Consideration of success probability and performance optimization in non-Gaussian continuous variable quantum teleportation

Chandan Kumar\textsuperscript{1,*} and Shikhar Arora\textsuperscript{1,†}

\textsuperscript{1}Department of Physical Sciences, Indian Institute of Science Education and Research Mohali, Sector 81 SAS Nagar, Panjab 140306 India.

Non-Gaussian operations have been shown to enhance the fidelity of continuous variable quantum teleportation. However, a disadvantage of these non-Gaussian operations is that they are probabilistic in nature. In this article, we study the trade-off between teleportation fidelity and success probability for optimal performance of the teleportation protocol, which to the best of our knowledge, has never been studied before. To this end, we first derive a unified expression for the Wigner characteristic function describing three non-Gaussian states, photon subtracted, photon added, and photon catalyzed two-mode squeezed vacuum states. We then utilize it to obtain the fidelity of teleportation for input coherent and squeezed vacuum states using the aforementioned non-Gaussian resource states. We optimize the product of the relative enhancement in fidelity and the probability of state preparation by tuning the transmissivity of the beam splitters involved in implementing non-Gaussian operations. This leads to a scenario that can be effectively implemented in a lab to enhance fidelity. It turns out that among all the considered non-Gaussian resource states, the symmetric one-photon subtracted TMSV state is the most advantageous. We provide the associated optimal squeezing and beam splitter transmissivity values for the considered non-Gaussian resource states, which will be of significant interest to the experimental community. We also consider the effect of imperfect photon detectors on teleportation fidelity. Further, we expect the derived Wigner characteristic function to be useful in state characterization and other quantum information processing protocols.

\section{I. INTRODUCTION}

A two-mode squeezed vacuum (TMSV) state is generally employed as a resource state for various continuous variable (CV) quantum information processing (QIP) protocols including quantum teleportation \cite{1}, quantum dense coding \cite{2}, and entanglement swapping \cite{3}. However, due to experimental limitations, it is difficult to produce high squeezed states \cite{4}, which puts an upper bound on the performance of quantum protocols. An alternative is to perform non-Gaussian operations such as photon subtraction (PS), photon addition (PA), and photon catalysis (PC) on the TMSV state, which can ameliorate the nonclassicality and entanglement content of the TMSV state. Further, non-Gaussian states including photon-subtracted TMSV (PSTMSV), photon-added TMSV (PATMSV), and photon-catalyzed TMSV (PCTMSV) states have been shown to improve the performance of various QIP protocols, such as quantum key distribution \cite{5–10}, quantum metrology \cite{11–15}, and quantum teleportation \cite{16–22}. This article refers to the PSTMSV, PATMSV, and PCTMSV states collectively as “NGTMSV states” and PS, PA, and PC operations as “non-Gaussian operations.”

These non-Gaussian operations generated by heralding schemes using photon number resolving detectors (PNRDs) \cite{23, 24} are probabilistic in nature \cite{25}. The success probability represents the fraction of successful non-Gaussian operations per trial; hence, it quantifies resource utilization. When considering the enhancement in quantum features such as nonclassicality, entanglement, and teleportation fidelity, it is necessary to account for the corresponding success probability. For example, the fidelity for teleporting input coherent state using PSTMSV resource states maximizes in the unit transmissivity limit, where the success probability approaches zero \cite{26}. Therefore, focusing on maximizing fidelity renders a highly undesirable scenario for the experimental implementation of non-Gaussian quantum teleportation. To achieve optimal performance, we must trade off the enhancement in quantum features against the success probability. Few research studies on non-Gaussian entanglement have already considered the success probability of the involved non-Gaussian operations \cite{27–29}. Further, preparation of non-Gaussian states such as Schrödinger cat states \cite{30–32} and a phase sensitivity study in Mach-Zehnder interferometer using non-Gaussian states \cite{33} have also taken success probability into account. However, such a study in quantum teleportation has not yet been undertaken. This article considers the success probability and finds an optimal balance between teleportation fidelity and success probability (resource utilization).

To this end, we derive the unified Wigner characteristic function of the NGTMSV states, which is then used to derive the analytical expression for the fidelity of teleporting input coherent and squeezed vacuum states. It should be noted that calculations involving non-Gaussian states are more complicated than those involving Gaussian states. Additionally, the experimental scheme based on beam splitters and PNRDs dramatically increases the

\* chandan.quantum@gmail.com
\† shikhar.quantum@gmail.com
challenge for theoretical investigation as we have to incorporate the free parameters associated with these devices in the analytical expressions.

We optimize the transmissivities of beam splitters to maximize the teleportation fidelity. While the PSTMSV and PCTMSV states can teleport input coherent and squeezed vacuum states, the PATMSV states can only teleport input squeezed vacuum states with large squeezing. We then study the difference between the fidelity of the NGTMSV states and that of the TMSV state, $\Delta F_{\text{NG}}$, which provides us insights into the magnitudes of the relative advantages provided by the different NGTMSV states.

The maximization of teleportation fidelity renders a highly undesirable scenario in terms of success probability (resource utilization). To overcome such a situation, we consider the success probability of non-Gaussian operations and the probability of generating multi-photon Fock states; and maximize the product of $\Delta F_{\text{NG}}$ and probability of state preparation. This leads to a scenario that provides enhanced performance with optimal resource utilization. We also provide the optimal squeezing of the resource states and transmissivities of the beam splitters for different NGTMSV states.

The analysis reveals that the Sym 1-PSTMSV state is the most advantageous of all the considered non-Gaussian states. This can be attributed to the fact that Fock states, which are generated probabilistically [34–38], are not required for PA operation. In contrast, they are required for PA and PC operations, which effectively decreases the success probability of PA and PC operations. Finally, we investigate the consequences of imperfect photon detectors on teleportation fidelity. The results reveal that although the imperfect detectors lower the teleportation fidelity, it is still advantageous to implement non-Gaussian operations on the TMSV state.

The unified Wigner characteristic function of the NGTMSV states derived in this article is of independent interest and great importance in its own right. As far as we know, this expression does not exist in the literature. We expect it will help handle similar calculational challenges arising in various non-Gaussian CV QIP protocols. We have explicitly provided the optimal transmissivities, which shall be highly relevant to experimentalists in achieving higher performance and resource optimization in non-Gaussian quantum teleportation.

The rest of the paper is organized as follows. In Sec. II, we obtain a general expression for the Wigner characteristic function of the NGTMSV states. In Sec. III, we provide a comprehensive study of the teleportation of input coherent and squeezed vacuum states. In Sec. IV, we analyze the effect of imperfect detectors on teleportation fidelity. Finally, we conclude with a discussion in Sec. V, where we outline the implications and future aspects of the current work. In the Appendix, we briefly describe the phase space description of the CV systems relevant to this article.

II. WIGNER CHARACTERISTIC FUNCTION OF THE NGTMSV STATES

\[
\begin{array}{c}
\text{TMSV} \\
A_1 \overset{F_1}{\rightarrow} T_1 \overset{|m_1\rangle \rightarrow}{\rightarrow} A'_1 \\
|m_1\rangle \rightarrow F_1 \\
|m_2\rangle \rightarrow F_2 \\
A_2 \overset{F_1'}{\rightarrow} T_2 \overset{|m_2\rangle \rightarrow}{\rightarrow} A'_2
\end{array}
\]

FIG. 1. Experimental setup for the preparation of non-Gaussian TMSV state. The TMSV state is mixed with ancilla modes initiated to Fock states $|m_1\rangle$ and $|m_2\rangle$ using beam splitters. Simultaneous detections of $n_1$ and $n_2$ photons in the output modes corresponding to the ancilla modes herald successful non-Gaussian operations on both modes.

The experimental setup for the generation of NGTMSV states is shown in Fig. 1. We consider a TMSV state labeled by $A_1$ and $A_2$. We represent the modes $A_1$ and $A_2$ by the quadrature operators $(\hat{q}_1, \hat{p}_1)^T$ and $(\hat{q}_2, \hat{p}_2)^T$, respectively. These quadrature operators are related to the annihilation and creation operators of the $i$th through the relation $\hat{a}_i = (\hat{q}_i + i\hat{p}_i)/\sqrt{2}$ and $\hat{a}_i^\dagger = (\hat{q}_i - i\hat{p}_i)/\sqrt{2}$. The TMSV state is obtained by the action of two-mode squeezing operator on two single-mode vacuum state [39–41]:

\[|\psi\rangle_{A_1,A_2} = \exp[r(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)]|0\rangle_1|0\rangle_2, \]

where $r$ is the squeezing parameter. We also consider two auxiliary modes labeled by $F_1$ and $F_2$ and initiated to Fock states $|m_1\rangle$ and $|m_2\rangle$, respectively. We represent the modes $F_1$ and $F_2$ by the quadrature operators $(\hat{q}_3, \hat{p}_3)^T$ and $(\hat{q}_4, \hat{p}_4)^T$, respectively. Mode $A_1$ ($A_2$) is mixed with mode $F_1$ ($F_2$) using beam-splitter of transmissivity $T_1$ ($T_2$). For convenience in calculating teleportation fidelity, we work in the phase space formalism using Wigner characteristic function. The Wigner characteristic function for a density operator $\hat{\rho}$ of an $n$-mode quantum system can be calculated as follows:

\[\chi(\Lambda) = \text{Tr}[\hat{\rho} \exp(-i\Lambda^T \Omega \xi)], \]

where $\xi = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)^T$, $\Lambda = (A_1, A_2, \ldots, A_n)^T$ with $A_i = (\tau_i, \sigma_i)^T \in \mathbb{R}^2$ and $\Omega$ is the symplectic form of an $n$-mode quantum system. Prior to the beam splitter operations, the Wigner characteristic function of the four-mode system is given by

\[\chi_{F_1A_1A_2F_2}(\Lambda) = \chi_{A_1A_2}(A)\chi_{|m_1\rangle}(A_3)\chi_{|m_2\rangle}(A_4), \]

where $\chi_{|m_1\rangle}(A_3)$ is the Wigner characteristic function of the Fock state $|m_1\rangle$ and $\chi_{|m_2\rangle}(A_4)$ is the Wigner characteristic function of the Fock state $|m_2\rangle$. The four
modes get entangled as a result of the mixing of the modes by the two beam splitters collectively represented by the symplectic transformation matrix \( B(T_1, T_2) = B_{A_1, T_1} \otimes B_{A_2, T_2} \) where \( B_i(T) \) is the beam splitter operation given in Eq. (A5) of the Appendix A. The evolved \( \chi_{A_1'}(T_1, T_2)^{-1} \lambda \).

PNRDs, represented by the positive-operator-valued measure (POVM) \( \{ \Pi_{n_1} = |n_1 \rangle \langle n_1|, \Pi_{n_2} = |n_2 \rangle \langle n_2|-\Pi_{n_2} \} \), are used to measure the transformed auxiliary modes \( F_1' \) and \( F_2' \). The simultaneous click of the POVM elements \( \Pi_{n_1} \) and \( \Pi_{n_2} \) heralds successful non-Gaussian operations on both modes. The post-measurement state corresponds to the unnormalized Wigner characteristic function of the NGTMSV states:

\[
\chi_{A_1|A_2} = \frac{1}{2(2\pi)^2} \int d^2A_3 d^2A_4 \chi_{F_1'A_1F_2'A_2}(A) \chi_{|n_1\rangle\langle n_1|} \chi_{|n_2\rangle\langle n_2|}.
\]

We can appropriately choose input and measured photons in the auxiliary modes and perform different non-Gaussian operations. For instance, by choosing \( m_i < n_i \), \( m_i > n_i \), and \( m_i = n_i \) we can perform PS, PA, and PC operations on mode \( A_i \), respectively. These operations on the TMSV state result in the generation of non-Gaussian states, abbreviated as PSTMSV, PATMSV, and PCTMSV states. In our analysis, we consider \( m_1 = m_2 = 0 \) and \( n_1 = n_2 = 0 \) for PS and PA operations, respectively. Further, we can perform asymmetric and symmetric non-Gaussian operations on the TMSV state as illustrated in Table I. Here we note that we perform the asymmetric non-Gaussian operations on mode \( A_2 \) of the TMSV state, and therefore, only beam splitter transmissivity \( T_2 \) appears in relevant expressions.

### Table I. Restrictions on the input photons (\( m_i \)) and the number of photons measured (\( n_i \)) in the auxiliary modes for various asymmetric and symmetric non-Gaussian operations on the TMSV state.

| Operations | Input | Detected |
|------------|-------|----------|
| Asym n-PS  | 0, 0  | 0, n     |
| Asym n-PA  | 0, n  | 0, 0     |
| Asym n-PC  | 0, n  | 0, n     |
| Sym n-PS   | 0, 0  | n, n     |
| Sym n-PA   | n, n  | 0, 0     |
| Sym n-PC   | n, n  | n, n     |

Using the Wigner characteristic function of the Fock state (A9) and integrating Eq. (5), we get

\[
\chi_{A_1' A_2} = \hat{F}_1 \exp (A^T M_1 A + u^T M_2 u + u^T M_3 u),
\]

where the column vectors \( A \) and \( u \) are defined as \( (\tau_1, \sigma_1, \tau_2, \sigma_2)^T \) and \( (u_1, v_1, u_2, v_2, u_1', v_1', u_2', v_2') \) respectively, and the matrices \( M_1 \), \( M_2 \), and \( M_3 \) are given in Eqs. (B1), (B3) and (B5) of the Appendix B. Further, the differential operator \( \hat{F}_1 \) is defined as

\[
\hat{F}_1 = 2^{-\frac{(m_1+n_1+n_2)}{2}} \frac{\partial^{n_1} \partial^{m_1} \partial^{m_2} \partial^{m_2}}{\partial \partial \partial \partial} \frac{m_1!m_2!n_1!n_2!}{\partial u_1^{m_1} \partial v_1^{m_1} \partial u_2^{m_2} \partial v_2^{m_2} (u_2 = v_2 = v_2 = 0 \text{ or } u_1 = v_1 = v_1 = v_1 = 0)}.
\]

The normalization factor corresponding to Eq. (6) represents the probability of success of non-Gaussian operations in both the modes and is evaluated as

\[
P_{NG} = \chi_{A_1|A_2}^{NG} |_{\tau_1 = \sigma_1 = \tau_2 = \sigma_2 = 0} = \hat{F}_1 \exp (u^T M_3 u).
\]

The normalized Wigner characteristic function \( \chi_{A_1|A_2}^{NG} \) of NGTMSV states is obtained as

\[
\chi_{A_1|A_2}^{NG} (\tau_1, \sigma_1, \tau_2, \sigma_2) = (P_{NG})^{-1} \chi_{A_1|A_2}^{NG} (\tau_1, \sigma_1, \tau_2, \sigma_2).
\]

Wigner characteristic function of several special states can be obtained from Eq. (9) as limiting cases. By taking the limit \( T_1 \to 1 \) and \( T_2 \to 1 \) in the symmetric PS case with \( m_1 = m_2 = 0 \), we obtain the Wigner characteristic function of the ideal PSTMSV state \( \hat{a}_1^{m_1} \hat{a}_2^{m_2} \) (TMSV).

### III. Teleportation Using NGTMSV Resource States

Having derived the Wigner characteristic function of the NGTMSV states, we proceed to derive the fidelity for teleporting input coherent and squeezed vacuum states. We follow the Braunstein-Kimble (BK) protocol for teleporting an unknown input quantum state between two distant physical systems. To begin with, an entangled resource is shared between Alice and Bob. An unknown input quantum state to be teleported is provided to Alice. The density operator of the entangled resource state and the unknown input state is represented by \( \rho_{A_1' A_2'} \) and \( \rho_{in} \), respectively. Their representation in terms of Wigner characteristic function are \( \chi_{A_1' A_2'} (A_1, A_2) \) and \( \chi_{in} (A_{in}) \), respectively.

Alice combines her mode and the single-mode input state using a balanced beam splitter. After that, the two output modes of the beam splitter are subjected to homodyne measurement by Alice, and the results are classically communicated to Bob. Based on the results, Bob
displaces his mode $A_2'$, and the resultant mode is denoted by ‘out’. The mode ‘out’ corresponds to the teleported state. The Wigner characteristic function allows us to write the teleported state as a product of the input state and the entangled resource state [42]:

$$\chi_{\text{out}}(\tau_2, \sigma_2) = \chi_{\text{in}}(\tau_2, \sigma_2)\chi_{A'_1 A'_2}(\tau_2, -\sigma_2, \tau_2, \sigma_2). \quad (10)$$

We define fidelity of teleportation as the overlap between the single mode input state $\rho_{\text{in}}$ and the teleported state $\rho_{\text{out}}$ to quantify the success of the protocol:

$$F = \text{Tr}[\rho_{\text{in}} \rho_{\text{out}}],$$

$$= \frac{1}{2\pi} \int d^2 \Lambda_2 \chi_{\text{in}}(\Lambda_2)\chi_{\text{out}}(-\Lambda_2). \quad (11)$$

It has been shown that a maximum fidelity of $1/2$ can be achieved without using a shared entangled state for teleporting input coherent state [43, 44]. Hence, successful quantum teleportation is marked by the magnitude of the fidelity rising above the classical limit of $1/2$. We require an infinitely entangled resource state to achieve perfect teleportation with unit fidelity. Further, the fidelity should be greater than $2/3$ for secure teleportation, i.e., for ensuring that any copy of the teleported state is not available with an eavesdropper [45–47].

### A. Teleporting an input coherent state

We now move on to compute the fidelity for teleporting an input coherent state via NGTMSV resource states (9). Using the Wigner characteristic function of the coherent state [Eq. (A14) of Appendix A], the fidelity can be evaluated using Eq. (11), which turns out to be

$$F^{\text{NG coh}} = \tilde{F}_1 \exp \left( u^T M_4 u \right), \quad (12)$$

where the matrix $M_4$ is given in Eq. (B7) of the Appendix B. By taking the limit $T_1 \to 1$ and $T_2 \to 1$ with $m_1 = n_1$, $m_2 = n_2$ in Eq. (12), we obtain the fidelity of quantum teleportation using the TMSV resource state:

$$F^{\text{TMSV}} = \frac{1 + \tanh r}{2}. \quad (13)$$

After deriving the analytical expression of fidelity for teleporting input coherent state, we now proceed to numerical investigation of the fidelity. We first numerically optimize the transmissivities of the beam splitters to maximize the fidelity, and the results are shown in Fig. 2.

While symmetric PSTMSV states show an advantage over the TMSV state, asymmetric PSTMSV states underperform as compared to the TMSV state. Further, the optimal transmissivity is one in this case; hence, the results correspond to ideal PSTMSV states.

We then observe that neither symmetric nor asymmetric PATMSV states improve the performance compared to the TMSV state. In this case, the optimal transmissivity also turns out to be one; therefore, these results correspond to ideal PATMSV states. We note that the quantum teleportation using ideal PSTMSV and PATMSV states as a resource has already been investigated in Refs. [18, 22].

Finally, we observe that only symmetric PCTMSV states enhance fidelity over the TMSV state. As can be seen in Fig. 2(c), this enhancement is observed till a certain squeezing threshold beyond which the fidelity is optimized at unit transmissivity. As we can see from the schematic in Fig. 1 that the output state at unit transmissivity in the case of PC operation is simply the
TMSV state. Therefore, the fidelity at unit transmissivity is equal to that of the TMSV state. This optimization of fidelity for PCTMSV states has been performed in Ref. [21]; however, their results for PCTMSV states show a lower fidelity than the TMSV state in the region where our results show equal fidelity for PCTMSV and TMSV states. Further, the Asym 1,2-PCTMSV state also enhances the fidelity compared to the TMSV state. We note that Asym 1,2-PCTMSV state can be generated by setting the parameters as \( m_1 = n_1 = 1 \) and \( m_2 = n_2 = 2 \).

1. Relative enhancement in fidelity

In the previous section, we studied the absolute performance of the NGTMSV states. Let us consider, for instance, the case of PSTMSV resource states [Fig. 2(a)], where the fidelity is maximized for high values of squeezing. For large values of squeezing, the original TMSV state performs almost the same as the PSTMSV state. Therefore, performing the non-Gaussian operation in the high squeezing range is not preferable. In order to find the squeezing value where implementing non-Gaussian operation provides the maximum advantage, we define a figure of merit as the difference in teleportation fidelity between the NGTMSV states and the TMSV state as

\[
\Delta F_{\text{NG}} = F_{\text{NG}} - F_{\text{TMSV}}. \tag{14}
\]

The squeezing parameter that renders maximum \( \Delta F_{\text{NG}} \) corresponds to the maximum advantage in implementing non-Gaussian operations. We optimize \( \Delta F_{\text{NG}} \) over the transmissivity parameters and since \( F_{\text{TMSV}} \) is independent of transmissivity, the optimized value of \( \Delta F_{\text{NG}} \) can be simply given by

\[
\Delta F_{\text{NG}}^{\text{opt}} = F_{\text{NG}}^{\text{opt}} - F_{\text{TMSV}}. \tag{15}
\]

We plot \( \Delta F_{\text{NG}}^{\text{opt}} \) as a function of squeezing parameter in Fig. 3 for the symmetric PSTMSV and PCTMSV states. While \( \Delta F_{\text{opt}} \) for PSTMSV states is maximized at an intermediate squeezing, \( \Delta F_{\text{opt}} \) for PCTMSV states is maximized in the limit of zero squeezing. We have not considered PATMSV states as they do not provide any advantage over the original TMSV state. We shall see later in this article that PATMSV states may be advantageous over the TMSV state in the teleportation of an input squeezed vacuum state.

As evident from Eq. (15), the optimal transmissivities of the beam splitters maximizing \( F_{\text{NG}} \) and \( \Delta F_{\text{NG}} \) are same. As mentioned earlier, the fidelity is maximized in the unit transmissivity limit for the PSTMSV and PATMSV states, irrespective of the squeezing. For the PCTMSV states, the optimal transmissivities as a function of the squeezing parameter are shown in Fig. 4(a). While the fidelity for the Sym \( n \)-PCTMSV states is maximized for \( T_1 = T_2 \), the fidelity for the Asym 1,2-PCTMSV state is maximized when \( T_1 \neq T_2 \).

\[ \text{FIG. 3. The optimized difference of teleportation fidelity between the NGTMSV and the TMSV states, } \Delta F_{\text{NG}}^{\text{opt}}, \text{ as a function of the squeezing parameter for different non-Gaussian states. The transmissivities have been optimized to maximize } \Delta F_{\text{NG}}. \]

\[ \text{FIG. 4. (a) Optimal beam splitter transmissivities and (b) corresponding probabilities as a function of the squeezing parameter for different PCTMSV states. The beam splitter transmissivities and corresponding probabilities have been truncated at the minimum squeezing where the TMSV state and the PCTMSV state have the same fidelity.} \]

The success probability of non-Gaussian operations, \( P_{\text{NG}} \), which can be thought of as the fraction of successful non-Gaussian operations per trial, quantifies the resource utilization. For PSTMSV and PATMSV states,
the success probability approaches zero in the unit transmissivity limit. Here the resource utilization approaches zero; therefore, this scenario is highly undesirable from an experimental point of view. Further, for the PCTMSV states, we have shown the success probability corresponding to the optimal transmissivity as a function of the squeezing parameter in Fig. 4(b). As we observed in Fig. 3, the maximum enhancement in fidelity is obtained in the zero squeezing limit for the PCTMSV resource states, but the corresponding success probability approaches zero. Therefore, this situation is also undesirable.

2. Relative enhancement in fidelity per trial

As we saw in the previous section, working at optimal squeezing and transmissivity parameters maximizing $\Delta F_{\text{NG}}$ may not represent the best scenario. To visualize this explicitly, we show $\Delta F_{\text{PS}}$ and $P_{\text{PS}}$ as a function of the transmissivity for the Sym 1-PSTMSV resource state in Fig. 5. While $\Delta F_{\text{PS}}$ is maximized in the unit transmissivity limit, the success probability approaches zero, which is an undesirable scenario for the generation of non-Gaussian states. To find an optimal scenario, we can trade-off between the success probability and $\Delta F_{\text{PS}}$ by adjusting the transmissivity.

![Graph of $\Delta F_{\text{PS}}$ and $P_{\text{PS}}$ vs $T$](image)

**FIG. 5.** The fidelity difference $\Delta F_{\text{PS}}$ between the Sym 1-PSTMSV and the TMSV states, and preparation probability $P_{\text{PS}}$ of the Sym 1-PSTMSV state as a function of transmissivity $T$ of the beam splitter. The squeezing of the resource state is taken to be $r = 0.5$.

Similarly, to find the optimal scenario for different NGTMSV states, we can trade-off between the success probability and $\Delta F_{\text{NG}}$ by adjusting the transmissivities of the beam splitters for a given squeezing. In the following analysis, we find optimal transmissivity parameters that renders the product of $\Delta F_{\text{NG}}$ and $P_{\text{NG}}$ maximum.

We plot the optimized product $(P \times \Delta F)^{\text{NG}}$ as a function of squeezing in Fig. 6. The results reveal that 1-Sym PC operation provides the maximum advantage when the success probability is taken into account. Further, among the considered non-Gaussian operations in Fig. 6, all different PC operations outperform the PS operations.

![Graph of $(P \times \Delta F)^{\text{NG}}$ vs $r$](image)

**FIG. 6.** The optimized product $(P \times \Delta F)^{\text{NG}}$ as a function of the squeezing parameter. The transmissivities have been optimized to maximize the product.

Figure 7 shows the optimal beam splitter transmissivities and the corresponding success probabilities, maximizing the product $P_{\text{NG}} \times \Delta F_{\text{NG}}$ as a function of the squeezing parameter. We notice that the success probability of the Sym 1-PSTMSV state is $\approx 3\%$ at the squeezing parameter maximizing the product of $\Delta F_{\text{NG}}$ and $P_{\text{NG}}$. Similarly, the success probability of the Sym 1-PCTMSV state is $\approx 4\%$. These success probabilities are significantly larger than those obtained while maximizing $\Delta F_{\text{NG}}$. By working at optimal conditions, these non-Gaussian operations can be effectively implemented in a lab for fidelity enhancement.

We compare our results with previous investigations on entanglement which considered the optimization of $\Delta E_{\text{NG}} \times P_{\text{NG}}$, where ‘E’ stands for entanglement [26]. It was shown that 1-Asym PC operation provides maximum advantages for low squeezing while 1-Asym PA operation provides maximum advantages for high squeezing.

As shown in Fig. 1, the auxiliary modes are initialized to Fock states in PA and PC operations, in contrast to vacuum states ($|00\rangle_{A_1A_2}$) in PS operation. These Fock states can be generated by photon-number measurement on one mode of the TMSV state. The success probability of producing Fock state $|m\rangle$ is

$$P_{\{m\}} = (1 - \lambda^2)\lambda^{2m}, \quad \text{with } \lambda = \tanh r. \quad (16)$$

Therefore, the effective probability is defined as

$$P_{\text{eff}} = (P_{\{m_1\}} P_{\{m_2\}}) P_{\text{NG}}. \quad (17)$$

Parameters maximizing the product $\Delta F_{\text{NG}} \times P_{\text{eff}}$ enable us to assess the efficacy of the different non-Gaussian resource states. To obtain the optimal parameters, we optimize the product $P_{\text{PC}} \times \Delta F_{\text{PC}}$ over the transmissivity parameters and the result is shown in Fig. 8. For 1-Sym PC operation, the maximum value of the optimized product $P_{\text{eff}} \times \Delta F_{\text{opt}}$ is $\approx 5 \times 10^{-5}$. In contrast, for 1-Sym PS operation, the maximum value of the optimized product $P \times \Delta F_{\text{opt}}$ is $\approx 1 \times 10^{-3}$. Therefore, 1-Sym...
The transmissivities have been optimized to maximize the product. We take the probability $P_{m_1} P_{m_2}$ to generate Fock states into account, the effective probability $P_{\text{eff}} = (P_{m_1} P_{m_2})^{\text{PMSV}}$ for PA and PC operations becomes very low as compared to PS operation. To see this explicitly, we have added Table II, showing the magnitudes of $\Delta F$ and $P_{\text{eff}}$ at optimal parameters [Figs. 6 and 8].

While the fidelity is of the same order for 1-Sym PS and 1-Sym PC operations [21, 22], the magnitude of $P_{\text{eff}} = (P_{11} P_{11})^{\text{PC}}$ (for PC operation) is one order low as compared to that of $P_{\text{PS}}$ and $P_{\text{PC}}$. This renders the PS operation more advantageous compared to PC operation.

Upon accounting for the probability of Fock state generation $P_{m_1} P_{m_2}$, the entanglement study of Ref. [26] observes that 1-Asym PC operation remains the maximum advantageous operation for low squeezing; however, for high squeezing, 1-Asym PS operation provides the maximum advantage.

### Table II. Maximum value of the product $(P \times \Delta F)_\text{opt}$ and corresponding $P$ and $\Delta F$.

| Operation      | $(P \times \Delta F)_\text{max}$ | $P$ ≈ $0.03$ | $\Delta F$ ≈ $0.04$ |
|----------------|----------------------------------|--------------|----------------------|
| 1-Sym PS       | $1 \times 10^{-5}$               | 1-Sym PC     | $3 \times 10^{-5}$   |
| 1-Sym PC       | $5 \times 10^{-5}$               | 1-Sym PC     | $5 \times 10^{-5}$   |

### B. Teleporting an input squeezed vacuum state

We now derive the analytical fidelity expression for teleporting an input squeezed vacuum state with squeezing $\epsilon$ using NGTMSV resource states. The expression of the Wigner characteristic function of squeezed vacuum state is given in Eq. (A15) of the Appendix A. Using the general formula for the fidelity of teleportation (11), we can evaluate the fidelity in this case, which turns out to be

$$F_{\text{sqv}} = \exp \left( u^T M_5 u \right),$$

where the matrix $M_5$ is given in Eq. (B9) of the Appendix B. Setting the limit $T_1 \rightarrow 1$ and $T_2 \rightarrow 1$ with $m_1 = n_1$, $m_2 = n_2$ in Eq. (18) yields the fidelity of teleporting an input squeezed vacuum state using TMSV resource state:

$$F_{\text{TMSV}} = \left[ \frac{1 + \tanh (r + \epsilon)}{2} \right]^{1/2} \left[ \frac{1 + \tanh (r - \epsilon)}{2} \right]^{1/2}.$$

On putting $\epsilon = 0$ in Eq. (19), we obtain the fidelity of teleporting an input coherent state using TMSV resource state (13).

We now numerically analyze the fidelity of teleporting an input squeezed vacuum state. We first optimize the
FIG. 9. Optimized fidelity for teleporting input squeezed vacuum state as a function of the squeezing parameter for (a) PSTMSV states, (b) PATMSV states, and (c) PCTMSV states. The transmissivities of the beam splitters are optimized in order to maximize the fidelity. We have set the squeezing of the input squeezed vacuum state as $\epsilon = 1.7$.

We now analyze the fidelity as a function of the squeezing $\epsilon$ of the input squeezed vacuum state. While the absolute fidelity decreases as $\epsilon$ is increased for NGTMSV and TMSV resource states, interesting observations can be made by analyzing $\Delta F_{\text{opt}}^{\text{NG}}$ as a function of $\epsilon$. In Fig. 12, we plot $\Delta F_{\text{opt}}^{\text{NG}}$ as a function of squeezing $\epsilon$ of the input squeezed vacuum state for different squeezing of the resource states. As $\epsilon$ increases in the case of Sym 1-PSTMSV resource state, the optimized difference $\Delta F_{\text{opt}}^{\text{PS}}$
FIG. 11. The optimized difference of teleportation fidelity between the NGTMSV and the TMSV states, $\Delta F_{\text{NG opt}}$, as a function of the squeezing parameter for different non-Gaussian states. The transmissivities have been optimized to maximize $\Delta F_{\text{NG}}$. We have set the squeezing of the input squeezed vacuum state as $\epsilon = 1.7$.

improves, attains a maximum value, and then starts decreasing. The Sym 1-PATMSV resource state outperforms the TMSV state in a region of $\epsilon$ indicated by the positive value of the optimized difference $\Delta F_{\text{PA opt}}$. Here too, we notice that as $\epsilon$ increases, the optimized difference $\Delta F_{\text{PA opt}}$ improves, attains a maximum value, and then starts decreasing. For Sym 1-PCTMSV resource state, the optimized difference $\Delta F_{\text{PC opt}}$ continuously decreases as $\epsilon$ is increased. One interesting behavior observed for different NGTMSV states is that the fidelity is almost constant for small $\epsilon$, and the inflection in fidelity is observed for higher $\epsilon$. Therefore, the analysis for input coherent state will resemble that of input squeezed vacuum state for small $\epsilon$ ($< 0.5$).

2. Relative enhancement in fidelity per trial

We now consider the success probability of non-Gaussian operations as well as the probability of the generation of Fock states. Since the success probability is independent of $\epsilon$, the analysis in Sec. III B 1 demonstrates the trend for quantities $(P \times \Delta F)^{\text{NG}}_{\text{opt}}$ or $(P_{\text{eff}} \times \Delta F)^{\text{NG}}_{\text{opt}}$ with respect to the squeezing $\epsilon$ of the input squeezed vacuum state.

We now examine $P^{\text{NG}} \times \Delta F^{\text{NG}}_{\text{opt}}$ as a function of the squeezing of resource states. The results are shown in Fig. 13. We observe that the 1-Sym PS operation provides a maximum advantage when the success probability is taken into account. This contrasts with the teleportation of input coherent state, where 1-Sym PC operation provides the maximum advantage. However, we note that the 1-Sym PC operation is just a little behind the 1-Sym PS operation in the case of input squeezed vacuum state teleportation.

We have also shown the optimal beam splitter transmissivities maximizing the product $P^{\text{NG}} \times \Delta F^{\text{NG}}_{\text{opt}}$ as a function of squeezing parameter in Fig. 14.

Finally, we plot the optimized product $(P_{\text{eff}} \times \Delta F)^{\text{NG}}_{\text{opt}}$ as a function of the squeezing parameter in Fig. 15. We see that the 1-Sym PA operation outperforms the 1-Sym PC operation. However, 1-Sym PS operation outperforms both 1-Sym PA and 1-Sym PC operations, which can be seen explicitly in Table III.

While the fidelity is of the same order for the considered non-Gaussian operations [21, 22], the magnitude of $P_{\text{eff}}$ for PA and PC operation is one order less compared
1-Sym PS \((P \times \Delta F)_{\text{NG}}^{\text{opt}}\) and corresponding \(P\) and \(\Delta F\).

| Operation       | \((P \times \Delta F)_{\text{max}}\) | \(P\)   | \(\Delta F\)   |
|-----------------|-----------------|-------|--------|
| 1-Sym PS        | \(\approx 1.6 \times 10^{-3}\)    | \(\approx 0.03\) | \(\approx 0.05\) |
| 1-Sym PA        | \(\approx 0.9 \times 10^{-3}\)    | \(\approx 0.03\) | \(\approx 0.03\) |
| 1-Sym PA        | \(\approx 5 \times 10^{-5}\)     | \(P_{\text{eff}} \approx 0.0016\) | \(\Delta F \approx 0.03\) |
| 1-Sym PC        | \(\approx 1.5 \times 10^{-3}\)    | \(P \approx 0.04\) | \(\Delta F \approx 0.03\) |
| 1-Sym PC        | \(\approx 2.6 \times 10^{-5}\)    | \(P_{\text{eff}} \approx 0.0014\) | \(\Delta F \approx 0.02\) |

\(P_{\text{NG}}\), the probability of the considered non-Gaussian operations. This renders the PS operation to be more advantageous as compared to PA and PC operations.

IV. IMPERFECT DETECTORS

In the preceding sections, we have considered the perfect photon detector, i.e., unit quantum efficiency detectors, employed in the implementation of non-Gaussian operations [Fig. 1]. In this section, we investigate the effects of imperfect detectors, i.e., non-unit quantum efficiency detectors, on the teleportation fidelity. An imperfect detector of efficiency \(\eta\) (without dark counts) can be modeled as a perfect detector of unit efficiency preceded by a beam splitter of transmissivity \(\eta\) [48, 49].

We first analyze the fidelity of teleporting coherent state \(F_{\text{coh}}^{\text{NG}}\) as a function of squeezing parameter \(r\) for different efficiencies \(\eta\) of the detector in Fig. 17. The considered resource states are 1-Sym PSTMSV state and 1-Sym PCTMSV state in Fig. 17(a) and (b), respectively. The results reveal that the fidelity decreases with decreasing efficiency; however, the fidelity of the non-Gaussian resource states can outperform the TMSV state in a certain squeezing range. We also observe that the detector’s inefficiency has a less detrimental effect on the fidelity of
1-Sym PSTMSV than the 1-Sym PCTMSV state. Further, the effect becomes more significant at larger squeezing values.

FIG. 17. Fidelity of teleporting coherent state as a function of squeezing parameter $r$ for different efficiencies of the detector. The considered resource state is (a) 1-Sym PSTMSV state and (b) 1-Sym PCTMSV state. The values of the beam splitter transmissivities involved in the non-Gaussian operation have been taken to be (a) $T_1 = T_2 = 0.8$ and (b) $T_1 = T_2 = 0.2$.

We now turn to the analysis of the fidelity of teleporting squeezed vacuum state $F_{\text{sqv}}^\text{NG}$ as a function of squeezing parameter $r$ for different detector efficiencies $\eta$ in Fig. 18. The considered resource states are 1-Sym PSTMSV state, 1-Sym PATMSV state, and 1-Sym PCTMSV state in Fig. 18(a), (b), and (c), respectively. While 1-Sym PSTMSV and 1-Sym PCTMSV resource states yield higher fidelity than TMSV states, 1-Sym PATMSV states do not provide any advantages over TMSV even at $\eta = 0.9$.

If the detector efficiency is low, the teleportation fidelity can drop significantly. However, single photon detectors with over 90% efficiency are experimentally available [50–52], and the corresponding drop in fidelity is not significant; this renders the non-Gaussian operation advantageous. In addition to the imperfect photon detectors, imperfect homodyne measurements and dissipation due to interaction with environments need to be taken into account. These problems will be taken up somewhere else.

FIG. 18. Fidelity of teleporting squeezed vacuum state as a function of squeezing parameter $r$ for different efficiencies of the detector. The considered resource state is (a) 1-Sym PSTMSV state, (b) 1-Sym PATMSV state, and (c) 1-Sym PCTMSV state. The values of the beam splitter transmissivities involved in the non-Gaussian operation have been taken to be (a) $T_1 = T_2 = 0.8$, (b) $T_1 = T_2 = 0.9$ and (c) $T_1 = T_2 = 0.2$.

V. CONCLUSION

We have systematically investigated the optimal conditions for the teleportation of input coherent and squeezed vacuum states using various non-Gaussian resource states while considering the probabilistic preparation of resource states. Our work complements and extends the non-Gaussian quantum teleportation studies performed in Refs. [18, 20–22]. While fidelity enhance-
ment using NGTMSV resource states was demonstrated in Refs. [18, 20–22], but the probabilistic considerations were overlooked. Neglecting the success probability can lead to undesirable scenarios. For instance, the ideal PSTMSV state (studied in Refs. [18, 22]), which maximizes the teleportation fidelity, has a success probability approaching zero (Fig. 5). Therefore, this scenario is highly non-optimal from the point of view of resource utilization. In this article, we have provided a systematic treatment for the optimal performance of the teleportation protocol by trade-off between the teleportation fidelity and the probability of state preparation.

To summarize the main results, the investigation of the fidelity difference between NGTMSV and TMSV states shows that the PSTMSV states provide a maximum advantage at intermediate values of squeezing parameters, whereas, at small squeezing, the PCTMSV states provide a maximum advantage. Further, taking the preparation probability of non-Gaussian states into account, we obtain optimal conditions that can be effectively implemented in a lab to enhance fidelity. Furthermore, the symmetric 1-PSTMSV state is the most beneficial resource state among the considered non-Gaussian states. We have also provided the optimal squeezing and beam splitter transmissivity values maximizing the performance. We believe these results will be consequential in the experimental realization of non-Gaussian teleportation protocols.

Another significant result of our paper is the derivation of a unified analytical expression of the Wigner characteristic function of the NGTMSV states, which does not exist in the literature as far as we know. This derived Wigner characteristic function depends on the transmissivity of the beam splitters and the squeezing of the resource states. Further, by choosing the input photon state and the number of detected photons in the auxiliary modes, which also appear in the Wigner characteristic function, one can subtract, add, or catalyze an arbitrary number of photons from the TMSV state. We also examined the effects of imperfect photon detectors on the teleportation fidelity. While the fidelity decreases due to imperfect detectors, the availability of high efficiency photon detectors renders the implementation of non-Gaussian operations on TMSV state advantageous.

The unified Wigner characteristic function will be useful in evaluating two-mode squeezing [53], Mandel-Q parameter [54], studying antibunching effects [55], non-Gaussianity [56] and nonlocality [57] and its considerations in entanglement distillation, entanglement swapping, and quantum illumination protocols. Furthermore, our work may inspire several future investigations identifying optimal conditions in various non-Gaussian QIP protocols.

ACKNOWLEDGEMENT

Both the authors thank Rishabh and Mohak Sharma for a careful reading of the final version of the draft. C.K. acknowledges the financial support from DST/ICPS/QuST/Theme-1/2019/General Project number Q-68.

Appendix A: Brief description of CV systems and its phase space description

Our system of interest is an $n$-mode CV system, whose $i^{\text{th}}$ mode can be expressed by a pair of Hermitian quadrature operators $\hat{q}_i$ and $\hat{p}_i$. We arrange these $n$ pairs in the form of a column vector as $[58–62]$

$$\hat{\xi} = (\hat{\xi}_i) = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)^T, \quad i = 1, 2, \ldots, 2n.$$  

This permits us to write the bosonic commutation relation between them compactly as $(h=1)$

$$[\hat{\xi}_i, \hat{\xi}_j] = i\Omega_{ij}, \quad (i, j = 1, 2, \ldots, 2n),$$

where $\Omega$ is the $2n \times 2n$ matrix given by

$$\Omega = \bigoplus_{k=1}^{n} \omega = \begin{pmatrix} \omega & \cdots & \omega \\ \cdots & \ddots & \cdots \\ \omega & \cdots & \omega \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (A3)$$

The photon annihilation and creation operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ ($i = 1, 2, \ldots, n$) are given as

$$\hat{a}_i = \frac{1}{\sqrt{2}}(\hat{q}_i + i\hat{p}_i), \quad \hat{a}_i^\dagger = \frac{1}{\sqrt{2}}(\hat{q}_i - i\hat{p}_i). \quad (A4)$$

We shall be concerned with two symplectic operations discussed below [58, 61].

**Beam splitter operation:** The two mode beam splitter operation acts on the quadrature operators $\xi = (\hat{q}_i, \hat{p}_i, \hat{q}_j, \hat{p}_j)^T$ of a two mode system as follows:

$$B_{ij}(T) = \begin{pmatrix} \sqrt{T} \mathbb{1}_2 & \sqrt{1-T} \mathbb{1}_2 \\ -\sqrt{1-T} \mathbb{1}_2 & \sqrt{T} \mathbb{1}_2 \end{pmatrix}, \quad (A5)$$

where $\mathbb{1}_2$ is the $2 \times 2$ identity matrix.

**Two mode squeezing operation:** The two mode squeezing operation acts on the quadrature operators $(\hat{q}_i, \hat{p}_i, \hat{q}_j, \hat{p}_j)^T$ as follows:

$$S_{ij}(r) = \begin{pmatrix} \cosh r \mathbb{1}_2 & \sinh r \mathbb{Z} \\ \sinh r \mathbb{Z} & \cosh r \mathbb{1}_2 \end{pmatrix}, \quad (A6)$$

where $\mathbb{Z} = \text{diag}(1, -1)$. The TMSV state is obtained by the action of two mode squeezing operator on two single mode vacuum state.
1. Phase space description

Working with Wigner characteristic function turns out to be convenient in quantum teleportation. We can find the Wigner characteristic function corresponding to a density operator $\rho$ of an $n$-mode quantum system using the relation

$$\chi(\Lambda) = \text{Tr}[\rho \exp(-i\Lambda^T\Omega\xi)],$$  \hspace{1cm} (A7)

where $\xi = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)^T$, $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_n)^T$ with $\Lambda_i = (\tau_i, \sigma_i)^T \in \mathbb{R}^2$. For instance, Eq. (A7) can be used to evaluate the Wigner characteristic function of a single mode Fock state $|n\rangle$:

$$\chi_{|n\rangle}(\tau, \sigma) = \exp\left[-\frac{\tau^2}{4} - \frac{\sigma^2}{4}\right] L_n\left(\frac{\tau^2}{2} + \frac{\sigma^2}{2}\right),$$ \hspace{1cm} (A8)

where $L_n(x)$ is the Laguerre polynomial. Writing the above equation in terms of exponential generating function, we get

$$\chi_{|n\rangle}(\tau, \sigma) = \exp\left[-\frac{\tau^2}{4} - \frac{\sigma^2}{4}\right] \tilde{F} e^{2st + s(\tau + i\sigma) - t(\tau - i\sigma)},$$ \hspace{1cm} (A9)

with

$$\tilde{F} = \frac{1}{2^{n!}} \frac{\partial^n}{\partial s^n} \frac{\partial^n}{\partial \tau^n} (\bullet)_{s=t=0}.$$ \hspace{1cm} (A10)

We define the first order moments for an $n$-mode CV system as

$$d = \langle \hat{\xi} \rangle = \text{Tr}[\rho\hat{\xi}].$$ \hspace{1cm} (A11)

Further, the second order moments can be written in the form of a real symmetric $2n \times 2n$ matrix, known as covariance matrix:

$$V = (V_{ij}) = \frac{1}{2} \{[\Delta \hat{\xi}_i, \Delta \hat{\xi}_j]\},$$ \hspace{1cm} (A12)

where $\Delta \hat{\xi}_i = \hat{\xi}_i - \langle \hat{\xi}_i \rangle$, and $\{ \ , \ \}$ denotes anti-commutator.

A special class of states, whose Wigner characteristic function is a Gaussian, are known as Gaussian states. Such states can be uniquely specified via its first and second order moments. The general formula for the Wigner characteristic function (A7) simplifies as follows for Gaussian states [61, 63]:

$$\chi(\Lambda) = \exp[-\frac{1}{2} \Lambda^T(\Omega V \Omega^T)\Lambda - i(\Omega d)^T\Lambda],$$ \hspace{1cm} (A13)

where $d$ and $V$ represents the displacement vector and the covariance matrix of the Gaussian state. The Wigner characteristic function of a single mode coherent state with displacement $d = (dx, dp)^T$ evaluates to

$$\chi_{\text{coh}}(\Lambda) = \exp\left[-\frac{1}{4}(\tau^2 + \sigma^2) - i(\tau dp - \sigma dx)\right].$$ \hspace{1cm} (A14)

The Wigner characteristic function of a single mode squeezed vacuum state turns out to be

$$\chi_{\text{svq}}(\Lambda) = \exp\left[-\frac{1}{4}(\tau^2 e^{2r} + \sigma^2 e^{-2r})\right].$$ \hspace{1cm} (A15)

Let $U(S)$ represent the infinite dimensional unitary representation for a homogeneous symplectic transformation $S$. Given the density operator transformation rule as $\rho \rightarrow U(S)\rho U(S)^\dagger$, the transformation of the displacement vector, covariance matrix and Wigner characteristic function turns out to be [58, 61, 63]

$$d \rightarrow Sd, \quad V \rightarrow SVS^T, \quad \text{and} \quad \chi(\Lambda) \rightarrow \chi(S^{-1}\Lambda).$$ \hspace{1cm} (A16)

Appendix B: Matrices appearing in the Wigner characteristic function, and the fidelity of teleportation using NGTMSV states.

1. Wigner characteristic function of the NGTMSV states

Here we provide the explicit forms of the matrices $M_1$, $M_2$ and $M_3$, which appear in the Wigner characteristic function (6) of the NGTMSV states. The matrix $M_1$ is given as

$$M_1 = -\frac{1}{4a_0} \begin{pmatrix} a_1 & 0 & -a_2 & 0 \\ 0 & a_1 & 0 & a_2 \\ -a_2 & 0 & a_1 & 0 \\ 0 & a_2 & 0 & a_1 \end{pmatrix},$$ \hspace{1cm} (B1)

where,

$$a_0 = \beta^2 - \alpha^2 t_1^2 t_2^2,$$
$$a_1 = \beta^2 + \alpha^2 t_1^2 t_2^2,$$
$$a_2 = 2\alpha \beta t_1 t_2.$$ \hspace{1cm} (B2)

Here $t_i = \sqrt{T_i}$ ($i = 1, 2$). Further $\alpha = \sinh r$ and $\beta = \cosh r$. The matrix $M_2$ is given by

$$M_2 = \frac{1}{a_0} \begin{pmatrix} b_1 & ib_1 & b_2 & -ib_2 \\ -b_1 & ib_1 & -b_2 & -ib_2 \\ b_3 & -ib_3 & b_4 & ib_4 \\ -b_3 & -ib_3 & b_4 & -ib_4 \end{pmatrix},$$ \hspace{1cm} (B3)

where,

$$b_1 = \beta^2 r_1,$$
$$b_2 = -\alpha \beta r_1 t_2,$$
$$b_3 = -\alpha \beta r_2 t_1,$$
$$b_4 = \beta^2 r_2,$$
$$b_5 = -\alpha^2 r_1 t_1 t_2^2,$$
$$b_6 = \alpha \beta r_1 t_2,$$
$$b_7 = \alpha \beta r_2 t_1,$$
$$b_8 = -\alpha^2 r_2 t_1 t_2^2.$$ \hspace{1cm} (B4)
Here $r_i = \sqrt{T_i}$ ($i = 1, 2$). The matrix $M_3$ is given by

$$
M_3 = \frac{1}{a_0} \begin{pmatrix}
0 & c_1 & 0 & c_2 & 0 & c_3 & 0 & c_4 & 0 \\
c_1 & 0 & 0 & c_2 & 0 & c_3 & 0 & c_4 & 0 \\
c_2 & 0 & 0 & c_5 & 0 & c_6 & 0 & c_7 & 0 \\
0 & c_2 & c_5 & 0 & 0 & c_6 & 0 & c_7 & 0 \\
0 & c_3 & c_6 & 0 & 0 & c_8 & 0 & c_9 & 0 \\
c_3 & 0 & 0 & c_6 & 0 & c_8 & 0 & c_9 & 0 \\
c_4 & 0 & 0 & c_7 & 0 & c_9 & 0 & c_{10} & 0 \\
0 & c_4 & c_7 & 0 & 0 & c_9 & 0 & c_{10} & 0 \\
0 & c_4 & 0 & c_7 & 0 & c_9 & 0 & c_{10} & 0
\end{pmatrix},
$$

where,

$$
c_1 = \beta^2 r_1^2, \quad c_6 = -\alpha \beta r_1 r_2 t_2,
$$

$$
c_2 = \alpha \beta r_1 r_2 t_2, \quad c_7 = \beta^2 t_2 - \alpha^2 t_1^2 t_2,
$$

$$
c_3 = \beta^2 t_1 - \alpha^2 t_1^2 t_2, \quad c_8 = \alpha r_1 r_2 t_2,
$$

$$
c_4 = -\alpha \beta r_1 r_2 t_2, \quad c_9 = \alpha r_1 r_2 t_2,
$$

$$
c_5 = \beta^2 r_2^2, \quad c_{10} = \alpha^2 r_2^2 t_1.
$$

2. Fidelity for input coherent state using NGTMSV states

The explicit form of matrix $M_4$ appearing in the fidelity of teleportation of input coherent state using NGTMSV states (12) is

$$
M_4 = \frac{1}{d_0} \begin{pmatrix}
0 & d_1 & 0 & 0 & d_2 & c_4 & 0 \\
d_1 & 0 & 0 & d_2 & 0 & 0 & c_4 \\
d_2 & 0 & 0 & d_1 & 0 & 0 & c_3 \\
0 & d_1 & c_5 & 0 & d_2 & 0 & c_3 \\
0 & d_2 & c_6 & 0 & c_5 & 0 & d_3 \\
0 & d_2 & c_6 & 0 & c_8 & 0 & d_4 \\
c_4 & 0 & d_3 & 0 & d_4 & 0 & c_10 \\
0 & c_4 & d_3 & 0 & d_4 & 0 & c_{10}
\end{pmatrix},
$$

where,

$$
d_0 = 2\beta (\beta - \alpha t_1 t_2),
$$

$$
d_1 = \beta^2 r_1 r_2,
$$

$$
d_2 = \beta (2\beta t_1 - \alpha t_2 (t_2^2 + 1)),
$$

$$
d_3 = \beta (2\beta t_2 - \alpha t_1 (t_2^2 + 1)),
$$

$$
d_4 = \alpha \beta r_1 r_2 (2\beta - \alpha t_1 t_2).
$$

3. Fidelity for input squeezed vacuum state using NGTMSV states

The explicit form of matrix $M_5$ appearing in the fidelity of teleportation of input coherent state using NGTMSV states (18) is

$$
M_5 = \frac{1}{e_0} \begin{pmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\
e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \\
e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \\
e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} \\
e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} & e_{13} \\
e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} & e_{13} & e_{14} \\
e_8 & e_9 & e_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} \\
e_9 & e_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\
e_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} \\
e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} & e_{18} \\
e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} & e_{18} & e_{19} \\
e_{13} & e_{14} & e_{15} & e_{16} & e_{17} & e_{18} & e_{19} & e_{20}
\end{pmatrix},
$$

where,

$$
e_0 = 2 (a_0 \delta + a_1),
$$

$$
e_1 = -2 \beta \gamma r_1^2,
$$

$$
e_2 = 2 \beta \gamma r_1^2 \left( \frac{a_0}{a_3} + \delta \right),
$$

$$
e_3 = \beta r_1 r_2 \left( \delta \left( \frac{a_0}{\sqrt{a_3}} + \alpha t_1 t_2 \right) + \frac{a_1}{\sqrt{a_3}} - \alpha t_1 t_2 \right),
$$

$$
e_4 = \beta^2 \gamma r_1 r_2,
$$

$$
e_5 = -\alpha \beta \gamma r_1 r_2 t_2,
$$

$$
e_6 = \frac{c_0}{a_0} \left( 1 + \alpha^2 t_2^2 - \alpha^2 r_1^2 t_2^2 a_0 a_0 + a_0 a_0 \frac{a_0}{a_0} \right),
$$

$$
e_7 = -\alpha \beta r_1 r_2 t_1 \left( \frac{a_0}{a_3} + \delta \right),
$$

$$
e_8 = \alpha \beta \gamma r_1 r_2 t_1,
$$

$$
e_9 = -\alpha \beta \gamma r_1 r_2 t_1,
$$

$$
e_10 = \beta^2 r_2^2 \left( \frac{a_0}{a_3} + \delta \right),
$$

$$
e_11 = -\alpha \beta r_1 r_2 t_2 \left( \frac{a_0}{a_3} + \delta \right),
$$

$$
e_12 = \alpha \beta r_1 r_2 t_2,
$$

$$
e_13 = -\alpha \beta \gamma r_2^2 t_1,
$$

$$
e_14 = \frac{c_0}{a_0} \left( 1 + \alpha^2 r_1^2 - \alpha^2 r_2^2 t_1^2 a_0 a_0 + a_0 a_0 \frac{a_0}{a_0} \right),
$$

$$
e_15 = -\alpha^2 \beta \gamma r_1^2 t_2,
$$

$$
e_16 = \alpha^2 r_2^2 t_2 \left( \frac{a_0}{a_3} + \delta \right),
$$

$$
e_17 = \alpha r_1 r_2 \left( \frac{2 \beta^2}{\sqrt{a_3}} - \alpha t_1 t_2 \frac{a_0}{a_3} + \frac{a_1}{\sqrt{a_3}} + \beta \right),
$$

$$
e_18 = \alpha^2 \beta \gamma r_1 r_2 t_1 t_2,
$$

$$
e_19 = -\alpha^2 \beta \gamma r_2^2 t_1,
$$

$$
e_20 = \alpha^2 r_2^2 t_2 \left( \frac{a_0}{a_3} + \delta \right),
$$

with $\gamma = \sinh(2e)$, $\delta = \cosh(2e)$ and $a_3 = (a_1 - a_2)$. (B10)
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