Comparative Study of Homotopy Analysis and Renormalization Group Methods on Rayleigh and Van der Pol Equations

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Abstract

A comparative study of the Homotopy Analysis method and an improved Renormalization Group method is presented in the context of the Rayleigh and the Van der Pol equations. Efficient approximate formulae as functions of the nonlinearity parameter $\varepsilon$ for the amplitudes $a(\varepsilon)$ of the limit cycles for both these oscillators are derived. The improvement in the Renormalization group analysis is achieved by invoking the idea of nonlinear time that should have significance in a nonlinear system.

Key Words: Rayleigh Van der Pol Equation, Homotopy Analysis Method, Renormalization Group

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1 Introduction

The study of non-perturbative methods for nonlinear differential equations is of considerable recent interest. Among the various well known singular perturbation techniques such as multiple scale analysis, method of boundary layers, WKB method and so on \cite{1,2}, the recently developed homotopy analysis method (HAM) \cite{3,4} and the Renormalization group method (RGM) \cite{5,6,7,8} appear to be very attractive. The aim of these new improved methods is to derive in an unified manner uniformly valid asymptotic quantities of interest for a given nonlinear dynamical problem. Although formulated almost parallelly over the past decades or so, relative strength and weakness of these two approaches have yet to be investigated systematically. The purpose of this paper is to undertake a comparative study of HAM and RGM in the context of the Rayleigh equation

\[ \ddot{y} + \varepsilon \left( \frac{1}{3} \dot{y}^3 - \dot{y} \right) + y = 0 \]  \hspace{1cm} (1)

and the Van der Pol equation

\[ \ddot{x} + \varepsilon \dot{x} \left( x^2 - 1 \right) + x = 0 \]  \hspace{1cm} (2)

where the dots are used to designate the derivative with respect to time $t$. The Rayleigh and the Van der Pol (VdP) equations represent two closely related nonlinear systems and

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have found significant applications in the study of self excited oscillations arising in biology, acoustics, robotics, engineering etc. [9, 10]. It is easy to observe that differentiating (1) with respect to time \( t \) and putting \( \dot{y}(t) = x(t) \) we obtain (2). Both these systems have unique isolated periodic orbit (limit cycle). The amplitude of a periodic oscillation \( y(t) \) (or \( x(t) \)) is generally defined by \( \max |y(t)| \) (or \( \max |x(t)| \)) over the entire cycle. It is well known that the naive perturbative solutions of these equations are meaningful when \( 0 < \varepsilon \ll 1 \) and yields the asymptotic value \( a(\varepsilon) \approx 2 \) of the amplitude for the limit cycle correctly. For \( \varepsilon \gg 1 \), simple analysis based on singular perturbation theory also yields the asymptotic amplitude for the relaxation oscillation as \( a(\varepsilon) \approx 2 \) for the VdP equation. However, the conventional perturbative approaches fail when \( \varepsilon \) is finite. One of the aim of this paper is to determine efficient approximate formulae for the amplitude of the limit cycle for the above systems by both HAM and RGM. Lopez et al [4] have reported an efficient formula for the amplitude of the limit cycle by HAM. We note here that a key difference in Rayleigh and VdP oscillators is the fact that with increase in input energy (voltage), the amplitude of the Rayleigh periodic oscillation increases, when that of the VdP oscillator remains almost constant at the value 2, with possible increase in the corresponding frequency only. For large \( \varepsilon \) (\( \gg 1 \)) relaxation oscillations, on the other hand, the Rayleigh system shows up a rather fast building up and slow subsequent release of internal energy, when the VdP models the reverse behaviour, with slow rise and fast drop in the accumulated energy.

As remarked above, HAM and RGM are formulated to determine the uniformly valid global asymptotic behaviours of relevant dynamical quantities like amplitude, period, frequency etc. related to periodic solutions of these equations for finite values of \( \varepsilon \), by devising efficient methods in eliminating divergent secular terms of the naive perturbation theory. HAM seems to have the advantage of yielding uniformly convergent solutions of very high order in the nonlinearity parameter \( \varepsilon \) utilizing a freedom in the choice of a free parameter \( h \). The computation of higher order term could be facilitated by symbolic computational algorithms. This method is used to obtain good approximate solutions for the VdP equation by a number of authors [4, 11]. Lopez et al [4] derived efficient formulae for estimating the amplitude of the limit cycle of the VdP equation for all values of \( \varepsilon > 0 \). Although, HAM is now considered to be an efficient method in the study of non-perturbative asymptotic analysis, it is recently pointed out [12] that this method might fail even in some innocent looking nonlinear problems.

The RGM, on the other hand, has a rich history, being originally formulated for managing divergences in the quantum field theory and later having deep applications in phase transitions and critical phenomena in statistical mechanics. Subsequently, Chen et al [5, 6] successfully translated the RG formalism into the study of nonlinear differential equations. It is noted that RGM is more efficient and accurate than conventional singular perturbative approaches in obtaining global informations from a naive perturbation series in \( \varepsilon \). It is also recognized that RGM generated expansions yield \( \varepsilon \)-dependent space/time scales naturally, when conventional approaches normally require invoking such scales in an ad hoc manner. The pertubative RGM, however, appears to have the limitation that the computation of higher order renormalized solutions could be quite involved and tedious. More serious is the inability of assuring the convergence of the renormalized expansions for large nonlinearity parameter. Further, there is still no evidence in the literature that RGM could be employed successfully to asymptotic estimation of the amplitude of an isolated periodic orbit for all values of \( \varepsilon \) as was reported for HAM [4].

Here we report analytic expressions of the amplitude of the periodic solutions of both the Rayleigh equation (1) and the VdP equation (2) as functions of \( \varepsilon \). We have made a comparative study of these two sets of formulae using both HAM and RGM. The HAM contains a control parameter \( h = h(\varepsilon) \) which controls the convergence of the approximation to the numerically computed exact value of the amplitude for all values of \( \varepsilon \). Suitable choice of \( h \) can control the relative percentage error.
The original RGM gives an approximation to the exact solution for small values of \( \varepsilon \). We report here the RG solution upto order 3. To the authors’ knowledge this seems to be the first higher order computation other than second order computations reported so far by various authors \([6, 8]\). A comparison of the amplitude of the periodic orbit with the exact computations reveals that even the present higher order perturbative approximations fails to give accurate estimation for moderate values of \( \varepsilon \). As the higher order RG computations are quite laborious and inefficient, it is very unlikely that higher order computations of amplitude would improve the quality of the estimated amplitude of the limit cycle. Further, in RGM one does not have the resource of a free parameter equivalent to \( h(\varepsilon) \) of HAM to improve the convergence of the RG expansions.

A major contribution in the present study is to propose an improved RGM (IRGM). In IRGM, we advocate the concept of nonlinear time \([13, 14, 15]\) that extends the original RG idea of eliminating the divergent secular term of the form \((t - t_0) \sin t\), where \( t_0 \) is the initial time, of the naive perturbation series for the solution of the nonlinear problem, by exploiting the arbitrariness in fixing the initial moment \( t_0 \). The original prescription rests on introducing new initial time \( \tau \) in the form \((t - \tau + \tau - t_0) \sin t\) and to allow the renormalized amplitude \( R = R(\tau) \) and phase \( \theta = \theta(\tau) \) of the renormalized solution to depend on the new parameter, viz, \( \tau - t_0 \) so that the original naive perturbative, constant values of amplitude \( R_0 \) and phase \( \theta_0 = 0 \) (say) ‘flow’ following the RG flow equations of the form

\[
\frac{dR}{d\tau} = f(R, \varepsilon), \quad \frac{d\theta}{d\tau} = g(R, \varepsilon) \tag{3}
\]

The functions in the right hand sides of the RG flow equations, in general, should depend both on \( R \) and \( \theta \), besides the explicit \( \varepsilon \) dependence. We suppress the \( \theta \) dependence for simplicity that should suffice for our present analysis of the Rayleigh and VdP equations (c.f. equations \([30], [31]\)). The flow equations are derived from the consistency condition that the actual renormalized solution \( y(t, \tau) \) should be independent of the arbitrary initial adjustment \( \tau: \frac{\partial y}{\partial \tau} = 0 \). The final form of the uniformly valid RG solution \( y_R(t) \) is obtained by setting \( \tau = t \) that eliminates the secular terms. Let us remark here that the actual convergence of the RG expansions is not well addressed and should require further investigations. Moreover, estimation of asymptotic amplitude for a limit cycle as \( t \to \infty \), for instance, from the perturbation expansion of \( f \) is expected to fail for \( \varepsilon \geq O(1) \).

In the framework of nonlinear time, we suppose the arbitrary initial time \( \tau \) to depend explicitly on the nonlinearity parameter (coupling strength) \( \varepsilon \) of the nonlinear equation, so that one can write \( \tau/\varepsilon = \varepsilon^h \) where \( h = h(\varepsilon, t) \) is a slowly varying (almost constant), free (asymptotic) control parameter for \( t \to \infty \) and \( \varepsilon \to \) either to 0 or \( \infty \), to be utilized judiciously to improve the convergence and non-perturbative global asymptotic behaviour of the original RG proposal \((h < 0 \text{ for } 0 < \varepsilon < 1)\). The secular terms in the naive perturbation series would now be altered instead as \((t - \tau/\varepsilon + \tau/\varepsilon - t_0) \sin t\) and we obtain the new RG flow equations in the form

\[
\frac{dR}{d\tau} = f_0(R) \left(1 + O(\varepsilon^2)\right), \quad \frac{d\theta}{d\tau} = g_1(R) \left(1 + O(\varepsilon^3)\right) \tag{4}
\]

where \( f_0(R) \) and \( g_1(R) \) are nonzero, minimal order \( R \) dependent terms in the respective perturbations series. We next make the key assumption that any possible divergent \( t \) dependence in the control parameter \( h \) is absorbed in the divergence of the higher order perturbation series so that in the asymptotic limit \( t \to \infty \), one expects the finite, non-perturbative flow equations directly for the periodic orbit of the nonlinear system

\[
\frac{da}{d\tau} = f_0(a), \quad \frac{d\theta}{d\tau} = \varepsilon g_1(a) \tag{5}
\]

where \( \tau = \varepsilon^{h_{RG}(\varepsilon)} \), and \( h_{RG}(\varepsilon) \) is a finite \( t \) independent control parameter and \( a(\varepsilon) = \lim_{t \to \infty} R(\varepsilon, t) \) is the \( \varepsilon \)- dependent amplitude of the limit cycle. A simple quadrature formula
should then relate the control parameter \( h_{RG} \) with the amplitude \( a(\varepsilon) \). As a consequence, adjusting the control parameter \( h_{RG} \) suitably, one can generate an efficient algorithm to estimate the amplitude \( a(\varepsilon) \) that would compare well with the exact values, up to any desired accuracy. It will transpire that the control parameter \( h_{RG}(\varepsilon) \) must respect some asymptotic conditions depending on the characteristic features of a particular relaxation oscillation (c.f. Section 4). It follows that the idea of nonlinear time offers one with a robust formalism for global asymptotic analysis for a general nonlinear system that might even be advantageous in many respects as compared to HAM. The application of nonlinear time in HAM will be considered separately.

The paper is organized as follows. In Section 2 we have deduced the solution to the equation (1) by HAM. In Section 3 we compute the classical RG solution up to \( O(\varepsilon^3) \) order and compare estimated values of the limit cycle amplitude with the exact values. The improved RG method is presented in Section 4. This introduces a control parameter \( h_{RG} \) in the RG analysis. In Subsection 4.1 approximate analytic formulae are deduced for the amplitudes of the limit cycles of the Rayleigh and the VdP equations. Efficient match with the exact values can be obtained by appropriate choice of \( h_{RG} \). We conclude in Section 5.

2 Computation of Amplitude by HAM

The Homotopy Analysis method proposed by Liao [3, 11] is used to obtain the solution of non-linear equation even if the problem does not contain a small or large parameter. HAM always gives a family of functions at any given order of approximation. An auxiliary parameter \( h \) is introduced in HAM to control the convergence region of approximating series involved in this method to the exact solution. HAM is based on the idea of homotopy in topology. In simple language, it involves continuous deformation of the solution of a linear ordinary differential equation (ODE) to that of desired nonlinear ODE. The solution of linear ODE gives a set of functions called base functions. One advantage of HAM is that it can be used to approximate a nonlinear problem by efficient choice of different sets of base functions. A suitable choice of the set of base functions and the convergence control parameter can speed up the convergence process.

In this paper we consider the self-excited system (1), which can be written as the ODE

\[
\ddot{U}(t) + \varepsilon \left( \frac{1}{3} \dot{U}^3(t) - \dot{U}(t) \right) + U(t) = 0, \quad t \geq 0
\]  

where the dot denotes the derivative with respect to the time \( t \). A limit cycle represents an isolated periodic motion of a self-excited system. This is an isolated closed curve \( \Gamma \) (say) in the phase plane so that any path in its suitable small neighbourhood starting from a point, specified by some given initial condition, ultimately converges to (or diverge from) \( \Gamma \). Consequently, this periodic motion represented by limit cycle is independent of initial conditions. It, however, involves the frequency \( \omega \) and the amplitude \( a \) of the oscillation. Therefore, without loss of generality, we consider an initial condition

\[
U(0) = a, \quad \dot{U}(0) = 0.
\]  

In [6], an alternative initial condition i.e. \( U(0) = 0, \dot{U}(0) = a \) was considered. Let, with slight abuse of notations,

\[
\tau = \omega t \quad \text{and} \quad U(t) = a u(\tau),
\]

so that (6) and (7) respectively become

\[
\omega^2 u''(\tau) + \varepsilon \left( \frac{1}{3} a^2 \omega^2 u'^2(\tau) - 1 \right) \omega u'(\tau) + u(\tau) = 0
\]  

4
and
\[ u(0) = 1, \quad u'(0) = 0. \]  
(9)

Since the limit cycle represents a periodic motion, so we suppose that the initial approximation to the solution \( u(\tau) \) to (5) can be taken as
\[ u_0(\tau) = \cos \tau \]

Let, \( \omega_0 \) and \( a_0 \) respectively denote the initial approximations of the frequency \( \omega \) and the amplitude \( a \).

We consider a linear operator
\[ L[\phi(\tau, p)] = \omega_0^2 \left[ \frac{\partial^2 \phi(\tau, p)}{\partial \tau^2} + \phi(\tau, p) \right] \]  
(10)
so that for the coefficients \( C_1 \) and \( C_2 \)
\[ L(C_1 \sin \tau + C_2 \cos \tau) = 0 \]  
(11)
We further consider a nonlinear operator
\[ N[\phi(\tau, p), \Omega(p), A(p)] = \Omega(p)^2 \left[ \frac{\partial^2 \phi(\tau, p)}{\partial \tau^2} + \phi(\tau, p) \right] - \Omega(p) \left( \frac{\partial \phi(\tau, p)}{\partial \tau} \right)^3 + \phi(\tau, p). \]  
(12)

Next, we construct a homotopy as
\[ H[\phi(\tau, p), h, p] = (1 - p) L[\phi(\tau, p) - u_0(\tau)] - h p N[\phi(\tau, p), \Omega(p), A(p)] \]  
(13)
where \( p \in [0, 1] \) is the embedding parameter and \( h \) a non-zero auxiliary (control) parameter used to improve the convergence of series expansions. Setting \( H[\phi(\tau, p), h, p] = 0 \) we obtain zero-th order deformation equation
\[ (1 - p) L[\phi(\tau, p) - u_0(\tau)] - h p N[\phi(\tau, p), \Omega(p), A(p)] = 0 \]  
(14)
subject to the initial conditions
\[ \phi(0, p) = 1, \quad \frac{\partial \phi(\tau, p)}{\partial \tau} \bigg|_{\tau=0} = 0. \]  
(15)
Clearly, as \( p \) increases from \( p = 0 \) to \( p = 1 \), (14) changes from \( L[\phi(\tau, p) - u_0(\tau)] = 0 \) to \( N[\phi(\tau, p), \Omega(p), A(p)] = 0 \) and as a consequence \( \phi(\tau, 0) = u_0(\tau) = \cos \tau \) to the exact solution \( \phi(\tau, 1) = u(\tau) \), so does \( \Omega(p) \) from \( \omega_0 \) to exact frequency \( \omega \) and \( A(p) \) from \( a_0 \) to the exact amplitude \( a \). It can be shown that assuming \( \phi(\tau, p), \Omega(p), A(p) \) analytic in \( p \in [0, 1] \) so that
\[ u_k(\tau) = \frac{1}{k!} \frac{\partial^k}{\partial p^k} \phi(\tau, p) \bigg|_{p=0}, \quad \omega_k = \frac{1}{k!} \frac{\partial^k}{\partial p^k} \Omega(p) \bigg|_{p=0}, \quad a_k = \frac{1}{k!} \frac{\partial^k}{\partial p^k} A(p) \bigg|_{p=0} \]  
(16)
we have,
\[ u(\tau) = \sum_{k=0}^{\infty} u_k(\tau) \]  
(17)
\[ \omega = \sum_{k=0}^{\infty} \omega_k \]  
(18)
\[ a = \sum_{k=0}^{\infty} a_k \]  
(19)
where \( u_k(\tau) \) are solutions of the \( k \)-th order deformation equation

\[
\mathcal{L} [u_k(\tau) - \chi_k u_{k-1}(\tau)] = h \, R_k(\tau)
\]

subject to the initial conditions

\[
u_k(0) = 0, \quad u_k'(0) = 0 \quad (21)
\]

in which

\[
R_k(\tau) = \frac{1}{(k-1)!} \left. \frac{\partial^{k-1}}{\partial p^{k-1}} \mathcal{N} [\phi(\tau, p), \Omega(p), A(p)] \right|_{p=0}
\]

\[
= \sum_{n=0}^{k-1} u''_{k-1-n}(\tau) \sum_{j=0}^{n} \omega_{j} \omega_{n-j} + u_{k-1}(\tau)
\]

\[
+ \frac{\varepsilon}{3} \sum_{n=0}^{k-1} \sum_{i=0}^{n} \left( \sum_{r=0}^{i} a_r a_{i-r} \right) \times \left( \sum_{s=0}^{n-i-s} \sum_{h=0}^{n-i-s} \omega_{s} \omega_{h} \omega_{n-i-s-h} \right)
\]

\[
\times \left( \sum_{j=0}^{k-1-n} u'_{j}(\tau) \sum_{m=0}^{k-1-n-j} u'_{m}(\tau) u'_{k-1-n-j-m}(\tau) \right) - \varepsilon \sum_{n=0}^{k-1} \omega_n u'_{k-1-n}(\tau)
\]

and

\[
\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 1. \end{cases}
\]

To ensure that the solution to the \( k \)-th order deformation equation (20) do not contain the secular terms \( \tau \sin \tau \) and \( \tau \cos \tau \) the coefficients of \( \sin \tau \) and \( \cos \tau \) in the expressions of \( R_k \) in (22) must vanish giving successive values of \( \omega_k \) and \( a_k \).

The linear equation \( \mathcal{L}(\phi(\tau, p)) = 0 \) represents a simple harmonic motion with frequency 1. So, we choose the initial guess of \( \omega \) as \( \omega_0 = 1 \). Again, by perturbation method [1] we find \( a \to 2 \) as \( \varepsilon \to 0 \). So, we choose the initial guess of \( a \) as \( a_0 = 2 \). Solving the differential equations given by (14), (15), (20), (21) and avoiding the generation of secular terms in each iteration we obtain

\[
u_1(\tau) = -\frac{1}{24} h \varepsilon \sin 3\tau + \frac{1}{8} h \varepsilon \sin \tau, \quad \omega_1 = -\frac{1}{16} h \varepsilon^2, \quad a_1 = \frac{1}{8} h \varepsilon^2
\]

\[
u_2(\tau) = \left( \frac{1}{384} h^2 \varepsilon^3 - \frac{1}{24} h^2 \varepsilon - \frac{1}{24} h \varepsilon \right) \sin 3\tau - \frac{1}{64} h^2 \varepsilon^2 \cos 3\tau
\]

\[
+ \frac{1}{64} h^2 \varepsilon^2 \cos \tau + \left( \frac{1}{8} h^2 \varepsilon - \frac{1}{128} h^2 \varepsilon^3 + \frac{1}{8} h \varepsilon \right) \sin \tau
\]

so that

\[
R_1 = \frac{1}{3} \varepsilon \sin 3\tau
\]

\[
R_2 = \frac{1}{24} \left[ 3 h \varepsilon^2 \cos 3\tau + \left( 8 h \varepsilon - \frac{1}{2} h \varepsilon^3 \right) \sin 3\tau \right]
\]

Computing \( R_k \) successively, we can find the successive expressions of \( u_k(\tau) \), \( \omega_k \) and \( a_k \). The first order approximation to the amplitude in (19) is

\[
a \approx a_0 + a_1 = 2 + \frac{1}{8} h \varepsilon^2 = a_E(\varepsilon) \quad \text{(say)}.
\]

(24)
Figure 1: The exact amplitude of Rayleigh Equation (by solid line) and its approximation $a_E(\varepsilon)$ given by (27) (by bold points) for $0 < \varepsilon \leq 50$.

The above first order expression for the amplitude involves as yet arbitrary control parameter $h$. Lopez et al. [4] proposed specific $\varepsilon$-dependent expressions for $h$ to obtain an efficient formula for the VdP limit cycle amplitude. They made the proposal that $h$, besides being continuous, must also vanish in the limits of $\varepsilon \to 0$ and $\varepsilon \to \infty$ to reproduce the zeroth order perturbative solutions. In our application of HAM for the Rayleigh limit cycle amplitude, we have chosen a different set of base functions and so can weaken the condition considerably, both on the continuity and the asymptotic limit $\varepsilon \to \infty$. From careful inspections of the graph of the exact amplitude (Fig.1), it turns out that an appropriate ansatz for the control parameter $h$ is given by

$$h = \frac{1}{0.5 + \varepsilon b(\varepsilon)}$$

(25)

where, $b(\varepsilon)$ is taken as the step function in the domain $0 < \varepsilon \leq 50$ as follows

| $\varepsilon$ | $0 < \varepsilon \leq 4$ | $4 < \varepsilon \leq 5$ | $5 < \varepsilon \leq 7$ | $7 < \varepsilon \leq 8$ | $8 < \varepsilon \leq 9$ |
|--------------|----------------|----------------|----------------|----------------|----------------|
| $b(\varepsilon)$ | 0.162 | 0.165 | 0.168 | 0.171 | 0.174 |
| $\varepsilon$ | $9 < \varepsilon \leq 11$ | $11 < \varepsilon \leq 15$ | $15 < \varepsilon \leq 20$ | $20 < \varepsilon \leq 30$ | $30 < \varepsilon \leq 50$ |
| $b(\varepsilon)$ | 0.176 | 0.179 | 0.181 | 0.183 | 0.185 |

With this particular form of $h$, we are able to find an analytic approximation $a_E(\varepsilon)$ to the numerically computed exact value $a = a(\varepsilon)$ in the domain $0 < \varepsilon \leq 50$ with maximum relative percentage error $\left| \frac{a_E(\varepsilon) - a(\varepsilon)}{a(\varepsilon)} \times 100 \right|$ less than 1%. Obviously, better accuracy fit can be obtained by considering finer subdivisions in the definition of $b(\varepsilon)$. We remark that a piece-wise continuous $\varepsilon$ dependence of $h$ as above is admissible in the framework of HAM.

Since the exact graph of $a(\varepsilon)$ is almost a straight line for sufficiently large $\varepsilon$ ($7 < \varepsilon \leq 50$), we can reduce the number of steps to 4 only. Let us choose

$$h = \frac{8m}{\varepsilon} - \frac{56m}{\varepsilon^2} + \frac{8c}{\varepsilon^2} - \frac{16}{\varepsilon^2}, \quad 7 < \varepsilon \leq 50$$

(26)

so that (27) becomes

$$a_E(\varepsilon) = \begin{cases} 
2 + \frac{1}{8} \left( \frac{1}{0.5+0.102\varepsilon} \right) \varepsilon^2, & 0 < \varepsilon \leq 4 \\
2 + \frac{1}{8} \left( \frac{1}{0.5+0.165\varepsilon} \right) \varepsilon^2, & 4 < \varepsilon \leq 5 \\
2 + \frac{1}{8} \left( \frac{1}{0.5+0.168\varepsilon} \right) \varepsilon^2, & 5 < \varepsilon \leq 7 \\
m(\varepsilon - 7) + c, & 7 < \varepsilon \leq 50 
\end{cases}$$

(27)
where \( m \) and \( c \) are computed from the exact solution as
\[
m = \frac{a(50) - a(7)}{50 - 7} = 0.657692 \quad \text{and} \quad c = a(7) = 5.63108
\]
keeping the maximum relative percentage error \( \left| \frac{a_E(\varepsilon) - a(\varepsilon)}{a(\varepsilon)} \right| \times 100 \) less than 1%. The plot of \( a_E(\varepsilon) \) given by (27) is shown by bold points in Figure 1 (explicit discontinuities of \( h \) at \( \varepsilon = 4, 5 \) and 7 are not visible at the resolution of the plotted figure). As remarked above, Lopez et. al. [4] proposed that a reasonable property for \( h \) would be to vanish in the limits as \( \varepsilon \to 0 \) and \( \varepsilon \to \infty \). However, from (25) and (26) we observe that a suitable approximation to the amplitude of Rayleigh equation can be obtained even if \( h \) do not follow this property. The graph of \( h(\varepsilon) \) is given in Figure 2 for \( 0 < \varepsilon \leq 50 \) (discontinuity in \( h \) is not visible at the level of resolution in the figure).

To summarize, one can obtain more accurate approximate formula by suitable choices of the control parameter \( h(\varepsilon) \) up to any desired level of accuracy. We also note that a piecewise continuous control parameter \( h \) enables us to obtain good approximation by solving only the first order deformation equation. However, the first order HAM estimated amplitude \( a(\varepsilon) \) is \( O(\varepsilon^2) \).

We report the estimation of the amplitude of the limit cycle for the Rayleigh and VdP equations by the improved RG method in Subsection 4.1. We do not undertake the computation of the VdP amplitude by HAM separately, as that was already reported by Lopez et al [4].

### 3 Computation of Amplitude by RG Method

The Renormalization Group method (RGM) introduced by Chen, Goldenfeld and Oono (CGO) [5, 6] gives a unified formal approach to derive asymptotic expansions for the solutions of a large class of nonlinear ODEs. The RG method is used in solid state physics, quantum field theory and other areas of physics. One advantage of RGM is that it starts from naive perturbation expansion of a problem and is expected to yield automatically the gauge functions such as fractional powers of \( \varepsilon \) and logarithmic terms in \( \varepsilon \) in the renormalized expansion. One does not require to have any prior knowledge to prescribe these unexpected gauge functions in an ad hoc manner. DeVille et. al. [7] have introduced an algorithmic approach for RGM which we adopt for the following application. As it will transpire the RGM appears to be deficient in estimating amplitude of a periodic orbit because of the absence of any free control parameter. In a latter section we have improved this RGM to incorporate a control parameter similar to HAM and derive efficient
estimations of amplitudes of both the Rayleigh and VdP equations. However, before the introduction of the improved RG method (IRGM), we first discuss the conventional RG method, given by DeVille et. al. and use it to obtain amplitude and phase equations for the Rayleigh equation (1). These equations are already obtained in [6, 7] to the order method, given by DeVille et. al. and use it to obtain amplitude and phase equations for conventional estimations of amplitudes of both the Rayleigh and VdP equations. However, before the introduction of the improved RG method (IRGM), we first discuss the conventional RG method, given by DeVille et. al. and use it to obtain amplitude and phase equations for the Rayleigh equation (1). These equations are already obtained in [6, 7] to the order method.

Substituting the naive expansion

\[ y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \varepsilon^3 y_3(t) + \cdots \]

in (1), we find at each order

\[ O(1) : \ddot{y}_0 + y_0 = 0 \]
\[ O(\varepsilon) : \ddot{y}_1 + y_1 = \dot{y}_0 - \frac{1}{3} \dot{y}_0^3 \]
\[ O(\varepsilon^2) : \ddot{y}_2 + y_2 = \dot{y}_1 - \dot{y}_0^2 \dot{y}_1 \]
\[ O(\varepsilon^3) : \ddot{y}_3 + y_3 = \dot{y}_2 - \dot{y}_0 \dot{y}_2 - \dot{y}_1 \dot{y}_0 \]

The solutions are

\[ y_0(t) = A e^{i(t-t_0)} + c.c. \]
\[ y_1(t) = \frac{1}{2} i A^3 e^{i(t-t_0)} + \frac{1}{2} A (1 - A A^*) (t - t_0) e^{i(t-t_0)} - \frac{1}{24} A^3 e^{3i(t-t_0)} + c.c \]
\[ y_2(t) = \left( \frac{1}{32} A^3 - \frac{3}{64} A^4 A^* \right) e^{i(t-t_0)} + \left( -\frac{1}{24} i A^4 A^* + \frac{1}{16} A^3 (A^*)^2 + \frac{1}{48} i A^3 + \frac{1}{48} i A^2 (A^*)^3 - \frac{1}{8} i A \right) (t - t_0) e^{i(t-t_0)} + \left( \frac{3}{8} A^3 (A^*)^2 - \frac{1}{2} A^2 A^* + \frac{1}{8} A \right) (t - t_0)^2 e^{i(t-t_0)} + \left( \frac{3}{64} A^4 A^* - \frac{1}{32} A^3 + \frac{1}{192} A^5 \right) e^{3i(t-t_0)} - \frac{1}{16} i A^3 (1 - A A^*) (t - t_0) e^{3i(t-t_0)} - \frac{1}{192} A^5 e^{5i(t-t_0)} + c.c. \]
\[ y_3(t) = \left( -\frac{1}{384} i A^6 A^* + \frac{37}{1536} i A^5 (A^*)^2 + \frac{1}{2304} i A^5 + \frac{1}{512} i A^4 (A^*)^3 - \frac{7}{256} i A^4 A^* - \frac{1}{128} i A^3 \right) e^{i(t-t_0)} + \left( \frac{1}{1152} i A^6 A^* + \frac{5}{128} A^5 (A^*)^2 - \frac{19}{1152} A^4 (A^*)^3 + \frac{11}{384} A^3 (A^*)^4 \right) (t - t_0) e^{i(t-t_0)} + \left( \frac{3}{8} i A^5 (A^*)^2 - \frac{3}{32} i A^4 (A^*)^3 - \frac{1}{21} A^4 A^* - \frac{1}{32} i A^3 (A^*)^4 \right) (t - t_0)^2 e^{i(t-t_0)} + \left( \frac{3}{32} i A^3 (A^*)^2 + \frac{1}{192} i A^3 + \frac{1}{16} i A^2 A^* + \frac{1}{48} i A^2 (A^*)^3 - \frac{1}{16} i A \right) (t - t_0)^3 e^{i(t-t_0)} + \left( \frac{1}{4608} i A^7 + \frac{7}{512} i A^6 A^* - \frac{37}{1536} i A^5 (A^*)^2 - \frac{1}{128} i A^5 - \frac{1}{512} i A^4 (A^*)^3 \right) e^{3i(t-t_0)} + \left( \frac{1}{96} A^6 A^* + \frac{7}{64} A^5 (A^*)^2 + \frac{1}{128} A^5 + \frac{1}{384} A^4 (A^*)^3 \right) (t - t_0) e^{3i(t-t_0)} + \left( -\frac{1}{64} i A^5 (A^*)^2 + \frac{1}{8} i A^4 A^* - \frac{3}{64} i A^3 \right) (t - t_0)^2 e^{3i(t-t_0)} + \left( \frac{17}{2304} i A^5 - \frac{17}{1536} i A^6 A^* - \frac{5}{4608} i A^7 \right) e^{5i(t-t_0)} + \left( \frac{5}{384} A^6 A^* - \frac{5}{384} A^5 \right) (t - t_0) e^{5i(t-t_0)} \]
\[ + \frac{1}{1152} i A^7 e^{7i(t-t_0)} + c.c. \]

We choose the homogeneous parts to the solutions \(y_1, y_2\) and \(y_3\) in such a manner that the solutions vanish at the initial time \(t_0\), i.e. \(y_1(t_0) = y_2(t_0) = y_3(t_0) = 0\). Next, we renormalize the integration constant \(A\) and create a new renormalized quantity \(A\) as

\[ A = A + a_1 \varepsilon + a_2 \varepsilon^2 + a_3 \varepsilon^3 + O(\varepsilon^4) \]

where the coefficients \(a_1, a_2, a_3, \ldots\) are chosen to absorb the homogeneous parts of the solutions \(y_1, y_2, \ldots\). Choosing

\[ a_1 = -\frac{i}{24} A^3, \quad a_2 = -\frac{A^3}{32} \left( 1 - \frac{3}{2} A A^* + \frac{1}{6} A^2 \right), \]

\[ a_3 = \frac{1}{1152} i A^7 - \frac{17}{1536} i A^6 A^* + \frac{17}{2304} i A^5 - \frac{37}{1536} i A^5 (A^*)^2 + \frac{7}{256} i A^4 A^* + \frac{1}{128} i A^3 \]

we obtain

\[ y_0(t) = A e^{i(t-t_0)} + c.c. \]

\[ y_1(t) = \left( \frac{1}{2} A (1 - AA^*) (t - t_0) e^{i(t-t_0)} - \frac{1}{24} i A^3 e^{3i(t-t_0)} \right) + c.c. \]

\[ y_2(t) = \left( \frac{1}{16} i A^3 (A^*)^2 - \frac{5}{2} i A \right) (t - t_0) e^{i(t-t_0)} + \frac{1}{8} A (AA^* - 1) (3 AA^* - 1) (t - t_0)^2 e^{i(t-t_0)} \]

\[ + \left( \frac{1}{32} A^4 A^* - \frac{9}{16} A^3 \right) e^{3i(t-t_0)} + \frac{1}{32} i A^3 (AA^* - 1) (t - t_0) e^{3i(t-t_0)} \]

\[ - \frac{1}{192} A^5 e^{5i(t-t_0)} + c.c. \]

\[ y_3(t) = \left( -\frac{128}{128} + \frac{1}{64} A^3 (A^*)^2 \right) (t - t_0) e^{i(t-t_0)} \]

\[ + \left( -\frac{3}{32} i A^4 (A^*)^3 + \frac{9}{16} A^3 (A^*)^2 + \frac{1}{16} i A^2 A^* - \frac{1}{16} i A \right) (t - t_0)^2 e^{i(t-t_0)} \]

\[ + \left( -\frac{5}{16} A^4 (A^*)^3 + \frac{9}{16} A^3 (A^*)^2 - \frac{13}{32} A^2 A^* + \frac{1}{32} i A \right) (t - t_0)^3 e^{i(t-t_0)} \]

\[ + \left( -\frac{37}{1536} i A^5 (A^*)^2 + \frac{1}{128} i A^3 + \frac{7}{256} i A^4 A^* \right) e^{3i(t-t_0)} \]

\[ + \left( -\frac{7}{64} A^5 (A^*)^2 + \frac{2a}{1536} i A^4 A^* - \frac{1}{16} A^3 \right) (t - t_0) e^{3i(t-t_0)} \]

\[ + \left( -\frac{5}{64} i A^5 (A^*)^2 + \frac{1}{8} i A^4 A^* - \frac{3}{64} i A^3 \right) (t - t_0)^2 e^{3i(t-t_0)} \]

\[ + \left( \frac{17}{2304} i A^5 - \frac{17}{1536} i A^6 A^* \right) e^{5i(t-t_0)} + \left( \frac{5}{384} A^6 A^* - \frac{5}{384} A^5 \right) (t - t_0) e^{5i(t-t_0)} \]

\[ + \frac{1}{1152} i A^7 e^{7i(t-t_0)} + c.c. \]

**Remark 1** DeVille [7] have obtained same result correct up to \(O(\varepsilon^3)\). However, their computed expression of \(a_2\) is not correct. We have made the correction in the expression of \(a_2\).

We observe that each of \(y_1(t), y_2(t), y_3(t)\) contains secular terms. As a consequence the solution

\[ y(t) = y_0(t) + y_1(t) \varepsilon + y_2(t) \varepsilon^2 + y_3(t) \varepsilon^3 + O(\varepsilon^4) \]

becomes divergent as \(t \to \infty\). To regularize the perturbation series using RGM an arbitrary time \(\tau\) is introduced and \(t - t_0\) is split as \((t - \tau) + (\tau - t_0)\). The terms containing \(\tau - t_0\) is observed in the renormalized counterpart \(A\) of the constant of integration \(A\). Since the final solution should not depend upon the choice of the arbitrary time \(\tau\), so

\[ \left. \frac{\partial y}{\partial \tau} \right|_{\tau=t} = 0 \quad (28) \]
for any $t$. However, DeVille et. al. [7] have simplified this condition and proposed an equivalent condition as

$$\left. \frac{\partial y}{\partial t_0} \right|_{t_0=t} = 0 \quad (29)$$

We note that renormalized counterpart $A$ is no longer a constant of motion in RGM. The RG condition (29) is developed in such a manner that one need to differentiate the terms containing $e^{i(t-t_0)}$, $e^{-i(t-t_0)}$, $(t-t_0)e^{i(t-t_0)}$ and $(t-t_0)e^{-i(t-t_0)}$ and thereafter substituting $t_0 = t$ the resultant expression is equated to zero. The other terms related to higher harmonics are not involved in RG condition. Simplifying RG condition (29) we get

$$\partial \frac{A}{A} \partial t = -\frac{1}{2} A (\AA^*) - 1 \frac{1}{8} i A \left( 1 - \frac{1}{2} A^2 (A^*)^2 \right) \varepsilon^2 - \frac{1}{64} A^3 (A^*)^2 \left( \frac{13}{2} A A^* - 11 \right) \varepsilon^3 = 0$$

to the order $O(\varepsilon^4)$. Taking $A = \frac{R}{2} e^{i\theta}$ we obtain corresponding amplitude and phase flow equations to the order $O(\varepsilon^4)$ as

$$\frac{dR}{dt} = \frac{1}{2} R \left( 1 - \frac{R^2}{4} \right) \varepsilon + \frac{1}{1024} R^5 \left( 11 - \frac{13}{8} R^2 \right) \varepsilon^3 + O(\varepsilon^4) \quad (30)$$

$$\frac{d\theta}{dt} = 1 - \frac{1}{8} \left( 1 - \frac{R^4}{32} \right) \varepsilon^2 + O(\varepsilon^4) \quad (31)$$

To the authors’ knowledge these higher order flow equations are reported for the first time in the literature. We remark here that by separating the explicit $t$ dependence of $\theta$ by writing $A = \frac{R}{2} e^{i(t+\theta)}$, one would retrieve the $O(\varepsilon^3)$ order phase flow equation of [6]. We remark also that above flow equations also match exactly with flow equations of the Van der Pol equation [8]. Although not done explicitly, we expect that the $O(\varepsilon^4)$ VdP flow equations would also have the equivalent forms.

Solving the amplitude equation (30) by numerical method and taking the limit as $t \to \infty$ so that for a fixed value of $\varepsilon$ we have $R \to a_{RG}(\varepsilon)$, the approximation of the amplitude of limit cycle of Rayleigh equation (1) by RGM, we obtain Figure 3 representing $\varepsilon$ dependence of the amplitude $a_{RG}$ by solid lines.

![Graph of $a_{RG}(\varepsilon)$](a)

![Graph of $a_{RG}(\varepsilon)$](b)

Figure 3: Graph of $a_{RG}(\varepsilon)$ (by solid lines) correct up to $O(\varepsilon^4)$ and compared with exact graph of $a(\varepsilon)$ (by dotted lines) for $0 < \varepsilon \leq 5$ in (a) and for $0 < \varepsilon \leq 20$ in (b).

Thus we observe that the RG flow equation to the order $O(\varepsilon^4)$ for the amplitude does not produce good approximation to the exact solution for moderate values of $\varepsilon$. 

11
4 Improved RG Method: Nonlinear Time

In RGM an arbitrary time \( \tau \) is introduced in between current time \( t \) and the initial time \( t_0 \) so that \( t - t_0 = (t - \tau) + (\tau - t_0) \) in order to remove the divergent terms in the naive perturbation expansion for the solution of the given differential equation. The solution is renormalized by suitable choice of the constants of integration to remove the terms containing \( (\tau - t_0) \) and keeping the terms having \( (t - \tau) \). Since the solution should be independent of the arbitrary time \( \tau \), the RG condition

\[
\left. \frac{\partial y}{\partial \tau} \right|_{\tau=t} = 0
\]

is applied to the renormalized solution. However, in the previous section we have seen that the method fails to produce good approximations to the exact solution for \( \varepsilon \sim O(1) \). Our target is not only to remove the divergent terms in the solution but also to introduce some control parameter \( h(\varepsilon) \) which can control the RG solution in such a manner that this solution ultimately converges to the exact solution. Moreover, our another goal is to achieve this accuracy by merely solving the differential equation to a minimal order of the expansion parameter, viz., up to \( O(\varepsilon^2) \) or less.

Since the basic idea is to split the time difference \( t - t_0 \) by introduction of an arbitrary time, so we can write \( t - t_0 = \left( t - \frac{\tau}{\varepsilon} \right) + \left( \frac{\tau}{\varepsilon} - t_0 \right) \). From now on let us assume that \( 0 \ll \varepsilon \ll 1 \). The case \( \varepsilon > 1 \) will be commented upon later. The constants of integration can be renormalized in order to remove the terms containing \( \left( \frac{\tau}{\varepsilon} - t_0 \right) \) from the solution keeping the terms containing \( \left( t - \frac{\tau}{\varepsilon} \right) \). Finally analogous to the classical RG method we put \( t = \frac{\tau}{\varepsilon} \), i.e. \( \tau = \varepsilon t \), in

\[
\left. \frac{\partial y}{\partial \tau} \right|_{\tau=t} = 0
\]

(32)

giving rise to an improved form of the RG flow equation to remove secular terms involving \( \left( t - \frac{\tau}{\varepsilon} \right) \). So far the improved method does not produce any qualitative new result compared to the RGM and so we must get the same phase and amplitude equation as deduced in Section 3.

We next proceed one step further. As stated already in the Introduction, we now exploit the possibility of extending the original linear time \( t \) dependence of \( \tau \) viz., \( \tau = t \) of RGM in removing the explicit divergences by a nonlinear dependence \( \tau = \varepsilon t \) along with the additional condition that \( \tau \to \tau_0(\varepsilon) \) as \( t \to \infty \) following the scales \( t \sim \varepsilon^{-n} \), \( n = 1, 2, \ldots \).

It follows that for a given nonlinear differential system, such a nonlinear time dependence always exists and nontrivial, provided one invokes a duality principle transferring nonlinear influences from the far asymptotic region into the finite observable sector in a cooperative manner \[14, 15\].

In fact, as the linear time \( t \to \infty \) following the above hierarchy of scales, there exists \( \tilde{t}_n \) such that \( 1 \ll (\varepsilon t)^n < \varepsilon^{-n} < \tilde{t}_n \) and satisfying the inversion law \( \tilde{t}_n/\varepsilon^{-n} \propto \varepsilon^{-n}/(\varepsilon t)^n \).

This inversion law makes a room for transfer of effective influences from nonobservable sector \( t > \varepsilon^{-n} \) to the observable sector \( t < \varepsilon^{-n} \) bypassing the dynamically generated singular points denoted by the scales \( \varepsilon^{-n} \). Let \( \tilde{t}(t) = \lim_{n \to \infty} (\tilde{t}_n)^{1/n} \) so that \( \varepsilon t < \varepsilon^{-1} < \tilde{t}(t) \) and \( \tilde{t}/\varepsilon^{-1} \propto \varepsilon^{-1}/(\varepsilon t) \). Define \( h_0(\varepsilon) = \lim_{n \to \infty} \log_{\varepsilon^{-n}} \tilde{t}_n/\varepsilon^n \). Then it follows that

\[
\tau = \lim_{n \to \infty} \varepsilon t = \varepsilon^{-h_{\text{RG}}(\varepsilon)}, \quad \varepsilon < 1
\]

(33)
as \( t \to \varepsilon^{-1} \), where we have absorbed a proportionality constant in the definition of \( \tau \). Moreover, \( h_{\text{RG}}(\varepsilon) = 1 - h_0(\varepsilon) \). The scaling exponent \( h_0(\varepsilon) \) here encodes the effective
cooperative influence of far asymptotic sector \( t > \varepsilon^{-n} \) into the observable sector \( 1 < t < \varepsilon^{-n} \) by the inversion mediated duality principle. Notice that, in the absence of the said duality the linear time \( t \) can in principle attain the scale \( \varepsilon^{-1} \) (say), and as a consequence \( h_0 = 0 \), retrieving the ordinary scaling of \( \tau = \varepsilon^{-1} t \) as \( t \sim \varepsilon^{-2} \). This also establishes that the scaling exponent \( h_0 (\varepsilon) \) is well defined and can exist nontrivially i.e. \( h_0 \sim O(1) \) in a nonlinear problem. As a consequence, the RG control parameter \( h_{RG} \) can be of both the signs, with relatively small numerical value in fully developed nonlinear systems \( \varepsilon \gg 1 \), but with possible \( O(1) \) variations for \( \varepsilon \sim O(1) \) or less.

The above construction actually tells somewhat more. Corresponding to the first generation scales \( \varepsilon^{-n} \), one can, in fact, have the second generation nonlinear scales \( \tau_m = \varepsilon^m t = \varepsilon^{h_{RG}(\varepsilon)} \) with \( h_{RG} = h_{RG} \). The nonlinear time \( \tau \) now stands for these hierarchy of scales \( \{ \tau_m \} \). Consequently, as the linear time \( t \) approaches \( \infty \) through the first generation scales, the slowly varying nonlinear time \( \tau \) approaches either to \( \infty \) or 0 at slower and slower rates as represented by the numerically small RG scaling exponents \( h_m(\varepsilon) \), each of which remains almost constant over longer and longer intervals of \( \varepsilon^{-1} \) (as \( \varepsilon^{-1} \to \infty \)).

Let us remark that for \( \varepsilon > 1 \), we consider instead the first generation scales as \( \varepsilon^n \), and the duality is invoked for variables satisfying \( t/\varepsilon < \varepsilon < \tilde{t}(t) \) so that the asymptotic scaling variable is derived as \( \tau = \varepsilon^{h_{RG}(\varepsilon)} \), \( \varepsilon > 1 \) where \( h_{RG} = 1 - h_0 \).

It now follows, from the above general remarks on the behaviour of \( h_{RG} \), that the nonlinear time \( \tau \) actually approaches \( 0 \) or \( \infty \) as \( \tau \sim (\log \varepsilon)^{-\alpha} \) or \( \tau \sim (\log \varepsilon)^{\alpha} \), \( \alpha > 0 \) respectively as \( \varepsilon \to \infty \). However, one must have \( \tau = \varepsilon^{-h_{RG}(\varepsilon)} \to \infty \) as \( \varepsilon \to 0 \). An example of the asymptotic behaviours of \( h_{RG} \) is given by \( \tau_m = \varepsilon^{\pm \alpha m} \log \varepsilon \) for \( \varepsilon \to 0 \), which one expects to verify explicitly in evaluation of asymptotic quantities, such as amplitude of a period cycle, in a nonlinear system.

In the IRGM, we exploit this nontrivial scaling information to rewrite the asymptotic RG flow equations in the limit \( t \to \infty \)

\[
\frac{da}{d\tau} = \frac{1}{2} a \left( 1 - \frac{a^2}{4} \right) \quad (34)
\]

\[
\varepsilon \frac{d\psi}{d\tau} = 1 - \frac{1}{8} \left( 1 - \frac{a^4}{32} \right) \varepsilon^2 \quad (35)
\]

for the amplitude \( a = a(\varepsilon) \) and the phase \( \psi = \psi(\varepsilon) \) of the limit cycle of both the Rayleigh and Van der Pol equations. These equations are exact and encode the full non-perturbative informations of the relevant dynamical quantities viz, \( a \) and \( \psi \). The conventional perturbative RG flow equations in the linear time \( t \) is now extended into the non-perturbative flow equations in the nontrivial scaling variable \( \tau = \varepsilon^{h_{RG}(\varepsilon)} \) involving the nonlinearity parameter \( \varepsilon > 1 \). The RG estimated approximate formulae for the amplitude \( a(\varepsilon) \) for the Rayleigh and Van der Pol limit cycles are obtained from the equation \( (34) \) when appropriate boundary condition, derived either from exact computations or from perturbative analysis, is used for \( \varepsilon \to \infty \) or at any suitable finite value.

### 4.1 Approximate Formula for Amplitude

We shall now use the above asymptotic amplitude flow equation \( (34) \) to find analytic approximations of the amplitudes of the limit cycle for both the Rayleigh and Van der Pol equations.

By a direct integration, one obtains from \( (34) \)

\[
\ln(a^2 - 4) - 2 \ln a = -\varepsilon^{h_{RG}} - 0.87953 \quad (36)
\]

as the Rayleigh limit cycle amplitude where we use the boundary condition the value \( a = 2.17271 \) for \( \varepsilon = 1 \). It follows immediately that for suitable choices of the control
parameter $h_{RG}$ one can achieve efficient matching for the estimated amplitude $a_E(\varepsilon)$. For example, using the HAM generated approximate formula (27) for $a_E(\varepsilon)$, we can determine the control parameter $h_{RG}(\varepsilon)$ by the formula

$$h_{RG} = \frac{1}{\ln \varepsilon} \ln \left\{ \left| \ln \left( \frac{a^2}{a^2 - 4} \right) - 0.87953 \right| \right\}$$

(37)

In Figure 4, we display the typical piece-wise smooth form of $h_{RG}(\varepsilon)$ given by (37) for $0 < \varepsilon \leq 5$ in (a) and for $0 < \varepsilon \leq 50$ in (b).

Figure 4: The graph of $h_{RG}(\varepsilon)$ used for approximation of the amplitude of the Rayleigh equation (1) by HAM given by (37) for $0 < \varepsilon \leq 5$ in (a) and for $0 < \varepsilon \leq 50$ in (b).

the Rayleigh limit cycle amplitude that would reproduce the HAM generated amplitude with relative error less that 1%. Clearly, the graph reveals variability of $h_{RG}$ for moderate values of $\varepsilon$, but the variability dies out fast for larger values $\varepsilon$, as expected.

We recall that the corresponding graph of the exact computed values of VdP amplitude $a(\varepsilon)$, on the other hand, has a hump like shape with a maximum roughly at $\varepsilon \approx 2.0235$ and having the asymptotic limits 2 as $\varepsilon \to 0$ and $\infty$. Lopez et al [4] obtained HAM generated approximate formula for the VdP amplitude with relative error less than 0.05% at the order $O(\varepsilon^4)$. It is interesting to note that the RG generated formula (36) can reproduce the exact computed values of the VdP amplitude with error less than 0.05% directly from only the first order RG flow equation. To achieve this goal we first intuitively guess an estimated piecewise smooth formula for the estimated amplitude $a_E$ by

$$a_E(\varepsilon) = \begin{cases} 
1.998 + \frac{0.015}{8.121 e^{-2.139 \varepsilon} + 0.512 e^{0.043 \varepsilon}} & 0 < \varepsilon < 3 \\
2.0025 + \frac{0.031}{0.5 e^{-0.033(\varepsilon-2.183)} + 1.869 e^{0.087(\varepsilon-6.376)}} & 3 \leq \varepsilon \leq 50
\end{cases}$$

(38)

keeping the maximum relative percentage error $\left| \frac{a_E(\varepsilon) - a(\varepsilon)}{a(\varepsilon)} \times 100 \right|$ less than 0.05%.

This shows that the approximation is quite accurate. The graph of $a_E(\varepsilon)$ is compared with the exact values in Figure 5.

Using this efficient formula for the VdP amplitude, we then obtain the RG flow equation in the form

$$\ln (a^2 - 4) - 2 \ln a = -\varepsilon^{h_{RG}} - 4.08785$$

(39)

where we use the boundary condition $a = 2.0086$ for $\varepsilon = 1$ for the VdP amplitude. Inverting this equation, we finally obtain the corresponding RG control parameter

$$h_{RG} = \frac{1}{\ln \varepsilon} \ln \left\{ \left| \ln \left( \frac{a_E^2}{a_E^2 - 4} \right) - 4.08785 \right| \right\}$$

(40)
Figure 5: The exact amplitude of Van der Pol Equation (2) (by solid line) and its approximation \( a_E(\varepsilon) \) given by (38) (by bold points) for \( 0 < \varepsilon \leq 50 \).

Figure 6: The graph of \( h_{RG}(\varepsilon) \) used for approximation (40) of the amplitude of the Van der Pol equation (2) for \( 0 < \varepsilon \leq 4 \) in (a) and for \( 0 < \varepsilon \leq 50 \) in (b).

Figure 6 displays the piecewise smooth variation of \( h_{RG} \) with \( \varepsilon \). The rapid \( O(1) \) variation for moderate values of \( \varepsilon \) is evident in Fig.6(a). As expected, \( h_{RG} \) dies out fast for larger values of \( \varepsilon \). However, a change in sign is noticed here already for \( \varepsilon > 20 \) (Figure 6(b)). One expects many more such small scale sign variations as \( \varepsilon \to \infty \). This particular form of the control parameter \( h_{RG} \), in turn, would reproduce the VdP amplitude with relative error less than 0.05%. As this level of accuracy is achieved only at the order \( O(\varepsilon) \), the improved RGM may be considered to be more efficient and advantageous compared to the HAM.

Alternatively, the amplitude equation (33) can be inverted as

\[
a(\tau) = \frac{a(\tau_0)}{\sqrt{e^{-\tau} + \frac{a^2(\tau_0)}{4}(1-e^{-\tau})}}
\]

where \( \tau_0 \) is a sufficiently large value of \( \tau = \varepsilon h_{RG} \). For the VdP equation \( \tau \sim (\log \varepsilon)^\alpha \) and for the Rayleigh equation \( \tau \sim (\log \varepsilon)^{-\alpha} \) for \( \varepsilon \to \infty \) and \( \alpha > 0 \). By adjusting suitably the values of \( \alpha \) over appropriate intervals on \( \varepsilon \) one should be able to obtain efficient matching with the exact values of \( a(\varepsilon) \).

To summarize, the recipe for deriving approximate formula for limit cycle amplitude of a nonlinear system can be stated as follows: Determine the first order (perturbative) RG flow equation for amplitude in the nonlinear time \( \tau \). This will yield an explicit formula for amplitude \( a(\varepsilon) \) as a function of the nonlinearity parameter \( \varepsilon \) and the control parameter \( h_{RG} \). Efficient match with the exact amplitude can be achieved by suitable choice (guess)
of the control parameter $h_{RG}$ or $\alpha$. Alternatively, determine an efficient formula for $a(\varepsilon)$ by inspection or expert guess. Then determine the control parameter $h_{RG}$ by an inversion of the estimated amplitude $a_E(\varepsilon)$ as in equation (40) (and Fig.6).

5 Concluding Remarks

In this paper we have presented a comparative study of the homotopy analysis method and the Renormalization Group method. The approximate formulae for the amplitudes of the limit cycles of the Rayleigh and the Van der Pol systems are derived using both the methods and are compared with the exact results. It turns out that the higher order perturbative calculations based on the conventional Renormalization group method would fail to give efficient formula for the limit cycle amplitudes for these nonlinear oscillators. However, an improved version of the Renormalization group analysis exploiting a novel concept of nonlinear time is shown to yield efficient amplitude formulae for all values of $\varepsilon$. The improved RG method is found to be more efficient and simpler in comparison to the Homotopy analysis method at least at the present level of our analysis. More detailed analysis of the nonlinear time formalism in several other nonlinear systems are necessary for a more conclusive answer.

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