SPHERICAL HALL ALGEBRAS OF CURVES AND HARDER-NARASIMHAN STRATAS

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Abstract. We show that the characteristic function $1_{S^\alpha}$ of any Harder-Narasimhan strata $S^\alpha \subset \text{Coh}_X^\alpha$ belongs to the spherical Hall algebra $H_{sph}^X$ of a smooth projective curve $X$ (defined over a finite field $F_q$). We prove a similar result in the geometric setting: the intersection cohomology complex $IC(S^\alpha)$ of any Harder-Narasimhan strata $S^\alpha \subset \text{Coh}^{\alpha}_{X}$ belongs to the category $\mathcal{Q}^{X}$ of spherical Eisenstein sheaves of $X$. We show by a simple example how a complete description of all spherical Eisenstein sheaves would necessarily involve the Brill-Noether stratas of $\text{Coh}^{\alpha}_{X}$.

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0. Introduction

Let $X$ be a smooth projective curve defined over a finite field $F_q$. To such a curve is associated, through a general formalism developed by Ringel, a Hopf algebra $H^X$ (called the Hall algebra of $X$). As a vector space, $H^X$ consists of all finitely supported functions on the set of (isomorphism classes of) coherent sheaves over $X$, and the (co)product encodes the structure of the extensions between coherent sheaves over $X$ (see e.g. [S2]).

Hall algebras were first considered by Ringel in the context of representations of quivers. He showed that a certain natural subalgebra $C^\ell Q$ of the Hall algebra $H^Q$ of a quiver $Q$ is isomorphic to the quantized enveloping algebra $U^+_q(g)$ of a Kac-Moody algebra $g$ attached to $Q$ ([R2]). This discovery paved the way for a completely new approach to the theory of quantum groups based on the representation theory of quivers (see e.g. [KS], [N], [VV],...). One of the most important development is the work of Lusztig who considered a geometric version of the Hall algebra of a quiver, in which functions are replaced by constructible sheaves (on moduli stacks); this gave rise to the theory canonical bases of quantum groups, whose impact in algebraic and geometric representation theory is well-known (see [L2]).

In the context of smooth projective curves, Hall algebras first appeared in pioneering work of Kapranov, in relation to the theory of automorphic forms over function fields (see [K1]). He observed some striking similarities between the Hall algebras of curves and quantum affine algebras $U^+_q(g)$ (more precisely between the functional equations for Eisenstein series over function fields and the so-called Drinfeld relations in quantum affine algebras). This analogy becomes very precise for $X = \mathbb{P}^1$ (see [BK]). Motivated by the theory of quantum loop algebras, we introduced in [S1] a natural subalgebra $H_{sph}^X$ of $H^X$ which we call the spherical Hall algebra of $X$: in the language of automorphic forms, $H_{sph}^X$ is generated by the Fourier coefficients of all Eisenstein series induced from the trivial character of a maximal torus. Inspired by work of Laumon, we also singled out a category $\mathcal{Q}^{X}_{sph}$ of simple perverse sheaves over the moduli stacks $\text{Coh}^{\alpha}_{X}$ of coherent sheaves over $X = \mathbb{P}^1 \otimes F_q$. These simple perverse sheaves (which we call spherical Eisenstein sheaves) are expected to provide (by means of the Faisceaux-Fonctions correspondence) a canonical basis for the spherical Hall algebras $H_{sph}^X$. From [SV2] it is also natural
to expect $Q_X$ to play a role in the geometric Langlands program for local systems in the formal neighborhood of the trivial local system.

The spherical Hall algebras $H_X$ and the spherical Eisenstein sheaves $Q_X$ of several low-genus (possibly orbifold) curves were computed in a series of papers (see [S1], [BS], [S3], [S4]) where they were shown to yield interesting quantum loop algebras (such as quantum toroidal algebras or spherical Cherednik algebras of type $A$) equipped with some canonical bases. Much less is known for higher genus curves; a combinatorial realization of $H_X^{sph}$ as a shuffle algebra is given in [SV2] for an arbitrary curve, but it is rather hard to analyze directly algebraically.

In this note, as a first step towards understanding these higher genus spherical Hall algebras, we exhibit an explicit class of elements in $H_X^{sph}$ when $X$ is of genus $g > 1$. Namely we prove (see Theorem 3.1) that the characteristic functions of all the Harder-Narasimhan stratas $S_\alpha$ belong to $H_X^{sph}$ (see Section 2 for notations). We also give a geometric version of the same result: the intersection cohomology complex $IC(S_\alpha)$ of any Harder-Narasimhan strata is a spherical Eisenstein sheaf (Theorem 5.1). As we show by an example, these classes of functions (resp. simple perverse sheaves) come very far from exhausting the whole of $H_X^{sph}$ (resp. $Q_X$) : a full description would (at least) involve the various Brill-Noether loci in $\text{Coh}_{\nu,d}(X)$ (see Remark 5.3.).

1. Spherical Hall algebras of curves

1.1. Let $X$ be a connected smooth projective curve of genus $g$ defined over the finite field $\mathbb{F}_q$. We will assume here that $g > 1$, although most of the results proved in this note hold for rational and elliptic curves as well (see [S3], [S4]). Let $\text{Coh}(X)$ stand for the category of coherent sheaves over $X$. Let us denote by $\langle \cdot, \cdot \rangle$ the Euler form on the Grothendieck group $K_0(X)$, and let $K'_0(X) = K_0(X)/\text{rad} \langle \cdot, \cdot \rangle$ be the numerical Grothendieck group of $X$. We have $K'_0(X) = \mathbb{Z} \text{rk} \oplus \mathbb{Z} \text{deg}$ where $\text{rk}$ and $\text{deg}$ are the rank and degree functions respectively. The Euler form on $K'_0(X)$ is given by

\[ \langle F, G \rangle = (1 - g) \text{rk}(F) \text{rk}(G) + \left| \frac{\text{deg}(F)}{\text{deg}(G)} \right| \]

1.2. Let us briefly recall the definition of the Hall algebra $H_X$ of $X$. We refer the reader to [S2, Lecture 4] for more details. Let $\mathcal{I}_X$ stand for the set of all coherent sheaves over $X$. Put

$H_X = \{ f : \mathcal{I}_X \rightarrow \mathbb{C} \mid \# \text{supp } f < \infty \} = \bigoplus_{\mathcal{F} \in \mathcal{I}} \mathbb{C} 1_{\mathcal{F}}$

where $1_{\mathcal{F}}$ is the characteristic function of the point $\mathcal{F} \in \mathcal{I}$. Let us fix a square root $v$ of $q^{-1}$. The multiplication in $H_X$ is defined by the following formula

\[ (f \cdot g)(\mathcal{R}) = \sum_{\mathcal{N} \subseteq \mathcal{R}} v^{-\langle \mathcal{R}/\mathcal{N} \rangle} f(\mathcal{R}/\mathcal{N}) g(\mathcal{N}) \]

and the comultiplication is

\[ \Delta(f)(\mathcal{M}, \mathcal{N}) = v^{\langle \mathcal{M}, \mathcal{N} \rangle/|\text{Ext}^1(\mathcal{M}, \mathcal{N})|} \sum_{\mathcal{X}_\xi \in \text{Ext}^1(\mathcal{M}, \mathcal{N})} f(\mathcal{X}_\xi) \]

where $\mathcal{X}_\xi$ is the extension of $\mathcal{N}$ by $\mathcal{M}$ corresponding to $\xi$. Notice that the coproduct $\Delta$ takes values in a completion $H_X \hat{\otimes} H_X$ of the tensor space $H_X \otimes H_X$ only (see e.g. [BS, Section 2]). The triple $(H_X, \cdot, \Delta)$ is not a (topological) bialgebra, but it becomes one if we suitably twist the
coproduct. For this we introduce an extra subalgebra \( \mathcal{K} = \mathbb{C}[\kappa_{r,d}], (r, d) \in \mathbb{Z}^2 \), and we define an extended Hall algebra \( \tilde{H}_X = H_X \otimes \mathcal{K} \) with relations

\[
\kappa_{r,d} \kappa_{s,t} = \kappa_{r+s,d+t}, \quad \kappa_{0,0} = 1, \quad \kappa_{r,d} 1_M \kappa_{r,d}^{-1} = \nu^{-2r(1-g)\text{rk}(M)} 1_M.
\]

The new coproduct is given by the formulas

\[
\tilde{\Delta}(\kappa_{r,d}) = \kappa_{r,d} \otimes \kappa_{r,d},
\]

\[
\tilde{\Delta}(f) = \sum_{M,N} \Delta(f)(M,N) 1_M \kappa_{r,d}^{2} \otimes 1_N.
\]

Then \( (\tilde{H}_X, \cdot, \tilde{\Delta}) \) is a topological bialgebra.

The Hall algebras \( H_X \) and \( \tilde{H}_X \) are \( \mathbb{Z}^2 \)-graded (by the class in the numerical Grothendieck group). We will sometimes write \( \Delta_{\alpha,\beta} \) or \( \tilde{\Delta}_{\alpha,\beta} \) in order to specify the graded components of the coproduct.

**1.3.** We will especially be interested in the *spherical* subalgebra \( H_X^{\text{sph}} \) of \( H_X \), which is defined as follows. For any \( d \in \mathbb{Z} \) let \( \text{Pic}^d(X) \) stand for the (finite) set of line bundles over \( X \) of degree \( d \), and let us set

\[
1^d_{\text{Pic}} = \sum_{\mathcal{L} \in \text{Pic}^d(X)} 1_{\mathcal{L}}.
\]

Next, let \( d \geq 1 \) and let \( \text{Tor}^d(X) \) stand for the (finite) set of all torsion sheaves over \( X \) of degree \( d \). We now set

\[
1^d_{\text{Tor}} = \sum_{\mathcal{T} \in \text{Tor}^d(X)} 1_{\mathcal{T}}.
\]

The spherical Hall algebra \( H_X^{\text{sph}} \) is generated by the elements \( \{1^d_{\text{Pic}} | d \in \mathbb{Z} \} \cup \{1^d_{\text{Tor}} | d \geq 1 \} \). In an effort to unburden the notation, and because this should not cause any confusion here, we will simply write \( U_X \) for \( H_X^{\text{sph}} \). The spherical Hall algebra contains two natural subalgebras, namely \( U_X^\nu \) which is generated by \( \{1^d_{\text{Pic}} | d \in \mathbb{Z} \} \), and \( U_X^0 \) which is generated by \( \{1^d_{\text{Tor}} | d \geq 1 \} \). Moreover, the multiplication map gives an isomorphism \( U_X^\nu \otimes U_X^\nu \rightarrow U_X \) (see e.g. [SV1], Section 6).

### 2. Harder-Narasimhan stratifications

**2.1.** Let us now briefly recall the various notions related to semistability of coherent sheaves over curves. We refer to [HN], [SS] for more details. We fix a smooth projective curve \( X \) of genus \( g \) as in Section 1.1. The slope of a coherent sheaf \( \mathcal{F} \) over \( X \) is defined to be

\[
\mu(\mathcal{F}) = \frac{\text{deg}(\mathcal{F})}{\text{rank}(\mathcal{F})} \in \mathbb{Q} \cup \{\infty\}.
\]

The sheaf \( \mathcal{F} \) is said to be *semistable of slope* \( \nu \) if \( \mu(\mathcal{F}) = \nu \) and if \( \mu(\mathcal{G}) \leq \nu \) for any subsheaf \( \mathcal{G} \) of \( \mathcal{F} \). If the above condition holds with \( < \) instead of \( \leq \) then we say that \( \mathcal{F} \) is *stable*. We denote by \( \mathcal{C}_\nu \) the full subcategory of \( \text{Coh}(X) \) whose objects are semistable sheaves of slope \( \nu \). The categories \( \mathcal{C}_\nu \) are abelian and artinian. The simple objects of \( \mathcal{C}_\nu \) are precisely given by the stable sheaves of slope \( \nu \).

The fundamental properties of the categories \( \mathcal{C}_\nu \) are listed below.

**Proposition 2.1.** The following hold:

i) \( \text{Hom}(\mathcal{C}_\nu, \mathcal{C}_\eta) = 0 \) if \( \nu > \eta \),

ii) \( \text{Ext}(\mathcal{C}_\nu, \mathcal{C}_\eta) = 0 \) if \( \eta > \nu + 2(g-1) \).
iii) any coherent sheaf $F$ possesses a unique filtration
\[(2.1) \quad 0 \subseteq F_i \subseteq \cdots \subseteq F_1 = F\]
satisfying the following conditions: $F_i/F_{i+1}$ is semistable for all $i$ and
\[\mu(F_i/F_2) < \cdots < \mu(F_{i-1}/F_i) < \mu(F_i).\]

The filtration (2.1) is called the Harder-Narasimhan (or HN filtration) of $F$. We also define the HN-type of $F$ to be $HN(F) = (\alpha_1, \ldots, \alpha_l)$ with $\alpha_i = F_i - F_{i+1}$. Here $\mathcal{G} = (\text{rank}(\mathcal{G}), \text{deg}(\mathcal{G})) \in \mathbb{Z}^2$ is the class of a sheaf $\mathcal{G}$ in the (numerical) Grothendieck group of $\text{Coh}(X)$—see Section 1.1. Note that the weight $\alpha := \alpha_1 + \cdots + \alpha_l$ of the HN type of $F$ is equal to $\mathcal{G}$.

It is convenient to view an HN type $(\alpha_1, \ldots, \alpha_l)$ as a polygon as follows:

![Harder-Narasimhan polygon of weight $\alpha$.](image)

This polygon, called the $HN$ polygon of $F$, is convex by construction. The following useful result is a consequence of Proposition 2.1 (see e.g. [S5], Theorem 2).

**Proposition 2.2.** Let $F$ be a coherent sheaf over $X$ of class $\alpha \in \mathbb{Z}^2$. Let $0 \subseteq F_1 \subseteq \cdots \subseteq F_1 = F$ be the HN filtration of $F$. Let $\mathcal{G}$ be a subsheaf of $F$ of class $\gamma$. Then

i) the point $\beta := \alpha - \gamma$ lies above the HN polygon of $F$;

ii) if moreover $\beta$ is a vertex of the HN polygon of $F$, that is if $\gamma = \alpha_i + \cdots + \alpha_l$ for some $1 \leq i \leq l$, then $\mathcal{G} = F_i$.

**2.2.** We may stratify the set $\mathcal{I}_X$ of all isomorphism classes of coherent sheaves over $X$ by the HN-type and write $\mathcal{I}_X = \bigsqcup_{\alpha} S_\alpha$ where $\alpha$ runs through the set of all possible HN types, i.e. $tuples \alpha = (\alpha_1, \ldots, \alpha_l)$ with $\alpha_i \in (\mathbb{Z}^2)^+$ and $\mu(\alpha_1) < \cdots < \mu(\alpha_l)$. Here $(\mathbb{Z}^2)^+ = \{(r, d) \in \mathbb{Z}^2 \mid r \geq 1 \text{ or } r = 0, d > 0 \}$. For instance, if $\alpha = (\alpha)$ then $S_\alpha$ is the set of isomorphism classes of semistable sheaves of class $\alpha$. Let us denote by $1_{S_\alpha} \in \mathbb{H}_X$ the characteristic function of the set of sheaves of a fixed HN type $\alpha$. Since $X$ is defined over a finite field, $S_\alpha$ is finite for any $\alpha$ hence $1_{S_\alpha}$ is a well-defined element of $\mathbb{H}_X$. For $\alpha \in (\mathbb{Z}^2)^+$ we will simply denote by $1_\alpha^6$ the characteristic function of $S_\alpha$.

From the uniqueness of the HN filtration of a given coherent sheaf we easily deduce

**Proposition 2.3.** For any HN type $\alpha = (\alpha_1, \ldots, \alpha_l)$ we have
\[1_{S_\alpha} = \sum_{\gamma < \alpha} 1_{S_\alpha}^6 \cdot 1_{S_\alpha}^6 \cdot 1_{S_\alpha}^6.\]

We use the stratification by HN type to define a completion of the Hall algebra $\mathbb{H}_X$ as follows. For $n \in \mathbb{Z}$ let us write $\alpha \geq n$ if $\alpha = (\alpha_1, \ldots, \alpha_l)$ with $\mu(\alpha_1) \geq n$. Let $\mathcal{C}_{\geq n}$ be the full subcategory
of \( \text{Coh}(X) \) generated by \( \mathbf{C}_\nu \) for all \( \nu \geq n \). By definition, the HN type \( \alpha \) of a sheaf \( F \) satisfies \( \alpha \geq n \) if and only if \( F \in \mathbf{C}_{\geq n}. \)

The set of HN types of a fixed weight \( \alpha \) satisfying \( \alpha \geq n \) is finite for any \( n \). Let \( H_X^{\leq n}[\alpha] \) be the subspace of \( H_X[\alpha] \) consisting of functions supported on the complement of \( \bigcup_{\alpha \geq n} S_\alpha \).

It is a subspace of \( H_X[\alpha] \) of finite codimension. Moreover there are some obvious inclusions \( H_X^{\leq m}[\alpha] \to H_X^{\leq n}[\alpha] \) for any \( m < n \). Put \( H_X^{\leq \alpha}[\alpha] = H_X[\alpha]/H_X^{\leq n}[\alpha] \). This is a finite dimensional space. We put

\[
\hat{H}_X[\alpha] = \lim_{\to} H_X^{\leq \alpha}[\alpha], \quad \hat{H}_X = \bigoplus_{\alpha} \hat{H}_X[\alpha].
\]

Note that \( \hat{H}_X[\alpha] = \{ f : \mathcal{I}_\alpha \to \mathbb{C} \} = \prod_{\mathcal{I} \in \mathcal{I}_\alpha} \mathbb{C}_{1\mathcal{I}} \) as a vector space, where we have denoted by \( \mathcal{I}_\alpha \subset \mathcal{I}_X \) the set of all coherent sheaves of class \( \alpha \). It is shown in [BS], Section 2 that the product and coproduct are well-defined in the limit and endow \( \hat{H}_X \) with the structure of a (twisted) bialgebra.

Consider the elements

\[
1_\alpha = \sum_{\mathcal{I} \in \mathcal{I}_\alpha} 1_{\mathcal{I}}, \quad 1^\text{vec}_\alpha = \sum_{\mathcal{V} \in \mathcal{V}_\alpha} 1_{\mathcal{V}}
\]

where the second sum ranges over all (isomorphism classes of) vector bundles of class \( \alpha \). These are both elements of \( \hat{H}_X \). As a direct corollary of Proposition 2.3 we have the following identities:

\[
1_\alpha = \sum_{\alpha \in X_\alpha} v_{\sum_{i<j}(\alpha_i,\alpha_j)} 1_{\alpha_1}^g \cdots 1_{\alpha_s}^g, \quad 1^\text{vec}_\alpha = \sum_{\alpha \in Y_\alpha} v_{\sum_{i<j}(\alpha_i,\alpha_j)} 1_{\alpha_1}^g \cdots 1_{\alpha_s}^g,
\]

where \( X_\alpha \) is the set of all HN types of weight \( \alpha \) and \( Y_\alpha \) is the set of all HN types \( \alpha = (\alpha_1, \ldots, \alpha_l) \) of weight \( \alpha \) for which \( \mu(\alpha_i) < \infty \).

3. Characteristic functions of semistables

3.1. Our aim in this section is to prove the following theorem:

**Theorem 3.1.** For any \( \alpha \in (\mathbb{Z}^2)^+ \) we have \( 1_\alpha^g \in U_X \).

Our proof of Theorem 3.1 hinges on the following preliminary result. Let us denote by \( \hat{U}_X \) the completion of \( U_X \) in \( \hat{H}_X \) (i.e. \( \hat{U}_X[\alpha] = \lim_{\to} U_X[\alpha]/(U_X[\alpha] \cap H_X^{\leq n}[\alpha]) \)).

**Proposition 3.2.** For any \( \alpha \in (\mathbb{Z}^2)^+ \) we have \( 1_\alpha^g \in \hat{U}_X \).

**Proof.** We may use Reineke’s inversion formula (see [R], Section 5.) to write

\[
1_\alpha^g = \frac{1}{2} \sum_{\beta} (-1)^{\sum_{i<j}(\beta_i,\beta_j)} 1_{\beta_1}^g \cdots 1_{\beta_s}^g
\]

where the sum ranges over all tuples \( \beta = (\beta_1, \ldots, \beta_s) \) of elements of \( (\mathbb{Z}^2)^+ \) satisfying \( \mu(\sum_{i=k}^s \beta_i) > \mu(\alpha) \) for all \( k = 1, \ldots, s \). The above sum converges in \( \hat{H}_X \). Since \( \hat{U}_X \) is a subalgebra of \( \hat{H}_X \), the proposition will be proved if we can show that \( 1_\alpha \in \hat{U}_X \) for all \( \alpha \). Furthermore, because \( 1_\alpha = \sum_{l \geq 0} l^{\text{rank}(\alpha)} 1_{\alpha - (0,l)}^\text{vec} 1_{(0,l)}^g \) and \( 1_{(0,l)} \in U_X \) for all \( l \), it suffices in fact to prove that \( 1^\text{vec}_\alpha \in \hat{U}_X \) for all \( \alpha \).

Let us write \( \alpha = (r,d) \) and argue by induction on the rank \( r \). The cases \( r = 0, 1 \) are obvious so let \( r > 1 \) and let us assume that

\[
1_{(r',d)} \in \hat{U}_X, \quad 1^\text{vec}_{(r',d)} \in \hat{U}_X
\]
for all \( r' < r \). We have to show that for any \( d \in \mathbb{Z} \) and any \( n \in \mathbb{Z} \) it holds
\[
1^\vec{v}_{r,d} \in \text{U}_X + \text{H}^n_X.
\]

Let us fix \( n \) and argue by induction on \( d \). If \( d < nr \) then no vector bundle of rank \( r \) and degree \( d \) may belong to \( C_{\geq n} \) and hence have an HN type \( \alpha \geq n \). Therefore \( 1^\vec{v}_{r,d} \in \text{H}^n_X \). Now let us fix some \( d \) and assume that (3.2) holds for all \( d' < d \).

Choose \( N < n - 2(g - 1) \) and let us consider the product \( 1_{r-1,d-N} \cdot 1^\vec{v}_{1,N} \). By definition, we have
\[
1_{r-1,d-N} \cdot 1^\vec{v}_{1,N} = \sum_{\mathcal{F}} c_{\mathcal{F}}[\mathcal{F}]
\]
where
\[
c_{\mathcal{F}} = \nu^{-(r-1,d-N),(1,N)} \sum_{\mathcal{L} \in \text{Pic}^N(X)} \frac{\# \{ \mathcal{L} \hookrightarrow \mathcal{F} \}}{\# \text{Aut}(\mathcal{L})} = \nu^{-(r-1,d-N),(1,N)} \sum_{\mathcal{L} \in \text{Pic}^N(X)} \frac{\# \{ \mathcal{L} \hookrightarrow \mathcal{F} \}}{\nu^{r-2} - 1}.
\]

Let us decompose \( \mathcal{F} = \mathcal{V}_r \oplus T_r \) into a direct sum of a vector bundle and a torsion sheaf, and let us assume that \( \mathcal{F} \in C_{\geq n} \). Then \( \mathcal{F} \in C_{\geq 2(g-1)+N} \) and thus \( \text{Ext}(\mathcal{L}, \mathcal{F}) = 0 \) by Serre duality. This in turn implies that \( \dim \text{Hom}(\mathcal{L}, \mathcal{F}) = ((1,N),(r,d)) \). Any nonzero map from a line bundle to a vector bundle is an embedding. From this we deduce that
\[
\# \{ \mathcal{L} \hookrightarrow \mathcal{F} \} = \nu^{-2\dim \text{Hom}(\mathcal{L}, \mathcal{F})} = \nu^{-2((1,N),(r,d))} = \nu^{-2\deg(T_r)}.
\]

The important point is that this only depends on \( \deg(T_r) \). From this discussion we deduce that there exists nonzero constants \( c_l \) for \( l \geq 0 \) such that
\[
1_{r-1,d-N} \cdot 1^\vec{v}_{1,N} \in c_0 1_{r,d} \cdot 1_{0,l} + \text{H}^n_X.
\]

We may rewrite this last equation as
\[
c_0 1^\vec{v}_{r,d} \in 1_{r-1,d-N} \cdot 1^\vec{v}_{1,N} - \sum_{l=1}^{d-rn} c_l 1^\vec{v}_{r,d-l} \cdot 1_{0,l} + \text{H}^n_X.
\]

Now, by our two induction hypotheses we have \( 1_{r-1,d-N} \in \hat{\text{U}}_X \) and \( 1^\vec{v}_{r,d-l} \in \hat{\text{U}}_X \) for all \( l \geq 1 \). But then (3.2) holds as well. We are done. \( \square \)

**Proof of Theorem 3.7.** We have to show that \( 1^\vec{v} \) belongs to \( \text{U}_X \), and not only to \( \hat{\text{U}}_X \). By Proposition 3.3, there exists for any \( n \) an element \( v_n \in \text{H}^n_X \) such that \( \deg(v_n) := 1^\vec{v} + v_n \in \text{U}_X \). We may further decompose \( v_n = \sum_{\alpha} v_n^{(\alpha)} \) according to the HN type \( \alpha \). The set of \( \alpha \) for which \( v_n^{(\alpha)} \) is nonzero is finite since \( v_n \in \text{H}^n_X \). Our proof is based on the following two lemmas.

**Lemma 3.3.** There exists \( n \ll 0 \) such that for any HN type \( \alpha = (\alpha_1, \ldots, \alpha_l) \) of weight \( \alpha \) satisfying \( \mu(\alpha_{i+1}) - \mu(\alpha_i) > 2(g - 1) \) for some \( 1 \leq i \leq l \).

**Proof.** Let \( \alpha = (\alpha_1, \ldots, \alpha_l) \) be as above. We have \( \deg(\alpha) = \text{rank}(\alpha_1)\mu(\alpha_1) + \cdots + \text{rank}(\alpha_l)\mu(\alpha_l) \).

If \( \mu(\alpha_i) < n \) and \( \mu(\alpha_{i+1}) - \mu(\alpha_i) \leq 2(g - 1) \) for all \( i \) then
\[
\deg(\alpha) = \text{rank}(\alpha)n + \sum_{i=2}^{l} 2(g - 1)(l - 1)\text{rank}(\alpha_i) < \text{rank}(\alpha)\left( n + 2(g - 1)\sum_{i=1}^{\text{rank}(\alpha)} + 1 \right),
\]
which is impossible for \( n \) sufficiently negative. \( \square \)
Lemma 3.4. Let $\mathcal{F} \in \text{Coh}(X)$ be a coherent sheaf of class $\alpha$ and of HN type $(\alpha_1, \ldots, \alpha_l)$. Assume that $\mu(\alpha_{i+1}) - \mu(\alpha_i) > 2(g - 1)$ for some $i$. Then $1_\mathcal{F} = m \circ \Delta_{\beta, \gamma}(1_\mathcal{F})$ for $\beta = \alpha_1 + \cdots + \alpha_i$, $\gamma = \alpha_{i+1} + \cdots + \alpha_l$.

Proof. Let $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_1 = \mathcal{F}$ be the HN filtration of $\mathcal{F}$. Since $\mathcal{F}_{i+1} \subset C_{\geq \mu(\alpha_{i+1})}$ and $\mathcal{F}/\mathcal{F}_{i+1} \subset C_{\leq \mu(\alpha_i)}$ while $\mu(\alpha_{i+1}) - \mu(\alpha_i) > 2(g - 1)$ we have $\text{Ext}(\mathcal{F}_{i+1}, \mathcal{F}/\mathcal{F}_{i+1}) = 0$ (see Proposition 2.1). It follows that $\mathcal{F} \cong \mathcal{F}_{i+1} \oplus \mathcal{F}/\mathcal{F}_{i+1}$. Moreover, $1_{\mathcal{F}/\mathcal{F}_{i+1}} 1_{\mathcal{F}_{i+1}} = v_\alpha^{\mathcal{F}/\mathcal{F}_{i+1}, \mathcal{F}_{i+1}}$ since there is a unique subsheaf of $\mathcal{F}$ isomorphic to $\mathcal{F}_{i+1}$. Hence Lemma 3.4 will be proved once we show that $\Delta_{\beta, \gamma}(1_\mathcal{F}) = v_\alpha^{\mathcal{F}/\mathcal{F}_{i+1}, \mathcal{F}_{i+1}}$. But this last equation is a consequence of the fact that there exists a unique subsheaf of $\mathcal{F}$ of class $\gamma$, namely $\mathcal{F}_{i+1}$ (see Proposition 2.2). $\square$

We are now ready to finish the proof of Theorem 3.1. Let us choose some $n \ll 0$ as in Lemma 3.3. Let $\mathcal{A}$ be the (finite) set of all $\alpha$ for which $v_n^{\beta, \alpha}$ is nonzero and let $\underline{\alpha}^0$ be the lower boundary of the convex hull of elements of $\mathcal{A}$.

**Figure 2.** The convex hull of a set of HN polygons.

Thus $\underline{\alpha}^0 = (\alpha^0_1, \ldots, \alpha^0_m)$ is also a convex path in $\mathbb{Z}^2$ of weight $\alpha$. Moreover $\mu(\alpha^0_i) < n$ so that the conclusion of Lemma 3.3 applies. Choose $i$ such that $\mu(\alpha^0_{i+1}) - \mu(\alpha^0_i) > 2(g - 1)$ and set $\beta = \alpha^0_1 + \cdots + \alpha^0_i$, $\gamma = \alpha^0_{i+1} + \cdots + \alpha^0_m$. By Lemma 3.4 $\Delta_{\beta, \gamma}(1_\mathcal{F}) = 0$ for all sheaves $\mathcal{F}$ whose HN polygon doesn’t lie below the segment $\beta$.

**Figure 3.** Choice of the vertex $\beta$.

This implies that $\Delta_{\beta, \gamma}(v_n^{\beta, \underline{\alpha}}) = 0$ for all HN types $\underline{\alpha}$ whose associated polygon doesn’t pass through the point $\beta$. Furthermore, by Lemma 3.4 again, $m \circ \Delta_{\beta, \gamma}(v_n^{\beta, \underline{\alpha}}) = v_n^{\beta, \underline{\alpha}}$ for any HN type $\underline{\alpha}$ whose polygon does pass through $\beta$. Hence

$$m \circ \Delta_{\beta, \gamma}(u_{\underline{\alpha}}) = m \circ \Delta_{\beta, \gamma}(1^\beta_\alpha + \sum_{\underline{\alpha}} v_n^{\beta, \underline{\alpha}}) = \sum_{\underline{\alpha} \in \mathbb{Z}_\beta^m} v_n^{\beta, \underline{\alpha}}.$$
where \( Z_\beta \) is the set of all HN types passing through \( \beta \). Because \( u_\alpha \) belongs to \( U_X \), which is stable under the coproduct, we deduce that \( \sum_{\alpha \in Z_\beta} v_{\alpha, \beta} \) belongs to \( U_X \) as well. Hence the same holds for \( u'_\alpha = 1^\alpha + \sum_{\alpha \neq \beta} v_{\alpha, \beta} \). Notice that \( u'_\alpha \) contains strictly fewer terms than \( u_\alpha \). Arguing as above repeatedly we obtain better and better approximations of \( 1^\beta \) by elements of \( U_X \) until we arrive at \( 1^\beta \in U_X \). Theorem 3.5 is proved.

The combination of Theorem 3.1 and Proposition 2.3 yields the following result:

**Corollary 3.5.** For any HN type \( \alpha \) we have \( 1_{\alpha, \beta} \in U_X \).

**Remark 3.6.** The above proof actually shows that \( \hat{U}_X \cap H_X = U_X \).

### 4. Spherical Eisenstein sheaves

**4.1.** Let us set \( \overline{X} = X \otimes \mathbb{F}_q \). For \( \alpha \in K_0'(\overline{X}) = \mathbb{Z}^2 \), let \( \text{Coh}\alpha_X \) stand for the moduli stack parametrizing coherent sheaves of class \( \alpha \) over \( \overline{X} \). This is a smooth irreducible stack, which is locally of finite type (see e.g. [L1]). It carries a Harder-Narasimhan stratification \( \text{Coh}\alpha_X = \bigcup_{\beta} S_{\alpha, \beta} \) similar to the one existing for \( \mathcal{I}_X \), and each locally closed substack \( S_{\alpha, \beta} \) is smooth and of finite type.

We will now define, following [S3], a certain category of simple perverse sheaves over the stacks \( \text{Coh}\alpha_X \). For this we consider the following induction diagrams, for \( \alpha, \beta \in K_0'(X) \):

\[
\begin{array}{ccc}
\text{Coh}_\alpha^X \times \text{Coh}_\beta^X & \xrightarrow{p_1} & \text{Coh}_\alpha^X \\
p_2 & & \downarrow \\
\text{Coh}_{\alpha+\beta}^X & & \\
\end{array}
\]

where \( \mathcal{E}_{\alpha, \beta} \) is the stack classifying inclusions \( G \subset F \) of a coherent sheaf \( G \) over \( \overline{X} \) of class \( \beta \) into a coherent sheaf \( F \) over \( \overline{X} \) of class \( \alpha + \beta \); and where the maps \( p_1, p_2 \) are given by the functors \( G \subset F \mapsto (F/G, G) \) and \( G \subset F \mapsto F \). The morphism \( p_1 \) is smooth while \( p_2 \) is proper and representable (see [L1]).

Let \( D^b(\text{Coh}_\alpha^X) \) be the bounded derived category of constructible \( \mathbb{Q}_l \)-sheaves over \( \text{Coh}_\alpha^X \). We define induction and restriction functors as

\[
m : D^b(\text{Coh}_\alpha^X \times \text{Coh}_\beta^X) \rightarrow D^b(\text{Coh}_{\alpha+\beta}^X)
\]

\[
[p] \mapsto p_2p_1^*([p][\text{dim } p_1]),
\]

and

\[
\Delta : D^b(\text{Coh}_{\alpha+\beta}^X) \rightarrow D^b(\text{Coh}_\alpha^X \times \text{Coh}_\beta^X)
\]

\[
[p] \mapsto p_1p_2^*([p][\text{dim } p_2]).
\]

By the Decomposition Theorem of [BBD], \( m \) preserves the subcategory of semisimple complexes of geometric origin. Both of the above functors are associative in the appropriate sense. We will sometimes write \( \mathbb{P} \otimes \mathbb{Q} \) for \( m(\mathbb{P} \otimes \mathbb{Q}) \). For \( \alpha \in K_0'(X) \), let \( 1_\alpha = \bigoplus_{[\alpha]} \text{dim } \text{Coh}_\alpha^X \) be the constant complex over \( \text{Coh}_\alpha^X \). We will call a product of the form

\[
L_{\alpha_1, \ldots, \alpha_r} = 1_{\alpha_1} \ast \cdots \ast 1_{\alpha_r}
\]

a *Lusztig sheaf*. It is a semisimple complex. We let \( \mathcal{P}_X = \bigcup_{\alpha} \mathcal{P}^\alpha \) stand for the set of all simple perverse sheaves appearing in some Lusztig sheaf \( L_{\alpha_1, \ldots, \alpha_r} \) where for all \( \alpha_i = (r_i, d_i) \) we have...
5. IC sheaves of Harder-Narasimhan strata

The purpose of this section is to prove the following result:

**Theorem 5.1.** For any Harder-Narasimhan type $\alpha$ we have $IC(S_{\alpha}) \in \mathcal{P}_X$.

This can be viewed as a direct geometric analog of Theorem 3.1. We will first establish the following special case:

**Proposition 5.2.** For any $\alpha \in K'_0(X)$ we have $1_{\alpha} \in \mathcal{P}_X$.

**Proof.** We argue by induction on the rank $r$ of $\alpha$. If $r = 1$ then $\alpha = (1, d)$ for some $d$ and the definition of $1_{\alpha} \subset \mathcal{P}_X$. Let us fix some $\alpha$ of rank $r > 1$ and let us assume that $1_{\beta}$ belongs to $\mathcal{P}_X$ for all $\beta$ of rank strictly less than $r$. Let us choose some $d \leq \mu(\alpha)$ and write $\beta = \alpha - (1, d)$. Then the proof of Proposition 5.2.

Consider the cartesian diagram obtained by restricting (5.1) to the open strata $S_{\alpha}$. We argue by induction on the rank $r$ of $\alpha$. If $r = 1$ then $\alpha = (1, d)$ for some $d$ and the definition of $1_{\alpha} \subset \mathcal{P}_X$. Let us fix some $\alpha$ of rank $r > 1$ and let us assume that $1_{\beta}$ belongs to $\mathcal{P}_X$ for all $\beta$ of rank strictly less than $r$. Let us choose some $d \leq \mu(\alpha)$ and set $\beta = \alpha - (1, d)$. Then the proof of Proposition 5.2. □
We are now in position to prove Theorem 5.1. Let $\mathbf{a} = (\alpha_1, \ldots, \alpha_l)$ be some HN type of weight $\alpha = \sum \alpha_i$. By Proposition 5.2 all the perverse sheaves $IC(S_{\alpha_i}) = 1_{\alpha_i}$ belong to $P_X$. We will show that $IC(S_{\mathbf{a}})$ belongs to $P_X$ as well by proving that

$$\text{Hom}(IC(S_{\mathbf{a}}), 1_{\alpha_1} \ast \cdots \ast 1_{\alpha_l}) \neq \{0\}. \quad (5.4)$$

For this, consider the (iterated) induction diagram

$$\begin{align*}
\text{Coh}^\alpha_X \times \cdots \times \text{Coh}^\alpha_X &\xrightarrow{p_1} \mathcal{E}^{(l)}_{\alpha_1, \ldots, \alpha_l} \xrightarrow{p_2} \text{Coh}^\alpha_X.
\end{align*} \quad (5.5)$$

We claim that $S_{\mathbf{a}}$ is open and dense in $\text{Im}(p_2)$. Indeed, set $\mathcal{E}^{(l)}_{\alpha_1, \ldots, \alpha_l} = p_1^{-1}(S_{\alpha_1} \times \cdots \times S_{\alpha_l})$. It is an open dense subset of $\mathcal{E}^{(l)}_{\alpha_1, \ldots, \alpha_l}$ and by construction $p_2(\mathcal{E}^{(l)}_{\alpha_1, \ldots, \alpha_l}) = S_{\mathbf{a}}$. Since $p_2$ is continuous, $p_2^{-1}(\text{Coh}^\alpha_X \setminus S_{\mathbf{a}})$ is an open substack of $\mathcal{E}^{(l)}_{\alpha_1, \ldots, \alpha_l}$ which does not intersect $\mathcal{E}^{(l)}_{\alpha_1, \ldots, \alpha_l}$. But this means that $p_2^{-1}(\text{Coh}^\alpha_X \setminus S_{\mathbf{a}})$ is empty, i.e. that $\text{Im}(p_2) \subset S_{\mathbf{a}}$ as wanted.

By definition, $1_{\alpha_1} \ast \cdots \ast 1_{\alpha_l} = p_2(1_{\mathcal{E}^{(l)}_{\alpha_1, \ldots, \alpha_l}})$. This is a semisimple complex and, by the above, $S_{\mathbf{a}}$ is open in its support. Therefore

$$\text{Hom}(IC(S_{\mathbf{a}}), 1_{\alpha_1} \ast \cdots \ast 1_{\alpha_l}) = \text{Hom}(1_{S_{\mathbf{a}}}, j_2^*(1_{\alpha_1} \ast \cdots \ast 1_{\alpha_l})) = \text{Hom}(1_{S_{\mathbf{a}}}, j_2^*p_2!(1_{\mathcal{E}^{(l)}_{\alpha_1, \ldots, \alpha_l}}))$$

where $j_2 : S_{\mathbf{a}} \to \text{Coh}^\alpha_X$ denote the inclusion. Observe that by the uniqueness of the Harder-Narasimhan filtration $F_1 \subset \cdots \subset F_\ell = F$ of a coherent sheaf $F \in S_{\mathbf{a}}$, the projective map $p_2$ restricts to an isomorphism $p_2^{-1}(S_{\alpha_2}) \simeq S_{\alpha_2}$. By base change, $j_2^*p_2!(1_{\mathcal{E}^{(l)}_{\alpha_1, \ldots, \alpha_l}}) = 1_{S_{\mathbf{a}}}$. But then $\text{Hom}(IC(S_{\mathbf{a}}), 1_{\alpha_1} \ast \cdots \ast 1_{\alpha_l}) = \mathbb{Q}$, and (5.4) follows. Theorem 5.1 is proved.

**Remark 5.3.** The collection of simple perverse sheaves $\{IC(S_{\mathbf{a}})\}_\mathbf{a}$ by no means exhausts of all $Q_X$. We illustrate this by a very simple example showing that one has at least to consider the various Brill-Noether stratas $W^k_{r,d}$ of the stacks $\text{Coh}^{r,d}$ (see e.g. [ACGH] or [K2]; note that we use slightly different notation). Let $(r, d) = (2, 0)$, and let us consider the direct summands of $1_{(1,0)} \ast 1_{(1,0)}$. The stack $S_{2,0} \subset \text{Coh}^{2,0}$ of semistable bundles may be stratified as follows: $S_{2,0} = W^0_{2,0} \cup W^1_{2,0} \cup W^2_{2,0} \cup U$ where

$$\begin{align*}
W^0_{2,0} &= \{V \in S_{2,0} \mid \text{Hom}(\mathcal{L}, V) = \{0\}, \forall \mathcal{L} \in \text{Pic}^0_X\},
W^1_{2,0} &= \{V \in S_{2,0} \mid \exists! \mathcal{L} \in \text{Pic}^0_X, \text{Hom}(\mathcal{L}, V) = \mathbb{Q}\},
U &= \{V \in S_{2,0} \mid \exists \mathcal{L}, \mathcal{L}' \in \text{Pic}^0_X, \mathcal{L} \neq \mathcal{L}', V \simeq \mathcal{L} \oplus \mathcal{L}'\},
W^2_{2,0} &= \{V \in S_{2,0} \mid \exists! \mathcal{L} \in \text{Pic}^0_X, \text{Hom}(\mathcal{L}, V) = \mathbb{Q}^2\}.
\end{align*}$$

The strata $W^1_{2,0}$ consists of semistable vector bundles which are nontrivial extensions

$$0 \longrightarrow \mathcal{L} \longrightarrow V \longrightarrow \mathcal{L}' \longrightarrow 0$$

of two degree zero line bundles $\mathcal{L}, \mathcal{L}'$; the strata $W^2_{2,0}$ consists of semistable bundles of the form $V \simeq \mathcal{L}^\oplus 2$ for some degree zero line bundle $\mathcal{L}$. Moreover, $W^0_{2,0}$ is open dense, $W^2_{2,0}$ is closed and we have inclusions of strata closures $W^1_{2,0} \supset U \supset W^2_{2,0}$. The restriction

$$p'_2 : p_2^{-1}(S_{2,0}) \to S_{2,0}$$

of the proper map $p_2$ in the induction diagram (1.1) corresponding to $1_{(1,0)} \ast 1_{(1,0)}$ respects the above stratification. The following table lists the dimension of each strata as well as the type of fiber:
It follows from the above table that $p'_2$ is a small resolution of the closure $\overline{W^1_{2,0}}$ of $W^1_{2,0}$, and hence that $p'_2(\overline{\mathrm{Pic}^{-1}(\mathcal{S}_{2,0})}) = IC(\overline{W^1_{2,0}})$. Considering induction products of the form $1_{1,d} \ast 1_{1,-d}$ for $d = 1, \ldots, 2g - 2$ we obtain elements of $\mathcal{P}_X$ supported on other nontrivial Brill-Noether type stratas of $\mathcal{S}_{2,0}$.

**Remark 5.4.** The method of proof of Theorem 5.1 is readily transposable to the context of quivers. Let $\overline{Q}$ be a quiver and let us assume that it contains no oriented cycles. Lusztig defined in [2] a set $Q_{\overline{Q}}$ of simple perverse sheaves over the moduli stacks $\mathcal{M}_{\overline{Q}, \alpha}$, $\alpha \in \mathcal{K}_0(\overline{Q})$. To any linear form $\Theta : \mathcal{K}_0(\overline{Q}) \to \mathbb{C}$ (the ‘stability parameter’) one may attach a slope function $\mu_{\Theta}$ on $\mathcal{K}_0(\overline{Q})$ and a Harder-Narasimhan stratification $\mathcal{M}_{\Theta} = \bigcup_{\alpha} S_{\alpha}$ (see [L1]). Then

**Theorem 5.5.** For any stability parameter $\Theta$ and any Harder-Narasimhan strata $S_{\alpha}$ we have $IC(S_{\alpha}) \in \mathcal{P}_{\overline{Q}}$.

Note that the analog of Proposition 5.2 holds since $\overline{Q}$ has no oriented cycles. Theorem 5.4 also holds in the context of quivers, where it is a simple consequence of Reineke’s inversion formula (R1).

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