AN UPPER BOUND FOR THE FIRST HILBERT COEFFICIENT OF
GORENSTEIN ALGEBRAS AND MODULES

SABINE EL KHOURY, MANOJ KUMMINI, AND HEMA SRINIVASAN

Abstract. Let \( R \) be a polynomial ring over a field and \( M = \bigoplus_n M_n \) a finitely generated graded \( R \)-module, minimally generated by homogeneous elements of degree zero with a graded \( R \)-minimal free resolution \( F \). A Cohen-Macaulay module \( M \) is Gorenstein when the graded resolution is symmetric. We give an upper bound for the first Hilbert coefficient, \( e_1 \), in terms of the shifts in the graded resolution of \( M \). When \( M = R/I \), a Gorenstein algebra, this bound agrees with the bound obtained in [ES09] in Gorenstein algebras with quasi-pure resolution. We conjecture a similar bound for the higher coefficients.

1. Introduction

Let \( k \) be any field and \( R = \bigoplus R_i \) be a polynomial ring over \( k \), finitely generated in degree one. Let \( M = \bigoplus_i M_i \) be a finitely generated graded \( R \)-module. We write \( H(M, i) := \dim_k M_i \) for the Hilbert function of \( M \). For \( i \gg 0 \), \( H(M, i) = P_M(i) \) where \( P_M(x) \) is a polynomial called the Hilbert polynomial of \( M \). The degree of \( P_M(x) \) is \( \dim M - 1 \). We can write

\[
P_M(x) = \sum_{i=0}^{d-1} (-1)^i e_i \left( \frac{x + d - 1 - i}{x} \right) = \frac{e_0}{(d-1)!} x^{d-1} + \ldots + (-1)^{d-1} e_{d-1}
\]

where the coefficients \( e_i \) are non-negative integers (depending on \( M \)), called the Hilbert coefficients of \( M \). The first one, \( e_0 \), also denoted by \( e \), is the multiplicity of \( M \).

We now introduce the notion of a Gorenstein \( R \)-module, generalizing the notion of a Gorenstein quotient ring of \( R \). Assume that \( M \) is generated minimally by elements of degree 0. Let

\[
F = F(M) : 0 \to \bigoplus_{j=t_s}^{T_s} R(-j)^{\beta_{s,j}} \xrightarrow{\delta_s} \ldots \to \bigoplus_{j=t_i}^{T_i} R(-j)^{\beta_{i,j}} \xrightarrow{\delta_i} \ldots \to \bigoplus_{j=t_1}^{T_1} R(-j)^{\beta_{1,j}} \xrightarrow{\delta_1} R^{\beta_0} \to M
\]

be a minimal graded free resolution of \( M \) as an \( R \)-module. We assume that \( \beta_{i,j,t_i} \neq 0 \) and \( \beta_{i,T_i} \neq 0 \) for every \( i \). The codimension of \( M \), denoted codim \( M \), is the height of the annihilator of \( M \). Note that \( s \geq \text{codim} \ M \), with equality if and only if \( M \) is Cohen-Macaulay. We say that \( M \) is Gorenstein if it is Cohen-Macaulay and if \( F \) is self-dual, i.e., \( \text{Hom}_R(F, R) \simeq R \) after appropriate shifts.

The numbers \( \beta_{ij} \) appearing in \( F \) are called the graded Betti numbers of \( M \); note that \( \beta_{i,j} \) is the number of copies of \( R(-j) \) that appear at homological degree \( i \) in any minimal graded free resolution of \( M \), and that \( \beta_{i,j} = \dim_k \text{Tor}_i^R(k, M)_j \). If \( M \) is Gorenstein, then \( t_s = T_s \) and \( \beta_{ij} = \beta_{s-i,N-j} \) for \( N = T_s \).

There has been a lot of work on bounding the multiplicity and the Hilbert coefficients of finitely generated modules, in terms of the minimal and maximal shifts of their minimal free
resolutions. C. Huneke and Srinivasan (see [[HS98], Conjecture 1]) and of J. Herzog and Srinivasan (see [[HS98], Conjecture 2]) proposed upper and lower bounds for the multiplicity in terms of $t_i$ and $T_i$. These conjectures were proven by using the Boij and Söderberg in full. In [HZ09], Herzog and Zheng extend the result on the multiplicity to all coefficients in the Cohen Macaulay case, and found upper and lower bounds for the $e_i$ in the sense of [HS98]. In the Gorenstein case, when the resolution is quasi-pure, i.e., $t_i \geq T_i - 1$ for all $i$, the duality of the resolution is used to find sharper bounds for the multiplicity and all Hilbert coefficients, see [Sri98] and [ElS12] respectively. When the resolution is not quasi-pure, the duality of the resolution can be picked up in the Betti table from the Boij and Söderberg decomposition. This gives a generalization for the upper bound of the multiplicity to all Gorenstein algebras, see [EKS] for instance. In this paper, we extend the upper bound found in [EKS] to the first coefficient $e_1$, and prove the following theorem:

**Theorem 1.1.** Let $M$ be a finitely generated graded Gorenstein $R$-module, minimally generated by homogeneous elements of degree zero. Let $s = \text{codim } M$ and $k = \lfloor \frac{s}{2} \rfloor$. Let $\beta_0(M)$ denote the minimal number of generators of $M$ and $F(M)$ be the graded resolution of $M$ as above. Set

$$\tilde{t}_i = \begin{cases} 
\min \{T_i, \left\lfloor \frac{t_i}{s^2} \right\rfloor \}, & i = 1, \ldots, k; \\
\max \{t_i, \left\lceil \frac{t_i}{s^2} \right\rceil \}, & i = k + 1, \ldots, s.
\end{cases}$$

Then

$$e_1(M) \leq \frac{\beta_0(M)}{(s + 1)!} \prod_{i=1}^{s} \tilde{t}_i \sum_{i=1}^{s} (\tilde{t}_i - i).$$

When the resolution is quasi-pure, $t_i \geq T_i - 1$ and this coincides with the bounds in [ES09]. Other authors gave also bounds for the first Hilbert coefficient $e_1$, see [HH], [RV10], [RV05], and [E05]. We compare our bounds to that of Rossi-Valla in [RV10] [RV05], and Elias in [E05], then conjecture that our result can be extended to all coefficients.

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2. **Notation**

Let $\mathbb{k}$ be a field and $R = \mathbb{k}[x_1, \ldots, x_n]$ be dimensional polynomial ring over $\mathbb{k}$ with deg $x_i = 1$ for all $1 \leq i \leq n$. Let $M$ be a finitely generated graded $R$-module. Suppose $M$ is Gorenstein then the duality of the resolution can be recorded as follow

If codim $M = 2k + 1$ then

$$0 \to R(-d_s) \to \bigoplus_{j=1}^{\beta_1} R(-(d_s - d_{1j})) \to \ldots \to \bigoplus_{j=1}^{\beta_k} R(-(d_s - d_{kj})) \to \bigoplus_{j=1}^{\beta_1} R(-d_{kj}) \to \ldots \to \bigoplus_{j=1}^{\beta_1} R(-d_{1j}) \to R^{\beta_0}$$
and if \( \text{codim } M = 2k \) then
\[
0 \to R(-d_s) \to \bigoplus_{j=1}^{\beta_1} R(-(d_s - d_{1j}) \to \ldots \to \bigoplus_{j=1}^{\beta_2/2=r_k} R(-(d_s - d_{kj})) \oplus \bigoplus_{j=1}^{\beta_3/2=r_k} R(-d_{kj}) \to \ldots \to \bigoplus_{j=1}^{\beta_1} R(-d_{1j}) \to R^{3k}
\]

The minimal shifts in the resolution are:
\[
t_i = \begin{cases} 
\min_j d_{ij} & 1 \leq i \leq k \\
d_s - \max_j d_{s-i,j} & k + 1 \leq i < s \\
d_s & i = s
\end{cases}
\]
and the maximal shifts in the resolution are:
\[
T_i = \begin{cases} 
\max_j d_{ij} & 1 \leq i \leq k \\
d_s - \min_j d_{s-i,j} & k + 1 \leq i < s \\
d_s & i = s
\end{cases}
\]

The graded Betti numbers of \( M \) denoted by \( \beta_{i,j}(M) \) are the number of copies of \( R(-d_{ij}) \) that appear at homological degree \( i \), in a minimal \( R \)-free resolution of \( M \). We have
\[
\sum_j \beta_{i,j} = \begin{cases} 
\beta_i & i \leq k \\
\beta_{s-i} & k + 1 \leq i < s \\
1 & i = s
\end{cases}
\]
We think of the collection \( \{\beta_{i,j}(M) : 0 \leq i \leq n, j \in \mathbb{Z}\} \) as an element
\[
\beta(M) = (\beta_{i,j}(M))_{0 \leq i \leq n, j \in \mathbb{Z}} \in \mathbb{B} := \bigoplus_{i=0}^{n} \bigoplus_{j \in \mathbb{Z}} \mathbb{Q},
\]
and call it the Betti table of \( M \). In general, a rational Betti table \( \beta \) is an element \( \beta = (\beta_{i,j})_{0 \leq i \leq n, j \in \mathbb{Z}} \in \mathbb{B} \) such that:
(i) for all \( 0 \leq i \leq n \), \( \beta_{i,j} = 0 \) for finitely many \( j \),
(ii) for all \( i > 0 \) and for all \( j \), if \( \beta_{i,j} \neq 0 \) then there exists \( j' < j \) such that \( \beta_{i-1,j'} \neq 0 \).

Let \( \beta = (\beta_{i,j})_{0 \leq i \leq n, j \in \mathbb{Z}} \) be a rational Betti table. Its length is \( \max\{i : \beta_{i,j} \neq 0 \text{ for some } j\} \).

**Vandermonde matrices.** Given \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \) a sequence of real numbers, we denote the following Vandermonde determinants by
\[
V_t = V_t(\alpha_1, \alpha_2, \ldots, \alpha_k) = \begin{vmatrix} 
1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_k \\
\alpha_1^2 & \alpha_2^2 & \ldots & \alpha_k^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{k-2} & \alpha_2^{k-2} & \ldots & \alpha_k^{k-2} \\
\alpha_1^{k-1+t} & \alpha_2^{k-1+t} & \ldots & \alpha_k^{k-1+t} 
\end{vmatrix}
= \prod_{1 \leq j < i \leq k} (\alpha_i - \alpha_j) \sum_{\gamma_1 + \gamma_2 + \ldots + \gamma_k = t} (\alpha_1^{\gamma_1} \cdot \alpha_2^{\gamma_2} \ldots \alpha_k^{\gamma_k})
\]
We denote \( V_0 \) by \( V \).
Remark 2.1. \(V_t(\alpha_1, \alpha_2, \ldots, \alpha_k) \geq 0\) if the sequence is in ascending order.

A degree sequence of length \(s\) is an increasing sequence \(d = (d_0 < d_1 < \cdots < d_s)\) of integers. An \(R\)-module \(M\) is said to have a pure resolution of type \(d\) if its Betti table \(\beta_{ij} \neq 0\) if and only if \(j = d_i, 1 \leq i \leq s\). By the Herzog-Kühl equations [HK84], \(\beta(M)\) is a positive rational multiple of the pure Betti table, which we denote by \(\beta(d)\), given by:

\[
(2.2) \quad \beta(d)_{i,j} = \begin{cases} 
\frac{1}{\prod_{l \neq i} |d_l - d_i|}, & 0 \leq i \leq s \text{ and } j = d_i \\
0, & \text{otherwise.}
\end{cases}
\]

We call the Betti table \(\beta(d)\) defined in (2.2), the pure Betti table associated to \(d\). For \(0 \leq i \leq s\), write \(\beta_i(d) = \beta(d)_{i,d_i}\).

Eisenbud, Floystad and Weyman [EFW11, Theorem 0.1] (in characteristic zero) and Eisenbud and Schreyer [ES09, Theorem 0.1] showed that for all degree sequences \(d\), there is a Cohen-Macaulay \(R\)-module \(M\) such that \(\beta(M)\) is a rational multiple of \(\beta(d)\). Moreover, for all \(R\)-modules \(M\), \(\beta(M)\) can be written as a non-negative rational combination of the \(\beta(d)\) [BS08]; if we take a saturated chain of degree sequences, (in the set of all degree sequences, a saturated chain with respect to the partially order given by by point-wise comparison) then the non-negative rational coefficients in the decomposition are unique [ES09, Theorem 0.2].

2.1. Self-dual resolutions and symmetrized Betti tables. Let \(\beta\) be a Betti table. Let \(s\) and \(N\) be integers. We say that \(\beta\) is \((s, N)\)-self-dual if \(\beta_{i,j} = \beta_{s-i,N-j}\) for all \(i, j\). We say that \(\beta\) is self-dual if there exist \(s\) and \(N\) such that \(\beta\) is \((s, N)\)-self-dual. If \(\beta\) is self-dual, then \(s\) is the length of \(\beta\) and \(N = \max\{j : \beta_{s,j} \neq 0\} + \min\{j : \beta_{0,j} \neq 0\}\).

Definition 2.3. Let \(d = (d_0 < \cdots < d_s)\) be a degree sequence and \(N \geq d_0 + d_s\). Let \(d^{\vee,N} = (N-d_s < \cdots < N-d_0)\). Denote the pure Betti table associated to \(d^{\vee,N}\) by \(\beta^{\vee,N}(d)\). Similarly, set \(\beta^{\vee,N}_{i,j}(d) = \beta^{\vee,N}(d)_{i,N-d_{s-i}}\). Let \(\beta_{\text{sym}}(d, N) = \beta(d) + \beta^{\vee,N}(d)\). We call \(\beta_{\text{sym}}(d, N)\) the symmetrized pure Betti table, given by symmetrizing \(d\) with respect to \(N\).

Peskine and Szpiro [PS74] showed that for a finitely generated graded \(R\)-module \(M\) with \(s = \text{codim } M\),

\[
\sum_{i=0}^{\text{pd } M} (-1)^i \sum_j d_{ij}^{t} = \begin{cases} 
0 & \text{if } 0 \leq t < s \\
(-1)^s(s!)e(M) & \text{if } t = s.
\end{cases}
\]

Moreover a similar expression to all coefficients was given in [E09, Lemma 3.2] and [ELS12, Theorem 3.2], for every finitely generated graded \(R\)-module \(M\) of codimension \(s\). We have,

\[
\sum_{r=0}^{t} (-1)^{t-r} \nu_{t-r} \sum_{i=0}^{\text{pd } M} (-1)^i \sum_{j=1}^{b_i} d_{ij}^{t+r} = (-1)^s(s+l)!e_l(M) \text{ if } t = s.
\]

with \(\nu_{l-r} = \sum_{1 \leq \xi_1 < \xi_2 < \cdots < \xi_{l-r} \leq s+l-1} \xi_1 \xi_2 \cdots \xi_{l-r} \text{ and } \nu_0 = 1\).

Let \(d\) be a degree sequence of length \(s\). Since the pure Betti table \(\beta(d)\) is, up to multiplication by a rational number, the Betti table of a Cohen-Macaulay \(R\)-module of codimension
s, we see that \( \sum_{i=0}^{s} (-1)^i \beta_i(d) d_i^l = 0 \) for all \( 0 \leq l < s \). Further, by a direct computation, we can see that \( \sum_{i=0}^{s} (-1)^i \beta_i(d) d_i^{s+r} = (-1)^s \frac{V_r(d_1,...,d_s)}{V(d_1,...,d_s)} = (-1)^s \sum_{\alpha_1+...+\alpha_s=r} d_1^{\alpha_1}d_2^{\alpha_2}...d_s^{\alpha_s} \).

**Definition 2.4.** Let \( d = (d_1, \ldots, d_s) \) be a degree sequence and \( \beta(d) \) and \( \beta(d'^N) \) be as in Definition 2.3. Then we define, \( e_l \) as follows:

\[
(s+l)!e_l(\beta(d)) = \sum_{r=0}^{l} (-1)^l \nu_{l-r} \sum_{\alpha_1+...+\alpha_s=r} d_1^{\alpha_1}d_2^{\alpha_2}...d_s^{\alpha_s}
\]

and

\[
(s+l)!e_l(\beta(d'^N)) = \sum_{r=0}^{l} (-1)^l \nu_{l-r} \sum_{\alpha_1+...+\alpha_s=r} (N-d_{s-1})^{\alpha_1}(N-d_{s-2})^{\alpha_2}...(N-d_0)^{\alpha_s}
\]

Note that the above sums can be factored as follows.

**Lemma 2.5.** For any sequence of integers \( y_1 < y_2 < \ldots < y_s \)

\[
\sum_{r=0}^{l} (-1)^l \nu_{l-r} \sum_{\alpha_1+...+\alpha_s=r} y_1^{\alpha_1}y_2^{\alpha_2}...y_s^{\alpha_s} = \sum_{1 \leq i_1 \leq \ldots \leq i_t \leq s} \prod_{t=1}^{l} (y_{i_t} - (i_t + t - 1))
\]

where \( \nu_{l-r} = \prod_{1 \leq \beta_1 < \ldots < \beta_{l-r} \leq s+l-1} \beta_1 \cdots \beta_{l-r} \).

This has been proved in [ELS12, Lemma 4.8] and [E09, Theorem 4.2]. We repeat it for the sake of completeness.

**Proof.** For any given tuples \( 1 \leq \gamma_1 \leq \cdots \leq \gamma_{t} \leq s \) with \( 0 \leq r \leq l \), we have \( 1 \leq \beta_1 \leq \cdots \leq \beta_{l-r} \leq s \) such that \( \{ \gamma_1, \ldots, \gamma_{r} \} \cup \{ \beta_1, \cdots, \beta_{l-r} \} = \{ i_1, \cdots, i_t \} \). In the product \( \sum_{r=0}^{l} \prod_{t=1}^{l} (y_{i_t} - (i_t + t - 1)) \), the coefficient of \( \prod_{1 \leq \gamma_1 \leq \ldots \leq \gamma_{s \leq \gamma_s \leq \gamma_{s+t-1}} \beta_1 \cdots \beta_{l-r} = \nu_{l-r} \) since \( i_t + t - 1 \) is strictly increasing until \( i_t + l - 1 \) is strictly increasing until \( i_t + l - 1 \).

Since \( \beta_{\text{sym}}(d,N) = \beta(d) + \beta(d'^N) \), we see that

\[
(s+l)!e_l(\beta_{\text{sym}}(d,N)) = \sum_{1 \leq i_1 \leq \ldots \leq i_t \leq s} \prod_{t=1}^{l} (d_{i_t} - (i_t + t - 1)) + \sum_{1 \leq i_1 \leq \ldots \leq i_t \leq s} \prod_{t=1}^{l} (N-d_{s-i_t} - (i_t + t - 1))
\]

The Betti table of a Gorenstein module can be decomposed into a non-negative rational combination of symmetrized pure Betti tables, by [EKS]

**Proposition 2.7.** [EKS]/[Proposition 2.4] Let \( M \) be finitely generated graded Cohen-Macaulay \( R \)-module with codim \( M = s \), generated minimally by homogeneous elements of degree zero.
Suppose that $\beta(M)$ is self-dual. Let $N = T_s = t_s$. Then there exist degree sequences $d^\alpha, 0 \leq \alpha \leq a$ for some $a \in \mathbb{N}$ and positive rational numbers $r_\alpha, 0 \leq \alpha \leq a$ such that
\[
\beta(M) = \sum_{\alpha=0}^{a} r_\alpha \beta_{\mathrm{sym}}(d^\alpha, N).
\]
Moreover,
(i) the $d^\alpha$ are degree sequences of length $s$ and they are not $(s, N)$-dual to each other.
(ii) $d^{\alpha+1} > d^\alpha$ for all $0 \leq \alpha \leq a - 1$.
(iii) $N \geq d^\alpha + d^{a-\alpha}$ for all $\alpha$ and $i$, or equivalently, $d^\alpha \leq (d^{\alpha})^{v,N}$ for all $\alpha$.

As a consequence, there is a formula for all the Hilbert coefficients, $e_\ell$. Let $M, N, d^\alpha$ be as in the theorem above.
\[
e_\ell(M) = \sum_{\alpha=0}^{a} r_\alpha \beta_{\mathrm{sym}}(d^\alpha, N).
\]

Using this, one can get a stronger upper bound for the multiplicity $e_0$ of Gorenstein modules \cite{EKS}\[Theorem 3.1\]. That is,
\[
e(M) \leq \frac{\beta_0(M)}{s!} \prod_{i=1}^{k} \min \left\{ T_i, \left\lceil \frac{t_s}{2} \right\rceil \right\} \prod_{i=k+1}^{s} \max \left\{ t_i, \left\lceil \frac{t_s}{2} \right\rceil \right\}.
\]

Again, when the resolution is quasi-pure this generalizes the bound in \cite{EKS}. This paper is an attempt to generalize this to higher Hilbert coefficients. We prove an analogous bound for the first coefficient, $e_1$ and conjecture a bound for the higher coefficients $e_i, i \geq 2$.

### 3. Upper bound for $e_1$

**Notation 3.1.** We define some notation used in the statements and proofs of our results. Let $d = (0, d_1, \ldots, d_s)$ be a non-decreasing sequence of integers.

(i) For a sequence $d' = (0, d'_1, \ldots, d'_s)$ of integers, we write $d < d'$ if $d \neq d'$ and $d_i \leq d'_i$ for every $1 \leq i \leq s$.

(ii) $\bar{d} = (\bar{d}_1, \ldots, \bar{d}_s)$ be the (non-decreasing) sequence
\[
\bar{d}_i = \begin{cases} 
\min \{d_s - d_{s-i}, \left\lfloor \frac{d_s}{2} \right\rfloor \}, & i = 1, \ldots, k; \\
\max \{d_i, \left\lceil \frac{d_s}{2} \right\rceil \}, & i = k+1, \ldots s.
\end{cases}
\]

(iii) Suppose that $d_s \geq d_i + d_{s-i}$ for every $0 \leq i \leq s$. Write
(a) $b_d = \beta_0(d) + \beta_0(d^{v,d_s}).$
(b) $\Psi_d = \prod_{i=1}^{s} d_i.$
(iv) For $1 \leq l \leq s$, define
\[
f_l(\bar{d}) = f_l(\bar{d}_1, \ldots, \bar{d}_s) = \sum_{1 \leq i_1 \leq \cdots \leq i_l \leq s} \prod_{t=1}^{l} (\bar{d}_{i_t} - (i_t + t - 1)).
\]
Set $f_0(\bar{d}_1, \ldots, \bar{d}_s) = 1.$
This section is devoted to the proof of Theorem 3.2, which is about finding an upper bound for the first Hilbert coefficient of Gorenstein algebras.

**Theorem 3.2.** Let $M$ be a finitely generated graded Gorenstein $R$-module, minimally generated by homogeneous elements of degree zero. Let $s = \text{codim} M$ and $k = \lfloor \frac{s}{2} \rfloor$. Let $\beta_0(M)$ denote the minimal number of generators of $M$. For $0 \leq i \leq s$, write $t_i = t_i(M) = \min\{j : \text{Tor}_i^R(k,M)_j \neq 0\}$ and $m = (t_i)_{i=0,...,s}$. Then

$$e_1(M) \leq \frac{\beta_0(M)}{(s+1)!} \Psi_m f_1(\tilde{t}).$$

This upper bound coincides with the upper bound of Gorenstein algebras with quasi-pure resolutions found in [EIS12]. In order to prove this theorem, we find the upper bound for $e_1(\beta_{\text{sym}}(d, N))$ of a symmetrized pure Betti table then use the Boij and Söderberg decomposition in order to generalize our result to $e_1(M)$. To proceed, we first need the following lemma,

**Lemma 3.3.** Let $d$ and $d'$ be degree sequences such that $d_0 = 0$ and $d < d' \leq (d')^{\vee,ds} < d^{\vee,ds}$. Then $\Psi_d f_1(d) \geq \Psi_d f_1(d')$.

**Proof.** By induction on $\sum_i d'_i - d_i$, we may assume, without loss of generality, that there exists $j$ such that $d'_j = d_j + 1$ and $d'_i = d_i$ for all $i \neq j$. Moreover, if $1 \leq i \leq s - k - 1$, then $d_i$ does not figure in the expression for $\Psi_d f_1(d)$, so we may assume that $j \geq s - k$. Additionally, $j \leq s - 1$. We rewrite $\Psi_d$ as

$$\Psi_d = \prod_{i=s-k}^{s-1} \min \left\{ d_s - d_i, \left\lfloor \frac{d_s}{2} \right\rfloor \right\} \prod_{i=k+1}^{s} \max \left\{ d_i, \left\lfloor \frac{d_s}{2} \right\rfloor \right\}.$$

We note that

$$f_1(d) = \sum_{1 \leq i_1 \leq s} (d_{i_1} - 1) = \sum_{1 \leq i \leq s} d_i - \left( \begin{array}{c} s+1 \\ 2 \end{array} \right).$$

Two cases arise: $j < k + 1$ and $j \geq k + 1$. The first case is possible if and only if $s = 2k$ and $j = k$. In this case, $d_k$ appears only once in (3.4), and since $d_k \leq d_s - d_{s-k} = d_s - d_k$, we get $d_s - d_k \geq \left\lfloor \frac{d_s}{2} \right\rfloor$. By the hypothesis that $d' \leq (d')^{\vee,ds}$, $d_s - d_k - 1 \geq d_k + 1$, so $d_s - d_k - 1 \geq \left\lfloor \frac{d_s}{2} \right\rfloor$. We get $\min \{d_s - d_k, \left\lfloor \frac{d_s}{2} \right\rfloor \} = \min \{d_s - d_k - 1, \left\lfloor \frac{d_s}{2} \right\rfloor \} = \left\lfloor \frac{d_s}{2} \right\rfloor$. In the expression $f_1(d')$, only $d_s - d_k - 1$ is involved, so we get $f_1(d') \Psi_d = f_1(d)(\Psi_d)$.

In the second case (i.e., $j \geq k + 1$), $d_j$ appears twice in in (3.4). We need to show that $\Psi_d f_1(d') \leq \Psi_d f_1(d)$.

If $d_j < \left\lfloor \frac{d_s}{2} \right\rfloor$, then $d_s - d_j > \left\lfloor \frac{d_s}{2} \right\rfloor$. So we get that $d_j + 1 \leq \left\lceil \frac{d_s}{2} \right\rceil$ and $d_s - d_j - 1 \geq \left\lceil \frac{d_s}{2} \right\rceil$. Hence, $\min \{d_s - d_j, \left\lfloor \frac{d_s}{2} \right\rfloor \} = \min \{d_s - d_j - 1, \left\lceil \frac{d_s}{2} \right\rceil \} = \left\lfloor \frac{d_s}{2} \right\rfloor$ and $\max \{d_j, \left\lfloor \frac{d_s}{2} \right\rfloor \} = \max \{d_j, \left\lfloor \frac{d_s}{2} \right\rfloor \} = \left\lfloor \frac{d_s}{2} \right\rfloor$. As a result we obtain $f_1(d') \Psi_d = f_1(d)(\Psi_d)$.
Let us now consider the case where \( d_j \geq \left\lceil \frac{d_j}{2} \right\rceil \) so \( d_j + 1 \geq \left\lceil \frac{d_j}{2} \right\rceil \). This implies that \( d_s - d_j - 1 < d_s - d_j \leq \left\lceil \frac{d_j}{2} \right\rceil \). It suffices to show that

\[
\frac{\Psi_d f_1(\tilde{d})}{\Psi_{\tilde{d}} f_1(d)} = \frac{(d_s - d_j - 1)(d_j + 1)f_1(\tilde{d})}{(d_s - d_j)d_j f_1(d)} \leq 1.
\]

We compute the difference between the numerator and the denominator. We write

\[
(d_s - d_j - 1)(d_j + 1)f_1(\tilde{d}) - (d_s - d_j)d_j f_1(d) = -(2d_j - d_s + 1)f_1(\tilde{d}) + (d_s - d_j - 1)(d_j + 1)(f_1(\tilde{d}) - f_1(\tilde{d}))
\]

where

\[
f_1(\tilde{d}) = \sum_{1 \leq i \leq s} \tilde{d}_i - \left(\frac{s + 1}{2}\right) = s \left\lfloor \frac{d_s}{2} \right\rfloor - \left(\frac{s + 1}{2}\right)
\]

and

\[
f_1(\tilde{d}') = \sum_{1 \leq i \leq s} \tilde{d}_i' - \left(\frac{s + 1}{2}\right) = s \left\lfloor \frac{d_s}{2} \right\rfloor - 2 - \left(\frac{s + 1}{2}\right)
\]

So Equation 3.5 is equal to:

\[-(2d_j - d_s + 1)(s \left\lfloor \frac{d_j}{2} \right\rfloor - \left(\frac{s + 1}{2}\right)) + (d_s - d_j - 1)(d_j + 1)(-2)\]

We know that \( d_j \geq \left\lceil \frac{d_j}{2} \right\rceil \) and \( s \left\lfloor \frac{d_j}{2} \right\rfloor - \frac{s(s+1)}{2} = s(\left\lfloor \frac{d_j}{2} \right\rfloor - \frac{s+1}{2}) \geq 0 \) since \( d_s \geq s + 2 \). Hence \( (d_s - d_j - 1)(d_j + 1)f_1(\tilde{d}') - (d_s - d_j)d_j f_1(d) \leq 0 \), and the proof is done. \( \blacksquare \)

The next proposition is needed for the proof of Theorem 3.2. Let \( d = (0, d_1, \ldots, d_s) \) be a degree sequence such that \( d_s \geq d_i + d_{s-i} \) for all \( 0 \leq i \leq s \). We show that the first coefficient \( e_1(\beta_{\text{sym}}(d, d_s)) \) of symmetrized pure sequences satisfy the upper bound in our main theorem. When \( l = 1 \), we write from equation 2.6:

\[
(s + l)!e_1(\beta_{\text{sym}}(d, d_s)) = \sum_{1 \leq i_1 \leq s} (d_{i_1} - i_1) + \sum_{1 \leq i_1 \leq s} (d_s - d_{s-i_1} - i_1)
\]

\[
= f_1(d) + f_1(d^\vee,d_s)
\]

We note that

\[
f_1(d) = \sum_{1 \leq i \leq s} d_i - \left(\frac{s + 1}{2}\right);
\]

\[
f_1(d^\vee,d_s) = s d_s - \sum_{1 \leq i \leq s} d_i - \left(\frac{s + 1}{2}\right).
\]

**Proposition 3.7.** Let \( d = (0, d_1, \ldots, d_s) \) be a degree sequence such that \( d \leq d^\vee,d_s \). Then

\[
f_1(d) + f_1(d^\vee,d_s) \leq b_d \Psi_{\tilde{d}} f_1(\tilde{d}).
\]
Proof. We again prove this by induction on \( \sum_i ((d^{\nu,d_s})_i - d_i) = \sum_i (d_s - d_{s-i} - d_i) \), which is non-negative by our hypothesis. If \( \sum_i (d_s - d_{s-i} - d_i) = 0 \) (equivalently, \( d = d^{\nu,d_s} \)), then \( \tilde{d} = d \), so the assertion follows from noting that \( b_d \Psi_d \geq 2 \) [EKS, Proposition 3.5].

If \( d < d^{\nu,d_s} \), then there exists \( j \geq k \) such that \( d_j < d_s - d_{s-j} \). Pick \( j \) to be maximal with this property. Note that \( j < s \), so \( d'_j = d_s \). Since \( d_j < d_s - d_{s-j} \), \( d_{j+1} = d_s - d_{s-j-1} \) and \( d_{s-j} > d_{s-j-1} \), we see that \( d' := (0, d_1, \ldots, d_{j-1}, d_j + 1, d_{j+1}, \ldots, d_s) \) is a degree sequence, that \( d' \leq (d')^{\nu,d_s} \) and that \( \sum_i (d_s - d_{s-i} - d_i) > \sum_i (d_s - d'_{s-i} - d'_i) \). Hence by induction

\[
\frac{1}{f_1(d')} + \frac{1}{f_1((d')^{\nu,d_s})} \leq b_{d'} \Psi_{d'} f_1(\tilde{d}).
\]

Therefore \( f_1(d') + f_1((d')^{\nu,d_s}) = f_1(d) + f_1(d^{\nu,d_s}) \). We now show that \( f_1(\tilde{d'}) = f_1(\tilde{d}) \). To this end, we consider various cases:

(i) \( j < s - k \): Then \( \tilde{d'} = \tilde{d} \), so \( f_1(\tilde{d'}) = f_1(\tilde{d}) \).

(ii) \( j = s - k \): We consider two sub-cases.

(a) \( s = 2k \): Then

\[
(d')_i = \begin{cases} 
\tilde{d}_i, & 1 \leq i \leq s, i \neq k \\
\min \left\{ d_s - d_k - 1, \left\lfloor \frac{d_k}{2} \right\rfloor \right\}, & i = k 
\end{cases}
\]

Note that since \( d_k < d_s - d_k, d_s - d_k - 1 \geq \left\lfloor \frac{d_k}{2} \right\rfloor \), so \( (d')_k = \tilde{d}_k = \left\lfloor \frac{d_k}{2} \right\rfloor \). Therefore \( d' = d \) and hence \( f_1(d') = f_1(\tilde{d}) \).

(b) \( s = 2k + 1 \): Then

\[
(d')_i = \begin{cases} 
\tilde{d}_i, & 1 \leq i \leq s, i \neq k \text{ and } i \neq k + 1 \\
\min \left\{ d_s - d_{k+1} - 1, \left\lfloor \frac{d_{k+1}}{2} \right\rfloor \right\}, & i = k \\
\max \left\{ d_{k+1} + 1, \left\lfloor \frac{d_{k+1}}{2} \right\rfloor \right\}, & i = k + 1 
\end{cases}
\]

Therefore \( f_1(d') = f_1(\tilde{d}) \).

(iii) \( j > s - k \): Then

\[
(d')_i = \begin{cases} 
\tilde{d}_i, & 1 \leq i \leq s, i \neq j \text{ and } i \neq s - j \\
\min \left\{ d_s - d_j - 1, \left\lfloor \frac{d_j}{2} \right\rfloor \right\}, & i = s - j \\
\max \left\{ d_j + 1, \left\lfloor \frac{d_j}{2} \right\rfloor \right\}, & i = j 
\end{cases}
\]

Therefore \( f_1(d') = f_1(\tilde{d}) \).

Now note that \( b_d \Psi_d \geq b_{d'} \Psi_{d'} \) [EKS, Proof of Proposition 3.5, p.126]. This completes the proof of the Proposition.

Proof of Theorem 3.2. Pick degree sequences \( d^\alpha \) and non-negative rational numbers \( r_\alpha \) as in Proposition 2.7. We need to show that \((s + 1)!e_1(M) \leq \beta_0(M)\Psi_d f_1(\tilde{t}) \). We get this as follows:
\[(s + 1)!e_1(M) = (s + 1)! \sum_{\alpha} r_{\alpha} e_1(\beta_{\text{sym}}(d^\alpha)) \]

\[= \sum_{\alpha} r_{\alpha} \left( f_1(d^\alpha) + f_1((d^\alpha)^{\vee, d_s}) \right) \text{ (by (3.6))} \]

\[\leq \sum_{\alpha} r_{\alpha} b_{d^\alpha} \Psi_{d^\alpha} f_1(d^\alpha) \text{ (by Proposition 3.7)} \]

\[\leq \left( \sum_{\alpha} r_{\alpha} b_{d^\alpha} \right) \Psi_t f_1(t) \text{ (by Lemma 3.3)} \]

\[= \beta_0(M) \Psi_t f_1(t). \] (3.8)

Next we give examples for the bound that we found and we compare our result to the bound found by Rossi-Valla in [RV05, Theorem 3.2] and Elias in [E05, Theorem 2.3]. We obtain a much sharper bound.

**Example 3.9.** Let \( R = \mathbb{k}[x, y, z, w, t, u, v] \) and \( I = (y^2, z^2, w^2, x^3 - zwt, x^2y - wt^2, x^2z, xyz - x^2t, yzt - xt^2, yt^2, zt^2, t^3) \) a homogeneous Gorenstein ideal of \( R \). Note that \( e_1(R/I) = 90 \), and the Betti diagram is given as follow:

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| total: | 1 | 12 | 28 | 29 | 12 | 1 |
| 0: | 1 . . . . . |
| 1: | . 8 14 9 2 . |
| 2: | . 8 14 9 2 . |
| 3: | . 8 14 9 2 . |
| 4: | . . . . 3 . |
| 5: | . . . . . 1 |

We have \( T_1 = 3, T_2 = 5, t_3 = 5, t_4 = 7, \) and \( d_5 = T_5 = t_5 = 10 \) and \( \frac{d_5}{2} = 5 \). By theorem 3.2, \( e_1(R/I) \leq \frac{1}{6!} f_1(T_1, T_2, t_3, t_4, t_5)T_1T_2t_3t_4t_5, \) which implies that

\[ e_1(R/I) \leq \frac{1}{6!} (2 + 3 + 2 + 3 + 5)3.5.7.10 = 109.375 \]

The bound found by Elias in [E05, Theorem 2.3] is equal to \( \left( \frac{e_0}{2} \right) - \left( \frac{\mu(I) - d}{2} \right) \) where \( \mu(I) = 11 \) is the minimal number of generator of \( I \) and \( d = 7 \) the dimension of \( R \). Since \( e_0 = 90 \), then the bound in [E05] gives \( e_1(R/I) \leq 624. \)

In the above example, the minimal free resolution of \( R/I \) is quasi-pure. This case was done in [ElS12]. In the next example, \( R/I \) has a non quasi-pure minimal free resolution.

**Example 3.10.** Let \( R = \mathbb{k}[x, y, z, w, t, u, v] \) and \( I = (wt, zt, xt, zw, yw, xw, yz, xz, x^2y + yt^2, x^3 - y^3 + yt^2 - t^3, w^5 + t^5, z^5 - t^5) \) a homogeneous Gorenstein ideal of \( R \). Note that \( e_1(R/I) = 65 \), and the Betti diagram is given as follow:

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| total: | 1 | 12 | 29 | 29 | 12 | 1 |
| 0: | 1 . . . . . |
| 1: | . 8 14 9 2 . |
| 2: | . 2 . 4 2 . |

\[ e_1(R/I) \leq \frac{1}{6!} (2 + 3 + 2 + 3 + 5)3.5.7.10 = 109.375 \]
We have \( T_1 = 5, T_2 = 6, t_3 = 4, t_4 = 5, \) and \( d_5 = T_3 = t_5 = 10 \) and \( \frac{d}{2} = 5 \). By theorem 3.2, \( e_1(R/I) \leq \frac{1}{6!} f_1(T_1, \frac{d_2}{2}, \frac{d_3}{2}, t_4, t_5) T_1 \frac{d_5}{2} \frac{d_4}{2} t_4 t_5 \), which implies that
\[
e_1(R/I) \leq \frac{1}{6!} (4 + 3 + 2 + 1 + 5) 5.5.5.5.10 \simeq 130.20833
\]

Whereas the bound found by Rossi-Valla in [RV05, Theorem 3.2] is equal to \( \left( \frac{e_0}{2} \right) - \left( \frac{\mu(I)}{2} - d \right) - \lambda(R/I) + 1 \) where \( \mu(I) = 12 \) is the minimal number of generator of \( I \), \( d = 8 \) the dimension of \( R \), and \( \lambda(R/I) = e_0 - e_1 + e_2 \). We note that \( e_0(R/I) = 26 \) and \( e_2(R/I) = 68 \). Hence the bound in [RV05] gives \( e_1(R/I) \leq 285 \). 

Huneke and Hanumanthu [HH] find a bound for \( e_1 \) in a slightly different setting. For a CM local ring \( (R, m) \) of dim \( d \) and an ideal \( I \) contained in \( m^k \), for some \( k > 2 \), let
\[
\lambda(R/I^{n+1}) = \sum_i (-1)^i e_i(I) \binom{n+d-i}{d}.
\]
In [HH] corollary 3.7, they show that \( e_1 \leq \binom{(e_0-k)}{2} \).

Finally, we conjecture that the above upper bound can be extended to all Hilbert coefficients \( e_i \)'s. We believe the following is true.

**Conjecture 3.11.** Let \( M \) be a finitely generated graded Gorenstein \( R \)-module, minimally generated by homogeneous elements of degree zero. Let \( s = \text{codim} M \) and \( k = \left\lfloor \frac{r}{2} \right\rfloor \). Let \( \beta_0(M) \) denote the minimal number of generators of \( M \) and write \( m = (t_i)_{i=0,...,s} \). Then
\[
e_j(M) \leq \frac{\beta_0(M)}{(s+1)!} \Psi e_j(t),
\]
for all \( 0 \leq j \leq d - 1 \).

**Example 3.12.** In Example 3.10, we considered the ideal \( I = (wt, zt, xt, zw, yw, wx, yz, xz, x^2y + yt^2, x^3 - y^3 + yt^2 - t^3, w^5 + t^5, z^5 - t^5) \) where \( T_1 = 5, T_2 = 6, t_3 = 4, t_4 = 5, d_5 = T_3 = t_5 = 10 \) and \( \frac{d}{2} = 5 \). We have \( e_2(R/I) = 68 \), and we note that
\[
e_2(R/I) \leq \frac{1}{6!} 150.5.5.5.10 \simeq 1302
\]

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Department of Mathematics, American University of Beirut, Beirut, Lebanon.
Email address: se24@aub.edu.lb

Chennai Mathematical Institute, Siruseri, Tamilnadu 603103 India.
Email address: mkummini@cmi.ac.in

Department of Mathematics, University of Missouri, Columbia, Missouri, USA.
Email address: hema@math.missouri.edu