SOME ASPECTS OF 2d INTEGRABILITY

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We study the KdV and Burgers nonlinear systems and show in a consistent way that they can be mapped to each other through a strong requirement about their evolutions’ flows to be connected. We expect that the established mapping between these particular systems should shed more light towards accomplishing some unification’s mechanism for KdV hierarchy’s integrable systems.

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GENERAL MOTIVATIONS

An interesting subject which have been studied recently from different point view deals with the field of non linear integrable systems. These are exactly solvable models exhibiting a very rich structure in lower dimensions and are involved in in many areas of modern sciences and more particularly in mathematics and physics.

From the physics point of view, integrable systems are known to play crucial role in describing physical phenomena in many areas such as condensed matter physics, hydrodynamics, plasma physics, high energy physics, nonlinear optics and so on.

Non linear integrable models, are associated to systems of non linear differential equations which can be solved exactly. Solving such kind of differential equations in general is not an easy job, we are constrained to introduce rigorous backgrounds such as the theory of pseudo-differential operators, Lie algebra and some physical methods such as the scattering inverse method and the related Lax formulation.

The particularity of 2d integrable systems is due to the pioneering role that they deserve to the nonlinear KdV differential equation. We focus in this work to study some properties related to this prototype nonlinear differential equation and show how one can reinforce its central role. We guess that such an objective is possible since a mapping between the KdV and the Burgers’s nonlinear systems is possible by means of the Miura transformation connecting the Lax operators of the two systems and a constrained requirements about the associated evolution flows. The existing mapping is expected to shed more light towards an accomplishment of the unification’s mechanism.

PSEUDO-DIFFERENTIAL OPERATORS

One way to introduce pseudo-differential operators is by using the so called KP hierarchy. It’s a defined as an infinite set of differential equations. These equations are in their turn defined through a pseudo-differential operator Q of the form

\[ Q = \partial + q_0 \partial^{-1} + q_1 \partial^{-2} + \ldots \]  

where \( \partial \) denotes \( \frac{\partial}{\partial x} \). Another kind of pseudo-differential operators can be obtained by using the KdV Lax operator

\[ \mathcal{L}(t) = \partial^2 + u_2(x,t) \]  

with \( u_2(x,t) \equiv u_2(x,t_3,t_5,...) \) is the KdV potential exhibiting a conformal weight \( s = 2 \). A typical example of pseudo-differential operators that can emerge from \( \mathcal{L}(t) \) is given by its square root \( \mathcal{L}^+(t) \). This is an infinite series in inverse powers of \( \partial \) namely,

\[ \mathcal{L}^+(t) = \partial + \frac{u}{2} \partial^{-1} - \frac{u'}{4} \partial^{-2} + (\frac{u''}{8} - \frac{u^2}{8}) \partial^{-3} + \ldots \]  

Note that \( \mathcal{L}^+(t) \) is an operator of weight \( |\mathcal{L}^+(t)| = 1 \). The KdV operator \( \mathcal{L}(t) \equiv \mathcal{L}_2 \) is also known to be the essential key towards building the so called 2--reduced KP hierarchy or KdV hierarchy whose form is given by

\[ \frac{\partial \mathcal{L}}{\partial t_{2k+1}} = [\mathcal{L}_{2k+1}^{2k+1}, \mathcal{L}] \]  

We have to precise that the following conventions notations are used:

- The prime derivative \( u' \) is with respect to the variable \( x \), ie \( u' = \frac{\partial u}{\partial x} \).
- The \( t_{2k+1} \) describe an infinite number of evolution parameters associated with the KdV hierarchy. The first parameter is \( t_1 = x \).
• $\mathcal{L}_{+}^{2k+1}$ is a local operator which means also the restriction to only positive part of the $(2k+1)$th power of the formal pseudo-differential operator $\mathcal{L}_{+}^{\frac{1}{2}}$. As an example $\mathcal{L}_{+}^{\frac{1}{2}} = \partial$.

• $\mathcal{L}_{+}^{2k+1}$ is an object of weight $|\mathcal{L}_{+}^{2k+1}| = (2k+1)$,

• Given the explicit form of $\mathcal{L}_{+}^{\frac{1}{2}}$ one can easily determine the form of $\mathcal{L}_{+}^{2k+1}$ by proceeding as follows $\mathcal{L}_{+}^{2k+1} = \mathcal{L}^{k} \mathcal{L}_{+}^{\frac{1}{2}} = (\partial^{2} + u_{2})^{k} \mathcal{L}_{+}^{\frac{1}{2}}$

Lax pair formalism

The principal idea, due to Lax, of this formalism rests on our interest to solve any given nonlinear system. One should emphasize that the Lax formalism is intimately related to the well known inverse scattering method (ISM). In fact given a nonlinear evolution equation, the principal tasks is to find a linear operator whose eigenvalues are constant under the nonlinear evolution. This is one of the success of the ISM. By the way, we refer the reader to [2] for more details about important aspects of integrable models.

The linear case

Let’s consider a linear evolution equation described by a time independent Hamiltonian $H$. The question consists in finding operators whose expectation values are preserved with time. Assume that $X$ is an operator satisfying such a property, then from the point of view of Heisenberg picture, $X(t)$ is required to be unitarily equivalent to $X(0)$ such that

$$U^{+}(t)X(t)U(t) = X(0),$$

where $U(t)$ is the time evolution operator given by

$$U(t) = exp(-iHt).$$

Straightforward computations, based on the derivation of the last equation from both sides, lead to

$$\frac{\partial X(t)}{\partial t} = i [X(t), H].$$

Requesting for the expectation value of $X(t)$ (Its eigenvalue) to be time independent is compatible with this equation. Furthermore, we have

$$\frac{\partial U(t)}{\partial t} = iHU(t) = BU(t)$$

where $B = -iH$ is an anti Hermitian operator $B^{+} = -B$ and

$$U^{+}(t)U(t) = 1,$$

The nonlinear case

We will follow the same steps relatives to the previous linear case and consider a nonlinear evolution equation such that

$$\mathcal{L}(u(x,t)) = \mathcal{L}(t).$$

denote the linear operator that we should determine with $u(x,t)$ is the dynamical variable in $(1+1)$ dimensions. In the case of the water waves, for example, this particular variable is interpreted as being the height of the wave above the water surface. One can also show that $u(x,t)$ can exhibits a quantum number, namely the conformal weight $s$, depending on the order of the KdV hierarchy. For consistency requirements, one assumes that $\mathcal{L}(t)$ is Hermitian and that its eigenvalues are independent of $t$. To do so, one also suppose the existence of an unitary operator $\bar{U}(t)$ such that

$$U^{+}(t)\mathcal{L}(t)\bar{U}(t) = \mathcal{L}(0).$$

The same steps followed previously lead to the following form of the evolution equation

$$\frac{\partial \mathcal{L}(t)}{\partial t} = [B(t), \mathcal{L}(t)]$$

One can then conclude that for $\mathcal{L}(t)$ to be isospectral it must satisfy a relation similar to the linear case obtained previously namely eq(4). The essential goal is then to find a linear operator $\mathcal{L}(t)$ in $u(x,t)$ and a second one $B$, satisfying as before $BU(t) = \frac{\partial U(t)}{\partial t}$, not necessary linear in such a way that the commutator $[B(t), \mathcal{L}(t)]$ reproduces the evolution of the dynamical variable $u(x,t)$. This means that the eigenvalues of $\mathcal{L}(t)$ are independent of $t$. We have

$$\mathcal{L}(t)\psi(t) = \lambda \psi(t).$$

where $\psi(t) = U(t)\psi(0)$ and the evolution with $t$ gives

$$\frac{\partial \psi(t)}{\partial t} = \frac{\partial U(t)}{\partial t} \psi(0) = B(t)\psi(t)$$

Definition:

If they exist, the operators $B(t)$ and $\mathcal{L}(t)$ are called the Lax pair associated to a given nonlinear evolution equation.

Properties:

• The Lax pair $B(t)$ and $\mathcal{L}(t)$ operators are important in the sense that they can constitute a guarantee for integrability of the original nonlinear evolution equation.

• The existence of $B(t)$ and $\mathcal{L}(t)$ means that the nonlinear evolution equation is linearizable.
The linear form of the evolution equation is given by
\[ \frac{\partial L(t)}{\partial t} = [B(t), L(t)]. \]
The Lax pair \( B(t) \) and \( L(t) \) play a central role in finding the solution of the evolution equation.
The operator \( L \) is usually called the lax operator, fixing the integrable model while \( B \) is the Hamiltonian of the system which is the local fractional power of \( L \).

**TOWARDS A KDV-HIERARCHY’S UNIFICATION**

**The Burgers Equation**

We present in this subsection the nonlinear Burgers system. Actually, our interest in this equation comes from its several important properties that we give as follows:

1. The Burgers equation is defined on the \((1+1)\)-dimensional space time. In the standard pseudo-differential operator formalism, this equation is associated to the following \( L \)-operator
   \[ L_{Burg} = \partial + u_1(x, t) \quad (15) \]
   where the function \( u_1 \) is the Burgers potential of conformal weight \(|u_1| = 1\). Using our convention notations, we can set \( L_{Burg} \in \Sigma_1^{(0,1)} \).

2. With respect to the previous \( L \)-operator, the non-linear differential equation of the Burgers equation is given by
   \[ \dot{u}_1 + \alpha u_1 u'_1 + \beta u''_1 = 0, \quad (16) \]
   where \( \dot{u} = \partial_{t_{Burg}} u \) and \( u' = \partial_x u \). The dimensions of the underlying objects are given by \([t_{Burg}] = -2 = -[\partial_{t_{Burg}}] \), \([x] = -1 \). One can then set \( t_{Burg} \equiv t_2 \).

3. On the commutative space-time, the Burgers equation can be derived from the Navier-Stokes equation and describes real phenomena, such as the turbulence and shock waves. In this sense, the Burgers equation draws much attention amongst many integrable equations.

4. It can be linearized by the Cole-Hopf transformation. The linearized equation is the diffusion equation and can be solved by Fourier transformation for given boundary conditions.

**Proposition 2:**
The Burgers Lax operator \( L_{Burg} \) is a local differential operator obtained through the following truncation of the KP pseudo-differential operator, namely
\[ L_{KP} = \partial + u_1 + u_2 \partial^{-1} + u_3 \partial^{-2} + \ldots \]
The local truncation is simply given by
\[ \Sigma_1^{(\infty,1)} \rightarrow \Sigma_1^{(0,1)}, \quad (17) \]
such that to any KP pseudo operator \( L_{KP} \in \Sigma_1^{(\infty,1)} \)
\[ L_{KP} \rightarrow \partial + u_1 = L_{Burg} \equiv (L_{KP})_+ \quad (18) \]

**Remark:**
\[ L_{Burg} \in \Sigma_1^{(0,1)} \equiv \Sigma_1^{(\infty,1)} \bigoplus \Sigma_1^{(\infty,-1)}, \]

**The KdV system**

The KdV equation plays a central role in 2d integrable systems. We present here below some of its remarkable properties

1. The KdV operator is given by
   \[ L_{K,dv} = \partial^2 + u_2(x, t) \quad (19) \]
   where the function \( u_2 \) is the KdV potential of conformal weight \(|u_2| = 2\). Using the same convention notations, we can set \( L_{K,dv} \in \Sigma_2^{(0,2)}/\Sigma_2^{(1,1)} \).

2. The Lax equation associated to the \( L_{K,dv} \)-operator is given by
   \[ \frac{\partial L}{\partial t_3} = [L, (L^2)_+], \quad (20) \]
   where straightforward calculations show that
   \[ (L^2)_+ = \partial^3 + \frac{3}{4}(\partial_u u_2 + u_2 \partial) + O(\partial^{-1}) \quad (21) \]

3. Explicit form of the previous Lax equation gives
   \[ \dot{u}_2 = 6 u_2 u'_2 + u''_2, \quad (22) \]
   which is nothing but the KdV equation where \( \dot{u} = \partial_{t_{K,dv}} u \). The dimensions of the underlying objects are given by \([t_{K,dv}] = -3 = -[\partial_t] \) and \([u_2] = 2 \). One can then set \( t_{K,dv} \equiv t_3 \).

4. Let’s consider the KdV equation \( \dot{u}_2 = 6 u_2 u'_2 + u''_2 \). Modulo the following scalings
   \[ \partial_{t_3} \rightarrow \frac{1}{4} \partial_{t_3}, \quad u_2 \rightarrow \frac{t_3}{4} u_2 \]
   this equation maps to an equivalent form, namely
   \[ \dot{u}_2 = \frac{3}{2} u_2 u'_2 + u''_2 \]
a form that we find in some related works (see for instance \[2\]).

Remarks:

1. For any given function \( f(u) \), the action of the \( \partial \)-derivation on this function is given by

\[
\partial f = f' + f \partial
\]

with \( f' \equiv \partial_x f \).

2. In eq\((\), \( (\partial u_2 + u_2 \partial) \equiv \frac{3}{4}(u_2' + 2u_2 \partial)\)

Proposition 3:
The Burgers Lax operator \( L_{\text{Burg}} \) is a local differential operator obtained through the following truncation of the KP pseudo-differential operator, namely

\[
L_{KP} = \partial + u_1 + u_2 \partial^{-1} + u_3 \partial^{-2} + ....
\]

The local truncation is simply given by

\[
\Sigma_{1}^{(-\infty, 1)} \to \Sigma_{1}^{(1, 0)}, \quad (23)
\]

such that to any KP pseudo operator \( L_{KP} \in \Sigma_{1}^{(-\infty, 1)} \)

\[
L_{KP} \mapsto \partial + u_1 = L_{\text{Burg}} \equiv (L_{KP})_{+} \quad (24)
\]

Remark:

\[
L_{\text{Burg}} \in \Sigma_{1}^{(0, 1)} \equiv [\Sigma_{1}^{(-\infty, 1)}]_{+} \equiv \Sigma_{1}^{(-\infty, 1)}/\Sigma_{1}^{(-\infty, -1)},
\]

The KdV-Burgers mapping

This section will be devoted to another significant aspect of integrable models, namely their possible unification. In some sense, one focuses to study the possibility to establish the existence of a law allowing transitions between known integrable systems, more notably those belonging to the generalized KdV hierarchy. The encouraging facts to follow such a way is that these models share at least the integrability’s property.

The principal focus, for the moment, is on the models discussed previously namely the KdV and Burgers systems. The idea to connect the two models was originated from the fact that integrability for the KdV system is something natural due to the possibility to connect with 2\(d\) conformal symmetry. We think that the strong backgrounds of conformal symmetry can help shed more light about integrability of the Burgers systems if one knows how to establish such a connection\[2\].

On the other hand, it is clear that these models are different due to the fact that for KdV system the Lax operator as well as the associated field \( u_2(x, t) \) are of conformal weights 2, whereas for the Burgers system, \( L_{\text{Burg}} \) and \( u_1 \) are of weight 1.

Our goal is to study the possibility of transition between the two spaces \( \Sigma_{2}^{(0, 2)}/\Sigma_{2}^{(1, 1)} \) and \( \Sigma_{1}^{(0, 1)} \) corresponding respectively to the two models. This transition, once it exists, should lead to extract more information on these models and also on their integrability.

To start, let us consider the following property

Proposition 4:
Let us consider the Burgers Lax operator \( L_{\text{Burg}}(u_1) = \partial + u_1 \in \Sigma_{1}^{(0, 1)} \). For any given \( s_{2} \) KdV operator \( L_{\text{KdV}}(u_2) = \partial^2 + u_2 \) belongings to the space \( \Sigma_{2}^{(0, 2)}/\Sigma_{2}^{(1, 1)} \), one can define the following mapping

\[
\Sigma_{1}^{(0, 1)} \mapsto \Sigma_{2}^{(0, 2)}/\Sigma_{2}^{(1, 1)}, \quad (25)
\]

in such a way that

\[
L_{\text{Burg}}(u_1) \to L_{\text{KdV}}(u_2) \equiv L_{\text{Burg}}(u_1) \times L_{\text{Burg}}(-u_1). \quad (26)
\]

We know that the space \( \Sigma_{2}^{(0, 2)} \) of KdV Lax operators of weight \( s = 2 \) is different from the one of operators of weight \( s = 1 \) namely \( \Sigma_{1}^{(0, 1)} \). What we are assuming in this proposition is a strong constraint leading to connect the two spaces. This constraint is also equivalent to set

\[
\Sigma_{2}^{(0, 2)}/\Sigma_{2}^{(1, 1)} \equiv \Sigma_{1}^{(0, 1)} \otimes \Sigma_{1}^{(0, 1)} \quad (27)
\]

Next we are interested in discovering the crucial key behind the previous proposition. For this reason, we underline that this mapping is easy to highlight through the well known Miura transformation

\[
L_{KdV} = \partial^2 + u_2 = (\partial^1 + u_1) \times (\partial^1 - u_1) \quad (28)
\]

giving rise to

\[
u_2 = -u_1^2 - u_1'. \quad (29)
\]

This is an important property since one has the possibility to realize the KdV \( s_{2} \) field \( u_2 \) in term of the Burgers field \( u_1 \) and its derivative \( u_1' \). This realization shows among other an underlying nonlinear behavior in the KdV field \( u_2 \) given by the quadratic term \( u_1^2 \).

However, proposition 1 can have a complete and consistent significance only if one manages to establish a connection between the differential equations associated to the two systems. Arriving at this stage, note that besides the principal difference due to conformal weight,
we stress that the two nonlinear evolutions equations: 
\( \dot{u}_2 = \frac{3}{2} uu' + \theta^2 u''' \) and 
\( \frac{1}{20} u'' + 2(1 - \eta) uu' - \xi u''' = 0 \)

of KdV and Burgers systems respectively are distinct by 
a remarkable fact that is the KdV flow 
\( t_{KdV} \equiv t_3 \) and 
the Burgers one \( t_{Burgers} \equiv t_2 \) have different conformal 
weights: \( [t_{KdV}] = -3 \) whereas \( [t_{Burgers}] = -2 \).

In order to be consistent with the objective of proposition 1, based on the idea of the possible link between the 
two integrable systems, presently we are constrained to 
circumvent the effect of proper aspects specific to both 
equations and consider the following second property:

**Proposition 5**

By virtue of the Burgers-KdV mapping and dimensional 
arguments, the associated flows are related through the 
following ansatz

\[ \partial t_2 \rightarrow \partial t_3 \equiv \partial t_2 \partial_x + \alpha \partial x^3 \]  

(30)

for an arbitrary parameter \( \alpha \).

With respect to assumption (i), relating the two 
evolution derivatives \( \partial t_2 \) and \( \partial t_3 \) belonging to Burgers 
and KdV’s hierarchies respectively, one should expect 
some strong constraint on the Burgers and KdV cur-
rents.

We have to identify the following three differential 
equations

\[ \partial t_3 u_2 = 6u_2 u'_2 + u'''_2, \]
\[ = -2u_3 \partial t_3 u_1 - \partial t_3 u'_1, \]  

(31)
\[ = \partial t_3 u'_2 + \alpha u'''_2. \]

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