Bianchi type-I bulk viscous cosmological model with inhomogeneous equation of state

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Abstract: In the present paper we investigate Bianchi type-I Cosmological model with inhomogeneous equation of state (EoS) in the presence of variable bulk viscous fluid (ζ) in the framework of general theory of relativity. In order to find exact solutions of the Einstein’s field equations, we have assumed a special law of variation for Hubble parameter H i.e. \( H \propto \frac{1}{\sqrt{\alpha}} \). The nature of model is discussed for \( \alpha \neq 0 \) and \( \alpha = 0 \). Some physical and kinematical aspects of model are also discussed for both the conditions.

Keywords: Bianchi type-I cosmology, Inhomogeneous EoS, Cosmological term(Λ), and variable \( \zeta \).

1. Introduction

In modern cosmology, it is great challenge to explain the accelerated expansion of the late universe. Many researchers state that our universe is accelerated which is giving a new direction of research in cosmological studies. It is clear that, the universe contains dark energy (DE) and dark matter (DM). Several theories have been proposed to describe nature of dark energy. The different dynamical DE models were introduced because the ΛCDM model faces the coincidence problem. In the study of celebrated DE problem [1] the ideal fluid with particular equation of state remains to be the possibility to describe the current cosmic acceleration. There are several examples of ideal fluid equation of state may be taken for this purpose: constant EoS with negative pressure, general EoS [2, 3], inhomogeneous EoS [4-9], imperfect EoS [2]. The ideal fluid with inhomogeneous equation of state was introduced by Nojiri and Odinstov [11-13].

Lately various discussions on the cosmological constant problem and consequence on cosmology with time varying cosmological constant are studied by [14]. It is the motivation part to introduce Λ term is to compose the age parameter and density parameter of the universe with recent observational data. The cosmological constant Λ enable itself to be compatible with observation and we see that a many researchers have done works to study the theoretical foundation of the variable cosmological constant and also investigate its model properties.

In 1940 Eckart [15] formulated the relativistic thermodynamic theory of dissipative fluid. Misner 1968 [16] was the first to introduce viscosity concept in cosmology. The phenomenon of bulk viscosity arises due to fluid fast expansion or contraction that the system goes out of thermal equilibrium. Therefore this bulk viscous fluid produces acceleration in the expansion of the universe without DE [17-
Many authors studied the concept of bulk viscosity to observe early time inflation and late time acceleration [20-24].

At the early stages of evolution of the universe Bianchi type space-time play a vital role in the discussion of cosmology and cosmological models. These are well known nine types of Bianchi models which are necessarily spatially homogeneous. Out of these Bianchi models some models tends towards isotropy at arbitrary large times and allows the formation of galaxies and develops intelligent life. Whereas Bianchi types II, VI, VIII and IX shows non flat models. Bianchi type I spacetime with inhomogeneous equation of state and variable bulk viscous fluid.

The above discussion and investigation inspire us to study Bianchi type-I cosmological model with inhomogeneous EoS in the presence of variable bulk viscous fluid.

2. The Metric and the field equations
We consider the spatially homogeneous and isotropic Bianchi type-I metric in the form

\[ ds^2 = -dt^2 + X_1^2 dx^2 + X_2^2 dy^2 + X_3^2 dz^2 \]  

(1)

Where \( X_1, X_2 \) and \( X_3 \) are the metric potentials and are functions of cosmic time \( t \) only.

The Einstein’s field equation are given by,

\[ R_{ij} - \frac{1}{2} R g_{ij} = -T_{ij} \]  

(2)

Here we assume energy momentum tensor in the form of bulk viscosity as follows:

\[ T_{ij} = (\rho + \bar{p}) u_i u_j + \bar{p} g_{ij} \]  

(3)

where \( \rho \) being matter density and \( \bar{p} \) is the effective pressure define as follows

\[ \bar{p} = p - 3\zeta H \]  

(4)

now \( p \) is the pressure, \( H \) is Hubble parameter and \( \zeta \) is the coefficient of bulk viscosity in the quadratic form

\[ \zeta = \zeta_0 H^2 + \zeta_1 H + \zeta_2 \]

The field equations (2) for the line element (1) with the help of (3) can be written as

\[ \frac{\dot{X}_2}{X_2} + \frac{\dot{X}_3}{X_3} + \frac{\dot{X}_2 X_3}{X_2 X_3} = -(p - 3\zeta H) \]  

(5)

\[ \frac{\dot{X}_3}{X_3} + \frac{\dot{X}_1}{X_1} + \frac{\dot{X}_3 X_1}{X_3 X_1} = -(p - 3\zeta H) \]  

(6)

\[ \frac{\dot{X}_1}{X_1} + \frac{\dot{X}_2}{X_2} + \frac{\dot{X}_1 X_2}{X_1 X_2} = -(p - 3\zeta H) \]  

(7)

\[ \frac{\dot{X}_2 X_3}{X_1 X_2} + \frac{\dot{X}_2 X_3}{X_2 X_3} + \frac{\dot{X}_1 X_3}{X_1 X_3} = \rho \]  

(8)

where an over dot denotes differentiation w.r.t time \( t \).
Now the energy conservation equation gives
\[ \dot{\rho} + (\rho + p) \left[ \dot{x}_1 x_1 + \dot{x}_2 x_2 + \dot{x}_3 x_3 \right] = 0 \] (9)

For the line element Bianchi type-I the average scale factor is defined as follows
\[ R^3 = X_1 X_2 X_3 \] (10)

The physical and kinematical parameters for equation (1) are spatial volume (V), expansion scalar (θ), shear scalar (σ^2), deceleration parameter (q) and anisotropic parameter (A) defined by,
\[ V = X_1 X_2 X_3 \] (11)
\[ \theta = 3H = [\dot{x}_1 x_1 + \dot{x}_2 x_2 + \dot{x}_3 x_3] \] (12)
\[ \sigma^2 = \frac{1}{2} \left[ \frac{\dot{x}_1^2}{x_1^2} + \frac{\dot{x}_2^2}{x_2^2} + \frac{\dot{x}_3^2}{x_3^2} \right] - \frac{\theta^2}{6} \] (13)
\[ q = -1 - \frac{\ddot{H}}{H^2} \] (14)
\[ \bar{A} = \frac{1}{3} \sum_{i=1}^{3} \left[ \frac{H_i - H}{H} \right]^2 \] (15)

We consider the EoS of the form,
\[ p = (\gamma - 1)\rho + \Lambda \quad , \gamma > 1 \] (16)

3. Solution of the Model

Now simplifying (5)-(7) and using equation (10), one can obtain the following
\[ \frac{\dot{x}_1}{x_1} = \frac{\dot{R}}{R} + \frac{(2C_0 + C_1)}{3R^3} \], \[ \frac{\dot{x}_2}{x_2} = \frac{\dot{R}}{R} + \frac{(C_1 - C_0)}{3R^3} \], \[ \frac{\dot{x}_3}{x_3} = \frac{\dot{R}}{R} - \frac{(C_0 + 2C_1)}{3R^3} \] (17)

where \( C_0 \) & \( C_1 \) are the constants of integration.

After integrating equation (17) reduces,
\[ X_1 = l_0 R \exp \left\{ \frac{(2C_0 + C_1)}{3} \int \frac{dt}{R^3} \right\} \] (18)
\[ X_2 = l_1 R \exp \left\{ \frac{(C_1 - C_0)}{3} \int \frac{dt}{R^3} \right\} \] (19)
\[ X_3 = l_2 R \exp \left\{ -\frac{(C_0 + 2C_1)}{3} \int \frac{dt}{R^3} \right\} \] (20)

where \( l_0, l_1 \) and \( l_2 \) are the integrating constants.

The number of unknowns in the field equations is more than the number of equation, therefore to obtain the solution, we can assume a special law of variation for Hubble parameter as follows:
\[ H = \frac{m_0}{R^\alpha} \] (21)

Where \( m_0 > 0 \) & \( \alpha \geq 0 \) are constants
Equation (21) becomes,

\[ R = \begin{cases} \left[ m_0 (t + t_0) \right]^{\frac{1}{a}}, & \alpha \neq 0 \\ \exp[m_0 (t - t_c)], & \alpha = 0 \end{cases} \] (22)

Where \( t_0 \) and \( t_c \) are constants of integration.

**Case I: \( \alpha \neq 0 \)**

With the help of (22), Equation (18)-(20) becomes,

\[
X_1 = l_0 \left[ m_0 (t + t_0) \right]^{\frac{1}{a}} \exp \left\{ N_1 \left[ m_0 (t + t_0) \right]^{\frac{a-3}{a}} \right\}
\]

(23)

\[
X_2 = l_1 \left[ m_0 (t + t_0) \right]^{\frac{1}{a}} \exp \left\{ N_2 \left[ m_0 (t + t_0) \right]^{\frac{a-3}{a}} \right\}
\]

(24)

\[
X_3 = l_2 \left[ m_0 (t + t_0) \right]^{\frac{1}{a}} \exp \left\{ -N_3 \left[ m_0 (t + t_0) \right]^{\frac{a-3}{a}} \right\}
\]

(25)

Where,

\[ N_1 = \frac{2c_0 + C_1}{3m_0(a-3)}, \quad N_2 = \frac{c_1 - c_0}{3m_0(a-3)}, \quad N_3 = \frac{c_0 + 2c_1}{3m_0(a-3)} \]

By using transformations \( t + t_0 = T, l_0 x = X, l_1 y = Y, l_2 z = Z \), the line element (1), reduces

\[
ds^2 = -dT^2 + (a m_0 T)^{2/3} \left\{ \exp 2 \left[ N_1 (a m_0 T)^{\frac{a-3}{a}} \right] dX^2 + \exp 2 \left[ N_2 (a m_0 T)^{\frac{a-3}{a}} \right] dY^2 + \exp 2 \left[ -N_3 (a m_0 T)^{\frac{a-3}{a}} \right] dZ^2 \right\}
\]

(26)

**Case 2: \( \text{For} \alpha = 0 \),**

we obtain the following solution,

\[
X_1 = l_0 \exp[m_0 (t - t_c)] - (Z_0) e^{-3m_0(t - t_c)}
\]

(27)

\[
X_2 = l_1 \exp[m_0 (t - t_c)] - (Z_1) e^{-3m_0(t - t_c)}
\]

(28)

\[
X_3 = l_2 \exp[m_0 (t - t_c)] + (Z_2) e^{-3m_0(t - t_c)}
\]

(29)

Where,

\[ Z_0 = \frac{2c_0 + C_1}{9m_0}, \quad Z_1 = \frac{c_1 - c_0}{9m_0}, \quad Z_2 = \frac{c_0 + 2c_1}{9m_0} \]

Using suitable transformation, we get

\[
ds^2 = -dT^2 + \exp(2m_0 T) \left\{ \exp[-2Z_0 e^{-3m_0 T}] dX_1^2 + \exp[-2Z_1 e^{-3m_0 T}] dX_2^2 + \exp[-2Z_2 e^{-3m_0 T}] dX_3^2 \right\}
\]

(30)

4. **Physical and Geometrical properties of the model**

By using equation (5)-(8), we can write in the following simple form of \( H \) & \( q \)

\[ \bar{p} = (2q - 1)H^2 - \sigma^2 \]

(31)
And
\[ \rho = 3H^2 - \sigma^2 \] (32)

Now adding (30) and (31), we get the expression for \( \Lambda \)
\[ \Lambda = 2(q + 1)H^2 + \zeta \theta - 2\sigma^2 - \gamma \rho \] (33)

**For the Case-I**, we obtain
\[ H_1 = N_1 m_0(\alpha - 3)(\alpha m_0 T)^{-3/\alpha} + m_0 (\alpha m_0 T)^{-1} \] (34)
\[ H_2 = N_2 m_0(\alpha - 3)(\alpha m_0 T)^{-3/\alpha} + m_0 (\alpha m_0 T)^{-1} \] (35)
\[ H_3 = N_3 m_0(\alpha - 3)(\alpha m_0 T)^{-3/\alpha} + m_0 (\alpha m_0 T)^{-1} \] (36)

The expression for Hubble parameter is,
\[ H = m_0(\alpha m_0 T)^{-1} \] (37)

The expression for expansion scalar is,
\[ \theta = H_1 + H_2 + H_3 = 3m_0 (\alpha m_0 T)^{-1} \] (38)

The expression for shear scalar is,
\[ \sigma^2 = \frac{k^2}{3} (m_0 \alpha T)^{-6/\alpha} \] (39)

The expression for anisotropic parameter is,
\[ \bar{A} = \frac{2k^2}{\eta m_0^2} (\alpha m_0 T)^{-2\alpha-6} \] (40)

The expression for bulk viscosity is,
\[ \zeta = \zeta_0 m_0^2 (\alpha m_0 T)^{-2} + \zeta_1 m_0 (\alpha m_0 T)^{-1} + \zeta_2 \] (41)

The expression for density is given by,
\[ \rho = 3(\alpha T)^{-2} - \frac{k^2}{3} (\alpha m_0 T)^{-6/\alpha} \] (42)

From equation (33), the expression for cosmological term \( \Lambda \) is given by
\[ \Lambda = (2\alpha - 3\gamma)(\alpha T)^{-2} - (2 - \gamma) \frac{k^2}{3} (\alpha m_0 T)^{-6/\alpha} + 3[\zeta_2 (\alpha m_0 T)^{-1} + [\zeta_0 (\alpha T)^{-1} + \zeta_1](\alpha T)^{-2}] \] (43)

Hence the expression for pressure is given by,
\[ p = \frac{(2\alpha-3)}{a^2 T^2} - \frac{k^2}{3(\alpha m_0)^2} \frac{1}{t^2} + \frac{2\alpha}{a^2 T^2} + 3 \left\{ \zeta_2 \frac{1}{m_0 \alpha t} + \left( \zeta_0 \frac{1}{t^2} + \zeta_1 \right) \frac{1}{a^2 T^2} \right\} \] (44)

The expression for deceleration parameter is,
\[ q = \alpha - 1 \] (45)

**For the Case-II**, we obtain
The expression for expansion scalar is,
\[ \theta = H_1 + H_2 + H_3 = 3m_0 \]  
(49)

The expression for shear scalar is,
\[ \sigma^2 = \frac{k^2}{3} e^{-6m_0 T} \]  
(50)

The expression for anisotropic parameter is,
\[ \tilde{A} = \frac{2k^2}{9m_0^2} e^{-6m_0 T} \]  
(51)

The expression for bulk viscosity is given by,
\[ \zeta = \zeta_0 m_0^2 + \zeta_1 m_0 + \zeta_2 = Q(\text{constant}) \]  
(52)

The expression for density is given by,
\[ \rho = 3m_0^2 - \frac{k^2}{3} e^{-6m_0 T} \]  
(53)

The expression for cosmological term \( \Lambda \) is given by
\[ \Lambda = (2\alpha - 3\gamma)m_0^2 - (2 - \gamma) \frac{k^2}{3} e^{-6m_0 T} + 3\{\zeta_2 m_0 + \{\zeta_0 m_0 + \zeta_1\}m_0^2\} \]  
(54)

Hence the expression for pressure is given by,
\[ p = (2\alpha - 3)m_0^2 - \frac{k^2}{3} e^{-6m_0 T} + 3\{\zeta_2 m_0 + \{\zeta_0 m_0 + \zeta_1\}m_0^2\} \]  
(55)

5. Conclusion

- In the present paper we have analyzed Bianchi type-I cosmological model with inhomogeneous equation of state in the presence of bulk viscous fluid. To get the solution here we have assumed special law of Hubble parameter i.e., \( H \propto \frac{1}{R^\alpha} \). Depending upon the \( \alpha \) in this paper we discussed the two cases, Case I \( \alpha \neq 0 \) and Case II \( \alpha = 0 \).
- From expression (38) and (42) it is clear that the expansion scalar(\( \theta \)) and energy density \( \rho \) are decreasing functions of time \( t \). Also it is found that the Hubble parameter \( H \) decreases with time and is constant at present day.
- It is observed that for \( \alpha = 1 \), the deceleration parameter vanishes.
- In case I the spatial volume \( V \) gradually increases as time \( t \) increases.
- From the expression (38) & (39) we observed that the ratio \( \frac{a}{t} \) becomes zero when \( t \) tends to infinity, provided \( \alpha < 3 \). Therefore the resultant reaches to isotropy for infinite value of \( T \). Hence the model represent shearing, non-rotating and expanding model of universe with a big bang start at late times.
- The value of cosmological term \( \Lambda \) from the expression (43) is found to be very small and positive which supported by results from Supernovae observations obtained by high-Z Supernova team and Supernova cosmological project [27-30]. Also density approaches to a small positive value as per recent observation.
- In Case II when $T$ increases, the scale factor $X_1, X_2, X_3$ and spatial volume $V$ increases exponentially whereas the pressure $p$, energy density $\rho$, cosmological term $\Lambda$, anisotropy parameter $\tilde{A}$, shear scalar $\tilde{\sigma}$ decreases.
- The interesting part of case II is, the expansion scalar ($\theta$) and bulk viscosity $\zeta$ are constant throughout the evolution of universe.
- For $t \to \infty$, the scalar factors and volume also tends to infinity and anisotropy parameter becomes zero.
- In this way we have presented the details of both the cosmological models in the framework of Bianchi type-I.

6. References

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