ROBBA’S METHOD ON EXPONENTIAL SUMS

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Abstract. In this article, we use Robba’s method to give an estimate of the Newton polygon for the $L$-function and we can draw the Newton polygon in some special cases.

1. Introduction

The basic objects of this study are exponential sums on a torus of dimension $n$ defined over a finite field $k$ with $\text{char}(k) = p$. Our methods are based on the work of Dwork, Adolphson and Sperber. In [4], Robba gives an explicit calculation of one variable twisted exponential sums. In fact, his method can be applied to the case of multi-variables.

Let $ζ_p$ be a primitive $p$-th root of unity. Let $ψ$ be the additive character of $k$ given by $ψ(t) = ω_{Trk/k_p}(t)$. Let $f$ be a Laurent polynomial and write

$$f = \sum_{i=1}^{N} a_i x^{w_i} \in k[x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}].$$

We assume that $a_i \neq 0$ for all $i$. Define exponential sums

$$S_i(f) = \sum_{x \in T^n(k_i)} ψ(Tr_{k_i/k}(f(x))),$$

where $k_i$ are the extensions of $k$ of degree $i$. The $L$-function is defined by

$$L(f, t) = \exp \left( \sum_{i=1}^{\infty} S_i(f) t^i / i \right).$$

In [1, section 2], Adolphson and Sperber use Dwork’s method to prove that $L(f, t)^{(-1)^{n-1}}$ is a polynomial when $f$ is nondegenerate. Moreover, they give a low bound of the Newton polygon of $L(f, t)^{(-1)^{n-1}}$ in [1, section 3], which we call Hodge polygon in this article. In our study, we want to give a more precise result about the Newton polygon when $f$ has only $n$ terms, that is $N = n$. Note that if we assume that $J = (w_1, \cdots, w_n)$ is invertible in $M_n(\mathbb{R})$, we can found a solution
b = (b_1, \cdots, b_n) \in \bar{k}^\times$ such that $a_ib^{w_i} = 1$ for all $i$. From now on, we assume that $(p, \det J) = 1$, $k = \mathbb{F}_p$ and 

$$f = \sum_{i=1}^n x^{w_i}.$$ 

Let $\Delta(f)$ be the Newton polyhedron at $\infty$ of $f$ which is defined to be the convex hull in $\mathbb{R}^n$ of the set $\{w_j\}_{j=1}^n \cup \{(0, \cdots, 0)\}$ and let $C(f)$ be the convex cone generated by $\{w_j\}_{j=1}^n$ in $\mathbb{R}^n$. Let $\text{Vol}(\Delta(f))$ be the volume of $\Delta(f)$ with respect to Lebesgue measure on $\mathbb{R}^n$. We say $f$ is nondegenerate with respect to $\Delta(f)$ if for any face $\sigma$ of $\Delta(f)$ not containing the origin, the Laurent polynomials $\frac{\partial f}{\partial x^i}, i = 1, \cdots, n$ have no common zero in $(\bar{k}^\times)^n$, where $f = \sum_{w_j \in \sigma} a_j x^{w_j}$. Set $M(f) = C(f) \cap \mathbb{Z}^n$. Note that $(p, \det J) = 1$ implies that $f$ is nondegenerate. Since we have assumed that $J$ is invertible, any element $u \in M(f)$ can be uniquely written

$$u = \sum_{i=1}^n r_i w_i. \tag{1.1}$$

We define a weight on $M(f)$

$$w(u) := \sum_{i=1}^n r_i.$$

Note that the set of all elements $u \in M(f)$ such that all $0 \leq r_i < 1$ in the expression (1.1) form a fundamental domain of the lattice $M(f)$. Denote it by $S(\Delta)$. Note that $\text{card}(S(\Delta)) = n! \text{Vol}(\Delta(f)) = \det(J)$ and $(p, \det J) = 1$ imply that $S(\Delta)$ has a natural $p$-action. For any $u = r_1 w_1 + \cdots + r_n w_n \in S(\Delta)$, define

$$p.u = \sum_{i=1}^n \{pr_i\} w_i,$$

where $\{pr_i\}$ is the fractional part of $pr_i$ for each $i$. We say $S(\Delta)$ is $p$-stable under weight function if $w(u) = w(p.u)$ for any $u \in S(\Delta)$. Now we give our main result.

**Theorem 1.1.** Suppose that $f = x^{w_1} + \cdots + x^{w_n}$ with $w_i \in \mathbb{Z}^n$ and $(p, \det J) = 1$. The Newton polygon of $L(f, t)^{(n-1)^{n-1}}$ coincides with the Hodge polygon of $\Delta(f)$ if and only if $S(\Delta)$ is $p$-stable under weight function.

Wan uses the Gauss sum to give an explicit formula of the $L$-function for the diagonal Laurent polynomial. Then he uses Stickelberger’s theorem to give a proof of above theorem. See [5, Theorem 3.4].
this article, we use Robba’s method to prove above theorem. Indeed, Robba’s method can also be applied to prove [1, Theorem 3.10] and it is easier than the method used in [1, §3].

2. P-ADIC ESTIMATES

Let $\mathbb{Q}_p$ be the $p$-adic numbers. Let $\Omega$ be the completion of the algebraic closure of $\mathbb{Q}_p$. Denote by “ord” the additive valuation on $\Omega$ normalized by $\text{ord}(p) = 1$. The norm on $\Omega$ is given by $|u| = p^{-\text{ord}(u)}$ for any $u \in \Omega$.

Note that there is an integer $M$ such that $w(M(f)) \subset \frac{1}{M}\mathbb{Z}$. In [1, section 1], Adolphson and Sperber introduce a filtration on $R(f) := k[x^M(f)]$ given by

$$R(f)_{i/M} = \left\{ \sum_{u \in M(f)} b_u x^u | w(u) \leq i/M \text{ for all } u \text{ with } b_u \neq 0 \right\}.$$  

The associated graded ring is

$$\bar{R} = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \bar{R}^i/M,$$

where

$$\bar{R}^i/M = R(f)_{i/M} / R(f)_{(i-1)/M}.$$  

For $1 \leq i \leq n$, let $\bar{f}_i$ be the image of $x_i \frac{\partial f}{\partial x_i} \in R(f)_1$ in $\bar{R}^1$. Let $\bar{I}$ be the ideal generated by $\bar{f}_1, \ldots, \bar{f}_n$ in $\bar{R}$. By [1, Theorem 2.14] and [1, Theorem 2.18], $\bar{f}_1, \ldots, \bar{f}_n$ in $\bar{R}$ form a regular sequence in $\bar{R}$ and $\dim_k \bar{R}/\bar{I} = n! \text{Vol}(\Delta(f))$. For each integer $i$, we have a decomposition

$$(2.1) \quad \bar{R}^i/M = \bar{V}^i/M \oplus (\bar{R}^i/M \cap \bar{I}).$$  

Set $a_i = \dim_k \bar{V}^i/M$.

For a non-negative integer $l$, set

$$W(l) = \text{card} \left\{ u \in M(f) | w(u) = \frac{l}{M} \right\}.$$  

Note that this is a finite number for each $l$. Set

$$H(i) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} W(i - lM).$$  

Lemma 2.1. With the notation above. Suppose that $f$ is nondegenerate. Then $H(i) = a_i$ for all integer $i \geq 0$. Moreover, we have

$$H(k) = 0 \text{ for } k > nM, \quad \sum_{k=0}^{nM} H(k) = n! \text{Vol}(\Delta(f)).$$
Proof. By [1, Theorem 2.14], \( \{ \bar{f}_i \}_{i=1}^n \) form a regular sequence in \( \bar{R} \). So

\[
P_{\bar{R}/\bar{I}}(t) = P_{\bar{R}}(t)(1 - t^M)^n,
\]

where \( P_{\bar{R}/\bar{I}} \) (resp. \( P_{\bar{R}} \)) is the Poincaré series of \( \bar{R}/\bar{I} \) (resp. \( \bar{R} \)). On the other hand, we have

\[
P_{\bar{R}/(\bar{f}_1, \ldots, \bar{f}_n)} = \sum_{i=0}^{\infty} a_it^i, \quad P_{\bar{R}}(t) = \sum_{i=0}^{\infty} W(i)t^i.
\]

Hence

\[
a_i = \sum_{l=0}^{n} (-1)^l \binom{n}{l} W(i - lM) = H(i).
\]

The second assertion follows from [3, Lemma 2.9].

Note that \( \bar{R}/\bar{I} \) has a finite basis \( S = \{ x_u | u \in S(\Delta) \} \) and \( \text{card}(S) = n! \text{Vol}(\Delta(f)) \).

Definition 2.2. The Hodge polygon \( HP(\Delta) \) of \( \Delta(f) \) is defined to be the convex polygon in \( \mathbb{R}^2 \) with vertices \((0, 0)\) and

\[
\left( \sum_{k=0}^{m} H(k), \frac{1}{M} \sum_{k=0}^{m} kH(k) \right).
\]

Consider the Artin-Hasse exponential series: \( E(t) = \exp \left( \sum_{i=0}^{\infty} \frac{p^i}{p^i} \right) \).

By [2, Lemma 4.1], the series \( \sum_{i=0}^{\infty} \frac{p^i}{p^i} \) has a zero at \( \gamma \in \Omega \) such that \( \text{ord} \gamma = 1/(p - 1) \) and \( \zeta_p \equiv 1 + \gamma \mod \gamma^2 \). Set

\[
\theta(t) = E(\gamma t) = \sum_{i=0}^{\infty} c_it^i.
\]

The series \( \theta(t) \) is a splitting function in Dwork’s terminology [2, §4a]. In particular, we have \( \text{ord} c_i \geq i/(p - 1) \), \( \theta(t) \in \mathbb{Q}_p(\zeta_p)[[t]] \) and \( \theta(1) = \zeta_p \).

Fix an \( M \)-th root \( \bar{\gamma} \) of \( \gamma \) in \( \Omega \). Let \( K = \mathbb{Q}_p(\bar{\gamma}) \), and \( \mathcal{O}_K \) the ring of integers of \( K \). Let \( \hat{a}_j \in K \) be the Techmüller lifting of \( a_j \) and set

\[
\hat{f}(x) = \sum_{j=1}^{N} \hat{a}_j x^{\rho_j} \in K[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}].
\]

Consider the following spaces of \( p \)-adic functions

\[
B_0 = \left\{ \sum_{u \in M(f)} A_u \bar{\gamma}^{Mw(u)}x^u | A_u \in \mathcal{O}_K, A_u \to 0 \text{ as } u \to 0 \right\}.
\]
\[ B = \left\{ \sum_{u \in M(f)} A_u \, \gamma_{Mw(u)} \, x^u | A_u \in K, A_u \to 0 \text{ as } u \to 0 \right\}. \]

Set \( \gamma_l = \sum_{i=0}^{l} \gamma^{p^i} / p^i \), \( h(t) = \sum_{i=0}^{\infty} \gamma_l t^i \). Define

\[ H(x) = \sum_{j=1}^{n} h(x^{w_j}), \quad F_0(x) = \prod_{i=1}^{n} \theta(x^{w_i}) = \sum_{v \in M(f)} h_v x^v. \]

Define an operator \( \psi \) on formal Laurent series by

\[ \psi \left( \sum_{u \in \mathbb{Z}^n} a_u x^u \right) = \sum_{u \in \mathbb{Z}^n} a_{pu} x^u. \]

Let \( \alpha = \psi \circ F_0(x) \). For \( i = 1, \cdots, n \), define operators

\[ E_i = x_i \partial / \partial x_i, \quad \hat{D}_i = E_i + E_i(H) \]

By [1, Corollary 2.9], we have

\[ L(f, t)^{(-1)^{n-1}} = \text{det}(1 - t\alpha|B/\sum_{i=1}^{n} \hat{D}_i B). \]

By [1, Therorem 2.18, Theorem A.1], \( S = \{ x^u \}_{u \in S(\Delta)} \) is a free basis of \( B/\sum_{i=1}^{n} \hat{D}_i B \). For any \( u \in M(f), u' \in S(\Delta) \), define \( A(u, u') \) by the relations

\[ x^u \equiv \sum_{u' \in S(\Delta)} A(u, u') x^{u'} \mod \sum_{i=1}^{n} \hat{D}_i B. \]

For any \( u, u' \in S(\Delta) \), define \( \gamma(u, u') \) by the relations

\[ \alpha(x^u) \equiv \sum_{u' \in S(\Delta)} \gamma(u, u') x^{u'} \mod \sum_{i=1}^{n} \hat{D}_i B. \]

The main purpose is to give estimate for the \( p \)-adic valuations of the coefficients \( \gamma(u, u') \).

For any \( u \in M(f) \), there is a unique \( u' \in S(\Delta) \) such that

\[ u \in S_{u'} = \left\{ u' + \sum_{i=1}^{n} \mathbb{Z}_{\geq 0} w_i \right\}. \]

Set \( R_{u'} = \{ \xi = \sum a_u x^u \in B_0 | u \in S_{u'} \} \).

**Lemma 2.3.** For any \( u \in M(f) \), we have \( A(u, u') = 0 \) if \( u \notin S_{u'} \), \( \text{ord}(A(u, u')) \geq \frac{w(u') - w(u)}{p-1} \) if \( u \in S_{u'} \).
Proof. The first assertion follows from the facts that
\[ B_0 = \bigoplus_{u' \in S(\Delta)} R_{u'} \]
and \( \hat{D}_i(R_{u'}) \subset R_{u'} \) for any \( i \) and \( u' \). Suppose that \( u \in S_{u'} \). By [1, Proposition 3.1], there exist \( A \in O_K \) and \( \xi_1, \ldots, \xi_n \in B_0 \) such that
\[ \tilde{\gamma}_i^{Mw(u)}x^u = A\tilde{\gamma}_i^{Mw(u')}x^{u'} + \sum_{i=1}^n \hat{D}_i\xi_i. \]
Hence, we have
\[ \text{ord}(A(u, u')) = \text{ord}(A\tilde{\gamma}_i^{Mw(u') - Mw(u)}) \geq \frac{w(u') - w(u)}{p - 1}. \]
\( \square \)

**Proposition 2.4.** For any \( u, u' \in S(\Delta) \), we have
\[ \text{ord}(\gamma(u, u')) = \begin{cases} +\infty & \text{if } p.u' - u \neq 0, \\ \frac{pw(u' - w(u))}{p-1} & \text{if } p.u' - u = 0. \end{cases} \]
\( \text{ord}(\gamma(u, u')) = +\infty \) means that \( \gamma(u, u') = 0. \)

Proof. Note that
\[ \alpha(x^u) = \psi(x^uF_0(x)) = \sum_{v \in M(f)} h_{pv-u}x^v \]
\[ = \sum_{u' \in S(\Delta)} \sum_{v \in M(f)} h_{pv-u}A(v, u')x^{u'} \mod \sum_{i=1}^n \hat{D}_iB. \]
By Lemma 2.3, \( A(v, u') = 0 \) when \( v \notin S_{u'} \). Hence, we have
\[ (2.2) \quad \gamma(u, u') = h_{pv'-u} + \sum_{v \in M(f) - S(\Delta)} h_{pv-u}A(v, u'). \]
Assume that \( v = u' + \sum_{i=1}^n s_iw_i \) with \( s_i \in \mathbb{Z}_{\geq 0} \). Note that
\[ h_{pv-u} = \prod_{j=1}^n c_{k_j}, \]
where \( (k_1, \ldots, k_n) \in \mathbb{Z}_{\geq 0}^n \) satisfies the equation
\[ (2.3) \quad \sum_{i=1}^n k_iw_i = pv - u = pu' - u + p \sum_{i=1}^n s_iw_i. \]
If \( p.u' - u \neq 0 \), the above equation has no integer solution which implies that \( \gamma(u, u') = 0. \) If \( p.u' - u = 0 \), suppose that \( pu' - u = r_1w_1 + \cdots + r_nw_n \)
with $r_i \in \mathbb{Z}_{\geq 0}$ for all $i$. Note that $r_i \leq p - 1$ for all $i$ and $w(pu' - u) = pw(u') - w(u) = r_1 + \cdots + r_n$. By (2.3), we have $k_i = r_i + ps_i$ for each $i$. Hence, by Lemma 2.3 and the estimate $\text{ord}(c_i) \geq \frac{i}{p-1}$, we have

$$\text{ord}(h_{p^u - u}A(v, u')) \geq \sum_{i=1}^{n} \frac{k_i - s_i}{p-1} = \sum_{i=1}^{n} \frac{s_i}{p-1} + \frac{pw(u') - w(u)}{p-1}. $$

If $v \notin S(\Delta)$, there is some $i$ such that $s_i > 0$, we have

$$\text{ord}(h_{p^u - u}A(v, u')) > \frac{pw(u') - w(u)}{p-1}. $$

If $v = u' \in S(\Delta)$, we have $k_i = r_i \leq p - 1$ for all $i$. Note that

$$\theta(t) \equiv \exp(\gamma t) \mod t^p. $$

We have $\text{ord}(c_i) = \text{ord}(\frac{i}{p}) = \frac{i}{p-1}$ for any $i \leq p - 1$. Hence

$$\text{ord}(h_{p^{u' - u}}) = \sum_{i=1}^{n} \text{ord}(c_i) = \frac{1}{p-1} \sum_{i=1}^{n} r_i = \frac{pw(u') - w(u)}{p-1}. $$

By (2.2), we have

$$\text{ord}(\gamma(u, u')) = \text{ord}(h_{p^{u' - u}}) = \frac{pw(u') - w(u)}{p-1}. $$

\[ \square \]

**Theorem 2.5.** Suppose that $f = \sum_{j=1}^{n} x^{w_j}$ and $(p, \det J) = 1$. The Newton polygon of $L(T^n, f, t)^{(-1)^{n-1}}$ coincides with the Hodge polygon $HP(\Delta)$ if and only if $S(\Delta)$ is $p$-stable under weight function.

**Proof.** By [1, Corollary 3.11], the Newton polygon of $L(T^n, f, t)^{(-1)^{n-1}}$ lies above the Hodge polygon of $HP(\Delta)$ with same endpoints and the matrix $\Gamma := (\gamma(u, u'))_{u, u' \in S(\Delta)}$ is invertible. By Proposition 2.4, $\gamma(u, u') \neq 0$ if and only if $p.u' - u = 0$. Hence there is exactly one non zero element in every column and row of $\Gamma$. Let $S(d, u)$ be the orbit of $u$ under the $p$-action with exactly $d$ elements. Suppose that $S(d, u) = \{u_1, \cdots, u_d\}$, where $u_i = p^{i-1}.u$. By Proposition 2.4, we have

$$\alpha(x^{u_1}, \cdots, x^{u_d}) = (x^{u_1}, \cdots, x^{u_d})
\begin{pmatrix}
0 & \gamma_{21} & \cdots \\
0 & 0 & \cdots \\
\vdots & \vdots & \ddots \\
\gamma_{1d} & \cdots & 0
\end{pmatrix}
$$

where $\gamma_{ij} = \gamma(u_i, u_j)$. Thus

$$\det(1 - \alpha t) = \prod_{S(d, u) \neq \emptyset} (1 - t^d \lambda_u),$$
where the above product runs through all the obits of \( S(\Delta) \) under the \( p \)-action and 
\[
\lambda_u = \gamma_1 \gamma_2 \cdots \gamma_{d-1} \quad \text{with}
\]
\[
\text{ord}(\lambda_u) = \text{ord}(\gamma_1 \gamma_2 \cdots \gamma_{d-1})
\]
\[
= \frac{pw(u_d) - w(u_1)}{p - 1} + \cdots + \frac{pw(u_{d-1}) - w(u_d)}{p - 1}
\]
\[
= \sum_{i=0}^{d-1} w(p^i u).
\]
Set \( f_{u,d} = 1 - t^d \lambda_u \) and
\[
g_{u,d} = \prod_{i=0}^{d-1} (1 - tp^i w(p^i u)).
\]
Note that the Newton polygon of \( f_{u,d} \) always lies above the Newton polygon of \( g_{u,d} \) and the Newton polygon of the polynomial \( \prod_{S(d,u)} g_{u,d} \) is \( HP(\Delta) \). Hence \( HP(\Delta) \) coincides with the Newton polygon of \( \det(1 - \alpha t) \) if and only if the Newton polygons of \( g_{u,d} \) and \( f_{u,d} \) coincide for each \( u \).

When \( S(\Delta) \) is \( p \)-stable under weight function. We have \( w(u) = w(p.u) = \cdots = w(p^{d-1}.u) \) for each \( u \). Hence, the Newton polygons of \( g_{u,d} \) and \( f_{u,d} \) coincide for each \( u \).

Conversely, if the Newton polygons of \( g_{u,d} \) and \( f_{u,d} \) coincide for each \( u \). Since both polygons have same endpoints, we have \( w(u) = w(p.u) = \cdots = w(p^{d-1}.u) \) for each \( u \). Hence \( S(\Delta) \) is \( p \)-stable under weight function.

\[\square\]

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