Current Interactions from the One-Form Sector of Nonlinear Higher-Spin Equations

O.A. Gelfond¹,² and M.A. Vasiliev²

¹ Federal Scientific Centre Scientific Research Institute of System Analysis of Russian Academy of Science, Nakhimovsky prospect 36-1, 117218, Moscow, Russia
² I.E. Tamm Department of Theoretical Physics, Lebedev Physical Institute, Leninsky prospect 53, 119991, Moscow, Russia

Abstract

The form of higher-spin current interactions in the sector of one-forms is derived from the nonlinear higher-spin equations in $AdS_4$. Quadratic corrections to higher-spin equations are shown to be independent of the phase of the parameter $\eta = \exp i\varphi$ in the full nonlinear higher-spin equations. The current deformation resulting from the nonlinear higher-spin equations is represented in the canonical form with the minimal number of space-time derivatives. The non-zero spin-dependent coupling constants of the resulting currents are determined in terms of the higher-spin coupling constant $\eta \bar{\eta}$. Our results confirm the conjecture that (anti-)self-dual nonlinear higher-spin equations result from the full system at $(\eta = 0) \bar{\eta} = 0$. 
## Contents

1 Introduction 8

2 Preliminaries
   2.1 Higher-spin equations in AdS$_4$ 6
   2.2 Perturbative analysis 8
   2.3 Current interactions 9
   2.4 Quadratic corrections in the zero-form sector 11

3 Main results 12

4 Derivation details
   4.1 Summary of main steps 14
   4.2 Field redefinition in the $\eta^2$ sector 15
   4.3 From nonlocal to local deformation in the $\eta\bar{\eta}$ sector 17
   4.4 Derivation of the canonical form of current interactions 18
      4.4.1 Flat limit rescalings 18
      4.4.2 Current decomposition 19
      4.4.3 Canonical currents 20

5 Current contribution to dynamical equations 23
   5.1 Spin 0 24
   5.2 Spin 1/2 24
   5.3 Maxwell equations 25
   5.4 Spin 3/2 27
   5.5 Spin two 28
   5.6 Higher spins 30
      5.6.1 Integer spins 30
      5.6.2 Half-integer spins 31

6 Conclusion 32

Appendix A. Useful formulas 33

Appendix B. Alternative redefinitions 34

References 37
1 Introduction

Though nonlinear field equations for massless higher-spin (HS) fields in various dimensions are available for a long time \([1, 2, 3, 4]\) their structure beyond the linearized level still is not fully understood. As shown in \([5, 6, 7, 8]\), HS interactions consistent with HS gauge symmetries contain higher derivatives though no higher derivatives appear at the quadratic level in a maximally symmetric background geometry \([9, 10]\).

Along with the fact, that a consistent HS theory containing a propagating field of any spin \(s > 2\) should necessarily contain an infinite tower of HS fields of infinitely increasing spins originally indicated by the analysis of HS symmetries in \([7, 8]\) and later shown to follow from the structure of HS symmetry algebras \([11, 12]\), this implies that any HS gauge theory is somehow nonlocal.

Appearance of higher derivatives in interactions demands a dimensionful coupling constant \(\rho\) which was identified in \([13, 14]\) with the radius of the background \((A)dS\) space. Resulting higher-derivative vertices allow no meaningful flat limit in agreement with numerous no-go statements ruling out nontrivial interactions of massless HS fields in Minkowski space \([13, 14]\) (see \([17]\) for more detail and references). Geometric origin of the dimensionful parameter \(\rho\) has an important consequence that any HS gauge theory with unbroken HS symmetries does not allow a parametric low-energy analysis with respect to a large scale parameter like Plank energy or \(\alpha'\) because the rescaled covariant derivatives \(D = \rho D\) in the expansion in powers of derivatives

\[
\sum_{n,m=0}^{\infty} a_{nm} D^n \phi D^m \phi + \ldots
\]  

\((1.1)\)

cannot be treated as small since, being non-commutative in the background \(AdS\) space-time of curvature \(\rho^{-2}\), they have commutator of order one, \([D, D] \sim 1\). As a result, all terms in \((1.1)\) may give comparable contributions. Whether expansion \((1.1)\) is local or not depends on the behavior of the coefficients \(a_{nm}\) at \(n, m \to \infty\). If at most a finite number of coefficients \(a_{n,m}\) is nonzero, field redefinition \((1.1)\) is genuinely local.

Importance of the proper definition of locality was originally stressed in \([3]\) where it was shown that by a field redefinition involving expansion of the form \((1.1)\) it is possible to get rid of the currents from 3d HS field equations. Recently this issue was reconsidered in \([18]\), where a proposal was put forward on the part of the problem associated with the exponential factors resulting from so-called inner Klein operators while the structure of the pre-exponential factors was only partially determined. The issue of locality was also analyzed in \([19, 20, 21]\). Focusing on the lowest-order current-type interactions the authors of \([19, 21]\) failed to find the appropriate scheme of the analysis of nonlinear HS equations, arriving at the conclusion that it may be hard to distinguish between local and nonlocal frames in the setup of \([2]\).

On the other hand, in \([22]\) a simple field redefinition was found that brings the quadratic corrections to the field equations in the sector of zero-forms to the canonical form of local current interactions found originally in \([23]\). The field redefinition of \([22]\) had a simple form, bringing the HS equations to the local form in the lowest order (in what follows this field redefinition as well as its extension to the one-form
sector considered in this paper will be referred to as proper). As explained in \[22\], this field redefinition is unique within the natural Ansatz expressing the separation of variables between the sectors of left and right spinors in the theory. Let us stress that the nonlocal field redefinitions applied in this paper as well as in \[22\] is not a principle issue but rather a technical tool relating the proper local formulation obtained with the originally known (improper) nonlocal one.

One of the surprising outputs of the analysis of \[22\] was that nonlinear HS equations properly reproduce usual current interactions of higher spins with the coupling constant independent of the phase parameter $\phi$ distinguishing between inequivalent HS equations. The aim of this paper is to extend the results of \[22\] to the sector of equations on HS one-form gauge potentials bringing their right-hand-sides to the standard local current form. We will show that indeed there exists a choice of field variables leading to the proper result and compute the coupling coefficients in front of different currents on the right-hand-side of HS equations. Let us stress that this choice of variables is uniquely determined by that of \[22\] up to the gauge transformations of HS gauge fields and local field redefinitions. Again, as in the zero-form sector, the resulting coupling constants turn out to be independent of the phase $\phi$.

The obtained results provide a basis for the analysis of locality in HS theory along the lines of \[24\] where it is shown in particular that the field redefinition found in \[22\] is the only proper one, hence leading to unambiguous results for the HS current coupling constants. Moreover, as stressed in \[24\], the necessity of the nonlocal field redefinitions found in \[22\] is a consequence of the improper choice of resolution operator in the process of solving the nonlinear HS equations in the sector of auxiliary $Z$-variables while an alternative choice of the “local resolution operator” leads directly to the correct local result of \[22\]. The results of this paper are anticipated to shed light on the form of the local resolution operator in the one-form sector as well.

There are two important consequences of the independence of the HS coupling of the phase $\phi$ which is the phase of the complex conjugated parameters

$$\eta = |\eta| \exp i \phi, \quad \bar{\eta} = |\eta| \exp -i \phi$$

in the nonlinear HS equations of \[2\]. One is that in \[2\] it was conjectured that the $4d$ HS theory with (\[\eta\]) $\bar{\eta} = 0$ describes an (anti-)self-dual HS theory. The conclusion reached in \[22\] and in this paper that the current interaction terms are proportional to $\eta \bar{\eta}$ implies in particular that all of them do not contribute to the purely self-dual sector, which result is anticipated because, as is well known, no nontrivial amplitude can be constructed in the purely self-dual sector. Hence, our results provide a nontrivial support for the conjecture of \[2\] on the (anti-)self-dual HS theory. The study of the latter theory which, as we show, needs a special choice of dynamical variables identified in this paper to the second order, is itself very interesting.

Another interesting direction is the HS holography (see e.g. \[25\]-\[35\]). The nontrivial question here is that the parity non-invariant HS theories with general
phase $\varphi$ were conjectured in [30, 31] to be dual to certain parity breaking Chern-Simons boundary theories. Though this conjecture seemingly contradicts the conclusion that the HS cubic vertices are independent of $\varphi$, in [22] it was argued that this is not the case and there is precise matching of the obtained $\varphi$-independent HS vertices with the structure of the boundary three-point function found in [33].

The analysis of [22] was performed in terms of certain boundary conditions on the HS connections, which may look unusual from the perspective of the standard approach. In [36] this question is reconsidered in a more standard way with the same final result.

The rest of the paper is organized as follows. In Section 2.1 we recall relevant features of the the full nonlinear HS equations in $AdS_4$ and their perturbative analysis. In Section 2.2 the quadratic corrections are discussed. General structure of current interactions is recalled in Section 2.3. Section 2.4 recalls the analysis of quadratic corrections in the zero-form sector, including the field redefinition of [22] bringing the quadratic corrections in this sector to the local form. Section 3 summarizes the main results. In Section 4 quadratic corrections in the one-form sector are considered. Specifically, the quadratic corrections from nonlinear equations are found in Section 4.2. It is shown that, modulo exact forms, the sum of quadratic corrections resulting from the zero-form redefinition with those coming from the nonlinear equations contains only $\eta \bar{\eta}$ terms. The appropriate field redefinition, that brings the $\eta \bar{\eta}$-proportional quadratic corrections in the one-form sector to the local form, is found in Section 4.3. Flat limit rescalings are recalled in Section 4.4. The local field redefinition bringing the local quadratic corrections in the one-form sector to the form allowing flat limit is also presented here. In Section 5, it is shown in detail how the resulting second-order corrections of unfolded equations bring currents to the right-hand sides of the Fronsdal-like dynamical equations for massless fields. Conclusions and perspectives are briefly discussed in Section 6. Some useful formulae are collected in Appendix A. Appendix B presents details of the derivation of field redefinitions in the one-form sector.

2 Preliminaries

2.1 Higher-spin equations in $AdS_4$

$4d$ nonlinear HS equations have the form [2]

$$dW + W \ast \wedge W = -i\theta_\alpha \wedge \theta^\alpha (1 + \eta B \ast \varpi \ast k) - i\bar{\theta}_{\dot{\alpha}} \wedge \bar{\theta}^{\dot{\alpha}} (1 + \bar{\eta} B \ast \bar{\varpi} \ast \bar{k}),$$  

$$dB + W \ast B - B \ast W = 0.$$  

(2.1) 

Here $d = dx^m \frac{\partial}{\partial x^m}$ is the space-time de Rham differential (onwards wedge symbol is omitted). $B(Z; Y; K|x)$ and $W(Z; Y; K|x)$ are fields of the theory which depend both on space-time coordinates $x$ and on spinorial variables $Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})$ and $Z^A = (z^\alpha, \bar{z}^{\dot{\alpha}})$ where the spinor indices $\alpha$ and $\dot{\alpha}$ take two values. The noncommutative star product $\ast$ acts on functions of $Y$ and $Z$

$$ (f \ast g) (Z; Y) = \int d^4U d^4V f (Z + U; Y + U) g (Z - V; Y + V) e^{iU_A V^A},$$  

(2.3)
where $U_A V^A = U^A V_B \epsilon_{AB} = u^\alpha v^\beta \epsilon_{\alpha\beta} + \bar{u}^\bar{\alpha} \bar{v}^{\bar{\beta}} \epsilon_{\bar{\alpha}\bar{\beta}}$ and $\epsilon_{AB}$ is the $sp(4)$-invariant symplectic form built from the $sp(2)$-forms $\epsilon_{\alpha\beta}, \epsilon_{\bar{\alpha}\bar{\beta}}$. Integration measure $dU^A dV^B$ in (2.1) is normalized so that $f \star 1 = f$, i.e., the factor of $\frac{1}{(2\pi)^4}$ is implicit.

Inner Klein operators $\kappa$ and $\bar{\kappa}$

$$\kappa := \exp \left( i z_\alpha y^\alpha \right), \quad \bar{\kappa} := \exp \left( i \bar{z}_{\bar{\alpha}} \bar{y}^{\bar{\alpha}} \right)$$

(2.4)

have the properties

$$\kappa \star \kappa = 1, \quad \kappa \star f (z^\alpha, y^\alpha) = f (-z^\alpha, -y^\alpha) \star \kappa,$$

$$f (y, z) \star \kappa = f (-z, -y) e^{iz_\alpha y^\alpha},$$

(2.5) (2.6)

and analogously for $\bar{\kappa}$.

$B$ is a zero-form, while $W$ is a one-form in the space-time differential $dx^m$ and auxiliary differential $\theta^A$ dual to $Z^A$

$$W(Z; Y; K|x) = dx^m W_m(Z; Y; K|x) + \theta^A S_A.$$  

(2.7)

All differentials anticommute

$$\{dx^m, dx^n\} = \{dx^m, \theta^A\} = \{\theta^A, \theta^B\} = 0.$$  

(2.8)

In addition to the inner Klein operators of the star-product algebra there is also a pair of outer Klein operators $k$ and $\bar{k}$ which have similar properties

$$k \star k = 1, \quad k \star f (z^\alpha, y^\alpha; \theta^\alpha) = f (-z^\alpha, -y^\alpha; -\theta^\alpha) \star k.$$  

(2.9)

However, being anticommutative with (anti)holomorphic $\theta$ differentials, they admit no star-product algebra realization. The fields $W(Z; Y; K|x)$ and $B(Z; Y; K|x)$ depend on the exterior Klein operators. (Relations (2.9) provide a definition of the $\star$-product for $k$ and $\bar{k}$.) The sector of physical fields is represented by $B(Z; Y; K|x)$ linear in $k$ or $\bar{k}$, while $W(Z; Y; K|x)$ instead depends on $k\bar{k}$.

For topological sector this is the other way around. The latter is truncated away in this paper. $\eta$ in (2.1) is a free complex parameter which can be normalized to be unimodular $|\eta| = 1$ hence representing the phase factor freedom $\eta = \exp i\varphi$.

Background $AdS_4$ space of radius $\lambda^{-1} = \rho$ is described by a flat $sp(4)$ connection $w = (\omega_L^{\alpha\beta}, \bar{\omega}_L^{\bar{\alpha}\bar{\beta}}, h^{\alpha\bar{\beta}})$ containing Lorentz connection $\omega_L^{\alpha\beta}$, $\bar{\omega}_L^{\bar{\alpha}\bar{\beta}}$ and vierbein $h^{\alpha\bar{\beta}}$ that obey the flatness conditions

$$R^{\alpha\beta} = 0, \quad \bar{R}^{\bar{\alpha}\bar{\beta}} = 0, \quad R^{\alpha\bar{\alpha}} = 0,$$

(2.10)

where

$$R^{\alpha\beta} = d\omega_L^{\alpha\beta} + \omega_L^{\alpha\gamma} \omega_L^{\beta\gamma} - \lambda^2 H^{\alpha\beta}, \quad \bar{R}^{\bar{\alpha}\bar{\beta}} = d\bar{\omega}_L^{\bar{\alpha}\bar{\beta}} + \bar{\omega}_L^{\bar{\alpha}\bar{\gamma}} \bar{\omega}_L^{\bar{\beta}\bar{\gamma}} - \lambda^2 \bar{H}^{\bar{\alpha}\bar{\beta}},$$

(2.11)

$$R^{\alpha\bar{\beta}} = dh^{\alpha\bar{\beta}} + \omega_L^{\alpha\gamma} h^{\beta\bar{\gamma}} + \bar{\omega}_L^{\bar{\alpha}\bar{\beta}} h^{\alpha\bar{\gamma}},$$

(2.12)
where
\[ H^{\alpha\beta} = \langle h^\alpha h^\beta \rangle, \quad \tilde{H}^{\dot{\alpha}\dot{\beta}} = \langle h^{\dot{\alpha}} h^{\dot{\beta}} \rangle, \quad h^{\alpha\dot{\alpha}} h^{\beta\dot{\beta}} = -\frac{1}{2} \varepsilon^{\alpha\beta} \tilde{H}^{\dot{\alpha}\dot{\beta}} - \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} H^{\alpha\beta}. \] (2.13)

In the first nontrivial order
\[ B_1(Z; Y; K|x) = C(Y; K|x). \] (2.14)

Thus, nonlinear HS equations (2.1), (2.2) give rise to the doubled set of massless fields
\[ C(Y; K|x) = C^{1|0}(Y|x)k + C^{0|1}(Y|x)\bar{k}. \] (2.15)

According to [2], nontrivial part of linearized equations (2.1), (2.2) has the form of so-called Central-on-shell theorem originally found in [3]
\[ \mathcal{D}_{ad}\omega(y, \bar{y}; K|x) = L(C), \]
\[ \mathcal{D}_{tw}C(y, \bar{y}; K|x) = 0, \]
(2.16) (2.17)
where the term \( L(C) \) linear in \( C \) reads as
\[ L(C) = \frac{i\lambda}{4} \left( \eta H^{\alpha\beta\delta} \partial_\alpha \partial_\beta \partial_\delta C(0, y; K|x) \ast k + \eta H^{\alpha\beta\delta} \partial_\alpha \partial_\beta C(y, 0; K|x) \ast \bar{k} \right), \] (2.18)
spin-s one-form \( \omega \) is
\[ \omega(y, \bar{y}; K|x) = \frac{1}{2i} \sum_{m,n \geq 0} \frac{1}{m!n!} \omega_{\alpha_1...\alpha_n, \tilde{\beta}_1...\tilde{\beta}_m}(K|x) y^{\alpha_1} ... y^{\alpha_n} \bar{y}^{\tilde{\beta}_1} ... \bar{y}^{\tilde{\beta}_m}, \] (2.19)
with \( n + m = 2(s - 1) \) (for \( s \geq 1 \), spin-s zero-form \( C(y, \bar{y}; K|x) \) \( (2.13) \)
\[ C^{ij}(y, \bar{y}|x) = \frac{1}{2i} \sum_{m,n \geq 0} \frac{1}{m!n!} C_{ij}^{\alpha_1...\alpha_n, \beta_1...\beta_m}(x) y^{\alpha_1} ... y^{\alpha_n} \bar{y}^{\beta_1} ... \bar{y}^{\beta_m}, \] (2.20)
has \( |n - m| = 2s \), and
\[ \mathcal{D}_{ad}\omega(y, \bar{y}; K|x) := D^L \omega(y, \bar{y}; K|x) + \lambda H^{\alpha\beta} \left( y_\alpha \bar{y}_\beta + \partial_\alpha \bar{y}_\beta \right) \omega(y, \bar{y}; K|x), \] (2.21)
\[ \mathcal{D}_{tw}C(y, \bar{y}; K|x) := D^L C(y, \bar{y}; K|x) - i\lambda H^{\alpha\beta} \left( y_\alpha \bar{y}_\beta - \partial_\alpha \bar{y}_\beta \right) C(y, \bar{y}; K|x), \] (2.22)
\[ D^L f(y, \bar{y}; K|x) := df(y, \bar{y}; K|x) + \left( \omega^{\alpha\beta}_L y_\alpha \partial_\beta + \omega^{\dot{\alpha}\dot{\beta}}_L \bar{y}_\dot{\alpha} \partial_\dot{\beta} \right) f(y, \bar{y}; K|x), \] (2.23)
\[ \partial_\alpha = \frac{\partial}{\partial y^\alpha}, \quad \partial_\dot{\alpha} = \frac{\partial}{\partial \bar{y}^\dot{\alpha}}, \quad d = dx^n \frac{\partial}{\partial x^n}. \]
Equations (2.19), (2.17) are equivalent to usual massless Fronsdal equations [3, 10] supplemented by an infinite set of auxiliary fields and constraints. The Fronsdal fields are contained in the frame-like fields \( \omega_{\alpha_1...\alpha_n, \beta_1...\beta_m}(x) \) with \( n = m \) for bosons and \( |n - m| = 1 \) for fermions.

Our goal is to find the second-order corrections to Central-on-shell theorem resulting from nonlinear equations (2.1), (2.2). To simplify formulae in the sequel we set \( \lambda = 1. \)
2.2 Perturbative analysis

To start a perturbative expansion one has to fix some vacuum solution to (2.1), (2.2). Eq. (2.2) can be solved by setting the vacuum value of $B$ to zero

$$B_0 = 0.$$ 

A natural vacuum solution for (2.2) is

$$W_0 = \phi_{AdS} + Z_A \theta^A ,$$ 

(2.24)

where $\phi_{AdS}$ is the space-time $sp(4)$ connection one-form describing the $AdS_4$ background

$$\phi_{AdS} = -\frac{i}{4} \left( \omega^{\alpha\beta} y_\alpha y_\beta + \omega^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + 2 h^{\alpha\beta} y_\alpha \bar{y}_\beta \right)$$ 

(2.25)

with Eq. (2.10) taking the form

$$d\phi_{AdS} + \phi_{AdS}^* \phi_{AdS} = 0.$$ 

(2.26)

To the second order, nonlinear equations (2.1), (2.2) have the form

$$\Delta_{ad} W_2 + W_1 * W_1 = -i \eta B_2 * k * \bar{\nu} * \theta_\alpha \theta^\alpha - i \bar{\eta} B_2 * \bar{k} * \nu * \bar{\theta}_\dot{\alpha} \bar{\theta}^{\dot{\alpha}} ,$$ 

(2.27)

$$\Delta_{tw} B_2 + [W_1, B_1]_* = 0 ,$$

where $B_2$ and $W_2$ are second-order fields,

$$\Delta_{ad} f := df + [\phi_{AdS}, f]_* - 2i dZ f$$ 

(2.28)

$$\Delta_{tw} f := df - \frac{i}{2} \left[ \omega^{AB} Y_A Y_B, f \right]_* - \frac{i}{2} \left\{ h^{AB} Y_A Y_B, f \right\}_* - 2i dZ f ,$$

$$dZ := \theta^A \frac{\partial}{\partial \theta^A} .$$

More precisely, the expansion should be interpreted as a filtration, i.e., $W_2$ and $B_2$ contain all terms up to order two rather than just the second-order part.

Equations on the $Z$-independent part (cohomology) of the $B_2$-field, $C(Y; K|x)$ (2.15), are [38]:

$$D_{tw} C = -\mathcal{H}_{tw} (W_1 * C - C * \pi (W_1)) ,$$ 

(2.29)

where $\pi (y, \bar{y}) = (-y, \bar{y})$ and

$$W_1 = \omega (Y) - i \Delta_{ad}^* (\eta C * k * \bar{\nu} * \theta_\alpha \theta^\alpha + \bar{\eta} C * \bar{k} * \nu * \bar{\theta}_\dot{\alpha} \bar{\theta}^{\dot{\alpha}}).$$ 

(2.30)

The second-order part of $B_2$

$$B_2 (Z; Y; K) = -\Delta_{tw}^* (W_1 * C - C * W_1) + C ,$$

$$\Delta_{tw} C = 0$$

contributes to the equation for the $Z$-independent part $\omega$ of $W_2$

$$D_{ad} \omega = L(C) - \mathcal{H}_{ad} (W_1 * W_1 + i \eta B_2 * k * \bar{\nu} * \theta_\alpha \theta^\alpha + i \bar{\eta} B_2 * \bar{k} * \nu * \bar{\theta}_\dot{\alpha} \bar{\theta}^{\dot{\alpha}})$$ 

(2.31)

with $L(C)$ (2.18). For the reader’s convenience manifest formulae for $\mathcal{H}_{tw}, \Delta_{tw}^*, \mathcal{H}_{ad}$ and $\Delta_{ad}^*$ of [38] are collected in Appendix A.
2.3 Current interactions

Schematically, Eqs. (2.27) have the form

\[ \mathcal{D}_{a\dot{d}\omega} + \omega * \omega - L(C) = G(w, J) + Q(C, \omega), \]
(2.32)

\[ \mathcal{D}_{tw}C + [\omega, C]_s = F(w, J), \]
(2.33)

where \( w = (\omega_L, \bar{\omega}_L, h) \) and the current \( J \) is identified with the bilinear combinations of \( C \)

\[ J(y^1, y^2; \bar{y}^1, \bar{y}^2; K|x) := C(y^1, \bar{y}^1; K|x)C(y^2, \bar{y}^2; K|x). \]
(2.34)

So defined \( J(y^1, y^2, \bar{y}^1, \bar{y}^2; K|x) \) verifies the current equation

\[ \left( D_L - i\hbar^{\alpha\dot{\alpha}} \left( y^1_{\alpha} \dot{y}^1_{\dot{\alpha}} - y^2_{\alpha} \dot{y}^2_{\dot{\alpha}} - \partial_{1\alpha} \bar{\partial}_{1\dot{\alpha}} + \partial_{2\alpha} \bar{\partial}_{2\dot{\alpha}} \right) \right) J(y^1, y^2; \bar{y}^1, \bar{y}^2; K|x) = 0 \]
(2.35)

at the convention that derivatives \( \partial_{1\alpha}(\bar{\partial}_{1\dot{\alpha}}) \) and \( \partial_{2\alpha}(\bar{\partial}_{2\dot{\alpha}}) \) over the first and second undotted(dotted) spinorial arguments of \( J \) are defined to anticommute with \( k(\bar{k}): \)

\[ \partial_2 A(w_1)kB(w_2) = -A(w_1)k\partial_2 B(w_2) , \quad \bar{\partial}_2 A(w_1)\bar{k}B(w_2) = -A(w_1)\bar{k}\bar{\partial}_2 B(w_2). \]
(2.36)

(The presence of either \( k \) or \( \bar{k} \) in the first factor of \( C \) leads to the change of a relative sign between the sector of \( Y_1 \) and \( Y_2 \) in (2.33) upon the Klein operator in the first factor of \( C(y^1, \bar{y}^1; K|x) \) is moved through the second factor.)

By virtue of Eq. (2.15) the bilinear current has the form

\[ J(Y^1; Y^2; K|x) = \sum_{j, l=0,1} C^{j, l-j}(Y^1|x)k^j\bar{k}^{1-j}C^{l, 1-l}(Y^2|x)k^l\bar{k}^{1-l}. \]
(2.37)

Nontrivial currents, that cannot be expressed via space-time derivatives of the others, are identified with the primary components of the conformal module realized by \( J(Y^1; Y^2; K|x) \) (see e.g. \[39\] and references therein). These are the conserved currents found originally in [10].

A simple but important fact, which follows from the analysis of \[39\], is that the \( \omega \)-dependent terms on the l.h.s. of Eq. (2.33) and usual interactions with gauge invariant currents contribute to different sectors of the equations. Namely, let \( s \) be the spin of the field \( C \) in the first term of (2.33) while \( s_1 \) and \( s_2 \) be spins of the constituent fields of \( J \sim CC \). Then the \( \omega \)-dependent terms can be non-zero only at \( s < s_1 + s_2 \).

Hence the \( F \)-terms in (2.33) corresponding to the usual gauge invariant (i.e., \( \omega \)-independent) current interactions are in the region

\[ s \geq s_1 + s_2. \]
(2.38)

In other words, the gauge invariant currents \( J \) built from the zero-forms \( C \) have spin \( s \) obeying (2.38). Note that this conclusion is in agreement with the results of [17] where the currents with spins beyond the region (2.38) were built in terms of the gauge connections \( \omega \). On the other hand, this argument does not apply to the
matter fields of spins $s < 1$ having no associated gauge fields represented by $\omega$ on the left-hand-side of (2.33) in which case currents $J$ (which are now gauge invariant for a trivial reason) can be built from the zero-forms $C$ obeying $s_1, s_2 \leq \frac{1}{2}$.

Recall, that the equations in the one-form sector can be decomposed into the sum of spin-$s$ eigenvectors of the operator

$$\hat{s} = y^\alpha \partial_\alpha + \bar{y}^\dot{\alpha} \bar{\partial}_{\dot{\alpha}}$$

(2.39)

with positive integer eigenvalues $2(s + 1)$.

To analyze local current deformations it is convenient to use the mutually commutative algebras $^v\mathfrak{sl}_2$ (vertical) and $^h\mathfrak{sl}_2$ (horizontal) [2], which are dual to the rank-two covariant derivative (2.33), mapping a solution of the current equation to a solution. Thus, solutions of the current equation form a $^v\mathfrak{sl}_2 \otimes ^h\mathfrak{sl}_2$-module. In this paper we will use $^v\mathfrak{sl}_2$ with the generators

$$f_+ = y_1^2 y_2^\gamma - \bar{\partial}_{1\gamma} \bar{\partial}_{2\gamma}, \quad f_- = \partial_{1\gamma} \partial_{2\gamma} - \bar{y}_1^2 \bar{y}_2^\gamma,$$

$$f_0 = y_1^2 \partial_{1\alpha} + y_2^2 \partial_{2\alpha} - \bar{y}_1^2 \bar{\partial}_{1\dot{\alpha}} - \bar{y}_2^2 \bar{\partial}_{2\dot{\alpha}},$$

$$[f_0, f_-] = -2f_-, \quad [f_0, f_+] = 2f_+, \quad [f_+, f_-] = f_0.$$

Note that the following useful formula

$$[f_+, \exp(af_-)] = a \exp(af_-)(f_0 - af_-)$$

(2.41)

is a consequences of Eq. (2.40) and the relation

$$[A, \exp B] = \int_0^1 dt \exp tB [A, B] \exp(1 - t)B.$$

Any function (formal series) $F(y^1, y^2, \bar{y}^1, \bar{y}^2)$ can be decomposed into the sum of eigenvectors $F_n(y^1, y^2, \bar{y}^1, \bar{y}^2)$ of the operator $f_0$ (2.40) with integer eigenvalues $2n$,

$$F = \sum_n F_n, \quad f_0 F_n(y^1, y^2, \bar{y}^1, \bar{y}^2) = 2n F_n(y^1, y^2, \bar{y}^1, \bar{y}^2).$$

(2.42)

A projection of $F$ to the eigenvector with eigenvalue $2n$ will be denoted $F_n$. Eigenvectors $F_n(y^1, y^2, \bar{y}^1, \bar{y}^2)$ form an $^v\mathfrak{sl}_2$-module. From (2.40) it follows that

$$[(f_-)^k F]_{n-k} = (f_-)^k F_n, \quad [(f_+)^k F]_{n+k} = (f_+)^k F_n.$$  

(2.43)

For a bilinear current $J_n$, $n$ is the sum of helicities of the constituent fields

$$J_n(y^1, y^2, \bar{y}^1, \bar{y}^2; K|x) := \sum_{h_1 + h_2 = n} C_{h_1}(y^1, \bar{y}^1; K|x) C_{h_2}(y^2, \bar{y}^2; K|x),$$

(2.44)

where $C_{h_j}$ is a helicity-$h_j$ field. Note that $n$ should not be confused with the helicity of $J_n$. For instance scalar constituent fields with $n = 0$ generate currents of any helicity (spin).
2.4 Quadratic corrections in the zero-form sector

In this section we summarize results of [22] on the computation of current interactions in the zero-form sector.

Quadratic deformation to equations on the zero-forms $C$ resulting from (2.1), (2.2) by virtue of the procedure explained in Section 2.2 has the form [22]

$$D_{tw} C + [\omega , C] + \mathcal{H}_\eta (J) + \mathcal{H}_\eta (J) = 0 ,$$

(2.45)

where $\omega$ stands for the first-order ($i.e.$, not containing the vacuum part $w$) part of the $Z$-independent part of HS connection

$$\omega (Y; K|x) = W(0; Y; K|x) , \quad C(Y; K|x) = B(0; Y; K|x) .$$

(2.46)

The quadratic deformation is

$$\mathcal{H}_\eta (J) = -i \eta \int DsDT \exp i S_A T^A \int_0^1 d\tau \left[ h(s, \tau \bar{y} - (1 - \tau) \bar{t}) J(\tau s, -(1 - \tau)y + t, \bar{y} + \bar{s}, \bar{y} + \bar{t}; K) 
- h(t, \tau \bar{y} - (1 - \tau) \bar{s}) J((1 - \tau) y + s, \tau t, \bar{y} + \bar{s}, \bar{y} + \bar{t}; K) \right] * k ,$$

(2.47)

$$\mathcal{H}_\eta (J) = -i \tilde{\eta} \int DsDT \exp i S_A T^A \int_0^1 d\tau \left[ h(\tau y - (1 - \tau) t, \bar{s}) J(y + s, y + t, \tau \bar{s}, -(1 - \tau) \bar{y} + \bar{t}; K) 
- h(\tau y - (1 - \tau) t, \bar{t}) J(y + s, y + t, (1 - \tau) \bar{y} + \bar{s}, \tau \bar{t}; K) \right] * \bar{k} ,$$

(2.48)

where

$$\eta (a, \bar{b}) := \epsilon^a_\beta a_\alpha \bar{b}_\beta .$$

(2.49)

Formulae (2.47), (2.48) follow from (2.22) and (A.2)-(A.5). The integration over $S$ and $T$ in (2.47), (2.48) brings infinite tails of contracted indices which, by virtue of the free unfolded equations (2.17), (2.22), effectively induce an infinite expansion in higher space-time derivatives of the constituent fields. The field redefinition of [22]

$$C(Y; K|x) \rightarrow C(Y; K|x) + \Phi_\eta (J) + \Phi_\eta (J)$$

(2.50)

with

$$\Phi_\eta (J) = \frac{1}{2} \eta \int DsDT \exp i S_A T^A \int d^3 \tau \prod_{i=1}^3 \theta (\tau_i) \delta (\sigma) J(\tau_3 s + \tau_1 y, t - \tau_2 y, \bar{y} + \bar{s}, \bar{y} + \bar{t}; K) * k ,$$

(2.51)

$$\Phi_\eta (J) = \frac{1}{2} \tilde{\eta} \int DsDT \exp i S_A T^A \int d^3 \bar{\tau} \prod_{i=1}^3 \theta (\bar{\tau}_i) \delta (\bar{\sigma}) J(s + y, t + y, \bar{\tau}_1 \bar{y} + \bar{\tau}_3 \bar{s}, - \bar{\tau}_2 \bar{y} + \bar{t}; K) * \bar{k} ,$$

(2.52)

$$\sigma = 1 - \sum_{i=1}^3 \tau_i , \quad \bar{\sigma} = 1 - \sum_{i=1}^3 \bar{\tau}_i ,$$

(2.53)
replaces $\mathcal{H}$ in equation (2.43) by $\mathcal{H}^{\text{loc}}$

\[
\mathcal{H}^{\text{loc}}_{\eta \text{cur}}(J) = \frac{1}{2} \eta \exp (i[\bar{\partial}_1 \bar{\partial}_2 \beta]) \int_0^1 d\tau h(y, (1 - \tau)\bar{\partial}_1 - \tau \bar{\partial}_2) J(\tau y, -(1 - \tau)y, \bar{y}, \bar{y}; K) \ast k, 
\]

(2.54)

\[
\mathcal{H}^{\text{loc}}_{\bar{\eta} \text{cur}}(J) = \frac{1}{2} \bar{\eta} \exp i[\partial_1 \bar{\partial}_2 \beta] \int_0^1 d\tau h((1 - \tau)\partial_1 - \tau \partial_2, \bar{y}) \bar{J}(y, y, \tau \bar{y}, -(1 - \tau)\bar{y}; K) \ast \bar{k}. 
\]

(2.55)

This current deformation is local since, containing only one type of contractions between spinor indices, for any given spins $s, s_1, s_2$ it contains a finite number of derivatives.

Our goal is to extend these results to current deformation in the one-form sector.

3 Main results

The main result of this paper consists of the derivation of local current interactions from the nonlinear HS equations. We actually obtained the two forms of local current interactions, referred to as natural and canonical. The natural form is simpler but contains some higher-derivative terms. The canonical form is a bit more involved, but contains currents with the minimal number of derivatives. Being related by a local field redefinition, the two forms are physically equivalent. For the reader’s convenience we present here both of them. Let us stress that both of these current deformations are proportional to $\eta \bar{\eta}$, being independent of the phase of $\eta$.

The natural form of deformed equations is

\[
\mathcal{D}_{\text{ad}} \omega + \omega \ast \omega = L(C) + Q(C, \omega) + \Gamma^{\text{loc}}_{\eta \bar{\eta}}(J), 
\]

(3.1)

\[
\mathcal{D}_{\text{tw}} C + [\omega, C]_* = -\mathcal{H}^{\text{loc}}_{\eta \text{cur}}(J) - \mathcal{H}^{\text{loc}}_{\bar{\eta} \text{cur}}(J) 
\]

(3.2)

with $L(C)$ (2.18), $\mathcal{H}^{\text{loc}}$ (2.54), (2.55),

\[
Q(C, \omega) = \eta \int dSdT \exp(iS_A T_A) \int_0^1 d\tau \Bigl\{ h(t, t\bar{\tau} - \bar{s}) \omega ((1 - \tau)y + s, \bar{y} + \bar{s}) C(\tau t, \bar{y} + \bar{t}; K|x) + h(s, s\tau - \bar{t}) C(-\tau s, \bar{y} + \bar{s}; K|x)\omega(-(1 - \tau)y - t, \bar{y} + \bar{t}) \Bigr\} \ast k + \text{c.c.} 
\]

(3.3)

and

\[
\Gamma^{\text{loc}}_{\eta \bar{\eta}} = \frac{i}{8} \eta \bar{\eta} \int \frac{d^4 \tau}{\tau^4} \delta(1 - \tau_3 - \tau_4) \delta(1 - \tau_1 - \tau_2) \theta(\tau_1) \theta(\tau_2) \theta(\tau_3) \theta(\tau_4) 
\]

(3.4)

\[
\Bigl\{ \bar{H}^{\alpha \beta} \bar{\partial}_1 \bar{\partial}_2 \partial^{\alpha \beta} \exp i\tau_3 \bar{\partial}_1 \bar{\partial}_2 \partial^{\alpha \beta} J(\tau_1 y, -\tau_2 y, \tau_4 \bar{y}, -\tau_4 \tau_1 \bar{y}; K) + H^{\alpha \beta} \partial_1 \partial_2 \partial^{\alpha \beta} J(\tau_1 y, -\tau_4 \tau_2 y, \tau_2 \bar{y}, -\tau_1 \bar{y}; K) \Bigr\}, 
\]

where $\partial_1, \partial_2, \bar{\partial}_1, \bar{\partial}_2$ are, respectively, derivatives over the first, second, third and fourth spinorial arguments of $J$ with upper indices. Note that the $\omega \ast \omega$ and $\omega C$ terms (3.3) are local since $\omega(y, \bar{y})$ is polynomial in $y$ and $\bar{y}$ for any finite spin.
Being defined for currents respecting condition \((2.38)\), the canonical form of deformed equations is
\[
\mathcal{D}_{ad}\omega + \omega \ast \omega = L(C) + Q(C, \omega) + \Gamma_{\eta \bar{\eta}}^{loc}|_{s<s_1+s_2}(J) + \Gamma_{\eta \bar{\eta}}^{can}(J), \quad (3.5)
\]
\[
\mathcal{D}_{tw}C + [\omega, C] = -\mathcal{H}_{\eta cur}^{loc}(J) - \mathcal{H}_{\eta cur}^{loc}(J) + \mathcal{D}_{tw}B^{sum}(J). \quad (3.6)
\]

The current deformation in the zero-form sector consists of two parts. The first one, \(- (\mathcal{H}_{\eta cur}^{loc}(J) + \mathcal{H}_{\eta cur}^{loc}(J))\), is the same as in Eq. (3.2). The second one defined in Eq. (3.52) does not contribute to the dynamical equations considered in Section 3 except for the spin-one case (see Section 5.3).

The current deformation in the one-form sector consists of two parts. The first one, \(\Gamma_{\eta \bar{\eta}}^{loc}|_{s<s_1+s_2}(J)\), is the projection of \(\Gamma_{\eta \bar{\eta}}^{loc}|_{s<s_1+s_2}(J)\) to the region of gauge dependent deformation. The second one, \(\Gamma_{\eta \bar{\eta}}^{can}(J)\), which is explicitly defined in Eq. (3.7), is just the gauge invariant deformation of [23].

The canonical current deformation in the one-form sector \(\Gamma_{\eta \bar{\eta}}^{can}(J)\) is
\[
\Gamma_{\eta \bar{\eta}}^{can}(J) = \frac{i}{8\eta\bar{\eta}} \int \frac{d\rho}{\rho} \frac{d^4\tau}{\tau_1} \delta'(1 - \rho_1 - \rho_2)\delta'(1 - \tau_3 - \tau_4)\delta'(1 - \tau_1 - \tau_2) \quad (3.7)
\]
\[
\times \left(1 - \delta(\rho_2) \right) \left[ \prod_{\alpha} \prod_{\beta} \right] \frac{dw}{2\pi i w} - 2(|s|_2 \right)^2 \int d\omega \exp (i\bar{\omega} \bar{\omega}) \left[ H^{a\bar{a}} \bar{\partial}_a \bar{\partial}_\beta \right.
\]
\[
J(\tau_1, w^{-1}, \rho_1, \tau_2, y_1, \rho_2, w^{-2}) w, (-\rho_1, \tau_1, y_1, \bar{\omega}, w; K)
\]
\[
+ \int ds dt H^{a\bar{a}} \partial_a \partial_\beta (\rho_1) [2(|s|_2 \right)^{2} \exp i(s_\gamma t^{-1}(w^{-2} \rho_1 \tau_3 - 1) + t_\gamma (\rho_1 \tau_1 t_\gamma + \tau^{-1}_3 \tau_2 s_\gamma) w) \]}
\[
J(s w^{-1}, \rho_1, \tau_3, t w^{-1}, (\tau_2 y + \bar{\omega}) w, (-\tau_1 y + w^{-2} \rho_2 \bar{\omega}) w; K) + c.c.,
\]
where \(\partial_a := \frac{\partial}{\partial w_\alpha}\), \(\partial_\alpha := \frac{\partial}{\partial \bar{w}_\alpha}\), and the measure \(\frac{dw}{2\pi i w} w^{-2(|s|_2 \right)^{2}}\) differs for bosonic currents with integer \(|s|_2 = 0\) and fermionic ones with half-integer \(|s|_2 = 1\). Note that the factors of \((1 - w^{-2} \rho_2 \rho_1 \tau_3)\) resulting from the Gaussian integration over \(s_\alpha\) and \(t_\alpha\) should be expanded in power series in \(w^{-2} \rho_2 \rho_1 \tau_3\).

As discussed in Section 5.6, formula (3.7) reproduces the two types of 4d cubic vertices found by Metsaev in [11]. The vertices containing in \(J\) the constituent fields of helicities of the same sign describe the AdS deformation of the Minkowski Lagrangian vertices with \(s + s_1 + s_2\) space-time derivatives. Those with constituent fields of opposite helicity signs describe the AdS deformation of the Minkowski vertices with \(s + |s_1 - s_2|\) derivatives (recall that we assume that \(s \geq s_1 + s_2\)). Moreover, as shown in [12], the resulting cubic vertices precisely reproduce the coefficients found by Metsaev in [13] from the analysis of quartic vertices.

Note that our vertex contains both parity even and parity odd parts, which appear in HS models with general \(\eta\) upon transition to the genuine Weyl tensors as explained in [14]. More precisely, this is true for the vertices with \(s + s_1 + s_2\) derivatives while those with the minimal number of derivatives remain parity even for any \(\eta\). Note that parity-odd vertices in four dimensions were considered in [14] for spin three and in [15] for general spin.
4 Derivation details

4.1 Summary of main steps

Before going into details of derivation of our results in the rest of this section, we briefly summarize the main steps.

In Section 4.2 it is shown that the $C^2$-deformation in the sector of one-forms resulting from the standard approach to nonlinear HS equations is $\mathcal{G} = \mathcal{G}_{\eta^2} + \mathcal{G}_{\bar{\eta}^2}$, i.e., $\mathcal{G}_{\eta\bar{\eta}} = 0$. The bilinear field redefinition in the zero-form sector (2.50) induces via the linear part $L(C)$ (2.18) of the Central-on-shell theorem the quadratic correction $\Gamma := \Gamma_{\eta^2} + \Gamma_{\bar{\eta}^2} + \Gamma_{\eta\bar{\eta}} + \Gamma_{\bar{\eta}^2} + Q(\omega, C)$. (4.1)

Here the nonzero $\omega C$-deformation resulting from the nonlinear HS equations in the sector of one-forms can be represented in the form

$$Q(C, \omega) = -\mathcal{H}_{ad} \left[ -i\omega \ast \Delta_{ad}^* \left( C \ast \eta \gamma \right) - i\Delta_{ad}^* \left( C \ast \eta \gamma \right) \ast \omega \right] + c.c.,$$

where

$$\gamma = k * \varepsilon * \theta_{a} \theta^{a}, \quad \bar{\gamma} = \bar{k} * \bar{\varepsilon} * \bar{\theta}_{a} \bar{\theta}^{a}.$$ (4.2)

The computation of $Q$ is straightforward leading to (3.3).

Then in Section 4.2 it is shown that

$$-\mathcal{G}_{\eta^2} - \mathcal{G}_{\bar{\eta}^2} + \Gamma_{\eta^2} + \Gamma_{\bar{\eta}^2} = \mathcal{D}_{ad}(\bar{\Omega} + \Psi)$$ (4.3)

with forms $\bar{\Omega}$ (1.12) and $\Psi$ (4.18). Upon the field redefinition $\omega := \omega - (\bar{\Omega} + \Psi)(J)$, the remaining quadratic terms in the one-form sector turn out to be proportional to $\eta\bar{\eta}$

$$\Gamma_{\eta\bar{\eta}}(J) = \frac{i}{8} \int dSdT \exp iS_A T^A \int d^3\tau d^3\tau$$

$$\left\{ \Upsilon \delta(\sigma) \delta'(\bar{\sigma}) \delta(\tau_1) \delta(\tau_2) \tilde{H}^{\alpha\beta} \bar{\partial}_a \bar{\partial}_\beta + \Upsilon \delta'(\sigma) \delta(\bar{\sigma}) \delta(\bar{\tau}_1) \delta(\bar{\tau}_2) H^{\alpha\beta} \partial_a \partial_\beta \right\}$$

$$J(\tau_3s + \tau_1y; t - \tau_2 y, \bar{\tau}_3 \bar{s} + \bar{\tau}_1 \bar{y}; \bar{t} - \bar{\tau}_2 \bar{y}; K),$$ (4.4)

where, abusing terminology, here and below we use the shorthand notation $\Upsilon$ for a product of $\theta(\tau)$ for all necessary homotopy parameters $\tau, \bar{\tau}, \rho$ etc., i.e., those to which no $\delta^{(m)}(\tau), \delta^{(k)}(\bar{\tau}), \delta^{(n)}(\rho) \ldots$ is associated,

$$\Upsilon := \prod_j \theta(\tau_{kj}) \theta(\bar{\tau}_{ij}) \theta(\rho_{lj}) \ldots.$$ (4.5)

In Section 4.3 such $X(J)$ (4.23) is found that $\mathcal{D}_{ad} X = \Gamma_{\eta\bar{\eta}} - \Gamma_{\eta\bar{\eta}}^{loc}$ with $\Gamma_{\eta\bar{\eta}}^{loc}$ (3.4). Upon the field redefinition $\omega \rightarrow \omega - X(J)$, the quadratic terms in the one-form sector acquire the local form $\Gamma_{\eta\bar{\eta}}^{loc}$. However, as explained in Section 4.4.2, it contains higher-derivative terms with the coefficients divergent in the flat limit. Finally, in
Section 4.4.3 we find such local one-form $\Lambda^{\text{sum}}$ and zero-form $B^{\text{sum}}$ that, upon the field redefinition

$$\omega \to \omega + \Lambda^{\text{sum}}(J), \quad C \to C + B^{\text{sum}}(J), \quad (4.6)$$

the quadratic terms in the one-form sector take gauge invariant canonical current deformation form $\Gamma_{\eta \bar{\eta}}^{\text{can}}$ with the minimal number of derivatives, that admits a proper flat limit.

### 4.2 Field redefinition in the $\eta^2$ sector

According to (A.3), (A.11), the $C^2$-deformation resulting from the nonlinear HS equations in the sector of one-forms can be represented in the form

$$\mathcal{G} := \mathcal{H}_{\text{ad}}\left\{\Delta^{*}_{\text{ad}}(C * (\eta \gamma + \bar{\eta} \bar{\gamma})) * \Delta^{*}_{\text{ad}}(C * (\eta \gamma + \bar{\eta} \bar{\gamma}))\right\} \quad (4.7)$$

Decomposing $\mathcal{G}$ (4.7) in powers of $\eta$ and $\bar{\eta}$ as $\mathcal{G} = G_{\eta^2} + G_{\eta \bar{\eta}} + G_{\bar{\eta} \bar{\eta}}$ it is not hard to see by virtue of (A.2)-(A.5) that $G_{\eta \bar{\eta}} = 0.$

Consider

$$G_{\eta^2} = \eta^2 \mathcal{H}_{\text{ad}}\left\{\Delta^{*}_{\text{tw}}\left([\Delta^{*}_{\text{ad}}(C * \gamma), C]_* \right) * \gamma + \Delta^{*}_{\text{ad}}(C * \gamma) * \Delta^{*}_{\text{ad}}(C * \gamma)\right\}, \quad (4.8)$$

where by virtue of (A.2)-(A.5) and (2.3), (2.5)

$$\mathcal{H}_{\text{ad}}\left(\Delta^{*}_{\text{tw}}\left([\Delta^{*}_{\text{ad}}(C * \gamma), C]_* \right) * \gamma\right) = \frac{i}{8} \int_{0}^{1} d\tau \int dS dT \exp(iS_A T^A) \quad (4.9)$$

$$\left(\exp(i(\tau s-t)\alpha y^\alpha)\right)\left\{\omega_L^{-\alpha}_{\gamma} \omega_L^{-\beta}_{\gamma} (\tau s-t)_{\alpha}(\tau s-t)_{\beta} + 2\hbar^{-\alpha}_{\gamma} \omega_L^{-\beta}_{\gamma} (\bar{t}-\bar{s})_{\alpha}(\tau s-t)_{\beta}\right\}

\quad - \left\{\exp(i(\tau s-t)\alpha y^\alpha) - 1\right\} H^{\beta}_{\alpha} (\bar{t}-\bar{s})_{\alpha}(\bar{t}-\bar{s})_{\beta}\right] J(\tau s, t, \bar{y} + \bar{s}, \bar{y} + \bar{t}; K|x) ,$$

$$\mathcal{H}_{\text{ad}}\left\{\Delta^{*}_{\text{ad}}(C * \gamma) * \Delta^{*}_{\text{ad}}(C * \gamma)\right\} = \frac{1}{8} \int dS dT \int_{0}^{1} d\tau_1 \int_{0}^{1} d\tau_2 \exp i \left(S_A T^A + (\tau_1 s_{\beta} - \tau_2 t_{\beta}) y^\beta\right)$$

$$+ 2\omega_L^{-\alpha}_{\gamma} (\tau_1 t_{\gamma} - 1)(\tau_2 s_{\alpha} t_{\delta} \tau_{\delta} - \tau_1 s_{\alpha} s_{\delta} t_{\delta} - \tau_2 t_{\alpha} t_{\delta} t_{\delta}) - 2\omega_L^{-\alpha}_{\gamma} h^{\alpha \beta} \tau_1 \tau_2 s_{\delta} t_{\delta} (t_{\beta} s_{\delta} + s_{\delta} t_{\beta}) \quad (4.10)$$

$$- 2\hbar^{-\alpha}_{\gamma} \omega_L^{-\beta}_{\gamma} (\bar{t}_1 \tau_2 - 1) (\tau_2 s_{\alpha} t_{\gamma} \tau_{\delta} \tau_{\delta} - \tau_1 s_{\alpha} s_{\delta} s_{\delta} - \tau_2 t_{\alpha} t_{\delta} t_{\delta}) - H^{\alpha \beta} s_{\alpha} t_{\beta} (s_{\delta} \tau_{\delta} - 2i)(\tau_1 \tau_2 - 1)$$

$$- 2\hbar^{-\alpha}_{\gamma} \omega_L^{-\beta}_{\gamma} (\tau_1 \tau_2 + 1)(s_{\alpha} \tau_{\delta} \tau_{\delta} - \tau_1 s_{\alpha} s_{\delta} - \tau_2 t_{\alpha} t_{\delta} t_{\delta}) J(\tau_1 s, \tau_2 t, \bar{y} + \bar{s}, \bar{y} + \bar{t}; K|x) , \quad (4.11)$$

with $J$ (2.34).

First of all, the part of $G_{\eta^2}$ containing $\omega_L^{\alpha \beta}$ has to be eliminated. To this end we set

$$\bar{\Omega} = \omega_L^{\alpha \beta} \Omega_{\alpha \beta} \quad (4.12)$$
\[ \Omega_{\delta \beta} = \frac{i\eta^2}{4} \int dSdT s_{\delta \beta} \int_0^1 d\tau_1 \tau_1 \int_0^1 d\tau_2 \tau_2 \exp i(S_A T^A + (\tau_1 s_{\gamma} - \tau_2 t_{\gamma}) y^\gamma) J(\tau_1 s, \tau_2 t, \bar{y} + \bar{s}, \bar{y} + \bar{t}; K|x). \] 

(4.13)

From (2.13), (2.21) it follows that
\[ \mathcal{D}_{ad} \tilde{\Omega} = \omega_{L}^{\beta} \gamma \omega_{L}^{\gamma} \delta \Omega_{\beta \delta} + H^{\alpha \beta} \Omega_{\alpha \beta} - \omega_{L}^{\alpha \beta} \mathcal{D}_{ad} \Omega_{\alpha \beta}. \] 

(4.14)

Using the useful identity
\[ \int ds dt \int_0^1 d\tau \exp(is_{\alpha} t^\alpha) \left\{ F(\tau s) (\tau)^{n-1} (n-2-is_{\beta} t^\beta) \frac{\partial}{\partial \tau} ((\tau)^n F(\tau s)) \right\} = 0, \] 

(4.15)

by virtue of (2.21) and (2.22) it is not hard to see that \( G_{\eta^2} \) can be represented in the form
\[ G_{\eta^2} = \mathcal{D}_{ad} \tilde{\Omega} + \mathcal{D}_{ad} \Psi + G'_{\eta^2}, \] 

(4.16)

where
\[ G'_{\eta^2} = -i \frac{\eta^2}{8} H^{\alpha \beta} \int_0^1 d\tau \int dSdT \exp(iS_A T^A) (t-s)_{\alpha} (t-s)_{\beta} J(\tau s, t, \bar{y} + \bar{s}, \bar{y} + \bar{t}; K), \] 

(4.17)

\[ \Psi = -i \frac{\eta^2}{4} h(s, \bar{s}) \int_0^1 d\tau_1 \tau_1 \int_0^1 d\tau_2 \int dSdT \exp i(S_A T^A + (\tau_1 s_{\gamma} - \tau_2 t_{\gamma}) y^\gamma) J(\tau_1 s, \tau_2 t, \bar{y} + \bar{s}, \bar{y} + \bar{t}; K). \] 

(4.18)

It should be stressed that both in \( G_{\eta^2} \) (4.18) and in \( \Psi \) (4.18) the dependence on the right spinors \( \bar{y}^\alpha \) remains unaffected by the homotopy integrals, \textit{i.e.,} these variables are only affected by the star product in the right sector. This Ansatz is specific for the separation of variables approach applied in [22] to the zero-form sector, where it was shown to lead to the unique local solution, and extended in this paper to the one-form sector. Modulo gauge transformations and local changes of variables, field redefinition (4.18) is also the only one respecting the separation of variables and giving the local result.

Analogously, modulo \( \mathcal{D}_{ad} \)-exact terms, the \( \bar{\eta}^2 \)-deformation is
\[ G'_{\bar{\eta}^2} = -i \frac{\eta^2}{8} H^{\alpha \beta} \int_0^1 d\bar{\tau} \int dSdT \exp(iS_A T^A) (t-s)_{\alpha} (t-s)_{\beta} J(y + s, y + t, \bar{r} \bar{s}, \bar{r} \bar{t}; K|x). \] 

Thus, upon the field redefinition
\[ \omega(y, \bar{y}; K|x) \rightarrow \omega(y, \bar{y}; K|x) + \tilde{\Omega} + \Psi \]

Eq. (2.31) acquires the form
\[ \mathcal{D}_{ad} \omega + \omega * \omega = L(C) + Q(C, \omega) - G'_{\eta^2} - G'_{\bar{\eta}^2}. \] 

(4.20)
The field redefinition in the zero-form sector \((2.50)\) induces the quadratic correction on the r.h.s. of \((4.20)\) by virtue of \((2.18)\)

\[
\Gamma := i \frac{\eta}{8} \bar{H}^{\alpha \beta} \partial_\alpha \partial_\beta \left\{ \Phi_\eta + \bar{\Phi}_\eta \right\} (J)(y, 0; K|x) + i \frac{\eta}{8} \bar{H}^{\dot{\alpha} \dot{\beta}} \bar{\partial}_\dot{\alpha} \bar{\partial}_\dot{\beta} \left\{ \Phi_\eta + \bar{\Phi}_\eta \right\} (J)(0, \bar{y}; K|x),
\]

which can be decomposed as \(\Gamma := \Gamma_{\eta^2} + \Gamma_{\bar{\eta}\eta} + \Gamma_{\bar{\eta}^2}\). Upon integration over \(\tau_1\) and \(\tau_2\)

\[
\Gamma_{\eta^2} = i \frac{\eta^2}{8} \int dSdT \exp i S_A T^A \int d\tau_3 \theta(\tau_3) \theta(1 - \tau_3) \bar{H}^{\dot{\alpha} \dot{\beta}} \bar{\partial}_\dot{\alpha} \bar{\partial}_\dot{\beta} J(\tau_3 s, t; \bar{s} + \bar{y}, \bar{t} + \bar{y}; K|x)
\]

just compensates \(-G'_{\eta^2} (4.17)\) in \((4.20)\). Hence, the full \(\eta^2\)-deformation is zero. Analogously, the full \(\bar{\eta}^2\)-deformation is also zero. Hence the nonlinear deformation in the one-form sector takes the form

\[
D_{\text{ad}} \omega + \omega \ast \omega = L(C) + Q(C, \omega) + \Gamma_{\eta \bar{\eta}}(J),
\]

with \(L(C) (2.18)\), \(Q(C, \omega) (3.3)\) and \(\Gamma_{\eta \bar{\eta}}(J) (4.4)\). Clearly, \(\Gamma_{\eta \bar{\eta}}(J)\) is not local.

The fact that there exists a field redefinition of the one- and zero-forms bringing the \(\eta^2\)-terms in the one-form sector to zero is not a priori obvious. Remarkably to reach this result one has just to apply the shift in the zero-form sector found in \([22]\). In other words, the alternative way to deduce the shift of \([22]\) is to demand that the \(\eta^2\)-terms in the one-form sector should be zero.

### 4.3 From nonlocal to local deformation in the \(\eta \bar{\eta}\) sector

Now we show, that there exists a proper field redefinition bringing deformation \((4.4)\) to the local form. Due to the gauge freedom for one-form HS gauge fields there exist many equivalent representatives for the same local current. An alternative field redefinition that contains integration over a simplex in the space of homotopy parameters is presented in Appendix B. It should be stressed that the field redefinition found in this section is unique modulo gauge transformations and further local transformations because the zero-form equations found in \([22]\) are demanded to be unaffected.

The problem will be solved in two steps. First, in this section we will introduce a \(D_{\text{ad}}\)-exact shift bringing the deformation to the local form. Second, in Section 4.4, a local field redefinition eliminating both the terms divergent in the flat limit and those contributing to the torsion-like HS curvature will be found in the sector of gauge invariant currents obeying \((2.38)\).

Let

\[
X(J) = i \frac{\eta \bar{\eta}}{8} \int d^3 \tau d^3 \bar{\tau} \delta(1 - \tau_3 - \bar{\tau}_2) \delta(1 - \bar{\tau}_3 - \tau_2) \delta'(1 - \tau_1 - \bar{\tau}_1) h(\partial, \bar{\partial})
\]

\[
\frac{(1 - \tau_3 \bar{\tau}_3)}{\tau_2 \bar{\tau}_2} \exp i \left( \tau_3 \bar{\partial}_1 \partial_2 + \bar{\tau}_3 \bar{\partial}_1 \partial_2 \right) J(\tau_2 \bar{\tau}_1 y, -\tau_2 \bar{\tau}_1 y, \bar{\tau}_2 \bar{\tau}_1 \bar{y}, -\tau_2 \bar{\tau}_1 \bar{y}; K|x)
\]

\[
X(J) = \frac{1}{8} \int d^3 \tau d^3 \bar{\tau} \delta(1 - \tau_3 - \bar{\tau}_2) \delta(1 - \bar{\tau}_3 - \tau_2) \delta'(1 - \tau_1 - \bar{\tau}_1) h(\partial, \bar{\partial})
\]

\[
\frac{(1 - \tau_3 \bar{\tau}_3)}{\tau_2 \bar{\tau}_2} \exp i \left( \tau_3 \bar{\partial}_1 \partial_2 + \bar{\tau}_3 \bar{\partial}_1 \partial_2 \right) J(\tau_2 \bar{\tau}_1 y, -\tau_2 \bar{\tau}_1 y, \bar{\tau}_2 \bar{\tau}_1 \bar{y}, -\tau_2 \bar{\tau}_1 \bar{y}; K|x)
\]
with $\Upsilon$ (4.3). Using (2.13), straightforward differentiation yields

$$
\mathcal{D}_{ad} \mathcal{X} = \frac{i}{8} \bar{\eta} \bar{\eta} \tilde{H}^{\alpha \beta} \bar{\partial}_\alpha \bar{\partial}_\beta \int_0^1 d\tau_3 \frac{\partial}{\partial \tau_3} \int d\bar{\tau}_3 \bar{\tau}_1 \Upsilon \delta(1-\tau_3-\bar{\tau}_2) \delta'(1-\tau_1-\bar{\tau}_1) \left\{ \frac{(1-\tau_3 \bar{\tau}_3)^2}{\bar{\tau}_2^2} \right\} (4.24)
$$

$$
\exp \left\{ i \left( \tau_3 \partial_1 \partial_2 \alpha + \bar{\tau}_3 \bar{\partial}_1 \bar{\partial}_2 \bar{\alpha} \right) J \left( (1-\tau_3)\tau_1 y, -(1-\tau_3)\bar{\tau}_1 \bar{y}, \bar{\tau}_2 \bar{\tau}_1 \bar{y}, -\bar{\tau}_2 \tau_1 \bar{y}; K|x \right) \right\} + c.c.
$$

Note that the $\bar{\tau}_2$-poles in Eq. (4.24) are fictitious due to the differentiations $\bar{\partial}_\alpha \bar{\partial}_\beta$.

Hence performing integration over $\tau_3$ one obtains

$$
\mathcal{D}_{ad} \mathcal{X}(J) = \Gamma^\text{loc}(J) - \Gamma^\text{loc}_{\bar{\eta} \bar{\eta}}(J) \tag{4.25}
$$

with

$$
\Gamma^\text{loc}(J) = \frac{i}{8} \bar{\eta} \bar{\eta} \int d^3 \tau \Upsilon \delta(1-\tau_3-\tau_4) \delta'(1-\tau_1-\tau_2) \tag{4.26}
$$

$$
\left\{ \tilde{H}^{\alpha \beta} \bar{\partial}_\alpha \bar{\partial}_\beta \exp \left\{ i \left( \partial_1 \partial_2 \alpha + \tau_3 \bar{\partial}_1 \bar{\partial}_2 \bar{\alpha} \right) J(0,0,\tau_4 \bar{\tau}_2 \bar{y}, -\tau_4 \tau_1 \bar{y}; K|x) \right\} + \tilde{H}^{\alpha \beta} \partial_\alpha \partial_\beta \exp \left\{ i \left( \tau_3 \partial_1 \partial_2 \alpha + \bar{\tau}_3 \bar{\partial}_1 \bar{\partial}_2 \bar{\alpha} \right) J(\tau_4 \tau_1 y, -\tau_4 \tau_2 y, 0, 0; K|x) \right\} \right\}
$$

and $\Gamma^\text{loc}_{\bar{\eta} \bar{\eta}}(J)$ (3.4). By virtue of the following simple formula

$$
\int d^3 \tau \delta'(1-\tau_1-\tau_2-\tau_3) \theta(\tau_1) \theta(\tau_2) \theta(\tau_3) f(\tau_1, \tau_2, \tau_3) \tag{4.27}
$$

$$
= \int d^3 \tau \delta'(1-\tau_1-\tau_2) \delta(1-\tau_3-\tau_4) \theta(\tau_1) \theta(\tau_2) \theta(\tau_3) \theta(\tau_4) f(\tau_1 \tau_4, \tau_2 \tau_4, \tau_3) \tag{4.28}
$$

$\Gamma^\text{loc}$ (4.26) coincides with $\Gamma_{\bar{\eta} \bar{\eta}}$ (1.4). Therefore, by a field redefinition

$$
\omega \rightarrow \omega - \mathcal{X}(J) \tag{4.28}
$$

with $\mathcal{X}(J)$ (1.23), the $\bar{\eta} \bar{\eta}$-current deformation in the one-form sector is reduced to $\Gamma^\text{loc}_{\bar{\eta} \bar{\eta}}(J)$.

The following comment is now in order. The deformation $\Gamma_{\bar{\eta} \bar{\eta}}$ (1.4) consists of two terms. One can check that each of these terms is $\mathcal{D}_{ad}$-closed. Originally, we anticipated that each of these terms can be brought to the local form by a field redefinition in the form of some homotopy integral. However, we failed to proceed this way. This is in agreement with the final field redefinition represented by a single term (1.23), providing one more evidence of the uniqueness of the proposed scheme.

Being local, deformation (3.4) does not reproduce the canonical deformation of (1.4). Hence the difference between the two forms of local currents should be an improvement, i.e., to bring deformation (3.4) to the canonical form we have to perform a further local field redefinition.

### 4.4 Derivation of the canonical form of current interactions

#### 4.4.1 Flat limit rescalings

To take the flat limit it is necessary to perform certain rescalings. To this end, following to (10), it is useful to introduce notations $A_\pm$ and $A_0$ so that the eigenvalues
of the helicity operator $\frac{1}{2} \left( y^\alpha \frac{\partial}{\partial y^\alpha} - \overline{y}^i \frac{\partial}{\partial \overline{y}^i} \right)$ are positive on $A_+(y, \overline{y} | x)$, negative on $A_-(y, \overline{y} | x)$, and zero on $A_0(y, \overline{y} | x)$. Using the decomposition

$$A(y, \overline{y} | x) = A_+(y, \overline{y} | x) + A_-(y, \overline{y} | x) + A_0(y, \overline{y} | x),$$  \hspace{1cm} (4.29)

the rescalings are introduced differently in the adjoint and twisted adjoint modules

$$\tilde{A}^{ad}(y, \overline{y} | x) = A_+(\lambda^{\frac{1}{2}} y, \lambda^{-\frac{1}{2}} \overline{y} | x) + A_-(\lambda^{-\frac{1}{2}} y, \lambda^{\frac{1}{2}} \overline{y} | x) + A_0(y, \overline{y} | x),$$  \hspace{1cm} (4.30)

$$\tilde{A}^{tw}(y, \overline{y} | x) = A(\lambda^{\frac{1}{2}} y, \lambda^{\frac{1}{2}} \overline{y} | x).$$

For the rescaled variables, the flat limit $\lambda \to 0$ of the adjoint and twisted adjoint covariant derivatives (2.21) and (2.22) gives

$$D_{fi}^{ad} \tilde{A}(y, \overline{y} | x) = D^L \tilde{A}(y, \overline{y} | x) + \left( h(\overline{y}, \overline{\partial}) \tilde{A}_-(y, \overline{y} | x) + h(\partial, \overline{y}) \tilde{A}_+(y, \overline{y} | x) \right),$$  \hspace{1cm} (4.31)

$$D_{fi}^{tw} \tilde{A}(y, \overline{y} | x) = D^L \tilde{A}(y, \overline{y} | x) + i h(\partial, \overline{\partial}) \tilde{A}(y, \overline{y} | x).$$  \hspace{1cm} (4.32)

The flat limit of the unfolded massless equations results from (2.16) via the substitution of $D^L$ and $h^{\alpha\dot{\alpha}}$ of Minkowski space along with the replacement of $D^{ad}$ and $D^{tw}$ by $D_{fi}^{ad}$ and $D_{fi}^{tw}$, respectively. The resulting field equations describe free HS fields in Minkowski space. Let us stress that, looking somewhat unnatural in the two-component spinor notation, prescription (4.30) is designed just to give rise to the theory of Fronsdal [9] and Fang and Fronsdal [10] (for more detail see [46]).

Note that, although the contraction $\lambda \to 0$ with the rescaling (4.30) is consistent with the free HS field equations, negative powers of $\lambda$ survive in the full nonlinear equations upon rescaling (4.30), making the Minkowski background unreachable in the nonlinear HS gauge theories of [13, 2, 4]. Nevertheless, the HS interactions with gauge invariant currents considered in this paper do admit a proper flat limit.

### 4.4.2 Current decomposition

Firstly let us represent $\Gamma^{loc}_{\eta \overline{\eta}}$ (3.4) as

$$\Gamma^{loc}_{\eta \overline{\eta}}(J) = \Gamma^{\geq loc}_{\eta \overline{\eta}}(J) + \Gamma^{\leq loc}_{\eta \overline{\eta}}(J),$$  \hspace{1cm} (4.33)

where

$$\Gamma^{\geq loc}_{\eta \overline{\eta}}(J) := \Gamma^{loc}_{\eta \overline{\eta}}(J)|_{s \geq s_1 + s_2}, \hspace{1cm} \Gamma^{\leq loc}_{\eta \overline{\eta}}(J) := \Gamma^{loc}_{\eta \overline{\eta}}(J)|_{s \leq s_1 + s_2}$$

are projections of $\Gamma^{loc}_{\eta \overline{\eta}}$ to the respective regions of spins.

To find canonical gauge invariant current deformation, that admits a proper flat limit, let us decompose $\Gamma^{loc}_{\eta \overline{\eta}}$ (4.33) as a sum of eigenvectors of two mutually commuting operators $\hat{s}$ (2.39) and $f_0$ (2.40) i.e.,

$$\Gamma^{\geq loc}_{\eta \overline{\eta}}(J) = \frac{i}{8} \eta \overline{\eta} \sum_{s \geq 1} \sum_{-s \leq n \leq s} \Gamma^{loc}(J)|_{s,n}.\hspace{1cm} (4.34)$$
where
\[ \Gamma^{loc}(J)|_{s,n} = 2(s-1)\Gamma^{loc}(J)|_{s,n} \]
\[ f_0 \Gamma^{loc}(J)|_{s,n} = 2nf^{loc}(J)|_{s,n}. \]

To this end, using (2.40) along with the fact that \( y_\alpha y^\alpha = y^\alpha y_\alpha = 0 \), it is convenient to represent \( \Gamma^\gamma_{n\eta} \) in a slightly different form resulting from the substitution
\[ \partial_\alpha \partial_2^\alpha \rightarrow f_-, \quad \bar{\partial}_1 \partial_2^\alpha \rightarrow -f_+, \]
where, introducing the contour integrations over cycles close to zero,
\[
\Gamma^{loc}(J)|_{s,n} = \frac{i}{8} \int \frac{d^4 \tau}{\tau_4} \delta(1-\tau_3-\tau_4)\delta'(1-\tau_1-\tau_2) \int \frac{dw}{2\pi i w^{2s+1}} \int \frac{dv}{2\pi i v^{2n+1}} \left\{ \begin{array}{l}
\hat{H}^\alpha\bar{\alpha}\bar{\beta}_\beta \exp(-i\tau_3 f_+ v^2) J(\tau_1 y v w, -\tau_2 y v w, \tau_1 \bar{y} v^{-1} w, -\tau_4 \bar{y} v^{-1} w; K)
+ H^\alpha\bar{\alpha}\bar{\beta}_\beta \exp(i\tau_3 f_- v^{-2}) J(\tau_1 y v w, -\tau_2 y v w, \tau_2 \bar{y} v^{-1} w, -\tau_1 \bar{y} v^{-1} w; K) \end{array} \right\}.
\]

For any \(|n| \leq s\), \( \Gamma^{loc}(J)|_{s,n} \) is a consistent deformation, that is gauge invariant since inequality (2.38) holds by construction.

So defined \( \Gamma^{loc}(J)|_{s,n} \) describes the spin-\( s \) contribution of \( J_n \) in the one-form sector since the total degree in \( y \) and \( \bar{y} \) is 2(s − 1). Moreover, \( \Gamma^{loc}(J)|_{s,n} \) projects currents to the components \( J_n \) obeying \( f_0 J_n = 2nJ_n \), i.e., \( \Gamma^{loc}(J)|_{s,n} \equiv \Gamma^{loc}(J_n)|_{s,n} \).

However, only deformations \( \Gamma^{loc}(J_n)|_{s,n} \) with \(|n| \leq \frac{1}{2}\) admit a proper flat limit (for more detail see [23]). On the other hand, using analogues of the manifest formulae for trivial deformations of [23] it will be shown that, up to a numerical coefficient, a current deformation \( \Gamma^{loc}(J_n)|_{s,n} \) at \( s \geq 2, s \geq n > \frac{1}{2} \) in AdS4 is equivalent modulo a local field redefinition to \( \Gamma^{loc}((f_-)^n J_n)|_{s,0} \) for integer \( n \) and \( \Gamma^{loc}((f_-)^{-n+\frac{1}{2}} J_n)|_{s,\frac{1}{2}} \) for half-integer \( n \). Analogously, a current deformation associated with \( J_n \) at \(-\frac{1}{2} > n > -s\) is equivalent up to a numerical coefficient to \( \Gamma^{loc}((f_+)^{-n} J_n)|_{s,0} \) for integer \( n \) and \( \Gamma^{loc}((f_+)^{-n+\frac{1}{2}} J_n)|_{s,-\frac{1}{2}} \) for half-integer \( n \).

Thus the proper strategy for reducing a local current interaction to the canonical form that admits flat limit is to add improvements involving \( f_\pm \) to remove all contributions of currents with \(|n| > \frac{1}{2}\) to achieve that the resulting deformation would only involve canonical currents \( J_m \sim (f_\pm)^{|m|} J_n \) with \( m = \pm \frac{1}{2} \) or 0.

Note that this procedure simultaneously removes contributions to the r.h.s. of the HS torsion-like tensor for integer spins, proportional to \( y^{-1}y^{-1} \). Indeed, \( J_0 \) does not contribute to the torsion-like terms because of the pre-factors \( \hat{T}^\alpha\bar{\alpha}\bar{\beta}_\beta \partial_\alpha \partial_2^\beta \) and \( H^\alpha\bar{\alpha}\bar{\beta}_\beta \partial_\alpha \partial_2^\beta \) in (1.35). Hence, canonical currents do not contribute to torsion.

4.4.3 Canonical currents

Our aim is to find such local one-form \( \Lambda^{sum} \) and zero-form \( B^{sum} \) that the field redefinition (4.6) reduces the current interactions to the canonical form (3.7).
Let $J$ be a solution to current equation (2.33). Consider the one-form

$$
\Lambda(J) = \frac{i}{8} \bar{\eta} \eta h(\partial, \bar{\partial}) \int_0^1 \frac{d\tau_3}{1 - \tau_3} \int d^2 \tau \delta'(1 - \tau_1 - \tau_2) \theta(\tau_1) \theta(\tau_2) \exp(i \tau_3 f_\eta J)(((1 - \tau_3) \tau_1 y + y^1, \cdots, (1 - \tau_3) \tau_2 y + y^2, \tau_2 \bar{y} + \bar{y}^1, -\tau_1 \bar{y} + \bar{y}^2; K|x)|_{y^\prime = \bar{y}^\prime = 0}
$$

analogous to the form $\Omega$ introduced in Appendix D of [23] for a similar purpose. (Note that the fictitious pole in $1 - \tau_3$ is compensated by the $y$-differentiation $\partial$ in $h(\partial, \bar{\partial})$.) Differentiation of $\Lambda(J)$ gives upon $\tau_3$-integration in the $H$-dependent term

$$
\mathcal{D}_{ad} \Lambda(J) = -\frac{i}{8} \bar{\eta} \eta \int d^2 \tau \delta'(1 - \tau_1 - \tau_2) \theta(\tau_1) \theta(\tau_2)
$$

$$
\left\{ H^{\alpha \beta} \partial_\alpha \partial_\beta \left( \exp(\epsilon f_\eta J)(y^1, y^2, \tau_2 \bar{y} + \bar{y}^1, -\tau_1 \bar{y} + \bar{y}^2; K|x) 
\right.
\right.

- J((1 - \tau_3) \tau_1 y + y^1, -(1 - \tau_3) \tau_2 y + y^2, \tau_2 \bar{y} + \bar{y}^1, -\tau_1 \bar{y} + \bar{y}^2; K|x) \right\}
|_{y^\prime = \bar{y}^\prime = 0},
$$

where

$$
\tilde{f}_- := (\tau_2 \bar{y} + \bar{y}^1)_{\bar{\alpha}}(-\tau_1 \bar{y} + \bar{y}^2)^{\alpha} + \partial_{\bar{\alpha}} \partial_\alpha.
$$

Taking into account Eqs. (2.13)-(2.23), (2.41) and using decomposition

$$
\Lambda(J) = \frac{i}{8} \bar{\eta} \eta \sum_{s \geq 1} \sum_{|n| \leq s} \Lambda(J)|_{s,n},
$$

$$
\Lambda(J)|_{s,n} = h(\partial, \bar{\partial}) \int_0^1 \frac{d\tau_3}{1 - \tau_3} \int d^2 \tau \delta'(1 - \tau_1 - \tau_2) \bar{\Upsilon} \int \frac{dw}{2\pi i w^{2s+1}} \int \frac{dv}{2\pi i v^{2n+1}} \exp(i \tau_3 f_\eta J)(((1 - \tau_3) \tau_1 y v w + v y^1, -(1 - \tau_3) \tau_2 y v w + v y^2, \tau_2 \bar{y} w v - \bar{y} w v^1, -\tau_1 \bar{y} w v - \bar{y}^2 v; K|x)|_{y^\prime = \bar{y}^\prime = 0},
$$

straightforward computation analogous to that of Appendix D of [23] yields

$$
\mathcal{D}_{ad} \Lambda(J)|_{s,n} = (s - n - 1) \Gamma_{\eta \eta}^{\geq loc}(J)|_{s,n} - \Gamma_{\eta \eta}^{\geq loc}([-i f_+ J])|_{s,n+1} + \frac{i}{8} \bar{\eta} \eta \bar{\Upsilon} H^{\alpha \beta} \partial_\alpha \partial_\beta B(J)|_{s,n}
$$

with $\Gamma_{\eta \eta}^{\geq loc}(J)|_{s,n}$ (4.35) and

$$
B(J)|_{s,n} = -\int d^2 \tau \delta'(1 - \tau_1 - \tau_2) \bar{\Upsilon} \int \frac{dw}{2\pi i w^{2s+1}} \int \frac{dv}{2\pi i v^{2n+1}} \exp(\epsilon f_\eta J)(y^1, y^2, \tau_2 \bar{y} w v - \bar{y} w v^1, -\tau_1 \bar{y} w v - \bar{y}^2 v; K|x)|_{y^\prime = \bar{y}^\prime = 0}.
$$

By virtue of (4.39), for $s - n > 1$, the field redefinition

$$
\omega \rightarrow \omega + \frac{i}{8} \eta \bar{\eta} \frac{1}{s - n - 1} \Lambda(J)|_{s,n}, \quad C \rightarrow C + \frac{1}{2} \frac{\eta \bar{\eta}}{s - n - 1} \bar{\Upsilon} B(J)|_{s,n},
$$

(4.41)
replaces \( \Gamma^{\geq \text{loc}}_{\bar{n}n}(|J|) \) in Eq. (4.22) by

\[
\frac{1}{(s - n - 1)} \Gamma^{\geq \text{loc}}_{\bar{n}n}(-i f_+ J) \big|_{s,n+1}.
\] (4.42)

Since \( f_+ J \) again satisfies current equation, this procedure can be repeated. For \(-s \leq n \leq -1\), to bring the current to the canonical form one needs \( k = [\{n\}] \equiv |n| - \{n\} \).

For any \( s \), the field redefinition

\[
\omega \rightarrow \omega + \Lambda^{-}(s, n), \quad C \rightarrow C + B^{-}(s, n)
\] (4.43)

with

\[
\Lambda^{-}(s, n) = \frac{i}{8} \bar{\eta} \int_{\bar{\eta}} \frac{dv v}{2\pi i(1 - v)} \int \frac{d^4 \tau d^2 \rho}{\rho^2 T_4} \delta(1 - \rho_1 - \rho_2) \delta'(1 - \tau_1 - \tau_2) \delta(1 - \rho_3 - \rho_4) \Upsilon (4.47)
\]

\[
\begin{array}{c}
h(\partial, \bar{\partial}) \exp(i\rho_1^{-1} \tau_4 f_-) \exp(i\rho_2 v^2((\tau_4 \tau_1 y_+^\alpha + y_+^\alpha)(-\tau_4 \tau_2 y_+^\alpha + y_+^\alpha) + \bar{\partial}_1 \bar{\partial}_2 \bar{\alpha})) \\
J((\tau_4 \tau_1 y + y_+^1)v, (-\tau_4 \tau_2 y + y_+^2)v, (\rho_1 \tau_2 y + y_+^1)v^1, (-\rho_1 \tau_1 y + y_+^2)v^1; K)|_{\rho^j = y_+^j = 0},
\end{array}
\]

\[
B^{-} = \frac{1}{2} \bar{\eta} \int_{\bar{\eta}} \frac{dv v}{2\pi i(1 - v)} \int \frac{d^2 \rho}{\rho^2 T_4} \delta(1 - \rho_1 - \rho_2) \Upsilon' (1 - \tau_1 - \tau_2)
\] (4.48)

exp\((i\rho_1^{-1} f_-) \exp(-i\rho_2 f_+ v^2)J(y_+^1v, y_+^2v, (\rho_1 \tau_2 y + y_+^1)v^1, (-\rho_1 \tau_1 y + y_+^2)v^1; K)|_{y^j = y_+^j = 0},
\]

where the measure \( \frac{dv}{2\pi i(1 - v)} \equiv \frac{dv}{2\pi i(v^2 + v^3 + \ldots)} \) implies summation over such \( J \) that \( f_0 \)-eigenvalue of \( f_+ J \) is smaller than \(-1\). One can see, that negative degrees in \( \rho_1 \) in (4.47) and (4.48) do not survive upon integration over \( v \).

The resulting canonical current is

\[
\Gamma^{-} = \frac{1}{8} \bar{\eta} \int_{\bar{\eta}} \frac{dw}{2\pi iw} w^{-2|s|_2} \int \frac{d^2 \rho d^4 \tau}{\rho_1 T_4^2} \delta'(1 - \rho_1 - \rho_2) \delta(1 - \tau_3 - \tau_4) \delta'(1 - \tau_1 - \tau_2)
\] (4.49)

\[
\Upsilon \left[ \bar{H}^{\alpha\beta} \bar{\partial}_\alpha \bar{\partial}_\beta \exp \left(-i(\rho_1 \tau_3 + w^2 \rho_2) f_+ f_- \right) \right.
\]

\[
J((\tau_1 y + y_+^1)w^1, (-\tau_2 y + y_+^2)w^1, (\rho_1 \tau_2 y + y_+^1)w, (-\rho_1 \tau_1 y + y_+^2)w; K)
\]

\[
+ \bar{H}^{\alpha\beta} \partial_\alpha \partial_\beta (\rho_1)_{2|s|_2} \exp (i\rho_1 \tau_3 f_-) \exp (-i\rho_2 f_+ w^2) \]

\[
J((\rho_1 \tau_4 y + y_+^1)w^1, (-\rho_1 \tau_4 y + y_+^2)w^1, (\tau_4 y + y_+^1)w, (-\tau_4 y + y_+^2)w; K) \right|_{y^j = y_+^j = 0},
\]
where
\[
\tilde{f}_+ := (\rho_1 \tau_1 y_\alpha + y_\alpha)(\rho_1 \tau_2 y^{2\alpha} - y^{2\alpha}) - \bar{\partial}_1 \bar{\partial}_2 \hat{\alpha}.
\]

The measure \(\frac{dw}{2\pi i w} w^{-|2s|_2}\) is different for bosonic currents with integer \(s\) \((|2s|_2 = 0)\) and fermionic ones with half-integer \(s\) \((|2s|_2 = 1)\).

The complex conjugated case with \(-i\tilde{f}_+ = i\tilde{f}_-\) is analogous. The respective generating functions can be obtained from \((4.47)-4.49) via the replacement
\[
if_- \leftrightarrow -if_+ , \quad y \leftrightarrow \bar{y} , \quad H \leftrightarrow \bar{H},
\]
that, abusing terminology, will be referred to as c.c. though the sign of the overall factor of \(i\) is not changed.

Summarizing, by virtue of Eq. \((4.39)\), local field redefinition \((4.4)\) with
\[
\Lambda^{\text{sum}}(J) = \Lambda^{-} + \text{c.c.} , \quad (4.51)
\]
\[
B^{\text{sum}}(J) = \bar{B}^{-} + \text{c.c.} , \quad (4.52)
\]
with \(\Lambda^{-} \quad (4.47)\), \(\bar{B}^{-} \quad (4.48)\) leads to deformed equations \((3.3)\), \((3.6)\). More precisely, taking into account that both \(\Gamma^{-}\) and its complex conjugate contain the same \(\rho_2\)-independent term, to avoid the double counting, we add to \(\Gamma^{-}(J) + \text{c.c.}\) the term proportional to \(\delta(\rho_2)\) obtaining \(\Gamma^{\text{sum}}(J)\) in the form \((3.7)\) upon the substitution of \(f_{\pm} \quad (2.40)\) and reformulation of the final result in the form of a Gaussian integral.

Note that there is a freedom in the field redefinition in the zero-form sector, that does not affect corrections to dynamical field equations in the one-form sector. Indeed, addition to \(B^{\text{sum}}(J) \quad (4.52)\) of any field containing a factor of \(y\bar{y}\) does not affect Eq. \((2.16)\). For instance, for \(s = 1, n = \pm 1\) it is convenient to use a field redefinition of this type to obtain the conventional form of the Maxwell equations as discussed in Section 5.3.

5 Current contribution to dynamical equations

In this section we derive explicit form of the current contribution to the r.h.s. of massless equations of different spins that follows from the nonlinear HS equations. Note that the results of this section extend the variety of examples of current interactions explicitly presented in \((2.35)\) to all sets of spins respecting inequality \((2.38)\).

For the future convenience we will use the following decompositions
\[
A(y_{1,2}, \bar{y}_{1,2}|x) = \sum_{m,\bar{m}} A_{m,\bar{m}}(y_{1,2}, \bar{y}_{1,2}|x) , \quad B(y, \bar{y}|x) = \sum_{m,\bar{m}} B_{m,\bar{m}}(y, \bar{y}|x) , \quad (5.1)
\]
with
\[
(y^{1\beta} \partial_{1\beta} + y^{2\beta} \partial_{2\beta}) A_{m,\bar{m}}(y_{1,2}, \bar{y}_{1,2}|x) = m A_{m,\bar{m}}(y_{1,2}, \bar{y}_{1,2}|x) ,
\]
\[
(y^{1\beta} \partial_{1\beta} + y^{2\beta} \partial_{2\beta}) A_{m,\bar{m}}(y_{1,2}, \bar{y}_{1,2}|x) = \bar{m} A_{m,\bar{m}}(y_{1,2}, \bar{y}_{1,2}|x) ,
\]
\[
y^{1\beta} \partial_{\beta} B_{m,\bar{m}}(y, \bar{y}|x) = m B_{m,\bar{m}}(y, \bar{y}|x) , \quad y^{2\beta} \partial_{\bar{\beta}} B_{m,\bar{m}}(y, \bar{y}|x) = \bar{m} B_{m,\bar{m}}(y, \bar{y}|x) .
\]
Recall, that we consider currents \((2.37)\) that are bilinear in fields represented as series \((2.20)\) in \(y\) and \(\bar{y}\).
5.1 Spin 0

As mentioned in Section 2.3, gauge invariant currents $J$ of spin zero are built from the zero-forms $C$ carrying $s_1 = s_2 = \frac{1}{2}$ (the $s = 0$ and $s = \frac{1}{2}$ conformal currents are not conserved since the fields of spins $s = 0$ and $s = \frac{1}{2}$ are not gauge).

By virtue of (2.22), (2.54) and (2.55), Eq. (3.6) yields for $J = J_{\pm 1}$

$$D^{L}_{a\dot{a}}C(K|x) + iC_{a\dot{a}}(K|x) = 0,$$

(5.2)

$$D^{L}_{a\dot{a}}C_{\beta\dot{\beta}}(K|x) + iC_{a\beta\dot{\beta}}(K|x) - i\lambda\varepsilon_{\dot{\alpha}\beta}\varepsilon_{\alpha\beta}C(K|x)$$

$$-\frac{i}{4}\eta \exp i[\bar{\partial}_{1\beta}\bar{\partial}_{2\beta}]]\varepsilon_{\dot{\alpha}\beta}(\bar{\partial}_{1\beta} + \bar{\partial}_{2\beta})(\bar{\partial}_{2\alpha} - \bar{\partial}_{1\alpha})J_{1}(y^{1}, \bar{y}^{1}; \bar{y}^{2}; K|x) \ast k|_{y^{1}} = 0 = 0$$

+ $i\eta \exp i[\bar{\partial}_{1\beta}\bar{\partial}_{2\beta}]\varepsilon_{\dot{\alpha}\beta}(\bar{\partial}_{1\beta} + \bar{\partial}_{2\beta})(\bar{\partial}_{2\alpha} - \bar{\partial}_{1\alpha})J_{-1}(y^{1}, \bar{y}^{1}; \bar{y}^{2}; K|x) \ast k|_{y^{1}} = 0.$

Contracting indices one obtains by virtue of (2.37) that the respective contribution of the currents $J_{\pm 1}$ bilinear in the fields $C$ with $s_1 = s_2 = \frac{1}{2}$ is

$$D^{L}_{a\dot{a}}D^{L}_{a\dot{a}}\sum_{j=0,1} C^{j,1-j}(x)k^{j}k^{1-j} + \eta \sum_{j, k=0,1} (-1)^{j}C^{j,1-j}(x)C^{k,1-l}(x)k^{j+l}k^{1-l-j}$$

$$+ \eta \sum_{j, k=0,1} (-1)^{j}C^{j,1-j}(x)C^{k,1-l}(x)k^{j+l}k^{1-l-j} = 0$$

just reproducing Yukawa interaction since $C(\alpha)$ and $C(\dot{\alpha})$ are dynamical spin-1/2 fields. Note that a $C^{2}$-deformation, that one might naively expect in the spin-zero sector, does not appear in agreement with the fact shown in [39] that the interactions with gauge invariant currents are conformal in $d = 4$, while the $C^{2}$-deformation is not.

5.2 Spin 1/2

By virtue of (2.22), taking into account (2.54), (1.52) and (1.47), Eq. (3.6) yields

$$D_{a\dot{a}}C_{\gamma}(K|x) + iC_{\gamma a\dot{a}}(K|x) + \frac{\eta}{4}\varepsilon_{\gamma a}(\bar{\partial}_{1\dot{a}} - \bar{\partial}_{2\dot{a}}) \exp i[\bar{\partial}_{1\beta}\bar{\partial}_{2\beta}]\varepsilon_{\dot{\alpha}\beta}J(K|x) \ast k|_{y^{1}} = 0 = 0.$$

(5.4)

Hence

$$D_{a\dot{a}}C^{\alpha}(K|x) -\frac{\eta}{2}(\bar{\partial}_{1\dot{a}} - \bar{\partial}_{2\dot{a}}) \exp i[\bar{\partial}_{1\beta}\bar{\partial}_{2\beta}]\varepsilon_{\dot{\alpha}\beta}J(K|x) \ast k|_{y^{1}} = 0.$$

(5.5)

Substitution of bilinear currents $J_{\pm \frac{1}{2}}$ (2.37) built from the fields of spins 0 and 1/2 gives the Yukawa interaction in the spin-1/2 sector

$$D_{a\dot{a}}\sum_{j=0,1} C^{j,1-j}(x)k^{j}k^{1-j} - \frac{1}{2}\eta \sum_{j, k=0,1} C^{j,1-j}(x)C^{k,1-l}(x)k^{j+l}k^{1-l-j}$$

$$+ \frac{1}{2}\eta \sum_{j, k=0,1} (-1)^{j}C^{j,1-j}(x)C^{k,1-l}(x)k^{j+l}k^{1-l-j} = 0.$$
Analogously,

\[
D^L_{\alpha\delta} \sum_{j=0,1} C^{j,1-j\dot{\alpha}}(x) k^{j} \bar{k}^{1-j} - \frac{1}{2} \bar{\eta} \sum_{j,l=0,1} C^{j,1-j\alpha}(x) C^{l,1-l}(x) k^{1+l+j} \bar{k}^{l-j} + \frac{1}{2} \bar{\eta} \sum_{j,l=0,1} (-1)^j C^{j,1-j}(x) C^{l,1-l}(x) k^{l+j} \bar{k}^{1-l-j} = 0. \tag{5.7}
\]

5.3 Maxwell equations

Eq. (3.5) still reads as (2.16)

\[
D_{ad\omega}(0,0; K|x) = \frac{i}{4} \left( \bar{\eta} H^{\dot{\alpha}\dot{\beta}} \bar{\partial}_\alpha \bar{\partial}_\beta C(0, \bar{y}; K|x) * k + \bar{\eta} H^{\alpha\beta} \partial_\alpha \partial_\beta C(y, 0; K|x) * \bar{k} \right)|_{y=\bar{y}=0}. \tag{5.8}
\]

This identifies \( \bar{\eta} C_{\alpha\beta}(x) \) and \( \bar{\eta} \mathcal{C}_{\alpha\dot{\beta}}(x) \), respectively, with the self-dual and anti-self-dual parts of the Maxwell field strength.

Using a freedom in local field redefinitions in the zero-form sector mentioned in Section 4.4.3 it is convenient to change \( B^{\text{sum}} \) \( \text{(5.9)} \) to \( \tilde{B}^{\text{sum}} \)

\[
B^{\text{sum}} \rightarrow \tilde{B}^{\text{sum}} = B'(J_1) + \bar{B}'(J_{-1}), \tag{5.9}
\]

where

\[
B'(J_1) = \frac{1}{2} \eta \int d^2 \tau \delta'(1 - \tau_1 - \tau_2) J_1(\tau_1 y, -\tau_2 y; \bar{y}, \bar{y}; K) * \bar{k},
\]

\[
\bar{B}'(J_{-1}) = \frac{1}{2} \bar{\eta} \int d^2 \tau \delta'(1 - \tau_1 - \tau_2) J_{-1}(y, -\tau_1 \bar{y}; \tau_2 \bar{y}; K) * k.
\]

Recall, that \( n \) in \( J_n \) \( \text{(2.44)} \) is the sum of helicities of the constituent fields. Also note that the additional \( J_{\pm 1} \)-dependent local field redefinition \( \text{(5.3)} \) was not discussed in [22] where the contribution of currents \( J_{\pm 1} \) was not considered.

Evidently, by virtue of \( \text{(4.52)} \) and \( \text{(5.9)} \)

\[
H^{\alpha\beta} \partial_\alpha \partial_\beta B'(J_1)(y, 0; K|x) \equiv 4 H^{\alpha\beta} \partial_\alpha \partial_\beta B^{\text{sum}}(J_1)(y, 0; K|x), \tag{5.10}
\]

\[
\bar{H}^{\dot{\alpha}\dot{\beta}} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} \bar{B}'(J_{-1})(0, \bar{y}; K|x) \equiv 4 \bar{H}^{\dot{\alpha}\dot{\beta}} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} \bar{B}^{\text{sum}}(J_{-1})(0, \bar{y}; K|x). \tag{5.11}
\]

Hence, in the \( s = 1 \) sector Eq. \( \text{(3.6)} \) is equivalent to

\[
\mathcal{D}_{tw} C = -\mathcal{H}_{\eta}^{\text{loc, cur}}(J_{-1}) - \mathcal{H}_{\eta}^{\text{loc, cur}}(J_1) + \mathcal{D}_{tw} B'(J_1) + \mathcal{D}_{tw} \bar{B}'(J_{-1}). \tag{5.12}
\]

The following useful formula results from \( \text{(5.9)} \) by the straightforward computation

\[
D_{tw} B'(J_1) = \frac{i}{2} \eta h^{\alpha\dot{\alpha}} \int d^2 \tau \delta(1 - \tau_1 - \tau_2) \left( -\tau_1 y_\alpha \bar{\partial}_{\dot{2}\dot{\alpha}} + \tau_2 y_\alpha \bar{\partial}_{\dot{1}\dot{\alpha}} \right) \left\{ \partial_1 \gamma \partial_2 \gamma J \right\}(\tau_1 y, -\tau_2 y; \tau_2 \bar{y}, -\tau_1 \bar{y}; K) * k. \tag{5.13}
\]
Deformed equation (5.12) in the $s = 1$ sector yields by virtue of Eqs. (2.54), (2.55), (5.9) and (5.13)

\[ D^L_{\alpha\alpha}C_{\beta\gamma}(K|x) + iC_{\beta\gamma\alpha\dot{\alpha}}(K|x) - \frac{1}{2} \eta [\varepsilon_{\alpha\beta} \partial_\gamma + \varepsilon_{\gamma\alpha} \partial_\beta] \int_0^1 d\tau (\bar{\partial}_2 - (1 - \tau) \bar{\partial}_1 \dot{\alpha}) \]

\[ \{ J_0 + i\bar{\partial}_{\beta} \bar{\partial}_2 \dot{J}_{-1} - i\partial_\beta \partial_\gamma J_1 \}(\tau y, -(1 - \tau) y, \bar{y}, \bar{y}; K|x) * k \mid_{y = \bar{y} = 0} = 0. \]

Contracting indices we obtain from (5.14)

\[ D^L_{\alpha\beta}C_{\beta\gamma}(0, 0; K|x) + \frac{3}{2} \eta \int_0^1 d\tau (\bar{\partial}_2 - (1 - \tau) \bar{\partial}_1 \partial_\gamma) \mid_{y = \bar{y} = 0} = 0. \]

Analogously,

\[ D^L_{\gamma\delta}C_{\beta\gamma}(0, 0; K|x) - \frac{3}{2} \bar{\eta} \int_0^1 d\tau (\bar{\partial}_2 - (1 - \tau) \bar{\partial}_1 \partial_\gamma) \mid_{y = \bar{y} = 0} = 0. \]

Hence, performing integration over $\tau$, from (5.15) and (5.16) it follows for $D^L = h^{\gamma\delta}D^L_{\gamma\delta}$ that

\[ \eta D^L_{\gamma\delta}C_{\beta\gamma}(0, 0; K|x) * \bar{k} + \bar{\eta} D^L_{\alpha\beta}C_{\beta\gamma}(0, 0; K|x) * k \]

\[ = \frac{1}{2} \eta \bar{\eta} (2\bar{\partial}_2 \partial_2 - \bar{\partial}_1 \partial_1 + 2\bar{\partial}_1 \partial_2) J_0(y^1, y^2, \bar{y}^1, \bar{y}^2; K|x) \mid_{y = \bar{y} = 0}. \]

Using identities

\[ H^{\alpha\beta}h^{\gamma\delta} = \varepsilon^{\alpha\beta}H^{\gamma\delta} + \varepsilon^{\gamma\delta}H^{\alpha\beta}, \]

\[ H^{\dot{\alpha}\dot{\beta}}h^{\gamma\delta} = -\varepsilon^{\dot{\alpha}\dot{\beta}}H^{\gamma\delta} - \varepsilon^{\gamma\delta}H^{\dot{\alpha}\dot{\beta}}, \]

where $H^{\alpha\gamma}$ are the frame three-forms, we obtain

\[ H^{\alpha\beta}h^{\gamma\delta}D^L_{\gamma\delta}C_{\alpha\beta} = 2H^{\beta\gamma}D^L_{\alpha\gamma}C_{\alpha\beta}, \]

\[ \bar{H}^{\dot{\alpha}\dot{\beta}}h^{\gamma\delta}D^L_{\gamma\delta}C_{\dot{\alpha}\dot{\beta}} = -2\bar{H}^{\dot{\alpha}\dot{\beta}}D^L_{\alpha\dot{\beta}}C_{\dot{\alpha}\dot{\beta}}. \]

Hence (5.17) yields

\[ D^L_{\gamma\delta} \left( \bar{\eta} H^{\alpha\beta}C_{\alpha\beta}(K|x) * k - \eta \bar{H}^{\dot{\alpha}\dot{\beta}}C_{\dot{\alpha}\dot{\beta}}(K|x) * \bar{k} \right) \]

\[ = \bar{\eta} \eta H^{\alpha\beta}(2\bar{\partial}_2 \partial_2 - \bar{\partial}_1 \partial_1 + 2\bar{\partial}_1 \partial_2) J_0(y^1, y^2, \bar{y}^1, \bar{y}^2; K|x) \mid_{y = \bar{y} = 0}. \]

just reproducing the Maxwell equations with a nonzero current.

Substitution of bilinear $J_0$ (2.37) that by virtue of inequality (2.38) is built from scalars or spinors gives

\[ \sum_{j=0,1} D^L \left( \bar{\eta} H^{\alpha\beta}C^{j,1-j}_{\alpha\beta}(x)k^{1+j}k^{1-j} - \eta \bar{H}^{\dot{\alpha}\dot{\beta}}C^{j,1-j}_{\dot{\alpha}\dot{\beta}}(x)k^{j}k^{2-j} \right) \]

\[ = \bar{\eta} \eta \sum_{j,\dot{\alpha}} H^{\gamma\delta}(2(-1)^jC^{j,1-j}_{\gamma\delta}(x)C^{d,1-l}_{\gamma\delta}(x) - C^{j,1-j}_{\gamma\delta}(x)C^{d,1-l}_{\gamma\delta}(x)) \]

\[ - C^{j,1-j}_{\dot{\alpha}\gamma}(x)C^{d,1-l}_{\dot{\alpha}\gamma}(x) + 2(-1)^jC^{j,1-j}_{\dot{\alpha}\gamma}(x)C^{d,1-l}_{\dot{\alpha}\gamma}(x)) \]

\[ k^{l+k}k^{2-l-k} \]
5.4 Spin 3/2

Using decomposition (5.21) and Eqs. (3.7), we obtain from Eq. (3.3) along with (2.21), (2.22)
\[ D^L \omega_{0,1}(0, \bar{y}; K|x) + h(\partial, \bar{y}) \omega_{1,0}(y, 0; K|x) \]
\[ = \eta^{-1} H^{\alpha \beta} \partial_\alpha \partial_\beta C(0, \bar{y}; K|x) \ast \bar{k} + H^{\alpha \beta} \partial_\alpha \partial_\beta \Lambda^3_{3/2} J_{2,1}(y^j, \bar{y}^i; K|x)|_{y^j=\bar{y}^i=0}, \]
\[ D^L \omega_{1,0}(y, 0; K|x) + h(y, \partial) \omega_{0,1}(0, \bar{y}; K|x) \]
\[ = \eta^{-1} H^{\alpha \beta} \partial_\alpha \partial_\beta C(y, 0; K|x) \ast k + H^{\alpha \beta} \partial_\alpha \partial_\beta \Lambda^3_{3/2} J_{1,2}(y^j, \bar{y}^i; K|x)|_{y^j=\bar{y}^i=0}, \]
where
\[ \Lambda^3_{3/2} = \frac{1}{2} \int d^2 \tau \theta(\tau_1) \theta(\tau_2) \delta'(1-\tau_1-\tau_2)(\tau_1 N_1 - \tau_2 N_2)^2, \]
\[ \Lambda^3_{3/2} = \frac{1}{2} \int d^2 \tau \theta(\tau_1) \theta(\tau_2) \delta'(1-\tau_1-\tau_2)(\tau_1 N_1 - \tau_2 N_2)^2, \]
\[ N_j = y^\alpha \partial_j \alpha, \quad \bar{N}_j = \bar{y}^\alpha \partial_j \bar{\alpha} \]
and, according to (2.38), (2.42) and (3.7),
\[ J_{2,1} = \frac{i}{8} \eta \left( J_{1/2} + \frac{i \partial_1 \gamma \partial_2 \gamma}{2} J_{3/2} \right), \quad J_{1,2} = \frac{i}{8} \left( J_{-1/2} + \frac{i \partial_1 \gamma \partial_2 \gamma}{2} J_{3/2} \right). \]
Representing one-forms \( \omega_{j,k} \) as
\[ \omega_{j,k} = h^{\alpha \beta} \omega_{\alpha \beta j,k} \]
Eqs. (5.21), (5.22), (5.23) yield spin-3/2 massless equations in AdS_4 in the form
\[ \left( \partial_\gamma D^L_{\alpha \beta} \omega_{0,1} \gamma \right) (0, \bar{y}; K|x) - \partial_\alpha \omega_{1,0} \gamma (y, 0; K|x) \right)|_{y=\bar{y}=0} \]
\[ = \frac{i}{8} \eta \left( \frac{1}{3} \partial_1 \partial_1 \partial_1 + \partial_2 \partial_2 \partial_1 - \partial_1 \partial_2 \partial_1 \partial_2 - \frac{2}{3} \partial_1 \partial_2 \partial_2 + \frac{2}{3} \partial_1 \partial_2 \partial_2 \partial_2 \right) \omega_{\alpha \gamma} \]
\[ \left( J_{1/2} (y^j, \bar{y}^i; K|x) + \frac{i \partial_1 \gamma \partial_2 \gamma}{2} J_{3/2} (y^j, \bar{y}^i; K|x) \right)|_{y^j=\bar{y}^i=0} \]
and complex conjugated.

Using inequality (2.38), substitution of bilinear \( J \) (2.37) gives the Rarita-Schwinger
equation with supercurrents on the right-hand side

\[
\left( \partial_\beta D^L_{\alpha \beta} \omega_{0,1} \partial^\beta (0, \bar{y}; K|x) - \partial_\alpha \omega_{1,0} \partial^\alpha (y, 0; K|x) \right) \big|_{y=\bar{y}=0} = \left( i \frac{\eta}{4} \sum_{j=0,1} \left\{ C^{j,1-j}_{\alpha \beta}(x) C^{l,1-l}_{\alpha \alpha}(x) + \frac{i}{2} (-1)^{j} \Lambda^{j,1-j}_{\alpha \beta}(x) C^{l,1-l}_{\alpha \alpha}(x) \right\} \right) k^{l+j-\bar{k}-j} \]

and complex conjugated.

### 5.5 Spin two

From Eq. (5.3), we obtain by virtue of (5.7)

\[
D^{ad} \omega(y, \bar{y}; K|x) = \left( H^{\alpha \beta} \partial_\alpha \partial_\beta \Lambda^2_{0} J_{2,2}(y^j, \bar{y}^j; K|x) + \bar{H}^{\alpha \beta} \partial_\alpha \partial_\beta \Lambda^2_{0} J_{2,2}(y^j, \bar{y}^j; K|x) \right) \big|_{y=\bar{y}=0} + \frac{i}{4} \left( \eta \bar{H}^{\alpha \beta} \partial_\alpha \partial_\beta \bar{C}(0, \bar{y}; K|x) + \bar{\eta} H^{\alpha \beta} \partial_\alpha \partial_\beta C(y, 0; K|x) \right),
\]

where

\[
J_{2,2} = \frac{i}{8} \sum_{0 \leq n \leq 2} \frac{1}{(1+n)!} \left[ (i \partial_1 \gamma_2)^n J_n + (i \bar{\partial}_1 \gamma_2)^n J_{-n} \right],
\]

\[
\Lambda^2_{0} = \frac{1}{4!} \sum_{k=0}^{2} \sum_{m=0}^{2} \frac{(m+k)!}{(2-k)!k!(2-m)!m!} (N_1)^m (-N_2)^{2-m} (-\bar{N}_2)^k (\bar{N}_1)^{2-k}.
\]

In terms of decomposition (5.1), this gives in particular

\[
D^L \omega^{0,2}(0, \bar{y}; K|x) = -\bar{H}^{\alpha \beta} \bar{y}_\alpha \bar{\partial}_\alpha \omega^{1,1}(y, \bar{y}; K|x) + H^{\alpha \beta} \partial_\alpha \partial_\beta \Lambda^2_{0} J_{2,2}(y^j, \bar{y}^j; K|x) \big|_{y=\bar{y}=0} + \frac{\eta}{4} \bar{H}^{\alpha \beta} \partial_\alpha \partial_\beta \bar{C}(0, \bar{y}; K|x) \ast \bar{k},
\]

\[
D^L \omega^{2,0}(y, 0; K|x) = -\bar{H}^{\alpha \beta} y_\alpha \partial_\alpha \omega^{1,1}(y, \bar{y}; K|x) + H^{\alpha \beta} \partial_\alpha \partial_\beta \Lambda^2_{0} J_{2,2}(y^j, \bar{y}^j; K|x) \big|_{y=\bar{y}=0} + \frac{\eta}{4} H^{\alpha \beta} \partial_\alpha \partial_\beta C(y, 0; K|x) \ast k.
\]
Hence from (5.30), (5.37) and (5.38) it follows
\[ \partial_{\hat{\beta}} \partial_{\hat{\beta}} \Lambda_{0}^{2} \mathcal{J}_{2,2} \bigg|_{y^i = \bar{y}^i = 0} = \bar{\eta}_{\hat{\beta}} \partial_{\hat{\beta}} \omega_{0,2,\hat{\beta}} , \]  \hspace{1cm} (5.34)
\[ \partial_{\hat{\beta}} \partial_{\hat{\beta}} \Lambda_{0}^{2} \mathcal{J}_{2,2} \bigg|_{y^i = \bar{y}^i = 0} = \eta_{\hat{\beta}} \partial_{\hat{\beta}} \omega_{0,2,\hat{\beta}} \]  \hspace{1cm} (5.35)
giving the linearized Einstein equations
\[ \partial_{\alpha} \partial_{\alpha} D^L_{\hat{\beta} \hat{\beta}} \omega_{0,2,\hat{\beta}} (0, \bar{y}; K|x) - 2 \partial_{\hat{\beta}} \partial_{\hat{\beta}} \omega_{0,1,\hat{\beta}} (y, \bar{y}; K|x) = \partial_{\hat{\beta}} \partial_{\hat{\beta}} \partial_{\hat{\beta}} \partial_{\hat{\beta}} \Lambda_{0}^{2} \mathcal{J}_{2,2} (y^i, \bar{y}^i; K|x) \bigg|_{y^i = \bar{y}^i = 0} \]  \hspace{1cm} (5.36)
accounting the contribution of the stress tensor.

From (5.31) it follows
\[ \partial_{\alpha} \partial_{\alpha} D^L_{\hat{\beta} \hat{\beta}} \omega_{0,2,\hat{\beta}} (0, \bar{y}; K|x) = \frac{1}{2} \partial_{\hat{\beta}} \partial_{\hat{\beta}} \omega_{0,1,\hat{\beta}} (y, \bar{y}; K|x) \]  \hspace{1cm} (5.37)
Substitution of \( J_{2,2} \) (5.37) into \( J_{2,2} \) yields
\[ J_{2,2} = \frac{i}{8} \eta_{\hat{\beta}} \sum_{\hat{\alpha} \hat{\beta} \hat{\gamma} \hat{\delta}} \left\{ C_{\hat{\gamma} \hat{\delta}}^{\hat{\beta} \hat{\beta}} C_{\hat{\beta} \hat{\beta}}^{\hat{\gamma} \hat{\delta}} + \frac{i(-1)^{i} C_{\hat{\beta} \hat{\beta}}^{\hat{\gamma} \hat{\delta}} C_{\hat{\beta} \hat{\beta}}^{\hat{\gamma} \hat{\delta}}}{3!} \right\} \]  \hspace{1cm} (5.38)
Hence from (5.30), (5.37) and (5.38) it follows
\[ \partial_{\alpha} \partial_{\alpha} D^L_{\hat{\beta} \hat{\beta}} \omega_{0,2,\hat{\beta}} (0, \bar{y}; K|x) - 2 \partial_{\hat{\beta}} \partial_{\hat{\beta}} \omega_{0,1,\hat{\beta}} (y, \bar{y}; K|x) = \]  \hspace{1cm} (5.39)
with the stress tensor of massless fields of spins 0, 1/2 and 1. Ellipses denotes other currents, that depend on massless fields of spins \( s \leq 2 \) and respect (5.38).
5.6 Higher spins

5.6.1 Integer spins

Using the decomposition (5.4) for $\omega$ and Eqs. (3.7), it follows from (3.5) that (2.21) yields

\begin{equation}
D^L\omega_{s-1,s-1}(y, \bar{y}; K|x) = h(\partial, \bar{y})\omega_{s,s-2}(y, \bar{y}; K|x) + h(y, \bar{\partial})\omega_{s-2,s}(y, \bar{y}; K|x),
\end{equation}

\begin{equation}
D^L\omega_{s-2,s-2}(y, \bar{y}; K|x) = -h(y, \bar{\partial})\omega_{s-1,s-1}(y, \bar{y}; K|x)
- h(\partial, \bar{y})\omega_{s+1,s-3}(y, \bar{y}; K|x) + \bar{H}^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}\Lambda^s_0 J_{s,s}(y^j, \bar{y}^\bar{j}; K|x)\Big|_{y^j = \bar{y}^\bar{j} = 0},
\end{equation}

\begin{equation}
D^L\omega_{s-2,s}(y, \bar{y}; K|x) = -h(y, \bar{\partial})\omega_{s-1,s-1}(y, \bar{y}; K|x)
- h(\partial, \bar{y})\omega_{s-3,s+1}(y, \bar{y}; K|x) + H^{\alpha\beta}\partial_\alpha\partial_{\beta}\Lambda^s_0 J_{s,s}(y^j, \bar{y}^\bar{j}; K|x)\Big|_{y^j = \bar{y}^\bar{j} = 0},
\end{equation}

\begin{equation}
J_{s,s} = i\frac{\eta\bar{\eta}}{8} \left\{ \sum_{0 \leq n \leq s} \left[ \frac{(i\partial_1 \tilde{\partial}_2^\gamma)^{|n|}}{(s + |n| - 1)!} J_n \right] + \sum_{0 < |n| \leq s} \left[ \frac{(i\tilde{\partial}_1 \partial_2^\gamma)^{|n|}}{(s + |n| - 1)!} J_n \right] \right\},
\end{equation}

\begin{equation}
\Lambda^s_0 = \int d^2\tau(\tau_1 \theta(\tau_2)\delta'(1 - \tau_1 - \tau_2)) (\tau_1 N_1 - \tau_2 N_2)^s (\tau_2 N_1 - \tau_1 N_2)^s (s - 1)! \frac{s}{s!}
\end{equation}

From here it follows that

\begin{equation}
h^{\gamma\bar{\eta}}h^{\alpha\bar{\alpha}}D^L_{\gamma\bar{\gamma}}\omega_{s-1,s-1}\omega_{-1,\alpha} = -h(\partial, \bar{y})h^{\alpha\bar{\alpha}}\omega_{s,s-2}\omega_{-2,\alpha}\omega_{-1,\alpha},
\end{equation}

\begin{equation}h^{\gamma\bar{\eta}}h^{\alpha\bar{\alpha}}D^L_{\gamma\bar{\gamma}}\omega_{s}s-2\omega_{-2,\alpha} = -h(y, \bar{\partial})h^{\alpha\bar{\alpha}}\omega_{s,s-1}\omega_{-1,\alpha} + \bar{H}^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}\Lambda^s_0 J_{s,s}\Big|_{y^j = \bar{y}^\bar{j} = 0},
\end{equation}

\begin{equation}h^{\gamma\bar{\eta}}h^{\alpha\bar{\alpha}}D^L_{\gamma\bar{\gamma}}\omega_{s-2,s}\omega_{s,0,\alpha} = -h(\partial, \bar{y})\omega_{s-1,s}\omega_{-1,\alpha} + \bar{H}^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}\Lambda^s_0 J_{s,s}\Big|_{y^j = \bar{y}^\bar{j} = 0}.
\end{equation}

Hence

\begin{equation}D^L_{\alpha\bar{\gamma}}\omega_{s-2,s}\omega_{s,0,\alpha} = \bar{y}_\alpha\partial_\alpha\omega_{s-1,s-1}\omega_{-1,\alpha} - y_\alpha\partial_\alpha\omega_{s-3,s+1}\omega_{-1,\alpha} + \bar{\partial}_\alpha\partial_\alpha\Lambda^s_0 J_{s,s}\Big|_{y^j = \bar{y}^\bar{j} = 0},
\end{equation}

\begin{equation}D^L_{\beta\bar{\gamma}}\omega_{s-2,s}\omega_{s,0,\beta} = -y_\beta\partial_\beta\omega_{s-1,s-1}\omega_{-1,\beta} - \bar{y}_\beta\partial_\beta\omega_{s-3,s+1}\omega_{-1,\beta} + \partial_\alpha\partial_\beta\Lambda^s_0 J_{s,s}\Big|_{y^j = \bar{y}^\bar{j} = 0}.
\end{equation}

Integrating over $\tau$ in (5.44) and substituting $J_{s,s}$ (2.34) into $J_{s,s}$ (5.43) one obtains

\begin{equation}\Lambda^s_0 J_{s,s} = i\eta\bar{\eta}\frac{(s - 2)!}{8(s2)!} \sum_{k,m \in [0,s]} (m + k)!(2s - m - k)! (N_1)^m (-N_2)^{s-m} (-\bar{N}_1)^k (\bar{N}_2)^{s-k} (s - k)! k! (s - m)! m! \left\{ \sum_{0 \leq n \leq s} \frac{1}{(s + n - 1)!} (i\partial_1 \tilde{\partial}_2^\gamma)^{|n|} \left[ \sum_{j,l=0}^1 C^{j,1-j}(Y^1|x)k^j\bar{k}^{1-j}C^{l,1-l}(Y^2|x)k^l\bar{k}^{1-l} \right]\right\} Y_{j,\bar{l}} = 0.
\end{equation}
In terms of decomposition (5.1), for any half-integer

5.6.2 Half-integer spins

In terms of decomposition (5.1), for any half-integer \( s > 1 \), it follows from Eqs. (5.7), (3.4) that

\[
D^L \omega_{[s-1,s]}(y, \bar{y}; K|x) = -h(y, \bar{y}) \omega_{[s-2,s+1]}(y, \bar{y}; K|x) - h(\bar{y}, \bar{y}) \omega_{s-1,s-1}(y, \bar{y}; K|x) + H^{\alpha \beta} \partial_{\alpha} \partial_{\beta} \Lambda^s \int_{s+\frac{1}{2}}^{s-\frac{1}{2}} (y^j, \bar{y}^j; K|x)
\]

and complex conjugated, where

\[
J = \int d^2 \tau \theta(\tau_1) \rho(\tau_2) (1 - \tau_1 - \tau_2) \frac{\left(\tau_1 N_1 - \tau_2 N_2\right)^{s+\frac{1}{2}} \left(\tau_2 N_1 - \tau_1 N_2\right)^{s-\frac{1}{2}}}{(s - \frac{1}{2})(s + \frac{1}{2})}.
\]

Hence

\[
D^L_{\alpha \gamma} \omega_{[s-1,s]} \alpha \gamma = -y_\alpha \delta_{\beta} \omega_{[s-2,s+1]} + 1 \alpha \beta - \bar{y}_\beta \partial_{\alpha} \omega_{[s-1,s-1]} + \bar{\alpha} \beta + \partial_{\alpha} \partial_{\beta} \Lambda^s \bar{J}_{s+\frac{1}{2}, s-\frac{1}{2}}(y^j, \bar{y}^j; K|x)
\]

Integrating over \( \tau \) in (5.53) and substituting \( J \) (2.37) into \( J_{s,s} \) yields by virtue of (5.52), (5.53)

\[
-L^s \int_{s+\frac{1}{2}}^{s-\frac{1}{2}} (y^j, \bar{y}^j; K|x) \left| y^j = \bar{y}^j = 0 \right.
\]

and complex conjugated

\[
\sum_{k, m=0}^{s} \frac{(m + k)! (2s + m - k)!}{(s + \frac{1}{2} - k)!(s - \frac{1}{2} - m)!m!} (N_1)^k (-N_2)^{s+\frac{1}{2}-k} (-N_2)^{m (N_1)^{s-\frac{1}{2}-m}} (i \partial_1 \rho_2) \left| \right. Y^j = 0.
\]
Substitution of $\Lambda^s \frac{1}{\gamma} \mathcal{J}_{s+s-s} (5.53)$ into (5.54) gives the half-integer spin equations in $AdS_4$ with the conformal currents. Complex conjugated equations are analogous.

Projecting away the terms, with the extra fields $\omega_{[s]+1,[s]+1}$ and $\omega_{[s]-2,[s]-2}$ by contracting free indices with $y^\alpha y^\alpha$ and $\bar{y}^\bar{\beta} \bar{y}^\bar{\beta}$, respectively, gives the Fang-Fronsdal field equations [10] in $AdS_4$ with the conformal currents on the right-hand-sides.

6 Conclusion

We have derived current sources to the right-hand side of field equations on massless fields of all spins resulting from the nonlinear field HS equations of [2]. Our results extend those obtained by one of us [22] for current interactions in the zero-form sector. The derivation agrees with that of [22] in many respects.

First of all, in agreement with the conclusion of [22], the bilinear (current) corrections turn out to be independent of the phase of the parameter $\eta$ in the HS theory, depending only on $\eta \bar{\eta}$. Naively this contradicts the standard expectation that the HS theory with different phases of $\eta$ correspond to different boundary conformal theories with Chern-Simons fields. However, as explained in [22, 36], the proper dependence on the phase of $\eta$ in the HS $AdS/CFT$ correspondence results from the phase-dependence of the linear terms in the HS equations upon transition to the genuine Weyl tensors. For general $\eta$ our vertex contains both parity even and parity odd parts, which appear in HS models with general $\eta$.

We not only reduced the HS current interactions to the local form with finite number of derivatives for any three spins, but also found its canonical form with the minimal number of derivatives and zero HS torsion. The resulting coupling constants are nonzero being uniquely determined in terms of the single HS coupling constant $\eta \bar{\eta}$.

The detailed computation of the resulting boundary correlators is presented in [36] based on the zero-form results of [22] (for a special case see also [18]). The results of this paper allow one to extend this analysis to the one-form (i.e., gauge field) sector checking in particular whether the nonlinear deformation of this paper matches the cubic vertex derived in [19] from the holographic analysis for the $A$-model. This problem was considered in [12] where it shown that the coefficients in the vertices resulting from our analysis match those of [13, 49]. Let us stress that the current interactions derived in this paper extend the vertex of [13, 14, 49] to the parity non-invariant vertices holographically dual to the HS theory with an arbitrary phase parameter $\eta$ (see [36, 22]).

The conclusion that the contribution to the currents proportional to $\eta^2$ should vanish fits the conjecture of [2] that the HS theory with $(\eta = 0) \bar{\eta} = 0$ is the (anti-)self-dual HS gauge theory. In that case it should describe the zero-form curvatures with only positive or only negative helicities which cannot contribute to nontrivial currents for the same reason why the amplitudes with helicities of the same sign cannot be nonzero. Hence, our results confirm the conjecture that the HS theory at $(\eta = 0) \bar{\eta} = 0$ is (anti)self-dual.
The obtained results provide a basis for understanding the proper general setup for the systematic derivation of minimally nonlocal perturbative corrections to non-linear HS equations. This issue is considered in [24]. The form of the results obtained in [22] and in this paper demonstrates in particular that this prescription should allow a proper formulation in the geometric terms of polyhedra associated with the integration parameters $\tau_i$.

Acknowledgements

We are grateful to Slava Didenko for critical comments on the manuscript, Nikita Misuna and Ruslan Metsaev for useful discussions, and Karapet Mkrtchyan for the correspondence. We would like to thank for hospitality the MIAPP programme “Higher Spin Theory and Duality” in May 2016 and Sergey Kuzenko, the School of Physics at the University of Western Australia, in November 2016 during the various stages of the project, as well as the Galileo Galilei Institute for Theoretical Physics (GGI) for the hospitality and INFN for partial support during the completion of this work, within the program New Developments in AdS3/CFT2 Holography. The work of MV is partially supported by the ARC Discovery Project DP160103633 and by a grant from the Simons Foundation. We acknowledge a partial support from the Russian Basic Research Foundation Grant No 17-02-00546.

Appendix A. Useful formulas

In the analysis of HS perturbations it is convenient to use the following generalized beta-function formula:

$$\int d\tau^m \delta^{(k)} \left( 1 - \sum_{i=1}^m \tau_i \right) \prod_{i=1}^m \theta(\tau_i) \tau_i^{\tau_i} = \prod_{i=1}^m n_i! \left( \sum_{i=1}^m n_i + m - 1 - k \right)! , \quad \forall n_i, k \geq 0 . \quad (A.1)$$

Now we reproduce some of formulas of [35] most relevant to the analysis of this paper. Let $\phi^{AB} Y_A Y_B = 4i\phi_{AdS}$, where $\phi_{AdS}$ is defined in (2.25). Then

$$\Delta_{ad} A (Z; Y; \theta) = \frac{i}{2} \int d\mu d\varphi d\chi dU dV \int_0^1 \frac{d\tau}{\tau} \exp \left\{ \chi_A \varphi^A + iU_A V^A \right\} \quad (A.2)$$

$$\exp \left\{ \mu \tau \chi_A Z^A + \frac{i}{2} (1 - \tau) \phi^{BC} \chi_B U_C \right\} A (\tau Z; Y + V; \tau \theta + \varphi) , \quad (A.3)$$

$$\mathcal{H}_{ad} J (Z; Y; \theta) = \int d\varphi d\chi dU dV \exp \left\{ \chi_A \varphi^A + iU_B V^B + \frac{i}{2} \phi^{BC} \chi_B U_C \right\} A (0; Y + V; \varphi) , \quad (A.3)$$
\[ \Delta^*_\omega A(Z; Y; \theta) *k = \frac{i}{2} \int d\mu d\sigma d\rho dU dV dP dQ \int_0^1 \frac{d\tau}{\tau} \] (A.4)

\[ \exp \{ \mu \tau \sigma A Z^A + \sigma A \rho^A + i U_A V^A + i P_A Q^A \} \]

\[ \exp \left\{ - \frac{i}{2} (1 - \tau) \omega^{AB} \sigma_A V_B + \frac{i}{2} (1 - \tau) h^{AB} \sigma_A \left( Y_B + \frac{1}{2} U_B + \frac{1}{2} (1 + \tau) Q_B \right) \right\} \]

\[ A(\tau Z + P; Y + U; \tau \theta + \rho) *k, \]

\[ \mathcal{K}_{tw} A(Z; Y; \theta) *k = \int d\sigma d\rho dU dV dP dQ \exp \{ \rho_A \sigma^A + i U_A V^A + i P_A Q^A \} \] (A.5)

\[ \exp \left\{ - \frac{i}{2} \omega^{AB} \sigma_A V_B + \frac{i}{2} h^{AB} \sigma_A \left( Y_B + \frac{1}{2} U_B + \frac{1}{2} Q_B \right) \right\} A(P; Y + U; \rho) *k. \]

**Appendix B. Alternative redefinitions**

Let

\[ X(J) = \int d^3\tau d^3\bar{\tau} h^{a\bar{a}} X_{a\bar{a}}(\tau, \bar{\tau}) \exp \left( \tau_3 \bar{\partial}_{1a} \partial_{2a} + \tau_3 \bar{\partial}_{1\bar{a}} \partial_{2\bar{a}} \right) J(\tau y, -\tau_2 y, \bar{\tau} y, -\bar{\tau} \bar{y}; K|x), \] (B.1)

where

\[ X_{a\bar{a}} = a y_{a\bar{a}} + y_a \sum_i \bar{b}_i \bar{\partial}_{i\bar{a}} + \sum_i b_i \partial_{i a} y_{\bar{a}} + \sum_{i,j} g_{ij} \partial_{i a} \bar{\partial}_{j \bar{a}}. \] (B.2)

We will look for a solution to

\[ \mathcal{D}_{ad} X = \Gamma_{\eta\bar{\eta}} + G^{\text{loc}}, \] (B.3)

where \( \Gamma_{\eta\bar{\eta}} \) is given by (1.4), while \( G^{\text{loc}} \) is some local vertex to be found.

For simlicity we set \( \eta\bar{\eta} = -4 \) in (4.4). Denote

\[ A_j := i \Delta_{j a} + \Delta_{3 \bar{b}_j}, \quad B = \Delta_1 b_1 - \Delta_2 b_2, \quad \bar{b}_2 = b_1, \quad \bar{b}_1 = b_2, \] (B.4)

\[ G_{kj} = \Delta_3 \bar{g}_{kj} + i \Delta_k \bar{b}_j, \quad F_j = \Delta_1 g_{1j} - \Delta_2 g_{2j}, \quad \bar{g}_{2j} = g_{1j}, \quad \bar{g}_{1j} = g_{2j}, \]

\[ \frac{\partial}{\partial \tau_j} := \Delta_j, \quad \frac{\partial}{\partial \bar{\tau}_j} := \bar{\partial}_j. \]

By virtue of the Fierz (Schouten) relations expressing the two-componentness of spinorial indices in the form

\[ (i \bar{y}_{\bar{a}} \frac{\partial}{\partial \tau_3} + \bar{\partial}_{1\bar{a}} \frac{\partial}{\partial \tau_2} + \bar{\partial}_{2\bar{a}} \frac{\partial}{\partial \tau_1}) \exp \left( \tau_3 \partial_{1a} \partial_{2a} + \tau_3 \bar{\partial}_{1\bar{a}} \bar{\partial}_{2\bar{a}} \right) J(\tau y, -\tau_2 y, \bar{\tau} y, -\bar{\tau} \bar{y}; K|x) = 0. \] (B.5)

Then

\[ \mathcal{F} \exp \left( \tau_3 \partial_{1a} \partial_{2a} + \tau_3 \bar{\partial}_{1\bar{a}} \bar{\partial}_{2\bar{a}} \right) J(\tau y, -\tau_2 y, \bar{\tau} y, -\bar{\tau} \bar{y}; K|x) = 0 \]

34
In this setup for any $F$ of the form

$$\mathcal{F} = \frac{1}{2} h_\mu h^{\mu\beta} \left( \alpha \bar{y}_\beta + \beta_1 \bar{\partial}_1 \bar{y}_\beta + \beta_2 \bar{\partial}_2 \bar{y}_\beta \right) \left( i \bar{y}_\alpha \frac{\partial}{\partial \bar{\tau}_3} + \bar{\partial}_1 \alpha \frac{\partial}{\partial \bar{\tau}_2} + \bar{\partial}_2 \alpha \frac{\partial}{\partial \bar{\tau}_1} \right). \quad (B.6)$$

In this setup

$$\left( D_{ad}X + \mathcal{F} \right) |_{\mathcal{H}} = -\frac{1}{2} R^{\dot{a}\dot{b}} \int \int d^3 \bar{\tau} d^3 \tau \quad (B.7)$$

$$\left\{ B \bar{y}_\dot{a} + F_j \bar{\partial}_j \dot{a} \right\} \left[ (i \tau \bar{\tau}_1 - i \tau_2 \bar{\tau}_3) \bar{y}_\beta + (\bar{\tau}_1 + \tau_2 \bar{\tau}_3) \bar{\partial}_1 \beta - (\bar{\tau}_2 + \tau_1 \bar{\tau}_3) \bar{\partial}_2 \beta \right]$$

$$+ i \left[ A_1 \bar{y}_\dot{a} + G_{1j} \bar{\partial}_j \dot{a} \right] \left[ (\tau_1 + \tau_3 \bar{\tau}_2) \bar{y}_\beta + i (\tau_3 \bar{\tau}_3 - 1) \bar{\partial}_1 \beta \right]$$

$$+ i \left[ A_2 \bar{y}_\dot{a} + G_{2j} \bar{\partial}_j \dot{a} \right] \left[ (\bar{\tau}_2 + \tau_3 \bar{\tau}_1) \bar{y}_\beta + i (\tau_3 \bar{\tau}_3 - 1) \bar{\partial}_2 \beta \right]$$

$$- \left( i \bar{y}_\dot{a} \bar{\partial}_3 \bar{\tau}_2 + \bar{\partial}_1 \dot{a} \bar{\tau}_2 + \bar{\partial}_2 \dot{a} \bar{\tau}_1 \right) \left( \alpha \bar{y}_\beta + \beta_1 \bar{\partial}_1 \bar{y}_\beta + \beta_2 \bar{\partial}_2 \bar{y}_\beta \right) \right\}$$

$$\exp \left( \tau_3 \bar{\partial}_1 \dot{a} \alpha + \bar{\tau}_3 \bar{\partial}_3 \dot{a} \beta \right) J(\tau_1 y, -\tau_2 y, \bar{\tau}_1 \bar{y}, -\bar{\tau}_2 \bar{y}; K|x) .$$

We found two solutions to this problem. The solution I, being technically more involved, which uses Fierz relations in full generality, is simpler methodologically. The solution II, which is based on the results of [23], has simpler form but contains a $\delta$-function of some nonlinear argument that demands a proper definition eventually leading to the simple expression (B.23).

**Solution I**

One can make sure that the following coefficients solve the problem: (the factor of $\Upsilon$ is implicit)

$$a = -\frac{1}{2} \left\{ \delta(\tau_3) \left[ (\bar{\tau}_1 + \tau_2 \bar{\tau}_3) \tau_1 \delta(\tau_2) + (\bar{\tau}_2 + \tau_1 \bar{\tau}_3) \tau_2 \delta(\bar{\tau}_1) \right] \right\} \delta(\sigma) \delta(\bar{\sigma}) , \quad (B.8)$$

$$b_1 = \frac{i}{2} (\tau_1 + \tau_3 \bar{\tau}_2) \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) , \quad b_2 = \frac{i}{2} (\tau_2 + \tau_3 \bar{\tau}_1) \delta(\tau_1) \delta(\sigma) \delta(\bar{\sigma}) ,$$

$$\bar{b}_1 = \frac{i}{2} (\bar{\tau}_1 + \bar{\tau}_3 \tau_2) \delta(\bar{\tau}_2) \delta'(\sigma) \delta(\bar{\sigma}) , \quad \bar{b}_2 = \frac{i}{2} (\bar{\tau}_2 + \bar{\tau}_3 \tau_1) \delta(\bar{\tau}_1) \delta'(\sigma) \delta(\bar{\sigma}) ,$$

$$g_{12} = (-1 + \tau_3 \bar{\tau}_3) \delta(\tau_2) \delta(\tau_1) \delta(\sigma) \delta(\bar{\sigma}) , \quad g_{21} = (-1 + \tau_3 \bar{\tau}_3) \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) ,$$

$$g_{11} = \frac{1}{2} \left[ (1 - \tau_3 \bar{\tau}_3) \delta(\tau_2) \delta'(\sigma) \delta(\bar{\sigma}) + \delta(\tau_2) \delta(\sigma) \delta'(\bar{\sigma}) - \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) \right] ,$$

$$g_{22} = \frac{1}{2} \left[ (1 - \tau_3 \bar{\tau}_3) \delta(\tau_1) \delta'(\sigma) \delta(\bar{\sigma}) + \delta(\tau_1) \delta(\sigma) \delta'(\bar{\sigma}) - \delta(\tau_1) \delta(\sigma) \delta(\bar{\sigma}) \right] ,$$

$$\alpha = \frac{i}{2} \left[ \bar{\tau}_1 (\tau_1 + \tau_3 \bar{\tau}_2) \delta(\tau_2) + \bar{\tau}_2 (\tau_2 + \tau_3 \bar{\tau}_1) \delta(\tau_1) \right] \delta'(\sigma) \delta(\bar{\sigma}) , \quad (B.9)$$

$$\beta_1 = \frac{1}{2} \left[ \bar{\tau}_1 (\bar{\tau}_1 - \bar{\tau}_2) \delta(\tau_1) \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) + (1 - \bar{\tau}_3 \tau_3) \bar{\tau}_1 \delta(\tau_2) \delta'(\sigma) \delta(\bar{\sigma}) \right] ,$$

$$\beta_2 = \frac{1}{2} \left[ \bar{\tau}_2 (\bar{\tau}_2 - \bar{\tau}_1) \delta(\tau_1) \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) + (1 - \bar{\tau}_3 \tau_3) \bar{\tau}_2 \delta(\tau_1) \delta'(\sigma) \delta(\bar{\sigma}) \right] ,$$

35
\[
\alpha = \frac{i}{2} \left[ \tau_1 (\bar{\tau}_1 + \bar{\tau}_3 \bar{\tau}_2) \delta(\bar{\sigma}_2) + \tau_2 (\bar{\tau}_2 + \bar{\tau}_3 \bar{\tau}_1) \delta(\bar{\tau}_1) \right] \delta(\sigma) \delta'(\bar{\sigma}), \tag{B.10}
\]

\[
\bar{\beta}_1 = \frac{1}{2} \left[ \tau_1 (\tau_1 - \tau_2) \delta(\tau_1) \delta(\sigma) \delta(\bar{\sigma}) + (1 - \tau_3 \bar{\tau}_3) \tau_1 \delta(\bar{\tau}_2) \delta(\sigma) \delta'(\bar{\sigma}) \right],
\]

\[
\bar{\beta}_2 = \frac{1}{2} \left[ \tau_2 (\tau_2 - \tau_1) \delta(\tau_1) \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) + (1 - \tau_3 \bar{\tau}_3) \tau_2 \delta(\bar{\tau}_1) \delta(\sigma) \delta'(\bar{\sigma}) \right],
\]

where

\[
\sigma = 1 - \sum_{i=1}^{3} \tau_i, \quad \bar{\sigma} = 1 - \sum_{i=1}^{3} \bar{\tau}_i. \tag{B.11}
\]

The resulting local vertex is

\[
G^{ loc}_{\alpha \alpha} \frac{1}{\alpha} \int \int d^3 \bar{\tau} d^3 \tau \chi_{\alpha \alpha} \exp \left( \tau_3 \partial_1 \alpha \partial_2 \alpha + \bar{\tau}_3 \bar{\partial}_1 \alpha \bar{\partial}_2 \alpha \right) J(\tau_1 y, -\tau_2 y, \bar{\tau}_1 \bar{y}, -\bar{\tau}_2 \bar{y}; \alpha | x)
\]

\[
g^{ loc}_{\alpha \alpha} = \frac{1}{2} [\bar{\alpha} + \bar{\alpha}] \left( \delta(\tau_1) + \delta(\tau_2) \right) \delta(\sigma) \delta'(\bar{\sigma}) \bar{\partial}_1 \alpha \bar{\partial}_2 \alpha + \frac{1}{2} \delta(\tau_3) \left( \delta(\bar{\tau}_1) + \delta(\bar{\tau}_2) \right) \delta'(\sigma) \delta(\bar{\sigma}) \bar{\partial}_1 \alpha \bar{\partial}_2 \alpha
\]

\[
- \frac{1}{2} \delta(\tau_3) \delta(\sigma) \delta(\bar{\sigma}) \left( \delta(\tau_1) \bar{\partial}_1 \alpha + \delta(\bar{\tau}_1) \bar{\partial}_2 \alpha \right) \bar{\partial} \delta'(\sigma) \delta(\bar{\sigma}) \left( \delta(\bar{\tau}_2) \bar{\partial}_2 \alpha + \delta(\bar{\tau}_2) \bar{\partial}_1 \alpha \right)
\]

\[
+ \frac{i}{2} \delta(\tau_3) \tau_1 \left( \delta(\bar{\tau}_1) \bar{\partial}_1 \alpha + \delta(\bar{\tau}_1) \bar{\partial}_2 \alpha \right) \bar{\partial} \delta'(\sigma) \delta(\bar{\sigma}) \left( \delta(\bar{\tau}_2) \bar{\partial}_2 \alpha + \delta(\bar{\tau}_2) \bar{\partial}_1 \alpha \right)
\]

\[
- \frac{i}{2} \delta(\tau_3) \left[ \tau_1 (\tau_1 - \tau_2) \delta(\tau_1) \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) + \tau_1 \delta(\tau_1) \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) \right] \bar{\partial}_1 \alpha \bar{\partial}_2 \alpha
\]

\[
+ \delta(\bar{\tau}_3) \left[ \frac{1}{2} \left( \bar{\tau}_1 (\tau_1 + \bar{\tau}_3 \bar{\tau}_2) \delta(\tau_1) \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) \right) \bar{\partial}_1 \alpha \bar{\partial}_2 \alpha
\]

\[
- \frac{1}{2} \tau_2 \tau_1 \left( \delta(\bar{\tau}_1) \delta(\tau_1) \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) \right) \bar{\partial}_1 \alpha \bar{\partial}_2 \alpha
\]

\[
- \frac{i}{2} \delta(\tau_3) \left[ \tau_2 (\bar{\tau}_2 - \bar{\tau}_1) \delta(\tau_2) \delta(\tau_1) \delta(\sigma) \delta(\bar{\sigma}) + \tau_2 \delta(\tau_1) \delta(\tau_2) \delta(\sigma) \delta(\bar{\sigma}) \right] \bar{\partial}_1 \alpha \bar{\partial}_2 \alpha.
\]

The complex conjugated case is analogous. This solution is less useful than Solution II obtained using another Ansatz.

**Solution II**

Setting \( a = b_i = \bar{b}_i = 0 \) in (B.2) as well as \( \alpha = \beta_i = \bar{\beta}_i = 0 \) in (B.6), and considering \( g_{ij} \) proportional to \( \delta(\tau_1 \bar{\tau}_1 - \tau_2 \bar{\tau}_2) \) one can make sure that

\[
g_{11} = \left\{ \bar{\tau}_1 (\bar{\tau}_2 + \bar{\tau}_3 \bar{\tau}_2) \delta(\sigma) \delta'(\bar{\sigma}) + \tau_1 (\bar{\tau}_2 + \bar{\tau}_3 \bar{\tau}_1) \delta'(\sigma) \delta(\bar{\sigma}) - \tau_1 \bar{\tau}_1 \delta(\sigma) \delta(\bar{\sigma}) \right\} \delta(Z) \Upsilon, \tag{B.13}
\]

\[
g_{21} = -\left\{ \bar{\tau}_1 (\bar{\tau}_2 + \bar{\tau}_3 \bar{\tau}_2) \delta(\sigma) \delta'(\bar{\sigma}) + \tau_2 (\bar{\tau}_2 + \bar{\tau}_3 \bar{\tau}_1) \delta'(\sigma) \delta(\bar{\sigma}) - \tau_2 \bar{\tau}_1 \delta(\sigma) \delta(\bar{\sigma}) \right\} \delta(Z) \Upsilon, \tag{B.14}
\]

\[
g_{22} = \left\{ \bar{\tau}_2 (\bar{\tau}_1 + \bar{\tau}_3 \bar{\tau}_2) \delta(\sigma) \delta'(\bar{\sigma}) + \tau_1 (\bar{\tau}_1 + \bar{\tau}_3 \bar{\tau}_2) \delta'(\sigma) \delta(\bar{\sigma}) - \tau_2 \bar{\tau}_2 \delta(\sigma) \delta(\bar{\sigma}) \right\} \delta(Z) \Upsilon, \tag{B.15}
\]

\[
g_{12} = -\left\{ \bar{\tau}_2 (\bar{\tau}_1 + \bar{\tau}_3 \bar{\tau}_2) \delta(\sigma) \delta'(\bar{\sigma}) + \tau_1 (\bar{\tau}_1 + \bar{\tau}_3 \bar{\tau}_2) \delta'(\sigma) \delta(\bar{\sigma}) - \tau_1 \bar{\tau}_2 \delta(\sigma) \delta(\bar{\sigma}) \right\} \delta(Z) \Upsilon, \tag{B.16}
\]

\[
Z = \tau_1 \bar{\tau}_1 - \tau_2 \bar{\tau}_2. \tag{B.17}
\]

36
substituted into $X(J)$ (B.1) solve equation (B.3).

Because of the factor of $\delta(Z)$, $D_{ad}X(J)$ contains distributions like $\delta(\tau_i)\theta(\tau_i)$ that may be ill defined at $\tau_i = 0$. One can see, however, that by the substitution

$$
\tau_1 \to (1 - \tau_3)\tau_1, \quad \tau_2 \to (1 - \tau_3)\tau_2, \quad \bar{\tau}_1 \to (1 - \bar{\tau}_3)\tau_1, \quad \bar{\tau}_2 \to (1 - \bar{\tau}_3)\tau_1,
$$

expression (B.1) acquires a nice form (4.23) which can be independently checked to solve the problem.

References

[1] M. A. Vasiliev, Phys. Lett. B 243 (1990) 378.
[2] M. A. Vasiliev, Phys. Lett. B 285 (1992) 225.
[3] S. F. Prokushkin and M. A. Vasiliev, Nucl. Phys. B 545 (1999) 385 [hep-th/9806236].
[4] M. A. Vasiliev, Phys. Lett. B 567 (2003) 139 [hep-th/0304049].
[5] A. K. H. Bengtsson, I. Bengtsson, and L. Brink, Nucl. Phys. B 227 (1983) 31.
[6] A. K. H. Bengtsson, I. Bengtsson, and L. Brink, Nucl. Phys. B 227 (1983) 41.
[7] F. A. Berends, G. J. H. Burgers, and H. Van Dam, Z. Phys. C 24 (1984) 247.
[8] F. A. Berends, G. J. H. Burgers, and H. van Dam, Nucl. Phys. B 260 (1985) 295.
[9] C. Fronsdal, Phys. Rev. D 18 (1978) 3624; D 20 (1979) 848.
[10] J. Fang and C. Fronsdal, Phys. Rev. D 18 (1978) 3630; D 22 (1980) 1361.
[11] E. S. Fradkin and M. A. Vasiliev, Annals Phys. 177 (1987) 63.
[12] M. A. Vasiliev, Fortschr. Phys. 36, 33 (1988).
[13] E. S. Fradkin and M. A. Vasiliev, Phys. Lett. B 189 (1987) 89–95.
[14] E. S. Fradkin and M. A. Vasiliev, Nucl. Phys. B 291 (1987) 141.
[15] S. R. Coleman and J. Mandula, Phys. Rev. 159 (1967) 1251–1256.
[16] C. Aragone and S. Deser, Phys. Lett. B 86 (1979) 161.
[17] X. Bekaert, N. Boulanger and P. Sundell, Rev. Mod. Phys. 84 (2012) 987 [arXiv:1007.0435 [hep-th]].
[18] M. A. Vasiliev, JHEP 1506 (2015) 031 [arXiv:1502.02271 [hep-th]].
[19] N. Boulanger, P. Kessel, E. D. Skvortsov and M. Taronna, J. Phys. A 49 (2016) no.9, 095402 [arXiv:1508.04139 [hep-th]].
[20] X. Bekaert, J. Erdmenger, D. Ponomarev and C. Sleight, JHEP 1511 (2015) 149 [arXiv:1508.04292 [hep-th]].
[21] E. D. Skvortsov and M. Taronna, JHEP 1511 (2015) 044 [arXiv:1508.04764 [hep-th]].
[22] M. A. Vasiliev, JHEP **1710** (2017) 111 [arXiv:1605.02662 [hep-th]].
[23] O. A. Gelfond and M. A. Vasiliev, J. Exp. Theor. Phys. **120** (2015) 3, 484 [arXiv:1012.3143 [hep-th]].
[24] M. A. Vasiliev, JHEP **1801** (2018) 062 [arXiv:1707.03735 [hep-th]].
[25] B. Sundborg, Nucl. Phys. Proc. Suppl. **102** (2001) 113 [arXiv:hep-th/0103247].
[26] E. Witten, talk at the John Schwarz 60-th birthday symposium, http://theory.caltech.edu/jhs60/witten/1.html
[27] E. Sezgin and P. Sundell, Nucl. Phys. B **644** (2002) 303 [Erratum-ibid. B **660** (2003) 403] [arXiv:hep-th/0205131].
[28] I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **550** (2002) 213 [arXiv:hep-th/0210114].
[29] S. Giombi and X. Yin, JHEP **1009** (2010) 115 [arXiv:0912.3462 [hep-th]].
[30] O. Aharony, G. Gur-Ari and R. Yacoby, JHEP **1203** (2012) 037 [arXiv:1110.4382 [hep-th]].
[31] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia and X. Yin, Eur. Phys. J. A **72** (2012) 2112 arXiv:1110.4386 [hep-th].
[32] J. Maldacena and A. Zhiboedov, J. Phys. A **46** (2013) 214011 arXiv:1112.1016 [hep-th].
[33] J. Maldacena and A. Zhiboedov, Class. Quant. Grav. **30** (2013) 104003 [arXiv:1204.3882 [hep-th]].
[34] M. A. Vasiliev, J. Phys. A **46** (2013) 214013 [arXiv:1203.5554 [hep-th]].
[35] S. Giombi and X. Yin, J. Phys. A **46** (2013) 214003 [arXiv:1208.4036 [hep-th]].
[36] V. E. Didenko and M. A. Vasiliev, Phys. Lett. B **775** (2017) 352 [arXiv:1705.03440 [hep-th]].
[37] M. A. Vasiliev, Ann. Phys. (NY) **190** (1989) 59.
[38] V. E. Didenko, N. G. Misuna and M. A. Vasiliev, JHEP **1607** (2016) 146 [arXiv:1512.04405 [hep-th]].
[39] O. A. Gelfond and M. A. Vasiliev, Theor. Math. Phys. **187** (2016) no.3, 797 [Teor. Mat. Fiz. **187** (2016) no.3, 401] [arXiv:1510.03488 [hep-th]].
[40] F. A. Berends, G. J. H. Burgers and H. van Dam, Nucl. Phys. B **271** (1986) 429.
[41] R. R. Metsaev, Nucl. Phys. B **759**, 147 (2006) [hep-th/0512342].
[42] N. Misuna, Phys. Lett. B **778** (2018) 71 [arXiv:1706.04605 [hep-th]].
[43] R. R. Metsaev, Mod. Phys. Lett. A **6** (1991) 359.
[44] N. Boulanger, S. Leclercq and S. Cnockaert, Phys. Rev. D **73** (2006) 065019 [hep-th/0509118].
[45] E. Conde, E. Joung and K. Mkrtchyan, JHEP **1608** (2016) 040 [arXiv:1605.07402 [hep-th]].
[46] M.A. Vasiliev, *Nucl.Phys.* **B793** (2008) 469, arXiv:0707.1085 [hep-th].

[47] P. A. Smirnov and M. A. Vasiliev, Theor. Math. Phys. **181** (2014) no.3, 1509 [arXiv:1312.6638 [hep-th]].

[48] E. Sezgin, E. D. Skvortsov and Y. Zhu, JHEP **1707** (2017) 133 [arXiv:1705.03197 [hep-th]].

[49] C. Sleight and M. Taronna, Phys. Rev. Lett. **116** (2016) no.18, 181602 [arXiv:1603.00022 [hep-th]].