Generalized Painlevé-Gullstrand descriptions of Kerr-Newman black holes

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Abstract. Generalized Painlevé-Gullstrand coordinates for stationary axisymmetric spacetimes are constructed explicitly; and the results are applied to the Kerr-Newman family of rotating black hole solutions with, in general, non-vanishing cosmological constant. Our generalization is also free of coordinate singularities at the horizon(s); but unlike the Doran metric it contains one extra function which is needed to ensure all variables of the metric remain real for all values of the mass, charge, angular momentum and cosmological constant.

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1. Overview

The Doran metric[1] is another coordinatization of the Kerr-Newman solution[2, 3] which describes a charged rotating black hole. The Doran metric can be considered to be the extension of the Painlevé-Gullstrand(PG)[4, 5] description of a black hole from spherically symmetric to axisymmetric spacetime. These descriptions have the advantage of being free from coordinate singularities at the horizon(s). The Kerr solution, which was discovered much later, is not a straightforward generalization of the Schwarzschild solution. Similarly the explicit Doran metric is a comparatively recent description, but it has found its way into several investigations of black hole physics. Constant-time Doran slicings of the ergosurface in non-extremal black holes have been demonstrated to be free of conical singularities at the poles[6]. Calculations of Hawking radiation[7], and also neutrino asymmetry due to the interaction of fermions and rotating black holes[8] have also made explicit use of the Doran metric. Other authors have proposed to utilize the Doran metric to extend or generalize spherically symmetric results to the context of rotating black holes[9, 10, 11].

Recently it has been demonstrated[12] that there is an obstruction to the implementation of flat PG slicings for spherically symmetric spacetimes; and the insistence on spatial flatness can lead to PG metrics which involve complex metric variables. The corresponding vierbein fields are then not related to those of the standard spherically symmetric metric by physical Lorentz boosts. Since the Doran metric contains spherically symmetric PG solution as a special case, it will be afflicted with the same problems (this will be explicitly illustrated later on). In discussions of black hole evaporation using the Parikh-Wilczek method[13], the insistence on spatially flat PG coordinates can give rise to spurious contributions which are ambiguous and problematic both to the computation of the tunneling rate and to the universality of the results. In a more general context, the appearance of complex quantities in the Doran metric causes unnecessary complications, and gives rise to difficulties and ambiguities in the physical interpretations. These problems can be avoided altogether by using a less restrictive form of constant-time slicing which generalizes the Doran metric. In this brief note we demonstrate how this goal can be realized explicitly. Although it is possible to introduce many parameters through the freedom of local Lorentz transformations which relate vierbein one-forms with the same metric, our generalized PG description is “optimal” in that only one additional function, $A(r, \theta)$, is needed, and introduced, to avoid all the troubles.

Throughout this short note geometric units $G = c = 1$ and the $(-+++) \text{ convention}$ for the spacetime signature are adopted.

2. Generalized Painleve-Gullstrand metrics for stationary axisymmetric spacetimes

We seek a generalization of the Doran metric by first assuming that the vierbein one-forms, $e^A_{GD}$ ($A = 0, 1, 2, 3$), can be expressed as

$$e^A_{GD} = \{ A(dt + \delta_t d\theta), Bdr + C(dt + \delta_t d\theta) + D(d\phi + \delta_\phi d\theta), \rho d\theta, E(d\phi + \delta_\phi d\theta) \}, \quad (1)$$
with the PG coordinates defined as
\[
\begin{align*}
dt_p &= dt + Jdr + \delta_t d\theta \\
d\phi_p &= d\phi + Kdr + \delta_\phi d\theta.
\end{align*}
\]
\(dt_p\) and \(d\phi_p\) being exact differentials require \(\delta_t = \int \frac{\partial J}{\partial r} dr\) and \(\delta_\phi = \int \frac{\partial K}{\partial r} dr\). The most general stationary axisymmetric metric is defined on five arbitrary functions of \(r\) and \(\theta\), and it has been discussed by Chandrasekhar [14] [15]. The metric is compatible with the vierbein
\[
e_{ax}^A = \{ A_s dt, \ B_s^{-1} dr, \ \rho d\theta, \ C_s (d\phi - \Omega dt) \}.
\]
To produce the same metric, \(e_{GD}^A\) and \(e_{ax}^A\) must be related by a Lorentz transformation. This can be realized by requiring \(A^A_B = \{ e_{GD}^A \}^{-1} \) to satisfy \(A^T \eta A = \eta\). This procedure fixes the functions \(B, C, D, E, J, K\) in terms of the variables in \(e_{ax}^A\) as
\[
\begin{align*}
J &= \pm \sqrt{A^2 - A_s^2}, \\
B &= \frac{C}{\sqrt{A^2 - A_s^2}}, \\
K &= \frac{\sqrt{A^2 - A_s^2}}{A_s B}, \\
C &= \sqrt{A^2 - A_s^2 + C_s^2 A_s^2}, \\
D &= \frac{C_s \Omega}{A_s^2 B}, \\
E &= \frac{A_s C}{A_s B}. \tag{3}
\end{align*}
\]
However the function \(A(r, \theta)\) is so far still unrestricted and completely free. In fact it expresses the freedom in the choice of the local Lorentz frame, as can be seen from the explicit relation
\[
e_{GD} = \frac{A_s B_s A^A J}{0} \frac{B_s A^A J}{0} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot e_{ax}. \tag{4}
\]
The choice of the prefix \(\pm\) sign of \(J\) in (3) determines the definition of \(dt_p\) introduced earlier. The main result is the generalization of the Doran metric for stationary axisymmetric spacetimes can thus be expressed as
\[
ds^2 = \eta_{AB} e_{GD}^A e_{GD}^B = -A_s^2 dt_p^2 + \left[ \frac{A_s C}{A_s B_s \sqrt{A^2 - A_s^2}} dr + C \left( dt_p - \frac{C_s^2 \Omega}{C_s^2} d\phi_p \right) \right]^2 + \rho^2 d\theta^2 + E_s^2 d\phi_p^2. \tag{5}
\]
Moreover, all the variables in the metric are explicitly real provided
\[
A^2 > A_s^2. \tag{6}
\]
Referring to our earlier observation of the relation between the vierbein one-forms [4], this criterion is also precisely the condition which ensures \(e_{GD}^A\) and \(e_{ax}^A\) are related by a physical Lorentz transformation. As long as \(\frac{\Omega}{\rho}\) remains finite, the metric is regular (this occurs for example at the horizon(s) of Kerr-Newman solutions at which both \(A_s\) and \(B_s\) approach zero). Note that the generalized PG coordinates discussed in this section are valid for stationary axisymmetric spacetimes, regardless of whether or not they are solutions of Einstein’s equations.

3. Generalized Doran metric and Kerr-Newman solutions

We now restrict our attention to the Kerr-Newman family of black hole solutions of Einstein’s equations with angular momentum \(aM\) and charge \(Q\). Expressed in the Boyer-Lindquist coordinates [8]
\[
ds^2 = \frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dt^2 - 2a \frac{\Delta}{\rho^2} (R^2 - \Delta) \sin^2 \theta dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \Sigma^2 \sin^2 \theta d\phi^2, \tag{7}
\]

with \(R^2 = r^2 + a^2, \rho^2 = r^2 + a^2 \cos^2 \theta, \Delta = R^2 - 2Mr + Q^2\) and \(\Sigma^2 \rho^2 = R^4 - a^2 \Delta \sin^2 \theta\). In fact our discussion can also be applied to Kerr-Newman metrics with non-vanishing cosmological constant [6]. The vierbein can be assumed to be
\[
e_{BL}^A = \left\{ \frac{\Delta}{\rho} (dt - a \sin^2 \theta d\phi), \ \frac{\rho}{\sqrt{\Delta}} dr, \ \rho d\theta, \ \frac{\sin \theta}{\rho} (R^2 d\phi - a dt) \right\}. \tag{9}
\]

\[\text{† See, for instance, derivations in Refs. [16] [17] based on the requirement of separable Hamilton-Jacobi equation for the geodesics; or for instance Ref [18] for a heuristic derivation based on Lorentz boosts of Minkowski spacetime expressed in the spheroidal coordinates.}

\[\text{§ See, for instance, Ref [19] in the extension to spacetimes with non-vanishing cosmological constant \(\Lambda\).}
\]

\[
ds^2 = -\frac{\Delta \rho^2 \Xi_0^2}{\Xi^2 (R^4 \Xi_0^2 - a^2 \Delta \sin^2 \theta)} dt^2 + \frac{\rho^2}{\Delta} dr^2 + \frac{\rho^2}{\Xi_0^2} d\theta^2 + \frac{R^4 \Xi_0^2 - a^2 \Delta \sin^2 \theta}{\rho^2 \Xi_0^2} \left( d\phi - \frac{a (R^2 \Xi_0^2 - \Delta)}{R^4 \Xi_0^2 - a^2 \Delta \sin^2 \theta} dt \right)^2 \sin^2 \theta. \tag{8}
\]

\[\text{with } \Delta = R^2 (1 - \frac{1}{3} M^2 r) - 2Mr + Q^2, \ \Xi = \left( 1 + \frac{a^2}{3} \right) \text{ and } \Xi_0 = \sqrt{1 + \frac{2a^2}{3} \cos^2 \theta}.\]
Matching with (3), we identify
\[ A_s = \frac{\sqrt{a}}{\rho}, \quad B_s = \frac{\sqrt{a}}{\rho}, \quad C_s = \Sigma \sin \theta, \quad \Omega = \frac{a(R^2 - \Delta)}{\Sigma \rho}; \]
consequently
\[ J = \pm \frac{\rho^2 \Theta \Delta}{\Delta}, \quad B = \pm \frac{\rho C}{\Delta \sqrt{4R^2 - \Delta}}, \quad D = -\frac{\Omega^2 \sin^2 \theta}{\rho}, \quad K = \pm \frac{\rho^2 \Theta \Delta}{\Delta \sqrt{4R^2 - \Delta}}, \quad C = \sqrt{A^2 + \Omega^2 \Sigma^2 \sin^2 \theta - \frac{\Delta}{\rho}}. \tag{11} \]
Furthermore it can be shown that
\[ e_{ax} = L_3 (\frac{\Delta a \sin \theta}{R^2}) e_{BL}, \tag{12} \]
wherein \( L_3 \) is a local Lorentz boost in the \( \epsilon^3 \)-direction with rapidity \( \beta \) given by \( \tan \beta = \frac{\Delta a \sin \theta}{R^2} \). Substitution of these explicit values in (10) and (11) into (5) yields the generalized of the Doran metric (which contains the additional function \( A \)) for Kerr-Newman black holes as
\[ ds^2_{GD} = -A^2 dt^2_p + \left( A^2 + \Omega^2 \Sigma^2 \sin^2 \theta - \frac{\Delta}{\Sigma^2} \right) \left[ \frac{\rho dr}{\sqrt{A^2 \Sigma^2 - \Delta}} \left( C \right) \right]^2 \pm \left( dt_p - \frac{a(R^2 - \Delta) \sin^2 \theta}{\rho} \frac{d\phi_P}{A^2 + \Omega^2 \Sigma^2 \sin^2 \theta - \frac{\Delta}{\rho}} \right)^2 \]
\[ + \rho^2 d\theta^2 + \frac{\sin^2 \theta \left( A^2 \Sigma^2 - \Delta \right)}{A^2 + \Omega^2 \Sigma^2 \sin^2 \theta - \frac{\Delta}{\rho}} d\phi_P^2. \tag{13} \]
All the variables in the metric are real iff \( A^2 \Sigma^2 - \Delta = A \rho^{-2} \left[ R^4 - \Delta (R^2 - \rho^2) \right] - \Delta > 0 \), which is the specialization of the condition (6). This can be guaranteed by choosing an appropriate \( A \) which has so far been unrestricted. The metric is also explicitly non-singular at the horizon(s) (\( \Delta = 0 \)). When \( A = 1 \), we recover the Doran metric (11) which is compatible with the vierbein
\[ e_{Do} = \left\{ dt_p, \frac{\rho}{R} dr \pm \frac{\sqrt{R^2 - \Delta}}{\rho} (dt_p - \sin^2 \theta d\phi_P), \rho, R \sin \theta d\phi_P \right\}, \tag{14} \]
\[ e_{Do} = L_3 \left( \frac{\pm a \sin \theta}{R} \right), L_1 \left( \pm \frac{\sqrt{R^2 - \Delta}}{R} \right), e_{BL}; \tag{15} \]
with \( J = R^2 K = \pm \frac{R \sqrt{R^2 - \Delta}}{R} \). However when \( A = 1 \), the criterion (6) becomes \( R^2 - \Delta = Q^2 - 2Mr > 0 \) which does not hold for \( r < Q^2 / 2M \). For the case with non-vanishing cosmological constant in (8),
\[ A_s = \frac{\sqrt{\Delta \rho^2_a}}{\sqrt{R^2 - \Delta}}, \quad B_s = \frac{\sqrt{\Delta}}{\rho^2_a}, \quad C_s = \frac{\sqrt{R^2 - \Delta}}{\rho^2_a} \sin \theta, \quad \Omega = \frac{a(R^2 - \Delta)}{\sqrt{R^2 - \Delta}}, \sin \theta \]
and the criterion \( A^2 - \Delta > 0 \) cannot be guaranteed for arbitrary values of \( r, \theta, M, a, Q, \Lambda \) if \( A \) is set to unity.

For the case of non-rotating black holes with \( a = 0 \), our generalized Doran metric reduces to the form discussed previously in Ref.[12]
\[ ds^2 = -A^2 dt^2_p + \left( A^{-1} dr \pm \sqrt{A^2 - f dt^2_P} \right)^2 + r^2 d\Omega^2; \]
wherein constant-\( t_P \) 3-geometries are fully characterized by the eigenvalues of the Ricci tensor \( R^i_j \) which can be shown to be repeated \( \frac{1}{\rho^2} \left( 1 - A^2 - \frac{1}{2} \rho \partial_r, A^2 \right) \) and \( \frac{1}{2} \partial_r, A^2 \). These hypersurfaces are conformally flat iff the Cotton–York tensor vanishes[20], which is equivalent to the condition \( 1 - A^2 + r^2 \partial_r, A^2 + r^2 A \Delta \partial_r, A = 0 \). The explicit eigenvalues of constant-\( t_P \) 3-geometries are quite complicated in the more general axisymmetric metrics; but one may resort to the analytic proof that there is no conformally flat slicing for stationary rotating black holes[21]. In the usual PG metric with flat slicing, \( A = 1 \), and criterion (6) does not always hold; but this deficiency can be remedied by appropriate choices of \( A \) which may not in general lead to spatial flatness[12].

4. Remarks

It can be deduced that the elimination of coordinate singularities at the horizon(s) is achieved in generalized PG coordinates through Lorentz boosts with infinite rapidity at the horizon(s). This can be verified explicitly from the identities (4), (12) and (15) which relate the vierbein one-forms of the original metric with coordinate singularities to those of the generalized PG metrics. However, how one approaches the infinite boost at the horizon is quite arbitrary, and this freedom is utilized in our generalization which is a minimal extension in that only one extra function \( A \) is introduced to achieve trouble-free real metrics which are explicitly non-singular at the horizon(s).

An alternative to the Doran metric proposed by Natario[22] recently is
\[ e_{Nat}^A = \left\{ dt_p, \frac{\rho}{\Sigma} (dr + \nu dt_P), \rho \delta \theta, (d\phi_P + \delta \theta - \Omega dt_P) \xi \sin \theta \right\}; \]
with $dt_P = dt + \alpha dr$, $d\phi_P = d\phi + \alpha \Omega dr - \delta d\theta$, $v = R\rho^{-2}\sqrt{R^2 - \Delta}$, $\alpha = \frac{R^2}{\Delta}$ and $\delta = -\int \alpha \frac{\partial \Omega}{\partial \rho} dr$. The Lorentz transformation relating it to the Kerr-Newman metric in Boyer-Lindquist coordinates is given by

$$c_{\text{Nat}} = L_1 \left( \frac{R\sqrt{R^2 - \Delta}}{\rho\Sigma} \right) \cdot L_3 \left( \frac{\alpha \sin \theta}{R^2} \right) \cdot e_{\text{BL}}.$$ 

However, without the benefit of our extra function $A$ in a generalized metric, the problem with this scheme is again the condition $R^2 - \Delta > 0$ for physical Lorentz boosts and real metric variables cannot always be ensured.

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