(\theta, \varphi) = I \circ \tilde{L}_1,\theta(\varphi).
\end{equation}
Here the right-hand side means a map $\mathbb{B} \ni t \mapsto I(t\tilde{L}_1,\theta(\varphi))$. We used (4.5), (4.7) and (4.11).

By quotient rule for Fréchet derivatives,
\begin{align*}
\partial_2 \tilde{F}(\theta, \varphi) &= \frac{I \circ \tilde{L}_\theta(\varphi) \cdot D \tilde{L}_\theta(\varphi) - \tilde{L}_\theta(\varphi) \cdot D \left[ I \circ \tilde{L}_\theta \right] (\varphi)}{\left( I \circ \tilde{L}_\theta(\varphi) \right)^2} - D \text{id}(\varphi).
\end{align*}
This implies the second claim of this lemma by (4.4) and (4.11). The first claim also can be proven in a similar manner, together with (4.7) and (4.12).

On the other hand, $\tilde{L}_0(0) = L(h_0) = h_0$, and thus $I \circ \tilde{L}_0(0) = I(h_0) = 1$. Similarly, $I \circ L(\psi) = I(\psi) = 0$ for each $\psi \in \text{Ker}(I)$. Moreover, by the assumption on $g$, we have that
\begin{equation}
I \circ \tilde{L}_{1,0}(0)(\sigma\omega) = \langle L_\omega(g_\omega h_{0,\omega}), 1_M \rangle = \mathbb{E}_{\mu_\omega}[g_\omega] = 0 \quad \mathbb{P}\text{-almost surely.}
\end{equation}
The last claim of Lemma 4.5 immediately follows from these observations. 

As a final preparation to apply the implicit function theorem to $\tilde{F}$ around $(\theta, \varphi) = (0, 0)$, we check that $\partial_2 \tilde{F}(0, 0) : \text{Ker}(I) \to \text{Ker}(I)$ is bijective. We first show injectivity. Arguing by contradiction, we assume that
\begin{equation}
\partial_2 \tilde{F}(0, 0)(\psi_1) = \partial_2 \tilde{F}(0, 0)(\psi_2)
\end{equation}
for some $\psi_1, \psi_2 \in \text{Ker}(I)$ with $\psi_1 \neq \psi_2$. Let $\psi = \psi_1 - \psi_2$. Then, $\psi$ is a nonzero mapping in $\text{Ker}(I)$, and it follows from Lemma 4.3 that $L(\psi) = \psi$. That is, $\psi_\omega \in \mathcal{F}_{0,\omega}$ $\mathbb{P}$-almost surely. On the other hand, $\psi_\omega \notin \mathcal{C} h_{0,\omega}$ because $(\psi_\omega, 1_M) = 0$ while $\langle h_{0,\omega}, 1_M \rangle = 1$. This contradicts to (UG), and we get that $\partial_2 \tilde{F}(0, 0)(\psi)$ is injective.

Next we show surjectivity. We note that a linear operator $N : \text{Ker}(I) \to \text{Ker}(I)$ given by
\begin{equation}
N\varphi = - \sum_{n=0}^{\infty} L^n\varphi \quad \text{for} \quad \varphi \in \text{Ker}
\end{equation}
is well-defined due to the assumption (UG) and the fact that $I \circ L = I$ on $L^\infty(\Omega, \mathcal{B})$. Therefore, since it is easy to see that
\begin{equation}
(\partial_2 \tilde{F}(0, 0))^{-1} = N \quad \text{on} \quad \text{Ker}(I)
\end{equation}
by Lemma 4.5, we obtain the surjectivity.

Note that $\tilde{F}(0, 0) = 0$. It follows from the implicit function theorem for Banach spaces together with the estimates in this subsection that there is a small ball $\mathbb{B}$ in
\(\mathbb{C}\) centered at 0 and a \(\mathcal{C}^1\) map \(\eta : \mathcal{B} \to \text{Ker}(I)\) such that \(\tilde{F}(\theta, \eta(\theta)) = 0\). Hence, by the observation in (1.2), if we let

\[
(4.15) \quad h(\theta) = h_0 + \eta(\theta) \quad \text{and} \quad \lambda(\theta)(\omega) = \langle M(\theta, h(\theta))(\sigma \omega), 1_M \rangle
\]

for \(\theta \in \mathcal{B}\) and \(\omega \in \Omega\), then \(h : \mathcal{B} \to L^\infty(\Omega, \mathcal{B})\) and \(\lambda : \mathcal{B} \to L^\infty(\Omega, \mathbb{C})\) are \(\mathcal{C}^1\) maps and satisfy the first equality in item (1) Theorem 4.2 and it \(\mathbb{P}\)-almost surely holds that

\[
(4.16) \quad h(0) = h_0 \quad \text{and} \quad \lambda(0)(\omega) = \langle L_{0,\omega} h_0, 1_M \rangle = \langle h_0, 1_M \rangle = 1
\]

by (UG). This completes the proof of the item (1) of Theorem 4.2.

Next we show the first assertion of the item (2) of Theorem 4.2. By applying the implicit function theorem for Banach spaces to \(\tilde{F}\), we have

\[
(4.17) \quad h'(0)(t)(\omega) = \eta'(0)(t)(\omega) = -\left(\partial_2 \tilde{F}(0,0)\right)^{-1} \left(\partial_1 \tilde{F}(0,0)(t)\right)(\omega)
\]

\[
= -tN \circ L_{1,0}(h_0)(\omega)
\]

\[
= t \sum_{n=1}^\infty L^{(n)}_{\sigma^{-n} \omega}(h_{0,\sigma^{-n} \omega} \circ g_{\sigma^{-n} \omega})
\]

for each \(t \in \mathcal{B}\) and \(\mathbb{P}\)-almost every \(\omega \in \Omega\) by Lemma 4.5 and (4.14). Since \(N\) is a bounded operator on \(\text{Ker}(I)\) and \(L_{1,0}(h_0) \in \text{Ker}(I)\) by the assumption on \(g\), we have

\[
(4.18) \quad h'(0)(t) \in \text{Ker}(I) \quad \text{for all} \ t \in \mathcal{B}.
\]

Using these estimates, we can show the following.

**Lemma 4.6.** For each \(\theta, t \in \mathcal{B}\) and \(\mathbb{P}\)-almost every \(\omega \in \Omega\),

\[
(4.19) \quad \lambda'(\theta)(t)(\omega) = \langle tL_{1,\theta}(h(\theta))(\sigma \omega), 1_M \rangle + \langle L_\theta(h'(\theta)(t))(\sigma \omega), 1_M \rangle.
\]

In particular, we have

\[
(4.20) \quad \lambda'(0)(t)(\omega) = 0.
\]

**Proof.** For each \(\theta \in \mathcal{B}\) and \(\varphi \in \text{Ker}(I)\), we define \(\lambda(\theta, \varphi) \in L^\infty(\Omega, \mathbb{C})\) by

\[
\lambda(\theta, \varphi)(\omega) = \langle M(\theta, \varphi)(\sigma \omega), 1_M \rangle \quad (\omega \in \Omega).
\]

Then, for any \(\theta, t \in \mathcal{B}, \varphi \in \text{Ker}(I)\) and \(\mathbb{P}\)-almost every \(\omega \in \Omega\), it follows from the assumption (A2) and \(1_M \in \mathcal{E}\) (given in Subsection 2.1) that

\[
|\lambda(\theta + t, \varphi)(\omega) - \lambda(\theta, \varphi)(\omega) - \langle tL_{1,\theta}(\varphi)(\sigma \omega), 1_M \rangle| \leq C_M \|M(\theta + t, \varphi)(\sigma \omega) - M(\theta, \varphi)(\sigma \omega) - tL_{1,\theta}(\varphi)(\sigma \omega)\|
\]

with \(C_M = \|1_M\|_\mathcal{E} > 0\). Therefore, due to (4.7), we get

\[
(4.21) \quad \partial_1 \lambda(\theta, \varphi)(t)(\omega) = \langle tL_{1,\theta}(\varphi)(\sigma \omega), 1_M \rangle.
\]

Similarly, by virtue of (4.4) and the assumption (A2) and \(1_M \in \mathcal{E}\), we can calculate \(\partial_2 \lambda(\theta, \varphi)\) as

\[
(4.22) \quad \partial_2 \lambda(\theta, \varphi)(\psi)(\omega) = \langle L_\theta(\psi)(\sigma \omega), 1_M \rangle.
\]

The first assertion of Lemma 4.6 immediately follows from (4.21) and (4.22) together with the chain rule for Fréchet derivatives. The second assertion also immediately follows from the assumption on \(g\), (4.6) and (4.18). \(\square\)
Let $\hat{\Lambda}(\theta)(\omega) = \log |\lambda(\theta)(\omega)|$ for each $\theta \in \mathbb{C}$ and $\omega \in \Omega$. Then, since $\hat{\Lambda}(\theta) = \frac{1}{2} \log(\lambda(\theta)\overline{\lambda}(\theta))$ one can check that (by the chain rule for Fréchet derivatives)

\begin{equation}
(4.23)
\hat{\Lambda}'(\theta) = \frac{\lambda'(\theta)\overline{\lambda}(\theta) + \lambda(\theta)\overline{\lambda}'(\theta)}{2|\lambda(\theta)|^2} = \Re\left(\frac{\lambda'(\theta)\overline{\lambda}(\theta)}{|\lambda(\theta)|^2}\right)
\end{equation}

under the identification in Remark 4.1, where $\Re(z) = (z + \overline{z}) / 2$ is the real part of a complex number $z$. Therefore, by virtue of (4.10) and (4.20) we have that

$$\hat{\Lambda}(0) = 0 \quad \text{and} \quad \hat{\Lambda}'(0) = 0,$$

which completes the proof of the first assertion in the item (2) of Theorem 4.2.

4.3. Second order regularity. For $(\theta, \varphi) \in \mathbb{C} \times L^\infty(\Omega, \mathcal{B})$, we denote by $\partial^2_2 \tilde{M}(\theta, \varphi)$ the Fréchet derivative of $\partial^1_2 \tilde{M}(\cdot, \varphi) : \mathbb{C} \to L(\mathbb{C}, \Ker(1))$ at $\theta \in \mathbb{C}$. Note that $\partial^2_2 \tilde{M}(\cdot, \varphi)$ is a map from $\mathbb{C}$ to $L(\mathbb{C}, L(\mathbb{C}, \Ker(1)))$. In a similar manner, we define $\partial_2 \partial^1_2 \tilde{M}(\theta, \varphi)$, $\partial_1 \partial^2_2 \tilde{M}(\theta, \varphi)$ and $\partial^2_2 \tilde{M}(\theta, \varphi)$.

**Lemma 4.7.** For each $\theta \in \mathbb{C}$ and $\varphi \in L^\infty(\Omega, \mathcal{B})$,

\begin{equation}
(4.24)
\partial^2_1 \tilde{M}(\theta, \varphi)(s)(t) = tsL_{2, \theta}(\varphi) \quad (s, t \in \mathbb{C}),
\end{equation}

\begin{equation}
(4.25)
\partial_2 \partial^1_2 \tilde{M}(\theta, \varphi)(\psi)(t) = tL_{1, \theta}(\psi) \quad (\psi \in \Ker(1), t \in \mathbb{C}).
\end{equation}

Furthermore, for each $\theta \in \mathbb{C}$ and $\varphi \in L^\infty(\Omega, \mathcal{B})$,

$$\partial_1 \partial^2_2 \tilde{M}(\theta, \varphi)(t)(\psi) = tL_{1, \theta}(\psi) \quad (t \in \mathbb{C}, \psi \in \Ker(1)),\,$$

$$\partial^2_2 \tilde{M}(\theta, \varphi)(\phi)(\psi) = 0 \quad (\phi, \psi \in \Ker(1)).$$

**Proof.** One can prove (4.24) (resp. (4.25)) from the formula (4.17) as the proof of (4.7) (resp. (4.14)), see also [17, Lemma B.11]. The proof of the later equalities also follows from the formula (4.4) by a similar argument (cf. [17, Lemma B.10]).

It follows from (4.5) that $\mathbf{I}$ is of class $\mathcal{C}^2$, and so is $\tilde{F} : \mathbb{B} \times \Ker(1) \to \Ker(1)$ due to the form of $\tilde{F}$ given in (4.11), (4.3). Hence, by the implicit function theorem, $h : \mathbb{B}$ given in (4.13) is of class $\mathcal{C}^2$.

Furthermore, in a manner similar to one in the proof of Lemma 4.6, it follows from (4.19) and Lemma 4.7 together with assumption (A2) and $1_M \in \mathcal{E}$, we have

$$\lambda''(\theta)(t)(s)(\omega) = \langle tsL_{2, \theta}(h(\theta))(s\omega), 1_M \rangle + \langle tL_{1, \theta}(h'(\theta)(s))(s\omega), 1_M \rangle + \langle sL_{1, \theta}(h'(\theta)(t))(s\omega), 1_M \rangle + \langle L_\theta(h''(\theta)(t)(s))(s\omega), 1_M \rangle$$

all $\theta, t, s \in \mathbb{C}$ and $\mathbb{P}$-almost every $\omega \in \Omega$. On the other hand, since $h$ is a $\mathcal{C}^2$ map from $\mathbb{B}$ to $\Ker(1)$, $h''(0)(t)(s) \in \Ker(1)$ for all $t, s \in \mathbb{B}$ and

$$\langle L_\theta(h''(\theta)(t)(s))(s\omega), 1_M \rangle = \langle h''(0)(t)(s)(\omega), 1_M \rangle = 0.$$
So, by \((4.6)\) and \((4.17)\),

\[
\lambda''(0)(t)(s)(\omega) = ts\mathbb{E}_{\mu_{\omega}}[g_{\omega}^2] + 2ts\sum_{n=1}^{\infty} E_{\mu_{\sigma^{-n}\omega}}[h_{0,\sigma^{-n}\omega}g_{\sigma^{-n}\omega}] + 2ts\sum_{n=1}^{\infty} E_{\mu_{\sigma^{-n}\omega}}[g_{\sigma^{-n}\omega}g_{\sigma^{-n}\omega}]
\]

Thus, it follows from the \(\sigma\)-invariance of \(P\) that \(E_P[\lambda''(0)] = V\) under the identification of Remark 4.1 (recall \((1.7)\)).

On the other hand, it follows from \((4.23)\) that (by chain and quotient rules for Fréchet derivatives)

\[
\hat{\Lambda}''(\theta) = \Re\left(\frac{\lambda''(\theta)}{\lambda'(\theta)} - \left(\frac{\lambda'(\theta)}{\lambda(\theta)}\right)^2\right).
\]

Since \(g_{\omega}\) is a real-valued function, \((4.16), (4.20)\) and \((4.26)\) leads to that

\[
\hat{\Lambda}''(0) = \Re(\lambda''(0)) = \lambda''(0) \quad P\text{-almost surely.}
\]

These observations immediately imply the later assertion of the item (3) of Theorem 4.2.

4.4. Relation between \(\Lambda_{\theta}\) and \(\hat{\Lambda}(\theta)\). We first show

\[
(4.27) \quad \mathbb{E}_P[\hat{\Lambda}(\theta)] \leq \Lambda_{\theta}
\]

for every \(\theta\) with sufficiently small absolute value. It follows from the item (1) of Theorem 4.2 and (UG) that for every \(\theta\) with sufficiently small absolute value and \(P\)-almost every \(\omega \in \Omega\),

\[
\Lambda_{\theta} \geq \lim_{n \to \infty} \frac{1}{n} \log \left\| L_{\theta,\omega}^{(n)} h_{\theta,\omega} \right\| \geq \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |\lambda_{\theta,\sigma^j\omega}|
\]

and the claim immediately follows from the definition of \(\hat{\Lambda}\) and the ergodicity of \((\theta, P)\).

Let \(B = \mathcal{F}_{\theta,\omega} \oplus \mathcal{H}_{\theta,\omega}\) be the closed splitting for \(L_{\theta}\) given in Subsection 4.1. Note that \(\dim(\mathcal{F}_{\theta,\omega}) = 1\) due to (UG). On the other hand, it immediately follows from \((2.11)\) that for each \(\omega \in \Omega\), \(\theta \mapsto L_{\theta,\omega} : C \to L(B)\) is continuous. Hence, by the semi-continuity argument of Lyapunov exponents of operator cocycles restricted on finite dimensional spaces (cf. [17, Theorem 3.12]), we get that

\[
(4.28) \quad \dim(\mathcal{F}_{\theta,\omega}) = 1
\]

for every \(\theta\) with sufficiently small absolute value and \(P\)-almost every \(\omega \in \Omega\).

Finally, the item (3) of Theorem 4.2 immediately follows from \((4.27), (4.28)\) and the item (1) of Theorem 4.2.
4.5. **Regularity of adjoint cocycles.** Note that the map from $\mathbb{N} \times \Omega \times \mathcal{B}$ to $\mathcal{B}$ given by

$$ (n, \omega, u) \mapsto \mathcal{L}^0_{n, \sigma} \circ \mathcal{L}^0_{n+1, \sigma} \circ \cdots \circ \mathcal{L}^0_{0, \sigma} \circ u $$

satisfies the cocycle property over the driving system $\sigma^{-1} \colon \Omega \to \Omega$. Hence, as the cocycle $(\mathcal{L}_0, \sigma)$ in Subsection 4.2, we define $J : L^\infty(\Omega, \mathcal{B}) \to L^\infty(\Omega, \mathcal{C})$ and $\mathcal{L}_0^*: L^\infty(\Omega, \mathcal{B}') \to L^\infty(\Omega, \mathcal{C})$ by

$$ J(u)(\omega) = u_\omega(h_{0,\omega}), \quad \mathcal{L}_0^*(u)(\omega) = \mathcal{L}_{\theta,\omega}^* u_{\sigma \omega} $$

for $u \in L^\infty(\Omega, \mathcal{B}')$ and $\omega \in \Omega$. By (UG), it holds that

$$ \mathbf{J} \circ \mathcal{L}_0^* u(\omega) = J(u)(\sigma \omega) \quad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega, $$

so $\mathcal{L}_0^*$ preserves $\text{Ker}(J)$. Let $w_0 \in L^\infty(\Omega, \mathcal{B}')$ such that

$$ \mathcal{L}_0^* w_0 = w_0 \quad \text{and} \quad \mathbf{J}(w_0) = 1 \quad \text{in } \mathbb{P}\text{-almost surely} $$

(its existence will be proven in Lemma 4.8 below). In a similar manner to Subsection 4.2 (by using the argument in the proof of Lemma 4.9 below), one can find a small neighborhood $\mathcal{B}$ of 0 in $\mathcal{C}$ such that $\mathbf{G}(\theta, u) : \Omega \to \mathcal{B}$ given by

$$ \mathbf{G}(\theta, u) = \frac{\mathcal{L}_0^*(u)}{\mathbf{J} \circ \mathcal{L}_0^*(u)} - u $$

is well-defined and in $\text{Ker}(\mathbf{J})$ for any $(\theta, u) \in \mathcal{B} \times (w_0 + \text{Ker}(\mathbf{J}))$. Define $\tilde{\mathbf{G}} : \mathcal{B} \times \text{Ker}(\mathbf{J}) \to \text{Ker}(\mathbf{J})$ by

$$ \tilde{\mathbf{G}}(\theta, u) = \mathbf{G}(\theta, w_0 + u) \quad \text{for } (\theta, u) \in \mathcal{B} \times \text{Ker}(\mathbf{J}), $$

and its derivatives $\partial_\theta \tilde{\mathbf{G}}, \partial_\omega \tilde{\mathbf{G}}$ as in Subsection 4.2.

As in Subsection 4.2, we will apply the implicit function theorem to $\tilde{\mathbf{G}}$. Necessary estimates for it is the following.

**Lemma 4.8.** There is a constant $C_2 > 0$ and $\rho \in (0, 1)$ such that for any $u \in \text{Ker}(\mathbf{J})$ and $n \geq 0$,

$$ \| (\mathcal{L}_0^0)^n(u) \|_{L^\infty(\Omega, \mathcal{B}')} \leq C_2^* \rho^n \| u \|_{L^\infty(\Omega, \mathcal{B}')}. $$

Furthermore, there exists $w_0 \in L^\infty(\Omega, \mathcal{B}')$ satisfying (4.29).

**Proof.** Let $\Pi_0 : \mathcal{B} \to \text{Ker}(\mathbf{I})$ denote the projection onto $\text{Ker}(\mathbf{I})$ along the subspace spanned by $h_{0,\omega}$. By (UG), we can take a positive number $K > 0$ such that

$$ 1 \leq \|h_{0,\omega}\| \leq K \quad \text{in } \mathbb{P}\text{-almost surely}. $$

Let $n_1$ be an integer such that $K^{-1} - C_2^* \rho^{n_1} > 0$, and $c$ a positive number given by

$$ c = \frac{K^{-1} - C_2^* \rho^{n_1}}{C_0^*}. $$

We will show that

$$ \text{ess sup} \| \Pi_\omega \| \leq 2c^{-1}. $$

Note that the invariant splitting $\mathcal{B} = \mathcal{F}_{0,\omega} \oplus \mathcal{A}_{0,\omega}$ with $\text{dim}(F_{0,\omega}) = 1$ with respect to the cocycle $(\mathcal{L}_0, \sigma)$ (given in Subsection 4.1) is written as $\mathcal{F}_{0,\omega} = \mathcal{C} h_{0,\omega}$ and $\mathcal{A}_{0,\omega} = \{ \varphi \in \mathcal{B} \mid \langle \varphi, 1_M \rangle = 0 \}$. Let $\gamma_\omega$ be a real number given by

$$ \gamma_\omega = \inf \| ah_{0,\omega} + \varphi \| : a \in \mathbb{C}, \varphi \in \mathcal{A}_{0,\omega}, \| ah_{0,\omega} \| = \| \varphi \| = 1. $$
Then it is straightforward to see that \( \| \Pi_\omega \| \leq 2 \gamma^{-1} \) (cf. [16] Lemma 1). On the other hand, by (A1), for any \( a \in \mathbb{C} \) and \( \varphi \in \mathcal{R}_{0,\omega} \),
\[
\| ah_{0,\omega} + \varphi \| \geq \frac{\| L_{0,\omega}^{(n)} (ah_{0,\omega} + \varphi) \|}{C_0^{n_1}} \geq \frac{\| L_{0,\omega}^{(n)} (ah_{0,\omega}) \| - \| L_{0,\omega}^{(n)} \varphi \|}{C_0^{n_1}}.
\]
So, if \( \| ah_{0,\omega} \| = \| \varphi \| = 1 \), then it follows from (UG) that
\[
\| ah_{0,\omega} + \varphi \| \geq \frac{1}{C_0} \left( \frac{\| h_{0,\omega} \|}{\| h_{0,\omega} \|} - C_2 \rho^a \right) \geq c,
\]
and we get (4.31).

It follows from (UG) and (4.31) that for any \( \theta \in \mathbb{R} \),
\[
\| (L_0^*)^n (u) (\omega) \|_{\mathcal{B}'} = \sup_{\varphi \in \mathcal{B}, \| \varphi \| \leq 1} \left| u(\sigma^n \omega)(L_0^{(n)} \varphi) \right|
= \sup_{\varphi \in \mathcal{B}, \| \varphi \| \leq 1} \left| u(\sigma^n \omega)(L_0^{(n)} \Pi_\omega \varphi) \right|
\leq C_2 \rho^a \| \Pi_\omega \| \| u \|_{L^\infty(\Omega, \mathcal{B}')} \leq 2c^{-1} C_2 \rho^a \| u \|_{L^\infty(\Omega, \mathcal{B}')}.
\]
and the first assertion of Lemma 4.8 is proven with \( C_2 = 2c^{-1} C_2 \).

Let \( \mathcal{F}_{0,\omega}^* \) be the subspace of \( \mathcal{B} \) given by
\[
\mathcal{F}_{0,\omega}^* = \{ u \in \mathcal{B} \mid u(\varphi) = 0 \text{ for all } \varphi \in \mathcal{R}_{0,\omega} \}.
\]
Then, \( \dim(\mathcal{F}_{0,\omega}^*) = 1 \) due to [17] Lemma 2.6. Let \( w_{0,\omega} \in \mathcal{F}_{0,\omega}^* \) such that \( J(w_0) = 1 \).
If we denote by \( \tilde{\lambda}_\omega \) the complex number such that \( L_{0,\omega}^{(n)} w_{0,\omega} = \tilde{\lambda}_\omega w_{0,\omega} \), then
\[
(4.32) \quad \tilde{\lambda}_\omega = \lambda_\omega w_{0,\omega} (h_{0,\omega}) = L_{0,\omega}^{*} w_{0,\omega} (h_{0,\omega}) = w_{0,\sigma} (L_{0,\omega} h_{0,\omega}) = w_{0,\sigma} (h_{0,\omega}) = 1.
\]
Furthermore, by (UG), for each \( a \in \mathbb{C} \) and \( \varphi \in \mathcal{R}_{0,\omega} \),
\[
(4.33) \quad w_{0,\omega} (ah_{0,\omega} + \varphi) = a = \langle ah_{0,\omega} + \varphi, 1_M \rangle.
\]
Hence, due to (A1) we \( \mathbb{P} \)-almost surely have
\[
\| w_{0,\omega} \|_{\mathcal{B}'} \leq \| 1_M \|_{\mathcal{B}}.
\]
By the assumption \( 1_M \in \mathcal{B} \), we obtain the second assertion of Lemma 4.8. \( \square \)

We need the following lemma to apply the implicit function theorem to \( \tilde{G} \). For each \( \theta \in \mathbb{C} \), define \( L_{1,\theta}^* : L^\infty(\Omega, \mathcal{B}') \to L^\infty(\Omega, \mathcal{B}') \) by
\[
(\tilde{L}_{1,\theta}^*)^* (u)(\omega)(\varphi) = v(\sigma \omega)(L_{1,\theta}^* \varphi)
\]
for \( u, v \in L^\infty(\Omega, \mathcal{B}') \), \( \omega \in \Omega \) and \( \varphi \in \mathcal{B} \).

**Lemma 4.9.** For each \( \theta \in \mathbb{B} \) and \( u \in \text{Ker}(J) \),
\[
\partial_1 \tilde{G}(\theta, u) = \frac{J \circ \tilde{L}_{1,\theta}^* (u) - \tilde{L}_{1,\theta}^* (u) \cdot J \circ \tilde{L}_{1,\theta}^* (u)}{(J \circ \tilde{L}_{1,\theta}^* (u))^2}
\]
and
\[
\partial_2 \tilde{G}(\theta, u) = \frac{J \circ \tilde{L}_{1,\theta}^* (u) \cdot L_{1,\theta}^* - \tilde{L}_{1,\theta}^* (u) \cdot J \circ L_{1,\theta}^* - \text{id.}}{(J \circ L_{1,\theta}^* (u))^2}
\]
In particular,
\[
\partial_2 \tilde{G}(0, 0) = \tilde{L}^* - \text{id.}
\]
Hence, it follows from (4.9) that
\[
\theta_0 = \theta_0(u) + \omega_0 \in \mathcal{B}_0 \cap \mathcal{B}_1(\theta_0, \omega)
\]
and so, the proof of the item (4) of Theorem 4.2.

\[\square\]

Proof. Define \( M^* : \mathbb{C} \times L^\infty(\Omega, \mathcal{B}') \to L^\infty(\Omega, \mathcal{B}') \) by \( M^*(\theta, u) = L^*_\theta(u) \), and \( \hat{M}^* : \mathbb{C} \times L^\infty(\Omega, \mathcal{B}') \to L^\infty(\Omega, \mathcal{B}') \) by \( \hat{M}^*(\theta, u) = M^*(\theta, u_0 + u) \). Since \( J \) and \( L^*_\theta \) are linear with \( \theta \in \mathbb{C} \),
\[
(DJ)(u) = J \quad \text{and} \quad \partial_\theta \hat{M}^*(\theta, u) = \hat{L}^*_\theta.
\]
for all \( \theta \in \mathbb{C} \) and \( u \in L^\infty(\Omega, \mathcal{B}') \).

We will show that
\[
\partial_\theta \hat{M}^*(\theta, u)(t) = t\hat{L}^*_{1,\theta}
\]
for all \( t, \theta \in \mathbb{C} \) and \( u \in L^\infty(\Omega, \mathcal{B}') \). For each \( \omega \in \Omega \) and \( \varphi \in \mathcal{B} \) with \( \|\varphi\| = 1 \),
\[
| (M^*(\theta + t, u) - M^*(\theta, u) - \tau \hat{L}^*_{1,\theta}(u))(\omega)(\varphi) | = | u(\varphi) (L_{\theta+t,\omega} - L_{\theta,\omega} - tL_{1,\theta,\omega})(\varphi) | \leq ||u||_{L^\infty(\Omega, \mathcal{B}')} \| (L_{\theta+t,\omega} - L_{\theta,\omega} - tL_{1,\theta,\omega})(\varphi) \|.
\]
Hence, it follows from (4.34) that
\[
limit_{t \to 0} \frac{\| (M^*(\theta + t, u) - M^*(\theta, u) - \tau \hat{L}^*_{1,\theta}(u)) \|_{L^\infty(\Omega, \mathcal{B}')}}{|t|} = 0,
\]
and we get (4.35).

On the other hand, by (4.34) and (4.33), we have that
\[
J \circ \tilde{L}^*_{1,0}(0)(\omega)(\varphi) = w_{0,\sigma}(L_{1,0,\omega}h_{0,\omega}) = I \circ \tilde{L}^*_{1,0}(0)(\sigma)(\varphi) = 0
\]
for \( \mathbb{P} \)-almost every \( \omega \in \Omega \). Therefore, Lemma 4.9 follows from (4.34), (4.35) in a similar manner to the proof of Lemma 4.5.

Now we can check that \( \partial_\theta \tilde{G}(0,0) \) is bijective by virtue of Lemma 4.5 in a manner similar to the proof of the bijectivity of \( \partial_\theta \tilde{F}(0,0) \) : \( \text{Ker}(J) \to \text{Ker}(J) \) (by using \( \text{dim}(\mathcal{F}_{\omega}^* \setminus \mathcal{F}_0^* ) = 1 \) shown in the proof of Lemma 4.5 instead of \( \text{dim}(\mathcal{F}_0^* ) = 1 \) from (UG), and (4.33) in Lemma 4.5 instead of (2.4) in (UG)).

Note that \( \tilde{G}(0,0) = 0 \). It follows from the implicit function theorem for Banach spaces together with the estimates in this subsection that there is a small ball \( \mathcal{B} \) in \( \mathbb{C} \) centered at 0 and a \( \mathcal{C}^1 \) map \( \xi : \mathcal{B} \to \text{Ker}(J) \) such that \( \tilde{G}(\theta, \xi(\theta)) = 0 \). Hence, by the definition of \( \tilde{G} \) and the fact that \( h : \mathcal{B} \to L^\infty(\Omega, \mathcal{B}') \) is a \( \mathcal{C}^1 \) map, one can check that if we let \( \tilde{w}_\theta = w_\theta + \xi(\theta), \quad w_\theta,\omega = \frac{\tilde{w}_\theta}{w_\theta}(h_{\theta,\omega}) \quad \text{and} \quad \tilde{\lambda}_\theta,\omega = \frac{(J \circ L^*_\theta(\tilde{w}_\theta))\omega \tilde{w}_\theta,\omega (h_{\theta,\omega})}{\tilde{w}_\theta,\omega(h_{\theta,\omega})} \)
for \( \theta \in \mathcal{B} \) and \( \omega \in \Omega \), then \( w : \mathcal{B} \to L^\infty(\Omega, \mathcal{B}') \) and \( \tilde{\lambda} : \mathcal{B} \to L^\infty(\Omega, \mathbb{C}) \) are \( \mathcal{C}^1 \) maps and satisfy
\[
\tilde{L}^*_{\theta,\omega} \tilde{w}_\theta,\omega = (J \circ L^*_\theta(\tilde{w}_\theta))\omega \tilde{w}_\theta,\omega \quad \mathbb{P}\text{-almost surely}
\]
and so,
\[
\tilde{L}^*_{\theta,\omega} w_{\theta,\omega} = \frac{\tilde{L}^*_{\theta,\omega} \tilde{w}_\theta,\omega}{\tilde{w}_\theta,\omega(h_{\theta,\omega})} = \frac{(J \circ L^*_\theta(\tilde{w}_\theta))\omega \tilde{w}_\theta,\omega}{\tilde{w}_\theta,\omega(h_{\theta,\omega})} = \tilde{\lambda}_\theta,\omega \tilde{w}_\theta,\omega \quad \mathbb{P}\text{-almost surely}.
\]
It also holds by construction that \( \tilde{w}_\theta,\omega(h_{\theta,\omega}) = 1 \quad \mathbb{P}\text{-almost surely} \). Therefore, repeating the argument in (4.32), we can see that \( \tilde{\lambda}_\theta,\omega = \lambda_{\theta,\omega} \). This completes the proof of the item (4) of Theorem 4.2.
4.6. Central limit theorem. Now we can prove Theorem 2.6 by using Theorem 4.2. By Lévy’s continuity theorem, it suffices to show that

\[
\lim_{n \to \infty} E_{\mu,\omega} \left[ e^{it(S_n(\omega))} \right] = e^{-\frac{t^2}{2}}
\]

for all \( t \in \mathbb{R} \) and \( \mathbb{P} \)-almost every \( \omega \in \Omega \). On the other hand, if we define \( I_{n,\omega} : \mathbb{C} \to \mathbb{C} \) by

\[
I_{n,\omega}(\theta) = E_{\mu,\omega} \left[ e^{i\theta(S_n(\omega))} \right], \quad \theta \in \mathbb{C},
\]

then

\[
E_{\mu,\omega} \left[ e^{it(S_n(\omega))} \right] = I_{n,\omega} \left( \frac{it}{\sqrt{n}} \right),
\]

and it is straightforward to see that

\[
I_{n,\omega}(\theta) = \langle L^{(n)}_{\theta,\omega} h_{\theta,\omega}, 1 \rangle
\]

(recall (2.11), (2.10) and (UG); cf. [17, Lemma 3.3]). Let \( B = B_{\theta,\omega} \oplus R_{\theta,\omega} \) be the invariant splitting for the cocycle \( (L_\theta, \sigma) \) given in Subsection 4.1. Then, it follows from [17, Lemma 2.6] that \( B_{\theta,\omega} = \Pi_{\theta,\omega} B \) and \( R_{\theta,\omega} = \tilde{\Pi}_{\theta,\omega} B \), where \( \Pi_{\theta,\omega} \) and \( \tilde{\Pi}_{\theta,\omega} \) are bounded operators on \( B \) given by

\[
\Pi_{\theta,\omega} \varphi = w_{\theta,\omega}(\varphi) h_{\theta,\omega}, \quad \tilde{\Pi}_{\theta,\omega} \varphi = \varphi - \Pi_{\theta,\omega} \varphi \quad \text{for} \ \varphi \in B
\]

(their boundedness are ensured by Theorem 4.2).

Lemma 4.10. There are constants \( K > 0 \) and \( 0 \leq r \leq 1 \) such that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) and every \( \varphi \in B_{\theta,\omega} \) and \( \psi \in B \),

\[
\left\| L^{(n)}_{\theta,\omega} \varphi \right\| \leq K r^n \| \varphi \| \quad \text{and} \quad \left\| \tilde{\Pi}_{\theta,\omega} \psi \right\| \leq K \| \psi \|.
\]

Proof. The second inequality Lemma 4.10 immediately follows from the continuity of \( h : \mathbb{B} \to L^\infty(\Omega, \mathcal{B}) \) and \( w : \mathbb{B} \to L^\infty(\Omega, \mathcal{B}') \) in Theorem 4.2, so we only prove the first inequality. Recall \( C_2 > 0 \) and \( \rho \in (0, 1) \) given in (UG). Take \( r \in (\rho, 1) \) and \( n_0 \geq 1 \) such that \( C_2 \rho^{n_0} < r^{n_0} \), so that

\[
\text{ess sup}_{\omega \in \Omega} \left\| L^{(n_0)}_{\theta,\omega} \right\| < r^{n_0}
\]

due to (UG). Since the map \( \theta \mapsto L_\theta \) from \( \mathbb{B} \) to the space of bounded operators on \( L^\infty(\Omega, \mathcal{B}) \) with the norm topology is continuous by the argument in (2.11), together with the continuity of \( h \) and \( w \), for any \( \theta \) with sufficiently small absolute value, we have

\[
\text{ess sup}_{\omega \in \Omega} \left\| L^{(n_0)}_{\theta,\omega} \right\| < r^{n_0}.
\]

Hence, for each \( n \geq 1 \), if we write \( n = kn_0 + \ell \) with \( 0 \leq \ell \leq n_0 - 1 \), then for \( \mathbb{P} \)-almost every \( \omega \in \Omega \),

\[
\left\| L^{(n)}_{\theta,\omega} \right\| \leq \left\| L^{(\ell)}_{\theta,\sigma^{kn_0}\omega} \right\| \prod_{j=0}^{k-1} \left\| L^{(n_0)}_{\theta,\sigma^{jn_0}\omega} \right\| \leq \left( \frac{K_1}{r} \right)^\ell r^n,
\]

where \( K_1 = \| L_{\theta,\omega} \| \) which is bounded by (A1) and (2.11). Therefore, the first inequality of Lemma 4.10 holds with \( K = \max\{1, (K_1/r)^{n_0}\} \). \( \square \)
By (A2), (UG), (4.39) and Lemma 4.10

\[
(4.40) \quad \left| I_{n,\omega}(\theta) - \left( \prod_{j=0}^{n-1} \lambda_{\theta,\sigma^j} \omega \right) w_{\theta,\omega}(h_{0,\omega}) \langle h_{\theta,\omega}, 1_M \rangle \right| = \left| \left( L_{\theta,\omega}^{(n)} \right) \mathbb{P}_{\theta,\omega} h_{0,\omega}, 1_M \right| \leq K^2 \| h_0 \|_{L^\infty(\Omega, \mathbb{F})} \| f \|_r n.
\]

Note that

\[
(4.41) \quad w_{\theta,\omega}(h_{0,\omega}) \langle h_{\theta,\omega}, 1_M \rangle \to 1 \quad (n \to \infty)
\]

by Theorem 4.2 and (UG). Furthermore, it follows from Theorem 4.2 that

\[
\log \left( \prod_{j=0}^{n-1} \lambda_{\theta,\sigma^j} \omega \right) = \sum_{j=0}^{n-1} \Lambda(\theta)(\sigma^j \omega) = \sum_{j=0}^{n-1} \Lambda''(0)(\sigma^j \omega) \theta^2 + n \cdot o_{\omega}(\theta^2),
\]

where \( o_{\omega}(\theta^2) \) satisfies that \( \text{ess sup}_{\omega}(\|o_{\omega}(\theta^2)/\theta^2\|) \to 0 \) as \( \theta \to 0 \). On the other hand,

\[
\sum_{j=0}^{n-1} \Lambda''(0)(\sigma^j \omega) \left( \frac{it}{\sqrt{n}} \right)^2 = \frac{-i^2}{n} \sum_{j=0}^{n-1} \Lambda''(0)(\sigma^j \omega).
\]

Therefore, by Birkhoff’s ergodic theorem together with Theorem 4.2

\[
(4.42) \quad \lim_{n \to \infty} \frac{1}{n} \log \prod_{j=0}^{n-1} \lambda_{\theta,\sigma^j} \omega = e^{-\frac{\theta^2}{2}},
\]

This together with (4.38), (4.40) and (4.41) leads to (4.36), which completes the proof of Theorem 2.6.

4.7. Large deviation principle. We give the proof of Theorem 2.7. Recall (4.37) for \( I_{n,\omega} \) with \( n \geq 1 \) and \( \omega \in \Omega \). By the Gärtner-Ellis theorem 32, it suffices to show that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \),

\[
\mathbb{R} \ni \theta \mapsto \lim_{n \to \infty} \frac{1}{n} \log I_{n,\omega}(\theta)
\]

is a strictly convex \( \mathcal{C}^1 \) function with vanishing derivative at \( \theta = 0 \). As in (4.40), we get

\[
I_{n,\omega}(\theta) = \left( \prod_{j=0}^{n-1} \lambda_{\theta,\sigma^j} \omega \right) w_{\theta,\omega}(h_{0,\omega}) \langle h_{\theta,\omega}, 1_M \rangle + \left( L_{\theta,\omega}^{(n)} \right) \mathbb{P}_{\theta,\omega} h_{0,\omega}, 1_M \langle h_{\theta,\omega}, 1_M \rangle.
\]

On the other hand, due to Lemma 4.10 and (4.41), one can find a constant \( K_0 > 0 \) such that

\[
K_0^{-1} \leq \left| w_{\theta,\omega}(h_{0,\omega}) \langle h_{\theta,\omega}, 1_M \rangle + \frac{\left( L_{\theta,\omega}^{(n)} \right) \mathbb{P}_{\theta,\omega} h_{0,\omega}, 1_M \langle h_{\theta,\omega}, 1_M \rangle}{\prod_{j=0}^{n-1} \lambda_{\theta,\sigma^j} \omega} \right| \leq K_0,
\]

so that

\[
\lim_{n \to \infty} \frac{1}{n} \log I_{n,\omega}(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \left( \prod_{j=0}^{n-1} \lambda_{\theta,\sigma^j} \omega \right) = \mathbb{E}_{\theta} \left[ \bar{\Lambda}(\theta) \right] \quad \mathbb{P} \text{-almost surely}
\]
by Birkhoff’s ergodic theorem. Therefore, it follows from Theorem 4.2 that \( \mathbb{R} \ni \theta \mapsto \lim_{n \to \infty} \frac{1}{n} \log I_{n, \omega}(\theta) \) is \( \mathbb{P} \)-almost surely a strictly convex \( C^1 \) function with vanishing derivative at \( \theta = 0 \), and this completes the proof of Theorem 2.7.

4.8. **Local central limit theorem.** We give the proof of Theorem 2.8. By a standard density argument (cf. (35)), one can see that it suffices to show that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) and every \( u \in L^1(\mathbb{R}) \), if the Fourier transform \( \hat{u} \) of \( u \) has compact support, then

\[
\lim_{n \to \infty} \sup_{z \in \mathbb{R}} \left| \frac{nV}{2\pi} \int \left( u(z + (S_n g)(\cdot)) \right) d\mu_{\omega} - \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \int u(\zeta) d\zeta \right| = 0.
\]

Fix \( u \in L^1(\mathbb{R}) \) for which the support of \( \hat{u} \) is included in \([-\delta_0, \delta_0]\) with some \( \delta_0 > 0 \). Given \( \delta \in (0, \delta_0) \), define \( A_1 \) and \( A_2 \) by

\[
A_1 = \frac{nV}{2\pi} \int_{|t| < \delta} e^{itz} \hat{u}(t) I_{n, \omega}(it) dt, \quad A_2 = \frac{nV}{2\pi} \int_{\delta \leq |t| \leq \delta_0} e^{itz} \hat{u}(t) I_{n, \omega}(it) dt.
\]

Then, by the Fourier inverse formula and Fubini’s theorem, it holds

\[
\frac{nV}{2\pi} \int u(z + (S_n g)(\cdot)) d\mu_{\omega} = \frac{nV}{2\pi} \int \left( \frac{1}{2\pi} \int \hat{u}(t) e^{itz} (S_n g)(z) dt \right) \mu_{\omega}(dx) = A_1 + A_2.
\]

On the other hand, it is straightforward to see that

\[
\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \int u(\zeta) d\zeta = \frac{\hat{u}(0)}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = B_1 + B_2,
\]

where

\[
B_1 = \frac{nV}{2\pi} \int_{|t| < \delta} e^{itz} \hat{u}(0) e^{-\frac{z^2}{2}} dt, \quad B_2 = \frac{nV}{2\pi} \int_{|t| \geq \delta} e^{itz} \hat{u}(0) e^{-\frac{z^2}{2}} dt.
\]

Finally, we decompose \( A_1 - B_1 \) into three parts \( s_1, s_2 \) and \( s_3 \) given by

\[
s_1 = \frac{nV}{2\pi} \int_{|t| < \delta} e^{itz} \hat{u}(t) \left( I_{n, \omega}(it) - \left( \prod_{j=0}^{n-1} \lambda_{it, \sigma^j} \right) w_{it, \omega}(h_{0, \omega}) \langle h_{it, \omega} 1_M \rangle \right) dt,
\]

\[
s_2 = \frac{nV}{2\pi} \int_{|t| < \delta} e^{itz} \hat{u}(t) \left( \prod_{j=0}^{n-1} \lambda_{it, \sigma^j} \right) (w_{it, \omega}(h_{0, \omega}) \langle h_{it, \omega} 1_M \rangle - 1) dt,
\]

\[
s_3 = \frac{nV}{2\pi} \int_{|t| < \delta} e^{itz} \left( \hat{u}(t) \left( \prod_{j=0}^{n-1} \lambda_{it, \sigma^j} \right) - \hat{u}(0) e^{-\frac{z^2}{2}} \right) dt,
\]

so that (4.43) immediately follows from that

\[
\lim_{n \to \infty} \sup_{z \in \mathbb{R}} (|A_2| + |B_2| + |s_1| + |s_2| + |s_3|) = 0
\]

for some \( \delta \in (0, \delta_0) \).

**Estimate of** \( A_2, B_2, s_1 \). By using condition (L) together with (A2) and (UG), we can easily get the desired estimate of \( A_2 \):

\[
\sup_{z \in \mathbb{R}} |A_2| \leq \frac{nV}{2\pi} 2(\delta_0 - \delta) \| \hat{u} \|_{L^\infty} C_1 C_\omega \rho^n \| h_0 \|_{L^\infty(\Omega, \mathbb{R})} \| 1_M \|_{\mathcal{E}},
\]

which goes to 0 as \( n \to \infty \).
Furthermore, it holds that
\[
\sup_{z \in \mathbb{R}} |B_2| \leq \frac{\hat{u}(0)\sqrt{V}}{2\pi} \int_{|t| \geq \delta \sqrt{n}} e^{-\frac{t^2}{4n}} \, dt,
\]
which, due to the dominated convergence theorem and the integrability of the map 
\( t \mapsto e^{-\frac{t^2}{4n}} \), converges to zero as \( n \to \infty \).

Finally it follows from \((4.40)\) that, if \( \delta \) is sufficiently small, then
\[
\lim_{n \to \infty} \sup_{z \in \mathbb{R}} |s_1| \leq \lim_{n \to \infty} \frac{\sqrt{n}V}{2\pi} 2\delta \| \hat{u} \|_{L^\infty} K^2 \| h_0 \|_{L^\infty(\Omega, \theta)} \| 1_M \| e^{r^2n} = 0.
\]

**Estimate of** \( s_2, s_3 \). The following lemma is important to get the desired estimates of \( s_2 \) and \( s_3 \).

**Lemma 4.11.** For any sufficiently small \( \delta > 0 \) and \( \mathbb{P} \)-almost every \( \omega \in \Omega \), there exists \( n_\omega \in \mathbb{N} \) such that for all \( n \geq n_\omega \) and \( t \in \mathbb{R} \) with \( |t| \leq \delta \sqrt{n} \),
\[
\lim_{j \to \infty} \frac{1}{\lambda_{n_\omega}^{\infty, \sigma_j \omega}} \leq e^{-\frac{t^2V}{n}}.
\]

**Proof.** Recall the proof of \((4.26)\) for \( \Lambda \). Let \( R \) denote the remainder of \( \Lambda \) of order 2 in the sense that \( \Lambda(\theta)(\omega) = \Lambda''(0)(\omega) \theta^2 + R(\theta)(\omega) \) holds, so that we have
\((4.45)\)
\[
\log \prod_{j=0}^{n-1} \lambda_j^{\infty, \sigma_j \omega} = -\frac{t^2}{2} \mathbb{R} \left( \frac{1}{n} \sum_{j=0}^{n-1} \Lambda''(0)(\sigma_j \omega) \right) + \mathbb{R} \left( \sum_{j=0}^{n-1} R(it/\sqrt{n})(\sigma_j \omega) \right).
\]
Let \( \delta \) be a positive number such that \( \| R(\theta) \|_{L^\infty(\Omega)} \leq \frac{V}{8} |\theta|^2 \) whenever \( |\theta| \leq \delta \).

On the one hand, by \((4.42)\), we have that
\[
\lim_{n \to \infty} \mathbb{R} \left( \frac{1}{n} \sum_{j=0}^{n-1} \Lambda''(0)(\sigma_j \omega) \right) = V \quad \mathbb{P} \text{-almost surely},
\]
and thus for \( \mathbb{P} \)-almost every \( \omega \) there exists \( n_\omega \in \mathbb{N} \) such that
\[
-\frac{t^2}{2} \mathbb{R} \left( \frac{1}{n} \sum_{j=0}^{n-1} \Lambda''(0)(\sigma_j \omega) \right) \leq -\frac{t^2V}{4} \quad \text{for } n \geq n_\omega.
\]
On the other hand, for \( t \in \mathbb{R} \) with \( |t| \leq \delta \sqrt{n} \), it holds that
\[
\left| \sum_{j=0}^{n-1} R(it/\sqrt{n})(\sigma_j \omega) \right| \leq \sum_{j=0}^{n-1} \frac{V}{8} \frac{|it/\sqrt{n}|^2}{V} = \frac{t^2V}{8},
\]
and thus
\[
\mathbb{R} \left( \sum_{j=0}^{n-1} R(it/\sqrt{n})(\sigma_j \omega) \right) \leq \frac{t^2V}{8}.
\]
These estimates together with \((4.45)\) completes the proof. \( \square \)

Now the desired estimate of \( |s_2| \) can be shown. Recall that \( w_{0, \omega}(h_{0, \omega}) = 1 \) and \( w_\theta \) is differentiable by Theorem 4.12, so that there exists a constant \( C > 0 \) such that \( |w_{\theta, \omega}(h_{0, \omega}) - 1| \leq C|\theta| \) for every \( \theta \) in a neighborhood of 0 in \( \mathbb{C} \). Similar reasoning from the facts about \( h_\theta \) in Theorem 4.12 leads to that \( |w_{\theta, \omega}(h_{0, \omega}) - 1| \leq C|\theta| \) for every
θ in a neighborhood of 0 in C, with some constant C > 0. Therefore, it follows from Lemma 4.11 that when n is sufficiently large and δ is sufficiently small, supzs∈R |s2| is bounded by

\[
\frac{\sqrt{V}}{2\pi} \int_{|t|<\delta \sqrt{n}} |\hat{u} \left( \frac{t}{\sqrt{n}} \right) \left( \prod_{j=0}^{n-1} \lambda_{\frac{m_j}{m_j^2}, \sigma_j} \left( h_{0, \omega} \right) (h_{0, \omega} \omega) - 1 \right) dt |
\]

with some constant C > 0, so we conclude that limn→∞ supzs∈R |s2| = 0.

Furthermore,

\[
\sup_{z \in R} |s_3| \leq \frac{\sqrt{V}}{2\pi} \int_{|t|<\delta \sqrt{n}} \left| \hat{u} \left( \frac{t}{\sqrt{n}} \right) \left( \prod_{j=0}^{n-1} \lambda_{\frac{m_j}{m_j^2}, \sigma_j} \left( h_{0, \omega} \right) (h_{0, \omega} \omega) \right) - \hat{u}(0) e^{-\lambda_{\frac{m_j}{m_j^2}, \sigma_j} t^2} \right| dt,
\]

which is bounded by 2\||\hat{u}|L^\infty e^{-\frac{\lambda_{\frac{m_j}{m_j^2}, \sigma_j}}{n}} for any sufficiently large n ∈ N and small δ > 0
due to Lemma 4.11. Hence, the dominated convergence theorem is applicable: since |

\[
\| \hat{u} \left( \frac{t}{\sqrt{n}} \right) \left( \prod_{j=0}^{n-1} \lambda_{\frac{m_j}{m_j^2}, \sigma_j} \left( h_{0, \omega} \right) (h_{0, \omega} \omega) \right) - \hat{u}(0) e^{-\lambda_{\frac{m_j}{m_j^2}, \sigma_j} t^2} \|
\]

→ 0 as n → ∞ by virtue of the continuity of \( \hat{u} \) and (4.32), we get limn→∞ supzs∈R |s3| = 0. This completes the proof of

Theorem 2.8.

APPENDIX A. THE PARTIAL CAPTIVITY CONDITION

In this appendix, we briefly recall the partial captivity condition. Let T = T(E, τ) be the U(1) extension of an expanding map E : S^1 → S^1 over τ ∈ C(S^1) with r ≥ 2 (given by (1.3) with T, E, τ in place of T_0, E_0, τ_0). Fix some R > \|τ\|∞. Then the corresponding cone \( \mathcal{X}_R = \{(\xi, \eta) ∈ \mathbb{R}^2 \mid |\eta| ≤ \max_{E(x)-1} |\xi| \} \) is (forward) invariant under the Jacobian matrix

\[
DT(z) = \begin{pmatrix}
E'(x) & 0 \\
\tau'(x) & 1
\end{pmatrix}
\]

z = (x, s) ∈ T^2.

For z ∈ T^2 and n ≥ 1, let us consider the images of \( \mathcal{X}_R \) by DT^n in T^2, i.e.

\[
DT^n(\zeta) \mathcal{X}_R \text{ for } \zeta ∈ T^{-n}(z)
\]

It is not difficult to see that τ is cohomologous to a constant if and only if all the above cones have a line in common at every point z ∈ T^2 and n ≥ 1. Thus we naturally come to the idea of considering transversality between the above cones. As a way to quantify this notion, we set

\[
n(τ) = \lim_{n→∞} \left( \sup_{z ∈ T^2} \sup_{v ∈ \mathbb{R}^2, |v| = 1} \#\{ζ ∈ T^{-n}(z) \mid v ∈ DT^n(ζ) \mathcal{X}_R\} \right)^{1/n}
\]

(see [56] for the existence of the limit). Note that n(τ) ≤ k with equality when τ is cohomologous to a constant. We say that T = T(E, τ) is partially captive if n(τ) = 1.

Theorem A.1 ([56]). Let r ≥ 2 and suppose that the expanding map E : S^1 → S^1 is fixed. For every q > 1, there is an open dense subset \( \mathcal{V}_q \subset \mathcal{C}^r(S^1) \) such that if \( \tau ∈ \mathcal{V}_q \) then

\[
n(τ) < q.
\]
Consequently, there is a residual subset $\mathcal{R} \subset C^r(S^1)$ such that $T = T_{(E, \tau)}$ is partially captive for every $\tau \in \mathcal{R}$.

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