On cohomology of almost complex 4-manifolds

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Abstract. Based on recent work of T. Draghici, T.-J. Li and W. Zhang, we further investigate properties of the dimension $h_J^-$ of the $J$-anti-invariant cohomology subgroup $H_J^-$ of a closed almost Hermitian 4-manifold $(M, g, J, F)$ using metric compatible and fundamental 2-form compatible almost complex structures. We prove that $h_J^-$ is 0 for generic almost complex structures $J$ on $M$. We also prove that $h_J^-$ is constant for almost complex structures $J$ on $M$ that are compatible with the same fundamental 2-form $F$.

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1 Introduction

For an almost complex manifold $(M, J)$, T.-J. Li and W. Zhang [30] introduced subgroups, $H_J^+$ and $H_J^-$, of the real degree 2 de Rham cohomology group $H^2(M, \mathbb{R})$, as the sets of cohomology classes which can be represented by $J$-invariant and $J$-anti-invariant real 2-forms, respectively. Let us denote by $h_J^+$ and $h_J^-$ the dimensions of $H_J^+$ and $H_J^-$, respectively.

It is interesting to consider whether or not the subgroups $H_J^+$ and $H_J^-$ induce a direct sum decomposition of $H^2(M, \mathbb{R})$. In the case of direct sum decomposition, $J$ is said to be $C^\infty$ pure and full (see Definition 2.2).

This is known to be true for integrable almost complex structures $J$ which admit compatible Kähler metrics on compact manifolds of any dimension. In this case, the induced decomposition is nothing but the classical real Hodge-Dolbeault decomposition of $H^2(M, \mathbb{R})$ (see [7]).

Note that there are topological obstructions to the existence of almost complex structures on an even dimensional manifold. For a closed 4-manifold, a necessary condition is that $1 - b_1 + b^+$ be even [7], where $b_1$ is the first Betti number and $b^+$ is the number of positive eigenvalues of the quadratic form on $H^2(M, \mathbb{R})$ defined by the cup product; hence the condition is either $b_1$ be even and $b^+$ odd, or $b_1$ be odd and $b^+$ even.

It is a well-known fact that any closed complex surface with $b^+$ odd is Kähler. This was originally obtained from the classification theory, but direct proofs have been given (cf. [12, 25]).

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In dimension 4, it was proved by T. Draghici, T.-J. Li and W. Zhang \[17\] that on any compact almost complex 4-manifold \((M, J)\), \(J\) is \(C^\infty\) pure and full. Further in \[18\], they computed the subgroups \(H^+_J\) and \(H^-_J\) and their dimensions \(h^+_J\) and \(h^-_J\) for almost complex structures metric related to an integrable one. Using Gauduchon metrics \[21, 22\], they proved that the almost complex structures \(\tilde{J}\) with \(h^-_{\tilde{J}} = 0\) form an open dense set in the \(C^\infty\)-topology in the space of almost complex structures metric related to an integrable one (\[18\] Theorem 1.1). Based on this, they made two conjectures about the dimension \(h^-_J\) of \(H^-_J\) on a compact 4-manifold: Conjecture 2.4 in \[18\] asserts that \(h^-_J\) vanishes for generic almost complex structures \(J\), and Conjecture 2.5 in \[18\] asserts that an almost complex structure \(J\) with \(h^-_J \geq 3\) is necessarily integrable. In particular, they have confirmed Conjecture 2.4 for 4-manifolds with \(b^+ = 1\) (see \[18, Theorem 3.1\]).

In this paper we confirm Conjecture 2.4 in \[18\] completely (see Theorem 1.1 below). For this, let us consider an almost Hermitian manifold \((M, g, J, F)\) where \(J\) is an almost complex structure on a \(2n\)-manifold \(M\), \(g\) an almost Hermitian metric which is \(J\)-compatible, namely, \(g(JX, JY) = g(X, Y)\) for all vector fields \(X\) and \(Y\), and \(F\) is the fundamental 2-form defined by \(F(X, Y) = g(JX, Y)\) (hence \(F\) is also \(J\)-compatible).

Let \(\mathcal{J}\) be the space of all almost complex structures on \(M\) and denote by \(\mathcal{J}^c_g\), \(\mathcal{J}^c_F\) and \(\mathcal{J}^{tF}\) respectively the spaces of \(g\)-compatible, \(F\)-compatible and \(F\)-tame almost complex structures on \(M\); namely,

- \(\mathcal{J}^c_g = \{ J \in \mathcal{J} \mid g(JX, JY) = g(X, Y), \forall X, Y \in TM\}\),
- \(\mathcal{J}^c_F = \{ J \in \mathcal{J} \mid F(JX, JY) = F(X, Y), \forall X, Y \in TM\}\),
- \(\mathcal{J}^{tF} = \{ J \in \mathcal{J} \mid F(JX, JX) > 0, \forall X \in TM, X \neq 0\}\).

It is well known that \(\mathcal{J}^c_g\), \(\mathcal{J}^c_F\) and \(\mathcal{J}^{tF}\) are contractible Fréchet spaces. See \[5, 9, 10, 17, 18, 32, 33\] for details.

Using \(g\)-compatible almost complex structures, we prove that for generic almost complex structures \(J\), the dimension \(h^-_J\) vanishes.

**Theorem 1.1.** Let \(M\) be a closed 4-manifold admitting almost complex structures. Then the set of almost complex structures \(J\) on \(M\) with \(h^-_J = 0\) is an open dense subset of \(\mathcal{J}\) in the \(C^\infty\)-topology.

Lejmi in \[28\] studied the existence of a smooth family of extremal almost Kähler metrics compatible with a fixed symplectic form under the assumption of the invariance of the dimension of \(J\)-anti-invariant cohomology for almost complex structures \(J\) compatible with the same symplectic form.

We shall prove (see Theorem 1.2 below) that, given any closed almost Hermitian 4-manifold \((M, g, J, F)\), the dimension \(h^-_J\) is constant for all \(\tilde{J} \in \mathcal{J}^{tF}\); in other words, the dimension of \(J\)-anti-invariant cohomology is stable under deformation of almost complex structures in \(\mathcal{J}^{tF}\).

**Theorem 1.2.** Suppose \((M, g, J, F)\) is a closed almost Hermitian 4-manifold. If \(\tilde{J} \in \mathcal{J}^{tF}\), then \(h^-_{\tilde{J}} = h^-_J\).
Note that the dimension $h^-_J$ is not constant under deformation of almost complex structure $J$ in $\mathcal{J}_F^c$ (see Remark 1.5). When $J$ is integrable, it follows from [17] Proposition 2.15] that $h^+_J = h^{1,1}_g$ and $h^-_J = 2h^{2,0}_g$, where $h^{1,1}_g$ and $h^{2,0}_g$ are (complex) dimensions of Dolbeault cohomology $H^{1,1}_J$ and $H^{2,0}_J$, respectively. Note that $h^{2,0}_g$ is also called the geometric genus of the compact complex surface. As noted in [17], together with the signature theorem ([7, Theorem 2.7]), this implies that if $b_1$ is even, then $h^+_J = b^- + 1$ and $h^-_J = b^+ - 1$, and that if $b_1$ is odd, then $h^+_J = b^-$ and $h^-_J = b^+$. Hence, for a complex surface $(M, J)$, $h^-_J$ is a topological invariant, equal to $b^+$ or $b^- - 1$. Thus, by Theorem 1.2 we have:

**Corollary 1.3.** If $(M, g, J, F)$ is a closed complex surface and $\tilde{J} \in \mathcal{J}_F^c$, then

$$h^-_{\tilde{J}} = h^-_J = \begin{cases} b^+ - 1, & \text{if } b_1 \text{ is even,} \\ b^+, & \text{if } b_1 \text{ is odd.} \end{cases}$$

**Remark 1.4.** In general, an almost complex structure which is $F$-compatible with an integrable one may be non-integrable. Indeed, Kim [23] proved that every closed symplectic 4-manifold $(M, F)$ admits an $F$-compatible almost Kähler metric of negative scalar curvature. On the other hand, for a closed Kähler surface $(M, g, J, F)$, we have the following estimate for the total scalar curvature:

$$\int_M S(g) \, d\mu_g \leq 4\pi c_1(J) \cup [F],$$

with equality if and only if the structure is Kähler, where $c_1(J)$ is the first Chern class of $(M, J)$ depending only on the homotopy class of $J$, $S(g)$ is the scalar curvature of $g$, and $[F]$ is the cohomology class of $F$ ([11, 17, 8, 11, 23, 36]). It follows that (cf. [20, 31, 37]) there exist no negatively scalar curved Kähler metrics on rational complex surfaces and $S^2$-bundles, although there exist on them negatively scalar curved almost Kähler metrics by Kim’s results. In particular, A.-K. Liu [31] has classified symplectic 4-manifolds with

$$c_1(M) \cup [F] = \int_M c_1(J) \wedge F > 0$$

and there are no examples beyond the standard ones furnished by rational complex surfaces and $S^2$-bundles. Since $c_1 = 0$ for $K3$ surfaces and torus $T^4$, by Kim’s result, these surfaces give examples with $b^+ \geq 3$ which admit non-integrable complex structures $J$ with $h^-_J = b^+ - 1$. Hence one can drop the assumption in [28] Theorem 1.1] that $h^-_J$ is invariant under deformation of almost complex structure $J$ compatible with a fixed symplectic 2-form.

We thus propose the following modification of Conjecture 2.5 in [18].

**Question 1.5.** Suppose $(M, g, J, F)$ is a closed almost Hermitian 4-manifold with $h^-_J \geq 3$. Does there exist an integrable $\tilde{J} \in \mathcal{J}_F^c$? 

By Corollary 1.3 we have the following question.
Question 1.6. Suppose \((M, g, J, F)\) is a closed almost Hermitian 4-manifold with \(h_J^- = b^+\) if \(b_1\) is odd, and \(h_J^- = b^+ - 1\) if \(b_1\) is even. Does there exist an integrable \(\tilde{J} \in J_c^c\)?

For compact complex surfaces \((M, J)\), \(b_+\) is stable under the blowing up operation. By Corollary 1.3, this gives the invariance of \(h_J^-\) under blowing up. Similarly, one can do the blowing up operation on almost complex 4-manifolds (cf. [7, 32]). Thus it is reasonable to ask the following question.

Question 1.7. Is \(h_J^-\) stable under the blowing up operation for non-integrable almost complex structures \(J\) on almost complex 4-manifolds?

The rest of the paper is organized as follows. In §2 we recall definitions and preliminary results mainly as given in [13]. In §3 we briefly discuss constructions of \(g\)- and \(F\)-compatible almost complex structures. Finally in §4 we give proofs of Theorems 1.1 and 1.2.

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2 Definitions and Preliminaries

Suppose \((M, J)\) is a closed almost complex 4-manifold. One can construct a \(J\)-invariant Riemannian metric \(g\) on \(M\), that is, \(g(JX, JY) = g(X, Y)\) for all tangent vector fields \(X\) and \(Y\) on \(M\). Such a metric \(g\) is called an almost Hermitian metric for \((M, J)\). This then in turn gives a \(J\)-compatible non-degenerate 2-form \(F\) on \(M\) by \(F(X, Y) = g(JX, Y)\), called the fundamental 2-form. Such a quadruple \((M, g, J, F)\) is called a closed almost Hermitian 4-manifold. Thus an almost Hermitian structure on \(M\) is a triple \((g, J, F)\). If the almost complex structure \(J\) is integrable, the triple \((g, J, F)\) is called a Hermitian structure. If the 2-form \(F\) is closed (i.e., symplectic), then the triple \((g, J, F)\) is called an almost Kähler structure. When the two conditions hold simultaneously, the \((g, J, F)\) defines a Kähler structure on \(M\). A metric will be called Hermitian, almost Kähler, or Kähler if it admits a compatible corresponding structure.

Let \(\Omega^2(M)\) denote the space of smooth 2-forms on \(M\), that is, the \(C^\infty\) sections of the bundle \(\Lambda^2(M)\). Then \(J\) acts on \(\Omega^2(M)\) as an involution by

\[\alpha(\cdot, \cdot) \mapsto \alpha(J\cdot, J\cdot), \quad \alpha \in \Omega^2(M).\]

This gives the vector bundle splitting

\[\Lambda^2 = \Lambda^+_J \oplus \Lambda^-_J,\]

where the bundles \(\Lambda^\pm_J\) are defined by

\[\Lambda^\pm_J = \{\alpha \in \Lambda^2 \mid \alpha(J\cdot, J\cdot) = \pm \alpha(\cdot, \cdot)\}.\]
Denote by $\Omega_J^+$ and $\Omega_J^-$, respectively, the $C^\infty$ sections of the bundles $\Lambda_J^+$ and $\Lambda_J^-$, that is, the spaces of $J$-invariant and $J$-anti-invariant 2-forms. For $\alpha \in \Omega^2(M)$, denote by $\alpha_J^+$ and $\alpha_J^-$, respectively, the $J$-invariant and $J$-anti-invariant components of $\alpha$ with respect to the decomposition (2.2).

**Remark 2.1.** Note that $\Lambda_J^-$ is a vector bundle of rank two and $\Lambda_J^-$ inherits an almost complex structure, still denoted by $J$, defined by

$$(J\alpha)(X,Y) = -\alpha(JX,Y), \quad \alpha \in \Lambda_J^-.$$ 

It is well known that, when $J$ is integrable, $\beta \in \Omega_J^-$ if and only if $J\beta \in \Omega_J^-$. Conversely, if $(M,J)$ is a connected almost complex 4-manifold and there exists nonzero $\beta \in \Omega_J^-$ such that $J\beta \in \Omega_J^-$, then $J$ is integrable (see [34]).

T.-J. Li and W. Zhang defined in [30] the $J$-invariant and $J$-anti-invariant cohomology subgroups $H_J^\pm$ of $H^2(M;\mathbb{R})$.

**Definition 2.2.** (cf. [30, 17]) Let $\mathcal{Z}^2$ denote the space of closed 2-forms on $M$ and set

$$Z_J^+ := \mathcal{Z}^2 \cap \Omega_J^+, \quad Z_J^- := \mathcal{Z}^2 \cap \Omega_J^-.$$ 

Define the $J$-invariant and $J$-anti-invariant cohomology subgroups $H_J^\pm$ by

$$H_J^\pm = \{ a \in H^2(M;\mathbb{R}) \mid \text{there exists } \alpha \in Z_J^\pm \text{ such that } a = [\alpha] \}.$$ 

We say $J$ is $C^\infty$-pure if $H_J^+ \cap H_J^- = \{0\}$, $C^\infty$-full if $H_J^+ + H_J^- = H^2(M;\mathbb{R})$, and thus $J$ is $C^\infty$-pure and full if and only if

$$H^2(M;\mathbb{R}) = H_J^+ \oplus H_J^-.$$ 

**Proposition 2.3.** [17, Theorem 2.3] For any closed almost complex 4-manifold $(M,J)$, the almost complex structure $J$ is $C^\infty$-pure and full.

**Remark 2.4.** Recently, Angella and Tomassini [2] showed that the Iwasawa manifold of dimensions 6 admits complex structures which are neither $C^\infty$-pure nor $C^\infty$-full. Other interesting examples appear in [3], showing, in particular, that the notions of $C^\infty$-pure and of $C^\infty$-full are not related. The first 6-dimensional examples of (non-integrable) almost complex nilmanifolds which are neither $C^\infty$-pure nor $C^\infty$-full were given by Fino and Tomassini [19]. Draghici, Li and Zhang [18] proved that $C^\infty$-pure property no longer holds even for Kähler $J$, if one gives up the compactness of the manifolds.

Since $(M,g,J,F)$ is a closed almost Hermitian 4-manifold, the Hodge star operator $*_g$ gives the well-known self-dual, anti-self-dual splitting of the bundle of 2-forms:

$$\Lambda^2 = \Lambda^+_g \oplus \Lambda^-_g.$$ 

We denote by $\Omega^\pm_\Lambda$ the spaces of sections of $\Lambda^\pm_\Lambda$, and by $\alpha^+_g$ and $\alpha^-_g$ respectively the self-dual and anti-self-dual components of a 2-form $\alpha$. Since the Hodge-de Rham Laplacian commutes with $*_g$, the decomposition (2.4) holds for
the space $\mathcal{H}_g$ of harmonic 2-forms as well. By Hodge theory, this induces cohomology decomposition by the metric $g$:

$$H^2(M; \mathbb{R}) = \mathcal{H}_g = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-.$$  \hfill (2.5)

As in Definition 2.2, one defines

$$H^\pm_g = \{ a \in H^2(M; \mathbb{R}) \ | \ \exists \alpha \in Z^\pm_g := Z^2 \cap \Omega_g^\pm \text{ such that } a = [\alpha] \}. \hfill (2.6)$$

For any $\alpha_1, \alpha_2 \in Z^\pm_g$ such that $[\alpha_1] = [\alpha_2]$, we have $\alpha_1, \alpha_2 \in H^\pm_g$ and

$$\|\alpha_1 - \alpha_2\|^2 = \int_M (\alpha_1 - \alpha_2) \wedge * (\alpha_1 - \alpha_2) = \pm \int_M (\alpha_1 - \alpha_2)^2 = 0. \hfill (2.7)$$

Hence

$$H^\pm_g = Z^\pm_g = H^\pm_g$$

and (2.5) can be written as

$$H^2(M; \mathbb{R}) = H^+_g \oplus H^-_g. \hfill (2.8)$$

There are the following relations between the decompositions (2.2) and (2.4) on an almost Hermitian 4-manifold:

$$\Lambda^+_j = \mathbb{R} \cdot F \oplus \Lambda^-_j, \hfill (2.9)$$

$$\Lambda^+_g = \mathbb{R} \cdot F \oplus \Lambda^-_j, \hfill (2.10)$$

$$\Lambda^+_j \cap \Lambda^+_g = \mathbb{R} \cdot F, \quad \Lambda^-_j \cap \Lambda^-_g = \{0\}. \hfill (2.11)$$

See [16] for details. It is easy to see that $H^-_j \subset \mathcal{H}_g^+$ and $\mathcal{H}_g^- \subset H^+_j$.

Let $b_2$ be the second Betti number, $b^+$ the self-dual Betti number, and $b^-$ the anti-self-dual Betti number of $M$; thus $b_2 = b^+ + b^-$. If $[\alpha_1] = [\alpha_2] \in H^-_j$ with $\alpha_1, \alpha_2 \in Z^-_j$, then by (2.7) and (2.10), we have $\alpha_1 = \alpha_2$. Thus, for a closed almost Hermitian 4-manifold $(M, g, J, F)$, there hold (see [17]):

$$H^-_j = Z^-_j, \quad h^+_j + h^-_j = b_2, \quad h^+_j \geq b^-, \quad h^-_j \leq b^+. \hfill (2.12)$$

Lejmi [27] recognizes $Z^-_j$ as the kernel of an elliptic operator on $\Omega^-_j$.

**Lemma 2.5.** [27, 28] Let $(M, g, J, F)$ be a closed almost Hermitian 4-manifold. Let operator $P : \Omega^-_j \to \Omega^-_j$ be defined by

$$P(\psi) = P^-_j (d\delta \psi) = (d\delta \psi)^-_j,$$

where $P^-_j : \Omega^2 \to \Omega^-_j$ is the projection, $\delta$ is the codifferential operator with respect to metric $g$. Then $P$ is a self-adjoint strongly elliptic linear operator with kernel the $g$-harmonic $J$-anti-invariant 2-forms. Hence,

$$\Omega^-_j = \ker P \oplus P^-_j (d\Omega^1) = H^-_j \oplus P^-_j (d\Omega^1).$$
In the case when \((M, J)\) is a closed complex surface, by using Gauduchon metrics on \((M, J)\), we have that the subgroups \(H^+_J\) are nothing but the (real) Dolbeault cohomology groups (see [2, 7, 17]):

\[
H^+_J = H^{1,1}_J \cap H^2(M; \mathbb{R}), \quad H^-_J = (H^{0,2}_J \oplus H^{2,0}_J) \cap H^2(M; \mathbb{R}). \tag{2.13}
\]

This gives the following relations between \(h^+_J\) and \(b^\pm\) for complex surfaces.

**Proposition 2.6.** [17, Proposition 2.15] If \((M, J)\) is a closed complex surface, then \(h^+_J\) are topological invariants of \(M\). Precisely, if \(b_1\) is even, then \(h^+_J = b^- + 1\) and \(h^-_J = b^+ - 1\), while if \(b_1\) is odd, then \(h^+_J = b^-\) and \(h^-_J = b^+\).

Let us denote the dimensions of \(H^{1,1}_J\) and \(H^{2,0}_J\) by \(h^{1,1}_J\) and \(h^{2,0}_J\), respectively. It follows from Proposition 2.6 that

\[
h^+_J = h^{1,1}_J, \quad h^-_J = 2h^{2,0}_J.
\]

Together with the signature theorem [7], we get

\[
h^+_J = \begin{cases} b^- + 1, & \text{if } b_1 \text{ is even}, \\ b^-, & \text{if } b_1 \text{ is odd}; \end{cases} \quad h^-_J = \begin{cases} b^+, & \text{if } b_1 \text{ is even}, \\ b^-, & \text{if } b_1 \text{ is odd}. \end{cases}
\]

Let \(H^{+,-}_J\) denote the subgroup of \(\mathcal{H}^+_g\) which is orthogonal to \(H^-_J\) with respect to the cup product; that is,

\[
H^{+,-}_J := \{ \omega \in Z^+_g \mid \int_M \omega \wedge \alpha = 0 \quad \forall \alpha \in Z^-_J \}. \tag{2.14}
\]

The following lemma will be used in §4.

**Lemma 2.7.** [17, Lemmas 2.4 and 2.6] Let \((M, g)\) be a closed Riemannian 4-manifold. If \(\alpha \in \Omega^+_g\) and \(\alpha = \alpha_h + d\theta + \delta \psi\) is its Hodge decomposition, then

\[
(d\theta)^+_g = (\delta \psi)^+_g \quad \text{and} \quad (d\theta)^-_g = -(\delta \psi)^-_g.
\]

Moreover, 2-form \(\alpha - 2(d\theta)^+_g = \alpha_h\) is harmonic and \(\alpha + 2(d\theta)^-_g = \alpha_h + 2d\theta\). In particular, if \((M, g, J, F)\) is a closed almost Hermitian 4-manifold and \(\alpha \in H^{+,-}_J\), then \(\alpha = fF + (d\theta)^-_J\) for some function \(f \not\equiv 0\) and \(\alpha - d\theta \in Z^+_J\).

**Remark 2.8.** As a direct consequence of Lemmas 2.5 and 2.7 we have

\[
\mathcal{H}^+_g = H^-_J \oplus H^{+,-}_J, \\
H^+_J = H^{+,-}_J \oplus \mathcal{H}^-_g.
\]

### 3 \(g\)- and \(F\)-compatible almost complex structures

In this section, let us fix an almost complex manifold \((M, g, J, F)\) and briefly describe constructions of \(g\)- and \(F\)-compatible almost complex structures on \(M\) (note that \(g\)-compatible almost complex structures are also called \(g\)-related ones [17, 18]); for details, see [6, 9, 27].
Recall that for a closed almost Hermitian 4-manifold \((M, g, J, F)\), \(g\) and \(F\) are \(J\)-compatible, that is, \(g(J\cdot, J\cdot) = g\), \(F(J\cdot, J\cdot) = F\) and \(F = g(J\cdot, \cdot)\). By a direct calculation, we have

\[
F \wedge F = 2d\mu_g,
\]

where \(d\mu_g\) is the volume form of \(M\) determined by \(g\).

Let us recall that the almost complex structure \(J\) acts on the cotangent bundle \(T^*M\) by \(J \cdot \alpha(X) = -\alpha(JX)\), where \(\alpha\) is a 1-form and \(X\) a vector field on \(M\). Hence \(J\) induces an action \(J \otimes J\) on \(\otimes^2 T^*M\), still denoted by \(J\). A section \(\psi\) of the bundle \(\otimes^2 T^*M\) admits an orthogonal splitting

\[
\psi = \psi^{J, +} + \psi^{J, -}
\]

as the sum of \(J\)-invariant and \(J\)-anti-invariant parts, where

\[
\psi^{J, +} (\cdot, \cdot) = \frac{1}{2} (\psi(\cdot, \cdot) + \psi(J\cdot, J\cdot)),
\]

\[
\psi^{J, -} (\cdot, \cdot) = \frac{1}{2} (\psi(\cdot, \cdot) - \psi(J\cdot, J\cdot)).
\]

In particular, the bundle of 2-forms decomposes under the action of \(J\) as:

\[
\Lambda^2(M) = \Lambda^+ J(M) \oplus \Lambda^- J(M),
\]

and the symmetric tensor bundle of type \((2, 0)\) decomposes as:

\[
S^2(M) = S^+ J(M) \oplus S^- J(M).
\]

Let us first construct \(g\)-compatible almost complex structures using sections of the bundle of \(J\)-anti-invariant 2-forms. Given \(\alpha \in \Omega_{-J}^2\), define tensor field \(K_\alpha\) of type \((1, 1)\) by

\[
g(X, K\alpha Y) = \alpha(X, Y).
\]

It can be checked that \(K_\alpha\) and \(JK\alpha\) are skew-adjoint. It follows that \(\text{Id} + JK\alpha\) is invertible \([26]\). Define tensor field \(J_\alpha\) of type \((1, 1)\) and 2-form \(F_\alpha\) by

\[
J_\alpha := (\text{Id} + JK\alpha)^{-1} J(\text{Id} + JK\alpha), \quad F_\alpha := g(J_\alpha \cdot, \cdot).
\]

The \(J_\alpha\) and \(F_\alpha\) so defined have the following properties.

**Proposition 3.1.** \([26]\) Proposition 1.5] Given \(\alpha \in \Omega_{-J}^2\), we define the norm function \(|\alpha|\) in \(C^\infty(M)\) of \(\alpha\) by \(\alpha \wedge \alpha = 2|\alpha|^2 d\mu_g\), i.e.,

\[
|\alpha|^2 = (\alpha \wedge \alpha)/(2d\mu_g).
\]

Then the \(J_\alpha\) and \(F_\alpha\) defined by \([3.3]\) satisfy:

\[
g(J_\alpha X, J_\alpha Y) = g(X, Y),
\]

\[
J_\alpha = \frac{1 - |\alpha|^2}{1 + |\alpha|^2} J - \frac{2}{1 + |\alpha|^2} K_\alpha,
\]

\[
F_\alpha = \frac{1 - |\alpha|^2}{1 + |\alpha|^2} F + \frac{2}{1 + |\alpha|^2} \alpha.
\]
Remark 3.2. The $J_\alpha$'s are sometimes called $g$-related almost complex structures (cf. [17, 18]). Note that the fibre bundle $\mathcal{Z}$ of all $g$-compatible almost complex structures is called the twistor space of $(M, g, J)$ (see [1]):

$$\mathcal{Z} = \{ J \in \text{SO}(TM) \mid J^2 = -\text{id} \} = \text{SO}(TM)/U(2);$$

so the twistor fibration $\pi : \mathcal{Z} \to M$ is an $S^2$ bundle. By using $g$-related almost complex structures one can study the twistor space of Riemannian 4-manifolds, complex structures on Riemannian 4-manifolds, Gromov-Witten invariants for Kähler surfaces and the dimension of $J$-anti-invariant cohomology of closed almost complex 4-manifolds (cf. [17, 18, 26, 34]).

Now we construct $F$-compatible almost complex structures by using $J$-anti-invariant symmetric tensor fields of type $(2,0)$. Suppose $h$ is a $J$-anti-invariant symmetric tensor field of type $(2,0)$, that is,

$$h(X,Y) = h(Y,X), \quad h(JX,JY) = -h(X,Y)$$

for all vector fields $X$ and $Y$. Define tensor field $S_h$ of type $(1,1)$ by

$$g(X,S_hY) = h(X,Y). \quad (3.9)$$

Then we have $tS_h = S_h$, since

$$g(X,S_hY) = h(X,Y) = h(Y,X) = g(Y,S_hX) = g(S_hX,Y).$$

Similarly, we have $S_h = JS_hJ$, that is,

$$S_hJ + JS_h = 0. \quad (3.10)$$

It follows that, for $k = 1, 2, \ldots$,

$$S_h^{2k-1}J + JS_h^{2k-1} = 0, \quad S_h^{2k}J - JS_h^{2k} = 0. \quad (3.11)$$

Define tensor field $J_h$ of type $(1,1)$ by

$$J_h := J e^{S_h} = e^{-S_h} J. \quad (3.12)$$

Then $J_h^2 = J e^{S_h} e^{-S_h} J = J^2 = -\text{Id}$. Define tensor field $g_h$ of type $(2,0)$ by

$$g_h(X,Y) = g(X,e^{S_h}Y). \quad (3.13)$$

Then $g_h$ is $J_h$-compatible, since

$$g_h(J_hX,J_hY) = g(J e^{S_h}X, e^{S_h} e^{-S_h} J Y) = g(J e^{S_h}X, JY) = g(e^{S_h}X,Y) = g(X,e^{S_h}Y) = g_h(X,Y). \quad (3.14)$$

We claim that $F$ is $J_h$-compatible. Indeed, we have

$$F(J_hX,J_hY) = g(J \circ J \circ e^{S_h}X, e^{-S_h} J Y) = -g(e^{S_h}X, e^{-S_h} J Y) = -g(X,Y) = g(JX,Y) = F(X,Y). \quad (3.15)$$
It follows that (with $g_1^h$ and $g_2^h$ defined by the corresponding sums)

$$g_h(X,Y) = g(X,Y) + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} g(X, S_h^{2k+1} Y) + \sum_{k=1}^{\infty} \frac{1}{(2k)!} g(X, S_h^{2k} Y)$$

$$= g(X,Y) + g_1^h(X,Y) + g_2^h(X,Y).$$

By (3.11), we have

$$g_1^h(X,Y) = -g_1^h(JX,JY), \quad g_2^h(X,Y) = g_2^h(JX,JY).$$

**Proposition 3.3.** Let $h$ be a $J$-anti-invariant symmetric tensor field of type $(2,0)$, and let $S_h$ be the tensor field of type $(1,1)$ defined by $g(X, S_h Y) = h(X,Y)$. Then $S_h + JS_h + S_h J = 0$. Furthermore,

$$J_h := Je^{S_h} = e^{-S_h} J$$

is an $F$-compatible almost complex structure and the corresponding almost Hermitian metric $g_h(\cdot, \cdot) := F(\cdot, J_h \cdot) = g(\cdot, e^{S_h} \cdot)$ is $J_h$-compatible.

Note that $J_h$ is also called an $F$-calibrated almost complex structure (cf. [5]), and $g_h$ an associated metric of $F$ (cf. [9] [10]). Audin [5] gives another construction of $F$-compatible almost complex structures. For any symmetric tensor field $S$ of type $(1,1)$ satisfying $\|S\| < 1$ and $JS + SJ = 0$, tensor field

$$J_S := J(Id + S)(Id - S)^{-1}$$

(3.16)

is an $F$-compatible almost complex structure.

**Remark 3.4.** $J^F_\mathbb{R}$ is the space of sections of a bundle over $(M,J)$ with fibre the Siegel upper half-space $\text{Sp}(4,\mathbb{R})/U(2)$, which has an $\text{Sp}(4,\mathbb{R})$-invariant Kähler metric. Using deformation of $F$-compatible almost complex structures, many authors have studied extremal almost Kähler metrics and almost Kähler metrics of negative scalar curvature (see [13] [14] [15] [23] [27] [28]).

## 4 Proofs of Theorems 1.1 and 1.2

In this section we present proofs of Theorems 1.1 and 1.2.

We first prove Theorem 1.2 by deforming almost complex structures within the $F$-compatible ones.

**Proof of Theorem 1.2** Suppose $(M,g,J,F)$ is a closed almost Hermitian 4-manifold. Note that, by (2.9)–(2.11), we have the following (pointwise) orthogonal decomposition

$$\Lambda^2 = \mathbb{R}F \oplus \Lambda^-_J \oplus \Lambda^+_J.$$

(4.1)

We define the subgroup $H^{F,\perp}$ of $H^2(M;\mathbb{R})$ by

$$H^{F,\perp} = \{a \in H^2(M;\mathbb{R}) \mid \exists \alpha \in \mathbb{Z}^2 \text{ such that } a = [\alpha], \alpha \wedge F = 0\}.$$
Given $J_1, J_2 \in J^\infty_F$, let $g_i = F(\cdot, J_i)$, $i = 1, 2$. By (4.1), we have

$$\Lambda^2 = \mathbb{R}F \oplus \Lambda^+_J \oplus \Lambda^-_J, \quad i = 1, 2.$$  

It follows that $F$ is pointwise orthogonal to $\Lambda^+_J \oplus \Lambda^-_J$, $i = 1, 2$. Hence $F$ is orthogonal to $H^+_J \oplus H^-_J$, and thus $H^+_J \oplus H^-_J \subseteq H^{F, \perp}_J$, $i = 1, 2$.

On the other hand, if $0 \neq [\alpha] \in H^{F, \perp}_J \subset H^2(\mathcal{M}; \mathbb{R})$ and $\alpha \wedge F = 0$, then, by Lemma 2.7 and Remark 2.8, we see that $\alpha$ is orthogonal to $H^+_J \oplus H^-_J$ with respect to cup product. Then $[\alpha] \in H^+_J \oplus H^-_J$ and hence $H^+_J \oplus H^-_J \supseteq H^{F, \perp}_J$, $i = 1, 2$. Thus $\alpha$ is orthogonal to $H^{F, \perp}_J$. Therefore $H^{F, \perp}_J$ is pointwise orthogonal to $\Lambda^+_J \oplus \Lambda^-_J$, $i = 1, 2$. Hence $H^{F, \perp}_J = H^+_J \oplus H^-_J$, $i = 1, 2$. Since $\dim H^-_J = b^-$, $i = 1, 2$, it follows that $h^+_J = h^+_J$. This completes the proof of Theorem 1.1.  

To prove Theorem 1.1, let us first describe the $C^\infty$-topology on the space $J^\infty$ of $C^\infty$ almost complex structures on $\mathcal{M}$. For $k = 0, 1, 2, \cdots$, the space $J^k$ of $C^k$ almost complex structures on $\mathcal{M}$ has a natural separable Banach manifold structure. The natural $C^\infty$-topology on $J^\infty$ is induced by the sequence of $C^k$ semi-norms $\|\cdot\|_k$, $k = 0, 1, 2, \cdots$. With this $C^\infty$-topology, $J^\infty$ is a Fréchet manifold. A complete metric which induces the $C^\infty$-topology on $J^\infty$ is defined by

$$d(J_1, J_2) = \sum_{k=0}^\infty \frac{\|J_1 - J_2\|_k}{2^k(1 + \|J_1 - J_2\|_k)}.$$  

For details, see [5, 18, 33].

**Proof of Theorem 1.1.** Let $(M, g, J, F)$ be a closed almost Hermitian 4-manifold. Note that $H^+_J \subset H^+_g$ and hence $h^+_J \leq b^+$. We may assume $b^+ \geq 2$, since the case where $b^+ = 1$ has been proved by T. Draghici, T.-J. Li and W. Zhang (cf. [18, Theorem 3.1]).

To prove the density statement, we may consider a family $J_t$, $t \in (0, 1)$ of almost complex structures on $\mathcal{M}$ which is a deformation of $J$, that is, $J_t \to J$ as $t \to 0$.

If $h^+_J = 0$, then as noted in [18], we can establish path-wise semi-continuity property for $h^\pm$ which follows directly from Lemma 2.5 and a classical result of Kodaira and Morrow ([24, Theorem 4.3]) showing the upper semi-continuity of the kernel of a family of elliptic differential operators. Therefore $h^+_J = 0$ for small $t$.

We now assume that $h^-_J \geq 1$. Let us write $m := h^-_J$ and $l := b^+ - m$. We shall construct a family of $g$-compatible almost complex structures $\{J_c\} \subset J^\infty_g$ where $c$ are cut-off functions to be chosen such that $h^-_{J_c} = 0$ and $J_c \to J$ in the $C^\infty$-topology as $c \to 0$.

First, suppose that $m(= h^-_J) < b^+$. Then $H^{-\perp}_J \neq \emptyset$ and

$$l = \dim H^{-\perp}_J = b^+ - m \geq 1.$$  

(4.3)
For each nonzero $[\omega] \in H_\perp^{-1}$ we set
\[ f_\omega := \langle \omega, F \rangle \in C^\infty(M). \] (4.4)

Then, by Lemma 2.7, $f_\omega \not\equiv 0$. Set
\[ S_J := \{ [\omega] \in H_\perp^{-1} | \int_M \omega^2 = 1 \}. \] (4.5)

Then $S_J$ is a sphere of dimension $b^+ - m - 1 = l - 1$. Define a function $V : S_J \to \mathbb{R}$ as follows: for any $[\omega] \in S_J$,
\[ V(\omega) := \text{vol} (M \setminus f^{-1}_\omega(0)) = \int_{M \setminus f^{-1}_\omega(0)} d\mu_g. \] (4.6)

Then $V$ is a continuous function on $S_J$. When $H_\perp^{-1}$ is non-empty,
\[ \delta_J := \inf_{\omega \in S_J} V(\omega) > 0 \] (4.7)
since $S_J$ is compact. Let $\alpha_1, \ldots, \alpha_m \in Z_J^-$ be such that $[\alpha_1], \ldots, [\alpha_m]$ is an orthonormal basis of $H_\perp$ with respect to cup product. Choose $[\omega_1], \ldots, [\omega_l] \in H_\perp^{-1}$, $l = b^+ - m$
such that $[\alpha_1], \ldots, [\alpha_m], [\omega_1], \ldots, [\omega_l]$ form an orthonormal basis of $H_\perp$ with respect to cup product; namely, for $1 \leq i \leq m$ and $1 \leq j \leq l$,
\[ \int_M \omega_i^2 = \int_M \alpha_j^2 = 1, \quad \int_M \omega_i \wedge \alpha_j = 0, \] (4.8)
and, for $1 \leq i_1 \neq i_2 \leq m$ and $1 \leq j_1 \neq j_2 \leq l$,
\[ \int_M \omega_{i_1} \wedge \omega_{i_2} = \int_M \alpha_{j_1} \wedge \alpha_{j_2} = 0. \] (4.9)

To complete the proof of the density statement in Theorem 1.1, we need the following lemma which is a special case of a theorem of Bär.

**Lemma 4.1.** [6 Main Theorem] Let $M$ be a closed Riemannian 4-manifold. Then every harmonic 2-form $\alpha$ on $M$ has the unique continuation property. Hence if $\alpha \not\equiv 0$, then its nodal set $\alpha^{-1}(0)$ has empty interior; in fact, $\alpha^{-1}(0)$ has Hausdorff dimension $\leq 2$.

By Lemma 4.1, the set $\bigcup_{i=1}^m \alpha_i^{-1}(0)$ has Hausdorff dimension $\leq 2$. Hence
\[ M' := \bigcap_{i=1}^m (M \setminus \alpha_i^{-1}(0)) \]
is an open submanifold of $M$ of full volume: $\text{vol}(M') = \text{vol}(M)$. Choose an open set $U \subset M'$ such that $\text{vol}(U) < \delta_J$. Then $\alpha_j|_U$, $1 \leq i \leq m$ are nonzero sections of $\Lambda_\perp^{-1} U$.

We now construct a new $g$-compatible almost complex structure on $M$ (cf. [17, 26]). Choose a cut-off function $c_1$ such that $\text{supp} c_1 \subset U$ and
\[ |c_1 \alpha| < 1. \] (4.10)
Then, by Proposition 3.1, we get
\[ J_{c_1} := J_{c_1 \alpha_1} = \frac{1 - |c_1 \alpha_1|^2}{1 + |c_1 \alpha_1|^2} J - \frac{2}{1 + |c_1 \alpha_1|^2} K_{c_1 \alpha_1}, \] (4.11)
\[ F_{c_1} := F_{c_1 \alpha_1} = \frac{1 - |c_1 \alpha_1|^2}{1 + |c_1 \alpha_1|^2} F + \frac{2}{1 + |c_1 \alpha_1|^2} c_1 \alpha_1. \] (4.12)

Thus it is easy to see that, as \( c_1 \to 0, \)
\[ J_{c_1} \to J, \quad F_{c_1} \to F. \] (4.13)

Note that \( J_{c_1} \in J_g^c \cap J_F^c. \)

We claim that \( H_{J_{c_1}}^- \subset H_J^- \). Since \( J_{c_1} \) is \( g \)-compatible, we have
\[ H_{J_{c_1}}^- \subset H_g^+ = H_J^- \oplus H_J^{-1}. \] (4.14)

Given any nonzero \( \beta \in Z_{J_{c_1}}^{-} \), there exist real constants \( \xi_i \) and \( \eta_j \) such that
\[ \beta = \sum_{i=1}^l \xi_i \omega_i + \sum_{j=1}^m \eta_j \alpha_j, \] (4.15)
where \( 1 \leq i \leq l \) and \( 1 \leq j \leq m \). Without loss of generality, we may assume that
\[ \int_M \beta^2 = \sum_{i=1}^l \xi_i^2 + \sum_{j=1}^m \eta_j^2 = 1. \] (4.16)

Obviously, \( \langle \beta, F_{c_1} \rangle = 0. \) Restricted to \( M \setminus \text{supp} c_1 \), we have \( F_{c_1} = F \). On \( M \setminus \text{supp} c_1 \), we get
\[ \langle \beta, F_{c_1} \rangle|_{M \setminus \text{supp} c_1} = \sum_{i=1}^l \xi_i \langle \omega_i, F \rangle|_{M \setminus \text{supp} c_1} = 0. \] (4.17)

If \( \sum_{i=1}^l \xi_i \omega_i \in H_J^{-1} \) is nontrivial, then we put
\[ \beta_1 = \frac{\sum_{i=1}^l \xi_i \omega_i}{(\int_M (\sum_{i=1}^l \xi_i \omega_i)^2)^{1/2}}. \] (4.18)

Obviously, \( \beta_1 \in S_J \). By (4.17) and (4.18), \( f_{\beta_1} = \langle \beta_1, F \rangle \equiv 0 \), restricted to \( M \setminus \text{supp} c_1 \). Hence \( M \setminus f_{\beta_1}^{-1}(0) \subset \text{supp} c_1 \subset U \). It follows that
\[ V(\beta_1) = \text{vol}(M \setminus f_{\beta_1}^{-1}(0)) \leq \text{vol}(U) < \delta_J, \] (4.19)
contradicting the definition of \( \delta_J \) (see (4.7)). Therefore \( \xi_i = 0 \) for \( 1 \leq i \leq l \) and \( \beta = \sum_{j=1}^m \eta_j \alpha_j \). Thus we have proved that if \( H_J^- \neq \emptyset \) then \( H_{J_{c_1}}^- \subset H_J^- \).

Remark 4.2. In [18], T. Draghici, T.-J. Li and W. Zhang have considered \( g \)-related almost complex structures: if almost complex structures \( J \) and \( \tilde{J} \) are \( g \)-related then \( \Lambda_J^- + \Lambda_{\tilde{J}}^- \subset \Lambda_{g}^+ \) and hence \( H_J^- + H_{\tilde{J}}^- \subset H_{g}^+ \).

Secondly, if \( H_J^- = \mathcal{H}^+ \) then we construct any \( J_{c_1} \) and \( F_{c_1} \) such that \( H_{J_{c_1}}^- \subset H_J^- \). In summary, we have obtained the following
Proposition 4.3. Let \((M, g, J, F)\) be a closed almost Hermitian 4-manifold. If \(h_J \geq 1\), then we can construct a \(g\)-compatible almost complex structure \(J_{c_1}\) where the volume of \(\text{supp } c_1\) is small enough, so that
\[ H^{-}_{J_{c_1}} \subset H^{-}_J. \]

The following observation is the key for the computation of \(h_{J_{c_1}}\).

Proposition 4.4. ([18, Proposition 3.7]) Suppose \(J\) and \(J'\) are \(g\)-related almost complex structures on a closed 4-manifold \(M\), with \(J' \not\equiv \pm J\). Then
\[ \dim(H^{-}_J \cap H^{-}_{J'}) \leq 1. \]

Let us return to the proof of the density statement in Theorem 1.1. Since \(H^{-}_{J_{c}} \subset H^{-}_J\), it follows from Propositions 4.3 and 4.4 that \(h^{-}_{J_{c_1}} \leq 1\). Without loss of generality, we may suppose that \(h^{-}_{J_{c_1}} = 1\). Choose \([\alpha] \in H^{-}_J\) such that \(\int_M \alpha^2 = 1\). We then have \(\dim H^{-}_{J_{c_1}} = b^+ - 1\), and by the same reason as that for (4.7),
\[ \delta_{J_{c_1}} := \inf_{\omega \in S^{-}_{J_{c_1}}} V(\omega) > 0. \quad (4.20) \]
Choose a cut-off function \(c_2\) such that \(\text{supp } c_2 \subset M \setminus \alpha^{-1}(0)\) (by Lemma 4.1) and that
\[ \text{vol}(\text{supp } c_2) < \delta_{J_{c_1}}. \quad (4.21) \]
Construct
\[ F_{J_{c_2}} = f_1 F_{J_{c_1}} + c_2 \alpha \]
such that \(F_{J_{c_2}} \wedge F_{J_{c_2}} = 2d\mu_g\). It is easy to see that \(J\) and \(J_{c_2}\) are both \(g\)-compatible (\(g\)-related); thus \(J_{c_2} \in J^c_g \cap J^F_k\).

We claim that \(h^{-}_{J_{c_2}} = 0\). Otherwise, there exists nonzero \(\beta \in H^{-}_{J_{c_2}}\). Then, by the above construction and by Proposition 4.3, \(H^{-}_{J_{c_2}} \subset H^{-}_{J_{c_1}}\); hence \(\beta \in H^{-}_{J_{c_1}}\), \(\beta = \eta \alpha\), \(\eta \neq 0\), and
\[ \langle \beta, F_{c_2} \rangle \equiv 0. \quad (4.22) \]
Restricted to \(\text{supp } c_2\), we have \(F_{c_2} = f_1 F_{c_1} + c_2 \alpha\). It follows that
\[ \langle \beta, F_{c_2} \rangle|_{\text{supp } c_2} = \langle \eta \alpha, c_2 \alpha \rangle = 2\eta c_2 |\alpha|^2 d\mu_g. \quad (4.23) \]
Thus \(\alpha^{-1}(0) \supset \text{supp } c_2\), contradicting Lemma 4.1 Hence \(h^{-}_{J_{c_2}} = 0\).

This completes the proof of the density statement in Theorem 1.1.

Remark 4.5. By the proof above, in contrast to Theorem 1.2, for any closed almost Hermitian 4-manifold \((M, g, J, F)\), we can find a family of almost complex structures \(\{J_{c_2}\} \subset J^c_g \cap J^F_k\) on \(M\) which depend on both of the cut-off functions \(c_1\) and \(c_2\) such that \(h^{-}_{J_{c_2}} = 0\) and \(J_{c_2} \to J\) in the \(C^\infty\)-topology as \(c_1 \to 0\) and \(c_2 \to 0\) simultaneously.
It remains to prove the openness statement in Theorem 3.1. The case $b^+ = 1$ is proved in [18]. Suppose $b^+ \geq 2$. Suppose that $J_k \to J$ as $k \to \infty$ and that

$$m_k := h_{J_k}^{-} \geq 1.$$  

We need to prove that $h_{J}^{-} \geq 1$. Suppose $h_{J}^{-} = 0$. Then $b^+ = h_{J}^{-} + b^-$. Let $g$ be a $J$-compatible metric and set $F = g(J, \cdot)$. Let $\psi^1, \ldots, \psi^{b^+} \in H^+_g$ be a cup-product orthonormal basis, that is,

$$\int_M \psi^i \wedge \psi^j = \delta_{ij}, \quad 1 \leq i, j \leq b^+.$$  

(4.24)

Note that $H^+_g = H^+_J \oplus H^{-1}_J$. Since $h_{J}^{-} = 0$, by Lemma 4.7, $\psi^i = f^i F + P^-_J d\theta^i$, where $\theta^i \in \Omega^1$. Set

$$g_k = \frac{1}{k}(g(\cdot, \cdot) + g(J_k, \cdot, \cdot)), \quad F_k = g_k(J_k, \cdot, \cdot).$$  

Then $(g_k, J_k, F_k) \to (g, J, F)$ as $k \to \infty$. Since $m_k = h_{J_k}^{-} \geq 1$, we may choose an orthonormal basis

$$\{\omega^i(k)\}_{1 \leq i \leq b^+ - m_k} \cup \{\alpha^l(k)\}_{b^+ - m_k + 1 \leq l \leq b^+}$$  

of $H^+_{g_k}$ with respect to cup product. Denote by $\triangle^k$ the Hodge-de Rham Laplace operator associated to $g_k$ and by $G^k$ the Green operator associated to $\triangle^k$. Then, as done in [18] Proof of Theorem 3.1, 

$$\psi^i = (\psi^i - G^k(\triangle^k \psi^i)) + G^k(\triangle^k \psi^i) = \psi^i_{h, k} + \psi^i_{\text{ex}, k},$$  

(4.25)

with $\psi^i_{h, k} := \psi^i - G^k(\triangle^k \psi^i)$ the $g_k$-harmonic part and $\psi^i_{\text{ex}, k} := G^k(\triangle^k \psi^i)$ the $g_k$-exact part. Thus $\psi^i_{h, k} \to \psi^i$ and $\psi^i_{\text{ex}, k} \to 0$ as $k \to \infty$. Moreover, if $(\psi^i_{h, k})^+$ denotes the $g_k$-self-dual part of $\psi^i_{h, k}$, then we still have $(\psi^i_{h, k})^+ \to \psi^i$ as $k \to \infty$ since $\psi^i$ is $g$-self-dual harmonic form. So there exist $a^i_1(k) \in \mathbb{R}$, $1 \leq l \leq b^+$ such that

$$(\psi^i_{h, k})^+ = \sum_{l=1}^{b^+ - m_k} a^i_1(k) \omega^l(k) + \sum_{l=b^+ - m_k + 1}^{b^+} a^i_1(k) \alpha^l(k)$$  

(4.26)

and

$$\langle (\psi^i_{h, k})^+, F_k \rangle = \sum_{l=1}^{b^+ - m_k} a^i_1(k) \langle \omega^l(k), F_k \rangle =: f^i(k).$$  

(4.27)

Since $F_k \to F$ and $(\psi^i_{h, k})^+ \to \psi^i$ as $k \to \infty$, we have $f^i(k) \to f^i$ as $k \to \infty$. Note that $\psi^i = f^i F + P^-_J d\theta^i$. By (4.26), we have

$$P^-_J d\theta^i(k) = P^-_J \sum_{l=1}^{b^+ - m_k} a^i_1(k) d\theta^l(k),$$  

(4.28)
where $\theta^i(k) \in \Omega^1(M)$. Note that $\psi^i \in H^{-1}_{-\infty} = \mathcal{H}_g$, $i = 1, \cdots, b^+$. Then

$$(\psi^i_{h,k})^+ := f^i(k)F_k + P^{-1}_{j_k}d\theta^i(k) \in \mathcal{H}_g^+ \setminus H^{-1}_{-\infty}$$ \tag{4.29}$$

and $(\psi^i_{h,k})^+ \rightarrow \psi^i$, $(\psi^i_{h,k})^+ - (\psi^i_{h,k})^+ \rightarrow 0$ as $k \rightarrow \infty$. Hence

$$\int_M (\psi^i_{h,k})^+ \wedge (\psi^i_{h,k})^+ \rightarrow \int_M \psi^i \wedge \psi^i = 1 \tag{4.30}$$

as $k \rightarrow \infty$. On the other hand,

$$(\psi^i_{h,k})^+ = (\psi^i_{h,k})^+ + \sum_{l=b^+-m_k+1}^{b^+} a^i_l(k)\alpha^l(k). \tag{4.31}$$

It follows that

$$\int_M (\psi^i_{h,k})^+ \wedge (\psi^i_{h,k})^+$$

$$= \int_M (\psi^i_{h,k})^+ \wedge (\psi^i_{h,k})^+ + \int_M \sum_{l=b^+-m_k+1}^{b^+} a^i_l(k)\alpha^l(k) \wedge \sum_{l=b^+-m_k+1}^{b^+} a^i_l(k)\alpha^l(k)$$

$$= \int_M (\psi^i_{h,k})^+ \wedge (\psi^i_{h,k})^+ + \sum_{l=b^+-m_k+1}^{b^+} (a^i_l(k))^2$$

$$\ge \int_M (\psi^i_{h,k})^+ \wedge (\psi^i_{h,k})^+ + (a^i_{b^+}(k))^2,$$

since $h^{-1}_{-\infty} \ge 1$ by a direct computation. Note that

$$\int_M \psi^i \wedge \psi^j = \delta_{ij} \text{ and } (\psi^i_{h,k})^+ \rightarrow \psi^i \text{ as } k \rightarrow \infty, \text{ for } 1 \le i \le b^+.$$ Thus,

$$\int_M (\psi^i_{h,k})^+ \wedge (\psi^i_{h,k})^+ = \sum_{l=1}^{b^+} a^i_l(k)a^l_i(k) \rightarrow \delta_{ij} \tag{4.33}$$

as $k \rightarrow \infty$. Denote by $A(k)$ the $b^+ \times b^+$ matrix: $A(k) = (a^i_l(k))$. Then

$$B(k) = A(k)A(k)^\dagger = \left(\sum_{l=1}^{b^+} a^i_l(k)\delta^i_l(k)\right) \rightarrow I_{b^+} \tag{4.34}$$

as $k \rightarrow +\infty$. Thus $A(k) \rightarrow A \in \text{SO}(b^+)$ as $k \rightarrow \infty$. Hence $A^\dagger \in \text{SO}(b^+)$ and

$$\sum_{l=1}^{n} a^i_l(k)a^l_m(k) \rightarrow \delta_{lm} \tag{4.35}$$
as \( k \to \infty \). In particular, as \( k \to \infty \),

\[
\sum_{i=1}^{n} a_i(k)^2 \to 1.
\]  

(4.36)

This contradicts (4.32). Thus we get \( h_{\tilde{J}} \geq 1 \).

Thus the almost complex structures \( J \) with \( h_{\tilde{J}} = 0 \) form an open subset of \( J \) in the \( C^\infty \)-topology. This completes the proof of Theorem 1.1.

\[ \square \]

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