Gradient flow and the Wilsonian renormalization group flow

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The gradient flow is the evolution of fields and physical quantities along a dimensionful parameter \( t \), the flow time. We give a simple argument that relates this gradient flow and the Wilsonian renormalization group (RG) flow. We then illustrate the Wilsonian RG flow on the basis of the gradient flow in two examples that possess an infrared fixed point, the 4D many-flavor gauge theory and the 3D \( O(N) \) linear sigma model.

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1. Introduction and the basic idea

The gradient flow \([1-5]\) is the evolution of fields and physical quantities along a dimensionful parameter \(t\), the flow time; the flow acts as the “coarse-graining” as \(t > 0\) becomes large. These two features of the gradient flow are common to the Wilsonian renormalization group (RG) flow \([6]\) in a broad sense, provided that the flow time is identified with the renormalization scale. In fact, it has sometimes been indicated that the gradient flow and the Wilsonian RG flow can be identified in some ways \([7-10]\); see also Refs. \([11-14]\) for related studies. In this paper, we give a simple argument that relates the gradient flow and the Wilsonian RG flow; our argument is somewhat similar to that of Ref. \([7]\). We then illustrate the Wilsonian RG flow on the basis of the gradient flow in two examples that possess an infrared fixed point, the 4D many-flavor gauge theory and the 3D \(O(N)\) linear sigma model.

Our idea is very simple. We take the following flow equations for the gauge potential \(A_\mu(x)\) and for the Dirac fields \(\tilde{\psi}(x)\) and \(\tilde{\psi}(x)\):

\[
\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x), \quad B_\mu(t = 0, x) = A_\mu(x), \tag{1.1}
\]

\[
\partial_t \chi(t, x) = \Delta \chi(t, x), \quad \chi(t = 0, x) = \tilde{\psi}(x), \tag{1.2}
\]

\[
\partial_t \tilde{\chi}(t, x) = \tilde{\chi}(t, x) \Delta, \quad \tilde{\chi}(t = 0, x) = \tilde{\psi}(x). \tag{1.3}
\]

Let us consider the correlation function of operators composed of the flowed fields:

\[
\langle \mathcal{O}_1(t_1, x_1) \cdots \mathcal{O}_N(t_N, x_N) \rangle. \tag{1.4}
\]

Let us also suppose that we have a set of (a generally infinite number of) coupling constants \(\{g_i\}\) with which the correlation function is computed\(^2\). We consider the mapping in this space of the coupling constants induced by the Wilsonian RG flow,

\[
\{g_i\} \rightarrow \{g_i(\xi)\}, \tag{1.5}
\]

where \(\xi\) parametrizes the RG flow. This RG flow can be characterized by the scaling relation\(^3\)

\[
\langle \mathcal{O}_1(e^{2\xi} t_1, e^\xi x_1) \cdots \mathcal{O}_N(e^{2\xi} t_N, e^\xi x_N) \rangle_{\{g_i\}} = Z(\xi) \langle \mathcal{O}_1(t_1, x_1) \cdots \mathcal{O}_N(t_N, x_N) \rangle_{\{g_i(\xi)\}}, \tag{1.6}
\]

where \(Z(\xi)\) is the multiplicative renormalization factor and the subscript implies that the correlation function is evaluated with respect to the set of coupling constants. Compare this relation with, for instance, Eqs. (7.10) and (7.15) of Ref. \([6]\). Note that the flow time has the mass dimension \(-2\) instead of \(-1\). The advantage of this characterization of the Wilsonian RG flow is that this scaling relation itself can be written down even for gauge theory for which the momentum cutoff is incompatible with the gauge invariance (at least naively). In

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\(^1\)Here, the covariant derivative on the gauge field are defined by \(D_\mu = \partial_\mu + [B_\mu, \cdot]\); the field strength is defined by \(G_{\mu\nu}(t, x) = \partial_\mu B_\nu(t, x) - \partial_\nu B_\mu(t, x) + [B_\mu(t, x), B_\nu(t, x)]\). The Laplacians on the Dirac fields is defined by \(\Delta \equiv D_\mu D^\mu\), and \(\tilde{\Delta} \equiv \tilde{D}_\mu \tilde{D}^\mu\) from the covariant derivatives on the Dirac fields, \(D_\mu = \partial_\mu + B_\mu\) and \(\tilde{D}_\mu = \tilde{\partial}_\mu - B_\mu\). We will occasionally use notation such as \(A_\mu(x) = A_\mu^a(x) T^a\) by using the generator of the gauge group, \(T^a\).

\(^2\)We implicitly assume the presence of the ultraviolet cutoff.

\(^3\)Here, we neglect a possible non-trivial mixing of operators under the RG flow, for notational simplicity.
particular, for the one-point function of an operator that does not require the multiplicative renormalization,

$$\left\langle O_1(e^{2\xi}t) \right\rangle_{\{g_i\}} = \left\langle O_1(t) \right\rangle_{\{g_i(\xi)\}} ,$$

where we have omitted the argument $x$ assuming translational invariance in the $x$-space. Hence, assuming that the correspondence,

$$\{\langle O_i(t) \rangle \} \leftrightarrow \{g_i(\xi)\} \quad (1.7)$$

arising from Eq. (1.7) is one to one, we can use the one-point functions $\{\langle O_i(t) \rangle \}$ instead of the coupling constants $\{g_i(\xi)\}$. Of course, this idea is well known for the case of the gauge coupling constant [3]:

$$g^2(\mu = 1/\sqrt{8t}) \propto t^2 \langle G_{\mu \nu} G_{\mu \nu}(t) \rangle .$$

(1.9)

In what follows, we illustrate the idea (1.8) in theories in which several coupling constants play an interesting role; we will observe the flow of relevant and irrelevant coupling constants around an RG fixed point through the correspondence (1.8). We hope that our present consideration will be useful for more difficult models for which an infrared non-trivial fixed point can be concluded only non-perturbatively.

2. 4D $N_f$-flavor gauge theory and the Banks–Zaks fixed point

Our first example is the 4D vector-like gauge theory with $N_f$-flavor Dirac fermions with the degenerate mass $m$. As the operators in Eq. (1.8), we take (as the one corresponding to the gauge coupling [3]):

$$O_1(t, x) \equiv \frac{8(4\pi)^2 t^2}{3 \dim(G)} G^a_{\mu \nu}(t, x) G^a_{\mu \nu}(t, x),$$

(2.1)

and

$$O_2(t, x) \equiv \frac{\bar{\chi}(t, x)\chi(t, x)}{t^{1/2} \langle \bar{\chi}(t, x)\chi(t, x) \rangle} \equiv \frac{(4\pi)^2 t^{3/2}}{2 \dim(R) N_f} \frac{\hat{\chi}(t, x)\hat{\chi}(t, x)}{\chi(t, x)} .$$

(2.2)

The flowed gauge field and its local products such as $O_1(t, x)$ do not receive any multiplicative renormalization [4]. On the other hand, although the flowed Dirac field is multiplicatively renormalized [5], the renormalization of local products is simply determined by the number of Dirac fields in the product. Thus, $O_2(t, x)$ in Eq. (2.2) also does not receive multiplicative renormalization because of the division by the expectation value [15].

In the present system, one observes the so-called Banks–Zaks infrared fixed point [16, 17] if one uses the two-loop approximation of the beta function. We introduce the running gauge

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4 The generators $T^a$ (a runs from 1 to $\dim(G)$) of the gauge group $G$ are anti-Hermitian and the structure constants are defined by $[T^a, T^b] = i f^{abc} T^c$. Quadratic Casimirs are defined by $f_{bcd} f_{bde} = C_2(G) \delta^{cd}$ and, for a gauge representation $R$, $\text{tr}_R(T^a T^b) = -T(R) \delta^{ab}$ and $T^a T^a = -C_2(R) I$. We also denote $\text{tr}_R(1) = \dim(R)$. 

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coupling $\bar{g}(\mu)$ and the running mass parameter $\bar{m}(\mu)$ in the $\overline{\text{MS}}$ scheme, respectively, by

$$\left[b_0 \bar{g}(\mu)^2\right]^{\frac{-b_1/(2b_0)}{b_0 \bar{g}(\mu)^2}} \exp \left[-\frac{1}{2b_0 \bar{g}(\mu)^2}\right] = \frac{\Lambda}{\mu}, \quad (2.3)$$

$$\bar{m}(\mu) = M \left[2b_0 \bar{g}(\mu)^2\right]^{\frac{d_0/(2b_0)}{b_0 \bar{g}(\mu)^2}}, \quad (2.4)$$

where

$$b_0 = \frac{1}{(4\pi)^2} \left[\frac{11}{3} C_2(G) - \frac{4}{3} T(R) N_f \right], \quad (2.5)$$

$$b_1 = \frac{1}{(4\pi)^4} \left\{ \frac{34}{3} C_2(G)^2 - \left[4C_2(R) + \frac{20}{3} C_2(G) \right] T(R) N_f \right\}, \quad (2.6)$$

$$d_0 = \frac{1}{(4\pi)^2} 6C_2(R), \quad (2.7)$$

and $\Lambda$ and $M$ are RG invariant mass scales. In terms of these running parameters, we have the one-point function,

$$\langle O_1(t) \rangle = \bar{g}(1/\sqrt{8t})^2 \left[1 + \bar{g}(1/\sqrt{8t})^2 \left(\frac{1}{(4\pi)^2} K_1(t) + \frac{\bar{g}(1/\sqrt{8t})^4}{(4\pi)^4} K_2 \right) \right], \quad (2.8)$$

where

$$K_1(t) = \left(\frac{11}{3} \gamma_E + \frac{52}{9} - 3 \ln 3 \right) C_2(G) + \left[\frac{4}{3} \gamma_E - \frac{8}{9} + \frac{8}{3} \ln 2 + 16 \bar{m}(1/\sqrt{8t})^2 t \right] T(R) N_f, \quad (2.9)$$

and

$$K_2 = 8(4\pi)^2 \left\{ -0.0136423(7) C_2(G)^2 \right. \right.$$

$$\left. + [0.006440 134(5) C_2(R) - 0.008688 4(2) C_2(G)] T(R) N_f \right.$$

$$\left. + 0.000936 117 T(R)^2 N_f^2 \right\}. \quad (2.10)$$

Equation (2.8) for the massless case was obtained in Ref. [3] and for general mass cases in Ref. [18]; we have retained only the leading mass correction in Eq. (2.9) (as given in Eq. (2.34) of Ref. [18]). Although this treatment of the mass correction, which is also adopted in Eq. (2.11), is approximate, this makes the resulting RG equations (2.12) and (2.13) quite simple and illustrative, so here we content ourselves with this approximate treatment.

On the other hand, to the one-loop order, $\langle O_2(t,x) \rangle$ is given by

$$\langle O_2(t) \rangle = \bar{m}(1/\sqrt{8t}) t^{1/2} \left\{ 1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2} \left[3 \gamma_E + 4 + 2 \ln 2 - \ln(432) \right] C_2(R) \right\}. \quad (2.11)$$

We now take the flow time derivatives of Eqs. (2.8) and (2.11). By using Eqs. (2.3) and (2.4) (or the corresponding RG equations) and eliminating the running parameters in favor of
one-point functions, we arrive at

\[ t \frac{d}{dt} \langle O_1(t) \rangle = b_0 \langle O_1(t) \rangle^2 + b_1 \langle O_1(t) \rangle^3 + \frac{1}{(4\pi)^2} 16T(R)Nf \langle O_1(t) \rangle^2 \langle O_2(t) \rangle^2, \]

\[ t \frac{d}{dt} \langle O_2(t) \rangle = \frac{1}{2} [1 + d_0 \langle O_1(t) \rangle] \langle O_2(t) \rangle. \]

From these equations, it is clear that \( \langle O_1(t) \rangle \) and \( \langle O_2(t) \rangle \) can be used as parameters in the coupling constant space. Note that the RG coefficients \( b_0, b_1 \) and \( d_0 \) are universal. In the infrared limit \( t \to \infty \), \( \langle O_2(t) \rangle \neq 0 \) corresponds to a relevant coupling \( \langle O_2(t) \rangle \to \infty \) around the Banks-Zaks fixed point at \( (\langle O_1(t) \rangle, \langle O_2(t) \rangle) = (-b_0/b_1, 0) \).

### 3. 3D \( O(N) \) linear sigma model at large \( N \) and the Wilson–Fisher fixed point

Our second example is the 3D \( O(N) \) linear sigma model that possesses the so-called Wilson–Fisher fixed point [19] in the infrared limit. The gradient flow of an operator in this system in relation to the Wilsonian RG flow was studied in detail in Ref. [10] and the Wilson–Fisher fixed point was observed. Actually, our present study was partially motivated by the study of Ref. [10]. We will consider the RG flow in the 2D coupling constant space in which there is one direction of the relevant operator around the fixed point (in Ref. [10], only 1D space along the irrelevant coupling is considered). We will work out the large-\( N \) approximation to the order of our concern. So, we first recapitulate the solution of the model in the large-\( N \) approximation for later use.

#### 3.1. The solution in the large-\( N \) approximation

The Euclidean action of the 3D \( O(N) \) linear sigma model is given by

\[ S = \int d^3x \left\{ \frac{1}{2} \partial_{\mu} \phi_i(x) \partial^{\mu} \phi_i(x) + \frac{1}{2} m_0^2 \phi_i(x) \phi_i(x) + \frac{1}{8N} \lambda_0 [\phi_i(x) \phi_i(x)]^2 \right\}, \]

where \( i = 1, \ldots, N \). We introduce the effective action, i.e., the generating functional of the 1PI correlation functions, as

\[ \Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3x_1 \cdots d^3x_n \phi_{i_1}(x_1) \cdots \phi_{i_n}(x_n) \Gamma_{i_1 \cdots i_n}^{(n)}(x_1, \ldots, x_n), \]

where \( \Gamma_{i_1 \cdots i_n}^{(n)}(x_1, \ldots, x_n) \) are the vertex functions. We also introduce the Fourier transformation:

\[ \Gamma_{i_1 \cdots i_n}^{(n)}(x_1, \ldots, x_n) = \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_n}{(2\pi)^3} e^{-ip_1x_1-\cdots-ip_nx_n} \tilde{\Gamma}_{i_1 \cdots i_n}^{(n)}(p_1, \ldots, p_n)(2\pi)^3 \delta(p_1 + \cdots + p_n). \]

\[^6\text{We assume } b_1 < 0; \text{ note that } \langle O_1(t) \rangle > 0 \text{ by definition.}\]
The large-$N$ approximation in this model is well known and, at the leading order of the approximation, by using the auxiliary field method for instance, we have
\[ \tilde{\Gamma}^{(2)}_{i_1i_2}(p_1, p_2) = \delta_{i_1i_2}(p_1^2 + M^2), \]  
(3.4)
\[ \tilde{\Gamma}^{(4)}_{i_1i_2i_3i_4}(p_1, p_2, p_3, p_4) = \delta_{i_1i_2}\delta_{i_3i_4} \left[ \frac{\lambda_0}{N} \left( 1 + \frac{\lambda_0}{8\pi} \frac{1}{\sqrt{(p_1^2 + p_2^2)^2}} \arctan \left( \frac{1}{2} \sqrt{\frac{(p_1^2 + p_2^2)^2}{M^2}} \right) \right)^{-1} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right]. \]  
(3.5)
In these expressions, the "physical" mass $M$ is given by the solution to the so-called gap equation,
\[ M^2 + \frac{\lambda_0}{8\pi} M = m_0^2 + \frac{1}{4\pi^2} \lambda_0 \Lambda, \]  
(3.6)
with $\Lambda$ being the momentum cutoff.
In the present model, the renormalized parameters in the mass-independent renormalization scheme can be defined as
\[ m_0^2 = Z_m m^2 + \delta m_0^2, \quad \lambda_0 = Z_\lambda \lambda. \]  
(3.7)
As Eq. (3.4) shows, there is no need of the wave function renormalization in the leading order of the large-$N$ approximation. We fix the renormalization constants $Z_m$, $\delta m_0^2$, and $Z_\lambda$ by imposing the following renormalization conditions at the renormalization scale $\mu$:
\[ \tilde{\Gamma}^{(2)}_{i_1i_2}(p_1, p_2) \bigg|_{p_1^2 = p_2^2 = 0, m^2 = 0} = 0, \]  
(3.8)
\[ \tilde{\Gamma}^{(2)}_{i_1i_2}(p_1, p_2) \bigg|_{p_1^2 = p_2^2 = 0, m^2 = \mu^2} = \mu^2, \]  
(3.9)
\[ \tilde{\Gamma}^{(4)}_{i_1i_2i_3i_4}(p_1, p_2, p_3, p_4) \bigg|_{p_i \cdot p_j = \mu^2 \delta_{ij} - \frac{1}{2} \mu^2 (1 - \delta_{ij}), m^2 = \mu^2} = \delta_{i_1i_2}\delta_{i_3i_4} \frac{\lambda}{N} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4). \]  
(3.10)
From the first two relations, we have
\[ \delta m_0^2 = -\frac{1}{4\pi^2} \lambda_0 \Lambda, \quad Z_m = 1 + \frac{1}{8\pi} \frac{\lambda_0}{\mu}, \]  
(3.11)
and from the last renormalization condition,
\[ \frac{\lambda}{\mu} = \frac{\lambda_0}{\mu} \left( 1 + \frac{\sqrt{3}}{96} \frac{\lambda_0}{\mu} \right)^{-1}. \]  
(3.12)
This gives rise to the beta function\[^7\]
\[ \beta \left( \frac{\lambda}{\mu} \right) \equiv \left( \mu \frac{\partial}{\partial \mu} \right) \frac{\lambda}{\mu} = -\frac{\lambda}{\mu} + \frac{\sqrt{3}}{96} \left( \frac{\lambda}{\mu} \right)^2. \]  
(3.13)
We note that the slopes of the beta function at two zeros of the beta function (fixed points) are given by
\[ \beta' \left( \frac{\lambda}{\mu} = 0 \right) = -1, \quad \beta' \left( \frac{\lambda}{\mu} = \frac{96}{\sqrt{3}} \right) = +1, \]  
(3.14)
respectively.
\[^7\] The subscript 0 in $(\mu \frac{\partial}{\partial \mu})_0$ implies that the derivative is taken while the bare parameters are kept fixed.
On the other hand, from the above relations, we have

\[
m^2 / \mu^2 = \left(1 + \frac{1}{8\pi \mu} \lambda_0 \right)^{-1} \left( m_0^2 / \mu^2 + \frac{1}{4\pi^2 \mu^2} \lambda_0 \Lambda \right),
\]
and

\[
\left( \mu \frac{\partial}{\partial \mu} \right)_0 m^2 / \mu^2 = -2 \left[ 1 + \left( \frac{3}{2} \frac{1}{8\pi} - \frac{\sqrt{3}}{96} \right) \frac{\lambda}{\mu} \right] m^2 / \mu^2.
\]

This RG equation becomes quite simple in terms of the parameter \( M \) defined by Eq. (3.6):

\[
\left( \mu \frac{\partial}{\partial \mu} \right)_0 M / \mu = -M / \mu.
\]

3.2. The flowed system and the RG flow

We now examine the picture (1.8) in the present model. We first have to introduce the flow equation for the scalar field \( \phi_i(x) \). The simplest choice is

\[
\partial_t \phi_i(t,x) = \partial_\mu \partial_\mu \phi_i(t,x), \quad \phi_i(t=0,x) = \phi_i(x).
\]

We refer the reader to Ref. [20] for the renormalizability of the flowed scalar theory. With the above choice, the correlation functions of the flowed field \( \phi_i(t,x) \) can be obtained from those of \( \phi_i(y) \) simply substituting \( \phi_i(t,x) \) by

\[
\phi_i(t,x) = \int d^3 y \int \frac{d^3 p}{(2\pi)^3} e^{ip(x-y)} e^{-tp^2} \phi_i(y).
\]

We thus have, for instance,

\[
\langle \phi_i(t,x) \phi_i(t,x) \rangle = N t^{-1/2} \int \frac{d^3 p}{(2\pi)^3} \frac{e^{-2p^2}}{p^2 + M^2 t} + O((1/N)^0) \quad \text{as} \quad t \to 0
\]

\[
\langle \partial_\mu \phi_i(t,x) \partial_\mu \phi_i(t,x) \rangle = N \frac{1}{(8\pi)^3} t^{-3/2} - M^2 \langle \phi_i(t,x) \phi_i(t,x) \rangle + O((1/N)^0),
\]

and

\[
\left[ \phi_i(t,x) \phi_i(t,x) \right]^2 - \left( 1 + \frac{2}{N} \right) \langle \phi_i(t,x) \phi_i(t,x) \rangle^2 \quad \text{as} \quad t \to 0
\]

\[
= -N \lambda_0 t^{-1/2} \prod_{i=1}^4 \int \frac{d^3 p_i}{(2\pi)^3} \frac{e^{-p_i^2}}{p_i^2 + M^2 t} \left( 2\pi \right)^3 \delta(p_1 + p_2 + p_3 + p_4) \times \left[ 1 + \frac{\lambda_0 t^{1/2}}{8\pi} \frac{1}{\sqrt{(p_1 + p_2)^2}} \arctan \left( \frac{1}{2} \sqrt{\frac{(p_1 + p_2)^2}{M^2 t}} \right) \right]^{-1} + O((1/N)^0).
\]

Note that in these expressions, momentum variables are dimensionless.
It is convenient to introduce a new field variable,
\[ \tilde{\varphi}_i(t, x) \equiv \sqrt{\frac{N}{2(2\pi)^{3/2} t^{1/2}}} \varphi_i(t, x) \sim t^{-0} \varphi_i(t, x) + O(1/N), \quad (3.32) \]
by analogy with Eq. (2.2), which is free from the wave function renormalization. Using this new variable, we define dimensionless operators,
\[ O_1(t, x) \equiv -\frac{4(2\pi)^{3}}{N} t [\tilde{\varphi}_i(t, x)\tilde{\varphi}_i(t, x)]^2 + (N + 2), \quad (3.25) \]
\[ O_2(t, x) \equiv \frac{16\pi}{N} t^{3/2} \partial_\mu \tilde{\varphi}_i(t, x) \partial_\mu \tilde{\varphi}_i(t, x) - \frac{1}{(2\pi)^{1/2}}. \quad (3.26) \]
Then, we have
\[ \langle O_1(t) \rangle = \lambda_0 t^{1/2} \left[ \int \frac{d^3 p}{(2\pi)^3} \frac{\lambda_0^{1/2}}{p^2 + M^2 t} \right]^{-2} \]
\[ \times \prod_{i=1}^{4} \left( \int \frac{d^3 p_i}{(2\pi)^3} \frac{\lambda_0^{1/2}}{p_i^2 + M^2 t} \right) (2\pi)^3 \delta(p_1 + p_2 + p_3 + p_4) \]
\[ \times \left[ 1 + \frac{1}{8\pi} \frac{1}{\sqrt{(p_1 + p_2)^2}} \arctan \left( \frac{1}{2} \sqrt{\frac{(p_1 + p_2)^2}{M^2 t}} \right) \right]^{-1} \quad (3.26) \]
\[ \overset{t \to 0}{\longrightarrow} K \lambda_0 t^{1/2} \quad (3.27) \]
\[ \overset{t \to \infty}{\longrightarrow} K' \lambda_0 \frac{1}{M} \left( 1 + \frac{1}{16\pi M} \right)^{-1} \frac{1}{M^3 t^{3/2}}, \quad (3.28) \]
where
\[ K = 32\pi^3 \prod_{i=1}^{4} \left( \int \frac{d^3 p_i}{(2\pi)^3} \frac{e^{-p_i^2}}{p_i^2} \right) (2\pi)^3 \delta(p_1 + p_2 + p_3 + p_4) \approx 0.289 432, \quad (3.29) \]
\[ K' = 512\pi^3 \prod_{i=1}^{4} \left( \int \frac{d^3 p_i}{(2\pi)^3} \frac{e^{-p_i^2}}{p_i^2} \right) (2\pi)^3 \delta(p_1 + p_2 + p_3 + p_4) = \frac{1}{(4\pi)^{3/2}}, \quad (3.30) \]
and
\[ \langle O_2(t) \rangle = \frac{1}{8\pi^2} \left[ \int \frac{d^3 p}{(2\pi)^3} \frac{e^{-2p^2}}{p^2 + M^2 t} \right]^{-1} - \left( \frac{8}{\pi} \right)^{1/2} M^2 t - \frac{1}{(2\pi)^{1/2}} \quad (3.31) \]
\[ \overset{t \to 0}{\longrightarrow} M t^{1/2} \quad (3.32) \]
\[ \overset{t \to \infty}{\longrightarrow} \left( \frac{2}{\pi} \right)^{1/2} - \frac{3}{(8\pi)^{1/2}} \frac{1}{M^2 t}. \quad (3.33) \]
The asymptotic behaviors (3.27) and (3.32) show that the initial condition of the flow is given by the parameters 0 and 0. In Figs. 1 and 2, we depict the RG flow lines in the space of \( \langle O_1(t) \rangle \) and \( \langle O_2(t) \rangle \) obtained numerically. We confirmed that the point indicated by the

\[ \text{If one sets } M \to 0 \text{ first, Eq. (3.26) yields } \langle O_1(t) \rangle \to K \lambda_0 t^{1/2}, \quad \langle O_1(t) \rangle \to 1.42596, \text{ while from Eq. (3.31), } \langle O_2(t) \rangle \equiv 0. \]
red point \([\langle O_1(t) \rangle, \langle O_2(t) \rangle] = (1.425,96,0)\) is an infrared fixed point that can be identified with the Wilson–Fischer fixed point \((\lambda_*/\mu = 96/\sqrt{3}\) in Eq. (3.14)). From the figures, we see that \(\langle O_2(t) \rangle\) basically corresponds to the relevant coupling around the fixed point; \(\langle O_1(t) \rangle\) to the irrelevant coupling.

![Fig. 1](image)

**Fig. 1** The RG flow in the space of \(\langle O_1(t) \rangle\) and \(\langle O_2(t) \rangle\). The arrows indicate how the point \((\langle O_1(t) \rangle, \langle O_2(t) \rangle)\) changes as \(t\) increases. The red point is the infrared fixed point.

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