ON THE HASSE PRINCIPLE FOR FINITE GROUP SCHEMES
OVER GLOBAL FUNCTION FIELDS

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Abstract. Let $K$ be a global function field of characteristic $p > 0$ and let $M$ be a (commutative) finite and flat $K$-group scheme. We show that the kernel of the canonical localization map $H^1(K, M) \longrightarrow \prod_{\text{all } v} H^1(K_v, M)$ in flat (fppf) cohomology can be computed solely in terms of Galois cohomology. We then give applications to the case where $M$ is the kernel of multiplication by $p^m$ on an abelian variety defined over $K$.

1. Statement of the main theorem

Let $K$ be a global field and let $\overline{K}$ be a fixed algebraic closure of $K$. Let $K^s$ be the separable closure of $K$ in $\overline{K}$ and set $G_K = \text{Gal}(K^s/K)$. Further, for each prime $v$ of $K$, let $\overline{K}_v$ be the completion of $\overline{K}$ at a fixed prime $\overline{v}$ of $\overline{K}$ lying above $v$ and let $K_v^s$ denote the completion of $K^s$ at the prime of $K^s$ lying below $\overline{v}$. Set $G_v = \text{Gal}(K_v^s/K_v)$. If $M$ is a commutative, finite and flat $K$-group scheme, let $H^i(G_K, M(K^s))$ (respectively, $H^i(G_v, M(K_v^s))$) denote the Galois cohomology group $H^i(G_K, M)$ (respectively, $H^i(G_v, M)$) in Galois cohomology, has been discussed in [12, 6]. See also [9], §I.9, pp. 117-120. However, if $K$ is a global function field of characteristic $p > 0$ and $M$ has $p$-power order, the injectivity of the canonical localization map $\beta^1(K, M): H^1(K, M) \rightarrow \prod_{\text{all } v} H^1(K_v, M)$ in flat (fppf) cohomology has not been discussed before (but see [7], Lemma 1, for some particular cases). In this paper we investigate this problem and show that the injectivity of $\beta^1(K, M)$ depends only on the finite $G_K$-module $M(K^s)$, which may be regarded as the maximal étale $K$-subgroup scheme of $M$. Indeed, let $\Pi^1(K, M) = \text{Ker } \beta^1(K, M)$ and set $\Pi^1(G_K, M) = \text{Ker } [H^1(G_K, M) \rightarrow \prod_{\text{all } v} H^1(G_v, M)]$. Then the following holds.

Main Theorem. Let $K$ be a global function field of characteristic $p > 0$ and let $M$ be a commutative, finite and flat $K$-group scheme. Let $v$ be any prime of $K$. Then the inflation map $H^1(G_K, M) \rightarrow H^1(K_v, M)$ induces an isomorphism

\[ \text{Ker } [H^1(G_K, M) \rightarrow H^1(G_v, M)] \cong \text{Ker } [H^1(K, M) \rightarrow H^1(K_v, M)]. \]

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In particular, $\text{III}^1(K, M) \simeq \text{III}^1(G_K, M)$.

Thus the Hasse principle holds for $M$, i.e., $\text{III}^1(K, M) = 0$, if, and only if, the Hasse principle holds for the $G_K$-module $M(K^s)$. An interesting example is the following. Let $A$ be an ordinary abelian variety over $K$ such that the Kodaira-Spencer map has maximal rank. Then $A_{p^n}(K^s) = 0$ for every integer $m \geq 1$ \cite{17}, Proposition, p.1093, and we conclude that the Hasse principle holds for $A_{p^n}$.

The theorem is proved in the next Section. In Section 3, which concludes the paper, we develop some applications.

2. Proof of the main theorem

We keep the notation introduced in the previous Section. In addition, we will write $(\text{Spec } K)_{\text{fl}}$ for the flat site on $\text{Spec } K$ as defined in \cite{3}, II.1, p.47, and $H^r((\text{Spec } K)_{\text{fl}}, M)_{\text{fppf}}$ will denote $H^r((\text{Spec } K)_{\text{fl}}, M)$. We will see presently that, in fact, $H^r(K, M) \simeq H^r((\text{Spec } K)_{\text{fppf}}, M)$.

By a theorem of M.Raynaud (see \cite{13} or \cite{1}, Theorem 3.1.1, p.110), there exist abelian varieties $A$ and $B$ defined over $K$ and an exact sequence of $K$-group schemes

\begin{equation}
0 \to M \xrightarrow{\iota} A \xrightarrow{\psi} B \to 0,
\end{equation}

where $\iota$ is a closed immersion. For $r \geq 0$, let $\iota^{(r)} : H^r(K, M) \to H^r(K, A)$ and $\psi^{(r)} : H^r(K, A) \to H^r(K, B)$ be the maps induced by $\iota$ and $\psi$. The long exact flat cohomology sequence associated to \text{(1)} yields an exact sequence

\begin{equation}
0 \to \text{Coker} \psi^{(r-1)} \to H^r(K, M) \to \text{Ker} \psi^{(r)} \to 0,
\end{equation}

where $r \geq 1$. Since the groups $H^r(K, A)$ and $H^r(K, B)$ coincide with the corresponding étale and fppf cohomology groups \cite{3}, Theorem III.3.9, p.114, and \cite{5}, Theorem 11.7, p.180, we conclude that $H^r(K, M) = H^r((\text{Spec } K)_{\text{fppf}}, M)$.

Lemma 2.1. Let $v$ be any prime of $K$. If $A$ is an abelian variety defined over $K$, then $A(K^s) = A(\overline{K}) \cap A(K_v^s)$, where the intersection takes place inside $A(\overline{K}_v)$.

Proof. Let $F/K$ be a finite subextension of $K^s/K$ and let $F_v$ denote the completion of $F$ at the prime of $F$ lying below $v$. Choose an element $t \in F$ such that $F_v = k((t))$, where $k$ is the field of constants of $F$, and let $m \geq 1$ be an integer. Then $F_v^{p^m} = k((t^{p^m})) = \sum_{i=1}^{\frac{p^m}{p^m-1}} F_v \subset F_v^{p^m} F_v$, whence $F_v^{p^m} = F_v^{p^m} F_v$. Now let $a \in \overline{K}$ be inseparable over $K$. Then there exists an integer $m \geq 1$ and an extension $F/K$ as above such that $K(a) = F^{p^m}$. Consequently $a \in K(a) \cdot F_v = F_v^{p^m} F_v = F_v^{p^m}$, whence $a$ is also inseparable over $K_v$. This shows that $K^s = \overline{K} \cap K_v^s$. Now let $V \subset \mathbb{A}_K^n$ be an affine $K$-variety and let $P = (x_1, \ldots, x_n) \in V(\overline{K}) \cap V(K_v^s)$. Then each $x_i \in \overline{K} \cap K_v^s = K^s$, whence $P \in V(K^s)$. Thus $V(K^s) = V(\overline{K}) \cap V(K_v^s)$, and the lemma is now clear since $A$ is covered by affine $K$-varieties. \qed
If \( v \) is a prime of \( K \), we will write \( \psi_v = \psi \times_{\Spec K} \Spec K_v \). Since \( H^1(K^s, A) = H^1(K_v^s, A) = 0 \), the exact sequence \([1]\) yields a commutative diagram

\[
\begin{array}{ccc}
B(K_s)/\psi(A(K_s)) & \sim & H^1(K^s, M) \\
\downarrow & & \downarrow \\
B(K_v^s)/\psi_v(A(K_v^s)) & \sim & H^1(K_v^s, M).
\end{array}
\]

**Lemma 2.2.** Let \( v \) be a prime of \( K \). Then the canonical map

\[
B(K^s)/\psi(A(K^s)) \to B(K_v^s)/\psi_v(A(K_v^s))
\]

is injective.

**Proof.** Write \( M = \Spec R \), where \( R \) is a finite \( K \)-algebra, and identify \( M(\overline{K}_v) \) with \( \Hom_{K_v}(K_v \otimes_K R, \overline{K}_v) \). If \( s \in M(\overline{K}_v) \), then the image of the composition \( f: R \to K_v \otimes_K R \xrightarrow{\sim} \overline{K}_v \) is a finite \( K \)-algebra and so, in fact, a finite field extension of \( K \). Consequently, \( \tilde{f} \) factors through some \( f \in \Hom_K(R, \overline{K}) = M(\overline{K}) \). This implies that \( M(\overline{K}_v) = M(\overline{K}) \). Now let \( P \in B(K^s) \cap \psi_v(A(K_v^s)) \subseteq B(K_v^s) \) and let \( Q \in A(K_v^s) \) be such that \( P = \psi_v(Q) \). Since \( A(\overline{K}) \xrightarrow{\psi} B(\overline{K}) \) is surjective, there exists an \( R \in A(\overline{K}) \) such that \( \psi(R) = P \). Then \( R - Q \in M(\overline{K}_v) = M(\overline{K}) \). This shows that \( Q \in A(\overline{K}) \cap A(K_v^s) = A(K^s) \), by the previous lemma. Thus \( P = \psi(Q) \in \psi(A(K^s)) \), as desired.

\[\square\]

The above lemma and diagram \([2]\) show that the localization map \( H^1(K^s, M) \to H^1(K_v^s, M) \) is injective. The main theorem is now immediate from the exact commutative diagram

\[
\begin{array}{cccc}
0 & \to & H^1(G_K, M) & \to H^1(K, M) & \to H^1(K^s, M) \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & H^1(G_v, M) & \to H^1(K_v, M) & \to H^1(K_v^s, M),
\end{array}
\]

whose rows are the inflation-restriction exact sequences in flat cohomology \([16]\), p.422, line -12.

### 3. Applications

Let \( K \) and \( M \) be as in the previous Section. We will write \( K(M) \) for the subfield of \( K^s \) fixed by \( \ker [G_K \to \Aut(M(K^s))] \). We note that the Hasse principle is known to hold for \( M(K^s) \) under any of the following hypotheses:

(a) \( \Gal(K(M)/K) \subseteq \Aut(M(K^s)) \) is cyclic. See \([9]\), Lemma I.9.3, p.118.
(b) \(M(K^s)\) is a simple \(G_K\)-module such that \(pM(K^s) = 0\) and \(\text{Gal}(K(M)/K)\) is a \(p\)-solvable group, i.e., \(\text{Gal}(K(M)/K)\) has a composition series whose factors of order divisible by \(p\) are cyclic. See [9], Theorem I.9.2(a), p.117.

(c) There exists a set \(T\) of primes of \(K\), containing the set \(S\) of all primes of \(K\) which split completely in \(K(M)\), such that \(T \setminus S\) has Dirichlet density zero and \([K(M): K] = \text{l.c.m.}\{[K(M)_v: K_v]: v \in T\}\). See [11], Theorem 9.1.9(iii), p.528.

In this Section we focus on case (a) above when \(M = A_{p^m}\) is the \(p^m\)-torsion subgroup scheme of an abelian variety \(A\) defined over \(K\). More precisely, we are interested in the class of abelian varieties \(A\) such that \(A_{p^m}(K^s)\) is cyclic, for then \(\text{Gal}(K(A_{p^m})/K) \twoheadrightarrow \text{Aut}(A_{p^m}(K^s))\) is cyclic as well if \(p\) is odd or \(m \leq 2\) and (a) applies. Clearly, this class contains all ordinary abelian varieties \(A\) such that the associated Kodaira-Spencer map has maximal rank since, as noted in Section 1, \(A_{p^m}(K^s)\) is in fact zero. To find more examples, recall that \(A_{p^m}(K) \simeq (\mathbb{Z}/p^m\mathbb{Z})^f\) for some integer \(f\) (called the \(p\)-rank of \(A\)) such that \(0 \leq f \leq \dim A\). Thus, if \(f \leq 1\), then \(A_{p^m}(K^s)\) is cyclic. Clearly, the condition \(f \leq 1\) holds if \(A\) is an elliptic curve, but there exist higher-dimensional abelian varieties \(A\) having \(f \leq 1\). See [13], §4.

**Remark 3.1.** Clearly, \(\text{Gal}(K(A_{p^m})/K)\) may be cyclic even if \(A_{p^m}(K^s)\) is not. For example, let \(k\) be the (finite) field of constants of \(K\), let \(A_0\) be an abelian variety defined over \(k\) and let \(A = A_0 \times_{\text{Spec} K} \text{Spec} \mathcal{K}\) be the constant abelian variety over \(K\) defined by \(A_0\). Then \(A(K^s)_{\text{tors}} = A_0(K)\), and it follows that \(\text{Gal}(K(A_{p^m})/K) \simeq \text{Gal}(k'/k)\) for some finite extension \(k'/k\). Consequently \(\text{Gal}(K(A_{p^m})/K)\) is cyclic and the Hasse principle holds for \(A_{p^m}(K^s)\).

We will write \(A\{p\}\) for the \(p\)-divisible group attached to \(A\), i.e., \(A\{p\} = \varinjlim \{A_{p^m}\}_{m=1}^{\infty}\). If \(B\) is an abelian group, \(B(p) = \bigcup_m B_{p^m}\) is the \(p\)-primary component of its torsion subgroup, \(B^\wedge = \varprojlim B/p^m\) is the adic completion of \(B\) and \(T_p B = \varprojlim B_{p^m}\) is the \(p\)-adic Tate module of \(B\). Further, if \(B\) is a topological abelian group, \(B^D\) will denote \(\text{Hom}_{\text{cont.}}(B, \mathbb{Q}/\mathbb{Z})\) endowed with the compact-open topology, where \(\mathbb{Q}/\mathbb{Z}\) is given the discrete topology.

Let \(X\) denote the unique smooth, projective and irreducible curve over the field of constants of \(K\) having function field \(K\). If \(A\) is an abelian variety over \(K\), we will write \(\mathcal{A}\) for the Néron model of \(A\) over \(X\).

The following statement is immediate from the main theorem and the above remarks.

**Proposition 3.2.** Let \(A\) be an abelian variety defined over \(K\) and let \(m\) be a positive integer. Assume that \(A_{p^m}(K^s)\) is cyclic. Assume, in addition, that \(m \leq 2\) if \(p = 2\). Then the localization map in flat cohomology

\[
H^1(K, A_{p^m}) \rightarrow \prod_{v} H^1(K_v, A_{p^m})
\]
is injective.

The next lemma confirms a long-standing and widely-held expectation.

**Lemma 3.3.** $H^2(K, A) = 0$.

*Proof.* Since $H^2(K_v, A) = 0$ for every $v$ [9], Theorem III.7.8, p.285, it suffices to check that $\mathbb{H}^2(A) = 0$. For any integer $n$, there exists a canonical exact sequence of flat cohomology groups

$$0 \to H^1(K, A)/n \to H^2(K, A_n) \to H^2(K, A)_n \to 0.$$  

Since the Galois cohomology groups $H^i(K, A)$ are torsion in degrees $i \geq 1$ and $\mathbb{Q}/\mathbb{Z}$ is divisible, the direct limit over $n$ of the above exact sequences yields a canonical isomorphism $H^2(K, A) = \lim_{\to} H^2(K, A_n)$. An analogous isomorphism exists over $K_v$ for each prime $v$ of $K$, and we conclude that $\mathbb{H}^2(A)$ is canonically isomorphic to $\lim_{\to} \mathbb{H}^2(A_n)$. Now, by Poitou-Tate duality [9], Theorem I.4.10(a), p.57, and [4], Theorem 4.8, the latter group is canonically isomorphic to the Pontryagin dual of $\mathbb{H}^1(A_n)$, where $A^t$ is the dual abelian variety of $A$. Thus, it suffices to show that $\lim_{\to} \mathbb{H}^1(A_t) = 0$. Let $U$ be the largest open subscheme of $X$ such that $A_t$ extends to a finite and flat $U$-group scheme $\mathcal{A}_t$. For each closed point $v$ of $U$, let $\mathcal{O}_v$ denote the completion of the local ring of $U$ at $v$. Now let $V$ be any nonempty open subscheme of $U$. By the computations at the beginning of [9], III.7, p.280, and the localization sequence [8], Proposition III.1.25, p.92, there exists an exact sequence

$$0 \to H^1(U, \mathcal{A}_t) \to H^1(V, \mathcal{A}_t) \to \bigoplus_{v \in U \setminus V} H^1(K_v, A_t)/H^1(\mathcal{O}_v, \mathcal{A}_t).$$

Taking the direct limit over $V$ in the above sequence and using [4], Lemma 2.3, we obtain an exact sequence

$$0 \to H^1(U, \mathcal{A}_t) \to H^1(K, A_t) \to \prod_{v \in U} H^1(K_v, A_t)/H^1(\mathcal{O}_v, \mathcal{A}_t),$$

where the product extends over all closed points of $U$. The exactness of the last sequence shows the injectivity of the right-hand vertical map in the diagram below

$$\begin{array}{ccccccccc}
0 & \to & H^1(U, \mathcal{A}_t) & \to & H^1(K, A_t) & \to & H^1(K, A_t)/H^1(U, \mathcal{A}_t) & \to & 0 \\
0 & \to & \prod_{v \in U} H^1(\mathcal{O}_v, \mathcal{A}_t) & \to & \prod_{v \in U} H^1(K_v, A_t) & \to & \prod_{v \in U} H^1(K_v, A_t)/H^1(\mathcal{O}_v, \mathcal{A}_t). & \to & 0 \\
\end{array}$$

We conclude that $\mathbb{H}^1(U, A_t) := \text{Ker} \left[ H^1(K, A_t) \to \prod_{v \in U} H^1(K_v, A_t) \right]$ equals

$\mathbb{H}^1(U, A_t) := \text{Ker} \left[ H^1(U, \mathcal{A}_t) \to \prod_{v \in U} H^1(\mathcal{O}_v, \mathcal{A}_t) \right]$. 

□
Now, it is shown in [10], Propositions 5 and 6, that \( \lim_{\leftarrow n} H^1(U, \mathcal{A}_n) = 0 \), whence \( \lim_{\leftarrow n} \prod_{v \notin U} H^1(K_v, A_n) = 0 \). Now the exact sequence
\[
0 \to \prod_{v \notin U} H^1(K_v, A_n) \to \prod_{v \notin U} H^1(K_v, A_n)
\]
shows that \( \lim_{\leftarrow n} \prod_{v \notin U} H^1(K_v, A_n) = 0 \), as desired. \( \Box \)

Remark 3.4. With the notation of the above proof, the kernel-cokernel exact sequence [9], Proposition I.0.24, p.16, for the pair of maps
\[
H^1(K, A_n^t) \to \bigoplus_{v} H^1(K_v, A_n^t) \to \bigoplus_{v} H^1(K_v, A^t)
\]
yields an exact sequence
\[
0 \to \prod_{v} H^0(K_v, A^t) / n.
\]

This injectivity was claimed in [3], p.300, line -8, but the “proof” given there is inadequate and should be replaced by the above one.

**Proposition 3.5.** Let \( A \) be an abelian variety over \( K \) and let \( A^t \) be the corresponding dual abelian variety. Assume that \( A_{p^m}^t(K^s) \) is cyclic, where \( m \) is a positive integer such that \( m \leq 2 \) if \( p = 2 \). Then the localization maps
\[
H^2(K, A_{p^m}) \to \bigoplus_{v} H^2(K_v, A_{p^m})
\]
and
\[
H^1(K, A) / p^m \to \bigoplus_{v} H^1(K_v, A) / p^m
\]
are injective.

**Proof.** The injectivity of the first map is immediate from Proposition 3.2 and Poitou-Tate duality [1], Theorem 4.8. On the other hand, the lemma and the long exact flat cohomology sequence associated to \( 0 \to A_{p^m} \to A_{p^m} \to A \to 0 \) over \( K \) and over \( K_v \) for each \( v \) identifies the second map of the statement with the first, thereby completing the proof. \( \Box \)

Remark 3.6. When \( A \) is an elliptic curve over a number field and \( p \) is any prime, the injectivity of the second map in the above proposition was first established in [2], Lemma 6.1, p.107.
Let $\prod'_v H^1(K_v, A_{p^m})$ denote the restricted product of the groups $H^1(K_v, A_{p^m})$ with respect to the subgroups $H^1(O_v, A_{p^m})$.

**Proposition 3.7.** Let $A$ be an abelian variety over $K$ and let $m$ be a positive integer such that $m \leq 2$ if $p = 2$. Assume that both $A_{p^m}(K^s)$ and $A_{t^m}(K^s)$ are cyclic. Then there exist exact sequences

$$0 \to A_{p^m}(K) \to \prod_{v} A_{p^m}(K_v) \to H^2(K, A_{p^m})^D \to 0,$$

$$0 \to H^1(K, A_{p^m}) \to \prod'_v H^1(K_v, A_{p^m}) \to H^1(K, A_{p^m})^D \to 0$$

and

$$0 \to H^2(K, A_{p^m}) \to \bigoplus_{v} H^2(K_v, A_{p^m}) \to A_{p^m}(K) \to 0.$$

**Proof.** This is immediate from Propositions 3.2 and 3.5 and the Poitou-Tate exact sequence in flat-cohomology [4], Theorem 4.11. \qed

Finally, set

$$\text{Sel}(A)_{p^m} = \text{Ker} \left[ H^1(K, A_{p^m}) \to \bigoplus_{v} H^1(K_v, A) \right]$$

and define $T_p \text{Sel}(A) = \lim_{\leftarrow m} \text{Sel}(A)_{p^m}$. Further, recall the group

$$\text{III}^1(A) = \text{Ker} \left[ H^1(K, A) \to \bigoplus_{v} H^1(K_v, A) \right].$$

Now define $H^1(K, T_p A^t) = \lim_{\leftarrow m} H^1(K, A_{p^m}^t)$.

**Corollary 3.8.** Under the hypotheses of the proposition, there exist canonical exact sequences

$$0 \to \text{III}^1(A)(p) \to \prod_{v} A(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1(K, T_p A^t)^D \to (T_p \text{Sel}(A^t))^D \to 0,$$

$$0 \to T_p \text{III}^1(A) \to (\prod_{v} A(K_v)^\wedge)/A(K)^\wedge \to H^1(K, A^t\{p\})^D$$

and

$$0 \to T_p \text{Sel}(A) \to \prod_{v} A(K_v)^\wedge \to H^1(K, A^t\{p\})^D.$$
Proof. Let \( m \geq 1 \) be an integer. The exact commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & A(K)/p^m & \longrightarrow & H^1(K, A_{p^m}) & \longrightarrow & H^1(K, A)_{p^m} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{v} A(K_v)/p^m & \longrightarrow & \prod_{v} H^1(K_v, A_{p^m}) & \longrightarrow & \bigoplus_{v} H^1(K_v, A)_{p^m} & \longrightarrow & 0,
\end{array}
\]

yields an exact sequence of profinite abelian groups

\( \text{(3)} \quad 0 \to \prod H^1(A)_{p^m} \to \prod H^1(A)/p^m \to H^1(K, A_{p^m}) \to \prod H^1(A)_{p^m} \to 0, \)

where \( \text{B}_m(A) = \text{Coker} \left[ H^1(K, A)_{p^m} \to \bigoplus_{v} H^1(K_v, A)_{p^m} \right] \). By the main theorem of [3], \( \lim_{\rightarrow} \text{B}_m(A) \simeq (T_p \text{Sel}(A))^D \) and the first exact sequence of the statement follows by taking the direct limit over \( m \) in (3). On the other hand, since the inverse limit functor is exact on the category of profinite groups [15], Proposition 2.2.4, p.32, the inverse limit over \( m \) of the sequences (3) is the second exact sequence of the statement. The third exact sequence follows from the second and the exact commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & A(K)^{\wedge} & \longrightarrow & T_p \text{Sel}(A) & \longrightarrow & T_p \prod H^1(A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A(K)^{\wedge} & \longrightarrow & \prod_{v} A(K_v)^{\wedge} & \longrightarrow & (\prod_{v} A(K_v)^{\wedge}) / A(K)^{\wedge} & \longrightarrow & 0.
\end{array}
\]

\( \square \)

4. Concluding remarks

Let \( K \) and \( M \) be as in Section 1 and assume that \( M \) has \( p \)-power order. Further, let \( M^* \) be the Cartier dual of \( M \). Since \( \prod H^1(K, M^*) \simeq \prod H^1(G_K, M^*) \) and there exists a perfect pairing of finite groups

\( \text{(4)} \quad \prod H^1(K, M^*) \times \prod H^2(K, M) \to \mathbb{Q}/\mathbb{Z} \)

[4], Theorem 4.8, it is natural to expect a Galois-cohomological description of \( \prod H^2(K, M) \). Note, however, that the natural guess \( \prod H^2(K, M) \simeq \prod H^2(G_K, M) \) is incorrect, since the latter group is zero (because the \( p \)-cohomological dimension of \( G_K \) is \( \leq 1 \)). To obtain the correct answer, we proceed as follows. Since \( H^i(K^s, A) = H^i(K^s, B) = 0 \) for all \( i \geq 1 \), the exact sequence [11] shows that \( H^i(K^s, M) = 0 \) for all \( i \geq 2 \). On the other hand, \( H^i(G_K, M) = 0 \) for all \( i \geq 2 \) as well, since \( \text{cd}_p(G_K) \leq 1 \). Now the exact sequence of terms of low degree belonging to
the Hochschild-Serre spectral sequence \( H^i(G_K, H^j(K^s, M)) \Rightarrow H^{i+j}(K, M) \) yields a canonical isomorphism

\[
H^2(K, M) \simeq H^1(G_K, H^1(K^s, M)) \simeq H^1(G_K, B(K^s)/\psi(A(K^s))).
\]

Analogous isomorphisms exist over \( K_v \) for every prime \( v \) of \( K \), and we conclude that

\[
\mathrm{III}^2(K, M) \simeq \mathrm{III}^1(G_K, B/\psi(A)).
\]

For example, if \( M = A_{p^m} \) for some abelian variety \( A \) over \( K \), then \( \mathrm{III}^2(K, A_{p^m}) \simeq \mathrm{III}^1(G_K, A/p^m) \) and the pairing (4) takes the form

\[
\mathrm{III}^1(G_K, A_{p^m}^t) \times \mathrm{III}^1(G_K, A/p^m) \rightarrow \mathbb{Q}/\mathbb{Z}.
\]

In particular, if \( A_{p^m}^t(K^s) \) is cyclic with \( m \leq 2 \) if \( p = 2 \), then Proposition 3.2 applied to \( A^t \) and the perfectness of the above pairing yield \( \mathrm{III}^1(G_K, A/p^m) = 0 \), i.e., the Hasse principle holds for the \( G_K \)-module \( A(K^s)/p^m \).

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