Stationary solutions of the Navier–Stokes–Fourier system in planar domains with impermeable boundary

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Abstract

The existence of weak solutions to the Navier–Stokes–Fourier system describing the stationary states of a compressible, viscous, and heat conducting fluid in bounded 2D-domains is shown under fairly general and physically relevant constitutive relations. The equation of state of a real fluid is considered, where the admissible range of density is confined to a bounded interval (hard sphere model). The transport coefficients depend on the temperature in a general way including both gases and liquids behavior. The heart of the paper are new *a priori* bounds resulting from Trudinger–Moser inequality.

**Keywords:** Navier–Stokes–Fourier system, stationary solution, inhomogeneous boundary conditions, Trudinger–Moser inequality

**Contents**

1 Introduction 2
2 Main result 5
   2.1 Constitutive equations, external forces 5
   2.2 Weak formulation 5
   2.3 Existence of weak solutions 6
3 Approximate system 7
4 Uniform bounds on approximate solutions 8
   4.1 Entropy equation 8
   4.2 Trudinger–Moser inequality 10
      4.2.1 Bounds on the temperature gradient 10
      4.2.2 Estimates of $\vartheta$ near absolute zero 12

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Stationary states of a compressible, viscous, and heat conducting fluid contained in $\Omega$ are described through the phase variables - the density $\rho \equiv \rho(x)$, the velocity field $u \equiv u(x)$, and the (absolute) temperature $\vartheta \equiv \vartheta(x)$ - satisfying the Navier–Stokes–Fourier system:

\begin{align}
\text{div}(\rho u) &= 0, \quad (1.1a) \\
\text{div}(\rho u \otimes u) + \nabla p(\rho, \vartheta) &= \text{div}(S(\vartheta, \nabla u)) + \rho f, \quad (1.1b) \\
\text{div}\left(\left[\frac{1}{2}\rho|u|^2 + \rho e(\rho, \vartheta)\right]u\right) &+ \text{div}(p(\vartheta, \vartheta)u) + \text{div}(q(\vartheta, \nabla \vartheta)) = \text{div}(S(\vartheta, \nabla u)u) + \rho f \cdot u + \rho G. \quad (1.1c)
\end{align}

We suppose the existence of the entropy $s \equiv s(\rho, \vartheta)$ related to the internal energy $e(\rho, \vartheta)$ and the pressure $p(\rho, \vartheta)$ through Gibbs’ equation:

\begin{equation}
\vartheta Ds = De + pD\left(\frac{1}{\rho}\right). \quad (1.2)
\end{equation}

The viscous stress tensor $S$ is given by Newton’s rheological law:

\begin{equation}
S(\vartheta, \nabla u) \equiv \mu(\vartheta)[\nabla u + (\nabla u)^T - \text{div}u I] + \lambda(\vartheta)\text{div}u I, \quad (1.3)
\end{equation}

while the heat flux $q \equiv q(\vartheta, \nabla \vartheta)$ obeys Fourier’s law:

\begin{equation}
q \equiv -\kappa(\vartheta)\nabla \vartheta. \quad (1.4)
\end{equation}
Here $(\cdot)^T$ and $I$ denote the transpose of a tensor and the identity matrix, respectively. The functions $f \equiv f(x)$ and $G \equiv G(x)$ represent the external force and external heat source, respectively. The problem is closed by the set of boundary conditions:

$$
\begin{align*}
\mathbf{u} |_{\partial \Omega} &= \bar{\mathbf{u}} \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n} |_{\partial \Omega} = 0, \\
\mathbf{q} \cdot \mathbf{n} |_{\partial \Omega} &= L(\bar{\vartheta} - \bar{\vartheta}) |_{\partial \Omega} \quad \text{with} \quad L > 0,
\end{align*}
$$

where $\bar{\vartheta}$ is a prescribed “threshold” temperature. The interested reader may consult the monograph by Galavotti [10] or [7, Chapter 1] for physical background of the problem $\text{(1.1)}$–$\text{(1.5)}$.

Our goal is to show the existence of admissible weak solutions to the Navier–Stokes–Fourier system under fairly general conditions imposed on the constitutive relations. Following [4], we consider the equation of state (EOS) describing real fluids:

$$
p(\rho, \vartheta) \to \infty \quad \text{as} \quad \rho \to \bar{\rho} \quad \text{with} \quad \bar{\rho} > 0 \text{ - a positive constant.}
$$

In particular for gases, the EOS is usually written in the form

$$
p(\rho, \vartheta) = g\vartheta(B_0 + B_1(\rho, \vartheta)),
$$

where the term $B_1$ represents the deviation from the standard Boyle–Mariotte law active in the degenerate area. Accordingly, we focus on EOS that can be written in the following form

$$
p(\rho, \vartheta) = g\vartheta h(\rho), \quad h(0) = h_0 > 0, \quad h(\rho) \to \infty \quad \text{as} \quad \rho \to \bar{\rho}.
$$

Note that Kolafa EOS [12] as well as Carnahan–Starling EOS [3] can be written as $\text{(1.6)}$. In accordance with Gibbs’ relation $\text{(1.2)}$, the associated internal energy $e$ is a function of the temperature only. Here we suppose

$$
e(\rho, \vartheta) \equiv c_v \vartheta \quad \text{with} \quad c_v > 0.
$$

More general EOS can be handled by the same approach. This issue is discussed briefly in the concluding part.

The available literature concerning stationary states of the Navier–Stokes–Fourier system is rather limited. Besides the results concerning smooth solutions arising as small perturbations of known static states, see Piasecki and Pokorný [16], Plotnikov, Ruban and Sokolowski [17, 18], there is a series of papers by Novotný, Pokorný, and their collaborators concerning the existence of weak solutions for problems with large data (external forces), [15]. The main novelties achieved in the present paper compared to the above mentioned results may be summarized as follows:

- General EOS of the form $\text{(1.6)}$ can be handled. In particular, the pressure vanishes for $\vartheta \to 0$, which is particularly relevant to gases. There is no need to add a ”cold pressure” component independent of $\vartheta$.

- The class of transport coefficients includes

$$
\mu(\vartheta) \approx (1 + \vartheta^\alpha), \quad \lambda(\vartheta) \approx (1 + \vartheta^\alpha), \quad \kappa(\vartheta) \approx (1 + \vartheta^\alpha), \quad 0 \leq \alpha < 1.
$$

This is relevant for both gases $\alpha = \frac{1}{2}$ and liquids $\alpha = 0$. Moreover, they are all of the same order that corresponds to finite Prandtl number.
The inhomogeneous boundary conditions for the velocity \((1.5a)\) are included.

The main new ingredient of our analysis is the Trudinger–Moser inequality available for Sobolev functions in \(W^{1,2}(\Omega)\) if \(\Omega \subset \mathbb{R}^2\) is a planar domain. The key point is the estimate of the temperature in the form

\[
\int_{\Omega} \exp(\omega|\log(\vartheta)|^2) \, dx < \infty \quad \text{for certain } \omega > 0
\]

resulting from boundedness of the associated entropy production rate. Obtaining \((1.9)\), however, is not completely straightforward due to the presence of external driving represented by the non–homogeneous boundary condition \((1.5a)\). Relation \((1.9)\) is obtained via a non–standard compactness argument based on the possibility of extending the field \(\bar{u}\) in \(\Omega\) with sufficiently small norm. Compactness of the density is then obtained by a combination of the method proposed by Lions \([13]\) based on the monotonicity of the pressure, and Commutator Lemma originally introduced in \([6]\) to handle the time dependent viscosity coefficients. The reader is also referred to \([7, \text{Lemma 3.6, p. 100}]\).

The paper is organized as follows. In Section 2, we collect the necessary preliminary material, formulate principal hypotheses, and state the main result. The existence proof follows a multi–level approximate scheme introduced in Section 3. It consists in:

- introducing artificial viscosity to regularize the equation of continuity \((1.1a)\) (small parameter \(\varepsilon\));
- discretizing the momentum equation \((1.1b)\) by means of a Galerkin approximation (dimension \(N\) of the approximate space);
- replacing the total energy balance \((1.1c)\) by the internal energy equation (small parameter \(\delta > 0\) to augment viscosity and thermal conductivity);
- truncating the singular pressure (truncation parameter \(R\)).

In Section 4 we establish uniform bounds on the family of approximate solutions noting that their existence can be shown in a manner similar to \([14]\). This amounts to deriving the associated entropy balance, the validity of which can be seen as an admissibility condition imposed on the class of weak solutions. In Section 5 we perform the limit in the Galerkin approximation. At this level, the internal energy equation is replaced by the total energy balance, and the system is augmented by the entropy inequality. In Section 6 we derive the pressure estimates based on the application of the so–called Bogovskii operator and then relax the truncation. As a result, the density here and hereafter is bounded above by \(\bar{\rho}\). In Section 7 we perform the vanishing viscosity limit in the equation of continuity. This a delicate but nowadays rather well understood process, where Lions’ method \([13]\) based on compactness of the effective viscous flux is combined with the commutator technique introduced in \([6]\). Finally, in Section 8 we remove the regularizing terms depending on a small parameter \(\delta\). In particular, we perform in full generality the estimates leading to the crucial bound \((1.9)\). The paper is concluded by a short discussion in Section 9.
2 Main result

We introduce the principal hypotheses on the data and state our main result. Here and hereafter, we use the symbol
\[ a \lesssim b \] if there exists a constant \( c > 0 \) such that \( a \leq cb \)
\[ a \approx b \] if \( a \lesssim b \) and \( b \lesssim a \).

For \( h \in L^1(\Omega) \), we denote its integral mean by
\[ \{h\}_m \overset{\text{def}}{=} \frac{1}{|\Omega|} \int_\Omega h \, dx. \]

We also use the symbol \( c_D \) to denote a generic positive constant depending only on the data (domain, boundary conditions, constitutive relations). When there is no confusion, we will use simply the notation \( X(\Omega) \) instead of \( X(\Omega; Y) \) where \( X \) is a functional space and \( Y \) a vectorial space.

2.1 Constitutive equations, external forces

The given external fields \( f, G, \bar{\vartheta} \) satisfy the following assumptions:
\[ f \in L^\infty(\Omega; \mathbb{R}^2), \quad G \in L^\infty(\Omega), \quad G \geq 0, \quad \bar{\vartheta} \in L^\infty(\partial\Omega), \quad \text{ess inf}_{\partial\Omega} \bar{\vartheta} > 0. \] (2.2)

We consider the pressure in the following form: there exists a constant \( \bar{\rho} > 0 \) such that
\[ p(\rho, \vartheta) \overset{\text{def}}{=} \rho \vartheta h(\rho) \]
where \( h \in C^0([0, \bar{\rho}) \cap C^1(0, \bar{\rho}), \quad h(0) = h_0 > 0, \quad h'(\rho) \geq 0, \]
\[ \lim_{\rho \to \bar{\rho}} h(\rho) = \infty. \] (2.3a)

In accordance with Gibbs’ relation (1.2), the internal energy is taken in the form:
\[ e(\rho, \vartheta) \overset{\text{def}}{=} c_v \vartheta \]
with \( c_v > 0 \), (2.4)

The transport coefficients \( \mu, \lambda, \kappa \) are continuously differentiable functions of the temperature satisfying
\[ \mu(\vartheta) \approx (1 + \vartheta^\alpha), \quad \lambda(\vartheta) \approx (1 + \vartheta^\alpha), \quad \kappa(\vartheta) \approx (1 + \vartheta^\alpha) \quad \text{with} \quad 0 \leq \alpha < 1. \] (2.5)

The boundary condition (1.5a) are determined via a field \( \bar{u} \) satisfying
\[ \bar{u} \in W^{1,\infty}(\Omega; \mathbb{R}^2), \quad \text{div}(\bar{u}) = 0, \quad \bar{u} \cdot n_{|\partial\Omega} = 0. \] (2.6)

2.2 Weak formulation

Let \( M \) be such that \( 0 < M < |\Omega|\bar{\rho} \) and denote by \( \varrho_M \overset{\text{def}}{=} \frac{M}{|\Omega|}, \) \( 0 < \varrho_M < \bar{\rho} \). Let \( s \overset{\text{def}}{=} s(\varrho, \vartheta) \) be the entropy derived from (2.3), (2.4) via Gibbs’ relation (1.2), leading to
\[ s(\varrho, \vartheta) = c_v \log(\vartheta) - \int_{\varrho_M}^\varrho \frac{h(z)}{z} \, dz. \] (2.7)

The weak formulation of the Navier–Stokes–Fourier system (1.1), with the boundary conditions (1.5a) and (1.5b) reads:
• Equation of continuity

\[ \int_\Omega \rho u \cdot \nabla \varphi \, dx = 0 \text{ for any } \varphi \in W^{1,\infty}(\Omega). \] (2.8)

• Momentum balance

\[ \int_\Omega \left[ \rho u \otimes u : \nabla \varphi + p(\varrho, \vartheta) \text{div} \varphi \right] \, dx = \int_\Omega S(\vartheta, \nabla u) : \nabla \varphi \, dx - \int_\Omega \rho f \cdot \varphi \, dx \] (2.9a)

for any \( \varphi \in W^{1,\infty}_0(\Omega; \mathbb{R}^2) \),

\[ u_{|_{\partial \Omega}} = \bar{u}_{|_{\partial \Omega}} \text{ in the sense of traces.} \] (2.9b)

• Total energy balance

\[ \int_\Omega \left( \frac{1}{2} \rho |u|^2 + \rho e(\varrho, \vartheta) \right) u \cdot \nabla \varphi \, dx + \int_\Omega p(\varrho, \vartheta) u \cdot \nabla \varphi \, dx + \int_\Omega q \cdot \nabla \varphi \, dx \]

\[ - \int_\Omega S(\vartheta, \nabla u) u \cdot \nabla \varphi \, dx = \int_\Omega L(\vartheta - \bar{\vartheta}) \varphi \, dS - \int_\Omega \varphi G \, dx - \int_\Omega \varphi f \cdot u \, dx \]

\[ + \int_\Omega \rho (u \otimes u) : \nabla (\varphi \bar{u}) \, dx + \int_\Omega p(\varrho, \vartheta) \text{div} (\varphi \bar{u}) \, dx \]

\[ + \int_\Omega \rho f \cdot (\varphi \bar{u}) \, dx - \int_\Omega S(\vartheta, \nabla u) : \nabla (\varphi \bar{u}) \, dx \] (2.10)

for any \( \varphi \in W^{1,\infty}(\Omega) \). Notice that the total energy balance is obtained by multiplying (1.1b) and (1.1c) by \( \varphi \bar{u} \) and \( \varphi \), respectively.

• Entropy inequality

\[ \int_\Omega \varphi \frac{1}{\vartheta} \left[ S(\vartheta, \nabla u) : \nabla u + \frac{\kappa(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right] \, dx + \int_\Omega \frac{\partial}{\partial \vartheta} G \varphi \, dx \]

\[ \leq \int_{\partial \Omega} \varphi L \left( 1 - \frac{\bar{\vartheta}}{\vartheta} \right) \, dS - \int_\Omega \left( \frac{q(\vartheta, \nabla \vartheta)}{\vartheta} \right) \cdot \nabla \varphi \, dx - \int_\Omega (\rho s(\varrho, \vartheta) u) \cdot \nabla \varphi \, dx \] (2.11)

for any \( \varphi \in W^{1,\infty}(\Omega) \) satisfying \( \varphi \geq 0 \).

2.3 Existence of weak solutions

We are ready to state our main result on the existence of weak solutions to the Navier–Stokes–Fourier system.

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain of class \( C^{2+\nu} \). Let the data \( f, G, \bar{\vartheta}, \) and \( \bar{u} \) satisfy the hypotheses (2.2) and (2.6). Let the pressure \( p \) and the internal energy \( e \) satisfy (2.3) and (2.4). Let the transport coefficients \( \mu, \lambda, \) and \( \kappa \) be continuously differentiable functions of \( \vartheta \) satisfying (2.5), with

\[ 0 \leq \alpha < 1. \] (2.12)
Then for any $M > 0$, the Navier–Stokes–Fourier system \((2.8)-(2.11)\) admits a solution $[\varrho, \mathbf{u}, \vartheta]$ satisfying

$$M = \int_{\Omega} \varrho \, dx.$$  \hspace{1cm} (2.13)

The solution belongs to the class:

- $\varrho \in L^\infty(\Omega)$, $0 \leq \varrho < \bar{\varrho}$ a.e in $\Omega$,
- $\mathbf{u} \in W^{1,r}(\Omega; \mathbb{R}^2)$ for any $1 \leq r < 2$,
- $\vartheta^\beta \in W^{1,r}(\Omega)$ for any $1 \leq r < 2$, $\beta \in \mathbb{R}$, $\vartheta > 0$ a.e in $\Omega$, $\log(\vartheta) \in W^{1,2}(\Omega)$.

The rest of the paper is devoted to the proof of Theorem 2.1.

### 3 Approximate system

Let us define the following cut–off function:

$$T \in C^\infty(\mathbb{R}), \quad T(\varrho) = \begin{cases} \varrho & \text{if } 0 \leq \varrho \leq \bar{\varrho}, \\ \in [-2\bar{\varrho}, 2\bar{\varrho}] & \text{otherwise}. \end{cases}$$

The solutions will be obtained through a multi–level approximate system including regularization of various types:

\begin{align*}
\text{div}(\varrho \mathbf{u}) &= \varepsilon \Delta \varrho - \varepsilon (\varrho - \varrho_M), \quad \nabla \varrho \cdot \mathbf{n}_{\partial \Omega} = 0, \hspace{1cm} (3.1a) \\
\int_{\Omega} \left[ \frac{1}{2} T(\varrho)(\mathbf{u} \otimes \mathbf{u} : \nabla \varphi - \mathbf{u} \cdot \nabla \mathbf{u} \cdot \varphi) + p_R(\varrho, \vartheta) \text{div} \varphi \right] \, dx &= \int_{\Omega} \left[ S_\delta(\vartheta, \nabla \mathbf{u}) : \nabla \varphi - T(\varrho) f \cdot \varphi \right] \, dx \hspace{1cm} (3.1b)
\end{align*}

for any $\varphi \in X_N$, $(\mathbf{u} - \bar{\mathbf{u}}) \in X_N$;

\begin{align*}
\text{div}(\varrho_R(\vartheta, \check{\vartheta}) \mathbf{u}) + \text{div} q_\delta(\vartheta, \nabla \vartheta) &= S_\delta(\vartheta, \mathbf{u}) : \nabla \mathbf{u} - p_R(\varrho, \vartheta) \text{div} \mathbf{u} + \varrho(G + \varepsilon), \hspace{1cm} (3.2a) \\
q_\delta \cdot \mathbf{n}_{\partial \Omega} &= L[\vartheta - \bar{\vartheta}], \hspace{1cm} (3.2b)
\end{align*}

with

$$S_\delta(\vartheta, \nabla \mathbf{u}) = \mu_\delta(\vartheta)(\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \text{div} \mathbf{u} \mathbf{I}) + \lambda_\delta(\vartheta) \text{div} \mathbf{u} \mathbf{I} \quad \text{and} \quad q_\delta(\vartheta, \nabla \vartheta) = -\kappa_\delta(\vartheta) \nabla \vartheta. \hspace{1cm} (3.3)$$

Furthermore, we have

$$\{\varrho\}_m = \varrho_M.$$  \hspace{1cm} (3.4)

There are four parameters: the dimension on the Galerkin approximation space $N$, the artificial viscosity (mass transport) coefficient $\varepsilon$, the truncation parameter $R$, and perturbations by regularizing terms depending on $\delta$.

The specific form of the convective term in \((3.1b)\) is borrowed from Novotný and Pokorný \[14\]. Moreover, using the same method as in \[14\] we can show that the approximate system \((3.1)-(3.3)\) admits regular (strong) solution whenever

$$\varepsilon > 0, \quad N < \infty, \quad R < \infty, \quad \delta > 0.$$  \hspace{1cm} (3.5a)
\( \kappa_\delta(\vartheta) = \kappa(\vartheta) + \delta(\vartheta^\beta + \vartheta^{-1}), \quad \beta \geq 2. \) \tag{3.5b}

and also
\[
\mu_\delta(\vartheta) \overset{\text{def}}{=} \mu(\vartheta) + \delta \vartheta^a \quad \text{and} \quad \lambda_\delta(\vartheta) \overset{\text{def}}{=} \lambda(\vartheta) + \delta \vartheta^a \quad \text{for some } \alpha < a < 1.
\] \tag{3.6}

Moreover there exist \( \underline{\vartheta} \) and \( \bar{\vartheta} \) (depending on \( \varepsilon, N, R \) and \( s \)) such that
\[
0 < \underline{\vartheta} \leq \vartheta \leq \bar{\vartheta} \text{ in } \Omega, \quad \text{see } [4], \text{ and } [8] \text{ for details.}
\] \tag{3.7}

The truncated pressure is defined as
\[
p_R \overset{\text{def}}{=} \varrho \partial h_R(\varrho) \quad \text{and} \quad e_R \overset{\text{def}}{=} e = c_v \vartheta,
\] \tag{3.8a}

\[
h_R(\varrho) \overset{\text{def}}{=} \begin{cases} h(\varrho) & \text{if } 0 < \varrho < \bar{\varrho} - \frac{1}{R}, \\ h(\bar{\varrho} - \frac{1}{R}) + h'(\bar{\varrho} - \frac{1}{R})(\varrho - (\bar{\varrho} - \frac{1}{R})) & \text{otherwise.} \end{cases}
\] \tag{3.8b}

Accordingly, we may define the entropy \( s_R \) by the following identity
\[
s_R(\varrho, \vartheta) \overset{\text{def}}{=} c_r \log(\vartheta) - \int_{\varrho M}^{\varrho} \frac{h_R(z)}{z} \, dz,
\]

such that the following Gibbs’ relation is satisfied
\[
\vartheta Ds_R = De_R + p_R D(\frac{1}{\varrho}). \tag{3.9}
\]

4 Uniform bounds on approximate solutions

Our goal is to establish uniform bounds for the approximate solutions solving \( 3.1 \)–\( 3.3 \).

4.1 Entropy equation

As \( \vartheta > 0 \), the internal energy balance \( 3.2a \) divided on \( \vartheta \) reads:
\[
\frac{1}{\vartheta} \text{div}(\varrho \vartheta \mathbf{u})e_R(\varrho, \vartheta) + \frac{1}{\vartheta} \varrho \nabla e_R \cdot \mathbf{u} - \frac{1}{\vartheta} \text{div}(\kappa_\delta(\vartheta) \nabla \vartheta)
\]
\[
= \frac{1}{\vartheta} s_\delta(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - \frac{1}{\vartheta} \varrho p_R(\varrho, \vartheta) \text{div} \mathbf{u} + \frac{\varrho}{\vartheta}(G + \varepsilon),
\]
in other words, we have
\[
\frac{1}{\vartheta} \left[ s_\delta(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \left( \frac{\kappa_\delta(\vartheta)}{\vartheta} \right) |\nabla \vartheta|^2 \right] + \frac{\varrho}{\vartheta}(G + \varepsilon)
\]
\[
= \text{div} \left( \frac{\varrho}{\vartheta} \vartheta e_R(\varrho, \vartheta) + \frac{1}{\vartheta} \varrho \nabla e_R \cdot \mathbf{u} + \frac{1}{\vartheta} \varrho p_R(\varrho, \vartheta) \text{div} \mathbf{u} \right)
\]
\[
= \text{div} \left( \frac{\varrho}{\vartheta} \vartheta e_R - \frac{1}{\vartheta^2} \varrho p_R(\varrho, \vartheta) \nabla \varrho \right) \cdot \mathbf{u} + \frac{1}{\vartheta} \text{div}(\varrho \mathbf{u}) \left( e_R(\varrho, \vartheta) + \frac{p_R(\varrho, \vartheta)}{\varrho} \right). \tag{4.1}
\]
In view of hypothesis (3.9), the relation (4.1) can be written in the following form:

\[
\frac{1}{\vartheta} \left[ S_{\delta}(\vartheta, \nabla u) : \nabla u + \frac{\kappa_{s}(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right] + \frac{\varrho}{\vartheta}(G + \varepsilon) = \text{div} \left( q_{\delta}(\vartheta, \nabla \vartheta) \right) + \text{div}(\phi S_{R}(\vartheta, \vartheta) u) + \text{div}(\phi u) \left( \frac{e_R(\vartheta, \vartheta)}{\vartheta} - s_R(\vartheta, \vartheta) + \frac{p_R(\vartheta, \vartheta)}{\vartheta} \right). \tag{4.2}
\]

Finally, by using (3.1a), we may conclude

\[
\frac{1}{\vartheta} \left[ S_{\delta}(\vartheta, \nabla u) : \nabla u + \frac{\kappa_{s}(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right] + \frac{\varrho}{\vartheta}(G + \varepsilon) = \text{div} \left( q_{\delta}(\vartheta, \nabla \vartheta) \right) + \text{div}(\phi S_{R}(\vartheta, \vartheta) u) + \varepsilon \left( \Delta \vartheta - (\vartheta - \vartheta_M) \right) \left( \frac{e_R(\vartheta, \vartheta)}{\vartheta} - s_R(\vartheta, \vartheta) + \frac{p_R(\vartheta, \vartheta)}{\vartheta} \right). \tag{4.3}
\]

The desired uniform bounds follow by integrating (4.3) over \( \Omega \), we get

\[
\int_{\Omega} \frac{1}{\vartheta} \left[ S_{\delta}(\vartheta, \nabla u) : \nabla u + \frac{\kappa_{s}(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right] \, dx + L \int_{\partial \Omega} \frac{\varrho}{\vartheta} \, dS + \varepsilon \int_{\Omega} \left( \vartheta - \vartheta_M \right) \frac{p_R(\vartheta, \vartheta)}{\vartheta} \, dx
\]

\[
+ \int_{\Omega} \frac{\varrho}{\vartheta}(G + \varepsilon) \, dx = L|\partial \Omega| + \varepsilon \int_{\Omega} \Delta \vartheta \left( \frac{e_R(\vartheta, \vartheta)}{\vartheta} - s_R(\vartheta, \vartheta) + \frac{p_R(\vartheta, \vartheta)}{\vartheta} \right) \, dx
\]

\[
- \varepsilon \int_{\Omega} \left( \vartheta - \vartheta_M \right) \left( \frac{e_R(\vartheta, \vartheta)}{\vartheta} - s_R(\vartheta, \vartheta) + \frac{p_R(\vartheta, \vartheta)}{\vartheta} \right) \, dx. \tag{4.4}
\]

Furthermore, using Gibbs’ relation (3.9) and the boundary condition \( \nabla \vartheta \cdot \mathbf{n}_{\partial \Omega} = 0 \), we find

\[
\varepsilon \int_{\Omega} \Delta \vartheta \left( \frac{e_R(\vartheta, \vartheta)}{\vartheta} - s_R(\vartheta, \vartheta) + \frac{p_R(\vartheta, \vartheta)}{\vartheta} \right) \, dx = -\varepsilon \int_{\Omega} \nabla \vartheta \cdot \nabla \left( \frac{e_R(\vartheta, \vartheta)}{\vartheta} - s_R(\vartheta, \vartheta) + \frac{p_R(\vartheta, \vartheta)}{\vartheta} \right) \, dx
\]

\[
= -\varepsilon \int_{\Omega} \frac{1}{\vartheta} \frac{\partial p_R}{\partial \vartheta}(\vartheta, \vartheta) |\nabla \vartheta|^2 \, dx + \varepsilon \int_{\Omega} \frac{1}{\vartheta} \left[ e_R(\vartheta, \vartheta) + \frac{p_R(\vartheta, \vartheta)}{\vartheta} - \frac{\varrho}{\vartheta} \frac{\partial p_R}{\partial \vartheta}(\vartheta, \vartheta) \right] \nabla \vartheta \cdot \nabla \log(\vartheta) \, dx. \tag{4.5}
\]

Consequently, relation (4.4) gives rise to

\[
\int_{\Omega} \frac{1}{\vartheta} \left[ S_{\delta}(\vartheta, \nabla u) : \nabla u + \frac{\kappa_{s}(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right] \, dx + L \int_{\partial \Omega} \frac{\varrho}{\vartheta} \, dS + \varepsilon \int_{\Omega} \left( \vartheta - \vartheta_M \right) \frac{p_R(\vartheta, \vartheta)}{\vartheta} \, dx
\]

\[
+ \varepsilon \int_{\Omega} \frac{1}{\vartheta} \frac{\partial p_R}{\partial \vartheta}(\vartheta, \vartheta) |\nabla \vartheta|^2 \, dx + \int_{\Omega} \frac{\varrho}{\vartheta}(G + \varepsilon) \, dx = L|\partial \Omega| + \varepsilon \int_{\Omega} \nabla \vartheta \cdot \nabla \log(\vartheta) \, dx
\]

\[
- \varepsilon \int_{\Omega} \left( \vartheta - \vartheta_M \right) \left[ e_R(\vartheta, \vartheta) + \frac{p_R(\vartheta, \vartheta)}{\vartheta} - \frac{\varrho}{\vartheta} \frac{\partial p_R}{\partial \vartheta}(\vartheta, \vartheta) \right] \nabla \vartheta \cdot \nabla \log(\vartheta) \, dx. \tag{4.6}
\]

Thus, using the specific form (4.5) of the pressure and energy truncations, we may infer that

\[
\int_{\Omega} \frac{1}{\vartheta} \left[ S_{\delta}(\vartheta, \nabla u) : \nabla u + \frac{\kappa_{s}(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right] \, dx + L \int_{\partial \Omega} \frac{\varrho}{\vartheta} \, dS + \varepsilon \int_{\Omega} \left( \vartheta - \vartheta_M \right) h_R(\vartheta) \, dx
\]

\[
+ \varepsilon \int_{\Omega} \left[ h_R(\vartheta) + \frac{h_R(\vartheta)}{\vartheta} \right] |\nabla \vartheta|^2 \, dx + \int_{\Omega} \frac{\varrho}{\vartheta}(G + \varepsilon) \, dx = L|\partial \Omega| + \varepsilon c_v \int_{\Omega} \nabla \vartheta \cdot \nabla \log(\vartheta) \, dx
\]

\[
- \varepsilon \int_{\Omega} \left( \vartheta - \vartheta_M \right) \left( \int_{\vartheta_M}^{\vartheta} \frac{h_R(z)}{z} \, dz \right) \, dx + \varepsilon c_v \int_{\Omega} \left( \vartheta - \vartheta_M \right) \left( \log(\vartheta) - \{\log(\vartheta)\}_{m} \right) \, dx. \tag{4.6}
\]
Note that, in accordance with (4.9), the entropy reads
\[
s(\varrho, \vartheta) = c_0 \log(\vartheta) - s_{R,h}(\varrho) \quad \text{and} \quad s_{R,h}'(\varrho) = \frac{1}{\varrho} h_R(\varrho). \tag{4.7}
\]

We determine now some uniform bounds based on the entropy. First observe that
\[
h_R'(\varrho) + \frac{h_R(\varrho)}{\varrho} \geq c_D > 0 \quad \text{independently of } R.
\]

Returning to (4.6), we may deduce that
\[
c \quad \text{where the constant } c_D \text{ is independent of the parameters } N, R, \varepsilon, \text{ and } \delta. \text{ Notice that} \]
\[
(\varrho - \varrho_M) \int_\Omega \frac{h(z)}{z} \, dz \geq 0 \quad \text{and} \quad \int_\Omega (\varrho - \varrho_M)(h_R(\varrho) - h_R(\varrho_M)) \, dx \geq 0.
\]

### 4.2 Trudinger–Moser inequality

In this section, we derive rather strong bounds on the temperature based on the Trudinger–Moser inequality.

#### 4.2.1 Bounds on the temperature gradient

Integrating (3.2a) over \( \Omega \) and using (3.4) and (2.2), we find
\[
\int_\Omega \frac{1}{\varrho} \left[ S_\delta(\varrho, \nabla u) : \nabla u + \frac{\kappa_\delta(\varrho)}{\varrho} |\nabla \varrho|^2 \right] \, dx + L \int_{\partial \Omega} (\vartheta - \varrho) \, dS + \varepsilon \int_\Omega (\varrho - \varrho_M) \left( \int_{\varrho_M}^\varrho \frac{h_R(z)}{z} \right) \, d\varrho \, dx
\]
\[
+ \varepsilon \int_\Omega (\varrho - \varrho_M)(h_R(\varrho) - h_R(\varrho_M)) \, dx + \varepsilon \int_\Omega \left[ h_R'(\varrho) + \frac{h_R(\varrho)}{\varrho} \right] |\nabla \varrho|^2 \, dx + \int_\Omega \frac{\varrho}{\varrho} (G + \varepsilon) \, dx \leq c_D,
\]
where the constant \( c_D \) is independent of the parameters \( N, R, \varepsilon, \text{ and } \delta \).

Next, taking \( u - \bar{u} \) as a test function in the momentum equation (3.1b), we get
\[
\int_\Omega \left[ S_\delta(\varrho, \nabla u) : \nabla u - p_R(\varrho, \vartheta) \text{div} u \right] \, dx + \int_\Omega \varrho (G + \varepsilon) \, dx
\]
\[
\leq \int_\Omega \left[ S_\delta(\varrho, \nabla u) : \nabla u - p_R(\varrho, \vartheta) \text{div} u \right] \, dx + c_D.
\]

and, consequently, we have
\[
L \int_{\partial \Omega} (\vartheta - \bar{\vartheta}) \, dS \leq \int_\Omega \left[ \frac{1}{2} T(\varrho)(u \cdot \nabla u \cdot \bar{u} - u \otimes u : \nabla \bar{u}) + S_\delta(\vartheta, \nabla u) : \nabla \bar{u} + T(\varrho)f \cdot (u - \bar{u}) \right] \, dx,
\]
\[
\lesssim \left( 1 + \int_\Omega \left[ |u \cdot \nabla u| + |u|^2 + |S_\delta(\varrho, \nabla u)| \right] \, dx \right).
\]
Our goal is to show that all the integrals on the right-hand side of (4.9) can be controlled by \( \| \vartheta \|_{L^q(\Omega)}^{\ell} \) for some \( q < \infty \) and \( \ell < 1 \). Let us first observe that

\[
\int_{\Omega} |(u - \bar{u}) \cdot \nabla (u - \bar{u})| \, dx \leq \int_{\Omega} |u - \bar{u}| \, |\nabla u - \nabla \bar{u}| \, dx.
\]

Then the Korn’s inequality leads to

\[
\| \nabla u - \nabla \bar{u} \|_{L^r(\Omega)} \leq c_K(r) \| D u - D \bar{u} \|_{L^r(\Omega)}, \quad 1 < r < \infty,
\]

and the Poincaré’s inequality gives

\[
\| u - \bar{u} \|_{L^1(\Omega)} \leq c_P \| \nabla u - \nabla \bar{u} \|_{L^r(\Omega)}, \quad s_1 \leq \frac{2r}{2 - r} \quad \text{and} \quad s_1 = \frac{r}{r - 1}.
\]

Consequently, there is \( \frac{4}{3} < r < 2 \) such that

\[
\int_{\Omega} |(u - \bar{u}) \cdot \nabla (u - \bar{u})| \, dx \lesssim \| D u - D \bar{u} \|_{L^r(\Omega)}^2,
\]

whence

\[
\int_{\Omega} |u \cdot \nabla u| \, dx \lesssim (1 + \| D u \|_{L^2(\Omega; \mathbb{R}^4)}^2) \quad \text{for some} \quad \frac{4}{3} < r < 2.
\]

On the other hand, the entropy estimates (4.8) together with hypothesis (3.6) yield to

\[
\| \frac{1}{\sqrt{\vartheta}} D u \|_{L^2(\Omega; \mathbb{R}^4)}^2 + \| \sqrt{\delta} \vartheta^{\alpha - \frac{3}{2}} D u \|_{L^2(\Omega; \mathbb{R}^4)}^2 \leq c_D.
\]

According to Hörder’s inequality, we obtain

\[
\| D u \|_{L^r(\Omega; \mathbb{R}^4)} = \frac{1}{\sqrt{\delta}} \| \vartheta^{\frac{1 - \alpha}{2}} \sqrt{\delta} \vartheta^{\frac{3 - \alpha}{2}} D u \|_{L^r(\Omega; \mathbb{R}^4)} \lesssim \frac{1}{\sqrt{\delta}} \| \vartheta^{\frac{1 - \alpha}{2}} \|_{L^r(\Omega)} \lesssim \frac{1}{\sqrt{\delta}} \| \vartheta \|_{L^r(\Omega)}^{\frac{1 - \alpha}{2}} \cdot \frac{1}{s} + \frac{1}{2} = \frac{1}{r}.
\]

Combining (4.12) and (4.14), we get

\[
\int_{\Omega} |u \cdot \nabla u| \, dx \leq c(\delta) \left( 1 + \| \vartheta \|_{L^r(\Omega)}^{\frac{1 - \alpha}{2}} \right) \quad \text{for some} \quad 1 < s < \infty.
\]

Similarly, we deduce

\[
\int_{\Omega} |u|^2 \, dx \leq c(\delta) \left( 1 + \| \vartheta \|_{L^r(\Omega)}^{\frac{1 - \alpha}{2}} \right) \quad \text{for some} \quad 1 < s < \infty.
\]

Finally, using once again the entropy bound (4.8) as well as (4.13), the last integral in (4.9) can be controlled as follows:

\[
\int_{\Omega} |S_{\delta}(\vartheta, \nabla u)| \, dx = \frac{1}{\sqrt{\delta}} \| \left( 1 + \vartheta^\alpha \right)^{-\frac{1}{2}} \sqrt{\delta} \left( 1 + \vartheta^\alpha \right)^{-\frac{1}{2}} D u \|_{L^r(\Omega; \mathbb{R}^4)} \leq c_1(\delta) \left( 1 + \| \vartheta \|_{L^2(\Omega)}^{\frac{1 - \alpha}{2}} \right) \leq c_2(\delta) \left( 1 + \| \vartheta \|_{L^2(\Omega)}^{\frac{1 - \alpha}{2}} \right).
\]

Summing up (4.19)–(4.17), we obtain

\[
\int_{\partial \Omega} \vartheta \, dS \leq c(\delta) \left( 1 + \| \vartheta \|_{L^r(\Omega)}^{\ell} \right) \quad \text{for some} \quad \ell < 1, \quad 1 < s < \infty.
\]

Seeing that
In accordance with hypothesis (3.5b) and the entropy estimates (4.8), we get
\[ \| \nabla \vartheta \|_{L^2(\Omega; \mathbb{R}^2)} \leq c(\delta). \]

- The Poincare’s inequality leads to
\[ \| \vartheta \|_{W^{1,2}(\Omega)} \lesssim \| \nabla \vartheta \|_{L^2(\Omega; \mathbb{R}^2)} + \| \vartheta \|_{L^1(\partial \Omega)}. \]

- The Sobolev embedding gives
\[ \| \vartheta \|_{L^s(\Omega)} \leq c(s) \| \vartheta \|_{W^{1,2}(\Omega)} \] for any \( 1 < s < \infty \).

- It comes from (4.8) and (4.18) that
\[ \| \vartheta \|_{W^{1,2}(\Omega)} \leq c(\delta) \text{ and } \| \log(\vartheta) \|_{W^{1,2}(\Omega)} \leq c(\delta). \] (4.19)

### 4.2.2 Estimates of \( \vartheta \) near absolute zero

We apply the Trudinger–Moser inequality, more specifically the Sobolev embedding
\[ W^{1,2}(\Omega) \hookrightarrow L_\Phi(\Omega), \quad \Omega \subset \mathbb{R}^2 \text{ a bounded Lipschitz domain}, \]
where \( L_\Phi \) is the Orlicz space with the generating function \( \Phi(z) = \exp(z^2) - 1 \), see e.g. Adams [1, Chap. 8]. In particular,
\[ \| v \|_{L_\Phi(\Omega)} \leq c_s \| v \|_{W^{1,2}(\Omega)}, \]
where \( \| \cdot \|_{L_\Phi(\Omega)} \) is the associated Luxemburg norm. In particular,
\[ \int_\Omega \exp\left( \frac{|v|^2}{c_s \| v \|^2_{W^{1,2}(\Omega)}} \right) - 1 \, dx \leq \int_\Omega \exp\left( \frac{|v|^2}{\| v \|^2_{L_\Phi(\Omega)}} \right) - 1 \, dx \leq 1 \text{ for any } v \in W^{1,2}(\Omega), \; v \neq 0. \] (4.20)

In view of the bounds established in (4.19), we may apply (4.20) to \( v = \log(\vartheta) \) which leads to
\[ \int_\Omega \exp(\omega(\delta)|\log(\vartheta)|^2) \, dx \leq 1 + |\Omega| \text{ for certain } \omega(\delta) > 0, \] (4.21)
and it follows that
\[ \| \vartheta^\beta \|_{L^s(\Omega)} \leq c(s, \beta, \delta) \text{ for any } 1 \leq s < \infty, \; \beta \in \mathbb{R}. \] (4.22)

Putting together (4.8), (4.19) and (4.22), the following uniform bounds for the approximate solutions depending only on the parameter \( \delta > 0 \) is obtained:
\[ \| \vartheta \|_{W^{1,2}(\Omega)} + \| \log(\vartheta) \|_{W^{1,2}(\Omega)} \leq c(\delta), \]
\[ \| \vartheta^\beta \|_{L^s(\Omega)} \leq c(\delta, \beta, s) \text{ for any } 1 \leq s < \infty, \; \beta \in \mathbb{R}, \] (4.23)
\[ \| u \|_{W^{1,r}(\Omega)} \leq c(\delta, r), \; 1 \leq r < 2. \]

In addition, we record the bounds on the density can be deduced from the entropy bounds (4.8), the standard elliptic estimates applied to the approximate equation of continuity (3.1a) that depend on \( \varepsilon \) and the fact that \( u \) is bounded in \( W^{1,r}(\Omega) \), namely we have
\[ \| \vartheta \|_{W^{2,q}(\Omega)} \leq c(\varepsilon, \delta, q) \text{ for any } 1 \leq q < 2. \] (4.24)
5 Limit $N \to \infty$

Keeping $R > 0$, $\varepsilon > 0$, $\delta > 0$ fixed, we perform the limit $N \to \infty$. Let $\{\varrho_N, u_N, \vartheta_N\}_{N \geq 1}$ be a sequence of solutions to (3.1)–(3.3). In view of the uniform estimates summarized in (4.23) and (4.24) and passing to the subsequences, if necessary, we find

\begin{align*}
\varrho_N &\rightharpoonup \varrho \text{ weakly in } W^{2,q}(\Omega) \text{ for any } 1 \leq q < 2, \quad (5.1a) \\
\vartheta_N &\rightharpoonup \vartheta \text{ weakly in } W^{1,2}(\Omega), \quad (5.1b) \\
\vartheta_N^\beta &\rightharpoonup \vartheta^\beta \text{ weakly in } W^{1,r}(\Omega) \text{ and } \vartheta_N^\beta \to \vartheta^\beta \text{ in } L^s(\Omega) \text{ for any } \beta \in \mathbb{R}, 1 \leq r < 2, 1 \leq s < \infty, \quad (5.1c) \\
u_N &\rightharpoonup u \text{ weakly in } W^{1,r}(\Omega; \mathbb{R}^2) \text{ for any } 1 \leq r < 2, \quad (5.1d)
\end{align*}
as $N \to \infty$.

5.1 Equation of continuity

According to (5.1) and some standard compactness arguments for Sobolev spaces, we easily deduce

\[ \mathrm{div}(\varrho u) = \varepsilon \Delta \varrho - \varepsilon (\varrho - \varrho_M), \quad \nabla \varrho \cdot n|_{\partial \Omega} = 0, \quad \varrho \geq 0, \quad \frac{1}{|\Omega|} \int_{\Omega} \varrho \, dx = \varrho_M. \]

As uniform bounds on the velocity gradient are no longer available, the limit density may vanish at certain point of $\Omega$.

5.2 Momentum balance

Using similar arguments as above, we perform the limit $N \to \infty$ in the momentum equation (3.1b), we find

\[ \int_{\Omega} \left[ \frac{1}{2} T(\varrho)(u \otimes u : \nabla \varphi - u \cdot \nabla u \cdot \varphi) + p_R(\varrho, \vartheta) \mathrm{div} \varphi \right] \, dx \]

\[ = \int_{\Omega} \left[ S_{\delta} (\vartheta, \nabla u) : \nabla \varphi - T(\varrho)f \cdot \varphi \right] \, dx \]

for any $\varphi \in W^{1,q}_0(\Omega; \mathbb{R}^2)$, $q > 2$, $(u - \bar{u}) \in W^{1,r}_0(\Omega; \mathbb{R}^2)$, $1 \leq r < 2$. Notice that the relation (5.3) is first obtained for any $\varphi \in X_M$, $M$ fixed and then it is extended to all $\varphi \in W^{1,q}_0(\Omega; \mathbb{R}^2)$ by using a density argument.

5.3 Total energy balance

The arguments giving rise to the total energy balance are much more delicate. The first step consists to pass to the weak formulation of the internal energy equation (3.2a) and (3.2b) to get

\[ \int_{\Omega} \varrho_N e_R(\varrho_N, \vartheta_N) u_N \cdot \nabla \varphi \, dx + \int_{\partial \Omega} L(\vartheta_N - \bar{\vartheta}) \varphi \, dS + \int_{\Omega} \kappa_{\delta}(\vartheta_N) \nabla \vartheta_N \cdot \nabla \varphi \, dx \]

\[ = \int_{\Omega} \left[ S_{\delta} (\vartheta_N, \nabla u_N) : \nabla u_N - p_R(\varrho_N, \vartheta_N) \mathrm{div} u_N \right] \varphi \, dx + \int_{\Omega} \varrho_N (G + \varepsilon) \varphi \, dx \]

(5.4)
for any $\varphi \in W^{1,\infty}(\Omega)$. Then $u_N - \bar{u} \in X_N$ is used as a test function in the approximate momentum equation \((3.15)\) giving

$$
\int_{\Omega} \left[ \frac{1}{2} T(\varrho_N)(u_N \cdot \nabla u_N \cdot \bar{u} - u_N \otimes u_N : \bar{u}) \right] dx
= \int_{\Omega} \left[ S(\varrho_N, \nabla u_N) : \nabla(\varrho_N - \bar{u}) - p_R(\varrho_N, \varrho_N) \text{div} u_N \right] dx - \int_{\Omega} T(\varrho_N) f \cdot (u_N - \bar{u}) dx.
$$

Taking the sum of \((5.3)\) with \((5.4)\) in the case where $\varphi = -1$, we obtain

$$
\int_{\partial\Omega} L(\bar{\varphi} - \varrho_N) dS + \int_{\Omega} \left[ \frac{1}{2} T(\varrho)(u \cdot \nabla u \cdot \bar{u} - u \otimes u : \bar{u}) \right] dx
= - \int_{\Omega} S(\varrho, \nabla u) : \nabla \bar{u} dx - \int_{\Omega} \varrho(G + \varepsilon) dx - \int_{\Omega} T(\varrho) f \cdot (u - \bar{u}) dx.
$$

Letting $N \to \infty$ we may therefore infer that

$$
\int_{\partial\Omega} L(\bar{\varphi} - \varrho_N) dS + \int_{\Omega} \left[ \frac{1}{2} T(\varrho)(u \cdot \nabla u) - u \otimes u : \bar{u} \right] dx
= - \int_{\Omega} S(\varrho, \nabla u) : \nabla \bar{u} dx - \int_{\Omega} \varrho(G + \varepsilon) dx - \int_{\Omega} T(\varrho) f \cdot (u - \bar{u}) dx.
$$

We consider now the internal energy \((3.4)\) and we assume that the test function $\psi \in W^{1,\infty}(\Omega)$ with $\psi \geq 0$, we find

$$
\int_{\Omega} S(\varrho_N, \nabla u_N) : \nabla \psi dx = - \int_{\Omega} \varrho_{\text{eq}}(\varrho_N, \varrho_N) u_N \cdot \nabla \psi dx + \int_{\partial\Omega} L(\bar{\varphi} - \varrho_N) \bar{\varrho} dS
+ \int_{\Omega} \kappa(\varrho_N) \nabla \varrho_N \cdot \nabla \psi dx + \int_{\Omega} p_R(\varrho_N, \varrho_N) \text{div} u_N \psi dx - \int_{\Omega} \varrho_N(G + \varepsilon) \psi dx.
$$

Letting $N \to \infty$ and using the weak lower semi-continuity of convex functions, we obtain

$$
\int_{\Omega} S(\varrho, \nabla u) : \nabla \psi dx \leq - \int_{\Omega} \varrho_{\text{eq}}(\varrho, \varrho) u \cdot \nabla \psi dx + \int_{\partial\Omega} L(\bar{\varphi} - \varrho) \bar{\varrho} dS
+ \int_{\Omega} \kappa(\varrho) \nabla \varrho \cdot \nabla \psi dx + \int_{\Omega} p_R(\varrho, \varrho) \text{div} u \psi dx - \int_{\Omega} \varrho(G + \varepsilon) \psi dx
$$

for any $\psi \in W^{1,\infty}(\Omega)$, $\psi \geq 0$. In particular, we have

$$
S(\varrho, \nabla u) : \nabla \psi \in L^1(\Omega) \quad (5.8).
$$

The bound \((5.8)\) allows us to consider the momentum balance \((5.3)\) with the test function

$$
\varphi \overset{\text{def}}{=} \varphi(u - \bar{u}) \text{ with } \varphi \in C^1(\bar{\Omega}),
$$
yielding to the following identity:

\[
\int_{\Omega} \varphi \mathcal{S}_\delta(\vartheta, \nabla \psi) : \nabla \psi \ dx \\
= \int_{\Omega} \left[ \frac{1}{2} T(\vartheta)(u \otimes u : \nabla[\varphi(u - \bar{u})] - \varphi u \cdot \nabla u \cdot (u - \bar{u}) + p_R(\vartheta, \vartheta) \text{div[} \varphi(u - \bar{u})] \right] \ dx \\
+ \int_{\Omega} \varphi T(\vartheta)f \cdot (u - \bar{u}) \ dx + \int_{\Omega} \mathcal{S}_\delta(\vartheta, \nabla \psi) \cdot \nabla \varphi \cdot (\bar{u} - u) \ dx + \int_{\Omega} \varphi \mathcal{S}_\delta(\vartheta, \nabla u) : \nabla \bar{u} \ dx \\
= \int_{\Omega} \frac{1}{2} T(\vartheta)(u \otimes u \cdot \nabla \varphi \cdot (u - \bar{u}) + \varphi u \cdot \nabla u \cdot \bar{u} - \varphi(u \otimes u) : \nabla \bar{u}) \ dx \\
+ \int_{\Omega} p_R(\vartheta, \vartheta) \nabla \varphi \cdot (u - \bar{u}) \ dx + \int_{\Omega} \varphi p_R(\vartheta, \vartheta) \text{div} u \ dx \\
+ \int_{\Omega} \varphi T(\vartheta)f \cdot (u - \bar{u}) \ dx + \int_{\Omega} \mathcal{S}_\delta(\vartheta, \nabla u) \cdot \nabla \varphi \cdot (\bar{u} - u) \ dx + \int_{\Omega} \varphi \mathcal{S}_\delta(\vartheta, \nabla u) : \nabla \bar{u} \ dx.
\]

Thus taking \( \varphi = \psi \geq 0 \) and using (5.7), we get

\[
\int_{\Omega} \frac{1}{2} T(\vartheta)(u \otimes u \cdot \nabla \psi \cdot (u - \bar{u}) + \psi u \cdot \nabla u \cdot \bar{u} - \psi(u \otimes u) : \nabla \bar{u}) \ dx + \int_{\Omega} p_R(\vartheta, \vartheta) \nabla \psi \cdot (u - \bar{u}) \ dx \\
+ \int_{\Omega} \psi T(\vartheta)f \cdot (u - \bar{u}) \ dx + \int_{\Omega} \mathcal{S}_\delta(\vartheta, \nabla u) \cdot \nabla \psi \cdot (\bar{u} - u) \ dx + \int_{\Omega} \psi \mathcal{S}_\delta(\vartheta, \nabla u) : \nabla \bar{u} \ dx \\
\leq -\int_{\Omega} \varrho e_R(\vartheta, \vartheta) u \cdot \nabla \psi \ dx + \int_{\partial\Omega} L(\vartheta - \bar{\vartheta}) \psi dS + \int_{\Omega} \kappa_\delta(\vartheta) \nabla \vartheta \cdot \nabla \psi \ dx - \int_{\Omega} \varrho(G + \varepsilon) \psi \ dx
\]

for any \( \psi \in C^1(\Omega) \) with \( \psi \geq 0 \). Let us consider now \( 0 \leq \psi \leq 1 \). Taking \( (1 - \psi) \geq 0 \) as a test function in (5.9), we obtain

\[
-\int_{\Omega} \frac{1}{2} T(\vartheta)(u \otimes u \cdot \nabla \psi \cdot (u - \bar{u}) + \psi u \cdot \nabla u \cdot \bar{u} - \psi(u \otimes u) : \nabla \bar{u}) \ dx \\
-\int_{\Omega} \frac{p_R(\vartheta, \vartheta)}{2} \nabla \psi \cdot (u - \bar{u}) \ dx - \int_{\Omega} \psi T(\vartheta)f \cdot (u - \bar{u}) \ dx - \int_{\Omega} \mathcal{S}_\delta(\vartheta, \nabla \psi) \cdot \nabla \varphi \cdot (\bar{u} - u) \ dx \\
-\int_{\Omega} \psi \mathcal{S}_\delta(\vartheta, \nabla u) : \nabla \bar{u} \ dx \leq \int_{\Omega} \varrho e_R(\vartheta, \vartheta) u \cdot \nabla \psi \ dx - \int_{\partial\Omega} L(\vartheta - \bar{\vartheta}) \psi dS - \int_{\Omega} \kappa_\delta(\vartheta) \nabla \vartheta \cdot \nabla \psi \ dx \\
+ \int_{\Omega} \varrho(G + \varepsilon) \psi dx + \int_{\Omega} \frac{1}{2} T(\vartheta)(u \otimes u : \nabla \bar{u} - u \cdot \nabla u \cdot \bar{u}) \ dx + \int_{\Omega} T(\vartheta)f \cdot (u - \bar{u}) \ dx \\
-\int_{\Omega} \mathcal{S}_\delta(\vartheta, \nabla u) : \nabla \bar{u} \ dx + \int_{\partial\Omega} L(\vartheta - \bar{\vartheta}) \ dS - \int_{\Omega} \varrho(G + \varepsilon) \ dx.
\]

However, by virtue of (5.6), the sum of the integrals on the right–hand side of (5.10) that do not contain \( \psi \) vanishes. Consequently, (5.10) is reduced to (5.9) with the opposite inequality. Therefore, we conclude
\[ (5.11) \] holds as an equality, namely we have

\[
\int_{\Omega} \left( \frac{1}{2} T(\varrho)|\mathbf{u}|^2 + \varrho \varepsilon R(\varrho, \vartheta) \right) \mathbf{u} \cdot \nabla \varphi \, dx + \int_{\Omega} p_R(\varrho, \vartheta) \mathbf{u} \cdot \nabla \varphi \, dx + \int_{\Omega} \mathbf{q}_\delta \cdot \nabla \varphi \, dx \\
- \int_{\Omega} S_\delta(\vartheta, \nabla \mathbf{u}) \cdot \mathbf{u} \cdot \nabla \varphi \, dx = \int_{\partial \Omega} L(\vartheta - \bar{\vartheta}) \varphi \, dS - \int_{\Omega} \varphi (G + \varepsilon) \, dx - \int_{\Omega} \varphi T(\varrho) \mathbf{f} \cdot \mathbf{u} \, dx \\
+ \int_{\Omega} \frac{1}{2} T(\varrho) (\mathbf{u} \otimes \mathbf{u}) : \nabla (\varphi \bar{\mathbf{u}}) - \mathbf{u} \cdot \nabla \varphi \cdot (\varphi \bar{\mathbf{u}}) \, dx + \int_{\Omega} p_R(\varrho, \vartheta) \text{div} (\varphi \bar{\mathbf{u}}) \, dx \\
+ \int_{\Omega} T(\varrho) \mathbf{f} \cdot (\varphi \bar{\mathbf{u}}) \, dx - \int_{\Omega} S_\delta(\vartheta, \nabla \mathbf{u}) : \nabla (\varphi \bar{\mathbf{u}}) \, dx
\]

for any \( \varphi \in W^{1,\infty}(\Omega) \). The integral equality \((5.11)\) represents a weak formulation of the total energy balance. Note that for \( \varphi \in W^{1,4}_0(\Omega; \mathbb{R}^2) \), the momentum equation \((5.3)\) can be used to eliminate the integrals containing \( \bar{\mathbf{u}} \) which gives

\[
\int_{\Omega} \left( \frac{1}{2} T(\varrho)|\mathbf{u}|^2 + \varrho \varepsilon R(\varrho, \vartheta) \right) \mathbf{u} \cdot \nabla \varphi \, dx + \int_{\Omega} p_R(\varrho, \vartheta) \mathbf{u} \cdot \nabla \varphi \, dx + \int_{\Omega} \mathbf{q}_\delta \cdot \nabla \varphi \, dx \\
- \int_{\Omega} S_\delta(\vartheta, \nabla \mathbf{u}) \cdot \mathbf{u} \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi (G + \varepsilon) \, dx - \int_{\Omega} \varphi T(\varrho) \mathbf{f} \cdot \mathbf{u} \, dx
\]

for any \( \varphi \in W^{1,\infty}_0(\Omega) \). Note that \((5.12)\) can be formally interpreted as the energy equation, namely we have

\[
\text{div} \left[ \left( \frac{1}{2} T(\varrho)|\mathbf{u}|^2 + \varrho \varepsilon R(\varrho, \vartheta) \right) \mathbf{u} \right] + \text{div}(p_R(\varrho, \vartheta) \mathbf{u}) + \text{div} \mathbf{q}_\delta = \text{div}(S_\delta(\vartheta, \nabla \mathbf{u}) \cdot \mathbf{u}) + T(\varrho) \mathbf{f} \cdot \mathbf{u} + \varrho(G + \varepsilon).
\]

### 5.4 Entropy inequality

We conclude the limit passage \( N \to \infty \) by reporting the entropy inequality:

\[
\int_{\Omega} \frac{1}{\varrho} \left[ S_\delta(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \frac{\kappa_\delta(\vartheta)}{\vartheta} |\nabla \varphi|^2 \right] \, dx + \int_{\Omega} \frac{\varrho}{\vartheta} (G + \varepsilon) \varphi \, dx \\
\leq \int_{\partial \Omega} \varphi L (1 - \bar{\vartheta}) \, dS - \int_{\Omega} \left( \mathbf{q}_\delta(\vartheta, \nabla \vartheta) \right) \cdot \nabla \varphi \, dx - \int_{\Omega} \left( \varrho s_R(\varrho, \vartheta) \mathbf{u} \right) \cdot \nabla \varphi \, dx + \varepsilon \int_{\Omega} \varphi (\Delta \varrho - (\varrho - \varrho M)(\varrho h_R(\varrho) - s_R(\varrho, \vartheta))) \, dx
\]

for any \( \varphi \in W^{1,\infty}(\Omega), \varphi \geq 0 \), that can be easily deduced from \((5.13)\) as well as the convergence in \((5.1)\).

### 6 Limit \( R \to \infty \)

Our next goal is to let the truncation parameter \( R \to \infty \) and thus to establish some uniform estimates on the density. Let \( \{\varrho_R, \mathbf{u}_R, \vartheta_R\}_{R>0} \) be the associated sequence of solutions to the problem \((5.2), (5.3), (5.11)\), satisfying also the entropy inequality \((5.14)\). For fixed values of the parameters \( \varepsilon \) and \( \delta \), the estimates \((4.23)\) and \((4.24)\) remain valid. Then passing to the subsequences, if necessary, we find

\[
\varrho_R \rightharpoonup \varrho \text{ weakly in } W^{2,q}(\Omega) \text{ for any } 1 \leq q < 2,
\]

\[(6.1a)\]
\[ \vartheta_R \to \vartheta \text{ weakly in } W^{1,2}(\Omega), \quad \text{(6.1b)} \]
\[ \vartheta^\beta_R \to \vartheta^\beta \text{ weakly in } W^{1,r}(\Omega) \text{ and } \vartheta^\beta_R \to \vartheta^\beta \text{ in } L^s(\Omega) \text{ for any } \beta \in \mathbb{R}, \ 1 \leq r < 2, \ 1 \leq s < \infty, \quad \text{(6.1c)} \]
\[ \mathbf{u}_R \to \mathbf{u} \text{ weakly in } W^{1,r}(\Omega; \mathbb{R}^2) \text{ for any } 1 \leq r < 2, \quad \text{(6.1d)} \]
as \( R \to \infty \). Moreover, we assume that
\[ h'_R \left( \bar{\varrho} - \frac{1}{R} \right) \to \infty \text{ as } R \to \infty. \quad \text{(6.2)} \]

### 6.1 Pressure estimates

We derive bounds on the pressure \( p_R(\varrho_R, \vartheta_R) \), uniform for \( R \to \infty \). To this end, we introduce an inverse operator \( \mathcal{B} \) to \( \text{div} \) constructed by Bogovskii [2]. The following properties of \( \mathcal{B} \) are nowadays rather standard, we refer to the monograph by Galdi [9] for the proofs.

- The operator \( \mathcal{B} \) satisfies
  \[ \text{div}(\mathcal{B}[h]) = h \quad \text{and} \quad \mathcal{B}[h]|_{\partial \Omega} = 0 \text{ for any } h \in L^q(\Omega), \ 1 < q < \infty \quad \text{and} \quad \int_{\Omega} h \, dx = 0. \]

- \( \mathcal{B} \) can be extended to functions \( h \in L^q(\Omega), \)
  \[ \norm{\mathcal{B}[h]}_{W^{1,q}_0(\Omega; \mathbb{R}^2)} \leq c(q) \norm{h}_{L^q(\Omega)}. \quad \text{(6.3)} \]

- If \( h \in L^q(\Omega), \ 1 < q < \infty, \ h = \text{div} \mathbf{g}, \ \mathbf{g} \in L^r(\Omega; \mathbb{R}^2) \) such that \( \mathbf{g} \cdot \mathbf{n}|_{\partial \Omega} = 0 \), then
  \[ \norm{\mathcal{B}[h]}_{L^r(\Omega; \mathbb{R}^2)} \leq c(p, r) \norm{\mathbf{g}}_{L^p(\Omega; \mathbb{R}^2)}. \quad \text{(6.4)} \]

The first step is to consider
\[ \varphi = \mathcal{B}[\varrho - \varrho_M] \]
as a test function in the approximate momentum equation (5.3), we get
\[
\int_{\Omega} (p_R(\varrho_R, \vartheta_R) - p_R(\varrho_M, \vartheta_R))(\varrho_R - \varrho_M) \, dx + \int_{\Omega} p_R(\varrho_M, \vartheta_R) \varrho_R \, dx \\
= \int_{\Omega} \left[ \frac{1}{2} T(\varrho_R)(\mathbf{u}_R \cdot \nabla \mathbf{u}_R) \cdot \nabla \mathcal{B}[\varrho_R - \varrho_M] - \mathbf{u}_R \otimes \nabla \mathcal{B}[\varrho_R - \varrho_M] \right] \, dx \\
+ \int_{\Omega} \left[ S(\varrho_R, \nabla \mathbf{u}_R) : \nabla \mathcal{B}[\varrho_R - \varrho_M] - T(\varrho_R) \mathbf{f} \cdot \mathcal{B}[\varrho_R - \varrho_M] \right] \, dx + \int_{\Omega} p_R(\varrho_M, \vartheta_R) \varrho_M \, dx.
\]

Observe that
\[
(p_R(\varrho_R, \vartheta_R) - p_R(\varrho_M, \vartheta_R))(\varrho_R - \varrho_M) = \vartheta_R(\varrho_R h_R(\varrho) - \varrho_M h_R(\varrho_M))(\varrho_R - \varrho_M).
\]

In virtue of the construction of the truncation \( h_R \) specified in (3.8) and (6.2), we have
\[
\frac{\partial}{\partial \varrho} (\varrho h_R(\varrho)) = h_R(\varrho) + \varrho h'_R(\varrho) \sim \varrho \text{ for all } \varrho \geq 0,
\]
Notice that
\[ |\varrho R h_R(\varrho_R) - \varrho M h_R(\varrho_M)| \lesssim \int_{\varrho_M}^{\varrho_R} z \, dz \lesssim \frac{1}{2} (\varrho_R - \varrho_M)^2, \]
which implies that
\[ \varrho_R |\varrho_R - \varrho_M|^3 \lesssim (pR(\varrho_R, \varrho_R) - pR(\varrho_M, \varrho_R))(\varrho_R - \varrho_M). \]
Consequently, we may rewrite inequality (6.5) in the following form
\[ \frac{1}{2} \int_{\Omega} \left( \frac{1}{2} T(\varrho_R)(u_R \cdot \nabla u_R \cdot B[\varrho_R - \varrho_M] - u_R \otimes u_R : \nabla B[\varrho_R - \varrho_M]) \right) \, dx \]
\[ \lesssim \int_{\Omega} \left[ \frac{1}{2} T(\varrho_R)(u_R \cdot \nabla u_R \cdot B[\varrho_R - \varrho_M] - u_R \otimes u_R : \nabla B[\varrho_R - \varrho_M]) \right] \, dx \]
\[ + \int_{\Omega} \left[ \delta(\varrho_R, \nabla u_R) : \nabla B[p_R - \{p_R\}_m] - T(\varrho_R) f : B[p_R - \{p_R\}_m] \right] \, dx \]
\[ + \int_{\Omega} p_R(\varrho_M, \varrho_R) \varrho_M \, dx. \] (6.6)
Keeping in mind the uniform bounds in (6.1), it is easy to control the integrals on the right–hand side of (6.6) by those on the left–hand side. Indeed, in view of the properties of the operator $B$ listed in (6.3) and (6.4), we have
\[ \|B[\varrho_R - \varrho_M]\|_{L^4(\Omega; \mathbb{R}^2)} + \|
abla B[\varrho_R - \varrho_M]\|_{L^4(\Omega; \mathbb{R}^4)} \]
\[ \lesssim \|\varrho_R - \varrho_M\|_{L^4(\Omega)} = \|\varrho_R^{\frac{1}{4}} \varrho_R^{\frac{1}{4}}(\varrho_R - \varrho_M)\|_{L^4(\Omega)} \leq c(\delta, q) \|\varrho_R^{\frac{1}{4}}(\varrho_R - \varrho_M)\|_{L^4(\Omega)} \]
for any $1 \leq q < 3$.
Consequently, in view of the uniform bounds (6.1), we deduce from (6.6) that the integrals on the left–hand side are bounded uniformly for $R \to \infty$, which implies that
\[ \frac{1}{2} \|\varrho_R^{\frac{1}{3}}(\varrho_R - \varrho_M)\|_{L^4(\Omega)} + \int_{\Omega} p_R(\varrho_R, \varrho_R)(\varrho_R - \varrho_M) \, dx \leq c(\delta) + \int_{\Omega} p_R(\varrho_M, \varrho_R)(\varrho_R - \varrho_M). \]
From (6.1) we may deduce that $\int_{\Omega} p_R(\varrho_M, \varrho_M)(\varrho_R - \varrho_M) \, dx$ is controlled by $\|\varrho_R^{\frac{1}{3}}(\varrho_R - \varrho_M)\|_{L^4(\Omega)}^3$. By considering two cases: $\varrho_R \leq \varrho_M + \eta$ and $\varrho_R > \varrho_M + \eta$, respectively, with $\eta > 0$ small enough, we finally get
\[ \{p_R^{\gamma}\}_m \overset{\text{def}}{=} \frac{1}{|\Omega|} \int_{\Omega} p_R^{\gamma}(\varrho_R, \varrho_R) \, dx \leq c(\delta), \ 0 \leq \gamma \leq 1. \] (6.7)
The second step consists to repeat the same procedure with
\[ B[p_R^{\gamma} - \{p_R\}_m], \ 0 < \gamma < 1. \]
Similarly using (6.7), we deduce that
\[ \int_{\Omega} p_R^{\gamma+1} \, dx \leq \int_{\Omega} \left[ \frac{1}{2} T(\varrho_R)(u_R \cdot \nabla u_R \cdot B[p_R^{\gamma} - \{p_R\}_m] - u_R \otimes u_R : \nabla B[p_R^{\gamma} - \{p_R\}_m]) \right] \, dx \]
\[ + \int_{\Omega} \left[ \delta(\varrho_R, \nabla u_R) : \nabla B[p_R^{\gamma} - \{p_R\}_m] - T(\varrho_R) f : B[p_R^{\gamma} - \{p_R\}_m] \right] \, dx. \] (6.8)
Once again using the bounds (6.1) combined with the properties of the operator $B$, we may infer that all integrals on the right–hand side of (6.8) can be controlled, modulo a multiplicative constant, by the following norm
\[ \|p_R^{\gamma}\|_{L^q(\Omega)} \] as soon as $q > 2$.
Thus for $q = \frac{5+1}{2}$, we may conclude that
\[ \int_{\Omega} p^{\gamma+1}_R \, dx \leq c(\delta) \left( 1 + \|p_R\|_{L_{\gamma+1}(\Omega)} \right), \]
and, consequently, we get
\[ \int_{\Omega} p^{\gamma+1}_R (\varrho_R, \vartheta_R) \, dx \leq c(\delta, \gamma), \quad 0 < \gamma < 1 \text{ uniformly for } R \to \infty. \] (6.9)
Finally, writing
\[ \varrho_R h_R(\varrho_R) = \vartheta_R^{-1} p_R(\varrho_R, \vartheta_R), \]
we also obtain
\[ \int_{\Omega} |\varrho_R h_R(\varrho_R)|^\omega \, dx \leq c(\delta, \omega), \quad 0 < \omega < 2 \text{ uniformly for } R \to \infty. \] (6.10)

### 6.2 Convergence and the limit system

With (6.1), (6.9) and (6.10) at hand, it is standard to perform the limit for $R \to \infty$ in the system of approximate equations. Moreover, as the limit pressure is singular at $\bar{\varrho}$ (see hypothesis (2.3)), we deduce from (6.10) that
\[ 0 \leq \varrho < \bar{\varrho} \text{ a.e in } \Omega \quad \text{and} \quad \|p(\varrho, \vartheta)\|_{L^p(\Omega)} \leq c(\delta, \omega) \text{ for any } 1 \leq \omega < 2, \] (6.11)

\[ \text{cf. also \cite{4}.} \]

Accordingly, the limit system of equations reads as follows:

\[
\begin{align*}
\text{div}(\varrho \mathbf{u}) & = \varepsilon \Delta \varrho - \varepsilon (\varrho - \varrho_M), \quad \nabla \varrho \cdot \mathbf{n}_{\partial \Omega} = 0, \quad 0 \leq \varrho < \bar{\varrho}, \quad \frac{1}{|\Omega|} \int_{\Omega} \varrho \, dx = \varrho_M, \quad (6.12a) \\
\int_{\Omega} \left[ \frac{1}{2} \varrho (\mathbf{u} \otimes \mathbf{u} : \nabla \varphi - \mathbf{u} \cdot \nabla \varphi + p(\varrho, \vartheta) \text{div} \varphi \right] \, dx & = \int_{\Omega} \left[ \delta(\vartheta, \nabla \mathbf{u}) : \nabla \varphi - \varrho \mathbf{f} \cdot \nabla \varphi \right] \, dx \quad (6.12b)
\end{align*}
\]

for any $\varphi \in W^{1,q}_0(\Omega; \mathbb{R}^2)$, $q > 2$. Notice that $(\mathbf{u} - \bar{\mathbf{u}}) \in W^{1,r}_0(\Omega; \mathbb{R}^2)$, $1 \leq r < 2$ and

\[
\begin{align*}
\int_{\Omega} \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho \nu(\varrho, \vartheta) \mathbf{u} \cdot \nabla \varphi \, dx + \int_{\Omega} p(\varrho, \vartheta) \mathbf{u} \cdot \nabla \varphi \, dx + \int_{\Omega} \mathbf{q}_s \cdot \nabla \varphi \, dx \\
- \int_{\partial \Omega} \delta(\vartheta, \nabla \mathbf{u}) \cdot \mathbf{u} \cdot \nabla \varphi \, dx & = \int_{\partial \Omega} L(\vartheta - \bar{\vartheta}) \varphi \, dS - \int_{\Omega} \varrho \nu (G + \varepsilon) \varphi \, dx - \int_{\Omega} \varrho \varphi \mathbf{f} \cdot \mathbf{u} \, dx \\
+ \int_{\Omega} \varrho \mathbf{f} \cdot (\varphi \mathbf{u}) \, dx - \int_{\Omega} \delta(\vartheta, \nabla \mathbf{u}) : \nabla (\varphi \mathbf{u}) \, dx & \quad (6.13)
\end{align*}
\]

for any $\varphi \in W^{1,\infty}(\Omega)$; together with the entropy inequality

\[
\begin{align*}
\int_{\Omega} \frac{1}{\varrho} \left[ \delta(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \frac{\kappa \delta(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right] \, dx + \int_{\Omega} \varrho (G + \varepsilon) \varphi \, dx \\
\leq \int_{\partial \Omega} \varrho L \left( 1 - \frac{\bar{\vartheta}}{\vartheta} \right) \, dS - \int_{\Omega} \left( \frac{\delta(\vartheta, \nabla \vartheta)}{\vartheta} \right) : \nabla \varphi \, dx - \int_{\Omega} (p_s(\varrho, \vartheta) \mathbf{u}) \cdot \nabla \varphi \, dx \quad (6.14)
\end{align*}
\]

\[
+ \varepsilon \int_{\Omega} \varphi (\Delta \varrho - (\varrho - \varrho_M))(\varrho h(\varrho) - s(\varrho, \vartheta)) \, dx
\]
for any \( \varphi \in W^{1,\infty}(\Omega) \), \( \varphi \geq 0 \).

7 Limit \( \varepsilon \to 0 \)

The process \( \varepsilon \to 0 \) is crucial as it requires strong convergence of the approximate densities. We use the approach proposed by Lions in [13], based on the monotonicity of the pressure, combined with the Commutator Lemma, introduced in [6], to handle the temperature fluctuations of the viscosity coefficients. Keeping \( \delta > 0 \) fixed, we consider a family \( \{ \rho_\varepsilon, u_\varepsilon, \vartheta_\varepsilon \}_{\varepsilon > 0} \) of solutions of the approximate system (6.12). Given the available \( \delta \)-dependent estimates derived in the preceding part, passing to the subsequences, if necessary, we find

\[
\rho_\varepsilon \rightharpoonup \rho \text{ weakly-}^* \text{ in } L^\infty(\Omega) \text{ with } 0 \leq \rho \leq \bar{\rho}, \tag{7.1a}
\]

\[
\vartheta_\varepsilon \rightharpoonup \vartheta \text{ weakly in } W^{1,2}(\Omega), \tag{7.1b}
\]

\[
\vartheta_\varepsilon^\beta \rightharpoonup \vartheta^\beta \text{ weakly in } W^{1,r}(\Omega) \text{ and } \vartheta_\varepsilon^\beta \to \vartheta^\beta \text{ in } L^s(\Omega) \text{ for any } \beta \in \mathbb{R}, 1 \leq r < 2, 1 \leq s < \infty, \tag{7.1c}
\]

\[
u_\varepsilon \rightharpoonup \nu \text{ weakly in } W^{1,r}(\Omega; \mathbb{R}^2) \text{ for any } 1 \leq r < 2, \tag{7.1d}
\]
as \( \varepsilon \to 0 \). Moreover, as a consequence of (6.11), we have

\[
p_\varepsilon = p(\rho_\varepsilon, \vartheta_\varepsilon) \to p(\rho, \vartheta) \text{ weakly in } L^\omega(\Omega) \text{ for any } 1 \leq \omega < 2. \tag{7.2}
\]

### 7.1 Strong convergence of approximate densities

Our goal is to show, up to a suitable subsequence,

\[
\rho_\varepsilon \to \rho \text{ a.e in } \Omega. \tag{7.3}
\]

The proof is based on monotonicity of the pressure in the density variable, cf. hypothesis (2.3). Similarly to [4], we show that

\[
p(\rho, \vartheta)\rho = \bar{p}(\rho, \vartheta)\rho \text{ a.e in } \Omega, \tag{7.4}
\]

where the bar is used to denote a weak limit of the corresponding composition. In view of the strong convergence of the temperature in (7.1), relation (7.4) gives rise to

\[
\vartheta \cdot \rho h(\rho)\rho = \vartheta \cdot h(\rho)\rho \rho, \tag{7.5}
\]

but since \( \vartheta > 0 \) almost everywhere in \( \Omega \), this yields

\[
\rho h(\rho)\rho = \bar{h}(\rho)\rho \rho \text{ a.e in } \Omega. \tag{7.6}
\]

The function \( \rho \mapsto \rho h(\rho) \) being (strictly) increasing, cf. (2.3), this implies (7.3), exactly as in [4].

Following the approach of Lions [13], we derive (7.4) from the effective viscous flux identity. To this end, we first perform the limit in the momentum equation (6.12b):

\[
\int_\Omega \left[ \frac{1}{2} \delta \varphi \cdot \nabla \varphi - \varphi \cdot \nabla \varphi \cdot \varphi + \bar{p}(\rho, \vartheta) \text{div } \varphi \right] dx = \int_\Omega \left[ \mathcal{S}_\delta(\vartheta, \nabla \varphi) : \nabla \varphi - \varphi f_\varepsilon \cdot \varphi \right] dx \tag{7.5}
\]
for any $\varphi \in W^{1,q}_0(\Omega;\mathbb{R}^2)$, $q > 2$, $(u - \bar{u}) \in W^{1,r}_0(\Omega;\mathbb{R}^2)$, $1 \leq r < 2$. Note that $\mathbb{S}_\delta(\vartheta, \nabla u) = \mathbb{S}_\delta(\vartheta, \nabla u)$ thanks to the strong convergence of the approximate temperatures.

Now, we repeat the same process with the test function

$$\varphi = \phi \nabla \Delta^{-1}[\phi \theta_\varepsilon]$$

where $\phi \in C^\infty_c(\Omega)$, $0 \leq \phi \leq 1$, and $\Delta^{-1}$ is the inverse of the Laplace operator defined by means of the Green function on $\mathbb{R}^2$. Plugging $\varphi$ in (6.12b), performing the limit and regrouping terms in the limit expression, we find

$$\int_{\Omega} \left[ \frac{1}{2} (\rho u \otimes u : \nabla (\phi \nabla \Delta^{-1}[\phi \vartheta]) - \frac{\rho u \cdot \nabla u}{\rho} \cdot \nabla \Delta^{-1}[\phi \vartheta] + \phi f \cdot \nabla \Delta^{-1}[\phi \vartheta]) \right] \, dx
$$

$$= \int_{\Omega} \left[ \mathbb{S}_\delta(\vartheta, \nabla u) : \nabla (\phi \nabla \Delta^{-1}[\phi \vartheta]) - p(\rho, \vartheta) \text{div}(\phi \nabla \Delta^{-1}[\phi \vartheta]) \right] \, dx. \quad (7.6)$$

Note that, thanks to the regularizing properties of the operator $\Delta^{-1}$, we have

$$\phi \nabla \Delta^{-1}[\phi \theta_\varepsilon] \to \phi \nabla \Delta^{-1}[\phi \vartheta]$$

(strongly) in $C^0(\bar{\Omega})$. (7.7)

Finally, we use the quantity

$$\varphi = \phi \nabla \Delta^{-1}[\phi \vartheta]$$

as a test function in the limit equation (7.5), we get

$$\int_{\Omega} \left[ \frac{1}{2} (\rho u \otimes u : \nabla (\phi \nabla \Delta^{-1}[\phi \vartheta]) - \frac{\rho u \cdot \nabla u}{\rho} \cdot \nabla \Delta^{-1}[\phi \vartheta] + \phi f \cdot \nabla \Delta^{-1}[\phi \vartheta]) \right] \, dx
$$

$$= \int_{\Omega} \left[ \mathbb{S}_\delta(\vartheta, \nabla u) : \nabla (\phi \nabla \Delta^{-1}[\phi \vartheta]) - p(\rho, \vartheta) \text{div}(\phi \nabla \Delta^{-1}[\phi \vartheta]) \right] \, dx. \quad (7.8)$$

Now, we compare the terms on the right–hand sides of (7.6), (7.8). As the velocity converges strongly, we have

$$\overline{\rho u \otimes u : \nabla (\phi \nabla \Delta^{-1}[\phi \vartheta])} = u \cdot \overline{\rho u \cdot \nabla (\phi \nabla \Delta^{-1}[\phi \vartheta])}$$

$$\overline{\rho u \otimes u : \nabla (\phi \nabla \Delta^{-1}[\phi \vartheta])} = u \cdot \overline{\rho u \cdot \nabla (\phi \nabla \Delta^{-1}[\phi \vartheta])}.$$

Next, we observe, exactly as in [4] that

$$\overline{\rho u \cdot \nabla (\phi \nabla \Delta^{-1}[\phi \vartheta])} = \overline{\rho u \cdot \nabla (\phi \nabla \Delta^{-1}[\phi \vartheta])}.$$ 

(7.9)

To this end, we use Div–Curl Lemma (see Tartar [19]), we get

$$\text{curl} \nabla (\phi \nabla \Delta^{-1}[\phi \theta_\varepsilon]) = 0,$$

and

$$\text{div}(\theta_\varepsilon u_\varepsilon) = \varepsilon \Delta \theta_\varepsilon - \varepsilon (\theta_\varepsilon - \theta_M).$$
Now, we take $\varphi = 1$ in the entropy inequality \((6.14)\), we find
\[
\int_\Omega \frac{1}{\vartheta} \left[ \mathcal{S}_\delta(\vartheta, \nabla \mathbf{u}_\varepsilon) : \nabla \mathbf{u}_\varepsilon + \frac{\kappa_\delta(\vartheta)}{\vartheta} |\nabla \vartheta_\varepsilon|^2 \right] dx + \int_\Omega \frac{\vartheta}{\vartheta}(G + \varepsilon) dx \\
+ \varepsilon \int_\Omega \left( h(\vartheta_\varepsilon) + \vartheta_\varepsilon h'(\vartheta_\varepsilon) + \frac{h(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right) |\nabla \vartheta_\varepsilon|^2 dx \\
+ \varepsilon \int_\Omega (\vartheta_\varepsilon - \vartheta M) \left( \vartheta_\varepsilon h(\vartheta_\varepsilon) - \vartheta M h(\vartheta M) + \int_{\partial M}^\varepsilon \frac{h(z)}{z} dz \right) dx \\
\leq \int_{\partial \Omega} L \left( 1 - \frac{\vartheta}{\vartheta_\varepsilon} \right) dS + -\varepsilon c_v \int_\Omega \nabla \vartheta_\varepsilon \cdot \nabla \log(\vartheta_\varepsilon) dx + \varepsilon \int_\Omega (\vartheta_\varepsilon - \vartheta M) c_v \log(\vartheta_\varepsilon) dx.
\]
In particular, we deduce
\[(7.10)\]

We may deduce from \((6.13)\) and \((7.11)\) that div$(\vartheta_\varepsilon \mathbf{u}_\varepsilon) \to 0$ in $W^{-1,2}(\Omega)$ then div$(\vartheta_\varepsilon \mathbf{u}_\varepsilon)$ belongs to a compact set in $W^{-1,2}(\Omega)$. Thus relation \((7.9)\) follows directly from Div–Curl Lemma.

Comparing \((7.6)\) and \((7.8)\), we obtain
\[
\int_\Omega \left[ \mathcal{S}_\delta(\vartheta, \nabla \mathbf{u}) : \nabla \left( \phi \nabla \Delta^{-1}[\varphi \vartheta] \right) - p(\vartheta, \vartheta) \text{div}(\phi \nabla \Delta^{-1}[\varphi \vartheta]) \right] dx \\
= \int_\Omega \left[ \mathcal{S}_\delta(\vartheta, \nabla \mathbf{u}) : \nabla \left( \phi \nabla \Delta^{-1}[\varphi \vartheta] \right) - p(\vartheta, \vartheta) \text{div}(\phi \nabla \Delta^{-1}[\varphi \vartheta]) \right] dx
\]
that can be simplified via \((7.7)\) to
\[(7.12)\]

Our plan consists in replacing $\nabla \Delta^{-1} \nabla : (\phi \mathcal{S}_\delta(\vartheta, \nabla \mathbf{u}))$ by $(\mu_\delta(\vartheta) + g(\vartheta)) \text{div} \mathbf{u}$ in the identity \((7.12)\) where $g$ is a polynomial increasing function. To this end, write
\[
\nabla \Delta^{-1} \nabla : (\phi \mathcal{S}_\delta(\vartheta, \nabla \mathbf{u})) = (\nabla \Delta^{-1} \nabla : (\phi \mathcal{S}_\delta(\vartheta, \nabla \mathbf{u})) - (\mu_\delta(\vartheta) + g(\vartheta)) \text{div} \mathbf{u}) + (\mu_\delta(\vartheta) + g(\vartheta)) \text{div} \mathbf{u}.
\]
The expression in the curly brackets is a commutator of the pseudo–differential operator $\nabla \Delta^{-1} \nabla$ with multiplication by a function of $\vartheta$. It enjoys extra compactness properties exploited in [6]. We report the following result that can be see as a version of the abstract results of Coifman and Meyer [5]:

**Lemma 7.1 (Commutator Lemma).** Let $w \in W^{1,r}(\mathbb{R}^N)$ and $\mathbf{V} \in L^q(\mathbb{R}^N; \mathbb{R}^N)$ be given fields,
\[
1 < r < N, \quad 1 < q < \infty, \quad \frac{1}{r} + \frac{1}{q} < 1 + \frac{1}{N}.
\]
Then for any $s$ satisfying
\[
\frac{1}{r} + \frac{1}{q} - \frac{1}{N} < \frac{1}{s} < 1,
\]
there exists $\beta \in (0,1)$ such that
\[
\|\nabla \Delta^{-1} \nabla : [w \mathbf{V}] - w \nabla \Delta^{-1} \nabla \cdot [\mathbf{V}]\|_{W^{3,s}(\mathbb{R}^N; \mathbb{R}^N)} \lesssim \|w\|_{W^{1,r}(\mathbb{R}^N)} \|\mathbf{V}\|_{L^q(\mathbb{R}^N; \mathbb{R}^N)}.
\]
The Navier–Stokes–Fourier system with impermeable boundary

We apply Lemma 7.1 to
\[ w = \phi \mu_{\delta} (\vartheta \epsilon), \quad \mathbf{V} = \nabla u_{\epsilon}, \quad i = 1, 2, \quad N = 2, \quad r < 2, \quad q < 2, \]
and we deduce the strong convergence of the commutator in L2-norm. Accordingly, we may deduce from (7.12) the desired relation:
\[
\int_{\Omega} \left[ \phi^2 \rho_{\delta}(\vartheta) \vartheta - \rho(\vartheta) \vartheta \right] d\mathbf{x} = \int_{\Omega} \phi^2 \left( \mu_{\delta}(\vartheta) + g(\vartheta)) (\rho_{\text{div}} \mathbf{u} - \rho \text{div} \mathbf{u}) \right] d\mathbf{x}. \tag{7.13}
\]
Relation (7.13) is called Lions’ identity. One can deduce (7.14), and, consequently, the strong convergence of the approximate densities from (7.13). The details of this procedure are detailed in [4].

7.2 Convergence and the limit system

Once strong convergence of the densities has been established, it is straightforward to pass to the limit in the approximate equations. Note that \( \sqrt{\varepsilon} \nabla \varphi \) is bounded in the L2-norm uniformly for \( \varepsilon \to 0 \). Consequently, letting \( \varepsilon \to 0 \) in (7.12)–(7.14), we obtain
\[
\int_{\Omega} \varrho \mathbf{u} \cdot \nabla \varphi d\mathbf{x} = 0 \quad \text{for any } \varphi \in W^{1,\infty}(\Omega), \quad 0 \leq \varrho < \bar{\varrho}, \quad \frac{1}{|\Omega|} \int_{\Omega} \varrho d\mathbf{x} = \varrho_M, \tag{7.14a}
\]
\[
\int_{\Omega} \left[ \frac{1}{2} \varrho (\mathbf{u} \otimes \mathbf{u} : \nabla \varphi - \varrho \cdot \nabla \mathbf{u} \cdot \varphi) + p(\vartheta, \vartheta) \text{div} \varphi \right] d\mathbf{x} = \int_{\Omega} \left[ \mathcal{S}_{\delta}(\vartheta, \nabla \mathbf{u}) : \nabla \varphi - \varrho \mathbf{f} \cdot \varphi \right] d\mathbf{x} \tag{7.14b}
\]
for any \( \varphi \in W^{1,q}_{0}(\Omega; \mathbb{R}^2) \), \( q > 2 \). Since \( (\mathbf{u} - \bar{\mathbf{u}}) \in W^{1,r}_{0}(\Omega; \mathbb{R}^2) \), \( 1 \leq r < 2 \), we have
\[
\int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\vartheta, \vartheta) \right) \mathbf{u} \cdot \nabla \varphi d\mathbf{x} + \int_{\Omega} p(\vartheta, \vartheta) \mathbf{u} \cdot \nabla \varphi d\mathbf{x} + \int_{\Omega} q_{\delta} \cdot \nabla \varphi d\mathbf{x} - \int_{\Omega} \mathcal{S}_{\delta}(\vartheta, \nabla \mathbf{u}) \cdot \mathbf{u} \cdot \nabla \varphi d\mathbf{x} = \int_{\Omega} L(\vartheta - \bar{\vartheta}) \varphi d\mathbf{x} - \int_{\partial \Omega} \varrho G \varphi dS - \int_{\Omega} \varrho \mathbf{f} \cdot \mathbf{u} d\mathbf{x} \tag{7.15}
\]
for any \( \varphi \in W^{1,\infty}(\Omega) \); and the entropy inequality
\[
\int_{\Omega} \varphi \int_{\partial \Omega} \left[ \mathcal{S}_{\delta}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \frac{\kappa_{\delta}(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right] d\mathbf{x} + \int_{\Omega} \varrho \varphi G \varphi d\mathbf{x} \leq \int_{\partial \Omega} \varphi L \left( 1 - \frac{\vartheta}{\bar{\vartheta}} \right) dS - \int_{\Omega} \left( \frac{q_{\delta}(\vartheta, \nabla \vartheta)}{\vartheta} \right) \cdot \nabla \varphi d\mathbf{x} - \int_{\Omega} (g_{\delta}(\vartheta, \vartheta)) \cdot \nabla \varphi d\mathbf{x}. \tag{7.16}
\]
for any \( \varphi \in W^{1,\infty}(\Omega) \), \( \varphi \geq 0 \).

8 Limit \( \delta \to 0 \)

Our ultimate goal is to perform the limit \( \delta \to 0 \) recovering the weak formulation of the original problem. This can be done in a similar way as in the preceding section, however, we must establish the necessary uniform bound independent of \( \delta \). As the bounds based on the entropy inequality (7.15) hold uniformly for \( \delta \to 0 \), we must only establish the bounds on the temperature similar to those obtained in Section 7.2.
8.1 Uniform bounds

Let \( \{ \vartheta_\delta, u_\delta, \delta \} \) be a sequence of approximate solutions solving (7.14)–(7.16). Taking \( \varphi = 1 \) as a test function in the total energy balance (7.15), similarly to Section 4.2, we obtain

\[
\int_{\partial \Omega} \vartheta_\delta \, dS \leq c_D + \int_{\Omega} \varrho_\delta f \cdot u_\delta \, dx - \int_{\Omega} \frac{1}{2} \varrho_\delta ((u_\delta \otimes u_\delta) : \nabla \bar{u} - u_\delta \cdot \nabla \bar{u}) \, dx \\
- \int_{\Omega} \varrho_\delta f \cdot \bar{u} \, dx + \int_{\Omega} \mathcal{S}_\delta(\vartheta_\delta, \nabla u_\delta) : \nabla \bar{u} \, dx.
\]

Moreover, the equation of continuity (7.14) can be used to rewrite the convective term, we get

\[
\int_{\partial \Omega} \vartheta_\delta \, dS \leq c_D + \int_{\Omega} \varrho_\delta f \cdot (u_\delta - \bar{u}) \, dx + \int_{\Omega} \varrho_\delta u_\delta \cdot \nabla u_\delta \cdot \bar{u} \, dx + \int_{\Omega} \mathcal{S}_\delta(\vartheta_\delta, \nabla u_\delta) : \nabla \bar{u} \, dx. \tag{8.1}
\]

Next, taking \( \varphi = 1 \) in the entropy inequality (7.16), we find

\[
\int_{\partial \Omega} \frac{1}{\vartheta_\delta} \left[ \mathcal{S}_\delta(\vartheta_\delta, \nabla u_\delta) : \nabla \bar{u} + \frac{\kappa_\delta(\vartheta_\delta)}{\vartheta_\delta} |\nabla \vartheta_\delta|^2 \right] \, dx + \int_{\Omega} \varrho_\delta G \, dx \leq \int_{\partial \Omega} L \left( 1 - \frac{\varphi}{\vartheta_\delta} \right) \, dS. \tag{8.2}
\]

Our goal, similarly to Section 4.2, is to control all integrals on the right-hand side of (8.1) by means of a suitable norm of \( \vartheta_\delta \). First observe that, by virtue of (8.10) and (8.11), we obtain

\[
\left| \int_{\Omega} \varrho_\delta f \cdot (u_\delta - \bar{u}) \, dx \right| \lesssim \| D u_\delta - D \bar{u} \|_{L^r(\Omega; \mathbb{R}^4)} \leq \| D u_\delta \|_{L^r(\Omega; \mathbb{R}^4)} + \| D \bar{u} \|_{L^r(\Omega; \mathbb{R}^4)}
\]

\[
= \| \vartheta_\delta^{\frac{1}{r}} \varrho_\delta^{\frac{1}{2}} D u_\delta \|_{L^r(\Omega; \mathbb{R}^4)} + \| D \bar{u} \|_{L^r(\Omega; \mathbb{R}^4)} \lesssim \| \vartheta_\delta^{\frac{1}{r}} \|_{L^q(\Omega)} + \| D \bar{u} \|_{L^r(\Omega; \mathbb{R}^4)} \tag{8.3}
\]

for some \( 1 < r < 2, s \geq 1 \). Note that according to the entropy estimates (8.2), the norm \( \vartheta_\delta^{\frac{1}{r}} D u_\delta \) is bounded in the \( L^2 \)-norm. Next, we handle the integral

\[
\left| \int_{\Omega} \mathcal{S}_\delta(\vartheta_\delta, \nabla u_\delta) : \nabla \bar{u} \, dx \right| \lesssim \| \nabla \bar{u} \|_{L^q(\Omega; \mathbb{R}^4)} \| (1 + \delta \vartheta_\delta) D u_\delta \|_{L^r(\Omega)} \cdot \frac{1}{q} + \frac{1}{r} = 1,
\]

where we focus on the case \( \alpha = 0 \) in (1.8) as otherwise the estimates would be the same as in Section 4.2. In view of the entropy estimates (8.2), we have

\[
\| \vartheta_\delta^{\frac{1}{r}} D u_\delta \|_{L^2(\Omega; \mathbb{R}^4)} + \| \vartheta_\delta^{\frac{n+1}{2}} D u_\delta \|_{L^2(\Omega; \mathbb{R}^4)} \leq c_D. \tag{8.4}
\]

Consequently, by interpolation, we get

\[
\| (1 + \delta \vartheta_\delta^2) D u_\delta \|_{L^r(\Omega)} \lesssim \| D u_\delta \|_{L^r(\Omega; \mathbb{R}^4)} + \| \delta \vartheta_\delta^2 D u_\delta \|_{L^r(\Omega; \mathbb{R}^4)}
\]

\[
= \| \vartheta_\delta^{\frac{2}{r}} \vartheta_\delta^{\frac{1}{2}} D u_\delta \|_{L^r(\Omega; \mathbb{R}^4)} + \| \vartheta_\delta^{\frac{n+1}{2}} \vartheta_\delta^{\frac{n+1}{4}} D u_\delta \|_{L^r(\Omega; \mathbb{R}^4)} \lesssim \| \vartheta_\delta^{\frac{1}{r}} \|_{L^q(\Omega)} + \| \vartheta_\delta^{\frac{n+1}{2}} \|_{L^r(\Omega)}
\]

for some \( s \geq 1 \) as soon as \( 1 \leq r < 2 \). Thus we may infer that

\[
\left| \int_{\Omega} \mathcal{S}_\delta(\vartheta_\delta, \nabla u_\delta) : \nabla \bar{u} \, dx \right| \lesssim \| \nabla \bar{u} \|_{L^q(\Omega; \mathbb{R}^4)} \left( \| \vartheta_\delta^{\frac{1}{r}} \|_{L^q(\Omega)} + \| \vartheta_\delta^{\frac{n+1}{2}} \|_{L^r(\Omega)} \right) \text{ for some } s \geq 1 \text{ if } q > 2. \tag{8.5}
\]
Finally, we have to estimate the integral
\[ \left| \int_{\Omega} g_{\delta} u_{\delta} \cdot \nabla u_{\delta} \cdot \bar{u} \, dx \right| \lesssim \| u \|_{L^q(\Omega; \mathbb{R}^2)} \| u_{\delta} \cdot \nabla u_{\delta} \|_{L^r(\Omega; \mathbb{R}^2)}, \quad \frac{1}{q} + \frac{1}{r} = 1. \]

Furthermore, we may notice that
\[ \| u_{\delta} \cdot \nabla u_{\delta} \|_{L^r(\Omega; \mathbb{R}^2)} \lesssim \left( \| (u_{\delta} - \bar{u}) \cdot \nabla (u_{\delta} - \bar{u}) \|_{L^r(\Omega; \mathbb{R}^2)} + \| \bar{u} \|_{W^{1,2r}(\Omega; \mathbb{R}^2)}^2 + c(\delta) \right). \]

Next, we use (8.4) and proceed exactly as in Section 4.2 to conclude
\[ \left| \int_{\partial \Omega} \vartheta_{\delta} \, dS \right| \lesssim \| \vartheta_{\delta} \|_{L^s(\Omega)} + \| \bar{u} \|_{W^{1,q}(\Omega; \mathbb{R}^2)}^2 \text{ for some } q, s \geq 1. \] (8.6)

Summing up (8.1), (8.3), (8.5) and (8.6) to get
\[ \int_{\partial \Omega} \vartheta_{\delta} \, dS \lesssim 1 + \| \vartheta_{\delta} \|_{L^s(\Omega)}^2 + \| \bar{u} \|_{W^{1,q}(\Omega; \mathbb{R}^4)}^4 \]
\[ + \| \bar{u} \|_{W^{1,q}(\Omega; \mathbb{R}^4)}^2 \left( \| \vartheta_{\delta} \|_{L^s(\Omega)}^2 + \sqrt{\delta} \| \vartheta_{\delta} \|_{L^s(\Omega)} \right) + \| \bar{u} \|_{L^q(\Omega; \mathbb{R}^2)} \| \vartheta_{\delta} \|_{L^s(\Omega)} \] (8.7)

for some finite \( s, q \geq 1 \).

At this stage, we need the following extension lemma proved in [4, Lemma A1].

**Lemma 8.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded Lipschitz domain. Let \( \bar{u} \in W^{1,p}(\Omega; \mathbb{R}^2), 1 < p < \infty \), be given such that \( \bar{u} \cdot \mathbf{n}_{\partial \Omega} = 0 \). Let \( q \) be given such that \( 1 < q < \frac{2p}{2-p} \), if \( p < 2 \) and \( q > 1 \) arbitrary finite otherwise. Then for any \( \omega > 0 \), there exists \( \bar{u}_{\omega} \in W^{1,p}(\Omega; \mathbb{R}^N) \) with the following properties:

- \( \bar{u}_{\omega} = \bar{u} \) on \( \partial \Omega \) in the sense of traces,
- \( \text{div} \bar{u}_{\omega} = 0 \) in \( \Omega \),
- \( \| \bar{u}_{\omega} \|_{L^q(\Omega; \mathbb{R}^2)} < \omega \),
- \( \| \bar{u}_{\omega} \|_{W^{1,p}(\Omega; \mathbb{R}^2)} \leq c(\omega, p, q) \| \bar{u} \|_{W^{1,p}(\Omega; \mathbb{R}^2)} \).

The idea is to replace \( \bar{u} \) by \( \bar{u}_{\omega} \) in the energy balance (7.15), and, subsequently in (8.7), to make the coefficient \( \| \bar{u}_{\omega} \|_{L^q(\Omega; \mathbb{R}^2)} \) multiplying the highest power of the norm of \( \vartheta_{\delta} \) small enough. Then the uniform bound on \( \vartheta_{\delta} \) is obtained from (8.2) and (8.7) via a compactness argument. To carry out this program, some preliminaries are necessary. The first may be seen as a direct consequence of the Sobolev embedding \( W^{1,2} \hookrightarrow L^6 \) already used in Section 4.2.

**Lemma 8.2.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded Lipschitz domain. There exists a function
\[ \chi \overset{\text{def}}{=} \chi(\Lambda_1, \Lambda_2, s) : [0, \infty)^2 \times [1, \infty) \to \mathbb{R} \]

with the following property: If \( \psi > 0 \) a.e in \( \Omega \) and there exist \( \Lambda_1, \Lambda_2 \geq 0 \) such that
\[ \| \nabla \log(\psi) \|_{L^2(\Omega)} \leq \Lambda_1 \quad \text{and} \quad \int_{\partial \Omega} \psi \, dS \leq \Lambda_2, \]
then
\[ \| \psi \|_{L^s(\Omega)} \leq \chi(\Lambda_1, \Lambda_2, s). \]
Next, we show the following:

**Lemma 8.3.** Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Let $\Lambda_1 \geq 0$, $Z \geq 0$, $s \geq 1$, $\beta \in (0,1)$, and $\omega > 0$ be such that

$$\omega \chi(\Lambda_1,1,s) < 1,$$

where $\chi$ is the function identified in Lemma 8.2. Then there exists $C \overset{\text{def}}{=} C(\Lambda_1,Z,s,\beta,\omega)$ such that

$$\int_{\partial \Omega} \psi \, dS \leq C$$

for any $\psi$, $\psi > 0$ almost everywhere in $\Omega$ satisfying

$$\|\nabla \log(\psi)\|_{L^2(\Omega)} \leq \Lambda_1 \text{ and } \int_{\partial \Omega} \psi \, dS \leq Z \left(1 + \|\psi\|_{L^s(\Omega)}^\beta\right) + \omega \|\psi\|_{L^s(\Omega)}.$$

**Proof.** Arguing by contradiction, we suppose that there is a sequence $\{\psi_n\}_{n=1}^\infty$ such that

$$\psi_n > 0 \text{ a.e in } \Omega,$$

$$\|\log(\psi_n)\|_{L^2(\Omega)} \leq \Lambda_1,$$  

$$\int_{\partial \Omega} \psi_n \, dS \leq Z \left(1 + \|\psi_n\|_{L^s(\Omega)}^\beta\right) + \omega \|\psi_n\|_{L^s(\Omega)},$$

$$b_n \overset{\text{def}}{=} \int_{\partial \Omega} \psi_n \, dS \to \infty \text{ as } n \to \infty.$$  

Consider the normalized sequence

$$\xi_n \overset{\text{def}}{=} \frac{\psi_n}{\int_{\partial \Omega} \psi_n \, dS}.$$

We have

$$\int_{\partial \Omega} \xi_n \, dS = 1 \text{ and } \|\nabla \log(\xi_n)\|_{L^2(\Omega)} = \|\nabla \log(\psi_n)\|_{L^2(\Omega)} \leq \Lambda_1.$$

It follows from Lemma 8.2 that

$$\|\xi_n\|_{L^s(\Omega)} \leq \chi(\Lambda_1,1,s).$$

Dividing (8.8c) on $b_n = \int_{\partial \Omega} \psi_n \, dS$, we obtain

$$1 = \int_{\partial \Omega} \xi_n \, dS \leq Z \left(\frac{1}{b_n} + \frac{1}{b_n^{1-\beta}} \|\xi_n\|_{L^s(\Omega)}^\beta\right) + \omega \|\xi_n\|_{L^s(\Omega)}$$

$$\leq Z \left(\frac{1}{b_n} + \frac{1}{b_n^{1-\beta}} \chi^\beta(\Lambda_1,1,s)\right) + \omega \chi(\Lambda_1,1,s) \to \omega \chi(\Lambda_1,1,s) < 1 \text{ as } n \to \infty,$$

which is a contradiction. 

We apply Lemmas 8.2 and 8.3 to $\psi = \vartheta_{\delta_1}$, $\Lambda_1$ determined by means of the entropy estimates (8.2), and $s$, $\beta$, $Z$ as in (8.7). In accordance with Lemma 8.1, we fix $\bar{u} = \bar{u}_\omega$ so that

$$c_D \|u_\omega\|_{L^q(\Omega;\mathbb{R}^2)} < \omega.$$
in (8.7). In accordance with Lemma 8.3, we conclude that
\[ \int_{\Omega} \vartheta \delta \, dx \leq c_D \text{ uniformly for } \delta \to 0, \]
which implies that
\[ \| \log(\vartheta) \|_{W^{1,2}(\Omega)} \leq c_D. \] (8.9)

### 8.2 Convergence

At this stage, the same machinery used in Section 4.2 allows us to conclude that
\[ \| \vartheta \beta \delta \|_{L^s(\Omega)} \leq c(\beta, s) \text{ for any } \beta \in \mathbb{R}, 1 \leq s < \infty, \]
\[ \| \nabla \vartheta \delta \|_{L^r(\Omega)} + \| \nabla \log(\vartheta) \|_{L^2(\Omega)} \leq c(r) \text{ for any } 1 \leq r < 2, \]
\[ \| u_\delta \|_{W^{1,r}(\Omega; \mathbb{R}^2)} \leq c(r) \text{ for any } 1 \leq r < 2. \] (8.10)

The uniform bounds (8.10), together with
\[ 0 \leq \varrho_\delta < \bar{\varrho} \text{ a.e in } \Omega \]
are strong enough to perform the limit passage in the equations by using the same arguments as in Section 7. We have completed the proof of Theorem 2.1.

### 9 Concluding remarks

We have considered the EOS of the form
\[ p(\varrho, \vartheta) \overset{\text{def}}{=} g \varrho h(\varrho) \quad \text{and} \quad e(\varrho, \vartheta) \overset{\text{def}}{=} c_v \vartheta. \]

In view of the fact that the density is a priori bounded and the rather strong estimates on the temperature, the result may be extended to more general pressure law including finite "virial series perturbation" of the form
\[ p(\varrho, \vartheta) \overset{\text{def}}{=} g \varrho h(\varrho) + \sum_{m=1}^{M} b_m(\vartheta) \varrho^\alpha_m, \quad \alpha_m \geq 0, \quad 0 \leq b_m(\vartheta) \sim \varrho^{-\beta} + \varrho^\beta, \quad \beta \geq 0. \]

Monotonicity of the pressure with respect to the density plays a crucial for stationary problems therefore the method cannot be adapted to pressure laws that are non–monotone with respect to the density.

The asymptotic behavior of the transport coefficients could be possibly relaxed to
\[ \mu(\vartheta) \approx (1 + \vartheta^{\alpha_1}), \quad \lambda(\vartheta) \approx (1 + \vartheta^{\alpha_2}), \quad \kappa(1 + \vartheta^{\alpha_3}), \quad 0 \leq \alpha_i < 1. \]

In view of the estimates in Section 7 however, the sublinear growth seems essential.

The proof depends heavily on the estimates (11.9) pertinent to planar domains. Extension to the 3-D case would be definitely limited by the available a priori bounds on the temperature and possibly require stronger hypothesis imposed on both the EOS and the transport coefficients.
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