Casimir-Polder forces in the presence of the cosmic photon heat bath

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Abstract

We study the effect of a photon background at finite temperature $T$ on the Van der Waals interactions among neutral bodies. It turns out that the long-range Casimir-Polder force is unaffected for distances much less than $T^{-1}$ and strongly enhanced for distances much above $T^{-1}$. 
The retarded dispersion potentials among neutral atoms or molecules have been studied since long time ago. Indeed, Casimir and Polder were the first to obtain the correct potential for the Van der Waals interaction of two neutral systems with no permanent electric dipole moment [1]. Their potential differed at large distances from the non-relativistic Coulomb interaction derived by London [2]. These results were obtained using old fashioned perturbation theory and an important step forward was the introduction of modern quantum field theory techniques. In this way a wide variety of phenomena associated to dispersion forces was systematically studied [3]. Two-neutrino-exchange forces, first discussed by Feinberg and Sucher [4], or spin independent forces arising from double (pseudo)scalar exchange [5] provide examples of the activity in this field.

The asymptotic form for the Casimir-Polder potential, generalised in ref. [6] to include magnetic effects, can be derived from the phenomenological lagrangian density [7]

\[ \mathcal{L} = -g_1 \partial_{\alpha} \phi \partial^{\beta} \phi F_{\alpha \gamma} F_{\beta \gamma} - g_2 \phi^2 F^2 \]

where \( \phi \) is a scalar field, \( F_{\alpha \beta} \) is the electromagnetic field-strength, and \( g_1 = \frac{\alpha E + \alpha B}{2m} \) and \( g_2 = -\frac{m \alpha B}{4} \). The relevant Feynman diagram is drawn in Fig.1 and the resulting potential is

\[ V_{CP}(r) = -\frac{\left[23(\alpha_E^a \alpha_E^b + \alpha_B^a \alpha_B^b) - 7(\alpha_E^a \alpha_B^b + \alpha_E^b \alpha_B^a)\right]}{(4\pi)^3 r^7}. \]

Figure 1: Interaction between neutral systems arising from electric and magnetic susceptibilities.

In a photon populated medium, such as the cosmic microwave background radiation (MWBR), one of the photons in the double exchange can be supplied by the thermal bath. In the static limit, i.e. momentum transfer \( q \simeq (0, \vec{Q}) \), where matter is supposed to be at rest in the microwave background radiation (MWBR) frame, the potential is related to the Feynman amplitude in Fig.1 through

\[ V(r) = \frac{1}{(2\pi)^2 r} \int dQ Q \sin Qr \frac{i T(Q)}{4m_a m_b}. \]
In this last formula, $Q \equiv |\vec{Q}|$ and $\mathcal{T}(Q) \equiv \mathcal{T}(q = (0, \vec{Q}))$ with

$$
\mathcal{T}(q) = \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \delta^4(k + k' - q)d(k, k') D_F(k^2) D_F(k'^2)
$$

(4)

where $g^{\mu\nu} D_F(k^2)$ stands for the photon propagator and the explicit expression for the function $d(k, k')$ is \[8]\:

$$
d(k, k') = 8 g_1^a g_2^b K_{\mu\nu,\rho\sigma}(k) K^{\mu\nu,\rho\sigma}(k')
$$

\[9\]

$$
+ \left[ 4 g_1^a g_2^b (p_0^a p_0^b + p_0^a p_0^b) K^{\alpha\nu,\rho\sigma}(k) K^{\beta\nu,\rho\sigma}(k') + (a \leftrightarrow b) \right]
$$

\[9\]

$$
+ g_1^a g_2^b \left[ (p_0^a p_0^b + p_0^a p_0^b) K^{\alpha\nu,\rho\kappa}(k) K^{\beta\nu,\rho\kappa}(k') \sqrt{4} \delta(k') (p_0^b p_0^b + p_0^b p_0^b) + (k \leftrightarrow k') \right]
$$

(5)

with $K_{\mu\nu,\rho\sigma}(k) = \bar{k}_\mu k_\rho g_{\nu\sigma} - \bar{k}_\nu k_\rho g_{\mu\sigma} - \bar{k}_\mu k_\sigma g_{\nu\rho} + \bar{k}_\nu k_\sigma g_{\mu\rho}$.

The propagation of photons in a medium at temperature $T$ is described by the Green function (in Feynman gauge)

$$
g^{\mu\nu} D_F(k, T) = -g^{\mu\nu} \left[ (k^2 + i\epsilon)^{-1} - 2\pi i\delta(k^2)n(T, k^0) \right]
$$

(6)

where $n(T, k^0)$ is the Bose-Einstein distribution function for the background photons\[3\]. Of course, use of the first piece in $D_F(k, T)$ above gives the zero temperature vacuum result in equation \[2\]. In a photon background, a contribution to the long range force can arise because a photon in the thermal bath may be excited and de-excited back to its original state in the course of the double scattering process. This effect is described by the crossed terms contained in $\mathcal{T}(q)$ that involve the thermal piece of one photon propagator along with the vacuum piece of the other photon propagator. This thermal component of the Feynman amplitude can be written as

$$
\mathcal{T}_T(q) = -i \int \frac{d^4k}{(2\pi)^3} \left[ \frac{1}{k^2} d(k, -(k - q)) n(T, k^0 - q^0) \delta \left( (k - q)^2 \right) 
$$

\[7\]

$$
+ \frac{1}{(k - q)^2} d(k, -(k - q)) n(T, k^0) \delta(k^2) \right] 
$$

(7)

where we used $\delta^4(k + k' - q)$ to integrate over $k'$. A shift of variable $k - q \rightarrow k$ in the first term of the sum in the previous expression leads to the more compact form,

$$
\mathcal{T}_T(q) = -i \int \frac{d^4k}{(2\pi)^3} n(T, k^0) \delta(k^2) \left[ \frac{d(k, -(k - q))}{(k - q)^2} + \frac{d(k + q, -k)}{(k + q)^2} \right].
$$

(8)

\[1\] Following \[1\], to compute the $T$-dependent effects, we need only the real part of $i\mathcal{T}(Q)$ correctly given by using \[3\], (see ref. \[3\]), which is the $1 - 1$ component of the full 2-dimensional matrix propagator used in the real time approach to finite temperature field theory \[1\].
We integrate over the photon energy $\omega \equiv k^0$ with the help of the Dirac delta and we are left with an integral over three-momentum $\vec{k}$. The integral over the azimuthal angle in $\vec{k}$-space is trivial and supplies a factor $2\pi$. Hence, the thermal part of equation (3) can be written as,

$$V_T(r) = \frac{1}{32m_am_b\pi^4r} \int_0^\infty \frac{\omega d\omega}{e^{\omega/T} - 1} \int_{-1}^1 dz \int_0^\infty dQ \sin Qr \left[ \frac{d(k, -(k - q))}{2\omega z - Q} - \frac{d(k + q, -k)}{2\omega z + Q} \right]_{k^0=\omega}.$$  \tag{9}

In the denominators of this formula we already took the static limit $q \simeq (0, \vec{Q})$. It is understood also that the non relativistic limit for functions $d(k, -(k - q))$ and $d(k + q, -k)$ has been taken. That is, we set $s = (p_a + p_b)^2 \simeq (m_a + m_b)^2$ and $t = (p_a - p_a')^2 \simeq -Q^2$, masses $m_a, m_b$ being the largest energy scales in the system, and use them to evaluate the scalar products of momenta contained in (5). The process is tedious but straightforward. Let us now illustrate with some detail the calculation of the component of the potential proportional to $g^a_2 g^b_2$. The rest is calculated along similar lines. The non-relativistic reduction of the square bracket in (9) gives in this case:

$$\left[ \frac{d(k, -(k - q))}{2\omega z - Q} - \frac{d(k + q, -k)}{2\omega z + Q} \right]_{k^0=\omega}^{(2,2)} = \frac{128Q^3\omega^2z^2}{4\omega^2z^2 - Q^2}. \tag{10}$$

Inspection of the potential (9) with the explicit form (10) in the integrand shows that the integral over $Q$ is ill-defined. It diverges for large $Q$. This is no surprise because the effective lagrangean (1) is not renormalisable. However, we are interested only on the long range (i.e. distances large compared to the dimensions of the atoms) behaviour of the potential and we may perform a finite number of short distance subtractions without modifying the asymptotic behaviour of the potential. For details of how this is done in the vacuum case see [3, 4]. Here we adopt the prescription $\int_0^\infty dQ \sin Qr \frac{Q^3}{(2\omega z)^2 - Q^2} \equiv -\frac{d^2}{dr^2} \int_0^\infty dQ \sin Qr \frac{Q}{(2\omega z)^2 - Q^2}$, where the integral on the right is perfectly defined and gives $-\frac{\pi}{2} \cos 2\omega z r$. This procedure will isolate the asymptotic piece of the potential. Hence, our potential reads,

$$V_T^{(2,2)}(r) = \frac{2g^a_2 g^b_2}{m_am_b\pi^3r} \frac{d^2}{dr^2} \int_0^\infty \frac{\omega^3 d\omega}{e^{\omega/T} - 1} \int_{-1}^1 z^2 dz \sin 2\omega z r. \tag{11}$$

The $z$-integral is easily done and $V_T^{(2,2)}(r)$ is,

$$V_T^{(2,2)}(r) = \frac{g^a_2 g^b_2}{m_am_b\pi^3r} \frac{d^2}{dr^2} \left[ \frac{1}{r^3} \left( -1 + r \frac{d}{dr} - \frac{r^2}{2} \frac{d^2}{dr^2} \right) \right] \int_0^\infty \frac{d\omega}{e^{\omega/T} - 1} \sin 2r\omega \right]. \tag{12}$$

The remaining thermal integral is found upon using the relation,

$$\frac{1}{e^{\omega/T} - 1} = \sum_{n=1}^\infty e^{-n\omega/T} \tag{13}$$
so that, once the $r$-derivatives are performed, the final result boils down to

$$V_T^{(2,2)}(r) = \sum_{n=1}^{\infty} \frac{3\alpha_a^a \alpha_b^b}{8\pi^3 r^7} (2rT)^6 \left[ -n^4 + 10n^2(2rT)^2 - 5(2rT)^4 \right] \left[ n^2 + (2rT)^2 \right]^5$$

(14)

where we reintroduced the magnetic polarisabilities of the particles $a$ and $b$.

Some more painful algebra incorporating all effects (electrical, magnetic and mixed) leads to the total finite temperature potential

$$V_T(r) = -\frac{1}{32\pi^3 r^7} \left\{ \left[ 23(\alpha_a^a \alpha_B^b + \alpha_E^a \alpha_B^b) - 7(\alpha_a^a \alpha_B^b + \alpha_a^a \alpha_B^b) \right] S_0(2rT) 
+ \left\{ 46(\alpha_E^a \alpha_B^b + \alpha_B^a \alpha_B^b) - 14(\alpha_B^a \alpha_B^b + \alpha_B^a \alpha_B^b) \right\} S_1(2rT) 
+ \left\{ 11(\alpha_B^a \alpha_B^b + \alpha_E^a \alpha_E^b) + 5(\alpha_E^a \alpha_B^b + \alpha_B^a \alpha_E^b) \right\} S_2(2rT) \right\}$$

(15)

with $S_l(x) \equiv x^{10-2l} \sum_{n=1}^{\infty} \frac{n^{2l}}{(n^2+x^2)^{2l}}$.

We can sum over $n$ by realising that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2} = \frac{1}{2x^2} (\pi x \coth \pi x - 1)$$

(16)

and calculating $\sum_{n=1}^{\infty} \frac{1}{(n^2+x^2)^2}$, $\sum_{n=1}^{\infty} \frac{n^2}{(n^2+x^2)^3}$ and $\sum_{n=1}^{\infty} \frac{n^4}{(n^2+x^2)^4}$ by differentiation of equation (16) with respect to the parameter $x^2$ and by trivial algebraic manipulation. Once this is done we may obtain analytic expressions for the functions $S_{l=0,1,2}(2rT)$ that enter the potential:

$$S_0(x) = \frac{1}{768} \left[ -384 + 105x\pi + 105x\pi \text{csch}(x\pi) 
+ \left( 105x^2\pi^2 + 90x^3\pi^3 + 40x^4\pi^4 + 8x^5\pi^5 \right) \text{csch}^2(x\pi) 
+ \left( 90x^3\pi^3 + 24x^5\pi^5 \right) \text{csch}^3(x\pi) 
+ \left( 60x^4\pi^4 + 40x^5\pi^5 \right) \text{csch}^4(x\pi) 
+ 24x^5\pi^5 \text{csch}^5(x\pi) \right]$$

(17)

$$S_1(x) = \frac{x\pi}{768} \left[ 15 + 15 \text{csch}(x\pi) 
+ \left( 15x\pi + 6x^2\pi^2 - 8x^3\pi^3 - 8x^4\pi^4 \right) \text{csch}^2(x\pi) 
+ \left( 6x^2\pi^2 - 24x^4\pi^4 \right) \text{csch}^3(x\pi) 
+ \left( -12x^3\pi^3 - 40x^4\pi^4 \right) \text{csch}^4(x\pi) 
- 24x^4\pi^4 \text{csch}^5(x\pi) \right]$$

(18)
and
\[
S_2(x) = \frac{x\pi}{768} \left[ 9 + 9 \text{csch}(x\pi) 
+ \left( 9x\pi - 6x^2\pi^2 - 24x^3\pi^3 + 8x^4\pi^4 \right) \text{csch}^2(x\pi) 
+ \left( -6x^2\pi^2 + 24x^4\pi^4 \right) \text{csch}^3(x\pi) 
+ \left( -36x^3\pi^3 + 40x^4\pi^4 \right) \text{csch}^4(x\pi) 
+ 24x^4\pi^4 \text{csch}^5(x\pi) \right].
\] (19)

Now, our potential is only valid for distances much larger than the dimensions of the objects involved. For the hydrogen atom, for instance, one can estimate that distances should be on the order or larger than \(10^2 \text{Å}\) \[3, 4\]. But the MWBR temperature sets an additional distance scale: \(\sim (3K)^{-1} \sim 1 \text{mm}\). So we may obtain approximate forms for the potential in two limits, i.e. \(10^2 \text{Å} \leq r \ll 1 \text{mm}\) and \(r \gg 1 \text{mm}\).

In the first case, for distances \(r\) such that \(rT \ll 1\) the potential is
\[
V_T(r) \simeq -\frac{1}{64\pi^3 r^7} \pi(rT) \left[ 32(\alpha_B^a\alpha_B^b + \alpha_E^a\alpha_E^b) - 12(\alpha_B^a\alpha_E^b + \alpha_E^a\alpha_B^b) \right].
\] (20)

On the other hand, when \(rT \gg 1\), we have
\[
V_T(r) \simeq -\frac{1}{64\pi^3 r^7} \pi(rT) 12(\alpha_B^a\alpha_B^b + \alpha_E^a\alpha_E^b).
\] (21)

Obviously, in the \(r \ll 1 \text{mm}\) regime, the effects on the original Casimir-Polder potential due to the MWBR heat bath are negligible and behaving as \(r^{-6}\), but in the very large distance domain, the dominant potential is the temperature dependent one whose \(r\)-behaviour is also \(r^{-6}\). This is incidentally the behaviour of the potential both in the old London model \[2\] and in the Casimir-Polder approach at very short distances \[4\] (e.g. \(1 \text{Å} < r \ll 10^2 \text{Å}\), in the hydrogen case) which we did not bother to consider when displaying the asymptotic potential in equation (2). For the intermediate region (\(r \sim O(1 \text{mm})\)), where both potentials are present with comparable strengths, we may plot the exact result instead of writing down the explicit form of equation (15). This is conveniently shown in Fig.2 where we display \(V_T(r)/V_{CP}(r)\) as a function of distance. We see clearly how, in the region of interest, the curve interpolates between a steep linear behaviour on the small \(r\) side and again a linearly growing function on the large side of \(r\).

We would like to close this paper with a few comments. Firstly, from the theoretical standpoint, the effect of the MWBR is definitely there: the Casimir-Polder force among
Figure 2: Ratio between the potential \( V_T \), at \( T \sim 2.72K \) corresponding to the cosmic MWBR, and the zero temperature Casimir-Polder potential \( V_{CP} \) when \( \alpha_{a,b}^B \gg \alpha_{a,b}^E \) as e.g. for two \( H \) atoms.

neutral bodies gets an additional contribution because these bodies sit in a background of cosmic photons. We feel that the effect should be calculated although, admittedly, the strength of this force is far beyond present experimental capabilities. Indeed, generic potentials with a power-law fall-off \( r^{-n} \) have been experimentally scrutinised in the laboratory at various distance scales. Even for the shortest ranges explored, the limits on the strength of potentials with \( n \geq 5 \) are very poor [10]. Finally, we would like to point out that when searching for extra forces in the laboratory (like the ones hypothesised in different completions of the Standard Model [11]) one has to be aware of the real effects that are present in the system. The Casimir-Polder force exists and is good to know that it is not modified deep in the sub-millimeter domain.

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