Non-iterative smearing correction for the actuator line method

V. G. Kleine$^{1,2}$, A. Hanifi$^1$, and D. S. Henningson$^1$

$^1$KTH Royal Institute of Technology, FLOW, Department of Engineering Mechanics, Stockholm, Sweden
$^2$Instituto Tecnológico de Aeronáutica, Division of Aeronautical and Aerospace Engineering, São José dos Campos - SP, Brazil

Abstract

The actuator line method (ALM) is extensively used in wind turbine and rotor simulations. However, its original uncorrected formulation overestimates the forces near the tip of the blades and does not reproduce well forces on translating wings. The recently proposed smearing correction for the ALM is a correction based on physical and mathematical properties of the simulation that allows for a more accurate and general ALM. So far, to correct the forces on the blades, the smearing correction depended on an iterative process at every time step, which is usually slower, less stable and less deterministic than direct methods. In this work, a non-iterative process is proposed and validated. First, we propose a formulation of the non-linear lifting line that is equivalent to the ALM with smearing correction, showing that their results are practically identical for a translating wing. Then, by linearizing the lifting line method, the iterative process of the correction is substituted by the direct solution of a small linear system. No significant difference is observed in the results of the iterative and non-iterative corrections, both in wing and rotor simulations. Additional contributions of the present work include the use of a more accurate approximation for the velocity induced by a smeared vortex segment and the implementation of a free-vortex wake model to define the vortex sheet, that contribute to the accuracy and generality of the method. The results present here may motivate the adoption of the ALM by other communities, for example, in fixed-wing applications.

1 Introduction

The actuator line method (ALM) was developed to represent blades or wings as lifting lines in numerical simulations of the Navier-Stokes equations (Sørensen & Shen 2002). It allows the reduction of computational costs by replacing the geometry of the blades by lines carrying body forces calculated using the local velocity and airfoil data (Sørensen et al. 2015).

In order to accurately calculate the forces, the local velocity needs to be corrected near the tip of the blades. This correction has been a mildly controversial issue in the past, with different proposed models (Shen et al. 2005, Sørensen 2016) and even arguments that it should not be needed (Martinez et al. 2012, Sørensen 2016) due to the fact that the ALM creates tip vortices that should lead to lower loads near the tip. However, the overestimation of loads near the tip on the numerical results indicates that a correction is needed (Sørensen 2016, Dağ 2017).

Some recent discoveries have shed some light on this apparent controversy. Dağ & Sørensen (2020) (originally in (Dağ 2017)) observed that the bound vortex created by the actuator line showed a Gaussian vorticity distribution equals to a Lamb-Oseen vortex model (Lamb 1932, Oseen 1911, Saffman 1992). The authors then conjectured that this pattern would be extended to the vortex sheet and proposed a correction based on this model to approximate the velocity in the ALM to the velocity induced by the singular vorticity distribution predicted by a discretized Prandtl’s lifting line. The simulations of rotors showed clear improvements compared to the uncorrected ALM. The method brings the forces smoothly to zero near the tip and the hub of the blades. The results for a translating planar wing show that the results approximate a lifting line method for this case. Meyer Forsting et al. (2019a) compared the results of a similar correction with the results of a lifting line technique for rotating blades with and without a viscous core model. They showed that the forces of the ALM without correction agree with a vortex method with finite core size, while the ALM with a smearing correction agrees with the vortex method using ideal vortices.
The mathematical connection between the vortices generated by the ALM and a Lamb-Ossen vortex model has been proven by Forsythe et al. (2015) for the bound vortex and Martínez-Tossas & Meneveau (2019) for the vortex sheet of a translating wing. They showed that it is a consequence of the convolution of the discrete forces with a Gaussian kernel, necessary to distribute the forces and avoid numerical instabilities. The correction of (Martínez-Tossas & Meneveau 2019), termed “subfilter-scale velocity correction” or “filtered actuator line model”, is, in essence, a variant of a smearing correction. It was applied to a wind turbine by (Stanly et al. 2022).

Caprace et al. (2019) modified Prandtl’s lifting line by distributing the vorticity by using a Gaussian kernel. Even though this work does not concern directly the ALM, some connections between the conclusions and ALM were inferred by the authors. In this work, a variable smoothing parameter approaching zero near the tip leads to improved results, a result also observed in ALM works (Shives & Crawford 2013, Jha et al. 2014, Jha & Schmitz 2018). However, in the ALM, a low value of smoothing parameter \( \varepsilon \) may lead to numerical instabilities. Since the minimum \( \varepsilon \) is usually around 2 to 4 times the grid spacing (Troldborg 2009, Kleusberg 2019), depending on the numerical method, a variable smoothing parameter may impose a very refined grid near the tips.

For this reason, corrections that take into account the difference between the vorticity created by the convoluted forces and the vorticity created by discrete singular forces, such as proposed by (Da˘g & Sørensen 2020), (Martínez-Tossas & Meneveau 2019) and (Meyer Forsting et al. 2019a), seem to be more cost-effective than reducing \( \varepsilon \). Some techniques to further reduce the computational cost of the smearing correction are explored by Meyer Forsting et al. (2020).

Meyer Forsting et al. (2019b) investigated the wake created by the actuator line with smearing correction, confirming that the smearing parameter continues to have an influence on the wake, despite its influence on the forces at the blades being greatly reduced.

All of these corrections apply an iterative process at each time step. At each iteration, the velocity or circulation is updated using a relaxation iterative process. From our experience, the choice of a relaxation factor close to unity can lead to numerical instability while a low relaxation factor increases the number of iterations and can increase the run time of the simulation.

In this work, we propose a direct way of computing the smearing correction, based on a linearized version of the lifting line. Using this method, the correction at each time step is found by solving a linear system of equations of the order \( N \), where \( N \) is the number of actuator line control points. In order to achieve this, two formulations of the discretized lifting line method based on the actuator line are presented: the non-linear formulation and its linearized version.

Additionally, some other aspects of the smearing correction are discussed, such as the choice of correction function and formation of the vortex sheet. Regarding these aspects, we aim to keep the method as general as possible. Previous works (Da˘g & Sørensen 2020, Martínez-Tossas & Meneveau 2019, Meyer Forsting et al. 2019a) already showed that the smearing correction can make the forces calculated by the ALM closely match the results of the lifting line method. By introducing fewer assumptions, avoiding especially assumptions related to rotating blades, the method could be applied to other problems beyond rotor simulations, such as simulations of fixed-wing aircrafts.

This work is structured as follows. First, we present the general idea of the actuator line method with smearing correction in Section 2. Then, in Section 3, we draw on the theoretical work of (Martínez-Tossas & Meneveau 2019) to develop the specific correction for velocities induced by vortex segments and describe the formation of the vortex sheet. The lifting line method based on the actuator line is presented in Section 4. The linearization of this lifting line method (Section 4.2) is the basis for the non-iterative smearing correction detailed in Section 5. Results of the simulations of a translating wing and the NREL 5-MW wind turbine are shown in Section 7. Finally, the main conclusions are summarized in Section 8.

2 The actuator line method

The incompressible Navier-Stokes equation written in primitive variables (pressure \( \rho \) and velocity \( \mathbf{u} \)) are (Mikkelsen 2003, Troldborg 2009):

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla \rho + \nu \nabla^2 \mathbf{u} + \mathbf{f}
\]  

(1)

where \( \rho \) and \( \nu \) are the density and kinematic viscosity of the fluid, respectively. The body force term, \( \mathbf{f} \), in the case of the actuator line method, is used to model the turbine (Sørensen & Shen 2002, Mikkelsen 2003, Troldborg 2009). The body forces are based on the two-dimensional force per spanwise unit length,
The body force is given by

\[ \mathbf{F} = (F_l, F_d) = \left( \frac{1}{2} \rho u_{rel}^2 c C_l, \frac{1}{2} \rho u_{rel}^2 c C_d \right) \]  

where \( F_l \) and \( F_d \) are the lift and drag forces (lift is perpendicular to the relative velocity and drag is parallel to the relative velocity, see figure 1), calculated from the relative velocity \( u_{rel} \), the local chord \( c \) and the two-dimensional lift and drag coefficients, \( C_l \) and \( C_d \), which are obtained from the airfoil data at the local Reynolds number and angle of attack \( \alpha \) (calculated using the local relative velocity).

The dimensions of \( \mathbf{F}_{2D} \) are not consistent with the dimensions of the body force \( \mathbf{f} \). As an intermediate step, an idealized three-dimensional body force \( \mathbf{f}^i \) is defined based on \( \mathbf{F}_{2D} \), where at each cross-section the body force is given by

\[ \mathbf{f}^i = \frac{1}{\rho} \mathbf{F}_{2D} \delta(y) \delta(z), \]

where \( \delta \) is the Dirac delta function, which can be interpreted as the limit of the Gaussian function

\[ \delta(y) = \lim_{\varepsilon \to 0} \frac{1}{\pi^{1/2} \varepsilon} e^{-\frac{y^2}{\varepsilon^2}}. \]

This body force \( \mathbf{f}^i \), where the superscript \( ^i \) indicates the ideal case, is concentrated at the origin of the local reference system (figure 1). Considering all cross-sections of a wing, the space of non-zero \( \mathbf{f}^i \) defines a line, which can be interpreted as the limit of the actuator line method, when the smearing parameter \( \varepsilon \) goes to zero. It is easy to see that \( \mathbf{f}^i \) is singular at the origin but its integration in the two-dimensional plane gives \( \mathbf{F}_{2D}/\rho \).

In the ALM, to avoid numerical problems related to singularities, the forces need to be distributed smoothly on several mesh points. The forces are usually projected into the domain by the convolution with a regularization kernel \( \eta \):

\[ \mathbf{f} = \mathbf{f}^i \ast \eta \]  

where \( \ast \) indicates the convolution of the two functions. Usually this regularization kernel is taken as a three-dimensional Gaussian kernel. Considering the same constant Gaussian width in the three directions:

\[ \eta(x, y, z) := \frac{1}{\pi^{3/2} \varepsilon^3} \exp \left( -\frac{x^2 + y^2 + z^2}{\varepsilon^2} \right) = \eta_x(x) \eta_y(y) \eta_z(z) \]  

where

\[ \eta_x(x) := \frac{1}{\pi^{1/2} \varepsilon} \exp \left( -\frac{x^2}{\varepsilon^2} \right) \]

and analogously for \( \eta_y \) and \( \eta_z \) (symbol := denoting equal to by definition). Non-uniform and anisotropic kernels have also been proposed (Mikkelsen 2003, Shives & Crawford 2013, Martínez-Tossas et al. 2017, Churchfield et al. 2017, Jha & Schmitz 2018, Cormier et al. 2021), but these are not investigated in the present work.

This presentation of the convolution operation in the actuator line method using an idealized body force formed by Dirac delta functions as an intermediate step may seem like an unusual approach, but is mathematically equivalent to the method of classical works (Sorensen & Shen 2002, Mikkelsen 2003) (save for possible typos regarding the density term, which is not relevant for incompressible flows), which employ one-dimensional integrals. This procedure, which was already applied by Martínez-Tossas & Meneveau (2019), formalizes the connection between the two-dimensional forces and the convolution in three dimensions, which is relevant for Section 3.
Similarly to previous studies (Martínez-Tossas & Meneveau 2019, Dağ & Sørensen 2020, Meyer Forsting et al. 2019a), we focus on the lift force, which has a greater influence on the forces of the turbine, and leave the effects related to the smearing of the drag force for future studies.

### 2.1 Lift force in the basic formulation of the actuator line method

In the general formulation of the actuator line (Sorensen & Shen 2002), the lift force $F_l$ at each spanwise section $j$ of the wing (force per spanwise length) is obtained from airfoil data from:

$$ F_{lj} = \frac{1}{2} \rho (u_z^2 + u_y^2) c_j C_l(\alpha_j) \quad (8) $$

where $c_j$ is the local chord of the airfoil and $C_l(\alpha_j)$ is the local lift coefficient, obtained from airfoil data at an angle of attack $\alpha_j$. The local reference system of figure 1 is used here. The velocities $u_x$ and $u_y$, relative to the actuator line, are obtained from the CFD (computational fluid dynamics) simulation at the control point position, $x_j$ of the actuator line segment. The lift coefficient $C_l(\alpha_j)$ is calculated by the interpolating tabulated airfoil coefficients using the effective local angle of attack

$$ \alpha_j = \alpha_{ej} + \arctan \left( \frac{u_x}{u_z} \right) \quad (9) $$

where $\alpha_{ej}$ is the geometric angle of attack given by the twist and incidence of the wing (or, in the case of rotating blades, the opposite of the local pitch angle (Troldborg 2009)). The formula for the effective angle of attack (equation (9)) may differ on different implementations of the actuator line, due to different reference systems. In particular, for rotating blades, a system of reference based on normal and tangential velocities to the rotating plane (corresponding to the $y$ and $z$-directions in the present work) is usually preferred and the rotation velocity of the actuator line $\Omega r$ is explicitly included in the formulation (see, e.g. (Troldborg 2009)). In the formulation described in the present work, the velocity of the actuator line is not explicit, because $u_x$ and $u_y$ are defined as relative velocities.

If needed, the circulation at each spanwise section can be calculated from the Kutta–Joukowski theorem

$$ \Gamma_j = \frac{F_{lj}}{\rho \sqrt{(u_z^2 + u_y^2)}} = \frac{1}{2} \sqrt{(u_z^2 + u_y^2)} c_j C_l(\alpha_j). \quad (10) $$

Usually, some form of tip or smearing correction is applied to this basic formulation. The smearing corrections employed in the present work are described in Sections 2.2 and 5.

### 2.2 Iterative smearing correction

In the actuator line method with smearing correction, the velocities $u_x$ and $u_y$, obtained from the CFD simulation at control point $x_j$, are summed to the “missing velocities” $u_w^m$ and $u_w^m$ to arrive at the corrected velocities $u_x^c$ and $u_y^c$:

$$ u_x^c(x_j) = u_x(x_j) + u_w^m(x_j) \quad (11) $$

$$ u_y^c(x_j) = u_y(x_j) + u_w^m(x_j). \quad (12) $$

The “missing velocities” are defined as the difference between the velocities induced by the vortices created by the actuator line and “ideal” vortices. The concept of what is considered an “ideal” vortex varies between the works. Dağ & Sørensen (2020) and Meyer Forsting et al. (2019a) consider a vortex filament with infinitesimal core, while Martínez-Tossas & Meneveau (2019) considers the vortex created using the optimum smearing parameter developed in (Martínez-Tossas et al. 2017). A viscous core model with the radius of the vortex core evolving in time could also be easily implemented. If there is the possibility of the ideal vortices impinging the actuator lines, it might be useful to employ some desingularization method. In the present work, we adopt the definition of ideal vortices as vortex filaments with infinitesimal core.

The procedure to calculate the missing velocities differs slightly in different works but follows the same general idea, outlined in the steps below. The detailed procedures are presented in each work (Dağ & Sørensen 2020, Martínez-Tossas & Meneveau 2019, Meyer Forsting et al. 2019a, 2020), and the implementation employed in the current study is detailed in the following sections.

The approaches of (Martínez-Tossas & Meneveau 2019, Meyer Forsting et al. 2019a) apply the relaxation factor to the velocity. In this work, we apply the relaxation factor to the circulation, because it is a
The steps of the iterative smearing correction are, for each time-step:

1. Start from a guess of the circulation distribution, usually the circulation from the previous time-step;
2. Form the vortex sheet by:
   - prescribing the vortex sheet, for example, by assuming helical or horseshoe vortices; or
   - employing a free-vortex wake method, for example, by advecting the vortices with the CFD velocity or by a combination of the CFD velocities and the velocities induced by the free vortices.
3. Calculate the missing velocities \( u^m_j(x_j) \) and \( u^m_\theta(x_j) \) at every control point, by computing the difference between the velocities induced by ideal vortices (superscript \(^i\)) and vortices with Gaussian core:
   \[
   u^m_j(x_j) = u^i_j(x_j) - u^\theta_j(x_j),
   \]
   \[
   u^m_\theta(x_j) = u^i_\theta(x_j) - u^\theta_j(x_j).
   \]
   Find the local corrected velocities according to equations (11) and (12);
4. Calculate the effective angle of attack using equation (9);
5. Find the local lift coefficient \( C_l \), interpolating from the airfoil data table;
6. Calculate the new value of circulation \( \Gamma^{new}_j \) using equation (10);
7. Update the current value of circulation using a relaxation factor \( r \):
   \[
   \Gamma_j = r\Gamma^{new}_j + (1-r)\Gamma^{old}_j
   \]
8. Restart from step (2) if the local velocity affects the formation of the vortex sheet or from step (3) otherwise, using the value of circulation calculated in the previous step. Iterate until a chosen convergence criterion is reached.

## 3 Vorticity created by body forces and the missing velocity

### 3.1 Vorticity created by body forces

Some of the development of Martínez-Tossas & Meneveau (2019) is reproduced in this section, Section 3.2 and 3.3, for completeness and, most importantly, in order to contextualize and justify the proposed correction function of Section 3.4. Neglecting viscous effects in a steady flow, the vorticity equation in steady flow becomes
\[
u \cdot \nabla \omega = \omega \cdot \nabla u + \nabla \times f.
\] (16)

Equation (16) is linearized considering uniform flow \( U = (0,0,U_z) = \text{const.} \) for a straight wing with only lift forces, that can be considered to act aligned with the y-axis, \( f = (0,f_z,0) \):

\[
U_z \frac{\partial \omega_x}{\partial z} = \frac{\partial f_z}{\partial x} \implies \omega_x = \frac{f_z}{U_z}
\] (17)

\[
U_z \frac{\partial \omega_z}{\partial z} = \frac{\partial f_z}{\partial x} \implies \omega_z = \int_{-\infty}^{z} \frac{1}{U_z} \frac{\partial f_z}{\partial x} dz
\] (18)

while \( \omega_x, \omega_z \gg \omega_y \approx 0 \).

From the Kutta–Joukowski theorem and equation (3), the body forces in an ideal case with concentrated lift forces can be written as
\[
f'_y(x,y,z) = \frac{F_i(x)}{\rho} \delta(y) \delta(z) = -U_z \Gamma(x) \delta(y) \delta(z),
\] (19)

where \( \Gamma(x) \) is the circulation distribution along the wing and \( \delta \) is the Dirac delta function.
From equations (17) and (18), the bound vortex along the wing is
\[
\omega'_x = \Gamma(x)\delta(y)\delta(z)
\] (20)
and the infinitesimal vortex sheet shed by the wing is
\[
\omega'_z = -\int_{-\infty}^{z} \frac{\partial}{\partial x} \delta(y)\delta(z)dz = -\frac{\partial}{\partial x} \delta(y)H(z) = \begin{cases} 0 & \text{for } z < 0 \\ -\frac{1}{2} \frac{\partial}{\partial x} \delta(y) & \text{for } z = 0 \\ -\frac{1}{2} \frac{\partial}{\partial x} \delta(y) & \text{for } z > 0 \end{cases},
\] (21)
where \(H(z)\) is the Heaviside step function with the half-maximum convention.

In the ALM, the forces are usually projected in the domain by the convolution with a Gaussian kernel \((\eta, \text{ from equation } (6))\). The convolution with a Gaussian function (also called Weierstrass transform (Widder & Hirschman 1955)) transforms the body forces and, by consequence, the vorticity generated by the forces. Considering the projection of the force of equation (19)
\[
f_z(x, y, z) = -(U_z \Gamma(x)\delta(y)\delta(z) \ast \eta(x, y, z) = -(U_z (\Gamma \ast \eta_x) (x)) \eta_y(y) \eta_z(z)
\] (22)
the force itself is distributed as a Gaussian function in the y-z-plane, what implies the distribution of spanwise vorticity as a Gaussian function
\[
\omega_x = -\frac{f_x}{U_z} = (\Gamma \ast \eta_x) \frac{1}{\pi \epsilon^2} e^{-\frac{y^2 + z^2}{2\epsilon^2}}.
\] (23)
The convolution also alters the vortex sheet
\[
\omega_z = \int_{-\infty}^{z} \frac{1}{U_z} \frac{\partial f_z}{\partial x} dz = -\left(\frac{\partial \Gamma}{\partial x} \ast \eta_x\right) \eta_y H_{\epsilon}(z)
\] (24)
where the function
\[
H_{\epsilon}(z) = \frac{\text{erf}(\frac{z}{\epsilon}) + 1}{2}
\] (25)
is defined as a smeared (mollified) Heaviside step function (Caprace et al. 2019). This function is the convolution of the Heaviside function with the Gaussian kernel.

3.2 Vorticity shed by discontinuous distributions of circulation

In numerical methods, usually the force is discretized and represented by interpolating functions inside segments. Since we are interested in a correction for the ALM, we analyze distributions of circulation that can be constructed using sum of functions, with the possibility of being discontinuous at the boundary of the segments. This approach is slightly different from (Martínez-Tossas & Meneveau 2019), but follows the same principles.

For each segment \(j\) we consider a circulation function is \(\Gamma_j(x)\) for x-positions between the boundaries of the segment, \(x_j \leq x \leq x_{j+1}\), and zero outside. The circulation function \(\Gamma(x)\) along the whole actuator line, taking into account all \(N\) segments, is:
\[
\Gamma(x) = \sum_{j=1}^{N} (H(x - x_{j-}) - H(x - x_{j+})) \Gamma_j(x).
\] (26)
where it is implicit that the average \(\bar{\Gamma}(x) = (\Gamma_j(x) + \Gamma_{j-1}(x))/2\) is considered at the boundary of the segments.

From equations (20) and (23), the bound vortex is:
\[
\omega'_x = \sum_{j=1}^{N} (H(x - x_{j-}) - H(x - x_{j+})) \Gamma_j(x) \delta(y)\delta(z)
\] (27)
\[
\omega_x = \sum_{j=1}^{N} \left( ((H(x - x_{j-}) - H(x - x_{j+})) \Gamma_j(x) \ast \eta_x) \eta_y(y) \eta_z(z) \right).
\] (28)
From equations (21) and (24), the vortex sheet is:

\[
\omega_z = - \sum_{j=1}^{N} \left[ \delta(x-x_j) \Gamma_j(x_j) - \delta(x-x_j) \Gamma_j(x_j) \right] + (H(x-x_j) - H(x-x_j)) \frac{\partial \Gamma_j}{\partial x} \delta(y) H(z) \tag{29}
\]

\[
\omega_z = - \sum_{j=1}^{N} \left[ \eta_x(x-x_j) \Gamma_j(x_j) - \eta_x(x-x_j) \Gamma_j(x_j) \right] + \left[ \left( H(x-x_j) - H(x-x_j) \right) \frac{\partial \Gamma_j}{\partial x} \right] \eta_y(y) H(z) \tag{30}
\]

The first two terms of equation (29) are the singular semi-infinite vortices generated by the discontinuities in the circulation at the boundary of the segment, in positions \(x_j\) and \(x_j\). These terms in equation (30) become semi-infinite Lamb-Oseen vortices centered in \(x_j\) and \(x_j\).

A relevant case for discontinuous distributions of circulation is when the circulation is approximated as constant along each segment, \(\Gamma_j(x) = \Gamma_j\). For this case the bound vorticity is

\[
\omega'_z = \sum_{j=1}^{N} (H(x-x_j) - H(x-x_j)) \delta(y) \delta(z) \Gamma_j \tag{31}
\]

\[
\omega_z = \sum_{j=1}^{N} (H_e(x-x_j) - H_e(x-x_j)) \eta_y(y) \eta_z(z) \Gamma_j \tag{32}
\]

where we can notice that the vorticity not only spread as a Gaussian in the directions normal to the actuator line (terms \(\eta_y\) and \(\eta_z\)) but also spread outside the boundaries of the line in the spanwise direction (terms \(H_e\)).

If the circulation is approximated as constant along each segment, \(\partial \Gamma_j/\partial x = 0\), hence the third term of equations (29) and (30) is null everywhere. In this case, only the first two terms shed vorticity to the wake. This case regresses to a discretized Prandtl’s lifting line in the case of equation (29) and to a lifting line with vortices with Gaussian core in the case of equation (30):

\[
\omega'_z = - \sum_{j=1}^{N} \delta(x-x_j) \delta(y) H(z) \Gamma_j -\delta(x-x_j) \delta(y) H(z) \Gamma_j \tag{33}
\]

\[
\omega_z = - \sum_{j=1}^{N} \left[ \eta_x(x-x_j) \eta_y(y) H_e(z) \Gamma_j - \eta_x(x-x_j) \eta_y(y) H_e(z) \Gamma_j \right] \tag{34}
\]

### 3.3 Velocity induced by a Semi-infinite Lamb-Oseen vortex on the actuator line

The velocity induced by the vortex sheet of equation (34) is different from the ideal case of equation (33). The velocity induced by a vorticity distribution is given by (Saffman 1992):

\[
u^v(x) = \frac{1}{4\pi} \int \frac{\omega(x') \times (x-x')}{|x-x'|^3} dx'. \tag{35}
\]

In the case of the velocity induced by a semi-infinite line vortex such as each term of equation (33) \((\omega'_z = \delta(x-x_j) \delta(y) H(z) \Gamma_j)\) on a position \(x = (x, 0, 0)\):

\[
u^{vi}_y(x) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(x'-x_j) \delta(y') H(z') \Gamma_j \hat{e}_z \times (x-x') \frac{dx'dy'dz'}{|x-x'|^3} = \frac{\Gamma_j}{4\pi(x-x_j)} \tag{36}
\]
The velocity induced by a smeared semi-infinite vortex is given by the corresponding terms of equation (34)

\[
\frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \eta_x(x' - x_{j^*}) \eta_z(y') \eta_z(z') \frac{H_e(z') \Gamma_j \delta_z \times (x - x')}{|x - x'|^3} dx' dy' dz' + \text{Velocity induced by cross-section of Gaussian vorticity distribution}
\]

(37)

which is equal to the velocity induced by a semi-infinite Lamb-Oseen vortex.

It can be noticed that the final formula of equation (37) differs from (36) only by the exponential factor. The missing velocity due to this vortex is then

\[
u^\text{mis} = \frac{\Gamma_j}{4\pi(x - x_{j^*})} (1 - e^{-\frac{(x - x_{j^*})^2}{\pi^2}}).
\]

The missing velocity when summed to the velocity sampled from the CFD domain reconstructs the case as if the velocity were induced by ideal vortices instead of Lamb-Oseen vortices. As already discussed by Martínez-Tossas & Meneveau (2019), this serves as proof that the correction proposed by Dağ & Sørensen (2020) can be derived from first principles for the case of a translating wing.

If the implementation of the ALM considers constant circulation inside each segment, the correction proposed by Dağ & Sørensen (2020) tends to a lifting line method, according to the linear approximation, as proposed by Dağ & Sørensen (2020) and Meyer Forsting et al. (2019a) (putting aside other simplifications, as discussed in Sections 3.4 and 3.5). If the implementation of the ALM considers a non-constant distribution of circulation inside each segment, the correction by Dağ & Sørensen (2020) is not identical to the discrete lifting line method, because of the third term of equations (29) and (30). Interestingly, also the discretized implementation of (Martínez-Tossas & Meneveau 2019) is not exactly identical (although very similar) to the classical lifting line method, because it implicitly considers the vortices located at the control points (equation 5.7 of that reference), not at the boundaries of segments, as would be usual in a classical discretized lifting line.

The bound vortex is also affected by the convolution with the Gaussian kernel, as seen in equation (32). However, it does not affect the forces in most cases, since most geometries have straight wings or blades and a straight bound vortex does not induce velocity on itself. For multi-blade configurations, the blades are usually too distant for the correction to make any difference (distance is several times \(\delta\)). A possible exception may be the hub, however, the circulation at the hub is low and the forces are not as relevant for performance. As a consequence, the effect of the convolution on the vorticity generated and the implementation of corrections for bound vorticity have been, justifiably, neglected.

The formula to implement analytically the convolution of the Gaussian function with a force distribution formed by segments of constant force is shown in Appendix A.1.

### 3.4 Velocity induced by a smeared vortex segment

Equation (38) has been shown to represent the missing velocity for a semi-infinite straight Lamb-Oseen vortex at the actuator line, when the origin of the semi-infinite vortex is in the same plane of the actuator line. However, in the method of (Dağ & Sørensen 2020, Meyer Forsting et al. 2019a) the correction is also applied for the velocity induced by a finite-length segment of vortex with a Lamb-Oseen distribution of vorticity. Hence, it is relevant to study the velocity induced by a cross-section of Gaussian vorticity distribution. We define the term inside the brackets in equation (37) as function \(\phi(x - x_{j^*}, z')\):

\[
\phi(x - x_{j^*}, z') := \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{(x - x')}{(x - x')^2 + y'^2 + z'^2} e^{-\frac{(x' - x_j)^2 + y'^2}{\pi^2 \delta^2}} dx' dy'.
\]

(39)

In Dağ & Sørensen (2020), the velocity induced by the cross-section of Gaussian vorticity distribution, normalized by \(\Gamma_j/(4\pi)\), is modelled as a modified version of the second formula of equation (36):

\[
\phi(x - x_{j^*}, z') := \frac{(x - x_j)}{(x - x_j)^2 + z'^2} \left(1 - e^{-\frac{(x - x_j)^2 + z'^2}{\pi^2 \delta^2}}\right)
\]

(40)
These models are compared to the numerical integration of relation (39). It can be noticed that there are significant discrepancies between the models and the value of the function \( \phi \). Despite the errors observed for \( \phi^{mpr} \), it would give the same result after the integration along \( z \) for a straight semi-infinite vortex of equation (37). This is the case for the basic wing test cases of (Meyer Forsting et al. 2019a), that produce semi-infinite vortices. However, according to figure 2, this is probably due to an undercorrection for lower values of \( z'/\epsilon \) being compensating by overcorrection for higher values of \( z'/\epsilon \). It should be noted that the function \( \phi^{ds} \) does not give the same result for semi-infinite vortices and, from figure 2, it always undercorrects. This indicates that the use of modeled functions \( \phi^{ds} \) or \( \phi^{mpr} \) would introduce errors (if the vortices are not straight semi-infinite in the case of \( \phi^{mpr} \)).

Such errors are not expected to be present in the application of the correction of (Martínez-Tossas & Meneveau 2019) to straight wings in uniform flow. However, the application of the formula of (Martínez-Tossas & Meneveau 2019) to the case of rotating blades without adaptations, such as performed in (Stanly et al. 2022), may incur even greater errors, since the helical vortex structure formed by the blades is being modeled by a vortex sheet formed by straight horseshoe vortices.

Nevertheless, the good results of (Dağ & Sørensen 2020) and (Meyer Forsting et al. 2019a) show that their corrections improve the results when compared to ALM without correction or with other heuristic tip corrections, even using the approximate value of \( \phi \). A possible explanation for that is based on the observation that the missing velocity is directly related to the difference \( \phi^t - \phi \). For the vortices near the blades which tend to dominate the correction, the value of \( \phi^t \) is much larger than \( \phi \), then the exact value of \( \phi \) is not as relevant, as long as the order of magnitude is the same. The dominance of the vortices near the blades is also a possible explanation for the improved results observed by (Stanly et al. 2022), even considering straight horseshoe vortices instead of helical vortices. The estimate of the order of magnitude
The velocity induced by a smeared vortex segment, is given by:

\[ \omega_z = \delta(x - x_{j,\text{c}}) \delta(y) \left( H(z - z_{j,\text{c}}) - H(z - z_{j}) \right) \Gamma_j. \]

Figure 3 shows a vortex filament of length \( z_{j,\text{c}} - z_{j} = 10\epsilon \). Isocontours and volume rendering indicate normalized vorticity \( \omega \pi \varepsilon^2 / \Gamma_j \), given by equation (44). Black line indicates the segment of vortex filament with concentrated vorticity.

We define a “smeared vortex segment” as the smeared equivalent of the straight segment of vortex filament, with concentrated vorticity.

This smeared vortex segment has a Gaussian vorticity distribution in the core, similar to the Lamb-Oseen vortex, but its magnitude is modulated by smeared Heaviside functions centered at the boundaries of the segment \( H(z - z_{j,\text{c}}) \) and \( H(z - z_{j}) \). Figure 3 shows a vortex filament of length 10\epsilon with concentrated (infinite) vorticity in black and the vorticity distribution of the corresponding smeared vortex segment.

The velocity induced by a vortex filament segment at \( x = (x, 0, 0) \) is given by (Katz & Plotkin 1991)

\[
\begin{align*}
\mathbf{u}_y (x) & = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(x' - x_{j,\text{c}}) \delta(y') \left( H(z' - z_{j,\text{c}}) - H(z' - z_{j}) \right) dxdydz' \\
& = \frac{\Gamma_j}{2\pi(x - x_{j,\text{c}})} \left[ \frac{z_{j,\text{c}}}{(x - x_{j,\text{c}})^2 + z_{j,\text{c}}^2} - \frac{z_j}{(x - x_{j})^2 + z_j^2} \right].
\end{align*}
\]

The velocity induced by a smeared vortex segment, is given by:

\[
\begin{align*}
\mathbf{u}_y (x) & = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \eta_s(x' - x_{j,\text{c}}) \eta_s(y') \left( H(z' - z_{j,\text{c}}) - H(z' - z_{j}) \right) dxdydz' \\
& = \frac{\Gamma_j}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{2} \left( \text{erf} \left( \frac{z' - z_{j,\text{c}}}{\epsilon} \right) - \text{erf} \left( \frac{z' - z_{j}}{\epsilon} \right) \right) \delta(x - x_{j,\text{c}}) d\epsilon dz'.
\end{align*}
\]
Defining
\[ \Phi(x - x_{j*}, z_{j*}) := \int_{-\infty}^{+\infty} \frac{1}{2} \left( \text{erf} \left( \frac{z' - z_{j*}}{\varepsilon} \right) \right) \phi(x - x_{j*}, z') dz' \]  
we can calculate the induced velocity of equation (46) as
\[ u_i^v(x) = \frac{\Gamma_j}{4\pi} \left( \Phi(x - x_{j*}, z_{j*}) - \Phi(x - x_{j*}, z_{j*}) \right). \]  

Analogously, \( \Phi'(x - x_{j*}, z_{j*}) \) can be defined as
\[ \Phi'(x - x_{j*}, z_{j*}) := \lim_{\varepsilon \to 0} \Phi(x - x_{j*}, z_{j*}) = \frac{1}{(x - x_{j*})} \left( -\frac{z_{j*}}{\sqrt{(x - x_{j*})^2 + z_{j*}^2}} \right). \]

The values of \( \Phi(x - x_{j*}, z_{j*}) \) were calculated by integrating numerically equation (47) and, contrary to what could be presumed from the approximations of (Dağ & Sørensen 2020) and (Meyer Forsting et al. 2019a), the value of \( \Phi(x - x_{j*}, z_{j*}) \) do not correspond to \( \Phi'(x - x_{j*}, z_{j*}) \) multiplied by the exponential functions shown in equations (40) or (41). Nevertheless, similarly to the case of semi-infinite vortices:
\[ \lim_{z_{j*} \to \infty} \Phi(x - x_{j*}, z_{j*}) = \Phi'(x - x_{j*}, z_{j*}) \left( 1 - e^{-\frac{(x - x_{j*})^2}{2\varepsilon^2}} \right) \]  

For this reason, in the present work, for each vortex segment, the missing velocity is calculated as
\[ u_i^v(x) = u_i^{vi}(x) - u_i^v(x) = \frac{\Gamma_j}{4\pi} \left( \Phi'(x - x_{j*}, z_{j*}) - \Phi'(x - x_{j*}, z_{j*}) \right) - \left( \Phi(x - x_{j*}, z_{j*}) - \Phi(x - x_{j*}, z_{j*}) \right). \]

from tabulated values of \( \Phi \). The same correction is applied to the bound vortices, taking into account the different orientations of the vortices. The details about the tabulated data and the three-dimensional implementation are found in Appendix A.2.

### 3.5 Forming the vortex sheet

Forming the vortex sheet is an intrinsic part of both the ALM with smearing correction and the lifting line method. It should be noted, however, that the methods are not going to be mathematically identical even if the vortex sheet is formed with the same procedure in both methods. In the actuator line, the induced velocity is formed partly by the smeared vortices created in the CFD simulation and partly by the vortex sheet of the smearing correction, which is formed by one of the strategies detailed below (that may be more or less dependent on the results from the simulation). Hence, even if the vortex sheet from the smearing correction is identical to the vortex sheet of the lifting line method, the induced velocities can differ slightly, because the vortex sheet created in the CFD simulation may be different from the vortex sheet of the lifting line method. This phenomenon can be observed in the results of Section 7.1, where its effect is noticeable but negligible.

The formation of the vortex sheet for the smearing correction is different among past works. The method of (Martínez-Tossas & Meneveau 2019) does not require an explicit vortex sheet, since the correction is based on analytical semi-infinite vortices (see Section 3.3) for a straight wing under uniform flow. However, the same correction was applied by (Stanly et al. 2022), which would mean that the helical vortex structure is implicitly modeled by horseshoe vortices. Errors are expected to be introduced by this approximation, however, the order of magnitude of these errors has not been estimated.

In (Dağ & Sørensen 2020), the helical vortices were imposed considering the pitch of each of the vortex segments based on the local relative flow angle at the position where the vortex is released. In (Meyer Forsting et al. 2019a) a fixed helical wake was assumed, based on the near-wake model of (Pirring et al. 2016, 2017), and the bookkeeping of the position of released vortices is only necessary for one of the factors involved in the smearing correction. The need for bookkeeping was later removed by Meyer Forsting et al. (2020), reducing computational cost with negligible effect on the forces.

In the present work, we implement a free-vortex wake method within our simulation, a strategy already suggested by Dağ & Sørensen (2020), assuming that the vortices are advected by the flow velocity.
particles are used to follow the position of the boundaries of the vortex segments. Also, the past values of circulation, that are present in the wake, are directly used, which is a more realistic situation than considering the whole wake as having the value of circulation that the wing currently assumes. This method has the drawback of requiring bookkeeping of the circulation and position of the vortices emitted in previous time steps. Nevertheless, it has the advantage of requiring no \textit{ad hoc} assumption and therefore the method maintains its generality.

There is an ambiguity regarding using the corrected velocity or the uncorrected velocity sampled directly from the numerical simulation, to estimate the position of vortices. However, this difference is of second order for the velocities along the actuator line, as shown in Appendix B. Using the corrected velocity might even introduce errors or numerical instability if used throughout the vortex sheet, because of the singularity of ideal vortices: vortices recently released might suffer the effect of the singularity of the bound vortex and curved vortices might induce numerical instabilities.

Hence, in the current implementation, the tracing particles that define the boundaries of the vortex filaments are advected by the velocities sampled from the simulation, without correction. This reduces the number of operations needed so that the correction velocity is only calculated on the control points of the actuator line, not on other points of the vortex sheet. More relevant for our non-iterative method, the vortex sheet can be advected before the computation of the correction velocity at a certain time step.

For its lower computational cost, a first-order Euler time-integration method is used to advect the tracing particles. In figure 4(a), a schematic representation of the advection process of the tracing particles is shown for \( n_w = 12 \) tracing particles per boundary of actuator line segment. The information related to timesteps older than \( n - n_w \) is not stored (where \( n \) is the current timestep).

In order to save computational resources, not all tracing particles are kept. The vortices released in the last \( n_{nw} \) time steps are always maintained in the memory. However, a group of vortices is fused if the distance between the tracing particles is below \( d_v \), with the circulation taken as the average of the circulation of the vortices fused together. For example, in the schematic representation of figure 4(b), the circulation of the vortex defined by tracing particles released at timesteps \( n - 5 \) and \( n - 8 \) is the average

Figure 4: Schematic representation of the advection process, from timestep \( n \) to timestep \( n + 2 \). The points indicate the tracing particles that are released at a certain time step and the arrows represent the vortices. For simplification purposes, the vortex sheet is represented as a grid-like structure, however, in the actual implementation, the particles are free to move in any direction and are advected by velocities taken from the numerical simulation. (a) Simple advection of vortices for \( n_w = n_{nw} = 12 \). (b) Advection of vortices with fusion for \( n_w = 8 \) and \( n_{nw} = 4 \).
of the circulation of timesteps $n-5$, $n-6$ and $n-7$. The length of the vortex wake changes in time, as shown in figure 4(b), but this is not considered to be a problem if the length is large enough. Compared to the simple advection process, it requires fewer tracing particles for the same wake length, while still maintaining high discretization in the near wake.

The strategy of fusing vortices guarantees a minimum wake length of $(n_w - n_{nw} - 1)d_w$ near the tip of the blades, the region of interest and where the advection velocity is larger. For the current simulations, the choice of parameters is $n_w = 50$, $n_{nw} = 10$ and $d_w = e/2$, guaranteeing a wake length of approximately 20$e$ near the tip. Analysis of the correction of equation (51) indicates that errors involved in discarding vortices farther than this length are negligible. The wake length is lower near the hub but this region is less relevant for the performance of the turbine. A parameter study for the simulation of the NREL 5-MW turbine of Section 7.2 showed that doubling $n_w$ and $n_{nw}$ had negligible effects on the circulation.

4 Lifting line method based on the actuator line method

4.1 Non-linear iterative lifting line

The formulation of the actuator line presented in Sections 2.1 and 2.2 is very similar to a non-linear lifting line method (Anderson Jr 1991) (see also Phillips & Snyder (2000)). Hence, we can develop a numerical non-linear lifting line method consistent with the actuator line method. While Anderson Jr (1991) uses the undisturbed uniform inflow velocity ($U_{\infty}$) to calculate the circulation, we use the local velocity, basing our method on equation (10), in an approach similar to Phillips & Snyder (2000). Also, in this formulation, the undisturbed velocities $U_{x}$ and $U_{y}$ (velocities as if the wing was not present in the flow) can change along the span, a feature paramount for the application of the actuator line to rotating blades. The local velocities $u_{x}$ and $u_{y}$, which are previously unknown, can be calculated from the undisturbed velocities and the velocities induced by the vortex system, $u_{x}^{*}$ and $u_{y}^{*}$, at control point $x_{j}$, given by

$$\begin{align*}
u_{x}(x_{j}) &= U_{x}(x_{j}) + u_{x}^{*}(x_{j}) \quad (52) \\
u_{z}(x_{j}) &= U_{z}(x_{j}) + u_{z}^{*}(x_{j}). \quad (53)
\end{align*}$$

The steps of the non-linear lifting line are:

1. Start from a guess of the circulation distribution, for example, $\Gamma_{j} = 0$ for every control point, or applying equation (10) with $u_{x} = U_{x}$ and $u_{y} = U_{z}$;

2. Form the vortex sheet by:
   - prescribing the vortex sheet, for example, by assuming helical or horseshoe vortices; or
   - employing a free-vortex wake method.

3. Calculate the velocity induced by the bound vortex and the vortex sheet at every control point, $u_{x}^{*}(x_{j})$ and $u_{z}^{*}(x_{j})$. Find the local velocities according to equations (52) and (53);

4. Calculate the effective angle of attack using equation (9);

5. Find the local lift coefficient $C_{L}$, interpolating from the airfoil data table;

6. Calculate the new value of circulation $\Gamma_{j}^{new}$ using equation (10);

7. Update the current value of circulation using a relaxation factor $r$:

$$\Gamma_{j} = r\Gamma_{j}^{new} + (1-r)\Gamma_{j}^{std} \quad (54)$$

8. Restart from step (2) if the local velocity affects the formation of the vortex sheet or from step (3) if a prescribed vortex sheet is considered, using the value of circulation calculated in the previous step. Iterate until a chosen convergence criterion is reached.
4.2 Linearized lifting line

Equation (10) can be linearized around the undisturbed velocities using the Taylor series expansion:

\[
\Gamma_j = \frac{1}{2} U_0 c_j C_l(\alpha_{0j}) + \frac{1}{2} \left( \frac{U_y u_{yi}^v}{U_0} + \frac{U_z u_{zi}^v}{U_0} \right) c_j C_l(\alpha_{0j})
+ \frac{1}{2} U_0 c_j \frac{\partial C_l}{\partial \alpha}(\alpha_{0j}) \left( \frac{U_y u_{yi}^v}{U_0} - \frac{U_z u_{zi}^v}{U_0} \right)
\]

(55)

where

\[
U_0 := \sqrt{U_x^2 + U_y^2},
\]

(56)

\[
\alpha_{0j} := \alpha_{gj} + \arctan \left( \frac{U_y}{U_z} \right).
\]

(57)

and \( \frac{\partial C_l}{\partial \alpha}(\alpha_{0j}) \) is the slope of the lift coefficient evaluated at angle \( \alpha_{0j} \). Defining

\[
\Gamma_{0j} := \frac{1}{2} U_0 c_j C_l(\alpha_{0j}),
\]

(58)

the expression (55) can be written as:

\[
\Gamma_j = \Gamma_{0j} + \frac{1}{2} c_j \left[ \left( C_l(\alpha_{0j}) \frac{U_y}{U_0} + \frac{\partial C_l}{\partial \alpha}(\alpha_{0j}) \frac{U_z}{U_0} \right) u_{yi}^v + \left( C_l(\alpha_{0j}) \frac{U_z}{U_0} - \frac{\partial C_l}{\partial \alpha}(\alpha_{0j}) \frac{U_y}{U_0} \right) u_{zi}^v \right].
\]

(59)

Defining

\[
b_{yj} := \frac{1}{2} c_j \left( C_l(\alpha_{0j}) \frac{U_y}{U_0} + \frac{\partial C_l}{\partial \alpha}(\alpha_{0j}) \frac{U_z}{U_0} \right)
\]

(60)

\[
b_{zj} := \frac{1}{2} c_j \left( C_l(\alpha_{0j}) \frac{U_z}{U_0} - \frac{\partial C_l}{\partial \alpha}(\alpha_{0j}) \frac{U_y}{U_0} \right)
\]

(61)

we arrive at

\[
\Gamma_j = \Gamma_{0j} + b_{yj} u_{yi}^v + b_{zj} u_{zi}^v.
\]

(62)

where we see that coefficients \( b_{yj} \) and \( b_{zj} \) are a measure of the sensitivity of the circulation to a change in the velocities in \( y \) and \( z \) directions.

To avoid an iterative process, another relationship between the induced velocities and circulation must be used. This relationship is the one described in step (3) of the iterative process of Section 4.1. The velocities induced at control point \( j \) by the vortex system \( k \) (which includes the bound vortex and the vortex sheet created by the segment \( k \)) can be written as:

\[
u_{yi}^k(\mathbf{x}_j) = a_{y,jk}\Gamma_k
\]

(63)

\[
u_{zi}^k(\mathbf{x}_j) = a_{z,jk}\Gamma_k
\]

(64)

where \( a_{y,jk} \) and \( a_{z,jk} \) are called the influence coefficients. The values of the influence coefficients are obtained from the Biot-Savart law and depend only on the shape of the vortex filament and its relative position to the control point. The formula for the influence coefficient for some common vortex filament shapes can be found in (Katz & Plotkin 1991). Then, the total induced velocities are found by considering all \( N \) vortex systems created by the \( N \) segments:

\[
\begin{bmatrix}
u_{yi}^1(\mathbf{x}_1) \\
u_{yi}^2(\mathbf{x}_2)
\vdots
\nu_{yi}^N(\mathbf{x}_N)
\end{bmatrix}
= \begin{bmatrix}
a_{y,11} & a_{y,12} & \cdots & a_{y,1N} \\
a_{y,21} & a_{y,22} & \cdots & a_{y,2N}
\vdots & \ddots & \vdots & \\
a_{y,N1} & a_{y,N2} & \cdots & a_{y,NN}
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 \\
\Gamma_2
\vdots
\Gamma_N
\end{bmatrix}
\]

(65)

that can be written in matrix form as

\[
u_{yi} = A_y \Gamma
\]

(66)

and analogously for \( u_{zi}^v \):

\[
u_{zi} = A_z \Gamma.
\]

(67)
Equation (62) can also be written in matrix form:

\[
\Gamma = \Gamma_0 + b_y \odot u_y^{vi} + b_z \odot u_z^{vi} = \Gamma_0 + (\text{diag}(b_y)A_y)\Gamma + (\text{diag}(b_z)A_z)\Gamma \tag{68}
\]

where \( \odot \) denotes the element-wise product and \( \text{diag}(b) \) is the square diagonal matrix formed by the elements of column vector \( b \). The circulation is finally found by solving the linear system

\[
(I - \text{diag}(b_y)A_y - \text{diag}(b_z)A_z)\Gamma = \Gamma_0, \tag{69}
\]

where \( I \) is the identity matrix. Therefore, equation (69) provides a direct way to calculate the circulation along the wing, instead of the iteration of Section 4.1. If needed, the induced velocities can be found using relations (66) and (67). Note that this formulation is quite general, it is valid for \( U_y \) or \( U_z \) that changes along the span and is not restricted to horseshoe vortices. Nevertheless, this formulation is reduced to one of the classical implementations of the numerical lifting line for \( U_y = 0 \) and constant \( U_z \), when an ideal airfoil and horseshoe vortices are employed, as can be seen in Appendix C.

5 Non-iterative smearing correction based on the lifting line method

The steps to apply the smearing correction (Section 2.2) are basically equal to the steps of the non-linear lifting line described in Section 4.1, except for step (3), which becomes the calculation of the missing velocities, and step (2), which allows the formation of the vortex sheet using data from the CFD simulation. Therefore, a non-iterative smearing correction based on the formulas of the linearized lifting line described in Section 4.2 can be devised.

A minor difference in the process of the iterative lifting line and the smearing correction is the possibility to start from a previous time step, which provides a better guess for the circulation. This minor difference, however, is what allows the linearized, non-iterative, formulation of the smearing correction to have good results even if the airfoil lift coefficient is not linear w.r.t. the angle of attack.

The first phase of the method is to find an approximate solution to linearize around. This is achieved by applying the first iterative steps of Section 2.2 only once:

1. Start from the circulation distribution \( \Gamma^{n-1} \) from the previous time-step \( n - 1 \);
2. Form the vortex sheet. In the current implementation, by advecting the vortices with the CFD velocity;
3. Calculate the missing velocities \( u_y^m(x_j, \Gamma^{n-1}) \) and \( u_z^m(x_j, \Gamma^{n-1}) \) at every control point, by computing the difference between the velocities induced by ideal vortices and Lamb-Oseen vortices

\[
\begin{align*}
    u_y^m(x_j, \Gamma^{n-1}) &= u_y^{vi}(x_j, \Gamma^{n-1}) - u_y(x_j, \Gamma^{n-1}) \tag{70} \\
    u_z^m(x_j, \Gamma^{n-1}) &= u_z^{vi}(x_j, \Gamma^{n-1}) - u_z(x_j, \Gamma^{n-1}) \tag{71}
\end{align*}
\]

Find the local corrected velocities, \( u_y^c(x_j, \Gamma^{n-1}) \) and \( u_z^c(x_j, \Gamma^{n-1}) \), according to equations (11) and (12);

We linearize around the velocities calculated in step (3). Representing the variables around which the functions are linearized by \( \dagger \):

\[
\begin{align*}
    U_y^\dagger := u_y^c(x_j, \Gamma^{n-1}) &= u_y(x_j) + u_y^m(x_j, \Gamma^{n-1}) \tag{72} \\
    U_z^\dagger := u_z^c(x_j, \Gamma^{n-1}) &= u_z(x_j) + u_z^m(x_j, \Gamma^{n-1}) \tag{73}
\end{align*}
\]

Analogously to Section 4.2

\[
\begin{align*}
    U_0^\dagger := \sqrt{(U_z^\dagger)^2 + (U_y^\dagger)^2} \tag{74} \\
    \alpha^\dagger_{0j} := \alpha_{0j} + \arctan \left( \frac{U_y^\dagger}{U_z^\dagger} \right) \tag{75} \\
    \Gamma^\dagger_{0j} := \frac{1}{2} U_0^\dagger c_j C_l(\alpha^\dagger_{0j}) \tag{76}
\end{align*}
\]
Figure 5: Schematic representation of equations (77) and (87). The missing velocity is the sum of the missing velocity due to the vortex sheet emitted in the previous time-steps (leftmost figure of the second row) and the missing velocity due to the bound vortex and the vortex sheet emitted in the current time-step, which can also be splitted into two terms, given by $A_{\gamma,mc} \Gamma^{n-1}$ and $A_{\gamma,mc} \Delta \Gamma$ (central and rightmost figures of the second row). The two first terms are known and the only unknown term is $\Delta u_\gamma^{mc}$.

Phase two of the linearization process involves finding another linear relation between the corrected velocity and the circulation. In step (2), it is assumed that the missing velocity is not relevant for the formation of the vortex sheet, as discussed in Section 3.5, and the CFD velocity is used directly to advect the vortices. This implies that the current value of circulation does not affect the geometry of the vortex sheet, what simplifies the method and is consistent with a first-order approximation.

The unknown missing velocity for the value of circulation at the current time step, $\Gamma^n$, can be divided in two components:

$$u_y^m(x_j, \Gamma^n) = u_y^{mp}(x_j, \Gamma^n) + u_y^{mc}(x_j, \Gamma^n)$$  \hspace{1cm} (77)

$$u_z^m(x_j, \Gamma^n) = u_z^{mp}(x_j, \Gamma^n) + u_z^{mc}(x_j, \Gamma^n)$$ \hspace{1cm} (78)

where $u_y^{mp}$ and $u_z^{mp}$ are the velocities induced by the vortex sheet emitted in the previous time steps and $u_y^{mc}$ and $u_z^{mc}$ are the velocities induced by the vortex sheet emitted in the current time step and by the bound vortices. A schematic representation is shown in figure 5.

The value of circulation at the current time step does not affect the vortex wake emitted in the previous time steps, hence, the velocity induced by its vortices is already computed:

$$u_y^{mp}(x_j, \Gamma^n) = u_y^{mp}(x_j, \Gamma^{n-1}) = u_y^{mp}(x_j)$$ \hspace{1cm} (79)

$$u_z^{mp}(x_j, \Gamma^n) = u_z^{mp}(x_j, \Gamma^{n-1}) = u_z^{mp}(x_j).$$ \hspace{1cm} (80)

However, the current value of circulation, $\Gamma^n$ (currently unknown), affects directly the velocity induced by the vortex sheet emitted in the current time step and by the bound vortices. We write the velocities induced at control point $j$ by the vortex system $k$ which includes the bound vortex and the vortex sheet created by the segment $k$ at the current time step, as:

$$u_y^{mc}(x_j, \Gamma_k^n) = a_{\gamma,j,k}^{mc} \Gamma_k^n$$ \hspace{1cm} (81)

$$u_z^{mc}(x_j, \Gamma_k^n) = a_{\gamma,j,k}^{mc} \Gamma_k^n$$ \hspace{1cm} (82)

where $a_{\gamma,j,k}^{mc}$ and $a_{\gamma,j,k}^{mc}$ are the influence coefficients obtained considering only the bound vortex and the vortex sheet emitted at the current time step (see figure 5). It should be noted that the coefficients $a_{\gamma,j,k}^{mc}$ and $a_{\gamma,j,k}^{mc}$ should already be known to perform step (3) of the usual iterative version of the smearing correction (Section 2.2). Equations (81) and (82) written in matrix form are:

$$u_\gamma^{mc}(\Gamma^n) = A_{\gamma}^{mc} \Gamma^n$$ \hspace{1cm} (83)
The CFD solver, what makes its solution computationally inexpensive.

just like a conventional lifting line method has good results despite starting from a completely wrong first iteration of an iterative smearing correction. If the lift coefficient has a linear relation with the angle being the difference between the circulation of the previous time step and the circulation obtained in the figure $\Delta$ much greater than even if the lift coefficient is not linear. The method is only expected to perform poorly if the lift coefficient velocities calculated at the first iteration of a conventional smearing correction. The differences in the

where $\Delta$, the circulation is calculated by equation (11), (72) and (87), the corrected velocity $u_y^c$ can be written as

$$u_y^c(\Gamma^n) = u_y + u_y^m(\Gamma^{n-1}) + \Delta u_y^{mc}$$

Analogously for $u_z^m$ and $u_z^c$

$$u_z^m(\Gamma^n) = u_z^m + u_z^{mc}(\Gamma^{n-1}) + A_z^{mc} \Delta \Gamma$$

Finally, the linear equation can be found following the steps of Section 4.2. Similar to equation (59), the linearization of equation (10) becomes:

$$\Gamma^n_j = \Gamma_{0j}^n + \frac{1}{2} c_j \left[ C_l(a_{0j}) \frac{U_j^0}{U_{0j}} + \frac{\partial C_l}{\partial a}(a_{0j}) \frac{U_j^0}{U_{0j}} \right] \Delta u_j^{mc} + \left[ C_l(a_{0j}) \frac{U_j^0}{U_{0j}} - \frac{\partial C_l}{\partial a}(a_{0j}) \frac{U_j^0}{U_{0j}} \right] \Delta u_j^{mc}$$

which can be written in matrix form as

$$\Gamma^n = \Gamma_0^n + (\text{diag}(b_y^\top) A_y^{mc}) \Delta \Gamma + (\text{diag}(b_z^\top) A_z^{mc}) \Delta \Gamma$$

where $b_y^\top$ and $b_z^\top$ can be found using the corresponding version of equations (60) and (61). The difference of the circulation $\Delta \Gamma^c$ can be found by solving the linear system

$$\left( I - (\text{diag}(b_y^\top) A_y^{mc}) - (\text{diag}(b_z^\top) A_z^{mc}) \right) \Delta \Gamma = \Gamma_0^n - \Gamma^{n-1}. $$

The circulation is calculated by $\Gamma^n = \Gamma^{n-1} + \Delta \Gamma$ and the corrected velocities are calculated using equations (88) and (90), by a simple matrix-vector multiplication.

Note that we solve for the difference of the circulation $\Delta \Gamma$, with the right-hand side of the equation being the difference between the circulation of the previous time step and the circulation obtained in the first iteration of an iterative smearing correction. If the lift coefficient has a linear relation with the angle of attack, this method is expected to give good results, even if the differences in the circulation are high, just like a conventional lifting line method has good results despite starting from a completely wrong circulation distribution.

However, the strength of the method is in the fact that the functions are linearized around the velocities calculated at the first iteration of a conventional smearing correction. The differences in the velocities and circulation are expected to be small in most simulations, what justifies this linearization, even if the lift coefficient is not linear. The method is only expected to perform poorly if the lift coefficient slope changes with the angle of attack and the flow conditions vary greatly between time steps. For these extreme cases, the linear method presented here can be used to accelerate the convergence of an iterative method. The linear system is relatively small, of size $N$, usually several orders of magnitude smaller than the CFD solver, what makes its solution computationally inexpensive.

For multiple turbines, each turbine can be solved in isolation, since the distance between turbines is much greater than $\epsilon$. In fact, if the distance between the blades is much larger than $\epsilon$ or if forces at the hub are not relevant, the system of equations can be divided and each blade solved independently. This last simplification was not implemented for the results of Section 7.

It is worth remembering that the current reference system is related to the local reference system of figure 1. Equation (91), for example, depends only on the local variables. The only term that takes into
account the contributions of other sections of the blades are the influence coefficients, hence, these are defined first in an absolute system of reference and then transformed into the local system of reference. Also, there is no constrain regarding the values or direction of the velocity (as opposed to the classical linearization of the lifting line method, which requires constant $U_z$ and $U_z \gg U_x$). Therefore, this method is directly applicable to any system of reference, including the case of rotating blades, both in a fixed frame of reference and a rotating frame of reference.

All the aspects regarding the formation of the vortex sheet and calculation of the induced velocity are accounted for in matrices $A_y^{m_0}$ and $A_z^{m_0}$. This means that the method of previous works (Da˘g & Sørensen 2020, Martínez-Tossas & Meneveau 2019, Meyer Forsting et al. 2019a, 2020, Stanly et al. 2022) could benefit from the procedure outlined in this section, just by forming matrices $A_y^{m_0}$ and $A_z^{m_0}$ compatible with their respective methods. In particular, if desirable, all of the methods for increasing the speed of calculations detailed in (Meyer Forsting et al. 2020) could be easily implemented and would only affect the coefficients of these matrices.

For the methods that prescribe the shape of the vortex sheet, the separation of the missing velocity in the components of equations (77) and (78) could be skipped, however, it would reduce errors relative to the linearization if implemented. This means that the non-iterative strategy should also work even without bookkeeping of previous values of circulation. A mixed strategy that could be used to reduce the cost is to use a prescribed wake to calculate $u_y^{m_0}(\Gamma^{m-1})$ considering the circulation along the whole wake as $\Gamma^{m-1}$ (substituting older values of $\Gamma$ in figure 5 by $\Gamma^{m-1}$) and another prescribed near wake to define $A_y^{m_0}$, reducing the magnitude of the influence coefficients (compared to the case without any bookkeeping).

6 Numerical method

6.1 Numerical solver of Navier-Stokes equations with the actuator line

The incompressible three-dimensional Navier-Stokes equations are solved in the spectral-element code Nek5000 (Fischer et al. 2008). The system of equations is expressed in weak form and the solution is expanded in terms of basis and test functions on each of these elements (Offermans et al. 2020). In this work, the basis for the velocity space are seventh-order Lagrange polynomial interpolants on Gauss-Lobatto-Legendre quadrature points in each element. The $P_N - P_{N-2}$ formulation is used (Maday & Patera 1989). For temporal discretization, a third-order implicit/explicit scheme (BDF3/EXT3) (Fischer 2003) is employed. Filtering of the higher modes is applied to stabilize the simulation (Fischer & Mullen 2001).

In the current simulations, an adaptive mesh refinement (AMR) strategy with a spectral error indicator (Offermans 2019, Offermans et al. 2020, Tanarro et al. 2020) is used to reduce the computational cost. We choose to force maximum discretization of the grid around the actuator lines, independently of the error indicator, to guarantee adequate discretization for the chosen value of the smearing parameter.

The actuator line was previously implemented in Nek5000 (Kleusberg et al. 2016, 2017, Kleusberg 2019) with Prandtl’s tip correction. The code was extensively validated, showing good agreement between numerical and experimental results (Kleusberg et al. 2017, Mühle et al. 2018, Kleusberg 2019, Kleusberg et al. 2020). Since the focus of those studies were on the wake and on total forces, the errors caused by the approximation of the tip correction were not considered to greatly affect the results. The low dissipation and dispersion of the spectral element method make it well suited for stability analysis, hence, this code was previously used in several vortex stability studies (Kleusberg 2019, Kleusberg et al. 2019, Kleine et al. 2019, 2022).

The iterative and direct smearing corrections of Sections 2 and 5, with the correction velocity detailed in Appendix A.2, were implemented in the code. For the iterative smearing correction, the circulation is considered converged when $\|\Gamma^{m_{ew}} - \Gamma^{m_{d}}\|/\|\Gamma^{m_{ew}}\| < 10^{-5}$. Also implemented is the analytical form of the convolution of the Gaussian function with a force distribution formed by segments of constant force presented in Appendix A.1.

For the NREL 5-MW reference turbine (Jonkman et al. 2009), the airfoil is defined only for a few spanwise positions in the original reference. In (Martínez-Tossas et al. 2018) two different approaches have been employed by different codes, one considering abrupt changes in the airfoil and the other by considering interpolation of coefficients between sections. In the present implementation, the airfoil data from (Jonkman et al. 2009) is interpolated between sections, as performed by the DTU code EllipSys3D in (Martínez-Tossas et al. 2018), making the force curves smoother and avoiding problems with the discontinuities.
Figure 6: Reference system and schematic view of the computational domain. The center of the turbine/wing is equidistant to the upper, lower and lateral boundary conditions. Distances to the inflow ($L_{zin}$) and to the outflow ($L_{zout}$) may be different. The center of the turbine/wing is located at the origin of the coordinate system.

| Span, $R$ | Chord, $c$ | $\alpha_g$ (rad) | $\alpha_g$ (deg) | $U$ | $Re$ | $(L_x, L_y, L_z)$ | $L_{zin}$ |
|-----------|-----------|----------------|----------------|-----|-----|----------------|---------|
| 1         | 0.1       | 1/(2\pi)      | 9.189          | (0,0,1) | $10^5$ | (12,12,12) | 6       |

Table 1: Parameters of the simulation of the wing with constant chord in uniform flow.

The non-iterative method requires the value of the lift coefficient slope. A third-order polynomial interpolation of $C_l$ is used to avoid discontinuities in the lift coefficient slope. The tabulated airfoil lift coefficient is first modeled using a shape-preserving piecewise cubic polynomial (The MathWorks 2022b). Then, the lift coefficient slope is modeled by the piecewise quadratic polynomial obtained from the analytical derivative of the cubic polynomial. The piecewise coefficients of the polynomials are given as parameters to Nek5000.

A schematic representation of the computational domain can be seen in figure 6. Dirichlet boundary conditions are used for the inflow, upper, lower and lateral boundary conditions. The natural outflow boundary condition is imposed at the outlet.

6.2 Lifting line method

The non-linear iterative lifting line method of Section 4.1 was implemented considering horseshoe vortices aligned with the z-direction. The same discretization of the actuator line method is used. Since the lifting line method is much faster to run, a stricter convergence criteria is used. The iteration is considered converged when the difference of circulation at every control point is lower than $10^{-8}$ of the mean absolute value of circulation:

$$\max_{1 \leq j \leq N} \left( \frac{1}{|\Gamma_j^{\text{new}}|} \sum_{j=1}^{N} |\Gamma_j^{\text{old}}| \right) < 10^{-8}. \quad (94)$$

7 Results

7.1 Comparison with the lifting line method

The results of the simulation of the actuator line method (ALM) with smearing correction were compared with the results of a non-linear iterative lifting line for a planar straight wing with constant chord in uniform flow. The parameters of the geometry and simulation are presented in Table 1. Except where explicitly stated, all values are non-dimensionalized by the span ($R$), the velocity at infinity ($U_{ref} = U_z = 1$) and the density ($\rho$).

An ideal airfoil without drag and with lift coefficient $C_l = 2\pi\alpha$ is chosen (considering the standard non-dimensionalization of $C_l$ using the chord). The angle of attack $\alpha_g = 1/(2\pi)$ is such that a lift coefficient of $C_l(\alpha_g) = 1$ and a circulation of $\Gamma_0 = 0.05$ would be expected for a two-dimensional simulation (case with zero induced velocity).

The average grid spacing in the region of the actuator line is $\Delta x = 0.01786$. The results for two values of smearing parameter are compared: $\varepsilon = 3.5\Delta x = R/16$ and $\varepsilon = 7\Delta x = R/8$. The first value has been
chosen based on parametric studies of the current implementation of the actuator line (Kleusberg 2019).

Because of symmetry, just the results for half of the wing are shown. As can be seen in figure 7, the agreement is excellent for all cases. The difference in the induced velocity \( u_y \) is in the order of \( 10^{-4} \) (0.01\% of the undisturbed velocity), an agreement much better than the ones reported by Da˘g & Sørensen (2020) and Meyer Forsting et al. (2019a). The difference is in the order of the square of \( u_z \), which is consistent with a first-order method and, therefore, a better agreement was not expected.

Martínez-Tossas & Meneveau (2019) also achieved close agreements. However, it does not look that their errors are lower than \( 10^{-4} \). However, this comparison might not be fair, we presume that it would not be possible to reach errors in the order of \( 10^{-4} \) with their choice of parameters, because their simulation is very constrained. We estimate the errors due to the presence of the upper and lower boundary conditions in that work to be in the order of \( 10^{-2} \) or higher. Even for our simulation, with \( L_y = 12 \), we estimate the errors due to the boundary conditions to be in the order of \( 5 \cdot 10^{-5} \). Despite this, we assumed that enlarging the domain to reduce the errors due to the boundary conditions is unnecessary for the main point of this validation.

The other works are mentioned not as a criticism, but to bring context to the current values, showing the strength of the method. For most practical applications of the actuator line, the errors of the other methods can also be considered negligible.

We believe that the remarkable agreement can be mainly explained by three factors. First, the convolution of the force with the Gaussian kernel was performed analytically using the equation of Appendix A.1, considering constant force in a segment, which is the case that is mathematically equivalent to the lifting line method. Second, the lifting line method was carefully constructed to be consistent with the actuator line method. Third, the use of the velocity induced by a smeared vortex segment of Section 3.4 has lower associated errors when compared to other strategies where the vortices are also discretized.

Because the flow has reached the steady-state, no difference was expected between the iterative and the direct method for this case, what is confirmed by the results. The difference between the results for the two values of the smearing parameter is negligible. This is exactly the aim of the smearing correction: to make the forces on the blades insensitive to variations of the smearing correction and approach the lifting line method, which is the limit \( \varepsilon \to 0 \).

The velocity in the \( z \)-direction is also shown in figure 7, because it influences the circulation. The effect of the smearing correction on \( u_z^\infty \) is minimal. For most points, the missing velocity \( u_z^\infty \) is at least one order of magnitude lower than the difference observed in figure 7(d). The only exception is the point closest to the tip for \( \varepsilon = 3.5 \Delta x \), where the missing velocity and the corrected velocity are comparable. Hence, the differences observed are mainly due to the velocity sampled from the numerical simulation. The reason for the differences is probably the discrepancies in the vortex sheet created in the CFD domain and the vortex sheet of the lifting line method, as mentioned in the first paragraph of Section 3.5. The prescribed horseshoe vortices of the lifting line method do not induce velocity in the \( z \)-direction. In the CFD simulation, the vortices are free. As a consequence, the downwash from the wing inclines the vortex sheet, which induces velocity in the negative \( z \)-direction. A more advanced lifting line method might be able to capture those effects. Hence, the differences in \( u_z \) (and its consequence in \( \Gamma \)) can be, at least partially, attributed to a limitation of the current implementation of the lifting line method.

The relative differences in the values of circulation are in the order of \( 10^{-3} \) (differences in the order of 0.1\% of \( \Gamma_0 \)). This difference is negligible for practical purposes, but worth discussing. A difference \( \delta \Gamma \) in the circulation is related to differences \( \delta u_y \) and \( \delta u_z \) according to the equation

\[
\frac{\delta \Gamma}{\Gamma_0} = \frac{b_y}{\Gamma_0} \delta u_y + \frac{b_z}{\Gamma_0} \delta u_z,
\]

obtained in a derivation analogous to the equation (59). For this case, this relationship is simplified to

\[
\frac{\delta \Gamma}{\Gamma_0} = \frac{1}{C_l(a_{ij})} \partial C_l(a_{ij}) \frac{\delta u_y}{U_0} + \frac{\delta u_z}{U_0} = \frac{2\pi}{U_0} \frac{\delta u_y}{U_0} + \frac{\delta u_z}{U_0},
\]

where \( U_0 = U_z = 1 \). One important conclusion from equation (96) is that both differences in \( u_y \) and \( u_z \) affect \( \Gamma \), with the weight of the \( u_z \) term depending on the local lift coefficient. For most cases \( C_l \approx 1 \) or lower, thus, differences in \( u_y \) cause differences in \( \Gamma \) with one order of magnitude greater, while differences in \( u_z \) affect \( \Gamma \) with the same order of magnitude. However, in our simulations, the order of magnitude of the difference in \( u_z \) is greater than in \( u_y \). For \( \varepsilon = 7 \Delta x \), the differences in \( u_y \) and \( u_z \) compensate each other and the differences in \( \Gamma \) are lower. For the case with \( \varepsilon = 3.5 \Delta x \), the differences compound to cause higher differences in \( \Gamma \). For the lower smearing parameter, given that the relative differences are in the order of \( 10^{-3} \) even for positions far from the tip, it can be concluded that the difference does not come
Figure 7: Comparison of the results for the straight wing in uniform flow. LL - Non-linear iterative lifting line method of Section 4.1; ALM$^d$ - ALM with non-iterative correction of Section 5; ALM$^i$ - ALM with iterative correction of Section 2.2. (a) Corrected velocity in $y$-direction. (b) Difference between $y$-velocities obtained by the ALM and LL. (c) Corrected velocity in $z$-direction. (d) Difference between $z$-velocities obtained by the ALM and LL. (d) Circulation. (e) Relative difference between circulation obtained by the ALM and LL.
from the smearing correction, because the missing velocity is negligible for these regions. One possible reason for this difference, for $\varepsilon = 3.5\Delta x$, is that it comes from one of the errors inherent of the numerical method when the vortex core is not well represented by the grid resolution.

The circulation for the higher value of the smearing parameter agrees better with the lifting line. This apparently contradictory result has two simple explanations. First, the vortex core is larger, so the representation of the vorticity is better, reducing numerical errors. Second, for higher values of $\varepsilon$ the results depend less on the data from the simulation and more on the smearing correction. In the limit $\varepsilon \to \infty$, no information from the numerical simulation is used and the force is determined only by the correction, that regresses identically to the lifting line method. Nevertheless, increasing the smearing parameter is not usually beneficial, because the value of $\varepsilon$ has other effects on the simulation beyond its influence on the forces, as discussed in Section 7.2.

Before the development of the smearing correction, the use of the ALM was almost exclusively restricted to simulations of rotating blades, probably due to its inability to accurately reproduce the induced velocities, which affected the lift of translating wings. The remarkable agreement achieved here, which further confirms the good agreement of previous works (Dağ & Sørensen 2020, Martínez-Tossas & Menneveau 2019, Meyer Forsting et al. 2019a), might encourage the adoption of the ALM for other applications beyond rotating blades. For example, the aeronautics community could use it to simulate wings or the lifting surfaces of aircrafts. The ALM enables the integration of lifting lines with a CFD solver, allowing more complex configurations and flow conditions than a classical lifting line method. The low grid requirement is a clear advantage over traditional CFD methods that employ wall boundary conditions for the wing. Also, the free-vortex method described here maintains the generality of the method, so diverse configurations are allowed, and it does not suffer from the instability problems of traditional vortex filament methods (Leishman 2000).

### 7.2 Unsteady rotor simulations

The NREL 5-MW wind turbine subjected to a sheared inflow was simulated using the iterative and direct smearing corrections. The undisturbed wind speed varies in the vertical direction ($y$), in order to simulate a condition in which the circulation of the blade changes every time step. Hence, for every time step of this simulation, the initial guess of circulation is different from the final value of circulation, both for the iterative and the non-iterative corrections.

The domain was reduced to $L_x = L_y = 8$ to make the inflow velocity $U_z = y/5 + 1$ always positive and avoid problems with the outflow boundary condition, which is not stable if the velocity is negative. The distance from the inflow to the blades, $L_{cin} = 4$, was also reduced, maintaining the distance from the blades to the outflow $L_{out} = 6$. Since this is a comparative study, the reduction of the domain does not affect any analysis.

The parameters of the simulation are presented in table 2. Except where explicitly stated, all values are non-dimensionalized by the radius ($R$), the velocity at the center of the turbine ($U_{ref} = U(0,0,0) = 1$) and the density ($\rho$). Hence, the geometry, detailed in (Jonkman et al. 2009), was non-dimensionalized by the radius. For this choice of reference values, the tip speed ratio (which is the ratio of $\Omega R$ and the velocity at the center of the turbine) is equal to the angular velocity of the blade, $\Omega = 7.55$. The average grid spacing in the region of the actuator line is the same as the simulation of the straight wing, $\Delta x = 0.01786$.

| Radius, $R$ | $U(x,y,z)$ | $U_{ref}$ | $\Omega$ | $Re$ | $(L_x, L_y, L_z)$ | $L_{cin}$ |
|------------|------------|------------|----------|------|------------------|----------|
| 1          | (0,0,y/5 + 1) | 1          | 7.55     | $5 \cdot 10^4$ | (8,8,10) | 4           |

Table 2: Parameters of the simulation of the NREL 5-MW wind turbine in sheared flow.

In figure 8 the circulation and forces at time $t = 12T$, where $T$ is the period of rotation, are compared for the blade that has null azimuthal angle at this time (blade aligned with the x-axis). The evolution in time of the circulation of the control point closest to the tip of this blade is shown in figure 9.

As can be seen in figures 8 and 9, the agreement is very good for all cases. The iterative and the direct methods have results that are practically identical. The differences in the value of circulation are lower than $10^{-5}$ if the same smearing parameter $\varepsilon$ is used. This confirms that the non-iterative procedure does not introduce errors for this unsteady flow.

The differences between the circulation for the two values of $\varepsilon$ are in the order of 1% of the value of the circulation. The differences in the forces are of the same order of magnitude. Similar differences have already been observed in previous studies (Meyer Forsting et al. 2019b). These differences are believed
Figure 8: Circulation and forces along the radial direction \( r \) at time \( t/T = 12 \), for the blade positioned at null azimuthal angle. (a) Lift force. (b) Drag force. (a) Circulation. (a) Difference of circulation, taking as reference the circulation of ALM with \( \varepsilon = 3.5\Delta x \), normalized by the maximum of the circulation.

Figure 9: Evolution in time of the circulation at the point closest to the tip. The data is saved with a period of \( T/20 \), which is higher than the actual time step in the simulation. The correction was turned off for the first 2 points.
to be in the order of magnitude of errors due to the actuator line approximation. For example, in the code comparison of (Martinez-Tossas et al. 2018), larger differences were observed between different implementations of the actuator line. Hence, for practical purposes, such differences can be considered negligible.

The bookkeeping of position and circulation of the free-wake (see Section 3.5) begins at the first time step. But the application of the correction is turned on after $t = T/20$. The same orders of magnitude of differences were observed in the figure 9, even for the points right after the correction is turned on (after the first 2 points).

It should be noted that the effects of the smearing parameter can never be completely eliminated. The smearing correction has the objective of correcting the forces at the blades. Possibly, a better understanding of the role of drag, viscous effects, unsteadiness or other phenomena might reduce even further the errors caused by the Gaussian smearing. However, some errors cannot be eliminated, because the discretization used in actuator line simulations does not usually support vortices with very low vortex core size. Hence, the vorticity that is shed is still smeared. The smeared vortices create a wake that is different depending on the choice of $\varepsilon$, as can be seen in (Meyer Forsting et al. 2019b), especially in the near-wake. As a consequence, differences in the wake can have an effect on the forces. It has been seen in equations (95) and (96) that errors in the velocity have a first order effect in the circulation. If the modified near-wake has a slightly different blockage profile, then the velocities at the turbine and, consequently, the forces would be affected.

8 Conclusions

The conception of the smearing correction by Dağ & Sørensen (2020) (originally in (Dağ 2017)), the formal analysis provided by Martínez-Tossas & Meneveau (2019) and its improvement and analogy to a viscous lifting line by Meyer Forsting et al. (2019a, 2020) were great advancements to the actuator line method. These steps provided not only a more accurate, general and reliable method but also an understanding of the mathematical and physical reasons for the errors of the uncorrected actuator line. However, these methods resort to iterative procedures with a relaxation factor, which are generally slower, less stable and less deterministic than direct methods.

In the present work, a non-iterative smearing correction for the actuator line method is proposed and validated. Based on the linearization of a lifting line method, the iterative procedures of previous works have been substituted by the direct solution of a small linear system. For the cases tested, no significant difference is observed in the results of the iterative method and the non-iterative method. All differences are presumed to be orders of magnitude lower than the accuracy of the actuator line method.

The smearing correction reduces the effect of the smearing parameter $\varepsilon$ on the forces. However, differences, which are in the order of the errors of the actuator line approximation, were observed in the forces and circulation for different smearing parameters. This indicates that the smearing parameter still affects the simulation, what agrees with the observation that smearing has an effect on the near wake (Meyer Forsting et al. 2019b). Nevertheless, for practical applications, the differences observed in the forces can be considered negligible.

Another contribution of the present work includes the use of a correction function based on the velocity induced by a smeared vortex segment, obtained from a table look-up method, what eliminates the need for approximations of the correction function. Also, we implement a free-vortex wake model to define the vortex sheet based on the CFD velocities. This keeps the generality of the method and makes it applicable to several configurations without the need for ad hoc assumptions. If computational cost is prioritized, one could use approximate functions, prescribe the wakes, or apply one of the methods for increasing the speed from Meyer Forsting et al. (2020). The focus of this study is on presenting and validating the technique. Many options to reduce the computational cost even further could be implemented. Additionally, the optimal choice of parameters for a desired accuracy is a matter of investigation. For these reasons, the analysis of the computational cost of different choices of options and parameters of the correction is left for future studies.

Additionally, by carefully constructing a non-linear lifting line method and an actuator line method that are consistent with each other, we arrive at differences in the induced velocity that are in the order of $10^{-4}$ for a planar wing in a steady uniform flow, considerable better than differences reported in the literature. The good agreement serves as an a posteriori justification for the linearization of the lifting line method, in addition to the proof that these methods are mathematically identical if the circulation in each segment of the actuator line is assumed constant.
Nevertheless, the main contribution is the new direct strategy that substitutes the iterative procedure of previous corrections. In the method proposed here, separating the contribution of the vortex sheet released in the past and the vortex sheet released at the current time-step is used to reduce the influence of the current value of circulation, which improves the results of a linearized method. However, this separation is not essential to the method. The methods proposed in previous works can also benefit from solving a linear system, either by considering the solution directly or to accelerate the convergence of an iterative procedure.

So far, the actuator line method has been used primarily by the rotor aerodynamics community, possibly because of its known limitations to replicate the forces on non-rotating wings. The use of the smearing correction with a general formulation removes this limitation and reproduces the results of a lifting line method, as shown here and in previous works. This could motivate other communities, such as the aeronautical community, to also adopt this technique to simulate wings, taking advantage of the lower grid requirements allowed by the ALM, in flow conditions more complex than the ones allowed by the lifting line method.

Acknowledgements

The computations were performed on resources provided by the Swedish National Infrastructure for Computing (SNIC) at the High Performance Computing Center North (HPC2N). This work was conducted within StandUp for Wind. V.G.K. thanks KTH Engineering Mechanics for partially funding this work.

Declaration of interests

The authors report no conflict of interest.

A Implementation of the formulas in three-dimensional form

A.1 Convolution of a segment of constant force with Gaussian function

In Section 3.2, it is shown that the ALM with smearing correction formed by segments of constant circulation tends to a discretized lifting line method, according to the linearization of the vorticity equation. However, in the primitive variables formulation of the Navier-Stokes equation, the circulation is not explicitly included, the body force is. If we assume that the velocity is approximately constant along a segment, then a constant circulation in a segment implies a constant body force in the segment. This assumption is an approximation, since the local velocity changes along the blades for a rotating blade. Nevertheless, this assumption is consistent with the linear theory (Section 3.1) and is a good approximation if the length of the segment is small compared to the radius.

For such spanwise distribution of force, we present the formula to calculate, analytically, the body force \( f \) at position \( x \), taking into account all segments \( j \) with boundaries at positions \( x_{j_1} \) and \( x_{j_2} \) with constant force \( F_{2D}(x_j) \), is

\[
f(x) = \frac{1}{\rho} \sum_{j=1}^{N} F_{2D}(x_j) \frac{1}{\pi \sigma^2} \exp \left( -\frac{d^2}{\sigma^2} \right) \frac{\left( \text{erf} \left( \frac{x-x_{j_1}}{\sigma} \right) - \text{erf} \left( \frac{x-x_{j_2}}{\sigma} \right) \right)}{2},
\]

where \( l_+ \) is the length of the projection of the distance from \( x \) to the point \( x_{j_1} \) onto the direction of the segment:

\[
l_+ = \frac{(x-x_{j_1}) \cdot (x_{j_1}-x_{j_2})}{\| (x_{j_1}-x_{j_2}) \|},
\]

\( l_- \) is the length of the projection of the distance to the point \( x_{j_2} \):

\[
l_- = \frac{(x-x_{j_2}) \cdot (x_{j_1}-x_{j_2})}{\| (x_{j_1}-x_{j_2}) \|} = l_+ + \| (x_{j_1}-x_{j_2}) \|
\]

and \( d \) is the distance of \( x \) to the segment (normal to the direction of the segment), given by

\[
d = \frac{\| (x_{j_1}-x_{j_2}) \times (x-x_{j_1}) \|}{\| (x_{j_1}-x_{j_2}) \|} = \sqrt{\| x-x_{j_1} \|^2 - l_+^2}.\]
A.2 Velocity induced by a smeared vortex segment from tabulated data

The velocity induced at point $\mathbf{x}$ by a smeared vortex segment, defined in Section 3.4 with boundaries located at $\mathbf{x}_{j-}$ and $\mathbf{x}_{j+}$, is calculated as

$$
u(x) = \frac{\Gamma_j}{4\pi\varepsilon} (\Phi(d^+, l_+^*) - \Phi(d^-, l_-^*)) \hat{e}_x$$

(101)

where the superscript * indicates values non-dimensionalized by $\varepsilon$ ($d^* = d/\varepsilon$, $l_*^+ = l_+^*/\varepsilon$, $l_*^- = l_-^*/\varepsilon$) and $\hat{e}_x$ is the unitary vector in the direction normal to the plane formed by the segment and point of interest

$$\hat{e}_x = \frac{(x_{j-} - x_{j+}) \times (x - x_{j+})}{\| (x_{j-} - x_{j+}) \times (x - x_{j+}) \|}$$

(102)

The non-dimensional value of $\Phi$ was tabulated by evaluating numerically the triple integral of equation (47) (and equation (39)) in MATLAB (The MathWorks 2022a), assuming $\varepsilon = 1$. Due to the antisymmetry of the function, only positive values for $l_*^+$ and $l_*^-$ are needed in the table. For negative values, the relationship $\Phi(d^+, l^*) = -\Phi(d^-, l^*)$ is used. For values of $d^*$ higher than the tabulated values, no correction is needed, because $\Phi \approx \Phi^*$ (due to $d \gg \varepsilon$) and for values of $l_*^+$ and $l_*^-$ higher than the tabulated values, equation (50) is used:

$$\Phi(d^*, l^*) \approx \Phi^*(d^*, l^*) \left( 1 - e^{-d^2} \right)$$

(103)

where, from equation (49),

$$\Phi^*(d^*, l^*) = -\frac{l^*}{d^* \sqrt{d^2 + l^2}}$$

(104)

B Advection velocity of the vortex sheet

The vortex sheet generated in the smearing correction is determined by the circulation and the position of the vortices. Due to the vortex transport theorem, the points that define the vortex sheet are treated as passive particles and are advected with the flow velocity. There is an ambiguity regarding using the corrected velocity, $\nu^*$, or the uncorrected velocity $\nu$, sampled directly from the numerical simulation, to define the position of vortices.

This ambiguity can be partially attributed to the fact that the method with a smearing correction has better results than the uncorrected actuator line, independently of the advection velocity, as can be seen in (Dağ & Sørensen 2020), that showed that even a prescribed wake had good results, and (Stanly et al. 2022), that used simple straight horseshoe vortices instead of helical vortices for rotating blades.

In order to understand the effect of the advection velocity, we consider a semi-infinite vortex, whose apex is located at $(x_{j-}, 0, 0)$ and calculate the induced velocity at position $(x, 0, 0)$. For conciseness, in this section, when the position is omitted in the velocity function, it is assumed that it corresponds to velocities at position $(x_{j-}, 0, 0)$.

The smeared vortices created by the body forces in the CFD simulations are advected by the velocity $\nu$ ($\nu^*$ is not known by the CFD solver). Assuming a semi-infinite smeared vortex that is advected with velocity $\nu(x_{j-})$, equation (37) is modified to

$$u^v_{y}(x) = \frac{\Gamma_j}{4\pi(x - x_{j-})} \left( 1 - e^{-\frac{(x-x_{j-})^2}{\varepsilon^2}} \right) \frac{u_z}{\sqrt{u_y^2 + u_z^2}}$$

(105)

and the velocity induced by an ideal semi-infinite vortex advected by the corrected velocity $\nu^*(x_{j-})$ is

$$u^v_{y}(x) = \frac{\Gamma_j}{4\pi(x - x_{j-})} \frac{u_z^*}{\sqrt{u_y^2 + u_z^*^2}}$$

(106)
The missing velocity term in the y-direction, $u_y^{m_{ij}}(x)$, due to this semi-infinite vortex then would be

$$u_y^{m_{ij}}(x) = u_y^{v_{ij}}(x) - u_y^{u_{ij}}(x)$$

$$= \frac{\Gamma_j}{4\pi(x-x_{ij})} \left[ \frac{u_x^c}{\sqrt{u_x^2 + u_z^2}} - \left(1 - e^{-(x-x_{ij})^2} \right) \frac{u_z}{\sqrt{u_x^2 + u_z^2}} \right]$$

$$= \frac{\Gamma_j}{4\pi(x-x_{ij})} e^{-(x-x_{ij})^2} \frac{u_z}{\sqrt{u_x^2 + u_z^2}} + \frac{\Gamma_j}{4\pi(x-x_{ij})} \left[ \frac{u_x^c}{\sqrt{u_x^2 + u_z^2}} - \frac{u_z}{\sqrt{u_x^2 + u_z^2}} \right]$$

$$(107)$$

In this case, the vortices also induce velocity in the z-direction. Analogously

$$u_z^{m_{ij}}(x) = \frac{\Gamma_j}{4\pi(x-x_{ij})} e^{-(x-x_{ij})^2} \frac{-u_y}{\sqrt{u_x^2 + u_z^2}} + \frac{\Gamma_j}{4\pi(x-x_{ij})} \left[ \frac{-u_y}{\sqrt{u_x^2 + u_z^2}} - \frac{-u_y}{\sqrt{u_x^2 + u_z^2}} \right]$$

$$(108)$$

The missing velocity $u_n^{m_{ij}}(x)$ in the direction normal to the velocity vector $(u_x(x_{ij}), u_z(x_{ij}))$ is:

$$u_n^{m_{ij}}(x) = u_y^{v_{ij}}(x) - u_y^{u_{ij}}(x) \frac{u_y}{\sqrt{u_x^2 + u_z^2}}$$

$$= \frac{\Gamma_j}{4\pi(x-x_{ij})} e^{-(x-x_{ij})^2} \frac{u_z}{\sqrt{u_x^2 + u_z^2}} + \frac{\Gamma_j}{4\pi(x-x_{ij})} \left[ \frac{u_x^c}{\sqrt{u_x^2 + u_z^2}} - \frac{u_y}{\sqrt{u_x^2 + u_z^2}} \right]$$

$$= \frac{\Gamma_j}{4\pi(x-x_{ij})} e^{-(x-x_{ij})^2} \frac{u_x^c}{\sqrt{u_x^2 + u_z^2}} + \frac{\Gamma_j}{4\pi(x-x_{ij})} \left[ \frac{-u_y}{\sqrt{u_x^2 + u_z^2}} - \frac{-u_y}{\sqrt{u_x^2 + u_z^2}} \right]$$

$$(109)$$

where the approximation came from the Taylor series expansion of $u_x^c(x_{ij})$ and $u_z^c(x_{ij})$ using equations (11) and (12). We note that the second term is of second order with respect to $u_n^m(x_{ij})$ and $u_n^m(x_{ij})$.

In a similar manner, it can be shown that the effect on the missing velocity in the direction of the velocity vector $(u_x(x_{ij}), u_z(x_{ij}))$ is of first order with respect to $u_n^m(x_{ij})$ and $u_n^m(x_{ij})$. However, the difference in the velocity in the direction of the velocity vector has a second-order influence on the angle of attack, if the directions of the velocities at $x$ and $x_{ij}$ are close, as is the case for the most relevant vortices for a certain control point.

From this, we conclude that the angle of attack is indifferent to the choice of advection velocity, as a first order approximation. Therefore, if the difference between corrected and sampled velocity is small, the missing velocity in the normal direction for a semi-infinite vortex, $u_n^{m_{ij}}(x)$, can be calculated with the formula described in previous works (Dağ & Sørensen 2020, Martínez-Tossas & Meneveau 2019, Meyer Forsting et al. 2019a):

$$u_n^m(x) = \frac{\Gamma_j}{4\pi(x-x_{ij})} e^{-(x-x_{ij})^2}$$

$$(110)$$

using the velocity sampled at the CFD domain to advect the vortices.

The error in this approximation is of the same order of magnitude as a conventional lifting line method applied to a straight wing that assumes that horseshoe vortices are perfectly aligned with the undisturbed flow. Since the error is of second order in magnitude, the error in advecting the vortex sheet by the velocities sampled from the CFD domain is negligible, as long as the missing velocity is much lower than the total velocity. Hence, in the current implementation, the tracing particles that define the vortex filaments are advected by the CFD velocities.

C Classical lifting line method

The linearization of the lifting line method, performed in Section 4.2 gives the equation

$$(I - \text{diag}(b_y)\mathbf{A}_y - \text{diag}(b_z)\mathbf{A}_z)\Gamma = \Gamma_0.$$
This formulation is equivalent to a classical lifting line under the following conditions:

1. \( U_y = 0; \)
2. Constant \( U_z = U_\infty; \)
3. Ideal airfoil: \( C_l = 2\pi\alpha; \)
4. Straight wing with spanwise direction aligned with the \( x \) direction;
5. Horseshoe vortices aligned with \( U_\infty. \)

Conditions (4) and (5) imply \( A_z = 0. \) Applying the other conditions:

\[
\begin{align*}
  b_{yj} &= \frac{1}{2} c_j 2\pi \\
  \Gamma_{0j} &= \frac{1}{2} U_\infty c_j 2\pi\alpha_{gj}
\end{align*}
\]

Hence, equation (111) becomes

\[
- \frac{1}{2\pi} (\text{diag}(c/2))^{-1} \Gamma + A_y \Gamma = -U_\infty \alpha_g
\]

Each term of equation (114) can be directly compared to a term in equation 8.11 of (Katz & Plotkin 1991):

\[
- \frac{\Gamma(x)}{2\pi \left( \frac{c(x)}{2} \right)} = -\frac{1}{4\pi} \int_{x_{\min}}^{x_{\max}} \frac{-\partial \Gamma(x')}{\partial x} \frac{dx'}{x-x'} = -U_\infty \alpha_g
\]

where the notation of (Katz & Plotkin 1991) was modified to be consistent with our notation. The term \(-U_\infty \alpha_g\) is the classical right-hand side term, given by the non-penetration condition at the control point (collocation point) in a classical lifting line (see sections 5.6 and 8.1.2 of (Katz & Plotkin 1991)). The term \( A_y \Gamma \) is the velocity induced by the vortex sheet (since the bound vortex has no contribution in this case), that is the discrete version of the integral of equation (115). The remaining term

\[
- \frac{\Gamma_j}{2\pi \frac{c_j}{2}}
\]

is the induced velocity of an infinite vortex with circulation \( \Gamma_j \) on a point distanced \( c_j/2 \). This is equivalent to an infinite bound vortex located at the quarter of chord and a collocation point at \( 3/4 c_j \), which is the classical position for the control point. This confirms that the classical numerical lifting line is a simplification of the derivation of Section 4.2.

References

Anderson Jr, J. (1991), Fundamentals of Aerodynamics, McGraw-Hill Inc.

Caprace, D.-G., Chatelain, P. & Winckelmans, G. (2019), ‘Lifting line with various mollifications: theory and application to an elliptical wing’, AIAA Journal 57(1), 17–28.

Churchfield, M. J., Schreck, S. J., Martinez, L. A., Meneveau, C. & Spalart, P. R. (2017), An advanced actuator line method for wind energy applications and beyond, in ‘35th Wind Energy Symposium’, p. 1998.

Cormier, M., Weihing, P. & Lutz, T. (2021), ‘Evaluation of the effects of actuator line force smearing on wind turbines near-wake development’, Journal of Physics: Conference Series 1934(1), 012013.

Da˘ g, K. O. (2017), Combined pseudo-spectral/actuator line model for wind turbine applications, PhD thesis, DTU Technical University of Denmark.

Da˘ g, K. O. & Sørensen, J. N. (2020), ‘A new tip correction for actuator line computations’, Wind Energy 23(2), 148–160.
Fischer, P. (2003), ‘Implementation considerations for the oifs/characteristics approach to convection problems’.

Fischer, P. F., Lottes, J. W. & Kerkemeier, S. G. (2008), ‘nek5000 web page’, https://nek5000.mcs.anl.gov/.

Fischer, P. & Mullen, J. (2001), ‘Filter-based stabilization of spectral element methods’, Comptes Rendus de l’Académie des Sciences-Series I-Mathematics 332(3), 265–270.

Forsythe, J. R., Lynch, E., Polsky, S. & Spalart, P. (2015), Coupled flight simulator and cfd calculations of ship airwake using kestrel, in ‘53rd AIAA Aerospace Sciences Meeting’, p. 0556.

Jha, P. K., Churchfield, M. J., Moriarty, P. J. & Schmitz, S. (2014), ‘Guidelines for volume force distributions within actuator line modeling of wind turbines on large-eddy simulation-type grids’, Journal of Solar Energy Engineering 136(3).

Jha, P. K. & Schmitz, S. (2018), ‘Actuator curve embedding—an advanced actuator line model’, Journal of Fluid Mechanics 834.

Jonkman, J., Butterfield, S., Musial, W. & Scott, G. (2009), Definition of a 5-mw reference wind turbine for offshore system development, Technical report, National Renewable Energy Laboratory. Tech. Rep. NREL/TP-500-38060.

Katz, J. & Plotkin, A. (1991), Low-speed aerodynamics: From wing theory to panel methods, McGraw-Hill Inc.

Kleine, V. G., Franceschini, L., Carmo, B. S., Hanifi, A. & Henningson, D. S. (2022), ‘The stability of wakes of floating wind turbines’, Physics of Fluids. In press.

Kleine, V., Kleusberg, E., Hanifi, A. & Henningson, D. S. (2019), ‘Tip-vortex instabilities of two in-line wind turbines’, Journal of Physics: Conference Series 1256(1), 012015.

Kleusberg, E. (2019), Wind-turbine wakes-Effects of yaw, shear and turbine interaction, PhD thesis, KTH Royal Institute of Technology.

Kleusberg, E., Benard, S. & Henningson, D. S. (2019), ‘Tip-vortex breakdown of wind turbines subject to shear’, Wind Energy 22(12), 1789–1799.

Kleusberg, E., Mikkelsen, R. F., Schlatter, P., Ivanell, S. & Henningson, D. S. (2017), ‘High-order numerical simulations of wind turbine wakes’, J. Phys.: Conf. Series 854, 012025.

Kleusberg, E., Sarmast, S., Schlatter, P., Ivanell, S. & Henningson, D. S. (2016), ‘Actuator line simulations of a joukowsky and tjæreborg rotor using spectral element and finite volume methods’, Journal of Physics: Conference Series 753(8), 082011.

Kleusberg, E., Schlatter, P. & Henningson, D. S. (2020), ‘Parametric dependencies of the yawed wind-turbine wake development’, Wind Energy 23(6), 1367–1380.

Lamb, H. (1932), Hydrodynamics, University Press.

Leishman, G. J. (2000), Principles of helicopter aerodynamics, Cambridge university press.

Maday, Y. & Patera, A. T. (1989), ‘Spectral element methods for the incompressible navier-stokes equations’, IN: State-of-the-art surveys on computational mechanics (A90-47176 21-64). New York pp. 71–143.

Martinez, L., Leonardi, S., Churchfield, M. & Moriarty, P. (2012), A comparison of actuator disk and actuator line wind turbine models and best practices for their use, in ‘50th AIAA Aerospace Sciences Meeting including the New Horizons Forum and Aerospace Exposition’, p. 900.

Martínez-Tossas, L. A., Churchfield, M. J. & Meneveau, C. (2017), ‘Optimal smoothing length scale for actuator line models of wind turbine blades based on gaussian body force distribution’, Wind Energy 20(6), 1083–1096.
Martinez-Tossas, L. A., Churchfield, M. J., Yilmaz, A. E., Sarlak, H., Johnson, P. L., Sørensen, J. N., Meyers, J. & Meneveau, C. (2018), ‘Comparison of four large-eddy simulation research codes and effects of model coefficient and inflow turbulence in actuator-line-based wind turbine modeling’, *Journal of Renewable and Sustainable Energy* **10**(3), 033301.

Martinez-Tossas, L. A. & Meneveau, C. (2019), ‘Filtered lifting line theory and application to the actuator line model’, *Journal of Fluid Mechanics* **863**(NREL/JA-5000-72646).

Meyer Forsting, A. R., Pirrung, G. R. & Ramos-García, N. (2019a), ‘A vortex-based tip/smearing correction for the actuator line’, *Wind Energy Science* **4**(2), 369–383.

Meyer Forsting, A. R., Pirrung, G. R. & Ramos-García, N. (2019b), ‘The wake of an actuator line with a vortex-based tip/smearing correction in uniform and turbulent inflow’, *Journal of Physics: Conference Series* **1256**(1), 012020.

Meyer Forsting, A. R., Pirrung, G. R. & Ramos-García, N. (2020), ‘Brief communication: A fast vortex-based smearing correction for the actuator line’, *Wind Energy Science* **5**(1), 349–353.

Mikkelsen, R. (2003), Actuator disc methods applied to wind turbines, PhD thesis, DTU Technical University of Denmark.

Mühle, F., Schottler, J., Bartl, J., Furtzynski, R., Evans, S., Bernini, L., Schito, P., Draper, M., Guggeri, A., Kleusberg, E., Henningson, D. S., Hölling, M., Peinke, J., Adaramola, M. S. & Sætran, L. (2018), ‘Blind test comparison on the wake behind a yawed wind turbine’, *Wind Energy Sci.* **3**(2), 883–903.

Offermans, N. (2019), Aspects of adaptive mesh refinement in the spectral element method, PhD thesis, KTH Royal Institute of Technology.

Offermans, N., Peplinski, A., Marin, O. & Schlatter, P. (2020), ‘Adaptive mesh refinement for steady flows in nek5000’, *Computers & Fluids* **197**, 104352.

Oseen, C. (1911), ‘Über wirbelbewegung in einer reibenden flüssigkeit, ark’, *Ark. Mat. Astro. Fys.* **7**.

Phillips, W. F. & Snyder, D. (2000), ‘Modern adaptation of prandtl’s classic lifting-line theory’, *Journal of Aircraft* **37**(4), 662–670.

Pirrung, G. R., Madsen, H. A., Kim, T. & Heinz, J. (2016), ‘A coupled near and far wake model for wind turbine aerodynamics’, *Wind Energy* **19**(11), 2053–2069.

Pirrung, G. R., Madsen, H. A. & Schreck, S. (2017), ‘Trailed vorticity modeling for aeroelastic wind turbine simulations in standstill’, *Wind Energy Science* **2**(2), 521–532.

Saffman, P. G. (1992), *Vortex dynamics*, Cambridge university press.

Shen, W. Z., Sørensen, J. N. & Mikkelsen, R. (2005), ‘Tip loss correction for actuator/navier–stokes computations’, *J. Sol. Energy Eng.* **127**(2), 209–213.

Shives, M. & Crawford, C. (2013), ‘Mesh and load distribution requirements for actuator line cfd simulations’, *Wind Energy* **16**(8), 1183–1196.

Sørensen, J. N. (2016), *General momentum theory for horizontal axis wind turbines*, Vol. 4 of *Research Topics in Wind Energy*, Springer.

Sørensen, J. N., Mikkelsen, R. F., Henningson, D. S., Ivanell, S., Sarmast, S. & Andersen, S. J. (2015), ‘Simulation of wind turbine wakes using the actuator line technique’, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* **373**(2035), 20140071.

Sørensen, J. N. & Shen, W. Z. (2002), ‘Numerical modeling of wind turbine wakes’, *J. Fluids Eng.* **124**(2), 393–399.

Stanly, R., Martinez-Tossas, L. A., Frankel, S. H. & Delorme, Y. (2022), ‘Large-eddy simulation of a wind turbine using a filtered actuator line model’, *Journal of Wind Engineering and Industrial Aerodynamics* **222**, 104868.
Tanarro, Á., Mallor, F., Offermans, N., Peplinski, A., Vinuesa, R. & Schlatter, P. (2020), ‘Enabling adaptive mesh refinement for spectral-element simulations of turbulence around wing sections’, Flow, Turbulence and Combustion 105(2), 415–436.

The MathWorks, I. (2022a), ‘integral3: Numerically evaluate triple integral’, https://se.mathworks.com/help/matlab/ref/integral3.html. Accessed: 2022-04-08.

The MathWorks, I. (2022b), ‘interp1: 1-d data interpolation (table lookup)’, https://se.mathworks.com/help/matlab/ref/interp1.html. Accessed: 2022-04-08.

Troldborg, N. (2009), Actuator line modeling of wind turbine wakes, PhD thesis, DTU Technical University of Denmark.

Widder, D. V. & Hirschman, I. I. (1955), Convolution Transform, Princeton University Press.