The Characteristic Class of a Lie Algebra Ideal, Contact Structures and the Poisson Algebra of Basic Functions

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Abstract

Some cohomology classes associated with an ideal in a Lie algebra, the Poisson structure on the algebra of basic functions for a contact structure, its Poisson cohomologies and geometric (pre)quantization are considered and investigated from the algebraic point of view.
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1 Introduction

A simple situation when we have a Lie algebra \( L \) which is a module over a commutative and associative algebra \( A \), and an ideal \( V \) in it, gives rise to many interesting structures. In this paper we consider some generalization of the well-known characteristic classes of a fiber bundle from this point of view. This classes, in this case, are the elements of the cohomology space of \( L \) with values in the homology space of \( V \). Also we define a Poisson structure on the basic algebra (i.e. the algebra of such elements \( a \in A \) that \( V(a) = 0 \)) and then consider the case of the contact manifold from this point of view. Investigate the Poisson cohomologies of this Poisson structure and its prequantization.

2 Lie Algebra Cohomologies with Values in a Module

We start from review of some definitions and facts from the cohomology theory of Lie algebras. Let \( L \) be a Lie algebra, \( A \) be an associative and commutative algebra over the field of complex (or real) numbers and \( S \) be a \( A \)-module over the algebra \( A \).

Definition 1 The triple \( (L, S, A) \) is said to be a Lie module over the Lie algebra \( L \) if the following conditions are satisfied:

1. \( L \) is a module over the algebra \( A \) and there is a Lie algebra homomorphism \( \phi : L \to \text{Der}(A) \), which is also a homomorphism of \( A \)-modules, such that for any \( a \in A \) and \( X,Y \in L \), we have: \( [X,aY] = X(a)Y + a[X,Y] \). Here \( \text{Der}(A) \) denotes the space of derivations of the algebra \( A \) and \( X(a) \) denotes \( \phi(X)(a) \).

2. There is a Lie algebra homomorphism \( \psi : L \to \text{End}_C(S) \), which is also a homomorphism of \( A \)-modules, such that for any \( X \in L \), \( s \in S \) and \( a \in A \), we have: \( X(as) = X(a)s + aX(s) \). Here \( \text{End}_C(S) \) denotes the space of \( C \)-linear mappings from \( S \) to itself and \( X(s) \) denotes \( \psi(X)(s) \).

Sometimes, for brevity, the term “Lie module over \( L \)” will be used for the \( A \)-module \( S \).

For any integer number \( m \geq 1 \), let us denote the space of \( A \)-multilinear mappings from \( L \times \cdots \times L \) to the Lie module \( S \), by \( \Omega^m_A(L, S) \). We also set
that $\Omega^0_A(L, S) = S$ and $\Omega_A(L, S) = \bigoplus_{m=0}^{\infty} \Omega^m_A(L, S)$.

For any $\omega \in \Omega^m_A(L, S)$ and $\{X_1, \cdots, X_{m+1}\} \subset L$, define $d\omega$ by the well-known Koszul formula:

$$
(d\omega)(X_1, \cdots, X_{m+1}) = \sum_i (-1)^{i-1} X_i \omega(X_1, \cdots, \hat{X}_i, \cdots, X_{m+1}) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots)
$$

Lemma 1 If the $A$-module $L$ is projective then the following three conditions are equivalent:

1. For any $a \in A$ and $s \in \Omega^0_A(L, S) = S$: $d(as) = d(a)s + ad(s)$.

2. The $A$-module $S$ is a Lie module over $L$.

3. For any integer $m \geq 1$ and $\omega \in \Omega^m_A(L, S)$, the mapping $d\omega : L^{m+1} \rightarrow S$ is $A$-multilinear (i.e., $(d\omega)(aX_1, \cdots, X_{m+1}) = a(d\omega)(X_1, \cdots, X_{m+1}))$.

Proof. It is clear that the conditions 1 and 2 are equivalent, because for any $X \in L$, we have that:

$$(d(as))(X) = X(as) = X(a)s + aX(s) = (d(a)s + ad(s))(X).$$

Also, it can be verified by direct calculation that from the condition 2 (or, which is the same, from 1) follows the condition 3. Now, suppose that the condition 3 is true. In this case For any $X, Y \in L, a \in A$ and $\omega \in \Omega^1_A(L, S)$, we have:

$$(d\omega)(X, aY) = a(d\omega)(X, Y) \Rightarrow X\omega(aY) - aY\omega(X) - \omega([X, aY]) = aX\omega(Y) - aY\omega(X) - a\omega([X, Y]) \Rightarrow X(a\omega(Y)) - aY\omega(X) - X(a)\omega(Y) - a\omega([X, Y]) =$$

$$= aX\omega(Y) - aY\omega(X) - a\omega([X, Y]) \Rightarrow X(a\omega(Y)) = X(a)\omega(Y) + aX(\omega(Y))$$
Because it is assumed that the \( A \)-module \( L \) is projective, for any \( s \in S \) can be found such \( \omega \in \Omega^1_A(L, S) \) and \( Y \in L \), that \( \omega(Y) = s \). Therefore, we obtain that \( X(as) = X(a)s + aX(s) \), i.e., \( S \) is a Lie module over the Lie algebra \( L \).

Hence, we have that if the \( A \)-module \( S \) is a Lie module over the Lie algebra \( L \), the operator \( d \), carries the space \( \Omega^m_A(L, S) \) into \( \Omega^{m+1}_A(L, S) \), which implies that the pair \( (\Omega_A(L, S), d) \) is a differential complex.

### 3 Lie Algebra Homologies and Supercommutator

Let \( g \) be a Lie algebra and a module over a commutative and associative algebra \( B \). If the Lie algebra bracket in \( g \) is bilinear for the elements of \( B \) ([\( x, ay \] = \( a[x, y] \), \( \forall x, y \in g \) and \( a \in B \)) one has a well-defined boundary operator \( \delta : \wedge_B^m(g) \to \wedge_B^{m-1}(g) \), where \( \wedge_B \) denotes the exterior product of a \( B \)-module by itself:

\[
\delta(x_1 \wedge \cdots \wedge x_m) = \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_m, \ m > 1
\]

and \( \delta(x) = 0, \ \forall \ \{x, x_1, \cdots, x_m\} \subset g \)

The homology given by the boundary operator \( \delta \) is known as the homology of the Lie algebra \( g \) with coefficients from the algebra \( B \).

There is a useful relation between the coboundary operator \( \delta \) and the Schouten-Hijenhuis bracket on the exterior algebra \( \wedge_B^m(g) \): \( \forall u \in \wedge_B^m(g) \) and \( \forall v \in \wedge_B^m(g) \), we have

\[
[u, v] = \delta(u) \wedge v + (-1)^m u \wedge \delta(v) - \delta(u \wedge v)
\]

(1)

It can be said that the supercommutator \([\cdot, \cdot]\) measures the deviation of the boundary operator \( \delta \) from being an antidifferential of degree -1 (see [4]). From the formula (1), easily follows that the induced supercommutator on the homology space \( H_\bullet(g, B) \) is trivial: for \( u \in \wedge_B^m(g) \) and \( v \in \wedge_B^m(g) \), such that \( \delta(u) = \delta(v) = 0 \), we have \([u, v] = -\delta(u \wedge v)\), which implies that the homology class of the element \([u, v]\) is trivial.

From the formula (1), also follows that the homology space \( H_\bullet(g, B) \) does not inherits the exterior algebra structure from \( \wedge_B(g) \): if the elements \( u, v \in \wedge_B(g) \) are closed, then we have that \( \delta(u \wedge v) = -[u, v] \).
4 The Homology Space of a Lie Algebra Ideal

Let $L$ be a Lie algebra and $A$ be a commutative and associative algebra over $\mathbb{C}$ (or $\mathbb{R}$). Suppose that the pair $(L, A)$ satisfies the condition $lI$ from the definition \[\text{from the definition 1.}\]

Let $V \subseteq L$ be an ideal in the Lie algebra $L$ (i.e., $\forall v \in V$ and $\forall x \in L$: $[v, x] \in V$). Define the subalgebra $A_V$, in the algebra $A$, as

$$A_V = \{a \in A | v(a) = 0, \forall v \in V\}$$

and the subspace $V'$ in $L$, as

$$V' = \left\{ \sum_{i=1}^{n} a_i v_i \mid n \in \mathbb{N} \ a_i \in A_V, \ v_i \in V, \ i = 1, \cdots, n \right\}$$

The latter can be defined as the minimal $A_V$-submodule of $L$, containing $V$.

For any $x \in L$, $v \in V$ and $a \in A_V$, we have

$$v(x(a)) = [v, x](a) + x(v(a)) = 0 \Rightarrow x(a) \in A_V$$

which implies that the subalgebra $A_V$ is invariant under the action of the elements of $L$. Therefore, we have that $[x, av] = x(a)v + a[x, v] \in V'$, which means that $V'$ is also an ideal in the Lie algebra $L$. We call the ideal $V'$ the complement of the ideal $V$.

From the fact that $V$ is a subspace of $V'$ follows that $A_{V'} \subseteq A_V$, but on the other hand, for any $a, b \in A_V$ and $v \in V$, we have that $(bv)(a) = b \cdot v(a) = 0$, which means that any $a \in A_V$ is also an element of $A_{V'}$. Hence, the two algebras $A_V$ and $A_{V'}$, coincide.

The ideal $V$ in the Lie algebra $L$ will be called complete, if $V = V'$. Further, by default, we assume that the ideal $V$ is complete (i.e., $V$ is a module over the the algebra $A_V$).

Consider the homology space of the Lie algebra $V$ with coefficients from $A_V$. One can define an action of the Lie algebra $L$ on the $A_V$-module $H_\bullet(V, A_V)$: for $X \in L$ and $v \in V$ such that $\delta(v) = 0$, let $X([v]) = [[X, v]]$, where $[\cdot]$ denotes the homology class and $[\cdot, \cdot]$ denotes the supercommutator.

**Proposition 2** The action of the Lie algebra $L$ on the homology space $H_\bullet(V, A_V)$ is correctly defined and gives a Lie module structure on the $A_V$-module $H_\bullet(V, A_V)$. 

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Proof. As it follows from the formula [1], we have that for any \( X \in L \) and a closed element \( u \in \wedge A(V) \): \([X, u] = -\delta(X \wedge u)\), which implies that \([X, u]\) is closed. If the element \( u \) is exact and \( u = \delta(v) \), then we have

\[
\delta([X, v]) = \delta(-X \wedge \delta(v) - \delta(X \wedge v)) = -\delta(X \wedge u) = -[X, u]
\]

which implies that the element \([X, u]\) is exact. From these two facts follows that the closed and exact element in the exterior algebra \( \wedge A(V) \) are invariant under the supercommutator with the elements of the Lie algebra \( L \). Therefore, the action, \( X([u]) = [[X, u]] \), of \( L \) on the homology space \( H_\bullet(V, A_V) \) is correctly defined. By definition of the algebra \( A_V \) and the properties of the supercommutator, easily follow that for any \( a \in A_V \): \((aX)([u]) = aX([u])\) and \( X(a[u]) = X([au]) = [[X, au]] = X(a)[u] + aX([u])\); which implies that the \( A_V \)-module \( H_\bullet(V, A_V) \) is a Lie module over the Lie algebra \( L \). ■

It follows from the above proposition, that for each integer \( n \geq 0 \), one can consider the the following differential complex \( (\Omega_{A_V}(L, H_n(V, A_V)), d) \), which gives the Lie algebra cohomology of \( L \) with values in the \( A_V \)-module \( H_n(V, A_V) \).

As it was mentioned early, the induced supercommutator on each homology space \( H_n(V, A_V) \), \( n = 0, \cdots, \infty \), is trivial. Therefore, the action of the Lie algebra \( V \) on each \( H_n(V, A_V) \) is trivial, which implies that the submodule of the \( A_V \)-module \( \Omega_{A_V}(L, H_\bullet(V, A_V)) \), consisting of such forms \( \omega \), that \( i_v(\omega) = 0 \), \( \forall v \in V \), is invariant under the action of the differential \( d (i_v(\omega) = 0 \Rightarrow i_v(d\omega) = 0, \ \forall v \in V) \). Hence, one can consider the subcomplex \( (\Omega_{A_V}(L, H_\bullet(V, A_V))_0, d) \) of the differential complex \( (\Omega_{A_V}(L, H_\bullet(V, A_V)), d) \), where

\[
\Omega_{A_V}(L, H_\bullet(V, A_V))_0 = \{ \omega \in \Omega_{A_V}(L, H_\bullet(V, A_V)) \mid i_v(\omega) = 0, \ \forall v \in V \}\n\]

Actually, the complex \( (\Omega_{A_V}(L, H_\bullet(V, A_V))_0, d) \) is canonically isomorphic to the complex \( (\Omega_{A_V}(L/V, H_\bullet(V, A_V)), d) \).

5 The Characteristic Class of a Lie Algebra Ideal

Let us denote by \( \Omega_{A_V}(L, H_\bullet(V, A_V))_1 \) the quotient module

\[
\Omega_{A_V}(L, H_\bullet(V, A_V))/\Omega_{A_V}(L, H_\bullet(V, A_V))_0
\]

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For any integer \( n \geq 0 \), we have the following short exact sequence:

\[
0 \longrightarrow \Omega_{A^*}(L/V, H_n(V, A^*)) \longrightarrow \Omega_{A^*}(L, H_n(V, A^*)) \overset{\pi}{\longrightarrow} \\
\pi \longrightarrow \Omega_{A^*}(L, H_n(V, A^*))_1 \longrightarrow 0
\]

which induces the standard homomorphism of cohomology spaces

\[
\sigma_n : H^*(L, H_n(V, A^*))_1 \longrightarrow H^{*+1}(L/V, H_n(V, A^*))
\]

where \( H^*(L, H_n(V, A^*))_1 \) denotes the cohomology space of the complex \( (\Omega_{A^*}(L, H_n(V, A^*)), d) \) and \( \sigma_n \) is defined as: \( \sigma_n([\pi(\omega)]) = [d\omega] \), for \( \omega \in \Omega_{A^*}(L, H_n(V, A^*)) \), such that \( d(\pi(\omega)) = 0 \) (\( \Rightarrow \pi(d\omega) = 0 \) \( \Rightarrow d\omega \in \Omega_{A^*}(L/V, H_n(V, A^*)) \)); here \([d\omega] \) denotes the cohomology class of the form \( d\omega \) in \( H^{*+1}(L/V, H_n(V, A^*)) \).

Consider the special case when \( n = 1 \). Let us denote the homology space \( H_1(V, A^*) \) by \( H_V \). So, we have an exact sequence

\[
0 \longrightarrow \Omega_{A^*}(L/V, H_V) \longrightarrow \Omega_{A^*}(L, H_V) \overset{\pi}{\longrightarrow} \Omega_{A^*}(L, H_V)_1 \longrightarrow 0 \quad (2)
\]

and the homomorphism: \( \sigma_1 : H^*(L, H_V)_1 \longrightarrow H^{*+1}(L/V, H_V) \).

There is a one-to-one correspondence between the set of splittings of the following short exact sequence

\[
0 \longrightarrow V \longrightarrow L \overset{p}{\longrightarrow} L/V \longrightarrow 0
\]

and the set of such homomorphisms \( \alpha : L \longrightarrow V \), that \( \alpha(v) = v, \ \forall v \in V \) (projection operator). Any such projection operator \( \alpha \), defines a 1-form \( \tilde{\alpha} \in \Omega_{A^*}^1(L, H_V) \):

\[
\tilde{\alpha}(X) = [\alpha(X)], \ \forall X \in L
\]

**Lemma 3** For any two projections \( \alpha, \beta : L \longrightarrow V \), the forms \( \pi(\tilde{\alpha}) \) and \( \pi(\tilde{\beta}) \) in \( \Omega_{A^*}^1(L, H_V)_1 \), are equal and closed.

**Proof.** For such \( \alpha \) and \( \beta \) we have: \( (\alpha - \beta)(v) = v - v = 0, \ \forall v \in V \), which implies that \( i_v(\tilde{\alpha} - \tilde{\beta}) = 0 \Rightarrow \tilde{\alpha} - \tilde{\beta} \in \Omega_{A^*}^1(L/V, H_V) \Rightarrow \pi(\tilde{\alpha}) = \pi(\tilde{\beta}) \).

For any \( v \in V \) and \( X \in L \), we have:

\[
(d\tilde{\alpha})(v, X) = v(\tilde{\alpha}(X)) - X(\tilde{\alpha}(v)) - \tilde{\alpha}([v, X]) = \\
= -X([v]) - ([v, X]) - [([v, v]) - ([v, X])] = 0
\]
Therefore, we obtain that $i_v(d\tilde{\alpha}) = 0$, $\forall v \in V$, which means that the form $d\tilde{\alpha}$ belongs to the submodule $\Omega^2_{AV}(L/V, H_V)$, or equivalently: $\pi(d\tilde{\alpha}) = d\pi(\tilde{\alpha}) = 0 \Rightarrow \pi(\tilde{\alpha})$ is closed in $\Omega^1_{AV}(L, H_V)_1$.

**Remark 1** $d\tilde{\alpha} - d\tilde{\beta} = d(\tilde{\alpha} - \tilde{\beta}) \Rightarrow$ the forms $\tilde{\alpha}$ and $\tilde{\beta}$ are cohomolog-ical in $\Omega^2_{AV}(L/V, H_V)$, and their cohomology class is exactly $\sigma_1(\pi(\tilde{\alpha})) \in H^2(L/V, H_1(V, AV))$.

As it follows from the above lemma, for any projection homomorphism $\alpha : L \rightarrow V$, the element $\pi(\tilde{\alpha})$ in $\Omega^1_{AV}(L, H_V)_1$ is one and the same. Let us denote this element (and also its cohomology class in $H^1(L, H_V)_1$) by $\Delta$. The element $\sigma_1(\Delta)$ in $H^2(L/V, H_1(V, AV))$ we call the characteristic class of the Lie algebra ideal $V$ in $L$.

**Remark 2** It is clear that if $V$ is an ideal in the Lie algebra $L$, then the space $[V, V] = \left\{ \sum_i a_i[u_i, v_i] \mid a_i \in AV, u_i, v_i \in V \right\}$ is also an ideal in $L$ (an in $V$, too). We can consider the Lie algebra $\tilde{L} = L/[V, V]$ and its commutative subalgebra $\tilde{V} = V/[V, V]$, which is also an ideal in $L$. It is clear that the homology space $H_1(V, AV)$ is the same as $\tilde{V}$ and the quotient $\tilde{L}/\tilde{V}$ is the same as $L/V$. One can consider the following short exact sequence

$$0 \rightarrow \tilde{V} \rightarrow \tilde{L} \rightarrow L/V \rightarrow 0 \ (3)$$

As the Lie algebra $\tilde{V}$ is commutative, in this situation, for any connection form $\alpha : \tilde{L} \rightarrow \tilde{V}$ (a splitting of the short exact sequence $3$), its curvature form coincides with $d\alpha \in \Omega^2_{AV}(L/V, \tilde{V})$, and the cohomology class of the form $d\alpha$ in $d\alpha \in \Omega^2_{AV}(L/V, \tilde{V})$ coincides with the characteristic class of the ideal $V$. Therefore, to study the properties of the characteristic class of a Lie algebra ideal, we can consider the case when the ideal is commutative.

### 6 The Case of a Fiber Bundle

The construction described in the previous section could be used in the case of fiber bundles, which in the classical case gives the well-known characteristic classes of this fiber bundle. In this section we consider this case from the algebraic point of view.
Let $S$ be a module over the algebra $A$. Denote by $Diff^1(S)$ the space of differential operators on $S$, of order $\leq 1$. That is: any $X \in Diff^1(S)$ is a $C$-linear (or $R$-linear, if $A$ is an algebra over $R$) map $X : S \to S$, such that, for any two $a, b \in A$, one has $[[X, mul(a)], mul(b)] = 0$, where $mul(a)$ (and $mul(b)$) denotes the “multiplication by $a$ (and $b$)” operator on the $A$-module $S$. From this definition easily follows that for any $a \in A$, the operator $[X, mul(a)] : S \to S$ is an element of $End_A(S)$, which is the space of $A$-module endomorphisms of $S$. Let us denote by $mul(A)$ the subspace of $End_A(S)$ generated by the operators of the type $mul(a)$, $a \in A$; and by $trans(S)$ the subspace of $Diff^1(S)$ consisting of such $X \in Diff^1(S)$ that $[X, mul(a)] \in mul(A)$, $\forall a \in A$. It is clear that $trans(S)$ is a Lie algebra.

**Remark 3** In the case when $S$ is a $C^\infty(M)$-module of smooth sections of a vector bundle over some smooth manifold $M$, $trans(S)$ is the Lie algebra of the Lie group of automorphisms of this vector bundle.

It is easy to verify that for any fixed element $X \in trans(S)$ the mapping $mul(A) \ni mul(a) \mapsto [X, mul(a)] \in mul(A)$ is a derivation operator on $mul(A)$ (i.e., satisfies the Leibnitz’s rule). For simplicity, assume that $A = mul(A)$ (otherwise, one can reduce $A$ to $mul(A)$). Let us denote the map $trans(S) \ni X \mapsto [X, \cdot] \in Der(A)$ by $\pi$.

**Definition 2** We call the $A$-module $S$ a fiber bundle module if the Lie algebra homomorphism $\pi : trans(S)/End_A(S) \to Der(A)$ is an epimorphism.

As it follows from this definition, for any $X \in trans(S)$, $s \in S$ and $a \in A$ we have $X(as) = \pi(X)(a) \cdot s + a \cdot X(s)$.

One has the following short exact sequence of Lie algebra homomorphisms which is also an exact sequence of $A$-module homomorphisms:

$$0 \to End_A(S) \equiv aut(S) \to trans(S) \stackrel{\pi}{\longrightarrow} Der(A) \to 0 \quad (4)$$

In the classical differential geometry, this short exact sequence is known as the Atiyah sequence (see [1]).

A connection on the fiber bundle $(S, \pi)$ is a splitting of the short exact sequence $\pi : Der(A) \ni X \mapsto \nabla_X \in trans(S)$. From the definition of the space $trans(S)$, follow the well-known properties of the connection:

$$\nabla_{aX}(s) = a\nabla_X(s)$$

and

$$\nabla_X(as) = \pi(\nabla_X)(a) \cdot s + a \cdot \nabla_X(s) = X(a) \cdot s + a \cdot \nabla_X(s)$$
for any \( a \in A, X \in \text{Der}(A) \) and \( s \in S \).

Any such splitting (connection) is equivalent to a homomorphism of \( A \)-modules \( \alpha : \text{trans}(S) \to \text{aut}(S) \), such that \( \alpha(U) = U, \forall U \in \text{aut}(S) \). The map \( X \mapsto \nabla_X \) is a Lie algebra homomorphism if and only if, the kernel of the map \( \alpha \) is a Lie subalgebra in \( \text{trans}(S) \). The deviation of \( \ker(\alpha) \) from being a Lie algebra is a 2-form on \( \text{trans}(S) \) with values in \( \text{aut}(S) \), in the classical differential geometry known as the curvature of the connection \( \alpha \):

\[
[X - \alpha(X), Y - \alpha(Y)] = [X, Y] - [X, \alpha(Y)] - [Y, \alpha(X)] + [\alpha(X), \alpha(Y)]
\]

As \( \text{aut}(S) \) is a kernel of a Lie algebra homomorphism, it is an ideal in the Lie algebra \( \text{trans}(S) \), therefore, we have that

\[
[X, \alpha(Y)], [Y, \alpha(X)], [\alpha(X), \alpha(Y)] \in \text{aut}(S)
\]

This implies that

\[
\alpha([X - \alpha(X), Y - \alpha(Y)]) =
\]

\[
= \alpha([X, Y]) - [X, \alpha(Y)] + [Y, \alpha(X)] + [\alpha(X), \alpha(Y)] =
\]

\[
= -(d\alpha)(X, Y) + [\alpha(X), \alpha(Y)]
\]

which, itself, implies that the homomorphism \( \alpha : \text{trans}(S) \to \text{aut}(S) \) induces a splitting of the exact sequence as an exact sequence of Lie algebra homomorphisms iff the form \( \omega(X, Y) = (d\alpha)(X, Y) - [\alpha(X), \alpha(Y)] \) is 0.

Applying the construction described in the previous section, to this situation, we have the following: for a connection form \( \alpha : \text{trans}(S) \to \text{aut}(S) \), which is a homomorphism of \( A \)-modules, and defines a splitting of the exact sequence consider the induced 1-form with values in the 1-homology space \( \tilde{\alpha} : \text{trans}(S) \to H_1(\text{aut}(S)) \). Its differential, \( d\tilde{\alpha} \) is a 2-form on the Lie algebra \( \text{trans}(S)/\text{aut}(S) \) which is isomorphic to \( \text{Der}(A) \). The cohomology class of the form \( d\tilde{\alpha} \) in \( \Omega^2_A(\text{Der}(A), H_1(\text{aut}(S))) \) is independent of a choice of the connection \( \alpha \).

This approach gives an interesting interpretation of the classical case, where we have a vector bundle over a smooth manifold. Consider this case in more details.

Let \( M \) be a smooth manifold, and \( \pi : E \to M \) be a complex vector bundle with Hermitian structure; \( A = C^\infty(M) \) and \( S = \Gamma(E) \) be the space
of smooth sections of the bundle \( \pi : E \rightarrow M \) (which is also a module over the algebra \( A \)). In this situation, let \( \text{trans}(S) \) be the Lie algebra of the group of unitary transformations of the fiber bundle \( \pi : E \rightarrow M \) and \( \text{aut}(S) \) be the Lie algebra of the group of unitary automorphisms of this fiber bundle (i.e., such unitary transformations that are identical on the base of the fiber bundle).

It is clear that \( \text{aut}(S) \) is the same as the space of sections of the bundle \( \text{aut}(\pi) : \text{aut}(E) \rightarrow M \), the fiber of which at a point \( x \in M \) is the space of anti-Hermitian operators on the Hermitian complex vector space \( \pi^{-1}(x) \). And the homology space \( H_1(\text{aut}(S)) \) is the same as the space of sections of the fiber bundle \( \tilde{\pi} : \tilde{E} \rightarrow M \), the fiber of which at a point \( x \in M \) is the space of anti-Hermitian operators on the Hermitian complex vector space \( \pi^{-1}(x) \). Here \( u(\pi^{-1}(x)) \) denotes the Lie algebra of anti-Hermitian operators on the space \( \pi^{-1}(x) \). If the fiber of the bundle \( \pi : E \rightarrow M \) is finite-dimensional, then the homology space \( H_1(u(\pi^{-1}(x))) \) is one-dimensional and the homology class of the element \( i \cdot \text{Id} \), where \( \text{Id} \) is the identical operator on the space \( \pi^{-1}(x) \), gives a canonical basis of \( H_1(u(\pi^{-1}(x))) \) as a vector space over \( \mathbb{R} \). Therefore, the homology space \( H_1(\text{aut}(S)) \) is canonically isomorphic to the algebra \( A = C^\infty(M) \) and the characteristic class of the ideal \( \text{aut}(S) \) in the Lie algebra \( \text{trans}(S) \) is the element of the De Rham cohomology space \( H^2(\text{trans}(S)/\text{aut}(S), H_2(\text{aut}(S))) \cong H^2(\text{Der}(A), A) \cong H^2(M, \mathbb{R}) \). This characteristic class, of course, coincides with the first Chern class of the fiber bundle \( \pi : E \rightarrow M \). Other classes are obtained by the characteristic polynomials on \( \text{aut}(S) \), which, in fact, are polynomials on \( H_1(\text{aut}(S)) \).

7 Connection Conserving Infinitesimal Transformations

Let a triple \((L, S, A)\), where \( L \) is a Lie algebra, \( A \) is a commutative, associative algebra over \( \mathbb{C} \) or \( \mathbb{R} \) and \( S \) is a module over the algebra \( A \), is a Lie module (see Definition 1). Let \( \alpha \) be a 1-form on \( L \) with values in the module \( S \), i.e., \( \alpha \in \Omega^1_A(L, S) \). Denote by \( L_\alpha \) the subspace of \( L \) consisting of such elements \( X \in L \), that \( \mathcal{L}_X \alpha = 0 \), where the operator \( \mathcal{L}_X \) is an operator of Lie derivation: \( \mathcal{L}_X = i_X \circ d + d \circ i_X \). It can be verified by direct calculations that the operators \( \mathcal{L}_X \) and \( i_Y \), for any \( X, Y \in L \) satisfy the conditions \( [\mathcal{L}_X, d] = 0 \) and \( [\mathcal{L}_X, i_Y] = i_{[X,Y]} \), from which follows that the mapping \( X \mapsto \mathcal{L}_X \) is a Lie algebra homomorphism \( (\mathcal{L}_X, \mathcal{L}_Y) = \mathcal{L}_{[X,Y]} \). This implies that the subspace
$L_{\alpha}$ is a Lie subalgebra in $L$.

**Lemma 4** The kernel of the restricted mapping $\alpha : L_{\alpha} \rightarrow S$ is a Lie algebra ideal in $L_{\alpha}$.

**Proof.** For any $V \in \ker(\alpha)$ and $X \in L_{\alpha}$, we have

\[
L_{V}\alpha = 0 \Rightarrow i_{V}(d\alpha) = 0;
\]

\[
(L_{X}\alpha)(V) = 0 \Rightarrow V\alpha(X) + (d\alpha)(X, V) = 0 \Rightarrow V\alpha(X) = 0;
\]

\[
\begin{align*}
(d\alpha)(X, V) &= X\alpha(V) - V\alpha(X) - \alpha([X, V]) = 0 \Rightarrow \\
& \Rightarrow \alpha([X, V]) = 0 \Rightarrow [X, V] \in \ker(\alpha).
\end{align*}
\]

The above lemma implies that the quotient space $L_{\alpha}/(\ker(\alpha) \cap L_{\alpha})$ inherits the Lie algebra structure from $L_{\alpha}$. Therefore, we have a Lie algebra structure on $\alpha(L_{\alpha}) \subset S$.

**Lemma 5** For any two elements $X, Y \in L_{\alpha}$, we have that $\alpha([X, Y]) = (d\alpha)(X, Y)$.

**Proof.** The condition $L_{X}\alpha = L_{Y}\alpha = 0$, implies:

\[
(L_{X}\alpha)(Y) = Y\alpha(X) + X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) = 0
\]

and

\[
(L_{Y}\alpha)(X) = X\alpha(Y) + Y\alpha(X) - X\alpha(Y) - \alpha([Y, X]) = 0
\]

consequently

\[
X\alpha(Y) - \alpha([X, Y]) = Y\alpha(X) + \alpha([X, Y]) = 0 \Rightarrow \\
\Rightarrow X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) = \alpha([X, Y]) \Rightarrow \\
\Rightarrow (d\alpha)(X, Y) = \alpha([X, Y])
\]
Lemma 6 If $S = A$ and there exists such element $\eta \in L_\alpha$ that $\alpha(\eta) = 1$ then the subset $\alpha(L_\alpha)$ is a subalgebra in the algebra $A$.

Proof. Under these conditions, any $X \in L_\alpha$ can be represented as $X = a \cdot \eta + X'$, where $a \in A$ and $X' \in \ker(\alpha)$: $X' = X - \alpha(X) \cdot \eta$. The equality $L_X \alpha = 0$ implies: $L_{X'} \alpha = -da$ ($\iff i_{X'}(da) = -da$). If $a = \alpha(X)$ and $b = \alpha(Y)$, then $X = a \cdot \eta + X_a$ and $Y = b \cdot \eta + X_b$, where $X_a, X_b \in \ker(\alpha)$ (though the correspondence $a \mapsto X_a$, generally, is not single-valued mapping). Consider the element $W = ab \cdot \eta + (aX_b + bX_a)$ and verify that $L_W \alpha = 0$. From the property of the Lie derivation: $L_{aX} \omega = a \cdot L_X \omega + da \wedge i_X(\omega)$, follows that

$$L_W \alpha = d(ab) - adb - bda = d(ab) - d(ab) = 0$$

Hence, we obtain that, for the element $W = ab \cdot \eta + (aX_b + bX_a)$: $\alpha(W) = ab$ and $L_W \alpha = 0$, which, itself, implies that if $a, b \in \alpha(L_\alpha)$, then $ab \in \alpha(L_\alpha)$. As $1 \in \alpha(L_\alpha)$, we obtain that $\alpha(L_\alpha)$ is a subalgebra of $A$, with unit.

Let us denote the bracket corresponding to the induced Lie algebra structure on $\alpha(L_\alpha)$, by $\{ , \}$.

The condition $L_{\eta} \alpha = 0$ implies $i_{\eta}(da) = 0$. This fact, together with the lemma [3], gives that, for any $a, b \in \alpha(L_\alpha)$, we have that $\{a, b\} = (da)(X_a, X_b)$. This equality, together with the lemma [3], implies the following

Proposition 7 If $S = A$ and there exists such element $\eta \in L_\alpha$ that $\alpha(\eta) = 1$, then $\alpha(L_\alpha)$ is a subalgebra in the algebra $A$, and the algebra $\alpha(L_\alpha)$ together with the bracket induced from $L_\alpha$ is a Poisson algebra.

Proof. For $a, b, c \in \alpha(L_\alpha)$ we have $X = a\eta + X_a, Y = b\eta + X_b, Z = c\eta + X_c \in L_\alpha$. We have, also, that the element in $L_\alpha$ corresponding to $ab \in \alpha(L_\alpha)$ is $W = bc\eta + (bX_c + cX_b)$. Therefore:

$$\{a, bc\} = d\alpha(X_a, bX_c + cX_b) = b(d\alpha)(X_a, X_c) + c(d\alpha(X_a, X_b)) =$$

$$= b\{a, c\} + \{a, b\}c$$

Further we consider this situation, in more details, for the case of a contact manifold.
8 The Canonical Bivector and Vector Fields on a Contact Manifold

A pair \((M, \alpha)\), where \(M\) is a \(2n + 1\)-dimensional smooth manifold and \(\alpha\) is a differential 1-form on it, such that \(\alpha_x \wedge (d\alpha)_x^n \neq 0\) for all points \(x \in M\) is called a contact manifold with contact structure given by the form \(\alpha\).

For any point \(x \in M\), let

\[\ker(\alpha)_x = \{u \in T_x M \mid \alpha_x(u) = 0\}\]

and

\[\ker(d\alpha)_x = \{u \in T_x M \mid (d\alpha)_x(u, \cdot) = 0\}\]

The condition \(\alpha_x \wedge (d\alpha)_x^n \neq 0\), implies that \(T_x M = \ker(\alpha)_x \oplus \ker(d\alpha)_x\), \(\forall x \in M\).

Denote the differential form \(d\alpha\) by \(\omega\) and define the homomorphism of fiber bundles \(\tilde{\omega} : TM \rightarrow T^*M\) as

\[\tilde{\omega}_x(u)(v) = \omega_x(u, v), \quad \forall x \in M, \forall u, v \in T_x M\]

This homomorphism induces the homomorphism of \(C^\infty(M)\)-modules from the space of vector fields on the manifold \(M\) to the space of differential 1-forms on \(M\). We denote this homomorphism also by \(\tilde{\omega} : V^1(M) \rightarrow \Omega^1(M)\). The latter could be extended to the tensor fields of higher degrees, and we obtain the homomorphism of the graded exterior algebras, from the space of antisymmetric covariant tensor fields to the space of differential forms on the manifold \(M\)

\[\wedge \tilde{\omega} = \oplus_{i=0}^\infty \wedge^i \tilde{\omega} : V(M) = \oplus_{i=0}^\infty V^i(M) \rightarrow \Omega(M) = \oplus_{i=0}^\infty \Omega^i(M)\]

Where \(V^i(M), i = 0, \ldots, \infty\) denotes the space of covariant tensor fields (multivector fields) of degree \(i\).

As \(V^1(M) = \ker(\alpha) \oplus \ker(\omega)\), the map \(\tilde{\omega} : \ker(\alpha) \rightarrow \Omega^1(M)\) is a monomorphism. The kernel of the map \(\wedge \tilde{\omega}\) is the ideal generated by \(\ker(\omega)\), and the map

\[\wedge \tilde{\omega} : \wedge \ker(\alpha) \rightarrow \Omega(M)\]

is also a monomorphism.

According to the Darboux theorem, there exists a local coordinate system \(x_0, x_1, \ldots, x_{2n}\), on the contact manifold \(M\), such that the contact form \(\alpha\) in
this coordinate system is expressed as
\[ \alpha = dx_0 + \sum_{i=1}^{n} x_{2i-1} dx_{2i} \]
Such a coordinate system is called the canonical coordinate system. This implies that in such coordinate system \( \omega = \sum_{i=1}^{n} \wedge dx_{2i-1} dx_{2i} \) and the mapping \( \tilde{\omega} : V^1(M) \longrightarrow \Omega^1(M) \) can be written as
\[ \tilde{\omega} \left( \frac{\partial}{\partial x_0} \right) = 0, \; \tilde{\omega} \left( \frac{\partial}{\partial x_{2i-1}} \right) = dx_{2i}, \; \tilde{\omega} \left( \frac{\partial}{\partial x_{2i}} \right) = -dx_{2i-1} \quad i = 1 \ldots n. \]

**Lemma 8** The differential form \( \omega \) is an element of the space \( \text{Im}(\wedge^2 \tilde{\omega}) \), for the mapping \( \wedge^2 \tilde{\omega} : V^2(M) \longrightarrow \Omega^2(M) \).

**Proof.** From the representations of the form \( \omega \) and the mapping \( \tilde{\omega} \) in the canonical coordinate system, easily follows that the map \( \wedge^2 \tilde{\omega} : V^2(M) \longrightarrow \Omega^2(M) \), carries the bivector field \( w = \sum_{i=1}^{n} \frac{\partial}{\partial x_{2i-1}} \wedge \frac{\partial}{\partial x_{2i}} \) into the form \( \omega \):
\[ \tilde{\omega}(w) = \sum_{i=1}^{n} \tilde{\omega} \left( \frac{\partial}{\partial x_{2i-1}} \wedge \frac{\partial}{\partial x_{2i}} \right) = \sum_{i=1}^{n} dx_{2i} \wedge (-dx_{2i-1}) = \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i} \]

It is clear that \( V^2(M) = (\ker(\omega) \wedge V^1(M)) \oplus (\wedge^2 \ker(\alpha)) \), and \( \ker(\wedge^2 \omega) = \ker(\omega) \wedge V^1(M) \). Therefore, the projection of the bivector field \( w \) on the subspace \( \wedge^2 \ker(\alpha) \) is also carried in the form \( \omega \), by the map \( \wedge^2 \tilde{\omega} \). Let us denote this projection by \( \mu \) (though the bivector field \( w \) could be only local, its projection is a global bivector field). The representation of \( \mu \) in the canonical coordinate system is
\[ \mu = \sum_{i=1}^{n} \frac{\partial}{\partial x_{2i-1}} \wedge \left( \frac{\partial}{\partial x_{2i}} - x_{2i-1} \frac{\partial}{\partial x_0} \right) \quad (5) \]

To summarize, we can state that there exists a unique, canonical bivector field \( \mu \in \wedge^2 \ker(\alpha) \) such that \( \wedge^2 \tilde{\omega}(\mu) = \omega \). We call the bivector field \( \mu \), the canonical bivector field on the contact manifold \((M, \alpha)\).
Any bivector field, on a smooth manifold, defines a bracket on the algebra of the smooth functions on that manifold, so does \( \mu \)

\[
\{f, g\} = (df \wedge dg)(\mu), \quad \forall f, g \in C^\infty(M)
\]

The same bracket can be uniquely defined by its property (see [4]):

\[
\{f, g\} \alpha \wedge \omega^n = ndf \wedge dg \wedge \alpha \wedge \omega^{n-1}
\]

It is clear that such bracket is antisymmetric and biderivative, but to satisfy the Jacoby identity, the bivector field \( \mu \) needs to be involutive, i.e. satisfy the condition \([\mu, \mu] = 0\) (see [8]). Here, the bracket \([\cdot, \cdot]\) denotes the super-commutator operation on the Lie superalgebra of antisymmetric covariant tensor fields. The coordinate representation of the bivector field \( \mu \) (see the formula 5), implies that \([\mu, \mu] = \partial_{x_0} \wedge W\), where \( W \) is some bivector field; therefore, the bracket \( \{\cdot, \cdot\} \) cannot satisfy the Jacoby identity on the entire algebra \( C^\infty(M) \).

For any bivector field \( \mu \) and any triplet of functions \( f, g, h \in C^\infty(M) \), we have the following equality for the bracket defined by \( \mu \) (see [4]):

\[
\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = \frac{1}{2}(df \wedge dg \wedge dh)([\mu, \mu])
\]

As the supercommutator \([\mu, \mu]\) is an element of the ideal generated by \( \ker(\omega) \), we have that the following subalgebra

\[
\mathcal{F}_B(M) = \{\varphi \in C^\infty(M) \mid X(\varphi) = 0, \forall X \in \ker(\omega)\}
\]

is the subalgebra (maximal) of the commutative algebra \( C^\infty(M) \), which is closed under the bracket defined by the canonical bivector field \( \mu \) and on which the Jacoby identity is satisfied. That is, the algebra \( \mathcal{F}_B(M) \) is a Poisson algebra under the bracket defined by the canonical bivector field.

Further, we call the elements of the submodule \( \ker(\alpha) \), the horizontal vector fields and the elements of the submodule \( \ker(\omega) \), the vertical vector fields. The algebra \( \mathcal{F}_B(M) \) is called the algebra of basic functions on the contact manifold \((M, \alpha)\) (see [7]).

For any \( w \in V^m(M) \) and \( \beta \in \Omega^n(M) \), where \( m \geq n \), we denote by \( \tilde{w}(\beta) \) the element of the space \( V^{m-n}(M) \) such that for any \( \lambda \in \Omega^{m-n}(M) : \lambda(\tilde{w}(\beta)) = (\beta \wedge \lambda)(w) \).

For the canonical bivector field \( \mu \), we have that \([\mu, \mu] \in V^3(M) \), therefore: \( \tilde{[\mu, \mu]}(\omega) \in V^1(M) \). Let us denote the vector field \(-[\mu, \mu](\omega) \in V^1(M) \) by

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η. Find the representation of this vector field in the canonical coordinate system:

\[ [\mu, \mu] = - \left( \sum \frac{\partial}{\partial x_{2i-1}} \wedge \frac{\partial}{\partial x_{2i}}, - \sum x_{2i-1} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_0} \right) = \]

\[ = \sum \frac{\partial}{\partial x_{2i}} \wedge \frac{\partial}{\partial x_{2i-1}} \wedge \frac{\partial}{\partial x_0} \Rightarrow [\tilde{\mu}, \mu](\omega) = - \frac{\partial}{\partial x_0} \Rightarrow \eta = \frac{\partial}{\partial x_0} \]

The vector field η can be uniquely characterized by the properties: \( i_\eta \alpha = 1, \ i_\eta \omega = 0, \) and is known as the canonical vector field on the contact manifold \((M, \alpha)\) (see [4]).

It is clear that the module \( \ker(\omega) \) is one-dimensional, therefore, the vector field η can be considered as its basis, after which the Poisson algebra of the basic functions, \( \mathcal{F}_B(M) \), can be defined as \( \mathcal{F}_B(M) = \ker(\eta) \), where \( \eta : C^\infty(M) \to C^\infty(M) \) is a derivation operator.

9 Some Properties of Basic Functions and Invariant Vector Fields

For any \( f \in C^\infty(M) \), we can consider the vector field \( \tilde{\mu}(df) \). Let us formulate and verify some properties of the vector fields of the type \( \tilde{\mu}(df) \), \( f \in C^\infty(M) \).

**Lemma 9** For any \( f \in C^\infty(M) \), the vector field \( \tilde{\mu}(df) \) is horizontal.

**Proof.** By definition of the operator \( \tilde{\mu} : \Omega^1(M) \to V^1(M) \), we have the following: \( \alpha(\tilde{\mu}(df)) = (df \wedge \alpha)(\mu) \), but \( \mu \) is an element of \( \wedge^2 \ker(\alpha) \), which implies that \( (df \wedge \alpha)(\mu) = 0 \). ■

**Proposition 10** The equality \( df = -\omega(\tilde{\mu}(df), \cdot) \) is true if and only if \( f \) is a base function.

**Proof.** If for some vector field \( X \) we have that \( df = -\omega(X, \cdot) \), then \( \eta(f) = -\omega(X, \eta) = 0 \). Therefore, in this case \( f \in \mathcal{F}_B(M) \). Now, the task is to verify that if \( f \in \mathcal{F}_B(M) \) then \( df = -\omega(\tilde{\mu}(df), \cdot) \). Using the canonical coordinate
system, we obtain:

\[ f \in \mathcal{F}_B(M) \implies \frac{\partial f}{\partial x_0} = 0 \implies \]

\[ \tilde{\mu}(df) = \sum \left( \frac{\partial f}{\partial x_{2i-1}} \frac{\partial}{\partial x_{2i}} - \frac{\partial f}{\partial x_{2i}} \frac{\partial}{\partial x_{2i-1}} \right) - \sum x_{2i-1} \frac{\partial f}{\partial x_{2i-1}} \frac{\partial}{\partial x_0} \implies \]

\[ \omega(\tilde{\mu}(df), \cdot) = -\frac{\partial f}{\partial x_{2i}} dx_{2i} - \frac{\partial f}{\partial x_{2i-1}} dx_{2i-1} = -df \]

To summarize, we can state that: for any function \( f \in C^\infty(M) \), the equality \( df = -\omega(X, \cdot) \) for some vector field \( X \) is true if and only if \( f \in \mathcal{F}_B(M) \); and as the form \( \omega \) is non-degenerated on the submodule of horizontal vector fields, in this case, there exists one and only one such horizontal vector field \( X \) which equal to \( \tilde{\mu}(df) \).

**Corollary 1** For the Poisson algebra \( \mathcal{F}_B(M) \) the following equality is true \( \{f, g\} = \omega(\tilde{\mu}(df), \tilde{\mu}(dg)) \).

Let us denote by \( V^1_1(M) \) the set of vector fields on the manifold \( M \) commuting with \( \eta : V^1_1(M) = \{X \in V^1(M) \mid [\eta, X] = 0\} \), and call them the invariant vector fields (see [3]). It is clear that \( V^1_1(M) \) is a Lie subalgebra in \( V^1(M) \) and a module over the algebra \( \mathcal{F}_B(M) \).

**Lemma 11** For any \( f \in \mathcal{F}_B(M) \) the horizontal vector field \( \tilde{\mu}(df) \) is an element of the space \( V^1_1(M) \).

**Proof.** The proof easily follows from the coordinate representation of the vector field \( \tilde{\mu}(df) \):

\[ \tilde{\mu}(df) = \sum \left( \frac{\partial f}{\partial x_{2i-1}} \frac{\partial}{\partial x_{2i}} - \frac{\partial f}{\partial x_{2i}} \frac{\partial}{\partial x_{2i-1}} \right) - \sum x_{2i-1} \frac{\partial f}{\partial x_{2i-1}} \frac{\partial}{\partial x_0} \]

The subalgebra \( \mathcal{F}_B(M) \) in \( C^\infty(M) \) is invariant under the action of the elements of the space \( V^1_1(M) \):

\[ \varphi \in \mathcal{F}_B(M), \ X \in V^1_1(M) \implies \eta(X(\varphi)) = [\eta, X](\varphi) + X(\eta(\varphi)) = 0 \implies \]

\[ \implies X(\varphi) \in \mathcal{F}_B(M) \]
Hence, we have a natural homomorphism of Lie algebras
\[ \pi : V^1_I(M) \longrightarrow \text{Der}(\mathcal{F}_B(M)) \]
where \(\text{Der}(A)\) denotes the Lie algebra of derivations for any algebra \(A\):
\[ \text{Der}(A) = \{ X : A \longrightarrow A \mid X \text{ is linear, and} \quad X(ab) = X(a)b + aX(b), \quad \forall a, b \in A \} \]
The kernel of the homomorphism \(\pi\) is the subalgebra
\[ \mathcal{F}_B(M) \cdot \eta = \{ \varphi \cdot \eta \mid \varphi \in \mathcal{F}_B(M) \} \]
Consider the following short exact sequence
\[ 0 \longrightarrow \mathcal{F}_B(M) \cdot \eta \hookrightarrow V^1_I(M) \xrightarrow{\pi} \text{Im}(\pi) \longrightarrow 0 \quad (6) \]
The submodule \(V^1_I(M) \cap \ker(\alpha)\) gives a splitting of the short exact sequence \(\mathfrak{3}\) but this splitting is not a Lie algebra homomorphism, because, for any two horizontal vector fields \(X, Y\), we have \(\alpha([X,Y]) = -\omega(X,Y)\). Any splitting of the short exact sequence \(\mathfrak{3}\) is equivalent to a choice of a \(\mathcal{F}_B(M)\)-submodule \(\mathcal{H}\) of the module \(V^1_I(M)\). This choice, itself, is equivalent to a 1-form \(\beta\) on \(V^1_I(M)\) with values in \(\mathcal{F}_B(M)\). The form \(\beta\), actually, is the projection operator on the submodule \(\mathcal{F}_B(M) \cdot \eta\), corresponding to the decomposition \(V^1_I(M) = (\mathcal{F}_B(M) \cdot \eta) \oplus \mathcal{H}\). The form \(\beta\) can be characterized by the following properties: \(i_\eta \beta = 1\) and \(L_\eta(\beta) = 0\), where \(L_\eta\) denotes the Lie derivation: \(L_\eta = d \circ i_\eta + i_\eta \circ d\) (for \(X \in V^1_I(M)\):
\[ (L_\eta(\beta))(X) = (d\beta)(\eta,X) = \eta \underbrace{\beta(X)}_{1} - X \underbrace{\beta(\eta)}_{0} - \beta([\eta,X]) = 0. \]
For each point \(x \in M\), the subalgebra \(V^1_I(M)\) gives the entire tangent space at this point, which implies that the form \(\beta : V^1_I(M) \longrightarrow \mathcal{F}_B(M)\) could be extended to a differential 1-form on the manifold \(M\).

Let us denote by \(\Omega_B(M)\) the subspace of the space \(\Omega(M)\) consisting of such differential forms \(\theta \in \Omega(M)\) that \(i_\eta \theta = L_\eta(\theta) = 0\). Such forms are called the basic forms (see \(\mathfrak{3}\)). For any basic form \(\theta\), we have: \(L_\eta(d\theta) = dL_\eta(\theta) = 0\) and \(i_\eta d\theta = L_\eta(\theta) - d\eta \theta = 0\); therefore, the subspace \(\Omega_B(M)\) is invariant under the action of the differential \(d\). Hence, we can talk about subcomplex \((\Omega_B(M), d)\) of the De Rham complex \((\Omega, d)\). It is clear that \(\Omega_B(M)\) is closed.
under the exterior multiplication. Let us denote the cohomology algebra of the complex $(\Omega_B(M), d)$ by $H_B(M)$ (basic cohomology, see [3]).

Any differential 1-form $\beta$ on the manifold $M$, corresponding to some splitting of the short exact sequence $\overline{6}$ (i.e., with properties $\beta(\eta) = 1$ and $L_\eta(\beta) = 0$) defines a cohomology class in $H^2_B(M)$ corresponding to the differential form $\theta = d\beta$. Any two such forms $\beta_1$ and $\beta_2$ define one and the same cohomology class in $H^2_B(M)$: $d\beta_1 - d\beta_2 = d(\beta_1 - \beta_1), \beta_1 - \beta_1 \in \Omega^1_B(M)$.

**Proposition 12** The exact sequence $\overline{6}$ of $\mathcal{F}_B(M)$-module homomorphisms can be split as the exact sequence of Lie algebra homomorphisms, if and only if the cohomology class of the form $\omega = d\alpha$ in $H^2_B(M)$ is trivial.

**Proof.** If the splitting defined by some form $\beta$ is such that the submodule $\ker(\beta)$ is a Lie subalgebra, then for any $X,Y \in \ker(\beta)$, we have that $(d\beta)(X,Y) = 0$, which, together with the fact that $V^1_I(M) = (\mathcal{F}_B(M) \cdot \eta) \oplus \ker(\beta)$ and the properties of the form $\beta$, implies that $d\beta = 0$. Consider the differential form $\gamma = \alpha - \beta$. It is clear that $\gamma \in \Omega^1_B(M)$ and $d\gamma = d\alpha = \omega$. ■

**10 Poisson Cohomologies of Basic Multivector Fields**

For any integer $n \geq 0$, let us denote by $V^n_I(M)$ the subspace of the space $V^n(M)$, consisting of such elements $w$ that $[\eta, w] = 0$, where the bracket $[\cdot, \cdot]$ denotes the supercommutator on the Lie superalgebra $V(M)$. Let us denote the graded space $\bigoplus_{i=0}^{\infty} V^i_I(M)$ by $V^I(M)$. We call the elements of the space $V^I(M)$ the *invariant multivector fields*. The space $V^I(M)$ is a module over the algebra $\mathcal{F}_B(M)$, which follows from the properties of the Schouten bracket: $[\eta, \varphi w] = \eta(\varphi)w + \varphi[\eta, w] = 0$ for $w \in V^I(M)$ and $\varphi \in \mathcal{F}_B(M)$. From the property $(-1)^{[u][w]}[[u, v], w] + (-1)^{[v][w]}[[v, w], u] + (-1)^{[w][v]}[[w, u], v] = 0$, follows that the space $V^I(M)$ is closed under the operation of the supercommutator; and from the property $[u, v \wedge w] = [u, v] \wedge w + (-1)^{[u][v]}v \wedge [u, w]$ follows that it is closed under the exterior product. To summarize, we can state that $V^I(M)$ inherits the exterior algebra and superalgebra structures from $V(M)$.

The canonical bivector field $\mu$ is an element of $V^2_I(M)$, because $[\eta, \mu] = 0$, which can be easily verified by using of the representations of $\eta$ and $\mu$ in the canonical coordinate system.
Lemma 13 The ideal generated by the canonical vector field $\eta \in V^1_1(M)$ in the exterior algebra $V_1(M)$ is also an ideal under the supercommutator.

Proof. For any $v, w \in V_1(M)$, we have:

$$[w, \eta \wedge v] = [w, \eta] \wedge v + (-1)^{|w|+1} \eta \wedge [w, v] = (-1)^{|w|+1} \eta \wedge [w, v] \in \eta \wedge V_1(M)$$

The canonical bivector field $\mu$ defines an operator of degree $+1$:

$$\delta_{\mu} : V(M) \longrightarrow V(M), \quad \delta_{\mu}(v) = [\mu, v]$$

This is the well-known coboundary operator in the case of Poisson manifolds; and it is well-known also that the property $\delta_{\mu} \circ \delta_{\mu} = 0$ is equivalent to $[\mu, \mu] = 0$. But in this case $[\mu, \mu] \neq 0$, which implies that the operator $\delta_{\mu}$ is not coboundary on $V(M)$.

Lemma 14 The subalgebra $V_1(M)$ in $V(M)$ is invariant under the action of the operator $\delta_{\mu}$.

Proof. For any $w \in V_1(M)$, we have the following

$$[\eta, \delta_{\mu}(w)] = [\eta, [\mu, w]] = \pm [\eta, [\mu, w]] = 0$$

which implies that $\delta_{\mu}(w) \in V_1(M)$.

Lemma 15 The ideal generated by the canonical vector field $\eta \wedge V_1(M)$ is invariant under the action of the operator $\delta_{\mu}$.

Proof. Actually, the statement of this lemma is tautological, because as it was mentioned $\mu \in V_2^2(M)$ and the ideal $\eta \wedge V_1(M)$ is an ideal also under the supercommutator. Therefore,

$$\delta_{\mu}(\eta \wedge V_1(M)) = [\mu, \eta \wedge V_1(M)] \subset \eta \wedge V_1(M)$$

Lemma 16 The multivector field $[\mu, \mu]$ is an element of the ideal $\eta \wedge V_1(M)$.
**Proof.** This fact can be easily verified by using the representation of $[\mu, \mu]$ in the canonical coordinate system (see the formula 3):

$$[\mu, \mu] = \sum \left( \frac{\partial}{\partial x_{2i}} \wedge \frac{\partial}{\partial x_{2i-1}} \right) \wedge \frac{\partial}{\partial x_0}$$

As the submodule $\eta \wedge V_I(M)$ is a superalgebra ideal in $V_I(M)$, we have that the quotient $V_I(M)/(\eta \wedge V_I(M))$ is also a superalgebra under the super-commutator induced from $V_I(M)$. As $[\mu, \mu] \in \eta \wedge V_I(M)$, we have that the induced operator on the quotient algebras

$$[\delta_\mu] : V_I(M)/(\eta \wedge V_I(M)) \rightarrow V_I(M)/(\eta \wedge V_I(M))$$

$$[\delta_\mu](p(w)) = p([\mu, w])$$

where $p(w)$ denotes the equivalency class of the element $w \in V_I(M)$ in the quotient $V_I(M)/(\eta \wedge V_I(M))$, is a coboundary operator.

For simplicity, we denote the operator $[\delta_\mu]$ by $\delta_\mu$ and the quotient algebra $V_I(M)/(\eta \wedge V_I(M))$ by $V_B(M)$; and call its elements the basic multivector fields. So, the algebra of the basic multivector fields, together with the operator $\delta_\mu$, is a graded differential algebra.

The definition of the graded exterior algebras $\Omega_B(M)$ and $V_B(M)$, depend only on the canonical vector field $\eta$, and not on the contact form $\alpha$; but the definition of the operator $\delta_\mu : V_B(M) \rightarrow V_B(M)$ depends not only on the vector field $\eta$ but on some bivector field $\mu$ (which is defined by the contact form), with the properties: $[\eta, \mu] \in I_\eta$ and $[\mu, \mu] \in I_\eta$, where $I_\eta$ is the ideal in the exterior algebra $V_B(M)$ generated by the element $\eta$.

The representation of any basic form $\theta$ on the contact manifold $M$, in the canonical local coordinate system is:

$$\theta = \sum_{i_p \neq 0, \forall p \in \{1, \ldots, k\}} \varphi_{i_1, \ldots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad \text{where} \quad \frac{\partial \varphi_{i_1, \ldots, i_k}}{\partial x_0} = 0$$

and the representation of any invariant multivector field $w$ in this coordinate system is:

$$w = \sum \psi_{j_1, \ldots, j_l} \frac{\partial}{\partial x_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}, \quad \text{where} \quad \frac{\partial \psi_{j_1, \ldots, j_l}}{\partial x_0} = 0$$
Take into consideration this representations, the space \( V_B(M) \), locally, could be identified with the set of such \( w \) that

\[
w = \sum_{j_\mu \neq 0, \forall \mu \in \{1, \ldots, l\}} \psi_{j_1, \ldots, j_l} \frac{\partial}{\partial x_{j_1}} \land \cdots \land \frac{\partial}{\partial x_{i_l}} \land \frac{\partial \psi_{j_1, \ldots, j_l}}{\partial x_0} = 0
\]

From these easily follows that the restriction of the map

\[
\land \tilde{\omega} : V(M) \rightarrow \Omega(M)
\]

on the subalgebra of invariant multivector fields, takes its values in the subalgebra of the basic forms \( \Omega_B(M) \).

The kernel of the map \( \land \tilde{\omega} : V_I(M) \rightarrow \Omega_B(M) \) is the ideal generated by the canonical vector field \( \eta \).

**Proposition 17** The homomorphism of the exterior superalgebras

\[
\land \tilde{\omega} : V_B(M) = V_I(M)/(\eta \land V_I(M)) \rightarrow \Omega_B(M)
\]

is an isomorphism of the differential complexes \((V_B(M), \delta)\) and \((\Omega_B(M), d)\).

**Proof.** By using of the canonical local coordinates, it is easy to verify that

\[
d \circ (\land \tilde{\omega}) = (\land \tilde{\omega}) \circ \delta,\text{ and the inverse map is}
\]

\[
p \circ (\land \tilde{\mu}) : \Omega_B(M) \rightarrow V_B(M)
\]

where \( p : V_I(M) \rightarrow V_B(M) \) is the quotient map and \( \tilde{\mu} : \Omega^1(M) \rightarrow V^1(M) \) is defined as: \( \theta(\tilde{\mu}(\gamma)) = (\gamma \land \theta)(\mu), \theta, \gamma \in \Omega^1(M) \).

The isomorphism \( \land \tilde{\omega} \) carries the equivalency class of the bivector field \( \mu \) in the differential form \( \omega = d\alpha \), therefore the proposition [17], together with the proposition [12], implies that the exact sequence \([1]\) can be split by a homomorphism of \( F_B(M) \)-modules and Lie algebras if and only if the cohomology class of \( \mu \) in \( V^2_B(M) \) is trivial; i.e., there exists such vector field \( V \in V^1_I(M) \) that \( ([\mu, V] - \mu) \in \eta \land V_I(M) \).

11 Geometric Quantization of the Poisson Algebra of Basic Functions

The contact form \( \alpha \) on the contact manifold \((M, \alpha)\), defines a mapping from the Lie algebra of the invariant vector fields to the Poisson algebra of basic functions:

\[
\alpha : V^1_I(M) \rightarrow F_B(M)
\]

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This mapping is surjective, because $\mathcal{F}_B(M) \cdot \eta$ is a subalgebra of $V^1_I(M)$, and on this subalgebra $\alpha(\varphi \cdot \eta) = \varphi$, for each $\varphi \in \mathcal{F}_B(M)$. But $V^1_I(M) \xrightarrow{\alpha} \mathcal{F}_B(M)$ is not a Lie algebra homomorphism, because the subalgebra $\mathcal{F}_B(M) \cdot \eta$ is commutative Lie subalgebra in $V^1_I(M)$, while the Lie algebra $(\mathcal{F}_B(M), \{ , \})$, generally, is not commutative.

Hence, we have the following short exact sequence of homomorphisms of $\mathcal{F}_B(M)$-modules:

$$0 \rightarrow \ker(\alpha)_I \hookrightarrow V^1_I(M) \xrightarrow{\alpha} \mathcal{F}_B(M) \rightarrow 0 \quad (7)$$

where $\ker(\alpha)_I$ denotes the submodule of invariant, horizontal vector fields: $V^1_I(M) \cap \ker(\alpha)$. Now consider some simple, general situation. Let $A$, $B$ and $C$ be linear spaces, and

$$0 \rightarrow A \hookrightarrow B \xrightarrow{\pi} C \rightarrow 0$$

be a short exact sequence of linear maps, and let $s_0 : C \rightarrow B$ be some splitting of this exact sequence. In this situation, there is a one-to-one correspondence between the set of splittings of the above short exact sequence and the set of linear maps from $C$ to $A$: if $\varphi : C \rightarrow A$ is a linear map, then the map $s = s_0 + \varphi : C \rightarrow B$ is a splitting, and vice versa, if $s : C \rightarrow B$ is a splitting, then $\varphi = s - s_0$ is a linear map from $C$ to $A$.

In our case, we have two maps: $f \mapsto \tilde{\mu}(df)$ from $\mathcal{F}_B(M)$ to $\ker(\alpha)_I$ and $f \mapsto f \cdot \eta$ from $\mathcal{F}_B(M)$ to $V^1_I(M)$. The second one is a splitting of the short exact sequence $\overset{[6]}{[6]}$. From these two maps, as it was discussed, can be constructed a new splitting of the short exact sequence $\overset{[7]}{[7]}$

$$\mathcal{F}_B(M) \ni f \mapsto \hat{f} = f \cdot \eta + \tilde{\mu}(df) \in V^1_I(M)$$

**Proposition 18** The mapping $f \mapsto \hat{f}$ is a Lie algebra monomorphism from the Poisson algebra of basic functions, $\mathcal{F}_B(M)$, to the Lie algebra of invariant vector fields $V^1_I(M)$, and the image of this mapping coincides with the subalgebra of such vector fields $X$ in $V^1_I(M)$ that $L_X(\alpha) = 0$.

**Proof.** It is clear that the set $\{X \in V^1_I(M) \mid L_X(\alpha) = 0\}$ is a Lie subalgebra in the Lie algebra $V^1_I(M)$. If $X = f \cdot \eta + \tilde{\mu}(df)$ for some basic function $f$, then we have $L_X(\alpha) = df + (d\alpha)(\tilde{\mu}(df), \cdot) = 0$ which follows from the equality $df = -(d\alpha)(\tilde{\mu}(, \cdot), \cdot)$, for the basic functions.
If $X \in V^1(M)$, then $X$ can be represented as $X = f \cdot \eta + X'$, where $X'$ is a horizontal vector field. The equality $L_X(\alpha) = 0$ implies $df + (d\alpha)(X', \cdot) = 0$, but as it was shown, for any basic function $f \in \mathcal{F}_B(M)$, there is one and only one horizontal vector field $X'$ such that $df = -\eta(X', \cdot)$, and this vector field is $\tilde{\mu}(df)$. Therefore, we obtain that $X = f \cdot \eta + \tilde{\mu}(df)$. ■

We can consider the map $X \mapsto \alpha(X)$, not only on the invariant vector fields but extend it to the Lie algebra of all vector fields on the contact manifold $M$: $V^1(M) \xrightarrow{\alpha} C^\infty(M)$. This map is a surjective too: $\alpha(\varphi \eta) = \varphi$, $\forall \varphi \in C^\infty(M)$. Hence, we can consider the short exact sequence

$$
0 \longrightarrow \ker(\alpha) \hookrightarrow V^1(M) \xrightarrow{\alpha} C^\infty(M) \longrightarrow 0 \tag{8}
$$

In this case, we have not defined a Lie algebra structure on the commutative algebra $C^\infty(M)$ (the bivector field $\mu$ gives such structure only on the subalgebra of the basic functions), but the mapping $f \mapsto \tilde{f} = f \cdot \eta + \tilde{\mu}(df)$ can be extended to the entire $C^\infty(M)$. To describe the image of this mapping, let us review some notions from the theory of contact manifolds.

**Definition 3 (Infinitesimal Contact Transformation)** A vector field $X$ on the contact manifold $(M, \alpha)$ is called an infinitesimal contact transformation, if exists such function $\varphi \in C^\infty(M)$, that $L_X(\alpha) = \varphi \cdot \alpha$ (see [7]).

Let us denote the set of infinitesimal contact transformations by $\text{cont}(M, \alpha)$.

**Definition 4 (Infinitesimal Contact Automorphism)** A vector field $X$ on the contact manifold $(M, \alpha)$ is called an infinitesimal contact automorphism, if $L_X(\alpha) = 0$ (see [7]).

Let us denote the set of infinitesimal contact automorphisms by $\text{cont}_0(M, \alpha)$.

It is clear, that $\text{cont}(M, \alpha)$ is a Lie algebra and $\text{cont}_0(M, \alpha)$ is its subalgebra. The Proposition 18 states that $\text{cont}_0(M, \alpha)$ is isomorphic to the Lie algebra of the basic functions, via the mapping

$$
\mathcal{F}_B(M) \ni f \mapsto \tilde{f} = f \cdot \eta + \tilde{\mu}(df) \in \text{cont}_0(M, \alpha)
$$

**Proposition 19** The mapping $f \mapsto \tilde{f} = f \cdot \eta + \tilde{\mu}(df)$, is a bijection between the set of all smooth functions on the contact manifold $M$ and the set $\text{cont}(M, \alpha)$.  

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Proof. Let $X = \widehat{f}$, for some $f \in C^\infty(M)$. Consider the form $L_X(\alpha)$. As the vector field $\tilde{\mu}(df)$ is always horizontal (see Lemma 4), we have that $\alpha(\tilde{\mu}(df)) = 0$, and $L_X(\alpha) = df + \omega(\tilde{\mu}(df), \cdot)$. In the canonical local coordinate system we have the following

$$df = \frac{\partial f}{\partial x_0} dx_0 + \sum \left( \frac{\partial f}{\partial x_{2i-1}} dx_{2i-1} + \frac{\partial f}{\partial x_{2i}} dx_{2i} \right) \Rightarrow$$

$$\tilde{\mu}(df) = \sum \left( x_{2i-1} \frac{\partial f}{\partial x_0} \frac{\partial}{\partial x_{2i-1}} + \frac{\partial f}{\partial x_{2i}} \frac{\partial}{\partial x_{2i}} \right) \Rightarrow \omega(\tilde{\mu}(df), \cdot) = \sum x_{2i-1} \frac{\partial f}{\partial x_0} dx_{2i-1} - \sum \left( \frac{\partial f}{\partial x_{2i-1}} dx_{2i-1} + \frac{\partial f}{\partial x_{2i}} dx_{2i} \right)$$

$$\Rightarrow df + \omega(\tilde{\mu}(df), \cdot) = \frac{\partial f}{\partial x_0} (dx_0 + \sum x_{2i-1} dx_{2i}) = \eta(f) \cdot \alpha$$

which implies that any vector field of the type $\widehat{f} = f \cdot \eta + \tilde{\mu}(df)$ is an element of the Lie algebra $cont(M, \alpha)$.

Now, let $X$ be any vector field from $cont(M, \alpha)$. The vector field $X$ can be decomposed as the sum of its vertical and horizontal components: $X = f \cdot \eta + W$. The condition $L_X(\alpha) = \varphi \cdot \alpha$, implies: $df + \omega(W, \cdot) = \varphi \cdot \alpha$. But it was shown that $df + \omega(\tilde{\mu}(df), \cdot) = \eta(f) \cdot \alpha$. Therefore, $\omega(W - \tilde{\mu}(df), \cdot) = (\varphi - \eta(f)) \cdot \alpha$. Putting $\eta$ in the both sides of this equality gives

$$\varphi - \eta(f) = 0 \Rightarrow \omega(W - \tilde{\mu}(df), \cdot) = 0$$

The vector fields $W$ and $\tilde{\mu}(df)$ are horizontal, therefore, such is their difference. As the form $\omega$ is non-degenerated on the module of horizontal vector fields, we obtain that $W = \tilde{\mu}(df)$. \qed

So, we obtained that the elements of the Lie algebra $cont(M, \alpha)$ are the vector fields of the type $X = f \cdot \eta + \tilde{\mu}(df)$, where $f \in C^\infty(M)$; and the function $\varphi \in C^\infty(M)$, for which $L_X(\alpha) = \varphi \cdot \alpha$ is the function $\eta(f)$ (or, which is the same: $\eta(\alpha(X))$). In the case of the basic function, we have that $\eta(f) = 0$, and consequently: $L_{\widehat{f}}(\alpha) = 0$.

The homomorphism of Lie algebras $\pi : V^1(M) \rightarrow Der(\mathcal{F}_B(M))$ can be restricted to the subalgebra of infinitesimal contact automorphisms. We call the image $\pi(cont_0(M, \alpha)) \subset Der(\mathcal{F}_B(M))$ the dynamical vector fields, and denote this Lie subalgebra under $dyn(M, \alpha)$. For any two basic functions $f$ and $\varphi$, we have that $f(\varphi) = (f \cdot \eta + \tilde{\mu}(df))(\varphi) = \tilde{\mu}(df)(\varphi) = \omega(\tilde{\mu}(d\varphi), \tilde{\mu}(df))$. As the differential 2-form $\omega$ is non-degenerated on the space of horizontal
vector fields, and for each point \( x \in M \), the subspace of the tangent space \( T_xM \), generated by the set vector fields of the type \( \tilde{\mu}(d\varphi) \), \( \varphi \in \mathcal{F}_B(M) \), coincides with the entire horizontal subspace, we obtain that

\[
(\hat{f}(\varphi) = 0, \ \forall \varphi \in \mathcal{F}_B(M)) \iff \tilde{\mu}(df) = 0 \iff df = 0
\]

Therefore, the kernel of the Lie algebra homomorphism

\[
\pi : \text{cont}_0(M, \alpha) \rightarrow \text{dyn}(M, \alpha)
\]

is \( \{ f \cdot \eta \mid f = \text{const} \in \mathbb{R} \} \equiv \mathbb{R} \cdot \eta \). So, we have the following short exact sequence:

\[
0 \rightarrow \mathbb{R} \cdot \eta \hookrightarrow \text{cont}_0(M, \alpha) \overset{\pi}{\rightarrow} \text{dyn}(M, \alpha) \rightarrow 0
\]

Assume that the \( M \) is a compact manifold, and introduce a scalar product on the space of complex-valued smooth functions on the manifold \( M \) as follows:

\[
\langle \phi, \psi \rangle = \int_M \phi \overline{\psi} \cdot \alpha \wedge \omega^n, \ \phi, \psi \in C^\infty(M)_\mathbb{C}
\]

Let us denote the corresponding Hilbert space under \( \mathcal{H}(M) \). A vector field \( X \), on the manifold \( M \), defines an operator on the Hilbert space \( \mathcal{H}(M) \), via the derivation: \( \varphi \mapsto X(\varphi) \).

**Lemma 20** If \( X \in \text{cont}_0(M, \alpha) \), then the corresponding derivation operator \( X : \mathcal{H}(M) \rightarrow \mathcal{H}(M) \) is antisymmetric for the scalar product \( \langle \cdot, \cdot \rangle \).

**Proof.** Let us denote the volume form \( \alpha \wedge \omega^n \) by \( V \). For any function \( f \in C^\infty(M)_\mathbb{C} \), we have the following:

\[
L_X(f \cdot V) = i_X(\overbrace{d(f \cdot V)}^0) + d(i_X(f \cdot V)) = d(i_X(f \cdot V))
\]

On the other hand, the operator of Lie derivation is a differential operator of degree 0, that is: \( L_X(f \cdot V) = X(f) \cdot V + f \cdot L_X(V) \). But by definition of the Lie algebra \( \text{cont}_0(M, \alpha) \), we have that \( L_X(V) = 0 \). Therefore, we obtain:

\[
X(f) \cdot V = d(i_X(f \cdot V))
\]
If $f = \phi \psi$, for some $\phi, \psi \in \mathcal{H}(M)$, we have that
\[
\langle X(\phi), \psi \rangle + \langle \phi, X(\psi) \rangle = \int_M d(i_X(\phi \psi) \cdot V) = 0 \iff \langle X(\phi), \psi \rangle = -\langle \phi, X(\psi) \rangle
\]
i.e., the operator $X : \mathcal{H}(M) \longrightarrow \mathcal{H}(M)$ is antisymmetric. ■

To summarize, we can state that we have a representation of the Poisson algebra of basic functions on the contact manifold $(M, \alpha)$, in the Lie algebra of antisymmetric operators on the Hilbert space $\mathcal{H}(M)$:
\[
\mathcal{F}_B(M) \ni f \mapsto \hat{f} : \mathcal{H}(M) \longrightarrow \mathcal{H}(M)
\]
It is clear that the Hilbert space $\mathcal{H}(M)$ is too large for this representation, to be irreducible. Let us make one “small” step to the reduction of this representation. For any $h \in \mathbb{R}$, let
\[
\mathcal{H}_h(M) = \{ \varphi \in \mathcal{H}(M) \mid \eta(\varphi) = ih \cdot \varphi \}
\]
As any operator of the type $\hat{f}$ commutes with $\eta$, the subspace $\mathcal{H}_h(M)$ is invariant for the operators $\hat{f}, f \in \mathcal{F}_B(M)$. For such operators, on the subspace $\mathcal{H}_h(M)$, we have $\hat{f}(\phi) = (f \cdot \eta + \tilde{\mu}(df))(\phi) = ih f \phi + \tilde{\mu}(df)(\phi)$. The operator $\hat{f}$ consists of the vertical and horizontal parts: $f \cdot \eta$ and $\tilde{\mu}(df)$. As smaller the absolute value of the real number $h$ is, as closer the operator $\hat{f}$ is to its horizontal part on the space $\mathcal{H}_h(M)$.

The homomorphism from the Poisson algebra $\mathcal{F}_B(M)$ to the Lie algebra $\text{cont}_0(M, \alpha)$: $f \mapsto \hat{f}$, is not a prequantization, because the operators $\hat{f}$ are antisymmetric ant not Hermitian. To “correct” this situation, we can multiply them by some imaginary complex number: $o(f) = ih \cdot f$. After this, redefine the commutator on the set $ih \cdot \text{cont}_0(M, \alpha)$, so that the obtained bracket be a Lie algebra structure:
\[
\{o(f), o(g)\} = \frac{1}{ih} [o(f), o(g)] = ih[\hat{f}, \hat{g}]
\]
After this, the pair $(ih \cdot \text{cont}_0(M, \alpha), \{, \})$ is a Lie algebra and the mapping $f \mapsto o(f)$ is a Lie algebra homomorphism. On the Hilbert space $\mathcal{H}_h(M)$, the operator $o(f)$, is $o(f) = f - \frac{\tilde{\mu}(df)}{h}$, where $f$ denotes the ”multiplication by $f$“ operator: $\varphi \mapsto f \cdot \varphi$.

In “good” cases, when the contact structure is regular, i.e., the space of orbits of the canonical vector field $\eta$ is separable, the contact manifold $M$
can be considered as the total space of a principal bundle over the space of orbits of the canonical vector field $\eta$. The structure group of this bundle is the circle $S^1 \cong U(1)$, and the form $A = 2\pi i \alpha$ is a connection form on this principal bundle. The base of this bundle, is a symplectic manifold, with symplectic form $dA$ (see [2]). The Hilbert space $\mathcal{H}_h(M), \ h \in \mathbb{R}$ is canonically isomorphic to the space of sections of the associated complex line bundle, via the representation of the group $U(1)$: $\varphi_h : U(1) \rightarrow U(1), \ \varphi_h(a) = a^h$. 
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