Abstract

Seven new solutions to the interior static and spherically symmetric Einstein’s field equations (EFE) are found and investigated. These new solutions are a generalisation of the quadratic density fall-off profile of the Tolman VII solution. The generalisation involves the addition of anisotropic pressures and electric charge to the density profile. Of these new solutions three are found to obey all the necessary conditions of physical acceptability, including linear stability under radial perturbations, and causality of the speed of pressure waves inside the object. Additionally an equation of state can be found for all the physically viable solutions. The generalised pulsation equation for interior solutions to the EFE that include both electric charge and pressure anisotropy is derived and used to determine the stability of the solutions. However the pulsation equation found is general and can be used for all new solutions that contain these ingredients.
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Chapter 1

Introduction

We motivate the work done in this thesis starting from very general notions. We attempt to find physically relevant exact interior solutions to Einstein’s Field equations (EFE) in their static and spherically symmetric case. The energy-momentum is assumed to come from a charged fluid having anisotropic pressures. New solutions that can be used to model compact objects (Neutron stars, Strange stars, etc.) are found. The equation of state is obtained as a result of the solution, and under additional assumptions that fix the parameter values to ones close to nuclear densities for example, might be useful for nuclear physics considerations.

1.1 General Relativity

First put forward in 1915 by Albert Einstein, General Relativity (GR) introduces the idea that gravitation is not a force, but rather, it is a consequence of curvature in a four dimensional space-time [40]. Space-time itself was introduced earlier (1905) by Einstein as part of the framework of special relativity (SR) to help elucidate the then pressing problem of the constancy of the speed of light with respect to aether, viz. the result of the Michelson-Morley experiment. GR, according to Wheeler [86], can be summarised by:

Matter tells space how to curve; Space tells matter how to move.

GR is a physical theory that has produced a number of predictions over the years. The Schwarzschild solution [115] to the full GR equations predicted the existence black-holes, which have now been observed indirectly. The currently accepted cosmological model [57, 122] the Friedmann–Lemaître–Robertson–Walker metric is a solution to the fundamental
equation of GR –the Einstein field equations (EFE)– and has been tested by combining observations from the WMAP [68] and Planck [104] satellites.

There have also been several other direct observational test of GR. The very first one that convinced people to start taking the theory seriously was the prediction by the theory of the perihelion shift in the orbit of Mercury. Gravitational waves [41] which were predicted barely a year after the main theory was put forward were detected a few months ago [80], confirming GR once again. Similarly, the gravitational lensing effect predicting that massive objects bend light rays and measured famously by Eddington [39] in 1919 also confirmed GR and brought it its fame initially. All these attest to the fact that classical GR is a well established physical theory that predicts measurable quantities.

The areas in which GR still has predictions that have not been tested completely are the strong regimes where potentials that scale like $M/R$ are very large. Then the non-linearities of the EFE become important and methods that can take these non-linearities into account become more valuable. The detection of gravitational waves tested some of these strong regime predictions through numerical methods modelling the space-time around the black-holes producing the waves [80]. Another method of getting predictions for the strong regime is through perturbation methods that go beyond the linearised EFE. These methods have been successful in calculating Love numbers and subsequent gravitational waves from systems of compact astrophysical objects [29, 30]. However these are not the methods this thesis investigates. Instead we approach the problem of strong gravitational fields by constructing exact solution to the EFE with a particular application toward understanding the structure of compact objects.
1.2 Exact Solutions

The reason for looking at exact solutions, aside from the fact that they are mathematically interesting, is that they also provide a baseline against which both of the methods mentioned above can be compared. Exact solutions like the Schwarzschild Interior solution [115] are still being used, despite being non-physical for this very reason: at best these exact solutions provide a limit on certain physical parameters, and at worse they guide the understanding into the behaviour of the gravitational field. For this reason the construction of exact solutions can be more rewarding than just the mathematical exercise. Some of the solutions found can be used for physical modelling and thereafter to predict measurable quantities about physical systems. Upon measurement, the truthfulness of the model can then be ascertained, or denied.

The history of the hunt of exact solutions to the interior EFE is a long and interesting one. We briefly outline this history in Section 2.5 and refer the reader to [120] for a more complete list and history. Of interest for this thesis is that of the roughly 130 static spherically symmetric perfect fluid solutions to the EFE that were known in 1998, only 9 were deemed to be physically viable [31, 43]. The criteria for physical viability are simple constraints from within the framework of GR and classical physics, and do not have any quantum-mechanical component to them. Once an interior solution has been deemed to behave physically, it can be used to predict measurables/observables of the system. For one of the known solutions deemed to be physically relevant in [31], we extract both masses and radii of the compact object modelled in Chapter 3. The solution considered is the Tolman VII solution [125], and we find that it indeed predicts masses and radii that are in line with current observations [111]. This not only demonstrates that the Tolman VII solution is a viable model for physical systems, it also shows that GR’s strong field solutions are accurate to the limit of our current observations.
capabilities in these systems. Another prediction that this model produces is an equation of state (EOS) for the matter inside the star. This EOS is obtained without any quantum mechanical assumptions, and while there is no direct way to test the accuracy of this EOS, that the observed mass and radii of neutron can be obtained without detailed microphysics, suggests that the sensitivity of the bulk properties of compact stars to the differences in many nuclear models is very low.

The above is the primary motivation for finding **new physically motivated solutions** to the EFE that can be used to model compact objects. Once these solutions have been found, all the predictions stemming from their use as model for these systems can be investigated. Furthermore more elaborate numerical and perturbative methods can be used and compared with the new exact solutions found. For example, the calculation of Love numbers in binary systems and the generation of gravitational waves in these same systems could greatly benefit from exact interior solutions.

### 1.3 This work

The solutions presented in [31] are all uncharged with a perfect fluid matter as the source. In Chapter 4, we generalize the source to include anisotropic pressures in the fluid in Section 4.1 and then additionally include electric charge in the source in Section 4.2. The physical reasoning behind these additions is provided in Chapter 2, with a brief historical overview for these types of solutions. Chapter 4 is the mathematical component of this work, and does not investigate the physical validity of the solutions found. It is only concerned with the mathematical consistency of the solutions to the EFE. During this process, we find seven new solutions, and generate a few solutions that had already been found before, when certain of our parameters are set to zero. A summary of the solution landscape is given in Figure 2.1.

A question that is often asked is whether static solutions to the EFE are stable. The answer is
unknown, but Chandrasekhar in a seminal paper [23] attempted to find whether the general perfect fluid solution was stable under radial perturbations, and came up with a “pulsation equation.” The frequencies of the normal modes of this equation then determines whether the solutions are stable under radial linear perturbations: real frequencies corresponding to “breathing modes” and imaginary ones to unstable expanding or contracting ones. When we began considering this for our new solutions, we found that the pulsation equation for our case—a charged fluid with anisotropic pressure—was not available. This was the basis of Chapter 5, where we investigate the stability of the solutions we found in Chapter 4 by first deriving a general pulsation equation. We then show that for certain parameter choices, our solutions are stable, and for which choices the solutions are not.

Chapter 6 instead analyses the new solutions found from the perspective of physical acceptability. We list the criteria for physical acceptability in Chapter 6 and use them in Section 6.1.2 onwards on all the new solutions and conclude that of the seven solutions we found, three have promising characteristics that make them physically interesting. We discuss these solutions extensively. While we are unable to provide exact cut off values for some parameters that distinguish between physical and unphysical solutions we can provide some important relations in terms of general inequalities.

As a summary, the new aspects of this work are

1. The construction of seven new solutions to the EFE with various combinations of the anisotropic pressures and charge.

2. The derivation of a stability equation for radial perturbations that can be applied to all new solutions with charge and/or anisotropic pressures. The pulsation equations for the simpler cases of zero charge, or zero pressure anisotropy are recovered when those parameters are set to zero, attesting to the accuracy of our derivation. We use the derived equation to prove that the solutions we are interested in are indeed stable.
3. The analysis of the seven new solutions, and the conclusion that three might be physically viable.

The new physical solutions we found could potentially be used to model astrophysical objects, deduce EOS for these compact stars, calculate Love numbers in binary systems, and even infer gravitational wave spectra of radiating neutron stars from the EOS. These are all avenues for further work.
Chapter 2

Preliminaries

We look at the ingredients that make up the EFE. We then look at some known interior solutions to the EFE, and how our work fits in the overall picture of finding physical solutions to model stars.

2.1 Definition of terms

This chapter uses the definitions and theorems stated in the Appendix A. Basing all work to follow on Einstein’s theory of gravity, Einstein’s Field equations (EFE) can be written as

\[ G_{ab} = \kappa T_{ab}. \] (2.1)

The next sections introduces all the elements needed to interpret and use equation (2.1). The assumptions and notations that we will use are the following:

1. We use geometrical units throughout this thesis, unless otherwise stated. Geometrical units are used to simplify most of the equations which would otherwise have the physical constants \( G \), Newton’s gravitational constant, and \( c \), the speed of light appear in various factors throughout. The use of geometrical units imply that \( G = c = 1 \), so that these constants no longer appear in the equations. In chapters 4 and 6 we relax this assumption to calculate values in SI units by putting back the value of the constants to \( G = 6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2} \), and \( c = 3.00 \times 10^8 \text{ m} \cdot \text{s}^{-1} \). Table 2.1 provides conversion factors to and from geometrical units to SI units in the main text of this chapter.

2. The matter coupling constant (See Appendix A) \( \kappa = 8\pi \).
3. The metric signature (for a definition of signature see Appendix A) we use throughout is $(+,-,-,-)$.

4. Latin indices are used for space-time tensor indices and take the values 0,1,2, and 3. Greek indices span 1,2,3 only instead and are used for spatial tensor indices.

| Physical quantity | SI unit | Geometrical unit | Conversion factor $(\times)$ |
|-------------------|---------|------------------|-------------------------------|
| Length            | m       | m                | $\frac{1}{c}$               |
|                   |         | s                | $c^{-1}$                     |
|                   |         | kg               | $c^2G^{-1}$                  |
| Time              | s       | s                | $1$                          |
|                   |         | m                | $c$                          |
|                   |         | kg               | $c^3G^{-1}$                  |
| Mass              | kg      | kg               | $1$                          |
|                   |         | m                | $Gc^{-2}$                    |
|                   |         | s                | $Gc^{-3}$                    |
| Energy            | kg·m$^2$·s$^{-2}$ | kg         | $c^{-2}$                     |
|                   |         | m                | $Gc^{-4}$                    |
|                   |         | s                | $Gc^{-5}$                    |
| Density           | kg·m$^{-3}$ | m$^{-2}$   | $Gc^{-2}$                    |
| Pressure          | kg·m$^{-1}$·s$^{-2}$ | m$^{-2}$ | $Gc^{-4}$                    |
| Speed             | m·s$^{-1}$ | unit-less | $c^{-1}$                     |
| Mass/Radius       | kg·m$^{-1}$ | unit-less       | $Gc^{-2}$                  |
| Electric charge   | A·s     | m                | $\frac{1}{c^2} \sqrt{\frac{G}{4\pi\varepsilon_0}}$ |

*Table 2.1: Conversion from SI units to Geometrical units for relevant physical quantities.*

2.2 Geometry

In general relativity, space-time is modelled by a (3+1)–dimensional **Lorentzian manifold**. The space-time is additionally endowed with a symmetric metric $g_{ab}$ that is used to measure lengths and angles. In GR the manifold also has a symmetric metric connection (and therefore no torsion). The definition of a manifold requires some additional ideas from set theory (notions of sets, subsets, elementary set operations, the real line, $\mathbb{R}$, etc.) which we shall assume and not state explicitly, and other definitions (taken mostly from references [25, 26].
which we give in Appendix A. Next we introduce the geometrical quantities that are needed in the EFE.

2.2.1 The metric

The metric is the dynamical quantity that specifies the general relativity component of our models for stars. We will be considering static models endowed with spherical symmetry. As a result the sixteen components of the general space-time metric will be reduced to four components, two of which will have arbitrary coefficients, that depend solely on one spatial coordinate. As a result, Einstein’s equations of general relativity are greatly simplified into a set of ordinary differential equations (ODEs), whose solutions and interpretations will be the main thrust of this thesis. Depending on our assumptions about matter, and electromagnetic fields we shall have either three coupled ODEs (isotropic matter without electric charge), or four ODEs (anisotropic matter without electric charge), or five coupled ODEs (anisotropic matter with electric charge). Each case will be treated separately and solved to yield viable physical models that can be used to model compact stars.

As mentioned in appendix A naively the metric should have sixteen\((=4 \times 4)\) components. Symmetry of the metric, that is \(g_{ab} = g_{ba}\) reduces this to 10 independent components. In this section we show how additional constraints will reduce these ten components to two only. Doing so will require the application of symmetry arguments most easily done with Lie derivatives, the definitions of which are given in appendix A.

2.2.2 Symmetry

The Lie derivative of the metric along vector field \(\xi\) is

\[
\mathcal{L}_\xi g_{ab} = \xi^c g_{ab,c} + \xi^c_{,b} g_{ac} + \xi^c_{,a} g_{cb},
\]

so that Killing’s equation becomes the above being identically equal to zero. For staticity, we want our first killing vector \(T\) to be \(\partial_t\), or in an adapted frame with usual spherical
type coordinates \((x_0, x_1, x_2, x_3) \equiv (t, r, \theta, \varphi)\): \(T = (1, 0, 0, 0)\). As a result of the constant components of this vector, we can immediately write \(T^a \, _b = 0\) as a result of which the Lie derivative above reduces to

\[
\mathcal{L}_T g_{ab} = T^c g_{ab, c} = g_{ab, 0} = 0.
\]

From this we deduce that our metric components can have no dependencies on the \(t\) coordinate at all, so that \(g_{ab}(t, r, \theta, \varphi) \equiv g_{ab}(r, \theta, \varphi)\).

For spherical symmetry, we require three additional Killing vectors which generate \(SO(3)\) and whose Lie brackets are cyclic with each other as in Appendix A. As given in the example involving the unit 2-sphere there, the following three vectors in the spherical adapted frames satisfy the conditions for spherical symmetry:

\[
P = -\sin \varphi \frac{\partial}{\partial \theta} - \cot \theta \cos \varphi \frac{\partial}{\partial \varphi} := L_1
\]

\[
Q = \cos \varphi \frac{\partial}{\partial \theta} - \cot \theta \sin \varphi \frac{\partial}{\partial \varphi} := L_2,
\]

\[
R = \frac{\partial}{\partial \varphi} := L_3
\]

As in the definition of spherical symmetry, given completely in Appendix A, the above vectors obey the cyclic Lie structure \([L_i, L_j] = e^{ijk} L_k\).

To impose symmetry on the metric, we can now simply use Killing’s equation on any linear combination of the \(L\) vectors given above. For \(\xi = xL_1 + yL_2 + zL_3\), we require that

\[
\mathcal{L}_\xi g_{ab} = \xi^c g_{ab, c} + \xi^c \, _b g_{ac} + \xi^c \, _a g_{cb} = 0,
\]

and the long calculation [116] that results end in the metric being split into two blocks, the first block containing exclusively the 2-sphere coordinates \(\theta\) and \(\varphi\), in the form

\[
ds_{2\text{-sphere}}^2 = r^2 \, d\Omega = f^2(r, t)(d\theta^2 + \sin^2 \theta \, d\varphi^2),
\]
where \( f(r, t) \) is an arbitrary function; and the other part containing the remaining two coordinates \( t \) and \( r \) through

\[
ds^2 = g_{AB} \, dx^A \, dx^B,
\]

with \( g_{AB} \) independent of the angles \( \theta \) and \( \varphi \). Since we also have the Killing vector \( T \) to contend with, the off diagonal terms of \( g_{AB} \) have to be zero. Similarly, \( \partial_t f(r, t) = 0 \), so that without loss of generality and through imposition of spherical symmetry, the metric’s 10 components are reduced to only four diagonal ones:

\[
ds^2 = e^{\nu(r)} \, dt^2 - e^{\lambda(r)} \, dr^2 - r^2 \, d\Omega^2. \tag{2.3}
\]

With the complete set of of Killing vectors, \( T, P, Q \) and \( R \) that we imposed on the metric, we can also impose the same symmetry requirements on the other tensors of the theory. This results in “collineation theory” which we consider next when we apply this idea on the matter and electromagnetic fields.

This section introduced the metric tensor we will be using throughout this work and reduced its ten components into only four. From methods presented in Appendix \( \text{[A]} \) we now have the complete geometrical description of the system we are considering, since all the geometrical tensors can now be computed in terms of the metric.

2.2.3 The connection coefficients

As given in the Appendix \( \text{[A]} \) in general relativity the metric is compatible with the Levi–Civita connection, which means that they can all be computed once the metric is specified. From the form of the metric \( 2.3 \), a lengthy but straight-forward calculation yields the following non-zero connection coefficients,

\[
\Gamma^r_{tt} = \frac{1}{2} e^{\nu - \lambda} \frac{d\nu}{dr}, \quad \Gamma^t_{tr} = \frac{1}{2} \frac{d\nu}{dr}, \quad \Gamma^r_{rr} = \frac{1}{2} \frac{d\lambda}{dr}, \quad \Gamma^r_{\theta\theta} = - r e^{-\lambda}. \tag{2.4a}
\]

\[
\Gamma^\theta_{\theta\varphi} = \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \quad \Gamma^r_{\varphi\varphi} = - r e^{-\lambda} \sin^2 \theta, \quad \Gamma^\theta_{\varphi\varphi} = - \sin \theta \cos \theta, \quad \Gamma^\varphi_{\theta\varphi} = \cot \theta. \tag{2.4b}
\]
In the above instead of using numbers for the indices, we have used the names of the four coordinates we used in metric (2.3) following the following convention: $0 \equiv t, 1 \equiv r, 2 \equiv \theta,$ and $3 \equiv \varphi.$

2.2.4 The Einstein tensor

The culmination of the geometrical calculations yields the Einstein Tensor. The whole procedure usually involves the computation of the Riemann tensor, contracting it into the Ricci tensor and scalars, and then the computation of the Einstein tensor. We will not show all these steps, and instead refer the reader to either the numerous tomes that contain all this information [34, 65, 126], or to analytical packages like 'GRTensor' [102] on Maple\textsuperscript{TM} or 'ctensor' on Maxima\textsuperscript{TM} [82], which allow such calculations. The Einstein tensor $G_{ab}$ for the metric (2.3) is given by

\begin{align}
G_{tt} &= \frac{e^{-\lambda}}{r^2} \left( -1 + e^\lambda + r \frac{d\lambda}{dr} \right), \\
G_{rr} &= \frac{e^{-\lambda}}{r^2} \left( -1 - e^\lambda + r \frac{d\nu}{dr} \right), \\
G_{\theta\theta} &= \frac{e^{-\lambda}}{4r} \left[ 2 \frac{d\nu}{dr} - 2 \frac{d\lambda}{dr} - r \left( \frac{d\nu}{dr} \right) \left( \frac{d\lambda}{dr} \right) + r \left( \frac{d\nu}{dr} \right)^2 + 2r \frac{d^2\nu}{dr^2} \right], \\
G_{\varphi\varphi} &= \frac{e^{-\lambda}}{4r} \left[ 2 \frac{d\nu}{dr} - 2 \frac{d\lambda}{dr} - r \left( \frac{d\nu}{dr} \right) \left( \frac{d\lambda}{dr} \right) + r \left( \frac{d\nu}{dr} \right)^2 + 2r \frac{d^2\nu}{dr^2} \right].
\end{align}

This concludes the geometrical considerations of this Chapter. The next section looks instead at the other side of the EFE, concerned with the source terms of gravitation. Thus we tackle what the models are made up of: matter and the electromagnetic field.

2.3 Source terms

The other side of the Einstein equations involves sources of curvature. These sources can be matter, or fields, and we look at each of these in the next Subsections.
2.3.1 Matter

In GR matter is expressed through the stress-energy tensor. We give a lengthy and complete derivation from first principles starting from a Newtonian picture of the form of the stress-energy tensor we use in this thesis in Appendix A. The salient points of this derivation is that in our symmetry case from the previous Section 2.2.2 the stress-energy tensor for the matter we are considering can be reduced into the form

\[
T^{i}_{j} = \begin{pmatrix}
\rho & 0 & 0 & 0 \\
0 & -p_r & 0 & 0 \\
0 & 0 & -p_\perp & 0 \\
0 & 0 & 0 & -p_\perp
\end{pmatrix},
\]

or in the more compact form \cite{78, 88}

\[
T^{ab} = (\rho + p_\perp)u^a u^b - p_\perp g^{ab} + (p_r - p_\perp)n^a n^b,
\]

where \(u^a\) is the four-velocity normalized so that \(u_a u^a = 1\), and \(n^a\) is a space-like unit vector in the radial direction, with \(n_a n^a = -1\), these being chosen so that \(n_a u^a = 0\). The quantities \(\rho, p_r,\) and \(p_\perp\) are the energy density, radial pressure, and angular pressure respectively, and the latter only exists in the case where we consider anisotropic pressures. When considering the case with isotropy only, we have \(p_\perp = p_r\), with subsequent simplifications of the above expressions.

The reasoning behind anisotropic pressures will be given in Section 2.5.2 and has to do with the use of multiple species to model the matter component of the star. An interesting consequence of spherical symmetry not usually considered is that the matter collineation induced by the Killing fields impose that the two pressures \(p_\perp\) and \(p_r\) be equal at the centre of the coordinate system, where \(r = 0\). This criterion needs not be satisfied if the model is not spherically symmetric, but in this work all our models are strictly spherically symmetric, and we make sure that \(p_r(r = 0) = p_\perp(r = 0)\).
Next we look at the electromagnetic field, and how we include it in our models.

2.3.2 Electromagnetic fields

General relativity like special relativity was brought about by Einstein thinking about the constancy of the speed of light in different frames. Light even during his time was modelled as an electromagnetic wave, underpinned by Maxwell’s theory. Since the origin of general relativity is so closely related to Maxwell’s equations, it should come as no surprise that these two theories are completely compatible, and the inclusion of electric charge in Einstein’s theory is not complicated. In the static case we are considering we define $F_{ab}$ as the electromagnetic field strength tensor (Faraday tensor) introduced in Appendix A through

$$F_{ab} = \partial_{[a}A_{b]} = \partial_a A_b - \partial_b A_a,$$

where $A_a$ is the electromagnetic 4-potential. In the static case we shall be considering in this thesis, we impose a gauge where $A_a = (A_0, A_\mu)$ with $A_\mu = 0$. This is because in the static case magnetic fields do not exist, and hence the magnetic vector potential vanishes. The $A_0$ component encodes the electric field, and in the case of the charged sphere for example, we get $A_0 = -q/r$, the same as the classical result.

The energy–momentum associated with the electromagnetic field is then given through

$$T^i_j = \frac{1}{4\kappa} \left( F^i c F^c_j - \frac{1}{4} \eta^i_j F^{cd} F_{cd} \right).$$

For the assumed electric field of $A = (-q/r, 0, 0, 0)$, this yields

$$T^i_j = \begin{pmatrix}
\frac{q^2}{\kappa r^4} & 0 & 0 & 0 \\
0 & \frac{q^2}{\kappa r^4} & 0 & 0 \\
0 & 0 & -\frac{q^2}{\kappa r^4} & 0 \\
0 & 0 & 0 & -\frac{q^2}{\kappa r^4}
\end{pmatrix},$$

where $\kappa = 8\pi$ is the gravitational coupling constant.

Minimal coupling of matter and fields as we mentioned in Appendix A then results in the full
matter–electromagnetic energy momentum tensor of the form

\[ T^i{}_j = \begin{pmatrix} \rho + \frac{q^2}{\kappa r^4} & 0 & 0 & 0 \\ 0 & -p_r + \frac{q^2}{\kappa r^4} & 0 & 0 \\ 0 & 0 & -p_\perp - \frac{q^2}{\kappa r^4} & 0 \\ 0 & 0 & 0 & -p_\perp - \frac{q^2}{\kappa r^4} \end{pmatrix}. \tag{2.6} \]

and this tensor encodes all the source terms that “generate” gravitation. The interesting conclusion that can be drawn from this discussion is that there is a simple way to include electromagnetic fields and matter in the same framework, and once specified, the solution to the EFE with the source term will yield consistent solutions incorporating both matter and charge.

### 2.4 Exterior solutions

Einstein’s field equations are expressed in terms of the Einstein’s tensor \( G_{ab} \) we just defined. In the static case this tensor is similar to the Laplacian of the scalar potential \( \nabla^2 \phi \) of electrodynamics in that it also consists of at most second derivatives of the gravitational “potential” \( g_{ab} \), and that it can be solved for both the vacuum case (Laplace’s equation), and the case where source terms exist (Poisson’s equation). With this general picture in mind, we discuss the two vacuum solutions we will pursue as the external solutions we need to match in the different cases we will find solutions for. These solutions will assume \( T^{ab} = 0 \), or \( T^{ab} = T^{ab}_{\text{electromagnetic}} \) in Subsections 2.4.1 and 2.4.2 respectively.

#### 2.4.1 The Schwarzschild exterior solution

This solution was found by Schwarzschild in 1916. His derivation of the equation is complicated because he used Cartesian coordinates instead of the more natural spherical coordinates for this spherically symmetric solution. Here we will just state the main lines of another derivation popularized by Droste [35].
The assumptions that lead to the Schwarzschild exterior solution are

1. A spherically symmetric and static space-time. As we showed in the Section 2.2.2, this implies that the metric is reduced to equation (2.3).

2. This is a vacuum solution, so there is no source terms, therefore Einstein’s tensor is annulled, the set of equation to be solved thus become

\[ G_{ab} = 0. \]

As a result of these assumptions, the complete set of linear ODEs to be solved become the set (2.5) equated to zero. Algebraic manipulation of this set then results into the simple linear equation for \( e^{-\lambda} \):

\[-e^{-\lambda} \left(1 - r \frac{d\lambda}{dr}\right) + 1 = 0 \implies \frac{d}{dr} (e^{-\lambda}) + \frac{e^{-\lambda}}{r} = \frac{1}{r}.\]

Solving the latter differential equation then results in

\[ e^{-\lambda} = 1 + \frac{A}{r}, \quad \text{and therefore} \quad e^\nu = B \left(1 + \frac{A}{r}\right).\]

with \( A \) and \( B \) arbitrary integration constants. A simple coordinate rescaling can be used to set the constant \( B = 1 \). Boundary conditions requiring that this solution be compatible with Newtonian gravity then forces \( A = -2M \), where \( M \) is the mass of the object perceived at some distance. Then the interpretation of this solution is the external gravitational field of some mass \( M \) at the symmetry centre of the solution. The complete Schwarzschild exterior metric hence becomes

\[ ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta \, d\varphi^2\right). \quad (2.7)\]

This exterior is usually used to model non-rotating and uncharged black-holes since the interior can be left unspecified, and is modelled through the total mass only. This is one of the consequences of Birkhoff’s theorem.
The Schwarzschild exterior solution will be the solution we want to match the interior solutions we find in this thesis when we have uncharged matter. The physical reason behind this: to first order the metric (2.7) gives classical Newtonian gravity, and to second order predicts the perihelion shift of mercury [41]. Any object, from a black-hole to the sun, and going through the “middle” case of a compact object like a neutron star exerts gravity in a similar way. Therefore the compact objects we model should also behave similarly.

Mathematically the uniqueness of this solution is ensured by Birkhoff’s theorem [10, 25] which we now state without proof.

**Theorem 1.** *Any spherically symmetric solution of the vacuum field equation \( G_{ab} = 0 \), must be static and asymptotically flat. This means that the exterior solution must be given by the Schwarzschild metric.*

This theorem is physically unexpected since in Newtonian theory, staticity is unrelated to spherical symmetry. Thus the Schwarzschild solution is the only possible solution for a spherically symmetric, asymptotically flat vacuum space-time.

2.4.2 The Reissner–Nordström solution

When the EFE are combined with the Maxwell’s equations which we just investigated in 2.3.2 the vacuum metric solution (2.7) can be generalised to an electro-vacuum solution: The Reissner–Nordström solution which includes the electrical charge. This solution was found by Reissner and Nordström after whom it is named [92, 112].

In this solution the electromagnetic 4-potential is given by

\[
A_\mu = 0, \quad A_0 = -\frac{Q}{r},
\]

since the field is static (no magnetic potential), and assumed to come from a sphere yielding the \(1/r\) electric potential. This is in line with the spherical symmetry of the solution.
A similar solution method to the one we showed for the Schwarzschild solution can be used to find the Reissner–Nordström solution. However, in this case we will have a source term coming from the charge that we assume the central body to process. As a result, the EFE cannot be equated to zero, and have to be equated to the energy-momentum tensor components \( T^{\alpha\beta} = \text{diag}(-Q^2/r^4, -Q^2/r^4, Q^2/r^4, Q^2/r^4) \). The additional terms do not change the solution procedure drastically and the final Reissner–Nordström metric then becomes

\[
\frac{2M}{r} - \frac{Q^2}{2r^2} < 1,
\]

This metric is smooth and Lorentzian with \( t \) a time variable as long as

\[
ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{2r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{2r^2}\right)^{-1} \left(1 - \frac{2M}{r} + \frac{Q^2}{2r^2}\right) \left(1 - \frac{2M}{r} + \frac{Q^2}{2r^2}\right) dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right). \quad (2.8)
\]

This metric is smooth and Lorentzian with \( t \) a time variable as long as

\[
\frac{2M}{r} - \frac{Q^2}{2r^2} < 1,
\]

to keep the metric from changing signature. For large values of \( r \) the term \( Q^2/r^2 \) decreases more rapidly than \( 2M/r \), and seems to suggest that charge effects are more difficult to observe directly in astrophysical objects. However as we shall see in our interior solutions, electric charge also affects \( M \) in non-trivial ways.

This is the metric to which we shall match when we are considering charged interior solution to the EFE. The matching of the mass and charge at the boundary then ensures the consistency of the metric throughout the space-time.

### 2.5 Interior Solutions

Interior solutions refer to equations which solve the EFE while connecting to some exterior (usually cosmological) solution. Together the two provide a complete picture of the local region inside and outside some matter that is gravitating. When we look for solutions, this “interior” quality is modelled by two different factors

1. The field equations will contain matter, usually in the form of fluids, and the exterior solution will either be more fluid, or empty space.
2. The junction condition must match the interior structure to some exterior ones without any singularities occurring at the junction. This junction is usually modelled as a hypersurface (called the matching hypersurface), and on this hypersurface, the interior and exterior metric must match, and be at least of class $C^1$.

These consistency conditions are important and without their satisfaction, the models built cannot be deemed to be mathematically consistent.

We now present a brief history of the process of finding interior solutions, and how our work fits into the overall solution landscape already present in the field.

2.5.1 Perfect fluid matter

The first interior solution was found in 1916 by Schwarzschild and this solution was matched to an exterior solution found by the same person, now called the Schwarzschild exterior solution. This solution modelled the matter content through an incompressible fluid of constant density $\rho$, and for this reason is nowadays deemed unphysical \[31, 43\]. The reasoning behind the unphysicality is that the speed of pressure waves (sound waves) in the Schwarzschild interior solution is infinite, in contradiction with relativity. Following its discovery, a number of studies into its properties were carried out. Of note is the re-expression of this solution’s metric in isotropic coordinates \[132\], and the extension, albeit in a perturbative manner to finite but constant speeds of sounds in \[19\]. The latter reference is also interesting in that it is the first time that an equation of state (EOS) of the form $p = \rho - \rho_b$ is considered directly. This form of EOS is important since now the speed of pressure waves is exactly equal to the speed of light since $v_s = \frac{dp}{d\rho} = 1$, instead of infinite as in the case of Schwarzschild interior.

Later it was shown that the Schwarzschild interior metric is the unique spherically symmetric and static metric that was also conformally flat \[52\] (Admitting the conformal Killing vector in addition to the one we imposed on our metric.) The importance of the Schwarzschild inte-
rior solution cannot be denied: It has been used to model, and limit physical characteristics of compact stars since its discovery, and the well-known Buchdahl limit of $M/R < 4/9$, is a result that depend crucially on its existence and properties [18]. Recently we showed [110] that one of the solution this thesis looks into closely: the Tolman VII solution, is a possible extension of the Schwarzschild interior metric.

A different set of interior solutions to the EFE was found in 1939 by Tolman. All these solutions were matched to the Schwarzschild exterior solution [125] and while all the matter quantities were not computed, some had interesting enough properties that they were either rediscovered [36, 37] or used under different names [83]. Some of this set were well known solutions: The Einstein universe, the Schwarzschild interior we just talked about, and the Schwarzschild–de Sitter solution [111]. Others have been generalized in various ways: an early attempt by [133], followed by various others. An incomplete but comprehensive list is given in [31, 43].

From then on there have been many other sets and classes of interior solutions. Most of these have numerous problems and are not interpretable as physical objects. Usual issues include the divergence and asymptotic behaviour of the matter variables: the density being infinite somewhere in the model being a common occurrence. In the same year, 1939, Oppenheimer and Volkoff derived an equation that has since been named the Tolman–Oppenheimer–Volkoff equation of general relativistic hydrodynamic stability [93].

This equation revolutionized the task of finding EOSs for compact object. This is because for the first time, it allowed a method other than brute mathematical intuition to be used to model these compact objects. If one had some idea of an equation of state, then one could impose that equation of state onto the geometry and get observable values like masses and radii of the objects. This line of approach was and remains popular in many places, particularly with those who model neutron star structure. The hope there is that neutron stars will give us a glimpse into what neutron-rich matter behaves like [74]. However certain recent results
seem to indicate that detailed quantum mechanical descriptions are not very important, since the masses and radii of neutron stars seem to be insensitive to the differences between most of the nuclear EOS that are investigated [134].

2.5.2 Anisotropic matter

All the interior solutions we have mentioned so far are uncharged and admit only one radial pressure. The interest in non-radial pressures began in the 1970’s, when anisotropic pressures with \( p_\perp \neq p_r \) started to be considered [17]. Later physical interpretations of these unequal pressures started to appear [7, 78], and the interpretation that the anisotropic pressure was just an expression for the existence of multiple perfect fluids minimally coupled to each other in the interior gained acceptance. Many solutions [28, 44] containing anisotropic pressures were found and used to model compact objects, and the analysis led to a number of discoveries. Of note is the discovery [12, 17, 106] that by having anisotropic pressures, one was no longer subject to the Buchdahl limit of \( M/R < 4/9 \).

At the same time, the generalized TOV equation which includes the anisotropic pressure was derived, and the realisation that this equation looked very much like the classical Newtonian hydrostatic equilibrium equation [59] was reached. This led to a number of investigations into the structure and stability of Newtonian stars [32, 33] to be done, since the similarity to the relativistic case meant that result for the Newtonian case could be extended to the relativistic case easily. A classic review [60] by Herrera and Santos during the same time brought anisotropic solutions to the forefront, and interest in such solutions particularly for gravitational wave generation, and Love numbers computation became a popular study. In this thesis, the relationship between the two pressures defines a measure of anisotropy. We define \( p_r(r) - p_\perp(r) = \Delta(r) \), generally in the beginning, and then continue on to define \( \Delta = \beta r^2 \), in certain solutions. As a result, by modifying the value of \( \beta \), a constant parameter, we change the anisotropic pressure indirectly. The physical reasoning being the expression
of \( \Delta \) will be explained in the relevant section which investigate that particular case.

2.5.3 Charged matter

From the charged EFE it is clear that if we have non-zero source terms (from an electromagnetic energy-momentum tensor,) the exterior solution to be matched becomes the Reissner-Nordström exterior solution. This affects the interior solution to be found as the electromagnetic repulsion of the charges will contribute to the overall energy, and thus mass of the object. The Reissner-Nordström solution provides a relationship between the mass and the charge for the interior solution to be consistent, and either \( M \) or \( Q \) could be set to zero. However this is not always possible in the interior, as was shown by Bonnor [13], since positing zero \( M \) necessitates a negative mass-energy density in the interior, an unphysical result. Having an interior solution as we will shortly demonstrate allows for obtaining more relationships between the charge and mass, and we shall do so in the course of this thesis. Following the discovery of several charged solutions in the 1950’s by Papapetrou, interest to find even more of these solutions grew, and many more solutions to match the exterior Reissner-Nordström solution have since been found. Bonnor [14] found another regular charged solution in 1965. In this solution the assumption that the mass density be equal to the charge density was utilised. Taking inspiration on this, in one of our solution, we instead assume that the anisotropy measure matches the charge density. This solution showed that is was indeed possible to have a consistent solution in which repelling charges could balance the attracting gravitational field, in general relativity. Many other solutions having singularities were also found during the same time period however. Some tried to remove, and/or transform away these singularities, meeting with meagre success. Soon after however, a host of solutions consistently matching to the Reissner-Nordström exterior were found [27, 45, 70, 118]. Some of these solutions also tried imposing the strong energy conditions \( \rho + 3p > 0 \), to various conclusions. Some were not able to
impose it, as some solutions did not allow for this condition to hold for any choice of $\rho$. Of note is the solution by Kyle and Martin which one of our solutions reduces to when the anisotropic factor $\beta = 0$. Ivanov summarizes many of these results and how to obtain them from first principles in an extensive review in Ref. [65], and we use the latter for many of our derivations and reasoning.

2.6 The solution landscape

Having introduced all the main ingredients that are used in this thesis, we now provide a logical flow for how we approach the finding of solutions that might be physical. In Figure 2.1 we look at the decision flow in terms of the parameters $\beta$ encoding the anisotropy, and $k$ encoding the electric charge to generate the new solutions we found. This diagram summarises the solution landscape around our solutions as well, showing for example how through setting certain parameters to zero, we can also generate already known solutions.
Figure 2.1: The solution landscape explored in this thesis. Lightly shaded boxes are the new solutions described in this work, and darker ones are the older known solutions. In the online coloured version, red-bordered boxes are solutions with isotropic pressure, blue-bordered ones are uncharged solutions with anisotropic pressures only and green-bordered ones are charged with anisotropic pressures.
2.7 Sturm-Liouville systems

A detour into the mathematical theory of Sturm–Liouville systems is required for our section on the stability of our models. Indeed, the overall stability of interior solutions, while clear on general physical intuitions is a lot more complicated to prove mathematically. At fault is the non-linear structure of the EFE, which causes perturbations in any matter fields to change every single equation, up to the metric functions, through a complicated propagation process. In this thesis we assess the linear stability of our models, and do not proceed further. Studies have shown that a range of different non-linear instabilities are also possible, but their analysis would require a lot more work.

The TOV equation and the EFE can be reduced in certain cases to a Sturm–Liouville form. The general form the linear stability equation for the EFE reduces to:

\[
\left\{ \frac{d}{dr} \left[ P(r) \frac{df}{dr} \right] + \left( Q(r) + \sigma^2 W(r) \right) f + R \right\} = 0. \tag{2.9}
\]

With this form, the most important result of the theory, the ordered spectrum theorem cannot be used directly, unless the function \( R \) vanishes. We show how this is achieved in our derivation and we then use Theorem 7 from Appendix A to conclude that if the fundamental mode of vibration, corresponding to the first eigenvalue \( \sigma_1^2 \) is positive, then so will all the higher modes, so that no modes will cause the star to have its radius perturbed away from equilibrium, proving the radial stability of the star. We provide this result in a table that investigates different parameter values to see which results in stable, and which in unstable stars.

2.8 Astronomical observation of compact objects

Astronomical observations of compact objects (neutron stars, pulsars, black holes) have been possible for quite some time now [94]. During the last decade both X-ray and \( \gamma \)–ray tele-
scopes have provided large datasets about precise timings of pulsars, and from these a number of measurements, most of them not independent of the underlying model behind these objects, have been possible.

A recent review by [Ozel and Freire](#) claims that precise masses of approximately 35 neutron stars are known, and the radii of about 10 of these are known to varying degrees of precision. These measurements already place constraints on the possible EOS of neutron matter, since the very heavy and small (having large $M/R$ values) stars eliminate many of the softer EOS that predict much lower maximum $M/R$ values than observed.

Since that initial discovery and observation, a number of theoretical modelling, usually based on the TOV equation together with some type of effective field theory or the newer quantum chromodynamics (QCD) calculations have also been used to understand the structure of these stars. However the theory aspect of the problem is fraught with difficulties. For example, the composition of matter at high densities, and in particular the density limit at which matter has to be modelled as quarks instead of nucleons is not known. Also unknown are the relative effects of the presence of boson condensates and strangeness in such high density matter. Work in particle accelerators working with neutron rich nuclei hope to find some of these parameters, however even those nuclei do not come close to what a neutron star’s composition is thought to be, and their validity to neutron star matter is contentious.

However this problem is approached, once certain approximations about the different components of the neutron matter has been made, an EOS can be deduced, and from utilising this EOS in the TOV equation, a mass to radius ($M - R$) curve corresponding to the EOS can be drawn. This curve can then be matched to observations of neutron stars, and appropriate conclusions drawn. This can be seen as a test of both GR in the strong field regime (through the use of the TOV equation,) and as a test of the EOS of cold ultra-dense matter.
2.8.1 Mass measurements

Most of the neutron star mass measurements come from pulsars in binary system. These constitute about 10% of all known pulsars for a total of 250. Of these, most are termed as “recycled” pulsars because throughout their lives, they have accrued mass from their companions. This mass transfer usually increases the mass of the pulsar, but the clearest signal of this process happening is the spin-up of the the pulsar [94]. Accompanying this spin-up is also the counter-intuitive reduction in the magnetic field associated with the pulsar, and indeed the mechanism leading to this reduction is very poorly understood.

All the pulsar masses that we have come from binary systems, and method employed usually involved pulsar timings of some sort. Through these accurate timings, much about the orbit of the pulsar can be inferred, and once the Keplerian orbital parameters are obtained, measuring the mass of the system becomes easy.

The nature of the companion changes the method of measurement of the pulsar itself (as opposed to the total mass of the whole binary system,) and some of the numerous methods used to measure their masses are:

1. Straight forward pulsar timings for pulsars in binary pulsar systems (the companion is a pulsar too). This has been used for example in [81].

2. Shapiro delay which is the delay in the reception of the radio pulses associated with the pulsars on Earth due to the propagation of the radio signal in the curved space-time near the companion star. This is akin to the “lensing” of radio waves. This is used with systems where the pulsar has a white dwarf companion, for example in [113]

3. Spectroscopic mass measurements, which involves the studying of the Balmer lines of hydrogen produced in the companion’s atmosphere. For this method to be used, the companion has to be optically bright. This has been used for example in [2].
These do not provide the complete extent of the methods, but do give an idea how complicated this field is, and how dependant the methods are on the companions.

We provide a list of pulsar masses obtained through various methods in Figure 2.2, which is taken from the review [94].

Masses of these pulsars are the “easy” measurements. Neutron stars are incredibly compact, and measuring their radii, which is typically of the order of a few kilometres is even more challenging. We look into the methods and observations of these next.

2.8.2 Radius measurements

The field of radii measurements of pulsars has only been active during the past decade, and the method employed for these measurements is based on the detection of thermal emission from the surface of the star either to measure its apparent angular size or to detect the effects of the neutron-star space-time on this emission to extract the radius information [94].

The major method that has produced reliable results for the radii measurements so far is spectroscopic measurements involving the determination of the angular sizes of the pulsars by measuring the thermal flux from the pulsar, modelling the spectrum to determine the effective temperature, and combining this with a distance measurement to obtain an apparent radii. The fact that neutron stars gravitationally lens their own emission make this process arduous [108]. Since some pulsars spin very quickly, the space-time around them can no longer be described by the Schwarzschild metric (a non-rotating solution), and this introduces other complications. The magnetic fields associated with the pulsars stream the flux from the pulsar, which can then cause a large temperature difference between different points on the pulsar. This makes the inferring of the temperature difficult. All these complications have to be either modelled, or sources that exhibit the least of these chosen for this method to work. This method was worked reasonably well in a few quiescent low mass X-ray binaries (qLXMBs) [51,76]. Quiescence refers to LXMBs which cease to accrete, or accrete at a
Figure 2.2: Figure taken from the review [94] by Ozel and Freire. The references that give the latest mass measurements are included in the article above, and we will not reproduce it here.
very low rate. Because of this low rate, the thermal emission from these stars can be observed without too much interference.

The major source of the radiation coming from LXMBs is a process known as Type-I X-Ray bursts. In these, accreted matter to the pulsar undergoes a helium flash\(^1\) that consumes the

\(^1\)A helium flash is a nuclear fusion reaction where a large quantity of helium is converted into carbon
accreted matter over the whole surface of the star. The luminosity of the star rises rapidly ($\sim 1\text{s}$) when this happens, but the energy is quickly radiated away ($\sim 50\text{s}$). In some cases the luminosity reaches the Eddington luminosity limit where the radiation pressure matches the gravitational force. When the Eddington limit is exceeded and the photosphere gets lifted from the surface of the star, the luminosity then behaves in a typical way that is well understood, and allows to infer and constrain the $M - R$ values of the pulsars.

The second method used for radii measurements rely on the periodic brightness oscillations that spinning neutron stars undergo. These oscillations are due to the temperature anisotropies on the surface of the star. The amplitude and spectra of the emissions depend on the neutron star space-time, and on the temperature profile of the stellar surface. Theoretical models of this emerging radiation can be used to constrain the mass and radii of the star. These theoretical models started with non-spinning neutron stars [100], to which the Doppler shifts and aberration expected from the spinning were added [84, 107]. Effects like frame dragging and the oblateness of the star were also subsequently added [21, 87]. Once the models were obtained and deemed accurate, constraints on mass and radii could be obtained.

These two main methods yielded constraints on both mass and radii, which we now show on the $M - R$ diagrams in Figure 2.3. Statistical analyses of the errors stem from the numerous assumptions and models used, and for this reason each measurement is represented as a “patch” in the $M - R$ plot.

This preliminary section introduces all the notions that the remaining chapters will use directly, without introduction. The first part of this work will look at a known solution to the interior EFE given in (2.1). Then we find new physical solutions to this EFE in two more general cases in Chapter 4. Then the linear stability of these solutions is investigated in Chapter 5 and a discussion of the possibility to use the solutions as models in the future through a triple alpha process. This reaction is associated with a large release of energy leading to thermal runaway (positive feed-back) and thus a rise in temperature.
investigated in Chapter 6
Chapter 3

The Tolman VII solution, an example

We apply the Tolman, and/or Ivanov procedure to the spherically symmetric and static case, without anisotropic pressure or electric charge to give an overview and the simplest example possible for our plan of attack before generalizing to the more complicated cases. We also present the type of analysis that we hope will be possible for the new solutions. In so doing, the Tolman VII solution for a static perfect fluid sphere to the Einstein equations is re-examined and a closed form class of equations of state (EOS) is deduced for the first time. These EOS allow further analysis to be carried out, leading to a viable model for compact stars with arbitrary boundary mass density to be obtained. Explicit application of causality conditions places further constraints on the model, and recent observations of masses and radii of neutron stars prove to be within the predictions of the model. The adiabatic index predicted is $\gamma \geq 2$, but self-bound crust solutions are not excluded if we allow for higher polytropic indices in the crustal regions of the star. The solution is also shown to obey known stability criteria often used in modelling such stars. It is argued that this solution provides realistic limits on models of compact stars, maybe even independently of the type of EOS, since most of EOS usually considered do show a quadratic density falloff to first order, and this solution is the unique exact solution that has this property.

3.1 Introduction

The construction of exact analytic solutions to the Einstein equations has had a long history, nearly one hundred years to be more precise. However in spite of the fact that the
total number of solutions is large \([69]\) and growing, only a small subset of those solutions can be thought of as having any physical relevance. Most solutions exhibit mathematical pathologies or violate simple principles of physics (energy conditions, causality, etc.) and are therefore not viable descriptions of any observable or potentially observable phenomena. Indeed works that review exact solutions and their properties demonstrate the difficulties associated with constructing solutions that might be relevant to gravitating systems that actually exist in our Universe. Even in the simplest case of exact analytic solutions for static, spherically symmetric fluid spheres, it has been shown that less than ten percent of the many known solutions can be considered as describing a realistic, observable object. For example Delgaty and Lake using computer algebra methods reviewed over 130 solutions and found that only nine could be classified as physically relevant \([31]\). A similar study by Finch and Skea \([43]\) arrived at the same conclusion. The latter review also introduced a classification that further reduced the number of physically relevant solutions to those that had exact analytic equations of state of the form \(p = p(\rho)\) where \(p\) is the fluid pressure and \(\rho\) is the matter density. This class of solutions was called “interesting solutions”.

In 1939 Tolman introduced a technique for constructing solutions to the static, spherically symmetric Einstein equations with material fluid sources \([125]\). That method led to eight exact analytic expressions for the metric functions, the matter density and in some cases the fluid pressure. Beginning with an exact analytic solution for one of the two metric functions, an expression for the mass density could be obtained by integration. With such expressions for the density and the first metric function in hand, the analytic expression for the second metric function could be obtained. This often required an appropriate change of the radial variable to obtain a simple integral. All functions could then be written as explicit functions of the radial coordinate \(r\). While the fluid pressure could, in principle, be obtained from the metric and density functions, Tolman chose not to evaluate the fluid pressure in some cases due to the fact that to do so would lead to rather mathematically complicated expressions
that might be difficult to interpret.

Of the eight solutions presented in his paper, three were already known (the Einstein universe, Schwarzschild–de Sitter solution, and the Schwarzschild constant density solution), most of the others "describe situations that are frankly unphysical, and these do have a tendency to distract attention from the more useful ones." [66]. One, the so-called Tolman VII solution appeared to have some physical relevance but this was one of the solutions for which no explicit expression for the pressure was given.

The Tolman VII solution has been rediscovered a number of times and has appeared under different names, the Durgapal [36, 37] and the Mehra solutions being two examples. That these solutions can be used to describe realistic physical systems has been noted by many authors including those of the two review papers mentioned above [31, 43]. It has been used as an exact analytic model for spherically symmetric stellar systems and additional research has investigated its stability properties [9, 89, 90]. While these later works were able to obtain the complicated expressions for the fluid pressure as a function of the radial coordinate, according to Finch and Skea it still was not one of the “interesting solutions” since it lacked an explicit expression for the equation of state. The choice of parameters that has been taken by different authors in order to completely specify the solution in many ways prevented the immediate interpretation of the physical conditions described by the solution. The reasons mentioned above are not sufficient to use or classify the Tolman VII solution as a physically viable one. Instead we seek physical motivations for the viability of this solution, and indeed we find these in many forms:

(i) From a Newtonian point of view, simple thermodynamic arguments yield polytropes of the form $p(\rho) = k \rho^\gamma$, (here $\gamma$ is the adiabatic index sometimes written in terms of the polytropic index $n$, $\gamma = 1 + 1/n$, and $k$ is known as the adiabatic constant that can vary from star to star) as viable models for neutron matter. When coupled with Newtonian hydrodynamic stability and gravitation, the result is the Lane-Emden differential equation for the density
profile, $\rho(r)$. Solutions of the latter, obtained numerically, or in particular cases ($\gamma = \infty$, 2, or 1.2) exactly, all have a distinctive density falloff from the centre to the edge of the Newtonian star. This is a feature we wish physical solutions to have. Furthermore this distinctive falloff is quadratic in the rescaled radius [62], suggesting that even in the relativistic case, such a falloff would be a good first approximation to model realistic stars, which have a proper thermodynamic grounding.

(ii) Looking at viable exact relativistic solutions to the Einstein equations, the one used extensively before 1939 and even much later, was the Schwarzschild interior solution. This solution has the feature that the density is constant throughout the sphere, and is not physical: the speed of sound (pressure) waves in its interior is infinite. However this solution provides clear predictions about the maximum possible mass of relativistic stars in the form of the Buchdahl limit [18]: $M \leq 4R/9$. The next best guess in this line of reasoning of finding limiting values from exact solutions would be to find an exact solution with a density profile that decreases with increasing radius, since a stability heuristic for stars demands that $\frac{d\rho}{dr} \leq 0$, as expected from (i) in the Newtonian case. Extension to the relativistic Lane-Emden equation also requires [62] that $\frac{d\rho}{dr}\big|_{r=0} = 0$, a property Tolman VII has.

(iii) Additionally an extensive review of most EOS used from nuclear physics to model neutron stars concluded that a quadratic falloff in the density is a very close approximation to most such nuclear models [74]: the differences between drastically different nuclear models from Tolman VII being only minor if only the density profiles were compared. Since Tolman VII is precisely the unique exact solution to the full Einstein field equations that exhibits a quadratic falloff in the density profile, we believe that it captures much of what nuclear models have to say about the overall structure of relativistic stars.

These three reasons taken together make a strong case for considering the Tolman VII solution as the best possible exact solution that is capable of describing a wide class of EOS for
neutron stars. At the very least it is as good a candidate that captures first order effects in density of *most* nuclear model EOS, and at best it is the model that all realistic nuclear models tend to, while including features like self-boundedness naturally, as we shall show.

The purpose of this Chapter is to re-examine the Tolman VII solution by introducing a set of constant parameters that we believe provide a more intuitive understanding of the physical content of the solution. In addition the solution now becomes a member of the set of “interesting solutions” since we provide an explicit expression for the EOS. The EOS will allow for further exploration of the predictions of the solution as well as a description of the material that makes up the star. The imposition of both causality conditions where the speed of sound in the fluid never exceeds the speed of light, and different boundary conditions will provide further restrictions on the parameters associated with the solution. What this all leads to is a complete analytic model for compact stars that can be used to compare with recent observations of neutron star masses and radii. That the Tolman VII solution is consistent with all measurements leads to the conclusion that this exact solution is not only physically relevant but may be one physically realized by nature.

This chapter is divided as follows: following the brief introduction presented in this section, we re-derive the Tolman VII solution in section 3.2, paying particular attention to the pressure expression in physically more intuitive variables. We then invert the density equation and use the pressure expression just found to derive an EOS in section 3.3 where we also carry out an analysis of the said EOS. We will then proceed to contrast the two different types of physical models that the solution admits in section 3.4 where we will also show how qualitative differences arise in the stars’ structure and quantitative ones appear in the predicted values of the adiabatic indices of the fluid. We shall then provide brief concluding remarks in section 3.5.
3.2 The Tolman solution

Beginning with a line element in terms of standard areal (Schwarzschild) coordinates for a static and spherically symmetric metric:

\[ ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \]

the Einstein equations for a perfect fluid source lead to three ordinary differential equations for the two metric variables \( \nu, \lambda \), and the two matter variables \( \rho \) and \( p \). However these variables will not be the most practical ones to carry out our analysis. Instead we introduce two different metric functions, \( Z(r) = e^{-\lambda(r)} \) and \( Y(r) = e^{\nu(r)/2} \), as derived in Ivanov [65]. The reason for these new metric variables is that with our subsequent density assumption, this will allow easier linearisation of the differential equations. The Einstein equations then reduce to the following set of three coupled ordinary differential equations (ODEs) for the four variables \( Z, Y, p, \) and \( \rho \):

\[
\begin{align*}
\kappa \rho &= e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{1}{r^2} - \frac{Z}{r^2} - \frac{1}{r} \frac{dZ}{dr}, \quad (3.1a) \\
\kappa p &= e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{2Z}{rY} \frac{dY}{dr} + \frac{Z}{r^2} - \frac{1}{r^2}, \quad (3.1b) \\
\kappa p &= e^{-\lambda} \left( \frac{\nu''}{2} - \frac{\nu' \lambda'}{4} + \frac{(\nu')^2}{4} + \frac{\nu' - \lambda'}{2r} \right) = \frac{Z}{Y} \frac{d^2Y}{dr^2} + \frac{1}{2Y} \frac{dY}{dr} \frac{dZ}{dr} + \frac{Z}{rY} \frac{dY}{dr} + \frac{1}{2r} \frac{dZ}{dr}. \quad (3.1c)
\end{align*}
\]

Where the primes (') denote differentiation with respect to \( r \), and \( \kappa \) is equal to \( 8\pi \), since we use natural units where \( G = c = 1 \). The first two equations (3.1a) and (3.1b) can be added together to generate the simpler equation

\[ \kappa(p + \rho) = \frac{2Z}{rY} \frac{dY}{dr} - \frac{1}{r} \frac{dZ}{dr}, \quad (3.2) \]

which will be useful later on. To solve this set of ODEs, we shall assume a specific form for the energy density function that we claim has physical merit:

\[ \rho = \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right], \quad (3.3) \]
where the constants \( r_b \) represents the boundary radius as mentioned previously, \( \rho_c \) represents the central density at \( r = 0 \), and \( \mu \) is a “self-boundness” dimensionless parameter, that will span values between zero and one, so that when it is equal to zero, we have a sphere of constant density. This form of the density function for \( \mu > 0 \) is physically realistic since it is monotonically decreasing from the centre to the edge of the sphere, in contrast to the constant density exact solution (Schwarzschild interior) frequently used to model such objects. Additionally we will need boundary conditions for the system: since we eventually want to match this interior solution to an external metric. Since the vacuum region is spherically symmetric and static, the only candidate by Birkhoff’s theorem is the Schwarzschild exterior solution. The Israel-Darmois junction conditions for this system can be shown to be equivalent to the following two conditions \[124\] as is derived in Appendix A:

\[
p(r_b) = 0, \quad \text{and,} \quad Z(r_b) = 1 - \frac{2M}{r_b} = Y^2(r_b). \tag{3.4b}
\]

Where \( M = m(r_b) \) is the total mass of the sphere as seen by an outside observer, and \( m(r) \) is the mass function defined by

\[
m(r) = 4\pi \int_0^r \rho(\bar{r})\bar{r}^2d\bar{r}. \tag{3.5}
\]

Furthermore we will also require the regularity of the mass function, that is for mass function to be zero at the \( r = 0 \) coordinate, from physical considerations: \( m(r = 0) = 0 \). On imposing \(3.4b\), we can immediately write \( Z \) in terms of the parameters appearing in the density assumption:

\[
Z(r) = 1 - \left( \frac{\kappa \rho_c}{3} \right) r^2 + \left( \frac{\kappa \mu \rho_c}{5r_b^2} \right) r^4 =: 1 - br^2 + ar^4. \tag{3.6}
\]

In contrast Tolman assumed the last equation, and then obtained the density function from equation \(3.1a\). The physical constants \( \mu, \rho_c, \) and \( r_b \) occur frequently enough in the combinations above that we will also use \( a \) and \( b \) as defined above when convenient. The solution
method for these ODEs leading to the Tolman VII solution have been given in multiple references [83, 125], and we briefly sketch it. Essentially two variable transformations convert the set of ODEs into a simple harmonic differential equation, and back-substitution to the original variables solves the system. The first step in this procedure in the change of variable from \( r \) to \( x = r^2 \), as a result of which the derivative terms have to be changed too.

The set of equation (3.1) with this variable change then becomes

\[
\begin{align*}
\kappa \rho &= \frac{1}{x} - \frac{Z}{x} - 2 \frac{dZ}{dx}, \\
\kappa p &= \frac{4Z}{Y} \frac{dY}{dx} + \frac{Z}{x} - \frac{1}{x}, \\
\kappa p &= \frac{dZ}{dx} + \frac{1}{Y} \left( 4 \frac{dY}{dx} + 4xZ \frac{d^2Y}{dx^2} + 2x \frac{dY}{dx} \right).
\end{align*}
\]

(3.7a) (3.7b) (3.7c)

We see that by subtracting equation (3.7c) from equation (3.7b), we get the second order differential equation

\[
\frac{Z}{x} - \frac{1}{x} - \frac{4xZ}{Y} \frac{d^2Y}{dx^2} - \frac{2x}{Y} \left( \frac{dY}{dx} \right) \left( \frac{dZ}{dx} \right) - \frac{dZ}{dx} = 0,
\]

which is further simplified by multiplying by \(-Y/(2x)\), resulting in

\[
2Z \frac{d^2Y}{dx^2} + \left( \frac{dY}{dx} \right) \left( \frac{dZ}{dx} \right) + Y \left( \frac{1}{2x^2} - \frac{Z}{2x^2} + \frac{1}{2x} \frac{dZ}{dx} \right) = 0.
\]

Solving the system in this form is equivalent to solving the whole system, and since we have already integrated the first Einstein equation (3.1a), and have an expression for \( Z \) in equation (3.6) in terms of \( r \) and equivalently \( x \), we substitute those now in the above differential equation to get

\[
Z \frac{d^2Y}{dx^2} + \frac{1}{2} \left( \frac{dZ}{dx} \right) \frac{dY}{dx} + Y \left[ \frac{a}{4} \right] = 0.
\]

(3.8)

This last equation has a middle cross term involving a first derivative of the metric function \( Y \) which we want to find. From the form of the equation we see that if we could transform the above equation with a substitution to get rid of the cross term, we would end up with a
second order equation of the simple harmonic type. It turns out that the variable change that permits this is

\[ \xi = \int_0^x \frac{dx}{\sqrt{Z(x)}} \Rightarrow \frac{d\xi}{dx} = \frac{1}{\sqrt{Z(x)}}, \quad (3.9) \]

Where we retain the integral form of the equation. Transforming our dependent variable \( x \), to \( \xi \) and taking into account that the derivatives with respect to \( x \) will transform according to

\[ \frac{dA}{dx} = \frac{dA}{d\xi} \frac{d\xi}{dx} = \sqrt{Z} \frac{dA}{d\xi}, \quad (3.10a) \]

\[ \frac{d^2A}{dx^2} = \frac{d}{dx} \left( \frac{dA}{d\xi} \frac{d\xi}{dx} \right) = \frac{d^2\xi}{dx^2} \frac{dA}{d\xi} + \left( \frac{d\xi}{dx} \right)^2 \frac{d^2A}{d\xi^2} = \frac{1}{Z} \frac{d^2A}{d\xi^2} = \frac{1}{Z^3} \frac{dA}{dx} \frac{dA}{d\xi}, \quad (3.10b) \]

where \( A \) is a dummy function of both \( x \) and \( \xi \), we simplify the differential equation (3.8) into

\[ Z \left\{ \frac{1}{Z} \frac{d^2Y}{d\xi^2} - \frac{1}{2Z\sqrt{Z}} \frac{dZ}{dx} \frac{dy}{d\xi} \right\} + \frac{1}{2} \frac{dZ}{dx} \left( \frac{1}{\sqrt{Z}} \frac{dy}{d\xi} \right) + \left( \frac{a}{4} \right) Y = 0, \]

which is indeed in simple harmonic form, as we wanted. We can as a result immediately identify three different classes of solutions depending on the value of the constant \( \phi^2 = a/4 \), summarized below:

| \( \phi^2 \)    | \( Y(\xi) \)                                          | Solution’s name                     |
|----------------|-------------------------------------------------------|------------------------------------|
| \( \phi^2 < 0 \) | \( c_1 \exp\left(\sqrt{-\phi^2} \xi\right) + c_2 \exp\left(-\sqrt{-\phi^2} \xi\right) \) | Bayin [5]                           |
| \( \phi^2 = 0 \) | \( c_1 + c_2 \xi \)                                   | Schwarzschild interior             |
| \( \phi^2 > 0 \) | \( c_1 \cos(\phi \xi) + c_2 \sin(\phi \xi) \)        | Tolman VII                         |

Table 3.1: The different solutions that can be generated through different values of the parameter \( \phi \).

Once we pick a value for \( \phi \), the solution is completely specified. Applying the boundary conditions will then permit us to find the value of the parameters. At this stage, we could using the interpretation scheme given previously for the constants \( \mu \), \( \rho_c \), and \( r_b \), to figure out the sign for \( \phi \). From equation (3.6), we find that since all the constants involved in \( a/4 \) are positive definite, \( \phi^2 \) must be so too, and hence we are forced to pick the third solution, i.e. Tolman’s, if we want to model physically realistic stars. We also note that were \( \mu \) to take
the limiting value of zero, we would not only have to pick the the second solution, which is Schwarzschild’s interior solution, but also assume a constant density, a well known aspect of this particular solution.

The complete Tolman VII solution is specified with the two functions below, together with the previously given density function $Y(\xi)$, and the metric function $Z$ in equation (3.6),

$$Y(\xi) = c_1 \cos(\phi \xi) + c_2 \sin(\phi \xi), \quad \text{with} \quad \phi = \sqrt{\frac{a}{4}}, \quad (3.11)$$

where we have used $\xi$, but not actually given an explicit expression for it in terms of $r$. The actual form is found by performing the integral (3.9), which can be solved by consulting an integrals’ table [50], however insight into the form of the integral is gained through an Euler substitution of the form $\sqrt{Z(\bar{x})} = \bar{t} - 1$, so that the denominator and Jacobian of the transformation become

$$\bar{t} - 1 = \frac{t^2 - bt + a}{t^2 - a} \quad \text{and} \quad \frac{d\bar{t}}{dt} = \frac{-2(t^2 + a - bt)}{(t^2 - a)^2}.$$ This allows the integrand to be expressed as

$$\xi(\bar{r}) = \int_{\bar{r}_0}^{\bar{r}} -2(t^2 - bt + a) \left( \frac{t^2 - a}{t^2 - bt + a} \right) dt = \int_{\bar{r}_0}^{\bar{r}} -2dt \left( \frac{t^2 - a}{t^2 - bt + a} \right),$$

which requires another substitution to be solved. A number of different substitutions would give different equivalent forms of the integral, however because of the positive sign of $a$, the most useful substitution is $t^2 = a \coth^2 \alpha$, with resulting Jacobian $dt = \sqrt{a}(-\csch^2 \alpha)d\alpha$. These reduce the denominator of the previous integrand to $a \csch^2 \alpha$, so that the final form of $\xi$ is

$$\xi(r) = \frac{2}{\sqrt{a}} \coth^{-1} \left( \frac{t}{\sqrt{a}} \right) \bigg|_{\bar{r}_0}^{\bar{r}_0} = \frac{2}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right), \quad (3.12)$$

where the last equality comes from back-substituting the multiple variable changes done before. Sometimes the equivalent form of the above equation, in terms of logarithms is more
useful, and in this form the above is expressed as

\[ \xi(r) = \frac{1}{\sqrt{a}} \left[ \log(b + 2\sqrt{a}) - \log(b - 2ar^2 + 2\sqrt{aZ}) \right], \quad (3.13) \]

from the well known hyperbolic identity \( \coth^{-1} x \equiv \frac{1}{2} \left[ \log \left( \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} \right) \right], \) for \( x \neq 0. \) Now that we have the full solution of the metric functions, we can compute the pressure through the relation below, obtained from a simple rearrangement and variable change of (3.2):

\[ \kappa p(r) = 4 \sqrt{Z} \frac{dY}{d\xi} - \frac{1}{r} \frac{dZ}{dr} - \kappa \rho, \quad (3.14) \]

resulting in the very complicated looking,

\[ \kappa p(r) = \frac{4\phi[c_2 \cos (\phi \xi) - c_1 \sin (\phi \xi)]\sqrt{1 - br^2 + ar^4}}{c_1 \cos (\phi \xi) + c_2 \sin (\phi \xi)} - 4ar^2 + 2b - \kappa \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right]. \quad (3.15) \]

So far we have not found the expressions of any of the two integration constants \( c_1 \) and \( c_2. \) To find those, we need to apply the boundary conditions explicitly and to do so we perform the variable changes \( r \to x \to \xi \) on equation (3.2),

\[ \kappa(p + \rho) = \frac{2Z}{Y} \frac{dY}{dr} - \frac{1}{r} \frac{dZ}{dr} \xrightarrow{r \to x} 4 \frac{Z}{Y} \frac{dY}{dx} - 2 \frac{dZ}{dx} \xrightarrow{x \to \xi} \frac{4\sqrt{Z}}{Y} \frac{dY}{d\xi} - 2 \frac{dZ}{dx}. \]

With this equation, together with the boundary condition (3.4b), we have

\[ \kappa(p + \rho) \bigg|_{x=x_b} = \frac{4\sqrt{Z(x_b)}}{Y(x_b)} \bigg|_{\xi=\xi_b} \frac{dY}{d\xi} \bigg|_{\xi=\xi_b} - 2 \frac{dZ}{dx} \bigg|_{x=x_b}, \]

where all the \( b \)-subscripted variables are the values at the boundary. However since according to the second boundary condition (3.4a), the pressure has to vanish at the boundary, the latter equation simplifies to

\[ \kappa \rho \bigg|_{x=x_b} = \frac{4}{\xi=\xi_b} \frac{dY}{d\xi} - 2 \frac{dZ}{dx} \bigg|_{x=x_b}, \]

which can be further simplified and rearranged as

\[ \frac{dY}{d\xi} \bigg|_{\xi=\xi_b} = \frac{b - ax_b}{4} = \frac{\kappa \rho_c}{4} \left( \frac{1}{3} - \mu \right) =: \alpha. \quad (3.16) \]
Since the ODE for $Y$ is second order, we also need a further constraint equation. This is simply going to be condition (3.4b) restated as

$$Y(x = x_b) = \sqrt{1 - bx_b + ax_b^2} = \sqrt{1 - \kappa \rho c r_b^2 \left( \frac{1}{3} - \frac{\mu}{5} \right)} =: \gamma$$

(3.17)

These two equations (3.16) and (3.17) constitute the complete Cauchy’s boundary condition on $Y$. We now only need to simplify our integration constants $c_1$ and $c_2$ with these, to specify the solution completely in terms of the parameters we chose initially. To do so we re-express the metric function and its derivatives in terms of their solutions, yielding two simultaneous equations:

$$\left. \frac{dY}{d\xi} \right|_{\xi = \xi_b} = \phi [c_2 \cos (\phi \xi_b) - c_1 \sin (\phi \xi_b)] = \alpha \Rightarrow c_2 \cos (\phi \xi_b) - c_1 \sin (\phi \xi_b) = \alpha/\phi,$$

(3.18)

$$Y(\xi = \xi_b) = c_2 \sin (\phi \xi_b) + c_1 \cos (\phi \xi_b) = \gamma.$$  

(3.19)

This system can be solved by first multiplying (3.18) by $\cos (\phi \xi_b)$, and (3.19) by $\sin (\phi \xi_b)$, and adding the equations obtained: yielding $c_2$. Similarly switching the multiplicands and performing a subtraction instead yields $c_1$, both of which we now give.

$$c_1 = \gamma \cos (\phi \xi_b) - \frac{\alpha}{\phi} \sin (\phi \xi_b),$$  

(3.20)

$$c_2 = \gamma \sin (\phi \xi_b) + \frac{\alpha}{\phi} \cos (\phi \xi_b).$$  

(3.21)

We note that all the constants employed in the expressions of the integration constants are ultimately in terms of the set of parameters $\Pi$ we initially chose, viz. $\Pi = \{\rho_c, r_b, \mu\}$. This completes the specification of the full Tolman VII solution in the new constant scheme.

Another quantity we wish to consider is the adiabatic speed of pressure(sound) waves that the fluid can sustain. The usual definition of this quantity in perfect fluids is $v^2 = d\rho/d\rho$. However, we will find it convenient to find an expression of this speed directly from the differential equations, since the expression and functional form, while completely equivalent is simpler to work with. We notice first that from the expression of the density (3.3), we can
obtain the derivative
\[
\frac{d\rho}{dr} = -\frac{2\mu \rho_c}{r_b^2} r,
\]
which is zero at \( r = 0 \), or if one of the 2 parameters \( \mu = 0 \) or \( \rho_c = 0 \). For the other equation, we use the conservation of the energy momentum tensor \( \nabla_a T^a_b = 0 \), which as we have shown before reduces to
\[
\frac{dp}{dr} = -\frac{\nu'(p + \rho)}{2} = -\frac{(p + \rho)}{Y} \frac{dY}{dr},
\]
in the \( b = 0 \) case. These two expressions can be used to find \( dp/d\rho \) for every value of \( r \) but the centre, so that
\[
v^2 = \left( \frac{dp}{dr} \right) \left/ \frac{d\rho}{dr} \right. = \frac{r_b^2}{4\mu \rho_c} \frac{\nu'(p + \rho)}{r} = \frac{r_b^2 (p + \rho)}{2\mu \rho_c r Y} \frac{dY}{dr}.
\]
(3.23)
Since we have expressions for all the terms in this formula, we also have a closed form for the speed of sound.

The bulk modulus \( K \) of a fluid is a measure of the resistance of a fluid to change its volume under an applied pressure. For perfect fluids it is related to the speed of sound (pressure waves) in the media through \( K = \rho v^2 \). This is also a quantity which we calculate for the fluid in the interior, and this calculation show us that the material we are dealing with has no earthly analogue, since the order of magnitude of the bulk modulus is much higher than any currently known substance.

The next step to understanding this solution is to investigate the behaviour of the different physical variables we have. However before we can do that, we have to specify values for our parameters. We will use different values of the parameters, and each time we will specify the values being used. The primary motivation for the values we will be using is that we ultimately wish to model compact astrophysical objects. As a result central densities \( \rho_c \) of \( 1 \times 10^{15} \text{g} \cdot \text{cm}^{-3} \) will be typical. Similarly radii \( r_b \) of \( 1 \times 10^6 \text{cm} \) will often be used for the same reason. As can be seen from the density profile (3.3), the latter decreases quadratically with the radial coordinate, as we show in figure (3.1a)
As this point we can interpret the effect of varying the parameter $\mu$ on the density profile: It is changing the surface density from a zero value when $\mu = 1$ to increasingly higher densities as $\mu$ is decreasing. In the literature [74], models having zero surface densities have been named “natural”, and those with non-vanishing surface densities have been called “self-bound.” As a result we will call $\mu$ the “self-boundness” parameter that will allow us to change the surface density in our models.

![Figure 3.1: The matter variables including the density, pressure, speed of sound, and bulk modulus inside the star. The parameter values are $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$, $r_b = 1 \times 10^4 \text{ m}$ and $\mu$ taking the various values shown in the legend](image)

Similarly, the complicated expression of the pressure that we have obtained can also be plotted. Of importance here is the fact that while the densities might not vanish at the boundary $r_b$, the pressure for all parameter values must do so according to our boundary condition (3.4a). This is eminently clear in figure 3.1b where we see the pressures associated with
the density curves shown in figure 3.1a. Similarly the speed of sound and bulk modulus, all associated with the matter content in the star, can be plotted and we show this in figure 3.1c and 3.1d respectively.

The other variables that solving our differential equations yield are the metric coefficients $Z(r)$ and $Y(r)$. We show both of these next in figures 3.2b and 3.2a respectively, again for different values of the self-boundness $\mu$. Equivalently we could give the metric coefficients in Schwarzschild form: the form most often used in the literature for specifying static spherically symmetric models. We do so for now the sake of completeness, giving $\lambda(r)$ in figure 3.2d and $\nu(r)$ in figure 3.2c respectively.

**Figure 3.2:** The variation of the metric variables with the radial coordinate inside the star: we show both Ivanov’s $Y(r)$, and $Z(r)$, and the more generic $\lambda(r)$, and $\nu(r)$. The parameter values are $\rho_c = 1 \times 10^{18}$ kg$ \cdot$ m$^{-3}$, $r_b = 1 \times 10^4$ m and $\mu$ taking the various values shown in the legend.
3.3 The equation of state

The nice feature of our density assumption (3.3) is that it can be inverted to easily obtain $r$ as a function of $\rho$, which allows us to generate an equation of state (EOS) for this solution. We give the full equation of state below, before starting to analyse it:

$$p(\rho) = -\frac{1}{20\pi h_1 h_2} \left\{ h_1 - h_2 \sqrt{-2f_1 \cot^2 f_2 + 4\pi h_1 h_2 \rho} \right\},$$

where $f_1(\rho)$ and $f_2(\rho)$ are functions of the density:

$$f_1(\rho) = 50 - 3 \left( \frac{h_1}{h_2} \right)^2 - \frac{4\pi h_1^2}{h_2}\rho + 32\pi^2 h_1^2 \rho^2,$$

and

$$f_2(\rho) = \frac{1}{2} \ln \left[ \frac{\sqrt{8f_1 h_2} + h_1 - 16\pi h_1 h_2 \rho}{20h_2 C} \right].$$

The constants $h_1$ and $h_2$ are determined by the central density and $\mu$, as follows:

$$h_1 = r_b \sqrt{\frac{5}{2\pi \rho_c \mu}} \quad \text{and} \quad h_2 = \frac{3}{8\pi \rho_c},$$

while the constant $C$ is expressible as a complicated function of the parameters only, in terms of the auxiliary variables $\sigma$, and $\chi$,

$$C = \left( 1 - \frac{h_1}{4h_2} \right) \sqrt{\frac{h_1(4h_2 - h_1)}{8r_b^2 h_2 - h_1^2}} \exp \left[ \arctan \left( \frac{\chi}{\sigma} \right) \right],$$

with,

$$\chi = 4 \sqrt{h_2(4h_2 r_b^4 - h_1^2 r_b^2 + h_1^2 h_2)},$$

$$\sigma = 16h_2 r_b^2 + 8\pi \rho_c h_1^2 h_2 (1 - \mu) - 2h_1^2.$$

We note here is that no assumption about the nature of matter, except for the very general thermodynamic prescription of a perfect fluid has gone into this solution. Everything else, and in particular the equation of state was obtained solely by virtue of the field equations and
the density profile (3.3). With the equation of state, it is a simple matter to find the derivative \( dp/d\rho \) for the speed of pressure waves.

The redshift of light emanating from a star as perceived by distant observers is another quantity that can potentially be measured. This quantity can also be calculated in our model, from the relation

\[
zs = \left(1 - \frac{2m(r_b)}{r_b}\right)^{-\frac{1}{2}} - 1.
\]

We show this value at the surface of the star for different values of \( \mu \) in figure 3.3 next.

![Figure 3.3: The redshift \( z \) at the surface of the sphere for different values of \( \mu \).](image)

3.4 Physical models

The expression for the EOS is somewhat complicated, but it is not without physical interpretation, contrary to what Tolman [125] thought in 1939:

The dependence of \( p \) on \( r \), with \( e^{-\lambda/2} \) and \( e^{-\nu} \) explicitly expressed in terms of \( r \),
is so complicated that the solution is not a convenient one for physical considerations.

Something that immediately becomes clear is possibility of two separate interpretations for an EOS. Both \( p(\rho; \Pi) \) for \( \rho_b = \rho_c(1 - \mu) \leq \rho \leq \rho_c \), with the values of the elements of \( \Pi \), in particular \( \rho_c \) fixed (henceforth called EOS1); and \( p(\rho = \rho_c; \Pi) \), with the parameters of \( \Pi \) varying between limits imposed by causality (EOS2) could be candidates. In the literature, both interpretations have been used, and sometimes even interchanged. However, each has a completely different content in that the first interpretation expresses how the pressure of the fluid changes in moving from the centre of the star \( r = 0, \rho = \rho_c \), to the boundary \( r = r_b, \rho_b = \rho_c(1 - \mu) \). The second interpretation by contrast looks closely at the fluid material itself and how the pressure at a certain point in the star changes as the density of the fluid at the centre changes. At this point in our derivation, we have not yet imposed any causality condition on any expressions.

We first carry out an analysis of EOS1, and find surprisingly that to a high degree of accuracy, the variation of \( p(\rho; \Pi) \), with \( \rho \), and equivalently \( r \), is very close to that of a polytrope of the form \( p = k \rho^\gamma - p_0 \). This relation is very obvious from the shape of the curve in the “natural” \( \mu = 1 \), case as is seen in figure 3.4.

Models employing polytropic perfect fluids use similar values for the adiabatic index \( \gamma \), as what we find for a range of different values of parameters \( \Pi \). We show this in figure 3.5 which treats \( \gamma \) as a continuous variable defined by \( \gamma = \frac{d(\log p)}{d(\log \rho)} \) and can be understood as the slope of the previous log–log graph.

From this figure it becomes evident how both types of stars have an interior structure well described by a polytrope with index close to 2.5. The “self-bound” stars exhibit the existence of an envelope consisting of material that is considerably stiffer than that found in the interior. Physically this is intuitive: for fixed \( \rho_c \) and \( r_b \) the self bound stars will become more and more massive as \( \mu \) decreases. The increasing boundary density discontinuity requires a
Figure 3.4: (Colour online) Log–log plot of pressure versus density for neutron star models determined by different $\mu$, but same $\rho_c$, and $r_b$. The densities and pressures are in cgs units, and the $\Pi$ is fixed by the following: $r_b = 10^6$ cm, $\rho_c = 10^{15}$ g · cm$^{-3}$. Since pressure is a decreasing function of distance from the centre, large densities indicate points closer to the centre of the star.

stiffer exterior mass distribution to maintain the equilibrium condition.

Now turning to the second way to characterize the EOS, concentrating on the behaviour of the fluid material itself, independent of the geometry of the star, we determine how different physical quantities depend on the values of the central density $\rho_c$. The total mass–energy is defined as,

$$M = 4\pi \int_0^{r_b} \bar{r}^2 \rho(\bar{r}) d\bar{r} = \frac{4\pi r_b^3 \rho_c (5 - 3\mu)}{15}.$$  \hspace{1cm} (3.24)

The mass is important since it is the only directly and reliably measurable quantity we obtain from neutron star observations. Lattimer and Prakash [74, 75, 77] and others [48, 49] have ruled out certain EOS based on mass and spin measurement of neutron stars. The former have also used Tolman VII, to constrain other EOS based on nuclear micro-physics, and have even postulated that Tolman VII could be used as a guideline discriminating between viable
Figure 3.5: (Colour online) The adiabatic index variation from the centre to the boundary of the star for different values of the parameter $\mu$. The other parameter values are the same as those in FIG. 3.4.

and non-viable EOS [77]. If this postulate is true, now that we have the complete Tolman VII EOS1, we can apply the causality condition, independent of measurements first, and compare with the previous references [48, 77].

We do this in figure 3.6 where we superimpose the result of [48], on our own analysis of the whole solution space $\Pi$. The surface shown is that of values at which the speed of sound $v_s = \left( \sqrt{\frac{dp}{d\rho}} \right)_{r=0}$, at the centre of the fluid sphere just reaches the speed of light. This is a sufficient condition for the solution to be causal since $v_s$ is a monotonically decreasing function of $r$ in the sphere. Any point located below this surface has coordinate values for $M, \rho_c, \mu$ that represents a valid causal solution to the Tolman VII differential equations. The orange line is the previous result obtained by Glendenning [48] from rotational considerations.
Figure 3.6: (Colour online) The mass of possible stars just obeying causality. The grey surface obeys the equation $v_s(r = 0) = c$. Every point below the surface is a possible realization of a star, and we can potentially read off the mass, central density, and $\mu$ value of that star. The numbered lines represent stars with the same mass that are causal, i.e. they are projections of the causal surface onto the $\rho_c–\mu$ plane. Glenndenning’s curve is shown in orange and represents a limit in the natural case only, and according to our results is acausal, being above our surface. The $\mu = 1$ plane’s intersection with our graph is the graph given in [77], and here too our prediction is more restrictive.

Imposing causality to constrain the parameter space $\Pi$, is not a new idea. However having an explicit EOS allows one to easily generate the causal surface shown above in figure 3.6. Previously the usual way to denote different EOS has been to calculate the compactness ratio, given by $\tau = \frac{GM}{c^2 r_b}$. We found that even in the case of Tolman VII, this is a stable quantity to characterize a star since the values of $\tau$ for large parameter variations $\Pi$ is relatively constant. This means that even though we might change the value for $\Pi$ of the stars, the ones bordering on causality share very similar compactness, albeit one that is lower than that previously thought possible. We show how this compactness $\tau$, varies with $\mu$, in figure 3.7. The previous maximal compactness was about 0.34 from rotational and causality criteria [77]. Our analysis shows that $\tau$ should be below 0.3 for all possible stars, if Tolman VII is a valid
Figure 3.7: (Colour online) The compactness as a function of the self-boundness parameter $\mu$. This plot was generated by varying $r_b$ from 4 km to 20 km for fixed $\mu$ and finding $\rho_c$ and subsequently the compactness each time, such that the sound speed was causal at the centre of the star. The curve shown is a polynomial fit, and the box-and-whisker plots (very small in green) show the variation of $\tau$ for fixed $\mu$, but different $r_b$. The very small whiskers justify the pertinence of $\tau$ as a useful measure in the analysis of the behaviour of the model.

physical model for stars.

Recently measurements of the radius of a limited number neutron stars have been obtained [53, 54, 95–97, 119, 123]. These are shown along with some other stars of known mass in figure 3.8. We also superimpose a few of the limiting causal curves obtained for different values of $\mu$ from Tolman VII, to show that Tolman VII is not ruled out by observational results, even though it predicts lower compactness than most nuclear models. However the lines shown are on the edge of causality, that is they are the counterparts of those on the surface of figure 3.6. Since all observations of compactness are bounded by the most extreme Tolman VII model we claim that the solution is actually realized by compact stars in nature.
Figure 3.8: (Color online) The mass $M$ in solar units versus radius $r_b$ in kilometres of a few stars for which these values have been measured. We use error bars to denote observational uncertainties, and coloured bands in the case where only the mass is known.

3.5 Conclusion

Thus a complete analysis of the Tolman VII solution was carried out and it was found that it is a completely valid solution with a huge potential for modelling physical objects. The EOS this solution predicts has been found, and in certain regimes behaves very much like a polytrope with an adiabatic index of 2.5. Using the EOS, we are able to compute the speed of pressure waves, and imposing causality on the latter results in a more restrictive limit on the maximum compactness of fluid spheres allowable by classical general relativity. The solution is also stable under radial perturbations, since the speed of these pressure waves is finite and
monotonically decreasing from the centre outwards, thus satisfying the stability criterion in [1]. If we believe as in ref [77] that Tolman VII is an upper limit on the possible energy density $\rho_c$, for a given mass $M$, some known models [75] will have to be reconsidered.
Chapter 4

New Solutions

We solve our coupled system of differential equations, under two different assumptions, and deduce expressions for the metric functions, and pressures. We then apply boundary conditions to these solutions and deduce all integration constants in terms of parameters that are physically meaningful. We then look at possibilities for using similar methods for finding new solutions.

Following the exposition of the Tolman VII [125] solution in the previous chapter, we now generalize this solution to generate new exact solutions to the Einstein’s interior equations. We feel that Tolman VII is a good candidate for such a generalization procedure since by itself Tolman VII obeys conditions for physical viability. Presumably, generalizations of the solution that maintain this physical viability will be possible, and this is what we attempt to do in this chapter.

This chapter has two major sections. In Section 4.1 we generalize the field equations to include an anisotropic pressure, while maintaining spherical symmetry. This has the advantage of introducing one additional degree of freedom in the types of functions we can posit for the matter quantities, thus making the generalization straightforward. In Section 4.2 we solve the Einstein–Maxwell system by including electric charge in our matter quantities. Charged models might seem like a strange concept since it is expected that astrophysical objects will be charge neutral. However it is still interesting to see what kind of additional structure charge introduces in stellar models. Finally in section 4.4 we tentatively suggest avenues for finding new solutions by using similar methods.
4.1 Uncharged case with anisotropic pressures

In this section, we generalize the Tolman VII solution by introducing an anisotropic pressure. In Appendix A we explain how the energy-momentum tensor \( T_{ab} \) changes under this new assumption: the components of the pressure, which we assumed to be the same in all directions must now be generalized to two different functions, which for intuitive reasons we will call \( p_r \) for the radial pressure component, and \( p_\perp \) for the angular pressure component. Our starting metric functions do not change from the original ones, since we are not relaxing our spherical symmetry axiom. As a result of these, our energy-momentum now becomes

\[
T^i_j = \begin{pmatrix}
\rho & 0 & 0 & 0 \\
0 & -p_r & 0 & 0 \\
0 & 0 & -p_\perp & 0 \\
0 & 0 & 0 & -p_\perp
\end{pmatrix},
\]

and the EFE corresponding to the above reduce to the following set

\begin{align}
\kappa \rho &= e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{1}{r^2} - \frac{Z}{r^2} - \frac{1}{r} \frac{dZ}{dr}, \\
\kappa p_r &= e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{2Z}{rY} \frac{dY}{dr} + \frac{Z}{r^2} - \frac{1}{r^2}, \\
\kappa p_\perp &= e^{-\lambda} \left( \frac{\nu''}{2} - \frac{\nu' \lambda'}{4} + \frac{(\nu')^2}{4} + \frac{\nu' - \lambda'}{2r} \right) = \frac{Z}{Y} \frac{d^2Y}{dr^2} + \frac{1}{2Y} \frac{dY}{dr} \frac{dZ}{dr} + \frac{Z}{rY} \frac{dY}{dr} + \frac{1}{2r} \frac{dZ}{dr}.
\end{align}

We note that equation (4.2c) is different from the previous (3.1c), since as we now have two pressure components, this third equation of this set is in terms of the new pressure. To find the solution of these ODEs, we will follow a similar method to the previous chapter, to be able to get a solution of the same form. In particular the ansatz for the density (3.3) that we used previously will be the same. As a result the first ODE is solved in the exact same way as the previous chapter. The boundary conditions will be expressed in the exact same way as
in chapter 3, a non-intuitive result we will show in due course. Schematically, we have:

\[
\rho = \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] \quad \rightarrow \quad Z(r) = 1 - \left( \frac{\kappa \rho_c}{3} \right) r^2 + \left( \frac{\kappa \mu \rho_c}{5 r_b^2} \right) r^4 =: 1 - br^2 + ar^4. \tag{4.3}
\]

The solution to the second and third equation is complicated by the inequality of the two equations (4.2b) and (4.2c). In Tolman VII, we equated these two equations: (3.1a) and (3.1c), but here we are forced to take the difference between the two, and call the new quantity the "measure of anisotropy" \( \Delta \):

\[
\kappa \Delta = \kappa (p_r - p_\perp) = \frac{Z}{rY} (\frac{dY}{dr}) - \frac{Z}{Y} (\frac{d^2Y}{dr^2}) - \frac{1}{2Y} (\frac{dZ}{dr}) (\frac{dY}{dr}) - \frac{1}{2r} (\frac{dZ}{dr}) + \frac{Z}{r^2} - \frac{1}{r^3}. \tag{4.4}
\]

This equation can be rearranged and simplified into a second order ODE for \( Y \), which can then be solved with our usual series of variable transformations:

\[
2r^2Z \left( \frac{d^2Y}{dr^2} \right) + \left[ r^2 \left( \frac{dZ}{dr} - 2rZ \right) \left( \frac{dY}{dr} \right) + \left[ 2 + 2r^2 \Delta - 2Z + r \frac{dZ}{dr} \right] Y = 0. \tag{4.5}
\]

The second order ODE will have \( \Delta \) as an undetermined function, which when set to zero transforms the ODE into the Tolman VII one for \( Y \) we had solved previously: in this aspect this is a generalization of the Tolman VII solution. The next step in the solution is the variable transformation \( x = r^2 \), where care must be taken to transform the derivatives to the appropriate form. A straight forward derivation yields \( \frac{d}{dr} \equiv 2\sqrt{x} \frac{d}{dx} \), and similarly \( \frac{d^2}{dr^2} \equiv 4x \frac{d^2}{dx^2} + 2 \frac{d}{dx} \). Applying these to the above equation 4.5 results in

\[
2rZ \left( 4x \frac{d^2Y}{dx^2} + 2 \frac{dY}{dx} \right) + \left( 2\sqrt{x} \frac{dY}{dx} \right) \left( 2x^{3/2} \frac{dZ}{dx} - 2\sqrt{x} Z \right) + \left[ 2 + 2x \Delta + \sqrt{x} \left( 2\sqrt{x} \frac{dZ}{dx} - 2Z \right) \right] Y = 0,
\]

which can be rearranged into

\[
8x^2Z \frac{d^2Y}{dx^2} + \left( 4xZ + 4x^2 \frac{dZ}{dx} - 4x^2Z \right) \frac{dY}{dx} + 2 \left( 1 + x\Delta + x \frac{dZ}{dx} - Z \right) Y = 0,
\]
a clear simplification of some of the cross terms appearing in the coefficient of the first derivative of $Y$. At this stage, dividing by $8x^2$ will tidy up our equation into

$$Z \frac{d^2 Y}{dx^2} + \left( \frac{1}{2} \frac{dZ}{dx} \right) \frac{dY}{dx} + \left( \frac{1 + x\Delta + x \frac{dZ}{dx} - Z}{4x^2} \right) Y = 0. \quad (4.6)$$

The second step of the solution procedure involves another variable change from $x$ to $\xi$ which is defined through

$$\xi = \int_0^x \frac{dx}{\sqrt{Z(x)}} \Rightarrow \frac{d\xi}{dx} = \frac{1}{\sqrt{Z}}. \quad (4.7)$$

This induces a change in the $x-$derivatives, so that we have $\frac{d}{dx} \equiv \frac{1}{\sqrt{Z(x)}} \frac{d}{d\xi}$, and $\frac{d^2}{dx^2} \equiv \frac{1}{Z} \frac{d^2}{d\xi^2} - \frac{1}{2Z^{3/2}} \frac{dZ}{dx} \frac{d}{d\xi}$. The actual expression for $\xi$ in terms of $x$ will be derived later on when it becomes useful.

Applying these changes to our differential equation (4.6) results in the elimination of the first derivative term for $Y$, further simplifying the second order ODE:

$$Z \left\{ \frac{1}{Z} \frac{d^2 Y}{d\xi^2} - \frac{1}{2Z^{3/2}} \frac{dZ}{dx} \frac{dY}{d\xi} \right\} + \frac{1}{2} \frac{dZ}{dx} \left( \frac{1}{\sqrt{Z}} \frac{dY}{d\xi} \right) + \left( \frac{1 + x\Delta + x \frac{dZ}{dx} - Z}{4x^2} \right) Y = 0.$$  

At this stage, except for the coefficient of $Y$, we have a simple equation. However from (4.3) we already have expressions for both $Z(x)$ and $\frac{dZ}{dx}$ with which we can reduce that last coefficient into a simple form consisting of our initial parameters only, yielding

$$\left( \frac{1 + x\Delta + x \frac{dZ}{dx} - Z}{4x^2} \right) \rightarrow \left( \frac{1 + x\Delta + x(ax - b) - (1 - bx + ax^2)}{4x^2} \right) = \left( \frac{a}{4} + \frac{\Delta}{4x} \right),$$

so that the ODE to be solved for $Y$ finally becomes

$$\frac{d^2 Y}{d\xi^2} + \left( \frac{a}{4} + \frac{\Delta}{4x} \right) Y = 0. \quad (4.8)$$

This equation would be very easy to solve if we had a constant term for the coefficient in brackets. As mentioned previously, $\Delta$ is a function we can pick and is a measure of anisotropy between the pressures in our model. From spherical symmetry we must have both the radial pressure $p_r$ and the tangential pressure $p_\perp$ be equal at the centre, resulting
in $\Delta$ having to be equal to zero when $x = r^2 = 0$. The energy conditions impose additional constraints on the absolute value that the pressures can take, and we will have to ensure compliance with the energy conditions later when we have the complete expression for both pressures. However, the requirement that $\Delta(r = 0) = 0$ suggests that setting $\Delta = \beta x$ might be a good candidate for a physical solution since one of the constraints is automatically taken care of, while considerably simplifying our ODE. Imposing this results in a simple harmonic ODE:

$$\frac{d^2Y}{d\xi^2} + \left(\frac{a + \beta}{4}\right) Y = 0,$$

whose solutions we can write immediately in terms of $\phi^2 = (a + \beta)/4$ in the following table, which also redirects us to the relevant section where the specific solution is looked into in detail.

| $\phi^2$ | $Y(\xi)$ | Solution’s analysis |
|----------|-----------|---------------------|
| $\phi^2 < 0$ | $c_1 \cosh\left(\sqrt{-\phi^2} \xi\right) + c_2 \sinh\left(-\sqrt{-\phi^2} \xi\right)$ | section 4.1.3 |
| $\phi^2 = 0$ | $c_1 + c_2 \xi$ | section 4.1.1 |
| $\phi^2 > 0$ | $c_1 \cos \left(\phi \xi\right) + c_2 \sin \left(\phi \xi\right)$ | section 4.1.2 |

Table 4.1: The different solutions that can be generated through different values of the parameter $\phi$. The integration constants $c_1$, and $c_2$ are determined by our two boundary conditions.

In the next sections we will analyse the different possibilities offered by this extension to anisotropic pressures, considering the different ones separately.

4.1.1 The $\phi^2 = 0$ case

When $\phi = 0$, the only possibility is for $\beta = -a = -\frac{\kappa \mu \rho}{5r_b}$, which is either negative when all the constants in the previous expression are positive definite: the case we will consider now, or zero when $\mu = 0$. The latter case reduces to the Schwarzschild interior solution on which there is much historical [126, 131] and contemporary literature [34, 129], and so we will not look at it in detail. For the $\beta \neq 0$ case, we have $p_\perp = p_r - \Delta = p_r + ax$, and the angular pressure is thus larger than the radial pressure everywhere but at the centre. We
now apply our two boundary conditions to solve for the integration constants. From last chapter’s arguments, and remembering that the boundary conditions come from imposing matching conditions on the interior and exterior metric through the use of the equation relating pressure and density (3.2) which is unchanged even in the anisotropic case, we have

- \( \frac{dY}{d\xi} \bigg|_{\xi=\xi_b} = \alpha \), where we can compute the \( \xi \)-derivative for \( Y \) from its expression. This results in \( c_2 = \alpha \).

- \( Y \big|_{\xi=\xi_b} = \gamma \Rightarrow c_1 + c_2 \xi_b = \gamma \), as a result of which we have \( c_1 = \gamma - \frac{2\alpha}{\sqrt{b}} \arcoth \left( \frac{1+\gamma}{r_b \sqrt{b}} \right) \).

A plot of the metric functions 4.1 will show the matching of the values and slopes of the metric functions at the radius \( r_b \), as expected from the matching to the Schwarzschild exterior metric.

**Figure 4.1:** Application of the boundary conditions resulting in the value and slope matching of the metric function at \( r = r_b \) for the \( \phi = 0 \) case. The parameter values are \( \rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3} \), \( r_b = 1 \times 10^4 \text{ m} \) and \( \mu = 1 \)
We can now give expressions for all quantities, since our system of equations has been completely solved. Starting with the density ansatz,

\[ \rho(r) = \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right], \quad (4.9) \]

which leads to an expression for the first metric function \( Z(r) \),

\[ Z(r) = 1 - \left( \frac{\kappa \rho_c}{3} \right) r^2 + \left( \frac{\kappa \mu \rho_c}{5r_b^2} \right) r^4. \quad (4.10) \]

The solution of the second metric function after the variable changes and substitutions give

\[ Y(r) = \gamma + \frac{2\alpha r_b}{\sqrt{\kappa \rho_c \mu / 5}} \left[ \text{arcoth} \left( \frac{1 - \sqrt{Z(r)}}{r^2 \sqrt{\kappa \rho_c \mu / 5}} \right) \right] - \text{arcoth} \left( \frac{1 - \gamma}{r_b \sqrt{\kappa \rho_c \mu / 5}} \right), \quad (4.11) \]

where the constants \( \alpha \) and \( \gamma \) are given in terms of the initial set of parameters \( r_b, \rho_c \) and \( \mu \) through

\[ \alpha = \frac{1}{4} \left( \frac{\kappa \rho_c}{3} - \frac{\kappa \rho_c \mu}{5} \right) = \frac{\kappa \rho_c (5 - 3\mu)}{60}, \quad (4.12) \]
\[ \gamma = \sqrt{1 - \left( \frac{\kappa \rho_c}{3} \right) r_b^2 + \left( \frac{\kappa \mu \rho_c}{5r_b^2} \right) r_b^4} = \sqrt{1 + \frac{\kappa \rho_c r_b^2 (3\mu - 5)}{15}}, \quad (4.13) \]
\[ \beta = -a = \frac{-\kappa \rho_c \mu}{5r_b^2}. \quad (4.14) \]

The two pressures can similarly be given in terms of the above variables. The radial pressure can be computed from the second Einstein equation \[4.2b\] in a straightforward manner to yield

\[ \kappa p_r(r) = \frac{2\kappa \rho_c}{3} - \frac{4\kappa \rho_c \mu r^2}{5r_b^2} - \kappa \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + \left( \frac{\kappa \rho_c}{3} - \frac{\kappa \rho_c \mu}{5} \right) \gamma + \frac{2\alpha r_b}{\sqrt{\kappa \rho_c \mu / 5}} \left[ \text{arcoth} \left( \frac{1 - \sqrt{Z(r)}}{r^2 \sqrt{\kappa \rho_c \mu / 5}} \right) \right] - \text{arcoth} \left( \frac{1 - \gamma}{r_b \sqrt{\kappa \rho_c \mu / 5}} \right), \quad (4.15) \]

and similarly the tangential pressure is easily written in terms of the above as

\[ p_\perp(r) = p_r - \beta r^2 = p_r + \frac{\kappa \rho_c \mu}{5r_b^2} r^2. \quad (4.16) \]
This completes the solution, since we have given all the functions in our ODEs in terms of the constants found in our ansatz and our coordinate variable only. If an equation of state for this solution is required, we could invert the density relation (3.3), to get an expression for \( r \) in terms of \( \rho \). Simple substitution in the expressions we have for the pressures (4.15) and (4.16) will then give us the equation of state for both pressures \( p_t(\rho) \), and \( p_\perp(\rho) \), a process similar to what we did in the previous chapter.

4.1.2 The \( \phi^2 > 0 \) case

When \( \phi^2 > 0 \), we must have that \( a + \beta > 0 \), which can only mean that \( \beta > -a \). Since we have an expression for \( a \), we get \( \beta > -\frac{\kappa \mu \rho_c}{2 r_b^5} \), which allows \( \beta \) to have negative values, since the fraction in the last expression is positive definite. We can also write expressions for the derivative of \( Y \) by direct computation, which will allow us to apply boundary conditions to solve for our integration constants as we show now:

\[
\left. \frac{dY}{d\xi} \right|_{\xi = \xi_b} = \phi \left[ c_2 \cos (\phi \xi_b) - c_1 \sin (\phi \xi_b) \right] = \alpha, \quad \text{and solving this results in an equation for } c_1 \text{ and } c_2 \text{ in the form of,}
\]

\[
c_2 \cos (\phi \xi_b) - c_1 \sin (\phi \xi_b) = \frac{\alpha}{\phi}, \quad \text{and,}
\]

\[
Y \left|_{\xi = \xi_b} = \gamma \Rightarrow c_2 \sin (\phi \xi_b) + c_1 \cos (\phi \xi_b) = \gamma.
\]

We solve this coupled system for \( c_1 \) and \( c_2 \) by the usual process of elimination by multiplication by the appropriate trigonometric function, and this yields

\[
c_2 = \gamma \sin (\phi \xi_b) + \frac{\alpha}{\phi} \cos (\phi \xi_b)
\]

\[
c_1 = \gamma \cos (\phi \xi_b) - \frac{\alpha}{\phi} \sin (\phi \xi_b).
\]

A plot of the metric functions (4.2) at this point will show the matching of of the values and slopes of the metric functions at the radius \( r_b \), as expected from the Schwarzschild metric:

The complete solution for the \( Y \)−metric function in this case is thus
Figure 4.2: Application of the boundary conditions resulting in the value and slope matching of the metric function at \( r = r_b \), for the \( \phi > 0 \) case. The parameter values are \( \rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^4 \text{ m} \) and \( \mu = 1 \), with \( \beta \) given in the legend.

\[
Y(r) = \left( \gamma \cos (\phi \xi_b) - \frac{\alpha}{\phi} \sin (\phi \xi_b) \right) \cos \left( \frac{2\phi}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right) + \\
+ \left( \gamma \sin (\phi \xi_b) + \frac{\alpha}{\phi} \cos (\phi \xi_b) \right) \sin \left( \frac{2\phi}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right),
\]

(4.17)

which then allows us to write the matter variables \( p_{\perp} \) and \( p_r \) as

\[
\kappa p_r(r) = \frac{2\kappa \rho_c}{3} - \frac{4\kappa \rho_c \mu^2}{5r_b^2} - \kappa \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + 4\phi \sqrt{1 - br^2 + ar^4} \times \\
\times \left[ \frac{\gamma \sin (\phi \xi_b) + \frac{\alpha}{\phi} \cos (\phi \xi_b)}{\gamma \sin (\phi \xi_b) + \frac{\alpha}{\phi} \cos (\phi \xi_b)} \right] \cos (\phi \xi) - \frac{\gamma \cos (\phi \xi_b) - \frac{\alpha}{\phi} \sin (\phi \xi_b)}{\gamma \cos (\phi \xi_b) - \frac{\alpha}{\phi} \sin (\phi \xi_b)} \sin (\phi \xi)
\]

(4.18)

and

\[
p_{\perp}(r) = p_r - \beta r^2.
\]

(4.19)

The variables in the above expressions for this case are given by:

\[
\alpha = \frac{\kappa \rho_c (5 - 3\mu)}{60}, \quad \beta > -\frac{\kappa \mu \rho_c}{5r_b^2},
\]
\[ \gamma = \sqrt{1 + \frac{\kappa \rho c r_b^2 (3\mu - 5)}{15}}, \quad \phi^2 = \frac{3\beta + 4\kappa \rho c}{12}, \]

which completes the solution. As with the previous examples, and in particular Tolman VII, we can invert the density relation and generate an equation of state.

### 4.1.3 The \( \phi^2 < 0 \) case

When \( \phi^2 < 0 \), we must have that \( a + \beta < 0 \), which can only mean that \( \beta < -a \). Since we have an expression for \( a \), we get \( \beta < -\frac{\kappa \rho c}{\partial r_b} \), which forces \( \beta \) to have negative values only, since the fraction in the last expression is positive definite. We can also write expressions for the derivative of \( Y \) by direct computation, which will allow us to apply boundary conditions to solve for our integration constants as we show now:

- \( \frac{dY}{d\xi} \bigg|_{\xi = \xi_b} = \phi [c_2 \cosh (\phi \xi_b) + c_1 \sinh (\phi \xi_b)] = \alpha \), and solving this results in an equation for \( c_1 \) and \( c_2 \) in the form of, \( c_2 \cosh (\phi \xi_b) + c_1 \sinh (\phi \xi_b) = \frac{\alpha}{\phi} \), and,

- \( Y \big|_{\xi = \xi_b} = \gamma \Rightarrow c_2 \sinh (\phi \xi_b) + c_1 \cosh (\phi \xi_b) = \gamma \).

We solve this coupled system for \( c_1 \) and \( c_2 \) by the usual process of elimination by multiplication by the appropriate trigonometric function, and this yields

\[
\begin{align*}
c_2 &= \frac{\alpha}{\phi} \cosh (\phi \xi_b) - \gamma \sinh (\phi \xi_b) \\
c_1 &= \gamma \cosh (\phi \xi_b) - \frac{\alpha}{\phi} \sinh (\phi \xi_b),
\end{align*}
\]

A plot of the metric functions at this point will show the matching of the values and slopes of the metric functions at the radius \( r_b \), as expected from the Schwarzschild metric in Figure 4.3. The complete solution for the \( Y \)–metric function in this case is thus

\[
Y(r) = \left( \gamma \cosh (\phi \xi_b) - \frac{\alpha}{\phi} \sinh (\phi \xi_b) \right) \cosh \left( \frac{2\phi}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right) + \\
+ \left( \frac{\alpha}{\phi} \cosh (\phi \xi_b) - \gamma \sinh (\phi \xi_b) \right) \sinh \left( \frac{2\phi}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right),
\]

(4.20)
Figure 4.3: Application of the boundary conditions resulting in the value and slope matching of the metric function at \( r = r_b \), for the \( \phi < 0 \) case. The parameter values are \( \rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^4 \text{ m} \) and \( \mu = 1 \), with \( \beta \) given in the legend.

which then allows us to write the matter variable \( p_r \) as

\[
\kappa p_r(r) = \frac{2\kappa \rho_c}{3} - \frac{4\kappa \rho_c \mu r^2}{5r_b^2} - \kappa \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + 4\phi \sqrt{1 - b r^2 + a r^2} \times \\
\left[ \frac{\alpha}{\phi} \cosh \left( \phi \xi_b \right) - \gamma \sinh \left( \phi \xi_b \right) \right] \cosh (\phi \xi) + \left[ \gamma \cosh \left( \phi \xi_b \right) - \frac{\alpha}{\phi} \sinh \left( \phi \xi_b \right) \right] \sinh (\phi \xi)
\]

and \( p_\perp \), the tangential pressure through the above as

\[
p_\perp(r) = p_r - \beta r^2.
\]

The Greek variables in the above expressions for this case are given by:

\[
\begin{align*}
\alpha &= \frac{\kappa \rho_c (5 - 3\mu)}{60} \\
\gamma &= \sqrt{1 + \frac{\kappa \rho_c r_b^2(3\mu - 5)}{15}} \\
\beta &= \frac{-\kappa \mu \rho_c}{5r_b^2} \\
\phi^2 &= \frac{3\beta + 4\kappa \rho_c}{12}.
\end{align*}
\]
which completes the solution. As with the previous examples, and in particular Tolman VII, we can invert the density relation and generate an equation of state.

4.2 Charged case with anisotropic pressures

In this section we investigate electrically charged solutions. As has been noted by numerous authors \[65\ \[72\ \[128\], in the static limit, this does not change the difficulty of solving the EFE, since we add a Maxwell differential equation for the electric charge that can immediately be integrated and incorporated into a global charge that is seen from the outside only through the Reissner-Nordström external metric. The EFE do not change drastically either, and a similar solution procedure to the one already employed can be used to great effect. We will give the full Einstein-Maxwell field equations (EFME) before showing how we solve then to get new solutions:

The energy-momentum tensor $T_{ab}$ for the static electromagnetic field is obtained from the Faraday tensor $F_{ab}$ through

$$
T_{ab}^{\text{EM}} = g_{ac}F^{cd}F_{db} - \frac{1}{4}g_{ab}F^{cd}F_{cd}.
$$

As mentioned in Appendix A the Faraday tensor in our case is

$$
F_{ab} = \begin{pmatrix}
0 & -\frac{qY}{r^2\sqrt{Z}} & 0 & 0 \\
\frac{qY}{r^2\sqrt{Z}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

(4.23)

and this allows us to write the total stress-energy, $T^{\text{Total}}$ as

$$
T^i_j = \begin{pmatrix}
\rho + \frac{q^2}{kr^4} & 0 & 0 & 0 \\
0 & -p_r + \frac{q^2}{kr^4} & 0 & 0 \\
0 & 0 & -p_\perp - \frac{q^2}{kr^4} & 0 \\
0 & 0 & 0 & -p_\perp - \frac{q^2}{kr^4}
\end{pmatrix}.
$$

(4.24)
As a result the EFME become the set

\[
\kappa \rho + \frac{q^2}{r^4} = e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{1}{r^2} - \frac{Z}{r^2} - \frac{1}{r} \frac{dZ}{dr},
\]

(4.25a)

\[
\kappa p_r - \frac{q^2}{r^4} = e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{2Z}{rY} \frac{dY}{dr} + \frac{Z}{r^2} - \frac{1}{r^2},
\]

(4.25b)

\[
\kappa p_{\perp} + \frac{q^2}{r^4} = e^{-\lambda} \left( \frac{\nu''}{2} - \frac{\nu' \lambda'}{4} + \frac{(\nu')^2}{4} + \frac{\nu' - \lambda'}{2r} \right) = \frac{Z}{rY} \frac{d^2Y}{dr^2} + \frac{1}{2Y} \frac{dY}{dr} \frac{dZ}{dr} + \frac{Z}{rY} \frac{dY}{dr} + \frac{1}{2r} \frac{dZ}{dr},
\]

(4.25c)

and as in the Tolman VII case, adding the first two equations to each other results in a simpler equation that will make applying boundary conditions easier:

\[
\kappa (p_r + \rho) = e^{-\lambda} \left( \frac{\nu'}{r} + \frac{\lambda'}{r} \right) = \frac{2Z}{rY} \frac{dY}{dr} - \frac{1}{r} \frac{dZ}{dr}.
\]

(4.26)

We go through the same procedure to simplify these equation, except for a crucial additional step: instead of using only equation (4.3) as the initial ansatz, we revert to Tolman’s initial ansatz about the metric function. He used \( Z(r) = 1 - br^2 + ar^4 \), as we shall, the reason being that by not using a density ansatz right away we do not have to posit a charge ansatz either, leaving us free until we have an idea about the physics. However since we already have an interpretation for the density function we have been using, we also wish to keep this. To bridge these concerns we segue into some physical considerations first.

Considering that we have a spherical object, classical physics suggests that most of the charge should be lying on the outer surface of the sphere. In GR since charge also contributes to the gravitation, we expect at least something similar to the classical picture, although we would expect non-zero but lesser charge in the interior. A good guess would be to have the charge be a monotonically increasing function of the radial coordinate, since then most of the charge is concentrated towards the surface. Additionally having the charge be a power of the radial coordinate is extremely convenient in finding a solution to our differential equation as we will see. Therefore for the time being, we append to our initial density the ansatz, \( q(r) = kr^n \), with \( n > 0 \).
This initial ansatz for $Z$ can be fed into the RHS of our first differential equation (4.25a), which results in
\[ \kappa \rho + \frac{q^2}{r^4} = 3b - 5ar^2. \]

Consistency, and the desire to keep the procedure to solving this system of equation the same as before then demands that the LHS of the differential equation also be a quadratic function with zero linear term. This can be seen as the “reason” for postulating the density (4.3) we did before, which had this same structure. Also, due to the structure of this differential equation we are forced to either pick either $q(r) = kr^2$, in which case we will have
\[ 3b - 5ar^2 = (\kappa \rho_c + k^2) - \frac{\kappa \rho_c \mu}{r_b^2} r^2, \]
or pick $q(r) = kr^3$, which results in
\[ 3b - 5ar^2 = \kappa \rho_c - \left(\frac{\kappa \rho_c \mu}{r_b^2} - k^2\right) r^2. \]

We can then read off $a$ and $b$ in either case, however if we continue our procedure of defining an anisotropy measure, and performing the same variable changes shown in the previous sections to simplify the equation for the $Y$ metric function, we quickly find out that the first choice of $q(r) = kr^2$ yields a differential equation for $Y$ that is not soluble with elementary functions\(^1\) contrary to our initial wish. We therefore discard this choice and instead restrict ourselves to $q(r) \propto r^3$ only. Then the variable changes go through as before and the differential equation for $Y$ reduces to:
\[ \frac{d^2Y}{d\xi^2} + \frac{Y}{4} \left( a + \frac{\Delta}{x} - \frac{2q^2}{x^3} \right) = 0. \] (4.27)

From this equation, it is easy to see that setting the value of $q$ to zero results in the uncharged anisotropic second order differential equation we had previously. Before we attempt to solve

\(^1\)The coefficient of $Y$ in the second order ODE after variable changes still contains a $1/r^2$ term, turning the problem into a variable coefficient one. Once additional assumptions about $a$ have been made, a solution in terms of hypergeometric functions is possible, but the assumption about $a$ renders the solution physically uninteresting.
this equation however we have to discuss the boundary and the junction conditions. As mentioned previously, the correct exterior solution to be matched in the Einstein-Maxwell case in the external vacuum Reisner-Nordström metric. This metric in Schwarzschild-type coordinates is given by

\[
\begin{align*}
\text{d}s^2 &= \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \text{d}t^2 - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} \text{d}r^2 - r^2 \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \right),
\end{align*}
\]

(4.28)

where \(M\) is the mass function and \(Q\) is the total electric charge, both enclosed by the interior metric and perceived to external observers. These quantities (see Appendix A or [65] for details) are given by

\[
\begin{align*}
M &= 4\pi \int_0^{r_b} \left( \rho(r) + \frac{q^2(r)}{8\pi r^4} \right) r^2 \text{d}r \quad \text{and}, \quad Q = 4\pi \int_0^{r_b} \sigma(r) \sqrt{Z(r)} r^2 \text{d}r, \quad (4.29)
\end{align*}
\]

where \(\rho(r)\) is the mass density associated with the interior solution, and we similarly define \(\sigma(r)\) as the charge density associated with the interior solution, and which is related to the \(q(r)\) we have in our energy-momentum tensor \(T^\text{EM}_{ab}\) by construction through \(Q^2(r_b) = q^2(r_b)\). This last equation also encodes the charged part of the junction conditions required of our differential equations. We note here that the mass function has been defined differently here than in the previous cases. A discussion on why this is the case can be found in the appendix [A].

We are now in a position to be able to solve the differential equation for \(Y\). As seen previously the simple-harmonic form of (4.27) under certain conditions allow for simplifications. Again we define \(\Phi\), a temporary variable different from the previous sections through

\[
4\Phi^2 = a + \frac{\Delta}{x} - \frac{2q^2}{x^3},
\]

requiring that \(\Phi\) be a number will ensure that the solution of our differential equation be simple. Requiring \(\Delta = 0\), and thus “switching off” anisotropy is the easiest thing to try, and doing so leaves us with

\[
4\Phi^2 = a - \frac{2q^2}{x^3}.
\]
Here, \( q = 0 \) gives us back Tolman’s solution as expected and is a solution we already considered. However \( 2q^2 = 2k^2x^3 \) looks promising since this would allow \( 4\Phi^2 = a - 2k^2 \), a pure number, and be of the same form \( q \propto r^3 \) we required before. However the solution by Kyle and Martin \[72\] reduces to this same assumption and is analysed in their article, so that we must look elsewhere, and consider the case \( \Delta \neq 0 \). This case allows for two immediate possibilities:

- **requiring** \( \Delta x = \frac{2q^2}{x^3} \) effectively “anisotropises” the electric charge allowing the latter to contribute to the anisotropy only, and considerably simplifying the solution to \( Y \). We will look at this solution is Section\[4.2.1\]

- If instead we ask that \( \Delta = \beta x \), and \( 2q^2 = 2k^2x^3 \), we get \( 4\Phi^2 = a + \beta - 2k^2 \), which allows an analysis very similar to what we did in the previous section since \( \Phi^2 \) can then be of either sign. We will look at these possibilities in Sections\[4.2.2, 4.2.3, 4.2.4\]

As with the Tolman VII case we need boundary conditions to find a complete closed form solution. We implement this next, and determine the integration constants in our solutions. The boundary conditions here are not very different from the previous case. We recall that in Tolman VII we required that the pressure at the fluid–vacuum interface vanish and that the metric coefficients be compatible with the Schwarzschild coefficients through \( (3.4) \). Here the first requirement is the same when applied to the radial pressure only, and the compatibility of metric coefficients is with Reissner–Nordström instead:

\[
p_r(r_b) = 0, \quad \text{and,} \quad Z(r_b) = 1 - \frac{2M}{r_b} + \frac{Q^2}{r_b^2} = Y^2(r_b). \tag{4.30a}
\]

Considering \( (4.26) \), we find an expression for the radial pressure as

\[
\kappa p_r = \frac{2Z}{rY} \frac{dY}{dr} - \frac{1}{r} \frac{dZ}{dr} - \kappa \rho \quad \text{as} \quad r \to x \quad \frac{4Z}{Y} \frac{dY}{dx} - \frac{2}{r} \frac{dZ}{dx} - \kappa \rho \quad \text{as} \quad x \to \xi \quad \frac{4\sqrt{Z}}{Y} \frac{dY}{d\xi} - \frac{2}{r} \frac{dZ}{dx} - \kappa \rho.
\]
Applied at the boundary \( r = r_b \), conditions (4.30) result in

\[
\kappa p_r(r_b) = 0 = 4 \sqrt{Z(r_b)} \left. \frac{dY}{d\xi} \right|_{\xi = \xi_b} - 2 \left. \frac{dZ}{dx} \right|_{x = x_b} - \kappa \rho(r_b),
\]

so that we have an “easy-to-use” equivalent condition on the derivative of \( Y \),

\[
\kappa \rho(r_b) = 4 \left. \frac{dY}{d\xi} \right|_{\xi = \xi_b} - 2 \left. \frac{dZ}{dx} \right|_{x = x_b} \Rightarrow \left. \frac{dY}{d\xi} \right|_{\xi = \xi_b} = \frac{1}{4} \left[ \frac{\kappa \rho_c}{3} - \frac{\kappa \rho_c \mu}{5} - \frac{4 k^2 r_b^2}{5} \right] =: \alpha. \tag{4.31}
\]

For the second condition we re-express equation (4.30b) in terms of \( Y \) as

\[
Y(r_b) = \sqrt{Z(r_b)} = \sqrt{1 + \frac{\kappa \rho_c r_b^2 (3\mu - 5)}{15} - \frac{k^2 r_b^4}{5}} =: \gamma \tag{4.32}
\]

and subsequent application of the value and slope condition on the \( Y \) metric function form a Cauchy boundary pair and results in unique integration constants for the \( Y \) metric function in terms of the auxiliary constants \( \alpha \) and \( \gamma \), defined through the above equality.

### 4.2.1 Anisotropised charge

In this section we analyse the solution to the EFME if we require that the electric charge and anisotropy be related to each other through the relation \( \Delta = 2 (q/x)^2 \), where we take the functional form \( q \propto r^3 = kr^3 \) as mentioned before. This particular choice simplifies the differential equation for our \( Y \) metric function allowing us to write an expression for the solution analogous to the Tolman VII solution for \( Y \) directly as

\[
Y(\xi) = c_1 \cos (\Phi \xi) + c_2 \sin (\Phi \xi), \quad \text{with} \quad \Phi = \sqrt{\frac{a}{4}}. \tag{4.33}
\]

However we have to keep in mind that this solution is fundamentally different from Tolman VII which was a solution to the Einstein’s system of equation and not the Einstein–Maxwell system. This fact comes in through three different ways

1. The charge in this system is non-zero, unlike the Tolman VII solution, where \( Q = 0 \).

2. The presence of anisotropic pressure in the solution means that \( p_\perp \) is not the same as the radial pressure \( p_r \). This is clear if we remember that \( \Delta \neq 0 \) here.
3. Also, this solution will have to be matched to the Reissner-Nordström metric outside
the sphere, as opposed to the Schwarzschild solution for Tolman VII.

If we take care to ensure these conditions, we have a fully-fledged new solution to the EMFE,
onto which we can apply boundary conditions (4.31) and (4.32).

• The first condition on the derivative results in

\[
\frac{dY}{d\xi}\bigg|_{\xi=\xi_b} = \Phi [c_2 \cos (\Phi \xi_b) - c_1 \sin (\Phi \xi_b)] = \alpha,
\]

which can be rearranged to yield an equation for \(c_1\) and \(c_2\) in terms of previously
defined constants: \(c_2 \cos (\Phi \xi_b) - c_1 \sin (\Phi \xi_b) = \frac{\alpha}{\Phi}\).

• The second condition also gives us a similar equation:

\[Y(r_b) = c_1 \cos (\Phi \xi_b) + c_2 \sin (\Phi \xi_b) = \gamma,\]

and together this pair of equations can be solved for \(c_1\) and \(c_2\) through simple algebraic
manipulation to give

\[
c_2 = \gamma \sin (\Phi \xi_b) + \frac{\alpha}{\Phi} \cos (\Phi \xi_b)
\]

\[
c_1 = \gamma \cos (\Phi \xi_b) - \frac{\alpha}{\Phi} \sin (\Phi \xi_b),
\]

A plot of the metric functions show us that indeed the conditions stated above are satisfied.
The complete solution where the anisotropy and the charge compensate for each other thus
becomes

\[
Y(r) = \left(\gamma \cos (\Phi \xi_b) - \frac{\alpha}{\Phi} \sin (\Phi \xi_b)\right) \cos \left(\frac{2\Phi}{\sqrt{a}} \coth^{-1} \left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}}\right)\right) +
\]

\[
+ \left(\gamma \sin (\Phi \xi_b) + \frac{\alpha}{\Phi} \cos (\Phi \xi_b)\right) \sin \left(\frac{2\Phi}{\sqrt{a}} \coth^{-1} \left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}}\right)\right),
\]

which then allows us to write the matter variables \(p_{\perp}\) and \(p_r\) as
Figure 4.4: Application of the boundary conditions resulting in the value and slope matching of the metric function at \( r = r_b \), for \( \Phi \neq 0 \), but where anisotropy compensates the charge. The parameter values are \( \rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^4 \text{ m} \) and \( \mu = 1 \)

\[
\kappa p_r(r) = \frac{2\kappa \rho_c}{3} - \frac{4}{5} \left( \frac{\kappa \rho_c \mu}{r_b^2} - k^2 \right) r^2 - \kappa \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + 4\Phi \sqrt{1 - br^2 + ar^4} \times \\
\times \left\{ \left[ \gamma \sin (\Phi \xi_b) + \frac{a}{\Phi} \cos (\Phi \xi_b) \right] \left[ \cos (\Phi \xi) - \left[ \gamma \cos (\Phi \xi_b) - \frac{a}{\Phi} \sin (\Phi \xi_b) \right] \sin (\Phi \xi) \right] \right. \\
\left. \left[ \gamma \sin (\Phi \xi_b) + \frac{a}{\Phi} \cos (\Phi \xi_b) \right] \sin (\Phi \xi) + \left[ \gamma \cos (\Phi \xi_b) - \frac{a}{\Phi} \sin (\Phi \xi_b) \right] \cos (\Phi \xi) \right\}, \tag{4.35}
\]

and

\[
p_\perp(r) = p_r - \Delta = p_r - 2k^2r^2 \tag{4.36}
\]

The variables in the above expressions for this case are given by:

\[
\alpha = \frac{\left( \kappa \rho_c (5 - 3\mu) - 12k^2r_b^2 \right)}{60}, \quad \Delta(r) = 2k^2r^2 = \frac{qr}{2k}, \\
\gamma = \sqrt{1 + \frac{\kappa \rho_c r_b^2 (3\mu - 5)}{15} - \frac{k^2r_b^2}{5}}, \quad \Phi^2 = \frac{1}{4} \left( \frac{\kappa \rho_c \mu}{r_b^2} - k^2 \right),
\]

which completes the solution. As can be seen, we could express the solution in terms of \( q \), or \( \Delta \) exclusively as expected, since these two functions are not independent in this particular
solution. As with the previous example, we can invert the density relation and generate an equation of state. The total mass and charge of the object modelled by this solution is obtained through \((4.29)\), and for this particular case, these equations simplify to

\[ M = 4\pi \rho_c r_b^3 \left( \frac{1}{3} - \frac{\mu}{5} \right) + \frac{k^2 r_b^5}{10}, \quad \text{and,} \quad Q = k r_b^3. \tag{4.37} \]

The last equation can be used to determine the charge density \(\sigma(r)\), since from \((4.29)\) we have

\[ \int_0^{r_b} \bar{r}^2 \, d\bar{r} \left[ 4\pi \sigma(\bar{r}) \sqrt{Z(\bar{r})} \right] = Q = k r_b^3 = \int_0^{r_b} \bar{r}^2 \, d\bar{r} \left[ 3k \right]. \]

Direct comparisons of terms yield the charge density

\[ \sigma(r) = \frac{3k}{4\pi \sqrt{Z(r)}}. \tag{4.38} \]

This completes the solution for this case. We now turn to the case where we have both charge and anisotropy independently of each other. As we mentioned previously, this will require a thorough analysis of the different combinations of charge and anisotropy, and how those conspire to change the character of the second differential equation we have.

The full Anisotropy and charged solution

Inspired by the previous sections, and building upon all the simplifications and discussions so far we look directly at the second order differential equation for the \(Y\) metric function in the form of \((4.27)\). This equation contains a number of assumptions, all of which we have discussed before. Of particular interest in finding a general solution will be the bracketed terms since different values or functions in the brackets will lead to fundamentally different solution type for \(Y\) independently of the form of \(Z\). As mentioned earlier also, for simplicity we pick functions for \(\Delta\) that give pure numbers for \(\Delta/x\), and the form of \(q\) being determined previously through the choice of \(Z\) also gives us a pure number for \(q/x^3\). These choices are reflected in the simplified form of equation \((4.27)\) which becomes

\[ \frac{d^2Y}{d\xi^2} + \frac{Y}{4 \left( a + \beta - 2k^2 \right)} = \frac{d^2Y}{d\xi^2} + \Phi^2 Y = 0. \tag{4.39} \]
The only choice remaining for the different forms of $Y$ thus depends on the overall sign of the term in brackets, $\Phi^2$. In this section we will provide conditions and the form of the complete solutions for the different possibilities in the three sub-sections below.

4.2.2 The $\Phi^2 = 0$ case

The fact that the coefficient of $Y$ in equation (4.39) contains terms of either sign immediately points us to the possibility of choosing the terms to annihilate the bracket completely. For this to happen we have to choose $2k^2 = a + \beta$, somehow making the charge contribution to be compensated by the anisotropy (through $\Delta$, and hence $\beta$) and density (through $a$) to yield the simplest anisotropic charged solution of this class. This choice is the crux of this special solution, allowing us to express the anisotropy measure $\Delta \propto \beta$ in terms of the charge $q \propto k$.

Since the term in brackets vanishes, the solution for $Y$ is the simple linear $Y = c_1 + c_2 \xi$, with $c_1$ and $c_2$ our integration constants. Applying boundary conditions on this solution then results in

$$\left. \frac{dY}{d\xi} \right|_{\xi=\xi_b} = c_2 = \alpha := \frac{1}{4} \left( \frac{\kappa \rho_c}{3} - \frac{3\kappa \rho_c \mu}{11} - \frac{4r_b^2 \beta}{11} \right),$$

and

$$c_1 + c_2 \xi_b = \gamma := \sqrt{1 + r_b^2 \kappa \rho_b \left( \frac{2\mu}{11} - \frac{1}{3} \right) - \beta r_b^4},$$

which can be solved together algebraically to give the value of $c_1$. This completes the solution for $Y$ in this particular case.

The $Z$ metric function still fixed by the Tolman assumption is $Z = 1 - br^2 + ar^4$, with however different values of $a$, and $b$ than previously. In this particular case these are given by

$$a = \frac{2}{11} \left( \frac{\kappa \mu \rho_c}{r_b^2} - \frac{\beta}{2} \right), \quad \text{and} \quad b = \frac{\kappa \rho_c}{3}.$$

Clearly in this case, because of the equation connecting $\beta$, $a$ and $k$, we can express these
Figure 4.5: Application of the boundary conditions resulting in the value and slope matching of the metric function at \( r = r_b \), for the \( \Phi = 0 \) case. The parameter values are \( \rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^4 \text{ m} \) and \( \mu = 1 \).

Constants in terms of each other, and require only two to completely specify the solution. We show this feature, and the consistent matching boundary in figure 4.5.

Once we have the two metric functions, all other quantities are determined, in particular the radial pressure \( p_r \) is given by

\[
p_r = \frac{1}{\kappa} \left[ \frac{4c_2\sqrt{1 - br^2 + ar^4}}{c_1 + c_2\xi} + 2b - 4ar^2 \right] - \rho(r),
\]

and the tangential pressure \( p_\perp \), in turn is \( p_\perp = p_r - \Delta/\kappa \), giving

\[
p_\perp = p_r - \frac{\beta x}{\kappa}.
\]

The mass \( M \) and charge \( Q \) seen from the exterior, which are still given by (4.37) result in

\[
M = 4\pi \rho_c r_b^3 \left( \frac{1}{3} - \frac{7\mu}{55} \right) + \frac{\beta r_b^5}{22}, \quad \text{and,} \quad Q = r_b^3 \sqrt{\left( \frac{5\beta c}{11} + \frac{\kappa\mu \rho_c}{11r_b^2} \right)}.
\]

This completes the solution for this particular case, and a summary of all the functions and constants used in this results is given in Appendix B.
4.2.3 The $\Phi^2 < 0$ case

For this to happen we need $(a + \beta - 2k^2)/4 < 0$, turning our ODE for $Y$ into a simple harmonic type equation with the “wrong” sign. As a result we expect a solution in terms of hyperbolic functions, in this case given by

$$Y = c_1 \cosh (\Phi \xi) + c_2 \sinh (\Phi \xi).$$

In this particular case, we will not have a simplification wherein the charge could be compensated completely by the anisotropy or mass, and we are forced to deal with all three components. We however have that the charge contribution will exceed the mass and anisotropy contribution (since $2k^2 > a + \beta$) and this lead us to believe that such a solution has very little chance of being physical. We will however reconsider it in detail and come to a conclusion on its viability as a physical solution later.

We apply boundary conditions to this solution to obtain the values of the constants $c_1$ and $c_2$ through

1. $\left. \frac{dY}{d\xi} \right|_{\xi=\xi_b} = \Phi [c_2 \cosh (\Phi \xi_b) + c_1 \sinh (\Phi \xi_b)] = \alpha$, and

2. $Y(\xi_b) = c_2 \sinh (\Phi \xi_b) + c_1 \cosh (\Phi \xi_b) = \gamma$.

Then using a procedure very similar to that of previous sections we obtain for the integration constants

$$c_2 = \frac{\alpha}{\Phi} \cosh (\Phi \xi_b) - \gamma \sinh (\Phi \xi_b), \quad (4.40)$$

$$c_1 = \gamma \cosh (\Phi \xi_b) - \frac{\alpha}{\Phi} \sinh (\Phi \xi_b) \quad (4.41)$$

We show the matching boundary conditions at the boundary in figure 4.6 and note that in this case we need both $\beta$ and $k$ to completely specify one particular solution.

This then completes the solution for the $Y$ metric coefficient.
Figure 4.6: Application of the boundary conditions resulting in the value and slope matching of the metric function at $r = r_b$ for $\Phi < 0$. The parameter values are $\rho_c = 1 \times 10^{18}$ kg $\cdot$ m$^{-3}$, $r_b = 1 \times 10^4$ m and $\mu = 1$

4.2.4 The $\Phi^2 > 0$ case

For this to happen we need $(a + \beta - 2k^2)/4 > 0$, turning our ODE for $Y$ into a simple harmonic type equation. As a result we expect a solution in terms of trigonometric functions, in this case given by

$$Y = c_1 \cos (\Phi \xi) + c_2 \sin (\Phi \xi).$$

In this particular case, we will not have a simplification wherein the charge could be compensated completely by the anisotropy or mass, and we are forced to deal with all three components. We however have that the charge contribution will be less than the mass and anisotropy contribution (since $2k^2 < a + \beta$) and this lead us to believe that this will be the most promising physically acceptable candidate in terms of new solutions. We will investigate this solution, and the remaining ones, in detail and come to a conclusion on their viability as a physical solution later.
We apply boundary conditions to this solution to obtain the values of the constants \( c_1 \) and \( c_2 \) through

1. \( \frac{dY}{d\xi} \bigg|_{\xi=\xi_b} = \Phi \left[ c_2 \cos (\Phi \xi_b) - c_1 \sin (\Phi \xi_b) \right] = \alpha \), and

2. \( Y(\xi_b) = c_2 \sin (\Phi \xi_b) + c_1 \cos (\Phi \xi_b) = \gamma \).

Then using a procedure very similar to that of previous sections we obtain for the integration constants

\[
\begin{align*}
    c_2 &= \gamma \sin (\Phi \xi_b) + \frac{\alpha}{\Phi} \cos (\Phi \xi_b), \\
    c_1 &= \gamma \cos (\Phi \xi_b) - \frac{\alpha}{\Phi} \sin (\Phi \xi_b). 
\end{align*}
\]

(4.42) \hspace{1cm} (4.43)

We show the matching boundary conditions at the boundary in figure 4.7 and note that in this case also we need both \( \beta \) and \( k \) to completely specify one particular solution.

**Figure 4.7:** Application of the boundary conditions resulting in the value and slope matching of the metric function at \( r = r_b \) for \( \Phi < 0 \). The parameter values are \( \rho_c = 1 \times 10^{18} \text{ kg m}^{-3} \), \( r_b = 1 \times 10^4 \text{ m} \) and \( \mu = 1 \).
4.3 Possibilities for other solutions

While keeping the ansatz for $Z$ fixed, but adding both anisotropy and charge to the system of equations, we managed to tease out new solutions for the EMFE. From working with the equations it is clear to us that since the choices for the charge function are not arbitrary if we want to maintain the form of $Z$, we can only realistically modify the anisotropy choice. We note here that the crux of our solution finding method stems from equation (4.27), which we then convert by judicious choices to a simple harmonic equation with no forcing or damping. Being restricted by the charge $q$ which has to be a cubic function not only for the existence of a simple $Y$ solution, but also crucially for $Z$, we can isolate this part of the equation immediately into

$$\frac{d^2 Y}{d\xi^2} - \frac{k^2}{2} Y + \frac{Y}{4} \left( a + \Delta \frac{x}{x} \right) = 0.$$ 

Of course picking uncharged solutions does away with both $k$ and $q$, and if we want uncharged solutions, this is the way we would proceed. However, if we want charged solutions, we will have to modify the terms in brackets in such a way as to keep a simple form for the $Y$ ODE.

The following discussion will to be heavily influenced by choosing linear ODEs with constant coefficients that are straightforward generalizations of the harmonic oscillator equation. We hope that this will give a simple way of extending this type of work to larger classes of physically relevant solutions to spherical static stars.

Adding linear first derivative terms in the ODE for $Y$ we could presumably posit more complicated forms for $\Delta$, for example $\Delta = f(\xi, Y; \frac{dY}{d\xi})x/Y$, for some particular choice of the $f$ function. However we have to keep in mind the criterion that $\Delta$ has to satisfy: it has to vanish at $x = r = \xi = 0$, If we manage to pick $f$ such that this is true, we will get other new solutions, with possibly new features to be explored.

As an example for this approach we pick $f$ to be $f = g \frac{dY}{d\xi}$, for some constant $g$. This converts
our undamped ODE into a damped one, whose solutions can be classified according to the schemes usual to solving second order ODEs of that form. Judicious choice of the value of $g$ will then ensure that the discriminant of the ODE is in the appropriate range to admit exponentially decaying envelopes to sinusoids (the usual characteristic of damped harmonic systems,) for the metric function $Y$. The difficulty in this type of approach will then be in ensuring that when the system has been solved that both $p_r$ and $p_\perp$ behave in physically expected ways.

Picking $f$ to be a forcing-type term as is usually encountered in forced electrical oscillators, will yield another potential class of solutions. Furthermore the frequency of the forcing term could be tuned to different values depending on how we want $\Delta$ to behave. As an example of this, we could pick $f = g \sin (\sqrt{\frac{a^2}{4} - k^2 + \epsilon \xi})$, so that we force our solution for $Y$ around its natural frequency, depending on the exact value of $\epsilon$. Again we would have to ensure that the behaviour of the physical variables be consistent with our starting assumptions, but hopefully this should be possible by tuning the frequency of the forcing, or by restricting the value of key parameters like $a$ or $k$. The fact that the latter two parameters have different signs should be helpful in this endeavour.

We have provided two distinct examples of how new solutions could be generated from our work, and clear paths to checking consistency of the new solutions. We will not look at any of these newer solutions in detail or check whether they have already been discovered, or even discuss their stability, but these are things that would have to be done if they are to be used for modelling physical objects.

### 4.4 Conclusion

In this chapter we provided the generalization we used to extend the Tolman VII ansatz models involving electric charge and anisotropic pressures. We computed exact analytic so-
olutions for different cases, and valid for different values of charge, and anisotropy, recovering the original solutions in degenerate cases, as expected. We also applied boundary conditions on the metric functions $Y$ and $Z$ to complete the closed form solutions in terms of variables that can be \textit{physically} interpreted. We obtained the expressions for the matter variables: the density $\rho$, the pressures $p_r$ and $p_\perp$, the electric charge $Q$, and the mass $M$ for our models. We did not show or mentions possible issues like stability (see Chapter 5) or divergences and non physicality in the matter variable (see Chapter 6) waiting for the next chapters for these clarifications. This chapter should be regarded as the mathematical component of solution finding, and model building for our solutions. The physics \textit{per-se} will be in the analysis Chapter 6 mostly where we will provide conditions and applicability criteria for each solution in detail, and predict measurables like masses, radii, and charges for our models. Comparisons with recent observations will also be done then.
Chapter 5

Stability analysis

We investigate the general stability theory of spherically static and symmetric space-times, as applied to stellar objects.

Spherically static and symmetric objects have been studied, and their stability analysed for quite some times under different circumstances. The general theory of stability in relativity is made complicated since many variables can change at the same time. Therefore maintaining consistency can be a difficult task. The most complete derivation, that of Chandrasekhar will be extended to include the case of anisotropic pressures, and electric charges in the following sections.

5.1 Introduction

The stability analysis of solutions to Einstein’s equation has a long history. If these solutions are to be used for physical modelling applications, the need to demonstrate that the solution is indeed stable becomes even more important. Global existence and uniqueness of solutions are usually the other aspects of solutions that are deemed to be as important as stability, but while the former two have been shown to be true, global stability of solutions is still an open question.

In this chapter we aim to show a very restricted version of stability: we plan to show that the solutions we have presented so far are indeed stable locally. Global stability issues are not considered in this Chapter. We proceed by first analysing some heuristic methods of determining stability. Since these produce contradictions we then continue with a full-blown linear perturbation analysis of our solutions.
5.2 Heuristics

Stability analysis based on perturbing the governing equations is usually a lengthy process, even when the equations we deal with are simple. In relativity the differential equations we start with are not simple, and are also coupled. Thus perturbing these equations and finding the linear stability of the system requires lengthy calculations. Over the years a number of heuristics have been developed to determine whether a relativistic system will be stable or not. These heuristics work most of the time, however there do exist cases where they do not hold, and a formal proof of linear stability is the only sure way of determining stability.

5.2.1 The static stability criterion

A heuristic based on [56] and widely used in the literature [55] states that for a star to be stable, it has to satisfy

\[
\frac{dM}{d\rho_c} > 0. \tag{5.1}
\]

However, as noted in many places [55, 56, 135], this is only a necessary condition, and is not sufficient to ensure stability, hence our classification of it as a heuristic.

In our case, we only ever have one expression for the density, and hence the mass is always the same function, given by equations (3.5), (4.29), and (6.1). The latter two equations include the mass contribution from the electric charge too, but that does not change how we implement this condition. Given these equations, since \( M = m(r_b) \), by taking the derivative of the latter we obtain

\[
\frac{dM}{d\rho_c} = 4\pi r_b^3 \left( \frac{1}{3} - \frac{\mu}{5} \right),
\]

onto which we can impose the positivity condition quite easily to yield

\[
\mu < \frac{5}{3}, \tag{5.2}
\]

since \( r_b > 0 \) for all stars consider. The above condition (5.2) is automatically satisfied since our starting assumption on \( \mu \) was that is was never going to be more than unity, ensur-
ing that at least this heuristic is always satisfied by all our solutions, anisotropy or charge notwithstanding.

5.2.2 The Abreu–Hernández–Núñez (AHN) criterion

This heuristic is based on [1], which analyses the “cracking” instability in anisotropic pressure models: the precise case we are dealing with in our solutions. The method consists in comparing the speed of pressure waves in the two principal directions of the spherically symmetric star: the radial sound speed with the tangential sound speed, and then based on those values at particular points in the sphere, we could potentially conclude whether the model is stable or unstable under cracking instability.

Cracking as a concept was introduced previously by Herrera in 1992. It involves the possibility of “breaking up” the fluid sphere due to the appearance of total radial forces of different signs, and hence in different directions, at different points in the star. It should be mentioned that this has never been observed, but that under suitable physical assumptions, it is a likely scenario, and was investigated as such in both [58] and [1]. This process is potentially a source of instability and is characterized most easily through the speed of pressure waves.

The main message of [1] is that if the tangential speed of pressure wave, \( v_{s\perp}^2 = \frac{d\rho}{d\rho} \), is larger that the radial speed of pressure waves, \( v_{sr}^2 = \frac{d\rho}{d\rho} \), then this could potentially result in cracking instabilities to occur in the star, rendering the latter unstable.

We shall explicitly check whether this occurs in our models, and hence classify our solutions according to the AHN scheme. We would normally start by testing this condition on the Tolman VII solution, however because “cracking” only occurs in anisotropic models [58], Tolman VII is automatically stable under “cracking” instabilities. In the other solutions that we have constructed, the quantity that becomes important is \( \Delta \), since it measures the difference in the tangential and radial pressures (4.4). The AHN condition then reduces to

\[
v_{sr}^2 - v_{s\perp}^2 < 0 \quad \Rightarrow \quad \frac{d\Delta}{d\rho} < 0.
\]

(5.3)
The assumption for $\Delta$ in all the new solution we presented has been that $\Delta = \beta x$, so that the above condition simplifies to

$$\beta \frac{dx}{d\rho} < 0 \implies \frac{\beta}{d\rho} \frac{1}{dx} < 0.$$

Since the density expression we use is the same in all the solutions, we can easily simplify the latter equation through (4.9), and remembering that $x = r^2$, we finally get

$$-\frac{\beta r^2}{\mu} < 0.$$

Since all the constants, except for $\beta$, in the above expression are positive definite, we have a prescription on the latter from the AHN prescription: we will have no cracking instability in our solutions if

$$\beta > 0.$$  \hspace{1cm} (5.4)

This concludes the application of this method on our solution. Once we start analysing these solutions in Chapter 6, we can impose the condition on $\beta$ to ensure no cracking instabilities.

5.2.3 Ponce De Leon’s criterion

This method is mostly concerned with the behaviour of the Weyl tensor for the solution, to conclude whether a certain model is more or less stable than another comparable model. It should be noted that this is a comparative method: nothing is said about the absolute stability: only the relative stability as compared to another model/solution can be obtained.

The exact method of performing this comparison was given in [106]. This method starts by calculating a function of the metric variables, called $W$, in [106]. This function, defined in terms of the $\lambda$ and $\nu$ metric variables in the original article, is given by

$$W(r) := \frac{r^2 Z'(Y - rY') + 2rZ(rY' - Y - r^2Y'') + 2rY}{12Y}$$  \hspace{1cm} (5.5)

in our metric variables $Z$ and $Y$, the primes (') denoting derivatives with respect to $r$. Through the use of Einstein’s equations, this purely geometrical quantity can be rewritten.
in terms of the matter variables. The complete derivation of this equivalence is in Ref \[106\],
and we will not give it here, but the result reads:

\[ W(r) \equiv m(r) - \frac{4\pi r^3}{3} \left( T^t_t + T^r_r - T^\theta_\theta \right). \] (5.6)

In addition to being easier to calculate than (5.5), this expression (5.6) can be interpreted quite
simply, particularly in our variables. Plugging in expressions of the energy momentum tensor
from equation (4.24), and the mass function from (4.29) for the most general expressions of
these quantities, we obtain

\[ W(r) = \frac{4\pi r^3}{3} \left\{ r^2 \left[ \frac{2}{5} \left( \rho c^2 \right) - \frac{3k^2}{8\pi} \right] + \left( \frac{k^2}{8\pi} + \Delta \right) \right\}, \] (5.7)

where we note that because of the different signs associated with the terms, \( W(r) \) can be of
either sign. The stability argument then proceed by comparing the value of \( W \) for spheres
having the same masses and radii, and then concluding that the lower the value of \( W \), the
more stable the corresponding sphere. The full argument as to why \( W \) can be used in such
a fashion is very long and given in full in \[106\], and touched upon in \[109\]. In both of these
references the relationship of \( W \) to the Weyl tensor \( C_{abcd} \), and to the Newmann-Penrose
Weyl scalar \( \Psi_2 \) is emphasized so that this stability criterion becomes less strange, but we
shall not go into details here.

Applying this criterion to all our solutions results in a different expression of \( W \) in all the
sub-classes of solution, and we summarize this in table \[5.1\] from which it is immediately
clear that adding anisotropy in the form of \( \Delta \neq 0 \), changes the value of \( W \), and depending
on the sign of \( \beta \) we can get \( W \) to increase or decrease. If we admit the “no cracking” heuristic
condition (5.4), we will have that any addition of anisotropy will increase \( W \). By contrast since
charge only comes in the form of \( k^2 \) in the expression for \( W \), all the charge terms contribute
positive quantities. However, since in the general expression of \( W \), the charge term occurring
with a negative sign is larger in magnitude, electric charge has a capacity of reducing \( W \).
| Solution Name                                      | Specific case | $W(r)$                                                                 |
|---------------------------------------------------|---------------|------------------------------------------------------------------------|
| Tolman VII                                        | $\Delta = k = 0$ | $\frac{8\pi \mu \rho_c r^5}{15 r_b^5}$                               |
| Anisotropic TVII                                  | $\Delta = \beta r^2, k = 0$ | $\frac{4\pi r^5}{3} \left[ \frac{2}{5} \left( \frac{\rho_c \mu}{r_b^2} \right) + \beta \right]$ |
| Anisotropic TVII with charge                     | $\Delta = 2k^2 r^2$ | $\frac{4\pi r^3}{3} \left\{ r^2 \left[ \frac{2}{5} \left( \frac{\rho_c \mu}{r_b^2} \right) + k^2 \left( 2 - \frac{3}{8\pi} \right) \right] + \frac{k^2}{8\pi} \right\}$ |
| Anisotropic TVII with charge                     | $\Delta = \beta r^2, k \neq 0$ | $\frac{4\pi r^3}{3} \left\{ r^2 \left[ \frac{2}{5} \left( \frac{\rho_c \mu}{r_b^2} \right) - \frac{3k^2}{8\pi} + \beta \right] + \frac{k^2}{8\pi} \right\}$ |

Table 5.1: The different expressions of the $W$ function for our different classes of solutions

From this heuristic we therefore conclude that addition of charge stabilizes the star, and addition of anisotropy destabilizes it. The exact effects of both however, and how these two interact with each other can only be guessed at this point. We shall elucidate this in the next section where we perform a full radial perturbation stability analysis of this system.

5.3 Radial Perturbation Analysis

We will follow Chandrasekhar in the initial phase of our derivation, with corrections from Chandrasekhar’s own erratum, and Knutsen and Pedersen. However since we will be considering a more general form of the energy-momentum tensor: one that admits both electric charge and anisotropic pressures, the later part of the derivation will be more cumbersome.

The most extensive use of Chandrasekhar’s derivations was by Tooper, who considered a number of models before integrating the Chandrasekhar’s pulsation equations numerically for polytropes of various orders to obtain the normal mode frequencies in both general relativity, and in post-Newtonian approximations. Many other authors have subsequently investigated stability. For example Negi checks the stability of self-bound Tolman VII solutions, and determines that this cannot be stable, however with hand-waving arguments involving
the “type independance’ property of mass ’M’(sic).” We prove later in this chapter that the self-bound Tolman VII solution can be stable.

The extension to anisotropic models was first done by Hillebrandt and Steinmetz, who studied the dynamic stability of anisotropic models numerically to find the eigenfrequencies of the normal modes for the pulsation equation [61]. They found that the same method (Sturm–Liouville eigenmode analysis) employed by Chandrasekhar could be extended to anisotropic pressures. However the work being numerical in nature, the type of anisotropy had to be specified, and Hillebrandt and Steinmetz only considered very specific types of anisotropy. They also tried to look at more general non-radial perturbations, but they only did so for the Newtonian case, citing that “Since the anisotropic term in the equilibrium equations is of purely Newtonian origin [...], we will discuss non-radial pulsations only in the Newtonian approximation.” The authors concluded by stating that anisotropic models are as stable as the isotropic stars, while allowing for a greater concentration of mass in the star.

Many other authors have worked on the issue of stability, with a marked preference given to the Newtonian stars which are deemed complicated enough that most of the physics would be similar when it comes to stability considerations. Of note is the article by Lieb and Yau, who derive the Newtonian hydrodynamic stability condition from a quantum mechanical point of view, even extending the analysis to boson stars [79]. By contrast Sharif and Azam look at the proper relativistic equations throughout up to and including the matching conditions to a Reissner-Nordström exterior, but then look at the pulsation and contractions in a Newtonian and a post-Newtonian limit, concluding that both anisotropy and electric charge affect the collapse, pulsations, and hence stability of the star, with charge reducing the instability of the star [117].

EOS with quasi-local components (where the EOS depends on quasi-local variables such as the average density, total mass or total radius, as well as local ones) have also been investigated intensively, for example in [63]. While technically more difficult because the boundary
conditions on the pulsation equations are now quasi-local, these EOS allow the density to increase outwards in violation of Buchdahl’s assumption of $d\rho/dr \leq 0$, leading to the maximum compactness [18] of $M/R \leq 4/9$, while maintaining the stability of the star with the anisotropy.

The most complete treatment of stability in anisotropic stars was that of Dev and Gleiser where the pulsation equation was obtained for the general anisotropic case, after a lengthy treatment of both Newtonian and relativistic stars. Due the the length of some of the equations, a number of typographic errors are present, but a very comprehensive section on different examples of anisotropy concludes that anisotropy, for the most part stabilizes stars [33].

We wish to extend this result to the electrically charged case.

Charged models have received similar treatment over the years: Stettner studied a constant mass density model with constant charge density on the surface of the star and concluded from an analysis which follows Chandrasekhar closely that in certain cases when the charge is not too large, the system is stabilized with inclusion of the electric charge [121]. Glazer by contrast looks at a completely general charged isotropic model in the spirit of Chandrasekhar until he obtains the pulsation equation, but then only applies his result to a dust solution by Bonnor to conclude that electric charge decreases the minimum radius at which dynamical stability is possible.

The only work that tried to extend the pulsation formalism to both anisotropic and electric charge was that of Esculpi and Alomá, however we could not use their result since they specialized their pulsation equation with a restrictive assumption for the type of anisotropy: one where $\Delta = Cp$. Furthermore the interior solution in their work has to admit conformal symmetry [42], which we do not have. We have tried to use notation consistent with the mentioned works as much as possible, and thoroughly checked our expressions, but due to the length of the involved equations, typographic errors are inevitable.

Following in the footsteps of Chandrasekhar, Dev and Gleiser and Esculpi and Alomá we
now proceed to perturb the metric with a non-zero radial four velocity $v$, whose time integral will be the perturbation control parameter $\zeta$. This radial perturbation will cause the stress-energy tensor to be non-diagonal, and as a result all the matter and metric variables will be perturbed by an amount that can be related to this radial velocity perturbation. Each of these perturbations will be consistently expanded to first order (linear perturbation analysis) in terms of $\zeta$ and unperturbed quantities. Conservation of baryon number inside the star is then used as a condition on an undetermined equation of state to close the system into a differential equation of the form

$$-rac{\partial^2 \zeta}{\partial t^2} = \mathcal{L} \zeta,$$

where $\mathcal{L}$ is some differential operator. As is usual in eigenmode analysis, $\zeta$ is then assumed to have a time dependence of the form $\zeta = e^{i \sigma t}$, and substitution in the above differential equation (5.8) results in a Sturm-Liouville type problem for the frequency $\sigma$. The analysis of the spectrum of $\sigma$, for specific test functions of $\zeta$ gives a natural ordering of eigenfrequencies, and the sign of the leading eigenfrequency determines whether the solution is stable or not. We will follow this prescription in what follows.

As is usual in deriving the pulsation equation for non-static, spherically symmetric metric, we will assume the following form for the line element:

$$d s^2 = e^\nu \, d t^2 - e^\lambda \, d r^2 - r^2 \left( d \theta^2 + \sin^2 \theta \, d \varphi^2 \right). \quad (5.9)$$

From this metric we will immediately be able to write down the Einstein equations:

$$\frac{8 \pi G}{c^4} T_{i j} = G_{i j}, \quad (5.10)$$

which in component form explicitly give:

$$- \frac{8 \pi G}{c^4} T^0_0 = e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = - \frac{1}{r^2} \left( r e^{-\lambda} \right)' + \frac{1}{r^2}, \quad (5.11a)$$

$$- \frac{8 \pi G}{c^4} T^1_1 = - e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2}, \quad (5.11b)$$
\[-8\pi G c^4 T^2_2 = -e^{-\lambda} \left( \frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2} \right) + e^{-\nu} \left( \frac{\dot{\lambda}\dot{\lambda}}{4} + \frac{\ddot{\nu}}{2} + \frac{\dot{\lambda}^2}{4} \right) \] (5.11c)

\[-8\pi G c^4 T^1_0 = -e^{-\lambda} \frac{\dot{\lambda}}{r}. \] (5.11d)

Here the primes (') and dots (\(\dot{}\)) refer to derivatives with respect to the radial coordinate \(r\), and time coordinate \(t\), respectively. It should also be noted that the coordinates are denoted in two separate but equivalent ways in this derivation: \(x^i \equiv (x^0, x^1, x^2, x^3) \equiv (t, r, \theta, \varphi)\). If we assert the same symmetry conditions on the energy momentum tensor, we have to assume a \(T^i_j\) of the following form, as discussed in previous chapters:

\[ T^i_j = (p_r + \rho) u^i u_j - \delta^i_j - (p_\perp - p_r) n^i n_j + \frac{1}{4\pi} (F_{jk} F^{ki} + \frac{1}{4} \delta^i_j F_{ab} F^{ab}), \] (5.12)

where \(\delta^i_j\) is the Kronecker delta, \(u^i\) and \(u_j\) are the contravariant and co-variant space-like four-velocities, defined through \(u^i = \frac{dx^i}{ds}\), so that \(u_i u^i = 1\), and \(n_i\) is a time-like four-velocity so that \(n_i n^i = -1\). The last part of the equation incorporates the electromagnetic part of the energy-momentum tensor derived from the Faraday tensor \(F_{ab} = A_a;b - A_b;a\), with \(A_a\) the usual electromagnetic four-potential. Since we are considering the static and spherically symmetric case with anisotropic pressure, we will have two distinct pressures: the radial pressure \(p_r\), and the tangential pressure \(p_\perp\); while \(\rho\) denotes the energy density, and the only non-zero component of the vector potential \(A_a\) is the time component, so that \(A_a = (A_0, 0, 0, 0)\). Further we will also have that the frame velocities are such that the angular four-velocities, \(u^2\) and \(u^3\) vanish. This results in the energy momentum tensor having the following form:

\[ T^i_j = \begin{pmatrix}
\rho + \eta & 0 & 0 & 0 \\
0 & -p_r + \eta & 0 & 0 \\
0 & 0 & -p_\perp - \eta & 0 \\
0 & 0 & 0 & -p_\perp - \eta
\end{pmatrix}. \] (5.13)

Here, as derived in Appendix A, \(\eta = \frac{e^{-(\nu + \lambda)}}{8\pi} (F_{01})^2\).
We note here that the form of the equations we are using is slightly more general than the previous versions \((3.1)\), or \((4.25)\) which did not include the time derivative terms. The reason for this more general form is that we want to be able to perturb our solutions in time, and this is impossible to do with the static set of Einstein’s equations we used before.

5.3.1 Simplifying the Einstein’s equations

The equations \((5.11a)\) and \((5.11b)\) can be combined to give the following more workable form:

\[
\frac{e^{-\lambda}}{r} \frac{\partial}{\partial r} (\lambda + \nu) = \frac{8\pi G}{c^4} \left( T_{1}^{1} - T_{0}^{0} \right).
\]

(5.14)

The equations \((5.11)\) and not independent, but rather are related through the Bianchi identities: \(T_{j;i} = 0\). Explicitly these give rise to the following two equations:

\[
\frac{\partial T_{0}^{0}}{\partial t} + \frac{\partial T_{1}^{1}}{\partial r} + \frac{1}{2} (T_{0}^{0} - T_{1}^{1}) \frac{\partial \lambda}{\partial t} + T_{1}^{1} \left( \frac{1}{2} \frac{\partial (\lambda + \nu)}{\partial r} + \frac{2}{r} \right) = 0,
\]

(5.15a)

and,

\[
\frac{\partial T_{0}^{1}}{\partial t} + \frac{\partial T_{1}^{1}}{\partial r} + \frac{1}{2} T_{1}^{1} \frac{\partial (\lambda + \nu)}{\partial t} + \frac{1}{2} \left( T_{1}^{1} - T_{0}^{0} \right) \frac{\partial \nu}{\partial r} + \frac{1}{r} \left( 2T_{1}^{1} - T_{2}^{2} - T_{3}^{3} \right) = 0.
\]

(5.15b)

in the \(j = 0, 1\) cases respectively. These equations will allow us to perform a number of simplifications later on.

The static case

For fluid balls that are in hydrostatic equilibrium, additionally, there is no dependence of any of the fields on the time coordinate \(t\). All of the above equations then simplify, as a result of the following zero-subscripted time independent variables replacing the general time dependant ones:

\[
T_{j}^{i} = \begin{pmatrix}
\rho_{0} + \eta_{0} & 0 & 0 & 0 \\
0 & -p_{r0} + \eta_{0} & 0 & 0 \\
0 & 0 & -p_{\perp0} - \eta_{0} & 0 \\
0 & 0 & 0 & -p_{\perp0} - \eta_{0}
\end{pmatrix},
\]

(5.16)
with the metric functions, $\lambda(r, t) = \lambda_0(r)$, and $\nu(r, t) = \nu_0(r)$. Additionally, all the frame four velocities also vanish by choice, except for $u^0$. We will also choose units so that $c = G = 1$, so as to avoid carrying all the $G$ and $c$ terms through this long calculation. As a result of these simplifications, the equations given above as (5.11), (5.12), (5.14), and (5.15) simplify to the following set:

\[
\begin{align*}
\frac{d(r e^{-\lambda_0})}{dr} &= 1 - 8\pi r^2 (\rho_0 + \eta_0), \quad (5.17a) \\
\frac{e^{-\lambda_0} d\nu_0}{r \, dr} &= \frac{1}{r^2} (1 - e^{-\lambda_0}) + 8\pi (p_{r0} - \eta_0), \quad (5.17b) \\
\frac{d(p_{r0} - \eta_0)}{dr} &= -\frac{1}{2} (p_{r0} + \rho_0) \frac{d\nu_0}{dr} + \frac{2}{r} (p_{\perp0} - p_{r0}) + \frac{4\eta_0}{r}, \quad (5.17c) \\
\frac{e^{-\lambda_0}}{r} \frac{d(\nu_0 + \lambda_0)}{dr} &= 8\pi (p_{r0} + \rho_0). \quad (5.17d)
\end{align*}
\]

If we want to investigate the stability of these equations, we will have to perform a time dependent perturbation on them, keeping in mind that the perturbed equations will still obey the full Einstein’s equations (5.11). Since we will be introducing time dependent fields in the pressure $p$, the density $\rho$, the metric coefficients $\nu$, and $\lambda$, and the four-velocities $u^i$, we will need the full set of the time dependant equations (5.11).

In linear stability analysis, it is common to expand every perturbed expression to first order consistently. This is what we will strive to do in the following, starting first with expressions for the time four-velocity:

\[
u^0 = \frac{dt}{ds} = \sqrt{\frac{dt^2}{ds^2}} = e^{-\nu/2}.
\]

The control variable we will be using to do our perturbation expansion is $v = \frac{dr}{dt}$. Expressing the radial four-velocity in terms of this variable, we get

\[
u^1 = \frac{dr}{ds} = \frac{dr}{dt} \cdot \frac{dt}{ds} = v e^{-\nu/2}, \quad \text{since} \quad \frac{ds}{dt} \neq 0.
\]

The respective co-variant versions of the velocities are obtained through the metric since $u_i = g_{ij} u^j$. Since the metric is diagonal, this expression simplifies considerably for both
radial and time four-velocities resulting in

\[ u_0 = g_{00} u^0 \approx e^{\nu/2} \quad \text{and} \quad u_1 = g_{11} u^1 = v e^{\lambda - \nu/2}. \]

5.3.2 Perturbing the static case

We are now ready to perturb the fields, keeping in mind that any terms that are second order or higher in the perturbation will be discarded. This process involves the consistent substitution of \( r \rightarrow r + \delta r = r + \zeta \), leading to \( \lambda \rightarrow \lambda_0 + \delta \lambda \), and \( \nu \rightarrow \nu_0 + \delta \nu \). As a result we will be getting the following

\[
\begin{align*}
    u^0 &= e^{(\nu_0 + \delta \nu)/2} \approx e^{-\nu_0/2}, \\
    u^1 &= v e^{-(\nu_0 + \delta \nu)/2} \approx v e^{-\nu_0/2}, \\
    u_0 &\approx e^{\nu_0/2}, \\
    u_1 &\approx v e^{\lambda_0 - \nu_0/2}.
\end{align*}
\]  

(5.18)

It might seem that terms of the first order are also being culled in the above, particularly in equation (5.18a), and (5.18c), but since the four-velocity \( u^0 \) always occurs in a product in all the fields we are considering here, instead of carrying the first order term continuously, and lengthen an already tedious process, we use only the zeroth order approximation, for these expressions.

By using the energy-momentum equation (5.12), we can find the corresponding perturbed energy-momentum introduced by the perturbed fields mentioned above. A straightforward substitution of the four-velocities results in perturbed pressures, \( p_r \rightarrow p_{r0} + \delta p_r \), and \( p_\perp \rightarrow p_{\perp 0} + \delta p_\perp \); perturbed energy density fields, \( \rho \rightarrow \rho_0 + \delta \rho \); and perturbed electromagnetic fields
\( \eta \rightarrow \eta_0 + \delta \eta \), resulting in,

\[
T^i_j = \begin{pmatrix}
A & B & 0 & 0 \\
- e^\lambda_0 B & C & 0 & 0 \\
0 & 0 & D & 0 \\
0 & 0 & 0 & D
\end{pmatrix},
\]

with,

\[
\begin{aligned}
A &= \rho_0 + \eta_0 + \delta \rho + \delta \eta, \\
B &= (p_{r0} + \rho_0) v, \\
C &= \eta_0 - p_{r0} + \delta \eta - \delta p_r, \\
D &= -\eta_0 - p_{\perp 0} - \delta \eta - \delta p_{\perp}.
\end{aligned}
\] (5.19)

This expression, through construction, is to first order, as we wanted, since the pressure and density are the fields we will be concerned with mostly. With this new perturbed equation in mind, and the full Einstein equations, we find that equation (5.17a) has a very similar perturbed form, viz

\[
\frac{\partial (r e^{-\lambda_0 - \delta \lambda})}{\partial r} = 1 - 8 \pi r^2 (\rho_0 + \delta \rho + \eta + \delta \eta).
\]

The total derivative is transformed into a partial one since now \( \delta \lambda \) also depends on time.

Simplifying this expression to first order results in the following:

\[
\begin{aligned}
\frac{\partial (r e^{-\lambda_0 - \delta \lambda})}{\partial r} &= 1 - 8 \pi r^2 (\rho_0 + \eta_0) - 8 \pi r^2 (\delta \rho + \delta \eta), \\
\frac{\partial (r e^{-\lambda_0} (1 - \delta \lambda + \cdots))}{\partial r} &= - 1 + 8 \pi r^2 (\rho_0 + \eta_0) = -8 \pi r^2 (\delta \rho + \delta \eta), \\
\frac{d(r e^{-\lambda_0})}{dr} &= 1 + 8 \pi r^2 (\rho_0 + \eta_0) - \frac{\partial}{\partial r} (r e^{-\lambda_0} \delta \lambda) = -8 \pi r^2 (\delta \rho + \delta \eta).
\end{aligned}
\]

This simplification give us the relation between the perturbation in energy density and electromagnetic field, and one of the metric coefficient:

\[
\frac{\partial}{\partial r} (r e^{-\lambda_0} \delta \lambda) = 8 \pi r^2 (\delta \rho + \delta \eta)
\] (5.20)

Similarly the perturbed equation relating the other metric coefficients, and the pressure resulting from perturbing (5.17b), becomes

\[
\frac{e^{-\lambda_0} e^{-\delta \lambda}}{r} \frac{\partial}{\partial r} (\nu_0 + \delta \nu) = \frac{1}{r^2} \left( 1 - e^{-\lambda_0} e^{-\delta \lambda} \right) + 8 \pi (p_{r0} + \delta p_r - \eta_0 - \delta \eta)
\]
On expanding and removing the zeroth order terms in accordance to the static equation (5.17b) results in the simplification:

\[
e^{-\lambda_0} \frac{1}{r} \left(1 - \delta \lambda + \cdots\right) \left(\frac{d\nu_0}{dr} + \frac{\partial \delta \nu}{\partial r}\right) \simeq \frac{1}{r^2} \left[1 - e^{-\lambda_0} \left(1 - \delta \lambda + \cdots\right)\right] + 8\pi (p_{r0} - \eta_0) + 8\pi (\delta p_r - \delta \eta).
\]

Rearranging this equation, and collecting terms that constitute the zeroth order form then gives,

\[
e^{-\lambda_0} \frac{1}{r} \frac{d\nu_0}{dr} - \frac{1}{r^2} (1 - e^{-\lambda_0}) - 8\pi (p_{r0} - \eta_0) = \frac{1}{r^2} (e^{-\lambda_0} \delta \lambda) + 8\pi \delta p - \frac{e^{-\lambda_0}}{r} \frac{\partial \delta \nu}{\partial r} + e^{-\lambda_0} r \delta \lambda \frac{d\nu_0}{dr} + e^{-\lambda_0} \frac{1}{r} \frac{\partial \delta \nu}{\partial r} + 8\pi (\delta p_r - \delta \eta).
\] (5.21)

We now remove all terms that are higher than first order. Derivatives of the perturbations are taken to be first order, but products of perturbations are of second order, and we neglect them. The previous equation reorganized thus gives:

\[
e^{-\lambda_0} \frac{1}{r} \frac{\partial \delta \nu}{\partial r} - \frac{e^{-\lambda_0}}{r} \delta \lambda \frac{d\nu_0}{dr} - \frac{e^{-\lambda_0}}{r} \frac{\partial \delta \nu}{\partial r} = \frac{e^{-\lambda_0}}{r^2} \delta \lambda + 8\pi (\delta p_r - \delta \eta),
\]

which finally results in an equation relating the perturbations only:

\[
e^{-\lambda_0} \frac{1}{r} \left(\frac{\partial \delta \nu}{\partial r} - \delta \lambda \frac{d\nu_0}{dr}\right) = \frac{e^{-\lambda_0}}{r^2} \delta \lambda + 8\pi (\delta p_r - \delta \eta).
\] (5.22)

In the static set of equations (5.17), we notice that since none of the variables depend explicitly on time, the off-diagonal terms of the Einstein equations, and energy momentum tensor vanish. In the perturbed set, this is unfortunately not the case, and we have to deal with the additional equation (5.11d), contingent upon the static version where

\[
-8\pi T^0_1 = \frac{e^{-\lambda_0} \dot{\lambda}_0}{r} = 0.
\] (5.23)

If we perturb equation (5.11d), use the linearised expression for $T^0_1$ from equation (5.19), linearise the rest, and remove the contribution from the static part (5.23), we get the following
simplifications:

\[-8\pi T^0_1 = -\frac{e^{-(\lambda_0 + \delta \lambda)}}{r} \frac{\partial}{\partial t} (\lambda_0 + \delta \lambda),\]

\[8\pi (p_r + \rho_0) v = -\frac{e^{\lambda_0}}{r} \frac{[1 - \delta \lambda + \cdots]}{\partial t} (\lambda_0 + \delta \lambda),\]

\[\simeq \frac{e^{-\lambda_0}}{r} \frac{d \lambda_0}{dt} (\delta \lambda - 1) - \frac{e^{-\lambda_0}}{r} \frac{\partial \delta \lambda}{\partial t} + \frac{e^{-\lambda_0}}{r} \frac{\delta \lambda}{\partial t} + \frac{1}{r} \frac{\partial}{\partial t} (\nu_0 + \delta \lambda).\]

The final equation relating the velocity \(v\) to the metric perturbations is then:

\[-8\pi (p_r + \rho_0) v = \frac{e^{-\lambda_0}}{r} \frac{\partial \delta \lambda}{\partial t}.\]  \hspace{1cm} (5.24)

We can also use the Bianchi identities (5.15) to get an equation relating all the different perturbations. To achieve this, we first look at how the static condition simplified this equation, and then generate the time dependant version with the perturbation.

The static case with assumptions (5.16) transforms the first Bianchi identity (5.15a) into equation (5.17c). The other Bianchi identity with the perturbed quantities results in

\[
\frac{\partial}{\partial t} \left[ -e^{\lambda_0 - \nu_0} (\rho_0 + p_r) v \right] + \frac{2}{r} (\eta_0 - p_r + \delta \eta - \delta p_r + \eta_0 + p_{\perp 0} + \delta \eta + \delta p_{\perp 0})\\
+ \frac{1}{2} \left[ -e^{\lambda_0 - \nu_0} (\rho_0 + p_r) v \right] \frac{\partial}{\partial t} (\nu_0 + \delta \nu + \lambda_0 + \delta \lambda) + \frac{\partial}{\partial r} (\eta_0 - p_r + \delta \eta - \delta p_r)\\
+ \frac{1}{2} (\eta_0 - p_r + \delta \eta - \delta p_r - \rho_0 - \eta_0 - \delta \rho - \delta \eta) \frac{\partial}{\partial r} (\nu_0 + \delta \nu) = 0.
\]

If we rearrange this equation and cancel the time-derivatives of static quantities, and additionally realize that the quantity \(v\) is already first order, we can simplify the above into:

\[-e^{\lambda_0 - \nu_0} (\rho_0 + \rho_0) \frac{\partial v}{\partial t} + \frac{\partial}{\partial r} (\eta_0 - p_r + \delta \eta - \delta p_r) - \frac{1}{2} (p_r + \rho_0) \frac{dv_0}{dr} - \frac{1}{2} (p_r + \rho_0) \frac{\partial \delta \nu}{\partial r} = 0.\]

If we rearrange this equation and cancel the time-derivatives of static quantities, and additionally realize that the quantity \(v\) is already first order, we can simplify the above into:

\[-e^{\lambda_0 - \nu_0} (p_r + \rho_0) \frac{\partial v}{\partial t} + \frac{\partial}{\partial r} (\eta_0 - p_r + \delta \eta - \delta p_r) - \frac{1}{2} (p_r + \rho_0) \frac{dv_0}{dr} - \frac{1}{2} (p_r + \rho_0) \frac{\partial \delta \nu}{\partial r} = 0.\]

\[-\frac{1}{2} [e^{\lambda_0 - \nu_0} (p_r + \rho_0) v] \frac{d}{dt} (\nu_0 + \lambda_0) + \frac{\partial}{\partial r} (\delta \nu + \delta \lambda) - \frac{1}{2} (\delta p_r + \delta \rho) \frac{\partial (\nu_0 + \delta \nu)}{\partial r} = 0.\]

Upon rearrangement, and isolating the parts of the equation that correspond to the static case to get a further simplification, we obtain
\[-e^{\lambda_0-\nu_0}(p_0 + \rho_0) \frac{\partial v}{\partial t} - \frac{1}{2}(\rho_0 + p_{r0}) \frac{\partial \nu_0}{\partial t} - \frac{\partial}{\partial r}(p_{r0} - \eta_0) + \frac{2}{r}(p_{\perp0} - p_{r0} + 2\eta_0) \]
\[-\frac{\partial}{\partial r}(\delta p_r - \delta \eta) + \frac{2}{r}(\delta p_r + \delta p_{\perp}) - \frac{1}{2}(p_{r0} + \rho_0) \frac{\partial \delta \nu}{\partial r} - \frac{1}{2}(\delta p_r + \delta \rho) \frac{\partial \nu_0}{\partial r} = 0.\]

The final constraint equation relating all the perturbations, obtained from the second Bianchi identity simplifies to,
\[e^{\lambda_0-\nu_0}(p_{r0} + \rho_0) \frac{\partial v}{\partial t} + \frac{\partial}{\partial r}(\delta p_r - \delta \eta) + \frac{1}{2}(\delta p_r + \delta \rho) \frac{d\nu_0}{dr} + \frac{1}{2}(p_{r0} + \rho_0) \frac{d\delta \nu}{dr} + \frac{2}{r}(\delta p_r - \delta p_{\perp}) = 0.\]

(5.25)

The first Bianchi identity imposes no further constraints on the perturbations, instead it regenerates one of the previous equations. We will now be using this second equation as our starting point and find all the terms in it by other methods. First we will find expressions for the metric perturbations \(\delta \nu\), and \(\delta \lambda\).

5.3.3 Lagrangian description and partial integration

If we now shift our attention to a Lagrangian description as opposed to the Eulerian one we have been using thus far, we can define a Lagrangian displacement in terms of the velocity \(v\).

Let \(\zeta\) represent such a displacement with respect to the time coordinate, \(x^0 = t\). Then we can define \(v = \frac{\partial \zeta}{\partial t}\), and rewrite equation (5.24) in terms of this displacement. Doing this allows us to immediately integrate the latter equation to:
\[-8\pi(p_{r0} + \rho_0) \frac{\partial \zeta}{\partial t} = \frac{e^{-\lambda_0}}{r} \frac{\partial \delta \lambda}{\partial t},\]
\[-8\pi(p_{r0} + \rho_0) \int \frac{\partial \zeta}{\partial t} dt = \frac{e^{-\lambda_0}}{r} \int \frac{\partial \delta \lambda}{\partial t} dt,\]
\[-8\pi(p_{r0} + \rho_0) \zeta = \frac{e^{-\lambda_0}}{r} \delta \lambda.\]

(5.26)

This equation can be further simplified through the use of another relation we have already found, viz equation (5.17d). Substituting this equation in the previous (5.26) gives:
\[\frac{e^{-\lambda_0}}{r} \delta \lambda = \frac{e^{-\lambda_0}}{d} \left[ \frac{d}{dr}(\nu_0 + \lambda_0) \right] \zeta,\]
giving us the final form of an equation relating the perturbation $\delta \lambda$ to the static variables, and Lagrangian displacement:

$$\delta \lambda = -\zeta \left[ \frac{d}{dr} (\nu_0 + \lambda_0) \right].$$  \hspace{1cm} (5.27)

We now simplify the relationship between the perturbed density $\delta \rho$, and the perturbed metric coefficient $\delta \lambda$, viz equation (5.20), with the relation just obtained (5.26),

$$\frac{\partial (r e^{-\lambda_0} \delta \lambda)}{\partial r} = 8\pi r^2 (\delta \rho + \delta \eta), \quad \text{with} \quad \delta \lambda = -8\pi (p_{r0} + \rho_0) \zeta r e^{\lambda_0},$$

to obtain the following simplification,

$$\frac{\partial}{\partial r} \left\{ r e^{-\lambda_0} \left[ -8\pi (p_{r0} + \rho_0) \zeta r e^{\lambda_0} \right] \right\} = 8\pi r^2 (\delta \rho + \delta \eta),$$

$$\frac{\partial}{\partial r} \left\{ -r^2 \zeta (p_{r0} + \rho_0) \right\} = r^2 (\delta \rho + \delta \eta),$$

finally giving us the equation relating the perturbed density in terms of the non perturbed variables:

$$\delta \rho + \delta \eta = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \zeta (p_{r0} + \rho_0) \right).$$ \hspace{1cm} (5.28)

The above compact form of the equation can be expanded and further simplified into an alternative version involving the expanded derivative on the right hand side. This is done to isolate perturbed quantities from static ones explicitly, as follows:

$$\delta \rho + \delta \eta = -\frac{1}{r^2} \left\{ r^2 \zeta \frac{\partial (p_{r0} + \rho_0)}{\partial r} + (p_{r0} + \rho_0) \frac{\partial (r^2 \zeta)}{\partial r} \right\},$$

$$\delta \rho + \delta \eta = -\zeta \frac{dp_{r0}}{dr} - \zeta \frac{d\rho_0}{dr} - \left( \frac{p_{r0} + \rho_0}{r^2} \right) \frac{\partial (r^2 \zeta)}{\partial r}. \hspace{1cm} (5.29)$$

Recalling that the simplified static Bianchi identity results in an expression for the static pressure derivative through equation (5.17c), we can substitute the first right hand term in the above equation, and together with the redefinition of anisotropy measure, $\Pi = p_\perp - p_r$, we have

$$\delta \rho + \delta \eta = -\zeta \frac{d\rho_0}{dr} - \zeta \left\{ -\frac{1}{2} (p_{r0} + \rho_0) \frac{d\nu_0}{dr} + \frac{2}{r} \Pi_0 + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right\} - \frac{p_{r0} + \rho_0}{r^2} \frac{\partial}{\partial r} (r^2 \zeta),$$

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which is easily rearranged into the more convenient,

\[ \delta \rho + \delta \eta = -\zeta \frac{d\rho_0}{dr} - \frac{p_{r0} + \rho_0}{r^2} \left\{ \frac{\partial (\zeta r^2)}{\partial r} - \frac{\zeta r^2}{2} \frac{d\nu_0}{dr} \right\} - \zeta \left[ \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right]. \]

We are now in a position to obtain a compact form of the above equation by multiplying the term in braces by unity: \( e^{\nu_0/2} e^{-\nu_0/2} = 1 \) as shown:

\[ \delta \rho = -\zeta \frac{d\rho_0}{dr} - \frac{(p_{r0} + \rho_0) e^{\nu_0/2}}{r^2} \left\{ e^{-\nu_0/2} \frac{\partial (r^2 \zeta)}{\partial r} - \frac{\zeta r^2}{2} \frac{e^{-\nu_0/2} d\nu_0}{dr} \right\} - \zeta \left[ \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right] - \delta \eta. \]

This last non-intuitive step is needed to factorize the derivative on the right hand side into the compact form we are looking for,

\[ \delta \rho = -\zeta \frac{d\rho_0}{dr} - \frac{(p_{r0} + \rho_0) e^{\nu_0/2}}{r^2} \left\{ \frac{\partial}{\partial r} (e^{-\nu_0/2 r^2 \zeta}) \right\} - \zeta \left[ \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right] - \delta \eta. \quad (5.30) \]

We now proceed in a similar fashion to obtain another expression, this time for the other perturbed metric coefficient \( \delta \nu \). To achieve this we first notice that we already have a promising candidate, viz equation (5.22). If we were to substitute for the \( \delta \lambda \) terms in this equation from the result (5.26), we get,

\[ \frac{e^{-\lambda_0}}{r} \frac{\partial \delta \nu}{\partial r} = 8\pi \left\{ \delta p_r - \delta \eta - \frac{\zeta (p_{r0} + \rho_0)}{r} \right\} + \frac{d\nu_0}{dr} \left\{ -8\pi (p_{r0} + \rho_0) \zeta \right\}, \]

upon collecting like terms we get the following more workable form of the equation:

\[ \frac{e^{-\lambda_0}}{r} \frac{\partial \delta \nu}{\partial r} = 8\pi \left\{ \delta p_r - \delta \eta - (p_{r0} + \rho_0) \zeta \left\{ \frac{d\nu_0}{dr} + \frac{1}{r} \right\} \right\}. \quad (5.31) \]

Remembering from equation (5.17d), that we have an expression for the first term in the above equation, we can do one more substitution in this equation to get

\[ (p_{r0} + \rho_0) \frac{\partial \delta \nu}{\partial r} = \left\{ \delta p_r - \delta \eta - (p_{r0} + \rho_0) \zeta \left\{ \frac{d\nu_0}{dr} + \frac{1}{r} \right\} \right\} \frac{d(\lambda_0 + \nu_0)}{dr}. \quad (5.32) \]

Thus far we have obtained the expressions for the perturbations of the metric functions \( \lambda \) and \( \nu \), and of the matter density \( \rho \) when the radius of the star is changed. Next we find the
perturbation of the electric field $\delta \eta$ in terms of the static quantities following the work of Glazer[46] who considers the Maxwell’s source equation:

$$\frac{\partial}{\partial x^a} \left( \sqrt{(-g)} F^{ab} \right) = -\frac{4\pi}{c} \sqrt{(-g)} J^b, \quad (5.33)$$

with $g = -(r^2 \sin \theta)^2 e^{(\nu + \lambda)}$ the metric determinant, and $J^a = \epsilon c u^a$ the 4-current density

with epsilon being the charge density.

In the static case the only non-zero components of $F_{ab}$ are $F_{01} = -F_{10} = E$, the electric field, so that equation (5.33) reduces to

$$\frac{\partial}{\partial t} \left( \sqrt{(-g)} F^{10} \right) = -\frac{4\pi}{\epsilon} \sqrt{(-g)} \epsilon u^1, \quad (5.34)$$

also since $F^{ab} = g^{ac} g^{bd} F_{cd}$, and $g^{ab}$ is diagonal, we are left with $F^{01} = g^{11} g^{00} F_{01}$, resulting in $F^{01} = -e^{-(\lambda + \nu)} E$. When the radial coordinate is perturbed, the electric field changes in such a way as to cause $E \to E_0 + \delta E$, and similarly the metric coefficients $\lambda \to \lambda_0 + \delta \lambda$, and $\nu \to \nu_0 + \delta \nu$.

Then both sides of equation (5.34) can be simplified separately: the LHS giving

$$\frac{\partial}{\partial t} \left( \sqrt{(-g)} F^{10} \right) = \frac{\partial}{\partial t} \left( -r^2 \sin \theta e^{(\lambda + \nu)/2} e^{-(\lambda + \nu)} (E) \right),$$

$$= -r^2 \sin \theta \frac{\partial}{\partial t} \left[ e^{-(\lambda_0 + \delta \lambda + \nu_0 + \delta \nu)/2} (E_0 + \delta E) \right],$$

$$= -r^2 e^{-(\lambda_0 + \nu_0)/2} \sin \theta \frac{\partial}{\partial t} \left( \left( 1 - \frac{\delta \lambda}{2} - \frac{\delta \nu}{2} + \cdots \right) (E_0 + \delta E) \right),$$

$$= -r^2 e^{-(\lambda_0 + \nu_0)/2} \sin \theta \frac{\partial}{\partial t} \left( \delta E - E_0 \frac{\delta \lambda}{2} - E_0 \frac{\delta \nu}{2} \right) + O(\delta^2).$$

Similarly since the velocity $u^1$, following (5.18b), goes to $v e^{-\nu_0/2} = \frac{\partial \zeta}{\partial t} e^{-\nu_0/2}$, under radial perturbations, and the change density $\epsilon \to \epsilon_0 + \delta \epsilon$, the RHS of equation (5.34) gives

$$-4\pi \sqrt{(-g)} \epsilon u^1 = -4\pi r^2 \sin \theta e^{(\lambda + \nu)/2} \epsilon v e^{-\nu_0/2},$$

$$= -4\pi r^2 \sin \theta e^{(\lambda_0 + \delta \lambda + \nu_0 + \delta \nu)/2} (\epsilon_0 + \delta \epsilon) \frac{\partial \zeta}{\partial t} e^{-\nu_0/2},$$

$$= -4\pi r^2 \sin \theta e^{\lambda_0/2} (1 + \delta \nu + \delta \lambda + \cdots) (\epsilon_0 + \delta \epsilon) \frac{\partial \zeta}{\partial t}.$$
\[-4\pi r^2 \sin \theta e^{\lambda_0/2} \epsilon_0 \frac{\partial \zeta}{\partial t} + O(\delta^2).\]

Identifying both sides of the equation

\[-r^2 e^{-(\lambda_0 + \nu_0)/2} \sin \theta \partial \left( \delta E - E_0 \frac{\delta \lambda}{2} - E_0 \frac{\delta \nu}{2} \right) = -4\pi r^2 \sin \theta e^{\lambda_0/2} \epsilon_0 \frac{\partial \zeta}{\partial t}\]

then allows us to conclude after a time integration that to first order,

\[2 \frac{\delta E}{E_0} - \delta \lambda - \delta \nu \frac{8\pi}{E_0} e^{(\lambda_0 + \nu_0/2)} \epsilon_0 \zeta, \quad (5.35)\]

As seen in appendix A, the electromagnetic part of the stress energy tensor can be expressed as \((T^0)_\text{EM} = \eta = e^{-(\lambda+\nu)} \frac{(F_{10})^2}{8\pi}\), which under a radial perturbation is transformed as

\[\eta_0 + \delta \eta = \frac{e^{-(\lambda_0 + \delta \lambda + \nu_0 + \delta \nu)}}{8\pi} \left( E_0 + \delta E \right)^2, \]

\[= \frac{e^{-(\lambda_0 + \nu_0)}}{8\pi} (E_0)^2 \left( 1 - \delta \nu - \delta \lambda + \cdots \right) \left( 1 + \frac{E_0}{E_0} \right)^{\frac{2}{3}} \]

\[= \frac{e^{-(\lambda_0 + \nu_0)}}{8\pi} (E_0)^2 \left( 1 + 2 \frac{E_0}{E_0} - \delta \nu - \delta \lambda \right) + O(\delta^2).\]

The terms in brackets being the same as equation \((5.35)\), we can immediately write

\[\eta_0 + \delta \eta = \frac{e^{-\lambda_0 + \nu_0} \eta_0 + \frac{e^{(\lambda_0 + \nu_0/2)} \epsilon_0 \zeta}}{8\pi} (E_0)^2 + \frac{e^{(\lambda_0 + \nu_0/2)} \epsilon_0 \zeta}, \]

allowing us to deduce that the perturbation of the stress energy component of the electromagnetic field is given by

\[\delta \eta = E_0 \epsilon_0 e^{-\nu_0/2} \zeta, \quad (5.36)\]

We now have all the perturbations of the field quantities, except for the pressures. The latter require a constitutive relation of the material and the one we will use is baryon conservation, to be able to close the system of equations and come up with a complete set of perturbation equations for all the matter and metric fields.
5.3.4 Baryon number conservation

An equation of state will involve a fixed number of baryons, since we will be considering the static case. This number will obviously depend on the other state variables in a non-trivial way. In the most general static case we will have \( \tilde{N}(\rho, p_r, p_{\perp}, \eta, r) \). However, in all models we will be analysing we will have a known dependence of the perpendicular pressure \( p_{\perp} \), on the radial pressure \( p_r \) and the radial parameter, so that we can simplify the baryon number as \( \tilde{N}(\rho, p_r, \eta, r) \). Furthermore the charge density \( \eta \), will as seen previously, depend on the mass density \( \rho \), and the radial parameter, so that \( \eta(\rho, r) \), then without loss of generality we can have the baryon number as \( N(\rho, p_r, r) \). In whichever way \( N \) is introduced, the scalar baryon number, \( N \) has to be conserved in any radial perturbation. The way this is expressed in general relativity, as seen previously (A.85) is

\[
(Nu^k)_{;k} = 0,
\]

where \( u^k \) is the four-velocity. Upon expansion this equation results in

\[
0 = \frac{\partial}{\partial x^k} (Nu^k) + Nu^k \frac{\partial}{\partial x^k} (\log \sqrt{-g}),
\]

\[
= \frac{\partial(Nu^0)}{\partial t} + \frac{\partial(Nu^1)}{\partial r} + Nu^1 \left( \frac{\nu' + \lambda'}{2} + \frac{2}{r} \right) + Nu^0 \left( \frac{\dot{\nu} + \dot{\lambda}}{2} \right),
\]

as seen in (A.86). Again considering the four-velocities introduced previously as a result of radial perturbation, viz (5.18), and expanding all the derivative of the products, results in

\[
0 = e^{-\nu/2} \frac{\partial N}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (Nvr^2 e^{-\nu/2}) + \frac{N e^{-\nu/2} \partial \lambda}{2} + \frac{N v e^{-\nu/2} \partial (\lambda + \nu)}{2},
\]

after simplification. The next step in obtaining the variation of the baryon number as a result of metric and field perturbations is to replace all perturbed variables with their closed form, and expand consistently to first order. This has to be carried out for all the terms in the above equation, and after a tedious but straightforward process, we obtain the following surviving first-order forms:

\[
e^{-\nu/2} \frac{\partial N}{\partial t} \rightarrow e^{-\nu_0/2} \frac{\partial (\delta N)}{\partial t}
\]
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( N v r^2 e^{-\nu/2} \right) \rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left( N_0 v r^2 e^{-\nu_0/2} \right)
\]
\[
\frac{N e^{-\nu/2} \partial \lambda}{2 \frac{\partial}{\partial t}} \rightarrow \frac{N_0 e^{-\nu_0/2}}{2} \frac{\partial (\delta \lambda)}{\partial t}
\]
\[
\frac{N v e^{-\nu/2} \partial (\lambda + \nu)}{2 \frac{\partial}{\partial r}} \rightarrow \frac{N_0 e^{-\nu_0/2} v}{2} \frac{\partial}{\partial r} (\lambda_0 + \nu_0),
\]
resulting in the first order perturbed baryon conservation equation to read:

\[
e^{-\nu_0/2} \frac{\partial (\delta N)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( N_0 r^2 e^{-\nu_0/2} \right) + \frac{N_0 e^{-\nu_0/2}}{2} \frac{\partial (\delta \lambda)}{\partial t} + \frac{N_0 e^{-\nu_0/2} v}{2} \frac{\partial}{\partial r} (\lambda_0 + \nu_0) = 0. \quad (5.38)
\]

This equation can be readily time-integrated, once the Eulerian velocity \( v \) is replaced by the corresponding Lagrangian displacements, \( \frac{d \zeta}{dt} \). Once this substitution is made, and the equation integrated, we have

\[
\delta N + \frac{e^{\nu_0/2}}{r^2} \frac{\partial}{\partial r} \left( N_0 r^2 e^{-\nu_0/2} \right) + \frac{N_0 e^{\nu_0/2}}{2} \frac{\partial (\delta \lambda)}{\partial t} + \frac{N_0 e^{-\nu_0/2} v}{2} \frac{\partial}{\partial r} (\lambda_0 + \nu_0) = 0, \quad \text{from (5.27)}
\]

finally resulting in

\[
\delta N = -\zeta \frac{d N_0}{d r} - \frac{N_0 e^{\nu_0/2}}{r^2} \frac{\partial}{\partial r} (r^2 e^{-\nu_0/2}). \quad (5.39)
\]

From the functional form of the baryon number, any variation to \( N \) resulting from metric perturbations, will come from both the radial pressure \( p_r \) and the mass density \( \rho \) allowing us to write

\[
\delta N = \frac{\partial N}{\partial \rho} \delta \rho + \frac{\partial N}{\partial p_r} \delta p_r, \quad \text{and} \quad dN_0 = \frac{\partial N}{\partial \rho} d\rho_0 + \frac{\partial N}{\partial p_r} d p_r, \quad (5.40)
\]

where the first equation involves perturbations, and the second one is the differential form of the baryon number equation. We already have an expression for the density (5.28) and baryon number (5.39) perturbation, and using those in the previous equation, we can find the resulting perturbation in the radial pressure. In order to do this, first we substitute (5.28) and (5.39) in (5.40), and solve for \( \delta p_r \). This results in

\[
\frac{\partial N}{\partial p_r} \delta p_r = -\zeta \frac{d N_0}{d r} - N_0 C - \frac{\partial N}{\partial \rho} \left\{ -\zeta \frac{d \rho_0}{d r} - (p_r + \rho_0) C \right\}
\]
\[
- \zeta \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right) - \delta \eta \right\}.
\]

Combining all the terms in \( \zeta \), and replacing the multiply occurring expression given by 
\[
e^{\nu_0/2} \frac{\partial}{\partial r} \left( r^2 \zeta e^{-\nu_0/2} \right)
\] with the place-holder \( B \) to make the equations more concise, we realize 
that the first right hand term in the next equation is recognizable as the expression \( \frac{\partial N}{\partial p_r} \frac{dp_{r_0}}{dr} \) 
from equation (5.40):

\[
\frac{\partial N}{\partial p_r} \delta p_r = -\zeta \left( \frac{dN_0}{dr} - \frac{\partial N}{\partial \rho} \frac{d\rho_0}{dr} \right) - B \left( N_0 - (p_{r_0} + \rho_0) \frac{\partial N}{\partial \rho} \right)
+ \frac{\partial N}{\partial \rho} \left[ \zeta \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right].
\]

Hence dividing this equation throughout by \( \frac{\partial N}{\partial p_r} \), since we are assuming that \( \frac{\partial N}{\partial p_r} \) is never 
equal to zero, we can solve for the perturbation in the radial pressure brought about by 
metric perturbations:

\[
\delta p_r = -\zeta \frac{\partial N}{\partial p_r} \frac{dp_{r_0}}{dr} - \frac{Bp_{r_0}}{p_{r_0} (\partial N/\partial p_r)} \left( N_0 - (p_{r_0} + \rho_0) \frac{\partial N}{\partial \rho} \right) + 
+ \frac{\partial N}{\partial \rho} \left[ \zeta \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right],
\]

\[
= -\zeta \frac{dp_{r_0}}{dr} - B\gamma p_{r_0} + \frac{\partial p_{r_0}}{\partial \rho_0} \left[ \zeta \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right].
\]

Replacing the place-holder \( B \) with its proper expression, we have,

\[
\delta p_r = -\zeta \frac{dp_{r_0}}{dr} - \gamma \frac{e^{\nu_0/2}p_{r_0} \partial}{\partial r} \left( r^2 \zeta e^{-\nu_0/2} \right) + \frac{\partial p_{r_0}}{\partial \rho_0} \left[ \zeta \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right]. 
\] \hspace{1cm} (5.41)

where we have defined the adiabatic index \( \gamma \), following Chandrasekhar as

\[
\gamma = \frac{1}{p_{r_0} (\partial N/\partial p_r)} \left( N_0 - (p_{r_0} + \rho_0) \frac{\partial N}{\partial \rho} \right) = \frac{1}{p_{r_0}} \left( N_0 \frac{\partial p_r}{\partial N} - (p_{r_0} + \rho_0) \frac{\partial p_r}{\partial \rho} \right). 
\] \hspace{1cm} (5.42)

Now that we have an expression for the perturbed pressure, we can continue solving for the 
pulsation equation, which required \( \delta p_r \) to be simplified.
5.3.5 Separation of variables

We now have in the form of equations (5.32) and (5.25), a set of constraints that the perturbations we are considering must obey. In order to separate the time dependence from the spatial dependence in all of these perturbations, we will assume the usual form of radial oscillations expected in this model, and postulate that all the perturbed fields can be expressed as the following:

\[ \zeta(r,t) \rightarrow \zeta(r) e^{i\sigma t} \implies v(r,t) = \frac{\partial \zeta}{\partial t} = i\sigma \zeta e^{i\sigma t}, \]  

(5.43)

as a result of which,

\[ \frac{dv}{dt} = -\sigma^2 \zeta e^{i\sigma t}. \]  

(5.44)

The pressures and density are assumed to follow similar time evolution, with the same frequency \( \sigma \) as above, and this results in the following postulates:

\[ \delta p_r \rightarrow \delta p_r e^{i\sigma t}, \quad \delta p_\perp \rightarrow \delta p_\perp e^{i\sigma t}, \quad \delta \rho \rightarrow \delta \rho e^{i\sigma t}, \]  

(5.45)

The metric coefficients will be similarly affected through this time dependence, and we thus get

\[ \delta \lambda \rightarrow \delta \lambda e^{i\sigma t}, \quad \delta \nu \rightarrow \delta \nu e^{i\sigma t}. \]  

(5.46)

The electromagnetic field component of the stress-energy will similarly pulsate through

\[ \delta \eta(r,t) = E_0\epsilon_0 e^{-\nu_0 t/2} \zeta(r,t) = E_0\epsilon_0 e^{-\nu_0 t/2} \zeta e^{i\sigma t}. \]  

(5.47)

5.3.6 The pulsation equation

We now have closed forms for all the different perturbations in equation (5.25). We want to deduce an equation for how the spatial dependence of the perturbations are constrained by the Einstein’s equations, to first order. In what follows the time dependence of all variables have been eliminated through the separation of variables method we have used previously, to give:


e^{\lambda_0 - \nu_0} (p_{r0} + \rho_0) \left[ -\sigma^2 \zeta e^{i\sigma r} \right] + \frac{\partial}{\partial r} \left[ (\delta p_r - \delta \eta) e^{i\sigma r} \right] + \frac{1}{2} \frac{d\nu_0}{dr} (\delta p_r + \delta \rho) e^{i\sigma r}

+ \frac{p_{r0} + \rho_0}{2} \frac{d(\delta \nu)}{dr} + \frac{2}{r} (\delta p_r - \delta p_\perp) e^{i\sigma r} = 0.

Since the oscillation frequency $\sigma$, does not depend on the spatial variables, it commutes with the derivatives in the above expression to yield:

$$
\sigma^2 e^{\lambda_0 - \nu_0} (p_{r0} + \rho_0) \zeta = \frac{\partial}{\partial r} (\delta p_r - \delta \eta) + \frac{d\nu_0}{dr} (\delta p_r + \delta \rho) + \frac{p_{r0} + \rho_0}{2} \frac{d(\delta \nu)}{dr} + \frac{2}{r} (\delta p_r - \delta p_\perp). 
$$

(5.48)

This equation can be further reduced if we recall that equation (5.32) gives us an expression for the partial derivative of one of the metric perturbations $\delta \nu$. Substituting the latter in the above, and rearranging terms results in the equation (5.49)

$$
\sigma^2 e^{\lambda_0 - \nu_0} (p_{r0} + \rho_0) \zeta = \frac{d}{dr} (\delta p_r - \delta \eta) + \left( \frac{2}{2} \right) \frac{d\nu_0}{dr} (\frac{\delta p_r + \delta \rho - \delta \eta}{2}) \frac{d\lambda_0}{dr} + \left( \frac{\delta p_r - \delta \eta}{2} + \frac{\delta p_r - \delta \eta}{2} \right) \frac{d\lambda_0}{dr} +

- \frac{p_{r0} + \rho_0}{2} \zeta \left( \frac{d\nu_0}{dr} + \frac{1}{r} \right) \left( \frac{d\lambda_0}{dr} + \frac{d\nu_0}{dr} \right) + \frac{2}{r} (\delta p_r - \delta p_\perp) 
$$

(5.49)

This equation is the one that we will be using to test the stability under first order perturbations (linear stability analysis) of any isotropic new interior solutions we will be using. Since everything in the above equation is in terms of the static variables, and all the terms are separately known in closed form, we can continue simplifying it into a workable Sturm-Liouville type problem.

The first person to do this was Chandrasekhar, and we will follow his method to express the above expression in two different parts. The first part will consist of all the parts that Chandrasekhar had to deal with in his derivation. Even this part will not be exactly what Chandrasekhar had, since our expressions for the terms in this equation, (e.g. $\delta p_r$), have additional contributions from anisotropy and electric charge not present in the original. However, intuitively we can see that the end result should be reducible to Chandrasekhar’s in the limit of zero anisotropic pressure and charge. With this general guideline in mind, we proceed
and systematically transfer all additional terms not present in Chandrasekhar into an “extra” part, like so

\[ \sigma^2 e^{\lambda_0 - \nu_0} (p_{r0} + \rho_0) \zeta = \underbrace{\text{Chandrasekhar}}_{\text{extra}} + \ldots. \]

We will show this division in the following equations by boxing the terms present in Chandrasekhar’s derivation, to keep track of how we are advancing in our simplification.

Equation (5.49) will first be separated as

\[ \sigma^2 e^{\lambda_0 - \nu_0} (p_{r0} + \rho_0) \zeta = \frac{\partial \delta p_r}{\partial r} + \delta p_r \left( \frac{d\nu_0}{dr} + \frac{1}{2} \frac{d\lambda_0}{dr} \right) + \frac{\delta \rho}{2} \frac{d\nu_0}{dr} + \frac{2}{r} (\delta p_r - \delta p_\perp) \]

\[ - \frac{1}{2} (p_{r0} + \rho_0) \zeta \left( \frac{d\nu_0}{dr} + \frac{1}{r} \right) \left( \frac{d\lambda_0}{dr} + \frac{d\nu_0}{dr} \right) - \frac{\delta \eta}{2} \left( \frac{d\nu_0}{dr} + \frac{d\lambda_0}{dr} \right) - \frac{\partial \delta \eta}{\partial r}, \quad (5.50) \]

where the boxed terms are present in Chandrasekhar’s derivation. Simplifying these only, i.e. substituting the closed forms for \( \delta p_r \) from equation (5.41) we have

\[ \frac{\partial}{\partial r} \delta p_r = \frac{\partial}{\partial r} \left\{ - \zeta p'_{r0} - \gamma B p_{r0} + \frac{\partial p_{r0}}{\partial \rho_0} \left[ \zeta \left( \frac{2\Pi}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right] \right\} \]

\[ = \frac{\partial}{\partial r} \left\{ - \zeta p'_{r0} - \gamma B p_{r0} \right\} \frac{\partial}{\partial \rho_0} \left[ \frac{\partial p_{r0}}{\partial \rho_0} \left[ \zeta \left( \frac{2\Pi}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right] \right\}, \]

and additionally,

\[ \delta p_r \left( \frac{d\nu_0}{dr} + \frac{1}{2} \frac{d\lambda_0}{dr} \right) \]

\[ = \left\{ - \zeta p'_{r0} - \gamma B p_{r0} + \frac{\partial p_{r0}}{\partial \rho_0} \left[ \zeta \left( \frac{2\Pi}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right] \right\} \left( \frac{d\nu_0}{dr} + \frac{1}{2} \frac{d\lambda_0}{dr} \right) \]

\[ + \frac{\partial p_{r0}}{\partial \rho_0} \left[ \zeta \left( \frac{2\Pi}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right] \left( \frac{d\nu_0}{dr} + \frac{1}{2} \frac{d\lambda_0}{dr} \right). \]

similarly with \( \delta \rho \) from equation (5.30), we get

\[ \frac{\delta \rho}{2} \frac{d\nu_0}{dr} = \frac{1}{2} \frac{d\nu_0}{dr} \left[ - \zeta \frac{d\rho_0}{dr} - e^{\nu_0/2} (p_{r0} + \rho_0) \frac{\partial}{\partial r} (e^{\nu_0/2} \frac{r^2}{2} \zeta) - \zeta \left( \frac{2\Pi}{r} + \frac{4\eta_0}{r} + \frac{d\eta_0}{dr} \right) - \delta \eta \right] \]

\[ = \frac{1}{2} \frac{d\nu_0}{dr} \left[ - \zeta \frac{d\rho_0}{dr} - e^{\nu_0/2} (p_{r0} + \rho_0) \frac{\partial}{\partial r} (e^{\nu_0/2} \frac{r^2}{2} \zeta) \right]. \]
This last equation is the final piece needed in the expression we started with, i.e. equation (5.50), we have the intermediate form of the pulsation equation, separated into the boxed part which Chandrasekhar derived, and the unboxed part resulting from anisotropic pressure and electric change densities,

\[ \sigma^2 e^{\lambda_0 - \nu_0} (p_{r0} + \rho_0) \zeta = \frac{\partial}{\partial r} (-\zeta p_{r0}' - \gamma B p_{r0}) - (\zeta p_{r0}' + \gamma B p_{r0}) \left[ \frac{d\nu_0}{dr} + \frac{1}{2} \frac{d\lambda_0}{dr} \right] \]

\[ + \frac{\partial}{\partial r} \left\{ \frac{d\rho_0}{\rho_0} \left[ \zeta \left( \frac{2 \Pi}{r} + \frac{4 \eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right] \right\} - \frac{1}{2} \frac{d\nu_0}{dr} \left[ \zeta \left( \frac{2 \Pi}{r} + \frac{4 \eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right] \]

\[ + \frac{1}{2} \left( p_{r0} + \rho_0 \right) \zeta \left[ \frac{d\nu_0}{dr} + \frac{1}{r} \left( \frac{d\lambda_0}{dr} + \frac{d\nu_0}{dr} \right) \right] + \frac{\partial p_{r0}}{\partial \rho_0} \left\{ \zeta \left( \frac{2 \Pi}{r} + \frac{4 \eta_0}{r} + \frac{d\eta_0}{dr} \right) + \delta \eta \right\} \left\{ \frac{d\nu_0}{dr} + \frac{1}{2} \frac{d\lambda_0}{dr} \right\} + \frac{2}{r} (\delta p_{r} - \delta p_{\perp}). \]

We notice immediately that there are sets of constants that appear often. In the interest of economy of equation length, we introduce two new auxiliary variables, \((\delta p_{r} - \delta p_{\perp}) \equiv \delta \Pi\), and \(\zeta (\frac{2 \Pi}{r} + \frac{4 \eta_0}{r} + \frac{d\eta_0}{dr}) + \delta \eta \equiv A\), while at the same time replacing some of the explicit \(r\)–derivatives, \(\frac{d}{dr}\) with primes (\('\)'). This reduces the pulsation equation to

\[ \sigma^2 e^{\lambda_0 - \nu_0} (p_{r0} + \rho_0) \zeta = \frac{\partial}{\partial r} \left( -\zeta p_{r0}' - \gamma B p_{r0} \right) - \zeta p_{r0}' \left( \frac{\lambda_0'}{2} + \nu_0 \right) - \gamma B p_{r0} \left( \nu_0' + \frac{\lambda_0'}{2} \right) \]

\[ + \frac{\partial}{\partial r} \left( A \frac{\partial p_{r0}}{\partial \rho_0} \right) + A \frac{\partial p_{r0}}{\partial \rho_0} \left( \frac{\lambda_0'}{2} + \nu_0 \right) - A \frac{\nu_0'}{2} + \frac{1}{2} \left( p_{r0} + \rho_0 \right) \zeta \left( \nu_0' + \frac{1}{r} \right) \left( \nu_0' + \frac{1}{r} \right) \]

\[ + \frac{\nu_0'}{2} \left[ -\zeta \frac{d\rho_0}{dr} - e^{\frac{\nu_0}{2} / (p_{r0} + \rho_0)} \frac{\partial}{\partial r} \left( e^{-\frac{\nu_0}{2} / (p_{r0} + \rho_0)} \right) \right] - \frac{\delta \eta}{2} (\nu_0' + \lambda_0') - \frac{\delta \eta}{2} + \frac{2}{r} \delta \Pi. \quad (5.51) \]

Here Chandrasekhar factorizes the boxed terms (the only ones he had) into a form that can be cast into a Sturm-Liouville problem. We will proceed similarly, and note in passing that while our boxed expressions and Chandrasekhar’s match, since our metric coefficients mean
different things, (since our energy-momentum tensor is anisotropic and admits non-zero charge) we will have to be creative in factorizing these expressions. Synthesizing all the boxed elements and reorganizing, we get

\[
\frac{\partial (\xi' p_{\tau 0})}{\partial r} - \gamma B p_{\tau 0} \left( \nu'_0 + \frac{\lambda'_0}{2} \right) - \frac{1}{2} (p_{\tau 0} + \rho_0) \varsigma \left( \nu'_0 + \frac{1}{r} \right) (\lambda'_0 + \nu'_0) + 2 \left( \frac{\lambda'_0 + \nu'_0}{2} \right).
\]

The first step in the factorization process is to notice that the second and third terms in the above equation can be written as a total derivative of a suitably chosen exponential, here 

\[
e^{-\frac{1}{2} (\lambda'_0 + \nu'_0) r^2} \frac{d}{dr} \left[ e^{\frac{1}{2} (\lambda'_0 + \nu'_0)} (\gamma B p_{\tau 0}) \right],
\]

as can be readily checked by expansion of the latter. With this factorization we get

\[
\frac{\partial (\xi' p_{\tau 0})}{\partial r} - e^{-\frac{1}{2} (\lambda'_0 + \nu'_0) r^2} \frac{d}{dr} \left[ e^{\frac{1}{2} (\lambda'_0 + \nu'_0)} (\gamma B p_{\tau 0}) \right] - \frac{1}{2} (p_{\tau 0} + \rho_0) \varsigma \left( \nu'_0 + \frac{1}{r} \right) (\lambda'_0 + \nu'_0) \\ - \frac{\nu'_0}{2} \left[ \varsigma \frac{d \rho_0}{dr} + e^{\frac{1}{2} (\lambda'_0 + \nu'_0)} \frac{\partial}{\partial r} \left( e^{-\frac{1}{2} (\lambda'_0 + \nu'_0) r^2} \right) - 2 \frac{\lambda'_0 + \nu'_0}{2} \right].
\]

The penultimate term is now expanded completely, and we add zero to the equation in a creative way, as shown:

\[
\frac{\partial (\xi' p_{\tau 0})}{\partial r} - e^{-\frac{1}{2} (\lambda'_0 + \nu'_0) r^2} \frac{d}{dr} \left[ e^{\frac{1}{2} (\lambda'_0 + \nu'_0)} (\gamma B p_{\tau 0}) \right] - \frac{1}{2} (p_{\tau 0} + \rho_0) \varsigma \left( \nu'_0 + \frac{1}{r} \right) (\lambda'_0 + \nu'_0) \\ - \frac{\nu'_0}{2} \left[ \varsigma \frac{d \rho_0}{dr} + (p_{\tau 0} + \rho_0) \left( \frac{2 \varsigma}{r} + \frac{d \varsigma}{dr} - \frac{\varsigma}{2} \frac{d \nu_0}{dr} \right) \right] - \varsigma \frac{p'_{\tau 0}}{2} \left( \frac{\lambda'_0}{2} + \nu'_0 \right) \\ + \frac{\nu'_0}{2} \left[ \varsigma p'_{\tau 0} + \varsigma \eta'_0 + \frac{2 \varsigma}{r} (p_{\tau 0} - p_{\tau 0} + 2 \eta_0) \right] \\ - \varsigma \frac{p'_{\tau 0}}{2} \left( \frac{\lambda'_0}{2} + \nu'_0 \right) \right] \right). 
\]

We immediately notice that the terms we are adding explicitly contain quantities that would not have been present in an uncharged and isotropic set of equations. However this new form will allow us to eliminate additional terms and factorize the expression a little bit more
into

\[ - \frac{\partial (\zeta'_{r0})}{\partial r} - e^{-(\lambda_0 + 2\nu_0)/2} \frac{d}{dr} \left[ e^{(\lambda_0 + 2\nu_0)/2} \gamma B_{p0} \right] - \frac{1}{2} (p_{r0} + \rho_0) \zeta \left( \nu_0' + \frac{1}{r} \right) (\lambda_0' + \nu_0') \\
- \frac{\nu_0'}{2} \left\{ \frac{d}{dr} \left[ \zeta (\rho_0 + p_{r0}) \right] + \frac{2\zeta}{r} (p_{r0} + \rho_0) - \zeta \left[ \frac{d p_{r0}}{d r} - \frac{2}{r} (p_{\perp0} - p_{r0} + 2\eta_0) - \frac{d \eta_0}{d r} + \frac{p_{r0} + \rho_0 d \nu_0}{2} \right] + \zeta \eta_0' + \frac{2\zeta}{r} (p_{\perp0} - p_{r0} + 2\eta_0) \right\} + \zeta p_{r0}' \left( \lambda_0' + \nu_0' \right) \]

which has the static Einstein expression (5.17e) spanning the second and third lines (slashed) and which is equal to zero. This can thus be removed, to give

\[ - \frac{\partial (\zeta'_{r0})}{\partial r} - e^{-(\lambda_0 + 2\nu_0)/2} \frac{d}{dr} \left[ e^{(\lambda_0 + 2\nu_0)/2} \gamma B_{p0} \right] - \frac{1}{2} (p_{r0} + \rho_0) \zeta \left( \nu_0' + \frac{1}{r} \right) (\lambda_0' + \nu_0') \\
- \frac{\nu_0'}{2} \left( \zeta (\rho_0 + p_{r0}) \right)' + \frac{2\zeta}{r} (p_{r0} + \rho_0) + \zeta \eta_0' + \frac{2\zeta}{r} (p_{\perp0} - p_{r0} + 2\eta_0) \right\} + \zeta p_{r0}' \left( \lambda_0' + \nu_0' \right) \]

We now reorganize the above equation to put the extra terms due to anisotropy and charge separately, employing the boxed notation to reference which part was Chandrasekhar’s, and which parts got added in our equations:

\[ - \frac{\partial (\zeta'_{r0})}{\partial r} - e^{-(\lambda_0 + 2\nu_0)/2} \frac{d}{dr} \left[ e^{(\lambda_0 + 2\nu_0)/2} \gamma B_{p0} \right] - \frac{1}{2} (p_{r0} + \rho_0) \zeta \left( \nu_0' + \frac{1}{r} \right) (\lambda_0' + \nu_0') \\
- \frac{\nu_0'}{2} \left\{ \zeta (\rho_0 + p_{r0}) \right\}' + \frac{2\zeta}{r} (p_{r0} + \rho_0) - \zeta \eta_0' \right\} - \zeta p_{r0}' \left( \lambda_0' + \nu_0' \right) \]

Following these simplifications, we are now in a position to recombine the two parts of the pulsation equation (boxed and unboxed) to give rise to the full pulsation equation:

\[ \sigma^2 e^{\lambda_0 - \nu_0} (p_{r0} + \rho_0) \zeta = - \frac{\partial (\zeta'_{r0})}{\partial r} - e^{-(\lambda_0 + 2\nu_0)/2} \left[ e^{(\lambda_0 + 2\nu_0)/2} \gamma B_{p0} \right]' - \frac{\delta \eta_0'}{2} (\nu_0' + \lambda_0') + \frac{1}{2} \frac{d \Pi}{d r} \\
- \frac{1}{2} (p_{r0} + \rho_0) \zeta \left( \nu_0' + \frac{1}{r} \right) (\lambda_0' + \nu_0') - \frac{\nu_0'}{2} \left\{ \zeta \eta_0' + \frac{2\zeta}{r} (p_{\perp0} - p_{r0} + 2\eta_0) \right\} - \zeta p_{r0}' \left( \lambda_0' + \nu_0' \right) \]

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expressions that can be obtained from rearranging (5.17c) to

\[
\frac{-\nu_0'}{2} \left[ \zeta (\rho_0 + p_{r0})' + \frac{2\zeta}{r} (p_{r0} + \rho_0) \right] + \frac{\partial}{\partial r} \left\{ A \frac{\partial p_{r0}}{\partial \rho_0} \right\} + A \frac{\partial p_{r0}}{\partial \rho_0} \left\{ \frac{\lambda_0}{2} + \nu_0' \right\} - \frac{A}{2} \nu_0' - (\delta \eta)'
\]

The same argument used previously involving factorization through an exponential can be applied to two terms involving the \( A \)s that are under-braced, to give

\[
\sigma^2 e^{\lambda_0-\nu_0} (p_{r0} + \rho_0) \zeta = -\frac{\partial (\zeta p_{r0}')}{\partial r} - \zeta p_{r0}' \left( \frac{\lambda_0}{2} + \nu_0' \right) - \frac{1}{2} (p_{r0} + \rho_0) \zeta \left( \nu_0' + \frac{1}{r} \right) \left( \lambda_0' + \nu_0' \right)
\]

\[
-\frac{\nu_0'}{2} \left\{ \zeta (p_{r0} + \rho_0)' + \frac{2\zeta}{r} (p_{r0} + p_{\perp0} + 2\eta_0) \right\} - \frac{\delta \eta}{2} (\nu_0' + \lambda_0') - \frac{A}{2} \nu_0' - (\delta \eta)' + \frac{2}{r} \delta \Pi
\]

\[
+ e^{-(\lambda_0+2\nu_0)/2} \left[ e^{(\lambda_0+2\nu_0)/2} A \frac{\partial p_{r0}}{\partial \rho_0} \right]' - e^{-(\lambda_0+2\nu_0)/2} \left[ e^{(\lambda_0+2\nu_0)/2} \gamma B p_{r0} \right]'.
\]

The next stage of the simplification is the substitution of the pressure derivative with the expressions that can be obtained from rearranging (5.17c) to

\[
\frac{\nu_0'}{2} = \frac{1}{p_{r0} + \rho_0} \left( A - \frac{\delta \eta}{\zeta} - p_{r0}' \right) \iff p_{r0}' = \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \eta_0 - \frac{\nu_0'}{2} (p_{r0} + \rho_0).
\]

This allows the first four terms of (5.52) to be combined since many common terms appear after this substitution. We now show this step term by term before combining everything: the first derivative becomes

\[
-\frac{\partial (\zeta p_{r0}')}{\partial r} = -\frac{\partial}{\partial r} \left[ \zeta \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \eta_0 - \frac{\nu_0'}{2} (p_{r0} + \rho_0) \right) \right] = -\left[ \zeta \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \eta_0 \right) \right]' + \zeta \nu_0'' (p_{r0} + \rho_0) + \frac{\nu_0'}{2} \frac{\delta \Pi}{(p_{r0} + \rho_0)',}
\]

which we combine with the second term,

\[
-\zeta \left( \frac{\lambda_0}{2} + \nu_0' \right) p_{r0} = \frac{\zeta}{2} (p_{r0} + \rho_0) \left( (\nu_0')^2 + \frac{\nu_0' \lambda_0'}{2} \right) - \zeta \left( \frac{\lambda_0}{2} + \nu_0' \right) \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \eta_0 \right),
\]

and the third,

\[
-\frac{1}{2} (p_{r0} + \rho_0) \zeta \left( \nu_0' + \frac{1}{r} \right) \left( \lambda_0' + \nu_0' \right) = -\frac{\zeta}{2} (p_{r0} + \rho_0) \left[ (\nu_0')^2 + \nu_0' \lambda_0 + \frac{\lambda_0}{r} + \nu_0' \right],
\]

with the forth,
\[-\frac{\nu'_0}{2}\left\{\left[\zeta(p_{\rho_0} + p_{r_0})\right]' + \zeta\eta'_0 + \frac{2\zeta}{r}(\rho_0 + p_{\perp 0} + 2\eta_0)\right\} = - \frac{\nu'_0}{2}\left[\zeta(p_{r_0} + \rho_0)\right]' - \frac{\zeta\eta'_0\nu'_0}{2}\]

\[-\frac{\nu'_0}{r}\left(\rho_0 + p_{r_0} - \Pi_0 + 2\eta_0\right) = - \frac{\nu'_0}{2}\left[\zeta(p_{r_0} + \rho_0)\right]' - \frac{\zeta\eta'_0\nu'_0}{r} - \frac{\zeta}{r}\nu'_0(p_{r_0} + \rho_0) + \frac{\zeta}{r}\nu'_0(\Pi_0 - 2\eta_0),\]

the resulting equation then becomes

\[-\frac{\nu'_0}{2}\left[\zeta(p_{r_0} + \rho_0)\right]' + \zeta\eta'_0 + \frac{2\zeta}{r}(\rho_0 + p_{\perp 0} + 2\eta_0)\right\} = - \frac{\nu'_0}{2}\left[\zeta(p_{r_0} + \rho_0)\right]' - \frac{\zeta\eta'_0\nu'_0}{2}\]

\[-\frac{\nu'_0}{r}\left(\rho_0 + p_{r_0} - \Pi_0 + 2\eta_0\right) = - \frac{\nu'_0}{2}\left[\zeta(p_{r_0} + \rho_0)\right]' - \frac{\zeta\eta'_0\nu'_0}{r} - \frac{\zeta}{r}\nu'_0(p_{r_0} + \rho_0) + \frac{\zeta}{r}\nu'_0(\Pi_0 - 2\eta_0),\]

By using a rearranged equation (5.11c) in the static case, we obtain terms very similar to the first brackets in the RHS of (5.53),

\[
\frac{16\pi G}{c^4}(p_{\perp 0} + \eta_0)e^{\lambda_0} = \left(\nu''_0 - \frac{\nu'_0\lambda'_0}{2} + \frac{(\nu'_0)^2}{2} + \frac{\nu'_0}{r} - \frac{\lambda'_0}{r}\right),
\]

and furthermore, the terms on the second line of equation (5.53) can be combined to produce another $A$, which we defined previously. Together these simplifications can be substituted into equation (5.53) to get a pulsation equation into the form

\[
\sigma^2 e^{\lambda_0-\nu_0}(p_{r_0} + \rho_0)\zeta = \frac{\zeta}{2}(p_{r_0} + \rho_0)\left(\frac{16\pi G}{c^4}(p_{\perp 0} + \eta_0)e^{\lambda_0} - \frac{(\nu'_0)^2}{2} - \frac{4\nu'_0}{r}\right) +
\]

\[-\frac{\zeta\nu'_0}{2}\left(\frac{4\eta_0}{r} - \frac{2\Pi_0}{r} + \frac{\nu'_0}{r}\right) + \frac{\delta\eta}{2}\nu'_0 - \frac{A}{2}\nu'_0 + \frac{2}{r}\delta\Pi +
\]

\[-\left(\frac{\lambda'_0}{2} + \nu'_0\right)\left[\zeta\left(\frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \eta'_0\right) + \delta\eta\right] - \left[\zeta\left(\frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \eta'_0\right) + \delta\eta\right]' +
\]

\[+ e^{-(\lambda_0+2\nu_0)/2}\left[\frac{16\pi G}{c^4}(p_{\perp 0} + \eta_0)(p_{r_0} + \rho_0)\zeta e^{\lambda_0} - \frac{\zeta}{4}(p_{r_0} + \rho_0)\nu'_0\left(\nu'_0 + \frac{8}{r}\right)\right].\]
\[ -\frac{\nu_0'}{2} \left[ \zeta \left( \frac{4\eta_0}{r} - \frac{2\Pi_0}{r} + \eta'_0 \right) + A - \delta \eta \right] + \frac{2}{r} \delta \Pi + e^{-(\lambda_0 + 2\nu_0)/2} \left\{ e^{(\lambda_0 + 2\nu_0)/2} A \left[ \frac{\partial \rho_0}{\partial \rho_0} - 1 \right] \right\}' - e^{-(\lambda_0 + 2\nu_0)/2} \left[ e^{(\lambda_0 + 2\nu_0)/2} \gamma \beta \rho_0 \right]' \] 

The term on the second line of the above simplifies too, to convert our pulsation equation into

\[ \sigma^2 e^{\lambda_0 - \nu_0} (p_{r0} + \rho_0) \zeta = \frac{8\pi G}{c^4} (p_{l0} + \eta_0)(p_{r0} + \rho_0) \zeta e^{\lambda_0} - \frac{\zeta}{4} (p_{r0} + \rho_0) \nu'_0 \left( \nu'_0 + \frac{8}{r} \right) + \] 

\[ + e^{-(\lambda_0 + 2\nu_0)/2} \left\{ e^{(\lambda_0 + 2\nu_0)/2} A \left[ \frac{\partial \rho_0}{\partial \rho_0} - 1 \right] \right\}' - e^{-(\lambda_0 + 2\nu_0)/2} \left[ e^{(\lambda_0 + 2\nu_0)/2} \gamma \beta \rho_0 \right]' + \] 

\[ - \nu'_0 \zeta \left( \frac{4\eta_0}{r} + \eta'_0 \right) + \frac{2}{r} \delta \Pi \] (5.55)

We continue the simplification by substituting equation (5.17c) again in the last term of the first line of equation (5.55), to get in a partial step that

\[ - \frac{\zeta}{4} (p_{r0} + \rho_0) \nu'_0 \left( \nu'_0 + \frac{8}{r} \right) = \frac{\zeta}{4} (p_{r0} + \rho_0) \nu'_0 \left( \nu'_0 + \frac{8}{r} \right) - \frac{2}{r} (p_{r0} + \rho_0) \nu'_0 \] 

\[ - \frac{\nu'_0}{2} \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \eta'_0 \right) \] (5.56)

The above equation is then combined with the last term on the first line of (5.55) and the combination of the last two terms of the latter results in

\[ \frac{p_{r0}'}{p_{r0} + \rho_0} (A - \delta \eta) - \frac{\zeta (p_{r0}')^2}{p_{r0} + \rho_0} - \zeta \left( \frac{\nu'_0}{2} \right) \left( \frac{2\Pi_0}{r} + \frac{12\eta_0}{r} + 3\eta'_0 \right) - \frac{4}{r} (A - \delta \eta) + \frac{4p_{r0}' \zeta}{r} \] 

after some tedious algebra. The boxed terms are again the only ones in Chandrasekhar’s analysis. Again substituting (5.17c) in above, and simplifying only the unboxed new terms at this point, we get

\[ \frac{4\zeta p_{r0}'}{p_{r0} + \rho_0} \left( \frac{\Pi_0}{r} + \frac{4\eta_0}{r} + \eta'_0 \right) - \frac{4\zeta}{r} \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \eta'_0 \right) - \] 

\[ - \frac{\zeta}{p_{r0} + \rho_0} \left( \frac{2\Pi_0}{r} + \frac{4\eta_0}{r} + \eta'_0 \right) \left( \frac{2\Pi_0}{r} + \frac{12\eta_0}{r} + 3\eta'_0 \right), \]
allowing us to write down the final form of the pulsation equation with metric coefficients appearing only in the exponentials, and with the terms resulting from anisotropy and charge appearing prominently only from the static contributions,

\[ \sigma^2 e^{\lambda - \nu} (p_r + \rho) \zeta = \frac{8\pi G}{c^4} (p_{r0} - \Pi_0 + \eta_0)(p_{r0} + \rho_0) \zeta e^{\lambda} + \frac{2 \delta \Pi - \frac{\zeta(p'_r)^2}{p_{r0} + \rho_0}}{p_{r0} + \rho_0} e^{-\left(\lambda_0 + 2\nu_0\right)/2} \left\{ e^{\left(\lambda_0 + 2\nu_0\right)/2} \left[ \frac{2 \Pi_0}{r} + \frac{4 \eta_0}{r} + \eta'_0 \right] + \delta \eta \left[ \frac{\partial p_{r0}}{\partial \rho_0} - 1 \right] \right\}' - \frac{4 \delta \Pi - \frac{\zeta(p'_r)^2}{p_{r0} + \rho_0}}{p_{r0} + \rho_0} e^{-\left(\lambda_0 + 2\nu_0\right)/2} \left[ e^{\left(\lambda_0 + 2\nu_0\right)/2} \gamma \frac{p_{r0}}{r^2} \left( r^2 \zeta e^{-\nu_0/2} \right) \right]' + \frac{4 \zeta p'_r \zeta}{r} \right\} - \frac{4 \Pi_0}{r} \left( \frac{2 \Pi_0}{r} + \frac{4 \eta_0}{r} + \eta'_0 \right) \left( \frac{2 \Pi_0}{r} + \frac{12 \eta_0}{r} + 3 \eta'_0 \right). \tag{5.57} \]

Since only static zero subscripted values appear in the above, we may without confusion remove all the zero-subscripts from the values in all subsequent expressions. We now check this completely general equation against special cases to check for consistency with the literature.

First, if we were to set \( \Pi = 0 \) for isotropy and \( \eta = 0 \) for no charge in the above, we immediately get

\[ \sigma^2 e^{\lambda - \nu} (p_r + \rho) \zeta = \frac{8\pi G}{c^4} (p_r + \rho) \zeta e^{\lambda} - \frac{\zeta(p'_r)^2}{p_r + \rho} e^{-\left(\lambda + 2\nu\right)/2} \left[ e^{\left(\lambda + 2\nu\right)/2} \gamma \frac{p_r}{r^2} \left( r^2 \zeta e^{-\nu/2} \right) \right]' + \frac{4 \zeta p'_r \zeta}{r} \]

exactly as expected from Chandrasekhar's result. Similarly, setting just \( \Pi = 0 \) for isotropy, but \( \eta \neq 0 \) for electric charge results in

\[ \sigma^2 e^{\lambda - \nu} (p_r + \rho) \zeta = \frac{8\pi G}{c^4} (p_r + \eta) (p_r + \rho) \zeta e^{\lambda} - e^{-\left(\lambda + 2\nu\right)/2} \left[ e^{\left(\lambda + 2\nu\right)/2} \gamma \frac{p_r}{r^2} \left( r^2 \zeta e^{-\nu/2} \right) \right]' \]

\[ + e^{-\left(\lambda + 2\nu\right)/2} \left\{ e^{\left(\lambda + 2\nu\right)/2} \left[ \frac{4 \eta}{r} + \eta' \right] + \delta \eta \left[ \frac{\partial p_r}{\partial \rho} - 1 \right] \right\}' - \frac{\zeta(p'_r)^2}{p_r + \rho} + \frac{4 \zeta p'_r}{p_r + \rho} \left( \frac{4 \eta}{r} + \eta' \right) - \frac{\zeta(p'_r)^2}{p_r + \rho} \left( \frac{4 \eta}{r} + \eta' \right) \left( \frac{12 \eta}{r} + 3 \eta' \right), \]

which is equivalent to Glazer's pulsation equation, with a slight notation change \( (\epsilon \to \rho_r) \) and different factorization of terms. In contrast setting \( \eta = 0 \) for no electric charge but \( \Pi \neq 0 \)
for anisotropic pressures results in

\[
\sigma^2 e^{\lambda - \nu} (p_r + \rho) \zeta = \frac{8\pi G}{c^4} (p_r - \Pi)(p_r + \rho) \xi e^{\lambda} + \frac{2}{r} \delta \Pi - \frac{\zeta (p_r')^2}{p_r + \rho}
\]

\[
+ e^{-(\lambda + 2\nu)/2} \left\{ e^{(\lambda + 2\nu)/2} \left( \frac{2\Pi}{r} \left[ \frac{\partial p_r}{\partial \rho} - 1 \right] \right) \right\} - e^{-(\lambda + 2\nu)/2} \left[ e^{(\lambda + 3\nu)/2} \frac{p_r}{r^2} \left( r^2 \xi e^{\nu/2} \right) \right]'
\]

\[
+ 4\zeta p_r' \frac{\zeta}{r} + 4\zeta p_r' \frac{(\Pi)}{r} - 4\zeta \left( \frac{2\Pi}{r} \right) + \frac{\zeta}{p_r + \rho} \left( \frac{4\Pi^2}{r^2} \right).
\]

as given in [Dev and Gleiser](#) with the caveat that the latter use a slightly different definition of the anisotropy so that \( \Pi \to -\Pi \), make a few sign mistakes along the way, and use the \( G = c = 1 \) normalization.

With the final form of the pulsation equation (5.57), and the boundary conditions ensuring that the radial pulsations are such that

\[
\zeta = 0 \quad \text{at} \quad r = 0 \quad \text{or equivalently} \quad \zeta \sim r \quad \text{as} \quad r \to 0,
\]

so that the fluid incurs no radial motion when at the centre, and

\[
\delta p_r = 0 \quad \text{at} \quad r = r_b
\]

in accordance with the definition of the boundary of the star, and the Israel-Darmois condition, the radial stability of the star reduces to an eigenvalue problem for the pulsation frequency \( \sigma \) with amplitude \( \zeta \). In [Chandrasekhar](#), this equation is integrated, after multiplication by the integrating factor \( r^2 \xi e^{(\lambda + \nu)/2} \), and then over the whole range of \( r \), giving

\[
\sigma^2 \int_0^{r_b} r^2 e^{(3\lambda - \nu)/2}(\rho - p_r)\zeta^2 \, dr = \frac{8\pi G}{c^4} \int_0^{r_b} r^2 (p_r - \Pi)(p_r + \rho) e^{(3\lambda + \nu)/2} \zeta^2 \, dr
\]

\[
+ 2 \int_0^{r_b} r e^{(\lambda + \nu)/2} (\delta \Pi) \zeta \, dr - \int_0^{r_b} r^2 e^{(\lambda + \nu)/2}(\rho', p_r + \rho) e^{(3\lambda + \nu)/2} \zeta^2 \, dr
\]

\[
+ \int_0^{r_b} r^2 e^{-\nu/2} \left\{ e^{(\lambda + 4\nu)/2} \left[ \frac{2\Pi}{r} + \frac{4\eta}{r} \right] r + \frac{\eta'}{r} \right\} \zeta \, dr
\]

\[
- \int_0^{r_b} \frac{\gamma p_r}{r^2} \left[ e^{(\lambda + 3\nu)/2} \right] \gamma p_r (r^2 \xi e^{\nu/2}) \zeta \, dr
\]

\[
+ \int_0^{r_b} r^2 p_r' \frac{(\Pi)}{p_r + \rho} \left( \frac{2\Pi}{r} + \frac{4\eta}{r} \right) e^{(\lambda + \nu)/2} \, dr - 4 \int_0^{r_b} r e^{(\lambda + \nu)/2} \left( \frac{2\Pi}{r} + \frac{4\eta}{r} \right) \zeta^2 \, dr
\]




\[- \int_0^{r_b} \frac{r^2 \zeta^2}{p_r + \rho} e^{(\lambda + \nu)/2} \left( \frac{2\Pi}{r} + \frac{4\eta}{r} + \eta' \right) \left( \frac{2\Pi}{r} + \frac{12\eta}{r} + 3\eta' \right) \, dr. \quad (5.60)\]

Integration by parts generates the struck out term, while the boundary conditions cause the former to vanish. The other integrals that have not been simplified have to be integrated for each specific solution, once all the different metric functions \(\lambda\) and \(\nu\), pressure \(p_r\), density \(\rho\), anisotropy \(\Pi\), and charge \(\eta\) have been specified. The amplitude of the radial oscillation \(\zeta\), also needs to be specified, and in the literature, different trial functions such that the boundary conditions are satisfied are picked, while making sure that the \(\zeta\)'s simplify the integrals at the same time.

However, Bardeen et al. rewrite the pulsation equation in a canonical Sturm-Liouville form first, and for completeness, we obtain this form too. The differential equation (5.57) can be multiplied by the integrating factor \(r^2\zeta e^{(\lambda + \nu)/2}\), explicitly shown in the following after having additionally imposed geometrical units such that \(G = c = 1\), to get

\[\sigma^2 r^2 e^{(3\lambda - \nu)/2}(\rho + p_r)\zeta^2 - 8\pi r^2(p_r - \Pi + \eta)(p_r + \rho) e^{(3\lambda + \nu)/2} \zeta^2 - 2r e^{(\lambda + \nu)/2}(\delta\Pi)\zeta \]

\[+ \frac{r^2 \zeta^2}{p_r + \rho} e^{(\lambda + \nu)/2}(p_r')^2 - 4rp_r' e^{(\lambda + \nu)/2} \zeta^2 + e^{-\nu/2} \left[ e^{(\lambda_0 + 3\nu_0)/2} \gamma \frac{p_{r0}}{r^2} (\nu_0^2 e^{(\nu_0)/2}) \right]' r^2 \]

\[- r^2 e^{-\nu/2} \left\{ e^{(\lambda + 2\nu)/2} \left[ \zeta \left( \frac{2\Pi}{r} + \frac{4\eta}{r} + \eta' \right) + \delta\eta \right] \left[ \frac{\partial p_r}{\partial \rho} - 1 \right] \right\}' \zeta \]

\[- 4r^2 \zeta^2 p_r' \left( \frac{\Pi}{r} + \frac{4\eta}{r} + \eta' \right) e^{(\lambda + \nu)/2} + 4r e^{(\lambda + \nu)/2} \left( \frac{2\Pi}{r} + \frac{4\eta}{r} + \eta' \right) \zeta^2 \]

\[+ \frac{r^2 \zeta^2}{p_r + \rho} e^{(\lambda + \nu)/2} \left( \frac{2\Pi}{r} + \frac{4\eta}{r} + \eta' \right) \left( \frac{2\Pi}{r} + \frac{12\eta}{r} + 3\eta' \right) = 0. \quad (5.61)\]

From the above equation we can deduce a generalized Sturm-Liouville form, i.e. equation (5.61) can be put in the form

\[f \left\{ \frac{d}{dr} \left[ P(r) \frac{df}{dr} \right] + (Q(r) + \sigma^2 W(r)) f + R \right\} = 0, \quad (5.62)\]

for the function \(f \neq 0\), since it encodes the radial perturbations which cannot vanish in a perturbation calculations, and which is defined through \(\zeta = \frac{e^{\nu/2} f(r)}{r^2}\). This substitution then
This would be a simple Sturm-Liouville problem if and only if $R$ could be absorbed in $Q$, and we will see that with our assumptions about $\delta \Pi$ and $\delta \eta$ this is indeed the case. Then, since $f \neq 0$, the parenthesized part of equation (5.62) vanishes, and we retrieve a Sturm-Liouville problem, with weight $W$ so that associated with this problem are the orthogonal eigenfunctions corresponding to the different eigenfrequencies. The orthogonality relation obeyed by this equation is then

$$
\int_0^{r_b} e^{(3\lambda-\nu)/2}(\rho + p_r) r^2 \zeta^i \zeta^j \, dr = \delta^{ij} = \int_0^{r_b} e^{(3\lambda+\nu)/2} \frac{(\rho + p_r)}{r^2} f^i f^j \, dr = \delta^{ij},
$$

(5.64)

with $\delta^{ij}$ being the Kronecker delta and $\zeta^i$ or equivalently $f^i$ the eigenfunctions associated with eigenfrequency $\sigma^i$. Similarly the boundary conditions that need to be satisfied by the functions $f$ stemming from the BCs on $\zeta$ from (5.58) and (5.59) now become

$$
f(r = 0) \sim r^3, \quad \text{and} \quad \delta p_r(r = r_b) = 0.
$$

(5.65)

This weight function, and the BCs will be useful when we start computing the integrals in the next section.

Before ending this section, we recap what has been achieved so far. We provide in the above equations the complete first order radial pulsation equation valid for all solutions admit-
ting both electric charge and pressure anisotropy, in all their generality. If one were to find
new spherically symmetric and static solution through various means [15, 73], and even in-
clude electric fields, and/or anisotropic pressures in those solutions, then one could use equa-
tion (5.60) to investigate its stability right-away, without going through the lengthy deriv-
ation we just presented. The result is general enough to be used in spherically symmetric and static cases where

1. The baryon number $N$ and the electric charge density $\eta$ are both functions of the mass
density $\rho$, the radial pressure $p_r$, and the radial coordinate $r$ only;

2. the anisotropic pressure $p_\perp$ is a function of the radial pressure $p_r$ and the radius $r$ only.

If these are satisfied, then this first order pulsation equation (5.60), and the associated eigen-
function orthogonality relation (5.64) can be used to test the model against radial perturba-
tions.

5.4 Applying the stability criterion on our solutions

In our solutions, we have expressions for the metric functions $\nu, \lambda$, and matter functions
$\rho, p_r$. However due to the complicated and lengthy expressions involved, it will be more
convenient to pick test functions $\zeta$ that simplify the integrals of (5.60) without requiring
explicit expansion of those functions. Furthermore the expressions for $\Pi$ and $\eta$ are simple
enough in our solutions that their computation will not be overly taxing, and we will simplify
those. In particular $\Pi = -\Delta$ in our solutions, with $\Delta = \beta r^2$, that is always some polynomial
of $r$ allows the computation of derivatives such as $\Pi'$ easily. As a result,

$$\delta \Pi = \Pi' \delta r = -2\beta r \zeta.$$  (5.66)

We also calculated the perturbation equation for the electromagnetic component of the stress
tensor, $\eta$ in equation (5.36). In our solutions we assume that $\eta = q^2/(\kappa r^4)$, with $q^2 = k^2 r^6$. 122
As a result, we have \( \eta = (kr)^2/\kappa \), allowing us to find the expression for \( \eta' = 2k^2r/\kappa \). To calculate the perturbation in \( \eta \), we proceed through equation (5.36) which states that 

\[ \delta \eta = \zeta E_0 e^{-\nu/2} \]

Substituting all the relevant quantities results in

\[ \delta \eta = 6k^2/\kappa r \zeta \] (5.67)

and using the fact that the static electric field is \( E_0 = (q e^{(\lambda_0+i\nu)/2})/r^2 \), while the static charge density is \( \epsilon_0 = [6k/\kappa] e^{-\lambda_0/2} \), from equation (4.38), we can finally proceed to simplify the integral equation (5.60).

Indeed, with these two ingredients, we can calculate most of the terms in the integral equation (5.60), which simplifies to

\[
\sigma^2 \int_0^{r_b} r^2 e^{(3\lambda-\nu)/2}(\rho + p_r) \zeta^2 \, dr = \kappa \int_0^{r_b} r^2 p_r (p_r + \rho) e^{(3\lambda+\nu)/2} \zeta^2 \, dr \\
+ \kappa \int_0^{r_b} r^4 \left( \beta + \frac{k^2}{\kappa} \right) (p_r + \rho) e^{(3\lambda+\nu)/2} \zeta^2 \, dr + \int_0^{r_b} \frac{\gamma p_r}{r^2} (r^2 \zeta e^{-\nu/2})^2 e^{(\lambda+3\nu)/2} \, dr \\
- \int_0^{r_b} \frac{r^2 \zeta^2}{p_r + \rho} e^{(\lambda+\nu)/2}(p'_r)^2 \, dr + \int_0^{r_b} r^2 e^{-\nu/2} \left\{ e^{(\lambda+2\nu)/2} \zeta_r \left[ \frac{12k^2}{\kappa} - 2\beta \right] \left[ \frac{\partial p_r}{\partial \rho} - 1 \right] \right\}' \, dr \\
+ \int_0^{r_b} 4r^3 e^{(\lambda+\nu)/2} \zeta_r \left( \frac{6k^3}{\kappa} - \beta \right) e^{(\lambda+\nu)/2} \, dr - 8 \int_0^{r_b} r^2 \zeta^2 e^{(\lambda+\nu)/2} \left( \frac{3k^2}{\kappa} - \beta \right) \, dr \\
+ 4 \int_0^{r_b} r p'_r e^{(\lambda+\nu)/2} \zeta^2 \, dr - \int_0^{r_b} 4r^4 e^{(\lambda+\nu)/2} \left( \frac{3k^2}{\kappa} - \beta \right) \left( \frac{9k^2}{\kappa} - \beta \right) \, dr. \tag{5.68}
\]

In this form, finding the frequency \( \sigma \) depends on guessing a correct test function that will allow the computation of the integrals of the above equation. Chandrasekhar could do this with much less effort since he did not have as many terms to satisfy at the same time. If we try to copy and adapt the method of Esculpi and Alomà [42], we find that the same test functions do not yield analytic closed form integrals for our case. Since we only wish to find the frequency and do not require the eigenfunctions for some \( \zeta \), we turn to numerical integration. However, since this is the case, we decided to use Bardeen, Thorne, and Meltzer’s formulation, since theirs is a clearer formulation for numerical work [3]. We do this next.
5.4.1 Numerical integration to obtain the fundamental frequency

To calculate the the fundamental mode we follow \cite{3} instead, since in their formulation, simple numerical integration obviates the need to guess an accurate test function for $\zeta$: usually a hard procedure. Before being able to use their result however, we have to absorb the $R$ term in equation (5.71) into $Q$ or $P$. since we have expressions for the perturbations of the electric field and anisotropy now in the form of (5.67) and (5.66) respectively, we proceed to simplify $R$. The first term containing the perturbed anisotropy simplifies as

$$-\frac{2(\delta \Pi)}{r} e^{(\lambda+2\nu)/2} f = \frac{4\beta}{r^2} e^{(\lambda+3\nu)/2} f^2,$$

which can then be absorbed into $Q$, since it contains the $f^2$ term. The second term in $R$ can similarly be simplified as

$$- f \left\{ \left[ \frac{e^{(\lambda+3\nu)/2}}{r^2} \left( \frac{2\Pi}{r} + \frac{4\eta}{r} + \eta' \right) f + e^{(\lambda+2\nu)/2} \delta \eta \right] \left[ \frac{d}{d\rho} \left( \frac{d}{d\rho} - 1 \right) \right] \right\}' =$$

$$f \left[ 2\beta - \frac{12k^2}{\kappa} \right] \left\{ \left[ \frac{d}{d\rho} \left( \frac{d}{d\rho} - 1 \right) \right] \right\}' =$$

$$f \left[ \left( \frac{d}{d\rho} \left( \frac{d}{d\rho} - 1 \right) \right) \left( \frac{e^{(\lambda+3\nu)/2}}{r} f' + \frac{e^{(\lambda+3\nu)/2}}{r} \left( \frac{d}{d\rho} - 1 \right) f \right) \right],$$

so that the second term can be absorbed in $Q$. However, since the first term of the above equation cannot be easily eliminated in this general form, we will have to find an additional integrating factor before being able to reduce it to a simple Sturm-Liouville form. We therefore use ideas from Horvat, Ilijić, and Marunović who look at a slightly different problem involving quasi-local quantities \cite{63} to be able to simplify this more generalized problem.

We first expand out the pulsation equation again and express it as a canonical second order PDE of the form

$$C_2 f'' + C_1 f' + C_0 f = -\sigma^2 f,$$  \hspace{1cm} (5.69)\footnote{As was pointed out by Dr. Gene Couch through private communication, it is the form of $R$ after substituting the expressions of $\delta \Pi$ and $\delta \eta$ from our solutions that allow for the existence of this integrating factor, Were we to have had the general expression for $R$, this step should not have been as straight forward. Indeed we are not claiming that a general nonlinear differential equation with any $R$ is liable to this same simplification method.}
with

\[ C_0 = -8\pi \left[ p_r + \left( \beta + \frac{k^2}{\kappa} \right) r^2 \right] e^\nu - \frac{24 e^{\nu - \lambda}}{p_r + \rho} \left( \frac{k^2}{\kappa} - \frac{\beta}{2} \right) + \frac{(p_r')^2 e^{\nu - \lambda}}{(p_r + \rho)^2} - \frac{4p_r' e^{\nu - \lambda}}{r(p_r + \rho)} + \frac{4r p_r' e^{\nu - \lambda}}{(p_r + \rho)^2} \left( \frac{6k^2}{\kappa} - \beta \right) + \frac{4r^2 e^{\nu - \lambda}}{(p_r + \rho)^2} \left( \frac{3k^2}{\kappa} - \beta \right) \left( \frac{9k^2}{\kappa^2} - \beta \right) + \frac{2r^2}{(p_r + \rho)} e^{(3\lambda + \nu)/2} \left( \beta - \frac{6k^2}{\kappa} \right) \left[ \frac{e^{(\lambda + 3\nu)/2}}{r} \left( \frac{\partial p_r}{\partial \rho} - 1 \right) \right]' \]

\[ C_1 = \left( \frac{\gamma p_r e^{(\lambda + 3\nu)/2}}{r^2} \right) \frac{r^2 e^{-(3\lambda + \nu)/2}}{p_r + \rho} + \frac{2r e^{\nu - \lambda}}{p_r + \rho} \left( \beta - \frac{6k^2}{\kappa} \right) \left( \frac{dp_r}{d\rho} - 1 \right) \]

\[ C_2 = \frac{P}{W} = \frac{\gamma p_r e^{\nu - \lambda}}{p_r + \rho}. \]

Then by multiplying equation (5.69) by another integrating factor

\[ F(r) = \exp \left( \int_0^r \frac{C_1(\bar{r}) - C_2'(\bar{r})}{C_2(\bar{r})} \, d\bar{r} \right) \implies F'(r) = \frac{C_1(r) - C_2'(r)}{C_2(r)} F(r), \]

a factor that depends crucially on both the anisotropy \( \beta \) and charge \( k \). We see that (5.69) then becomes

\[ (C_2 F) f'' + (C_1 F) f' + (C_0 F) f = -\sigma^2 F f, \]

which is factorizable into a true Sturm-Liouville equation (5.62) with the following coefficients:

\[ P = FC_2 = \frac{\gamma p_r e^{(\lambda + 3\nu)/2}}{r^2} \left\{ \exp \left[ \left( \beta - \frac{6k^2}{\kappa} \right) \int_0^r \frac{2\bar{r}}{\gamma p_r} \left( \frac{\partial p_r}{\partial \rho} - 1 \right) \, d\bar{r} \right] \right\}, \]

\[ Q = FC_0 = \left\{ \exp \left[ \left( \beta - \frac{6k^2}{\kappa} \right) \int_0^r \frac{2\bar{r}}{\gamma p_r} \left( \frac{\partial p_r}{\partial \rho} - 1 \right) \, d\bar{r} \right] \right\} \times \left\{ -\frac{4p_r' e^{(3\nu + \lambda)/2}}{r^3} \right\} + \frac{8\pi e^{3(\lambda + \nu)/2}}{r^2} \left[ p_r + \left( \beta + \frac{k^2}{\kappa} \right) r^2 \right] \left( p_r + \rho \right) - \frac{24 e^{(3\nu + \lambda)/2}}{r^2} \left( \frac{k^2}{\kappa} - \frac{\beta}{2} \right) + \frac{4 e^{(3\nu + \lambda)/2}}{p_r + \rho} \left( \frac{3k^2}{\kappa} - \beta \right) \left( \frac{9k^2}{\kappa^2} - \beta \right) - \frac{4p_r' e^{(3\nu + \lambda)/2}}{r(p_r + \rho)} \left( \frac{6k^2}{\kappa} - \beta \right) + 2\left( \beta - \frac{6k^2}{\kappa} \right) \left[ \frac{e^{(\lambda + 3\nu)/2}}{r} \left( \frac{\partial p_r}{\partial \rho} - 1 \right) \right]' + \frac{e^{(3\nu + \lambda)/2}(p_r')^2}{r^2(p_r + \rho)}, \]

\[ W = F = \frac{(p_r + \rho) e^{(3\lambda + \nu)/2}}{r^2} \left\{ \exp \left[ \left( \beta - \frac{6k^2}{\kappa} \right) \int_0^r \frac{2\bar{r}}{\gamma p_r} \left( \frac{\partial p_r}{\partial \rho} - 1 \right) \, d\bar{r} \right] \right\}, \]

\[ R = 0 \quad \text{in true Sturm-Liouville canonical form}, \]

(5.71)
since \((FC_2)^\prime \prime + (FC_0)f = -\sigma^2 F f\), yields (5.70) after simplification, as can be checked explicitly by expansion.

We can therefore and finally use the main result of Sturm-Liouville theory (refer to appendix \([A]\)) which directly gives the fundamental frequency of normal modes of our model. If this fundamental mode is positive, then all the other modes are too, and the model is deemed stable. This fundamental frequency is given \([3]\) by the maximal value of

\[
\sigma_0^2 = \int_0^{r_b} \left[ P \left( \frac{d f}{d r} \right)^2 - Q f^2 \right] dr,
\]

(5.72)

where \(f\) is taken to be a function that obeys the normalization condition

\[
\int_0^{r_b} W f^2 dr = 1,
\]

(5.73)

and the BCs given in (5.65). Finding a function \(f\) that obeys the normalization condition (5.73) is \textit{a priori} difficult. So if we wish to eschew this additional condition, we could instead modify (5.72) to

\[
\sigma_0^2 = \frac{\int_0^{r_b} \left[ P \left( \frac{d f}{d r} \right)^2 - Q f^2 \right] dr}{\int_0^{r_b} W f^2 dr},
\]

(5.74)

but now require that the normalization on different eigenfunctions \(f_i\) to the Sturm-Liouville system (5.71) be such that the weight \(W\) annihilates the integral:

\[
\int_0^{r_b} W f_i f_j dr = 0, \quad \text{whenever} \quad i \neq j.
\]

We are free to choose different functions \(f\), however the true value of the fundamental frequency will only be obtained with a proper eigenfunction of the above equation. Since we only want to test for stability, and do not want the actual frequency of the fundamental mode, we will choose the simplest \(f\) possible, e.g. \(f = r^3\), in accordance with (5.65), and then find the value of \(\sigma_0\) by numerically computing the integrals involved in equation (5.74), with the coefficients \(P, Q\) and \(W\) given by (5.71). We did not write our own integral methods, instead relying on the proved and tested QUADPACK suite of integration routines available in MAXIMA. These can deal with all type of integrals, even oscillating ones through a
| Solution                  | Parameters               | $\sigma_0^2$ (Hz) | Stable? |
|---------------------------|--------------------------|-------------------|---------|
| Natural Tolman VII        | $\mu = 1$                | 54.8              | Y       |
|                           | $\rho_c = 7.43 \times 10^{-10}$ m$^{-2}$ |                   |         |
|                           | $r_b = 1 \times 10^4$ m   |                   |         |
|                           | $k = 0$                  |                   |         |
|                           | $\beta = 0$              |                   |         |
| Self-bound Tolman VII     | $\mu = 0.7$              | 77.6              | Y       |
|                           | $\rho_c = 7.43 \times 10^{-10}$ m$^{-2}$ |                   |         |
|                           | $r_b = 1 \times 10^4$ m   |                   |         |
|                           | $k = 0$                  |                   |         |
|                           | $\beta = 0$              |                   |         |
| Tolman VII with anisotropy| $\mu = 1$                | 91.4              | Y       |
|                           | $\rho_c = 7.43 \times 10^{-10}$ m$^{-2}$ |                   |         |
|                           | $r_b = 1 \times 10^4$ m   |                   |         |
|                           | $k = 0$                  |                   |         |
|                           | $\beta \sim a/3 = 1 \times 10^{-17}$ m$^{-4}$ |            |         |
| Tolman VII with anisotropy| $\mu = 1$                | 821               | Y       |
|                           | $\rho_c = 7.43 \times 10^{-10}$ m$^{-2}$ |                   |         |
|                           | $r_b = 1 \times 10^4$ m   |                   |         |
|                           | $k = 0$                  |                   |         |
|                           | $\beta \sim 10a/3 = 1 \times 10^{-16}$ m$^{-4}$ |         |         |
| Self-bound Tolman VII with anisotropy | $\mu = 0.7$          | 139               | Y       |
|                           | $\rho_c = 7.43 \times 10^{-10}$ m$^{-2}$ |                   |         |
|                           | $r_b = 1 \times 10^4$ m   |                   |         |
|                           | $k = 0$                  |                   |         |
|                           | $\beta \sim a/3 = 1 \times 10^{-17}$ m$^{-4}$ |            |         |
| Tolman VII with anisotropy and charge | $\mu = 1$          | 91.1              | Y       |
|                           | $\rho_c = 7.43 \times 10^{-10}$ m$^{-2}$ |                   |         |
|                           | $r_b = 1 \times 10^4$ m   |                   |         |
|                           | $k = 1 \times 10^{-10}$ m$^{-2}$ |                   |         |
|                           | $\beta \sim a/3 = 1 \times 10^{-17}$ m$^{-4}$ |            |         |
| Tolman VII with anisotropy and charge | $\mu = 1$          | 215               | Y       |
|                           | $\rho_c = 7.43 \times 10^{-10}$ m$^{-2}$ |                   |         |
|                           | $r_b = 1 \times 10^4$ m   |                   |         |
|                           | $k = 9 \times 10^{-10}$ m$^{-2}$ |                   |         |
|                           | $\beta \sim a/3 = 1 \times 10^{-17}$ m$^{-4}$ |            |         |
| Tolman VII with anisotropy and charge | $\mu = 1$          | -1360             | N       |
|                           | $\rho_c = 7.43 \times 10^{-10}$ m$^{-2}$ |                   |         |
|                           | $r_b = 1 \times 10^4$ m   |                   |         |
|                           | $k = 5 \times 10^{-9}$ m$^{-2}$ |                   |         |
|                           | $\beta \sim a/3 = 1 \times 10^{-17}$ m$^{-4}$ |            |         |

Table 5.2: Eigenfrequency of the fundamental mode of various solutions with different parameter values, and their stability
number of methods designed to robustly integrate and constrain the numerical errors within tight bounds. We provide the MAXIMA routine we used in appendix C and here provide a table 5.2 with different parameter values, and an approximate fundamental mode frequency obtained for those parameter values. We note that the positivity of this fundamental frequency is the only criterion needed to prove the stability of the solution for those parameter values, and that the value of $\sigma$ is only approximately equal to the true eigenfrequency, since our calculations depend on the test function $f$ we chose. However the sign of the fundamental frequency is correct (i.e. positive), since if even one test function gives a positive value, we can be certain that some others will too since the true eigenfrequency is the maximum value of all possible $\sigma$ when we span the $L^2$ function (Lebesgue) space, according to theorem 7 given in appendix A.

5.5 Conclusion

In this chapter we have shown that all of our new solutions, together with the original Tolman VII solution are stable under first order radial perturbations, for certain values of the parameters. Of course, for “excessive” charge or anisotropy, the stability is compromised, as expected. This results adds to the heuristics we discussed in the beginning, and complements the conclusions reached that our solutions can indeed be stable. If any instability is to occur in these solutions, while the charge and anisotropy are within “normal” ranges, it will occur from second order effects, from non-linear perturbations, or from non-radial pulsations: all of which are beyond the scope of this work.

Our final calculations were done numerically, and a better way to approach this would have

\footnote{Using large $k \sim 1 \times 10^{-9} m^{-2}$ resulted in the $Q$ integral not converging suggesting that even the QUADPACK routines have trouble dealing with the integrals when $\beta \sim 6k^2/\kappa$, since then integrating factor, and hence the integrand accumulate many numerical errors. Furthermore, it also seems that any charge at all, without anisotropy distabilises the star, suggesting that purely charged solutions are unstable. Since we did the calculations numerically, we can only guess the maximum $k$ value allowed, but further work should be able to find it.}
been to find a suitable test function $\zeta$ that would have allowed an expression of the fundamental frequency to be obtained as a function of both $\Pi$ and $k$. This would have placed constraints on the maximum allowed anisotropy and charge of our model. We did not proceed in this direction because such an endeavour would have taken more time, and we were only interested in showing the viability of our model for modelling stars. This approach should lead to interesting results in charge and anisotropic bounds in the future.

The proof of stability relied on finding the normal mode frequency of linear radial oscillations which were obtained in a generalized way from the non-static Einstein equations. Our general expressions are valid for charged and anisotropic solutions too: to our knowledge, this generalized formulation, in both Bardeen et al.’s formulation in the form of equations (5.71) and (5.62); and Chandrasekhar’s formulation in the form of equation (5.61) is new, and will be useful to prove the stability for all static solutions, new and old that contain electric charge and/or anisotropic pressures.

Recently there has been a renewed interest in the stability of neutron stars, particularly because pulsation in such stars could potentially produce measurable gravitation waves. Krüger et al., for example [71] look at polar modes of perturbations in the stars, a considerably more difficult problem, to see if gravitational modes would be generated. The pulsation equation we generated is only for radial modes, but could presumably be extended to study the seismology of stars in a similar fashion. In a similar vein, Chirenti et al. look at $f$—mode (fundamental modes) oscillation as we do, but pay more attention to the damping of the mode, thus requiring more than just a linear approximation for the perturbation calculations, in the search of universal relations over many EOS, for gravitational waves again. This could also be a way to continue this work [24].

On this note, we move on to look at the predictions of the new solutions in the next chapter.
Chapter 6

Analysis of new solutions

We investigate the solutions we found previously in chapter 4 and deduce the behaviour of the matter variables, and metric functions. We interpret these in view of using these solutions to model compact stars, and to achieve this goal, we test a number of criteria that is believed to be necessary for these mathematical solutions to be viable as models of actual astrophysical objects. We also compute observables such as the masses and radii of the models and compare them with observed values of neutron star masses and radii, showing that some of our models might indeed describe actual stars.

6.1 Properties of the fluid as described by the new solutions

We will now analyse the solutions found previously, starting with those that include anisotropic pressures, and moving to the charged ones with anisotropic pressures later. Along the way we will provide conditions that the parameters must satisfy to abide by the constraints of physicality we impose. Other constraints stemming from the definiteness of the metric functions will also be used to further restrict the values of our parameters. Just as a reminder, and as introduced in chapter 3, a brief interpretation of our ansätze in terms of physically meaningful concepts will be discussed.

6.1.1 The free parameters

The fundamental ansatz we have used consistently is the Tolman VII ansatz that assumes a specific form of the metric variable \( Z = 1 - br^2 + ar^4 \). In Tolman’s solution, the coefficients \( a \) and \( b \) of this quartic function correspond to well defined combinations of three different
physical values: the central density $\rho_c$, the coordinate radius of the boundary $r_b$, and the self-boundness parameter $\mu$. In the more general charged case however, the charge density $k$ also determines the $a$ coefficient, and this changes the simple and straight-forward interpretation we had in Tolman VII.

The other parameter that can be interpreted is $\beta$, which governs the difference between the tangential pressure $p_{\perp}$ and the radial pressure $p_r$. In the isotropic cases for example, $\beta = 0$, and in one specific case which we called “anisotropised charge,” $\beta$ can be used to express the charge parameter $k$, so that only one of these last two variables is independant, and thus enough to specify that particular solution.

This list of parameters, \{\rho_c, r_b, \mu, \beta\} for the uncharged case, and \{\rho_c, r_b, \mu, \beta, k\} for the charged case are free by construction. They correspond to the exact number of parameters expected for the system of differential equations associated with each case and therefore all the integration constants used in the solutions can be expressed uniquely in terms of the respective set of parameters only. As a result of this construction, and immediate interpretation of the constants, we can already impose naive restrictions on the values of these parameters.

In particular, to model a realistic star, the central density has to be positive definite so that $\rho_c > 0$, or else we would be talking about matter having undefined characteristics. Similarly the boundary radius of stars has to be positive definite too, and we immediately get $r_b > 0$. To get similar bounds on $\mu$, we investigate the expression for mass density that is also common to all our solutions, in the form of equation 3.3, which we rewrite here

$$\rho = \rho_c \left[1 - \mu \left(\frac{r}{r_b}\right)^2\right].$$

Clearly if $\mu$ is negative, we will have increasing mass density with increasing $r$. This is not what we would like for a stable configuration of matter\(^1\) so we restrict $\mu$ to values that are greater than zero. Similarly, by modelling a star’s interior, we will naturally restrict the

\(^1\)It is Weinberg who famously said "It is difficult to imagine that a fluid sphere with a larger density near the
coordinate \( r \) to values less than the boundary, so that \( r \leq r_b \). As a result, \( r/r_b \leq 1 \), implying that having \( \mu > 1 \) will again result in negative densities, which we want to avoid. As a result, we will also restrict \( 0 \leq \mu \leq 1 \).

Naively this is unfortunately as far as we can go. To restrict the values of these parameters, and the other constants further we will need additional constraints.

6.1.2 Constraints for physical relevance

We will list a series of constraints that have been discussed in the literature [20, 31, 38, 43]. These are simple and “obvious” criteria that gravitationally stable spherical balls of matter should have. These in one form or another have been used to restrict parameter values allowed by interior solutions to Einstein’s equations. We shall use this set of criteria in the following sections to understand and interpret the solutions we have found. The list we will use is:

(i) The metric coefficients must be regular (not be singular, and be at least differentiable) everywhere including at the centre of the star when \( r = 0 \).

(ii) The metric functions must match an exterior solution (the Schwarzschild exterior or the Reissner-Nordström) to the Einstein equations at the boundary where \( r = r_b \).

(iii) The integrated “observables” including total charge, total mass, and proper radius must correspond to those parameters in the exterior vacuum solution matched at the boundary.

(iv) The radial pressure \( p_r \) must be positive and finite everywhere inside the fluid, including the centre \( r = 0 \).

(v) The radial pressure must vanish at the boundary, \( p_r(r_b) = 0 \).

surface than near the centre could be stable” [130]. He was proved wrong with certain anisotropic models [63], but the statement remains a good rule of thumb.
(vi) The tangential pressure must be equal to the radial pressure at the centre of the star,
\[ p_r(r = 0) = p_\perp(r = 0) = p_c \implies \Delta(r = 0) = 0. \]

(vii) All three, the pressures \( p_r \) and \( p_\perp \), and the density \( \rho \) must be decreasing functions of \( r \) so that their first derivatives with respect to \( r \) is negative everywhere, except possibly at \( r = 0 \), and at \( r = r_b \), where it could be zero.

(viii) The energy strong condition, the most restrictive one of the energy conditions [57] states that for realistic matter, with our type of energy-momentum, we must have that
\[ \sum_\alpha p_\alpha + \rho \geq 0, \]

(ix) The speed of pressure (sound) waves \( v_s = \sqrt{\frac{d p_r}{d \rho}} \) is causal in the interior, so that in geometrical units, \( 0 \leq v_s \leq 1 \).

(x) The speed of sound decreases monotonically with increasing coordinate \( r \) [1].

We shall now look at these conditions in detail, and determine which ones can be implemented without an explicit solution. These we will apply directly, and then in specialized sections we will look at those conditions that require the full solutions, and constrain the latter further in their respective sections.

6.1.3 Implementing the constraints

The first condition [i] requires the complete analytic form of both metric functions and as a result we have to wait before we can implement it completely. Of interest in implementing this condition is the capacity to express the integration constants in terms of the elements of our parameter list. In all the solutions we consider, the \( Z \) metric coefficient is expressible in the form \( 1 - br^2 + ar^4 \), where \( a \) and \( b \) are slightly different functions of the parameter list depending on the solution we look at. However for all the solutions this metric function is equal to unity at \( r = 0 \), and as a result this condition is automatically satisfied. In view of this, we can modify this particular constraint to read
(i) The $Y$ metric coefficient must be regular (not be singular, and be at least differentiable) everywhere including at the centre of the star when $r = 0$.

Considering the next two constraints (ii) and (iii), these were imposed as one of the boundary conditions used to generate our solutions: indeed, our computation of the integration constants $c_1$ and $c_2$ crucially depended on these particular assumptions, and all the solutions we have proposed so far automatically obey these constraints. As a result we do not impose these constraints explicitly again, using them instead to check the consistency of the final solutions at the last stage in the form of “inside-and-outside” metric plots for different parameter values. Of physical importance however is the value of the external observables mentioned in condition (iii). In our cases these correspond to the mass $M = m(r_b)$, electric charge $Q = q(r_b)$, and proper radius $R = \int_0^{r_b} \sqrt{\frac{\dd r}{Z(r)}}$. Since these quantities depend only on the $Z$ metric function (and the $Z$ metric function does not change drastically from solution to solution), we can compute these quantities right away to get

$$M = m(r_b) = 4\pi \rho_c r_b^3 \left( \frac{1}{3} - \frac{\mu}{5} \right) + \frac{k^2 r_b^5}{10}, \quad (6.1)$$

for the mass. The charge is simply given by

$$Q = kr_b^3, \quad (6.2)$$

since we defined the charge density indirectly through an integral incorporating the metric function instead in equation (4.38). As a result the difficulty arises in calculating the charge density of the fluid sphere instead of the total charge. For the proper radius of the sphere we have instead

$$R = \int_0^{r_b} \frac{\dd \tilde{r}}{\sqrt{1 - \tilde{r}^2 + a\tilde{r}^4}} = \frac{1}{2\sqrt{a}} \left[ \log \left( \frac{2\sqrt{a(1 - br_b^2 + ar_b^4)} + 2ar_b^2 - b}{2\sqrt{a} - b} \right) \right], \quad (6.3)$$

where we have used integration tables from Reference [50] to compute the final form of the integrand. It is immediately clear that since $R$ must be well-defined, we must have that
2\sqrt{a} > b$, which translates into a well defined constraint on our physically interpretable parameters. However as we mentioned before, the exact values of $a$ and $b$ depend on which solutions we are considering. As a result the conditions (i) and (iii) reduce to $2\sqrt{a} > b$.

We now consider condition (iv) and to implement it we look at the expression of the pressure in our solutions. Without specifying a solution, consider equation (4.26), which gives us

$$\kappa p_r = \frac{2(1 - br^2 + ar^4)}{r} \left( \frac{1}{V} \frac{dY}{dr} \right) + 2b - 4ar^2 - \kappa \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] > 0.$$ 

However even this form of the equation proves insufficiently simple to be able to deduce any constraints directly from it. We therefore leave condition (iv) for consideration later when we have a more definite form of the pressure.

Condition (v) is already implemented as the second boundary condition in chapter 4 and all our solution obey it by construction. This condition is technically equivalent to the Israel-Darmois junction condition on the metric and derivatives as has been shown for example in [85, 105], and as we discussed in Appendix A.

The next condition (vi) concerns the tangential pressure, and is due to spherical symmetry. The only way to admit an anisotropic pressure, while still having spherical symmetry is to ensure that the pressures $p_r$ and $p_\perp$ are equal at the centre of the star. This forces our anisotropy measure $\Delta$ to vanish at $r = 0$. By construction, in all solutions, we posited $\Delta = \beta r^2$, which satisfies this condition, since $\Delta = 0$ at $r = 0$.

The strong energy condition (viii) has to be used to provide a constraint on the type of matter we can have in our solutions. This is easily implemented once we have expressions of the density and pressures, and we cannot really implement it until we have specified parameter values for those quantities. We will show how we test this in each solution’s section, and just add here that in both the anisotropic case and the charged anisotropic cases this condition reduces to $p_r + 2p_\perp + \rho \geq 0$, so that this condition unfortunately tell us nothing about the electric charge, or charge densities involved.
Causality is one of the important conditions, and we implement this through condition (ix) by enforcing that the speed of pressure (sound) waves in the fluid not propagate at arbitrary speeds. The speed of light, $c$, in vacuum is taken to be unity in all our calculations, and to impose this condition we require that the speed of sound waves be less than one. To help us impose this, we first have to find an expression for the speed of these waves.

The speed of sound

As we saw in appendix A, the speed of pressure waves is given by $v^2 = \frac{dp}{d\rho}$. We can either invert the density relation we have and derive an equation of state once we have obtained the pressure from the $Y$ metric coefficient, or we could simply compute $v^2 = \frac{dp}{dr} / \frac{d\rho}{dr}$. However in doing the latter we have to be careful, since we are mostly interested in the behaviour of the speed at the centre where $r = 0$, and the density relation has a turning point there so that $\frac{d\rho}{dr} = 0$ when $r = 0$. In view of this, we carefully proceed by taking the limit $\lim_{r \to 0} \left( \frac{dp}{dr} / \frac{d\rho}{dr} \right)$, which we then check against the actual expression obtained for the speed of sound from the equation of state: This we did in the Tolman VII case, where luckily both methods give the same result showing that the “short-cut” evaluation is actually valid even at $r = 0$.

We assume for the time being that a similar result holds for the more general cases we will be considering in this chapter: the reason being that the mathematical structure of the new solutions is not very different from Tolman VII. However we keep in mind that this must be checked later on when we have a full expression for the pressures. The reason for wanting to evaluate the speed of sound without specifying a solution for the metric functions in detail is two–fold: a) doing this now ensures a certain independence from the more inconvenient details (values ranges of constants) we will have to deal with later on, b) were we to get a

\[2\text{The reason for being interested in the value at } r = 0 \text{ is simply because the additional constraints we have make it so that the speed of sound is monotonically decreasing with increasing } r. \text{ Since the maximum value of the speed of sound occurs at the centre (an expected result, from an intuitive Newtonian picture), we should compute it there to impose causality more efficiently.}\]
finite speed that is unconditionally larger than the speed of light, we could reject a solution class right here, without going through a complicated calculation involving a solution that is obviously unphysical.

We provide such a derivation now, starting from the definition of the “measure of anisotropy,” \( \Delta = \kappa(p_r - p_\perp) \) in chapter 4. However, we note that we are dealing with the charged case in the most general formulation of the problem, ending up with more terms in this equation than in the previous chapter. Also this derivation is more transparent in the original metric variables \( \lambda \) and \( \nu \), so we re-express everything in this equivalent set:

\[
\Delta = e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} - e^{-\lambda} \left( \frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{(\nu')^2}{4} + \frac{\nu' - \lambda'}{2r} \right) + \frac{2q^2}{r^4}. \tag{6.4}
\]

We need to simplify, rearrange and factorise this equation first, and recognise that some parts can be expressed as the derivative of the pressure variable. This derivation yielding the famous Tolman–Oppenheimer–Volkoff (TOV) equation in the uncharged and isotropic case was obtained by Oppenheimer and Volkoff [93], and we follow in their footsteps here.

We multiply equation (6.4) by \(-2/r\), and move the charge term on the left hand side to give

\[
\frac{4q^2}{r^5} - \frac{2\Delta}{r} = \frac{2}{r} \left[ e^{-\lambda} \left( \frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{(\nu')^2}{4} + \frac{\nu' - \lambda'}{2r} \right) - e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} \right].
\]

Next we want to group expressions that could be factorised as the derivative of a product, and choose \( e^{-\lambda} \) as a common factor to give

\[
\frac{4q^2}{r^5} - \frac{2\Delta}{r} = e^{-\lambda} \left( \frac{\nu''}{r} - \frac{\nu'}{r^2} - \frac{2}{r} \right) - \lambda' e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{2r} \right) + e^{-\lambda} \left( \frac{\nu'}{2} \right) \left( \frac{\nu'}{r} \right) + \frac{2}{r^3},
\]

we then add zero to the second bracket in the above equation in the form shown to be able to isolate product derivatives in the next step as shown,

\[
\frac{4q^2}{r^5} - \frac{2\Delta}{r} = e^{-\lambda} \left( \frac{\nu''}{r} - \frac{\nu'}{r^2} - \frac{2}{r} \right) - \lambda' e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} + \frac{\nu'}{2r} \right) + e^{-\lambda} \left( \frac{\nu'}{2} \right) \left( \frac{\nu'}{r} \right) + \frac{2}{r^3},
\]

\[
= e^{-\lambda} \frac{d}{dr} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) + \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) \frac{d}{dr} \left( e^{-\lambda} \right) - \frac{d}{dr} \left( \frac{1}{r^2} \right) + \frac{e^{-\lambda} \nu'}{2} \left( \frac{\nu'}{r} + \frac{\lambda'}{r} \right),
\]

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This last step allows us to factorise the derivative terms into an expression resembling the second Einstein’s equation (4.25b) with the other term being related to equation (4.26):

\[
\frac{4q^2}{r^5} - \frac{2\Delta}{r} = \frac{d}{dr} \left[ e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r} \right] + \frac{\nu'}{r} \left[ e^{-\lambda} \left( \frac{\nu' + \lambda'}{r} \right) \right].
\]

Substituting the latter two in the above equation thus results in a generalised TOV equation having both charge and anisotropic pressures (through \( \Delta \))

\[
\frac{4q^2}{r^5} - \frac{2\Delta}{r} = \frac{d}{dr} \left( \kappa p_r - \frac{q^2}{r^4} \right) + \frac{\nu'\kappa(p_r + \rho)}{2},
\]

upon rearranging we have the final form of the generalised TOV equation which has the coveted radial pressure derivative in terms of other variables:

\[
\frac{dp_r}{dr} = \frac{2qq'}{\kappa r^4} - \frac{2\Delta}{\kappa r} - \frac{\nu'}{2} (p_r + \rho).
\] (6.5)

The density \( \rho \) does not change from Tolman’s in our solutions previously, so that we can calculate its \( r \)-derivative to get

\[
\frac{d\rho}{dr} = -\frac{2\rho_c \mu r}{r_b^2}.
\] (6.6)

From these two, we can compute the formal speed of sound, technically valid only for \( r \neq 0 \), but extensible to that case too,

\[
v_s^2 = \left( \frac{dp_r}{dr} \right) \left( \frac{d\rho}{dr} \right) = \left( \frac{r_b^2}{\kappa \rho_c \mu} \right) \left[ \frac{\nu'\kappa (p_r + \rho)}{4r} - \frac{qq'}{r^5} + \frac{\Delta}{r^2} \right].
\] (6.7)

This equation allows us to understand how adding different parameters and assumptions like charge and anisotropic pressures modify the speed of sound in a very intuitive manner. For zero charge \( q \), and zero anisotropy \( \Delta \), we get back the same speed of sound as in the Tolman VII case from equation (3.23) as expected. For models with charge only, we expect the speed of sound to be lower than the same model with no charge, suggesting a “softening” of the equation of state in the presence of electric charge. Similarly addition of some positive (negative) anisotropy “stiffens” (“softens”) the equation of state, making the speed of pressure
waves larger (smaller) than the model would have had in the isotropic case. Finally with both charge and anisotropic pressures, the contributions of each could somehow conspire to not change the overall “stiffness” of the EOS, with the “stiffening” effect of anisotropy contributing more (since it scales $\sim 1/r^2$) than the softening of the electric charge (which scales $\sim 1/r^5$) if the anisotropy is positive.

After this derivation of an important quantity that will be useful in the analysis done in this chapter, we can continue looking at the criteria of physical applicability. Condition (x) is easily implemented from our newly crafted speed of sound expression. This condition demands that $dv_{\nu}/dr < 0$ so that the speed of sound is maximum at the centre of the star.

From equation (6.7), it is clear that without prior knowledge of $\nu'$ or $p_r$, this is difficult to implement, and hence we wait until we have full solutions in order to use this condition to extract restrictions on our parameters.

The next condition (vii) is easier to implement and check since we already have derivatives of all the relevant variables. The density derivative is always given by (6.6), and is obviously always negative since the the other parameters in the equation are positive. The pressure derivative given by (6.5) is more complicated, however we know from the Tolman VII solution that the last term in that equation is always negative. Anisotropic contributions through positive $\Delta$ will only make this derivative more negative, so that should not cause any problems. However picking negative $\Delta$’s might offset the last Tolman term, and this particular condition gives us that in the uncharged case,

$$\Delta < \frac{\kappa \nu'}{4} (p_r + \rho) r \quad \Rightarrow \quad \beta < \frac{\kappa \nu'}{4r} (p_r + \rho)$$

which gives us an idea as to the range $\beta$ can take, if it is negative. A similar argument applies for the charge, and the inequality above becomes more complicated in the most general case where we could presumably have negative $\beta$ and large charge that could potentially force the pressure derivative to become positive in the star. These are cases we have to ensure against
when we get our solutions.

The final part of this condition concerns the tangential pressure $p_{\perp}$. Its $r$-derivative has to be negative too, and since $p_{\perp} = p_r - \Delta$, we only have to check that

$$\frac{dp_{\perp}}{dr} = \frac{dp_r}{dr} - \frac{d\Delta}{dr} < 0 \Rightarrow \frac{dp_r}{dr} < \frac{d\Delta}{dr}.$$  

Since $\Delta = \beta r^2$ in our solutions, we want $\beta > \frac{1}{2r} \frac{dp_r}{dr}$, a condition we can combine with the previous one in the right circumstances.

This concludes our discussion of the conditions for physical relevance. We should note that most of these depend on the final form of the metric functions $Y$ and equivalently $\nu$ to be implemented, but promise to restrict our parameter space depending on the type of solution.

We should also keep in mind that this list is not exhaustive: other more stringent criteria might become important, for example from stability analysis, or from more accurate thermodynamics.

In the next section, we go into the details of each solution in full, check the behaviour of the matter and metric functions, and by ensuring that the conditions above hold, restrict the range of applicability of the solutions we found to interesting physical cases when possible.

## 6.2 The solutions

We will look at each solution in turn, and determine what parameter values are valid for each. For reference of which solution is being discussed, please refer to the relevant section in chapter [4] and for a quick list of the various functions and expressions please refer to appendix [B]. We start with the anisotropic uncharged generalisations to Tolman VII.

### 6.2.1 The $\phi^2 = 0$ case from [4.1.1]

This is a special case of the general anisotropic pressure case, where the pressure anisotropy $\beta$ is fixed to the value of $-a$. To help us in evaluating all the different conditions, it will be
helpful to compute first the expressions for quantities at both the centre of the star when \( r = 0 \) and at the matter–vacuum boundary, when \( r = r_b \). The \( Z \) metric for example gives

\[
Z(0) = 1, \quad \text{and} \quad Z(r_b) = 1 - \kappa \rho_c r_b^2 \left( \frac{1}{3} - \frac{\mu}{5} \right). \tag{6.8}
\]

Since \( Z \) is the metric function as it appears in the line element, it cannot change sign, so that \( Z(r_b) > 0 \). This allows us to use the second equation above to conclude that

\[
\rho_c < \frac{15}{(5 - 3\mu)\kappa r_b^2}, \tag{6.9}
\]

a relation that will be needed later.

To check our first condition [0] we have to ensure that the metric function \( Y \) is regular in the interior of the solution. The expression for \( Y \) is given in equation (4.11) as

\[
Y(r) = \gamma + \frac{2\alpha r_b}{\sqrt{\kappa \rho_c \mu/5}} \left[ \operatorname{arcoth} \left( \frac{1 - \sqrt{Z(r)}}{\kappa r_b} \right) - \operatorname{arcoth} \left( \frac{1 - \gamma}{r_b \sqrt{\kappa \rho_c \mu/5}} \right) \right].
\]

Evaluating this metric function at \( r = r_b \) annihilates the square bracket, which makes \( \gamma \), a finite number as the answer for \( Y(r_b) \). For the value of \( Y(0) \), we have to take a formal limit of the function in the first \( \operatorname{arcoth} \), since substitution evaluation results in an undefined \( 0/0 \). Using l’Hôpital’s rule on this function results in

\[
\lim_{r \to 0} \left[ \operatorname{arcoth} \left( \frac{1 - \sqrt{Z(r)}}{r_b \sqrt{\kappa \rho_c \mu/5}} \right) \right] = \lim_{r \to 0} \left[ \operatorname{arcoth} \left( \frac{-1/2 Z^{-1/2}}{2r} \sqrt{\kappa \rho_c \mu/5} \right) \right] = \operatorname{arcoth}(\infty) = 0,
\]

the second limit depending on \( Z(0) = 1 \). As a result of this \( Y(0) \) reduces to the finite value of

\[
Y(0) = \gamma - \frac{2\alpha r_b}{\sqrt{\kappa \rho_c \mu/5}} \left[ \operatorname{arcoth} \left( \frac{1 - \gamma}{r_b \sqrt{\kappa \rho_c \mu/5}} \right) \right], \tag{6.10}
\]

so that we are reasonably sure that since \( Y \) is regular at both \( r = 0 \) and \( r = r_b \), the extreme values of \( r \), it remains so for every value of \( 0 \leq r \leq r_b \), proving that condition [0] is satisfied by this solution.
We now show a plot of these metric functions in their two different forms for specific parameter values in figure 6.1. Turning to conditions (ii)–(iii), which reduce to $2\sqrt{a} > b$, we have for this particular case that

$$2\sqrt{\frac{\kappa \mu \rho_c}{5r_b^2}} > \frac{\kappa \rho_c}{3} \implies \rho_c < \frac{36\mu}{5\kappa r_b^2},$$

a limit on the maximum value of the central density for a given type of star (spanning from natural with $\mu = 1$ to various other ones with different “self-boundness”) at a given radius $r_b$. The only additional assumption that went into this relation is the positive values of all parameters in the above equation. We keep this in mind to restrict our parameter space later.

The next condition (iv) concerning the positive definiteness of the pressure is harder to im-
To test this, we first write down the pressure in terms of known variables as

\[
p_{r}(r) = \frac{2\kappa \rho_{c}}{3} - \frac{4\kappa \rho_{c} \mu r^2}{5r_b^2} - \kappa \rho_{c} \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + \\
\left( \frac{\kappa \rho_{c}}{3} - \frac{\kappa \rho_{c} \mu}{5} \right) \frac{\sqrt{1 - \frac{5\rho_{c} r^2}{3} + \frac{\kappa \rho_{c} \mu r^4}{5r_b^2}}}{\gamma + \frac{2\alpha r_b}{\sqrt{\kappa \rho_{c} \mu / 5}}} \arcoth \left( \frac{1}{\gamma \frac{r_b \sqrt{\kappa \rho_{c} \mu / 5}}{Z(r)}} \right) - \arcoth \left( \frac{1 - \gamma}{r_b \sqrt{\kappa \rho_{c} \mu / 5}} \right)
\]

(6.12)

First we remember that the pressure \( p_{r} \) is zero at the boundary, and the expression we have should give this same result. We can consider this a consistency check on our arithmetic, and indeed, since \( \gamma = \sqrt{Z(r_b)} \), and the vanishing of the square brackets in the denominator of (6.12), we are left with the simple

\[
p_{r}(r_b) = \frac{2\kappa \rho_{c}}{3} - \frac{4\kappa \rho_{c} \mu}{5} - \kappa \rho_{c} \left[ 1 - \mu \right] + \left( \frac{\kappa \rho_{c}}{3} - \frac{\kappa \rho_{c} \mu}{5} \right) = 0,
\]

confirming that at least the expression of the pressure is consistent.

Evaluating this expression at \( r = 0 \), results in having to evaluate \( Y(0) \), which we already found before in equation (6.10), and \( Z(0) \) which is just one. With these, we get that

\[
p_{r}(0) = \kappa \rho_{c} \left( \frac{5 - 3\mu}{15 \left\{ \gamma - \frac{2\alpha r_b}{\sqrt{\kappa \rho_{c} \mu / 5}} \left( \frac{1 - \gamma}{r_b \sqrt{\kappa \rho_{c} \mu / 5}} \right) \right\} - \frac{1}{3} \right)
\]

an expression that is complicated enough that we cannot immediately infer its behaviour. We will see this happen many times in the course of this chapter, and as a result we will depend heavily on graphing these functions to determine their behaviour. Since our parameter space is four dimensional in the uncharged chase, we will be forced to assume specific values for certain parameters, and we will of course use the constraints we already have to pick these values consistently.

Since we are interested in modelling neutron stars, whose typical radii \( r_b \) is of the order of tens of kilometres, and that have typical central densities of the order of nuclear densities, we
use these as baseline values and compute the pressures for different values of $\mu$. We check first that our previous inequalities are satisfied: inequality (6.9) gives that $\rho_c < 1.6 \times 10^{18} \text{kg} \cdot \text{m}^{-3}$ when $\mu = 0$, and $\rho_c < 4.0 \times 10^{18} \text{kg} \cdot \text{m}^{-3}$ when $\mu = 1$. The second inequality (6.11) instead gives, $\rho_c < 0$ when $\mu = 0$, scaling linearly with it, and $\rho_c < 3.9 \times 10^{18} \text{kg} \cdot \text{m}^{-3}$ when $\mu = 1$. Thus taking the more restrictive inequality to dictate the value of $\rho_c$, we pick $\rho_c = 1 \times 10^{18} \text{kg} \cdot \text{m}^{-3}$ and $r_b = 10 \text{km}$, as they satisfy both inequalities even up to $\mu = 0.6$, and plot the radial pressure for these choices in figure 6.2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.2.png}
\caption{The radial pressure in the interior of the star for this solution. The parameter values are $\rho_c = 1 \times 10^{18} \text{kg} \cdot \text{m}^{-3}$, $r_b = 1 \times 10^4 \text{m}$ and $\mu$ varies between 1 and 0.6.}
\end{figure}

It is immediately clear that the pressures are negative, something no normal fluid should have, even though every precaution in choosing values for our parameters was taken. However this is a well known issue with interior solutions: more often than not, and very clearly here, the solutions found behave non-physically. Declaring this solution unphysical, having failed criterion (iv) no further analysis of this solution will be carried out.
6.2.2 The $\phi^2 < 0$ case from 4.1.3

In this case, $\beta$ is no longer fixed to one value as previously: instead it takes on a range of possible values and as long as the inequality $\beta < -\frac{\kappa \rho_c}{5r_b^2}$ is satisfied, the value of $\phi$ will be appropriate for this solution. We first consider the metric functions in their two forms for different value of $\mu$ through plots in figure 6.3.

![Graphs of Y(r), Z(r), Lambda(r), and Nu(r) metric functions](image)

**Figure 6.3:** Variation of metric variables $Y(r)$, and $Z(r)$ in Ivanov’s formulation, and $\lambda(r)$, with $\nu(r)$ in the usual formulation, with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18}$ kg · m$^{-3}$, $r_b = 1 \times 10^3$ m, $\mu$ taking the various values shown in the legend, and $\beta$ being set to $-2a$.

Considering the form of $Z$, it is clear that changing the value of $\beta$ will not affect it. The metric function $Y$ however is a different story and we show in the next figure how it changes for different values of the anisotropy factor. We forgo a direct analysis of the limiting behaviour of the algebraic expressions (which can be found in Appendix B) since the latter are quite
complicated, consisting of products of hyperbolic functions.

As it stands we see that both metric functions are well-behaved in the interior for a range of parameter values, so that we can be reasonably sure that condition [i] is satisfied.

![Graphs](image-url)

(a) The $Y(r)$ metric function, $\mu = 1$
(b) The $Y(r)$ metric function, $\mu = 0.6$

**Figure 6.4:** Variation of the $Y$ metric variables with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18}$ kg $\cdot$ m$^{-3}$, $r_b = 1 \times 10^4$ m, $\mu$ is set to 1 on the left, and 0.6 on the right, and $\beta$ is set to the various values shown in the legend.

The discussion at the beginning of the previous section is still valid, and in particular the inequalities (6.9) and (6.11) must still hold in view of [ii]–[iii] which yield the exact same results as the $\phi = 0$ case. To ensure that both these conditions remain satisfied, we next look at the pressure in view of condition [iv]. Here too the algebraic expression is very complicated, and instead we provide graphs of the behaviour for varying parameters $\mu$ in figure 6.6 and $\beta$ in figure 6.5.

It is clear that our boundary condition requiring that the pressure vanish at the boundary is working, as is evident in all the pressure graphs we are showing. It should be noted that even though $\beta$ is extremely important in the tangential pressure $p_\perp$ component, its contribution to the radial pressure $p_r$ is not zero: an unintuitive result stemming from the non-linearity of the EFE. From these four figures it is immediately clear that for sensible values of the parameters the radial pressures are all negative. This result destroys the viability of this solution as a whole. We therefore look further for the other case of this solution.
6.2.3 The $\phi^2 > 0$ case from 4.1.2

In this case too, $\beta$ is no longer fixed to one value: instead it takes on a range of possible values and as long as the inequality $\beta > -\frac{\kappa \mu \rho_c}{5 r_b^4}$ is satisfied, the value of $\phi$ will be appropriate for this solution. Because of the above inequality, this solution offers us the possibility of having different signs for $\beta$. Since the latter could be negative while still having a positive $\phi^2$. As a result, we need to investigate the effect of the sign of $\beta$ on our solutions too.

We first consider the metric functions in their two forms for different value of $\mu$ through
plots in figure 6.7. These are all well behaved for various $\mu$s, but in this case also we need to check if a similar behaviour holds for various anisotropy factor $\beta$, and indeed we see that this is so in figure 6.8. Since both metric functions are continuous in the regions we want, we conclude that condition (i) is satisfied for this solution.

The discussion leading to inequalities (6.9) and (6.11) must still hold in view of conditions (ii) and (iii) which yield the exact same results as the $\phi \leq 0$ cases. Ensuring that both these conditions remain satisfied, we next look at the pressure in view of condition (iv).

We see in figure 6.9 that even for widely differing positive values of $\beta$, we still see positive pressures. However even for the small negative value of $\beta = -a/2$, we get a negative radial pressure, suggesting that even though the solution is valid for those negative values of $\beta$,
any $\beta < 0$ will yield negative pressures in the natural case where $\mu = 1$. Furthermore, since that same value of $\beta$ produces positive radial pressures when $\mu = 0.6$, it is probable that the extreme value of $\beta$ is $\mu$ dependant.

As a side note here, we can try to consider how $\beta$ is dictating the type of solution we have. When $\beta$ is zero, this solution reduces to Tolman VII which has perfectly physical pressures and densities. When $\beta \leq -a$, as in the previous two sections 6.2.2 and 6.2.1 all the radial pressures were negative. Therefore the region $-a \leq \beta \leq 0$, is the problematic one which we
should investigate in more detail.

In the next figure 6.10 we check if the same is true for different values of $\mu$, and find that this is mostly so. These two sets of figures thus confirms that there are parameter values for which we can get both positive pressures and densities, very much like normal matter in this particular solution, while emphasizing that even smaller values of $|\beta|$ than previously yield negative pressures in the natural case if negative. This confirms that we will have to find a way to constrain $\beta$ if we were to try to use this solution as a model for stars.

This is however the first time we obtain positive pressures with the new solutions, and the similarity to Tolman VII, which we use to get this solution is finally paying off. We now need to proceed onto checking the other conditions for physical viability. This will take the form of finding a range of values for $\beta$,

![Figure 6.10: Variation of the radial pressure with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18}$ kg $\cdot$ m$^{-3}$, $r_b = 1 \times 10^4$ m, $\beta$ is set to $-a/5$ on the left, and $5a$ on the right, and $\mu$ is set to the various values shown in the legend](image)

The next conditions (v) and (vi) are true as can be seen both by construction in the boundary conditions, and through the radial pressure pressure graphs we have shown so far in figures 6.10 and 6.9. To show that the construction of $p_\perp$, in the form of $p_r - p_\perp = \beta r^2$, we could, as have for example [4] and others, use the first zero of the pressure to fix the radius of the star, i.e. define $r_b$ such that $r_b = \min \{ r \in \mathbb{R}^+ | p_r(r) = 0 \}$, and thus not deal with the part where the radial pressure gets to negative values. However this will not work in this case because of our imposed boundary matching conditions which has to occur at the specific $r_b$ that has been already specified. The other conditions about the derivative of the pressure also start failing if we were to admit this here.
also leads to condition (vi), we provide a plot of the tangential pressure and the radial pressure for a few stars, noting the equal values of the former at the centre of the star where \( r = 0 \). This plot also shows that while the radial pressure always vanishes at the boundary radius \( r = r_b \), the tangential pressure can take on negative values, which might sound problematic for ordinary matter, and we investigate this aspect by invoking the next condition (viii) involving the energy conditions. Before however, we check the condition (vii), which involves the derivative of the matter variables.

The strong energy condition states that as long as the sum of all the pressures and the energy densities is positive, we can be certain that the energy condition is satisfied and that such matter might plausibly exist. In the anisotropic case, this condition reduces to \( 2p_\perp + p_r + \rho \geq 0 \), and using the definition of \( p_\perp \) in terms of \( p_r \) and \( \beta \) from equation (4.19), this relation imposes a constraint on \( \beta \) in the form of

\[
2\beta r^2 \leq 3p_r + \rho \quad \Rightarrow \quad \beta \leq \frac{3p_r + \rho}{2r^2}.
\]

As a result, we are now forced to pick values of \( \beta \) within this range if we want to talk about physical stars, and we will endeavour to ensure that this holds true in the final solutions we use. To clarify what this range implies, we should attempt to find an extremum for \( \beta \) with
the expressions we have for the densities and pressures, however the above equation has $\beta$ on both sides since $p_r$ depends on trigonometric functions of $\phi$, which in turn contains $\beta$. As a result we are left with a transcendental equation to solve, and no closed-form solution is possible. We should also note that in the form in which we have given the energy condition, both the pressure and density have the same geometrical units of $[L]^{-2}$, which is a relief since we are adding them, and not the SI–units we have been using on our plots. In those units, $\rho$ and $p_r$ have values closer to each other, making the evaluation of the inequalities even more important.

Figure 6.12: Variation of speed of sound with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18}$ kg $\cdot$ m$^{-3}$, $r_b = 1 \times 10^4$ m, $\mu$ is fixed in the bottom two plots at one on the left, and 0.6 on the right, but takes on the values in the legend for the top graphs. $\beta$ is set respectively to $a$ and $2a$ in the left and right top plots, but varies in the bottom ones.

If we were to use equation (4.26), which gives us a ready expression of both density and
pressure, we can compute the inequality (6.13) above as
\[
\beta \leq \frac{3}{2\kappa r^2} \left( \frac{2Z \, dY}{rY \, dr} - \frac{1}{r} \frac{dZ}{dr} \right) - \frac{\rho}{r^2},
\]
whose right-hand-side can be evaluated. The logical places to evaluate it are where \( p_\perp \) is

(a) the largest, that is where \( r = r_b \), and when we do so, we are left with
\[
\beta \leq \frac{3}{\kappa r_b^2} \left[ 2(1 - br_b^2 + ar_b^4) \left( \frac{1}{rY} \frac{dY}{dr} \right) \right]_{r=r_b} + 2a - 4br_b^2 - \frac{\rho c(1 - \mu)}{r_b^2} \tag{6.14}
\]

(b) or equal to the radial pressure, i.e. where \( r = 0 \) and when we evaluate this, we get
\[
\beta \leq \frac{3}{\kappa} \left[ \frac{2}{r^2} \left( \frac{1}{rY} \frac{dY}{dr} \right) \right]_{r=0} - \frac{2a}{r^2} - \frac{\rho c}{r^2} \tag{6.15}
\]
expressions which depend crucially on the value of the term in round parentheses, and hence the complicated \( Y \) metric functions that include \( \beta \). The second expression (6.15) is useless\(^4\) because the \( rs \) in the denominator causes the expression to diverge even if the term is round brackets were finite.

However from the form of the first equation (6.14), if we were to find a way to evaluate some approximation to \( (1/rY) \frac{dY}{dr} \) we could easily find a suitable range for \( \beta \). We keep this in mind since for the time being, we have no other constraints that we can use.

To continue with our program of applying the constraints to our equations, we turn to condition (ix) which requires our speed of pressure waves in the interior to be less than the speed of light. First we show, to give an idea about how the speed of sound changes with the different parameters \( \beta \) and \( \mu \), plots of this speed for differing parameter values in figure 6.12

We see clearly that for certain parameter values the speed at the centre is larger than the speed of light, telling us that causality is being violated. A formal analysis should allow us to cull parameter values that get us such violations.

\(^4\)Instead of an evaluation, the limit of this expression could be taken as \( r \to 0 \). This limit does not necessarily exist, but once a solution with parameters is specified, this equation could be used to check the parameters’ validity. I only noticed this after an examiner pointed it out, and did not use this condition in the subsequent analysis.
To implement this, we can use equation \([6.7]\) directly. In this particular case, the equation has no electric charge, so that \(q = q' = 0\), and \(\Delta = \beta r^2\). Therefore the speed of sound condition reduces to
\[
\left( \frac{r_b^2}{\kappa \rho_c \mu} \right) \left[ \frac{\nu' \kappa (p_r + \rho)}{4r} + \beta \right] \leq 1,
\]
(6.16) in geometrical units. Re-expressing equation (6.16) in terms of the metric variable \(Y\) instead through the use of (4.26), we have
\[
\left( \frac{r_b^2}{\kappa \rho_c \mu} \right) \left\{ \frac{1}{2} \left( \frac{1}{rY} \frac{dY}{dr} \right) \left[ 2Z \frac{dY}{dr} \frac{rY}{dr} \right] - \frac{1}{r} \frac{dZ}{dr} + \beta \right\} \leq 1.
\]
Upon rearranging, and considering that the factor \(1/(rY) \frac{dY}{dr}\) occurs very frequently, we define
\[
\psi := \frac{1}{rY} \frac{dY}{dr}, \quad \text{with} \quad \psi_0 := \left( \frac{1}{rY} \frac{dY}{dr} \right) \bigg|_{r=0}, \quad \text{and} \quad \psi_b := \left( \frac{1}{rY} \frac{dY}{dr} \right) \bigg|_{r=r_b},
\]
where the second expression is to be understood as a formal limit, and we get the quadratic inequality
\[
\left( \frac{r_b^2}{\kappa \rho_c \mu} \right) \left\{ \beta + Z \psi^2 - (2ar^2 - b) \psi \right\} \leq 1.
\]
(6.17)
The highest value of the speed of sound should occur at the centre of the star since both the density and pressure are maximal there, so we evaluate the above (6.17) at \(r = 0\), such that \(Z = 1\), to get
\[
\left( \frac{r_b^2}{\kappa \rho_c \mu} \right) \left( \beta + \psi_0^2 + b \psi_0 \right) \leq 1 \quad \Rightarrow \quad \psi_0^2 + b \psi_0 + \left( \beta - \frac{\kappa \rho_c \mu}{r_b^2} \right) \leq 0
\]
This inequality can be solved by first finding the roots of the quadratic above,
\[
\psi_{0\pm} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \beta + \frac{\kappa \rho_c \mu}{r_b^2}},
\]
to finally get
\[
\psi_{0-} \leq \psi_0 \leq \psi_{0+}.
\]
(6.18)
We finally have an equation for the range of $\psi_0$, and recall that we needed this information before in equation (6.15). Proceeding similarly, but evaluating the speed of sound at the boundary instead, we get the analogous inequality $\psi_b^- \leq \psi_b \leq \psi_b^+$, with

$$\psi_b^\pm = \frac{(2ar^2_b - b)}{2Z_b} \pm \sqrt{\frac{(2ar^2_b - b)^2 - 4Z_b \left( \beta - \frac{r_c^2}{\kappa,\mu} \right)}{2Z_b}},$$

where $Z_b = 1 - br^2_b + ar^4_b$. This is a considerably more complicated expression, but we can consider these two conditions to be our causality criteria on all variables involved, however we will only test them once we start specifying parameters for the star. Furthermore, considering inequality (6.14), which has $\psi_b$ as one of its terms, we finally have a complete prescription to figure out if the value of $\beta$ we are using is within the range of physical acceptability.

The square-root including $\beta$ in the expression of $\psi_b^\pm$ makes computation unwieldy, but once one specifies the set $\{\rho_c, \mu, r_b\}, \beta$ is completely constrained.

Continuing on with the next constraint, we need the speed of sound derived earlier to be decreasing with increasing radius, i.e. $\frac{dv}{dr} < 0$. We can easily write down this condition, but from figure 6.12 it is clear that for whatever parameter we might choose, even when these parameters do not obey the causality criterion, the speed of sound is a decreasing function.

This concludes the implementation of the constraint onto this particular solution. We should note here that this particular solution can potentially satisfy all the criteria we have. Additional restrictions on $\beta$ were also found, in particular $\beta > 0$, to prevent negative radial pressures is absolutely necessary. Additional restrictions depending on the value of the other parameter in the set $\{\mu, r_b, \rho_c\}$ were also found, and will be checked when value for those are picked. Having spelled out all the conditions in this section, we move onto the next solutions: the charged ones.
6.2.4 The anisotropised charge case, $\Phi^2 \neq 0$

In this special case of the more general charged anisotropic case, the charge is chosen so that it annihilates the contribution of the anisotropy so that the matter density is the only term in the differential equation for $Y$.

We start by ensuring that the metric functions are well behaved in the star, as per condition (i). All the charged solutions have a slightly different definition of the $Z$ metric function, as compared to the uncharged solutions. In this particular case, as we saw in chapter 4, it is given by

$$Z(r) = 1 - \left(\frac{\kappa \rho_c}{3}\right) r^2 + \frac{1}{5} \left(\frac{\kappa \rho_c \mu}{r_b^2} - k^2\right) r^4. \quad (6.19)$$

Considering the value of this metric function at $r = 0$, and $r = r_b$, we get

$$Z(0) = 1, \quad \text{and} \quad Z_b = Z(r_b) = 1 - \kappa \rho_c r_b^2 \left(\frac{1}{3} - \frac{\mu}{5}\right) - \frac{k^2 r_b^4}{5},$$

a result different from the uncharged case due to the presence of electric charge through $k$. Since this metric function cannot change sign, we can use the second equation to obtain a constraint on the maximum charge:

$$k^2 < \frac{1}{r_b^2} \left[\frac{5}{r_b^2} - \kappa \rho_c \left(\frac{5}{3} - \mu\right)\right]. \quad (6.20)$$

Plugging in the typical values we usually use for the central density, boundary radius and self-boundness, viz. $\{\mu = 1, \rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^4 \text{ m}\}$, we find that the above inequality (6.20) gives us $|k| < 2.2 \times 10^{-8} \text{ m}^{-2}$. The charge density $k$ is in geometrical units, having been normalised by a factor of $\sqrt{G/(4\pi c^4 \epsilon_0)}$, Einstein’s constant multiplied by Coulomb’s. When we will go back to physical units to make connections to observed values, this is the factor we will use to convert to say S.I. charge units. In particular the above limit corresponds to a charge density of $|k| < 2.6 \times 10^9 \text{ C} \cdot \text{m}^{-3}$. As a result, for a model having a radius of $r_b = 10 \text{ km}$, the total maximal electric charge $Q = kr_b^3 = 2.6 \times 10^{21} \text{ C}$. This limit varies non-linearly with $\mu$, but the maximum value of $k$ is not very sensitive to the change in $\mu$. 

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The next condition we check comes from the necessity for the proper radius $R$ to exist. As we derived earlier in the section following (6.3), this means that $2\sqrt{a} > b$, and using the expressions for these constants for this solution, we get straight-forwardly that

$$k^2 < \kappa \rho_c \left( \frac{\mu}{r_b^2} - \frac{5\kappa \rho_c}{36} \right).$$

The value of $k^2$ must clearly be positive, so that the term in brackets must be positive. This immediately yields $\rho_c < \frac{36\mu}{5\kappa r_b^2}$, the same constraint on density as for the uncharged case. The inequality on $k$ can then be supplemented with the usual values of $\rho_c$, and $r_b$, to yield a function of $\mu$, which we plot next in figure 6.13. As we can see from the figure and inequalities only values of $k$ below both the red and blue lines can be used as a valid charge for models having densities and radii specified in the legend. In this case, there can be no charge for models having low $\mu$. This trend is seen for all parameter values.

We next check the second metric function’s behaviour: $Y$ is given by equation (4.34), which is complicated. Instead we give plots of the two metric function. As we see in figure 6.14 all

Figure 6.13: The limiting value of $k$ for different $\mu$: the blue line corresponds to equation (6.20), the red line to equation (6.21), where values of the central density, $\rho_c = 1 \times 10^{18}$ kg $\cdot$ m$^{-3}$, and the boundary radius $r_b = 10000$ m have been supplied to produce the lines.
Figure 6.14: Variation of metric variables with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18} \text{kg} \cdot \text{m}^{-3}$, $r_b = 1 \times 10^4 \text{m}$, $k = 3 \times 10^{-9} \text{m}^{-2}$ and $\mu$ taking the various values shown in the legend.

The metric functions are well behaved for the range of parameters we picked.

Next we look at the behaviour of the radial pressure $p_r$ in the star. The expression of the former was previously given in (4.35), and here we only provide a plot of the radial pressure for differing parameter values instead in figure 6.15.

As we can see immediately, the radial pressures get to negative values in the star for low values of $\mu$, and the “natural” case is also plagued by this feature. The effect is only enhanced with higher charge values $k$, as is obvious in the top right pane 6.15b. The bottom panes in 6.15 vary $k$ instead for fixed $\mu$, and shows that there must exist some critical relation between $k$ and $\mu$ in this model that will allow the pressure inside the star to be always positive. We now try to find this critical value. Considering the shape of the radial pressure
Figure 6.15: Variation of the radial pressure with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18}$ kg m$^{-3}$, $r_b = 1 \times 10^4$ m, $\beta$ is fixed by $k$ in this solution and $\mu$ is given in the legend for the top plots, and $k$ is given in the bottom ones.

graphs, we will have negative pressures if when we solve $p_r(r) = 0$, for $r$, the solution be less than $r_b$. However solving the complicated equation (4.35) is impossible analytically. The alternative way we could try finding the critical value of $k$ is through the derivative of the radial pressure, whose expression we already have. This is possible because from physical consideration, we need $p_r(r = 0) > 0$, and from the boundary conditions also that $p_r(r = r_b) = 0$. The only way to have negative pressure is therefore to have a turning point at $r = r_t$, where $0 < r_t < r_b$. If $r_t$ then exists, we are assured that the pressure has become negative somewhere. Equation (6.5) gives us the expression of the pressure derivative. We can simplify this equation in our particular case, from $q = kr^3$, and $\Delta = 2k^2r^2$ to give the
condition for a turning point in the pressure if

$$0 = \frac{dp_r}{dr} = \frac{2k^2r}{\kappa} - \frac{\nu'}{2}(p_r + \rho).$$  \hfill (6.22)

This equation is only true for some particular $r$ and if and only if the pressure becomes negative somewhere in the region we are interested in. We will use it to check for valid values of $k$.

Next we turn to the tangential pressure in figure 6.16 where we see a similar trend. The tangential pressure however can be negative, so we cannot further constrain our parameters just yet. To do that we turn to our next constraint (vii) which tells us that all our matter fields must be decreasing function of the radial coordinate. We only need concern ourselves with the pressures, and indeed the previous condition we derived had this same flavour, and came about from the non monotonicity of the pressure. We can hence be confident that condition (6.22) above is exactly the constraint we require for this physical condition to hold.

The next constraint (viii) about the energy conditions is more interesting. We want the dominant energy condition to hold, and in this particular case, this translates to

$$3p_r + \rho - 4k^2r^2 \geq 0.$$  \hfill (6.23)
This will also have to be tested when we are ready to try to model stars, and will hopefully provide valuable insight into the possible values of $k$.

If we now consider the speed of pressure waves according to (ix), we find that in this case because $q = kr^3$, and $\Delta = 2k^2r^2$, the expression reduces to

$$v_s^2 = \left(\frac{r_b^2}{\kappa \rho_0 \mu}\right) \left[ \frac{\nu' \kappa (p_r + \rho)}{4r} + k^2 \right], \quad (6.24)$$

a speed that is larger that the Tolman VII case by precisely the charge factor $k^2$. As a result we expect the parameter value that we can use in this case to have to be slightly lower than previously, since we still want to maintain causality: $v_s = c$, at the centre of the star.

We show the behaviour of the speed of sound in figure 6.17. The other conditions seen previously will be useful in determining constraints on $k$, and hopefully this one can be used for the other parameters, as we did in Tolman VII. Similarly the next condition on the derivative of the speed of sound will also be useful to restrict the other parameters. We can see from the form of (6.24) that the derivative with respect to $r$ is not going to be affected by the additional $k^2$ factor in that expression. However this is incorrect since $\nu'$ is $k$ dependent too in this solution, but since we can see the behaviour of the speed of sound in figure 6.17 we see that this condition is mostly satisfied, except in 6.17c for large values of charge which we had already determined to be too large.

This completes the constraints section of this solution. Like the previous solution, this solution looks to be viable too as a model for neutron stars, if we consider the “self-bound” ones with $\mu \neq 1$, since not abiding by this constraint gives us negative radial pressures. We will look at this solution in detail too in the next section.
Figure 6.17: Variation of speed of sound with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$, $r_b = 1 \times 10^4 \text{ m}$, $\mu$ is fixed in the bottom two plots at one on the left, and 0.6 on the right, but takes on the values in the legend for the top graphs. $k$ is set respectively to $1 \times 10^{-9} \text{ m}$ and $3 \times 10^{-9} \text{ m}$ in the left and right top plots, but varies in the bottom ones.

6.2.5 The charged case, $\Phi^2 = 0$, derived in [4.2.2]

In this charged and anisotropic case, we annihilate the coefficient of $Y$ in our differential equation to get the simplest solution of the charged case. By doing so, we express the measures of charge $k$, anisotropy $\beta$, and Tolman VII parameters $\{\rho_c, \mu, r_b\}$ in terms of each other. The metric function for $Y$ then becomes a simple linear function of the radial-like coordinate $\xi$. We now look at the behaviour of this solution, while applying all the constraints, up to the point at which we run into unphysical behaviour.

The first constraint we look at is the regularity of the metric $Z$, and in particular its unchanging sign from the centre of the star where its value is $Z(0) = 1$. We require that even at the
boundary radius, this metric function remain positive, and this yields

$$Z(r_b) = 1 - \left( \frac{\kappa \rho_c r_b^2}{3} \right) + \frac{2}{11} \left( \kappa \mu \rho_c r_b^2 - \frac{\beta r_b^4}{2} \right) > 0,$$

which on simplification results in an immediate limit on $\beta$:

$$\beta < \frac{11}{2r_b^2} \left[ 1 + \kappa \rho_c r_b^2 \left( \frac{2\mu}{11} - \frac{1}{3} \right) \right].$$

(6.25)

With our typical values for the constants above, $\{\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 10^4 \text{ m}\}$ we get a function of $\mu$, graphed in figure 6.18. For the next constraint we turn to the fact that the value of the proper radius has to exist, and as discussed previously, this means that $2\sqrt{a} > b$.

In this particular case, this relation gives us that

$$\beta < \kappa \rho_c \left( \frac{2\mu}{r_b^2} - \frac{11\kappa \rho_c}{36} \right),$$

(6.26)

another more restrictive inequality on $\beta$. From this one it is clear that for some values of $\mu$ we will need zero or even negative $\beta$. We keep this in mind as we proceed, and for the time being restrict $\mu > 0.4$, so as to have positive $\beta$ only.

![Figure 6.18: The limiting value of $\beta$ for different $\mu$. Here the parameters are taken to be $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$, and the boundary radius $r_b = 10000 \text{ m}$, and show the two different constraints we are looking at.](image-url)
Next we look at the metric functions, and their general behaviour, for the restricted values of $\beta$ we just found. This is shown in figure 6.19. The only striking feature here is the strange behaviour of the $Y$ and hence $\nu$ metric function, whose derivatives seem to be of either sign. This is a sign of trouble, since the metric derivatives have to behave smoothly, and here it seems that for some parameter values, the metric could be constant. Suspecting unphysicality, we turn to the next condition (iv) which requires positive pressures. We show in figure 6.20 how the pressure changes while varying both the value of the anisotropy $\beta$, and self-boundness $\mu$, but unfortunately find that in both cases, we only get negative pressures. As a result, we forgo this particular solution as unphysical, and do not waste time ensuring any of the other conditions hold.

Figure 6.19: Variation of metric variables with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^4 \text{ m}, \beta = 5 \times 10^{-17} \text{ m}$ and $\mu$ taking the various values shown in the legend.
6.2.6 The charged case, $\Phi^2 < 0$, derived in [4.2.3]

We now turn to this more general case where none of the constants are specified or fixed at the beginning. To get to this class of solutions, we have to ensure that $a + \beta - 2k^2 < 0$, as mentioned previously. The numerical values, and ranges surmised for these constants from the previous sections will provide a guideline for what value we pick initially for our plots, but we will go through a formal derivation from the conditions here too, to check if any of the conditions yield different constraints.

The definiteness of the metric functions and observables, or conditions [i]–[iii] yield the same constraints on $k$ as as (6.20) and (6.21), since $Z$ does not depend explicitly on $\beta$. As a result we maintain a charge density limit of $|k| < 1.2 \times 10^8 \text{ C} \cdot \text{m}^{-3}$, for the usual parameter values of $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$, and $r_b = 10^4 \text{ m}$. We now give plots of the metric functions for this particular case in figure [6.21] where we notice that they are all well behaved. The surprising change in sign for the $\nu$ metric function is of no concern, since it is $e^{\nu}$ that appears in the line element, and that function does not change sign. Next we look at condition [iv] on the pressure. From the general trend of how these solutions have worked so far, we suspect that we will get negative pressures, and indeed this is exactly what we find, as exemplified in

![Figure 6.20: Variation of the radial pressure with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$, $r_b = 1 \times 10^4 \text{ m}$, $\beta$ is fixed to $5 \times 10^{-17}$ in left panel and $\mu$ is given in the legend, while for the right panel $\mu$ is fixed to 1, and $k$ varies as shown in the legend](image-url)
Figure 6.21: Variation of metric variables with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18} \text{kg} \cdot \text{m}^{-3}$, $r_b = 1 \times 10^4 \text{m}$, $\beta = -2 \times 10^{-16} \text{m}$, $k = 1 \times 10^{-9} \text{m}^{-2}$ and $\mu$ taking the various values shown in the legend.

As a result we leave off this solution as being unphysical, and turn to the next one that looks more promising since it is a very straightforward generalisation of Tolman VII with charge and anisotropic pressures as the "bells-and-whistle."
Figure 6.22: Variation of the radial pressure with the radial coordinate inside the star. The parameter values are \( \rho_c = 1 \times 10^{18} \, \text{kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^{4} \, \text{m}, \beta \) is fixed to \((-2 \times 10^{-16}) \times \mu\) in left and middle panel, while it is fixed to \((-2 \times 10^{-16})\) in the right one. \( k \) is fixed to \(1 \times 10^{-9} \, \text{m}^{-2}\) in the right and left plots, and changes according to the legend in the middle one. \( \mu \) is fixed to 1 in the middle and right panels, and varies as shown in the legend in the left panel.

6.2.7 The charged anisotropic case with \( \Phi^2 > 0 \), derived in 4.2.4

In this case too, \( \beta \) is no longer fixed to one value: instead it takes on a range of possible values and as long as the inequality \( \beta > \frac{1}{5} \left( 11k^2 - \frac{\kappa \rho_c}{r_b^6} \right) \) is satisfied, the value of \( \Phi \) will be appropriate for this solution. Because of the above inequality, this solution offers us the possibility of having different signs for \( \beta \): since the latter could be negative and we would still have a positive \( \Phi^2 \). However we remember the previous case where we only had anisotropy, and negative \( \beta \)'s only gave us negative pressures. Here the situation seems even worse because the charge \( k \) already comes with a negative sign, suggesting we might run into trouble right at the beginning.
We start as we have been doing by looking at the behaviour of the metric functions in figure 6.23 and find that they are all well behaved, because they do not show sign changes for example. The condition on the existence of the proper radius reduces to the same as previously, viz. equation (6.21), and we keep this in mind as we proceed here too.

Next we look at the radial pressures. Since in this case we have all of anisotropic pressures, electric charge, and self-boundness, we show how the pressure varies with all these parameters in figure 6.24. We expect from the structure of the equations that at some critical values of each of $k$, $\mu$ and $\beta$, none independent of each other, for the pressure to become negative. However we notice that for the range of parameters we chose in the plots in 6.24 the pressure is surprisingly, but advantageously never negative. This further strengthens our perception

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{metric_functions.png}
\caption{Variation of metric variables with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$, $r_b = 1 \times 10^4 \text{ m}$, $\beta = 2 \times 10^{-16} \text{ m}$, $k = 1 \times 10^{-9} \text{ m}^{-2}$ and $\mu$ taking the various values shown in the legend.}
\end{figure}
that this solution will be well suited as the model for an actual physical object.

![Graphs showing variations of radial pressure with radial coordinate inside the star.](image)

**Figure 6.24:** Variation of the radial pressure with the radial coordinate inside the star. The parameter values are $\rho_c = 1 \times 10^{18}$ kg·m$^{-3}$, $r_b = 1 \times 10^4$ m, $\beta$ is fixed to $(2 \times 10^{-16})$ in the two left plots, while it is varied in two different ways in the right ones. $k$ is fixed to $1 \times 10^{-9}$ m$^{-2}$ in (a), (b) and (d), but changes according to the legend in (c). $\mu$ is fixed to 1 in the bottom two panels, and varies as shown in the legend in the top panels. Even though we do not see negative radial pressures, this is dependant on some particular choices, and we will derive conditions for this not to hold momentarily. Before doing so, we take a look at the tangential pressures in 6.25, which can be negative without implying unphysicality, and as expected we do see negative tangential pressures. We first figure out the limiting values of $k$, since this comes directly from the $Z$ metric function, through constraint (i). This results in the same values as before, i.e. $|k| < 1.55 \times 10^{-9}$ m$^{-2}$. The next two constraints (iii) and (ii) were also already implemented and results in the same parameter ranges as before. From the graphs of the radial pressures we show in 6.24, it is also clear that
Figure 6.25: Variation of the radial pressure with the radial coordinate inside the star. The parameter values are \( \rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3} \), \( r_b = 1 \times 10^4 \text{ m} \), \( \beta \) is fixed to \( 2 \times 10^{-16} \) in the two left plots, while it is varied in two different ways in the right ones. \( k \) is fixed to \( 1 \times 10^{-9} \text{ m}^{-2} \) in (a), (b) and (d), but changes according to the legend in (c). \( \mu \) is fixed to 1 in the bottom two panels, and varies as shown in the legend in the top panels.

The constraints concerning the positivity of the radial pressures hold. However since we still have no constraints on the parameter values like \( \beta \), we proceed as before and invoke the energy condition (viii) since all the previous ones have clearly been satisfied by some suitable choice of parameters. In this case, we get the constraint equation on \( \beta \), and \( k \) to be

\[
\beta r^2 + \frac{q^2}{r^4} \leq \frac{\rho + 3p_r}{2}.
\]

Since we have been positing \( q = kr^3 \) throughout this solution, this can be further simplified into

\[
\beta + k^2 \leq \frac{\rho + 3p_r}{2r^2}.
\]  
(6.27)
We are in a position to evaluate this inequality in full, since we have the full solution now, knowing all the metric components. The procedure we employed before for the anisotropised charged works here too, and we are left instead with a relation on both $k$ and $\beta$, instead of just $\beta$. In this case these relations give instead

$$\beta + k^2 \leq \frac{3}{2\kappa r^2} \left( \frac{2Z}{rY} \frac{dY}{dr} - \frac{1}{r} \frac{dZ}{dr} \right) - \frac{\rho}{r^2}. \tag{6.28}$$

Again evaluating this expression both at the boundary and at the centre we get the following:

(a) where $r = r_b$,

$$\beta + k^2 \leq \frac{3}{\kappa r_b^2} \left[ 2(1 - br_b^2 + ar_b^4) \left( \frac{1}{rY} \frac{dY}{dr} \right) \bigg|_{r=r_b} + 2a - 4br_b^2 \right] - \frac{\rho c(1 - \mu)}{r_b^2} \tag{6.29}$$

(b) where $r = 0$, we get

$$\beta + k^2 \leq \frac{3}{\kappa} \left[ \frac{2}{r^2} \left( \frac{1}{rY} \frac{dY}{dr} \right) \bigg|_{r=0} - \frac{2a}{r^2} \right] - \frac{\rho c}{r^2} \tag{6.30}$$

expressions which depend crucially on the value of the term in round parentheses, and hence the complicated $Y$ metric functions that include both $\beta$, and $k^2$.

As in the previous section, we use the criteria on the speed of sound to constrain the terms in brackets. We show plots of the speed of sound in figure 6.26 and if we consider the expressions of this same speed, in this general case, we have from equation (6.7), since $\Delta = \beta r^2$, and $q = kr^3 \implies q' = 3kr^2$,

$$\left( \frac{r_b^2}{\kappa \rho_c \mu} \right) \left[ \frac{\nu' \kappa (p_r + \rho)}{4r} - 3k^2 + \beta \right] \leq 1. \tag{6.30}$$

We re-express this in terms of the previously defined variable $\psi$, in a similar vein as previously after substituting for $\nu'$ to get an inequality on $\psi$:

$$\left( \frac{r_b^2}{\kappa \rho_c \mu} \right) \left\{ \beta - 3k^2 + Z\psi^2 - (2ar^2 - b)\psi \right\} \leq 1.$$
Figure 6.26: Variation of speed of sound with the radial coordinate inside the star. The parameter values are \( \rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^4 \text{m}, \mu \) is fixed in the bottom two plots at one, but takes on the values in the legend for the top graphs. \( k \) is set to \( 1 \times 10^{-9} \) m top plots, but varies in the bottom right one. \( \beta \) is fixed to \( 5 \times 10^{-17} \) in the top left, and is proportional to \( \mu \) with the same proportionality factor in the right one.

We expect from the shape of the velocity plots to have the maximum speed at the centre of the star, so we evaluate the above equation at \( r = 0 \Rightarrow Z = 1 \), to get

\[
\left( \frac{r_b^2}{\kappa \rho_c \mu} \right) \left[ \beta - 3k^2 + \psi_0^2 + b\psi_0 \right] \leq 1 \Rightarrow \psi_0^2 + b\psi_0 + \left( \beta - 3k^2 - \frac{\kappa \rho_c \mu}{r_b^2} \right) \leq 0.
\]

We solve this inequality for \( \psi_0 \), and get that \( \psi_0^- \leq \psi_0 \leq \psi_0^+ \) with

\[
\psi_{0 \pm} = \frac{-b}{2} \pm \sqrt{3k^2 - \beta + \frac{b^2}{4} + \frac{\kappa \rho_c \mu}{r_b^2}}.
\]

For the usual values we use in our plots for example: \( \rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^4 \text{m}, \mu = 1, \) and \( k = 1 \times 10^9 \text{m}^{-2} \), the inequality results in \( \beta < 3.65 \times 10^{-6} \), a less restrictive
constraint on $\beta$, than the previous ones we had in other solutions: as a result we can “crank-up” the anisotropy in this particular solution to larger values, while still maintaining the energy conditions.

6.3 Application of the models to physical objects

In the last section, we investigated all the classes of the new solutions we derived previously in detail, and found out that only three specific ones give us sensible values for the physical matter variables. In this section we look at these three viable solutions in greater detail by

(a) deriving an equation of state for each solution. This equation of state comes directly from our assumptions, and general relativity: no matter interactions being assumed.

(b) We find the masses, radii, and total electric charge each solution admits, and how changing parameters change these observables.

(c) We restrict the parameters for each solutions, so that within the restricted parameter ranges, these solutions behave physically with no (unphysical characteristics)

(d) We compare our solutions with known observed values of radii and masses of visible astrophysical objects.

6.3.1 The equation of state

We derived the equation of state of the Tolman VII solution and presented it in 3.3. This was possible because of the simple nature of the density relation which is easily invertible. As a result, all instances of $r$ in the expressions of the pressures can be converted to some function of $\rho$, through the inverted equation (3.3):

$$r(\rho) = r_b \sqrt{\frac{1}{\mu} \left( 1 - \frac{\rho}{\rho_c} \right)}.$$  \hspace{1cm} (6.31)
With this prescription applied to the pressures we already found in the previous section for the three specifically physical solutions, we arrive at the equation of state for each of these solutions. A note of caution: these equations of states were obtained from a purely general relativistic method, with no nuclear physics assumptions, however as we will show shortly, they still predict values for observables that are in line with all current measurements from neutron stars, were the TOV method applied to them.

The solution with anisotropy only

The first equation of state we look at is for the anisotropic uncharged case, whose pressure is given by (4.18). With the inverted density relation the equation of state is obtained by replacing each occurrence of $r$ through (6.31). Since the expression is the exact same one as before, we will here give the expressions of all the components of the $p_r$ function in terms of $\rho$ instead of rewriting the full pressure again. The $Z$ metric function in this solution is given by

$$Z(\rho) = 1 + \frac{\kappa r_b^2}{5\mu} \left( \frac{\rho^2}{\rho_c} - \frac{\rho}{3} - \frac{2\rho_c}{3} \right).$$

(6.32)

All the evaluated constants like $a, b, \alpha, \beta, \gamma$ and $\Phi$ remain the same, but the expression of $\xi$ does change into

$$\xi(\rho) = 2r_b \sqrt{\frac{5}{\kappa \mu \rho_c}} \text{arcoth} \left( \frac{\sqrt{15\mu \rho_c - \kappa r_b^2 (2\rho_c^2 + \rho_c \rho - 3\rho^2)}}{r_b \sqrt{3\kappa (\rho_c - \rho)}} \right).$$

(6.33)

when put in terms of $\rho$. With these two functions in terms of $\rho$, the equation of state is easily written from equation (4.18) by straight forward substitution. The resulting expression as can be guessed from the length of the previous two equations is very long, and we will not attempt to write it down in full here. We however give plots of what the equation of state look like in figure 6.27, where we also notice some interesting features that merit discussion.

The first feature that immediately jumps at us is that the plot has all the EOS curves with $\mu \neq 1$ starting at non-zero densities. This is easily understood since self bound stars with
\[ \rho = (\times 10^{18} \text{ kg} \cdot \text{m}^{-3}) \]

\[ \mu = 1.0 \]
\[ \mu = 0.9 \]
\[ \mu = 0.8 \]
\[ \mu = 0.7 \]
\[ \mu = 0.6 \]

(a) The radial pressure, \( \beta = 1 \times 10^{-16} \)

(b) The radial pressure, \( \mu = 1 \)

**Figure 6.27**: Variation of the radial pressure with the density inside the star. The parameter values are \( \rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^4 \text{m} \), \( \beta \) is set to \( 1 \times 10^{-16} \) on the left, and varies on the right, and \( \mu \) is set to one on the right but takes the various values shown in the legend on the left.

\( \mu \neq 1 \) do not have boundaries at zero density. This also means that the full functional form of the EOS (whose plot we have culled at zero pressure) extends to pressures for densities lower than the boundary density for that star, and hence to negative pressures. Of course, such regions do not exist in our models, since we match our solution to an exterior metric before that happens. However it appears in the plots shown because we are plotting the pressures at densities that these particular stars do not have “access” to.

Another feature that is obvious is that this negative pressure does not occur in 6.27b. This is simply because all the EOS shown there are “natural” with \( \mu = 1 \), and hence are valid up to zero densities. The other characteristic of the second figure is how drastically the magnitude of the pressure function changes by changing the value of \( \beta \), something that suggests that for high enough anisotropies, the energy conditions might be violated: a conclusion we arrived at previously, through a more pedestrian approach.

The solution where charge compensates anisotropy

In this solution, \( \beta \) is compensated by \( k \), so that the expressions of the functions and constants, in particular \( a \), and hence \( Z \) do change a bit. In this particular case, the metric function \( Z \) is
thus given by
\[
Z(\rho) = 1 - \frac{k^2 r_b^4}{5\mu^2 \rho_c^2} (\rho - \rho)^2 + \frac{\kappa \rho_c^2}{5\mu} \left( \frac{\rho^2}{3} - \frac{\rho}{3} - \frac{2\rho_c}{3} \right). \tag{6.34}
\]
which has the same components as the previous (6.32), with an additional piece containing the charge \(k\). Similarly, the proper radius \(\xi\) is also changed, and its expression is given by
\[
\xi(\rho) = 2r_b \sqrt{\frac{5}{\kappa \mu \rho_c - k^2 r_b^2}} \times \arcoth \left[ \frac{\sqrt{3k^2 r_b^4 (\rho_c - \rho)^2 + \kappa \mu \rho_c (3\rho^2 - \rho \rho_c - 2\rho_c^2) + 15\mu^2 \rho_c^2}}{r_b (\rho_c - \rho) \sqrt{3k \mu \rho_c - 3k^2 r_b^2}} \right]. \tag{6.35}
\]
which is also similar to the previous (6.33) except for the additional \(k\) factors. Putting these two expressions in the pressure (4.35), we get the equation of state of this solution whose plots we show next in 6.28. Here also we notice particular trends in the two plots, very similar to the previous solution: changing \(\mu\) prevents all the densities from being “accessible” to the solution as before. The trend on the right panel 6.28b is striking in its regularity: the effect of charge \(k\) on the pressure is made clear, increasing the charge has a similar effect as increasing the self-boundness inasmuch as certain densities become inaccessible, but it does so in such a way that the shape of the EOS does not change.

![Figure 6.28: Variation of the radial pressure with the density inside the star. The parameter values are \(\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_b = 1 \times 10^4 \text{ m}, k\) is set to \(1 \times 10^{-9}\) on the left, and varies on the right, and \(\mu\) is set to one on the right but takes the various values shown in the legend on the left](image)
The solution where both charge and anisotropy exists
Since the metric function $Z$ does not depend on the anisotropy, having both anisotropy
and charge does not change the functional form of $Z$ from the previous case, and we still
have (6.34) as the expression of $Z(\rho)$. The same argument applied to $\xi(\rho)$, since the difference
in the pressure expressions come from the other constants that have $\beta$ in them along
with $k$ in this particular case.

(a) The radial pressure, $\beta = 1 \times 10^{-16}, k = 1 \times 10^{-9}$

(b) The radial pressure, $\mu = 1, \beta = 1 \times 10^{-16}$

(c) The radial pressure, $\mu = 1, k = 1 \times 10^{-9}$

Figure 6.29: Variation of the radial pressure with the density inside the star. The parameter values
are $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}, r_h = 1 \times 10^{4} \text{m}, \beta$ is set to $1 \times 10^{-16}$ on the left, and varies on the
right, and $\mu$ is set to one on the right but takes the various values shown in the legend on the left

6.3.2 Observables: masses, and radii
As in Tolman VII, in this section we use causality, i.e. the criterion that the speed of sound
inside the star not exceed the speed of light to limit the maximum possible masses, radii, and
electric charge that the models we are pursuing can admit.

To implement this constraint, we find the expression for the speed of sound in the star, and from the shape of the equation of state, \( p(\rho) \), we know that the speed of sound is positive definite, and a maximum at the centre of the star. We find this expression as a function of the free parameters at the centre of the star, for all the models we have.

Once this expression was found, we varied the parameters \( \mu \) and \( r_b \) to find the maximum possible value for the central density \( \rho_c \) for which the speed of sound \( v_s \) is just causal at the centre, in Tolman VII. This allowed us then to calculate the resulting mass of the star, since all the three parameters were known. This method works in Tolman VII where those are the only parameters completely determining the solution. In the new models however, this is no longer the case.

We now specialize to the different solutions we have, and consider the application of stability to each separately, and in doing so, encounter a number of complications. Indeed, the very fact that made the finding of solutions easier: the greater number of parameters that could be freely given, now hinders a straight forward physical interpretation, since the speed of pressure waves is now dependant on all of the new parameters too. As a reminder, equation (6.7) when taken to the limit of \( r = 0 \), the centre reduces to

\[
v_s(r = 0, \mu, \rho_c, r_b, k, \beta) = \left( \frac{r_b^2}{\kappa \rho_c \mu} \right) \left[ \beta - 3k^2 + \psi_0^2 + \frac{\kappa \rho_c}{3} \psi_0 \right],
\]

where

\[
\psi_0 = \lim_{r \to 0} \left( \frac{1}{r^2} \frac{dY}{dr} \right).
\]

\( \psi \) being the complicated part of these expressions we only show plots of how one might go about restricting these parameter ranges.

The Anisotropic case only

In the anisotropic new solution we found, we have the anisotropy factor \( \beta \), but no charge \( k \).

To present how this additional parameter changes the observables, we first investigate how
the speed of sound $v_s$ changes with the different parameters and $\beta$. Since we are concerned mostly with causality, and because we are using natural units, the zeros of the function $v_s^2 - 1$ give the parameter value we want to get the coordinates of the causality surface. This is more clearly shown in the Figures 6.30 where we show how $v_s^2 - 1$ behaves for certain fixed parameters chosen to be within the range of realistic stars, while another parameter varies on the $x-$axis. Figure 6.30 shows different values of the parameters $\rho_c$, $\beta$, $r_b$, and $\mu$ where the causality function $v_s^2 - 1$ crosses the horizontal axis. These solutions to the function give parameter ranges for which the speed of sound is causal at the centre, and hence everywhere in the star. Any value of the parameters that allow for negative values of the function are

Figure 6.30: Variation of the function $v_s^2 - 1$ at the centre of the star with different parameters being varied. In panel (a) the parameters are chosen as $r_b = 3 \times 10^3 m$ and $\mu = 1$; In (c) they are $\rho_c = 1 \times 10^{-9} m^{-2}$, $\mu = 1$; In (d) they are $\rho_c = 4 \times 10^{-9} m^{-2}$, $r_b = 3 \times 10^3 m$; In (b) they are $r_b = 3 \times 10^3$, $\mu = 1$. These values were chosen after analysis of the different shapes of the curves in terms of the different parameters.
causal parameter choices, and can potentially be used to model a compact object.

We notice that the parameter change induce non-trivial parameter range changes. The graphs shown have been chosen to have parameters in specially picked ranges to emphasize the issues that are involved in finding appropriate ranges for the models. In particular, notice that in Figure 6.30b, where the central density $\rho_c$ is varied for different values of the anisotropic parameter $\beta$, or vice-versa in 6.30a, result in the solution for the causality function to range through many different values.

The interpretation that can be afforded to the strange shape of 6.30a is that for low central densities, the stiffness of the star has to be huge resulting in huge pressure wave velocities, violating causality. These models are also unstable, since there is not enough mass to hold the star together. In the middle range between the two solutions – where the curves intersect the horizontal axis – we have central densities that are big enough to hold the star together, and small enough to maintain the stiffness low so that the speed of pressure wave is not too high. This is the range of $\rho_c$ we are interested in to model physical stars.

In Figure 6.30b we see how $\beta$ changes both the shape/slope of the velocity profile. In this diagram by contrast, any value of $\beta$ corresponding to the causality function being positive is rejected, and only lower values of $\beta$ are then used.

When considering variations of the boundary radius $r_b$, in Figure 6.30d, we notice that stars with larger radii, for fixed central densities and anisotropies are closer to the causality limit. Furthermore, higher anisotropies in larger values of $\beta$ ensures that the maximum radius $r_b$ that can be used is smaller than without anisotropy.

If variations in the self-boundedness $\mu$ is sought instead, we see that the natural case is almost always causal (at least for the chosen parameter range,) but the closer one gets to the Schwarzschild interior solution with $\mu \to 0$, the less causal the same models become.

An alternative way to look at these parameter spaces is through three-dimensional plots. Next we plot the same type of surface as Figure 3.6 but with different values of the anisotropies.
Additionally we also show another set of three-dimensional plots, for the natural $\mu = 1$ case but with varying values of anisotropy beta. These plots give an idea as to how the anisotropy parameter alone changes the masses and maximum central densities for causal stars.

In the first series of three dimensional plots, we show the mass in solar units, the central mass density and the self-boundness parameter $\mu$ on the $z$, $x$ and $y$ axes respectively. The surface shows the triplets that make the star just causal at the centre of the star, and since the speed of sound is a monotonic function of the radial parameter, this implies that the speed of sound is always causal in the star.

![Figure 6.31: The causality surfaces for a variety of values of $\beta$](image_url)

In Figure 6.31, we first plot the isotropic Tolman VII solution’s causality surface. Note that any triplet underneath that causality curve represents a viable physical star having parameters that form a causal star. The other two surfaces are for two different anisotropic parameters $\beta$, and we see that the higher the anisotropy, the lower the causality surface, implying that those stars have lower maximum masses typically than the Tolman VII solution, for the same parameters. However since we are usually mostly concerned with stars that are not on the edge of causality, this should not be a problem to model more complex anisotropic stars.
The “anisotropised charge” case

Here since only one of the parameters of either \(k\) or \(\beta\) remain, we choose one, and express the other one in terms of it. Here for convenience we chose \(k = \sqrt{\beta/2}\), and vary \(\beta\) according to the legend. This particular value of charge ensures that all the anisotropic pressure is accounted for by the charge as explained in detail in Section 4.2.1.

In Figure 6.33 we plot the same causality function as in the above section with respect to the different parameters, with the charge always fixed to match the anisotropic pressure. As a result of varying \(\rho_c\), in Figure 6.33a we see how the different values of the causality function can be. Again we are only interested in the range \(|v_s^2| < 1|\), and more specifically the solution of \(v_s^2 - 1 = 0\) for the limiting value of \(\rho_c\) for causality. The initial parts of the plot mimic the plot 6.30a we had previously when charge was not important. We see that for some anisotropies/charge the initial part of the function—for lower value of \(\rho_c\)—is always above the zero–axis, so that no plausible value of low central density is admissible. This is physically intuitive since for large charge, we would require the mass to be large enough to gravitationally compensate for electromagnetic repulsion. For higher densities however, we run into a different problem, where \(v_s\) is no longer real, i.e. \(v_s^2 < 0\) sometimes. These are obviously unphysical and cannot be used. However certain values of the central density in the higher ranges are admissible for causal solutions, and these are the ones we would
choose to model physical objects. By contrast the behaviour of causality is monotonic
parameter values where the star is causal is expected instead of the “sheet” type surface we saw previously.

The monotonicity of the last [6.33d] suggests that once again most natural cases are causal, and as the anisotropic parameter is increased, the possibility of a causal star decreases, even if it were natural with $\mu = 1$. This is to be expected since self-bound stars, with charge are even less physically plausible than just self-bound ones in quark stars.

This concludes the analysis of this particular solution. We see that parameter ranges for a causal, “anisotropised” charged model are possible, since the pressures can be chosen to be positive everywhere in the star as shown in Section 4.2.1 and these models can be causal as we just showed. These being the more stringent criteria that physical stars have to obey, we can conclude that these model can be viable for modelling physical stars.

The general case with both charge and anisotropy

When both the charge and anisotropic parameter can be varied, there is more possibilities for unwanted behaviour in the speed of sound, and causality to occur, as we now see. In the set of plots shown in Figure 6.34 we find how the causality depends on the initial parameters of this solution.

In 6.34a for example, we find that the trend we noticed in 6.33a is only accentuated in that the initial part of the curves do not even cross the horizontal axis, thus reducing the range of applicable central densities that could potentially be used for modelling purposes. The other striking feature is that for all values of $\beta$ there exists certain densities that have imaginary speeds of sound. This makes the choice of the parameter set that can be used tricky to specify exactly.

In the next plot 6.34b we see that as with the previous case 6.33b we have a monotonic dependence of the causality with respect to $\beta$, although different $\rho_c$ do change the shape of the curves. Of note here is that for high values of $\rho_c$, as suggested by Figure 6.34a, the speed
(a) Varying $\rho_c$

(b) Varying $\beta$

(c) Varying $r_b$

(d) Varying $\mu$

(e) Varying $k$

(f) Varying $k$

**Figure 6.34**: Variation of the function $v_s^2 - 1$ at the centre of the star with different parameters being varied. In all the plots, $k$ is specified separately from $\beta$ as expected in this model. In panel (a) the parameters are chosen as $r_b = 3 \times 10^3 \text{ m}, k = 1 \times 10^{-8} \text{ m},$ and $\mu = 1$; in (c) they are $\rho_c = 1 \times 10^{-9} \text{ m}^{-2}, \mu = 1$ and $k = 1 \times 10^{-8} \text{ m}^{-2}$; in (d) they are $\rho_c = 4 \times 10^{-9} \text{ m}^{-2}, r_b = 3 \times 10^3 \text{ m}$ and $k = 7 \times 10^{-9} \text{ m}^{-2}$; in (b) they are $r_b = 3 \times 10^3, \mu = 1,$ and $k = 7 \times 10^{-9} \text{ m}^{-2}$; in (f) they are $r_b = 3 \times 10^3, \mu = 1,$ and $\beta = 5 \times 10^{-16} \text{ m}$; in (e) they are $r_b = 3 \times 10^3, \mu = 1,$ and $\beta = 5 \times 10^{-16} \text{ m}$; in (f) they are $r_b = 3 \times 10^3, \mu = 1,$ and $\rho_c = 4 \times 10^{-9} \text{ m}$. These values were chosen after analysis of the different shapes of the curves in terms of the different parameters.
of sound is non-causal for all values of $\beta$, rendering that particular set of parameter values impossible to use for modelling causal stars.

In Figure 6.34c we see that indeed for certain values of $r_b$, an asymptote exists in the causality function. As a result, only the values of $r_b$ that are very small (smaller than actual neutron stars) are admissible, or, for low enough $\beta$, some large radii are still valid, however at even larger radii, these same models start having imaginary speeds of sounds, so that a very small set of $r_b$ is actually admissible in the end. This plot, together with Figure 6.34e and 6.34b, accentuate the difficulty in specifying a definite range of values where the model is causal.

The monotonicity of the speed with $\mu$ in Figure 6.34d makes interpretation easy, and as previously, we see that natural models have a better chance of being causal. However high values of $\beta$ ensures that no value of $\mu$ can be used for any models we want.

The next two plots 6.34e and 6.34f finally show how by changing the electric charge the model can go from causal initially, to non-causal through an asymptote, but come back to causality with higher values of charge, independently of the anisotropy factor $\beta$. This is very counter-intuitive, as one would expect that higher charge would cause the stiffness of the star to increase considerably. However since in general relativity, charge also contributed to the energy density, this stiffening is not permanent and the star starts becoming causal again for higher values of charge.

This concludes this section, which looked at the difficulties in giving strict values for ranges of the parameters we are looking at. Even in these simplistic models that we produced, the behaviour of the causality with respect to any of the the parameters is complicated and has to be approached with care, since certain sets of parameter values can push the star into non-causality, and therefore probably instabilities.
6.4 Solution validity: parameter value ranges

Using both stability and causality, it should be possible to restrict the values of the parameter set $\rho_c, r_b, \mu, k, \text{and } \beta$. In the previous sections we have already given a number of inequalities that restrict the values of these parameters. However in both Chapter 5 and Section 6.3.2 we found out that except for the inequalities mentioned, no fixed value for either the charge $k$ or anisotropy parameter $\beta$ can be specified. As a result, even specifying a range of values for each parameter is impossible, however a well established algorithm in the form of the list of inequalities given previously can be used to get bounds on each parameter, once others are stated. These bounds are also parameter dependent as we showed in the form of plot in figures 6.30, 6.33 and 6.34.

From the surface plots we showed that the range of values of masses and radii produced by the stars. The maximum masses for the just causal stars is around four solar masses. This is higher than the measured mass of compact objects for the same radii, implying that any triplet of parameter values below the surface, generating a mass smaller than the highest one possible can be modelled through these models and equations of state.

We can therefore conclude that all the 3 models, we have given EOS figures for: the anisotropic model with $\Phi^2 > 0$, the “anisotropised charged” model, and the charged anisotropic model with $\Phi^2 > 0$ are viable models that can be used to model compact objects. The values for $\beta$ that can be used in the first model is typically around $1 \times 10^{-17} \text{ m}^{-4}$ in geometrical units. This corresponds to about $1 \times 10^{27} \text{ Pa} \cdot \text{ m}^{-2}$. Since we picked a function from a mathematical point of view, there is no fundamental quantity we associate with such a unit, except that $\beta r^2$ is a pressure.

The typical values of $k$ that we have been using are $k = 3 \times 10^{-9} \text{ m}^{-2}$. This correspond to total charges $q = kr_b^3 = 3 \times 10^3 \text{ m}$ in geometrical units. Converting to SI units we get that the typical charges that can be associated with the model is around $q = 3 \times 10^{20} C$, for causal
stars.

6.5 Comparison with actual observation

The table in Figure 2.2 gives masses of neutron stars, with the maximum mass being around 2.5 solar masses. This means that any of our models, including Tolman VII can provide reasonable models for them. What more the models we provide are causal, and exact. If gravitational wave calculations, or Love number calculations in neutron star binaries have to be carried out, then our models which from a relativistic perspective have all the attributes of physical relevance should be considered, since many of the calculations will be greatly simplified, with an exact solution in hand.

The same arguments apply to the radii measurements. All of our models have used radii of about 10 km, but the parameters can be changed down so that we have even more compact star, with $r_b \sim 5$ km. While the latter are close to just being non-causal, they are still not ruled out by the measurements in Figure 2.3 If these objects need to be modelled, one of our solutions can be used there too.

The prediction that we do get from our models are the typical values of the anisotropy parameter, and the total charge that a compact object can admit. While direct measurement of these quantities is not currently feasible, these are definite predictions that could be used to infer the viability of our models.
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Appendix A

We state the definitions and mathematical machinery necessary for general relativity, and the tools we eventually use in this thesis.

A.1 Geometry

We introduce all the geometry needed for this work in detail starting from set theory.

A.1.1 Topology

We will not delve into a full topological machinery. This would take us too far into fundamental complexities of topological spaces that we will not encounter in our application. Instead, we give the bare minimum required to understand the definitions used later.

A collection $U$ of subsets of a set $X$ defines a topology on $X$ if $U$ contains

- the empty set and the set $X$ itself,
- the union of every one of its sub-collections,
- the intersection of every one of its finite sub-collections.

The sets in $U$ are then called the open sets of the topological space $(X, U)$.

A neighbourhood of a point $x$ in $X$ is a set $N(x)$ containing an open set which contains the point $x$. A family of neighbourhoods of $x$ introduces a notion of “nearness to $x$.” A topological space is Hausdorff if any two distinct points possess disjoint neighbourhoods. All spaces we will use in this thesis will be Hausdorff, and this definition is meant to discriminate against certain topological spaces that would not have properties useful for our purposes.

A collection $\{U_i\}$ of open subsets of $X$ is a covering if each element in $X$ belongs to at least one $U_i$. This means that $\bigcup_i U_i = X$. If the system $\{U_i\}$ has a finite number of elements, the
covering is said to be finite. A subcovering of the covering $\mathcal{U}$ is a subset of $\mathcal{U}$ which is itself a covering.

The covering $V = \{V_i\}$ is a refinement of the covering $\mathcal{U} = \{U_i\}$ if for every $V_i$ there exists a $U_j$ such that $V_i \subset U_j$. Thus this new finer cover is in some sense smaller that the original cover. A covering $\mathcal{U}$ is locally finite if for every point $x$ there exists a neighbourhood $N(x)$, which has a non-empty intersection with only a finite number of members of $\mathcal{U}$.

A subset $A \subset X$ is compact if it is Hausdorff and if every covering of $A$ has a finite subcovering.

If $(X, U)$ and $(Y, V)$ are two topological spaces, we can build a product space, denoted by $X \times Y$, such that elements of $X \times Y$ come from both $X$ and $Y$ in the following way: $X \times Y \equiv \{(x,y) : x \in X, y \in Y\}$. We also need a collection of subsets to define this product space topology, and for those we pick all subsets of $X \times Y$ which can be expressed as unions of the sets of the form $U_1 \times V_1$ with $U_1 \in U$ and $V_1 \in V$.

We now state a theorem, due to Tychonoff that will allow us to eventually define tensors as objects on these topological spaces.

**Theorem 2.** Let $(X, U)$ and $(Y, V)$ be compact topological spaces. Then the product space $X \times Y$ is compact in the product topology. This result holds even if we take the product of infinitely many compact topological spaces.

We shall not prove this theorem, but will make use of its implications very commonly.

### A.1.2 Mappings

A mapping $f$ from a set $X$ to a set $Y$ associates every $x$ in $X$ to a uniquely determined element $y = f(x) \in Y$. Different notations depending on whether the sets, or the elements are the purpose of the discussion exist in literature. If the sets themselves are being considered, then it is usual to see $f : X \rightarrow Y$. If it is the elements that are being considered,
$f : x \mapsto y = f(x)$ is more usual. Mappings are also called functions, however we will differentiate between the two terms, reserving function for a more restricted form of mapping.

A **composite mapping** of two the mappings \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) is the mapping \( g \circ f : X \rightarrow Z \) such that \( x \mapsto g(f(x)) \).

For some \( M \) a subset of \( X \), The symbol \( f(M) \) denotes the subset \( \{ f(x) : x \in M \} \) of \( Y \); \( f(M) \) is the **image** of \( M \) under the mapping \( f \). Similarly for some \( N \subset Y \), the symbol \( f^{-1}(N) \) denotes the subset \( \{ x : f(x) \in N \} \); \( f^{-1}(N) \) is the **inverse image** of \( N \) under the mapping \( f \).

Now we turn to a classification of functions that is going to be important subsequently:

If for every \( y \in f(X) \) there is only one \( x \in X \) such that \( f(x) = y \) then \( f \) has an **inverse mapping**, \( f^{-1} \) and is said to be **one-one** or **injective**. This is usually notated as \( f^{-1} : f(X) \rightarrow X \) or \( f^{-1} : y \mapsto x = f^{-1}(y) \).

The mapping \( f \) is said to map \( X \) **onto** \( Y \) if \( f(X) = Y \). Then \( f \) is also called **surjective**.

The mapping \( f \) is a **bijection** if it is both one-one and onto.

A mapping \( f \) from a topological space \( X \) to a topological space \( Y \) is **continuous** at \( x \in X \) if given any neighbourhood \( N \subset Y \) of \( f(x) \) there exists a neighbourhood \( M \) of \( x \in X \) such that \( f(M) \subset N \). \( f \) is continuous on \( X \) if it is continuous at all points \( x \) on \( X \).

A **homeomorphism** \( f : X \rightarrow Y \) is a bijection \( f \) which is bicontinuous, i.e. both \( f \) and \( f^{-1} \) are continuous.

### A.1.3 Manifolds

We can finally define a manifolds using these previous ideas.

An \( n \)-dimensional topological **manifold** is a Hausdorff topological space such that every point has a neighbourhood homeomorphic to \( \mathbb{R}^n \). As such this definition is terse and not immediately useful. The Hausdorff property is necessary to restrict pathological topologies from our models, homeomorphism of local neighbourhoods to a euclidean space \( \mathbb{R}^n \) ensures
the existence of local coordinates, and a topological space is a more general structure than the pre-relativistic Euclidean space.

A **chart** \((U, \varphi)\) of a manifold \(M\) is an open set \(U\) of \(M\), called the domain of the chart, together with a homeomorphism \(\varphi : U \to V\) of \(U\) onto an open set \(V\) in \(\mathbb{R}^n\). The coordinates \((x^1, x^2, ..., x^n)\) of the image \(\varphi(x) \in \mathbb{R}^n\) of the point \(x \in U \subset X\) are called the **coordinates** of \(x\) in the chart \((U, \varphi)\). A chart \((U, \varphi)\) is also called a **local coordinate system**. This takes care of local coordinates, but not of coordinate transformations. The next definition, by introducing compatibility conditions, allows coordinate transformations.

An **atlas** of class \(C^k\) on a manifold \(X\) is a set \(\{(U_a, \varphi_a)\}\) of charts of \(X\) such that the domains \(\{U_a\}\) cover \(X\) and the homeomorphisms satisfy the following compatibility condition. The maps \(\varphi_b \circ \varphi_a^{-1} : \varphi_a(U_a \cap U_b) \to \varphi_b(U_a \cap U_b)\) are maps of open sets of \(\mathbb{R}^n\) into \(\mathbb{R}^n\) of class \(C^k\). In other terms, when \((x^i)\) and \((y^i)\) are the coordinates of \(x\) in the charts \((U_a, \varphi_a)\) and \((U_b, \varphi_b)\) respectively, the mapping \(\varphi_b \circ \varphi_a^{-1}\) is given in \(\varphi_a(U_a \cap U_b)\) by \(n\) real valued \(C^k\) functions of \(n\) variables, \((x^i) \mapsto y^i = f^i(x^i)\). This property is easier to visualize than read, and figure [A.1] makes it clear what is happening, and how the overlapping of two coordinate systems is dealt with in the theory. With these definitions, we can now talk of “interesting” manifolds that we use in general relativity. A topological manifold \(X\) together with an equivalence class of \(C^k\) atlases is a \(C^k\) structure on \(X\), and we call \(X\) a **\(C^k\) manifold**. A **differential manifold** is a manifold such that the maps \(\varphi_b \circ \varphi_a^{-1}\) of open sets \(\mathbb{R}^n\) into \(\mathbb{R}^n\) are differentiable, but the expressions differential manifold and **smooth manifold** are often used to mean a \(C^k\) manifold where \(k\) is large enough for the given context. It is usual to use infinitely differentiable functions, belonging to the \(C^\infty\) class in physics. However sometimes we are more interested in analytic functions, of class \(C^\omega\). An **analytic** function \(f\) can be expanded in a Taylor’s series about the point of analyticity \(x_0\), and converges to the function value, \(f(x_0)\) in some neighbourhood of \(x_0\).

A **diffeomorphism** \(f : X \to Y\) is a bijection \(f\) which is continuously differentiable (of class
Figure A.1: The compatibility criterion given in the text that allows charts to overlap, and so allow coordinate transformations to be possible generically.

$C^1$). Nevertheless homeomorphisms of class $C^1$ are not necessarily diffeomorphisms, as the simple counterexample of $f : \mathbb{R} \to \mathbb{R} : x \mapsto y = x^3$, which is a class $C^1$ homeomorphism shows. In the latter example, $f^{-1} : y \mapsto x^{1/3}$ is continuous but not differentiable at $x = 0$.

A.1.4 Calculus

Calculus on manifolds is complicated because the implicit function theorem does not hold in these spaces without more assumptions [26]. Here we will assume that our spaces have the required properties ensuring the existence of differentiable functions.

A function $f$ on an $n$ dimensional manifold $X$ is a mapping $f : X \to \mathbb{R}$, specified by $f : x \mapsto f(x)$. Its representative in local coordinates of the chart $(U, \varphi)$ is a function on an open set of $\mathbb{R}^n$, defined through $f_\varphi := f \circ \varphi^{-1} : (x^i) \mapsto f(\varphi^{-1}(x^i))$. 

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The function \( f \) is **differentiable** at \( x \) if \( f_\varphi \) is differentiable at \( \varphi^{-1}(x) \). This definition is chart independent if \( X \) is a differential manifold. The **gradient**, also called **differential**, of \( f \) is represented in a chart by the partial derivatives of \( f_\varphi \). If \((U_a, \varphi_a)\) and \((U_b, \varphi_b)\) are two charts containing point \( x \), it holds that at \( x \)

\[
\frac{\partial f_\varphi_a}{\partial x^i_a} = \frac{\partial f_\varphi_b}{\partial x^j_b} \frac{\partial x^i_b}{\partial x^j_a}.
\]  

(A.1)

This equivalence relation allows us to call the differential of \( f \) a covariant vector, which we will define formally later.

A differential mapping \( f \) between differentiable manifolds, the source \( X \) of dimension \( n \) and the target \( Y \) of dimension \( m \), that is \( f : X \to Y \), is defined analogously. The differential at \( x \in X \) is represented in a chart at \( x \in X \) and a chart at \( f(x) \in Y \) by a linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \).

**A.1.5 Vectors**

Once we have a differential manifold, we can define vectors and tensors to characterise differential properties of objects. However we have to be careful because we do not want our definitions to be reliant on the local coordinates: we want generic vectors and tensors as geometrical objects intrinsically tied to the manifold. This is done by defining a tangent vector space \( T_x(X) \) at each \( x \in X \), such that the tangent vector space “linearise” the manifold locally around \( x \). Many equivalent definitions of tangent vectors, and their spaces exist, but we will use the one most convenient for our purposes. We will assume the basic axioms of a vector space, although that these are satisfied can be proved formally from our definitions.

A **tangent vector** \( v_x \) to a differential manifold at a point \( x \) is an equivalence class of triplets \((U_a, \varphi_a, v_\varphi_a)\) where \((U_a, \varphi_a)\) are charts containing \( x \), while \( v_\varphi_a = (v^i_\varphi_a), i = 1, \ldots, n \), are vectors in \( \mathbb{R}^n \). The equivalence relation is given by

\[
v^i_\varphi_a = v^j_\varphi_b \frac{\partial x^i_a}{\partial x^j_b}.
\]
where $x^i_a$ and $x^i_b$ are local coordinates in the charts $(U_a, \varphi_a)$ and $(U_b, \varphi_b)$ respectively. The vector $v_\varphi \in \mathbb{R}^n$ is the representative of the vector $v$ in the chart $(U, \varphi)$.

The vector $v$ is attached to the manifold by the assumption that the numbers $v^i_\varphi$ are the components of $v_\varphi$ in the frame of $\mathbb{R}^n$ defined by the tangent to the coordinate curves, where only one coordinate varies. This definition is compatible to the equivalence relation given above, and is usually expressed as the short expression

$$v := v^i \frac{\partial}{\partial x^i} = v^i \partial_i.$$

(A.2)

Tangent vectors at $x$ make up a vector space, the tangent space of $X$ at $x$ denoted by $T_x X$. An arbitrary set of $n$ linearly independent tangent vectors constitute a frame at $x$. The natural frame associated to a chart $(U, \varphi)$ is the set of $n$ vectors $e^i_{(\varphi)}$, $i = 1, ..., n$, such that $e^i_{(\varphi), \varphi} = \delta^i_j$. These $n$ vectors are the tangent vectors to the images in $X$ of the coordinate curves of the chart. The numbers $v^i_{\varphi_a}$ are then the components of the vector $v$ in the natural frame.

With this definition of a vector, we can generalise the notion to vector fields. The general idea is to associate a vector at each point $x$ of the manifold $X$.

Formally, A vector field on $X$ assigns a tangent vector at $x \in X$ to each point $x$, the tangent vector belonging to the tangent spaces $T_x X$. Since each point $x$ has its own tangent space, vector operations on manifolds are not trivial, as the field has to pick a vector from a different tangent space for each different point. A complete definition would however require additional concepts, and so we will not introduce them, and instead refer the reader to [26].

The relations governing vector (A.2) show that given a differentiable function $f$ on $X$, the quantity $v(f)$ defined for points $x$ in the domain $U$ of chart $(U, \varphi)$ by

$$v(f) := v^i_\varphi \frac{\partial f_\varphi}{\partial x^i_\varphi},$$

(A.3)

is chart independent. $v$ defines a mapping between differentiable functions. From this definition, the linearity of $v$, expressed as $v(f + g) = v(f) + v(g)$, can be proved. $v$ is
also a **derivation** since in addition to linearity it satisfies the **Leibniz law**, expressed as $v(fg) = fv(g) + v(f)g$. In the same vein, $v(f)$ is called the derivation of $f$ along vector $v$. If we take for $v$ a vector of a natural frame $e_{\varphi,(i)}$, so that $v^j = \delta^j_i$ then the quantity $v(f)$ reduces to the simple expression

$$v(f) \equiv e_{\varphi,(i)}(f) = \frac{\partial f(\varphi^{-1})}{\partial x^i_\varphi}.$$ 

In general the differential operator associated to the vector $e_{(i)}$ of an arbitrary frame is called a **Pfaff derivative**, and is denoted by $\partial_i$. In the natural frame, this corresponds to the partial derivative.

Given a $C^\infty$ map $h : X \to Y$, we define a map $h_* : T_xX \to T_{h(x)}Y$, called the **induced linear map**, or **push-forward** which maps tangent vectors of a curve $\gamma$ at $x \in X$ to the tangent vector to the curve $h(\gamma)$ at $h(x) \in Y$.

### A.1.6 Curves

Here the notion of a curve will become useful because identifying vectors with tangent vectors to curves give us back the intuitive understanding we have for vectors. A **parametrized curve** $\gamma$ on a manifold $X$ is a continuous mapping from an open interval $I \subset \mathbb{R}$ into $X$ specified by $\gamma : I \to X$ such that $\gamma : \lambda \mapsto \gamma(\lambda)$. The curve is oriented in the direction of increasing $\lambda$. A provable, but not quite obvious consequence of this definition is that a curve is invariant under reparametrizations which preserve orientation. As a result continuous, smooth and monotonously increasing mappings $I \to I' : \lambda \mapsto \lambda'$ preserve curves.

Once we have both the notions of curves, and vector fields, we turn to an **integral curve** of $v$ in $X$, which is a curve $\gamma$ in $X$ such that at each point $x$ on $\gamma$, the tangent vector is $v_x$. This integral curve is **complete** if it is defined for all values of $\lambda \in I \subset \mathbb{R}$. A set of complete integral curves of a vector field is a **congruence**. The concept of an integral curve in this
context comes about because in the curve \( \gamma \) defined above, it holds that

\[
\frac{d\gamma(\lambda)}{d\lambda} = v(\gamma(\lambda)) \quad \lambda \in I \subset \mathbb{R},
\]

which is an ordinary differential equation whose solution is the integral curve. If we were using a dynamical systems’ terminology, the same integral curve would be called a **trajectory**. By using the following theorem which can be proved through existence and uniqueness of solutions to exact differential equations, we find that trajectories are unique. Formally,

**Theorem 3.** Suppose \( v \) is a \( C^r \) vector field on the manifold \( X \), then for every \( x \in X \), there exists an integral curve of \( v \), given by \( \lambda \mapsto \gamma(x, \lambda) \), such that

1. \( \gamma(\lambda, x) \) is defined for some \( \lambda \) belonging to some interval \( I(x) \subset \mathbb{R} \), containing \( \lambda = 0 \), and is of class \( C^{r+1} \) there.
2. \( \gamma(0, x) = x \) for every \( x \in X \).
3. This curve is unique: Given \( x \in X \) there is no \( C^1 \) integral curve of \( v \) defined on an interval strictly greater than \( I(x) \), and passing through \( x \).

The same uniqueness theorem also ensures that in a given congruence, no curves will intersect, since intersection would require more than one possible curve through a given point. The fundamental reason we introduced curves is that curves provide a natural way to map a manifold onto itself. To see how this happens, consider \( \lambda \) and \( \mu \), two parameters belonging to the same \( I \subset \mathbb{R} \), such that the sum \( \lambda + \mu \in I \). Then since \( \gamma(\mu, x) \) is a point on one trajectory, we can consider the point \( \gamma(\lambda, \gamma(\mu, x)) \), which must clearly be another point further “down” the same trajectory. We can thus identify \( \gamma(\lambda, \gamma(\mu, x)) = \gamma(\lambda + \mu, x) \).

Since each curve of a congruence is a one–dimensional set of points (parametrised by \( \lambda \), say,) the set of all curves of a congruence to an \( n \)–dimensional manifold is an \( (n + 1) \)–dimensional smooth manifold, \( \Sigma_v \).
The mapping \( \gamma : \Sigma_v \rightarrow X : (x, \lambda) \mapsto \gamma(\lambda, x) \) is called the flow of the vector field \( v \). If both \( X \) and \( v \) are of the same differentiable class, then so is the flow. Now, because the constituents of \( \Sigma_v \) are both defined on open neighbourhoods, and because of the form of the map involved, for every \( x_0 \in X \), there must be a neighbourhood \( N(x_0) \subset X \), and also an interval \( I(x_0) \subset \mathbb{R} \) on the product of which \( \gamma \) is defined. Since products of smooth open neighbourhoods are also smooth and open, if both \( X \) and \( v \) are smooth, the domain of \( \Sigma_v \), denoted by \( N(x_0) \times I(x_0) \), must be smooth and open, by theorem 2. With this flow map, we can define a local transformation of \( X \) generated by the vector field \( v \) through \( \gamma(\lambda, \cdot) \equiv \gamma_\lambda : x \mapsto \gamma(\lambda, x) \) defined on \( N(x_0) \subset X \) for \( \lambda \in I(x_0) \). Under this mapping, a point \( x \in N(x_0) \) goes to a point \( \gamma_\lambda(x) \in X \) along the integral curve of \( v \) at \( x \), the location of \( \gamma_\lambda(x) \) along the curve being determined by the curve parameter \( \lambda \). Pictorially, the situation look like figure A.2, where the different flows are shown. We are now in a position to define global transformations on manifolds. We do this by extending the domain of our curve parameter \( \lambda \) to the whole real line. However we take note that the interval \( I(x_0) \subset \mathbb{R} \) depends on \( N(x_0) \) in general. The intersection \( I \) of all the intervals \( I(x_0) \) corresponding to a set of neighbourhoods \( \{N(x_0)\} \) covering \( X \) may be empty and this is a case we want to avoid. However, if \( X \) is compact, \( I \) is never empty, since it is then given by a finite intersection, by

\[ \begin{array}{c}
\text{Figure A.2: } \gamma_\lambda \text{ maps each dot into a cross on the same integral curve, as does } \gamma_\mu. \text{ We also show how compositions of flows work with } \gamma_{\lambda+\mu}. \\
\end{array} \]
the definition of compactness. Therefore when \( I \) is not empty, then \( \gamma_\lambda \) with \( \lambda \in I \) defines a **global transformation** of \( X \). Moreover, we can now extend \( \gamma_\lambda \) for all \( \lambda \in \mathbb{R} \) through the relation \( \gamma_{\lambda+\mu} = \gamma_\lambda \circ \gamma_\mu \), as we show pictorially in figure [A.2]. We can also define an inverse transformation for each \( \gamma_\lambda \), denoted by \( \gamma_{-\lambda} \), which undoes the flow \( \gamma_\lambda \).

As a result of the existence of such a global transformation on compact manifolds, we have

**Theorem 4.** A smooth vector field on a manifold \( X \), which vanishes outside a compact set \( K \subset X \), generates a one parameter group of diffeomorphisms of \( X \).

This theorem allows for points to be “dragged” along congruences globally on the manifold, and this process is sometimes called **Lie dragging**.

Points are not the only things that can be dragged with a congruence of curves. If a function \( f \) is defined on a manifold, then the group of diffeomorphisms generated by a vector field defined on the manifold defines a new function \( f_\lambda^* \). This works by carrying \( f \) along the congruence: if a point \( x \) on a certain integral curve is mapped to a point \( y \), a parameter value \( \lambda \) away on the same curve, (as in figure [A.2]) then the new function \( f_\lambda^* \) has the same value at \( y \) as \( f \) had at \( x \), that is \( f(x) = f_\lambda^*(y) \). If the value of \( f_\lambda^*(y) \) takes the same value as \( f(y) \), so that the dragged function and the original function have the same value at the same point, the function is said to be **invariant** under the mapping. If additionally this condition holds for all values of \( \lambda \), the function \( f \) is said to be **Lie dragged**, and then since \( f \) does not seem to depend on any motion along the congruence, we see that \( df/d\lambda = 0 \).

Vector fields can also be dragged along congruences, and this notion can be used to define the **Lie derivative** of a vector field.

### A.1.7 Forms

Before considering tensors, it is convenient to define a dual vector space to the tangent space. The **cotangent space** \( T_x^* \) to \( X \) is the dual of \( T_x \), that is the space of 1-forms on \( T_x \). These form a vector space of **covariant vectors**. Covariant vectors are geometrical objects independent
of the choice of coordinates. The components of a covariant vector at $x \in X$, in a chart $(U, \varphi)$ containing $x$, is a set of $n$ numbers $\omega_i, i = 1, \ldots, n$. Under a change of chart from $(U, \varphi_a)$ to $(U, \varphi_b)$ the numbers $\omega_i$ transform through

$$\omega_i = \omega_j \frac{\partial x^j_a}{\partial x^i_b}. \quad (A.4)$$

Covariant vectors can also be defined through the equivalence relation (A.4), analogous to the definition of contravariant vectors given earlier. A **coframe** is a set of $n$ linearly independent covariant vectors. The **natural coframe** is the set of differentials $dx^i$ of the coordinate functions $x \mapsto x^i$. A frame (sets of $n$ vectors $e_{(i)}$, $i = 1, \ldots, n$) and coframe (sets of $n$ 1-forms $\theta^{(j)}$, $j = 1, \ldots, n$) are **dual frames** if

$$\theta^{(j)} e_{(i)} = \delta^j_i, \quad \text{the Kronecker symbol.}$$

The differential of a differentiable function $f$ is a covariant vector field denoted by $df$, and is called an exact 1-form.

Like vectors, forms can also be Lie dragged and therefore Lie derived.

**A.1.8 Tensors**

Now that we have introduced both contravariant vectors and covariant ones (1-forms,) we can define more general tensors. However before doing so, we will introduce two conventions that we will be using as from now on. The first is called the **abstract index notation**, and was introduced by Penrose [101] to write formulae using representatives of vector and tensor fields in arbitrary frames, instead of the geometric objects themselves. These formulae are equivalent to the geometric ones, and tell us how the equivalence classes of representatives behave. As a result, geometric objects, when we will look at them will not have indices, but formulae involving these geometric objects will have indices to make calculations easier.

The second convention is called the **Einstein summation convention**. Stated simply, whenever the same letter index appears both upstairs and downstairs in formulae, an implied
sum over that index is assumed. This is done to de-clutter our formulae from the numerous summation signs that would otherwise be needed to denote contraction: a tensor operation. With these two conventions, we define tensors through their transformation properties as follows: A covariant \( p \)-tensor at a point \( x \in X \) is a \( p \) multilinear form on the \( p \) direct product of the tangent space \( T_x X \). Similarly contravariant tensors are multilinear forms on the direct products of the cotangent space \( T^*_x X \). As an example a covariant 2-tensor \( T \) at \( x \in X \) is an equivalence class of triplets \((U_I, \varphi_I, T_{\varphi I, ij})\), \( i, j = 1, ..., n \), with the equivalence relation allowing for the components to change from chart \((U, \varphi)\) to \((U, \varphi')\) through

\[
T_{i'j'} = \frac{\partial x^h}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{j'}} T_{hk}. \tag{A.5}
\]

A contravariant tensor would then transform analogously through

\[
T^{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} T^{hk}. \tag{A.6}
\]

The space of covariant[contravariant] 2-tensors at \( x \) is denoted by \( T^*_x \otimes T^*_x \) [respectively \( T_x \otimes T_x \).] The natural basis of this space associated to the chart with local coordinates \( x^i \) is denoted \( dx^i \otimes dx^j \) [respectively \( e_{(i)} \otimes e_{(j)} \)]. Therefore the covariant 2-tensor \( T \) is given by

\[
T = T_{ij} \, dx^i \otimes dx^j,
\]

where \( dx^i \otimes dx^j \) is the covariant 2-tensor, bilinear form on \( T_x X \otimes T_x X \), such that for any pair of vectors \( u \) and \( v \) with natural frame components \( u^i \) and \( v^i \) respectively, it holds that

\[
(dx^i \otimes dx^j)(u, v) = u^i v^j.
\]

The tensor direct product \( S \otimes T \) of a \( p \)-tensor \( S \) and a \( q \)-tensor \( T \) is a \( p + q \) tensor with components defined by products of components. Although products of components are commutative, tensor products are non-commutative, and \( S \otimes T \) and \( T \otimes S \) are different objects belonging to different spaces. As an example, if \( T \) is a contravariant 2-tensor, and \( S \) a covariant vector, the mixed tensor product \( S \otimes T \) is written as

\[
(S \otimes T)_{ij}^{\,jk} = S_i T^{jk}.
\]
The **contracted product** or a $p$ contravariant tensor and a $q$ covariant tensor is a tensor of order $p + q - 2$ whose components are obtained by summing over a repeated index appearing once upstairs and once downstairs. For example, we can contract the tensors $S$ and $T$ above in these different ways: $W^j = S_i T^{ij} = \sum_i T^{ij} S_i$, or $V^a = S_j T^{aj} = \sum_j S_j T^{aj}$, where we have eschewed Einstein’s convention for clarity.

From these definitions, it can be proved that certain properties of tensors are intrinsic, and this independent of coordinates or charts. The symmetry and antisymmetry properties of similarly places indices (i.e. all indices being considered being either all upstairs, or downstairs) is intrinsic, as is that of a tensor vanishing.

Like vectors and forms, we can define **tensor fields** as tensors at $x$ for each point $x \in X$. Differentiability is defined again on the charts, and the notion of a $C^k$ tensor is chart independent if the underlying manifold is at least of class $C^{k+1}$.

The tensor fields are defined on structures called fibre bundles. The basic notion to define a fibre bundle is that of a fibre.

A **bundle** is a triple $(E, B, \pi)$ consisting of two topological spaces $E$ and $B$ and a continuous surjective mapping $\pi : E \to B$. The space $B$ is called the base. We will restrict ourselves to situations in which the topological spaces $\pi^{-1}(x)$, for all $x \in B$ are homeomorphic to a space $F$. Then $\pi^{-1}(x)$ is called the **fibre** at $x$ and is denoted $F_x$. The space $F$ itself is called the **typical fibre**. If the bundle also has certain additional structure involving a group of homeomorphisms of $F$ and a covering of $B$ involving open sets, it is called a fibre bundle.

If $F$ is a vector space and the group is the linear group, the fibre bundle is called a **vector bundle**.

A formal definition of a fibre bundle will take us too far into category theory, and the simple notion above will suffice for our purpose. As a example we state the relevant spaces for

1. **a tangent bundle**. Let $T(X^n)$ be the space of pairs $(x, v_x)$ for all $x$ in the differential manifold $X^n$ and all $v_x \in T_x(X^n)$, the tangent space of $x$. This space of pairs can be
given the following fibre bundle structure \((T(X^n), X^n, \pi, \text{GL}(n, \mathbb{R}))\):

- the fibre \(F_x\) at \(x\) is \(T_x(X^n)\),
- the typical fibre \(F\) is \(\mathbb{R}^n\),
- the projection \(\pi : (x, v_x) \mapsto x\),
- the covering of \(X^n\) is \(\{U_j; \{U_j, \psi_j\}\}\) is an atlas of \(X^n\),
- the coordinates of a point \(p = (x, v_x) \in \pi^{-1}(U_j) \subset T(X^n)\) are
  \[(x^1, \ldots, x^n, v^1_x, \ldots, v^n_x)\],
- the structural group \(G\) is \(\text{GL}(n, \mathbb{R})\), the group of linear automorphisms of \(\mathbb{R}^n\) whose matrix representations is the set of \(n \times n\) matrices with non-vanishing determinant.

2. a **tensor bundle** of order \(s = (p + q)\). Let \(T(X)\) be the space of pairs \((x, R_x)\) for all \(x\) in the differential manifold \(X\) and all \(R_x \in, \bigotimes_{i=1}^p (T_x X); \bigotimes_{j=1}^q (T^*_x X)\) the set of vector spaces of \(x\). This space of pairs can be given the following fibre bundle structure \((T(X), X, \pi, \text{GL}(n, \mathbb{R}));\):

- the fibre \(F_x\) at \(x\) is \(T(X)\), i.e. each the \(n\)–components representation of elements of the \(p\) tangent and \(q\) cotangent spaces.
- the typical fibre \(F\) is \(\bigotimes_{i=1}^s \mathbb{R}^n\),
- the projection \(\pi : (x, R_x) \mapsto x\), mapping the tensor to its point,
- the covering of \(X\) is \(\{U_j; \{U_j, \psi_j\}\}\) is an atlas of \(X\),
- the coordinates of a point \(p = (x, R_x) \in \pi^{-1}(U_j) \subset T(X)\) are
  \[(x^1, \ldots, x^n, R^1_x, \ldots, R^n_x)\],
  the \(n^s\) real components of the tensor associated with each point.
• the structural group $G$ is $GL(n, \mathbb{R})$, the group of linear automorphisms of $\mathbb{R}^n$
whose representations are multidimensional objects.

We now have notions of vectors, forms, tensors and Lie dragging on differential manifolds. What we still lack to characterise the geometry completely enough to do physics are the notions of distance and parallelism. These two concepts will allow us to eventually talk about curvature, and from there lead us to Einsteinian relativity. We start with the notion of distance through the definition of a metric, and introduce the two different types of derivatives we use in this thesis: the Lie, and the covariant derivative.

A.1.9 Derivative of tensors

In this Section we discuss the various ways calculus can be done on tensor fields. Calculus is difficult in general manifolds because each point has its own tangent fibre on which the tensors are defined, and since calculus is involved with the comparing of objects at different points, a method of mapping fibres in the fibre bundles is needed. There are various ways to achieve this. Here, we use a completely operational approach involving tensor components instead of the abstract geometrical objects. However the 2 approaches are equivalent. We follow mostly the Refs. [25], [34], and [114].

Lie dragging introduces the concept of the Lie derivative, and parallel transport introduces the covariant derivative. Before going into the details however, the notation used throughout this thesis concerning derivatives is as follows:

1. The ordinary partial derivative will be denoted with a comma in the index. Thus for a tensor $T_{ab}$ for example,

   \[ T_{ab,c} = \frac{\partial T_{ab}}{\partial x^c}. \]

2. An equivalent notation that will be used for partial derivatives when it is convenient is the $\partial$ notation. In this notation, derivatives w.r.t the coordinate $x^c$ is written as $\partial_c$, and
for example the above equation is written
\[ \frac{\partial T_{ab}}{\partial x^c} = \partial_c T_{ab}. \]

3. The Lie derivative induced by a vector field \( u \) with components \( u^i \) will be given by
\[
(\mathcal{L}_u T)_{ab} = u^i \partial_i T_{ab} + T_{ib} \partial_a u^i + T_{ai} \partial_b u^i = u^i T_{ab,i} + T_{ib} u^i a + T_{ai} u^i b.
\]

4. The covariant derivative of tensor \( T_{ab} \) induced by a connection whose components are given as \( \Gamma^a_{bc} \) will be written either with the \( \nabla \) symbol or with a semicolon (;) thus,
\[
\nabla_c T_{ab} = T_{ab,c} = T_{ab,c} - \Gamma_{ac}^i T_{ib} - \Gamma_{bc}^i T_{ai} = \partial_c T_{ab} - \Gamma_{ac}^i T_{ib} - \Gamma_{bc}^i T_{ai}.
\]

Now we look into Lie derivatives.

A.1.10 The Lie derivative

The basic idea behind the notion of a Lie derivative is the following: Consider a vector field \( v \) in the same neighbourhood \( N \), in a manifold \( M \), and two points \( p, q \in N \). Then the vector field assigns a vector to each of \( p \) and \( q \), named \( v(p) \), and \( v(q) \), respectively. Choose a congruence of curves defined by another vector field \( u \) which has one curve through \( p \) that also goes through \( q \). Then \( v(p) \) can be Lie dragged along the curve defined by \( u \) up to \( q \). If the one parameter group of diffeomorphism generated by \( Y \), is denoted by \( f_t \), the “dragged along” point \( p \) is \( q = f_t(p) \), and therefore the vector at point \( q \) can also be denoted as \( v(q) = v(f_t(p)) \). Then the \( v(q) \) Lie dragged vector at \( p \) is \( f_t^{-1}(v(f_t(p))) \). Both of \( v(p) \) and \( f_t^{-1}(v(f_t(p))) \) are now vectors at point \( p \), and hence in the same tangent space, so that vector operations can be performed on them. Indeed, by subtracting the two vectors above, and the addition of a limit process on the parameter \( t \), the Lie derivative of vector \( v \) may be defined through Lie dragging along the curve defined through vector \( u \) by
\[
\mathcal{L}_u v(p) =: \lim_{t \to 0} [(f_t^{-1}(v(f_t(p)))) - v(p)].
\]

(A.7)
This derivative produces another vector \( w \), which in component form is given by

\[
 w^j = (L_u v)^j = u^i \frac{\partial v^j}{\partial x^i} - v^i \frac{\partial u^j}{\partial x^i}.
\]

The same procedure can be applied to forms, and tensors, both of which can be Lie dragged and compared along the same congruence of curves. We will here just provide the general expression that can be used to compute this derivative. For a 1-forms \( \alpha \), the derivative produces another 1-form \( \beta \),

\[
 \beta_j = (L_u \alpha)_j = u^i \frac{\partial \alpha_j}{\partial x^i} + \alpha_i \frac{\partial u^j}{\partial x^i}.
\]

For the general \( p \) contravariant and \( q \) covariant tensor, the mathematical expression is complicated, but if the tensor is given in component form, following the above prescription for vectors and 1-forms, the first derivative is the partial derivative of the tensor components contracted with the vector. We then get an additional term for each upstairs index, of the form \( T^{a_1, \ldots, a_p}_{\alpha_1, \ldots, \alpha_q} u^i \) and which carries a positive sign. Another additional term of the form \( T^{a_1, \ldots, a_p}_{b_1, \ldots, b_q} \partial u_i \) is then added with a negative sign for each downstairs index.

\[
 T^{a_1, \ldots, a_p}_{b_1, \ldots, b_q} = (L_u T)^{a_1, \ldots, a_p}_{b_1, \ldots, b_q} = u^i \partial_i T^{a_1, \ldots, a_p}_{b_1, \ldots, b_q}
 - \sum_{i=1}^{p} T^{a_1, \ldots, a_p}_{b_1, \ldots, b_q} \partial_i u^i
 + \sum_{i=1}^{q} T^{a_1, \ldots, a_p}_{b_1, \ldots, b_q} \partial_b u^i.
\]

We now have a prescription for the computation of the Lie derivatives. We state some important properties of the Lie derivative. Note that a vector field is needed for its computation and construction.

1. It is **type-preserving** in that the Lie derivative of a \((p + q)\) tensor is also a \((p + q)\) tensor.
2. As can be derived from its construction, it is linear:

\[ \mathcal{L}_u(\lambda v^a + \mu w^a) = \lambda \mathcal{L}_u v^a + \mu \mathcal{L}_u w^a, \]

for \( \lambda \) and \( \mu \) constants.

3. It follows the Leibniz law, just like the ordinary derivative so that

\[ \mathcal{L}_u Y^a Z_{bc} = Y^a(\mathcal{L}_u Z_{bc}) + (\mathcal{L}_u Y^a)Z_{bc}. \]

4. It commutes with contraction, so that for example, if the contraction \( S_a T^a_b = Q_b \)

\[ S_a(\mathcal{L}_u T)^a_b = \mathcal{L}_u(S_a T^a_b) = \mathcal{L}_u Q_b \]

5. The Lie derivative of a scalar field \( \phi \) is given by

\[ \mathcal{L}_u \phi = u^a \partial_a \phi, \]

i.e. the contraction of the vector components with the gradient of the scalar field: the standard directional derivative in the direction of the vector field.

Lie derivatives are useful because they allow the definition of isometries and symmetries of tensor, and in this thesis of the metric. We shall make use of these notions in the appropriate section below.

A.1.11 The covariant derivative

We now introduce another derivative that is used in this thesis, the covariant derivative. We will do so in an operational manner in component form, neglecting the deeper mathematical underpinnings of parallel transport which would take longer than we have to introduce.

Consider a contravariant vector field \( X^a(x) \) evaluated at a point \( Q \) with coordinates \( x^a + \delta x^a \), near a point \( P \), with coordinates \( x^a \). Then by using Taylor’s theorem, the vector field’s

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\[ \text{near } P \text{, with coordinates } x^a \text{. Then by using Taylor’s theorem, the vector field’s} \]
components at $Q$ can be expressed as $X^a(x + \delta x) = X^a(x) + \delta x^b \partial_b X^a$ to first order. Denoting the second term by

$$
\delta X^a(x) = \delta x^b \partial_b X^a = X^a(x + \delta x) - X^a(x),
$$

(A.8)

we find that this quantity is not tensorial: It involves the subtraction of vectors at different points of the manifold, and hence belonging to different fibres. We proceed to define a tensorial derivative by introducing a vector at $Q$ which is parallel to $X^a$ at $P$. Since $x^a + \delta x^a$ is close to $x^a$, we assume that the parallel vector differs from $X^a$ by a small amount denoted by $\tilde{\delta}X^a(x)$. $\tilde{\delta}X^a$ is not tensorial since it also involves the difference of vectors at 2 different points. However we can construct another quantity, the difference between the first (A.8) and the parallel vector’s deviation through

$$
\delta X^a(x) - \tilde{\delta}X^a(x) = X^a(x) + \delta X^a(x) - [X^a(x) + \tilde{\delta}X^a(x)],
$$

is tensorial. To do so, consider that $\delta X^a(x)$ would be zero if either $\delta x$, or $X^a(x)$ vanishes. As a result assuming that $\tilde{\delta}X^a$ be linear in both is the first step. This implies that there are objects (multiplicative factors) which we will call $\Gamma^a_{bc}$ such that

$$
\tilde{\delta}X^a(x) = -\Gamma^a_{bc} X^b(x) \delta x^c
$$

(A.9)

where the negative sign is introduced to agree with convention. We have therefore introduced $n^3$ functions $\Gamma^a_{bc}$ on the manifold. The transformation properties of these objects is what the rest of this section looks into. However once these have been introduced, a covariant derivative of $X^a$ can be defined through the limiting process

$$
X^a_{;c} = \lim_{\delta x^c \to 0} \frac{X^a(x + \delta x) - [X^a(x) + \tilde{\delta}X^a(x)]}{\delta x^c}.
$$

(A.10)

This is the difference between the vector $X^a$ at $Q$ and the vector parallel to $X^a$ at $P$ parallel transported to $Q$. This notion of parallel transport can be made rigorous. This limiting

---

2 This is what the rigorous notion of a linear connection establishes.
process then gives an expression for computing the covariant derivative of a vector as

$$X^a;_c = \partial_c X^a + \Gamma^a_{bc} X^b.$$  \hspace{2cm} (A.11)

The requirement that this object $X^a;_c$ be a $(1+1)$ rank tensor through the use of equation (A.5) and (A.6) then give the transformation properties of the objects $\Gamma^a_{bc}$ which are not tensors, but are called affine connections. The transformation properties on changing coordinate systems ($x^a \to x'^a$) is

$$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^f} \frac{\partial x^f}{\partial x'^d} \Gamma^d_{ef} - \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^d} \frac{\partial^2 x'^a}{\partial x'^d \partial x^e}. \hspace{2cm} (A.12)$$

A manifold that has a connection defined on it is called an affine manifold.

We also define the covariant derivative of a scalar field $\phi$ to be the same as the ordinary derivative through $\nabla_c \phi = \phi;_c = \partial_c \phi$. As a result, the covariant derivatives of forms can be defined as well since, vectors and forms contract to give rise to scalars, and demanding that the covariant derivative obey the Leibniz law yields

$$X^a;_c = \partial_c X^a - \Gamma^b_{ac} X^b.$$ 

The name covariant derivative comes from the fact that one additional covariant index get attached to the object being derived. Indeed the covariant derivative is not type-preserving like the Lie derivative, since a $(p + q)$ tensor’s covariant derivative is the $(p + q + 1)$ tensor given by

$$S^{a_1...,a_p}_{b_1,...,b_q,b_{q+1}} = T^{a_1...,a_p}_{b_1,...,b_q,c} = \partial_c T^{b_1,...,b_q} + \sum_{i=1}^{p} \Gamma^{a_i}_{dc} T^{a_1,...,d,...,a_p}_{b_1,...,b_q} - \sum_{i=1}^{q} \Gamma^{d}_{b_i,c} T^{a_1,...,a_p}_{b_1,...,d,...,b_q}.$$ 

From the transformation properties of the connection $\Gamma^a_{bc}$, it can be deduced that the difference between the connections with covariant index swapped, i.e. $\Gamma^a_{bc}$ and $\Gamma^a_{cb}$ gives a tensor.
This is because the last term of equation (A.12) vanishes upon subtraction, and the resulting tensor is called the **torsion tensor** and is given by

\[ T^{a}_{bc} = \Gamma^{a}_{[bc]} = \Gamma^{a}_{bc} - \Gamma^{a}_{cb}. \]

The covariant derivative is the most useful derivative from a physical point of view, since all the conservation laws can be most succinctly expressed in terms of this derivative. A usual rule of thumb given to students when going from special to general relativity is that most differential equations in SR involving partial derivatives get “promoted” to covariant derivatives, everything else being unchanged. We make use of this derivative in the geometrical aspects in the next section, and in computing conservation laws in a later Section.

We now have two notions of derivatives on the manifold. We proceed by defining a measure of distance and angle through the metric.

### A.1.12 The metric

A metric will be an object that associates a notion of distance between two points. There are many equivalent ways of doing this, and a good place to start is with Pythagoras’ theorem which allow us to calculate rectilinear distances between points in Euclidean space. Here because we only have a general, not necessarily euclidean, manifold, we generalise this idea of rectilinear distance to apply only infinitesimally.

A **Riemannian manifold** is a smooth manifold \( X \) together with a continuous 2-covariant tensor field \( g \), called the **metric tensor**, such that

1. \( g \) is symmetric. We expect this for the usual distance function, and require it here too, since the distance between two points does not depend on which point we consider first.

2. for each \( x \in X \), the bilinear form \( g_x \) is non-degenerate. This means that \( g_x(v, w) = 0 \) for all \( v \in T_x(X) \) if and only if \( w = 0 \). This requirement ensures that the metric is
invertible.

Such a manifold is said to possess a Riemannian structure. It is a **proper** Riemannian manifold if we have further that \( g_x(v, v) > 0 \) for all possible \( v \in T_x(X) \) such that \( v \neq 0 \). If the manifold is not proper, we call it a **pseudo-Riemannian** manifold endowed with an **indefinite metric**.

The condition 2 above is necessary if we want the metric to have an inverse, and can be expressed more conveniently in component form. In a local coordinate chart, \( g_x(v, w) \) is written as \( g_{ij}v^iw^j \), and the non-degeneracy requirement above implies that the determinant of \( g \), with elements \( g_{ij} \) does not vanish in any chart. This property is not dependent on the choice of the charts since local coordinate changes, \((x^i) \to (x'^i)\) result in \( g \) transforming through

\[
g_{ij'} = \frac{\partial x^h}{\partial x'^i} \frac{\partial x^k}{\partial x'^j} g_{hk} = [\Lambda^k_{ij'}][\Lambda^h_{ij'}] g_{hk}.
\]

If initially we require \( \text{Det}(g') \neq 0 \), then since

\[
\text{Det}(g) = \text{Det}(g') \text{Det}(\Lambda^k_{ij'}) \text{Det}(\Lambda^h_{ij'}),
\]

the determinant of \( g \) is never zero for local coordinate transformations.

The inverse of the matrix \((g_{ij})\) is denoted \((g^{ij})\) and defines the components of a contravariant symmetric 2-tensor. Both the metric and its inverse can be used to respectively “lower” and “raise” indices on other tensors. This works by contraction with the metric of the covariant or contravariant components of the geometrical object involved. As an example, the vector \( v \) having contravariant components \((v^i)\), can be expressed as to the object \( v \) having covariant components \((v_i)\), related to the contravariant components through \( v_i = g_{ij}v^j \), and vice-versa by \( v^i = g^{ij}v_j \). As mentioned previously, the kernel \( v \) represents the geometrical object that has covariant and contravariant components depending on the context, as required in the abstract index notation. Similarly, we can build **mixed** tensors out of purely covariant or contravariant ones, e.g. \( T^i{}_j = g_{aj}T^{ia} \). Since we defined the metric \( g_{ij} \) and its inverse \( g^{ij} \) as
matrix inverses, it holds trivially that $g_{ij}g^{ik} = \delta_j^k$, where $\delta_j^k$ is the Kronecker symbol. The **norm** of a vector $v \in T_x(X)$ is only defined if we have a metric, and is given by

$$|v|^2 = g_x(v, v) = g_{ij}v^iv^j.$$ (A.13)

If the our manifold is Riemannian proper, then this norm will always be positive, and if $|v| = 0$, we call $v$ a **null** vector. We will take a null vector to be orthogonal to itself. At each point $x \in X$ the null vectors can be imagined to form a cone in $T_x(X)$ called the **null cone**.

### A.1.13 Metric signature and orthogonality of vectors

Before we can turn to physics, we investigate the types of relations that the metric give us, once it is defined. The norm of a vector $v \in T_x(X)$, and equivalently the scalar from the quadratic form $g_{ij}v^iv^j$ in some chosen basis can be expressed as a sum of $k$ positive squares and $n - k$ negative ones, where $n$ is the dimension of the manifold $X$

$$g_{ij}v^iv^j = \sum_{i=1}^k (v^i)^2 - \sum_{i=k+1}^n (v^i)^2.$$ 

The number $k$ is then called the **index** of the quadratic form, and is independent of the basis. Then then number $k - (n - k)$ is called the **metric signature**. Since we have defined the metric $g$ to be continuous, the index and thus the signature of the metric is the same at each point $x \in X$, and one can speak of the signature of the whole manifold. The index of a proper Riemannian manifold is the same as its dimension $n$. This follows simply from our two metric axioms. A pseudo-Riemannian metric $g$ is called a **Lorentzian metric** if the signature of the quadratic form is $(+, - , -, \ldots, -)$. In the case of a Lorentzian manifold we denote its dimensions by $n + 1$ and we use Latin letters $(a, b, \ldots, = 0, 1, \ldots, n)$ for labelling local coordinates and tensor components, and use Greek letter $(\alpha, \beta, \cdots = 1, \ldots, n)$ for the spatial components.

In this thesis, because of the assumptions of classical general relativity, we restrict the dimension of the Lorentzian manifold we look at to 4, the index to 1, for a signature for -2. We
will choose to have the time component of our metric be positive, and the spatial ones to be
negative.
A vector \( v \in T_x(X) \) such that \( g_{ab}v^av^b < 0 \), that is one outside the null cone is called \textbf{spacelike}. A vector \( u \in T_x(X) \), such that \( g_{ab}v^av^b > 0 \), that is inside the null cone is called \textbf{timelike}. The null cone \( C_x \) is made up of two half-cones. If one of the half-cone is chosen
and called the future half-cone \( C^+_x \), then the tangent space \( T_x(X) \) is said to be \textbf{time ori-
ented}. A timelike vector in \( C^+_x \) is said to be \textbf{future-directed}; a timelike vector in \( C^-_x \) is said
to be \textbf{past-directed}.
The metric itself is usually written in the natural coordinate frame, and as a tensor usually
expressed through
\[
g = g_{ab} \, dx^a \, dx^b. \tag{A.14}
\]
As mentioned earlier, the metric is used to define length, surface and volume measures on a
manifold. The \textbf{length} of a parametrized curve \( \tau \mapsto x(\tau) \) joining two point of manifold \( M \)
with parameters \( \tau_1 \) and \( \tau_2 \) is
\[
l := \int_{\tau_1}^{\tau_2} g_{ab}(x(\tau)) \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \, d\tau \tag{A.15}
\]
The curve \( s \mapsto x(s) \) is said to be parametrized by arc length if
\[
\left| g_{ab}(x(\tau)) \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \right| = 1. \tag{A.16}
\]
Since we now have a metric tensor through equation \( \text{(A.14)} \), we use the component form of
vectors and tensors to perform calculations. The norm \( \text{(A.13)} \) of a vector having components
\( X^a \) is thus
\[
|X|^2 = g_{ab}X^aX^b.
\]
For two vectors \( X^a \) and \( Y^a \), neither null, the angle between then is defined through the
cosine of the angle between then. This is given by
\[
\cos(X, Y) = \frac{g_{ab}X^aY^b}{\sqrt{g_{cd}X^cX^d} \sqrt{g_{ef}Y^eY^f}}.
\]
The next step in getting to GR is the concept of geodesics. We define a \textbf{timelike metric geodesic} between points \( p, q \in X \), as the privileged curve joining the two points whose length is \textit{stationary} under small variations that vanish at the endpoints. This length may this be a maximum, a minimum, or a saddle point. To implement this condition, we require the calculus of variations, and the Euler–Lagrange equation, which we are going to assume without giving details. We refer the interested reader to \cite{26, 34} for details. The Euler–Lagrange (E–L) equation here has to be applied to the length which behaves as the action in equation (A.15); the metric is the generalised coordinate; and the curve parameter \( \tau \), the independent variable. The application of the E–L equation results in the equation of motion which is this case is the equation of a geodesic in a general Lorentzian manifold:

\begin{equation}
\frac{g_{ab}}{\tau^2} \frac{d^2 x^b}{d \tau^2} + \left\{ b c, a \right\} \frac{d x^b}{d \tau} \frac{d x^c}{d \tau} = \left( \frac{d^2 l}{d s^2} / \frac{d s}{d u} \right) \frac{d x_b}{d \tau} .
\end{equation}

(A.17)

The quantities denoted by \( \{ ab, c \} \) are known as the \textbf{Christoffel symbols of the first kind}, and are given by

\begin{equation}
\{ ab, c \} = \frac{1}{2} ( g_{bc, a} + g_{bc, a} - g_{ab, c} ) .
\end{equation}

(A.18)

The equation (A.17) can be simplified by using arc-length parametrisation (A.16), and finding an expression for the derivatives on the RHS in (A.17), which can easily be done with the definitions we have. This results in the geodesic equation simplifying to

\begin{equation}
\frac{d^2 x^a}{d l^2} + \left\{ a \right\} \frac{d x^b}{d \tau} \frac{d x^c}{d \tau} = 0 .
\end{equation}

(A.19)

with the \( \left\{ a \right\} \) being the \textbf{Christoffel symbols of the second kind} given by

\begin{equation}
\left\{ a \right\} = g^{ad} \{ bc, d \} = \frac{1}{2} g^{ad} ( g_{bd, c} + g_{cd, b} - g_{bc, d} ) .
\end{equation}

(A.20)

We can now define the \textbf{Einstein metric}, which is the special choice of the metric so that the connection is the same as the Christoffel symbol of the second kind. If this choice is made (the two objects transform similarly), the connection is called a \textbf{metric connection}. With
this particular choice we have

\[ \Gamma^a_{bc} = \left\{ \begin{array}{c} a \\ b \\ c \end{array} \right\} = \frac{1}{2} g^{ad} (g_{bd,c} + g_{cd,b} - g_{bc,d}) \cdot \quad (A.21) \]

As a result, \( \Gamma^a_{bc} \) is automatically symmetric, with \( \Gamma^a_{bc} = \Gamma^a_{cb} \) and the torsion \( T^a_{bc} \) vanishes.

The consequence of this identification, of the geodesic coefficients \( \{ a \} \) on the one hand and the connection \( \Gamma^a_{bc} \) on the other, means that the covariant derivative of the metric is zero

\[ \nabla_c g_{ab} = g_{ab,c} = 0, \]

as can be readily computed. We now have chosen the manifold, its dimension, the metric, the metric connection to be the ingredients of the theory of classical GR.

At each point of a curved Lorentzian manifold, we can choose a coordinate system such that at that point the metric is locally Minkowski. This is known as the local flatness theorem, and we state it here without proof. The interested reader is referred to Refs. [105, 114] where the theorem is proved through a first order Taylor expansion of the metric coefficients around point \( P \) and comparison with the curved metric transformation laws.

**Theorem 5.** For a given point \( P \) in space-time it is always possible to find a coordinate system \( x^a' \) such that

\[ g_{a'b'}(P) = \eta_{a'b'}, \quad \text{and} \quad \Gamma^a_{b'c'}(P) = 0. \]

where \( \eta_{a'b'} = \text{diag} (1, -1, -1, -1) \) is the **Minkowski metric** of flat space. Such a coordinate system is called a **Lorentz frame** at \( P \). The physical interpretation of this theorem leads directly to Einstein’s equivalence principle which states that free-falling observers do not see any gravitational effects in their immediate vicinity. Note however, that the derivatives of the connection coefficients \( \partial_d \Gamma^a_{b'c'} \) and therefore the second derivatives of the metric \( g_{a'b',c'd'} \) are not zero.

This theorem is used in the following sections, and sometimes even assumed without explicit warning.
We have a metric algebraically symmetric in its indices which reduces the number of components of $g_{ab}$ from $n^2 = 16$ to $\frac{1}{2}n(n+1) = 10$. However we need to be able to enforce physical symmetries on this metric, and other tensors. We will now use Lie derivatives to show how symmetries are enforced.

A.1.14 Symmetry

An isometry of a Lorentzian manifold $(V,g)$ is a diffeomorphism $f$ which leaves the metric $g$ invariant; that is $f^* g = g$. A metric is invariant by a 1-parameter group of isometries generated by a vector field $X$ if its Lie derivative with respect to $X$ vanishes. This is expressed as

$$\mathcal{L}_X g_{ab} = X^i g_{ab,i} + g_{ib} X_{,a}^i + g_{ai} X_{,b}^i = 0,$$

(A.22)

if $X$ is the vector field that generates the symmetries we are concerned with. Equation (A.22) is called Killing’s equation, and the associated vector field a Killing field.

This vector field that generates the symmetries can be any of the numerous ones we usually see in physics: for example there exists suitable vectors encoding time invariance, Lorentz invariance, spherical symmetry, etc. In this thesis we will be concerned with spherical symmetry and staticity and therefore briefly define what this entails.

A family of hypersurfaces is given by the equation $f(x^a) = \mu$, where the different members of the family have different values of $\mu$. This is similar to the usual concept of surfaces in Cartesian 3-space, but here generalised to manifolds. We can define a covariant vector field $N_a$ of vectors normal to the hypersurface by $N_a = \frac{\partial f}{\partial x^a}$. In the same analogy, these are the gradient of the surfaces in 3-space. This normal vector can be made to be of unit length by the usual process of dividing by its norm if the normal vector is nowhere null. This is defined through $n_a = \frac{N_a}{|N_a N^a|}$. Then depending on the type of hypersurface, the unit normal vector
A vector field $X^a$ is said to be **hypersurface-orthogonal** if it is everywhere orthogonal to the family of hypersurfaces, and proportional to the normal vector $n_a$ everywhere, so that $X_a = \lambda(x)n_a$.

A space-time is said to be **stationary** if and only if it admits a timelike Killing vector field. If the vector field is additionally hypersurface-orthogonal, the space-time is called **static**. In a static space-time, there exists a coordinate system **adapted** to the timelike Killing field above in which the metric is time-independent and has no cross-terms in the line element involving the time.

A space-time is said to be **spherically symmetric** if and only if it admits three linearly independent spacelike Killing vector fields $X^a$, $Y^a$ and $Z^a$ whose orbits are closed, and which obey the following relations

$$[X^a, Y^a] = Z^a, \quad [Y^a, Z^a] = X^a, \quad [Z^a, X^a] = Y^a.$$  

The Lie brackets $[X, Y]$ being defined through $[X, Y] = \mathcal{L}_X Y - \mathcal{L}_Y X$. These vectors are usually picked in a Cartesian frame so that

$$X = x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2} \quad \text{(A.24)}$$

$$Y = x^3 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^3} \quad \text{(A.25)}$$

$$Z = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \quad \text{(A.26)}$$

The adapted coordinates to these vectors are the usual spherical coordinates. These vectors together also generate the $SO(3)$ symmetry Lie group.
A.1.15 Curvature and the Riemann tensor

Curvature is going to play an important part in this thesis. Geometrically curvature is signalled by the non-commutativity of the covariant derivatives. This essentially means that a vector that has been parallel transported along a closed loop up to its starting point in no longer the same. This deviation of the 2 vectors is an effect of curvature.

The covariant derivative, unlike the partial derivative is not commutative. We define the commutator of a tensor $T^a_{\ldots b_{\ldots}}$ to be

$$\nabla_c \nabla_d T^a_{\ldots b_{\ldots}} - \nabla_d \nabla_c T^a_{\ldots b_{\ldots}}.$$

To compute the curvature we use the definition above and calculate the commutator of a vector directly. After a lengthy process, we obtain

$$\nabla_c \nabla_d X^a - \nabla_d \nabla_c X^a = (\partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma^e_{bc} \Gamma^a_{ed}) X^b + (\Gamma^e_{cd} - \Gamma^e_{dc}) \nabla_e X^a,$$

$$= R^a_{\ bcd} X^b + T^e_{\ cd} \nabla_e X^a.$$ (A.27)

The last equation uses the definition of the torsion, and also defines the Riemann tensor as the $(1+3)$ tensor $R^a_{\ bcd}$ that measures the curvature as a vector moves around a loop. If additionally we choose a metric connection \([A.21]\), the torsion vanishes and the expression of the Riemann tensor \([A.27]\) simplifies to

$$R^a_{\ bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma^e_{bc} \Gamma^a_{ed}.$$ (A.28)

Since the metric connection depends on the metric’s first derivatives as given in equation \([A.21]\), the Riemann tensor depends on the first and second derivatives of the metric. This becomes important when the boundary conditions have to be applied on Einstein’s equations which depend on the Riemann tensor, and hence up to the second derivative of the metric.

The Riemann tensor has a number of algebraic symmetries inherent in it. We state a few here, as these can easily be proved from the definitions we have given so far.
(i) It is antisymmetric in its last two indices so that $R_{abcd} = -R_{badc}$. 

(ii) A symmetric connection, and zero torsion leads to the following identity $R_{[abcd]} = R_{abcd} + R_{adbc} + R_{acdb} = 0$.

(iii) When all the indices of the Riemann tensor are lowered $R_{abcd} = g_{ad} R_{dbcd}$, the interchange of the first and last pair of indices do not change the tensor, that is $R_{abcd} = R_{cdab}$.

(iv) The only way the above identity can work is if the Riemann tensor is antisymmetric in its first 2 indices, and indeed it is, $R_{abcd} = -R_{bacd}$.

All these symmetries reduce the number of components of the Riemann tensor from the naive $n^4 = 256$ to $\frac{1}{12} n^2 (n^2 - 1) = 20$ independent components. The additional symmetries we will have on the metric tensor in this thesis reduce these even further.

The Bianchi identities can be stated in terms of the Riemann tensor directly. The contracted form of these identities are used in this thesis to simplify some of the differential equations, and in the derivation of the TOV equation. The identities read

$$R_{de[bc;a]} = \nabla_a R_{debc} + \nabla_c R_{deab} + \nabla_b R_{deca} = 0. \quad (A.29)$$

Once the Riemann tensor has been defined on the manifold, we build the Einstein tensor from different contractions of the tensor.

A.1.16 The Ricci tensor

The Ricci tensor is obtained by contracting the Riemann tensor in its first and third indices. The resulting tensor is a rank 2 tensor given by

$$R_{ab} = g^{cd} R_{dabc} = R_{acb}. \quad (A.30)$$

This is a symmetric tensor since $R_{ab} = R_{ba}$, as can be confirmed from the symmetry of $R_{abcd}$ in its interchange of the first and last pair of indices.
A.1.17 The Ricci scalar

The Ricci scalar is generated by a further contraction of the Ricci tensor (A.30). It is given by

\[ R = G^{ab} R_{ab} = R^a_a. \]  \hspace{1cm} (A.31)

It is the trace of the Ricci tensor, and can be considered as the average curvature in a certain sense. It is used to define Einstein’s tensor which we do next.

A.1.18 The Einstein tensor

Finally the Einstein tensor is defined as

\[ G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R. \]  \hspace{1cm} (A.32)

in terms of the Ricci tensor and the Ricci scalar. This is the tensor that the Einstein field equations are written with, and encode all the geometrical aspects that are directly influenced by matter and fields. The curvature, and metric are all affected through climbing up the contraction “ladder” just presented.

The Einstein tensor obeys the contracted Bianchi identity, given by

\[ \nabla_b G^b_a = G^b_a; b = 0. \]

We now turn to the other part of the Einstein field equations, having completed the geometrical aspects of it. The next part concerns the source of this geometrical curvature, matter and fields.

A.1.19 The Weyl tensor

The Weyl tensor \( C_{abcd} \) is defined through

\[ C_{abcd} = R_{abcd} - g_{a[c} R_{d]b} + g_{b[c} R_{d]a} + \frac{R}{3} g_{a[c} g_{d]b}, \]  \hspace{1cm} (A.33)
and satisfies the symmetries (i), (ii), and (iv) of the Riemann tensor. Additionally it is trace-free in all of its indices, and is hence also known as the “trace-free” part of the Riemann tensor. Under conformal transformation of the metric, $\bar{g}_{ab} \rightarrow \Omega^2 g_{ab}$, the Weyl tensor remains invariant, and sometimes this tensor is also called the \textbf{conformal tensor}.

A.2 Matter

To our best knowledge, matter is discontinuous at all scales, and quantum mechanics confirms this. However most of the time we are not concerned with this discontinuity and the advantages obtained by “smoothing” out these discontinuities are so many, that it is usual to describe matter in physics through a fluid. This approximation is usually valid only when we want to look at the behaviour of volumes big enough that quantum effects do not come into play, and small enough that arguments based on the infinitesimal are valid. Therefore if we can define measurable quantities associated with some finite volume (at whatever scale) at space-time events, we deem our model for matter to be continuous. This has to be taken with the grain of salt that the model breaks down when quantum effects come into play, but this breaking down only occurs at scales we are unconcerned with, and the macroscopic picture (with quantities corresponding to averages over microscopic ones) remains more or less faithful to reality. In this thesis we will model matter as a fluid.

A fluid is a model of matter where the only interaction possible between fluid elements occur at the interface between the elements, if no external forces are acting \cite{25}. These interactions might be of any type, including slipping, compression, pushing, etc. In this thesis we will be looking at three specific cases of fluids: dust, perfect fluids and fluids with anisotropic pressures. Dust is the simplest with no possible interaction between fluid elements. The next are perfect fluids which have minimal interactions between fluid elements, and the most complicated of the three is a simple generalization of perfect fluids.
To be able to use matter as a source of gravitation (an idea we wish to preserve from Newtonian physics), we need an object, preferably tensorial, that encodes matter and its aspects (energy, momentum, temperature, enthalpy). This will then allow us to specify a source for the Einstein equations. We proceed in the footsteps of Tolman [126] who derives a tensor encoding aspects of matter from very general considerations of a continuum. The reason we do not start with the relativistic dynamics of particles is that it is not possible to derive the equations of the continuum, in either Newtonian or relativistic physics from particle dynamics without a fair bit of quantum mechanics [8, 99, 126]. A more modern approach to this problem is given in [114] and while many details are eschewed, the concept of a momentarily comoving frame of reference (MCFR), which we use extensively is explained in detail.

A.2.1 Newtonian Analysis

The first aspect of matter we wish to capture is momentum and its conservation. In a continuum, at any point we can define nine quantities, usually called the stress matrix that give both the tangential and perpendicular components of the force acting on an imaginary surface at that point. We will label these quantities with two indices (as with a matrix). The first index will correspond to the direction in which the component of the force is acting, and the second index will refer to the direction normal to the surface to which the force component is acting. In a Cartesian coordinate system this will correspond to

\[
t_{ij} = \begin{pmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{pmatrix},
\]

(A.34)

where e.g. \( t_{zy} \) refers to the z-component of the force acting on a surface oriented in the y-direction: the imaginary surface that spans part of the \( xy \)-plane at the point. This force is caused by the material around the point in question and is thought of in the above example to be due to matter present at lower y-coordinate values. We show this in detail in figure A.3.
The major source of confusion in this set up is the direction of the normal for the imaginary surfaces. These normals are degenerate and could be in two opposing directions, however since we have not imposed any external forces on our continuum thus far, we expect Newton’s third law to hold, and we impose this by requiring that the force at the same point, but on the plane with the normal opposite to the one from the first surface be the same. Thus in our diagram, parallel surfaces of the cube would have opposite forces in the absence of volumetric external forces, as shown by both the complementary colour and notation: $t_i(-j)$ refers to the force in the $i$ direction with respect to the surface normal to the $(-j)$ direction.

**Figure A.3:** *The convention for specifying directions of the components of the stress tensor in Cartesian coordinates*

In the presence of external volumetric forces (electromagnetic, gravitational, etc.) the forces on parallel surfaces will not cancel each other as above. Instead the difference between these
forces will tell us about how the external forces are acting on the volume under consideration. Hence instead of \( t_{zy} \) on the upper surface and \( t_{z(-y)} = -t_{zy} \) on the lower one, we will have \(- (t_{zy} + \frac{\partial t_{zy}}{\partial y})\) on the bottom surface. The partial derivative will give us a measure of how the external volumetric forces are acting, and by summing all the contributions in say the \( z \) direction in particular, we get

\[
f_z = -\frac{\partial t_{zx}}{\partial x} - \frac{\partial t_{zy}}{\partial y} - \frac{\partial t_{zz}}{\partial z},
\]

the total external force in the \( z \) direction, per unit volume. This argument applies to the other directions, and we can immediately conclude that

\[
f_i = -\frac{\partial t_{ij}}{\partial x_j} = t_{ij,j}, \tag{A.35}
\]

where we are using Einstein’s notation again. We wish to relate this force to the change in momentum of the volume under consideration in an attempt to get back a dynamical rule like Newton’s second law, and to do so we will introduce, following Tolman [8, 126] a momentum volume density, whose component in the \( i \) direction is denoted by \( g_i \). The rate of change of this momentum density will be related to the force \( f_i \) through the usual relation,

\[
f_i \delta V = \frac{d}{dt} (g_i \delta V), \tag{A.36}
\]

where \( \delta V \) is the volume of the cube we are considering. Combining the two relations (A.36) and (A.35) we obtain

\[
-\frac{\partial t_{ij}}{\partial x_j} \delta V = \frac{d}{dt} (g_i \delta V) = \frac{dg_i}{dt} \delta V + g_i \frac{d(\delta V)}{dt}, \tag{A.37}
\]

which can immediately be simplified since the momentum density can change either instantaneously at one point or by the movement of the fluid element. This is usually expressed though a material derivative, but we will not pursue this matter further than to simplify the first term of the right hand side in terms of the velocities of the fluid element defined though
\[ u_i = \frac{\partial x_i}{\partial t}. \]

Therefore we have

\[ \frac{dg_i}{dt} = \frac{\partial g_i}{\partial t} + \frac{\partial g_i}{\partial x_j} \frac{\partial x_j}{\partial t} = \frac{\partial g_i}{\partial t} + u_j \frac{\partial g_i}{\partial x_j}. \quad (A.38) \]

Similarly, the volume element itself changes as it moves and the way each surface of our initial cube moves with the velocities defined above allows us to determine the quantitative change through

\[ \frac{d}{dt}(\delta V) = \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \delta V = \frac{\partial u_j}{\partial x_j} \delta V. \quad (A.39) \]

The derivative of the velocities appear instead of the velocities only since both parallel surfaces in our initial cube move with different velocities. Substituting the above results \((A.39)\) and \((A.38)\) in \((A.37)\), we get the final dynamical equation of motion of our fluid element in terms of the momentum density

\[ -\frac{\partial t_{ij}}{\partial x_j} = \frac{\partial g_i}{\partial t} + u_j \frac{\partial g_i}{\partial x_j} + g_i \frac{\partial u_j}{\partial x_j} = \frac{\partial g_i}{\partial t} + \frac{\partial}{\partial x_j} (g_i u_j). \quad (A.40) \]

This becomes our equation of momentum conservation. Additionally we require an equation of mass conservation, which in this framework we will take as the conservation of mass density, \(\rho\). Typically, we expect mass to exit or enter our cube through a momentum flow in and out of the surfaces of the cube. This means that

\[ -\frac{\partial M}{\partial t} = -\frac{\partial}{\partial t} \int \rho \, dV = \oint_{\partial V} g_i n^i \, dA = \int g_i n^i \, dV \]

The negative sign comes from the fact that the momentum density is taken to be pointing outward from the cube in our definition, and thus results in a reduction in mass of our initial volume; \(n_i\) is the normal vector in the \(i\) direction, pointing outwards too. Straightforward application of Gauss’s theorem on the surface integral then results in the following conservation law for the mass density:

\[ -\frac{\partial \rho}{\partial t} = \frac{\partial g_i}{\partial x_i} \quad (A.41) \]
All the above discussion is valid in Newtonian physics. We now wish to generalise it to a relativistic framework, and to do so we require the transformation properties of our dynamical variables under Lorentz transformations, since according to the first principle of relativity, the form of the above laws are invariant in all frames under uniform motion. To do so we refer to the transformation laws involving mass, force, momentum and area, and deduce the transformation laws for the stress tensor.

A.2.2 Special relativistic generalisation

In order to set up the problem, we will require two coordinate systems, related to each other through a proper motion given by a velocity vector $\vec{V}$. The first system $S$, will be assumed to be oriented (with no loss of generality) so that the fluid is moving along one particular direction, say the $x-$direction, with no components in the other spatial directions. We call the velocity of the fluid element with respect to the $S-$system, $\vec{u}$. In order to simplify the derivation of the transformation properties we shall also assume that the second coordinate system $S^0$ is also moving in the $x-$direction with respect to $S$ with the same velocity, so that in $S^0$ the components of the fluid element is given by $\vec{u} = (u^0_x, u^0_y, u^0_z) = \vec{0}$. This frame $S^0$ is what is referred to in [114] as the MCRF.

Since we have expressions for the stress components $t^0_{ij}$ in a rest frame: which we will identify with $S^0$, we can use the Lorentz transformations to generate the expressions of these quantities in the frame $S$. To do this we first transform all the forces (A.35) to the moving frame $S$, with the velocity $\vec{u}$ relating the proper motions between the two frames, to obtain

$$f_x = f^0_x, \quad f_y = f^0_y \sqrt{1 - (u/c)^2}, \quad f_z = f^0_z \sqrt{1 - (u/c)^2},$$  \hspace{1cm} (A.42)

since the proper motion is in the $x-$direction. Similarly, the surface areas of the cube faces that are normal to the $y-$ and $z-$ axes will be contracted, with the areas normal to the
direction of motion remaining the same, so that

\[ A_x = A_x^0, \quad A_y = A_y^0 \sqrt{1 - (u/c)^2}, \quad A_z = A_z^0 \sqrt{1 - (u/c)^2}. \]  

(A.43)

From the definition of the stress tensor we used previously (A.34) of force per unit area, and correspondence we just derived from the Lorentz transformations (A.42), and (A.43), we infer that the stress components in the \( S \) frame become:

\[
\begin{pmatrix}
  t_{0xx}^0 & t_{0xy}^0 & t_{0xz}^0 \\
  t_{0yx}^0 \sqrt{1 - (u/c)^2} & t_{0yy}^0 & t_{0yz}^0 \\
  t_{0zx}^0 \sqrt{1 - (u/c)^2} & t_{0zy}^0 & t_{0zz}^0
\end{pmatrix}
\]

(A.44)

The above equation is surprising in a two ways. First, we notice that since the velocity relating the frames is only in the \( x \)-direction, equation (A.44) is specialized for that particular case. Secondly it is quite surprising that while the stress tensor is a symmetrical array in the rest frame, that is \( t_{ij}^0 = t_{ji}^0 \), it is quite clear that even more so in the case of general velocities between the frames, the transformed stress array will not be symmetrical, so that in general relativistic fluids, \( t_{ij} \neq t_{ji} \). However now we have a way to relate the stresses measured by an observer at rest with respect to the fluid, with stresses measured in an arbitrary frame.

To get a complete tensorial description of matter, we should also include the transformations of energy-momentum \( g_i \) and density \( \rho \) between frames. From the expressions we have obtained before, we should then, following Tolman find an expression for the momentum, calculate the force acting on the stressed fluid, and thus calculate the work done, and energy change on our moving cube in terms of its mass, energy, velocity and stresses. The complication that arises however is that the change in momentum is not only due to the motion of the cube, but also due to the work done by the stress on the moving faces, and hence the volume of the cube. We shall initially assume as we did previously that the velocity is only
in the \( x \)-direction, so that the momentum density is given by

\[
\vec{g} = \begin{pmatrix}
g_x \\
g_y \\
g_z
\end{pmatrix} = \begin{pmatrix}
\rho u + \frac{t_{xx} u}{c^2} \\
\frac{t_{xy} u}{c^2} \\
\frac{t_{xz} u}{c^2}
\end{pmatrix}, \tag{A.45}
\]

where only the stress components \( t_{xj} \), with \( j = x, y, z \) are chosen since the cube is moving in the \( x \)-direction only. The products \( t_{xj} u \) thus give the energy density flow in the \( j \) direction due to the stresses and we divide by \( c^2 \) to make the units match. As is evident and expected, the momentum density in the \( i \) direction is also affected by stresses perpendicular to that direction in relativity.

Since the cube we are considering is infinitesimal, integrating the above \( \text{(A.45)} \) gives the total momentum change in some volume \( V \) as

\[
\vec{G} = \begin{pmatrix}
G_x \\
G_y \\
G_z
\end{pmatrix} = \begin{pmatrix}
\frac{E + V t_{xx} u}{c^2} \\
\frac{t_{xy} V}{c^2} u \\
\frac{t_{xz} V}{c^2} u
\end{pmatrix},
\]

which allows us to use Newton’s second law to find the force exerted on the volume \( V \) to change its velocity \( u \) in the \( x \)-direction as

\[
\vec{F} = \frac{d}{dt} \begin{pmatrix}
G_x \\
G_y \\
G_z
\end{pmatrix} = \frac{d}{dt} \begin{pmatrix}
\frac{E + V t_{xx} u}{c^2} \\
\frac{t_{xy} V}{c^2} u \\
\frac{t_{xz} V}{c^2} u
\end{pmatrix}. \tag{A.46}
\]

To calculate the work done on the stressed volume, we start with the initial volume in the observer’s rest frame, characterised by \( (V^0, t_{ij}^0, E^0) \), and bring it from rest up to some velocity by an adiabatic acceleration (boost), so that the observer is also moving with the accelerated material. The volume gets Lorentz contracted in the direction of motion through

\[
V = V^0 \sqrt{1 - (u/c)^2}, \tag{A.47}
\]
and throughout the boost, according to (A.44), the stresses transform through $t_{xx} = t_{xx}^0$. Therefore the change in energy of the volume, which comes from both the force accelerating the volume, and the work done by the stresses to contract the volume in the $x$–direction is

$$\frac{dE}{dt} = F_x \frac{dx}{dt} - t_{xx} \frac{dV}{dt}.$$ (A.48)

This can be expanded with the expression for the force from (A.46), and by holding $t_{xx}$ constant as per the transformation law into

$$\frac{dE}{dt} = \frac{dE}{dt}(u/c)^2 + E \frac{u}{c^2} \frac{du}{dt} + t_{xx} \left( \frac{u}{c} \right)^2 \frac{dV}{dt} + t_{xx} \frac{u}{c^2} \frac{du}{dt} V - t_{xx} \frac{dV}{dt},$$ (A.49)

after which it can be factorised into

$$\left(1 - \frac{u^2}{c^2}\right) \frac{d}{dt}(E + t_{xx}V) = \left(E + t_{xx}V\right) \frac{u}{c^2} \frac{du}{dt}.$$

To get the energy change in the moving frame, we have to integrate from zero velocity at time $t = 0$ to a velocity of $u' = u$, at $t = t'$, after rearrangement into logarithmic integrals:

$$\frac{1}{E + t_{xx}V} \frac{d}{dt}(E + t_{xx}V) = -\frac{1}{2} \frac{-2u' \frac{du'}{dt}}{\left(1 - \left(\frac{u'}{c}\right)^2\right)},$$

giving rise to

$$\log \left(E + t_{xx}V\right)_{t=0,u'=0}^{t=t',u'=u} = -\frac{1}{2} \int_{t=0,u'=0}^{t=t',u'=u} \frac{-2u' \frac{du'}{dt}}{\left(1 - \left(\frac{u'}{c}\right)^2\right)} \, dt = \log \left[1 - \left(\frac{u}{c}\right)^2\right]^{-1/2},$$

which is then simplified into the simple

$$E + t_{xx}V = \frac{E^0 + t_{xx}^0 V^0}{\sqrt{1 - (u/c)^2}}.$$ (A.50)

The zero-superscripted variables being evaluated in the rest observer frame prior to the boost.

With equation (A.50), we can now deduce the transformation of energy densities in a continuous fluid due to both relativistic motion, and stresses. To do so we first convert the energy $E$ to an energy density by dividing by $c^2 V$, followed by a conversion of all the volumes $V$ into $V^0$ through (A.47), such that equation (A.50) is converted into the equivalent

$$\rho = \frac{\rho_0 + t_{xx}^0 u^2 / c^4}{1 - (u/c)^2}. $$ (A.51)
In the above (A.51), $\rho_{00}$ is the proper energy density in the cube at rest (i.e. in the observer’s frame, such that $\rho_{00} = E^0/V^0$.)

We can now write down the momentum densities in the moving frame by combining equations (A.51), (A.54), and (A.45) to get

$$\vec{g} = \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix} = \begin{pmatrix} c^2 \rho_{00} + t_{0x}^0 & \frac{u}{1-(u/c)^2} & \frac{u}{c^2} \\ \frac{t_{0y}^0}{\sqrt{1-(u/c)^2}} & \frac{u}{c^2} \\ \frac{t_{0z}^0}{\sqrt{1-(u/c)^2}} & \frac{u}{c^2} \end{pmatrix},$$

(A.52)

This equation is important because it will allow us to compute the mass density $\rho$ and momentum density $g_i$, at some point in a medium moving with a velocity $u$ in terms of this velocity, and the values of the densities $\rho_{00}$ and stresses $t_{ij}^0$ measured by an observer moving with the fluid element, if the velocity $u$ is oriented along the $x-$direction. From the form of the equations, extension to other more general directions immediately becomes possible, and can even be done by inspection, but we will not give those expressions here. Before continuing onto the transformations and definitions of pressures in this fluid, we remember that no quantum mechanical considerations has gone into this derivation: only basic Newtonian thermodynamics and relativity principles have been used.

### A.2.3 Pressures

We mentioned before that surprisingly the relativistic stress matrix is no longer symmetrical. However it is a known feature of continuum mechanics that the symmetry of the stress tensor helps in the interpretation, and indeed with many calculations in Newtonian mechanics.

We now look for a convenient relativistic framework that will achieve the same end. The formalism as presented so far is complete inasmuch it allows the calculation of all stresses and energies in frames moving with respect to each other. To retrieve symmetrical matrices and then tensors, we first define “absolute stresses,” or pressures in this continuum. We
consider the array of quantities given by

\[ p_{ij} = t_{ij} + g_i u_j, \]  
(A.53)

where \( t_{ij} \) are the stresses defined previously in (A.34), \( g_i \) are the momentum densities defined in (A.45), and \( u_i \) are the velocity components of the continuum at that point. If we place ourselves in the observer frame moving with the fluid, according to the symmetry of the stresses \( t_{ij} \) in the rest frame, we automatically have

\[ p^0_{ij} = p^0_{ji} = t^0_{ij} = t^0_{ji}, \]

since the \( u_i = 0 \). In the moving frame however, here again assumed to be moving in the \( x \)-direction, we have the \( p_{ij} \) transform to

\[
p_{ij} = \begin{pmatrix}
P^0_{xx} + p_{00} u^2 & \sqrt{1-(u/c)^2} p^0_{xy} & \sqrt{1-(u/c)^2} p^0_{xz} \\
\sqrt{1-(u/c)^2} p^0_{yx} & p^0_{yy} & \sqrt{1-(u/c)^2} p^0_{yz} \\
\sqrt{1-(u/c)^2} p^0_{zx} & \sqrt{1-(u/c)^2} p^0_{zy} & p^0_{zz}
\end{pmatrix},
\]  
(A.54)
as can be computed directly through equation (A.53). As we can see this array is symmetrical, as we set out to do. If we now go back to equations (A.51) and (A.52), we can re-express them in terms of the pressures instead of the stresses to get,

\[
\begin{pmatrix}
g_x \\
g_y \\
g_z
\end{pmatrix} = \begin{pmatrix}
c^2 \rho_{00} + p^0_{xx} u \\
\sqrt{1-(u/c)^2} \frac{u}{c^2} \\
\sqrt{1-(u/c)^2} \frac{u}{c^2}
\end{pmatrix}, \quad \text{and,} \quad \rho = \rho_{00} + p^0_{xx} u^2 / c^4 \frac{1}{1-(u/c)^2}.
\]  
(A.55)

As a result of re-expressing everything in terms of the pressures instead, the equation of motion (A.40) can be put in a similar form as the continuity equation (A.41), the simple

\[ \frac{\partial p_{ij}}{\partial x_j} + \frac{\partial g_i}{\partial t} = 0. \]  
(A.56)

This simple form of both of these equations is the clue that leads to the generalisation to 4-quantities required for a relativistic treatment. The only difference in the general case where
the fluid is not assumed to be moving only along the $x-$direction is that the transformations seen above get more complicated, the general idea of how these work having been made clear.

However before an extension to four dimensional quantities, a note on terminology. As mentioned previously, the quantities $t_{ij}$ are called stresses, and corresponds to the forces one side of the imaginary cube pictured previously exerts on another potion of the fluid. This aspect is sometimes used to call the $t_{ij}$ the “relative stresses,” particularly in continuum mechanics of solids. By contrast since the $p_{ij}$ take into account the total momentum densities at every point of the fluid, independently of the surroundings, in one particular coordinate system, they correspond to “absolute stresses” in the same terminology. Now we generalize these notions to four dimensional space.

A.2.4 Four dimensional quantities

The relativistic treatment we started in the previous section suggests that both the pressures $p_{ij}$ and the density $\rho_{00}$, by transforming according to the Lorentz transformations of spatial and time like variables respectively can be combined in one four-tensor. To do so we consider proper coordinates that move along with the fluid as previously such that the fluid has zero spatial velocity in these coordinates. If we have $(t, x, y, z) \equiv (x_0^0, x_0^1, x_0^2, x_0^3)$, as coordinates, the spatial velocities are expressed as

$$\frac{dx_0^1}{ds} = \frac{dx_0^2}{ds} = \frac{dx_0^3}{ds} = 0,$$

where $s$ is the proper distance along the fluid world line. In such a system we can define an energy-momentum tensor through the components of the pressures and the density such that this tensor transforms according to the general diffeomorphic invariance we expect of
tensors,

$$T_{0}^{ab} = \begin{pmatrix} c^2 \rho_{00} & 0 & 0 & 0 \\ 0 & p_{xx}^{0} & p_{xy}^{0} & p_{xz}^{0} \\ 0 & p_{yx}^{0} & p_{yy}^{0} & p_{yz}^{0} \\ 0 & p_{zx}^{0} & p_{zy}^{0} & p_{zz}^{0} \end{pmatrix}, \quad \text{with,} \quad T^{ab} = \frac{\partial x^a}{\partial x'_0} \frac{\partial x^b}{\partial x'_d} T_{0}^{cd}, \quad (A.57)$$

the second equation permitting the transformation of the quantities in the observer frame to any other frame at any other point. Since we just stated a definition without really proving that the new tensor $T$ really transforms like one, we can check that this is indeed the case. To do so, we start with an observer with coordinates $(x_0^0, x_1^0, x_2^0, x_3^0)$, who perceives the fluid to be at rest, and hence has an energy momentum tensor given by $\text{(A.57)}$ and transform to a frame where the fluid is moving parallel to the $x$–axis with velocity $u$, as we did previously, but at the same point, the coordinates transform according the Lorentz transformation with respect to a velocity of $-u$ through

$$\begin{pmatrix} x_0^0 \\ x_1^1 \\ x_2^2 \\ x_3^3 \end{pmatrix} = \begin{pmatrix} x_0^0 + xu_0^0c \sqrt{1 - (u/c)^2} \\ \frac{x_1^1 + xu_0^0c}{\sqrt{1 - (u/c)^2}} \\ \frac{x_2^2 + xu_0^0c}{\sqrt{1 - (u/c)^2}} \\ x_3^3 \end{pmatrix}. \quad (A.58)$$

This then allows the computation of the derivatives that we had in the second equation of $\text{(A.57)}$, the symmetric matrix:

$$\frac{\partial x^a}{\partial x^c_0} = \begin{pmatrix} \frac{1}{\sqrt{1 - (u/c)^2}} & \frac{u/c}{\sqrt{1 - (u/c)^2}} & 0 & 0 \\ \frac{u/c}{\sqrt{1 - (u/c)^2}} & \frac{1}{\sqrt{1 - (u/c)^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (A.59)$$
which in turn allows the computation of the energy-momentum tensor in the moving frame with equation (A.57), since we now possess all the pieces of the latter. The computation gives

\[
T^{ab} = \begin{pmatrix}
\frac{c^2 \rho_0 + p_{0x} u^2 / c^2}{1 - (u/c)^2} & \frac{c^2 \rho_0 + p_{0y} u^2 / c^2}{1 - (u/c)^2} & \frac{p_{0x}}{\sqrt{1 - (u/c)^2}} & \frac{p_{0y}}{\sqrt{1 - (u/c)^2}} & \frac{p_{0z}}{\sqrt{1 - (u/c)^2}} \\
\frac{c^2 \rho_0 + p_{0y} u^2 / c^2}{1 - (u/c)^2} & \frac{c^2 \rho_0 + p_{0x} u^2 / c^2}{1 - (u/c)^2} & \frac{p_{0x} u + p_{0y}}{\sqrt{1 - (u/c)^2}} & \frac{p_{0y} u + p_{0x}}{\sqrt{1 - (u/c)^2}} & \frac{p_{0z}}{\sqrt{1 - (u/c)^2}} \\
\frac{p_{0x} u}{\sqrt{1 - (u/c)^2}} & \frac{p_{0y} u}{\sqrt{1 - (u/c)^2}} & \frac{p_{0x} p_{0y}}{p_{0z}} & \frac{p_{0y} p_{0z}}{p_{0x}} & \frac{p_{0z} p_{0z}}{p_{0x}} \\
\frac{p_{0x} p_{0y}}{p_{0z}} & \frac{p_{0y} p_{0z}}{p_{0x}} & \frac{p_{0z} p_{0z}}{p_{0x}} & \frac{p_{0z} p_{0z}}{p_{0x}} & \frac{p_{0z} p_{0z}}{p_{0x}} \\
\end{pmatrix},
\]

(A.60)
directly from the tensor equations, without any notions of momentum gain or loss through the cubes faces being evident. Upon comparing with the actual momentum flow equations (A.54), and (A.55), we can easily see by inspection that the above equation can be expressed very simply in the form

\[
T^{ab} = \begin{pmatrix}
c^2 \rho & c g_x & c g_y & c g_z \\
c g_x & p_{xx} & p_{xy} & p_{x} \\
c g_y & p_{yx} & p_{yy} & p_{y} \\
c g_z & p_{zx} & p_{zy} & p_{z} \\
\end{pmatrix},
\]

(A.61)
just as we wanted the forces, densities and momenta to transform in any frame: this proves that indeed the object $T^{ab}$ is a well defined tensor, and we have also shown that it summarizes all the mechanical fluid properties of interest in any Lorentzian frame within the scope of special relativity. This will serve as a starting point for the general relativistic notion of a stress-energy, or energy-momentum tensor.

As a summary, the components of the energy-momentum tensor can be interpreted as

1. $T^{00}$ is the energy density of the fluid.

2. $T^{0\alpha}$ is the energy flux across the surface of the cube associated with the observer.

3. $T^{\alpha 0}$ is the momentum density across the surface of the same cube. In all our considerations, and through construction $T^{0\alpha} = T^{\alpha 0}$. 

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4. $T^\alpha\beta$ are the pressures, or stresses compensated by momentum flux.

All of the above only holds in the rest frame where the observer moves with the fluid. In moving frames, these quantities get Lorentz-transformed and their straightforward interpretation become problematic, in the same way as the concepts of space and time separately become problematic.

Before we extend this formalism to general relativity however, we should conveniently notice that the conservation equations (A.56), and (A.41) can be expressed in the short form of

$$\frac{\partial}{\partial x^a} T^{ab} = 0 \implies T^{ab}_{\ ,a} = 0,$$

(A.62)

an equation valid only in Lorentzian frames. As a result it is a tensorial equation that was in terms of partial derivatives only because of the procedure of our derivation in the special relativistic case.

However, through the correspondence referred to previously, partial derivatives being “promoted” to covariant ones in general relativity, where the coordinate transformations are more general, we would obtain the equivalent general relativistic postulate encapsulating all of the matter equations through

$$T^{ab}_{\ ,a} \longrightarrow T^{ab}_{\ ;a} = \frac{\partial}{\partial x^a} T^{ab} + \Gamma^a_{cd} T^{db} + \Gamma^b_{ad} T^{ad},$$

(A.63)

this being the beginning of the generalization to general relativity of the continuum.

A.2.5 The energy momentum tensor for specific fluids

We now look at different types of fluids that will be used in this thesis. We only look at simplified fluids, mostly because of the symmetry requirements we have. More general fluids can be reduced to these simple cases in the symmetries we consider, and the proof of this follows the same proof that allows the reduction of the components of the metric tensor given in the Section

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Dust

**Dust** is also called *incoherent matter* because neighbouring fluid elements (components of the dust) do not exert any force whatsoever on each other. As a result, all the dust elements exert no stress, or pressure on each other, so that they can be characterised by the energy density of the fluid only. The energy-momentum tensor can then be expressed in the very simple coordinate independent, and coordinate dependant forms respectively:

\[
T = \rho \vec{u} \otimes \vec{u} \implies T^{ab} = \rho u^a u^b,
\]

where \( u^a = \frac{dx^a}{ds} \), the proper velocity of the fluid.

Perfect fluids

A **perfect fluid** is a mechanically continuous medium that is incapable of exerting any transverse stresses on other fluid elements. This is equivalent to demanding that the fluid admits no viscosity, since viscous forces are always tangential to the surfaces of the cubes in the fluid element picture we had previously. Therefore non-diagonal elements of the stress energy array have to vanish. Furthermore, the no viscosity criteria being global to the fluid, this has to hold at every single point. It is easy to see that the only array that admits this even through coordinate transformations is one that is diagonal in all frames, and linear algebra ensures that only a tensor that is a multiple of the identity array holds this property \([114]\). Therefore in the MCFR, together with having no transverse momenta, no non-diagonal pressures are present either, so that

\[
T^{ab}_0 = \begin{pmatrix}
\rho_0 & 0 & 0 & 0 \\
0 & p_0 & 0 & 0 \\
0 & 0 & p_0 & 0 \\
0 & 0 & 0 & p_0
\end{pmatrix}.
\]

\[(A.65)\]
Non-perfect fluids

**Non-perfect fluids** are the most general case of continuous media one can look into, however this is only possible in reduced symmetry systems. Many classes of these fluids exist: fluids admitting heat fluxes and heat conduction would have non-zero momenta components in the MCFR: \( T_{0}^{\alpha \beta} = T_{0}^{\alpha 0} \neq 0 \), for example. Other examples like ones admitting **anisotropic pressures** will be of prime interest to us, and can be given simply in the MCFR as

\[
T_{0}^{\alpha \beta} = \begin{pmatrix}
\rho_{00} & 0 & 0 & 0 \\
0 & p_{r} & 0 & 0 \\
0 & 0 & p_{t} & 0 \\
0 & 0 & 0 & p_{t}
\end{pmatrix},
\]

where the only difference from a perfect fluid is the unequal pressures in the diagonal entries. How this particular characteristic comes about is complicated, but we will investigate them and their structure only after introducing a general relativistic formulation.

A.2.6 The matter continuum in general relativity

The previous section’s analysis of the continuum can be extended to the general relativistic cases through the canonical “promotion” of the derivative operators. However many interpretation of the tensor components can be carried out through the notions given previously in the special relativistic case. Notions like angular momentum: which we shall not be concerned with, and mass: which we will have to consider, being a few examples. However since we already have a local conservation equation of energy density and of momentum through the energy momentum tensor in the MCFR, we shall continue and generalize all these concepts to the general relativistic case.

In addition to the promotion of the derivative operators, the other canonical transformation that can be carried out is the promotion of the Lorentzian metric \( \eta^{ab} \) to the general metric \( g^{ab} \).
We start with the special relativistic perfect fluid and investigate how its general expression changes in moving frames. Since the MCFR is a Lorentzian frame, locally the metric in this frame is $\eta^{ab}$, and the energy-momentum tensor for a perfect fluid is still $T_{0}^{ab}$ given by equation (A.65), while the velocity of the fluid in the MCFR is $u_{0}^{a} = (1, 0, 0, 0)^{\top}$.

To get the general form of the energy-momentum in arbitrary frames, we proceed by the general tensor transformation rule,

$$T^{ab} = \frac{\partial x^{a}}{\partial \tilde{x}_{0}^{a}} \frac{\partial x^{b}}{\partial \tilde{x}_{0}^{b}} T_{0}^{cd},$$

which simplifies due to $T_{0}^{ab}$ being diagonal into

$$T^{ab} = \frac{\partial x^{a}}{\partial \tilde{x}_{0}^{a}} \frac{\partial x^{b}}{\partial \tilde{x}_{0}^{b}} \rho_{00} + \frac{\partial x^{a}}{\partial \tilde{x}_{0}^{1}} \frac{\partial x^{b}}{\partial \tilde{x}_{0}^{1}} p_{0} + \frac{\partial x^{a}}{\partial \tilde{x}_{0}^{2}} \frac{\partial x^{b}}{\partial \tilde{x}_{0}^{2}} p_{0} + \frac{\partial x^{a}}{\partial \tilde{x}_{0}^{3}} \frac{\partial x^{b}}{\partial \tilde{x}_{0}^{3}} p_{0}. \quad (A.67)$$

The velocity vector transforms according to

$$u^{a} = \frac{\partial x^{a}}{\partial \tilde{x}_{0}^{a}} u_{0}^{c} \Rightarrow \frac{dx^{a}}{ds} = \frac{\partial x^{a}}{\partial \tilde{x}_{0}^{a}},$$

while the metric in a general frame is given by

$$g^{ab} = \frac{\partial x^{a}}{\partial \tilde{x}_{0}^{a}} \frac{\partial x^{b}}{\partial \tilde{x}_{0}^{b}} \eta^{cd} = \left( \begin{array}{cccc} \frac{\partial x^{a}}{\partial \tilde{x}_{0}^{a}} \frac{\partial x^{b}}{\partial \tilde{x}_{0}^{b}} & 0 & 0 & 0 \\ 0 & -\frac{\partial x^{a}}{\partial \tilde{x}_{0}^{1}} \frac{\partial x^{b}}{\partial \tilde{x}_{0}^{1}} & 0 & 0 \\ 0 & 0 & -\frac{\partial x^{a}}{\partial \tilde{x}_{0}^{2}} \frac{\partial x^{b}}{\partial \tilde{x}_{0}^{2}} & 0 \\ 0 & 0 & 0 & -\frac{\partial x^{a}}{\partial \tilde{x}_{0}^{3}} \frac{\partial x^{b}}{\partial \tilde{x}_{0}^{3}} \end{array} \right).$$

These two relations when substituted in (A.67) yield the coordinate invariant general energy momentum tensor in arbitrary frames:

$$T^{ab} = (\rho_{00} + p_{0}) u^{a} u^{b} - g^{ab} p_{0} \Rightarrow T = (\rho + p) \tilde{u} \otimes \tilde{u} - pg^{-1}, \quad (A.68)$$

in terms of the fluid velocity in those frames, and the metric at those point. This will be the stating point for most of the work in this thesis.
Non-perfect fluids

We will also be concerned with non-perfect fluids in this thesis. The reasons leading to their consideration are many, but stem from the fact that assuming local pressure isotropy in the energy-momentum tensor is an oversimplification due to a perfect fluid assumption, and does not follow from spherical symmetry. Local anisotropy is more interesting since \[60\]

(i) Anisotropic models can naturally incorporate charged distributions, with the anisotropy proportional to some static charge distribution \[11\]. (ii) Any solution requiring more than one perfect fluid matter component with minimal interaction has to be modelled with an anisotropic stress tensor \[6, 7, 78\]. Anisotropy would allow the possibility of different species interacting with each other, instead of just one homogeneous fluid species. Cases like quark stars and neutron stars above would require this for example, since these require multiple types of particles for any form of quantum stability. (iii) An isotropic energy-momentum tensor means that all interaction of the fluid with itself has to be modelled with one barotropic equation of state, \( p(\rho) \). This might not provide enough degrees of freedom if complex interactions is to take place in the fluid. All of these reasons suggest that more complicated fluid profiles that perfect fluids with one barotropic equation of state would be useful in modelling actual physical objects. Such generalizations could occur in many directions: Some include conductivity terms in the energy momentum tensor in an attempt to model conductive fluids. Others embed the interior solutions in an external magnetic field, and generate a different energy momentum tensor through this external field.

We will instead follow Letelier and more recently Boonserm et al. who couple multiple fluids minimally to generate anisotropy. Letelier’s method is more involved mathematically, but is more transparent. Starting with two perfect fluids labelled with their proper future oriented velocities \( u, v \), prescribed by our coordinate independent model of fluids \(A.68\), we use
the additive property\footnote{The total energy momentum tensor of multiple fluids is just the addition of the individual energy momentum tensors of the separate fluids. This simple but counter-intuitive canonical relation comes from the alternative definition of the tensor in the Lagrangian formulation of general relativity, where the energy momentum tensor behaves very much like a Lagrange density, and is thus additive. This is investigated in Section A.5.}

\[ T^{ab}(u + v) = T^{ab}(u) + T^{ab}(v), \quad (A.69) \]

of the energy momentum tensor to build the total energy momentum tensor of the system:

\[ T^{ab}(u) = (\rho + p)u^a u^b - pg^{ab}, \quad (A.70a) \]
\[ T^{ab}(v) = (\tau + q)v^a v^b - qg^{ab}, \quad (A.70b) \]

where \( \rho \) and \( \tau \) are the rest energy densities, while \( p \) and \( q \) are the pressures of the fluids \( u \) and \( v \) respectively. Therefore \( T^{ab}(u + v) \) is the energy momentum tensor of the combination of the two fluids. Additionally we have the usual normalization conditions of the fluid velocities \( u_a u^a = 1 \), and \( v^a v_a = 1 \), with \( v_a \neq u_a \).

The consistency conditions for energy momentum tensors (A.69) is

\[ G^{ab}_{;a} = 0 \implies T^{ab}_{;a}(u + v) = 0, \quad (A.71) \]

for the final energy momentum \( u + v \). These however are not enough to completely determine all the unknowns of the system (A.69), and we need additional conditions to close the system. To see why this is, we consider the fact that (A.69) has as unknowns \( g^{ab} \) with 10 independent components, \( u^a \) and \( v^a \) each having 3 independent components (since the normalisation conditions reduce the 4 components by one each,) and 4 from the matter variables \( \rho, \tau, p \) and \( q \), for a total of 20 unknowns. As constraints, we have the Einstein equations (2.1) which provide a total of 10 constraint equations, and the Bianchi identities (A.71) which provide 4. To close the system we need to provide 6 more equations. (i) First we assume a form of minimal coupling for the individual energy momentum equations so that

\[ T^{ab}_{;a}(u) = 0, \quad \text{and}, \quad T^{ab}_{;a}(v) = 0. \quad (A.72) \]
Since we already have constraints on the metric, these two equations only provide constraints on the matter variables individually, for 4 equations in total from (A.72). The form of the above equations also ensures that (A.71) is automatically satisfied. (ii) Additionally we will also assume that there exist an equation of state relating the state variables of each fluids in the form of

\[ h_1(p, \rho) = 0, \quad \text{and,} \quad h_2(q, \tau) = 0 \quad (A.73) \]

for another two constraints, closing the system: so that we have 20 unknowns, and 20 constraints. With these two assumptions: minimal coupling, and the existence of an equation of state, we now try to find a simple form of the energy momentum for a combination of two perfect fluids. What we want is for \( T^{ab}(u + v) \) to be expressed as

\[ T^{ab}(u + v) = DU^aU^b + Q^{ab}, \quad (A.74) \]

where \( U^a \) is normalised such that \( U^aU_a = 1 \), like the previous 4-velocities, in some coordinate system, and \( Q^{ab} \) is a stress tensor as we defined previously as \( p^{ij} \) in (A.53). It obeys the normalisation condition \( Q^{ab}U_a = 0 \), as expected from the stress tensor. We also want \( D > 0 \), since we want it to correspond to a rest energy density.

Considering (A.69), and using (A.70a) and (A.70b) to expand it, we get

\[ T^{ab}(u + v) = (\rho + p)u^a u^b + (\tau + q)v^a v^b - (p + q)g^{ab}. \]

Now, following the insight of Letelier, and using the diffeomorphic invariance of general relativity, we transform to a different coordinate system, in such a way that the four velocities transform as

\[ u^a \rightarrow \tilde{u}^a = u^a \cos \alpha + v^a \sqrt{\left(\frac{\tau + q}{\rho + p}\right)} \sin \alpha, \quad (A.75a) \]

\[ v^a \rightarrow \tilde{v}^a = -u^a \sqrt{\left(\frac{\rho + p}{\tau + q}\right)} \sin \alpha + v^a \cos \alpha, \quad (A.75b) \]
where $\alpha$ is undetermined as of yet. By substituting the expressions of the transformed velocities in the energy-momentum tensor, we find that the rotation indeed preserves the tensor, that is,

$$T^{ab}(u, v) = T^{ab}(\tilde{u}, \tilde{v}) \quad \text{(A.76)}$$

We now pick the new coordinate system so that it allows us to easily interpret the new velocities. We choose $\tilde{u}^a$ to be a timelike vector, and $\tilde{v}^a$ to be a spacelike vector respectively. As a result $\tilde{u}$ and $\tilde{v}$ are orthogonal, so that $\tilde{u}^a \tilde{v}_a = 0$. Demanding that this condition be satisfied uniquely determines the “angle” $\alpha$ in the coordinate transformation (A.75) to be

$$\tan (2\alpha) = 2u^a v_a \sqrt{(\rho + p)(\tau + q)} \over \rho + p - \tau - q.$$

In addition to the spacelike and timelike nature of the new velocities, the fact that all 4-velocities must be future oriented implies additionally that $\tilde{u}^a \tilde{u}_a > 0$ and $\tilde{v}^a \tilde{v}_a < 0$ respectively. With this transformation completely specified, we can now express the tensors in (A.74) in terms of the rotated vectors. A straight forward calculations then gives that

$$U^a := \frac{\tilde{u}^a}{\sqrt{\tilde{u}^a \tilde{u}_a}}, \quad \text{(A.77a)}$$
$$V^a := \frac{\tilde{v}^a}{\sqrt{-\tilde{v}^a \tilde{v}_a}}, \quad \text{(A.77b)}$$
$$D := T^{ab} U_a U_b = (\rho + p)\tilde{u}^a \tilde{u}_a - (p + q), \quad \text{(A.77c)}$$
$$S := T^{ab} V_a V_b = (p + q) - (\tau + q)\tilde{v}^a \tilde{v}_a, \quad \text{(A.77d)}$$
$$T := p + q, \quad \text{(A.77e)}$$

To understand what $D$ and $S$ mean, consider that by expanding (A.77c) and (A.77d) with the expressions of $\tilde{u}$, and $\tilde{v}$, we explicitly get

$$D = \frac{1}{2}(\rho - p + \tau - q) + \frac{1}{2} \sqrt{(p + \rho - \rho + \tau + q)^2 + 4(p + \rho)(q + \tau)(\rho + p - \tau - q)^2 - 1}, \quad \text{(A.78a)}$$
$$S = -\frac{1}{2}(\rho - p + \tau - q) + \frac{1}{2} \sqrt{(p + \rho - \rho - \tau - q)^2 + 4(p + \rho)(q + \tau)(\rho + p - \tau - q)^2} \quad \text{(A.78b)}$$
which we notice to be both positive since the terms in the square root is always larger than the first term. Therefore the interpretation of $D$ and $S$ as a density and a stress respectively is not far-fetched. With this in mind, we re-express the energy momentum tensor given in (A.76) through

$$T^{ab} = (D + A)U^aU^b + (S - A)V^aV^b - Ag^{ab} =: DU^aU^b + Q^{ab}$$  \hspace{1cm} (A.79)$$
in the rotated velocities, and extract the quantity $Q^{ab}$ given by

$$Q^{ab} = SV^aV^b + A\left(U^aU^b - V^aV^b - g^{ab}\right).$$  \hspace{1cm} (A.80)$$

Another rotation of the coordinate system to the canonical tangent space of the metric, where $g^{ab} = \eta^{ab}$, the Minkowski metric, which is always possible at a point according to theorem (5), then diagonalizes (A.80) into

$$Q^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix},$$  \hspace{1cm} (A.81)$$
since in the new coordinates, we have $U^a = \delta^a_0$, and $V^a = \delta^a_1$. This is the result we are after, since in this rotated system, we have just expressed the mixture of two ideal fluid into an anisotropic energy-momentum tensor, by just exploiting the coordinate transformations allowed by general relativity. Thus

$$T^{ab}(u + v) = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix},$$

with $D > 0$, the rest energy density, $S = p_r > 0$ the pressure in the $\delta^a_1$ direction: corresponding to the radial direction in spherical coordinates, and another pressure $A = p_\perp < S$,
from (A.78). This last pressure has to be perpendicular to $\delta^q_a$ direction and we identify it with the other non-radial direction in spherical symmetry. We also note that since $u$ and $v$ both have to be zero at $r = 0$, the centre by symmetry, $A = S$ there too, in accordance to anisotropic but spherically symmetric pressures. A similar derivation, which we will not give since it is simpler and follows this exact same procedure for the combination of a perfect fluid (interacting matter) and a null fluid (dust). This alternative derivation produces different expressions for $D, S$ and $A$, but the diagonalization proceeds through exactly in the same way, and we can say that any combination of more that one fluid, perfect or null can be transformed into one anisotropic fluid with calculable properties. This ends this section on fluids, and we have justified the physical relevance of anisotropic fluids for the modelling of physical stars, if the latter can be thought as to be a combination of perfect fluid species interacting minimally with each other.

### A.3 Electromagnetic fields

Electromagnetic fields can be added to general relativity in a similar manner as matter was. Matter was introduced in section A.2 through an energy-momentum tensor, $T_{\alpha\beta}$. In classical Maxwell theory, an energy-momentum tensor can be constructed in terms of the known electric and magnetic fields. The simplest way to achieve this is to use the potential formulation of the electromagnetic field. In this formulation the electromagnetic field is encoded in a 4-vector $A$, whose components are given as $A_{\alpha} = (\phi/c, -\vec{A})$, where $\phi$ is the electric potential and $\vec{A}$ the magnetic vector potential. From this 4-potential, we can then define the **Faraday tensor**

$$F_{\alpha\beta} = \partial_{[\alpha}A_{\beta]} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}. \quad (A.82)$$

Then the addition of Maxwell’s equations to those of the Einstein’s field equations (EFE) can
be done canonically through a transformation involving the substitution of the flat Lorentz metric $\eta_{ab}$, which is a natural component of Maxwell’s theory into a general metric tensor $g_{ab}$. Following this substitution, the Maxwell’s equations which in standard tensorial notation in flat space-time can be expressed as

$$\partial_{[a}F_{bc]} = 0,$$  \hspace{1cm} (A.83a)

$$\partial_a (\eta^{ac} \eta^{bd} F_{cd}) = \partial_a F^{ab} = \mu_0 j^b,$$ \hspace{1cm} (A.83b)

get transformed into the Maxwell’s equation for curved space

$$\nabla_b F^{ab} = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} F^{ab}) = j^a,$$ \hspace{1cm} subject to \hspace{1cm} $\nabla_a j^a = 0,$  \hspace{1cm} (A.84a)

$$\partial_{[a}F_{bc]} = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = \nabla_{[a}F_{bc]} = 0,$$ \hspace{1cm} (A.84b)

where $j^b$ is the current density 4-vector. We see that the partial derivatives are canonically promoted to covariant derivatives one as is discussed in Section A.1.11.

The coupling of matter with the electromagnetic field is most easily done in a Lagrangian formulation, and we investigate this in Section A.5.

A.4 Energy conditions and conservation laws

We now have all the structure and matter variables to talk about energy, and conservation laws, and how these are implemented. In general relativity total energy is not well defined in a coordinate independent manner. For this reason, the energy conservation conditions are expressed through the energy-momentum tensor and timelike vectors. We state the three main energy conditions usually used in the literature, and in this thesis:

1. The **weak energy condition** is satisfied if the energy-momentum tensor $T_{ab}$ satisfies

$$T_{ab} X^a X^b \geq 0,$$

for all timelike vectors $X$. 

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2. The **strong energy condition** is satisfied if instead
\[
\left( T_{ab} - \frac{1}{2}g_{ab}T_c^c \right) X^a X^b \geq 0.
\]

This also called the Ricci positivity condition, because the tensor that is contracted with the timelike vectors is the Ricci tensor if Einstein equations hold.

3. The **dominant energy condition** is satisfied if the energy-momentum tensor is such that the vector \(-T^a_b X^b\) is timelike and future directed for all timelike and future directed vectors \(X\).

We now state how conservation laws are expressed in curved space-time, and provide a few formulae which are useful for their computations. As stated in Section [A.1.11](#), we use covariant derivatives in GR where we would have used partial derivatives in SR.

A scalar \(\phi\) is locally conserved if \(\nabla_a \phi = \phi_{,a} = 0\). Since covariant derivatives of scalars are defined to be the same as partial derivatives, the latter equation reduces to \(\partial_a \phi = \phi_{,a} = 0\).

A vector \(X^a\) is locally conserved if
\[
\nabla_a X^a = X^a_{,a} = X^a_{,a} + \Gamma^a_{ab} X^b = 0.
\]

(A.85)

This is also called the covariant divergence of the vector.

A tensor \(T^{ab}\) is locally conserved if \(\nabla_a T^{ab} = T_a^{\;\;b,a} + \Gamma^a_{ac} T^{cb} + \Gamma^a_{ab} T^{bc} = 0\).

A common theme in the above equations is the appearance of the contracted connection coefficients \(\Gamma^a_{ab}\). A convenient method to compute these is now given. From the metric compatibility of \(\Gamma^a_{bc}\), we can use the definition of the connection in equation (A.21). Contracting this equation, we have
\[
\Gamma^a_{ab} = \Gamma^a_{ba} = \frac{1}{2}g^{am}\partial g_{am} \frac{\partial x^b}{\partial x^a} = \frac{1}{2g} \frac{\partial g}{\partial x^b} = \frac{\partial \log \sqrt{-g}}{\partial x^b}.
\]

(A.86)
The second equation is due to the symmetry of the connection (there is no torsion in GR), the third from a simple contraction of equation (A.21). The next equality results from the definition of the metric determinant \( g = \det (g) \). From this definition, we can easily calculate \( \partial_c g = gg^{ab} \partial_c g_{ab} \), from which the fourth equation results. The final equation, the one we use in chapter 5 can be seen to be true from a simple differentiation identity. In this form, equation (A.86) is very useful for the computation of conserved quantities and divergences.

### A.5 Lagrangian approach

Another way to approach general relativity is through a Lagrangian approach, and we introduce this here because it makes transparent how different fields, in particular the electromagnetic field can be included into the EFE without a complicated process.

The action of a gravitational field is geometrical in nature and is encoded in the **Einstein Lagrangian** given by \( \mathcal{L}_G = \sqrt{-g} R \), where \( R \) is the Ricci curvature scalar defined in equation (A.31). Since we need both the metric and the Ricci scalar, all the notions of manifolds, connections, and covariant derivatives are also needed. The Einstein Lagrangian is a functional dependant on the metric and its first and second derivatives, since \( R \) which is defined in terms of the Riemann tensor which in turn depends on the second derivatives of \( g_{ab} \). As a result of this dependence, the Euler-Lagrange equations of the action

\[
I = \int \Omega \mathcal{L}_G(g_{ab}, g_{ab,c}, g_{ab,cd}) \, d\Omega,
\]

with respect to the metric is

\[
\frac{\delta \mathcal{L}_G}{\delta g_{ab}} = \frac{\partial \mathcal{L}_G}{\partial g_{ab}} - \left( \frac{\partial \mathcal{L}_G}{\partial g_{ab,c}} \right)_{,c} + \left( \frac{\partial \mathcal{L}_G}{\partial g_{ab,cd}} \right)_{,cd} = 0.
\]

The calculation of all the terms in equation (A.88) is very lengthy and cumbersome, and we only state that the final answer is indeed the Einstein tensor density

\[
\mathcal{L}_G^{ab} = \frac{\delta \mathcal{L}_G}{\delta g_{ab}} = -\sqrt{-g} G^{ab},
\]

(A.89)
with $G^{ab}$ the Einstein tensor defined in equation (A.32).

In the presence of matter and electromagnetic fields, since we have a Lagrangian theory now, we only need modify the Lagrangian to include the different fields, with some coupling. For matter we include a matter Lagrangian of the form $\mathcal{L}_M$, which couples to the gravitational Lagrangian through the coupling constant $\kappa$ manifestly in the action:

$$I = \int_{\Omega} (\mathcal{L}_G + \kappa \mathcal{L}_m) \, d\Omega.$$ \hspace{1cm} (A.90)

The variation of each Lagrangian then gives the full Einstein equations if we define the energy-momentum tensor to be

$$\frac{\delta \mathcal{L}_M}{\delta g^{ab}} = \sqrt{-g} T^{ab},$$ \hspace{1cm} (A.91)

since from (A.89), we get the corresponding geometrical part. Together the variation of the whole action (A.90) gives the full Einstein equations $G^{ab} = \kappa T^{ab}$. We already derived the full matter $T^{ab}$ in the Section 2.3.1 and we use this as the $T^{ab}$ to generate the matter Lagrangian $\mathcal{L}_M$.

To couple with an electromagnetic field, we only need to include the Lagrangian $\mathcal{L}_E$ for the electromagnetic field. Classical Maxwell field theory already has an answer ready for this Lagrangian, and it is in terms of the metric $g_{ab}$ and the Faraday tensor $F_{ab}$. It is given by

$$\mathcal{L}_E(A_a, F_{ab}) = \frac{\sqrt{-g}}{8\pi} g^{ac} g^{bd} F_{ab} F_{cd},$$ \hspace{1cm} (A.92)

where $g$ is the metric determinant. Then the equation of motion resulting from the Euler-Lagrange equation and the action

$$I = \int_{\Omega} \mathcal{L}_E \, d\Omega,$$

give the Maxwell equations, and the definition for the Faraday tensor in terms of the 4-potentials. In the above

$$\frac{\delta \mathcal{L}_E}{\delta g_{ab}} = \frac{\partial \mathcal{L}_E}{\partial g_{ab}} = \sqrt{-g} \left( -g^{cd} F_c^a F_d^b + \frac{1}{4} F_{cd} F^{cd} g^{ab} \right).$$ \hspace{1cm} (A.93)

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The electromagnetic stress-energy tensor can then be written as

\[
T^{ab} = \frac{1}{4\pi} \left( -g^{cd} F^a_c F^b_d + \frac{1}{4} F_{cd} F^{cd} g^{ab} \right),
\]

(A.94)

following the identification of

\[
\frac{\delta \mathcal{L}_E}{\delta g_{ab}} = -\sqrt{-g} T_{ab}^{\text{Electromagnetic}}
\]

in line with equation (A.91) for the matter Lagrangian \( \mathcal{L}_M \).

Once the stress-energy tensor of the electromagnetic field has been specified, coupling the stress-energy of matter and of the electromagnetic field through minimal coupling is achieved through

\[
T_{ab}^{\text{Total}} = T_{ab}^{\text{Electromagnetic}} + T_{ab}^{\text{Matter}}.
\]

(A.95)

This is possible since we write the complete action of the total system: curvature, matter and electromagnetic field as

\[
I = \int_{\Omega} (\mathcal{L}_G + \kappa \mathcal{L}_M + \mathcal{L}_E) \, d\Omega,
\]

(A.96)

then variation of the whole action gives the full Einstein equations with the total stress energy tensor as in (A.95).

Because of the contracted Bianchi identity requiring that \( \nabla_a G^a_b = 0 \), from Einstein’s equation we must have \( \nabla_a T^{ab} = 0 \). We constructed the matter \( T_{ab} \) to behave this way explicitly, and the electromagnetic one obeys this equation because of the form of the Maxwell’s equation. Indeed, we have \( T^{ab}_{\ ,b} = 0 \quad \Rightarrow \quad T^{ab}_{\ ,b} = 0 \) for the \( T_{\text{Electromagnetic}}^{ab} \), and therefore as is required from the EFE,

\[
\nabla_b T^{ab}_{\ \text{Total}} = 0.
\]

This completes the introduction to Einstein’s theory of gravitation as used in this thesis.

Most of the aspects introduced here is used in one part or another of the main text, and an index provides the relevant section where definitions and theorems may be found.
A.6 Differential equations

All Einstein equations are partial differential equations (PDEs). However, our symmetry requirements (spherical symmetry and staticity) force the equations to simplify into ordinary differential equations (ODEs), except when we have to consider a relaxation of the static condition with pulsations for the stability analysis in Chapter 5. In Subsection A.6.1 we consider how to find and apply boundary conditions on the EFE we have to solve. In Chapter 5 we use Sturm-Liouville theory to determine linear stability of the EFE for our solutions, and in Subsection A.6.2 we sketch the main results that will be useful for this conclusion.

A.6.1 Israel-Darmois junction conditions

This Section is heavily influenced from [64, 85, 105] where these conditions are extensively treated. The EFE form a set of PDEs. Taking the cue from classical mechanics, where a complete solutions to the equations of motion can be obtained uniquely from the initial values on the positions and velocities, we expect that in equations involving the metric $g_{ab}$, initial conditions on $g_{ab}$ and $g_{ab,t}$ should be sufficient to find unique solutions. This is essentially correct, but not very helpful in practice where no such information is usually available.

The initial value problem of GR begins with the selection of a spacelike hypersurface $\Sigma$ which represents an ‘instant of time’. This hypersurface can be chosen freely, and we place an arbitrary system of coordinates $y^\alpha$ on it. (Note, we are using Greek indices spanning 1,2,3 only here, because we are on a hypersurface of 3-dimensions)

We can define a normal vector to the hypersurface through (A.23). Additionally we can define a metric intrinsic to the hypersurface $\Sigma$ by restricting the line element to displacements confined to the hypersurface. If we define curves $x^\alpha = x^\alpha(y^\alpha)$ on the hypersurface, we have that the vectors

$$e^\alpha_\alpha = \frac{\partial x^\alpha}{\partial y^\alpha},$$

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are tangent to the curves contained in $\Sigma$. As a result we have trivially that $e^a_\alpha n_a = 0$, that is the tangent vectors on the hypersurface are normal to the hypersurface orthogonal vector $n_a$. For displacements within $\Sigma$ we have

$$ds^2 = g_{ab} dx^a dx^b = g_{ab} \left( \frac{\partial x^a}{\partial y^\alpha} dy^\alpha \right) \left( \frac{\partial x^b}{\partial y^\beta} dy^\beta \right) = h_{\alpha\beta} dy^\alpha dy^\beta,$$

where

$$h_{\alpha\beta} := g_{ab} e^a_\alpha e^b_\beta$$

is the **induced metric**, also called the **first fundamental form**, of the hypersurface. The inverse metric can also be expressed in terms of the induced metric and the normal vector through

$$g^{ab} = \epsilon n^a_n^b + h^{\alpha\beta} e^a_\alpha e^b_\beta,$$

where $h^{\alpha\beta}$ is the metric on the hypersurface.

Having defined an intrinsic metric on the hypersurface $\Sigma$, it should come to no surprise that both a metric connection, and a covariant derivative on the hypersurface is possible. We just state this, and refer the reader to the beginning of the appendix which treats all this in a dimension independent fashion. The extension to lower dimensional spaces should not be difficult.

The next object we define is the **extrinsic curvature** of the hypersurface. This three-tensor (on the hypersurface) $K_{\alpha\beta}$ is defined in terms of vectors not all present on the hypersurface, hence the name. It characterises how the hypersurface is embedded into the higher dimensional space it is a surface in. We have

$$K_{\alpha\beta} := n_{a;b} e^a_\alpha e^b_\beta.$$
\(K_{\alpha\beta}\) is also called the \textbf{second fundamental form} of the hypersurface. From the definition, we can show that the extrinsic curvature is symmetric, \(K_{\alpha\beta} = K_{\beta\alpha}\).

In terms of the first and second fundamental forms of the hypersurface \(\Sigma\), we have a complete characterisation in terms of intrinsic properties, and embedding properties. We can now continue investigating the initial value problem. The space-time metric \(g_{ab}\) when evaluated on \(\Sigma\) has some components that characterise displacements outside of the hypersurface (e.g. \(g_{tt}\) if \(\Sigma\) as in our case is a surface of constant \(t\)). The initial values of these components cannot be given from the intrinsic geometrical properties of \(\Sigma\) alone.

The initial data for the \(g_{ab}\) corresponding to the ‘positions’ in our analogy have to come from the first fundamental form \(h_{\alpha\beta}\) of the chosen hypersurface (for a total of 6 components). The data for the remaining four components of \(g_{ab}\) are expressed in the choice of the hypersurface and its arbitrary coordinate system.

Similarly the initial data for \(g_{ab,t}\) corresponding to the ‘velocities’ in our analogy, come from the second fundamental form \(K_{\alpha\beta}\) of \(\Sigma\). Together these provide the initial data for the initial value problem of GR. In the complete space-time these cannot be arbitrarily specified and have to obey the EFE.

We can now state the form of the boundary conditions for the junction of two space-times and their corresponding metrics. The question one asks when solving these equations is the following: A hypersurface \(\Sigma\) partitions spacetime into two regions \(\mathcal{V}^+\) and \(\mathcal{V}^-\). In \(\mathcal{V}^+\) the metric is \(g^{+}_{ab}\) expressed in coordinates \(x^a_+\) and in \(\mathcal{V}^-\), it is \(g^{-}_{ab}\) expressed in coordinates \(x^a_-\). How do we get a consistent solution for the whole space-time, and more precisely, what conditions must be put on the metrics \(g^{+}_{ab}\) and \(g^{-}_{ab}\), to ensure that \(\mathcal{V}^+\) and \(\mathcal{V}^-\) are joined smoothly at \(\Sigma\).

We will state the answer without proof, with the explanation hinging on the fundamental forms defined above, since details will not be used in this thesis. Before however we introduce
the notation $[A]$ for any object $A$ to mean

$$[A] := A(V^+) \big|_\Sigma - A(V^-) \big|_\Sigma,$$

i.e. the difference in the value of $A$, an object defined on both sides of the hypersurface. $[A]$ can be considered as the “jump” in the value of $A$ as one crosses the hypersurface.

To answer the question, we state

**Theorem 6.** The Israel-Darmois junction condition states that the two space times $V^+$ and $V^-$ are joined smoothly at $\Sigma$, in their metric structure, and the full space-time obeys the EFE if

1. the intrinsic curvature induced by both metrics on $\Sigma$,

   $$[h_{\alpha\beta}] = 0,$$

2. the extrinsic curvature induced by both metrics on $\Sigma$,

   $$[K_{\alpha\beta}] = 0.$$

*If the extrinsic curvature is not the same on both sides, additional details about the surface must be considered.*

In our case, we shall match an interior solution we find to the Schwarzschild exterior solution. Since we are in a static spacetime, $n_a$ is a timelike killing vector that is hypersurface orthogonal. This simplifies the calculation of both $h_{\alpha\beta}$ and $K_{\alpha\beta}$, for both the Schwarzschild exterior metric in $V^-$,

$$ds^2 = g_{ab}^{-} dx^a dx^b = \left( 1 - \frac{2M}{r} \right) dt^2 - \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2 d\Omega^2,$$

and the interior solution’s metric in $V^+$,

$$ds^2 = g_{ab}^{+} dx^a dx^b = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\Omega^2.$$

The computation of the two conditions is still lengthy, and results in
1. the condition on \([h_{\alpha\beta}]\) reducing to matching the metric function on both sides and specifying that

\[
m(r = 0) = \frac{dm}{dr} \bigg|_{(r=0)} = 0,
\]

leading to the definition of \(m(r)\) given through

\[
e^{-\lambda(r_b)} = e^{\nu(r_b)} = 1 - \frac{2M}{r_b}, \quad \text{with} \quad M = m(r_b) = 4\pi \int_0^{r_b} \rho(\bar{r}) \bar{r}^2 d\bar{r}.
\]

2. The computation on \([K_{\alpha\beta}]\) is lengthy and results in

\[
p(r_b) = 0,
\]

after simplifications, as was shown in [85].

We now have the two boundary condition on the system of ODE for the solution of the complete space-time.

### A.6.2 Sturm-Liouville theory

A Sturm–Liouville problem consists in finding eigenvalues \(\sigma^2\) and eigenfunctions \(f(x)\) for the differential equation

\[
\frac{d}{dx} \left[ P(x) \frac{df}{dx} \right] - Q(x)f(x) + \sigma^2 W(x)f(x) = 0, \quad (a < x < b), \quad (A.98)
\]

satisfying the boundary conditions at \(a\) and \(b\) given through

\[
\begin{align*}
\alpha_1 f(a) + \alpha_2 \left. \frac{df}{dx} \right|_a &= 0, \quad \alpha_1^2 + \alpha_2^2 > 0, \quad (A.99a) \\
\beta_1 f(b) + \beta_2 \left. \frac{df}{dx} \right|_b &= 0, \quad \beta_1^2 + \beta_2^2 > 0, \quad (A.99b)
\end{align*}
\]

In our work we also use the generalised S–L equation, given instead by

\[
\frac{d}{dx} \left[ P(x) \frac{df}{dx} \right] - Q(x)f(x) + \sigma^2 W(x)f(x) = R(x, f), \quad (a < x < b). \quad (A.100)
\]
Both of these equations have been investigated in the past, and for our purposes, the reduction of (A.100) into (A.98), through the absorption of $R(x, f)$ into $Q(x)$ is what we strive for. Once we have the equation in the form of (A.98), we express the latter in a variational form, in terms of functionals, so that an Euler-Lagrange technique can be applied to it. As can be checked by expansion and simplification, if the functional
\[
I[f(x)] = \int_a^b \left[ p(x) \left( \frac{df}{dx} \right)^2 + Q(x)f^2 \right] \, dx,
\]
is minimized through the Euler-Lagrange method, subject to the condition that the functional
\[
J[f(x)] = \int_a^b W(x)f^2 \, dx = \text{constant}, \tag{A.101}
\]
is a constant, the eigenvalue $\sigma^2$ appears as a Lagrange multiplier. Since (A.101) is the normalisation condition for $f(x)$ under a weight function $W(x)$, the variation technique is equivalent to minimizing the functional
\[
K[f(x)] = \frac{I[f(x)]}{J[f(x)]}.
\]
Once this functional has been defined, we can use the main theorem of the theory given below,

**Theorem 7.** A Sturm–Liouville problem (A.98) is **regular** if $P(x) > 0, W(x) > 0$, and $P(x), P'(x), Q(x), W(x)$ are all continuous functions over the finite interval $[a, b]$, and additionally satisfy the boundary conditions given through (A.99). If the Sturm–Liouville problem is regular, then the eigenvalues $\sigma_i^2$ of (A.98) are real and can be ordered such that
\[
\sigma_1^2 < \sigma_2^2 < \sigma_3^2 < \cdots < \sigma_n^2 < \cdots \to \infty.
\]

To each eigenvalue $\sigma_i^2$, there corresponds a unique eigenfunction $f_i(x)$, which is called the $i^{th}$ **fundamental solution** satisfying the regular Sturm-Liouville problem. Furthermore, the normalized eigenfunctions form an orthonormal basis in the Hilbert space $L^2([a, b])$ with weight
$W(x)$ and norm

$$\int_{b}^{a} f_m(x) f_n(x) W(x) \, dx = \delta_{nm}. $$
Appendix B

We provide a complete reference of all the new solutions mentioned in the thesis in the form of the metric functions associated with each, and the different classes of parameters each solution is valid for, with simplifications. This is meant as complete reference of the solutions and a useful “cheat-sheet” for looking up values of parameters mid-text.

B.1 The Tolman VII solution, with anisotropic pressure and no charge

This section gives the expressions for the uncharged case with $k = 0$ but with anisotropy, $\beta \neq 0$. 


### B.1.1 The $\phi^2 = 0$ case

| $\phi^2$ | 0 |
|----------|---|
| $\mu$    | $0 < \mu \leq 1$ |
| $Z$      | $1 - \left( \frac{\kappa \rho_c}{3} \right) r^2 + \left( \frac{\kappa \mu \rho_c}{5r_b^2} \right) r^4$ |
| $Y$      | $\gamma + \frac{2\alpha r_b}{\sqrt{\kappa \rho_c \mu/5}} \left[ \text{arcoth} \left( \frac{1 - \sqrt{Z(r)}}{\frac{\kappa \rho_c}{5r_b^2}} \right) - \text{arcoth} \left( \frac{1 - \gamma}{r_b \sqrt{\kappa \rho_c \mu/5}} \right) \right]$ |
| $\rho$   | $\rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right]$ |
| $a$      | $\frac{\kappa \mu \rho_c}{5r_b^2}$ |
| $b$      | $\frac{\kappa \rho_c}{3}$ |
| $\xi$    | $\frac{2}{\sqrt{a}} \text{arcoth} \left( \frac{1 + \sqrt{1 - b r^2 + a r^4}}{\sqrt{a} r^2} \right)$ |
| $c_1$    | $\gamma - \frac{2\alpha}{\sqrt{b}}$ |
| $c_2$    | $\alpha$ |
| $\alpha$ | $\frac{\kappa \rho_c (5 - 3\mu)}{60}$ |
| $\beta$  | $-a$ |
| $\gamma$ | $\sqrt{1 + \kappa \rho_c r_b^2 (3\mu - 5)}$ |
| $p_r$    | $\frac{2\kappa \rho_c}{3} - \frac{4\kappa \rho_c \mu r^2}{5r_b^2} - \kappa \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + \left( \frac{\kappa \rho_c}{3} - \frac{\kappa \rho_c \mu}{5} \right) \sqrt{1 - \frac{\kappa \rho_c}{3} r^2 + \frac{\kappa \mu \rho_c \mu}{5r_b^2} r^4}$ |
| $p_\perp$ | $p_r + \frac{\kappa \rho_c \mu}{5r_b^2} r^2$ |
B.1.2 The $\phi^2 < 0$ case

| $\phi^2$ | $a + \beta < 0$ |
|----------|----------------|
| $\mu$    | $0 < \mu \leq 1$ |
| $Z$      | $1 - \left( \frac{\kappa \rho_c}{3} \right) r^2 + \left( \frac{\kappa \mu \rho_c}{5 r_b^2} \right) r^4$ |
| $Y$      | $\left[ \gamma \cosh (\phi_b \xi) - \frac{\alpha}{\phi} \sinh (\phi_b \xi) \right] \cosh \left[ \frac{2 \phi}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right]$ + $\left[ \frac{\alpha}{\phi} \cosh (\phi_b \xi) - \gamma \sinh (\phi_b \xi) \right] \sinh \left[ \frac{2 \phi}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right]$ |
| $\rho$   | $\rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right]$ |
| $a$      | $\frac{\kappa \mu \rho_c}{5 r_b^2}$ |
| $b$      | $\frac{\kappa \rho_c}{3}$ |
| $\xi$    | $\frac{2}{\sqrt{a}} \arcoth \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{\sqrt{a} r^2} \right)$ |
| $c_1$    | $\frac{\alpha}{\phi} \cosh (\phi_b \xi) - \gamma \sinh (\phi_b \xi)$ |
| $c_2$    | $\gamma \cosh (\phi_b \xi) - \frac{\alpha}{\phi} \sinh (\phi_b \xi)$ |
| $\alpha$ | $\frac{\kappa \rho_c (5 - 3 \mu)}{60}$ |
| $\beta$  | $< -a \implies < -\frac{\kappa \mu \rho_c}{5 r_b^2}$ |
| $\gamma$ | $\sqrt{1 + \frac{\kappa \rho_c r_b^2 (3 \mu - 5)}{15}}$ |
| $p_r$    | $\frac{1}{\kappa} \left\{ \frac{2 \kappa \rho_c}{3} - \frac{4 \kappa \rho_c \mu r^2}{5 r_b^2} - \kappa \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + 4 \phi \sqrt{Z} \times \left[ \frac{\alpha}{\phi} \cosh (\phi_b \xi) - \gamma \sinh (\phi_b \xi) \right] \cosh (\phi \xi) + \left[ \gamma \cosh (\phi_b \xi) - \frac{\alpha}{\phi} \sinh (\phi_b \xi) \right] \sinh (\phi \xi) \right\}$ |
| $p_\perp$| $p_r - \beta r^2$ |
### B.1.3 The $\phi^2 > 0$ case

| $\phi^2$ | $a + \beta > 0$ |
|-----------|----------------|
| $\mu$     | $0 < \mu \leq 1$ |
| $Z$       | $1 - \left(\frac{\kappa \rho_c}{3}\right) r^2 + \left(\frac{\kappa \mu \rho_c}{5 r_b^2}\right) r^4$ |
| $Y$       | $\left[\gamma \cos (\phi \xi_b) - \frac{\alpha}{\phi} \sin (\phi \xi_b)\right] \cos \left[\frac{2 \phi}{\sqrt{a}} \coth^{-1}\left(\frac{1 + \sqrt{1 - br^2 + a r^4}}{r^2 \sqrt{a}}\right)\right] + \left[\frac{\alpha}{\phi} \cos (\phi \xi_b) + \gamma \sin (\phi \xi_b)\right] \sin \left[\frac{2 \phi}{\sqrt{a}} \coth^{-1}\left(\frac{1 + \sqrt{1 - br^2 + a r^4}}{r^2 \sqrt{a}}\right)\right]$, |
| $\rho$    | $\rho_c \left[1 - \mu \left(\frac{r}{r_b}\right)^2\right]$ |
| $a$       | $\left(\frac{\kappa \mu \rho_c}{5 r_b^2}\right)$ |
| $b$       | $\frac{\kappa \rho_c}{3}$ |
| $\xi$     | $\frac{2}{\sqrt{a}} \arcoth \left(\frac{1 + \sqrt{1 - br^2 + a r^4}}{\sqrt{a} r^2}\right)$ |
| $c_1$     | $\frac{\alpha}{\phi} \cos (\phi \xi_b) + \gamma \sin (\phi \xi_b)$ |
| $c_2$     | $\gamma \cos (\phi \xi_b) - \frac{\alpha}{\phi} \sin (\phi \xi_b)$ |
| $\alpha$  | $\frac{\kappa \rho_c (5 - 3 \mu)}{60}$ |
| $\beta$   | $> -a \implies > -\frac{\kappa \mu \rho_c}{5 r_b^2}$ |
| $\gamma$  | $\sqrt{1 + \frac{\kappa \rho_c r_b^2 (3 \mu - 5)}{15}}$ |
| $p_r$     | $\frac{1}{\kappa} \left\{\frac{2 \kappa \rho_c}{3} - \frac{4 \kappa \rho_c \mu r^2}{5 r_b^2} - \kappa \rho_c \left[1 - \mu \left(\frac{r}{r_b}\right)^2\right]\right\} + 4 \phi \sqrt{Z} \times \left[\gamma \sin (\phi \xi_b) + \frac{\alpha}{\phi} \cos (\phi \xi_b)\right] \cos (\phi \xi) - \left[\gamma \cos (\phi \xi_b) - \frac{\alpha}{\phi} \sin (\phi \xi_b)\right] \sin (\phi \xi)$ |
| $p_\perp$ | $p_r - \beta r^2$ |
### B.2 The Tolman VII solution, with anisotropic pressure and charge

#### B.2.1 Anisoprtised charge, with charge matching anisotropy, but $Φ^2 ≠ 0$

| $Φ^2$ | $Φ^2 = \frac{1}{20} \left(\frac{κρ_cμ}{r_b^2} - k^2\right)$ |
|-------|--------------------------------------------------|
| $μ$   | $0 < μ ≤ 1$                                      |
| $Z$   | $1 - \left(\frac{κρ_c}{3}\right) r^2 + \frac{1}{5} \left(\frac{κρ_cμ}{r_b^2} - k^2\right) r^4$ |
| $Y$   | $(γ \cos(Φξ_b) - \frac{α}{Φ} \sin(Φξ_b)) \cos\left(\frac{2Φ}{\sqrt{a}} \coth^{-1}\left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}}\right)\right) +$ |
|       | $+ (γ \sin(Φξ_b) + \frac{α}{Φ} \cos(Φξ_b)) \sin\left(\frac{2Φ}{\sqrt{a}} \coth^{-1}\left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}}\right)\right)$ |
| $ρ$   | $ρ_c \left[1 - μ \left(\frac{r}{r_b}\right)^2\right]$ |
| $a$   | $\frac{1}{5} \left(\frac{κρ_cμ}{r_b^2} - k^2\right)$ |
| $b$   | $\frac{κρ_c}{3}$ |
| $ξ$   | $\frac{2}{\sqrt{a}} \arcoth\left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{\sqrt{a} r^2}\right)$ |
| $c_1$ | $γ \cos(Φξ_b) - \frac{α}{Φ} \sin(Φξ_b)$ |
| $c_2$ | $γ \sin(Φξ_b) + \frac{α}{Φ} \cos(Φξ_b)$ |
| $α$   | $(κρ_c(5 - 3μ) - 12k^2 r_b^2) \frac{1}{60}$ |
| $Δ$   | $2k^2 r^2 = \frac{qr}{2k}$ |
| $γ$   | $\sqrt{1 + \frac{κρ_c r_b^2 (3μ - 5)}{15} - \frac{k^2 r_b^2}{5}}$ |
| $p_r$ | $\frac{2κρ_c}{3} - \frac{4}{5} \left(\frac{κρ_cμ}{r_b^2} - k^2\right) r^2 - κρ_c \left[1 - μ \left(\frac{r}{r_b}\right)^2\right] + 4Φ \sqrt{Z} \times$ |
|       | $\left\{ \begin{array}{l} \gamma \sin(Φξ_b) + \frac{α}{Φ} \cos(Φξ_b) \cos(Φξ) - \left[γ \cos(Φξ_b) - \frac{α}{Φ} \sin(Φξ_b)\right] \sin(Φξ) \\ \gamma \sin(Φξ_b) + \frac{α}{Φ} \cos(Φξ_b) \sin(Φξ) + \left[γ \cos(Φξ_b) - \frac{α}{Φ} \sin(Φξ_b)\right] \cos(Φξ) \end{array} \right\}$ |
| $p_⊥$ | $p_r - 2k^2 r^2$ |
B.2.2  The $\Phi^2 = 0$ case

| $\Phi^2$ | 0 |
|----------|---|
| $\mu$    | $0 < \mu \leq 1$ |
| $Z$      | $1 - \left( \frac{\kappa \rho_c}{3} \right) r^2 + \frac{2}{11} \left( \frac{\kappa \mu \rho_c}{r_b^2} - \frac{\beta}{2} \right) r^4$ |
| $Y$      | $c_1 + c_2 \xi$ |
| $\rho$   | $\rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right]$ |
| $a$      | $\frac{2}{11} \left( \frac{\kappa \mu \rho_c}{r_b^2} - \frac{\beta}{2} \right)$ |
| $b$      | $\frac{\kappa \rho_c}{3}$ |
| $\xi$    | $\frac{2}{\sqrt{a}} \arcoth \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{\sqrt{a} r^2} \right)$ |
| $c_1$    | $\gamma - \alpha \xi_b$ |
| $c_2$    | $\alpha$ |
| $k$      | $\sqrt{(a + \beta)/2}$ |
| $\alpha$ | $\frac{1}{4} \left( \frac{\kappa \rho_c}{3} - 3\kappa \rho_c \mu - \frac{4r_b^2 \beta}{11} \right)$ |
| $\beta$  | $2k^2 - a$ |
| $\gamma$ | $\sqrt{1 + r_b^2 \kappa \rho_b} \left( \frac{2 \mu}{11} - \frac{1}{3} \right) - \frac{\beta r_b^4}{11}$ |
| $p_r$    | $\frac{1}{\kappa} \left[ \frac{4c_2 \sqrt{1 - br^2 + ar^4}}{c_1 + c_2 \xi} + 2b - 4ar^2 \right] - \rho(r)$ |
| $p_\perp$| $p_r - \frac{\beta r^2}{\kappa}$ |
### B.2.3 The $\Phi^2 < 0$ case

| $\Phi^2$ | $a + \beta - 2k^2 < 0$ |
| --- | --- |
| $\mu$ | $0 < \mu \leq 1$ |
| $Z$ | $1 - \left(\frac{\kappa\rho_c}{3}\right) r^2 + \frac{1}{5} \left(\frac{\kappa\rho_c\mu}{r_b^2} - k^2\right) r^4$ |
| $Y$ | $\left[\gamma \cosh(\Phi \xi_b) - \frac{\alpha}{\Phi} \sinh(\Phi \xi_b)\right] \cosh \left[\frac{2\Phi}{\sqrt{a}} \coth^{-1} \left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2\sqrt{a}}\right)\right] + \frac{\alpha}{\Phi} \cosh(\Phi \xi_b) - \gamma \sinh(\Phi \xi_b) \sinh \left[\frac{2\Phi}{\sqrt{a}} \coth^{-1} \left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2\sqrt{a}}\right)\right]$, |
| $\rho$ | $\rho_c \left[1 - \mu \left(\frac{r}{r_b}\right)^2\right]$ |
| $a$ | $\frac{1}{5} \left(\kappa\mu\rho_c \frac{r^2}{r_b^2} - k^2\right)$ |
| $b$ | $\frac{\kappa\rho_c}{3}$ |
| $\xi$ | $\frac{2}{\sqrt{a}} \arcoth \left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{\sqrt{a} r^2}\right)$ |
| $c_1$ | $\frac{\alpha}{\Phi} \cosh(\Phi \xi_b) - \gamma \sinh(\Phi \xi_b)$ |
| $c_2$ | $\gamma \cosh(\Phi \xi_b) - \frac{\alpha}{\Phi} \sinh(\Phi \xi_b)$ |
| $\alpha$ | $\frac{1}{4} \left(\frac{\kappa\rho_c}{3} - \frac{\kappa\rho_c\mu}{5} - \frac{4k^2 r_b^2}{5}\right)$ |
| $\beta$ | $\leq 2k^2 - a \implies \frac{1}{5} \left(11k^2 - \frac{\kappa\mu\rho_c}{r_b^2}\right)$ |
| $\gamma$ | $\sqrt{1 + \kappa\rho_c r_b^2 \left(\frac{\mu}{5} - \frac{1}{3}\right) - \frac{k^2 r_b^4}{5}}$ |
| $p_r$ | $\frac{1}{\kappa} \left\{\frac{2\kappa\rho_c}{3} - \frac{4\kappa\rho_c\mu r^2}{5r_b^2} - \kappa\rho_c \left[1 - \mu \left(\frac{r}{r_b}\right)^2\right]\right\} + 4\Phi \sqrt{Z} \times$ |
| $\{\gamma \cosh(\Phi \xi_b) - \frac{\alpha}{\Phi} \sinh(\Phi \xi_b)\} \cosh(\Phi \xi) + \left[\gamma \cosh(\Phi \xi_b) - \frac{\alpha}{\Phi} \sinh(\Phi \xi_b)\right] \sinh(\Phi \xi)\}$ |
| $p_\perp$ | $p_r - \beta r^2$ |
### B.2.4 The $\Phi^2 > 0$ case

| $\Phi^2$ | $a + \beta - 2k^2 > 0$ |
|----------|-------------------|
| $\mu$    | $0 < \mu \leq 1$  |
| $Z$      | $1 - \left(\frac{\kappa\mu}{3}\right) r^2 + \frac{1}{5} \left(\frac{\kappa\mu}{r_b^2} - k^2\right) r^4$ |
| $Y$      | $\left[\gamma \cos (\Phi \xi_b) - \frac{\alpha}{\Phi} \sin (\Phi \xi_b)\right] \cos \left[\frac{2\Phi}{\sqrt{a}} \coth^{-1} \left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2\sqrt{a}}\right)\right] + \left[\frac{\alpha}{\Phi} \cos (\Phi \xi_b) + \gamma \sin (\Phi \xi_b)\right] \sin \left[\frac{2\Phi}{\sqrt{a}} \coth^{-1} \left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2\sqrt{a}}\right)\right]$, |
| $\rho$   | $\rho_c \left[1 - \mu \left(\frac{r}{r_b}\right)^2\right]$ |
| $a$      | $\frac{1}{5} \left(\frac{\kappa\mu}{r_b^2} - k^2\right)$ |
| $b$      | $\frac{\kappa\rho_c}{3}$ |
| $\xi$    | $\frac{2}{\sqrt{a}} \arcoth \left(\frac{1 + \sqrt{1 - br^2 + ar^4}}{\sqrt{a} r^2}\right)$ |
| $c_1$    | $\frac{\alpha}{\Phi} \cos (\Phi \xi_b) + \gamma \sin (\Phi \xi_b)$ |
| $c_2$    | $\gamma \cos (\Phi \xi_b) - \frac{\alpha}{\Phi} \sin (\Phi \xi_b)$ |
| $\alpha$ | $\frac{1}{4} \left(\frac{\kappa\rho_c}{3} - \frac{\kappa\mu}{5} - \frac{4k^2 r_b^2}{5}\right)$ |
| $\beta$  | $> 2k^2 - a \implies > \frac{1}{5} \left(11k^2 - \frac{\kappa\mu}{r_b^2}\right)$ |
| $\gamma$ | $\sqrt{1 + \kappa\rho_c r_b^2 \left(\frac{\mu}{5} - \frac{1}{3}\right) - \frac{k^2 r_b^4}{5}}$ |
| $p_r$    | $\frac{1}{\kappa} \left\{\frac{2\kappa\rho_c}{3} - \frac{4\kappa\rho_c r^2}{5r_b^2} - \kappa\rho_c \left[1 - \mu \left(\frac{r}{r_b}\right)^2\right] + 4\Phi \sqrt{Z} \times \left[\gamma \sin (\Phi \xi_b) + \frac{\alpha}{\Phi} \cos (\Phi \xi_b)\right] \cos (\Phi) - \left[\gamma \cos (\Phi \xi_b) - \frac{\alpha}{\Phi} \sin (\Phi \xi_b)\right] \sin (\Phi)\right\}$ |
| $p_\perp$| $p_r - \beta r^2$ |
Appendix C

We provide the source code for some selected MAXIMA functions, definitions and procedures that were used in the writing of this thesis.

C.1 Stability routines

The type of input file MAXIMA accept resembles the following, and for this program, the output consists of the integrals, with their uncertainties, and we also provide those below

```
Listing C.1: Stability routines

/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 13.04.2 ] */

/* [wxMaxima: input start ] */
/* reset all variables */
kill(all);

/* for numerical methods, newton_raphson, etc.
   Used for finding critical densities */
load(newton1)$

/* Set up constant boundaries, and ranges;
   allows certain simplifications to occur */
assume(rho_c >0, r_b >0, mu>=0 and mu<=1, r >=0, a >0, b >0, d >0)$

/* coordinate transformations */
x(r):= r^2$
xi(r):= -1/sqrt(a) * log((b-2*a*x(r)+2*sqrt(a)*sqrt(1-b*x(r)+a*x(r)+2))/((b+2*sqrt(a))))$

/* metric, matter variables and their derivatives for
   TVII if beta=k=0, and more general ones otherwise */
Z(r):= 1-b*x(r)+a*x(r)+2$
Y(r) := c2*sin(phi*xi(r)) + c1*cos(phi*xi(r))$
lamb(r):= -log(Z(r))$
```
nu(r) := 2 * log(Y(r))

rho(r) := rho_c * (1 - mu * (r / r_b)^2)

p_r(r) := (2 * Z(r) / (r * Y(r)) * diff(Y(r), r) + 1 / r * diff(Z(r), r) - kappa * rho(r))

r(rho) := r_b * sqrt((1 - rho / rho_c) / mu)

/* Polytropic index */
dp_drdrho := subst(rho = rho(r), diff(p_r(r(h)), rho))

Gamma(r) := (rho(r) + p_r(r)) / p_r(r) * dp_drdrho(r)

/* Constant values. Needed for numeric computations */
cnu := mu = 1

crho_c := rho_c = 1.0e18 * 7.42591549e-28

cr_b := r_b = 1e4

ck := k = 1e-10

cbeta := beta = 0

c_kappa := kappa = 8 * pi

/* TVIIca substitutions */
c_a := a = (kappa * rho_c * mu / r_b^2 - k^2) / 5

c_b := b = kappa * rho_c / 3

ca := alpha = (kappa * rho_c / 3 - kappa * rho_c * mu / 5 - 4 * k^2 * r_b^2 / 5) / 4

cphi := phi = sqrt(a + beta - 2 * k^2) / 2

cZb := Zb = Z(r_b)

cxib := xib = xi(r_b)

cxb := xb = x(r_b)

cc1 := c1 = sqrt(Zb) * cos(phi * xib) - alpha / phi * sin(phi * xib)

cc2 := c2 = alpha / phi * cos(phi * xib) + sqrt(Zb) * sin(phi * xib)

cxi := xi = xi(r)

cx := x = x(r)

cY := Y = Y(r)

cZ := Z = Z(x)

crho := rho = rho(r)

cr := r = r(rho)

c_lam := lamb = lamb(r)

cnu := nu = nu(r)

/* Substitution list applied in this specific order to generate graphs */
\* /\* rho_c, mu, and r_b free */\* \\
subls: [cY, cZ, crho] \* \\
tovalls: [cc1, cc2, cZb, cphi, calpha, cxib, cxi, cxb, ca, cb] \* \\
constls: [kappa=8*%pi*G/c^2, G=1,c=1] \* \\
ultls: [rho_c=3/(16*%pi), r_b=1.] \* \\
cnoParams: append(subls, tovalls, constls) \* \\
/

/\* Definitions of integrands, and integrating factors used to build 
Sturm-Liouville (SL) coefficients */

Ip_r(r) := subst(cnoParams, p_r(r))\* 
Ip_rprime(r) := subst(cnoParams, dp_rdrho(r))\* 
Igamma(r) := subst(cnoParams, Gamma(r))\* 
Ifacd(r) := 2*r/(Igamma(r)*Ip_r(r)) *(Ip_rprime(r) - 1)\* 
IfacN(r) := subst([ckappa, cmu, crho_c, ck, cr_b, cbeta], Ifacd(r))\* 
Ifac(r, params):= 
  block([llim:1, ulim:r, const: params], 
      prefac := subst(const, beta - 6*k^2/kappa), 
      exp(prefac = quad_qags(IfacN(t), t, llim, ulim)[1]) )\* 

SL_sqBraPrime(r) := diff(((exp((lamb(r)+3*nu(r))/2)*(Ip_rprime(r)-1)/r), r)\* 

SL_P(r) := Gamma(r)*p_r(r)*exp(((lamb(r)+3*nu(r))/2)/r^2* ( 
          Ifac(r,[ckappa, cmu, crho_c, ck, cr_b, cbeta]) 
      )\* 

SL_Q(r) := Ifac(r,[ckappa, cmu, crho_c, ck, cr_b, cbeta])*( 
            -4*exp((3*nu(r)+lamb(r))/2)/r^3*diff(p_r(r), r) + 
            -8*%pi*exp(3*(lamb(r)+nu(r))/2)/r^2*(p_r(r) + (beta+k^2/kappa)*r^2)*(p_r(r)+rho(r)) + 
            -24*exp((3*nu(r)+lamb(r))/2)/r^2*(k^2/kappa - beta/2) + 
            4*exp((3*nu(r)+lamb(r))/2)/(p_r(r)+rho(r))*((3*k^2/kappa-3*beta))^2 + 
            -4*diff(p_r(r), r)*exp((3*nu(r)+lamb(r))/2)/(r*(p_r(r)+rho(r)))*(9*k^2/kappa-beta) + 
            2*(beta - 6*k^2/kappa)*SL_sqBraPrime(r) + 
            (diff(p_r(r), r))^2*exp((3*nu(r)+lamb(r))/2)/(r^2*(p_r(r)+rho(r))) \* 

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101 )$
102
103 SL_W(r) := \text{Ifac}(r,[ckappa,cmu,crho_c,ck,cr_b,cbeta])*(
104 (\text{p}_r(r)\cdot\text{rho}(r))\cdot\exp((3\cdot\text{lamb}(r)+\text{nu}(r))/2)/r^2)
105 )$
106
107 /* [wxMaxima: input end ] */
108
109 /* [wxMaxima: input start ] */
110 /* testing plots, and functions */
111
112 appendList: append(subls, tovalls, consts, \{cmu,crho_c,ck,cr_b,cbeta\})$
113
114 PSL_P : subst(appendList,SL_P(r))$
115 PSL_Q : subst(appendList,SL_Q(r))$
116 PSL_W : subst(appendList,SL_W(r))$
117
118 plot2d(PSL_Q,\{r,1,10000\});
119 /* [wxMaxima: input end ] */
120
121 /* [wxMaxima: input start ] */
122 /* Evaluate the integrals with different function f's*/
123
124 appendList: append(subls, tovalls, consts, \{cmu,crho_c,ck,cr_b,cbeta\})$
125
126 TestF(r) := r^3$
127 TestdiffF(r):= \text{diff}(TestF(r),r)$
128
129 ulim: 10000$
130 llim: 0$
131
132 firstnIntegrand(r) := subst(appendList,SL_P(r)*(TestdiffF(r)^2))$
133 secondnIntegrand(r) := subst(appendList,SL_Q(r)*(TestF(r)^2))$
134 denomIntegrand(r) := subst(appendList,SL_W(r)*(TestF(r)^2))$
135
136 fInt : quad_qags(firstnIntegrand(r),r,llim,ulim);
137 sInt : quad_qags(secondnIntegrand(r),r,llim,ulim);
138 tInt : quad_qags(denomIntegrand(r),r,llim,ulim);
139
140 sqfrequency0 : (fInt[1]−sInt[1])/tInt[1];
The output of this routine then gives the frequencies associated with each solution.

**Listing C.2: Stability output**

```plaintext
(1) [2727.549758809813, 4.21059264169478*10^−7, 21, 0]
(2) [1159.023634396673, 5.878771063296232*10^−7, 21, 0]
(3) [8.581280570195294*10^−9, 4.979393690689304, 21, 0]
(4) 1.827846218967577*10^−7
```

As mentioned previously, the integration routine used is from the QUADPACK [103] function `quag_qags`, which provides the output shown. The last line of the output is the value of the fundamental frequency squared, and its value for different parameter values are shown in the main text of this thesis.

### C.2 Tensor routines

These routines were used to calculate the components of tensors, particularly the Einstein tensor. We show a sample here, working with a static spherically symmetric metric as given in equation (2.3).

**Listing C.3: Tensor routines**

```plaintext
NIL
(%o3) done
(%i4)
(%o5) done
(%i1) load(c tensor);
(%o1) /usr/share/maxima/5.37.2/share/tensor/c tensor.mac
```
10 (%i2) cset up ();
11 (%o2) done
12
13 (%i3) write file ("x.txt");
14 (%o3) done
15 (%o4) done
16
17 (%i5) n = n - 1
18 (%e) n
19 (%r) r
20
21 (%i5) mcs = ------------
22 1, 1, 2 2
23
24
25 (%i6) mcs = --
26 1, 2, 1 2
27
28
29 (%i7) mcs = --
30 2, 2, 2 2
31
32
33 (%i8) mcs = --
34 2, 3, 3 r
35
36
37 (%i9) mcs = --
38 2, 4, 4 r
39
40
41 (%i10) mcs = -- %e r
42 3, 3, 2
43
44 (%i11) mcs = ------------
45
46 285
\[
\begin{align*}
3, 4, 4 & \sin(\theta) \\
-1 & 2 \\
\text{(\%t12)} & \text{mcs} = -\%e \ r \sin(\theta) \\
4, 4, 2 \\
\text{(\%t13)} & \text{mcs} = -\cos(\theta) \sin(\theta) \\
4, 4, 3 \\
\text{(\%o13)} & \text{done} \\
\text{(\%i14)} & \\
\] \\
\begin{bmatrix}
\%e & n \\
\%e & r \\
0 & 0 & 0 \\
2 & & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{bmatrix} \\
\text{mcs} = [ n ] \\
1 \begin{bmatrix}
r \\
-0 & 0 & 0 \\
2 & & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{bmatrix} \\
\text{mcs} = [ 2 ] \\
2 \begin{bmatrix}
r \\
-0 & 0 & 0 \\
2 & & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{bmatrix} \\
\] \\
\] \\
\]
\[
\begin{bmatrix}
1 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & r & 0 & 0 \\
\end{bmatrix}
\]

**mcs =**

\[
\begin{bmatrix}
3 & -1 & 0 & 0 \\
0 & r & 0 & 0 \\
\cos(\theta) & 0 & 0 & 0 \\
\sin(\theta) & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 2 & 0 & 0 \\
0 & r & \sin(\theta) - \cos(\theta) \sin(\theta) & 0 \\
\cos(\theta) & 0 & 0 & 0 \\
\sin(\theta) & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cos(\theta) & 0 & 0 & 0 \\
\sin(\theta) & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 \\
0 & r & \sin(\theta) - \cos(\theta) \sin(\theta) & 0 \\
\end{bmatrix}
\]

(%o14) done

(%i15) (%o15) done

(%i16) \texttt{einstein = einstein}

(%o16) done

(%i17) -1 1

(%o17) \texttt{((-1) + r)}

r

\texttt{ein = matrix([------------------------, 0, 0, 0], [}}
\[
\begin{align*}
\text{riem} & = (-\frac{1}{r} + \frac{2}{r} + \frac{2}{r} + \frac{2}{r} + \frac{2}{r}) + (-\frac{1}{r} + \frac{2}{r} + \frac{2}{r} + \frac{2}{r} + \frac{2}{r}) \\
& = \left[ 0, 2 \right] , \\
& = \left[ 0, 0, 4 \right] , \\
& = \left[ 0, 0, 0, 4 \right] , \\
\text{done}
\end{align*}
\]
riem = $-\frac{1}{2}$ \(1, 4, 1, 4\) \(2\ r\)

\((-1\ n\) + (n) + 2\ n\)

riem = $-\frac{1}{2}$ \(2, 2, 1, 1, 4\)

riem = $-\frac{1}{2}$ \(2, 3, 2, 3\) \(2\ r\)

riem = $-\frac{1}{2}$ \(2, 4, 2, 4\) \(2\ r\)

riem = $-\frac{1}{2}$ \(3, 3, 1, 1\) \(2\)

riem = $-\frac{1}{2}$ \(3, 3, 2, 2\) \(2\)

riem = $-\frac{1}{2}$ \(3, 4, 3, 4\)

riem = $-\frac{1}{2}$ \(3, 4, 3, 4\)

% e n r sin (theta)
C.3 Plotting routines

These routines were used to generate data for the plot we produced in this thesis. We only show a sample of the files used.

**Listing C.4: Data generating routines**

```plaintext
/* wxMaxima batch file version 1 ] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 13.04.2 ] */

/* [wxMaxima: input start ] */
kill(all);
assume(a>0, b>0, mu<1 and mu>0, x_b>0, rho_c >0, r >0)$
x(r) := r^2$

xi(x) := 2/sqrt(a) * acoth( (1+sqrt(Z(x)))/(sqrt(a)*x) )$
Z(x) := 1 - b*x + a*x^2$
Z_x(x) := diff(Z(x),x)$
M(r) := 4*%pi*r_b^3*(1/3-mu/5)*rho_c + k^2*r_b^5/10$
Q(r) := k*r_b^3$
Y(xi) := c1*cosh(phi*xi) + c2*sinh(phi*xi)$
Y_xi(xi) := diff(Y(xi),xi)$
Y_r(r) := diff(Y(xi(x(r))),r)$
```

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Y_\_rr (r) := diff(Y_r(r), r)
Y_xi_r (r) := diff(Y_xi(x(r)) , r)
rho (r) := rho_c*(1-\mu*(r/r_b)^2)
r(rho) := r_b*sqrt((1-rho/rho_c)/\mu)
lamb (x) := -log(Z(x))$
nu (xi) := 2*log(Y(xi))$
nu_r (r) := diff(nu(xi(x(r))), r)$
Zsurf(mu) := (1 - 2*M(r_b)/r_b) * (-1/2) - 1$
elami(x) := 1/Z(x)$$
enui(xi) := (Y(xi))^2$
elambo(r) := 1/(1 - 2*G*M(r_b)/(r*c^2))$
enuo(r) := (1 - 2*G*M(r_b)/(r*c^2))$
ca := a = (\kappa*\rho_c*\mu/r_b^2 - k^2)/5$
cb := b = \kappa*\rho_c/3$
cphi := \phi = sqrt((-a+\beta - 2*k^2)/2)$
calpha := \alpha = (\kappa*\rho_c/3 - \kappa*\rho_c*\mu/5 - 4*k^2*r_b^2/5)/4$
cgamma := \gamma = sqrt(Z_b)$
cZb := Zb = Z(xb)$
cxib := xib = xi(xb)$
cxb := xb = x(r_b)$
cc1 := c1 = gamma*cosh(\phi*xib) - alpha/\phi*sinh(\phi*xib)$
cc2 := c2 = alpha/\phi*cosh(\phi*xib) - gamma*sinh(\phi*xib)$
cxi := xi = xi(x)$
cx := x = x(r)$
cY := Y = Y(xi)$
cY_xi := Y_xi = Y_xi(x)$
cZ := Z = Z(x)$
cY_r := Y_r = Y_r(r)$
cZ_x := Z_x = Z_x(x)$
crho := rho = rho(r)$
cr := r = r(rho)$
csigma := rho = exp(sigma)$
clambi := lamb = lamb(x)$
cnu := nu = nu(xi)$
\[ \text{cnu}_r: \quad \text{nu}_r = \text{nu}_r(r) \]
\[ \text{cElambi}: \quad \text{elambi} = \text{elambi}(x) \]
\[ \text{cEnui}: \quad \text{enui} = \text{enui}(xi) \]
\[ \text{cElambo}: \quad \text{elambo} = \text{elambo}(r) \]
\[ \text{cEnuo}: \quad \text{enuo} = \text{enuo}(r) \]

\[ v_2: \quad v_2 = c^2 * \frac{Y_r/Y}{2 * \sqrt{Z} * \frac{Y_xi}{Y} - \frac{Z_x}{r_b^2 / (\rho_c * \mu * \kappa)}} \]
\[ \text{pr}: \quad pr = c^2 * \frac{4 * \sqrt{Z} * \frac{Y_xi}{Y} - 2 * \frac{Z_x}{\kappa} - \kappa * \rho}{\kappa} \]
\[ \text{pt}: \quad pt = \text{rhs(pr)} - c^2 * \beta * x / \kappa \]
\[ \text{Bul}: \quad K = \rho * \text{rhs}(v_2) \]
\[ \text{Zsurf}: \quad \text{Zsurf} = \text{Zsurf}(\mu) \]

myterminal: "pdfeairo enhanced transparent dashed linewidth 3"

subs: \[ \{ cY, cY_r, cY_xi, cZ, cZ_x, crho \} \]
tovalls: \[ \{ cc1, cc2, cphi, calpha, cgamma, cZb, cxib, cxi, cx, cb, ca, cb \} \]
tovallsEOS: \[ \{ cc1, cc2, cZb, cd, cbeta, cxib, cxi, cx, cb, ca, cb, cr \} \]
consts: \[ \{ kappa = 8 * \%pi * G / c^2, G = 6.673 e-11, c = 2.99792458 e8 \} \]
ultls: \[ \{ rho_c = 1 e18, r_b = 10000 \} \]
ultlsmuHalf: \[ \{ rho_c = 1 e18, mu = 0.9, r_b = 10000 \} \]
noMu: \[ \{ rho_c = 1 e18, r_b = 10000 \} \]

*) wxMaxima: input end *)

*) wxMaxima: input start *)

*) Boundary metric compatibility plot *

comp_post: "set key center left; set xlabel 'r (m)'; set ylabel 'metric coefficients'; set grid;"

my_file: "/home/Ambrish/Documents/Phd/Text/NewSolutions/Figures/
chargedAnisotropicMetricBoundaryPhiN.data"

tovalls: \[ \{ cc1, cc2, cphi, calpha, cgamma, cZb, cxib, cxi, cx, cb, ca, cb \} \]
constb: \[ \{ beta = -1 e-16, ca, k = 8 e-9, r_b = 1 e4, rho_c = 7.42564845 e-28 * 1 e18, mu = 1, kappa = 8 * \%pi \} \]
const2b: \[ \{ beta = -5 e-17, ca, k = 6 e-9, r_b = 1 e4, rho_c = 7.42564845 e-28 * 1 e18, mu = 1, kappa = 8 * \%pi \} \]
pGrb: \[ \text{subst(append(tovalls, constb)), Z(x(r))} \]
pGttb: \[ \text{subst(append(tovalls, constb)), Y(xi(x(r)))*2} \]
pGr2b: \[ \text{subst(append(tovalls, const2b)), Z(x(r))} \]
pGtt2b: \[ \text{subst(append(tovalls, const2b)), Y(xi(x(r)))*2} \]
plot2d([pGrrb, pGttb, pGrr2b, pGtt2b],[r, 0, 1e4],
[xlabel, 'r'],
[ylabel, 'Metric coefficients 1/g_{11} and g_{00}'],
[legend, '{1/g_{rr}}', '{/ Symbol b}=-1e-16,k=8e-9','g_{tt}'],
[legend, '{1/g_{rr}}', '{/ Symbol b}=-5e-17,k=6e-9','g_{tt}'],
[plot_format, gnuplot_pipes],
[gnuplot_out_file, my_file],
[gnuplot_postamble, comp_post]);

Density plot

density_post: "set key bottom left; set xlabel 'r (m)'; set ylabel 'density (kgm^{-3})'; set grid
;
my_file: "/home/Ambrish/Documents/Phd/Text/Analysis/Figures/chargedAnisotropicPhiN/density.data"

data"$

pD60: subst(append(constls, tovalls, ultls,[mu=0.6]), rhs(crho))$
pD70: subst(append(constls, tovalls, ultls,[mu=0.7]), rhs(crho))$
pD80: subst(append(constls, tovalls, ultls,[mu=0.8]), rhs(crho))$
pD90: subst(append(constls, tovalls, ultls,[mu=0.9]), rhs(crho))$
pD100: subst(append(constls, tovalls, ultls,[mu=1]), rhs(crho))$

plot2d([pD60, pD70, pD80, pD90, pD100],[r, 0, 10000],
[legend, '{/ Symbol m} = 0.6', '{/ Symbol m} = 0.7', '{/ Symbol m} = 0.8', '{/ Symbol m} = 0.9', '{/ Symbol m} = 1.0'],
[plot_format, gnuplot_pipes],
[gnuplot_out_file, my_file],[gnuplot_postamble, density_post])$

Radial Pressure plot, beta=-2e-16, k=1e-9 */

density_post: "set key top left; set xlabel 'r (m)'; set ylabel 'pressure (Pa)'; set grid;"$

my_file: "/home/Ambrish/Documents/Phd/Text/Analysis/Figures/chargedAnisotropicPhiN/presureRbetaAK9.data"$

cbeta: beta = -2e-16*mu$
ck : k = 1e-95

pP60: subst(append(subls,tovalls,constls,ultls,[cbeta, ck, mu=0.6]), rhs(pr))$
pP70: subst(append(subls,tovalls,constls,ultls,[cbeta, ck, mu=0.7]), rhs(pr))$
pP80: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=0.8]), rhs(pr))

pP90: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=0.9]), rhs(pr))

pP100: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=1]), rhs(pr))

plot2d([pP60,pP70,pP80,pP90,pP100],[r,0,10000],
[legend,"[/Symbol m] = 0.6","[/Symbol m] = 0.7","[/Symbol m] = 0.8","[/Symbol m] = 0.9","[/Symbol m] = 1.0"],
[plot_format, gnuplot_pipes],
[gnuplot_out_file, my_file],[gnuplot_postamble, pressure_post])$

/* wxMaxima: input end */

/* Radial Pressure plot, mu=1, beta=−2e−16 */

pressure_post: "set key top left; set xlabel 'r (m)'; set ylabel 'pressure (Pa)'; set grid;"

my_file: "/home/Ambrish/Documents/Phd/Text/Analysis/Figures/chargedAnisotropicPhiN/
pressureRbetaMu1.data"

cbeta: beta = −2e−16*mu$

cmu: mu=1$

pP60: subst(append(subls,tovalls,constls,ultls,[cbeta,k=1e-9,cmu]), rhs(pr))

pP70: subst(append(subls,tovalls,constls,ultls,[cbeta,k=2e-9,cmu]), rhs(pr))

pP80: subst(append(subls,tovalls,constls,ultls,[cbeta,k=3e-9,cmu]), rhs(pr))

pP90: subst(append(subls,tovalls,constls,ultls,[cbeta,k=4e-9,cmu]), rhs(pr))

pP100: subst(append(subls,tovalls,constls,ultls,[cbeta,k=5e-9,cmu]), rhs(pr))

plot2d([pP60,pP70,pP80,pP90,pP100],[r,0,10000],
[legend,"k = 1 \times 10^{-9}","k = 2 \times 10^{-9}","k = 3 \times 10^{-9}","k = 4 \times 10^{-9}","k = 5 \times 10^{-9}"],
[plot_format, gnuplot_pipes],
[gnuplot_out_file, my_file],[gnuplot_postamble, pressure_post])$

/* wxMaxima: input end */

/* Radial Pressure plot, mu=1, beta=−2e−16 */

pressure_post: "set key top left; set xlabel 'r (m)'; set ylabel 'pressure (Pa)'; set grid;"

my_file: "/home/Ambrish/Documents/Phd/Text/Analysis/Figures/chargedAnisotropicPhiN/
pressureRk9Mu1.data"

/* wxMaxima: input start */

/* Radial Pressure plot, mu=1, beta=−2e−16 */

pressure_post: "set key top left; set xlabel 'r (m)'; set ylabel 'pressure (Pa)'; set grid;"

my_file: "/home/Ambrish/Documents/Phd/Text/Analysis/Figures/chargedAnisotropicPhiN/
pressureRk9Mu1.data"

cmu: mu = 1$
\( ck : \quad k = 1 \times 10^{-9} \)

\[
pP60 : \quad \text{subst(append(subs,tovalls,constls,ultls,[beta = -2e-16,ck,cmu]), rhs(pr)))}
\]

\[
pP70 : \quad \text{subst(append(subs,tovalls,constls,ultls,[beta = -3e-16,ck,cmu]), rhs(pr)))}
\]

\[
pP80 : \quad \text{subst(append(subs,tovalls,constls,ultls,[beta = -4e-16,ck,cmu]), rhs(pr)))}
\]

\[
pP90 : \quad \text{subst(append(subs,tovalls,constls,ultls,[beta = -5e-16,ck,cmu]), rhs(pr)))}
\]

\[
pP100 : \quad \text{subst(append(subs,tovalls,constls,ultls,[beta = -6e-16,ck,cmu]), rhs(pr)))}
\]

\[
\text{plot2d([pP60,pP70,pP80,pP90,pP100],[r,0,10000],}
\]
\[
[\text{legend,"{/ Symbol b} = -2 \times 10^\{-16\}" , "{/ Symbol b} = -3 \times 10^\{-16\}" , "{/ Symbol b} = -4 \times }\]
\[
10^\{-16\}" , "{/ Symbol b} = -5 \times 10^\{-16\}" , "{/ Symbol b} = -6 \times 10^\{-16\}]},
\]
\[
[\text{plot_format, gnuplot_pipes}],
\]
\[
[\text{gnuplot_out_file, my_file}], [\text{gnuplot_postamble, pressure_post}]$
\]

\[
\text{/* [wxMaxima: input end ] */}
\]

\[
\text{/* [wxMaxima: input start ] */}
\]

\[
\text{/* Tangential Pressure plot */}
\]

\[
\text{pressure_post : "set key top left; set xlabel "r (m)"; set ylab..."}
\]
\[
\text{my_file : "/home/Ambrish/Documents/Phd/Text/Analysis/Figures/chargedAnisotropicPhiN/pressureT...}
\]

\[
\text{data"}$
\]

\[
pP60 : \quad \text{subst(append(subs,tovalls,constls,ultls,[mu=0.6]), rhs(pt)))}
\]

\[
pP70 : \quad \text{subst(append(subs,tovalls,constls,ultls,[mu=0.7]), rhs(pt)))}
\]

\[
pP80 : \quad \text{subst(append(subs,tovalls,constls,ultls,[mu=0.8]), rhs(pt)))}
\]

\[
pP90 : \quad \text{subst(append(subs,tovalls,constls,ultls,[mu=0.9]), rhs(pt)))}
\]

\[
pP100 : \quad \text{subst(append(subs,tovalls,constls,ultls,[mu=1]), rhs(pt)))}
\]

\[
\text{plot2d([pP60,pP70,pP80,pP90,pP100],[r,0,10000],}
\]
\[
[\text{legend,"{/ Symbol m} = 0.6" , "{/ Symbol m} = 0.7" , "{/ Symbol m} = 0.8" , "{/ Symbol m} = 0.9" , "{/}
\]
\[
\text{Symbol m} = 1.0"]},
\]
\[
[\text{plot_format, gnuplot_pipes}],
\]
\[
[\text{gnuplot_out_file, my_file}], [\text{gnuplot_postamble, pressure_post}]$
\]

\[
\text{/* [wxMaxima: input end ] */}
\]

\[
\text{/* [wxMaxima: input start ] */}
\]

\[
\text{/* Y metric plot */}
\]

\[
\text{Y_post : "set key top right; set xlabel "r (m)"; set ylab..."}
\]
\[
\text{my_file : "/home/Ambrish/Documents/Phd/Text/Analysis/Figures/chargedAnisotropicPhiN/Ymetric...}
\]

\[
data"}$
\]
cbeta: beta = -2e-16*mu$
ck: k=1e-9$

pY60: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=0.6]), rhs(cY))$
pY70: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=0.7]), rhs(cY))$
pY80: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=0.8]), rhs(cY))$
pY90: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=0.9]), rhs(cY))$
pY100: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=1]), rhs(cY))$

plot2d([pY60,pY70,pY80,pY90,pY100],[r,0,10000],
[legend ,"{/Symbol m} = 0.6","{/Symbol m} = 0.7","{/Symbol m} = 0.8","{/Symbol m} = 0.9","{/Symbol m} = 1.0"],
[plot_format ,gnuplot_pipes],
[gnuplot_out_file ,my_file],[gnuplot_postamble ,Y_post])$
/* [wxMaxima: input end ] */

/* [wxMaxima: input start ] */
/* Z metric plot */
Z_post: "set key top right ; set xlabel 'r (m) ' ; set ylabel 'metric function , Z(r)' ; set grid ;"$
my_file: "/home/Ambrish/Documents/Phd/Text/Analysis/Figures/chargedAnisotropicPhiN/Zmetric.data"$
cbeta: beta = -2e-16*mu$
ck: k=1e-9$

pZ60: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=0.6]), rhs(cZ))$
pZ70: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=0.7]), rhs(cZ))$
pZ80: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=0.8]), rhs(cZ))$
pZ90: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=0.9]), rhs(cZ))$
pZ100: subst(append(subls,tovalls,constls,ultls,[cbeta,ck,mu=1]), rhs(cZ))$

plot2d([pZ60,pZ70,pZ80,pZ90,pZ100],[r,0,10000],
[legend ,"{/Symbol m} = 0.6","{/Symbol m} = 0.7","{/Symbol m} = 0.8","{/Symbol m} = 0.9","{/Symbol m} = 1.0"],
[plot_format ,gnuplot_pipes],
[gnuplot_out_file ,my_file],[gnuplot_postamble ,Z_post])$
/* [wxMaxima: input end ] */

/* [wxMaxima: input start ] */
Lamb_post: "set key top left;set xlabel 'r (m)';set ylabel 'metric function, \( l \{ /Symbol l \}(r) \);set grid;"$

my_file: "/home/Ambrish/Documents/Phd/Text/Analysis/ Figures/chargedAnisotropicPhiN/Lmetric.dat"$

cbeta: beta = -2e-16*mu$

ck: k=1e-9$

plamb60: subst(append(subls,tovalls,consts,ultls,[cbeta,ck,mu=0.6]),rhs(clambi))$

plamb70: subst(append(subls,tovalls,consts,ultls,[cbeta,ck,mu=0.7]),rhs(clambi))$

plamb80: subst(append(subls,tovalls,consts,ultls,[cbeta,ck,mu=0.8]),rhs(clambi))$

plamb90: subst(append(subls,tovalls,consts,ultls,[cbeta,ck,mu=0.9]),rhs(clambi))$

plamb100: subst(append(subls,tovalls,consts,ultls,[cbeta,ck,mu=1]),rhs(clambi))$

plot2d([plamb60,plamb70,plamb80,plamb90,plamb100],[r,0,10000],[legend, "\{ /Symbol m \} = 0.6", "\{ /Symbol m \} = 0.7", "\{ /Symbol m \} = 0.8", "\{ /Symbol m \} = 0.9", "\{ /Symbol m \} = 1.0"]$, [plot_format, gnuplot_pipes], [gnuplot_out_file, my_file],[gnuplot_postamble,Lamb_post])$

/* [wxMaxima: input end ] */$

/* [wxMaxima: input start ] */

exp(lamb) metric plot IN and OUT$

mIO_post: "set key top left;set xlabel 'r (m)';set ylabel 'metric function, exp(lamb)';set grid;"$

my_file: "/home/Ambrish/Documents/Phd/Text/NewSolutions/ Figures/LIOmetricPhiN.dat"$

plamb60i: subst(append(subls,tovalls,consts,ultls,[mu=0.6]),unit_step(r_b-r)*rhs(cElambi))$

plamb70i: subst(append(subls,tovalls,consts,ultls,[mu=0.7]),unit_step(r_b-r)*rhs(cElambi))$

plamb80i: subst(append(subls,tovalls,consts,ultls,[mu=0.8]),unit_step(r_b-r)*rhs(cElambi))$

plamb90i: subst(append(subls,tovalls,consts,ultls,[mu=0.9]),unit_step(r_b-r)*rhs(cElambi))$

plamb100i: subst(append(subls,tovalls,consts,ultls,[mu=1]),unit_step(r_b-r)*rhs(cElambi))$

/* [wxMaxima: input end ] */
With the data generated, I then used gnuplot, a plotting program to plot the data. A sample of the gnuplot file is given below

**Listing C.5: Gnuplot routines**

```
set term pdfcairo enhanced transparent dashed
set output '/home/Ambrish/Documents/Phd/Text/Analysis/Figures/chargedAnisotropicPhiP/EOSbeta1k9.pdf
set style line 1 lc rgb '#0000FF' lt 1 dt 1
set style line 2 lc rgb '#3300CC' lt 2 dt 2
set style line 3 lc rgb '#5500AA' lt 3 dt 3
set style line 4 lc rgb '#770088' lt 4 dt 4
set style line 5 lc rgb '#990066' lt 5 dt 5
```
set style line 6 lc rgb '#BB0044' lt 6 dt 6
set style line 7 lc rgb '#AA0022' lt 7 dt 7
set style line 8 lc rgb '#FF0000' lt 8 dt 8

set multiplot

set key bottom right
set xlabel '{/Symbol\m}264|\264|10^{18}|\264|m^{−3}) '
set ylabel 'pressure\264|\264|10^{34}|\264|Pa '
set size ratio 0.618

set yrange [0:]
set xrange [0:]

plot 'EOSbeta1k9.data' using ($1/1e18):($2/1e34) index 0 with lines ls 1 title '{/Symbol\m}ω=ω
1.0\ ', \n      using ($1/1e18):($2/1e34) index 1 with lines ls 2 title '{/Symbol\m}ω=ω
0.9\ ', \n      using ($1/1e18):($2/1e34) index 2 with lines ls 3 title '{/Symbol\m}ω=ω
0.8\ ', \n      using ($1/1e18):($2/1e34) index 3 with lines ls 4 title '{/Symbol\m}ω=ω
0.7\ ', \n      using ($1/1e18):($2/1e34) index 4 with lines ls 5 title '{/Symbol\m}ω=ω
0.6\ '
set size 0.5
set origin 0.15,0.45
set xrange[0.8:0.9]
set yrange[4.5:]
unset key
set xtics 0.02
set xlabel "
set ylabel "
set grid x,y

plot 'EOSbeta1k9.data' using ($1/1e18):($2/1e34) index 0 with lines ls 1 title '{/Symbol\m}ω=ω
1.0\ ', \n      using ($1/1e18):($2/1e34) index 1 with lines ls 2 title '{/Symbol\m}ω=ω
0.9\ ', \n
```
unset multiplot
```

```
  " using ($1/1e18)$2/1e34 index 2 with lines ls 3 title "{/Symbol\textsubscript{m}}_{\text{\textomega}}\text{" \n  0.8", \n  " using ($1/1e18)$2/1e34 index 3 with lines ls 4 title "{/Symbol\textsubscript{m}}_{\text{\textomega}}\text{" \n  0.7", \n  " using ($1/1e18)$2/1e34 index 4 with lines ls 5 title "{/Symbol\textsubscript{m}}_{\text{\textomega}}\text{" \n  0.6"
```

300
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