\( \mathcal{N}(p, q, s) \)-TYPE SPACES IN THE UNIT BALL OF \( \mathbb{C}^n \)

BIN YANG HU AND SONGXIAO LI†

Abstract. In this paper, we consider a new class of space, called \( \mathcal{N}(p, q, s) \)-type spaces, in the unit ball \( \mathbb{B} \) of \( \mathbb{C}^n \). We study some basic properties, Hadamard gaps, Hadamard products, Random power series, Korenblum’s inequality, Gleason’s problem, atomic decomposition of \( \mathcal{N}(p, q, s) \)-type spaces. Moreover, we also establish several equivalent characterizations, including Carleson measure characterization and various derivative characterizations. Finally, we also characterize the distance between Bergman-type spaces and \( \mathcal{N}(p, q, s) \)-type spaces, Riemann-Stieltjes operators and multipliers on \( \mathcal{N}(p, q, s) \)-type spaces.

Contents

1. Introduction 2
2. Basic properties of \( \mathcal{N}(p, q, s) \)-type spaces 6
2.1. Basic structure 6
2.2. The closure of all polynomials in \( \mathcal{N}(p, q, s) \)-type spaces 8
2.3. Description given by Green’s function 12
3. Hadamard gaps in \( \mathcal{N}(p, q, s) \)-type spaces 15
4. Carselon measure, Hadamard products and Random power series 21
4.1. Carselon measure 21
4.2. Hadamard products 33
4.3. Random power series 36
5. Characterizations of \( \mathcal{N}(p, q, s) \)-type spaces 39
5.1. Various derivative characterizations 39
5.2. Korenblum’s inequality of \( \mathcal{N}(p, q, s) \)-type spaces 49
5.3. Derivative-free, mixture and oscillation characterizations 52
6. Atomic decomposition and Gleason’s problem for \( \mathcal{N}(p, q, s) \)-type spaces 58

Date: July 9, 2018.
2010 Mathematics Subject Classification. 32A05, 32A36.
Key words and phrases. \( \mathcal{N}(p, q, s) \)-type spaces, Hadamard gap, atomic decomposition, Carleson measure, Riemann-Stieltjes operator.
† Corresponding author.
1. Introduction

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$ with $\mathbb{S}$ as its boundary and $H(\mathbb{B})$ the collection of all holomorphic functions in $\mathbb{B}$. $H^\infty$ denotes the Banach space consisting of all bounded holomorphic functions in $\mathbb{B}$ with the norm $\|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)|$. For $l > 0$, the Bergman-type space $A^{-l}(\mathbb{B})$ is the space of all $f \in H(\mathbb{B})$ such that

$$|f|_l = \sup_{z \in \mathbb{B}} |f(z)|(1 - |z|^2)^l < \infty.$$ 

Let $A^{-l}_0(\mathbb{B})$ denote the closed subspace of $A^{-l}(\mathbb{B})$ such that $\lim_{|z| \to 1} |f(z)|(1 - |z|^2)^l = 0$.

Denote $dV$ the normalized volume measure over $\mathbb{B}$ and for $\alpha > -1$, the weighted Lebesgue measure $dV_\alpha$ is defined by

$$dV_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dV(z)$$

where $c_\alpha = \frac{\Gamma(n+\alpha+1)}{\pi^{\frac{n}{2}}\Gamma(\alpha+1)}$ is a normalizing constant so that $dV_\alpha$ is a probability measure on $\mathbb{B}$.

For $\alpha > -1$ and $p > 0$, the weighted Bergman space $A^p_\alpha$ consists of all $f \in H(\mathbb{B})$ satisfying

$$\|f\|_{p,\alpha} = \left[ \int_{\mathbb{B}} |f(z)|^p dV_\alpha(z) \right]^{1/p} < \infty.$$ 

It is well-known that when $1 \leq p < \infty$, $A^p_\alpha$ is a Banach space and when $0 < p < 1$, $A^p_\alpha$ becomes a complete metric space. We refer the reader to the books [6, 44, 47] for more information.

Let $\Phi_a(z)$ be the automorphism of $\mathbb{B}$ for $a \in \mathbb{B}$, i.e.,

$$\Phi_a(z) = \frac{a - P_aoz - saQ_aoz}{1 - \langle z, a \rangle},$$

where $P_ao$, $Q_ao$, and $s_ao$ are coefficients.
where \( s_a = \sqrt{1 - |a|^2} \), \( \mathcal{P}_a \) is the orthogonal projection into the space spanned by \( a \) and \( Q_a = I - \mathcal{P}_a \) (see, e.g., \cite{47}). For \( p > 0 \), the space \( \mathcal{N}_p \) on \( \mathbb{B} \) was studied in \cite{8}, i.e.,

\[
\mathcal{N}_p = \{ f \in H(\mathbb{B}) : \sup_{a \in \mathbb{B}} \left( \int_{\mathbb{B}} |f(z)|^p (1 - |\Phi_a(z)|^2)^{p/2} dV(z) \right)^{1/2} < \infty \}
\]

The little space of \( \mathcal{N}_p \)-space, denoted by \( \mathcal{N}_p^0 \), which consists of all \( f \in \mathcal{N}_p \) such that

\[
\lim_{|a| \to 1} \int_{\mathbb{B}} |f(z)|^p (1 - |\Phi_a(z)|^2)^{p/2} dV(z) = 0.
\]

Let \( d\lambda(z) = \frac{dV(z)}{(1 - |z|^2)^{n+1}} \). Then \( d\lambda \) is Möbius invariant (see, e.g., \cite{30}), which means,

\[
\int_{\mathbb{B}} f(z) d\lambda(z) = \int_{\mathbb{B}} f \circ \phi(z) d\lambda(z)
\]

for each \( f \in L^1(\lambda) \) and \( \phi \) an automorphism of \( \mathbb{B} \).

For \( p > 0 \) and \( q, s > 0 \), in this paper we consider the \( \mathcal{N}(p, q, s) \)-type space as follows:

\[
\mathcal{N}(p, q, s) := \{ f \in H(\mathbb{B}) : \|f\| < \infty \},
\]

where

\[
\|f\|^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z).
\]

The corresponding little space, denoted by \( \mathcal{N}^0(p, q, s) \), is the space of all \( f \in \mathcal{N}(p, q, s) \) such that

\[
\lim_{|a| \to 1} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) = 0.
\]

It should be noted that when \( p = 2, q = n + 1, s > 0 \), \( \mathcal{N}(2, n + 1, s) \) coincides the \( \mathcal{N}_{ns} \)-space, as well as the little space, in particular, by \cite{8} Theorem 2.1, when \( p = 2, q = n + 1, s > 1 \),

\[
\mathcal{N}(2, n + 1, s) = A^{-\frac{n+1}{2}}(\mathbb{B}).
\]

Moreover, as we will show later (see, e.g., Remark \cite{5,6}), the \( \mathcal{N}(p, q, s) \)-spaces coincide with \( F(p, q, t) \)-spaces by the identification

\[
\mathcal{N}(p, q, s) = F(p, p + q - n - 1, ns).
\]

Moreover,

\[
\mathcal{N}^0(p, q, s) = F_0(p, p + q - n - 1, ns).
\]
Let \(0 < p < \infty, 0 \leq t < \infty, -1 < q + t < \infty, -1 - n < q < \infty\). Recall that an \(f \in H(\mathbb{B})\) is said to belong to the \(F(p, q, t)\)-space if
\[
|f(0)|^p + \sup_{a \in \mathbb{B}} \int_\mathbb{B} |\nabla f(z)|^p (1 - |z|^2)^q g^t(z, a) dV(z) < \infty,
\]
and an \(f \in H(\mathbb{B})\) is said to belong to the \(F_0(p, q, t)\)-space if
\[
\lim_{|a| \to 1} \int_\mathbb{B} |\nabla f(z)|^p (1 - |z|^2)^q g^t(z, a) dV(z) = 0,
\]
where
\[
\nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right) \quad \text{and} \quad g(z, a) = \log \frac{1}{|\Phi_a(z)|}.
\]
The \(F(p, q, t)\)-spaces were first introduced by R. Zhao on the unit disk in \(\mathbb{C}\) in [42], and later studied on the unit disk and the unit ball \(\mathbb{B}\) of \(\mathbb{C}^n\) by various authors (see, e.g., [17, 29, 40, 41] and the reference therein). However, to the best of our knowledge, currently there are few results in the theory of \(F(p, q, t)\)-spaces in the unit ball due to the complexity of the parameters \(p, q\) and \(t\), as well as the high dimension.

As is well-known from the literature, the weighted Bergman space \(A^p_\alpha\) has a lot of nice properties. Moreover, many concrete operators (including composition operators, Toeplitz operators, Hankel operators, Riemann-Stieltjes operators and etc.) on weighted Bergman spaces have been completely characterized. However, the Dirichlet type space \(D^p_\alpha\) does not. Here, we say an \(f \in H(\mathbb{B})\) belongs to the Dirichlet type space, denoted by \(D^p_\alpha\), if
\[
\int_\mathbb{B} |\nabla f(z)|^p (1 - |z|^2)^\alpha dV(z) < \infty.
\]
Here \(0 < p < \infty, \alpha > -1\). Note that when \(\alpha = p - n - 1, p > n\), the Dirichlet type space becomes the classical Besov space \(B_p\). (see, e.g., [18, 47] for more information on these spaces). Inspired by this fact, it is natural for us to consider the \(N(p, q, s)\)-spaces.

Another important feature of \(N(p, q, s)\)-spaces is that for any \(\frac{n-1}{n} < s < \frac{n}{n-1}, n \geq 2\),
\[
Q_s \subseteq N(2, 1, s).
\]
For detailed studies of \(Q_s\) spaces on the unit disk and the unit ball, we refer the readers to the monographs [25, 37, 38]. The above claim follows from a comparison of these two classes of function spaces. More precisely, for \(Q_s\) space, we have
(a). When \(1 < s < \frac{n}{n-1}\), \(Q_s = \mathcal{B}\) (the Bloch space);
\( N(p, q, s) \)-TYPE SPACES IN THE UNIT BALL OF \( \mathbb{C}^n \)

(b). When \( \frac{n-1}{n} < s \leq 1 \), \( f \in Q_s \) if and only if

\[
\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\tilde{\nabla} f(z)|^2 (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty,
\]

where \( (\tilde{\nabla} f)(z) = \nabla (f \circ \Phi_z)(0) \) denotes the invariant gradient of \( \mathbb{B} \) (see, e.g., [20]). In fact, \( Q_s = F(2, 2 - (n + 1), ns) \).

On the other hand, for \( N(2, 1, s) \)-space, we have

(a'). When \( s > 1 \), \( N(2, 1, s) = A_{-\frac{1}{2}}(\mathbb{B}) = B^{3/2} \) (the \( 3/2 \)-Bloch space);

(b'). When \( \frac{n-1}{n} < s \leq 1 \), \( f \in N(2, 1, s) \) if and only if

\[
\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\tilde{\nabla} f(z)|^2 (1 - |z|^2)(1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty.
\]

The assertion (a') follows from Proposition 2.1, Proposition 2.4 and Lemma 5.8, while assertion (b') follows from Corollary 5.5. In fact, \( N(2, 1, s) = F(2, 3 - (n + 1), ns) \). From these facts, it is easy to see that the desired inclusion holds.

Moreover, if we allow \( q = 0 \) in the definition of the \( N(p, q, s) \)-spaces, that is the endpoint case, then from the proofs of Propositions 2.1 and 2.4 one can easily check that \( N(p, 0, s) \subseteq H^\infty, p \geq 1, s > 0 \), while the equality holds when \( s > 1 \) (see [13] for the case of the unit disk). Combining this observation with the fact that \( Q_s = B \) when \( 1 < s < \frac{n}{n-1} \) and \( Q_s = \{ \text{constant functions} \} \) when \( s \geq \frac{n}{n-1} \), we can also see that the \( N(p, q, s) \)-spaces are independent of the \( Q_s \) spaces and of their own interest.

The aim of this paper is to systematically study \( N(p, q, s) \)-spaces in the unit ball, and we believe the methods developed in this paper, as well as some new features of \( N(p, q, s) \)-type spaces and \( N^0(p, q, s) \)-type spaces, can be generalized to \( F(p, q, s) \)-spaces and \( F_0(p, q, s) \)-spaces.

In this paper, we study various properties of \( N(p, q, s) \)-type spaces, including some basic properties, Hadamard gaps, Hadamard products, Gleason’s problem, Random power series, Korenblum’s inequality, atomic decomposition of \( N(p, q, s) \)-type spaces. We also establish several equivalent characterizations, including Carleson measure characterization and various derivative characterizations. Finally, we investigate the distance between Bergman-type spaces and \( N(p, q, s) \)-type spaces, Riemann-Stieltjes operators and multipliers on \( N(p, q, s) \)-type spaces.

Throughout this paper, for \( a, b \in \mathbb{R} \), \( a \lesssim b \) (\( a \gtrsim b \), respectively) means there exists a positive number \( C \), which is independent of \( a \) and \( b \), such that \( a \leq Cb \) (\( a \geq Cb \), respectively). Moreover, if both \( a \lesssim b \) and \( a \gtrsim b \) hold, then we say \( a \simeq b \).
2. Basic properties of $\mathcal{N}(p, q, s)$-type spaces

2.1. Basic structure. We first show that $\mathcal{N}(p, q, s)$ is a functional Banach space (see, e.g., [2]) when $p \geq 1$ and $q, s > 0$. For $a \in \mathbb{B}$ and $0 < R < 1$, define

$D(a, R) = \Phi_a (\{z \in \mathbb{B} : |z| < R\}) = \{z \in \mathbb{B} : |\Phi_a(z)| < R\}.$

The following result plays an important role in the sequel.

**Proposition 2.1.** Let $p \geq 1$ and $q, s > 0$. The point evaluation $K_z : f \mapsto f(z)$ is a continuous linear functional on $\mathcal{N}(p, q, s)$. Moreover, $\mathcal{N}(p, q, s) \subseteq A^{-\frac{p}{q}}(\mathbb{B})$.

**Proof.** Denote $\mathbb{B}_{1/2} := \{z : |z| < \frac{1}{2}\}$. For each $f \in \mathcal{N}(p, q, s)$ and $a_0 \in \mathbb{B}$, we have

$$\|f\|^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)$$

$$\geq \int_{D(a_0, 1/2)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_{a_0}(z)|^2)^{ns} d\lambda(z)$$

$$\geq C_{n,s} \int_{D(a_0, 1/2)} |f(z)|^p (1 - |z|^2)^q d\lambda(z)$$

$$= C_{n,s} \int_{\mathbb{B}_{1/2}} |f(\Phi_{a_0}(w))|^p (1 - |\Phi_{a_0}(w)|^2)^q d\lambda(w)$$

(change variable $z = \Phi_{a_0}(w)$)

$$= C_{n,s} \int_{\mathbb{B}_{1/2}} |f(\Phi_{a_0}(w))|^p (1 - |a_0|^2)^q (1 - |w|^2)^{q-n-1} |1 - \langle a_0, w \rangle|^{2q} dV(w)$$

$$\geq C_{q,s,n} (1 - |a_0|^2)^q \int_{\mathbb{B}_{1/2}} |f(\Phi_{a_0}(w))|^p dV(w)$$

$$\geq C'_{q,s,n} (1 - |a_0|^2)^q |f(a_0)|^p \quad (|f \circ \Phi_{a_0}(\cdot)|^p \text{ is subharmonic}),$$

where $C_{n,s}$ is a constant depending on $n$ and $s$ and $C_{q,s,n}$ and $C'_{q,s,n}$ are some constants depending on $q$, $s$ and $n$. Hence, for any $z \in \mathbb{B}$, we have

$$|f(z)| \lesssim \frac{\|f\|}{(1 - |z|^2)^{\frac{q}{p}}},$$

which implies the point evaluation is a continuous linear functional, as well as, $\mathcal{N}(p, q, s) \subseteq A^{-\frac{p}{q}}(\mathbb{B})$. \qed

Letting $q = n + 1$ and $p = 2$ in Proposition 2.1, we get [8] Theorem 2.1, (a) as a particular case.

**Corollary 2.2.** For $p > 0$, $\mathcal{N}_p(\mathbb{B}) \subseteq A^{-\frac{p}{q}}(\mathbb{B})$. 
\[ N(p,q,s)-\text{type spaces in the unit ball of } \mathbb{C}^n \]

**Theorem 2.3.** Let \( p \geq 1 \) and \( q, s > 0 \). \( N(p,q,s) \) is a functional Banach space.

**Proof.** It is clear that \( N(p,q,s) \) is a normed vector space with respect to the norm \( \| \cdot \| \). It suffices to show the completeness of \( N(p,q,s) \). Let \( \{ f_m \} \) be a Cauchy sequence in \( N(p,q,s) \). From this, by Proposition 2.4, it follows that \( \{ f_m \} \) is a Cauchy sequence in the space \( H(\mathbb{B}) \), and hence it converges to some \( f \in H(\mathbb{B}) \). It remains to show that \( f \in N(p,q,s) \). Indeed, there exists a \( \ell_0 \in \mathbb{N} \) such that for all \( m, \ell \geq \ell_0 \), it holds \( \| f_m - f_\ell \| \leq 1 \). Take and fix an arbitrary \( a \in \mathbb{B} \), by Fatou's lemma, we have

\[
\int_\mathbb{B} |f(z) - f_{\ell_0}(z)|^p(1 - |z|^2)^q(1 - |\Phi_a(z)|^2)^{ns}d\lambda(z) \\
\leq \lim_{\ell \to \infty} \int_\mathbb{B} |f_{\ell}(z) - f_{\ell_0}(z)|^p(1 - |z|^2)^q(1 - |\Phi_a(z)|^2)^{ns}d\lambda(z) \\
\leq \lim_{\ell \to \infty} \| f_\ell - f_{\ell_0} \|^p \leq 1,
\]

which implies that

\[
\| f - f_{\ell_0} \| = \sup_{a \in \mathbb{B}} \int_\mathbb{B} |f(z) - f_{\ell_0}(z)|^p(1 - |z|^2)^q(1 - |\Phi_a(z)|^2)^{ns}d\lambda(z) \leq 1,
\]

and hence \( \| f \| \leq 1 + \| f_{\ell_0} \| < \infty \). Thus, combining this with the fact that the point evaluation on \( N(p,q,s) \) is a continuous linear functional, we conclude that \( N(p,q,s) \) is a functional Banach space. \( \Box \)

Next we show that when \( p \geq 1 \), \( q > 0 \) and \( s > 1 \), \( N(p,q,s) = A^{-\frac{q}{p}}(\mathbb{B}) \). More precisely, we have the following result.

**Proposition 2.4.** Let \( p \geq 1 \), \( q, s > 0 \). If \( s > 1 - \frac{q-kp}{n} \), \( k \in \left(0, \frac{q}{p}\right] \), then \( A^{-k}(\mathbb{B}) \subseteq N(p,q,s) \). In particular, when \( s > 1 \), \( N(p,q,s) = A^{-\frac{q}{p}}(\mathbb{B}) \).

**Proof.** Suppose \( p \geq 1 \), \( q > 0 \) and \( s > 1 - \frac{q-kp}{n} \) for some \( k \in \left(0, \frac{q}{p}\right] \). Since \( s > 1 - \frac{q-kp}{n} \), we have \( q + ns - n - 1 - kp > -1 \), then by [47, Theorem 1.12], for each \( a \in \mathbb{B} \), we have

\[
\int_\mathbb{B} \frac{(1 - |z|^2)^q + ns - n - 1 - pk}{|1 - \langle a, z \rangle|^{2ns}}dV(z) \lesssim \begin{cases} 
\text{bounded in } \mathbb{B}, & \text{if } ns + pk < q; \\
\log \frac{1}{1 - |a|^2}, & \text{if } ns + pk = q; \\
(1 - |a|^2)^{q - ns - pk}, & \text{if } ns + pk > q,
\end{cases}
\]

which implies, there exists a positive constant \( C \) such that

\[
(2.1) \sup_{a \in \mathbb{B}} (1 - |a|^2)^{ns} \int_\mathbb{B} \frac{(1 - |z|^2)^q + ns - n - 1 - pk}{|1 - \langle a, z \rangle|^{2ns}}dV(z) \leq C.
\]
Let \( f \in A^{-k}(\mathbb{B}) \). By (2.1), we have

\[
\|f\|^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi(z)|^2)^{ns} d\lambda(z)
\]

\[
= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^{pk} (1 - |z|^2)^{q-pk} (1 - |\Phi(z)|^2)^{ns} d\lambda(z)
\]

\[
\leq |f|^p \sup_{a \in \mathbb{B}} (1 - |a|^2)^{ns} \int_{\mathbb{B}} \frac{(1 - |z|^2)^q |ns - n - 1 - pk|}{1 - \langle a, z \rangle^2} dV(z)
\]

\[
\leq C |f|^p,
\]

which implies \( A^{-k}(\mathbb{B}) \subseteq \mathcal{N}(p, q, s) \).

Now if \( s > 1 \), then in particular, we can take \( k = \frac{q}{p} \) and hence by the above argument, we have \( A^{-\frac{q}{p}}(\mathbb{B}) \subseteq \mathcal{N}(p, q, s) \). Combing this fact with Proposition 2.1, we get the desired result. \( \square \)

Letting \( q = n + 1 \) and \( p = 2 \) in Proposition 2.4, we get [8, Theorem 2.1, (b)] as a particular case.

### 2.2. The closure of all polynomials in \( \mathcal{N}(p, q, s) \)-type spaces

**Proposition 2.5.** Let \( p \geq 1 \) and \( q, s > 0 \). Then \( \mathcal{N}^0(p, q, s) \) is a closed subspace of \( \mathcal{N}(p, q, s) \) and hence \( \mathcal{N}^0(p, q, s) \) is a Banach space.

**Proof.** First we note that it is trivial to see that \( \mathcal{N}^0(p, q, s) \) is a subspace of \( \mathcal{N}(p, q, s) \) and hence it suffice to show that \( \mathcal{N}^0(p, q, s) \) is complete. Suppose \( \{f_n\} \) is a Cauchy sequence in \( \mathcal{N}^0(p, q, s) \), by Theorem (2.3) there is a limit \( f \in \mathcal{N}(p, q, s) \) of \( \{f_n\} \). For any \( \varepsilon > 0 \), there exists a \( N \in \mathbb{N} \), such that when \( n > N \), \( \|f - f_n\| < (\frac{\varepsilon}{2^p})^{1/p} \). Take \( n_0 > N \), since \( f_{n_0} \in \mathcal{N}^0(p, q, s) \), there exists a \( \delta \in (0, 1) \) such that when \( \delta < |a| < 1 \),

\[
\sup_{\delta < |a| < 1} \int_{\mathbb{B}} |f_{n_0}(z)|^p (1 - |z|^2)^q (1 - |\Phi(z)|^2)^{ns} d\lambda(z) < \frac{\varepsilon}{2^p}.
\]

Hence, we have

\[
\sup_{\delta < |a| < 1} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi(z)|^2)^{ns} d\lambda(z)
\]

\[
\leq 2^{p-1} \|f - f_{n_0}\|^p
\]

\[
+ 2^{p-1} \sup_{\delta < |a| < 1} \int_{\mathbb{B}} |f_{n_0}(z)|^p (1 - |z|^2)^q (1 - |\Phi(z)|^2)^{ns} d\lambda(z) < \varepsilon,
\]

which implies that \( f \in \mathcal{N}^0(p, q, s) \). \( \square \)

**Theorem 2.6.** Suppose \( f \in \mathcal{N}(p, q, s) \) with \( ns + q > n \). Then \( f \in \mathcal{N}^0(p, q, s) \) if and only if \( \|f_r - f\| \to 0 \), \( r \to 1 \), where \( f_r(z) = f(rz) \) for all \( z \in \mathbb{B} \).
\( \mathcal{N}(p,q,s) \)-TYPE SPACES IN THE UNIT BALL OF \( \mathbb{C}^n \)

**Proof. Necessity.** Suppose \( f \in \mathcal{N}^0(p,q,s) \). This implies that for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that with \( \delta > |a| < 1 \), we have

\[
(2.2) \quad \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^n s d\lambda(z) < \frac{\varepsilon}{3 \cdot 2^{m+p-1}}.
\]

Furthermore, by Schwarz-Pick Lemma (see, e.g., [30, Theorem 8.1.4]), we have

\[
(2.3) \quad |\Phi_{ra}(rz)| \leq |\Phi_a(z)|, \text{ for all } r \in (0, 1) \text{ and } a, z \in \mathbb{B}.
\]

Now take and fix \( \delta_0 \in (\delta, 1) \). Consider \( r \) satisfying \( \max \left\{ \frac{1}{n}, \frac{2}{n} \right\} < r < 1 \). In this case, for all \( a \in \mathbb{B} \) with \( |a| \in (\delta_0, 1) \), by (2.2) and (2.3), we have

\[
\int_{\mathbb{B}} |f(rz) - f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^n s d\lambda(z)
\]

\[
\leq \int_{\mathbb{B}} |f(rz)|^p (1 - |rz|^2)^q (1 - |\Phi_{ra}(rz)|^2)^n s d\lambda(z)
\]

\[
= \left( \frac{1}{r} \right)^{2n} \int_{\mathbb{B}} |f(w)|^p (1 - |w|^2)^q (1 - |\Phi_{ra}(w)|^2)^n s d\lambda(w)
\]

\[
\leq 4^n \int_{\mathbb{B}} |f(w)|^p (1 - |w|^2)^q (1 - |\Phi_{ra}(w)|^2)^n s d\lambda(w) < \frac{\varepsilon}{3 \cdot 2^{p-1}}.
\]

On the other hand, since \( ns + q > n \), we have \( ns + q - n - 1 > -1 \). By Proposition 2.6, \( f_r \) converges to \( f \) as \( r \to 1 \), in the norm topology of the weighted Bergman space \( A^p_{ns+q-n-1}(\mathbb{B}) \). This implies that there exists a \( r_1 \in (0, 1) \), such that for \( 1 < r < 1 \), we have

\[
\int_{\mathbb{B}} |f(rz) - f(z)|^p (1 - |z|^2)^{ns+q-n-q} dV(z) < \frac{(1 - \delta_0)^{2ns} \cdot \varepsilon}{3}.
\]

Hence, for \( |a| \leq \delta_0 \) and \( r_1 < r < 1 \), we have

\[
\sup_{|a| \leq \delta_0} \int_{\mathbb{B}} |f(rz) - f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^n s d\lambda(z)
\]

\[
= \sup_{|a| \leq \delta_0} \left\{ (1 - |a|^2)^{ns} \int_{\mathbb{B}} |f(rz) - f(z)|^p \frac{(1 - |z|^2)^{ns+q-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) \right\}
\]

\[
\leq \sup_{|a| \leq \delta_0} \int_{\mathbb{B}} |f(rz) - f(z)|^p \frac{(1 - |z|^2)^{ns+q-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z)
\]

\[
\leq \frac{1}{(1 - \delta_0)^{2ns}} \int_{\mathbb{B}} |f(rz) - f(z)|^p (1 - |z|^2)^{ns+q-n-1} dV(z) < \frac{\varepsilon}{3}.
\]
Consequently, for all \( r \) with \( \max \left\{ \frac{1}{2}, \frac{r}{\delta_0}, r_1 \right\} < r < 1 \), we have

\[
\| f_r - f \|^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f_r(z) - f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
\leq \left( \sup_{|a| \leq \delta_0} + \sup_{\delta_0 < |a| < 1} \right) \int_{\mathbb{B}} |f(rz) - f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
\leq \frac{\varepsilon}{3} + 2^{p-1} \sup_{\delta_0 < |a| < 1} \int_{\mathbb{B}} (|f(rz)|^p + |f(z)|^p) (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
< \frac{\varepsilon}{3} + 2^{p-1} \left( \frac{\varepsilon}{3 \cdot 2^{p-1}} + \frac{\varepsilon}{3 \cdot 2^n} \right) \quad \varepsilon,
\]

which shows that \( \| f_r - f \| \to 0 \) as \( r \to 1 \).

**Sufficiency.** First we show that for each \( r \in (0, 1) \) and each \( f \in \mathcal{N}(p, q, s) \), then \( f_r \in \mathcal{N}(p, q, s) \). By [47, Theorem 1.12], we have

\[
\int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+ns-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) \simeq \begin{cases} 
\text{bounded in } \mathbb{B}, & \text{if } ns < q; \\
\frac{1}{\log \frac{1}{1-|a|^2}}, & \text{if } ns = q; \\
(1 - |a|^2)^{q-ns}, & \text{if } ns > q,
\end{cases}
\]

which implies

\[
\sup_{a \in \mathbb{B}} (1 - |a|^2)^{ns} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+ns-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) < \infty.
\]

Hence, we have

\[
\| f_r \|^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f_r(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
\leq \left( \sup_{z \in \mathbb{B}} |f(rz)| \right)^p \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
= \left( \sup_{z \in \mathbb{B}} |f(rz)| \right)^p \sup_{a \in \mathbb{B}} (1 - |a|^2)^{ns} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+ns-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z),
\]

which is finite. Thus, \( f_r \in \mathcal{N}(p, q, s) \). Moreover, we have

\[
\int_{\mathbb{B}} |f_r(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
\leq M \int_{\mathbb{B}} (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z),
\]

for some \( M > 0 \) (this is due to the boundedness of the function \( f(z) \) on \( \{ |z| \leq r \} \)).

Noting that \( ns + q > n \) and by the estimation on the term
\[ \mathcal{N}(p,q,s) \text{-TYPE SPACES IN THE UNIT BALL OF } \mathbb{C}^n \]

\[ \int_{B} \frac{(1-|z|^2)^q(1-|\Phi_\alpha(z)|^2)^{ns}}{|1-(a,z)|^{2ns}} dV(z) \] above, we have

\[ \int_{B} (1-|z|^2)^q(1-|\Phi_\alpha(z)|^2)^{ns} d\lambda(z) \rightarrow 0, \text{ as } |a| \rightarrow 1, \]

which implies that \( f_r \in \mathcal{N}^0(p,q,s) \) for all \( r \in (0,1) \).

Now suppose that \( \|f_r - f\| \rightarrow 0 \) as \( r \rightarrow 1 \). Then by the fact that \( \mathcal{N}^0(p,q,s) \) is a closed subspace of \( \mathcal{N}(p,q,s) \), it follows that \( f \in \mathcal{N}^0(p,q,s) \).

As a corollary of Theorem 2.6, we obtain the following result.

**Corollary 2.7.** The set of polynomials is dense in \( \mathcal{N}^0(p,q,s) \) when \( ns + q > n \).

**Proof.** By Theorem 2.6, for any \( f \in \mathcal{N}^0(p,q,s) \), we have

\[ \lim_{r \rightarrow 1} \|f_r - f\| = 0. \]

Since each \( f_r \) can be uniformly approximated by polynomials, and moreover, by the proof of Theorem 2.6, the sup-norm in \( B \) dominates the \( \mathcal{N}(p,q,s) \)-norm, we conclude that every \( f \in \mathcal{N}^0(p,q,s) \) can be approximated in the \( \mathcal{N}(p,q,s) \)-norm by polynomials.

However, generally, it is not true that \( \mathcal{N}(p,q,s) \) contains all the polynomials. Precisely, we have the following result.

**Proposition 2.8.** Let \( p \geq 1 \) and \( q, s > 0 \). Then the set of polynomials are contained in \( \mathcal{N}(p,q,s) \) if and only if \( ns + q > n \).

**Proof.** The sufficiency clearly follows from Theorem 2.6 and Corollary 2.7. Next we prove the necessity. Assume \( ns + q \leq n \) and denote \( \alpha = n - ns - q, \alpha \geq 0 \). We claim that in this case, the constant function \( F(z) = 1, z \in B \) does not belong to \( \mathcal{N}(p,q,s) \). Indeed,

\[ \|F\|^p = \sup_{a \in B} \int_{B} (1-|z|^2)^q(1-|\Phi_\alpha(z)|^2)^{ns} d\lambda(z) \]

\[ = \sup_{a \in B} (1-|a|^2)^{ns} \int_{B} \frac{(1-|z|^2)^q^{ns-n-1}}{|1-(a,z)|^{2ns}} dV(z) \]

\[ \geq \int_{B} \frac{1}{(1-|z|^2)^{\alpha+1}} dV(z) \quad \text{(Put } a = 0) \]

\[ \approx \int_{0}^{1} \frac{r^{2n-1}}{(1-r^2)^{1+\alpha}} dr \geq \int_{1/2}^{1} \frac{1}{(1-r)^{1+\alpha}} dr \]

\[ = \infty, \]

which is a contradiction. Hence, we get the desired result. \qed
2.3. Description given by Green’s function. In [33] and [34], the invariant Green’s function is defined as
\[ G(z, a) = g(\Phi_a(z)), \]
where
\[ g(z) = \frac{n + 1}{2n} \int_{|z|}^{1} (1 - t^2)^{n-1} t^{-2n+1} dt. \]

The following property of \( g \) is important (see, e.g., [24]).

**Proposition 2.9.** Let \( n \geq 2 \) be an integer. Then there are positive constants \( C_1 \) and \( C_2 \) such that for all \( z \in \mathbb{B}\setminus\{0\} \),
\[ C_1 (1 - |z|^2)^n |z|^{-2(n-1)} \leq g(z) \leq C_2 (1 - |z|^2)^n |z|^{-2(n-1)}. \]

For \( p \geq 1, q, s > 0 \) and \( n \geq 2 \), we define the following \( N_*(p, q, s) \)-type space:
\[ N_*(p, q, s) := \left\{ f \in H(\mathbb{B}) : \|f\|_p^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) < \infty \right\}, \]
where the corresponding little space is defined as
\[ N_0(p, q, s) := \left\{ f \in N_*(p, q, s) : \lim_{|a| \to 1} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) = 0 \right\}. \]

Noting that by Proposition 2.9, it is clear that \( \| \cdot \| \lesssim \| \cdot \|_* \), that is, \( N_*(p, q, s) \subseteq N(p, q, s) \). Combining this fact with the proof of Proposition 2.1 and Theorem 2.3 and Proposition 2.5 we have the following result.

**Theorem 2.10.** For \( p \geq 1, q, s > 0 \) and \( n \geq 2 \). \( N_*(p, q, s) \) is a functional Banach space and \( N_0(p, q, s) \) is a closed subspace of \( N_*(p, q, s) \). Moreover, \( N_*(p, q, s) \subseteq A^{-\frac{n}{n-1}}(\mathbb{B}) \).

However, generally, it is not true that \( N_*(p, q, s) \) contain all the polynomials, for example, by Proposition 2.8 and the fact that \( \| \cdot \| \lesssim \| \cdot \|_* \), we know that when \( ns + q \leq n \), \( F \notin N_*(p, q, s) \), where \( F(z) = 1, z \in \mathbb{B} \).

We are interested in the following question: when does \( N_*(p, q, s) \) contain the set of polynomials? We have the following result.

**Proposition 2.11.** Let \( p \geq 1, q, s > 0 \) and \( n \geq 2 \). Then the set of polynomials are contained in \( N_*(p, q, s) \) if and only if \( ns + q > n \) and \( s < \frac{n}{n-1} \).
Proof. Necessity. We prove it by contradiction. Consider the constant function $F(z) = 1, z \in \mathbb{B}$. By Proposition 2.9 we have

$$\|F\|_s^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |z|^2)^q G^s(z, a) d\lambda(z) \geq \int_{\mathbb{B}} (1 - |z|^2)^q G^s(z) d\lambda(z) = \int_{\mathbb{B}} (1 - |z|^2)^q d\lambda(z) = \int_{\mathbb{B}} (1 - |z|^2)^q dV(z) = I_1 + I_2,$$

where

$$I_1 = \int_{\mathbb{B}_{1/2}} \frac{(1 - |z|^2)^q}{|z|^{2(n-1)s}} dV(z)$$

and

$$I_2 = \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} \frac{(1 - |z|^2)^q}{|z|^{2(n-1)s}} dV(z).$$

We consider two cases.

Case I: $ns + q \leq n$. We have $I_2 = \infty$, which is a contradiction. Indeed,

$$I_2 \simeq \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} (1 - |z|^2)^q dV(z) \simeq \int_{1/2}^1 \frac{1}{(1 - r)^{n+1 - q - ns}} dr = \infty.$$

Case II: $s \geq \frac{n}{n-1}$. We claim in this case, $I_1 = \infty$, which contradicts to the assumption that $F \in \mathcal{N}_s(p, q, s)$. Indeed, we have

$$I_1 \simeq \int_{\mathbb{B}_{1/2}} \frac{1}{|z|^{2(n-1)s}} dV(z) \simeq \int_0^{1/2} \frac{1}{r^{2(n-1)s - 2n+1}} dr = \infty.$$

Sufficiency. Suppose $ns + q > n, s < \frac{n}{n-1}$ and $P$ is a polynomial defined on $\mathbb{B}$. Then we have

$$\|P\|_s^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |P(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \leq \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |z|^2)^q G^s(z, a) d\lambda(z) (P \text{ is bounded on } \mathbb{B})$$

$$\simeq \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |z|^2)^q \frac{(1 - |\Phi_a(z)|^2)^{ns}}{|\Phi_a(z)|^{2(n-1)s}} d\lambda(z)$$

$$= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |\Phi_a(w)|^2)^q \frac{(1 - |w|^2)^{ns}}{|w|^{2(n-1)s}} d\lambda(w)$$

(change variable $w = \Phi_a(z)$)

$$= \sup_{a \in \mathbb{B}} (1 - |a|^2)^q \int_{\mathbb{B}} \frac{(1 - |w|^2)^{q+ns-n-1}}{|w|^{2(n-1)s}|1 - \langle a, w \rangle|^{2q}} dV(w).$$
For each \( a \in \mathbb{B} \), consider the term
\[
\int_{\mathbb{B}} \frac{(1 - |w|^2)^{q + ns - n - 1}}{|w|^{2(n-1)s} |1 - \langle a, w \rangle|^{2q}} dV(w) = I_{1,a} + I_{2,a},
\]
where
\[
I_{1,a} = \int_{\mathbb{B}_{1/2}} \frac{(1 - |w|^2)^{q + ns - n - 1}}{|w|^{2(n-1)s} |1 - \langle a, w \rangle|^{2q}} dV(w)
\]
and
\[
I_{2,a} = \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} \frac{(1 - |w|^2)^{q + ns - n - 1}}{|w|^{2(n-1)s} |1 - \langle a, w \rangle|^{2q}} dV(w).
\]

For \( I_{1,a} \), by the proof of necessary part, we have
\[
I_{1,a} \simeq \int_{\mathbb{B}_{1/2}} \frac{1}{|w|^{2(n-1)s}} dV(w) < M,
\]
for some \( M > 0 \), which is independent of the choice of \( a \). Thus, we have \( \sup_{a \in \mathbb{B}} (1 - |a|^2)^q I_{1,a} < \infty \).

For \( I_{2,a} \), by [47, Theorem 1.12], we have
\[
I_{2,a} \simeq \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} \frac{(1 - |w|^2)^{q + ns - n - 1}}{|w|^{2(n-1)s} |1 - \langle a, w \rangle|^{2q}} dV(w)
\]
\[
\lesssim \int_{\mathbb{B}} \frac{(1 - |w|^2)^{q + ns - n - 1}}{|1 - \langle a, w \rangle|^{2q}} dV(w)
\]
\[
\simeq \begin{cases} 
\text{bounded in } \mathbb{B}, & \text{if } ns > q; \\
\log \frac{1}{1 - |a|^2}, & \text{if } ns = q; \\
(1 - |a|^2)^{ns - q}, & \text{if } ns < q,
\end{cases}
\]

which implies \( \sup_{a \in \mathbb{B}} (1 - |a|^2)^q I_{2,a} < \infty \).

Combing the above estimations, we see that \( \|P\|_* < \infty \), which implies the desired result. \( \square \)

The above proposition provide us a hint on describing the \( N(p, q, s) \)-spaces by using the invariant Green’s function. More precisely, we have the following result.

**Theorem 2.12.** Let \( p \geq 1, q, s > 0 \) and \( n \geq 2 \). If \( s < \frac{n}{n-1} \), then \( N(p, q, s) = N_s(p, q, s) \). In particular, if \( 1 < s < \frac{n}{n-1} \), then \( N(p, q, s) = N_s(p, q, s) = A^{-\frac{q}{p}}(\mathbb{B}) \).

**Proof.** Clearly, \( N_s(p, q, s) \subseteq N(p, q, s) \) and hence it suffices to show \( N(p, q, s) \subseteq N_s(p, q, s) \) when \( s < \frac{n}{n-1} \). Take \( f \in N_s(p, q, s) \). For each
$a \in \mathbb{B}$, we have
\[
\int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z)
\]
\[
\simeq \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q \frac{(1 - |\Phi_a(z)|^2)^{ns}}{|\Phi_a(z)|^{2(n-1)s}} d\lambda(z)
\]
\[
= \int_{\mathbb{B}} |f(\Phi_a(w))|^p (1 - |\Phi_a(w)|^2)^q \frac{(1 - |w|^2)^{ns}}{|w|^{2(n-1)s}} d\lambda(w)
\]
\[
\text{(change variable } w = \Phi_a(z)\text{)}
\]
\[
= J_{1,a} + J_{2,a},
\]
where
\[
J_{1,a} = \int_{\mathbb{B}_{1/2}} |f(\Phi_a(w))|^p (1 - |\Phi_a(w)|^2)^q \frac{(1 - |w|^2)^{ns}}{|w|^{2(n-1)s}} d\lambda(w)
\]
and
\[
J_{2,a} = \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} |f(\Phi_a(w))|^p (1 - |\Phi_a(w)|^2)^q \frac{(1 - |w|^2)^{ns}}{|w|^{2(n-1)s}} d\lambda(w).
\]
For $J_{2,a}$, we have
\[
J_{2,a} \lesssim \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} |f(\Phi_a(w))|^p (1 - |\Phi_a(w)|^2)^q (1 - |w|^2)^{ns} d\lambda(w)
\]
\[
\leq \int_{\mathbb{B}} |f(\Phi_a(w))|^p (1 - |\Phi_a(w)|^2)^q (1 - |w|^2)^{ns} d\lambda(w)
\]
\[
= \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \leq \|f\|^p.
\]
For $J_{1,a}$, by Proposition 2.1, $s < \frac{n}{n-1}$ and the proof of Proposition 2.11 we have
\[
J_{1,a} \leq \|f\|^p \int_{\mathbb{B}_{1/2}} \frac{(1 - |w|^2)^{ns-n-1}}{|w|^{2(n-1)s}} dV(w)
\]
\[
\lesssim \|f\|^p \int_{\mathbb{B}_{1/2}} \frac{1}{|w|^{2(n-1)s}} dV(w) \leq M \|f\|^p,
\]
for some $M > 0$.

Hence, combining the estimations on $J_{1,a}$ and $J_{2,a}$, we have, for each $a \in \mathbb{B},$
\[
\int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \lesssim \|f\|^p,
\]
which implies $\|f\|_* \lesssim \|f\|$, that is, $\mathcal{N}(p, q, s) \subseteq \mathcal{N}_s(p, q, s)$ if $s < \frac{n}{n-1}$.

Finally, when $1 < s < \frac{n}{n-1}$, the desired result follows from the above argument and Proposition 2.11. \qed
The following result is straightforward from Theorem \ref{thm:2.12}

**Corollary 2.13.** Let \( n \geq 2 \) and \( 0 < s < \frac{n}{n-1} \). Then

\[
\mathcal{N}_{ns} = \mathcal{N}(2, n + 1, s) = \mathcal{N}_*(2, n + 1, s).
\]

Moreover, if \( 1 < s < \frac{n}{n-1} \), then \( \mathcal{A} = \mathcal{N}(2, n + 1, s) = \mathcal{N}_*(2, n + 1, s) \).

Our next result shows that \( \mathcal{N}^*_*(p, q, s) \) is trivial if \( s \geq \frac{n}{n-1} \).

**Proposition 2.14.** Let \( p \geq 1, q, s > 0 \) and \( n \geq 2 \). If \( s \geq \frac{n}{n-1} \), then \( \mathcal{N}^*_*(p, q, s) \) only contains the zero function.

**Proof.** We prove it by contradiction. Assume there exists a \( a_0 \in \mathbb{B} \), such that \( |f(a_0)| \geq \delta > 0 \). Then there exists a \( r > 0 \), such that

\[
|f(\Phi_{a_0}(w))| \geq \frac{\delta}{2}, \quad |w| < r.
\]

Therefore we have

\[
\|f\|_*^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z)
\]

\[
\geq \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a_0) d\lambda(z)
\]

\[
\geq \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q \frac{(1 - |\Phi_{a_0}(z)|^2)^{ns}}{|\Phi_{a_0}(z)|^{2(n-1)s}} d\lambda(z)
\]

\[
= \int_{\mathbb{B}} |f(\Phi_{a_0}(w))|^p (1 - |\Phi_{a_0}(w)|^2)^q \frac{(1 - |w|^2)^{ns+n-1}}{|w|^{2(n-1)s}} dV(w)
\]

(change variable \( z = \Phi_{a_0}(w) \))

\[
\geq \int_{|w| < r} |f(\Phi_{a_0}(w))|^p (1 - |\Phi_{a_0}(w)|^2)^q \frac{(1 - |w|^2)^{ns+n-1}}{|w|^{2(n-1)s}} dV(w)
\]

\[
\geq \left( \frac{\delta}{2} \right)^p (1 - |a_0|^2)^q \int_{|w| < r} (1 - |w|^2)^{ns+q-1} dV(w)
\]

\[
\geq \int_{|w| < r} \frac{1}{|w|^{2(n-1)s}} dV(w) \geq \int_0^r \frac{1}{t^{2(n-1)s-2n+1}} dt = \infty,
\]

which is a contradiction. \( \Box \)

**Remark 2.15.** By Theorem \ref{thm:2.12} and Proposition \ref{prop:2.14}, it is clear that \( \mathcal{N}_*(p, q, s) \)-type space is a special case of \( \mathcal{N}(p, q, s) \)-type space. Hence, in the sequel, we will focus our interest on \( \mathcal{N}(p, q, s) \)-type spaces.
3. Hadamard Gaps in $\mathcal{N}(p,q,s)$-Type Spaces

A holomorphic function $f$ on $\mathbb{B}$ written in the form

$$f(z) = \sum_{k=0}^{\infty} P_{n_k}(z),$$

where $P_{n_k}$ is a homogeneous polynomial of degree $n_k$, is said to have Hadamard gaps if for some $c > 1$ (see, e.g., [31]),

$$n_{k+1}/n_k \geq c, \forall k \geq 0.$$

Given a Hadamard gap series, we are interested in the following question: for $p \geq 1$ and $q, s > 0$, when does this Hadamard gap series belong to the $\mathcal{N}(p,q,s)$ spaces?

Observing that a constant function has Hadamard gaps, and hence by Proposition 2.8 or Proposition 2.11, we always assume that the condition $ns + q > n$ holds, that is, $s > 1 - \frac{q}{n}$. Moreover, note that by Proposition 2.4, if $s > 1$, then $\mathcal{N}(p,q,s) = A^q_0(\mathbb{B})$ and for this case, it was already studied by the authors in [7, Theorem 2.5]. Hence, we also assume that $s \leq 1$ in this section.

To formulate our main result in this section, we denote

$$M_k = \sup_{\xi \in \mathbb{S}} |P_{n_k}(\xi)| \quad \text{and} \quad L_{k,p} = \left( \int_{\mathbb{S}} |P_{n_k}(\xi)|^p d\sigma(\xi) \right)^{1/p}, \quad p \geq 1,$$

where $d\sigma$ is the normalized surface measure on $\mathbb{S}$, that is, $\sigma(\mathbb{S}) = 1$. Clearly for each $k \geq 0$ and $p \geq 1$, $M_k$ and $L_{k,p}$ are well-defined.

We have the following result.

**Theorem 3.1.** Let $p \geq 1, q > 0$ and $\max \{0, 1 - \frac{q}{n}\} < s \leq 1$ and $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$ with Hadamard gaps. Consider the following statements

(a) $\sum_{k=0}^{\infty} \frac{1}{2k(n_k+q-n)} \left( \sum_{2^k \leq n_j < 2^{k+1}} M_j^p \right) < \infty$;

(b) $f \in \mathcal{N}^0(p,q,s)$;

(c) $f \in \mathcal{N}(p,q,s)$;

(d) $\sum_{k=0}^{\infty} \frac{1}{2k(n_k+q-n)} \left( \sum_{2^k \leq n_j < 2^{k+1}} L_{j,p}^p \right) < \infty$.

We have $(a) \implies (b) \implies (c) \implies (d)$.

**Proof.** $(a) \implies (b)$. Suppose that $(a)$ holds. First, we prove that $f \in \mathcal{N}(p,q,s)$. For $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$, using the polar coordinates and
Lemma 1.8], we have

\[ \|f\|^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left( \sum_{k=0}^{\infty} |P_n(z)| \right)^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \]

\[ \leq \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left( \sum_{k=0}^{\infty} |P_n(z)| \right)^p (1 - |a|^2)^{ns} (1 - |z|^2)^{ns+q-n-1} |1 - \langle a, z \rangle|^{2ns} d\lambda(z) \]

\[ = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left( \sum_{k=0}^{\infty} |P_n(z)| \right)^p (1 - |a|^2)^{ns} \left( \sum_{n=0}^{\infty} |P_n(r\xi)| \right)^q |1 - \langle a, z \rangle|^{2ns} d\lambda(z) \]

Since for each \( k \in \mathbb{N} \), \( P_n \) is homogeneous, we have for each \( \xi \in \mathbb{S} \),

\[ \sum_{k=0}^{\infty} |P_n(r\xi)| = \sum_{k=0}^{\infty} |P_n(\xi)|^{n^k} \leq \sum_{k=0}^{\infty} M_{n_k} \]

Moreover, for each \( a \in \mathbb{B} \), by Lemma 1.12, we have

\[ \int_{\mathbb{S}} \frac{1}{1 - \langle r\xi, a \rangle}^{2ns} d\sigma(\xi) = \int_{\mathbb{S}} \frac{1}{1 - \langle \xi, ra \rangle}^{2ns} d\sigma(\xi) \]

\[ \simeq \begin{cases} \text{bounded in } \mathbb{B}, & \text{if } s < \frac{1}{2}; \\ \log \frac{1}{1-|a|^2} \leq \log \frac{1}{1-|a|^2}, & \text{if } s = \frac{1}{2}; \\ (1 - |r|^2|a|^2)^{n-2ns} \leq (1 - |a|^2)^{n-2ns}, & \text{if } \frac{1}{2} < s < 1, \end{cases} \]

which implies, there exists a positive constant \( C \) such that

\[ \sup_{a \in \mathbb{B}} (1 - |a|^2)^{ns} \int_{\mathbb{S}} \frac{1}{1 - \langle r\xi, a \rangle}^{2ns} dV(z) \leq C. \]

Thus, by (3.1), (3.2) and [47 Theorem 1], we have

\[ \|f\|^p \simeq \int_{0}^{1} \left( \sum_{k=0}^{\infty} M_{n_k} r^{n_k} \right)^p (1 - |r|^2)^{ns+q-n-1} dr \]

\[ \simeq \sum_{k=0}^{\infty} \frac{1}{2^{n(ns+q-n)}} \left( \sum_{2^k \leq n_j < 2^{k+1}} M_{j} \right)^p. \]

Since \( f \) is in the Hadamard gaps class, there exists a constant \( c > 1 \) such that \( n_{j+1} \leq cn_j \) for all \( j \geq 0 \). Hence, the maximum number of \( n_j \)'s between \( 2^k \) and \( 2^{k+1} \) is less or equal to \( \lceil \log_2 c \rceil + 1 \) for \( k = 0, 1, 2, \ldots. \)
Since for every $k \geq 0$, by Hölder inequality,
\[
\left( \sum_{2^k \leq n_j < 2^{k+1}} M_j \right)^p \leq \left( [\log c_2] + 1 \right)^{p-1} \left( \sum_{2^k \leq n_j < 2^{k+1}} M_j^p \right),
\]
Thus, we have
\[
\|f\|^p \lesssim \sum_{k=0}^\infty \frac{1}{2^{k(ns+q-n)}} \left( \sum_{2^k \leq n_j < 2^{k+1}} M_j^p \right),
\]
which implies $f \in \mathcal{N}(p, q, s)$.

Next, we prove that $f \in \mathcal{N}(p, q, s)$. Put
\[
f_m(z) = \sum_{k=0}^m P_{nk}(z), m \in \mathbb{N},
\]
which is bounded in $\mathbb{B}$. Thus, by the proof of Theorem 2.6, we know that for each $m \in \mathbb{N}, f_m \in \mathcal{N}^0(p, q, s)$. Moreover, by Corollary 2.7, $\mathcal{N}^0(p, q, s)$ is closed and the set of all polynomials is dense in $\mathcal{N}^0(p, q, s)$, and hence it suffices to show that $\|f_m - f\| \to 0$ as $m \to \infty$. By (3.3), we have
\[
\|f_m - f\|^p \lesssim \sum_{k=0}^m \frac{1}{2^{k(ns+q-n)}} \left( \sum_{2^k \leq n_j < 2^{k+1}} M_j^p \right),
\]
where $m' = \left\lfloor \frac{m+1}{[\log c_2]+1} \right\rfloor$. The result follows from condition (a) and (3.4).

(b) $\implies$ (c). It is obvious.

(c) $\implies$ (d). Suppose $f \in \mathcal{N}(p, q, s)$. As the proof in [31, Theorem 1], we have
\[
\|f\|^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left( \sum_{k=0}^\infty P_{nk}(z) \right)^p (1 - |z|^2)^q(1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]
\[
\geq \int_{\mathbb{B}} \left( \sum_{k=0}^\infty P_{nk}(z) \right)^p (1 - |z|^2)^{q+ns-n-1} dV(z)
\]
\[
\geq \int_{\mathbb{S}} \left( \sum_{k=0}^\infty \frac{1}{2^{k(ns+q-n)}} \sum_{2^k \leq n_j < 2^{k+1}} |P_{nk}(\xi)|^p \right) d\sigma(\xi)
\]
\[
= \sum_{k=0}^\infty \frac{1}{2^{k(ns+q-n)}} \left( \sum_{2^k \leq n_j < 2^{k+1}} L_j^p \right),
\]
which implies the desired result. \qed
Letting $q = n + 1, p = 2$ and $0 < s < 1$ in Theorem 3.1 we get [7 Theorem 2.1] as a particular case. Generally, condition (d) does not imply condition (a), an example can be found in [7, Remark 2.2].

Next we consider some special cases when all the conditions in Theorem 3.1 are equivalent.

In [35, Corollary 1], for $p > 0$, the authors constructed a sequence of homogeneous polynomials \( \{W_k\}_{k \in \mathbb{N}} \) satisfying $\deg(W_k) = k$, 
\[
\sup_{\xi \in \mathbb{S}} |W_k(\xi)| = 1 \quad \text{and} \quad \int_{\mathbb{S}} |W_k(\xi)|^p d\sigma(\xi) \geq C(p, n),
\]
where $C(p, n)$ is a positive constant depending on $p$ and $n$.

An immediate corollary of Theorem 3.1 is stated as follows.

**Corollary 3.2.** Let $p \geq 1$, $q > 0$ and $\max \{0, 1 - \frac{q}{n}\} < s \leq 1$ and $f(z) = \sum_{k=0}^{\infty} a_k W_{nk}(z)$ with Hadamard gaps, where $a_k \in \mathbb{C}, k \geq 0$. Then the following statements are equivalent.

(a) \[
\sum_{k=0}^{\infty} \frac{1}{2^k(n+q-n)} \left( \sum_{2^k \leq n_j < 2^{k+1}} |a_j|^p \right) < \infty;
\]
(b) $f \in \mathcal{N}^0(p, q, s)$;
(c) $f \in \mathcal{N}(p, q, s)$.

**Proof.** The desired result follows from the fact that for each $k \geq 0$, $M_k \simeq L_{k,p}$. \( \square \)

Let $p \geq 1, q > 0$ and $\max \{0, 1 - \frac{q}{n}\} < s_1 < s_2 \leq 1$. It is clear that
\[
(3.5) \quad \mathcal{N}(p, q, s_1) \subseteq \mathcal{N}(p, q, s_2) \subseteq A^{-\frac{q}{p}}(\mathbb{B}).
\]
The second application of our main result in this section is to show that the inclusions in (3.5) is strict.

**Corollary 3.3.** Let $p \geq 1, q > 0$ and $\max \{0, 1 - \frac{q}{n}\} < s_1 < s_2 \leq 1$. Then
\[
\mathcal{N}(p, q, s_1) \subsetneq \mathcal{N}(p, q, s_2) \subsetneq A^{-\frac{q}{p}}(\mathbb{B}).
\]

**Proof.** First we prove that $\mathcal{N}(p, q, s_2) \subsetneq A^{-\frac{q}{p}}(\mathbb{B})$. Consider the series $f_1(z) = \sum_{k=0}^{\infty} 2^{\frac{kq}{p}} W_{2k}(z)$. On one hand, by [7, Theorem 2.5], $f_1 \in A^{-\frac{q}{p}}$. On the other hand,
\[
\sum_{k=0}^{\infty} \frac{1}{2^k(ns_2+q-n)} \left( \sum_{2^k \leq n_j < 2^{k+1}} \left| 2^{\frac{kq}{p}} \right|^p \right) = \sum_{k=0}^{\infty} \frac{1}{2^k(ns_2-n)} = \infty,
\]
which, by Corollary 3.2, implies $f_1 \notin \mathcal{N}(p, q, s_2)$. 

An immediate corollary of Theorem 3.1 is stated as follows.

**Corollary 3.2.** Let $p \geq 1, q > 0$ and $\max \{0, 1 - \frac{q}{n}\} < s \leq 1$ and $f \in \mathcal{N}(p, q, s)$.

Then the following statements are equivalent.

(a) \[
\sum_{k=0}^{\infty} \frac{1}{2^{k(n+q-n)}} \left( \sum_{2^k \leq n_j < 2^{k+1}} |a_j|^p \right) < \infty;
\]
(b) $f \in \mathcal{N}^0(p, q, s)$;
(c) $f \in \mathcal{N}(p, q, s)$.

**Proof.** The desired result follows from the fact that for each $k \geq 0$, $M_k \simeq L_{k,p}$. \( \square \)

Let $p \geq 1, q > 0$ and $\max \{0, 1 - \frac{q}{n}\} < s_1 < s_2 \leq 1$. It is clear that
\[
(3.5) \quad \mathcal{N}(p, q, s_1) \subseteq \mathcal{N}(p, q, s_2) \subseteq A^{-\frac{q}{p}}(\mathbb{B}).
\]
The second application of our main result in this section is to show that the inclusions in (3.5) is strict.

**Corollary 3.3.** Let $p \geq 1, q > 0$ and $\max \{0, 1 - \frac{q}{n}\} < s_1 < s_2 \leq 1$. Then
\[
\mathcal{N}(p, q, s_1) \subsetneq \mathcal{N}(p, q, s_2) \subsetneq A^{-\frac{q}{p}}(\mathbb{B}).
\]

**Proof.** First we prove that $\mathcal{N}(p, q, s_2) \subsetneq A^{-\frac{q}{p}}(\mathbb{B})$. Consider the series $f_1(z) = \sum_{k=0}^{\infty} 2^{\frac{kq}{p}} W_{2k}(z)$. On one hand, by [7, Theorem 2.5], $f_1 \in A^{-\frac{q}{p}}$. On the other hand,
\[
\sum_{k=0}^{\infty} \frac{1}{2^k(ns_2+q-n)} \left( \sum_{2^k \leq n_j < 2^{k+1}} \left| 2^{\frac{kq}{p}} \right|^p \right) = \sum_{k=0}^{\infty} \frac{1}{2^k(ns_2-n)} = \infty,
\]
which, by Corollary 3.2, implies $f_1 \notin \mathcal{N}(p, q, s_2)$. 

Next we show that \( \mathcal{N}(p, q, s_1) \subsetneq \mathcal{N}(p, q, s_2) \). Consider the series

\[
f_2(z) = \sum_{k=0}^{\infty} 2^{\frac{k(n_1+q-n)}{p}} W_{2k}(z).
\]

On one hand, by Corollary 3.2,

\[
\sum_{k=0}^{\infty} \frac{1}{2^{k(n_2+q-n)}} \left( \sum_{2^k \leq n_j < 2^{k+1}} \left| 2^{\frac{k(n_1+q-n)}{p}} \right|^p \right) = \sum_{k=0}^{\infty} \frac{1}{2^{kn_2(s_2-s_1)}} < \infty,
\]

which implies that \( f_2 \in \mathcal{N}(p, q, s_2) \). On the other hand,

\[
\sum_{k=0}^{\infty} \frac{1}{2^{k(n_1+q-n)}} \left( \sum_{2^k \leq n_j < 2^{k+1}} \left| 2^{\frac{k(n_1+q-n)}{p}} \right|^p \right) = \sum_{k=0}^{\infty} 1 = \infty,
\]

which, again by Corollary 3.2, implies \( f_2 \notin \mathcal{N}(p, q, s_1) \). \( \square \)

Letting \( q = n + 1, p = 2 \) and \( 0 < s \leq 1 \) in Corollary 3.3, we get the corresponding results in [9] as a particular case.

**Remark 3.4.** From the above corollary, it is also straightforward to see that for any \( k \in \left( 0, \frac{q+ns-n}{p} \right) \), we have

\[
A^{-k}(\mathbb{B}) \not\subset \mathcal{N}(p, q, s).
\]

Indeed, we can find an \( s' \) satisfying \( k < \frac{q+s'-n}{p} < \frac{q+ns-n}{p} \). By the above corollary and Corollary 3.2, it follows that

\[
A^{-k}(\mathbb{B}) \subseteq \mathcal{N}(p, q, s') \subsetneq \mathcal{N}(p, q, s),
\]

which implies the desired claim.

## 4. Carselon measure, Hadamard products and Random power series

### 4.1. Carselon measure.

First we give an equivalent expression of \( \mathcal{N}(p, q, s) \)-norm by Carleson measures. Recall that for \( \xi \in \mathbb{S} \) and \( r > 0 \), a Carleson tube at \( \xi \) is defined as (see, e.g., [17])

\[
Q_r(\xi) = \{ z \in \mathbb{B} : |1 - \langle z, \xi \rangle| < r \}.
\]

Moreover, we denote \( Q(\xi, r) = \{ w \in \mathbb{S} : |1 - \langle w, \xi \rangle| < r \} \).

A positive Borel measure \( \mu \) in \( \mathbb{B} \) is called a \( p \)-Carleson measure if there exists a constant \( C > 0 \) such that

\[
\mu(Q_r(\xi)) \leq Cr^p
\]
for all \( \xi \in S \) and \( r > 0 \). Moreover, if

\[
\lim_{r \to 0} \frac{\mu(Q_r(\xi))}{r^p} = 0
\]

uniformly for \( \xi \in S \), then \( \mu \) is called a vanishing \( p \)-Carleson measure.

The following result describes a relationship between functions in \( \mathcal{N}(p, q, s) \) as well as \( \mathcal{N}^0(p, q, s) \) and Carleson measures.

**Proposition 4.1.** Let \( f \in H(\mathbb{B}) \) and \( p \geq 1, q, s > 0 \), and \( d\mu_{f,p,q,s}(z) = |f(z)|^p(1 - |z|^2)^{q+ns}d\lambda(z) \). The following assertions hold.

1. \( f \in \mathcal{N}(p, q, s) \) if and only if \( d\mu_{f,p,q,s} \) is an \((ns)\)-Carleson measure;
2. \( f \in \mathcal{N}^0(p, q, s) \) if and only if \( d\mu_{f,p,q,s} \) is a vanishing \((ns)\)-Carleson measure.

Moreover,

\[
\|f\|_p \simeq \sup_{0 < r < 1} \sup_{\xi \in S} \frac{\mu_{f,p,q,s}(Q_r(\xi))}{r^{ns}}
\]

(4.1)

\[
= \sup_{0 < r < 1} \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p(1 - |z|^2)^{q+ns}d\lambda(z).
\]

**Proof.** (1) Note that for \( f \in \mathcal{N}(p, q, s) \), we can write

\[
\|f\|_p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p \frac{(1 - |a|^2)^{ns}(1 - |z|^2)^{q+ns-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z)
\]

\[
= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left( \frac{1 - |a|^2}{|1 - \langle a, z \rangle|^2} \right)^{ns} d\mu_{f,p,q,s}(z).
\]

Then (1) is obtained, by [44, Theorem 45]. Moreover, the equation (4.1) also follows from [44, Theorem 45].

(2) This is a consequence of the “little-oh version” of [44, Theorem 45]. □

4.1.1. Embedding relationship with weighted Bergman space. For \( k > 0 \) and \( f \) holomorphic in \( \mathbb{B} \), we denote

\[
M_k(r, f) = \left( \int_S |f(r\xi)|^k d\sigma(\xi) \right)^{1/k}, 0 \leq r < 1.
\]
By using this expression, we can rewrite the norm of \( \| \cdot \|_{k, \rho} \) of the weighted Bergman space \( A^k_{\rho} \) with \( k \geq 1 \) and \( \rho > -1 \) as follows.

\[
\| f \|_{k, \rho} \simeq \left( \int_0^1 r^{2n-1} (1 - r^2)^\rho M_k^k(r, f) dr \right)^{1/k}
\]

\[
\simeq \left( \int_0^1 r^{2n-1} (1 - r)^\rho M_k^k(r, f) dr \right)^{1/k}.
\]

As an application of Proposition 4.1, we establish the following embedding relation between \( A^k_{\rho} \) and \( \mathcal{N}(p, q, s) \) with some proper condition on \( k \) and \( \rho \). Note that, by Proposition 2.8 and the fact that the set of all polynomials belongs to \( A^k_{\rho} \), it is natural for us to assume that \( ns + q > n \).

We have the following result.

**Proposition 4.2.** Let \( p \geq 1, q > 0 \) and \( s > \max \left\{ 0, 1 - \frac{a}{b} \right\} \). Then the following assertions hold.

1. If \( 0 < s < 1 \), then for \( \max\{0, q - n\} < \rho < \frac{a + ns - n}{1 - s} \), we have

\[
\| f \| \lesssim \| f \|_{\frac{p(n+\rho)}{q}, \rho - 1}, \quad \text{that is, } A_{\rho - 1} \subseteq \mathcal{N}(p, q, s);
\]

2. If \( s \geq 1 \), then for \( \rho > \max\{0, q - n\} \), we have \( \| f \| \lesssim \| f \|_{\frac{p(n+\rho)}{q}, (\rho - 1)} \),

\[
\quad \text{that is, } A_{\rho - 1} \subseteq \mathcal{N}(p, q, s).
\]

**Proof.** The proof for (2) is a simple modification of (1) and hence we omit the proof for (2) here. Suppose \( 0 < s < 1 \). Note that since \( \rho < \frac{a + ns - n}{1 - s} \), we have

\[
(q + ns - n - \rho) \cdot \frac{n + \rho}{n + \rho - q} + \rho > 0.
\]

By [47 Corollary 5.24] and Hölder’s inequality, for fixed \( \xi \in S \) and \( 0 < r < 1 \), we have

\[
I_{r, \xi} = \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{q + ns} d\lambda(z)
\]

\[
\simeq \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{q + ns - n - \rho} dV_{\rho - 1}(z)
\]

\[
\leq \frac{1}{r^{ns}} \left( \int_{Q_r(\xi)} |f(z)|^{\frac{p(n+\rho)}{q}} dV_{\rho - 1}(z) \right)^{\frac{q}{n+\rho}}
\]

\[
\cdot \left( \int_{Q_r(\xi)} (1 - |z|^2)^{(q + ns - n - \rho)} \cdot \frac{n + \rho}{n + \rho - q} dV_{\rho - 1}(z) \right)^{\frac{n + \rho - q}{n + \rho}}
\]
Remark 4.4. In Proposition [4.1], in particular, putting \( \rho = q + ns - n \), we get \( A^{p+q}_{q+ns-n-1} \subseteq \mathcal{N}(p, q, s) \). Moreover, it is clear that \( \| \cdot \|_{p,q+ns-n-1} \leq \| \cdot \| \), that is, \( \mathcal{N}(p, q, s) \subseteq A^{p+q}_{q+ns-n-1} \). Combining this fact with Proposition [4.2], we have, for \( p \geq 1, q > 0 \) and \( s > \max \{0, 1 - \frac{2}{n}\} \), the following embedding relation holds

\[
A^{p+q}_{q+ns-n-1} \subseteq \mathcal{N}(p, q, s) \subseteq A^{p}_{q+ns-n-1}.
\]
4.1.2. Embedding relationship with weighted Hardy space. Recall that for \( t > 0 \), the Hardy space \( H^t \) consists of all holomorphic functions \( f \) in \( \mathbb{B} \) such that (see, e.g., [47])

\[
\|f\|_{H^t} = \sup_{0<r<1} M_t(r,f) < \infty.
\]

It is well-known that when \( 1 \leq t < \infty \), \( H^t \) is a Banach space with the norm \( \| \cdot \|_{H^t} \); if \( 0 < t < 1 \), \( H^t \) is a complete metric space.

More generally, for \( \alpha > 0 \) and \( \beta \geq 0 \), the weighted Hardy space \( H^t_{\alpha, \beta} \) is defined as follows.

\[
H^t_{\alpha, \beta} = \left\{ f \in H(\mathbb{B}) : \|f\|_{H^t_{\alpha, \beta}} = \sup_{0<r<1} (1-r)^{\beta}M_\alpha(r,f) < \infty \right\}.
\]

It is known that when \( \alpha \geq 1 \), \( H^t_{\alpha, \beta} \) is a Banach space with the norm \( \| \cdot \|_{H^t_{\alpha, \beta}} \) (see, e.g., [32]). Moreover, the little weighted Hardy space \( H^t_{\alpha,0} \) is the space of all \( f \in H^t_{\alpha, \beta} \) such that

\[
\lim_{r \to 1} (1-r)^{\beta}M_\alpha(r,f) = 0.
\]

It is easy to see that \( H^t_{\alpha,0} \subseteq H^t_{\alpha, \beta} \).

As an application of Proposition 4.1, we have the following result.

**Proposition 4.5.** Let \( p \geq 1, q > 0 \) and \( s > \max \{0,1-n/p\} \). The following statements hold.

1. If \( 0 < s < 1 \), then \( H^{\alpha}_{p,q,n} \subseteq \mathcal{N}(p,q,s) \), where \( \max \{p,\frac{nq}{q} \} \leq \alpha < \frac{1}{1-s} \);

2. If \( s \geq 1 \), then \( H^{\alpha}_{p,q,n} \subseteq \mathcal{N}(p,q,s) \), where \( \alpha \geq \max \{p,\frac{nq}{q} \} \). In particular, if \( \alpha \geq \max \{p,\frac{nq}{q} \} \), then \( H^{\alpha}_{p,q,n} \subseteq A^{-\frac{n}{q}} \).

**Proof.** The proof for (2) is a simple modification of (1) and hence we omit the proof for (2) here.

Note that for fixed \( \xi \in \mathbb{S} \) and \( 0 < r < 1 \), if \( z \in Q_r(\xi) \), by the argument in Corollary 4.3, we have

\[
1 - r < |z| < 1.
\]

We consider two different cases.

**Case I: \( \alpha = p \).** First we note that by condition, \( p = \alpha \geq \frac{nq}{q} \), which implies \( q \geq n \). For each \( \xi \in \mathbb{S} \) and \( 0 < r < 1 \), by (4.3), we have
\[ I_{r, \xi} = \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{q + ns} d\lambda(z) \]
\[ = \frac{1}{r^{ns}} \int_{\{z \in \mathbb{B} : |1 - \langle z, \xi \rangle| < r\}} |f(z)|^p (1 - |z|^2)^{q + ns - n - 1} dV(z) \]
\[ \lesssim \frac{1}{r^{ns}} \int_{1-r}^1 (1 - t^2)^{q + ns - n - 1} \left( \int_{Q(\xi, r)} |f(\gamma t)|^p d\sigma(\gamma) \right) dt \]
\[ = \frac{1}{r^{ns}} \int_{1-r}^1 (1 - t^2)^{q + ns - n - 1} \frac{(1 - t)^{q - n} M^p(t, f)}{(1-t)^{q - n}} dt \]
\[ \leq \|f\|_{H^q_{\frac{p}{2} - \frac{n}{\alpha}}}^p \cdot \frac{1}{r^{ns}} \int_{1-r}^1 (1 - t)^{ns - 1} dt \simeq \|f\|_{H^q_{\frac{p}{2} - \frac{n}{\alpha}}}^p. \]

Thus, we have
\[ \|f\|^p = \sup_{0 < r < 1, \xi \in \mathbb{S}} I_{r, \xi} \lesssim \|f\|_{H^q_{\frac{p}{2} - \frac{n}{\alpha}}}^p, \]
which implies the desired result.

Case II: \( \alpha > p \). For each \( \xi \in \mathbb{S} \) and \( 0 < r < 1 \), by Hölder inequality and \([17, \text{Lemma 4.6}] \), we have
\[ \int_{Q(\xi, r)} |f(\gamma t)|^p d\sigma(\gamma) \leq \sigma(Q(\xi, r))^{1 - \frac{p}{\alpha}} \left( \int_{Q(\xi, r)} |f(\gamma t)|^\alpha d\sigma(\gamma) \right)^{\frac{p}{\alpha}} \]
\[ \lesssim \frac{r^{n - \frac{np}{\alpha}} (1 - t)^{q - \frac{np}{\alpha}} M^p(t, f)}{(1-t)^{q - \frac{np}{\alpha}}} \]
\[ \leq \frac{r^{n - \frac{np}{\alpha}} \|f\|_{H^q_{\frac{p}{2} - \frac{n}{\alpha}}}^p}{(1-t)^{q - \frac{np}{\alpha}}}. \]

Then, by (4.3) and previous calculation, we have
\[ I_{r, \xi} \lesssim \frac{1}{r^{ns}} \int_{1-r}^1 (1 - t^2)^{q + ns - n - 1} \left( \int_{Q(\xi, r)} |f(\gamma t)|^p d\sigma(\gamma) \right) dt \]
\[ \lesssim r^{n - \frac{np}{\alpha} - ns} \|f\|_{H^q_{\frac{p}{2} - \frac{n}{\alpha}}}^p \int_{1-r}^1 (1 - t)^{q + ns - n - 1} dt \]
\[ = r^{n - \frac{np}{\alpha} - ns} \|f\|_{H^q_{\frac{p}{2} - \frac{n}{\alpha}}}^p \int_{1-r}^1 (1 - t)^{ns + \frac{np}{\alpha} - n - 1} dt \]
\[ \simeq \|f\|_{H^q_{\frac{p}{2} - \frac{n}{\alpha}}}^p. \]

Thus,
\[ \|f\|_p \simeq \sup_{0 < r < 1, \xi \in \mathbb{S}} I_{r, \xi} \lesssim \|f\|_{H^q_{\frac{p}{2} - \frac{n}{\alpha}}}^p. \]
and hence the desired result follows. □

The following result describes the behavior of Hadamard gap series in $H^\alpha_{\beta}$, whose idea comes from [19, Theorem 1].

**Proposition 4.6.** Let $\alpha > 0$, $\beta > 0$ and $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ with Hadamard gaps. Then the following statements hold true:

1. $f \in H^\alpha_{\beta}$ if and only if $\sup_{k \geq 1} \frac{L_{k,\alpha}}{n_k^\beta} < \infty$.

2. $f \in H^\alpha_{\beta,0}$ if and only if $\lim_{k \to \infty} \frac{L_{k,\alpha}}{n_k^\beta} = 0$.

Here $L_{k,\alpha} = (\int_{S} |P_{n_k}(\xi)|^\alpha d\sigma(\xi))^\frac{1}{\alpha}$.

**Proof.** (1) **Necessity.** Let $f \in H^\alpha_{\beta}$. By [30, Proposition 1.4.7] and [19, Lemma 1], we have, for each $k \in \mathbb{N}$,

\[
\int_{S} |f(r\xi)|^\alpha d\sigma(\xi) = \int_{S} \left( \int_{0}^{2\pi} |f(r\xi e^{i\theta})|^{\alpha} \frac{d\theta}{2\pi} \right) d\sigma(\xi) = \int_{S} \left( \int_{0}^{2\pi} \left| \sum_{k=0}^{\infty} P_{n_k}(r\xi e^{i\theta}) \right|^{\alpha} \frac{d\theta}{2\pi} \right) d\sigma(\xi) \geq r^{\alpha n_k} \int_{S} |P_{n_k}(\xi)|^{\alpha} d\sigma(\xi).
\]

Hence,

\[
(1 - r)^\beta r^{n_k} L_{k,\alpha} \leq (1 - r)^\beta M_\alpha(r, f) \leq \|f\|_{H^\alpha_{\beta}}.
\]

Choosing $r = 1 - \frac{1}{n_k}$ and using the well-known inequality $(1 + \frac{1}{m})^{m+1} \leq 4, m \in \mathbb{N}$, we obtain

\[
\sup_{k \in \mathbb{N}} \frac{L_{k,\alpha}}{n_k^\beta} \leq C \|f\|_{H^\alpha_{\beta}},
\]

as desired.

**Sufficiency.** Suppose that $\sup_{k \in \mathbb{N}} \frac{L_{k,\alpha}}{n_k^\beta} < \infty$. For a fixed $r \in (0, 1)$, we have

\[
\sum_{k=0}^{\infty} r^{\alpha n_k} n_k^{\alpha \beta} = \left( \sum_{k=0}^{\infty} r^{\alpha n_k} n_k^{\alpha \beta} \right) \cdot \left( \sum_{s=0}^{\infty} r^{\alpha s} \right) = \sum_{t=0}^{\infty} \left( \sum_{n_j \leq t} n_j^{\alpha \beta} \right) r^{\alpha t}.
\]

Since,

\[
\lim_{k \to \infty} \frac{k^{\alpha \beta} k!}{(\alpha \beta)(\alpha \beta + 1) \ldots (\alpha \beta + k)} = \Gamma(\alpha \beta), \alpha \beta > 0,
\]
we have
\[ \sup_{k \in \mathbb{N}} \left( \frac{k^{\alpha \beta} k!}{(k + \alpha \beta)(k + \alpha \beta - 1) \ldots (\alpha \beta + 1)} \right) \leq M, \]
where \( M \) is some positive number depending on \( \alpha \) and \( \beta \). Hence, for each \( k \geq 0 \),
\[ \frac{k^{\alpha \beta}}{(-1)^k \binom{-\alpha \beta - 1}{k}} = \frac{k^{\alpha \beta} k!}{(-1)^k (-\alpha \beta - 1)(-\alpha \beta - 2) \ldots (-\alpha \beta - k)} \]
\begin{equation}
\end{equation}
\[ = \frac{k^{\alpha \beta} k!}{(k + \alpha \beta)(k + \alpha \beta - 1) \ldots (\alpha \beta + 1)} \leq M, \]
which implies
\[ \sum_{n_j \leq t} n_j^{\alpha \beta} \leq (-1)^t \binom{-\alpha \beta - 1}{t} \frac{M c^{\alpha \beta}}{c^{\alpha \beta} - 1}. \]
Hence, by (4.8), we have
\[ \sum_{k=0}^{\infty} r^{\alpha \beta} n_k^{\alpha \beta} \leq \sum_{r=0}^{\infty} (-1)^t \binom{-\alpha \beta - 1}{t} r^{\alpha t} = \frac{1}{(1 - r^{-\alpha})^{-\alpha \beta + 1}}, \]
which implies for \( \alpha, \beta > 0 \),
\[ (1 - r^{-\alpha})^{\alpha \beta} \sum_{k=0}^{\infty} r^{\alpha \beta} n_k^{\alpha \beta} \lesssim 1. \]
Now we consider two different cases.

Case I: \( \alpha \in (0, 2] \). From (4.4), (4.9) and by using the known inequality
\[ \left( \sum_{k=1}^{\infty} a_k \right)^q \leq \sum_{k=1}^{\infty} a_k^q, \]
where \( a_k \geq 0, k \in \mathbb{N}, q \in [0, 1] \), we have that

\[
\|f\|_{H^\alpha_\beta}^\alpha \simeq \sup_{0 < r < 1} (1 - r)^{\alpha \beta} \int_S \left( \sum_{k=0}^{\infty} |P_{n_k}(\xi)|^2 r^{2n_k} \right)^{\alpha/2} d\sigma(\xi)
\]

\[
\leq \sup_{0 < r < 1} (1 - r)^{\alpha \beta} \int_S \left( \sum_{k=0}^{\infty} |P_{n_k}(\xi)|^{\alpha \beta r^{\alpha n_k}} \right) d\sigma(\xi)
\]

\[
= \sup_{0 < r < 1} (1 - r)^{\alpha \beta} \cdot \sum_{k=0}^{\infty} r^{\alpha n_k} L_{k,\alpha}^\alpha
\]

\[
\lesssim \sup_{0 < r < 1} (1 - r)^{\alpha \beta} \cdot \sum_{k=0}^{\infty} r^{\alpha n_k} n_k^{\alpha \beta}
\]

\[
\lesssim \sup_{0 < r < 1} \left( \frac{1 - r}{1 - r^\alpha} \right)^{\alpha \beta} < \infty,
\]

which implies the desired result.

**Case II**: \( \alpha > 2 \). For each \( r \in (0, 1) \), by Minkowski's inequality and (4.9), we have

\[
\left[ \int_S \left( \sum_{k=0}^{\infty} |P_{n_k}(\xi)|^2 r^{2n_k} \right)^{\alpha/2} d\sigma(\xi) \right]^{\frac{2}{\alpha}} \leq \sum_{k=0}^{\infty} \left( \int_S |P_{n_k}(\xi)|^2 r^{2n_k} \right)^{\frac{\alpha}{2}} d\sigma(\xi)
\]

\[
= \sum_{k=0}^{\infty} r^{2n_k} L_{k,\alpha}^2 \lesssim \sum_{k=0}^{\infty} r^{2n_k} n_k^{2\beta}.
\]

Thus, by (4.4) and (4.9),

\[
\|f\|_{H^\alpha_\beta}^\alpha \simeq \sup_{0 < r < 1} (1 - r)^{\alpha \beta} \int_S \left( \sum_{k=0}^{\infty} |P_{n_k}(\xi)|^2 r^{2n_k} \right)^{\alpha/2} d\sigma(\xi)
\]

\[
\leq \sup_{0 < r < 1} (1 - r)^{\alpha \beta} \left( \sum_{k=0}^{\infty} r^{2n_k} L_{k,\alpha}^2 \right)^{\frac{\alpha}{2}}
\]

\[
\lesssim \sup_{0 < r < 1} (1 - r)^{\alpha \beta} \left( \sum_{k=0}^{\infty} r^{2n_k} n_k^{2\beta} \right)^{\frac{\alpha}{2}}
\]

\[
\lesssim \sup_{0 < r < 1} \left( \frac{1 - r}{1 - r^2} \right)^{\alpha \beta} < \infty,
\]

and hence \( f \in H^\alpha_\beta \).
(2) **Necessity.** Let \( f \in H^{\alpha, \beta}_{\delta, 0} \). Then for every \( \varepsilon > 0 \), there is a \( \delta > 0 \), such that

\[
(1 - r)^\beta M_\alpha(r, f) < \varepsilon
\]

whenever \( \delta < r < 1 \). From the first inequality in (4.5) and (4.12), we have

\[
(1 - r)^\beta r^{n_k} L_{k, \alpha} < \varepsilon
\]

for each \( k \in \mathbb{N} \) and \( r \in (\delta, 1) \). Choosing \( r = 1 - \frac{1}{n_k} \), where \( n_k > \frac{1}{1-\delta} \), we obtain

\[
\frac{L_{k, \alpha}}{n_k^\beta} < 4\varepsilon.
\]

From this and since \( \varepsilon \) is an arbitrary positive number, it follows that

\[
\lim_{k \to \infty} \frac{L_{k, \alpha}}{n_k^\beta} = 0.
\]

**Sufficiency.** Suppose \( \lim_{k \to \infty} \frac{L_{k, \alpha}}{n_k^\beta} = 0 \). Take and fix a \( \varepsilon > 0 \), there is a \( k_0 \in \mathbb{N} \) such that

\[
L_{k, \alpha} < \varepsilon n_k^\beta \quad \text{for } k \geq k_0.
\]

Fix the \( k_0 \) chosen above. Then, there exists a \( \delta > 0 \), such that when \( \delta < r < 1 \),

\[
(1 - r)^\alpha \sum_{k=0}^{k_0} L_{k, \alpha}^\alpha < \varepsilon.
\]

Again, we consider two cases as the proof in part (1).

**Case I:** \( \alpha \in (0, 2] \). By (4.5) and the proof in part (1), for \( r \in (\delta, 1) \), we have

\[
(1 - r)^\alpha M_\alpha^\alpha(r, f) \simeq (1 - r)^\alpha \int_S \left( \sum_{k=0}^{\infty} |P_{n_k} (\xi)|^2 r^{2n_k} \right)^{\alpha/2} d\sigma(\xi)
\]

\[
\leq (1 - r)^\alpha \cdot \left[ \sum_{k=0}^{k_0} + \sum_{k=k_0+1}^{\infty} r^{\alpha n_k} L_{k, \alpha}^\alpha \right]
\]

\[
\leq \varepsilon + (1 - r)^\alpha \cdot \sum_{k=k_0+1}^{\infty} r^{\alpha n_k} L_{k, \alpha}^\alpha
\]

\[
\leq \varepsilon + \varepsilon (1 - r)^{\alpha \beta} \cdot \sum_{k=k_0+1}^{\infty} r^{\alpha n_k} n_k^\alpha
\]

\[
\lesssim \varepsilon \cdot \left( 1 + \sup_{0 < r < 1} \left( \frac{1 - r}{1 - r^2} \right)^{\alpha \beta} \right) \lesssim \varepsilon,
\]
which implies
\[
\lim_{r \to 1^{-}} (1 - r)^\beta M_\alpha(r, f) = 0
\]
and hence \( f \in H^\beta_{\alpha, 0} \) as desired.

**Case II:** \( \alpha > 2 \). The implication for case \( p \geq 2 \) follows similarly, from (1.10), (4.11) and the known inequality
\[
(a + b)^{\frac{a}{2}} \leq 2^{\frac{a}{2} - 1} (a^{\frac{a}{2}} + b^{\frac{a}{2}}), \quad a, b \geq 0.
\]
Hence, we omit the detail here. \( \square \)

As a corollary of Proposition 4.6, we can show that when \( \frac{q}{p} - \frac{n}{\alpha} > 0 \), the inclusion in Proposition 4.5 is strict.

**Corollary 4.7.** Let \( p, q, s \) and \( \alpha \) satisfy the condition in Proposition 4.5. If
\[
\frac{q}{p} - \frac{n}{\alpha} > 0,
\]
then the inclusion in Proposition 4.5 is strict.

**Proof.** First we take a sequence of homogeneous polynomials \( \{W_k\}_{k \in \mathbb{N}} \) satisfying deg\( (W_k) = k \),
\[
\sup_{\xi \in S} |W_k(\xi)| = 1, \quad \text{and} \quad \int_S |W_k(\xi)|^p d\sigma(\xi) \geq C(p, n) > 0.
\]
Note that since \( p \leq \alpha \), by Hölder’s inequality, we have
\[
\int_S |W_k(\xi)|^p d\sigma(\xi) \leq \left( \int_S |W_k(\xi)|^\alpha d\sigma(\xi) \right)^\frac{p}{\alpha},
\]
that is
\[
(4.13) \quad \int_S |W_k(\xi)|^\alpha d\sigma(\xi) \geq \left( \int_S |W_k(\xi)|^p d\sigma(\xi) \right)^\frac{p}{\alpha} \geq C^\frac{p}{\alpha}(p, n).
\]
Note that when \( s > 1 \), \( \mathcal{N}(p, q, s) = A^{-\frac{q}{p}} \), and hence we consider three cases.

**Case I:** \( s > 1 \). Consider the series \( f_1(z) = \sum_{k=0}^{\infty} 2^{kq} W_{2^k}(z) \), which by [7, Theorem 2.5], belongs to \( A^{-\frac{q}{p}} \). On the other hand, for each \( k \in \mathbb{N} \), by (4.13),
\[
\frac{L_{k, \alpha}}{2^{k(\frac{q}{p} - \frac{n}{\alpha})}} = \left( \int_S \left| 2^{kq} W_{2^k}(z) \right|^\alpha d\sigma(\xi) \right)^\frac{1}{\alpha} \geq C^\frac{1}{\alpha}(p, n) 2^{k\frac{q}{p}}.
\]
Hence, we have
\[
\sup_{k \geq 0} \frac{L_{k, \alpha}}{2^{k(\frac{q}{p} - \frac{n}{\alpha})}} = \infty,
\]
which implies that $f_1 \notin H^{\alpha}_{\frac{q}{p} - \frac{n}{\alpha}}$.

Case II: $s = 1$. Take any $\varepsilon \in (0, \frac{np}{\alpha})$ and consider the series $f_2(z) = \sum_{k=0}^{\infty} 2^\frac{k(q-\varepsilon)}{p} W_{2k}(z)$. On one hand, we have
\[
\sum_{k=0}^{\infty} \frac{1}{2^{\frac{ns}{p} - n}} \left| 2^\frac{k(q-\varepsilon)}{p} \right|^p = \sum_{k=0}^{\infty} \frac{1}{2^{\frac{k\varepsilon}{p}}} < \infty,
\]
which by Corollary 3.2 implies that $f_2 \in \mathcal{N}(p, q, 1)$. However, for each $k \in \mathbb{N}$, by (4.13),
\[
\frac{L_{k,\alpha}}{2^k(\frac{q}{p} - \frac{n}{\alpha})} = \left( \int_{\mathbb{S}} \left| 2^\frac{k(q-\varepsilon)}{p} W_{2k}(z) \right|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} \geq C^\frac{1}{p}(p, n) 2^k(\frac{q}{p} - \frac{n}{\alpha}).
\]
Hence, we have
\[
\sup_{k \geq 0} \frac{L_{k,\alpha}}{2^k(\frac{q}{p} - \frac{n}{\alpha})} = \infty,
\]
which implies that $f_2 \notin H^{\alpha}_{\frac{q}{p} - \frac{n}{\alpha}}$.

Case III: $0 < s < 1$. Since $\alpha < \frac{p}{1-s}$, we have $\frac{s-1}{p} + \frac{1}{\alpha} > 0$. Take any $\varepsilon \in (0, n(s - 1) + \frac{np}{\alpha})$ and consider the series $f_3(z) = \sum_{k=0}^{\infty} 2^\frac{k(s+q-n-\varepsilon)}{p} W_{2k}(z)$. On one hand, we have
\[
\sum_{k=0}^{\infty} \frac{1}{2^{\frac{kn(s+q-n) - n}{p}}} \left| 2^\frac{k(s+q-n-\varepsilon)}{p} \right|^p = \sum_{k=0}^{\infty} \frac{1}{2^{\frac{k\varepsilon}{p}}} < \infty,
\]
which by Corollary 3.2 implies that $f_3 \in \mathcal{N}(p, q, s)$. However, for each $k \in \mathbb{N}$, by (4.13),
\[
\frac{L_{k,\alpha}}{2^k(\frac{q}{p} - \frac{n}{\alpha})} = \left( \int_{\mathbb{S}} \left| 2^\frac{k(s+q-n-\varepsilon)}{p} W_{2k}(z) \right|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} \geq C^\frac{1}{p}(p, n) 2^k(\frac{q}{p} - \frac{n}{\alpha} - \frac{\varepsilon}{p}).
\]
Hence, we have
\[
\sup_{k \geq 0} \frac{L_{k,\alpha}}{2^k(\frac{q}{p} - \frac{n}{\alpha})} = \infty,
\]
which implies that $f_3 \notin H^{\alpha}_{\frac{q}{p} - \frac{n}{\alpha}}$. □
4.2. Hadamard products. An important application of those Carleson property of \( N(p, q, s) \)-type spaces is to study Hadamard product in them.

We first set up some basic notations which shall be used in this subsection. For \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{Z}_+^n, \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n \), we let

\[
\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n), \quad z^\eta = z_1^{\eta_1} \cdots z_n^{\eta_n},
\]

\[
|\eta| = \eta_1 + \cdots + \eta_n, \quad \eta! = \eta_1! \cdots \eta_n!,
\]

\[
\partial_j = \frac{\partial}{\partial_j}, \quad 1 \leq j \leq n, \quad \partial^\eta = \partial_1^{\eta_1} \cdots \partial_n^{\eta_n}.
\]

The Bloch space on the unit ball, denoted by \( \mathcal{B} \), is defined as the space of \( H(B) \) for which

\[
\|f\|_B = |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2)|\nabla f(z)| < \infty.
\]

Since \( \mathbb{B} \) is a complete Rheinhardt domain in \( \mathbb{C}^n \), i.e. \( z \in \mathbb{B} \) implies \( z \cdot \xi = (z_1 \xi_1, \ldots, z_n \xi_n) \in \mathbb{B} \) for every \( \xi \in \bar{U} \), where \( U \) is the unit polydisc in \( \mathbb{C}^n \). Then any \( f \in H(\mathbb{B}) \) has a unique power series

\[
f(z) = \sum_\eta a_\eta z^\eta, \quad z \in \mathbb{B},
\]

with \( a_\eta = \frac{\partial^\eta f(0)}{\eta!}, \eta \in \mathbb{Z}_+^n \). So \( H(\mathbb{B}) \) may be regarded as a space of multi-index sequence \( \{a_\eta\} \).

For \( f(z) = \sum_\eta a_\eta z^\eta, g(z) = \sum_\eta b_\eta z^\eta \in H(\mathbb{B}) \) and \( d > 0 \), the \( d \)-Hadamard product of \( f \) and \( g \) is defined as follows (see, e.g., [1, 21]).

\[
(f \ast g)_d(z) = \sum_\eta \omega_\eta(d)a_\eta b_\eta z^\eta,
\]

where

\[
\omega_\eta(d) = \frac{\eta! \Gamma(n + d)}{\Gamma(n + d + |\eta|)} \quad \eta \in \mathbb{Z}_+^n.
\]

Let us denote \( H(U) \) the collection of all holomorphic functions in \( U \) and \( H^\infty(U) \) the Banach space consisting of all bounded holomorphic functions in \( U \). The interesting feature of this product is that it is lying not only in \( H(\mathbb{B}) \) but also in \( H(U) \) and for \( f, g \in H(\mathbb{B}) \) and any \( 0 \leq r < 1 \),

\[
(4.14) \quad (f \ast g)_d(rz) = \langle f_r, g^*_d \rangle_d := \int_{\mathbb{B}} f(rw)g(z \cdot \overline{w})dV_{d-1}(w),
\]

for any \( z \in U \), where

\[
f_r(z) = f(rz), \quad g^*(z) = \overline{g(z)} \quad \text{and} \quad f_\xi(z) = f(\xi \cdot z), \quad \xi \in \mathbb{B},
\]
Moreover, if also \( p_3 = \infty \), i.e. \( p_2 = p'_1 = p_1/(p_1-1) \), then \( \|f * g\|_\infty \leq \|f\|_{p_1,d-1}\|g\|_{p'_1,d-1} \). In particular, \( (f * g)_d \in H^\infty(U) \) if \( f \in A_{p_1-1}^{p'_1} \) and \( g \in A_{d-1}^{p'_1} \).

**Proposition 4.9.** Let \( p, r \geq 1, q, d > 0 \) and \( s > \max \{0, 1 - \frac{q}{n} \} \). Then

1. \( (f * g)_d \in \mathcal{N}(p, q, s) \) if \( f \in A_{d-1}^r \) and \( g \in A_{d-1}^{r'} \), where \( r > 1 \) and \( r' = \frac{r}{r-1} \);
2. \( (f * g)_d \in \mathcal{N}(p, q, s) \) if \( f \in A_{d-1}^1 \) and \( g \in B \).

**Proof.** (1) Since \( s > 1 - \frac{q}{n} \), by Proposition 2.8 and Lemma 4.8, we have
\[
\|(f * g)_d\| \lesssim \|(f * g)_\infty\| \leq \|f\|_{r,d-1}\|g\|_{r',d-1} < \infty,
\]
which implies the desired result.

(2) By the proof of 21 Theorem 1], we have
\[
|\langle f * g_d(z) \rangle| \lesssim \|g\|_B\|f\|_{1,d-1},
\]
which, again, by the condition \( s > 1 - \frac{q}{n} \), implies
\[
\|(f * g)_d\| \lesssim \|(f * g)_\infty\| \lesssim \|g\|_B\|f\|_{1,d-1}.
\]
Hence the desired result follows. \( \square \)

By using the embedding relation between \( \mathcal{N}(p, q, s) \) and \( A_{p}^{k} \), we have the following result.

**Proposition 4.10.** Let \( p, q, s, \rho \) satisfy the conditions in Proposition 4.2. Then for \( r_1, r_2 \geq 1 \) with satisfying
\[
1 + \frac{q}{p(n + \rho)} = \frac{1}{r_1} + \frac{1}{r_2},
\]
we have \( (f * g)_\rho \in \mathcal{N}(p, q, s) \) if \( f \in A_{\rho-1}^{r_1} \) and \( g \in A_{\rho-1}^{r_2} \).

**Proof.** By Proposition 4.2 and Lemma 4.8, we have
\[
\|(f * g)_\rho\| \lesssim \|(f * g)_\rho\|_{\frac{p(n+\rho)}{q},(\rho-1)} \leq \|f\|_{r_1,(\rho-1)}\|g\|_{r_2,(\rho-1)},
\]
which implies the desired result. \( \square \)

**Corollary 4.11.** Let \( p \geq 1, q > 0 \) and \( s > \max \{0, 1 - \frac{q}{n} \} \). Then
\( (f * g)_{q+ns-n} \in \mathcal{N}(p, q, s) \) if \( f \in \mathcal{N}(p, q, s) \) and \( g \in A_{q+ns-n-1}^{p(q+ns-n)} \).
Proof. By Proposition 4.10, we have

\[ \| (f \ast g)(q+ns-n) \| \lesssim \| f \|_{p, (q+ns-n-1)} \| g \|_{p(q+ns-n), q+ns-n-1} \]

\[ \lesssim \| f \| \| g \|_{p(q+ns-n), q+ns-n-1}. \]

Hence, we get the desired result. \( \square \)

The following lemma gives an estimation of the term \( M_\alpha(r, (f \ast g)_d) \), which gives us another description of Hadamard products via the embedding relation between \( \mathcal{N}(p, q, s) \) and \( H_\beta^\alpha \).

Lemma 4.12. Let \( \alpha \geq 1, r \in [0, 1), d > 0 \) and \( f, g \in H(\mathbb{B}) \). Then

\[ M_\alpha(r, (f \ast g)_d) \leq M_\alpha(\sqrt{r}, g)\| f \|_{1, d-1}. \]

Proof. Without the loss of generality, we assume that \( \| f \|_{1, d-1} < \infty \). Thus, by Proposition 3.1, (i) and the fact that the integral mean of subharmonic function over sphere is an increasing function of multi-radius, we have

\[
M_\alpha(r, (f \ast g)_d) = \left( \int_S |(f \ast g)_d(r, \xi)|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} \\
= \left( \int_S |(f \ast g)_d(\sqrt{r} \cdot \sqrt{r} \xi)|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} \\
= \left( \int_B \int_B f(\sqrt{r} w) g(\sqrt{r} \xi \cdot \overline{w}) dV_{d-1}(w) \right)^{\frac{\alpha}{2}} \\
\leq \int_B |f(\sqrt{r} w)| \left( \int_S |g(\sqrt{r} \xi \cdot \overline{w})|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} dV_{d-1}(w) \\
\quad \text{(By Minkowski’s inequality)} \\
\leq \int_B |f(\sqrt{r} w)| \left( \int_S |g(\sqrt{r} \xi \cdot \overline{w})|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} dV_{d-1}(w) \\
\quad \text{(Since } |g(z)|^\alpha \text{ is subharmonic)} \\
\leq M_\alpha(\sqrt{r}, g)\| f \|_{1, d-1}.
\]

\( \square \)

Proposition 4.13. Let \( d > 0 \) and \( p, q, s, \alpha \) satisfy the conditions in Proposition 4.5. Then we have \( (f \ast g)_d \in \mathcal{N}(p, q, s) \) if \( f \in A_{d-1}^1 \) and \( g \in H_{\frac{p}{q}}^\alpha \).

\[ \mathcal{N}(p, q, s) \text{-TYPE SPACES IN THE UNIT BALL OF } \mathbb{C}^n \]
Proof. For any \( f \in A^1_{d-1} \) and \( g \in H^\alpha (B) \), by Proposition 4.5 and Lemma 4.12 we have

\[
\| (f \ast g)_d \| \lesssim \sup_{0 < r < 1} (1 - r)^{\frac{2n}{p} - \frac{n}{\alpha}} M_\alpha (r, (f \ast g)_d) \\
\leq \| f \|_{1,d-1} \cdot \sup_{0 < r < 1} (1 - r)^{\frac{2n}{p} - \frac{n}{\alpha}} M_\alpha (\sqrt{r}, g) \\
\lesssim \| f \|_{1,d-1} \cdot \sup_{0 < r < 1} (1 - \sqrt{r})^{\frac{2n}{p} - \frac{n}{\alpha}} M_\alpha (\sqrt{r}, g) \\
= \| f \|_{1,d-1} \cdot \| g \|_{H^\alpha (B)}^{\frac{2n}{p} - \frac{n}{\alpha}}.
\]

which implies \((f \ast g)_d \in \mathcal{N}(p, q, s)\). \qed

4.3. Random power series. A second application of the above characterization of \( \mathcal{N}(p, q, s) \)-type spaces by Carleson measure is to study the random power series. The behavior of the random power series on \( Q_s \) spaces was studied in [15].

Let \( \{ \varepsilon_\alpha(w) \} \) be a Bernoulli sequence of random variables on a probability space \((\Omega, \mathcal{A}, P)\). In particular, this sequence is independent, and each \( \varepsilon_\alpha(w) \) takes the values 1 and \(-1\) with probability \( \frac{1}{2} \) each. A well-known example of such a Bernoulli sequence is the Rademacher functions, which are defined as

\[
\{ r_j(t) \}_{j \in \mathbb{N}} = \{ \text{sgn}(\sin(2^j \pi t)) \}_{j \in \mathbb{N}}, \quad t \in [0, 1].
\]

It is easy to check that the \( r_j \)'s are mutually independent random variables on \([0, 1]\). We refer the readers to the excellent book [5] for some details for Rademacher functions. For \( f \) a holomorphic function in \( B \) with Taylor expansion \( f(z) = \sum_\alpha a_\alpha z^\alpha \), the randomization of \( f \) is defined as

\[
f_\omega(z) = \sum_\alpha \varepsilon_\alpha(\omega) a_\alpha z^\alpha.
\]

The following result gives a sufficient condition for \( f_\omega \) belonging to the \( \mathcal{N}(p, q, s) \)-type spaces.

**Proposition 4.14.** Let \( p \geq 1, q > 0, s > \max \{0, 1 - \frac{2}{n}\} \), \( \rho \) be some positive number satisfying

\[
\begin{cases}
\max\{0, q - n\} < \rho < \frac{q + ns - n}{1 - s}, & 0 < s < 1; \\
\rho > \max\{0, q - n\}, & s \geq 1
\end{cases}
\]

and \( k = \frac{p(n + \rho)}{q} \). Let further, \( f(z) = \sum_\alpha a_\alpha z^\alpha \in H(B) \). If \( \{ |a_\alpha| \omega_{\alpha,k} \} _\alpha \in \ell^{\min\{2,k\}} \),
then \( f_\omega \in \mathcal{N}(p, q, s) \) for almost every \( \omega \in \Omega \), where for each multi-index \( \alpha \),

\[
\omega_{\alpha, k} = \left( \int_{\mathbb{S}} |\xi^\alpha|^k d\sigma(\xi) \right)^{\frac{1}{k}}.
\]

**Proof.** By Proposition 4.2, it suffices to show that \( f_\omega \in A^{k}_{p-1} \) for almost every \( \omega \in \Omega \), that is, it suffices to show that

\[
P \left\{ \omega \in \Omega : \int_0^1 M_k^k(r, f_\omega)(1 - r)^{p-1} dr < \infty \right\} = 1.
\]

Indeed, by Fubini’s theorem, we have

\[
E \left( \int_0^1 M_k^k(r, f_\omega)(1 - r)^{p-1} dr \right)
\]

\[
= \int_0^1 (1 - r)^{p-1} \left[ \int_\Omega \int_{\mathbb{S}} |f_\omega(r\xi)|^k d\sigma(\xi) dP \right] dr
\]

\[
= \int_0^1 (1 - r)^{p-1} \left[ \int_{\mathbb{S}} \int_\Omega |f_\omega(r\xi)|^k dP d\sigma(\xi) \right] dr
\]

\[
= \int_0^1 (1 - r)^{p-1} \left[ \int_{\mathbb{S}} \int_\Omega \left| \sum_{\alpha} \varepsilon_{\alpha}(\omega) a_\alpha r^{|\alpha|} \xi^\alpha \right|^k dP d\sigma(\xi) \right] dr
\]

\[
\lesssim \int_0^1 (1 - r)^{p-1} \left[ \int_{\mathbb{S}} \left( \sum_{\alpha} |a_\alpha|^2 r^{2|\alpha|} |\xi^\alpha|^2 \right)^{\frac{k}{2}} d\sigma(\xi) \right] dr,
\]

where the last inequality follows from Khintchine’s inequality.

**Case I:** \( 0 < k \leq 2 \). Since \( \frac{k}{2} \leq 1 \), we have

\[
E \left( \int_0^1 M_k^k(r, f_\omega)(1 - r)^{p-1} dr \right)
\]

\[
\lesssim \int_0^1 (1 - r)^{p-1} \left[ \int_{\mathbb{S}} \left( \sum_{\alpha} |a_\alpha|^k r^{k|\alpha|} |\xi^\alpha|^k \right) d\sigma \right] dr
\]

\[
\lesssim \sum_{\alpha} |a_\alpha|^k \omega_{\alpha, k}^k
\]

\[
< \infty.
\]
Case II: $k > 2$. By Minkowski’s inequality, we have

$$E \left( \int_0^1 M_k^p(r, f_\omega)(1 - r)^{p-1} dr \right)$$

$$\lesssim \int_0^1 (1 - r)^{p-1} \left[ \sum_\alpha \left( \int_S |a_\alpha|^k r^k |\xi^\alpha|^k d\sigma(\xi) \right)^{\frac{q}{k}} \right]^{\frac{1}{q}} dr$$

$$= \int_0^1 (1 - r)^{p-1} \left( \sum_\alpha \{|a_\alpha|^2 r^2 |\omega_{\alpha,k}^2| \}^{\frac{k}{q}} \right) dr$$

$$\leq \left( \sum_\alpha |a_\alpha|^2 \omega_{\alpha,k}^2 \right)^{\frac{k}{q}} < \infty.$$ 

Thus, for both cases, we have

$$E \left( \int_0^1 M_k^p(r, f_\omega)(1 - r)^{p-1} dr \right) < \infty,$$

which implies (4.15) clearly.

Conversely, we have the following result.

**Proposition 4.15.** Let $p \geq 1, q > 0$ and $s > \max \{0, 1 - \frac{q}{n}\}$. Let further, $f(z) = \sum_\alpha a_\alpha z^\alpha \in H(\mathbb{B})$. If there exists some $a_0, \epsilon > 0$ such that for any $a > a_0$, the following decay condition

$$P\{\omega : \|f_\omega\| > a\} \lesssim a^{-1-\epsilon}$$

holds, then

$$\left\{ |a_\alpha| w_{\alpha,p} B_\tau(2n + p|\alpha|, q + ns - n) \right\}_{\alpha} \in \ell^\infty.$$ 

Here $B(\cdot, \cdot)$ is the Beta function.

**Proof.** By Remark [4.4] it is known that $N(p, q, s) \subseteq A_{q+ns-n-1}^p$ and hence $f_\omega \in A_{q+ns-n-1}^p$ for almost every $\omega \in \Omega$. Thus, for any $a > a_0$, we have

$$P \left\{ w : \int_0^1 r^{2n-1}(1 - r)^{q+ns-n-1} M_p^p(r, f_\omega) dr > a \right\} \lesssim a^{-1-\epsilon},$$

which clearly implies $E(f_\omega) < \infty$ by writing the expectation into an integration with respect to distribution function. Thus, for each $\alpha$, we
have

\[ \infty > E \left( \int_0^1 r^{2n-1}(1 - r)^{q + ns - n - 1} M_p(r, f_\omega) dr \right) \]

\[ = \int_0^1 r^{2n-1}(1 - r)^{q + ns - n - 1} \left[ \int_{\mathbb{S}} \int_{\Omega} |f_\omega(r_\xi)|^p dP d\sigma(\xi) \right] dr \]

\[ = \int_0^1 r^{2n-1}(1 - r)^{q + ns - n - 1} \left[ \int_{\mathbb{S}} \int_{\Omega} \sum_\alpha \varepsilon_\alpha(\omega) a_\alpha r^{|\alpha|} |\xi_\alpha|^p dP d\sigma(\xi) \right] dr \]

\[ \geq \int_0^1 r^{2n-1}(1 - r)^{q + ns - n - 1} \left[ \int_{\mathbb{S}} \left( \sum_\alpha |a_\alpha|^2 r^{|\alpha|} |\xi_\alpha|^2 \right)^{\frac{p}{2}} d\sigma(\xi) \right] dr \]

(by Khintchine’s inequality)

\[ \geq \int_0^1 r^{2n-1}(1 - r)^{q + ns - n - 1} \left[ \int_{\mathbb{S}} |a_\alpha|^p w_{\alpha, p}^p B(2n + p|\alpha|, q + ns - n) \right] dr \]

which implies the desired result. \( \square \)

5. Characterizations of \( N(p, q, s) \)-type spaces

Recall that by Proposition 4.1, \( f \) belongs to \( N(p, q, s) \) is equivalent to \( d\mu = d\mu_{f, p, q, s}(z) = |f(z)|^p (1 - |z|^2)^{q + ns} d\lambda(z) \) is an \( (ns) \)-Carleson measure. In this section, we extend this result and establish several characterizations of the \( N(p, q, s) \)-norm.

5.1. Various derivative characterizations. Let us recall several notations first. For \( f \in H(\mathbb{B}), z \in \mathbb{B} \). Let

\[ \nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right) \]

denote the complex gradient of \( f \) and let \( \tilde{\nabla} f \) denote the invariant gradient of \( \mathbb{B} \), i.e., \( (\nabla f)(z) = \nabla (f \circ \Phi_z)(0) \). Moreover, we write

\[ Rf(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z) \]

as the radial derivative of \( f \) (see, e.g., [47]) and for \( 1 \leq i, j \leq n \),

\[ T_{i,j} f(z) = \bar{z}_j \frac{\partial f}{\partial z_i} - \bar{z}_i \frac{\partial f}{\partial z_j} \]

as the tangential derivative of \( f \) (see, e.g., [12]). We need the following lemmas.
Lemma 5.1. Let $\xi \in S$ and $0 < \delta < 1$. Then there exists some $M > 0$, which is independent of $\delta$, such that
\[ \bigcup_{z \in Q_\delta(\xi)} D\left(z, \frac{1}{4}\right) \subseteq Q_{M\delta}(\xi). \]

Proof. Take $w \in \bigcup_{z \in Q_\delta(\xi)} D\left(z, \frac{1}{4}\right)$, which implies $w \in D\left(z, \frac{1}{4}\right)$ for some $z \in Q_\delta(\xi)$. Since $|\Phi_w(z)| < \frac{1}{4}$, by [17, Proposition 1.21 and Lemma 2.20], there exists some $M' > 0$ independent of $z$ and $w$, such that
\[
\frac{1}{M'} \leq \frac{1 - |w|^2}{1 - |z|^2} \leq M'.
\]
Hence, we have
\[
|1 - \langle w, \xi \rangle|^{1/2} \leq |1 - \langle w, z \rangle|^{1/2} + |1 - \langle z, \xi \rangle|^{1/2}
\leq \delta^{1/2} + \left(\frac{(1 - |z|^2)(1 - |w|^2)}{1 - |\Phi_w(z)|^2}\right)^{1/4}
\leq \delta^{1/2} + 2M'^{1/4}\delta^{1/2}.
\]
The result follows by taking $M = (1 + 2M'^{1/4})^2$.

Lemma 5.2. For $f \in H(\mathbb{B})$, there exists a constant $C > 0$, such that
\[
|Rf(z)| \leq \frac{C}{(1 - |z|^2)^{1/2}} \int_{D\left(z, \frac{1}{4}\right)} \sum_{i<j} |T_{i,j}f(w)|d\lambda(w), \ \forall z \in \mathbb{B}.
\]

Proof. The proof of this lemma is a simple modification of [11, Lemma 2] and hence we omit it here.

We have the following result.

Theorem 5.3. Let $f \in H(\mathbb{B})$ and $p \geq 1, q > 0$ and $s > \max \{0, 1 - \frac{q}{p}\}$. The following statements are equivalent:

1. $f \in \mathcal{N}(p,q,s)$ or equivalently, $d\mu_1 = |f(z)|^p(1 - |z|^2)^{q+ns}d\lambda(z)$ is an $(ns)$-Carleson measure;
2. $d\mu_2 = |\nabla f(z)|^p(1 - |z|^2)^{p+q+ns}d\lambda(z)$ is an $(ns)$-Carleson measure;
3. $d\mu_3 = |\nabla f(z)|^p(1 - |z|^2)^{q+ns}d\lambda(z)$ is an $(ns)$-Carleson measure;
4. $d\mu_4 = |\nabla f(z)|^p(1 - |z|^2)^{p+q+ns}d\lambda(z)$ is an $(ns)$-Carleson measure;
5. $d\mu_5 = \left(\sum_{i<j} |T_{i,j}f(z)|^p(1 - |z|^2)^{q+ns}\right)^{1/p}\lambda^s\lambda^{1/2}d\lambda(z)$ is an $(ns)$-Carleson measure.
\textit{Proof.} Note that the equivalence between (1) and (2) follows from [41, Theorem 3.2] and [44, Theorem 45]. Moreover, since

\[(1 - |z|^2)|Rf(z)| \leq (1 - |z|^2)|\nabla f(z)| \leq \tilde{\nabla} f(z), \quad z \in \mathbb{B},\]

it is clear that (3) \(\implies\) (2) \(\implies\) (4). Furthermore, the identity

\[|z|^2|\tilde{\nabla} f(z)|^2 = (1 - |z|^2) \left( (1 - |z|^2) |Rf(z)|^2 + \sum_{i<j} |T_{i,j} f(z)|^2 \right)\]

implies that (3) follows from (4) and (5). Therefore, it suffices to show the equivalence between (4) and (5).

(5) \(\implies\) (4). Suppose \(d\mu_5\) is an \((ns)\)-Carleson measure. First we note that for any \(z \in \mathbb{B}\),

\[(5.2) \quad \int_{D(z,\frac{1}{4})} d\lambda(w) < K,\]

for some \(K\) independent of the choice of \(z\). Indeed, by [30, 2.2.7], we have

\[
\int_{D(z,\frac{1}{4})} d\lambda(w) = \int_{D(z,\frac{1}{4})} \frac{1}{(1 - |w|^2)^{n+1}} dV(w) \\
\approx \frac{1}{(1 - |z|^2)^{n+1}} \int_{D(z,\frac{1}{4})} dV(w) \quad \text{by (5.1)} \\
\approx \frac{1}{(1 - |z|^2)^{n+1}} \cdot (1 - |z|^2)^{n+1} = 1.
\]

By Lemmas 5.1 and 5.2 (5.1) and (5.2), for any \(\xi \in S\) and \(\delta \in (0,1)\),

\[
\int_{Q_\delta(\xi)} |Rf(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) \\
\leq \int_{Q_\delta(\xi)} \left( \int_{D(z,\frac{1}{4})} \sum_{i<j} |T_{i,j} f(w)| d\lambda(w) \right)^p (1 - |z|^2)^{\frac{p}{2}+q+ns} d\lambda(z) \\
\leq \int_{Q_\delta(\xi)} \sum_{i<j} \left( \int_{D(z,\frac{1}{4})} |T_{i,j} f(w)| d\lambda(w) \right)^p (1 - |z|^2)^{\frac{p}{2}+q+ns} d\lambda(z)
\]
\[
\leq \int_{Q_s(\xi)} \left( \sum_{i<j} \int_{D(z, \frac{1}{4})} |T_{i,j}f(w)|^p d\lambda(w) \right) (1 - |z|^2)^{\frac{p}{2} + q + ns} d\lambda(z)
\]
\[
\leq \int_B \sum_{i<j} |T_{i,j}f(w)|^p \chi_{\cup Q_s(\xi)} D(\frac{z}{4}) (w) \times \\
\left( \int_{Q_s(\xi)} \chi_D \left( z, \frac{1}{4} \right) \left( 1 - |z|^2 \right)^{\frac{p}{2} + q + ns} d\lambda(z) \right) d\lambda(w)
\]
\[
\leq \int_B (1 - |w|^2)^{\frac{p}{2} + q + ns} \sum_{i<j} |T_{i,j}f(w)|^p \chi_{\cup Q_s(\xi)} D(\frac{z}{4}) (w) \times \\
\left( \int_{Q_s(\xi)} \chi_D \left( z, \frac{1}{4} \right) d\lambda(z) \right) d\lambda(w)
\]
\[
\leq \int_{Q_{\delta s}(\xi)} \left( 1 - |w|^2 \right)^{\frac{p}{2} + q + ns} \sum_{i<j} |T_{i,j}f(w)|^p d\lambda(w)
\]
\[
\leq \int_{Q_{\delta s}(\xi)} \left( 1 - |w|^2 \right)^{\frac{p}{2} + q + ns} \sum_{i<j} |T_{i,j}f(w)|^p d\lambda(w)
\]
\[
\leq \delta^{ns} \quad \text{(since } d\mu_5 \text{ is an } (ns)\text{-Carleson measure)}.
\]

Hence, we get the desired result.

(4) \implies (5). Suppose \( d\mu_4 \) is an \((ns)\)-Carleson measure. From
\[
f(z) - f(0) = \int_0^1 \frac{d}{dt} f(tz) dt = \int_0^1 \frac{Rf(tz)}{t} dt,
\]
we see that for \( 1 \leq i, j \leq n \),
\[
T_{i,j}f(z) = \int_0^1 T_{i,j}(Rf(tz)) \frac{dt}{t} = \int_0^1 \frac{(T_{i,j}f)(tz)}{t} dt.
\]
Hence, it suffices to prove for each \( 1 \leq i, j \leq n \),
\[
\left| \int_0^1 \frac{(T_{i,j}f)(tz)}{t} dt \right|^p (1 - |z|^2)^{\frac{p}{2} + q + ns} d\lambda(z)
\]
is an \((ns)\)-Carleson measure.

Note that by [47 Corollary 5.24] and the fact that \( \frac{p}{2} + q + ns - n - 1 > -1 \), we have, for any \( \xi \in \mathbb{S} \) and \( 0 < \delta < 1 \),
\[
\int_{Q_s(\xi)} \left| \int_0^{1/2} \frac{(T_{i,j}f)(tz)}{t} dt \right|^p (1 - |z|^2)^{\frac{p}{2} + q + ns} d\lambda(z)
\]
\[
\leq \int_{Q_s(\xi)} \left( 1 - |z|^2 \right)^{\frac{p}{2} + q + ns - n - 1} dV(z) \sim r^{\frac{p}{2} + q + ns} \leq r^{ns}
\]
Thus, we only need to show for $1 \leq i, j \leq n$,

\[
\left( \int_{1/2}^{1} |(T_{i,j}Rf)(tz)|dt \right)^p (1 - |z|^2)^{\frac{p}{2} + q + ns}d\lambda(z)
\]

is an $(n.s)$-Carleson measure.

By the proof of [11, Theorem 1], for any $\gamma \geq 0$, we have

\[
(5.3) \quad \int_{1/2}^{1} |(T_{i,j}Rf)(tw)|dt \lesssim \int_{B} \frac{(1 - |z|^2)^{\gamma}|Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma+\frac{2}{p}}}dV(z), \ w \in \mathbb{B}.
\]

Now we consider two cases.

**Case I:** $p > 1$. Let $p'$ be the conjugate of $p$. Take and fix two positive numbers $\gamma$ and $\rho$, such that

\[
\max \left\{ 0, 1 - \frac{p'}{2} \right\} < 2p' \rho < 1, \ \gamma > \max \{ (p + p')\rho, p + q + ns - n - 1 - pp \}.
\]

Then for $w \in \mathbb{B}$, by [17, Theorem 1.12], we have

\[
\begin{align*}
&\left( \int_{B} \frac{(1 - |z|^2)^{\gamma}|Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma+\frac{2}{p}}}dV(z) \right)^p \\
= &\left( \int_{B} \frac{(1 - |z|^2)^{\frac{\gamma}{p}}(1 - |z|^2)^{\frac{1}{p'}}|Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma+\frac{2}{p}}}dV(z) \right)^p \\
= &\left( \int_{B} \frac{(1 - |z|^2)^{\frac{\gamma}{p}+\rho}(1 - |z|^2)^{\frac{1}{p'}-\rho}|Rf(z)|}{|1 - \langle z, w \rangle|^{\frac{np}{p'p}+\rho+\frac{1}{2}}}dV(z) \right)^p \\
\leq &\left( \int_{B} \frac{(1 - |z|^2)^{\frac{\gamma}{p}+pp}|Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma-pp}}dV(z) \right)^p \left( \int_{B} \frac{(1 - |z|^2)^{\gamma-pp}}{|1 - \langle z, w \rangle|^{n+\gamma+pp+\frac{2}{p'}}}dV(z) \right)^{p/p'} \\
\geq &\left( \int_{B} \frac{(1 - |z|^2)^{\gamma+pp}|Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma-pp}}dV(z) \right)^p (1 - |w|^2)^{1-2p'p - \frac{2}{p'}}^{p-1} \\
= &\left( \int_{B} \frac{(1 - |z|^2)^{\gamma+pp}|Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma-pp}}dV(z) \right)^p (1 - |w|^2)^{\frac{2}{p'-pp} - 2p'p - 1}.
\end{align*}
\]
Thus, for any $\xi \in S$ and $rl \in (0, 1)$,
\[
\int_{Q_r(\xi)} \left( \int_{1/2}^1 \left| (T_{i,j}Rf)(tw) \right| dt \right)^p (1 - |w|^2)^{\frac{p}{2} + q + ns} d\lambda(w)
\]
\[
\geq \int_{Q_r(\xi)} \left( \int_{B} \frac{(1 - |z|^2)^{\gamma + pp}|Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma-pp}} dV(z) \right)^p (1 - |w|^2)^{\frac{p}{2} + q + ns} d\lambda(w)
\]
\[
\geq \int_{Q_r(\xi)} \left( \int_{B} \frac{(1 - |z|^2)^{\gamma + pp}|Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma-pp}} dV(z) \right)^p (1 - |w|^2)^{p+q+ns-2pp-1} d\lambda(w)
\]
\[
+ \sum_{j=1}^{\infty} \int_{Q_r(\xi)} \left\{ \int_{Q_{1/2r+1,\xi}(\xi) \setminus Q_{1/2r,\xi}(\xi)} \frac{(1 - |z|^2)^{\gamma + pp}|Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma-pp}} dV(z) \right\} (1 - |w|^2)^{p+q+ns-2pp-1} d\lambda(w)
\]
\[= I_1 + I_2.
\]
- **Estimation of $I_1$.**
Since $2p'\rho < 1$ and $s > 1 - \frac{2}{n}$, it follows that
\[p + q + ns - n - 2pp - 2 > p - 2p\rho - 2 > -1.
\]
Hence, by [47, Theorem 1.12] and Fubini’s theorem, we have
\[
I_1 \leq \int_{Q_{2r}(\xi)} \frac{(1 - |z|^2)^{\gamma + pp}|Rf(z)|^p}{|1 - \langle z, w \rangle|^{n+\gamma-pp}} dV(z)
\]
\[
\leq \int_{Q_{2r}(\xi)} \frac{(1 - |z|^2)^{\gamma + pp}|Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma-pp}} dV(z)
\]
\[
\leq \int_{Q_{2r}(\xi)} |Rf(z)|^p (1 - |z|^2)^{\gamma + pp + q + ns - \gamma - pp - n - 1} dV(z)
\]
\[
= \int_{Q_{2r}(\xi)} |Rf(z)|^p (1 - |z|^2)^{\gamma + q + ns} d\lambda(z)
\]
\[
\leq r^{ns}.
\]
- **Estimation of $I_2$.**
First we note that for \( w \in Q_r(\xi) \) and
\[
z \in Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi), j \in \mathbb{N},
\]
we have
\[
|1 - \langle z, w \rangle|^{1/2} \geq |1 - \langle z, \xi \rangle|^{1/2} - |1 - \langle w, \xi \rangle|^{1/2} \geq (2^{j/2} - 1)r^{1/2},
\]
which implies for \( w \in Q_r(\xi) \) and \( z \in Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi), j \in \mathbb{N} \), we have \( |1 - \langle z, w \rangle| \geq \frac{2^{j/2}r}{100} \). Since \( \gamma + p\rho > p + q + ns - n - 1 \), we put \( \beta = \gamma + pp - p - q - ns + n + 1 > 0 \), and hence \( n + \gamma - pp = p + q + ns + \beta - 2pp - 1 \). Moreover, since \( 2p'\rho < 1 \), we have \( 2pp + 1 - p - q < 0 \).

Thus, using the fact that \( p + q + ns - 2pp - n - 2 > -1 \) and [17, Corollary 5.24], it follows that

\[
I_2 = \sum_{j=1}^{\infty} \left\{ \int_{Q_r(\xi)} \int_{Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi)} \frac{(1 - |z|^2)^{\beta}}{|\langle z, w \rangle|^{\beta}} \left( \frac{(1 - |z|^2)^{p+q+ns-n-1} |Rf(z)|^p}{|1 - \langle z, w \rangle|^{p+q+ns-2pp-1}} dV(z) (1 - |w|^2)^{p+q+ns-2pp-1} d\lambda(w) \right) \right\}
\]

\[
\geq \sum_{j=1}^{\infty} \left\{ (2^j r)^{2pp+1-p-q-ns} \int_{Q_r(\xi)} (1 - |w|^2)^{p+q+ns-2pp-1} \left( \int_{Q_{2^{j+1}r}(\xi)} (1 - |z|^2)^{p+q+ns} |Rf(z)|^p d\lambda(z) \right) d\lambda(w) \right\}
\]

\[
\geq \sum_{j=1}^{\infty} \left\{ (2^j r)^{2pp+1-p-q-ns} (2^{j+1})^{ns} \int_{Q_r(\xi)} (1 - |w|^2)^{p+q+ns-2pp-1} d\lambda(z) \right\}
\]

\[
\geq \sum_{j=1}^{\infty} (2^j r)^{2pp+1-p-q-ns} (2^{j+1})^{ns} \int_{Q_r(\xi)} (1 - |w|^2)^{p+q+ns-2pp-1} d\lambda(z)
\]

\[
\geq \left( \sum_{j=1}^{\infty} 2^j (2pp+1-p-q) \right) r^{ns} \lesssim r^{ns}.
\]

Finally, combining the estimations of \( I_1 \) and \( I_2 \), we get the desired result.
Case II: $p = 1$.
Take and fix a positive number $\gamma$, such that $\gamma > q + ns - n$, and put $\beta = \gamma - q - ns + n$, which implies that

$$n + \gamma + \frac{1}{2} = \beta + q + ns + \frac{1}{2}.$$ 

For any $\xi \in \mathbb{S}$ and $r \in (0, 1)$, we have

$$
\int_{Q_r(\xi)} \left( \int_{1/2}^1 |(T_{i,j} Rf)(tw)| dt \right) (1 - |w|^2)^{\frac{1}{2} + q + ns} d\lambda(w) \\
\lesssim \int_{Q_r(\xi)} \left( \int_{\mathbb{S}} \frac{(1 - |z|^2)^\gamma |Rf(z)|}{|1 - \langle z, w \rangle|^{n + \gamma + \frac{1}{2}}} dV(z) \right) (1 - |w|^2)^{\frac{1}{2} + q + ns} d\lambda(w) \\
= \int_{Q_r(\xi)} (1 - |w|^2)^{\frac{1}{2} + q + ns} \left( \int_{Q_{2r}(\xi)} \frac{(1 - |z|^2)^\gamma |Rf(z)|}{|1 - \langle z, w \rangle|^{n + \gamma + \frac{1}{2}}} dV(z) \right) d\lambda(w) \\
+ \sum_{j=1}^{\infty} \int_{Q_r(\xi)} \left\{ \left( \int_{Q_{2^j r}(\xi) \setminus Q_{2^{j-1}}(\xi)} \frac{(1 - |z|^2)^\gamma |Rf(z)|}{|1 - \langle z, w \rangle|^{n + \gamma + \frac{1}{2}}} dV(z) \right) (1 - |w|^2)^{\frac{1}{2} + q + ns} d\lambda(w) \right\} \\
= J_1 + J_2.
$$

- **Estimation of $J_1$.**
By Fubini’s theorem and \[47\] Theorem 1.12], we have

$$J_1 = \int_{Q_r(\xi)} (1 - |w|^2)^{q + ns - n - \frac{1}{2}} \left( \int_{Q_{2r}(\xi)} \frac{(1 - |z|^2)^\gamma |Rf(z)|}{|1 - \langle z, w \rangle|^{n + \gamma + \frac{1}{2}}} dV(z) \right) dV(w) \\
\lesssim \int_{Q_{2r}(\xi)} |Rf(z)|(1 - |z|^2)^\gamma \left( \int_{\mathbb{S}} \frac{(1 - |w|^2)^{q + ns - n - \frac{1}{2}}}{|1 - \langle z, w \rangle|^{n + \gamma + \frac{1}{2}}} dV(w) \right) dV(z) \\
\approx \int_{Q_{2r}(\xi)} |Rf(z)|(1 - |z|^2)^\gamma (1 - |z|^2)^{q + ns - n - \gamma} dV(z) \\
= \int_{Q_{2r}(\xi)} |Rf(z)|(1 - |z|^2)^{1 + q + ns} d\lambda(z) \lesssim r^{ns}.
$$

- **Estimation of $J_2$.**
By our choice of $\gamma$ and [47 Corollary 5.24], we have

$$J_2 = \sum_{j=1}^{\infty} \int_{Q_r(\xi)} \int_{Q_{2^j+1}(\xi)\setminus Q_{2^j}(\xi)} \frac{(1 - |z|^2)^\beta}{1 - \langle z, w \rangle^{q+ns+\frac{q}{2}}} dV(z) (1 - |w|^2)^{\frac{1}{2} + q + ns} d\lambda(w)$$

$$\lesssim \sum_{j=1}^{\infty} \left\{ (2^j r)^{-q - ns - \frac{q}{2}} \int_{Q_r(\xi)} (1 - |w|^2)^{\frac{1}{2} + q + ns} \right\}$$

$$\lesssim \sum_{j=1}^{\infty} (2^j r)^{-q - ns - \frac{q}{2}} (2^j + 1)^{ns} \int_{Q_r(\xi)} (1 - |w|^2)^{\frac{1}{2} + q + ns} d\lambda(w)$$

$$\lesssim \sum_{j=1}^{\infty} (2^j r)^{-q - ns - \frac{q}{2}} (2^j + 1)^{ns} r^{\frac{1}{2} + q + ns} \lesssim r^{ns}.$$

The desired result follows from the estimations on $J_1$ and $J_2$. □

Correspondingly, we have the following result for $N^0(p, q, s)$ spaces.

**Theorem 5.4.** Let $f \in H(\mathbb{B})$, $p \geq 1$, $q > 0$ and $s > \max \{0, 1 - \frac{2}{n}\}$. The following statements are equivalent:

1. $f \in N^0(p, q, s)$ or equivalently, $d\mu_1 = |f(z)|^p (1 - |z|^2)^{q + ns} d\lambda(z)$ is a vanishing $(ns)$-Carleson measure;

2. $d\mu_2 = |\nabla f(z)|^p (1 - |z|^2)^{p + q + ns} d\lambda(z)$ is a vanishing $(ns)$-Carleson measure;

3. $d\mu_3 = |\overline{\nabla} f(z)|^p (1 - |z|^2)^{q + ns} d\lambda(z)$ is a vanishing $(ns)$-Carleson measure;

4. $d\mu_4 = |R f(z)|^p (1 - |z|^2)^{p + q + ns} d\lambda(z)$ is a vanishing $(ns)$-Carleson measure;

5. $d\mu_5 = \left( \sum_{i<j} |T_{i,j} f(z)|^p \right) (1 - |z|^2)^{\frac{p}{2} + q + ns} d\lambda(z)$ is a vanishing $(ns)$-Carleson measure.

**Proof.** The proof of this theorem is a simple modification of Theorem [5.3] and hence we omit it here. □
Moreover, combining Theorem 5.3, [44, Theorem 45] and [41, Theorem 3.2], we have the following characterizations of $N(p, q, s)$-norm. In particular, we have:

**Corollary 5.5.** Let $f \in H(B), p \geq 1, q > 0$ and $s > \max\{0, 1 - \frac{2}{n}\}$. The following quantities are equivalent.

1. $I_1 = \sup_{a \in B} \int_B |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)$;

2. $I_2 = |f(0)|^p + \sup_{a \in B} \int_B |\nabla f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)$;

3. $I_3 = |f(0)|^p + \sup_{a \in B} \int_B |R f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)$;

4. $I_4 = |f(0)|^p + \sup_{a \in B} \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)$;

5. $I_5 = |f(0)|^p + \sup_{a \in B} \int_B \left( \sum_{i < j} |T_{i,j} f(z)|^p \right) (1 - |z|^2)^{\frac{p}{t} + q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)$.

**Remark 5.6.** From the above results, it is easy to see that $N(p, q, s) = F(p, p + q - n - 1, ns)$ and $N^0(p, q, s) = F_0(p, p + q - n - 1, ns)$.

An alternative proof of the equivalence of (1), (2), (3) and (4) in Corollary 5.3 can be found in [40], which depends on a careful estimation of the quantity

$I_{w,a} = \int_B \frac{(1 - |z|^2)^{\delta}}{|1 - \langle z, w \rangle|^t |1 - \langle z, a \rangle|^r} dV(z)$,

where $w, a \in B$, $\delta > -1$, $t, r \geq 0$ and $r + t - \delta > n + 1$. However, their approach does not cover the last quantity $I_5$. 
5.2. Korenblum’s inequality of $N(p, q, s)$-type spaces. We denote $f_r(z) = f(rz)$ for $f$ being a holomorphic function and $0 \leq r < 1$. Recall that the Korenblum’s inequality usually refers to the following inequality:

$$
\|g_r\|_{BMOA} \leq \|g\|_B \sqrt{\left| \log(1 - r^2) \right|},
$$

where $g \in H(\mathbb{D})$, BMOA is the space of bounded mean oscillation of holomorphic functions and $B$ is the Bloch space. By [14], the above estimation is sharp.

As an application of our main result in this section, we study the Korenblum’s inequality for $N(p, q, s)$-type spaces.

The following description via higher radial derivative is straightforward from Corollary 5.5.

**Corollary 5.7.** Let $f \in H(\mathbb{B})$, $p \geq 1$, $q > 0$, $s > \max \{0, 1 - \frac{2}{n}\}$ and $m \in \mathbb{N}$. Then $f \in N(p, q, s)$ if and only if

$$
\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |R^m f(z)|^p (1 - |z|^2)^{mp+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty.
$$

We need the following preliminaries. Let $\alpha > 0$. The $\alpha$-Bloch space $B^\alpha$ is the space of all $f \in H(\mathbb{B})$ such that $\sup_{a \in \mathbb{B}} (1 - |z|^2)^{\alpha} |R f(z)| < \infty$ and the little $\alpha$-Bloch space $B_0^\alpha$ consists of those $f \in H(\mathbb{B})$ satisfying $\lim_{|z| \to 1} (1 - |z|^2)^{\alpha} |R f(z)| = 0$. It is well-known that $B^\alpha$ becomes a Banach space with the norm

$$
\|f\|_{B^\alpha} = |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2)^{\alpha | R f(z) |}.
$$

We denote

$$
K_1 = |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2)^{\alpha | \nabla f(z) |}
$$

and

$$
K_2 = |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2)^{\alpha - 1 | \nabla f(z) |}.
$$

It is known that when $\alpha > 0$, $K_1$ and $\|f\|_{B^\alpha}$ are equivalent and when $\alpha > \frac{1}{2}$, the same is true for both $K_1$ and $K_2$ (see, e.g., [47], Theorem 7.1 and Theorem 7.2).

The $\alpha$-Bloch space $B^\alpha$ has a close relationship with the Bergman-type space $A^{-p}(\mathbb{B})$. It is a classical result that for $p > 0$, we have (see, e.g., [47])

$$
A^{-p}(\mathbb{B}) = B^{p+1}.
$$

However, we did not find a reference for the proof of the above result, and hence we give its proof here for the readers’ convenience.
Lemma 5.8. Suppose $p > 0$. Then $f$ is in $B^{p+1}$ if and only if $|f(z)|(1-|z|^2)^p$ is bounded in $\mathbb{B}$; $f$ is in $B_0^{p+1}$ if and only if $(1-|z|^2)^p|f(z)| \to 0$ as $|z| \to 1$.

Proof. (i) Necessity. Let $f \in B^{p+1}$. Take and fix some $\alpha > p + 1$. It is clear that $Rf \in A^{1+\alpha}$. Hence, by the proof of [47, Theorem 2.16] and [30, Proposition 1.4.10], we have

$$|f(z) - f(0)| \lesssim \int_{\mathbb{B}} \frac{(1-|w|^2)^\alpha |Rf(w)|dV(w)}{|1 - \langle z, w \rangle|^{n+\alpha}} \leq \|f\|_{B^{p+1}} \int_{\mathbb{B}} \frac{(1-|w|^2)^{\alpha-p-1}}{|1 - \langle z, w \rangle|^{n+1+(\alpha-p-1)+p}} dV(w) \lesssim \frac{\|f\|_{B^{p+1}}}{(1-|z|^2)^p},$$

which implies the desired claim.

Sufficiency. If $(1-|z|^2)^p|f(z)| \leq M$ for some constant $M > 0$, then by [47, Theorem 2.2],

$$f(z) = \int_{\mathbb{B}} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+p}}dV_p(w).$$

Thus, by [30 Proposition 1.4.10], we get

$$|Rf(z)| = (n+p+1) \left| \int_{\mathbb{B}} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+2+p}} \left( \sum_{k=1}^n z_k \bar{w}_k \right) dV_p(w) \right| \lesssim \int_{\mathbb{B}} |f(w)|(1-|w|^2)^p dV(w) \leq M \int_{\mathbb{B}} \frac{1}{|1 - \langle z, w \rangle|^{n+1+p+1}} dV(w) \lesssim \frac{1}{(1-|z|^2)^{p+1}},$$

as desired.

(ii) The second assertion follows from (i) and the fact that $B_0^{p+1}$ and $A_0^{-p}(\mathbb{B})$ are the closure of the polynomials in $B^{p+1}$ and $A^{-p}(\mathbb{B})$ respectively. □

Theorem 5.9. Let $p \geq 1, q > 0, s > \max\{0,1-\frac{q}{n}\}$ and $0 \leq r < 1$. Then for any $f \in H(\mathbb{B})$, we have

$$\|f_r\| \leq \begin{cases} C|f|_{\frac{p}{s}} \cdot \frac{1}{(1-r)^{\frac{p}{s}}} & s > 1 \\ C|f|_{\frac{p}{s}} \cdot \frac{1}{(1-r)^{\frac{(n+1-s)}{p}}} & s \leq 1, \end{cases}$$
where $C$ is some constant independent of $r$.

Proof. When $s > 1$, it is clear that the $\mathcal{N}(p, q, s)$-norm is equivalent to the $A^{-\frac{2}{s}}(\mathbb{B})$-norm, from which, the desired result follows trivially. Now we assume $s \leq 1$.

Take and fix the smallest $m \in \mathbb{N}$ such that $mp + q - n - 1 > 0$. First note that an easy calculation shows that $R(f_r)(z) = Rf(rz) = (Rf)_{r}(z), \forall z \in \mathbb{B}$, which implies

\[(5.4) \quad R^m(f_r)(z) = (R^m f)_r(z), \quad \forall z \in \mathbb{B}.\]

Next, for any $a \in \mathbb{B}$, apply the maximum modulus principle in $D(ar, r)$ to the function $z \mapsto z \Phi_a(z), \forall z \in D(ar, r)$, it follows that

\[(5.5) \quad |\Phi_a(z)| \geq |z|,
\]

since $\Phi_a$ maps $S$ homeomorphically to itself for each $a \in \mathbb{B}$.

Moreover, for any fixed $a \in \mathbb{B}$, we claim that

\[(5.6) \quad \int_{D(ar, r)} (1 - |z|^2)^{ns - n - 1} dV(z) \leq C \cdot \frac{r^{2n}}{(1 - r^2)^{2(n + 1 - ns)}}.
\]

where $C$ is independent of both $a$ and $r$. Indeed, for any $z \in D(ar, r)$, we have $|\Phi_a(z)| < r$, which implies $1 - |\Phi_a(z)|^2 > 1 - r^2$, that is

\[
\frac{(1 - |ra|^2)(1 - |z|^2)}{|1 - \langle z, ra \rangle|^2} > 1 - r^2.
\]

Thus, for $z \in D(ra, r)$, we have

\[
1 - |z|^2 \geq \frac{(1 - r^2)|1 - \langle z, ra \rangle|^2}{1 - |ra|^2} \gtrsim (1 - r^2)(1 - |ra|^2).
\]

Thus, by [47] Lemma 1.23 and the fact that $ns - n - 1 < 0$, we have

\[
\int_{D(ar, r)} (1 - |z|^2)^{ns - n - 1} dV(z) \\
\lesssim (1 - r^2)^{ns - n - 1}(1 - |ra|^2)^{ns - n - 1} V(D(ar, r)) \\
= (1 - r^2)^{ns - n - 1}(1 - |ra|^2)^n \cdot \frac{r^{2n}}{(1 - r^4|a|^2)^{n+1}} \\
\leq r^{2n}(1 - r^2)^{ns - n - 1}(1 - r^4|a|^2)^{ns - n - 1} \\
\lesssim \frac{r^{2n}}{(1 - r^2)^{2(n + 1 - ns)}}.
\]
Therefore, by (5.4), (5.5) and (5.6), we have
\[
\|f_r\|_p^p = \sup_{a \in B} \int_\mathbb{B} |R^m f_r(z)|^p (1 - |z|^2)^{mp+q} (1 - |\Phi_a(z)|^2)^ns d\lambda(z)
\]
\[
= \sup_{a \in B} \int_\mathbb{B} |R^m f(rz)|^p (1 - |z|^2)^{mp+q-n-1} (1 - |\Phi_a(z)|^2)^ns dV(z)
\]
\[
= \sup_{a \in B} \int_{|w|<r} |R^m f(w)|^p \left(1 - \left|\frac{w}{r}\right|^2\right)^{mp+q-n-1} \left(1 - \left|\Phi_a\left(\frac{w}{r}\right)\right|^2\right)^ns \frac{dV(w)}{r^{2n}}
\]
(\text{change variable with } w = rz)
\[
= \sup_{a \in B} \int_{D(ar,r)} |R^m f \circ \Phi_{ra}(u)|^p \left(1 - \left|\frac{\Phi_{ra}(u)}{r}\right|^2\right)^{mp+q-n-1} \left(1 - \left|\Phi_a\left(\frac{\Phi_{ra}(u)}{r}\right)\right|^2\right)^ns \left(1 - \frac{|ra|^2}{1 - \langle u, ra \rangle^2}\right)^{n+1} \frac{dV(u)}{r^{2n}}
\]
(\text{change variable with } u = \Phi_{ar}(w))
\[
\leq \sup_{a \in B} \int_{D(ar,r)} |R^m f \circ \Phi_{ra}(u)|^p \left(1 - \left|\frac{\Phi_{ra}(u)}{r}\right|^2\right)^{mp+q-n-1} \left(1 - \frac{|ra|^2}{1 - \langle u, ra \rangle^2}\right)^{n+1} \frac{dV(u)}{r^{2n}}
\]
\[
\lesssim |f|_p^p \sup_{a \in B} \int_{D(ar,r)} \left(1 - \frac{|u|^2}{1 - \langle u, ra \rangle^2}\right)^{ns-n-1} \frac{dV(u)}{r^{2n}}
\]
(by [44, Lemma 15] and Corollary 5.8)
\[
\lesssim |f|_p^p \cdot \frac{1}{(1 - r^2)^{2(n+1-ns)}},
\]
which implies the desired result. \(\square\)

5.3. Derivative-free, mixture and oscillation characterizations.
As a second application of Corollary 5.5, we study some other new derivative-free, mixture and oscillation characterizations to \(N(p,q,s)\)-spaces, whose idea comes from [20].

We need the following lemma, which was proved in [26].

Lemma 5.10. Suppose \(\alpha > -1, p > 0, 0 \leq \beta < p + 2\) and \(f \in H(\mathbb{B})\). Then \(f \in A^p_\alpha\) if and only if
\[
K(f) = \int_{\mathbb{B}} |f(z)|^{p-\beta} |\nabla f(z)|^\beta dV(z) < \infty.
\]
Moreover, the quantities \( \|f\|_{p,\alpha}^p \) and \( |f(0)|^p + K(f) \) are comparable for \( f \in H(B) \).

**Theorem 5.11.** Suppose \( f \in H(B), p \geq 1, q > 0, s > \max \{0, 1 - \frac{q}{n}\}, 0 \leq \beta < p + 2 \) and \( \alpha > q + ns - n - 1 \). Then \( f \in \mathcal{N}(p, q, s) \) if and only if

\[
M = \sup_{a \in B} \int_B \int_B \frac{|f(z) - f(w)|^{p-\beta} |\nabla f(z)|^{\beta} (1 - |w|^2)^q}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} \ dx_V(z) dx_V(w) < \infty.
\]

**Proof.** Sufficiency. Suppose \( M < \infty \). Note that by

\[
|\nabla (f \circ \Phi_w)(z)| = |\nabla f(\Phi_w(z))|
\]

and \( \alpha > q + ns - n - 1 > -1 \), we have

(5.7)

\[
M = \sup_{a \in B} \int_B (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} \left( \int_B |F_w(z)|^p |\nabla F_w(z)|^\beta dV_\alpha(z) \right) d\lambda(w),
\]

where \( F_w = f \circ \Phi_w - f(w), w \in B \). By Lemma 5.10

\[
M \asymp \sup_{a \in B} \int_B (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} \left( \int_B |F_w(z)|^p dV_\alpha(z) \right) d\lambda(w).
\]

Note that by [47, Lemma 2.4], we have, for any \( w \in B \),

\[
|\nabla f(w)|^p = |\nabla (f \circ \Phi_w)(0)|^p = |\nabla (f \circ \Phi_w - f(w))(0)|^p \lesssim \int_B |f \circ \Phi_w(z) - f(w)|^p dV_\alpha(z) = \int_B |F_w(z)|^p dV_\alpha(z).
\]

Hence

\[
\infty > M \gtrsim \sup_{a \in B} \int_B |\nabla f(w)|^p (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w),
\]

which, by Corollary 5.5, implies \( f \in \mathcal{N}(p, q, s) \).

**Necessity.** Suppose \( f \in \mathcal{N}(p, q, s) \). First by Lemma 5.10, we see that the quantities

\[
\int_B |F_w(z)|^p dV_\alpha(z) \quad \text{and} \quad \int_B |\nabla F_w(z)|^p dV_\alpha(z)
\]

...
are comparable. Hence, by (5.7), we have
\[
M = \int_B (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} \left( \int_B |F_w(z)|^{p-\beta} |\tilde{\nabla} F_w(z)|^\beta dV_\alpha(z) \right) d\lambda(w)
\]
\[
\lesssim \int_B \left( \int_B |\tilde{\nabla} F_w(z)|^p dV_\alpha(z) \right) (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w)
\]
\[
= \int_B \left( \int_B |\tilde{\nabla} f(\Phi_w(u))|^p dV_\alpha(u) \right) (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w)
\]
\[
= \int_B \left( \int_B |\tilde{\nabla} f(z)|^p (1 - |\Phi_w(z)|^2)^{n+1+\alpha} d\lambda(z) \right)
(1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w)
\]
\[
\leq \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \cdot I,
\]
where
\[
I = \sup_{a,z \in \mathbb{B}} \int_B \frac{(1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} (1 - |\Phi_w(z)|^2)^{n+1+\alpha} d\lambda(w)}{(1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} (1 - |\Phi_z(w)|^2)^{n+1+\alpha} d\lambda(w)}
\]
\[
= \sup_{a,z \in \mathbb{B}} \int_B \frac{(1 - |\Phi_z(u)|^2)^q (1 - |\Phi_a(\Phi_z(u))|^2)^{ns} (1 - |u|^2)^{n+1+\alpha} d\lambda(u)}{(1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} (1 - |\Phi_z(u)|^2)^{n+1+\alpha} d\lambda(u)}
\]
\[
\approx \sup_{a,z \in \mathbb{B}} \int_B \frac{(1 - |u|^2)^q (1 - |u|^2)^{ns} dV_\alpha(u)}{|1 - \langle u, a \rangle|^{2q+2} |1 - \langle \Phi_a(z), u \rangle|^{2ns} dV_\alpha(u)}
\]
\[
\leq \sup_{a,z \in \mathbb{B}} \left\{ \left( \int_B \frac{(1 - |u|^2)^{q+ns} dV_\alpha(u)}{|1 - \langle u, a \rangle|^{2(q+ns)} dV_\alpha(u)} \right)^{\frac{ns}{q+ns}} \cdot \left( \int_B \frac{(1 - |u|^2)^{q+ns} dV_\alpha(u)}{|1 - \langle u, \Phi_z(a) \rangle|^{2(q+ns)} dV_\alpha(u)} \right)^{\frac{ns}{q+ns}} \right\}
\]
\[
< \infty.
\]
Here, we use the fact that \( q + ns - n - 1 - \alpha < 0 \) in the last inequality.
Thus, we get
\[
\int_B (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} \left( \int_B |F_w(z)|^{p-\beta} |\tilde{\nabla} F_w(z)|^\beta dV_\alpha(z) \right) d\lambda(w) \lesssim \|f\|^p,
\]
which implies the desired result. \(\square\)

In particular, taking \( \beta = 0 \), we get the following result.
Theorem 5.12. Suppose $f \in H(B), p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and $\alpha > q + ns - n - 1$. Then $f \in \mathcal{N}(p, q, s)$ if and only if

$$\sup_{a \in B} \int_B \int_B \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^s dV_a(w) dV_a(z) < \infty.$$ 

We also have the following description.

Theorem 5.13. Suppose $f \in H(B), p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and $0 < r < 1$. Then the following statements are equivalent:

(1) $f \in \mathcal{N}(p, q, s)$;

(2) $$\sup_{a \in B} \int_B \left( \frac{1}{V(D(z, r))} \int_{D(z, r)} |f(z) - f(w)|(1 - |z|^2)^{\frac{q}{p}} (1 - |w|^2)^{\frac{q}{p}}
\left(1 - |\Phi_a(z)|^{\frac{q}{p}} (1 - |\Phi_a(w)|^{\frac{q}{p}}\right)^{\frac{n}{2}} dV(w) \right)^p d\lambda(z) < \infty;$$

(3) $$\sup_{a \in B} \int_B \left( \sup_{w \in D(z, r)} |f(z) - f(w)|(1 - |z|^2)^{\frac{q}{p}} (1 - |w|^2)^{\frac{q}{p}}
\left(1 - |\Phi_a(z)|^{\frac{q}{p}} (1 - |\Phi_a(w)|^{\frac{q}{p}}\right)^{\frac{n}{2}} d\lambda(z) < \infty;$$

(4) There exists some $c$ satisfying $1 < c < \frac{1}{r}$, such that

$$\sup_{a \in B} \int_B \left( \frac{1}{V(D(z, cr))} \int_{D(z, cr)} |f(z) - f(w)|^p |(1 - |z|^2)^{\frac{q}{p}} (1 - |w|^2)^{\frac{q}{p}}
\left(1 - |\Phi_a(z)|^{\frac{q}{p}} (1 - |\Phi_a(w)|^{\frac{q}{p}}\right)^{\frac{n}{2}} dV(w) \right) d\lambda(z) < \infty;$$

Proof. $(3) \implies (2)$. This implication is obvious.

$(2) \implies (1)$. When $z \in D(a, r), a, z \in B$, we have (see, e.g., [47])

$$1 - |z|^2 \sim 1 - |a|^2 \sim 1 - |a, z| \sim |1 - \langle a, z \rangle|^{n+1} \sim V(D(a, r))$$

as well as (see, e.g., [37], (2.20))

$$|1 - \langle z, u \rangle| \sim |1 - \langle a, u \rangle|, \ \forall u \in B.$$

By the inequality

$$(1 - |z|^2) |\nabla f(z)| \lesssim \frac{1}{V(D(z, r))} \int_{D(z, r)} |f(z) - f(w)| dV(w), \forall z \in B,$$
(see, e.g., [20]), we have
\[
|\nabla f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns}
\]
\[
\lesssim \left( \frac{1}{V(D(z, r))} \int_{D(z, r)} |f(z) - f(w)| (1 - |z|^2)^{\frac{q}{2}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{2}} dV(w) \right)^p
\]
\[
\approx \left( \frac{1}{V(D(z, r))} \int_{D(z, r)} |f(z) - f(w)| (1 - |z|^2)^{\frac{q}{2p}} (1 - |w|^2)^{\frac{n}{p}}
\]
\[
(1 - |\Phi_a(z)|^2)^{\frac{ns}{2p}} (1 - |\Phi_a(w)|^2)^{\frac{ns}{2p}} dV(w) \right)^p.
\]
Integrating with respect to \(z\) over \(B\) on both sides and taking the supremum over \(a\), we get
\[
\sup_{a \in B} \int_B |\nabla f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z),
\]
which implies \(f \in \mathcal{N}(p, q, s)\).

(1) \(\Rightarrow\) (4). Indeed, for this assertion, we can show that for each
\[1 \leq c < \frac{1}{r}, (4)\] is satisfied. Take and fix some \(c \in (1, \frac{1}{r})\). From Lemma
5.10 and the fact that
\[
|\tilde{\nabla} f(z)|^2 = (1 - |z|^2)(|\nabla f(z)|^2 - |R f(z)|^2),
\]
we have
\[
\int_{D(z, cr)} |f(w)|^p dV(w) \lesssim \int_{D(z, cr)} (1 - |w|^2)^p |\tilde{\nabla} f(w)|^p dV(w) + |f(z)|^p.
\]
Hence, we have
\[
\int_B \left( \frac{1}{V(D(z, cr))} \int_{D(z, cr)} |f(z) - f(w)|^p (1 - |z|^2)^{\frac{q}{2}} (1 - |w|^2)^{\frac{n}{2}} dV(w) \right) d\lambda(z)
\]
\[
\lesssim \int_B \left( \frac{1}{V(D(z, cr))} \int_{D(z, cr)} (1 - |w|^2)^{p+q} |\tilde{\nabla} f(w)|^p (1 - |\Phi_a(w)|^2)^{ns} dV(w) \right) d\lambda(z)
\]
\[
\approx \int_B \int_B \chi_{D(z, cr)}(w) |\tilde{\nabla} f(w)|^p (1 - |w|^2)^{p+q} (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w) d\lambda(z)
\]
\[
= \int_B \chi_{D(w, cr)}(z) \left( \int_B |\tilde{\nabla} f(w)|^p (1 - |w|^2)^{p+q} (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w) \right) d\lambda(z)
\]
\[
\lesssim \|f\|^p < \infty.
\]

(4) \(\Rightarrow\) (3). Suppose there exists some \(c\) satisfying (4). For any
\(f \in H(B)\), by the subharmonicity and [47] Proposition 1.21 and Lemma
Theorem 5.14. Suppose \( f \in H(\mathbb{B}), p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\} \) and \( 0 < r < 1 \). Then the following statements are equivalent:

1. \( f \in \mathcal{N}(p, q, s); \)

From Theorem 5.13 (5.8) and (5.9), we easily get the following result.
\[
\sup_{a \in B} \int_B \left( \frac{1}{V(D(z, r))} \int_{D(z, r)} |f(z) - f(w)|(1 - |z|^2)^{\frac{p}{q}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{p}} dV(w) \right)^p d\lambda(z) < \infty; \\
\sup_{a \in B} \int_B \left( \sup_{w \in D(z, r)} |f(z) - f(w)|(1 - |z|^2)^{\frac{p}{q}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{p}} \right)^p d\lambda(z) < \infty; \\
\text{(4) There exists some } c \text{ satisfying } 1 < c < \frac{1}{r}, \text{ such that} \\
\sup_{a \in B} \int_B \left( \frac{1}{V(D(z, cr))} \int_{D(z, cr)} |f(z) - f(w)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{\frac{ns}{p}} \right)^p d\lambda(z) < \infty.
\]

Remark 5.15. Our earlier estimates in Theorems 5.11, 5.12, 5.13 and 5.14 are pointwise estimates with respect to \(a \in B\), so if we replace \(\sup_{a \in B}\) and \(< \infty\) by \(\lim_{|a| \to 1}\) and \(= 0\), respectively, we obtain the corresponding characterizations of \(N^0(p, q, s)\). Hence, we omit the details of the proof.

6. Atomic decomposition and Gleason’s problem for \(N(p, q, s)\)-type spaces

In this section, we will focus on the decomposition of functions in \(N(p, q, s)\)-type spaces, which is an important concept and is a useful tool in studying such kind of function spaces.

6.1. Atomic decomposition. First, we give some preliminaries. Recall that for \(z, w \in \mathbb{B}\), the Bergman metric can be written as (see, e.g. [47, Proposition 1.21])

\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + |\Phi_z(w)|}{1 - |\Phi_z(w)|}.
\]

Moreover, for \(r > 0\) and \(z \in \mathbb{B}\), the set \(E(z, r) = \{w \in \mathbb{B} : \beta(z, w) < r\}\) is a Bergman metric ball at \(z\). Note that by a simple calculation, we have

\[
|\Phi_z(w)| = \tanh \beta(z, w), \ z, w \in \mathbb{B},
\]

which implies that \(E(z, r) = D(z, \tanh r)\).
Lemma 6.1. [47, Theorem 2.23] There exists a positive integer $N$ such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}$ in $\mathbb{B}$ with the following properties:

1. $\mathbb{B} = \bigcup_k E(a_k, r)$;
2. The sets $E(a_k, r/4)$ are mutually disjoint;
3. Each point $z \in \mathbb{B}$ belongs to at most $N$ of the sets $E(a_k, 4r)$.

Lemma 6.2. [47, Lemma 2.28] Take and fix a sequence $\{a_k\}$ chosen according to Lemma 6.1 with $r$ the separation constant. Then for each $k \geq 1$ there exists a Borel set $E_k$ satisfying the following conditions:

1. $E(a_k, r/4) \subset E_k \subset E(a_k, r)$ for every $k$;
2. $E_k \cap E_j = \emptyset$ for $k \neq j$;
3. $\mathbb{B} = \bigcup_k E_k$.

Lemma 6.3. Suppose $p \geq 1, q > 0, s > \max\{1, 1 - \frac{q}{n}\}$ and $\{a_k\} \subset \mathbb{B}$ is a chosen sequence according to Lemma 6.1 with the separation constant $r \in (0, 1]$. Then

$$d\mu_1 = \sum_k |c_k|^p (1 - |a_k|^2)^{q+ns} \delta_{a_k} dV(z)$$

is an $(ns)$-Carleson measure if and only if

$$d\mu_2 = \sum_k \frac{|c_k|^p}{(1 - |a_k|^2)^p} (1 - |z|^2)^{p+q+ns} \chi_k(z) d\lambda(z)$$

is an $(ns)$-Carleson measure, where $\chi_k$ is the characteristic function of $E(a_k, r)$.

**Proof.** For any $a \in \mathbb{B}$, by [47, Lemma 2.20, (2.20)] and [30, 2.2.7], we have

$$\int_{\mathbb{B}} \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle^2} \right)^{ns} d\mu_2(z)$$

$$= \int_{\mathbb{B}} \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle^2} \right)^{ns} \left( \sum_k \frac{|c_k|^p}{(1 - |a_k|^2)^p} (1 - |z|^2)^{p+q+ns} \chi_k(z) \right) d\lambda(z)$$

$$= \sum_k \frac{|c_k|^p}{(1 - |a_k|^2)^p} \int_{D(a_k, r)} \frac{(1 - |a|^2)^{ns}(1 - |z|^2)^{p+q+ns-n-1}}{|1 - \langle z, a \rangle|^{2ns}} dV(z)$$

$$\approx \sum_k \frac{|c_k|^p}{(1 - |a_k|^2)^{n+1-q-ns}} \cdot \frac{(1 - |a|^2)^{ns}}{|1 - \langle a_k, a \rangle|^{2ns}} V(D(a_k, r))$$
\[
\sum_{k} |c_k|^p (1 - |a_k|^2)^{q+ns} \cdot \frac{(1 - |a|^2)^{ns}}{|1 - \langle a_k, a \rangle|^{2ns}} \\
= \int_{B} \left( \frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{ns} d\mu_1(z).
\]

The desired result follows from [44, Theorem 45].

Fix a parameter \(b > n\) and let \(\alpha = b - (n+1)\). We also fix a sequence \(\{a_k\}\) chosen according to Lemma 6.1 with separation constant \(r\) and a sequence of Borel measurable sets \(\{E_k\}\) with each \(E_k\) satisfying the condition in Lemma 6.2. Recall that the operator \(T\) associated to \(\{a_k\}\) is as follows:

\[
T(f)(z) = \int_{B} \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} f(w) dV(w),
\]

where \(f\) is some Lebesgue measurable function.

Moreover, take a finer “lattice” \(\{a_{kj}\}\) with separation constant \(\gamma\) in the Bergman metric than \(\{a_k\}\) with a sequence of Borel measurable sets \(\{E_{kj}\}\) chosen according to [47, Page 64] and define

\[
S(f)(z) = \sum_{k,j} V_{\alpha}(E_{kj}) f(a_{kj}) \left(1 - \langle z, a_{kj} \rangle \right)^{b},
\]

where \(f \in H(B)\). Note that \(\{a_{kj}\}\) also satisfies the conditions in Lemma 6.1. We refer the reader to the excellent book [47] for the detailed information about such a decomposition of \(B\) into Bergman metric balls.

The following result indicates a deep relationship between \(T\) and \(S\).

**Lemma 6.4.** [47, Lemma 3.22] There exists a constant \(C > 0\), independent of the separation constant \(r\) for \(\{a_k\}\) and the separation constant \(\gamma\) for \(\{a_{kj}\}\), such that

\[
|f(z) - S(f)(z)| \leq C\sigma T(|f|)(z)
\]

for all \(f \in H(B)\) and \(z \in B\), where \(\sigma = \gamma + \frac{\tanh(\gamma)}{\tanh(r)}\).

We have the following result considering the behaviour of \(T\), which follows the methods in [47, Theorem 5.26] and [27, Lemma 1].

**Lemma 6.5.** Let \(f\) be some Lebesgue measurable function. Suppose \(p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}\) and \(t > n - p - q - ns\). Then we have

1. If \(p > 1, b > \frac{n+1}{p'} + \frac{q+ns+t}{p} + 1\) and \(|f(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z)\) is an \((ns)\)-Carleson measure, where \(p'\) is the conjugate of \(p\), then
   \[
   |T(f)(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z)
   \]
is also an \((ns)\)-Carleson measure.

(2) If \(p = 1, b > 1 + q + ns + t\) and \(|f(z)|(1 - |z|^2)^{1+q+ns+t}d\lambda(z)\) is an \((ns)\)-Carleson measure, then

\[
|T(f)(z)|(1 - |z|^2)^{1+q+ns+t}d\lambda(z)
\]

is also an \((ns)\)-Carleson measure.

**Proof.** The proof of (2) is a simple modification of (1), which turns out to be much more easier than (1), and hence we omit it here.

For any \(\xi \in \mathbb{S}\) and \(\delta \in (0, 1)\), we have

\[
\frac{1}{\delta^{ns}} \int_{Q_{3}(\xi)} |T(f)(z)|^p(1 - |z|^2)^{p+q+ns+t}d\lambda(z) 
\leq \frac{1}{\delta^{ns}} \int_{Q_{3}(\xi)} \left( \int_{B} \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} |f(w)|dV(w) \right)^p (1 - |z|^2)^{p+q+ns+t}d\lambda(z)
\]

\[
= \frac{1}{\delta^{ns}} \int_{Q_{3}(\xi)} \left( \left[ \int_{Q_{2}(\xi)} + \sum_{j=1}^{\infty} \int_{A_j} \right] \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} |f(w)|dV(w) \right)^p \]

\[
(1 - |z|^2)^{p+q+ns+t}d\lambda(z) 
\leq \frac{2^{p-1}}{\delta^{ns}} \int_{Q_{3}(\xi)} \left( \int_{Q_{2}(\xi)} \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} |f(w)|dV(w) \right)^p \]

\[
(1 - |z|^2)^{p+q+ns+t}d\lambda(z) 
+ \frac{2^{p-1}}{\delta^{ns}} \int_{Q_{3}(\xi)} \left( \sum_{j=1}^{\infty} \int_{A_j} \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} |f(w)|dV(w) \right)^p \]

\[
(1 - |z|^2)^{p+q+ns+t}d\lambda(z) = I_1 + I_2,
\]

where \(A_j = \{w \in \mathbb{B} : 2^j\delta \leq |1 - \langle w, \xi \rangle| < 2^{j+1}\delta\}, i = 1, 2, \ldots\).

- **Estimation of \(I_1\).**
  Consider the integral operator \(M\) induced by \(K(z, w)\),

\[
Mh(z) = \int_{\mathbb{B}} K(z, w)h(w)dV(w), z \in \mathbb{B},
\]

where

\[
K(z, w) = \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} \frac{q+ns+t-1}{p} (1 - |z|^2)^{1+q+ns+t-n-1} \frac{|1 - \langle z, w \rangle|^b}{|1 - \langle z, w \rangle|^b}, \quad z, w \in \mathbb{B}.
\]
We claim that $M$ is a bounded operator on $L^p(\mathbb{B}, dV)$. Indeed, consider the function $g(z) = (1 - |z|^2)^{-\frac{1}{p+p'}}$. On one hand, we have

$$\int_{\mathbb{B}} K(z, w)g^p(w) dV(w)$$

$$= (1 - |z|^2)^{1 + \frac{2 + n + s + t - n - 1}{p}} \int_{\mathbb{B}} \frac{(1 - |w|^2)^{b - \frac{n+1}{p'}} - \frac{q + n + s + t - 1 - \frac{p'}{p+p'}}{p+p'}}{|1 - \langle z, w \rangle|^b} dV(w)$$

$$= (1 - |z|^2)^{1 + \frac{2 + n + s + t - n - 1}{p}} \int_{\mathbb{B}} \frac{(1 - |w|^2)^{b - \frac{n+1}{p'}} - \frac{q + n + s + t - 1 - \frac{p'}{p+p'}}{p+p'}}{|1 - \langle z, w \rangle|^b} dV(w),$$

where $c_1 = 1 + \frac{q + n + s + t - n - 1}{p} + \frac{p'}{p+p'}$. Note that by our assumption,

$$b - \frac{n+1}{p'} - \frac{q + n + s + t - 1 - \frac{p'}{p+p'}}{p+p'} > -1$$

and

$$c_1 = 1 + \frac{q + n + s + t - n - 1}{p} + \frac{p'}{p+p'} > 0,$$

and hence by [30], Proposition 1.4.10],

$$\int_{\mathbb{B}} K(z, w)g^p(w) dV(w) \lesssim (1 - |z|^2)^{1 + \frac{2 + n + s + t - n - 1}{p}} \cdot (1 - |z|^2)^{-\frac{q + n + s + t - 1 - \frac{p'}{p+p'}}{p+p'}} = g^p(z).$$

On the other hand, we have

$$\int_{\mathbb{B}} K(z, w)g^p(z) dV(z)$$

$$= (1 - |w|^2)^{b - \frac{n+1}{p'}} - \frac{q + n + s + t - 1 - \frac{p'}{p+p'}}{p+p'} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{1 + \frac{2 + n + s + t - n - 1}{p}} - \frac{p}{p+p'}}{|1 - \langle z, w \rangle|^b} dV(z)$$

$$= (1 - |w|^2)^{b - \frac{n+1}{p'}} - \frac{q + n + s + t - 1 - \frac{p'}{p+p'}}{p+p'} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{1 + \frac{2 + n + s + t - n - 1}{p}} - \frac{p}{p+p'}}{|1 - \langle z, w \rangle|^b} dV(z),$$

where $c_2 = b + \frac{p}{p+p'} - \frac{n+1}{p'} - 1 - \frac{q + n + s + t}{p}$. By our assumption again, it follows that

$$1 + \frac{q + n + s + t - n - 1}{p} - \frac{p}{p+p'} > -1$$

and

$$c_2 = b + \frac{p}{p+p'} - \frac{n+1}{p'} - 1 - \frac{q + n + s + t}{p} > 0.$$
Thus, we have

\[ \int_{B} K(z, w)g^p(z) dV(z) \]

\[ \lesssim (1 - |w|^2)^{b+\frac{n+1}{p} - \frac{n+1}{pr+1} - 1} (1 - |w|^2)^{-\frac{b}{pr+1} + \frac{n+1}{pr+1} + \frac{n+1}{p}} = g^p(w). \]

The boundedness of \( M \) on \( L^p(\mathbb{B}, dV) \) is clear by Schur’s test (see, e.g., [47, Theorem 2.9]). Put

\[ h(w) = |f(w)|(1 - |w|^2)^{1+ \frac{n+1}{p} - \frac{n-1}{pr+1}} \chi_{Q_{2\delta}^{\delta p}(\xi)}(w), \ w \in \mathbb{B}. \]

It is easy to see \( h \in L^p(\mathbb{B}, dV) \) since \( |f(z)|^p (1 - |z|^2)^{p+qs+t} d\lambda(z) \) is an \((ns)\)-Carleson measure. Moreover, we have

\[ \frac{1}{(2\delta)^{ns}} \|h\|_{L^p}^p = \frac{1}{(2\delta)^{ns}} \int_{Q_{2\delta}^{\delta p}(\xi)} |f(w)|^p (1 - |w|^2)^{p+qs+t} d\lambda(w) < \infty. \]

Therefore,

\[ I_1 \leq \frac{2^{p-1}}{\delta^{ns}} \int_{B} \left( \int_{B} K(z, w)h(w) dV(w) \right)^p dV(z) \]

\[ = \frac{2^{p-1}}{\delta^{ns}} \|Mh\|_{L^p}^p \lesssim \frac{1}{\delta^{ns}} \|h\|_{L^p}^p < \infty. \]

**Estimation of \( I_2 \).**

Note that for \( z \in Q_{\delta}^{\delta p}(\xi) \) and \( w \in A_j \), we have

\[ |1 - \langle z, w \rangle|^{\frac{q}{p}} \geq |1 - \langle \xi, w \rangle|^{\frac{q}{p}} - |1 - \langle \xi, z \rangle|^{\frac{q}{p}} > \frac{1}{2} (\sqrt{2} - 1) 2^{j} \delta^{\frac{q}{p}}. \]

Moreover, for each \( j \geq 1 \), consider the term

\[ I_{3,j} = \int_{Q_{2^{j+1}\delta}^{\delta p}(\xi)} |f(w)|(1 - |w|^2)^{b-n-1} dV(w). \]

Using Hölder’s inequality and [47, Corollary 5.24], we have

\[ I_{3,j} \leq \left( \int_{Q_{2^{j+1}\delta}^{\delta p}(\xi)} |f(w)|^p (1 - |w|^2)^{p+qs+t} d\lambda(z) \right)^{\frac{1}{p}} \]

\[ \times \left( \int_{Q_{2^{j+1}\delta}^{\delta p}(\xi)} (1 - |w|^2)^{p'(b-n-1) - p'(1 + \frac{n+1}{p} - \frac{n-1}{pr+1})} dV(w) \right)^{\frac{1}{p'}} \]

\[ \lesssim (2^{j+1}\delta)^{b-1 - \frac{n+1}{p}} (2^{j+1}\delta)^{\frac{ns}{p}} \]

\[ \times \left( \int_{Q_{2^{j+1}\delta}^{\delta p}(\xi)} |f(w)|^p (1 - |w|^2)^{p+qs+t} d\lambda(z) \right)^{\frac{1}{p'}} \]

\[ \lesssim (2^{j+1}\delta)^{b-1 - \frac{n+1}{p}}. \]
Thus, we have

\[ I_2 \lesssim \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} \left( \sum_{j=1}^{\infty} \frac{1}{(2^j \delta)^b} \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|(1 - |w|^2)^{b-n-1}dV(w) \right)^p(1 - |z|^2)^{p+q+ns+t-n-1}dV(z) \]

\[ \simeq \delta^{p+q+t} \left( \sum_{j=1}^{\infty} \frac{1}{(2^j \delta)^b} \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|(1 - |w|^2)^{b-n-1}dV(w) \right)^p \]

\[ \lesssim \delta^{p+q+t} \left( \sum_{j=1}^{\infty} \frac{1}{(2^j \delta)^b (2^{j+1}\delta)^{b-1-\frac{q+s}{p}}} \right)^p \lesssim \left( \sum_{j=1}^{\infty} \frac{1}{2^{j(1+\frac{q+s}{p})}} \right)^p < \infty. \]

Finally, combining the estimations on \( I_1 \) and \( I_2 \), it is clear that for any \( \xi \in S \) and \( \delta \in (0, 1) \),

\[ \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} |T(f)(z)|^p(1 - |z|^2)^{p+q+ns+t}d\lambda(z) < \infty, \]

which implies \( |T(f)(z)|^p(1 - |z|^2)^{p+q+ns+t}d\lambda(z) \) is an \((ns)\)-Carleson measure.

We are now ready to establish our main results in this section.

**Theorem 6.6.** Suppose \( p \geq 1, q > 0, s > \max \{0, 1-\frac{q}{n}\} \) and \( b > \left\{ \begin{array}{ll} \frac{n+1}{p'} + \frac{q+ns}{p}, & p > 1; \\ q + ns, & p = 1. \end{array} \right. \)

(1) Let \( \{a_k\} \) be a sequence satisfying the conditions in Lemma 6.1 with the separation constant \( r \in (0, 1) \). If \( \{c_k\} \) is a sequence such that the measure \( \sum_k |c_k|^p(1 - |a_k|^2)^{q+ns}\delta_{a_k} \) is an \((ns)\)-Carleson measure, then the function

\[ f(z) = \sum_k c_k \left( \frac{1 - |a_k|^2}{1 - \langle z, a_k \rangle} \right)^b \]

belongs to \( \mathcal{N}(p, q, s) \).

(2) There exists a sequence \( \{a_k\} \) in \( B \) such that \( \mathcal{N}(p, q, s) \)-type spaces consists exactly of function of the form

\[ f(z) = \sum_k c_k \left( \frac{1 - |a_k|^2}{1 - \langle z, a_k \rangle} \right)^b, \]

where the sequence \( \{c_k\} \) has the property that \( \sum_k |c_k|^p(1 - |a_k|^2)^{q+ns}\delta_{a_k} \) is an \((ns)\)-Carleson measure.
Proof. Without the loss of generality, we may assume \( p > 1 \).

(1) For each \( k \geq 1 \), let \( E_k = E \left( a_k, \frac{r}{4} \right) \). Consider the function

\[
u(z) = \sum_{k=1}^{\infty} \frac{|c_k| \chi_k(z)}{1 - |a_k|^2},
\]

where \( \chi_k \) is the characteristic function of the set \( E_k \). By Lemma 6.1, the sets \( E_k \) are mutually disjoint, the measure \( |\nu(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) \) can be written as

\[
\sum_{k=1}^{\infty} \left( \frac{|c_k| \chi_k(z)}{1 - |a_k|^2} \right)^p (1 - |z|^2)^{p+q+ns} d\lambda(z)
\]

which by Lemma 6.3 is an \((ns)\)-Carleson measure.

Let \( T \) be the operator with the parameter \( b + 1 \). By Lemma 6.5 since \( |\nu(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) \) is an \((ns)\)-Carleson measure, it follows that \( |T(\nu)(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) \) is also an \((ns)\)-Carleson measure.

From the proof of [47, Lemma 5.28], we know that \( |Rf(z)| \lesssim T(\nu)(z) \), which implies that

\[
|Rf(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z)
\]

is also an \((ns)\)-Carleson measure. The desired result follows from Theorem 5.3.

(2) Let \( X \) be the function space consist \( f \in H(\mathbb{B}) \) satisfying

\[
\|f\|_X^p = \sup_{0 < \delta < 1, \xi \in S} \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} |f(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) < \infty.
\]

It is easy to check that \( X \) becomes a Banach space when equipped with the norm \( \| \cdot \|_X \) (see, e.g., [47, Page 185]). Hence, by Lemma 6.5 the operator \( T \) defined in (6.1) with parameter \( b + 1 \) is bounded on \( X \).

Take and fix a sequence \( \{b_k\} \) satisfying the condition in Lemma 6.1 with separation constant \( r \) and a finer “lattice” \( \{b_{kj}\} \) with separation constant \( \gamma \) satisfying that

\[
C\sigma\|T\| < 1,
\]

where \( C \) and \( \sigma \) are defined in Lemma 6.4. Let \( S \) be the linear operator defined in (6.2) with the parameter \( b + 1 \) associated to \( \{b_{kj}\} \). Thus, by Lemma 6.4 again, we can get

\[
\|f - Sf\|_X \leq C\sigma\|T(|f|)\|_X \leq C\sigma\|T\|\|f\|_X < \|f\|_X,
\]
which implies the operator \( I - S \) is bounded on \( X \) with operator norm strictly less than 1, where \( I \) is the identity operator. Hence, by [36, Theorem 1.5.2], \( S \) is invertible on \( X \).

Fix \( f \in \mathcal{N}(p, q, s) \) and let \( g = R^{a,1}f \), where \( a = b - (n + 1) \) and \( R^{a,1} \) is a linear partial differential operator of order 1 (see, e.g., [47, Proposition 1.15]). By [47, Proposition 1.15] and Theorem 5.3, \( g \in X \).

Since \( S \) is invertible on \( X \), there exists a function \( h \in X \) such that \( g = Sh \). Thus \( g \) admits the representation

\[
g(z) = \sum_{k,j} \frac{V_\beta(E_{kj})h(b_{kj})}{(1 - \langle z, b_{kj} \rangle)^{b+1}}.
\]

where \( \beta = (b + 1) - (n + 1) = b - n \). Applying the inverse of \( R^{a,1} \) to both sides with [47, Proposition 1.14], we obtain

\[
f(z) = \sum_{k,j} \frac{V_\beta(E_{kj})h(b_{kj})}{(1 - \langle z, b_{kj} \rangle)^{b}}.
\]

Let

\[
c_{kj} = \frac{V_\beta(E_{kj})h(b_{kj})}{(1 - |b_{kj}|^2)^b}, \quad k \geq 1, 1 \leq j \leq J,
\]

where \( J \) is some integer depending on \( \gamma \) (see, e.g., [47, Page 64]) and hence we can write

\[
f(z) = \sum_{k,j} c_{kj} \left( \frac{1 - |b_{kj}|^2}{1 - \langle z, b_{kj} \rangle} \right)^b.
\]

It remains for us to show that the measure

\[
\sum_{k,j} |c_{kj}|^p (1 - |b_{kj}|^2)^{q+ns} \delta_{b_{kj}}
\]

is an \((ns)\)-Carleson measure. Since

\[
V_\beta(E_{kj}) \leq V_\beta(E_k) \simeq (1 - |b_{kj}|^2)^{n+1+\beta} = (1 - |b_k|^2)^{b+1} \simeq (1 - |b_{kj}|^2)^{b+1},
\]

where the last estimation follows from the fact that \( b_{kj} \in E(b_k, r), 1 \leq j \leq J \), it suffices to show that the measure

\[
d\mu = \sum_{k,j} (1 - |b_{kj}|^2)^{p+q+ns} |h(b_{kj})|^p \delta_{b_{kj}}
\]

is an \((ns)\)-Carleson measure.
By Lemma 6.2 we know that $E(b_{kj}, z)$ are mutually disjoint. Using Lemma 2.20, (2.20) and Lemma 2.24, we have for any $a \in B$,

\[
\int_{B} \left( \frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{ns} d\mu(z)
\]

\[
= \sum_{k,j} \left( \frac{1 - |a|^2}{|1 - \langle b_{kj}, a \rangle|^2} \right)^{ns} |h(b_{kj})|^{p}(1 - |b_{kj}|^2)^{p + q + ns}
\]

\[
\leq \sum_{k,j} \int_{E(b_{kj}, z)} \left( \frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{ns} |h(z)|^{p}(1 - |z|^2)^{p + q + ns} d\lambda(z)
\]

\[
< \infty,
\]

where the last inequality follows from the fact that $f \in X$ and Theorem 45. Using Theorem 45 again, we conclude that the measure $\mu$ is an $(ns)$-Carleson measure. The proof is complete. \qed

Similarly, we have the following description for $\mathcal{N}(p,q,s)$ with regarding its atomic decomposition. First we observe that we have the following “little” version for Lemma 6.5

**Lemma 6.7.** Let $f$ be some Lebesgue measurable function. Suppose $p \geq 1$, $q > 0$, $s > \max \{0, 1 - \frac{q}{n} \}$ and $t > n - p - q - ns$. Then we have

1. If $p > 1$, $b > \frac{n+1}{p} + \frac{q+ns+t}{p} + 1$ and $|f(z)|^{p}(1 - |z|^2)^{p + q + ns + t} d\lambda(z)$ is a vanishing $(ns)$-Carleson measure, where $p'$ is the conjugate of $p$, then

\[
|T (f)(z)|^{p} (1 - |z|^2)^{p + q + ns + t} d\lambda(z)
\]

is also a vanishing $(ns)$-Carleson measure.

2. If $p = 1$, $b > 1 + q + ns + t$ and $|f(z)|((1 - |z|^2)^{1+q+ns+t} d\lambda(z)$ is a vanishing $(ns)$-Carleson measure, then

\[
|T (f)(z)|(1 - |z|^2)^{1+q+ns+t} d\lambda(z)
\]

is also a vanishing $(ns)$-Carleson measure.

**Proof.** We would only consider the case $p > 1$ again. By our assumption, for any $\varepsilon > 0$, there exists a $\delta_0 > 0$, such that the estimate

\[
(6.3) \quad \frac{1}{\delta^{ns}} \int_{Q_{\delta}(x)} |f(z)|^{p}(1 - |z|^2)^{p + q + ns + t} d\lambda(z) < \varepsilon
\]
holds uniformly for \( \xi \in \mathcal{S} \) when \( \delta < \delta_0 \). From the proof of Lemma 6.5 we have

\[
I_1 \lesssim \frac{1}{\delta^{ns}} \int_{Q_{2\delta}(\xi)} |f(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z),
\]

which, combining with (6.3), implies when \( \delta < \frac{\delta_0}{2} \), we have

\[
I_1 \lesssim 2^{ns} \varepsilon.
\]

Now we estimate \( I_2 \). For the chosen \( \varepsilon \), there exists a \( J_0 \in \mathbb{N} \), such that

\[
\sum_{j=J_0+1}^{\infty} \frac{1}{2^j (1+\frac{q+t}{p})} < \varepsilon^{1/p}.
\]

From the proof of Lemma 6.5 we have

\[
I_2 \lesssim \frac{1}{\delta^{ns}} \int_{Q_{\delta}(\xi)} \left( \sum_{j=1}^{J_0} \frac{1}{(2j\delta)^b} \int_{Q_{2j+1}\delta}(\xi) |f(w)|(1 - |w|^2)^{b-n-1} dV(w) \right)^p
\]

\[
(1 - |z|^2)^{p+q+ns+t-n-1} dV(z)
\]

\[
\lesssim \frac{1}{\delta^{ns}} \int_{Q_{\delta}(\xi)} \left( \sum_{j=1}^{J_0} \frac{1}{(2j\delta)^b} \int_{Q_{2j+1}\delta}(\xi) |f(w)|(1 - |w|^2)^{b-n-1} dV(w) \right)^p
\]

\[
(1 - |z|^2)^{p+q+ns+t-n-1} dV(z)
\]

\[
+ \frac{1}{\delta^{ns}} \int_{Q_{\delta}(\xi)} \left( \sum_{j=J_0+1}^{\infty} \frac{1}{(2j\delta)^b} \int_{Q_{2j+1}\delta}(\xi) |f(w)|(1 - |w|^2)^{b-n-1} dV(w) \right)^p
\]

\[
(1 - |z|^2)^{p+q+ns+t-n-1} dV(z)
\]

\[
= J_1 + J_2.
\]

• **Estimation of \( J_1 \).**

  Take \( \delta < \frac{\delta_0}{2^{J_0}+1} \). Then by (6.3) and the estimation of \( I_{3,j}, j \geq 1 \), we have for \( j = 1, 2, \ldots, J_0 \),

\[
I_{3,j} \lesssim (2^{j+1}\delta)^{b-1-\frac{q+t}{p}} \left( \frac{1}{(2^{j+1}\delta)^{ns}} \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|^p (1 - |w|^2)^{p+q+ns+t} d\lambda(z) \right)^{\frac{1}{p}}
\]

\[
\lesssim (2^{j+1}\delta)^{b-1-\frac{q+t}{p}} \varepsilon^{\frac{1}{p}}.
\]
Thus, we see that
\[ J_1 \lesssim \frac{\varepsilon}{\delta^{ns}} \int_{Q_\delta(\xi)} \left( \sum_{j=1}^{J_0} \left( \frac{1}{(2^j \delta)^{1+\frac{q+ns}{p}}} \right)^p \right) (1 - |z|^2)^{p+q+ns+t-n-1}dV(z) \]
\[ \lesssim \varepsilon \cdot \left( \sum_{j=1}^{J_0} \frac{1}{2^j(1+\frac{q+ns}{p})} \right)^p \lesssim \varepsilon. \]

- **Estimation of \( J_2 \).**

By the proof in Lemma 6.5, it is easy to see that
\[ J_2 \lesssim \left( \sum_{j=J_0+1}^{\infty} \frac{1}{2^j(1+\frac{q+ns}{p})} \right)^p < \varepsilon. \]

Hence, it follows that \( I_2 \lesssim \varepsilon \), which implies the desired result. \( \square \)

By using the last lemma, we get the following theorem considering the atomic decomposition on \( \mathcal{N}^0(p,q,s) \)-type space, whose proof is straightforward from Theorem 6.6, and hence we omit the detail here.

**Theorem 6.8.** Suppose \( p \geq 1, q > 0, s > \max\{0, 1 - \frac{n}{a}\} \) and
\[ b > \begin{cases} \frac{n+1}{p'} + \frac{2+ns}{p}, & p > 1; \\ q + ns, & p = 1. \end{cases} \]

1. Let \( \{a_k\} \) be a sequence satisfying the conditions in Lemma 6.7 with the separation constant \( r \in (0,1) \). If \( \{c_k\} \) is a sequence such that the measure \( \sum_k |c_k|^p(1 - |a_k|^2)^{q+ns}\delta_{a_k} \) is a vanishing \((ns)\)-Carleson measure, then the function
\[ f(z) = \sum_k c_k \left( \frac{1 - |a_k|^2}{1 - \langle z, a_k \rangle} \right)^b \]
belongs to \( \mathcal{N}^0(p,q,s) \).

2. There exists a sequence \( \{a_k\} \) in \( \mathbb{B} \) such that \( \mathcal{N}^0(p,q,s) \)-type spaces consists exactly of function of the form
\[ f(z) = \sum_k c_k \left( \frac{1 - |a_k|^2}{1 - \langle z, a_k \rangle} \right)^b, \]
where the sequence \( \{c_k\} \) has the property that \( \sum_k |c_k|^p(1 - |a_k|^2)^{q+ns}\delta_{a_k} \) is a vanishing \((ns)\)-Carleson measure.

We already see that the atomic decomposition on \( \mathcal{N}(p,q,s) \)-type space is closely related to the choice of \( \{a_k\} \). We say that a sequence \( \{a_k\} \) of distinct points with in \( \mathbb{B} \) is an \( r\)-lattice in the Bergman metric.
if it satisfies Lemma 6.1 with separation constant $r$. We need the following lemma.

**Lemma 6.9.** Let $\{z_n\}$ be a sequence of points on $\mathbb{B}$, $\alpha > -1$ and $f \in A^1_\alpha(\mathbb{B})$. Let $\{a_n\}$ be an $r$-lattice in the Bergman metric. Then there exists a constant $C_1 > 0$ depending only on $r$ and $\alpha$ so that

$$\|f\|_{1,\alpha} \geq C_1 \sum_n (1 - |a_n|^2)^{n+1+\alpha} |f(a_n)|. \tag{6.4}$$

Furthermore, there exists some $r_0 > 0$ and a constant $C_2 > 0$ depending only on $r$ and $\alpha$ so that

$$\|f\|_{1,\alpha} \leq C_2 \sum_n (1 - |a_n|^2)^{n+1+\alpha} |f(a_n)|, \tag{6.5}$$

if $0 < r < r_0$.

**Proof.** The first part of the above lemma is a particular case of [13, Lemma 1.5] and the second part is proved in [22, Theorem 2]. \hfill $\Box$

Using the above lemma, we have the following result.

**Theorem 6.10.** Let $p \geq 1$, $q > 0$, $s > \max\{0, 1 - \frac{q}{n}\}$, $\alpha > -1$ and $\{a_n\}$ be an $r$-lattice in the Bergman metric in $\mathbb{B}$. Then for any $\{c_n\} \in \ell^\infty$,

$$f(z) = \sum_n c_n \cdot \left( \frac{1 - |a_n|^2}{1 - \langle z, a_n \rangle} \right)^{n+1+\alpha} \tag{6.6}$$

belongs to $\mathcal{N}(p, q, s)$. Moreover, there is an $r_0 > 0$ such that every $f \in \mathcal{N}(p, q, s)$ has the form (6.6) for some $\{c_n\} \in \ell^\infty$ if $r < r_0$.

**Proof.** Let $\{a_n\}$ be an $r$-lattice in the Bergman metric in $\mathbb{B}$. Moreover, since $s > 1 - \frac{q}{n}$, there exists some $k_0 \in \left(0, \frac{s}{p}\right]$, such that

$$s > 1 - \frac{q - k_0p}{n},$$

which, by Proposition 2.4, implies $A^{-k_0}(\mathbb{B}) \subseteq \mathcal{N}(p, q, s)$.

Then $T$, defined as follows, is a bounded linear operator from $A^1_\alpha(\mathbb{B})$ to $\ell^1$,

$$Tf = \{(Tf)_n\} = \{(1 - |a_n|^2)^{n+1+\alpha} f(a_n)\}, \quad f \in A^1_\alpha(\mathbb{B}),$$

where the boundedness of $T$ is due to (6.4) under $\{z_n\}$ being an $r$-lattice. Thus $T^*$, by [47, Theorem 7.6], the adjoint operator of $T$ is a bounded linear operator from $\ell^\infty(= (\ell^1)^*)$ to $A^{-k_0}(= \mathcal{B}^{k_0+1} = (A^1_\alpha)^*)$, 

...
where $B^{k_0+1}$ is the $(k_0 + 1)$-Bloch space. Moreover, $T^*$ can be written as follows.

$$\langle Tf, y \rangle = \langle f, T^*y \rangle, \quad f \in A^1_\alpha(\mathbb{B}), \ y \in \ell^\infty,$$

where the $\langle \cdot, \cdot \rangle$ is just the usual inner product between $\ell^1$ and $\ell^\infty$.

To compute $T^*$, we take

$$y = e_n, \quad (e_n)_m = \begin{cases} 1, & m = n \\ 0, & m \neq n. \end{cases}$$

So, for $f \in A^1_\alpha(\mathbb{B})$,

$$\langle Tf, e_n \rangle = (Tf)_n = (1 - |a_n|^2)^{n+1+\alpha} f(a_n) = (1 - |a_n|^2)^{n+1+\alpha} \langle f, K_{a_n} \rangle_{\alpha+k_0},$$

where $K_{a_n}(z) = \frac{1}{(1 - \langle z, a_n \rangle)^{n+1+\alpha}}$ is the reproducing kernel for $A^1_\alpha(\mathbb{B})$ and $\langle \cdot, \cdot \rangle_{\alpha+k_0}$ is the integral pair defined in [47 Theorem 7.6]. Hence

$$T^*e_n = (1 - |z_n|^2)^{n+1+\alpha} K_{a_n}(z)$$

and

$$T^*y = \sum_n c_n \cdot \left( \frac{1 - |a_n|^2}{1 - \langle z, a_n \rangle} \right)^{n+1+\alpha}, \quad \text{for} \ y = \{c_n\} \in \ell^\infty,$$

i.e. the function in the form (6.6) is in $A^{-k_0}$. Since $A^{-k_0} \subseteq \mathcal{N}(p, q, s)$, which implies the desired result.

Now we turn to show the second part. Noting that by [47 Theorem 7.6] again, $(A^1_\alpha)^* = B^{\frac{1}{q}+1} = A^{-\frac{2}{q}}(\mathbb{B})$, we also can regard $T^*$ as a bounded linear operator from $\ell^\infty$ to $A^{-\frac{2}{q}}(\mathbb{B})$, where in the sequel, we denote $T^*$ as $S^*$.

In fact, it is only necessary to claim $S^*$ to be surjective. However, $S^*$ is onto if and only if $S : A^1_\alpha(\mathbb{B}) \mapsto \ell^1$ is bounded below (see, e.g., [4, Theorem A, Page 194]). By Lemma 6.9 there exists an $r_0 > 0$ such that $S$ is bounded below if $\{a_n\}$ is an $r$-lattice with $0 < r < r_0$, that is to say, there is an $r_0 > 0$, such that every $f \in A^{-\frac{2}{r}}(\mathbb{B})$ has the form (6.6) for some $\{c_n\} \in \ell^\infty$. Finally, by Proposition 2.1 we note that $\mathcal{N}(p, q, s) \subseteq A^{-\frac{2}{r}}(\mathbb{B})$. The proof is complete.

By modifying the proof of the above theorem a bit, we easily get the following corollary.
Corollary 6.11. Let \( p > 0, \alpha > -1 \) and \( \{a_n\} \) be an \( r \)-lattice in the Bergman metric in \( \mathbb{B} \). Then for any \( \{c_n\} \in \ell^\infty \),
\[
(6.7) \quad f(z) = \sum_n c_n \cdot \left( \frac{1 - |a_n|^2}{1 - \langle z, a_n \rangle} \right)^{n+1+\alpha}
\]
belongs to \( A^{-p}(\mathbb{B}) \). Moreover, there is an \( r_0 > 0 \) such that every \( f \in A^{-p}(\mathbb{B}) \) has the form \((6.7)\) for some \( \{c_n\} \in \ell^\infty \) if \( r < r_0 \).

6.2. The solvability of Gleason’s problem. In the second half of this section, we study the Gleason’s problem on \( \mathcal{N}(p,q,s) \)-type spaces, which, finally turns out to be an application of Lemma 6.5.

Let \( X \) be a space of holomorphic functions on a domain \( \Omega \) in \( \mathbb{C}^n \). The Gleason’s problem for \( X \) is the following statement: Given \( a \in \Omega, f \in X, \) and \( f(a) = 0 \), do there exist functions \( f_1, \ldots, f_n \) in \( X \) such that
\[
f(z) = \sum_{k=1}^n (z_k - a_k) f_k(z)
\]
for all \( z \) in \( \Omega \)?

The case when \( X \) is the Bergman space or the Bloch space was considered in [46]. In the present survey, we focus on the case \( X = \mathcal{N}(p,q,s) \) and \( \Omega = \mathbb{B} \). We have the following results.

Proposition 6.12. Suppose \( p \geq 1, q > 0 \) and \( s > \max\{0, 1 - \frac{q}{n}\} \).
Then there exist bounded linear operators \( A_1, \ldots, A_n \) on \( \mathcal{N}(p,q,s) \) such that
\[
f(z) - f(0) = \sum_{k=1}^n z_k A_k f(z), \quad \forall f \in \mathcal{N}(p,q,s), z \in \mathbb{B}.
\]
Here
\[
A_k f = \int_0^1 \frac{\partial f}{\partial z_k}(tz) dt.
\]

Proof. Note that for any \( f \) holomorphic in \( \mathbb{B} \), we have
\[
f(z) - f(0) = \sum_{k=1}^n z_k \int_0^1 \frac{\partial f}{\partial z_k}(tz) dt.
\]
Thus, it suffices to show that the operator \( A_k \) is bounded on \( \mathcal{N}(p,q,s) \) for each \( k \in \{1, \ldots, n\} \). Without the loss of generality, we may assume \( p > 1 \) since the proof for the case \( p = 1 \) follows exactly the same line as the case \( p > 1 \).
Take any $f \in \mathcal{N}(p, q, s)$, and hence $f \in A^{-\frac{q}{p}}(\mathbb{B})$. Moreover, take and fix $\alpha > \max \left\{ \frac{q}{p} - 1, \frac{q+ns-n-1}{p} \right\}$. It is easy to see that $f \in A^1_{\alpha}$, and hence by [47, Theorem 2.2]

$$f(z) = \int_{\mathbb{B}} \frac{f(w)dV_{\alpha}(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad \forall z \in \mathbb{B},$$

which implies that

$$\frac{\partial f}{\partial z_k}(z) = C(\alpha) \int_{\mathbb{B}} \frac{\bar{w}_k(1 - |w|^2)^{\alpha}f(w)dV(w)}{(1 - \langle z, w \rangle)^{n+2+\alpha}},$$

where $C(\alpha)$ is some constant which only depends on $\alpha$. Then we have

$$|A_kf(z)| = \left| \int_0^1 \frac{\partial f}{\partial z_k}(tz)dt \right| \leq \left| \int_0^1 \left( \int_{\mathbb{B}} \frac{\bar{w}_k(1 - |w|^2)^{\alpha}f(w)dV(w)}{(1 - t\langle z, w \rangle)^{n+2+\alpha}} \right) dt \right|$$

$$\leq \left| \int_{\mathbb{B}} \frac{\bar{w}_k(1 - |w|^2)^{\alpha}f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \left( \int_0^1 \frac{1}{(1 - t\langle z, w \rangle)^{n+2+\alpha}}dt \right) dV(w) \right|$$

$$\leq \left| \int_{\mathbb{B}} \frac{\bar{w}_k(1 - |w|^2)^{\alpha}f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \frac{1 - (1 - \langle z, w \rangle)^{n+1+\alpha}}{\langle z, w \rangle} dV(w) \right|.$$

Note that

$$\frac{1 - (1 - \langle z, w \rangle)^{n+1+\alpha}}{\langle z, w \rangle}$$

is a polynomial in $z$ and $\bar{w}$. Thus, we have

$$|A_kf(z)| \lesssim \int_{\mathbb{B}} \frac{(1 - |w|^2)^{\alpha}|f(w)|}{|1 - \langle z, w \rangle|^{n+1+\alpha}} dV(w).$$

Now in Lemma 6.5 put $t = -p$. Then by the choice of $\alpha$, we have

$$|A_kf(z)|^p(1 - |z|^2)^{q+ns}d\lambda(z)$$

is an $(ns)$-Carleson measure, which, by Proposition 4.1, implies the desired result. \hfill \Box

**Corollary 6.13.** Suppose $p \geq 1, q > 0$ and $s > \max \left\{ 0, 1 - \frac{q}{n} \right\}$. Then there exist bounded linear operators $A_1, \ldots, A_n$ on $\mathcal{N}(p, q, s)$ such that

$$f(z) = \sum_{k=1}^n (z_k - b_k)A_kf(z), \quad z \in \mathbb{B}$$

for all $f \in \mathcal{N}(p, q, s)$ with $f(b) = 0, b = (b_1, \ldots, b_n) \in \mathbb{B}$. 

Proof. Note that
\[ f(z) = f(z) - f(b) = \int_0^1 \left( \frac{d}{dt} f(b + t(z - b)) \right) dt \]
\[ = \sum_{k=1}^n (z_k - b_k) \int_0^1 \frac{\partial f}{\partial z_k} (b + t(z - b)) dt = \sum_{k=1}^n (z_k - b_k) A_k f(z). \]

A similar argument as in Proposition 6.12 with choosing the same \( \alpha \) shows that
\[ |A_k f(z)| \leq \int_B (1 - |w|^2)^\alpha |f(w)| \left| \frac{(1 - \langle b, w \rangle)^{n+1+\alpha} - (1 - \langle z, w \rangle)^{n+1+\alpha}}{\langle b, w \rangle - \langle z, w \rangle} \right| dV(z) \]
\[ \lesssim \int_B (1 - |w|^2)^\alpha |f(w)| \left| 1 - \langle z, w \rangle \right|^{n+1+\alpha} dV(w), \]

since the quantity
\[ \frac{(1 - x)^{n+1+\alpha} - (1 - y)^{n+1+\alpha}}{x - y} \]
is clearly bounded when \( |x|, |y| \leq 2 \). The rest of the proof follows exactly the same as Proposition 6.12. \hfill \Box

Corollary 6.14. Suppose \( p \geq 1, q > 0, s > \max \{0,1 - \frac{2}{n}\} \) and \( m \in \mathbb{N} \). Then there exist bounded linear operators \( A_\alpha (|\alpha| = m) \) on \( \mathcal{N}(p,q,s) \) such that
\[ f(z) = \sum_{|\alpha|=m} (z - b)^\alpha A_\alpha f(z), \quad z \in \mathbb{B} \]
for all \( f \in \mathcal{N}(p,q,s) \) with \( (D^\beta f)(b) = 0, b = (b_1, \ldots, b_n) \in \mathbb{B} \), where \( \beta \) is the multi-index satisfying \( |\beta| < m \). Moreover, we have
\[ A_\alpha f(z) = C_\alpha \int_0^1 (1 - t)^m (D^\alpha f)(b + t(z - b)) dt, \quad \forall z \in \mathbb{B}. \]

where \( C_\alpha \) is some constant only depending on \( \alpha \).

Proof. We prove the result by induction. The base case is exactly Corollary 6.13. Suppose the statement is valid when \( |\alpha| = m \). Take any \( f \in \mathcal{N}(p,q,s) \) with \( (D^\beta f)(b) = 0, |\beta| < m + 1 \), then by induction assumption and the fact that \( A_\alpha f(b) = 0, \forall |\alpha| = m \) (this follows from
(6.8), we have
\[
 f(z) = \sum_{|\alpha|=m} (z-b)^\alpha A_\alpha f(z)
 = \sum_{|\alpha|=m} (z-b)^\alpha \left( A_\alpha f(b) + \sum_{k=1}^n (z_k-b_k)A_k(A_\alpha f)(z) \right)
 = \sum_{|\alpha|=m} \sum_{k=1}^n (z-b)^\alpha (z_k-b_k)A_k(A_\alpha f)(z),
\]
which clearly can be written into the form of
\[
 \sum_{|\gamma|=m+1} (z-b)^\gamma A_\gamma f(z).
\]
The boundedness of $A_\gamma$ follows from the facts that $A_\gamma$ can be written as a finite sum
\[
 \sum_{|k|=1,|\alpha|=m,\alpha+e_k=\gamma} A_k \circ A_\alpha
\]
and both $A_k$ and $A_\alpha$ are bounded on $\mathcal{N}(p, q, s)$.

To prove (6.8), it suffices to show that for each $k$ and $\alpha$, the operator $A_k \circ A_\alpha$ is also of this form. Indeed, we have
\[
 A_k(A_\alpha f)(z) = \int_0^1 \frac{\partial(A_\alpha f)}{\partial z_k}(b+v(z-b))dv
 = C_\alpha \int_0^1 \int_0^1 t(1-t)^m (D^{\alpha+e_k} f)(b+tv(z-b))dtdv
 = C_\alpha \int_0^1 \int_0^1 (1-t)^m (D^{\alpha+e_k} f)(b+u(z-b))dudt
\]
\[
\text{(change variables with } t=t, u=tv)\]
\[
 = C_\alpha \int_0^1 (D^{\alpha+e_k} f)(b+u(z-b)) \left( \int_u^1 (1-t)^m dt \right) du
\]
\[
\text{(by Fubini’s theorem)}
\]
\[
 = C_\alpha' \int_0^1 (1-u)^{m+1} (D^\gamma f)(b+u(z-b))du,
\]
which implies the desired result. \qed
7. Distance between $A^{-\frac{q}{p}}(\mathbb{B})$ spaces and $\mathcal{N}(p, q, s)$-type spaces

Recall that from Theorem 2.1, we have, for $p \geq 1$ and $q, s > 0$,

$$\mathcal{N}(p, q, s) \subseteq A^{-\frac{q}{p}}(\mathbb{B}).$$

A natural question can be asked is: for any $f \in A^{-\frac{q}{p}}(\mathbb{B})$, what can we say about the distance between $f$ and $\mathcal{N}(p, q, s)$ with regarding $\mathcal{N}(p, q, s)$ as a subspace of $A^{-\frac{q}{p}}(\mathbb{B})$? In this section, we will focus on this question.

We denote the distance in $A^{-\frac{q}{p}}(\mathbb{B})$ of $f$ to $\mathcal{N}(p, q, s)$ by $d(f, \mathcal{N}(p, q, s))$,

$$d(f, \mathcal{N}(p, q, s)) = \inf_{g \in \mathcal{N}(p, q, s)} |f - g|_{\frac{q}{p}},$$

where $| \cdot |_{\frac{q}{p}}$ is the norm defined on $A^{-\frac{q}{p}}(\mathbb{B})$. Moreover, for $f \in H(\mathbb{B})$ and $\varepsilon > 0$, let

$$\Omega_\varepsilon(f) = \{z \in \mathbb{B} : |f(z)|(1 - |z|^2)^{\frac{q}{p}} \geq \varepsilon\}.$$

We have the following result.

**Theorem 7.1.** Suppose $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and $f \in A^{-\frac{q}{p}}(\mathbb{B})$. Then the following quantities are equivalent:

1. $d_1 = d(f, \mathcal{N}(p, q, s));$
2. $d_2 = \inf\{\varepsilon : \chi_{\Omega_\varepsilon(f)}(z)(1 - |z|^2)^{ns}d\lambda(z) \text{ is an (ns)-Carleson measure}\};$
3. $d_3 = \inf\left\{\varepsilon : \sup_{a \in \mathbb{B}} \int_{\Omega_\varepsilon(f)} |f(z)|^p(1 - |z|^2)^{q}(1 - |\Phi_a(z)|^2)^{ns}d\lambda(z) < \infty\right\}.$

**Proof.** (1) $d_1 \lesssim d_2.$

Without the loss of generality, we may assume that $p > 1$. Let $\varepsilon$ be a positive number such that $\chi_{\Omega_\varepsilon(f)}(z)(1 - |z|^2)^{ns}d\lambda(z)$ is an $(ns)$-Carleson measure. Since $f \in A^{-\frac{q}{p}}(\mathbb{B})$, we have

$$\sup_{z \in \mathbb{B}} |f(z)|(1 - |z|^2)^{\frac{q}{p}} < \infty.$$

Take and fix some $\alpha > \max\left\{\frac{q}{p} - 1, \frac{q + ns - n - 1}{p}\right\}$. It is easy to see that $f \in A^1_\alpha$, and hence by [17, Theorem 2.2], we have

$$f(z) = \int_\mathbb{B} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+\alpha}}dV_\alpha(w).$$
Let
\[ f_1(z) = \int_{\Omega_{c}(f)} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} dV_{\alpha}(w) \]
and
\[ f_2(z) = \int_{B \setminus \Omega_{c}(f)} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} dV_{\alpha}(w). \]

It is clear that both \( f_1 \) and \( f_2 \) are holomorphic functions in \( \mathbb{B} \) and \( f(z) = f_1(z) + f_2(z) \). We have the following claims.

\( \bullet \) \( \| f_2 \|_p \leq C \varepsilon \) for some constant \( C > 0 \).

For any \( z \in \mathbb{B} \), by [30, Proposition 1.4.10], we have
\[
|f_2(z)| \leq \int_{\mathbb{B} \setminus \Omega_{c}(f)} \frac{|f(w)|}{|1 - \langle z, w \rangle|^{n+1+\alpha}} dV_{\alpha}(w) \\
\lesssim \varepsilon \int_{\mathbb{B}} \frac{(1 - |w|^2)^{\frac{\alpha - \frac{2}{p}}{p}}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} dV_{\alpha}(w) \\
\lesssim \varepsilon (1 - |z|^2)^{-\frac{\alpha}{p}}
\]
which implies the desired claim.

\( \bullet \) \( f_1 \in N(p, q, s) \).

By Theorem 5.3, it suffices to show that \( |f_1(z)|^p (1 - |z|^2)^{q + ns} d\lambda(z) \) is a \( (ns) \)-Carleson measure. Note that for \( z \in \mathbb{B} \), we have
\[
|f_1(z)| \lesssim \int_{\Omega_{c}(f)} \frac{|f(w)|(1 - |w|^2)^{\frac{\alpha}{p}}(1 - |w|^2)^{\alpha - \frac{2}{p}}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} dV(w) \\
\lesssim \int_{\Omega_{c}(f)} \frac{(1 - |w|^2)^{\alpha - \frac{2}{p}}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} dV_{\alpha}(w) \\
eq \int_{\mathbb{B}} \frac{(1 - |w|^2)^{\alpha}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \chi_{\Omega_{c}(f)}(w) \frac{\chi_{\Omega_{c}(f)}(w)}{(1 - |w|^2)^{\frac{2}{p}} dV_{\alpha}(w)}.
\]
Let
\[ g(w) = \frac{\chi_{\Omega_{c}(f)}(w)}{(1 - |w|^2)^{\frac{2}{p}}}, \]
and hence we have
\[
|g(w)|^p (1 - |w|^2)^{q + ns} d\lambda(w) = \chi_{\Omega_{c}(f)}(w) (1 - |w|^2)^{ns} d\lambda(w),
\]
which is an \( (ns) \)-Carleson measure by our assumption. Now in Theorem 6.5 let \( t = -p \) and \( b = n + 1 + \alpha \), it is easy to check that
\[ t = -p > n - p - q - ns \]
and
\[ b = n + 1 + \alpha > \frac{n + 1}{p'} + \frac{q + ns}{p}. \]
Hence, the operator $T$ with parameter $n+1+\alpha$ sends the $(ns)$-Carleson measure
\[ |g(z)|^p(1 - |z|^2)^{q+ns}d\lambda(z) \]
to
\[ |Tg(z)|^p(1 - |z|^2)^{q+ns}d\lambda(z) \]
\[ = \left[ \int_{\mathbb{B}} \frac{(1 - |w|^2)^\alpha}{1 - \langle z, w \rangle^{n+1+\alpha}} \cdot \frac{\chi_{\Omega_{c}(f)}(w)}{(1 - |w|^2)^\frac{q}{p}} dV(w) \right]^p (1 - |z|^2)^{q+ns}d\lambda(z), \]
which, by Theorem 6.5, is also an $(ns)$-Carleson measure. Finally, since
\[ |f_1(z)| \lesssim \int_{\mathbb{B}} \frac{(1 - |w|^2)^\alpha}{1 - \langle z, w \rangle^{n+1+\alpha}} \cdot \frac{\chi_{\Omega_{c}(f)}(w)}{(1 - |w|^2)^\frac{q}{p}} dV(w), \]
it follows that $|f_1(z)|^p(1 - |z|^2)^{q+ns}d\lambda(z)$ is an $(ns)$-Carleson measure. Hence, the claim is proved.

Note that $f_1 \in A^{-\frac{q}{p}}(\mathbb{B})$ since $\mathcal{N}(p, q, s) \subseteq A^{-\frac{q}{p}}(\mathbb{B})$. Thus, we have
\[ d_1 = d(\mathcal{N}(p, q, s)) \leq |f - f_1|_p = |f_2|_p \lesssim \varepsilon. \]
Finally, by letting $\varepsilon$ tends to $d_2$, we get the desired result.

(2) $d_2 \leq d_3$.

Take and fix an $\varepsilon > 0$ such that
\[ \sup_{a \in \mathbb{B}} \int_{\Omega_{c}(f)} |f(z)|^p(1 - |z|^2)^q(1 - |\Phi_a(z)|^2)^{ns}d\lambda(z) < \infty. \]
Since $|f(z)|^p(1 - |z|^2)^q \geq \varepsilon^p$ for $z \in \Omega_{c}(f)$, it follows that
\[ \sup_{a \in \mathbb{B}} \int_{\Omega_{c}(f)} (1 - |\Phi_a(z)|^2)^{ns}d\lambda(z) \]
\[ = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left( \frac{1 - |a|^2}{1 - \langle z, a \rangle^2} \right)^{ns} \chi_{\Omega_{c}(f)}(z)(1 - |z|^2)^{ns}d\lambda(z) < \infty, \]
which, by Theorem 6.5, implies $\chi_{\Omega_{c}(f)}(z)(1 - |z|^2)^{ns}d\lambda(z)$ is an $(ns)$-Carleson measure. Hence,
\[ \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\Omega_{c}(f)} |f(z)|^p(1 - |z|^2)^q(1 - |\Phi_a(z)|^2)^{ns}d\lambda(z) < \infty \right\} \]
\[ \subseteq \left\{ \varepsilon : \chi_{\Omega_{c}(f)}(z)(1 - |z|^2)^{ns}d\lambda(z) \text{ is an (ns)-Carleson measure} \right\}, \]
which implies $d_2 \leq d_3$.

(3) $d_3 \leq d_1$.

It suffices to show that
\[ (d_1, \infty) \subseteq \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\Omega_{c}(f)} |f(z)|^p(1 - |z|^2)^q(1 - |\Phi_a(z)|^2)^{ns}d\lambda(z) < \infty \right\}. \]
Take and fix any \( \varepsilon > d_1 \), then there exists a \( f_1 \in \mathcal{N}(p, q, s) \), such that

\[
|f - f_1|_{\frac{q}{p}} < \frac{d_1 + \varepsilon}{2}.
\]

By triangle inequality, we have for \( z \in \Omega_{\varepsilon}(f) \),

\[
|f_1(z)|(1 - |z|^2)^{\frac{q}{p}} \geq |f(z)|(1 - |z|^2)^{\frac{q}{p}} - |f(z) - f_1(z)|(1 - |z|^2)^{\frac{q}{p}} \geq \varepsilon - \frac{d_1 + \varepsilon}{2} = \varepsilon - \frac{d_1}{2}.
\]

Thus, we have

\[
\sup_{a \in \mathbb{B}} \int_{\Omega_{\varepsilon}(f)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
\leq \sup_{a \in \mathbb{B}} \int_{\Omega_{\varepsilon}(f)} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
\leq \frac{2^p}{(\varepsilon - d_1)^p} \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f_1(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
< \infty,
\]

which implies the desired inclusion. The proof is complete. \( \square \)

Noting that by Corollary \[\text{Corollary 5.5}\] we can express the \( \mathcal{N}(p, q, s) \)-norm by using complex gradient and radial derivative respectively. By using these equivalent norms, we can express \( d(f, \mathcal{N}(p, q, s)) \) via different forms.

More precisely, from the view of Lemma \[\text{Lemma 5.8}\] it is clear that we can express \( d(f, \mathcal{N}(p, q, s)) \) as

\[
\inf_{g \in \mathcal{N}(p, q, s)} \|f - g\|_{\mathcal{B}^{\frac{q}{p} + 1}}.
\]

Now for \( f \in H(\mathbb{B}) \) and \( \varepsilon > 0 \), we denote

\[
\tilde{\Omega}_{\varepsilon}(f) = \{ z \in \mathbb{B} : |Rf(z)|(1 - |z|^2)^{\frac{q}{p} + 1} \geq \varepsilon \}.
\]

By using the above expression, we have the following result.

**Theorem 7.2.** Suppose \( p \geq 1, q > 0, s > \max \{0, 1 - \frac{2}{n}\} \) and \( f \in A^{-\frac{q}{p}}(\mathbb{B}) = \mathcal{B}^{\frac{q}{p} + 1} \). Then the following quantities are equivalent:

1. \( d_1 = d(f, \mathcal{N}(p, q, s)) \);
2. \( d_4 = \inf \{ \varepsilon : \chi_{\tilde{\Omega}_{\varepsilon}(f)}(z)(1 - |z|^2)^{ns} d\lambda(z) \text{ is an } (ns)-\text{Carleson measure} \} \);
3. \( d_5 = \inf \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_{\varepsilon}(f)} |Rf(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\} \);
\( d_6 = \inf \left\{ \varepsilon : \sup_{a \in B} \int_{\tilde{\Omega}_\varepsilon(f)} |\nabla f(z)|^p(1 - |z|^2)^{p+q}(1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\} ; \)

\( d_7 = \inf \left\{ \varepsilon : \sup_{a \in B} \int_{\tilde{\Omega}_\varepsilon(f)} |\nabla f(z)|^p(1 - |z|^2)^q(1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\} . \)

Proof. (1) \( d_1 \lesssim d_4 \).

The proof for this part is similar to the proof of \( d_1 \lesssim d_2 \) in Theorem 7.1. Again, we may assume that \( p > 1 \). Let \( \varepsilon \) be a positive number such that \( \chi_{\tilde{\Omega}_\varepsilon(f)}(1 - |z|^2)^{ns} d\lambda(z) \) is an \((ns)\)-Carleson measure. Since \( f \in B^{q,p+1}_\alpha \), we have

\[ \sup_z Rf(z)|(1 - |z|^2)^{\frac{q}{p}+1} < \infty. \]

Take and fix some \( \alpha > \max \{ \frac{q}{p}, \frac{q+ns-n-1}{p} + 1 \} \). It is easy to see that \( Rf(z) \in A^1_\alpha \), and hence by [17, Theorem 2.2], we have

\[ Rf(z) = \int_B Rf(w) dV_\alpha(w) \left( \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} - 1 \right) dV_\alpha(w), \quad z \in B. \]

Since \( Rf(0) = 0 \), we have

\[ Rf(z) = \int_B Rf(w) \left( \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} - 1 \right) dV_\alpha(w), \quad z \in B. \]

It follows that

\[ f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} dt = \int_B Rf(w) L(z, w) dV_\alpha(w), \]

where the kernel

\[ L(z, w) = \int_0^1 \left( \frac{1}{(1 - t\langle z, w \rangle)^{n+1+\alpha}} - 1 \right) dt \]

Let \( f(z) = f_1(z) + f_2(z) \), where

\[ f_1(z) = f(0) + \int_{\tilde{\Omega}_\varepsilon(f)} Rf(w) L(z, w) dV_\alpha(w) \]

and

\[ f_2(z) = \int_{B \setminus \tilde{\Omega}_\varepsilon(f)} Rf(w) L(z, w) dV_\alpha(w) \]

We have the following claims.

- \( \|f_2\|_{B^{q,p+1}} \leq C \varepsilon \) for some constant \( C > 0 \).
Since
\[ RL(z, w) = \int_0^1 \frac{(n + 1 + \alpha) \langle z, w \rangle}{(1 - t \langle z, w \rangle)^{n+\alpha+2}} = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha} - 1}, \]
we have
\[ |Rf_2(z)| = \left| \int_{\mathbb{B} \setminus \tilde{\Omega}_\epsilon(f)} Rf(w) RL(z, w) dV_\alpha(w) \right| \]
\[ \lesssim \varepsilon \int_{\mathbb{B} \setminus \tilde{\Omega}_\epsilon(f)} (1 - |w|^2)^{\alpha - \frac{2}{p} - 1} \cdot \left( \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha} - 1} \right) dV(w) \]
\[ \leq \varepsilon \cdot \left( \int_{\mathbb{B}} |1 - \langle z, w \rangle|^{n+1+\alpha - \frac{2}{p} - 1 + \frac{q}{p} + 1} dV(w) + 1 \right) \]
\[ \lesssim \frac{\varepsilon}{(1 - |z|^2)^{\frac{2}{p} + 1}}, \]
which implies the desired claim.

• \( f_1 \in \mathcal{N}(p, q, s) \).

By Theorem 5.3, it suffices to show that \( |Rf_1(z)|^p(1 - |z|^2)^{p+q+ns} d\lambda(z) \) is a \((ns)\)-Carleson measure. Note that for \( z \in \mathbb{B} \), we have
\[ |Rf_1(z)| = \left| \int_{\tilde{\Omega}_\epsilon(f)} Rf(w) RL(z, w) dV_\alpha(w) \right| \]
\[ \leq \int_{\tilde{\Omega}_\epsilon(f)} |Rf(w)| \left( \frac{1}{|1 - \langle z, w \rangle|^{n+1+\alpha} + 1} \right) dV_\alpha(w) \]
\[ = I_1 + I_2, \]
where
\[ I_1 = \int_{\tilde{\Omega}_\epsilon(f)} \frac{|Rf(w)|}{|1 - \langle z, w \rangle|^{n+1+\alpha}} dV_\alpha(w) \]
and
\[ I_2 = \int_{\tilde{\Omega}_\epsilon(f)} |Rf(w)| dV_\alpha(w). \]
First, we note that by our choice of \( \alpha \), it follows that
\[ I_2 \lesssim \int_{\mathbb{B}} |Rf(w)|(1 - |w|^2)^{\frac{n+1}{p} + (1 - |w|^2)^{\alpha - \frac{2}{p} - 1} dV(w) \]
\[ \lesssim \int_{\mathbb{B}} (1 - |w|^2)^{\alpha - \frac{2}{p} - 1} dV(w) \lesssim 1. \]
Next, we estimate $I_1$. Note that
\[
I_1 \approx \int_{\tilde{\Omega}_z(f)} \frac{|Rf(w)|(1 - |w|^2)\frac{2}{p} + 1(1 - |w|^2)^\alpha \frac{2}{p} - 1}{|1 - \langle z, w \rangle|^{n+1+\alpha}} dV(w)
\]
and hence we have
\[
|g(w)|^p(1 - |w|^2)p + q + ns d\lambda(w) = \chi_{\tilde{\Omega}_z(f)}(w)(1 - |w|^2)^{ns} d\lambda(w),
\]
which is an $(ns)$-Carleson measure by our assumption. Now in Theorem 6.5 let $t = 0$ and $b = n + 1 + \alpha$, it is easy to check that
\[
t = 0 > n - p - q - ns
\]
and
\[
b = n + 1 + \alpha > \frac{n + 1}{p'} + \frac{q + ns}{s} + 1.
\]
Hence, the operator $T$ with parameter $n + 1 + \alpha$ sends the $(ns)$-Carleson measure
\[
|g(z)|^p(1 - |z|^2)p + q + ns d\lambda(z)
\]
to
\[
|Tg(z)|^p(1 - |z|^2)p + q + ns d\lambda(z)
\]
\[
\left| \int_B \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \cdot \frac{\chi_{\tilde{\Omega}_z(f)}(w)}{(1 - |w|^2)^\frac{2}{p} + 1} dV(w) \right|^p (1 - |z|^2)p + q + ns d\lambda(z),
\]
which, by Theorem 6.5 is also an $(ns)$-Carleson measure. Finally, we have
\[
|Rf_1(z)| \leq I_1 + I_2 \leq \left| \int_B \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \cdot \frac{\chi_{\tilde{\Omega}_z(f)}(w)}{(1 - |w|^2)^\frac{2}{p} + 1} dV(w) \right| + 1
\]
and it follows that $|Rf_1(z)|^p(1 - |z|^2)p + q + ns d\lambda(z)$ is an $(ns)$-Carleson measure. Hence, the claim is proved.

Note that $f_1 \in A^{-\frac{2}{p}}(B) = B^{\frac{2}{p}+1}$ since $\mathcal{N}(p, q, s) \subseteq A^{-\frac{2}{p}}(B) = B^{\frac{2}{p}+1}$. Thus, we have
\[
d_1 = d(f, \mathcal{N}(p, q, s)) \lesssim \|f - f_1\|_{B^{\frac{2}{p}+1}} = \|f_2\|_{B^{\frac{2}{p}+1}} \lesssim \varepsilon.
\]
Finally, by letting $\varepsilon$ tends to $d_2$, we get the desired result.
(2) $d_4 \leq d_5$. 
The proof for this part is almost the same as the proof for \( d_2 \leq d_3 \) in Theorem 7.1 and hence we omit it here.

(3) \( d_5 \leq d_6 \leq d_7 \).

This assertion is follows by the inequality (see, e.g., \([47, \text{Lemma } 2.14}\])
\[
|Rf(z)|(1 - |z|^2) \leq |\nabla f(z)|(1 - |z|^2) \leq |\tilde{\nabla} f(z)|.
\]

(4) \( d_7 \leq d_1 \).

It suffices to show that
\[
(\Omega_{d_1, \infty}) \subset \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_{\varepsilon}(f)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^s d\lambda(z) < \infty \right\}.
\]

Take and fix any \( \varepsilon > d_1 \), there exists a function \( f_1 \in \mathcal{N}(p, q, s) \) such that
\[
\|f - f_1\|_{B^{q,p+1}} < \frac{d_1 + \varepsilon}{2}.
\]

Then, by triangle inequality, we have for \( z \in \tilde{\Omega}_{\varepsilon}(f) \),
\[
|\tilde{\nabla} f_1(z)|(1 - |z|^2)^\frac{s}{p} \geq |Rf_1(z)|(1 - |z|^2)^\frac{s}{p+1} \geq |Rf(z)|(1 - |z|^2)^\frac{s}{p+1} - |R(f - f_1)(z)|(1 - |z|^2)^\frac{s}{p+1} \geq \varepsilon - d_1 \frac{d_1 + \varepsilon}{2} = \frac{\varepsilon - d_1}{2}.
\]

Thus, it follows that
\[
\sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_{\varepsilon}(f)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^s d\lambda(z) \leq \sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_{\varepsilon}(f)} (1 - |\Phi_a(z)|^2)^s d\lambda(z) \leq \left( \frac{2}{\varepsilon - d_1} \right)^p \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\nabla f_1(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^s d\lambda(z) < \infty,
\]

where in the last inequality, we use Corollary 5.5 and in the first inequality, we use the fact that \( 1 + \frac{q}{p} > 1 \) and hence by our previous remark,
\[
|f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2)^\frac{s}{p} |\tilde{\nabla} f(z)|
\]

becomes an equivalent norm of \( B^{\frac{p}{s}+1} \). Therefore we get the desired result. \( \square \)

**Corollary 7.3.** Suppose \( p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{p}\} \) and \( f \in A^{-\frac{q}{p}}(\mathbb{B}) \). Then the following conditions are equivalent:

(1) \( f \) is in the closure of \( \mathcal{N}(p, q, s) \) in \( A^{-\frac{q}{p}}(\mathbb{B}) \).
\( \chi_{\Omega_\varepsilon(f)}(z) (1 - |z|^2)^{ns} d\lambda(z) \) is an \((ns)\)-Carleson measure for every \( \varepsilon > 0 \);

\( \chi_{\Omega_\varepsilon(f)}(z) (1 - |z|^2)^{ns} d\lambda(z) \) is an \((ns)\)-Carleson measure for every \( \varepsilon > 0 \);

\[
\sup_{a \in B} \int_{\Omega_\varepsilon(f)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda < \infty
\]

for every \( \varepsilon > 0 \);

\[
\sup_{a \in B} \int_{\Omega_\varepsilon(f)} |Rf(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty.
\]

for every \( \varepsilon > 0 \);

\[
\sup_{a \in B} \int_{\Omega_\varepsilon(f)} |\nabla f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty
\]

for every \( \varepsilon > 0 \);

\[
\sup_{a \in B} \int_{\Omega_\varepsilon(f)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty
\]

for every \( \varepsilon > 0 \).

For the “little-oh” version, we denote the distance in \( A^{-\frac{q}{p}}(\mathbb{B}) \) of \( f \) to \( \mathcal{N}^0(p, q, s) \) by \( d(f, \mathcal{N}^0(p, q, s)) \), that is

\[
d(f, \mathcal{N}^0(p, q, s)) = \inf_{g \in \mathcal{N}^0(p, q, s)} |f - g|_{\mathbb{B}}.
\]

We have the following result.

**Theorem 7.4.** Suppose \( p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{p}\} \) and \( f \in A^{-\frac{q}{p}}(\mathbb{B}) \). Then the following conditions are equivalent:

1. \( e_1 = d(f, A_0^{-\frac{q}{p}}(\mathbb{B})) \);
2. \( e_2 = d(f, \mathcal{N}^0(p, q, s)) \);
3. \( e_3 = \inf\{\varepsilon : \chi_{\Omega_\varepsilon(f)}(z)(1 - |z|^2)^{ns} d\lambda(z) \text{ is a vanishing (ns)-Carleson measure}\} \);
4. \( e_4 = \inf\{\varepsilon : \chi_{\Omega_\varepsilon(f)}(z)(1 - |z|^2)^{ns} d\lambda(z) \text{ is a vanishing (ns)-Carleson measure}\} \);
(5) \[ e_5 = \inf \left\{ \varepsilon : \lim_{|a| \to 1} \int_{\Omega_\varepsilon(f)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) = 0 \right\}; \]

(6) \[ e_6 = \inf \left\{ \varepsilon : \lim_{|a| \to 1} \int_{\tilde{\Omega}_\varepsilon(f)} |Rf(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) = 0 \right\}; \]

(7) \[ e_7 = \inf \left\{ \varepsilon : \lim_{|a| \to 1} \int_{\tilde{\Omega}_\varepsilon(f)} |\nabla f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) = 0 \right\}; \]

(8) \[ e_8 = \inf \left\{ \varepsilon : \lim_{|a| \to 1} \int_{\tilde{\Omega}_\varepsilon(f)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) = 0 \right\}. \]

**Proof.** Since both \( A_0^{-\frac{p}{q}}(\mathbb{B}) \) and \( \mathcal{N}_0(p, q, s) \) are the closure of all the polynomials in \( A_0^{-\frac{p}{q}}(\mathbb{B}) \) and \( \mathcal{N}(p, q, s) \) respectively, the equivalence between \( e_1 \) and \( e_2 \) is obvious.

Moreover, in the proof of Theorems 7.1 and 7.2 by interchanging the role of \( d(f, \mathcal{N}(p, q, s)) \) to \( d(f, \mathcal{N}_0(p, q, s)) \), \( \sup_{a \in \mathbb{B}} \) to \( \lim_{|a| \to 1} \) and applying Lemma 6.7 instead of Lemma 6.5, we can get the equivalence of \( e_2, e_3, \ldots, e_8 \).

\[ \square \]

8. **RIEMANN-STIELTJES OPERATORS AND MULTIPLIERS**

In this section, we will study the behavior of Riemann-Stieltjes operators on \( \mathcal{N}(p, q, s) \)-type spaces, which can be interpreted as “half” of the multiplication operator.

Precisely, let \( g \) be a holomorphic function on \( \mathbb{B} \). Denote the **Riemann-Stieltjes operators** \( T_g \) and \( L_g \) with symbol \( g \) as

\[ T_g f(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \ z \in \mathbb{B}; \]

\[ L_g f(z) = \int_0^1 g(tz) Rf(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \ z \in \mathbb{B}. \]

Clearly, the Riemann-Stieltjes operator \( T_g \) can be viewed as a generalization of the well-known Cesàro operator. It is also easy to see that the multiplication operator \( M_g \) are determined by

\[ M_g f(z) = g(z) f(z) = g(0) f(0) + T_g f(z) + L_g f(z), \quad f \in H(\mathbb{B}), \ z \in \mathbb{B}. \]
In general, these operators are usually referred as the integral operators, which have been studied under various settings (see, e.g., [3, 10, 17, 39]).

For the purpose to study the boundedness and compactness of the Riemann-Stieltjes operators on $N(p, q, s)$-spaces, we introduce the following non-isotropic tent type space $T^{\infty}_{m,l}$ of all $\mu$-measure functions $f$ on $\mathbb{B}$ satisfying

$$
\|f\|_{T^{\infty}_{m,l}(\mu)} = \sup_{\xi \in \mathbb{S}, \delta > 0} \left( \frac{1}{\delta^m} \int_{Q_\delta(\xi)} |f|^m d\mu \right)^{\frac{1}{m}} < \infty,
$$

where $Q_\delta(\xi) = \{z \in \mathbb{B} : |1 - \langle z, \xi \rangle| < \delta\}$ for $\xi \in \mathbb{S}$ and $\delta > 0$. The tent-type space is a very powerful tool in studying some deep properties of function spaces, for example, in Section 6, we take the advantage of this tent-type space to study the atomic decomposition of $N(p, q, s)$-spaces with $d\mu = (1 - |z|^2)^{p+q+ns} d\lambda(z)$. Finally, based on the setting at the beginning of Section 4, we denote

$$
\|\mu\|_{CM_p} = \sup_{\xi \in \mathbb{S}, \delta > 0} \frac{\mu(Q_\delta(\xi))}{\delta^p}.
$$

8.1. **Embedding theorem of $N(p, q, s)$-spaces into the tent space.** We need the following well-known lemma (see, e.g., [14, Theorem 45]).

**Lemma 8.1.** Suppose $n + 1 + \alpha > 0$ and $\mu$ is a positive Borel measure on $\mathbb{B}$. Then the following conditions are equivalent.

(a) There exists a constant $C > 0$ such that

$$
\mu(Q_r(\xi)) \leq C r^{n+1+\alpha}
$$

for all $\xi \in \mathbb{S}$ and all $r > 0$.

(b) For each $s > 0$ there exists a constant $C > 0$ such that

$$
\int_{\mathbb{B}} \frac{(1 - |z|^2)^s d\mu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} \leq C
$$

for all $z \in \mathbb{B}$.

(c) For some $s > 0$ there exists a constant $C > 0$ such that the inequality in (1.1) holds for all $z \in \mathbb{B}$.

The following lemma plays an important role in the sequel.

**Lemma 8.2.** Let $p \geq 1, q > 0, s > \max \{0, 1 - \frac{4}{n}\}$ and for a fixed $w \in \mathbb{B}$, put

$$
K_w(z) = \frac{1 - |w|^2}{(1 - \langle z, w \rangle)^{1+s+p}}, \quad z \in \mathbb{B}.
$$

Then $\sup_{w \in \mathbb{B}} \|K_w\| \lesssim 1$. 
Proof. For any fixed \( w, a \in \mathbb{B} \), we consider two different cases.

Case I: \( s > 1 \).

By [30, Proposition 1.4.10] and the inequality

\[
|1 - \langle z, w \rangle| > \max\{1 - |z|, 1 - |w|\}, \forall z, w \in \mathbb{B},
\]

we have

\[
\int_{\mathbb{B}} |K_w(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
= \int_{\mathbb{B}} \frac{(1 - |w|^2)^p (1 - |z|^2)^{q-n-1}}{|1 - \langle z, w \rangle|^{p+q}} \cdot \frac{(1 - |a|^2)^{ns} (1 - |z|^2)^{ns}}{|1 - \langle z, a \rangle|^{2ns}} dV(z)
\]

\[
\leq (1 - |a|^2)^{ns} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{ns-n-1}}{|1 - \langle z, a \rangle|^{n+1+(ns-n-1)+ns}} dV(z) < \infty.
\]

Case II: \( s \leq 1 \). We may assume that \( s < 1 \) since the proof for the case \( s = 1 \) is similar to the case \( s < 1 \). Now we take and fix an \( L \) satisfying

\[
\max \left\{ 1, \frac{n}{q} \right\} < L < \frac{1}{1 - s}
\]

and \( \gamma = q - \frac{n+1}{L} \). Again, using [30, Proposition 1.4.10] and the fact that \( n + 1 + (q + ns - n - 1 - \gamma)L' = nsL' \), where \( L' \) is the conjugate of \( L \), we have

\[
\int_{\mathbb{B}} |K_w(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
= \int_{\mathbb{B}} \frac{(1 - |w|^2)^p (1 - |z|^2)^{\gamma}}{|1 - \langle z, w \rangle|^{p+q}} \cdot \frac{(1 - |a|^2)^{ns} (1 - |z|^2)^{\gamma+ns-n-1-\gamma}}{|1 - \langle z, a \rangle|^{2ns}} dV(z)
\]

\[
\leq \left( \int_{\mathbb{B}} \frac{(1 - |w|^2)^{pL} (1 - |z|^2)^{\gamma L}}{|1 - \langle z, w \rangle|^{(p+q)L}} dV(z) \right)^{\frac{1}{L}}
\]

\[
\cdot \left( \int_{\mathbb{B}} \frac{(1 - |a|^2)^{nsL'} (1 - |z|^2)^{(q+ns-n-1-\gamma)L'}}{|1 - \langle z, a \rangle|^{2nsL'}} dV(z) \right)^{\frac{1}{L'}} < \infty,
\]

where in the last inequality, we use the facts that \( \gamma L > -1 \) and \( (q + ns - n - 1 - \gamma)L' > -1 \). \( \square \)
Note that for $p \geq 1$ and $\alpha > -1$, by \[47\text{ Lemma 2.24}\] and \[5.8\], we have

\[
|f(z)|^p \lesssim \frac{1}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,1/2)} |f(w)|^p dV_{\alpha}(w)
\]

\[
\approx \int_{D(z,1/2)} \frac{|f(w)|^p dV_{\alpha}(w)}{1-\langle z, w \rangle^{n+1+\alpha}}
\]

\[
\leq \int_{\mathbb{B}} \frac{|f(w)|^p dV_{\alpha}(w)}{1-\langle z, w \rangle^{n+1+\alpha}}.
\]

**Lemma 8.3.** Let $p \geq 1, q > 0, s > \max \{0, 1 - \frac{2}{n}\}$, $t \geq s + \frac{2}{n}$ and $\mu$ be an $(nt)$-Carleson measure. Then for any fixed $\xi \in \mathbb{S}$ and $0 < \delta \leq 2$, we have

\[
\int_{Q_{4\delta}(\xi)} \int_{\mathbb{B}} \frac{|f(w)|^p(1-|w|^2)^\alpha}{1-\langle z, w \rangle^{n+1+\alpha}} dV(w) d\mu(z) \lesssim \delta^{nt-q} \|\mu\|_{CMnt} \|f\|^p
\]

for some $\alpha > nt - n - 1$.

**Proof.** Denote

\[
I = \int_{Q_{4\delta}(\xi)} \int_{Q_{4\delta}(\xi)} \frac{|f(w)|^p(1-|w|^2)^\alpha}{1-\langle z, w \rangle^{n+1+\alpha}} dV(w) d\mu(z)
\]

\[
= \int_{Q_{4\delta}(\xi)} \int_{Q_{4\delta}(\xi)} \frac{|f(w)|^p(1-|w|^2)^\alpha}{1-\langle z, w \rangle^{n+1+\alpha}} dV(w) d\mu(z) + \sum_{j=2}^{\infty} \int_{Q_{4\delta}(\xi)} \int_{A_j} \frac{|f(w)|^p(1-|w|^2)^\alpha}{1-\langle z, w \rangle^{n+1+\alpha}} dV(w) d\mu(z)
\]

\[
= I_1 + I_2,
\]

where $A_1 = Q_{4\delta}(\xi)$ and $A_j = Q_{4\delta}\setminus Q_{4\delta-2\delta}(\xi), j \geq 2$.

- **Estimation of $I_1$.**

Note that for $w \in Q_{4\delta}(\xi)$, we have $1-|w| \leq |1-\langle w, \xi \rangle| < 4\delta$, which implies $(1-|w|^2)^{nt-q-ns} \lesssim \delta^{nt-q-ns}$ since $t \geq s + \frac{2}{n}$. Thus, by Proposition 4.6, Lemma 8.1 and Fubini’s theorem, we have

\[
I_1 = \int_{Q_{4\delta}(\xi)} \left[ \int_{Q_{4\delta}(\xi)} \frac{(1-|w|^2)^\alpha}{1-\langle z, w \rangle^{n+1+\alpha}} d\mu(z) \right] |f(w)|^p dV(w)
\]

\[
= \int_{Q_{4\delta}(\xi)} \left[ \int_{Q_{4\delta}(\xi)} \frac{(1-|w|^2)^{\alpha+n+1-q-ns}}{1-\langle z, w \rangle^{n+1+\alpha}} d\mu(z) \right] |f(w)|^p(1-|w|^2)^{q+ns} d\lambda(w)
\]

\[
= \int_{Q_{4\delta}(\xi)} \left[ \int_{Q_{4\delta}(\xi)} \frac{(1-|w|^2)^{\alpha+n+1-nt+(nt-q-ns)}}{1-\langle z, w \rangle^{n+1+\alpha-nt+nt}} d\mu(z) \right] |f(w)|^p(1-|w|^2)^{q+ns} d\lambda(w)
\]
Using Lemma 4.1 and (8.3), we have

\[ \int_{Q_{4\delta}(\xi)} \left[ \int_{Q_{\delta}(\xi)} \frac{(1 - |w|^2)^{\alpha+n-1} - t}{|1 - \langle z, w \rangle|^{\alpha+n+1} + nt} d\mu(z) \right] |f(w)|^p (1 - |w|^2)^q d\lambda(w) \]

\[ \leq \delta^{nt-q-ns} \|\mu\|_{\mathcal{CM}_nt} \int_{Q_{4\delta}(\xi)} |f(w)|^p (1 - |w|^2)^q d\lambda(w) \]

\[ \leq \delta^{nt-q} \|\mu\|_{\mathcal{CM}_nt} \|f\|^p. \]

- **Estimation of I_2.**

Note that by [30, Proposition 5.1.2], for \( j \geq 2, z \in Q_{3\delta}(\xi) \) and \( w \in A_j \), we have

\[ |1 - \langle w, z \rangle|^{\frac{1}{2}} \geq |1 - \langle w, \xi \rangle|^{\frac{1}{2}} - |1 - \langle z, \xi \rangle|^{\frac{1}{2}} \geq (4j^{-1}\delta)^{\frac{1}{2}} - \delta^{\frac{1}{2}} \geq 2^{j-2}\delta^{\frac{1}{2}} \]

and for \( w \in Q_{4\delta}(\xi) \),

\[ (8.3) \quad 1 - |w|^2 \simeq 1 - |w| \leq |1 - \langle w, \xi \rangle| < 4^j \delta. \]

Moreover, for each \( j \geq 2 \), consider the term

\[ I_{2,j} = \int_{Q_{4\delta}(\xi)} \int_{A_j} |f(w)|^p (1 - |w|^2)^q dV(w) d\mu(z). \]

Using Lemma 4.1 and (8.3), we have

\[ I_{2,j} \leq \frac{1}{(4^j \delta)^{n+1+\alpha}} \int_{Q_{4\delta}(\xi)} \int_{A_j} |f(w)|^p (1 - |w|^2)^q dV(w) d\mu(z) \]

\[ \leq \frac{(4^j \delta)^{\alpha+n+1-q-ns}}{(4^j \delta)^{n+1+\alpha}} \int_{Q_{4\delta}(\xi)} \int_{Q_{4\delta}(\xi)} |f(w)|^p (1 - |w|^2)^q d\lambda(w) d\mu(z) \]

\[ = \frac{\mu(Q_{4\delta}(\xi))}{(4^j \delta)^q} \cdot \frac{1}{(4^j \delta)^{ns}} \int_{Q_{4\delta}(\xi)} |f(w)|^p (1 - |w|^2)^q d\lambda(w) \]

\[ \leq \frac{\delta^{nt-q}}{4^j \delta^q} \|\mu\|_{\mathcal{CM}_nt} \|f\|^p. \]

Thus,

\[ I_2 = \sum_{j=2}^{\infty} I_{2,j} \leq \sum_{j=2}^{\infty} \frac{\delta^{nt-q}}{4^j} \|\mu\|_{\mathcal{CM}_nt} \|f\|^p \leq \delta^{nt-q} \|\mu\|_{\mathcal{CM}_nt} \|f\|^p. \]

Combining both estimation of \( I_1 \) and \( I_2 \), the proof is complete. \( \Box \)

We are now ready to establish the main result in this section.

**Theorem 8.4.** Let \( p \geq 1, q > 0, s > \max \{0, 1 - \frac{q}{n}\}, t \geq s + \frac{2}{n} \) and \( \mu \) be a positive Borel measure on \( \mathbb{B} \). Then the identity operator

\[ I : \mathcal{N}(p,q,s) \rightarrow T_{p,(t-\frac{2}{n})}^{\infty}(\mu) \]

is bounded if and only if \( \mu \) is an \( (nt) \)-Carleson measure.
Proof. Sufficiency. Suppose \( \mu \) is an \((nt)\)-Carleson measure. Take some \( \alpha > nt - n - 1 \geq n \left( s + \frac{q}{n} \right) - n - 1 = q + ns - n - 1 > -1 \). By (8.2), for \( f \in \mathcal{N}(p, q, s) \), we have
\[
|f(z)|^p \lesssim \int_B \frac{|f(w)|^p(1-|w|^2)^\alpha dV(w)}{|1-\langle z, w \rangle|^{n+1+\alpha}}.
\]
Thus, for any \( \xi \in \mathbb{S} \) and \( \delta > 0 \), by Lemma 8.3, we have
\[
\frac{1}{\delta^{nt-q}} \int_{Q_\delta(\xi)} |f(z)|^p d\mu(z) \lesssim \frac{1}{\delta^{nt-q}} \int_{Q_\delta(\xi)} \int_B \frac{|f(w)|^p(1-|w|^2)^\alpha dV(w)}{|1-\langle z, w \rangle|^{n+1+\alpha}} d\mu(z) \lesssim \|\mu\|_{CM_{nt}} \|f\|^p,
\]
which implies the desired result.

Necessity. Suppose the identity operator \( I : \mathcal{N}(p, q, s) \to T_{\mu, (1-\delta)}^\infty \) is bounded. First, note that \( 1 \in \mathcal{N}(p, q, s) \) and hence for any \( \delta > 0 \), by the boundedness of \( I \), we have
\[
\frac{1}{\delta^{nt-q}} \int_{Q_\delta(\xi)} d\mu(z) \lesssim 1,
\]
which, in particular, implies that \( \mu \) is a finite measure on \( \mathbb{B} \).

For any \( \xi \in \mathbb{S} \) and \( 0 < \delta < 1 \), we consider the function
\[
K_{(1-\delta)\xi}(z) = \frac{1-|1-\delta|^2}{(1-\langle z, (1-\delta)\xi \rangle)^{1+\frac{2}{p}}}, \quad z \in \mathbb{B}.
\]
Note that for \( z \in Q_\delta(\xi) \), we have
\[
(8.4) \quad |1-\langle z, (1-\delta)\xi \rangle| \leq |1-\langle z, \xi \rangle| + |\langle z, \delta \xi \rangle| \leq 2\delta
\]
and
\[
(8.5) \quad |1-\langle z, (1-\delta)\xi \rangle| \geq 1 - (1-\delta)|\langle z, \xi \rangle| \geq 1 - (1-\delta)|z||\xi| \geq \delta,
\]
which implies that
\[
(8.6) \quad |K_{(1-\delta)\xi}(z)| \simeq \frac{2-\delta}{\delta^{\frac{2}{p}}}, \quad \xi \in \mathbb{S}, \ 0 < \delta < 1.
\]
Thus, by the boundedness of \( I \) and Lemma 8.2, we have
\[
\frac{1}{\delta^{nt-q}} \int_{Q_\delta(\xi)} |K_{(1-\delta)\xi}(z)|^p d\mu \lesssim \|K_{(1-\delta)\xi}\|^p \lesssim 1,
\]
which implies
\[
\sup_{\xi \in \mathbb{S}, \delta > 0} \frac{\mu(Q_\delta(\xi))}{\delta^{nt}} \lesssim 1,
\]
Thus, the boundedness of $T$ and hence $\mu$ is an $(nt)$-Carleson measure. \qed

8.2. Behavior of the Riemann-Stieltjes operators on $\mathcal{N}(p, q, s)$-type spaces. In this subsection, by using the embedding theorem in the previous subsection, we study the boundedness and compactness of the Riemann-Stieltjes operators on $\mathcal{N}(p, q, s)$-type spaces.

\textbf{Theorem 8.5.} Let $p \geq 1, q > 0, s_2 \geq 1 > \max \{0, 1 - \frac{2}{n}\},$ $g \in H(\mathbb{B})$ and $d\mu_{g,p,q,s_2} = |Rg(z)|^p (1 - |z|^2)^{p+q+n s_2} d\lambda(z)$. Then

(1) $T_g : \mathcal{N}(p, q, s_1) \mapsto \mathcal{N}(p, q, s_2)$ is bounded if and only if $\mu_{g,p,q,s_2}$ is an $(ns_2 + q)$-Carleson measure;

(2) $L_g : \mathcal{N}(p, q, s_1) \mapsto \mathcal{N}(p, q, s_2)$ is bounded if and only if $\|g\|_{\infty} < \infty$;

(3) $M_g : \mathcal{N}(p, q, s_1) \mapsto \mathcal{N}(p, q, s_2)$ is bounded if and only if $\mu_{g,p,q,s_2}$ is an $(ns_2 + q)$-Carleson measure and $\|g\|_{\infty} < \infty$.

\textbf{Proof.} (1) Note that $R(T_g f)(z) = f(z) Rg(z)$ and hence for any $\xi \in \mathbb{S}$ and $\delta > 0$, we have

$$
\frac{1}{\delta^{ns_2}} \int_{Q_\delta(\xi)} |R(T_g f)(z)|^p (1 - |z|^2)^{p+q+n s_2} d\lambda(z)
= \frac{1}{\delta^{ns_2}} \int_{Q_\delta(\xi)} |f(z)|^p |Rg(z)|^p (1 - |z|^2)^{p+q+n s_2} d\lambda(z)
= \frac{1}{\delta^{n(s_2 + \frac{2}{n} - \frac{2}{s})}} \int_{Q_\delta(\xi)} |f(z)|^p |Rg(z)|^p (1 - |z|^2)^{p+q+n s_2} d\lambda(z).
$$

Thus, the boundedness of $T_g$ is equivalent to the boundedness of $I : \mathcal{N}(p, q, s_1) \mapsto \mathcal{T}_{p, q, s_2}(\mu_{g,p,q,s_2})$, which, by Theorem \textbf{8.4}, is equivalent to $\|\mu_{g,p,q,s_2}\|_{C,M_{ns_2 + q}} < \infty$.

(2) \textbf{Sufficiency.} Suppose $\|g\|_{\infty} < \infty$, then

$$
\frac{1}{\delta^{ns_2}} \int_{Q_\delta(\xi)} |R(L_g f)(z)|^p (1 - |z|^2)^{p+q+n s_2} d\lambda(z)
= \frac{1}{\delta^{ns_2}} \int_{Q_\delta(\xi)} |g(z)|^p |Rf(z)|^p (1 - |z|^2)^{p+q+n s_2} d\lambda(z)
\lesssim \|g\|_{\infty}^p \|f\|_{\mathcal{N}(p,q,s_2)}^p \leq \|g\|_{\infty}^p \|f\|_{\mathcal{N}(p,q,s_1)}^p.
$$

The boundedness of $L_g$ follows by taking the supremum on both sides of the above inequality over all $\xi \in \mathbb{S}$ and $\delta > 0$.

\textbf{Necessity.} Suppose $L_g : \mathcal{N}(p, q, s_1) \mapsto \mathcal{N}(p, q, s_2)$ is bounded. Take and fix $w \in \mathbb{B}$ with $|w| > \frac{2}{3}$, and further, we take $\xi = w/|w|$. Using \textbf{8.30}, Proposition 5.1.2, it is easy to see that there exists a $\delta \in (0, 1)$,
such that
\[(8.7) \quad D\left(w, \frac{1}{2}\right) \subseteq Q_\delta(\xi) \quad \text{and} \quad 1 - |w|^2 \simeq \delta.\]

Recall that \(K_w(z) = \frac{1 - |w|^2}{(1 - \langle z, w \rangle)^{1+\frac{p}{q}}} \) and an easy calculation shows that
\[(8.8) \quad R(K_w)(z) = \left(1 + \frac{q}{p}\right) \frac{(1 - |w|^2)\langle z, w \rangle}{(1 - \langle z, w \rangle)^{2+\frac{q}{p}}}.
\]

Moreover, when \(z \in D\left(w, \frac{1}{2}\right)\), we have
\[(8.9) \quad V\left(D\left(w, \frac{1}{2}\right)\right) \simeq (1 - |w|^2)^{n+1},\]

and (see, e.g., \([20, 47]\))
\[(8.10) \quad 1 - |w|^2 \simeq 1 - |z|^2 \simeq |1 - \langle z, w \rangle|, \quad z \in D\left(w, \frac{1}{2}\right).
\]

Also note that for \(z \in D\left(w, \frac{1}{2}\right)\), we have
\[1 - |\Phi_w(z)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \langle z, w \rangle|^2} > \frac{3}{4},\]

and hence
\[1 - |\langle z, w \rangle| \leq |1 - \langle z, w \rangle| < \frac{2}{\sqrt{3}}(1 - |w|^2)^{\frac{1}{2}}(1 - |z|^2)^{\frac{1}{2}} \leq \frac{2}{\sqrt{3}}(1 - |w|^2)^{\frac{1}{2}} < \frac{2\sqrt{15}}{9},\]

which implies \(|\langle z, w \rangle| > 1 - \frac{2\sqrt{15}}{9}\), i.e. \(|\langle z, w \rangle| \simeq 1\). Thus, by \((8.8), (8.9), (8.10), [47, Lemma 2.24]\) and the boundedness of \(L_g\), we have
\[
\begin{align*}
|g(w)|^p & \lesssim \frac{1}{V\left(D\left(w, \frac{1}{2}\right)\right)} \int_{D\left(w, \frac{1}{2}\right)} |g(z)|^p V(z) \\
& \lesssim \frac{1}{(1 - |w|^2)^{n+1}} \int_{D\left(w, \frac{1}{2}\right)} |g(z)|^p V(z) \\
& \lesssim \frac{1}{\delta^{ns_2}} \int_{D\left(w, \frac{1}{2}\right)} \frac{|g(z)|^p |\langle z, w \rangle|^p (1 - |w|^2)^p}{|1 - \langle z, w \rangle|^{2p+q}} (1 - |z|^2)^{p+q+ns_2} d\lambda(z) \\
& \lesssim \frac{1}{\delta^{ns_2}} \int_{Q_\delta(\xi)} \frac{|g(z)|^p |\langle z, w \rangle|^p (1 - |w|^2)^p}{|1 - \langle z, w \rangle|^{2p+q}} (1 - |z|^2)^{p+q+ns_2} d\lambda(z) \\
& \lesssim \frac{1}{\delta^{ns_2}} \int_{Q_\delta(\xi)} |g(z)|^p |R(K_w)(z)|^p (1 - |z|^2)^{p+q+ns_2} d\lambda(z) \\
& \lesssim |L_g(K_w)|_{\mathcal{N}(p,q,s_2)}^p \lesssim |L_g|^p |K_w|^p_{\mathcal{N}(p,q,s_1)} \lesssim 1,
\end{align*}
\]
which implies $|g(w)| \lesssim 1$ for $|w| > \frac{2}{3}$. The desired result then follows from the maximum modulus principle.

(3) Sufficiency. Recall that for $f \in H(\mathbb{B})$, we have

$$M_g f(z) = g(z) f(z) = g(0) f(0) + T_g f(z) + L_g f(z).$$

Thus, the sufficient part is obvious from (1) and (2), as well as the inequality

$$|f(z)| \lesssim \frac{\|f\|_{N(p,q,s)}}{(1-|z|^2)^{\frac{1}{p}}}, \quad z \in \mathbb{B},$$

which follows from (2.1).

Necessity. Suppose $M_g : \mathcal{N}(p,q,s_1) \mapsto \mathcal{N}(p,q,s_2)$ is bounded. Again, for any fixed $w \in \mathbb{B}$, it is clear that $K_w \in \mathcal{N}(p,q,s_1)$ and hence $gK_w \in \mathcal{N}(p,q,s_2)$. Using (8.11) and Lemma 8.2 we have

$$|g(z)K_w(z)| \lesssim \frac{|M_g K_w|_{N(p,q,s_2)}}{(1-|z|^2)^{\frac{1}{p}}} \lesssim \frac{|M_g|}{(1-|z|^2)^{\frac{1}{p}}}, \quad z \in \mathbb{B}.$$

Put $z = w$ and we get $|g(w)| \leq \|M_g\|$ for all $w \in \mathbb{B}$. Thus, $\|g\|_{\infty} < \infty$, which, by (2), implies $L_g : \mathcal{N}(p,q,s_1) \mapsto \mathcal{N}(p,q,s_2)$ is bounded. Consequently, $T_g f = M_g f - L_g f - f(0)g(0)$ gives the boundedness of $T_g : \mathcal{N}(p,q,s_1) \mapsto \mathcal{N}(p,q,s_2)$, which implies $\mu_{g,p,q,s_2}$ is an $(ns_2 + q)$-Carleson measure.

Note that when $s_1 = s_2 = s$, clearly if $g \in H^{\infty}(\mathbb{B})$, $M_g$ is bounded on $\mathcal{N}(p,q,s)$. Keeping this trivial but crucial fact in mind, we can refine our result above as follows.

**Theorem 8.6.** Let $p \geq 1, q > 0, s_2 \geq s_1 > \max\left\{0, 1 - \frac{2}{p}\right\}$ and $g \in H(\mathbb{B})$ with $\mu_{g,p,q,s_2}$ defined as above. Consider the following statements:

1. $\|g\|_{\infty} < \infty$;
2. $L_g : \mathcal{N}(p,q,s_1) \mapsto \mathcal{N}(p,q,s_2)$ is bounded;
3. $M_g : \mathcal{N}(p,q,s_1) \mapsto \mathcal{N}(p,q,s_2)$ is bounded;
4. $T_g : \mathcal{N}(p,q,s_1) \mapsto \mathcal{N}(p,q,s_2)$ is bounded;
5. $\mu_{g,p,q,s_2}$ is an $(ns_2 + q)$-Carleson measure.

We have (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) $\iff$ (5).

**Proof.** Clearly, (1) $\iff$ (2), (4) $\iff$ (5), (3) $\implies$ (1) and (3) $\implies$ (4) are shown in Theorem 8.5. It suffices to show (1) $\implies$ (3). Indeed, since $\|g\|_{\infty} < \infty$, we have $M_g$ is bounded on $\mathcal{N}(p,q,s_2)$, which, by Theorem 8.5 with $s_1 = s_2$, implies that $\mu_{g,p,q,s_2}$ is an $(ns_2 + q)$-Carleson measure. Thus, both $L_g$ and $T_g$ are bounded, and hence condition (3) is established. \(\square\)
To study the compactness, we need the following lemma, whose proof is standard by using Montel’s theorem and Fatou’s lemma, and hence we omit it here.

**Lemma 8.7.** Let \( p \geq 1, q > 0 \) and \( s_2 \geq s_1 > \max\{0, 1 - \frac{q}{n}\} \) and \( g \in H(B) \). Then the following statements are equivalent:

(i) \( T_g \) (respectively \( L_g \)) is a compact operator from \( \mathcal{N}(p, q, s_1) \) to \( \mathcal{N}(p, q, s_2) \);

(ii) For every bounded sequence \( \{f_j\} \) in \( \mathcal{N}(p, q, s_1) \) such that \( f_j \to 0 \) uniformly on compact sets of \( B \), then the sequence \( \{T_g(f_j)\} \) (respectively \( L_g(f_j) \)) converges to zero in the norm of \( \mathcal{N}(p, q, s_2) \).

For \( \xi \in S \) and \( \delta > 0 \), set

\[
D'_\delta(\xi) = \{ \gamma \in S : |1 - \langle \gamma, \xi \rangle| < \delta \},
\]

which is known as the nonisotropic metric ball on \( S \) with radius \( \delta^{1/2} \). Note that \( D'_\delta = S \) when \( \delta > 2 \). (see, e.g., [30, Page 65]). Using this conception, we can define the following tent

\[
\hat{Q}_\delta(\xi) = \left\{ z \in B : \frac{z}{|z|} \in D'_\delta(\xi), 1 - \delta < |z| < 1 \right\}.
\]

It is known that \( Q_\delta(\xi) \subset \hat{Q}_{4\delta}(\xi) \subset Q_{16\delta}(\xi) \) (see, e.g., [16, Theorem 4.1.6]). Hence in the definition of (vanishing) Carleson measure, we can replace \( Q_\delta(\xi) \) by \( \hat{Q}_\delta(\xi) \).

We have the following important covering lemma of \( D'_\delta(\xi) \) (see, e.g., [28, Lemma 3.3]).

**Lemma 8.8.** Given any natural number \( m \), there exists a natural number \( N \) such that every non-isotropic ball of radius \( \delta \leq 2 \), can be covered by \( N \) non-isotropic balls of radius \( \delta/m \). Moreover, \( N \) can be taken as

\[
\frac{\Gamma(n + 1)}{4\Gamma^2 \left( \frac{n}{2} + 1 \right)} \cdot \left( 2m + \frac{1}{2} \right)^n.
\]

**Lemma 8.9.** Let \( f \in H^\infty \). Then for any \( z_1, z_2 \in B \),

\[
|f(z_1) - f(z_2)| \leq 2\|f\|_\infty |\Phi_{z_1}(z_2)|.
\]

(see, e.g., [28, Lemma 3.4]).

**Theorem 8.10.** Let \( p \geq 1, q > 0, s_2 \geq s_1 > \max\{0, 1 - \frac{q}{n}\} \), \( g \in H(B) \) and \( d\mu_{g,p,q,s_2} = |Rg(z)|^p(1 - |z|^2)^{p+q+ns_2}d\lambda(z) \). Then

1. \( T_g : \mathcal{N}(p, q, s_1) \mapsto \mathcal{N}(p, q, s_2) \) is compact if and only if \( \mu_{g,p,q,s_2} \) is a vanishing \( (ns_2 + q) \)-Carleson measure;
2. \( L_g : \mathcal{N}(p, q, s_1) \mapsto \mathcal{N}(p, q, s_2) \) is compact if and only if \( g = 0 \);
3. \( M_g : \mathcal{N}(p, q, s_1) \mapsto \mathcal{N}(p, q, s_2) \) is compact if and only if \( g = 0 \).
Proof. (1) Sufficiency. Suppose $\mu_{g,p,q,s_2}$ is a vanishing $(n s_2 + q)$-Carleson measure. Let $\{f_j\}$ be any bounded sequence in $\mathcal{N}(p, q, s_1)$ and $f_j \to 0$ uniformly on compact sets of $\mathbb{B}$. By Lemma 8.7, it suffices to prove $\lim_{j \to \infty} \|T_g f_j\|_{\mathcal{N}(p,q,s_2)} = 0$.

Let $\chi_E$ denote the characteristic function of a set $E$ of $\mathbb{B}$. For $r \in (0,1)$, define the cut-off measure $d\mu_r = \chi_{\{z \in \mathbb{B} : |z| > r\}} d\mu_{g,p,q,s_2}$ and for fixed $\xi \in \mathbb{S}$ and $0 < \delta < 1$, we have

$$
\frac{1}{\delta^{n s_2}} \int_{Q_\delta(\xi)} |R(T_g f_j)(z)|^p (1 - |z|^2)^{p + q + ns} d\lambda(z) 
= \frac{1}{\delta^{n s_2}} \int_{Q_\delta(\xi)} \left| f_j(z) \right|^p |R g(z)|^p (1 - |z|^2)^{p + q + ns} d\lambda(z) 
= \frac{1}{\delta^{n s_2}} \int_{Q_\delta(\xi)} \left| f_j(z) \right|^p d\mu_{g,p,q,s_2}(z) 
= \frac{1}{\delta^{n s_2}} \int_{Q_\delta(\xi)} \left| f_j(z) \right|^p \chi_{\{z \in \mathbb{B} : |z| \leq r\}} d\mu_{g,p,q,s_2}(z) + \frac{1}{\delta^{n s_2}} \int_{Q_\delta(\xi)} \left| f_j(z) \right|^p d\mu_r(z) 
= J_{1,r} + J_{2,r}.
$$

By the proof of Theorem 8.4, we have for any $j \in \mathbb{N}$,

$$
J_{2,r} = \frac{1}{\delta^{n s_2 + \frac{d}{n} - \frac{2}{m}}} \int_{Q_\delta(\xi)} \left| f_j(z) \right|^p d\mu_r(z) 
\lesssim \|\mu_r\|_{cM_{n s_2 + q}} \|f_j\|^p_{\mathcal{N}(p,q,s_1)} \lesssim \|\mu_r\|_{cM_{n s_2 + q}}.
$$

Claim:

$$
\|\mu_r\|_{cM_{n s_2 + q}} \to 0 \quad \text{as} \quad r \to 1^-.
$$

Indeed, for any $\varepsilon > 0$, by our assumption, there exists a $\delta_0 > 0$, such that

$$
\mu_{g,p,q,s_2} \left( \tilde{Q}_{\delta_1}(\xi) \right) < \varepsilon \delta_1^{n s_2 + q},
$$

for all $\delta_1 \leq \delta_0$ and $\xi \in \mathbb{S}$ uniformly. If $\delta \leq \delta_0$, it is clear that

$$
(8.12) \quad \mu_r \left( \tilde{Q}_\delta(\xi) \right) \leq \mu_{g,p,q,s_2} \left( \tilde{Q}_\delta(\xi) \right) < \varepsilon \delta^{n s_2 + q}.
$$

If $\delta > \delta_0$, take and fix $m = \left\lfloor \frac{\delta}{\delta_0} \right\rfloor + 1 \leq \frac{2\delta}{\delta_0}$, where $\lfloor \cdot \rfloor$ is the floor function. Note that $\frac{\delta}{m} < \delta_0$. Then by Lemma 8.8, we have $Q_{\delta}^\prime$ can be covered by $N$ balls $Q_{\delta^m}^\prime$ on $\mathbb{S}$ with $N \simeq m^n$. Thus, by the definition of $\tilde{Q}_\delta(\xi)$, it follows that

$$
\tilde{Q}_\delta \cap \left\{ z \in \mathbb{B} : |z| > 1 - \frac{\delta_0}{m} \right\} \subset \bigcup_N \tilde{Q}_{\delta^m}.
$$
Putting \( r_0 = 1 - \frac{\delta_0}{m} \), and for \( r_0 < r < 1 \), we have

\[
\mu_r(\hat{Q}_0(\xi)) \leq \mu_r \left( \bigcup_N \hat{Q}_{\delta/m} \right) \leq \mu_r \left( \bigcup_N \hat{Q}_{\delta_0} \right) \leq \sum_N \mu_r(\hat{Q}_{\delta_0}) \leq \sum_N \mu_{g,p,q,s_2}(\hat{Q}_{\delta_0}) \leq N \varepsilon \delta_0^{n_{s_2} + q} \lesssim \delta_0^{n_{s_2} + q} \delta_0^{n_{s_2} + q},
\]

where in the last inequality, we use the fact that \( s_2 > 1 - \frac{q}{n} \). Combining this estimation with (8.12), we prove the claim.

Now for any \( \varepsilon > 0 \), by the above claim and estimation on \( J_{2,r} \), there exists a \( r_1 \in (0, 1) \), such that when \( r_1 \leq r < 1 \) and \( j \in \mathbb{N} \), we have

\[
J_{2,r} = \frac{1}{\delta^{ns_2}} \int_{Q_\delta(\xi)} |f_j(z)|^p d\mu_r(z) < \varepsilon.
\]

Fix \( r_1 \). Noting that for \( J_{1,r_1} \), we have

\[
J_{1,r_1} = \frac{1}{\delta^{ns_2}} \int_{Q_\delta(\xi)} |f_j(z)|^p |Rg(z)|^p \chi_{\{z \in \mathbb{B} : |z| \leq r_1\}} (1 - |z|^2)^{p + q + ns_2} d\lambda(z)
\]

\[
= \sup_{|z| \leq r_1} |f_j(z)Rg(z)|^p \cdot \frac{1}{\delta^{ns_2}} \int_{Q_\delta(\xi)} (1 - |z|^2)^{q + ns_2} d\lambda(z)
\]

\[
\lesssim \|1\|_{N(p,q,s_2)}^p \cdot \sup_{|z| \leq r_1} |f_j(z)Rg(z)|^p
\]

\[
\lesssim \sup_{|z| \leq r_1} |f_j(z)Rg(z)|^p,
\]

where in the last inequality, we use [7, Propostion 2.8] and 1 refers to the constant function \( f(z) = 1, \forall z \in \mathbb{B} \). Since \( f_j \to 0 \) as \( j \to \infty \) uniformly on the compact subset of \( \mathbb{B} \), there exists some \( j_0 \in \mathbb{N} \), such that when \( j > j_0 \), we have \( J_{1,r_1} \lesssim \varepsilon \). Combining the above estimation on \( J_{1,r_1} \) and \( J_{2,r_2} \), we get the desired result.

**Necessity.** Suppose \( T_g : N(p,q,s_1) \to N(p,q,s_2) \) is compact. For any \( \xi \in \mathbb{S}, \delta_j \to 0 \) as \( j \to \infty \), we consider the functions

\[
K_{(1-\delta_j)}(z) = \frac{1 - |1 - \delta_j|^2}{(1 - \langle z, (1 - \delta_j)\xi \rangle)^{1+\frac{q}{p}}}, \quad z \in \mathbb{B}.
\]

By Lemma [8.2], it is clear that \( \sup_{j \in \mathbb{N}} \|K_{(1-\delta_j)}\|_{N(p,q,s_1)} \lesssim 1 \). Moreover, it is easy to see that \( K_{(1-\delta_j)} \to 0 \) uniformly on compact subsets of \( \mathbb{B} \).
as \( j \to \infty \). Thus, using (8.6), we have

\[
\frac{\mu_{g,p,q,s_2}(Q_{\delta_j}(\xi))}{\delta_j^{n_{s_2}+q}} = \frac{1}{\delta_j^{n_{s_2}+q}} \int_{Q_{\delta_j}(\xi)} d\mu_{g,p,q,s_2}(z)
\]

\[
\leq \frac{1}{\delta_j^{n_{s_2}}} \int_{Q_{\delta_j}(\xi)} |K_{(1-\delta_j)\xi}(z)|^p d\mu_{g,p,q,s_2}(z)
\]

\[
= \frac{1}{\delta_j^{n_{s_2}}} \int_{Q_{\delta_j}(\xi)} |K_{(1-\delta_j)\xi}(z)|^p |Rg(z)|^p (1-|z|^2)^{p+q+s} d\lambda(z)
\]

\[
= \frac{1}{\delta_j^{n_{s_2}}} \int_{Q_{\delta_j}(\xi)} |R(T_gK_{(1-\delta_j)\xi})(z)|^p (1-|z|^2)^{p+q+s} d\lambda(z)
\]

\[
\lesssim \|T_g(K_{(1-\delta_j)\xi})\|_{N_{(p,q,s_2)}},
\]

which, by Lemma 8.7, converges to 0 as \( j \to \infty \). Thus, \( \mu_{g,p,q,s_2} \) is a vanishing \((n_{s_2}+q)\)-Carleson measure.

(2) The sufficiency is obvious and we only verify the necessity. By Theorem 8.5 (ii), the compactness of \( L_g \) implies that \( g \in H^\infty \).

We prove the statement by contradiction. Assume that \( g \) is not identically equal to 0, i.e., there exists some \( w_0 \in \mathbb{B} \) such that \( |g(w_0)| = \varepsilon_0 > 0 \). By Lemma 8.9 we have

\[
|g(z_1) - g(z_2)| \leq 2\|g\|_{\infty} |\Phi_{z_1}(z_2)|, \quad z_1, z_2 \in \mathbb{B}.
\]

This inequality implies that there is a sufficient small \( r > 0 \) such that for any \( a \in \mathbb{B} \), \( |g(z)| \geq \frac{\varepsilon_0}{2} \) for all \( z \) satisfying \( |\Phi_a(z)| < r \). Fix the \( r \) chosen above and note that the choice of \( r \) only depends on \( g \).

By the maximum modulus principle, we can take a sequence \( \{w_m\}_{m \geq 1} \) of points in \( \mathbb{B} \) with \( |w_m| \to 1 \) as \( m \to \infty \), such that

\[
\max \{|w_0|, r^{1/2}\} < |w_1| < \cdots < |w_m| < \cdots < 1,
\]

and

\[
|g(w_m)| \geq \varepsilon, \quad \forall m \geq 1.
\]

Putting \( \xi_j = \frac{w_j}{|w_j|} \), \( j \in \mathbb{N} \) and applying the same argument in (8.7), we can find a sequence \( \{\delta_j\}_{j \in \mathbb{N}} \) such that

\[
D(w_j, r) \subseteq Q_{\delta_j}(\xi_j) \quad \text{and} \quad 1 - |w_j|^2 \simeq \delta_j, \quad j \in \mathbb{N}.
\]

Thus, the above argument implies for each \( j \in \mathbb{N} \), \( |g(z)| \geq \frac{\varepsilon_0}{2} \) for those \( z \) satisfying \( |\Phi_{w_j}(z)| < r \).
Again, we consider the test functions
\[ K_j(z) = \frac{1 - |w_j|^2}{(1 - \langle z, w_j \rangle)^{1 + \frac{2}{p}}}, \quad z \in \mathbb{B}, j \in \mathbb{N}. \]

It is clear that \( \sup_{j \in \mathbb{N}} \|K_j\|_{N(p,q,s_2)} \leq 1 \) and \( K_j \) converges to 0 as \( j \to \infty \) uniformly on compact subset of \( \mathbb{B} \). Thus, by Lemma 8.7, we have \( \|L_{q}(K_j)\|_{N(p,q,s_2)} \to 0 \) as \( j \to \infty \).

Note that \( |1 - \langle z, w_j \rangle| \approx \delta_j \) for \( z \in Q_{\delta_j}(\xi_j) \) and \( |\langle z, w_j \rangle| \approx 1 \) for \( z \in D(w_j, r) \). Indeed, the first assertion follows from (8.4) and (8.5) and for the second one, we have for \( z \in D(w_j, r) \),

\[ 1 - |\Phi_{w_j}(z)|^2 = \frac{(1 - |w_j|^2)(1 - |z|^2)}{|1 - \langle z, w_j \rangle|^2} > 1 - r^2, \]

and hence

\[ 1 - |\langle z, w_j \rangle| \leq |1 - \langle z, w_j \rangle| < \frac{1}{\sqrt{1 - r^2}}(1 - |w_j|^2)^{\frac{1}{2}}(1 - |z|^2)^{\frac{1}{2}} \leq \frac{1}{\sqrt{1 - r^2}}(1 - |w_j|^2)^{\frac{1}{2}} \leq \sqrt{\frac{1}{1 + r}}, \]

which implies the desired assertion.

Thus, for each \( j \in \mathbb{N} \), by (8.8), we have

\[ \|L_{q}(K_j)\|_{N(p,q,s_2)}^p \geq \frac{1}{\delta_j^{ns_2}} \int_{Q_{\delta_j}(\xi_j)} |R(K_j)(z)|^p |g(z)|^p (1 - |z|^2)^{p+q+ns_2} d\lambda(z) \]

\[ \geq \frac{1}{\delta_j^{ns_2}} \int_{Q_{\delta_j}(\xi_j)} \frac{(1 - |w_j|^2)^p |\langle z, w_j \rangle|^p}{|1 - \langle z, w_j \rangle|^{2p+q}} |g(z)|^p (1 - |z|^2)^{p+q+ns_2} d\lambda(z) \]

\[ \geq \frac{1}{\delta_j^{ns_2}} \int_{|\Phi_{w_j}(z)| < r} \frac{(1 - |w_j|^2)^p |\langle z, w_j \rangle|^p}{|1 - \langle z, w_j \rangle|^{2p+q}} |g(z)|^p (1 - |z|^2)^{p+q+ns_2} d\lambda(z) \]

\[ \geq \left( \frac{\varepsilon_0}{2} \right)^p \cdot \frac{(1 - |w_j|^2)^{ns_2-n-1}}{\delta_j^{ns_2}} \cdot V(D(w_j, r)) \geq \left( \frac{\varepsilon_0}{2} \right)^p, \]

which contradicts to Lemma 8.7

(3) Again the sufficiency is obvious and for the necessity, we suppose that \( M_j : \mathcal{N}(p, q, s_1) \to \mathcal{N}(p, q, s_2) \) is compact. Then by Theorem 8.5 we have \( g \in H^\infty \). Let \( \{w_j\}_{j \geq 1} \) be a sequence in \( \mathbb{B} \) such that \( |w_j| \to 1 \), and

\[ K_j(z) = \frac{1 - |w_j|^2}{(1 - \langle z, w_j \rangle)^{1 + \frac{2}{p}}}, \quad z \in \mathbb{B}, j \in \mathbb{N} \]
as above. Then \( \sup_{j \in \mathbb{N}} \| K_j \|_{\mathcal{N}(p,q,s)} \lesssim 1 \) and \( K_j \) converges to 0 as \( j \to \infty \) uniformly on compact subset of \( \mathbb{B} \). Thus, \( \| M_g(K_j) \|_{\mathcal{N}(p,q,s)} \to 0 \) as \( j \to \infty \) since \( M_g \) is compact. Since 

\[
|g(z)K_j(z)| = |M_g(K_j)(z)| \lesssim \frac{\| M_g(K_j) \|_{\mathcal{N}(p,q,s)}}{(1 - |z|^2)^{\frac{2}{p}}}, \quad \forall z \in \mathbb{B},
\]

by letting \( z = w_j \), we get

\[
|g(w_j)| \lesssim \| M_g(K_j) \|_{\mathcal{N}(p,q,s)},
\]

hence \( g(w_j) \to 0 \) as \( j \to \infty \). Since \( g \) is bounded holomorphic function on \( \mathbb{B} \), it follows that \( g = 0 \). \( \square \)

8.3. Multipliers of \( \mathcal{N}(p,q,s) \)-type spaces. In this subsection, we discuss the pointwise multipliers of the \( \mathcal{N}(p,q,s) \)-type spaces. Let \( X, Y \) be two spaces of holomorphic functions in \( \mathbb{B} \). We call \( \varphi \in H(\mathbb{B}) \) a pointwise multiplier from \( X \) to \( Y \) if

\[
\varphi f \in Y
\]

for all \( f \in X \). The collection of all such functions \( \varphi \) is denoted by \( M(X,Y) \). When \( X = Y \), the set \( M(X,Y) \) is denoted simply by \( M(X) \). It is clear that from Theorem 8.6, we have

\[
(8.13) \quad M(\mathcal{N}(p,q,s)) = H^\infty.
\]

We are interested in the following question: what can we say about \( M(X,\mathcal{N}(p,q,s)) \) if \( X \) is replaced by another function space?

Recall that in Corollary 4.3 we have shown that if \( q > n \), then \( A^p_{q-n-1} \subseteq \mathcal{N}(p,q,s) \). Thus, it turns out that it is a natural choice if we consider the case when \( X \) is the Bergman space \( A^p_\alpha \), in particular for the case \( \alpha = p - n - 1 \). However, before we go further, we would like to remove the restriction \( \alpha > -1 \) in the definition of the Bergman space first with considering a general type of Bergman space.

For any positive \( p \) and real \( \alpha \), we let \( N \) be the smallest nonnegative integer such that \( pN + \alpha > -1 \) and recall that the general Bergman space (still denote it as \( A^p_\alpha \)) is defined as the collection of all \( f \in H(\mathbb{B}) \) such that

\[
\| f \|_{p,\alpha} := |f(0)| + \left( \int_{\mathbb{B}} (1 - |z|^2)^{pN} |R^N f(z)|^p dV_\alpha(z) \right)^{1/p} < \infty.
\]

Once again, the \( A^p_\alpha \) becomes a Banach space when \( p \geq 1 \) and a complete metric space when \( 0 < p < 1 \). Moreover, for \( \beta \) real, let \( k \) be the
smallest nonnegative integer greater than $\beta$ and define the holomorphic Lipschitz spaces by

$$\Lambda_\beta := \left\{ f \in H(\mathbb{B}) : \| f \|_{\Lambda_\beta} = |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2)^{k-\beta}|R^k f(z)| < \infty \right\}.$$

We have the following observation, which generalizes our previous Corollary 4.3.

**Lemma 8.11.** Let $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$, $m$ the smallest nonnegative integer such that $mp + q - n > 0$ and $0 < \varepsilon < mp + q - n$. Then we have

$$\Lambda_{n+\varepsilon-q} \subseteq A_{q-n-1} \subseteq \bigcap_{s > \max\{0, 1 - \frac{q}{n}\}} N^0(p, q, s).$$

**Proof.** For the first inclusion, first we note that $m$ is the smallest nonnegative integer such that $m - n + \varepsilon - q > 0$. Then we have

$$\| f \|_{p, q-n-1} \simeq |f(0)|^p + \int_{\mathbb{B}} |R^m f(z)|^p (1 - |z|^2)^{mp+q-n-1} dV(z)$$

$$= |f(0)|^p + \int_{\mathbb{B}} |R^m f(z)|^p (1 - |z|^2)^{mp+q-n-\varepsilon} (1 - |z|^2)^{\varepsilon-1} dV(z)$$

$$\lesssim \| f \|_{\Lambda_{n+\varepsilon-q}},$$

which implies the desired result.

Let $f \in A_{q-n-1}$. We have to show $f \in \bigcap_{s > \max\{0, 1 - \frac{q}{n}\}} N^0(p, q, s)$. Clearly, by Theorem 5.4, it suffices to show for any $s > \max\{0, 1 - \frac{q}{n}\},$

$$|R^m f(z)|^p (1 - |z|^2)^{mp+q+ns} d\lambda(z)$$

is a vanishing $(ns)$-Carleson measure. Thus, for any $\xi \in \mathbb{S}$ and $\delta \in (0, \frac{1}{10m})$, we have

$$\frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} |R^m f(z)|^p (1 - |z|^2)^{mp+q+ns} d\lambda(z)$$

$$\leq \frac{1}{\delta^{ns}} \int_{\mathbb{Q}_{4\delta}(\xi)} |R^m f(z)|^p (1 - |z|^2)^{mp+q+ns} d\lambda(z)$$

$$\lesssim \int_{1-4\delta} r^{2n-1}(1 - r^2)^{mp+q-n-1} \left( \int_{D_{4\delta}(\xi)} |R^m f(r\xi)|^p d\sigma(\xi) \right) dr$$

$$= \int_{\mathbb{Q}_{4\delta}(\xi)} |R^m f(z)|^p (1 - |z|^2)^{mp+q-n-1} dV(z),$$
which clearly converges to zero uniformly with respect to all $\xi \in S$ as $\delta \to 0$. The proof is complete. \hfill $\square$

Remark 8.12. From the above lemma, it is clear that by (8.13)

$$H^\infty \subseteq M(\Lambda_{q_n-1}, \mathcal{N}(p, q, s)) \subseteq M(\Lambda_{n+q-1}, \mathcal{N}(p, q, s)).$$

8.3.1. The space $M(A_{p,q,s}^\alpha, \mathcal{N}(p, q, s))$.

Lemma 8.13. Let $p \geq 1$ and $\alpha, b \in \mathbb{R}$ satisfying that $b$ is neither $0$ nor a negative integer, and

$$b > n + \frac{\alpha + 1}{p}.$$ 

Let further, for each $w \in \mathbb{B}$,

$$J_w(z) = \frac{(1 - |w|^2)^{b-n+1+\alpha p}}{(1 - \langle z, w \rangle)^b}.$$ 

Then $\sup_{w \in \mathbb{B}} \|J_w\|_{p,\alpha} \leq 1$.

Proof. The above lemma is an easy consequence of the atomic decomposition for $A_{p,q}^\alpha$ (see, e.g., [44, Theorem 32]) and hence the proof is omitted here. \hfill $\square$

The following result gives a general description of the space of multipliers $M(A_{p,q,s}^\alpha, \mathcal{N}(p, q, s))$.

Theorem 8.14. Let $q > 0$ and $s > \max\{0, 1 - \frac{2}{n}\}$. Then we have the following assertions:

1. If $p \geq 1$ and $n + 1 + \alpha < 0$, then

$$M(A_{p,q}^\alpha, \mathcal{N}(p, q, s)) = \mathcal{N}(p, q, s).$$

2. If $p = 1$ and $n + 1 + \alpha = 0$, then

$$M(A_{p,q}^\alpha, \mathcal{N}(p, q, s)) = \mathcal{N}(p, q, s).$$

3. If $p > 1$ and $n + 1 + \alpha = 0$, then

$$M(A_{p,q}^\alpha, \mathcal{N}(p, q, s)) \subseteq \mathcal{N}(p, q, s).$$

Conversely, if $\varphi \in H(\mathbb{B})$ satisfies

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\varphi(z)|^p \left(\frac{2}{1 - |z|^2}\right)^{p-1} (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty,$$

then we have $\varphi \in M(A_{p,q}^\alpha, \mathcal{N}(p, q, s))$.

4. If $p \geq 1$ and $n + 1 + \alpha > 0$, then

(a) If $n + 1 + \alpha > q$, then $M(A_{p,q}^\alpha, \mathcal{N}(p, q, s))$ only contains the constant function $\varphi(z) \equiv 0$;
(b) If \( n + 1 + \alpha = q \), then \( M(A_\alpha^p, N(p, q, s)) = H^\infty \);

(c) If \( n + 1 + \alpha < q \), then 

\[ N(p, q - n - 1 - \alpha, s) \subseteq M(A_\alpha^p, N(p, q, s)) \subseteq N(p, q, s) \cap A^{\frac{q - (n + 1 + \alpha)}{p}}(B). \]

Proof. (1)&(2) It is clear that 

\[ M(A_\alpha^p, N(p, q, s)) \subseteq N(p, q, s) \]

since the constant function \( \varphi(z) \equiv 1 \) is in \( A_\alpha^p \).

Conversely, by \[44, \text{Thereom 21, 22(a)}\], we know that all the functions in \( A_\alpha^p \) are bounded under the assumption of (1) and (2), which implies the desired result.

(3) The first claim is clear. Now take any \( \varphi \in H(B) \) satisfying (8.15). To prove \( \varphi \in M(A_\alpha^p, N(p, q, s)) \), it suffices to show that \( \varphi f \in N(p, q, s) \) for any \( f \in A_{p-n-1} \). Indeed, by \[44, \text{Theroem 22(b)}\], we have for any \( \xi \in S \) and \( \delta > 0 \), 

\[
\frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} |\varphi(z)f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z) 
\leq \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} |\varphi(z)|^p \left( \log \frac{2}{1 - |z|^2} \right)^{p-1} (1 - |z|^2)^{q+ns} d\lambda(z) 
< \infty,
\]

where the last inequality follows from (8.15) and Lemma 8.1. By (4.1), we get the desired result.

(4) Take \( g \in M(A_\alpha^p, N(p, q, s)) \). By Lemma 8.13, we have \( gJ_w \in N(p, q, s), \forall w \in B \), and 

\[
\|gJ_w\| \leq \|M_g\| \|J_w\|_{p,\alpha} \leq \|M_g\|.
\]

Thus, by Proposition 2.1, for each fixed \( w \in B \),

\[
|g(z)J_w(z)| \lesssim \frac{\|gJ_w\|}{(1 - |z|^2)^{\frac{q+ns}{p}}} \lesssim \frac{1}{(1 - |z|^2)^{\frac{q+ns}{p}}}, \quad \forall z \in B.
\]

Let \( z = w \). We have 

(8.16) \[ |g(w)|(1 - |w|^2)^{\frac{q-n-1-\alpha}{p}} \lesssim 1. \]

(4a) The desired claim follows from the maximum modulus principle since it is clear that \( \lim_{|w| \to 1^-} |g(w)| = 0 \) by (8.16).

(4b) From (8.14), we already know that \( H^\infty \subseteq M(A_{q-n-1}^p, N(p, q, s)) \), while \( M(A_{q-n-1}^p, N(p, q, s)) \subseteq H^\infty \) clearly follows from (8.16).

(4c) The second inclusion clearly follows from (8.16) and the fact that the constant function \( \varphi(z) \equiv 1 \) belongs to \( A_\alpha^p \). We have to show
the first inclusion, that is, for any fixed \( g \in \mathcal{N}(p,q-n-1-\alpha,s) \), the multiplication operator \( M_g \) is bounded from \( A^p_\alpha \) to \( \mathcal{N}(p,q,s) \).

Indeed, for any \( f \in A^p_\alpha \), we have
\[
\|gf\|_{\mathcal{N}(p,q,s)}^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)g(z)|^p (1 - |z|^2)^q (1 - |\Phi_\alpha(z)|^2)^{ns} d\lambda(z)
\leq \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |g(z)|^p (1 - |z|^2)^q \cdot \frac{\|f\|_{p,\alpha}^p}{(1 - |z|^2)^{n+1+\alpha}} (1 - |\Phi_\alpha(z)|^2)^{ns} d\lambda(z)
\leq \|g\|_{\mathcal{N}(p,q-n-1-\alpha,s)} \|f\|_{p,\alpha}^p.
\]

Here, in the first inequality, we use the following estimation
\[
|f(z)| \leq \frac{C\|f\|_{p,\alpha}}{(1 - |z|^2)^{\alpha + \frac{1+\alpha}{p}}},
\]
where \( f \in A^p_\alpha \) where \( p > 0 \) and \( n + 1 + \alpha > 0 \) (see, e.g. [44, Theorem 20]).

Similarly, we have the following description on the multipliers between \( A^p_\alpha \) and \( \mathcal{N}^0(p,q,s) \).

**Theorem 8.15.** Let \( q > 0 \) and \( s > \max \{0, 1 - \frac{2}{n} \} \). Then we have the following assertions:

1. If \( p \geq 1 \) and \( n + 1 + \alpha < 0 \), then
   \( M(A^p_\alpha, \mathcal{N}^0(p,q,s)) = \mathcal{N}^0(p,q,s) \).

2. If \( p = 1 \) and \( n + 1 + \alpha = 0 \), then
   \( M(A^p_\alpha, \mathcal{N}^0(p,q,s)) = \mathcal{N}^0(p,q,s) \).

3. If \( p > 1 \) and \( n + 1 + \alpha = 0 \), then
   \( M(A^p_\alpha, \mathcal{N}^0(p,q,s)) \subseteq \mathcal{N}^0(p,q,s) \).

Conversely, if \( \varphi \in H(\mathbb{B}) \) satisfies [8,13] and
\[
\lim_{|a| \to 1} \int_{\mathbb{B}} |\varphi(z)|^p \left( \log \frac{2}{1 - |z|^2} \right)^{n-1} (1 - |z|^2)^q (1 - |\Phi_\alpha(z)|^2)^{ns} d\lambda(z) = 0,
\]
then we have \( \varphi \in M(A^p_\alpha, \mathcal{N}^0(p,q,s)) \).

4. If \( p \geq 1 \) and \( n + 1 + \alpha > 0 \), then
   (a) If \( n + 1 + \alpha > q \), then \( M(A^p_\alpha, \mathcal{N}^0(p,q,s)) \) only contains the constant function \( \varphi(z) \equiv 0 \);
   (b) If \( n + 1 + \alpha = q \), then \( M(A^p_\alpha, \mathcal{N}^0(p,q,s)) = H^\infty \);
   (c) If \( n + 1 + \alpha < q \), then

\( \mathcal{N}^0(p,q-n-1-\alpha,s) \subseteq M(A^p_\alpha, \mathcal{N}^0(p,q,s)) \subseteq \mathcal{N}^0(p,q,s) \cap A^{-\frac{q-(n+1+\alpha)}{p}}(\mathbb{B}). \)
Proof. The proof for the above theorem is an easy modification of Theorem 8.14 and hence is omitted here. □

Remark 8.16. From the above theorem, we can conclude that \( H^\infty = M(\mathcal{N}^0(p, q, s)) \). Indeed, \( H^\infty \subseteq M(\mathcal{N}^0(p, q, s)) \) is clearly by the definition. Conversely, by the above theorem, we have

\[
M(\mathcal{N}^0(p, q, s)) \subseteq M(A^{p-n-1}_q, \mathcal{N}^0(p, q, s)) = H^\infty,
\]

which implies the desired result.

8.3.2. The space \( M(\Lambda_\beta, \mathcal{N}(p, q, s)) \). Next, we study the space of multipliers between \( \Lambda_\beta \) and \( \mathcal{N}(p, q, s) \)-type spaces. We need the following lemma, which gives an integral representation of the functions in \( \Lambda_\beta \).

Lemma 8.17. \([44, Theorem 17]\) Suppose \( f \in H(B) \) and \( \beta \) is real. If \( \Re \gamma > -1 \) and \( n + \gamma - \beta \) is not a negative integer, then \( f \in \Lambda_\beta \) if and only if there exists a function \( g \in L^\infty(B) \) such that

\[
f(z) = \int_B \frac{g(w)dV_\gamma(w)}{(1 - \langle z, w \rangle)^{n+1+\gamma-\beta}}
\]

for \( z \in \mathbb{B} \).

Theorem 8.18. Let \( p \geq 1, q > 0 \) and \( s > \max\{0, 1 - \frac{q}{n}\} \). Then we have the following results.

1. If \( \beta > 0 \), then \( M(\Lambda_\beta, \mathcal{N}(p, q, s)) = \mathcal{N}(p, q, s) \).
2. If \( \beta = 0 \), then \( M(\Lambda_\beta, \mathcal{N}(p, q, s)) \subseteq \mathcal{N}(p, q, s) \). Conversely, if \( \varphi \in H(\mathbb{B}) \) satisfying

\[
\sup_{a \in \mathbb{B}} \int_B |\varphi(z)|^p \left( \log \frac{2}{1 - |z|^2} \right)^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{n} d\lambda(z) < \infty,
\]

then \( \varphi \in M(\Lambda_\beta, \mathcal{N}(p, q, s)) \).

Proof. (1) Once again, since clearly the constant function \( \varphi(z) \equiv 1 \) belongs to \( \Lambda_\beta \), we have \( M(\Lambda_\beta, \mathcal{N}(p, q, s)) \subseteq \mathcal{N}(p, q, s) \).

Let \( \varphi \in \Lambda_\beta \). We claim that \( \varphi \in H^\infty \). Indeed, by Lemma 8.17 and \([30, Proposition 1.4.10]\), we have

\[
|\varphi(z)| \lesssim \int_B \frac{(1 - |w|^{2})^{n}dV(z)}{(1 - \langle z, w \rangle)^{n+1+\gamma-\beta}} < \infty,
\]

where \( \gamma \) is some constant satisfying the requirement in Lemma 8.17. This clearly implies that for any \( f \in \mathcal{N}(p, q, s) \), we have \( f\varphi \in \mathcal{N}(p, q, s) \). The proof is complete.

(2) The first assertion is clear. To show the second assertion, first we note that when \( \beta = 0 \), \( \Gamma_\beta = B \), the classical Bloch space. Let \( \varphi \in H(\mathbb{B}) \)
satisfying (8.17) and \(f \in \mathcal{B}\). We need to show that \(f \varphi \in \mathcal{N}(p, q, s)\).

First we claim that \(\varphi \in \mathcal{N}(p, q, s)\). Indeed,

\[
\|\varphi\|^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\varphi(z)|^2 (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
= \sup_{a \in \mathbb{B}} \left( \int_{|z| < 0.9} + \int_{0.9 < |z| < 1} \right) |\varphi(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
\lesssim \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\varphi(z)|^p \left( \log \frac{2}{1 - |z|^2} \right)^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty.
\]

Thus, by the growth estimation of \(f \in \mathcal{B}\), we have

\[
\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)\varphi(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
\lesssim \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\varphi(z)|^p \left( \log \frac{2}{1 - |z|^2} \right)^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z)
\]

\[
< \infty.
\]

We get the desired result. \(\square\)

Finally, we consider the case when \(\beta < 0\). Let \(l = -\beta\). Note that by the definition of \(\Lambda_\beta\), in the present case, we have

\[
\Lambda_\beta = A^{-l}(\mathbb{B}).
\]

Thus, in the sequel, instead of using the notation \(M(\Lambda_\beta, \mathcal{N}(p, q, s))\), we consider the space \(M(A^{-l}(\mathbb{B}), \mathcal{N}(p, q, s))\). First we note that for each \(w \in \mathbb{B}\), the function

\[
L_w = \frac{(1 - |w|^2)^l}{(1 - \langle z, w \rangle)^{2l}}
\]

belongs to \(A^{-l}(\mathbb{B})\) and \(\sup_{w \in \mathbb{B}} |L_w| \leq 1\).

We have the following result.

**Theorem 8.19.** Let \(p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}\) and \(l > 0\). Then we have the following assertions:

1. If \(l > \frac{q}{p}\), then \(M(A^{-l}(\mathbb{B}), \mathcal{N}(p, q, s))\) only contains the constant function \(\varphi(z) \equiv 0\).
2. If \(l = \frac{q}{p}\), then \(M(A^{-l}(\mathbb{B}), \mathcal{N}(p, q, s)) = H^\infty\).
3. If \(0 < l < \frac{q}{p}\), then

\[
\mathcal{N}(p, q - pl, s) \subset M(A^{-l}(\mathbb{B}), \mathcal{N}(p, q, s)) \subset \mathcal{N}(p, q, s) \cap A^{-l-\frac{q}{p}}.
\]
Proof. Take any \( g \in M(A^{-l}(\mathbb{B}), \mathcal{N}(p, q, s)) \). Then we have \( gL_w \in \mathcal{N}(p, q, s), \forall w \in \mathbb{B} \). We get

\[
|g(z)L_w(z)| \leq \frac{\|gL_w\|}{(1 - |z|^2)^\frac{2}{p}} \leq \frac{\|M_g\|\|L_w\|}{(1 - |z|^2)^\frac{2}{p}}.
\]

Let \( z = w \). We have

\[
(8.18) \quad |g(w)| \lesssim (1 - |w|^2)^{l - \frac{2}{p}}.
\]

(1) It is clear that by \((8.18)\), \( \lim_{|w| \to 1} |g(w)| = 0 \), which implies \( g(z) \) equals to 0 everywhere by maximum modulus principle.

(2) The inclusion \( M(A^{-l}(\mathbb{B}), \mathcal{N}(p, q, s)) \subseteq H^\infty \) clearly follows from \((8.18)\). For the converse direction, let \( \varphi \in H^\infty \) and \( f \in A^{-\frac{2}{p}}(\mathbb{B}) \), and we have

\[
\|f\varphi\| \leq \|\varphi\|_{H^\infty}\|f\| \leq \|\varphi\|_{H^\infty}\|f\|_p < \infty,
\]

where the second inequality follows from Proposition 2.1.

(3) The second inclusion follows clearly from \((8.18)\), and hence we omit the proof here. For the first inclusion, we take and fix a \( g \in \mathcal{N}(p, q - pl, s) \), and then for any \( f \in A^{-l}(\mathbb{B}) \), we have

\[
\|gf\|_{\mathcal{N}(p, q, s)}^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |g(z)f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^ns d\lambda(z)
\]

\[
\lesssim \sup_{z \in \mathbb{B}} \int_{\mathbb{B}} |g(z)|^p (1 - |z|^2)^q \cdot \frac{|f|^p}{(1 - |z|^2)^{pl}} \cdot (1 - |\Phi_a(z)|^2)^ns d\lambda(z)
\]

\[
\leq \|g\|_{\mathcal{N}(p, q - pl, s)}^p \|f\|_p^p,
\]

which implies the desired result. \( \square \)

The corresponding results for \( \mathcal{N}^0(p, q, s) \) again are immediate by an easy modification, and hence we only state the result as follows.

**Theorem 8.20.** Let \( p \geq 1, q > 0, s > \max \{0, 1 - \frac{2}{n}\} \) and \( \beta, l \geq 0 \). Then we have the following results.

1. If \( \beta > 0 \), then \( M(\Lambda_\beta, \mathcal{N}^0(p, q, s)) = \mathcal{N}^0(p, q, s) \).
2. If \( \beta = 0 \), then \( M(\Lambda_\beta, \mathcal{N}^0(p, q, s)) \subseteq \mathcal{N}^0(p, q, s) \). Conversely, if \( \varphi \in H(\mathbb{B}) \) satisfying \((8.17)\) and

\[
\lim_{|a| \to 1} \int_{\mathbb{B}} |\varphi(z)|^p \left( \log \frac{2}{1 - |z|^2} \right)^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^ns d\lambda(z) = 0,
\]

then \( \varphi \in M(\Lambda_\beta, \mathcal{N}^0(p, q, s)) \).
3. If \( l > \frac{2}{p} \), then \( M(\Lambda_\beta, \mathcal{N}^0(p, q, s)) \) only contains the constant function \( \varphi(z) \equiv 0 \).
\( \mathcal{N}(p, q, s) \)-TYPE SPACES IN THE UNIT BALL OF \( \mathbb{C}^n \)

(4) If \( l = \frac{q}{p} \), then \( M(A^{-l}(\mathbb{B}), \mathcal{N}^0(p, q, s)) = H^\infty \).

(5) If \( 0 < l < \frac{q}{p} \), then

\[
\mathcal{N}^0(p, q - pl, s) \subseteq M(A^{-l}(\mathbb{B}), \mathcal{N}^0(p, q, s)) \subseteq \mathcal{N}^0(p, q, s) \cap A^{-(l - \frac{q}{p})}(\mathbb{B}).
\]

As a conclusion of this subsection, we have the following result with consider the cases \( \alpha = q - n - 1 \) and \( \beta = -l = -\frac{q}{p} \).

**Theorem 8.21.** Let \( p \geq 1, q > 0, s > \max \{0, 1 - \frac{q}{n}\} \) and \( \varphi \in H(\mathbb{B}) \). The following statements are equivalent:

(a) \( \varphi \in H^\infty \);
(b) \( \varphi \mathcal{N}(p, q, s) \subseteq \mathcal{N}(p, q, s) \);
(c) \( \varphi \mathcal{N}_0^0(p, q, s) \subseteq \mathcal{N}^0(p, q, s) \);
(d) \( \varphi A^p_{q-n-1} \subseteq \mathcal{N}(p, q, s) \);
(e) \( \varphi A^p_{q-n-1} \subseteq \mathcal{N}^0(p, q, s) \);
(f) \( \varphi A^{-\frac{q}{p}}(\mathbb{B}) \subseteq \mathcal{N}(p, q, s) \);
(g) \( \varphi A^{-\frac{q}{p}}(\mathbb{B}) \subseteq \mathcal{N}^0(p, q, s) \).

**Acknowledgments.** The corresponding author was supported by the Macao Science and Technology Development Fund (No.083/2014/A2) and NSF of China (No. 11471143 and No.11720101003).

**References**

[1] J. Burbea and S. Li, Weighted Hadamard products of holomorphic functions in the unit ball, *Pacific J. Math.* 168 (1995), 235–270.

[2] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[3] D. C. Chang, S. Li and S. Stević, On some integral operators on the unit polydisk and the unit ball, *Taiwanese J. Math.* 11(5) (2007), 1251–1286.

[4] P. Duren and A. Schuster, *Bergman Space*, Mathematical Surveys and Monographs, Vol. 100, American Mathematical Society, Providence, RI, 2004.

[5] L. Grafakos, *Classical Fourier Analysis*, Third Edition, Graduate Texts in Math., no 249, Springer, New York, 2014.

[6] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces*, Springer-Verlag, 2000.

[7] B. Hu and S. Li, \( N_p \)-type functions with Hadamard gaps in the unit ball, *Complex Var. Elliptic Equ.* 61 (2016), 843–853.

[8] B. Hu and Le Hai Khoi, Weighted composition operators on \( N_p \) spaces in the unit ball, *C. R. Acad. Sci. Paris, Ser. I* 351 (2013), 719–723.

[9] B. Hu, Le Hai Khoi and Trieu LE, On structure of \( N_p \) space in the unit ball, *preprint*, 2015.

[10] Z. Hu, Extended Cesáro operators on mixed norm spaces, *Proc. Amer. Math. Soc.* 131(7) (2003), 2171–2179.

[11] Z. Hu, \( Q_p \) spaces in the unit ball of \( \mathbb{C}^n \) with \( \frac{n-1}{n} < p \leq 1 \), *J. Baoji Univ. Arts Sci. Math. Colloq. Chin. Univ.* 1 (2004), 19–27.
[12] M. Jevtić, On the Carleson measure characterization of BMO functions on the unit sphere, *Proc. Amer. Math. Soc.* **123** (1995), 3371–3377.

[13] M. Jevtić, X. Massaneda and P. Thomas, Interpolating sequences for weighted Bergman spaces of the unit ball, [arXiv:math.CV/9511202](http://arxiv.org/abs/math.CV/9511202) v1, 21 Nov 1995.

[14] B. Korenblum, BMO estimations and radial growth of Bloch functions, *Bull. Amer. Math. Soc. (New Series)*, **12** (1985), 99–102.

[15] B. Li and C. Ouyang, Randomization of $Q_p$ spaces on the unit ball of $\mathbb{C}^n$. *Sci. China Ser. A, suppl.* **48** (2005), 306–317.

[16] B. Li and C. Ouyang, Higher radial derivative of functions of $Q_p$ spaces and its applications, *J. Math. Anal. Appl.* **327** (2007) 1257–1272.

[17] S. Li, Riemann-Stieltjes operators from $F(p,q,s)$ to $\alpha$-Bloch space on the unit ball, Vol. 2006, Article ID 27874, *J. Inequal. Appl.* (2006), 14 pages.

[18] S. Li, Some new characterizations of Dirichlet type spaces on the unit ball of $\mathbb{C}^n$, *J. Math. Anal. Appl.* **324** (2006) 1073–1083.

[19] S. Li and S. Stević, Weighted-Hardy functions with Hadamard gaps on the unit ball, *Appl. Math. Comput.* **212** (2009) 222–233.

[20] S. Li and H. Wulan, Characterizations of $Q_p$ spaces in the unit ball of $\mathbb{C}^n$, *J. Math. Anal. Appl.* **360** (2009), 689–696.

[21] Z. Lou and H. Wulan, Characterizations of Bloch functions in the unit ball of $\mathbb{C}^n$, I, *Bull. Austral. Math. Soc.* **68** (2003), 205–212.

[22] D. Luecking, Closed range restriction operators on weighted Bergman spaces, *Pacific J. Math.* **110** (1984), 145–160.

[23] M. Mateljević and M. Pavlović, $L^p$-behavior of power series with positive coefficients and Hardy spaces, *Proc. Amer. Math. Soc.* **87** (1983), 309–316.

[24] C. Ouyang, W. Yang and R. Zhao, Characterizations of Bergman spaces and Bloch space in the unit ball of $\mathbb{C}^n$, *Trans. Amer. Math. Soc.* **347** (1995), 4301–4313.

[25] C. Ouyang, W. Yang and R. Zhao, Möbius invariant $Q_p$ spaces associated with the Green function on the unit ball, *Pacific J. Math.* **182** (1998), 69–99.

[26] M. Pavlović and K. Zhu, New characterizations of Bergman spaces, *Ann. Acad. Sci. Fenn. Math.* **33** (2008), 87–99.

[27] R. Peng and C. Ouyang, Decomposition theorems for $Q_p$ spaces with small scale $p$ on the unit ball of $\mathbb{C}^n$, *Acta Math. Sci. Ser. B.* **30** (2010), 1419–1428.

[28] R. Peng and C. Ouyang, Riemann-Stieltjes operators and multipliers on $Q_p$ spaces in the unit ball of $\mathbb{C}^n$, *J. Math. Anal. Appl.* **377** (2011), 180–193.

[29] F. Pérez-González, J. Rättyä, Forelli-Rudin estimates, Carleson measures and $F(p, q, s)$-functions, *J. Math. Anal. Appl.* **315** (2006), 394–414.

[30] W. Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Springer-Verlag, New York, 1980.

[31] S. Stević, A generalization of a result of Choa on analytic functions with Hadamard gaps, *J. Korean Math. Soc.* **43** (2006), 579–591.

[32] S. Stević and S. Ueki, Weighted composition operators and integral-type operators between weighted Hardy spaces on the unit ball, *Discrete Dyn. Nature Soc.* Volume 2009, Article ID 952831, 21 pages.

[33] M. Stoll, A characterization of Hardy spaces on the unit ball of $\mathbb{C}^n$, *J. London Math. Soc.* **48** (1993), 126–136.

[34] D. Ullrich, Radial limits of $\mathcal{M}$-subharmonic functions, *Trans. Amer. Math. Soc.* **292** (1985), 501–518.
\[ N(p,q,s) \text{-TYPE SPACES IN THE UNIT BALL OF } \mathbb{C}^n \]

[35] D. Ullrich, Radial divergence in BMOA, Proc. London Math. Soc. 68 (1994), 145–160.

[36] A. William, A Short Course on Spectral Theory, Springer-Verlag, New York, 2002.

[37] J. Xiao, Holomorphic Q classes, Lecture Notes in Mathematics, 1767. Springer-Verlag, Berlin, 2001.

[38] J. Xiao, Geometric Q_p functions, Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[39] J. Xiao, Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball, J. London. Math. Soc. 70 (2) (2004), 199-214.

[40] X. Zhang, C. He and F. Cao, The equivalent norms of \( F(p,q,s) \) space in \( \mathbb{C}^n \), J. Math. Anal. Appl. 401 (2013), 601–610.

[41] X. Zhang, J. Xiao, Z. Hu, Y. Liu, D. Xiong and Y. Wu, Equivalent characterization and application of \( F(p,q,s) \) space in \( \mathbb{C}^n \), Acta Math. Sinica, Ser.A. 54 (2011), 1029–1042.

[42] R. Zhao, On a general family of function spaces, Ann. Acad. Sci. Math. Diss. 105 (1996).

[43] R. Zhao, Pointwise multipliers from weighted Bergman spaces and Hardy spaces to weighted Bergman spaces, Ann. Acad. Sci. Math. 29 (2004), 139–150.

[44] R. Zhao and K. Zhu, Theory of Bergman spaces on the unit ball, Memoires de la SMF 115 (2008).

[45] K. Zhu, Bloch type spaces of analytic functions, Rocky Mountain J. Math. 23 (1993), 1143–1177.

[46] K. Zhu, The Bergman spaces, the Bloch spaces and Gleason’s problem, Trans. Amer. Math. Soc. 309 (1988), 253–268.

[47] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Springer-Verlag, 2004.

Bingyang Hu: Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388, USA.

E-mail address: bhu32@wisc.edu

Songxiao Li: Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, 610054, Chengdu, Sichuan, P.R. China.

Institute System Engineering, Macau University of Science and Technology, Avenida Wai Long, Taipa, Macau.

E-mail address: jyulsx@163.com