We discuss the recent progress in computing the $D$-meson decay constant and $D \to \pi \ell \nu$ form factors from the lattice QCD simulations with $N_f = 2$ dynamical using Wilson quarks. We report $f_D = 201(20) \text{ MeV}$ and $F_+(1 \text{ GeV}^2)/f_D = 4.04(78) \text{ GeV}^{-1}$ at $a \simeq 0.08 \text{ fm}$.

1 Introduction and lattice setup

Accurate determinations of the Cabibbo–Kobayashi–Maskawa (CKM) couplings provide an essential test of the Standard Model. The most straightforward method for their extraction is through the leptonic and/or semileptonic meson decays. However, such an extraction from the experimentally measured decay widths requires a reliable information about the hadronic quantities, namely the decay constants and the form factors. Such an information is expected to be provided by the QCD simulations on the lattice.

Many lattice calculations of these quantities have been attempted over the past twenty years. Until a few of years ago, all computations were performed in the quenched approximation in which the effect of virtual quark loops is neglected. Recent progress allowed us to move to the unquenched case in which at least $N_f = 2$ dynamical quarks are present in the QCD vacuum fluctuations. The lattice quark action that are being used nowadays are $O(a)$–improved so that the systematic errors are $O((am)^2)$. In this write-up we present new results for the charmed decays using an improved Wilson action with $N_f = 2$ degenerate sea quarks. The results reported here refer to the simulations made at $a \simeq 0.08 \text{ fm}$ for three different values of the sea quark mass corresponding to $m_{\pi qq} \simeq 770 \text{ MeV}$, 600 MeV and 400 MeV. Other unquenched results relevant to the leptonic and semileptonic decays were obtained by using the staggered quark action. Since the formal proof of validity of the staggered formulation is still missing, the study based on Wilson quarks is more than needed.

2 Leptonic decays

2.1 Hadronic matrix element

The simplest way to determine the CKM matrix element $|V_{cd}|$ is via the leptonic decay $D \to \ell \nu$ with $\ell = \tau, \mu, e$. The decay width is given by

$$
\Gamma(D^+ \to \ell^+ \nu_\ell) = |V_{cd}|^2 \frac{G_F^2}{8\pi} m_D m_\ell \left( 1 - \frac{m_\ell^2}{m_{D^+}^2} \right) f_D^2, \quad (1)
$$
where $G_F$ is the Fermi constant and the decay constant $f_D$ parametrizes the hadronic matrix element:

$$\langle 0| A_\mu | D(p) \rangle = i f_D p_\mu ,$$  \hspace{1cm} (2)$$

with $A^\mu = \bar{c} \gamma^\mu \gamma^5 d$. The theoretical uncertainty in eq. (1) is entirely due to $f_D$. For $\ell = e$ or $\mu$ this decay mode has been recently accurately measured. On the lattice, $f_D$ is extracted from the asymptotic behavior of the 2-point Green function, i.e.

$$C^{(2)}_\mu(t) = \sum_{\vec{x}} \langle 0| (A_\mu)_{\vec{0},0} (\bar{c} \gamma_5 q)_{\vec{x},t} | 0 \rangle \xrightarrow{t \gg 0} \langle 0| A_\mu | D(0) \rangle \times \frac{Z_D}{2m_D} e^{-m_D t} ,$$  \hspace{1cm} (3)$$

where $A_\mu$ is the appropriately renormalized axial current. $Z_D$ is evaluated from:

$$\sum_{\vec{x}} \langle 0| (\bar{c} \gamma_5 q)_{\vec{0},0} (\bar{c} \gamma_5 q)_{\vec{x},t} | 0 \rangle \xrightarrow{t \gg 0} \frac{Z_D}{2m_D} e^{-m_D t} .$$  \hspace{1cm} (4)$$

### 2.2 Computation of $f_D$ and chiral extrapolations

We focus to the unquenched case in which the valence and the sea quark masses are equal. Thus, for each of our directly accessible light quark masses, we compute $f_{Dq}$ which then needs to be extrapolated to the physical $f_{Dd} \equiv f_D$. That extrapolation can be made either linearly in $m_q$, or by using the expression derived in heavy meson chiral perturbation theory (HMχPT): \( f_{Dq} \sqrt{m_{Dq}} = \Phi_0 \left[ 1 - \frac{3}{4} \frac{1 + 3g^2}{4\pi f_0} \log m_{\pi qq}^2 + c^\Phi m_{\pi qq}^2 \right] . $$  \hspace{1cm} (5)$$

In this formula, $\Phi_0$ and $c^\Phi$ are the fit parameters, $f_0$ is the pion decay constant in the chiral limit determined on the same lattice, while $g$ is related to the coupling between the heavy meson doublet, $(D, D^*)$ and the soft pion, i.e.

$$\langle D(k)\pi(p, \lambda) | D^* (p, \lambda) \rangle = \langle \epsilon^\lambda . q \rangle g_{D^*D\pi} = \langle \epsilon^\lambda . q \rangle \frac{2\sqrt{m_D m_{D^*}}}{f_\pi} g ,$$  \hspace{1cm} (6)$$

were $\epsilon^\lambda$ is the polarization vector of $D^*$. Its value has been previously computed in the static limit $g \simeq 0.5$, and with the propagating charm quark $g \simeq 0.6$, both values leading to the large factor multiplying the logarithm in eq. (5). That would drive the extrapolation of $f_{Dq} \sqrt{m_{Dq}}$ way below the result obtained through the linear extrapolation, thus to large systematic errors. It is then more convenient to consider the ratio $f_D/f_\pi$ in which the chiral logarithmic term is halved:
\[
\frac{f_{D_{q}}}{f_{\pi_{qq}}} = \frac{\Phi_{0}}{f_{0}\sqrt{m_{D_{q}}}} \left[ 1 + \frac{15 - 9g^{2}}{2m_{\pi_{qq}}^{2}} \log \frac{m_{\pi_{qq}}^{2}}{c_{1}} \right],
\]

where \( c_{1} \) is a fit parameter. From fig. (1), we see that linear and \( \chi \)-log fits are very consistent and by using both values for \( g \) mentioned above, we get \( f_{D}/f_{\pi} = 1.50(24) \) \( \chi \)-log and \( f_{D}/f_{\pi} = 1.52(17) \) linear. With the physical pion decay constant \( f_{\pi} = 139.6 \text{ MeV} \), we arrive at

\[
f_{D} = 201(22) \left( \frac{+4}{-9} \right) \text{ MeV}.
\]

3 Semileptonic decays

3.1 Hadronic matrix element

\(|V_{cd}| \) can also be extracted by studying the partial or total decay width of the semileptonic decay \( D \rightarrow \pi \ell \nu_{\ell} \) \( (\ell = e, \mu) \), which has been measured in various recent experiments\(^8\). The differential decay width is given by

\[
d\Gamma/dq^{2}(D \rightarrow \pi \ell \nu_{\ell}) = |V_{cd}|^{2} \frac{G_{F}^{2}}{192\pi^{2}m_{D}^{2}} \lambda^{3/2}(q^{2}) \left| F_{+}(q^{2}) \right|^{2},
\]

where \( q \) is the momentum transfer and \( \lambda(q^{2}) = (q^{2} + m_{\pi}^{2} - m_{D}^{2})^{2} - 4m_{D}^{2}m_{\pi}^{2} \). The vector form factor \( F_{+}(q^{2}) \) parametrizes the hadronic matrix element of the weak current

\[
\langle \pi(k) | (V - A)_{\mu} | D(p) \rangle = \left( p + k - q \frac{m_{D}^{2} - m_{\pi}^{2}}{q^{2}} \right) F_{+}(q^{2}) + q_{\mu} \frac{m_{D}^{2} - m_{\pi}^{2}}{q^{2}} F_{0}(q^{2}),
\]

where \( q = p - k \) and \( F_{+}(0) = F_{0}(0) \). The contribution of the scalar form factor \( F_{0}(q^{2}) \) to the decay width comes with a \( m_{\pi}^{2} \)-factor and therefore can be neglected. The form factors are extracted from the behavior of the following 3–point correlation functions:

\[
C^{(3)}_{\mu}(\vec{q}, \vec{t}, t_{s}) = \sum_{E_{s}E_{t}} \langle 0 | (\overline{\tau} \gamma_{5} q)_{\mu, 0} (V_{\mu})_{\ell, \bar{\ell}} (\overline{\tau} \gamma_{5} q)_{\ell, t_{s}} e^{-i(\vec{q} \cdot \vec{x} - \vec{t} \cdot \vec{y})} | 0 \rangle \sqrt{2E_{s}E_{t}} e^{-E_{s}t} \langle \pi(\vec{k}) | V_{\mu} | D(\vec{p}) \rangle \sqrt{2E_{D}} e^{-E_{D}(t_{s} - t)},
\]

computed on the lattice. The axial current does not contribute due to the parity conservation in QCD. Extracting the matrix element from a fit of \( C^{(3)}_{\mu} \) requires the knowledge of several independent quantities \( (Z_{\pi, D}, \text{renormalization constants and } O(a) \text{ improvement coefficients of the vector current}) \) whose error lower the accuracy of \( F_{+}(q^{2}) \). This can be improved by using the strategies we previously studied in\(^3\) based on ratios of such 3–point correlators and twisted boundary conditions. In fig. (2) we show the results obtained with the 1st strategy.

3.2 Computation of \( F_{+}(q^{2} = 1 \text{ GeV}^{2}) \) and chiral extrapolations

The differential decay width in eq. (9) is experimentally measured for various values of \( q^{2} \). To extract \( |V_{cd}| \), one then needs a single \( F_{+}(q^{2}) \)-value. We choose \( q^{2} = 1 \text{ GeV}^{2} \) where both theoretical and experimental errors are under a reasonable control. The extrapolation of \( F_{+}(1 \text{ GeV}^{2}) \) toward the physical value leads to a difficulty similar to what we discussed above. To get around that difficulty, we use HM\( \chi \)PT fits for the ratio \( F_{+}(q^{2})/f_{D} \) where large deviations due to \( m_{\pi}^{2} \log m_{\pi}^{2} \) terms are reduced\(^10\). We get

\[
\frac{F_{+}(1 \text{ GeV}^{2})}{f_{D}} = 4.32(56) \text{ GeV}^{-1} \text{ (HM\( \chi \)PT fit)} \quad 3.76(54) \text{ GeV}^{-1} \text{ (Linear fit)},
\]

where we also quote our result obtained by using the naive linear extrapolation. The difference of the two is an estimate of the systematic uncertainty of the extrapolation procedure.
Figure 2: The $q^2$-dependence of the vector (left) and the scalar (right) form factors relevant to $D \to \pi \ell \nu$ decay for 3 different pion masses, accessible directly from our lattices.

4 Summary

In this short note, we reported on the progress in determining the key hadronic quantities entering the leptonic and semileptonic decays on the lattice by using the Wilson quarks. A better control over the systematic uncertainties is achieved if one chooses judiciously the ratios in which various sources of uncertainties cancel out, or are diminished. In the case of semileptonic decays, also the use of twisted boundary conditions is very important. The quenched experience suggests that the $\mathcal{O}(a^2)$ artifacts are reasonable\cite{11}, but to that end, more work is needed.

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