On Penrose inequality in holography

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The recent holographic deduction of Penrose inequality only assumes null energy condition while the weak or dominant energy condition is required in usual geometric proof. This paper makes a step toward filling up gap between these two approaches. For planar/spherically symmetrically asymptotically Schwarzschild anti-de Sitter (AdS) black holes, we give a purely geometric proof for Penrose inequality by assuming the null energy condition. We also point out that two naive generalizations of charged Penrose inequality are not generally true and propose two new candidates. When the spacetime is asymptotically AdS but not Schwarzschild-AdS, the total mass is defined according to holographic renormalization and depends on scheme of quantization. In this case, the holographic argument implies that the Penrose inequality should still be valid but this paper use concrete example to show that whether the Penrose inequality holds or not will depend on what kind of quantization scheme we employ.

I. INTRODUCTION

In general relativity, there are many famous and universal inequalities, such as the Penrose inequality \([1, 2]\), the positive mass theorem \([3, 4]\), the second law of black holes \([5, 6]\) and so on. As a theoretical test, Penrose inequality is related to the establishment of cosmic censorship. Specifically, given the ADM mass/energy \(M\) of a 4-dimensional asymptotically flat spacetime which contains a black hole as the initial data and denoting \(A\) to be minimal area of surface enclosing the apparent horizon \(\sigma\), Penrose inequality states that the spacetime’s total mass \(M\) should be at least \(\sqrt{A/16\pi}\) and the saturation appears only when the exterior is Schwarzschild. Note that \(A\) is defined as the minimal area of surface enclosing the apparent horizon \(\sigma\). As pointed out by Ref. \([7]\), the apparent horizon area, in general, may not satisfy the Penrose inequality. Penrose’s argument \([8]\) is as following: If we wait a very long time, the black hole will eventually settle down to a Kerr solution. In Kerr solution, the relationship between the black hole’s mass \(M_{\text{kerr}}\) and the area of event horizon \(A_{\text{ev}}\) is \(M_{\text{kerr}} \geq \sqrt{A_{\text{ev}}/16\pi}\). Under this evolution, the black hole’s mass which is described by Bondi mass can not increase. Assuming cosmic censorship and appropriate energy conditions, the apparent horizon either lies within or coincides with the event horizon. Combining with the second law of black holes that states the area of the event horizon can’t decrease, Penrose got his inequality immediately:

\[
M \geq \sqrt{\frac{A}{16\pi}}. \tag{1}
\]

It is worth noting that the above argument is based on a lot of mathematical or physical assumptions. Although mathematicians have proven that Penrose inequality is true in some certain cases, there is no general proof for Penrose inequality (see e.g. Refs. \([9, 10]\)).

Taking the same argument, we can also conjecture Penrose inequality for 4-dimensional asymptotically AdS spacetime \([11]\):

\[
M \geq \left(\frac{A}{16\pi}\right)^{\frac{1}{2}} + \frac{1}{2\ell_{\text{AdS}}} \left(\frac{A}{4\pi}\right)^{\frac{1}{2}}, \tag{2}
\]

where \(M\) is the total mass/energy defined according to holographic renormalization and \(A\) is the minimal area to enclose apparent horizon \(\sigma\) for this asymptotically AdS spacetime. Here we have absorbed the Casimir energy \([12]\) into the definition of total mass. When the black hole is described by Schwarzschild-AdS solution, the inequality take equal sign. Another way to rephrase this conjecture is, given the same mass, the minimal area to enclose apparent horizon is bounded by the area of AdS Schwarzschild black hole’s horizon. In holography, the bulk’s geometry is dual to the two asymptotically boundary’s QFT state \([13, 14]\). For each boundary’s reduced density matrix, its holographic entropy is proportional to the area of black hole’s apparent horizon \([15, 16]\). Moreover, given same total mass \(M\), Ref. \([17]\) shows via holography that boundary’s QFT state dual to Schwarzschild-AdS black hole has the maximum entropy. Consequently, the AdS Penrose inequality can get from the above basic holography’s argument:

\[
A \leq \max A = A_{\text{sch}}. \tag{3}
\]

Here \(A_{\text{sch}}\) stands for the horizon area of Schwarzschild-AdS black hole with same total energy. This idea was recently used by Ref. \([18]\) to argue the Penrose inequality in asymptotically AdS spacetime. We note that the Refs. \([17, 18]\) though discussed the AdS black hole of which the cross-section of event horizon has spherical topology, their discussions are regardless the topology of event horizon. Thus, if one follows their discussions, one shall obtain generalized Penrose inequalities of asymptotically AdS black hole with planar or hyperbolic topologies.

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For charged black holes, one might wonder that the charged generalization for Penrose inequality can be argued with Penrose’s original idea. However, this does not work for the charged cases. Given the initial data mass $M$ and charge $Q$, the relation between initial total mass $M$, the final black hole mass $M_{\text{RN}}$ and the final event horizon area $A_{\text{ev}}$ before using the second law of black holes is

$$M \geq M_{\text{RN}} = \left( \frac{A_{\text{ev}}}{16\pi} \right)^{\frac{3}{2}} + \frac{1}{2} \frac{A_{\text{ev}}}{4\pi} + \frac{Q^2}{2} \frac{\sqrt{4\pi}}{A_{\text{ev}}} .$$

(4)

here we assume that no charge can be radiated away. Assuming cosmic censorship and appropriate energy conditions, the event horizon area $A_{\text{ev}}$ is larger than the minimal area $A$ of surface enclosing the apparent horizon according to the second law of black holes, i.e. $A_{\text{ev}} \geq A$. If “right-hand side of the inequality (4) is a monotonically increasing function of area”, then we would obtain the charged generalization proposed by Ref. [11]

$$M \geq \left( \frac{A}{16\pi} \right)^{\frac{3}{2}} + \frac{1}{2} \frac{A}{4\pi} + \frac{Q^2}{2} \frac{\sqrt{4\pi}}{A} .$$

(5)

Unfortunately, such monotonicity is not so obvious. Thus, one should not be surprised if the charged generalization (5) is broken in some cases. In fact, counterexamples of (5) has been reported by Refs. [19, 20] for the case $\ell_{\text{AdS}} \to \infty$. Though the original idea of Penrose’s is invalid, the logic of holographic argument proposed by Ref. [18] still works. If such holographic argument was really true, the minimal area $A$ would be bounded by the event horizon area of Reissner-Nordström (RN) black holes if fixing the total mass $M$ and charge $Q$

$$A(M, Q) \leq A_{\text{RN}}(M, Q) .$$

(6)

This is a different generalization of Penrose inequality and was holographically argued to be true by Ref. [18].

Although the recent holographic argument for Penrose inequality does not need to assume the cosmic censorship, it requires matters to satisfy the null energy condition in the bulk[18]. However, in recent years people has proved Penrose inequality in some certain cases, including the asymptotically AdS spacetimes, which requires that dominant or weak energy condition [11, 21, 22]. Both dominant and weak energy conditions are stronger than the null energy condition. This forms a gap between the holographic argument and current geometric proofs on Penrose inequality in asymptotically AdS spacetime. If matters decay rapidly enough near the AdS boundary, the total mass $M$ and minimal area $A$ are geometrically well defined. The Penrose inequality in this case becomes a purely geometric inequality. Since the argument of Ref. [18] uses the conjecture of holography, if its conclusion is true, then it is necessary to ask: is it possible to find a purely geometric proof for AdS Penrose inequality under null energy condition without referring to the unproved conjecture of holographic principle?

According to holographic argument for charged black holes, RN black holes will have the maximum entropy so that the charged generalization is given by Eq. (6). If setting the AdS radius $\ell_{\text{AdS}}$ to infinity, the charged generalization (6) will following generalized Penrose inequality in asymptotically flat spacetime

$$\left( \frac{A}{16\pi} \right)^{1/2} \leq \frac{1}{2} \left[ M + \sqrt{M^2 - Q^2} \right] .$$

(7)

However, there are also counterexamples [19, 20] for such charged generalization (7). So we believe that the charged generalization (6) from holographic argument is not general true under null energy condition. What is the correct generalization for charged case? In addition, such counterexamples also give us enough motivation and necessity to seek purely geometric checks for the conclusions obtained from holographic principle.

So far we have assumed that matters decay rapidly enough near the AdS boundary and black holes in fact are asymptotically Schwarzschild-AdS black holes. Both total mass $M$ and the area of minimal surface $A$ are determined by the bulk’s geometry in this case [24]. If the matters do not decay rapidly enough, the spacetime may be still asymptotically AdS but not asymptotically Schwarzschild-AdS. In this case, the situation will be complicated. For instance, the existence of matter on AdS boundary will contribute to the total mass for asymptotically AdS black holes according to holographic renormalization, see e.g. Refs. [25, 26]. Since total mass obtained from holographic renormalization is not determined by bulk geometry, the null energy condition in the bulk can still guarantee the Penrose inequality?

This paper aims to answer above questions (at least partially). For static asymptotically Schwarzschild-AdS black holes, we prove that the null energy condition can guarantee the Penrose inequality only for planar/spherical horizon geometry cases, but to guarantee the inequality for hyperbolically symmetric case we have to assume weak energy condition. A concrete counterexample is given to show the Penrose inequality is broken for hyperbolic horizon geometry under null energy condition. This implies that the conclusions of

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1 Penrose inequality focus on black hole’s horizon and its exterior. More precisely, the matter should satisfy the null energy condition in the black hole’s exterior.

2 "Asymptotically Schwarzschild-AdS" is stronger than "asymptotically AdS", see Ref. [23]

3 In the following proof, we will not distinguish between ADM mass and Bondi mass since the ADM mass is equal to Bondi mass for static asymptotically Schwarzschild-AdS black holes.

4 For stationary solutions, the apparent horizon will coincide with the event horizon. So the minimal area to enclose apparent horizon $A$ is just the area of event horizon.
Refs. [17, 18] implicitly depend on the topology of event horizon, though more detailed reason is still unclear for us. For charged black hole, as we have explained that the naive generalization (5) and holographic version (6) are both incorrect. We then propose two kinds of charged generalization of Penrose inequality. As we mentioned before, the total mass is very subtle for asymptotically AdS black holes. This paper follows the standard holographic renormalization procedure [27, 28] and obtain the holographic mass as the total mass. Without loss of generality, we construct an asymptotically AdS black hole coupled to a scalar field to check the Penrose inequality in holography. When the source of scalar field is nonzero, the spacetime is asymptotically AdS but not asymptotically Schwarzschild-AdS. In this case, we find that the null energy condition is not enough to guarantee the Penrose inequality. Exactly speaking, whether the inequality holds or not in this case depends on what kind of quantization scheme we employ.

The organization of this paper is as follows. In section II, given the metric ansatz for static $(d+1)$-dimensional asymptotically Schwarzschild-AdS black holes, with null energy condition, we find the null energy condition guarantee the Penrose inequality only for spherically and planar symmetric black holes. In section III, we propose two types of charged generalization for Penrose inequality and prove them in static planar and spherically symmetric cases. In section IV, we construct a 4-dimensional Einstein-scalar gravity and numerically check the Penrose inequality with two different quantization schemes for scalar field sector.

II. PROOF OF PENROSE INEQUALITY WITH NULL ENERGY CONDITION

In this section, we will proposed a general version of Penrose inequality in $(d+1)$-dimensional asymptotically AdS spacetime with null energy condition and prove it under spherically/planar/hyperbolic symmetric cases. But before the general proof, we firstly revisit the Penrose inequality (1)–(2) in 4-dimensional spacetime and give some comments about them. In this paper, we will consider three kinds of topologies for event horizon, which are denoted by the parameter $k$. For asymptotically flat spacetime, only the spherical topology ($k = +1$) can exist in a black hole solution. However, in asymptotically AdS spacetime, the black holes have three topologies for the cross-section of its event horizon, i.e. the spherical ($k = 1$)/planar ($k = 0$)/hyperbolic ($k = -1$) topologies. The Penrose inequality then should be generalized into

$$M \geq \left( \frac{A}{16\pi} \right)^{\frac{1}{d}} k + \frac{1}{2} \ell_{\text{AdS}}^2 \left( \frac{A}{4\pi} \right)^{\frac{2}{d}} ,$$

As pointed by Ref. [11, there is nonzero Casimir energy [12] for spherical and hyperbolic topologies. Here we have absorbed the Casimir energy into the definition of total mass to simplify our notations. For the hyperbolic and planar geometries, the volume of cross section will be infinite, which will lead the inequality (8) meaningless. However, for a static asymptotically AdS spacetime, we can always choose coordinate gauge so that the leading term of metric near the AdS boundary has following form

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\Sigma^2_{k,d-1} ,$$

Here $d\Sigma_{k,d-1}$ is the transverse metric of unit sphere/planar/hyperbolic defined by Eq. (15). Denote $\Omega_{k,d-1} := \int d\Sigma_{k,d-1}$. For event horizon, we can always define an “effective” radius $r_h$ according to equation

$$A = \Omega_{k,d-1} r_h^{d-1} .$$

Similarly, we can introduce a “mass density parameter” $f_0$ according to

$$M = \frac{(d-1)\Omega_{k,d-1}}{16\pi} f_0^d$$

Although both the total mass $M$ and area $A$ are infinite in planar or hyperbolic cases, we should notice that the mass parameter $f_0^d$ and the horizon radius $r_h$ are always finite. The Penrose inequality (8) then can be reorganized in term of following inequality (See, e.g. [11, 21, 22])

$$\frac{1}{\ell_{\text{AdS}}^2} + \frac{k}{r_h^2} - \frac{f_0^d}{r_h^d} \leq 0$$

for general dimensional and all three different topologies of horizon. We then propose following conjecture for static asymptotically Schwarzschild-AdS black holes

**Conjecture 1.** For a static asymptotically Schwarzschild-AdS black hole, if (1) Einstein equation is satisfied, (2) Matter’s energy momentum tensor $T_{\mu\nu}$ satisfies null energy condition, and (3) the cross section of event horizon has spherical or planar topology, then the inequality (12) is true and the saturation appears only if the exterior of event horizon is Schwarzschild-AdS.

Note that the parameters $f_0$ and $r_h$ in asymptotically Schwarzschild-AdS black hole will be determined completely by the bulk geometry, so we expect there should be a geometrical proof without referring to the conjecture of AdS/CFT. If we recall the inequality (12), the conjecture 1 then implies

$$f_0^d \geq r_h^d \left( \frac{1}{\ell_{\text{AdS}}^2} + \frac{k}{r_h^2} \right) .$$

Combining with the definition for total mass (11), we can see that the Penrose inequality is the stronger version of the positive energy theorem, if the cross-section of event horizon has planar or spherical topology. This is interesting and seemingly surprising since the local energy density could be negative under null energy condition. In
following, we will give such geometrical proof in spherically/planar symmetric static cases. We will also give a detailed counterexample to show that inequality (12) can be broken in hyperbolic topology if we impose only null energy condition.

A. Einstein equation in spherically/planar/hyperbolically symmetric cases

For spherical/planar/hyperbolic symmetric geometries, the metric ansatz for asymptotically (\(d + 1\))-dimensional black holes is given by

\[
ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2_{k,d-1},
\]

(14)

here \(k = 0, \pm 1\) represents different symmetric cases

\[
d\Sigma^2_{k,d-1} = \begin{cases} d\Omega^2_{d-1} = d\theta^2 + \sin^2 \theta d\Omega^2_{d-2}, & \text{for } k = +1 \\ d\ell^2_{d-1} = \sum_{i=1}^{d-1} dx_i^2, & \text{for } k = 0 \\ d\Xi^2_{d-1} = d\theta^2 + \sinh^2 \theta d\Omega^2_{d-2}, & \text{for } k = -1 \end{cases}
\]

(15)

There is an event horizon\(^5\) for black hole at \(r = r_h\) which is the largest root of \(f(r) = 0\). Outermost horizon condition will lead to \(f'(r_h) \geq 0\), i.e. the derivative of blackening factor \(f(r)\) with respect to \(r\) at the horizon \(r_h\) is nonnegative. From the perspective of thermal ensemble, the temperature of black holes is nonnegative because the temperature is given by

\[
T = \frac{e^{-\chi(r_h)/2}}{4\pi} f'(r_h) \geq 0.
\]

(16)

In the following proof and following sections, we will set \(\ell_{\mathrm{AdS}} = 1\) for convenience. The energy momentum tensor \(T^\mu_\nu\) has a form

\[
8\pi T^\mu_\nu = \text{diag} \{ -\rho(r), p_r(r), p_T(r), p_T(r), \cdots, p_T(r) \}.
\]

(17)

The Einstein equation shows following three independent equations

\[
f' = \frac{d-2}{r} - \frac{2}{d-1} \dot{r} - \frac{(d-2)f}{r},
\]

(18a)

\[
\chi' = -\frac{2r}{(d-1)f} (\rho + p_r),
\]

(18b)

\[
p'_r = \frac{(d-2)\dot{r} + 2(d-1)\dot{p}_r - d\dot{p}_T}{2r} - \frac{\dot{p}_r + \dot{\rho}}{(d-1)rf}(\dot{r}^2 + (d-2)(d-1)k/2),
\]

(18c)

here

\[
\dot{\rho} = \rho - \frac{d(d-1)}{2}, \quad \dot{p}_r = p_r + \frac{d(d-1)}{2}, \quad \dot{p}_T = p_T + \frac{d(d-1)}{2}.
\]

(19)

the extra factor \(\frac{d(d-1)}{2}\) is contributed by the cosmological constant term \(\Lambda g_{\mu\nu}\) in Einstein equation. As is known to all, in order to match the asymptotically AdS boundary condition, the two functions \(f(r)\) and \(\chi(r)\) must follow the asymptotically behaviors as

\[
limit_{r \to \infty} \frac{f(r)}{r^2} = 1, \quad \lim_{r \to \infty} \frac{\chi(r)}{r^2} = 0.
\]

(20)

But the "asymptotically AdS" is not enough for our proof. Moreover, we need the matter decays rapidly near the AdS boundary \(r \to \infty\), so that the functions \(f(r)\) and \(\chi(r)\) satisfy following asymptotically Schwarzschild-AdS boundary conditions

\[
\lim_{r \to \infty} \frac{f(r)}{r^2} = 1 + \frac{k}{r^2} - f^d_0/r^d + \cdots, \quad \lim_{r \to \infty} \frac{\chi(r)}{r^2} = \chi_0/r^{d+\alpha} + \cdots, \quad \alpha > 0.
\]

(21)

By virtue of the asymptotic behavior for \(f(r)\) and \(\chi(r)\), we can define a "\(\text{quasi-local mass}\)" for an equal-r surface

\[
m(r) = \frac{k}{d} \left[ r^{d-2} + X(r) \right] + \frac{r^{d+1}e^{\chi/2}}{2d} \left( \frac{f - e^{-\chi}}{r^2} \right),
\]

(22)

One can check that \(\frac{\chi'}{\chi} \leq 0\), \(m' \geq 0\) and vice versa. The boundary condition (21) also implies \(\chi(r) \geq 0\) and \(X(r) \geq 0\) outside the horizon, which we will use in the following proof. Above all, \(m' \geq 0\) and \(m(\infty) = f^d_0/2\) imply that

\[
m(r) \leq m(\infty) = f^d_0/2.
\]

(23)

At the horizon we have \(f'(r_h) \geq 0\), so Eq. (22) implies

\[
m(r_h) \geq \frac{k}{d} \left[ r_h^{d-2} + X(r_h) \right].
\]

(24)

We can use energy density \(\rho\) and transverse pressure density \(p_T\) to express \(m'(r)\). One can verify

\[
m'(r) = \frac{r^{d-1}e^{-\chi/2}}{d} (\dot{\rho} + \dot{p}_T) = \frac{r^{d-1}e^{-\chi/2}}{d} (\rho + p_T).
\]

(25)

If the matter satisfies null energy condition, combining with Eqs. (18b) and (25), we can directly conclude that

\[
\chi' \leq 0, \quad m' \geq 0
\]

(26)

and vice versa. The boundary condition (21) also implies \(\chi(r) \geq 0\) and \(X(r) \geq 0\) outside the horizon, which we will use in the following proof. Above all, \(m' \geq 0\) and \(m(\infty) = f^d_0/2\) imply that

\[
m(r) \leq m(\infty) = f^d_0/2.
\]

(27)

We can conclude that the "\(\text{quasi-local mass}\)" \(m(r)\) is a monotonically increasing function outside the black hole, which takes the minimum value \(\frac{k}{d} \left[ r_h^{d-2} + X(r_h) \right]\) at the horizon and the maximum value \(f^d_0/2\) on the AdS boundary. Let’s discuss three different horizon topologies, respectively.

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\(^5\) For a static solution, the outermost horizon will coincide with the event horizon [29, 30].

\(^6\) For instance \(k = +1, d = 3\), the total mass \(M\) is equal to \(m(\infty) = f^d_0/2\). This is why we call \(m(r)\) "\(\text{quasi-local mass}\)."
B. Planar Geometry

For planar horizon case \( k = 0 \), the expression of \( m(r) \) is reduced to
\[
m(r) = \frac{r^{d+1}e^{\chi/2}}{2d} \left( \frac{f e^{-\chi}}{r^2} \right)'.
\]
(29)
Solving \( f e^{-\chi}/r^2 \) in terms of \( m(r) \) and \( \chi(r) \), we obtain
\[
\frac{f e^{-\chi}}{r^2} = 2d \int_{r_h}^r \frac{m(x)e^{-\chi(x)/2}}{x^{d+1}} dx,
\]
(30)
When \( r \to \infty \), boundary condition indicates that
\[
1 = 2d \int_{r_h}^\infty \frac{m(x)e^{-\chi(x)/2}}{x^{d+1}} dx,
\]
(31)
If the null energy condition is satisfied, then we have \( 0 \leq e^{-\chi(r)/2} \leq 1 \). Combined with \( 0 \leq m(r) \leq f_0^d/2 \) in planar case, the above equation becomes an inequality
\[
1 = 2d \int_{r_h}^\infty \frac{m(x)e^{-\chi(x)/2}}{x^{d+1}} dx \leq d \int_{r_h}^\infty \frac{f_0^d}{x^{d+1}} dx = f_0^d/r_h^d,
\]
(32)
which is the Penrose inequality (12) for the planar symmetric case.

The inequality is saturated only if all unequal signs take the equal sign, which means \( \chi = 0 \) and \( m = f_0^d/2 \). From Eq. (30), we can solve \( f(r) \) in terms of \( \chi(r) \) and \( m(r) \) which both are constant in this case. The solution is
\[
f(r) = r^2(1 - f_0^d/r^d), \quad \rho = p_r = p_T = 0,
\]
(33)
which is exactly the metric of Schwarzschild-AdS black hole. Thus, we conclude that: for planar symmetric static asymptotically Schwarzschild-AdS back holes, if the null energy condition is satisfied, then Penrose inequality is true and its saturation appears only if the black hole is Schwarzschild-AdS black hole.

C. Spherical Geometry

In this subsection, we will consider spherical symmetric \( (k = 1) \) case. We still firstly solve function \( f(r) \) in terms of \( m(r) \) and \( \chi(r) \) and obtain
\[
\frac{f(r)e^{-\chi(r)}}{r^2} = \frac{2}{f_h^d} \int_{r_h}^r \frac{[dm(y) - X(y) - y^{d-2}] e^{-\chi(y)/2}}{y^{d+1}} dy,
\]
(34)
When \( r \) evolves to \( \infty \), the left hand of Eq. (34) becomes unit one
\[
1 = 2 \int_{r_h}^\infty \frac{[dm(x) - X(x) - x^{d-2}] e^{-\chi(x)/2}}{x^{d+1}} dx
\]
(35)
Similar to planar symmetric case, we will focus on the integral on the right hand of Eq. (34). From Eq. (28), we find \( m(r_h) \geq 0 \) because \( X(r) \geq 0 \). Combining it with \( m(r) \leq m(\infty) = f_0^d/2 \), then we obtain
\[
1 \leq 2 \int_{r_h}^\infty \frac{(df_0^d/2 - x^{d-2}) e^{-\chi(x)/2}}{x^{d+1}} dx.
\]
(36)
Let \( r_0 \) to be the root of \( df_0^d/2 - x^{d-2} = 0 \), the condition \( \chi' \leq 0 \) insures
\[
(df_0^d/2 - x^{d-2}) \left[ e^{-\chi(r)/2} - e^{-\chi(r_0)/2} \right] \leq 0
\]
(37)
which leads to
\[
\int_{r_h}^\infty \frac{(df_0^d/2 - x^{d-2}) e^{-\chi(r)/2}}{x^{d+1}} dx \leq e^{-\chi(r_0)/2} \int_{r_h}^\infty \frac{df_0^d/2 - x^{d-2}}{x^{d+1}} dx.
\]
(38)
Combining this result in Eq. (36), then yields
\[
1 \leq e^{\chi(r)/2} \int_{r_h}^\infty \frac{df_0^d/2 - x^{d-2}}{x^{d+1}} dx = f_0^d/r_h^d - \frac{1}{r_h^d},
\]
(39)
and Penrose inequality (12) follows. The inequality is saturated only if \( \chi = 0 \) and \( m = f_0^d/2 \), which leads to
\[
f(r) = r^2(1 + 1/r^2 - f_0^d/r^d), \quad \rho = p_r = p_T = 0
\]
(40)
Thus, we conclude that: static asymptotically Schwarzschild-AdS back hole with spherical symmetry, if null energy condition is satisfied, then the Penrose inequality is true and its saturation appears only if the black hole is Schwarzschild-AdS black hole. We then have proved the Conjecture 1 in the spherically and planar symmetrical cases.

D. Broken Case: Hyperbolic Geometry

For hyperbolic symmetric case, the null energy condition can not guarantee Penrose inequality (12). To verify this conclusion, we will give a concrete counterexample\(^7\). We note that Eq. (34) becomes
\[
\frac{f(r)e^{-\chi(r)}}{r^2} = 2 \int_{r_h}^r \frac{[dm(y) + X(y) + y^{d-2}] e^{-\chi(y)/2}}{y^{d+1}} dy.
\]
(41)
Now let us take \( r_h = 1, d = 3 \) and
\[
e^{-\chi/2} = \frac{4 + \tanh(r - 5)}{5}, \quad m(r) = f_0^d/2 \approx -0.15105, \quad (42)
\]
\(^7\) In view of [31], the validity of the Penrose inequality in general hyperbolic cases seems to be rather unlikely.
\(^8\) For hyperbolic black holes, the mass parameter \( f_0^d \) can take the negative value.
we can get the expression of $X(r)$

$$X(r) = \frac{1}{5} \left[ \ln \cosh(r - 5) - r + \ln 2 \right] + 1,$$

(43)

and two functions $f(r)e^{-\chi(r)/r^2}$ and $\chi(r)$ which are shown in Fig. 1. Since $\chi' < 0$ and $m' = 0$, the null energy condition is guaranteed. However, let us check the sign of $\left[ 1 - \frac{1}{r_h} - \frac{f_d}{r_h^2} \right]$, we find that

$$1 - \frac{1}{r_h^2} - \frac{f_d}{r_h^2} = -\frac{f_d}{r_h^2} > 0.$$

(44)

So we can conclude that the Penrose inequality (12) is broken when $k = -1$.

Null energy condition does not require the energy density $\rho$ nonnegative but the sum of energy density and pressure density nonnegative. If matter satisfies the weak energy condition, it can be proved that the inequality is true and its saturation appears only in the Schwarzschild-AdS black hole at least for the maximally symmetric and static cases. See appendix A for details.

### III. CHARGED BLACK HOLES

Before we discuss the charged generalization of AdS Penrose inequality, we first consider the asymptotically charged flat case, which can be regarded as the limit of $\ell_{\text{AdS}} \to \infty$. With the total mass $M$ and charge $Q$ as initial data, there are two naive charged generalizations in 4-dimensional spacetime

$$M \geq \sqrt{\frac{A}{16\pi}} + Q^2 \sqrt{\frac{\pi}{A}},$$

(45)

and a weaker version

$$\left( \frac{A}{16\pi} \right)^{1/2} \leq \frac{1}{2} \left[ M + \sqrt{M^2 - Q^2} \right].$$

(46)

here the saturation appears only if the black holes are RN black holes. Hold on, when we introduce the charge $Q$ as a initial data as well as $M$, the definition of $Q$ is vague. For general charged black holes, $Q(r)$ is defined as

$$\frac{1}{2\Omega_{k,d-1}} \int_{S_r} F_{\mu\nu} dS^{\mu\nu} = Q(r),$$

(47)

$S_r$ is an equal-$r$ surface and $\Omega_{k,d-1}$ denotes the dimensionless volume of the relevant horizon geometry (15). For RN black holes, charge $Q(r)$ is a constant outside the black holes. This is because there is no charge outside RN black holes. For general cases, $Q(r)$ is dependent on $r$, because matters outside the black hole usually also carry charge. If interpreting $Q^2$ in inequality (46) as the square of total charge, i.e. $Q^2(\infty)$, we shall find that the inequality (46) is not always true. See Refs. [19, 20] for a counterexample. This reminds us that the charged generalization for Penrose inequality needs to be treated carefully and naive generalizations (5) and (6) are both incorrect in general. In this section, we will conjecture two different types of charged generalization for Penrose inequality.

#### A. The First Type of Generalization

We will separate the energy momentum tensor $T_{\mu\nu}$ into two parts,

$$T_{\mu\nu} = T^{(M)}_{\mu\nu} + T^{(a)}_{\mu\nu}$$

(48)

where the energy momentum tensor $T^{(M)}_{\mu\nu}$ for Maxwell field is defined as

$$T^{(M)}_{\mu\nu} = 2 \left( F_{\mu} \sigma F_{\nu} \sigma - \frac{g_{\mu\nu}}{4} F_{\sigma\tau} F^{\sigma\tau} \right),$$

(49)

and $T^{(a)}_{\mu\nu}$ stands for other parts in $T_{\mu\nu}$. To present our generalized Penrose inequality in static charged black holes, we need a little more preparation.

Denote $\gamma$ to be a co-dimensional-2 spacelike surface. Denote that $l^\mu$ to be future-directed infalling null geodesic vectors field which are normal to $\gamma$ and satisfies $\xi^\mu l_\mu = -1$, where $\xi^\mu$ is the Killing vector standing for static geometry and normalized at infinity by $\xi^\mu \xi_\mu = -1$. We denote the expansion $\theta(l)$ for $l^\mu$. Then we define $Q^2_m$ to be

$$Q^2_m = \inf_S \left[ \frac{(1 - d)Q^2(\gamma)S(\gamma)}{r_s \int_\gamma \theta(l) dS} \right].$$

(50)

$r_s$ is an "effective" radius which satisfies

$$r_s^{-d-1}\Omega_{k,d-1} = S(\gamma),$$

(51)

here $S(\gamma)$ denotes the area of $\gamma$. We now propose the first type of charged generalization:

![Graph showing values of $f(r)e^{-\chi(r)/r^2}$ and $\chi(r)$](image-url)

**Figure 1.** The values of $f(r)e^{-\chi(r)/r^2}$ and $\chi(r)$ shown in Fig. 1. We can see that $f(r)e^{-\chi(r)/r^2}$ and $\chi(r)$ have the expected asymptotic behavior, but the inequality is still broken.
Conjecture 2. For an asymptotically Schwarzschild-AdS black hole, if (1) Einstein equation is satisfied, (2) $\mathcal{I}^{(o)}_{\mu\nu}$ satisfies null energy condition, and (3) the cross section of event horizon has spherical or planar topology, then the charged generalization of Penrose inequality reads

$$1 + \frac{k}{r_h} - \frac{f_{o}^{d}}{r_h^d} + \frac{2Q_m}{(d-1)(d-2)r_h^{2d-2}} \leq 0 .$$

(52)

The saturation appears only in RN black holes.

This one is very similar to the generalization proposed by Ref. [11], however, $Q_m$ here in general will be different from the total charge.

To support our this generalization, we will give the proof for spherically and planar symmetric cases. Under coordinates gauge (14), the expansion $\theta_{(i)}$ for $l^\mu$ is given by

$$\theta_{(i)} = (1-d)\frac{e^{\chi/2}}{r}$$

(53)

and the Maxwell field strength tensor has a form

$$F_{\mu\nu} = - \frac{Q(r)e^{\chi/2}}{r^{d-1}} (dt)_\mu \wedge (dr)_\nu .$$

(54)

The nonvanishing components of surface element $dS_{\mu\nu}$ read

$$dS_{01} = - dS_{10} = e^{-\chi/2}r^{d-1}d\Sigma_{k,d-1} .$$

Substituting this result into the definition (50), we can obtain the expression of $Q_m^2$

$$Q_m^2 = \min \left[ Q^2(r)e^{-\chi/2}, \right] \text{, for } r \geq r_h$$

(56)

In order to prove this inequality (52), the key step is to separate the energy density $\rho$ and pressure density $\{p_r, pr\}$ into two parts respectively

$$\rho = \frac{Q^2}{r^{2d-2}} + \rho^{(o)}, \quad p_r = - \frac{Q^2}{r^{2d-2}} + p_r^{(o)}, \quad pr = \frac{Q^2}{r^{2d-2}} + p_T^{(o)} .$$

(57)

Since we require that $\rho^{(o)} + p_r^{(o)} \geq 0$ and $\rho^{(o)} + p_T^{(o)} \geq 0$, then we obtain that

$$\rho + p_r \geq 0 , \quad \rho + pr \geq \frac{2Q^2}{r^{2d-2}} .$$

(58)

To prove the inequality (52), we introduce a new "quasi-local mass" $\tilde{m}(r)$

$$\tilde{m}(r) = m(r) + \int_r^\infty \frac{2Q^2e^{-\chi/2}}{dy^{d-1}} dy ,$$

(59)

so that the derivative of $\tilde{m}(r)$ is always nonnegative

$$\tilde{m}'(r) = \frac{r^{d-1}e^{-\chi/2}}{d} \left( \rho + pr - \frac{2Q^2}{r^{2d-2}} \right) \geq 0 .$$

(60)

and when $r \to \infty$

$$\tilde{m}(\infty) = m(\infty) = df_0^d/2 .$$

(61)

This implies

$$\tilde{m}(r) \leq df_0^d/2 .$$

(62)

Recall the definition of old "quasi-local mass" (22), we substitute the expression of $m(r)$ into our new "quasi-local mass" (59)

$$\tilde{m}(r) = \frac{k}{d} \left[ r^{d-2} + X(r) \right] + \frac{r^{d+1}e^{\chi/2}}{2d} \left( \frac{f_e^{-\chi}}{r^2} \right)'$$

$$+ \int_r^\infty \frac{2Q^2e^{-\chi/2}}{dy^{d-1}} dy ,$$

(63)

Here $k = 0$ and 1, which stands for planar and spherically symmetry respectively. Following the standard procedure in section II, we should solve $fe^{-\chi/r^2}$ in terms of $m(r)$ and $\chi(r)$

$$2\int_{r_h}^r \left[ d\tilde{m}(r) - \int_x^\infty \frac{2Q^2e^{-\chi/2}}{y^{d-1}} dy - kX(x) \right]$$

$$- kx^{d-2}e^{-\chi(x)/2}x^{-(d+1)} dx = \frac{f(r)e^{-\chi(r)}}{r^2} ,$$

(64)

Recall that $f(r_h)' \geq 0$ because the surface of $r = r_h$ is outermost horizon, so we can obtain

$$\left[ d\tilde{m}(r_h) - \int_{r_h}^\infty \frac{2Q^2e^{-\chi/2}}{y^{d+1}} dy - kX(r_h) - kr^{d-2}_h \right] e^{-\chi(r_h)/2}$$

$$\geq 0 .$$

(65)

Due to $X(r) \geq 0$ and $e^{-\chi(r_h)/2} \geq 0$, the above inequality (65) becomes

$$\frac{d\tilde{m}(r_h) - \int_{r_h}^\infty \frac{2Q^2e^{-\chi/2}}{y^{d+1}} dy - kr^{d-2}_h}{r^{d+1}_h} \geq 0 .$$

(66)

We shall see that, the above inequality which is defined at the horizon $r_h$ plays a decisive role in the following proof. In particular, the inequality (66) restrict the evaluation relationship between total mass $M$ and $Q_m^2$. For the convenience of our proof, we define an auxiliary function $W(r)$

$$W(r) = df_0^d/2 - \frac{2Q_m}{(d-2)r^{2d-2}} - kr^{d-2} .$$

(67)

Combine it with the Eqs. (56), (62) and (66) and we will obtain

$$\frac{W(r)}{r^{d+1}} \geq \frac{d\tilde{m}(r) - \int_{r_h}^\infty \frac{2Q^2e^{-\chi/2}}{y^{d+1}} dy - kr^{d-2}_h}{r^{d+1}_h} \geq 0 .$$

(68)

Particularly, at horizon $r = r_h$ we have

$$\frac{W(r_h)}{r^{d+1}} \geq \frac{d\tilde{m}(r_h) - \int_{r_h}^\infty \frac{2Q^2e^{-\chi/2}}{y^{d+1}} dy - kr^{d-2}_h}{r^{d+1}_h} \geq 0 .$$

(69)
due to inequality (66). Thus the horizon \( r_h \) is limited by the value of function \( W(r) \).

Back to Eq. (64), the left hand of Eq. (64) will become unit one when \( r \) evolves to \( \infty \), which is the boundary condition of asymptotically AdS spacetime

\[
1 = 2 \int_{r_h}^{\infty} \left[ d\bar{n}(x) - \int_{x}^{\infty} \frac{2Q^2 e^{-\frac{r}{2}}}{y^d-1} dy - kX(x) - kx^{d-2} e^{-\chi(x)/2} x^{-(d+1)} dx \right].
\]

(70)

Through Eqs. (28) (59), we find \( \bar{n}(r_h) \geq 0 \). Combining it with \( \bar{n}(r) = \frac{f_0^d}{2} \) and \( X(r) \geq 0 \), we then obtain

\[
1 \leq 2 \int_{r_h}^{\infty} \frac{df_0^d/2 - \int_{x}^{\infty} \frac{2Q^2 e^{-\frac{r}{2}}}{y^d-1} dy - kx^{d-2} e^{-\chi(x)/2}}{x^{d+1}} dx.
\]

(71)

Using the inequality (68), we see that the above inequality becomes

\[
1 \leq 2 \int_{r_h}^{\infty} \frac{W(x)e^{-\chi(x)/2}}{x^{d+1}} dx.
\]

(72)

Above inequality implies that the maximum value of \( W(r) \) must be positive

\[
\max W(r) = \max \left[ \frac{df_0^d/2 - \frac{2Q^2}{(d-2)y^d-2} - kx^{d-2}}{x^{d+1}} \right] > 0,
\]

(73)

otherwise the above integration Eq. (72) will be negative.

For \( k = 1 \), it’s obvious that when \( x^{d-2} = r_\Delta^{d-2} = \sqrt{\frac{2Q^2}{d-2}} \), \( W(r) \) takes the maximum value \( df_0^d/2 - 2\sqrt{\frac{2Q^2}{d-2}} \).

\[
\max W(r) = W(r_\Delta) = \frac{df_0^d}{2} - 2\sqrt{\frac{2Q^2_m}{d-2}} > 0.
\]

(74)

There are two points \( r_1, r_2 \) which are the roots of \( W(r) = 0 \),

\[
\begin{align*}
    r_1^{d-2} &= \frac{df_0^d}{4} - \sqrt{(\frac{df_0^d}{4})^2 - \frac{2Q^2_m}{d-2}}; \\
    r_2^{d-2} &= \frac{df_0^d}{4} + \sqrt{(\frac{df_0^d}{4})^2 - \frac{2Q^2_m}{d-2}}.
\end{align*}
\]

(75)

As we can see that \( r_1 \leq r_h < r_2 \) from Fig. 2 because \( W(r_h) \geq 0 \).

\[\text{FIG. 2. The schematic diagram of function } W(r) \text{ for } k = 1. \text{ The value of function } W(r) \text{ at the horizon } r_h \text{ must be nonnegative.}\]

We separate the interval \( [r_h, \infty) \) into two parts \( [r_h, r_2) \) and \( [r_2, \infty) \). Then there are two different situations.

- \( r_h \leq r < r_2 \)
  
  Then \( \chi' \leq 0 \) insures \( e^{-\chi(r)/2} - e^{-\chi(r_2)/2} \leq 0 \) and \( W(r) \geq 0 \). We then have

\[
W(r) \left[ e^{-\chi(r)/2} - e^{-\chi(r_2)/2} \right] \leq 0,
\]

(76)

- \( r_2 \leq r \)
  
  We see \( e^{-\chi(r)/2} - e^{-\chi(r_2)/2} \geq 0 \) and \( W(r) \leq 0 \), so still have inequality (76).

We see that inequality (76) is true for all \( r \in [r_h, \infty) \). This leads to

\[
1 \leq 2 \int_{r_h}^{\infty} \frac{W(r)e^{-\chi(r)/2}}{r^{d+1}} dr \leq 2e^{-\chi(r_2)/2} \int_{r_h}^{\infty} \frac{W(r)}{r^{d+1}} dr.
\]

(77)

Multiplying \( e^{-\chi(r_2)/2} \) into above inequality, we finally obtain

\[
1 \leq e^{\chi(r_2)/2} \leq 2 \int_{r_h}^{\infty} \frac{W(r)}{r^{d+1}} dx = \frac{f_0^d}{r_h^d} - \frac{1}{r_h^d} - \frac{2Q^2_m}{(d-1)(d-2)r_h^{2d-2}}.
\]

(78)

For \( k = 0 \), \( W(r) \) is a monotonically increasing function and so that

\[
\max W(r) = W(r \to \infty) = \frac{df_0^d}{2} > 0.
\]

(79)

There is a point \( r_0 \) satisfies \( W(r_0) = 0 \)

\[
\frac{df_0^d}{d(d-2)f_0^d}.
\]

(80)

Combining \( W(r_h) \geq 0 \) with \( W'(r) \geq 0 \) for planar case, we obtain that \( W(r) \geq 0 \) for \( r \geq r_h \). The inequality (72)
becomes
\[
1 \leq 2 \int_{r_h}^{\infty} \left( \frac{W(x)e^{-\chi(x)/2}}{x^{d+1}} \right) dx \leq 2 \int_{r_h}^{\infty} \frac{W(x) dx}{x^{d+1}}
\]
\[
= \frac{f_0^d}{r_h^d} - \frac{2Q_m^2}{(d-1)(d-2)r_h^{2d-2}}.
\]
(81)

Combined two symmetric cases, the first type of charged generalization (52) for Penrose inequality is followed.

Recall the whole proof, the saturation for charged inequality appears if \( \chi(r) = 0, \tilde{m}(r) = \int_0^d/2 \) and \( Q(r) = Q_m \). This implies charged density \( j(r) = 0 \) and the exterior is a RN black hole.

### B. The Second Type of Generalization

The inequality (52) is not expressed in term of the boundary quantities of asymptotically AdS spacetime. It would be more satisfactory if we could use boundary quantities to express the charged generalization of Penrose inequality since such version of Penrose inequality can be interrupted as the inequality of dual boundary field theory according to holography. In order to alleviate the contradiction between charged generalization for Penrose inequality and the basic idea from the holography, this paper will propose the second type of charged Penrose inequality. Considering the Maxwell equation with source \( J_\nu \)
\[
\nabla^\mu F_{\mu\nu} = 4\pi J_\nu .
\]
(82)

We introduce the gauge potential which satisfies
\[
F_{\mu\nu} = (dA)_{\mu\nu} .
\]
(83)

To find the generation in general case, we will separate the energy momentum tensor \( T_{\mu\nu}^{(o)} \) into following form
\[
T_{\mu\nu}^{(o)} = \tilde{T}_{\mu\nu}^{(o)} - 8\pi \left( g_{\mu\nu} J_\rho A^\rho - \frac{1}{2} J_\mu A_\nu - \frac{1}{2} J_\nu A_\mu \right) ,
\]
(84)

so the total energy momentum tensor reads
\[
T_{\mu\nu} = T_{\mu\nu}^{(M)} + \tilde{T}_{\mu\nu}^{(o)} - 8\pi \left( g_{\mu\nu} J_\rho A^\rho - \frac{1}{2} J_\mu A_\nu - \frac{1}{2} J_\nu A_\mu \right) ,
\]
(85)

In static case, the gauge potential \( A_\mu \) and charge density \( J_\mu \) have following form
\[
A_\mu \propto \xi_\mu , \quad J_\mu \propto \xi_\mu .
\]
(86)

here the potential \( \Phi := A_\mu \xi^\mu \) and charged density \( j := J_\mu \xi_\mu \). In holography, the potential \( \Phi_\infty \) which is defined on the AdS boundary\(^{10} \) is interpreted as chemical potential. Given the initial data \( f_0^d/2 \) and \( \Phi_\infty \) and take the gauge \( \Phi = 0 \) at event horizon, we have the following conjecture

\[\textbf{Conjecture 3.} \text{ For an asymptotically Schwarzschild-AdS black hole, if (1) Einstein equation is satisfied, (2) } \tilde{T}_{\mu\nu}^{(o)} \text{ and } T_{\mu\nu} \text{ are both satisfy null energy condition, (3) charge of black hole and charge density } j \text{ have same sign, and (4) the cross section of event horizon has spherical or planar topology, then the charged generation of Penrose inequality reads}
\]
\[
1 \leq \frac{f_0^d}{r_h^d} - \frac{k}{r_h^{2d-2}} - \frac{2(d-2)\Phi_\infty^2}{(d-1)r_h^2}.
\]

and the saturation appears only if the exterior of event horizon is AdS-RN.

Differing from naive generalizations (5) and (6), here we use chemical potential to replace the charge. Like previous sections, to support this conjecture we will prove it under the spherically or planar symmetric spacetime.

Under the same coordinates gauge (14), the gauge potential \( A_\mu \) and charge density \( J_\mu \) have following form
\[
A_\mu = \Phi(r)(dt)_\mu , \quad J_\mu = j(r)(dt)_\mu .
\]
(88)

According to Eq. (54), we obtain
\[
\frac{Q e^{-\chi/2}}{r^{d-1}} = \Phi'.
\]
(89)

The Maxwell equation reads
\[
\left( \Phi' e^{\chi/2} r^{d-1} \right)' = \frac{4\pi j e^{\chi/2} r^{d-1}}{f}.
\]
(90)

The energy density \( \rho \) and pressure density \( \{p_r, p_T\} \) now is replaced by
\[
\rho = \Phi'^2 e^\chi + \tilde{\rho}^{(o)} ,
\]
\[
p_r = -\Phi'^2 e^\chi + \frac{8\pi \Phi j e^\chi}{f} + \tilde{\rho}^{(o)} ,
\]
\[
p_T = \Phi'^2 e^\chi + \frac{8\pi \Phi j e^\chi}{f} + \tilde{\rho}^{(o)} .
\]
(91)

Null energy condition requires \( \rho + p_r \geq 0, \rho + p_T \geq 0 \), then we obtain
\[
\frac{8\pi \Phi j e^\chi}{f} + \tilde{\rho}^{(o)} + \tilde{p}^{(o)} \geq 0 , \quad 2\Phi'^2 + \frac{8\pi \Phi j e^\chi}{f} + \tilde{\rho}^{(o)} + \tilde{p}^{(o)} \geq 0 .
\]
(92)

A new "quasi-local mass" \( \tilde{m}(r) \) is defined as
\[
\tilde{m}(r) = m(r) - \frac{2}{d} (\Phi Q) ,
\]
(93)

so that the derivative of \( \tilde{m}(r) \) is
\[
\tilde{m}'(r) = m'(r) - \frac{2}{d} \left( \Phi' e^{\chi/2} r^{d-1} \right)' ,
\]
\[
= m'(r) - \frac{2}{d} e^{\chi/2} r^{d-1} \Phi'^2 - \frac{2}{d} \Phi \left( e^{\chi/2} r^{d-1} \Phi' \right)' ,
\]
(94)

\(^{10}\) In this paper, we abbreviate \( \Phi(\infty) \) as \( \Phi_\infty \).
Substituting Eq. (25), Eq. (92) and Eq. (90) into above equation and we obtain
\[ m'(r) = \frac{r^{d-1}e^{-\chi/2}}{d} \left( \bar{\rho}^{(o)} + \bar{p}_T^{(o)} \right) \geq 0 \]  
(95)

So we obtain \( m(\infty) = f_0^d/2 - \frac{2}{\eta}(\Phi_\infty Q_\infty) \). Let us rephrase the Maxwell equation
\[ Q' = \frac{4\pi j e^{\chi/2}r^{d-1}}{f} . \]  
(96)

Because \( Q(r_h) \) and \( j \) have same sign, we can take \( Q(r_h) \geq 0 \) and \( j \geq 0 \) without losing generality and so charge \( Q(r) \) will always nonnegative
\[ Q(r) \geq 0 . \]  
(97)

According to the relationship between \( Q \) and \( \Phi \) (89), we can obtain
\[ \Phi' \geq 0 . \]  
(98)

In holography, we generally set the value of potential \( \Phi \) at the horizon equal to zero as a gauge fixing
\[ \Phi(r_h) = 0 . \]  
(99)

After the gauge fixing, we can directly obtain the chemical potential \( \Phi_\infty \) on the AdS boundary. In order to compare with RN black holes, we rephrase the Maxwell equation
\[ (\Phi r^{d-1})' = \left( \frac{4\pi j}{f} - \frac{\chi'\Phi'}{2} \right) r^{d-1} . \]  
(100)

Let use denote \( \Phi_{RN}(r) \) to the gauge potential of RN black holes with same horizon and chemical potential, \( \text{i.e.} \ \Phi_{RN}(r) \) satisfies \( \Phi_{RN}(r_h) = 0, \Phi_{RN}(\infty) = \Phi_\infty \) and
\[ (\Phi_{RN} r^{d-1})' = 0 . \]  
(101)

We define \( \Delta \Phi = \Phi - \Phi_{RN} \),
\[ (\Delta \Phi r^{d-1})' = \left( \frac{4\pi j}{f} - \frac{\chi'\Phi'}{2} \right) r^{d-1} \geq 0 . \]  
(102)

Since \( j \geq 0, \chi' \leq 0 \) and \( \Phi' \leq 0 \). Then the "maximal principle" shows that the maximum of \( \Delta \Phi \) can only attain at endpoints. Thus, we have
\[ \max \Delta \Phi = \Delta \Phi(r_h) = \Delta \Phi(\infty) = 0 , \]  
(103)

So the relationship between the \( \Phi \) and \( \Phi_{RN} \) is
\[ 0 \leq \Phi \leq \Phi_{RN} . \]  
(104)

As usual, we solve function \( f(r) \) in terms of \( m(r) \) and \( \chi(r) \)
\[ f(r) e^{-\chi(r)} \]
\[ = 2 \int_{r_h}^{r} \left[ d\tilde{n}(x) + 2(\Phi Q) - kX(x) - kx^{d-2} \right] e^{-\chi(x)/2} dx , \]  
(105)
We substitute (110) into the above inequality
\[ 1 \leq 2 \int_{r_h}^{\infty} \frac{\bar{W}(x)e^{-\chi(x)/2}}{x^{d+1}} \, dx , \] (116)
which is very similar to the proof of the first type charged generalization. The only difference is the coefficient of \( 1/\mu^{d-2} \) in the auxiliary function. So we can take the same discussion in subsection III A and then obtain\(^{11}\)
\[ 1 \leq 2 \int_{r_h}^{\infty} \frac{\bar{W}(x)e^{-\chi(x)/2}}{x^{d+1}} \, dx \leq 2e^{-\chi(r_2)/2} \int_{r_h}^{\infty} \frac{\bar{W}(x)}{x^{d+1}} \, dx , \] (117)
which yields
\[ \frac{f_0^d}{r_h^d} - \frac{k}{r_h^2} - 2\Phi_\infty Q_\infty = 2 \int_{r_h}^{\infty} \frac{\bar{W}(x)}{x^{d+1}} \, dx \geq e^{\chi(r_2)/2} \geq 1 . \] (118)
The next step is to find the relation between \( \Phi_\infty \) and \( Q_\infty \). Near the infinity we have following asymptotic expansions for \( \Phi \),
\[ \Phi = \Phi_\infty - \frac{Q_\infty}{(d-2)r^{d-2}} + \ldots \] (119)
We have already known that \( \Phi \leq \Phi_{RN} \) for all \( r \geq r_h \), then Eq. (119) and (110) implies
\[ \frac{Q_\infty}{d-2} \geq \Phi r_{h}^{d-2} \] (120)
After a long journey, we finally reach the expected charged generalization
\[ 1 \leq \frac{f_0^d}{r_h^d} - \frac{k}{r_h^2} - 2\Phi_\infty Q_\infty \leq \frac{f_0^d}{r_h^d} - \frac{k}{r_h^2} - \frac{2(d-2)\Phi_\infty^2}{(d-1)r_h^2} . \] (121)
To saturate this inequality, we see from Eqs. (95), (113) and (115) that \( j(r) = \chi(r) = 0 \) and \( \tilde{m}(r) = f_0^d/2 \). The zero charge density and \( \chi = 0 \) shows that \( \Phi(r) = \Phi_{RN}(r) \). This leads to
\[ f(r) = 1 + \frac{k}{r^2} - \frac{f_0^d}{r_h^d} + \frac{2Q_\infty^2}{(d-1)(d-2)r^{2d-2}} \] (122)
and so the bulk geometry is a RN black hole.

IV. PENROSE INEQUALITY AND SCHEME OF QUANTIZATION

In above sections, we assume that the bulk geometry is asymptotically Schwarzschild AdS so that all the quantity, especially, the total mass is defined only by bulk geometry. In holography, when the dual field theory has nonzero external source, the total mass cannot be read directly from the bulk metric. Instead, we have to use the so called “holographic renormalization” approach to find the total mass. In this case, our above proofs is invalid. It is interesting to ask: can we still obtain the Penrose inequality in such case if the bulk matters satisfy the null energy condition? In this section, we will consider the asymptotically AdS black hole with scalar field \( \phi \) as a concrete example.

A. The Model

For \((d+1)\)-dimensional Einstein-scalar gravity, the theory’s action reads
\[ S = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} \left[ R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right] . \] (123)
Considering the static asymptotically AdS black hole with spherical/planar/hyperbolic horizon geometry, the ansatz is same as Eq. (14)
\[ ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{d-1}^2 . \] (124)
In order to satisfy asymptotically AdS boundary condition, the function \( f(r), \chi(r) \) must satisfy following conditions at AdS boundary \( r \to \infty \):
\[ f(r) = r^2 + \ldots , \quad \chi(r) = \chi_0/r^\alpha + \ldots \quad \alpha > 0 . \] (125)
If \( \phi(r) = 0 \) at any \( r \), the scalar potential \( V(\phi) \) will return to \( -d(d-1) \) so that the theory (123) is pure AdS gravity. Without loss of generality, assuming \( \phi(r \to \infty) \to 0 \), we choose the potential function as
\[ V(\phi) = -d(d-1) - \frac{1}{2} m^2 \phi^2 + \mathcal{O}(\phi^4) \] (126)
near the boundary.\(^{12}\) The parameter \( m \) is the mass of the scalar field. In holography, the mass-squared of the scalar field can be negative, but above the Breitenlohner-Freedman bound \( m^2_{BF} \)\(^{13}\)
\[ m^2 > m^2_{BF} = \frac{d^2}{4} . \] (127)
According to the action (123), equations of motion are following as:
\[ \nabla_\mu \nabla^\mu \phi - \partial_\phi V = 0 , \] (128)
\(^{11}\) Like the first type of charged, if \( \max W(r) \leq 0 \), the inequality will be broken. One can verify the mass parameter \( f_0^d \) have a inequality relation:
\[ df_0^d/2 - 2\sqrt{2\Phi_\infty Q_\infty r_h^{d-2}} \geq 0 \] for \( k = 1 \) and \( d_0^d/2 \geq 0 \) for \( k = 0 \).
\(^{12}\) Recall we have taken the AdS radius \( \ell_{AdS} \) equal to unit in the opening of this paper.
\(^{13}\) It was first derived in Refs. [32, 33]. Loosely speaking, the negative mass-squared below the Breitenlohner-Freedman bound \( m^2 < m^2_{BF} \) will lead to instability.
\[
\mathcal{R}_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} = \frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{1}{2} \left( -\frac{1}{2} \nabla_{\rho} \nabla^\rho \phi - V(\phi) \right) g_{\mu\nu}.
\]

(129)

One can check that the scalar hairy black hole solution satisfy the null energy condition. Substituting the ansatz (124) into the above equations, we can obtain

\[
\phi'' + \left( \frac{f' - \delta + \frac{d-1}{r}}{f} \right) \phi' - \frac{1}{f} \partial_\phi V = 0,
\]

(130a)

\[
\frac{2}{r} \frac{f'}{f} - \frac{\chi'}{r} + \frac{2}{d-1} \frac{V}{f} + \frac{2(d-2)(f-k)}{r^2 f} = 0.
\]

(130b)

Near the AdS boundary, the scalar field has such asymptotic form

\[
\phi(r) = \frac{\phi_s}{r^{d-\Delta}} (1 + \cdots) + \frac{\phi_v}{r^{\Delta}} (1 + \cdots),
\]

(131)

where \( \phi_s \) and \( \phi_v \) are coefficients of leading terms and \( \Delta \) is the conformal dimension of the dual operator. There is the usual relationship [13, 34] between \( \Delta \) and \( m^2 \)

\[
\Delta = \left( d + \sqrt{d^2 + 4m^2} \right) / 2.
\]

(132)

In order to get the order of every expansion coefficient, we should expand the metric (124) at large \( r \), and substitute the expansion of both scalar field \( \phi \) and metric into the equations of motion (130). Then, given the boundary condition, we can solve these coefficients order by order. However, it depends on the specific form of the potential \( V(\phi) \) in such solving process. With loss of generality, we consider the specific model in 4-dimensional spacetime with planar horizon geometry \( k = 0 \) to illustrate the key feature. We take the scalar potential function [35] as

\[
V(\phi) = -6 - \frac{4}{\delta^2} \sinh \left[ \frac{\delta \phi}{2} \right]^2
\]

(133)

where \( \delta \) is a constant. One can check the boundary’s asymptotic form of potential

\[
V(\phi) = -6 - \phi^2 + \mathcal{O} \left( \phi^4 \right)
\]

(134)

here \( \Delta = 2 \) and \( m^2 = -2 \) which satisfies the Breitenlohner-Freedman bound (127). Then, near the AdS boundary, we expand the metric that is determined by functions \( f(r) \) and \( \chi(r) \):

\[
f(r) = r^2 \left[ 1 + \frac{\phi_s^2}{4r^2} - \frac{\phi_v^3}{3r^3} + \mathcal{O} \left( \frac{1}{r^4} \right) \right],
\]

(135a)

\[
\chi(r) = \frac{\phi_s^2}{4r^2} + 2 \frac{\phi_s \phi_v}{3r^3} + \mathcal{O} \left( \frac{1}{r^4} \right).
\]

(135b)

**B. Numerical check on Penrose inequality**

If we want to obtain the right holographic stress tensor \( T^\mu_{\nu} \) through the well-defined variational principle, the Gibbons-Hawking-York boundary term [36, 37] should be added to the action (123)

\[
S_{\text{GHY}} = \lim_{r \to \infty} \frac{1}{8 \pi G} \int d^3 x \sqrt{-h} K
\]

(136)

here \( h_{ij} \) is the induced metric on AdS boundary, and \( K \) is the trace of second fundamental form \( K_{ij} \)

\[
K_{ij} = -\frac{1}{2} \mathcal{L}_n h_{ij}, \quad K = h^{ij} K_{ij},
\]

(137)

where \( n^a \) is the outward pointing unit vector normal to AdS boundary. Because the action is still divergent in AdS boundary after adding Gibbons-Hawking-York boundary term, we should introduce a boundary cosmological constant to regulate the infinity. Following standard holographic renormalization scheme [12, 27, 38], the counter term for gravitational sector in this case is given by

\[
S_{\text{c.t.}} = \lim_{r \to \infty} \frac{1}{16 \pi G} \int d^3 x \sqrt{-h} (-4)
\]

(138)

Since mass satisfying \( -\frac{\phi^2}{\delta^2} < m^2 < 1 - \frac{\phi^2}{\delta^2} \), there are two different renormalization schemes [28, 39, 40] for scalar field \( \phi(r) \) sector. For instance, if we treat \( \phi_s \) as the source, we must fix the value of \( \phi_s \) on the AdS boundary which is referred as standard quantization for \( \phi(r) \). Then, we should add the following counter term

\[
S_{\phi_s} = \lim_{r \to \infty} \frac{1}{16 \pi G} \int d^3 x \sqrt{-h} \left( -\frac{1}{2} \phi^2 \right).
\]

(139)

However, if we fix the value of \( \phi_v \) on the boundary, the counter term we need to add is different from previous one

\[
S_{\phi_v} = \lim_{r \to \infty} \frac{1}{16 \pi G} \int d^3 x \sqrt{-h} \left[ \phi(n^\mu \partial_\mu \phi) + \frac{1}{2} \phi^2 \right].
\]

(140)

Then, we obtain the regulated action \( \tilde{S} \),

\[
\tilde{S} = S + S_{\text{GHY}} + S_{\text{c.t.}} + S_{\phi_{s,v}}.
\]

(141)

So far, \( \tilde{S} \) is finite when \( r \to \infty \). Then we can obtain the holographic stress tensor

\[
T^\mu_{\nu} = \frac{1}{16 \pi G r \to \infty} \lim r^2 (K h_{\mu\nu} - K_{\mu\nu} - 2 h_{\mu\nu})
\]

\[
+ h_{\mu\nu} \times \left\{ -\frac{1}{2} \phi^2 \left( \phi(n^\mu \partial_\mu \phi) + \frac{1}{2} \phi^2 \right) \right\}.
\]

(142)

Substituting the asymptotic expansions into \( T^\mu_{\nu} \), we can obtain the value of \( tt \) component

\[
16 \pi G T_{tt} = \begin{cases} 
2 f_0^2 + \phi_s \phi_v & \text{fix} \phi_s, \\
2 f_0^2 + 2 \phi_s \phi_v & \text{fix} \phi_v.
\end{cases}
\]

(143)
The new mass parameter $f_0^3$ which is defined by holographic mass/energy is relevant to the value of $T_{tt}$. In this case, $f_0^3$ is given by

$$f_0^3 / 2 = 4\pi G T_{tt} = \left\{ \begin{array}{ll} f_0^3 / 2 + \phi_s \phi_v / 4 & \text{fix } \phi_s \\ f_0^3 / 2 + \phi_s \phi_v / 2 & \text{fix } \phi_v \end{array} \right. \quad (144)$$

Fixing $r_h = 1$ and $\delta = 1$, we can solve the equations of motion (130) numerically and then read the data $(f_0^3, \phi_s, \phi_v)$ from the asymptotic form of Eqs. (135) on the boundary.

![Graph showing $f_0^3 / 2$ with two renormalization schemes](image)

In Fig. 3, independent variable is the value of $\phi(r_h)$ which is the value of $\phi$ at horizon. Under the same $\phi(r_h)$, it’s obvious that the value of holographic mass $M$ is different while employing two kinds of quantization schemes. In this case, the Penrose inequality is given by

$$\frac{4\pi M}{\Omega_0^2} = \frac{f_0^3}{2} = \frac{r_h}{2} \geq \frac{1}{2}. \quad (145)$$

The inequality is guaranteed as $\phi(r_h)$ increases if we fix $\phi_s$ on the boundary. Otherwise, the inequality will be broken if we fix $\phi_v$ on the boundary. This means that the Penrose inequality is not generally true if we use alternative quantization in holography.

V. SUMMARY

The recent holographic deduction of Penrose inequality only assumes null energy condition while the weak or dominant energy condition is required in usual geometric proof. This paper tries to make a step toward filling up gap between these two approaches. We first discussed the AdS Penrose inequality and null energy condition from the viewpoint of pure geometry. For asymptotically Schwarzschild-AdS black hole, the matter decays fast enough so that we can read the total mass directly from asymptotically expansion of bulk metric near the AdS boundary. By the virtue of this property, we defined a "quasi-local mass" (22) which satisfies $m(r) = M$ on the AdS boundary. What’s more, due to its particular form, the derivative of $m(r)$ is non-negative (25) which is guaranteed by the null energy condition. Our proof indicates that the null energy condition can guarantee the Penrose inequality for black holes with planar/spherical symmetries, as expected from the holographic argument of Ref. [18]. The holographic argument of Ref. [18] also implies that null energy condition could guarantee the Penrose inequality for hyperbolically symmetric black hole, however, we find a counterexample (42) to shows that this is not true. These results inspire us to conjecture that the null energy condition can guarantee the Penrose inequality in asymptotically Schwarzschild AdS black hole only when the cross-section of horizon has planar or spherical topology.

Next we proposed two kinds of charged generalization for the inequality for charged black holes. The holographic argument of Ref. [18] implies the naive generalization (6) provided null energy condition. However, counterexamples in static spherically symmetric case have been found independently in Refs. [19, 20] for such naive generalization. After re-examining the charge $Q$ in the inequality (46), we proposed the first type of generalization which interprets $Q$ in the inequality as $Q_m$ (50). However, the $Q_m$ is not defined at boundary and so cannot be interrupted as a physical quantity of dual boundary field theory according to AdS/CFT correspondence. Thus, we proposed the second version (87) of charged Penrose inequality, in which the charge $Q$ is replaced by the chemical potential $\Phi_{\infty}$. We then gives the proofs for such two generalizations in spherically and planar symmetric cases.

Furthermore, we find that the null energy condition is not enough to guarantee the inequality in holography if the bulk geometry is asymptotically AdS but not asymptotically Schwarzschild AdS. In order to make this argument more explicit, this paper constructed the asymptotically AdS black holes coupled to a scalar field. Following the holographic renormalization, we find that different quantizations for $\phi(r)$ will lead to different values of holographic mass (144). Then we give strong numerical evidence to show that whether the Penrose inequality holds or not will depend on quantization scheme. We note that the arguments of Refs. [17, 18] are regardless the topologies of horizon and the quantization scheme. Thus, such holographic arguments would lead to similar conclusions for different topologies and quantization schemes. However, our geometric proofs and concrete examples show that, if we only impose null energy condition, the whether generalized Penose inequality in asymptotically AdS spacetime is true or not will strongly depend on the topologies of horizon and the quantization schemes.

In present paper, though we propose the conjectures for general static case, we can only give the proofs in
spherically and planar symmetric cases. It’s worthy to examine our conjectures in inhomogeneous cases. What’s more, it is well known that in general relativity there are many other definitions of mass, such as Kamor mass and ADM mass and so on. In this paper, we used the holographic renormalization to define the total mass in Penrose inequality. The question is how much the mass of different definitions influence the structure of Penrose inequality, which is still open. Nevertheless, this paper only consider the scalar hairy black holes. It is also interesting to consider other types of black holes, such as vector hairy black holes.

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Appendix A: Proof of Penrose Inequality with Weak Energy Condition

In the section II, we have considered the Penrose inequality by assuming null energy condition. Null energy condition does not require the energy density $\rho$ nonnegative but the sum of energy density and pressure density nonnegative. If matter satisfies the weak energy condition, we can find the Penrose inequality follows directly since $\rho$ is always nonnegative. Just like previous procedure, let us define a "quasi-local mass" which is known as the Hawking mass [41, 42]

$$m(r) = \frac{r^{d-2}(r^2 + k - f)}{2} \quad (A1)$$

when $r \to \infty$, one can check $m(r)$ is equal to one half of mass parameter $f_0^d$

$$m(\infty) = f_0^d/2. \quad (A2)$$

Take the derivative of $m(r)$ with respect to $r$

$$m'(r) = \frac{(d - 2)k + dr^2 - (d - 2)f - rf'}{r^{3-d}} \quad (A3)$$

Integrate the left and right sides of Eq. (A3) from the horizon $r_h$ to $\infty$:

$$m(\infty) - m(r_h) = \int_{r_h}^{\infty} \frac{(d - 2)k + dr^2 - (d - 2)f - rf'}{r^{3-d}} \, dr. \quad (A4)$$

From Eq. (18a), the expression of $\rho$ is

$$\rho = \frac{(d - 1) [(d - 2)k + dr^2 - (d - 2)f - rf']}{2r^2} \geq 0. \quad (A5)$$

Combining with weak energy condition, we can obtain

$$m(\infty) - m(r_h) = f_0^d/2 - \frac{r_h^d + kr_h}{2} \geq 0, \quad (A6)$$

so that the Penrose inequality (12) follows. In order to see why the weak energy condition can guarantee the Penrose inequality with no difficulty, we express the integral (A4) in terms of energy density $\rho$

$$m(\infty) - m(r_h) = \int_{r_h}^{\infty} \frac{2\rho}{d-1} r^{d-1} \, dr. \quad (A7)$$

If we interpret $\rho$ as the mass density for "quasi-local mass" $m(r)$, the $m(\infty)$ must greater than or equal to $m(r_h)$ due to $\rho \geq 0$. By virtue of construction of "quasi-local mass" (A1), we can see more clearly that Penrose inequality is a stronger version of positive energy theorem for planar and spherical cases.

We conclude that the inequality is saturated only if $\chi(r) = 0$ and $m(r) = m(\infty) = f_0^d/2$, which leads to

$$f(r) = r^2(1 + k/r^2 - f_0^d/r_h^d), \quad \rho = p_r = p_T = 0. \quad (A8)$$

We can see that, if weak energy condition is satisfied, then the Penrose inequality in all three different topologies is true. The saturation appears only if the black hole is Schwarzschild-AdS black hole.

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