The Barnes $\zeta$–function, sphere determinants and Glaisher-Kinkelin-Bendersky constants

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Abstract

Summations and relations involving the Hurwitz and Riemann $\zeta$–functions are extended first to Barnes $\zeta$–functions and then to $\zeta$–functions of general type. The analysis is motivated by the evaluation of determinants on spheres which are treated both by a direct expansion method and by regularised sums. Comments on existing calculations are made. It is suggested that the combination $\zeta'(-n) + H_n \zeta(-n)$, where $H_n$ is a harmonic number, should be taken as more relevant than just $\zeta'(-n)$. This leads to a Kaluza–Klein technique, providing a determinant interpretation of the Glaisher–Kinkelin–Bendersky constants which are then generalised to arbitrary $\zeta$–functions. This technique allows an improved treatment of sphere determinants.

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1. Introduction

The Barnes $\zeta$–function, [1], has enjoyed only sporadic mathematical interest since it was first introduced in 1903, [2–4]. It generalises the Hurwitz $\zeta$–function, has number theory applications and involves a natural generalisation of the Euler $\Gamma$–function, examples of which had already appeared, [5–7] and were developed more directly by Bendersky [8].

Some of these works discuss regularised products, [9–12], and the related Laplacian determinants, [13–15], on various spaces from varying points of view. Explicit results are usually obtained only when the eigenvalue spectrum is known or sufficiently powerful geometrical or analytical information is available.

As might be expected, particularly detailed investigations have been made in the case of spheres. Explicit $d$-sphere determinant expressions are given, and plotted, in [16], obtained by a direct method, ancillary to a discussion of the orbifold quotient, $S^d/\Gamma$. Choi [17] analyses the 3-sphere using a factorisation technique founded on the method in Voros’ basic paper [14] while Choi and Srivastava [18] improve this calculation. Quine and Choi [19], with a systematic method, give explicit forms for the full $d$-sphere. Kumagai [20] attempts to give a corrected version of Vardi’s analysis, [13], for the $d$-sphere but does not calculate beyond the 4-sphere. Further remarks are made later.

During the course of these investigations, identities and relations appear which we would like to systematise in a certain way and enlarge upon. Some considerable amount of work in this area has appeared in the mathematical literature in the past few years and we wish to draw attention to relevant work by physicists which is often overlooked and which might be useful.

Applications of the Barnes function in physics embrace the Casimir effect around cosmic strings [21], on orbifolded spheres, [22,23], integrable field theories [24–27], and statistical mechanics, [28]. The Barnes $\zeta$–function also arises for the higher-dimensional harmonic oscillator and is useful in connection with Bose-Einstein condensation and trapping [29,30].

3 As noted in [16], it is clear there is some error in Vardi’s manipulations. This can be traced to two simple oversights on p.504 of [13]. The first is in the proof that there are no log $k$ terms. The upper limit on the $k$ sum cannot be changed from $n−1$ to $n−2$ because there is a contribution when $d=0$. The second slip is an incorrect interchange of the $d$ and $r$ summations. Vardi produces a general formula for the $d$–sphere determinant in terms of the derivatives of the Riemann $\zeta$–function at negative integers, cf [16,19], which he then converts into multiple $\Gamma$–functions, $\Gamma_n(1/2)$. 
The structure of this paper is as follows. The Barnes $\zeta$–function is introduced via scalar $\zeta$–functions on spheres. The determinant is next looked at, which focuses attention on certain constructs and relations. These further motivate an investigation of the *general* $\zeta$–function, $\sum \lambda^{-s}$, and we then link up with the regularised product approach giving a Kaluza-Klein interpretation of generalised Glaisher-Kinkelin-Bendersky constants. The paper proceeds as a series of generalisations.

Determinants appear frequently in field and string theory. No attempt will be made to justify their computation nor to detail their general history. For the 2-sphere the first computations were by Hortacsu et al [31] and by Weisberger [32,33]. The topic of $\zeta$–functions on spheres, and symmetric spaces in general, also has a long record. (See Camporesi, [34], for a useful survey and results.)

2. The sphere $\zeta$–function and the Barnes $\zeta$–function.

The general definition of the Barnes $\zeta$–function is,

$$
\zeta_d(s, a|d) = \frac{i \Gamma(1 - s)}{2\pi} \int_L d\tau \prod_{i=1}^d \frac{\exp(-a\tau)(-\tau)^{s-1}}{(1 - \exp(-d_i\tau))} = \sum_{m=0}^{\infty} \frac{1}{(a + m.d)^s}, \quad \text{Re } s > d,
$$

where we refer to the components, $d_i$, of the $d$-vector, $d$, as the *degrees* or *quasi-periods*. For simplicity, we assume that the $d_i$ are real and positive. If $a$ is zero, the origin, $m = 0$, is to be excluded. The contour, $L$, is the standard Hankel one.

On the $d$-sphere, consider the Laplacian–type operator,

$$
-\Delta + \xi R - \alpha^2,
$$

where $\xi = (d - 1)/4d$ and $\alpha$ is a parameter introduced for convenience of discussing several specific cases at once. The value $\alpha = 0$ corresponds to conformal coupling in $d + 1$ dimensions, $\alpha = 1/2$ to conformal coupling in $d$-dimensions and $\alpha = (d - 1)/2$ to minimal coupling *i.e.* just the operator $-\Delta$. In the first case the eigenvalues are perfect squares,

$$
\frac{1}{4}(m + d - 2)^2, \quad m = 1, 3, \ldots.
$$

It is best to think of the mode set on the full sphere as the union of Dirichlet and Neumann mode sets on a *hemisphere*, despite the apparent extra complication. The
reason is that these individual $\zeta$–functions are Barnes functions, with all degrees equal to unity, $d = 1$, as shown in [22]. Specifically, for $\alpha = 0$,
\[
\zeta_N(s) = \zeta_d(2s, (d - 1)/2 | d),
\]
\[
\zeta_D(s) = \zeta_d(2s, \sum d_i - (d - 1)/2 | d),
\]
where the parameters in the $\zeta$–functions have been left general because, although for the full–sphere, and hemisphere, we need only unit degrees, we wish to retain the general case for as long as possible as it applies for the other orbifold factorings of the sphere\(^4\).

If the sum over the vector $m$ in (1) is performed as far as possible, it is easy to regain the standard eigenvalues, (3), together with the usual degeneracies.

For the general operator (2) the $\zeta$–function to define is clearly,
\[
\zeta(s, a, \alpha; d) = \sum_{m=0}^{\infty} \frac{1}{((a + m.d)^2 - \alpha^2)^s}, \tag{4}
\]
and if $d = 1$, giving the hemisphere, we have,
\[
\zeta_{HS}(s) = \sum_{m=0}^{\infty} \frac{1}{((m + d - 1)\frac{m + d - 1}{d - 1})^s}. \tag{5}
\]
From this, it is easy to confirm that the sum of the Dirichlet and Neumann $\zeta$–functions on the hemisphere does equal the full–sphere $\zeta$–function [16]. A technical point is that for minimal coupling and Neumann conditions, the origin, $m = 0$, is to be omitted from the sum.

There are at least three approaches to the continuation of the $\zeta$–function (5). One is to use a summation formulae, such as Plana’s (see [35] for this specific case). The second is that used by Minakshisundaram and later by Candelas and Weinberg [36] in a treatment of sphere $\zeta$–functions and employs a Bessel function identity to perform the summation over $m$. The third method involves an expansion in the associated parameter, $\alpha^2$, and, after some manipulation with the binomial coefficient, the problem is thrown onto the continuation of a series of Hurwitz $\zeta$–functions, a standard matter \(^5\).

\(^4\) In this case the $d_i$ are the integer degrees associated with the polytope symmetry group, $\Gamma$. For the simplest (cyclic) case, $\mathbb{Z}_q$, $d_1 = q$ with the rest unity.

\(^5\) For $\alpha = 0$, i.e. the Barnes $\zeta$–function, Barnes [37] gives the reduction to a finite sum of Hurwitz $\zeta$–functions and remarks that it could be made fundamental for the theory when $d = 1$. This reduction has been rediscovered in many, more recent discussions of the multiple $\Gamma$–function, restricted, as they are, to just this case. In particular, on p.432, Barnes gives an expression for $\log \Gamma_n$ essentially equivalent to formulae of Vardi [13] and Kanemitsu et al, [38]. A similar reduction is also possible for rational degrees.
The first approach is not practicable for general degrees, \( d \neq 1 \), (4), since there are several genuine summation variables. The second method yields the Bessel function expression
\[
\zeta(s, a, \alpha \mid d) = {\sqrt{\pi}} \frac{\Gamma(s)}{\prod_{i=1}^{d} (1 - \exp(-d_i \tau))} \left( \frac{\tau}{2\alpha} \right)^{s-1/2} I_{s-1/2}(\alpha \tau),
\]
which could be continued to give the analogue of (1) or treated as in Candelas and Weinberg [36] and Chodos and Myers [39] to enable values at negative \( s \) to be computed.

This is a practical method of obtaining a continuation of the sphere \( \zeta \)–function but as our main interest is really with the Barnes function we apply the third procedure and expand in \( \alpha^2 \). One then encounters the continuation of an infinite series of Barnes \( \zeta \)–functions, suitable only for special values of \( s \) but sufficient for our purposes.

3. Sphere determinants.

The first step is a standard expansion in powers of \( \alpha^2 \), (cf [40]),
\[
\zeta(s, a, \alpha \mid d) = \sum_{r=0}^{\infty} \alpha^{2r} \frac{s(s + 1) \ldots (s + r - 1)}{r!} \zeta_d(2s + 2r, a \mid d),
\]
which allows certain information to be extracted, but which does not constitute a complete continuation. The principle being applied here is expansion in terms of ‘known’ \( \zeta \)–functions and the continuation of the Barnes \( \zeta \)–function will be assumed in principle to be already achieved by Barnes with the numerical computation of any particular case looked upon as a soluble technical challenge.

Barnes has given the values of \( \zeta_d(s, a \mid d) \) for \( s \) a nonpositive integer, and also the residues and remainders at the poles \( s = 1, \ldots, d \), in terms of generalised Bernoulli functions rapidly computed by recursion. Hence, for example, one can easily find the values \( \zeta(-n, a, \alpha \mid d) \) for \( n = 0, -1, \ldots, [23] \).

The derivative at 0 also follows, but not so directly. Firstly from (7)
\[
\zeta'(0, a, \alpha \mid d) = 2\zeta_d'(0, a \mid d) + \sum_{r=1}^{\infty} \alpha^{2r} \left( \frac{R_{2r}(d)}{r} + \frac{1}{2} N_{2r}(d) \frac{1}{k} \right) + \sum_{r=u+1}^{\infty} \alpha^{2r} \zeta_d(2r, a \mid d),
\]

4
The notation is that \( N_r(d) \) is the residue and \( R_r(d) \) the remainder defined by the behaviour at the known Barnes poles

\[
\zeta_d(s + r, a \mid d) \to \frac{N_r(d)}{s} + R_r(d) \quad \text{as } s \to 0,
\]

where \( 1 \leq r \leq d \) and \( u = d/2 \) if \( d \) is even and \( u = (d - 1)/2 \) if \( d \) is odd.

The problem is the evaluation of the infinite series on the right.

### 4. \( \zeta \)-function sums.

Expression (8) concentrates our attention on the sum of Barnes \( \zeta \)-functions,

\[
\sum_{r=u+1}^{\infty} \frac{\alpha^{2r}}{r} \zeta_d(2r, a \mid d),
\]

which can be obtained from

\[
\sum_{r=d+1}^{\infty} \frac{(-\alpha)^r}{r} \zeta_d(r, a \mid d),
\]

by averaging over \( \pm \alpha \).

For the Hurwitz \( \zeta \)-function \((d = 1)\) this sum is standard (e.g. [41] p.276) but the analysis extends to the more general case as shown by Barnes who has, ([1] p.424 eq(1)), (cf [43]),

\[
\sum_{r=d+1}^{\infty} \frac{(-\alpha)^r}{r} \zeta_d(r, a \mid d) = \log \frac{\Gamma_d(a + \alpha)}{\Gamma_d(a)} - \sum_{r=1}^{d} \frac{\alpha^r}{r!} \psi_d^{(r)}(a).
\]

The sum on the left is over those \( \psi \) functions defined by the basic summation formula while that on the right contains \( \psi \)'s that have to be regularised.

In this paper we maintain Barnes’ original notation so that the multiple \( \Gamma \)- and \( \psi \)-functions are defined by

\[
\zeta'_d(0, a \mid d) = \log \frac{\Gamma_d(a)}{\rho_d(d)}, \quad \psi_d^{(q)}(a) = \frac{\partial^q}{\partial a^q} \log \Gamma_d(a),
\]

6 It may happen that a residue vanishes.
7 More complicated sums involving the Hurwitz \( \zeta \)-function have been extensively investigated by Srivastava and others, e.g. [42]. Several methods have been adopted and can give different ‘final’ expressions leading to identities.
where the \((d + 1)\)th \(Γ\)-modular form, \(ρ_d\), is given by

\[
\lim_{a \to 0} \left[ ζ'_d(0, a \mid d) + \log a \right] = -\log ρ_d(d).
\] (14)

Before generalising (12), we derive it by the method of [23] for a reason explained later.

The integral representation of the Barnes \(ζ\)-function allows the left-hand sum in (12) to be written, using an intermediate regularisation,

\[
\sum_{r=d+1} \frac{(-α)^r}{r} Γ(r) \int_0^∞ dτ τ^{r-1} K(τ)
= \lim_{s \to 0} \int_0^∞ dτ \left( \exp(-ατ) - \sum_{r=0}^d \frac{(-ατ)^r}{r!} \right) τ^{s-1} K(τ)
= \lim_{s \to 0} \left( Γ(s)ζ_d(s, a + α \mid d) - \sum_{r=0}^d \frac{(-α)^r}{r!} Γ(s + r)ζ_d(s + r, a \mid d) \right),
\] (15)

where the ‘heat-kernel’ \(K(τ)\) is defined by

\[
K(τ) = \frac{\exp(-aτ)}{\prod_{i=1}^d (1 - \exp(-d_i τ))}.
\]

Since the total quantity in (15) is finite, the individual pole terms that arise in the \(s \to 0\) limit must cancel yielding the identity between generalised Bernoulli polynomials,

\[
ζ_d(0, a + α \mid d) - ζ_d(0, a \mid d) = \sum_{r=1}^d \frac{(-α)^r}{r} N_r(d).
\] (16)

The finite remainder is the required result and equals,

\[
ζ'_d(0, a + α \mid d) - ζ'_d(0, a \mid d) - \sum_{r=1}^d \frac{(-α)^r}{r} R_r(d)
- γ(ζ_d(0, a + α \mid d) - ζ_d(0, a \mid d)) - \sum_{r=1}^d \frac{(-α)^r}{r} (\psi(r) + \gamma) N_r(d),
\]

which, in view of (16), yields

\[
\sum_{r=d+1}^∞ \frac{(-α)^r}{r} ζ_d(r, a \mid d)
= ζ'_d(0, a + α \mid d) - ζ'_d(0, a \mid d) - \sum_{r=1}^d \frac{(-α)^r}{r} R_r(d) - \sum_{r=1}^d \frac{(-α)^r}{r} (\psi(r) + \gamma) N_r(d)
= \log \frac{Γ_d(a + α)}{Γ_d(a)} - \sum_{r=1}^d \frac{(-α)^r}{r} (R_r(d) + H_{r-1} N_r(d)),
\] (17)
where \( H_r \) is the harmonic number, \( H_r = \sum_{k=1}^{r} 1/k \), with \( H_0 = 0 \).

To compare with (12) the form of the remainder \( R_r(d) \) given in Barnes is
\[
R_r(d) = (-1)^r \left( \frac{1}{(r-1)!} \psi_d^{(r)}(a) - N_r(d) H_{r-1} \right),
\]
and we see that the two expressions for the sum, (12) and (17), agree.

We will return to the sphere derivative (8) in section 8, but proceed to generalise the sum (12) by differentiating with respect to \( \alpha \), multiplying by \( \alpha^\lambda \) and integrating back to get,
\[
\sum_{r=d+1}^{\infty} \frac{(-1)^r \alpha^{r+\lambda}}{r+\lambda} \zeta_d(r, a | d) = \int_0^\alpha dw \, w^\lambda \left( \psi_d^{(1)}(a+w) - \psi_d^{(1)}(a) \right)
- \sum_{r=2}^{d} \frac{\alpha^{r+\lambda}}{(r+\lambda)(r-1)!} \psi_d^{(r)}(a).
\]

This formula generalises one given essentially by Nash and O’Connor [44] for the Hurwitz case, \( d = 1 \), when it follows after simple geometric summation. (See also Choi and Nash [45].)

Clearly one could continue playing the same game and derive similar summations but further progress can be made with (18) when \( \lambda = n = 0, 1, 2, \ldots \). We concentrate on the integral on the right-hand side which we write in the form
\[
\int_0^\alpha dw \, w^n \psi_d^{(1)}(a+w) = \alpha^n \log \Gamma_d(a+\alpha) - \delta_{n0} \log \Gamma_d(a) - n \int_0^\alpha dw \, w^{n-1} \log \Gamma_d(a+w),
\]
with the understanding that the final term vanishes when \( n = 0 \).

5. Moments of \( \log \Gamma \) and \( \psi \).

As a preliminary, we discuss integrals of the type
\[
\int_0^\alpha dw \, w^{n-1} \log \Gamma_d(w), \quad \int_0^\alpha dw \, w^n \psi_d^{(1)}(w),
\]
which can be treated by the method in [46] since the algebraic technique applies equally well to the Barnes \( \zeta \)-function, or indeed to any ‘modified’ \( \zeta \)-function of the form,
\[
\zeta(s, w) = \sum_m \frac{1}{(\lambda_m + w)^s},
\]
\[8\] This occurs in many places and \( w \) could be thought of as a \((\text{mass})^2\) or as a Laplace transform/resolvent variable.
as it relies solely on iteration of the basic relation
\[
\frac{\partial \zeta(s, w)}{\partial w} = -s\zeta(s + 1, w) .
\] (22)

The formulae in [46] give indefinite integrals. We here choose the definite forms, (20) with a lower limit of zero. In the Barnes case, the only difference is that \(\log \sqrt{2\pi}\) becomes \(\log \rho_d\) but this can be avoided formally by extending the summation in [46] eq.(109) down to \(l = 0\) and then making some algebraic transformations to arrive at the result,

\[
\frac{1}{n!} \int_0^\alpha dw \, w^n \psi_d^{(1)}(w) = -\sum_{i=1}^d \frac{(-1)^n}{n!} (\zeta_{d+1-i}(-n, d_i | d_i) + H_n \zeta_{d+1-i}(-n, d_i | d_i)) + \sum_{l=0}^n \frac{n^{-l}}{(n-l)!} \frac{(-1)^l}{l!} (\zeta_d(-l, \alpha | d) + H_l \zeta_d(-l, \alpha | d)) .
\] (23)

The first term on the right-hand side of (23) comes from the lower limit, \(w = 0\), which we have treated by iterating the basic recursion,

\[
\zeta_d(s, w | d) = \zeta_d(s, w + d_s | d) + \zeta_{d-1}(s, w | d_s) ,
\] (24)

where \(d_s\) is any degree and \(d_s\) is the \((d-1)\)-vector obtained by omitting the \(d_s\) component from \(d\). Equation (24) has been iterated on \(d\) down to \(\zeta_0(s, w) = 1/w^s\). For convenience we have chosen the \(d_s\) to be \(d_1, d_2, \ldots, d_d\) taken in turn. The notation in (23) then is that the vector \(d_i\) has components \((d_i, \ldots, d_d)\), e.g. \(d_1 = d\) and \(d_d\) is just the single number \(d_d\). Finally we let \(w\) tend to zero. Incidentally, (24) reveals why \(d_s\) is a quasi–period.9

Adamchik derives the result (23) for the simpler Hurwitz function \((d = 1)\) from a series expression for \(\log \Gamma(1 + x)\). Espinosa and Moll [47] use recursion to arrive at the Hurwitz results.

6. The general \(\zeta\)–function.

It is clear from its structure that (12) is the regularised form of an eigenvalue sum and its derivation can be paralleled formally for the general \(\zeta\)–function, (21),

9 For the standard Hurwitz \(\zeta\)–function, the single quasi-period is 1, while for the general case, (21), there are none.
the precise form of the heat-kernel in (15) not being required. The result is exactly equation (17), rewritten in a slightly different notation,

\[
\sum_{r=[\mu]+1}^{\infty} \frac{(-\alpha)^r}{r} \zeta(r, w) = \log \frac{\Gamma(w + \alpha)}{\Gamma(w)} - \sum_{r=1}^{[\mu]} \frac{(-\alpha)^r}{r} \left( FP\zeta(r, w) + H_{r-1} N_r(w) \right), \tag{25}
\]

where \( \mu \) is the order of the infinite set \( \lambda_m \), which could be any sequence of numbers, e.g. [14], but which most commonly arises as the spectrum of a self-adjoint operator, typically the Laplacian on a Riemannian manifold when \( \mu = d/2 \), where \( d \) is the dimension of the manifold. However, one should not limit the meaning of the \( \lambda_m \) to this case. As they have arisen here, they could be the quantities \( m_d \) in (4) and might be interpreted as the spectrum of a pseudo-differential operator, such as the square root, \( \Delta^{1/2} \), or something similar. The \( m_d \) can be realised as the eigenvalues of the pseudo–operator \( -i d |\nabla| \) on the \( d \)-torus, the degrees, \( d_i \), being the inverse radii and \( \mu \) being the dimension, \( d \). In symmetrical cases, for example when several degrees coincide, a partially spherical realisation is possible, in accordance with section 2.

The pole residues are given in terms of the coefficients in the short–time expansion of the heat-kernel, \( \sum \exp \left( - (\lambda_m + w) \tau \right) \) (including the mass-squared term, \( w \)) in a standard way,

\[
N_r(w) = \frac{C_{\mu-r}(w)}{(r-1)!}, \tag{26}
\]

The finite part, \( FP\zeta(r, w) \), is just another symbol for the remainder, \( R_r \), at the possible \( s = r \) pole,

\[
\zeta(s + r, w) \rightarrow \frac{N_r(w)}{s} + R_r(w) \quad \text{as} \quad s \rightarrow 0. \tag{27}
\]

For a given sequence, \( \lambda_m \), there may be no relevant poles at \( s = r \in \mathbb{Z} \) so that \( FP\zeta(r, w) = \zeta(r, w) \) and there would be no need to separate the summations as in (25).

The \( \Gamma \)-functions in (25) are defined in the usual manner, cf (14), by

\[
\zeta'(0, w) = \log \frac{\Gamma(w)}{\rho}, \quad \log \rho = -\zeta'(0), \tag{28}
\]

\[10\] The identity, (16), becomes a standard one between heat-kernel coefficients.

\[11\] An alternative derivation using regularised sums is contained in the next section.
with the ‘massless’ $\zeta$–function, denoted by a tilde, 

$$\tilde{\zeta}(s) = \sum' \frac{1}{\lambda_m^s},$$

any zero modes being omitted. With this convention, one could set $\tilde{\zeta}(s) = \zeta(s,0)$.

If $\psi$–functions are also defined in the usual fashion,

$$\psi^{(q)}(w) = \frac{\partial^q}{\partial w^q} \log \Gamma(w),$$

so that for $q > \mu$,

$$\psi^{(q)}(w) = (-1)^{q}(q-1)! \zeta(q, w),$$

the general formula, (25), looks exactly like the Barnes formula, (12),

$$\sum_{r=1}^{\infty} \frac{(-\alpha)^r}{r} \zeta(r, w) = \log \frac{\Gamma(w + \alpha)}{\Gamma(w)} - \sum_{r=1}^{[\mu]} \frac{\alpha^r}{r!} \psi^{(r)}(w),$$

leading to the generalisation of (18), for example,

$$\sum_{r=1}^{\infty} \frac{(-1)^{r} \alpha^{r+\lambda}}{r + \lambda} \zeta(r, w) = \int_{0}^{\alpha} dw^\lambda \left( \psi^{(1)}(w + v) - \psi^{(1)}(w) \right)$$

$$- \sum_{r=2}^{[\mu]} \frac{\alpha^{r+\lambda}}{(r + \lambda) (r - 1)!} \psi^{(r)}(w).$$

One can also easily produce the formula analogous to (23) for the general $\zeta$–function, (21),

$$\frac{1}{n!} \int_{0}^{\alpha} dw \, w^n \psi^{(1)}(w) = - \frac{(-1)^n}{n!} \left( \tilde{\zeta}'(-n) + H_n \tilde{\zeta}(-n) \right)$$

$$+ \sum_{l=0}^{n} \frac{\alpha^{n-l}}{(n-l)!} \frac{(-1)^l}{l!} \left( \zeta'(-l, \alpha) + H_l \zeta(-l, \alpha) \right).$$

From the above results one can conclude that the combination $\zeta'(-n, \alpha) + H_n \zeta(-n, \alpha)$ is significant. The best way of inverting (32) for this quantity is to compute the multiple integral

$$\frac{(-1)^n}{n!} \int_{0}^{\alpha} dw \, (\alpha - w)^n \psi^{(1)}(w) = \frac{(-1)^n}{n!} \left( \zeta'(-n, \alpha) + H_n \zeta(-n, \alpha) \right)$$

$$- \sum_{l=0}^{n} \frac{(-\alpha)^{n-l}}{(n-l)!} \frac{(-1)^l}{l!} \left( \tilde{\zeta}'(-l) + H_l \tilde{\zeta}(-l) \right),$$
where we have adopted the convention\textsuperscript{12} that
\[
\int_{0}^{\alpha} dw \psi^{(1)}(w) = \log \Gamma(\alpha)
\]
so that (23) and (32) hold for \(n = 0\). We confirm this for the Barnes case, eqn. (23), the right-hand side of which is, at \(n = 0\),
\[
\zeta'_{d}(0, \alpha | d) - \sum_{i=1}^{d} \zeta'_{d+1-i}(0, d_i | d_i).
\]
Using the definitions (13), this can be written
\[
- \log \left( \frac{\rho_{d}(d_1)}{\Gamma_{d}(\alpha | d_1)} \cdot \frac{\Gamma_{d-1}(d_2 | d_2)}{\rho_{d-1}(d_2)} \cdot \ldots \cdot \frac{\Gamma_{1}(d_d | d_d)}{\rho_{1}(d_d)} \right)
\]
\[
= \log \Gamma_{d}(\alpha | d_1)
\]
where we have used the fact that \(\Gamma_{d}(d_* | d) = \rho_{d-1}(d_*)\) and \(\Gamma_{1}(d_d | d_d) = 1\). This is really a check of algebraic accuracy only, since these relations follow from the recursion (24).

7. Regularised products and sums.

With (25), or (30), contact has been made with the notion of regularised products and sums (see e.g. [14,9,10]) because the left-hand side is nothing other than the Weierstrass regularisation, \(\sum^{*} \log \left( 1 + \alpha / (\lambda m + w) \right)\), of the eigenvalue sum \(\sum \log \left( 1 + \alpha / (\lambda m + w) \right)\),
\[
\sum_{r=[\mu]+1}^{\infty} \frac{(-\alpha)^r}{r} \zeta(r, w) = - \sum_{m}^{*} \log \left( 1 + \frac{\alpha}{\lambda m + w} \right)
\]
\[
\equiv - \sum_{m} \left( \log \left( 1 + \frac{\alpha}{\lambda m + w} \right) + P\left( - \alpha / (\lambda m + w) \right) \right),
\]
with
\[
P(x) = x + \frac{x^2}{2} + \ldots + \frac{x^{[\mu]}}{[\mu]}.
\]
This amounts to the subtraction of the first \([\mu]\) terms in the \(\alpha\)-Taylor expansion of \(\log \left( 1 + \alpha / (\lambda m + w) \right)\).

\textsuperscript{12}Adamchik [48] appears to do the same thing in his Proposition 3.
The Barnes equation, (12), is a generalised canonical product expression for the multiple Γ–function.

Equation (8) is, of course, an example of (25) with λₘ the eigenvalues of a Laplace-type operator. In fact, differentiation of the α–Taylor series of (21), with \( w \rightarrow w + \alpha \), constitutes another approach to the derivation of (25) (cf [40] eqn.(9) for the Hurwitz function). In the general case we may refer to (25) as Voros’ relation, [14], eqn (4.12).

Equation (25) can be obtained by differentiation via a slightly different route, [49,43]. First, one defines the Weierstrass α–regularised sum,

\[
\zeta^*(s, w, \alpha) \equiv \sum_m^* \frac{1}{(\lambda_m + w + \alpha)^s} = \sum_m \left( \frac{1}{(\lambda_m + w + \alpha)^s} - \frac{1}{(\lambda_m + w)^s} \right) - \sum_{k=1}^M \left( -s \right)^k \frac{\alpha^k}{(\lambda_m + w)^{k+s}},
\]

(35)

where sufficient terms in the α–Taylor series have been removed in order to ensure convergence for any selected value of \( s \). The integer \( M \) determines how many terms are to be subtracted. In terms of the order, \( \mu \), this means that \( M = \lfloor \mu - s \rfloor \).

Differentiation with respect to \( s \), and setting \( s \) to zero, produces

\[
\zeta^*'(0, w, \alpha) = -\sum_m^* \ln \left(1 + \alpha/(\lambda_m + w)\right).
\]

(36)

Further, the summation over \( \lambda_m \) in \( \zeta^*(s, w + \alpha) \), (35), can be performed to give the continuation

\[
\zeta^*(s, w, \alpha) = \zeta(s, w + \alpha) - \zeta(s, w) - \sum_{k=1}^M \left( -s \right)^k \alpha^k \zeta(s + k, w).
\]

(37)

In particular

\[
\zeta^*'(0, w, \alpha) = \zeta'(0, w + \alpha) - \zeta'(0, w) - \frac{\partial}{\partial s} \sum_{k=1}^{\lfloor \mu \rfloor} \left( -s \right)^k \alpha^k \zeta(s + k, w) \bigg|_{s=0},
\]

(38)

which is just (25), after evaluation of the final term.

---

13 This construction is slightly different from that used by Quine et al [9] in an equivalent analysis. They set \( M = \lfloor \mu \rfloor \) for all \( s \). This is just sufficient to encompass \( s = 0 \). Our definition is more flexible, as we will see, and is in keeping with the work of Dikii [50] and Watson [51].

14 It is therefore consistent to write \( \zeta^* = \zeta^* \). Note also that \( \zeta^*(s, w, \alpha) \neq \zeta^*(s, \alpha, w) \).
This slicker derivation avoids the use of the heat-kernel form of the ζ-function, which really acted only as an intermediary. The present approach can be thought of as the Mellin transform of the previous one.

A similar technique, in a particular case, is used by Nash and O’Connor [44] App.A.

8. Sphere determinants again.

The general sum in (25) was motivated by the example (11), needed in the computation of a specific (sphere) determinant. The use made of it depends on how much is known about the spectrum. For the direct computation of the sphere determinants, obtained by combining (8), (10), (11) and (17), the expression in (17) is calculationally explicit since the relevant quantities can be found from the properties of the Barnes ζ-function. In [23,16] the final expressions were given in terms of the derivatives of the Riemann ζ-function at negative integers.\(^{15}\) In the sphere combination, the nonlocal Barnes remainders, \(R_r\), cancel, in contrast to the computation of Casimir energies, for example.

The further analysis of the hemisphere derivative, (8), can be carried out less specifically by starting from

\[
Z(s, w, \alpha) = \sum_m \frac{1}{\left(\lambda_m + w)^2 - \alpha^2\right)^s} \tag{39}
\]

instead of from (4). Then

\[
Z'(0, w, \alpha) = 2\zeta'(0, w) + \sum_{r=1}^{[\mu/2]} \frac{\alpha^{2r}}{r} \left(R_{2r}(w) + \frac{1}{2}N_{2r}(w)H_{r-1}\right) + \sum_{r=\lceil\mu/2\rceil+1}^{\infty} \frac{\alpha^{2r}}{r} \zeta(2r, w),
\]

where \(\zeta(s, w)\) is the general \(\zeta\)-function, (21).

\(^{15}\) The calculation uses the aforementioned reduction of the Barnes function for unit degrees into Hurwitz functions. However, what constitutes a ‘final’ formula is arguable since the Barnes form could justifiably be regarded as the end, seeing that what remains is ‘merely’ a numerical or cosmetic affair. For the full–sphere, there is the minor point that one has to compute both the Dirichlet and Neumann hemisphere determinants.
The final sum in (40) follows from the sum in (25) by averaging,

\[ \sum_{r=[\mu/2]+1}^{\infty} \frac{\alpha^{2r}}{r} \zeta(2r, w) = \log \frac{\Gamma(w + \alpha) \Gamma(w - \alpha)}{\Gamma^2(w)} - \sum_{r=1}^{\lfloor \mu/2 \rfloor} \frac{\alpha^{2r}}{r} \left( R_{2r}(w) + H_{2r-1}N_{2r}(w) \right), \]

(41)

and the combination with (40) yields the mentioned cancellation of the \( \zeta(s, w) \) remainders, and also of the \( \log \Gamma^2(w) \), leaving the formal, but definite, expression

\[ Z'(0, w, \alpha) = \log \frac{\Gamma(w + \alpha) \Gamma(w - \alpha)}{\rho^2} - \sum_{r=1}^{\lfloor \mu/2 \rfloor} \frac{\alpha^{2r}}{r} H_{r-1}^{O}N_{2r}(w) \]

(42)

where \( H_{r}^{O} \) is the odd harmonic number, \( H_{r}^{O} = \sum_{k=0}^{r} 1/(2k + 1) \).

For the \( d \)-hemisphere, this result is given in [23] and, as mentioned there, it illustrates the fact that the determinant of a product is not the product of the determinants, at least not if the determinant is defined by \( \zeta \)-function regularisation. The eigenvalues in (39) factorise,

\[ (\lambda_{m} + w)^2 - \alpha^2 = (\lambda_{m} + w - \alpha) (\lambda_{m} + w + \alpha) \]

(43)

and the first term on the right-hand side of (42) gives the product of the determinants of the individual factors. The remainder is a correction or ‘anomaly’ which was first noticed in physical contexts by Allen [52] and by Chodos and Myers [53] and has attracted more recent mathematical, and physical, attention. It is trivially zero when \( \alpha = 0 \).

For the hemisphere, \( \lambda_{m} = m.d \) and \( \zeta(s, w) \) is the Barnes \( \zeta \)-function with unit degrees. If the Neumann and Dirichlet hemisphere expressions are added, so as to give the full-sphere result, the anomaly contributions cancel in odd dimensions, as can be specifically checked. Actually, for odd-dimensional closed manifolds, this vanishing is a general result and follows from properties of the heat-kernel expansion coefficients.

Everything is quite definite in (42), which could be taken as the final answer. Expressions for the hemisphere Laplacian determinants \( \alpha = (d-1)/2 \) were given in [23] in terms of the Barnes function. However, numerical calculation might require one to go further and express everything in terms of the Hurwitz or Riemann \( \zeta \)-function. (This is arguable.) The details are in [16] where a concrete formula is...
produced,

\[ \zeta_{S_d}(0) = \frac{1}{(d-1)!} \sum_{k=0}^{d-1} (1 - (-1)^{d+k}) \left( S_{d-1}^{(k)} + S_d^{(k+1)} \right) \zeta_R(-k) \]

\[ - \sum_{r=1}^{u} \frac{(d-1)^{2r}}{2^{2r} r^2} H_r - N_{2r}^R(d) + \log(d-1). \]  (44)

Here, \( S_j^{(k)} \) are Stirling numbers and \( N^R \) is the sum of the Dirichlet and Neumann hemisphere \( \zeta \)-function residues,

\[ N_{2r}^R(d) = N_{2r}^N(d) + N_{2r}^D(d). \]

These residues are given in terms of generalised Bernoulli polynomials for which there exists the handy calculational form,

\[ N_{2r}^R(d) = \frac{2^{2r-d-1}}{(d-1)!(2r-2)!} \frac{d^{2r-2}}{2} \prod_{i=1}^{d-2} (x-i) \bigg|_{x=(d-1)/2}, \]

and it is easily confirmed that this is zero for odd \( d \), although we actually know this at an earlier stage.

The graph of the results, up to dimension 23, for the Laplacian determinant shows a curious difference between odd and even dimensions. The values diverge as \( d \) increases. This might not be surprising as there are fundamental differences between odd and even dimensional spaces, spheres especially. Other values of \( \alpha \) in (2) can be treated in a like manner.

We now make some comments on related, full-sphere calculations. Choi and Srivastava [18], following Voros, use (25) directly for the two– and three–sphere. The Laplacian eigenvalues are written in the usual form, and the standard binomial degeneracies are used. Equation (25), with \( w = 0 \), is then taken as an equation for \( \log \Gamma(\alpha) \) in terms of \( \log \Gamma(0) \), which is easily found,

\[ \log \Gamma(\alpha) = \log \Gamma(0) + \sum_{r=1}^{[\mu]} (-\alpha)^r \left( R_r + H_{r-1} N_r \right) - \sum_{m}^* \log \left( 1 + \frac{\alpha}{\lambda_m} \right). \]  (45)

The correction terms require \( R_r \) and \( N_r \) to be determined from the same \( \alpha = 0 \) information, which is also relatively straightforward. The most awkward part is the evaluation of the last term, \( i.e. \) the regularised sum. This is obtained from the definition, (34), after using some specific \( \zeta \)-function summations. Although
quite workable for small dimensions, this method obscures the general nature of the cancellations that must occur.

In an interesting paper, Quine and Choi [19] express the determinants through regularised sums and produce an explicit formula for the $d$-sphere using a method comparable to our own, [16], and obtain the equivalents of (42) and (44). They do not split the mode problem into Dirichlet and Neumann hemispheres.

**Bessel approach to the general $\zeta$-function.**

An alternative continuation for the $\zeta$-function, (39), uses the Bessel identity mentioned in section 2. This gives

$$Z(s, w, \alpha) = \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty d\tau K(\tau) \left( \frac{\tau}{2\alpha} \right)^{s-1/2} I_{s-1/2}(\alpha \tau),$$

(46)

where $K(\tau)$ is the ‘cylinder’ heat-kernel

$$K(\tau) = \sum_m e^{-(\lambda_m + w)\tau}.$$

The terminology is a reflection of the fact that the squares, $(\lambda_m + w)^2$, are usually the eigenvalues of a second order Laplace-type operator so that $K(\tau)$ is the heat-kernel of a square-root (pseudo)-operator, with different locality properties and possible log $\tau$ terms in its expansion.

The form (46) has been exploited by Bytsenko and Williams [54] in connection with the multiplicative anomaly.

9. Implicit eigenvalues.

In the sphere example, the sum, (11), is simply an intermediate calculational quantity. It can be made to play a more important role in the evaluation of determinants especially when the spectrum is not known explicitly.

The key idea, [49], is to turn (30) around to give a formula, this time for log $\Gamma(w)$,

$$\log \Gamma(w) = \sum^\ast \log \left( 1 + \frac{\alpha}{\lambda_m + w} \right) + \log \Gamma(w + \alpha) - \sum_{r=1}^{[\mu]} \frac{\alpha^r}{r!} \psi_r^{(r)}(w),$$

(47)

and then to observe, trivially, that the right-hand side has to be independent of $\alpha$. It thus can be calculated at any convenient value. In our work [49,55,43,56]
the natural, infinite mass, limit $\alpha \to \infty$, was selected. The $\alpha$–dependence has to disappear from the asymptotic form and what remains must be $\log \Gamma(w)$. For example the last term in (47) can be disregarded since its $\alpha$–dependence is explicit, in finite terms. Note that one is not required to know anything about the nature of the $\psi^{(r)}$.

The asymptotic behaviour of $\log \Gamma(w + \alpha)$ as $\alpha \to \infty$ has been determined ([14,57,58], for example) in terms of the heat-kernel coefficients, $C_n(w)$, and contains no $\alpha$–independent terms, apart from the normalisation, $\log \rho$. Hence we obtain the equation

$$\text{Det} D^{-1} = \frac{\Gamma}{\rho} \approx \lim_{\alpha \to \infty} \prod^* \left(1 + \frac{\alpha}{\lambda_m}\right) \approx \lim_{\alpha \to \infty} \prod m \left(1 + \frac{\alpha}{\lambda_m}\right) \exp \sum m P(-\alpha/\lambda_m)$$

(48)

with the understanding that only the $\alpha$–independent part of the right-hand side is to be retained. The notation has been streamlined a little by dropping explicit reference to the parameter $w$ and taking the $\lambda_m$ as the eigenvalues of the generic operator $D$. Keeping only the $\alpha$–independent part means that the $P$-polynomial bit of (48) is usually irrelevant.

This equation is particularly valuable when the eigenvalues are given implicitly as the roots of some equation, $F(\alpha) = 0$, since, under certain conditions which are often satisfied, the Mittag-Leffler theorem says that $F(\alpha) \sim \prod \lambda (1 + \alpha/\lambda)$ and the asymptotic behaviour of $F$ is often a known episode in special function theory, if one is lucky.

In practice, the method is not necessarily straightforward as the function $F$ often occurs after separation of variables and there are remaining summations to be dealt with, but it has been applied to $d$–balls for scalars and higher spin. The method also works nicely for bounded Möbius corners [43], where, one might note, the sum (12) occurs with $\alpha$ the parameter in the Robin boundary condition.

There are several variants of the above technique but all involve the asymptotic behaviour of special functions. The spherical cap has been successfully treated by Barvinsky et al. [59], using the asymptotics of Legendre functions and the ball has been dealt with in various ways that all require Olver–Bessel asymptotics, e.g. [60–62].
10. A Kaluza-Klein interpretation.

The previous development suggests that we make $\zeta'(-n, \alpha) + H_n \zeta(-n, \alpha)$ the subject of equation (33) but rather than the straight derivatives we try to retain a Barnesian formulation and rewrite this quantity in terms of a new $\Gamma$–function.

The best way of doing this is to introduce the $\zeta$–function that yields this combination as its derivative at 0. This has formal and manipulative benefits. Therefore define the $\zeta$–function,

$$
\zeta^{(n)}(s, w) = \frac{\Gamma(s - n)}{\Gamma(s)} \zeta(s - n, w),
$$

so that

$$
\zeta^{(n)'}(0, w) = \frac{(-1)^n}{n!} \left( \zeta'(-n, w) + H_n \zeta(-n, w) \right).
$$

The new $\Gamma$–function and modulus, $\Gamma^{(n)}(w)$ and $\rho^{(n)}$, are defined by

$$
\log \frac{\Gamma^{(n)}(w)}{\rho^{(n)}} = \zeta^{(n)'}(0, w),
$$

and

$$
\log \rho^{(n)} = - \lim_{w \to 0} \left( \zeta^{(n)'}(0, w) + b_0 \log w \right)
$$

where

$$
= - \tilde{\zeta}^{(n)'}(0)
$$

so that $\Gamma^{(0)}(w) = \Gamma(w)$ and $\rho^{(0)} = \rho$. In (52), $b_0$ is the number of zero $\lambda_m$ modes.

Corresponding $\psi$–functions are defined by

$$
\psi^{(n,q)}(w) = \frac{\partial^q}{\partial w^q} \log \Gamma^{(n)}(w),
$$

(to complete the formalism).

Relation (22) gives the recursion on the additional dimensions,

$$
\frac{\partial}{\partial w} \zeta^{(n)'}(0, w) = - \zeta^{(n-1)'}(0, w).
$$

The basic idea is to set $\zeta^{(n)}(s, w)$ in place of $\zeta(s, w)$ in formal manipulations and then, if desired, return to $\zeta(s, w)$ via (49).
Equation (33) now reads in the new notation,

\[
\log \Gamma^{(n)}(\alpha) = \frac{(-1)^n}{n!} \int_0^{\alpha} dw (\alpha - w)^n \psi^{(1)}(w) - \sum_{l=0}^{n-1} \frac{(-\alpha)^{n-l}}{(n-l)!} \log \rho^{(l)}. \tag{55}
\]

An interpretation of (49) is the following. When the \(\lambda_m\) are the eigenvalues of the Laplacian on a manifold, \(\mathcal{M}\), the derivative, \(\tilde{\zeta}'(0)\), is, up to a factor, the one-loop effective action of scalar quantum field theory on the ‘space-time’, \(\mathcal{M}\). If a mass is added then one requires \(\zeta'(0, m^2)\).

In the same situation, the quantity we have denoted by \(\log \left( \frac{\Gamma^{(n)}(w)/\rho^{(n)}}{\rho^{(n)}} \right)\) is, again up to a factor, the effective action density on the Kaluza-Klein manifold, \(\mathbb{R}^{2n} \times \mathcal{M}\). (It is a density in the noncompact factor, \(\mathbb{R}^{2n}\).) It therefore can be looked upon, when exponentiated, as determining the functional determinant density on this manifold. The Kaluza-Klein eigenvalues are \(\lambda_m + k^2 + w\) where \(k\) is a 2n-dimensional real vector and the integral over \(k\) is taken with a certain \(4\pi\) volume normalisation to make (49) true.

The situation with which we are most concerned is when the \(\lambda_m\) are the eigenvalues of a linear pseudo-operator on \(\mathcal{M}\), as mentioned earlier for the Barnes \(\zeta\)-function. The ‘Kaluza-Klein’ eigenvalues are \(\lambda_m + k^2 + w\), where \(k\) is a real \(n\)-vector, with \(0 \leq k_i \leq \infty\). The ‘Kaluza-Klein’ manifold, this time, is \(\mathbb{R}^n \times \mathcal{M}\).

In the specific Barnes case, taking \(\mathcal{M}\) to be the \(d\)-torus, the Kaluza-Klein manifold \(\mathbb{R}^n \times \mathcal{M}\) can be thought of as a \((d+n)\)-torus with \(n\) infinite radii.

11. Generalised Glaisher-Kinkelin-Bendersky constants.

The usual Glaisher-Kinkelin-Bendersky constants, \(A_p\), are defined, [8], as the \(N\)-independent parts of the sums,

\[
\log A_p = \lim_{N \to \infty} \sum_{m=1}^{N} m^p \log m \biggl|_{N\text{-independent}} = \lim_{N \to \infty} \log \prod_{m=1}^{N} m^{m^p} \biggl|_{N\text{-independent}} , \tag{56}
\]

which, for \(p = 0\), is Stirling’s formula (with the Stirling constant, \(A_0 = \sqrt{2\pi}\)).

The original, [63], Glaisher constant, \(A_1 = \exp \left( \frac{1}{12} - \zeta'_R(-1) \right)\), has surfaced occasionally in statistical physics, \(e.g. [64]\), App.B, as noted by Voros, [14] and the Bendersky constants in vacuum quantum field theory, \(e.g. [65]\).

It is clear from (56) that \(\log A_p\) is related to the Riemann \(\zeta\)-function, \(\zeta'_R(-p)\). The exact connection can be obtained by going via the Hurwitz function, noting
that, directly from the definition,

\[
\log A_p = \lim_{N \to \infty} \left( \zeta_R'(-p, N + 1) - \zeta_R'(-p, 1) \right)
\]

(57)

and then using, without thought, the standard, large \( w \) expansion of \( \zeta_R(s, w) \), [66], to arrive at

\[
\log A_p = -\zeta_R'(-p) - H_p \zeta_R(-p).
\]

(58)

This simple formula is given by Adamchik [48].

In our notation, and normalisation, (see (50)),

\[
\log A_p = -p!(-1)^p \zeta_R^{(p)}(0),
\]

and, in this way, one can give the usual Glaisher–Kinkelin–Bendersky constants a massless Kaluza–Klein determinantal interpretation.

The \( \Gamma \)-modular forms, \( \rho^{(n)} \), (52), are ‘new’ constants, and generalised Glaisher–Kinkelin–Bendersky numbers,

\[
G_n = \rho^{(n)}
\]

can be defined in the general situation to mimic the result (58), which holds for the Hurwitz \( \zeta \)-function case (when \( \zeta \) is the Riemann \( \zeta \)-function, \( \zeta_R \)).

In terms of the \( \Gamma \)-function, the modular form could be defined as the \( w \)-independent part of \( \log \Gamma^{(n)}(w) \) as \( w \to \infty \), so that our new constants are,

\[
\log G_n = \log \rho^{(n)} = \lim_{w \to \infty} \log \Gamma^{(n)}(w)
\]

\( w \)-independent,

(59)

illustrating the formal similarity to (56).

For the Barnes case, explicit forms follow from (23),

\[
\log G_n^{(d)}(d) = -\left( \frac{-1}{n!} \right)^n \sum_{i=1}^{d} \left( \zeta_{d+1-i}^{(d)}(-n, d_i \mid d_i) + H_n \zeta_{d+1-i}(-n, d_i \mid d_i) \right).
\]
12. Kaluza-Klein regularised sums.

The use of the Kaluza–Klein \( \zeta^{(n)}(s, w) \) in order to facilitate the construction of the sphere derivative, \( \zeta'(-n, a, \alpha | d) \), (see (4)), is contained in [67]. The presentation of this in [68] is adapted here to a treatment of the general \( \zeta \)-function, (21), using regularised sums.

The combination (49) can be introduced into equation (37) to give

\[
\zeta^{(n)\ast}(s, w, \alpha) = \zeta^{(n)}(s, w + \alpha) - \zeta^{(n)}(s, w) - \sum_{k=1}^{M} \left(\frac{-s}{k}\right) \alpha^k \zeta^{(n)}(s + k, w),
\]

so that

\[
\zeta^{(n)\ast}'(0, w, \alpha) = \zeta^{(n)'}(0, w + \alpha) - \zeta^{(n)'}(0, w) - \frac{\partial}{\partial s} \sum_{k=1}^{[\mu]+n} \left(\frac{-s}{k}\right) \alpha^k \zeta^{(n)}(s + k, w) \bigg|_{s=0}.
\]

Evaluation of the last term, produces immediately the equation corresponding to (25),

\[
\sum_{r=[\mu]+n+1}^{\infty} \frac{(-\alpha)^r}{r} \zeta^{(n)}(r, w) = \log \frac{\Gamma^{(n)}(w + \alpha)}{\Gamma^{(n)}(w)} - \sum_{r=1}^{[\mu]+n} \frac{(-\alpha)^r}{r} \left( R^{(n)}_r + H^{(n)}_{r-1} N^{(n)}_{r-1} \right)
\]

\[
= \log \frac{\Gamma^{(n)}(w + \alpha)}{\Gamma^{(n)}(w)} - \sum_{r=1}^{[\mu]+n} \alpha^r \psi^{(n,r)}(w),
\]

where the residue and remainder are defined by

\[
\zeta^{(n)}(s + r, w) \to \frac{N^{(n)}_r(w)}{s} + R^{(n)}_r(w) \quad \text{as} \quad s \to 0.
\]

The equality,

\[
\zeta^{(n)\ast}'(0, w, \alpha) = \sum_{r=[\mu]+n+1}^{\infty} \frac{(-\alpha)^r}{r} \zeta^{(n)}(r, w),
\]

is shown exactly as in the usual case \( (n = 0) \) by summing over \( \lambda_m \) and integrating over \( k \) after the \( \alpha \)-expansion of \( \log \left( 1 + \alpha / (k^2 + \lambda_m + w) \right) \), or of \( \log \left( 1 + \alpha / (k.1 + \lambda_m + w) \right) \) depending on the interpretation of the eigenvalues.

Equation (62) extends (30) and is our final, formal generalisation.
If \( \zeta^{(n)} \) is viewed only as an intermediate quantity one should return to the basic \( \zeta \)-function, \( \zeta(s, w) = \zeta^{(0)}(s, w) \), and express the \( N_r^{(n)} \) and \( R_r^{(n)} \) in terms of the \( N_r \) and \( R_r \) in order to rewrite the last summation in (62).

From the definition (49), the poles of \( \zeta^{(n)}(s, w) \) at \( s = r \) in the summation range, \( 1 \leq r \leq [\mu] + n \), divide into two sets, those due to \( \Gamma(s-n)/\Gamma(s) \), i.e. \( r = 1, \ldots, n \), and those coming, possibly, from the \( \zeta \)-function, i.e. \( r = n+1 \ldots, [\mu] + n \).

A straightforward calculation produces,

\[
\sum_{r=1}^{[\mu]+n} \frac{(-\alpha)^r}{r} \left( R_r^{(n)} + H_{r-1}N_r^{(n)} \right)
\]

\[
= \sum_{r=1}^{n} \frac{(-\alpha)^r}{r!} \frac{(-1)^{n-r}}{(n-r)!} \left( \zeta'(r-n, w) + H_{n-r} \zeta(r-n, w) \right)
\]

\[+
\sum_{r=n+1}^{[\mu]+n} \frac{(-\alpha)^r r - n - 1}{r!} \left( R_{r-n}(w) + H_{r-n-1}N_{r-n}(w) \right),
\]

and so equation (62) can be written entirely in terms of the original \( \zeta \)-function, \( \zeta(s, w) \).

After some shift in the summation variables and multiplication by factors, one finds the relation,

\[
\sum_{k=1}^{[\mu]+n} \frac{(-\alpha)^k}{k} \left( \frac{n+k}{n} \right)^{-1} \zeta(k, w)
\]

\[= \alpha^{-n} \left( \zeta'(-n, w + \alpha) + H_n \zeta(-n, w + \alpha) \right)
\]

\[\sum_{k=0}^{n} \alpha^{-k} \left( \frac{n}{k} \right) \left( \zeta'(-k, w) + H_k \zeta(-k, w) \right)
\]

\[+ \sum_{k=1}^{[\mu]} \frac{(-\alpha)^k}{k} \left( \frac{n+k}{n} \right)^{-1} \left( R_k(w) + H_{k-1}N_k(w) \right),
\]

which can of course be proved directly without difficulty. One way is given in [38], p.13, for the simpler Hurwitz \( \zeta \)-function (\( \mu = 1 \)).

**13. Conclusion and comments.**

In this work, using the Barnes \( \zeta \)-function, we have presented improved and generalised expressions related to recent, and not so recent, work concerning sums of \( \zeta \)-functions which arise incidentally in the computation of sphere determinants.
We have concentrated on the full-sphere results but the power of the Barnes function shows up when computing quantities on the orbifold factors, $S^d/\Gamma$, where $\Gamma$ is a polytope symmetry group. We leave these considerations, as well as other factorings such as lens spaces, for another time.

Extensions of our results to forms and spinors, cf [46], is straightforward and has topological and possible M-theory applications, [69].

A determinant interpretation of the Glaisher–Kinkelin–Bendersky constants has been given, based on the Adamchik form, (58). These have been generalised to the Barnes $\zeta$–function case and also to the more general $\zeta$–function, (21).
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