CLOSURE OF STEINBERG FIBERS AND AFFINE DELIGNE-LUSZTIG VARIETIES

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Abstract. We discuss some connections between the closure $\bar{F}$ of a Steinberg fiber in the wonderful compactification of an adjoint group and the affine Deligne-Lusztig varieties $X_w(1)$ in the affine flag variety. Among other things, we describe the emptiness/nonemptiness pattern of $X_w(1)$ if the translation part of $w$ is quasi-regular. As a by-product, we give a new proof of the explicit description of $\bar{F}$, first obtained in [9].

Introduction

0.1. Let $k$ be an algebraic closure of a finite field $F_q$, $L = k((\epsilon))$ be the formal Laurent series and $\mathfrak{o} = k[[\epsilon]]$ be the ring of formal power series. Let $\sigma$ be an automorphism on $L$ defined by $\sigma(\sum a_n \epsilon^n) = \sum a_q \epsilon^n$.

Let $G$ be a simple algebraic group over $k$, split over $F_q$. We fix opposite Borel subgroups $B$ and $B^-$. Let $K = G(\mathfrak{o})$ be a maximal bounded subgroup of the loop group $G(L)$. Let $I$ (resp. $I'$) be the inverse image of $B^-$ (resp. $B$) under the projection map $K \mapsto G$ sending $\epsilon$ to 0. The automorphism $\sigma$ on $L$ induces an automorphism on $G(L)$, which we still denote by $\sigma$.

For any element $w$ in the extended affine Weyl group $\hat{W}$ of $G(L)$, the affine Deligne-Lusztig variety $X_w(1)$ is defined by

$$X_w(1) = \{ g \in G(L)/I; g^{-1}\sigma(g) \in I\hat{w}I \}.$$

0.2. One interesting question is to determine when $X_w(1)$ is empty. This is related to the reduction of Shimura varieties. See the survey papers by Rapoport [18] and Haines [8].

Reuman [19] gave a conjecture on the emptiness/nonemptiness pattern of $X_w(1)$ if $w$ lies in the lowest two-sided cell of $\hat{W}$ (i.e. the union of shrunken Weyl chambers). The emptiness is proved by Görtz, Haines, Kottwitz and Reuman in [7, Proposition 9.5.4] and the nonemptiness is proved in a joint work with Görtz in [5]. A partial result on the nonemptiness is also obtained by Beazley in [1].

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For the elements outside the lowest two-sided cell, a conjecture is given in [7, Conjecture 1.1.1] and it is proved in [7, Theorem 1.1.2] that the emptiness occurs as predicted. But it is unknown if $X_w(1)$ is nonempty for the remaining elements $w$.

**0.3.** In this paper, we will study the emptiness/nonemptiness pattern of $X_w(1)$ from a different point of view. We relate this problem to the problem of describing the closure $\bar{F}$ of Steinberg fiber in the wonderful compactification $\bar{G}$. The latter problem was solved in an earlier paper [9].

There is a specialization map from the loop group $G(L)$ to the wonderful compactification $\bar{G}$, introduced by Springer in [20]. We will show in the paper that this map gives nice correspondences between the decomposition of $G(L)$ into $I' \times I'$-orbits and the decomposition of $\bar{G}$ into $B \times B$-orbits, and between the decomposition of $G(L)$ into $K$-stable pieces and the decomposition of $\bar{G}$ into $G$-stable pieces. Moreover, this map connects the emptiness/nonemptiness pattern of $X_w(1)$ to the emptiness/nonemptiness of the intersection of the closure of $\bar{F}$ with certain $B \times B$-orbit in $\bar{G}$.

The main purpose of this paper is to prove that if the translation part of $w$ is quasi-regular (see §4.1 for the definition), then $X_w(1)$ is empty (resp. nonempty) if and only if the corresponding intersection in $\bar{G}$ is empty (resp. nonempty). This includes a large fraction of the cases outside the lowest two-sided cell and the generic cases in the lowest two-sided cell.

The precise statements are Proposition 3.4 (for the emptiness) and Theorem 4.1 (for the nonemptiness). Some special cases are stated in Corollary 5.3 without using the wonderful compactification.

It would be interesting to connect the emptiness/nonemptiness pattern here with [7, Conjecture 1.1.1].

**0.4.** We now review the content of this paper in more detail.

In section 1, we recall the definition of the wonderful compactification and Springer’s specialization map $G(L) \to \bar{G}$. The correspondence between the decomposition of $G(L)$ into $I' \times I'$-orbits and the decomposition of $\bar{G}$ into $B \times B$-orbits follows easily from the definition. In section 2, we discuss the correspondence between the decomposition of $G(L)$ into $K$-stable pieces and the decomposition of $\bar{G}$ into $G$-stable pieces. The closure relation of the $K$-stable pieces is also obtained. In section 3, we discuss some connections between the affine Deligne-Lusztig varieties and the closure of Steinberg fibers in $\bar{G}$. We also give a new proof of [9, Theorem 4.3]. In section 4, we prove our main result on the emptiness/nonemptiness pattern for affine Weyl group element with quasi-regular translation part. Further discussions on the intersection of the closure of $\bar{F}$ with certain $B \times B$-orbit in $\bar{G}$ will be discussed in section 5.
1. Some decompositions on $\tilde{G}$ and $G(L)$

1.1. Let $B$ be a Borel subgroup of $G$, $B^-$ be an opposite Borel subgroup and $T = B \cap B^-$. Let $X$ be the coroot lattice and $Y$ be the coweight lattice. We denote by $Y^+$ the set of dominant coweights and $X^+ = X \cap Y^+$. Let $(\alpha_i)_{i \in S}$ be the set of simple roots determined by $(B, T)$. Let $\Phi$ (resp. $\Phi^+$, $\Phi^-$) be the set of roots (resp. positive roots, negative roots). We denote by $W$ the Weyl group $N(T)/T$. For $i \in S$, we denote by $s_i$ the simple reflection corresponding to $i$. For $w \in W$, we denote by $\text{supp}(w)$ the set of simple reflections occurring in a reduced expression of $w$. For $w \in W$, we choose a representative $\tilde{w}$ in $N(T)$.

For any $J \subset S$, let $P_J \supset B$ be the standard parabolic subgroup corresponding to $J$ and $P_J^- \supset B^-$ be the opposite parabolic subgroup. Let $L_J = P_J \cap P_J^-$ be a Levi subgroup. Let $\Phi_J$ be the roots of $L_J$, i.e., the roots spanned by $\alpha_j$ for $j \in J$.

For any parabolic subgroup $P$, let $U_P$ be the unipotent radical of $P$. We simply write $U$ for $U_B$ and $U^-$ for $U_{B^-}$.

For any $J \subset S$, let $\rho^J$ be the dominant coweight with $\langle \rho^J, \alpha_i \rangle = \begin{cases} 1, & \text{if } i \in J \\ 0, & \text{if } i \notin J \end{cases}$.

We simply write $\rho^\lambda$ for $\rho^\lambda_S$.

1.2. Let $\tilde{W} = N(T(L))/(T(L) \cap I)$ be the extended affine Weyl group of $G(L)$. It is known that $\tilde{W} = W \ltimes Y = \{w\varepsilon^\chi; w \in W, \chi \in Y\}$. We call $\chi$ the translation part of $w\varepsilon^\chi$. Let $l : \tilde{W} \to \mathbb{N} \cup \{0\}$ be the length function. For $x = w\varepsilon^\chi \in \tilde{W}$, let $x = w\varepsilon^\chi$ be a representative in $N(T(L))$.

Let $W_a = W \ltimes X \subset \tilde{W}$ be the affine Weyl group. Set $\tilde{S} = S \cup \{0\}$ and $s_0 = \varepsilon^{\theta^\vee} s_\theta$, where $\theta$ is the largest positive root of $G$. Then $(W_a, \tilde{S})$ is a Coxeter system. For any $J \subset S$, let $W_J$ be the subgroup of $W_a$ generated by $J$ and $\tilde{W}^J$ be the set of minimal length coset representative of $\tilde{W}/W_J$. In the case where $J \subset S$, we write $W^J$ for $\tilde{W}^J \cap W$.

For $w \in \tilde{W}^J$, set $I(J, w) = \text{max}\{K \subset J; \forall i \in K, \exists j \in K \text{ such that } ws_i = s_j w\}$.

In particular, an element $w \in \tilde{W}^S$ is of the form $x\varepsilon^{-\lambda}$, where $\lambda \in Y^+$ and $x \in W^{I(\lambda)}$ and $I(S, x\varepsilon^{-\lambda}) = I(I(\lambda), x)$, here $I(\lambda) = \{i \in S; \langle \lambda, \alpha_i \rangle = 0\}$.

1.3. Let $\tilde{G}$ be the wonderful compactification of $G$ ([4], [25]). It is an irreducible, smooth projective $(G \times G)$-variety with finitely many $G \times G$-orbits $Z_J$ indexed by the subsets $J$ of $S$. We follow [21, section 2] for the description of $Z_J$. 

Let $\lambda$ be a dominant coweight. Since $\bar{G}$ is complete, \( \lambda(0) = \lim_{\epsilon \to 0} \lambda(\epsilon) \) is a well-defined point of $\bar{G}$. Moreover, if $\mu$ is another dominant coweight with $I(\lambda) = I(\mu)$, then $\lambda(0) = \mu(0)$. Set $h_J = \lambda(0)$ for dominant coweight $\lambda$ with $I(\lambda) = J$. This is a base point of $Z_J$. The map $(x, y) \mapsto (x, y) \cdot h_J$ induces an isomorphism from the quotient space $(G \times G) \times_{P_J \times P_J} L_J/Z(L_J)$ to $Z_J$.

1.4. Now we recall two partitions of $\bar{G}$.

For $J \subset S$, $x \in W^J$ and $y \in W$, set
\[
[J, x, y] = (B\dot{x}, B\dot{y}) \cdot h_J.
\]
Then $[J, x, y]$ is a $B \times B$-orbit in $\bar{G}$. By [2] and [20, Lemma 1.3],
\[\tag{1}
\bar{G} = \sqcup_{J \subset S, x \in W^J, y \in W} [J, x, y].
\]

Let $G_\Delta$ be the diagonal image of $G$ in $G \times G$. For $J \subset S$ and $w \in W^J$, set
\[
Z_{J,w} = G_\Delta \cdot [J, w, 1].
\]
We call $Z_{J,w}$ a $G$-stable piece of $\bar{G}$. By [16, 12.3] and [9, Prop 2.6],
\[\tag{2}
\bar{G} = \sqcup_{J \subset S} \sqcup_{w \in W^J} Z_{J,w}.
\]

The following properties will also be used in this paper.

(3) For $J \subset S$ and $w \in W^J$, $\overline{Z_{J,w}} = \sqcup_{K \subset J, w' \in W^J, w' \supseteq uwu^{-1}} Z_{K,w'}$ for some $u \in W_J Z_{K,w'}$. See [10, Theorem 4.5].

(4) Let $J \subset S$ and $w \in W^J$. If $w$ is a Coxeter element, then $G$ acts transitively on $Z_{J,w}$. See [23, Corollary 3.8].

1.5. Since $\bar{G}$ is complete, the inclusion $\circlearrowleft \rightarrow L$ induces a bijection from $G(\circlearrowleft)$ to $G(L)$. Now the specialization $\epsilon \mapsto 0$ defines a map $s : G(L) \rightarrow \bar{G}$. In particular, for any $g \in G(L)$, $s(g) \in \bar{G}$. This is the specialization map introduced in [22, 2.1]. In particular, we have that $s(K) = G$ and $s(I^\prime) = B$.

Notice that any element in $\bar{W}$ can be written in a unique way as $xe^\lambda y^{-1}$, where $\lambda \in Y^+$, $x \in W^{I(\lambda)}$ and $y \in W$. We have the following decompositions on $G(L)$,
\[\tag{1}
G(L) = \sqcup_{\lambda \in Y^+} K e^\lambda K = \sqcup_{\lambda \in Y^+, x \in W^{I(\lambda)}, y \in W} I^\prime \dot{x} e^\lambda \dot{y}^{-1} I^\prime.
\]

We have that
\[
s(K e^\lambda K) = (s(K), s(K)) \cdot s(e^\lambda) = (G, G) \cdot h_{I(\lambda)} = Z_{I(\lambda)},
\]
\[
s(I^\prime \dot{x} e^\lambda \dot{y}^{-1} I^\prime) = (s(I^\prime) \dot{x}, s(I^\prime) \dot{y}) \cdot h_{I(\lambda)} = [I(\lambda), x, y].
\]

Thus by the decomposition (1) and the decomposition 1.4 (1), we have that

(2) For any $J \subset S$, $s^{-1}(Z_J) = \sqcup_{\lambda \in Y^+, I(\lambda) = J} K e^\lambda K$.
(3) For any $J \subset S, x \in W^J, y \in W$, $s^{-1}([J, x, y]) = \sqcup_{\lambda \in Y^+, I(\lambda) = J} I^\prime \dot{x} e^\lambda \dot{y}^{-1} I^\prime$. 
2. \textit{K}-stable pieces in $G(L)$ and $G$-stable pieces in $\tilde{G}$

2.1. An analogue of $G$-stable pieces in the affine case is introduced in [17]. For any $w \in \tilde{W}^S$, set
\[ K_w = K \cdot I \hat{w} I, \quad K_{w, \sigma} = K \cdot \sigma I \hat{w} I, \]
where $\cdot$ is the usual conjugation action of $G(L)$, $g \cdot g' = gg'g^{-1}$ and $\cdot \sigma$ is the $\sigma$-twisted conjugation action of $G(L)$, $g \cdot \sigma g' = gg'(g)^{-1}$. We call $K_w$ a $K$-stable piece of $G(L)$ and $K_{w, \sigma}$ a $\sigma$-twisted $K$-stable piece of $G(L)$.

This definition is different from the one in [17]. However, one can show in the same way as in [9, Prop 2.6] that the two definitions are equivalent. Since the equivalence of the two definition is not used in this paper, we skip the details.

2.2. We first recall some properties about “partial conjugation action” of $\tilde{W}$. Although the results were proved for affine Weyl groups in [11], it is easy to see that they also hold for extended affine Weyl groups.

Let $J \subset \tilde{S}$. We consider the conjugation action of $\tilde{W}$ on $\tilde{W}$, $x \cdot y = xyx^{-1}$ for $x \in \tilde{W}$ and $y \in \tilde{W}$. For $w \in \tilde{W}^S$, set
\[ [w] = W \cdot (wW_{I(S,w)}) = W \cdot (W_{I(S,w)}w). \]

By [11, Corollary 2.6], $\tilde{W} = \bigsqcup_{w \in \tilde{W}^S} [w]$.

Given $w, w' \in \tilde{W}$ and $i \in S$, we write $w \overset{s_i}{\rightarrow} w'$ if $w' = s_iws_i$ and $l(w') \leq l(w)$. If $w = w_0, w_1, \ldots, w_n = w'$ is a sequence of elements in $\tilde{W}$ such that for all $k$, we have $w_{k-1} \overset{s_j}{\rightarrow} w_k$ for some $j \in S$, then we write $w \rightarrow_S w'$. We write $w \approx_S w'$ if $w \rightarrow_S w'$ and $w' \rightarrow_S w$.

By [11, Proposition 3.4], we have the following properties:

1. (1) for any $w \in \tilde{W}$, there exists a minimal length element $w' \in \tilde{W} \cdot w$ such that $w \rightarrow_S w'$. Moreover, we may take $w'$ to be an element of the form $vw_1$ with $w_1 \in \tilde{W}^S$ and $v \in W_{I(S,w)}$.

2. (2) Let $O$ be a $W$-orbit of $\tilde{W}$ that contains $w \in \tilde{W}^S$. Then $w' \approx_S w$ for any minimal length element $w' \in O$.

2.3. By [11, Corollary 4.5], for any $W$-orbit $O$ of $\tilde{W}$ and $v \in O$, the following conditions are equivalent:

1. (1) $v$ is a minimal element in $O$ with respect to the restriction to $O$ of the Bruhat order on $\tilde{W}$.

2. (2) $v$ is an element of minimal length in $O$.

We denote by $O_{\min}$ the set of elements in $O$ satisfy the above conditions.

As in [11, 4.7], we have a natural partial order $\leq_S$ on $\tilde{W}^S$ defined as follows:

Let $w, w' \in \tilde{W}^S$. Then $w \leq_S w'$ if for some (or equivalently, any) $v' \in (W \cdot w')_{\min}$, there exists $v \in (W \cdot w)_{\min}$ such that $v \leq v'$. 


In general, for \( w \in \tilde{W}^S \) and \( w' \in \tilde{W} \), we write \( w \leq_S w' \) if there exists \( v \in (\tilde{W} \cdot w)_{\min} \) such that \( v \leq w' \). By [11, Lemma 4.4], if \( w'' \xrightarrow{S} w' \) and \( w \leq_S w' \), then \( w \leq_S w'' \).

The main purpose of this section is to prove the following correspondence between \( K \)-stable pieces in \( G(L) \) and \( G \)-stable pieces in \( \tilde{G} \).

**Theorem 2.1.** Let \( * \) be the involution on \( W \) defined by \( w^* = w_0 w w_0 \), where \( w_0 \) is the longest element in \( W \). Then

1. For any \( \lambda \in Y^+ \) and \( x \in W^I(\lambda) \), \( s(K_{x_\lambda}) = Z_{I,-w_0(\lambda), x^*} \).
2. For any \( J \subset S \) and \( x \in W^J \), \( s^{-1}(Z_{J,x}) = \bigcup_{\lambda \in Y^+, I(\lambda) = -w_0} K_{x^*, x^*} \).

(1) We have that

\[
s(K_{x_\lambda}) = s(K \cdot I^* x_{\lambda} I) = s(K \cdot \tilde{w}_0 I \tilde{x}_0 \tilde{w}_0 e^{-w_0 \lambda} \tilde{w}_0 I \tilde{x}_0^{-1})
= s(K \cdot I^* x_{\lambda}^* e^{-w_0 \lambda} I') = s(K) \cdot s(I') \tilde{x}_0 s(e^{-w_0 \lambda}) s(I')
= G_{I'}(\tilde{B} \tilde{x}_0, B) \cdot h_{I,-w_0(\lambda)} = Z_{I,-w_0(\lambda), x^*}.
\]

The proof of part (2) will be given in \( \S 2.4 \).

**Lemma 2.2.** Let \( w, w' \in \tilde{W} \) and \( i \in S \) with \( w' = s_i w s_i \). Then

1. If \( l(w) = l(w') \), then \( K \cdot I \tilde{w} I = K \cdot I \tilde{w}' I \).
2. If \( l(w') < l(w) \), then \( K \cdot I \tilde{w} I = K \cdot I \tilde{w}' I \cup K \cdot I s_i \tilde{w}' I \)

If \( w = w' \), then the statement is obvious. Now suppose that \( w \neq w' \).

By [3, Lemma 1.6.4], \( l(w') \leq l(w) \) implies that \( s_i w < w \) or \( ws_i < w \).

If \( s_i w < w \), then

\[
K \cdot I \tilde{w} I = K \cdot I \tilde{s}_i I \tilde{s}_i \tilde{w} I = K \cdot I \tilde{s}_i \tilde{w} I \tilde{s}_i I = \begin{cases} K \cdot I \tilde{w}' I, & \text{if } l(w') = l(w) \\ K \cdot I \tilde{w}' I \cup K \cdot I \tilde{s}_i \tilde{w}' I, & \text{otherwise} \end{cases}
\]

If \( s_i w > w \) and \( ws_i < w \), then \( l(w') = l(w) \) and \( K \cdot I \tilde{w} I = K \cdot I \tilde{w}' I \).

**Lemma 2.3.** Let \( w, w' \in \tilde{W} \). Then

1. If \( w \xrightarrow{S} w' \), then \( K \cdot I \tilde{w} I \subset K \cdot I \tilde{w}' I \cup \cup_{w \in W_{W, d(v) < l(w)}} K \cdot I \tilde{x} I \).
2. If \( w \approx_S w' \), then \( K \cdot I \tilde{w} I = K \cdot I \tilde{w}' I \).

By definition, there exists a finite sequence \( w = w_0 \xrightarrow{i_1} w_1 \xrightarrow{i_2} \ldots \xrightarrow{i_m} w_n = w' \), where \( i_j \in S \) for all \( j \). We prove the lemma by induction on \( m \).

For \( m = 0 \) this is clear. Now assume that \( m > 0 \) and the statements hold for \( m - 1 \). By the previous lemma, \( K \cdot I \tilde{w} I \subset K \cdot I \tilde{w}_1 I \cup \cup_{x \in W_{W, d(x) < l(w)}} K \cdot I \tilde{x} I \). Notice that \( l(w_1) \leq l(w) \). By induction hypothesis, \( K \cdot I \tilde{w}_1 I \subset K \cdot I \tilde{w}' I \cup \cup_{x \in W_{W, d(x) < l(w)}} K \cdot I \tilde{x} I \). Hence \( K \cdot I \tilde{w} I \subset K \cdot I \tilde{w}' I \cup \cup_{x \in W_{W, d(x) < l(w)}} K \cdot I \tilde{x} I \).

If moreover, \( w \approx_S w' \), then \( l(w_1) = l(w) \) and \( w_1 \approx_S w' \). By induction hypothesis, \( K \cdot I \tilde{w} I = K \cdot I \tilde{w}_1 I = K \cdot I \tilde{w}' I \).
Lemma 2.4. Let \( w \in \hat{W}^S \) and \( v \in W_{I(S,w)} \). Then \( K \cdot I\hat{w}I \subset K_w \).

Set \( J = I(S,w) \). Notice that \( I = (B^- \cap L_J)I_J \), where \( I_J \) is the inverse image of \( U_{I_J} \) under the map \( K \to G \). Then \( I_J \) is normal in \( I \) and \( \hat{w}(B^- \cap L_J)\hat{w}^{-1} = B^- \cap L_J \). We have that
\[
I\hat{w}I = I_J(B^- \cap L_J)\hat{w}(B^- \cap L_J)I_J = I_J(B^- \cap L_J)\hat{w}(B^- \cap L_J)\hat{w}I_J \\
\subset I_JL_J\hat{w}I_J.
\]
The map \( l \mapsto \hat{w}l\hat{w}^{-1} \) is an automorphism on \( L_J \) and \( \hat{w}(B^- \cap L_J)\hat{w}^{-1} = B^- \cap L_J \). By [24, Lemma 7.3], \( L_J\hat{w} = \{ll'\hat{w}^{-1}; l \in L_J, l' \in B^- \cap L_J \} \). So
\[
L_J \cdot I\hat{w}I_J = \{II_J(B^- \cap L_J)\hat{w}I_Jl^{-1}; l \in L_J \}
= \{I_Jl(B^- \cap L_J)\hat{w}^{-1}I_J; l \in L_J \} = I_JL_J\hat{w}I_J
\]
and \( K \cdot I\hat{w}I \subset K \cdot L_JI_J\hat{w}I_J = K \cdot I\hat{w}I_J \subset K \cdot I\hat{w}I = K_w \). \( \square \)

Proposition 2.5. Let \( w \in \hat{W} \). Then
\[
\overline{K \cdot \hat{w}I} = K \cdot \overline{\hat{w}I} = \bigcup_{w' \in W^S, w' \leq S \leq w} K_{w'}.
\]

Define the action of \( I \) on \( K \times G(L) \) by \( i \cdot kg = (ki^{-1},igi^{-1}) \). Let \( K \times_I G(L) \) be the quotient space. Then the map \( K \times G(L) \to G(L), (k,g) \mapsto kgk^{-1} \) induces a map \( f : K \times_I G(L) \to G(L) \). Notice that each fiber is isomorphic to \( K/I \cong G/B \). Thus \( K \cdot \overline{\hat{w}I} = f(K \times_I \overline{\hat{w}I}) \) is closed in \( G(L) \). Hence \( \overline{K \cdot \hat{w}I} = K \cdot \overline{\hat{w}I} \).

If \( w' \in W^S \) with \( w' \leq S \leq w \), then there exists an element \( w'' \in (W \cdot w')_{\min} \) such that \( w'' \leq w \). By 2.2 (2), \( w'' \approx w' \). So by Lemma 2.3, \( K_{w'} = K \cdot Iw''I \subset K \cdot \overline{\hat{w}I} \). Now we prove by induction on \( l(w) \) that
\[
(a) \ K \cdot I\hat{w}I \subset \bigcup_{w' \in W^S, w' \leq S \leq w} K_{w'}.
\]

If \( w \in (W \cdot w)_{\min} \), then by 2.2 (1), \( w \approx vw_1 \) for some \( w_1 \in \hat{W}^S \) and \( v \in W_{I(S,w_1)} \). Then \( w_1 \leq S \leq w \). By Lemma 2.4, \( K \cdot I\hat{w}I \subset K_{w_1} \). If \( w \notin (W \cdot w)_{\min} \), then by 2.2 (1), there exists \( w_1 \approx w \) and \( i \in S \) with \( s_iw_1s_i \mapsto s_1w_1s_i \) and \( l(s_iw_1s_i) < l(w) \). By Lemma 2.2 and Lemma 2.3, \( K \cdot I\hat{w}I = K \cdot I\hat{w}_1I \subset K \cdot I\hat{w}_1I \cup K \cdot I\hat{w}_1I \). Since \( l(s_iw_1s_i) < l(w) \), we have that \( s_iw_1s_i < s_1w_1 < w_1 \). Thus for any \( w' \in W^S \), if \( w' \leq S \leq s_iw_1s_i \) or \( w' \leq S \leq s_1w_1 \), then \( w' \leq S \leq w_1 \). (a) follows from induction hypothesis.

Now \( K \cdot \overline{\hat{w}I} = \bigcup_{x \leq w} K \cdot I\hat{x}I \subset \bigcup_{w' \in W^S, w' \leq S \leq x} K_{w'} = \bigcup_{w' \in W^S, w' \leq S \leq x} K_{w'} \).

The proposition is proved.

Proposition 2.6. We have that \( G(L) = \bigcup_{w \in \hat{W}^S} K_w \).

Remark. This is essentially contained in [17, 1.4]. Here we give a different proof.

By Proposition 2.5,
\[
G(L) = \bigcup_{x \leq w} K \cdot I\hat{x}I = \bigcup_{x \leq w} K \cdot I\hat{x}I = \bigcup_{x \leq w} K \cdot \overline{\hat{x}I} \\
= \bigcup_{x \leq w} \bigcup_{w' \in W^S, w' \leq S \leq x} K_{w'} = \bigcup_{w \in \hat{W}^S} K_w.
\]
Let \( w, w' \in \tilde{W}^S \). Then \( w = x e^{-\lambda} \) and \( w' = x' e^{-\lambda'} \) for some \( \lambda, \lambda' \in Y^+ \) and \( x \in W^I(\lambda), x' \in W^I(\lambda') \). Suppose that \( K_w \cap K_{w'} \neq \emptyset \). Notice that \( K_w \subset K e^{-\lambda} K \) and \( K_{w'} \subset K e^{-\lambda'} K \). By the decomposition \( G(L) = \bigsqcup_{\mu \in Y^+} K e^{-\mu} K \), we have that \( \lambda = \lambda' \). We also have that \( s(K_w) \cap s(K_{w'}) \neq \emptyset \). Since \( s(K_w) = Z_{-w_0 I(\lambda)} \cdot x \cdot s(\lambda) \) and \( s(K_{w'}) = Z_{-w_0 I(\lambda') \cdot x' \cdot s(\lambda')} \), by the \( G \)-stable-piece decomposition 1.4 (2), we must have that \( x = x' \). Thus \( w = w' \). The Proposition is proved. \( \square \)

**Corollary 2.7.** Let \( w \in \tilde{W}^S \). Then \( \overline{K_w} = \bigsqcup_{w' \in \tilde{W}^S, w' \in s_w K_w} \).

### 2.4. Proof of Theorem 2.1 (2).

By part (1), \( K w e^{-\lambda} \subset s^{-1}(Z_{J,w}) \) for any \( \lambda \in Y^+ \) with \( l(\lambda) = -w_0 J \). Now let \( g \in s^{-1}(Z_{J,w}) \), then by Proposition 2.6, \( g \in K_w \) for some \( w \in \tilde{W}^S \). Again by part (1), \( s(K_w) \) is a \( G \)-stable piece in \( G \). Hence \( s(K_w) = Z_{J,w} \) and \( w \) must be of the form \( x e^{-\lambda} \) for any \( \lambda \in Y^+ \) with \( l(\lambda) = -w_0 J \).

#### 2.5. All the above results remain valid for the \( \sigma \)-twisted \( K \)-stable pieces of \( G(L) \). In fact, we have stronger result below for \( K_{w,\sigma} \) than Lemma 2.4 for \( K_w \).

**Lemma 2.8.** Let \( w \in \tilde{W}^S \) and \( v \in W_{I(S,w)} \). Then \( K \cdot \sigma I \hat{w} \hat{l} = K_{w,\sigma} \).

Set \( J = I(S,w) \). By the proof of Lemma 2.4, \( I \hat{w} \hat{l} = I_J(B^− \cap L_J) \hat{w} \hat{l} \cap L_J) \hat{w} \hat{l} \). By Lang’s theorem, \( L_J = \{ i \hat{w} \hat{l} \sigma(l)^{-1} \hat{w} \hat{l} ; l \in L_J \} \)
and \( L_J = \{ i \hat{w} \hat{l} \sigma(l)^{-1} \} \). Therefore \( L_J \cdot \sigma i \hat{w} \hat{l} = L_J \cdot \sigma i \hat{w} \hat{l} \cap L_J \hat{w} \hat{l} \). Taking \( v = 1 \), then \( L_J \cdot \sigma i \hat{w} \hat{l} = L_J \cdot \sigma i \hat{w} \hat{l} \).

Hence \( L_J \cdot \sigma i \hat{w} \hat{l} = L_J \cdot \sigma i \hat{w} \hat{l} \) and \( K \cdot \sigma i \hat{w} \hat{l} = K_{w,\sigma} \).

**Lemma 2.9.** Let \( w \in \tilde{W} \). Then \( K \cdot \sigma i \hat{w} \hat{l} \) is a union of \( \sigma \)-twisted \( K \)-stable pieces.

We prove by induction on \( l(w) \).

If \( w \in (W \cdot w)_{\min} \), then by 2.2 (1), \( w \approx v w_1 \) for some \( w_1 \in \tilde{W}^S \) and \( v \in W_{I(S,w_1)} \). Then \( K \cdot \sigma i \hat{w} \hat{l} = K \cdot \sigma i \hat{w} \hat{l} = K_{w_1,\sigma} \) is a \( \sigma \)-twisted \( K \)-stable piece. If \( w \notin (W \cdot w)_{\min} \), then by 2.2 (1), there exists \( w_1 \approx w \) and \( i \in S \) with \( w_1 \xrightarrow{i} s_i w_1 s_i \) and \( l(s_i w_1 s_i) < l(w) \). Then \( K \cdot \sigma i \hat{w} \hat{l} = K \cdot \sigma i \hat{w} \hat{l} = K \cdot \sigma i \hat{w} \hat{l} \cup K \cdot \sigma i \hat{w} \hat{l} \cup K \cdot \sigma i \hat{w} \hat{l} \). Since \( l(s_i w_1 s_i) l(s_i w_1 s_i) < l(w) \), \( K \cdot \sigma i \hat{w} \hat{l} \) and \( K \cdot \sigma i \hat{w} \hat{l} \) are unions of \( \sigma \)-twisted \( K \)-stable pieces. Hence \( K \cdot \sigma i \hat{w} \hat{l} \) is a union of \( \sigma \)-twisted \( K \)-stable pieces.

**2.6.** By the same argument as in [12, Corollary 2.6], the subset \( G_\sigma(B \hat{x}, B) \cdot h_J \subset G \) is a single \( G_\sigma \)-orbit, here \( G_\sigma = \{(g, \sigma(g)) ; g \in G \} \), \( J \subset S \) and \( x \in W^J \). Thus using the specialization map \( s : G(L) \rightarrow G \), one can show that \( K \cdot \sigma i \hat{w} \hat{l} \) is a single orbit of \( K_\sigma(U_K \times U_K) \) for any \( w' \in \tilde{W}^S \), here \( K_\sigma = \{(g, \sigma(g)) ; g \in K \} \subset K \times K \) and \( U_K \) is the inverse image of \( 1 \in G \) under the projection map \( K \rightarrow G(L) \) sending \( \epsilon \) to 0. This gives another proof of the above Lemma. We omit the details.
3. ADLV AND THE CLOSURE OF STEINBERG FIBERS IN $\bar{G}$

We first discuss some equivalence conditions for the nonemptiness of an affine Deligne-Lusztig variety $X_w(1) = \{g \in G(L)/I; g^{-1}\sigma(g) \in I\dot{w}I\}$.

**Proposition 3.1.** Let $w \in \tilde{W}$ and $t \in I \cap T(L)$. Then the following conditions are equivalent:

1. $X_w(1) \neq \emptyset$;
2. $U(L) \cap K \cdot I\dot{w}I \neq \emptyset$;
3. $tU(L) \cap K \cdot I\dot{w}I \neq \emptyset$;
4. $\dot{x}^{-1}U(L)\dot{x} \cap I\dot{w}I \neq \emptyset$ for some $x \in W$.

**Remark.** The equivalence of (1) and (4) is essentially contained in [6, Section 6].

By definition, $X_w(1) \neq \emptyset$ if and only if $g^{-1}\sigma(g) \in I\dot{w}I$ for some $g \in G(L)$. Using the decomposition $G(L) = \cup_{v \in \tilde{W}} U(L)\dot{v}I$, this is equivalent to $\dot{v}^{-1}u^{-1}\sigma(u)\sigma(\dot{v}) \in I\dot{w}I$ for some $u \in U(L)$ and $v \in \tilde{W}$. By [15, 1.3], $\{u^{-1}\sigma(u); u \in U(L)\} = U(L)$. Notice that $\sigma(\dot{v}) \in \dot{v}T$. Hence $1 \in G(L) \cdot _{\sigma} I\dot{w}I$ if and only if $\dot{v}^{-1}U(L)\dot{v} \cap I\dot{w}I \neq \emptyset$ for some $v \in \tilde{W}$. Write $v$ as $v = e^{\lambda}x$ for $x \in W$ and $\lambda \in Y$. Then $\dot{v}^{-1}U(L)\dot{v} = \dot{x}^{-1}e^{-\lambda}U(L)e^{\lambda}\dot{x} = \dot{x}^{-1}U(L)\dot{x}$. Thus (1) is equivalent to (4).

By the same argument, $tU(L) \cap G(L) \cdot I\dot{w}I \neq \emptyset$ if and only if $\dot{x}^{-1}tU(L)\dot{x} \cap I\dot{w}I \neq \emptyset$ for some $x \in W$. Since $t \in I \cap T(L)$, this is equivalent to the condition (4). On the other hand, (4) implies (3) and (3) implies that $tU(L) \cap G(L) \cdot I\dot{w}I \neq \emptyset$. Hence (3) is equivalent to (4). Taking $t = 1$, we obtain the equivalence between (2) and (4). □

**Corollary 3.2.** Let $w \in \tilde{W}$. Then $X_w(1) \neq \emptyset$ if and only if $X_{w^{-1}}(1) \neq \emptyset$.

By Proposition 3.1, if $X_w(1) \neq \emptyset$, then $U(L) \cap K \cdot I\dot{w}I \neq \emptyset$. Applying the inverse map, we have that $U(L) \cap K \cdot \dot{w}^{-1}I \neq \emptyset$. Again by Proposition 3.1, $X_{w^{-1}}(1) \neq \emptyset$. □

3.1. Let $St : G \to T/W$ be the Steinberg map, i.e., $St(g)$ is the $W$-orbit in $T$ that contains an element conjugate to the semisimple part of $g$. The fibers of this map are called Steinberg fibers. It is known that each Steinberg fiber is of the form $G \cdot tU$ for some $t \in T$. An example of Steinberg fiber is the unipotent variety $\mathcal{U}$ of $G$. Some other examples are regular semisimple conjugacy classes of $G$.

Let $F$ be a Steinberg fiber and $\bar{F}$ be its closure in $\bar{G}$. The following description of $\bar{F}$ was first obtained in [9, Theorem 4.3 & 4.5] using a case-by-case check. A more conceptual proof was obtained later in [14]. Springer also gave a different proof in [21]. Both [14] and [21] uses some properties of proper intersection. Below we give a group-theoretic proof in positive characteristic based on the connection between loop groups and group compactifications.
Theorem 3.3. Let $F$ be a Steinberg fiber of $G$. Then

$$\widetilde{F} - F = \sqcup_{J \neq S} \sqcup_{w \in W^J, \text{supp}(w) = S} Z_{J,w}.$$ 

The part $\widetilde{F} - F \subset \sqcup_{J \neq S} \sqcup_{w \in W^J, \text{supp}(w) = S} Z_{J,w}$ follows from the fact that the elements in $\widetilde{F} - F$ are represented by nilpotent endomorphisms. We refer to [21, 3.3(b)] for the details.

By 1.4 (3), in order to show that $\sqcup_{J \neq S} \sqcup_{w \in W^J, \text{supp}(w) = S} Z_{J,w} \subset \widetilde{F}$, it suffices to show that $Z_{J,w} \subset \widetilde{F}$ for any $J \neq S$ and Coxeter element $w \in W^J$.

Assume that $F = G_\Delta \cdot tU$ for some $t \in T$. By [22, 1.4], $\widetilde{F} = G_\Delta \cdot t\bar{U}$. Let $J \neq S$ and $w \in W^J$ be a Coxeter element of $W$. By 1.4 (4), $Z_{J,w}$ is a single $G$-orbit. So it suffices to show that $Z_{J,w} \cap t\bar{U} \neq \emptyset$.

Pick $\lambda \in X^+$ with $I(-w_0\lambda) = J$. Since $w$ and $w^*$ are Coxeter elements, there exists $\mu \in Y \otimes \mathbb{Z} \mathbb{Q}$ such that $\mu - (w^*)^{-1}\mu = \lambda$. Let $m \in \mathbb{N}$ with $m\mu \in Y$. Then $m\mu - (w^*)^{-1}(m\mu) = m\lambda$ and $w^*\epsilon^{-m\lambda}$ is conjugated to $w^*$ by $\epsilon^m$. Therefore, $\dot{w}^* \in G(L) \cdot \sigma \dot{w}^* \epsilon^{-m\lambda}$. By Lang’s theorem, $1 \in K \cdot \sigma \dot{w}^*$. Hence $1 \in G(L) \cdot \sigma \dot{I} \dot{w}^* \epsilon^{-m\lambda} I$. By Proposition 3.1, $tU(L) \cap K \cdot \dot{w}^* \epsilon^{-m\lambda} \neq \emptyset$. Applying the specialization map $s : G(L) \to \bar{G}$, we have that $t\bar{U} \cap Z_{J,w} \neq \emptyset$. This finishes the proof. 

The following result relates the emptiness problem of ADLV to the description of the closure of a Steinberg fiber in $\bar{G}$.

Proposition 3.4. Let $J \subsetneq S$, $x \in W^J$ and $y \in W$. If $[J,x,y] \subset \sqcup_{w \in W^J, \text{supp}(w) \neq S} Z_{J,w}$, then for any $\lambda \in Y^+$ with $I(\lambda) = J$, we have that $X_{\lambda^{-\epsilon} \cdot y^{-1}}(1) = \emptyset$.

Let $\delta$ be the diagram automorphism on $G$ whose induced permutation on $S$ equals $-w_0$. Then $\delta$ also gives an automorphism on $G(L)$ which we still denote by $\delta$. Then $\delta(1) = 1$ and $\delta(I) = I$. So $\delta(I \dot{x} \epsilon^{-\lambda} \cdot \dot{y}^{-1} I) = I \dot{x}^* \epsilon^{w_0\lambda}(\dot{y}^*)^{-1} I$.

If $X_{\lambda^{-\epsilon} \cdot y^{-1}}(1) \neq \emptyset$, then $X_{\lambda^{-\epsilon} \cdot y^{-1}}(1) \neq \emptyset$ and $1 \in G(L) \cdot \sigma \dot{x} \epsilon^{w_0\lambda}(\dot{y}^*)^{-1} I$. By Corollary 3.1, $U(L) \cap K \cdot \dot{x} \epsilon^{w_0\lambda}(\dot{y}^*)^{-1} I \neq \emptyset$. As in $\S 2.4$, $K \cdot \dot{x} \epsilon^{w_0\lambda}(\dot{y}^*)^{-1} I = K \cdot I \dot{x} \epsilon^{w_0\lambda}(\dot{y}^*)^{-1} I = K \cdot I \dot{x} \epsilon^{w_0\lambda}(\dot{y}^*)^{-1} I$. Thus, $K \cdot U(L) \cap I \dot{x} \epsilon^{w_0\lambda}(\dot{y}^*)^{-1} I \neq \emptyset$. Applying the specialization map, we have that $G_\Delta \cdot \bar{U} \cap [J,x,y] \neq \emptyset$. That is impossible by the previous theorem. 

4. Main result

4.1. Note that $\sqcup_{w \in W_a} I \dot{w} I$ is a normal subgroup of $G(L)$ that contains 1. It is easy to see that if $X_w(1) \neq \emptyset$ for some $w \in W$, then we must have that $w \in W_a$, i.e., the translation part of $w$ is $X$.

Now let $\lambda \in X$. We call $\lambda$ quasi-regular if for any $\alpha \in \Phi$, either $\langle \lambda, \alpha \rangle = 0$ or $|\langle \lambda, \alpha \rangle| \geq (\langle \rho^\vee, \theta \rangle + 2)^{|S|+1}$. Any affine Weyl group element with quasi-regular translation part is of the form $x \epsilon^{-\lambda} \cdot y^{-1}$, where $\lambda \in X^+$ with $\langle \lambda, \alpha_i \rangle \geq (\langle \rho^\vee, \theta \rangle + 2)^{|S|+1}$ for any $i \notin I(\lambda)$, $x \in W^{I(\lambda)}$ and...
$y \in W$. Our main result below describes the emptiness/nonemptiness pattern of $X_w(1)$ if the translation part of $w$ is quasi-regular.

**Theorem 4.1.** Let $J \subseteq S$, $x \in W^J$, $y \in W$ and $\lambda \in X^+$ with $I(\lambda) = J$. Assume that $\lambda$ is quasi-regular. Then $X_{x^\lambda}^{-1}(1) \neq \emptyset$ if and only if $[J, x, y] \not\subseteq \cup_{w \in W, \text{supp}(w) \neq S} Z_{J, w}$.

The “only if” is proved in Proposition 3.4. The proof of “if” part will be given in §4.2. Our strategy is as follows. First, we use the Proposition 4.2 below to reduce to problem to elements in $\tilde{\mathcal{S}}$, using the technique of “partial conjugation action” introduced in [11]. Then in Proposition 4.4 we reduce the elements in $\tilde{\mathcal{S}}$ with quasi-regular translation part to some elements for which the nonemptiness is already known. The trick we use here is similar to the “P-operators” introduced in [13]. Because of the quasi-regular condition, the case here is easier to handle than in loc. cit.

**Proposition 4.2.** Let $\lambda \in Y^+$ with $I(\lambda) = J \subseteq S$. Let $x, w \in W^J$ and $y \in W$. Then the following conditions are equivalent:

1. $[J, x, y] \cap Z_{J, w} \neq \emptyset$;
2. $I \dot{x} \cos^{\cdot \lambda} y^{-1} I \cap K_{w \cdot \lambda} \neq \emptyset$;
3. $I \dot{x} \cos^{\cdot \lambda} y^{-1} I \cap K_{w \cdot \lambda, \sigma} \neq \emptyset$;
4. $K_{w \cdot \lambda, \sigma} \subseteq K \cdot \sigma I \dot{x} \cos^{\cdot \lambda} y^{-1} I$.

We first prove the equivalence of (1) and (2).

Let $\delta$ be the automorphism of $G(L)$ defined in the proof of Proposition 3.4. Then

$$G_{\Delta} \cdot [J, x, y] = s(K \cdot I \dot{x} \cos^{\cdot \lambda} y^{-1} I') = s(K \cdot I \dot{x} \cos^{\cdot \lambda} y^{-1} I) = s(\delta(K \cdot I \dot{x} \cos^{\cdot \lambda} y^{-1} I)),$$

$$Z_{J, w} = G_{\Delta} \cdot [J, w, 1] = s(\delta(K \cdot I \dot{w} \cos^{\cdot \lambda} I)) = s(\delta(K_{\lambda})).$$

So if $I \dot{x} \cos^{\cdot \lambda} y^{-1} I \cap K_{w \cdot \lambda} \neq \emptyset$, then $K \cdot I \dot{x} \cos^{\cdot \lambda} y^{-1} I \cap K_{w \cdot \lambda} \neq \emptyset$ and $s(\delta(K \cdot I \dot{x} \cos^{\cdot \lambda} y^{-1} I)) \cap s(\delta(K_{w \cdot \lambda})) \neq \emptyset$. Thus $G_{\Delta} \cdot [J, x, y] \cap Z_{J, w} \neq \emptyset$ and $[J, x, y] \cap Z_{J, w} \neq \emptyset$. On the other hand, if $I \dot{x} \cos^{\cdot \lambda} y^{-1} I \cap K_{w \cdot \lambda} = \emptyset$, then by Proposition 2.6, $K \cdot \sigma I \dot{x} \cos^{\cdot \lambda} y^{-1} I \subseteq \cup_{w' \in W^J \neq w} K_{w' \cdot \lambda}$. Hence

$$G_{\Delta} \cdot [J, x, y] = s(\delta(K \cdot I \dot{w} \cos^{\cdot \lambda} y^{-1} I)) \subseteq \cup_{w' \in W^J \neq w} \cup_{w' \in Z_{J, w}} s(\delta(K_{w' \cdot \lambda})).$$

Therefore $[J, x, y] \cap Z_{J, w} = \emptyset$.

By Lemma 2.9, $K \cdot \sigma I \dot{x} \cos^{\cdot \lambda} y^{-1} I$ is a union of $\sigma$-twisted $K$-stable pieces. Hence (3) is equivalent to (4).

Set $\tilde{w} = x^\lambda y^{-1}$. Now we prove the equivalence of (2) and (3) by induction on $I(\tilde{w})$.

If $\tilde{w} \in (W \cdot \tilde{w})_{\text{min}}$, then by 2.2 (1), $\tilde{w} \approx v\tilde{w}_1$ for some $\tilde{w}_1 \in \tilde{W}^S$ and $v \in W_{I(\tilde{w})_{\text{min}}}$. By Lemma 2.4 and Lemma 2.8, $K \cdot I \tilde{w} \cdot I \subseteq K_{\tilde{w}_1}$ and $K \cdot \sigma I \tilde{w} \cdot I = K_{\tilde{w}_1, \sigma}$. By Proposition 2.6, the $K$-stable pieces form a disjoint union of $G(L)$. Hence (2) is equivalent to $w \cos^{\cdot \lambda} = \tilde{w}_1$. Similarly, (3) is equivalent to $w \cos^{\cdot \lambda} = \tilde{w}_1$. 

Therefore $[J, x, y] \cap Z_{J, w} = \emptyset$. 

By Lemma 2.9, $K \cdot \sigma I \dot{x} \cos^{\cdot \lambda} y^{-1} I$ is a union of $\sigma$-twisted $K$-stable pieces. Hence (3) is equivalent to (4).

Set $\tilde{w} = x^\lambda y^{-1}$. Now we prove the equivalence of (2) and (3) by induction on $I(\tilde{w})$.

If $\tilde{w} \in (W \cdot \tilde{w})_{\text{min}}$, then by 2.2 (1), $\tilde{w} \approx v\tilde{w}_1$ for some $\tilde{w}_1 \in \tilde{W}^S$ and $v \in W_{I(\tilde{w})_{\text{min}}}$. By Lemma 2.4 and Lemma 2.8, $K \cdot I \tilde{w} \cdot I \subseteq K_{\tilde{w}_1}$ and $K \cdot \sigma I \tilde{w} \cdot I = K_{\tilde{w}_1, \sigma}$. By Proposition 2.6, the $K$-stable pieces form a disjoint union of $G(L)$. Hence (2) is equivalent to $w \cos^{\cdot \lambda} = \tilde{w}_1$. Similarly, (3) is equivalent to $w \cos^{\cdot \lambda} = \tilde{w}_1$. 

Therefore $[J, x, y] \cap Z_{J, w} = \emptyset$. 

By Lemma 2.9, $K \cdot \sigma I \dot{x} \cos^{\cdot \lambda} y^{-1} I$ is a union of $\sigma$-twisted $K$-stable pieces. Hence (3) is equivalent to (4).
If $\bar{w} \notin (W \cdot \bar{w})_{\min}$, then by 2.2 (1), there exists $\bar{w}_1 \approx \bar{w}$ and $i \in S$ with $\bar{w}_1 \rightarrow s_i \bar{w}_1 s_i$ and $l(s_i \bar{w}_1 s_i) < l(\bar{w})$. By Lemma 2.2, $K \cdot I \bar{w} I = K \cdot I \bar{w}_1 I = K \cdot I s_i \bar{w}_1 I \cup K \cdot I s_i \bar{w}_1 s_i I$. Similarly, $K \cdot I \bar{w} I = K \cdot I s_i \bar{w} I \cup K \cdot I s_i \bar{w}_0 I$. Since $l(s_i \bar{w}_1 s_i) < l(\bar{w})$, by induction hypothesis, $I s_i \bar{w}_1 I \cap K_{\omega^e, \sigma} = \emptyset$ if and only if $I s_i \bar{w}_0 I \cap K_{\omega^e, \sigma} = \emptyset$ and $I s_i \bar{w}_1 s_i I \cap K_{\omega^e, \sigma} = \emptyset$ if and only if $I s_i \bar{w}_0 I \cap K_{\omega^e, \sigma} = \emptyset$. So $K \cdot I \bar{w} I \cap K_{\omega^e, \sigma} = \emptyset$ if and only if $K \cdot I \bar{w} I \cap K_{\omega^e, \sigma} = \emptyset$. The equivalence of (2) and (3) is proved.

**Lemma 4.3.** Let $x, y \in W$. Assume that for any $\alpha \in \Phi^+$ with $y^{-1} \alpha \in \Phi^-$, $x \alpha \in \Phi^-$. Then $l(xy) = l(x) - l(y)$.

We prove by induction on $l(y)$. For $y = 1$ this is clear. Suppose that $l(y) \geq 1$. Then there exists $i \in S$ such that $y^{-1} \alpha_i \in \Phi^-$. Thus $y = s_i y'$ for some $y' \in W$ with $l(y') = l(y) - 1$. We have that $x \alpha_i \in \Phi^-$. Hence $x = x's_i$ for some $x' \in W$ with $l(x') = l(x) - 1$.

Now for any $\alpha \in \Phi^+$ with $y^{-1} \alpha \in \Phi^-$, either $\alpha = \alpha_i$ or $s_i \alpha \in \Phi^+$ with $(y')^{-1} (s_i \alpha) \in \Phi^-$. By our assumption, $x'(s_i \alpha) = x \alpha \in \Phi^-$. Therefore for any $\alpha' \in \Phi^+$ with $(y')^{-1} \alpha' \in \Phi^-$, we have that $\alpha' \neq \alpha_i$ and $x' \alpha' \in \Phi^-$. By induction hypothesis, $l(xy) = l(x'y') = l(x') - l(y') = l(x) - l(y)$.

**Proposition 4.4.** Let $J \subset S$. Then for any $x \in W^J$ and $y \in W$ with $y^{-1} x \in W^J$ and $\supp(y^{-1} x) = S$ and any $\lambda \in X^+$ with $I(\lambda) = J$ and $\langle \lambda, \alpha_i \rangle \geq (\langle \rho^\vee, \theta \rangle + 2)^{|J| + 1}$ for any $i \notin J$, we have that $X_{x \gamma - y^{-1} x}(1) \neq \emptyset$.

We prove by induction on $|J|$. Note that

$$l(x \epsilon^{-\lambda} y^{-1}) = l(\epsilon^\lambda) - l(x) + l(y) = l(x \epsilon^{-M \rho_{\gamma^\vee}} y^{-1} + l(\epsilon^{-\lambda + M \rho_{\gamma^\vee}} y^{-1}),$$

where $M = (\langle \rho^\vee, \theta \rangle + 2)^{|J|}$. So

(a) $K \cdot I x \epsilon^{-\lambda} y^{-1} I = K \cdot I x \epsilon^{-M \rho_{\gamma^\vee}} I \epsilon^{-\lambda + M \rho_{\gamma^\vee}} y^{-1} I$ 

$$= K \cdot I \epsilon^{-\lambda + M \rho_{\gamma^\vee}} y^{-1} I \epsilon^{-M \rho_{\gamma^\vee}} y^{-1} I$$

$$\supset K \cdot I \epsilon^{-\lambda + M \rho_{\gamma^\vee}} y^{-1} x \epsilon^{-M \rho_{\gamma^\vee}} y^{-1} I$$

$$= K \cdot I \epsilon^{-\lambda + M \rho_{\gamma^\vee}} y^{-1} x \epsilon^{-M \rho_{\gamma^\vee}} y^{-1} I$$

Now $\lambda - M \rho_{\gamma^\vee} + y^{-1} x M \rho_{\gamma^\vee} = v \gamma$ for some $\gamma \in X^+$ and $v \in W^I(\gamma)$. If $v = 1$, then $\epsilon^{-\lambda - M \rho_{\gamma^\vee} + y^{-1} x M \rho_{\gamma^\vee}} y^{-1} = \epsilon^{-\gamma} y^{-1} x$. By [5], $X_{x \gamma - y^{-1} x}(1) \neq \emptyset$. Hence by Corollary 3.2, $X_{x \gamma - y^{-1} x}(1) \neq \emptyset$. By (a), $1 \in K \cdot I \epsilon^{-\gamma} y^{-1} x I \subset K \cdot I x \epsilon^{-\lambda} y^{-1} I$. Thus $X_{x \gamma - y^{-1} x}(1) \neq \emptyset$ and the Proposition holds in this case.

Now we consider the case where $v \neq 1$. For any $\alpha \in \Phi^+ - \Phi_f^+$,

(b) $\langle v \gamma, \alpha \rangle = \langle \lambda - M \rho_{\gamma^\vee}, \alpha \rangle + \langle y^{-1} x M \rho_{\gamma^\vee}, \alpha \rangle$

$$\geq \langle \rho^\vee, \theta \rangle + 2)^{|J| + 1} - M + M \langle \rho_{\gamma^\vee}, x^{-1} y \alpha \rangle$$

$$\geq \langle \rho^\vee, \theta \rangle + 2)^{|J| + 1} - M - M \langle \rho^\vee, \theta \rangle = M.$$
Let $\alpha \in \Phi^+$ with $v^{-1}\alpha \in \Phi^-$. Then $\langle v\gamma, \alpha \rangle = \langle \gamma, v^{-1}\alpha \rangle \leq 0$. However, if $\langle v\gamma, \alpha \rangle = 0$, then $v^{-1}\alpha \in \Phi^-_{\gamma}$ and $\alpha = v(v^{-1}\alpha) \in \Phi^-$. That is a contradiction. Hence

$$\langle v\gamma, \alpha \rangle = \langle \lambda - M\rho^\gamma_{S-J}, \alpha \rangle + M\langle \rho^\gamma_{S-J}, xy^{-1}\alpha \rangle < 0.$$  

By (b), $\alpha \in \Phi^+_J$. Therefore $v \in W_J$. Since $\lambda - M\rho^\gamma_{S-J}$ and $\rho^\gamma_{S-J}$ are dominant, $xy^{-1}\alpha < 0$. By the previous Lemma, we have that, $l(v^{-1}y^{-1}x) = l(y^{-1}x) - l(v)$. Since $y^{-1}x \in W_J$, $v^{-1}y^{-1}x \in W_J$.

Now $\varepsilon - (\lambda - M\rho^\gamma_{S-J} + y^{-1}xM\rho^\gamma_{S-J})y^{-1}x = v\varepsilon - v^{-1}y^{-1}x$. We check that

(c) $I(\gamma) \subsetneq J$.

Let $i \in S$ with $\langle \gamma, \alpha_i \rangle = 0$. Then $\langle v\gamma, v\alpha_i \rangle = 0$. Hence $v\alpha_i \in \Phi_J$ and $i \in J$. If $I(\gamma) = J$, then $\langle v\gamma, \alpha \rangle = \langle \gamma, v^{-1}\alpha \rangle = 0$ for any $\alpha \in \Phi^+_J$. By (b), $\langle v\gamma, \alpha \rangle \geq 0$ for $\alpha \in \Phi^+$ and $v\gamma$ is dominant. Since $\gamma$ is dominant, $v\gamma = \gamma$. Notice that $v \in W^I(\gamma)$. Then $v = 1$. That is a contradiction. (c) is proved.

(d) $\langle \gamma, \alpha_i \rangle \geq (\langle \rho^\gamma, \theta \rangle + 2)I(\gamma)_{i+1}$ if $i \notin I(\gamma)$.

If $i \notin J$, then $\langle \gamma, \alpha_i \rangle = \langle v\gamma, v\alpha_i \rangle$. We have that $v\alpha_i \in \Phi^+ - \Phi^+_J$.

By (b), $\langle \gamma, \alpha_i \rangle \geq M$. If $i \in J - I(\gamma)$, then $\langle \gamma, \alpha_i \rangle = \langle v\gamma, v\alpha_i \rangle > 0$.

Notice that $\langle v\gamma, v\alpha_i \rangle = \langle \lambda - M\rho^\gamma_{S-J}, v\alpha_i \rangle + M\langle y^{-1}x\rho^\gamma_{S-J}, v\alpha_i \rangle$ and $\langle \lambda - M\rho^\gamma_{S-J}, v\alpha_i \rangle = 0$. So $\langle y^{-1}x\rho^\gamma_{S-J}, v\alpha_i \rangle \geq 1$ and $\langle \gamma, \alpha_i \rangle \geq M$. (d) is proved.

(e) $v^{-1}y^{-1}xv \in W^I(\gamma)$ and $\text{supp}(v^{-1}y^{-1}xv) = S$.

We have shown that $v^{-1}y^{-1}x \in W_J$ and $v \in W_J \cap W^I(\gamma)$. Then $v^{-1}y^{-1}x \in W^I(\gamma)$ and $l(v^{-1}y^{-1}x) = l(v^{-1}y^{-1}x) + l(v)$. So

$$\text{supp}(v^{-1}y^{-1}xv) = \text{supp}(v^{-1}y^{-1}x) \cup \text{supp}(v) \supset \text{supp}(v \cdot v^{-1}y^{-1}x)$$

$$= \text{supp}(y^{-1}x) = S.$$  

(e) is proved.

By induction hypothesis for $(v, v^{-1}y^{-1}x, \gamma)$, $X_{v^{-1}y^{-1}x}(1) \neq \emptyset$. By (a), $1 \in K \cdot \sigma I\varepsilon^{-1}y^{-1}xI \subset K \cdot \sigma I\varepsilon^{-1}y^{-1}I$. Thus $X_{x^{-1}\gamma^{-1}(1)} \neq \emptyset$ and the lemma holds in this case.

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4.2. **Proof of Theorem 4.1.** Assume that $[J, x, y] \cap Z_{w, w} \neq \emptyset$ for some $w \in W_J$ with $\text{supp}(w) = S$. By Proposition 4.2, $K \cdot \sigma I\varepsilon^{-1}I \subset K \cdot \sigma I\varepsilon^{-1}y^{-1}I$. By Proposition 4.4, $1 \in K \cdot \sigma I\varepsilon^{-1}I$. Hence $1 \in K \cdot \sigma I\varepsilon^{-1}y^{-1}I$ and $X_{x^{-1}\gamma^{-1}(1)} \neq \emptyset$.

5. **Further Discussions**

In this section we discuss in more details the condition $[J, x, y] \notin \sqcup_{w \in W_J, \text{supp}(w) \neq S} Z_{w, w}$.

**Lemma 5.1.** Let $J \subset S$, $x \in W_J$ and $y \in W$. If $\text{supp}(y^{-1}x) \neq S$, then $[J, x, y] \subset \sqcup_{w \in W_J, \text{supp}(w) \neq S} Z_{w, w}$.  

Suppose that \( \text{supp}(y^{-1}x) = J' \neq S \). Let \( \lambda \in Y^+ \) with \( I(\lambda) = J \). By Proposition 4.2, it suffices to prove that \( K \cdot I\hat{x}\epsilon^{-\lambda}\hat{y}^{-1}I \subset \sqcup_{w \in W^{J'} \cap \text{supp}(w) \neq S} K_{uw^{-\lambda}} \).

We may assume that \( x = u_1v_1 \) and \( y = u_2v_2 \) for \( u_1, u_2 \in W^{J'} \) and \( v_1, v_2 \in W_J \). Since \( x \in yW_J \), we must have that \( u_1 = u_2 \). Since \( l(u_1^{-1}x) = l(v_1) = l(x) - l(u_1) \), we have that \( v_1 \in W^J \). Then
\[
l(x^{-\lambda}y^{-1}) = l(x^{-\lambda}) + l(y) = l(x^{-\lambda}) + l(v_2^{-1}) = l(x^{-\lambda}v_2^{-1}) + l(u_2^{-1});
\]
\[
l(v_1^{-\lambda}v_2^{-1}) = l(\epsilon^{-\lambda}v_2^{-1}) - l(v_1) = l(\epsilon^{-\lambda}v_2^{-1}) - l(x) + l(u_1) = l(u_1) + l(x^{-\lambda}v_2^{-1}).
\]
Hence
\[
K \cdot I\hat{x}\epsilon^{-\lambda}\hat{y}^{-1}I = K \cdot I\hat{x}\epsilon^{-\lambda}\hat{y}^{-1}I\hat{u}_2^{-1}I = K \cdot I\hat{u}_2^{-1}I\hat{x}\epsilon^{-\lambda}\hat{y}^{-1}I
\]
\[
= K \cdot I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}I.
\]
We prove by induction on \( l(v_2) \) that
\[
(a) \quad K \cdot I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}I \subset \sqcup_{w \in W^{J'} \cap \text{supp}(w) \neq S} K_{uw^{-\lambda}}.
\]

Let \( v_2 = s_{i_1} \cdots s_{i_n} \) be a reduced expression. For \( v_2 = 1 \) this is clear. Assume now that \( n > 0 \) and (a) holds for all \( v_2' \) with \( l(v_2') < n \) but (a) fails for \( v_2 \). Then we have that
\[
K \cdot I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}I = K \cdot I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}s_{i_1}I\hat{s}_{i_1}I = K \cdot I\hat{s}_{i_1}I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}s_{i_1}I
\]
\[
\subset K \cdot I\hat{s}_{i_1}I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}s_{i_1}I \cup K \cdot I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}s_{i_1}I.
\]
By induction hypothesis, \( K \cdot I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}s_{i_1}I \subset \sqcup_{w \in W^{J'} \cap \text{supp}(w) \neq S} K_{uw^{-\lambda}} \). So \( K \cdot I\hat{s}_{i_1}I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}s_{i_1}I \notin \sqcup_{w \in W^{J'} \cap \text{supp}(w) \neq S} K_{uw^{-\lambda}} \). By induction hypothesis, \( s_{i_1}v_1 \notin W^J \). Hence \( s_{i_1}v_1 = v_1s_{j_1} \) for some \( j_1 \in J \). Also \( v_1, s_{i_1} \in W^J \). So \( j_1 \in J' \). Now
\[
K \cdot I\hat{s}_{i_1}I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}s_{i_1}I = K \cdot I\hat{v}_1\epsilon^{-\lambda}s_{j_1}\hat{v}_2^{-1}s_{i_1}I \notin \sqcup_{w \in W^{J'} \cap \text{supp}(w) \neq S} K_{uw^{-\lambda}}.
\]
Again by induction hypothesis, \( s_{i_1}v_2s_{j_1} > s_{i_1}v_2 \) and \( l(s_{i_1}v_2s_{j_1}) = n \). Now apply the same argument to \( s_{i_1}v_2s_{j_1} \) instead of \( v_2 \), we have that \( s_{i_2}v_1 = v_1s_{j_2} \) for some \( j_2 \in J \cap J' \) and \( l(s_{i_2}s_{i_1}s_{j_2}s_{j_1}) = n \). Repeat the same procedure, one can show that for \( 1 \leq k \leq n \) and \( m \in \mathbb{N} \), \( s_{i_k}v_1^m = v_1^m s_j \) for some \( j \in J \cap J' \). In other words, \( \text{supp}(v_2) \subset I(J, v_1) = I(S, v_1e^{-\lambda}) \). Now by Lemma 2.4, \( K \cdot I\hat{v}_1\epsilon^{-\lambda}\hat{v}_2^{-1}I \subset K_{v_1e^{-\lambda}} \). That is a contradiction. Therefore (a) always holds and the Lemma is proved.

\[\square\]

**Lemma 5.2.** Let \( J \subset S, x \in W^J \) and \( y \in W \). If \( \text{supp}(uy^{-1}xu^{-1}) = S \) for all \( u \in W_J \), then there exists \( w \in W^J \) with \( \text{supp}(w) = S \) and \( [J, x, y] \cap Z_{J,w} \neq \emptyset \).

By [11, Corollary 2.6], there exists \( u \in W_J, w \in W^J \) and \( v \in W_{I(J,w)} \) such that \( uy^{-1}xu^{-1} = wv \). By our assumption, \( \text{supp}(wv) = S \). By the
proof of [12, Prop 3.2], supp(w) = S. Now
\[(\dot{x}, y) \cdot h_J = (\dot{x}^{-1}, y^{-1}) \cdot h_J \in G_\Delta(u^{\dot{y}^{-1}}\dot{x}^{-1}, T) \cdot h_J \]
\[= G_\Delta(\dot{w}, T) \cdot h_J \subseteq G_\Delta : [J, w, v^{-1}].\]

By [10, Proposition 1.10], \((\dot{x}, y) \cdot h_J \in Z_{J,w}\). The Lemma is proved. \(\square\)

Corollary 5.3. (1) Let \(\lambda \in Y^+\) with \(I(\lambda) = J \subseteq \emptyset\). Let \(x \in W^J\) and \(y \in W\). If \(\text{supp}(y^{-1}x) \neq S\), then \(X_{x^{-1}y^{-1}}(1) = \emptyset\).

(2) Let \(J \subseteq \emptyset\), \(x \in W^J\) and \(y \in W\). Let \(\lambda \in X^+\) with \(I(\lambda) = J\) and \(\langle \lambda, \alpha_i \rangle \geq (\langle \rho^Y, \theta \rangle + 2)^{|J|+1}\). If \(y = 1\) and \(\text{supp}(x) = S\) or \(\text{supp}(uy^{-1}ux^{-1}) = S\) for all \(u \in W_J\), then \(X_{x^{-1}y^{-1}}(1) \neq \emptyset\).

Remark. Part (1) was first proved in [7, Proposition 9.4.4]. Here we give a different proof.

Part (1) follows from Proposition 3.4 and Lemma 5.1. The case where \(y = 1\) and \(\text{supp}(x) = S\) in part (2) follows from Proposition 4.4 and the case where \(\text{supp}(uy^{-1}ux^{-1}) = S\) for all \(u \in W_J\) follows from Theorem 4.1 and Lemma 5.2. \(\square\)

5.1. In fact, the condition \(\text{supp}(uy^{-1}ux^{-1}) = S\) for all \(u \in W_J\) can’t be replaced by \(\text{supp}(y^{-1}x) = S\) in general. The following is an example.

Let \(G = \text{PGL}_4\), \(J = \{1\}\), \(x = s_2s_1s_3s_2\) and \(y = s_3s_2\). Then \(\text{supp}(y^{-1}x) = S\) but \(\text{supp}(s_1y^{-1}xs_1) = \{2, 3\}\). In this case, one can show that \(\{1\}, x, y] \subset Z_{(1),s_3s_2}\). So \(X_{x^{-1}y^{-1}}(1) = \emptyset\) for all \(\lambda \in X^+\) with \(I(\lambda) = J\).

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