C-axis Optical Sum Rule in Josephson Coupled Vortex State

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Observed violations of the c-axis optical sum rule can give important information on deviations from in-plane Fermi liquid behavior and on the nature of interlayer coupling between adjacent copper oxide planes. Application of a magnetic field perpendicular to these planes is another way to probe in-plane dynamics. We find that the optical sum rule is considerably modified in the presence of the c-axis magnetic field. Interlayer correlation of pancake vortices is involved in the sum rule modification; however, details of the vortex distribution in the plane are less important.

I. INTRODUCTION

The conductivity sum rule is one of the most useful methods to analyze optical properties of high-\(T_c\) cuprates. The conventional sum rule\(^1\) states that the missing spectral weight \(\Delta N\) under the real part of the optical conductivity between superconducting and normal state is equal to the superfluid density \(\rho_s\). However, violations of the conventional sum rule along the c-axis have been observed in some high-\(T_c\) cuprates, and are related to the change in kinetic energy\(^2\) on entering the superconducting state. Since c-axis response reflects in-plane dynamics\(^3\), the c-axis conductivity sum rule and the corresponding superfluid density are worthwhile in investigating to understand characteristics of CuO\(_2\) planes as well as for their own importance and interest.

We have previously studied effects of an in-plane magnetic field on the c-axis sum rule and on the superfluid density under the assumption that the in-plane field freely penetrates between CuO\(_2\) planes, and found that such a field could not easily change the c-axis sum rule. The reason for this is that the in-plane field is not directly pair breaking and it is only the induced vortex due to Josephson tunneling that reduces the superfluid density. Usually the conductivity sum rule is not changed by a perturbation since the c-axis kinetic energy remains unchanged in many cases. In some notion of interlayer coupling theory, superconductivity is related to the kinetic energy; namely, it might be kinetic energy-driven.\(^4\)

In this respect the changes in the kinetic energy and in the c-axis sum rule suggest fundamental information on superconductivity. Therefore, the c-axis magnetic field is more interesting because it gives rise to a shift in the quasiparticle energy spectrum, and serves as a direct pair breaker.

When a c-axis magnetic field (\(H\)) is above \(H_{c1}\) and below \(H_{c2}\) in a magnitude, where \(H_{c1}\) (\(H_{c2}\)) is the lower (upper) critical field, and the anisotropy of the Dirac cone \(\alpha_D = v_F/v_G\) is much larger than one, where \(v_F\) (\(v_G\)) is the Fermi (gap) velocity, the semiclassical approach\(^5\) can be applied. Since \(H_{c1} < H < H_{c2}\), vortices are still well separated and it is an extended quasiparticle state that dominantly determines the properties of the in-plane dynamics for a d-wave superconductor. For \(\alpha_D \approx 1\), quantum mechanical effect becomes important; however, this is not the case for the high-\(T_c\) cuprates, for which \(\alpha_D > 1\).

In this paper we consider the effect of a magnetic field oriented perpendicular to the CuO\(_2\) planes on the c-axis sum rule and on the corresponding superfluid density. We apply a semiclassical approximation to a vortex state in a two dimensional d-wave superconductor for \(H_{c1} < H < H_{c2}\) and \(\alpha_D > 1\), and include in the calculation only the energy shift due to the circulating supercurrents around the vortex cores. We assume that the core occupies only a small part of the single vortex unit cell, and that it is the energy shift of the quasiparticles outside the cores that is most important. We study both the the case of correlated and uncorrelated pancake vortices, and find that the c-axis sum rule depends on the interlayer correlation of pancake vortices while details of the intralayer vortex distribution is less significant.

II. FORMALISM

We begin with the Hamiltonian \(\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_c\), where \(\mathcal{H}_0\) describes a d-wave superconductivity, and \(\mathcal{H}_c = \sum_{i\sigma} t_{\perp} \left[ c_{i1\sigma}^+ c_{i2\sigma} + c_{i2\sigma}^+ c_{i1\sigma} \right]\) is interlayer coupling\(^6\) therefore, an electron transfers from a site \(i\) in plane 1 to the same site \(i\) in plane 2. In the semiclassical approximation, a quasiparticle sitting on the supercurrent circulating around a vortex has an energy shift, the so-called Doppler shift, and the magnitude of the supercurrent is inversely proportional to the distance from the center of the vortex. As illustrated in Fig1. (a), for the correlated vortex state a quasiparticle sitting on, say, the slow supercurrent transfers to the next plane without change in momentum and, therefore, still sits on the slow supercurrent loop as before. Consequently, we expect that the kinetic energy will not change during the interlayer hopping. Note that the c-axis physics is determined by the in-plane dynamics as we pointed out earlier \(\text{See also Eq. (7)}\). On the other hand, for the uncorrelated vortex state, Fig1. (b), a quasiparticle on the slow supercurrent loop sits on the fast supercurrent after hop-
ping; therefore, the kinetic energy changes. The detailed calculation requires the in-plane distribution of pancake vortices and it will be explained later.

The optical conductivity follows from the appropriate Kubo formula:

\[
\sigma_{c}(\omega) = (i/\omega) \left[ \Pi_{cext}(\omega) - c^2 d \langle \mathcal{H}_c \rangle \right]
\]

where \( \Pi_{cext}(\omega) \) is the analytic continuation for the current-current correlation in Matsubara representation, and \( \langle \mathcal{H}_c \rangle \) represents the thermal average of the \( c \)-axis kinetic energy. Here \( c \) and \( d \) are the electron charge and the interlayer spacing, respectively. The superfluid density is determined by Kramers-Kronig relation:

\[
\rho_s = 4\pi \lim_{\omega \to 0} \left[ \omega \text{Im} \sigma_c(\omega) \right].
\]

In terms of the real part of the conductivity \( \sigma_{cR}(\omega) \), the \( c \)-axis conductivity sum rule reads:

\[
\rho_s = \Delta N - 4\pi e^2 d \left[ \langle \mathcal{H}_c \rangle^R - \langle \mathcal{H}_c \rangle^d \right],
\]

where \( \Delta N = 8 \int_0^\infty d\omega \left[ \sigma_{cR}^R(\omega) - \sigma_{cR}^d(\omega) \right] \) and the superscripts \( n \) and \( s \) denote the normal and superconducting state, respectively. In the integral defining \( \Delta N \), the upper limit is sufficiently large so as to include all optical transitions of importance to the problem but not interband effects. The kinetic energy change \( 4\pi e^2 d \left[ \langle \mathcal{H}_c \rangle^R - \langle \mathcal{H}_c \rangle^d \right] \) will be denoted as \( \Delta K \). Therefore, in the presence of a magnetic field \( (H) \), the sum rule becomes \( \rho_s(H) = \Delta N(H) - \Delta K(H) \), and it can be written as

\[
\frac{\Delta N(H)}{\rho_s(H)} \simeq 1 + \frac{\Delta K(0)}{\rho_s(0)} \left[ 1 - \frac{\delta \rho_s}{\rho_s(0)} \right] + \frac{\delta K_s}{\rho_s(0)}
\]

where \( \rho_s(H) = \rho_s(0) + \delta \rho_s \) and \( \delta K_s = \Delta K(H) - \Delta K(0) \). As can be easily seen, the sum rule change due to the field depends not only on \( \delta K_s \) but also on \( \delta \rho_s \).

### III. MAGNETIC FIELD EFFECTS ON THE KINETIC ENERGY AND THE SUM RULE

It is easy to understand that the \( c \)-axis magnetic field has no effect on the interlayer hopping for a Fermi liquid in the overdoped regime while the absence of a well-established theory of the pseudogap state would seemingly cause ambiguity in the validity of our approach for the pseudogap regime. However, recently it has been observed that the pseudogap is insensitive to the magnetic field. This implies that even for the underdoped cuprates, a semiclassical approach may be applicable to the sum rule calculation and the kinetic energy change is determined only by the field effect on the superconducting state; namely, the Doppler shift. Here, we would like to emphasize that we do not calculate the zero field kinetic energy difference \( \Delta K(0) \) for the pseudogap state because a model for the pseudogap would be required to do so; instead, we base our analysis on experimental observations for this case. We also point out that effects of the Doppler shift are considered only on the superconducting state not on the pseudogap state.\[3]

Let us first calculate the kinetic energy change \( \delta K_s \). In the semiclassical approximation, the fermionic Matsubara frequencies \( i\omega_n \) in the superconducting \((2 \times 2)\) Green function \( \hat{G}(k, i\omega_n) \) are shifted by \( v_s(r) \cdot \mathbf{k} \), where \( v_s(r) \) is the superfluid velocity at \( r \) in a plane and \( \mathbf{k} \) the quasiparticle momentum, that is, the \((2 \times 2)\) Green function in Nambu space is \( \hat{G}(k, i\omega_n - v_s(r) \cdot \mathbf{k}) \), and an average over the vortex unit cell is to be carried out. Denoting \( \bar{c} = v_s(r) \cdot \mathbf{k} \), the \((2 \times 2)\) Green function is given by

\[
\hat{G}(k, i\omega_n - \bar{c}) = \frac{(i\omega_n - \bar{c}) \tau_3 + \Delta_k \tau_1 + \frac{\xi_k}{\bar{c}} \tau_2}{(i\omega_n - \bar{c})^2 - \xi_k^2 - \Delta_k^2}.
\]

Here \( \Delta_k \) is a \( d \)-wave superconducting gap such that the quasiparticle energy \( E_k = \sqrt{\xi_k^2 + \Delta_k^2} \) in zero field, and \( \tau_3 \)'s are the Pauli matrices in spin space. In terms of \( \hat{G}(k, i\omega_n - \bar{c}) \), the \( c \)-axis kinetic energy in the vortex state is given at position \( r_1 \) by \( K_s(r_1, r_2) = \sum_k K_s(k; \bar{c}_1, \bar{c}_2) \), where \( \bar{c}_1 = v_{s1}(r_1) \cdot \mathbf{k} \) with

\[
K_s(k; \bar{c}_1, \bar{c}_2) = \frac{CT}{4} \sum_{\omega_n} \text{Tr} \left[ \tilde{\tau}_3 \hat{G}(k, i\omega_n - \bar{c}_1) \tilde{\tau}_3 \hat{G}(k, i\omega_n - \bar{c}_2) \right],
\]

where \( C = 32\pi e^2 d r_1^2 \). The total kinetic energy is obtained after \( K_s(r) \) is appropriately averaged over a vortex unit cell of radius \( R \) \( i.e. \ K_s(H) = (1/A) \int_{r \leq R} dr K^s(r) \) with \( A = \pi R^2 \) the area of the unit cell. Actually, the correct averaging procedure depends on whether or not the pancake vortices from one plane to the next are correlated or are completely uncorrelated. For the correlated case \( v_{s1}(r_1) = v_{s2}(r_2) \) and there is a simple space average over \( r \) while in the uncorrelated case we need to introduce two separate uncorrelated space variable \( v_{s1}(r_1) \) and \( v_{s2}(r_2) \) and take independent averages over \( r_1 \) and \( r_2 \) \( i.e. \) \( (1/A^2) \int_{r_1 \leq R} dr_1 \int_{r_2 \leq R} dr_2 \).

We consider the case of uncorrelated pancakes distributed randomly in each plane so that \( v_{s1}(r_1) \cdot \mathbf{k} \) and \( v_{s2}(r_2) \cdot \mathbf{k} \) are to be averaged independently and separately. In a \( d \)-wave superconductor it is the nodal quasiparticles that dominate the low \( T \) or the small \( H \) response. We take \( T \to 0 \) and work with the difference in kinetic energy with field on and off \( i.e. \ K^s(H) - K^s(0) \). Note that this quantity is determined only by the field effect on the superconducting state in our consideration. We then apply a nodal approximation. In carrying out the sum over \( \mathbf{k} \) we make a coordinate transformation to \( \mathbf{p} \) with the nodal point \( \mathbf{k}_n \) on the Fermi surface playing the role of the origin for \( \mathbf{p} \). We then integrate over a nodal region and the integration over \( \mathbf{p} \) should not be sensitive to the region of integration provided \( |\mathbf{p}| \) is taken to be of order \( p_0 \sim O(\Delta_0) \). With these simplification we can introduce a \( \delta \)-function \( \delta (\bar{c}_1 - v_{s1}(r_1) \cdot \mathbf{k}_n) \) into the expression for \( K_s(H) - K_s(0) \) and define

\[
\frac{\delta K_s}{\rho_s(0)} = \frac{\delta K_s}{\rho_s(0)}.
\]
\[
\mathcal{P}(\epsilon_1) = \frac{1}{A} \int d\mathbf{r}_1 \delta(\epsilon_1 - \mathbf{v}_{s1}(\mathbf{r}_1) \cdot \mathbf{k}_n), \tag{6}
\]

to obtain
\[
K_*(H) - K_*(0) = \int_{-\infty}^{\infty} \left[ \mathcal{P}(\epsilon_1) \mathcal{P}(\epsilon_2) \times (\Psi(\epsilon_1, \epsilon_2) - \Psi(0, 0)) \right] \tag{7}
\]

where \(\Psi(\epsilon_1, \epsilon_2) = \sum_k K_0(k; \epsilon_1, \epsilon_2)\). The matrix sum in \(\Psi(\epsilon_1, \epsilon_2)\) can be done to get
\[
\Psi(\epsilon_1, \epsilon_2) = \frac{C}{T} \sum_k \left[ \sum_{\omega_n} \left( \frac{\epsilon_k^2}{E_k} \left. \frac{\partial f(z)}{\partial \epsilon} \right|_{\epsilon + E_k} + \left. \frac{\partial f(z)}{\partial \epsilon} \right|_{\epsilon - E_k} \right) + \frac{\Delta_k^2}{E_k^2} \left( \frac{f(z) + f(\epsilon_k)}{\epsilon + 2E_k} - \frac{f(z) - f(\epsilon_k)}{\epsilon - 2E_k} \right) \right], \tag{8}
\]

with \(G(k, \omega_n)\) and \(F(k, \omega_n)\) the superconducting state ordinary and anomalous amplitude, respectively. For simplicity we will treat the pure case, and after some algebra we derive \(\Psi(\epsilon_1, \epsilon_2)\) as follows:
\[
\Psi(\epsilon_1, \epsilon_2) = \frac{C}{4} \sum_k \left\{ \frac{\epsilon_k^2}{E_k^2} \left[ \frac{f_+ - f_-}{\epsilon_2} + \frac{f_+ - f_-}{\epsilon_1} \right] \right. + \frac{\Delta_k^2}{E_k^2} \left( \frac{f_+ - f_-}{\epsilon_2} + \frac{f_+ - f_-}{\epsilon_1} \right), \tag{9}
\]

where \(f_{\pm} = f(\epsilon_{\pm} \pm E_k)\) is the Fermi function, and \(\epsilon_{12} = \epsilon_1 - \epsilon_2\). Note the symmetry properties: \(\Psi(\epsilon_1, \epsilon_2) = \Psi(\epsilon_2, \epsilon_1) = \Psi(-\epsilon_1, -\epsilon_2)\) and \(\Psi(-\epsilon_1, \epsilon_2) = \Psi(\epsilon_1, -\epsilon_2)\).

The limit \(\epsilon_1 \to \epsilon_2\) is of interest because it enters the correlated pancake case for which \(\mathbf{v}_{s1}(\mathbf{r}_1) \cdot \mathbf{k} = \mathbf{v}_{s2}(\mathbf{r}_2) \cdot \mathbf{k}\). In this case the average for the kinetic energy over a single \(\mathcal{P}(\epsilon)\) is
\[
K_*(H) - K_*(0) = \int_{-\infty}^{\infty} d\epsilon \mathcal{P}(\epsilon) \left[ \Psi(\epsilon, \epsilon) - \Psi(0, 0) \right], \tag{10}
\]

where
\[
\Psi(\epsilon, \epsilon) = \frac{C}{4} \sum_k \left\{ \frac{\epsilon_k^2}{E_k^2} \left[ \frac{\partial f(z)}{\partial \epsilon} \right|_{\epsilon + E_k} + \left. \frac{\partial f(z)}{\partial \epsilon} \right|_{\epsilon - E_k} \right) + \frac{\Delta_k^2}{E_k^2} \left( f(\epsilon + E_k) - f(\epsilon - E_k) \right) \right\}. \tag{11}
\]

In the nodal coordinate system \(E_k = \sqrt{\rho_1^2 + \rho_2^2} = p\) and \(\sum_k = \sum_{\text{node}} J \int p dp d\theta\), where \(\theta\) is a polar angle and
\(J = \left[ (2\pi)^2 v_F g C \right]^{-1}\). At zero temperature the Fermi function become \(\Theta\)-function i.e. \(f(z) \to \Theta(\Theta)\) such that \(\Theta(\epsilon) = 0\) for \(\epsilon > 0\) and otherwise \(\Theta(\epsilon) = 1\). Substituting \(\Psi(\epsilon, \epsilon)\) into Eq. (10), we obtain
\[
K^*(H) - K^*(0) = 2C \int_{-\infty}^{\infty} d\mathcal{P}(\epsilon) J \int p dp d\theta
\]

\(\times \left\{ \frac{p^2}{E^2} \left[ -\delta(\epsilon + p) - \delta(\epsilon - p) \right] \right. \]
\(+ \left[ \Theta(-\epsilon - p) - \Theta(-\epsilon + p) + 1 \right] \frac{p^2}{E^2} \right\}. \tag{12}
\]

The first \(\delta\)-function and \(\Theta\)-function in Eq. (12) do not survive as they give zero contribution. The first integral in Eq. (12) over \(p\) gives a contribution proportional to \(\epsilon\). The second integral is limited to the range \(p \in (\epsilon, p_0)\), but its range can be extended to \((0, p_0)\) by including the contribution of the first integral. This combined contribution cancels the last integral in Eq. (12) to give zero. As pictorially illustrated in Fig. (1), we analytically showed that \(K_*(H) = K_*(0)\) i.e. the out-of-plane magnetic field has no effect on the \(c\)-axis kinetic energy regardless of the form of \(\mathcal{P}(\epsilon)\) in the correlated vortex state. This does not mean, however, that there is no change in the sum rule given by Eq. (8). Instead it means that the sum rule is modified only by a possible change in superfluid density. The calculation of \(\delta \rho_s\) is similar to the above [See later explanation]. It turns out that \(\rho_s(H) - \rho_s(0) = 4\pi C J \int_{-\infty}^{\infty} d\epsilon \mathcal{P}(\epsilon)\), where \(\rho_s(0) = \mathcal{P}(0)/2\) with the density of states at the Fermi level \(N(0)\), and we obtain \(\delta \rho_s / \rho_s(0) = -(2/\Delta_0) \int_{-\infty}^{\infty} d\epsilon \mathcal{P}(\epsilon)\).

Recently, Vekhter et al. have given analytic expressions for different models of the vortex distribution function \(\mathcal{P}(\epsilon)\). They consider a Gaussian \(\mathcal{P}_G(\epsilon)\) and two possible vortex liquid model \(\mathcal{P}_{L1}(\epsilon)\) and \(\mathcal{P}_{L2}(\epsilon)\):

\[
\mathcal{P}_G(\epsilon) = \frac{1}{\sqrt{\pi} E_H} \exp \left( -\frac{\epsilon^2}{E_H^2} \right) \tag{13}
\]
\[
\mathcal{P}_{L1}(\epsilon) = \frac{1}{2} \frac{E_H^2}{(E_H^2 + \epsilon^2)^{3/2}} \tag{14}
\]
\[
\mathcal{P}_{L2}(\epsilon) = \frac{1}{\pi E_H} \left[ \arccos \left( \frac{1}{\sqrt{2\epsilon / E_H}} + 1 \right) \right] \times \left( \frac{E_H^3}{\epsilon^4} + \frac{3E_H^5}{4\epsilon^6} - \frac{3E_H^6}{4\epsilon^6} \right). \tag{15}
\]

The magnetic energy \(E_H\) is given by \(v_F / (2R)\), where the radius of the vortex unit cell \(R = \sqrt{\Phi_0 / \pi H}\) with \(\Phi_0\) the flux quantum. Using the model \(\mathcal{P}(\epsilon)\), we obtain \(\delta \rho_s / \rho_s(0) = -\alpha_1 E_H / \Delta_0\). where \(\alpha_1 = 1/\sqrt{\pi}\) for \(G\) and \(1\) for \(L1\), and \(0.84\) for \(L2\). Note that all distributions give the same linear dependence on \(E_H / \Delta_0\) with slightly different coefficients, and \(H\) reduces the superfluid density as we expect.

Our main results so far are that in the correlated vortex case there is no change in kinetic energy on application of a magnetic field \(H\) perpendicular to the layers, independent of the vortex distribution function \(\mathcal{P}(\epsilon)\). In all cases considered, the superfluid density is reduced by the magnetic field, and \(\delta \rho_s / \rho_s(0)\) is linearly proportional to \(E_H / \Delta_0\) with a coefficient of order 1. The form of \(\mathcal{P}(\epsilon)\) determines the exact value of the coefficient. We consider two interesting regimes of doping for the correlated
vortex case ($\delta K_s = 0$). For the overdoped case we assumed that the in-plane motion can be described by a Fermi liquid. In this case, it has been found previously that in zero field $\Delta N(0)/\rho_s(0) = 1$, i.e., the c-axis sum rule is conventional, which is consistent with experimental observations. It arises because $\Delta K(0) = 0$ in this instance. Therefore, it follows from Eq. (2) that $\Delta N(H)/\rho_s(H) = 1$ also, i.e., there is no change in the sum rule induced by the external field.

As we mentioned earlier, we do not rely on a specific theory to calculate the sum rule for the underdoped regime. Instead we use experimental observations on the underdoped cuprates. In this case the zero field sum rule is observed to be about a half; namely, $\Delta N(0)/\rho_s(0) \approx 1/2$, which also follows directly in the preformed pair model for the pseudogap. This means that $\Delta K(0)/\rho_s(0) = -1/2$ in Eq. (2). When we calculate $\delta K_s$, we take into account only effects of the Doppler shift due to the magnetic field, again, based on experimental results. Consequently, we obtain $\Delta N(H)/\rho_s(H) = 1/2 + \delta \rho_s/2 \rho_s(0) = 1/2 - \alpha_1 E_H/2 \Delta_0$, where $\alpha_1 = 1/\sqrt{\pi}$ for G, 1 for L1, and 0.84 for L2. The sum rule originally equal to 1/2 is further reduced by the magnetic field.

Now, we return to Eq. (2) for the uncorrelated vortex state with Eq. (2) defining $\Psi(\epsilon_1, \epsilon_2)$. In this case $K^s(H) - K^s(0)$ becomes

$$K^s(H) - K^s(0) = 2 \int_0^\infty d\epsilon_1 d\epsilon_2 \mathcal{P}(\epsilon_1) \mathcal{P}(\epsilon_2) [\Psi(\epsilon_1, \epsilon_2) - \Psi(0, 0)]$$

$$+ 2 \int_0^\infty d\epsilon_1 d\epsilon_2 \mathcal{P}(\epsilon_1) \mathcal{P}(\epsilon_2) [\Psi(\epsilon_1, -\epsilon_2) - \Psi(0, 0)]. \quad (16)$$

On application of the nodal approximation to the difference $\Psi(\epsilon_1, \epsilon_2) - \Psi(0, 0)$ in the limit $T \to 0$, we obtain

$$\delta K_s = \frac{C}{2} \frac{N(0)}{\Delta_0} \int_0^\infty d\epsilon_1 d\epsilon_2 \mathcal{P}(\epsilon_1) \mathcal{P}(\epsilon_2)$$

$$\times \left\{ \frac{\epsilon_1 \epsilon_2}{(\epsilon_1 + \epsilon_2)} - \frac{1}{2} (\epsilon_1 + \epsilon_2) \ln \left| \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \right| \right\}, \quad (17)$$

which is of order $E_H/\Delta_0$, and numerical calculations give $\delta K_s/\rho_s(0) = -\alpha_2 E_H/\Delta_0$, where $\alpha_2$ is 0.123 for G, 0.135 for L1, and 0.102 for L2. We also derive the change in superfluid density which follows in a similar fashion; namely, $\delta \rho_s = \int_0^\infty d\epsilon_1 d\epsilon_2 \mathcal{P}(\epsilon_1) \mathcal{P}(\epsilon_2) [\chi(\epsilon_1, \epsilon_2) - \chi(0, 0)]$, where

$$\chi(\epsilon_1, \epsilon_2) = \frac{C}{4} \sum_k \frac{\Delta^2}{E_k^2} \left[ \frac{f_{1+} - f_{2+}}{\epsilon_{12}} + \frac{f_{1-} - f_{2-}}{\epsilon_{12}} \right.$$  

$$- \frac{f_{1+} - f_{2-}}{\epsilon_{12} + 2E_k} - \frac{f_{1-} - f_{2+}}{\epsilon_{12} - 2E_k} \left\}, \quad (18)$$

It can be shown that the superfluid density reduction becomes

$$\delta \rho_s = -\frac{C}{2} \frac{N(0)}{\Delta_0} \int_0^\infty d\epsilon_1 d\epsilon_2 \mathcal{P}(\epsilon_1) \mathcal{P}(\epsilon_2)$$

$$\times \left\{ \frac{3(\epsilon_1 + \epsilon_2)^2 + \epsilon_1^2 + \epsilon_2^2}{2(\epsilon_1 + \epsilon_2)} + \frac{\epsilon_1 + \epsilon_2}{2} \ln \left| \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \right| \right\}, \quad (19)$$

and $\delta \rho_s/\rho_s(0) = -\alpha_3 E_H/\Delta_0$, where $\alpha_3$ is 0.34 for G, 0.72 for L1, and 0.63 for L2. For correlated vortex, we used $\chi(\epsilon, \epsilon) \chi(0, 0) = \rho_s(0)$.

The sum rule of the uncorrelated vortex state reads

$$\Delta N(H)/\rho_s(H) \approx 1 - \alpha_2 (E_H/\Delta_0)$$

for the overdoped, and

$$\Delta N(H)/\rho_s(H) \approx 1/2 - (\alpha_2 + \alpha_3/2)(E_H/\Delta_0)$$

for the underdoped regime. In both cases the sum rule is reduced by the presence of a magnetic field. In the overdoped case the reference is, however, 1 while for the underdoped case it is 1/2, and the coefficient of the reduction, which goes like $E_H/\Delta_0$, is larger for the underdoped case as compared with the overdoped case. Note that the magnetic energy $E_H$ goes like $\sqrt{H}$ in all cases.

**IV. CONCLUSIONS**

For correlated pancake vortices with an in-plane Fermi liquid and coherent c-axis coupling, there is no change in the c-axis optical sum rule when a magnetic field is applied perpendicular to the layers of a d-wave superconductor. The sum rule keeps its conventional value of one. At the same time the superfluid density in the c-direction decreases by an amount proportional to the square root of the magnitude of the external magnetic field ($\sqrt{H}$). If, however, the in-plane dynamics is unconventional, for example, it can be described by a pseudogap above $T_c$, the sum rule, which is equal to 1/2 when $H = 0$, gets reduced linearly in $\sqrt{H}$ as does the c-axis superfluid density. For uncorrelated vortices, there is always a reduction in the sum rule which is also proportional to $\sqrt{H}$. For the in-plane Fermi liquid case the sum rule is reduced below one, while for the unconventional case with a pseudogap it is reduced below 1/2. In both cases a reduction in the c-axis superfluid density accompanies the sum rule reduction.

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1 R. A. Ferrell and R. E. Glover, Phys. Rev. 109, 1398 (1958); M. Tinkham and R. A. Ferrell, Phys. Rev. Lett. 2, 331 (1959).
Generally speaking, one has to include all possible interlayer couplings. However, the universal value observed for c-axis quasiparticle conductivities for some cuprates such as Bi-2212 imply that coherent coupling with a constant amplitude dominates interlayer hopping. See W. Kim and J. P. Carbotte, Phys. Rev. B 63, 054526 (2001).

In a simple idea such as a preformed pair model, the electron pairing remains in the pseudogap state up to $T^*$. At $T_c$ phase coherence sets in and the superconducting state is assumed to be usual $d$-wave-like. See, for example, Refs. 4, 6, and Kim and Carbotte, Phys. Rev. B 62, 8661 (2000).

This is in accord with the preformed pair model. One could conceive of more complicated origin of the pseudogap in which its effect persists below $T_c$ as distinct from superconductivity and so modifies the supercurrents in a way not specifically treated here.

FIG. 1. Pictorial illustration for a correlated (a) and an uncorrelated (b) vortex state. $v_s$ is the supervelocity and the gray tubes are vortex cores.
(a) Correlated vortex

(b) Uncorrelated vortex