Localization of low-energy eigenfunctions in Šeba billiards

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We investigate localization of low-energy modes of the Laplacian with a point scatterer on a rectangular plate. We observe that the point scatterer acts as a barrier confining the low-level modes to one side of the plate while assuming the Dirichlet boundary condition at a point does not induce such type of localization. This low-energy phenomenon extends to higher modes as we increase the eccentricity of the plate.

I. INTRODUCTION

Suppose we have a two-dimensional structure such as a vibrating membrane, a vibrating plate or a quantum particle on a layer whose physical property is described by a partial differential equation. Then it is natural to ask how the geometry of the structure affects the properties of the waves on it. In particular, there has been considerable interest in eigenmodes localized in a specific region of the structure. For the Laplacian, it has been studied by Sapoval et al. [1–3] that irregular geometry induces a strong localization similarly as in the bi-Laplacian. A similar analysis on the bi-Laplacian \( \Delta^2 \) in a rectangular plate with the Dirichlet boundary condition studied by Sapoval et al. [1–3] that irregular geometry does not induce such type of localization. This low-energy phenomenon extends to higher modes as we increase the eccentricity of the plate.

On the other hand, we construct a point scatterer at \( x_0 = (x_0, y_0) \in \Omega \) as follows: First, restrict the domain of the Dirichlet Laplacian \( -\Delta \) to the functions vanishing at \( x_0 \in \Omega \). By theory of self-adjoint extension developed by Von Neumann, such a symmetric operator has a family of self-adjoint extensions \( -\Delta_{\alpha,x_0} \) with a parameter \( \alpha \in (-\infty, \infty] \). More precisely, let \( G_z \) be the integral kernel of the resolvent \( (-\Delta - z)^{-1} : L^2(\Omega) \to L^2(\Omega) \), namely,

\[
G_z(x, x') = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(x')}{E_n - z}
\]

so that for \( f \in L^2(\Omega) \),

\[
(-\Delta - z)^{-1} f(x) = \int_\Omega G_z(x, x') f(x') dx'.
\]

Then for \( z \in \rho(-\Delta_{\alpha,x_0}) \), the integral kernel of \( (-\Delta_{\alpha,x_0} - z)^{-1} : L^2(\Omega) \to L^2(\Omega) \) reads

\[
(-\Delta_{\alpha,x_0} - z)^{-1}(x, x') = G_z(x, x') + [\alpha - F(z)]^{-1} G_z(x_0, x') G_z(x, x_0)
\]

(1)

where

\[
F(z) = \sum_{n=1}^{\infty} \phi_n(x_0)^2 \left( \frac{1}{E_n - z} - \frac{E_n}{E_n^2 + 1} \right).
\]

The coupling constant \( \alpha \in (-\infty, \infty] \) can be considered as a parameter related to strength of the point scatterer. Note that the point scatterer annihilates as \( \alpha \to \pm \infty \) while it acts stronger when \( |\alpha| \ll \infty \). Now we consider the spectral property of Šeba billiards. Let \( \sigma(P) \) denote the spectrum of an operator \( P \) and let mult(\( z, P \)) denote the multiplicity of an eigenvalue \( z \in \sigma(P) \). As the Dirichlet Laplacian \( -\Delta \) has a purely discrete spectrum, so does \( -\Delta_{\alpha,x_0} \). In addition, some eigenvalues of \( -\Delta_{\alpha,x_0} \) remain in \( \sigma(-\Delta) \) regardless of the coupling constant \( \alpha \) while the others do not. Hence, for \( \alpha \in \mathbb{R} \), we divide \( \sigma(-\Delta_{\alpha,x_0}) \) into the following two types:

1. Perturbed eigenvalues: \( \sigma(-\Delta_{\alpha,x_0}) \setminus \sigma(-\Delta) \)
2. Unperturbed eigenvalues: \( \sigma(-\Delta_{\alpha,x_0}) \cap \sigma(-\Delta) \)

where each of them is obtained by different conditions as follows:
Theorem 1. For \( \alpha \in \mathbb{R} \), \( z \in \sigma(-\Delta_{0, x_0}) \setminus \sigma(-\Delta) \) if and only if

\[ \alpha = F(z). \]

Then \( \text{mult}(z, -\Delta_{0, x_0}) = 1 \) with the corresponding eigenfunctions

\[ \psi(x) = N^{-1} G_z(x, x_0) \]

where \( N = \|G_z(\bullet, x_0)\|_{L^2(\Omega)} \) is the normalization constant.

Theorem 2. Define \( \mu \) and \( \mu_0 \) as

\[
\begin{align*}
\mu(z) & \equiv \text{mult}(z, -\Delta) = \# \{ n \geq 1 \mid z = E_n \} \quad (3) \\
\mu_0(z) & \equiv \# \{ n \geq 1 \mid z = E_n, \phi_n(x_0) = 0 \}. \quad (4)
\end{align*}
\]

Then for \( \alpha \in \mathbb{R} \), \( z \in \sigma(-\Delta_{0, x_0}) \cap \sigma(-\Delta) \) if and only if

\[ \mu_0(z) \geq 1 \text{ or } \mu(z) \geq 2 \]

Also,

\[ \text{mult}(z, -\Delta_{0, x_0}) = \begin{cases} 
\mu(z), & \text{if } \mu_0(z) = \mu(z) \\
\mu(z) - 1, & \text{if } \mu_0(z) < \mu(z)
\end{cases} \]

with the corresponding eigenspaces

\[ \left\{ \sum_{z \in E_n} c_n \phi_n \right| \sum_{z \in E_n} c_n \phi_n(x_0) = 0, \quad c_n \in \mathbb{C} \right\}. \]

Proofs can be found in [6], Chapter 2, with generalized statements for a compact Riemannian manifold of dimension two or three. The coupling constant \( \alpha \) in Eq. (1) can be obtained by following the notations provided by Albeverio et al. [7]. Note that \( \alpha \) also corresponds to the inverse of the coupling constant \( v_B \) or \( v_F \) in Shigehara’s setting [8][9].

We may interpret Theorem 2 as that the Laplacian eigenfunctions vanishing at \( x_0 \) do not feel the presence of the point scatterer. So not only they remain as the eigenfunctions of \( -\Delta_{0, x_0} \), but also the associated eigenvalues stay in \( \sigma(-\Delta_{0, x_0}) \) for any \( \alpha \).

On the other hand, by combining Theorem 1 and 2, we obtain that the eigenvalues of the point scatterer are interlaced between those of the Dirichlet Laplacian. In other words, for \( \alpha \in (-\infty, \infty) \), let \( z_1(\alpha) \leq z_2(\alpha) \leq \cdots \) be the eigenvalues of \( -\Delta_{0, x_0} \). Then we have

\[ z_1(\alpha) \leq E_1 \leq z_2(\alpha) \leq E_2 \leq z_3(\alpha) \leq E_3 \cdots. \]

In addition, for \( n \geq 1 \),

\[
\begin{align*}
\lim_{\alpha \to -\infty} z_n(\alpha) &= E_n \\
\lim_{\alpha \to \infty} z_{n+1}(\alpha) &= E_n \\
\lim_{\alpha \to -\infty} z_1(\alpha) &= -\infty
\end{align*}
\]

### III. LOCALIZATION OF EIGENFUNCTIONS

In this section, we show several examples of perturbed eigenfunctions localized on a plate due to the point scatterer with a suitable coupling constant \( \alpha \in \mathbb{R} \).

Let \( \Omega = [0, a] \times [0, b] \) with \( \alpha = \sqrt{E} \) and \( b = 1/\sqrt{E} \) so every plate has unit area for any \( E > 0 \) which is the eccentricity of the plate. The unperturbed eigenfunctions obtained by Theorem 2 are independent of \( \alpha \) so they have no chance to be localized at all. In order to avoid such cases as much as possible, first we assume the eccentricity \( E \) irrational so that all \( E_n \)'s are non-degenerate. In addition, let \( \frac{a}{x_0} \) be irrational to minimize the case in which \( \phi_n \) vanishes at \( x_0 \). In this paper, we choose a specific value \( \frac{a}{x_0} = 2\pi \) (Figure 2). However, it should be noted that the qualitative property we observe also holds for other values of \( \frac{a}{x_0} \) as long as they are irrational.

By Theorem 2 if \( z_n(\alpha) \) is a perturbed eigenvalue of \( -\Delta_{0, x_0} \) then the corresponding normalized eigenfunction \( \psi_{n, \alpha} \in L^2(\Omega) \) satisfies the following \( L^2 \)-identity:

\[
\psi_{n, \alpha}(x) = N_{n, \alpha}^{-1} \sum_{n'=1}^{\infty} \frac{\phi_{n'}(x)\phi_{n'}(x_0)}{E_{n'} - z_n(\alpha)}
\]

FIG. 2: (Color online) Geometry of a point scatterer at \( x_0 = (x_0, y_0) \) (marked as \( \times \)) in \( \Omega \). The left part of the plate divided by \( x_0 \) is denoted by \( \Omega_1 = [0, x_0] \times [0, b] \).
where $N_{n,\alpha}$ is the $L^2$-normalization constant.

We now investigate the localization of the perturbed eigenfunctions given by Eq. (5) which depends on the mode number $n$, the coupling constant $\alpha$, and the eccentricity $E$. Among those three variables, we mainly concentrate on $n$ and $E$. It should be noted that $\alpha$ is chosen to maximize the localization property for each situation.

In order to quantify the localization of multiple modes with ease, we introduce two kinds of measurement: First, define the $L^2$-norm ratio $R_1(n, \alpha)$ as

$$R_1(n, \alpha) = \left(\int_{\Omega_1} |\psi_{n,\alpha}(x)|^2 dx\right)^{\frac{1}{2}},$$

(6)

where $\Omega_1 = [0, x_0] \times [0, b]$ denotes the left part of the plate divided by the point scatterer. In addition, let $A(n, \alpha)$ be the amplitude at $x_0$:

$$A(n, \alpha) = |\psi_{n,\alpha}(x_0)|.$$  

(7)

For simplicity, let us omit $E$ in those notations since it is already embedded in every $E_n$ and $\phi_n$ of Eq. (5).

Note that we assume that all eigenfunctions are $L^2$-normalized. Then $R(n, \alpha)$ measures the ratio of $L^2$-norm localized in $\Omega_1$. For instance, $R_1(n, \alpha) = 0$ and $R_1(n, \alpha) = 1$ imply that $\psi_{n,\alpha}$ is completely localized in $\Omega \setminus \Omega_1$ and $\Omega_1$, respectively. On the other hand, $A(n, \alpha)$ measures how much the point scatterer at $x_0$ attracts the amplitude of modes.

### A. Point scatterer acting as a barrier

Now we provide numerical results showing that the low-level eigenfunctions with $n \geq 2$ localize to the left or the right of $x_0$ where the point scatterer is located. In Figure 3, we compare some eigenfunctions localized by a point scatterer (right column) to those of the Dirichlet Laplacian (left column) where $E = 10\pi$. These modes are examples in which the point scatterer acts as a barrier confining the amplitude distribution to the left or right of itself.

Instead of presenting the amplitude distribution of every localized eigenfunction on the plate $\Omega$, let us draw a graph of the $L^2$-norm ratio $R_1(n, \alpha)$ as a function of the mode number $n$ for each $E$ fixed. The eigenfunction $\psi_{n,\alpha}$ will be considered to be localized in terms of $L^2$-norm ratio if $R_1(n, \alpha) < 0.1$ or $R_1(n, \alpha) > 0.9$.

Figure 4 compares $R_1(n, \alpha)$ of the first 500 eigenfunctions of $-\Delta_{n,\alpha}$ to those of the Dirichlet Laplacian where $E = \frac{\pi}{7}$ and $E = 10\pi$. For each $E$, $\alpha$ is chosen to maximize the number of localized modes. The blue points and red points represent the eigenvalues given by Theorem 1 and 2 respectively. Note that if the modes are localized completely to the right or the left of $x_0$ then all points in the graph will be polarized to either 0 or 1. When eccentricity is small ($E = \frac{\pi}{7}$), the point scatterer weakly perturbs the $L^2$-norm ratio of modes but it is hard to say these modes are localized enough. On the other hand, when eccentricity is large ($E = 10\pi$), one can observe a strong localization especially at the low-level modes. Video clips for continuous transition from Figure 4a to 4b and from 4c to 4d can be found at \url{http://math.berkeley.edu/~lmj0425/seba_PR_pi3.avi} and \url{http://math.berkeley.edu/~lmj0425/seba_PR_10pi.avi} respectively. Note that the Dirichlet Laplacian is equivalent to the point scatterer with $\alpha = \infty$.

Now we discuss how far the localization in terms of the $L^2$-norm ratio maintains its influence up to the higher-
level eigenfunctions. It has been proved by Keating et al. \[10\] that the eigenfunctions of Šeba billiards tend to localize around eight points in momentum space as the level of mode increases. In other words, the localization in position space we observe in this paper is an intermediate phenomenon that tends to diminish as the mode number increases.

However, one can observe that the localization effect extends to higher-level eigenfunctions as the eccentricity \( E \) increases. Figure 5 displays the number of localized modes out of the first 500 modes as a function of eccentricity \( E \). The coupling constant \( \alpha \) is chosen to maximize the number of localized modes for each \( E \). Therefore, we can conclude that the point scatterer induces a strong localization as a barrier confining the amplitude of low-level eigenfunctions to either \( \Omega_1 \) or \( \Omega \setminus \Omega_1 \).

**B. Point scatterer acting as an attractor**

On the other hand, the eigenfunction corresponding to the lowest eigenvalue \( z_1(\alpha) \in (-\infty, E_1) \) shows a different behavior: It tends to localize around \( x_0 \) so we can say the point scatterer attracts the amplitude of the first mode.

A numerical simulation indicates that the amplitude at \( x_0 \) mainly depends on the mode number. In particular, the first mode with the associated eigenvalue \( z_1(\alpha) \in (-\infty, E_1) \) tends to localize around \( x_0 \) as \( z_1(\alpha) \to -\infty \), or equivalently, as \( \alpha \to -\infty \). Figure 6 shows the eigenfunction of \( -\Delta_{\alpha,x_0} \) corresponding to \( z_1(\alpha) = -1.29 \cdot 10^4 \) with \( \alpha = -0.48 \). Since the amplitude localizes around the point scatterer evenly, our first criterion using the \( L^2 \)-norm ratio cannot detect this type of localization. So we introduce the second measurement \( A(n,\alpha) \), the amplitude of the mode at \( x_0 \), to investigate the behavior described above.

Figure 7 displays how the presence of the point scatterer with the coupling constant \( \alpha \) affects \( A(n,\alpha) \) of the first 5 modes where \( E = \frac{\pi}{3} \) (solid lines) and \( E = 10\pi \) (dashed lines). Regardless of the eccentricity, the amplitude of the first mode at \( x_0 \) blows up as \( \alpha \to -\infty \) but...
FIG. 7: (Color online) Amplitude of the first five modes of at $x_0$ for the coupling constant $\alpha \in [-5, 5]$. Solid lines and dashed lines correspond to the eccentricity $E = \frac{\pi}{3}$ and $E = 10\pi$, respectively. The first mode tends to localize around the point scatterer as $\alpha \to -\infty$ regardless of the eccentricity while the others maintain low amplitude for all $\alpha$’s.

such localization does not occur in the other modes. This can be justified by the Fourier series representation of the perturbed eigenfunction in Eq. (5) since for each $\phi_n$, the Fourier coefficients

$$\phi_n(x_0) \cdot \frac{E_n - z}{E_n - z}$$

gets relatively uniform as $z \to -\infty$. On the other hand, if $E_j \leq z \leq E_{j+1}$ for some $j \geq 1$, then the Fourier coefficients corresponding to $E_n$’s near $z$ prevail the summation which prevents the amplitude of higher modes from diverging at a certain point.

IV. CONCLUSION

We have shown that the point scatterer placed on a plate behaves as a barrier confining the low-energy eigenfunctions. Although it has been proved that such a localization property has to diminish as the mode number increases, we can increase the number of localized modes by elongating the plate. Note that the lowest eigenfunction should be excluded from this phenomenon since the point scatterer attracts its amplitude when the corresponding eigenvalue is large and negative regardless of the eccentricity of the plate.

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[1] D. Grebenkov and B. Nguyen, SIAM Review 55, 601 (2013).
[2] S. M. Heilman and R. S. Strichartz, Notices of the AMS 57, 624 (2010).
[3] B. Sapoval, S. Félix, and M. Filoche, The European Physical Journal Special Topics 161, 225 (2008).
[4] M. Filoche and S. Mayboroda, Phys. Rev. Lett. 103, 254301 (2009).
[5] P. Séba, Phys. Rev. Lett. 64, 1855 (1990).
[6] Y. Colin de Verdière, Annales de l’institut Fourier 32, 275 (1982).
[7] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, Solvable models in quantum mechanics, 2nd ed. (AMS Chelsea Publishing, Providence, RI, 2005) pp. xiv+488, with an appendix by Pavel Exner.
[8] T. Shigehara, Phys. Rev. E 50, 4357 (1994).
[9] T. Shigehara and T. Cheon, Phys. Rev. E 54, 1321 (1996).
[10] J. P. Keating, J. Marklof, and B. Winn, Journal of Mathematical Physics 51, 062101 (2010) arXiv:0909.3797 [math-ph].
[11] Z. Rudnick and H. Ueberschär, Communications in Mathematical Physics 316, 763 (2012) arXiv:1109.4582 [math.AP].
[12] H. Ueberschär, Royal Society of London Philosophical Transactions Series A 372, 2059 (2013) arXiv:1212.1086 [math-ph].
[13] P. de Vries, D. V. van Coevorden, and A. Lagendijk, Rev. Mod. Phys. 70, 447 (1998).
[14] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Fourier analysis, self-adjointness No. V. 2 (Academic Press, 1975).
[15] P. Exner and P. Šeba, Journal of Mathematical Physics 30 (1989).