Gravity Dual of Quantum Information Metric

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We study a quantum information metric (or fidelity susceptibility) in conformal field theories with respect to a small perturbation by a primary operator. We argue that its gravity dual is approximately given by a volume of maximal time slice in an AdS spacetime when the perturbation is exactly marginal. We confirm our claim in several examples.

Introduction

The microscopic understanding of black hole entropy in string theory by Strominger and Vafa\textsuperscript{1} implies that quantum information plays a crucial role to understand gravitational aspects of string theory. Indeed, quantum information theoretic considerations have provided various useful viewpoints in studies of AdS/CFT\textsuperscript{2} or more generally holography\textsuperscript{3}. Especially, the idea of quantum entanglement has turned out to be crucially involved in geometries of holographic spacetimes, as typical in the non-trivial topology of eternal black holes\textsuperscript{4}. To quantify quantum entanglement we can study the holographic entanglement entropy\textsuperscript{5}, which is given by the area of codimension two extremal surfaces. In the AdS/CFT, this area is equal to the entanglement entropy in conformal field theories (CFTs).

It is natural to wonder if there might be some other information theoretic quantities which are useful to develop studies of holography. As pointed out by Susskind in\textsuperscript{6} (see also\textsuperscript{7}), it is also intriguing to find a quantity in CFTs which is dual to a volume of a codimension one time slice in AdS. The time slice can connect two boundaries dual to the thermofield doubled CFTs, through the Einstein-Rosen bridge (see Fig.\textsuperscript{1}). In\textsuperscript{6}, it is conjectured that this quantity is related to a measure of complexity.

The main purpose of this letter is to point out a quantum information theoretic quantity which is related to the volume of a time slice. This quantity is called quantum information metric or Bures metric (see e.g.\textsuperscript{8}), which we will simply call the information metric. Here we mainly consider the information metric for pure states, though it can be defined for mixed states. Consider one parameter family of quantum states $|\Psi(\lambda)\rangle$ and perturb $\lambda$ infinitesimally $\lambda \rightarrow \lambda + \delta \lambda$. Then $G_{\lambda\lambda}$ is simply defined from the inner product between them as follows:

\begin{equation}
|\langle \Psi(\lambda) | \Psi(\lambda + \delta \lambda) \rangle| = 1 - G_{\lambda\lambda} \cdot (\delta \lambda)^2 + O((\delta \lambda)^3).
\end{equation}

This metric measures the distance between two infinitesimally different quantum states. Since the left-hand side of\textsuperscript{11} is called the fidelity, $G_{\lambda\lambda}$ is also called the fidelity susceptibility (see e.g. the review\textsuperscript{10} for applications to quantum phase transitions.).

We will argue that $G_{\lambda\lambda}$ when a $d+1$ dimensional CFT is deformed by an exactly marginal perturbation, parameterized by $\lambda$, is holographically estimated by

\begin{equation}
G_{\lambda\lambda} = n_d \cdot \frac{\text{Vol}(\Sigma_{\text{max}})}{R^{d+1}},
\end{equation}

where $n_d$ is an $O(1)$ constant and $R$ is the AdS radius. The $d+1$ dimensional space-like surface $\Sigma_{\text{max}}$ is the time slice with the maximal volume in the AdS which ends on the time slice at the AdS boundary(ies). See also\textsuperscript{9} for some other holographic interpretations of information metric.

Information Metric in CFT\textsubscript{$d+1$}

Now we introduce the information metric for quantum states in CFTs on $R^{d+1}$, whose Euclidean time and space coordinates are denoted by $\tau$ and $x$. We consider the inner product $\langle \Omega_1 | \Omega_2 \rangle$ between two states $|\Omega_1\rangle$ and $|\Omega_2\rangle$. $|\Omega_i\rangle$ ($i = 1, 2$) are ground states for the two Hamiltonians $H_i$ ($i = 1, 2$). We define their Euclidean lagrangians by $L_i$ ($i = 1, 2$) and their partition functions $Z_i$ ($i = 1, 2$).
The inner product is described by the path-integral:

$$\langle \Omega_2 | \Omega_1 \rangle = (Z_1 Z_2)^{-1/2} \int D\phi \exp[- \int d^4x (\int_{-\infty}^{0} d\tau \mathcal{L}_1 + \int_{0}^{\infty} d\tau \mathcal{L}_2)].$$

Assume the difference $\mathcal{L}_2 - \mathcal{L}_1$ is infinitesimally small and is written by using the primary operator $O(\tau, x)$ as

$$\mathcal{L}_2 - \mathcal{L}_1 \equiv \delta \mathcal{L} = \delta \lambda \cdot O(\tau, x). \quad (4)$$

Next, we rewrite (3) by using the expectation value $\langle \cdots \rangle$ in the vacuum state $|\Omega_1\rangle$:

$$\langle \Omega_2(\epsilon) | \Omega_1 \rangle \equiv \frac{(\exp[- \int_{-\epsilon}^{\epsilon} d\tau \int d^4x \delta \mathcal{L}])}{(\exp[-(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty})d\tau \int d^4x \delta \mathcal{L}])}$$

where $\delta \mathcal{L} \equiv \mathcal{L}_2 - \mathcal{L}_1$. Here we introduced the UV regularization $\epsilon$ by replacing the ground state $|\Omega_2\rangle$ with

$$|\tilde{\Omega}_2(\epsilon)\rangle \equiv \frac{e^{-\epsilon H_1}}{(|\Omega_2\rangle - e^{-2\epsilon H_1} |\Omega_2\rangle)^{1/2}}. \quad (6)$$

By performing perturbative expansions of (3) up to quadratic terms, we obtain

$$1 - \langle \tilde{\Omega}_2(\epsilon) | \Omega_1 \rangle = \frac{1}{2} \int_{-\epsilon}^{\epsilon} d\tau \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^4x \int d^4x' \delta \mathcal{L}(\tau, x) \delta \mathcal{L}(\tau', x'),$$

where we assumed the time reversal symmetry relation $\langle \delta \mathcal{L}(\tau, x) \delta \mathcal{L}(-\tau', x') \rangle = (\delta \mathcal{L}(\tau, x) \delta \mathcal{L}(\tau', x')).$

In this way, the information metric with respect to the $\lambda$ perturbation (11) is computed as

$$G_{\lambda \lambda} = \frac{1}{2} \int_{-\epsilon}^{\epsilon} d\tau \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^4x \int d^4x' O(\tau, x) O(\tau', x').$$

Note that up to now, our argument can be applied to any local operator $O$ in any quantum field theory. However, for simplicity, we would like to focus on the case where the spin of the primary field $O$ is zero and its total conformal dimension is $\Delta$ in this letter. We can generalize our analysis to a current or an energy stress tensor as we present the details in Appendix A and B.

The (normalized) two point function of the primary field takes the universal form

$$\langle O(\tau, x) O(\tau', x') \rangle = \frac{1}{(\tau - \tau')^2 + (x - x')^2} \Delta. \quad (8)$$

By plugging (8) into (14), when $d + 2 - 2\Delta < 0$ we obtain

$$G_{\lambda \lambda} = N_d \cdot V_d \cdot \epsilon^{d+2-2\Delta}, \quad (9)$$

where we define $N_d = \frac{\epsilon^{d+2-2\Delta} \text{Vol}^2(\gamma - d/2 - 1) \Gamma(\Delta - d/2 - 1)}{2(\Delta - d/2 - 1) \Gamma(\Delta)}$ and $V_d$ is the infinite volume of $R^d$. In particular, for a marginal perturbation we have $d + 2 - 2\Delta = -d$. On the other hand, when $d + 2 - 2\Delta \geq 0$, there exists an infrared (IR) convergence and we need an IR cut off $L$ for both the $\tau$ and $x$ integral. This leads to $G_{\lambda \lambda} \sim V_d \cdot L^{d+2-2\Delta}$ (agreeing with (14)), where for $d + 2 - 2\Delta = 0$ we regard $L^0$ as $\log(L/\epsilon)$.

**Holographic Computation**

Now we would like to turn to holographic calculations. For this we focus on the case where the perturbation (7) is exactly marginal $\Delta = d + 1$. This greatly simplifies the computation in the gravity dual. This is because gravity backgrounds dual to both of $|\Omega_i\rangle$ ($i = 1, 2$) are the pure AdS$_{d+2}$ with the same radius $R$. Such a gravity dual which interpolates two AdS spaces is called a Janus solution (11). The massless bulk scalar field dual to the exactly marginal operator $O$ is denoted by $\phi$.

Let us first study the AdS$_2$ Janus solution introduced in (12). This setup is defined by the action:

$$S = -\frac{1}{16\pi G_N} \int dz^3 \sqrt{g} \left( -g^{ab} \partial_a \phi \partial_b \phi + \frac{1}{R^2} \right). \quad (10)$$

The Janus solution is given by the metric

$$ds^2 = R^2 \left( dy^2 + f(y)ds^2_{AdS_2} \right), \quad f(y) = \frac{1}{2}(1 + \sqrt{1 - 2\gamma^2 \cosh(2y)}), \quad (11)$$

and the dilaton

$$\phi(y) = \gamma \int_{-\infty}^{y} \frac{dy}{f(y)} + \phi_1. \quad (12)$$

where $\gamma$ ($\leq \gamma_2$) is the parameter of Janus deformation. The metric of AdS$_2$ slice is given by $ds^2_{AdS_2} = (dz^2 + dx^2)/z^2$. $\phi_1 = \phi(-\infty)$ is dual to the coupling constant of the exactly marginal deformation for the ground state $|\Omega_1\rangle$. On the other hand, the value $\phi_2 = \phi(\infty)$ for the other ground state $|\Omega_2\rangle$ is obtained by performing the integral in (12) as $\phi_2 - \phi_1 = \sqrt{2} \arctan \left( \frac{1 - \sqrt{1 - 2\gamma^2}}{\sqrt{2}\gamma} \right) \approx \gamma$ when $\gamma \ll 1$.

By matching the asymptotic behavior of the metric (12) at the infinity $|y| = y_{\infty}$ with that of undeformed metric ($\gamma = 0$)

$$ds^2_{pure} = R^2 \left( dy^2 + \frac{1}{2}(1 + \cosh(2y))ds^2_{AdS_2} \right), \quad (13)$$

we find the following condition

$$\sqrt{1 - 2\gamma^2} \cosh(2y) = \epsilon^{2y_{\infty}}. \quad (14)$$

The on-shell action of (10) is evaluated by

$$S(\gamma) = \frac{R}{4\pi G_N} V_{AdS_2} \int_{-y_{\infty}}^{y_{\infty}} dy \left\{ \frac{1}{2}(1 + \sqrt{1 - 2\gamma^2 \cosh(2y)}) \right\} = \frac{R}{4\pi G_N} V_{AdS_2} \left\{ y_{\infty} + \frac{1}{2} \sqrt{1 - 2\gamma^2 \sinh(2y_{\infty})} \right\}. \quad (15)$$
where \( V_{AdS_2} = \int dx \int_0^\infty \frac{dv}{v^2} = \frac{\Gamma(1)}{\Gamma(3)} \) is the volume of \( AdS_2 \) with a unit radius.

By using the condition (14) at the infinity
\[
S(\gamma) - S(0) = \frac{R}{4\pi G_N} V_{AdS_2} \cdot (y_\infty - \dot{y}_\infty) = \frac{RV_1}{16\pi G_N\epsilon} \log \left( \frac{1}{1 - 2\gamma^2} \right) > 0.
\]

For small \( \gamma^2 \), we finally find
\[
|\langle \Omega_2 | \Omega_1 \rangle | = e^{-S(\gamma) - S(0)} \approx 1 - \frac{RV_1}{8\pi G_N\epsilon} \gamma^2.
\]

Therefore the information metric is estimated as follows
\[
G_{\gamma \gamma} = \frac{cV_1}{12\pi \epsilon},
\]
where we employed the holographic expression of the central charge \( c = \frac{R}{2\pi G_N} \). Since the normalization of scalar field \( \phi \) leads to the two point function of \( O \) which is proportional to the central charge \( c \) (i.e. \( \delta \lambda \propto \sqrt{\epsilon} \delta \phi = \sqrt{\epsilon} \gamma \)), we indeed obtain the advertised formula (2), where \( \Sigma \) is given by the \( AdS_2 \) slice \( \rho = 0 \) in \( AdS_3 \).

In order to study higher dimensional examples in a universal way, we would like to consider a simple holographic model, which turns out to give an excellent approximation to various explicit examples. This holographic model is obtained by identifying the exactly marginal deformation at the time slice \( \tau = 0 \) in \( CFT_{d+1} \) with a \( d + 1 \) dimensional defect brane \( \Sigma \) with a tension \( T \), which extends from the time slice on the \( AdS \) boundary to the bulk. This is similar to the holographic constructions in \[13, 13\]. In the gravity setup this is simply realized by adding the defect brane action
\[
S_{brane} = T \int_\Sigma \sqrt{g}.
\]

This prescription is consistent with the boundary (or defect) conformal symmetry in a way similar to the AdS/BCFT \[12\]. Again we can describe the perturbation \[1\] by the profile of a massless scalar field \( \phi \). When the deformation \( \delta \lambda \) is infinitesimally small, we have
\[
T \approx n_d \cdot \frac{(\delta \lambda)^2}{R^{d+1}},
\]
where \( n_d \) is an \( O(1) \) constant which is fixed from the normalization of two point function \[8\]. The Einstein equation shows that \( T \) is proportional to \( (\delta \lambda)^2 \), as the bulk stress tensor is quadratic with respect to the scalar field. The dependence on \( R \) can be explained by the dimensional reason or by comparing with the Janus computation. In this limit, we can treat the brane as a probe, ignoring its back-reaction. Finally we impose the equation of motion with respect to the brane embedding. This requires that the action \[13\] is extremized and thus in the Lorentzian signature, \( \Sigma \) is the maximal area surface \( \Sigma_{\text{max}} \) which ends on the time slice \( \tau = 0 \) at the \( AdS \) boundary. Together with (20), we reach our claim (2).

For a \( CFT_{d+1} \) on \( R^{d+1} \), the holographic formula (2) leads to
\[
G_{\lambda \lambda} = n_d V_d \int_0^\infty \frac{dz}{z^{d+1}} = n_d V_d \frac{\epsilon}{d^3 \epsilon},
\]
which indeed agrees with (9).

Similarly we can analyze the global \( AdS_{d+2} \), which is described by the metric
\[
ds^2 = R^2 \left( - (r^2 + 1) dt^2 + \frac{dr^2}{r^2 + 1} + r^2 d\Omega_2^2 \right),
\]
to obtain the information metric for a \( CFT_{d+1} \) on \( R \times S^d \).
\[
G_{\lambda \lambda} = n_d V_d \int_0^{r_\infty} \frac{r^d}{\sqrt{r^2 + 1}} dr = n_d V_d \epsilon + \cdots,
\]
where \( r_\infty \sim 1/\epsilon \) is dual to the UV cut off of the CFT. It might be curious that there appears a logarithmic divergent term \( \log r_\infty \) when \( d \) is even, i.e. odd dimensional CFTs. This logarithmic term is analogous to the boundary central charge in BCFT \[16\]. Also it is clear from \[23\] that \( G_{\lambda \lambda} \) is smaller than the flat space one \[24\] and this is due to the mass gap in CFTs on a compact space.

Another interesting example is the \( d + 2 \) dimensional \( AdS \) Schwarzschild black hole
\[
ds^2 = R^2 \left( \frac{dz^2}{h(z) z^2} - \frac{h(z)}{z^2} dt^2 + \sum_{i=1}^d \frac{dx_i^2}{z^2} \right),
\]
\[
h(z) \equiv 1 - (z/z_0)^{d+1},
\]
which is dual to the finite temperature CFT. The parameter \( z_0 \) is related the temperature \( T \) via \( z_0 = \frac{d+1}{\pi T} \). The information metric is computed as
\[
G_{\lambda \lambda} = n_d V_d \int_0^{z_0} \frac{dz}{\sqrt{h(z)} z^{d+1}} = \frac{n_d V_d}{d} \frac{1}{(1 + \frac{bd}{d+1})},
\]
\[
b_d \equiv -1 + d \int_0^1 dy \left( 1 - y^{d+1} + \sqrt{1 - y^{d+1}} \right)^{-1} = (d - 1) \sqrt[2d+1]{\frac{\Gamma}{\pi^{2d+1}}} \left( \frac{1 + \frac{bd}{d+1}}{1 + \frac{bd}{d+1}} \right) \geq 0.
\]

For example, we have \( b_1 = 0, b_2 \approx 0.70, b_3 \approx 1.31 \).

Finally we would like to study a time-dependent example in order to confirm our proposed formula (2) can be applied to such a non-trivial setup. For this purpose, we consider thermo-field double (TFD) description of finite temperature state in a two dimensional (2d) CFT:
\[
|\Psi_{TFD}(t)\rangle \propto e^{-i(H^{(A)} + H^{(B)})t} \sum_n e^{-\frac{\epsilon}{2}(H^{(A)} + H^{(B)})} |n\rangle_A |n\rangle_B,
\]
(26)
where \( H^{(A)} \) and \( H^{(B)} \) are the identical Hamiltonians for the first and second CFT of the TFD; the states \( |n\rangle_{A,B} \) are the unit norm energy eigenstates in the two CFTs. This TFD setup is dual to the extended geometry of eternal AdS black hole depicted in Fig[1].

We are interested in the inner product \( \langle \Psi^\prime_{TFD}(t)|\Psi^\prime_{TFD}(t) \rangle \) and the information metric \( G_{\lambda \lambda} \). Here, the state \( \langle \Psi^\prime_{TFD}(t) \rangle \) is the TFD state with a Hamiltonian \( H^{(A)} + H^{(B)} \) which is obtained by an infinitesimal exactly marginal \( \lambda \)-perturbation [1] with respect to each of \( H^{(A)} \) and \( H^{(B)} \) at the same time \( t \).

We argue this deformation is dual to introduce a defect brane \( \Sigma \) in the BTZ black hole as in Fig[1].

Let us compute \( G_{\lambda \lambda} \) in an Euclidean path-integral formalism of a 2d CFT. The two point function on \( S^2 \)brane \( \Sigma \) in the BTZ black hole as in Fig[1].

Here, the state \( \langle \Psi^\prime_{TFD}(t) \rangle \) is a thermal circle with periodicity \( \pi \). Then,

\[
G_{\lambda \lambda} = \int d\pi \cos \kappa \sqrt{\sin^2(2\kappa) - (\partial \kappa / \partial \tilde{t})^2}. \tag{33}
\]

We define \( \kappa_* \) \( (0 \leq \kappa_* < \pi/4) \) to be the value of \( \kappa \) where \( \partial \kappa / \partial \tilde{t} = 0 \). We can maximize the volume \( \text{Vol} \) and extend the solution into the region (I) in a way similar to [17]. In the end we obtain the following expression of \( \text{Vol}(\Sigma_{\text{max}}) \) as a function of \( t \) in terms of the parameter \( \kappa_* \):

\[
\text{Vol}(\Sigma) = R^d V_1 \int d\tilde{t} \cos \kappa \sqrt{\sin^2\kappa - (\partial \kappa / \partial \tilde{t})^2}. \tag{33}
\]

We can continue into the region (II) by setting \( \kappa = -i \rho \) and \( t = \tilde{t} + \pi \). In the region (II), if we specify \( \kappa \) by \( \kappa = \kappa(t) \) of \( \Sigma \), its volume is given by

\[
\text{Vol}(\Sigma) = R^d V_1 \int d\rho \sqrt{\sin^2\rho - (\partial \rho / \partial \tilde{t})^2}. \tag{33}
\]

We plotted the holographic result \( \text{Vol}(\Sigma)/(R^{d+1} V_1) \) (blue) with that of our CFT result \( 2G_{\lambda \lambda}/V_1 \) (red) as a function of the time \( t \) for the TFD state. They deviates only slightly. We set \( \beta = 2\pi \).

Fig. 2. We compare the finite part of our holographic result \( \text{Vol}(\Sigma)/(R^{d+1} V_1) \) (blue) with that of our CFT result \( 2G_{\lambda \lambda}/V_1 \) (red) as a function of the time \( t \) for the TFD state. They deviates only slightly. We set \( \beta = 2\pi \).

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We now turn to the holographic computation in the BTZ black hole, where the region (I) in Fig[1] is described by the metric (we set \( \beta = 2\pi \) for simplicity)

\[
ds^2 = R^2 (-\sin^2 \rho dt^2 + d\rho^2 + \cosh^2 \rho dx^2). \tag{32}
\]
\( G_{\lambda\lambda} \sim \frac{\text{Vol}(\Sigma_{\text{max}})}{G_{SR}} \), which is the same formula argued in [2] to measure the amount of complexity.

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[1] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” Phys. Lett. B 379 (1996) 99 [hep-th/9601029].
[2] J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113] arXiv:hep-th/9711200; G. ’t Hooft, arXiv:gr-qc/9310026; L. Susskind, J. Math. Phys. 36, 6377 (1995) arXiv:hep-th/9409089; D. Bigatti and L. Susskind, arXiv:hep-th/0602044.
[3] J. M. Maldacena, “Eternal black holes in anti-de Sitter,” JHEP 0304 (2003) 021 [hep-th/0106112].
[4] S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96 (2006) 181602 [hep-th/0603001]; JHEP 0608, 045 (2006) [hep-th/0605073]; V. E. Hubeny, M. Rangamani and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” JHEP 0707 (2007) 062 arXiv:0705.0016 [hep-th].
[5] L. Susskind, “Computational Complexity and Black Hole Horizons,” arXiv:1403.5995 [hep-th]; “Entanglement is not Enough,” arXiv:1411.0690 [hep-th].
[6] D. Stanford and L. Susskind, “Complexity and Shock Wave Geometries,” Phys. Rev. D 90 (2014) 12, 126007 [arXiv:1406.2678 [hep-th]].
[7] S. L. Braunstein and C. M. Caves, “Statistical distance and the geometry of quantum states,” Phys. Rev. Lett. 72 (1994) 3439.
[8] M. Nozaki, S. Ryu and T. Takayanagi, JHEP 1210 (2012) 193 [arXiv:1208.3469 [hep-th]]; M. Miyaji and T. Takayanagi, arXiv:1503.03542 [hep-th], to be published in PTEP; M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi and K. Watanabe, “cMERA as Surface/State Correspondence in AdS/CFT,” arXiv:1506.01353 [hep-th].
[9] S.-J. Gu, “Fidelity Approach to Quantum Phase Transitions,” Int. J. Mod. Phys. B 24 (2010) 4371.
[10] D. Bak, M. Gutperle and S. Hirano, “A Dilatonic deformation of AdS(5) and its field theory dual,” JHEP 0305 (2003) 072 [hep-th/0304129].
[11] D. Bak, M. Gutperle and S. Hirano, “Three dimensional Janus and time-dependent black holes,” JHEP 0702 (2007) 068 [hep-th/0701108].
[12] A. Karch and L. Randall, “Locally localized gravity,” JHEP 0105 (2001) 008 [hep-th/0011156]; “Open and closed string interpretation of SUSY CFT’s on branes with boundaries,” JHEP 0106 (2001) 063 [hep-th/0105132].
[13] T. Azeyanagi, A. Karch, T. Takayanagi and E. G. Thompson, “Holographic calculation of boundary entropy,” JHEP 0803 (2008) 054 arXiv:0712.1850 [hep-th].
[14] T. Takayanagi, “Holographic Dual of BCFT,” Phys. Rev. Lett. 107 (2011) 101602 [arXiv:1105.5165 [hep-th]]; M. Fujita, T. Takayanagi and E. Tomi, “Aspects of AdS/BCFT,” JHEP 1111 (2011) 043 [arXiv:1108.5152 [hep-th]].
[15] M. Nozaki, T. Takayanagi and T. Ugajin, “Central Charges for BCFTs and Holography,” JHEP 1206 (2012) 066 [arXiv:1206.1371 [hep-th]].
[16] T. Hartman and J. Maldacena, “Time Evolution of Entanglement Entropy from Black Hole Interiors,” JHEP 1305 (2013) 014 arXiv:1303.1080 [hep-th].
[17] H. Osborn and A. C. Petkos, “Implications of conformal invariance in field theories for general dimensions,” Annals Phys. 231 (1994) 311 [hep-th/9307010].

Appendix A: Information Metric for Gauge Field Perturbations in CFTs

We consider a CFT perturbed by the conserved current,

\[ S_{\text{tot}} = S_{\text{CFT}} + \int d^d x d\tau A_\mu(x) J^\mu(\tau, x), \]

where \( \partial_\tau \mu = 0 \). In this case the overlap in the leading order of the perturbation is given by

\[ 1 - \langle \Omega_2(\epsilon) | \Omega_1 \rangle = \frac{1}{2} \int d^d x d^d y \delta A^\mu(x) \delta A^\nu(y) J_{\mu\nu}(x, y) \]

where we have defined

\[ J_{\mu\nu}(x - y) \equiv \int_0^\infty d\tau \int_0^{-\epsilon} d\tau' \langle J_\mu(\tau, x) J_\nu(\tau', y) \rangle = \int_{-\epsilon}^\infty d(\tau - \tau')((\tau - \tau') - 2\epsilon) \langle J_\mu(\tau, x) J_\nu(\tau', y) \rangle. \]

We can calculate \( J_{\mu\nu} \) by using the explicit expression for the correlation function of the conserved currents [18],

\[ \langle J_\mu(\tau, x) J_\nu(\tau', y) \rangle = \left( \frac{C_V}{(\tau - \tau')^2 + (x - y)^2} \right) \times \left( \delta_{\mu\nu} - 2 \frac{(x - y) \mu (x - y) \nu}{(\tau - \tau')^2 + (x - y)^2} \right). \]
where $C_V$ is a constant determining the overall scale of this two point function. For $d > 1$, we obtain

\begin{equation}
J_{\tau\tau}(x)/C_V = (4\epsilon^2 + r^2)^{-d} \left[ -4\epsilon^2 \left( \frac{4\epsilon^2}{r^2} + 1 \right) \frac{\Gamma(1/2)}{2(2d)(d+1)/2} \right]^{d/2} \frac{\Gamma(d+1/2)}{\Gamma(d+1)} \left( \frac{d-2}{d} \right)^{d/2} \Gamma(d+1) - \frac{r^2}{4d} \Gamma(d+1)
\end{equation}

and

\begin{equation}
J_{\tau j}(x)/C_V = -2x_j \left[ \frac{2^{2-d}((4\epsilon^2 + r^2)^{-d} \left[ \frac{\pi}{r^2} \frac{\Gamma(1/2)}{2(2d)(d+1)/2} \right]^{d/2} \frac{\Gamma(d+1)}{\Gamma(d+1)} \left( \frac{d-2}{d} \right)^{d/2} \Gamma(d+1) - \frac{r^2}{4d} \Gamma(d+1) \right] \right)
\end{equation}

as

\begin{equation}
J_{ij}(x)/C_V = \delta_{ij} \left[ \frac{8\epsilon^2 d^2 \frac{\Gamma(d+1/2)}{\Gamma(d+1)} \left( \frac{d-2}{d} \right)^{d/2} \Gamma(d+1)}{2(2d)(d+1)/2} \right]^{d/2} \frac{\Gamma(d+1)}{\Gamma(d+1)} \left( \frac{d-2}{d} \right)^{d/2} \Gamma(d+1) - \frac{r^2}{4d} \Gamma(d+1)
\end{equation}

and

\begin{equation}
J_{\tau x}(x)/C_V = \arctan \left( \frac{2\epsilon}{x} \right) - \arctan \left( \frac{T}{x} \right) - \frac{x(2\epsilon - T)}{x^2 + T^2}.
\end{equation}

It is useful to perform the Fourier transformation: $J_{\mu \nu}(k) = \int_{-\infty}^{\infty} dxe^{-ikx} J_{\mu \nu}(x)$, which leads to

\begin{equation}
\hat{J}_{\tau \tau}(k)/C_V = -\hat{J}_{xx}(k)/C_V = \frac{\pi}{|k|} \left( e^{-|k|\theta} - e^{-2|k|\theta} \right) + \frac{T - 2\epsilon}{2\pi} \left( e^{kT \theta} - e^{-kT \theta} \right)
\end{equation}

and

\begin{equation}
\hat{J}_{\tau x}(k)/C_V = 0
\end{equation}

After we go back to the original Lorentz signature by the Wick rotation $\tau = it$, for $T \to \infty$ and $\epsilon \to 0$, we obtain

\begin{equation}
\hat{J}_{\mu \nu}(k)/C_V = \hat{J}_{xx}(k)/C_V = \frac{\pi}{|k|}, \quad \hat{J}_{\tau x}(k)/C_V = \frac{\pi}{k}.
\end{equation}

Thus the eigenvalues of $\hat{J}_{\mu \nu}/C_V$ are $\left( \frac{x}{|k|} \pm \frac{x}{k} \right) \geq 0$, and the information metric is non-negative.

**Appendix B: Information Metric for Metric Perturbations in CFTs**

In this section, we consider a CFT perturbed by the energy momentum tensor:

\begin{equation}
S_{tot} = S_{CFT} + \int d^4xdtdy \, \bar{y} \Pi(x)T^{\mu \nu}(\tau, x)
\end{equation}

In this case the overlap in leading order of the perturbation is given by

\begin{equation}
\langle \tilde{\Omega}_{2}(\epsilon) \rangle = 1 - \frac{1}{2} \int d^4xdy \, \delta g^{\mu \nu}(x)\delta g^{\sigma \rho}(y)Q_{\mu \nu \rho \sigma}(x-y)
\end{equation}

where we defined

\begin{equation}
Q_{\mu \nu \rho \sigma}(x-y) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} dt' \langle T_{\mu \nu}(x, t)T_{\rho \sigma}(y, t') \rangle.
\end{equation}

We can calculate $Q_{\mu \nu \rho \sigma}$ by using the explicit expression for the correlation functions of the energy momentum tensor $\Pi$,

\begin{equation}
\langle T_{\mu \nu}(x, t)T_{\rho \sigma}(y, t') \rangle = \frac{C_T^{\rho \sigma} \delta_{\mu \nu}}{((x-y)^2 + (t-t')^2)^{d+1}} R_{\mu \nu}(x, t : y, t') R_{\rho \sigma}(x, t ; y, t')
\end{equation}
where
\[ E_{\mu\nu;\xi}^T = \frac{1}{2} (\delta_{\mu\eta} \delta_{\nu\xi} + \delta_{\mu\xi} \delta_{\nu\eta}) - \frac{1}{d+1} \delta_{\mu\nu} \delta_{\xi\eta}, \] (51)

\[ R_{\mu\nu}(x \cdot t; y, t') = \delta_{\mu\nu} - \frac{2}{(x - y)_{\mu} (x - y)_{\nu}} (t - t')^2, \] (52)

and \( C_T \) is a constant determining the overall scale of this two point function, which is related to the central charge of the CFT.

In real space coordinates, we obtain
\[ Q_{\mu\nu\eta\xi}^T(x)/C_T = \frac{(r^2 + 4\epsilon^2)^{-d-1}}{2(d+1)(d+2)} \times \left[ (-2 + d)r^2 + 4(-2 + d + 2d^2)\epsilon^2 \right. \\
- 8(-1 + d)(1 + d)^2r^2 F_1 \left( \frac{1}{2}, 1; \frac{3}{2} + d; -\frac{r^2}{4\epsilon^2} \right) \\
\left. - 8(1 + 2d)^2 F_1 \left( \frac{3}{2} + d + \frac{\epsilon^2}{d} \right) \right], \]

\[ Q_{\mu\nu\xi^\dagger\xi^\dagger}(x)/C_T = \delta_{ij} \frac{- (r^2 + 4\epsilon^2)^{-d-1}}{4\Gamma(d+2)} \times \left[ (r^2 + 4\epsilon^2)\Gamma(d) - (r^2 - 4\epsilon^2)\Gamma(d+1) \\
+ 4^{-d-2}(r^2 + 4\epsilon^2)^{1+d}\Gamma(2 + d) \\
- 2F_1 \left( \frac{3}{2} + d; 1 + 2d \right) \\
+ 2\epsilon^2 F_1 \left( \frac{5}{2} + d; \frac{3}{2} + d; \frac{\epsilon^2}{\epsilon^2} \right) \\
\right] \left[ (r^2 - 4\epsilon^2)^{-d-1}(r^2 + 4\epsilon^2)^{-d} \Gamma(d+2) \\
- 2F_1 \left( \frac{3}{2} + d; \frac{3}{2} + d; \frac{\epsilon^2}{\epsilon^2} \right) \\
\frac{1}{2(3 + 2d)\epsilon^2} \right] \right], \]

\[ Q_{x^i x^j x^k}(x)/C_T = \frac{1}{32(d+1)(d+2)(2d+3)\epsilon^3\Gamma(d+3)} \times \left[ (r^2 + 4\epsilon^2)^{-d-1}(r^2 - 4\epsilon^2)^{-d} \Gamma(d+2) \\
- 2F_1 \left( \frac{3}{2}, 1 + d \right) 2F_1 \left( \frac{3}{2} + d; \frac{3}{2} + d; \frac{\epsilon^2}{\epsilon^2} \right) \\
+ 4x^i x^j x^k \right] \] (53)

\[ Q_{x^i x^j x^k}(x)/C_T = - \frac{1}{32(d+1)(d+2)(2d+3)\epsilon^3\Gamma(d+3)} \times \left[ (r^2 + 4\epsilon^2)^{-d-1}(r^2 - 4\epsilon^2)^{-d} \Gamma(d+2) \\
- 2F_1 \left( \frac{3}{2}, 1 + d \right) 2F_1 \left( \frac{3}{2} + d; \frac{3}{2} + d; \frac{\epsilon^2}{\epsilon^2} \right) \\
+ 4x^i x^j x^k \right] \] (55)
When only $g_{00}$ component is perturbed by a constant, the perturbed Hamiltonian is a constant multiple of the original Hamiltonian. Therefore the information metric for such deformation should be zero identically. We can explicitly confirm this fact by a integration
\[
\int d^d x \, Q_{0000}(x) = 0.
\]