Abstract. We adapt a basis of Habiro’s for the even Kauffman bracket skein module of \( S^1 \times D^2 \) to define bases for the even and odd skein modules of \( S^1 \times D^2 \) relative to two points. We discuss genus-1 tangle embedding, and define an even and odd version of the previously defined Kauffman bracket ideal for genus-1 tangles. These even and odd Kauffman bracket ideals are obstructions to tangle embeddings. Using our even and odd bases for the relative skein modules, we show how to compute a finite list of generators for the even and odd Kauffman bracket ideals of a genus-1 tangle. We do this explicitly for three genus-1 tangles. We relate these ideals to determinants of closures of genus-1 tangles.

1. Introduction

Let \( M \subseteq S^3 \) be a compact, oriented 3-manifold with boundary. Then an \((M, 2n)\)-tangle is 1-manifold with \( 2n \) boundary components properly embedded in \( M \). We refer to \((S^1 \times D^2, 2)\)-tangles as genus-1 tangles. An \((M, 2n)\)-tangle \( \mathcal{T} \) embeds in a link \( L \subseteq S^3 \) if there exists a complementary 1-manifold \( \mathcal{T}' \) with \( 2n \) boundary components in \( S^3 \setminus \text{Int}(M) \) such that upon gluing \( \mathcal{T}' \) to \( \mathcal{T} \) along their boundaries, we obtain a link isotopic to \( L \). Such a link is called a closure of \( \mathcal{T} \). We refer to \( \mathcal{T}' \) as the complementary 1-manifold of the closure. Note that if \( \mathcal{T} \) is a genus-1 tangle, then \( \mathcal{T}' \) is also a genus-1 tangle. The focus of this paper is genus-1 tangle embedding.

In [A, A3], the first author defined the notion of even and odd closures for any genus-1 tangle \( \mathcal{G} \). This definition depends on a choice of longitude \( l \) in the solid torus containing \( \mathcal{G} \). If we choose \( l \) to be the longitude pictured in Figure 1, then we may think of even and odd closures more intuitively as follows. Even (respectively, odd) closures are those
whose complementary 1-manifold passes through the hole of the solid torus containing $G$ an even (respectively, odd) number of times. For the remainder of this paper, when we discuss even (and odd) closures, we mean even (and odd) with respect to the longitude $l$.

We also choose our two marked points on the boundary of $S^1 \times D^2$ to be the endpoints of the tangle in this figure.

In [A2, A3], the first author defined the Kauffman bracket ideal of an $(M, 2n)$-tangle $T$ to be the ideal $I_T$ of $\mathbb{Z}[A, A^{-1}]$ generated by the reduced Kauffman bracket polynomials of all closures of $T$ which gave an obstruction to embedding. This ideal, in the case $(M, 2n) = (B^3, 4)$, was first studied by Przytycki, Silver and Williams [PSW].

The first author outlined a method for computing this ideal in the case of genus-1 tangles using skein theory techniques. In this paper, we define an even and odd version of the Kauffman bracket ideal for genus-1 tangles. The even Kauffman bracket ideal of a genus-1 tangle $G$ is the ideal $I^\text{even}_G$ generated by the reduced Kauffman bracket polynomials of all even closures of $G$. The odd Kauffman bracket ideal $I^\text{odd}_G$ is defined similarly. If an ideal is equal to $\mathbb{Z}[A, A^{-1}]$, we refer to that ideal as trivial.

The following proposition is an immediate consequence of these definitions.

**Proposition 1.1.** Let $G$ be a genus-1 tangle. If $I^\text{even}_G$ (respectively, $I^\text{odd}_G$) is non-trivial, then the unknot is not an even (respectively, odd) closure of $G$. More generally, if $L$ is an even (respectively, odd) closure of $G$, then the reduced Kauffman bracket polynomial of $L$ must lie in $I^\text{even}_G$ (respectively, $I^\text{odd}_G$).

We also note that if the ordinary Kauffman bracket ideal is non-trivial, then both the even and odd Kauffman bracket ideals must be non-trivial. However, the converse is not true. In §6 we give an example of a genus-1 tangle (see Figure 5) which has non-trivial even and odd Kauffman bracket ideals but trivial ordinary Kauffman bracket ideal. We also show that Krebes’ tangle [K], pictured in Figure 1, has trivial even ideal and non-trivial odd ideal, and give another example (in Figure 4) which has non-trivial even ideal and trivial odd ideal. In [A2, A3], we showed that the ordinary Kauffman bracket ideal of Krebes’ tangle $A$ is trivial.

In §5, we outline a method for computing a finite list of generators for the even and odd Kauffman bracket ideals, following the same basic procedure as in [A2, A3]. We use the graph basis defined in [A2, A3] as well as bases for the even and odd Kauffman bracket skein modules of $S^1 \times D^2$ relative to two points which we define in §3. These even
and odd bases are adapted from a definition of Habiro’s [Hab]. In the
§ we relate the even and odd ideals to the determinants of even and
odd closures, relating the results of this paper to those of [A].

2. Kauffman bracket skein modules

The Kauffman bracket polynomial of a framed link $D$, denoted by
$\langle D \rangle$, is an element of $\mathbb{Z}[A, A^{-1}]$ given by the following three relations,
where $\delta = -A^2 - A^{-2}$:

(i) $\langle \begin{array}{c} \vspace{0.05in} \\
\end{array} \ \rangle = A\langle \begin{array}{c} \\ \vspace{0.05in} \\
\end{array} \rangle + A^{-1}\langle \begin{array}{c} \\ \vspace{0.05in} \\
\end{array} \rangle$

(ii) $\langle \begin{array}{c} \\ \\
\end{array} \rangle = \delta \langle D' \rangle$.

(iii) $\langle \rangle = 1$.

We let $\langle D' \rangle$ denote the reduced Kauffman bracket polynomial of $D$;
that is, where $\langle D' \rangle = \langle D \rangle / \delta$.

The Kauffman bracket skein module of a 3-manifold $M$ is the $\mathbb{Z}[A, A^{-1}]$-
module $K_M$ generated by isotopy classes of framed links in $M$ 
modulo the Kauffman bracket relations above.

Of particular concern to us is the relative Kauffman bracket skein 
module. Let $M$ be a compact oriented 3-manifold with boundary and
a set of $m$ specified marked points on $\partial M$. Then the Kauffman 
bracket skein module of $M$ relative to the $m$ marked points is the $\mathbb{Z}[A, A^{-1}]$-
module $K_M(m)$ generated by isotopy classes of framed 1-manifolds
with boundary the marked points modulo the above Kauffman bracket 
relations. We can view any genus-1 tangle (equipped with the black-
board framing) as a skein element in $K(S^1 \times D^2, 2)$.

As in [A2, A3], we generalize the Hopf pairing on $K(S^1 \times D^2)$ defined
in [BHMV] to obtain the relative Hopf pairing $\langle \ , \rangle : K(S^1 \times D^2, 2) \times
K(S^1 \times D^2, 2) \to K(S^3) = \mathbb{Z}[A, A^{-1}]$. Given $a$ and $b$ in $K(S^1 \times D^2, 2)$,
we let
\[
\langle a, b \rangle = \langle \begin{tikzpicture}[scale=0.5]
\node (a) at (0,0) [shape=circle,draw,inner sep=0.5cm] {a};
\node (b) at (2,0) [shape=circle,draw,inner sep=0.5cm] {b};
\draw (a) to[bend left=45] (b);
\draw (a) to[bend right=45] (b);
\end{tikzpicture} \rangle
\]
where \(a\) and \(b\) lie in regular neighborhoods of the trivalent graphs.

If a genus-1 tangle \(G\) embeds in a link \(L \subseteq S^3\), then we can describe this tangle embedding using the relative Hopf pairing. We have that
\[
\langle L \rangle = \langle \mathcal{G}, \mathcal{G}' \rangle
\]
for some \(\mathcal{G}' \in K(S^1 \times D^2, 2)\).

3. Even and odd relative skein modules

As in [BHMV], we let \(z\) denote a standard band in \(K(S^1 \times D^2)\). As described in [Hab], one can obtain a \(\mathbb{Z}_2\)-graded algebra structure on the Kauffman bracket skein module \(K(S^1 \times D^2)\) by letting \(K^{\text{even}}(S^1 \times D^2)\) be the subalgebra of \(K(S^1 \times D^2)\) generated by \(z^2\) and \(K^{\text{odd}}(S^1 \times D^2)\) be \(zK^{\text{even}}(S^1 \times D^2)\). Then, one has that
\[
K(S^1 \times D^2) = K^{\text{even}}(S^1 \times D^2) \oplus K^{\text{odd}}(S^1 \times D^2).
\]

In fact [GH, p.105], as the Kauffman skein relations respect \(\mathbb{Z}_2\)-homology classes, the skein module of any 3-manifold \(M\) has a direct sum decomposition into submodules indexed by the set \(H_1(M, \text{marked points}, \mathbb{Z}_2)\). Thus we have a decomposition into even and odd submodules for the relative skein module \(K(S^1 \times D^2, 2)\).

Let \(u\) be a framed 1-manifold in \(S^1 \times D^2\) with two boundary components. Then we say that \(u\) is even (respectively, odd), if \(u\) intersects the disk highlighted in Figure 2 an even (respectively, odd) number of times.

Let \(K^{\text{even}}(S^1 \times D^2, 2)\) and \(K^{\text{odd}}(S^1 \times D^2, 2)\) be the submodules of \(K(S^1 \times D^2, 2)\) generated by all even and odd 1-manifolds, respectively. Then, we have that
\[
K(S^1 \times D^2, 2) = K^{\text{even}}(S^1 \times D^2, 2) \oplus K^{\text{odd}}(S^1 \times D^2, 2)
\]
which gives a \(\mathbb{Z}_2\)-graded algebra structure on \(K(S^1 \times D^2, 2)\). Note that if \(L\) is an even closure of a genus-1 tangle \(\mathcal{G}\), then the Kauffman bracket polynomial of \(L\) can be written as \(\langle L \rangle = \langle \mathcal{G}, \mathcal{G}' \rangle\) where \(\mathcal{G}' \in K^{\text{even}}(S^1 \times D^2, 2)\). The analogous statement is true for odd closures.

We let \(\phi_k = -A^{2k+2} - A^{-2k-2}\) (in [BHMV] and elsewhere this is denoted \(\lambda_i\)). The authors define a basis of \(K(S^1 \times D^2)\) in [BHMV] as follows. Let \(Q_n = \prod_{i=0}^{n-1} (z - \phi_i)\) for \(n \geq 0\). Habiro modifies this definition
in [Hab] to obtain a basis for the even submodule \( K_{\text{even}}(S^1 \times D^2) \) by defining \( S_n = \prod_{i=0}^{n-1} (z^2 - \phi_i^2) \) for \( n \geq 0 \). Note that \( S_n = Q_n \prod_{i=0}^{n-1} (z + \phi_i) \).

We adapt Habiro’s basis to obtain bases for the \( K_{\text{even}}(S^1 \times D^2, 2) \) and \( K_{\text{odd}}(S^1 \times D^2, 2) \). We refer to them as the even basis and odd basis, respectively, and they are defined as follows. The even basis consists of the following elements, where \( n \geq 0 \):

\[
\begin{align*}
x_n^{\text{even}} &= x_n \\
y_n^{\text{even}} &= y_n 
\end{align*}
\]

Similarly, the odd basis consists of the following elements, where \( n \geq 0 \):

\[
\begin{align*}
x_n^{\text{odd}} &= x_n \\
y_n^{\text{odd}} &= y_n 
\end{align*}
\]

That these are bases follows ultimately from the basis for \( K(S^1 \times D^2, 2) \) consisting of framed links described by isotopy classes of diagrams without crossings and without contractible loops in \( S^1 \times D^2 \). One also uses the fact that there is a triangular unimodular change of basis matrix over \( \mathbb{Z}[A, A^{-1}] \) relating the bases \( \{S_n\} \) and \( \{z^{2n}\} \) for \( K_{\text{even}}(S^1 \times D^2) \).

4. Graph basis of \( K_R(S^1 \times D^2, 2) \)

Trivalent graphs will be interpreted as in [A2, A3, GH, KL, MV]. Any unlabelled edge is assumed to be colored one. The colors of the three edges incident to a single vertex must form an admissible triple. Given non-negative integers \( a, b, \) and \( c \), the triple \((a, b, c)\) is admissible if \( |a-b| \leq c \leq a+b \) and \( a+b+c \equiv 0(\text{mod } 2) \). We use the notation of [KL]: \( \Delta_n, \theta(a,b,c), \text{Tet} \left[ \begin{array}{ccc} a & b & c \\ c & d & f \end{array} \right] \), and \( \lambda_c^{a,b} \).
We use the graph basis defined in [A2, A3]. Given a pair of non-negative integers \((i, \varepsilon)\) such that \(\varepsilon = i + 1\) or \(\varepsilon = i - 1\), let

\[
g_{i, \varepsilon} = \begin{array}{c}
\varepsilon \\
i
\end{array}
\]

Let \(R\) denote the ring \(\mathbb{Z}[A, A^{-1}]\) localized by inverting the quantum integers, and let \(K_R(M, m)\) denote the Kauffman bracket skein module of \(M\) relative to \(m\) points with coefficients in \(R\). According to [P, Theorem 2.3], we have that \(K_R(M, m) \cong K(M, m) \otimes R\), so we can essentially view \(K(M, m)\) as a subset of \(K_R(M, m)\). We make this distinction because when computing a finite list of generators for the even and odd Kauffman bracket ideals, we end up passing through \(K_R(S^1 \times D^2, 2)\) when using the doubling pairing defined in [A2, §2.3]. However, each of the generators we obtain is in fact an element of \(K(S^1 \times D^2, 2)\).

Recall, according to [HP], \(K_R(S^1 \times S^2)/\text{torsion}\) is isomorphic to \(R\). Let \(\psi : K_R(S^1 \times S^2) \rightarrow R\) be the epimorphism that sends the empty link to 1 \(\in R\). The doubling pairing is defined to be the symmetric pairing \(\langle \cdot, \cdot \rangle_D : K_R(S^1 \times D^2, 2) \times K_R(S^1 \times D^2, 2) \rightarrow R\) obtained by gluing two solid tori containing skein elements together via a certain orientation-reversing homeomorphism to obtain a skein element in \(S^1 \times S^2\), and evaluating this skein element under \(\psi\). Figure 3 illustrates the doubling pairing of two graph basis elements. The thick dark colored loop indicates where a 0-framed surgery is to be performed, converting \(S^3\) to \(S^1 \times S^2\). According to [A2, Theorem 2.4], the graph basis is orthogonal with respect to the doubling pairing.

5. Applications to genus-1 tangle embedding

Let \(G\) be a genus-1 tangle. We follow the same basic procedure as in [A2, A3] to obtain finite lists of generators for \(I^\text{even}_G\) and \(I^\text{odd}_G\).
First, we write $G$ as a linear combination of graph basis elements $G = \sum c_{i,\varepsilon}g_{i,\varepsilon}$. Since the graph basis is orthogonal, we have that $c_{i,\varepsilon} = \langle G, g_{i,\varepsilon} \rangle_D / \langle g_{i,\varepsilon}, g_{i,\varepsilon} \rangle_D$ and only finitely many $c_{i,\varepsilon}$ are non-zero.

We then use this linear combination to compute the relative Hopf pairing of $G$ with the even (respectively, odd) basis to obtain a generating set for $I_{G,\text{even}}$ (respectively, $I_{G,\text{odd}}$). The following results allow us to compute the relative Hopf pairing of the graph basis with the even and odd bases. The proof Lemma 5.1 below is similar to that of [A2, Lemma 4.1].

**Lemma 5.1.**

$$
S_n = \prod_{k=0}^{n-1} (\phi_i^2 - \phi_k^2)
$$

If $n = 0$, we interpret $\prod_{k=0}^{n-1} (\phi_i^2 - \phi_k^2)$ as 1.

Notice that $\prod_{k=0}^{n-1} (\phi_i^2 - \phi_k^2)$ is zero if $n > i$.

**Proposition 5.2.**

(i) $\langle g_{i,\varepsilon}, x_{n,\text{even}} \rangle = \theta(1, i, \varepsilon) \prod_{k=0}^{n-1} (\phi_i^2 - \phi_k^2)$.

(ii) $\langle g_{i,\varepsilon}, y_{n,\text{even}} \rangle = \phi_i (\lambda_{\varepsilon}^i)^{-1} \theta(1, i, \varepsilon) \prod_{k=0}^{n-1} (\phi_i^2 - \phi_k^2)$.

(iii) $\langle g_{i,\varepsilon}, x_{n,\text{odd}} \rangle = \phi_i \theta(1, i, \varepsilon) \prod_{k=0}^{n-1} (\phi_i^2 - \phi_k^2)$.

(iv) $\langle g_{i,\varepsilon}, y_{n,\text{odd}} \rangle = (\lambda_{\varepsilon}^i)^{-1} \theta(1, i, \varepsilon) \prod_{k=0}^{n-1} (\phi_i^2 - \phi_k^2)$.

Each of these is zero if $n > i$.

**Proof.** (i) We have from Lemma 5.1 that

$$
\langle g_{i,\varepsilon}, x_{n,\text{even}} \rangle = S_n \cdot \prod_{k=0}^{n-1} (\phi_i^2 - \phi_k^2) = \theta(1, i, \varepsilon) \prod_{k=0}^{n-1} (\phi_i^2 - \phi_k^2).
$$
(ii) Using Lemma 5.1, [A2, equations 2.2, 2.5] and \( \lambda^i_j = \lambda^j_i \),

\[
\langle g_i, x^\text{even}_n \rangle = \phi_i \prod_{k=0}^{n-1} (\phi^2_i - \phi^2_k) = \phi_i (\lambda^i_\varepsilon)^{-1}(\lambda^i_\varepsilon)^{-1} \prod_{k=0}^{n-1} (\phi^2_i - \phi^2_k) = \phi_i (\lambda^i_\varepsilon)^{-2}(1, i, \varepsilon) \prod_{k=0}^{n-1} (\phi^2_i - \phi^2_k). 
\]

(iii)

\[
\langle g_i, x^\text{odd}_n \rangle = \phi_i \prod_{k=0}^{n-1} (\phi^2_i - \phi^2_k) = \phi_i \theta(1, i, \varepsilon) \prod_{k=0}^{n-1} (\phi^2_i - \phi^2_k). 
\]

(iv)

\[
\langle g_i, y^\text{odd}_n \rangle = \phi_i \prod_{k=0}^{n-1} (\phi^2_i - \phi^2_k) = (\lambda^i_\varepsilon)^{-2}(1, i, \varepsilon) \prod_{k=0}^{n-1} (\phi^2_i - \phi^2_k). 
\]

\[\square\]

Recall, the Kauffman bracket polynomial of any even closure \( L \) of \( G \) can be written as \( \langle L \rangle = \langle G, G' \rangle \) where \( G' \in K^\text{even}(S^1 \times D^2, 2) \). So, \( \langle G, x^\text{even}_n \rangle / \delta \) and \( \langle G, y^\text{even}_n \rangle / \delta \) form a generating set for \( I^\text{even}_G \). Proposition 5.2 implies that only finitely many of these will be non-zero. Similarly, \( \langle G, x^\text{odd}_n \rangle / \delta \) and \( \langle G, y^\text{odd}_n \rangle / \delta \) form a finite generating set for \( I^\text{odd}_G \).
6. Examples

We compute the even and odd Kauffman bracket ideals for three tangles $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{H}$. In each of these computations, our first step is to compute the doubling pairing of the tangle in question with the graph basis. We leave out the full computation for the sake of brevity, but we follow the same procedure as in [A2, Appendix A]. We then write each basis. We leave out the full computation for the sake of brevity, but we compute the doubling pairing of the tangle in question with the graph basis. It turns out that due to admissibility conditions, $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{H}$ may all be written as $c_{0,1} g_{0,1} + c_{2,1} g_{2,1} + c_{2,3} g_{2,3}$ for some coefficients $c_{i,\varepsilon} \in R$. Thus we have using Proposition 5.2.

Lemma 6.1. If $G=\mathcal{A}$, $\mathcal{D}$, or $\mathcal{H}$, $\mathcal{I}^\text{even}_G$ is generated by $\langle G, x_i^\text{even} \rangle/\delta$ and $\langle G, y_i^\text{even} \rangle/\delta$ where $0 \leq i \leq 2$. Similarly, $\mathcal{I}^\text{odd}_G$ is generated by $\langle G, x_i^\text{odd} \rangle/\delta$ and $\langle G, y_i^\text{odd} \rangle/\delta$ where $0 \leq i \leq 2$.

For $G=\mathcal{A}$, $\mathcal{D}$ or $\mathcal{H}$, we followed the same procedure as in [A2, A3], to find $\mathcal{I}^\text{even}_A$ and $\mathcal{I}^\text{odd}_A$, using Proposition 5.2, Lemma 6.1 and Mathematica. One can verify directly that the claimed ideals are indeed non-trivial using the computations in §7.

6.1. Krebes’ Tangle $\mathcal{A}$. We consider the genus-1 tangle given by Krebes ($K$) pictured in Figure 1. We have that

$$\langle \mathcal{A}, g_i, \varepsilon \rangle_D =$$

where the sum is over all $j$, $k$, and $l$ such that the following triples are admissible: $(1, 1, i)$, $(1, 1, j)$, $(1, j, \varepsilon)$, $(1, 1, k)$, $(\varepsilon, k, 1)$, $(1, 1, l)$, and $(l, k, j)$. Admissibility conditions imply that 0 and 2 are the only possible admissible values for $j$, $k$, and $l$. Note that, if $i \neq 0, 2$, there are no such $j, k, l$, and the given sum is over an empty index set, and the value of the sum is zero. So $\langle \mathcal{A}, g_i, \varepsilon \rangle_D = 0$ unless $i = 0$ or $i = 2$.

The coefficients for $\mathcal{A}$ as a linear combination of the graph basis are $c_{0,1} = -1 - A^8 + A^{12}/1 + A^4$, $c_{2,1} = -1 - A^4 + A^{12} / A^6 + A^{10} + A^{17}$, and $c_{2,3} = 1.$

Proposition 6.2. The even Kauffman bracket ideal $\mathcal{I}^\text{even}_A$ of Krebes’ tangle $\mathcal{A}$ is trivial. The odd Kauffman bracket ideal of $\mathcal{A}$ is $\mathcal{I}^\text{odd}_A = \langle 9, 4 + A^4 \rangle$ which is non-trivial.
6.2. A small tangle, $\mathcal{D}$. We now consider the genus-1 tangle $\mathcal{D}$ pictured in Figure 4. In contrast to Krebes’ example, $\mathcal{D}$ has a non-trivial even Kauffman bracket ideal and a trivial odd Kauffman bracket ideal.

**Figure 4.** A genus-1 tangle, denoted by $\mathcal{D}$.

We have that

$$\langle \mathcal{D}, g_{i, \varepsilon} \rangle_{\mathcal{D}} = \sum_j \lambda_i^{-1} (\lambda_j^{-1})^{-3} \Delta_j \text{Tet} \left[ \begin{array}{ccc} 1 & 1 & j \\ 1 & \varepsilon & i \\ 1 & j & 1 \end{array} \right] \text{Tet} \left[ \begin{array}{ccc} 1 & i & \varepsilon \\ 1 & j & 1 \end{array} \right]$$

where the sum is over all integers $j$, such that the following are admissible triples; $(1, 1, i)$, $(1, 1, j)$, and $(1, \varepsilon, j)$. Admissibility conditions imply that 0 and 2 are the only possible admissible values for $j$ and $\langle \mathcal{D}, g_{i, \varepsilon} \rangle_{\mathcal{D}} = 0$ unless $i = 0$ or $i = 2$.

The coefficients for $\mathcal{D}$ as a linear combination of the graph basis are $c_{0,1} = \frac{1-A^4-A^{12}}{A^2+A^6}$, $c_{2,1} = \frac{1+A^8-A^{12}}{A^8+A^{12}+A^6}$, and $c_{2,3} = A^2$.

**Proposition 6.3.** The even Kauffman bracket ideal of $\mathcal{D}$ is $I^{\text{even}}_{\mathcal{D}} = \langle -9, -2 + A^4 \rangle$ which is non-trivial. The odd Kauffman bracket ideal $I^{\text{odd}}_{\mathcal{D}}$ of $\mathcal{D}$ is trivial.
6.3. A particularly interesting tangle, $\mathcal{H}$. We have that

$$\langle \mathcal{H}, g_{i, \varepsilon} \rangle_D = \sum_{j, k, l} \lambda_i^1 \lambda_j^1 \Delta_j \Delta_l \Delta_k \Delta_l \Theta \left[ \begin{array}{ccc} 1 & i & \varepsilon \\ 1 & j & 1 \\ 1 & k & 1 \\ 1 & l & 1 \end{array} \right] \Theta \left[ \begin{array}{ccc} 1 & k & \varepsilon \\ 1 & l & 1 \end{array} \right]$$

where the sum is over all $j, k, l$ such that the following triples are admissible: $(1, 1, i), (1, 1, j), (1, j, \varepsilon), (1, 1, k), (1, k, \varepsilon), (1, 1, l),$ and $(1, l, \varepsilon)$. Admissibility conditions imply that 0 and 2 are the only possible admissible values for $j, k,$ and $l$, and $\langle \mathcal{H}, g_{i, \varepsilon} \rangle_D = 0$ unless $i = 0$ or $i = 2$.

The coefficients for $\mathcal{H}$ as a linear combination of the graph basis are

$$c_{0,1} = -\frac{1 + 2A^4 - 3A^8 + 2A^{12} - 3A^{16} + 2A^{20} - A^{24} + A^{28}}{A^{18} + A^{22} + A^{26}},$$

$$c_{2,1} = -\frac{1 + A^4 - 2A^8 + 3A^{12} - 2A^{16} + 3A^{20} - 2A^{24} + A^{28}}{A^{18} + A^{22} + A^{26}},$$

and $c_{2,3} = A^4$.

**Proposition 6.4.** The even Kauffman bracket ideal of $\mathcal{H}$ is $I_{\mathcal{H}}^\text{even} = \langle 5, 1 + A^4 \rangle$ which is non-trivial. The odd Kauffman bracket ideal of $\mathcal{H}$ is $I_{\mathcal{H}}^\text{odd} = \langle 9, 4 + A^4 \rangle$ which is also non-trivial.

These corollaries follow immediately.

**Corollary 6.5.** The Kauffman bracket ideal $I_{\mathcal{H}}$ of the genus-1 tangle $\mathcal{H}$ is trivial.

**Corollary 6.6.** The genus-1 tangle $\mathcal{H}$ does not embed in the unknot.
Although $\mathcal{H}$ is not obstructed from embedding in the unknot by the ordinary Kauffman bracket ideal, the even and odd Kauffman bracket ideals do provide an obstruction.

7. Relation to Determinants

Let $\omega$ denote $e^{\frac{\pi i}{4}}$, and $\Omega : \mathbb{Z}[A, A^{-1}] \to \mathbb{Z}[\omega]$ be the ring epimorphism sending $A$ to $\omega$. Recall that the determinant $\det(L)$ of a link $L$ is the absolute value of the determinant of a Seifert matrix for $L$ symmetrized or, equivalently, the order of the first homology of the double branched cover of $S^3$ along $L$. If this homology group is infinite, the order is interpreted to be zero. One has that \cite[Prop. 1]{K}, $\det(L) = \omega^j \Omega(\langle L \rangle')$ for an integer $j$, chosen so that $\omega^j \Omega(\langle L \rangle')$ is a non-negative integer.

Proposition 7.1. Let $L$ be a closure of a genus-1 tangle $G$. Then $\det(L) \in \Omega(I_G) \cap \mathbb{Z}$. If $L$ is an even closure, $\det(L) \in \Omega(I_G^{\text{even}}) \cap \mathbb{Z}$. If $L$ is an odd closure, $\det(L) \in \Omega(I_G^{\text{odd}}) \cap \mathbb{Z}$.

For the ideals computed in the examples, we have (noting that $\Omega(A^4) = -1$ and $\Omega \cap \mathbb{Q} = \mathbb{Z}$):

\[
\Omega(I_A^{\text{odd}}) \cap \mathbb{Z} = \Omega(I_H^{\text{even}}) \cap \mathbb{Z} = \Omega(\langle 9, 4 + A^4 \rangle) \cap \mathbb{Z} = \langle 3 \rangle.
\]

\[
\Omega(I_D^{\text{even}}) \cap \mathbb{Z} = \Omega(\langle 9, -2 + A^4 \rangle) \cap \mathbb{Z} = \langle 3 \rangle.
\]

\[
\Omega(I_H^{\text{odd}}) \cap \mathbb{Z} = \Omega(\langle 5, 1 + A^4 \rangle) \cap \mathbb{Z} = \langle 5 \rangle.
\]

This gives a second proof of a result in \cite[Theorem 1.3]{A} which is the first sentence in:

Proposition 7.2. Let $L$ be an odd closure of $A$, then $\det(L) \equiv 0 \pmod{3}$. Let $L$ be an even closure of $D$, then $\det(L) \equiv 0 \pmod{5}$. If $L$ is an odd closure of $H$, $\det(L) \equiv 0 \pmod{5}$. If $L$ is an even closure of $H$, $\det(L) \equiv 0 \pmod{3}$.

Note for the tangle $F$ shown in \cite{A} to have $I_F = \langle 11, 4 - A^4 \rangle$, one has $\Omega(I_F \cap \mathbb{Z}) = \langle 1 \rangle$.

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