Stochastic models of exotic transport

Piotr Garbaczewski*
Institute of Physics, Pedagogical University,
pl. Słowiński 6, PL-65 069 Zielona Góra, Poland

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Abstract

Non-typical transport phenomena may arise when randomly driven particles remain in an active relationship with the environment instead of being passive. If we attribute to Brownian particles an ability to induce alterations of the environment on suitable space-time scales, those in turn must influence their further movement. In that case a general feedback mechanism needs to be respected. By resorting to a specific choice of the particle-bath coupling, an enhanced (super-diffusion) or non-dispersive diffusion-type processes are found to exist in generically non-equilibrium contexts.

1 Prerequisites

A simple Brownian motion, in its canonical (model) manifestations, does not seem to hide any surprises. It is our main goal in the present paper to reconsider those ingredients of the standard formalism which, while relaxed or slightly modified, would lead to conceptually new and possibly exotic (in the sense of being non-typical) features.

One of possible "defects" of the standard theory is that the Brownian motion is incapable to originate spatiotemporal patterns (structures) and rather washes them out, even if initially in existence. In this particular context, "active" Brownian particles were introduced, [1]-[5]. Most generally, they are supposed to remain in a feedback relationship with the environment (heat reservoir, thermostat, random medium, deterministic driving system or whatever else) through which they propagate. If regarded as a thermal bath, the environment is basically out of equilibrium. Obviously, a concrete meaning of being active relies on a specific choice of the model for the thermostat (deterministic or random, linear or nonlinear etc.) and the detailed particle-medium coupling mechanism.

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Another "defective" feature of the standard theory is rooted in its possible molecular (chaos) foundations where the major simplification is needed while passing from the Boltzmann collision theory to the novel kinetic framework set by the Kramers equation, cf. [6]. In the process, microscopic (molecular) energy-momentum conservation laws completely evaporate from the formalism. Then, there is no obvious way how to reconcile random exchanges of energy and momentum between the Brownian particle and the thermostat with the manifest isothermality requirement. An issue of the heat production/removal and resulting thermal inhomogeneities is normally ignored (cf. however [7]-[13] for a discussion of various thermostat models, and the notion of thermostatting in non-equilibrium dynamics).

That was the starting point of our discussion of the origins of isothermal flows and the ultimate usage of the third Newton law in the mean to justify the concept of the Brownian motion with a recoil (via feedback relationship with the bath), [7].

In conformity with the metaphor, [14]: "everything depends on everything else", we relax the habitual sharp distinction between an entirely passive particle (nonetheless performing a continual erratic motion) and its exclusively active (perpetually at rest and in equilibrium) driving agency - the thermostat.

In the present paper we take the following point of view on major signatures of the particle activity while moving at random due to external (environmental) reasons: any action upon the particle exerted by the environment induces (on suitable space and time scales) a compensating reaction in the carrier random medium. In loose terms of Refs. [1]-[3] we attribute to any (even single) Brownian particle an ability to generate perturbations of the medium (named a selfconsistent field in Ref. [1] and perturbations of noise in Ref. [7]) which in turn influence its further movement. That is a non-linear feedback relationship mentioned before: while inducing a random dynamics of a particle (action), the medium suffers alterations (reaction), which modify next stages of the particle motion (feedback). Such features are obviously alien to the standard formalism.

There is no experimentally reliable way to watch Brownian particles individually on time scales below say $\frac{1}{100}$s, nor get any insight into the associated fine detailed particle-bath interaction phenomena. The only realistic procedure to quantify observable features of the Brownian motion is to exploit its familiar Janus face (cf. [1]). Namely, the same mathematical formalism essentially applies to a single particle and to a statistical ensemble of identical non-interacting Brownian particles. Then a hydrodynamical analogy can be exploited on both levels of description, [15, 16], once we invoke a standard kinetic reasoning (used in passing from the Boltzmann equation to gas or liquid dynamics equations).

Effectively, we need to evaluate (conditional) averages over a statistical ensemble of particles, which (in loose terminology that follows Ref. [15], see also [17]) constitutes one component (noninteracting Brownian "gas") of a coupled two-component continuous system. Another component is the thermostat.

What are the ultimate kinetic features of the Brownian motion (diffusion process)
critically relies on the thermostat model i.e. specific mechanisms of the energy/heat exchange due to the coupling between the thermostat and the Brownian "gas". This particular issue we shall investigate in some detail in below, by resorting to somewhat nontypical (exotic in the Brownian motion context) methods.

2 Local conservation laws for the Brownian motion and Smoluchowski-type diffusion

It is useful to exploit a standard phase-space argument that is valid, under isothermal conditions, for a Markovian diffusion process taking place in (or relative to) a driving flow $\vec{w}(\vec{x}, t)$ with as yet unspecified dynamics nor concrete physical origin. (In particular, such a flow can receive an interpretation of a selfconsistent field generated in the environment by Brownian particles themselves, cf. Ref. [1].)

We account for an explicit external force (here, acceleration $\vec{K} = \vec{F}/m$) exerted upon diffusing particles, while not directly affecting the driving flow itself. Then, infinitesimal increments of phase-space random variables read:

$$d\vec{X}(t) = \vec{V}(t)dt$$

$$d\vec{V}(t) = \beta[\vec{w}(\vec{x}, t) - \vec{V}(t)]dt + \vec{K}(\vec{x})dt + \beta\sqrt{2D}d\vec{W}(t).$$

Following the leading idea of the Smoluchowski approximation, we assume that $\beta$ is large, and consider the process on time scales significantly exceeding $\beta^{-1}$. Then, an appropriate choice of the velocity field $\vec{w}(\vec{x}, t)$ may in principle guarantee the convergence of the spatial part $\vec{X}(t)$ of the process to the Itô diffusion process with infinitesimal increments:

$$d\vec{X}(t) = \vec{b}(\vec{x}, t)dt + \sqrt{2D}d\vec{W}(t).$$

In this case, the forward drift of the process reads $\vec{b}(\vec{x}, t) = \vec{w}(\vec{x}, t) + \frac{1}{\beta}\vec{K}(\vec{x})$. Notice that the $\beta^{-1}\vec{K}$ contribution can be safely ignored if we are interested in the dominant driving motion.

Throughout the paper we are interested in Markovian diffusion processes, which propagate respectively the phase-space or configuration space probability densities (weak solutions of stochastic differential equations are thus involved). In the configuration space variant, we deal with a Markovian stochastic process whose probability density $\rho(\vec{x}, t)$ evolves according to the standard Fokker-Planck equation

$$\partial_t \rho = D\Delta \rho - \nabla \cdot (\vec{b}\rho)$$

and the forward drift is not necessarily of the form dictated by Eqs. (1) and (2). We admit here more general forward drift functions, cf. Ref. [7], which do not allow for a simple additive decomposition and thus are capable to give account of nonlinearities in the particle-bath coupling.
One can easily transform the Fokker-Planck equation to the familiar form of the continuity equation (hydrodynamic mass conservation law) \( \partial_t \rho = -\nabla \cdot (\vec{v} \rho) \) by defining \( \vec{v} = \vec{b} - D \frac{\nabla \rho}{\rho} \). The current velocity \( \vec{v} \) obeys a local (momentum per unit of mass) conservation law which directly originates from the rules of the Itô calculus for Markovian diffusion processes, and from the first moment equation in the diffusion approximation of the Kramers theory.\[8,16]:

\[
\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = \nabla (\Omega - Q) .
\] (5)

While looking similar to the standard Euler equation appropriate for the lowest order hydrodynamical description of gases and liquids, this equation conveys an entirely different physical message.

First of all, for a class of forward drifts that are gradient fields the most general admissible form of an auxiliary potential \( \Omega(\vec{x}, t) \) reads:

\[
\Omega(\vec{x}, t) = 2D[\partial_t \phi + \frac{1}{2}(\frac{\vec{b}^2}{2D} + \nabla \cdot \vec{b})] .
\] (6)

Here \( \vec{b}(\vec{x}, t) = 2D\nabla \phi(\vec{x}, t) \). In reverse, by choosing a bounded from below continuous function to represent conservative force fields (after taking the gradient) i.e. otherwise arbitrary \( \Omega \), we can always disentangle the above (Riccati-type) identity with respect to the drift field.

Moreover, instead of the standard pressure term (consider a state equation \( P \sim \rho^\alpha, \alpha > 0 \)), there appears a contribution from more complicated (derivatives!) \( \rho \)-dependent potential \( Q(\vec{x}, t) \). It is is given in terms of the so-called osmotic velocity field \( \vec{u}(\vec{x}, t) = D\nabla \ln \rho(\vec{x}, t) \):

\[
Q(\vec{x}, t) = \frac{1}{2}\vec{u}^2 + D\nabla \cdot \vec{u} .
\] (7)

An equivalent form of the enthalpy-related potential \( Q \) is \( Q = 2D^2 \frac{\partial \rho^{\alpha/2}}{\rho^{\alpha/2}} \).

A general expression for the local diffusion current is \( \vec{j} = \rho \vec{v} = \rho(\vec{b} - D \nabla \rho) \). This local flow in principle may be experimentally observed for a cloud of suspended particles in a liquid. The current \( \vec{j} \) is nonzero in non-equilibrium situations and a non-negligible matter transport occurs as a consequence of the Brownian motion, on the ensemble average. We thus cannot avoid local heating/cooling phenomena that need to push the environment out of equilibrium. That leads to obvious temperature inhomogeneities, which are normally disregarded, cf. Ref. [8].

If the forward drift is interpreted as a gradient of a suitable function \( (b = 2D\nabla \phi = \nabla \Phi) \) and we take \( \Phi(\vec{x}, 0) \) as the initial data for the \( t \geq 0 \) evolution), then we have:

\[
\Omega = \partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + D\Delta \Phi .
\] (8)
If we decide that the above Hamilton-Jacobi-type equation is to be solved with respect to the field $\Phi(\mathbf{x}, t)$, its solution (and general solvability issue) relies on the choice of a bounded from below, continuous function $\Omega(\mathbf{x}, t)$, \textit{without} any a priori knowledge of forward drifts. Viewed that way, Eq. (7) sets limitations on admissible forms of the space-time dependence of any conceivable self-consistent field to be generated in the bath by Brownian particles (cf. Ref. [1]). There is no freedom at all for postulating various partial differential equations, if their solutions are to be interpreted as forward drifts of Markovian diffusion processes, [18].

There is also interesting to observe that a gradient field ansatz for the diffusion current velocity $\mathbf{v} = \mathbf{\nabla} S$ allows to transform the momentum conservation law of a Markovian diffusion process to the universal Hamilton-Jacobi form:

$$\Omega = \partial_t S + \frac{1}{2} |\mathbf{\nabla} S|^2 + Q$$  \hspace{1cm} (9)

where $Q(\mathbf{x}, t)$ was defined before as the "pressure/enthalpy"-type function. (That form looks deceivingly similar to the standard hydrodynamic conservation law, valid for liquids and gases at thermal equilibrium, where instead of $Q$ an enthalpy function normally appears.) By performing the gradient operation we recover the previous hydrodynamical form of the law.

In the above, the contribution due to $Q$ is a direct consequence of an initial probability measure choice for the diffusion process, while $\Omega$ alone does account for an appropriate forward drift of the process, playing at the same time the role of the volume force potential ($\mathbf{\nabla} Q = \mathbf{\nabla} P$ contributes to energy-momentum transfer effects through the boundaries of any volume).

3 Moment equations for the free Brownian motion: Getting out of conventions

The derivation of a hierarchy of local conservation laws (moment equations) for the Kramers equation can be patterned after the standard procedure for the Boltzmann equation, [6, 16, 15]. Those laws do not form a closed system and additional specifications (like the familiar thermodynamic equation of state) are needed to that end. In case of the isothermal Brownian motion, when considered in the large friction regime (e.g. Smoluchowski diffusion approximation), it suffices to supplement the Fokker-Planck equation by one more conservation law \textit{only} to arrive at a closed system.

To give a deeper insight into what really happens on the way from the phase-space theory of the Brownian motion to its approximate configuration-space (Smoluchowski) version, let us consider the familiar Ornstein-Uhlenbeck process (in velocity/momentum) in its extended phase-space form. For clarity of discussion, we discuss random dynamics for one degree of freedom only.
In the absence of external forces, the kinetic (Kramers-Fokker-Planck equation) reads:

$$\partial_t W + u \nabla_x W = \beta \nabla_u (Wu) + q \triangle_u W$$  \hspace{1cm} (10)

where $q = D\beta^2$. Here $\beta$ is the friction coefficient, $D$ will be identified later with the spatial diffusion constant, and provisionally we set $D = k_B T / m \beta$ in conformity with the Einstein fluctuation-dissipation identity.

The joint probability distribution (in fact, density) $W(x, u, t)$ for a freely moving Brownian particle which at $t = 0$ initiates its motion at $x_0$ with an arbitrary initial velocity $u_0$ can be given in the form of the maximally symmetric displacement probability law:

$$W(x, u, t) = W(R, S) = \left\{ \frac{4\pi^2 (FG - H^2)}{2(FG - H^2)} \right\}^{-1/2} \exp \left\{ -\frac{GR^2 - HRS + FS^2}{2(FG - H^2)} \right\}$$  \hspace{1cm} (11)

where $R = x - u_0(1 - e^{-\beta t})^{\beta^{-1}}$, $S = u - u_0 e^{-\beta t}$ while $F = \frac{D}{\beta}(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t})$, $G = D\beta(1 - e^{-2\beta t})$ and $H = D(1 - e^{-\beta t})^2$.

Marginal probability densities, in the Smoluchowski regime (we take for granted that time scales $\beta^{-1}$ and space scales $(D\beta^{-1})^{1/2}$ are irrelevant), take familiar forms of the Maxwell-Boltzmann $w(u, t) = \left( \frac{m^2 \pi k_B T}{2\pi} \right)^{1/2} \exp \left( -\frac{mu^2}{2k_B T} \right)$ and the diffusion kernel $w(x, t) = \left( \frac{4\pi D t}{2\pi} \right)^{-1/2} \exp \left( -\frac{x^2}{4Dt} \right)$ respectively.

A direct evaluation of the first and second local moment of the phase-space probability density:

$$<u> = \int du u W(x, u, t) = w(R) [(H/F)R + u_0e^{-\beta t}]$$  \hspace{1cm} (12)

$$<u^2> = \int du u^2 W(x, u, t) = \left( \frac{FG - H^2}{F} + \frac{H^2}{F^2} R^2 \right) \cdot (2\pi F)^{-1/2} \exp \left( -\frac{R^2}{2F} \right)$$  \hspace{1cm} (13)

after passing to the diffusion (Smoluchowski) regime, allows to recover the local (configuration space conditioned) moment $<u> = \frac{1}{w} <u>$ which reads

$$<u>_x = \frac{x}{2t} = -D \frac{\nabla w(x, t)}{w(x, t)}$$  \hspace{1cm} (14)

while for the second local moment $<u^2>_x = \frac{1}{w} <u^2>$ we arrive at

$$<u^2>_x = (D\beta - D/2t) + <u^2>$$  \hspace{1cm} (15)

By inspection one verifies that the transport (Kramers) equation for $W(x, u, t)$ implies local conservation laws:

$$\partial_t w + \nabla(<u>_x w) = 0$$  \hspace{1cm} (16)
and
\[ \partial_t (< u >_x w) + \nabla_x (< u^2 >_x w) = -\beta < u >_x w . \] (17)

By introducing (we strictly follow the moment equations strategy of the traditional kinetic theory of gases and liquids) the notion of pressure function \( P_{\text{kin}} \) (we choose another notation to make a difference with the previous notion \( P \) of the pressure function, cf. \( \nabla Q = \frac{\nabla \rho}{\rho} \)):
\[ P_{\text{kin}}(x,t) = (< u^2 >_x - < u >^2_x)w(x,t) \] (18)
we can analyze the local momentum conservation law
\[ (\partial_t + < u >_x \nabla) < u >_x = -\beta < u >_x - \frac{\nabla P_{\text{kin}}}{w} \] (19)

In the Smoluchowski regime the friction term is cancelled away by a counterterm coming from \( \frac{1}{w} \nabla P_{\text{kin}} \) so that
\[ (\partial_t + < u >_x \nabla) < u >_x = \frac{D}{2t} \frac{\nabla w}{w} = -\frac{\nabla P}{w} = -\nabla Q \] (20)
where \( P = D^2 w \Delta \ln w \).

For comparison with the notations (local conservation laws) of the previous section one needs to replace \( w(x,t) \) by \( \rho(x,t) \) and \( < u >_x \) by \( v(x,t) \) in all formulas that pertain to the Smoluchowski regime.

Further exploiting the kinetic lore, we can tell few words about the temperature of Brownian particles as opposed to the (possibly equilibrium) temperature of the thermal bath. Namely, in view of (we stay in the Smoluchowski regime) \( P_{\text{kin}} \sim (D\beta - \frac{D}{2t})w \) where \( D = \frac{k_B T}{m_m \beta} \), we can formally set:
\[ \Theta = \frac{P_{\text{kin}}}{w} \sim (k_B T - \frac{D}{2t}) < k_B T . \] (21)

That quantifies the degree of thermal agitation (temperature) of Brownian particles to be less than the thermostat temperature. Heat is continually pumped from the thermostat to the Brownian ”gas”, until asymptotically both temperatures equalize. This may be called a ”thermalization” of Brownian particles. In the process of thermalization the Brownian ”gas” temperature monotonically grows up until the mean kinetic energy of particles and that of mean flows asymptotically approach the familiar kinetic relationship:
\[ \int \frac{w}{2} (< u^2 >_x - < u >^2_x)dx = k_B T \] (22)

In view of this medium → particles heat transfer issue, one must be really careful while associating habitual thermal equilibrium conditions with essentially nonequilibrium phenomena, cf. Ref. [7] for more extended discussion.
4 Hydrodynamical reasoning: In search for exotic

Once local conservation laws were introduced, it seems instructive to comment on the essentially hydrodynamical features (compressible fluid/gas case) of the problem. Specifically, the "pressure" term $\nabla Q$ is here quite annoying from the traditional kinetic theory perspective. That is quite apart from the fact that our local conservation laws have a conspicuous Euler form appropriate for the standard hydrodynamics of gases and liquids. Let us stress that in case of normal liquids the pressure is exerted upon any control volume (droplet) by the surrounding fluid. We may interpret that as a compression of a droplet. In case of Brownian motion, we deal with a definite decompression: particles are driven away from areas of higher concentration (probability of occurrence). Hence, typically the Brownian "pressure" is exerted by the droplet upon its surrounding.

Following the hydrodynamic tradition let us analyze that "pressure" issue in more detail. We consider a reference volume (control interval, finite droplet) $[\alpha, \beta]$ in $R^d$ (or $\Lambda \subset R^d$ ) which at time $t \in [0, T]$ comprises a certain fraction of particles (Brownian "fluid" constituents). The time rate of particles loss or gain by the volume $[\alpha, \beta]$ at time $t$, is equal to the flow outgoing through the boundaries i.e.

$$- \partial_t \int_{\alpha}^{\beta} \rho(x,t) dx = \rho(\beta, t)v(\beta, t) - \rho(\alpha, t)v(\alpha, t)$$

which is a consequence of the continuity equation.

To analyze the momentum balance, let us allow for an infinitesimal deformation of the boundaries of $[\alpha, \beta]$ to have entirely compensated the mass (particle) loss or gain:

$$[\alpha, \beta] \rightarrow [\alpha + v(\alpha, t)\Delta t, \beta + v(\beta, t)\Delta t]$$

Effectively, we pass then to the locally co-moving frame. That implies

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_{\alpha + v(\alpha)\Delta t}^{\beta + v(\beta)\Delta t} \int_{\alpha}^{\beta} \rho(x,t) dx + \Delta t \int_{\alpha}^{\beta} (\partial_t \rho) dx + \int_{\beta}^{\beta + v(\beta)\Delta t} \rho(x,t) dx \right] = 0$$

Let us investigate what happens to the local matter flows $(\rho v)(x,t)$, if we proceed in the same way (leading terms only are retained):

$$\int_{\alpha + v(\alpha)\Delta t}^{\beta + v(\beta)\Delta t} (\rho v)(x,t) dx - \int_{\alpha}^{\beta} (\rho v)(x,t) dt \sim$$

$$\sim -(\rho v^2)(\alpha, t)\Delta t + (\rho v^2)(\beta, t)\Delta t + \Delta t \int_{\alpha}^{\beta} [\partial_t \rho v] dx$$
In view of local conservation laws we have \( \partial_t (\rho v) = -\nabla (\rho v^2) + \rho \nabla (\Omega - Q) \) and the rate of change of momentum associated with the control volume \([\alpha, \beta]\) is (here per unit of mass)

\[
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_{\alpha + v_2 \Delta t}^{\beta + v_2 \Delta t} (\rho v)(x, t + \Delta t) - \int_{\alpha}^{\beta} (\rho v)(x, t) \right] = \int_{\alpha}^{\beta} \rho \nabla (\Omega - Q) \, dx
\]  

(27)

However, \( \nabla Q = \frac{\nabla P}{\rho} \) and \( P = D^2 \rho \Delta ln \rho \). Therefore:

\[
\int_{\alpha}^{\beta} \rho \nabla (\Omega - Q) \, dx = \int_{\alpha}^{\beta} \rho \nabla Q \, dx - \int_{\alpha}^{\beta} \nabla P \, dx =
\]

\[
E[\nabla \Omega]_{\alpha}^{\beta} + P(\alpha, t) - P(\beta, t)
\]

(28)

Clearly, \( \nabla \Omega \) refers to the Euler-type volume force, while \( \nabla Q \) (or more correctly, \( P \)) refers to the “pressure” effects entirely due to the particle transfer rate through the boundaries of the considered volume.

The missing ingredient of our discussion is the time development of the kinetic energy of the matter flow \( \frac{1}{2}(\rho v^2) \) (per unit of mass) transported through the chosen volume. Let us therefore evaluate:

\[
\int_{\alpha + v_2 \Delta t}^{\beta + v_2 \Delta t} \frac{1}{2}(\rho v^2)(x, t + \Delta t) \, dx - \int_{\alpha}^{\beta} \frac{1}{2}(\rho v^2)(x, t) \, dt \sim
\]

\[
\sim -\frac{1}{2}(v^3)(\alpha, t) \Delta t + \frac{1}{2}(v^3)(\beta, t) \Delta t + \Delta t \int_{\alpha}^{\beta} [\partial_t \frac{1}{2}(\rho v^2)] \, dx
\]

(29)

We have \( \partial_t (\frac{1}{2} \rho v^2) = -\frac{1}{2} v^2 \nabla (\rho v) - \rho v \nabla (Q - \Omega + \frac{1}{2} v^2) = -\nabla (\frac{1}{2} \rho v^3) - \rho v \nabla Q - \Omega \). Consequently, the time rate of the kinetic energy (of the flow) loss/gain by the volume reads:

\[
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_{\alpha + v_2 \Delta t}^{\beta + v_2 \Delta t} \frac{1}{2}(\rho v^2)(x, t + \Delta t) - \int_{\alpha}^{\beta} \frac{1}{2}(\rho v^2)(x, t) \right] = \int_{\alpha}^{\beta} (\rho v) \nabla (\Omega - Q) \, dx
\]

(30)

In the integrand one immediately recognizes an expression (up to the mass parameter) for the power released/absorbed by the control volume at time \( t \).

In particular, by taking advantage of the identity \( \nabla Q = \frac{\nabla P}{\rho} \) we can rewrite the pure ”pressure” contribution as follows: \( -\int_{\alpha}^{\beta} \rho v \nabla Q \, dx = -\int_{\alpha}^{\beta} v \nabla P \, dx \) which clearly is a direct analog of the standard mechanical expression for the power release \( \frac{dE}{dt} = F \cdot v \).

Surely, one cannot interpret the above local averages as an outcome of an innocent operation of the random medium upon particles. There is a non-negligible transfer of energy and momentum to be accounted for in association with the Brownian motion, and a number of problems (when can we disregard temperature gradients?) pertaining to local heating and cooling phenomena suffered by the environment (in terms of local averages) should be consistently resolved.
5 Implementing the feedback

In contrast to molecular chaos derivations based on the Boltzmann equation, in case of the Brownian motion we have no access into the microscopic fine details of particle propagation. Instead, we need to postulate the mathematically reliable form of the dynamics, even if ending up with obvious artifacts of the formalism (like e.g. non-differentiable paths, or their infinite spatial variation on arbitrarily short time intervals, in case of the Wiener process or, even worse, white-noise input).

Our major hypothesis about an exotic particle-bath coupling pertains to local averages and is motivated by a fairly intuitive picture of the behaviour of suspended particles in thermally inhomogeneous media. Namely, it is well known that hot areas support much lower density of suspended particles (can even be free of any dust admixtures) than the lower temperature areas. Dynamically we can interpret this phenomenon as a repulsion of suspended particles by warm areas and an attraction by the cool ones, cf. Ref. [7].

In the course of our discussion we have spent quite a while on demonstrating that a non-trivial energy/heat exchange is completely ignored in the standard approach to the Brownian motion (even, if under suitable circumstances one has good reasons to do that). Effectively, the model dramatically violates basic conservation laws of physics. That derives from the assumption that the thermostat is perpetually in the state of rest (in the mean), in thermal equilibrium and thus free of any thermal currents.

The force/acceleration term (we turn back to the three-dimensional notation) \( \vec{\nabla}(\Omega - Q) \) appears in all formulas that refer to energy/momentum flows supported by local averages of the Brownian motion. Their sole reason is the action of the random medium upon particles which causes an expansion of the Brownian ”swarm” out of areas of higher concentration. That however needs a local cooling of the medium (Brownian particles are being ”thermalized” by increasing their mobility e.g. temperature). On sufficiently low time scales that should amount to an instantaneous reaction of the medium in terms of thermally induced currents: Brownian particles are driven back (attraction !) to the cooler areas i.e. float down the temperature gradients as long as they are non-vanishing, [19].

We shall bypass the thermal inhomogeneity issue by resorting to an explicit energy-momentum balance information available through local conservation laws for the Brownian motion.

If we regard the term \( \vec{\nabla}(\Omega - Q) \) as a quantification of the sole medium action upon particles, then the most likely quantification of the medium reaction should be exactly the opposite i.e. \( \vec{\nabla}(Q - \Omega) \).

Told otherwise, medium ”effort” to release momentum and energy from the Brownian ”droplet” (control volume), at a time rate determined by the functional form of \( \vec{\nabla}(\Omega - Q)(x, t) \), induces a compensating (in view of the heat deficit) energy and momentum delivery to that volume needed to remove the thermal inhomogeneity of the thermostat. As a consequence, Brownian particles propagate through the medium
which is no longer in the state of rest and develops intrinsic (mean, on the ensemble average) flows.

A mathematical encoding of this Brownian recoil principle [20], or third Newton law in the mean [7] hypothesis is rather well established.

The momentum conservation law for the process *with a recoil* (the reaction term replaces the decompressive action term) will read:

\[ \partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{\nabla}(Q - \Omega) \] (31)

implying that

\[ \partial_t S + \frac{1}{2} |\vec{\nabla}S|^2 - Q = -\Omega \] (32)

stands for the corresponding Hamilton-Jacobi equation, cf. [21], instead of ”normal” one. A suitable adjustment (re-setting) of the initial data is here necessary, cf. [7].

In the coarse-grained picture of motion we shall deal with a sequence of repeatable feedback scenarios realized on the Smoluchowski process time scale: the Brownian ”swarm” expansion build-up is accompanied by the parallel counter-flow build-up, which in turn modifies the subsequent stage of the Brownian ”swarm” migration (being interpreted to modify the forward drift of the process) and the corresponding, built-up anew counter-flow.

Perhaps surprisingly, we are still dealing with Markovian diffusion-type processes, [22]. The link is particularly obvious [23] if we observe that the new Hamilton-Jacobi equation can be formally rewritten in the previous form by introducing:

\[ \Omega_r = \partial_t S + \frac{1}{2} |\vec{\nabla}S|^2 + Q \] (33)

where \( \Omega_r = 2Q - \Omega \) and \( \Omega \) represents the previously defined potential function characterizing any Smoluchowski (or more general) diffusion process. It is however \( \Omega_r \) which would determine forward drifts of the Markovian diffusion process with a recoil. Those must come out from the Cameron-Martin-Girsanov identity \( \Omega_r = 2Q - \Omega = 2D[\partial_t \phi + \frac{1}{2}(\vec{b}^2 + \vec{\nabla} \cdot \vec{b})] \).

After complementing the Hamilton-Jacobi-type equation by the continuity equation, we again end up with a closed system of conservation laws. The system is badly nonlinear and coupled, but its linearisation can be immediately given in terms of an adjoint pair of Schrödinger equations with a potential \( \Omega \) (the imaginary unit \( i \) on the left-hand-side in below is not an error !), [22, 21, 23]. Indeed,

\[ i\partial_t \psi = -D\Delta \psi + \frac{\Omega}{2D} \psi \] (34)

with a solution represented in the polar form \( \psi = \rho^{1/2} \exp(iS) \) and its complex adjoint makes the job. The choice of \( \psi(x,0) \) sets here a solvable Cauchy problem. Notice that, in view of the Schrödinger-type linearization, for time-independent \( \Omega \) (conservative
forces) the total energy \( \int_{R^3} \left( \frac{v^2}{2} - Q + \Omega \right) \rho d^3x \) of the system is a conserved finite quantity. We thus reside within the framework of so-called finite energy diffusion processes, whose mathematical features received some attention in the literature. In particular, it is known that in the absence of volume forces, a superdiffusion appears, while harmonic volume forces allow for non-dispersive diffusion-type processes, [7].

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