The Calderón problem for the fractional magnetic operator

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Abstract
We introduce a fractional magnetic operator involving a magnetic potential and an electric potential. We formulate an inverse problem for the fractional magnetic operator. We determine the electric potential from the exterior partial measurements of the associated Dirichlet-to-Neumann map by using Runge approximation property.

Keywords: the Calderón problem, Weyl pseudo-differential operator, fractional magnetic operator, Runge approximation property

1. Introduction
The study of the Calderón problem for local operators dates back to 1980s. We refer readers to [1] for a report on the progress made in this area. As a variation of the classical Calderón problem, the Calderón problem for the magnetic Schrödinger operator

\[ (-i\nabla + A(x))^2 + q(x) \]

where \( A(x) \) is a vector-valued magnetic potential and \( q(x) \) is an electric potential, has been extensively studied in the past decades. See, for instance, [2–5]. In those articles, the authors considered the Dirichlet problem

\[ (-i\nabla + A)^2 u + qu = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = f \]

where \( \Omega \) is a bounded domain with smooth boundary. They determined both \( A \) (up to a gauge equivalence) and \( q \) from the knowledge of the associated Dirichlet-to-Neumann map (DN map)

\[ \Lambda_{A,q} : f \rightarrow (\partial_{\nu} + iA \cdot \nu)u_f|_{\partial\Omega} \]

where \( u_f \) is the unique solution of the Dirichlet problem and \( \nu \) is the unit outer normal on \( \partial\Omega \).
In recent years, the study of the Calderón problem for fractional operators has also been an active research field in mathematics. This study is motivated by problems involving anomalous diffusion and random processes with jumps in probability theory. The fractional Calderón problem was first introduced in [6] where the inverse problem for the fractional operator

\[ (-\Delta)^s + q \quad (0 < s < 1) \]

was studied. In [6], the authors considered the exterior Dirichlet problem

\[ ((-\Delta)^s + q)u = 0 \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega_c} = f \]

where \( \Omega_c := \mathbb{R}^n \setminus \bar{\Omega} \). By using the strong uniqueness property of \((-\Delta)^s\), the authors proved the Runge approximation property of \((-\Delta)^s + q\) and the fundamental uniqueness result. The potential \( q \) in \( \Omega \) can be determined from the exterior partial measurements of the DN map

\[ L_q : f \rightarrow (\Delta)^s f|_{\Omega_c}. \]

This result has been generalized. In [7], the authors considered the fractional elliptic operator

\[ \mathcal{L}^s := (\nabla \cdot (M(x)\nabla))^s \]

where \( M \) is a smooth, real symmetric matrix-valued function satisfying the uniformly elliptic condition. They formulated the Calderón problem for \( \mathcal{L}^s + q \) and proved the corresponding uniqueness theorem. Also see [8–12] for more results related with the fractional Calderón problem.

In this paper, we study the Calderón problem for the fractional magnetic operator \( \mathcal{L}_A^s + q \). Our operator \( \mathcal{L}_A^s \) is formally defined by

\[ \mathcal{L}_A^s u(x) := 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(\epsilon, x)} \left( u(x) - e^{i|x-y|A} \left( \frac{\Delta}{\nabla \cdot (M(x)\nabla)} \right) u(y) \right) K(x, y) \, dy \quad \text{(1)} \]

where \( K(x, y) \) is a function associated with the heat kernel \( p_t(x, y) \) defined in subsection 2.2. We will see \( \mathcal{L}_A^s + q \) actually generalizes \( \mathcal{L}^s + q \) later. Besides, \( \mathcal{L}_A^s \) generalizes of the fractional magnetic Laplacian

\[ (-\Delta)^s_A u(x) := c_{n,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(\epsilon, x)} \frac{u(x) - e^{i|x-y|A} \left( \frac{\Delta}{\nabla \cdot (M(x)\nabla)} \right) u(y)}{|x-y|^{n+2s}} \, dy \]

introduced in [13]. It was proved in [14] that \( (-\Delta)^s_A \) converges to the magnetic Laplacian \((\nabla - iA(x))^2\) as \( s \to 1^- \) in an appropriate sense. Hence our problem can be viewed as a generalization of the Calderón problem studied in [7] as well as a nonlocal analogue of the Calderón problem for the magnetic Schrödinger operator.

**Remark.** \((-\Delta)^s_A\) has the form of a Weyl pseudo-differential operator. In fact, we consider

\[ u_x : y \rightarrow e^{i(x-y)A} \left( \frac{\Delta}{\nabla \cdot (M(x)\nabla)} \right) u(y). \]

for each fixed \( x \). Since we have the equivalent singular integral and Fourier transform definition of \((-\Delta)^s\) (see for instance, [15])

\[ (-\Delta)^s u(x) = c_{n,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(\epsilon, x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u(\xi))(x) \]

In the above, \( \mathcal{F} \) represents the Fourier transform, and \( \mathcal{F}^{-1} \) its inverse.
then by considering the value of $(-\Delta)^s u_e$ at $x$, we can formally do the computation

$$(-\Delta)^s u(x) = (-\Delta)^s u_e(x) = (2\pi)^{-n} \int e^{i\xi \cdot x} \frac{1}{\xi^2} F u_e(\xi) \, d\xi$$

$$= (2\pi)^{-n} \int e^{i\xi \cdot y} \left( \xi^2 + \left(\frac{x+y}{2}\right)^2 \right) |\xi|^2 u(y) \, dy \, d\xi$$

$$= (2\pi)^{-n} \int e^{i\xi \cdot y} |\xi - A(x)|^2 u(y) \, dy \, d\xi.$$  

In particular, when $s = \frac{1}{2}$, the symbol $|\xi - A(x)|$ corresponds to the classical relativistic Hamiltonian for a spinless particle of zero mass under the influence of the magnetic potential $A(x)$ and $(-\Delta)^{3/2}$ is one of the quantized kinetic energy operators. See for instance [16, 17].

To formulate the Calderón problem for $L_A^\prime + q$, we assume $A \in L^\infty(\mathbb{R}^n)$ and $q$ is regular for $L_A^\prime$, i.e. the associated sesquilinear form $B_{A,q}$ is coercive on $H^s(\Omega) \times H^s(\Omega)$ where $H^s(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in the Sobolev space $H^s(\mathbb{R}^n)$ to ensure that the exterior Dirichlet problem

$$(L_A^\prime + q)u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = g$$

has a unique solution $u_e \in H^s(\mathbb{R}^n)$ for each $g \in H^s(\mathbb{R}^n)$ and the solution operator $P_{A,q} : g \mapsto u_e$ is bounded on $H^s(\mathbb{R}^n)$. Then we can introduce the DN map $\Lambda_{A,q} : H^s(\Omega_e) \to H^s(\Omega_e)\ast$, which is formally defined by

$$\Lambda_{A,q} g := L_A^\prime u_e|_{\Omega_e}.$$  

The following theorem is the main result in this paper.

**Theorem 1.1.** Suppose $A \in L^\infty(\mathbb{R}^n)$ and $q_j \in L^\infty(\Omega)$ are regular for $L_A^\prime$, $\Omega \cup \text{supp } A \subset B_r(0)$ for some $r > 0$, $W_j$ are open sets s.t. $W_j \subset \Omega_j$ and $W_j \cap \overline{B_{r_j}(0)} \neq \emptyset$ ($j = 1, 2$). If

$$\Lambda_{A,q_j} g|_{W_2} = \Lambda_{A,q_j} g|_{W_2}$$

for any $g \in C_c^\infty(W_1)$, then $q_1 = q_2$.

**Remark.** Here we determine the electric potential $q$ from the knowledge of DN map for a fixed magnetic potential $A$. The question whether we can determine $A, q$ simultaneously from $\Lambda_{A,q}$ is still open. The assumption $W_j \cap \overline{B_{r_j}(0)} \neq \emptyset$ looks unnatural but it is essential when we show the Runge approximation property of $L_A^\prime + q$ based on the strong uniqueness property of $L_A^\prime$ proved in [7] (see section 5 for more details).

A closely related but different work on the fractional Calderón problem can be found in [18] where the fractional gradient $\nabla^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n \times \mathbb{R}^n)$ defined by

$$\nabla^s u(x,y) := c_{n,s}(u(x) - u(y)) \frac{y-x}{|y-x|^{n+2+s+1}}$$

was considered. Based on the identity

$$\langle (-\Delta)^s u, v \rangle = \langle \nabla^s u, \nabla^s v \rangle,$$

the author defined the operator $(-\Delta)^s_A$ by

$$\langle (-\Delta)^s_A u, v \rangle := \langle (\nabla^s + A(x,y))u, (\nabla^s + A(x,y))v \rangle.$$
for a bivariate vector-valued function \( \mathcal{A}(x, y) \) and then defined the DN map \( \Lambda_{A,q} \) associated with the fractional operator \((-\Delta)^s A + q\). It was proved in [18] that \((A, q)\) (up to a gauge equivalence) can be determined from the exterior partial measurements of \( \Lambda_{A,q} \) under some appropriate assumptions on \( A \) and \( q \).

We remark that both the operator \( L_A^* + q \) here and the operator \((-\Delta)^s A + q\) in [18] have their own advantages. \( L_A^* + q \) generalizes a broader class of fractional operators while the uniqueness theorem for \((-\Delta)^s A + q\) is stronger (see theorem 1.1 and lemma 3.8 in [18]).

The rest of this paper is organized in the following way. In section 2, we summarize the background knowledge. In section 3, we give the rigorous definition of \( L_A^* \) in bilinear form. We rigorously define the exterior Dirichlet problem and the DN map associated with \( L_A^* + q \) in section 4, prove the Runge approximation property of \( L_A^* + q \) and the main theorem in section 5.

2. Preliminaries

Throughout this paper

- \( n \geq 2 \) denotes the space dimension and \( 0 < s < 1 \) denotes the fractional power;
- \( \Omega \) denotes a bounded Lipschitz domain and \( \Omega := \mathbb{R}^n \setminus \bar{\Omega} \);
- \( B_r(0) \) denotes the open ball centered at the origin with radius \( r > 0 \) and \( \overline{B}_r(0) \) denotes the closure of \( B_r(0) \);
- \( A : \mathbb{R}^n \to \mathbb{R}^n \) denotes a real vector-valued magnetic potential;
- \( q \) defined on \( \Omega \) denotes an electric potential;
- \( c, C, C', C_1, \cdots \) denote positive constants (which may depend on some parameters);
- \( \int \cdots \int = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \);
- \( X^* \) denotes the continuous dual space of \( X \) and write \( \langle f, u \rangle = f(u) \) for \( u \in X, f \in X^* \);
- \( \mathcal{S}'(\mathbb{R}^n) \) denotes the space of temperate distributions.

2.1. Fourier transform and Sobolev spaces

Our notations for the Fourier transform and Sobolev spaces are

\[
\mathcal{F}u(\xi) = \hat{u}(\xi) := \int e^{-ix\cdot\xi}u(x)\,dx, \quad H^s(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : \int (1 + |\xi|^2)^s|\hat{u}(\xi)|^2\,d\xi < \infty \right\}
\]

where \( t \in \mathbb{R} \). For \( 0 < s < 1 \), one of the equivalent forms of the norm \( \| \cdot \|_{H^s} \) is

\[
\|u\|_{H^s} := (\|u\|_{L^2}^2 + \int \int \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy)^{1/2}.
\]

We will use the natural identification \( H^{-s}(\mathbb{R}^n) = H^s(\mathbb{R}^n)^* \).

Given an open set \( U \) and a closed set \( F \) in \( \mathbb{R}^n \),

\[
H^s(U) := \{ u|_U : u \in H^s(\mathbb{R}^n) \}, \quad H^s_0(\mathbb{R}^n) := \{ u \in H^s(\mathbb{R}^n) : \text{supp} u \subset F \},
\]

\[
\hat{H}^s(U) := \text{the closure of } C^\infty_c(U) \text{ in } H^s(\mathbb{R}^n).
\]

Since \( \Omega \) is Lipschitz bounded, then \( \hat{H}^s(\Omega) = H^s_{\Omega}(\mathbb{R}^n) \) and we have the natural identification

\[
H^s(\mathbb{R}^n) / \hat{H}^s(\Omega) = H^s(\Omega)\cdot
\]
2.2. Spectral theory and heat kernels

We will not use any spectral theory in later sections. Our goal here is to provide some background knowledge of $K(x,y)$, which appears in (1).

For a fixed smooth real symmetric matrix-valued function $M(x) = (a_{ij}(x))$ satisfying the uniformly elliptic condition, i.e.

$$C_M^{-1}|\xi|^2 \leq \sum_{1 \leq i,j \leq n} a_{ij}(x)\xi_i\xi_j \leq C_M|\xi|^2, \quad x, \xi \in \mathbb{R}^n,$$

$\mathcal{L} := -\nabla \cdot (M(x)\nabla)$ is well-defined and symmetric on $C_c^\infty(\mathbb{R}^n)$. It is known (see for instance, [19, 20]) that $\mathcal{L}$ extends to be a non-negative, self-adjoint operator in $L^2(\mathbb{R}^n)$ with the domain

$$\text{Dom}(\mathcal{L}) = \{u \in H^1(\mathbb{R}^n) : \mathcal{L}u \in L^2(\mathbb{R}^n)\}$$

and $\text{spec} \mathcal{L} \subset [0, \infty)$. For $t \geq 0$, we define

$$e^{-t\mathcal{L}} := \int_0^\infty e^{-ts}dE_\lambda,$$

where $E_\lambda$ is the spectral resolution of $\mathcal{L}$. This is a family of bounded self-adjoint operators on $L^2(\mathbb{R}^n)$. It is known (see theorems 7.13 and 7.20 in [19, 20]) that, there exists a unique symmetric heat kernel $p_t(\cdot, \cdot)$ s.t. $p_t(x, y)$ is $C^\infty$-smooth jointly in $t > 0, x, y \in \mathbb{R}^n$ and

$$(e^{-t\mathcal{L}}f)(x) = \int p_t(x, y)f(y)dy, \quad x \in \mathbb{R}^n, t > 0, f \in L^2(\mathbb{R}^n).$$

Moreover, we have the following Gaussian bounds on $p_t(x, y)$ (see chapter 3 in [19])

$$c_1e^{-b_1 \frac{|x-y|^2}{t}} \leq p_t(x, y) \leq c_2e^{-b_2 \frac{|x-y|^2}{t}}, \quad x, y \in \mathbb{R}^n, t > 0.$$

Now we define

$$K(x, y) := C\int_0^\infty p_t(x, y)\frac{dt}{t^{n+\frac{n}{2}}},$$

By using the substitution $\alpha = \frac{|x-y|^2}{t}$, we can easily get the estimate

$$\frac{C_1}{|x-y|^{n+2}} \leq K(x, y) = K(y, x) \leq \frac{C_2}{|x-y|^{n+2}}, \quad x \neq y, x, y \in \mathbb{R}^n.$$

**Remark.** Note that if $M$ is the identity matrix, then we have

$$p_t(x, y) = \frac{1}{(4\pi t)^\frac{n}{2}}e^{-\frac{|x-y|^2}{4t}}, \quad K(x, y) = \frac{c}{|x-y|^{n+2}}.$$

It has been shown in [7] that

$$\mathcal{L}u(x) = 2\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} (u(x) - u(y))K(x, y)dy, \quad u \in H^1(\mathbb{R}^n)$$

so it is clear from (1) that $\mathcal{L}_A^\alpha = \mathcal{L}^\alpha$ when $A = 0.$
3. Fractional operator $\mathcal{L}_A^s$

Recall that in section 1 we gave the formal pointwise definition of $\mathcal{L}_A^s$ in (1). Now we do a formal computation to motivate the bilinear form definition of $\mathcal{L}_A^s$. For convenience we will write

$$E_A(x, y) = e^{i(x-y)A}\left(\frac{\cdot}{\cdot}\right)$$

when necessary. Note that

$$E_A(x, y) = E_A(\bar{y}, \bar{x}), \quad |E_A(x, y)| = 1, \quad K(x, y) = K(y, x)$$

so formally we have

\[
2 \int \left( \int_{\mathbb{R}^n \setminus B_i(x)} (u(x) - E_A(x, y)u(y))K(x, y) dy \right) \bar{v}(y) dx
\]

\[
= 2 \int \left( \int_{|x-y| > \epsilon} (u(x) - E_A(x, y)u(y))K(x, y) \bar{v}(y) dy \right) dx
\]

\[
= \int \left( \int_{|x-y| > \epsilon} (u(x) - E_A(x, y)u(y))K(x, y) \bar{v}(y) dy \right) dx
\]

\[
+ \int \left( \int_{|x-y| > \epsilon} (u(y) - E_A(x, y)u(x))K(y, x) \bar{v}(y) dx dy \right)
\]

\[
= \int \left( \int_{|x-y| > \epsilon} (u(x) - E_A(x, y)u(y))K(x, y) \bar{v}(y) dy \right) dx
\]

\[
- \int \left( \int_{|x-y| > \epsilon} (u(x) - E_A(x, y)u(y))K(x, y) E_A(x, y) \bar{v}(y) dx dy \right)
\]

\[
= \int \left( \int_{|x-y| > \epsilon} ((u(x) - E_A(x, y)u(y)) \bar{v}(x) - E_A(x, y) \bar{v}(y))K(x, y) dx dy \right).
\]

Let $\epsilon \to 0^+$, then formally we have

$$\langle \mathcal{L}_A^s u, \bar{v} \rangle = \int \int \left( u(x) - e^{i(x-y)A} \left(\frac{\cdot}{\cdot}\right) u(y) \right) \left( \bar{v}(x) - e^{i(x-y)A} \left(\frac{\cdot}{\cdot}\right) \bar{v}(y) \right) K(x, y) dx dy.$$

(5)

**Definition 3.1.** We define $\mathcal{L}_A^s$ by the bilinear form

$$\langle \mathcal{L}_A^s u, v \rangle := \int \int \left( u(x) - e^{i(x-y)A} \left(\frac{\cdot}{\cdot}\right) u(y) \right) \left( v(x) - e^{-i(x-y)A} \left(\frac{\cdot}{\cdot}\right) v(y) \right) K(x, y) dx dy.$$

(6)

It is clear from (6) that

$$\langle \mathcal{L}_A^s u, v \rangle = \langle \mathcal{L}_A^s v, u \rangle.$$

(7)
Remark. Note that by (3), (4) and (6), we have
\[
\langle \mathcal{L}_A^* u, v \rangle - \langle \mathcal{L}_A^* v, u \rangle = \int \int (u(x) - E_A(x,y)u(y))(v(x) - E_A(x,y)v(y))K(x,y)\,dxdy
- \int \int (v(x) - E_A(x,y)v(y))(u(x) - E_A(x,y)u(y))K(x,y)\,dxdy
= \int \int (E_A(x,y) - \overline{E_A(x,y)})u(x)v(y)K(x,y)\,dxdy
- \int \int (E_A(x,y) - \overline{E_A(x,y)})u(y)v(x)K(x,y)\,dxdy
= \int \int (E_A(x,y) - \overline{E_A(x,y)})u(x)v(y)K(x,y)\,dxdy
- \int \int (E_A(x,y) - \overline{E_A(x,y)})u(y)v(x)K(y,x)\,dydx
= 2 \int \int (E_A(x,y) - \overline{E_A(x,y)})u(x)v(y)K(x,y)\,dxdy
\]
so in general
\[
\langle \mathcal{L}_A^* u, v \rangle \neq \langle \mathcal{L}_A^* v, u \rangle.
\]
We claim that (6) is a bounded bilinear form on $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ for $A \in L^\infty(\mathbb{R}^n)$. To show this, we need to consider the norm $\| \cdot \|_{H_A^1}$ introduced in [13, 14].

Definition 3.2. The magnetic Sobolev norm $\| \cdot \|_{H_A^1}$ is defined by
\[
\|u\|_{H_A^1} := (\|u\|_{L^2}^2 + [u]_{H_A^1}^2)^{1/2}
\]
where
\[
[u]_{H_A^1} := \left( \int \int \frac{|u(x) - e^{i(x-y)\cdot A}(\overline{u})|}{|x-y|^{n+2s}}^2 \,dxdy \right)^{1/2}.
\]  
(8)

Clearly, $\| \cdot \|_{H_A^1} = \| \cdot \|_{L^2}$ when $A = 0$. In fact, we can show the equivalence between $\| \cdot \|_{H_A^1}$ and the classical $H^s$ norm for $A \in L^\infty(\mathbb{R}^n)$. The key estimate we will use is that
\[
|e^{i(x-y)\cdot A}(\overline{u}) - 1| \leq C \min \{1, |x-y|\}
\]
where $C$ depends on $\|A\|_{L^\infty}$.

Lemma 3.3. Suppose $0 < s < 1$ and $A \in L^\infty(\mathbb{R}^n)$, then $\| \cdot \|_{H_A^1} \sim \| \cdot \|_{H^s}$.

Proof. We only need to show that
\[
\|u\|_{H^s} - [u]_{H_A^1} \leq C \|u\|_{L^2}.
\]  
(9)
In fact, by using the identity
\[
|a|^2 - |b|^2 = (a - b)(a + \overline{b})b
\]
and (3), we have

\[ |[u]^2_{H^s} - [u]^2_{H^s_A}| = \left| \iint \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dxdy \right| = |f_1 + I_2| \]

where

\[ I_1 := \iint \frac{(E_A(x,y) - I)u(y)(u(x) - u(y))}{|x - y|^{n+2s}} \, dxdy \]

\[ I_2 := \iint \frac{(E_A(x,y) - I)u(y)(u(x) - E_A(x,y)u(y))}{|x - y|^{n+2s}} \, dxdy. \]

By Cauchy–Schwarz inequality we have

\[ |I_1| \leq \left( \iint \frac{|(E_A(x,y) - I)u(y)|^2}{|x - y|^{n+2s}} \, dxdy \right)^{1/2} \left( \iint \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dxdy \right)^{1/2} \]

\[ = \left( \left( \iint_{|x-y| \leq 1} |(E_A(x,y) - I)u(y)|^2 \, dxdy \right)^{1/2} \right) \left( \iint |u(x) - u(y)|^2 \, dxdy \right)^{1/2} \]

\[ \leq \left( \left( \iint_{|x-y| \leq 1} \frac{C^2|x - y|^2|u(y)|^2}{|x - y|^{n+2s}} \, dxdy \right)^{1/2} \right) \left( \iint |u(x) - u(y)|^2 \, dxdy \right)^{1/2} \]

\[ = \left( \int \left( \iint_{|x-y| \leq 1} \frac{C^2|x - y|^2|u(y)|^2}{|x - y|^{n+2s}} \, dxdy \right) |u(y)|^2 \, dy + \int \left( \iint_{|x-y| \geq 1} \frac{C^2|x - y|^2|u(y)|^2}{|x - y|^{n+2s}} \, dxdy \right) |u(y)|^2 \, dy \right)^{1/2} \]

\[ \leq C'\|u\|_{L^2} \|u\|_{H^s} \]

Similarly we can show

\[ |I_2| \leq C'\|u\|_{L^2} |[u]_{H^s_A}. \]

Hence, we have

\[ |[u]^2_{H^s} - [u]^2_{H^s_A}| \leq C'\|u\|_{L^2} ([u]_{H^s} + [u]_{H^s_A}), \]

which implies (9). \(\square\)

The boundness of \( L^s_A \) is now an immediate consequence of the lemma above.

**Proposition 3.4.** Suppose \(0 < s < 1\) and \(A \in L^\infty(\mathbb{R}^n)\), then

\[ L^s_A : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n) \]

is linear and bounded.

**Proof.** Since \(K(x,y) \sim 1/|x - y|^{n+2s}\), then by (5) and Cauchy–Schwarz inequality we have

\[ |\langle L^s_A u, \tilde{v} \rangle| \leq C[u]_{H^s} [v]_{H^{-s}} \leq C'\|u\|_{H^s} \|v\|_{H^{-s}} = C'\|u\|_{H^s} \|\tilde{v}\|_{H^{-s}}. \]

\(\square\)
4. Exterior Dirichlet problem and DN map

From now on we always assume \( A \in L^\infty(\mathbb{R}^n) \) and \( q \in L^\infty(\Omega) \).

**Definition 4.1.** The sesquilinear form associated with \( \mathcal{L}_A^* + q \) is defined by

\[
B_{A,q}(u, v) := \langle \mathcal{L}_A^* u, v \rangle + \int_{\Omega} quv, \quad u, v \in H^s(\mathbb{R}^n). \tag{10}
\]

The boundness of \( B_{A,q} \) follows from the boundness of \( \mathcal{L}_A^* \).

**Definition 4.2.** We say \( q \) is regular for \( \mathcal{L}_A^* \) if \( B_{A,q} \) is coercive on \( \tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega) \).

(7) implies \( q \) is regular for \( \mathcal{L}_A^* \) if and only if \( q \) is regular for \( \mathcal{L}_A \). Now we give a sufficient condition for \( q \) being regular.

**Proposition 4.3.** Suppose \( c \lesssim q \in L^\infty(\Omega) \) for some \( c > 0 \), then \( B_{A,q} \) is coercive on \( \tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega) \).

**Proof.** Since \( K(x, y) \sim 1/|x - y|^{n+2s} \), then by (5), (8) and lemma 3.3 we have

\[
B_{A,q}(u, u) \geq C\|u\|_{H^s}^2 + c\|u\|_{L^2}^2 \geq C\|u\|_{H^s}^2
\]

for \( u \in \tilde{H}^s(\Omega) \), so the coercivity holds. \( \square \)

4.1. Exterior Dirichlet problem

**Definition 4.4.** We say \( u \in H^s(\mathbb{R}^n) \) is a weak solution of the exterior Dirichlet problem

\[
\begin{cases}
(\mathcal{L}_A^* + q)u = f & \text{in } \Omega \\
u = g & \text{in } \Omega_e
\end{cases}
\tag{11}
\]

where \( f \in (\tilde{H}^s(\Omega))^* \) and \( g \in H^s(\mathbb{R}^n) \) if \( u \) satisfies \( u - g \in \tilde{H}^s(\Omega) \) and

\[
B_{A,q}(u, \phi) = f(\bar{\phi}), \quad \phi \in \tilde{H}^s(\Omega).
\]

**Proposition 4.5.** Suppose \( q \) is regular for \( \mathcal{L}_A \), then for each \( g \in H^s(\mathbb{R}^n) \), the problem

\[
\begin{cases}
(\mathcal{L}_A^* + q)u = 0 & \text{in } \Omega \\
u = g & \text{in } \Omega_e
\end{cases}
\tag{12}
\]

has a unique solution \( u_g \in H^s(\mathbb{R}^n) \) and the solution operator \( P_{A,q} : g \rightarrow u_g \) is bounded on \( H^s(\mathbb{R}^n) \).

**Proof.** By Lax–Milgram theorem, there exists an invertible bounded linear map \( f \rightarrow w_f \) from \( (\tilde{H}^s(\Omega))^* \) to \( \tilde{H}^s(\Omega) \) s.t. \( w_f \) satisfies

\[
B_{A,q}(w, \phi) = f(\bar{\phi}), \quad \phi \in \tilde{H}^s(\Omega).
\]

For any fixed \( g \in H^s(\mathbb{R}^n) \), let \( f = -(\mathcal{L}_A^* + q)g \), then \( u_g := w_f + g \) is the unique weak solution of (12) and the boundness of \( P_{A,q} \) on \( H^s(\mathbb{R}^n) \) is clear. \( \square \)
4.2. DN map

From now on we always assume \( q \) is regular for \( L_A \).

Let \( X := H'(\mathbb{R}^n)/H'(\Omega) = H'(\Omega_c) \) and \( \tilde{g} := \) the natural image of \( g \in H'(\mathbb{R}^n) \) in \( X \).

**Definition 4.6.** We define the Dirichlet-to-Neumann map \( \Lambda_{A,q} \) by

\[
\langle \Lambda_{A,q} \tilde{g}, \tilde{h} \rangle := B_{A,q}(u_g, \tilde{h}), \quad g, h \in H'(\mathbb{R}^n)
\]

(13)

where \( u_g = P_{A,q}g \).

Note that if \( g_2 - g_1 \in \tilde{H}'(\Omega) \) and \( h_2 - h_1 \in \tilde{H}'(\Omega) \), then \( u_{g_1} = u_{g_2} \) and

\[
B_{A,q}(u_{g_2}, h_2) - B_{A,q}(u_{g_1}, h_1) = B_{A,q}(u_{g_2} - u_{g_1}, h_2) + B_{A,q}(u_{g_1}, h_2 - h_1) = 0
\]

so \( \Lambda_{A,q} \) is well-defined. If \( g, h \) belong to the orthogonal complement of \( \tilde{H}'(\Omega) \) in \( H'(\mathbb{R}^n) \), then

\[
|\langle \Lambda_{A,q} \tilde{g}, \tilde{h} \rangle| \leq C\|u_g\|_{H'}\|h\|_{H'} \leq C'\|g\|_{H'}\|h\|_{H'} = C'\|\tilde{g}\|_X\|\tilde{h}\|_X
\]

so \( \Lambda_{A,q} : X \to X^* \) is bounded.

For convenience, we just write \( \Lambda_{A,q} g \) and \( \langle \Lambda_{A,q} g, h \rangle \) instead of \( \Lambda_{A,q} \tilde{g} \) and \( \langle \Lambda_{A,q} \tilde{g}, \tilde{h} \rangle \).

**Remark.** Roughly speaking, \( \Lambda_{A,q} g = L_A^* u_g|_{\Omega_c} \) since we can formally do the computation

\[
\langle \Lambda_{A,q} g, h \rangle = \int (\mathcal{L}_A^* u_g) h + \int_{\Omega_c} q u_g h
\]

\[
= \left( \int_{\Omega_c} (\mathcal{L}_A^* u_g) h + \int_{\Omega} q u_g h \right) = \int_{\Omega_c} (\mathcal{L}_A^* u_g) h.
\]

The following integral identity will be used in section 5 to prove the main theorem.

**Proposition 4.7.** Suppose \( q_{j} \) are regular for \( L_A^* \) \((j = 1, 2)\). For \( g_1, g_2 \in \tilde{H}'(\mathbb{R}^n) \), let \( u_j^+ := P_{A,q_j}(g_j) \) and \( u_j^- := P_{-A,q_j}(g_j) \), i.e. \( u_j^+ \) is the unique weak solution of

\[
\begin{cases}
(\mathcal{L}_A^* + q_j)u = 0 \quad \text{in } \Omega \\
u = g_j \quad \text{in } \Omega_c
\end{cases}
\]

(14)

and \( u_j^- \) is the unique weak solution of

\[
\begin{cases}
(\mathcal{L}_A^* + q_j)u = 0 \quad \text{in } \Omega \\
u = g_j \quad \text{in } \Omega_c
\end{cases}
\]

(15)

then we have

\[
\langle (\Lambda_{A,q_1} - \Lambda_{A,q_2}) g_1, g_2 \rangle = \int_{\Omega} (q_1 - q_2) u_1^+ u_2^-.
\]

(16)

**Proof.** By (7), (10) and (13), we have

\[
\langle \Lambda_{A,q} g, h \rangle = B_{A,q}(P_{A,q} \tilde{g}, \overline{P_{A,q} h}) = B_{-A,q}(P_{-A,q} h, \overline{P_{A,q} g}) = \langle \Lambda_{-A,q} h, g \rangle.
\]
Thus we have
\[
\langle (A_{\partial_1} - A_{\partial_2})g_1, g_2 \rangle = (A_{\partial_1}g_1, g_2) - (A_{-\partial_2}g_2, g_1)
\]
\[
= B_{\partial_1}(u_1^+, u_2^+) - B_{-\partial_2}(u_2^-, u_1^-)
\]
\[
= (L_+u_1^+, u_2^-) - (L_-u_2^-, u_1^+) + \int_\Omega (q_1 - q_2)u_1^+u_2^-
\]
\[
= \int_\Omega (q_1 - q_2)u_1^+u_2^-.
\]

5. Proof of the main theorem

The proof of theorem 1.1 relies on the Runge approximation property of $L^A_q$, which is based on the following strong uniqueness property.

**Proposition 5.1** (Theorem 1.2 in [7]). Suppose $0 < s < 1$ and $u \in H^s(\mathbb{R}^n)$. If both $u$ and $L^su$ vanish in a nonempty open set $W$, then $u = 0$ in $\mathbb{R}^n$.

The next lemma is the bridge from the strong uniqueness property of $L^s$ to the Runge approximation property of $L^A_q$.

**Lemma 5.2.** Suppose $\Omega \cup \text{supp } A \subset B_r(0)$ for some $r > 0$, $W$ is a nonempty open set s.t. $W \cap B_{3r}(0) = \emptyset$, then we have
\[
L^su|_W = L^A_qu|_W, \quad u \in \tilde{H}^s(\Omega).
\]

**Proof.** Let $u \in C_0^\infty(\Omega)$ and $v \in C_0^\infty(W)$. By (3), (4) and (6), we have
\[
\langle (L^s - L^A_q)u, v \rangle = \int \int [(u(x) - u(y))(v(x) - v(y)) - (u(x) - E_A(x, y)u(y))(v(x)
\]
\[
- E_A(x, y)v(y))]K(x, y) \, dx \, dy
\]
\[
= \int \int (E_A(x, y) - 1)u(x)v(x)K(x, y) \, dx \, dy + \int \int (E_A(x, y) - 1)u(x)v(y)K(x, y) \, dx \, dy
\]
\[
= \int \int (E_A(x, y) - 1)u(x)v(x)K(x, y) \, dx \, dy + \int \int (E_A(x, y) - 1)u(y)v(x)K(y, x) \, dy \, dx
\]
\[
= 2 \int \int (E_A(x, y) - 1)u(x)v(x)K(x, y) \, dx \, dy. \quad (17)
\]

Note that if $x \notin W$, then $v(x) = 0$; if $y \notin \Omega$, then $u(y) = 0$; if $x \in W$ and $y \in \Omega$, then
\[
\frac{|x + y|}{2} \geq \frac{|x| - |y|}{2} \geq \frac{3r - r}{2} = r,
\]
which implies $E_A(x, y) = 1$ in this case. Hence the integrand in (17) is always zero.

**Corollary 5.3.** Suppose $\Omega \cup \text{supp } A \subset B_r(0)$ for some $r > 0$, $W$ is an open set s.t. $W \cap B_{3r}(0) \neq \emptyset$. If
\[
u \in \tilde{H}^s(\Omega), \quad L^A_qu|_W = 0
\]
then $u = 0$ in $\mathbb{R}^n$. 

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**Proof.** By lemma 5.2, \( u = L^t u = 0 \) in \( W \backslash B_{2r}(0) \) so \( u = 0 \) in \( \mathbb{R}^n \) by proposition 5.1. □

Now we can prove the Runge approximation property of \( L^t_A \).

**Proposition 5.4.** Suppose \( \Omega \cup \text{supp } A \subset B_r(0) \) for some \( r > 0 \), \( W \) is an open set s.t. \( W \subset \Omega_r \) and \( W \backslash B_{3r}(0) \neq \emptyset \), then

\[
S := \{ P_{A,q} f | \Omega : f \in C^\infty_c(W) \}
\]

is dense in \( L^2(\Omega) \) where \( P_{A,q} \) is the solution operator defined in subsection 4.1.

**Proof.** By the Hahn–Banach theorem, it suffices to show that:

If \( v \in L^2(\Omega) \) and \( \int_{\Omega} v w = 0 \) for all \( w \in S \), then \( v = 0 \) in \( \Omega \).

In fact, for any given \( v \in L^2(\Omega) \), let \( \phi \) be the unique weak solution of

\[
\begin{cases}
(L^t_A + q)\phi = v & \text{in } \Omega \\
\phi = 0 & \text{in } \Omega_r.
\end{cases}
\]

(18)

then for any \( f \in C^\infty_c(W) \), we have

\[
\int_{\Omega} v P_{A,q} f = \langle v, P_{A,q} f - f \rangle = \langle (L^t_A + q)\phi, P_{A,q} f - f \rangle = \langle (L^t_A + q)(P_{A,q} f - f), \phi \rangle
\]

since \( P_{A,q} f - f \in \bar{H}^t(\Omega) \). Also note that

\[
\langle (L^t_A + q)P_{A,q} f, \phi \rangle = 0
\]

since \( P_{A,q} f \) is the solution operator and \( \phi \in \bar{H}^t(\Omega) \), so we have

\[
\int_{\Omega} v P_{A,q} f = -\langle (L^t_A + q)f, \phi \rangle = -\langle L^t_A f, \phi \rangle = -\langle L^t_{-A}\phi, f \rangle.
\]

Hence, if \( v \in L^2(\Omega) \) and \( \int_{\Omega} v w = 0 \) for all \( w \in S \), then the corresponding \( \phi \) satisfies

\[
\phi \in \bar{H}^t(\Omega), \quad L^t_{-A}\phi|_W = 0.
\]

This implies \( \phi = 0 \) in \( \mathbb{R}^n \) by corollary 5.3 and thus \( v = 0 \) in \( \Omega \). □

Now we are ready to prove theorem 1.1.

**Proof.** For any fixed \( \epsilon > 0 \) and \( f \in L^2(\Omega) \), by proposition 5.4 we can choose \( u_1^+ = P_{A,q_1}(g_1) \) for some \( g_1 \in C^\infty_c(W_1) \) s.t.

\[
\|u_1^+ - f\|_{L^2(\Omega)} \leq \epsilon
\]

and for this chosen \( u_1^+ \), we can choose \( u_2^- = P_{-A,q_2}(g_2) \) for some \( g_2 \in C^\infty_c(W_2) \) s.t.

\[
\|u_1^+\|_{L^2(\Omega)}\|u_2^- - 1\|_{L^1(\Omega)} \leq \epsilon.
\]

Now by (2) and (16), we have

\[
\int_{\Omega} (q_1 - q_2)u_1^+ u_2^- = 0
\]
so
\[
\left| \int_\Omega (q_1 - q_2)f \right| = \left| \int_\Omega (q_1 - q_2)(f - u_1^+ + (1 - u_2^-)) \right| \leq C\epsilon.
\]

Let \( \epsilon \to 0^+ \), then we have
\[
\int_\Omega (q_1 - q_2)f = 0.
\]

Since \( f \in L^2(\Omega) \) is arbitrary, then we can conclude that \( q_1 = q_2 \). \( \square \)

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