A SEQUENCE OF MATRIX VALUED ORTHOGONAL POLYNOMIALS ASSOCIATED TO SPHERICAL FUNCTIONS

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Abstract. The main purpose of this paper is to obtain an explicit expression of a family of matrix valued orthogonal polynomials \( \{P_n\}_n \), with respect to a weight \( W \), that are eigenfunctions of a second order differential operator \( D \). The weight \( W \) and the differential operator \( D \) were found in [12], using some aspects of the theory of the spherical functions associated to the complex projective spaces. We also find other second order differential operator \( E \) symmetric with respect to \( W \) and we describe the algebra generated by \( D \) and \( E \).

1. Introduction

The theory of the harmonic analysis on homogeneous spaces is closely connected with the theory of special functions. This is apparent, for example, on the two dimensional sphere \( S^2 = \text{SO}(3)/\text{SO}(2) \), where the harmonic analysis with respect to the action of the orthogonal group is contained in the classical theory of the spherical harmonics. In spherical coordinates the spherical functions are the Legendre polynomials \( P_n(\cos \theta) \). Also the zonal spherical functions of the sphere \( S^n = \text{SO}(n+1)/\text{SO}(n) \) are given, in spherical coordinates, in terms of Jacobi polynomials \( P_n^{(\alpha,\alpha)}(\cos \theta) \), with \( \alpha = (n-2)/2 \). More generally the zonal spherical functions on a Riemannian symmetric space of rank one can always be expressed in terms of the classical Gauss’ hypergeometric functions, in the case of compact spaces we get Jacobi polynomials.

As in the scalar case alluded above, in the matrix setting we also have these three ingredients: the theory of matrix valued spherical functions of any \( K \)-type, the matrix valued hypergeometric function and the theory of matrix valued orthogonal polynomials. In this paper we exhibit the interplay among these concepts in the case of the complex projective space \( \text{SU}(n+1)/\text{U}(n) \).

The theory of matrix valued spherical functions goes back to [14] and [5], based on the foundational papers of Godement and Harish-Chandra. In [6] we find explicit expressions for spherical functions, of any \( K \)-type associated

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to complex projective plane $P_2(\mathbb{C}) = SU(3)/U(2)$. This is accomplished by associating to a spherical function $\Phi$ on $G$ a vector valued function $H$ defined on a complex affine plane $\mathbb{C}^2$, whose entries are given in terms of a special class of generalized hypergeometric functions $p+1F_p$.

The matrix valued hypergeometric function was studied in [15]. Let $V$ be a $d$-dimensional complex vector space, and let $A$, $B$, and $C \in \text{End}(V)$. The hypergeometric equation is

$$z(1-z)F''(z) + (C-z(A+B+I))F'(z) - ABF(z) = 0.$$ (1)

If the eigenvalues of $C$ are not in $-\mathbb{N}_0$ we define the function

$$2F_1\left(\frac{A;B}{C}; z\right) = \sum_{m=0}^{\infty} \frac{z^m}{m!}(C; A; B)_m,$$

where the symbol $(C; A; B)_m$ is defined inductively by

$$(C; A; B)_0 = 1,$$

$$(C; A; B)_{m+1} = (C + m)^{-1}(A + m)(B + m)(C; A; B)_m, \quad m \geq 0.$$ The function $2F_1\left(\frac{A;B}{C}; z\right)$ is analytic on $|z| < 1$ with values in $\text{End}(V)$. Moreover if $F_0 \in V$ then $F(z) = 2F_1\left(\frac{A;B}{C}; z\right)F_0$ is a solution of the hypergeometric equation (1) such that $F(0) = F_0$. Conversely any solution $F$, analytic at $z = 0$ is of this form.

The theory of matrix valued orthogonal polynomials, without any consideration of differential equations goes back to [10] and [11]. In [2], the study of the matrix valued orthogonal polynomials which are eigenfunctions of certain second order differential operators was started. The first explicit examples of such polynomials are given in [7], [8] and [3].

Given a self adjoint positive definite matrix valued smooth weight function $W = W(t)$ with finite moments, we can consider the skew symmetric bilinear form defined for any pair of square matrix valued polynomial functions $P(t)$ and $Q(t)$ by the numerical matrix

$$(P, Q) = \int_{\mathbb{R}} P(t)W(t)Q^*(t)dt,$$

where $Q^*(t)$ denotes the conjugate transpose of $Q(t)$. This leads to the existence of a sequence of matrix valued orthogonal polynomials, that is a sequence $\{P_n(t)\}$, where $P_n$ is a polynomial of degree $n$ with non singular leading coefficients and $(P_n, P_m) = 0$ if $n \neq m$.

We also consider the skew symmetric bilinear form

$$(P, Q) = (P^*, Q^*)^*,$$ (2)

and we say that a differential operator $D$ is symmetric with respect to $W$ if

$$(DP, Q) = (P, DQ),$$ (3)

for all matrix valued polynomial functions $P$ and $Q$. 
Let $D$ be an ordinary linear differential operator with matrix valued polynomial coefficients of degree less or equal to the order of derivation. If $D$ is symmetric with respect to $W$ then any orthogonal sequence $\{P_n\}$, with respect to $(\cdot, \cdot)$, satisfies
\[
DP_n^* = P_n^*\Lambda_n,
\]
for some numerical matrix $\Lambda_n$.

Assume that the weight function $W = W(t)$ is supported in the interval $(a, b)$ and let $D$ be a second order differential operator of the form
\[
D = A_2(t) \frac{d^2}{dt^2} + A_1(t) \frac{d}{dt} + A_0(t),
\]
with matrix valued polynomial coefficients $A_j(t)$ of degree less or equal to $j$. In [8] (see also [3]) it is proved that the condition of symmetry for $D$ is equivalent to the following three differential equations
\[
A_2^* W = W A_2, \quad A_1^* W = -W A_1 + 2(W A_2)', \quad A_0^* W = W A_0 - (W A_1)' + (W A_2)'',
\]
with the boundary conditions
\[
\lim_{t \to x} W(t) A_2(t) = 0 = \lim_{t \to x} (W(t) A_1(t) - A_1^*(t) W(t)), \text{ for } x = a, b.
\]

Finding explicit solutions of these equations is a highly non trivial task. In [3] and [4] the authors give some families of examples. In [12] one finds, for each dimension, a three parameter family of pairs $\{W, D\}$ satisfying (6) and (7). These families arise from the representation theory of Lie groups. After the change of variable $u = 1 - t$, the main result in [12] reads:

**Theorem 1.1.** Let $\alpha, \beta > -1$, $0 < k < \beta + 1$ and $\ell \in \mathbb{N}$. Let $D$ be the differential operator defined by
\[
D = u(1 - u) \frac{d^2}{du^2} + (C - uU) \frac{d}{du} - V,
\]
with
\[
C = \sum_{i=0}^{\ell} (\beta + 1 + 2i) E_{ii} + \sum_{i=1}^{\ell} i E_{i,i-1}, \quad U = \sum_{i=0}^{\ell} (\alpha + \beta + \ell + i + 2) E_{ii},
\]
\[
V = \sum_{i=0}^{\ell} i(\alpha + \beta + i - k + 1) E_{ii} - \sum_{i=0}^{\ell-1} (\ell - i)(i + \beta - k + 1) E_{i,i+1}.
\]

Then the differential operator $D$ is symmetric with respect to the weight matrix $W(u) = (1 - u)^\alpha u^\beta Z(u)$ given by
\[
Z(u) = \sum_{i,j=0}^{\ell} \left( \sum_{r=0}^{\ell} \binom{\ell}{r} \binom{\ell+k-r-1}{\ell-r} (\beta-k+r) (1-u)^{\ell-r} u^{i+j} \right) E_{ij}.
\]
Remark. Here, and in other parts of the paper, we use $E_{ij}$ to denote the matrix with entry $(i, j)$ equal 1 and 0 otherwise.

This theorem is obtained from the first few steps in the explicit determination of all matrix valued spherical functions associated to the $n$-dimensional projective space $P_n(\mathbb{C}) = SU(n+1)/U(n)$. The idea, also used in [6], is to cook up from a matrix valued spherical function a function $H$ which depends on a single variable $u$. Using that the spherical functions are eigenfunctions of the Casimir operator of $SU(n + 1)$ we deduce that, after an appropriate conjugation, $H$ is an eigenfunction of an ordinary linear second order matrix valued differential operator $D$. The fact that this operator is symmetric with respect to the weight $W$ is a consequence of the fact that the Casimir operator is symmetric with respect to the $L^2$-inner product between matrix valued functions on $SU(n + 1)$. At this point some readers may find useful to consult references [5], [14] and [12].

One of the main purposes of this paper is to give explicit expressions of a sequence of orthogonal polynomials associated to the weight $W$ given in Theorem 1.1. This is accomplished by studying the vector space $V(\lambda)$ of all vector valued polynomial solutions of the hypergeometric equation $DF - \lambda F = 0$. This space is non trivial if and only if

$$\lambda = \lambda_j(w) = -w(w + \alpha + \beta + \ell + j + 1) - j(\alpha + \beta - k + 1 + j),$$

for some $w \in \mathbb{N}_0$ and $j = 0, 1, \ldots, \ell$. If the eigenvalues $\lambda_j(w)$ are all different then there exists a unique polynomial solution (up to scalars) of $DF = \lambda F$.

In Proposition 2.3 we compute, in the general case, the dimension of the space $V(\lambda)$. With this knowledge at hand, we construct a sequence of polynomials $\{P_w\}$, by choosing the $j$-th column of $P_w$ as a particular polynomial in $V(\lambda_j(w))$. In Theorem 2.4 we prove that $\{P_w\}$ is an orthogonal sequence of matrix valued polynomials such that $DP_w^* = P_w^* \Lambda_w(D)$, where $\Lambda_n(D)$ is the real valued diagonal matrix

$$\Lambda_w(D) = \sum_{0 \leq j \leq \ell} \lambda_j(w) E_{jj}.$$ 

The matrix spherical functions associated to $(G, K) = (SU(n+1), U(n))$ are eigenfunctions, not only of the Casimir operator, but also of any element in the algebra $D(G)^G$ of all differential operators in $G$ which are left and right invariant under multiplication by elements of $G$. In this case this algebra is a polynomial algebra in $n$ algebraically independent generators, one of them can be taken to be the Casimir operator of $G$. For $n = 2$, in [6] the explicit expression of this set of generators was given and two differential operators $D$ and $E$ which commute were obtained. For a general $n$ we do not have simple expressions for a complete set of generators of the algebra $D(G)^G$, beyond the Casimir operator. However in this paper we are able to find another second order differential operator $E$, which commutes with $D$ and such that it is symmetric with respect to $W$ (See Theorem 3.1). The way in which we obtain this operator is different to the one used in [12] and
it is inspired in the operator $\tilde{E}$ given in [13]. Here we only knew that such an operator should exist and after a trial and error process we find it and prove that it is symmetric.

The sequence of matrix valued orthogonal polynomial constructed in Theorem 2.4 $\{P_w\}$ also satisfies $EP_w^* = P_w^*\Lambda_w(E)$ with

$$\Lambda_w(E) = \sum_{j=0}^\ell (-w(w+\alpha+\beta+\ell+j+1)(\alpha-\ell+3j)$$

$$- j(j+\alpha+\beta-k+1)(\alpha+2\ell+3k))E_{jj}.$$ 

We also study the algebra generated by the differential operators $D$ and $E$. In Theorem 5.3 we prove that it is isomorphic to the affine algebra of the following union of lines in $\mathbb{C}^2$:

$$\prod_{j=0}^\ell (y-(\alpha-\ell+3j)x + 3j(\ell-j+k)(j+\alpha+\beta-k+1)).$$ 

Recently, in [1] this situation is considered in the case $\ell = 2$. The authors conjecture that the algebra generated by $D$ and $E$ coincide with the algebra of all differential operators that have the orthogonal polynomials $P_w$ as simultaneous eigenfunctions.

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2. Orthogonal polynomials associated to the pair $\{W, D\}$

The aim of this section is to give explicitly a sequence of matrix valued orthogonal polynomials associated to the weight function $W$ and the differential operator $D$ introduced in Theorem 1.1, i.e. we construct a sequence $\{P_w\}$ of orthogonal polynomials with respect to $W$, such that $DP_w^* = P_w^*\Lambda_w$, where $\Lambda_w(D)$ is a real diagonal matrix.

The columns $\{P_w^j\}_{j=0,\ldots,\ell}$ of $P_w^*$ are $\mathbb{C}^{\ell+1}$-valued polynomials such that $DP_w^j = \lambda_j(w)P_w^j$ and $(P_w^j, P_w'^j) = \delta_{w,w'}\delta_{j,j'}n_{w,j}$, for some positive real number $n_{w,j}$.

2.1. Polynomial solutions of $DF = \lambda F$. We start studying the $\mathbb{C}^{\ell+1}$-vector valued polynomial solutions of $DF = \lambda F$. We will find all polynomials $F(u)$ such that

$$(8) \quad u(1-u)F''(u) + (C - uU)F'(u) - (V + \lambda)F(u) = 0,$$

where the matrices $C, U, V$ are given in Theorem 1.1. This equation is an instance of a hypergeometric differential equation studied in [15]. Since the
eigenvalues of \( C \) are not in \( -\mathbb{N}_0 \) the function \( F \) is characterized by \( F_0 = F(0) \). For \( |u| < 1 \) it is given by

\[
F(u) = 2H_1(U;V;\lambda;u) \quad F_0 = \sum_{i=0}^{\infty} \frac{u^i}{i!} [C;U;V+\lambda]_i F_0, \quad F_0 \in \mathbb{C}^\ell,
\]

where the symbol \([C;U;V+\lambda]_i\) is defined inductively by

\[
[C;U;V+\lambda]_0 = 1, \\
[C;U;V+\lambda]_{i+1} = (C+i)^{-1} (i(U+i-1)+V+\lambda) [C;U;V+\lambda]_i,
\]

for all \( i \geq 0 \).

There exists a polynomial solution of (8) if and only if the coefficient \([C;U;V+\lambda]_i\) is singular for some \( i \in \mathbb{Z} \). Let us assume that \([C;U;V+\lambda]_{w+1}\) is singular and that \([C;U;V+\lambda]_w\) is not singular.

Since the matrix \((C+w)\) is invertible, we have that \([C;U;V+\lambda]_{w+1}\) is singular if and only if \((w(U+w-1)+V+\lambda)\) is singular. The matrix

\[
M_w = (w(U+w-1)+V+\lambda)
\]

is upper triangular and

\[
(M_w)_{j,j} = w(w+\alpha+\beta+\ell+j+1) + j(\alpha+\beta-k+1+j) + \lambda.
\]

Therefore \([C;U;V+\lambda]_{w+1}\) is singular if and only if

\[
\lambda = \lambda_j(w) = -w(w+\alpha+\beta+\ell+j+1) - j(\alpha+\beta-k+1+j),
\]

for some \( 0 \leq j \leq \ell \).

We will distinguish the cases when the eigenvalues \( \lambda_j(w) \) are all different (varying \( j \) or \( w \)) or when they are repeated. We start studying the polynomial solutions of (8) in the first case.

**Proposition 2.1.** Assume that all eigenvalues \( \lambda_j(w) \) are different. If \( \lambda = \lambda_j(w) \), for some \( j = 0, \ldots, \ell \), then there exists a unique \( F_0 \in \mathbb{C}^{\ell+1} \) (up to scalars) such that \( F(u) = 2H_1(U;V;\lambda;u) F_0 \) is a polynomial function. Moreover this polynomial is of degree \( w \).

**Proof.** We have already observed that for \( \lambda = \lambda_j(w) = -w(w+\alpha+\beta+\ell+j+1) - j(\alpha+\beta-k+1+j) \), the matrix \([C;U;V+\lambda]_{w+1}\) is singular. Then the function \( F(u) = \sum_{i=0}^{\infty} \frac{w^i}{i!} [C;U;V+\lambda]_i F_0 \) is a polynomial if and only if \( F_0 \) is a vector such that

\[
[C;U;V+\lambda]_w F_0 \in \ker(M_w);
\]

where \( M_w = w(U+w-1)+V+\lambda_j(w) \). The matrix \([C;U;V+\lambda]_w\) is invertible, hence \( F_0 \) is univocally determined by an element in the kernel of \( M_w \). We have that

\[
M_w = \sum_{0 \leq i \leq \ell} ((i-j)(\alpha+\beta-k+1+i+j+w)E_{ii} - (\ell-i)(\beta-k+1+i)E_{i,i+1}).
\]
Since all eigenvalues $\lambda_j(w)$ are different we have that $0 \neq \lambda_j(w) - \lambda_i(w) = (i-j)(\alpha + \beta - k + 1 + i + j + w)$ if $i \neq j$, hence the dimension of the kernel of $M_w$ is one. Explicitly $(x_0, x_1, \ldots, x_\ell) \in \ker(M_w)$ if and only if

$$
\begin{align*}
  x_i &= (-1)^{i+j} (\ell-i) \frac{\binom{\ell-i+k+1}{\alpha+\beta+j+i+w-k+1}}{\alpha+\beta+j+i+w-k+1} x_j & \text{for } i = 0, \ldots, j, \\
  x_{j+1} &= x_{j+2} = \cdots = x_\ell = 0,
\end{align*}
$$

(13)

where we use $(z)_r = z(z+1)\ldots(z+r-1)$, $(z)_0 = 1$.

Hence, up to scalar, $F_0$ is uniquely determined by (11) and it is clear that $F(u) = g\mathcal{H}_I(U; V^+; \lambda; \nu) F_0$ is a polynomial of degree $w$ with leading coefficient \( \lambda \mathcal{C}_I[C, U, V + \lambda_j(w)] w F_0 \). This completes the proof of the proposition.

Now we have to study the case when some eigenvalues are repeated, that is when there exist $w, w' \in \mathbb{N}_0$ and $0 \leq j, j' \leq \ell$ such that $\lambda_j(w) = \lambda_{j'}(w')$. We start observing the following facts.

**Lemma 2.2.** If $\lambda_j(w) = \lambda_{j'}(w')$ for some $w, w' \in \mathbb{N}_0$ and $0 \leq j, j' \leq \ell$ then

i) We have $w = w'$ if and only if $j = j'$.

ii) If $w' > w$ then $j > j' + 1$.

**Proof.** If $\lambda_j(w) = \lambda_{j'}(w')$ then

$$(w' - w)(\alpha + \beta + \ell + 1 + w + w' + j') + (j' - j)(\alpha + \beta - k + 1 + j + j' + w) = 0.$$  

In particular if $w' = w$, we have $(j' - j)(\alpha + \beta - k + 1 + j + j' + w) = 0$. We observe that $j \neq j'$ implies that $\alpha + \beta - k + 1 + j + j' + w > 0$, because $\alpha > -1, \beta - k + 1 > 0, j + j' \geq 1$ and $w \geq 0$.

Similarly if $j' = j$ we have $(w' - w)(\alpha + \beta + \ell + 1 + w + w' + j) = 0$. Since $\alpha > -1, \beta + \ell + 1 > 0$ and $w + w' + j + 1 \geq 1$ we obtain that $(\alpha + \beta + \ell + 1 + w + w' + j) > 0$ and therefore $w = w'$. This completes the proof of i).

For ii) we start from

$$(w' - w)(\alpha + \beta + \ell + 1 + w + w' + j') = (j - j')(\alpha + \beta - k + 1 + j + j' + w),$$

and we observe that the left hand side of this identity, as well as the factor $(\alpha + \beta - k + 1 + j + j' + w)$ are positive numbers, by hypothesis, then we have $j > j'$. Finally suppose that $j = j' + 1$ then $(w' - w)(\alpha + \beta + \ell + w + w' + j) = (\alpha + \beta - k + w + 2j)$, equivalently

$$(w' - w - 1)(\alpha + \beta + \ell + w + w' + j) = -(w' + \ell - j + k).$$

The left hand side is non negative while the right hand side is negative because $k > 0$, which is a contradiction.

Let $V(\lambda)$ be the vector space of all $\mathbb{C}^{\ell+1}$-vector valued polynomials such that $DP = \lambda P$. We observe that Proposition 2.1 said that if the eigenvalues $\lambda = \lambda_j(w)$ are all different the dimension of $V(\lambda)$ is one. The next proposition generalizes this result to the case when the eigenvalues $\lambda_j(w)$ are repeated.
Proposition 2.3. Let $\alpha, \beta > -1$, $0 < k < \beta + 1$ and let $\lambda = \lambda_j(w)$, for some $w \in \mathbb{N}_0$. Then

$$\dim \{ P \in V(\lambda) : \deg P \leq w \} = \text{card} \{ w' : 0 \leq w' \leq w, \lambda = \lambda_j(w'), \text{ for some } 0 \leq j' \leq \ell \}. \quad (14)$$

In particular

$$\dim V(\lambda) = \text{card} \{ (w,j) : \lambda = \lambda_j(w) \}. \quad \text{Proof.} \quad \text{We have already observed that for } \lambda = \lambda_j(w) \text{ the function } F = F(u) \text{ is a polynomial solution of } DF = \lambda F \text{ if and only if } F(u) = 2H_1(C, U, V + \lambda)F_0 \text{ with } F_0 \in C^{\ell+1} \text{ such that } [C, U, V + \lambda]wF_0 \in \ker(M_{w,j}), \text{ where}$$

$$M_{w,j} = \sum_{0 \leq i \leq \ell} ((i-j)(\alpha + \beta - k + 1 + i + j + w)E_{ii} - (\ell - j)(\beta - k + 1 + i)E_{i,i+1})$$

We have that $(i-j)(\alpha + \beta - k + 1 + i + j + w) \neq 0$ if $i \neq j$. Hence the dimension of $\ker(M_{w,j})$ is one. Moreover it is generated by $(x_0, \ldots, x_\ell) \in C^{\ell+1}$ such that

$$x_i = (-1)^{i+j} \binom{\ell-i}{\ell-j} \binom{\beta - k + 1 + i}{\alpha + \beta + j + w - k + 1} \quad \text{for } i = 0, \ldots, j - 1,$$

$$x_j = 1$$

$$x_{j+1} = x_{j+2} = \cdots = x_\ell = 0,$$

where we use $(z)_r = z(z+1) \cdots (z+r-1)$, $(z)_0 = 1$.

If the eigenvalue $\lambda$ is repeated $s$ times and $w_1 = \min\{w \in \mathbb{N}_0 : \lambda = \lambda_j(w), 0 \leq j \leq \ell \}$, using Lemma 2.2, we can assume that

$$\lambda = \lambda_{j_1}(w_1) = \cdots = \lambda_{j_s}(w_s)$$

with $w_1 < w_2 < \cdots < w_s$ and $j_1 > j_2 + 1$, $j_2 > j_3 + 1$, $\ldots$, $j_{s-1} > j_s$.

For $w = w_1$ and $j = j_1$ the matrix $[C, U, V + \lambda]w_1$ is invertible and $F_0$ is univocally determined by an element in $\ker(M_{w_1,j_1})$, which is one dimensional, thus proving (14) in this case.

Then to prove the proposition for any $w_r$ we proceed by induction on $1 \leq r \leq s$. Thus let us assume that for $2 \leq r \leq s$ we know that

$$\{ P \in V(\lambda) : \deg P \leq w_{r-1} \} = r - 1.$$

Let $M_r = M_{w_r,j_r}$. As we remarked $0 \neq P \in V(\lambda)$ is of degree $w_r$ if and only if $P_0 = P(0)$ satisfies $0 \neq [C, U, V + \lambda]w_rP_0 \in \ker(M_r)$.

Let

$$[C, U, V + \lambda]w_r = N_rM_{r-1} \cdots N_1M_0,$$

where $N_i$ are invertible matrices. The leading coefficient $P_r$ of such a $P$ is uniquely determined, up to scalar, by the condition

$$M_rN_rM_{r-1} \cdots N_1M_1N_0P_0 = 0,$$

because we may assume that

$$P_r = N_rM_{r-1} \cdots N_1M_1N_0P_0 = (x_0, \ldots, x_{j_r-1}, 1, 0, \ldots, 0).$$
Now let us prove that there exists $\tilde{P} \in V(\lambda)$ of degree $w_r$, by constructing one by downward induction.

Let $v_r = (x_0, \ldots, x_{j_r-1}, 1, 0, \ldots, 0) \in \ker(M_r)$ and let $b_r = N_r^{-1}v_r$. The equation $b_r = M_{r-1}v_{r-1}$ has a unique solution $v_{r-1}$ of the form $v_{r-1} = (z_0, \ldots, z_{j_r+1}, 0, \ldots, 0)$ because $b_r = (y_0, \ldots, y_{j_r+1}, 0, \ldots, 0)$ with $y_{j_r+1} \neq 0$ and $M_{r-1}$ is upper triangular with a unique zero in the main diagonal in the $j_r-1$-position. Similarly let $b_{r-1} = N_{r-1}^{-1}v_{r-1}$, then there exists a unique $v_{r-2} = (t_0, \ldots, t_{j_r+2}, 0, \ldots, 0)$ such that $M_{r-2}v_{r-2} = b_{r-1}$. In this way we construct the sequence $v_r, v_{r-1}, \ldots, v_0$ such that

$$v_r = N_r b_r = N_r M_{r-1}v_{r-1} = N_r M_{r-1}M_{r-2}v_{r-2} = \cdots$$

$$= N_r M_{r-1} \cdots N_1 M_1 N_0 v_0$$

Hence $\tilde{P} = \mathcal{P}_1(C, U, V + \lambda)v_r$ is a polynomial in $V(\lambda)$ of degree $w_r$.

Now we observe that

$$\{ P \in V(\lambda) : \deg P \leq w_r \} = \mathbb{C}\tilde{P} \oplus \{ P \in V(\lambda) : \deg P \leq w_{r-1} \}.$$

In fact it is clear that the right hand side is a direct sum contained in the left hand side. To prove the other inclusion we first observe that if $P \in V(\lambda)$ and $\deg P < w_r$ then, as we saw, $\deg P \leq w_{r-1}$. If $P \in V(\lambda)$ is of degree $w_r$ then the leading coefficient of $P$ is equal to the leading coefficient of $t\tilde{P}$ for some $t \in \mathbb{C}$. Therefore $P - t\tilde{P} \in \{ P \in V(\lambda) : \deg P \leq w_{r-1} \}$. This completes the proof of the proposition. \hfill \Box

2.2. Matrix valued orthogonal polynomials associated to $\{W, D\}$.

We want to construct a sequence $\{P^\alpha_w\}_{w \geq 0}$ of matrix valued orthogonal polynomials with respect to the weight function $W$, with degree of $P^\alpha_w$ equal to $w$, with non singular leading coefficient and that satisfies $DP^\alpha_w = P^\alpha_w \Lambda_w$, where $\Lambda_w(D)$ is a real diagonal matrix.

Then the columns $\{P^\alpha_{j,w}\}_{j=0,\ldots,\ell}$ of $P^\alpha_w$ are $\mathbb{C}^{\ell+1}$-valued polynomials such that $P^\alpha_0$ and $P^\alpha_{j,w}'$ are orthogonal to each other if $(j, w) \neq (j', w')$ and they satisfy that $DP^\alpha_{j,w} = \lambda_j(w)P^\alpha_{j,w}$, where

$$\lambda_j(w) = -w(w + \alpha + \beta + \ell + j + 1) - j(j + \alpha + \beta - k + 1),$$

for each $w \in \mathbb{N}_0$ and $j = 0, \ldots, \ell$.

If an eigenvalue $\lambda = \lambda_j(w)$ is not repeated, then we choose the unique $F_0 \in \mathbb{C}^{\ell+1}$ such that

$$[C, U, V + \lambda_j(w)]_w F_0 = \sum_{0 \leq i \leq j} (-1)^{i+j} \binom{\ell-i}{\ell-j} \frac{(\beta-k+1+i)_{j-i}}{(\alpha+\beta+j+i+w-k+1)_{j-i}} e_i,$$

where $e_i$ denotes the $i$-th vector in the canonical basis of $\mathbb{R}^{\ell+1}$. Then we take

$$P^\alpha_{j,w}(u) = 2H_1 \left( U; V + \lambda_j(w); C \right) [C, U, V + \lambda_j(w)]_w F_0$$

$$= \sum_{i=0}^{\infty} \frac{u^i}{i!} [C, U, V + \lambda_j(w)]_i F_0.$$
which is a polynomial function of degree \( w \) and satisfies

\[
DP_w^j(u) = \lambda_j(w)P_w^j(u).
\]

(See Proposition 2.1.)

If an eigenvalue \( \lambda = \lambda_j(w) \) is repeated we saw that,

\[
\lambda = \lambda_j_1(w_1) = \lambda_j_2(w_2) = \ldots = \lambda_j_s(w_s),
\]

with \( w_1 < w_2 < \cdots < w_s \) and \( j_r \geq j_{r+1} + 1 \), for \( 1 \leq r \leq s-1 \).

Let \( V_r = \{ P \in V(\lambda) : \deg P \leq w_r \} \), for \( 1 \leq r \leq s \). Then we saw, in the proof of Proposition 2.3 that

\[
0 \neq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_s
\]

with \( \dim V_s = s \). Now we take, for each \( 1 \leq r \leq s \)

\[
0 \neq P_{w_r}^j(u) = 2H_1 \left( U; V + \lambda_j(w) \right) F_{w_r}^j \in V_r \text{ orthogonal to } V_{r-1}.
\]

In this way, for each \( w \in \mathbb{N}_0 \) we have defined \( \ell + 1 \) orthogonal polynomial functions \( P^0_w, P^1_w, \ldots, P^\ell_w \) of degree \( w \).

**Theorem 2.4.** Let \( P_w(u) \) be the matrix whose rows are the vectors \( P^j_w(u) \).

Then the sequence \( \{ P_w(u) \}_{w \in \mathbb{N}_0} \) is an orthogonal sequence of matrix valued polynomials such that

\[
DP_w^*(u) = P_w^*(u)\Lambda_w,
\]

where \( \Lambda_w = \sum_{j=0}^\ell \lambda_j(w)E_{jj} \).

**Proof.** Let \( (w, j) \neq (w', j') \). If \( \lambda_j(w) \neq \lambda_{j'}(w') \) then \( (P^j_w, P^{j'}_{w'}) = 0 \) because \( D \) is symmetric. If \( \lambda_j(w) = \lambda_{j'}(w') \) then \( (P^j_w, P^{j'}_{w'}) = 0 \) by construction. Therefore the matrices \( P_w \) satisfies \( (P_w, P_{w'}) = 0 \) if \( w \neq w' \).

On the other hand we have that for each \( w = 0, 1, 2 \ldots \) the degree of \( P_w(u) \) is \( w \) and the leading coefficient of \( P_w \) is the non singular triangular matrix

\[
I + \sum_{s<r} (-1)^{r+s} \binom{\ell-s}{\ell-r} \frac{\alpha^s \beta^k}{\alpha-r+s+w-k-1} E_{rs}.
\]

This completes the proof of the theorem.

\( \square \)

3. The symmetry of the differential operator \( E \)

The aim of this section is to exhibit another second order ordinary differential operator which is symmetric with respect to the weight \( W \).

**Theorem 3.1.** Let \( \alpha, \beta > -1, 0 < k < \beta + 1 \) and \( \ell \in \mathbb{N} \). Let \( E \) be the differential operator defined by

\[
E = (1 - u)(Q_0 + uQ_1) \frac{d^2}{du^2} + (P_0 + uP_1) \frac{d}{du} - (\alpha + 2\ell + 3k)V,
\]

with

\[
Q_0 = \sum_{i=0}^\ell 3iE_{i,i-1},
\]

\[
Q_1 = \sum_{i=0}^\ell (\alpha - \ell + 3i)E_{ii},
\]

\[
\]
\[ P_0 = \sum_{i=0}^\ell ((\alpha + 2\ell)(\beta + 1 + 2i) - 3k(\ell - i) - 3i(\beta - k + i)) E_{ii} \]
\[- \sum_{i=0}^\ell i(3i + 3\beta - 3k + 3 + \ell + 2\alpha) E_{i,i-1}, \]
\[ P_1 = \sum_{i=0}^\ell -(\alpha - \ell + 3i)(\alpha + \beta + \ell + i + 2) E_{ii} \]
\[ + \sum_{i=0}^\ell 3(\beta - k + 1 + i)(\ell - i) E_{i,i+1}, \]
\[ V = \sum_{i=0}^\ell i(\alpha + \beta - k + 1 + i) E_{ii} - \sum_{i=0}^{\ell-1}(\ell - i)(\beta - k + 1 + i) E_{i,i+1}. \]

Then \( E \) is symmetric with respect to the weight matrix \( W(u) = (1-u)^\alpha u^\beta Z(u) \), where \( Z(u) \) is given by
\[ Z(u) = \sum_{i,j=0}^\ell \left( \sum_{r=0}^\ell \binom{\ell}{i} \binom{\ell}{j} \binom{\ell+k-1-r}{\ell-r} (\beta - k + r)(1-u)^{\ell-r} u^{i+j} \right) E_{ij}. \]

**Proof.**

We need to prove that the equations (17) and (17) are satisfied. The equations in (17) take the form
\[ (Q_0^* + uQ_1^*)Z - Z(Q_0 + uQ_1) = 0, \]
\[ (P_0^* + uP_1^*)Z + Z(P_0 + uP_1) - 2Z(Q_1 - Q_0 - 2uZQ_1) \]
\[ - 2(1-u)Z'(Q_0 + uQ_1) - \frac{(\beta(1-u) - \alpha u)}{u}Z(Q_0 + uQ_1) = 0, \]
\[ P_1^*Z + (P_0^* + uP_1^*)Z' - Z'(P_0 + uP_1) - ZP_1 \]
\[ + \frac{(\beta - \alpha)}{u}((P_0^* + uP_1^*)Z - Z(P_0 + uP_1)) \]
\[ - 2(\alpha + 2\ell + 3k)(ZV - V^*Z) = 0. \]

The \( ij \)-entry in the left hand side of the equation (17) is
\[ u(\alpha - \ell + 3i)z_{ij} + 3(i+1)z_{i+1,j} - u(\alpha - \ell + 3j)z_{ij} + 3(j+1)z_{i,j+1} = 0, \]
because it is easy to verify that
\[ (i+1)z_{i+1,j} - (j+1)z_{i,j+1} = u(j - i)z_{ij}. \]

In order to prove the identity (18) we compute the \( ij \)-entry of the matrices involved there:
\[ ((P_0^* + uP_1^*)Z)_{ij} = u^{i+j} \sum_{r=\max(i,j)}^\ell \binom{\ell}{i} \binom{\ell}{j} \binom{\ell+k-1-r}{\ell-r} (1-u)^{\ell-r} \]
\[ - (r-i)(3i + 3\beta + \ell + 2\alpha - 3k + 6) - (\alpha - \ell + 3i)(\alpha + \beta + \ell + i + 2) \]
\[ + u^{i+j} \sum_{r=\max(i-1,j-1)}^\ell \binom{r+1}{i} \binom{r+1}{j} \binom{\ell+k-2-r}{\ell-r} (1-u)^{\ell-r} \]
\[ - (r-i+1)(3i + 3\beta + \ell + 2\alpha - 3k + 6) + (\alpha - \ell + 3i)(\alpha + \beta + \ell + i + 2) \]
\[ + u^{i+j} \sum_{r=\max(i-1,j)}^\ell \binom{r}{i-1} \binom{\ell+k-1-r}{\ell-r} (1-u)^{\ell-r} 3(\ell - i + 1)(\beta + i - k), \]
\[(Z(P_0 + uP_1))_{ij} = u^{i+j} \sum_r \binom{r}{i} \binom{r}{j} (\beta + r - 1) (\ell + k - 1 - r) (1 - u)^{\ell - r} (P_0)_{jj}
\]

\[-(r - j) (3j + 3\beta + \ell + 2\alpha - 3k + 6) - (\alpha - \ell + 3j) (\alpha + \beta + \ell + j + 2) + u^{i+j} \sum_r \binom{r+1}{i} \binom{r+1}{j} (\beta + r) (\ell + k - 2 - r) (1 - u)^{\ell - r}
\]

\[(r - j + 1) (3j + 3\beta + \ell + 2\alpha - 3k + 6) + (\alpha - \ell + 3j) (\alpha + \beta + \ell + j + 2) + u^{i+j} \sum_r \binom{r}{i} \binom{r}{j} (\beta + r - 1) (\ell + k - 1 - r) (1 - u)^{\ell - r} 3(\ell - j + 1) (\beta + j - k),
\]

\[(Z(Q_1 - Q_0 - 2uQ_1))_{ij} = u^{i+j} \sum_r \binom{r}{i} \binom{r}{j} (\beta + r - 1) (\ell + k - 1 - r) (1 - u)^{\ell - r} (-\alpha + \ell + 3r)
\]

\[+ u^{i+j} \sum_r \binom{r+1}{i} \binom{r+1}{j} (\beta + r) (\ell + k - 2 - r) (1 - u)^{\ell - r} (3r + 3j + 2\alpha - 2\ell + 3),
\]

\[\left(\frac{(\beta(1-u) - \alpha u)}{u}\right) (Q_0 + uQ_1)_{ij} = u^{i+j} \sum_r \binom{r}{i} \binom{r}{j} (\beta + r - 1) (\ell + k - 1 - r) (1 - u)^{\ell - r} \alpha (\ell - \alpha - 3r)
\]

\[+ u^{i+j} \sum_r \binom{r+1}{i} \binom{r+1}{j} (\beta + r) (\ell + k - 2 - r) (1 - u)^{\ell - r} (1 + \beta) (3\alpha + \ell + 3r),
\]

\[(1 - u) Z'(Q_1 + uQ_0)_{i,j} = u^{i+j} \sum_r \binom{r}{i} \binom{r}{j} (\beta + r - 1) (\ell + k - 2 - r) (1 - u)^{\ell - r} (r - \ell) (\alpha - \ell + 3r)
\]

\[+ u^{i+j} \sum_r \binom{r+1}{i} \binom{r+1}{j} (\beta + r) (\ell + k - 2 - r) (1 - u)^{\ell - r} (3(r - j + 1)(\ell - r + i + j))
\]

\[+ (\alpha - \ell + 3j)(\ell - r + i + j - 1).
\]

By using the previous results we get that the identity (18) is equivalent to

\[\sum_{r=j}^{\ell} \binom{r}{i} \binom{r}{j} (\beta + r - 1) (\ell + k - 1 - r) (1 - u)^{\ell - r} \frac{3(r+1)(\ell + 2 - r - j + k)}{(r - i + 1)(\ell - j + 1)} + (1 - u)^{\ell - j + 1} \binom{r}{i} (\beta + r - 1) (\ell + k - 1 - j) 3(\ell - j + k)(j - i)
\]

\[-\sum_{r=j-1}^{\ell - 1} \binom{r+1}{i} \binom{r+1}{j} (\beta + r) (\ell + k - 2 - r) (1 - u)^{\ell - r} 3(\ell - r + k - 1)(2r + 2 - i - j)
\]

\[= 0,
\]

which easily follows.
In order to prove the identity (19) we compute
\[(ZV - V^*Z)_{ij} = (i - j)u^{i+j-1} \left( - \sum_r (\ell)_i (\ell)_j (\beta + r - 1) (\ell + k - 1 - r)(1 - u)^{-r}(\alpha + \ell - r + 1) \right) + \sum_r (\ell)_i (\ell)_j (\beta + r - 1) (\ell + k - 1 - r)(1 - u)^{-r}(\alpha + \beta + i + j + \ell - r + 1)) \right),
\]
\[(P_1^*Z - ZP_1)_{ij} = (i - j)u^{i+j-1} \left( \sum_r (\ell)_i (\ell)_j (\beta + r - 1) (\ell + k - 1 - r)(1 - u)^{-r}(\alpha + \ell - r + 1) \right) - \sum_r (\ell)_i (\ell)_j (\beta + r - 1) (\ell + k - 1 - r)(1 - u)^{-r}(4\alpha + 5\ell + 3k + 6 - 3r) \right),
\]
\[\left( \left( \frac{\beta(1 - u) - \alpha u}{u(1 - u)} \right)(P_0^* + uP_0^*)Z - Z(P_0 + uP_1) \right) + (P_0^* + uP_0^*)Z' \]
\[- Z'(P_0 + uP_1) \right)_{i,j} = u^{i+j-1}(i - j) \sum_r (\ell)_i (\ell)_j (\beta + r - 1) (\ell + k - 1 - r)(1 - u)^{-r} \]
\[2(\alpha + 2\ell + 3k)(\alpha + \ell - r + 2) + 2\alpha + \ell + 6 - 3k - 3r \]
\[- u^{i+j-1}(i - j) \sum_r (\ell)_i (\ell)_j (\beta + r - 1) (\ell + k - 1 - r)(1 - u)^{-r}(4\alpha + 5\ell + 3k + 6 - 3r) \right),
\]
Now it is easy to verify that (19) is satisfied.

Finally the boundary conditions (7) can be easily check and this concludes the proof of the theorem. \qed

4. The algebra of differential operators

Most of the results of this section are due to J. Tirao and they are taken from [16].

Let \( W = W(x) \) be a \( L \times L \) matrix weight function with finite moments and let \( \{P_n\} \) be any sequence of matrix valued orthogonal polynomials associated to a weight function \( W \).

Let
\[ V_n = \{F \in M_{L \times L}(\mathbb{C})[x] : \text{deg}(F) \leq n\} \]
be the set of all matrix valued polynomials in the variable \( x \) of degree less or equal to \( n \).

**Proposition 4.1.** We have the following decomposition of \( V_n \)
\[ V_n = \bigoplus_{j=0}^n P_j^* M_{L \times L}(\mathbb{C}). \]
Proof. It is clear that \( \sum_{j=0}^{n} P_j^* M_{L \times L}(\mathbb{C}) \) is a subspace of \( V_n \) and that for \( n = 0 \) they are the same. Let us denote by \( M_n \) the leading coefficient of \( P_n^* \).

If \( H = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_0 \) is a polynomial in \( V_n \) then \( H - P_n^* M_{n-1} A_n \) is a polynomial of degree \( \leq n - 1 \). Thus, by induction in \( n \) we obtain that
\[
H \in \sum_{j=0}^{n} P_j^* M_{L \times L}(\mathbb{C}).
\]

In order to prove that this sum is a direct sum we assume that \( P_0^* A_0^* + \cdots + P_n^* A_n^* = 0 \). By comparing, inductively the coefficients of \( x^n, x^{n-1}, \ldots, x^0 \) we obtain that \( A_n = \cdots = A_0 = 0 \).

Let \( D \) be the algebra of all differential operators of the form
\[
D = F_s(x) \frac{d^s}{dx^s} + F_{s-1}(x) \frac{d^{s-1}}{dx^{s-1}} + \cdots + F_1(x) \frac{d}{dx} + F_0(x)
\]
whith \( F_j \) a polynomial function of degree less or equal to \( j \).

**Theorem 4.2.** Let \( \{P_n\} \) be any sequence of matrix valued orthogonal polynomials associated to \( W \). If \( D \in \mathcal{D} \) is symmetric respect to \( W \) then \( DP_n^* = P_n^* \Lambda_n \), for some matrix \( \Lambda_n \).

**Remark.** We recall that \( D \) is symmetric with respect to \( W \) if \( \langle DP, DQ \rangle = \langle P, DQ \rangle \) for all \( P, Q \) polynomials. The sequence \( \{P_n\} \) is orthogonal with respect to \( (, ) \). The bilinear forms \( (, ) \) and \( (, ) \) are related by \( \langle P, Q \rangle = (P^*, Q^*)^* \).

**Proof.** Since \( D \in \mathcal{D} \) the operator \( D \) preserves the vector spaces \( V_n \), for each \( n \geq 0 \).

For \( n = 0 \) we have that \( DP_0^* \in V_0 \), thus \( DP_0^* = P_0^* \Lambda_0 \). By induction we assume that \( DP_j^* = P_j^* \Lambda_j \), for each \( 0 \leq j \leq n - 1 \). By Proposition [4.1] we have that \( DP_n^* = \sum_{i=0}^{n} P_i^* A_i \). Thus, for each \( 0 \leq j \leq n - 1 \) we have
\[
\langle DP_n^*, P_j^* \rangle = \sum_{i=0}^{j} \langle P_i^* A_i, P_j^* \rangle = \sum_{i=1}^{j} (P_i, P_j)^* A_i = (P_j, P_j)^* A_j.
\]

On the other hand, since \( D \) is symmetric we obtain
\[
\langle DP_n^*, P_j^* \rangle = \langle P_n^*, DP_j^* \rangle = \langle P_n^*, P_j^* \Lambda_j \rangle = ((P_n, P_j)^* \Lambda_j)^* = 0.
\]

Thus \( (P_j, P_j)^* A_j = 0 \) for each \( 0 \leq j \leq n - 1 \), which implies that \( A_j = 0 \) because the matrix \( (P_j, P_j) \) is non singular. Therefore \( DP_n^* = P_n^* \Lambda_n \) and this concludes the proof.

Given \( \{P_n\} \) any sequence of matrix valued orthogonal polynomials associated to the weight \( W \), we define
\[
\mathcal{D}(W) = \{ D \in \mathcal{D} : \exists DP_n^* = P_n^* \Lambda_n(D), \forall n \geq 0, \text{ for some matrix } \Lambda_n(D) \}.
\]

**Proposition 4.3.** We have
1. \( \mathcal{D}(W) \) is a subalgebra of \( \mathcal{D} \) which does not depend on the sequence \( \{P_n\} \).
2. For each \( n \in \mathbb{N}_0 \), the function \( \Lambda_n : \mathcal{D}(W) \rightarrow M_{L \times L}(\mathbb{C}) \) given by \( D \mapsto \Lambda_n(D) \) is a representation of the algebra \( \mathcal{D}(W) \).
(3) The family \( \{A_n\}_{n \geq 0} \) separates points of \( D(W) \). That is, if \( D_1 \) and \( D_2 \) are distinct points of \( D(W) \), then there exists \( n_0 \geq 0 \) such that 

\[ \Lambda_{n_0}(D_1) \neq \Lambda_{n_0}(D_2). \]

**Proof.** It is easy to verify that \( D(W) \) is a subalgebra of \( D \). To prove that it is independent of the sequence \( \{P_n\} \) we take other sequence of orthogonal polynomials \( \{Q_n\} \). Then \( Q_n = A_n P_n \), for some non singular matrix \( A_n \). Then we have \( DQ_n^* = DP_n^* A_n^* = P_n^* \Lambda_n(D) A_n^* = Q_n^* Y_n(D) \), where \( Y_n(D) = (A_n^*)^{-1} \Lambda_n(D) A_n^*. \)

If \( D_1 \) and \( D_2 \) are in \( D(W) \) then

\[ D_1 D_2 P_n^* = D_1 (P_n^* \Lambda_n(D_2)) = P_n^* \Lambda_n(D_1) \Lambda_n(D_2). \]

Hence \( \Lambda_n(D_1 D_2) = \Lambda_n(D_1) \Lambda_n(D_2) \).

Let us assume that there exists \( D \in D(W) \) such that \( \Lambda_n(D) = 0 \) for all \( n \geq 0 \). To prove (3) we have to verify that \( D = 0 \). By hypothesis we have that \( D = \sum_{i=0}^{s} F_i(x) \frac{d^i}{dx^i} \) satisfies \( DP_n^* = 0 \), for all \( n \geq 0 \). For \( n = 0 \) we obtain \( F_0 P_0^* = 0 \), thus \( F_0 = 0 \).

By induction, we may assume that \( F_i = 0 \) for \( 0 \leq i \leq j - 1 \), with \( j \leq s \). Then \( 0 = DP_j^* = \sum_{i=1}^{j} F_i(x) \frac{d^i}{dx^i} = F_j(x) j! M_j \), where \( M_j \) is the leading coefficient of \( P_j \), which is non singular. Therefore \( F_j = 0 \). This concludes the proof. \( \square \)

**Corollary 4.4.** The operators \( D_1 \) and \( D_2 \) in the algebra \( D(W) \) commute if and only if the matrices \( \Lambda_n(D_1) \) and \( \Lambda_n(D_2) \) commute for all \( n \in \mathbb{N}_0 \).

**Proof.** By Proposition 4.3 (3) we have that \( D_1 D_2 = D_2 D_1 \) if and only if \( \Lambda_n(D_1 D_2) = \Lambda_n(D_2 D_1) \) for all \( n \). From Proposition 4.3 (2) we get \( \Lambda_n(D_1 D_2) = \Lambda_n(D_1) \Lambda(D_2) \). \( \square \)

**Proposition 4.5.** Let \( \{Q_n\} \) be the sequence of monic orthogonal polynomials. Let \( D = \sum_{i=0}^{s} F_i(u) \frac{d^i}{du^i} \) such that \( DQ_n^* = Q_n^* \Gamma_n \). Then

\[ \Gamma_n = \sum_{0 \leq i \leq s} [n]_i A_i^* \quad \text{for all} \quad n \geq 0, \tag{22} \]

where \( A_i^* \) is the coefficient of \( x^i \) in the polynomial \( F_i \).

**Remark.** Here we are using the notation \( [n]_i = n(n-1) \ldots (n-i+1) \) for \( n \geq 1 \), and \( [n]_0 = 1 \), for \( n \geq 0 \).

**Proof.** From

\[ \sum_{0 \leq i \leq s} F_i(u) \frac{d^i}{du^i}(u) = Q_n^*(u) \Gamma_n, \]

by comparing the monomials of degree \( n \) we get \( \sum_{0 \leq i \leq s}[n]_i A_i^* = \Gamma_n \). \( \square \)

**Remark 4.6.** Observe that in particular, Proposition 4.5 implies that the eigenvalues \( \Gamma_n \) is a polynomial function on \( n \) of degree less or equal to \( \deg(D) \).
5. The operator \( E \)

5.1. \( D \) and \( E \) commute. In this subsection we use the results described in Section 4 to give an elegant proof of the fact that the operators \( D \) and \( E \) commute. Of course we also can verify this by making the explicit computations.

**Theorem 5.1.** The differential operators \( D \) and \( E \), introduced respectively in Theorems 1.1 and 3.1, commute.

**Proof.** From Theorem 3.1 the operator \( E \) is symmetric with respect to the weight \( W \). Thus \( E \) belongs to the algebra \( \mathcal{D}(W) \) defined in (21) (See Theorem 4.2). To see that \( D \) and \( E \) commute it is enough to verify that the corresponding eigenvalues commute. (See Corollary 4.4).

Let \( \{Q_n\} \) be the monic sequence of orthogonal polynomials. Then for any \( D \in \mathcal{D}(W) \), we have

\[
D Q_n^* = Q_n^* \Gamma_n(D),
\]

where the eigenvalue \( \Gamma_n(D) \) is given explicitly in terms of the coefficients of the differential operator \( D \) (see Proposition 4.5).

For the operators \( D \) and \( E \) introduced respectively in Theorems 1.1 and 3.1, these eigenvalues are

\[
\Gamma_n(D) = -n(U + n - 1) - V
\]

\[
\Gamma_n(E) = -n(n - 1)Q_1 + nP_1 - (\alpha + 2\ell + 3k)V,
\]

where the matrices \( U, V, Q_1 \) and \( P_1 \) are given in Theorems 1.1 and 3.1.

Explicitly we have

\[
\Gamma_n(D) = -\sum_{i=0}^{\ell} (n(n + \alpha + \beta + \ell + i + 1) + i(i + \alpha + \beta - k + 1)) E_{ii} + \sum_{i=0}^{\ell-1} (\ell - i)(\beta + i - k + 1) E_{i,i+1}
\]

\[
\Gamma_n(E) = -\sum_{i=0}^{\ell} (n(n - \ell + 3i)(n + \alpha + \beta + \ell + i + 1) + (\alpha + 2\ell + 3k)i(i + \alpha + \beta - k + 1)) E_{ii} + \sum_{i=0}^{\ell-1} (\ell - i)(\beta + i - k + 1)(\alpha + 2\ell + 3k + 3n) E_{i,i+1}.
\]

Now it is easy to verify that

\[
(23) \quad \Gamma_n(E) = (\alpha + 2\ell + 3k + 3n) \Gamma_n(D) + 3n(\ell + k + n)(n + \alpha + \beta + \ell + 1) I.
\]

Thus the matrix \( \Gamma_n(E) \) commutes with \( \Gamma_n(D) \) and by Corollary 4.4 we have that \( D \) and \( E \) commute. \( \square \)

5.2. The eigenfunctions of \( E \). In Subsection 2.2 we give a sequence \( \{P_w\}_w \) of matrix valued polynomials, which are orthogonal with respect to \( W \) and eigenfunctions of the differential operator \( D \). The rows \( P^l_w \) of \( P_w \) are orthogonal polynomials of degree \( w \) and they satisfy \( DP^l_w = \lambda_j(w) P^l_w \).

Since \( D \) and \( E \) commute, it follows that \( E \) preserves the eigenspaces of \( D \). Therefore if an eigenvalue \( \lambda = \lambda_j(w) \) has multiplicity one, then the vector valued polynomial \( P^l_w \) is also an eigenfunction of the differential operator \( E \).
In the next theorem, we prove that this is true, even if the multiplicity of an eigenvalue is bigger than one.

**Theorem 5.2.** The sequence \( \{P_w\}_w \) of orthogonal polynomials associated to the pair \( \{W, D\} \) satisfies

\[
EP_w^*(u) = P_w^*(u)\Lambda_w(E),
\]

where \( \Lambda_w(E) = \sum_{0 \leq j \leq \ell} \mu_j(w)E_{jj} \), and

\[
\mu_j(w) = -w(w + \alpha + \beta + \ell + j + 1)(\alpha - \ell + 3j) - j(j + \alpha + \beta - k + 1)(\alpha + 2\ell + 3k).
\]

**Proof.** Let \( \{Q^*_w\}_{w \geq 0} \) be the sequence of monic orthogonal polynomials. Since \( E \) is symmetric with respect to the weight \( W \), Theorem 4.2 says that

\[
EQ_w^* = Q_w^* \Gamma_w(E)
\]

for some matrix \( \Gamma_w(E) \). If \( Q_w^* = P_w^* A_w^* \) then we have that

\[
DP_w^* A_w^* = P_w^* A_w^* \Gamma_w(D)
\]

and

\[
EP_w^* A_w^* = P_w^* A_w^* \Gamma_w(E).
\]

Therefore

\[
\Lambda_w(D) = A_w^* \Gamma_w(D)(A_w^* )^{-1},
\]

\[
\Lambda_w(E) = A_w^* \Gamma_w(E)(A_w^* )^{-1}.
\]

Thus from (23) we obtain that

\[
\Lambda_w(E) = (\alpha + 2\ell + 3k + 3w)\Lambda_w(D) + 3w(\ell + k + w)(w + \alpha + \beta + \ell + 1)I.
\]

Observe that the fact that \( \Lambda_w(D) \) is a diagonal matrix implies that \( \Lambda_w(E) \) is diagonal. Moreover the eigenvalue \( \mu_j(w) = (\Lambda_w(E))_{jj} \) is given by

\[
\mu_j(w) = (\alpha + 2\ell + 3k + 3w)(-w(w + \alpha + \beta + \ell + j + 1)
- j(j + \alpha + \beta - k + 1)) + 3w(\ell + k + w)(w + \alpha + \beta + \ell + 1)
- w(w + \alpha + \beta + \ell + j + 1)(\alpha - \ell + 3j)
- j(j + \alpha + \beta - k + 1)(\alpha + 2\ell + 3k).
\]

This concludes the proof of the theorem. \( \square \)

### 5.3. The operator algebra generated by \( D \) and \( E \)

In this subsection we study the algebra generated by the differential operators \( D \) and \( E \).

Let \( \mathbb{C}[x, y] \) be the algebra of all polynomials in the variables \( x \) and \( y \) with complex coefficients.

**Theorem 5.3.** The algebra of differential operators generated by \( D \) and \( E \) is isomorphic to the quotient algebra \( \mathbb{C}[x, y]/\langle Q \rangle \), where \( \langle Q \rangle \) denotes the ideal generated by the polynomial

\[
Q(x, y) = \prod_{j=0}^{\ell} (y - (\alpha - \ell + 3j)x + 3j(\ell - j + k)(j + \alpha + \beta - k + 1)).
\]

**Proof.** The algebra of differential operators generated by \( D \) and \( E \) is isomorphic to the quotient algebra \( \mathbb{C}[x, y]/I \) where \( I = \{ p \in \mathbb{C}[x, y] : p(D, E) = 0 \} \).

Since \( \Lambda_w \) is a representation which separates points of \( D(W) \) (Proposition 4.3), we have that \( p(D, E) = 0 \) if and only if

\[
\Lambda_w(p(D, E)) = p(\Lambda_w(D), \Lambda_w(E)) = 0, \text{ for all } w.
\]
Moreover, since the matrices $\Lambda_w(D)$ and $\Lambda_w(E)$ are diagonal matrices, we have that $p(\Lambda_w(D), \Lambda_w(E)) = 0$ if and only if $p((\Lambda_w(D))_{jj}, (\Lambda_w(E))_{jj}) = 0$ for all $0 \leq j \leq \ell$. Thus the ideal $I$ is

$$I = \{ p \in \mathbb{C}[x, y] : p(\lambda_j(w), \mu_j(w)) = 0, \text{for } j = 0, 1, \ldots, \ell \}.$$  

Let $p_j(x, y)$ be the polynomial

$$p_j(x, y) = y - (\alpha - \ell + 3j)x + 3j(\ell - j + k)(j + \ell + \beta - k + 1).$$

It is easy to verify that $p_j(\lambda_j(w), \mu_j(w)) = 0$, for all $w \geq 0$. Therefore $Q(x, y) = \prod_{j=0}^{\ell} p_j(x, y)$ belongs to the ideal $I$.

On the other hand we have that any $f \in I$ vanishes in all points of the form $(x, y)$ with $y = (\alpha - \ell + 3j)x + 3j(\ell - j + k)(j + \ell + \beta - k + 1)$, for each $j = 0, \ldots, \ell$. In fact if we let,

$$a_j = \alpha - \ell + 3j \quad b_j = -3j(\ell - j + k)(j + \ell + \beta - k + 1) \quad (j = 0, 1, \ldots, \ell)$$

then we observe that the polynomial $f(x, a_jx + b_j)$ has infinitely many roots, because $f(\lambda_j(w), \mu_j(w)) = 0$ and $\mu_j(w) = a_j\lambda_j(w) + b_j$.

Any polynomial in $\mathbb{C}[x, y]$ is also a polynomial in $x$ and $y - ax - b$. Then it is clear that if $p(x, y) = 0$ in the line $y = ax + b$ then $p$ is divisible by $y - ax - b$.

Thus we have that if $f$ belongs to the ideal $I$ then $f \in \cap_{j=0}^{\ell} \langle p_j \rangle = \langle \prod_j p_j \rangle$. Therefore we have that the ideal $I$ is generated by the polynomial $Q(x, y)$, which concludes the proof of the Theorem. 

\[\square\]

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