Large-deviation theory for a Brownian particle on a ring: a WKB approach

Karel Proesmans\textsuperscript{1,2} and Bernard Derrida\textsuperscript{2}

\textsuperscript{1} Hasselt University, B-3590 Diepenbeek, Belgium
\textsuperscript{2} Collège de France, PSL, 11 place Marcelin Berthelot, F-75231 Paris Cedex 05, France
E-mail: Karel.Proesmans@uhasselt.be

Received 28 September 2018
Accepted for publication 19 December 2018
Published 1 February 2019

Abstract. We study the large deviation function of the displacement of a Brownian particle confined on a ring. In the zero noise limit this large deviation function has a cusp at zero velocity given by the Freidlin–Wentzell theory. We develop a WKB approach to analyse how this cusp is rounded in the weak noise limit.

Keywords: diffusion, stochastic particle dynamics
1. Introduction

Large deviations have a long history in the mathematical literature [1–3]. Over the last few decades, they have also become a central part of non-equilibrium statistical mechanics [4–6], in particular in the context of the fluctuation theorem [7].

One of the simplest models one can consider in the context of large-deviation theory is the Brownian particle dragged through a periodic potential [8–13]. In the long-time limit, the empirical velocity, $v$, of the Brownian particle satisfies a large-deviation principle:

$$I(v) = -\lim_{t \to \infty} \frac{1}{t} \ln P_t(x_t = vt),$$

where $P_t(x_t)$ is the probability distribution associated with the displacement $x_t$ after a time $t$. A general, exact expression for this large-deviation function does not exist, but several approximations have been derived to get to a solution [14–18]. Furthermore, related studies have been done in the context of e.g. first-passage time distributions [19] and underdamped dynamics [20]. In the low-noise limit, one can tackle the problem using the Freidlin–Wentzell theory [21–25]. This method is based on the fact that, in the aforementioned limit, one can calculate the large deviation function associated with trajectories. One can subsequently contract this large-deviation function of trajectories to obtain $I(v)$ [26].

Near $v = 0$, something odd happens; a ‘kink’ appears in the Freidlin–Wentzell large-deviation function [7, 14, 18, 22, 27]. Therefore, to get a precise value of $I(v)$ in this neighbourhood, we need to look at higher order contributions of the noise. To do
this, we use a tilted generator method [5]. This method focuses on finding the largest eigenvalue of a Schrödinger-like equation, which is generally hard to solve, but there exist methods known from quantum mechanics, such as diffusive Monte-Carlo methods [28, 29] and Rayleigh–Schrödinger perturbation theory [30], to obtain the lowest eigenvalue. Here, we will solve the equation in the low-but-finite-noise limit using a WKB approach. This approach allows us to understand how the kink of $I(v)$ is rounded in a weak-noise expansion.

We will start in section 2, by introducing the model and discussing some basic concepts of large-deviation theory. In section 3, we will review the Freidlin–Wentzell approach to derive $I(v)$ and discuss its limitations. We reproduce a number of existing results [18, 22], in order to connect them with our results of section 4. In particular, we will see that the large-deviation function of the velocity generally exhibits a cusp at zero velocity. In the main part of this paper (section 4) we use a WKB approach to calculate $I(v)$ or rather its Legendre transform $\mu(\lambda)$ in the case where the force vanishes nowhere i.e. the case where there is no metastable state. This will allow to analyse how the cusp in the large-deviation function is rounded by a small but finite noise. This analytic result is the main contribution of the present work. Finally, we end with conclusions and perspectives in section 5.

2. Model

The focus in this paper will be on a Brownian particle dragged through a periodic potential $V(x)$ (period 1) with a force $f$. For notational simplicity, we shall assume that $f \geq 0$ throughout this text. The particle ’feels’ an effective force equal to

$$F(x) = f - V'(x), \quad V(x + 1) = V(x)$$

and an associated effective potential

$$U(x) = V(x) - fx,$$

see figure 1. In this section, we shall construct the steady state associated with the position of the particle, and discuss how one can derive the large-deviation function associated with the displacement of the particle. Throughout this paper, we will mainly focus on periodic potentials with no extrema such as the one drawn on the right side of figure 1.

2.1. Steady-state distribution

The position $x(t)$ of the Brownian particle evolves on the infinite line according to an overdamped Langevin equation

$$\dot{x}(t) = -U'(x) + \eta(t),$$

where $\eta(t)$ is a Brownian motion,

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = \epsilon \delta(t - t'),$$

see figure 1. In this section, we shall construct the steady state associated with the position of the particle, and discuss how one can derive the large-deviation function associated with the displacement of the particle. Throughout this paper, we will mainly focus on periodic potentials with no extrema such as the one drawn on the right side of figure 1.
Large-deviation theory for a Brownian particle on a ring: a WKB approach

and $\epsilon$ is a measure for the strength of the noise. Associated with this Langevin equation, one can write a Fokker–Planck equation, describing the time-evolution of the probability distribution, $p_t(x)$ associated with $x(t)$:

$$\frac{d}{dt} p_t(x) = -\frac{d}{dx}(F(x)p_t(x)) + \epsilon \frac{d^2}{dx^2} p_t(x). \tag{6}$$

Although the distribution $p_t(x)$ broadens and spreads out over the whole real axis, the distribution $P_t(x)$, projected on the ring,

$$P_t(x) = \sum_{n \in \mathbb{Z}} p_t(x + n), \tag{7}$$

has a steady-state solution, $p_{ss}(x)$, that satisfies

$$-\frac{d}{dx}(F(x)p_{ss}(x)) + \epsilon \frac{d^2}{dx^2} p_{ss}(x) = 0. \tag{8}$$

Due to the periodicity, one has $p_{ss}(x) = p_{ss}(x + 1)$. This boundary condition fixes the solution of equation (8) [31]:

$$p_{ss}(x) = C \exp\left(-\frac{2}{\epsilon} U(x)\right) \times \left(\int_0^x dy \exp\left(\frac{2}{\epsilon} U(y)\right) + e^{2f} \int_x^1 dy \exp\left(\frac{2}{\epsilon} U(y)\right)\right), \tag{9}$$

where $C$ is a normalization constant. The average velocity of the particle is given by

$$\langle v \rangle = \frac{C\epsilon}{2} \left(e^{2f} - 1\right). \tag{10}$$

In the weak noise limit ($\epsilon$ small), the behaviour of the velocity depends on the strength of the external force and can be separated in two classes:

- If $f < \max V'(x)$, the effective potential $U(x)$ exhibits a local minimum and maximum, at $x = x_0$ and $x = x_1$ respectively (see the left panel of figure 1), leading to a meta-stable state for the particle at $x = x_0$. The particle generally spends most of its time in this metastable state and the average velocity is exponentially small [31],

Figure 1. One period of $U(x)$. On the left hand side, a case with a single metastable state. On the right hand side, a case without metastable state.
\[ \langle v \rangle \simeq \frac{\sqrt{-U''(x_0)U''(x_1)}}{2\pi} e^{\frac{2(U(x_0) - U(x_1))}{\epsilon}}. \] (11)

Analysing the whole \( x \) range in equation (9), one can also see that \( p_{ss}(x) \) is exponentially peaked at \( x = x_0 \), and exhibits a large-deviation principle in terms of \( \epsilon \), with non-analytic points at the values of \( x \) where the two terms in equation (9) have the same magnitude [23, 27, 32].

- If \( f > \text{max } V'(x) \), there are no local minima in the effective potential. Therefore, the probability distribution associated with the position of the particle is spread out over the ring and the average velocity of the particle stays finite for arbitrary small noise:

\[ \langle v \rangle \simeq \frac{1}{\int_{0}^{1} dy F(y)^{-1}}. \] (12)

As \( f > 0 \), the second term in equation (9) is dominant, leading to

\[ p_{ss}(x) \simeq \frac{C'}{F(x)}, \] (13)

where \( C' \) again is a normalisation constant.

### 2.2. Large-deviation theory

In the long-time limit, the measured velocity of the Brownian particle will always converge to the average velocity, equations (11) and (12). All other velocities become exponentially unlikely. This behaviour is described by the associated large-deviation function:

\[ I(v) = -\lim_{t \to \infty} \frac{1}{t} \ln P_t(x_t = vt), \] (14)

where \( x_t \) is the total displacement of the Brownian particle after time \( t \). In the following it will be more convenient to work with the cumulant-generating function \( \mu(\lambda) \) defined by

\[ \mu(\lambda) = \lim_{t \to \infty} \frac{1}{t} \ln \langle e^{t\lambda v} \rangle. \] (15)

From \( \mu(\lambda) \), one can uncover all cumulants associated with the displacement, as the \( n \)th derivative of \( \mu(\lambda) \) evaluated at \( \lambda = 0 \) is equal to the \( n \)th cumulant. The convexity of the large-deviation function allows one to extract it via a Legendre transform [5],

\[ I(v) = \max_{\lambda} (\lambda v - \mu(\lambda)) \quad ; \quad \mu(\lambda) = \max_{v} (\lambda v - I(v)). \] (16)

Therefore, one can determine the large-deviation function by first calculating the cumulant-generating function, \( \mu(\lambda) \), and then doing a Legendre transform.
The cumulant-generating function can be found as the largest eigenvalue of a ’tilted’ Fokker–Planck operator \([33, 34]\), see also appendix A,

\[
\mu(\lambda) r(x) = \lambda F(x) r(x) - \frac{d}{dx} \left(F(x) r(x)\right) \\
+ \frac{\epsilon}{2} \left(\lambda^2 r(x) - 2\lambda \frac{d}{dx} r(x) + \frac{d^2}{dx^2} r(x)\right),
\]

where \(r(x)\) is the associated eigenvector, which satisfies the periodic boundary condition \(r(x + 1) = r(x)\). This equation can be simplified by introducing

\[
s(x) = \exp \left(-\lambda x\right) r(x),
\]

leading to

\[
\mu(\lambda) s(x) = -\frac{d}{dx} \left(F(x) s(x)\right) + \frac{\epsilon}{2} \frac{d^2}{dx^2} s(x),
\]

with boundary condition

\[
s(x + 1) = e^{-\lambda} s(x).
\]

In this way, the eigenvalue equation, equation (19), does no longer explicitly depend on \(\lambda\), which only appears via the boundary condition, equation (20). As \(\mu(\lambda)\) is the largest eigenvalue of a tilted Fokker–Planck operator, it is also the largest eigenvalue of the adjoint operator,

\[
\mu(\lambda) \ell(x) = \lambda F(x) \ell(x) + F(x) \frac{d \ell(x)}{dx} \\
+ \frac{\epsilon}{2} \left(\lambda^2 \ell(x) + 2\lambda \frac{d \ell(x)}{dx} + \frac{d^2 \ell(x)}{dx^2}\right),
\]

which can also be simplified by defining \(m(x) = \exp (\lambda x) \ell(x)\):

\[
\mu(\lambda) m(x) = F(x) \frac{d}{dx} m(x) + \frac{\epsilon}{2} \frac{d^2}{dx^2} m(x).
\]

Interestingly, the left and right eigenvector have a physical interpretation \([33–36]\):

\[
\ell(x) r(x) = m(x) s(x) \sim P(x|v = \mu' (\lambda)).
\]

In words, this means that, up to a normalisation constant, the product of the left and right eigenvector is equal to the probability distribution associated with the position of the particle, conditioned to the average velocity \(v = \mu'(\lambda)\).

## 3. Freidlin–Wentzell theory

One way to try to obtain the large-deviation function, \(I(v)\), is via a Freidlin–Wentzell approach \([21]\), where one determines the most likely trajectory leading to the average velocity \(v\). In this section, we shall review this approach, which was earlier applied to
the model under study in [22, 25]. Whenever this approach holds, the large-deviation function is given, up to leading order in the noise strength, by

\[ I(v) \simeq \lim_{\tau \to \infty} \min_{{x(t)}} \frac{1}{2\epsilon\tau} \int_0^\tau dt \left( \dot{x}(t) - F(x(t)) \right)^2, \tag{24} \]

with the boundary conditions

\[ x(0) = 0, \quad \frac{x(\tau)}{\tau} = v, \tag{25} \]

where one takes the limit \( \tau \to \infty \). The optimal path in equation (24) can be obtained using Lagrangian techniques:

\[ \dot{x}(t)^2 = F(x(t))^2 + K, \tag{26} \]

where \( K \) is an integration constant, which can be determined by the boundary condition, equation (25). As \( K \) is a constant of motion, the velocity \( \dot{x}(t) \) is a function of the position \( x(t) \) only. Therefore, the time \( T \) for the particle to travel around the ring once is given by

\[ T = \int_0^1 \frac{dx}{|\dot{x}(t)|}. \tag{27} \]

This implies that the optimal trajectory is periodic, which leads to the following expression of \( I(v) \) in a parametric form [22]:

\[ I(v) \simeq \frac{v}{\epsilon} \int_0^1 dx \left( \frac{2F(x)^2 + K}{2\sqrt{F(x)^2 + K}} - F(x) \right), \tag{28} \]

with

\[ v^{-1} = T = \int_0^1 \frac{dx}{\sqrt{F(x)^2 + K}}, \tag{29} \]

for \( v > 0 \) and

\[ I(v) \simeq -\frac{v}{\epsilon} \int_0^1 dx \left( \frac{2F(x)^2 + K}{2\sqrt{F(x)^2 + K}} + F(x) \right), \tag{30} \]

\[ v^{-1} = -\int_0^1 \frac{dx}{\sqrt{F(x)^2 + K}}, \tag{31} \]

for \( v < 0 \). Using equation (16), one can also determine the cumulant generating function:

\[ \lambda = I'(v) \simeq \frac{1}{\epsilon} \int_0^1 dx \left( \pm \sqrt{F(x)^2 + K} - F(x) \right), \tag{32} \]

\[ \mu(\lambda) = \lambda v - I(v) = \frac{K}{2\epsilon}. \tag{33} \]
which gives an implicit equation for $\mu(\lambda)$:

$$
\epsilon \lambda = -f + \int_0^1 dx \sqrt{2\epsilon \mu(\lambda) + F(x)^2},
$$

(34)

where the sign associated with the integral is everywhere equal to the sign of $v$.

There is a peculiarity about this solution. Clearly for the square roots in the above equations (28)–(34) to be defined, one needs that $K \geq \min_x F(x)^2$ so that

$$
\mu(\lambda) \geq -\frac{F(x)^2}{2\epsilon} \quad \text{for all } x.
$$

(35)

Therefore the above expression (34) is only valid outside the following range for $\lambda$

$$
-f - \int_0^1 dx \sqrt{F(x)^2 - F(x^*)^2} \leq \epsilon \lambda < -f + \int_0^1 dx \sqrt{F(x)^2 - F(x^*)^2},
$$

(36)

where $F(x^*)^2$ is the minimal value of $F(x)^2$. If $F(x^*) = 0$, i.e. in the presence of meta-stable states, this unreachable range simplifies to

$$
-f - \int_0^1 dx |F(x)| \leq \epsilon \lambda < -f + \int_0^1 dx |F(x)|.
$$

(37)

In this range, $\mu(\lambda)$ will be exponentially small, as discussed in [18]. This also manifests itself in the large-deviation function, which has a 'cusp' around $v = 0$, see figure 2. Indeed, one sees from equations (28)–(31) that $K \to -F(x^*)^2$ as $v \to 0$ so that

$$
I(0^+) = I(0^-) = \frac{F(x^*)^2}{2\epsilon},
$$

(38)

and from equation (32) [18, 22]

$$
\epsilon I'(0^-) = -f - \int_0^1 dx \sqrt{F(x)^2 - F(x^*)^2} \neq -f + \int_0^1 dx \sqrt{F(x)^2 - F(x^*)^2} = \epsilon I'(0^+).
$$

(39)

To explore the range (36) or (37) one needs to study more carefully the limit $\mu \to -\frac{F(x^*)^2}{2\epsilon}$ and this will be done in the next section using a WKB approach.

It is clear from equation (26) that $|\dot{x}| = \sqrt{F^2(x) + K}$. As the time spent near position $x$ is proportional to $|\dot{x}|^{-1}$, the probability $P(x|v)$ of finding the particle in $x$, conditioned on a certain value of the empirical velocity $v$, is given by

$$
P(x|v) \simeq \frac{v}{\sqrt{F^2(x) + K}},
$$

(40)

where we used equations (29)–(31) to find the normalisation constant. We will come back to this below (see, equation (49)). This equation of course reduces to equation (13) in the limit $\lambda \to 0$ (i.e. $\mu \to 0$ and $K \to 0$).

Finally, we note (see equations (28) and (30)) that the large-deviation function satisfies the fluctuation theorem [7, 18, 37]:

https://doi.org/10.1088/1742-5468/aafa7e
I(v) = I(−v) − 2ν \int_0^1 dx F(x). \tag{41}

4. WKB approach when there is no metastable state

In this section, we obtain \( \mu(\lambda) \) by solving the eigenvalue equation (19), in the low-but-finite noise limit. To do this, we look for an eigenvector, in a WKB form

\[
    s(x) \simeq g(x) \exp \left( \frac{h(x)}{\epsilon} \right), \tag{42}
\]

where \( g(x) \) and \( h(x) \) are unknown functions, independent of \( \epsilon \).

As we expect from equation (33) that \( \mu(\lambda) = O(\epsilon^{-1}) \) plugging in equations (42) into (19) one gets:

\[
    \frac{2\mu \epsilon + 2F(x)h'(x) - h'(x)^2}{2\epsilon} + \frac{2F'(x)g(x) - g(x)h''(x) + 2F(x)g'(x) - 2g'(x)h'(x)}{2g(x)} = O(\epsilon). \tag{43}
\]

Solving this equation gives us a solution for \( s(x) \) (up to zero-th order in \( \epsilon \) in the prefactors):

\[
    s(x) = C_+ s_+(x) + C_- s_-(x) \tag{44}
\]

where

\[
    s_\pm(x) = \sqrt{1 \pm \frac{F(x)}{\sqrt{2\epsilon\mu + F(x)^2}}} \exp \left[ \frac{1}{\epsilon} \int_0^x dy \left( F(y) \pm \sqrt{2\epsilon\mu + F(y)^2} \right) \right]. \tag{45}
\]

Figure 2. Freidlin–Wentzell large-deviation function with \( F(x) = \cos(2\pi x) + f \), with (a) \( f = 1/2 \) and (b)\( f = 2 \). One sees on the left panel that in the presence of metastable states \( I(0) = 0 \). In both cases a cusp appears at \( v = 0 \).
Writing that \( s(x) \) and its derivatives satisfy the boundary condition, equation (20), implies that one of the two constants \( C_+ \) or \( C_- \) vanishes and fixes the value of \( \lambda \)

\[
\epsilon \lambda = - \left[ f \pm \int_0^1 dx \sqrt{2\epsilon \mu(\lambda) + F(x)^2} \right].
\]

(46)

One recovers that way the result from the previous section, equation (34).

Similarly one can write the solution of equation (22) for the left eigenvector in a WKB form

\[
m(x) = C'_+ m_+(x) + C'_- m_-(x)
\]

(47)

where

\[
m_{\pm}(x) = \sqrt{1 \mp \frac{F(x)}{\sqrt{2\epsilon \mu + F(x)^2}}} \exp \left[ -\frac{1}{\epsilon} \int_0^x dy \left( F(y) \pm \sqrt{2\epsilon \mu + F(y)^2} \right) \right]
\]

(48)

and again the boundary condition \( m(x+1) = e^{\lambda} m(x) \) forces one of the two constants \( C'_+ \) or \( C'_- \) to be zero and fixes the value of \( \lambda \) as in equation (46). Using equation (23) one sees that the probability of finding the particle in \( x \), conditioned on the velocity \( v \) is

\[
P(x|v = \mu'(\lambda)) \sim l(x)r(x) = m(x)s(x) \sim \frac{C}{\sqrt{2\epsilon \mu(\lambda) + F(x)^2}}.
\]

(49)

This is exactly what was obtained in equation (40) with the Freidlin–Wentzell approach.

Note that in contrast to the left and right eigenvector, equations (44)–(47), the probability distribution in equation (49) does not have any exponential factor, implying that the distribution is not heavily peaked at a certain value, but is relatively spread out over the entire ring, as was earlier pointed out in [18].

All the above calculations are valid as long as

\[
\mu + \frac{F(x^*)^2}{2\epsilon} = O(1),
\]

where \( F(x^*)^2 = \min_x F(x)^2 \). This can be seen as the prefactors in equations (45) and (48) diverge in the limit \( x \to x^* \) and \( \epsilon \mu \to -\frac{F(x^*)^2}{2} \).

In order to understand this limit, we consider now the case where \( F(x) \) does not vanish on the ring and has a single quadratic minimum \( F_0 \) at some position \( x_0 \)

\[
F(x) \simeq F_0 + F_1(x - x_0)^2 + O((x - x_0)^2)
\]

(50)

and we set

\[
\mu = -\frac{F_0^2}{2\epsilon} + \sqrt{2F_0F_1} \left( \nu - \frac{1}{2} \right)
\]

(51)

where \( \nu - \frac{1}{2} \) is of order 1 (or smaller) in the limit \( \epsilon \to 0 \).

In this range of values of \( \mu \), to solve the eigenvalue problem equation (19), we decompose the ring into three regions
Region I: \(0 < x < x_0\) and \(x_0 - x \gg \sqrt{\epsilon}\)

Region II: \(x_0 - x = O(\sqrt{\epsilon})\)

Region III: \(x_0 < x < 1\) and \(x - x_0 \gg \sqrt{\epsilon}\)

In regions I and III, one can use solutions analogous to equations (44) and (45) for the eigenvector \(s(x)\) solution of equation (19) (see equation (B.1)), whereas in region II the solution takes a scaling form

\[
s = \exp \left[ \frac{F_0(x - x_0)}{\epsilon} + \sqrt{\frac{F_0 F_1}{2}} \frac{(x - x_0)^2}{\epsilon} \right] G \left( (2F_0 F_1)^{1/4} \frac{(x - x_0)}{\sqrt{\epsilon}} \right). \tag{52}\]

When this form is injected into equation (19), one gets that \(G\) should satisfy

\[
\nu G = \frac{d}{dz} (zG) + \frac{1}{2} \frac{d^2 G}{dz^2}. \tag{53}\]

Our task then is to choose pairs of constants \(C_+\) and \(C_-\) of equation (44) in regions I and III and the appropriate solution of the Hermite equation, equation (53) for the asymptotics of equation (52) in region II to match with those of the solutions in regions I and III in the range \(\sqrt{\epsilon} \ll |x - x_0| \ll 1\). This is what we do in appendix B where we show that

\[
\mu + \frac{F_0^2}{2\epsilon} + \frac{\sqrt{2F_0 F_1}}{2} \approx \left( \frac{2F_0 F_1}{\sqrt{\epsilon}} \right)^{1/4} \left( e^{\lambda - D_4 + D_2} + e^{-\lambda + D_3 - D_1} \right) \tag{54}\]

where the constants \(D_1, D_2, D_3, D_4\) are given by equations (B.3), (B.4), (B.7) and (B.8). We see that as \(\lambda\) varies in the range (36), the \(\lambda\)-dependence of \(\mu(\lambda)\) is exponentially small, as was pointed out earlier in [18].

At the boundaries \(\epsilon \lambda = -\left( f \pm \int_0^1 dx \sqrt{F(x)^2 - 2\epsilon F_0} + O(\epsilon) \right)\), one can connect the two results, equations (46) and (54), via the formulas

\[
-\lambda + D_3 - D_1 = \ln \left( \frac{2\nu \sqrt{\pi}}{\Gamma(\nu)} \left( \frac{2F_0 F_1}{\epsilon^2} \right)^{\nu - \frac{1}{2}} \right), \tag{55}\]

near \(\epsilon \lambda \approx -f - \int_0^1 dx \sqrt{2\epsilon \mu(\lambda) + F(x)^2}\), and

\[
\lambda - D_4 + D_2 = \ln \left( \frac{2\nu \sqrt{\pi}}{\Gamma(\nu)} \left( \frac{2F_0 F_1}{\epsilon^2} \right)^{\nu - \frac{1}{2}} \right), \tag{56}\]

near \(\epsilon \lambda \approx -f + \int_0^1 dx \sqrt{2\epsilon \mu(\lambda) + F(x)^2}\). One can check that these equations are in agreement with equations (46) and (54) in the appropriate limit [25].

Finally, we return to the large-deviation function \(I(\nu)\). As discussed in the previous section, the large-deviation function away from \(\nu \approx 0\), can be described by equations (28)–(31). Near \(\nu = 0\) (in particular for \(|\nu| \ll \epsilon\)), one can now use equation (54) to determine the large deviation function. This gives
Large-deviation theory for a Brownian particle on a ring: a WKB approach

\[ I(v) \simeq \frac{F_0^2}{2\epsilon} + \frac{\sqrt{2F_0F_1}}{2} + (D_3 - D_1)v - \sqrt{v^2 + \frac{4(2F_0F_1)^3}{\epsilon\pi}}e^{-D_1+D_2+D_3-D_4} \]
\[ - v \ln \left( \frac{\sqrt{\pi\epsilon} \left( \sqrt{v^2 + \frac{4(2F_0F_1)^3}{\epsilon\pi}}e^{-D_1+D_2+D_3-D_4} - v \right)}{2 \left( 2F_0F_1 \right)^{\frac{3}{2}}} \right). \] 

In the limit where \( v > 0 \) and \( \ln v \gg -1/\epsilon \), this becomes

\[ \epsilon I(v) \simeq \frac{F_0^2}{2} + v \int_0^1 \! dx \left( -F'(x) + \sqrt{F(x)^2 - F_0^2} \right) + O(\epsilon), \] 

while \( v < 0 \) and \( \ln(-v) \gg -1/\epsilon \) leads to

\[ \epsilon I(v) \simeq \frac{F_0^2}{2} + v \int_0^1 \! dx \left( -F'(x) - \sqrt{F(x)^2 - F_0^2} \right) + O(\epsilon). \] 

In these two ranges one recovers the low-velocity limit in the Freidlin–Wentzell large-deviation function, equations (38) and (39). Therefore, one concludes that the results can be smoothly connected to each other.

5. Conclusion

In this paper, we have calculated in the low noise limit the large-deviation and cumulant-generating functions associated with the velocity of a Brownian particle on a ring. In all cases the large deviation function exhibits a cusp at zero velocity in the limit of zero noise (see figure 2). Our main progress is to calculate the leading order of the cumulant-generating function. Away from the region given by equation (36), this corresponds to the well-known Freidlin–Wentzell result [22]. Inside the region of equation (36), we have shown that the cumulant-generating function is given by equation (54). Furthermore, there exist boundaries between these two regions, where the cumulant-generating function is described by equations (55) and (56). By doing a Legendre transform, we are able to show that the associated large-deviation function is smooth near \( v = 0 \), in contrast to the lowest-order Freidlin–Wentzell large-deviation function, equation (28).

We limited our analysis to the case of a periodic force with no metastable state, in contrast to most previous works [14, 18, 22, 23], which mainly focused on potentials with metastable states. Our analysis can be extended to those cases. For example in the case of a single metastable state as in the left panel of figure 1, one would need to consider 5 regions: \( x < x_0 \), \( x \) close to \( x_0 \), \( x_0 < x < x_1 \), \( x \) close to \( x_1 \) and \( x_1 < x < 1 \) and one would calculate \( \mu \) by matching the asymptotics very much as we did in section 4 and appendix B.
Acknowledgment

KP was supported by the Flemish Science Foundation (FWO-Vlaanderen) travel grant V436217N and post-doctoral grant 12J2819N. We also thank Bertrand Eynard for very useful discussions on the WKB method.

Appendix A. The large deviation of the current and the deformed Fokker–Planck equation

In this appendix, we show how to derive equation (17). Similar equations have appeared in a number of earlier works (see [5] and the references therein). We briefly explain here the derivation.

After a short time interval $\Delta t$ one has

$$x(t + \Delta t) = x(t) + F(x(t))\Delta t + B$$

where $B$ is a Gaussian random variable satisfying

$$\langle B \rangle = 0 \quad ; \quad \langle B^2 \rangle = \epsilon \Delta t.$$  \hfill (A.2)

If $p_t(x|x_0)$ is the probability that $x_t = x$, given $x_0$, one has

$$p_{t+\Delta t}(x|x_0) = \int dx' \delta \left( x - x' - F(x') \Delta t - B \right) p_t(x'|x_0)$$

and

$$p_{t+\Delta t}(x|x_0) = \int dx'_0 \delta \left( x'_0 - x_0 - F(x_0) \Delta t - B \right) p_t(x|x'_0).$$

Taking the limit $\Delta t \to 0$ these equations become

$$\frac{dp(x|x_0)}{dt} = -\frac{d[F(x) p(x|x_0)]}{dx} + \frac{\epsilon}{2} \frac{d^2p(x|x_0)}{dx^2},$$

and

$$\frac{dp(x|x_0)}{dt} = F(x_0) \frac{dp(x|x_0)}{dx_0} + \frac{\epsilon}{2} + \frac{d^2p(x|x_0)}{dx_0^2},$$

with the initial condition

$$p_0(x|x_0) = \delta(x - x_0).$$

If one introduces the generating function

$$\tilde{P}_t(x|x_0) = \sum_n e^\lambda (x-x_0+n) p(x+n|x_0)$$

it satisfies

https://doi.org/10.1088/1742-5468/aa7a7e
Large-deviation theory for a Brownian particle on a ring: a WKB approach

\[ \frac{d\tilde{P}(x|x_0)}{dt} = \lambda F(x) \tilde{P}(x|x_0) - \frac{d[F(x)\tilde{P}(x|x_0)]}{dx} \]
\[ + \frac{\epsilon}{2} \left( \lambda^2 \tilde{P}(x|x_0) - 2\lambda \frac{d\tilde{P}(x|x_0)}{dx} + \frac{d^2\tilde{P}(x|x_0)}{dx^2} \right) \]  \hspace{1cm} (A.9)

and

\[ \frac{d\tilde{P}(x|x_0)}{dt} = \lambda F(x) \tilde{P}(x|x_0) + F(x_0) \frac{d\tilde{P}(x|x_0)}{dx_0} \]
\[ + \frac{\epsilon}{2} \left( \lambda^2 \tilde{P}(x|x_0) + 2\lambda \frac{d\tilde{P}(x|x_0)}{dx_0} + \frac{d^2\tilde{P}(x|x_0)}{dx_0^2} \right) \]  \hspace{1cm} (A.10)

with the initial condition

\[ \tilde{P}_0(x|x_0) = \delta(x - x_0). \]  \hspace{1cm} (A.11)

In the long time limit

\[ \tilde{P}_0(x|x_0) \sim e^{\mu(\lambda)t} r(x) \ell(x_0) \]  \hspace{1cm} (A.12)

\( r(x) \) and \( \ell(x) \) are the right and left eigenfunctions solution of the eigenvalue problem

\[ \mu(\lambda) r(x) = \lambda F(x) r(x) - \frac{d[F(x)r(x)]}{dx} \]
\[ + \frac{\epsilon}{2} \left( \lambda^2 r(x) - 2\lambda \frac{dr(x)}{dx} + \frac{d^2r(x)}{dx^2} \right) \]  \hspace{1cm} (A.13)

\[ \mu(\lambda) \ell(x) = \lambda F(x) \ell(x) + F(x) \frac{d\ell(x)}{dx} + \frac{\epsilon}{2} \left( \lambda^2 \ell(x) + 2\lambda \frac{d\ell(x)}{dx} + \frac{d^2\ell(x)}{dx^2} \right) \]  \hspace{1cm} (A.14)

**Appendix B. The matching of the asymptotics**

In this appendix, we analyse the situation (51) and we derive connection formulas between the expressions of the solution \( s(x) \) in the various regions.

- In Region I \((0 < x < x_0)\) one can write the solution \( s(x) \) of equation (19) as (see equations (44) and (45))

\[ s_1(x) = c_1 \sqrt{\frac{F(x)}{\sqrt{F(x)^2 - F_0^2}}} + 1 \exp \left[ \int_0^x dy \left( \frac{F(y) + \sqrt{F(y)^2 - F_0^2}}{\epsilon} + \frac{(\nu - \frac{1}{2}) \sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}} \right) \right] \]
\[ + c_2 \sqrt{\frac{F(x)}{\sqrt{F(x)^2 - F_0^2}} - 1} \exp \left[ \int_0^x dy \left( \frac{F(y) - \sqrt{F(y)^2 - F_0^2}}{\epsilon} - \frac{(\nu - \frac{1}{2}) \sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}} \right) \right]. \] \hspace{1cm} (B.1)

For \( x \to x_0 \) in this region I this leads to the following asymptotics
Large-deviation theory for a Brownian particle on a ring: a WKB approach

\[ s_1(x) \simeq \left( \frac{F_0}{2F_1} \right)^{\frac{1}{2}} \left( c_1 (x_0 - x)^{-\nu} \exp \left[ D_1 - \frac{F_0(x_0 - x)}{\epsilon} - \sqrt{\frac{F_0F_1(x_0 - x)^2}{2 \epsilon}} \right] + c_2 (x_0 - x)^{\nu-1} \exp \left[ D_2 - \frac{F_0(x_0 - x)}{\epsilon} + \sqrt{\frac{F_0F_1(x_0 - x)^2}{2 \epsilon}} \right] \right) \]

where

\[ D_1 = (\nu - \frac{1}{2}) \log x_0 + \int_0^{x_0} dy \left( \frac{F(y) + \sqrt{F(y)^2 - F_0^2}}{\epsilon} + \frac{\left( \nu - \frac{1}{2} \right) \sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}} - \frac{\nu - \frac{1}{2}}{x_0 - y} \right) \]

and

\[ D_2 = - \left( \nu - \frac{1}{2} \right) \log x_0 + \int_0^{x_0} dy \left( \frac{F(y) - \sqrt{F(y)^2 - F_0^2}}{\epsilon} - \frac{\left( \nu - \frac{1}{2} \right) \sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}} + \frac{\nu - \frac{1}{2}}{x_0 - y} \right). \]

• Similarly in region III \((x_0 < x < 1)\)

\[ s_{III}(x) = c_3 \sqrt{\frac{F(x)}{\sqrt{F(x)^2 - F_0^2}}} + 1 \exp \left[ - \int_x^1 dy \left( \frac{F(y) + \sqrt{F(y)^2 - F_0^2}}{\epsilon} + \frac{\left( \nu - \frac{1}{2} \right) \sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}} \right) \right] \]

\[ + c_4 \sqrt{\frac{F(x)}{\sqrt{F(x)^2 - F_0^2}}} - 1 \exp \left[ - \int_x^1 dy \left( \frac{F(y) - \sqrt{F(y)^2 - F_0^2}}{\epsilon} - \frac{\left( \nu - \frac{1}{2} \right) \sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}} + \frac{\nu - \frac{1}{2}}{x_0 - y} \right) \right]. \]

which gives as \(x \to x_0\)

\[ s_{III}(x) \simeq \left( \frac{F_0}{2F_1} \right)^{\frac{1}{2}} \left( c_3 (x - x_0)^{-\nu} \exp \left[ D_3 + \frac{F_0(x - x_0)}{\epsilon} + \sqrt{\frac{F_0F_1(x_0 - x)^2}{2 \epsilon}} \right] + c_4 (x_0 - x)^{\nu-1} \exp \left[ D_4 + \frac{F_0(x - x_0)}{\epsilon} - \sqrt{\frac{F_0F_1(x_0 - x)^2}{2 \epsilon}} \right] \right) \]

where

\[ D_3 = - \left( \nu - \frac{1}{2} \right) \log(1 - x_0) - \int_{x_0}^1 dy \left( \frac{F(y) + \sqrt{F(y)^2 - F_0^2}}{\epsilon} + \frac{\left( \nu - \frac{1}{2} \right) \sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}} - \frac{\nu - \frac{1}{2}}{y - x_0} \right) \]

and

\[ D_4 = \left( \nu - \frac{1}{2} \right) \log(1 - x_0) - \int_{x_0}^1 dy \left( \frac{F(y) - \sqrt{F(y)^2 - F_0^2}}{\epsilon} - \frac{\left( \nu - \frac{1}{2} \right) \sqrt{2F_0F_1}}{\sqrt{F(y)^2 - F_0^2}} + \frac{\nu - \frac{1}{2}}{y - x_0} \right). \]

• Finally in region II \((x - x_0 = O(\sqrt{\epsilon}))\) the solution is of the form equation (52) with the following asymptotics (see equations (C.2) and (C.3)):

\[ \text{for } (x - x_0)/\sqrt{\epsilon} \to -\infty \]

https://doi.org/10.1088/1742-5468/aa7a8e
Large-deviation theory for a Brownian particle on a ring: a WKB approach

\[ s_{II} \approx \exp \left[ \frac{F_0(x - x_0)}{\epsilon} \right] \left( V \left( \frac{2F_0 F_1}{\epsilon^2} \right)^{\frac{\nu-1}{2}} (x_0 - x)^{\nu-1} \exp \left[ \sqrt{\frac{F_0 F_1 (x - x_0)^2}{2 \epsilon}} \right] \right. \\
+ W \left( \frac{2F_0 F_1}{\epsilon^2} \right)^{\frac{-\nu}{2}} (x_0 - x)^{-\nu} \exp \left[ -\sqrt{\frac{F_0 F_1 (x - x_0)^2}{2 \epsilon}} \right] \right) \\
\text{and for } (x - x_0)/\sqrt{\epsilon} \to +\infty \\
\[ s_{II} \approx \exp \left[ \frac{F_0(x - x_0)}{\epsilon} \right] \left( V \left( \frac{2F_0 F_1}{\epsilon^2} \right)^{\frac{\nu-1}{2}} (x_0 - x)^{\nu-1} \exp \left[ \sqrt{\frac{F_0 F_1 (x - x_0)^2}{2 \epsilon}} \right] \right. \\
+ W'(x - x_0)^{-\nu} \left( \frac{2F_0 F_1}{\epsilon^2} \right)^{\frac{-\nu}{2}} \exp \left[ -\sqrt{\frac{F_0 F_1 (x - x_0)^2}{2 \epsilon}} \right] \right). \] (B.9)

Now using the boundary condition (20) one has

\[ c_3 = c_1 e^{-\lambda} ; \quad c_4 = c_2 e^{-\lambda} \] (B.11)

and matching the asymptotics, on the one hand equations (B.2) and (B.9) and on the other hand equations (B.6) and (B.10), one gets using equation (C.4) that \( \lambda \) should satisfy

\[ e^{2\lambda} - e^{\lambda} \left( \frac{X(\nu)}{\Gamma(\nu)} e^{D_3-D_2} + \frac{\Gamma(\nu)(1-Z^2)}{X(\nu)} e^{D_4-D_1} \right) + e^{D_4+D_3-D_1-D_2} = 0 \] (B.12)

where

\[ X(\nu) = 2^\nu \sqrt{\pi} \left( \frac{2F_0 F_1}{\epsilon^2} \right)^{\frac{\nu-1}{2}}. \] (B.13)

For \( \epsilon \) small, one has \( D_1 - D_2 \gg D_3 - D_1 \). This, combined with \( Z = 1 + O(\nu) \) and \( \nu \ll \epsilon^{-1} \) one can see that the term containing \( Z \) becomes negligible over the entire range the range (36), i.e.

\[ D_3 - D_1 < \lambda < D_4 - D_2. \]

This simplifies the above equation to

\[ \Gamma(\nu) = \frac{X(\nu)}{e^{\lambda-D_4+D_2} + e^{-\lambda+D_3-D_1}}. \] (B.14)

Generally, \( \lambda - D_4 + D_2 \) and \( -\lambda + D_3 - D_1 \) are of the order \( \epsilon^{-1} \), and in this regime the above equation can only be satisfied for \( \nu \ll 1 \), leading to

\[ \nu = \frac{(2F_0 F_1)^{\frac{1}{2}}}{\sqrt{\pi \epsilon}} (e^{\lambda-D_4+D_2} + e^{-\lambda+D_3-D_1}) \]. (B.15)

One can see that for \( \lambda \approx D_3 - D_1 \) or \( \lambda \approx D_4 - D_2 \) this simplification does no longer hold. In these regimes, one gets

\[ -\lambda + D_3 - D_1 = \ln \left( \frac{X(\nu)}{\Gamma(\nu)} \right) \] (B.16)
and
\[
\lambda - D_4 + D_2 = \ln \left( \frac{X(\nu)}{\Gamma(\nu)} \right),
\]  \tag{B.17}
respectively. For $\nu \gg 1$ this simplifies to
\[
\lambda + \int_0^1 dy F(y) + \frac{\sqrt{F(y)^2 - F_0^2}}{\epsilon} = \nu \log (\nu \epsilon)
\]  \tag{B.18}
and
\[
\lambda - \int_0^1 dy F(y) - \frac{\sqrt{F(y)^2 - F_0^2}}{\epsilon} = -\nu \log (\nu \epsilon).
\]  \tag{B.19}
This result can be verified by taking the limit to the boundary of equations (46), which leads to exactly the same result \[25\].

**Appendix C. On the asymptotics of the solution of equation (53)**

In this appendix we discuss some aspects of the connection formula of the asymptotics at $z \to +\infty$ and at $z \to -\infty$ of a solution $G$ of
\[
\nu G = \frac{d}{dz}(z G) + \frac{1}{2} \frac{d^2 G}{dz^2}.
\]  \tag{C.1}

For large $z$ one expects either $G \sim z^{\nu-1}$ or $G \sim e^{-z^2}z^{-\nu}$ and we want to relate the pair $V, W$ to the pair $V’, W’$ which characterizes the asymptotics at $\pm \infty$
\[
G \simeq V(-z)^{\nu-1}\left(1 + \frac{(\nu - 1)(\nu - 2)}{4 z^2} + \cdots\right) + W \frac{e^{-z^2}}{(-z)^\nu}\left(1 - \frac{\nu(\nu + 1)}{4 z^2} + \cdots\right) \quad \text{as} \quad z \to -\infty
\]  \tag{C.2}
and
\[
G \simeq V’z^{\nu-1}\left(1 + \frac{(\nu - 1)(\nu - 2)}{4 z^2} + \cdots\right) + W’ \frac{e^{-z^2}}{z^\nu}\left(1 - \frac{\nu(\nu + 1)}{4 z^2} + \cdots\right) \quad \text{as} \quad z \to +\infty.
\]  \tag{C.3}

The goal of this appendix is to show that
\[
V’ = -ZV + \frac{2^{\nu-1}\sqrt{\pi}}{\Gamma(\nu)}W \quad ; \quad W’ = \frac{(1 - Z^2)}{2^{\nu-1}\sqrt{\pi}}V + Z W
\]  \tag{C.4}
where
\[
Z = \cos(\pi \nu).
\]  \tag{C.5}

By expanding around $z = 0$, a general solution of equation (C.1) can be written as
\[
G = g G_3 + g’ G_4
\]  \tag{C.6}
where $G_3$ and $G_4$ are the even and the odd solutions
\[
G_3 = \sum_{n \geq 0} (-)^n z^{2n} \frac{\Gamma(2n - \nu)\Gamma(-\frac{\nu}{2})}{\Gamma(\nu)\Gamma(n - \frac{\nu}{2})(2n)!} = 1 + (\nu - 1)z^2 + \frac{(\nu - 1)(\nu - 3)}{6}z^4 + \cdots
\]
Large-deviation theory for a Brownian particle on a ring: a WKB approach

\[ G_4 = \sum_{n \geq 0} (-)^n z^{2n+1} \frac{2^{2n} \Gamma(n + 1 - \frac{\nu}{2})}{\Gamma(1 - \frac{\nu}{2})(2n + 1)!} = z + \frac{\nu - 2}{3} z^3 + \frac{(\nu - 2)(\nu - 4)}{30} z^5 + \cdots. \]

If one defines (assuming that \( \nu \) is not an integer or half an integer)

\[ G_1 = \int_{-\infty+it}^{\infty} e^{-z^2+tz-t^2} \, dt \quad \text{and} \quad G_2 = \int_{-\infty-it}^{\infty} e^{-z^2+tz-t^2} \, dt \]

one has

\[ g_1 = \Gamma \left( \frac{\nu}{2} \right) (1 - e^{i\pi \nu}) 2^{\nu-1} \quad ; \quad g_1' = \Gamma \left( \frac{\nu+1}{2} \right) (1 + e^{i\pi \nu}) 2^{\nu} \]

and

\[ g_2 = \Gamma \left( \frac{\nu}{2} \right) (1 - e^{-i\pi \nu}) 2^{\nu-1} \quad ; \quad g_2' = \Gamma \left( \frac{\nu+1}{2} \right) (1 + e^{-i\pi \nu}) 2^{\nu}. \]

Therefore

\[ G_3 = \frac{1}{2^{\nu} \Gamma \left( \frac{\nu}{2} \right) (1 - e^{i\pi \nu})} \left( G_1 - e^{i\pi \nu} G_2 \right) \quad \text{(C.7)} \]

\[ G_4 = \frac{1}{2^{\nu+1} \Gamma \left( \frac{\nu+1}{2} \right) (1 + e^{i\pi \nu})} \left( G_1 + e^{i\pi \nu} G_2 \right). \quad \text{(C.8)} \]

Because

\[ G_2 - G_1 = \int_{-\infty+it}^{\infty} e^{-z^2+tz-t^2} \, dt \quad \text{(C.9)} \]

and because this integral is dominated for large \( z \) by the neighborhood of \( t = 0 \) one has the following asymptotics for \( z \to +\infty \)

\[ G_2 - G_1 \sim (e^{i\pi \nu} - e^{-i\pi \nu}) e^{-z^2} \left( \frac{\Gamma(\nu)}{z^\nu} - \frac{\Gamma(\nu+2)}{4 z^{\nu+2}} + \cdots \right). \quad \text{(C.10)} \]

On the other hand for large positive \( z \) a saddle point calculation leads to

\[ G_1 \sim G_2 \sim 2^{\nu} \sqrt{\pi} z^{\nu-1} \left( 1 + \frac{(\nu - 1)(\nu - 2)}{4 z^2} + \cdots \right). \]

Then from equations (C.7) and (C.8) one gets for large positive \( z \)

\[ G_3 \simeq \frac{\sqrt{\pi}}{\Gamma \left( \frac{\nu}{2} \right)} z^{\nu-1} \quad ; \quad G_4 \simeq -\frac{\sqrt{\pi}}{2 \Gamma \left( \frac{\nu+1}{2} \right)} z^{\nu-1} \quad \text{(C.11)} \]

and from equation (C.9)
\[ \Gamma \left( \frac{\nu}{2} \right) 2^{\nu-1} G_3 - 2^\nu \Gamma \left( \frac{\nu+1}{2} \right) G_4 \simeq \Gamma(\nu) \frac{e^{-z^2}}{z^\nu}. \]

So if one postulates that for \( z \to +\infty \)

\[ G_3 = \frac{\sqrt{\pi}}{\Gamma(\frac{\nu}{2})} \left[ z^{\nu-1} \left( 1 + \frac{(\nu - 1)(\nu - 2)}{4z^2} \right) + \frac{\beta}{2^\nu-1} \frac{e^{-z^2}}{z^\nu} (1 + \ldots) \right] \quad (C.12) \]

\[ G_4 = \frac{\sqrt{\pi}}{2\Gamma(\frac{\nu+1}{2})} \left[ z^{\nu-1} \left( 1 + \frac{(\nu - 1)(\nu - 2)}{4z^2} \right) + \frac{\gamma}{2^\nu-1} \frac{e^{-z^2}}{z^\nu} (1 + \ldots) \right] \quad (C.13) \]

one should have

\[ \beta - \gamma = \frac{1}{\sqrt{\pi}} \Gamma(\nu). \quad (C.14) \]

In the above expressions \( \beta \) and \( \gamma \) are factors of subdominant terms and they are \textit{a priori} ill defined unless one specifies how the dominant divergent series is resummed.

A general solution of equation (C.1) can always be written as

\[ G = xG_3 + yG_4. \]

Then one has (see equations (C.12) and (C.13))

\[ V = \frac{\sqrt{\pi}}{\Gamma(\frac{\nu}{2})} x - \frac{\sqrt{\pi}}{2\Gamma(\frac{\nu+1}{2})} y \quad ; \quad W = \frac{2\sqrt{\pi}}{2^\nu \Gamma(\frac{\nu}{2})} x \beta - \frac{2\sqrt{\pi}}{2^\nu+1 \Gamma(\frac{\nu+1}{2})} y \gamma \]

\[ V' = \frac{\sqrt{\pi}}{\Gamma(\frac{\nu}{2})} x + \frac{\sqrt{\pi}}{2\Gamma(\frac{\nu+1}{2})} y \quad ; \quad W' = \frac{2\sqrt{\pi}}{2^\nu \Gamma(\frac{\nu}{2})} x \beta + \frac{2\sqrt{\pi}}{2^\nu+1 \Gamma(\frac{\nu+1}{2})} y \gamma. \]

Eliminating \( x \) and \( y \) one gets equation (C.4) where

\[ Z = \frac{\sqrt{\pi}}{\Gamma(\mu)} (\beta + \gamma). \quad (C.15) \]

So far \( Z \) is undetermined, and as mentionned earlier it depends on the way the dominant contribution is resummed in equations (C.12) and (C.13). This is related to Stokes phenomenon [38].

As for real positive \( z \) the solutions \( G_1 \) and \( G_2 \) are complex conjugates one can consider that their real part is by definition the resummed dominant contribution of the large \( z \) asymptotics. This implies (see equation (C.10))

\[ G_2 \simeq 2\sqrt{\pi} \frac{\Gamma(\nu)}{2} \left( e^{i\pi \mu} - e^{-i\pi \nu} \right) \frac{e^{-z^2}}{z^\nu} \]

\[ G_1 \simeq 2\sqrt{\pi} \frac{\Gamma(\nu)}{2} \left( e^{i\pi \nu} - e^{-i\pi \nu} \right) \frac{e^{-z^2}}{z^\nu}. \]

This gives equations (C.7), (C.8), (C.12) and (C.13)
Large-deviation theory for a Brownian particle on a ring: a WKB approach

\[ \beta = \frac{\Gamma(\nu)\left(1 + \cos(\pi \nu)\right)}{2\sqrt{\pi}} \quad \text{and} \quad \gamma = \frac{\Gamma(\nu)\left(-1 + \cos(\pi \nu)\right)}{2\sqrt{\pi}} \]

so that (see equation (C.15))

\[ Z = \cos(\pi \nu) \]

as in equation (C.5).

References

[1] Donsker F D and Varadhan S S 1975 Commun. Pure Appl. Math. 28 1–47
[2] Ellis R S 1988 Ann. Probab. 16 1496–1508
[3] Den Hollander F 2008 Large Deviations vol 14 (Providence, RI: American Mathematical Society)
[4] Derrida B 2007 J. Stat. Mech. P07023
[5] Touchette H 2009 Phys. Rep. 478 1–69
[6] Bertini L, De Sole A, Gabrielli D, Jona-Lasinio G and Landim C 2015 Rev. Mod. Phys. 87 593
[7] Lebowitz J L and Spohn H 1999 J. Stat. Phys. 95 333–65
[8] Derrida B 1983 J. Stat. Phys. 31 333–50
[9] Faucheux I P, Stolovitzky G and Libchaber A 1995 Phys. Rev. E 51 5239
[10] Speck T, Blickle V, Bechinger C and Seifert U 2007 Europhys. Lett. 79 30002
[11] Maes C, Netočný K and Wynants B 2008 Physica A 387 2675–89
[12] Chernyak Y V, Chertkov M, Malinin S V and Teodorescu R 2009 J. Stat. Phys. 137 109
[13] Masaharu S 2018 Physica A 501 126–33
[14] Mehl J, Speck T and Seifert U 2008 Phys. Rev. E 78 011123
[15] Lacoste D and Mallick K 2009 Phys. Rev. E 80 021923
[16] Nemoto T and Sasa S I 2011 Phys. Rev. E 83 030105
[17] Chetrite R and Touchette H 2015 J. Stat. Mech. P12001
[18] Nyawo P T and Touchette H 2016 Phys. Rev. E 94 032101
[19] Saito K and Dhar A 2016 Europhys. Lett. 114 50004
[20] Fischer L P, Pietzonka P and Seifert U 2018 Phys. Rev. E 97 022143
[21] Freidlin M I and Wentzell A D 1994 Random Perturbations of Hamiltonian Systems vol 523 (Providence, RI: American Mathematical Society)
[22] Speck T, Engel A and Seifert U 2012 J. Stat. Mech. P12001
[23] Faggionato A et al 2012 A representation formula for large deviations rate functionals of invariant measures on the one dimensional torus Ann. l’Inst. Henri Poincaré 48 212–34
[24] Bouchet F and Reyner J 2016 Generalisation of the Eyring–Kramers transition rate formula to irreversible diffusion processes Ann. Henri Poincaré 17 3499–532
[25] Tizón-Escamilla N, Lecomte V and Bérzin E 2019 J. Stat. Mech. 013201
[26] Graham R 1987 Macroscopic potentials, bifurcations and noise in dissipative systems Fluctuations and Stochastic Phenomena in Condensed Matter (Berlin: Springer) pp 1–34
[27] Baek Y and Kafri Y 2015 J. Stat. Mech. P08026
[28] Lecomte V and Tailleur J 2007 J. Stat. Mech. P03004
[29] Ray U, Chan G K I and Limmer D T 2018 Phys. Rev. Lett. 120 210602
[30] Baiés M, Maes C and Netočný K 2009 J. Stat. Phys. 135 57–75
[31] Risken H 1996 Fokker–Planck equation The Fokker–Planck Equation (Berlin: Springer) pp 63–95
[32] Graham R 1995 Fluctuations in the steady state 25 Years of Non-Equilibrium Statistical Mechanics (Berlin: Springer) pp 125–34
[33] Touchette H 2018 Physica A 504 5–19
[34] Derrida B and Sadhu T 2018 preprint (arXiv:1807.06543)
[35] Jack R I and Sollich P 2010 Prog. Theor. Phys. Suppl. 184 304–17
[36] Chetrite R and Touchette H 2015 Nonequilibrium markov processes conditioned on large deviations Ann. Henri Poincaré 16 2005–57
[37] Gallavotti G and Cohen E G D 1995 Phys. Rev. Lett. 74 2694
[38] Temme N M 2015 Asymptotic Methods for Integrals (Singapore: World Scientific)

https://doi.org/10.1088/1742-5468/aa7a7e