Variational principle and a perturbative solution of non-linear string equations in curved space

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Abstract
String dynamics in a curved space-time is studied on the basis of an action functional including a small parameter of rescaled tension $\varepsilon = \gamma/\alpha'$, where $\gamma$ is a metric parametrizing constant. A rescaled slow worldsheet time $T = \varepsilon \tau$ is introduced, and general covariant non-linear string equation are derived.

It is shown that in the first order of an $\varepsilon$–expansion these equations are reduced to the known equation for geodesic derivation but complemented by a string oscillatory term. These equations are solved for the de Sitter and Friedmann-Robertson-Walker spaces. The primary string constraints are found to be split into a chain of perturbative constraints and their conservation and consistency are proved. It is established that in the proposed realization of the perturbative approach the string dynamics in the de Sitter space is stable for a large Hubble constant $H (\alpha' H^2 \gg 1)$.

1 Introduction

In recent years much attention has been paid to studying the role of strings in cosmology [1-3]. Investigation of this problem is complicated by nonlinear character of string equations solvable for special types of metrics. Therefore in [3-6] an approach to studying approximate solutions of string equations using a perturbative expansion was initiated. This approach is based on the idea of an expansion of string solutions around the geodesic line of the string mass center described by a mass parameter $m$. A great deal of work has been done in this direction, and a considerable class of perturbative string equation solutions was found for different cosmological spaces [3-13 and Refs. there].

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An extensive application of this perturbative approach necessitates its further investigations. In particular, the nature of a small perturbative parameter and the procedure of its bringing into the string equations and constraints are important points for study. Moreover, it is appropriate to find a mechanism for fixing arbitrariness in the choice of the phenomenological mass parameter $m$, to define a relevant scale for measuring the worldsheet parameter $\tau$ and $\sigma$ which are the arguments of perturbative functions. In principle a well defined mass parameter attributed to the center mass trajectory may be absent. For example, in contrast to the case of the Minkowski space-time, where a particle is characterized by a fixed mass and spin, in a curved space-time a particle has some fixed eigenvalues of other Casimir operators. These operators are built of the generators of a symmetry group of the curved space-time and may be a complicated combination of the momentum and spin operators. To investigate the above mentioned problems seems to be important for the classification of cosmological spaces where a perturbative string dynamics is selfconsistent.

While studying this matter, a new representation for the string action including kinetic and potential terms of the string lagrangian as independent additive terms was considered in [14]. This representation comprises a rescaled string tension $\gamma/\alpha'$ as a small dimensionless parameter – a world sheet ”cosmological term”. The constant $\gamma$ in the rescaled tension $\gamma/\alpha'$ with the dimension $L^2$ ($\hbar = c = 1$) is a constant parameterizing the metric of a curved space-time. For example, for the de Sitter space $\gamma = H^{-2}$, where $H$ is the Hubble parameter. Using this representation for the Nambu-Goto string action the perturbative string equations were derived. The perturbative string equations [14], were shown to be transformed into the perturbative equations [4,5] after rescaling the worldsheet parameter $\sigma$ (or $\tau$) and fixing the phenomenological mass parameter by the value $m = 0$. These results point out to the existence of different realizations of the considered perturbative approach. So it becomes important to establish the regions for applicability of the different realizations and to understand the physical effects connected with them. In particular, it may occur that the perturbative string dynamics critically depends on the value of the phenomenological mass parameter $m$ for some type of the curved space-time. The de Sitter space just belongs to this case. Actually, as shown in [5], the perturbative string frequency modes in the de Sitter space are defined as $\omega_n = \sqrt{n^2 - (\alpha'Hm)^2}$ and become imaginary for large values of the Hubble constant $H$. This results in instabilities of the string dynamics in the realization of the perturbative approach considered in [4,5]. It follows from the above formula for $\omega_n$ that these instabilities must disappear, if the phenomenological parameter $m$ acquires zero value. This value is in exact accordance with the restriction of the perturbative scheme realized in [14]. Therefore it seems important to present a rigorous verification of the absence of instabilities in the realization of the perturbative approach proposed in [14], as well as to develop and substantiate this perturbative scheme itself.

Note also that in [16] a perturbative approach to strings using null string as zero approximation was considered. As a result the perturbative equations [16] did not

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1This representation is a natural generalization of the representation [15] to the case of an arbitrary curved space-time.
include any oscillatory terms in the first and second approximations. In view of the fact that the introduction of any arbitrary small tension should excite an oscillatory regime in the pattern of the string evolution, it becomes obvious that the realization [16] of the perturbative approach needs additional groundings.

A novelty of the present paper is the introduction of a rescaled slow worldsheet parameter $T = \varepsilon \tau$, where $\varepsilon = \gamma/\alpha'$ is a small dimensionless parameter presenting the rescaled string tension. The transition to the scale $T$ shows the degree of roughening of the string dynamics in the considered perturbative scheme. Using general covariant formulation of the perturbative equations and constraints we show that the string equations in the first approximation acquire the form of the geodesic deviation equations [22] complemented by an additional oscillatory term.

It is proved that primary non-perturbative constraints are split into a system of constraints for the perturbative functions. We show that the general covariantization procedure provides an essential simplification of these perturbative constraints. The proof of the consistency of these perturbative constraints and their conservation is presented. This proof becomes very simple in the proposed general covariant formulation. Further we find that the constraints of the first approximation functions are reduced to the condition of their orthogonality to the geodesic world trajectories of the zero approximation. We establish that the constraints in question may be considered as the initial data of the perturbative equations.

Considered is the application of the perturbative approach for a wide class of the Friedmann-Robertson-Walker universes. It is shown that their linearized equations of the first approximation have the form of the modified Bessel equations. Their exact solutions are found.

## 2 Rescaled tension as a perturbation in the Nambu-Goto action

As shown in [14,15], the Nambu-Goto string action in the curved space can be presented in the form

$$S = S_0 + S_1 = \int d\tau d\sigma \left[ \frac{\det(\partial_{\mu}x^M G_{MN}(x)\partial_{\nu}x^N)}{E(\tau, \sigma)} - \frac{1}{(\alpha')^2} E(\tau, \sigma) \right],$$

where $E$ is an auxiliary world-sheet density. The motion equation for $E$ produced by $S$ (1) is

$$E = \alpha' \sqrt{-\det g_{\mu\nu}},$$

$$g_{\mu\nu} = \partial_{\mu}x^M G_{MN}(x)\partial_{\nu}x^N,$$

The substitution of $E$ (2) into the functional $S$ (1) transforms the latter into the Nambu-Goto representation

$$S = -\frac{2}{\alpha'} \int d\tau d\sigma \sqrt{-\det (\partial_{\mu}x^M G_{MN}(x)\partial_{\nu}x^N)}$$

Thus, the representations (1) and (4) for the string action $S$ (1) are classically equivalent. Unlike the representation (4), the representation (1) includes the string
tension parameter $1/\alpha'$ as a constant at an additive world-sheet "cosmological" term playing the role of potential energy. Respectively, this term may be considered as a perturbative addition for the case of a week tension. But what are the measure units in terms of which the string tension is a small value?

To answer this question we are to consider one of dimensional parameter $\gamma$ or some combination of the parameters defining the metric of the curved space where the string moves. Without loss of generality put that $\gamma$ has the dimension $L^2(\hbar = c = 1)$. Then the value of the dimensionless combination

$$\varepsilon = \gamma/\alpha'$$

(5)
can be considered as a parameter characterizing the power of string tension. When $\varepsilon \ll 1$, or equivalently,

$$1/\alpha' \ll \gamma^{-1},$$

(6)

the tension $1/\alpha'$ should be considered as a weak one. For example, in the de Sitter space the Hubble parameter $H$ plays the role of $\gamma^{-1/2}$, and we consider tension as a weak one when

$$1/\alpha' \ll H^2$$

(7)

Of course, for the cases of more complicated background including additional fields such as members of supergravity multiplet, we get wider possibilities for the choice of a perturbative parameter.

A natural condition for appearance of $\varepsilon$ (5) in the representation (1) is the agreement that the string world coordinates $x^M$ are measured in terms of the metric parameter $\gamma$. Actually, if we choose dimensionless coordinates $\tilde{x}^M$ and the Lagrange multiplier $\tilde{E}$

$$x^M = \gamma^{1/2} \tilde{x}^M , \quad E = \gamma^2 \tilde{E}$$

(8)

the action $S$ (1) is presented in the form

$$S = \int d\tau d\sigma \left[ \det \left( \partial_{\mu} \tilde{x}^M G_{MN}(\tilde{x}) \partial_{\nu} \tilde{x}^N \right) \frac{E}{\tilde{E}} - \left( \frac{\gamma}{\alpha'} \right)^2 \tilde{E} \right]$$

(9)

containing the dimensionless parameter $\varepsilon$. In the case, when $x^M$ are measured by the constant $\alpha'$, i.e.

$$x^M = \sqrt{\alpha'} \tilde{x}^M , \quad E = \alpha'^2 \tilde{E}$$

(10)

the "cosmological term" in the representation of the action (1) loses the role of a perturbation term. If we prefer to work in terms of the original world coordinates $x^M$, then the condition for the measurement of $x^M$ in the units of the constant $\gamma$ is manifested by the choice of a worldsheet gauge fixing in the form [14]

$$E = -\gamma(\dot{x}^M G_{MN} \dot{x}^N)$$

(11)

In the gauge (11) complemented by the orthonormality condition

$$\left( \dot{x}^M G_{MN} \dot{x}^N \right) = 0$$

(12)
the variational Euler-Lagrange motion equations generated by $S$ (1) acquire the form [14]

$$
\ddot{x}^M - \left(\frac{\gamma}{\alpha'}\right)^2 \dot{x}^M + \Gamma^M_{PQ}(x) \left[ \dot{x}^P \dot{x}^Q - \left(\frac{\gamma}{\alpha'}\right)^2 \dot{x}^P \dot{x}^Q \right] = 0
$$

and contain the dimensionless parameter $\varepsilon (5)$. This parameter appears in another string constraint

$$
\left(\dot{x}^M G_{MN} \dot{x}^N\right) + \left(\frac{\gamma}{\alpha'}\right)^2 \left(\dot{x}^M G_{MN} \dot{x}^N\right) = 0
$$

which is additional to (12) and follows from Eqs.(2) and (11). Provided that $\frac{\gamma}{\alpha'} \ll 1$ Eqs.(14) can be considered as nonlinear equations with the small parameter $\varepsilon (5)$. Then we should seek for a solution of (13) and for the constraints (12) and (14) in the form of a series expansion in terms of $\varepsilon (5)$.

Physically the case $\varepsilon \ll 1$ corresponds to a strong gravitational field or, equivalently, to a large scalar space-time curvature $R^M_M$ measured in terms of the tension $1/\alpha'$. This is evident for the case of the de Sitter spaces where the condition (6) is equivalently presented in the form

$$
1/\alpha' \ll R^M_M,
$$

which shows that the elastic force of the string is less than the gravity force. In the limit of zero tension $\varepsilon = 0$ ($\alpha' \to \infty$) the action (1), constraints (12-14) and Eqs.(13) transform into the relations characterizing a tensionless string or a massless particle. The tensionless string moves translatally along the the geodesic lines of the considered space-time without any oscillations. In the case of $\varepsilon \ll 1$ a very small elastic force described by the terms with the $\sigma-$derivatives in (13-14) appears in addition to the external gravity force. Then the small string oscillations appear and each point of string gets an additional shift. But the amplitudes of these oscillator shifts are smaller than the paths caused by the translating movements. Thus these oscillations can be considered as small perturbations of the translating movements of the string points.

The perturbative oscillations are characterized by small frequencies and, subsequently, by large periods. A characteristic time scale of the oscillator periods is proportional to $1/\varepsilon$. This observation follows from the string equations (13) where the transition to a rescaled worldsheet proper time $T$

$$
T = \varepsilon \tau, \quad \frac{\partial}{\partial \tau} = \varepsilon \frac{\partial}{\partial T}, \quad \frac{\partial^2}{\partial \tau^2} = \varepsilon^2 \frac{\partial^2}{\partial T^2}
$$

is performed. Such a transition transforms Eqs.(13) and the constraints (12,14) into the standard form

$$
\ddot{x}^M_{TT} - \dot{x}^M_{T} + \Gamma^M_{PQ}(x) \left[ x^P_{,T} x^Q_{,T} - \dot{x}^P \dot{x}^Q \right] = 0
$$

$$
\left( x^M_{,T} \dot{x}^M_{,T} \right) = 0
$$

$$
x^M_{,TT} x^M_{,TT} + \dot{x}^M \dot{x}^M = 0,
$$


where $x^M_T \equiv \partial_T x^M$ and $x_{M,T} \equiv G_{MN} \partial_T x^N$. The transition to the slow worldsheet time $T$ (16) means an enlargement of the original world sheet time $\tau$ by $1/\varepsilon$ times. The choice of such large units for the worldsheet time leads to an essential roughening of the string motion pattern due to which an information on the microscopic string dynamics is lost. On the slow scale $T$ the string oscillations can be observed owing to a sufficient observation time. But the rescaling of the worldsheet time does not result in an increase of oscillation amplitudes in contrast to the translating movement length. Thus, after the exclusion of the small parameter $\varepsilon$ from the string equations and the constraints (17-19) the ratio of the oscillation amplitudes to the translation displacement plays the role of a small parameter. This allows to seek for the string equations solution on the scale $T$ in the form of a superposition of its large translating and small oscillation displacements

$$x^M = \varphi^M(T) + \varepsilon \psi^M(T, \sigma) + \varepsilon^2 \chi^M(T, \sigma) + \ldots (20)$$

The zero approximation functions $\varphi^M$ in the asymptotic series expansion (20) do not depend on the parameter $\sigma$ which enumerates different points of the string. Such a choice is explained by an assumption that differences in the displacements of the string points and the oscillation amplitudes have the same order of smallness. While comparing Eqs.(17) with the correspond ones in [4] we conclude that the latter transform into (17) after a formal change $\tau \rightarrow T$. This observation means that from the view point of the variational principle for the string action (1), the perturbative expansion [14] works starting from $\tau \geq 1/\varepsilon$.

The perturbative equations and constraints following from the exact ones (17-19) have been derived in [14] in terms of the worldsheet variables $(\tau, \xi)$, where $\xi = \sigma/\varepsilon$. The variables $(\tau, \xi)$ are connected with the variables $(T, \sigma)$ used here by the dilaton transformation defined by the parameter $\varepsilon$

$$T = \varepsilon \tau, \quad \sigma = \varepsilon \xi (21)$$

Usage of the variables $(\tau, \xi)$ in the perturbative description is not very convenient due to the appearance of $\varepsilon$ in the boundary conditions for closed string discussed here

$$x^M(T, \sigma = 0) = x^M(T, \sigma = 2\pi) (22)$$

This inconvenience disappears when the variables $(T, \sigma)$ (16-19) are introduced. Take into account the fact that the transformation (21) belongs to the two dimensional conformal group which is a local symmetry of the string equations and constraints. At the same time we find that the perturbative equations and constraints generated by Eqs.(17-19) are obtained from the correspond ones in [14] after the simple change $(\tau, \xi) \rightarrow (T, \sigma)$.

For the zero approximation functions $\varphi^M(T)$ we get the equations

$$\dot{\varphi}^M_T + \Gamma^M_{PQ}(\varphi) \varphi^P_T \varphi^Q_T = 0, (23)$$

$$\left( \varphi^M_T \varphi_M_T \right) \equiv \left( \varphi^M_T G_{MN}(\varphi) \varphi^N_T \right) = 0 (23')$$

The constraint (23') shows that the vector $\varphi^M_T$ is a light-like vector corresponding to 4-velocity of a massless particle moving along the geodesic line (23). Thus we find
that the variational principle applied to the action (1) fixes the value of the mass parameter \( m \) introduced in [4,5]. Later we will see that this fixation \( m = 0 \) leads to important consequences.

The equations and constraints for the first approximation functions \( \psi^M(T, \sigma) \) take the form

\[
\Delta^M_L \psi^L \equiv \psi^M_{,TT} - \psi^M + 2 \left[ \Gamma^M_{PQ}(\varphi) \psi^P_{,T} \psi^Q_{,T} + \frac{1}{2} \psi^L \partial_L \Gamma^M_{PQ} \varphi^P_{,T} \varphi^Q_{,T} \right] = 0,
\]

\[
(\varphi_{M,T} \psi^M_{,T}) + \frac{1}{2} \psi^L \left( (\varphi^M_{,T} \partial_L G_{MN} \varphi^N_{,T}) \right) = 0,
\]

\[
(\varphi_{M,T} \dot{\psi}^M) = 0
\]

Since the constraint (24') is a constraint for the initial data of Eqs.(24) we may integrate (24') with respect to \( \sigma \) and obtain

\[
(\varphi_{M,T} \psi^M) = C(T),
\]

because \( G_{ML} \varphi^L_{,T} \) do not depend on the variable \( \sigma \). The kinematic "constant" \( C(T) \) may be chosen equal to zero without loss of generality. This choice means that the perturbation \( \psi^M \) produced by the small string oscillations is orthogonal to the light-like geodesic line (23). Further we shall use the orthogonality constraint

\[
(\varphi_{M,T} \dot{\psi}^M) = 0
\]

instead of (25), and it is a new point in comparison with the results of [14].

Finally the equations and constraints for the functions of the second-order approximation \( \chi^M(T, \sigma) \) in terms of \( (T, \sigma) \) worldsheet variables get the form

\[
\Delta^M_L \chi^L \equiv \chi^M_{,TT} - \chi^M + 2 \left[ \Gamma^M_{PQ}(\varphi) \chi^P_{,T} \chi^Q_{,T} + \frac{1}{2} \psi^L \partial_L \Gamma^M_{PQ} \varphi^P_{,T} \varphi^Q_{,T} \right] = 0,
\]

\[
2 \left( \varphi_{M,T} \chi^M \right) + \chi^L \partial_L G_{MN} \varphi^M_{,T} \varphi^N_{,T} + \left[ (\psi^M_{,T} \psi^M_{,T}) + (\dot{\psi}^M \dot{\psi}^M) \right] +
\]

\[
2 \psi^L \partial_L G_{MN} \varphi^M_{,T} \chi^N_{,T} + \frac{1}{2} \psi^L \psi^K \partial_L G_{MN} \varphi^M_{,T} \varphi^N_{,T} = 0,
\]

\[
(\varphi_{M,T} \dot{\chi}^M) + (\psi_{M,T} \dot{\psi}^M) + \psi^L \partial_L G_{MN} \varphi^M_{,T} \dot{\chi}^N = 0
\]

Eqs.(23-26) should be complemented by the periodicity conditions for \( \psi^M \) and \( \chi^M \)

\[
\psi^M(T, \sigma = 0) = \psi^M(T, \sigma = 2\pi),
\]

\[
\chi^M(T, \sigma = 0) = \chi^M(T, \sigma = 2\pi)
\]

Eqs.(23-26) show that the effects caused by the appearance of a small tension manifest themselves starting from the first approximation (and conserve in the second one). On the scale \( T \), i.e., when \( \tau \geq 1/\varepsilon \), these effects have an oscillation character and agree with the qualitative picture described here and in [14].
3 General covariance and consistency of the perturbative constraints and equations

In this section we show a general covariant character of the perturbative scheme under discussion and prove the selfconsistency of the perturbative split chain of the equations and constraints (23-26).

The general covariant differential and derivative corresponding to a target space metric $G_{MN}(\varphi)$ are

\[ D^V_M = dV^M + d\varphi^P \Gamma^M_{PQ}(\varphi)V^Q \]
\[ D_T V^M = V^M_{,T} + \varphi^P_{,T} \Gamma^M_{PQ}(\varphi)V^Q \]  

(28)

The definition (28) turns out to present the geodesic equation (23) in the form

\[ D_T \varphi^M_{,T} = 0, \]  

(29)

To prove the conservation of the constraint (23′) let us differentiate it with respect to $\tau$ and get

\[ \partial_T \left( \varphi^M_{,T} G_{MN}(\varphi) \varphi^N_{,T} \right) = 2 \left( D_T \varphi^M_{,T} \cdot \varphi_{M,T} \right) + \varphi^M_{,T} \varphi^N_{,T} D_T G_{MN}(\varphi) \]  

(30)

The first and the second terms in (30) equal zero in view of (29), and the the well known property of $G_{MN}(\varphi)$

\[ D_T G_{MN} = 0, \]  

(31)

respectively. The motion equations (24) for the first order perturbative functions $\psi^M(T, \sigma)$ involve the differential operator $\Delta^M_L$

\[ \Delta^M_L \equiv \delta^M_L \left( D^2_T - D^2_{\sigma} \right) + 2 \varphi^P_{,T} \Gamma^M_{PQ}(\varphi) \partial_T + \varphi^P_{,T} \varphi^Q_{,T} \partial_L \Gamma^M_{PQ} \]  

(32)

Using the definition of $\Gamma^M_{PQ}$ [22] and their independence on $\sigma$ we find that the general covariant representation of $\Delta^M_L$ has the form

\[ \Delta^M_L = \delta^M_L \left( D^2_T - D^2_{\sigma} \right) - R^M_{PQL} \varphi^P_{,T} \varphi^Q_{,T}, \]  

(33)

where $R^M_{PQL}$ is the Riemann-Christoffel tensor

\[ \frac{1}{2} R^M_{PQL} = \partial_{[Q} \Gamma^M_{L]P} + \Gamma^N_{P[L} \Gamma^M_{Q]N} \]  

(34)

Note that $D^2_{\sigma} = \partial^2_{\sigma}$ since $\varphi^M(T)$ is independent on $\sigma$. By means of these observations and definitions the equations and constraints (24 – 24′) can be rewritten in the general covariant form

\[ \left( D^2_T - D^2_{\sigma} \right) \psi^M + R^M_{PQL} \varphi^P_{,T} \varphi^Q_{,T} \psi^L = 0 \]  

(35)

\[ \left( \varphi_{M,T} D_T \psi^M \right) = 0, \]  

(35′)

\[ \left( \varphi_{M,T} \psi^M \right) = 0, \]  

(35″)
In the absence of the term \( \mathcal{D}_\sigma^2 \psi^M \) in Eqs.(35) the latter acquire the form of the geodesic deviation equations [22]. The term \( \mathcal{D}_\sigma^2 \psi^M = \partial^2_\sigma \psi^M \) in (35) describes the contribution of the string elastic forces which push out the string points from the geodesic lines enumerated by the parameter \( \sigma \).

To prove the conservation of the constraints (35') we are to differentiate them

\[
\partial_T \left( \varphi_{M,T} \mathcal{D}_T \psi^M \right) = \left( \mathcal{D}_T \varphi_{M,T} \mathcal{D}_T \psi^M \right) + \left( \varphi_{M,T} \mathcal{D}_\sigma^2 \psi^M \right)
\]

(36)

The first term in (36) equals to zero because of the Eqs.(29). After using Eqs.(35) the second term in (36) takes the form

\[
\left( \varphi_{M,T} \mathcal{D}_T^2 \psi^M \right) = \left( \varphi_{M,T} \partial^2_\sigma \psi^M \right) - \varphi_{M,T} R^M_{PQL} \varphi^P_T \varphi^Q_T \psi^L = \partial^2_\sigma \left( \varphi_{M,T} \psi^M \right)
\]

(37)

and also goes to zero owing to the constraint (35'').

Similar reasoning should be used to prove the conservation of the constraint (35'') which takes the form

\[
\partial_T \left( \varphi_{M,T} \psi^M \right) = \left( \varphi_{M,T} \mathcal{D}_T \psi^M \right)
\]

(38)

after differentiation and taking into account of Eqs.(29) and (31). The right-hand side of (38)equals zero in view of the constraints (35'). Thus, the constraints (23') and (35', 35'') are consistent and conserved owing to the motion equations (29) and (35). Due to these properties the constraints (23) and (24' − 24'') can be considered as the constraints for initial data of Eqs.(23) and Eqs.(24). Generally covariant formulation of Eqs.(26) and (26' − 26'') for the second-order perturbative functions \( \chi^M(T, \sigma) \) should be studied by analogy, but here we restrict ourselves by considering the perturbative string dynamics in the first and second approximations.

As follows from the general covariant formulation (35), the string equations in the first approximation acquire a simple form of a covariant wave equation

\[
\left( \mathcal{D}_T^2 - \partial^2_\sigma \right) \psi^M = 0
\]

(39)

for the class of symmetric spaces characterized by the condition

\[
R^v_{MPQL} = \kappa (G_{MQ}G_{PL} - G_{ML}G_{PQ})
\]

(40)

Such a simplification is a consequence of the constraints (24'') and (23'), since

\[
R^M_{PQL} \varphi^P_T \varphi^Q_T \psi^L = \kappa \left[ \varphi^M_T \left( \psi^N_T \varphi^N_T \right) - \psi^M \left( \varphi^N_T \varphi^N_T \right) \right] = 0
\]

(41)

The de Sitter space is an important example of the class of symmetric spaces and will be studied below.

4 Stability of perturbative string oscillations in the de Sitter universe

Here we consider the application of the above considered realization of perturbative approach for the solution of the string equations in the Friedmann-Robertson-Walker cosmological spaces. The F-R-W metrics are characterized by the following quadratic form

\[
ds^2 = (dx^0)^2 - R^2(x^0)\delta_{ik} dx^i dx^k
\]

(42)
The solution of the zero approximation equations and constraints (23, 23') for the metric (42) is well-known and has the form (in notations of [11])

$$T = T_0 + \left( \dot{N}^0 \right)^{-1} \int_{\varphi_0(T_0)}^{\varphi(T)} dt R(t),$$

$$\varphi^i(\varphi_0) = \varphi^i(T_0) + \nu^i \int_{\varphi_0(T_0)}^{\varphi(T)} dt R^{-1}(t),$$

where $\nu^i = \dot{N}^i / \dot{N}^0$ and $\dot{N}^0(T_0)$ are the Cauchy initial data having the dimensionality $L$. In terms of these initial data the constraint (23') has the form

$$\dot{N}^M \dot{N}_M = 0, \quad \nu^i \nu^j = 1$$

and the tangent vectors $\varphi^M_T$ are

$$\varphi^0_T = \dot{N}^0 R^{-1}(\varphi_0), \quad \varphi^i_T = \dot{N}^i R^{-2}(\varphi_0)$$

Now we may proceed to the solution of Eqs.(35) and the constraints (35' – 35'').

The substitution of the velocity components into the constraints (35'') appears to resolve them

$$\psi^0 = R(\nu^i \psi^i)$$

To analyse the second constraint (35') we may use the expressions for non-zero components of $\Gamma^M_{PQ}(\varphi)$ for the F-R-W metric (42)

$$\Gamma^0_{ik}(\varphi) = \frac{1}{2} \delta_{ik} \partial_0 R^2,$$

where $\partial_0 \equiv \partial / \partial \varphi^0$. The covariant derivatives (28) for the F-R-W metric are given by

$$\mathcal{D}_T V^0 = V^0_T + \dot{N}^i(R\partial_0 R) V^i,$$

$$\mathcal{D}_T V^i = V^i_T + \dot{N}^0(R^{-2} \partial_0 R) V^i + \dot{N}^i(R^{-3} \partial_0 R) V^0$$

After the substitution of (48) into the constraint (35') we find that the latter takes the form

$$\mathcal{D}_T \psi^0 = R(\nu^i \mathcal{D}_T \psi^i)$$

and is identically satisfied by the solution (46).

Now let us leave the constraints and consider Eqs.(35). Using the definitions (47-48) and the solutions (45) we find that Eqs.(35) are transformed into the following ones

$$\psi^0_{TT} - \psi^0 + 2 R^{-1} \partial_0 R \dot{N}^i \psi^i_T + \dot{N}^i \dot{N}_i \left[ \partial_0^2 R + R^{-1} (\partial_0 R)^2 \right] R^{-3} \psi^0 = 0,$$

$$\psi^i_{TT} - \psi^i + 2 \dot{N}^0 R^{-2} \partial_0 R \psi^i_T + 2 \dot{N}^i R^{-3} \partial_0 R \psi^0_T +$$

$$+ 2 \dot{N}_0 R \dot{N}^i \left[ \partial_0^2 R - R^{-1} (\partial_0 R)^2 \right] R^{-4} \psi^0 = 0$$
After using the constraints (44) and (46) the equations (50-51) are transformed into the separated ones for the time $\psi^0$ and space $\psi^i$ components of $\psi^M$

$$\psi^{0}_{TT} - \psi'^0 + a\psi^0_{T} + b\psi^0 = 0,$$  \hspace{1cm}  (52)

$$\psi^{i}_{TT} - \psi'^i + a_{ij}\psi^j_{T} + b_{ij}\psi^j = 0,$$  \hspace{1cm}  (53)

where

$$a = 2\mathcal{N}^0 R^{-2}\partial_0 R, \quad b = \left(\mathcal{N}^0 R\right)^2 \partial_0 (R^{-1}\partial_0 R),$$

$$a_{ij} = 2\mathcal{N}^0 R^{-2}\partial_0 R(\delta_{ij} + \nu_i\nu_j),$$

$$b_{ij} = 2(\mathcal{N}^0)^2 (R^{-3}\partial_0^2 R)\nu_i\nu_j.$$  \hspace{1cm} (54)

Note that Eqs.(50-51) have been obtained without use of the constraint (46). Later it will be convenient to apply Eqs.(53) in some other form similar to that of (51)

$$\psi^{i}_{TT} - \psi'^i + a_i\psi^i_{T} + 2\nu_i\mathcal{N}^0 R^{-2}(R^{-1}\partial_0 R\psi^0)_{T} = 0$$  \hspace{1cm} (51')

Of course, Eqs.(52-53) are not independent in view of the usage of (46). It can be verified that Eqs.(53) are reduced to Eq.(52) after their multiplication by $\nu_i$, summing up and use of the constraint (46). This result is a justification test of the general conclusion concerning the consistency of the equations and constraints of the perturbative scheme.

Now we are ready to consider the most interesting case of the de Sitter space inflationary metrics with the conformal factor $R$ (42) equal to

$$R = e^{H\phi^0(T)}$$  \hspace{1cm} (55)

For this case the solution (43) acquires the form

$$\phi^0(T) = H^{-1}\ln [N^0(T + \Lambda)],$$

$$\phi^i(T) = q^i_0 - \nu^i H^{-1}e^{-H\phi^0(T)} = q^i_0 - \frac{\nu^i}{HN^0(T + \Lambda)},$$  \hspace{1cm} (56)

where the dimensionless constants $Hq^i_0$ and $N^0$ equal to

$$q^i_0 = \phi^i(T_0) + \nu^i H^{-1} e^{-H\phi^0(T_0)}, \quad N^0 = \mathcal{N}^0 H$$  \hspace{1cm} (57)

Eq.(56) shows that an asymptotic scale for the worldsheet time $T$ ($T \gg 1/\varepsilon$), where the considered perturbative scheme correctly works, corresponds to the asymptotic scale in the cosmic time $\phi^0$. The substitution of $R$ corresponding to the solutions (56)

$$R = N^0(T + \Lambda)$$  \hspace{1cm} (58)

into Eq.(54) gives $b = 0$, and Eq.(52) is reduced to

$$\psi^{0}_{TT} - \psi'^0 + 2(T + \Lambda)^{-1}\psi^0_{T} = 0$$  \hspace{1cm} (59)
The same substitution transforms Eqs.(53) into the equations

$$
\psi_{TT} - \psi'' + 2(T + \Lambda)^{-1}\psi_T = -2(N^0)^{-1}\nu'(T + \Lambda)^{-2}\psi^0_T
$$

(60)

We will see that after shifting $\psi^i$ into $\Theta^i$

$$
\Theta^i = \psi^i - \nu^i R^{-1}\psi^0 = \psi^i - \nu^i \left( N^0 (T + \Lambda) \right)^{-1}\psi^0
$$

(61)

Eqs.(60) are transformed into homogenous wave equations

$$
\Theta_{TT} - \Theta'' + 2(T + \Lambda)^{-1}\Theta_T = 0,
$$

(62)

which coincide with the Eq.(59) for $\psi^0$. The proof is based on an interesting property of the linear differential operator $\hat{L}$ in (59) and (60)

$$
\hat{L} = \partial_T^2 - \partial^2_\sigma + 2(T + \Lambda)^{-1}\partial_T,
$$

(63)

which is expressed by a commutator relation

$$
[\hat{L}, (T + \Lambda)^{-1}] = -2(T + \Lambda)^{-2}\partial_T
$$

(64)

Following from Eq.(64) is the relation

$$
\hat{L} \left( (T + \Lambda)^{-1}\psi^0 \right) = -2(T + \Lambda)^{-2}\psi^0_T,
$$

(65)

which proves the validity of Eqs.(62). As concerns the constraint (46), it takes a simple form

$$
\nu^i \Theta^i = 0
$$

(66)

Thus, we conclude that the perturbative string dynamics in the first approximation on $\varepsilon$ in the de Sitter space is described by Eqs.(59,62) and the constraint (66).

To solve Eq.(62) take into account the periodicity condition with respect to $\sigma$

$$
\Theta^i(T, 0) = \Theta^i(T, 2\pi), \quad \psi^0(T, 0) = \psi^0(T, 2\pi)
$$

(67)

and expand $\Theta^i$ in a Fourier series

$$
\Theta^i = \sum_{n=-\infty}^{\infty} A_n^i(T)e^{in\sigma}
$$

(68)

After the substitution of the expansion (68) into the Eq.(62) the latter transforms into the Bessel equation for the Fourier coefficients $a_n^i(T)$

$$
A_{n,TT}^i + \frac{2}{T + \Lambda} A_{n,T}^i + n^2 A_n^i = 0
$$

(69)

The general solution of Eqs.(69) for $n \neq 0$ is [23]

$$
A_n^i(T) = (T + \Lambda)^{-1/2} Z_{-1/2} (|n|(T + \Lambda)) = (T + \Lambda)^{-1/2} \left[ a_n^i J_{-1/2} (|n|(T + \Lambda)) + b_n^i Y_{-1/2} (|n|(T + \Lambda)) \right]
$$

(70)
where $J_{-1/2}(z)$ and $Y_{-1/2}(z)$ are the Bessel functions of the first and second type respectively

$$J_{-1/2}(z) = \left( \frac{2}{\pi z} \right)^{1/2} \cos z,
Y_{-1/2}(z) = J_{1/2}(z) = \left( \frac{2}{\pi z} \right)^{1/2} \sin z,$$

and $a^i_n$, $b^i_n$ are the initial data. For $n = 0$ the general solution of Eqs.(69) is

$$A^i_0 = b^i_0 + \frac{a^i_0}{T + \Lambda}$$

(72)

Then the general solution of Eqs.(69) will be the following

$$\Theta^i = \frac{1}{T + \Lambda} \left\{ a^i_0 + \sum_{n=-\infty \atop n \neq 0}^{\infty} \left( \frac{2}{|n|\pi} \right)^{1/2} \left[ a^i_n \cos |n|(T + \Lambda) + b^i_n \sin |n|(T + \Lambda) \right] e^{in\sigma} \right\}
+ b^i_0$$

(73)

Introducing new oscillator coefficients $\alpha^i_n$ and $\beta^i_n$

$$\alpha^i_n = (2\pi|n|)^{-1/2}(a^i_n + ib^i_n)e^{-i\Lambda},
\beta^i_n = (2\pi|n|)^{-1/2}(a^i_n - ib^i_n)e^{i\Lambda} \quad (n \neq 0),$$

$$\alpha^i_0 = a^i_0, \quad \alpha^i_n = \ast \alpha^i_n, \quad \beta^i_n = \ast \beta^i_n, \quad \beta^i_0 = 0,$$

we present the solution (73) in the form of independent left and right waves running along the closed string

$$\Theta^i(T, \sigma) = \frac{1}{T + \Lambda} \sum_{n=-\infty}^{\infty} \left[ \alpha^i_n e^{i\sigma(T-nT)} + \beta^i_n e^{i\sigma(T+nT)} \right] + b^i_0$$

(75)

The general solution of Eq.(59) has the same form

$$\psi^0(T, \sigma) = \frac{1}{T + \Lambda} \sum_{n=-\infty}^{\infty} \left[ \alpha^0_n e^{i\sigma(T-nT)} + \beta^0_n e^{i\sigma(T+nT)} \right] + b^0_0$$

(76)

The substitution of $\Theta^i$ (75) and $\psi^0$ (76) into the representation (61) and the constraint (66) leads to the solution for $\psi^i(T, \sigma)$

$$\psi^i(T, \sigma) = \frac{1}{T + \Lambda} \sum_{n=-\infty}^{\infty} \left[ \alpha^i_n e^{i\sigma(T-nT)} + \beta^i_n e^{i\sigma(T+nT)} \right] +$$

$$+ \frac{\nu^i}{N^0(T + \Lambda)^2} \sum_{n=-\infty}^{\infty} \left[ \alpha^0_n e^{i\sigma(T-nT)} + \beta^0_n e^{i\sigma(T+nT)} \right] + \frac{\nu^i}{N^0(T + \Lambda)} b^0_0 + b^i_0$$

(77)

together with the constraint for the oscillator coefficients $\alpha^i_n$ and $\beta^i_n$

$$\nu^i \alpha^i_n = \nu^i \beta^i_n = \nu^i b^i_0 = 0$$

(78')
Substituting the solution (55-56) and (76-77) into the expansion (20) we find

\[ x^0(T, \sigma) = (H^{-1}\ln N^0 + \varepsilon b_0^0) + \]

\[ + \left\{ H^{-1}\ln (T + \Lambda) + \frac{\varepsilon}{T + \Lambda} \sum_{n=\infty}^{\infty} \left( \alpha_n^0 e^{i(n-\sigma T)} + \beta_n^0 e^{i(n+\sigma T)} \right) \right\} + O(\varepsilon^2), \]

\[ x^i(T, \sigma) = (q_0^i + \varepsilon b_0^i) + \]

\[ + \frac{1}{N^0(T + \Lambda)} \left\{ \nu^i \left[ -H^{-1} + \varepsilon b_0^0 + \frac{\varepsilon}{N^0(T + \Lambda)} \sum_{n \neq 0} \left( \alpha_n^0 e^{i(n-\sigma T)} + \beta_n^0 e^{i(n+\sigma T)} \right) \right] + \]

\[ + \varepsilon \sum_{n \neq 0} \left( \alpha_n^i e^{in(\sigma - T)} + \beta_n^i e^{in(\sigma + T)} \right) \right\} + O(\varepsilon^2) \]

(78)

As follows from the representations (76) and (77), the perturbative corrections are connected with string oscillations in the directions orthogonal and tangent to the geodesic trajectory (56) of the zero approximation. The amplitudes of these oscillations are asymptotically small when \( T \gg 1 \) (or equivalently \( \tau \gg 1/\varepsilon \)) and the amplitudes of the longitudinal oscillations are essentially smaller than the amplitudes of the transversional oscillations. Therefore the former oscillations may be neglected.

At the considered large scale \( T \) the frequencies of perturbative oscillations coincide with the Nambu-Goto frequencies. At the original microscopic scale \( \tau \) these frequencies are rescaled by the parameter \( \varepsilon \) and become very small and equal to \( \varepsilon \), so we have

\[ \omega_n \bigg|_{T \text{ scale}} = n, \quad \omega_n \bigg|_{\tau \text{ scale}} = \varepsilon n \]

(79)

As a consequence, all these frequencies are stable at the considered large scale \( T \) or equivalently when \( H \gg 1/\sqrt{\alpha'} \) (7). In the above discussion we have already noted that the string instabilities discovered in [5] was a consequence of the formula

\[ \omega_n = \sqrt{n^2 - (\alpha'Hm)^2} \]

(80)

for the oscillator frequencies, where the constant \( m \) is a phenomenological mass parameter associated with the mass of the particle replacing the string in the zero approximation. We have shown here that in accordance with the variational principle based on the action \( S \) (1) the mass parameter \( m \) must be equal to zero, and this leads to the disappearance of these instabilities. It seems that the condition \( m = 0 \) points out that such a parameter may be absent in string dynamics at all. Indeed, it is known that a particle in de Sitter space has neither definite mass nor definite spin, but has a definite eigenvalues of two other Casimir operators. These eigenvalues are some combinations of usual mass and angular momentum.
5 Solution of the perturbative equations in Friedmann-Robertson-Walker universes

Here we shall study the perturbative equations (52-54) in the F-R-W universes with a power parametrization of the scalar factor \( R \)

\[ R = a(\varphi^0)^\alpha, \]  

(81)

where \( a \) is a metric constant with the dimensionality \( L^{-\alpha} \) and \( \alpha \) is an arbitrary parameter [3].

In this metric the solution (43) for the cosmic time \( \varphi^0(T) \) has the form

\[ \varphi^0(T) = A(T + \bar{\Lambda})^{1/1+\alpha}, \quad (\alpha \neq -1), \]  

(82)

where the constant \( A \) with the dimension \( L \) and the dimensionless constant \( \bar{\Lambda} \) are defined by the relations

\[ A = \left( (1 + \alpha) \bar{N}^0 a^{1/\alpha} \right)^{1/1+\alpha} a^{-1/\alpha}, \]

\[ \bar{\Lambda} = A^{-(1+\alpha)} \left( \varphi^0(T_0) \right)^{\alpha+1} - T_0 \]  

(83)

For the space world coordinate \( \varphi^i(T) \) the solution of Eq.(43) is

\[ \varphi^i \left( \varphi^0(T) \right) = q_0^i + \nu^i [(1 - \alpha)a]^{-1} \left( \varphi^0(T) \right)^{1/1+\alpha} \]  

(84)

for \( (\alpha \neq \pm 1) \). After the substitution of the representation (82) into Eq.(84) we find

\[ \varphi^i(T) = q_0^i + \nu^i B(T + \bar{\Lambda})^{1/1+\alpha}, \quad (\alpha \neq \pm 1), \]  

(85)

where the constants \( q_0^i \) and \( B \) are defined by

\[ q_0^i = \varphi^i(T_0) - \frac{\nu^i}{a(1 - \alpha)} \left( \varphi^0(T_0) \right)^{1-\alpha}, \quad B = \frac{A^{1-\alpha}}{a(1 - \alpha)} \]  

(86)

The special case \( \alpha = \pm 1 \) may be easy studied separately.

Now let us to study the perturbative equations (50 - 51'). The substitution of the solution (82-83) for \( R \)

\[ R = a A^\alpha(T + \bar{\Lambda})^{\alpha/1+\alpha} \quad (\alpha \neq -1) \]  

(87)

into the relations defining the coefficients \( a, b, a_{ij}, b_{ij} \) (54) gives the following expressions \( (\alpha \neq \pm 1) \)

\[ a = \frac{2\alpha}{1+\alpha} (T + \bar{\Lambda})^{-1}, \quad a_{ij} = \frac{2\alpha}{1+\alpha} (T + \bar{\Lambda})^{-1} (\delta_{ij} + \nu_i \nu_j) \]

(88)

\[ b = -\frac{\alpha}{(1+\alpha)^2} (T + \bar{\Lambda})^{-2}, \quad b_{ij} = \frac{2\alpha(\alpha - 1)}{(1+\alpha)^2} (T + \bar{\Lambda})^{-2} \nu_i \nu_j \]
Using (88) find that Eqs. (52,51′) can be written as
\[ \psi_0^{0,TT} - \psi_0^{0,\sigma\sigma} + \frac{2\alpha}{1 + \alpha} (T + \tilde{\Lambda})^{-1} \psi_0^{0,T} - \frac{\alpha}{(1 + \alpha)^2} (T + \tilde{\Lambda})^{-2} \psi_0^{0} = 0, \]  
(89)
\[ \psi_i^{i,TT} - \psi_i^{i,\sigma\sigma} + \frac{2\alpha}{1 + \alpha} (T + \tilde{\Lambda})^{-1} \psi_i^{i,T} + \nu^i(T + \tilde{\Lambda})^{-\frac{1+2\alpha}{1+\alpha}} r \psi_0^{0,T} - \nu^i(T + \tilde{\Lambda})^{-\frac{2+3\alpha}{1+\alpha}} s \psi_0^{0} = 0, \]  
(90)
where the constant coefficients \( r \) and \( s \) are defined as
\[ r = 2 \frac{\alpha}{(1 + \alpha)^{\frac{2+3\alpha}{1+\alpha}}} \left( a \cdot (N^0)^{\alpha} \right)^{-1/1+\alpha}, \]
\[ s = 2 \frac{\alpha}{(1 + \alpha)^{\frac{2+3\alpha}{1+\alpha}}} \left( a \cdot (N^0)^{\alpha} \right)^{-1/1+\alpha} = r/1 + \alpha \]
(91)
The coefficients \( r \) and \( s \) are dimensionless because they include the dimensional constants \( a \) and \( N^0 \) only in the dimensionless combination \( [a \cdot (N^0)^{\alpha}] \). Introducing a dimensionless constant \( \kappa \)
\[ \kappa = \left( (1 + \alpha) a^{1/\alpha} \tilde{N}^0 \right)^{\alpha/(\alpha+1)} \]
we can present the constraint (46) in the form
\[ \psi_0^{0} = \kappa(T + \tilde{\Lambda})^{\frac{\alpha}{\alpha+1}} \left( \nu^i \psi_i^{i} \right) \]
(92)
This constraint can be omitted now, because it was used for obtaining Eqs.(90). So if we substitute the constraint (92) in Eq.(89) then the latter transforms into a linear combination of Eqs.(90). In the limiting case when \( \alpha' \to \infty \) Eqs.(89-90) are reduced to the de Sitter equations (59) and (69), because the coefficients \( r \) and \( s \) (91) go to zero. These equations belong to the same class of the Bessel-like equations.

To solve Eq.(89) expand \( \psi_0^{0}(T, \sigma) \) in a Fourier series
\[ \psi_0^{0}(T, \sigma) = \tilde{A}_0^{0}(T) + \sum_{n\neq0} \tilde{A}_n^{0}(T) e^{i n \sigma}, \]
(93)
substitute the expansion (93) into Eq.(89) and get the equation for \( \tilde{A}_n^{0}(T) \)
\[ \tilde{A}_n^{0,TT} + \frac{2\alpha}{1 + \alpha} (T + \tilde{\Lambda})^{-1} \tilde{A}_n^{0,T} + \left[ n^2 - \frac{\alpha}{(1 + \alpha)^2} (T + \tilde{\Lambda})^{-2} \right] \tilde{A}_n^{0} = 0 \]
(94)
The general solution of Eq.(94) for the case \( n \neq 0 \) has the form (see [23])
\[ \tilde{A}_n^{0}(T) = (T + \tilde{\Lambda})^{-\frac{1}{2+\alpha}} Z_{-1/2} \left( |n|(T + \tilde{\Lambda}) \right) = \]
\[ = (T + \tilde{\Lambda})^{-\frac{1}{2+\alpha}} \left[ a_n^{0, J_{-1/2}} \left( |n|(T + \tilde{\Lambda}) \right) + b_n^{0, Y_{-1/2}} \left( |n|(T + \tilde{\Lambda}) \right) \right] \]
(95)
Respectively the general solution for \( \tilde{A}_0^{0}(T) \) corresponding the case \( n = 0 \) is
\[ \tilde{A}_0^{0}(T) = \tilde{a}_0^{0}(T + \tilde{\Lambda})^{-\frac{\alpha}{\alpha+1}} + \tilde{b}_0^{0}(T + \tilde{\Lambda})^{\frac{\alpha}{\alpha+1}} \]
(96)
The substitution of the solutions (95-96) into (93) gives the general solution of Eq.(89)

\[ \psi^0(T, \sigma) = (T + \tilde{\Lambda})^{\frac{2\alpha}{1+\alpha}} \sum_{n=-\infty}^{\infty} \left[ \tilde{a}^0_n e^{in(\sigma-T)} + \tilde{\beta}^0_n e^{in(\sigma+T)} \right] + \tilde{b}^0_0 (T + \tilde{\Lambda})^{\frac{1}{1+\alpha}} \]  

(97)

In the limiting case when \( \alpha' \to \infty \) the solution (97) reduces to the solution (76). The general solution (20) for the cosmic time coordinate \( x^0(T, \sigma) \) acquires the form

\[ x^0(T, \sigma) = \left[ \mathcal{A} + \varepsilon \tilde{b}^0_0 \right] (T + \tilde{\Lambda})^{\frac{1}{1+\alpha}} + \varepsilon (T + \tilde{\Lambda})^{-\frac{2\alpha}{1+\alpha}} \sum_{n=-\infty}^{\infty} \left[ \tilde{a}^0_n e^{in(\sigma-T)} + \tilde{\beta}^0_n e^{in(\sigma+T)} \right] + O(\varepsilon^2) \]

(98)

Having the solution (97) for \( \psi^0(T) \) we shall seek for the general solution of Eqs.(90) in the form of the Fourier series expansion

\[ \psi^i(T, \sigma) = \tilde{A}^i_0(T) + \sum_{n \neq 0} \tilde{A}^i_n(T) e^{in\sigma}, \]

(99)

Then the substitution of the expansions (99) and (93) into Eqs.(90) will give the equations for \( \tilde{A}^i_0(T) \) and \( \tilde{A}^i_n(T) \). We find that the equation for the zero mode \( \tilde{A}^i_0(T) \) turns out to be the following

\[ \dot{\tilde{A}}^i_0 + \frac{2\alpha}{1+\alpha} (T + \tilde{\Lambda})^{-1} \tilde{A}^i_0 = \nu^i r \tilde{a}^0_0 (T + \tilde{\Lambda})^{-\frac{2(1+\alpha)}{1+\alpha}} \]

(100)

After rewriting Eq.(100) in the form

\[ (T + \tilde{\Lambda})^{\frac{2\alpha}{1+\alpha}} \left[ (T + \tilde{\Lambda})^{\frac{2\alpha}{1+\alpha}} \tilde{A}^i_0 \right]_{,T} = \nu^i r \tilde{a}^0_0 (T + \tilde{\Lambda})^{-\frac{2(1+\alpha)}{1+\alpha}} \]

(101)

it is easily integrated, and its general solution is

\[ \tilde{A}^i_0 = \frac{1+\alpha}{2\alpha} \nu^i r \tilde{a}^0_0 (T + \tilde{\Lambda})^{\frac{2\alpha}{1+\alpha}} + \frac{1+\alpha}{1-\alpha} \tilde{C}^i_{01}(T + \tilde{\Lambda})^{\frac{1+\alpha}{1+\alpha}} + \tilde{C}^i_{02} \]

(102)

Similarly one can derive the equation for the n-th mode \( \tilde{A}^i_n(T) \)

\[ \dot{\tilde{A}}^i_n + \frac{2\alpha}{1+\alpha} (T + \tilde{\Lambda})^{-1} \tilde{A}^i_n = \nu^i r F_n, \]

(103)

where \( F_n \) is defined as

\[ F_n \equiv - (T + \tilde{\Lambda})^{-\frac{1+2\alpha}{1+\alpha}} \left[ \tilde{A}^0_{n,T} - \frac{\alpha}{1+\alpha} (T + \tilde{\Lambda})^{-1} \tilde{A}_n^0 \right] \]

(104)

The substitution of \( \tilde{A}_n^0 \) (95) into Eq.(104) allows to present \( F_n \) as

\[ F_n = \sqrt{\frac{2}{\pi |n|}} (T + \tilde{\Lambda})^{-\frac{2(1+\alpha)}{1+\alpha}} \{ \tilde{f}_1^0 \left[ \cos |n|(T + \tilde{\Lambda}) + |n|(T + \tilde{\Lambda}) \sin |n|(T + \tilde{\Lambda}) \right] + \tilde{f}_2^0 \left[ \sin |n|(T + \tilde{\Lambda}) - |n|(T + \tilde{\Lambda}) \cos |n|(T + \tilde{\Lambda}) \right] \}

(105)
The general solution of Eq.(103) is a sum of the general solution of the homogenous equation
\[
\tilde{B}^i_{n,T T} + \frac{2\alpha}{1 + \alpha}(T + \tilde{\Lambda})^{-1} B^i_{n,T} + n^2 \tilde{B}^i_{n} = 0
\]
and a particular solution of Eq.(103). The general solution of Eq.(106) is given by the expression
\[
\tilde{B}^i_{n} = (T + \tilde{\Lambda})^{\frac{1-\alpha}{2(1+\alpha)}} \left( |n|(T + \tilde{\Lambda}) \right) = \left( T + \tilde{\Lambda} \right)^{\frac{1-\alpha}{2(1+\alpha)}} \left[ b^i_{n} J_{\frac{1-\alpha}{2(1+\alpha)}} \left( |n|(T + \tilde{\Lambda}) \right) + \tilde{b}^i_{n} Y_{\frac{1-\alpha}{2(1+\alpha)}} \left( |n|(T + \tilde{\Lambda}) \right) \right],
\]
where \( J_{\frac{1-\alpha}{2(1+\alpha)}}(z) \) and \( Y_{\frac{1-\alpha}{2(1+\alpha)}}(z) \) are the Bessel functions of the first and second order respectively.

The general solution of Eqs.(103) is presented as a sum of the solution \( \tilde{B}^i_{n} \) (107) and a particular solution \( \tilde{H}^i_{n} \)
\[
\tilde{A}^i_{n} = \gamma_{n} \tilde{B}^i_{n} + \tilde{H}^i_{n}
\]
Eq.(103) with the right-hand side given by the expression (105) belongs to the set of exactly integrable inhomogenous Bessel-like equations. Therefore the particular solution \( \tilde{H}^i_{n} \) (108), depending on the parameter \( \alpha \), belongs to the set of one-parameter solutions discussed in [23]. That is why we do not dwell on the discussion of these particular solutions. Instead we shall show another way for the solution of Eqs.(90).

To this end notice that Eqs.(90) as well as Eqs.(51') preceding them, mix the space component \( \psi^i \) with the time component \( \psi^0 \) and its T-derivative. On the other hand, Eqs.(53) are equivalent to Eqs.(51') and contain only the space component \( \psi^i \). Thus, we can solve Eqs.(53) independently on \( \psi^0 \) and then use the constraint (46) (or (98)) for establishing a connection between the integration constants contained in the solutions for \( \psi^i \) and \( \psi^0 \).

To illustrate this possibility we consider the special case of initial data when the velocity \( \nu^i \) has only one non-zero component \( \nu^z \), i.e.
\[
\nu^i \equiv (\nu^x, \nu^y, \nu^z) \equiv (\nu^t, \nu^z) = (0, 0, 1)
\]
The general case of arbitrary initial data for \( \nu^i \) reduces to the case (109) after the fixation of the coordinate frame in the F-R-W space-time. Such a choice of the general covariant gauge is a correct operation in view of the general covariance of the perturbative scheme studied here. In the gauge (109) Eqs.(53) transform to the homogenous Bessel-like equations
\[
\psi^{t,T T} - \psi^{t,\sigma \sigma} + \frac{2\alpha}{1 + \alpha}(T + \tilde{\Lambda})^{-1} \psi^{t,\sigma} = 0,
\]
\[
\psi^{z,T T} - \psi^{z,\sigma \sigma} + \frac{4\alpha}{1 + \alpha}(T + \tilde{\Lambda})^{-1} \psi^{z,\sigma} = 0
\]
After the substitution of the \( \psi^{t} \) Fourier expansion
\[
\psi^{t}(T, \sigma) = \tilde{A}^{t}_{0}(T) + \sum_{n \neq 0} \tilde{A}^{t}_{n}(T)e^{in\sigma},
\]
into Eqs.(110) we get the solutions (102) and (107) for the Fourier components \( \tilde{A}^0(T) \) and \( \tilde{A}^i_n(T) \)
\[
\tilde{A}^0_n = \frac{1 + \alpha}{1 - \alpha} \tilde{C}_0^0(T + \tilde{\Lambda})^{\frac{1-\alpha}{1+\alpha}} + \tilde{C}_0^{02}, \tag{113}
\]
\[
\tilde{A}^i_n = (T + \tilde{\Lambda})^{\frac{1-\alpha}{2(1+\alpha)}} Z_n^{\frac{1-\alpha}{2(1+\alpha)}} \left( |n|(T + \tilde{\Lambda}) \right) = (T + \tilde{\Lambda})^{\frac{1-\alpha}{2(1+\alpha)}} \left[ \tilde{b}_n^{i*} J_{\frac{1-\alpha}{2(1+\alpha)}} \left( |n|(T + \tilde{\Lambda}) \right) + \tilde{b}_n^{i*} Y_{\frac{1-\alpha}{2(1+\alpha)}} \left( |n|(T + \tilde{\Lambda}) \right) \right], \tag{114}
\]
The substitution of the Fourier expansion of \( \psi^z \)
\[
\psi^z(T, \sigma) = \sum_n \tilde{A}^z_n(T) e^{in\sigma}, \tag{115}
\]
into Eq.(111) transforms the latter into the equation
\[
\tilde{A}^z_{n,TT} + \frac{4\alpha}{1 + \alpha} (T + \tilde{\Lambda})^{-1} \tilde{A}^z_{n,T} + \left[ n^2 - \frac{2\alpha(1 - \alpha)}{(1 + \alpha)^2} (T + \tilde{\Lambda})^{-2} \right] \tilde{A}^z_n = 0, \tag{116}
\]
which is similar to Eq.(94) for \( \psi^0 \). The general solution of Eq.(116) is
\[
\tilde{A}^z_0(T) = \tilde{a}^z_0(T + \tilde{\Lambda})^{\frac{2\alpha}{1+\alpha}} + \tilde{b}^z_0(T + \tilde{\Lambda})^{\frac{2\alpha}{1+\alpha}} \tag{117}
\]
for the zero mode of the expansion (115) and
\[
\tilde{A}^z_n(T) = (T + \tilde{\Lambda})^{\frac{2\alpha}{2(1+\alpha)}} Z_{-1/2} \left( |n|(T + \tilde{\Lambda}) \right) = (T + \tilde{\Lambda})^{\frac{2\alpha}{2(1+\alpha)}} \left[ \tilde{a}_n^z J_{-1/2} \left( |n|(T + \tilde{\Lambda}) \right) + \tilde{b}_n^z Y_{-1/2} \left( |n|(T + \tilde{\Lambda}) \right) \right] \tag{118}
\]
for the oscillatory modes.

Now let us return to the constraint (92) and substitute the solutions (113-114), (117-118) and (95-96) in this constraint. We find that the constraint (92) will be identically satisfied, if the integration constants are connected by the relations
\[
\tilde{a}^0_n = \kappa \tilde{a}_n^z, \quad \tilde{b}^0_n = \kappa \tilde{b}_n^z \tag{119}
\]
for all \( n \). Note that the constraint (92) does not restrict the integration constants contained in the solutions (113-114) which describe the string oscillations orthogonal to the initial velocity \( \nu^z \) (109). Taking into account the relations (119) we find that the solution for \( \psi^z \) can be presented in the form similar to the one given by (97).
\[
\psi^z(T, \sigma) = \kappa (T + \tilde{\Lambda})^{\frac{2\alpha}{1+\alpha}} \sum_{n=-\infty}^{\infty} \left[ \tilde{a}^0_n e^{i(n\sigma - T)} + \tilde{b}^0_n e^{i(n\sigma + T)} \right] + \kappa \tilde{b}^0_0(T + \tilde{\Lambda})^{\frac{2\alpha}{1+\alpha}} \tag{120}
\]
Substituting the solutions (85) and (120) into the perturbative expansions (20) we obtain the following solution for the world string coordinate \( x^z \)
\[
x^z(T, \sigma) = q^z_0 + \left[ B + \varepsilon \kappa \tilde{b}^0_0 \right] (T + \tilde{\Lambda})^{\frac{1-\alpha}{1+\alpha}} + \\
+ \varepsilon \kappa (T + \tilde{\Lambda})^{\frac{1-\alpha}{1+\alpha}} \sum_{n=-\infty}^{\infty} \left[ \tilde{a}^0_n e^{i(n\sigma - T)} + \tilde{b}^0_n e^{i(n\sigma + T)} \right] + O(\varepsilon^2) \tag{121}
\]
As follows from the representation (121), the zero mode of $\psi^z$ gives a correction to the translational movement whereas the oscillations of $z$ give rise to an additional oscillatory movement of the string coordinate $x^z$. Moreover in the asymptotic regime, when $T \gg 1$ (or equivalently $\tau \gg 1/\varepsilon$), the amplitude of the oscillation are smaller than the corresponding translation. This behaviour of $x^z$ is in agreement with the above-mentioned qualitative picture of the perturbative string dynamics in curved space-time.

Taking into account a weak tension also leads to the appearance of the translations (113) and the oscillations (114) in the transverse directions to the $\nu_i$ velocity (109). In the zero approximation the string motion in these transverse directions was absent. A general of the amplitudes of the string oscillations in $x, y$ and $z$ directions is thus asymptotic drop when the parameter $\alpha$ lies in the region $\alpha > 0$ or $\alpha < -1$.

At this point we stop our general discussion illustrating the applicability of the proposed realization for the perturbative approach. More detailed analysis of the perturbative string motions depends on the values of $\alpha$ and the initial data contained into the presented solutions. We shall return to this analysis in another paper.

6 Conclusion

We discuss here the problem of approximate solution of the string equations in curved space-time. A suitable representation for the string action with covariantly separated kinetic and potential terms is applied for this goal. Using the existence of a dimensional parameter in the metric of curved space a dimensionless parameter depending on the string tension is built. It is shown that the potential term in the string action can be treated as a perturbation for the case of smallness of this dimensionless parameter. At the same time this small parameter is appearing in the constraints and Euler-Lagrange variational equations and they can be reformulated into the chain of perturbative linear equations.

Established is the fact that the perturbative string equations for the de Sitter and the Friedmann-Robertson-Walker universes are reduced to the linear system of the exactly solvable modified Bessel equations. Moreover, the corresponding string constraints are transformed to the simple linear conditions for the Fourier coefficients in the expansions of the perturbative solutions. The proposed approximation selfconsistently describes the string dynamics on the scale of large values for the world-sheet time in the fixed gauge. The asymptotic non-trivial string motion has the character of damped oscillations with the amplitudes falling as a power of the slow worldsheet time. An interesting peculiarity of this perturbative description is the asymptotic stability of the string dynamics in the de Sitter space for a large Hubble constant.

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