EXISTENCE AND MULTIPlicity OF
Positive Periodic Solutions for a
Class of Second Order Damped
Functional Differential Equations
With Multiple Delays

Ping Liu\textsuperscript{1}, Yonghong Fan\textsuperscript{1,†} and Linlin Wang\textsuperscript{1}

Abstract By using the Krasnoselskii fixed point theorem, sufficient conditions are obtained for the existence and multiplicity of positive periodic solutions for a class of second order damped functional differential equations with multiple delays. Our results are a further expansion of the previous research results.

Keywords Periodic solutions, Green’s function, Krasnoselskii fixed point theorem.

MSC(2010) 35B10, 65L03.

1. Introduction

For the following equation

\[ x'' = f(t, x(t)), \quad t \in \mathbb{R}, \]

where \( f \in C(\mathbb{R}/\mathbb{T}, (0, +\infty)) \), there are many results \([3,18]\) on the periodic solution of this equation.

However, the systems controlled by feedback loops in engineering, predator-prey models in ecosystems \([8,12]\), and value laws in economics in real life all have the influence of delay factors, so the research on functional differential equations has already stepped into a climax period \([1,15,17]\). At the same time, many research methods have been considered, such as the upper and lower solutions method and monotone iterative technique \([10,16]\), fixed point theorems \([11,13,21]\) and so on \([5,9,14,19,20]\).

Jiang et al. \([10]\) studied the following periodic problem

\[ -x'' = f(t, x(t), x(t-\tau(t)) ), \quad t \in \mathbb{R}, \]

where \( f \in C(\mathbb{R}^3, \mathbb{R}), \tau \in C(\mathbb{R}, [0, +\infty)) \), and they are \( T \)-periodic functions. They established the existence results of \( T \)-periodic solutions by using monotone iterative technique.

\textsuperscript{†}The corresponding author. Email address: fanyh.1993@sina.com (Y. Fan)
\textsuperscript{1}School of Mathematics and Statistics Science, Ludong University, Yantai 264025, China
However, for many problems in real life, we only need to consider the properties of its positive periodic solution. In [21], Wu obtained the existence and multiplicity of the solutions to the following equation

\[ x'' + a(t)x = \lambda f(t, x(t-\tau_0(t)), x(t-\tau_1(t)), \ldots, x(t-\tau_n(t))), \quad t \in \mathbb{R}, \]

where \( a \in C(\mathbb{R}/\mathbb{T}, (0, +\infty)) \), \( f \in C(\mathbb{R}/\mathbb{T} \times [0, +\infty)^n, [0, +\infty)) \), \( \tau_i(t) \in C(\mathbb{R}/\mathbb{T}, \mathbb{R}) \), and \( a(t) \) satisfies the condition that \( 0 < a(t) < \frac{\pi^2}{4} \) for every \( t \in \mathbb{R} \).

Li et al. studied the following equation in [13]

\[ x'' + a(t)x = f(t, x(t-\tau_1(t)), \ldots, x(t-\tau_n(t))), \quad t \in \mathbb{R}, \]

where \( a \in C(\mathbb{R}/\mathbb{T}, (0, +\infty)) \), \( f \in C(\mathbb{R}/\mathbb{T} \times [0, +\infty)^n, [0, +\infty)) \), \( \tau_i(t) \in C(\mathbb{R}/\mathbb{T}, (0, +\infty)) \), they obtained the existence of positive periodic solution by using the first eigenvalue corresponding to the relevant linear operator and fixed-point index theory in cones.

In [11], Kang et al. considered the following equation with damped term

\[ x'' + h(t)x' + a(t)x = \lambda g(t)f(t, x(t-\tau(t))), \quad t \in \mathbb{R}, \]

where \( h \in C(\mathbb{R}/\mathbb{T}, [0, +\infty)), a \in C(\mathbb{R}/\mathbb{T}, \mathbb{R}), f \in C(\mathbb{R}/\mathbb{T} \times [0, +\infty), [0, +\infty)) \), \( \tau_i(t) \in C(\mathbb{R}/\mathbb{T}, \mathbb{R}), g \in C(\mathbb{R}/\mathbb{T}, [0, +\infty)) \). They obtained the existence and multiplicity of positive periodic solutions when the coefficients \( h(t), a(t) \) and \( g(t) \) satisfy \( \int_0^T h(\xi)d\xi > 0, \int_0^T a(\xi)d\xi > 0 \) and \( \int_0^T g(\xi)d\xi > 0 \), respectively, moreover, \( f \) is nondecreasing in the second variable.

Motivated by the above papers, in this paper, we study the existence, multiplicity of positive periodic solutions for the following equation

\[ x'' + h(t)x' + a(t)x = \lambda g(t)f(t, x(t-\tau_0(t)), x(t-\tau_1(t)), \ldots, x(t-\tau_n(t))), \quad (1.1) \]

where \( h \in C(\mathbb{R}/\mathbb{T}, \mathbb{R}), a \in C(\mathbb{R}/\mathbb{T}, \mathbb{R}), f \in C(\mathbb{R}/\mathbb{T} \times [0, +\infty), [0, +\infty)) \) and \( f(t, x_0, x_1, \ldots, x_n) > 0 \) for \((x_i \geq 0, 0 \leq i \leq n, (x_0, x_1, \ldots, x_n) \neq (0, 0, \ldots, 0)\), \( \tau_i(t) \in C(\mathbb{R}/\mathbb{T}, \mathbb{R}), g \in C(\mathbb{R}/\mathbb{T}, [0, +\infty)) \) and \( \int_0^T g(\xi)d\xi > 0, \lambda > 0 \) is a parameter.

Three highlights should be pointed out. Firstly, compared with the equation studied in [13, 21], we add the damping term \( h(t)x' \). Secondly, different from [11], the equation we studied has multiple delays. Thirdly, we relax the restrictions for the coefficients \( h(t) \) and \( a(t) \) in [11].

2. Preliminaries

If the unique solution of linear equation

\[ x'' + h(t)x' + a(t)x = 0, \quad (2.1) \]

associated to periodic boundary conditions

\[ x(0) = x(T), \quad x'(0) = x'(T) \quad (2.2) \]
is trivial, then it is nonresonant. By Fredholm’s alternative theorem, we know that when (2.1)-(2.2) is nonresonant,

\[ x'' + p(t)x' + q(t)x = f(t) \quad (2.3) \]

has a unique solution and it can be expressed as

\[ x(t) = \int_{0}^{T} G(t,\xi)l(\xi)d\xi, \]

where \( G(t,\xi) \) is the Green’s function of (2.1)-(2.2).

Next we assume that:

(A0) The Green’s function \( G(t,\xi) \) of system (2.1)-(2.2), is positive for all \( (t,\xi) \in [0,T] \times [0,T] \).

In general, condition (A0) is difficult to establish. However, through the anti-maximum principle established by Hakl and Torres (see [7]), Chu, Fan and Torres obtained that (A0) is true in [2]. Describe the above criterion by defining the following function

\[ \sigma(h)(t) = \exp(\int_{0}^{T} h(\xi)d\xi), \]

and

\[ \sigma_{1}(h)(t) = \sigma(h)(T) \int_{0}^{t} \sigma(h)(\xi)d\xi + \int_{t}^{T} \sigma(h)(\xi)d\xi. \]

**Lemma 2.1** (Corollary 2.6, [7]). If \( a(t) \neq 0 \) and the following two inequalities

\[ \int_{0}^{T} a(\xi)\sigma(h)(\xi)\sigma_{1}(-h)(\xi)d\xi \geq 0, \quad (H1) \]

and

\[ \sup_{0 \leq t \leq T} \left\{ \int_{t}^{t+T} \sigma(-h)(\xi)d\xi \int_{t}^{t+T} [a(\xi)]_{+}\sigma(h)(\xi)d\xi \right\} \leq 4 \quad (H2) \]

are satisfied, where \([a(\xi)]_{+} = \max\{a(\xi),0\}\). Then (A0) holds.

When (A0) holds, we always denote

\[ A = \min_{0 \leq \xi,\xi \leq T} G(t,\xi), \quad B = \max_{0 \leq \xi,\xi \leq T} G(t,\xi), \quad \sigma = A/B. \quad (2.4) \]

Obviously \( B > A > 0 \) and \( 0 < \sigma < 1 \).

Then, let \( X = C(\mathbb{R}/T\mathbb{Z},\mathbb{R}), \|x\| = \max\{|x(t)| : x(t) \in X, t \in [0,T]\} \), and \( P = \{x(t) \in X : x(t) \geq \sigma \|x\|, t \in [0,T]\} \). Moreover, for \( r > 0 \), let \( \Omega_{r} = \{x \in X, \|x\| < r\} \) and

\[ m(r) = \min\{f(t,x_{0},x_{1},...,x_{n}) : 0 \leq t \leq T, \sigma r \leq x_{i} \leq r, 0 \leq i \leq n\}; \]

\[ M(r) = \max\{f(t,x_{0},x_{1},...,x_{n}) : 0 \leq t \leq T, 0 \leq x_{i} \leq r, 0 \leq i \leq n\}. \]

Define operator:

\[ Q_{\lambda} x(t) = \lambda \int_{0}^{T} G(t,\xi)g(\xi)f(\xi,x(\xi-\tau(\xi)),x(\xi-\tau_{1}(\xi)),...,x(\xi-\tau_{n}(\xi)))d\xi. \]

Therefore, the fixed point of the operator equation \( x = Q_{\lambda} x \) is the \( T \)-periodic solution of (1.1).
Lemma 2.2. \( Q_\lambda : P \to P \) is completely continuous and \( Q_\lambda(P) \subset P \).

Proof. Since

\[
Q_\lambda x(t) \geq \lambda A \int_0^T g(\xi)f(\xi, x(\xi - \tau_0(\xi)), x(\xi - \tau_1(\xi)), \ldots, x(\xi - \tau_n(\xi)))d\xi,
\]

and

\[
\| Q_\lambda x(t) \| \leq \lambda B \int_0^T g(\xi)f(\xi, x(\xi - \tau_0(\xi)), x(\xi - \tau_1(\xi)), \ldots, x(\xi - \tau_n(\xi)))d\xi,
\]

therefore

\[
Q_\lambda x(t) \geq \lambda A \frac{\| Q_\lambda x(t) \|}{\lambda B} = \sigma \| Q_\lambda x(t) \|.
\]

Then, according to the Arscoli-Arzele theorem, \( Q_\lambda \) is completely continuous. The proof is completed.

Lemma 2.3. If \( x \in P \cap \partial \Omega_r \) for \( r > 0 \), then

\[
\lambda A m(r) T \int_0^T g(\xi)d\xi \leq \| Q_\lambda x(t) \| \leq \lambda B M(r) T \int_0^T g(\xi)d\xi.
\]

Proof. Since \( x \in P \cap \partial \Omega_r \), it is clear that \( \sigma r \leq x(t) \leq r \), that is

\[
Q_\lambda x(t) \geq \lambda A \int_0^T g(\xi)m(r)d\xi,
\]

hence

\[
\| Q_\lambda x(t) \| \geq \lambda A \int_0^T g(\xi)d\xi.
\]

And

\[
Q_\lambda x(t) \leq \lambda B \int_0^T g(\xi)M(r)d\xi,
\]

thus

\[
\| Q_\lambda x(t) \| \leq \lambda B M(r) \int_0^T g(\xi)d\xi.
\]

The proof is finished.

Lemma 2.4 ([4,6]). Let \( X \) be a Banach space and \( P \) be a close convex cone in \( X \). \( \Omega_1, \Omega_2 \) are bounded open subsets of \( X \), \( \theta \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2 \). \( Q : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P \) is a completely continuous operator. Assume that \( Q \) satisfies one of the following conditions:

(i) \( \| Qx \| \geq \| x \| \) for \( x \in P \cap \partial \Omega_1 \), \( \| Qx \| \leq \| x \| \) for \( x \in P \cap \partial \Omega_2 \);

(ii) \( \| Qx \| \leq \| x \| \) for \( x \in P \cap \partial \Omega_1 \), \( \| Qx \| \geq \| x \| \) for \( x \in P \cap \partial \Omega_2 \).

Then \( Q \) has at least one fixed point in \( P \cap (\bar{\Omega}_2 \setminus \Omega_1) \).

3. Main results

Let \( x = (x_0, x_1, \ldots, x_n) \in [0, +\infty)^{n+1} \), \( x \triangleq \max\{x_0, x_1, \ldots, x_n\} \).
Next, make the following assumptions about \( f \):

\[
f^0 = \limsup_{x \to 0^+} \max_{t \in [0,T]} \frac{f(t, x)}{x}, \quad f_\infty = \liminf_{x \to +\infty} \min_{t \in [0,T]} \frac{f(t, x)}{x}, \quad f_0 = \liminf_{x \to 0^+} \min_{t \in [0,T]} \frac{f(t, x)}{x}, \quad f_\infty = \limsup_{x \to +\infty} \max_{t \in [0,T]} \frac{f(t, x)}{x}.
\]

Assume that:

- \( j_0 \) = the number of zeros in set \( \{ f^0, f_\infty \} \); \( j_\infty \) = the number of infinities in set \( \{ f^0, f_\infty \} \);
- \( j'_0 \) = the number of zeros in set \( \{ f_0, f_\infty \} \); \( j'_\infty \) = the number of infinities in set \( \{ f_0, f_\infty \} \).

**Theorem 3.1.** Suppose that (A0) holds.

1. If \( j_0 = 1 \) or \( 2 \), when \( \lambda > \frac{1}{A_{m(1)} f_0 g(\xi) d\xi} > 0 \), equation (1.1) has at least \( j_0 \) positive \( T \)-periodic solution(s).
2. If \( j'_\infty = 1 \) or \( 2 \), when \( 0 < \lambda < \frac{1}{BM(1) f_0 g(\xi) d\xi} \), equation (1.1) has at least \( j'_\infty \) positive \( T \)-periodic solution(s).
3. If \( j'_0 = 0 \) or \( j_\infty = 0 \), there is no positive \( T \)-periodic solution to equation (1.1) for sufficiently large or sufficiently small \( \lambda > 0 \), respectively.

**Proof.** For \( \phi \in P \cap \partial \Omega_r \), define

\[
\Phi(t) = (\phi(t - \tau_0(t)), \phi(t - \tau_1(t)), \ldots, \phi(t - \tau_n(t)))
\]

and \( \bar{\Phi}(t) = \max_{0 \leq i \leq n} \{ \phi(t - \tau_i(t)) \} \).

1. Let \( r_1 = 1 \), by Lemma 2.3, we can obtain that there exists \( \lambda_0 = \frac{1}{A_{m(1)} f_0 g(\xi) d\xi} > 0 \), such that

\[
\| Q\lambda \phi \| \geq \lambda A_{m(1)} \int_0^T g(\xi) d\xi > \| \phi \|, \quad \phi \in P \cap \partial \Omega_1, \quad \lambda > \lambda_0.
\]

If \( f^0 = 0 \), then we have \( f(t, x) \leq \varepsilon \overline{\Phi} \) for \( 0 < \overline{\Phi} \leq r_2 \) and \( t \in [0, T] \), where \( \varepsilon > 0 \) satisfies \( \lambda \epsilon B \int_0^T g(\xi) d\xi < 1 \), and \( 0 < r_2 < r_1 = 1 \), obviously, \( \Omega_{r_2} \subset \Omega_1 \).

Then \( 0 < \sigma r_2 = \sigma \| \phi \| \leq \| \overline{\Phi(t)} \| \leq \| \phi \| = r_2 \), for all \( \phi \in P \cap \partial \Omega_{r_2} \), \( t \in [0, T] \), thus

\[
f(t, \Phi(t)) \leq \varepsilon \overline{\Phi(t)}.
\]

From the definition of \( Q\lambda \), for \( \phi \in P \cap \partial \Omega_{r_2} \), we can obtain

\[
\| Q\lambda \phi \| \leq \lambda \epsilon B \int_0^T g(\xi) \overline{\Phi(\xi)} d\xi \leq \lambda \epsilon B \| \phi \| \int_0^T g(\xi) d\xi < \| \phi \|.
\]

Thus, by Lemma 2.4(ii), the operator \( Q\lambda \) has at least one fixed point in \( P \cap (\overline{\Omega_1} \setminus \Omega_{r_2}) \).

If \( f_\infty = 0 \), then there exists \( H > 0 \), such that \( f(t, x) \leq \varepsilon \overline{\Phi} \) for \( \overline{\Phi} \geq H \) and \( t \in [0, T] \), where \( \varepsilon > 0 \) still satisfies \( \lambda \epsilon B \int_0^T g(\xi) d\xi < 1 \). Moreover, select \( r_3 = \max\{2, \frac{H}{\varepsilon} \} \), obviously, \( \Omega_1 \subset \Omega_{r_3} \).
Periodic solutions of the equation

Then \( \Phi(t) \geq \sigma \parallel \phi \parallel = \sigma r_3 \geq H \), for all \( \phi \in P \cap \partial \Omega_{r_3}, t \in [0, T] \), thus

\[
f(t, \Phi(t)) \leq \varepsilon \Phi(t).
\]

Then for \( \phi \in P \cap \partial \Omega_{r_3} \), we can obtain

\[
\parallel Q_{\lambda} \phi \parallel \leq \lambda \varepsilon B \parallel \phi \parallel \int_0^T g(\xi) d\xi < \parallel \phi \parallel.
\]

Thus, by Lemma 2.4(i), the operator \( Q_{\lambda} \) has at least one fixed point in \( P \cap (\overline{\Omega_{r_3}} \setminus \Omega_1) \).

Above all, if \( f^0 = 0 \) and \( f^\infty = 0 \), the operator \( Q_{\lambda} \) has at least two fixed points in \( P \cap (\overline{\Omega_{r_3}} \setminus \Omega_{r_2}) \), that is, (1.1) has at least two positive \( T \)-periodic solutions for \( \lambda > \lambda_0 \).

(2) Let \( r_1 = 1 \), by Lemma 2.3, we can obtain that there exists \( \lambda_0 = \frac{1}{BM(1) f_0^0 g(\xi) d\xi} > 0 \), such that

\[
\parallel Q_{\lambda} \phi \parallel \leq \lambda BM(1) \int_0^T g(\xi) d\xi < \parallel \phi \parallel, \quad \phi \in P \cap \partial \Omega_1, \quad 0 < \lambda < \lambda_0.
\]

If \( f_0 = \infty \), then we have \( f(t, x) \geq \eta x \) for \( 0 < x \leq r_2 \) and \( t \in [0, T] \), where \( \eta > 0 \) satisfies \( \lambda \eta A \int_0^T g(\xi) d\xi > 1 \), and \( 0 \leq \phi \leq 1 \), obviously, \( \Omega_{r_2} \subset \Omega_1 \).

Then \( 0 \leq \sigma r_2 = \sigma \parallel \phi \parallel \leq \Phi(t) \leq \parallel \phi \parallel = r_2 \), for all \( \phi \in P \cap \partial \Omega_{r_2}, t \in [0, T] \), thus

\[
f(t, \Phi(t)) \geq \eta \Phi(t).
\]

From the definition of \( Q_{\lambda} \), for \( \phi \in P \cap \partial \Omega_{r_2} \), we can obtain

\[
\parallel Q_{\lambda} \phi \parallel \geq \lambda \eta A \int_0^T g(\xi) \Phi(t) d\xi
\]

\[
\geq \lambda \eta A \parallel \phi \parallel \int_0^T g(\xi) d\xi > \parallel \phi \parallel.
\]

Thus, by Lemma 2.4(i), the operator \( Q_{\lambda} \) has at least one fixed point in \( P \cap (\overline{\Omega_1} \setminus \Omega_{r_2}) \).

If \( f^\infty = \infty \), then there exists \( H' > 0 \), such that \( f(t, x) \geq \eta \phi \) for \( \phi \geq H' \) and \( t \in [0, T] \), where \( \eta > 0 \) still satisfies \( \lambda \eta A \int_0^T g(\xi) d\xi > 1 \). Moreover, select \( r_3 = \max \{2, H'\} \), obviously, \( \Omega_1 \subset \Omega_{r_3} \).

Then \( \Phi(t) \geq \sigma \parallel \phi \parallel = \sigma r_3 \geq H' \), for all \( \phi \in P \cap \partial \Omega_{r_3}, t \in [0, T] \), thus

\[
f(t, \Phi(t)) \geq \eta \Phi(t).
\]

Then for \( \phi \in P \cap \partial \Omega_{r_3} \), we can obtain

\[
\parallel Q_{\lambda} \phi \parallel \geq \lambda \eta A \parallel \phi \parallel \int_0^T g(\xi) d\xi > \parallel \phi \parallel.
\]

Thus, by Lemma 2.4(ii), the operator \( Q_{\lambda} \) has at least one fixed point in \( P \cap (\overline{\Omega_{r_3}} \setminus \Omega_1) \).

Above all, if \( f_0 = \infty \) and \( f^\infty = \infty \), the operator \( Q_{\lambda} \) has at least two fixed points \( P \cap (\overline{\Omega_{r_3}} \setminus \Omega_{r_2}) \), that is, (1.1) has at least two positive \( T \)-periodic solutions for \( 0 < \lambda < \lambda_0 \).
(3) If \( j_0^* = 0 \), then \( f_0 > 0 \) and \( f_\infty > 0 \), that is, there exist positive constants \( \omega_1, \omega_2, r_1, r_2 \), where \( r_1 < r_2 \), such that
\[
\begin{align*}
  f(t, x) &\geq \omega_1 \mathfrak{f}, \quad \mathfrak{f} \in [0, r_1], \quad t \in [0, T]; \\
  f(t, x) &\geq \omega_2 \mathfrak{f}, \quad \mathfrak{f} \in [r_2, +\infty], \quad t \in [0, T].
\end{align*}
\]
Select \( c_1 = \min \{\omega_1, \omega_2, \min \{\frac{f(t,x)}{\mathfrak{f}} : t \in [0, T], \mathfrak{f} \in [r_1, r_2]\}\} \). Thus \( c_1 > 0 \), and
\[
f(t, x) \geq c_1 \mathfrak{f}, \quad \forall x \in [0, +\infty)^{n+1}, \quad t \in [0, T].
\]

Assume \( \varphi(t) \) is the fixed point of the operator \( Q_\lambda \), then \( Q_\lambda \varphi(t) = \varphi(t), t \in [0, T] \). Moreover, define \( \varphi' = (\varphi(t-\tau_0(t)), \varphi(t-\tau_1(t)), ..., \varphi(t-\tau_n(t))) \), thus \( f(t, \varphi') \geq c_1 \varphi' \).

On the other hand, there exists \( \lambda_0 = \frac{c_1 \sigma A}{c_1 \sigma A J_0^1 g(\xi) d\xi} \), such that
\[
\| \varphi \| = \| Q_\lambda \varphi \| \geq \lambda c_1 \sigma A \| \varphi \| \int_0^T g(\xi) d\xi > \| \varphi \|,
\]
for \( \lambda > \lambda_0 \). This is contradictory.

If \( j_\infty = 0 \), then \( f^0 < 0 \) and \( f_\infty < \infty \), that is, there exist positive constants \( \zeta_1, \zeta_2, r_1, r_2 \), where \( r_1 < r_2 \), such that
\[
\begin{align*}
  f(t, x) &\leq \zeta_1 \mathfrak{f}, \quad \mathfrak{f} \in [0, r_1], \quad t \in [0, T]; \\
  f(t, x) &\leq \zeta_2 \mathfrak{f}, \quad \mathfrak{f} \in [r_2, +\infty), \quad t \in [0, T].
\end{align*}
\]
Select \( c_2 = \max \{\zeta_1, \zeta_2, \max \{\frac{f(t,x)}{\mathfrak{f}} : t \in [0, T], \mathfrak{f} \in [r_1, r_2]\}\} \). Thus \( c_2 > 0 \), and
\[
f(t, x) \leq c_2 \mathfrak{f}, \quad \forall x \in [0, +\infty)^{n+1}, \quad t \in [0, T].
\]

Assume \( \psi(t) \) is the fixed point of the operator \( Q_\lambda \), then \( Q_\lambda \psi(t) = \psi(t), t \in [0, T] \). Moreover, define \( \psi' = (\psi(t-\tau_0(t)), \psi(t-\tau_1(t)), ..., \psi(t-\tau_n(t))) \), thus \( f(t, \psi') \leq c_2 \psi' \).

On the other hand, there exists \( \lambda_0 = \frac{c_2 B J_0^1 g(\xi) d\xi}{c_2 B J_0^1 g(\xi) d\xi} \), such that
\[
\| \psi \| = \| Q_\lambda \psi \| \leq \lambda c_2 B \| \psi \| \int_0^T g(\xi) d\xi < \| \psi \|,
\]
for \( 0 < \lambda < \lambda_0 \). This is also contradictory.

This proves the theorem.

\[ \square \]

**Corollary 3.1.** Suppose that (A0) holds.

(1) If there exists a \( c_1 > 0 \) such that \( f(t, x) \geq c_1 \mathfrak{f} \) for \( t \in [0, T], x \in [0, +\infty)^{n+1} \), when \( \lambda > \frac{1}{c_1 \sigma A J_0^1 g(\xi) d\xi} \), equation (1.1) has no positive \( T \)-periodic solution.

(2) If there exists a \( c_2 > 0 \) such that \( f(t, x) \leq c_2 \mathfrak{f} \) for \( t \in [0, T], x \in [0, +\infty)^{n+1} \), when \( 0 < \lambda < \frac{1}{c_2 B J_0^1 g(\xi) d\xi} \), equation (1.1) has no positive \( T \)-periodic solution.

**Theorem 3.2.** Suppose that (A0) holds and \( j_0^* = j_\infty^* = j_\infty = 0 \).

(1) If \( f^0 B < f_\infty \sigma A \), when \( \frac{1}{f_\infty \sigma A J_0^1 g(\xi) d\xi} < \lambda < \frac{1}{f^0 B J_0^1 g(\xi) d\xi} \), equation (1.1) has at least a positive \( T \)-periodic solution.

(2) If \( f_0 \sigma A > f^\infty B \), when \( \frac{1}{f_0 \sigma A J_0^1 g(\xi) d\xi} < \lambda < \frac{1}{f^\infty B J_0^1 g(\xi) d\xi} \), equation (1.1) has at least a positive \( T \)-periodic solution.
Thus, we have \( \phi_{0} \), then there exists \( \epsilon < f_{\infty} \), such that

\[
\frac{1}{(f_{\infty} - \epsilon) \sigma A \int_{0}^{T} g(\xi) d\xi} < \lambda < \frac{1}{(f^{0} + \epsilon) B \int_{0}^{T} g(\xi) d\xi},
\]

for the above \( \epsilon \), choose \( r_{1} > 0 \), such that \( f(t, x) \leq (f^{0} + \epsilon) \mathfrak{I} \) for \( \mathfrak{I} \in [0, r_{1}] \), \( t \in [0, T] \). Thus, for all \( \phi \in P \cap \partial \Omega_{r_{1}} \), we have \( 0 \leq \Phi(t) \leq r_{1} \), that is

\[
f(t, \Phi(t)) \leq (f^{0} + \epsilon) \Phi(t).
\]

Thus, we have

\[
\| Q_{\lambda} \phi \| \geq \lambda (f_{\infty} - \epsilon) A \| \phi \| \int_{0}^{T} g(\xi) d\xi \| \phi \|, \tag{3.3}
\]

for all \( \phi \in P \cap \partial \Omega_{r_{1}} \).

On the other hand, there exists \( H_{1} > 0 \), such that \( f(t, x) \geq (f_{\infty} - \epsilon) \mathfrak{I} \) for \( \mathfrak{I} \geq H_{1} \) and \( t \in [0, T] \). Moreover, select \( r_{2} = \max\{2r_{1}, \frac{H_{1}}{\mathfrak{I}}\} \), obviously, \( \Omega_{r_{2}} \subset \Omega_{r_{1}} \).

Then \( \Phi(t) \geq \sigma \| \phi \| = \sigma r_{2} \geq H_{1} \), for all \( \phi \in P \cap \partial \Omega_{r_{2}} \), \( t \in [0, T] \). Thus

\[
f(t, \Phi(t)) \geq (f_{\infty} - \epsilon) \Phi(t).
\]

Then, for \( \phi \in P \cap \partial \Omega_{r_{2}} \), we can obtain

\[
\| Q_{\lambda} \phi \| \geq \lambda (f_{\infty} - \epsilon) A \| \phi \| \int_{0}^{T} g(\xi) d\xi \| \phi \|.
\]

Thus, by Lemma 2.4(ii), the operator \( Q_{\lambda} \) has at least one fixed point in \( P \cap (\overline{(\Omega_{r_{2}} \setminus \Omega_{1})}) \), that is, (1.1) has at least a positive \( T \)-periodic solution for \( \frac{1}{f_{\infty} \sigma A \int_{0}^{T} g(\xi) d\xi} < \lambda < \frac{1}{f^{0} B \int_{0}^{T} g(\xi) d\xi} \).

(2) Assume \( f_{\infty} \sigma A > f^{\infty} B \), then \( f_{0} > f^{\infty} \), such that

\[
\frac{1}{(f_{0} - \epsilon) \sigma A \int_{0}^{T} g(\xi) d\xi} < \lambda < \frac{1}{(f^{\infty} + \epsilon) B \int_{0}^{T} g(\xi) d\xi},
\]

for the above \( \epsilon \), choose \( r_{1} > 0 \), such that \( f(t, x) \geq (f_{0} - \epsilon) \mathfrak{I} \) for \( \mathfrak{I} \in [0, r_{1}] \), \( t \in [0, T] \). Thus, for all \( \phi \in P \cap \partial \Omega_{r_{1}} \), we have \( 0 \leq \Phi(t) \leq r_{1} \), that is

\[
f(t, \Phi(t)) \geq (f_{0} - \epsilon) \Phi(t).
\]

Thus, we have

\[
\| Q_{\lambda} \phi \| \geq \lambda (f_{0} - \epsilon) A \| \phi \| \int_{0}^{T} g(\xi) d\xi \| \phi \|, \tag{3.4}
\]
for all $\phi \in P \cap \partial \Omega_{r_1}$.

On the other hand, there exists $H_2 > 0$, such that $f(t, x) \leq (f^\infty + \varepsilon)\pi$ for $\pi \geq H_2$ and $t \in [0, T]$. Moreover, select $r_2 = \max\{2r_1, \frac{H_2}{\sigma}\}$, obviously, $\Omega_{r_1} \subset \Omega_{r_2}$.

Then $\Phi(t) \geq \sigma \| \phi \| = \sigma r_2 \geq H_2$, for all $\phi \in P \cap \partial \Omega_{r_2}, t \in [0, T]$. Thus

$$f(t, \Phi(t)) \leq (f^\infty + \varepsilon)\Phi(t).$$

Then, for $\phi \in P \cap \partial \Omega_{r_2}$, we can obtain

$$\| Q \lambda \phi \| \leq \lambda (f^\infty + \varepsilon) B \| \phi \| \int_0^T g(\xi)d\xi < \| \phi \|.$$

Thus, by Lemma 2.4(i), the operator $Q \lambda$ has at least one fixed point in $P \cap (\Omega_{r_2} \setminus \Omega_1)$, which is the positive $T$-periodic solution of (1.1) for $\frac{1}{J_0} \int_0^T g(\xi)d\xi < \lambda < \frac{1}{f_0 B f(T)} \int_0^T g(\xi)d\xi$.

The proof is completed. \qed

**Corollary 3.2.** Suppose $h(t) \equiv 0, a(t) \not\equiv 0$, then (A0) holds if $\int_0^T a(\xi)d\xi \geq 0$ and $\int_0^T [a(\xi)]_+d\xi \geq \frac{1}{T}$.

4. Example

**Example 4.1.** Consider the following equations:

$$\phi'' + 2\phi' + \phi = \lambda (1 + \sin 8t) \frac{2 + \cos 8t}{2 + \phi(t - \tau(t))^n}, \quad n > 0, \quad (4.1)$$

where $h(t) = 2, a(t) = 1, g(t) = 1 + \sin 8t, f(t, x) = \frac{2 + \cos 8t}{2 + x^n}$, obviously, they are all $T = \frac{\pi}{4}$ periodic functions in $t$, moreover, $\tau(t)$ is an arbitrary $\frac{\pi}{4}$-periodic continuous function.

Through some calculations, the conditions of Lemma 2.1 are satisfied,

$$A = \frac{\frac{\pi}{4}}{[\exp(\frac{\pi}{4}) - 1]^2}, \quad B = \frac{\frac{\pi}{4} \exp(\frac{\pi}{4})}{[\exp(\frac{\pi}{4}) - 1]^2}, \quad \sigma = \exp(-\frac{\pi}{2}),$$

and

$$\int_0^\frac{\pi}{4} g(\xi)d\xi = \int_0^\frac{\pi}{4} (1 + \sin 8\xi)d\xi = \frac{\pi}{4},$$

$$m(1) = \min\{f(t, x), 0 \leq t \leq \frac{\pi}{4}, \exp(-\frac{\pi}{2}) \leq x \leq 1\} = \min\{\frac{2 + \cos 8t}{2 + x^n}, 0 \leq t \leq \frac{\pi}{4}, \exp(-\frac{\pi}{2}) \leq x \leq 1\} = \frac{1}{3},$$

$$M(1) = \max\{f(t, x), 0 \leq t \leq \frac{\pi}{4}, 0 \leq x \leq 1\} = \max\{\frac{2 + \cos 8t}{2 + x^n}, 0 \leq t \leq \frac{\pi}{4}, 0 \leq x \leq 1\} = \frac{3}{2}.$$
Periodic solutions of the equation

\[ f_\infty = \lim \inf_{x \to +\infty} \min_{t \in [0, \frac{\pi}{4}]} \frac{f(t, x)}{x} = \lim \inf_{x \to +\infty} \min_{t \in [0, \frac{\pi}{4}]} \frac{2 + \cos 8t}{x(2 + x^n)} = 0, \]

\[ f_0 = \lim \inf_{x \to 0^+} \min_{t \in [0, \frac{\pi}{4}]} \frac{f(t, x)}{x} = \lim \inf_{x \to 0^+} \min_{t \in [0, \frac{\pi}{4}]} \frac{2 + \cos 8t}{x(2 + x^n)} = \infty, \]

\[ f^\infty = \lim \sup_{x \to +\infty} \max_{t \in [0, \frac{\pi}{4}]} \frac{f(t, x)}{x} = \lim \sup_{x \to +\infty} \max_{t \in [0, \frac{\pi}{4}]} \frac{2 + \cos 8t}{x(2 + x^n)} = 0. \]

Thus, \( j_0 = 1, j'_\infty = 1 \), furthermore,

\[ \lambda_{01} = \frac{1}{A m(1)} \int_0^{\frac{\pi}{4}} g(\xi) d\xi = \frac{48[\exp(\frac{\pi}{4}) - 1]^2}{\pi^2}, \quad \lambda_{02} = \frac{1}{B M(1)} \int_0^{\frac{\pi}{4}} g(\xi) d\xi = \frac{32[\exp(\frac{\pi}{4}) - 1]^2}{3\pi^2 \exp(\frac{\pi}{2})}. \]

Therefore, by Theorem 3.1(1), Eq. (4.1) has at least a positive \( \frac{\pi}{4} \)-periodic solution for \( \lambda > \lambda_{01} = \frac{48[\exp(\frac{\pi}{4}) - 1]^2}{\pi^2} \), and by Theorem 3.1(2), Eq. (4.1) has at least a positive \( \frac{\pi}{4} \)-periodic solution for \( 0 < \lambda < \lambda_{02} = \frac{32[\exp(\frac{\pi}{4}) - 1]^2}{3\pi^2 \exp(\frac{\pi}{2})} \).

When \( n = 5, \tau = 0.7 \) and \( \lambda = 10 \), now \( \lambda > \lambda_{01} \), Figure 1 is the numerical simulation of Example 4.1.

![Figure 1. The numerical simulation of Example 4.1.](image)

**Example 4.2.** Now consider the following equations:

\[ \phi'' + 2\phi' + \phi = \lambda(1 + \sin 8t)\frac{\phi(t - \tau(t))^2(2 + \cos 8t)}{2 + \phi(t - \tau(t))^6}, \] (4.2)

note that \( f(t, x) = \frac{x^2(2 + \cos 8t)}{2 + x^n} \), moreover, \( \tau(t) \) is still an arbitrary \( \frac{\pi}{4} \)-periodic continuous function.

Now the conditions of Lemma 2.1 are still satisfied,

\[ A = \frac{\frac{\pi}{4}}{[\exp(\frac{\pi}{4}) - 1]^2}, \quad B = \frac{\frac{\pi}{4} \exp(\frac{\pi}{4})}{[\exp(\frac{\pi}{4}) - 1]^2}, \quad \sigma = \exp(-\frac{\pi}{2}), \]

and

\[ \int_0^{\frac{\pi}{4}} g(\xi) d\xi = \frac{\pi}{4}, \]

\[ \int_0^{\frac{\pi}{4}} (1 + \sin 8\xi) d\xi = \frac{\pi}{4}. \]
\[ m(1) = \min \{ f(t, x), 0 \leq t \leq \frac{\pi}{4}, \exp(-\frac{\pi}{2}) \leq x \leq 1 \} \]
\[ = \min \left\{ \frac{x^2(2 + \cos 8t)}{2 + x^6}, 0 \leq t \leq \frac{\pi}{4}, \exp(-\frac{\pi}{2}) \leq x \leq 1 \right\} \]
\[ = \frac{1}{2\exp(\pi) + \exp(-2\pi)}, \]

moreover,

\[ f^0 = \limsup_{x \to 0^+} \max_{t \in [0, \frac{\pi}{4}]} \frac{f(t, x)}{x} = \limsup_{x \to 0^+} \max_{t \in [0, \frac{\pi}{4}]} \frac{x^2(2 + \cos 8t)}{x(2 + x^6)} \]
\[ = \limsup_{x \to 0^+} \frac{x(2 + \cos 8t)}{2 + x^6} = 0, \]

\[ f^\infty = \limsup_{x \to +\infty} \max_{t \in [0, \frac{\pi}{4}]} \frac{f(t, x)}{x} = \limsup_{x \to +\infty} \max_{t \in [0, \frac{\pi}{4}]} \frac{x^2(2 + \cos 8t)}{x(2 + x^6)} \]
\[ = \limsup_{x \to +\infty} \frac{x(2 + \cos 8t)}{2 + x^6} = 0. \]

Thus, \( j_0 = 2, \)

\[ \lambda_{01} = \frac{1}{Am(1) \int_0^\frac{\pi}{4} g(\xi) d\xi} = \frac{16[\exp(\frac{\pi}{4}) - 1]^2[2\exp(\pi) + \exp(-2\pi)]}{\pi^2}. \]

Therefore, by Theorem 3.1(1), Eq.(4.2) has at least two positive \( \frac{\pi}{4} \)-periodic solutions for \( \lambda > \lambda_{01} = \frac{16[\exp(\frac{\pi}{4}) - 1]^2[2\exp(\pi) + \exp(-2\pi)]}{\pi^2}. \)

Acknowledgements

The author thanks the authors of the references for their useful inspirations, the reviewers and editors for their valuable suggestions and comments on this article.

References

[1] Z. Cheng and F. Li, Positive periodic solutions for a kind of second-order neutral differential equations with variable coefficient and delay, Mediterr. J. Math., 2018, 15(3), 134–152.
[2] J. Chu, N. Fan and P. J. Torres, Periodic solutions for second order singular damped differential equations, J. Math. Anal. Appl., 2012, 388(2), 665–675.
[3] J. Chu, X. Lin, D. O. Regan et al., Multiplicity of positive solutions to second order differential equations, Bull. Austral. Math. Soc., 2006, 73(02), 175–182.
[4] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.
[5] H. Gabsi, A. Ardjouni and A. Djoudi, Existence of positive periodic solutions of nonlinear neutral differential systems with variable delays, 2018, 64, 83–97.
[6] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.
[7] R. Hakl and P. J. Torres, Maximum and antimaximum principles for a second order differential operator with variable coefficients of indefinite sign, Appl. Math. Comput., 2011, 217(19), 7599–7611.
[8] J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[9] E. Hernández and S. Trofimchuk, *Traveling waves solutions for partial neutral differential equations*, Journal of Mathematical Analysis and Applications, 2020, 481(1), Article ID 123458.

[10] D. Jiang, J. J. Nieto and W. Zuo, *On monotone method for first and second order periodic boundary value problems and periodic solutions of functional differential equations*, J. Math. Anal. Appl., 2004, 289(2), 691–699.

[11] S. Kang and S. Cheng, *Periodic solution for second order periodic differential equations under scalable control*, Appl. Math. Comput., 2012, 218(18), 9138–9146.

[12] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, New York, 1993.

[13] Q. Li and Y. Li, *On the existence of positive periodic solutions for second-order functional differential equations with multiple delays*, Abstr. Appl. Anal., 2012, Article ID 929870.

[14] B. Mansouri, A. Ardjouni and A. Djoudi, *Periodicity and continuous dependence in iterative differential equations*, Rendiconti del Circolo Matematico di Palermo Series 2, 2019. DOI: 10.1007/s12215-019-00420-5.

[15] M. I. Muminov, *On the method of finding periodic solutions of second-order neutral differential equations with piecewise constant arguments*, Advances in Difference Equations, 2017, 336, 1–17.

[16] H. V. Ngo, *Existence results for extremal solutions of interval fractional functional integro-differential equations*, Fuzzy Sets and Systems, 2018, 347(15), 29–53.

[17] Z. Sabir, H. A. Wahab, M. Umar et al., *Stochastic numerical approach for solving second order nonlinear singular functional differential equation*, Applied Mathematics and Computation, 2019, 363(15), Article ID 124605.

[18] P. J. Torres, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations, 2003, 190(2), 643–662.

[19] C. Tunc and S. Erdur, *New qualitative results for solutions of functional differential equations of second order*, Discrete Dynamics in Nature and Society, 2018, DOI: 10.1155/2018/3151742.

[20] N. Wang, *Existence and uniqueness of periodic solutions for a kind of second-order neutral functional differential equation with delays*, Advances in Mathematical Physics, 2017, Article ID 9591087.

[21] Y. Wu, *Existence nonexistence and multiplicity of periodic solutions for a kind of functional differential equation with parameter*, Nonlinear Anal., 2009, 70(1), 433–443.