Solving the 3 + 1 GRMHD equations in the eXtended Conformally Flat Condition: the XNS code for magnetized neutron stars

N. Bucciantini\textsuperscript{1,2}, A.G. Pili\textsuperscript{3}, L. Del Zanna\textsuperscript{3,1,2}
\textsuperscript{1}INAF - Osservatorio Astrofisico di Arcetri, Firenze, Italy
\textsuperscript{2}INFN - Sezione di Firenze, Italy
\textsuperscript{3}Dipartimento di Fisica e Astronomia, Universit`a di Firenze, Italy

Abstract. High-energy phenomena in astrophysics involve quite generally a combination of relativistic motions and strong gravity. The simultaneous solution of Einstein equations and General Relativistic MHD equations is thus necessary to model with accuracy such phenomena. The so-called Conformally Flat Condition (CFC) allows a simplified treatment of Einstein equations, that can be particularly efficient in those contexts where gravitational wave emission is negligible, like core-collapse, or the formation/evolution of neutron stars. We have developed a set of codes to model axisymmetric MHD flows, in General Relativity, where the solution of Einstein equations is achieved with a semi-spectral scheme. Here, we will show how this framework is particularly well suited to investigate neutron star equilibrium models in the presence of strong magnetic fields and we will present the XNS code, that has been recently developed and here updated to treat poloidal and mixed configurations.

1. Introduction

Neutron stars (NSs) are the most compact objects in the universe endowed with an internal structure, and they are among the most studied objects in high energy astrophysics. NSs have very high surface magnetic fields, up to $10^{16}$ G for magnetars (Mereghetti 2008). It is this very strong magnetic field that is responsible for most of their phenomenology. This field is amplified during the NS formation, and in principle, in the interior it could be as high as $10^{18}$ G. Strongly magnetized NSs are at the base of the so called millisecond magnetar model for Long and Short Gamma Ray Bursts (GRBs) (Bucciantini et al. 2012; Metzger et al. 2011; Bucciantini et al. 2009). A strong magnetic field will inevitably introduce deformations of the NS, and this makes them ideal Gravitational Waves sources.

During formation it is reasonable to expect that the magnetic field will rapidly settle into an equilibrium configuration. Recently, models for relativistic magnetized stars have been presented either for a purely toroidal field by Kiuchi & Yoshida (2008, KY), and Frieben & Rezzolla (2012, FR) or for a purely poloidal magnetic field by Bocquet et al. (1995). However such configurations are unstable (Braithwaite & Spruit 2006; Braithwaite 2009). Stability requires a mixed configuration of toroidal and poloidal field usually referred as Twisted Torus (TT). TT models have been presented so far either in Newtonian regime (Lander & Jones 2009, 2012), or in General Relativity (GR)
but with a perturbative approach for the metric and/or magnetic terms (Ciolfi et al. 2009, 2010; Ciolfi & Rezzolla 2013).

The main difficulty in solving for magnetized equilibrium models in GR is due to the complexity of Einstein equations, if an exact solution is desired. Einstein equations reduce to a set of non-linear coupled elliptical partial differential equations, which can only be solved numerically. However it is well known that non-linear elliptical equations can be numerically unstable, depending on the way the non-linear terms are cast.

Here we present a novel approach to compute magnetized equilibrium models for NSs. Instead of looking for an exact solution of Einstein equations, we make the simplifying assumption that the Conformally Flat Condition (CFC) holds for the metric. This allows us to greatly simplify the equations to be solved, and to cast them in a form that is numerically stable (extended CFC, or XCFC), improving upon previous perturbative works where the metric was assumed to be spherically symmetric. This allows us to handle stronger fields and deformations. Results are indistinguishable from those obtained in the correct regime. This suggests that the simplification of our approach does not compromise the accuracy of the results, while greatly simplifying their computation. In this work we also describe the XNS code (Bucciantini & Del Zanna 2011), here extended to the case of poloidal fields.

2. Spacetime metric in the Conformally Flat Condition

Given a generic spacetime, the line element can be written as (Alcubierre 2008; Gourgoulhon 2012):

\[ ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \]

where \( \alpha \) is called the lapse function, \( \beta^i \) is the shift vector, \( \gamma_{ij} \) is the three-metric, and \( i, j = r, \theta, \phi \), if a spherical coordinate system \( x^\mu = (t, r, \theta, \phi) \) is chosen. The assumptions of stationarity and axisymmetry imply that all metric terms are only a function of \( r \) and \( \theta \). A metric is said to be conformally-flat (CFC) if \( \gamma_{ij} = \psi^4 \text{diag}(1, r^2, r^2 \sin^2 \theta) \), where \( \psi \) is called the conformal factor.

If the energy-momentum tensor \( T^{\mu\nu} \) of the matter (and fields) distribution satisfies the circularity relations:

\[ t_\mu T^{\mu[\nu} r^\nu \phi^4] = 0, \quad \phi_\mu T^{\mu[\nu} r^\nu \phi^4] = 0, \]

where square brackets indicates antisymmetrization with respect to enclosed indexes and we have defined \( \nu^\mu := (\partial_i)^\mu, \phi^\mu := (\partial_k)^\mu \), then the metric is quasi-isotropic, with \( \gamma_{ij} = \psi^4 \text{diag}(1, r^2, Ar^2 \sin^2 \theta) \) and \( \beta^i = (0, 0, 0) \).

The GRMHD stress-energy tensor reads

\[ T^{\mu\nu} = (e + p + b^2) u^\mu u^\nu - b^\mu b^\nu + (p + b^2/2) g^{\mu\nu}, \]

where \( e \) is the total energy density, \( p \) is the pressure, \( u^\mu \) is the 4-velocity of the fluid, and \( b^\mu := F^{\mu\nu} u_\nu \) is the magnetic field as measured in the comoving frame, and \( F^{\mu\nu} \) is the Faraday tensor (the asterisk indicates the dual). For the static configurations assumed here \( u_\mu = (-\alpha, 0, 0, 0) \), the magnetic field in the lab frame \( B^\mu = b^\mu \), and circularity holds in the case of purely toroidal \( \{B^\mu = (0, 0, 0, B^\phi)\} \) or purely poloidal \( \{B^\mu = (0, B^r, B^\phi, 0)\} \) fields.
For mixed TT configurations, deviations from circularity become relevant only for magnetic fields with strength $\sim 10^{19}$ G, unrealistically high even for extreme NSs. So in general one can safely assume circularity also in the TT case and a quasi-isotropic metric. However, even for highly deformed objects, i.e. for rotating NSs at the mass shedding limit, the value of $A$ deviates from 1 by no more than $10^{-4}$. A conformally flat metric thus appears to be a good approximation, better suitable to numerical solution.

The last assumption is that of a static spacetime, for which $\beta^i = 0$, leading to an extra condition called maximal slicing and to a further simplification of the Einstein equations. In this case, the equations for the two remaining unknowns $\psi$ and $\alpha$ are:

$$\Delta \psi = [-2\pi\psi^5(e + B^2/2)]^{1/5},$$ (4)

$$\Delta (\alpha \psi) = [2\pi(\psi^5(e + B^2/2) + \psi^3(6p + B^2)\psi^{-2})](\alpha \psi),$$ (5)

where $\Delta$ is the standard laplacian operator in spherical coordinates, and the source term have been renormalized to insure stability, according to the XCFC approach (Cordero-Carrión et al. 2009; Bucciantini & Del Zanna 2011).

3. Bernoulli Integral and Grad-Shafranov equation

The divergence-free condition for the magnetic field allows us to rewrite the poloidal components as derivatives of the $\phi$ component of the vector potential, $A_\phi$, and the imposed symmetries will allow us to define free functions of that quantity, defined on the so-called magnetic surfaces. The only non-vanishing equation of the static GRMHD system is the Euler equation in the presence of an external electromagnetic field:

$$\partial_i p + (e + p)\partial_i \ln \alpha = L_i := \epsilon_{ijk} J^j B_k,$$ (6)

where $L_i$ is the Lorentz force and $J^i = \alpha^{-1} \epsilon^{ijk} \partial_j (\alpha B_k)$ is the conduction current. Assuming a barotropic EoS $p = p(\rho)$, $e = e(\rho)$ ($\rho$ is the rest mass density), we find

$$\partial_i \ln h + \partial_i \ln \alpha = \frac{L_i}{\rho h},$$ (7)

where the specific enthalpy is $h := (e + p)/\rho$. Integrability requires

$$L_i = \rho h \partial_i M = \rho h \left( \frac{dM}{dA_\phi} \partial_i A_\phi \to \ln \left( \frac{h}{h_c} \right) + \ln \left( \frac{\alpha}{\alpha_c} \right) - \mathcal{M}(A_\phi) \right) = 0,$$ (8)

where constants are calculated at the star center. Moreover, the $\phi$ component of the Lorentz force, which must also vanish, implies

$$B_\phi = \alpha^{-1} I(A_\phi).$$ (9)

Introducing $\sigma := \alpha^2 \psi^4 r^2 \sin^2 \theta$ and $\tilde{A}_\phi := A_\phi/(r \sin \theta)$ and the new operator

$$\tilde{\Delta}_3 := \Delta - \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial r} + 2 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \tan \theta} \frac{\partial}{\partial \theta} - \frac{1}{r^2 \sin^2 \theta},$$ (10)

for which $\tilde{\Delta}_3 \tilde{A}_\phi = \Delta_3 A_\phi/(r \sin \theta)$ (it coincides with the $\phi$ component of the vector laplacian in spherical coordinates), after some calculations we retrieve the Grad-Shafranov equation for the magnetic flux function $A_\phi$:

$$\tilde{\Delta}_3 \tilde{A}_\phi + \frac{\partial A_\phi \partial \ln (\alpha \psi^{-2})}{r \sin \theta} + \psi^3 r \sin \theta \left( \rho h \frac{dM}{dA_\phi} + I \frac{dI}{dA_\phi} \right) = 0.$$ (11)
Figure 1. Left: variation of the circularization radius \( KY \) as a function of the maximum internal magnetic field, for a NS model with baryonic mass \( 1.68 M_\odot \). Lines represent the results by FR and KY, points are our results with the XNS code. Right: magnetic field in units \( 10^{18} G \) in the left part and density in units \( 10^{14} g \ cm^{-3} \) in the right part. The outer line is the stellar surface.

4. Numerical Scheme

For the non-linear Poisson-like equations Eq. 4-5 we employ a numerical algorithm (XECHO/XNS) fully presented in Bucciantini & Del Zanna (2011), to which the reader is referred for a complete description [see also Del Zanna et al. (2007); Bucciantini & Del Zanna (2013) for the treatment of the fluid MHD part]. Axisymmetric solutions are searched in terms of a series of spherical harmonics \( Y_l(\theta) \):

\[
q(r, \theta) := \sum_{l=0}^{\infty} [A_l(r)Y_l(\theta)].
\]

The Laplacian can then be reduced to a series of radial 2nd order boundary value ODEs for the coefficients \( A_l(r) \) of each harmonic, which are then solved using tridiagonal matrix inversion. This procedure is repeated until convergence, using in the source term the value of the solution computed at the previous iteration.

The Grad-Shafranov equation Eq. 11 can be reduced to the solution of a non-linear vector Poisson equation, which is formally equivalent to the equation for the \( \phi \) component of the shift-vector in the CFC approximation, for rotating systems. Again we use the same algorithm, with a combination of vector spherical harmonics decomposition for the angular part, and matrix inversion for the radial part (Bucciantini & Del Zanna 2011).

In all of our models we have used 20 spherical harmonics for the elliptic solvers and a grid in spherical coordinates in the domain \( r = [0, 30] \text{km} \), \( \theta = [0, \pi] \), with 250 points in the radial direction and 100 points in the angular one. We have verified that with these choices our results are fully converged.

5. Results

By choosing an appropriate form for the functions \( I(A_\phi), M(A_\phi) \) it is possible to obtain either purely toroidal, purely poloidal, or mixed field configurations. It is also possible to change the structure of these configurations, or equivalently change the related
current distributions. We have applied our XNS code and build several equilibrium sequences for various magnetic configurations.

An example of a purely toroidal configuration is shown in Fig.1, where the prolate shape is evident. In Fig.1 we also show a sequence of equilibrium models, with purely toroidal field, characterized by the same baryonic mass. Our results are compared with what was previously found in literature (KY,FR). We confirm the more recent results by FR against the previous one by KY. Note that FR and KY solve the same set of equations (in the exact regime), and they both claim convergence of the results, while we solve in the much simpler CFC. Our findings on one hand are indicative that using a set of equations that is guarantee to be numerically stable might be important to assure the correctness of the results, on the other confirm that the errors introduced by the CFC approximation are negligible.

For brevity we will here illustrate in detail only a TT model. In Fig.2 the magnetic field and density distribution of the TT configuration are shown. We found that usually the toroidal field is smaller than the poloidal one, but even when they are equal, the deformation is oblate, and only marginally different from cases of purely poloidal field. This is because the deformation is dominated by the central region, most of the energy is in the poloidal component, and fields confined at outer radii have marginal effects.

To summarize:

- the characteristic deformation induced by a purely toroidal field, fully confined below the stellar surface, is prolate: the magnetic field acts concentrating the internal layers of the star around its symmetry axis, causing, on the other hand, an expansion of the outer layers;
- given the same strength, magnetic fields concentrated in the outer part of the star, lead to smaller deformations, with respect to magnetic fields concentrated in the internal regions;
- a purely poloidal field, that in our case extends also outside the star, leads to oblate equilibrium configurations: the magnetic stresses act preferentially in the central regions, where the field peaks, leading to a flatter density profile perpendicular to the axis itself. We can also obtain doughnut-like configurations where the density maximum is not at the center;
- the presence of additional currents located in the outer layers of the stars, leads only to marginal changes in its structure, and on the shape of the magnetic field lines outside the stellar surface;
- for the same maximum magnetic field inside the star, purely poloidal configuration, are characterized by smaller deformations, than purely toroidal ones;
- we have computed Twisted-Torus configurations in the non-perturbative regime. The toroidal component can reach a strength comparable with the poloidal one but is energetically subdominant. The deformation are almost completely due to the poloidal field, acting on the interior;
- for a fixed central density, a higher magnetic field gives a higher eccentricity, a higher radius and a higher gravitational mass;
- the more compact configurations, having a higher central density, can support stronger magnetic fields, and show much smaller deformations.
Figure 2. Mixed TT configurations. Right panel: density distribution. Left panel: strength of the toroidal (left) and poloidal (right) magnetic field components, superimposed to magnetic field surfaces. The outer line if the stellar surface.

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