On the Complexity of Connected \((s, t)\)-Vertex Separator

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Abstract. We investigate the complexity of minimum connected \((s, t)\)-vertex separator \((s,t)\)-CVS and show that \((s, t)\)-CVS is \(\Omega(\log^{2+\epsilon} n)\)-hard for any \(\epsilon > 0\) unless NP has quasi-polynomial Las-Vegas algorithms, i.e., for any \(\epsilon > 0\) and for some \(\delta > 0\), \((s, t)\)-CVS is unlikely to have \(\delta \log^{2+\epsilon} n\)-approximation algorithm. We then present an interesting chordality dichotomy: we show that \((s, t)\)-CVS is NP-complete on graphs of chordality at least 5 and present a polynomial-time algorithm for \((s, t)\)-CVS on chordality 4 graphs. We also present a \(\lceil c/2 \rceil\)-approximation algorithm for \((s, t)\)-CVS on graphs with chordality \(c\). Finally, from the parameterized setting, we show that \((s, t)\)-CVS parameterized above the \((s, t)\)-vertex connectivity is \(W[2]\)-hard.

1 Introduction

The vertex or edge connectivity of a graph and the corresponding separators are of fundamental interest in Computer Science and Graph Theory. Many kinds of vertex separators, stable vertex separators [1], clique vertex separators [2], and \(\alpha\)-balanced separators [3] are of interest to the research community. As far as complexity results are concerned, finding a minimum vertex separator and a clique vertex separator are polynomial-time solvable, whereas, stable vertex separator and other constrained separators reported in [3] are NP-hard. This shows that imposing an appropriate constraint on the well-studied vertex separator problem makes the problem NP-hard. Interestingly, constrained vertex separators have received much attention in parameterized complexity as well [3]. In particular, Marx in [3] considered the parameterized complexity of constrained separators satisfying some hereditary properties. For example, clique separators and stable separators. It is shown in [3] that the above problems have an algorithm whose running time is \(f(k)n^{O(1)}\), where \(k\) is the size of a constrained separator. Algorithms of this nature are popularly known as fixed-parameter tractable algorithms with parameter as the solution size [5]. While many constrained vertex separators have attracted researchers from both classical and parameterized complexity, the related problem of finding a minimum connected \((s, t)\)-vertex separator is open. Light of [3], this question can also be looked at as finding a \((s, t)\)-vertex separator satisfying some non-hereditary property, like connectedness. Moreover, the results in [3] do not carry over to connected \((s, t)\)-vertex separator and the complexity of it remains open. With these motivations, in this paper, we focus our attention on the computational complexity of minimum connected \((s, t)\)-vertex separator \((s,t)\)-CVS.

Remark: The \((s, t)\)-CVS can also be motivated from the theory of graph minors. We observe that there is an equivalence between the computational problems of finding a minimum connected \((s, t)\)-vertex separator and a minimum set of edges whose contraction reduces the \((s, t)\)-vertex connectivity to one. It is important to note that the analogous computational problem of reducing the \((s, t)\)-edge connectivity to zero by a minimum number of edge deletions is polynomial-time solvable, because this is computationally equivalent to finding a minimum \((s, t)\)-cut and deleting all edges in it.

Our Results: In this paper, we consider connected undirected unweighted simple graphs. For a graph \(G\), let \((s, t)\) denote a fixed non-adjacent pair of vertices in \(G\). Throughout this paper, when we refer to edge contraction, we do not contract edges incident on \(s\) and edges incident on \(t\).

1. We establish a polynomial-time reduction from the Group Steiner Tree [ND12, 3] to \((s, t)\)-CVS. Consequently, it follows that there is no polynomial-time approximation algorithm with approximation factor \(\delta \log^{2+\epsilon} n\) for some \(\delta > 0\) and for any \(\epsilon > 0\), unless NP has quasi-polynomial Las-Vegas algorithms.
2. We then observe that on chordal graphs finding a minimum \((s, t)\)-CVS is polynomial-time solvable as every
minimal vertex separator is a clique. We show that deciding \((s, t)-\text{CVS}\) is NP-complete on chordality 5 graphs and on chordality 4 graphs \((s, t)-\text{CVS}\) is polynomial-time solvable. We also present a \(\frac{2}{5}\)-approximation algorithm for \((s, t)-\text{CVS}\) on graphs with chordality \(c\).

3. We then consider designing algorithms for \((s, t)-\text{CVS}\) whose running time is \(f(k)n^{O(1)}\) where \(k\) is the parameter of interest and \(f(k)\) is a function independent of \(n\). If the parameter of interest is the chordality \(c\) of the graph, then it follows from the above result that \((s, t)-\text{CVS}\) is unlikely to have an algorithm whose running time is \(f(c)n^{O(1)}, c \geq 5\), unless \(P=\text{NP}\). Whereas, on graphs of treewidth \(\omega\), we show the existence of an algorithm for \((s, t)-\text{CVS}\) with running time \(f(\omega)n^{O(1)}\), here treewidth is the parameter of interest. Algorithms with running time of this nature are well-studied in the literature and they are called fixed-parameter tractable algorithms in the theory of parameterized complexity \cite{5}. Further, an important lower bound for \((s, t)-\text{CVS}\) is the \((s, t)-\text{vertex connectivity}\) itself. It is now natural to consider the following parameterization: the size of a \((s, t)-\text{CVS}\) minus the \((s, t)-\text{vertex connectivity}\). This type of parameterization is known as above guarantee parameters \cite{7}. We show that \((s, t)-\text{CVS}\) parameterized above the \((s, t)-\text{vertex connectivity}\) is unlikely to be fixed-parameter tractable under standard parameterized complexity assumption, and in the terminology of parameterized hardness theory, it is hard for the complexity class \(W[2]\) in the \(W\)-hierarchy.

**Graph Preliminaries:** Notation and definitions are as per \cite{32, 39}. Let \(G = (V, E)\) be a connected undirected unweighted simple graph where \(V(G)\) is the set of vertices and \(E(G)\) is the set of edges. For \(S \subseteq V(G), G[S]\) denote the graph induced on the set \(S\) and \(G\setminus S\) is the induced graph on the vertex set \(V(G)\setminus S\). A vertex separator \(S \subseteq V(G)\) is called a \((s, t)-\text{vertex separator}\) if in \(G\setminus S, s\) and \(t\) are in two different connected components and \(S\) is minimal if no proper subset of it is a \((s, t)-\text{vertex separator}\). A minimum \((s, t)-\text{vertex separator}\) is a \((s, t)-\text{vertex separator}\) of least size. The \((s, t)-\text{vertex connectivity}\) denote the size of a minimum \((s, t)-\text{vertex separator}\). A connected \((s, t)-\text{vertex separator}\) \(S\) is a \((s, t)-\text{vertex separator}\) such that \(G[S]\) is connected and such a set \(S\) of least size is a minimum connected \((s, t)-\text{vertex separator}\). For a minimal \((s, t)-\text{vertex separator}\) \(S, C_s\) and \(C_t\) denote the connected components of \(G\setminus S\) such that \(s\) is in \(C_s\) and \(t\) is in \(C_t\). We let \(G\cdot e\) denote the graph obtained by contracting the edge \(e = \{u, v\}\) in \(G\) such that \(V(G\cdot e) = V(G)\setminus \{u, v\}\cup \{z_{uv}\}\) and \(E(G\cdot e) = \{\{u, x\} \mid \{u, x\} \in E(G)\cup \{x, y\} \in E(G)\} \cup \{\{x, y\} \mid \{x, y\} \in E(G)\}\). A graph is said to have chordality \(c\), if it contains no induced cycle of length at least \(c+1\). i.e. every cycle \(C\) of length at least \(c+1\) in \(G\) has a chord (an edge joining a pair of non consecutive vertices in \(C\)). An optimization problem \(\mathcal{P}\) is \(O(f(n))\)-hard if there exists a constant \(c > 0\) so that \(\mathcal{P}\) admits no \(c.f(n)\)-approximation algorithm, unless \(P=\text{NP}\) (or \(NP\) has quasi-polynomial Las-Vegas algorithms).

**Roadmap:** In Section 2 we analyze the complexity of \((s, t)-\text{CVS}\) and present various hardness results. In Section 3 we present an approximation algorithm and polynomial-time algorithms for \((s, t)-\text{CVS}\) in special graph classes.

## 2 Complexity of \((s, t)-\text{CVS}\): Classical and Parameterized Hardness

We first establish a classical hardness of \((s, t)-\text{CVS}\) by presenting a polynomial-time reduction from the Group Steiner tree to \((s, t)-\text{CVS}\). Moreover, the same reduction establishes an hardness of approximation for \((s, t)-\text{CVS}\). We then analyze \((s, t)-\text{CVS}\) on graphs with chordality \(c\) and show that it is NP-complete on chordality 5 graphs. We conclude this section by showing that \((s, t)-\text{CVS}\) parameterizing above the \((s, t)-\text{vertex connectivity}\) is \(W[2]\)-hard.

### 2.1 Classical Hardness: A Reduction from Group Steiner tree to \((s, t)-\text{CVS}\)

The decision version of \((s, t)-\text{CVS}\) is given below

**Instance:** A graph \(G\), a non-adjacent pair \((s, t)\), and \(q \in \mathbb{Z}^+\)

**Question:** Is there a \((s, t)-\text{vertex separator}\) \(S \subseteq V(G), |S| \leq q\) and \(G[S]\) is connected?

The Group Steiner tree problem can be stated as follows: given a connected undirected unweighted graph \(G\), an integer \(r\), and a collection of sets, which we call groups \(g_1, g_2, \ldots, g_l \subseteq V(G)\), the objective is to find a subtree \(T\) of \(G\) with at most \(r\) edges that contains at least one vertex from each group \(g_i\). We assume that
the groups are disjoint. The Group Steiner tree problem is a generalization of the Steiner tree problem and therefore, it is NP-complete.

We transform an instance $I = (G, g_1, g_2, \ldots, g_l) \subseteq V(G), r)$ of the Group Steiner tree to the corresponding instance $I' = (G', s, t, l + r + 1)$ of $(s, t)$-CVS as follows: $V(G') = V(G) \cup \{s, t\} \cup \{x_i \mid 1 \leq i \leq l\}$. $E(G') = E(G) \cup \{(s, x_i) \mid 1 \leq i \leq l\} \cup \{(t, x_i) \mid 1 \leq i \leq l\} \cup \{(x_i, y) \mid y \in g_i \text{ and } 1 \leq i \leq l\}$. An example is illustrated in Figure 1.

![An Instance of Group Steiner](image1)

**Fig. 1.** An instance of Group Steiner tree reduces to an instance of $(s, t)$-CVS

**Theorem 1.** For $I$ and $I'$ as defined above, $G$ has a Group Steiner tree with at most $r$ edges if and only if $G'$ has a $(s, t)$-CVS of size at most $r + 1 + l$.

**Proof.** We first prove the necessity. Given that $G$ has a Group Steiner tree $T$ with at most $r$ edges that contains at least one vertex from each group $g_i$. By the construction of $G'$, it is clear that the $(s, t)$-vertex connectivity is $l$. Therefore, any $(s, t)$-CVS in $G'$ has at least $l$ vertices. Clearly, these $l$ new vertices together with at most $r + 1$ vertices in $T$ form a $(s, t)$-CVS of size at most $r + 1 + l$ in $G'$. Conversely, by the construction of $G'$, any $(s, t)$-CVS $S$ of size at most $r + 1 + l$ must contain all $x_i$'s. i.e. $N_{G'}(s) \subseteq S$. This is true because $N_{G'}(s)$ is a $(s, t)$-vertex separator. Since $S$ is connected and $N_{G'}(s)$ is an independent set, it follows that by the construction $S \setminus N_{G'}(s)$ is connected. Moreover, $S$ must contain at least one element of $N_{G'}(x_i)$ for each $x_i$. Since $|S \setminus N_{G'}(s)| \leq r + 1$, any spanning tree on $S \setminus N_{G'}(s)$ is a Group Steiner tree with at most $r$ edges. Hence, the theorem follows.

As a consequence of the above theorem, it follows that $(s, t)$-CVS is NP-hard and it is easy to verify that $(s, t)$-CVS is in NP as certificate testing can be done in polynomial time using standard graph traversals. Therefore, $(s, t)$-CVS is NP-complete. We now show that our reduction establishes a stronger result: $(s, t)$-CVS is $\Omega(\log^{2-\epsilon} n)$-hard, for all $\epsilon > 0$ unless NP has quasi-polynomial Las-Vegas algorithms.

**Hardness of Approximation of $(s, t)$-CVS:** The Group Steiner tree problem with $l$ groups is at least as hard as the Set Cover problem, thus can not be approximated to a factor $o(\log l)$, unless $P = NP$. On the hardness of approximation due to [12], the following result is known: Group Steiner tree problem is $\Omega(\log^{2-\epsilon} n)$-hard, for all $\epsilon > 0$ unless NP has quasi-polynomial Las-Vegas algorithms. i.e. there is no polynomial-time approximation algorithm for Group Steiner tree with approximation factor $\delta \log^{2-\epsilon} n$ for some $\delta > 0$ and for any $\epsilon > 0$, unless NP has quasi-polynomial Las-Vegas algorithms. We now show that the above reduction is an approximation-ratio preserving reduction. Let $OPT_g$ and $OPT_c$ denote the size
of any optimum solution of the Group Steiner tree problem and the \((s, t)\)-CVS problem, respectively. Note that \(OPT_c = OPT_g + l\) and \(OPT_c \geq l\). Suppose there is an \((1 + \alpha)\) approximation algorithm for \((s, t)\)-CVS, where \(\alpha \leq \delta \log^{2 - \epsilon} n\), for some \(\delta, \epsilon > 0\). Then the size of the output of the algorithm is \((1 + \alpha)OPT_c = \alpha\delta \log^{2 - \epsilon} n\)

This implies \((1 + \alpha)OPT_g = (1 + \alpha)(OPT_g + l) \leq (1 + \alpha)(OPT_g + OPT_c) = 2(1 + \alpha)OPT_g\)

Now, this line of thought leaves open the complexity of \((s, t)\)-CVS in chordality \(c\) graphs. This is true because in chordal graphs, every minimal vertex separator is a clique \([8]\). Now, this line of thought leaves open the complexity of \((s, t)\)-CVS in chordality \(c\) graphs. A graph is said to have chordality \(c\), if there is no induced cycle of length at least \(c + 1\). Note that chordal graphs have chordality 3. We now show that \((s, t)\)-CVS on chordality 5 graphs is NP-complete.

**Theorem 2.** \((s, t)\)-CVS is NP-complete on chordality 5 graphs.

**Proof.** \((s, t)\)-CVS is in NP: Given a certificate on an input instance \((G, s, t, q)\) of \((s, t)\)-CVS, the certificate on Yes instances is a set \(S \subseteq V(G)\) which is a connected \((s,t)\)-vertex separator of cardinality at most \(q\). Clearly, \(S\) can be verified in polynomial time by standard reachability algorithms \([10]\).

\((s, t)\)-CVS is NP-hard: It is known from \([13]\) that Steiner tree problem on split graphs is NP-complete and this can be reduced in polynomial time to \((s, t)\)-CVS in chordality 5 graphs using the following construction. Note that any split graph \(G\) can be seen as a graph with \(V(G) = V_1 \cup V_2\) such that \(G[V_1]\) is a clique and \(G[V_2]\) is an independent set. Also, split graphs are a subclass of chordal graphs and hence have chordality 3. We now show that instances created by this transformation have chordality 5, i.e., in \(G\), any cycle \(C\) of length at least 6 has a chord. Clearly, \(C\) must contain either \(s\) or \(t\) but not both. Let \(\{s, u_1, \ldots, u_p\}, p \geq 5\) denote the ordering of vertices in \(C\).

**Case 1:** \(\{u_1, u_p\} \subseteq V_2\). Since \(G\) is a split graph, \(\{u_2, u_{p-1}\} \subset V_1\), and therefore, \(\{u_2, u_{p-1}\} \in E(G)\) which is a chord in \(C\).

**Case 2:** \(u_1 \in V_2\) and \(u_p \in V_1\). Clearly, \(u_2 \in V_1\) and \(\{u_2,u_p\} \in E(G)\), a chord in \(C\).

Therefore, we conclude that chordality of \(G\) is 5. A proof in the similar line of Theorem \([14]\) argues that \(G\) has a Steiner tree with at most \(r\) edges if and only if \(G'\) has a \((s, t)\)-CVS with at most \(r + 1\) vertices. As a consequence, it follows that \((s, t)\)-CVS in chordality 5 graphs is NP-hard. Thus, we conclude \((s, t)\)-CVS in chordality 5 graphs is NP-complete. \(\Box\)

### 2.3 \((s, t)\)-CVS Parameterized above the \((s, t)\)-vertex connectivity is \(W[2]\)-hard

We consider the following parameterization which is the size of \((s, t)\)-CVS minus the \((s, t)\)-vertex connectivity. Since the size of every \((s, t)\)-CVS is at least the \((s, t)\)-vertex connectivity, it is natural to parameterize above the \((s, t)\)-vertex connectivity and its parameterized version is defined below.

\[
\begin{array}{ll}
\text{Parameterized above the \((s, t)\)-vertex connectivity:} & \\
\text{Instance: A graph} & G, \text{a non-adjacent pair} \ (s, t) \text{ with} \ (s, t)\text{-vertex connectivity} \\
& k \text{ and} \ r \in \mathbb{Z}^+ \\
\text{Parameter:} & r \\
\text{Question: Is there a} \ (s, t)\text{-vertex separator} \ S \subseteq V(G), \ |S| \leq k + r \text{ such that} \\
& G[S] \text{ is connected?}
\end{array}
\]

We now show that there is no fixed-parameter tractable algorithm for \((s, t)\)-CVS parameterized above the \((s, t)\)-vertex connectivity. In order to characterize those problems that do not seem to admit a fixed-parameter tractable algorithms, Downey and Fellows defined a parameterized reduction and a hierarchy of intractable...
parameterized problem classes above FPT, the popular classes are $W[1]$ and $W[2]$. We refer [3] for details about parameterized reductions. We now present a parameterized reduction from parameterized Steiner tree problem to $(s, t)$-CVS parameterized above the $(s, t)$-vertex connectivity. This parameterized version of Steiner tree problem is shown to be $W[2]$-hard in [14].

| Parameterized Steiner tree problem: |
|-------------------------------------|
| Instance: A graph $G$, a terminal set $R \subseteq V(G)$, and an integer $p$ |
| Parameter: $p$ |
| Question: Is there a set of vertices $T \subseteq V(G) \setminus R$ such that $|T| \leq p$ and $G[R \cup T]$ is connected? $T$ is called Steiner set (Steiner vertices). |

**Theorem 3.** $(s, t)$-CVS Parameterized above the $(s, t)$-vertex connectivity is $W[2]$-hard.

**Proof.** Given an instance $(G, R, r)$ of Steiner tree problem, we construct the corresponding instance $(G', s, t, k, r)$ of $(s, t)$-CVS with the $(s, t)$-vertex connectivity $k = |R|$ as follows: $V(G') = V(G) \cup \{s, t\}$ and $E(G') = E(G) \cup \{\{s, v\} \mid v \in R\} \cup \{\{t, v\} \mid v \in R\}$. We now show that $(G, R, r)$ has a Steiner tree with at most $r$ Steiner vertices if and only if $(G', s, t, k, r)$ has a $(s, t)$-CVS of size at most $k + r$. For only if claim, $G$ has a Steiner tree $T$ containing all vertices of $R$ and at most $r$ Steiner vertices. By our construction of $G'$, to disconnect $s$ and $t$, we must remove the set $N_G'(s)$ which is $R$, as there is an edge from each element of $N_G'(s)$ to $t$. Since $G$ has a Steiner tree with at most $r$ Steiner vertices, implies that in $G'$, it guarantees a $(s, t)$-CVS of size at most $k + r$. For if claim, $G'$ has a $(s, t)$-CVS $S$ with at most $k + r$ vertices. Since the $(s, t)$-vertex connectivity is $k$ and $S$ is a $(s, t)$-vertex separator, from our construction of $G'$ it follows that $N_G'(s) \subseteq S$ and $k = |N_G'(s)|$. This implies that $G$ has a Steiner tree with $R = N_G'(s)$ as the terminal set and $S \setminus N_G'(s)$ as the Steiner vertices of size at most $r$. Hence the claim. $|V(G')| = |V(G)| + 2$ and $|E(G')| \leq |E(G)| + 2|V(G)|$ and the construction of $G'$ takes $O(|E(G)|)$. Clearly, the reduction is a parameter preserving parameterized reduction. Therefore, we conclude that deciding whether a graph has a $(s, t)$-CVS is $W[2]$-hard with parameter $r$. □

**Remark:** The natural parameter for $(s, t)$-CVS is the size of $(s, t)$-CVS. A recent result due to [15] shows that $(s, t)$-CVS parameterized by the size is fixed-parameter tractable.

### 3 Algorithms for $(s, t)$-CVS on Graphs of Bounded Chordality

The focus of this section is to present a polynomial-time approximation algorithm with ratio $\lceil \frac{2}{3} \rceil$ for $(s, t)$-CVS on graphs with chordality $c$ and a polynomial-time algorithm for $(s, t)$-CVS on chordality 4 graphs. Also, the existence of a polynomial-time algorithm for $(s, t)$-CVS on graphs of bounded treewidth is shown.

**Lemma 1.** Let $G$ be a graph of chordality $c$. For each minimal vertex separator $S$, for each $u, v \in S$ such that $\{u, v\} \notin E(G)$, there exists a path of length at most $\lceil \frac{c}{2} \rceil$ whose internal vertices are in $C_s$ or $C_t$, where $C_s$ and $C_t$ are components in $G \setminus S$ containing $s$ and $t$, respectively.

**Proof.** Suppose for some non-adjacent pair $\{u, v\} \subseteq S$, both $P_{uv}^1$ and $P_{uv}^2$ are of length more than $\lceil \frac{c}{2} \rceil$, where $P_{uv}^1$ and $P_{uv}^2$ are shortest paths from $u$ to $v$ whose internal vertices are in $C_s$ and $C_t$, respectively. Now, there is an induced cycle $C$ containing $u$ and $v$ such that $|C| > \lceil \frac{c}{2} \rceil + \lceil \frac{c}{2} \rceil = l$. However, this contradicts the fact that $G$ is of chordality $l$. □

#### 3.1 $\lceil \frac{c}{2} \rceil$-Approximation for $(s, t)$-CVS on Graphs with Chordality $c$

Let $OPT$ denote the size of any minimum $(s, t)$-CVS on chordality $c$ graphs. Clearly, $OPT \geq k$, where $k$ is the $(s, t)$-vertex connectivity. The description of approximation algorithm $ALG$ is as follows:

1. Compute a minimum $(s, t)$-vertex separator $S$ in $G$. $S = \{v_1, \ldots, v_k\}$ be an arbitrary ordering of vertices in $S$. 

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2. For each non-adjacent pair \( \{v_i, v_{i+1}\} \subseteq S, 1 \leq i \leq k-1 \), find a path \( P_{v_i,v_{i+1}} \) of length at most \( \left\lceil \frac{k}{2} \right\rceil \) whose internal vertices are in \( C_s \) or \( C_t \). Such a path exists as per Lemma \( \[1\] S' = \bigcup_{1 \leq i \leq k-1} V(P_{v_i,v_{i+1}}) \cup S \).

Observe that \( S' \) is a \((s,t)\)-CVS in \( G \). The upper bound on the size of \( S' \) output by \( ALG \) is: \(|S'| \leq k + (k - 1)(\left\lceil \frac{1}{2} \right\rceil) - 1 \). Therefore, approximation ratio \( \beta \) is
\[
\beta \leq \frac{k + (k - 1)(\left\lceil \frac{1}{2} \right\rceil) - 1}{k} = 1 + (1 - \frac{1}{2})(\left\lceil \frac{1}{2} \right\rceil) - 1 < 1 + (\left\lceil \frac{c}{2} \right\rceil - 1) = \left\lceil \frac{c}{2} \right\rceil
\]

### 3.2 \((s,t)\)-CVS in Chordality 4 Graphs is Polynomial Time

We have already mentioned that \((s,t)\)-CVS in chordality 3 graphs is polynomial time as every minimal vertex separator is a clique and due to Theorem \(2\) it is NP-complete on chordality 5 graphs. This observation leaves open the question of \((s,t)\)-CVS in chordality 4 graphs. We now present a structural result about minimal vertex separators in chordality 4 graphs, using which we show that \((s,t)\)-CVS in chordality 4 graphs is polynomial-time solvable. This is the dichotomy we mentioned in the abstract.

**Theorem 4.** Every minimal \((s,t)\)-vertex separator \( S \) in a chordality 4 graph \( G \) satisfies one of the following properties:

1. \( G[S] \) is connected.
2. Let \( \{X_1, \ldots, X_r\}, r \geq 2 \) denote the set of connected components in \( G[S] \) and \( V(X_i) \) denotes the vertex set of the component \( X_i \). In \( G \setminus S \), there exists \( u \in C_s \) and there exists \( v \in C_t \) such that for all \( 1 \leq i \leq r \), \( V(G)(u) \cap V(X_i) \neq \emptyset \) and \( V(G)(v) \cap V(X_i) \neq \emptyset \), where \( C_s \) and \( C_t \) denote the connected components in \( G \setminus S \) containing \( s \) and \( t \), respectively.

**Proof.** Our proof is by induction on \( n = |V(G)| \). It is easy to verify that the statement of the lemma is true when \( |V(G)| = 1 \), \( |V(G)| = 2 \), and \( |V(G)| = 3 \). Let us now assume all chordality 4 graphs on \( n - 1 \) vertices satisfy our claim. Consider a chordality 4 graph \( G \) on \( n \) vertices. Consider a minimal \((s,t)\)-vertex separator \( S \) with \(|S| = 1 \). Then \( S \) contains a cut vertex and our claim is true. If \(|S| = 2 \), then either \( G[S] \) is connected or we can find \( u \) in \( C_s \) such that \( S \subseteq N_G(u) \) and \( v \) in \( C_t \) such that \( S \subseteq N_G(v) \). Consider the case when \(|S| \geq 3 \).

Let \( C_s \) and \( C_t \) denote components in \( G \setminus S \) containing \( s \) and \( t \), respectively. Without loss of generality, we assume that both \( C_s \) and \( C_t \) contain at least two elements. Otherwise, it must be the case that \( S = N_G(s) \) or \( S = N_G(t) \).

**Case 1:** \( G[S] \) is not an independent set. Let \( e = \{x,y\} \) be an edge contained in a connected component \( X \) of \( G[S] \). Consider the graph \( G \cdot e \) obtained from \( G \) by contracting \( e \). Clearly, \( |V(G \cdot e)| = n - 1 \). Let \( S' = S \setminus \{x,y\} \cup \{z_{xy}\} \). Edges incident on \( x \) or \( y \) are now incident on \( z_{xy} \). Observe that \( S' \) is a minimal \((s,t)\)-vertex separator in \( G \cdot e \). If \( G[S'] \) is connected in \( G \cdot e \) then it implies that \( G[S] \) is connected in \( G \) as well. Otherwise, by the induction hypothesis, in \( G \cdot e \), there exists \( u \) and \( v \) with the desired property. In particular, \( V(X') \cap N_{G \cdot e}(u) \) and \( V(X') \cap N_{G \cdot e}(v) \) are non empty where \( X' = X \setminus \{x,y\} \cup \{z_{xy}\} \) and \( X \) is the connected component in \( S \) containing \( x \) and \( y \). Thus, both \( u \) and \( v \) have the desired property in \( G \) too.

**Case 2:** \( G[S] \) is an independent set. Now consider \( x, y \in S \). Consider the graph \( G \cdot xy \) obtained by contracting the non-adjacent pair \( \{x, y\} \). Let \( S' = S \setminus \{x,y\} \cup \{z_{xy}\} \) and edges incident in \( x \) or \( y \) are now incident on \( z_{xy} \). Observe that \( S' \) is a minimal \((s,t)\)-vertex separator in \( G \cdot xy \). Clearly, \( |V(G \cdot xy)| = |V(G)| - 1 \) and hence, by the induction hypothesis, in \( G \cdot xy \), there exists \( u \in C_s' \) and \( v \in C_t' \) satisfying our claim where \( C_s' \) and \( C_t' \) are connected components in \( G \cdot xy \setminus S' \) containing \( s \) and \( t \), respectively. Let \( S = \{x, y, u_1, \ldots, u_p\}, p \geq 1 \). We now prove in \( G \) the existence of vertex \( u \) in \( C_s \) satisfying our claim. If \( \{u, x\}, \{u, y\} \in E(G) \), then clearly \( u \in C_s \) is the desired vertex in \( G \). Otherwise, without loss of generality assume that \( x \notin N_G(u) \). Thus, \( S \setminus \{x\} \subseteq N_G(u) \). Let \( P_{ux} \) denote a shortest path between \( u \) and \( x \) such that the internal vertices are in \( C_s \). Consider the vertex \( w \) in \( P_{ux} \) such that \( \{x, w\} \in E(G) \). Such a \( w \) exists as \( S \) is a minimal \((s,t)\)-vertex separator in \( G \). If for all \( z \in S \), \( \{w, z\} \in E(G) \), then \( w \) is a desired vertex in \( C_s \). Otherwise, there exists \( z \in S \) such that \( \{w, z\} \notin E(G) \). Let \( P_{wu} \) denote the subpath of \( P_{ux} \) on the vertex set \( \{w, w_1, \ldots, w_q = u\}, q \geq 2 \). If for each \( 2 \leq i \leq q - 1 \), \( \{z, w_i\} \notin E(G) \), then \( P_{ux} \) form an induced cycle of length at least 5 in
Using the above two claims, and with the help of the following two key combinatorial observations on the contractible edges in this case, we know from Theorem 4, there exists a vertex $s,t$ where $G$ becomes a cut-vertex. Moreover, any edge contraction does not disconnect a graph which is already connected. Therefore, any shortest path $P_{u,v}$ between $s$ and $t$ in one of the components in $G - \{s,t\}$ is either connected or contains a cut-vertex. Hence, the claim is true for the base case. If suppose there exists a chord between $x$ and $y$ in $G$, then the proof is complete.

**Lemma 2.** Let $G$ be a chordality 4 graph with the $(s,t)$-vertex connectivity $k$. The size of any minimum $(s,t)$-CVS in $G$ is either $k$ or $k + 1$.

**Proof.** Note that any minimum $(s,t)$-CVS is of size at least $k$ as the $(s,t)$-vertex connectivity is $k$. If a minimum $(s,t)$-vertex separator itself is connected then we get a minimum $(s,t)$-CVS of size $k$. Otherwise, there exists a minimum $(s,t)$-vertex separator $S$ such that $G[S]$ is a collection of connected components. In this case, we know from Theorem 4 there exists a vertex $v$ in one of the components of $G \setminus S$ such that $S \subseteq N_G(v)$. Therefore, $S \cup \{v\}$ is a minimum $(s,t)$-CVS of size $k + 1$. Hence the claim.

Using the above two claims, and with the help of the following two key combinatorial observations on the structure of minimal vertex separators in chordality 4 graphs we show that $(s,t)$-CVS in chordality 4 graphs is polynomial-time solvable. We make use of the notion of contractible edges. Given a connected graph $G$ with $(s,t)$-vertex connectivity $k$, an edge $e \in E(G)$ is said to be contractible if $(s,t)$-vertex connectivity in $G \cdot e$ is at least $k$. Otherwise, $e$ is called non-contractible.

**Lemma 3.** Let $G$ be a connected graph and $S$ be a minimum $(s,t)$-vertex separator with $|S| \geq 2$. Let $F = \{u,v\} \cup S$ and $\{u,v\} \in E(G)$. $G[S]$ is connected if and only if the $(s,t)$-vertex connectivity in $G \cdot F$ (the graph obtained by contracting $F$ in $G$) is one. Further, $G \cdot F$ contains a cut-vertex.

**Proof.** Clearly each edge contained in $S$ is non-contractible and by contracting all non-contractible edges, $S$ becomes a cut-vertex. Moreover, any edge contraction does not disconnect a graph which is already connected. Therefore, $\kappa(G \cdot F) = 1$. We prove the converse by contradiction. Suppose, for all minimum $(s,t)$-vertex separator $S$, $G[S]$ contains at least two components. This implies that the graph resulting from contracting any sequence of non-contractible edges has $(s,t)$-vertex connectivity at least two. A contradiction to the fact $G \cdot F$ contains a cut-vertex. Hence the claim.

Using the above lemma, we can decide in polynomial time whether a chordality 4 graph with the $(s,t)$-vertex connectivity $k$ contains a $(s,t)$-CVS of size $k$ or $k + 1$. The approach is to contract all non-contractible edges and check whether the resulting graph contains a cut-vertex or not. If so, then the given chordality 4 graph contains a $(s,t)$-CVS of size $k$. Otherwise, any minimum $(s,t)$-vertex separator in $G$ together with the vertex $v$ in one of the components in $G \setminus S$ (due to Theorem 4) yields a $(s,t)$-CVS of size $k + 1$ in $G$. Our next combinatorial observation help us in finding a minimum $(s,t)$-CVS in polynomial time.

**Lemma 4.** Let $G$ be a chordality 4 graph with the $(s,t)$-vertex connectivity $k$ and $e = \{u,v\}$ be a non-contractible edge contained in minimum vertex separators $S$ and $S'$ such that $G[S]$ is connected and $G[S']$ is not connected. Then, for each $x \in S' \setminus S$, there exists $y \in S$, such that $\{x,y\} \in E(G)$.

**Proof.** Clearly $k \geq 3$. Let $S = \{x_1 = u, x_2 = v, \ldots, x_k\}$, $S' = \{y_1 = u, y_2 = v, \ldots, y_k\}$ and $S' \setminus S = \{y_1, \ldots, y_j\}, i \geq 3, j \leq k$. We prove by induction on $\kappa(G)$. For the base case, $\kappa(G) = 3$, i.e., $|S' \setminus S| = 1$. Suppose $y_k \in S' \setminus S$ is such that for any $x_j \in S$, $\{x_j, y_k\} \notin E(G)$. In particular, $\{x_3, y_3\} \notin E(G)$. This implies that any shortest path $P_{x_3y_3}$ between $x_3$ and $y_3$ is of length at least 2 in $G$. Since $S$ is a minimal vertex separator, there exists $u \in C_1$ such that $\{x_2, w\} \in E(G)$, where $C_1$ is a connected component in $G \setminus S$. Note that, the paths $P_{x_3y_3}, P_{uwz},$ and $P_{x_3y_3}$ induces a cycle of length at least 5 in $G$, where $P_{x_3y_3}$ is a shortest path whose internal vertices are in $S$ or in $C_2$, where $C_2(\neq C_1)$ is a connected component in $G \setminus S$. Observe that there can not be chords from $w$ to any internal vertex in $P_{x_3y_3}$ as $S'$ is a vertex separator in $G$. Hence, our assumption that $\{x_3y_3\} \notin E(G)$ is wrong. Therefore, the claim is true for the base case. If suppose there exists a chord between $x_2$ and some internal vertex, say $z$ in $P_{x_3y_3}$, then $\{x_1, x_2, z\}$ is a connected minimum
vertex separator and we run the above argument by considering $S = \{x_1, x_2, z\}$. We assume that our claim is true for all connected graphs with $\kappa(G) < k, k \geq 3$ and satisfying the premise of the lemma. Let $G$ be a chordality 4 graph with $\kappa(G) = k, k \geq 4$. Consider $S$ and $S'$ as defined in the premise of the lemma. Consider the graph $G'$ obtained from $G$ by contracting the pairs $\{x_k, x_{k-1}\}$ and $\{y_j, y_{j-1}\}$ and $p$ and $q$ are the newly created vertices due to contraction, respectively. Clearly, $\kappa(G') = k - 1$ as $S$ and $S'$ are minimum vertex separators of size $k - 1$. By our induction hypothesis in $G'$, for each $x \in S' \setminus S$, there exists $y \in S$, such that \{x, y\} $\in E(G')$. Since $q$ has a neighbour in $S$ in $G'$, at least one of $y_j$ or $y_{j-1}$ must have a neighbour in $S$ in $G$. If suppose in $G$, $y_{j-1}$ has a neighbour in $S$ and $y_j$ does not have a neighbour in $S$. An argument similar to the base case produces an induced cycle of length at least 5 containing $y_j$. This completes the induction and therefore, the lemma follows. 

\[\square\]

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**A polynomial-time algorithm to find $(s,t)$-CVS in chordality 4 graphs**

**Input:** Chordality 4 graph with the $(s,t)$-vertex connectivity $k$

1. Find the set $E_c$ of all contractible edges in $G$.
2. Contract $E_c$ and check the $(s,t)$-vertex connectivity in the resulting graph $G_c$.
3. If $\kappa(G_c) = 1$, then the size of minimum $(s,t)$-CVS is $k$. Otherwise it is $k + 1$.
4. If $\kappa(G_c) \geq 2$, then any minimum vertex separator $S$ in $G$ augmented with the vertex $v$ in one of the components in $G \setminus S$ such that $S \subseteq N_G(v)$ is a minimum $(s,t)$-CVS of size $k + 1$.
5. If $\kappa(G_c) = 1$, then there exists a connected minimum vertex separator. To obtain one such separator, perform the following:
6. For each non-contractible edge $e$ in $G$, contract $e$ and find a minimum vertex separator $S'$ in $G \cdot e$.
6a. If $G[S']$ is connected, then output the minimum vertex separator $S$ in $G$ corresponding to $S'$.
6b. If $G[S']$ is not connected, then check whether there exists a connected minimum vertex separator $S''$ in the neighbourhood of $S'$.

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**3.3 $(s,t)$-CVS Parameterized by treewidth is Fixed-parameter Tractable**

We transform the instance of $(s,t)$-CVS problem to the satisfiability of a formula in *monadic second order logic* (MSOL). Using Courcelle’s Theorem [17] that a problem over bounded treewidth graphs expressible in MSOL can be solved in linear time. A recent paper by Marx [18] uses the same approach to prove the fixed-parameter tractability of other constrained separator problems. We now present the description of monadic second order logic for $(s,t)$-CVS. The atomic predicates used are as follows: For a set $S \subseteq V(G)$, $S(v)$ denotes that $v$ is an element of $S$, and the predicate $E(u, v)$ denotes the adjacency between $u$ and $v$ in $G$. $T = \{s, t\}$ and $k$ is the upper bound on the size of the desired $(s,t)$-CVS. We construct the formula $\phi$ in MSOL as

$$\phi = \exists S (\text{AtMost}_k(S) \land \text{Separates}(S) \land \text{ConnectedSubgraph}(S))$$

Here the predicate $\text{AtMost}_k(S)$ is true if and only if $|S| \leq k$, $\text{Separates}(S)$ is true if and only if $S$ separates the vertices of $T$ in $G$, and $\text{ConnectedSubgraph}(S)$ is true if and only if $S$ induces a connected subgraph in $G$. We refer [18] for formulae $\text{AtMost}_k(S)$ and $\text{Separates}(S)$. The formula $\text{Connects}(Z, s, t)$ is due to [18], where $\text{Connects}(Z, s, t)$ is true if and only if in $G$, there is a path from $s$ and $t$ to all vertices of which belong to $Z$. $\text{ConnectedSubgraph}(S)$ is true if and only if for every subset $S'$ of $S$ there is an edge between $S'$ and $S' \setminus S'$.

- $\text{AtMost}_k(S) : \forall c_1, \ldots, \forall c_{k+1} \bigvee_{1 \leq i, j \leq k+1} (c_i = c_j)$

- $\text{Separates}(S) : \forall s \forall t \forall Z(T(s) \land T(t) \land (s = t) \land (s \land \neg S(s) \land \neg S(t) \land \text{Connects}(Z, s, t)) \to (\exists v (S(v) \land Z(v))))$

- $\text{Connects}(Z, s, t) : Z(s) \land Z(t) \land \forall P((P(s) \land \neg P(t)) \to (\exists v \exists w (Z(v) \land Z(w) \land P(v) \land \neg P(w) \land E(v, w))))$
ConnectedSubgraph$(S)$ : $\forall S' \subseteq S((S \neq S') \land (\exists u(S(u) \land S'(u)))) \rightarrow (\exists u \exists w(S(u) \land S(w) \land (u, w) \in E(G) \land S'(v) \land \neg S'(w)))$

This completes the observation that in bounded treewidth graphs, a minimum $(s,t)$-CVS can be found in linear time.

Concluding Remarks and Further Research:
In this paper, we have investigated the complexity of connected $(s,t)$-vertex separator ($(s,t)$-CVS) and shown that for every $\epsilon > 0$, $(s,t)$-CVS is $\Omega(\log^2 n)$-hard, unless NP has quasi-polynomial Las-Vegas algorithms. Also shown that $(s,t)$-CVS is NP-complete on graphs with chordality at least 5 and presented a polynomial-time algorithm for $(s,t)$-CVS on chordality 4 graphs. Moreover, parameterizing above $(s,t)$-vertex connectivity is $W[2]$-hard. An interesting problem for further research is to parameterize $(s,t)$-CVS by the $(s,t)$-vertex connectivity.

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