On the energy stability of Strang-splitting for Cahn-Hilliard

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Abstract

We consider a Strang-type second order operator-splitting discretization for the Cahn-Hilliard equation. We introduce a new theoretical framework and prove uniform energy stability of the numerical solution and persistence of all higher Sobolev norms. This is the first strong stability result for second order operator-splitting methods for the Cahn-Hilliard equation. In particular we settle several long-standing open issues in the work of Cheng, Kurganov, Qu and Tang [25].

1 Introduction

In this work we consider the Cahn-Hilliard equation ([2]) of the form:

\[
\begin{cases}
\partial_t u = \Delta(-\nu\Delta u + f(u)), & (t, x) \in (0, \infty) \times \Omega, \\
\left. u \right|_{t=0} = u_0.
\end{cases}
\]  

(1.1)

Here the main unknown \( u = u(t, x) : [0, \infty) \times \Omega \rightarrow \mathbb{R} \) denotes the concentration difference in a binary system. The parameter \( \nu > 0 \) is called the mobility coefficient and we fix it as a constant for simplicity. We take the nonlinear term \( f(u) = u^3 - u = F'(u) \), where \( F(u) = \frac{1}{4}(u^2 - 1)^2 \) is the standard double well. The minima of this potential are situated at \( u = \pm 1 \) which correspond to different phases or states. In order not to overburden the readers with various subtle technicalities, we take the spatial domain \( \Omega \) in (1.1) as the one-dimensional \( 2\pi \)-periodic torus \( \mathbb{T} = \mathbb{R} / 2\pi\mathbb{Z} = [-\pi, \pi] \). With some additional work our analysis can be extended to other physical dimensions \( d \leq 3 \). Throughout this note we shall consider mean zero initial data, i.e. \( \frac{1}{2\pi} \int_{\mathbb{T}} u_0 dx = 0 \). This is clearly invariant under the dynamics thanks to the mass conservation law. It

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follows that \( u(t, \cdot) \) has zero mean for all \( t > 0 \). The system (1.1) naturally arises as a gradient flow of a Ginzburg-Landau type energy functional \( E(u) \) in \( H^{-1} \), where

\[
E(u) = \int_{\Omega} \left( \frac{1}{2} \nu |\nabla u|^2 + F(u) \right) \, dx = \int_{\Omega} \left( \frac{1}{2} \nu |\nabla u|^2 + \frac{1}{4}(u^2 - 1)^2 \right) \, dx. \tag{1.2}
\]

For smooth solutions, the fundamental energy conservation law can be expressed as

\[
\frac{d}{dt} E(u(t)) + \| |\nabla|^{-1} \partial_t u \|_2^2 = \frac{d}{dt} E(u(t)) + \int_{\Omega} |\nabla(-\nu \Delta u + f(u))|^2 \, dx = 0. \tag{1.3}
\]

Consequently, one obtains a priori \( \dot{H}^1 \)-norm control of the solution for all \( t > 0 \). Since the scaling-critical space for CH is \( H^{-\frac{1}{2}} \) in 1D, the global wellposedness and regularity for \( H^1 \)-initial data follows from standard arguments.

The main purpose of this work is to establish strong stability of a second order Strang-type operator-splitting algorithm applied to the Cahn-Hilliard equation (1.1). Concerning the operator splitting approximation of (1.1), there is a lot of flexibility in designing the linear/nonlinear operators and the interwoven patterns of these operators. To fix the terminology, let \( a \) be a fixed initial data, and define for \( \tau > 0 \),

\[
S_L(\tau) a = e^{-\tau(\nu \Delta^2 + \Delta)} a.
\]

In yet other words, \( u = S_L(t)a \) solves the equation

\[
\begin{cases}
\partial_t u = -\nu \Delta^2 u - \Delta u, & 0 < t \leq \tau; \\
u |\nabla|^2 u = a.
\end{cases}
\tag{1.4}
\]

Let \( u = S_N(t)a \) solve the nonlinear problem

\[
\begin{cases}
\partial_t u = \Delta(u^3), & 0 < t \leq \tau; \\
u |\nabla|^2 u = a.
\end{cases}
\tag{1.5}
\]

Denote by \( u = u^P \) the exact PDE solution to (1.1) corresponding to initial data \( a \). A Strang-type approximation amounts to the approximation of the form:

\[
u^P(\rho, \cdot) = S_L(\frac{\tau}{2}) S_N(\tau) S_N(\frac{\tau}{2}) a + O(\tau^3). \tag{1.6}
\]

At the cost of some high regularity assumption on \( a \) and certain smallness of the time interval length \( \tau \), one can show that (1.6) holds in some Sobolev class. In practical numerical implementation, we need to iterate (1.6) \( n \)-times, that is

\[
u^P(n\tau, \cdot) \approx S_L(\frac{\tau}{2}) S_N(\tau) S_N(\frac{\tau}{2}) \cdots S_L(\frac{\tau}{2}) S_N(\tau) S_N(\frac{\tau}{2}) a. \tag{1.7}
\]

To justify the convergence and stability of the numerical approximation, a fundamental problem is to establish an estimate of the form

\[
\sup_{n \geq 1} \left\| S_L(\frac{\tau}{2}) S_N(\tau) S_N(\frac{\tau}{2}) \cdots S_L(\frac{\tau}{2}) S_N(\tau) S_N(\frac{\tau}{2}) a \right\|_{H^k} \lesssim 1, \tag{1.8}
\]

\begin{equation}
\text{\( n \) times}
\end{equation}
where we assume \( a \in H^k \) for some \( k \geq 1 \), and \( \tau \) is taken to be sufficiently small. This is by no means trivial since (1.4) in general only guarantees \( \| S_L(\tau^2) a \|_2 \leq e^{c\tau} \| a \|_2 \) and the nonlinear evolution (1.5) only gives control of the \( L^p \)-norm. Needless to say, the wellposedness of (1.5) in Sobolev class and control of the lifespan of the local solution also present various subtle technical difficulties. Another variation of the theme for the operator-splitting approximation of (1.1) goes as follows. Define for \( \tau > 0 \),

\[
S^{(1)}_L(\tau) a = e^{-\nu \Delta^2 \tau} a, \quad u = S^{(1)}_L(t) a
\]

solves the equation

\[
\begin{cases}
\partial_t u = -\nu \Delta^2 u, & 0 < t \leq \tau; \\
u \mid_{t=0} = a.
\end{cases}
\]

Let \( u = S^{(1)}_N(t) a \) solve the nonlinear problem

\[
\begin{cases}
\partial_t u = \Delta(u^3 - u), & 0 < t \leq \tau; \\
u \mid_{t=0} = a.
\end{cases}
\]

We then approximate \( u^P(\tau, \cdot) \) via the scheme

\[
u^P(\tau, \cdot) \approx S^{(1)}_L(\frac{\tau}{2}) S^{(1)}_N(\tau) S^{(1)}_L(\frac{\tau}{2}) a.
\]

One should note that although we have \( \| S^{(1)}_L(\tau^2) a \|_{H^k} \leq \| a \|_{H^k} \) for any \( k \geq 0 \). The nonlinear evolution \( S^{(1)}_N(\tau) \) no longer has contraction in \( L^p \). This brings essential technical difficulties for the stability analysis.

Due to these aforementioned technical obstructions, there were very few rigorous results on the analysis of the operator-splitting type algorithms for the Cahn-Hilliard equation and similar models. In [26], Gidey and Reddy considered a convective Cahn-Hilliard model of the form

\[
\partial_t u - \gamma \nabla \cdot h(u) + \epsilon^2 \Delta^2 u = \Delta(f(u)),
\]

where \( h(u) = \frac{1}{2}(u^2, u^2) \). By using operator-splitting, (1.12) were split into the hyperbolic part, nonlinear diffusion part and diffusion part respectively. Some conditional results concerning certain weak solutions were obtained in [26]. In [27], Weng, Zhai and Feng considered a viscous Cahn-Hilliard model of the form

\[
(1 - \alpha) \partial_t u = \Delta(-\epsilon^2 \Delta u + f(u) + \alpha \partial_t u),
\]

where the parameter \( \alpha \in (0, 1) \). Weng, Zhai and Feng considered a fast explicit Strang splitting and showed stability and convergence under the assumption that \( A = \| \nabla u^{\text{num}} \|_\infty^2, B = \| u^{\text{num}} \|_\infty^2 \) are bounded, and satisfy a technical condition \( 6A + 8 - 24B > 0 \) (see Theorem 1 on pp. 7 of [27]), where \( u^{\text{num}} \) denotes the numerical solution.

\(^1\) Most results in the literature are conditional one way or another in disguise.
The first genuine progress on the energy-stability analysis of the operator-splitting approximation of (1.1) were made in recent [13], where we considered a splitting approximation of (1.1) of the form:

\[ u^p(\tau, \cdot) \approx S_L^{(1)}(\tau)S_N^{(2)}(\tau)a. \]  

(1.14)

Here \( u = S_N^{(2)}(\tau)a \) solves

\[ \frac{u - a}{\tau} = \Delta(a^3 - a). \]  

(1.15)

By introducing a novel modified energy, we showed monotonic decay of the new modified energy which is coercive in \( H^1 \)-sense. Moreover we also obtained uniform control of higher Sobolev regularity. However, this line of analysis relies in an essential way the monotonicity of the modified energy and has no bearing on the second-order and higher case which have some intrinsic technical difficulties. In [25], Cheng, Kurganov, Qu and Tang considered the Strang splitting for the Cahn-Hilliard equation in the style of (1.6). Some conditional results were given in [25] but the rigorous analysis of energy stability has remained an outstanding open problem. The purpose of this work is to establish a completely new theoretical framework for the rigorous analysis of energy stability and higher-order Sobolev-norm stability for higher order operator-splitting method such as (1.6). Our first result reads as follows.

**Theorem 1.1.** Let \( \nu > 0 \) and consider the one-dimensional periodic torus \( \mathbb{T} = [-\pi, \pi] \). Assume the initial data \( u^0 \in H^{k_0}(\mathbb{T}) \) (\( k_0 \geq 1 \) is an integer) and has mean zero. Let \( \tau > 0 \) and define

\[ u^{n+1} = S_L(\frac{\tau}{2})S_N(\tau)S_L(\frac{\tau}{2})u^n, \quad n \geq 0. \]  

(1.16)

There exists a constant \( \tau_* > 0 \) depending only on \( \|u^0\|_2 \) and \( \nu \), such that if \( 0 < \tau < \tau_* \), then

\[ \sup_{n \geq 0} \|u^n\|_{H^{k_0}} \leq A_1 < \infty, \]  

(1.17)

where \( A_1 > 0 \) depends on \( (\|u^0\|_{H^{k_0}}, \nu, k_0) \).

Our second result establishes the convergence of the operator splitting approximation. Not surprisingly since this is a Strang-type splitting approximation, the convergence is second order in \( \tau \) on any finite time interval \([0, T]\).

**Theorem 1.2** (Convergence of the splitting approximation). Assume the initial data \( u^0 \in H^{40}(\mathbb{T}) \) with mean zero. Let \( u^n \) be defined as in Theorem 1.1. Let \( u \) be the exact PDE solution to (1.1) corresponding to initial data \( u^0 \). Let \( 0 < \tau < \tau_* \) as in Theorem 1.1. Then for any \( T > 0 \), we have

\[ \sup_{n \geq 1, n\tau \leq T} \|u^n - u(n\tau, \cdot)\|_{L^2(\mathbb{T})} \leq C \cdot \tau^2, \]  

(1.18)

where \( C > 0 \) depends on \( (\nu, \|u^0\|_{H^{40}}, T) \).
Remark 1.1. The regularity assumption on initial data can be lowered but we shall not dwell on this issue here for simplicity of presentation. One can also work out the convergence in higher Sobolev norms.

The rest of this paper is organized as follows. In Section 2 we set up the notation and collect some preliminary lemmas. In Section 3 we carry out the main analysis for the propagators. In Section 4 we complete the proofs of Theorem 1.1 and 1.2.

2 Notation and preliminaries

For any two positive quantities $X$ and $Y$, we shall write $X \lesssim Y$ or $Y \gtrsim X$ if $X \leq CY$ for some constant $C > 0$ whose precise value is unimportant. We shall write $X \sim Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold. We write $X \lesssim_{\alpha} Y$ if the constant $C$ depends on some parameter $\alpha$. We shall write $X = O(Y)$ if $|X| \lesssim Y$ and $X = O_{\alpha}(Y)$ if $|X| \lesssim_{\alpha} Y$.

We shall denote $X \ll Y$ if $X \leq cY$ for some sufficiently small constant $c$. The smallness of the constant $c$ is usually clear from the context. The notation $X \gg Y$ is similarly defined. Note that our use of $\ll$ and $\gg$ here is different from the usual Vinogradov notation in number theory or asymptotic analysis.

For any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we denote $|x| = |x|_2 = \sqrt{x_1^2 + \cdots + x_d^2}$, and $|x|_\infty = \max_{1 \leq j \leq d} |x_j|$. Also occasionally we use the Japanese bracket notation: $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$.

We denote by $\mathbb{T}^d = [-\pi, \pi]^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ the usual $2\pi$-periodic torus. For $1 \leq p \leq \infty$ and any function $f : x \in \mathbb{T}^d \to \mathbb{R}$, we denote the Lebesgue $L^p$-norm of $f$ as

$$\|f\|_{L^p(\mathbb{T}^d)} = \|f\|_{L^p(\mathbb{T})}$$

If $(a_j)_{j \in I}$ is a sequence of complex numbers and $I$ is the index set, we denote the discrete $l^p$-norm as

$$\|(a_j)_{j \in I}\|_{l^p(I)} = \|(a_j)\|_{l^p(I)} = \begin{cases} \left( \sum_{j \in I} |a_j|^p \right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \sup_{j \in I} |a_j|, & p = \infty. \end{cases} \tag{2.1}$$

For example, $\|\hat{f}(k)\|_{l^2(\mathbb{Z}^d)} = \left( \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 \right)^{\frac{1}{2}}$. If $f = (f_1, \cdots, f_m)$ is a vector-valued function, we denote $|f| = \sqrt{\sum_{j=1}^m |f_j|^2}$, and $\|f\|_p = \|(\sum_{j=1}^m f_j^2)^{\frac{1}{2}}\|_p$. We use similar convention for the corresponding discrete $l^p$ norms for the vector-valued case.

We use the following convention for the Fourier transform pair:

$$\hat{f}(k) = \int_{\mathbb{T}^d} f(x)e^{-ik \cdot x}dx, \quad f(x) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \hat{f}(k)e^{ik \cdot x}, \tag{2.2}$$
and denote for $0 \leq s \in \mathbb{R}$,
\[
\|f\|_{H^s} = \|f\|_{H^s(T^d)} = \|\|\nabla|^s f\|_{L^2(T^d)} \sim \|k|^s \hat{f}(k)\|_{L^2_Z},
\]
\[
\|f\|_{H^s} = \sqrt{\|f\|^2 + \|f\|_{H^s}^2} \sim \|k|^s \hat{f}(k)\|_{L^2_Z}.
\]

**Lemma 2.1.** Let $\nu > 0$, $d \geq 1$ and $\beta > 0$. Consider on the torus $T^d = [-\pi, \pi]^d$,
\[
K(x) = \mathcal{F}^{-1}(e^{-\beta(\nu|k|^4 - |k|^2)}) = e^{-\beta(\nu \Delta^2 + \Delta)} \delta_0,
\]
where $\delta_0$ is the periodic Dirac comb. Then for any $1 \leq p \leq \infty$,
\[
\|K\|_{L^p(T^d)} \leq c_{d,p,\nu} (1 + \beta^{-d(\frac{1}{4} - \frac{1}{4p})})^{-d_1},
\]
where $c_{d,p,\nu} > 0$ depends only on $(d, p, \nu)$ and $d_1 > 0$ depends only on $(d, \nu)$.

**Proof of Lemma 2.1.** We shall write $X \lesssim Y$ if $X \leq CY$ and $C$ depends on $(d, \nu, p)$.

Define $K_w(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i \xi \cdot x} e^{-\beta(\nu|\xi|^4 - |\xi|^2)} d\xi$. It is not difficult to check that
\[
|K_w(x)| \lesssim \langle x \rangle^{-10d} e^{-d_1 \beta} (1 + \beta^{-\frac{4}{4}}).
\]
Poisson summation gives
\[
K(x) = \sum_{l \in \mathbb{Z}^d} K_w(x + 2\pi l).
\]
The desired estimate then follows easily. 

**Lemma 2.2.** Let $d = 1$ and $\nu > 0$. Let $0 < \tau \leq 1$. Then for any $g \in L^4(T)$, we have
\[
\|e^{-\tau(\nu \partial_x^4 + \partial_x^2)} g\|_\infty \leq C_1 \tau^{-\frac{1}{10}} \|g\|_4,
\]
where $C_1 > 0$ depends on $\nu$. For any $g_1 \in L^\frac{4}{3}(T)$, we have
\[
\|\tau \partial_{xx} e^{-\tau(\nu \partial_x^4 + \partial_x^2)} g_1\|_\infty \leq C_2 \tau^\frac{1}{10} \|g_1\|_{\frac{4}{3}},
\]
where $C_2 > 0$ depends on $\nu$.

**Proof.** The first inequality follows from Lemma 2.1. For the second inequality denote
\[
K_2 = \mathcal{F}^{-1} \left( \tau |k|^2 e^{-\tau(\nu|k|^4 - k^2)} \right).
\]
We then have $\|K_2\|_{L^1(T)} \lesssim \|\hat{K}_2\|_{L^1_Z} \lesssim \tau^{-\frac{1}{10}}$ for $0 < \tau \leq 1$. 

3 Analysis of the propagators

Lemma 3.1 (One-step solvability of $S_N(\tau)$). Let $\nu > 0$. Suppose $a \in L^2(\mathbb{T})$ with zero mean and $\|a\|_2 \leq A_1 < \infty$ for some $A_1 > 0$. There exists $\tau_1 = \tau_1(A_1, \nu) > 0$ such that if $0 < \tau \leq \tau_1$, then there exists a unique solution $w \in C([0, \tau], H^2)$ to the equation

$$\begin{cases}
\partial_t w = \partial_{xx}(w^3), & 0 < t \leq \tau, \\
w|_{t=0} = e^{-\frac{1}{2}(\nu \partial_x^4 - \partial_x^2)} a.
\end{cases} \tag{3.1}$$

The solution $w$ satisfies

$$\max_{0 \leq t \leq \tau} \|w(t, \cdot)\|_2 \leq \|w(0, \cdot)\|_2 \leq e^{c_\nu \tau} \|a\|_2,$$

$$\max_{0 \leq t \leq \tau} \|w(t, \cdot)\|_4 \leq \|w(0, \cdot)\|_4 \leq c^{(1)}(1 + \tau^{-\frac{1}{2}}) \|a\|_2,$$

where $c_\nu > 0, c^{(1)} > 0$ depend only on $\nu$. Furthermore if $\|a\|_{H^k(\mathbb{T})} \leq A_2 < \infty$ for some integer $k \geq 1$ and $A_2 > 0$, then we also have the bound

$$\sup_{0 \leq t \leq \tau} \|w(t, \cdot)\|_{H^k} \leq C^{(1)}_{\nu, k, A_1, A_2} < \infty, \tag{3.2}$$

where $C^{(1)}_{\nu, k, A_1, A_2} > 0$ depends on $(\nu, k, A_1, A_2)$.

Proof. For the local existence of solution, we can work with a regularized problem

$$\partial_t w_\delta = -\delta \partial_x^4 w_\delta + \partial_{xx}((w_\delta)^3)$$

and take the limit $\delta \to 0$. The key point in the argument is to derive uniform-in-$\delta$ bounds on $\|\partial_{xx}w\|_2$. Here and below we drop the subscript $\delta$ for simplicity. Thanks to the benign nonlinear diffusion term, it is not difficult to check that

$$\frac{d}{dt} \|\partial_{xx}w\|_2^2 \lesssim \|w\|_{H^2}^3. \tag{3.3}$$

Note that

$$\|w(0, \cdot)\|_{H^2} \lesssim (1 + \tau^{-\frac{1}{2}}) \|a\|_2. \tag{3.4}$$

It suffices for us to choose $\tau$ sufficiently small such that

$$\tau(1 + \tau^{-\frac{1}{2}}) \|a\|_2 \ll 1. \tag{3.5}$$

We then obtain a local solution in $C([0, \tau], H^2)$. It is not difficult to check the uniqueness and the $L^2$, $L^4$ estimates. The estimate (3.2) follows from the $H^2$ bound and additional energy estimates. We omit the details. \qed
Lemma 3.2 \((O(1))-\text{step stability of } S_N(\tau)S_L(\tau/2) \text{ and } S_L(\tau/2)S_N(\tau)S_L(\tau/2))\). Let \(\nu > 0\). Suppose \(a \in L^2(\mathbb{T})\) with zero mean and \(\|a\|_2 \leq A_1 < \infty\) for some \(A_1 > 0\). Define \(u^0 = v^0 = a\) and

\[
v^{n+1} = S_N(\tau)S_L(\frac{\tau}{2})v^n, \quad n \geq 0;
\]

\[
u^{n+1} = S_L(\frac{\tau}{2})S_N(\tau)S_L(\frac{\tau}{2})u^n, \quad n \geq 0.
\]

There exists \(\tau_2 = \tau_2(A_1, \nu) > 0\) such that if \(0 < \tau \leq \tau_2\), then the following hold.

1. The iterates \(u^n, v^n\) are well-defined for all \(n \leq \frac{10}{\tau}\), and

\[
\sup_{1 \leq n \leq \frac{10}{\tau}} \left( \|u^n\|_2 + \|v^n\|_2 \right) \leq C^{(2)}_\nu A_1, \quad (3.6)
\]

where \(C^{(2)}_\nu > 0\) depends only on \(\nu\).

2. We have

\[
\sup_{\frac{1}{\tau} \leq n \leq \frac{10}{\tau}} \left( \|u^n\|_{H^40} + \|v^n\|_{H^40} \right) \leq C^{(3)}_{\nu, A_1}, \quad (3.7)
\]

where \(C^{(3)}_{\nu, A_1} > 0\) depends only on \((\nu, A_1)\).

3. If \(\|a\|_{H^1} \leq \bar{A}_1\) for some \(\bar{A}_1 < \infty\), then we also have

\[
\sup_{1 \leq n \leq \frac{10}{\nu}} \left( \|u^n\|_{H^1} + \|v^n\|_{H^1} \right) \leq \tilde{C}^{(3)}_{\nu, \bar{A}_1}, \quad (3.8)
\]

where \(\tilde{C}^{(3)}_{\nu, \bar{A}_1} > 0\) depends only on \((\nu, \bar{A}_1)\).

Proof. That the iterates \(u^n, v^n\) are well-defined along with the estimate [3.6] follows from Lemma 3.1. A key observation here is that \(\|u^n\|_2, \|v^n\|_2\) remains \(O(1)\) for \(n\tau \lesssim 1\). It suffices for us to show [3.7] for \(u^n\) since the estimates for \(v^n\) follow from it. To this end we rewrite

\[
u^{n+1} = S_L(\frac{\tau}{2})S_N(\tau)S_L(\frac{\tau}{2})u^n
\]

\[
= S_L(\frac{\tau}{2})(S_L(\frac{\tau}{2})u^n + \tau \partial_{xx} f^n), \quad (3.9)
\]

where

\[
f^n = \frac{1}{\tau} \int_0^\tau w_n(s)^3 ds, \quad (3.10)
\]
and \( w_n \) solves the PDE

\[
\begin{aligned}
\frac{\partial_t w_n}{w_n} &= \frac{\partial_{xx}(w_n^3)}{w_n}, \quad 0 < t \leq \tau; \\
\frac{w_n}{t=0} &= S_L(\frac{\tau}{2})u^n.
\end{aligned}
\] (3.11)

By Lemma 3.1, we have

\[
\sup_{n \tau \leq 10} \|f^n\|_3 \lesssim 1 + \tau^{-\frac{3}{16}}. \tag{3.12}
\]

Iterating (3.9), we obtain

\[
u u_{n+1} = S_L((n+1)\tau)u^0 + \tau \sum_{k=0}^n \partial_{xx} S_L((k+1)\frac{1}{2}\tau) f^{n-k}. \tag{3.13}
\]

The desired estimates then follow from bootstrapping smoothing estimates.

Lemma 3.3 (Almost steady states are benign). Let \( \nu > 0 \). Suppose \( f \in H^2(\mathbb{T}) \) has zero mean and satisfies

\[
\|\nu \partial_{xx} f - f^3 + \overline{f^3} + f\|_2 \leq 1, \tag{3.14}
\]

where \( \overline{f^3} \) denotes the average of \( f^3 \) on \( \mathbb{T} \). Then

\[
\|f\|_{H^{\omega}(\mathbb{T})} \leq C_{\nu}^{(4)}, \tag{3.15}
\]

where \( C_{\nu}^{(4)} > 0 \) depends only on \( \nu \). Furthermore if \( 0 < \tau \leq \tau^{(0)}(\nu) \) where \( \tau^{(0)}(\nu) > 0 \) is a sufficiently small constant depending only on \( \nu \), then

\[
E(S_L(\frac{\tau}{2})S_N(\tau) S_L(\frac{\tau}{2}) f) \leq C_{\nu}^{(5)}, \tag{3.16}
\]

where \( C_{\nu}^{(5)} > 0 \) depends only on \( \nu \).

Remark 3.1. We pick the constant 1 for convenience. If \( \|\nu \partial_{xx} f - f^3 + \overline{f^3} + f\|_2 \leq \epsilon_0 \) for some \( \epsilon_0 > 0 \), then we obtain \( \|f\|_{H^{\omega}(\mathbb{T})} \leq C_{\nu, \epsilon_0}^{(4)} \), where \( C_{\nu, \epsilon_0}^{(4)} > 0 \) depends on \( (\nu, \epsilon_0) \).

Proof. By a simple energy estimate, we have

\[
\nu \|\partial_x f\|_2^2 + \|f^2 - \frac{1}{2}\|_2^2 \leq \|f\|_2^2 + \frac{\pi}{2}. \tag{3.17}
\]

One can then obtain the \( H^1 \) bound. Bootstrapping yields the desired estimate for higher Sobolev norms. The estimate (3.16) also follows easily.
Lemma 3.4 (One-step strict energy dissipation for non-steady data). Let $\nu > 0$. Suppose $a \in H^{40}(\mathbb{T})$ and has zero mean. Assume

$$\|\nu \partial_{xx} a - a^3 + \overline{a^3} + a\|_2 \geq 1,$$

$$\|a\|_{H^{40}(\mathbb{T})} \leq B_1 < \infty,$$  \hfill (3.18)

where $B_1$ is a given constant, and $\overline{a^3}$ denotes the average of $a^3$ on $\mathbb{T}$. There exists $\tau_3 = \tau_3(\nu, B_1) > 0$ sufficiently small such that if $0 < \tau \leq \tau_3$, then

$$E(S_L(\frac{\tau}{2})S_N(\tau)S_L(\frac{\tau}{2})a) < E(a).$$  \hfill (3.19)

Proof. Denote by $u^P$ as the exact PDE solution corresponding to initial data $a$. We clearly have

$$E(u^P(\tau)) + \int_0^\tau \|\partial_x(\nu \partial_{xx} u^P - (u^P)^3 + u^P)\|_2^2 \, ds = E(a).$$  \hfill (3.20)

By Poincaré we have

$$\|\partial_x(\nu \partial_{xx} u^P - (u^P)^3 + u^P)\|_2 \geq \|\nu \partial_{xx} u^P - (u^P)^3 + \overline{(u^P)^3} + u^P\|_2.$$  \hfill (3.21)

Thanks to the high regularity assumption on $a$ and the usual local theory, we have

$$\sup_{0 \leq s \leq \tau} \left\| \nu \partial_{xx} u^P(s) - (u^P(s))^3 + \overline{(u^P(s))^3} + u^P(s) - \nu \partial_{xx} a + a^3 - \overline{a^3} - a \right\|_2 \lesssim \tau.$$  \hfill (3.22)

It follows that

$$E(u^P(\tau)) + \tau \|\nu \partial_{xx} a - a^3 + \overline{a^3} + a\|_2^2 + O(\tau^2) \leq E(a).$$  \hfill (3.23)

Thus for $\tau > 0$ sufficiently small we have

$$E(u^P(\tau)) + \frac{1}{2} \tau \leq E(a).$$  \hfill (3.24)

We now only need to check that

$$\|u^P(\tau) - S_L(\frac{\tau}{2})S_N(\tau)S_L(\frac{\tau}{2})a\|_{H^1(\mathbb{T})} = O(\tau^2).$$  \hfill (3.25)

Consider an implicit-explicit discretization:

$$\frac{w - a}{\tau} = -\nu \partial^3_x w - \partial^2_x w + \partial^2_x(a^3).$$  \hfill (3.26)

It is not difficult to check that

$$\|u^P(\tau) - w\|_{H^1} = O(\tau^2)$$  \hfill (3.27)
and
\[ w = (1 + \nu \tau \partial_x^4 + \tau \partial_x^2)^{-1} a + \tau (1 + \nu \tau \partial_x^4 + \tau \partial_x^2)^{-1} \partial_{xx}(a^3). \] (3.28)

On the other hand, we have
\[ \| S_L(\tau_2) S_N(\tau) S_L(\tau_2) a - S_L(\tau_2) (S_L(\tau_2) a + \tau \partial_{xx}(a^3)) \|_{H^1} = O(\tau^2). \] (3.29)

The desired result then follows from the estimates
\[ \| (1 + \nu \tau \partial_x^4 + \tau \partial_x^2)^{-1} a - S_L(\tau_2) a \|_{H^1} = O(\tau^2); \]
\[ \| (1 + \nu \tau \partial_x^4 + \tau \partial_x^2)^{-1} \partial_{xx}(a^3) - S_L(\tau_2) \partial_{xx}(a^3) \|_{H^1} = O(\tau). \]

**Theorem 3.1.** Let \( \nu > 0 \). Assume \( u^0 \in H^1(\mathbb{T}) \) with zero mean. Suppose \( \| u^0 \|_2 \leq \gamma_0 \) and \( \| u^0 \|_{H^1} \leq \gamma_1 \). Define
\[ u^{n+1} = S_L(\tau_2) S_N(\tau) S_L(\tau_2) u^n, \quad n \geq 0. \] (3.30)

There exists \( \tau_* = \tau_*(\nu, \gamma_0) > 0 \) sufficiently small such that if \( 0 < \tau \leq \tau_* \), then
\[ \sup_{n \geq 1} \| u^n \|_{H^1(\mathbb{T})} \leq F_{\nu, \gamma_1}^{(0)}, \] (3.31)

where \( F_{\nu, \gamma_1}^{(0)} > 0 \) depends only on \( (\nu, \gamma_1) \).

**Proof.** By Lemma 3.2 for some \( \tilde{\tau}_1(\nu, \gamma_0) > 0 \) sufficiently small and \( 0 < \tau \leq \tilde{\tau}_1(\gamma, \gamma_0) \), we have
\[ \sup_{1 \leq n \leq 10} \| u^n \|_2 + \sup_{1 \leq n \leq 10} (\| u^n \|_{H^4} + E(u^n)) \leq F_{\nu, \gamma_0}^{(1)}, \] (3.32)

where \( F_{\nu, \gamma_0}^{(1)} \) depends only on \( (\nu, \gamma_0) \).

Define
\[ G_{\nu, \gamma_0} = F_{\nu, \gamma_0}^{(1)} + C_{\nu}^{(5)} + 1, \] (3.33)

where \( C_{\nu}^{(5)} > 0 \) is the same as in (3.16).

**Claim:** For some \( \tilde{\tau}_2(\nu, \gamma_0) > 0 \) sufficiently small and \( 0 < \tau \leq \tilde{\tau}_2(\nu, \gamma_0) \), we have
\[ \sup_{n \geq 1} E(u^n) \leq G := G_{\nu, \gamma_0}, \] (3.34)

To prove the claim we argue by contradiction. The smallness condition \( 0 < \tau \leq \tilde{\tau}_2(\nu, \gamma_0) \) will be assumed in the argument below. The needed smallness of \( \tilde{\tau}_2(\nu, \gamma_0) \) can
be easily worked out from the conditions specified in the used lemmas such as Lemma 3.3 and so on.

Suppose \( n_0 \geq \frac{1}{\tau} \) is the first integer such that

\[
E(u^{n_0}) \leq G, \quad E(u^{n_0+1}) > G.
\]

(3.35)

Clearly by our choice of \( G \), we have \( n_0 \geq \frac{10}{\tau} \). By Lemma 3.3 we must have

\[
\| \nu \partial_{xx} u^{n_0} - (u^{n_0})^3 + (u^{n_0})^3 + u^{n_0} \|_2 > 1.
\]

(3.36)

Since \( n_0 \geq \frac{10}{\tau} \), we have \( E(u^{n_0-j_0}) \leq G \) for some integer \( \frac{1}{\tau} \leq j_0 \leq \frac{1}{\tau} + 2 \). By using smoothing estimates we obtain

\[
\| u^{n_0} \|_{H^{40}(\tau)} \leq C_{\nu,G},
\]

(3.37)

where \( C_{\nu,G} > 0 \) depends on \( (\nu, G) \). Since \( G \) depends on \( (\nu, \gamma_0) \), we have \( C_{\nu,G} \) depends only on \( (\nu, \gamma_0) \). By (3.36), (3.37) and Lemma 3.4 we obtain for sufficiently small \( \tau \) that

\[
E(u^{n_0+1}) < E(u^{n_0})
\]

(3.38)

which is clearly a contradiction to (3.35). Thus we have proved the claim. Finally the estimate (3.31) follows from a uniform \( H^1 \) estimate of \( \|u^n\|_{H^1} \) for \( 1 \leq n \leq \frac{1}{\tau} \) using the condition \( \|u^0\|_{H^1} \leq \gamma_1 \).

4 Proof of Theorem 1.1 and 1.2

Proof of Theorem 1.1. The \( H^1 \) estimate follows from Theorem 3.1. Higher order estimates follow from the smoothing estimates.

Proof of Theorem 1.2. Thanks to the uniform \( H^{40} \) estimates on the numerical solution and the exact PDE solution, we only need to check consistency. To simplify the notation, we shall denote

\[
L = -\nu \partial_x^4 - \partial_{xx}^2.
\]

Consistency for the propagator \( S_L(\frac{\tau}{2})S_N(\tau)S_L(\frac{\tau}{2}) \). We first note that if

\[
\begin{cases}
\partial_t w = \partial_{xx}(w^3), & 0 < t \leq \tau; \\
w\big|_{t=0} = b,
\end{cases}
\]

(4.2)

where \( w \) admits uniform control of its Sobolev norm, then

\[
w(\tau) = b + \tau \partial_{xx}(b^3) + \frac{1}{2} \tau^2 \partial_{xx}(3b^2 \partial_{xx}(b^3)) + O(\tau^3).
\]

(4.3)
Now if \( u = S_L(\frac{\tau}{2}) S_N(\tau) S_L(\frac{\tau}{2}) a \), then (below \( b = S_L(\frac{\tau}{2}) a \))

\[
\begin{align*}
  u &= S_L(\frac{\tau}{2}) \left( b + \tau \partial_{xx}(b^3) + \frac{1}{2} \tau^2 \partial_{xx}(3b^2 \partial_{xx}(b^3)) \right) + O(\tau^3) \\
  &= S_L(\tau) a + \tau S_L(\frac{\tau}{2}) \partial_{xx}((a + \frac{1}{2} \tau La)^3) + \frac{1}{2} \tau^2 \partial_{xx}(3a^2 \partial_{xx}(a^3)) + O(\tau^3) \\
  &= S_L(\tau) a + \tau \partial_{xx}(a^3) + \frac{1}{2} \tau^2 \partial_{xx} \left( L(a^3) + 3a^2 (La + \partial_{xx}(a^3)) \right) + O(\tau^3). \quad (4.4)
\end{align*}
\]

Reformulation of the exact PDE solution. Let \( u^P \) be the exact PDE solution to \((1.1)\) with initial data \( \tilde{a} \). We have

\[
\begin{align*}
  u^P(\tau) &= S_L(\tau) \tilde{a} + \int_{0}^{\tau} S_L(\tau - s) \partial_{xx}(u(s)^3) ds \\
  &= S_L(\tau) \tilde{a} + \int_{0}^{\tau} \left( 1 + (\tau - s)L \right) \partial_{xx}(u(s)^3) ds + O(\tau^3) \\
  &= S_L(\tau) \tilde{a} + \int_{0}^{\tau} \partial_{xx} \left( (\tilde{a} + s(L\tilde{a} + \partial_{xx}(\tilde{a}^3)))^3 \right) ds + \int_{0}^{\tau} (\tau - s)L \partial_{xx}(\tilde{a}^3) ds + O(\tau^3) \\
  &= S_L(\tau) \tilde{a} + \tau \partial_{xx}(\tilde{a}^3) + \frac{1}{2} \tau^2 \partial_{xx} \left( L(\tilde{a}^3) + 3\tilde{a}^2 (L\tilde{a} + \partial_{xx}(\tilde{a}^3)) \right) + O(\tau^3). \quad (4.5)
\end{align*}
\]

The desired estimate then clearly follows. \( \square \)

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