Topology in QCD
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1. Introduction

Topology is important in QCD largely due to the way it influences the propagation of quarks. Here it is the zero-modes of \( iD \) that are special; not the exact zero modes whose number per unit volume
\[
\frac{|Q|}{V} \propto \frac{1}{\sqrt{V}} \quad V \to \infty \to 0,
\]
but rather the (mixed) would-be zero modes that would have been exact zero-modes if their parent topological charges had been isolated. The total number of these is certainly \( \propto V \), if only from the small instantons whose density is calculable analytically. Lattice calculations find, in both SU(2) and SU(3), a substantial value for
\[
\langle Q^2 \rangle / V \sim (200 \text{ MeV})^4 \approx \frac{1}{fm^4},
\]
and this suggests that a lot of the topological charge density tends to cluster in lumps that are uncorrelated at larger distances – just like a ‘gas’ of instantons. So, for convenience, this is the language I shall use in this talk.

I will start with the aspect that is no doubt on the firmest theoretical ground: the large mass of the \( \eta' \) and the value of the topological susceptibility \( \chi_t \equiv \langle Q^2 \rangle / V \). I dwell on the quenched case because it provides one area in which the lattice calculations are in very good shape, even if there is not much new this year (probably because they are in such good shape ...). I will then move onto the possible role of instantons in driving the spontaneous breaking of chiral symmetry. Here the exciting news is that we have finally got lattice fermions that are good enough to address this question realistically. More speculative is the influence of topology on hadrons. This provides a motivation for trying to extract the properties of the ‘gas’ of instantons in the vacuum. There has been some interesting new work on the latter during the past year. Finally I turn to the role of instantons in confinement. The usual view is that there is no such role. I will discuss a recent calculation that claims the opposite. This will be the one area where fermions don’t appear at all.

2. \( Q, \chi_t \) and \( m_{\eta'} \)

Instantons provide a resolution of the \( \eta' \) puzzle. The mass of the \( \eta' \) can be related to the strength of topological fluctuations:
\[
\chi_t \equiv \frac{\langle Q^2 \rangle}{V} \sim \frac{m_{\eta'}^2 f_\pi^2}{2N_f} \sim (180 \text{ MeV})^4.
\]
This is to leading order in \( N_c \) and so the value of \( \chi_t \) is the quenched value. The practical application (i.e. MeV units) assumes that \( N_c = 3 \) is close to \( N_c = \infty \) and this is also something that one needs to confirm.

Let me start with some reassuring comments about calculating the topological charge \( Q \) of lattice gauge fields. We suppose the lattice spacing \( a \) is very small. Consider topological charges which are localised within a core of radius \( \rho \). For \( a \ll \rho \ll 0.5 fm \) these charges have an analytically calculable density, which turns out to be \( \rho^6 \) for SU(3). Large scales, say \( \rho \geq 0.5 fm \), are a non-perturbative problem. At the ultraviolet scale, \( \rho \sim a \), lattice artifacts may dominate. Now, the point to note is that during the Monte Carlo each step changes only one link matrix i.e. the fields within a volume \( \delta v \sim a^4 \). So the only
way we can change the value of $Q$ is for the core of
the topological charge to shrink over many Monte
Carlo steps from say $\rho \sim 0.2 f m$ down to $\rho \sim a$
and then within a hypercube and so out of the
lattice. Now because the density $\propto \rho^6$, the
region $a \ll \rho \ll 0.1 f m$ (say) will normally (in our
finite lattice volume of a couple of fermi) have
no instantons at all. So as an instanton traverses
this topological desert it will become very visi-
able – particularly once $\rho \sim few \times a$. At this
point the core will stick out above the
$\zeta$ although
in powers of $1/a$
malisations can be either calculated analytically
These additive \[6\] and multiplicative \[7\] renor-

ations, the above scenario can work for modest
$lattice calculations (using only those with at least
3 $\beta$ values) and compare their continuum extrap-
ations. I extrapolate the dimensionless ratio
$\chi_t^{1/4}/\sqrt{\sigma}$ ($\sigma$ = string tension)

$$\frac{\chi_t^{1/4}(a)}{\sqrt{\sigma}(a)} = \frac{\chi_t^{1/4}(0)}{\sqrt{\sigma}(0)} + ca^2 \sigma \quad (6)$$

using a common set of values for $\sigma$ (as listed in
\[8\]) so that any differences are differences in the
calculation of topology, not of the scale. I now
list the SU(3) calculations I use, and the results of
the corresponding continuum extrapolations.

• SU(3)

Pisa\[10\] Smearred version of $Q_L(x)$, calculated di-
rectly from Monte Carlo field average using eqn \[8\]
(5.9 $\leq \beta \leq 6.1$): $\chi_t^{1/4}/\sqrt{\sigma} = 0.464(23)$
Boulder\[11\] Algebraic lattice $Q$ on RG smoothed
fields (5.85 $\leq \beta \leq 6.1$): $\chi_t^{1/4}/\sqrt{\sigma} = 0.456(23)$
Oxford\[2\] $Q_L$ on cooled fields (5.7 $\leq \beta \leq 6.2$):
$\chi_t^{1/4}/\sqrt{\sigma} = 0.449(17)$
UKQCD\[13\] $Q_L$ on cooled fields (6.0 $\leq \beta \leq 6.4$):
$\chi_t^{1/4}/\sqrt{\sigma} = 0.448(50)$

• SU(2)

Pisa\[11\] Smearred version of $Q_L(x)$, calculated di-
rectly from Monte Carlo field averages using eqn \[8\]
(2.44 $\leq \beta \leq 2.57$): $\chi_t^{1/4}/\sqrt{\sigma} = 0.480(23)$
Boulder\[15\] lattice $Q$ on RG mapped fields ( 2.4 $\leq 
\beta \leq 2.6$): $\chi_t^{1/4}/\sqrt{\sigma} = 0.528(21)$
Oxford-Liverpool\[13\] $Q_L$ on cooled fields ( 2.2 $\leq \beta 
\leq 2.6$): $\chi_t^{1/4}/\sqrt{\sigma} = 0.480(12)$
Oxford\[17\] blocked geometric $Q$ directly on Monte
Carlo fields (2.3 $\leq \beta \leq 2.6$): $\chi_t^{1/4}/\sqrt{\sigma} = 0.480(18)$
Zurich\[16\] version of $Q_L$ on improved-cooled
fields (2.4 $\leq \beta \leq 2.6$): $\chi_t^{1/4}/\sqrt{\sigma} = 0.501(45)$

All these continuum limits are consistent with
each other despite the wide variety of methods
being used. They provide the following conserva-
We observe that the two values are close to each other suggesting that (for the pure gauge theory) SU(3) is indeed close to SU(∞). If we plug in the value $\sqrt{\sigma} \simeq 440 \pm 38\text{MeV}$ [4], then we find $\chi_{t}^{1/4} = 200 \pm 18\text{MeV}$ for SU(3). As we have seen this is roughly the density of topological fluctuations that is needed [2] to provide the $\eta'$ with its observed mass.

One can of course also calculate $\chi_{t}$ for ‘full QCD’. We expect that $\langle Q^{2} \rangle \to 0$ as $m_{q} \to 0$, since the zero modes ensure that det$\mathcal{D} = 0$ for $Q \neq 0$. In fact we know more: the anomalous Ward identities tell us that

$$\chi_{t} = \frac{m_{q}^{2}f_{\pi}^{2}}{n_{f}^{2}} + O(m_{q}^{2}) \propto m_{q}$$  \hspace{1cm} (8)$$

if we are in the phase in which chiral symmetry is spontaneously broken. By contrast, in a chirally symmetric phase we expect $\chi_{t} \propto m_{q}'$, and, of course, in the quenched case $\chi_{t} \propto m_{q}'$. So we can test the lattice calculations against this relation.

One has to be careful because the lattice spacing $a$ is both a function of $\beta$ and $m_{q}$. (For a large quark mass the running of the coupling will include the quark only for scales below $O(1/m_{q})$.) So in testing eqn 3 we should express all the quantities in terms of some physical quantity that is not expected to vary strongly with $m_{q}$, e.g. $r_{0}$ or $\sqrt{\sigma}$. The ($n_{f} = 2$) UKQCD calculations [19] reported at this meeting go one better by tuning $\beta$ with $m_{q}$ so that $r_{0}/a$ is independent of $m_{q}$. Such a calculation separates the $m_{q}$ dependence from any $a$ dependence. If one plots $r_{0}^{2}\chi_{t}$ versus $r_{0}^{2}m_{q}^{2}$ one finds [13] that the (three) points are consistent with eqn 4 but only if one decreases $f_{\pi}$ by about 20% from its physical value. That, I think, is pretty good. Less pretty are the large statistical errors. Despite the latter it is clear that the value of $r_{0}^{2}\chi_{t}$ has decreased from its quenched value and that a $\propto m_{q}^{n_{f}=2}$ behaviour is excluded. This contrasts with the $n_{f} = 2$ CP-PACS calculation [21] of $\chi_{t}/\sigma^{2}$ which strangely finds no $m_{q}$ dependence at all. On the other hand the $n_{f} = 2$ calculation of the Pisa group [21] does show some sign of a decrease as $m_{q} \to 0$.

The meson that (perhaps) most directly reflects topological fluctuations is the $\eta'$. CP-PACS has produced [21] a very nice calculation of $m_{\eta'}$. This is a tough calculation because such flavour-singlet hadrons simultaneously need the statistics of glueball calculations and expensive quark propagators. CP-PACS do a direct calculation that finds $m_{\eta'} \neq 0$ as $m_{q} \to 0$, in contrast to the $\propto m_{q}$ Goldstone behaviour one would expect if topological fluctuations were negligible. The value they obtain in the continuum chiral limit is $m_{\eta'} = 863(86)\text{MeV}$. This is for $n_{f} = 2$ and $n_{c} = 3$, so it is amusing to note that if we plug into eqn 3 the values $n_{f} = 2$, $f_{\pi} = 93\text{MeV}$ (since this should be insensitive to the strange quark) and $\chi_{t} = (200\text{MeV})^{4}$ (the lattice value) then we obtain $m_{\eta'} \simeq 860\text{MeV}$ as our expectation. This is promising and I hope CP-PACS will pursue this calculation; perhaps in the direction [22] of explicitly showing that it is dominated by the lowest modes of $i\mathcal{D}$. A related calculation has been performed by UKQCD [23]; and the Pisa group [24] has tried calculating the mass using correlators of the $(\bar{p} = 0)$ topological charge.

3. Chiral symmetry breaking

Let $\rho(\lambda)$ be the normalised spectral density of $i\mathcal{D}[A]$ averaged over gauge fields. Then we can express the chiral condensate as

$$\langle \bar{\psi}\psi \rangle = \lim_{m \to 0} \lim_{V \to \infty} \langle \bar{\psi}\psi \rangle_{m,V} = \lim_{m \to 0} \lim_{V \to \infty} \int_{0}^{\infty} \frac{2m\rho(\lambda,m)}{\lambda^{2} + m^{2}} d\lambda = \pi\rho(0)$$  \hspace{1cm} (9)$$

So chiral symmetry breaking requires a non-zero density of modes at $\lambda = 0$. Now if we remove all interactions, then each (anti)instanton has a zero-mode and so $\rho(\lambda) \propto \delta(\lambda)$. Interactions will spread these modes away from zero: however instantons are clearly a good first guess if what you want is $\rho(0) \neq 0$. Contrast it with the non-interacting limit of the perturbative vacuum – the free theory – where $\rho(\lambda) \propto \lambda^{3}$.

This idea, that instantons might drive chiral
symmetry breaking, is an old one \[3\]. There have been lattice calculations to test this idea, for example \[25\]. The calculations in \[25\], with staggered fermions in the pure gauge SU(2) vacuum, found that the chiral symmetry breaking disappeared if one removed the topological eigen-modes of \(i\mathcal{D}[A]\). While conclusive about the lattice theory, there was a question mark over the continuum theory: despite the lattice spacing being ‘small’ (e.g. \(\beta = 2.6\)) the \(|Q|\) zero-modes were no closer to zero than the other small modes of \(i\mathcal{D}[A]\). (See also \[26\].) This raises doubts about how continuum-like is the spectrum of the O\((V)\) mixed would-be zero modes; one really needs the lattice shift in the exact zero-modes to leave them small compared to the other small modes in a reasonably sized box. (The exact \(|Q| \sim \sqrt{V}\) zero modes become irrelevant in the thermodynamic limit.) This raises doubts about any lattice calculations of the influence of instantons on quarks; e.g hadron masses as well as chiral symmetry breaking.

So the exciting news here is, of course, ‘Ginsparg-Wilson’. In particular the Columbia group has pursued the domain wall variant and have produced some very pretty calculations \[27\] showing that even with a modest 5'th dimension one obtains ‘exact’ lattice zero-modes that are very much smaller than the other small modes of the Dirac spectrum in a reasonably sized box. (In practice they observe the \(\propto 1/m\) behaviour in \((\bar{\psi}\psi)_{m,V}\) that one obtains from such modes.) This provides an explicit demonstration that controlled calculations of the influence of topology on continuum hadronic physics can now be done.

There has also been related work with overlap fermions \[28\] and there has been a great deal of comparison with Random Matrix Theory, as well as other model calculations, which I hope will be reviewed elsewhere.

4. The instanton content of the vacuum

There has been a new calculation of the topological structure of the SU(2) vacuum \[29\]. The main novelty here is a modified cooling algorithm that is designed to cool out to a specified cooling radius \(r_c\). One finds that while some features, such as the average instanton size, vary weakly with \(r_c\), other quantities, such as the instanton density and average nearest instanton anti-instanton distance, vary rapidly with \(r_c\). (Not unlike the conclusions of \[30\] using ordinary cooling.) This is largely bad news if what you want is to obtain the detailed properties of the instanton ‘gas’ in the original vacuum, so that you can provide an input into phenomenological instanton calculations \[3\].

However one thing that is not understood is how much of this apparent variation with cooling is real and how much of it is actually the fault of the ‘pattern recognition’ algorithms that turn the topological charge density into an instanton ensemble. Here a gleam of hope comes from a re-analysis in \[31\] of \[30\]. Clearly in \(n_c\) usual cooling sweeps one expects roughly \(r_c \propto a \sqrt{n_c}\). Plotting the data of \[30\] for all \(\beta\) and \(n_c\) against the corresponding scaling variable, \[31\] finds a very nice scaling. More importantly, they find a range of \(r_c\) where the instanton properties become independent of \(r_c\). This range is narrow, so it needs more work. But it provides some hope ...

The instanton gases found on the lattice are usually denser, and with larger average instanton sizes, than the instanton liquid models \[5\] would like. So it is interesting that a recent calculation \[32\] of the quark physics (in a ‘toy model’) from the lattice instanton ensembles of \[30\] finds that the important part of the low-\(\lambda\) spectrum looks like that of a dilutish gas of narrower instantons. There is a simple reason for this: a large instanton has a large zero-mode that has a correspondingly small density. It will therefore have a small value when integrated over the small volume where the zero-mode of the small instanton resides. This leads to a small mixing between large and small instantons: they approximately decouple. This mechanism provides a possibility for reconciling the lattice and the instanton liquid.

Note also related work in \[33\] and \[34\].

5. Instantons and confinement

An isolated instanton affects a large Wilson loop weakly; it merely renormalises the Coulomb potential \[8\]. So one expects that an instanton
‘gas’ (no long range order) will not disorder a Wilson loop strongly enough to produce an area decay. The claim in [35] that random ensembles of instantons do produce linear confinement is therefore surprising. However we note that our analytic intuition really holds for Wilson loops that avoid instanton cores. In [35] the density distributions used are \( D(\rho) \propto 1/\rho^3 \) or \( 1/\rho^5 \). This corresponds to the ‘packing fraction’ \( \frac{1}{4} \int d\rho D(\rho)\rho^4 \) diverging! So these are very dense gases, and the Wilson loop will, throughout its length, pass in the middle of densely overlapping instanton cores. It may be that this will disorder the Wilson loop sufficiently to confine. This is relevant since lattice calculations suggest that instantons in the real vacuum are dense and highly overlapping. It is of course important to check that the field configurations being used, generated by the approximation of adding the individual \( A_I(x) \) (in singular gauge), do indeed accurately denote a ‘gas’ of topological charges, and that the confining disorder is not being produced by the breakdown of this standard linear addition ansatz. These and related [35,36] directions will be interesting to pursue further.

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