Privacy-Preserving Distributed Learning with Secret Gradient Descent

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Abstract

In many important application domains of machine learning, data is a privacy-sensitive resource. In addition, due to the growing complexity of the models, single actors typically do not have sufficient data to train a model on their own. Motivated by these challenges, we propose Secret Gradient Descent (SecGD), a method for training machine learning models on data that is spread over different clients while preserving the privacy of the training data. We achieve this by letting each client add temporary noise to the information they send to the server during the training process. They also share this noise in separate messages with the server, which can then subtract it from the previously received values. By routing all data through an anonymization network such as Tor, we prevent the server from knowing which messages originate from the same client, which in turn allows us to show that breaking a client’s privacy is computationally intractable as it would require solving a hard instance of the subset sum problem. This setup allows SecGD to work in the presence of only two honest clients and a malicious server, and without the need for peer-to-peer connections.

1 Introduction

Machine learning (ML) has shown impressive success across many domains, enabled by increasingly complex models requiring large amounts of training data. Ever more data is being collected, and it is collected in ever more places: from phones, smartwatches, Internet-of-things devices, cars, etc. At first glance, it seems as though this abundance of data should solve the problem of data-hungry models—if only we could pool the data from all the different places. The caveat is that much modern data is privacy-sensitive.

Hence, ironically, while the rate at which data is being produced is ever-increasing, data collection and pooling has in a certain sense become not easier but harder, both because of new data protection laws such as the GDPR [14] and because of more awareness in the wake of data scandals such as those surrounding Cambridge Analytica [28] or Strava [19].

A straightforward approach to mitigate privacy concerns would be to add noise to datasets before merging them. However, this would require so much noise that it would render the data essentially useless [11, 15]. There are more sophisticated ways of adding noise [20], but the fundamental tradeoff between privacy and utility always remains. A different possible tradeoff must be made between privacy on one side and communication cost on the other side; cryptographic methods generally belong to this domain. A downside of these methods is that they typically make trust assumptions on the clients or the server [6, 8].

Contributions: SecGD. In this paper we propose Secret Gradient Descent (SecGD), a method that allows for training arbitrary ML models via gradient-based methods on distributed data. It provides a fairly strong privacy guarantee with very small trust assumptions and can be extended to provide
differential privacy with very small noise addition. In standard distributed gradient descent, clients compute the gradient of the loss function on their local datasets in each step and send these gradients back to a central server for summation. The main idea of SecGD is to let clients add noise to their gradients prior to sending them in a way that makes them lose any information about the original gradients, thereby making them useless for the extraction of sensitive information about a client’s dataset. Their usefulness for model training is restored in a second step, where the negative value of the noise is sent to the server through an anonymization network such as Tor [31]. Adding this negative noise to the sum of the noisy gradients yields the sum of the original gradients. However, no single local gradient can be recovered, because in the anonymization network the noise vectors of all clients are mixed, making them unlinkable to a specific noisy local gradient.

Properties of SecGD. During the training process, the server learns nothing but the global gradient, that is, the sum over the local gradients that are computed on the individual distributed datasets. SecGD belongs to second kind of approaches mentioned in the beginning: it trades communication cost against privacy. An summary can be found in Sec. 4.5 These are the key properties of SecGD:

- Main idea: privacy by adding and canceling noise.
- Computational privacy guarantee: compromising privacy would require solving hard instances of the multidimensional subset sum problem, which is computationally intractable.
- Trust assumptions: at least two honest clients; an anonymization network such as Tor [31].
- Increase of communication compared to standard distributed gradient descent by only a logarithmic factor in the number of users and the number of parameters of the ML model.
- No need for peer-to-peer connections between clients.
- Noise-efficient extension to ensure differential privacy for the global gradient.
- Limitation to settings where clients have a stable Internet connection and are unlikely to drop out during a training iteration.

Despite the limitation mentioned last, SecGD applies to many situations of practical importance, e.g., multiple hospitals that each collect patient data and want to pool the data across hospitals to answer research questions; a franchise company that wants to analyze customer behavior via purchase logs in its different branches; or a vendor of server software that wants to improve its product’s performance by optimizing it for common usage patterns. In the last example, the software would collect usage statistics that the vendor would like to use to train an ML model on a central server.

2 Related Work

There are two inherently different approaches towards protecting privacy when training ML models on distributed data via gradient based methods: by adding noise or by using cryptographic or information theoretic approaches.

The former leads to differential privacy guarantees by adding independent noise to the gradients before sending them to the server [12]. However, in this standard form large amounts of noise need to be added since each client’s data needs to be protected individually without being able to rely on other clients. We discuss this topic in more detail in Sec. 5.4.

Cryptographic approaches based on homomorphic encryption [2,3] typically try to protect a private client’s data from an aggregating server but not from other clients. Clients encrypt their gradients with the same key before sending them to the server, which then performs model updates in the encrypted space. The resulting model can be decrypted by the clients. The encryption scheme of Shi et al. [30] allows the clients to encrypt their gradient with different keys, but requires a trusted setup phase with a trusted server, or communication between the clients. Methods based on generic secure multiparty computation use secret sharing [5], where a secret value is distributed between the clients in several parts such that a certain number of them is required to reconstruct it. These protocols require communication between the clients and often a large number of honest clients [6,9]. Methods without this last restriction [26] still need direct communication between clients, which might not be possible in, e.g., the setting from the introduction where the vendor of a server software wants to gather usage statistics, since especially database servers often only allow connections from a whitelist of IP addresses for security reasons. Bonawitz et al. [8] proposed a protocol that is based on a similar idea as ours, i.e., adding noise that cancels out later. However, it requires a trusted server to distribute keys in the setup phase. Trusting an already established distributed public infrastructure
An observation also used in other privacy protocols \cite{8} is that when computing a sum, it does not matter if one adds additional summands that sum up to 0. Instead of adding a 0 sum directly, our protocol as well. To update $w'$, the only information that server needs from the clients is $g'$, and we want to prevent it from learning more than what is absolutely necessary for the model updates.

Training goal and adversarial model. The server should have access to the global gradients $g'$ to train the ML model. At the same time, we want to prevent it from learning any of the local gradients $g'_i$, because they might contain sensitive information about a client’s dataset. We assume that the server is actively malicious: In order to break privacy and to learn any of the $g'_i$, it may deviate from the training protocol. In addition, the server may collude with any but two clients, which may deviate from the protocol as well.

Note that SecGD can be used to securely sum up any kind of vectors. We build our exposition around the gradient descent, though, since this is one of the most important applications. In gradient descent, the sum $g'$ of the local vectors might still contain sensitive information. Preventing their extraction is a problem orthogonal to ours, but in Sec. 5.4 we describe how a differentially private mechanism can be combined with SecGD to solve it in our case.

4 Proposed Solution: Secret Gradient Descent

An observation also used in other privacy protocols \cite{8} is that when computing a sum, it does not matter if one adds additional summands that sum up to 0. Instead of adding a 0 sum directly, our method operates in two steps: first, random vectors are subtracted from the gradients to obfuscate them, then the same vectors are added again.

4.1 Preprocessing

For our solution we need to represent the entries of the gradient vectors as elements from the group $\mathbb{Z}_{2^m}$, i.e., the integers modulo $2^m$ for an integer $m$. Here, $2^m$ is an upper bound on the entries of the global gradient $g'$, derived from an upper bound $2^m$ on the entries of the local gradients and the number $N$ of users: $m = \lceil \log(N) \rceil \bar{m}$. The clients can transform their real-valued gradient vectors to elements from $\mathbb{Z}_{2^m}$ in a preprocessing step that we detail in Appendix 4.1.

4.2 Protocol

Main idea. The idea behind SecGD is to first add noise to the gradients prior to sending them, let the server sum them up to obtain a noisy version of $g'$, and to then tell the server how much noise was added so that it can remove the noise from $g'$. However, the noise vectors of the different clients get mixed throughout the process, turning the reconstruction of any of the $g'_i$ into a computationally hard subset sum problem. So instead of sending one message containing $g'_i$, each client additionally generates $K$ independent random vectors $s'_{i1},\ldots,s'_{iK}$ and sends the following $K + 1$ messages:

$$g'_i := g'_i - \sum_{k=1}^{K} s'_{ik}, \quad s'_{i1}, \ldots, s'_{iK}.$$  

To obtain $g'$, the server simply has to sum up all messages it received in the $t$-th training round, so from the utility perspective nothing has changed over regular distributed gradient descent. What about privacy? If the server knows which of the messages were sent by the same client $i$, summing
We will discuss both of those in the following two paragraphs.

Package content. For making the messages unlinkable via the vectors they contain, we need to make them, or at least the $s_{ik}'$, all look the same. This can easily be done by sampling them independently from the same distribution. We, however, also do not want $g_i'$ itself to carry any information about $g_i'$. This could for example happen if $K$ were small and the $s_{ik}'$ were sampled from a distribution with small variance. This is why we choose the uniform distribution on $\mathbb{Z}_d^d$ for the additional summands, that is, $s_{ik}' \sim \mathcal{U}(\mathbb{Z}_d^d)$ i.i.d. As a consequence, $g_i'$, too, is uniformly distributed on $\mathbb{Z}_d^d$. Furthermore, any $K$-element subset of the $K + 1$ messages a client sends is statistically independent. In Sec. 5 we show that the information that still remains in the set of messages cannot be used by a computationally bounded adversary if we choose $K = dm/2$ (we do not have to send $dm/2$ vectors; cf. Sec. 4.3).

Metadata. There are two types of metadata that the server receives, (1) the IP address from which a package was sent and (2) the time at which a package arrives. To make this information useless, we first route all messages through an anonymization network, such as Tor [31], or through a similar proxy server infrastructure, thereby removing all information that was originally carried by the IP address. Second, to remove any information that the arrival times of packages might contain, we set the length of one training iteration to $n$ seconds and make clients send their packages not all at once, but spread them randomly over the $n$ seconds, thus making the packages’ arrival times useless for attacks. Without this measure, all update packages from the same user might be sent right after one another, and the server would receive groups of packages with larger breaks after each group and could thus assume that each such group contains all packages from exactly one user.

Malicious adversary. So far we worked under the assumption that the server is honest but curious, i.e., that it might try to infer additional information from the data it receives but that it at least honestly follows the protocol. Due to the nature of our protocol, the only way the server could deviate from it would be to send different parameter vectors or different additional organizational information (such as the length of a training period) to different clients. In order to prevent this, we give clients a way to recognize when the server violates the protocol: Instead of requesting the current parameter vectors and information once from the server, the clients request it multiple times per iteration. Only if they get the same response every time do they respond; otherwise they must assume an attack. Since the clients’ requests are routed through an anonymization network, the server cannot identify subsequent requests from the same client and cannot maliciously send the same spurious data every time. To reduce communication costs, the clients do not actually request the data multiple times, but only once in the beginning, and afterwards request hashes of it. As an even safer countermeasure, one could distribute the data that would otherwise be obtained directly from the server via a blockchain. This way, each user would be able to verify the integrity of the data they receive. In the initial setup phase, the server also has to tell each client the total number of clients $N$ so that they can compute $m$, the number of bits to use for their vectors. Lying about $N$, however, would not give the server any significant advantage since it would have to lie to all clients in the same way. We go more into detail about this in Sec. 5.3.

4.3 Improving Communication Efficiency

It is not necessary to send the $s_{ik}'$ as vectors, which would be of the same, potentially high, dimension as the gradient. Instead, the server and the clients agree on a common random number generator (RNG) beforehand, e.g., by hardcoding it. A client then generates $K$ seeds $S_{1i}',... ,S_{Ki}'$ and uses the RNG to compute $s_{1i}',... ,s_{Ki}'$. It then sends the vector $g_i'$ and the scalars $S_{1i}',... ,S_{Ki}'$, which are used by the server to compute $s_{1i}',... ,s_{Ki}'$ once again.

How many bits do we need for the seeds? Using seeds with fewer bits than the random vectors that are generated from them increases the probability of collisions, i.e., two users generating the same seeds and hence the same random vectors by chance. This might weaken the hardness guarantee in Sec. 5.2. As we will see later, only collisions between the vectors of two of the users are to be avoided. If $q$ is the number of bits used for the seeds, we can easily upper bound the collision probability $p$ by assuming that the event of the collision of any two seeds is independent of the event of the collision of any other seeds. We can then arrange the seeds in a list and compute the probability that the
We will first ignore possible information leakage from the global gradient and will describe a noise addition technique to prevent it in Sec. 5.4. Combining this with Thm. 1 yields a differential privacy guarantee for the global gradient.

**Theorem 1.** Assume that there exist at least two honest clients and that the set of subset sum problems with \( dm \) uniformly distributed \( d \)-dimensional vectors with encoding length \( m \) per entry and the sum consisting of \( dm/2 \) summands is computationally hard (cf. Sec. 5.7). Then a computationally bounded server is not able to prove that with positive probability for a given list of vectors \( h^1, \ldots, h^T \in \mathbb{Z}_{2^m}^d \) there exists a client \( t \) among the set of honest clients and an iteration \( t \) such that \( h^t = \bar{g}^t \), if this cannot be learned from \( g^t \) alone.
We say “with positive probability” because the server will not necessarily be able to definitely prove, even with unlimited computational power, that there exists a client that had a specific gradient in a specific iteration, but only whether this is possible or not, i.e., whether the answer to the decision problem from Sec. 5.1 is true or false. We further point out that the privacy guarantee against the server automatically yields the same privacy guarantee against other clients since we allow the server to collude with all but two clients.

**Intuition.** Thm. 1 is a strong guarantee: If the server is not even able to determine whether a specific client participated in the training, then it certainly will not be able to infer any other kind of information (e.g., a part of their dataset) about a client from the training data. This holds even in the case when the server has arbitrary side information about the client to be attacked, e.g., when the server knows some entries of a gradient and wants to infer the rest of its entries. For example, assume that each dataset corresponds to one person and that one of the features in the dataset is the age of the client, another one whether they have cancer. Then the server will not be able to tell whether one of the persons has a given age $z$, and hence definitely not whether a person of age $z$ has cancer or not.

The proof consists of first showing that the task of detecting a given gradient is equivalent to solving a certain instance of subset sum (Sec. 5.1) and then showing that these instances are hard (Sec. 5.2). In the proof we allow the server to know which of the messages sent by the users are of the type $\tilde{g}_t^i$ and which are of the type $s_{ik}^i$. This is the case when using the more efficient way of sending the $s_{ik}^i$ as random seeds instead of vector (see Sec. 4.3). Since the $s_{ik}^i$ arrive in a random order and are all independent and identically distributed, from the server’s perspective they just form one big multiset of messages, and the $\tilde{g}_t^i$ form another multiset, which we will together denote by $M' = \{(a_1', \ldots, a_N'), (b_1', \ldots, b_{NK}')\}$, where we use the notation $\{\cdot\}$ for multisets. The complexity guarantee we give is based on the multi-dimensional subset sum problem [13, 21].

**Definition 2** ($d$-dimensional Decisional Subset Sum Problem). *Given $n$ vectors $V = \{(v_1, \ldots, v_n)\}$ and a vector $w$ in $\mathbb{Z}_m^d$, decide whether there exists a submultiset $\tilde{V} \subset V$ such that $\sum_{v \in \tilde{V}} v = w$."

Note that we can reduce the search version of this problem (finding a suitable $\tilde{V}$) to the decision version and the other way around [24]. We will now show that the problem instances described in Thm. 1 are equivalent to a certain set of $d$-dimensional Subset Sum ($d$-SSS) instances, and will then show that these instances are computationally hard.
5.1 Equivalence to $d$-SSS

As we will explain in the last sentence of this subsection, we may assume w.l.o.g. that the server is only given the messages $M^0 = (\{(a_1^0, \ldots, a_{2m}^0)\}, \{(b_{1k}^0, \ldots, b_{2k}^0)\})$ from a single training iteration $t_0$, where all vector entries are encoded with $m$ bits. We assume further that at least two clients are honest, w.l.o.g. clients 1 and 2, and that in the case where one of the gradients that the server is searching for was sent in iteration $t_0$, it was sent by either client 1 or client 2. For the moment we will ignore the messages of all other clients. We hence work with the messages $((\{(a_1^0, a_{2m}^0)\}, \{(b_{1k}^0, \ldots, b_{2k}^0)\}))$, and the task of the adversarial server is to determine whether there exists a $K$-element submultiset $V$ of $\{b_{1k}^0, \ldots, b_{2K}^0\}$ such that either $a_1^0 + \sum_{i \in V} \tilde{v} = h^0$ or $a_{2m}^0 + \sum_{i \in V} \tilde{v} = h^0$. We can equivalently formulate this as proving or disproving the existence of a submultiset $\tilde{V}$ such that either $\sum_{i \in \tilde{V}} \tilde{v} = h^0 - a_1^0$ or $\sum_{i \in \tilde{V}} \tilde{v} = h^0 - a_{2m}^0$. Since the noisy gradients $a_1^0$ and $a_{2m}^0$ are uniformly random, the right sides of these two equations are uniformly random too. Thus, the server’s task is equivalent to solving a $d$-SSS problem with a multiset of uniformly random vectors, a uniformly random target sum $w$, and the additional constraint that the submultiset $\tilde{V}$ needs to be of cardinality $K$.

The messages of clients other than 1 or 2 are independent of those of clients 1 and 2 and can therefore be ignored as pure noise. Including them would only increase the chance of false positives, i.e., solutions to the $d$-SSS search problem that do not correspond to a set of vectors sent by a single client. Furthermore, the $d$-SSS instances resulting from different training iterations are clearly independent, so the assumption from the beginning of this subsection that the server only has access to the data from one training iteration can be made w.l.o.g.

5.2 Hardness Guarantee

We now show that the set of $d$-SSS instances from Sec. 5.1 which an adversary would have to solve, is computationally hard for the parameter choice $K = dm/2$. In the one-dimensional case and without the additional constraint, this has been done already in 1996 by Impagliazzo and Naor [21]. The proof can be extended to our setting. We first formalize the problem using a similar notation as Impagliazzo and Naor, but invert it: Whereas in their case the number of vectors $K$ is fixed, we fix the encoding length $m$ and write $K$ as a function of $m$.

**Definition 3.** Let $B = \{(b_1, \ldots, b_{2K(m)})\}$ be a multiset of vectors drawn uniformly and independently from $\mathbb{Z}_2^m$. SecGD SSS is the problem of inverting the function $f_B(S) = (B, \sum_{b \in S} b)$, where $S$ is a uniformly randomly drawn submultiset of $B$ with cardinality $K(m)$.

Note that $S$ can be represented as a vector $t \in \{0, 1\}^{2K(m)}$ with $L_1$-norm equal to $K(m)$ ($b_i \in S$ iff $t_i = 1$), which we will do in the following. We are interested in hard instances of this problem, depending on the number of vectors $2K(m)$, i.e., instances for which the function from Def. 3 is hard to invert. For this we use the usual definition of one-way functions [17] to sequences of functions $\{f_n\}$, where $f_n$ is used for inputs of length $n$ and may be random. In our case, $f_{n/2K(m)dm} = f_B$ for a multiset $B$ of $2K(m)$ $d$-dimensional random vectors and an encoding length of $m$.

**Definition 4 [21].** Let $\{f_n\}$ be a sequence of (potentially random) functions defined on $D_n \subset \{0, 1\}^n$ and let $f^* : \bigcup_n D_n \to \{0, 1\}^*$ be defined by its restrictions to the $D_n$: $f^*|_{D_n} = f_n$. $\{f_n\}$ is one-way if the following two conditions hold:

- $f^*(t)$ is computable in polynomial time for every $t \in \bigcup_n D_n$.
- Let $\{t_n\}$ be a sequence of uniformly random inputs, $t_n \sim \mathcal{U}(D_n)$ i.i.d. For every probabilistic polynomial-time algorithm $A$ (that attempts to invert $f^*$) and for all $c > 0$, $\Pr(f^*(A(f^*(t_n))) = f^*(t_n)) < n^{-c}$ for all sufficiently large $n$.

Since solving the search version of SecGD SSS is exactly the problem of inverting a function $f_n$, we call sets of instances for which the corresponding sequence $\{f_n\}$ is one-way hard [21]. In our case, sets of instances are defined by the number of messages as a function of the encoding length $2K(m)$. Using this definition, the hardest instances are those for which $2K(m) = dm$:

**Theorem 5** (cf. [21] Prop. 1.2).

1. Let $2K^*(m) \leq 2K(m) \leq dm$. If SecGD SSS is hard for $K^*(m)$, then it is also hard for $K(m)$.
2. Let $dm \leq 2K(m) \leq 2K^*(m)$. If SecGD SSS is hard for $K^*(m)$, then it is also hard for $K(m)$. 


For the proof of Thm. 5 we need to characterize the SecGD SSS instances in the two different cases. In the first case the function \( f_B \) is almost injective, while in the second case it is almost surjective with all values in its range occurring almost the same number of times.

**Definition 6** (cf. \[21\]). Let \( D \) be a probability distribution on \( \{0,1\}^n \). We say \( D \) is quasi-random within \( \epsilon \), if for all \( U \subset \{0,1\}^n \) we have that \( |\Pr_D(U) - |U|/2^n| < \epsilon \).

**Lemma 7** (cf. \[21\] Prop. 1.1).

1. Let \( 2K(m) \leq \text{cdm} \) for \( c < 1 \). Let \( B \) and \( S \) both be chosen uniformly at random. Except with probability exponentially small, there is no \( S' \neq S \) such that \( f_B(S) = f_B(S') \).
2. Let \( 2K(m) \geq \text{cdm} \) for \( c > 1 \). Let \( B \) be chosen uniformly at random. Except with probability exponentially small (w.r.t. the choice of \( B \)), the distribution given by \( f_B(S) \) for a randomly chosen \( S \) is quasi-random (w.r.t. \( S \)) within an exponentially small amount.

Thm. 5 can now easily be deduced from Lemma 7 (see \[21\]). Assume we have an algorithm that efficiently solves SecGD SSS for instances with \( 2K(m) \) vectors where the function \( K \) is chosen such that \( f_B \) is almost injective. Instances with \( 2K'(m) < 2K(m) \) vectors can be transformed into instances with \( 2K(m) \) vectors by removing enough of the least significant bits. This adds only few false positives, i.e., solutions to the inversion of \( f_B \), due to instances with \( 2K(m) \) vectors being almost injective. Similarly, if we have an algorithm for instances with \( 2K(m) \) vectors where \( f_B \) is almost uniform, we transform instances with \( 2K'(m) > 2K(m) \) vectors by adding random bits at the end. Every solution to the modified problem is a solution to the original problem, and we do not lose many solutions because the sums \( f_B(S) \) are almost uniformly distributed for \( 2K(m) \) vectors.

The proof of Lemma 7 can be found in the Appendix \[3\] and closely follows that of Prop. 1.1 from Impagliazzo and Naor \[21\]. For the first part we use a simple union bound while for the second part we rely on the Leftover Hash Lemma from Santha and Vazirani \[29\].

### 5.3 Hardness in Practice

One-dimensional SSS is an NP hard problem \[25\], its multi-dimensional generalization therefore is as well. However, multi-dimensional SSS has been mostly neglected by the research community so far, apart from a negative result about its approximability \[13\]. One-dimensional subset sum, on the other hand, has a long history of study. Similar to this paper, instances are typically characterized in terms of the ratio of the number of messages \( n \) and their encoding length \( l(n) \), where the optimal choice for security is \( l(n) = n \) \[21\]. For \( l(n) > 1.06n \), SSS can be transformed into a lattice shortest vector problem \[23\] \[10\] that can be solved efficiently for certain instances but is, like SSS, NP hard in the general case. For \( l(n) = O (\log(n)) \) there exists a very efficient dynamic programming solution \[16\]. Instances with \( l(n) = n \) are hard instances in the same sense as in our paper: The number of possible summands equals the number of bits per summand. For them the fastest algorithms still require exponential time. The fastest traditional algorithm runs in \( \tilde{O}(2^{0.291n}) \) \[4\], the fastest quantum algorithm in time \( \tilde{O}(2^{0.220n}) \) \[7\], where the notation \( \tilde{O} \) suppresses polynomial factors. Thus, despite significant efforts, no efficient algorithm has been found for the hardest set of one-dimensional SSS instances and it seems likely that this will also be the case for its multi-dimensional counterpart. Note that for small \( n \), the problem is still solvable. This translates to both \( d \) and \( m \) being small. Since \( m = [\log(N) / \tilde{m}] \) and the server could lie to the clients about \( N \), in the case of \( d \) being small, \( \tilde{m} \) has to be chosen sufficiently large. Because this choice is transparent to the clients, they are able to detect when the server chooses a too small value and can refuse to participate in the training.

### 5.4 Protecting the Global Gradient via Differential Privacy

Up to this point we allowed the server to learn the exact global gradient \( g' \) and ensured that the server cannot learn anything about the clients’ data beyond what can be learned from \( g' \). In practice this is often not sufficient to keep the data private. Just the value of the global gradient can already reveal the values of all local gradients \( g_i' \); for an example, see Appendix \[4\]. This problem can be solved by adding noise to the \( g_i' \) to ensure differential privacy \[12\]. Let us assume that we want to achieve a certain privacy level \( (\epsilon, \delta) \) and that the corresponding necessary variance when adding Gaussian noise is \( \sigma^2 \). In standard distributed gradient descent we would have to add \( \mathcal{N}(0, \sigma^2 I) \) to each \( g_i' \), resulting in \( \mathcal{N}(0, N \sigma^2 I) \) noise for \( g' \). When using SecGD, however, the local gradients are already protected and we only need to ensure differential privacy for the global gradient. If we assume that...
there are $\tilde{N}$ honest clients, then each of them only has to add $\mathcal{N}(0, \sigma^2/\tilde{N})$ to their $g'_i$ to ensure a final noise of at least $\mathcal{N}(0, \sigma^2\tilde{N})$. Thus, the noise added to $g'_i$ will be only $\mathcal{N}(0, (1 + \tilde{N} - \tilde{N})\sigma^2)$. A more detailed exposition can be found in Appendix [C].

6 Discussion

In this paper, we developed SecGD, a protocol for training ML models on distributed data via gradient based methods in a privacy-preserving fashion. It only requires two honest clients and no peer-to-peer connections between clients. We achieve our privacy guarantees by adding noise that cancels out later, and routing messages through an anonymization network. We use this network as a building block and are therefore not concerned with weaknesses of specific implementations such as TOR. Also, making as few assumptions as we do does not come without a cost. If a client drops out during a training round, the round has to be restarted, since the incomplete sum that the server receives is uniformly randomly distributed on the entire range $\mathbb{Z}_m$. Hence, SecGD is best suited for settings where clients have stable connections to the server. This restriction might be relaxed by drawing the noise $s'_i$ not from the uniform distribution but from a distribution with smaller variance. This would require adapting the complexity analysis and would yield differential privacy guarantees for the noisy gradients $g'_i$. We leave this for future work, as well as ensuring correctness, i.e., preventing clients from poisoning the training by sending forged gradients. We would further like to encourage research on the multi-dimensional subset sum problem, the basis for our privacy guarantee.

Appendices

A Preprocessing

To transform the real-valued gradient vectors into vectors with entries from $\mathbb{Z}_m$, there are two things that we need to do: (1) upper bound the gradient entries and (2) represent them as non-negative integers.

We first choose a number of bits $\tilde{m}$ to use for the representation of each entry, and the position of the decimal point, i.e., how many bits are used for the integer and how many for the fractional part. In the following we assume for simplicity and w.l.o.g. that all bits are used for the integer part. If there exists an a priori bound on the gradient entries, this bound naturally determines the number of integer bits. If no such bound is known, we can choose a large value for $\tilde{m}$, but might still encounter gradients $g'_i$ with one or more entries with absolute value larger or equal $2^{\tilde{m}}$. We project those gradients to the $L^\infty$ ball with radius $2^{\tilde{m}} - 1/2$ around 0: $g'_i \leftarrow (2^{\tilde{m}} - 1/2)/\|g'_i\|_\infty$. This operation, known as clipping, preserves the ratio of the gradient entries w.r.t. each other and is commonly used in differentially private ML [1] and also non-differentially private neural network training [22]. To obtain non-negative vectors, we replace each $g'_i$ by $g'_i + (2^{\tilde{m}} - 1/2) \mathbb{I}$, where $\mathbb{I}$ denotes the $d$-dimensional 1-vector. To reverse this transformation, the server can simply subtract $N(2^{\tilde{m}} - 1/2)$ from the final $g'$. Now all $g'_i$ lie in $[0, 2^{\tilde{m}} - 1]^d$. We still need to discretize them to integers. This can be done by, e.g., rounding the entries stochastically to the nearest integer, i.e., for an integer $i$ such that $i \leq r \leq i + 1$, we round $r$ to $i$ with probability $r - i$ and to $i + 1$ with probability $i + 1 - r$. In expectation, this doesn’t change the value of $r$. In the beginning we mentioned that we would work in $\mathbb{Z}_m$. We choose $m \geq \tilde{m} + \log(N)$ so that the sum over all local gradients $g'_i$ doesn’t exceed $2^m$.

We would like to remark that the discretization of the gradients comes with only a small decrease in model performance [18].

B Proof of Lemma 7

Lemma 7.

1. Let $2K(m) \leq cdm$ for $c < 1$. Let $B$ and $S$ both be chosen uniformly at random. Except with probability exponentially small, there is no $S' \neq S$ such that $f_B(S) = f_B(S')$. 

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2. Let $2K(m) \geq cdm$ for $c > 1$. Let $B$ be chosen uniformly at random. Except with probability exponentially small (w.r.t. the choice of $B$), the distribution given by $f_B(S)$ for a randomly chosen $S$ is quasi-random (w.r.t. $S$) within an exponentially small amount.

Proof:

1. Let $S' \neq S$ be a different $K(m)$-element subsets of the $2K(m)$-element set $B$ in $d$ dimensions with $2^m$ bits per dimension. Then $f_B(S)$ and $f_B(S')$ are independent uniformly random vectors, as mentioned earlier. There are $\binom{2K(m)}{K(m)} - 1 \leq 2^{2K(m)}$ possible choices for $S'$. Hence we have

$$\Pr(\exists S' \neq S : f_B(S) = f_B(S')) \leq \sum_{S' \neq S} \Pr(f_B(S) = f_B(S')) \leq 2^{2K(m)}2^{-dm} \leq 2^{-(1-c)dm}.$$ 

2. Because $f_B(t)$ and $f_B(t')$ are independent and uniformly random w.r.t. the choice of $B$ for $t \neq t'$, $\{f_B(B) : |B| = 2K(m)\}$ is a family of universal hash functions from $\{0,1\}^{2K(m)}$ to $\mathbb{Z}_2^d$. We can therefore apply the following lemma:

**Lemma 8** (Leftover Hash Lemma [29]). Let $U \subset \{0,1\}^n$, $|U| \geq 2^l$. Let $e > 0$ and let $F$ be an almost universal family of hash functions mapping $n$ bits to $l - 2e$ bits. Then the distribution $(f, f(u))$ is quasi-random within $1/2^e$ (on the set $F \times \{0,1\}^{l-2e}$), where $f$ is chosen uniformly at random from $F$, and $u$ uniformly from $U$.

Since we only allow subset sums where exactly half of the vectors is summed up, the domain of our hash functions is restricted to $U = \{v \in \{0,1\}^{2K(m)} : \|v\|_1 = K(m)\}$. This set has a cardinality of $\binom{2K(m)}{K(m)} \geq 2^{2K(m)}/(2K(m) + 1) = 2^{2K(m) - \log(2K(m) + 1)}$, whereas the domain of $f_B$ has cardinality $2^{dm}$. We thus get $e = 2K(m) - \log(2K(m) + 1) - dm \leq (c - 1 - O(\log(dm)/dm))dm$ for the $e$ from Lemma 8, which yields, for all $T \subset \{0,1\}^{dm}$:

$$\mathbb{E}_{u} \Pr(f_B(u) \in T) - \frac{|T|}{2^{dm}} < 2^{-(c-1-O(\log(dm)/dm))dm}.$$ 

Because this bound on the expectation w.r.t. $B$ is exponential, Markov’s inequality asserts that an exponential bound holds for all but an exponentially small fraction of all $B$:

$$\Pr_B(\Pr_u(f_B(u) \in T) - \frac{|T|}{2^{dm}} \geq 2^{-(c-1-O(\log(dm)/dm))dm}) \leq 2^{-(c-1-O(\log(dm)/dm))dm}.$$ 

\[ \square \]

### C Protecting the Global Gradient via Differential Privacy

Just the value of the global gradient itself — that we allow the server to learn — can already reveal all values of the local gradients, as the following example shows.

Assume that there are only two clients, the dataset of each one consisting of only one entry, a positive scalar feature value, $x_1 = 1$ and $x_2 = 2$, respectively. The loss function that should be minimized is $L(w,x) = \frac{1}{2} e^{wx} - w$, $w \in \mathbb{R}$ with $\partial L(w,x)/\partial w = e^{wx} - 1$. In the first iteration, the server sends the parameter $w = -1$, in the second iteration $w = 1$, and receives the global gradients $e^{-1} + e^{-2} - 2$ and $e + e^2 - 2$, respectively. From these two global gradients and with the knowledge that there are exactly two clients, the server can reconstruct the local gradients from the resulting system of equations

$$(1) \ e^{x_1} + e^{x_2} = e + e^2, \quad (2) \ e^{-x_1} + e^{-x_2} = e^{-1} + e^{-2},$$

which can be uniquely solved (up to exchanging $x_1$ and $x_2$) using the substitutions $\tilde{x}_1 = e^{x_1}$, $\tilde{x}_2 = e^{x_2}$. To prevent such leakage of information through the global gradient, we ensure differential privacy (DP) for this sum by adding noise.
were to use DP without our protocol, one would need to add $N$ variance of the noise is chosen based on the desired level of privacy, the number of training iterations

A randomized algorithm $\mathcal{M}$ with domain $\mathbb{N}[\mathcal{X}]$ for a
set $\mathcal{X}$ is $(\varepsilon, \delta)$-differentially private if, for all $\mathcal{S} \subseteq \text{Range}(\mathcal{M})$ and for all $x, y \in \mathbb{N}[\mathcal{X}]$ such that $\|x - y\|_1 \leq 1$,

$$\Pr(\mathcal{M}(x) \in \mathcal{S}) \leq e^\varepsilon \Pr(\mathcal{M}(y) \in \mathcal{S}) + \delta.$$  

Differential privacy means that an adversary that gets the output of algorithm $\mathcal{M}$ is not able to determine whether an individual in $\mathbb{N}[\mathcal{X}]$ participated in the data collection or not, even with arbitrary side information about $\mathcal{M}$ and the dataset.

Client level vs. sample level DP. While Thm. 1 from the main paper is concerned with hiding the participation of entire clients (with a dataset consisting of possibly many training samples), in most cases only the weaker guarantee of hiding the participation of an individual training sample is considered in the DP literature. Both can be combined with our protocol. The combination of client level DP with our protocol is straight-forward. When combining sample level DP with it, Thm. 1 does not directly apply. However, one can in this case assume a stronger server that knows the gradient of the client in whose dataset the attacked sample is potentially contained, and make the adversary’s task easier by only requiring them to detect whether this client participated in the training or not. Hence, the analogue of Thm. 1 for gradients of individual training samples holds. In the following we consider the more common case of sample level DP.

A standard way to achieve DP in gradient-based training is to add noise to the gradients. The variance of the noise is chosen based on the desired level of privacy, the number of training iterations and the sensitivity of the gradient to changes in the data. The very common Gaussian mechanism achieves $(\varepsilon, \delta)$-DP for a $d$-dimensional function $f$ by adding noise from the $d$-dimensional Gaussian $\mathcal{N}(0, \sigma^2 I)$ [12], where

$$\sigma^2 > \frac{\Delta_2}{\varepsilon} \sqrt{\frac{2 \ln(1.25/\delta)}{d}}.$$  

Here $\Delta_2$ is the $L_2$-sensitivity of $f$, i.e., how much its value changes at most in $L_2$-norm when one entry in the database that is used as its argument is changed. In our case this would be the maximum $L_2$-norm of the loss function $L$. We already have the bound $\sqrt{d}2^\delta$ on $L$ due to the bound of $2^\delta$ on all entries of the $d$-dim. gradients. This bound, however, could be very loose and we can enforce a tighter bound by using clipping w.r.t. the $L_2$-norm, as done in Sec. [4,1] for the $L_\infty$-norm. If one were to use DP without our protocol, one would need to add $\mathcal{N}(0, \sigma^2 I)$ to every client’s gradient, resulting in $\mathcal{N}(0, \sigma^2 I) = \mathcal{N}(0, N \sigma^2 I)$ total noise, because we need to ensure DP for every single summand of $g$. In contrast, with our protocol summands are not revealed to the server, and hence DP only needs to be ensured for the sum $g$. If we assume $\bar{N}$ honest clients, each client would hence only need to add $\mathcal{N}(0, \sigma^2 / \bar{N} I)$ to their gradient. Thus, by using SecGD, the amount of noise necessary to achieve DP for the global gradient reduces from $\mathcal{N}(0, N \sigma^2 I)$ to $\mathcal{N}(0, (1 + N - \bar{N}) \sigma^2 I)$.

DP over multiple training iterations can be ensured by using a composition theorem, e.g., [12] [1]. The effects of the addition of noise to the gradients on the model performance are well explored in the literature and have been shown to be small [1].

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