Vanishing of local cohomology and set-theoretically
Cohen–Macaulay ideals

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Abstract
In this paper, first, we generalize a result of Peskine–Szpiro on the relation between the
cohomological dimension and projective dimension. Then, we give conditions for the van-
ishing of local cohomology from local to global and vice versa. Our final goal in the pre-
sent paper is to examine the set-theoretically Cohen–Macaulay ideals to find some coho-
mological characterization of these kinds of ideals.

Keywords Local cohomology · Ring endomorphism · Set-theoretically Cohen–Macaulay
ideals · Linkage · Cohomological dimension · Projective dimension

Mathematics Subject Classification 13D45 · 14B15

1 Introduction
Throughout this paper, all rings are commutative and Noetherian with identity. For an ideal
I of a local ring \((R, \mathfrak{m})\), the local cohomology modules \(H^i_I(R)\) may be considered as the iso-
morphism \(H^i_I(R) = \lim Ext^i_R(R/I^t, R)\) for \(i \geq 0\). Moreover, hereafter \(\sqrt{I}\) will denote the rad-
cal of the ideal \(I\). On the other hand, given \((R, \mathfrak{m})\) a local ring, \(E\) will denote the injective hull of the residue field of \(R\), and \((-)^\vee\) will denote the Matlis duality functor \(\text{Hom}_R(-, E)\).

In their landmark paper, Peskine and Szpiro [29, Proposition 4.1] proved that
whenever \((R, \mathfrak{m})\) is a regular local ring containing a field of positive characteris-
tic and \(I \subset R\) is a Cohen–Macaulay ideal (i.e. the ring \(R/I\) is Cohen–Macaulay),
\(H^i_I(R) = 0\) for all \(i > \dim R - \text{depth}R/I\). An immediate implication of it is that the ine-
quality \(\text{cd}(R, I) \leq \text{pd}R/I\) holds, where cohomological dimension \(\text{cd}(R, I)\) resp. projective
dimension \(\text{pd}R/I\) of \(I\) is defined as

\[
\text{cd}(R, I) \leq \text{pd}R/I
\]

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\[ \text{cd}(R, I) = \min \{ i : H^i_I(R) = 0 \text{ for all } j > i \}, \]

resp.
\[ \text{pd} R/I = \sup \{ n | \text{Ext}^i_R(R/I, N) = 0, \text{ for all } R - \text{mod. } N \text{ and all } i \geq n + 1 \}. \]

Let \( \varphi : R \to R \) be a ring endomorphism, it induces a natural \( \varphi \) action on the local cohomology modules \( \varphi_* : H^i_I(R) \to H^i_{\varphi(I)\varphi}(R) \) via \( \varphi(r)\varphi_*(\eta) = \varphi_*(r\eta) \), where \( r \in R, \eta \in H^i_I(R) \). \( \sqrt{I} = \sqrt{\varphi(I)R} \) which is an endomorphism of the underlying Abelian group (details, including notation, are given in Sect. 2). Inspired of [29] and [23], as a generalization of Frobenius action, the mentioned action is an effective tool in the study of local cohomology modules as it has been used recently in [1, 5, 7, 33].

In Sect. 3, (cf. Theorem 3.3) we generalize the inequality \( \text{cd}(R, I) \leq \dim(R) - \text{depth} R/I \) using the action of \( \varphi \) on local cohomology (without any restriction on the characteristic of the ring).

Recall that \( I \) is called a set-theoretically complete intersection ideal if there are \( h = \text{height}(I) := \text{ht}(I) \) elements \( g_1, \ldots, g_s \in R \) such that they generate an ideal which has the same radical as \( I \); in other words, \( \text{ht}(I) = \text{ara}(I) \), where \( \text{ara}(I) \) denotes the arithmetic rank of \( I \). It is known \([6, 3.3.4]\) that, in general, \( \text{ht}(I) \leq \text{cd}(R, I) \leq \text{ara}(I) \), and, borrowing terminology from \([15]\), \( I \) is called cohomologically complete intersection, whenever \( \text{ht}(I) = \text{cd}(R, I) \). Notice that a set-theoretically complete intersection ideal is a cohomologically complete intersection one, but the converse is no longer true.

From the other point of view, the aforementioned result of Peskine and Szpiro says that \( I \) is cohomologically complete intersection, whenever it is a perfect ideal. Recently, Barbo has shown that their idea works for all Noetherian rings of positive characteristic (cf. \([36, \text{Corollary 2.2}]\)). On the other hand, a conjecture of Hartshorne on the relation between being a set-theoretically complete intersection curve in \( \mathbb{P}^3_k \) over a field \( k \) of characteristic 0 and Cohen–Macaulayness of its coordinate ring (see Discussion 3.6), motivated us to give the following Theorem as our main result in Sect. 3:

**Theorem 1.1** (cf. Theorem 3.7) Let \( (R, \mathfrak{m}) \) be a Gorenstein local ring, let \( I \subset R \) be an ideal, and let \( \varphi : R \to R \) be a ring endomorphism. Suppose that

(a) \( \varphi \) has the going down property;
(b) the ideals \( \{ \varphi^j(I)R \}_j \) form a descending chain cofinal with \( \{ I \}_j \).

Then the following statements are equivalent.

1. \( R/I \) is a Cohen–Macaulay ring.
2. \( \text{ht}(I) = \text{cd}(R, I) \) and \( \text{Ext}^i_R(R/\varphi^j(I)R, R) \longrightarrow \text{Ext}^i_R(R/\varphi^{j+1}(I)R, R) \) are injective of nonzero \( R \)-modules for all \( t \) and \( r := \text{depth} R/I \).

Second part of the paper is devoted to the study of set-theoretically Cohen–Macaulay ideals. Our very concrete motivation is the following question asked in [34].

**Question 1.2** Given an affine variety \( V \), does \( V \) support a Cohen–Macaulay scheme, i.e. whether there exists a Cohen–Macaulay ring \( R \) such that \( V \) is isomorphic to \( \text{Spec} R_{\text{red}} \)?
Let $X$ be a Cohen–Macaulay scheme. It fails to be true that $X_{\text{red}}$, the corresponding reduced scheme, is Cohen–Macaulay. In general, Singh and Walther defined an ideal $I$ in a regular local ring to be set-theoretically Cohen–Macaulay if there exists an ideal $J \subset R$ with $\sqrt{I} = \sqrt{J}$ such that the ring $R/J$ is Cohen–Macaulay. It is evident that set-theoretically complete intersection and Cohen–Macaulay radical ideals are set-theoretically Cohen–Macaulay but the converse is no longer true. It is a natural question to ask on the relation between Cohen–Macaulay and set-theoretically Cohen–Macaulay ideals.

Our first result in this direction is Proposition 4.2, which is a consequence of Theorem 3.7. Under the assumptions given in Proposition 4.2, the concepts of Cohen–Macaulayness and set-theoretically Cohen–Macaulayness are the same.

Not so much is known about the cohomological characterization of set-theoretically Cohen–Macaulay ideals. In Sect. 4, among other results we show that under certain conditions, set-theoretically Cohen–Macaulay ideals are the same as cohomologically complete intersections.

**Theorem 1.3** (cf. Theorem 4.4) Let $(R, \mathfrak{m})$ be a regular local ring, let $I \subseteq R$ be an ideal, and let $\varphi : R \to R$ be a ring endomorphism. Suppose that $\varphi$ is flat, and there is an ideal $J \subseteq R$ with $\sqrt{I} = \sqrt{J}$ such that $R/J$ is Cohen–Macaulay and $\{\varphi^t(J)R\}$, is cofinal with respect to $\{I^t\}$. Then, one has $\text{ht}(I) = \text{cd}(R, I)$.

Whenever the action of $\varphi$ on $R/\sqrt{I}$ is pure and flat, the equivalence between the cohomologically complete intersection and set-theoretically Cohen–Macaulay ideals in Theorem 4.4 holds, see Corollary 4.5.

Another way to study set-theoretically Cohen–Macaulayness of a variety is through linkage theory. Roughly speaking, the linkage is the study of two varieties where their union has nice properties. Through the use of this concept one may consider a variety linked with the second one which one understands better.

We cite the following paragraph from a fruitful paper of A. Martsinkovsky and J. R. Strooker [26] on the significance of the concept of linkage theory.

'It goes back to the late 19th and early 20th century, when M. Noether, Halphen, and Severi used it to study algebraic curves in $\mathbb{P}^3$. Linkage allows to pass from a given curve to another curve, related in a geometric way to the original one. Iterating the procedure one obtains a whole series of curves in the same linkage class. The usefulness of this technique is explained by two observations: (a) certain properties of the curve are preserved under linkage, and (b) the resulting curves may be simpler, and thus easier to handle, than the original one.'

In the same vein, Peskine and Szpiro in [30, Proposition 1.3] proved that in a Gorenstein local ring if $I$ and $J$ are linked ideals and $I$ is a Cohen–Macaulay ideal then so is $J$. The same result was proved by Schenzel in [31, Corollary 3.3] for Buchsbaum ideals. In this direction, in Corollary 5.9 we give conditions where the set-theoretically Cohen–Macaulay property can be shared between two linked ideals.

### 2 Notations and remarks

We start this section with the idea used by Singh and Walther in [33].

Let $A$ be a commutative Noetherian ring with a flat endomorphism $\varphi : A \to A$, and let $\mathfrak{a}$ be an ideal of $A$. We denote by $\varphi_\mathfrak{a}A$ the following $(A, A)$-bimodule: for any $r, r_1, r_2 \in A$. Springer
Let \( \Phi \) be the functor on the category of \( A \)-modules with \( \Phi(M) = \varphi_A \otimes_A M \). The iteration \( \Phi^t \) is the functor

\[
\Phi^t(M) = \varphi_A \otimes_A \Phi^{t-1}M, \quad t \geq 1,
\]

where \( \Phi^0 \) is interpreted as the identity functor; the reader will easily note that the flatness of \( \varphi \) is equivalent to the exactness of \( \Phi \). At once, one can realize that \( \Phi^t(M) = \varphi_t^* \otimes_A M \).

Let us notice that \( \Phi(A) \cong A \) given by \( \varphi^* r \mapsto \varphi(r) \). Furthermore, if \( M \) and \( N \) are \( A \)-modules, then [33, 2.6.1] there are natural isomorphisms

\[
\Phi(\text{Ext}^i_A(M, N)) \cong \text{Ext}^i_A(\Phi(M), \Phi(N)), \quad \text{for all } i \geq 0.
\]

Assume, in addition, that the ideals \( \{ \varphi^i(a)R \}_{i \geq 0} \) form a descending chain cofinal with the chain \( \{ a' \}_{i \geq 0} \), where \( a \) is an ideal of \( A \). Under these assumptions, one can easily check [33, page 291] that

\[
\Phi(H^i_a(A)) \cong H^i_{\varphi(a)}(A) \cong H^i_a(A), \quad \text{for all } i \geq 0.
\]

Now, let us fix our notations. Throughout this section, let \( R \) be a commutative Noetherian local ring and \( \varphi : R \rightarrow R \) be a local ring endomorphism. Suppose that \( I \) is an ideal of \( R \) and \( \{ \varphi^i(I)R \}_{i \geq 0} \) is a decreasing chain of ideals cofinal with \( \{ I \}_{i \geq 0} \). Keeping these assumptions, we introduce the following notations.

**Notation 2.1** Let \( \varphi : R \rightarrow R \) be a flat ring endomorphism (a ring map \( R \rightarrow S \) is called flat if \( S \) is flat as an \( R \)-module). In this case, we say that triple \( (R, I, \varphi) \) has (Flat) property.

Next result can be found in [28, Exercise 7.1].

**Lemma 2.2** Let \( A, B \) be two rings. If \( B \) is a faithfully flat \( A \)-algebra then for an \( A \)-module \( M \) one has \( B \otimes_A M \) is \( B \)-flat if and only if \( M \) is \( A \)-flat.

In Lemma 2.2, put \( B := \hat{R} \), the \( m \)-adic completion of local ring \( R \) and \( M = A := R \). Then for every flat ring endomorphism \( \varphi : R \rightarrow R \), the induced ring endomorphism \( \hat{\varphi} : \hat{R} \rightarrow \hat{R} \) is flat.

**Definition 2.3** For two rings \( A \) and \( B \), a ring map \( f : A \rightarrow B \) has the going-down property if, for any two primes \( p' \subseteq p \) of \( A \) and any prime \( P \) in \( B \) with \( f^{-1}(P) = p \), there is a prime \( p' \) of \( B \) such that \( f^{-1}(P \supseteq p) = p' \).

**Notation 2.4** Let \( \varphi : R \rightarrow R \) be a ring endomorphism satisfying the going down property. Suppose that \( I \) is an ideal of \( R \) with finite projective dimension. We say that the triple \( (R, I, \varphi) \) has (GD) property.

**Example 2.5**

(i) Notice that every flat homomorphism implies the going down property [27, (5.D) Theorem 4].
(ii) In general, an extension $A \subset B$ of domains, with $A$ normal and $B$ integral over $A$ satisfies going down property, for a ring homomorphism $f : A \to B$ [27, (5.E) Theorem 5].

(iii) Let $R$ be a commutative Noetherian ring, and let $R \to R$ be a ring homomorphism such that the induced map on $\text{Spec}(R)$ is the identity. Then, $\varphi$ satisfies the Going Down property.

(iv) Take into account that the Frobenius endomorphism of a regular local ring of positive characteristic and for any field $k$ and any integer $t \geq 2$, the $k$-linear endomorphism $\varphi(x_i) = x_i^t$ of $k[x_1, \ldots, x_n]$ are the prototypical examples of flat endomorphisms.

(v) For a given regular local ring $R$, by [28, Theorem 33.1] the endomorphism $\varphi : R \to R$ which is of finite length, is flat. A local homomorphism $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ is of finite length, if the following equivalent conditions hold:

(a) $B/\mathfrak{m}B$ is Artinian. (b) The only prime ideal of $B$ which contracts to $\mathfrak{m}$ is $\mathfrak{n}$.

Notation 2.6 Suppose that $\text{Ext}_R^t(R/\varphi^t(1)R, R) \to \text{Ext}_R^t(R/\varphi^{t+1}(1)R, R)$, is injective for all $i \geq 0$ and all $t \geq 0$. We say $(R, I, \varphi)$ has (Inj) property.

Definition 2.7 A ring homomorphism $f : A \to B$ is pure if the map $f \otimes 1 : A \otimes_A M \to B \otimes_A M$ is injective for each $A$-module $M$. If $A$ contains a field of prime characteristic $p$, then $A$ is $F$-pure if the Frobenius homomorphism $F : A \to A, a \mapsto a^p$ is pure, where $a \in A$.

Remark 2.8 Note that by virtue of [18, Lemma 6.2] purity of $\varphi : R \to R$ implies the same property for $\varphi : R/I \to R/\varphi(I)$ for all positive integer $t$. It follows from [33, Theorem 2.8] that, when $R$ is a regular ring and the triple $(R, I, \varphi)$ satisfies the (Flat) property, purity of $\varphi : R/I \to R/I$ implies that $(R, I, \varphi)$ has (Inj) property.

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$. There exists a $k$-linear endomorphism $\varphi : R \to R$ with $\varphi(x_i) = x_i^t$ for $1 \leq i \leq n$ and positive integer $t$. If $I \subset R$ is an ideal generated by square-free monomials, then there exists a pure endomorphism $\tilde{\varphi} : R/I \to R/I$, (cf. [33, Example 2.2]).

3 On a result of Peskine–Szpiro

We start by recalling properties of some homological invariants. Then discuss and examine their relationships using ring endomorphisms. Let $(R, \mathfrak{m})$ be a local ring and $I$ be an ideal of $R$ of finite projective dimension $(\text{pd}_R/I < \infty)$. The Auslander-Buchsbaum formula says that $\text{pd}_R R/I = \text{depth} R - \text{depth} R/I$. On the other hand, when $R$ is a complete local domain,
the Hartshorne–Lichtenbaum Theorem gives a necessary and sufficient condition to guarantee $\text{cd}(R, I) \leq \dim(R) - 1$, and Huneke and Lyubeznik in [19, Theorem 1.1] give conditions for $\text{cd}(R, I) \leq \dim R - 2$, where $R$ is a regular local ring containing a field. The interested reader may like to consult [4] and the references given therein for additional details.

Hereafter in this section, we focus on finding sufficient conditions for the equality $\text{cd}(R, I) = \dim R - \text{depth} R/I$.

**Lemma 3.1** (cf. [7, Lemma 3.4]) Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I \subset R$ an ideal of finite projective dimension. Assume that $\varphi : R \to R$ is a ring endomorphism satisfying the going down property and $J = \varphi(I)R$. Then $\text{depth} R/I \leq \text{depth} R/J$.

**Corollary 3.2** Let $(R, \mathfrak{m})$ be a Gorenstein local ring. Suppose that

(a) The triple $(R, I, \varphi)$ satisfies (Inj);
(b) For any $t \geq 0$, $\text{depth} R/\varphi^t(I)R \leq \text{depth} R/\varphi^{t+1}(I)R$;
(c) Projective dimension of $R/I$ is finite;

Then, the following statements are true.

1. $\text{depth}_R R/I = \text{depth}_R R/\varphi(I)R$, for all $t \geq 0$.
2. $R/I$ is Cohen–Macaulay if and only if $R/\varphi^t(I)R$, is Cohen–Macaulay for all $t \geq 0$.
3. $\text{Ass}_R R/I = \text{Ass}_R R/\varphi(I)R$, for all $t \geq 0$.

**Proof** Thanks to assumption (b) one may write

$$\text{depth}_R R/I \leq \text{depth}_R R/\varphi(I)R \leq \text{depth}_R R/\varphi^t(I)R \leq \ldots$$

On the other hand, suppose that $\text{depth}_R R/I = u$. As $(R, I, \varphi)$ has (Inj) property, one has a monomorphism, for each $t$,

$$\text{Ext}_R^{\dim(R) - u}(R/\varphi^t(I)R, R) \hookrightarrow \text{Ext}_R^{\dim(R) - u}(R/\varphi^{t+1}(I)R, R).$$

By applying the Matlis duality functor $(-)^\vee = \text{Hom}_R(-, E)$ to this monomorphism, one gets an epimorphism

$$\text{Ext}_R^{\dim(R) - u}(R/\varphi^t(I)R, R)^\vee \twoheadrightarrow \text{Ext}_R^{\dim(R) - u}(R/\varphi^{t+1}(I)R, R)^\vee.$$

Finally, using local duality [6, 11.2.5] the previous epimorphism is exactly the epimorphism

$$H^u_\mathfrak{m}(R/\varphi^{t+1}(I)R) \to H^u_\mathfrak{m}(R/\varphi^t(I)R),$$

of non zero modules, for all $t$. It implies that $\text{depth}_R R/\varphi^t(I)R = u$ for all $t$. This proves part (1).
Now, note that $\dim_R R/I = \dim_R R/\varphi'(I) R$, for all $t$, because $\{I^t\}_t$ is cofinal with $\{\varphi'(I) R\}_t$. In this way, part (2) follows combining part (1) jointly with Auslander–Buchsbaum formula.

Finally, note that for a prime ideal $p$ of $R$, $p \in \text{Ass}_R M$ if and only if $p R_p \in \text{Ass}_{R_p} M_p$ where $M$ is an $R$-module. Thus, one may assume that $p = m$. Hence, it is enough to prove that $m \in \text{Ass}_R R/I$ if and only if $m \in \text{Ass}_R R/\varphi'(I) R$, hence part (3) follows once again from part (1).

Now is the time to prove the first main result of this section, this result should be compared with item (1) of [10, Proposition 2.5];

**Theorem 3.3** Let $(R, m)$ be a Gorenstein local ring. Suppose that

(a) For any $t \geq 0$, $\text{depth}_R R/\varphi'(I) R \leq \text{depth}_R R/\varphi^{t+1}(I) R$;

(b) Projective dimension of $R/I$ is finite;

Then one has $\text{cd}(R, I) \leq \text{pd} R/I$.

If in addition, the triple $(R, I, \varphi)$ satisfies $(\text{Inj})$ property, then the equality holds.

**Proof** Suppose that $\text{pd} R/I = u$ for some integer $u$. By the Auslander–Buchsbaum formula $\text{depth} R/I = \dim R - u$. Now, combining this equality jointly with our assumption (a) one has

$$H^i_m(R/\varphi'(I) R) = 0 \text{ for all } i < \dim R - u \text{ and all } t.$$ 

This implies, again by local duality [6, 11.2.5], that

$$\text{Ext}^i_R(\text{dim}_R - i)(R/\varphi'(I) R, R)^\vee = 0 \text{ for all } i < \text{dim}(R) - u.$$ 

Since Matlis duality is a faithful functor, one has that the above vanishing is equivalent to

$$\text{Ext}^i_R(\text{dim}_R - i)(R/\varphi'(I) R, R) = 0 \text{ for all } i < \text{dim}(R) - u.$$ 

Finally, using the fact that $\{\varphi'(I) R\}_t$ is cofinal with respect to $\{I^t\}_t$, jointly with this Ext vanishing one obtains, for all $\dim(R) - i > u$, that

$$H^i_{\text{dim}_R - i}(R) = \lim_{\to} \text{Ext}^i_R(\text{dim}_R - i)(R/\varphi'(I) R, R) = 0.$$ 

This vanishing implies that $\text{cd}(R, I) \leq \text{pd} R/I$.

In order to prove the equality $\text{cd}(R, I) = \text{pd} R/I$, it is enough to prove $\text{cd}(R, I) \geq \text{pd} R/I$. It follows from the assumptions and Corollary 3.2 that the homomorphisms

$$H^i_{m}(\text{dim}_R - u)(R/\varphi^{t+1}(I) R) \to H^i_{m}(\text{dim}_R - u)(R/\varphi'(I) R),$$

induced from natural homomorphism $R/\varphi^{t+1}(I) R \to R/\varphi'(I) R$, are surjective of nonzero modules for all $t$. Then $\lim_{\to} H^i_{m}(\text{dim}_R - u)(R/I^t) \neq 0$ and $\lim_{\to} H^i_{m}(R/I^t) = 0$ for all $i < \dim R - u$.

To this end note that as $\{\varphi'(I) R\}_{t \geq 0}$ and $\{I^t R\}_{t \geq 0}$ are cofinal, one has $\lim_{\to} H^i_{m}(\text{dim}_R - u)(R/\varphi'(I) R) \cong \lim_{\to} H^i_{m}(\text{dim}_R - u)(R/I^t)$. 

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Finally, combining Local Duality jointly with the fact that contravariant Hom transforms inverse limits into direct limits and that the Matlis duality functor is faithful one gets that

\[ H^u_I(R) = \lim\limits_{\rightarrow} \text{Ext}^u_M(R/I', R) = \lim\limits_{\rightarrow} \text{Hom}_R(H^\text{dim} R-u_m(R/I'), E) \]

\[ \cong \text{Hom}_R(\lim\limits_{\rightarrow} H^\text{dim} R-u_m(R/I'), E) \neq 0. \]

Therefore, we have checked that \( H^u_I(R) \neq 0 \), hence \( u \leq \text{cd}(R, I) \) that is \( \text{cd}(R, I) \geq \text{pd}R/I \), as desired.

The first consequence of Theorem 3.3 we want to single out is the below:

**Corollary 3.4** Let \((R, m)\) be a Gorenstein local ring of prime characteristic \(p\), and let \(I \subseteq R\) be an ideal of finite projective dimension. Then, one has \( \text{pd}R/I \geq \text{cd}(R, I) \).

If, in addition, \(F\) denotes the Frobenius endomorphism of \(R\), and the triple \((R, I, F)\) satisfies the (Inj) property, then \(\text{pd}R/I = \text{cd}(R, I)\).

**Proof** Since \(F\) induces identity on \(\text{Spec}(R)\), it satisfies Going Down, in particular, it increases depth thanks to Lemma 3.1. Therefore, the result follows immediately from Theorem 3.3.

The first consequence of Theorem 3.3 we want to single out is the below:

**Corollary 3.5** Let \(R = k[x_1, \ldots, x_n]\) be a polynomial ring of \(n\) variables over a field \(k\) and \(I\) be a square-free monomial ideal. Then \(\text{pd}R/I = \text{cd}(R, I)\).

**Proof** Define the \(k\)-linear endomorphism \(\varphi : R \rightarrow R\) by \(x_i \mapsto x_i^2\) for \(1 \leq i \leq n\). It is a flat ring endomorphism and without loss of generality, after localizing at the maximal ideal \(m = (x_1, \ldots, x_n)\) we may assume that \(R\) is a local ring such that \((R, I, \varphi)\) satisfies the (GD) property. Hence, we are done by Theorem 3.3. To complete the proof note that by what we have indicated at Remark 2.8, the endomorphism \(R/I \rightarrow R/I\) is pure.

**Discussion 3.6** A special case of a conjecture of Hartshorne [13, page 126] is the following:

**Conjecture:** Let \(C\) be a curve in \(\mathbb{P}^3_k\) over a field \(k\) of characteristic 0. If \(C\) is a set-theoretic complete intersection, the curve \(\tilde{C}\) is arithmetically Cohen–Macaulay. I.e. the homogeneous coordinate ring \(k[x_1, \ldots, x_n]/I(C)\) is Cohen–Macaulay.

It is known that this conjecture is not true, see for instance [35]. To do so further and motivated by the preceding conjecture we consider the equivalent property between the Cohen–Macaulayness of \(R/I\) and the equality \(\text{ht}(I) = \text{cd}(R, I)\).

**Theorem 3.7** Let \((R, m)\) be a Gorenstein local ring. Suppose that

(a) The triple \((R, I, \varphi)\) has (GD) property;

(b) Projective dimension of \(R/I\) is finite;

Then the following statements are equivalent.
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(1) $R/I$ is a Cohen–Macaulay ring.

(2) $\text{ht}(I) = \text{cd}(R, I)$ and $\text{Ext}^t_R(R/\varphi^i(I), R) \twoheadrightarrow \text{Ext}^{t+1}_R(R/\varphi^i(I), R)$ are injective of nonzero $R$-modules for all $t$ and $r := \text{depth}(R/I)$.

**Proof** (1) ⇒ (2) First of all note that $\dim R/I = \dim R/I^t = \dim R/\varphi^i(I)$ for all $t \in \mathbb{N}$. Since $R/I$ is a Cohen–Macaulay ring, by virtue of Lemma 3.1, $\dim R/I = \text{depth}(R/I) = \text{depth}(R/\varphi^i(I))$ for all $t \in \mathbb{N}$. The first part follows from Theorem 3.3 and the Auslander-Buchsbaum Theorem. For the second part consider the short exact sequence

$$0 \to \varphi^i(I)R/\varphi^{i+1}(I)R \to R/\varphi^{i+1}(I)R \to R/\varphi^i(I)R \to 0.$$ 

Applying $H^i_m(\_)$ and use the Grothendieck’s Vanishing Theorem and the fact that $\dim \varphi^i(I)R/\varphi^{i+1}(I)R \leq r$ we obtain an epimorphism $H^i_m(R/\varphi^{i+1}(I)R) \twoheadrightarrow H^i_m(R/\varphi^i(I)R)$. Now, by local duality [6, 11.2.5] this surjection is equivalent to the surjection

$$\text{Ext}^{\dim(R) - r}(R/\varphi^{t+1}(I)R, R)^\vee \twoheadrightarrow \text{Ext}^{\dim(R) - r}(R/\varphi^t(I)R, R)^\vee.$$ 

Applying to this surjection the Matlis duality functor $(-)^\vee$ one gets an injection

$$\text{Ext}^{\dim(R) - r}(R/\varphi^i(I)R, R)^\vee \hookrightarrow \text{Ext}^{\dim(R) - r}(R/\varphi^{i+1}(I)R, R)^{\vee \vee}.$$ 

Now, consider the commutative square, where the vertical arrows are the natural homothety maps from a module to its Matlis bidual, and the horizontal arrows are the natural ones:

$$\begin{array}{ccc}
\text{Ext}^{\dim(R) - r}(R/\varphi^t(I)R, R) & \twoheadrightarrow & \text{Ext}^{\dim(R) - r}(R/\varphi^{t+1}(I)R, R) \\
\downarrow & & \downarrow \\
\text{Ext}^{\dim(R) - r}(R/\varphi^t(I)R, R)^{\vee \vee} & \twoheadrightarrow & \text{Ext}^{\dim(R) - r}(R/\varphi^{t+1}(I)R, R)^{\vee \vee}. \\
\end{array}$$

Notice that the vertical maps are injective, as seen in [6, 10.2.1 and 10.2.2]; moreover, the bottom one is also injective by what we have seen before, hence the top one is also injective, which is exactly what we wanted to prove.

(2) ⇒ (1) By the assumptions $\text{ht}(I) = \text{cd}(R, I)$, i.e. $H^i_I(R) = 0$ for all $i \neq \text{ht}(I)$. By local duality one has

$$\text{Hom}_R(H^i_I(R), E(R/m)) \cong \text{Hom}_R(\varprojlim_i \text{Ext}^i_R(R/I^t, R), E(R/m))$$

$$\cong \varprojlim_i \text{Hom}_R(\text{Ext}^i_R(R/I^t, R), E(R/m))$$

$$\cong \varprojlim_i H^i_m(R/I^t).$$

It means that for all $i \neq \text{ht}(I)$, $\varprojlim_i H^i_m(R/I^t) = 0$. Now, it is enough to prove that $\varprojlim_i H^i_m(R/I^t) \neq 0$. This follows from the fact that the homomorphisms
are surjective of nonzero $R$-modules for all $t$. \hfill \qed

As immediate consequence of Theorem 3.7 we obtain the following statement, which recovers and extends [11, Proposition 3.2]:

**Corollary 3.8** Let $(R, \mathfrak{m})$ be a Gorenstein local ring of prime characteristic $p$, and let $I \subseteq R$ be an ideal of finite projective dimension. Then, we have that $RI$ is Cohen–Macaulay if and only if $\dim(R/I) = \text{cd}(R, I)$ and $\Ext_R^j(R/I^{[p^j]}, R) \to \Ext_R^j(R/I^{[p^{j+1}]}, R)$ are injective of nonzero $R$-modules for all $t$ and $r := \text{depth}R/I$.

**Theorem 3.9** Let $(R, \mathfrak{m})$ be a Gorenstein local ring, let $I \subseteq R$ be an ideal, and let $R \to R$ be a flat ring endomorphism such that $\varphi(I)R$, is a descending chain of ideals that is cofinal with $\{I^t\}_t$, and $\varphi'(\mathfrak{m}R)$, is a descending chain of ideals that is cofinal with $\{\mathfrak{m'}^t\}_t$. Then, for any integer $j \geq 0$, $H_J^j(R) = 0$ if and only if there is an integer $t \gg 0$ such that the map

$$H_m^j(R/I) \to H_m^j(R/I)$$

is zero.

**Proof** Since $\varphi$ is flat and $\varphi'(I)R$, is cofinal with $\{I^t\}_t$, we have

$$H_J^j(R) = \lim_{t \to \infty} \Ext_R^j(R/\varphi'(I)R, R) = \lim_{t \to \infty} \Phi'(\Ext_R^j(R/I, R)).$$

The above equality shows that $H_J^j(R) = 0$ if and only if there is an integer $t \gg 0$ such that the natural map $\Ext_R^j(R/I, R) \to \Phi'(\Ext_R^j(R/I, R))$

Now, we apply the Matlis duality functor to this zero map; indeed, on the one hand the source of the map becomes $H_m^{\dim(R)-j}(R/I)$ just because of Local duality. On the other hand, using the isomorphism $\Phi(E) \cong E$ that is given by the assumption that $\{\varphi'(\mathfrak{m})R\}$, is cofinal with $\{\mathfrak{m'}^t\}_t$, jointly again with Local Duality one can check that the target is

$$\Phi'\left(\Ext_R^j(R/I, R)\right)^\vee = \text{Hom}_R(\Phi'(\Ext_R^j(R/I, R)), \Phi'E)$$

$$\cong \Phi'\text{Hom}_R(\Ext_R^j(R/I, R), E) \cong \Phi'H_m^{\dim(R)-j}(R/I).$$

Summing up, the above zero map implies (check) that the map $\varphi'$ on $H_m^{\dim(R)-j}(R/I)$ is also zero, which is what we finally wanted to prove. \hfill \qed

**Remark 3.10** Notice that the proof of the above Theorem is just the one sketched in [20, Theorem 22.1] for the case of the Frobenius map in prime characteristic in a regular local ring, see [23, Theorem 1.1] for full details. It can be regarded as a mild generalization of [33, Theorem 4.1].
4 Set-theoretically Cohen–Macaulay ideals

It is known that the radical of a Cohen–Macaulay ideal need not to be Cohen–Macaulay in general. A well-known evidence is an example due to Hartshorne [14] shows that whenever $k$ is a field of positive prime characteristic, the ideal

$$I = \ker(\varphi : k[x_1, x_2, x_3, x_4] \to k[s^4, s^3t, st^3, t^4])$$

via $x_1 \mapsto s^4, x_2 \mapsto s^3t, x_3 \mapsto st^3, x_4 \mapsto t^4$ is a non-Cohen–Macaulay set-theoretic complete intersection. However, there are some Cohen–Macaulay ideals having the same property for their radicals. For instance, if $I$ is a Cohen–Macaulay monomial ideal of a polynomial ring $R = k[x_1, \ldots, x_n]$ over a field $k$, then $\sqrt{I}$ is Cohen–Macaulay (cf. [17, Theorem 2.6]). Furthermore, principal and Veronese monomial ideals have such a property, [16, Theorem 3.2].

Let us recall the following definition in order to strengthen the above results.

**Definition 4.1** Let $(R, \mathfrak{m})$ be a regular local ring. An ideal $I \subset R$ is called set-theoretically Cohen–Macaulay if there exists an ideal $J \subset R$ with $\sqrt{I} = \sqrt{J}$ such that the ring $R/J$ is Cohen–Macaulay.

It is clear that Cohen–Macaulay radical ideals are set-theoretically Cohen–Macaulay. It should be noted that there exist also set-theoretic Cohen–Macaulay ideals which are not Cohen–Macaulay. Suppose $R = k[[x, y]]$ is a formal power series ring over a field $k$ of $x, y$. Put $I = (y) \cap (xy, y^2)$ and $J = \sqrt{I}$. It is clear that $R/J$ is Cohen–Macaulay but this is not the case for $R/I$.

Our first result in this direction is a consequence of Theorem 3.7 stating that under the assumptions given in Proposition 4.2 the concepts of Cohen–Macaulayness and set-theoretically Cohen–Macaulayness are the same.

**Proposition 4.2** Let $(R, \mathfrak{m})$ be a regular local ring with a flat ring endomorphism $\varphi : R \to R$. Further suppose that triples $(R, I, \varphi)$ and $(R, J, \varphi)$ have (Inj) property where $\sqrt{I} = \sqrt{J}$. Then $R/I$ is Cohen–Macaulay if and only if $R/J$ is Cohen–Macaulay.

**Proof** First note that as $(R, \mathfrak{m})$ is a regular local ring with a flat local ring endomorphism $\varphi : R \to R$, the dimension formula $\dim R + \dim R/\varphi(\mathfrak{m})R = \dim R$ implies that $\varphi(\mathfrak{m})R$ is $\mathfrak{m}$-primary. Then, since $R$ is a Noetherian ring and $\sqrt{I} = \sqrt{J}$, thus $\{F\}_{t \geq 0}$ and $\{J'\}_{t \geq 0}$ are cofinal. Now we are done by Theorem 3.7. \qed

As non trivial consequence of Proposition 4.2, we may recover [34, Lemma 3.1].

**Corollary 4.3** (Huneke) Let $(R, \mathfrak{m})$ be a regular local ring of positive characteristic $p > 0$ and $I$ an ideal of $R$. If the ring $R/I$ is $F$-pure, and $I$ is set–theoretically Cohen–Macaulay, then $I$ is a Cohen–Macaulay ideal.

**Proof** Since $I$ is a set-theoretically Cohen–Macaulay ideal, there is an ideal $J \subset R$ with $\sqrt{I} = \sqrt{J}$ such that $R/IJ$ is Cohen–Macaulay. Now, denoting by $F$ the Frobenius endomorphism of $R$, on the one hand the triple $(R, I, F)$ has (Inj) property because $R/I$ is $F$–pure by
assumption; on the other hand, the triple \((R, J, F)\) also has \((\text{Inj})\) property as consequence of Corollary 3.8. Keeping in mind all these facts, we conclude, thanks to Proposition 4.2, that \(I\) is Cohen–Macaulay.

According to Discussion 3.6 in Sect. 3, we observe that Cohen–Macaulayness of \(R/I\) is not equivalent to the ideal \(I\) being set-theoretically complete intersection. On the other hand, set-theoretically complete intersection ideals are set-theoretically Cohen–Macaulay but the converse is no longer true. See for instance [34]. We show that instead of set-theoretically complete intersection ideals one may regard cohomologically complete intersection ideals.

Next result (Theorem 4.4) shows the relation between set-theoretically Cohen–Macaulay and cohomologically complete intersection ideals. Notice that a set-theoretically complete intersection ideal is a cohomologically complete intersection one but the converse is no longer true.

**Theorem 4.4** Let \((R, \frak{m})\) be a regular local ring and the triple \((R, I, \varphi)\) has \((\text{Flat})\) property. Suppose that \(I\) is a set-theoretically Cohen–Macaulay ideal such that there is an ideal \(J \subseteq R\) with \(\sqrt{I} = \sqrt{J}\) such that \(R/J\) is Cohen–Macaulay and \(\{\varphi^t(J)R\}_t\) is cofinal with respect to \(\{I^t\}_t\). Then, one has \(\text{ht}(I) = \text{cd}(R, I)\).

**Proof** Preserving the assumptions and notations established above, we know that \(\sqrt{I} = \sqrt{J}\) and \(H^j_m(R/J) = 0\) for all \(i \neq \dim R/I\). As \(\varphi : R \to R\) is a flat ring endomorphism, then so does all its iterations. By applying the functor \(\Phi^t\) to \(H^j_m(R/J)\), one has \(H^j_m(R/\varphi^t(J)R) = 0\) for all integer \(t\) and all \(i \neq \dim R/I\). This implies, by local duality, that

\[
\text{Ext}^{\dim(R) - i}_R(R/\varphi^t(J)R, R)^\vee = 0 \quad \text{for all} \quad \dim(R) - i \neq \dim(R) - \dim(R/I),
\]

hence \(\text{Ext}^{\dim(R) - i}_R(R/\varphi^t(J)R, R) = 0\) for the same range of values because of the faithfulness of Matlis duality. Finally, combining this vanishing of Ext’s modules jointly with the fact that \(\{\varphi^t(J)R\}_t\) is cofinal with respect to \(\{I^t\}_t\), one has

\[
H^j_I(R) = \lim_{\longrightarrow t} \text{Ext}^{j}_R(R/\varphi^t(J)R, R) = 0 \quad \text{for all} \quad j \neq \text{ht}(I)
\]

It implies that \(\text{ht}(I) = \text{cd}(R, I)\).

Next result, which is a non–completely obvious consequence of Theorem 4.4 and Theorem 3.3, recovers and extends [11, Corollary 4.3].

**Corollary 4.5** Let \(R = k[x_1, \ldots, x_n]\) be a polynomial ring over a field \(k\) and let \(\frak{m} = (x_1, \ldots, x_n)\) be the maximal ideal. Suppose that \(I\) is an ideal such that \(\sqrt{I}\) is a square free monomial ideal. Then the following statements are equivalent.

(a) \(I\) is a set-theoretically Cohen–Macaulay ideal;
(b) \(I\) is a cohomologically complete intersection ideal.
Proof Let \( \varphi \) be the \( k \)-algebra endomorphism that maps any variable \( x_i \) to its square, which is a flat map; moreover, we also know that, for any ideal \( J \subseteq R \), \( \{ \varphi(J)R \} \) is cofinal with respect to \( \{ J' \} \). Now, we can appeal to Theorem 4.4 to conclude that (a) implies (b).

To prove the reverse statement, it is enough to show that \( H^i_m(R/\sqrt{I}) = 0 \) for all \( i \neq \dim R/I \). First of all, notice that, since the map \( \varphi \) induced on \( R/\sqrt{I} \) is pure, one has that the triple \((R, \sqrt{I}, \varphi)\) satisfies the (Inj) property (cf. Remark 2.8). In this way, the result follows immediately from Theorem 3.3.

The equivalence between the cohomologically complete intersection and set-theoretically Cohen–Macaulay ideals in Theorem 4.4 holds whenever \( R/\sqrt{I} \) is pure and flat.

We conclude this section with an investigation on the cohomological dimension of the intersection of set-theoretically Cohen–Macaulay ideals. Before it, we need the following Lemma.

Lemma 4.6 Let \((R, m)\) be a regular local ring and \( I, J \) two ideals of \( R \) with a flat ring endomorphism \( \varphi : R \to R \) such that \( \sqrt{I} = \varphi(I)R \) and \( \sqrt{J} = \varphi(J)R \). Then \( \sqrt{I} \cap J = \sqrt{\varphi(I \cap J)R} \).

Proof To prove, it is enough to note that by the flatness of \( \varphi \), one has \( \varphi(I \cap J)R = \varphi(I)R \cap \varphi(J)R \).

Proposition 4.7 Let \((R, m)\) be a regular local ring and the triples \((R, I, \varphi)\) and \((R, J, \varphi)\) have (Flat) property where \( \sqrt{I} \), \( \sqrt{J} \) are Cohen–Macaulay ideals with the same dimension \( d \). Further suppose that \( \sqrt{I} \cap \sqrt{J} = \sqrt{I} \sqrt{J} \) and \( d > \dim R/(\sqrt{I} + \sqrt{J}) \). Then \( \cd(R, I \cap J) \leq \ht(I) + \ht(J) - 1 \).

Proof As cohomological dimension is stable under taking radical of ideals, Lemma 4.6 and Theorem 3.3 imply that

\[
\cd(R, I \cap J) = \cd(R, \sqrt{I} \cap \sqrt{J}) \leq \dim R - \depth R/\sqrt{I} \cap \sqrt{J}.
\]

Now, we claim that

\[
\depth(R/\sqrt{I} \cap \sqrt{J}) \geq \depth(R/(\sqrt{I} + \sqrt{J})) + 1.
\]

Indeed, by [20, Lemma 8.7 (2)] we know that

\[
\depth(R/\sqrt{I} \cap \sqrt{J}) \geq \min\{\depth(R/\sqrt{I} \oplus R/\sqrt{J}), \depth(R/\sqrt{I} + \sqrt{J}) + 1\}.
\]

Moreover, using also [20, Lemma 8.7 (1)] we also have

\[
\depth(R/\sqrt{I} \cap \sqrt{J}) \geq \min\{\depth(R/\sqrt{I}), \depth(R/\sqrt{J}), \depth(R/\sqrt{I} + \sqrt{J}) + 1\}.
\]

From this last upper inequality we deduce, because of the assumption \( d > \depth R/(\sqrt{I} + \sqrt{J}) \), that

\[
\depth(R/\sqrt{I} \cap \sqrt{J}) \geq \depth(R/(\sqrt{I} + \sqrt{J})) + 1,
\]

and therefore
On the other hand, our assumption $\sqrt{I} \cap \sqrt{J} = \sqrt{I} \sqrt{J}$ implies that $\text{Tor}^R_i(R/\sqrt{I}, R/\sqrt{J}) = 0$. By rigidity (see either [21, Corollary 1] or [8, Corollary 2.5]) $\text{Tor}^R_i(R/\sqrt{I}, R/\sqrt{J}) = 0$ for all $i \geq 1$. Hence, by [3, Theorem 1.2] the depth formula holds for $(R/\sqrt{I}, R/\sqrt{J})$. Therefore, one has that

$$\text{depth}(R/\sqrt{I} + \sqrt{J}) = \text{depth}(R/\sqrt{I} \otimes_R (R/\sqrt{J})) = \text{depth}(R/\sqrt{J}) - \text{pd}(R/\sqrt{I}).$$

Combining this equality jointly with Auslander–Buchsbaum formula for $\text{pd}(R/\sqrt{I})$ one gets

$$\text{depth}(R/\sqrt{I} + \sqrt{J}) = -\dim R + \text{depth}(R/\sqrt{I}) + \text{depth}(R/\sqrt{J}).$$

Combining (4.1) and (4.2) we conclude that $\text{cd}(R, I \cap J) \leq \text{ht}(I) + \text{ht}(J) - 1$. 

We conclude this section with the following:

**Remark 4.8** We would like to mention that the upper bound of the cohomological dimension of the intersection of two ideals is far from being sharp, the interested reader may like to consult [24, Theorem 1.1 and Corollary 1.2], [9, Theorem 3.8], and the references given therein for further details. For lower bounds, one can consult, for instance, [6, 19.2.8].

## 5 Linkage

In the present section, we consider the concept of linkage. Roughly speaking, the linkage is the study of two subschemes where their union has nice properties. Exploiting this concept one may consider a subscheme linked with the second one which one understands better. To be more precise and from local algebra point of view we recall the definition of two linked ideals.

**Definition 5.1** Let $I$ and $J$ be two ideals of pure height $g$ of a local Gorenstein ring $(R, \mathfrak{m})$. The ideals $I$ and $J$ are (algebraically) linked by a complete intersection $\underline{x} := x_1, \ldots, x_g$ with $\langle \underline{x} \rangle \subseteq I \cap J$ if $I/\langle \underline{x} \rangle \cong \text{Hom}_R(R/J, R/\langle \underline{x} \rangle)$ and $J/\langle \underline{x} \rangle \cong \text{Hom}_R(R/I, R/\langle \underline{x} \rangle)$. We write it as $I \sim_{\langle \underline{x} \rangle} J$. For brevity we often write $I \sim J$ for $I \sim_{\langle \underline{x} \rangle} J$ when there is no ambiguity about the ideal $\langle \underline{x} \rangle$.

Ideals $I$ and $J$ are in the same linkage class if there is a sequence of links $J = I_0 \sim I_1 \sim \ldots \sim I_q = I$. An ideal $I$ is in the linkage class of a complete intersection if any of the ideals in the linkage class of $I$ are generated by a regular sequence. If $q$ is an even integer, we say that $J$ is in the even linkage class of $I$.

Let $I$ and $J$ be two ideals linked by a complete intersection $\epsilon$ of height $g$. Then

$$\text{Hom}_R(R/I, R/\epsilon) \cong \text{Hom}_{R/I}(R/I, R/\epsilon) \cong \text{Ext}_R^g(R/I, R) := K_{R/I}$$

and
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\[ \text{Hom}_R(R/J, R/c) \cong \text{Hom}_{R/c}(R/J, R) \cong \text{Ext}_R^i(R/J, R) := K_{R/J} \]

are the canonical modules of \( R/I \) resp. \( R/J \).

A ring \( R \) satisfies Serre’s condition \((S_r)\) if for all \( \mathfrak{p} \in \text{Spec}R \),
\[
\text{depth}_R \mathfrak{p} \geq \min\{r, \dim R_\mathfrak{p}\}.
\]

For an \( n \)-dimensional ring, being Cohen–Macaulay is equivalent to satisfying \((S_n)\).

**Lemma 5.2** [32, Theorem 1.14] Let \( M \) denote a finitely generated, equidimensional \( R \)-module with \( d = \dim M \), where \( R \) is a factor ring of a Gorenstein ring. Then for an integer \( r \geq 1 \) the following statements are equivalent:

(a) \( M \) satisfies condition \((S_r)\).

(b) The natural map \( M \rightarrow K_{K_M} \) is bijective (resp. injective for \( r = 1 \)) and
\[
H^i_m(K_M) = 0 \text{ for all } d - r + 2 \leq i < d.
\]

We will use the following facts on cohomological relations of linked ideas, later on.

**Lemma 5.3** Let \( I \) and \( J \) be two linked ideals of a local Gorenstein ring \((R, \mathfrak{m})\). Suppose that \( E := E(R/\mathfrak{m}) \) is the injective hull of \( R/\mathfrak{m} \) and \( d = \dim R/I = \dim R/J \).

(a) There exist a canonical exact sequence
\[
0 \rightarrow H^{d-1}_m(R/J) \rightarrow H^d_m(K_{R/J}) \rightarrow \text{Hom}_R(R/I, E) \rightarrow 0,
\]
and the canonical isomorphisms
\[
H^{i-1}_m(R/J) \cong H^i_m(K_{R/J}), \ i < d,
\]
(cf. [31, Lemma 4.2]).

(b) For an integer \( r \geq 2 \) the following statements are equivalent:

1. \( R/I \) satisfies \((S_r)\);
2. \( H^i_m(R/J) = 0 \) for all integers \( d - r < i < d \), (cf. [31, Theorem 4.1]).

(c) Assume that the local cohomology modules \( H^i_m(R/I) \) have finite length over \( R \) for all integers \( i = 0, 1, \ldots, d - 1 \). Then,
\[
H^{d-i}_m(R/J) = \text{Hom}_R(H^i_m(R/I), E), \text{ for all integers } i = 1, \ldots, d - 1,
\]
(cf. [35, Theorem 1.2, pp. 157] or [31, Corollary 3.3]).

**Remark 5.4** Let \( I, J, c \) be as before. Then \( \mathfrak{m} \not\subseteq \text{Ass}(R/I) \) if and only if \( \mathfrak{m} \not\subseteq \text{Ass}(R/J) \). To see this, without loss of generality assume that \( R/I \) is not Cohen–Macaulay. Suppose that \( \mathfrak{m} \not\subseteq \text{Ass}(R/I) \). From the above descriptions, one can obtain the following exact sequence
\[ 0 \rightarrow K_{R/J} \rightarrow R/c \rightarrow R/I \rightarrow 0. \]  

(5.1)

As \( \text{zd}(R/I) = \bigcup_{p \in \text{Ass}(R/I)} p \), we observe that \( m \not\subset \text{zd}(R/I) \) (\( m \) is not contained in the zero divisors of \( R/I \)), that is \( H^0_m(R/I) = 0 \). From the exact sequence (5.1), one has \( \text{depth} K_{R/I} = \text{depth} R/I \) (for detailed proof see [12, Proposition 3.3]). It implies that \( \text{depth} K_{R/J} > 1 \) once again, in the light of (5.1) we have \( H^1_m(R/c) = 0 \). Hence, we conclude that \( m \not\subset \text{Ass}(R/J) \). To this end note that \( \text{Ass}(R/I) \cup \text{Ass}(R/J) = \text{Ass}(R/c) \).

Of particular interest is that if \( I \) and \( J \) are in the same linkage class then what properties of \( I \) are shared by \( J \). For instance, in [30, Proposition 1.3] Peskine and Szpiro proved that if \( I \) and \( J \) are linked ideals and \( I \) is a Cohen–Macaulay ideal then so is \( J \). The same result was proved by Schenzel in [31, Corollary 3.3] for Buchsbaum ideals. In this direction, using the results of Schenzel (Lemma 5.3), with some mild assumptions, we show that between evenly linked ideals the Serre’s condition \( (S_r) \) can be shared.

**Proposition 5.5** Let \((R, m)\) be a Gorenstein local ring and \( r \geq 2 \) be an integer. Suppose that either

(a) \( I, J \) are two evenly linked ideals of \( R \), or
(b) \( I, J \) are two linked ideals of \( R \) and \( m \not\subset \text{Ass}(R/I) \).

Then, \( R/I \) satisfies \( (S_r) \) if and only if \( R/J \) is so.

**Proof** (a) Suppose that \( a \) is an ideal of \( R \) where \( I \sim a \sim J \). By virtue of Lemma 5.2, \( R/I \) satisfies \( (S_r) \) if and only if

\[ H^d_m(K_{R/I}) \rightarrow \text{Hom}_R(R/I, E(R/m)), \ d = \dim R/I, \]  

(5.2)

is bijective and \( H^i_m(K_{R/I}) = 0, \ d - r + 1 < i < d \). As \( I \) is linked to \( a \), by Lemma 5.3(a)

\[ 0 = H^{d-1}_m(R/a) \cong H^i_m(K_{R/I}), \ d - r + 1 < i < d, \]

and as \( a \) is linked to \( J \),

\[ H^{d-1}_m(R/a) \cong H^i_m(K_{R/J}), \ d - r + 1 < i < d. \]

From the exact sequence (5.2) and Lemma 5.3(a) one has \( H^{d-1}_m(R/a) = 0 \). Thus, by the following exact sequence

\[ 0 \rightarrow H^{d-1}_m(R/a) \rightarrow H^d_m(K_{R/I}) \rightarrow \text{Hom}_R(R/J, E(R/m)) \rightarrow 0, \]

one has the homomorphism \( H^d_m(K_{R/J}) \rightarrow \text{Hom}_R(R/J, E(R/m)) \) is bijective. Once again using Lemma 5.2 it turns out that \( R/I \) satisfies \( (S_r) \).

(b) First of all note that as \( m \not\subset \text{Ass}(R/I) \), by Remark 5.4 one has \( m \not\subset \text{Ass}(R/J) \), that is \( H^0_m(R/I) = 0 = H^0_m(R/J) \). Then, we are done by virtue of Lemma 5.3(b),(c).

Of particular interest is examining cohomological dimension of two linked ideals.
Proposition 5.6 Let \((R, m)\) be a Gorenstein local ring and \(I, J\) two evenly linked ideals of \(R\). Suppose that

(a) the triples \((R, I, \varphi)\) and \((R, J, \varphi)\) have (Flat) property;
(b) the triple \((R, I, \varphi)\) has (Inj) property;
(c) \(\varphi\) is of finite length.

Then for a given integer \(i\), \(H^i_I(R) = 0\) if and only if \(H^i_J(R) = 0\).

**Proof** Note that two linked ideals have the same height so for \(i \leq \text{ht}(I)\) we have nothing to prove. Hence, we may assume that \(i > \text{ht}(I)\).

Suppose that \(a\) is an ideal of \(R\) where \(I \sim a \sim J\). By what we have seen in the proof of Proposition 5.5, in the light of Lemma 5.3(a) one has \(H^j_m(R/I) \cong H^j_m(R/J)\) for all \(j < \dim R/I\). Hence, the claim follows from Theorem 3.9.

**Remark 5.7** It is noteworthy to mention that the Proposition 5.6 is no longer true for non-evenly linked ideals. To see this, let \(R = k[x_0, x_1, x_2, x_3]\) be a polynomial ring over an algebraically closed field \(k\). Let \(I = (x_0, x_1) \cap (x_2, x_3)\) be the defining ideal of the union of the two skew lines in \(\mathbb{P}^3\) and \(J = (x_0x_3 - x_1x_2, x_1^3 - x_0^2x_2, x_0x_2^2 - x_1^2x_3, x_1x_3^2 - x_1x_3)\) be the defining ideal of the twisted quartic curve in \(\mathbb{P}^3\). Then it is not hard to show that

\[ I \cap J = (x_0x_3 - x_1x_2, x_0x_2^2 - x_1x_3) \]

is a complete intersection. Therefore \(I\) is linked to \(J\) by \(c := I \cap J\), where \(H^1_I(R) \neq 0\) and \(H^2_J(R) = 0\).

Next, we consider the property of being set-theoretically Cohen–Macaulay between linked ideals.

**Theorem 5.8** Let \((R, m)\) be a regular local ring and let \(I, J\) be two linked ideals of \(R\). Suppose that

(a) \((R, I, \varphi)\) satisfies (Flat) property,
(b) \((R, I, \varphi)\) satisfies (Inj) property,
(c) \(m \notin \text{zd}(R/I)\).

If there exists a Cohen–Macaulay ideal \(b\) with \(\sqrt{b} = \sqrt{I}\) such that \(\{\varphi'(b)R\}\) is cofinal with respect to \(\{b'\}\), then \(J\) is Cohen–Macaulay.

**Proof** Note that, \(d := \dim R/\varphi(I) = \dim R/I = \dim R/J\), where the third equality follows by the fact that two linked ideals have the same dimension. We are going to show that \(H^i_m(R/J) = 0\) for all \(i < d\).

Let \(b\) be the ideal described in the assumptions. Then \(H^i_m(R/b) = 0\) for all \(i < d\). It then follows from Theorem 3.9 that \(H^{\dim R-i}_m(R) \cong H^{\dim R-i}_m(R) = 0\) for all \(\dim R - i > \dim R - d\). Now, using Theorem 3.9 again shows that \(H^i_m(R/I) = 0\) for all \(i < d\). Hence, Lemma 5.3(c) ensures that \(H^i_m(R/J) = 0\) for \(i = 1, \ldots, d - 1\). In this way, combining the previous
information jointly with Remark 5.4 it follows that $H^0_m(R/J) = 0$, this completes the proof.

\[\square\]

**Corollary 5.9** Let $(R, m)$ be a regular local ring and let $I, J$ be two linked ideals of $R$. Suppose that

(a) $(R, I, \varphi)$ and $(R, J, \varphi)$ satisfy (Flat) property,
(b) $m \not\subseteq \text{zd}(R/I)$,
(c) the induced ring endomorphisms $R/\sqrt{I} \to R/\sqrt{I}$ and $R/\sqrt{J} \to R/\sqrt{J}$ are pure and flat.

Then, if $I$ is set-theoretically Cohen–Macaulay then so is $J$.

**Proof** Theorem 5.8 ensures that $J$ is a Cohen–Macaulay ideal. Then, the equality $ht(J) = \text{cd}(R, J)$ holds by Theorem 3.7. Now, exploiting Theorem 4.4 and the proof of Corollary 4.5, one observe that $J$ is a set-theoretically Cohen–Macaulay ideal.

Notice that one can not remove the purity assumption from the above result, as the following example shows.

**Remark 5.10** Let $R$, $I$ and $J$ be as in Remark 5.7. Since depth $R/I = 1$ and dim $R/I = 2$, then $I$ is a non-Cohen–Macaulay radical ideal. As $I$ is a square-free ideal, $R/I$ is $F$-pure but this is not the case for $R/J$, because it is not reduced. On the other hand by virtue of Hartshorne [14] the ideal $J$ is set-theoretically Complete intersection and then it is set-theoretically Cohen–Macaulay.

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**References**

1. Àlvarez Montaner, J.: Lyubeznik table of sequentially Cohen–Macaulay rings. Commun. Algebra 43, 3695–3704 (2015)
2. Avramov, L., Iyengar, S., Miller, C.: Homology over local homomorphisms. Am. J. Math. 128(1), 23–90 (2006)
3. Auslander, M.: Modules over unramified regular local rings. Illinois J. Math. 5, 631–647 (1961)
4. Bhattacharyya, R.: A note on the second vanishing theorem. Available at arXiv:2004.02075pdf
5. Boix, A.F., Eghbali, M.: Annihilators of local cohomology modules and simplicity of rings of differential operators. Beitr. Algebra Geom. 59(4), 665–684 (2018)
6. Brodmann, M.P., Sharp, R.Y.: Local Cohomology: An Algebraic Introduction with Geometric Applications. Cambridge Studies in Advanced Mathematics, vol. 136, 2nd edn. Cambridge University Press, Cambridge (2013)
7. Banisaeed, E., Rahmati, F., Ahmadi-Amoli, K., Eghbali, M.: On the relation between formal grade and depth with a view toward vanishing of Lyubeznik numbers. Commun. Algebra 45(12), 5137–5144 (2017)
8. Celikbas, O., Wiegand, R.: Vanishing of Tor, and why we care about it. J. Pure Appl. Algebra 219(3), 429–448 (2015)
9. Dao, H., Takagi, S.: On the relationship between depth and cohomological dimension. Compos. Math. 152(4), 876–888 (2016)
10. Dao, H., De Stefani, A., Ma, L.: Cohomologically full rings. Int. Math. Res. Not. IMRN 17, 13508–13545 (2021)
11. Eghbali, M.: On set theoretically and cohomologically complete intersection ideals. Can. Math. Bull. 57(3), 477–484 (2014)
12. Eghbali, M., Shirmohammadi, N.: On cohomological dimension and depth under linkage. Commun. Algebra 45(3), 1134–1140 (2017)
13. Hartshorne, R.: Ample Subvarieties of Algebraic Varieties. Lecture notes in Mathematics, vol. 156. Springer, Berlin (1970)
14. Hartshorne, R.: Complete intersections in characteristic $p > 0$. Am. J. Math. 101(2), 380–383 (1979)
15. Hellus, M., Schenzel, P.: On cohomologically complete intersections. J. Algebra 320(10), 3733–3748 (2008)
16. Herzog, J., Hibi, T.: Cohen–Macaulay polymatroidal ideals. Eur. J. Combin. 27(4), 513–517 (2006)
17. Herzog, J., Takayama, Y., Terai, N.: On the radical of a monomial ideal. Arch. Math. 85, 397–408 (2005)
18. Hochster, M., Roberts, J.L.: Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay. Adv. Math. 13, 115–175 (1974)
19. Huneke, C., Lyubeznik, G.: On the vanishing of local cohomology modules. Invent. Math. 102, 73–93 (1990)
20. Iyengar, S.B., Leuschke, G.J., Leykin, A., Miller, C., Miller, E., Singh, A.K., Walther, U.: Twenty-Four Hours of Local Cohomology. Graduate Studies in Mathematics, vol. 87. American Mathematical Society, Providence (2007)
21. Lichtenbaum, S.: On the vanishing of Tor in regular local rings. Illinois J. Math. 10, 220–226 (1966)
22. Lyubeznik, G.: On the local cohomology modules $H_i^\alpha(R)$ ideals $\alpha$ generated by monomials in an $R$-sequence. In: Complete Intersections (Acireale, 1983). Lecture Notes in Mathematics, vol. 1092. Springer, Berlin (1984)
23. Lyubeznik, G.: On the vanishing of local cohomology in characteristic $p > 0$. Compos. Math. 142, 207–221 (2006)
24. Lyubeznik, G.: On some local cohomology modules. Adv. Math. 213(2), 621–643 (2007)
25. Majidi-Zolbanin, M., Miasnikov, N., Szpiro, L.: Entropy and flatness in local algebraic dynamics. Publ. Mat. 57, 509–544 (2013)
26. Martincikovsky, A., Strooker, J.R.: Linkage of modules. J. Algebra 271, 587–626 (2004)
27. Matsumura, H.: Commutative Algebra, 2nd edn. The Benjamin Cummings Publishing Company (1980)
28. Matsumura, H.: Commutative Ring Theory. Cambridge University Press (1987)
29. Peskine, C., Szpiro, L.: Dimension projective finie et cohomologie locale. Publ. Math. I.H.E.S. 42, 323–395 (1973)
30. Peskine, C., Szpiro, L.: Liaison des variétés algébriques. I. Invent. Math. 26, 271–302 (1974)
31. Schenzel, P.: Notes on liaison and duality. J. Math. Kyoto Univ. 22(3), 485–498 (1982)
32. Schenzel, P.: On the use of local cohomology in algebra and geometry. In: Elias, J., Giral, J.M., Miró-Roig, R.M., Zarzuela, S. (eds.) Six Lectures in Commutative Algebra, Proceed Summer School on Commutative Algebra at Centre de Recerca Matemtica. Program in Mathematics, vol. 166, pp. 241–292. Birkhäuser (1998)
33. Singh, A., Walther, U.: Local cohomology and pure morphisms. Illinois J. Math. 51(1), 287–298 (2007)
34. Singh, A., Walther, U.: On the arithmetic rank of certain Segre products. In: Commutative Algebra and Algebraic Geometry, Contemporary Mathematics, vol. 390, pp. 147–155. Amer. Math. Soc., Providence (2005)
35. Stückrad, J., Vogel, W.: Buchsbaum Rings and Applications. VEB Deutscher Verlag der Wissenschaften, Berlin (1986)
36. Varbaro, M.: Cohomological and projective dimensions. Compos. Math. 149, 1203–1210 (2013)

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