The word problem for some classes of Adian inverse semigroups

Muhammad Inam

Abstract We show that all of the Schützenberger complexes of an Adian inverse semigroup are finite if the Schützenberger complex of every positive word is finite. This enables us to solve the word problem for certain classes of Adian inverse semigroups (and hence for the corresponding Adian semigroups and Adian groups).

Keywords Inverse semigroups · Positive presentation · Cycle free presentation · Baumslag-Solitar presentation

1 Introduction

Groups and semigroups are often presented by a pair \( \langle X | R \rangle \), where \( X \) denotes the set of generators and \( R \) denotes the set of defining relations. If \( R \subseteq \{ (u, v) \mid u, v \in X^+ \} \), then the presentation \( \langle X | R \rangle \) is called a positive presentation. We consider positive presentations throughout this paper. We can construct two undirected graphs corresponding to a positive presentation \( \langle X | R \rangle \). These graphs are called the left graph and the right graph of the presentation and are denoted by \( LG(\langle X | R \rangle) \) and \( RG(\langle X | R \rangle) \) respectively. The vertices of \( LG(\langle X | R \rangle) \) and \( RG(\langle X | R \rangle) \) are labeled by the elements of the set \( X \). In \( LG(\langle X | R \rangle) \), we obtain an edge by joining the vertex labeled by the prefix letter of \( u \) with the vertex labeled by the prefix letter of \( v \), for all \( (u, v) \in R \). The graph \( RG(\langle X | R \rangle) \) is constructed dually by joining the vertex labeled by the suffix letter of \( u \) with the vertex labeled by suffix letter of \( v \), for all \( (u, v) \in R \). If there is no cycle in

Communicated by László Márki.

✉ Muhammad Inam
minam@huskers.unl.edu

1 Department of Mathematics, University of West Georgia, Carrollton, GA 30118, USA
the left graph and no cycle in the right graph of a presentation then the presentation is called a cycle free presentation or an Adian presentation. These presentations were first studied by Adian [1], where it is shown that every finitely presented Adian semigroup \( Sg \langle X | R \rangle \) embeds in the corresponding Adian group \( Gp(X | R) \). Later, John H. Remmers [6] generalized this embedding result to any (possibly nonfinite) Adian presentation. Unless stated otherwise, throughout this paper we consider our presentations to be Adian presentations.

A semigroup \( S \) is called an inverse semigroup if for every element \( a \in S \) there exists a unique element \( b \in S \) such that \( aba = a \) and \( bab = b \). This unique element \( b \) is called the inverse of \( a \) and is denoted by \( a^{-1} \). The natural partial order on the elements of an inverse semigroup \( S \) is defined as \( a \leq b \) if and only if \( a = aa^{-1}b \), for \( a, b \in S \). The congruence relation \( \sigma \) on \( S \) is defined as \( a\sigma b \) if and only if there exists an element \( c \in S \) such that \( c \leq a \) and \( c \leq b \). It turns out that the \( \sigma \) is the minimal group congruence on an inverse semigroup \( S \) and so \( S / \sigma \) is the maximal group homomorphic image of \( S \). If \( S \) is presented by a presentation \( Inv(X | R) \), then \( S / \sigma \) is isomorphic to the corresponding group \( Gp(X | R) \). Detailed proofs of these facts about inverse semigroups are provided in the text [3]. In order to be consistent with most of the literature about inverse semigroups, we abuse the notation of \( \sigma \) and we also denote the natural homomorphism from \( Inv(X | R) \) to \( Gp(X | R) \) by \( \sigma \). As usual, the set of idempotents of \( S \) is denoted by \( E(S) = \{ e \in S \mid e^2 = e \} \). An inverse semigroup \( S \) is called \( E \)-unitary if \( \sigma^{-1}(1) = E(S) \).

Stephen [7] introduced the notion of Schützenberger graphs to solve the word problem for inverse semigroups. If \( M = Inv(X | R) \) is an inverse semigroup then we may consider the corresponding Cayley graph \( \Gamma(M, X) \). The vertices of this graph are labeled by the elements of \( M \) and there exists a directed edge labeled by \( x \in X \cup X^{-1} \) from the vertex labeled by \( m_1 \) to the vertex labeled by \( m_2 \) if \( m_2 = m_1x \). The Cayley graph \( \Gamma(M, X) \) is not necessarily strongly connected, unless \( M \) happens to be a group, because it may happen that when there is an edge labeled by \( x \) from \( m_1 \) to \( m_2 \) there is not an edge labeled by \( x^{-1} \) from \( m_2 \) to \( m_1 \) (so, \( m_2 \neq m_1x^{-1} \)). The strongly connected components of \( \Gamma(M, X) \) are called the Schützenberger graphs of \( M \). For any word \( u \in (X \cup X^{-1})^* \) the strongly connected component of \( \Gamma(M, X) \) that contains the vertex labeled by \( u \) is the Schützenberger graph of \( u \) and it is denoted by \( \Sigma(M, X, u) \).

In [7] it is shown that the vertices of \( \Sigma(M, X, u) \) are precisely those vertices that are labeled by the elements of the \( R \)-class of \( u \), i.e., \( R_u = \{ m \in M \mid mm^{-1} = uu^{-1} \} \).

There exists a natural graph morphism (not necessarily injective) from each Schützenberger graph of \( M \) to the Cayley graph of the group \( G = M / \sigma \). Meakin showed that each Schützenberger graph of \( M \) embeds in the Cayley graph of \( M / \sigma \) if and only if \( M \) is \( E \)-unitary.

For any word \( u \in (X \cup X^{-1})^* \), it is useful to consider the Schützenberger automaton \( (uu^{-1}, \Sigma(M, X, u), u) \) with initial vertex \( uu^{-1} \in M \), terminal vertex \( u \in M \) and with the Schützenberger graph of \( u \) as the underlying graph of the Schützenberger automaton of \( u \). The language accepted by this automaton is a subset of \( (X \cup X^{-1})^* \) and will be denoted as \( L(u) \).

\[
L(u) = \{ w \in (X \cup X^{-1})^* \mid w \text{ labels a path from } uu^{-1} \text{ to } u \text{ in } \Sigma(M, X, u) \}.
\]
Here, $u$ and $w$ may be regarded both as elements of $(X \cup X^{-1})^*$ and as elements of $M$. Thus, $L(u)$ may be regarded as a subset of $(X \cup X^{-1})^*$ or as a subset of $M$.

The following result of Stephen [7] plays a key role in solving the word problem for inverse semigroups.

**Theorem 1.1** Let $M = \text{Inv}(X|R)$ and let $u, v \in (X \cup X^{-1})^*$.

(1) $L(u) = \{w \mid w \geq u \text{ in the natural partial order on } M\}$.

(2) The following are equivalent:

(i) $u = v$ in $M$.

(ii) $L(u) = L(v)$.

(iii) $u \in L(v)$ and $v \in L(u)$.

(iv) $(uu^{-1}, S\Gamma(M, X, u), u)$ and $(vv^{-1}, S\Gamma(M, X, v), v)$ are isomorphic as automata.

We briefly describe the iterative procedure described by Stephen [7] for building a Schützenberger graph. Let $\text{Inv}(X|R)$ be a presentation of an inverse monoid.

Given a word $u = a_1 a_2 \ldots a_n \in (X \cup X^{-1})^*$, the linear graph of $u$ is the birooted inverse word graph $(\alpha_u, \Gamma_u, \beta_u)$ consisting of the set of vertices

\[ V((\alpha_u, \Gamma_u, \beta_u)) = \{\alpha_u, \beta_u, \gamma_1, \ldots, \gamma_{n-1}\} \]

and edges

\[ (\alpha_u, a_1, \gamma_1), (\gamma_1, a_2, \gamma_2), \ldots, (\gamma_{n-2}, a_{n-1}, \gamma_{n-1}), (\gamma_{n-1}, a_n, \beta_u), \]

together with the corresponding inverse edges.

Let $(\alpha, \Gamma, \beta)$ be a birooted inverse word graph over $X \cup X^{-1}$. The following operations may be used to obtain a new birooted inverse word graph $(\alpha', \Gamma', \beta')$:

- **Determination or folding**: Let $(\alpha, \Gamma, \beta)$ be a birooted inverse word graph with vertices $v, v_1, v_2$, with $v_1 \neq v_2$, and edges $(v, x, v_1)$ and $(v, x, v_2)$ for some $x \in X \cup X^{-1}$.

  Then we obtain a new birooted inverse word graph $(\alpha', \Gamma', \beta')$ by taking the quotient of $(\alpha, \Gamma, \beta)$ by the equivalence relation that identifies the vertices $v_1$ and $v_2$ and the two edges. In other words, edges with the same label coming out of a vertex are folded together to become one edge.

- **Elementary $\mathcal{P}$-expansion**: Let $r = s$ be a relation in $R$ and suppose that $r$ can be read from $v_1$ to $v_2$ in $\Gamma$, but $s$ cannot be read from $v_1$ to $v_2$ in $\Gamma$. Then we define $(\alpha', \Gamma', \beta')$ to be the quotient of $\Gamma \cup (\alpha_s, \Gamma_s, \beta_s)$ by the equivalence relation that identifies vertices $v_1$ and $\alpha_s$ and vertices $v_2$ and $\beta_s$. In other words, we “sew” on a linear graph for $s$ from $v_1$ to $v_2$ to complete the other half of the relation $r = s$.

  An inverse word graph is deterministic if no folding can be performed and closed if it is deterministic and no elementary expansion can be performed over a presentation $\langle X|R \rangle$. Note that given a finite inverse word graph it is always possible to produce a determinized form of the graph, because determination reduces the number of vertices. So, the process of determination must stop after finitely many steps, We note also that the process of folding is confluent [7].
If \((\alpha_1, \Gamma_1, \beta_1)\) is obtained from \((\alpha, \Gamma, \beta)\) by an elementary \(\mathcal{P}\)-expansion, and 
\((\alpha_2, \Gamma_2, \beta_2)\) is the determined form of \((\alpha_1, \Gamma_1, \beta_1)\), then we write \((\alpha, \Gamma, \beta) \Rightarrow (\alpha_2, \Gamma_2, \beta_2)\) and say that \((\alpha_2, \Gamma_2, \beta_2)\) is obtained from \((\alpha, \Gamma, \beta)\) by a \(\mathcal{P}\)-expansion. The reflexive and transitive closure of \(\Rightarrow\) is denoted by \(\Rightarrow^*\).

For \(u \in (X \cup X^{-1})^*\), an approximate graph of \((uu^{-1}, \Sigma\Gamma(u), u)\) is a birooted inverse word graph \(A = (\alpha, \Gamma, \beta)\) such that \(u \in L[A]\) and \(y \geq u\) holds in \(M\) for all \(y \in L[A]\). Stephen showed in [7] that the linear graph of \(u\) is an approximate graph of \((uu^{-1}, \Sigma\Gamma(u), u)\). He also proved the following:

**Theorem 1.2** Let \(u \in (X \cup X^{-1})^*\) and let \((\alpha, \Gamma, \beta)\) be an approximate graph of \((uu^{-1}, \Sigma\Gamma(u), u)\). If \((\alpha, \Gamma, \beta) \Rightarrow^* (\alpha', \Gamma', \beta')\) and \((\alpha', \Gamma', \beta')\) is closed, then \((\alpha', \Gamma', \beta')\) is the Schützenberger automaton for \(u\).

In [7], Stephen showed that the class of all birooted inverse words graphs over \(X \cup X^{-1}\) is a co-complete category and that the directed system of all finite \(\mathcal{P}\)-expansions of a linear graph of \(u\) has a direct limit. Since the directed system includes all possible \(\mathcal{P}\)-expansions, this limit must be closed. Therefore, by 1.2, the Schützenberger graph of \(u\) is the direct limit.

**Full \(\mathcal{P}\)-expansion (a generalization of the concept of \(\mathcal{P}\)-expansion):** A full \(\mathcal{P}\)-expansion of a birooted inverse word graph \((\alpha, \Gamma, \beta)\) is obtained in the following way:

- Form the graph \((\alpha', \Gamma', \beta')\), which is obtained from \((\alpha, \Gamma, \beta)\) by performing all possible elementary \(\mathcal{P}\)-expansions of \((\alpha, \Gamma, \beta)\), relative to \((\alpha, \Gamma, \beta)\). We emphasize that an elementary \(\mathcal{P}\)-expansion may introduce a path labeled by one side of relation in \(R\), but we do not perform an elementary \(\mathcal{P}\)-expansion that could not be done to \((\alpha, \Gamma, \beta)\) when we do a full \(\mathcal{P}\)-expansion.
- Find the determinized form \((\alpha_1, \Gamma_1, \beta_1)\), of \((\alpha', \Gamma', \beta')\).

The birooted inverse word graph \((\alpha_1, \Gamma_1, \beta_1)\) is called the full \(\mathcal{P}\)-expansion of \((\alpha, \Gamma, \beta)\). We denote this relationship by \((\alpha, \Gamma, \beta) \Rightarrow_f (\alpha_1, \Gamma_1, \beta_1)\). If \((\alpha_n, \Gamma_n, \beta_n)\) is obtained from \((\alpha, \Gamma, \beta)\) by a sequence of full \(\mathcal{P}\)-expansions then we denote this by \((\alpha, \Gamma, \beta) \Rightarrow^*_f (\alpha_n, \Gamma_n, \beta_n)\).

We now expand the notion of Schützenberger graph to the Schützenberger complexes. The Schützenberger complexes were first defined by Steinberg in [8]. Later in [4], Steven Linblad made a small modification in Steinberg’s definition of Schützenberger complexes. In this paper, we are using Linblad’s definition of Schützenberger complexes. Let \(M = Inv(X|R)\) be an inverse monoid and \(m \in M\). The Schützenberger complex \(SC(m)\) for \(m \in M\) is defined as follows:

1. The 1-skeleton of \(SC(m)\) is the Schützenberger graph \(\Sigma\Gamma(m)\).
2. For each relation \((r, s) \in R\) and vertex \(v\), if \(r\) and \(s\) can be read at \(v\), then there is a face with boundary given by the pair of paths labeled by \(r\) and \(s\) starting from \(v\).

In similar manner, any of Stephen’s approximate graphs can be viewed as an approximate complex by sewing faces onto the approximate graph as described in (2) above. The direct limit of all approximate complexes for \(u\) will then be the Schützenberger complex for \(u\).
2 The word problem for Adian semigroups, Adian inverse semigroups and Adian groups

The following theorem was first proved by Adian in [1] for finite presentations. Later, it was generalized by Remmers to any Adian presentation, in [6].

**Theorem 2.1** An Adian semigroup $Sg\langle X|R\rangle$ embeds in the corresponding Adian group $Gp\langle X|R\rangle$.

From the embedding in Theorem 2.1, we can derive the fact that every Adian semigroup embeds in the corresponding Adian inverse semigroup, as proved in the following proposition.

**Proposition 1** An Adian semigroup $S = Sg\langle X|R\rangle$ embeds in the Adian inverse semigroup $M = Inv\langle X|R\rangle$.

**Proof** Let $\theta : S \to M$ be the natural homomorphism and $\phi : S = \langle X|R\rangle \to G = Gp\langle X|R\rangle$ be the natural homomorphism. $\phi$ is an injective homomorphism by 2.1. Note that $\phi = \sigma \circ \theta$. Since $\phi = \sigma \circ \theta$ and $\phi$ is injective, then $\theta$ must be injective. $\square$

**Conjecture 2.2** (Adian, 1976) The word problem for Adian semigroups is decidable.

**Remark 1** The word problem for one relation Adian semigroups is decidable. This is because Magnus [5] proved that the word problem for one relator groups is decidable and by Theorem 2.1, a one relation Adian semigroup embeds in the corresponding one relator Adian group.

**Proposition 2** The word problem for an Adian semigroup $S = Sg\langle X|R\rangle$ is decidable if the word problem for the corresponding Adian inverse semigroup $M = Inv\langle X|R\rangle$ is decidable. Furthermore, the word problem for the corresponding Adian group $G = Gp\langle X|R\rangle$ is decidable if both:

1. the word problem for the Adian inverse semigroup $M = Inv\langle X|R\rangle$ is decidable, and
2. the Adian inverse semigroup $M = Inv\langle X|R\rangle$ is E-unitary.

**Proof** By Proposition 1, $S$ embeds in $M$. Thus if (1) holds, then the word problem for $S$ is also decidable.

For any word $u \in (X \cup X^{-1})^*$, it immediately follows from (2) that $u = 1$ in $G$ if and only if $u$ is an idempotent in $M$. Thus if (1) and (2) both hold, we can check whether or not $u = 1$ in $G$ by checking the equality of the words $u = u^2$ in $M$. $\square$

The following theorem proved in [2] establishes the second hypothesis of Proposition 2.

**Theorem 2.3** Adian inverse semigroups are E-unitary.

We will prove the following theorem in the next section. This theorem will enable us to solve the word problem for some classes of Adian inverse semigroups.
The word problem for some classes of Adian inverse semigroups

Let $M = Inv\langle X|R \rangle$ be a finitely presented Adian inverse semigroup. Then the Schützenberger graph of $w$ is finite for all words $w \in (X \cup X^{-1})^*$ if and only if the Schützenberger graph of $w'$ is finite for all positive words $w' \in X^+$. 

For a finitely presented inverse semigroup $M = Inv\langle X|R \rangle$, a Schützenberger graph $\Gamma(w)$ is finite if and only if the corresponding Schützenberger complex $SC(w)$ is finite. Thus, the above Theorem 2.4 will also be true for Schützenberger complexes. That is, for a finitely presented Adian inverse semigroup, the Schützenberger complex of $w$ is finite for all words $w \in (X \cup X^{-1})^*$ if and only if the Schützenberger complex of $w'$ is finite for all positive words $w' \in X^+$. For the purpose of analyzing the word problem for inverse semigroups we are really only interested in the underlying Schützenberger graph and so we have stated the above theorem in the context of Schützenberger graphs. It turns out however that the concept of Schützenberger complexes is useful in proving some of the preliminary propositions and lemmas for this main theorem.

3 Main theorem

Let $Inv\langle X|R \rangle$ be an inverse semigroup. Then for any word $w \in (X \cup X^{-1})^*$, let $(\alpha_0, \Gamma_0(w), \beta_0)$ be the determinized form of the linear automaton for $w$. Then, the sequence of approximate graphs $\{(\alpha_n, \Gamma_n(w), \beta_n)|n \in \mathbb{N}\}$ obtained by full $P$-expansion over the presentation $\langle X|R \rangle$, converges to the Schützenberger graph of $w$ over the presentation $\langle X|R \rangle$. There exist graph homomorphisms, $\psi_n : \Gamma_n(w) \to \Gamma_{n+1}(w)$, such that $\psi_n(\alpha_n) = \alpha_{n+1}$ and $\psi_n(\beta_n) = \beta_{n+1}$, for all $n \in \mathbb{N}$. If we attach to $\Gamma_n(w)$ 2-cells corresponding to the relations in the obvious way, we obtain an approximate complex of $SC(w)$. We use the same notation, so that $\{(\alpha_n, \Gamma_n(w), \beta_n)|n \in \mathbb{N}\}$ becomes a sequence of approximate complexes that converges to $SC(w)$. We call a 2-cell to be an $n$-th generation 2-cell if it occurs in $(\alpha_n, \Gamma_n(w), \beta_n) \setminus (\psi_{n-1}(\alpha_{n-1}), \psi_{n-1}(\Gamma_{n-1}(w)), \psi_{n-1}(\beta_{n-1}))$, for all $n \in \mathbb{N}$. The following lemma is due to Steinberg [8,9].

Lemma 1 Let $M = Inv\langle X|R \rangle$ be an inverse semigroup and $w \in (X \cup X^{-1})^*$. Then the Schützenberger complex of $w$, $SC(w)$, is simply connected.

Lemma 2 Let $M = Inv\langle X|R \rangle$ be an Adian inverse semigroup and $w \in (X \cup X^{-1})^*$. Then the Schützenberger graph of $w$ contains no directed cycles of 1-cells.

Proof For every Schützenberger graph of an inverse semigroup $M$, there is a graph homomorphism into the Cayley graph of the maximal group image. In fact, Meakin showed that if $M$ is an $E$-unitary inverse semigroup then each Schützenberger graph embeds into the Cayley graph of the maximal group homomorphic image of $M$. It was proved in [2] that Adian inverse semigroups are $E$-unitary. So, $\Gamma(w)$ embeds into the Cayley graph of the group $G_M(X|R)$.

If $\Gamma(w)$ contains a directed cycle then the Cayley graph of $G_M(X|R)$ contains a directed cycle as well. We assume that that this directed cycle is labeled by some positive word $z \in X^+$. Then $z = 1$ and so there exists a Van Kampen diagram with
boundary labeled by \( z \). But this contradicts Lemma 2 (ii) of [2] which states that a Van Kampen diagram over an Adian presentation contains no directed cycles. In other words, for an Adian group \( G \) there is no positive word \( z \) that is equal to the identity element in \( G \). \( \square \)

The set of all edges (1-cells) of a graph (complex) whose tail vertex lies on a vertex labeled by \( v \) is denoted \( \text{Star}^o(v) \) and the set of all edges (1-cells) whose tip lies at a vertex \( v \) is denoted by \( \text{Star}^i(v) \).

The following lemma shows that if \( \text{Inv}(X|R) \) is an Adian inverse semigroup, then Stephen’s construction of approximations of the Schützenberger graph of a positive word only involves the elementary \( \mathcal{P} \)-expansion process and no folding at all.

**Lemma 3** Let \( M = \text{Inv}(X|R) \) be an Adian inverse semigroup and \( w \in X^+ \). Then no two edges fold together in Stephen’s process of constructing approximations of the Schützenberger graph of \( w \).

**Proof** We use induction on the number of elementary \( \mathcal{P} \)-expansions used in constructing an approximate graph.

At the beginning, before any elementary \( \mathcal{P} \)-expansion has been applied, we begin with the linear automaton of \( w \). Because \( w \in X^+ \) is a positive word, the linear automaton has no two consecutive edges oppositely oriented and so the linear automaton is already deterministic. No two edges need to be folded together to construct the first approximation \( \Gamma_0(w) \).

Suppose now that \( n \) elementary \( \mathcal{P} \)-expansions have been performed and no edge folding has been needed in constructing the approximate graph \( \Gamma_n(w) \). Suppose that \( (r, s) \) is a relation in \( R \) and \( r \) labels a path from vertex \( v \) to vertex \( v' \) in \( \Gamma_n(w) \) and that \( s \) does not label such a path from \( v \) to \( v' \). Let \( a_1, a_2, \ldots, a_n \) be the labels of the edges in \( \text{Star}^o(v) \). Since folding never occurred it must be that there are edges in \( LG(X|R) \) that comprise a tree spanning the vertices \( a_1, a_2, \ldots, a_n \) of \( LG(X|R) \). The word \( r \) starts with one of these \( a_i \), but \( s \) cannot start with one of the \( a_j \). Otherwise, we would get a cycle in \( LG(X|R) \) because the relation \( r = s \) would correspond to an edge in \( LG(X|R) \) between \( a_i \) and \( a_j \) that is not among the edges in the aforementioned spanning tree (because the relation \( r = s \) was not applied before at \( v \)). Thus, the sewing on of the relation \( r = s \) will not need to be followed by an edge folding at \( v \).

A dual argument, using \( RG(X|R) \), holds for incoming edges at vertices of \( \Gamma_n(w) \). \( \square \)

**Proposition 3** Let \( M = \text{Inv}(X|R) \) be an Adian inverse semigroup and \( w \in X^+ \). Then:

(i) \( \psi_n : (\alpha_n, \Gamma_n(w), \beta_n) \to (\alpha_{n+1}, \Gamma_{n+1}(w), \beta_{n+1}) \) is an embedding for all \( n \in \mathbb{N} \).

(ii) \( \Gamma(w) \) has exactly one source vertex \( \alpha \) and exactly one sink vertex \( \beta \), where \( (\alpha, \Gamma(w), \beta) \) is the Schützenberger automaton for \( w \).

(iii) For every vertex \( v \neq \alpha \) in \( (\alpha, \Gamma(w), \beta) \) there exists a positively labeled path in \( \Gamma(w) \) from \( \alpha \) to \( v \). For every vertex \( v \neq \beta \) there exists a positively labeled path in \( \Gamma(w) \) from \( v \) to \( \beta \).

(iv) Every positively labeled path in \( \Gamma(w) \) can be extended to a positively labeled transversal from \( \alpha \) to \( \beta \).
Proof (i) This follows immediately from Lemma 3. Since no foldings occur in the construction of $\Gamma_{n+1}(w)$ from $\Gamma_n(w)$, the images of two distinct vertices of the approximate graph $(\alpha_n, \Gamma_n(w), \beta_n)$ remain distinct under the map $\psi_n$, for all $n \in \mathbb{N}$.

(ii) From (i) and the fact that $(\alpha, \Gamma(w), \beta)$ is the direct limit of the sequence of approximate graphs $(\alpha_n, \Gamma_n(w), \beta_n)$ we can write $\psi_n(\alpha_n) = \alpha_{n+1} = \alpha$ and $\psi_n(\beta_n) = \beta_{n+1} = \beta$ for all $n \in \mathbb{N}$. We sew on a new positively labeled path (labeled by one side of a relation) to an approximate graph $(\alpha, \Gamma_n(w), \beta)$ only when we read a positively labeled path, labeled by the other side of the same relation, and each such expansion requires no subsequent folding. Since $\alpha$ and $\beta$ were the unique (and distinct) source and sink vertices of the linear automaton $(\alpha, \Gamma_0(w), \beta)$ and since no folding occurs in the construction of each $(\alpha, \Gamma_{n+1}(w), \beta)$ from $(\alpha, \Gamma_n(w), \beta)$ then $\alpha$ and $\beta$ remain the unique source and sink vertices for each approximate graph and thus for the limit graph $\Gamma(w)$.

(iii) Let $v \neq \alpha$ be a vertex in $\Gamma(w)$. Note that, as indicated in the proof of (i) and (ii), we can regard each approximate graph $(\alpha, \Gamma_n(w), \beta)$ to be a subgraph of the (possibly infinite) limit graph $(\alpha, \Gamma(w), \beta)$. For any vertex $v$ of $\Gamma(w)$ there is some $n$ such that $v$ first appears in the approximate graph $\Gamma_n(w)$. Since $(\alpha, \Gamma_n(w), \beta)$ is a subgraph of $(\alpha, \Gamma(w), \beta)$, it is sufficient to prove that the statement of (iii) is true for all approximate graphs. We’ll use induction on the number of elementary expansions used to construct an approximate graph. It is clear that for any vertex $v \neq \alpha$ of the linear automaton $(\alpha, \Gamma_0(w), \beta)$ there is a (unique) positively labeled path in $\Gamma_0(w)$ from $\alpha$ to $v$. Let $n \in \mathbb{N}$ and make the inductive assumption that for any vertex $v \neq \alpha$ of $\Gamma_n(w)$ there exists a positively labeled path in $\Gamma_n(w)$ from $\alpha$ to $v$. Let $(\alpha, \Gamma_n'(w), \beta)$ be the result of applying one elementary expansion by sewing on a path labeled by an $R$-word $r$ from vertex $v_1$ to vertex $v_2$ of $\Gamma_n(w)$. Supposing now that $v$ is a vertex in $\Gamma_n'(w)$, we may assume that $v$ lies on the path labeled by $r$ that was sewn on between vertices $v_1$ and $v_2$ in the expansion from $\Gamma_n(w)$ to $\Gamma_n'(w)$. Otherwise, the conclusion of (iii) would follow directly from the inductive assumption. Let $p_1$ be the path in $\Gamma_n(w)$ from $\alpha$ to $v_1$ that is guaranteed by the inductive assumption and let $p_2$ be the subpath that runs from $v_1$ to $v$ along the sewn-on $r$-labeled path. The concatenation of paths $p_1p_2$ is a positively labeled path in $\Gamma_n'(w)$ from $\alpha$ to $v$.

An entirely analogous proof guarantees the existence of a positively labeled path from arbitrary vertex $v$ to $\beta$ in $\Gamma(w)$, where $v \neq \beta$.

(iv) Let $p$ be a positively labeled path from vertex $u$ to vertex $v$ in $(\alpha, \Gamma(w), \beta)$. From (iii) we know that there is a positively labeled, or possibly empty, path $q_1$ from $\alpha$ to $u$ in $\Gamma(w)$. (The path $q_1$ would be the empty path if it happens that $\alpha = u$.) Likewise from (iii) there is a positively labeled, or possibly empty, path $q_2$ from $v$ to $\beta$ in $\Gamma(w)$. The concatenation of paths $q_1pq_2$ is a positively labeled path in $\Gamma(w)$ from $\alpha$ to $\beta$ that extends the path $p$. 

\[ \square \]

The word problem for $S = S_{G\langle X\mid R \rangle}$ is the question of whether there is an algorithm which given any two words $u, v \in X^+$, will determine whether $u = v$ in $S$. 

\[ \text{Springer} \]
For any two words $u, v \in X^+$, $u \equiv v$ if and only if there exists a transition sequence from $u$ to $v$,

$$u \equiv w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_n \equiv v,$$

for some $n \geq 0$,

where $w_{i-1} \rightarrow w_i$ represents that $w_i$ is obtained from $w_{i-1}$ by replacing one side of a relation $r$ (that happens to be a subword of $w_{i-1}$) with the other side $s$ of the same relation, for some $(r, s) \in R$. The above transition sequence is called a regular derivation sequence of length $n$ for the pair $(u, v)$ over the presentation $Sg\langle X | R \rangle$.

A semigroup diagram or $S$-diagram over a semigroup presentation $Sg\langle X | R \rangle$ for a pair of positive words $(u, v)$ is a finite, planar cell complex $D \subseteq \mathbb{R}^2$, that satisfies the following properties:

- The complex $D$ is connected and simply connected.
- Each edge (1-cell) is directed and labeled by a letter of the alphabet $X$.
- Each region (2-cell) of $D$ is labeled by the word $rs^{-1}$ for some defining relation $(r, s) \in R$.
- There is a distinguished vertex $\alpha$ on the boundary of $D$ such that the boundary of $D$ starting at $\alpha$ is labeled by the word $uv^{-1}$. $\alpha$ is a source in $D$ (i.e. there is no edge in $D$ with terminal vertex $\alpha$).
- There are no interior sources or sinks in $D$.

In [6], Remmers proved an analogue of Van Kampen’s Lemma for semigroups to address the word problem for semigroups.

**Theorem 3.1** Let $S = Sg\langle X | R \rangle$ be a semigroup and $u, v \in X^+$. Then there exists a regular derivation sequence of length $n$ for the pair $(u, v)$ over the presentation $Sg\langle X | R \rangle$ if and only if there is an $S$-diagram over the presentation $Sg\langle X | R \rangle$ for the pair $(u, v)$ having exactly $n$ regions.

**Proposition 4** Let $M = Inv\langle X | R \rangle$ be an Adian inverse semigroup and $w_1, w_2 \in X^+$ such that $w_1 \leq_M w_2$. Then:

(i) There exists a (planar) $S$-diagram corresponding to the pair of words $(w_1, w_2)$ that embeds in $SC(w_1)$.

(ii) $w_1 \equiv_M w_2$.

**Proof** (i) If $w_1 \leq_M w_2$, then $w_2 \in L(\alpha, \Gamma(w_1), \beta)$ (Stephen, [7]). So, $w_2$ labels a directed transversal from the vertex $\alpha$ to the vertex $\beta$. Since the transversals labeled by $w_1$ and $w_2$ are co-terminal and $SC(w_1)$ is simply connected by Lemma 1, then the closed path labeled by the word $w_1 w_2^{-1}$ is filled with finitely many 2-cells. Every 2-cell is two sided because $\langle X | R \rangle$ is an Adian presentation. So, we can obtain a regular derivation sequence from the word $w_1$ to the word $w_2$ from the complex $SC(w_1)$, over the presentation $\langle X | R \rangle$ in the following way.

Geometrically, we push the transversal labeled by $w_1$ across all first generation 2-cells that were contained in the closed path labeled $w_1 w_2^{-1}$ to obtain a new transversal labeled by $u_1 \in X^+$. Combinatorially, we have replaced some of the
non overlapping \(R\)-words that were subwords of the word \(w_1\) by the other side of the same relations. Then we push the transversal labeled by \(u_1\) across all generation 2-cells that were contained in the closed path labeled by \(w_1w_2^{-1}\) to obtain a new transversal labeled by \(u_2 \in X^+\). Again, we have just replaced some of the non overlapping \(R\)-words that were subwords of the word \(u_1\). This process eventually terminates because the closed path labeled by \(w_1w_2^{-1}\) contains only finitely many 2-cells. So, we obtain a regular derivation \(w_1 \to u_1 \to u_2 \to \ldots \to w_2\) over the presentation \(\langle X|R\rangle\). Hence there exists an \(S\)-diagram \(\mathcal{P}\) corresponding to this derivation sequence. Since no two edges fold together in \(SC(w_1)\), therefore \(\mathcal{P}\) embeds in \(SC(w_1)\).

(ii) This follows immediately from (i) and Theorem 3.1.

\[\square\]

**Lemma 4** Let \(M = Inv(X|R)\) be an Adian inverse semigroup, let \(w \in (X \cup X^{-1})^*\) and let \(w_1, w_2 \in X^+\) label two co-terminal paths in \(SC(w)\). Then there exists an \(S\)-diagram corresponding to the pair of words \((w_1, w_2)\) that embeds in \(SC(w)\).

**Proof** Since \(M\) is \(E\)-unitary, \(SC(w)\) embeds into the Cayley complex of the group \(G = Gp(X|R)\). So, the word \(w_1w_2^{-1}\) labels a closed path in the Cayley complex. Hence \(w_1w_2^{-1} = 1\). So, \(w_1 = w_2\). It follows from Theorem 2.1 that \(w_1 = w_2\) in \(Sg(X|R)\). So, there exists a regular derivation sequence \(D\) from \(w_1\) to \(w_2\) over the presentation \(S = Sg(X|R)\). The semigroup \(S\) embeds into the inverse semigroup \(M\), by Proposition 1. So the regular derivation sequence sequence \(D\) also holds in \(M\). The complex \(SC(w)\) is closed under elementary \(\mathcal{P}\)-expansion and folding and therefore all the 2-cells corresponding to this regular derivation sequence \(D\) already exist in \(SC(w)\) between the paths labeled by \(w_1\) and \(w_2\). Hence the \(S\)-diagram corresponding to the regular derivation sequence \(D\) embeds in \(SC(w)\).

\[\square\]

In general for any finite inverse semigroup presentation \((X, R)\), and for any word \(w \in (X \cup X^{-1})^*\), there are natural birouted graph morphisms for the sequence of approximate graphs given by Stephen’s procedure for approximating the birouted Schützenberger graph \((\alpha, \Gamma(w), \beta)\).

\[(\alpha_1, \Gamma_1(w), \beta_1) \to \cdots \to (\alpha_n, \Gamma_n(w), \beta_n) \to \to (\alpha_{n+1}, \Gamma_{n+1}(w), \beta_{n+1}) \to \cdots \to (\alpha, \Gamma(w), \beta)\]

It was established in Proposition 3 that if \(M = Inv(X|R)\) is an Adian inverse semigroup and \(w\) is a positive word, that is \(w \in X^+\), then all of the maps in the above sequence are actually embeddings. We may abuse the notation slightly in this case and for each approximate graph \(\Gamma_n(w)\) we denote the initial and terminal vertices simply as \(\alpha\) and \(\beta\).

\[(\alpha, \Gamma_1(w), \beta) \leftrightarrow \cdots \leftrightarrow (\alpha, \Gamma_n(w), \beta) \leftrightarrow (\alpha, \Gamma_{n+1}(w), \beta) \leftrightarrow \cdots \leftrightarrow (\alpha, \Gamma(w), \beta)\]

Recall that for a positive word \(w \in X^+\), a transversal of an approximate Schützenberger complex \((\alpha, \Gamma_k(w), \beta)\) is defined to be any positively labeled path in \(\Gamma_k(w)\) from \(\alpha\) to \(\beta\). Before proving the main theorem of this section we introduce the following definition of \(n\)-th generation transversal of a Schützenberger complex.
Definition 1 An \textit{n-th generation transversal} of the Schützenberger complex \((\alpha, \Gamma(w), \beta)\) is a positively labeled path from \(\alpha\) to \(\beta\) that can be read in the approximate complex \((\alpha, \Gamma_n(w), \beta)\) but cannot be read in \((\alpha, \Gamma_{n-1}(w), \beta)\), for some \(n \in \mathbb{N}\).

When studying certain problems involving an inverse monoid given by a finite presentation \((X, R)\), for example when considering the word problem for \(M = \text{Inv}(X|R)\), it is natural to first ask whether it might happen to be the case that all of the Schützenberger graphs of \(M\) are finite. The following theorem, our main theorem for this paper, shows that this question can be reduced to the question of whether or not all Schützenberger graphs of positive words are finite.

Theorem 3.2 Let \(M = \text{Inv}(X|R)\) be a finitely presented Adian inverse semigroup. Then the Schützenberger complex of \(w\) is finite for all words \(w \in (X \cup X^{-1})^*\) if and only if the Schützenberger complex of \(w'\) is finite for all positive words \(w' \in X^+\).

Idea of the proof: We assume that the Schützenberger graph of every positive word is finite and we let \(w\) be an arbitrary word, \(w \in (X \cup X^{-1})^*\). We will use induction on the number of edges in the Munn tree \(MT(w)\) to prove that the Schützenberger complex of \(w\) is finite. The essential part of the proof involves realizing \(SC(w)\) as the limit of a sequence of finite complexes, via the procedure of Stephen’s \(P\)-expansion. We begin with a finite inverse graph (complex) \(S\) that is closed under \(P\)-expansions relative to \((X|R)\). To the complex \(S\) we attach a single edge \(e\) that is labeled by some letter, say \(a \in X\). The resulting complex \(S_1 = S \cup \{e\}\) will in general not be closed under \(P\)-expansions. In a process similar to Stephen’s full \(P\)-expansion construction, we define a sequence of finite complexes \(S_1, S_2, \ldots\) that converges in the limit to \(SC(w)\). Our theorem will be proved if we can show that this limit is in fact a finite complex. Equivalently, we must prove that the sequence \(S_1, S_2, \ldots\) stabilizes after finitely many steps at some \(S_k\).

Proof We assume that \(SC(w')\) is finite for all positive words \(w' \in X^+\). Let \(w\) be an arbitrary word, \(w \in (X \cup X^{-1})^*\). We will show that \(SC(w)\) is finite, by applying induction on the number of edges in \(MT(w)\).

For the base of our induction, we suppose that \(MT(w)\) consists of only one edge, labeled say by \(a \in X\). Then by our assumption about positive words, \(SC(w) = SC(a)\) is finite.

For our induction hypothesis, we assume that \(SC(w_0)\) is finite for all words \(w_0 \in (X \cup X^{-1})^*\), whose Munn tree consist of \(k\) edges. Let \(w \in (X \cup X^{-1})^*\) be a word such that \(MT(w)\) consists of \(k + 1\) edges. We will show that \(SC(w)\) is finite.

Let \(\alpha\) be an extremal vertex of \(MT(w)\) (i.e., a leaf of the tree \(MT(w)\)), and let \(e\) be the edge of \(MT(w)\) that connects \(\alpha\) to the remainder of \(MT(w)\). Assume that \(e\) is a positively labeled edge with initial vertex \(\alpha\) and terminal vertex \(\beta\). The case when \(e\) is negatively labeled is dual. The sub-tree obtained by removing the edge \(e\) consists of \(k\) edges; we denote this sub-tree by \(T\) for our future reference. Note that the tree \(T\) is in fact the Munn tree of some word \(z \in (X \cup X^{-1})^*\). That is, \(T = MT(z)\) and there are \(k\) edges in \(MT(z)\). By the induction hypothesis, the Schützenberger complex \(S\) generated by \(T\), i.e., \(S = SC(z)\), is a finite complex. There exists a graph morphism \(\phi : T \to S\), and so we may regard the vertex \(\beta\) of \(T\) as a vertex in the complex.
$S = SC(z)$. We reattach the edge $e$ to the vertex $\beta$ of $S$ and denote the resulting finite complex by $S_1$, (see Fig. 1.) The finite complex $S = SC(z)$ was obtained from the subtree $T$ by sewing on relations from $R$, and since we reattached the edge $e$ in $S_1 = S \vee \{e\}$, then naturally $S_1$ may be regarded as an approximation to $SC(w)$. While the Schützenberger complex $S = SC(z)$ is closed under $\mathcal{P}$-expansion, the complex $S_1$ is not necessarily closed under $\mathcal{P}$-expansion. In particular, it is possible that there may be one or more relations $(r, s) \in R$ such that $r$ labels a path in $S_1$ that begins at vertex $\alpha$ and the other side of the relation, $s$, is not read in $S_1$ at $\alpha$. It is clear that the closure of $S_1$ under $\mathcal{P}$-expansion over $\langle X \mid R \rangle$ is the Schützenberger complex $SC(w)$.

If the edge $e$ (labeled say, by $a$) gets immediately identified by folding to an $a$-labeled edge of the finite complex $S$, then we are done, because in that case we would have $SC(w) = S$, which was assumed to be a finite complex. So, we assume that the edge $e$ does not get immediately identified by folding with any of the edges in the finite complex $S$. We extend the edge $e$ to all possible maximal positively labeled paths in $S_1$. There are only finitely many maximal positively labeled paths in $S_1$ with initial edge $e$ because $S_1$ is a finite complex that has no positively labeled cycles (by Lemma 2). We assume that these paths are labeled by $w_1, w_2, \ldots, w_n$, where $w_i \in X^+$ for $1 \leq i \leq n$. Each such $w_i$ labels a path from $\alpha$ to some vertex $\beta_i$ of $S$. (See Fig. 2).

In order to complete $S_1$ under elementary $\mathcal{P}$-expansion and folding, we first attach $SC(w_i)$ to the path labeled by $w_i$ in $S_1$, for all $i$, and denote the resulting finite complex by $S'_1$. (See Fig. 3.) Each complex $SC(w_i)$ for $1 \leq i \leq n$, is finite since each $w_i \in X^+$. Thus we obtain $S'_1$ by attaching to $S_1$ finitely many complexes, each of which is finite, and so $S'_1$ is finite.

The complex $S'_1$ is not necessarily determinized. That is, as a consequence of attaching the complexes $SC(w_i)$, there may now be vertices along the paths in $S'_1$ labeled by $w_i$ at which there exist two or more edges labeled by the same letter and so we must perform foldings to obtain a determinized graph (complex). We denote the determinized form of $S'_1$ by $S_2$. Since $S'_1$ is finite, then so is its determinized quotient $S_2$. Note however, as a consequence of folding $S'_1$ to $S_2$, that the complex $S_2$ may not be closed under $\mathcal{P}$-expansions.
And so, we iterate the procedure. In general, the complex $S_k$ may not be closed under $P$-expansion. That is, there may be a relation $(r, s) \in R$ and a path $p$ labeled by $r$ between some two vertices $v$ and $v'$ in $S_k$ such that the other side of the relation, $s$, does not label a path in $S_k$ between $v$ and $v'$. We refer to such a path $p$ in $S_k$ as an unsaturated path.

Note that in Stephen’s procedure we would at this stage simply attach a path labeled by $s$ between $v$ and $v'$ as one step in the $P$-expansion. In our setting it turns out, however, that we can “speed up” Stephen’s procedure. We will show that every such unsaturated path $p$ can be extended to a positively labeled path that begins at the vertex $\alpha$. Such a path can itself be extended to a maximal positively labeled path that begins at $\alpha$. So, instead of merely attaching a single cell along the unsaturated path $p$ that is labeled by $r$, we instead read the label, say $w_j$, of the maximal positively labeled path beginning at $\alpha$ that contains $p$ as a subpath and we attach the (finite) complex
SC(w_j) along this path. Thus in this one step we are attaching not only the one cell along the unsaturated path p, but also are attaching all cells that would arise from \( \mathcal{P} \)-expansions on the maximal path labeled by \( w_j \).

Our iterative procedure for constructing a sequence \( \{S_n\} \) of complexes can be summarized as follows. Suppose that \( S_k \) has been constructed. We first look for all positive words \( w_i \) that label a maximal positive path in \( S_k \) that starts at the vertex \( \alpha \) and that does not label a path starting at \( \alpha \) in \( S_{k-1}' \). We obtain \( S_k' \) by attaching \( SC(w_i) \) to each such maximal positive path in \( S_k \). Then we obtain \( S_{k+1} \) by determinizing \( S_k' \). Thus we obtain a sequence of finite complexes \( \{S_n\} \) that has \( SC(w) \) as its limit.

\[
S_1 \rightarrow \cdots \rightarrow S_k \rightarrow S_k' \rightarrow S_{k+1} \rightarrow \cdots
\]

For each maximal positive path with label \( w_i \) that starts at \( \alpha \) and does not exist in \( S_{k-1}' \), attach \( SC(w_i) \).

Fold. (Determinize.)

A question is whether we ever reach a graph \( S_k \) in the procedure such that all maximal positive paths that can be read at \( \alpha \) in \( S_k \) can already be read in \( S_{k-1}' \) at \( \alpha \). If this happens, then the sequence of complexes will stabilize at \( S_k \). Equivalently, we can ask whether we ever reach a complex \( S_k \) such that every maximal positive path in \( S_k \) is closed under \( \mathcal{P} \)-expansion. (We say that a maximal positive path \( p \) is closed under \( \mathcal{P} \)-expansion if for every relation \( (r, s) \in \mathcal{R} \), and for every subpath of \( p \) labeled by \( r \) between vertices \( v \) and \( v' \), then the other side of the relation, \( s \), already labels a path between \( v \) and \( v' \) in \( S_k \).)

To answer this question, we analyze how the process of folding \( S_k' \) to \( S_{k+1} \) affects positively labeled paths that already exist in \( S_k' \) and how new positively labeled paths may be created in \( S_{k+1} \) as a consequence of the folding process. We first note, for example, that when we fold \( S_k' \) to \( S_2 \) that each of the attached complexes \( SC(w_i) \) that we attached to \( S_1 \) will be embedded in \( S_2 \) after the folding process. This follows from a fact (see Stephen, [7]) about \( E \)-unitary semigroups: If \( M \) is \( E \)-unitary and \( w \) is any word, and a word \( w_i \) can be read along some path of the Schützenberger graph \( \Gamma(w) \), then the entire Schützenberger graph \( \Gamma(w) \) will occur as an embedded subgraph of \( \Gamma(w) \) along that path. Since the theorem we are proving assumes that the semigroup \( M \) has an Adian presentation, we know from the main theorem of [2] that \( M \) is \( E \)-unitary. Thus, each \( SC(w_i) \) embeds in \( S_2 \). Likewise, the original graph \( S \) is actually the Schützenberger graph \( \Gamma(z) \), where the word \( z \) labels a path in \( S_1 \), and so we know that the original graph \( S \) also remains embedded as a subgraph of \( S_2 \). In other words, no two vertices of any one graph \( SC(w_i) \) will become identified with each other and no two vertices of the original graph \( S \) will become identified with each other in the folding process that takes \( S_1' \) to \( S_2 \). Two vertices of \( S_1' \) will become identified in the folding process only if one of the vertices belongs to the original \( S \) and the other vertex belongs to one of the attached \( SC(w_i) \), or if the two vertices belong to \( SC(w_i) \) and \( SC(w_j) \), with \( i \neq j \). Everything that we just said about the process of folding \( S_1' \) to \( S_2 \) holds as well for the process of folding \( S_k' \) to \( S_{k+1} \). The original complex \( S \) and each complex \( SC(w_i) \), attached at any step of the iteration, will be embedded as subcomplexes of \( S_{k+1} \). The interest is in what new paths may be formed as a result of folding \( S_k' \) to \( S_{k+1} \).
Claim Suppose that $SC(w_i)$ is one of the complexes that was attached to $S'_k$ to form $S'_k$. Suppose, in the process of folding $S'_k \to S_{k+1}$, that a vertex $\gamma$ of $SC(w_i)$ gets identified, as a consequence of folding, with a vertex $\gamma'$ of the original complex $S$. Then we claim that every positively labeled path in $SC(w_i)$ from $\gamma$ to $\beta_i$ will get identified by the folding process with a path in $S$ from the vertex $\gamma'$ to $\beta_i$. Thus, every maximal positive path $p$ that begins at the vertex $\alpha$ in $S_{k+1}$ and cannot be read beginning at the vertex $\alpha$ in $S'_k$ will factor uniquely as $p = p_1p_2$, where $p_1$ is a path in some $SC(w_i)$ and $p_2$ is a path in the original complex $S$.

Proof of Claim Since $\gamma$ gets identified with $\gamma'$ through folding, we know (Stephen, [7]) that in $S'_k$ there is a path from $\gamma$ to $\gamma'$ labeled by a Dyck word $d$ that we may assume is of the form $d = ss^{-1}$, where $s \in (X \cup X^{-1})^*$. Further, we may assume that $s$ is a reduced word. There must be a vertex $\delta$ that lies on the path labeled by $w_i$ (the intersection of $SC(w_i)$ and $S$ in $S'_k$) so that $s$ labels a path in $SC(w_i)$ from $\gamma$ to $\delta$ and $s$ also labels a path in $S$ from $\gamma'$ to $\delta$. (See Figs. 4, 5, 6.) To prove the above claim we assume that $r_2$ labels an arbitrary positive path in $SC(w_i)$ from $\gamma$ to $\beta_i$. We need to prove that this path gets identified by folding with a path in $S$ that is labeled by $r_2$ from $\gamma'$ to $\beta_i$. The path in $SC(w_i)$ that is labeled by $r_2$ can be extended to an $n$th generation transversal $t$ of $SC(w_i)$, for some $n$. To complete our proof, we apply induction on the generation number, $n$, of the transversal of $t$ of $SC(w_i)$.

Suppose that the path labeled by $r_2$ lies on a 1st generation transversal $t$ of $SC(w_i)$. We assume that $t \equiv r_1r_2$ where $r_1$ labels the sub-path of $t$ from $\alpha$ to $\gamma$ and $r_2$ labels the sub-path of $t$ from $\gamma$ to $\beta_i$. Since $\gamma$ lies on a 1st generation transversal of $SC(w_i)$, we conclude that in the Dyck word path $ss^{-1}$ that goes from $\gamma$ to $\delta$ to $\gamma'$, it must be that either $s$ is a positive word or $s^{-1}$ is a positive word. We examine the following three cases.

Fig. 4 Case 1 for the first generation transversals of the proof of the Claim
Case 1. (See Fig. 4.) Suppose that the vertex $\delta$ is actually $\beta_i$, the terminal vertex of the path labeled by $w_i$, and suppose the Dyck word labeling the path from $\gamma$ to $\gamma'$ is the word $r_2r_2^{-1}$. In this case, the word $r_2$ labels the subpath of the transversal $t$ of $SC(w_i)$ from $\gamma$ to $\beta_i$ and the word $r_2$ also labels a path from $\gamma'$ to $\beta_i$. These two paths, both labeled by $r_2$, which meet at the vertex $\beta_i$, will fold together so that the path labeled by $r_2$ in $SC(w_i)$ from $\gamma$ to $\beta_i$ gets identified with the path in $S$ from $\gamma'$ to $\beta_i$. So in this case the statement of the claim obviously holds.

Case 2. (See Fig. 5.) We assume that the Dyck word path from $\gamma$ to $\delta$ to $\gamma'$ is labeled by $r_3r_3^{-1}$, where $r_3$ is a positive word, $r_3 \in X^+$. So, the path in $SC(w_i)$ from $\gamma$ to $\delta$ is labeled by $r_3$ and the path in $S$ from $\gamma'$ to $\delta$ is labeled by $r_3$. The
oppositely oriented paths labeled by $r_3$, which meet at $\delta$, become identified with each other through folding. We assume that the sub-path from $\delta$ to $\beta_i$, of the maximal path labeled by $w_i$, is labeled by $r_4 \in X^+$. Obviously, the path labeled by $r_4$ from $\delta$ to $\beta_i$ is in $SC(w_i)$ and the path labeled by $r_3$ from $\gamma$ to $\delta$ is also in $SC(w_i)$. Hence the path labeled by $r_3 r_4$ from $\gamma$ to $\beta_i$ is in $SC(w_i)$. The positive words $r_2$ and $r_3 r_4$ label two co-terminal paths in $SC(w_i)$. So, by Lemma 4 an $S$-diagram corresponding to the pair of words $(r_2, r_3 r_4)$ embeds in $SC(w_i)$. This $S$-diagram also embeds in $S$, because $S$ contains a path labeled by one side of this $S$-diagram, (namely, the path from $\gamma'$ to $\beta_i$ in $S$ that is labeled by $r_3 r_4$), and $S$ is closed under elementary $\mathcal{P}$-expansion. So, the two $S$-diagrams corresponding to the pair of words $(r_2, r_3 r_4)$ get identified with each other and our claim holds in this case. That is, the path in $SC(w_i)$ labeled by $r_2$ from $\gamma$ to $\beta_i$ gets identified with a path in $S$ from $\gamma'$ to $\beta_i$.

Case 3. (See Fig. 6.) We assume that the Dyck word path from $\gamma$ to $\delta$ to $\gamma'$ is labeled by $r_3^{-1} r_3$, where $r_3$ is a positive word. So, the path in $SC(w_i)$ from $\delta$ to $\gamma$ is labeled by $r_3$ and the path in $S$ from $\delta$ to $\gamma'$ is labeled by $r_3$. The oppositely oriented paths labeled by $r_3$, which meet at $\delta$, become identified with each other through folding. Now we have the path in $SC(w_i)$ labeled by $r_3 r_2$ from $\delta$ to $\beta_i$ in $SC(w_i)$. Again, we let the positive word $r_4$ be the label of the sub-path of the maximal path labeled by $w_i$ from $\delta$ to $\beta_i$. The words $r_3 r_2$ and $r_4$ label two co-terminal paths in $SC(w_i)$. So, by Lemma 4 an $S$-diagram corresponding to the pair of words $(r_3 r_2, r_4)$ embeds in $SC(w_i)$. This $S$-diagram also embeds in $S$, because $S$ contains a path labeled by one side of this $S$-diagram (namely, the path labeled by $r_4$) and $S$ is closed under elementary $\mathcal{P}$-expansion. Hence these two $S$-diagrams get identified with each other through folding. In particular, the path in $SC(w_i)$ labeled by $r_2$ from $\gamma$ to $\beta_i$ gets identified with a path in $S$ from $\gamma'$ to $\beta_i$, and so our claim follows in this case as well. This concludes the base case of the inductive proof of the claim.

We assume that our claim is true for all paths from $\gamma$ to $\beta_i$ that lie along a $(n-1)$-st generation transversal of $SC(w_i)$. We prove that our claim is true for all $n$-th generation transversals of $SC(w_i)$ as well.

Suppose that an arbitrary path from $\gamma$ to $\beta_i$ is labeled by the positive word $s_2$ and suppose that this path extends to an $n$-th generation transversal $t$ of $SC(w_i)$. Let $t \equiv s_1 s_2$ where $s_1$ labels the sub-path of $t$ from $\alpha$ to $\gamma$, and $s_2$ labels the sub-path of $t$ from $\gamma$ to $\beta_i$. Again, since the vertex $\gamma$ in $SC(w_i)$ gets identified with with the vertex $\gamma'$ in $S$ through folding, we know that there is a Dyck word $ss^{-1}$ that labels a path from $\gamma$ to $\gamma'$. Since $\gamma$ lies on an $n$-th generation transversal of $SC(w_i)$, the Dyck word path $ss^{-1}$ must pass through some vertex $\delta$ of $SC(w_i)$ that lies on an $(n-1)$-st generation transversal of $SC(w_i)$. We examine the following three cases.

Case 1. Suppose the vertex $\delta$ is the vertex $\beta_i$ and we read the Dyck word $s_2 s_2^{-1}$ in $S_k$ from $\gamma$ to $\gamma'$. Then we fold the oppositely oriented paths labeled by $s_2$. So in this case it is obvious that that path in $SC(w_i)$ labeled by $s_2$ gets identified with the path in $S$ from $\gamma'$ to $\beta_i$.

Case 2. (See Fig. 7.) Suppose $q$ denotes an $(n-1)$-st generation transversal of $SC(w_i)$ and the vertex $\delta$ lies on the transversal $q$. Assume also that $\delta$ has already been identified with a vertex of $S$ as a consequence of folding along the Dyck word $ss^{-1}$. In this case (Case 2), we assume that the portion of the Dyck word path from $\gamma$ to $\delta$ is labeled by a positive word $s_3 \in X^+$. Since $\delta$ has already been folded and identified
with a vertex of $S$, we have the Dyck word $s_3s_3^{-1}$ labeling a path from $\gamma$ to $\delta$ to $\gamma'$ in the partially folded $S'_k$. The path in $SC(w_i)$ labeled by $s_3$ from $\gamma$ to $\delta$ can be extended along the $(n - 1)$-st generation transversal $q$ to the vertex $\beta_i$. We assume that this path is labeled by $s_3s_4 \in X^+$, where $s_4 \in X^+$ labels the sub-path of $q$ from $\delta$ to $\beta_i$. By the induction hypothesis the sub-path of $q$ from $\delta$ to $\beta_i$ gets identified with a path in $S$. Thus we have a path in $SC(w_i)$ labeled by $s_3s_4$ from $\gamma$ to $\beta_i$ and we also have a path in $S$ labeled by $s_3s_4$ from $\gamma'$ to $\beta_i$. The positive words $s_2$ and $s_3s_4$ label two co-terminal paths in $SC(w_i)$. So by Lemma 4 an $S$-diagram corresponding to the pair of words $(s_2, s_3s_4)$ embeds in $SC(w_i)$. This $S$-diagram also embeds in $S$, because $S$ contains one side of this $S$-diagram and $S$ is closed under elementary $P$-expansion. So the two $S$-diagrams corresponding to the pair of words $(s_2, s_3s_4)$ get identified with each other. Hence, the path in $SC(w_i)$ labeled by $s_2$ gets identified with a path in $S$, and the claim holds in this case.

**Case 3.** (See Fig. 8.) Suppose $q$ denotes an $(n - 1)$-st generation transversal of $SC(w_i)$ and $\delta$ is a vertex on the transversal $q$ that has already been identified with a vertex of $S$. We also assume that there exists a path, labeled by $s_3 \in X^+$, from $\delta$ to $\gamma$ such that we can read the Dyck word $s_3^{-1}s_3$ from $\gamma$ to $\delta$ to $\gamma'$ in $S'_k$. We extend the path labeled by $s_3$ from $\delta$ to $\gamma$ along the $n$-th generation transversal $t$ to a positively labeled path (labeled by $s_3s_2 \in X^+$) from $\delta$ to $\beta_i$. By the induction hypothesis the sub-path of $q$ from the vertex $\delta$ to the vertex $\beta_i$ (say, labeled by $s_4 \in X^+$) is in $S$. Hence $s_3s_2$ and $s_4$ label two co-terminal paths in $SC(w_i)$ and the path labeled by $s_4$ is also in $S$. By Lemma 4 an $S$-diagram corresponding to the pair of words $(s_3s_2, s_4)$ embeds in $SC(w_i)$. This $S$-diagram also embeds in $S$, because $S$ contains one side of this $S$-diagram and $S$ is closed under elementary $P$-expansion. Thus the two $S$-diagrams corresponding to the pair of words $(s_3s_2, s_4)$ get identified with each other and the claim holds in this case as well. This completes the inductive step in the proof of the claim.  

$\square$
We perform all possible foldings mentioned in the proof of above claim in $S_1'$ and denote the resulting complex by $S_2$. In this folding process edges of the complex $S$ are folded with the edges of $SC(w_i)$ for some $i$. This folding process may create new maximal positively labeled paths starting from $\alpha$ that did not exist in $S_1'$.

For example, $S_1'$ may contain a path labeled by $r_3^{-1}r_5$ for some $r_5 \in X^+$ starting from the vertex $\beta_i$ as shown in the Fig. 9. After folding the path labeled by $r_3r_3^{-1}$, we can read a new positively labeled path starting from $\alpha$ and labeled by $r_1r_5$ which could not be read in $S_1'$.

It is also possible for example, that $S_1'$ may contain a path labeled by $r_4^{-1}r_5$ for some $r_5 \in X^+$ starting from the vertex $\delta$ as shown in the diagram (Fig. 10). After folding...
Fig. 10 An example showing the construction of new maximal positively labeled paths starting from $\alpha$ as a consequence of folding of edges in $S'_1$.

The path labeled by $r_4^{-1}r_4$, we can read a new positively labeled path starting from $\alpha$ and labeled by $r_1r_5$ which could not be read in $S'_1$.

The examples in Figs. 9 and 10 illustrate a principle that is important to keep in mind when we argue that it must happen that we eventually will reach a complex $S_k$ such that all maximal positive paths that can be read at $\alpha$ in $S_k$ can already be read at $\alpha$ in $S'_{k-1}$. In Fig. 9 assume that $\gamma$ is the first vertex read along the transversal of $SC(w_i)$ that is identified with a vertex of $S$ in the folding process. Then $r_1r_5$ labels a path that begins at $\alpha$ in $S_2$ but does not label a path that begins at $\alpha$ in $S'_{1}$. This new path was created at the cost of increasing the star set of the vertex $\gamma'$ in $S$. That is, after $\gamma$ is identified with $\gamma'$ in the folding process, the set of all labels of edges that can be read at $\gamma'$ in $S_2$ is more than could be read at $\gamma'$ in $S'_{1}$. In particular in Fig. 9, the label of the terminal edge of the path $r_1$ is in the star set of $\gamma'$ in $S_2$, but is not in the star set of $\gamma'$ in $S'_{1}$. Otherwise, $\gamma$ would not have been the first vertex along the transversal of $SC(w_i)$ to be identified with a vertex of $S$. In general, we can say that a new maximal positive path is created by the process of folding $S'_{1}$ to $S_2$ only if there is at least one vertex of $S$ whose star set is increased by the folding process.

Now we consider all those maximal positively labeled paths in $S_2$ that start from $\alpha$ and did not exist in $S'_1$. Since there are only finitely many such paths, we assume that these paths are labeled by $u_1, u_2, \ldots, u_m$ and for each $i$ the path labeled by $u_i$ terminates at some vertex $\gamma_i$. Note that each maximal path labeled by $u_i$ must terminate at a vertex $\gamma_i$ that is in the original complex $S$. For each $i$, the graph $SC(u_i)$ is finite, by hypothesis. We attach $SC(u_i)$ at the path labeled by $u_i$ for all $i$ and denote the resulting complex by $S'_2$.

If the paths labeled by $u_i$ and $u_j$ for some $i \neq j$ are co-terminal, then by the same argument as above $SC(u_i)$ and $SC(u_j)$ are isomorphic to each other as edge labeled graphs. So, they get identified with each other.

If for some $i$, $SC(u_i)$ contains a positively labeled transversal $t$ such that $\theta(\neq \gamma_i)$ is the first vertex of $t$ that gets identified with a vertex in $S$, then the entire positively labeled sub-path of $t$ from $\theta$ to $\gamma_i$ gets identified with a positively labeled path in $S$. This can be verified by using the same argument as above.
We perform all possible foldings in $S'_2$ and denote the resulting complex by $S_3$. This folding process may create new maximal positively labeled paths starting from $\alpha$ that did not exist in $S'_2$, but any such new maximal positive path must begin at $\alpha$ and end at a vertex in the original complex $S$. Furthermore, such a new path will arise only if there is at least one vertex of $S$ that has its star set increased by the folding process.

We keep repeating this process of sewing on finite Schützenberger complexes of positive words and folding until we reach the point at which we do not create new maximal positively labeled paths starting from $\alpha$ and terminating at a vertex in the complex $S$. We will in fact reach that point, for some folding $S'_k \rightarrow S_{k+1}$, because a new maximal positive path is created by the folding process only if there is at least one vertex in $S$ whose star set is increased by the folding process. The presentation $\langle X | R \rangle$ is a finite and so in particular, the generating set $X$ is finite. Each vertex can have at most $|X|$ incoming edges and $|X|$ outgoing edges. There are only finitely many vertices in the original complex $S$ and so it is not possible for star sets of vertices in $S$ to increase forever. Thus, the expansion process terminates in a finite complex. By construction, the resulting complex is deterministic and closed with respect to elementary $\mathcal{P}$-expansions, and so it is the Schützenberger complex of $w$. ⊓⊔

The following Corollary follows immediately from Theorem 3.2, because the Schützenberger complex, $SC(w)$, for any $w \in (X \cup X^{-1})^*$, is finite over a presentation $(X, R)$ if and only if the underlying graph (1-skeleton) of $SC(w)$ is finite over the same presentation.

**Corollary 1** Let $M = Inv\langle X | R \rangle$ be a finitely presented Adian inverse semigroup. Then the Schützenberger graph of $w$ is finite for all words $w \in (X \cup X^{-1})^*$ if and only if the Schützenberger graph of $w'$ is finite for all positive words $w' \in X^+$. 

4 Some applications of Theorem 3.2

4.1 The word problem for a sub-family of Adian inverse semigroups that satisfy condition (⋆)

**Definition 2** We say that a positive presentation $\langle X | R \rangle$ satisfies condition (⋆), if it satisfies the following two conditions:

1. No proper prefix of an $R$-word is a suffix of itself or any other $R$-word.
2. No proper suffix of an $R$-word is a prefix of itself or any other $R$-word.

If $\langle X | R \rangle$ is a finite Adian presentation that satisfies condition (⋆), then the set of relations $R$ consists of two types of relations. First, those relations which are of the form $(u, xvy)$, where $u$ and $v$ are $R$-words and $x, y \in X^+$. Second, those relations $(u, v) \in R$ where neither $u$ nor $v$ contains an $R$-word as a proper sub-word. We construct a directed graph corresponding to an Adian presentation that satisfies condition (⋆) as follows. We call this graph the bi-sided graph of the presentation $\langle X | R \rangle$. The bi-sided graph of a positive presentation is defined as follows.

**Definition 3** The bi-sided graph of the presentation $\langle X | R \rangle$ is a finite, directed, edge-labeled graph, denoted by $BS(X, R)$ satisfying
The vertex set of $BS(X, R)$ is the set of all $R$-words.

To define the edge set of $BS(X, R)$, let $u, v$ be two $R$-words (where it may happen that $u$ and $v$ are the same $R$-word). There is a directed edge from the vertex $u$ to the vertex $v$ if any of the following three conditions holds:

1. $(u, xvy) \in R$, for some $x, y \in X^+$. In this case, the directed edge from $u$ to $v$ is labeled by the ordered pair $(x, y)$.
2. $u \equiv xvy$, for some $x, y \in X^+$ and $u$ and $v$ are distinct $R$-words. In this case, the edge from $u$ to $v$ is labeled by the ordered pair $(x, y)$.
3. If $(u, v) \in R$ is such that neither $u$ nor $v$ contains any $R$-word as a proper subword, then there is an edge in the bi-sided graph between $u$ and $v$, pointing in both directions. This edge is labeled by $(\epsilon, \epsilon)$, where $\epsilon$ denotes the empty word.

In general, the bi-sided graph $BS(X, R)$ of an Adian presentation may contain closed paths.

**Example 1** The bi-sided graph of the Adian presentation $\langle a, b | aba = b \rangle$ contains a directed closed path (cycle), namely the loop consisting of the single edge labeled by $(a, a)$ from the vertex $b$ to itself. (See Fig. 11.) The presentation $\langle a, b | aba = b \rangle$ does not satisfy condition (⋆) either, because the $R$-word $aba$ has the letter $a$ as a prefix and as a suffix.

**Example 2** The bi-sided graph of the Adian presentation $\langle a, b, c, d, e, f, g, h, i, j, k | a = fbg, a = jck, b = hci, c = de \rangle$ contains an undirected closed path. (See Fig. 12.) The presentation $\langle a, b, c, d, e, f, g, h, i, j, k | a = fbg, a = jck, b = hci, c = de \rangle$ satisfies condition (⋆).

**Example 3** The bi-sided graph of the Adian presentation $\langle a, b, c, d, e, f, g, h, i, j, k, l, m | a = fcg, b = hci, c = de, l = jm^2k \rangle$ is a forest. (See Fig. 13). The presentation $\langle a, b, c, d, e, f, g, h, i, j, k, l, m | a = fcg, b = hci, c = de, l = jm^2k \rangle$ satisfies condition (⋆).

**Remark 2** For the remainder of this subsection we consider presentations $\langle X | R \rangle$ such that

\[
\begin{align*}
\text{Edge of type 1} \\
(a,a) \\
\text{Edge of type 2} \\
(b) \\
(aba) \\
\end{align*}
\]
Fig. 12 The bi-sided graph of \( \langle a, b, c, d, e, f, g, h, i, j, k | a = fb \, g, a = jck, b = hci, c = de \rangle \)

Fig. 13 The bi-sided graph of \( \langle a, b, c, d, e, f, g, h, i, j, k, l | a = fcg, b = hci, c = de, l = jm^2k \rangle \)

(1) \( \langle X | R \rangle \) is an Adian presentation.
(2) \( \langle X | R \rangle \) satisfies condition \((\ast)\).
(3) The bi-sided graph \( BS(X, R) \) is cycle-free. That is, there are no closed paths, directed or un-directed, in \( BS(X, R) \). In other words, \( BS(X, R) \) is a forest.

Note that if the bi-sided graph of a presentation \( \langle X | R \rangle \) is a forest, then every \( R \)-word labels a vertex of a connected component in \( BS(X, R) \) that is a tree. For an \( R \)-word \( u \), we let \( T_u \) denote the unrooted tree that contains the vertex \( u \) and we refer to \( T_u \) as
the bi-sided tree of \( u \). If \( u \) and \( v \) label two different vertices of the same bi-sided tree, then \( T_u \) and \( T_v \) denote the same unrooted, bi-sided tree.

**Definition 4** Let \( u \) be an \( R \)-word. We say that a vertex labeled by \( v \) of \( T_u \) is accessible from \( u \) if there exists a path labeled by \( v \) in \( SC(u) \).

Note that if there exists a directed edge from \( v_1 \) to \( v_2 \) in the bi-sided tree of \( u \) and \( v_1 \) is accessible from \( u \) then \( v_2 \) is also accessible from \( u \). Because if \( v_1 \) is accessible from \( u \), then there exists a path labeled by \( v_1 \) in \( SC(u) \). Since there exists a directed edge from \( v_1 \) to \( v_2 \) in the bi-sided tree of \( u \) therefore either \((v_1, xv_2y) \in R \) or \( v_1 \equiv xv_2y \) for some \( x, y \in X^+ \). In either case there exists a path labeled by \( v_2 \) in \( SC(u) \).

However, if \( v_2 \) is accessible from \( u \), and there is an edge in \( BS(X, R) \) from \( v_1 \) to \( v_2 \), then \( v_1 \) is not necessarily accessible from \( u \). Also, if a vertex \( v \) of \( T_u \) is not accessible from \( u \) then all of the vertices of \( T_u \) that lie after the vertex \( v \) going along any geodesic path from \( u \) to an extremal vertex of \( T_u \) are also not accessible.

**Lemma 5** Let \( M = Inv(X,R) \) be a finitely presented Adian inverse semigroup that satisfies condition \((\star)\) and (such that) \( BS(X, R) \) contains no closed paths. Then \( SC(u) \) is finite, for every \( R \)-word \( u \).

**Proof** We start from the linear automaton \((\alpha, \Gamma_0(u), \beta)\) and obtain the approximate complex \((\alpha, \Gamma_1(u), \beta)\) by applying full \( \mathcal{P} \)-expansion on \((\alpha, \Gamma_0(u), \beta)\).

In the construction of \((\alpha, \Gamma_1(u), \beta)\), we attach a path labeled by one side of a relation whose other side can be read in \((\alpha, \Gamma_0(u), \beta)\). In the linear automaton \((\alpha, \Gamma_0(u), \beta)\) we can precisely read \( u \) and all those \( R \)-words that are proper subwords of \( u \). Each relation of \( R \) with one side \( u \) is either of the form \((u, xvy)\) for some \( x, y \in X^+ \) and an \( R \)-word \( v \) or of the form \((u, v)\) where \( v \) contains no \( R \)-word as its proper subword. The first case corresponds to an edge of “type 1” in \( T_u \) and the second case corresponds to an edge of “type 2” in \( T_u \). Any \( R \)-word \( u_1 \) that is a proper subword of \( u \) corresponds to an edge of “type 2” in \( T_u \), where \( u_1 \equiv xu_1y \) for some \( x, y \in X^+ \). Thus, in performing Stephen’s \( \mathcal{P} \)-expansion to obtain \( \Gamma_1(u) \) from \( \Gamma_0(u) \), there is a precise correspondence between first generation transversals of \( SC(u) \) and the edges of \( T_u \) whose initial vertex is either labeled by \( u \) or labeled by an \( R \)-word that is a proper subword of \( u \).

If there exists an edge of type 1 or type 2 labeled by \((x, y)\) for some \( x, y \in X^+ \) from a vertex labeled by an \( R \)-word \( v \) to the vertex \( u \) in \( T_u \), then either \((v, xuy) \in R \) or \( v \equiv xuy \). We show that no generation 1 transversal contains a subpath labeled by \( v \). In other words, we show that \( v \) is inaccessible from \( u \).

Note that we cannot read the word \( xuy \) in the linear automaton \((\alpha, \Gamma_0(u), \beta)\) because \((\alpha, \Gamma_0(u), \beta)\) contains only one path from \( \alpha \) to \( \beta \) that is labeled by the word \( u \). So, if \((v, xuy) \in R \) then we cannot attach a path labeled by \( v \) to the linear automaton \((\alpha, \Gamma_0(u), \beta)\) and if \((v, xuy) \in R \) then we cannot read a path labeled by \( v \) in \((\alpha, \Gamma_0(u), \beta)\) because it is longer than the path labeled by \( u \).

We obtain the approximate graphs \((\alpha, \Gamma_2(u), \beta)\) by applying the full \( \mathcal{P} \)-expansion on \((\alpha, \Gamma_1(u), \beta)\). We observe that the second generation transversals of \( SC(u) \) are obtained as a consequence of attaching paths labeled by those \( R \)-words which label the terminal vertices of those edges of \( T_u \) whose initial vertex is either a first generation transversal or a proper subword of an \( R \)-word that labels a first generation transversal.
of $SC(u)$. If there exists an edge in $BS(X, R)$ labeled by $(x_1, y_1)$ for some $x_1, y_1 \in X^*$ with initial vertex labeled by $v_1$ and terminal vertex labeled by $v_2$ such that $x_1v_2y_1$ is a subword of an $R$-word that labels a first generation transversal, then we obtain a second generation transversal by sewing on a path labeled by $v_1$ from the initial vertex to the terminal vertex of the of the path $x_1v_2y_1$.

If there exists an edge of type 1 or type 2 labeled by $(x_1, y_1)$ with initial vertex $v_1$ and terminal vertex $v_2$ (i.e. either $(v_1, x_1v_2y_1) \in R$ or $v_1 \equiv x_1v_2y_1$) such that $v_2$ is a subword of an $R$-word that labels a first generation transversal but $x_1v_2y_1$ is not a subword of that transversal, then we cannot attach a path labeled by $v_1$ to $(\alpha, \Gamma_1(u), \beta)$. So, none of the vertices of $T_u$ that occur after the vertex $v_1$ going along a path from the vertex $u$ to an extremal vertex of $T_u$ will be accessible from $u$ in $SC(u)$.

We observe that when we apply a full $P$-expansion on $(\alpha, \Gamma_n(u), \beta)$ for some $n \in \mathbb{N}$, we cover some more vertices of $T_u$ that were not covered before in the sense that we add some new transversals that contain an $R$-word that labels a vertex of $T_u$ and that $R$-word does not label a path in $(\alpha, \Gamma_n(u), \beta)$. Since $T_u$ is a finite tree and none of the $R$-words labels two distinct vertices of $T_u$, therefore the process of applying full $P$-expansions must terminate after a finite number of steps. Hence $SC(u)$ is finite. \(\square\)

Lemma 6 Let $M = Inv\langle X | R \rangle$ be an Adian inverse semigroup. Let $u$ be an $R$-word and $z$ labels a proper suffix of a transversal $p$ of $SC(u)$. Then either
(i) $z$ contains an $R$ word that also labels a subpath of $p$, or
(ii) A prefix of $z$ is a suffix of some $R$-word that also labels a subpath of $p$.

Proof If $z$ does not contain an $R$-word that also labels a subpath of the transversal $p$ then the initial vertex of the path labeled by $z$ is not the initial vertex of any $R$-word that labels a subpath of $p$. Hence the initial vertex of $z$ lies between a pair of vertices of $p$ that are the initial and the terminal vertex of a subpath of $p$ that is labeled by an $R$-word. Hence a prefix of $z$ is a suffix of an $R$-word. \(\square\)

We also remark that a dual statement also holds for a prefix of a transversal of the Schützenberger complex of an $R$-word over an Adian presentation.

Remark 3 In Lemma 6, if the presentation $\langle X | R \rangle$ satisfies condition $(\star)$ and $z$ happens to be a prefix of an $R$-word, then only (i) holds, because (ii) violates the condition $(\star)$.

Similarly, in the dual statement to Lemma 6, if the presentation $\langle X | R \rangle$ satisfies condition $(\star)$ and $z$ happens to be a suffix of an $R$-word, then only (i) holds, because (ii) violates the condition $(\star)$.

Theorem 4.1 Let $M = Inv\langle X | R \rangle$ be a finitely presented Adian inverse semigroup that satisfies condition $(\star)$ and $BS(X, R)$ contains no closed path. Then $SC(w)$ is finite, for all $w \in (X \cup X^{-1})^*$.

Proof We just need to show that $SC(w)$ is finite for all $w \in X^+$ and then the above theorem follows from Theorem 3.2. So, we assume that $w \in X^+$ and we construct the linear automaton of $w$, $(\alpha, \Gamma_0(w), \beta)$.

It follows from condition $(\star)$, that no two distinct $R$-words will overlap with each other. However, an $R$-word can be a proper subword of another $R$-word. So, we can
uniquely factorize $w$ as $x_0u_1x_2u_2 \ldots u_nx_n$, where $x_i \in X^*$ and $u_i$’s are maximal $R$-words in the sense that none of the $u_i$’s are properly contained in another $R$-word that is also a subword of $w$.

It follows from Lemma 5, that $SC(u_i)$ is finite for all $1 \leq i \leq n$. So, we attach $SC(u_i)$ for all $1 \leq i \leq n$ to the corresponding paths labeled by $u_i$’s in $(\alpha, \Gamma_0(w), \beta)$ to construct $SC(w)$ and denote the resulting complex by $S_1$. It follows from Lemma 3, that no two edges get identified with each other as a consequence of attaching $SC(u_i)$ for all $1 \leq i \leq n$ to the linear automaton $(\alpha, \Gamma_0(w), \beta)$. If $S_1$ is closed under elementary $P$-expansion, then we are done. Otherwise, we will be able to read a finite number of $R$-words labeling the paths of $S_1$ where we can attach new 2-cells by sewing on paths labeled by the other sides of the corresponding relations.

We assume that $v_1, v_2, \ldots, v_m$ are the $R$-words that label the paths of $S_1$ where we can attach new 2-cells. Note that each of $v_i$ labels a vertex of $T_{u/j}$ for some $i$ and $j$, that was inaccessible from $u_j$ earlier. Because if $v_i$ labels a path in $S_1$ then by Lemma 6 and Remark 3 the path labeled by $v_i$ contains an $R$-word $r_j$ as a proper subword such that the $R$-word $r_j$ labels a path in $SC(u_j)$. In other words, $r_j$ labels an accessible vertex of $T_{u/j}$ from the vertex $u_j$. Since $r_j$ is a proper subword of $v_i$, therefore there exists an edge of type 2 in $T_{u/j}$ with initial vertex labeled by $v_i$ and the terminal vertex labeled by $r_j$. So, $T_{v_i}$ and $T_{u/j}$ represent the same tree for some $i$ and $j$. By Lemma 5, $SC(v_i)$ is finite for all $1 \leq i \leq m$ and covers some more vertices of $T_{u/j}$ in the sense that $SC(v_i)$ contains paths labeled by those $R$-word which also label some of the vertices of $T_{u/j}$, for some $1 \leq j \leq n$, that were not covered by $SC(u_j)$.

We attach $SC(v_i)$ to the paths labeled by $v_i$ for all $1 \leq i \leq m$ in $S_1$ and denote the resulting complex by $S_2$. No two edges get identified with each other in $S_2$ as a consequence of attaching $SC(v_i)$’s to $S_1$ by Lemma 3. If $S_2$ is closed under elementary $P$-expansion then we are done. Otherwise we repeat this process of attaching Schützenberger complexes of $R$-words and capturing more vertices of the trees $T_{u_i}$ for some $1 \leq i \leq n$. This process eventually terminates, because, each $T_{u_i}$ is a finite tree with all the paths labeled by distinct $R$-words and every $R$-word labels a vertex of exactly one tree. Hence, $SC(w)$ is a finite complex. \qed

\textbf{Remark 4} Let $M = Inv\langle X | R \rangle$ be an Adian inverse semigroup that satisfies condition $(\ast)$ and $BS(X, R)$ contains no closed path then the word problem for $M$ is decidable. It follows from Theorem 3.2 that the Schützenberger complex of every word $w \in (X \cup X^{-1})^*$ is finite over the presentation $(X | R)$. So, for any two given words $w_1, w_2 \in (X \cup X^{-1})^*$, we can easily check whether $w_1 \in L(w_2)$ and $w_2 \in L(w_1)$ or not.

\subsection*{4.2 The word problem for Inverse semigroups given by the presentation $\langle a, b | ab^m = b^n a \rangle$}

In this section we show that the word problem is decidable for the inverse semigroup given by the presentation $M = Inv\langle a, b | ab^m = b^n a \rangle$, where $m, n \in \mathbb{N}$. The word problem for the case $m = n$ follows from Corollary 6.6 of [7]. So, throughout this section we assume that $m > n$. The case $m < n$ follows from a dual argument. We can get an alternate proof for the case $n = m$ by following along same lines as in the case of $m < n$. \hfill \copyright Springer
**Lemma 7** The Schützenberger complex of a word $a^k b^t$ for $k, t \in \mathbb{N}$, over the presentation $\langle a, b | ab^m = b^n a \rangle$, is finite.

**Proof** We adopt a slightly different approach to construct $SC(a^k b^t)$. We draw edges labeled by $a$ horizontally and edges labeled by $b$ vertically. Then the linear automaton of $a^k b^t$, $(a_0, \Gamma_0(a^k b^t), \beta_0)$, is shown in Fig. 14. If $t < m$, then we cannot attach any 2-cell to $(a_0, \Gamma_0(a^k b^t), \beta_0)$. So, the above lemma is true for this case.

If $t \geq m$, then $t = q_1 m + r_1$, where $q_1$ is the quotient and $r_1$ is a remainder and $0 \leq r_1 < m$. We can attach $q_1$ 2-cells in the first column along the vertical segment labeled by $b^t$ of $(a_0, \Gamma_0(a^k b^t), \beta_0)$. After attaching all the 2-cells in the first column along the vertical segment labeled by $b^t$ we have created a new vertical segment labeled by $b^{nq_1}$, because there are total $q_1$ 2-cells and each 2-cells contains exactly $n$ edges on the newly attached side of the 2-cell.

If $nq_1 < m$ or $k = 1$ then the process of attaching new 2-cells terminates at this stage. If neither $nq_1 < m$ nor $k = 1$, then $nq_1 = q_2 m + r_2$, where $q_2$ is a quotient, $r_2$ is a remainder and $0 \leq r_2 < m$. So, We can attach a column of $q_2$ 2-cells along the vertical segment labeled by $b^{nq_1}$. This process of attaching columns of new 2-cells terminates after at most $k$ steps. So, $SC(a^k b^t)$ is a finite complex. \hfill $\Box$

**Remark 5** In the above construction of $SC(a^k b^t)$ in Lemma 7, every new maximal vertical segment contains fewer edges than the other vertical side of the same column of 2-cells.

**Lemma 8** The Schützenberger complex of a word $b^t a^k$ for $k, t \in \mathbb{N}$, over the presentation $\langle a, b | ab^m = b^n a \rangle$, is finite.

**Proof** We draw edges labeled by $a$ horizontally and edges labeled by $b$ vertically. Then the linear automaton of $b^t a^k$, $(a_0, \Gamma_0(b^t a^k), \beta_0)$, is shown in Fig. 15. If $t < n$, then we cannot attach any 2-cell to $(a_0, \Gamma_0(b^t a^k), \beta_0)$. So, the above lemma is true for this case.
The word problem for some classes of Adian inverse… 219

Fig. 15 Construction of $SC(b^t a^k)$

If $t \geq n$, then $t = q_1 n + r_1$, where $q_1$ is the quotient and $r_1$ is a remainder and $0 \leq r_1 < n$. We can attach $q_1$ 2-cells in the first column along the vertical segment labeled by $b^t$ of $(a_0, \Gamma_0(a^k b^t), \beta_0)$. After attaching all the 2-cells in the first column along the vertical segment labeled by $b^t$ we have created a new vertical segment labeled by $b^{mq_1}$, because there are total $q_1$ 2-cells and each 2-cells contains exactly $m$ edges on the newly attached side of the 2-cell.

If $k = 1$ then the process of attaching new 2-cells terminates at this stage. Otherwise $mq_1 = q_2 n + r_2$, where $q_2$ is a quotient, $r_2$ is a remainder and $0 \leq r_2 < n$. So, We can attach a column of $q_2$ 2-cells along the vertical segment labeled by $b^{mq_1}$. Clearly, this process of attaching columns of new 2-cells terminates after $k$ steps. So, $SC(b^t a^k)$ is a finite complex.

**Remark 6** In the above construction of $SC(b^t a^k)$ in Lemma 8 every new maximal vertical segment contains more edges than the other vertical side of the same column of 2-cells.

**Theorem 4.2** For all $w \in \{a, b, a^{-1}, b^{-1}\}^*$ the Schützenberger complex of a word $w$, $SC(w)$, over the presentation $\langle a, b | ab^m = b^n a \rangle$ is finite. Hence the word problem is decidable for $M$.

**Proof** Since $\langle a, b | ab^m = b^n a \rangle$ is an Adian presentation, we just need to show that $SC(w)$ for all $w \in \{a, b\}^+$ is finite, then Theorem 4.2 follows from Theorem 3.2.

If $w$ is of the form $a^k$ or $b^k$ for some $k \in \mathbb{N}$, then $SC(w)$ is finite.

We assume that $w \equiv a^{k_0} b^{t_0} a^{k_1} b^{t_1} \ldots a^{k_l} b^{t_l}$, where $k_0, t_0 \in \mathbb{N} \cup \{0\}$ and $k_i, t_j \in \mathbb{N}$ for $1 \leq i \leq l$ and $0 \leq j \leq l - 1$.

We construct $SC(w)$ by drawing the edges labeled by $a$ horizontally and the edges labeled by $b$ vertically. So, $(a_0, \Gamma_0(w), \beta_0)$ looks like the diagram shown in Fig. 16.
We attach $SC(a^k b^i)$ on the path labeled by $a^k b^i$ of $(\alpha_0, \Gamma_0(w), \beta_0)$ wherever it is possible to attach and we denote the resulting complex by $S_1$. No two edges get identified with each other as a consequence of attaching these finite complexes to $(\alpha_0, \Gamma_0(w), \beta_0)$ by Lemma 3. As a consequence of attaching these finite complexes to the $(\alpha_0, \Gamma_0(w), \beta_0)$, we have created at most $l - 1$ new maximal directed paths labeled by $a^k b^t$ for some $k, t \in \mathbb{N}$.

We attach finite complexes of the form $SC(a^k b^t)$ to every new maximal path labeled by a word of the form $a^k b^t$ for some $k, t \in \mathbb{N}$ in $S_1$. We denote the resulting complex by $S_2$. Again by Lemma 3, no two edges get identified with each other in $S_2$. We can read at most $l - 2$ new maximal directed paths in $S_2$ which are labeled by the words of the form $a^k b^t$ for some $k, t \in \mathbb{N}$. So, we repeat the process of attaching finite Schützenberger complexes of the words of the form $a^k b^t$ where ever it is possible to attach and denote the resulting complex by $S_3$. Note that this process of attaching finite Schützenberger complexes of the words of the form $a^k b^t$ eventually terminates after at most $l$ steps. We denote the resulting complex by $S'_3$.

Now in $S'_3$, on the other side of the path labeled by $w$, we attach finite Schützenberger complexes of the words $b^i a^{k+1}$ at the paths labeled by $b^i a^{k+1}$ for $0 \leq i \leq l - 1$, where ever it is possible to attach and denote the resulting complex by $S'_1$. By Lemma 3 no two edges in $S'_1$ get identified with each other as a consequence of attaching these finite complexes. As a consequence of attaching these finite complexes we have created at most $l - 1$ new maximal paths which are labeled by the words of the form $b^t a^k$ for some $k, t \in \mathbb{N}$. So, we repeat the process of attaching finite Schützenberger complexes of the words of the form $b^t a^k$ at the corresponding new paths in $S'_1$. We denote the resulting complex by $S'_2$. This process of attaching finite complexes of the words of the
form $b^l a^k$ terminates after at most $l$ steps and we obtain a finite complex which is closed under elementary $P$-expansion and folding. Hence, $SC(w)$ is a finite complex.

**Acknowledgements** The author of this paper is thankful to John Meakin and Robert Ruyle for their several useful suggestions.

**References**

1. Adian, S.I.: Defining relations and algorithmic problems for groups and semigroups. Proc. Steklov Inst. Math. 85, 1–152 (1966)
2. Inam, M., Meakin, J., Ruyle, R.: A structural property of Adian inverse semigroups. Semigroup Forum, 94(1), 93–103 (2017)
3. Lawson, M.V.: Inverse Semigroups. World Scientific Co. Pte. Ltd., Singapore (1998)
4. Linblad, S.P.: Inverse monoids presented by a single relator. PhD thesis, Dept. of Math., University of Nebraska-Lincoln (2003)
5. Magnus, W.: Das Identitätsproblem für Gruppen mit einer definierenden Relation. Math. Ann. 106, 295–307 (1932)
6. Remmers, J.H.: On the geometry of semigroup presentations. Adv. Math. 36, 283–296 (1980)
7. Stephen, J.B.: Presentations of inverse monoids. J. Pure Appl. Algebra 63, 81–112 (1990)
8. Steinberg, B.: A topological approach to inverse and regular semigroups. Pac. J. Math. 208(2), 367–396 (2003)
9. Steinberg, B.: A Sampler of a Topological Approach to Inverse Semigroups, Algorithms, Automata and Languages. Word Scientific, Singapore (2002)