Asymptotic analysis of the lattice Boltzmann method for generalized Newtonian fluid flows

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Abstract

In this article, we present a detailed asymptotic analysis of the lattice Boltzmann method with two different collision mechanisms of BGK-type on the D2Q9-lattice for generalized Newtonian fluids. Unlike that based on the Chapman-Enskog expansion leading to the compressible Navier-Stokes equations, our analysis gives the incompressible ones directly and exposes certain important features of the lattice Boltzmann solutions. Moreover, our analysis provides a theoretical basis for using the iteration to compute the rate-of-strain tensor, which makes sense specially for generalized Newtonian fluids. As a by-product, a seemingly new structural condition on the generalized Newtonian fluids is singled out. This condition reads as “the magnitude of the stress tensor increases with increasing the shear rate”. We verify this condition for all the existing constitutive relations which are known to us. In addition, it it straightforward to extend our analysis to MRT models or to three-dimensional lattices.

Keywords: Lattice Boltzmann method; generalized Newtonian fluid; asymptotic analysis; constitutive relation; construction criterion

1 Introduction

During the last two decades, the lattice Boltzmann method (LBM) has been developed into an effective and viable tool for simulating various fluid flow problems. It has been proved to be quite successful in simulating complex Newtonian fluid flows such as turbulent flows, micro-flows, multi-phase and multi-component flows, particulate suspensions, and interfacial dynamics. We refer to [1, 2, 3] for a comprehensive account of the method and its applications. Moreover, the potential of LBM in simulating flows of generalized Newtonian fluids, for which the dynamic shear viscosity depends on the shear rate [4], was shown by Aharonov and Rothman [5] as early as 1993. In recent years, this potential has attracted much attention [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

There are at least two kinds of LB-BGK models for generalized Newtonian fluids. In the first model, the relaxation time is not a constant anymore but depends on the shear rate. The non-Newtonian effects

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are embedded in the LBM through a dynamic change of the relaxation time. The second model has a constant relaxation time but its equilibrium distribution contains the shear rate.

The original goal of this paper is to extend the asymptotic analysis of LBM [17, 18] for classical Newtonian fluids to generalized ones. The analysis is based on the diffusive, instead of convective, scaling developed in [19] for the Boltzmann equation and in [17] for LB equations. It turns out that the diffusive scaling is the most natural choice if the LBE is viewed as a numerical solver of the incompressible Navier-Stokes equations. Unlike that based on the Chapman-Enskog expansion leading to the compressible Navier-Stokes equations (see, e.g. [11]), our analysis uses the Hilbert expansion and gives the incompressible equations directly. Meanwhile, this analysis exposes certain important features of the LB solutions. In particular, it provides a theoretical basis for using the Richardson extrapolation technique to improve the accuracy up to higher orders and for using the iteration [7, 12, 15] to compute the rate-of-strain tensor. The latter makes sense especially for generalized Newtonian fluids.

To achieve the above goal, we realize that a proper introduction of the lattice spacing in the equilibrium distribution functions are indispensable to recover the correct macroscopic equations. Moreover, we have to carefully dispose many possibly non-zero terms due to the dependence on the shear rate, which vanish for classical Newtonian fluids.

As a by-product of our analysis, a seemingly new structural condition on the generalized Newtonian fluids is singled out. This condition reads as “the magnitude of the stress tensor increases with increasing the shear rate” (magnitude of the rate-of-strain tensor). We verify this condition for all the existing constitutive relations which are known to us. By inductive reasoning, we suggest to take this structural condition as a criterion in constructing further constitutive relations for generalized Newtonian fluids.

In addition, let us mention that our analysis can be easily extended to the multiple-relaxation-time (MRT) models [20, 21, 22] or to three-dimensional lattices, although it is presented only for BGK models on the two-dimensional lattice D2Q9 [23].

This paper is organized as follows. In Section 2 we introduce the LBM together with the D2Q9 lattice and two collision terms for generalized Newtonian fluids. The asymptotic analysis is outlined in Section 3. In Section 4, we verify the structural condition for the existing constitutive relations known to us. Several conclusions are summarized in Section 5. Finally, an appendix is devoted to the technical details skipped in Section 3.

2 Lattice Boltzmann Method

The general form of the lattice Boltzmann method is

\[ f_i(x + c_i h, t + \delta t) - f_i(x, t) = \Omega_i(f_1, f_2, \cdots, f_N)(x, t) \]  (2.1)

for \( i = 0, 1, 2, \cdots, N \). Here \( N \) is a given integer, \( f_i = f_i(x, t) \) is the \( i \)-th density distribution function of particles at the space-time point \( (x, t) \), \( c_i \) is the \( i \)-th given velocity, \( h \) is the lattice spacing, \( \delta t \) is the time step, and \( \Omega_i = \Omega_i(f_1, f_2, \cdots, f_N) \) is the \( i \)-th given collision term. Motivated by the diffusive scaling
analyzed in [19] [17], we take
\[ \delta t = h^2 \]
in what follows.

For the sake of definiteness, we take the D2Q9 lattice throughout this paper. Namely, our problem is two-dimensional, \( N = 8 \) and
\[
c_i = \begin{cases} 
(0,0) & i = 0 \\
(1,0), (0,1), (-1,0), (0,-1) & i = 1, 2, 3, 4 \\
(1,1), (-1,1), (-1,-1), (1,-1) & i = 5, 6, 7, 8.
\end{cases}
\]

For \( i \in \{0, 1, 2, \cdots, 8\} \), we define
\[
\bar{i} = \begin{cases} 
0 & i = 0 \\
3, 4, 1, 2 & i = 1, 2, 3, 4 \\
7, 8, 5, 6 & i = 5, 6, 7, 8.
\end{cases}
\tag{2.2}
\]
It is clear that \( c_i = -c_{\bar{i}} \). In this sense, \( c_i \) is said to be odd.

We will discuss two different collision mechanisms of BGK-type for generalized Newtonian fluids [4]. Unlike for classical Newtonian fluids, the dynamic shear viscosity for the generalized ones is not a constant but depends on the gradient of the fluid velocity \( v \). Precisely, the viscosity \( \mu = \mu(I) \) is a non-negative function of \( I = \text{tr}(S^2) \) with \( S \) the rate-of-strain tensor or rate-of-deformation tensor
\[
S = \nabla v + (\nabla v)^t
\tag{2.3}
\]
where the superscript \( t \) indicates the transpose. The corresponding stress tensor \( T \) is given as
\[
T = \mu(I)S.
\]

The first collision term reads as
\[
\Omega_i = \frac{1}{\tau(S/h)} (f_i^{eq}(\rho, v) - f_i).
\tag{2.4}
\]
Here the relaxation time is taken as
\[
\tau(S) = 3\mu(I) + \frac{1}{2},
\]
while the equilibrium distribution is quite standard:
\[
f_i^{eq} = f_i^{eq}(\rho, v) = w_i[\rho + 3c_i \cdot v + \frac{9}{2}(c_i \cdot v)^2 - \frac{3}{2}v \cdot v]
\tag{2.5}
\]
(· indicates the inner product) with

\[ \rho = \sum_{i=0}^{8} f_i, \quad v = \sum_{i=0}^{8} c_i f_i \quad (2.6) \]

and weighted coefficients

\[ w_i = \frac{1}{36} \begin{cases} 
16 & i = 0 \\
4 & i = 1, 2, 3, 4 \\
1 & i = 5, 6, 7, 8.
\end{cases} \quad (2.7) \]

Note that \( w_i = \bar{w}_i \), that is, \( w_i \) is even.

For convenience, we decompose the equilibrium distribution above as

\[ f_{i}^{eq} = f_{iL}(\rho, v) + f_{iQ}(v, v) \quad (2.8) \]

with

\[ f_{iL}(\rho, v) := w_i (\rho + 3c_i \cdot v), \]
\[ f_{iQ}(v, v) := w_i \frac{9}{2} (c_i \cdot u)(v \cdot v) - \frac{3}{2} u \cdot v]. \quad (2.9) \]

Since \( c_i \) is odd and \( w_i \) is even, it is easy to see that

\[ \sum_i f_{iL}(\rho, v) \equiv \rho, \quad \sum_i f_{iQ}(u, v) \equiv 0, \]
\[ \sum_i c_i f_{iL}(\rho, v) \equiv v, \quad \sum_i c_i f_{iQ}(u, v) \equiv 0. \]

The second collision term is

\[ \Omega_i = \frac{1}{\tau} (f_{i}^{eq}(\rho, v; S, h) - f_i) \quad (2.10) \]

with a constant relaxation time \( \tau \). The equilibrium distribution is \([8, 11]\)

\[ f_{i}^{eq}(\rho, v; S, h) = f_{iL}(\rho, v) + f_{iQ}(v, v) + w_i h A(S : c_i'c_i) \quad (2.11) \]

with scalar function

\[ A(S) = \frac{3}{2}(\tau - \frac{1}{2}) - \frac{9}{7} \mu (tr S^2). \]

In (2.11) \( S : c_i'c_i \) is the standard contraction of two symmetric tensors \( S \) and \( c_i'c_i \).

It is remarkable that the lattice spacing \( h \) has been introduced in the two collision terms (2.4) and (2.10), which seems new.
3 Asymptotic Analysis

Motivated by Strang [24], we notice that the LB solution \( f_i(x, t; h) \) depends on the lattice spacing \( h \) which is small. Thus, we will seek an expansion of the form

\[
f_i(x, t; h) \sim \sum_{n \geq 0} h^n f_i^{(n)}(x, t). \tag{3.1}
\]

Referring to this expansion, (2.6) and (2.3), we introduce

\[
\rho^{(n)} := \sum_{i=0}^{\infty} f_i^{(n)}(x), \quad v^{(n)} := \sum_{i=0}^{\infty} c_i f_i^{(n)}(x), \quad S^{(n)} := (\nabla v^{(n)}) + (\nabla v^{(n)})^T \tag{3.2}
\]

and

\[
\rho_h \sim \sum_{n \geq 0} h^n \rho^{(n)}, \quad v_h \sim \sum_{n \geq 0} h^n v^{(n)}, \quad S_h \sim \sum_{n \geq 0} h^n S^{(n)}. \tag{3.3}
\]

Take \( f_i^{(0)} = w_i \) as the leading term in (3.1). It follows clearly from (3.2) together with the even/odd properties of \( w_i \) and \( c_i \) that

\[
\rho^{(0)} = 1, \quad v^{(0)} = 0, \quad S^{(0)} = 0.
\]

By using the Taylor expansion, we have

\[
f_i^{(n)}(x + c_i h, t + h^2) - f_i^{(n)}(x, t) \sim \sum_{l>0} \left( \frac{h^2 \partial_k + hc_i \cdot \nabla}{l!} \right)^l f_i^{(n)}(x, t). \tag{3.4}
\]

Since

\[
\sum_{l>0} \left( \frac{h^2 \partial_k + hc_i \cdot \nabla}{l!} \right)^l = \sum_{l>0} \sum_{m=0}^{l} \frac{1}{m! (l-m)!} (h^2 \partial_k)^m (hc_i \cdot \nabla)^{l-m} = \sum_{s>0} h^s D_{i,s}
\]

with \( D_{i,s} \) a differential operator, it follows from (3.4) and the constancy of \( f_i^{(0)} \) that

\[
f_i(x + c_i h, t + h^2) - f_i(x, t) = \sum_{n \geq 2} \sum_{l>0} \sum_{m \leq l} h^{n+l+m} \frac{\partial^m (c_i \cdot \nabla)^{l-m}}{m! (l-m)!} f_i^{(n)}(x, t)
\]

\[
= \sum_{n \geq 2} \sum_{s=1}^{n-1} h^n D_{i,s} f_i^{(n-s)}(x, t). \tag{3.5}
\]

For the first collision [24], we expand

\[
\tau(S_h/h) = \tau(S^{(1)}) + h S^{(2)} + \cdots \sim \sum_{n \geq 0} h^n F^{(n)}.
\]

It is not difficult to see that \( F^{(n)} \) is determined with \( S^{(l)} \) for \( l = 1, 2, \cdots, n + 1 \). In particular, \( F^{(0)} = \)
\( \tau(S^{(1)}) \). Thus, we may rewrite the LBM (2.1) as

\[
\sum_{r \geq 0} h^r F^{(r)} \sum_{n \geq 2} h^n \sum_{s=1}^{n-1} D_{i,s} f_i^{(n-s)} = -\sum_{n \geq 0} h^n [f_i^{(n)} - f_{iL}(\rho^{(n)}, v^{(n)})] + \sum_{n \geq 1} h^n \sum_{p+q=n} f_i Q(\rho^{(p)}, v^{(q)}).
\]

(3.6)

By equating the coefficient of \( h^k \) in the two sides of the last equation and using \( v^{(0)} = 0 \), we obtain

\[
h^0 : \quad f_i^{(0)} = f_{iL}(\rho^{(0)}, v^{(0)}) = f_{iL}(1, 0),
\]

(3.7)

\[
h^1 : \quad f_i^{(1)} = f_{iL}(\rho^{(1)}, v^{(1)}),
\]

(3.8)

\[
h^2 : \quad \tau(S^{(1)}) D_{i,1} f_i^{(1)} + f_i^{(2)} = f_{iL}(\rho^{(2)}, v^{(2)}) + f_i Q(\rho^{(1)}, v^{(1)}),
\]

(3.9)

\[
h^k : \quad \tau(S^{(1)}) \sum_{s=1}^{k-1} D_{i,s} f_i^{(k-s)} + F^{(1)} \sum_{s=1}^{k-2} D_{i,s} f_i^{(k-1-s)} + \cdots + F^{(k-2)} D_{i,1} f_i^{(1)} + f_i^{(k)}
\]

(3.10)

for \( k \geq 3 \). With this hierarchy of equations, the expansion coefficient \( f_i^{(k)} \) can be uniquely determined in terms of \((\rho^{(l)}, v^{(l)})\) for \( l = 1, 2, \ldots, k \). In contrast to the case [17, 18] for classical Newtonian fluids, there are more terms involving \( F^{(j)} \) with \( j \geq 1 \) in (3.10).

In Appendix, we will show that \( F^{(j)} = 0 \) with \( j \) odd and \((\rho^{(l)}, v^{(l)})\) can be inductively obtained by solving a hierarchy of quasilinear or linear partial differential equations. In particular, we can show that \( \rho^{(1)} \equiv 0 \) and \((\rho^{(2)}, v^{(1)})\) satisfies the following equations

\[
\nabla \cdot v^{(1)} = 0
\]

(3.11)

\[
\frac{\partial v^{(1)}}{\partial t} + \nabla \left[ \frac{\rho^{(2)}}{3} + v^{(1)} \cdot \nabla v^{(1)} \right] = \nabla \cdot [ \frac{1}{3} \tau(S^{(1)}) - \frac{1}{2} S^{(1)} ].
\]

Namely, \( v^{(1)} \) and \( \frac{\rho^{(0)}}{3} \) are the respect velocity and pressure of the generalized Newtonian fluid, for the relaxation time is taken as \( \tau(S) = 3\mu(I) + \frac{3}{2} \). Furthermore, we have

**Theorem 1** Assume \( \rho^{(2k+1)} \mid_{t=0} = 0 \) and \( v^{(2k)} \mid_{t=0} = 0 \) for \( k = 0, 1, 2, \ldots \), and the viscosity \( \mu = \mu(I) \) satisfies

\[
\mu(I) + 2 \min \left\{ \frac{d\mu(I)}{dt}, 0 \right\} I \geq 0.
\]

Then, for periodic boundary-value problems, the expansion coefficients possess the following nice property

\[
f_i^{(k)} = (-1)^k f_i^{(k)}.
\]

**Remark.** In the next section, the structural condition above

\[
\mu(I) + 2 \min \left\{ \frac{d\mu(I)}{dt}, 0 \right\} I \geq 0
\]

is shown to be equivalent to the statement that the magnitude of the stress tensor increases with increasing
the shear rate.

This theorem and the equations in (3.11) are both valid for the second collision mechanism (2.10) together with (2.11). The proof is given also in Appendix, where the expansion of $\tau(S/h)$ is replaced by

$$A(S/h) \sim \sum_{n \geq 0} h^n A^{(n)}(S^{(1)}, \ldots, S^{(n+1)})$$

and the analogue of (3.6) reads as

$$\tau \sum_{n \geq 2} h^n \sum_{s=1}^{n-1} D_i s f_i^{(n-s)} = - \sum_{n \geq 0} h^n [f_i^{(n)} - f_i L(\rho^{(n)}, v^{(n)})] + \sum_{n \geq 1} h^n \sum_{p+q=n} f_{iQ}(v^{(p)}, v^{(q)})$$

$$+ \sum_{n \geq 0} h^{n+1} \sum_{l+m=n} w_i A^{(l)} S^{(m)} : c_i^T c_i. \quad (3.12)$$

From the theorem above and (3.11), we see clearly that the density and velocity moments of the LB solution $f_i = f_i(x, t; h)$ can be expanded as

$$\rho_h := \sum_i f_i \sim \sum_{n \geq 0} h^n \rho^{(n)} = 1 + h^2 \rho^{(2)} + h^4 \rho^{(4)} + h^6 \rho^{(6)} + \ldots,$$

$$v_h := \sum_i c_i f_i \sim \sum_{n \geq 0} h^n v^{(n)} = h v^{(1)} + h^3 v^{(3)} + h^5 v^{(5)} + \ldots.$$

Thus we have

$$\frac{v_h}{h} - v^{(1)} = h^2 v^{(3)} + h^4 v^{(5)} + \ldots,$$

$$\frac{\rho_h - 1}{h^2} - \rho^{(2)} = h^2 \rho^{(4)} + h^4 \rho^{(6)} + \ldots. \quad (3.13)$$

These relations suggest that the rescaled LB moments $\frac{v_h}{h}$ and $\frac{\rho_h - 1}{3h^2}$ should be taken as approximations of the velocity $v^{(1)}$ and pressure $\rho^{(2)}/3$ of the generalized Newtonian fluids with second-order accuracy in space and first-order accuracy in time. This confirms the observation of [15] based on numerical simulations. They also provide a basis for using the Richardson extrapolation technique to improve the accuracy up to higher orders.

Furthermore, we can follow [25] to deduce from the theorem that

$$\tau(S/h)S = \frac{3 \sum_i (f^{eq}_i - f_i) c_i \otimes c_i}{h} + O(h^2).$$

This relation hints an alternative way to implement the LB method with the first collision mechanism, instead of computing the shear-rate tensor $S = \nabla v + (\nabla v)^T$ directly from $v$ with finite difference or other methods. This issue is absent for classical Newtonian fluids, where $\mu$ and thereby $\tau$ are constant. Indeed, we could use the relation

$$\Sigma = \frac{3 \sum_{i=1}^8 (f^{eq}_i - f_i) c_i \otimes c_i}{h^2 \tau(\Sigma)}$$

to compute $\Sigma = S/h$ and thereby $\tau(\Sigma)$ by iteration [7, 12, 14]. In this way, the computation of $S$ involves the LB solution only at the current lattice point.
4 A construction criterion

In this section, we verify the structural precondition of Theorem 1:

\[ \mu(I) + 2I \min\{\mu'(I), 0\} \geq 0 \]  

for some widely used constitutive relations for generalized Newtonian fluids, including the power-law model \cite{4}, Carreau model \cite{8}, Carreau-Yasuda model \cite{4}, a smoothed Bingham model \cite{26}, a smoothed Casson model \cite{27}, and so on. We only consider the smoothed Bingham and Casson models instead of the original ones, because the smoothness of viscosity is required in our analysis. Notice that up to now we have treated the dynamic shear viscosity \( \mu = \mu(I) \) as a function of the invariant \( I = tr(S^2) \). However, in the literature it is preferred to use the magnitude \( \dot{\gamma} = \sqrt{I/2} \) of the rate-of-strain tensor, instead of \( I \) itself.

Considering the viscosity \( \mu = \mu(\dot{\gamma}) \) as a function of \( \dot{\gamma} \), we have \( 2I\mu'(I) = \dot{\gamma} \frac{d}{d\dot{\gamma}} \mu(\dot{\gamma}) \). Thus the precondition in (4.1) becomes

\[ \mu(\dot{\gamma}) + \dot{\gamma} \min\{\frac{d}{d\dot{\gamma}} \mu(\dot{\gamma}), 0\} \geq 0. \]

Moreover, it is not difficult to see that the last inequality is equivalent to

\[ \frac{d}{d\dot{\gamma}} [\dot{\gamma} \mu(\dot{\gamma})] \geq 0, \]  

for the viscosity \( \mu(\dot{\gamma}) \) is non-negative. The last one says nothing but that \( \dot{\gamma} \mu(\dot{\gamma}) \) is a monotone increasing function of \( \dot{\gamma} \geq 0 \). Remark that the stress tensor \( F = \mu(\dot{\gamma})S \) and

\[ F : F = \mu^2 I = 2\mu(\dot{\gamma})^2 \dot{\gamma}^2. \]

The above condition is just that the magnitude of the stress tensor increases with increasing the magnitude of the rate-of-strain tensor.

Now we turn to several concrete models. The power-law model is given in \cite{4} as

\[ \mu(\dot{\gamma}) = \mu_p \dot{\gamma}^{n-1}, \]

which contains two parameters \( \mu_p \) and \( n \). Here \( \mu_p \) is the flow consistency coefficient and \( n \) is the power-law index of fluid. According to the index \( n \), the power-law fluid can be divided into three different types. The case \( n < 1 \) corresponds to a shear-thinning or pseudo-plastic fluid, which is widely used in practice, whereas \( n > 1 \) corresponds to a shear-thickening or dilatant fluid, and \( n = 1 \) reduces to the classical Newtonian fluid. Obviously, \( \dot{\gamma} \mu(\dot{\gamma}) \) is a monotone increasing function of \( \dot{\gamma} \geq 0 \) if \( n \geq 0 \).

Another model for shear-shinning fluids is the Carreau model which is preferred and used more widely in industrial applications than the power-law model. Its viscosity is given in \cite{8} as

\[ \mu(\dot{\gamma}) = \mu_c + (\mu_0 - \mu_c)[1 + (\lambda \dot{\gamma})^2]^{(n-1)/2} \quad \text{for } 0 < n \leq 1. \]
Here $\mu_0$ is the zero-shear-rate viscosity ($\dot{\gamma} \to 0$), $\mu_\infty$ is the infinity-shear-rate viscosity ($\dot{\gamma} \to \infty$), and $\lambda$ is a time constant. Notice that $\mu_0 > \mu_\infty$ for shear-shinning fluids. We compute
\[
\frac{d}{d\dot{\gamma}} [\dot{\gamma}\mu(\dot{\gamma})] = \mu_\infty + (\mu_0 - \mu_\infty)[1 + \lambda^2\dot{\gamma}^2]\frac{n-3}{2} (1 + n\lambda^2\dot{\gamma}^2) > 0
\]
for $0 < n \leq 1$ and $\mu_0 > \mu_\infty$. Namely, $\dot{\gamma}\mu(\dot{\gamma})$ is a monotone increasing function of $\dot{\gamma} \geq 0$.

A slightly generalization of the above model is called Carreau-Yasuda model [4]
\[
\mu(\dot{\gamma}) = \mu_\infty + (\mu_0 - \mu_\infty)[1 + (\lambda\dot{\gamma})^a]^{(1-n)/a},
\]
where the parameters have the same meaning as above and the new parameter $a$ is an extra material constant. Another model for shear-thinning is the Cross model [28]
\[
\mu(\dot{\gamma}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + (\dot{\gamma})^{(1-n)}}.
\]
These models can also describe shear-thickening fluids, where $n > 1$ and the meanings of $\mu_0, \mu_\infty$ exchange as follows
\[
\lim_{\dot{\gamma} \to 0} \mu(\dot{\gamma}) = \mu_\infty, \quad \lim_{\dot{\gamma} \to \infty} \mu(\dot{\gamma}) = \mu_0.
\]
It is easy to see that the structural condition holds also for these models.

The viscosity of the original Bingham model is not a continuous function of the shear rate. Such a discontinuous function is not suitable for numerical simulations [16]. In [26], Papanastasiou proposed the following model as a smoothed version of the Bingham model:
\[
\mu(\dot{\gamma}) = \frac{\tau_0}{\dot{\gamma}} (1 - e^{-m\dot{\gamma}}) + \eta_p.
\]
Here $\tau_0$ is the yield stress, $m$ is the stress growth exponent (regularization parameter), and $\eta_p$ is the plastic viscosity. Obviously, $\dot{\gamma}\mu(\dot{\gamma})$ is a monotone increasing function of $\dot{\gamma} \geq 0$.

Another discontinuous model is the Casson model [29]. Its following smoothed version
\[
\mu(\dot{\gamma}) = \left[ \sqrt{\frac{\tau_0}{\dot{\gamma}} (1 - e^{-\sqrt{m}\dot{\gamma}})} + \sqrt{\eta_p} \right]^2
\]
was introduced in [27]. Here the parameters $\tau_0, m$ and $\eta_p$ are same as in the last model. It is obvious that $\dot{\gamma}\mu(\dot{\gamma})$ is a monotone increasing function of $\dot{\gamma} \geq 0$.

We conclude this section with the Powell-Eyring model [28]
\[
\mu(\dot{\gamma}) = \mu_\infty + (\mu_0 - \mu_\infty)\frac{\sinh^{-1}(\lambda\dot{\gamma})}{\lambda\dot{\gamma}},
\]
where $\mu_0, \mu_\infty$ and $\lambda$ are material constants, $\lim_{\dot{\gamma} \to 0} \mu(\dot{\gamma}) = \mu_0$ and $\lim_{\dot{\gamma} \to \infty} \mu(\dot{\gamma}) = \mu_\infty$. Obviously, the magnitude of the stress tensor $\dot{\gamma}\mu(\dot{\gamma})$ for this model is also monotone increasing.
5 Summary

In this article we present a general methodology to conduct a detailed asymptotic analysis of the two LB-BGK models for generalized Newtonian fluids. We would like to point out that our analysis is quite different from that based on the Chapman-Enskog expansion, which leads to the compressible Navier-Stokes equations. Our analysis uses the Hilbert expansion and gives the incompressible equations directly. It can expose certain important features of the LB solutions. As shown in Section 3, such an analysis provides a theoretical basis not only for using the Richardson extrapolation technique to improve the accuracy up to higher orders, but also for using the iteration method to compute the the rate-of-strain tensor. The latter makes sense specially for generalized Newtonian fluids.

In contrast to the analysis for classical Newtonian fluids \[17\] \[18\], we have introduced the lattice spacing \( h \) into the equilibrium distributions in (2.4) and (2.10). More importantly, we have to deal with the possibly non-zero terms \( F^{(n)} \) in (3.10) and \( A^{(n)} \) in (3.12) properly.

As a by-product of our analysis, a seemingly new structural condition on the generalized Newtonian fluids is singled out. This condition is that the magnitude of the stress tensor increases with increasing the shear rate (magnitude of the rate-of-strain tensor). We verify this condition for all the existing constitutive relations which are known to us. By inductive reasoning, we suggest to take this structural condition as a criterion in constructing further constitutive relations for generalized Newtonian fluids.

Finally, let us mention that our analysis can be easily extended to the multiple-relaxation-time (MRT) models \[20\] \[21\] \[22\] or to three-dimensional lattices, although it is presented only for BGK models on the two-dimensional lattice D2Q9 \[23\].

Appendix

In this Appendix we derive the equations in (3.11) and prove Theorem 1 for the two LB models defined in (2.4) and (2.10). For this purpose, we will often use, without notice, the following simple facts that \( w_i \) is even, \( c_i \) is odd,

\[
\sum_{i=0}^{8} w_i = 1, \quad \sum_{i} w_i c_i c_i = \frac{1}{3} I_2
\]

with \( I_2 \) the unit matrix of order 2. From these facts we easily deduce that

\[
\sum_i f_{iL}(\rho, v) \equiv \rho, \quad \sum_i f_{iQ}(u, v) \equiv 0,
\]

\[
\sum_i c_i f_{iL}(\rho, v) \equiv v, \quad \sum_i c_i f_{iQ}(u, v) \equiv 0.
\]

The basic idea of our proofs is similar to that in \[17\] \[18\], but the possibly non-zero terms \( F^{(n)} \) in (3.10) and \( A^{(n)} \) in (3.12) have to be treated properly.
A1. The first model

We begin with the hierarchy of equations in (3.8)–(3.10):

\[ h^1: \quad f_i^{(1)} = f_{iL}(\rho^{(1)}, v^{(1)}), \]
\[ h^2: \quad \tau(S^{(1)}) D_{i,1} f_i^{(1)} + f_i^{(2)} = f_{iL}(\rho^{(2)}, v^{(2)}) + f_{iQ}(v^{(1)}, v^{(1)}), \]
\[ h^k: \quad \tau(S^{(1)}) \sum_{s=1}^{k-1} D_{i,s} f_i^{(k-s)} + F^{(1)} \sum_{s=1}^{k-2} D_{i,s} f_i^{(k-1-s)} + \cdots + F^{(k-2)} D_{i,1} f_i^{(1)} + f_i^{(k)} = f_{iL}(\rho^{(k)}, v^{(k)}) + \sum_{p+q=k} f_{iQ}(v^{(p)}, v^{(q)}), \]

for \( k \geq 3 \). Summing up two sides of (5.4) over \( i \) and using (5.2) together with \( D_{i,1} f_i^{(1)} = c_i \cdot \nabla \), we obtain

\[ 0 = \tau(S^{(1)}) \sum_i c_i \cdot \nabla f_i^{(1)} = \tau(S^{(1)}) \nabla \cdot v^{(1)}. \]

Here we have used the definitions of \( \rho^{(2)} \) and \( v^{(1)} \) given in (3.2). Thus we have

\[ \nabla \cdot v^{(1)} = 0 \] (5.6)

for \( \tau(S^{(1)}) = 3\mu(S^{(1)} : S^{(1)}) + \frac{1}{2} > 0 \). Secondly, we multiply (5.4) with \( c_i \) and sum up the resultant equality to obtain

\[ \tau(S^{(1)}) \sum_i c_i (c_i \cdot \nabla f_i^{(1)}) = 0. \]

Moreover, it follows from (5.1) and the expression of \( f_i^{(1)} \) given in (5.3) that

\[ \nabla \rho^{(1)} = 0. \]

Thus, \( \rho^{(1)} \) is spatially homogenous.

Similarly, we deduce from (5.5) with \( k = 3 \) that

\[ 0 = \tau(S^{(1)}) (\sum_i D_{i,2} f_i^{(1)} + \sum_i c_i \cdot \nabla f_i^{(2)}) + F^{(1)} \sum_i c_i \cdot \nabla f_i^{(1)} = \tau(S^{(1)}) (\sum_i D_{i,2} f_i^{(1)} + \sum_i c_i \cdot \nabla f_i^{(2)}), \]
\[ 0 = \tau(S^{(1)}) (\sum_i c_i D_{i,2} f_i^{(1)} + \sum_i c_i c_i \cdot \nabla f_i^{(2)}) + F^{(1)} \sum_i c_i c_i \cdot \nabla f_i^{(1)} = \tau(S^{(1)}) (\sum_i c_i D_{i,2} f_i^{(1)} + \sum_i c_i c_i \cdot \nabla f_i^{(2)}). \]

Thus, we have

\[ \sum_i D_{i,2} f_i^{(1)} + \sum_i c_i \cdot \nabla f_i^{(2)} = 0, \] (5.7)
\[ \sum_i c_i D_{i,2} f_i^{(1)} + \sum_i c_i c_i \cdot \nabla f_i^{(2)} = 0, \]

for \( \tau(S^{(1)}) > 0 \). Note that \( D_{i,2} = \partial_t + (c_i \cdot \nabla)^2/2 \) and \( \rho^{(1)} \) is spatially homogenous. The first equation in
that I use 

\[ \nabla \cdot f = 0 \]

Moreover, we use \( \nabla \cdot v^{(1)} = 0 \) to compute the two parts in the second equation in (5.4): 

\[
\sum_i c_i D_{i,3} f_i^{(1)} = \frac{\partial v^{(1)}}{\partial t} + \frac{1}{2} \sum_i c_i (c_i \cdot \nabla)^2 f_{1L}(\rho^{(1)}, v^{(1)}) = \frac{\partial \rho^{(1)}}{\partial t} + \frac{1}{6} \Delta \rho^{(1)},
\]

(5.8a)

\[
\sum_i c_i (c_i \cdot \nabla) f_i^{(2)} = \sum_i c_i (c_i \cdot \nabla) [f_{1L}(\rho^{(2)}, v^{(2)}) + f_{1Q}(v^{(1)}, v^{(1)}) - \tau(S^{(1)}) c_i \cdot \nabla f_i^{(1)}],
\]

\[
= \nabla \frac{\rho^{(2)}}{3} + v^{(1)} \cdot \nabla v^{(1)} - \nabla \frac{\tau(S^{(1)})}{3} S^{(1)}.
\]

(5.8b)

Here we have used the following identity 

\[
\sum_i w_i c_i (c_i \cdot \nabla) (c_i \cdot u)(c_i \cdot v) = \frac{1}{9} \left( \frac{2\partial_x (u_1 v_1) + \partial_y (u \cdot v) + \partial_z (u_2 v_1 + u_1 v_2)}{2\partial_x (u_2 v_2) + \partial_y (u \cdot v) + \partial_z (u_2 v_1 + u_1 v_2)} \right)
\]

for \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \). By combining (5.6) and (5.8), we arrive at (5.11).

Furthermore, we can consecutively deduce from (5.5) that

\[
\sum_i \sum_{s=1}^{k-1} D_{i,s} f_i^{(k-s)} = 0, \quad \sum_i \sum_{s=1}^{k-1} c_i D_{i,s} f_i^{(k-s)} = 0
\]

(5.9)

for \( k \geq 3 \). Observe that \( D_{i,s} = (-1)^s D_{i,s} \).

Now we prove Theorem 1 by induction on \( k \). For \( k = 0 \), the conclusion follows simply from the choice of \( f_i^{(k)} = w_i \) and the evenness of \( w_i \).

Assume \( f_i^{(l)} = (-1)^l f_i^{(l)} \) for \( l \leq k \). We show \( f_i^{(k+1)} = (-1)^{k+1} f_i^{(k+1)} \). The inductive assumption simply implies that \( v^{(l)} = 0 \) and thereby \( S^{(l)} = 0 \) for even \( l \leq k \). With this fact, we observe that the quadratic term in (5.8) is always even and vanishes for \( k \) odd. Indeed, for \( k \) odd we see from \( p + q = k \) that one of \( p \) and \( q \) must be even and thereby the corresponding \( v^{(p)} \) or \( v^{(q)} \) vanishes. Moreover, we recall that \( I_h = tr(S_h')/h^2 \) and compute the coefficient

\[
F^{(m)} = \frac{1}{m!} \sum_{j=1}^m d^j \tau(I_h) \bigg|_{h=0} \sum_{k_1 + \cdots + k_j = m} \frac{d^k I_h}{dh^{k_1}} \bigg|_{h=0} \frac{d^{k_2} I_h}{dh^{k_2}} \bigg|_{h=0} \cdots \frac{d^{k_j} I_h}{dh^{k_j}} \bigg|_{h=0}
\]

by the chain rule, which can be proved by induction on \( m \). From this formula and the expression of \( I_h \) it is easy to see that

\[
F^{(m)} = 0
\]
for all odd $m < k$ and
\[ F^{(k)} = \frac{d^r(I_0)}{dI} \mid_{h=0} S^{(1)} : S^{(k+1)} = \frac{d^r(I_0)}{dI} \mid_{h=0} S^{(1)} : S^{(k+1)}, \]  
(5.10)
for odd $k$. Thus, we deduce from (5.9) that
\[ f_i^{(k+1)} = f_{iL}(\rho^{(k+1)}, v^{(k+1)}) \equiv g_i^k = (-1)^{k+1} g_i^k, \]
\[ f_i^{(k+2)} = f_{iL}(\rho^{(k+2)}, v^{(k+2)}) - 2f_{iQ}(v^{(1)}, v^{(k+1)}) + \tau(S^{(1)})D_{i,1}f_i^{(k+1)} + F^{(k)}D_{i,1}f_i^{(1)} \equiv \tilde{g}_i^k = (-1)^k \tilde{g}_i^k. \]  
(5.11)
Consequently, it suffices to prove that $\rho^{(k+1)} = 0$ in case $k$ is even or $v^{(k+1)} = 0$ in case $k$ is odd, for $f_{iL}(\rho, v) = w_i(\rho + 3c_i \cdot v)$.  

For $k$ even, it follows from (5.9), the inductive assumption and the oddness of $g_i^k$ defined in (5.11) that
\[ 0 = \sum_i \sum_{s=1}^{k+1} c_i D_{i,s} f_i^{(k+2-s)} = \sum_i c_i D_{i,1} f_i^{(k+1)} = \sum_i c_i (c_i \cdot \nabla) [f_{iL}(\rho^{(k+1)}, v^{(k+1)}) + g_i^k] \]
\[ = \sum_i c_i (c_i \cdot \nabla) f_{iL}(\rho^{(k+1)}, v^{(k+1)}) = \nabla \frac{\rho^{(k+1)}}{3}. \]
Namely, $\rho^{(k+1)}$ is spatially homogeneous. Moreover, we deduce from (5.9) that
\[ 0 = \sum_i \sum_{s=1}^{k+2} D_{i,s} f_i^{(k+3-s)} = \sum_i D_{i,1} f_i^{(k+2)} + \sum_i D_{i,2} f_i^{(k+1)} = \partial_t \rho^{(k+1)} + \nabla \cdot v^{(k+2)}. \]
This, together with the initial condition $\rho^{(k+1)} \mid_{t=0} = 0$ and the periodicity of the problem, gives $\rho^{(k+1)} = 0$ and thereby $\nabla \cdot v^{(k+2)} = 0$.  

For $k$ odd, $(k - 1)$ is even and we have $\nabla \cdot v^{(k+2)} = 0$ as above. Moreover, we see from (5.9) and (5.11) that
\[ 0 = \sum_i c_i D_{i,2} f_i^{(k+1)} + \sum_i c_i D_{i,1} f_i^{(k+2)} \]
\[ = \sum_i c_i D_{i,2} [f_{iL}(\rho^{(k+1)}, v^{(k+1)}) + g_i^k] + \sum_i c_i D_{i,1} [f_{iL}(\rho^{(k+2)}, v^{(k+2)}) + 2f_{iQ}(v^{(1)}, v^{(k+1)}) - \tau(S^{(1)})D_{i,1}f_i^{(k+1)} + F^{(k)}D_{i,1}f_i^{(1)} + \tilde{g}_i^k] \]
\[ = \sum_i c_i D_{i,2} f_{iL}(\rho^{(k+1)}, v^{(k+1)}) + \sum_i c_i D_{i,1} f_{iL}(\rho^{(k+2)}, v^{(k+2)}) + 2f_{iQ}(v^{(1)}, v^{(k+1)}) - \tau(S^{(1)})D_{i,1}f_i^{(k+1)} - F^{(k)}D_{i,1}f_i^{(1)} \]
\[ = \frac{\partial v^{(k+1)}}{\partial t} + \frac{1}{6} \Delta v^{(k+1)} + \frac{1}{3} \nabla \rho^{(k+2)} + v^{(k+1)} \cdot \nabla v^{(1)} + v^{(1)} : \nabla v^{(k+1)} - \frac{1}{3} \nabla \cdot [\tau(S^{(1)})S^{(k+1)} + F^{(k)}S^{(1)}]. \]
Here the computations in (5.8) have been used in the last step. Then we have

\[ \nabla \cdot v^{(k+1)} = 0, \]

\[ \frac{\partial v^{(k+1)}}{\partial t} + \nabla \rho^{(k+2)} + v^{(1)} \cdot \nabla v^{(k+1)} + v^{(k+1)} \cdot \nabla v^{(1)} = \frac{1}{3} \nabla \cdot \left[ (\tau(S^{(1)}) - \frac{1}{2} ) S^{(k+1)} \right] + \frac{1}{3} \nabla \cdot [F^{(k)} S^{(1)}]. \tag{5.12} \]

Next we show that \( v^{(k+1)} = 0 \) by using the equations in (5.12) together with the initial condition \( v^{(k+1)}|_{t=0} = 0 \). Note that the equations in (5.12) with \( k \) odd are linear with respect to \( v^{(k+1)} \). Taking the inner product of the second equation in (5.12) with \( v^{(k+1)} \) and integrating the resultant equality, we get

\[ \frac{1}{2} \frac{d}{dt} \| v^{(k+1)} \|^2_{L^2} + \int_\Omega v^{(k+1)} \cdot (v^{(k+1)} \cdot \nabla v^{(1)}) dx = \int_\Omega v^{(k+1)} \cdot (\nabla v^{(1)} - \mu(I^{(1)}) S^{(1)}) dx + \frac{1}{3} \int_\Omega F^{(k)} S^{(1)} : \nabla v^{(k+1)} dx \]

\[ - \int_\Omega v^{(k+1)} (v^{(1)} \cdot \nabla v^{(1)}) dx - \int_\Omega \mu(I^{(1)}) S^{(k+1)} : \nabla v^{(k+1)} dx + \frac{1}{3} \int_\Omega F^{(k)} S^{(1)} : \nabla v^{(k+1)} dx \]

\[ \leq \| \nabla v^{(1)} \|_{L^\infty} \| v^{(k+1)} \|^2_{L^2} + \frac{1}{2} \int_\Omega \mu(I^{(1)}) S^{(k+1)} : S^{(k+1)} dx - \frac{1}{6} \int_\Omega F^{(k)} S^{(1)} : S^{(k+1)} dx \]

\[ \leq \| \nabla v^{(1)} \|_{L^\infty} \| v^{(k+1)} \|^2_{L^2} - \frac{1}{2} \int_\Omega \mu(I^{(1)}) S^{(k+1)} : S^{(k+1)} dx - \int_\Omega \mu'(I^{(1)}) S^{(1)} : S^{(k+1)} dx \]

\[ \leq \| \nabla v^{(1)} \|_{L^\infty} \| v^{(k+1)} \|^2_{L^2} - \frac{1}{2} \int_\Omega \mu(I^{(1)}) + 2 \min \{ \mu'(I^{(1)}), 0 \} I^{(1)} S^{(k+1)} : S^{(k+1)} dx. \]

Here the last inequality is due to the Cauchy-Schwartz inequality. Thanks to \( \mu(I) + 2 \min \{ \mu'(I), 0 \} I \geq 0 \), the last estimate leads to

\[ \frac{d}{dt} \| v^{(k+1)} \|^2_{L^2} \leq 2 \| \nabla v^{(1)} \|_{L^\infty} \| v^{(k+1)} \|^2_{L^2}. \]

This differential inequality together with the initial condition \( v^{(k+1)}|_{t=0} = 0 \) simply gives \( v^{(k+1)} = 0 \).

This completes the proof of Theorem 1 for the first model.
A2. The second model

This subsection is devoted to the second model. By equating the coefficient of $h^k$ in the two sides of (5.12) and using $v^{(0)} = 0$, we obtain the following equations

\begin{align*}
h^0 : & \quad f_i^{(0)}(0) = f_{iL}(\rho^{(0)}, v^{(0)}) = f_{iL}(1,0), \\
h^1 : & \quad f_i^{(1)}(0) = f_{iL}(\rho^{(1)}, v^{(1)}), \\
h^2 : & \quad \sum_{s=1}^{k-1} D_i, s f_i^{(k-s)} + f_i^{(2)} = f_{iL}(\rho^{(2)}, v^{(2)}) + f_{iQ}(v^{(1)}, v^{(1)}) + w_i A^{(0)} S^{(1)} : c_i^t c_i, \\
h^k : & \quad \tau \sum_{s=1}^{k-1} D_i, s f_i^{(k-s)} + f_i^{(k)} = f_{iL}(\rho^{(k)}, v^{(k)}) + \sum_{p+q=k} f_{iQ}(v^{(p)}, v^{(q)}) + \sum_{a+b=k-1} w_i A^{(a)} S^{(b)} : c_i^t c_i \tag{5.15}
\end{align*}

for $k \geq 3$. Note that

\begin{equation}
\sum_i w_i A^{(a)} S^{(b)} : c_i^t c_i = A^{(a)} S^{(b)} : \sum_i w_i c_i^t c_i = \frac{2}{3} A^{(a)} \nabla \cdot v^{(b)}. \tag{5.16}
\end{equation}

Summing up the two sides of (5.13) over $i$, we get

\begin{equation}
\tau \sum_i c_i \cdot \nabla f_i^{(1)} = \frac{2}{3} A^{(0)} \nabla \cdot v^{(1)}
\end{equation}

and thereby

\begin{equation}
(\tau - \frac{2}{3} A^{(0)}) \nabla \cdot v^{(1)} = 0.
\end{equation}

This gives

\begin{equation}
\nabla \cdot v^{(1)} = 0 \tag{5.17}
\end{equation}

for $A^{(0)} = A(S^{(1)})$ and $\tau - \frac{4}{3} A(S^{(1)}) = \frac{1}{2} + 3\mu(S^{(1)} : S^{(1)}) > 0$. On the other hand, we multiply the two sides of (5.13) by $c_i$ and sum up the resultant equalities over $i$ to obtain

\begin{equation}
\tau \sum_i c_i (c_i \cdot \nabla f_i^{(1)}) = 0
\end{equation}

and thereby

\begin{equation}
\nabla \rho^{(1)} = 0.
\end{equation}

Thus, $\rho^{(1)}$ is spatially homogenous.

Similarly, we deduce from (5.15) with $k = 3$ that

\begin{equation}
\tau \sum_i D_{i,1} f_i^{(1)} + \sum_i D_{i,1} f_i^{(2)} = \sum_i w_i A^{(0)} S^{(2)} : c_i^t c_i + \sum_i w_i A^{(1)} S^{(1)} : c_i^t c_i = \frac{2}{3} A^{(0)} \nabla \cdot v^{(2)} + \frac{2}{3} A^{(1)} \nabla \cdot v^{(1)}.
\end{equation}
Since $\nabla \cdot v^{(1)} = 0$ and $\rho^{(1)}$ is spatially homogenous, the last equation leads to
\[
\frac{\partial \rho^{(1)}}{\partial t} + \nabla \cdot v^{(2)} = \frac{2A^{(0)}}{3\tau}\nabla \cdot v^{(2)} \tag{5.18}
\]
and thereby
\[
\left(1 - \frac{2A^{(0)}}{3\tau}\right)^{-1} \frac{\partial \rho^{(1)}}{\partial t} = \nabla \cdot v^{(2)}
\]
for $1 - \frac{2A^{(0)}}{3\tau} A > 0$ and $A^{(0)} = A(S^{(1)})$. Integrating the last equation over the periodic domain and using the spatial homogeneousness of $\rho^{(1)}$ gives
\[
\frac{\partial \rho^{(1)}}{\partial t} \int \left(1 - \frac{2A^{(0)}}{3\tau}\right)^{-1} dx = 0.
\]
and thereby $\frac{\partial \rho^{(1)}}{\partial t} = 0$. This together with the initial condition $\rho^{(1)}|_{t=0} = 0$ implies that $\rho^{(1)} = 0$, $\nabla \cdot v^{(2)} = 0$. \tag{5.19}

On the other hand, we multiply the two sides of (5.15) with $k = 3$ by $c_i$ and sum up the resultant equalities over $i$ to obtain
\[
\tau \left(\sum_i c_i D_{i,2} f_i^{(1)} + \sum_i c_i D_{i,1} f_i^{(2)}\right) = 0.
\]
From this, we deduce as from (5.7) to (5.8) that
\[
\sum_i c_i D_{i,2} f_i^{(1)} = \frac{\partial v^{(1)}}{\partial t} + \frac{1}{6} \Delta v^{(1)}, \tag{5.20a}
\]
\[
\sum_i c_i D_{i,1} f_i^{(2)} = \frac{\nabla v^{(2)}}{3} + v^{(1)} \cdot \nabla v^{(1)} - \frac{\tau}{3} \Delta v^{(1)} + \frac{2}{9} \nabla \cdot (A^{(0)} S^{(1)}) \tag{5.20b}
\]
By combining these with (5.17), we arrive at (3.11).

Now we turn to prove Theorem 1 for the second model again by induction on $k$. For $k = 0$, the conclusion follows simply from the choice of $f_i^{(k)} = w_i$.

Assume $f_i^{(l)} = (-1)^l f_i^{(l)}$ for $l \leq k$. We show $f_i^{(k+1)} = (-1)^{k+1} f_i^{(k+1)}$. As in the previous subsection, we deduce from (5.15) that
\[
f_i^{(k+1)} - f_{iL}(\rho^{(k+1)}, v^{(k+1)}) \equiv g_i^k = (-1)^{k+1} g_i^k, \tag{5.21}
\]
Consequently, it suffices to prove that $\rho^{(k+1)} = 0$ in case $k$ is even or $v^{(k+1)} = 0$ in case $k$ is odd, for $f_{iL}(\rho, v) = w_i (\rho + 3c_i \cdot v)$.

For $k$ even, it follows from (5.15), the inductive assumption and the oddness of $g_i^k$ defined in (5.21)
that

\[ 0 = \sum_i \sum_{s=1}^{k+1} c_i D_{i,s} f_i^{(k+2-s)} = \sum_i c_i D_{i,1} f_i^{(k+1)} = \sum_i c_i (c_i \cdot \nabla) [f_{iL} (\rho^{(k+1)}, v^{(k+1)}) + g_i^k] = \sum_i c_i (c_i \cdot \nabla) f_{iL} (\rho^{(k+1)}, v^{(k+1)}) = \nabla \frac{\rho^{(k+1)}}{3}. \]

Namely, \( \rho^{(k+1)} \) is spatially homogeneous. Moreover, we deduce from (5.15) and (5.16) that

\[ \frac{2}{3 \tau} A^{(0)} \cdot v^{(k+2)} = \sum_i \sum_{s=1}^{k+2} D_{i,s} f_i^{(k+2-s)} = \sum_i D_{i,1} f_i^{(k+2)} + \sum_i D_{i,2} f_i^{(k+1)} = \partial_t \rho^{(k+1)} + \nabla \cdot v^{(k+2)}. \]

As from (5.18) to (5.19), this gives \( \rho^{(k+1)} = 0 \) and thereby \( \nabla \cdot v^{(k+2)} = 0. \)

For \( k \) odd, \( (k - 1) \) is even and we have \( \nabla \cdot v^{(k+1)} = 0 \) as above. Moreover, we see from (5.15) and (5.21) that

\[ 0 = \sum_i c_i D_{i,1} f_i^{(k+1)} + \sum_i c_i D_{i,2} f_i^{(k+2)} = \sum_i c_i D_{i,2} [f_{iL} (\rho^{(k+1)}, v^{(k+1)}) + g_i^k] + \sum_i c_i D_{i,1} [f_{iL} (\rho^{(k+2)}, v^{(k+2)}) + 2f_i \cdot \nabla \cdot v^{(k+2)} - \tau D_{i,1} f_i^{(k+1)} + w_i A^{(0)} S^{(k+1)} : c_i' c_i + \frac{\tau}{3} A^{(k)} S^{(1)} : c_i'] = \frac{\partial_t v^{(k+1)}}{\partial t} + \frac{1}{6} \Delta v^{(k+1)} + \frac{1}{3} \nabla \rho^{(k+2)} + v^{(k+1)} \cdot \nabla v^{(1)} + v^{(1)} \cdot \nabla v^{(k+1)} - \frac{2}{9} \nabla \cdot [A^{(0)} S^{(k+1)}] + \frac{2}{9} \nabla \cdot [A^{(k)} S^{(1)}]. \]

Here the computations in (5.20) have been used in the last step. Thus we have

\[ \nabla \cdot v^{(k+1)} = 0, \]

\[ \frac{\partial_t v^{(k+1)}}{\partial t} + \frac{1}{3} \nabla \rho^{(k+2)} + v^{(k+1)} \cdot \nabla v^{(1)} + v^{(1)} \cdot \nabla v^{(k+1)} = \nabla \cdot [\mu (S^{(1)} : S^{(1)}) S^{(k+1)}] - \frac{2}{9} \nabla \cdot [A^{(k)} S^{(1)}], \quad (5.22) \]

for

\[ \frac{1}{3} \tau - \frac{1}{2} + \frac{2}{9} A^{(0)} = \mu (S^{(1)} : S^{(1)}). \]

Since

\[ A^{(k)} = -9 \mu' (S^{(1)} : S^{(1)}) S^{(1)} : S^{(k+1)} \]

which is similar to \( F^{(k)} \) in (5.10), the equations in (5.22) is exactly same as those in (5.12) and, hence, we have \( v^{(k+1)} = 0. \)
This completes the proof.

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