Tiny Graviton Matrix Theory On Time-Dependent Background

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Abstract

In this article we construct a tiny graviton matrix model for type IIB string theory on a plane-wave background with null dilaton. For the linear null dilaton case, we analyze its vacuum and the excitation spectrum around the vacuum, and discuss the time-dependent fuzzy three-sphere solutions and their evolution. It turns out that at very late time the non-Abelian fuzzy degrees of freedom disappear, which indicates the appearance of perturbative strings.
1 Introduction

One of central problems in the study of string theory is to look for its non-perturbative definition. One decade ago, Banks, Fischler, Shenker and Susskind (BFSS) suggested that a matrix model could be a candidate for non-perturbative definition of string theory (M theory)\[^1\]. They constructed so-called BFSS matrix model on the flat background, and conjectured that large $N$ limit of their model is equivalent to uncompactified M theory in the infinite momentum frame. In [2, 3], the authors showed how to obtain BFSS model from discrete light-cone quantization (DLCQ) of M theory. The BFSS model is a $(0 + 1)$-dimensional super-Yang-Mills gauge theory, or a matrix model, and its Hamiltonian can be obtained from the nonrelativistic Hamiltonian of D-particles. Remarkably, there is another way to obtain this model by doing matrix regularization of super-membrane in flat space-time.

Since the construction of BFSS model, people has tried hard to construct the matrix models on general backgrounds. For the weakly curved spacetime, the authors in [4] studied the non-Abelian D-particle action, which could be essential to get the matrix model action. Even for this case, the complete D-particle action is still not clear partially due to notorious difficulty in constructing non-Abelian D-brane action. However, for one class of curved spacetime, so-called plane-wave background, the corresponding BMN matrix model was well-established and much better understood[5]. Similar to the matrix model in flat spacetime, the BMN model could be obtained from the matrix regularization of supermembrane on the background[6]. This suggest that the matrix regularization of super-membrane could be an important way to construct the matrix model in curved spacetime. For the general curved spacetime, the action of the supermembrane could not be determined completely[7]. However, it is remarkable, for a general class of plane-wave-like background, the matrix model could be constructed to all order[8].

There are also other nonperturbative definition of various kinds of string theory. For IIA theory, one non-perturbative definition of string theory is matrix string theory[9]. For IIB theory, one interesting construction is the IKKT matrix model [10], which is a $(0 + 0)$-dimensional theory. Quite recently, a $(0 + 1)$-dimensional matrix model associated to IIB theory on plane-wave background was proposed by Sheikh-Jabbari[11,12]. His observation is that the matrix models, including BFSS and BMN, can be understood as the theory of gravitons. In BFSS case, the D-particles are gravity waves in 11-dimensional flat spacetime, while in BMN case, the fundamental degrees of freedom are tiny graviton, which are membrane gravitons with very small

\[^1\]For an alternative proposal to the Tiny graviton matrix theory, see [12].
size. In other words, the DLCQ of M-theory in 11-dimensional background could be described by a theory of $N$ membrane graviton. This led Sheikh-Jabbari to conjecture that DLCQ of IIB superstring theory on plane-wave background is a $0 + 1$-dimensional matrix model, whose Hamiltonian can be written down through the action of tiny gravitons. In this case the tiny gravitons are very small spherical D3-branes, so the Hamiltonian can be obtained from light-cone matrix regularization and quantization of D3-brane Hamiltonian on the background. More study about tiny graviton matrix model can be found in [13] [14] [15].

The search for a non-perturbative definition of string theory is not only just to complete the formulation of the theory, but may also has profound physical applications. In particular, at the very early universe, the appropriate description of the physics there could be in terms of nonperturbative string degrees of freedom since the string coupling could be strong. One concrete realization of this idea is so called matrix big bang model and its cousins. The original matrix big bang is starting from the IIA string theory in null dilaton background[16]. Formally, this IIA string theory is exactly solvable perturbatively but since the string coupling becomes strong at null infinity, the perturbative string theory breaks down. It turned out that the matrix degrees of freedom, rather than the point particle or the perturbative string, could describe the physics near the null singularity. Following the prescription in [2] [3], a dual matrix string theory was proposed. The study of the plane-wave matrix big bang model via DLCQ was carried out very recently[17]. For a nice review on this model, its generalization and relevant references, please see [18].

In [19], one class of solvable IIB string theory in linear null dilaton background has been investigated. The background is

$$ds^2 = -2dx^+dx^- - \lambda(x^+)x_i^2dx^+dx^- + dx^idx^i,$$

$$\phi = \phi(x^+),$$

$$(F_5)_{1234} = (F_5)_{5678} = 4f(x^+),$$

(1.1)

in which one has to require

$$\lambda(x^+) = -\frac{1}{4}\phi''(x^+) + f^2(x^+e^{2\phi(x^+)},$$

(1.2)

to satisfy the equations of motion. This background keeps sixteen non-linearly realized supersymmetries. The background has continuous symmetry algebra $so(4) \oplus so(4) \oplus h(8)$, in which two $so(4)$ are the residual rotational symmetry algebra and $h(8)$ is the Heisenberg-type algebra.
Generically the background (1.1) has curvature singularity. To show this clear, it is better to work in the Einstein frame. For the linear null dilaton case we are interested in this paper,
\[ \phi = -cx^+ \quad \text{with } c > 0 \] (1.3)
the issue has been discussed carefully in section two of [19]. It has been shown that no matter what the background flux is the background (1.1) is geodesically incomplete and \( x^+ \to -\infty \) is a curvature singularity where an inertial observer will feel a divergent tidal force. The existence of singularity and the fact that the string coupling is also divergent at \( x^+ \to -\infty \) both suggest that the perturbative string degrees of freedom to which the graviton belong can not give correct description of the physics there, and instead nonperturbative string will take over and play a fundamental role. In other words, to describe the physics near \( x^+ \to -\infty \) calls for a nonperturbative definition of string theory, which might be a matrix model.

Similar to the usual plane-wave background, one can do light-cone quantization of perturbative string in the background (1.1). However, due to the absence of translational Killing vector along \( x^- \), the lightcone Hamiltonian is not conserved. Actually both the Hamiltonian and perturbative string states are time-dependent. Nevertheless, because this background is invariant under translation in the \( x^- \) direction, the compactification of \( x^- \) on a circle gives a conserved lightcone momentum. This fact also suggests that DLCQ of string and D-brane in this background is feasible. In this paper, we will follow the proposal in [11] to construct a time-dependent matrix model on the background (1.1).

This article is organized as follows. In section 2, after constructing the matrix model, we analyze the spectrum around the vacuum, and compare its spectrum with the spectrum of perturbative string on this background. In section 3, we look for and discuss the fuzzy three sphere solutions. In our cases, the fuzzy three-sphere solutions are time-dependent, with their radius varying with time. At last, we end with conclusions and some discussions. In the Appendix, we discuss the giant graviton in the background and show that its dynamics is the same as that of fuzzy three sphere.

## 2 The matrix model

To construct the matrix model, we follow the construction proposed in [11]. Let us start from the light-cone Hamiltonian of D3-brane. The Born-
Infeld and Chern-Simons action for D3-brane is
\[ S = \int d\tau d^3\sigma e^{-\phi} \sqrt{-\det(G_{\mu\nu}\partial_{\mu}X^\mu\partial_{\nu}X^\nu)} + \int C_4. \]  

(2.1)

Here \( G_{\mu\nu}, C_4 \) are the spacetime metric and Remond-Remond four-form potential of the background. The \( X^\mu \)'s are the embedding coordinates. The indices \( \mu', \nu' = 0, 1, 2, 3 \) label the D-brane’s world-volume coordinates, and \( i', j', \ldots \) denote the spatial ones.

In the light-cone gauge,
\[ X^+ = \tau, \ G_{\mu\nu}\partial_0 X^\mu\partial_\nu X^\nu = 0. \]

(2.2)

We use \( P_- \) to denote the conjugate momentum of \( X^- \), \( P^+ = -P_- \), and \( P_I \) to denote the conjugate momenta of \( X^I, I = 1, 2, \ldots, 8 \). Using the light-cone Hamiltonian density \( \mathcal{H} = P_-\partial_\tau X^- + P_I\partial_\tau X^I - \mathcal{L} \), we have the bosonic part of light-cone Hamiltonian:
\[ H_{bos} = \int d^3\sigma \left( \frac{1}{2P_+} (P_i^2 + P_a^2) + \frac{\lambda(\tau)P_+}{2}(X_i^2 + X_a^2) \right. \]
\[ + \frac{e^{-2\phi(\tau)}}{2P_+} \left( \frac{1}{3!} \{X^i, X^j, X^k\} \{X^i, X^j, X^k\} \right. \]
\[ + \frac{1}{3!} \{X^a, X^b, X^c\} \{X^a, X^b, X^c\} \]
\[ + \frac{1}{2!} \{X^i, X^j, X^a\} \{X^i, X^j, X^a\} \right. \]
\[ + \left. \frac{1}{2!} \{X^i, X^j, X^a\} \{X^i, X^j, X^a\} + \frac{1}{2!} \{X^i, X^a, X^b\} \{X^i, X^a, X^b\} \right) \]
\[ - \frac{f(\tau)}{6} \left( \epsilon^{ijkl} X^i \{X^j, X^k, X^l\} + \epsilon^{abcd} X^a \{X^b, X^c, X^d\} \right) \right], \]

(2.3)

where \( i, j, \ldots = 1, 2, 3, 4, a, b, \ldots = 5, 6, 7, 8 \). In the above relation the Nambu three brackets are used,
\[ \{X^I, X^J, X^K\} \equiv \epsilon^{ijkl} X^i \partial^J \partial^k X^J. \]

(2.4)

The fermionic part of the Hamiltonian can be obtained through superspace techniques. On the background, it is given by
\[ H_{fer} = \int d^3\sigma \left[ e^{\phi} \gamma^{\alpha\beta} \psi_{\alpha\beta} \right. \]
\[ + 2e^{-\phi} P_+ (\gamma^{\alpha\beta} (\sigma^{ij})^\alpha_{\delta} \{X^i, X^j, \psi_{\delta\beta}\} + \psi^{\alpha\beta} (\sigma^{ab})_{\alpha} \{X^a, X^b, \psi_{\delta\beta}\}) \]
\[ - e^{\phi} \gamma_{\alpha\beta} \psi^{\alpha\beta} - \frac{2e^{-\phi}}{P_+} (\gamma^{\alpha\beta} (\sigma^{ij})^\alpha_{\delta} \{X^i, X^j, \psi^{\alpha\beta}\}) \]
\[ + \psi^{\alpha\beta} (\sigma^{ab})_{\alpha} \{X^a, X^b, \psi^{\alpha\beta}\} \right]. \]

(2.5)

\(^2\text{We will always set } l_s = 1 \text{ in this paper.}\)
Here the $\psi$’s carry two spinor indices, corresponding to two $SO(4)(SU(2)_L \times SU(2)_R)$’s Weyl spinor representations respectively.

To do discrete light-cone quantization, the light-like direction should be compactified.

$$X^- = X^- + R_-.$$  \hspace{1cm} (2.6)

With this compactification, $P^+$ is quantized,

$$P^+ = \frac{J}{R_-},$$  \hspace{1cm} (2.7)

where $J \in \mathbb{Z}$.

Next one has to quantize Nambu three brackets in order to have a matrix regularization of the D3-brane Hamiltonian. It is well known that the quantization of Nambu odd brackets is very difficult, because some important properties of Nambu odd brackets will get lost if we apply the usual way of quantization of Nambu even brackets. In [11], Sheikh-Jabbari proposed a way to quantize the Nambu odd bracket by introducing a special constant matrix. For Nambu three brackets, the constant matrix is $\mathcal{L}_5$, and the quantization is to replace $\{F, G, K\}$ with $J[F_{J \times J}, G_{J \times J}, K_{J \times J}, \mathcal{L}_5]$, where

$$[F_1, F_2, F_3, F_4] = \frac{1}{24} \epsilon^{i_1 i_2 i_3 i_4} F_{i_1} F_{i_2} F_{i_3} F_{i_4}.  \hspace{1cm} (2.8)$$

The $\mathcal{L}_5$ is made out of the direct product of unit matrix $I_{4 \times 4}$ and $SO(4)$ chirality operator $\Gamma^5$. In this paper, we only need one property of it. Namely there exist matrices $X^i_{J \times J}$, $i = 1, 2, 3, 4$, which satisfy

$$[X^i_{J \times J}, X^j_{J \times J}, X^k_{J \times J}, \mathcal{L}_5] = -\frac{R^2}{J} \epsilon^{ijkl} X^l_{J \times J},$$

$$\sum_{i=1}^{4} (X^i_{J \times J})^2 = R^2 I_{J \times J}.  \hspace{1cm} (2.9)$$

Recalling that a three sphere of radius $R$ can be embedded in four dimensional space as

$$\{X^i, X^j, X^k\} = R^2 \epsilon^{ijkl} X^l,$$

$$\sum_{i=1}^{4} (X^i)^2 = R^2,  \hspace{1cm} (2.10)$$

we know that the matrices satisfying (2.9) is actually a realization of fuzzy three sphere $S^3_F$ of radius $R$. For the explicit construction, please see [11] and [23].
Now for the sector with $J$ units of light-cone momentum, we replace $X^I, P^I, \psi_{\alpha\beta}, \psi_{\dot{\alpha}\dot{\beta}}$ with $J \times J$ matrices.

\[
X^I \rightarrow X^I_{J \times J} \\
P^I \rightarrow J \Pi^I_{J \times J} \\
\psi \rightarrow \sqrt{J} \psi_{J \times J}.
\] (2.11)

In the following part of this paper, we will denote $X^I_{J \times J}, \Pi^I_{J \times J}, \psi_{J \times J}$ with $X^I, \Pi^I, \psi$. We wish this will not bring any confusion.

With all these preparation, we can finish constructing the matrix model for DLCQ of IIB string theory on background (1.1) by doing the replacements:

\[
\frac{1}{P^+} \int d^3 \sigma \rightarrow R_+ \text{ Tr} \\
\{F, G, K\} \rightarrow J[F, G, K, L_5].
\] (2.12) (2.13)

Then the matrix model Hamiltonian is

\[
H = R_+ \text{ Tr} \left\{ \frac{1}{2} \Pi^2_I + \frac{\lambda}{2 R_-^2} X^2_I + \frac{e^{-2\phi}}{2 \cdot 3!} [X^I, X^J, X^K, L_5] [X^I, X^J, X^K, L_5] \\
- \frac{f}{3! R_-} (\epsilon^{ijkl} X^i [X^j, X^k, X^l, L_5] + \epsilon^{abcd} X^a [X^b, X^c, X^d, L_5]) \\
+ \frac{e \phi}{R_-} (\psi^{\dot{\alpha}\beta} \psi_{\alpha\beta} - \psi^{\dot{\alpha}\beta} \psi_{\dot{\alpha}\beta}) + 2 e^{-\phi} (\psi^{\dot{\alpha}\beta} (\sigma^{ij})_{\alpha}^{\delta} [X^i, X^j, \psi_{\delta\beta}, L_5] \\
+ \psi^{\dot{\alpha}\beta} (\sigma^{ab})_{\alpha}^{\delta} [X^a, X^b, \psi^{\delta\beta}, L_5]) - 2 e^{-\phi} (\psi_{\dot{\alpha}\beta} (\sigma^{ij})_{\dot{\alpha}}^{\delta} [X^i, X^j, \psi^{\dot{\alpha}\beta}, L_5] \\
+ \psi_{\dot{\alpha}\beta} (\sigma^{ab})_{\dot{\alpha}}^{\delta} [X^a, X^b, \psi^{\delta\beta}, L_5]) \right\}
\] (2.14)

In this Hamiltonian, $\lambda, \phi, f$ depend on $\tau$, and satisfy

\[
\lambda(\tau) = -\frac{1}{4} \phi''(\tau) + f^2(\tau) e^{2\phi(\tau)}. \tag{2.15}
\]

Now we have constructed a $(0 + 1)$-dimensional matrix model of type IIB string theory on time-dependent background (1.1). The symmetry algebra of background (1.1) is $[so(4) \oplus so(4)] \oplus h(8)$, where $h(8)$ is Heisenberg-type algebra. So DLCQ of string theory on this background should respect $SO(4) \times SO(4)$ symmetry. From (2.14), it is easy to see that our matrix model do have this symmetry.

Generically the matrix model has time-dependent mass-squared $\lambda$ and the coupling constant is also time-dependent. In this paper, we are interested in the null linear dilaton background

\[
\phi = -c \tau, \quad c > 0. \tag{2.16}
\]
so that
\[ \lambda = f^2 e^{2\phi}. \]  
(2.17)

As the usual PP-wave case, the string theory limit could be
\[ J, R_\rightarrow \infty, \quad \text{with } P^+ \text{ being fixed.} \]  
(2.18)

In this limit, our matrix model should describe light-cone type IIB string theory on the background (1.1).

Because our matrix model is time-dependent, it is difficult to find a static solution. The unique static solution is the vacuum solution:
\[ X^i = X^a = \psi_{\alpha\beta} = \psi^{\dot{\alpha}\dot{\beta}} = 0. \]  
(2.19)

Let us consider the fluctuations around the vacuum. Expanding the Hamiltonian to the second order of the fluctuations around the vacuum, we obtain
\[
H^{(2)}_{tri} = \text{Tr}[(\frac{R_i}{2} \Pi_I^2 + \frac{\lambda}{2R_-} X_i^2) + (\frac{R^a}{2} \Pi_a^2 + \frac{\lambda}{2R_-} X_a^2)] \\
+ f e^{\phi} \psi^{\dagger \alpha\beta} \psi_{\alpha\beta} - f e^{\phi} \psi^{\dagger \dot{\alpha}\dot{\beta}} \psi_{\dot{\alpha}\dot{\beta}},
\]  
(2.20)

where we use \( X^i, X^a, \psi_{\alpha\beta}, \psi^{\dagger \alpha\beta} \) to denote the fluctuations.

The lowest excited states around the vacuum are
\[
\text{Tr}(\sqrt{\frac{\lambda}{2R_-}} X^I - i \sqrt{\frac{R_-}{2\lambda}} \Pi_I) \langle 0 |, \text{Tr} \psi^{\dagger \alpha\beta} | 0 \rangle, \text{Tr} \psi^{\dagger \dot{\alpha}\dot{\beta}} | 0 \rangle.
\]  
(2.21)

The bosonic excitations have energy
\[ E = \lambda^{1/2}, \]  
(2.22)

which depends only on the background geometry. And the fermionic excitations have energy
\[ E = f e^{\phi}, \]  
(2.23)

which depend on the background fluxes. If \( \phi = ax^+ + b, a, b \) are constant, the first excited bosonic and fermionic states will have the same energy. These energy are generically time-dependent except a few special cases. One of such exception is the usual plane-wave background, which has been discussed in (2.24). Another special example is
\[ \phi = -c \tau, \quad f = f_0 e^{c \tau}, \]  
(2.24)
with $\lambda = f_0^2 = \text{const.}$, so the first excited states have the fixed energy.

The higher excited states can be discussed similarly. As the first excited states, their energy will be generically time-dependent.

Though we have computed the energy of the excited states, they are only perturbational results in the matrix model. They are only trustable if the coupling in the matrix model be very small. If we define

$$\tilde{X}^I = \sqrt{\frac{1}{R_-}} X^I, \quad \tilde{\Pi}_I = \sqrt{R_- \Pi^I},$$

(2.25)

the bosonic part of the matrix model Hamiltonian about this vacuum is

$$H_{bos} = \text{Tr}\left\{ \frac{1}{2} \tilde{\Pi}_I^2 + \frac{\lambda}{2} \tilde{X}^2 \right\}$$

$$+ \frac{1}{2} \cdot 3! (R_- e^{-\frac{\Phi}{2}})^4 [\tilde{X}^I, \tilde{X}^J, \tilde{X}^K, \mathcal{L}_5] [\tilde{X}^I, \tilde{X}^J, \tilde{X}^K, \mathcal{L}_5]$$

$$- \frac{1}{3!} (R_- f_\tau^2)^2 (\epsilon^{ijkl}[\tilde{X}^i, \tilde{X}^j, \tilde{X}^k, \tilde{X}^l] + \epsilon^{abcd}[\tilde{X}^a, \tilde{X}^b, \tilde{X}^c, \tilde{X}^d, \mathcal{L}_5]) \right\}$$

(2.26)

The ’t Hooft coupling in (2.26) is

$$g = J R_\tau^2 e^{-\Phi} = \frac{J^3 e^{e\tau}}{(P^+)^2}$$

(2.27)

In the early time, $\tau \to -\infty$, the ’t Hooft coupling will be very weak. So we can trust the perturbation theory of matrix model. On the contrary, in the late time, when $\tau \to \infty$, the perturbation matrix calculation will be not convincing, but the perturbative string theory is well-defined then.

For the case (2.26), defining

$$a_I = \sqrt{\frac{f e^{-e\tau}}{2R_-}} X^I + i \sqrt{\frac{R_-}{2fe^{-e\tau}}} \Pi_I,$$

$$a_I^\dagger = \sqrt{\frac{f e^{-e\tau}}{2R_-}} X^I - i \sqrt{\frac{R_-}{2fe^{-e\tau}}} \Pi_I,$$

(2.28)

we obtain the normal-ordered free Hamiltonian

$$H_{tri}^{(2)} = \text{Tr}\left\{ f e^{-e\tau} (a_I^\dagger a_I + \psi^{\dagger a\beta} \psi_{a\beta} + \psi^{\dagger a\bar{\beta}} \psi_{a\bar{\beta}}) \right\}.$$

(2.29)

The zero-point energy is canceled between bosonic and fermionic sectors in the background (1.1). The free Hamiltonian is the same as the one with only zero modes in perturbative string vacuum. This suggests that in the
string theory limit, the static vacuum becomes the vacuum of string theory on plane-wave background with linear null dilaton.

In the tiny graviton matrix model on plane-wave background, if one takes the string theory limit, the 't Hooft coupling becomes very large, which suggests that the closed string will appear as non-perturbative objects of the model. In our case, the string theory limit is more subtle. Since the background is time-dependent, the matrix model and its 't Hooft coupling is also time-dependent. Strictly speaking, the matrix model is only well-defined in the very early time when the matrix coupling is small and the string coupling is large, and on the other hand the string theory is well-defined in the very late time when the string coupling is small and the matrix coupling is large. If we take the string theory limit at fixed time where the string coupling is not very large, the 't Hooft coupling becomes very large just like the plane-wave background cases, and the vacuum in matrix model changes to closed string vacuum..

3 The fuzzy three sphere solutions

In this section, we will discuss fuzzy three sphere solutions. In our time-dependent matrix model, there is no static fuzzy sphere solution and the only possible fuzzy three sphere solutions are time-dependent. Since our background keeps $SO(4) \times SO(4)$ symmetry, without losing generality, we take the following ansatz

$$X^a = \psi_{\alpha\dot{\beta}} = \psi_{\dot{\alpha}\beta} = 0,$$

$$X^i = S(\tau)X^i_0,$$

$$i = 1, 2, 3, 4, \quad a = 5, 6, 7, 8.$$ (3.1)

Here $X^i_0$’s are constant $J \times J$ matrices satisfying

$$[X^i_0, X^j_0, X^k_0, \mathcal{L}_5] = -\epsilon^{ijkl}X^l_0,$$

$$\sum_{i=1}^{4}(X^i_0)^2 = J, \quad (3.2)$$

such that

$$[X^i, X^j, X^k, \mathcal{L}_5] = -S^2(\tau)\epsilon^{ijkl}X^l,$$

$$\sum_{i=1}^{4}(X^i)^2 = S^2(\tau)J.$$ (3.3)
With this ansatz, the equation of motion for $X^i$ is

$$\ddot{X}^i + \lambda X^i - \frac{4R_- f}{3!} e^{ijkl}[X^j, X^k, X^l, \mathcal{L}_5]$$

$$- \frac{R^2 e^{-2\phi}}{2} [[X^i, X^j, X^k, \mathcal{L}_5], X^j, X^k, \mathcal{L}_5] = 0,$$

(3.4)

which gives the equation for $S(\tau)$,

$$\ddot{S}(\tau) + \lambda S - 4R_- f S^3(\tau) + 3R^2 e^{-2\phi} S^5(\tau) = 0.$$  

(3.5)

The linear term of the above equation stems from the mass-squared term in the matrix model, which is determined by the background geometry. The other two terms, cubic one and quintic one, are from the interaction terms in the matrix model. There are two parameters $\phi(\tau)$ and $f(\tau)$ in the equation. In this article, we discuss two cases whose corresponding perturbative string have been studied in [19]:

- Case 1: $\phi = -c\tau, f(\tau) = f_0 e^{c\tau}, c > 0, f_0 = const$;
- Case 2: $\phi = -c\tau, f(\tau) = f_0, c > 0, f_0 = const$.

In both cases, $\phi = -c\tau$, so (3.6) is

$$\ddot{S}(\tau) + f^2 e^{-2c\tau} S - 4R_- f S^3(\tau) + 3R^2 e^{2c\tau} S^5(\tau) = 0.$$  

(3.6)

It is very difficult to obtain analytic solutions of this equation. Using the numerical method to discuss the solution and combining the analysis of the early time and late time asymptotic behavior of the solution, we will obtain a good qualitative understanding of the solution.

For the first case, $\lambda = f_0^2$ is a constant. The equation for $S(\tau)$ is

$$\ddot{S}(\tau) + f_0^2 S(\tau) - 4R_- f_0 e^{c\tau} S^3(\tau) + 3R^2 e^{2c\tau} S^5(\tau) = 0.$$  

(3.7)

The solutions depend on three parameters, $R_-, f_0, c$. For a definite discussion, at first, we let

$$R_- = 2, \quad f_0 = 1, \quad c = 1.$$  

(3.8)

The figure is for the numerical solution between $\tau = -5$ and $\tau = 5$ if we set initial conditions

$$S(0) = 1, \quad \dot{S}(0) = 0.$$  

(3.9)
Figure 1: The solution of Case 1 when $\tau \in [-5, 5]$ and $R_- = 2, f_0 = 1, c = 1$. At late time, this solution vibrate faster with a smaller amplitude. In fact, its amplitude will shrink to zero very quickly.

From this figure, we can read out the behavior of $S(\tau)$ in $[-5, 5]$. When $\tau$ becomes larger, in other words, when $g_s$ becomes weaker, $S(\tau)$ vibrates more rapidly, and the amplitude becomes smaller. The figure can not tell us the whole story, because when $\tau \gg 1$, $S(\tau)$ vibrates so rapidly that the computer can not give right numerical solutions. Fortunately, when $|\tau|$ is large, we can analyze the solutions from their asymptotic behaviors.

At early time, if we take $\tau \to -\infty$ limit, the equation is dominated by the linear term and is approximated to be

$$\ddot{S}(\tau) + f_0^2 S(\tau) = 0,$$  

(3.10)

which has the solution

$$S(\tau) = A \sin (f_0 \tau + B).$$  

(3.11)

$A, B$ are constants which depend on initial conditions. So at early time, fuzzy three sphere is in a stable vibrating state.

At late time, when $\tau \to \infty$, the equation (3.7) is dominated by the interaction term and can be approximated to be

$$\ddot{S}(\tau) + 3R_-^2 e^{2c\tau} S^5(\tau) = 0.$$  

(3.12)

So at very late time, $S(\tau)$ will vibrate very rapidly around zero. The amplitude will also decay to zero very fast, at the rate of $e^{-\frac{c}{2}\tau}$. This behavior is expected. From matrix model point of view, at late time, the coupling
is stronger and the nonperturbative effect becomes essential and the fundamental closed string degree of freedom becomes important so the fuzzy effect is quite weak.

Therefore, we have the following picture. A fuzzy three sphere, vibrates in a stable rhythm in the early time. But after a long time evolution, at late time, the non-Abelian degrees of freedom of the matrix model become less important and the fundamental string degrees of freedom take it over.

Changing parameters in the equation will not change the whole picture but can postpone or accelerate the solutions to reach their asymptotic behaviors. If we change the setting, let

\[ R_{-} = 10, \quad f_0 = 1, \quad c = 1, \quad (3.13) \]

and still set the initial conditions

\[ S(0) = 1, \quad \dot{S}(0) = 0, \quad (3.14) \]

we obtain the figure 2.

Figure 2: The solution of Case 1 when \( \tau \in [-5, 5] \) and \( R_{-} = 10, f_0 = 1, c = 1 \). At late time, it vibrates more rapidly than the one in \( R_{-} = 2 \) case which is plotted in figure 1.

Comparing the figures (1) and (2), we find that at late time, \( S(\tau) \) vibrates more rapidly when \( R_{-} \) becomes larger. This is because the coefficient of the quintic term is larger, which suggests a stronger interaction and leads to faster vibration. In other words, a larger \( R_{-} \) makes fuzzy effect weaker. If
we let $f_0$ be larger, from (3.7), we can deduce that the fuzzy three sphere will shrink more slowly.

For the second case, $\lambda = f_0^2 e^{-2c\tau}$, the equation (3.5) is

$$\ddot{S}(\tau) + f_0^2 e^{-2c\tau} S - 4R_- f_0 S^3(\tau) + 3R_-^2 e^{2c\tau} S^5(\tau) = 0. \quad (3.15)$$

At first, we set $R_- = 2, f_0 = 1, c = 1$ just like (3.8). If we input

$$S(0) = 1, \dot{S}(0) = 0, \quad (3.16)$$

we have the figure 3 for $\tau \in [-5, 5].$

![Figure 3: The solution of Case 2 when $\tau \in [-5, 5]$ and $R_- = 2, f_0 = 1, c = 1.$ At late time, its asymptotic behavior is similar to the asymptotic behavior of solutions in Case 1. At early time, the solution has a similar asymptotic behavior to its late time asymptotic behavior.]

From this figure, we find that the late time asymptotic behavior of this solution is very similar to the one in Case 1. But the early time asymptotic behavior is very different. Just like Case 1, we can not obtain conclusions from figure when $|\tau| \gg 1$. So we must analyze the asymptotic behaviors of the solutions.

At late time, when $\tau \to \infty$, the equation (3.15) is approximated to be

$$\ddot{S}(\tau) + 3R_-^2 e^{2c\tau} S^5(\tau) = 0. \quad (3.17)$$

It is the same equation as (3.12). So the asymptotic behaviors of solutions in these two cases at late time are the same. We know at late time, $\lambda, g_s \to 0, f = f_0$ still is constant. So the decay of fuzzy three sphere is a natural phenomenon because the background now is nearly flat and the string coupling is weak and the matrix coupling is strong.
At very early time, the equation (3.15) is approximated to be
\[ \ddot{S}(\tau) + f_0^2 e^{-2\tau} S(\tau) = 0. \] (3.18)

This equation tells us that when \( \tau \to -\infty \), the asymptotic behavior of the solution of (3.15) will be very similar to its late time one. So the behavior shown in figure (3) is inevitable. To understand these behaviors, let us come back to equation (3.4). The most important term dominating the early time behavior, is \( \lambda X^3 \), which depends on the background geometry. Since \( \lim_{\tau \to -\infty} \lambda = \infty \) in this case, the large spatial extension reduces fuzzy effect. This phenomenon happens also in the matrix models on other time-dependent backgrounds [8, 24].

Nevertheless, we can read out the whole behavior of solutions from the figure (3), (3.17) and (3.18). If there is a very small fuzzy three sphere at early time, it will grow up as time goes by. And after it reaches its maximum, it becomes smaller and smaller. At late time, it vibrates fiercely and decays out.

We can also analyze the effect of changing parameters. If we use the setting
\[ R_- = 10, \quad f_0 = 1, \quad c = 1, \] (3.19)
we get the figure 4 for the solution between \( \tau = -5 \) and \( \tau = 5 \) with the initial conditions (3.16).

As before, a larger \( R_- \) makes the solution vibrate more rapidly when \( \tau \to \infty \), and makes the solution vibrate more slowly when \( \tau \to -\infty \) (Since we only plot the figure when \( \tau \in [-5,5] \), the early time asymptotic behavior of the solution is not evident in the figure 4). If we let \( f_0 \) take a larger value, however, we will find contrary effect.

Before ending our study of fuzzy three sphere, it would be interesting to discuss how the tiny graviton scale changes with time. In the appendix, we investigate the giant gravitons in the background (1.1). We learn that the size of giant graviton is changing with time:
\[ \frac{d^2r}{d\tau^2} + \lambda r - 4f(\frac{\mu}{P_-})r^3 + 3(\frac{\mu}{P_-})^2 r^5 e^{-2\phi(\tau)} = 0, \] (3.20)
where \( \mu = 2\pi^2 T_3 \) with \( T_3 \) being the tension of D3-brane and \( P_- \) is conserved lightcone momentum. In the case \( \phi = \text{constant} \), \( \lambda = f_0^2 \), there exist static giant gravitons with radius \( r_0^2 = f_0 \frac{P_-}{\mu} \) and \( r_0^2 = 3f_0 \frac{P_-}{\mu} \). The former one is the
Figure 4: The solution of Case 2 when $\tau \in [-5, 5]$ and $R_- = 10, f_0 = 1, c = 1$. At late time, this solution shrinks to zero much faster than the one when we set $R_- = 2, f_0 = 1, c = 1$. At early time, though its full asymptotic behavior can not be seen from the figure because we do not plot the figure of the solution when $\tau < -5$, we can confirm that its asymptotic behavior is similar to $R_- = 2$ case through analysis in this section.

well-studied stable giant graviton keeping half supersymmetries. The latter one is not stable. From this, one can read out the size of tiny graviton which carry one unit of lightcone momentum,

$$r_{\tiny{\text{tiny}}} = \frac{r_{\tiny{\text{pl}}}}{r_{\text{AdS}}}. \tag{3.21}$$

For the case $\phi$ being not a constant, after identifying $P_- = \mu/R_-$, we can find that the equation (3.20) takes the same form as the equation (3.5). This shows that the scale of tiny graviton changes with $\tau$ in the same way as the size of fuzzy three sphere. This is not a surprise since the tiny graviton is the fundamental parton of our matrix model. This fact could be understood as following: the fuzzy three sphere could be taken as the blow-up of the tiny gravitons, therefore its size is changing with the evolution of the scale of tiny graviton.

Actually the fact that the dynamics of fuzzy three sphere is in perfect agreement with the dynamics of giant graviton in the background (1.1) gives a consistent check that our matrix model on time-dependent background is correct.
4 Conclusions and discussions

In this paper, we constructed a matrix model for type IIB superstring theory on a plane-wave background with null dilaton \( (1.1) \). Our matrix model is time-dependent, have much different character from the tiny matrix model on plane-wave background. One of the differences is that the vacuum solution is the unique static solution. The spectrum about this vacuum are time-dependent in most generic cases. We compared our second order Hamiltonian of trivial vacuum to Hamiltonian of string \( \sigma \)-model on the linear null dilaton background, and found that the spectrum of perturbation of background in string theory could be produced from our second order Hamiltonian.

We also investigated the fuzzy three-sphere solutions in our matrix model and found that they must be time-dependent. The solutions depend on the background geometry and flux. In this paper, we discuss two cases with linear null dilaton. One is \( \phi = -c\tau, f(\tau) = f_0e^{c\tau}, f_0 = constant \). The other one is \( \phi = -c\tau, f(\tau) = f_0 = constant \). For these two cases, we could not obtain analytic solutions yet, but we investigated the solutions numerically and also analyzed their early and late time asymptotic behaviors. The results are interesting. In the first case, a fuzzy sphere vibrates stably at early time, but at late time it vibrates much more rapidly, then decays out. In the second case, at early time a very small fast-vibrating fuzzy sphere appears and grows up gradually to reach its steady maximum and then decays through rapid vibration at late time. The picture can be understood as following. The early time behavior of the fuzzy three-sphere is determined by the background geometry completely, so in the second case, the fuzzy degrees of freedom is diluted. On the other hand, the late time asymptotic behavior is determined by the interaction terms in the matrix model. The very strong interaction at the late time requires the nonperturbative degrees of freedom to replace the matrix degrees of freedom. This explain why the fuzzy sphere always decays out at the late time.

The matrix model constructed here is well-defined at the very early time where the coupling of the interaction is weak. It is dual to the late time perturbative string description. In other words, the closed string perturbative string will appear as the nonperturbative objects in the matrix model, and vice versa. The dual pictures give a complete description of the string theory in the linear null dilaton background.

In the matrix model on plane-wave background\([11]\), the spectrum around fuzzy sphere vacuum contain photon states, which are the gauge field on the giant graviton in the string theory limit. In our case, fuzzy sphere solutions are too complex and time-dependent. We did not go to similar discussions in this paper. We hope we can discuss these issues in the future.
One essential ingredient in the construction of tiny graviton matrix model is the existence of Nambu three-bracket in the D3-brane action, which allow us to do matrix regularization to obtain a matrix model. On the other hand, the Nambu three-bracket is a realization of infinite dimensional 3-algebra, which plays a central role in the recent study of multiple membrane theory by Bagger-Lambert [25, 26] and Gustavsson [27]. In [28, 29], it was pointed out that the space of functions on three dimensional manifold, with Nambu three bracket could be identified as an infinite dimensional 3-algebra. The Nambu three bracket is the realization of three-bracket of this infinite dimensional 3-algebra. Especially, it was proposed in [30] that the deformed BLG-theory on $R \times T^2$ gives tiny graviton matrix theory of Type IIB plane-wave. Very recently, in [32], the authors used the similar way to [11] as an matrix realization of “relaxed three-algebra”, which is a ‘3-algebra’ with relaxed closure and fundamental identity conditions. The relaxed three-algebra is a linear space of matrices with a relaxed three bracket, which is defined as $[T^a, T^b, T^c, T^-]$ with $T^-$ being a special fixed constant matrix. Using relaxed three-algebra, they tried to construct a Hermitian model to describe $n$ (which is a general number) membranes systems. These discoveries suggest a deep but not clear relation between multi-membrane theory and tiny graviton matrix model. It would be important to understand this issue better. And it is also interesting to discuss the multiple membrane in a time-dependent background [33].

Acknowledgments

The work was partially supported by NSFC Grant No. 10535060, 10775002, and NKBRPC (No. 2006CB805905). C would like to thank TIFR for the hospitality, where the project was finishing.

Appendix: Giant gravitons

In this appendix, we would like to study the giant gravitons in the background [1.1]. Let us introduce the polar coordinates $(r, \theta, \phi, \varphi)$ in the plane defined by the Cartesian coordinates $x_i, i = 1, 2, 3, 4$. In polar coordinates, the 5-form flux

$$(F_5)_{1234} = (F_5)_{+r\theta\phi\varphi} = 4f(x^+)^3r^3\sin^2 \theta \sin \phi$$

The relation between mass deformed M2 theory on $T^2$ and IIB DLCQ string on plane-wave was first pointed out and identified in [31].

18
with the relevant nonvanishing components of 4-form potential

\[ C_{+\theta\phi\phi} = f(x^+)r^4\sin^2\theta\sin\phi. \]  

(4.2)

Here we just focus on the case that the giant graviton being blown up in \( x_i, i = 1, 2, 3, 4 \). Obviously the case that the giant graviton being blown up in \( x_i, i = 5, 6, 7, 8 \) could be studied similarly.

The giant gravitons correspond to spherical D3-brane embedded in the background consistently. The D3-brane is wrapped around \( S^3 \) characterized by \( \theta, \phi, \varphi \). This allows us to identify the worldvolume coordinates of D3-branes to be \( \tau, \theta, \phi \) and \( \varphi \). To simplify the discussion, we can choose the gauge \( X^+ = \tau \) such that the only nontrivial embedding is \( r \) depending on \( \tau \). After this truncation, the action of the D3-brane reduce to that of a point particle:

\[ S = \mu \int d\tau [-e^{-\phi(\tau)}r^3(\tau)\sqrt{-G_{AB}\partial\tau X^A\partial\tau X^B + f(\tau)r^4(\tau)}], \]  

(4.3)

where \( \mu = 2\pi^2 T_3 \). As we have seen, the translation along \( x^- \) is still an isometry so its corresponding canonical momentum is a conserved quantity. And the canonical Hamiltonian is just \( H = -P_+ \), which is of the form

\[ H = \frac{P_r^2}{2P_+} + \frac{(\mu r^3e^{-\phi})^2}{2P_+} + \frac{\lambda r^2P_-}{2} - \mu fr^4, \]  

(4.4)

where \( P_r \) is the canonical momentum of \( r \). The Hamiltonian describes a particle with mass \( P_- \) moving in a time-dependent potential. The \( r \) is actually the size of the giant graviton, which is determined by the equation of motion

\[ \frac{d^2r}{d\tau^2} + \lambda r - 4f\frac{\mu}{P_-}r^3 + 3\left( \frac{\mu}{P_-} \right)^2e^{-2\phi}r^5 = 0. \]  

(4.5)

This equation is the same as \[3.5\] after identification \( R_- = \mu / P_- \). Therefore the dynamics of giant graviton is in exact agreement with fuzzy three sphere.

References

[1] T.Banks, W.Fischler, S.H.Shenker and L.Susskind, ‘M theory as a matrix model: a conjecture’, Phys.Rev.D 55,5112 (1997), [arXiv:hep-th/9610043].

\[4\] In our construction of matrix model, we start from a D3-brane action with \( T_3 = 1 \).
[2] N. Seiberg, ‘Why is the matrix model correct’, Phys.Rev.Lett.79, 3577, arXiv:hep-th/9710009.

[3] A. Sen, ‘D0 Branes on $T^n$ and Matrix Theory’, Adv.Theor.Math.Phys. 2 (1998) 51-59, arXiv:hep-th/9709220.

[4] W. Taylor and M. V. Raamsdonk, ‘Multiple D0-brane in weakly curved background’, Nucl.Phys.B 558. 63 (1999), arXiv:hep-th/9904095.

[5] D. Berenstein, J. Maldacena, H. Nastase, ‘Strings in flat space and pp waves from N=4 super Yang Mills’, JHEP 0204 (2002) 013, arXiv:hep-th/0202021.

[6] K. Dasgupta, M. M. Sheikh-Jabbari and M. Van Raamsdonk, “Matrix perturbation theory for M-theory on a pp-wave”, JHEP 05(2002)056, arXiv:hep-th/0205185.

[7] B. de Wit, K. Peeters and J. Plefka, ‘Superspace geometry for supermembrane backgrounds’, Nucl. Phys. B532(1998)99, arXiv:hep-th/9803209.

[8] B. Chen, “The time-dependent supersymmetric configurations in M-theory and matrix models”, Phys. Lett. B632(2006)393, arXiv:hep-th/0508191.

H. Chen and B. Chen, ‘Matrix model in a class of time dependent supersymmetric backgrounds’, Phys.Lett. B638 (2006) 74, arXiv:hep-th/0603147.

[9] R. Dijkgraaf, E. Verlinde, H. Verlinde, ‘Matrix String Theory’, Nucl.Phys. B500 (1997) 43-61, arXiv:hep-th/9703030.

[10] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, ‘A large-N reduced model as superstring’, Nucl.Phys.B 498. 467 (1997), arXiv:hep-th/9612115.

[11] M. M. Sheikh-Jabbari, ‘Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture’, JHEP 0409 (2004) 017, arXiv:hep-th/0406214.

[12] Y. Lozano and D. Rodriguez-Gomez ‘Type II pp-wave matrix models from point-like gravitons’, JHEP 0608:022,2006, arXiv:hep-th/0606057.
[13] M. M. Sheikh-Jabbari and M. Torabian, ‘Classification of All 1/2 BPS Solutions of the Tiny Graviton Matrix Theory’, JHEP 0504 (2005) 001, [arXiv:hep-th/0501001].

[14] M. Ali-Akbari, M. M. Sheikh-Jabbari and M. Torabian, ‘Extensions of $AdS_5 \times S^5$ and the Plane-wave Superalgebras and Their Realization in the Tiny Graviton Matrix Theory’, JHEP 0603 (2006) 065, [arXiv:hep-th/0512037].

[15] M. Ali-Akbari, M. M. Sheikh-Jabbari and M. Torabian, ‘Tiny Graviton Matrix Theory/SYM Correspondence: Analysis of BPS States’, Phys.Rev. D74 (2006) 066005, [arXiv:hep-th/0606117].

[16] B. Craps, S. Sethi, E. Verlinde, ‘A Matrix Big Bang’, JHEP 0510 (2005) 005, [arXiv:hep-th/0506180].

[17] M. Blau and M. O’Laughlin, ‘DLCQ and Plane Wave Matrix Big Bang Models’, [arXiv:0806.3255].

[18] B. Craps, “Big Bang Models in String Theory”, Class. Quant. Grav. 23(2006)S49-S81, [arXiv:hep-th/0605199].

[19] B. Chen, Y. He and P. Zhang, ‘Exact solvable model of superstring in plane-wave background with linear null dilaton’. Nucl.Phys.B 741. 269 (2006), [arXiv:hep-th/0509113].

[20] M. Cederwall, A. V. Gussich, E. W. Nilsson and A. Westerberg, ‘The Dirichlet super-three-brane in ten-dimensional type IIB supergravity’, Nucl.Phys. B 490. 163 (1997), [hep-th/9610148].

[21] R. R. Metsaev, A. A. Tseytlin, ‘Exactly solvable model of superstring in Ramond-Ramond plane wave background’, Phys.Rev. D65 (2002) 126004, [arXiv:hep-th/0202109].

[22] D. Sadri and M. M. Sheikh-Jabbari, ‘Giant hedge-hogs: spikes on giant gravitons’, Nucl.Phys.B 687. 161 (2004), [arXiv:hep-th/0312155].

[23] Z. Guralnik and S. Ramgoolam, ‘On the polarization of unstable D0-branes into non-commutative odd spheres’, JHEP 02(2001)032, [arXiv:hep-th/0101001].

S. Ramgoolam, ‘On spherical harmonics for fuzzy spheres in diverse dimensions’, Nucl. Phys. B 610(2001)461, [arXiv:hep-th/0105006].
[24] S.R. Das and J. Michelson, ‘pp wave big bangs: Matrix strings and shrinking fuzzy spheres’, Phys. Rev. D72(2005)086005, [arXiv:hep-th/0508068]. ‘Matrix membrane big bangs and D-brane production’, Phys.Rev.D73:126006,2006, [arXiv:hep-th/0602099].

[25] J.Bagger and N.Lambert, ‘Modeling Multiple M2’s’, Phys.Rev. D75 (2007) 045020, [arXiv:hep-th/0611108].

[26] J.Bagger and N.Lambert, ‘Gauge Symmetry and Supersymmetry of Multiple M2-Branes’, Phys.Rev. D77 (2008) 065008, [arXiv:hep-th/0711.0955].

[27] A.Gustavsson, ‘Algebraic structures on parallel M2-branes’, [arXiv:hep-th/0709.1260].

[28] P.Ho and Y.Matsuo, ‘M5 from M2’, [arXiv:hep-th/0804.3629]. P.Ho, Y.Imamura and Y.Matsuo, ‘M2 to D2 revisited’, [arXiv:hep-th/0805.1202]. P.Ho, Y.Imamura, Y.Matsuo and S.Shiba, ‘M5-brane in three-form flux and multiple M2-branes’, [arXiv:hep-th/0805.2898].

[29] I.A.Bandos, P.K.Townsend, ‘Light-cone M5 and multiple M2-branes’, [arXiv:hep-th/0806.4777]. ‘SDiff Gauge Theory and the M2 Condensate’, [arXiv:0808.1583].

[30] J. Gomis, A.J. Salim and F. Passerini, “Matrix Theory of Type IIB plane wave from membranes”, [arXiv:0804.2186].

[31] H. Lin, O. Lunin and J. Maldacena, ‘Bubbling AdS space and 1/2 BPS geometries’, JHEP 0410:025,2004, [arXiv:hep-th/0409174].

[32] M. Ali-Akbari, M.M. Sheikh-Jabbari and J. Simon, ‘Relaxed Three-Algebras: Their Matrix Representations and Implications for Multi M2-brane Theory’, [arXiv:hep-th/0807.1570].

[33] M. Blau and M. O’Laughlin, ‘Multiple M2-Branes and Plane Waves ’, [arXiv: 0806.3253].