Symplectic gyrokinetic Vlasov-Maxwell theory

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A new representation of electromagnetic gyrokinetic Vlasov-Maxwell theory is presented in which the gyrocenter symplectic structure contains the electric and magnetic field perturbations needed to yield the standard gyrocenter polarization and magnetization terms appearing in the gyrokinetic Maxwell equations, without the need of deriving the second-order gyrocenter Hamiltonian required for an energy-consistent standard Hamiltonian gyrokinetic Vlasov-Maxwell theory. The gyrocenter Hamilton equations, which are expressed in terms of a time-dependent gyrocenter Jacobian and a gyrocenter Poisson bracket that contains electromagnetic field perturbations, satisfy the Liouville property exactly. The self-consistent gyrokinetic Vlasov-Maxwell equations are derived from a variational principle, which also yields exact energy-momentum conservation laws (through the Noether method) that are verified explicitly.

I. INTRODUCTION

One of the hallmarks of Hamiltonian gyrokinetic Vlasov-Maxwell theory involves the gyrocenter polarization and magnetization effects appearing in the gyrokinetic Maxwell equations. In standard Hamiltonian gyrokinetic Vlasov-Maxwell theory, the first-order gyrocenter polarization and magnetization are derived by variations of the second-order gyrocenter Hamiltonian with respect to the first-order electric and magnetic fields, respectively. In a self-consistent Hamiltonian theory, the second-order gyrocenter Hamiltonian must also appear in the full gyrokinetic Vlasov equation in which second-order gyrocenter drifts must be retained in order to satisfy exact energy and momentum conservation laws.

The inclusion of a second-order gyrocenter Hamiltonian can be problematic for practical applications in gyrokinetic particle simulations, however, and alternate gyrokinetic models (such as the δf representation) are often preferred (see the recent reviews by Garbet et al. [2] and Krommes [3]). The purpose of the present work is to present a new gyrokinetic model that retains first-order gyrocenter polarization and magnetization effects in the gyrokinetic Maxwell equations without the need of keeping a second-order gyrocenter Hamiltonian.

Before deriving our new gyrokinetic Vlasov-Maxwell model, we introduce a generic gyrocenter extended one-form expressed in extended gyrocenter phase space \( Z^a = (\mathbf{X}, p||, J, \zeta, w, t) \):

\[
\Gamma_{gy} \equiv \left( -A_0 + p||b_0 + \Pi_{gy} \right) \cdot d\mathbf{X} + J d\zeta - w dt,
\]

where the unperturbed magnetic field is \( B_0 = \nabla \times A_0 = B_0 \hat{b}_0 \) and the gyrocenter symplectic momentum \( \Pi_{gy} \), which may contain higher-order guiding-center corrections [2] (i.e., at zeroth order in the gyrocenter analysis), includes first-order terms of the electromagnetic field/potential perturbations that are selected based on specific theoretical or numerical considerations. Here, the gyrocenter phase-space coordinates \( (\mathbf{X}, p||) \) are used to describe the reduced gyrocenter Hamiltonian dynamics, while the canonical pair \((w, t)\) are the energy-time coordinates used because of the explicit time-dependence of the electromagnetic-field perturbations considered in this work. The fast gyromotion is represented by the action-angle coordinates \((J, \zeta)\), where the gyroaction \( J \) is an invariant of the gyrocenter motion since the gyroangle \( \zeta \) is an ignorable coordinate of the reduced gyrocenter Hamiltonian dynamics.

For our present purposes, we will consider a first-order gyrocenter symplectic momentum \( \Pi_{1gy} \) that may depend on any combinations of the following first-order perturbation terms

\[
\begin{cases}
\text{(i)} & e (A_{1gc})/c \\
\text{(ii)} & (E_{1||gc}) \times (e b_0/\Omega) \\
\text{(iii)} & p|| (B_{1||gc})/B_0
\end{cases}
\]

or none at all (in the so-called Hamiltonian representation [3]), where the first-order perpendicular electric and magnetic vector fields \( E_{1||gc} = -\nabla_\perp \Phi_{1gc} \) and \( B_{1||gc} = \nabla_\perp A_{1||gc} \times \hat{b}_0 \) are expressed in terms of the lowest-order guiding-center and gyrokinetic orderings. Here, a perturbed field \( T_{gc}^{-1} f_1 = f_{1gc} = f_1(X + \rho_0, t) \) is transformed into a function on guiding-center phase space with the help of the push-forward Lie transform \( T_{gc}^{-1} = \exp(\rho_0 \cdot \nabla) \), which is expressed in terms of the lowest-order guiding-center transformation [3] (involving the gyroangle-dependent guiding-center gyroradius \( \rho_0 \)) and \((f_{1gc})(X, J)\) denotes the gyroangle-averaged part of \( f_{1gc} \). The parallel-symplectic scenario, where only the parallel component \( A_{1||gc} = b_0 \cdot A_{1gc} \) is kept in scenario (i), was initially considered by Hahm...
et al. [7] and was recently discussed by Brizard [8] within a variational formulation; the polarization-drift scenario (ii) was discussed by Wang & Hahn [9, 10], Leerink, Parra, & Heikkinen [11], and Heikkinen & Nora [12]; and the combined scenarios (i)-(ii) was discussed by Duthoit, Hahn, & Wang [13]. The scenario (iii), which involves adding the magnetic flutter velocity to the gyrocenter symplectic structure, has not yet been discussed in the literature until now. In a recent paper, Burby and Brizard [14] derived a gauge-invariant gyrokinetic theory by using the local minimal-coupling terms $A_1(X) \cdot \mathbf{d}X - \Phi_1(X) \cdot c \, dt$, instead of the non-local terms $(A_{1gc}) \cdot \mathbf{d}X - \langle \Phi_{1gc} \rangle \cdot c \, dt$ used here. Despite this difference, the gyrocenter equations of motion derived in the present paper are gauge-independent in the sense that only perturbed electric and magnetic fields appear explicitly.

The remainder of the paper is organized as follows. In Sec. II, we derive generic non-canonical gyrocenter Hamilton equations of motion based on the symplectic gyrocenter one-form [11]. After introducing the symplectic gyrocenter transformation in Sec. III, we calculate the first-order gyroangle-averaged gyrocenter polarization displacement $(\mathbf{T}_{gy}^{-1}(\mathbf{X} + \rho_0)) - \mathbf{X} = \epsilon \langle \rho_{gy} \rangle$ and use it to find a suitable expression for the first-order gyrocenter symplectic momentum $\Pi_{gy}$ so that gyrocenter polarization now appears solely in the gyrocenter symplectic one-form $\Pi_{gy}$. The gyrocenter Euler-Lagrange and Hamilton equations are also presented. Next, in Sec. IV, the variational derivation of the symplectic gyrokinetic Vlasov-Maxwell equations is presented, while the gyrokinetic energy-momentum conservation laws are presented and proved explicitly in Sec. V. Lastly, our work is summarized in Sec. VI and Bessel-function identities used in Sec. III are derived in App. A.

II. GYROCENTER SYMPLECTIC STRUCTURE

The gyrocenter Hamilton equations of motion $\dot{Z}^a = \{Z^a, \mathcal{H}_{gy}\}_{gy}$ are expressed in terms of the extended gyrocenter Hamiltonian $\mathcal{H}_{gy} \equiv H_{gy} - \omega$ and the extended gyrocenter Poisson bracket $\{ , \}_{gy}$, which are constructed by Lie-transform perturbation methods [1] directly from the gyrocenter phase-space transformation.

In order to construct the gyrocenter Poisson bracket $\{ , \}_{gy}$ from the gyrocenter extended one-form $\Pi_{gy}$, we construct an $8 \times 8$ Lagrange matrix $\omega_{gy}$ from the extended two-form $\omega_{gy} = d\Pi_{gy}$ constructed as the exterior derivative of the gyrocenter extended one-form $\Pi_{gy}$. From this matrix, we find the gyrocenter Jacobian

$$J_{gy} \equiv \sqrt{\det(\omega_{gy})} = \frac{e}{c} b_{gy}^* \cdot B_{gy}^* = \frac{e}{c} B_{\parallel gy} + b_{gy}^* \cdot \nabla \times \left( p_{\parallel} \hat{b}_0 + \Pi_{gy} \right),$$

where we use the definitions

$$b_{gy}^* \equiv \hat{b}_0 + \frac{\partial \Pi_{gy}}{\partial p_{\parallel}} \quad B_{gy}^* \equiv B_0 + (c/e) \nabla \times \left( p_{\parallel} \hat{b}_0 + \Pi_{gy} \right) \quad \left\{ \right\} \quad (4)$$

According to Eq. (2) [see Eq. (11)], the term $\partial \Pi_{gy} / \partial p_{\parallel}$ is perpendicular to $B_0$, so that $\hat{b}_0 \cdot b_{gy}^* = 1$.

We note that the gyrocenter Jacobian (3) is time-dependent, because of the presence of $\Pi_{gy}$, with

$$\frac{\partial J_{gy}}{\partial t} = \frac{\partial}{\partial p_{\parallel}} \left( \frac{\partial \Pi_{gy}}{\partial t} \right) \cdot \frac{e}{c} B_{gy}^* + b_{gy}^* \cdot \nabla \times \frac{\partial \Pi_{gy}}{\partial t} = \frac{\partial}{\partial p_{\parallel}} \left( \frac{\partial \Pi_{gy}}{\partial t} \cdot \frac{e}{c} B_{gy}^* \right) + \nabla \cdot \left( \frac{\partial \Pi_{gy}}{\partial t} \times b_{gy}^* \right), \quad (5)$$

where we used the relation

$$(c/e) \partial B_{gy}^*/\partial p_{\parallel} = \nabla \times b_{gy}^*, \quad (6)$$

which follows from the definitions (4).

A. Symplectic Poisson bracket

Assuming that the gyrocenter Jacobian $J_{gy}$ does not vanish (which is true under most general conditions), the Lagrange matrix $\omega_{gy}$ can be inverted, from which we construct the gyrocenter Poisson matrix $J_{gy}$, whose components $J_{gy}^{ab} = \{Z^a, Z^b\}_{gy}$ define the fundamental gyrocenter Poisson-bracket elements. Hence, according to this inversion procedure (which guarantees the standard properties of Poisson brackets since the condition $\nabla \cdot B_{gy}^* = 0$ is satisfied), we obtain the gyrocenter Poisson bracket

$$\{F, G\}_{gy} = \frac{\partial F}{\partial w} \frac{\partial^* G}{\partial t} - \frac{\partial F}{\partial t} \frac{\partial^* G}{\partial w} + \frac{\partial F}{\partial \zeta} \frac{\partial^* G}{\partial \zeta} - \frac{\partial F}{\partial \zeta} \frac{\partial^* G}{\partial \zeta} + \frac{e}{c J_{gy}} \cdot \left( \nabla^* F \frac{\partial G}{\partial p_{\parallel}} - \frac{\partial F}{\partial p_{\parallel}} \nabla^* G \right) - \frac{b_{gy}^*}{J_{gy}} \cdot \nabla^* F \times \nabla^* G, \quad (7)$$
where the modified spatial gradient and time-derivative operators are
\[ \nabla^* \equiv \nabla - \frac{\partial \Pi_{gy}}{\partial \Gamma_{gy}} \frac{\partial}{\partial \zeta} - \frac{\partial \Pi_{gy}}{\partial w} \] and \( \partial^* \equiv \frac{\partial}{\partial t} + \frac{\Pi_{gy}}{\Gamma_{gy}} \left( \frac{\partial \Pi_{gy}}{\partial t} \times \frac{\partial \Pi_{gy}}{\partial J} \right) \frac{\partial}{\partial \zeta}. \]

Since the gyrocenter Poisson matrix satisfies the Liouville property
\[ \frac{\partial}{\partial Z^a} \left( J_{gy} \{ Z^a, Z^b \}_{gy} \right) = 0, \] the gyrocenter Poisson bracket (7) may also be expressed in phase-space divergence form as
\[ \{F, G\}_{gy} = \frac{1}{J_{gy}} \frac{\partial}{\partial Z^a} \left( J_{gy} F \{ Z^a, G \}_{gy} \right). \] (9)

### B. Symplectic gyrocenter Hamilton equations

Using the Poisson bracket (7), we now write the gyrocenter Hamilton equations of motion \( \dot{Z}^a = \{ Z^a, H_{gy} \}_{gy} \):
\[
\dot{\mathbf{X}} = \frac{\mathbf{b}_{gy}^*}{J_{gy}} \times \left( \nabla H_{gy} + \frac{\partial \Pi_{gy}}{\partial \mathbf{p}} \right) + \frac{e}{c} \frac{\mathbf{B}_{gy}^*}{J_{gy}} \cdot \nabla \left( \frac{\partial H_{gy}}{\partial \mathbf{p}} \right), \] (10)
\[
\dot{\mathbf{p}} = -\frac{e}{c} \frac{\mathbf{B}_{gy}^*}{J_{gy}} \cdot \nabla H_{gy} + \frac{\partial \Pi_{gy}}{\partial t}, \] (11)

where the extended gyrocenter Hamiltonian is defined as
\[ H_{gy} = \frac{p^2}{2m} + J \Omega + e \Psi_{gy} - w \equiv H_{gy} - w. \] (12)

Here, the effective gyroangle-independent potential \( \Psi_{gy} \) depends on the perturbed electrostatic potential \( \Phi_1 \) and may depend on the perturbed vector potential \( \mathbf{A}_1 \) and magnetic field \( \mathbf{B}_1 \), depending on which representation is used. We note that, unless the Hamiltonian representation is chosen (for which \( \Pi_{1gy} = 0 \)), the gyrocenter Hamilton equations (10)-(11) will contain explicit partial time derivatives.

The remaining gyrocenter Hamilton equations are \( J = -\partial H_{gy}/\partial \zeta \equiv 0 \), which immediately follows from the gyroangle-independence of the gyrocenter Hamiltonian (see below), \( \zeta = \partial H_{gy}/\partial J - \dot{\mathbf{X}} \cdot \partial \Pi_{gy}/\partial J, \) \( t = -\partial H_{gy}/\partial w = 1 \), and \( \dot{w} = \partial H_{gy}/\partial t - \dot{\mathbf{X}} \cdot \partial \Pi_{gy}/\partial t \). Lastly, because of the Liouville property (8), the gyrocenter Hamilton equations (10)-(11) satisfy the Liouville Theorem:
\[ \frac{\partial J_{gy}}{\partial t} + \nabla \cdot \left( J_{gy} \dot{\mathbf{X}} \right) + \frac{\partial}{\partial \mathbf{p}} \left( J_{gy} \dot{\mathbf{p}} \right) = 0, \] (13)
where
\[
\nabla \cdot \left( J_{gy} \dot{\mathbf{X}} \right) = \nabla \times \mathbf{b}_{gy}^* \cdot \nabla H_{gy} + \nabla \cdot \left( \mathbf{b}_{gy}^* \times \frac{\partial \Pi_{gy}}{\partial \mathbf{p}} \right) + \frac{e}{c} \mathbf{B}_{gy}^* \cdot \nabla \left( \frac{\partial H_{gy}}{\partial \mathbf{p}} \right),
\]
\[
\frac{\partial}{\partial \mathbf{p}} \left( J_{gy} \dot{\mathbf{p}} \right) = -\frac{e}{c} \frac{\partial \mathbf{B}_{gy}^*}{\partial \mathbf{p}} \cdot \nabla H_{gy} - \frac{e}{c} \mathbf{B}_{gy}^* \cdot \nabla \left( \frac{\partial H_{gy}}{\partial \mathbf{p}} \right) - \frac{\partial}{\partial \mathbf{p}} \left( \frac{e}{c} \mathbf{B}_{gy}^* \cdot \frac{\partial \Pi_{gy}}{\partial \mathbf{p}} \right),
\]

from which we recover Eq. (5) when Eq. (6) is used.

### III. SYMPLLECTIC GYROCENTER PHASE-SPACE TRANSFORMATION

Using Lie-transform perturbation methods, the derivations of the gyrocenter symplectic one-form (1) and the gyrocenter Hamiltonian (12) proceed by a near-identity phase-space transformation (associated with a small parameter \( \epsilon \) that denotes the amplitude of the first-order perturbation fields) from the perturbed symplectic guiding-center one-form
\[ \Gamma_{gc} = \frac{\epsilon}{c} \left[ \mathbf{A}_0^* \cdot d\mathbf{X}_0 + \epsilon \mathbf{A}_{1gc} \cdot d(\mathbf{X}_0 + \mathbf{\rho}_0) \right] + J_0 d\zeta_0 - w_0 dt \equiv \Gamma_{0gc} + \epsilon \mathbf{A}_{1gc} \cdot d(\mathbf{X}_0 + \mathbf{\rho}_0), \] (14)
and the perturbed guiding-center Hamiltonian

\[ H_{gc} = \frac{p_{R0}^2}{2m} + J_0 \Omega_0 + \epsilon \Phi_{1gc} - w_0 \equiv H_{0gc} + \epsilon \Phi_{1gc}, \]  

(15)

where the guiding-center coordinates are \( Z_0^a = (X_0, p_{\parallel 0}, J_0, \zeta_0, w_0, t) \), and the term \((e/c) \Phi_{1}\) includes gyrogauging and higher-order guiding-center corrections \( \Phi_{1} \). We now look for the gyrocenter phase-space coordinates constructed as expansions in powers of \( \epsilon \):

\[ Z^a = Z_0^a + \epsilon G_1^a + \epsilon^2 \left( G_2^a + \frac{1}{2} G_1^b \frac{\partial G_1^a}{\partial Z_0^b} \right) + \cdots, \]  

(16)

where the generating vector field \( G_1^a \) generates the transformation at \( n \)th order. The fact that the dimensionless parameter \( \epsilon \ll 1 \) is small implies that the transformation is a near-identity transformation that is invertible:

\[ Z_0^a = Z^a - \epsilon G_1^a - \epsilon^2 \left( G_2^a - \frac{1}{2} G_1^b \frac{\partial G_1^a}{\partial Z_0^b} \right) + \cdots. \]  

(17)

We note that the Jacobian \( J_{gy} \) can be constructed from the unperturbed (guiding-center) Jacobian \( J_{0gc} \) as \( J_{gy} = J_{0gc} - \epsilon \frac{\partial S}{\partial J_{0gc}} G_1^a \) + ... .

The gyrocenter symplectic one-form \( \Gamma \) is constructed from the perturbed symplectic guiding-center one-form \( \Gamma_{gc} \) by Lie-transform method \( \hat{G} \) :

\[ \Gamma_{gy} = T_{gy}^{-1} \Gamma_{gc} + dS, \]  

(18)

where the gyrocenter push-forward operator \( T_{gy}^{-1} \equiv \cdots \exp(-\epsilon^2 L_2) \exp(-\epsilon L_1) \) is defined in terms of Lie derivatives \( (L_1, L_2, \cdots) \) that are generated by the generating vector fields \( (G_1, G_2, \cdots) \) and the gauge function \( S \equiv S_1 + \epsilon^2 S_2 + \cdots \) represents the generating function for the canonical part of the gyrocenter phase-space transformation.

Once the generating vector fields \( (G_1, G_2, \cdots) \) are obtained from the solution of Eq. \( (18) \), the gauge functions \( (S_1, S_2, \cdots) \) are determined from the solution of the gyrocenter Hamiltonian

\[ H_{gy} = \frac{p_{R0}^2}{2m} + J_0 \Omega_0 + \epsilon \Psi_{gy} - w \equiv T_{gy}^{-1} H_{gc}, \]  

(19)

where the effective potential \( \Psi_{gy} = \epsilon \Psi_{1gy} + \epsilon^2 \Psi_{1gy} + \cdots \) is gyroangle-independent up to an arbitrary order in \( \epsilon \), which therefore guarantees the exact invariance of the gyrocenter gyroaction \( J \) (even though it is an adiabatic invariant of the exact particle dynamics).

A. First-order analysis

At first order in the perturbation analysis \( (18) \), we find the first-order symplectic equation

\[ \Pi_{1gy} \cdot \frac{\partial X}{\partial Z^b} = \frac{e}{c} A_{1gc} \cdot \frac{\partial (X + \rho_0)}{\partial Z^b} - G_1^a \omega_{0ab} + \frac{\partial S_1}{\partial Z^b}, \]

from which we obtain the first-order components

\[ G_1^a \equiv \{ S_1, Z^a \}_0 + \frac{e}{c} A_{1gc} \cdot \{ X + \rho_0, Z^a \}_0 - \Pi_{1gy} \cdot \{ X, Z^a \}_0. \]  

(20)

where \( \{ \cdot, \cdot \}_0 \) denotes the unperturbed gyrocenter (guiding-center) Poisson bracket \( \hat{G} \) (with \( \epsilon = 0 \), obtained by inverting the unperturbed Lagrange matrix \( \omega_{0ab} \). The contributions in the first-order gyrocenter phase-space transformation generated by Eq. \( (20) \) include a canonical part (generated by \( S_1 \)) and the non-canonical substitution of the gyroangle-independent symplectic momentum \( \Pi_{1gy} \) after having removed the term \( A_{1gc} \cdot d(X_0 + \rho_0) \) from the perturbed symplectic structure \( \Pi_{1gy} \).

The canonical gauge function \( S_1 \) is determined from the first-order Hamilton equation

\[ \epsilon \Psi_{1gy} = \epsilon \psi_{1gc} + \Pi_{1gy} \cdot X_0 - \{ S_1, H_0 \}_0. \]  

(21)
where \( \dot{\mathbf{X}}_0 = \{ \mathbf{X}, \mathcal{H}_0 \}_0 = (p_\parallel / m) \hat{\mathbf{b}}_0 \) is the lowest-order unperturbed gyrocenter (guiding-center) velocity, and the effective first-order perturbation potential \( \psi_{1gc} \equiv \Phi_{1gc} - A_{1gc} \cdot \mathbf{v}_0 / c \) is expressed in terms of the lowest-order particle velocity \( \mathbf{v}_0 = (p_\parallel / m) \hat{\mathbf{b}}_0 + \Omega \partial \mathbf{p}_0 / \partial \zeta \). Since we want \( \Psi_{1gy} \) to be gyroangle-independent, it is defined as the gyroangle-averaged part of the right side of Eq. (21):

\[
e \Psi_{1gy} = e \langle \psi_{1gc} \rangle + \frac{p_\parallel}{m} \Pi_{1gy} \cdot \hat{\mathbf{b}}_0,
\]

where, by definition, we took \( \Pi_{1gy} \) to be gyroangle-independent. We now seek a new representation for the gyrokinetic Vlasov-Maxwell equations by choosing the gyrocenter symplectic momentum \( \Pi_{1gy} \) such that the gyrocenter polarization displacement has a specified form. In order to calculate \( \Pi_{1gy} \), we define \( S_1 \) in terms of indefinite gyroangle integrals of the right side of Eq. (23). We note that, to lowest order in the standard gyrokinetic ordering, \( \{ S_1, \mathcal{H}_0 \}_0 \equiv \Omega \partial S_1 / \partial \zeta \) so that \( S_1 \) can be explicitly obtained in terms of indefinite gyroangle integrals of the right side of Eq. (23). In addition, we note that the definition of the gyrocenter gauge function \( S_1 \) is independent of the choice of the gyrocenter symplectic momentum \( \Pi_{1gy} \). Hence, the definition of the gyrocenter gyroaction \( J \) is also independent of that choice. This equivalence of representations was discussed previously in the context of guiding-center theory [1] and gyrocenter theory [3].

### B. Symplectic gyrocenter polarization displacement

We now seek a new representation for the gyrokinetic Vlasov-Maxwell equations by choosing the gyrocenter symplectic momentum \( \Pi_{1gy} \), which is also independent of that choice.

\[
\rho_{1gy} \equiv - \mathcal{G}_1 \cdot \mathbf{d}(\mathbf{X} + \mathbf{p}_0) = \{ \mathbf{X} + \mathbf{p}_0, S_1 \}_0 + \Pi_{1gy} \times \frac{\hat{\mathbf{b}}_0}{eB_0},
\]

where the Poisson-bracket term is evaluated to lowest order in the guiding-center and gyrokinetic orderings [1]. If we now use the lowest-order solution to Eq. (23), which assumes that the background magnetic field is constant and uniform, we find

\[
\langle \rho_{1gy} \rangle = \left\langle \{ \rho_0, S_1 \}_0 \right\rangle + \Pi_{1gy} \times \frac{\hat{\mathbf{b}}_0}{eB_0} \simeq - \frac{\partial}{\partial J} \left\langle \rho_0 \frac{\partial S_1}{\partial \zeta} \right\rangle + \Pi_{1gy} \times \frac{\hat{\mathbf{b}}_0}{eB_0},
\]

where \( \{ \cdot, \cdot \} \) is the Poisson-bracket term.

In order to find an explicit expression for Eqs. (24) - (26), we need to evaluate the gyroangle-averaged operators \( \langle \rho_0 T_{gc}^{-1} \rangle \) and \( \langle \rho_0 \rho_0 T_{gc}^{-1} \rangle \), as well as their derivatives \( \partial \langle \rho_0 T_{gc}^{-1} \rangle / \partial J \) and \( \partial \langle \rho_0 \rho_0 T_{gc}^{-1} \rangle / \partial J \), which are computed in App. A. By replacing the first term in Eq. (26) with Eq. (A9),

\[
\frac{\partial}{\partial J} \left\langle \rho_0 \frac{\partial S_1}{\partial \zeta} \right\rangle = - \frac{e}{m\Omega_0^2} \left( \langle \mathbf{E}_{1gc} \rangle + \frac{p_\parallel \hat{\mathbf{b}}_0}{mc} \times \langle \mathbf{B}_{1gc} \rangle \right) + \frac{\mu}{m\Omega_0^2} \nabla_\perp \langle \{ B_{1||gc} \} \rangle - \frac{\hat{\mathbf{b}}_0}{B_0} \times \langle \mathbf{A}_{1\perp gc} \rangle,
\]

where the symbol \( \langle \{ \cdots \} \rangle \) was introduced by Porazik and Lin [15] to denote a gyro-surface average (see App. A), we obtain the first-order gyroangle-averaged gyrocenter displacement

\[
\langle \rho_{1gy} \rangle = \frac{e}{m\Omega_0^2} \left( \langle \mathbf{E}_{1gc} \rangle + \frac{p_\parallel \hat{\mathbf{b}}_0}{mc} \times \langle \mathbf{B}_{1gc} \rangle \right) + \left( \Pi_{1gy} - \frac{e}{c} \langle \mathbf{A}_{1gc} \rangle \right) \times \frac{\hat{\mathbf{b}}_0}{m\Omega_0} - \frac{\mu}{m\Omega_0^2} \nabla_\perp \langle \{ B_{1||gc} \} \rangle.
\]
In the present work, we choose the first-order gyroangle-averaged gyrocenter displacement
\( \langle \rho_{1gy} \rangle \equiv -\frac{\mu}{m\Omega_0^2} \nabla_\perp (\langle B_{1||gc} \rangle) \) (29)
as the first-order correction to the zeroth-order (guiding-center) gyrocenter polarization displacement \( \langle \rho_{0gy} \rangle \) [4, 16]
\( \langle \rho_{0gy} \rangle = \hat{b}_0 \times \dot{X}_{gc} = -\frac{1}{m\Omega_0^2} \left( \mu \nabla_\perp B_0 + \frac{p_\parallel}{m} \hat{b}_0 \cdot \nabla \right) \). (30)

Hence, we choose the first-order gyrocenter symplectic moment \( \Pi_{1gy} \) as
\( \Pi_{1gy} = \frac{e}{c} \langle A_{1gc} \rangle + \left( \langle E_{1gc} \rangle + \frac{p_\parallel}{mc} \times \langle B_{1gc} \rangle \right) \times \frac{e\hat{b}_0}{\Omega_0^2} \equiv \frac{e}{c} \langle A_{1gc} \rangle + \mathbf{P}_{1gy}, \) (31)
which incorporates all three scenarios introduced in Eq. (2). With this choice, the gyrocenter extended one form (1)
\( \Gamma_{gy} = \mathbf{P}_{gy} \cdot d\mathbf{X} + J d\zeta - w dt, \) (32)
where the gyrocenter canonical momentum is
\( \mathbf{P}_{gy} = \frac{e}{c} \left( \mathbf{A}_0^* + \epsilon \langle \mathbf{A}_{1gc} \rangle \right) + \epsilon \mathbf{P}_{1gy}. \) (33)

In addition, the first-order gyrocenter Hamiltonian (22) becomes
\( e \Psi_{1gy} = e \langle \Phi_{1gc} \rangle + \mu \langle \langle B_{1||gc} \rangle \rangle, \) (34)
which yields the gyrocenter Hamiltonian
\( H_{gy} = \frac{p_\parallel^2}{2m} + \mu \left( B_0 + \epsilon \langle \langle B_{1||gc} \rangle \rangle \right) + e\langle \Phi_{1gc} \rangle. \) (35)

C. Symplectic Euler-Lagrange and Hamilton gyrocenter equations

We are now ready to derive explicit gyrocenter equations of motion. First, the gyrocenter Lagrangian is defined as
\( L_{gy} = \mathbf{P}_{gy} \cdot \dot{\mathbf{X}} + J \dot{\zeta} - H_{gy}, \) (36)
where \( \mathbf{P}_{gy} \) and \( H_{gy} \) are defined in Eqs. (33) and (35), respectively. The symplectic gyrocenter Euler-Lagrange equations associated with arbitrary variations in \( (\mathbf{X}, p_\parallel, J) \) are, respectively,
\[ 0 = \epsilon \mathbf{E}_{gy}^* + \frac{e}{c} \mathbf{X} \times \mathbf{B}_{gy}^* - \dot{p}_\parallel \hat{b}_{gy}^*, \] (37)
\[ 0 = \dot{\mathbf{X}} \cdot \hat{b}_{gy}^* - p_\parallel/m, \] (38)
\[ 0 = \dot{\zeta} + \dot{\mathbf{X}} \cdot \partial \mathbf{P}_{gy}/\partial J - \partial H_{gy}/\partial J, \] (39)
where the effective gyrocenter electric field is
\( \epsilon \mathbf{E}_{gy}^* \equiv -\nabla H_{gy} - \frac{\partial \mathbf{P}_{gy}}{\partial t} = \epsilon e \left( \langle \mathbf{E}_{1gc} \rangle - \frac{d_0 \langle \mathbf{E}_{1gc} \rangle}{dt} \times \frac{\hat{b}_0}{\Omega_0} \right) - \mu \nabla \left( B_0 + \epsilon \langle \langle B_{1||gc} \rangle \rangle \right), \) (40)
and the effective gyrocenter magnetic field is
\( \mathbf{B}_{gy}^* = \nabla \times \left( \frac{e}{c} \mathbf{P}_{gy} \right) = \mathbf{B}_0^* + \epsilon \langle \mathbf{B}_{1gc} \rangle + \epsilon \nabla \times \left( \frac{e}{c} \mathbf{P}_{1gy} \right), \) (41)
with \( \mathbf{B}_0^* \equiv \nabla \times \mathbf{A}_0^* \) and
\( \hat{b}_{gy}^* = \frac{\partial \mathbf{P}_{gy}}{\partial p_\parallel} = \hat{b}_0 + \epsilon \langle \mathbf{B}_{1\perp gc} \rangle/B_0. \) (42)
Here, the perturbed electric field \( \langle E_{1gc} \rangle \) includes its inductive component \(-c^{-1} \partial_t (A_{1gc})\), and the lowest-order time derivative

\[
\frac{\partial}{\partial t} \left( (E_{1gc}) + \frac{p_\parallel b_0}{mc} \times (B_{1gc}) \right) \approx \frac{d_0 (E_{1gc})}{dt}
\]

is computed with the help of the gyrokinetic Faraday’s Law, with \( d_0/dt = \partial/\partial t + (p_\parallel/m) \hat{b}_0 \cdot \nabla \) (to lowest order in the gyrokinetic ordering). We note that the effective gyrocenter electromagnetic fields satisfy the Maxwell equations \( \nabla \cdot B_{gy} = 0 \) and \( \partial B_{gy}/\partial t + c \nabla \times E_{gy} = 0 \).

The gyrocenter Euler-Lagrange equations (37)-(38) can also be written in Hamiltonian form as

\[
\dot{X} = \{X, H_{gy}\}_{gy} = E_{gy}^* \times \frac{c b_{gy}^*}{B_{gy}^*} + \frac{p_\parallel B_{gy}^*}{m B_{gy}^*}, \quad (43)
\]

\[
\dot{p}_\parallel = \{p_\parallel, H_{gy}\}_{gy} = e E_{gy}^* \cdot \frac{B_{gy}^*}{B_{gy}^*}, \quad (44)
\]

where \( B_{gy}^* \equiv b_{gy}^* \cdot B_{gy}^* \). In the expression for the symplectic gyrocenter velocity (43), we find the perturbed \( E \times B \) velocity (defined in terms of the total magnetic field \( B_0 + e (B_{1gc}) \)), the polarization drift velocity (involving \( d_0 (E_{1gc})/dt \), and the total guiding-center drift velocity (i.e., the magnetic gradient and curvature drifts). We note that the parallel gyrocenter momentum \( p_\parallel \equiv mb_{gy}^* \cdot \dot{X} \) is explicitly defined in terms of Eq. (43) as the gyrocenter momentum along the total (perturbed) magnetic field. The identity

\[
\frac{p_\parallel \dot{p}_\parallel}{m} = e E_{gy}^* \cdot \frac{p_\parallel B_{gy}^*}{m B_{gy}^*} = e E_{gy}^* \cdot \dot{X} \quad (45)
\]

will be useful in our discussion of energy conservation.

### IV. GYROKINETIC VARIATIONAL PRINCIPLE

The Hamiltonian gyrokinetic Vlasov-Maxwell equations can be derived either from a Low-Lagrange [17], an Euler-Poincaré [18, 19], or an Euler-Poincaré [20] variational principle. In recent work, Brizard and Tronci [21] showed how the guiding-center Vlasov-Maxwell equations can be explicitly derived according to each of these variational principles. In the present work, the separation of background and perturbed electromagnetic fields introduces a low-frequency gyrokinetic space-time ordering that assumes that the non-uniform background fields are stationary and non-variational.

We are now ready to derive the gyrokinetic Vlasov-Maxwell equations from a variational principle \( \delta A_{gy} = 0 \), based on the gyrokinetic action functional [19]

\[
A_{gy} = - \int F_{gy} \mathcal{H}_{gy} d^8 \mathcal{Z} + \int \frac{d^4x}{8\pi} \left( \left| \mathbf{E} \right|^2 - \left| \mathbf{B} \right|^2 \right), \quad (46)
\]

where summation over particle species is implicitly assumed and the infinitesimal extended phase-space volume element \( d^8 \mathcal{Z} \) does not include the Jacobian \( J_{gy} \). Instead, the Jacobian is included in the definition of the gyrocenter extended Vlasov density

\[
F_{gy} = J_{gy} F = J_{gy} F \delta (w - H_{gy}), \quad (47)
\]

which also includes the energy delta function that enforces the constraint \( \mathcal{H}_{gy} = H_{gy} - w = 0 \) in extended gyrocenter phase space.

The variation of the gyrokinetic action functional yields

\[
\delta A_{gy} = - \int \left( \delta F_{gy} \mathcal{H}_{gy} + F_{gy} \delta \mathcal{H}_{gy} \right) d^8 \mathcal{Z} + \int \frac{d^4x}{4\pi} \left( \epsilon \delta \mathbf{E}_1 \cdot \mathbf{E} - \epsilon \delta \mathbf{B}_1 \cdot \mathbf{B} \right), \quad (48)
\]

where the electromagnetic variations \( \delta \mathbf{E}_1 \equiv - \nabla \delta \Phi_1 - c^{-1} \partial_t \delta \mathbf{A}_1 \) and \( \delta \mathbf{B}_1 \equiv \nabla \times \delta \mathbf{A}_1 \). only involve the perturbation potentials \( \delta \Phi_1, \delta \mathbf{A}_1 \), i.e., the background magnetic field \( \mathbf{B}_0 \) is held constant under field variations. The variation of the gyrocenter Hamiltonian (47):

\[
\delta \mathcal{H}_{gy} = \epsilon e \langle \delta \Phi_{1gc} \rangle + \epsilon \mu \hat{b}_0 \cdot \langle (\delta \mathbf{B}_{1gc}) \rangle \quad (49)
\]
is also expressed in terms of $\delta \Phi_1$ and $\delta \mathbf{B}_1$. The variation of the gyrocenter extended Vlasov density $\delta \mathcal{F}_{gy} = \delta \mathcal{J}_{gy} \mathcal{F} + \mathcal{J}_{gy} \delta \mathcal{F}$ is expressed as

$$
\delta \mathcal{F}_{gy} = \mathcal{F} \left( \frac{\partial \delta \Pi_{gy}}{\partial p_\parallel} + e \frac{c}{e} \mathbf{b}_* \cdot \mathbf{v} + b_z^* \nabla \times \delta \Pi_{gy} \right) + \mathcal{J}_{gy} \left( \{ \delta \mathcal{S}, \mathcal{F} \}_gy + \delta \Pi_{gy} \cdot \{ \mathbf{X}, \mathcal{F} \}_gy \right) \equiv - \frac{\partial}{\partial \mathcal{Z}^a} (\delta \mathcal{Z}^a \mathcal{F}_{gy}),
$$

where the virtual extended phase-space displacement

$$
\delta \mathcal{Z}^a = \left\{ \mathcal{Z}^a, \delta \mathcal{S} \right\}_gy - \delta \Pi_{gy} \cdot \left\{ \mathbf{X}, \mathcal{Z}^a \right\}_gy
$$

is defined in terms of a canonical part generated by $\delta \mathcal{S}$ and a non-canonical part generated by

$$
\delta \Pi_{gy} = \frac{e}{c} \epsilon \left\langle \delta \mathbf{A}_{1gc} \right\rangle + \epsilon \left( \left\langle \delta \mathbf{E}_{1gc} \right\rangle + \frac{p_\parallel \mathbf{b}_0}{mc} \times \left\langle \delta \mathbf{B}_{1gc} \right\rangle \right) \times \frac{e \mathbf{b}_0}{\Omega_0}.
$$

The first two variations in Eq. (48) can be combined

$$
- \delta (\mathcal{F}_{gy} \mathcal{H}_{gy}) = - \mathcal{J}_{gy} \{ \mathcal{F}, \mathcal{H}_{gy} \}_gy \delta \mathcal{S} + \epsilon \mathcal{F}_{gy} \delta L_{1gy} + \frac{\partial}{\partial t} (\mathcal{F}_{gy} \delta \mathcal{S}) + \nabla \cdot \left( \mathbf{X} \mathcal{F}_{gy} \delta \mathcal{S} \right) + \frac{\partial}{\partial p_\parallel} (\dot{p}_\parallel \mathcal{F}_{gy} \delta \mathcal{S}),
$$

where the variation of the gyrocenter Lagrangian (30)

$$
\delta L_{1gy} \equiv \frac{e}{c} \left\langle \delta \mathbf{A}_{1gc} \right\rangle \cdot \mathbf{X} - e \left\langle \delta \Phi_1 \right\rangle + \left( \left\langle \delta \mathbf{E}_{1gc} \right\rangle + \frac{p_\parallel \mathbf{b}_0}{mc} \times \left\langle \delta \mathbf{B}_{1gc} \right\rangle \right) \cdot \mathbf{\pi}_{gy} - \left\langle \left( \delta \mathbf{B}_{1gc} \right) + \mu \mathbf{b}_0, \right\rangle
$$

is expressed in terms of the gyrocenter electric-dipole moment

$$
\mathbf{\pi}_{gy} \equiv (e \mathbf{b}_0 / \Omega_0) \times \mathbf{X},
$$

which includes guiding-center [4] and gyrocenter [10] contributions. The Lagrangian variation term

$$
\int_{\mathcal{Z}} \mathcal{F}_{gy} \delta L_{1gy} = \int_{\mathcal{Z}} \left( \frac{1}{c} \delta \mathbf{A}_1 \cdot \mathbf{J}_{gy} - \delta \mathbf{\Phi}_1 \cdot \theta_{gy} + \delta \mathbf{E}_1 \cdot \mathbf{P}_{gy} + \delta \mathbf{B}_1 \cdot \mathbf{M}_{gy} \right)
$$

can be expressed in terms of the gyrocenter charge and current densities

$$
\left\{ \theta_{gy}, \mathbf{J}_{gy} \right\} \equiv \int_{\mathcal{Z}} \mathcal{F}_{gy} \langle \delta^3(\mathbf{x} + \rho_0 - \mathbf{x}) \rangle \left( e, e \mathbf{X} \right)
$$

and the gyrocenter polarization and magnetization

$$
\left\{ \mathbf{P}_{gy}, \mathbf{M}_{gy} \right\} \equiv \int_{\mathcal{Z}} \mathcal{F}_{gy} \left( \langle \delta^3(\mathbf{x} + \rho_0 - \mathbf{x}) \rangle \mathbf{\pi}_{gy}, -\mu \mathbf{b}_0 \langle \langle \delta^3(\mathbf{x} + \rho_0 - \mathbf{x}) \rangle \rangle + \langle \delta^3(\mathbf{x} + \rho_0 - \mathbf{x}) \rangle \mathbf{\pi}_{gy} \times \frac{p_\parallel \mathbf{b}_0}{mc} \right),
$$

where the delta function $\delta^3(\mathbf{x} + \rho_0 - \mathbf{x}) \equiv \delta_{gc}^3$ yields the standard guiding-center finite-Larmor-radius effects (see App. A for additional details). We note that the gyrocenter magnetization is the sum of the intrinsic magnetic-moment contribution and the moving electric-dipole contribution. The variation of the Maxwell Lagrangian density can be expressed as

$$
\delta \mathbf{E}_1 \cdot \mathbf{E} - \delta \mathbf{B}_1 \cdot \mathbf{B} = \delta \mathbf{A}_1 \cdot \left( \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} \right) + \mathbf{\Phi}_1 \left( \nabla \cdot \mathbf{E} \right) - \frac{\partial}{\partial t} \left( \frac{1}{c} \delta \mathbf{A}_1 \cdot \mathbf{E} \right) - \nabla \cdot \left( \delta \mathbf{F}_1 \cdot \mathbf{E} + \delta \mathbf{A}_1 \times \mathbf{B} \right),
$$

where $\delta \mathbf{F}_1 = -\nabla \mathbf{\Phi}_1 + c^{-1} \partial \delta \mathbf{A}_1 / \partial t$ and $\delta \mathbf{B}_1 = \nabla \times \delta \mathbf{A}_1$.

If we now combine Eqs. (53)-(58) into the variation of the gyrokinetic action functional (48): $\delta \mathcal{A}_{gy} \equiv \int \delta \mathcal{L}_{gy} d^4x$, we obtain the variation of the gyrokinetic Lagrangian density

$$
\delta \mathcal{L}_{gy} = - \int_{P} \mathcal{J}_{gy} \{ \mathcal{F}, \mathcal{H}_{gy} \}_gy \delta \mathcal{S} + \frac{e}{c} \frac{\partial \mathbf{\Phi}_1}{\partial t} \left( \nabla \cdot \mathbf{D}_{gy} - \frac{4\pi}{c} \theta_{gy} \right) + \frac{e}{4\pi} \delta \mathbf{A}_1 \cdot \frac{1}{c} \frac{\partial \mathbf{D}_{gy}}{\partial t} - \nabla \times \mathbf{H}_{gy} + \frac{4\pi}{c} \mathbf{J}_{gy} \right) \right) + \nabla \cdot \left( \int_{P} \mathbf{X} \mathcal{F}_{gy} \delta \mathcal{S} - \frac{e}{4\pi} (\delta \mathbf{\Phi}_1 \mathbf{D}_{gy} + \delta \mathbf{A}_1 \times \mathbf{H}_{gy}) \right),
$$

(59)
where the gyrocenter macroscopic electromagnetic fields are defined as

\[
D_{gy} \equiv \epsilon E_1 + 4\pi P_{gy}, \\
H_{gy} \equiv B_0 + \epsilon B_1 - 4\pi M_{gy}
\] (60)

and the variations \((\delta S, \delta \Phi_1, \delta A_1)\) are assumed to be arbitrary. Hence, variation with respect to \(\delta S\) yields the gyrokinetic Vlasov equation in extended phase space \(\{F, H_{gy}\}_{gy} = 0\). If we integrate \(J_{gy}\{F, H_{gy}\}_{gy}\) over the energy \(w\) coordinate, we find

\[
\begin{align*}
0 &= \int J_{gy}\{F, H_{gy}\}_{gy} \, dw = \int \frac{\partial}{\partial Z_i} \left( J_{gy} F \dot{Z}_i \right) \, dw \\
&= \frac{\partial (J_{gy} F)}{\partial t} + \nabla \cdot \left( J_{gy} F \dot{X} \right) + \frac{\partial}{\partial p_{||}} \left( J_{gy} F \dot{p}_{||} \right) \\
&= J_{gy} \left( \frac{\partial F}{\partial t} + \dot{X} \cdot \nabla F + \dot{p}_{||} \frac{\partial F}{\partial p_{||}} \right),
\end{align*}
\] (61)

where we have used the Liouville theorem [8] to obtain the last expression in order to recover the gyrokinetic Vlasov equation.

Next, the variation with respect to the electromagnetic potentials \((\delta \Phi_1, \delta A_1)\) yield the macroscopic gyrokinetic Maxwell equations

\[
\begin{align*}
\nabla \cdot D_{gy} &= 4\pi \rho_{gy}, \\
\nabla \times H_{gy} &= \frac{1}{c} \frac{\partial D_{gy}}{\partial t} + \frac{4\pi}{c} J_{gy},
\end{align*}
\] (62)

which can also be expressed as the microscopic Maxwell equations

\[
\begin{align*}
\nabla \cdot \epsilon E_1 &= 4\pi \left( \rho_{gy} - \nabla \cdot P_{gy} \right), \\
\nabla \times \left( B_0 + \epsilon B_1 \right) &= \epsilon \frac{\partial E_1}{\partial t} + \frac{4\pi}{c} \left( J_{gy} + \frac{\partial P_{gy}}{\partial t} + c \nabla \times M_{gy} \right),
\end{align*}
\] (63)

which are complemented by Faraday’s Law

\[
\frac{\partial B_1}{\partial t} + c \nabla \times E_1 = 0
\] (66)

and \(\nabla \cdot B_1 = 0\). Now that the gyrokinetic Vlasov-Maxwell equations [61]-[63] have been derived from a variational principle, we now use the remaining part of the gyrokinetic Lagrangian density variation [59] to derive exact conservation laws.

**V. SYMPLECTIC GYROKINETIC CONSERVATION LAWS**

The variational derivation of the reduced Vlasov-Maxwell equations guarantees that these reduced equations satisfy exact energy-momentum conservation laws [22]. In particular, the exact conservation of the gyrokinetic Vlasov-Maxwell energy [23] has played an important role in the numerical implementation of the energy-conserving gyrokinetic equations. The gyrokinetic angular-momentum conservation law (consistent with a variational principle) has so far only been discussed in the case of electrostatic potential fluctuations [23, 26]. In the present Section, we derive the gyrokinetic Noether energy-momentum equations and extract exact energy-momentum conservation laws for the gyrokinetic Vlasov-Maxwell equations [61]-[63].

For this purpose, we introduce the gyrokinetic Noether equation

\[
\delta L_{gy} = \frac{\partial}{\partial t} \left( \int_p F_{gy} \delta S - \frac{\epsilon}{4\pi c} \delta A_1 \cdot D_{gy} \right) + \nabla \cdot \left[ \int_p \dot{X} F_{gy} \delta S - \frac{\epsilon}{4\pi} \left( \delta \Phi_1 D_{gy} + \delta A_1 \times H_{gy} \right) \right],
\] (67)

where the variations are now explicitly expressed in terms of the space-time displacements \(\delta x\) and \(\delta t\):

\[
\begin{align*}
\delta S &= P_{gy} \cdot \delta x - w \delta t \\
\delta \Phi_1 &= E_1 \cdot \delta x - c^{-1} \partial \delta \chi_1 / \partial t \\
\delta A_1 &= E_1 c \delta t + \delta x \times B_1 + \nabla \delta \chi_1
\end{align*}
\]
with the gauge variation $\delta \chi_1$ defined as
$$\delta \chi_1 \equiv \Phi_1 c \delta t - A_1 \cdot \delta x.$$ Upon rearranging the gauge variation $\delta \chi_1$, using the identity
$$- \frac{\partial}{\partial t} (\nabla \chi_1 \cdot D_{gy}) + \nabla \cdot \left( \frac{\partial \delta \chi_1}{\partial t} D_{gy} - e \nabla \delta \chi_1 \times H_{gy} \right) = \frac{\partial}{\partial t} \left( \delta \chi_1 \nabla \cdot D_{gy} \right) - \nabla \cdot \left[ \delta \chi_1 \left( \frac{\partial D_{gy}}{\partial t} - e \nabla \times H_{gy} \right) \right],$$
with the macroscopic gyrokinetic Maxwell equations \(\text{[62]-[63]}\), we obtain the gauge-invariant form of the gyrokinetic Noether equation \(\text{[67]}\):
$$\delta \mathcal{L}_{gy} = \frac{\partial \delta \mathcal{N}_{gy}}{\partial t} + \nabla \cdot \delta \mathbf{\Gamma}_{gy}, \quad (68)$$
where the action-density variation is
$$\delta \mathcal{N}_{gy} = \int_p \mathcal{F}_{gy} \left( \delta \mathcal{S} + \frac{e}{c} \langle \delta \chi_{1gc} \rangle \right) - \left( \epsilon \mathbf{E}_1 \delta t + \frac{\delta \mathbf{x}}{c} \times \epsilon \mathbf{B}_1 \right) \cdot D_{gy} \frac{1}{4 \pi}, \quad (69)$$
and the action-density-flux variation is
$$\delta \mathbf{\Gamma}_{gy} = \int_p \mathbf{\dot{X}} \mathcal{F}_{gy} \left( \delta \mathcal{S} + \frac{e}{c} \langle \delta \chi_{1gc} \rangle \right) - \delta \mathbf{x} \cdot \left( \frac{e}{4 \pi} \mathbf{E}_1 D_{gy} \right) + \frac{e}{4 \pi} \left( \mathbf{E}_1 c \delta t + \delta \mathbf{x} \times \mathbf{B}_1 \right) \times H_{gy}. \quad (70)$$
Here, the gauge-invariant terms are
$$\delta \mathcal{S} + \frac{e}{c} \langle \delta \chi_{1gc} \rangle = \left( \mathbf{P}_{gy} - \frac{e}{c} \langle \mathbf{A}_{1gc} \rangle \right) \cdot \delta \mathbf{x} - \left( w - \epsilon e \langle \Phi_{1gc} \rangle \right) \delta t, \quad (71)$$
with
$$\mathbf{P}_{gy} - \frac{e}{c} \langle \mathbf{A}_{1gc} \rangle = \frac{e}{c} \mathbf{A}_0^* + \epsilon \left( \langle \mathbf{E}_{1gc} \rangle + \frac{p_{b0}}{mc} \langle \mathbf{B}_{1gc} \rangle \right) \times \frac{\mathbf{e}_{b0}}{\Omega_0} = \frac{e}{c} \mathbf{A}_0^* + \epsilon \mathbf{P}_{1gy},$$
$$w - \epsilon \langle \Phi_{1gc} \rangle = \left( w - H_{gy} \right) + \frac{p_{b0}^2}{2m} + \mu \left( B_0 + \epsilon \langle B_{1gc} \rangle \right) \equiv \left( w - H_{gy} \right) + K_{gy}.$$ We note that the guiding-center vector potential $\mathbf{A}_0^*$ is used to describe the unperturbed background magnetic field $\mathbf{B}_0^* = \nabla \times \mathbf{A}_0^*$ and is not subjected to gauge transformation.

A complete expression for the gyrokinetic Noether equation \(\text{[68]}\) also requires an explicit expression for the Lagrangian variation $\delta \mathcal{L}_{gy}$ on the right side of Eq. \(\text{[65]}\). For the derivation of the momentum-energy conservation laws, we consider the specific space-time variations of the gyrokinetic Lagrangian density
$$\delta \mathcal{L}_{gy} = - \left( \delta t \frac{\partial}{\partial t} + \delta \mathbf{x} \cdot \nabla \right) \left[ \frac{1}{8 \pi \epsilon} \left( e^2 |\mathbf{E}_1|^2 - |\mathbf{B}|^2 \right) \right] - \delta \mathbf{x} \cdot \left[ \nabla B_0 \cdot \frac{\mathbf{B}}{4 \pi} + \int_p \mathcal{J}_{gy} F \left( \nabla' K_{gy} - \nabla' \mathbf{P}_{gy} \cdot \mathbf{X} \right) \right], \quad (72)$$
where the gradient operator $\nabla'$ only takes into account the non-uniformity of the background magnetic field, i.e., the first-order fields $\langle \mathbf{E}_{1gc} \rangle$ and $\langle \mathbf{B}_{1gc} \rangle$ are frozen at a fixed position $\mathbf{x} = \mathbf{X} + \mathbf{p}_0$, so that, for example, $\nabla' K_{gy} = \mu \nabla B_0 + \mu \nabla \mathbf{e}_{b0} \cdot \langle \mathbf{B}_{1gc} \rangle$. In addition, the $w$-integration was performed to leave the standard gyrocenter Vlasov distribution $F(\mathbf{X}, p_{||}, \mu, t)$, with $\int_p$ now denoting an integration over $(p_{||}, \mu)$.

### A. Gyrokinetic energy conservation law

Since the background magnetic field $\mathbf{B}_0$ is time-independent, the total energy associated with the gyrokinetic Vlasov-Maxwell equations \(\text{[61]-[63]}\) is conserved. We derive the energy conservation law from the gyrokinetic Noether equation \(\text{[68]}\) by setting $\delta t \neq 0$ and $\delta \mathbf{x} = 0$, which yields the gyrokinetic energy conservation law
$$\delta \mathcal{E}_{gy} / \delta t + \nabla \cdot \mathbf{S}_{gy} = 0,$$ where the gyrokinetic energy density is
$$\mathcal{E}_{gy} = \int_p \mathcal{J}_{gy} F \left[ K_{gy} + \epsilon \mathbf{E}_1 \cdot \mathbf{P}_{gy} + \frac{1}{8 \pi \epsilon} \left( e^2 |\mathbf{E}_1|^2 + |\mathbf{B}|^2 \right) \right]$$
$$= \int_p \mathcal{J}_{gy} F \left[ \frac{p_{b0}^2}{2m} + \mu \left( B_0 + \epsilon \langle B_{1gc} \rangle \right) \right] + \epsilon \langle \mathbf{E}_{1gc} \rangle \cdot \left( \frac{e \mathbf{e}_{b0}}{\Omega_0} \times \mathbf{X} \right) + \frac{1}{8 \pi \epsilon} \left( e^2 |\mathbf{E}_1|^2 + |\mathbf{B}|^2 \right), \quad (73)$$
while the gyrokinetic energy-density flux is

\[ S_{gy} = \int P_{gy} F K_{gy} \dot{X} + \frac{e}{4\pi} \epsilon E_1 \times H_{gy}, \]  

(74)

where the polarization and magnetization \((P_{gy}, M_{gy})\) are defined in Eq. (57), with \(H_{gy}\) defined in Eq. (60). In the gyrokinetic energy density (73), we note that the gyrokinetic polarization \(P_{gy}\) includes the full gyrocenter velocity \(\dot{X}\) defined in Eq. (43), which is expressed in terms of the effective electric and magnetic fields \((E_1, H_{gy})\).

The explicit proof of energy conservation proceeds as follows. First, we begin with

\[ \frac{\partial E_{gy}}{\partial t} = \int P \left[ \frac{\partial (J_{gy} F)}{\partial t} K_{gy} + J_{gy} F \mu \frac{\hat{B}_0}{m} + \frac{\epsilon}{4\pi} \cdot \partial \| \right] + \frac{\epsilon E_1}{4\pi} \cdot \partial \| + \frac{\epsilon E_1}{4\pi} \cdot \partial \| + \frac{\epsilon B_1}{4\pi} \cdot \partial B_1. \]  

(75)

Using the phase-space divergence form (61) of the gyrokinetic Vlasov equation, the first term on the right can be expressed as

\[ \int P \left[ \frac{\partial (J_{gy} F)}{\partial t} K_{gy} = -\nabla \cdot \left( \left( \int P J_{gy} F K_{gy} \dot{X} \right) + \int P J_{gy} F \left[ \frac{\hat{B}_0}{m} + \hat{X} \cdot \nabla \mu \left( B_0 + \epsilon \langle B_1 \rangle \right) \right] \right), \]  

(76)

while, using the definition (57) of the gyrokinetic magnetization, the second term in Eq. (75) can be expressed

\[ \int P J_{gy} F \mu \frac{\hat{B}_0}{m} \cdot \epsilon \frac{\partial \langle B_1 \rangle}{\partial t} = -M_{gy} \epsilon \frac{\partial B_1}{\partial t} + \int P J_{gy} F \pi_{gy} \cdot \frac{\partial}{\partial t} \left( \frac{\hat{B}_0}{m} \times \epsilon \langle B_1 \rangle \right). \]  

(77)

By combining these expressions, Eq. (76) becomes

\[ \frac{\partial E_{gy}}{\partial t} = -\nabla \cdot \left( \left( \int P J_{gy} F K_{gy} \dot{X} \right) + \frac{\epsilon B_1}{4\pi} \cdot \epsilon \frac{\partial \|}{\partial t} + \frac{\epsilon E_1}{4\pi} \cdot \partial \| \right) \]

\[ + \int P J_{gy} F \left[ \pi_{gy} \cdot \frac{\partial}{\partial t} \left( \epsilon \langle E_{1gy} \rangle + \frac{p_{\|} \dot{B}_0}{mc} + \epsilon \langle B_{1gy} \rangle \right) + \frac{p_{\|} \dot{p}_{\|}}{m} + \hat{X} \cdot \nabla \mu \left( B_0 + \epsilon \langle B_1 \rangle \right) \right], \]  

(78)

where we introduced the definition of the polarization \(P_{gy}\) and the definitions (60) for the macroscopic fields \((D_{gy}, H_{gy})\).

Next, we use Faraday’s Law (68) to write

\[ \frac{H_{gy}}{4\pi} \cdot \epsilon \frac{\partial B_1}{\partial t} = -\frac{e H_{gy}}{4\pi} \cdot \nabla \times E_1 = -\nabla \cdot \left( \frac{e}{4\pi} \epsilon E_1 \times H_{gy} \right) - \frac{e E_1}{4\pi} \cdot c \nabla \times H_{gy}, \]

so that Eq. (78) becomes

\[ \frac{\partial E_{gy}}{\partial t} + \nabla \cdot S_{gy} = -\frac{e E_1}{4\pi} \cdot \left( c \nabla \times H_{gy} - \frac{\partial \|}{\partial t} \right) \]

\[ + \int P J_{gy} F \left[ \hat{X} \cdot \frac{\partial P_{gy}}{\partial t} + \frac{p_{\|} \dot{p}_{\|}}{m} + \hat{X} \cdot \nabla \mu \left( B_0 + \epsilon \langle B_1 \rangle \right) \right], \]  

(79)

where we reconstructed the gyrokinetic energy-density flux (74) on the left side of Eq. (78). Lastly, we use the macroscopic gyrokinetic Maxwell equation (63) to obtain

\[ \frac{\partial E_{gy}}{\partial t} + \nabla \cdot S_{gy} = \int P J_{gy} F \left( \frac{p_{\|} \dot{p}_{\|}}{m} - \hat{X} \cdot E_{gy} \right), \]  

(80)

where we introduced the definition (60) of the effective gyrocenter electric field \(E_{gy}\). Using the identity (45), we readily recover the exact gyrokinetic energy conservation law.

**B. Gyrokinetic Noether momentum equation**

Because the background magnetic field \(B_0\) considered in gyrokinetic Vlasov-Maxwell theory is weakly non-uniform (consistent with the guiding-center approximation), a general gyrokinetic Vlasov-Maxwell momentum conservation law does not exist. Indeed, according to Noether’s Theorem, momentum is conserved only in directions corresponding
to symmetries of the background magnetic field. Before we derive the gyrokinetic angular-momentum conservation law associated with an axisymmetric background magnetic field, we wish to show that the gyrokinetic Noether momentum equation, from which our exact angular-momentum conservation law will be derived, is consistent with the gyrokinetic Vlasov-Maxwell equations (61)-(63).

We begin with the gyrokinetic Noether momentum equation, obtained from Eq. (85) by setting \( \delta t = 0 \) and \( \delta \mathbf{x} \neq 0 \):

\[
\frac{\partial \mathbf{P}_g}{\partial t} + \nabla \cdot \mathbf{T}_g = \int_{\mathbf{P}} \mathcal{J}_{gy} F \left( \frac{e}{c} \mathbf{A}_0^* \cdot \mathbf{X} + \mathbf{P}_{1gy} \cdot \dot{\mathbf{X}} - \nabla \mathbf{K}_{gy} \right) - \nabla \mathbf{B}_0 \cdot \frac{\mathbf{B}}{4\pi},
\]

where the gyrokinetic canonical momentum density is defined as

\[
\mathbf{P}_g = \int_{\mathbf{P}} \mathcal{J}_{gy} F \left( \frac{e}{c} \mathbf{A}_0^* + \mathbf{P}_{1gy} \right) + \frac{\mathcal{D}_{gy}}{4\pi} \times \frac{e}{c} \mathbf{B}_1,
\]

and the gyrokinetic canonical stress tensor is defined as

\[
\mathbf{T}_g = \int_{\mathbf{P}} \mathcal{J}_{gy} F \left( \frac{e}{c} \mathbf{X} \cdot \mathbf{A}_0^* + \mathbf{P}_{1gy} \right) - \frac{\mathcal{E}_{gy}}{4\pi} \left( \mathbf{F} \frac{\mathbf{E}_1}{\mathbf{B}_1} + \mathbf{B}_1 \mathbf{H}_g \right)
\]

where \( \mathbf{I} \) denotes the identity matrix. We will return to the gyrokinetic Noether canonical momentum equation (81) when we derive the gyrokinetic canonical angular-momentum conservation law. We note that, while the gyrokinetic stress tensor (82) is manifestly not symmetric (e.g., because of polarization and magnetization), the exact conservation of the gyrokinetic angular-momentum will follow from the right side of Eq. (92) vanishing exactly.

1. Perturbed gyrokinetic Noether momentum equation

We first would like to show that Eq. (81) is an exact consequence of the gyrokinetic Vlasov-Maxwell equations (61)-(63). We begin with simplifying the gyrokinetic Noether canonical momentum equation (81) by using the phase-space divergence form (61) of the gyrokinetic Vlasov equation to obtain

\[
\frac{\partial \mathbf{P}_{1gy}}{\partial t} + \nabla \cdot \mathbf{T}_{1gy} = - \mathbf{I} \left( \int_{\mathbf{P}} \mathcal{J}_{gy} F \frac{e}{c} \mathbf{X} - \nabla \mathbf{K}_{gy} - \frac{\mathbf{F}}{4\pi} \right),
\]

which allows us to obtain the perturbed gyrokinetic Noether momentum equation

\[
\frac{\partial \mathbf{P}_{1gy}}{\partial t} + \nabla \cdot \mathbf{T}_{1gy} = e^{-1} \int_{\mathbf{P}} \mathcal{J}_{gy} F \left( \frac{e}{c} \mathbf{X} \times \mathbf{B}_0^* + \mathbf{P}_{1gy} \cdot \mathbf{K}_{gy} - \frac{\mathbf{F}}{4\pi} \right),
\]

where the perturbed gyrokinetic momentum density is defined as

\[
\mathbf{P}_{1gy} = \int_{\mathbf{P}} \mathcal{J}_{gy} F \mathbf{X} \mathbf{P}_{1gy} + \frac{\mathcal{D}_{gy}}{4\pi} \times \mathbf{B}_1,
\]

and the perturbed gyrokinetic stress tensor is defined as

\[
\mathbf{T}_{1gy} = \int_{\mathbf{P}} \mathcal{J}_{gy} F \mathbf{X} \mathbf{P}_{1gy} - \frac{1}{4\pi} \left( \mathbf{F} \frac{\mathbf{E}_1}{\mathbf{B}_1} + \mathbf{B}_1 \mathbf{H}_g \right) + \mathbf{I} \left[ \frac{e}{8\pi} \left( |\mathbf{E}_1|^2 + |\mathbf{B}_1|^2 \right) - \mathbf{B}_1 \cdot \mathbf{M}_g \right].
\]

We note that the first term on the right side of Eq. (83) includes the unperturbed form of the Euler-Lagrange equation (87): \( \mathbf{0} = (e/c) \mathbf{X}_0 \times \mathbf{B}_0^* - \nabla \mathbf{K}_{gy} - \hat{\mathbf{p}} \times \mathbf{B}_0^* \). Hence, in the absence of electromagnetic-field perturbations \( \epsilon = 0 \), the perturbed gyrokinetic Noether momentum equation (83) is identically satisfied.

The proof that the perturbed gyrokinetic Noether momentum equation (83) follows from the gyrokinetic Vlasov-Maxwell equations (61)-(63) resumes by evaluating the partial time derivative

\[
\frac{\partial \mathbf{P}_{1gy}}{\partial t} = - \int_{\mathbf{P}} \mathcal{J}_{gy} F \mathbf{X} \left( \frac{\partial}{\partial \mathbf{P}_{1gy}} \mathcal{J}_{gy} F \hat{\mathbf{p}} \right) + \frac{\partial}{\partial \mathbf{P}_{1gy}} \mathcal{J}_{gy} F \hat{\mathbf{p}} + \int_{\mathbf{P}} \mathcal{J}_{gy} F \frac{\partial \mathbf{P}_{1gy}}{\partial t}
\]

\[
+ \left( \nabla \times \mathbf{H}_g - \frac{4\pi}{c} \mathbf{J}_{gy} \right) \times \frac{\mathbf{B}_1}{4\pi} - \frac{\mathcal{D}_{gy}}{4\pi} \times (\nabla \times \mathbf{E}_1),
\]

(88)
which yields
\[ \frac{\partial \mathbf{P}_{1gy}}{\partial t} + \nabla \cdot \mathbf{T}_{1gy} = \int_{\mathbf{P}} \mathcal{J}_{gy} F \left( \frac{\partial \mathbf{P}_{1gy}}{\partial t} + \dot{\mathbf{X}} \cdot \nabla \mathbf{P}_{1gy} + \dot{p}_p \frac{\partial \mathbf{P}_{1gy}}{\partial p_p} \right) - \left( \vartheta_{gy} \mathbf{E}_1 + \frac{1}{c} \mathbf{J}_{gy} \times \mathbf{B}_1 \right) - \left( \nabla \mathbf{E}_1 \cdot \mathbf{P}_{gy} + \nabla \mathbf{B}_1 \cdot \mathbf{M}_{gy} \right) - \nabla \mathbf{B}_0 \cdot \frac{\mathbf{B}_1}{4\pi}. \] (89)

In order to complete our proof, we now need to show that the right sides of Eqs. (88) and (89) are indeed equal to each other. For this purpose, we introduce the identities
\[ \vartheta_{gy} \mathbf{E}_1 + \frac{1}{c} \mathbf{J}_{gy} \times \mathbf{B}_1 = \int_{\mathbf{P}} \mathcal{J}_{gy} F \left( \epsilon (\mathbf{E}_{1gc}) + \frac{\epsilon}{c} \dot{\mathbf{X}} \times (\mathbf{B}_{1gc}) \right), \] (90)
\[ \nabla \mathbf{E}_1 \cdot \mathbf{P}_{gy} + \nabla \mathbf{B}_1 \cdot \mathbf{M}_{gy} = \int_{\mathbf{P}} \mathcal{J}_{gy} F \left( (\nabla \mathbf{P}_{1gc} - \nabla \mathbf{P}_{1gc}) \cdot \dot{\mathbf{X}} - \mu \left( \nabla \langle \langle \mathbf{B}_{11} \rangle \rangle - \nabla \mathbf{b}_0 \langle \langle \mathbf{B}_{11} \rangle \rangle \right) \right), \] (91)
and by subtracting the right sides of Eqs. (88) and (89) from each other, we arrive at the identity
\[ 0 = \epsilon^{-1} \int_{\mathbf{P}} \mathcal{J}_{gy} F \left( \epsilon \mathbf{E}_{1gy}^* + \frac{\epsilon}{c} \dot{\mathbf{X}} \times \mathbf{B}_{1gy}^* - \hat{p}_p \mathbf{b}_{1gy}^* \right), \]
which is identically satisfied as a result of the gyrocenter Euler-Lagrange equation (37). Hence, we are guaranteed that the gyrokinetic parallel momentum equation will have the form
\[ \frac{\partial}{\partial t} \left( \int_{\mathbf{P}} \mathcal{J}_{gy} F \hat{p}_p \right) + \nabla \cdot \left( \int_{\mathbf{P}} \mathcal{J}_{gy} F \hat{X} \hat{p}_p \right) = \int_{\mathbf{P}} \mathcal{J}_{gy} F \hat{p}_p, \]
where \( \hat{p}_p \) is given by Eq. (41), which is either obtained by direct calculation or by taking the parallel component of the gyrokinetic Noether momentum equation (51).

2. Gyrokinetic angular-momentum conservation law

Assuming that the background magnetic field is axisymmetric, we now derive the gyrokinetic canonical angular-momentum conservation law by taking the scalar product of Eq. (51) with \( \partial \mathbf{x} / \partial \varphi \), where the toroidal angle \( \varphi \) is associated with rotations about the \( z \)-axis. Hence, the toroidal canonical angular-momentum density \( \mathbf{P}^*_{gy\varphi} \equiv \mathbf{P}^*_{gy} \cdot \partial \mathbf{x} / \partial \varphi \) satisfies the Noether canonical angular-momentum equation
\[ \frac{\partial \mathbf{P}^*_{gy\varphi}}{\partial t} + \nabla \cdot \left( \mathbf{T}^*_{gy\varphi} : \frac{\partial \mathbf{x}}{\partial \varphi} \right) = \mathbf{T}^*_{gy\varphi} : \nabla \left( \frac{\partial \mathbf{x}}{\partial \varphi} \right) - \frac{\partial \mathbf{B}_0}{\partial \varphi} \cdot \frac{\mathbf{B}}{4\pi} \]
\[ + \int_{\mathbf{P}} \mathcal{J}_{gy} F \left( \frac{\epsilon}{c} \frac{\partial \mathbf{A}^*_0}{\partial \varphi} \cdot \dot{\mathbf{X}} + \frac{\epsilon}{c} \frac{\partial \mathbf{P}_{1gy}}{\partial \varphi} \cdot \dot{\mathbf{X}} - \mu \frac{\partial \mathbf{b}_0}{\partial \varphi} \right) \langle \langle \mathbf{B}_{11} \rangle \rangle, \] (92)

where, under the assumption that the background magnetic field is axisymmetric, we have \( \partial \mathbf{B}_0 / \partial \varphi = 0 \) and, in what follows, we will use the identity \( \partial \mathbf{b}_0 / \partial \varphi = \hat{\mathbf{z}} \times \mathbf{b}_0 \), so that \( \mathbf{B} \cdot \partial \mathbf{B}_0 / \partial \varphi = \epsilon \mathbf{B}_1 \cdot (\hat{\mathbf{z}} \times \mathbf{b}_0) \).

The first term on the right of Eq. (92) denotes the contraction of the transpose of the gyrokinetic canonical stress tensor (52) with the anti-symmetric dyadic tensor \( \nabla (\partial \mathbf{x} / \partial \varphi) = \hat{\mathbf{R}} \varphi - \varphi \hat{\mathbf{R}} \), where \( \hat{\mathbf{R}} \equiv |\partial \mathbf{x} / \partial \varphi| \). Because of the anti-symmetry of \( \nabla (\partial \mathbf{x} / \partial \varphi) \), only the anti-symmetric part of \( \mathbf{T}^*_{gy\varphi} \) contributes in Eq. (92), where we find
\[ \mathbf{T}^*_{gy\varphi} : \nabla \left( \frac{\partial \mathbf{x}}{\partial \varphi} \right) = \hat{\mathbf{z}} \left[ \int_{\mathbf{P}} \mathcal{J}_{gy} F \dot{\mathbf{X}} \times \left( \frac{\epsilon}{c} \mathbf{A}_0^* + \epsilon \mathbf{P}_{1gy} \right) \right] - \frac{1}{4\pi} \left( \mathbf{D}_{gy} \times \mathbf{E}_1 + \mathbf{B}_1 \times \mathbf{M}_{gy} \right) \]
\[ = \hat{\mathbf{z}} \left[ \int_{\mathbf{P}} \mathcal{J}_{gy} F \dot{\mathbf{X}} \times \left( \frac{\epsilon}{c} \mathbf{A}_0^* + \epsilon \mathbf{P}_{1gy} \right) \right] + \mathbf{E}_1 \times \mathbf{P}_{gy} + \mathbf{B}_1 \times \mathbf{M}_{gy} - \hat{\mathbf{z}} \frac{4\pi}{4\pi} \langle \langle \mathbf{B}_{11} \rangle \rangle \right) = \int_{\mathbf{P}} \mathcal{J}_{gy} \left( \hat{\mathbf{z}} \cdot \left( \dot{\mathbf{X}} \times \mathbf{P}_{1gy} - \mu \mathbf{b}_0 \times \langle \langle \mathbf{B}_{11} \rangle \rangle \right) + \frac{\partial \mathbf{P}_{1gy}}{\partial \varphi} \cdot \dot{\mathbf{X}} \right) \]
\[ + \hat{\mathbf{z}} \cdot \left( \mathbf{E}_1 \times \mathbf{P}_{gy} + \mathbf{B}_1 \times \mathbf{M}_{gy} \right), \] (93)

Next, we write \( \partial \mathbf{A}_0^*/\partial \varphi = \hat{\mathbf{z}} \times \mathbf{A}_0^* \) and, after some cancellations, Eq. (92) becomes
\[ \frac{\partial \mathbf{P}^*_{gy\varphi}}{\partial t} + \nabla \cdot \left( \mathbf{T}^*_{gy\varphi} : \frac{\partial \mathbf{x}}{\partial \varphi} \right) = \epsilon \int_{\mathbf{P}} \mathcal{J}_{gy} F \left[ \hat{\mathbf{z}} \cdot \left( \dot{\mathbf{X}} \times \mathbf{P}_{1gy} - \mu \mathbf{b}_0 \times \langle \langle \mathbf{B}_{11} \rangle \rangle \right) + \frac{\partial \mathbf{P}_{1gy}}{\partial \varphi} \cdot \dot{\mathbf{X}} \right] \]
\[ + \hat{\mathbf{z}} \cdot \left( \mathbf{E}_1 \times \mathbf{P}_{gy} + \mathbf{B}_1 \times \mathbf{M}_{gy} \right), \] (94)
where
\[
\frac{\partial \mathbf{P}_{1gy}}{\partial \varphi} \cdot \dot{\mathbf{X}} = \left[ \left( \mathbf{E}_{1gc} + \frac{p_\parallel \hat{b}_0}{mc} \times \langle \mathbf{B}_{1gc} \rangle \right) \times \frac{\partial}{\partial \varphi} \left( \frac{e \hat{b}_0}{\Omega_0} \right) + \frac{p_\parallel}{mc} \left( \frac{\partial \hat{b}_0}{\partial \varphi} \times \langle \mathbf{B}_{1gc} \rangle \right) \times \frac{e \hat{b}_0}{\Omega_0} \right] \cdot \dot{\mathbf{X}}
\]

\[
= \frac{\partial}{\partial \varphi} \left( \frac{e \hat{b}_0}{\Omega_0} \right) \left[ \mathbf{X} \times \left( \mathbf{E}_{1gc} + \frac{p_\parallel \hat{b}_0}{mc} \times \langle \mathbf{B}_{1gc} \rangle \right) \right] + \frac{p_\parallel}{mc} \left( \langle \hat{b}_0 \times \langle \mathbf{B}_{1gc} \rangle \rangle \right) \cdot \pi_{gy}
\]

and
\[
\mathbf{E}_1 \times \mathbf{P}_{gy} + \mathbf{B}_1 \times \mathbf{M}_{gy} = \int \mathbf{J}_{gy} \mathbf{F} \left[ \left( \mathbf{E}_{1gc} \times \pi_{gy} - \mu \langle \mathbf{B}_{1gc} \rangle \times \hat{b}_0 \right) + \langle \mathbf{B}_{1gc} \rangle \times \left( \pi_{gy} \times \frac{p_\parallel \hat{b}_0}{mc} \right) \right].
\]

We now write
\[
\ddot{z} \cdot (\dot{\mathbf{X}} \times \mathbf{P}_{1gy}) = - \ddot{z} \cdot \left( \left( \mathbf{E}_{1gc} + \frac{p_\parallel \hat{b}_0}{mc} \times \langle \mathbf{B}_{1gc} \rangle \right) \times \pi_{gy} \right) - \frac{\partial}{\partial \varphi} \left( \frac{e \hat{b}_0}{\Omega_0} \right) \left[ \mathbf{X} \times \left( \mathbf{E}_{1gc} + \frac{p_\parallel \hat{b}_0}{mc} \times \langle \mathbf{B}_{1gc} \rangle \right) \right],
\]

so that, upon additional cancellations, Eq. (94) becomes
\[
\frac{\partial \mathbf{P}_{gy}^*}{\partial t} + \nabla \cdot \left( \mathbf{T}_{gy} \frac{\partial}{\partial \varphi} \right) = \mathbf{e} \ddot{z} \cdot \int \mathbf{J}_{gy} \mathbf{F} \left( \frac{p_\parallel}{mc} \left[ \mathbf{B}_{1gc} \times \left( \pi_{gy} \times \hat{b}_0 \right) + \pi_{gy} \times \left( \hat{b}_0 \times \langle \mathbf{B}_{1gc} \rangle \rangle \right) + \hat{b}_0 \times (\langle \mathbf{B}_{1gc} \rangle \times \pi_{gy}) \right],
\]

which yields the gyrokinetic canonical angular-momentum conservation law

\[
\frac{\partial \mathbf{P}_{gy}^*}{\partial t} + \nabla \cdot \left( \mathbf{T}_{gy} \frac{\partial}{\partial \varphi} \right) = 0,
\]

upon using the Jacobi identity \(\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) \equiv 0\) for the double vector product of any three arbitrary vector fields \((\mathbf{A}, \mathbf{B}, \mathbf{C})\).

The total toroidal angular-momentum density

\[
\mathbf{P}_{gy}^* = \int \mathbf{J}_{gy} \mathbf{F} \left[ \mathbf{P}_{gy} + \mathbf{e} \left( \mathbf{E}_{1gc} \times \frac{e \hat{b}_0}{\Omega_0} \frac{\partial \mathbf{X}}{\partial \varphi} + \frac{p_\parallel}{B_0} \langle \mathbf{B}_{1gc} \rangle \times \frac{\partial \mathbf{X}}{\partial \varphi} \right) \right] + \frac{\mathbf{D}_{gy}}{4\pi c} \times \mathbf{B}_1 \frac{\partial \mathbf{X}}{\partial \varphi}
\]

is the sum of the gyrocenter moment of the guiding-center toroidal angular-momentum \(\mathbf{P}_{gy} = - \mathbf{e}/c \psi + p_\parallel b_0 \varphi + \cdots\), which is defined with higher-order guiding-center corrections as \(\mathbf{A}\)

\[
\mathbf{P}_{gy} = \mathbf{e} \frac{\partial \mathbf{X}}{\partial \varphi} = - \mathbf{e} \left[ \mathbf{\psi} + \nabla \cdot \left( \frac{J}{2m \Omega_0} \nabla \mathbf{\psi} \right) \right] + p_\parallel b_0 \varphi - 2 J b_{0z}.
\]

the toroidal components of the perturbed \(E \times B\) velocity and magnetic-flutter momentum, and the toroidal component of the Minkowski electromagnetic momentum (which includes gyrocenter polarization effects). In the absence of magnetic-field perturbations, we recover the gyrokinetic toroidal angular-momentum density previously derived (without guiding-center corrections) in the electrostatic case \([24, 26]\).

VI. SUMMARY

A new set of gyrokinetic Vlasov-Maxwell equations was derived according to a symplectic representation in which polarization effects were inserted in the symplectic structure. This new symplectic representation allowed for the introduction of self-consistent gyrocenter polarization and magnetization in the gyrokinetic Maxwell equations \([62-63]\) without the need of retaining a second-order gyrocenter Hamiltonian in the gyrokinetic Vlasov equation \([64]\). The self-consistency of the gyrokinetic Vlasov-Maxwell equations \([61-63]\) is guaranteed by their variational derivation from the gyrokinetic action functional \([64]\). By applying the Noether method on this gyrokinetic variational principle, we were able to derive exact conservation laws for gyrokinetic energy as well as gyrokinetic toroidal angular-momentum under the assumption of a time-independent and axisymmetric background magnetic field. The numerical implementation of these gyrokinetic Vlasov-Maxwell equations remains to be explored and is well outside of the scope of the present paper.
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Appendix A: Bessel-function Identities

In this Appendix, we use Bessel-function identities [27] to derive Eq. (27). We begin with the operator identity

\[ \langle T_{gc}^{-1}(z) \rangle \equiv \langle \exp(\rho_0 \cdot \nabla_\perp) \rangle = J_0(z), \]  

(A1)

where \( J_\ell(z) \) denotes the \( \ell \)-th order Bessel function with the argument \( z \) defined from the relation

\[ z^2 \equiv -(2j/m\Omega_0)|\nabla_\perp|^2. \]  

(A2)

First, we derive an expression for \( \langle \rho_0 T_{gc}^{-1} \rangle \):

\[ \langle \rho_0 T_{gc}^{-1} \rangle = \frac{\partial(T_{gc}^{-1})}{\partial \nabla_\perp} = \frac{\partial_z}{\partial \nabla_\perp} J'_0(z) = \left(-\frac{2}{z} \frac{J_\perp}{m\Omega_0}\right) J'_0(z) = \frac{J_1(z)}{m\Omega_0} \left(\frac{J_1(z)}{z/2}\right) \nabla_\perp, \]  

(A3)

where we used \( J'_0(z) = -J_1(z) \). We can thus express

\[ -\frac{c\Omega_0}{\lambda} \left\langle \frac{\partial \rho_0}{\partial \xi} \cdot A_{1\perp} \right\rangle = \frac{m\Omega_0^2 \hat{b}_0}{\lambda} \cdot \left\langle \rho_0 T_{gc}^{-1} \right\rangle \times A_{1\perp} = \mu \left(\frac{J_1(z)}{z/2}\right) \hat{b}_0 \cdot \nabla_\perp \times A_{1\perp} \equiv \mu \left\langle \langle B_{1\|gc} \rangle \right\rangle, \]  

(A4)

where the symbol \( \langle \cdots \rangle \) was introduced by Porazik and Lin [15] to denote a gyro-surface average.

From Eq. (A3), we obtain

\[ \frac{\partial}{\partial J} \langle \rho_0 T_{gc}^{-1} \rangle = -\frac{\partial}{\partial J} \left( z J_1(z) \right)' \left[ \frac{\nabla_\perp}{|\nabla_\perp|^2} \right]^2 = -\frac{\partial z}{\partial J} \left( z J_0(z) \right) \frac{\nabla_\perp}{|\nabla_\perp|^2} \]  

\[ = -\frac{\partial (z^2/2)}{\partial J} J_0(z) \frac{\nabla_\perp}{|\nabla_\perp|^2} = \frac{J_0(z)}{m\Omega_0} \nabla_\perp = \frac{1}{m\Omega_0} \nabla_\perp \langle T_{gc}^{-1} \rangle, \]  

(A5)

where we used the Bessel relation \( (z J_1(z))' = z J_0(z) \). Using this relation, we also obtain the gyro-surface average identity

\[ \langle T_{gc}^{-1}(z) \rangle = \frac{2}{z^2} \int_0^z \langle T_{gc}^{-1}(\lambda) \lambda \rangle d\lambda = \frac{2}{z^2} \int_0^z J_0(\lambda) \lambda d\lambda = \frac{J_1(z)}{z/2}. \]  

(A6)

Next, we derive the expression for \( \langle \rho_0 \rho_0 T_{gc}^{-1} \rangle \):

\[ \langle \rho_0 \rho_0 T_{gc}^{-1} \rangle = -\frac{\partial}{\partial \nabla_\perp} \left[ \left( z J_1(z) \right)' \left[ \frac{\nabla_\perp}{|\nabla_\perp|^2} \right] \right] = -\left( z J_1(z) \right) \frac{\nabla_\perp}{|\nabla_\perp|^2} + \left( \frac{z^2}{2} J_2(z) \right) \frac{\nabla_\perp}{|\nabla_\perp|^2}, \]  

(A7)

where \( \mathbb{I} \equiv \mathbb{I} - \hat{b}_0 \hat{b}_0 \) and we used the recurrence relation

\[ z^2 J_2(z) = 2z J_1(z) - z^2 J_0(z). \]

We now derive the gyroaction derivative to find

\[ \frac{\partial}{\partial J} \langle \rho_0 \rho_0 T_{gc}^{-1} \rangle = -\frac{\partial z}{\partial J} \left( z J_1(z) \right)' \left[ \frac{\nabla_\perp}{|\nabla_\perp|^2} \right] + \frac{\partial z}{\partial J} \left( \frac{z^2}{2} J_2(z) \right) \frac{\nabla_\perp}{|\nabla_\perp|^2} \]  

\[ = -\frac{\partial (z^2/2)}{\partial J} J_0(z) \frac{\nabla_\perp}{|\nabla_\perp|^2} + \frac{\partial (z^2/2)}{\partial J} z J_1(z) \frac{\nabla_\perp}{|\nabla_\perp|^2} \]  

\[ = \frac{J_0(z)}{m\Omega_0} \mathbb{I} + \frac{2}{z m\Omega_0} \left( \frac{J}{m\Omega_0} \mathbb{I} \nabla_\perp \right), \]  

(A8)
where we used \((z^2 J_z(z))^\prime = z^2 J_1(z)\).

We now combine these results to obtain the formula found in Eq. (27):

\[
\frac{\partial}{\partial J} \left( \rho_0 \frac{\partial S_1}{\partial \zeta} \right) = \frac{e}{\Omega_0} \frac{\partial}{\partial J} \left( \rho_0 T^{-1}_{gc} \left( \Phi_1 - \frac{v_{||}}{c} A_{||} \right) \right) - \frac{e}{c} \frac{\partial}{\partial J} \left( \rho_0 \rho_0 T^{-1}_{gc} \right) \cdot \mathbf{b}_0 \times \mathbf{A}_{1\perp}
\]

\[
= \frac{e}{m \Omega_0^2} \left( \nabla_{\perp} \langle \Phi_{1gc} \rangle - \frac{v_{||}}{c} \nabla_{\perp} \langle A_{1||} \rangle \right) + \frac{\mu}{m \Omega_0^2} \nabla_{\perp} \langle \langle B_{1||} \rangle \rangle - \frac{\mathbf{b}_0}{B_0} \times \langle \mathbf{A}_{1\perp, gc} \rangle. \quad (A9)
\]