Inflationary Cosmology with a $R + \lambda R_{\mu\nu} R^{\mu\nu}/R$

Lagrangian

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Abstract

We consider an alternate fourth-order gravity Lagrangian which is non-analytic in the Ricci scalar, and apply it to a Robertson-Walker metric. We find vacuum solutions which undergo power-law Inflation. Once matter is introduced the theory behaves very much like ordinary General Relativity; except that the radiation evolution $a \sim t^{1/2}$ is not allowed since it corresponds to $R = 0$. We comment on the possibility of wormhole solutions in such a theory.

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1 Introduction

Inflationary universe models can be constructed by adding higher order corrections to the usual Einstein-Hilbert Lagrangian for gravity. The first such model was constructed using the trace anomaly of conformally coupled fields which is due to a term $\sim R^2 \log R/\mu$ ($\mu =$ renormalization scale) in the Lagrangian[1]. However, typical numbers and types of field in the early universe would not produce enough inflation, and any inflationary phase would finish too abruptly causing an inhomogeneous universe.

The model was modified by adding a term $\epsilon R^2$ to the Lagrangian which results in a quasi-de Sitter expansion[2-4]. The Hubble parameter decreases slowly for large $\epsilon$ before going into an oscillation phase which can reheat the universe[4,5]. The general Lagrangian so far considered has the form $\mathcal{L} = R + \epsilon R^2 + \beta R_{\mu\nu} R^{\mu\nu}$ (1)

The coefficients $\epsilon$ and $\beta$ have dimensions of [length]$^2$ and (in order for stability or to prevent tachyonic solutions) certain signs have to be taken. The $\beta$ term is only relevant for anisotropic models since in the isotropic case it just alters the $\epsilon$ coefficient. New gravitational waves have also been found in such Lagrangians[6].

Various other Lagrangians such as $R + \epsilon R^2 + \beta R^3$ and $\exp(\lambda R)$ have been considered and generally give power law inflation [7-9]: instead of exponential expansion there is typically an expansion law $a \sim t^m$. So long as $m > 1$ and expansion occurs for sufficient time the usual problems, which are solved by inflation, can still be solved.

Some papers have appeared concerning the stability and solutions of general Lagrangians $f(R)$[11-14]. It is sometimes claimed that such theories or their quasi-de Sitter solutions are unstable [11,12] but at least classically this is not the case. This was analyzed in detail in ref.[13] for the $R + \epsilon R^2$ theory and also found to match ordinary gravity as $\epsilon \to 0$. The so-called small perturbations considered in ref.[11] are actually abnormal: they correspond to a large $dH/dt$ ($H =$ Hubble parameter). In analogy with a scalar field inflationary model this $\dot{H}$ term is like a velocity causing you to shoot up the scalar potential - but it is eventually damped [15].

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3 Some works use the equivalent in 4 dimensions $R + aR^2 + bC^2$ Lagrangian, where $C$ is the Weyl tensor.
A recent problem has arisen in the case of Bianchi type-I models for the Lagrangian (1) for $\beta \neq 0$ so including the sign that is stable in the absence of anisotropy. The model is found to collapse due to the anisotropy [16]: this was also found by one of us previously using the equations presented in ref.[17] but not taken too seriously at the time since it conflicted with ref.[18]. Note that this is more restrictive than for the case of scalar field inflation, where spatial gradients can cause collapse[19] but not simply anisotropy[20]. Instead in this paper we consider an alternative higher order derivative theory, first proposed in ref.[21]

$$\mathcal{L} = R + \lambda R_{\mu\nu} R^{\mu\nu} / R + \tau R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} / R.$$  \hspace{1cm} (2)

Because the parameters are now dimensionless such a Lagrangian could help incorporate higher derivative theories into variable $G$ ones (eg. Brans-Dicke[22]) or those involving phase transitions c.f. induced gravity [23]. These terms in the Lagrangian might result when quantum gravity corrections are made. Although we are not aware of them being invoked from string theory, which is usually taken to give corrections of the form Lagrangian (1), this might change when the theory is better developed or is superseded.

In any case such a non-analytic Lagrangian is interesting in its own right. It is considerably more complex than Lagrangian (1) and requires us to look for solutions with $R \neq 0$. If it is found to have vacuum Inflationary solutions, it would be a further extension to the generality of higher derivative theories in having such Inflationary behaviour. Since we would expect that such corrections to the Lagrangian would be present in the early universe, when quantum gravity effects were dominating, they could help explain any inflationary epoch.

## 2 Field Equations

We consider an action of the form

$$S = \int \left[ (R + \lambda R_{\mu\nu} R^{\mu\nu} / R + \tau R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} / R) \sqrt{-g} + \frac{8\pi G}{c^2} L_m \right] d^4x.$$ \hspace{1cm} (3)

The field equations follow from variations of this action leading to

$$G^{\mu\nu} = C^{\mu\nu}_E + \lambda G^{\mu\nu}_1 + \tau G^{\mu\nu}_2 = -\frac{8\pi G}{c^2} T^{\mu\nu},$$ \hspace{1cm} (4)
where
\[ G_{E}^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R, \] (5)
\[ G_{1}^{\mu\nu} = g^{\mu\nu} \left( R^{\sigma\tau} / R \right)_{,\sigma\tau} + (R^{\mu\nu} / R)_{,\tau} - g^{\mu\nu} \left( R_{\alpha\beta} R^{\alpha\beta} / R^2 \right)_{,\tau} \]
\[ + \left[ (R^{\mu\sigma} / R)_{,\sigma} + (R^{\nu\rho} / R)_{,\rho} + (R_{\alpha\beta} R^{\alpha\beta} / R^2)_{,\mu\nu} \right] + R^{\mu\nu} \left( R_{\alpha\beta} R^{\alpha\beta} / R^2 - 2 R^{\mu\sigma} R^{\nu\rho} / R + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} / R, \right) \] (6)
\[ G_{2}^{\mu\nu} = 4 \left( R^{\mu\nu\rho\tau} / R \right)_{,\rho\tau} - g^{\mu\nu} \left( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} / R^2 \right)_{,\tau} \]
\[ + (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} / R^2)_{,\mu\nu} - \frac{2}{R} R^{\mu\sigma\tau\rho} R^{\nu\sigma\tau\rho} / R \]
\[ + R^{\mu\nu} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} / R^2 + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} / R \]. (7)

Since \( R^{\mu\nu} \) and \( R^{\alpha\beta\gamma\delta} \) include second order derivatives of the metric, the field equations are fourth-order partial differential equations, which is the same order as the normal higher order gravitational theory.

Before considering a Robertson-Walker metric we can find the equation for a maximally symmetric spacetime. Such a spacetime has
\[ R_{\mu\nu} = \frac{1}{4} g_{\mu\nu} R, \] (8)
\[ R_{\alpha\beta\gamma\delta} = - R \left( g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \right) / 12. \] (9)

Substituting (8) and (9) into (6) and (7), respectively, and since the covariant derivatives of \( g_{\mu\nu} \) are equal to zero, we have
\[ G_{1}^{\mu\nu} = R^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} / R^2 - 2 R^{\mu\sigma} R^{\nu\rho} / R \]
\[ + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} / R = \frac{1}{16} g^{\mu\nu} R, \] (10)
\[ G_{2}^{\mu\nu} = -2 R^{\mu\sigma\tau\rho} R^{\nu\sigma\tau\rho} / R + R^{\mu\nu} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} / R^2 \]
\[ + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} / R = \frac{1}{24} R g^{\mu\nu}. \] (11)

Substituting (10) and (11) into (4), we find
\[ \frac{1}{4} g^{\mu\nu} R (1 - \lambda / 4 - \tau / 6) = 8\pi G / c^2 T^{\mu\nu}, \] (12)
\[ R = \frac{8\pi G}{c^2} T / (1 - \lambda / 4 - \tau / 6) \] (13)
which is just the Einstein’s field equation in a maximally symmetric space except that the “Gravitational constant” is $G(1 - \lambda/4 - \tau/6)^{-1}$. When this constant is $\infty$ we can perhaps expect a de-Sitter solution even in the vacuum case i.e. when $T = 0$\[4\]

3 Robertson-Walker Case

For simplicity we consider only the Ricci squared term so setting $\tau = 0$. We also take a flat $k = 0$ Robertson-Walker metric:

$$ds^2 = -dt^2 + a^2 (dr^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2)$$ \hfill (14)

We use the conventions of Weinberg$^{[24]}$ such that the Ricci-scalar $R$ is related to the Hubble parameter $H$ as,

$$R = -12H^2 - 6\dot{H}$$ \hfill (15)

With this metric the field equations give just two independent equations. The trace equation

$$\left(\frac{2}{3} - \frac{1}{\lambda}\right) R + 24H^2 + 396\frac{H^4}{R} + 2016\frac{H^6}{R^2} + 96H^3 \frac{\dot{R}}{R^2} + 936H^5 \frac{\dot{R}^2}{R^3}$$

$$+216H^4 \frac{\dot{R}^2}{R^4} - 72H^4 \frac{\dot{R}}{R^3} = \frac{8\pi G}{\lambda} \left(\rho - 3p\right) \equiv \kappa_1$$ \hfill (16)

and the $(0, 0)$ component equation

$$\left(6 + \frac{3}{\lambda}\right) H^2 + 60\frac{H^4}{R} - 252\frac{H^6}{R^2} + 24H^3 \frac{\dot{R}}{R^2} - 72H^5 \frac{\dot{R}}{R^3}$$

$$= \frac{8\pi G}{\lambda} \rho \equiv \kappa_2$$ \hfill (17)

We have also included a perfect fluid matter source with equation of state $p = (\alpha - 1)\rho$ ($p$ and $\rho$ are the pressure and energy density respectively). The conservation equation is given by

$$\dot{\rho} = -3\alpha H \rho \Rightarrow \rho = \frac{\rho_0}{a^{3\alpha}}$$ \hfill (18)

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\[4\] As pointed out by the referee this argument is somewhat unsatisfactory, but does give a heuristic argument as to why de-Sitter solution are present.
A few well known equations of state are $\alpha = 4/3$ (radiation); $\alpha = 1$ (matter) and $\alpha = 0$ (cosmological constant).

Scaling properties of equations (16) and (17) with respect to changes in the time scale suggest we look for solutions for which $R$ and $H^2$ are proportional to each other i.e. to look for solutions for which

$$\sigma = -\frac{H^2}{R}$$

where $\sigma$ is a constant to be determined. The $-ve$ sign is chosen to keep $\sigma + ve$ in our notation, a plus sign would require a large velocity $\dot{H}$ and would not be expected to be inflationary cf. eq.(15). Using the relation eq.(15) and the ansatz eq.(19) gives for the Hubble parameter

$$H = \frac{H_0}{1 + H_0(2 - 1/6\sigma)t}$$

(20)

Since $a = a_0 exp(f H dt)$ this leads to solutions of the form

$$a(t) = a_0 \left(1 + H_0(2 - \frac{1}{6\sigma})t\right)^{\frac{6\sigma}{12\sigma - 1}}$$

(21)

with $H_o$ and $a_o$ the initial values of the Hubble parameter and scale factor respectively. Eq.(21) therefore includes solutions like $a \sim t^m$ with $m = 6\sigma/(12\sigma - 1)$.

If $(2 - 1/6\sigma) = 0$ then simply $H = H_0$ and

$$a = a_0 exp(tH_0)$$

(22)

This usual de-Sitter solution corresponds as expected to $\sigma = 1/12$ when $R = -12H^2$.

The ansatz when substituted into the Trace equation and the $(0,0)$ component equation, gives respectively

$$R \left(12\sigma^2 + 4\sigma - \frac{2}{3} + \frac{1}{\lambda}\right) = -\kappa_1$$

(23)

$$3\sigma R \left(12\sigma^2 + 4\sigma - \frac{2}{3} + \frac{1}{\lambda}\right) = -\kappa_2$$

(24)

**Vacuum Case**
We first consider the absence of matter i.e. $\kappa_1 = \kappa_2 = 0$. Since we require $R \neq 0$ we set the $\sigma$ expression to be zero.

$$12\sigma^2 + 4\sigma - \frac{2}{3} + \frac{1}{\lambda} = 0$$  \hspace{1cm} (25)

with solution

$$\sigma = -\frac{1}{6} + \frac{\sqrt{3}}{6} \left(1 - \frac{1}{\lambda}\right)^{\frac{1}{2}}$$  \hspace{1cm} (26)

The other solution of the quadratic equation can be ignored since we require $\sigma + ve$. This gives for $m$

$$m = \frac{-1 + \sqrt{3} \left(1 - \frac{1}{\lambda}\right)^{1/2}}{-3 + 2\sqrt{3} \left(1 - \frac{1}{\lambda}\right)^{1/2}}$$  \hspace{1cm} (27)

$\lambda$ positive:

Real i.e. non oscillatory solutions, require $\lambda > 1$; the de Sitter solution $m \to \infty$ occurs when $3 = 2\sqrt{3}(1 - 1/\lambda)^{1/2} \Rightarrow \lambda = 4$. We can plot the behaviour of $m$ against $\lambda$ in Fig.(1). For $\lambda > 4$ we have power law inflationary behaviour with $m \simeq 1.6$ as $\lambda \to \infty$.

We consider the stability of these solutions with respect to linearized perturbations in the appendix. We find that they are stable to such perturbations.

For $1 < \lambda < 4$ the universe is contracting i.e $m < 0$. There is also an oscillation region for $0 < \lambda \leq 1$ where $m$ becomes complex: but since we are primarily interested in inflationary behaviour we do not consider it further.

$\lambda$ negative:

As $\lambda$ decreases the expansion rate $m$ increases with $m \simeq 1.6$ as $\lambda \to -\infty$. To get inflationary behaviour (i.e. $m > 1$) requires $\lambda < -3$. Again these inflationary solutions are stable to linearized perturbations.

**Matter Case**

We now consider the addition of an energy density such that

$$\rho = \frac{\rho_0}{a^n}$$  \hspace{1cm} (28)

We are setting $n = 3\alpha$ cf. eq.(18). Sticking with our same ansatz the trace equation and $(0,0)$ equations gives respectively,

$$\left(\frac{2}{3} - \frac{1}{\lambda} - 4\sigma - 12\sigma^2\right)R = \frac{8\pi G \rho_0(4 - 3\alpha)}{c^2\lambda a^n}$$  \hspace{1cm} (29)
\[
\left( \frac{2}{3} - \frac{1}{\lambda} - 4\sigma - 12\sigma^2 \right) R = \frac{8\pi G}{c^2\lambda} \frac{\rho_0}{3\sigma a^n}
\]  
(30)

Equating the two gives
\[
\frac{1}{3\sigma} = 4 - 3\alpha
\]  
(31)

The solution has the expansion behaviour \( m = 2/n = 2/3\alpha \) which is the usual relationship in standard General Relativity.

Since for the solution eq.(21) we have
\[
H(t) = H_0 \left( \frac{a_0}{a(t)} \right)^{1/m}
\]  
(32)

the Ricci scalar can be written as
\[
R = -\frac{1}{\sigma} H_0^2 a_0^{2/m} a(t)^{-2/m}
\]  
(33)

where the initial values \( H_0, a_0 \) and \( \rho_0 \) are related by eq.(30).

We have to avoid \( R = 0 \) which occurs when \( 1/\sigma = 0 \) i.e. \( 4 \neq 3\alpha \) or \( n \neq 4 \). This is the equation of state for radiation whose scale factor behaves as \( a \sim t^{1/2} \). This is not allowed in this theory but a correction \( a \sim t^{1/2+\delta} \) with \( \delta \) small is.

**Cosmological constant**

This is a special case of the previously example with \( n = 0 \) i.e. \( \rho = \rho_0 \). Since we expect de-Sitter solutions we consider \( H = \text{constant} \). From the Trace and \((0,0)\) equations, we get

\[
12 \left( \frac{1}{\lambda} - \frac{1}{4} \right) H^2 = \kappa_1
\]  
(34)

\[
3 \left( \frac{1}{\lambda} - \frac{1}{4} \right) H^2 = \kappa_2
\]  
(35)

Therefore
\[
4 = \frac{\kappa_1}{\kappa_2} = \left( 1 - \frac{3\rho}{\rho} \right) \Rightarrow p = -\rho
\]  
(36)

A de-Sitter solution occurs for the equation of state \( p = -\rho \) or equivalently a cosmological constant \( \Lambda \sim 8\pi G\rho \)

Because \( H \) should be real the value of \( \lambda \) is constrained, from eq.(35)

\[
\left( 1 - \frac{\lambda}{4} \right) H^2 = \frac{8\pi G}{3c^2} \rho
\]  
(37)

therefore \( \lambda < 4 \) is required for such solutions.
4 Discussion and Conclusions

We have found that provided the value of $|\lambda|$ is sufficiently large the alternative higher order Lagrangian can have power law inflationary solutions, which are stable to linearized perturbations. Once matter is introduced, perhaps by a quantum process (c.f. particle production see e.g ref.[25]) in the early universe, the usual behaviour of General Relativity(GR)results. This is provided the matter is allowed to dominate while the tendency of the vacuum inflationary solution is to dilute any matter present. If instead quantum process proceed abruptly it might cause the inflationary phase to rapidly stop—a graceful exit problem. Until the coupling to matter was known it would be difficult to calculate the effects of particle creation and how such a scenario could proceed: whether such matter creation can cause the inflationary behaviour to gradually switch over to a normal expansion $a \sim t^p$ with $p < 1$. In any case, there have been claims that the usual inflationary models are unstable to quantum processes[26]. If this is the case then not having a “classical” ending to inflation need not be a drawback: inflation would end due to quantum instabilities and a mechanism to enable a switch to a classical non-inflationary phase is provided. A weak quantum instability would therefore appear to be a necessity, in this, and similar power law inflationary models weak in the sense that there is still a quantum probability of sufficient inflation occurring.

These quantum processes would also be required to heat the universe since there is no oscillating field present: this is also a problem with scalar field power law inflationary models where the field potential is too shallow. If such processes are not present or cause the inflationary expansion to prematurely stop then a second period of inflation would be required. The first period of inflation would still have contributed to the flatness and horizon problem so that the second inflationary stage (presumably due to a matter field) could be shorter than the usual $\sim 70$ e-foldings.

There is the problem that the radiation expansion $a \sim t^{1/2}$ cannot quite occur (this would set the Ricci scalar zero). This might conflict with the usual predictions of nucleosynthesis and so require that $\lambda$ be smaller.

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5Or at least enabled the universe to wait, without recollapsing prematurely c.f. ref.[27], till a lower energy scale inflationary mechanism could dominate.

6We have not yet calculated how $\lambda$ affects the nucleosynthesis rates and if any possible differences disappear as $\lambda \to 0$. 
We could still envisage that during the early epoch of the universe the value of $\lambda$ is large: so giving inflation. As quantum gravity effects weaken due to the expansion of the universe the value of $\lambda$ reduces, so effectively agreeing with GR later. We would hope to show that as $\lambda \to 0$ we would recover or agree with the results of GR. In this case $\lambda$ varies like a “running coupling constant” in particle physics due to the relevant energy scale present.

An alternative scenario would be for $\lambda$ to be simulated by the effect of extremely highly energetic matter e.g. a string epoch. $\lambda$ could then change suddenly rather like the induced gravity model c.f ref.[23] due to the different behaviour of the matter fields present (in the string epoch this could correspond to a phase transition ). This would require us to understand the matter fields present, and their coupling to the higher derivative terms, before we would know if they could set $\lambda$ small. Note that the field equations have been derived under the assumption that $\lambda$ is independent of time. If this assumption was dropped then the field equations would be more complicated.

Because the effective Newton’s constant $G_{eff}$ can change sign there is also the chance of finding wormhole solutions. This is similar to the wormhole found in the case of a conformally coupled scalar field which has $G_{eff} < 0$ in its vicinity[28,29]. Wormholes can be found in the usual higher derivative Lagrangian (1) if we take the “wrong sign” of epsilon[30,31]: this is related to the bounce (avoidance of the singularity) in such models for spatial curvature $k = -1[31]$.

In order to find wormholes (or bounces) in the Lagrangian presented here we would have to keep the spatial curvature $k$ in the equations. This would be rather more complicated as would the case of Bianchi models: these could perhaps be more realistically tackled using algebraic computing packages.

In conclusion we have obtained vacuum inflationary solutions for a non analytic higher order Lagrangian. This Lagrangian might contain certain aspects of the “correct quantum gravity” theory at high energy densities. Such solutions are further evidence that De Sitter or power law inflationary solutions are a general feature of higher order gravitational theories.

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Appendix

The question of stability of the solutions is clearly of importance. However, a comprehensive discussion of this topic would include such concepts as e.g. linearized, weak, asymptotic, and Liapunov stability. This would entail considerable work beyond the scope of this paper, so we limit ourselves to just an analysis of the linearized stability of the vacuum equations. For a general reference see eg.[32].

First consider the \((0, 0)\) eq.(17) rewritten in the form

\[
\dot{R} = \frac{1}{(1 - 3H^2/R)} \left( \frac{21}{2} H^3 - \frac{5}{2} HR - \frac{1}{8} (2 + \frac{1}{\lambda}) \frac{R^2}{H} \right) \equiv F_2(H, R) \tag{38}
\]

together with the expression for the Ricci scalar

\[
\dot{H} = -2H^2 - R/6 \equiv F_1(H, R) \tag{39}
\]

We consider perturbations \(h\) and \(r\) around the known solutions such that now

\[
H = \overline{H} + h \quad R = \overline{R} + r \tag{40}
\]

where \(\overline{H}\) and \(\overline{R}\) are the previously found solutions. The linearized equations for the perturbations are given by

\[
\dot{h} = \frac{\partial F_1(H, R)}{\partial H} h + \frac{\partial F_1(H, R)}{\partial R} r \tag{41}
\]

\[
\dot{r} = \frac{\partial F_2(H, R)}{\partial H} h + \frac{\partial F_2(H, R)}{\partial R} r \tag{42}
\]

Calculating the partial derivatives eventually leads to

\[
\dot{h} = -4h\overline{H} - r/6 \tag{43}
\]

\[
\dot{r} = \beta_1 h\overline{H}^2 + \beta_2 r\overline{H} \tag{44}
\]

where

\[
\overline{H}^{-1} = H_0^{-1} + \gamma t, \quad \gamma \equiv (2 - 1/6\sigma) \geq 0 \tag{45}
\]

and

\[
\beta_1 = \frac{1}{(1 + 3\sigma)^2} \left( 18 + 18\sigma + \frac{5}{\sigma} + \frac{1}{3\sigma^2} \right) \tag{46}
\]
\[
\beta_2 = \frac{1}{(1+3\sigma)^2} \left( -\frac{1}{2} - \frac{45}{2} \sigma - 45\sigma^2 + \frac{2}{3\sigma} \right). \quad (47)
\]

In deriving \( \beta_1, \beta_2 \) we have made use of eq.(25) to express \( \lambda \) in terms of \( \sigma \). Scaling to a new time parameter \( \tau \) such that \( d\tau/dt = \gamma \) and renaming \( u(\tau) = h(t) \) and \( v(\tau) = r(t) \) gives

\[
\dot{u}(\tau) = \frac{1}{\gamma} \left( -\frac{4u}{\tau} - \frac{v}{6} \right) \quad (48)
\]
\[
\dot{v}(\tau) = \frac{1}{\gamma} \left( \frac{\beta_1 u}{\tau^2} + \frac{\beta_2 v}{\tau} \right) \quad (49)
\]

We have to find solutions with \( \tau \equiv \frac{1}{\Pi^{-1}} \geq \tau_0 = H_0^{-1} \), with initial conditions \( u(\tau_0) \equiv u_0 \) and \( v(\tau_0) = v_0 \).

The nature of the equations suggests we look for solutions of the form

\[
u = u_0 \tau^{-\alpha} \quad v = v_0 \tau^{-\alpha-1}
\]

When substituted into the equations this ansatz leads to the linear system for \( u_0, v_0 \)

\[
(4 - \alpha \gamma) u_0 + v_0 / 6 = 0 \quad (51)
\]
\[
\beta_1 u_0 + (\beta_2 + (\alpha + 1) \gamma) v_0 = 0 \quad (52)
\]

Non-trivial solutions require that the determinant vanish i.e.

\[
(4 - \alpha \gamma)(\beta_2 + (\alpha + 1) \gamma) - \beta_1 / 6 = 0 \quad (53)
\]

or

\[
\alpha^2 - \left( \frac{4 - \beta_2}{\gamma} - 1 \right) \alpha - \left( \frac{4\beta_2 - \beta_1 / 6}{\gamma^2} + \frac{4}{\gamma} \right) = 0 \quad (54)
\]

with solutions

\[
\alpha_{1,2} = \frac{1}{2} \left[ \frac{4 - \beta_2}{\gamma} - 1 \right] \pm \sqrt{\frac{1}{4} \left[ \frac{4 - \beta_2}{\gamma} - 1 \right]^2 + \frac{4\beta_2 - \beta_1 / 6}{\gamma^2} + \frac{4}{\gamma} }, \quad (55)
\]

which can be written in the form (with obvious definitions for \( A, B \))

\[
\alpha_{1,2} = \frac{A}{\gamma} \pm \frac{1}{\gamma} \sqrt{A^2 + B}. \quad (56)
\]
For stability we require both $\alpha_1$ and $\alpha_2 > 0$ this occurs if the following inequalities are satisfied (recall $\gamma \geq 0$):

$$A > 0, \ B < 0, \ A^2 + B > 0$$  \hspace{1cm} (57)

Substituting for $\beta_1, \beta_2$ allows these inequalities to be written as

$$0 < \frac{1}{(1 + 3\sigma)^2} \left( \frac{7}{2} + 36\sigma + 63\sigma^2 - \frac{1}{2\sigma} \right)$$  \hspace{1cm} (58)

$$0 < \frac{1}{(1 + 3\sigma)^2} \left( 1 + 51\sigma + 108\sigma^2 + \frac{1}{18\sigma^2} - \frac{7}{6\sigma} \right)$$  \hspace{1cm} (59)

$$0 < (4 - \beta_2 - \gamma)^2 + \frac{1}{(1 + 3\sigma)^2} \left( -4 - 204\sigma - 432\sigma^2 + \frac{14}{3\sigma} - \frac{2}{9\sigma^2} \right).$$  \hspace{1cm} (60)

The value of $\sigma$ when inflationary behaviour occurs i.e. $\lambda > 4$ is constrained in the region

$$\frac{1}{12} \leq \sigma \leq \left( \frac{1}{6} \sqrt{3} - \frac{1}{6} \right) \approx 0.122$$  \hspace{1cm} (61)

where we have again used eq.(26) to relate $\sigma$ to $\lambda$. For $\lambda < -3$ the corresponding expression is

$$\frac{1}{6} (\sqrt{3} - 1) \leq \sigma \leq \frac{1}{6}$$  \hspace{1cm} (62)

With these values of $\sigma$ the inequalities are satisfied. The first two are easily satisfied while the third one has been checked numerically. In fact, the restriction to the inflationary regime is not strictly necessary since the inequalities are only violated in the oscillatory region $0 < \lambda \leq 1$.

The values of $\alpha_1, \alpha_2$ are therefore positive and the perturbations decay as $t$ or $\tau \to \infty$. Let us represent the solutions for the perturbations as $\omega(\alpha_1), \omega(\alpha_2)$, where $\omega \equiv \begin{pmatrix} u \\ v \end{pmatrix}$. These two independent solution of $\omega$ will span the solution space of eq.(48,49), which is a two dimensional vector space. This is in contrast with the solutions (which can be called $\Omega \equiv \begin{pmatrix} \mathcal{H} \\ \mathcal{R} \end{pmatrix}$) which go as $\mathcal{H} = 1/\tau, \mathcal{R} = -1/(\sigma \tau^2)$ (see eq.(19).

In order for the perturbations not to dominate they must decay more rapidly than their corresponding solutions i.e. we require $\alpha > 1$. Because $\alpha_2$ is less
than $\alpha_1$, $\omega(\alpha_2)$ will give the dominant behaviour as $\tau \to \infty$.

Now $\alpha_2$ is given by

$$\alpha_2 = \frac{A}{\gamma} - \frac{1}{\gamma} \sqrt{A^2 + B}.$$  \hfill (63)

Numerically it can be shown that $(A^2 + B) \simeq 0.2$ so approximately

$$\alpha_2 \simeq \frac{A}{\gamma} \equiv \frac{1}{2\gamma(1 + 3\sigma)^2} \left(\frac{7}{2} + 36\sigma + 63\sigma^2 - \frac{1}{2\sigma}\right) - \frac{0.2}{\gamma}.$$  \hfill (64)

For $\sigma$ in the inflationary range ($\lambda > 4$) we obtain

$$\sim 1.4 < \alpha_2 \leq \infty.$$  \hfill (65)

(For $\lambda < -3$ the corresponding expression is $1.4 < \alpha_2 < 1.8$.) We can therefore conclude that since $\alpha_2 > 1$ the perturbations $\omega(\alpha_2)$ will decay faster than their corresponding solutions $\Omega$.

Again this restriction to the inflationary range was not strictly necessary except for the oscillatory region. Although we do not try to analyze this region further, we would also hope that for stability in this region where $\alpha$ is complex, the real part i.e both $\alpha_{1,2}^{\text{real}}$ are positive.

We would also like to show at some stage that the inclusion of matter is likewise stable. Although we are fairly confident that this will be the case, since more rapidly expanding solutions are usually more prone to instability, it would require extensive further work to be certain.

The trace equation should also be considered but it can be shown that it only imposes restrictions on the initial values $u_0,v_0$ and so does not add further to the stability analysis. We can therefore conclude that the inflationary solutions are stable, at least to linearized perturbations. We can be fairly confident that other solutions (i.e oscillatory, matter dominated) will also be stable without explicitly doing each case.
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**Figures**

Fig.1
Plot of the expansion coefficient $m$ against the parameter $\lambda$. Power law inflation occurs for $\lambda \geq 4$ and $\lambda \leq -3$. 