Pebble-Intervals Automata and
$\text{FO}^2$ with Two Orders$^*$
(Extended Version)

Nadia Labai, Tomer Kotek, Magdalena Ortiz, and Helmut Veith†
TU Wien, Vienna, Austria

Abstract

We introduce a novel automata model, which we call pebble-intervals automata (PIA), and study its power and closure properties. PIAs are tailored for a decidable fragment of FO that is important for reasoning about structures that use data values from infinite domains: the two-variable fragment with one total preorder and its induced successor relation, one linear order, and an arbitrary number of unary relations. We prove that the string projection of every language of data words definable in the logic is accepted by a pebble-intervals automaton $A$, and obtain as a corollary an automata-theoretic proof of the $\text{EXPSPACE}$ upper bound for finite satisfiability due to Schwentick and Zeume.

1 Introduction

Finding decidable fragments of First Order Logic (FO) that are expressive enough for reasoning in different applications is a major line of research. A prominent such fragment is the two-variable fragment $\text{FO}^2$ of FO, which has a decidable finite satisfiability problem [22, 13] and is well-suited for handling graph-like structures. It captures many description logics, which are prominent formalisms for knowledge representation, and several authors have recently applied fragments based on $\text{FO}^2$ to verification of programs [16, 1, 8, 7, 27]. Unfortunately, $\text{FO}^2$ has severe limitations, e.g., it cannot express transitivity, and in the applications to verification above, it cannot reason about programs whose variables range over data values from infinite domains. This has motivated the exploration of decidable extensions of $\text{FO}^2$ with special relations which are not axiomatizable in $\text{FO}^2$. For example, finite satisfiability of $\text{FO}^2$ with a linear order was shown to be $\text{NEXPTIME}$-complete in [25], even in the presence of the induced successor relation [12], and equivalence relations have been used to

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model data values which can be tested for equality \[4, 3, 10, 2, 4\]. However, related extensions of \(\text{FO}_2\) with preorders easily become undecidable [3, 19]. Recently the logic \(\text{FO}_2(\leq, \preceq, S_2)\), that is \(\text{FO}_2\) with a linear order \(\leq\), a total preorder \(\preceq\) and its induced successor \(S_2\), and any number of unary relations from a finite alphabet, was shown to have an \(\text{EXPSPACE}\)-complete satisfiability problem [29]. This logic can compare data values in terms of which is smaller than which and whether they are consecutive in \(\preceq\), making it suitable to model linearly ordered data domains, and a good candidate for extending existing verification methods which use two-variable logics. We continue the study of \(\text{FO}_2(\leq, \preceq, S_2)\), and in particular, focus on a suitable automata model for it. Establishing a connection to suitable automata for fragments of \(\text{FO}\) that can talk about values from infinite domains is an active area of research. Automata are also important in automated verification, where they are used, for example, to reason about temporal properties of program traces [32, 9]. We make the following contributions:

- As an automata model for \(\text{FO}_2(\leq, \preceq, S_2)\) we propose \textit{pebble-intervals automata} (PIAs). Similarly to classical finite-state automata, PIAs are read-once automata for strings. However, they read the input in varying order. Using a fixed set of pebbles \(m = \{1, \ldots, m\}\), a PIA reads a position \(p\) by choosing three pebbles \(i, j, k \in m\) and non-deterministically moving \(k\) to position \(p\) between the positions of \(i\) and \(j\).

- We study the computational power and closure properties of PIAs. We describe a restricted class of PIAs that accept exactly the regular languages, and show that some context-free languages, and even languages which are not context-free, are accepted by PIAs. We prove that PIAs are effectively closed under union, concatenation, Kleene star, shuffle, and iterated shuffle, but not effectively closed under intersection, even with regular languages, nor under complement.

- We show that the emptiness problem for PIA is \(\text{NL}\)-complete if the number of pebbles is logarithmic in the size of the automaton, and is \(\text{PSPACE}\) in general.

- We show that PIAs contain \(\text{FO}_2(\leq, \preceq, S_2)\) in the following sense: for each sentence \(\psi\), there is a PIA whose language coincides with the projection language of \(\psi\), obtained by omitting \(\preceq\) and \(S_2\) from the structures satisfying \(\psi\).

- As a corollary, we get an automata-theoretic proof for \(\text{EXPSPACE}\) membership of finite satisfiability for \(\text{FO}_2(\leq, \preceq, S_2)\) that was established in [29].

# 2 Pebble-Intervals Automata

In this section, we introduce pebble-intervals automata (PIA). We study their emptiness problem, their expressive power, and closure properties of the languages they accept.

Let \([n] = \{1, \ldots, n\}\). A string of length \(n \geq 0\) over alphabet \(\Sigma\) is a mapping \(w : [n] \to \Sigma\), written also \(w = w(1) \cdots w(n)\). Note that \([0] = \emptyset\) and \(w : [0] \to \Sigma\) is the empty string \(\varepsilon\). We often use \(s, u, v,\) and \(w\) for strings, and \(|w|\) for the length of \(w\).

A PIA is equipped with a finite number \(m\) of pebbles. It begins its computation with no pebbles on the input \(w\), and uses \textsc{move} transitions to place and replace pebbles. In a \textsc{k-move}_{i,j} transition, the pebble \(k\) (which may or may not have been previously placed on \(w\)) is non-deterministically placed on a previously unread position in the interval
between pebbles $i$ and $j$. The input boundaries can be used as interval boundaries, e.g., a \text{MOVE}_{i,<}$ transition places pebble $k$ to the right of pebble $i$. For convenience we allow silent transitions that go to a new state without moving any pebbles. As pebbles can only be placed on unread positions, each position of $w$ is read at most once. In an accepting run all positions must be read, and the run must end at an accepting state.

**Definition 1 (Pebble-intervals automata).** A PIA $A$ is a tuple $(\Sigma, m, Q, q_{\text{init}}, F, \delta)$, where $\Sigma$ is the (finite) alphabet, $m \in \mathbb{N}$, $Q$ is the finite set of states, $q_{\text{init}} \in Q$ is the initial state, $F \subseteq Q$ are the accepting states, and $\delta \subseteq (Q \times Q) \cup (Q \times \text{MOVE}_m \times \Sigma \times Q)$ is the transition relation with \text{MOVE}_m = \{ \text{MOVE}_{i,j} \mid i \in [m] \cup \{\text{>}\}, j \in [m] \cup \{<\}, k \in [m], i \neq j \}$. We may omit $m$ when it is clear from the context. Transitions in $Q \times \text{MOVE} \times \Sigma \times Q$ are \text{MOVE} transitions, and transitions in $Q \times Q$ are silent transitions. The size of $A$ is $|\delta| + |\Sigma| + |Q|$.

The positions of $m$ pebbles on a string of length $n$ during a run of a PIA are described by an \textit{(m,n)}-pebble assignment, which is a function $\tau : [m] \to [n] \cup \{\_\}$ with either $\tau(i) \neq \tau(j)$ or $\tau(i) = \tau(j) = \_ \forall 1 \leq i < j \leq m$; the pebbles $j$ with $\tau(j) = \_ \forall i \in [m]$. By $\hat{\tau} : [m] \cup \{\_\} \to \{0\} \cup [n+1]$ we denote the extension of $\tau$ with $\hat{\tau}(\_\_ \_ \_ \_)$ $= 0$ and $\hat{\tau}(\_\_ \_ \_ \_ \_ \_ \_)$ $= n + 1$.

**Definition 2 (Semantics of PIAs).** Consider a PIA $A = (\Sigma, m, Q, q_{\text{init}}, F, \delta)$. A configuration of $A$ on string $u \in \Sigma^*$ is a triple $(q, \rho, N)$ where $q \in Q$ is the current state, $\rho : [m] \to [[u]] \cup \{\_\}$ is the current pebble assignment, and $N \subseteq [[u]]$ is the set of already-read positions. The initial configuration $\pi_{\text{init}}$ is $(q_{\text{init}}, \rho_{\text{init}}, \emptyset)$. A configuration $(q, \rho, N)$ is accepting if $q \in F$ and $N = [[u]]$. Let $\pi = (q, \rho, N)$ and $\pi' = (q', \rho', N')$ be configurations on $u$. We call them consecutive and write $\pi \overset{t}{\Rightarrow} \pi'$ if there exists a transition $t$ in $\delta$ such that each:

1. $t$ is a silent transition of the form $(q, q')$, $N = N'$, and $\rho = \rho'$; or
2. $t$ is a \text{MOVE} transition $(q, k\text{-MOVE}_{i,j}, u(\ell), q')$ with $\hat{\rho}(i) < \ell < \hat{\rho}(j)$ and $\ell \in [[u]] - N$, and additionally $\rho' = \rho|k \mapsto \ell$ and $N' = N \cup \{\ell\}$. That is, pebble $k$ is placed on position $\ell$ in the open interval between $i$ and $j$, reading the letter $u(\ell)$.

Let $\bar{t} = (t_1, \ldots, t_r)$ and $\bar{\pi} = (\pi_0, \ldots, \pi_r)$ be sequences of transitions and configurations. We call $(\bar{t}, \bar{\pi})$ a computation of $A$ on $u$ if $\pi_0 = \pi_{\text{init}}$ and $\pi_{i+1} \overset{t_i}{\Rightarrow} \pi_i$ for every $i \in [r]$, and write $\pi_0 \overset{\bar{t}}{\Rightarrow} \pi_r$. We call $(\bar{t}, \bar{\pi})$ accepting if $\pi_r$ is accepting. We write $\pi \overset{\bar{t}}{\Rightarrow} \pi'$ if $\pi \overset{\bar{t}}{\Rightarrow} \pi'$ for some $\bar{t}$. The automaton $A$ accepts $u$ if there is an accepting computation of $A$ on $u$. The set of all $u$ accepted by $A$ is denoted $L(A)$, and called a PI language.

**Computational power.** PIAs generalize standard non-deterministic finite-state automata. A PIA $A = (\Sigma, 1, Q, q_{\text{init}}, F, \delta)$ with one pebble is unidirectional if $q_{\text{init}}$ has no incoming transitions, and the \text{MOVE} transitions from other states use \text{MOVE}_{1,<}$ only.

**Proposition 1.** A language $L$ is accepted by a standard non-deterministic finite-state automaton iff $L = L(A)$ for a unidirectional PIA $A$ with the same number of states.
PI languages also contain non-regular languages, and even some non-context-free ones.

**Examples 1.** The following are examples of PI languages:

1. There is a PIA $A_{Dyck}$ with one pebble that accepts the Dyck language $L_{Dyck}$ of well-nested brackets, which is context-free but not regular. The alphabet has two letters $[\quad]$ and $\{\quad\}$, and the states are $q_1$ and $q_2$. The initial and only accepting state is $q_2$. The transition relation contains $(q_1, 1\text{-MOVE}_>,<,[\quad],q_1)$ and $(q_1, 1\text{-MOVE}_{1,1},\{\quad\},q_1)$. $A_{Dyck}$ accepts a string iff there are as many left as right brackets, and no prefix has more right than left brackets.

2. A similar one pebble PIA accepts the language $L_{\text{two}}$ of all strings of two types of parentheses, where each type is well-nested with respect to itself, but not necessarily to the other type. E.g., $(\quad)\] \in L_{\text{two}}$, but $[\quad) \notin L_{\text{two}}$. $L_{\text{two}}$ is not context-free.

3. $\{a^n$$b^n$#$c^n \mid n \geq 0\}$, which is not context-free, is accepted by a PIA with $3$ pebbles. Pebbles $1$ and $2$ read the $\$\$ and the $\#$, and then the PIA keeps doing the following: pebble $3$ reads an $a$ to the left of pebble $1$, a $b$ between pebbles $1,2$, and a $c$ to the right of pebble $2$.

4. $\{w\$w$w \mid w \in \{0,1\}^+\}$ is not context-free, and is accepted by a PIA with $3$ pebbles. Pebble $1$ reads the $\$\$, pebble $2$ reads a letter $\sigma$ to the left of pebble $1$, and pebble $3$ also reads $\sigma$ to the right of pebble $1$. Then the PIA repeats: (i) a letter $\sigma$ is non-deterministically chosen, (ii) pebble $2$ reads $\sigma$ between its current position and pebble $1$, and (iii) pebble $3$ reads $\sigma$ to the right of its current position. Similar languages are PI languages, e.g., $\{wwR$w$w \mid w \in \Sigma^*\}$, where $w^R$ is $w$ in reverse.

We conjecture that not all context-free language are PI languages; e.g., the Dyck language of two types of well-nested parentheses seems not to be PI.

**Closure properties.** We provide a construction of suitable PIAs in the appendix to show the following.

**Theorem 1.** The class of PI languages is effectively closed under union, concatenation, Kleene-$\star$, shuffle, and iterated shuffle. It is not effectively closed under intersection, even with regular languages, nor under complement.

From the construction used in the proof of the above theorem, we also obtain:

**Corollary 1.** The universality and inclusion problems for PIAs are undecidable.

**Emptiness.** For deciding whether $L(A) \neq \emptyset$ for a given PIA, we use feasible sequences of transitions, which are those that correspond to an actual computation of a PIA. One can show that for a given PIA with $m$ pebbles, $L(A) \neq \emptyset$ iff there is a feasible sequence of transitions $t$ of length at most $|A| \cdot 2^{O(m \log m)}$, and that the existence of the latter can be guessed and verified using a bounded amount of information (roughly a counter, two transitions, and two pebble assignments). This gives us the upper bounds below, which hold also if $A$ is not given explicitly, as long as $\delta$ can be computed non-deterministically in $\log(|A|)$ space. For the case where $A$ has $O(\log |A|)$ pebbles, NL-hardness follows from the same result for standard finite state automata and Prop.1

**Theorem 2.** If a PIA $A$ has $O(\log |A|)$ pebbles, its emptiness problem is NL-complete. In general, the emptiness problem for PIA is in PSPACE.
Related automata models.

Jumping finite automata \cite{20} are probably the closest to PIAs: they are essentially PIAs with one pebble, which is placed on an arbitrary unvisited position without specifying an interval. In the context of languages with infinite alphabets, various automata models have been proposed that run on data words: string words where values from an infinite domain are attached to each position. Register automata are finite-state machines on data words which use registers to compare whether data values are equal \cite{17, 23, 6}; their string projection languages are regular. Pebble automata \cite{23} use pebbles in a stack discipline to test for equality of data values. Data automata \cite{3, 14, 8} are an extension of register automata introduced to prove the decidability of satisfiability of \(\text{FO}^2\) on words with a linear order, a successor relation, and an equivalence relation. Their projection languages are accepted by multi-counter automata, which are finite automata on strings extended with counters, that are equivalent to Vector Addition Systems or Petri Nets \cite{11}. Class Memory Automata \cite{2} have the same expressive power as data automata. Variable Finite Automata \cite{15} extend finite state automata with variables from an infinite alphabet. Many works have studied these automata models and their variations, see \cite{30} and \cite{18, Chapter 4} for surveys.

3 PIAs and \(\text{FO}^2(\leq_1, \preceq_2, S_2)\)

To establish the relation between \(\text{FO}^2(\leq_1, \preceq_2, S_2)\) and PIAs, we need some preliminaries. Recall that a total preorder \(\preceq_2\) is a transitive total relation which can be seen as an equivalence relation whose equivalence classes are linearly ordered. We use \(x \sim_2 y\) as shorthand for \((x \preceq_2 y) \land (y \preceq_2 x)\). The induced successor relation \(S\) of a total preorder \(\preceq_2\) is such that \(S(x, y)\) if \(x \preceq_2 y\) and there is no \(z\) such that \(x \preceq_2 z \preceq_2 y\).

Two-variable logic (\(\text{FO}^2\)) is the restriction of \(\text{FO}\) to formulas that only use two variables \(x\) and \(y\), and \(\text{FO}^2(\leq_1, \preceq_2, S_2)\) is \(\text{FO}^2\) with a linear order \(\leq_1\), a total preorder \(\preceq_2\) and its induced successor \(S_2\), and any number of unary relations from a finite alphabet.

All structures and strings in this paper are finite. For a structure \(\mathcal{A}\), we denote its universe by \(\mathcal{A}\) and its size by \(|\mathcal{A}|\). The empty structure has \(\mathcal{A} = \emptyset\) and is denoted \(\emptyset\).

Data words. Let \(\Sigma\) a finite alphabet. Its extension for data words is \(\text{voc}_{\text{DW}}(\Sigma) = \langle \leq_1, \preceq_2, S_2, \sigma : \sigma \in \Sigma \rangle\). A data word over \(\Sigma\) is a finite \(\text{voc}_{\text{DW}}(\Sigma)\)-structure \(D\) with universe \(\sigma : \sigma \in \Sigma\) are interpreted as unary relations that partition \(D\). We use \(D, D', \ldots\) to denote data words. The empty word is denoted by \(\emptyset_{\text{DW}(\Sigma)}\), and the class of all data words over \(\Sigma\) by \(\text{DW}(\Sigma)\). A set of data words is called a data language.

Let \(\varphi_1, \varphi_2\) be \(\text{FO}^2(\text{voc}_{\text{DW}}(\Sigma))\) formulas. We write \(\varphi_1 \models_{\text{DW}(\Sigma)} \varphi_2\) if \(D \models \varphi_1\) implies \(D \models \varphi_2\) for every \(D \in \text{DW}(\Sigma)\), and define equivalence \(\equiv_{\text{DW}(\Sigma)}\) analogously. We may omit \(\Sigma\) if clear from context. The data value \(\text{value}_{D}(d)\) of an element \(d \in D\) is the number of equivalence classes \(E\) of \(\sim_2\) whose elements \(d' \in E\) satisfy \(d' \preceq_2 d\), and \(\max\text{val}_D = \max_{d \in D} \text{value}_{D}(d)\). The string projection of \(D\), denoted \(\text{string}(D)\), is the string \(w\) of length \(|w| = |D|\) where for all \(\ell \in [|w|]\), \(w(\ell) = \sigma\) if and only if \(D \models \sigma(d)\) where \(d\) is the unique element of \(D\) such that \(\ell = |\{d' \in D \mid D \models d' \leq_1 d\}|\). The projection of the empty structure \(\emptyset_{\text{voc}_{\text{DW}}(\Sigma)}\), and only of \(\emptyset_{\text{voc}_{\text{DW}}(\Sigma)}\), is \(\varepsilon\). The projection language of a data language \(\Delta\) is the string language \(L(\Delta) = \{w \mid w = \)
string(D) for some D ∈ ∆. If a formula ψ defines ∆, we write L(ψ) for L(∆).

Example 1. To avoid ambiguity, in our running examples we use underlined symbols.
Let Ξ = {{ξ₁,ξ₂} be a set of unary relations and let D be the data word with universe
Ξ = {a,b,c,d,e,f} where ≤₁ is the lexicographic order, the interpretation of ξ₁ is
(a,b,c,d,e,f), the interpretation of ξ₂ is {d,f}, and b ≤₂ a ≤₂ d ≤₂ e ≤₂ f. Note e.g. that D |= S₂(a,e) and D |= ¬S₂(b,e) ∧ (b ≤₂ e). The string projection of D is
string(D) = ξ₁ ξ₂ ξ₂ ξ₁ ξ₁ ξ₁.

The goal of this section is to prove the following theorem.

Theorem 3. If ψ is a FO²(≤₁, ≤₂, S₂) sentence, there is a PIA A with L(ψ) = L(A).
To prove this, we rely on the normal form defined next. A 1-type ν(x) over
vocDW(Σ) is a maximal consistent conjunction of atomic and negated atomic formulas
with the free variable x. A 2-type θ(x,y) is defined similarly. Given a FO²(vocDW(Σ))
formula ψ, we obtain a ϕ in normal form by taking the Scott Normal Form [14] Theorem
2.1] of ψ, and expanding the quantifier-free formulas to Disjunctive Normal Form,
in fact to disjunctions of 2-types θ. The Scott Normal Form of ψ introduces linearly
many new symbols, resulting in an extended Σ'. We let Ξ = {ξ₁ | a ∈ [A]} be
an alphabet containing a symbol for every 1-type over Σ'.

Theorem 4 (Normal Form). Let ψ ∈ FO²(vocDW(Σ)). Then there exist A, B, C ∈ N,
an alphabet Ξ = {ξ₁ | a ∈ [A]}, a formula ϕ ∈ FO²(vocDW(Ξ)) of the form
ϕ = ϕ₁ ∧ ϕ₂ ∧ a letter-to-letter substitution h : Ξ → Σ such that L(ψ) = h(ϕ).
ϕ₁ = ∀x∀y ∨ θ(x,y) ϕ₂ = ∀x ∀a|A| a∈A y ∈ Y h(ϕ₁) ∧ x ∈ X ϕ₃ = ∃y (θ(x,y))
with θ and θ₀ 2-types over vocDW(Ξ), and ϕ₄ = True if υ ∈ vocDW(Σ) ψ and
ϕ₅ = True if ψ ∈ vocDW(Σ) ∉ vocDW(Σ) ψ. Moreover, ϕ is computable in EXPSPACE
and is of length exponential in |ψ|.

We let Θ₃ = {θ₁ | a ∈ [A], b ∈ [B], c ∈ [C]} and Θ = Θ₃ ∪ Θ₃. Given a ∈ [A], a
witless type set for a is a choice of 2-types satisfying the right-hand side of the implication
ϕ₄. That is, a set of 2-types ω ⊆ Θ₃ that contains one θ₁ for every b ∈ [B],
representing a choice of the existential constraints an element needs to fulfill. Denote
by Ωₐ the set of witless type sets for a and let Ω = ∪ₐ∈[A] Ωₐ. For a witless type set
ω ∈ Ω, let ω(x) = ∃θ₃ ω(x,y) be its existential constraints. Note that ω(x) is always
satisfiable and that there is a unique letter ζω ∈ Ξ such that ω(x) |= DW(Ξ) ζω(x).

Example 2. Consider the following formula ϕ given in normal form
∀x∀y ∃x (ξ₂(x) ∧ ∨ x (ξ₂(x) → ∃y (θ₄(x,y) ∨ θ₅(x,y)) ∧ θ₆(x,y)) → ∃y (θ₄(x,y) ∨ θ₅(x,y)))
where χ(x, y) is the disjunction of 2-types equivalent to (ξ₂(x) ∧ ξ₂(y)) → x ~₂ y,
and the θ₄ are given as the following 2-types (omitted clauses are negated):
θ₄₁ = x <₁ y ∧ S₂(x,y) ∧ ξ₂(x) ∧ ξ₂(y) θ₄₂ = x <₁ y ∧ S₂(x,y) ∧ x <₂ y ∧ ξ₂(x) ∧ ξ₂(y) θ₄₃ = y <₁ x ∧ S₂(y,x) ∧ ξ₂(y) ∧ ξ₂(x) ∧ ξ₂(y)
θ₄₄ = x <₁ y ∧ S₂(x,y) ∧ ξ₂(x) ∧ ξ₂(y)
A data word satisfies ϕ if it is the empty structure, or (a) the largest element of ≤₁
has letter ξ₁, (b) the smallest element of ≤₁ has letter ξ₁, (c) all elements with ξ₂ have
maximal value, and (d) no element with ξ₁ has maximal value.
Note that $\mathcal{D} \models \varphi$. The projection language $L(\varphi)$ is the regular language with regular expression $\xi_1(\xi_2 + \xi_3)\xi_4 + \varepsilon$. We have $\Omega_3 = \{ h_1, h_2, h_3, h_4 \}$. For $\varphi$, we have $A = 2$, $B = 1$, and $C = 2$. The witness type sets of $\varphi$ are $\{ h_{111}, \}, \{ h_{112}, \}, \{ h_{211}, \}$, and $\{ h_{212} \}$, where $h_{111} = h_1$, $h_{112} = h_2$, $h_{211} = h_3$, and $h_{212} = h_4$. Hence, we have $\Omega = \{ \{ h_1 \}, \{ h_2 \}, \{ h_3 \}, \{ h_4 \} \}$, and $\xi(h_1) = \xi_1$, $\xi(h_2) = \xi_2$, $\xi(h_3) = \xi_3$, and $\xi(h_4) = \xi_4$.

We construct a PIA $\mathcal{A}^\varphi$ that accepts a string $w$ if it can be extended into a data word $\mathcal{D}$ that satisfies the normal form $\varphi$ of a given sentence $\psi$. Note that $\psi$ and $\varphi$ have different alphabets, but since there is a letter-to-letter substitution $h$ such that $L(\psi) = h(L(\mathcal{A}^\varphi))$, and PIAs are closed under letter-to-letter substitutions, this proves Theorem 5.

For constructing our PIA, we first focus on the existential part, i.e., whether $w$ can be extended into a $\mathcal{D}$ that satisfies $\varphi_3$. This is achieved in two steps: (S1) We reduce the existence of $\mathcal{D}$ to the existence of a sequence of consecutive task words, data words that store additional information of already satisfied vs. promised subformulas; the sequence should lead to a completed task word where all promises are fulfilled. (S2) We do not have a bound on the length of task words and their data values, so we use extremal strings to decide the existence of the desired sequence with the limited memory of PIAs. After these two steps, we introduce perfect extremal strings to guarantee the satisfaction of $\varphi_\forall$. Our PIA will then decide if a sequence of perfect extremal strings exists.

**Task words for $\varphi_3$** We start by defining task words, which are like data words but do more book-keeping. Additionally to data values, elements in task words are assigned tasks, which are witness type sets where each 2-type may be marked as completed if its satisfaction has already been established, or as promised otherwise. We reduce the satisfaction of $\varphi_3$ to the existence of a sequence of $T_1, \ldots, T_n$ of consecutive task words, where we keep assigning new data values and updating promised into completed tasks, until we reached a completed task word $T_n$.

**Definition 3 (Tasks).** For $\theta \in \Theta_3$, we call $C_0$ a completed task and $P_0$ a promised task. Let $\text{Tasks}_C = \{ C_0 \mid \theta \in \Theta_3 \}$, $\text{Tasks}_P = \{ P_0 \mid \theta \in \Theta_3 \}$ and $\text{Tasks} = \text{Tasks}_C \cup \text{Tasks}_P$.

For each task set $ts \subseteq \text{Tasks}$, there is at most one witness type set $\omega \in \Omega$ that $ts$ realizes, which means that for every $\theta \in \Theta_3$, (1) $|\{ P_0, C_0 \} \cap ts| \leq 1$, and (2) $|\{ P_0, C_0 \} \cap ts| = 1$ if and only if $\theta \in \omega$. If there is such an $\omega$, we denote it $\omega(ts)$, and call $ts$ an $\Omega$-realization. The set of all $\Omega$-realizations is $2^{\text{Tasks}}_\omega$, and $2^{\text{Tasks}_C}_\omega = 2^{\text{Tasks}}_\omega \cap 2^{\text{Tasks}_C}$ and $2^{\text{Tasks}_P}_\omega = 2^{\text{Tasks}}_\omega \cap 2^{\text{Tasks}_P}$.

**Example 3.** Since the witness type sets in $\Omega$ are singletons, so are the $ts \in 2^{\text{Tasks}}_\omega$. Let $ts_C^i = \{ C_{\theta} \}$ and $ts_P^i = \{ P_{\theta} \}$ for $i \in [4]$. Then we have $2^{\text{Tasks}}_\Omega = \{ ts_C^i \mid i \in [4] \} \cup \{ ts_P^i \mid i \in [4] \}$, and $\{ C_{\theta} \}$ and $\{ P_{\theta} \}$ are $\{ \xi_i \}$-realizations for $i \in [4]$.

$\mathcal{D}$-task words are data words that assign tasks to the elements of $\mathcal{D}$. More precisely, each $d \in D$ is assigned, instead of a letter $\xi_d$, a task set $ts$ that realizes a witness type set $\omega$ which contains $C_{\theta}$ for each $\theta \in \omega$ that $d$ satisfies, and $P_{\theta}$ for the remaining $\theta \in \omega$.
Definition 4 (Task word). Let $\mathcal{D}$ be a data word over $\Xi$. A $\mathcal{D}$-task word is a data word $\mathcal{T}$ over $2^{\text{Tasks}}$ that has the same universe and order relations as $\mathcal{D}$, and for every $d \in \mathcal{D}$ with $\mathcal{T} \models ts(d)$, (1) $\mathcal{D} \models \xi^\phi(ts)(d)$, and (2) for every $\theta \in \omega(ts)$, $C_\theta \in ts$ iff $\mathcal{D} \models \exists y \theta(d, y)$. A task word $\mathcal{T}$ is a $\mathcal{D}$-task word for some $\mathcal{D}$, and it is completed if $\mathcal{T} \models \varphi_\mathcal{D}$ for some existential constraints are satisfied and $\mathcal{T}$ is completed.

Example 4. We define a $\mathcal{D}$-task word $\mathcal{T}$; its vocabulary is $2^{\text{Tasks}}$, its universe is $\{a, b, c, d, e, f\}$, and $\leq_1$, $\leq_2$, and $S_2$ are the same as in $\mathcal{D}$. The interpretation of the letter $ts^c_1$ is $\{e\}$, that of $ts^c_2$ is $\{d\}$, that of $ts^c_3$ is $\{a, b, c\}$, and that of $ts^c_4$ is $\{f\}$; the other letters are empty. As $\mathcal{D} \models \varphi_\mathcal{D}$ all existential constraints are satisfied and $\mathcal{T}$ is completed.

The satisfaction of $\varphi_\mathcal{D}$ coincides with the existence of a completed task word.

Lemma 1. Let $\mathcal{D} \in \text{DW}(\Xi)$. There exists a completed $\mathcal{D}$-task word iff $\mathcal{D} \models \varphi_\mathcal{D}$.

We now characterize the notion of consecutive task words using trimmings.

Definition 5 (Trimming, consecutiveness). The trimming of a data word $\mathcal{D}$, denoted $\mathcal{D}^{\setminus 1}$, is the substructure of $\mathcal{D}$ induced by removing the elements with the maximal data value. For task words, trimmings are obtained by removing the elements with the largest data value and updating the tasks of the remaining elements correctly. That is, a trimming of a $\mathcal{D}$-task word $\mathcal{T}$ is a $\mathcal{D}^{\setminus 1}$-task word $\mathcal{T}^1$ such that $\omega(ts) = \omega(ts_1)$ for every $d$ and every $ts, ts_1 \in 2^{\text{Tasks}}$ with $\mathcal{T} \models ts(d)$ and $\mathcal{T}^1 \models ts_1(d)$. We say that $\mathcal{T}^1$ is a trimming of $\mathcal{T}$.

The trimming of a task word is unique, and we denote it $\mathcal{T}^{\setminus 1}$.

Example 5. $\mathcal{D}^{\setminus 1}$ is obtained from $\mathcal{D}$ by removing $d$ and $f$. The $\mathcal{D}^{\setminus 1}$-task word $\mathcal{T}^{\setminus 1}$ has universe $\{a, b, c, e\}$ and order relations as in $\mathcal{D}^{\setminus 1}$. Note that $d$ and $f$ contributed in $\mathcal{D}$ to the satisfaction of $\varphi_\mathcal{D}$, so $\mathcal{T}^{\setminus 1}$ has promised tasks and is no longer completed, with interpretations $ts^P = \{e\}$, $ts^C = \{a, b, c\}$, and the remaining letters empty. Note that the tasks for the shared elements of $\mathcal{T}$ and $\mathcal{T}^{\setminus 1}$ realize the same witness type sets.

We have achieved (S1): reducing satisfaction of $\varphi_\mathcal{D}$ to finding a sequence of task words.

Proposition 2. There is a data word $\mathcal{D} \models \varphi_\mathcal{D}$ if and only if there is a sequence $\mathcal{T}_1, \ldots, \mathcal{T}_n$ of consecutive task words, where $\mathcal{T}_n$ is a completed $\mathcal{D}$-task word.

Now to (S2): as the limited memory of PIAs hinders the manipulation of task words with unbounded length and data values, we operate on their extremal strings instead.

First, in data abstractions of task words, we do not distinguish all data values, but only the top layer elements with maximal value, the second to top layer, and the rest. We let Layers = \{1top, 2top, rest\}, and define the alphabet $\Gamma = \text{Layers} \times 2^{\text{Tasks}}$. We also define its restrictions to completed and promised tasks as $\Gamma^C = \text{Layers} \times 2^{\text{Tasks}^C}$ and $\Gamma^P = \text{Layers} \times 2^{\text{Tasks}^P}$, while $\Gamma_h = \{h\} \times 2^{\text{Tasks}^P}$ is the restriction of $\Gamma$ to some specific $h \in \text{Layers}$. For a symbol $\gamma = (h, ts)$ in $\Gamma$, we denote $ts(\gamma) = ts$ and $\omega(\gamma) = \omega(ts)$.
Definition 6 (Data abstraction). Let $T$ be a $D$-task word. For every $d \in D$, let $ts_d$ be such that $T \models ts_d(d)$, and let $A$ be the data word over $\Gamma$ with same universe and order relations as $T$, and with $A \models \gamma_h(d)$ where $\gamma_h = (h, ts_d)$ iff (a) $h = 1$ and $\text{value}_T(d) = \text{maxval}_D$, (b) $h = 2$ and $\text{value}_T(d) = \text{maxval}_D - 1$, or (c) $h = \text{rest}$ and $\text{value}_T(d) \in [\text{maxval}_D - 2]$. The data abstraction $\text{abst}(T)$ of $T$ is the string projection string$(A)$.

Extremal strings are obtained from data abstractions by keeping only the maximal and minimal positions in each layer with respect to the tasks. We extend to them the notions of consecutive and completed.

Definition 7 (extremal strings). Let $w \in \Gamma^*$. We define its extremal positions $\text{extPos}(w)$:

\[
\text{pos}_{\text{ext}, \theta}(w) = \{ \ell \in [w] | w(\ell) = (h, ts), \theta \in \omega(\ell) \}
\]

\[
\text{pos}_{\text{rest}, \theta}(w) = \{ \ell \in [w] | w(\ell) = (\text{rest}, ts), \theta \in \omega(\ell) \}
\]

\[
\text{extPos}_{\text{ext}, \theta}(w) = \{ \ell | \ell = \max(\text{pos}_{\text{ext}, \theta}(w)) \text{ or } \ell = \min(\text{pos}_{\text{ext}, \theta}(w)) \}
\]

If $\theta \models x \leq y$, $\text{extPos}_{\text{ext}, \theta}(w) = \{ \ell | \ell = \max(\text{pos}_{\text{ext}, \theta}(w)) \}
\]

If $\theta \models y < x$, $\text{extPos}_{\text{ext}, \theta}(w) = \{ \ell | \ell = \min(\text{pos}_{\text{ext}, \theta}(w)) \}
\]

$\text{extPos}(w) = \bigcup_{\theta \in \Theta_2} \text{extPos}_{\text{ext}, \theta}(w)$

If $\text{extPos}(w) = \{ \ell_1, \ldots, \ell_r \}$ and $\ell_1 < \cdots < \ell_r$, then the extremal string of $w$ is $w(\ell_1) \cdots w(\ell_r)$. $\text{EXT}(\Gamma) = \{ w | w \in \Gamma^* \}$ denotes the set of extremal strings. Note that $s = \text{ext}(w)$ implies $\text{ext}(s) = s$ and $\text{ext}(\varepsilon) = \varepsilon$. An extremal string $s$ is completed if $s \in \Gamma_C^\varepsilon$, or if $s = \varepsilon$ and $\theta_{\text{DW}(\Sigma)} \models \psi$. We occasionally write $\text{ext}(T)$ to mean $\text{ext}(\text{abst}(T))$.

A pair $s', s$ of extremal strings is consecutive if $s' = \text{ext}(T \setminus \{s\})$ and $s = \text{ext}(T)$ for some task word $T$.

For an extremal string $s$ and $\ell \in [|s|]$, the set of letters that can augment $s$ at position $\ell$ without being extremal is $\Gamma_{\text{not} \text{ext}}^h(s, \ell) = \{ \gamma \in \Gamma | ext(s) = ext(s(1) \cdots s(\ell - 1)\gamma s(\ell) \cdots s(|s|)) \}$, and we define $\Gamma_{\text{not} \text{ext}}^{\text{not} \text{ext}}(s, \ell) = \Gamma_h \cap \Gamma_{\text{not} \text{ext}}^h(s, \ell)$ for $h \in \text{Layers}$.

Example 6. Let $w$ be the following 6-letter string over $\Gamma = \text{Layers} \times 2^{\Omega_{\text{Tasks}}}$:

\[
\text{extPos}(w) = \{1, 3, 4, 5, 6\}, \text{ since the letter at position } 2 \text{ appears both to the left, at position } 1, \text{ and to the right, at position } 5, \text{ and } w = \text{ext}(w) \text{ is the substring obtained from } w \text{ by removing the non-extremal position } 2.\]

The concludes (S2), reducing the existence of $D$ to a sequence of extremal strings.

Corollary 2. There is a data word $D \models \varphi_\exists$ if and only if there is a sequence of consecutive extremal strings where the last one is completed.

Perfect extremal strings for $\varphi_\exists$ We define in the appendix a formula which intuitively `extracts' the 2-type of elements in a data word. Let $\alpha = (h_\alpha, ts_\alpha)$ and $\beta = (h_\beta, ts_\beta)$ in $\Gamma$ with at least one of them in $\Gamma_{1\text{top}}$. The formula $\text{perf}_{\alpha, \beta}(x, y)$ implies for every atomic formula either itself or its negation. For example, if $h_\alpha = h_\beta = 1$ and $\text{perf}_{\alpha, \beta}(x, y)$ implies $x \leq 2 y$, $y \leq 2 x$, $\neg S_2(x, y)$, and $\neg S_2(y, x)$. Hence for all $\alpha, \beta \in \Gamma$ with at least one of them in $\Gamma_{1\text{top}}$, there exists a 2-type $\theta(x, y)$ such that
perform_{\alpha,\beta}(x, y) \equiv_{DW(\Xi)} \theta(x, y). This allows us to describe the 2-type of elements in
task words via perform_{\alpha,\beta} formulas. For any two elements of the data word, there is a
(possibly iterated) trimming in which both appear and one of them has the maximal
data value, and their perfect formula, which is equivalent to their 2-type, determines
whether they satisfy the universal constraint \( \chi \). Thus we can ensure satisfaction of \( \chi \)
using perform_{\alpha,\beta}(x, y) formulas from all the trimmings.

**Definition 8** (Perfect string, perfect task word). Let \( w \in \Gamma^\ast \). We say \( w \) is a perfect
string if for every two positions \( \ell_1 < \ell_2 \) in \( w \) such that \( \{w(\ell_1), w(\ell_2)\} \cap \Gamma_{1\text{top}} \neq \emptyset \) we have
perform_{w(\ell_1), w(\ell_2)}(x, y) \models_{DW(\Xi)} \chi(x, y) \land \chi(y, x). Note that the empty string \( \varepsilon \) is
perfect. A task word \( T \) is perfect if it is empty, or if \( \text{ext}(T) \) and \( \text{ext}(T^{\backslash 1}) \) are perfect.

**Example 7.** Let \( \zeta = (2\text{top}, t_{s_2}^\xi) \) and \( \zeta = (1\text{top}, t_{s_2}^\xi) \). Then perform_{\zeta}(x, y) is given
by: perform_{\zeta}(x, y) = \xi_2(x) \land \xi_\zeta(y) \land (x < y) \land (x \not\leq y \land S_2(x, y)).
The 2-type \( \theta \) to which perform_{\zeta}(x, y) is equivalent over \( DW(\Xi) \) is given by the conjunction
of perform_{\zeta}(x, y) with \( \neg \xi_2(x) \land \neg \xi_\zeta(y) \land (y < x) \land (y \not\leq x) \land \neg S_2(y, x) \). We
have that \( w \) is a perfect string, and perform_{\zeta}(x, y) \models_{DW(\Xi)} \chi(x, y) \land \chi(y, x).

We characterize satisfiability in terms of perfect completed task words.

**Lemma 2.** Let \( T \) be a \( D \)-task word. \( T \) is perfect if and only if \( D \models \varphi \).

As a corollary of Lemma 2 and Lemma 1 we get:

**Proposition 3.** For every data word \( D \in DW(\Xi) \), \( D \models \varphi \) if and only if there exists a
perfect completed \( D \)-task word. There is \( \exists D \models \varphi \) if and only if there is a sequence of
consecutive perfect extremal strings where the last one is completed.

We are almost ready to define \( A^\varphi \). Intuitively, it will guess a sequence of extremal
strings as in Prop. 3 placing pebbles from an extremal string to a consecutive one.
This requires the automaton to verify consecutiveness, and to know which positions
in consecutive extremal strings correspond to the same position in the input. This is
easy if we have the underlying task word; indeed, given a task word \( T \) and an extremal
string \( s' = \text{ext}(T) \), there is a bijective mapping from the extremal elements of \( T \)
that \( s' \) stores, to their positions in \( s' \). The same holds for \( T^{\backslash 1} \) and \( s = \text{ext}(T^{\backslash 1}). \)
By composing these mappings after inverting the latter, and restricting its domain to
positions that remain extremal after updating the abstracted data values (that is, shifting
the top layer to second top, and the second top into the remaining layer), we obtain a
partial embedding from \( s \) to \( s' \) via \( T \) that keeps track of the matching positions; the
precise definition is in the appendix. But one major hurdle remains: these notions
are defined in terms of a task word \( T \), and our PIA cannot store task words, only
their extremal strings. We overcome this through a merely syntactic characterization
of consecutiveness, which can be verified without a concrete task word. This rather
technical step relies on the fact that if \( s, s' \) are consecutive, then \( s' \) can be obtained
by guessing a substring \( r \) that will get new data values, interleaving it into the proper
positions \( g \) of \( s \), which can also be guessed, and updating the abstracted data values.
Also the partial embedding that keeps track of matching the positions can be obtained
without a concrete \( T \), using \( r \) and \( g \).
Lemma 3. We can decide whether two given extremal strings $s, s'$ are consecutive in $\text{EXPSPACE}$. If they are, then we can also obtain in $\text{EXPSPACE}$ a partial embedding $\text{PEmb}_{s \rightarrow s'}$ from positions in $s$ to positions in $s'$ that coincides with the partial embedding from $s$ to $s'$ via $T$ for every task word $T$ such that $s' = \text{ext}(T)$ and $s = \text{ext}(T^{k_1})$.

The automaton. We give a high-level description of $\mathcal{A}^\circ = (\Xi, m+1, Q, q_{\text{init}}, \Gamma, \delta)$, and refer to App. [F] for a full definition. We have $m = 7 \cdot |\Theta_3|$: there is one pebble for each existential constraint in $\Theta_3$ and each layer in $\Gamma$, plus an additional pebble per constraint, and one designated pebble $m+1$ to read non-extremal positions. $Q = Q_e \cup Q_p$ has two types of states:

- $Q_e$ contains states $(s, \tau)$ with $s$ a perfect extremal string and $\tau$ an $(m+1, |s|)$-pebble assignment, which intuitively describes the assignment after reading $s$.
- $Q_p$ contains states of the forms $(s, \bar{s}, \tau, 0)$ and $(s, \bar{s}, \tau, 1)$ for every perfect extremal string $s$, non-empty prefix $\bar{s}$ of $s$, and $(m+1, |s|)$-pebble assignment $\tau$ that satisfies certain conditions that hold when only the prefix $\bar{s}$ has been read.

The initial state is $q_{\text{init}} = (\varepsilon, \rho_\perp) \in Q_e$ and the final states are $F = \{(s, \tau) \in Q_e \mid s \text{ is completed}\}$. The transition $\delta$ is roughly as follows. $\mathcal{A}^\circ$ should transition from $(s, \tau) \in Q_e$ to $(s', \tau') \in Q_e$ for consecutive $s, s'$, but since it can only move one pebble at a time, we have intermediate states in $Q_p$, which allow it to read $s'$ from left to right by iterating over all its prefixes. We start reading $s'$ by moving to $(s', s'(1), \tau'_0, 0) \in Q_p$, where $\tau'_0$ stores the pebble assignment induced by $\text{PEmb}_{s \rightarrow s'}$. Once the whole extremal string $s'$ has been read, we move to the next extremal state.

This finishes the construction of the automaton $\mathcal{A}^\circ$ with $L(\mathcal{A}^\circ) = L(\varphi)$, and thus the proof of Theorem 3. Concerning the upper bound on finite satisfiability, by Theorem 4 and $\text{EXT}(\Gamma) \subseteq \Gamma^{|\psi|}$, we get that $\mathcal{A}^\circ$ has size at most double exponential in $|\psi|$. For the $\text{EXPSPACE}$ upper bound, we need to show that the transition relation of $\mathcal{A}^\circ$ is $\text{EXPSPACE}$-computable (Lemma 8 in the appendix). This with Theorem 2 gives an alternative proof of the upper bound in [29]:

Corollary 3. Finite satisfiability of $\text{FO}^2(\leq_1, \leq_2, S_2)$ is in $\text{EXPSPACE}$.

Relation to the proof of Schwentick and Zeume [29]. Naturally, there are similarities between the techniques; our extremal strings and tasks are similar to their profiles and directional constraints. However, a key difference is that in their ‘geometric’ view, elements of the data word are assigned points $(a, b)$ in the plane with $a$ a position in $\leq_1$, and $b$ a data value. Existential constraints are indicated by marking the witnesses with the letters they should have, and many profiles in a consistent sequence can contain points with the same $a$ value. In contrast, our ‘temporal’ view arises from the computation of the PIA. We mark elements with existential constraints they need to satisfy and that they have satisfied, which is compatible with the read-once nature of PIA. It does not seem possible to use their proof techniques without modifying PIA to allow multiple readings of the input. The modified model would work for the logic-to-automata relation established here, but we suspect it would be too strong for the other direction.
4 Discussion and Conclusion

We introduced pebble-intervals automata (PIA) and studied their computational power. We proved that the projections of data languages definable in $\text{FO}^2(\leq_1, \lesseq_{\leq_2}, S_2)$ are PI languages, and as a by-product, obtained an alternative proof that finite satisfiability is in EXPSPACE. The main question that remains is the converse of our main result: whether every PI language is the projection of an $\text{FO}^2(\leq_1, \lesseq_{\leq_2}, S_2)$ definable data language. We believe this is the case. Our work also gives rise to other questions. We suspect that our results can be extended to $\omega$-languages, and we would like to adapt them to $C^2$, which extends $\text{FO}^2$ with counting quantifiers [26, 31]. We also plan to explore further the computational power of our automata model, for instance, to establish a pumping lemma that allows us to prove that some context-free languages are not PI languages.

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A Embeddings

Throughout the paper, we introduce definitions in which a data word or string $X$ is obtained from a data word or string $Y$ by applying an operation $Op$. The operation $Op$ induces an injective order-preserving function $F_{Op}$ relating each position or universe element in $X$ to a position or universe element in $Y$. $F_{Op}$ is called an embedding of $X$ into $Y$. By order-preserving we mean that whenever a pair $a \leq_X b$ of positions or universe elements is mapped by $F_{Op}$ to a pair $a' \leq_Y b'$, we have $a' \leq_Y b'$, where $\leq$ for $Z \in \{X, Y\}$ is as follows: (i) if $Z$ is a string, $\leq_Z$ is the order on natural numbers; (ii) if $Z$ is a data word, $\leq_Z$ is the order $\leq_1$ of $Z$. For instance, given a data word $D$ with universe $D$ of size $n$, the embedding of the string projection $w_D = \text{string}(D)$ of $D$ into $D$ assigns to every position $\ell \in [n]$ the element $d$ which satisfies $\ell = |\{d' \in D \mid D \models d' \leq_1 d\}|$. We denote the embedding of $X$ into $Y$ by $\text{Emb}_{Op,Y}$. In the case of the string projection, $\text{Emb}_{\text{string},D}$ is an injective function (indeed a bijection) from $[n]$ to $D$ which preserves the order $\leq_1$ of $D$ in terms of the order of positions in the projection. The inverse $\text{Emb}_{Op,Y}^{-1}$ of an embedding is a partial function. Given two operations $Op_1$ and $Op_2$, we denote the embedding resulting from their composition $X = Op_1(Op_2(Z))$, $\text{Emb}_{Op_2,Z} \circ \text{Emb}_{Op_1,Op_2}(Z)$, by $\text{Emb}_{Op_1 \circ Op_2, Z}$.

Example 1. $\text{Emb}_{\text{string},D} : [6] \rightarrow D$ is given by:

$\text{Emb}_{\text{string},D}(1) = a$ \hspace{1cm} $\text{Emb}_{\text{string},D}(4) = d$

$\text{Emb}_{\text{string},D}(2) = b$ \hspace{1cm} $\text{Emb}_{\text{string},D}(5) = e$

$\text{Emb}_{\text{string},D}(3) = c$ \hspace{1cm} $\text{Emb}_{\text{string},D}(6) = f$

Example 2. Let $w = \text{abst}(T)$ be the data abstraction of the task word $T$. Then $w$ is the 6-letter string over $\Gamma = \text{Layers} \times 2|\text{Tasks} |$ given by:

$(\text{rest}, ts^C_3)(\text{rest}, ts^C_3)(2\text{top}, ts^C_1)(\text{1top}, ts^C_3)(\text{rest}, ts^C_3)(1\text{top}, ts^C_4)$.

Let $w' = \text{abst}(T' \setminus 1)$. Then $w'$ is given by the string

$(\text{rest}, ts^P_3)(\text{rest}, ts^P_3)(1\text{top}, ts^P_1)(2\text{top}, ts^P_3)$.

Note that the length discrepancy between $w$ and $w'$ matches the number of elements removed in the trimming. The embedding $\text{Emb}_{\text{abst},T}$ is given by:

$\text{Emb}_{\text{abst},T}(1) = a$ \hspace{1cm} $\text{Emb}_{\text{abst},T}(4) = d$

$\text{Emb}_{\text{abst},T}(2) = b$ \hspace{1cm} $\text{Emb}_{\text{abst},T}(5) = e$

$\text{Emb}_{\text{abst},T}(3) = c$ \hspace{1cm} $\text{Emb}_{\text{abst},T}(6) = f$

The embedding $\text{Emb}_{\text{abst},T' \setminus 1}$ is given by:

$\text{Emb}_{\text{abst},T' \setminus 1}(1) = a$

$\text{Emb}_{\text{abst},T' \setminus 1}(2) = b$

$\text{Emb}_{\text{abst},T' \setminus 1}(3) = c$
Example 3. The embedding $\text{Emb}_{\text{ext}, \omega}$ is given by:

$$
\begin{align*}
\text{Emb}_{\text{ext}, \omega}(1) &= 1 \\
\text{Emb}_{\text{ext}, \omega}(2) &= 3 \\
\text{Emb}_{\text{ext}, \omega}(3) &= 4 \\
\text{Emb}_{\text{ext}, \omega}(4) &= 5 \\
\text{Emb}_{\text{ext}, \omega}(5) &= 6
\end{align*}
$$

The embedding $\text{Emb}_{\text{ext}, \omega}'$ is the identity function $\text{Emb}_{\text{ext}, \omega}'(i) = i$.

## B Proof of closure properties

We briefly recall the definitions of shuffle and iterated shuffle.

**Definition 9** (Iterated shuffle). Let $u, v \in \Sigma^*$. A string $w \in \Sigma^*$ (of length $|u| + |v|$) is called a shuffle of $u, v \in \Sigma^*$ if there is a set $S = \{i_1, \ldots, i_{|u|}\} \subseteq [|w|]$ such that $u$ is the string $w(i_1) \cdots w(i_{|u|})$, and $v$ is the string of the remaining letters of $w$. We denote $u \shuffle v$ the set of all strings that are a shuffle of $u$ and $v$, and define the shuffle of two languages $L, L' \subseteq \Sigma^*$ as $L \shuffle L' = \bigcup_{u \in L, v \in L'} u \shuffle v$. The iterated shuffle of a language $L$ is defined as $L^{i+1} = \bigcup_{i \geq 0} L^i$ where $L_0 = L$ and $L_{i+1} = L_i \shuffle L$.

**Theorem 5.** The class of PI languages is effectively closed under union, concatenation, Kleene-*, shuffle, and iterated shuffle.

**Proof.** Let $A_1$ and $A_2$ be PIAs that accept the languages $L(A_1)$ and $L(A_2)$, respectively. In the following, we describe an automaton $A$ which accepts the corresponding language operation.

**Union.** $L(A_1) \cup L(A_2)$ is accepted by a pebble-intervals automaton $A$ which on input $w$, uses a silent transition to move to the initial state of either $A_1$ or $A_2$ (intuitively, it guesses whether $w \in L(A_1)$ or $w \in L(A_2)$) and then simulates the corresponding automaton on $w$.

**Concatenation.** We show that $L(A_1)L(A_2)$ is accepted by a pebble-intervals automaton $A$. Essentially, $A$ guesses where the partition is in the input, then it simulates the corresponding automaton on each segment of the input. More precisely, given input $w$, if $\varepsilon \in L(A_1) \cap L(A_2)$, the automaton $A$ guesses whether $w$ is the empty string, and if so moves to an accepting state using a silent transition. Otherwise, $A$ places a pebble $p$ on an arbitrary position $\ell$ in $w$. $A$ remembers the letter $w(\ell)$ in its state, as it will need to simulate $A_2$ as if it has read the letter $w(\ell)$ in the first position in the input to $A_2$. $A$ then simulates $A_1$ on the prefix $w(1) \cdots w(\ell - 1)$ by replacing all move transitions $k$-\textbf{MOVE} with $k$-\textbf{MOVE}$_{k,p}$. Similarly, $A$ then simulates $A_2$ on the suffix $w(\ell) \cdots w(n)$ by replacing all move transitions $k$-\textbf{MOVE}$_{k,i}$ with $k$-\textbf{MOVE}$_{k,p,i}$. However, for this part of the input we need some new silent transitions to simulate $A_2$ placing a pebble on the position $p$ is on and reading the letter $w(\ell)$. $A$ accepts if both simulations accept.
Kleene-⋆. We show that $L(A_1)^*$ is accepted by a pebble-intervals automaton $A$. This automaton works similarly to the concatenation case, except it uses two pebble to enclose the substring it simulate an automaton on, and it non-deterministically chooses when to move on to the next segment of the word. The automaton $A$ guesses whether the input $w$ is the empty string or not. If so, it goes into an accepting state. Otherwise, $A$ begins to simulate $A_1$ on contiguous substrings of $w$ in rounds, using pebbles $p, p'$ to enclose the substrings. In the first round, $A$ places $p$ anywhere and then places $p'$ somewhere to the right of $p$. Then $A$ simulates $A_1$ on the substring between the pebbles by replacing $k$-MOVE$_{<,i}$ transitions with $k$-MOVE$_{p,i}$ transitions, and $k$-MOVE$_{i,>}$ transitions with $k$-MOVE$_{i,p'}$ transitions. In addition, $A$ uses new silent transitions to simulate $A_1$ placing pebbles on the positions of $p$ and $p'$. The subsequent rounds are similar, except that they begin with $A$ placing $p$ to the right of $p'$. Note that for a run to be successful, the placement of $p$ in the first round must be on the first position of the input, and in subsequent rounds, the placement of $p$ must be immediately to the right of $p'$. $A$ decides non-deterministically to finish a round or continue it whenever the simulated state of $A_1$ is accepting (otherwise $A$ must continue the round). $A$ is at an accepting state whenever $A_1$ is at an accepting state.

Shuffle. The automaton $A$ uses two disjoint sets of pebbles. $A$ uses one set of pebbles to simulate $A_1$ on some substring of $w$, which is not necessarily contiguous, and the other set of pebbles to simulate $A_2$ on the substring composed of the remaining positions. $A$ accepts if both simulations accept.

Iterated shuffle. This is similar to the case of the shuffle, except that $A$ performs several simulations of $A_1$ in rounds. At the beginning of each round, $A_1$ is set to its initial state, and all pebbles are considered available. $A$ simulates $A_1$ on an arbitrary subset of the unread positions of the input. If $A_1$ goes into an accepting state, $A$ non-deterministically chooses whether to continue the simulation of $A_1$, or whether a shuffled string was accepted by $A_1$ and this is the end of a round. In the latter case $A$ either starts a new round or it goes into an accepting state.

B.1 Proofs of non-closure

Theorem 6. The class of PI languages is not effectively closed under

1. intersection, even with regular languages, nor under

2. complement.

Proof. (Part 1.) We show that if pebble-intervals automata were effectively closed under intersection with regular languages, we could decide an undecidable problem. Namely, the Minsky halting problem. A Minsky machine \([21]\) is a sequence of labeled
instructions

0: \text{comm}_0
1: \text{comm}_1
: 
\vdots
n-1: \text{comm}_{n-1}
\text{HALT}: \text{halt}

where each of the first \( n \) instructions is either an \text{inc}_c\) instruction of the form:

\[ i: c := c+1; \text{goto } i+1 \]

or a conditional \text{dec}_c\( (j)\) instruction of the form:

\[ i: \text{if } c=0 \text{ then goto } j \]
\[ \text{else } c:=c-1; \text{goto } i+1 \]

and \( c \) is one of the registers of the machine.

A trace of a Minsky machine \( M \) is a sequence \( t_1, \ldots, t_m \) of labels where \( t_1 = 0 \), \( t_m = \text{HALT} \), and for \( 0 < k < m \), if \( t_{k-1} = i \) is an \text{inc}_c\) instruction, then \( t_k = i+1 \), and if it is a \text{dec}_c\( (j)\) instruction, then either \( t_k = i+1 \), or \( t_k = j \).

Note that the language of traces of a Minsky machine \( M \) is regular.

A trace \( t_1, \ldots, t_m \) is feasible if the following holds. If \( t_{k-1} = i \) for a \text{dec}_c\( (j)\) instruction, and \( j \neq i+1 \), then \( t_k = j \) iff \( c = 0 \) at step \( k-1 \) during the run of \( M \).

Note that \( c = 0 \) holds exactly at the points where the number of \text{inc}_c\) instructions is equal to the number of \text{dec}_c\) instructions.

The problem of deciding whether a given Minsky machine \( M \) with two registers \( c_0, c_1 \) halts when started with \( c_0 = c_1 = 0 \) is referred to here as the Minsky halting problem. It is known that the Minsky halting problem is undecidable \[28\]. In other words, it is undecidable whether there is a feasible trace of \( M \).

Now fix a Minsky machine \( M \) and denote the language of its traces by \( T \).

Let \( \Sigma = \{ \text{halt}, \text{inc}_c, \text{dec}_c, \text{jmp}_c \mid c \in \{c_0, c_1\} \} \). We define a mapping \( h : T \rightarrow \Sigma^* \) from traces to \( \Sigma^* \) which will also produce a regular language. Let \( t_1 \ldots t_m \) be a trace. For \( t_{k-1} = i \) where \( 0 < k < m \):  

- If \( i \) is an \text{inc}_c\) instruction, then \( f(t_{k-1}, t_k) = \text{inc}_c \).

- If \( i \) is a \text{dec}_c\( (j)\) instruction with \( j \neq i+1 \), then

\[
f(t_{k-1}, t_k) = \begin{cases} 
\text{dec}_c & \text{if } t_k = i+1, \\
\text{jmp}_c & \text{if } t_k = j 
\end{cases}
\]

- If \( i \) is a \text{dec}_c\( (i+1)\) instruction, then \( f(t_{k-1}, t_k) \in \{ \text{dec}_c, \text{jmp}_c \} \).

Now define \( h(t_1 \ldots t_m) \) as the string \( f(t_1, t_2) f(t_2, t_3) \ldots f(t_{m-1}, t_m) \) halt.

Denote the language resulting from applying \( h \) to all the traces by \( \text{Inst}_M = \{ h(t) \mid t \in T \} \), and note \( \text{Inst}_M \) is regular. This language describes the traces of a machine. In order to decide whether there exists a feasible trace of \( M \), we need...
to be able to test if the appearances of the \texttt{jmp} letter are in appropriate positions. For this purpose, we next define a pebble-intervals language by shuffling two pebble-intervals languages given by a context-free grammar, which can distinguish between appropriately-placed and inappropriately-placed \texttt{jmp} letters. The intersection of the languages will consist of the feasible traces, meaning it will be non-empty if and only if the given Minsky machine halts.

We now define a context free grammar which produces sequences of \texttt{inc} and \texttt{dec} which are well matched, along with \texttt{jmp} letters which are appropriately placed, that is, if \texttt{jmp} appear in position \(i\) of a string, then there is an equal number of \texttt{inc} and \texttt{dec} in the prefix containing the first \(i\) positions. Define the context free grammar \(\Gamma_c\) as follows:

\[
\begin{align*}
A & ::= B \text{ halt} \\
B & ::= BB | C BC | D \\
C & ::= \varepsilon | \text{ jmp} C \\
D & ::= \text{ inc} D \text{ dec} | DD | \varepsilon
\end{align*}
\]

**Claim 1.** The language produced by the grammar \(\Gamma_c\) is PI.

**Proof of Claim** Let \(A\) simulate \(A_{Dyck}\) on substrings of its input in order verify that every jump is preceded by the appropriate number of increments and decrements. These substrings are enclosed between pebbles 1 and 2 (or the ends of the input). The automaton \(A_{Dyck}\) is simulated using a disjoint set of pebbles. We describe a successful run of the automaton for one of the registers. The automaton places pebble 0 on the last position to read \texttt{halt}. Then it guesses whether there are any \texttt{jmp} on the input. If not, then it simply simulates \(A_{Dyck}\) on the whole input. Otherwise, pebble 1 is placed to the left of 0 to read \texttt{jmp}, and \(A_{Dyck}\) is simulated on the substring in the interval between the beginning of the input and pebble 1. Let \(i \in \{1, 2\}\) be the pebble placed at the beginning of the current round. In a new round, the automaton guesses whether there is \texttt{jmp} to the right of pebble \(i\), and if so, it moves pebble \(3 - i\) to the right of pebble \(i\) to read \texttt{jmp}, and simulates \(A_{Dyck}\) on the substring in the interval between \(3 - i\) and \(i\). If the automaton guesses there is no \texttt{jmp} to the right of pebble \(i\), it simulates \(A_{Dyck}\) on the remainder of the input, in the interval between \(i\) and 0 and does not enter any new rounds.

We resume the proof of Theorem. Define \(L_{c0}\) as the language produced by \(\Gamma_{c0}\), and \(L_{c1}\) as the language produced by \(\Gamma_{c1}\). Finally, define \(L\) as the shuffle between \(L_{c0}\) and \(L_{c1}\), and note that by Claim and Theorem it is a pebble-intervals language.

Therefore, intersecting \(L\) with \(\text{Inst}_M\) would result in exactly the feasible traces. Now assume for contradiction that pebble-intervals languages are effectively closed under intersection with regular languages. We describe a procedure for deciding the Minsky halting problem. Given \(M\), generate and automaton \(A_{\text{inst}}\) for \(\text{Inst}_M\). Using the assumed effective procedure for intersection with regular languages, now generate an automaton \(A_M\) for \(L_M = L \cap \text{Inst}_M\). To decide the Minsky halting problem for \(M\), test \(A_M\) for emptiness.

Since emptiness of pebble-intervals automata is decidable, we contradict undecidability of the Minsky halting problem, so we conclude that pebble-intervals languages are not effectively closed under intersection, even with regular languages.
(Part 2.) We show we can build an automaton for \((L_M)^c\) and conclude that if we could effectively build a complement automaton for any pebble-intervals language, then in particular we could build one for \(((L_M)^c)^c = L_M\), which we could test for emptiness and again solve the Minsky halting problem.

Automaton for \((L_M)^c\): since \(L_M = L \cap \text{Inst}_M\), we have that \((L_M)^c = L^c \cup (\text{Inst}_M)^c\). Note that if \(L^c\) and \((\text{Inst}_M)^c\) are PI languages, then by Theorem 5 their union \((L_M)^c\) is a pebble-intervals language. \((\text{Inst}_M)^c\) is regular and therefore also a pebble-intervals language, so it remains to show that \(L^c\) is accepted by a pebble-intervals automaton.

**Claim 2.** \(L^c\) is a pebble-intervals language.

**Proof of Claim 2.** Note that for every \(w \in L^c\), it holds that there is a prefix of \(w\) ending with some jmp such that the number of increments and decrements are not equal. The automaton guesses for which register this happens, the prefix, and whether it is the number of increments or decrements that is larger. Then it verifies its guesses using \(A_{Dyck}\).

This concludes the proof of Theorem 6.

**B.2 Proof of Corollary 1**

**Proof.** We have seen in the proof above that we can effectively build an automaton \(A\) for \((L_M)^c\). We have that \(A\) has a universal language if and only if \(L_M = \emptyset\). Thus if we could test for universality, we could again solve the Minsky halting problem.

Since universality easily reduces to inclusion, undecidability of the inclusion problem follows.

**C Proof of upper bound for emptiness**

We now prove the upper bounds of Theorem 2. For the rest of this section, assume a PIA \(A = (\Sigma, m, Q, \text{init}, F, \delta)\).

**Definition 10 (Feasible sequence of transitions).** An arrangement of pebbles \([m]\) is a linear order \(O = (O, R_\leq)\) with \(O \subseteq [m]\). Each \((m, n)\)-pebble assignment \(\rho\) induces an arrangement \(\sigma(\rho)\) with \(O = [m] - \rho^{-1}(\perp)\) and \(R_\leq(i, j)\) if and only if \(\rho(i) \leq \rho(j)\) for every \(i, j \in [m]\).

Let \(r \geq 0\) and let \(\bar{\nu} = (t_1, \ldots, t_r)\) be a sequence of transitions. We say \(\bar{\nu}\) is feasible if there is a sequence \(\bar{O} = (O_0, \ldots, O_r)\) of arrangements and a sequence \(\bar{q} = (q_0, \ldots, q_r)\) of states with \(q_{\text{init}} = q_0\), such that \(O_0\) is empty, and for every \(1 \leq \ell \leq r:\)

1. if \(t_\ell\) is a silent transition, then \(t_\ell = (q_{\ell-1}, q_\ell)\) and \(O_\ell = O_{\ell-1}\), and

2. if \(t_\ell\) is a MOVE transition, then there are \(k \in [m]\), \(i, j \in ([m] - \{k\}) \cup \{>, <\}\), and \(\sigma \in \Sigma\) with \(t_\ell = (q_{\ell-1}, k\cdot\text{MOVE}_{i,j}, \sigma, q_\ell)\) and the arrangement \(O_\ell = (O_\ell, R_{\leq,\ell})\) is such that \(O_\ell = O_{\ell-1} \cup \{k\}\), \(R_{\leq,\ell}(i', j')\) if and only if \(R_{\leq,\ell-1}(i', j')\) for all for \(i', j' \in [m] - \{k\}\), and additionally, if \(i \in [m]\) then \(R_{\leq,\ell}(i, k)\), and if \(j \in [m]\) then \(R_{\leq,\ell}(k, j)\).
The following lemma shows that the feasible sequences of transitions are exactly the ones corresponding to actual computations of the automaton.

**Lemma 4.** A sequence \( \bar{t} \) of transitions is feasible if and only if there is a string \( u \in \Sigma^* \) and a sequence of configurations \( \bar{\pi} \) such that \( (\bar{t}, \bar{\pi}) \) is a computation of \( \mathcal{A} \) on \( u \). Moreover, \( \bar{t} \) ends in an accepting state if and only if \( (\bar{t}, \bar{\pi}) \) is an accepting computation.

**Lemma 5.** There is a feasible sequence of transitions \( \bar{t} \) of length at most \( |\mathcal{A}| \cdot 2^{O(m \log m)} \) ending in an accepting state iff \( L(\mathcal{A}) \neq \emptyset \).

**Proof.** By Lemma 4 if \( \bar{t} \) is feasible and ends in an accepting state, then there is a sequence of configurations \( \bar{\pi} \) such that \( (\bar{t}, \bar{\pi}) \) is an accepting computation of \( \mathcal{A} \) on some string \( u \). Hence \( u \in L(\mathcal{A}) \).

Conversely, assume that \( L(\mathcal{A}) \neq \emptyset \), and let \( (\bar{t}, \bar{\pi}) \) be an accepting computation of minimal length on any string \( u \). Denote \( (\bar{t}, \bar{\pi}) = ((t_1, \ldots, t_r), \pi_0, \ldots, \pi_r) \). By Lemma 4 \( \bar{t} \) is feasible and ends in an accepting state. Let \( \bar{O} \) be as guaranteed for \( \bar{t} \) in Definition 10. Now assume for contradiction that there are two distinct \( h_1, h_2 \in [r] \) such that \( t_{h_1} = t_{h_2} \) and \( \mathcal{O}_{h_1} = \mathcal{O}_{h_2} \). Then \( (t_1, \ldots, t_{h_1}, t_{h_2+1}, \ldots, t_r) \) is a feasible sequence of transitions ending in an accepting state. Hence, there is a word \( u_{12} \) which is accepted by a computation of length \( r - (h_2 - h_1) \), in contradiction to the minimality of \( (\bar{t}, \bar{\pi}) \). Hence, the length of \( \bar{t} \) is at most \( |\delta| \cdot M \), where \( M \) is the number of arrangements \( \mathcal{O} \) of \( m \) pebbles. We have that \( M \leq 2^m \cdot m! \) since there are \( 2^m \) ways of choosing a subset \( O \subseteq [m] \) as the universe of \( \mathcal{O} \), and \( |O|! \leq m! \) ways to linearly order the set \( O \).

**C.1 Proof of Theorem 2**

**Proof.** Let \( \mathcal{A} = (\Sigma, m, Q, q_{\text{init}}, F, \delta) \). We non-deterministically attempt to guess a feasible sequence of transitions \( \bar{t} \) which ends at an accepting state. Due to the number of non-repeating sequences of arrangements, \( L(\mathcal{A}) \neq \emptyset \) if and only if there is a feasible sequence of transitions ending in an accepting state and sequences \( \bar{O} \) as in Definition 10 whose lengths are at most \( |\mathcal{A}| \cdot 2^{O(m \log m)} \). We only need to keep simultaneously a counter of the sequence length \( r \), two transitions \( t_h, t_{h+1} : h \in [r - 1] \) and two arrangements \( \mathcal{O}_h, \mathcal{O}_{h+1} \). The size of the representation of a transition is logarithmic in \( |\mathcal{A}| \). The size of the representation of an arrangement is \( O(m^2) \). Completeness for the case where the automaton has \( O(\log |\mathcal{A}|) \) pebbles follows from the NL-completeness of the emptiness problem of standard finite state automata and Prop. 1.

**D Proof of the normal form**

**Proof.** Here we prove Theorem 4. We denote the class of all finite structures over a vocabulary \( \text{voc} \) by \( \text{Str}(\text{voc}) \). To define \( h \), we first need to introduce two functions (called translations), \( \text{trans}_1 \) and \( \text{trans}_2 \). For simplicity we assume that the empty word satisfies \( \psi \), and therefore \( \varphi_e = \text{True} \). For the other case, we need to change the following by conjoining each of \( \varphi^0, \varphi^1, \) and \( \varphi^2 \) with \( \varphi_e = \exists x \text{ (True)} \).
The translation $\text{trans}_1$ Let

$$\varphi^0 = \forall x \forall y \chi^0(x, y) \land \bigwedge_{b=1}^B \forall x \exists y \chi^0_b(x, y)$$

be the Scott Normal Form of $\psi$ (see e.g. [14, Theorem 2.1]). The formula $\varphi^0$ is over the vocabulary $\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF}$ where $\text{voc}_{SNF}$ is a set of fresh unary relations. The formulas $\chi^0$, and $\chi^0_b$ are quantifier-free. The length of $\varphi^0$ is linear in that of $\psi$.

For every model $D \in \text{DW}(\Sigma)$ of $\psi$, there is a unique expansion of $D$ which satisfies $\varphi^0$. Let $\text{trans}_1 : \text{Str}(\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF}) \to \text{DW}(\Sigma)$ be the function which takes a $\text{Str}(\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF})$-structure to its reduct to $\text{voc}_{DW}(\Sigma)$. The following hold:

(i) For every $E \in \text{Str}(\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF})$, if $E \models \varphi^0$ then $\text{trans}_1(E) \models_{\text{DW}(\Sigma)} \psi$, and

(ii) For every $D \in \text{DW}(\Sigma)$, if $D \models_{\text{DW}(\Sigma)} \psi$ then there exists $E \in \text{Str}(\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF})$ such that $\text{trans}_1(E) = D$ and $E \models \varphi^0$.

Making the types explicit Next we define a formula $\varphi^1$ which is equivalent to $\varphi^0$. Let $A$ be a finite set such that $\{\nu_a \mid a \in A\}$ is the set of 1-types over $\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF}$. Every quantifier-free formula is equivalent to a disjunction of 2-types. Hence, there is a set $C$ whose size is at most the number of 2-types over $\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF}$ such that every conjunct $\forall x \exists y \chi^0(x, y)$ is equivalent to

$$\forall x \bigwedge_{a=1}^A \nu_a(x) \to \exists y \bigvee_{c=1}^C \beta_{abc}(x, y)$$

where $\beta_{abc}(x, y)$ is 2-type over $\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF}$ for every $a$, $b$, and $c$. There is a set $\Theta^0(x, y)$ of 2-types over $\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF}$ such that

$$\chi^0(x, y) = \bigvee_{\beta \in \Theta^0} \beta(x, y),$$

$$\varphi^0 = \forall x \forall y \bigvee_{\beta \in \Theta^0} \beta(x, y) \land \bigwedge_{b=1}^B \forall x \bigwedge_{a=1}^A \nu_a(x) \to \exists y \bigvee_{c=1}^C \beta_{abc}(x, y).$$

Let $\varphi^1 = \varphi^0 \land \varphi^3$, where

$$\varphi^1 = \forall x \bigwedge_{a=1}^A \nu_a(x) \to \bigwedge_{b=1}^B \nu_b(x) \to \bigvee_{c=1}^C \beta_{abc}(x, y).$$

We have $\varphi^1 \equiv \varphi^0$ and from (i) and (ii) hold:

(i) For every $E \in \text{Str}(\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF})$, if $E \models \varphi^1$ then $\text{trans}_1(E) \models \varphi^0$ and $\psi$, and

(ii) For every $D \in \text{DW}(\Sigma)$, if $D \models_{\text{DW}(\Sigma)} \psi$ then there exists $E \in \text{Str}(\text{voc}_{DW}(\Sigma) \cup \text{voc}_{SNF})$ such that $\text{trans}_1(E) = D$ and $E \models \varphi^1$.  

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The translation $\text{trans}_2$ Let $\Xi = \{\xi_a \mid a \in [A]\}$. For every 2-type $\beta$ over $\text{voc}_{\text{DW}}(\Sigma) \cup \text{voc}_{\text{SNF}}$, let $\beta^2$ be the 2-type over $\text{voc}_{\text{DW}}(\Xi)$ such that:

- For every $\alpha(x, y)$ which is one of $R(x, y)$ or $\neg R(y, x)$ for $R \in \{\leq_1, \leq_2, S_2\}$, $\beta(x, y) \models \alpha(x, y)$ if and only if $\beta^2(x, y) \models \alpha(x, y)$.
- For every $a \in [A]$ and $z \in \{x, y\}$, $\beta(x, y) \models \nu_a(z)$ if and only if $\beta^2(x, y) \models \xi_a(z)$.

Let $\varphi^2 \in \text{FO}^2(\text{voc}_{\text{DW}}(\Xi))$ be the formula obtained from $\varphi^1$ by replacing the 1-types $\nu_a(x)$ with $\xi_a(x)$ and the 2-types $\beta(x, y)$ and $\beta_{abc}(x, y)$ with $\beta^2(x, y)$ and $\beta^2_{abc}(x, y)$ respectively:

$$\varphi^2 = \varphi^2 \land \varphi^2$$

where

$$\varphi^2 = \forall x \forall y \bigwedge_{\beta \in \Theta^\Xi} \beta^2(x, y),$$

$$\varphi^2_{abc} = \forall x \bigwedge_{a=1}^A \xi_a(x) \rightarrow \bigwedge_{b=1}^B \exists y \bigwedge_{c=1}^C \beta^2_{abc}(x, y).$$

and where $\Theta^\Xi = \{\beta^2 \mid \beta \in \Theta^\Sigma\}$.

Finally, we define the translation $\text{trans}_2 : \text{DW}(\Xi) \rightarrow \text{Str}(\text{voc}_{\text{DW}}(\Sigma) \cup \text{voc}_{\text{SNF}})$. Let $D \in \text{DW}(\Xi)$ with universe $D$. We define $\text{trans}_2(D)$ as follows:

- The universe and order relations of $\text{trans}_2(D)$ are identical to those of $D$.
- For every $\xi_a \in \Xi$ and $d \in D$, if $D \models \xi_a(d)$ then $d$ has 1-type $\nu_a(x)$ in $\text{trans}_2(D)$.

Observe that for a data word $D \in \text{DW}(\Xi)$ with universe $D$, we have for all $d, d' \in D$ and every 2-type $\beta^2$, $D \models \beta^2(d, d')$ if and only if $\text{trans}_2(D) \models \beta(d, d')$. Hence, from (i) and (ii)

(i2) For every $E \in \text{DW}(\Xi)$, if $E \models_{\text{DW}(\Xi)} \varphi^2$ then $\text{trans}_2(E) \models \varphi^1$, and

(ii2) For every $D \in \text{Str}(\text{voc}_{\text{DW}}(\Sigma) \cup \text{voc}_{\text{SNF}})$, if $D \models \varphi^1$ then there exists $E \in \text{DW}(\Xi)$ such that $\text{trans}_2(E) = D$ and $E \models_{\text{DW}(\Xi)} \varphi^2$.

Let $h$ be a letter-to-letter substitution given as follows: for every $\xi_a \in \Xi$, $h(\xi_a) = \sigma_a$, where $\sigma_a$ is the unique letter in $\Sigma$ such that $\nu_a(x) \models \sigma_a(x)$. Let $\overline{h}$ be the function $\overline{h} : \text{DW}(\Xi) \rightarrow \text{DW}(\Sigma)$ such that for every $D \in \text{DW}(\Xi)$, $\overline{h}(D) = E$, where $E$ has the same universe and order relations as $D$, and where the interpretation $\sigma^E$ of $\sigma \in \Sigma$ in $E$ is $\bigcup_{\xi \in \overline{h}^{-1}(\sigma)} \xi^D$. Note that $\overline{h}$ is the composition of $\text{trans}_2$ and $\text{trans}_1$. Using (i2) and (ii2) we have:

(iii) For every $E \in \text{DW}(\Xi)$, $E \models_{\text{DW}(\Xi)} \varphi^2$ implies $\overline{h}(E) \models_{\text{DW}(\Sigma)} \psi$, and

(ii) For every $D \in \text{DW}(\Sigma)$, $D \models_{\text{DW}(\Sigma)} \psi$ implies the existence of $E \in \text{DW}(\Xi)$ such that $\overline{h}(E) = D$ and $E \models_{\text{DW}(\Xi)} \varphi^2$.

Let $\hat{h}$ be the function $\hat{h} : 2^\Xi \rightarrow 2^\Sigma$ which transforms every word $u$ in the input language by substituting the letters according to $h$. Now we can prove that $L(\psi) = \hat{h}(L(\varphi^2)).$ Observe that for $E \in \text{DW}(\Xi)$, $\hat{h}(\text{string}(E)) = \text{string}(\overline{h}(E))$. 

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Let \( u \in \hat{h}(L(\varphi^2)) \). There is some \( \mathcal{E} \in DW(\Xi) \) such that \( \mathcal{E} \models \varphi^2 \) and \( \{u\} = \hat{h}(<\text{string}(\mathcal{E})>) \) and hence \( u = \text{string}(\hat{h}(\mathcal{E})) \). By \([[\iota_h]] \hat{h}(\mathcal{E}) \models \psi \) and hence \( u \in \hat{h}(L(\varphi^2)) \).

Conversely, let \( u \in L(\psi) \). There is some \( \mathcal{D} \in DW(\Sigma) \) such that \( \mathcal{D} \models \psi \) and \( u = \text{string}(\mathcal{D}) \). By \([[\iota_h]]\) there is \( \mathcal{E} \in DW(\Xi) \) such that \( \hat{h}(\mathcal{E}) = \mathcal{D} \) and \( \mathcal{E} \models_{DW(\Xi)} \varphi^2 \). Hence, \( \text{string}(\mathcal{E}) \in L(\varphi^2) \). We have \( \hat{h}(<\text{string}(\mathcal{E})>) = \{\text{string}(\hat{h}(\mathcal{E}))\} = \{\{u\}\}, \) and hence \( u \in \hat{h}(L(\varphi^2)) \).

Given \( \psi \), the formula \( \varphi^0 \) can be computed in polynomial time in the length of \( \psi \). The size of \( \text{voc}_{SNF} \) is linear in the length of \( \psi \). W.l.o.g. we can assume that every symbol in \( \text{voc}_{DW(\Sigma)} \cup \text{voc}_{SNF} \) occurs in \( \psi \). Then the number of 1-types and 2-types over \( \text{voc}_{DW(\Sigma)} \cup \text{voc}_{SNF} \) is at most exponential in the length of \( \psi \), and the formulas \( \varphi^1 \) and \( \varphi^2 \) can be computed in exponential space. The lemma follows with the notation slightly simplified by replacing \( \beta^2 \) with \( \theta , \beta^2_{abc} \) with \( \theta_{abc} , \Theta _\Psi ^2 \) with \( \Theta _\Psi , \varphi _\beta ^2 \) with \( \varphi _\beta \wedge \forall x \wp _{\iota = 1} A \xi _\alpha (x) \rightarrow \wp _{b = 1} ^{B} \exists y \wp _{c = 1} ^{C} \theta _{abc} (x , y) \), and \( \varphi _\beta ^3 \) with \( \forall x \forall y \wp _{\theta _\Psi } \theta (x , y) \).

\[ \square \]

## E Proofs for the Automata-logic Connection

### E.1 Proof of Lemma [1]

**Proof.** If \( D \) is the empty word \( \theta _{DW(\Xi)} \), then the empty word \( \theta _{2^T_{\text{tasks}}} \) over the vocabulary \( 2^T_{\text{tasks}} \) is the unique \( D \)-task word. It is easy to verify that \( \theta _{2^T_{\text{tasks}}} \) is a completed \( D \)-task word if and only if \( D \models \varphi _3 \). From now on, we assume \( D \) is not the empty word. Note \( \mathcal{D} \models \varphi _2 \).

Assume \( D \models \varphi _3 \). Let \( \mathcal{T} \) have the same universe and order relations as \( D \). For every \( d \in D \), we define the unique \( ts _d \in 2^T_{\text{tasks}} \) for which \( \mathcal{T} \models ts _d (d) \) as follows. Let \( a \in [A] \) be such that \( D \models \xi _a (d) \). Since \( D \models \varphi _3 \), we have \( D \models \wp _{b = 1} ^{B} \exists y \wp _{c = 1} ^{C} \theta _{abc} (d , y) \). Therefore there exist \( d _b \in D : b \in B \) and \( c _b \in [C] : b \in [B] \) such that \( D \models \theta _{abc} (d , d _b) \) for every \( b \in [B] \). Let \( ts _d = \{C _\theta \mid \theta \in \omega _d \} \), and set \( \mathcal{T} \models ts _d (d) \).

The set \( \omega _d = \{\theta _{abc} \mid b \in [B]\} \) is a witness type set and \( ts _d \) realizes \( \omega _d \). Hence, \( \mathcal{T} \models \wp _{ts _d \in 2^T_{\text{tasks}}} \xi _a (d) \). For every \( d \in D , \mathcal{D} \models \xi _a (d) \), and for every \( \theta \in \omega _d , P _\theta \notin ts _d \) and \( \mathcal{D} \models \exists y \theta (d , y) \). Hence, \( \mathcal{T} \) is a \( D \)-task word.

For the other direction, let \( \mathcal{T} \) be completed \( D \)-task word. Let \( d \in D \) and \( a \in [A] \) such that \( \mathcal{D} \models \xi _a (d) \). Since \( \mathcal{T} \) is a completed task word, there exists \( ts _d \in 2^T_{\text{tasks}} \) such that \( \mathcal{T} \models ts _d (d) \). Let \( \omega _d \) be the witness type set such that \( ts _d \) realizes \( \omega _d \). There exist \( c _b \in [C] : b \in [B] \) such that \( \omega _d = \{\theta _{abc} \mid b \in [B]\} \) and \( ts _d = \{C _{\theta _{abc}} \mid b \in [B]\} \). Since \( \mathcal{T} \) is a task word, \( \mathcal{D} \models \exists y \theta (d , y) \).

Consequently, \( \mathcal{D} \models \wp _{b = 1} ^{B} \exists y \wp _{c = 1} ^{C} \theta _{abc} (d , y) \) for every \( d \in D \) and \( a \in [A] \), and hence \( \mathcal{D} \models \varphi _3 \).

\[ \square \]
Lemma 6. Let $\mathcal{T}$ be a $\mathcal{D}$-task word. There is a unique $\mathcal{D}^{\perp 1}$-task word $\mathcal{T}_1$ for which $\omega(ts) = \omega(ts_1)$ for each $ts, ts_1 \in 2^{\text{Tasks}}$ and each $d$ in the universe of $\mathcal{D}^{\perp 1}$ s.t. $\mathcal{T} \models ts(d)$ and $\mathcal{T}_1 \models ts_1(d)$.

Before proving the claim, we recall the conditions of Definition 3 in a $\mathcal{D}$-task word $\mathcal{T}$ of each $d \in D$ with $\mathcal{T} \models ts(d)$ satisfies:

1. $\mathcal{D} \models \xi^{\omega(ts)}(d)$.
2. for every $\theta \in \omega(ts)$, $P_\theta \in ts$ iff $\mathcal{D} \models \exists y \theta(d, y)$.
3. $C_\theta \in ts \implies \mathcal{D} \models \exists y \theta(d, y)$

Proof. First we show the existence of such $\mathcal{T}_1$. We denote the universe of $\mathcal{T}_1$ by $T_1$. Let $\mathcal{T}_1$ be the $\mathcal{D}^{\perp 1}$-task word given as follows. For every $d \in T_1$, let $ts \in 2^{\text{Tasks}}$ be such that $\mathcal{T}_1 \models ts(d)$, and let $ts_1$ be:

$$ts_1 = \{ C_\theta \mid \theta \in \omega(ts), \mathcal{D}^{\perp 1} \models \exists y \theta(d, y) \} \cup \{ P_\theta \mid \theta \in \omega(ts), \mathcal{D}^{\perp 1} \models \exists \exists y \theta(d, y) \}.$$

Then $\mathcal{T}_1 \models ts_1(d)$. Since $\omega(ts_1) = \omega(ts)$, Condition 2 in Definition 4 holds. Since $\mathcal{T}$ is a $\mathcal{D}$-task word, for every $d \in T_1$ we have $\mathcal{D} \models \xi^{\omega(ts)}(d)$, implying Condition 1 in Definition 4 holds.

It remains to show that $\mathcal{T}_1$ is unique. Assume for contradiction that there exists another $\mathcal{D}^{\perp 1}$-task word $\tilde{\mathcal{T}}$ satisfying the statement of the lemma. Then there is some $d \in D_1$ and distinct $ts_1, \tilde{ts} \in 2^{\text{Tasks}}$ such that $\mathcal{T}_1 \models ts_1(d)$ and $\tilde{\mathcal{T}} \models ts(d)$. We have $\omega(ts) = \omega(ts_1) = \omega(ts_1)$, and hence there is $\theta \in \omega(ts)$ such that either $P_\theta \in ts_1 - ts$ or $C_\theta \in ts_1 - ts$. In either case, since $\mathcal{T}_1$ satisfies Condition 2 in Definition 4 $\tilde{\mathcal{T}}$ does not satisfy Condition 2 in Definition 4 in contradiction to the assumption that $\tilde{\mathcal{T}}$ is a $\mathcal{D}^{\perp 1}$-task word.

E.3 Definitions and Proofs for Perfect Extremal Strings

**Definition 11** $(\text{perf}_{\alpha, \beta}(x, y))$. Let $\alpha = (h_\alpha, ts_\alpha)$ and $\beta = (h_\beta, ts_\beta)$ in $\Gamma$ with at least one of them in $\Gamma_{\text{top}}$. Using the subformulas in Table 7 we define $\text{perf}_{\alpha, \beta}(x, y) = \text{perf}_{\alpha}(x) \wedge \text{perf}_{\beta}(y) \wedge \bigwedge_{\text{bin} \in \{x_1, y_2, y_2, x_2\}} \text{perf}_{\alpha, \beta, \text{bin}}(x, y)$.

**Lemma 7.** Let $\mathcal{T}$ be a $\mathcal{D}$-task word with universe $D$ and let $w = \text{abst}(\mathcal{T})$. Let $d_1, d_2 \in D$ be such that $\mathcal{D} \models d_1 < d_2$ and $\text{max val}_D \in \{ \text{value}_D(d_1), \text{value}_D(d_2) \}$. Let $\ell_1$ and $\ell_2$ be such that $d_1$ is mapped to position $\ell_1$ in $\text{abst}(\mathcal{T})$ and $d_2$ is mapped to position $\ell_2$ in $\text{abst}(\mathcal{T})$. Then $\ell_1 < \ell_2$ and $\mathcal{D} \models \text{perf}_{w(\ell_1), w(\ell_2)}(d_1, d_2)$.

**Proof.** It is given that $\mathcal{D} \models \text{perf}_{w(\ell_1), w(\ell_2)}(d_1, d_2)$. Since $\text{Emb}_{\text{abst}},\mathcal{T}$ is order-preserving and $\mathcal{D} \models d_1 < d_2$, we get $\ell_1 < \ell_2$. Since $\mathcal{T}$ is a task word, there exist $ts_1, ts_2 \in 2^{\text{Tasks}}$ such that $\mathcal{T} \models ts_1(d_1)$ and $\mathcal{T} \models ts_2(d_2)$. By definition of a task word, we have that $\mathcal{D} \models \xi^{\omega(ts_1)}(d_1)$ and $\mathcal{D} \models \xi^{\omega(ts_2)}(d_2)$, and therefore $\mathcal{D} \models \text{perf}_{w(\ell_1)}(d_1) \wedge \text{perf}_{w(\ell_2)}(d_2)$.
We consider one of the cases for $\text{perf}_{w(t_1),w(t_2),\preceq_2}(x,y)$. The other cases can be treated analogously. If $\text{perf}_{w(t_1),w(t_2),\preceq_2}(x,y) = x \preceq_2 y$ and $w(t_1) \notin \Gamma_{1\text{top}}$, then $w(t_2) \in \Gamma_{1\text{top}}$. By definition of $w = \text{abst}(\mathcal{T})$, we have that $\text{value}_D(\text{Emb}_{\text{abst},\mathcal{T}}(t_1)) < \text{max val}_D$ and $\text{value}_D(\text{Emb}_{\text{abst},\mathcal{T}}(t_2)) = \text{max val}_D$. Hence, $D \models \text{perf}_{w(t_1),w(t_2),\preceq_2}(d_1,d_2)$.

We consider one of the cases for $\text{perf}_{w(t_1),w(t_2),S_2}(x,y)$. The other cases can be treated analogously. If $\text{perf}_{w(t_1),w(t_2),S_2}(x,y) = \neg S_2(x,y)$ then $w(t_1) \in \Gamma_{1\text{top}}$ while $w(t_2) \in \Gamma_{\text{rest}}$. By definition of $w = \text{abst}(\mathcal{T})$, we have that $\text{value}_D(\text{Emb}_{\text{abst},\mathcal{T}}(t_1)) = \text{max val}_D$ and $\text{value}_D(\text{Emb}_{\text{abst},\mathcal{T}}(t_2)) \leq \text{max val}_D - 2$. Therefore, we have $D \models \text{perf}_{w(t_1),w(t_2),S_2}(d_1,d_2)$. \hfill $\square$

**Lemma 8.** Consider $\alpha, \beta \in \Gamma$ such that at least one of them is in $\Gamma_{1\text{top}}$ and such that $\text{perf}_{\alpha,\beta}(x,y) \not\models_{\text{DW}(\Xi)} \chi(x,y) \land \chi(y,x)$. Then it holds that $\exists x \exists y \text{perf}_{\alpha,\beta}(x,y) \models_{\text{DW}(\Xi)} \neg \varphi_y$.

We first recall the following:

**Observation 1.** Let $\alpha, \beta \in \Gamma$ such that at least one of them is in $\Gamma_{1\text{top}}$. There exists a 2-type $\theta(x,y)$ such that $\text{perf}_{\alpha,\beta}(x,y) \equiv_{\text{DW}(\Xi)} \theta(x,y)$.

**Proof.** Since $\text{perf}_{\alpha,\beta}(x,y) \not\models_{\text{DW}(\Xi)} \chi(x,y) \land \chi(y,x)$, there exists $D$ and elements $d_1, d_2 \in D$ such that $D \models \text{perf}_{\alpha,\beta}(d_1,d_2)$ and $D \not\models \chi(d_1,d_2) \land \chi(d_2,d_1)$. By Observation 1, we know that for every data word $D'$ and every two elements $d'_1, d'_2 \in D'$ such that $D' \models \text{perf}_{\alpha,\beta}(d'_1,d'_2)$, the 2-type of $(d'_1,d'_2)$ is the same as the 2-type of $(d_1,d_2)$. Let us denote this 2-type by $\theta(x,y)$. Since $D \not\models \chi(d_1,d_2) \land \chi(d_2,d_1)$, we have that either $\theta(x,y) \not\in \Theta_{\mathcal{V}}$ or $\theta(y,x) \not\in \Theta_{\mathcal{V}}$ and therefore also $D' \not\models \chi(d'_1,d'_2) \land \chi(d'_2,d'_1)$. \hfill $\square$
E.4 Proof of Lemma 2

Proof.

Claim 3. Let \( w \in \Gamma^+ \). Then \( w \) is perfect if and only if \( \text{ext}(w) \) is perfect.

Proof. Let \( s = \text{ext}(w) \).

Assume that \( w \) is perfect. Let \( \ell_1 < \ell_2 \) be positions in \( s \) such that at least one of \( s(\ell_1), s(\ell_2) \) is in \( \Gamma_{1\text{top}} \). For \( i = 1, 2 \), let \( \ell'_i = \text{Emb}_{\text{ext}, w}(\ell_i) \). We have \( w(\ell'_i) = s(\ell_i) \) and \( \ell'_1 < \ell'_2 \). Therefore \( \text{perf}_{s(\ell_1), s(\ell_2)}(x, y) = \text{perf}_{w(\ell'_1), w(\ell'_2)}(x, y) \). Since \( w \) is perfect, \( \text{perf}_{s(\ell_1), s(\ell_2)}(x, y) \models_{\text{DW}(\Xi)} \chi(x, y) \land \chi(y, x) \).

Now assume that \( s \) is perfect. Let \( \ell'_1 < \ell'_2 \) be positions in \( w \) such that at least one of \( w(\ell'_1), w(\ell'_2) \) is in \( \Gamma_{1\text{top}} \). Denote \( \ell''_1 = \min \{ \ell \mid w(\ell) = w(\ell'_1) \} \) and \( \ell''_2 = \max \{ \ell \mid w(\ell) = w(\ell'_2) \} \) and note that \( \ell'_1, \ell''_2 \in \text{ext}(w) \). For \( i = 1, 2 \), let \( \ell_i \) be such that \( \ell''_i = \text{Emb}_{\text{ext}, w}(\ell_i) \). We have \( w(\ell'_i) = w(\ell''_i) = s(\ell_i) \) for \( i = 1, 2 \) and \( \ell'_1 < \ell'_2 \). Hence, \( \text{perf}_{w(\ell'_1), w(\ell'_2)}(x, y) = \text{perf}_{s(\ell_1), s(\ell_2)}(x, y) \), and since \( s \) is perfect, \( \text{perf}_{w(\ell'_1), w(\ell'_2)}(x, y) \models_{\text{DW}(\Xi)} \chi(x, y) \land \chi(y, x) \).

Assume \( T \) is perfect. Let \( d_1 \) and \( d_2 \) be distinct elements of \( D \). We show \( D \models \chi(d_1, d_2) \). Let \( e = \max \text{val}_D - \max \{ \text{value}_D(d_1), \text{value}_D(d_2) \} \). Let the universe of \( D_{\text{V}e} \) be \( D' \). Note that \( d_1, d_2 \in D' \) and that \( \max \text{val}_D_{\text{V}e} \in \{ \text{value}_D_{\text{V}e}(d_1), \text{value}_D_{\text{V}e}(d_2) \} \). Since \( T \) is a perfect task word, we have that \( \text{ext}(\text{abst}(T_{\text{V}e})) \) is a perfect string, and by Claim 3 so is \( w_e = \text{abst}(T_{\text{V}e}) \). By Lemma 7 either \( D_{\text{V}e} \models \text{perf}_{w_e(\ell_1), w_e(\ell_2)}(d_1, d_2) \) or \( D_{\text{V}e} \models \text{perf}_{w_e(\ell_2), w_e(\ell_1)}(d_2, d_1) \). In either case, since \( \text{abst}(T_{\text{V}e}) \) is perfect, we have that \( D_{\text{V}e} \models \chi(d_1, d_2) \). Since \( \chi(x, y) \) is quantifier-free and \( D_{\text{V}e} \) is a substructure of \( D \), we also have that \( D \models \chi(d_1, d_2) \).

For the other direction, assume \( D \models \varphi_\gamma \). Assume for contradiction that there is \( e \) such that \( \text{ext}(\text{abst}(T_{\text{V}e})) \) is not perfect. Then by Claim 4 \( w_e = \text{abst}(T_{\text{V}e}) \) is also not perfect. That is, there exist positions \( \ell_1 < \ell_2 \) in \( w_e \) such that at least one of \( w_e(\ell_1), w_e(\ell_2) \) is in \( \Gamma_{1\text{top}} \), and such that \( \text{perf}_{w_e(\ell_1), w_e(\ell_2)}(x, y) \not\models_{\text{DW}(\Xi)} \chi(x, y) \land \chi(y, x) \). For \( i = 1, 2 \), let \( d_i = \text{Emb}_{\text{abst}, T_{\text{V}e}}(\ell_i) \). We have \( d_1 < d_2 \). By the definition of \( \text{abst} \), \( \max \text{val}_D_{\text{V}e} \in \{ \text{value}_D_{\text{V}e}(d_1), \text{value}_D_{\text{V}e}(d_2) \} \). By Lemma 7 we have \( D_{\text{V}e} \models \text{perf}_{w_e(\ell_1), w_e(\ell_2)}(d_1, d_2) \) and by applying Lemma 8 with \( \alpha = w_e(\ell_1) \) and \( \beta = w_e(\ell_2) \), we have that \( D_{\text{V}e} \not\models \varphi_\gamma \). Since \( D_{\text{V}e} \) is a substructure of \( D \) and \( \varphi_\gamma \) is universal, we also have that \( D \not\models \varphi_\gamma \) in contradiction to our assumption.

E.4.1 A syntactic representation of consecutive extremal strings

The extremal string \( r_{\downarrow g} s^0 \) simulates the extension of a task word \( T^0 \) whose extremal string is \( s^0 \) by adding elements with a new maximal data value. The letters of these elements are determined by \( r \) and their placement in the linear order of \( T^0 \) is determined by \( g \).

We introduce some notation.

\[
\begin{align*}
pos_h(w) &= \{ \ell \in [|w|] \mid w(\ell) \in \Gamma_h \} \\
pos_{\leq 1\text{top}}(w) &= \{ \ell \in [|w|] \mid w(\ell) \in \Gamma_{2\text{top}} \cup \Gamma_{\text{rest}} \}
\end{align*}
\]
Definition 12 ($\downarrow_g s^0$). Let $s \in \Gamma^*$. We denote by $\downarrow s$ the string that is obtained from $s$ by substituting letters of the form $(1\text{top}, ts)$ with $(2\text{top}, ts)$ and letters of the form $(2\text{top}, ts)$ or $(\text{rest}, ts)$ with $(\text{rest}, ts)$. Let $s^0 \in \text{EXT}(\Gamma)$, $r \in \Gamma_{1\text{top}} \cap \Gamma^*$, and let $g : [n_r] \to [n_r + n_{v_0}]$ be a strictly monotone function. Then let $s^1 = r_{w_0} \downarrow \downarrow s^0$, where $\emptyset$, $v$ denotes the shuffle of $u = w(i_1) \cdots w(i_{|u|})$ and $v$ associated to the function $g$ that has $g(i) = i_j$ for all $1 \leq j \leq |u|$, see Definition[9].

We now define the string $r \downarrow_g s^0$, which is obtained from $s^1$ substituting every letter $(h, ts)$ at position $\ell \in [n_{s^1}]$ with a letter $(h, ts')$. Essentially, this replacement updates the tasks to take into account the new elements described by $r$ (and whose position is given by $g$). The letter $(h, ts')$ is such that $\omega(ts) = \omega(ts')$ and, for every $\theta \in \omega(ts)$, we have $C_0 \in ts'$ if $\theta$ is completed, that is, if any of the following hold:

1. $\theta$ was already completed: $C_0 \in ts$.

2. The necessary witness $\ell_2$ for $\ell$ was found to the right: $\theta \models_{\text{ext}}(\Xi) \ x \leq y$ and there is $\ell_2 \in [n_{s^1}]$ such that $\ell \leq \ell_2$, either $\ell \in \text{pos}_{1\text{top}}(s^1)$ or $\ell_2 \in \text{pos}_{1\text{top}}(s^1)$, and $\text{perf}_{\ell_1, \ell_2}(\ell, x, y) \equiv_{\text{ext}}(\Xi) \theta(x, y)$.

3. The necessary witness $\ell_2$ for $\ell$ was found to the left: $\theta \models_{\text{ext}}(\Xi) \ y < 1 \ x$ and there is $\ell_1 \in [n_{s^1}]$ such that $\ell_1 < \ell$, either $\ell_1 \in \text{pos}_{1\text{top}}(s^1)$ or $\ell \in \text{pos}_{1\text{top}}(s^1)$, and $\text{perf}_{\ell_1, \ell_2}(\ell, x, y) \equiv_{\text{ext}}(\Xi) \theta(x, y)$.

Otherwise, since $\theta$ was not completed, it remains as a promised task and we have $P_0 \in ts'$.

The pair $(s^0, r \downarrow_g s^0)$ of extremal strings is consecutive.

Lemma 9. Let $T_0$ be a $D_0$-task word, $s^0 = \text{ext}(T_0)$, $r \in \Gamma_{1\text{top}} \cap \Gamma^*$, and let $g : [n_r] \to [n_r + n_{v_0}]$ be a strictly monotone function. Then $s^0$ and $\text{ext}(r \downarrow_g s^0)$ are consecutive.

Proof. Let $s = \text{ext}(r \downarrow_g s^0)$. Without loss of generality, we may assume the universe $D_0$ of $D_0$ is disjoint from $\mathbb{N}$. Let $\bar{g}$ be as in the definition of $r_{w_0} s^0$. Let $D$ be the data word over $\Xi$ with universe $D_0 \cup [n_r]$ such that:

1. $D_0$ is the substructure of $D$ induced by $D_0$.

2. For every $\ell \in [n_r]$ and $r(\ell) = (h, ts)$, $D \models \xi(\omega(ts))(\ell)$.

3. For every $\ell_1, \ell_2 \in [n_r]$, $D \models \ell_1 \sim_\ell \ell_2$.

4. For every $d \in D_0$ and $\ell \in [n_r]$, $D \models d <_\ell \ell$.

5. For every $\ell_1, \ell_2 \in [n_r]$, $D \models \ell_1 \leq_\ell \ell_2$ if and only if $\ell_1 \leq_\ell \ell_2$.

6. For every $d \in D_0$, let $d' \in D_0$ be the maximal element of $\text{extElem}(\ell)$ with respect to $\leq_\ell$ such that $d' \leq_\ell d$, and let $\ell' \in [n_{s^1}]$ be such that $d' = \text{Emb}_{\text{ext}}(\ell)$.

For every $\ell_1 \in [n_r]$, $D \models d \leq \ell_1$ if and only if $\bar{g}(\ell') < g(\ell_1)$; if no such $d'$ exists for $d$ then $D \models d \leq_\ell \ell_1$. 28
Let $T$ be a $D$-task word such that, for every $d \in D_0$, there are $ts, ts_0 \in 2^{|T|}$ such that $\omega(ts) = \omega(ts_0)$, $T \models ts(d)$, and $T_0 \models ts_0(d)$. Clearly $\text{ext}(\text{abst}(T_{\downarrow 1})) = s^0$. Let $r, w_X, w_{X'}$, and $w_{X_{1\text{top}}}$ be as in Lemma 10. By the construction of $D$,

$$g = \text{Emb}_{\text{abst}, T}|_{X_{1\text{top}}} \circ \text{Emb}_{\text{abst}, T}|_{X}.$$ 

By Lemma 10 $\text{ext}(r \downarrow g s^0) = \text{ext}(\text{abst}(T))$, i.e., $s^0$ and $s$ are consecutive extremal strings.

Example 8. Applying $\downarrow$ to $g'$, we have

$$\downarrow g' = (\text{rest}, ts_1^p)(\text{rest}, ts_2^p)(\text{rest}, ts_3^p).$$

Let $s = (1\text{top}, ts_3^p)(1\text{top}, ts_4^p)$, and let $g : [2] \rightarrow [2 + 4]$ be $g(1) = 4$ and $g(2) = 6$. Then

$$s' = (\text{rest}, ts_1^p)(\text{rest}, ts_2^p)(\text{rest}, ts_3^p)(1\text{top}, ts_1^c)(1\text{top}, ts_2^c).$$

Observe that $\text{ext}(r \downarrow g s') = s$, and recall that $(s', s)$ are consecutive.

If we have the $D$-task word $T$ at hand, then we can obtain the string $r$ and the function $g$ for $s$ and $s^0$. In fact, obtaining $r$ is quite easy: it suffices to look at the elements with maximal data value, and substitute every letter $(1\text{top}, ts)$ with $(1\text{top}, ts_p)$ such that $ts_p = \{P_0 \mid \theta \in \omega(ts)\}$.

Lemma 10. Given a $D$-task word $T$ and two extremal strings $s$ and $s^0$ such that $s = \text{ext}(T)$ and $s^0 = \text{ext}(T_{\downarrow 1})$, we can effectively obtain an $r \in \Gamma_{1\text{top}} \cap \Gamma_D$ and $g : |s^0| \rightarrow |s^0| + |s|$ such that $s = \text{ext}(r \downarrow g s^0)$.

Proof. We first need some additional notation. Let

$$w = \text{abst}(T) \quad X' = \text{extElem}(T_{\downarrow 1})$$

$$X_{1\text{top}} = \text{Emb}_{\text{abst}, T}(\text{pos}_{1\text{top}}(w)) \quad X = X_{1\text{top}} \cup X'$$

$$w_X = \text{abst}(T|_X) \quad w_{X_{1\text{top}}} = \text{abst}(T|_{X_{1\text{top}}})$$

The string $r \in \Gamma_{1\text{top}} \cap \Gamma_D$ is obtained from $w_{X_{1\text{top}}}$ by substituting every letter $(1\text{top}, ts)$ with $(1\text{top}, ts_p)$ such that $ts_p = \{P_0 \mid \theta \in \omega(ts)\}$. The function $g$ is given by

$$g = \text{Emb}_{\text{abst}, T}|_{X_{1\text{top}}} \circ \text{Emb}_{\text{abst}, T}|_{X}.$$ 

Finally, let $w' = \text{abst}(T_{\downarrow 1})$, and $w_{X'} = \text{abst}(T|_{X'})$.

By Lemma 24 $\text{extElem}(T) \subseteq X$. Hence, $s = \text{ext}(w_X)$. We will prove that $w_X = r \downarrow g s^0$, and the lemma will follow.

Notice that $w_X = w_{X_{1\text{top}}} \cup w_{X'}$, $|w_{X_{1\text{top}}} = |r|$, $|w_{X'}| = |s^0|$, and $g$ is strictly monotone as the composition of two order-preserving functions. Hence, $s^0$, $r$, and
Given two extremal strings $s, s'$, whether they are consecutive can be decided in \textsc{ExpSpace}. If they are, then we can also obtain in \textsc{ExpSpace} a partial embedding $\text{PEmb}_r$ from positions in $s$ to positions in $s'$ that coincides with the partial embedding from $s$ to $s'$ via $\mathcal{T}$ for every task word $\mathcal{T}$ such that $s' = \text{ext}(\mathcal{T}^h)$ and $s = \text{ext}(\mathcal{T}^h)$.

\textbf{Proof.} Given $s_c$ and $s_{c+1}$, checking if $(s_c, s_{c+1})$ are consecutive in \textsc{ExpSpace} is done as follows. We iterate over all $r \in \Gamma_{\text{top}} \cap \Gamma_{\text{C}}$ such that $|r| \leq 7|\Theta_3|$ and over all strictly monotone functions $g : [n_r] \rightarrow [n_r + n_s]$. We search for such $r$ and $g$ for which $s_{c+1} = r \downarrow g s_c$ and answer to whether such $r$ and $g$ are found. Lemmas\cite{11} and\cite{9} guarantee the correctness of a semi-decision procedure behaving as above without restricting the length of $r$, and we show that if there is an $r$ such that $s_{c+1} = r \downarrow g s_c$, then there is such an $r$ with $|r| \leq 7|\Theta_3|$ (see App.\cite{E.6}).
E.5 Witnesses of task completion

Here we prove a few lemmas which provide witnesses to the completion of tasks based on existing witnesses to the same or other tasks.

**Lemma 12.** Let $\mathcal{D}$ be a data word and let $d, d'$ be elements with the same 1-type such that $\mathcal{D} \models d \leq_1 d' \land d \sim_2 d'$. Let $\theta \in \Theta_3$.

1. If $\theta(x, y) \models x \leq_1 y$ and $\mathcal{D} \models \exists y \theta(d', y)$, then $\mathcal{D} \models \exists y \theta(d, y)$.

2. If $\theta(x, y) \models y \leq_1 x$ and $\mathcal{D} \models \exists y \theta(d, y)$, then $\mathcal{D} \models \exists y \theta(d', y)$.

**Proof.**

1. Since $\mathcal{D} \models d \leq_1 d' \land d \sim_2 d'$, and $d, d'$ have the same 1-type, for any element $d''$ such that $\mathcal{D} \models d' \leq_1 d''$, the 2-type of $(d', d'')$ is the same as the 2-type of $(d, d'')$. Since $\theta(x, y) \models x \leq_1 y$, if $\mathcal{D} \models \exists y \theta(d', y)$ there exists some $d'' \geq_1 d'$ such that $\mathcal{D} \models \theta(d', d'')$. Therefore also $\mathcal{D} \models \theta(d, d'')$ and $\mathcal{D} \models \exists y \theta(d, y)$.

2. Analogous to the previous case.

**Lemma 13.** Let $\mathcal{T}$ be a $\mathcal{D}$-task word and let $d, d'$ be elements such that $\mathcal{D} \models d \leq_1 d' \land d \sim_2 d'$. Let $\mathcal{T} \models ts_d(d)$ and $\mathcal{T} \models ts_{d'}(d')$ such that $ts_d$ and $ts_{d'}$ realize the same witness type set. Let $\theta \in \Theta_3$.

1. If $\theta(x, y) \models x \leq_1 y$ and $C_\theta \in ts_d$, then $C_\theta \in ts_d$.

2. If $\theta(x, y) \models y \leq_1 x$ and $C_\theta \in ts_{d'}$, then $C_\theta \in ts_{d'}$.

**Proof.** If $ts_d$ and $ts_{d'}$ realize the same witness type set, then $d$ and $d'$ have the same 1-type in $\mathcal{D}$. Then the claim follows from Lemma [12] and the definition of task words.

**Lemma 14.** Let $\mathcal{D}$ be a data word and let $d, d'$ be elements with the same 1-type such that $\mathcal{D} \models d \leq_1 d', \text{value}_\mathcal{D}(d) \leq \text{maxval}_\mathcal{D} - 2$, and $\text{value}_\mathcal{D}(d') \leq \text{maxval}_\mathcal{D} - 2$. Let $\theta \in \Theta_3$ be such that $\mathcal{D}^{\neg 1} \nvdash \exists y \theta(d, y)$ and $\mathcal{D}^{\neg 1} \nvdash \exists y \theta(d', y)$.

1. If $\theta(x, y) \models x \leq_1 y$ and $\mathcal{D} \models \exists y \theta(d', y)$, then $\mathcal{D} \models \exists y \theta(d, y)$.

2. If $\theta(x, y) \models y \leq_1 x$ and $\mathcal{D} \models \exists y \theta(d, y)$, then $\mathcal{D} \models \exists y \theta(d', y)$.

**Proof.**

1. Let $\mathcal{D} \models \exists y \theta(d', y)$, and denote by $d''$ the element such that $\mathcal{D} \models \theta(d', d'')$. Since $\mathcal{D}^{\neg 1} \nvdash \exists y \theta(d', y)$, we conclude that $\text{value}_\mathcal{D}(d'') = \text{maxval}_\mathcal{D}$. Since both $d$ and $d'$ have data value at most $\text{maxval}_\mathcal{D} - 2$, it holds that $\mathcal{D} \models d \preceq_2 d'' \land \neg S_2(d, d'')$ and $\mathcal{D} \models d' \preceq_2 d'' \land \neg S_2(d', d'')$. Since $\mathcal{D} \models d \leq_1 d'$ and $\theta(x, y) \models x \leq_1 y$, also $\mathcal{D} \models d \leq_1 d''$, and all in all, the 2-type of $(d', d'')$ is the same the 2-type of $(d, d'')$. Therefore, $\mathcal{D} \models \theta(d, d'')$, implying $\mathcal{D} \models \exists y \theta(d, y)$.

2. Analogous to the previous case.

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Lemma 15. Let $\mathcal{T}$ be a $\mathcal{D}$-task word and let $d, d'$ be elements such that $\mathcal{D} \models d \leq d'$, $\text{value}_\mathcal{D}(d) \leq \maxval_\mathcal{D} - 2$, and $\text{value}_\mathcal{D}(d') \leq \maxval_\mathcal{D} - 2$. Let $\mathcal{T}^{\leq 1} \models ts_1(d)$ and $\mathcal{T}^{\leq 1} \models ts_1'(d')$ such that $\omega(ts_1) = \omega(ts_1')$. Let $\theta \in \Theta_3$ such that $P_0 \in ts_1 \cap ts_1'$. Finally, let $\mathcal{T} \models ts(d)$ and $\mathcal{T} \models ts'(d')$.

1. If $\theta(x, y) \models x \leq 1$ and $C_0 \in ts'$, then $C_0 \in ts$.
2. If $\theta(x, y) \models y \leq 1$ and $C_0 \in ts$, then $C_0 \in ts'$.

Proof. Since $ts_1$ and $ts_1'$ realize the same witness type set, $d$ and $d'$ have the same 1-type in $\mathcal{D}$. Since $P_0 \in ts_1 \cap ts_1'$, we have $\mathcal{D}^{\leq 1} \not\models \exists y \theta(d, y)$ and $\mathcal{D}^{\leq 1} \not\models \exists y \theta(d', y)$. Then the claim follows from Lemma 14 and the definition of task words.

Lemma 16 (Extremal witnesses). Let $\mathcal{T}$ be a $\mathcal{D}$-task word. Let $d, d_0$ be elements of $\mathcal{T}$ and $\theta \in \Theta_3$. If $\mathcal{D} \models \theta(d, d_0)$ and $\maxval_\mathcal{D} \in \{\text{value}_\mathcal{D}(d), \text{value}_\mathcal{D}(d_0)\}$, then there is an element $d' \in \text{extElem}(\mathcal{T})$ such that $\mathcal{D} \models \theta(d, d')$.

Proof. Let $w = \text{abst}(\mathcal{T})$. Let $w(\text{Emb}_{\text{abst}, \mathcal{T}}^{-1}(d)) = (h, ts)$, and let $w(\text{Emb}_{\text{abst}, \mathcal{T}}^{-1}(d_0)) = (h_0, ts_0)$. Let $\theta \in ts_0$. There are $d_{0, 1} \leq 1 d_0 \leq 1 d_{0, 2}$ such that

1. $d_{0, 1}, d_{0, 2} \in \text{extElem}_{\theta, h}(\mathcal{T})$,
2. $\mathcal{D} \models \xi^{\omega(ts_0)}(d_{0, 1})$ and $\mathcal{D} \models \xi^{\omega(ts_0)}(d_{0, 2})$,
3. $\text{value}_\mathcal{D}(d_0) = \text{value}_\mathcal{D}(d_{0, 1}) = \text{value}_\mathcal{D}(d_{0, 2})$ if $\text{value}_\mathcal{D}(d_0) \geq \maxval_\mathcal{D} - 1$,
4. both $\text{value}_\mathcal{D}(d_{0, 1}) \leq \maxval_\mathcal{D} - 2$ and $\text{value}_\mathcal{D}(d_{0, 2}) \leq \maxval_\mathcal{D} - 2$ if we have $\text{value}_\mathcal{D}(d_0) \leq \maxval_\mathcal{D} - 2$, and
5. $d_0, d_{0, 1}, d_{0, 2}$ have the same 1-type in $\mathcal{D}$.

Since $\maxval_\mathcal{D} \in \{\text{value}_\mathcal{D}(d), \text{value}_\mathcal{D}(d_0)\}$, the 2-types of $(d, d_0)$ and either $(d, d_{0, 1})$ or $(d, d_{0, 2})$ are equal depending on whether $\theta(x, y) \models x \leq 1 y$; for the case that $\text{value}_\mathcal{D}(d_0) \geq \maxval_\mathcal{D} - 1$ we use that $\mathcal{D} \models d_0 \sim_2 d_{0, 1}$ and $\mathcal{D} \models d_0 \sim_2 d_{0, 2}$, while the case that $\text{value}_\mathcal{D}(d_0) \leq \maxval_\mathcal{D} - 2$ is similar to the discussion in the proof of Lemma 14. Hence the lemma follows by setting either $d' = d_{0, 1}$ or $d' = d_{0, 2}$.

E.6 Lemmas for EXPSPACE

To guarantee we non-deterministically explore the entire transition relation, we use the following lemma to bound the search space:

Lemma 17. Let $\mathcal{T}$ be a $\mathcal{D}$-task word. Let $d$ be an element of $\mathcal{T}$ not in $\text{extElem}(\mathcal{T})$ such that $\text{value}_\mathcal{D}(d) = \maxval_\mathcal{D}$. Let $\mathcal{D}_{-d}$ and $\mathcal{T}_{-d}$ be the substructures of $\mathcal{D}$ respectively $\mathcal{T}$ obtained by removing $d$. Then $\mathcal{T}_{-d}$ is a $\mathcal{D}_{-d}$-task word and $\text{ext}(\text{abst}(\mathcal{T})) = \text{ext}(\text{abst}(\mathcal{T}_{-d}))$. 

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Assume that \( T_{-d} \) is not a task word. There are \( d' \in D, ts \in 2^{|\text{Tasks}|} \), and \( \theta \in \omega(ts) \) such that \( d \neq d', T \models ts(d) \), and \( C_\theta \in ts \), but for every element \( d'' \neq d \) of \( D_{-d}, D_{-d} \not\models \theta(d', d'') \). However, the non-existence of such \( d'' \) is in contradiction to Lemma 16. Since \( d \notin \text{extElem}(T) \), \( \text{extElem}(T) = \text{extElem}(T_{-d}) \). The string \( \text{abst}(T_{-d}) \) is the substring of \( \text{abst}(T) \) obtained by deleting the letter corresponding to \( d \), and hence \( \text{ext}(\text{abst}(T)) = \text{ext}(\text{abst}(T_{-d})) \).

**Lemma 18.** Let \( n = |\psi| \). The size of \( \mathcal{A} \) is at most double exponential in \( n \), while \( m \) is at most exponential in \( n \).

**Proof.** By Lemma 17, \( |\Xi| \) and \( |\Theta| \) and hence \( m \) are exponential in \( n \). The size of \( \Gamma \) is therefore double exponential in \( n \). We have \( \text{EXT}(\Gamma) \subseteq \Gamma^{\neg|\Theta|} \), which is double exponential in \( n \). Given an extremal string \( s \) of length \( n_s \), the number of \((m + 1, n_s)\)-pebble assignments is at most \((n_s + 1)^{m+1} \), which is double exponential in \( n \). Hence \( |Q| \) is double exponential. Since \( \delta \subseteq (Q \times Q) \cup (Q \times \text{MOVE}_{m+1} \times \Sigma \times Q) \), \( |\delta| \) is double exponential.

**Lemma 19.** \( \delta \) is \( \text{EXPSPACE}(|\mathcal{A}|) \)-computable.

**Proof.** Let \( n = |\psi| \). Using Lemma 18 the sizes of the representation of a state \( q \in Q \), a string \( s \in \Gamma^{\neg|\Theta|} \), a pebble \( k \in [m+1] \), and a letter \( \gamma \in \Gamma_{1\text{top}} \) are all at most exponential in \( n \). We can verify that \( s \) is an extremal string in exponential space by computing \( \text{extPos}(s) \) and verifying that \( |\text{extPos}(s)| = |s| \). The set \( \text{extPos}(s) \) can be computed by going over the string \( s \) and keeping track of the relevant minimum and maximum positions (of which there are an exponential number). Similarly, for \( \ell \in [|s|] \), one can verify that \( \gamma \in \Gamma_{1\text{top}}^{\text{notext}}(s, \ell) \).

Verifying that an extremal string \( s \) of length \( n_s \) is perfect is done as follows: for every two positions \( \ell_1 < \ell_2 \) of \( s \) such that \( \{\ell_1, \ell_2\} \cap \text{pos}_{1\text{top}}(s) \neq \emptyset \), it is straightforward to compute the formula \( \text{perf}_{s(\ell_1), s(\ell_2)}(x, y) \) in exponential space. Since the formula \( \text{perf}_{s(\ell_1), s(\ell_2)}(x, y) \) is a conjunction of atoms and negations of atoms, it is also easy to complete it to its equivalent 2-type \( \beta(x, y) \). We can then check that \( \beta(x, y), \beta(y, x) \in \Theta_{\psi} \).

We know from our previous discussion and Lemma 17 that it is possible to nondeterministically compute the extremal strings which are consecutive to \( s \), and hence the set of transitions leaving \( q \in Q \), in \( \text{EXPSPACE} \).

**F The automaton \( \mathcal{A}^p \)**

We define an automaton \( \mathcal{A}^p = (\Xi, m + 1, Q, q_{\text{init}}, F, \delta) \) where \( Q = Q_e \cup Q_p \) and \( m = 7 \cdot |\Theta| \). \( \mathcal{A}^p \) uses one pebble for each existential constraint in \( \Theta \) and each layer in \( \Gamma \), plus an additional pebble per constraint. It also uses the designated pebble \( m + 1 \) to read non-extremal positions. We describe its states and transitions next.

**Non-prefix states.** \( Q_e \) is the set of states of the form \( (s, \tau) \) with \( s \) a perfect extremal string and \( \tau \) an \((m + 1, |s|)\)-pebble assignment satisfying the following conditions, which hold when \( s \) has just been read:

1. The pebble \( m + 1 \) was not used to read \( s \), i.e., we have \( \tau(m + 1) = \bot \).
(e_2) every position of \( s \) has a pebble on it, i.e., if \( s \neq \varepsilon \), then \( |s| \leq \tau([m]) \), and

(e_3) if \( s = \varepsilon \), there are no pebbles on \( s \), i.e., \( \tau = \rho_{\perp} \), in other words, \( \tau([m+1]) = \{\perp\} \).

We would like the automaton to transition from \( (s, \tau) \in Q_e \) to \( (s', \tau') \in Q_e \) if \( s \) and \( s' \) are consecutive. But since the automaton can only move on pebble at a time, we need some intermediate steps. We therefore have another set of prefix states \( Q_p \) that the automaton uses to read extremal strings from left to right, by iterating over all their prefixes.

**Prefix states.** \( Q_p \) is the set of states of the forms \((s, \tilde{s}, \tau, 0)\) and \((s, \tilde{s}, \tau, 1)\) for every perfect extremal string \( s \), non-empty prefix \( \tilde{s} \) of \( s \), and \((m+1, |s|)\)-pebble assignment \( \tau \) satisfying similar conditions as before, but which now hold if only the prefix \( s \) has been read:

- \((e_4)\) \( \tau(m+1) = \perp \),
- \((e_5)\) \(|\tilde{s}|-1 \leq \tau([m])\), and
- \((e_6)\) pebbles beyond the current prefix had been placed previously, i.e., for every \( |\tilde{s}| \leq \ell \leq |s|, \ell \in \tau([m]) \) if and only if \( s(\ell) \not\in \Gamma_{1\text{top}} \).

The 0/1 flag in prefix states is used below for deciding where to place the \( m+1 \) pebble.

**Initial state.** The initial state is \( q_{\text{init}} = (\varepsilon, \rho_{\perp}) \in Q_e \).

**Final states.** The final states are \( F = \{(s, \tau) \in Q_e \mid s \text{ is completed}\} \).

For a state \( q \in Q_p \) of the form \((s, \tilde{s}, \tau, 0)\) or \((s, \tilde{s}, \tau, 1)\), or \( q \in Q_e \) of the form \((s, \tau)\), denote \( \tau_q = \tau \). For a state \( q \in Q_p \cup Q_e \), we say a pebble \( k \in [m+1] \) is available in \( q \) if \( \tau_q(k) = \perp \). Note that the pebble \( m+1 \) is available by \((e_4)\) and \((e_5)\).

Let \( q = (s, \tilde{s}, \tau_q, b) \) or \( q = (s, \tau_q) \) with \( |\tilde{s}| = n_s \). Since \( \tau_q \) is an \((m+1, n_s)\)-pebble assignment, there are at most \( n_s \) pebbles \( k \) with \( \tau_q(k) \neq \perp \). By the bound on the length of extremal strings, we have \( n_s \leq 7 \cdot |\Theta| < m \), therefore there is at least one pebble \( k \in [m] \) which is available in \( q \). All in all, we have:

**Observation 2.** For every \( q \in Q_p \cup Q_e \), there is a pebble \( k \in [m] \) such that both pebble \( k \) and pebble \( m+1 \) are available in \( q \).

**Transitions from prefix states.** We can now define \( \delta \). Let \( q = (s, \tilde{s}, \tau_q, b) \in Q_p \).

1. **Non-extremal transitions:** while reading a prefix of a task word, pebble \( m+1 \) iterates over non-extremal positions. For every letter \( \gamma \in \Gamma_{1\text{top}}^{\text{ext}}(s, |\tilde{s}|) \), we have that \((q, (m+1)\text{-MOVE}_{i\gamma, c}, \xi^{\omega(\gamma)}, q') \in \delta \) where \( q' = (s, \tilde{s}, \tau_q, 1) \) and \( i \) is equal to argmax_{0 \leq c < |\tilde{s}|} \{t \mid \tilde{\tau}_q(t) = \ell \} \) if \( b = 0 \), and is \( m+1 \) if \( b = 1 \).

2. **Extremal transitions:** if we are at a new 1-top position, we read it with an available pebble, and if the current position already has a pebble, a silent transition moves on. The automaton will now be on either the next prefix, or the next extremal state if \( s = \tilde{s} \), that is, the whole \( s \) has been read. Let \( \tau_q \) be \( \tau_q[k \mapsto |\tilde{s}|] \) if \( \tilde{s}(|\tilde{s}|) \in \Gamma_{1\text{top}} \), and be \( \tau_q \) if \( \tilde{s}(|\tilde{s}|) \not\in \Gamma_{1\text{top}} \). Let \( q' \) be \( (s, \tilde{s}(|\tilde{s}|+1), \tau_q, 0) \)
Lemma 20. Let \( L \). Proof of \( F \) or every Observation 3. Let \( \bar{q} \) be induced by \( \bar{q} = (\bar{q}, \rho, N) \) a configuration on \( u \). Let \( \bar{q} \) be a perfect \( 0 \)-pebble assignment, \( \tau \) an \( (m+1, n_s) \)-pebble assignment, \( q = (s, \bar{s}, \tau, b) \) or \( q = (s, \tau) \) be a state and \( \pi = (q, \rho, N) \) a configuration on \( u \).

\[ \pi \text{ is } \omega \text{-coherent if for every } \ell \in [m] \text{ such that } \tau(k) \neq \bot, \rho(k) = \text{Emb}_{\text{ext}}(\tau(k)). \]

Lemma 21. Let \( \mathcal{D} \) be a data word with string \( \mathcal{D} = u \) such that \( \mathcal{D} \models \varphi \). Then \( u \in L(\mathcal{A}^\omega) \).

We recall the following:

**Observation 3.** For every \( q \in Q_p \cup Q_e \), there is a pebble \( k \in [m] \) such that both pebble \( k \) and pebble \( m+1 \) are available in \( q \).

**Coherent configurations** Let \( \mathcal{D} \) be a data word with string \( \mathcal{D} = u \) and \( |u| = n \). Let \( \mathcal{T} \) be a perfect \( \mathcal{D} \)-task word, \( w = \text{abst}(\mathcal{T}) \), and \( s = \text{ext}(w) \) with \( |s| = n_s \). Let \( \rho \) be an \( (m+1, n) \)-pebble assignment, \( \tau \) an \( (m+1, n_s) \)-pebble assignment, \( q = (s, \bar{s}, \tau, b) \) or \( q = (s, \tau) \) be a state and \( \pi = (q, \rho, N) \) a configuration on \( u \).

\[ \pi \text{ is } \omega \text{-coherent if for every } \ell \in [m] \text{ such that } \tau(k) \neq \bot, \rho(k) = \text{Emb}_{\text{ext}}(\tau(k)). \]

**Proof.** Let \( C \in \mathbb{N} \) and \( 1 \leq \ell_1 < \cdots < \ell_C \leq n_u \) such that

\[ \{\ell_1, \ldots, \ell_C\} = \text{extPos}(w) \cup \text{pos}_{1\top}(w). \]

For every \( c \in \{0\} \cup [C] \), let \( N_c = \text{pos}_{1\top}(w) \cup \{\ell_1, \ldots, \ell_c\} \). We have \( N_{c+1} = N_c \cup \{\ell_c\} \) for \( c \neq 0 \). Let \( a_0 = 0 \). For every \( c \in \{0\} \cup [C-1] \), let:

\[ a_{c+1} = \begin{cases} a_c + 1, & \ell_{c+1} \in \text{extPos}(w) \\ a_c, & \ell_{c+1} \notin \text{extPos}(w) \end{cases} \]

Let \( \ell_{\max} = \max(\text{pos}_{1\top}(w)) \). Observe that \( \ell_C \in \text{extPos}(w) \), since it holds that \( \ell_{\max} \in \text{extPos}_{1\top}(\theta(w)) \) for every \( \theta \in \omega(w(\ell_{\max})) \). Consequently, \( s(1) \cdots s(\ell_c) = s \) if and only if \( c = C \). Similarly, \( \ell_1 \in \text{extPos}(w) \).

We give a construction of a computation \((\bar{t}, \bar{\pi})\) with transitions \( \bar{t} = (t_1, \ldots, t_C) \) and \( \omega \)-coherent configurations \( \bar{\pi} = (\pi_0, \ldots, \pi_C) \) on \( u \) such that \( \pi_0 = \pi \), and \( \pi_c \sim_{\omega} \pi_{c+1} \).
π_{c+1} for all c ∈ \{0\} ∪ [C - 1]. Let π_c = (q_c, \rho_c, N_c) where q_c = (s, s(1) \cdots s(a_c + 1), \tau_c, b_c) for c < C and q_C = (s, \tau_C). We construct t_{c+1} and π_{c+1} inductively for c ∈ \{0\} ∪ [C - 1] by dividing into cases as follows.

Assume \( \ell_{c+1} \in \text{extPos}(w) \). We have \( \ell_{c+1} = \text{Emb}_{\text{ext},w}(a_{c+1}) \) and hence \( w(\ell_{c+1}) = s(a_{c+1}) \). If \( c + 1 < C \), let \( b_{c+1} = 0 \).

1. Assume \( s(a_{c+1}) \notin \Gamma_{1\text{top}} \). Let \( t_{c+1} = (q_c,q_{c+1}),\rho_{c+1} = \rho_c, \) and \( \tau_{c+1} = \tau_c \).

Then \( t_{c+1} \in \delta \), \( \pi_c \overset{i_{c+1}}{\sim} \pi_{c+1} \), and \( \pi_{c+1} \) is a \( w \)-coherent configuration on \( u \).

2. Assume \( s(a_{c+1}) \in \Gamma_{1\text{top}} \). Let \( \tau_{c+1} = \tau_c[k \mapsto a_{c+1}] \). By Observation 3, there is some available pebble \( k \in [m] \) in \( q_c \). There exist \( i,j \in [m] \cup \{\top,\bot\} \) and

\[ t_{c+1} = (q_c,k\text{-MOVE}_{i,j},\xi^{w(a_{c+1})},q_{c+1}) \]

such that \( t_{c+1} \in \delta \). By the choice of \( i \) and \( j \) in the definition of an extremal transition from a prefix state to a prefix state in \( [F] \), \( \hat{\tau}_e(i) < a_{c+1} \leq \hat{\tau}_e(j) \). Since \( \pi_c \) is \( w \)-coherent and \( \text{Emb}_{\text{ext},w} \) is order-preserving, \( \hat{\rho}_e(i) < \ell_{c+1} \leq \hat{\rho}_e(j) \). We have \( \ell_{c+1} \notin N_c \), hence no pebble has been placed on \( \ell_{c+1} \). In particular we have \( \ell_{c+1} \neq \rho_e(j) \), implying that \( \hat{\rho}_e(i) < \ell_{c+1} \leq \hat{\rho}_e(j) \). Let \( \rho_{c+1} = \rho_e[k \mapsto a_{c+1}] \).

Since \( s(a_{c+1}) = w(\ell_{c+1}) \), \( \xi^{w(a_{c+1})} = \xi^{w(\ell_{c+1})} = u(\ell_{c+1}) \). Then \( \pi_c \overset{i_{c+1}}{\sim} \pi_{c+1} \) and \( \pi_c \) is \( w \)-coherent.

Now assume \( \ell_{c+1} \notin \text{extPos}(w) \). Then \( \ell_{c+1} \in \text{pos}_{1\text{top}}(w) \), \( \ell_{c+1} \notin N_c \), \( a_{c+1} = a_c \), and for every \( \theta \in \Theta \), there are \( 1 \leq \theta_{\ell,1} < \theta < \theta_{\ell,r} \leq C \) such that

\[ c_{\theta,1} = \max(\{ \hat{c} \in [C] \mid \ell_c \in \text{extPos}_{0\text{top}}(w) \cap [c - 1], \hat{c} \}) \]

Let \( s_{\text{pr}} \) and \( s_{\text{su}} \) be respectively the prefix and suffix of \( s \) given by \( s_{\text{pr}} = s(1) \cdots s(a_{c+1}) \) and \( s_{\text{su}} = s(a_{c+1}+1) \cdots s(n_s) \). We have \( s = s_{\text{pr}} s_{\text{su}} \) and \( \text{ext}(s) = \text{ext}(s_{\text{pr}} w(\ell_{c+1}) s_{\text{su}}) \), and hence \( w(\ell_{c+1}) \in \Gamma_{1\text{top}}(s, a_{c+1}) \). Let \( \tau_{c+1} = \tau_c \) and \( b_{c+1} = 1 \). There exists \( i \) such that

\[ (q_c,(m+1)\text{-MOVE}_{i,\bot},\xi^{w(\ell_{c+1})},q_{c+1}) \in \delta. \]

If \( b_c = 0 \), then by the choice of \( i \) in the definition of a non-extremal transition from a prefix state in \( [F] \), \( \hat{\tau}_e(i) \leq a_c \). Since \( \text{Emb}_{\text{ext},w} \) is order-preserving, \( \hat{\rho}_e(i) \leq \ell_c \) and hence \( \hat{\rho}_e(i) < \ell_{c+1} \). If \( b_c = 1 \), then the computation \((t_1, \ldots, t_c), (\pi_0, \ldots, \pi_c)\) has at least one non-extremal transition. Hence the pebble \( m + 1 \) was moved during this computation, i.e. \( \rho_c(m + 1) \in \{\ell_1, \ldots, \ell_c\} \). In either case, \( \hat{\rho}_e(i) < \ell_{c+1} \leq \hat{\rho}_e(\llo) \). Let \( \rho_{c+1} = \rho_c \), and note that \( \xi^{w(\ell_{c+1})} = u(\ell_{c+1}) \). Then \( \pi_c \overset{i_{c+1}}{\sim} \pi_{c+1} \), and since \( \tau_{c+1} = \tau_c \) and \( \rho_{c+1} \) agrees with \( \rho_c \) on all \( k \in [m] \), we have that \( \pi_{c+1} \) is \( w \)-coherent.

The lemma follows with \( t' = t_C \) and \( \pi' = \pi_C \).

\[ \square \]

\textbf{Induced sequences of extremal strings} Let \( D \) be a data word with \( h = \text{val}_D \), and \( T \) a \( D \)-task word. For every \( e \in \{0\} \cup [h] \), let \( s_e = \text{ext}(\text{abst}(T^{h-e})) \). We say the sequence \( s_0, \ldots, s_h \) is induced by \( T \). By Prop. 3.
Lemma 22. Let $\mathcal{D}$ be a data word with $h = \text{max val}_\mathcal{D}$ such that $\mathcal{D} \models \varphi$. There exists a perfect completed $\mathcal{D}$-task word $\mathcal{T}$ such that, if $s_0, \ldots, s_h$ is induced by $\mathcal{T}$, then $s_0 = \varepsilon$, $s_h$ is completed, and $(s_{e-1}, s_e)$ is consecutive for $e \in [h]$.

Proof. Since $\mathcal{D} \models \varphi$, by Proposition 5 there exists a perfect completed $\mathcal{D}$-task word $\mathcal{T}$. Let $s_0, \ldots, s_h$ be the sequence of extremal strings induced by $\mathcal{T}$. We have $s_0 = \text{ext}(\text{abst}(\mathcal{T}^{h})) = \varepsilon$. Since $\mathcal{T}$ is perfect, all the extremal strings in the sequence are perfect. Since $\mathcal{T}$ is completed, $s_h = \text{ext}(\text{abst}(\mathcal{T}))$ is completed. Since for $e \in [h]$ we have $\mathcal{T}^{h-e} = (\mathcal{T}^{h-e})^{-1}$, every pair $(s_{e-1}, s_e)$ is consecutive.

Proof of Lemma 19. Let $R = \text{max val}_\mathcal{D}$. By Lemma 22 there exists a perfect completed $\mathcal{D}$-task word $\mathcal{T}$ such that the sequence $s_0, \ldots, s_R$ of extremal strings induced by $\mathcal{T}$ satisfies that $s_0 = \varepsilon$, $s_R$ is completed, and $(s_{r-1}, s_r)$ is consecutive for $r \in [R]$. For every $r \in \{0\} \cup [R]$, let $w_r$ be the string projection of $\mathcal{D} \setminus R$ and let $w_r = \text{abst}(\mathcal{T}^{r-R})$. We have $s_r = \text{ext}(w_r)$. Since $\mathcal{T}$ is perfect, so are $\mathcal{T}^{r-R}$ and $s_r$.

Let $\pi_0 = \pi_{\text{init}}$. Observe that for every $r \in \{0\} \cup [R]$, $\pi_0$ is a $w_r$-coherent configuration on $u_r$. We construct a sequence of transitions $(t_1, \ldots, t_R)$ and a sequence of configurations $(\pi_1, \ldots, \pi_R)$ such that, for every $r \in [R]$, $\pi_r = ((s_r, \tau_r), \rho_r, [u_r])$ is a $w_r$-coherent configuration on $u_r$ and $\pi_0 \leadsto^* \pi_r$. We construct the transitions and configurations inductively as follows. For every $r \in [R]$, assume there are $t_r$ and $\pi_r$ as described above.

The universe of $\mathcal{T}^{R-r}$ is a subset of the universe of $\mathcal{T}^{R-(r+1)}$. We use the notation $\text{Emb}_{u_r \rightarrow u_{r+1}}$ for the embedding obtained by the composition of $\text{Emb}_{\text{abst}, \mathcal{T}^{R-r}}$ and $\text{Emb}_{\text{abst}, \mathcal{T}^{R-(r+1)}}$. The string $u_r$ is a substring of $u_{r+1}$. We denote by $\text{pos}_{(r+1)}$ the mapping of positions of $u_r$ to positions of $u_{r+1}$. Since the universe of $\mathcal{T}^{R-e}$ is equal to the universe of $\mathcal{D}^{R-e}$ for $e \in \{r, r+1\}$, $\text{Emb}_{u_r \rightarrow u_{r+1}} = \text{pos}_{(r+1)}$.

Let $\tilde{\pi}_r = ((s_r, \tau_r), \tilde{\rho}_r, \text{pos}_{\text{top}}(u_{r+1}))$ be a configuration on $u_{r+1}$ with $\tilde{\rho}_r(k) = \perp$ if $\rho_r(k) = \perp$, and $\tilde{\rho}_r(k) = \text{Emb}_{u_r \rightarrow u_{r+1}}(\rho_r(k))$ if $\rho_r(k) \neq \perp$. The semantics of PIA allow us to lift a computation from a substring to a string, thus $\pi_0 \leadsto^* w_{r+1} \tilde{\pi}_r$.

Let $\pi_{r+1}^{pr}$ be the configuration on $u_{r+1}$ given by $\pi_{r+1}^{pr} = ((s_{r+1}, s_{r+1}(1), \tau_{r+1}, 0), \tilde{\rho}_r, \text{pos}_{\text{top}}(w_{r+1}))$ where for every pebble $k \in [m]$, $\tau_{r+1}^{pr}(k)$ is $\perp$ if $\tau_r(k)$ is not in the image of $\text{PEmb}_{s_{r+1} \rightarrow s_r}$, and is $(\text{PEmb}_{s_{r+1} \rightarrow s_r})^{-1}((\tau_r(k)))$ otherwise. Let $t_{r+1} = ((s_r, \tau_r), (s_{r+1}, s_{r+1}(1), \tau_{r+1}), 0))$.

Since the pair $(s_r, s_{r+1})$ is consecutive, we get that $t_{r+1} \in \delta$ and $\pi_{r+1} \leadsto_{\text{top}}^{t_{r+1}} \pi_{r+1}^{pr}$, implying that $\pi_0 \leadsto_{\text{top}}^{t_{r+1}} \pi_{r+1}^{pr}$. We prove that $\pi_{r+1}^{pr}$ is $w_{r+1}$-coherent. $\pi_r$ is $w_r$-coherent by the assumption, hence $\rho_r = \text{Emb}_{u_r \rightarrow u_{r+1}}(\tau_r(k))$.

Let $k \in [m]$ be such that $\tau_{r+1}^{pr}(k) \neq \perp$. Then $\tau_r(k)$ is in the image of $\text{PEmb}_{s_{r+1} \rightarrow s_r}$, and in particular $\tau_r(k) \neq \perp$, and $\tau_{r+1}^{pr}(k)$ is given by:

$\text{Emb}_{\text{top}}(\mathcal{T}^{R-(r+1)}) (\text{Emb}_{\text{top}}(\mathcal{T}^{R-r}) (\tau_r(k))) = \text{Emb}_{(u_{r+1})} (\text{Emb}_{u_r \rightarrow u_{r+1}} (\rho_r(k)))$. 

We have
\[ \text{Emb}_{\text{ext}, w_{r+1}} (\tau_{r+1}^{\text{pr}} (k)) = \text{Emb}_{u_r \mapsto w_{r+1}} (\hat{\rho}_r (k)) = \hat{\rho}_r (k) \]

Hence, \( \tau_{r+1}^{\text{pr}} (k) \) is \( w_{r+1} \)-coherent.

Now we apply Lemma 21 with the data word \( D^{(r+1)} \) of size \( n_{r+1} \), the string projection \( u_{r+1} \), the perfect \( D^{(r+1)} \)-task word \( T^{(r+1)} \), the abstraction \( w_{r+1} \), the extremal string \( s_{r+1} \), and the configuration \( \tau_{r+1}^{\text{pr}} (k) \); we get that there are \( \tau_{r+1} \) and \( \rho_{r+1} \) such that \( \tau_{r+1} \) is a \( w_{r+1} \)-coherent configuration on \( u_{r+1} \) and \( \tau_{r+1}^{\text{pr}} (k) \sim u_{r+1} \) \( \tau_{r+1} \), and therefore \( \tau_0 \sim u_{r+1} \) \( \tau_{r+1} \). The lemma follows for the computation \( \tau_0 \sim u_{r+1} \) \( \tau_R \) since \( u = u_{r+1} \), \( n_R = n \), and \( (s_{r+1}, \tau_R) \in F \).

G  Proof of \( L (A^r) \subseteq L (\varphi) \)

Proof. Given \( w \in L (A^r) \), we build a data word \( D \) for it based on an accepting computation of \( A^r \) on \( w \). To show that \( D \models \varphi \), we prove the existence of a perfect completed \( D \)-task word based on the syntactic representation of consecutive extremal strings.

Let \((t_0, \bar{\pi})\) be an accepting computation. Let \( z \) be the number of transitions from a state in \( \{ q_{\text{init}} \} \cup Q_e \) to a state in \( Q_p \) in \( \bar{t} \). The computation can be broken down into parts as follows:

\[
\bar{\pi}_0 \sim w_{t_0} \bar{\pi}_1 \sim w_{t_1} \bar{\pi}_2 \sim w_{t_2} \ldots \sim w_{t_{h-1}} \bar{\pi}_h \sim w_{t_h} \bar{\pi}_f
\]

where the target state of any transition is in \( Q_e \) if and only if there is \( e \in [z] \) such that the transition is the last one in \( t_e \). The sequence \( \bar{t} \) is equal to the concatenation of \( t_0 \) and the sequences \( \bar{t}_1, \ldots, \bar{t}_z \). For every \( e \) and \( a \), let the state of \( \bar{\pi}_a \) be \((s_e, s_e(1) \ldots s_e(a), s_e(1) \ldots s_e(a), b_e)\).

For every transition \( t = t_a^{(e)} \), let \( \gamma_t \in \Gamma_{10}^{e} \cup \{ \varepsilon \} \) be:

1. if \( t = Q \), \( \gamma_t = \varepsilon \).

2. Otherwise, if \( t \) is a non-extremal transition, let \( \gamma_t \in \Gamma_{10}^{\text{not} (s_e, a_e)} \) such that \( t = Q \times \text{MOVE}_{m+1} \times \{ \xi_u \} \times Q \). If \( t \) is an extremal transition, let \( \gamma_t = s(a_e) \) (and note that we again have that \( t = Q \times \text{MOVE}_{m+1} \times \{ \xi_u \} \times Q \)).

Let \( \gamma_{10} = \gamma_{10}^{(e)} \cdots \gamma_{10}^{(e) \mid \text{len}(e)} \), where \( \text{len}(e) = |t_e| \). For every \( e \in [z] \), let \( v_e, u_e \) be the substrings of \( w \) which the automaton reads during the transitions \( t_e \) respectively \((l_0, l_1, \ldots, l_e) \). Let \( g_e \) be such that \( u_e \) is the shuffle of \( v_0 \ldots v_e \) relative to the positions in \( w \).

We prove the following by induction on \( e \): there is a data word \( D_e \) and a \( D_e \)-task word \( T_e \) such that for every \( p \leq e \) we have: (i) \( \text{string}(D_e^{\langle p \rangle}) = u_e \), (ii) \( \text{ext(abst}(T_e^{\langle p \rangle})) = s_{e-p} \), (iii) \( T_e^{\langle p \rangle} = T_{e-p} \) and \( T_e^{\langle p \rangle} \) is a \( D_e \)-task word, (iv) the universe of \( D_e \) is contained in \([e] \times \mathbb{N})

Since for every \( e \in [z] \), \( s_e \) is an extremal string appearing in a state in \( Q_p \), \( s_e \) is perfect. Therefore \( T_e \) is a perfect \( D \)-task word. Since the computation is accepting, the extremal string \( s_e \) is complete and therefore \( T_e \) is a completed \( D \)-task word. By Prop.3 we have \( D \models \varphi \).

\[ \square \]
We now prove inductively the following Lemma:

**Lemma 23.** Let \( w \in \Xi^* \) be accepted by \( \mathcal{A}^r \). Then \( w \in L(\varphi) \).

We assume the induction hypothesis for \( e - 1 \) and prove for \( e \). Let \( r_e \in \Gamma_{1\text{top}}^* \cap \Gamma_p^* \) be obtained from \( \gamma_e \), by setting all tasks to \( P \). Notice \( \psi_e = \xi_{r_e} = \xi_{\gamma_e} \). Let \( g_e : [n_{r_e}] \to [n_{r_e} + n_{s_e-1}] \) be given by \( g_e(\ell) = \ell + |\text{extPos}(\text{abst}(T_{e-1})) \cap [g'_e(\ell) - \ell]| \).

Recall that \( g_e : [n_{s_e-1}] \to [n_{r_e} + n_{s_e-1}] \). Note:

\[
\forall \ell \in [n_{r_e}], \quad \xi^{(r_e, s_e-1)}(g_e(\ell)) = (v_e \downarrow g'_e u_{e-1})(g'_e(\ell))
\]

\[
\forall \ell \in [n_{s_e-1}], \quad \xi^{(r_e, s_e-1)}(g_e(\ell)) = (v_e \downarrow g'_e u_{e-1})(g'_e(\ell))
\]

**Claim 4.** There is a data word \( D_e \) such that \( \text{string}(D_e) = u_e \), and the universe of \( D_e \) is contained in \([e] \times \mathbb{N}\), and there exists a \( D_e \)-task word \( T_e \) such that \( \text{ext}(\text{abst}(T_e)) = \text{ext}(r_e \downarrow g_e s_{e-1}) \) and for every \( p \leq e \), \( T_{e,p} = T_{e-1,p} \).

**Proof.** Let \( D_{e-1} \) be the universe of \( D_{e-1} \). Let \( D_e \) be the data word over \( \Xi \) with universe \( D_{e-1} \cup ([e] \times [n_{r_e}]) \) such that:

1. \( D_{e-1} \) is the substructure of \( D_e \) induced by \( D_{e-1} \).
2. For every \( \ell \in [n_{r_e}] \) and \( r_e(\ell) = (h, ts) \), \( D_e \models \xi^{(ts)}(e, \ell) \).
3. For every \( \ell_1, \ell_2 \in [n_{r_e}] \), \( D_e \models (e, \ell_1) \sim_2 (e, \ell_2) \).
4. For every \( d \in D_{e-1} \) and \( \ell \in [n_{r_e}] \), \( D_e \models d <_2 (e, \ell) \).
5. For every \( \ell_1, \ell_2 \in [n_{r_e}] \), \( D_e \models (e, \ell_1) \leq_1 (e, \ell_2) \) if and only if \( \ell_1 \leq \ell_2 \).
6. For every \( d \in D_{e-1} \) and \( \ell \in [n_{r_e}] \), \( D_e \models d \leq_1 (e, \ell) \) if and only if \( g'_e(\ell) \geq |\{d' \in D_{e-1} | D_{e-1} \models d' \leq_1 d\}| + \ell \).

Let \( T_e \) be the \( D_e \)-task word such that

- for every \( d \in D_{e-1} \), there are \( ts, ts' \in 2\Gamma_{1\text{tasks}} \) such that \( \omega(ts) = \omega(ts') \), \( T_e \models ts(d) \), and \( T_{e-1} \models ts'(d) \), and
- for every \( (e, \ell) \in D_e \), there are \( ts, ts' \in 2\Gamma_{1\text{tasks}} \) such that \( \omega(ts) = \omega(ts') \), \( r_e(\ell) = (1\text{top}, ts') \), and \( T_e \models ts(e, \ell) \).

By our construction, \( T_{e,p} = T_{e-1,p} \) for \( p = 1 \). For \( p > 1 \), this equality follows from the induction hypothesis.

Clearly \( \text{ext}(\text{abst}(T_{e,p})) = \text{ext}(\text{abst}(T_{e-1,p})) = s_{e-1} \), by the induction hypothesis. We apply Lemma [□] with \( D_e \) for \( D_e \), \( T_e \) for \( T_e \) and \( r_e \) for \( r_e \). Note that \( g \) in the lemma is \( g_e \), hence we get \( \text{ext}(\text{abst}(T_e)) = \text{ext}(r_e \downarrow g_e s_{e-1}) \).

Let \( D_{e,max} \) be the substructure of \( D_e \) which consists of the elements of \( D_e \) with maximal data value. By the definition of \( D_e \), we have \( \text{string}(D_{e,max}) = \xi^{r_e} = u_e \).

By induction, \( \text{string}(D_{e-1}) = u_{e-1} \). By the definition \( g'_e \), \( u_e = u_e \downarrow g'_e u_{e-1} \), and hence \( \text{string}(D_e) = u_e \).

\[\square\]
We prove:

In both cases, we divide into two cases:

• Assume the letter at position \( \ell + 1 \) in \( r'_e \) is in \( \Gamma_{\text{top}} \). Then the letter at position \( \ell + 1 \) in \( r'_e \) is not in \( \Gamma_{\text{top}} \). Let \( \ell_1 \) and \( \ell_2 \) be such that \( G_e(\ell_1) = G_e(\ell_2) = \ell + 1 \). Then \( F_e(\ell_1) > F_e(\ell_2) \) and \( G_e(\ell_2) > G_e(\ell_1) \).

• Assume the letter at position \( \ell + 1 \) in \( r'_e \) is in \( \Gamma_{\text{top}} \). Analogously to the previous case, we have \( G_e(\ell_1) > G_e(\ell_2) \) and \( F_e(\ell_2) > F_e(\ell_1) \).

In both cases, \( G_e(\ell_1) < G_e(\ell_2) \) if and only if \( F_e(\ell_2) < F_e(\ell_1) \).

Let \( \ell^{(e)}_c \) be the transition in which the automaton reads position \( \text{Emb}_{\text{ext}, r_e}(\ell) \) of \( v_e \). Let \( \ell^{(c)}_t \in [n_w] \) be the position of \( w \) which \( \ell^{(c)}_t \) reads. Let \( k \)-\textsc{Move}_{i,j} \) be the move in \( \ell^{(c)}_t \). There is a pebble \( k' = \tau_{e,c}^{-1}(F_e(\ell_2)) \) on the position in \( s_e \) corresponding to \( \ell_2 \) in \( s_{e-1}' \). Let \( \ell^{(w)}_t \in [n_w] \) be the position of \( k' \) in \( w \). \( G_e \) and \( G_e \) have disjoint images, hence \( G_e(\ell_1) \neq G_e(\ell_2) \), and similarly for \( F_e, F_e \). We divide into cases:

1. If \( G_e(\ell_1) < G_e(\ell_2) \), then \( F_e(\ell_2) < F_e(\ell_1) \). By the definition of the automaton, the pebble \( i \) is located to the left of \( \ell^{(w)}_t \), and \( i \) is either \( k' \) itself, or another pebble located to the right of \( k' \). Hence, \( \ell^{(w)}_t > \ell^{(w)}_c \). But since \( G_e(\ell_1) < G_e(\ell_2) \), we have \( g_e(\text{Emb}_{\text{ext}, r_e}(\ell_1)) < g_e(\text{Emb}_{\text{ext}, p_{e-1}}(\ell_2)) \). We have \( g'_e(\text{Emb}_{\text{ext}, p_{e-1}}(\ell_2)) < g'_e(\text{Emb}_{\text{ext}, p_{e-1}}(\ell_1)) \), and hence \( \ell^{(w)}_1 < \ell^{(w)}_2 \), in contradiction.

2. The case of \( g_e(\ell_1) > g_e(\ell_2) \) is analogous.

Hence \( F_e = G_e \) and \( s_e = \text{ext}(r_e \downarrow g_e, s_{e-1}) \), and the induction hypothesis holds.

To define partial embeddings, we use the following definition and lemma:

**Definition 13** (\( \text{extElem}(T) \)). Denote \( \text{extElem}(T) = \text{Emb}_{\text{abst}, T}(\text{extPos}(\text{abst}(T))) \) for an \( \mathcal{D} \)-task word \( T \). That is, the elements corresponding to positions in the extremal string of \( T \). For \( h \in \text{Layers} \), \( \theta \in \Theta_3 \), denote \( \text{extElem}_{h, \theta}(T) \) and \( \text{extElem}_{\theta}(T) \) similarly.

**Lemma 24.** Let \( T \) be a \( \mathcal{D} \)-task word. For every \( \theta \in \Theta_3 \):

1. \( \text{extElem}_{2\text{top}, \theta}(T) = \text{extElem}_{1\text{top}, \theta}(T^{h_1}) \).
2. \( \text{extElem}(T) \subseteq \text{extElem}_{2\text{top}, \theta}(T^{h_1}) \cup \text{extElem}_{\theta}(T^{h_1}) \), and
3. \( \text{extElem}_{\text{rest}, \theta}(T) \subseteq \text{extElem}_{2\text{top}, \theta}(T^{h_1}) \cup \text{extElem}_{\text{rest}, \theta}(T^{h_1}) \).

**Proof.** We prove:
1. \( \text{extElem}_{\text{2top}, \theta}(T) = \text{extElem}_{\text{1top}, \theta}(T^{h_1}) \)

2. \( \text{extElem}_{\theta}(T) \subseteq \text{extElem}_{\text{2top}, \theta}(T^{h_1}) \cup \text{extElem}_{\theta}(T^{h_1}) \)

3. \( \text{extElem}_{\text{rest}, \theta}(T) \subseteq \text{extElem}_{\text{2top}, \theta}(T^{h_1}) \cup \text{extElem}_{\text{rest}, \theta}(T^{h_1}) \)

Let \( w = \text{abst}(T) \) and \( w' = \text{abst}(T^{h_1}) \).

1. Let \( d \in D \). Since \( \text{value}_{T}(d) = \text{value}_{T^{h_1}}(d) + 1 \), we have \( w(\text{Emb}^{-1}_{\text{abst}, T}(d)) \in \Gamma_{2\text{top}} \) iff \( w'(\text{Emb}^{-1}_{\text{abst}, T^{h_1}}(d)) \in \Gamma_{1\text{top}} \). Using Lemma \( \text{Lemma}[5] \) we have for every \( \theta \in \Theta_3 \) that

\[
\text{Emb}_{\text{abst}, T}(\text{pos}_{2\text{top}, \theta}(w)) = \text{Emb}_{\text{abst}, T^{h_1}}(\text{pos}_{1\text{top}, \theta}(w')).
\]

Since \( \text{Emb}_{\text{abst}, T} \) and \( \text{Emb}_{\text{abst}, T^{h_1}} \) are order-preserving, for both functions \( \text{opt} = \max \) and \( \text{opt} = \min \) we have that the positions \( \ell_{\text{opt}} = \text{opt}(\text{pos}_{2\text{top}, \theta}(w)) \), \( \ell'_{\text{opt}} = \text{opt}(\text{pos}_{1\text{top}, \theta}(w')) \) are obtained under \( \text{abst} \) from the same \( T \) element

\[
d_{\text{opt}} = \text{Emb}_{\text{abst}, T}(\ell_{\text{opt}}) = \text{Emb}_{\text{abst}, T^{h_1}}(\ell'_{\text{opt}}).
\]

2. Let \( \ell \in \text{extPos}_{\theta}(w) \), \( d = \text{Emb}_{\text{abst}, T}(\ell) \), and let \( w(\ell) = (\text{rest}, ts_d) \). We have that \( P_\theta \in ts_d \). Let \( \ell' \) be such that \( d = \text{Emb}_{\text{abst}, T^{h_1}}(\ell') \), and \( w'(\ell') = (h, ts'_{d}) \) with \( h \in \{2\text{top}, \text{rest}\} \). By Lemma \( \text{Lemma}[6] \) \( \theta \in \omega(ts'_{d}) \). Let \( D' \) be the universe of \( D^{1} \). Since \( D^{1} \) is a substructure of \( D \) and \( d \in D' \), if \( D \not\models \exists y \theta(d, y) \) then also \( D^{1} \not\models \exists y \theta(d, y) \) and therefore also \( P_\theta \in ts'_{d} \). Assume \( \theta(x, y) \models x \leq y \). The case of \( \theta(x, y) \models y \leq x \) is analogous.

Assume for contradiction that \( \ell' \notin \text{extPos}_{2\text{top}, \theta}(w') \) and \( \ell' \notin \text{extPos}_{\theta}(w') \).

There exist \( d_1 \in D', \ell'_1 \in ||D'||, \) and \( h_1 \in \{2\text{top}, \text{rest}\} \) such that \( d_1 = \text{Emb}_{\text{abst}, T^{h_1}}(\ell'_1), w'(\ell'_1) \in \omega(h_1, \ell' < \ell'_1, D \models d < d_1, \) and:

(a) if \( w'(\ell') \in \Gamma_{2\text{top}} \), then \( h_1 = 2\text{top}, \ell'_1 \in \text{extPos}_{2\text{top}, \theta}(w'), \) and \( D \models d \sim_2 d_1; \)

(b) if \( w'(\ell') \in \Gamma_{\text{rest}} \), then \( h_1 = \text{rest}, \ell'_1 \in \text{extPos}_{\theta}(w') \) and \( w'(\ell'_1) \in \Gamma_{\text{rest}}. \)

Let \( \ell_1 \) be such that \( d_1 = \text{Emb}_{\text{abst}, T}(\ell_1) \). We have \( \ell < \ell_1 \) and \( w(\ell_1) \in \Gamma_{\text{rest}} \). Let \( w(\ell_1) = (\text{rest}, ts_{d_1}) \) and \( w'(\ell'_1) = (h_1, ts'_{d_1}) \). Since \( \ell'_1 \in \text{extPos}_{2\text{top}, \theta}(w') \cup \text{extPos}_{\theta}(w') \), we have \( \theta \in \omega(ts_{d_1}) \). By Lemma \( \text{Lemma}[8] \) we have \( \theta \in \omega(ts_{d_1}) \), and using that \( \ell \in \text{extPos}_{\theta}(w) \) we have \( C_\theta \in ts_{d_1} \) and hence \( D \models \exists y \theta(d_1, y) \). Since \( \theta \in \omega(ts_d) \cap \omega(ts_{d_1}) \), we have \( \xi^{\omega(ts_d)} = \xi^{\omega(ts_{d_1})} \), implying that \( d \) and \( d_1 \) have the same 1-type in \( D \).

(a) If \( w'(\ell') \in \Gamma_{2\text{top}} \), by Lemma \( \text{Lemma}[12] \) we have \( D \models \exists y \theta(d, y) \), in contradiction to \( P_\theta \in ts_d \).

(b) If \( w'(\ell') \in \Gamma_{\text{rest}} \), since \( \ell'_1 \in \text{extPos}_{\theta}(w') \) we have \( P_\theta \in ts'_{d_1} \) and hence \( D^{1} \not\models \exists y \theta(d_1, y) \). By Lemma \( \text{Lemma}[14] \) we have \( D \models \exists y \theta(d, y) \), in contradiction to \( P_\theta \in ts_d \).
3. For every $d \in D$, $\text{value}_D(d) = \text{value}_{D\backslash \{d\}}(d) - 1$. Hence, using Lemma 6, we have that for every $\theta \in \Theta_3$, the embedding $\text{Emb}_{\text{abst}, \mathcal{T}}(\text{pos}_{\text{rest}, \theta}(w))$ is given by

$$\text{Emb}_{\text{abst}, \mathcal{T} \backslash 1}(\text{pos}_{\text{ext}, \theta}(w) \cup \text{pos}_{\text{rest}, \theta}(w)).$$

Since $\text{Emb}_{\text{abst}, \mathcal{T}}$ and $\text{Emb}_{\text{abst}, \mathcal{T} \backslash 1}$ are order-preserving, for both functions $\text{opt} = \max$ and $\text{opt} = \min$ we have that the positions $\ell'_{\text{opt}} = \text{opt}(\text{pos}_{\text{ext}, \theta}(w) \cup \text{pos}_{\text{rest}, \theta}(w))$ and $\ell_{\text{opt}} = \text{opt}(\text{pos}_{\text{rest}, \theta}(w))$ are obtained under $\text{abst}$ from the same $\mathcal{T}$ element $d_{\text{opt}} = \text{Emb}_{\text{abst}, \mathcal{T}}(\ell_{\text{opt}}) = \text{Emb}_{\text{abst}, \mathcal{T} \backslash 1}(\ell'_{\text{opt}})$.

\[\Box\]

Let $(s', s)$ be a pair of consecutive extremal strings, and let $\mathcal{T}$ be a task word such that $s = \text{ext}(\mathcal{T})$ and $s' = \text{ext}(\mathcal{T} \backslash 1)$. We denote by $\text{PEmb}_{s \rightarrow s'}^\mathcal{T}$ the function from the set $s^{-1}(\Gamma - \Gamma_{1\text{top}})$ of positions $\ell$ for which $s(\ell) \notin \Gamma_{1\text{top}}$ to $[n_{s'}]$ defined as $\text{PEmb}_{s \rightarrow s'}^\mathcal{T}(\ell) = \text{Emb}_{\text{ext} \circ \text{abst}, \mathcal{T} \backslash 1}^{-1} \circ \text{Emb}_{\text{abst}, \mathcal{T}}^{-1}(\text{Emb}_{\text{ext} \circ \text{abst}, \mathcal{T}} \circ \text{Emb}_{\text{ext}, \mathcal{T}})(\ell)$. The function $\text{PEmb}_{s \rightarrow s'}^\mathcal{T}$ is well-defined: since $s(\ell)$ is not in $\Gamma_{1\text{top}}$, we get from Lemma 24 that $\text{Emb}_{\text{ext} \circ \text{abst}, \mathcal{T}}(\ell) \in \text{Emb}_{\text{ext} \circ \text{abst}, \mathcal{T} \backslash 1}([n_{s'}])$. We call $\text{PEmb}_{s \rightarrow s'}^\mathcal{T}$ a partial embedding via $\mathcal{T}$ since it is injective and order-preserving.

**Example 4.** $\text{PEmb}_{s \rightarrow s'}^\mathcal{T}$ has the domain $\{1, 2, 4\}$, and is equal to the composition:

$$\text{Emb}_{\text{ext} \circ \text{abst}, \mathcal{T} \backslash 1}^{-1} \circ \text{Emb}_{\text{abst}, \mathcal{T}}^{-1} \circ \text{Emb}_{\text{ext}, \mathcal{T}} \circ \text{Emb}_{\text{ext}, \mathcal{T}}.$$

The embeddings $\text{Emb}_{\text{abst}, \mathcal{T} \backslash 1}$ and $\text{Emb}_{\text{abst}, \mathcal{T}}$ were given in Example 2 and the embeddings $\text{Emb}_{\text{ext}, \mathcal{T} \backslash 1}$ and $\text{Emb}_{\text{ext}, \mathcal{T}}$ were given in Example 3. We have:

$$\text{PEmb}_{s \rightarrow s'}^\mathcal{T}(1) = 1,$$
$$\text{PEmb}_{s \rightarrow s'}^\mathcal{T}(2) = 3,$$
$$\text{PEmb}_{s \rightarrow s'}^\mathcal{T}(4) = 4.$$

**Definition 14 (Partial embedding).** Let $(s', s)$ be consecutive extremal strings. We denote by $\text{PEmb}_{s \rightarrow s'}^\mathcal{T}$ the function from $s^{-1}(\Gamma - \Gamma_{1\text{top}})$ to $[n_{s'}]$ defined as follows. Let $\mathcal{T}$ be a task word, and $s = \text{ext}(\mathcal{T})$ and $s' = \text{ext}(\mathcal{T} \backslash 1)$. Then we have $\text{PEmb}_{s \rightarrow s'}^\mathcal{T} = \text{PEmb}_{s \rightarrow s'}^\mathcal{T}$.

The following lemma shows that partial embeddings are well-defined:

**Lemma 25.** Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be task words, and let $s = \text{ext}(\text{abst}(\mathcal{T}_1)) = \text{ext}(\text{abst}(\mathcal{T}_2))$ and $s' = \text{ext}(\text{abst}(\mathcal{T}_1 \backslash 1)) = \text{ext}(\text{abst}(\mathcal{T}_2 \backslash 1))$. Then $\text{PEmb}_{s \rightarrow s'}^{\mathcal{T}_1} = \text{PEmb}_{s \rightarrow s'}^{\mathcal{T}_2}$.

**Proof of Lemma 25** Let $n_s = |s|$. Since $\text{ext}(\text{abst}(\mathcal{T}_1)) = \text{ext}(\text{abst}(\mathcal{T}_2))$, the domains of $\text{PEmb}_{s \rightarrow s'}^{\mathcal{T}_1}$ and $\text{PEmb}_{s \rightarrow s'}^{\mathcal{T}_2}$ are equal. Assume for contradiction that $\text{PEmb}_{s \rightarrow s'}^{\mathcal{T}_1} \neq \text{PEmb}_{s \rightarrow s'}^{\mathcal{T}_2}$. Let $\ell_s$ be the minimal position in $s$ such that $s(\ell_s) \notin \Gamma_{1\text{top}}$ and $\text{PEmb}_{s \rightarrow s'}^{\mathcal{T}_1}(\ell_s) \neq \text{PEmb}_{s \rightarrow s'}^{\mathcal{T}_2}(\ell_s)$. 

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In each row there is one element \( d \) of \( T_i^{h1} \) with \( i \in \{1, 2\} \). Each of the columns \( u \in \{w_1, w_2, s\} \) indicates the position of \( d \) in the string \( u \) according to the embedding \( \text{Emb}_{Op, T_i} \) with the appropriate operation \( Op = \text{abst} \) for \( u \in \{w_1, w_2\} \) and \( Op = \text{ext} \circ \text{abst} \) for \( u = s \). Each of the columns \( u' \in \{s, w_2'\} \) indicates the position of \( d \) in the string \( u' \) according to the embedding \( \text{Emb}_{Op, T_i^{h1}} \) with \( Op = \text{abst} \) for \( u' = w_2' \) and \( Op = \text{ext} \circ \text{abst} \) for \( u' = s \). Each of the elements and positions in row 1 (respectively row 3) are strictly smaller than the elements and positions in the same column in row 2 (respectively row 4). (The comparison of elements is with respect to \( \leq_{1} \).

| \( T_i^{h1} \) | \( T_i^{h1} \) | \( w_1 \) | \( w_2 \) | \( s \) | \( s' \) | \( w_2' \) |
|----------------|----------------|------|------|-----|-----|-------|
| smaller        | \( d_{1,2} \)  | \( \ell_{1,2} \) | \( \ell_{s',1} \) | \( \ell_{w_2',1} \) |
| larger         | \( d_{2} \)    | \( \ell_{2} \)  | \( \ell_{s} \)   | \( \ell_{s',2} \)  |
| smaller        | \( d_{1} \)    | \( \ell_{1} \)  | \( \ell_{s} \)   | \( \ell_{s',1} \)  |
| larger         | \( d_{2,1} \)  | \( \ell_{2,1} \) | \( \ell_{s} \)   | \( \ell_{s',2} \)  |

\( \text{PEmb}_{T_2 \rightarrow s}(\ell_s) \). For \( i = 1, 2 \), let \( w_i = \text{abst}(T_i), n_i = |w_i| \), and \( w_i' = \text{abst}(T_i^{h1}) \). Let \( \ell_i, d_i, \ell_{s',i}, \ell_{s'} \) be as follows:

\[
\begin{align*}
\ell_i &= \text{Emb}_{\text{ext}, w_i}(\ell_s) \\
\ell_{s',i} &= \text{Emb}_{\text{abst}, T_i}(\ell_i) \\
\ell_{s'} &= \text{PEmb}_{T_i^{h1} \rightarrow s}(\ell_s)
\end{align*}
\]

and note \( d_i = \text{Emb}_{\text{ext} \circ \text{abst}, T_i}(\ell_s) = \text{Emb}_{\text{ext} \circ \text{abst}, T_i^{h1} \circ \text{emb}}(\ell_{s',i}) \), and since \( s(\ell_s) \notin \Gamma_{\text{top}} \), \( d_i \) belongs to the universe of \( T_i^{h1} \). For distinct \( i, j \in \{1, 2\} \), let \( d_{i,j}, \ell_{i,j}, \ell_{w_2',1} \) be as follows:

\[
\begin{align*}
\ell_{w_2',i} &= \text{Emb}_{\text{ext}, w_2'}(\ell_{s',i}) \\
\ell_{i,j} &= \text{Emb}_{\text{abst}, T_j}(\ell_{w_2',i}) \\
\ell_{i,j} &= (\text{Emb}_{\text{abst}, T_j})^{-1}(d_{i,j})
\end{align*}
\]

and note \( d_{i,j} = \text{Emb}_{\text{ext} \circ \text{abst}, T_j}(\ell_{s',i}) \), and that \( d_{i,j} \) belongs to the universe of \( T_j^{h1} \) and hence to that of \( T_j \). W.l.o.g \( \ell_{s',1} < \ell_{s',2} \), and therefore by the order-preservation property of embeddings, \( T_1 \models \ell_1 < d_{2,1}, \ell_1 < \ell_{2,1}, T_2 \models \ell_{1,2} < d_{2} \), and \( \ell_{1,2} < \ell_{2} \) (see Table 2). Let

\[
\begin{align*}
\text{ts}(s(\ell_s)) &= \text{ts}_s \\
\text{ts}(w_i(\ell_i)) &= \text{ts}_i \\
\text{ts}(s'(\ell_{s',i})) &= \text{ts}_{s',i} \\
\text{ts}(w_j(\ell_{i,j})) &= \text{ts}_{i,j}
\end{align*}
\]

From the definition of \( \text{ext} \), \( ts_s = ts_1 = ts_2 \). By the definitions \( \text{ext} \) and \( \text{abst} \) and from Lemma 6, \( ts_s, ts_{s',1}, ts_{s',2}, ts_{1,2}, \text{and } ts_{2,1} \) all realize the same set-type.

Before continuing the proof of Lemma 25, we prove three claims.

**Claim 5.** Let \( (s', s) \) be a pair of consecutive extremal strings, and let \( T \) be a task word such that \( s = \text{ext}(\text{abst}(T)) \) and \( s' = \text{ext}(\text{abst}(T^{h1})) \). Let \( \ell \in [n_s] \). Then:
1. \( s(\ell) \in \Gamma_{\text{top}} \) if and only if \( s'(P_{\text{Emb}}_{s \to s'}^{\ell}(\ell)) \in \Gamma_{1\text{top}} \).

2. \( s(\ell) \in \Gamma_{\text{rest}} \) if and only if \( s'(P_{\text{Emb}}_{s \to s'}^{\ell}(\ell)) \in \Gamma_{2\text{top}} \cup \Gamma_{\text{rest}} \).

**Proof.** The claim follows from the definitions of \( P_{\text{Emb}}_{s \to s'}^{\ell} \), ext, and abst, and from Claim 24. \( \square \)

**Claim 6.** \( \ell_{12} \not\in \text{extPos}(w_2) \).

**Proof.** Assume for contradiction that \( \ell_{12} \in \text{extPos}(w_2) \). Then there exists \( \hat{\ell}_{12} \in [n_s] \) such that \( \hat{\ell}_{12} = \text{Emb}_{\text{ext},w_2}(\ell_{12}) \) and \( d_{12} = \text{Emb}_{\text{ext abst},\tau_2}(\hat{\ell}_{12}) \) and we have that \( P_{\text{Emb}}_{s \to s'}^{\hat{\ell}_{12}}(\ell_{12}) = \ell_{s'1} \). Since \( d_{12} < 1 \) \( d_2 \) we have \( s(\ell_{12}) \notin \Gamma_{\text{top}} \) and \( \hat{\ell}_{12} < \ell_s \). Since \( P_{\text{Emb}}_{s \to s'}^{\hat{\ell}_{12}} \) is injective, and \( P_{\text{Emb}}_{s \to s'}^{\hat{\ell}_{12}}(\ell_{s'1}) = \ell_{s'1} \), we have \( P_{\text{Emb}}_{s \to s'}^{\hat{\ell}_{12}}(\ell_{12}) \neq \ell_{s'1} \), in contradiction to the minimality of \( \ell_s \). \( \square \)

**Claim 7.** Let \( \bar{P}_0 \in ts_s, s(\ell_s) \in \Gamma_{\text{rest}}, \) and \( \ell_2 \in \text{extPos}_2(w_2) \). Then \( \bar{\theta}(x,y) = x \leq \bar{\theta} \) and \( \bar{P}_0 \in ts_{21} \).

**Proof.** Since \( s(\ell_s) \in \Gamma_{\text{rest}} \), by Claim 5 we have \( s'(\ell_{s'1}), s'(\ell_{s'2}) \in \Gamma_{2\text{top}} \cup \Gamma_{\text{rest}} \). Hence

\[
\text{value}_D(d_1), \text{value}_D(d_2), \text{value}_D(d_{21}), \text{value}_D(d_{12}) \leq \maxval_D - 2 = \maxval_{D|1} - 1
\]

and \( s(\ell_{12}), s(\ell_{21}) \in \Gamma_{\text{rest}} \). Since \( \omega(ts_s) = \omega(ts_{12}) = \omega(ts_{21}) \), we have \( \bar{\theta} \in \omega(ts_{12}) \cap \omega(ts_{21}) \). Since \( D|1 \) is a substructure of \( D \) and \( \bar{P}_0 \in ts_s \), we also have \( \bar{P}_0 \in ts_{s'1} \cap ts_{s'2} \).

Assume for contradiction that \( \bar{\theta}(x,y) = y < x \) if \( \bar{P}_0 \in ts_{21} \), then since \( \ell_{12} < \ell_2 \), we have \( \ell_2 \notin \text{extPos}_2(w_2) = \{ \ell \mid \ell = \min(\text{pos}_{\text{rest}}(P_\theta(w))) \} \) in contradiction. If \( C_\theta \in ts_{21} \), then \( \bar{P}_0 \in ts_{s'1} \cap ts_{s'2} \), and since \( \bar{\theta}(x,y) = y < x \), from Lemma 13 (with \( d = d_{12} \) and \( d' = d_{21} \)) it follows that \( C_\theta \in ts_s \), in contradiction to \( \bar{P}_0 \in ts_s \). Hence \( \bar{\theta}(x,y) = x \leq y \).

Since \( \bar{\theta} \in \omega(ts_{21}) \), either \( \bar{P}_0 \in ts_{21} \) or \( C_\theta \in ts_{21} \). If \( C_\theta \in ts_{21} \), then from Lemma 13 (with \( d = d_{12} \) and \( d' = d_{21} \)) it follows that \( C_\theta \in ts_s \), in contradiction to \( \bar{P}_0 \in ts_s \). Hence \( \bar{P}_0 \in ts_{21} \). \( \square \)

We are now ready to resume the proof of Lemma 25. We divide into cases depending on whether \( s(\ell_s) \in \Gamma_{2\text{top}} \) or \( s(\ell_s) \in \Gamma_{\text{rest}} \). First assume that \( s(\ell_s) \in \Gamma_{2\text{top}} \).

Since \( \ell_{s'1} = P_{\text{Emb}}_{s \to s'}(\ell_s) \), we have that \( s'(\ell_{s'1}) \in \Gamma_{1\text{top}} \) by Claim 5.

Now assume \( s(\ell_s) \in \Gamma_{\text{rest}} \). We have \( s'(\ell_{s'1}), s'(\ell_{s'2}) \in \Gamma_{2\text{top}} \cup \Gamma_{\text{rest}} \) by Claim 5, i.e. \( d_{12}, d_{21} \leq \maxval_D - 1 = \maxval_{D|1} - 2 \). Hence, \( s(\ell_{12}), s(\ell_{21}) \in \Gamma_{\text{rest}} \).

Next assume \( s(\ell_s) \in \Gamma_{\text{rest}} \). We have \( s'(\ell_{s'1}), s'(\ell_{s'2}) \in \Gamma_{2\text{top}} \cup \Gamma_{\text{rest}} \) by Claim 5. Hence, \( s(\ell_{s'1}), s(\ell_{s'2}) \in \Gamma_{\text{rest}} \).

Since \( \ell_{s'1} = \text{Emb}_{\text{ext},w_2}(\ell_s) \) we have that \( \ell_2 \in \text{extPos}(w_2) \). Let \( \theta \in \omega(ts_s) = \omega(ts_{12}) \).

By Claim 6 and using that \( s(\ell_{12}) \in \Gamma_{\text{rest}} \), there exists \( \hat{\ell}_{12} < \ell_{12} \) such that \( \hat{\ell}_{12} \in \text{extPos}_{\text{rest},\theta}(w_2) \). Let \( \ell_0, \theta \) be such that \( \ell_0 = \text{Emb}_{\text{ext},w_2}(\ell_{0\theta}) \). We have that \( \hat{\ell}_{12} < \ell_2 \), and hence \( \ell_{0\theta} < \ell_s \). Let \( s(\ell_{0\theta}) = (h_{0\theta}, ts_{0\theta}) \).

Since \( \hat{\ell}_{12} \in \text{extPos}_{\text{rest},\theta}(w_2) \) we
have \( \theta \in \omega(t_{s_0,\theta}) \). Let \( \ell_{3,\theta} \) be such that \( \ell_{2,1} = \text{Emb}_{\text{ext},w_1}(\ell_{3,\theta}) \). Since \( \ell_1 < \ell_{2,1} \) we have \( \ell_s < \ell_{3,\theta} \). We have \( \theta \in \omega(\ell_{2,1}) = \omega(\ell_{3,\theta}) \). Hence, \( \ell_2 \notin \text{extPos}_{\text{rest},\theta}(w_2) \) for all \( \theta \in \omega(ts) \). Consequently, there is \( \theta \) such that \( \ell_2 \in \text{extPos}(w_2) \) and \( P_\theta \in ts_2 = ts_s \).

By Claim 7 we have \( \theta(x,y) \models x \leq y \) and \( P_\theta \in ts_{2,1} \), and hence there is \( \ell_{\bar{\theta},w_1} \) such that \( \text{extPos}_{\bar{\theta}}(w_1) = \{ \ell_{\bar{\theta},w_1} \} \) and \( \ell_{\bar{\theta},w_1} = \max(\text{pos}_{\text{rest},P_\theta}(w_1)) \). We have \( P_\theta \in ts(w_1(\ell_{\bar{\theta},w_1})) \) and \( \ell_1 < \ell_{2,1} \leq \ell_{\bar{\theta},w_1} \). Let \( \ell_{\bar{\theta},s} \in [n_s] \) and \( \ell_{\bar{\theta},w_2} \in [n_2] \) be such that \( \ell_{\bar{\theta},w_1} = \text{Emb}_{\text{ext},w_1}(\ell_{\bar{\theta},s}) \) and \( \ell_{\bar{\theta},w_2} = \text{Emb}_{\text{ext},w_2}(\ell_{\bar{\theta},s}) \), then we have that \( P_\theta \in ts(s(\ell_{\bar{\theta},s})) = ts(w_1(\ell_{\bar{\theta},w_1})) = ts(w_2(\ell_{\bar{\theta},w_2})) \), \( \ell_s < \ell_{\bar{\theta},s} \), and \( \ell_2 < \ell_{\bar{\theta},w_2} \). Since \( u_1(\ell_{\bar{\theta},w_1}) \in \Gamma_{\text{rest}} \) and \( u_1(\ell_{\bar{\theta},w_1}) = s(\ell_{\bar{\theta},s}) = w_2(\ell_{\bar{\theta},w_2}) \), we have \( \ell_2 \notin \text{extPos}(w_2) = \{ \max(\text{pos}_{\text{rest},P_\theta}(w_2)) \} \), in contradiction.