On $a_2^{(1)}$ Reflection Matrices and Affine Toda Theories

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ABSTRACT

We construct new non-diagonal solutions to the boundary Yang–Baxter–Equation corresponding to a two-dimensional field theory with $U_q(a_2^{(1)})$ quantum affine symmetry on a half-line. The requirements of boundary unitarity and boundary crossing symmetry are then used to find overall scalar factors which lead to consistent reflection matrices. Using the boundary bootstrap equations we also compute the reflection factors for scalar bound states (breathers). These breathers are expected to be identified with the fundamental quantum particles in $a_2^{(1)}$ affine Toda field theory and we therefore obtain a conjecture for the affine Toda reflection factors. We compare these factors with known classical results and discuss their duality properties and their connections with particular boundary conditions.

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1 Introduction

There has been an increasing amount of effort towards the understanding of two-dimensional integrable field theories with reflecting boundaries in recent years. Apart from their role as toy models for higher dimensional field theories and their connections with open string theories, these models have also found some applications in real physical situations like impurity problems in condensed matter physics. Many interesting open questions remain in the classical as well as in the quantum theory of integrable models with reflecting boundaries.

Classically, one of the first questions to ask is whether there are any boundary conditions for a given integrable field theory which preserve integrability. This problem was extensively studied in a series of papers [1, 2] (for a recent review see [3]) in which the authors examined real coupling affine Toda field theories (ATFT) on a half-line \((-\infty < x \leq 0)\), defined by a Lagrangian of the form

\[ \mathcal{L}_B = \Theta(-x)\mathcal{L} - \delta(x)\mathcal{B}(\phi), \]

in which \(\mathcal{L}\) is the usual affine Toda Lagrangian and \(\mathcal{B}\) is some boundary potential, which is chosen to depend only on the field \(\phi\) and not on any time derivatives of \(\phi\). In particular, for the case of \(a_n^{(1)}\) ATFTs it was found that \(\mathcal{L}_B\) defines an integrable theory if the boundary potential takes the following form

\[ \mathcal{B} = \sum_{j=0}^{n} b_j e^{\alpha_j \frac{\phi}{\pi}}, \]

in which the \(\alpha_j\) are the simple roots of \(a_n^{(1)}\) and \(b_j\) are some boundary parameters. Surprisingly, it turned out that only in the simple case of the sinh–Gordon theory, which is \(a_1^{(1)}\) ATFT, the boundary parameters are free and hence there are an infinite number of integrable boundary conditions. Whereas in all other cases \((n \geq 2)\) only a finite number of integrable boundary potentials exist. For instance for the \(a_2^{(1)}\) ATFT on a half-line it was found that there are only nine possible boundary terms which lead to a classically integrable theory, namely

- either \(b_0 = b_1 = b_2 = 0\),
- or \(b_j = \pm 2\) \((j = 0, 1, 2)\).

In the quantum case all on-shell information of a two-dimensional integrable field theory is contained in its \(S\)-matrix. In the presence of a reflecting boundary the \(S\)-matrix has to be supplemented by a so-called reflection matrix, which describes the scattering of a particle from the boundary. Following earlier works by Cherednik [4] and Sklyanin [5] on factorisable scattering on a half-line, in a seminal paper [6] Ghoshal and Zamolodchikov studied the general requirements imposed by integrability on the reflection matrices. In particular they were able to solve the boundary Yang–Baxter–Equation and construct reflection matrices for the sine–Gordon theory on a half-line. One of the most difficult problems in the boundary sine-Gordon theory turned out to be the question of how to relate the free parameters appearing in the reflection matrices with the boundary parameters in the Lagrangian. This is still an open problem, even
though some progress has been achieved in a recent paper [7], in which it was noticed that the breather reflection factors of sine–Gordon theory [8] should be identical to the reflection factors for the particles in sinh–Gordon. Hence these reflection factors could be compared with results from perturbation theory.

On the other hand, since in any of the higher order $a_n^{(1)}$ ATFTs only a finite number of integrable boundary conditions exist, it would seem feasible that, if we could find the general form of the reflection factors for these theories, it should be fairly straightforward to associate them with particular boundary conditions. However, despite several attempts the construction of reflection factors for ATFTs on a half-line has been elusive so far.

Another interesting problem is the question of duality. The real affine Toda $S$-matrices were found to display an intriguing weak–strong coupling duality [9]. Up to now it has been unclear whether a similar duality will be present in the reflection matrices of ATFTs. However, recently it was suggested in [7] that in the sinh–Gordon theory duality relates the reflection factors of the Neumann boundary condition ($b_0 = b_1 = 0$) with those of the ‘positive’ boundary condition ($b_0 = b_1 = +2$). It would be interesting to see whether a similar relation could be true for any other ATFT.

One of the main results in this paper is a derivation of the reflection factors for the simplest ATFT with more than one particle, namely the theory based on the algebra $a_2^{(1)}$. This derivation uses an analogy with the $S$-matrices for ATFTs on the whole line. It has been realised over the past few years that the real Toda $S$-matrices can be constructed using exact $S$-matrices with quantum affine symmetries [10, 11, 12, 13]. These $U_q(\hat{g})$ invariant $S$-matrices, which were constructed using trigonometric $R$-matrices, have been conjectured to describe the scattering of the solitons in imaginary coupled ATFTs. It then turned out that the scalar bound states (breathers) in these theories can be identified with the fundamental Toda quantum particles and that the scattering amplitudes for the breathers are identical to the real Toda $S$-matrices after analytic continuation of the coupling constant. Here we introduce a similar construction for a $U_q(a_2^{(1)})$ invariant theory with reflecting constant. Here we introduce a similar construction for a $U_q(a_2^{(1)})$ invariant theory with reflecting boundary.

The layout of this paper is as follows. Section 2 briefly reviews the $U_q(a_2^{(1)})$ invariant $S$-matrix and the use of the Faddeev–Zamolodchikov algebra. Section 3 provides an introduction to the boundary Yang–Baxter–Equation and we construct solutions related to the $U_q(a_2^{(1)})$ invariant $S$-matrix. In section 4 we discuss the restrictions imposed on these solutions by boundary unitarity and boundary crossing, and we construct suitable overall scalar factors. Section 5 examines the implications of the boundary bootstrap conditions, and we use the bootstrap equations in order to compute reflection factors for the breathers. Finally in section 6 we give a conjecture for the reflection factors of real coupling $a_2^{(1)}$ ATFT on a half-line and discuss their duality properties. Two appendices provide some of the computational details which have been omitted from the main text.
2 The $U_q(a_2^{(1)})$ S-matrix

This is a brief review of the $U_q(a_2^{(1)})$ invariant trigonometric S-matrix which was studied in detail in \cite{10,12}. We consider a two-dimensional integrable field theory which displays a $U_q(a_2^{(1)})$ affine quantum symmetry. The theory contains two multiplets transforming under the two fundamental representations of $U_q(a_2^{(1)})$. Each of these multiplets contains three states, which we label by $A_1, A_2, A_3$ and $\overline{A}_1, \overline{A}_2, \overline{A}_3$, respectively. The S-matrix of the theory is an intertwining map on the two representation spaces (denoted by $V_1$ and $V_2$)

$$S_{a,b}(\theta) : V_a \otimes V_b \rightarrow V_b \otimes V_a , \quad (a, b = 1, 2) ,$$

in which $\theta$ is the rapidity difference of the incoming states. Instead of the $\theta$, from now on we will always use the following parametrisation of the rapidity variable:

$$\mu = -i \frac{3\lambda}{2\pi} \theta ,$$

in which $\lambda$ is a coupling constant parameter. Due to the quantum affine symmetry of the theory, the S-matrix can be constructed using the $U_q(a_2^{(1)})$ R-matrix in the principal gradation. This construction was first performed in \cite{14} (for more details see \cite{12}).

Rather than using the R-matrix, for our purposes it will prove more convenient to write the scattering theory in terms of a non-commutative algebra, the so called Faddeev–Zamolodchikov (FZ) algebra. The generators of this algebra are denoted by $A_i(\mu)$, $\overline{A}_i(\mu)$, which represent the fundamental states in the theory, and the possible scattering processes are expressed in terms of the following braiding relations:

$$A_j(\mu_1)A_j(\mu_2) = S^f(\mu_{12}) A_j(\mu_2)A_j(\mu_1) ,$$
$$A_j(\mu_1)A_k(\mu_2) = S^t(\mu_{12}) A_k(\mu_2)A_j(\mu_1) + S^{R(j,k)}(\mu_{12}) A_j(\mu_2)A_k(\mu_1) , \quad (j \neq k)$$

in which $\mu_{12} = \mu_1 - \mu_2$, and the elements of the S-matrix are given as

$$S^f(\mu) = F(\mu) ,$$
$$S^t(\mu) = \frac{\sin(\pi \mu)}{\sin(\pi(\lambda - \mu))} F(\mu) ,$$
$$S^{R(j,k)}(\mu) = \frac{\sin(\pi \lambda)}{\sin(\pi(\lambda - \mu))} e^{\nu(j,k)i\pi\frac{\mu}{3}} F(\mu) ,$$

in which

$$\nu(j,k) \equiv \begin{cases} +1 , & \text{if } (j,k) = (1,2), (2,3) \text{ or } (3,1) , \\ -1 , & \text{if } (j,k) = (2,1), (3,2) \text{ or } (1,3) . \end{cases}$$

The overall scalar factor $F(\mu)$ can be written as an infinite product of Gamma functions

$$F(\mu) = - \prod_{j=1}^{\infty} \frac{\Gamma(\mu + 3j\lambda - 2\lambda + 1) \Gamma(\mu + 3j\lambda - \lambda) \Gamma(-\mu + 3j\lambda - 3\lambda + 1) \Gamma(-\mu + 3j\lambda)}{\Gamma(-\mu + 3j\lambda - 2\lambda + 1) \Gamma(-\mu + 3j\lambda - \lambda) \Gamma(\mu + 3j\lambda - 3\lambda + 1) \Gamma(\mu + 3j\lambda)} .$$
Scattering processes involving the states $\overline{A}_i$ can be easily derived from (2.3) using crossing symmetry, e.g.

$$\overline{A}_j(\mu_1) A_j(\mu_2) = S^I(\frac{3}{2} \lambda - \mu_{12}) A_j(\mu_2) \overline{A}_j(\mu_1) + S^{R(j,k)}(\frac{3}{2} \lambda - \mu_{12}) A_k(\mu_2) \overline{A}_k(\mu_1),$$
$$\overline{A}_k(\mu_1) A_j(\mu_2) = S^T(\frac{3}{2} \lambda - \mu_{12}) A_j(\mu_2) \overline{A}_k(\mu_1).$$  \hspace{1cm} (2.7)

This $S$-matrix has been conjectured to describe the scattering of solitons in $a_2^{(1)}$ ATFT with imaginary coupling constant. In the following we will therefore often call the states $A_i$, $\overline{A}_i$ solitons. Apart from these fundamental solitons there are also bound states in the theory, which correspond to simple poles in the $S$-matrix. There are two kinds of bound states in the theory, namely scalar bound states or breathers and excited solitons. Here we are only interested in the lowest mass breathers, i.e. those bound states corresponding to the pole $\mu = 3\lambda - 1$ which appears in the cross channel of $S^I(\mu)$. In order to describe the scattering of these bound states we have to extend the FZ algebra by generators $B_1(\mu)$ and $\overline{B}_1(\mu)$ corresponding to the lowest breathers. It has been shown in [10] that these generators can be defined formally as

$$B_1(\frac{\mu_1 + \mu_2}{2}) = \lim_{\mu_2 - \mu_1 \rightarrow \frac{3}{2} \lambda - 1} \sum_{m=1}^{3} \alpha_m A_m(\mu_1) \overline{A}_m(\mu_2),$$
$$\overline{B}_1(\frac{\mu_1 + \mu_2}{2}) = \lim_{\mu_2 - \mu_1 \rightarrow \frac{3}{2} \lambda - 1} \sum_{m=1}^{3} \alpha_m \overline{A}_m(\mu_1) A_m(\mu_2),$$  \hspace{1cm} (2.8)

in which

$$\alpha_1 = e^{i\frac{\pi}{3}}, \quad \alpha_2 = 1, \quad \alpha_3 = e^{-i\frac{\pi}{3}}.$$  

The scattering of two of these breathers with each other can then be described by the braiding relation

$$B_1(\mu_1) B_1(\mu_2) = S_{B_1,B_1}(\mu_{12}) B_1(\mu_2) B_1(\mu_1),$$

and similarly for the conjugate breathers $\overline{B}_1$. One can compute the breather scattering amplitudes by using the bootstrap equations, and it was found that

$$S_{B_1,B_1}(\mu) = S_{\overline{B}_1,B_1}(\mu) = \left(1 \right) \left(\lambda \right) \left(-1 - \lambda \right),$$  \hspace{1cm} (2.9)

and

$$S_{B_1,\overline{B}_1}(\mu) = S_{\overline{B}_1,B_1}(\mu) = S_{B_1,B_1}(\frac{3}{2} \lambda - \mu),$$

in which we have used the bracket notation

$$\left(a \right) \equiv \frac{\sin(\frac{\pi}{3\lambda}(\mu + a))}{\sin(\frac{\pi}{3\lambda}(\mu - a))}. $$  \hspace{1cm} (2.10)
Finally we note that if we choose the parameter $\lambda$ to be related to the affine Toda coupling constant $\beta$ in the following way\footnote{Note, that this is not really a choice, but can be derived by considering non-local conserved charges in imaginary coupled ATFT.}

$$\lambda = \frac{4\pi}{\beta^2} - 1,$$

then it turns out that after analytic continuation, $\beta \rightarrow i\beta$, the breather scattering amplitudes \textcolor{red}{(2.9)} are identical to the $S$-matrix for real coupling $a_2^{(1)}$ ATFT \textcolor{red}{[9]}\footnote{For the general case of a $a_n^{(1)}$ theory we expect that solitons in the $a$th multiplet reflect into solitons in the charge conjugate $(n + 1 - a)$th multiplet.}. This establishes the fact that, just as in the sine–Gordon theory, the lowest mass breathers can be identified with the fundamental quantum particles in $a_2^{(1)}$ ATFT. This identification will later provide the basis for our conjecture of the reflection factors in $a_2^{(1)}$ ATFT on a half-line.

\section{The Boundary Yang–Baxter–Equation}

From now on we will deal with a $U_q(a_2^{(1)})$ invariant scattering theory in which the spatial coordinate is restricted to the half-line $-\infty < x \leq 0$. We assume that this theory is still integrable and that any scattering processes far away from the boundary are still described by the $U_q(a_2^{(1)})$ invariant $S$-matrix from the previous section. Incoming states moving towards the boundary at $x = 0$ will be reflected into outgoing states with negative rapidity. This reflection of states from the boundary will be described by so-called reflection matrices.

First of all, we will assume that the reflection of solitons is always multiplet changing, which means an incoming state of type $A_i$ can only reflect into a state of type $A_j$ and vice versa\footnote{Note, that this is not really a choice, but can be derived by considering non-local conserved charges in imaginary coupled ATFT.}. Hence the reflection matrices are maps on the representation spaces of the form

$$K(\mu) : V_1 \rightarrow V_2,$$

$$K(\mu) : V_2 \rightarrow V_1.$$

This assumption of multiplet changes for the reflection of solitons may seem rather unmotivated at this stage. However, some recent work by Gustav Delius \textcolor{red}{[15]} on classical soliton solutions in ATFTs on a half-line has shown that the only classical solutions possible are those in which a soliton is reflected into a corresponding antisoliton. Furthermore, this property will also lead to the fact that the breathers do not change after reflection, which is expected if we want to identify the lowest breathers with the fundamental quantum particles in ATFT. In appendix A we will briefly mention other known solutions to the BYBE which are not of the type \textcolor{red}{(3.1)}.

The main restriction on the explicit form of the reflection matrices \textcolor{red}{(3.1)} comes from the boundary Yang–Baxter–Equation (BYBE), which is the analogue of the Yang–Baxter–Equation
(YBE) in scattering theories on the full line. In our case the BYBE takes the following form:

\[
[I_2 \otimes K(\mu')] \cdot S_{1,2}(\mu + \mu') \cdot [I_1 \otimes K(\mu)] \cdot S_{1,1}(\mu - \mu') = S_{2,2}(\mu - \mu') \cdot [I_2 \otimes K(\mu)] \cdot S_{1,2}(\mu + \mu') \cdot [I_1 \otimes K(\mu')] ,
\]

in which \( I_a \) denotes the identity on \( V_a \) and both sides map \( V_1 \otimes V_1 \) into \( V_2 \otimes V_2 \). The corresponding equation for \( K(\mu) \) is the same with all indices 1 and 2 exchanged.

In order to explicitly solve these equations it is useful to write them in terms of the FZ algebra. Since \( V_1 \) and \( V_2 \) are three dimensional, we can write the reflection matrices as \( 3 \times 3 \) matrices \( (K_j^i)_{i,j=1,...,3} \). We then extend the FZ algebra (2.3) for the theory on a half-line by the following relations:

\[
A_j(\mu) B = K_j^k(\mu) A_k(-\mu) B ,
\]

\[
\overline{A}_j(\mu) B = \overline{K}_j^k(\mu) A_k(-\mu) B ,
\]

in which \( B \) denotes the boundary at \( x = 0 \) and summation over repeated indices is assumed. Figure 1 shows the diagrammatic representation of these simple soliton reflection processes. (In these and all following diagrams time is meant to be increasing up the page.)

\[
A_j(\mu) A_k(\mu) = S_{j,k}^{m,l}(\mu_{12}) A_l(\mu_2) A_m(\mu_1) ,
\]

in which the non-zero components of the \( S \)-matrix are

\[
S_{i,i}^{i,j}(\mu) = S^{I}(\mu) ,
\]

\[
S_{i,j}^{i,j}(\mu) = S^{T}(\mu) ,
\]

\[
S_{i,j}^{i,j}(\mu) = S^{R(i,j)}(\mu) , \quad \text{for} \quad i,j = 1,\ldots,3; \quad \text{and} \quad i \neq j .
\]
The matrix elements involving the scattering of antisolitons can again be obtained easily using crossing symmetry:

\[ S_{k,l}^{i,j}(\mu) = S_{l,j}^{k,i}(\frac{3}{2}\lambda - \mu), \quad \text{and} \quad S_{l,j}^{k,i}(\mu) = S_{l,k}(\mu). \]

We can therefore write the BYBE as a matrix equation

\[ K_j^k(\mu') S_{i,k}^{l,m}(\mu + \mu') K_m^l(\mu) S_{l,m}^{p,n}(\mu - \mu') = S_{i,j}^{k,l}(\mu - \mu') K_m^l(\mu) S_{k,m}^{p,n}(\mu + \mu') K_r^r(\mu'), \quad (3.6) \]

in which we sum over the indices \( k, l, m, n \). This equation can be illustrated by the equality of the two scattering diagrams in figure 2.

![Figure 2: The Boundary Yang–Baxter–Equation](image)

There are four free indices in expression (3.6) and we thus obtain 81 independent and mostly non-trivial equations for the nine unknown functions \( K_j^k(\mu) \). We therefore must resort to the use of some algebraic manipulation software in order to solve these equations. Most of the computations in this paper were done using MapleV. We only list the results here, details of the derivation of these solutions can be found in appendix A.

We have found essentially three different solutions to the \( a_2^{(1)} \) BYBE. There is one diagonal solution and two non-diagonal solutions, all of which contain several free coupling constant dependent parameters and are only determined up to an overall scalar factor. Let us start with the two non-diagonal solutions denoted by \( K^+(\mu) \) and \( K^-(\mu) \). They contain two free coupling constant dependent parameters \( g(\lambda) \) and \( h(\lambda) \), and written as matrices take the following form:

\[ K^\pm(\mu) = \begin{pmatrix} \kappa_{\pm}(\mu) \frac{h(\lambda)}{g(\lambda)} & e^{i\pi(\frac{3}{4} - \frac{\lambda}{4})} & \pm e^{-i\pi(\frac{3}{4} - \frac{\lambda}{4})} h(\lambda) \\ \pm e^{-i\pi(\frac{3}{4} - \frac{\lambda}{4})} & \kappa_{\pm}(\mu) \frac{g(\lambda)}{h(\lambda)} & e^{i\pi(\frac{3}{4} - \frac{\lambda}{4})} g(\lambda) \\ e^{i\pi(\frac{\lambda}{4} - \frac{3}{4})} h(\lambda) & \pm e^{-i\pi(\frac{\lambda}{4} - \frac{3}{4})} g(\lambda) & \kappa_{\pm}(\mu) h(\lambda) g(\lambda) \end{pmatrix} A^\pm(\mu), \quad (3.7) \]
in which \( \mathcal{A}^\pm(\mu) \) is an overall scalar factor,

\[
\kappa_+(\mu) = \frac{\sin(\pi(\mu - \frac{1}{3}))}{\sin(\pi \frac{1}{3})}, \quad \text{and} \quad \kappa_-(\mu) = i \frac{\cos(\pi(\mu - \frac{1}{3}))}{\sin(\pi \frac{1}{3})}.
\]

The reflection matrix \( \overline{K}^i_j(\mu) \) for the reflection of an incoming antisoliton into a soliton can be obtained by solving the following analogue of (3.6):

\[
\overline{K}^k_j(\mu') S^l_{i,k}(\mu + \mu') \overline{K}^m_i(\mu) S^n_{l,m}(\mu - \mu') = S^k_{i,j}(\mu - \mu') \overline{K}^n_i(\mu) S^p_{k,n}(\mu + \mu') \overline{K}^r_n(\mu') .
\]

This can be solved in a exactly the same way and we obtain

\[
\overline{K}^\pm(\mu) = \begin{pmatrix}
\kappa_\pm(\mu) \overline{T}(\lambda) & \pm e^{-i\pi(\frac{2}{3} - \frac{1}{3})} & e^{i\pi(\frac{2}{3} - \frac{1}{3})}\overline{T}(\lambda) \\
e^{i\pi(\frac{2}{3} - \frac{1}{3})} & \kappa_\pm(\mu) \overline{T}(\lambda) \overline{T}(\lambda) & \pm e^{-i\pi(\frac{2}{3} - \frac{1}{3})}\overline{T}(\lambda) \\
\pm e^{-i\pi(\frac{2}{3} - \frac{1}{3})}\overline{T}(\lambda) & e^{i\pi(\frac{2}{3} - \frac{1}{3})}\overline{T}(\lambda) & \kappa_\pm(\mu) \overline{T}(\lambda) \overline{T}(\lambda)
\end{pmatrix} \mathcal{A}^\pm(\mu) .
\]

Apart from these two non-diagonal solutions there is also a purely diagonal solution, i.e. \( K^i_i(\mu) = \overline{K}^i_i(\mu) = 0 \) (for all \( i \neq j \)). It can be seen easily that in this case the BYBE is trivially solved by

\[
K^d(\mu) = \begin{pmatrix}
1 & 0 & 0 \\
0 & d_1(\lambda) & 0 \\
0 & 0 & d_2(\lambda)
\end{pmatrix} \mathcal{A}^d(\mu) ,
\]

in which \( d_1(\lambda), d_2(\lambda) \) are free boundary parameters. Similarly, we can write a solution for the conjugate reflection matrix in the form

\[
\overline{K}^d(\mu) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \overline{d}_1(\lambda) & 0 \\
0 & 0 & \overline{d}_2(\lambda)
\end{pmatrix} \overline{\mathcal{A}}^d(\mu) ,
\]

So far no connection between the matrices \( K(\mu) \) and \( \overline{K}(\mu) \) exists. However, in order for these \( K \)-matrices to define consistent boundary scattering theories, several other conditions have to be satisfied. We will see in the subsequent sections that these conditions will not only provide a connection between \( K(\mu) \) and \( \overline{K}(\mu) \), but will also determine the overall scalar factors and restrict further the number of free parameters in the reflection matrices.
4 Boundary crossing and boundary unitarity

Exact $S$-matrices in two-dimensional integrable theories must satisfy a number of very restrictive conditions. These are known as the analytic $S$-matrix axioms, which include the requirements of $S$-matrix unitarity and crossing symmetry, as well as the bootstrap principle. It is because of these conditions that it is possible to construct $S$-matrices exactly for many integrable models. These $S$-matrix axioms all have analogues in the case of integrable theories with a boundary, which impose strong restrictions on the possible form of the reflection matrices. The conditions of boundary unitarity, boundary crossing and the boundary bootstrap were discussed in detail by Ghoshal and Zamolodchikov in [6]. In this section we will use these conditions in order to construct consistent reflection matrices related to the above solutions of the BYBE.

4.1 Boundary unitarity

The condition of boundary unitarity provides a connection between $K(\mu)$ and $\overline{K}(\mu)$, and takes the form

$$K^j_i(\mu)\overline{K}^k_j(-\mu) = \delta^k_i .$$

(4.1)

Putting our solutions from the previous section into this equation we first find that it can only be satisfied if $K^+(\mu)$, $K^-(\mu)$ and $K^d(\mu)$ are connected with $\overline{K}^+(\mu)$, $\overline{K}^-(\mu)$ and $\overline{K}^d(\mu)$, respectively. Furthermore, it emerges that (4.1) also puts a constraint on the free boundary parameters, namely for the non-diagonal solutions we find

$$g(\lambda)\overline{g}(\lambda) = h(\lambda)\overline{h}(\lambda) = 1 ,$$

(4.2)

and in the case of $K^d(\mu)$

$$d_1(\lambda)\overline{d}_1(\lambda) = d_2(\lambda)\overline{d}_2(\lambda) = 1 .$$

(4.3)

And finally, the conditions imposed on the overall scalar factors by (4.1) are

$$\mathcal{A}^+(\mu)\overline{\mathcal{A}}^+(\mu) = \frac{\sin^2\left(\frac{\pi}{2}\lambda\right)}{\sin \left(\pi(\mu + \frac{3}{4}\lambda)\right) \sin \left(\pi(\mu - \frac{3}{4}\lambda)\right)} ,$$

$$\mathcal{A}^-(\mu)\overline{\mathcal{A}}^-(\mu) = \frac{\sin^2\left(\frac{\lambda}{2}\right)}{\cos \left(\pi(\mu + \frac{3}{4}\lambda)\right) \cos \left(\pi(\mu - \frac{3}{4}\lambda)\right)} ,$$

$$\mathcal{A}^d(\mu)\overline{\mathcal{A}}^d(\mu) = 1 .$$

(4.4)

4.2 Boundary crossing

The boundary analogue of the crossing symmetry condition was called boundary cross-unitarity by Ghoshal and Zamolodchikov. In our notation this condition takes the form

$$K^j_i\left(\frac{3}{4}\lambda - \mu\right) = S^{k,l}_{j,i}(2\mu)K^k_l\left(\frac{3}{4}\lambda + \mu\right) ,$$

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Putting the solutions to the BYBE and the explicit expression for the $S$-matrix, as given in section 2, into these equations, we obtain a second set of equations determining the scalar factors:

\[
\begin{align*}
\mathcal{A}^+(-\mu + \frac{3}{4} \lambda) &= \frac{\mathcal{A}^+(-\mu + \frac{3}{4} \lambda)}{\mathcal{A}^+(\mu + \frac{3}{4} \lambda)} = \frac{-\sin(\pi (\mu + \frac{1}{2}))}{\sin(\pi (\mu - \frac{1}{2}))} F(2\mu), \\
\mathcal{A}^-(-\mu + \frac{3}{4} \lambda) &= \frac{\mathcal{A}^-(-\mu + \frac{3}{4} \lambda)}{\mathcal{A}^-(\mu + \frac{3}{4} \lambda)} = \frac{\cos(\pi (\mu + \frac{1}{2}))}{\cos(\pi (\mu - \frac{1}{2}))} F(2\mu), \\
\mathcal{A}^d(-\mu + \frac{3}{4} \lambda) &= \frac{\mathcal{A}^d(-\mu + \frac{3}{4} \lambda)}{\mathcal{A}^d(\mu + \frac{3}{4} \lambda)} = F(2\mu),
\end{align*}
\]

in which $F(\mu)$ is the $S$-matrix scalar factor (2.6). Here we already notice one major difference between this case and the case of the boundary sine–Gordon theory [6], in that none of the equations in (4.4, 4.6) contain any dependence on the free boundary parameters $g(\lambda), h(\lambda)$ or $d_1(\lambda), d_2(\lambda)$.

4.3 The overall scalar factors

We want to construct solutions to equations (4.4) and (4.6). In order to do this it will prove useful to separate these two sets of equations by making the following ansatz:

\[
\begin{align*}
\mathcal{A}^\epsilon(\mu) &= \sin(\pi \frac{\lambda}{2}) a_0^\epsilon(\mu) a_1(\mu), \\
\overline{\mathcal{A}}(\mu) &= \sin(\pi \frac{\lambda}{2}) \overline{a}_0^\epsilon(\mu) \overline{a}_1(\mu),
\end{align*}
\]

in which $\epsilon = +, -$ or $d$, such that $a_1(\mu)$, which is the same for all three cases, is determined by

\[
\left(\begin{array}{c}
\overline{a}_1(-\mu) \\
a_1(-\mu + \frac{3}{4} \lambda)
\end{array}\right) = \left(\begin{array}{c}
-1 \\
\frac{\overline{a}_1(-\mu + \frac{3}{4} \lambda)}{a_1(\mu + \frac{3}{4} \lambda)}
\end{array}\right) = -F(2\mu),
\]

and the $a_0^\epsilon(\mu)$ satisfy

\[
\left(\begin{array}{c}
a_0^+(-\mu) \\
a_0^+(-\mu + \frac{3}{4} \lambda)
\end{array}\right) = \left(\begin{array}{c}
\frac{-1}{\sin(\pi (\mu + \frac{3}{4} \lambda)) \sin(\pi (\mu - \frac{3}{4} \lambda))} \\
\frac{\overline{a}_0^+(-\mu + \frac{3}{4} \lambda)}{\overline{a}_0^+(\mu + \frac{3}{4} \lambda)}
\end{array}\right),
\]

and

\[
\left(\begin{array}{c}
a_0^-(-\mu) \\
a_0^-(-\mu + \frac{3}{4} \lambda)
\end{array}\right) = \left(\begin{array}{c}
\frac{-1}{\sin(\pi (\mu + \frac{3}{4} \lambda)) \sin(\pi (\mu - \frac{3}{4} \lambda))} \\
\frac{\overline{a}_0^-(-\mu + \frac{3}{4} \lambda)}{\overline{a}_0^-(\mu + \frac{3}{4} \lambda)}
\end{array}\right),
\]

\[
\left(\begin{array}{c}
a_0^d(-\mu) \\
a_0^d(-\mu + \frac{3}{4} \lambda)
\end{array}\right) = \left(\begin{array}{c}
\frac{-1}{\sin(\pi (\mu + \frac{3}{4} \lambda)) \sin(\pi (\mu - \frac{3}{4} \lambda))} \\
\frac{\overline{a}_0^d(-\mu + \frac{3}{4} \lambda)}{\overline{a}_0^d(\mu + \frac{3}{4} \lambda)}
\end{array}\right).
\]
and analogously

$$a_0^{-}(-\mu) = \frac{1}{\cos(\pi(\mu - \frac{3}{4}\lambda)) \cos(\pi(\mu - \frac{3}{4}\lambda))} \cdot$$

$$a_0^{-}(\mu + \frac{3}{4}\lambda) = \frac{a_0^{-}(\mu + \frac{3}{4}\lambda)}{\pi_0^{-}(\mu + \frac{3}{4}\lambda)} = -\frac{\cos(\pi(\mu + \frac{3}{4}\lambda))}{\cos(\pi(\mu - \frac{3}{4}\lambda))}, \quad (4.10)$$

$$a_0^{d}(\mu) a_0^{d}(-\mu) = \frac{1}{\sin^2(\pi \frac{3}{8})},$$

$$\frac{a_0^{d}(\mu + \frac{3}{4}\lambda)}{a_0^{d}(\mu + \frac{3}{4}\lambda)} = \frac{a_0^{d}(\mu + \frac{3}{4}\lambda)}{a_0^{d}(\mu + \frac{3}{4}\lambda)} = -1. \quad (4.11)$$

Solutions to equations of this type can be constructed in a similar way as the solutions for the scalar factors in imaginary ATFTs on the whole line. Let us concentrate on $a_0^{+}(\mu)$ first. It is easy to check that equations (4.9) can be solved formally by an expression of the form

$$a_0^{+}(\mu) = a_0^{+}(\mu) = \frac{1}{\sin(\pi(\mu - \frac{1}{4}))) \sin(\pi(\mu - \frac{1}{4}))} \prod_{j=1}^{\infty} f(\mu + 3j\lambda - 3\lambda)f(-\mu + 3j\lambda - \frac{3}{2}\lambda) f(-\mu + 3j\lambda)f(\mu + 3j\lambda - \frac{3}{2}\lambda) ,$$

for any function $f(\mu)$ with

$$f(\mu)f(-\mu) = \sin(\pi(\mu - \frac{1}{4})) \sin(\pi(-\mu - \frac{1}{4})) .$$

In order to avoid the occurrence of an infinite number of poles in the physical strip, the obvious choice for $f(\mu)$ is

$$f(\mu) = \frac{\pi}{\Gamma(\mu - \frac{1}{4})\Gamma(1 + \mu + \frac{3}{4})} .$$

Using the expression (2.6) for the S-matrix scalar factor the solution for $a_1(\mu)$ can be constructed in a similar way and we obtain

$$a_1(\mu) = a_1(\mu) = \prod_{k=1}^{\infty} \frac{\Gamma(2\mu + 6k\lambda - \frac{9}{2}\lambda + 1)\Gamma(2\mu + 6k\lambda - \frac{3}{2}\lambda)}{\Gamma(-2\mu + 6k\lambda - \frac{9}{2}\lambda + 1)\Gamma(-2\mu + 6k\lambda - \frac{3}{2}\lambda) \Gamma(2\mu + 6k\lambda - \frac{9}{2}\lambda + 1)\Gamma(2\mu + 6k\lambda - \frac{3}{2}\lambda) .} \quad (4.12)$$

If we use the abbreviation

$$G_{j}(\mu, a) = \frac{\Gamma(\mu + 3j\lambda + a)}{\Gamma(-\mu + 3j\lambda + a)} , \quad (4.13)$$

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the final result for the scalar factor can be written in the following compact form:

\[ A^+ (\mu) = \bar{A}^+ (\mu) = \frac{\sin(\frac{\pi \mu}{2})}{\sin(\pi(\mu - \frac{3}{4} \lambda))} \prod_{j=1}^{\infty} \frac{G_j(\mu, -\frac{7}{2} \lambda) G_j(\mu, -\frac{5}{2} \lambda + 1)}{G_j(\mu, -\frac{1}{4} \lambda) G_j(\mu, -\frac{11}{4} \lambda + 1)} \times \prod_{k=1}^{\infty} \frac{G_{2k}(2\mu, -\frac{3}{2} \lambda) G_{2k}(2\mu, -\frac{9}{2} \lambda + 1)}{G_{2k}(2\mu, -\frac{5}{2} \lambda) G_{2k}(2\mu, -\frac{7}{2} \lambda + 1)}. \] (4.14)

Analogously, for the case of \( K^- (\mu) \) we find

\[ A^- (\mu) = -\bar{A}^- (\mu) = \frac{\sin(\frac{\pi \mu}{2})}{\cos(\pi(\mu - \frac{3}{4} \lambda))} \left( -\frac{3}{4} \lambda \right) \prod_{j=1}^{\infty} \frac{G_j(\mu, -\frac{7}{4} \lambda + \frac{1}{2}) G_j(\mu, -\frac{5}{4} \lambda + \frac{1}{2})}{G_j(\mu, -\frac{1}{4} \lambda + \frac{1}{2}) G_j(\mu, -\frac{11}{4} \lambda + \frac{1}{2})} \times \prod_{k=1}^{\infty} \frac{G_{2k}(2\mu, -\frac{3}{2} \lambda) G_{2k}(2\mu, -\frac{9}{2} \lambda + 1)}{G_{2k}(2\mu, -\frac{5}{2} \lambda) G_{2k}(2\mu, -\frac{7}{2} \lambda + 1)}, \] (4.15)

in which we have used the bracket notation (2.10) as introduced in section 2. The solution for \( a^d_0 (\mu) \) is rather trivial and we obtain the scalar factor for the diagonal reflection matrices in the form

\[ A^d (\mu) = \bar{A}^d (\mu) = \left( -\frac{3}{4} \lambda \right) a_1 (\mu). \] (4.16)

As usual, the infinite products of Gamma functions in these overall scalar factors can alternatively be written in integral form. We find

\[ A^+ (\mu) = \frac{\sin(\frac{\pi \mu}{2})}{\sin(\pi(\mu - \frac{3}{4} \lambda))} \exp \int_0^{\infty} \frac{dt}{t} \left[ -2 \sinh(2\mu t) \mathcal{I}_1 (t) + \sinh(\mu t) \mathcal{I}_+ (t) \right], \]

\[ A^- (\mu) = \frac{\sin(\frac{\pi \mu}{2})}{\cos(\pi(\mu - \frac{3}{4} \lambda))} \left( -\frac{3}{4} \lambda \right) \exp \int_0^{\infty} \frac{dt}{t} \left[ -2 \sinh(2\mu t) \mathcal{I}_1 (t) + \sinh(\mu t) \mathcal{I}_- (t) \right], \]

\[ A^d (\mu) = \left( -\frac{3}{4} \lambda \right) \exp \int_0^{\infty} \frac{dt}{t} \left[ -2 \sinh(2\mu t) \mathcal{I}_1 (t) \right], \] (4.17)

in which

\[ \mathcal{I}_1 (t) = \frac{\sinh(\frac{\lambda}{2} t) \sinh((\lambda - \frac{1}{2}) t)}{\sinh(\frac{1}{2} t) \sinh(3 \lambda t)}, \]

\[ \mathcal{I}_+ (t) = \frac{\sinh(\frac{\lambda - 1}{2} t)}{\sinh(\frac{1}{2} t) \cosh(\frac{3}{2} \lambda t)}, \]

\[ \mathcal{I}_- (t) = \frac{\sinh(\frac{\lambda}{2} t)}{\sinh(\frac{1}{2} t) \cosh(\frac{1}{2} \lambda t)}. \] (4.18)
Of course, in all three cases there still exists an overall ambiguity. We can always multiply $A(\mu)$ and $\overline{A}(\mu)$ by a function $\sigma(\mu)$ which satisfies
\[ \sigma(\mu)\sigma(-\mu) = 1, \]
\[ \sigma(\mu + \frac{3}{4}\lambda) = \sigma(-\mu + \frac{3}{4}\lambda). \] (4.19)
In the next section the bootstrap condition is used to remove this ambiguity. Note also that all solutions we found satisfy $A(\mu) = \pm \overline{A}(\mu)$. There is however no obvious reason that this has to be so, and we will see below that there are more general scalar factors for which $A(\mu) \neq \pm \overline{A}(\mu)$.

5 The boundary bootstrap relations

The bootstrap principle for an integrable two-dimensional field theory states that simple $S$-matrix poles in the physical strip ($0 \leq \text{Im}(\theta) \leq \pi$) correspond to bound states of the two incoming particles. In the bulk theory the bootstrap equations relate the scattering amplitude of such a bound state to the product of the scattering amplitudes of the incoming states, thus allowing the construction of bound state $S$-matrices from the $S$-matrices of the fundamental states in the theory. Correspondingly, in theories on a half-line the so-called boundary bootstrap equations can be used to construct the amplitudes for the reflection of bound states from the boundary.

As mentioned earlier, the $U_q(a^{(1)}_2)$ invariant scattering theory contains two types of bound states, namely scalar bound states and bound states with non-zero topological charge, the so-called excited solitons. Therefore, we have to examine two types of bootstrap equations, the soliton bootstrap and the breather bootstrap.

5.1 The soliton bootstrap

Again, here we are only interested in the bound states with lowest mass. The lowest mass excited solitons are just the fundamental solitons themselves. In other words in the $U_q(a^{(1)}_2)$ invariant scattering theory two solitons can fuse into a single antisoliton and vice versa. This fusion corresponds to the physical strip pole at $\mu = \lambda$ (or $\theta = \frac{2\pi}{3}$) in $S_T(\mu)$. In [10] it was shown that this fusion can formally be expressed in terms of the FZ generators:

\[ \overline{A}_j(\frac{\mu_1 + \mu_2}{2}) = \lim_{\mu_2 - \mu_1 \to \lambda} \left[ A_k(\mu_1)A_l(\mu_2) + \gamma_{k,l}A_l(\mu_1)A_k(\mu_2) \right], \]
\[ A_j(\frac{\mu_1 + \mu_2}{2}) = \lim_{\mu_2 - \mu_1 \to \lambda} \left[ \overline{A}_l(\mu_1)\overline{A}_k(\mu_2) + \gamma_{k,l}\overline{A}_k(\mu_1)\overline{A}_l(\mu_2) \right], \] (5.1)
in which $(j, k, l)$ are even permutations of $(1, 2, 3)$ and $\gamma_{k,l} = e^{i\pi \frac{\nu(k,l)}{3}}$, where $\nu(k,l)$ was defined in (2.3).
By analogy with the bootstrap on the whole line, the boundary bootstrap expresses the fact that the reflection amplitude of two particles should be independent of whether their fusion into a bound state occurs before or after their reflection from the boundary. This can be illustrated in the equality of the two reflection processes depicted in figure 3. The corresponding soliton bootstrap equations provide a relation between $K(\mu)$ and $\overline{K}(\mu)$ and can therefore not only serve as a highly non-trivial check for our reflection matrices, but they also fix the CDD ambiguity (4.19) in the overall scalar factors. This will be important in the next section, since different choices of $\sigma(\mu)$ would lead to different breather reflection amplitudes.

\[ \sigma(\mu + \lambda/2)\sigma(\mu - \lambda/2) = \left(-\frac{\lambda}{4}\right)^2 \left(\frac{3}{4}\lambda\right)^2 \left(-\frac{5}{4}\lambda\right)^2 \sigma(\mu) . \]  

A simple solution to these equations can be written conveniently in the form

\[ \sigma^+(\mu) = \left(\frac{\lambda}{4}\right) \left(\frac{5}{4}\lambda\right) . \]  

For the cases of $K^-(\mu)$ and $K^d(\mu)$ the soliton bootstrap relations are satisfied without introducing any further factors. Furthermore, we find that the soliton bootstrap equations put another constraint on the free parameters. We obtain that these bootstrap equations can only be satisfied if we choose

\[ g(\lambda) = -\frac{1}{h(\lambda)} , \quad \text{(in $K^+$)} . \]
The diagonal reflection matrices are $U$ to one. We, therefore, obtain the final result for the $U_q(a_2^{(1)})$ invariant reflection matrices:

$$
g(\lambda) = \frac{i}{h(\lambda)}, \quad \text{ (in } K^{-}),$$
$$d_1(\lambda) = \frac{1}{d_2(\lambda)}, \quad \text{ (in } K^{d}),$$

which reduces the total number of continuous free parameters in each of the reflection matrices to one. We, therefore, obtain the final result for the $U_q(a_2^{(1)})$ invariant reflection matrices:

$$K^+(\mu) = \begin{pmatrix}
-\frac{\sin(\pi(\mu - \frac{1}{4}))}{\sin(\pi \frac{5}{4})} & e^{i\pi(\frac{5}{4} - \frac{i}{4})} & e^{-i\pi(\frac{5}{4} - \frac{i}{4})} \\
0 & -\cos(\pi(\mu - \frac{1}{4})) & \sin(\pi(\mu - \frac{1}{4})) \\
0 & 0 & 1
\end{pmatrix} \sigma^+(\mu) A^+(\mu),$$

$$K^-(\mu) = \begin{pmatrix}
\frac{\cos(\pi(\mu - \frac{1}{4}))}{\sin(\pi \frac{5}{4})} & e^{i\pi(\frac{5}{4} - \frac{i}{4})} & e^{-i\pi(\frac{5}{4} - \frac{i}{4})} \\
-\frac{\cos(\pi(\mu - \frac{1}{4}))}{\sin(\pi \frac{5}{4})} & \cos(\pi(\mu - \frac{1}{4})) & \sin(\pi(\mu - \frac{1}{4})) \\
e^{i\pi(\frac{5}{4} - \frac{i}{4})} & 0 & 0
\end{pmatrix} A^-(\mu),$$

The corresponding reflection matrices $\overline{K}^\pm(\mu)$ for the charge conjugate states can be written conveniently as

$$\overline{K}^\pm(\mu) = \mathcal{H}^\pm (K^\pm(\mu))^\top \mathcal{H}^\pm,$$

in which

$$\mathcal{H}^+ = \begin{pmatrix}
h^{-2}(\lambda) & 0 & 0 \\
0 & h^2(\lambda) & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \text{ and } \mathcal{H}^- = \begin{pmatrix}
i h^{-2}(\lambda) & 0 & 0 \\
0 & -i h^2(\lambda) & 0 \\
0 & 0 & -i
\end{pmatrix}.\]
However, as already mentioned in the previous section, there is another ambiguity in the overall scalar factors which cannot be removed by the requirements of the bootstrap equations. The above scalar factors all satisfy $A(\mu) = \pm A(\mu)$. Although this seems to be a reasonable assumption, it does not follow from the equations (4.4) and (4.6). We are thus free to define more general scalar factors

$$A_\epsilon(n)(\mu) \equiv \tau(\mu)^* A_\epsilon(\mu),$$

where $\epsilon$ can be $+, -$, or $d$. These new scalar factors satisfy the necessary unitarity and crossing conditions if we require that

$$\tau(\mu)\tau(-\mu) = 1,$$  \hspace{1cm} (5.10)

$$\tau(-\mu + \frac{3}{4}\lambda)\tau(\mu + \frac{3}{4}\lambda) = \tau(\mu),$$

$$\tau(-\mu - \frac{1}{2}\lambda)\tau(\mu + \frac{1}{2}\lambda) = \tau(\mu).$$  \hspace{1cm} (5.11)

However, we also need to ensure that they do not violate the soliton bootstrap relations. From the detailed computation in appendix B we can see that the soliton bootstrap equations will continue to be satisfied if we require

$$\tau(\mu - \frac{1}{2}\lambda)\tau(\mu + \frac{1}{2}\lambda) = \tau(\mu),$$

$$\tau(\mu - \frac{1}{2}\lambda)\tau(\mu + \frac{1}{2}\lambda) = \tau(\mu).$$  \hspace{1cm} (5.12)

The question remains whether any solutions to equations (5.10 – 5.12) exist. It is straightforward to see that there are in fact infinitely many such solutions which can be written in the following form:

$$\tau(\mu) = \left[ \eta \right] \left[ -\eta + \frac{3}{2}\lambda \right] \left[ \eta + \lambda \right]^{-1},$$

in which $[a] \equiv \sin\left(\frac{\pi}{3\lambda} (\mu + a)\right)$ and $\eta = \eta(\lambda)$ is an arbitrary function of $\lambda$. Equation (5.10) then implies

$$\tau(\mu) = \left[ -\eta - \frac{1}{2} \right] \left[ \eta - \frac{3}{2}\lambda \right]^{-1},$$

and is straightforward to check that this satisfies equations (5.11) and (5.12). We will discuss the implications of this ambiguity in the next section.

### 5.2 The breather bootstrap

The breather bootstrap equations are similar to the soliton bootstrap, but now the two incoming solitons fuse into a scalar bound state, the so-called breather, instead of another soliton. Again we only consider the lowest mass breathers whose FZ generators where defined in (2.8).

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The breather bootstrap equations on the boundary are depicted in figure 4. It is important to note here that the type of breather created depends on the ordering of the incoming soliton–antisoliton pair, which can also be seen from the definition (2.8). The picture in figure 4 now illustrates clearly the fact that the change of multiplets in the soliton reflection implies that the breather reflection is purely diagonal, i.e. a breather $B_1$ can only reflect into itself and not into its conjugate partner $\overline{B_1}$.

**Figure 4: The boundary breather bootstrap equations**

Using the FZ algebra relation it is now straightforward to evaluate the two scattering processes in figure 4, and we obtain the reflection amplitudes for the reflection of a breather $B_1$ from the boundary. The details of these computations can again be found in appendix B, where we obtained the following results

$$B_1(\mu)B = K_B^{(+)}(\mu)B_1(-\mu)B,$$

in which the boundary reflection factors corresponding to the three different reflection matrices are given by

$$K_B^{(+)}(\mu) = \left(\frac{1}{2}\right)\left(-\lambda\right)\left(\lambda - \frac{1}{2}\right),$$

$$K_B^{(-)}(\mu) = -\left(\frac{\lambda}{2}\right)\left(-\frac{\lambda}{2} - \frac{1}{2}\right)\left(-\frac{3}{2}\lambda + \frac{1}{2}\right),$$

$$K_B^{(d)}(\mu) = \left(-\frac{\lambda}{2} - \frac{1}{2}\right)\left(-\lambda\right)\left(-\frac{3}{2}\lambda + \frac{1}{2}\right).$$

Here we have again used the block notation as defined in (2.11).

Finally, let us discuss briefly how the inclusion of the more general scalar factors (5.9) would affect the final result of the breather reflection factors. From the breather bootstrap equations
we find that using the new scalar factors \( A_n(\mu), \tilde{A}_n(\mu) \) introduces an additional term

\[
k^{(\mu)}(\mu) \equiv \tau(\mu - \frac{3}{4} \lambda + \frac{1}{2})\tau(\mu + \frac{3}{4} \lambda - \frac{1}{2}) = \left(\eta - \frac{3}{4} \lambda + \frac{1}{2}\right)\left(-\eta + \frac{3}{4} \lambda + \frac{1}{2}\right)\left(-\eta - \frac{\lambda}{4} + \frac{1}{2}\right)\left(\eta + \frac{\lambda}{4} + \frac{1}{2}\right)
\]

into the lowest breather reflection factor \( K_B^{(\mu)}(\mu) \). Note that we can of course use any number of terms of the form (5.13) as our \( \tau(\mu) \), which would then introduce more than one free parameter into the breather reflection factors. However the inclusion of these more general scalar factors also introduces new poles into the reflection factors and it would therefore be necessary to investigate whether these additional physical strip poles can be explained in terms of physical scattering processes.

### 6 Real \( a^{(1)}_2 \) affine Toda field theory on a half-line

Affine Toda field theory with real coupling constant based on the affine Lie algebra \( a^{(1)}_2 \) is an integrable two-dimensional field theory which contains two (mass degenerate) scalar particles. Extending earlier works by Arinshtein et al. \[16\], exact \( S \)-matrices for this theory (and all other ATFTs based on simply laced algebras) have been constructed some years ago by Braden et al. in \[9\]. Some years later it was demonstrated in \[10\] that these \( S \)-matrices could also be obtained from the \( U_q(a^{(1)}_2) \) invariant \( S \)-matrix. It was shown that the scattering amplitudes of the lowest breathers were identical to the real affine Toda \( S \)-matrices after analytic continuation of the coupling constant from imaginary to real value.

In this section this identification of the breathers with the fundamental quantum particles will be used in order to provide a conjecture for the reflection factors in real \( a^{(1)}_2 \) ATFT. We therefore have to analytically continue the above breather reflection factors (5.16–5.18) from imaginary to real coupling constant \( \beta \). The connection between \( \lambda \) and the Toda coupling constant \( \beta \) must be the same as in the bulk theory and was given in (2.11). In order to compare our results with previous works on real ATFT we use instead of (2.10) a slightly different form of the bracket notation in this section

\[
\left( \frac{y}{\pi} \right) \equiv \frac{\sin\left(\frac{\theta}{2} + \frac{\nu \pi}{6}\right)}{\sin\left(\frac{\theta}{2} - \frac{\nu \pi}{6}\right)}.
\]

The coupling constant dependence in the real Toda \( S \)-matrices is usually given in terms of the function

\[
B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1 + \frac{\beta^2}{4\pi}},
\]

and thus analytic continuation \( \beta \rightarrow i\beta \) is equivalent to

\[
\lambda \rightarrow -\frac{2}{B}.
\]
Performing this analytic continuation in the three different breather reflection factors from the previous section, we obtain three possible reflection factors

\[ K_B^{(+)}(\mu) \rightarrow K_1^{(+)}(\theta) = \left(-2\right)\left(-\frac{B}{2}\right)\left(2 + \frac{B}{2}\right), \]

\[ K_B^{(-)}(\mu) \rightarrow K_1^{(-)}(\theta) = -\left(1\right)\left(\frac{B}{2} - 1\right)\left(3 - \frac{B}{2}\right), \]

\[ K_B^{(d)}(\mu) \rightarrow K_1^{(d)}(\theta) = \left(-2\right)\left(\frac{B}{2} - 1\right)\left(3 - \frac{B}{2}\right), \]

in which the \( K_1 \)'s are now the reflection amplitudes for the reflection of the first particle in real \( a_2^{(1)} \) ATFT on a half-line.

As an important check for these reflection factors we need to demonstrate their consistency with the boundary bootstrap. In \( a_2^{(1)} \) ATFT we have possible fusion processes \( 1 + 1 \rightarrow 2 \) and \( 2 + 2 \rightarrow 1 \), occurring at the rapidity difference \( \theta = \frac{2\pi}{3} \), and we thus require the following reflection bootstrap relation:

\[ K_2(\theta) = K_1(\theta + \frac{i\pi}{3}) K_1(\theta - \frac{i\pi}{3}) S_{11}(2\theta), \]  

(6.5)

in which \( S_{11}(\theta) \) is the \( a_2^{(1)} \) affine Toda \( S \)-matrix [9]

\[ S_{11}(\theta) = \left(2\right)\left(-2 + B\right)\left(-B\right), \]  

(6.6)

and therefore

\[ S_{11}(2\theta) = -\left(1\right)\left(-2\right)\left(-1 + \frac{B}{2}\right)\left(2 + \frac{B}{2}\right)\left(-\frac{B}{2}\right)\left(3 - \frac{B}{2}\right). \]

Using this expression and the above formulas for \( K_1(\theta) \) we find in all three cases

\[ K_2^{(c)}(\theta) = K_1^{(c)}(\theta), \]  

(6.7)

which is expected since the two particles in \( a_2^{(1)} \) ATFT are mass degenerate and the boundary conditions do not distinguish between the two particles[d]. Furthermore, the requirement of boundary crossing and unitarity implies

\[ K_1^{(c)}(\theta) K_2^{(c)}(\theta - i\pi) = S_{11}(2\theta), \]  

(6.8)

which again can be checked to be true for all three reflection factors in (6.4).

---

dSince the second quantum particle is identified with the conjugate breather \( \overline{B}_1 \), we could have checked relation (6.7) directly by performing the breather bootstrap on \( \overline{B}_1 \), and we would have found that \( K_B(\mu) = K_\overline{B}(\mu) \).
As already mentioned in section 2, one of the remarkable features of the $S$-matrices of real ATFTs is the fact that they display a weak-strong coupling duality, which means that the $S$-matrices in the simply-laced cases are invariant under the transformation

$$\beta \longrightarrow \frac{4\pi}{\beta}. \quad (6.9)$$

It has been a long standing problem to understand whether a similar duality holds for the theories on a half-line. In a number of previous attempts to construct affine Toda reflection factors $[17, 18, 19, 20]$ it was assumed that the theories on a half-line display the same weak-strong coupling duality, which means the reflection factors were expected to be invariant under the transformation $[5.9]$. However, in a recent paper $[7]$ it was pointed out that the form of the breather reflection factors in the sine–Gordon theory $[8]$ implies that the reflection factors in the sinh–Gordon case ($a_1^{(1)}$ ATFT) are not self-dual in general. Similarly, in the $a_2^{(1)}$ affine Toda case we now find that none of the three reflection factors in $[5.4]$ is self-dual. However, it turns out that the two factors $K_1^{(+)}(\theta)$ and $K_1^{(d)}(\theta)$ are dual to each other, i.e.

$$K_1^{(+)}(\theta) \longrightarrow K_1^{(d)}(\theta), \quad \text{as} \quad \beta \longrightarrow \frac{4\pi}{\beta}. \quad (6.10)$$

This supports the view expressed in $[7]$ that, rather than being a symmetry of any particular boundary theory, duality should relate theories with different boundary conditions to each other.

In order to decide which boundary conditions could correspond to the reflection factors $[5.4]$ we first need to check their semiclassical limits. Expanding the reflection factors in terms of $\beta^2$ we obtain

$$K_1^{(+)}(\theta) = 1 - \frac{1}{4} \cos \left( \frac{\theta}{2} \right) \beta^2 + O(\beta^4),$$

$$K_1^{(-)}(\theta) = 1 - \frac{1}{4} \cos \left( \frac{\theta}{2} \right) \beta^2 + O(\beta^4),$$

$$K_1^{(d)}(\theta) = -(-1) \left( -2 \right) + \frac{\tan \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} + \frac{\pi}{3} \right)}{8 \sin \left( \frac{\theta}{2} + \frac{\pi}{3} \right) - 4 \cos \left( \frac{3\theta}{2} + \frac{\pi}{6} \right) + 4 \sin \left( \frac{\theta}{2} \right)} \beta^2 + O(\beta^4). \quad (6.11)$$

In $[18]$ Kim used perturbative methods in order to construct affine Toda reflection matrices for the Neumann boundary condition. It turns out that his perturbative results$[18]$ for $a_2^{(1)}$ match the semiclassical limit of our $K_1^{+}(\theta)$ up to order $\beta^2$. We therefore conjecture that $K_1^{+}(\theta)$ is the reflection factor for $a_2^{(1)}$ ATFT on a half-line with Neumann boundary condition. In terms of the affine Toda boundary potential $[1.2]$ this corresponds to the case where $b_0 = b_1 = b_2 = 0$.

*Note that the conjectured reflection factor in $[18]$ is equal to $K_1^{(+)}(\theta)$ only up to order $\beta^2$, since the exact reflection factor in $[18]$ was assumed to be self-dual.*
We can also compare our result with the paper \cite{1} in which Corrigan et al. conjectured the form of some of the reflection factors in real coupling $a_2^{(1)}$ ATFT. In particular their ‘minimal’ conjecture for the reflection factor corresponding to the boundary condition with $b_0 = b_1 = b_2 = +2$ is equal to our $K_1^{(d)}(\theta)$. In a more recent paper \cite{21} Perkins and Bowcock calculated the $\mathcal{O}(\beta^2)$ quantum corrections to the classical reflection factor for this boundary condition. The result of this computation was also found to be in agreement with the semiclassical limit of our $K_1^{(d)}(\theta)$. This now suggests that the theory with Neumann boundary condition is dual to that with ‘positive’ boundary condition $b_0 = b_1 = b_2 = +2$, which matches the result obtained for the sine/sinh–Gordon theories in \cite{7}.

Unfortunately we do not have any simple explanation for the remaining reflection factor $K_1^{(-)}(\theta)$. The first problem is the fact that the classical limit of $K_1^{(-)}(\theta)$ is equal to 1 just like that of $K_1^{(+)}(\theta)$. However, classically only the reflection factor corresponding to the Neumann boundary condition should be unity. It therefore seems rather strange to have two different reflection factors with this property. The other puzzle with $K_1^{(-)}(\theta)$ is the question of its dual. The dual of $K_1^{(-)}(\theta)$ is of course in itself a perfectly reasonable reflection factor which satisfies all necessary constraints\footnote{In fact the dual of $K_1^{(-)}(\theta)$ is the ‘minimal’ reflection factor conjectured in \cite{1} to correspond to the boundary condition with $b_0 = b_1 = b_2 = -2$.}, but we are not able to derive it from the $U_q(a_2^{(1)})$ invariant reflection matrices.

Finally, let us briefly discuss the effect of the more general scalar factors \cite{5,3}. These introduce an additional term $k^{(\mu)}(\mu)$ of the form \cite{5,19} into the lowest breather reflection factors, which consequently introduces the following additional term into the particle reflection factors:

$$k^{(\mu)}(\mu) \rightarrow k^{(\mu)}(\theta) \equiv \left( B\eta - \frac{B}{2} + \frac{3}{2} \right) \left( B\eta + \frac{B}{2} - \frac{1}{2} \right) \left( -B\eta - \frac{B}{2} - \frac{3}{2} \right) \left( -B\eta + \frac{B}{2} + \frac{1}{2} \right). \quad (6.12)$$

We can check whether the inclusion of such a factor spoils the boundary crossing and bootstrap conditions. It can be seen that this is not the case from the following short computation:

$$k_2^{(\eta)}(\theta) k_2^{(\eta)}(i\pi - \theta) = 1.$$

In \cite{17} Sasaki noticed that a reflection factor multiplied by any function which satisfies the $S$-matrix bootstrap equations is again a consistent reflection factor itself. In our case this freedom is expressed in the possible inclusion of $k_1^{(\eta)}(\theta)$, since equation \cite{6,13} is identical to the $S$-matrix bootstrap equation of $a_2^{(1)}$ ATFT. Note that for $\eta = \frac{1}{2} - \frac{3m}{2B}$ the factor $k_1^{(\eta)}(\theta)$ is just the $S$-matrix $S_{11}(\theta)$ itself. In general the inclusion of these factors will violate condition \cite{6,7}, since only if $\eta = \frac{1+3m}{B}$, (for any integer $m$), do we get $k_1^{(\eta)}(\theta) = k_2^{(\eta)}(\theta)$. However, reflection factors which violate \cite{6,7} are not necessarily ruled out since it was discovered in \cite{22} that it is possible to generalise the boundary potential \cite{12} to include time derivatives of the fields, in which case mass degenerate particles may no longer have the same reflection factors.
Notice also that the inclusion of these factors does not provide us with a solution to the problems regarding the factor $K_1^{(-)}(\theta)$. First we notice that the inclusion of a factor $k_1^{(\eta)}$ does not change the classical limit of the reflection factor, because

$$k_1^{(\eta)} \rightarrow 1, \quad (\text{for } \beta \rightarrow 0).$$

And secondly, it is not possible to find a value for $\eta$ such that the reflection factor $K_1^{(-)}(\theta)k_1^{(\eta)}(\theta)$ would become self-dual. We expect that the inclusion of any of these additional factors can be ruled out, because they will introduce additional physical strip poles which cannot be explained in terms of physically allowed bound states.

### 7 Discussion

The main results of this paper are contained in equations (5.5), (5.6) and (6.4). We have found new non-diagonal solutions to the BYBE and used them to construct $U_q(a_2^{(1)})$ invariant reflection matrices. It was shown that these reflection matrices satisfy the necessary consistency requirements of boundary unitarity, boundary crossing and the boundary bootstrap. We then used the boundary bootstrap equations in order to derive reflection factors for the lowest breathers. Although the reflection matrices contained some free parameters, it turned out that all these free parameters disappear in the breather bootstrap and we end up with only a finite number of possible breather reflection matrices. Since the breathers can be identified with the real Toda quantum particles, this result is consistent with the classical result that $a_2^{(1)}$ ATFT permits only a finite number of integrable boundary conditions. However, we only found three different reflection factors, whereas there are nine integrable boundary conditions in the classical theory. This fact is consistent with the possibility that not all boundary conditions, which were found to be classically integrable, are also quantum integrable.

One of the most interesting discoveries is the fact that none of the reflection factors turned out to be self-dual under the weak-strong coupling duality (6.9). Instead we confirmed the conjecture that duality relates different boundary theories to each other. In particular we found that the two reflection factors corresponding to the Neumann and ‘positive’ boundary conditions are dual to each other.

Probably the most puzzling results concern the reflection factor $K_1^{(-)}(\theta)$ in (6.4). This factor has a classical limit of one, just like the factor $K_1^{(+)}(\theta)$, which corresponds to the Neumann boundary condition. However, we would expect that only the reflection factor corresponding to the Neumann boundary condition should be one in the classical limit. Is the second factor $K_1^{(-)}(\theta)$ not related to real coupling ATFT but to some other as yet unknown theory? Or could

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8 This would also confirm the work [23] in which the authors show that some of the classically integrable boundary conditions contain certain instabilities and therefore do not lead to consistent quantum theories.
it be possible that two different quantum theories correspond to the same classical boundary condition?

The other problem concerning the factor $K_1^{(-)}(\theta)$ is the fact that we were not able to find a solution to the BYBE which would lead to the dual of $K_1^{(-)}(\theta)$. However, we have no reason to assume that the weak-strong coupling duality must be present in the solutions to the BYBE. Even in the bulk theory it was found that weak-strong coupling duality does not exist in the trigonometric $S$-matrices, but appears somewhat mysteriously only on the level of the lowest breathers [10, 11].

Apart from trying to understand these issues concerning the factor $K_1^{(-)}(\theta)$ there are a number of other open questions. In particular, it would be interesting to study the pole structure of our reflection matrices and examine the appearance of possible boundary bound states. Furthermore, in the solutions to the BYBE we have not allowed the possibility of different boundary states. If the boundary itself carried some quantum number then the $K$-matrices would depend on two additional indices, i.e.

$$A_i(\mu)B_a = K_{i,a}^{j,b}(\mu) A_j(-\mu)B_b.$$  

In this case the BYBE would become a great deal more complicated and at this stage we are not able to provide a general solution. We do not know what effect the inclusion of these boundary states would have on the final results. However, it is interesting to note that in the sine–Gordon theory the solution of the simplest BYBE (i.e. that without boundary labels) seems to give all possible breather reflection factors.

Another open problem is the extension of this work to other algebras. We expect to perform a similar analysis for the case of $a_2^{(2)}$ ATFT in a forthcoming paper. The case of $a_2^{(2)}$ is interesting, because unlike in the $a_2^{(1)}$ theory we expect to find at least one free boundary parameter in the lowest breather reflection factors [4]. Apart from the theories based on $a_2^{(1)}$ and $a_2^{(2)}$, it is currently not possible to construct reflection matrices for any other algebra. The reason for this is that we do not know the exact matrix structure of any of the higher trigonometric $R$-matrices, which are usually given only in terms of their spectral decompositions. Rather than solving the BYBE step by step, it would be highly desirable to find a more general construction in order to find solutions. We hope that the explicit solutions in this paper will eventually lead to a better understanding of the general structure of solutions to the BYBE and thus provide a more general method for constructing reflection matrices associated with higher rank algebras.

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A Solving the Boundary Yang–Baxter–Equation

In this appendix we present some of the steps in the solution of the \( U_q(a_2^{(1)}) \) invariant BYBE. We only show the solution for \( K^\pm(\mu) \) (the solution for the conjugate reflection matrices \( \overline{K}^\pm(\mu) \) is completely analogous).

The BYBE was given in explicit matrix form in equation (3.3) and illustrated in figure 2, in which two incoming solitons \( A_i(\mu), A_j(\mu') \) reflect into two outgoing solitons \( \overline{A}_p(-\mu), \overline{A}_r(-\mu') \). Hence, relation (3.3) leads to 81 independent equations, which we label by \((i,j,p,r)\), for the nine unknown functions \( K_k^i(\mu) \), which are the matrix elements of the reflection matrices \( K^\pm(\mu) \).

For the sake of simplicity we can remove the overall \( S \)-matrix scalar factors from the BYBE, by dividing both sides of (3.3) by \( \frac{F(\mu-\mu')}{\sin(\pi(\lambda-\mu+\mu'))} \frac{F(\frac{\pi}{2}(\lambda-\mu-\mu'))}{\sin(\pi(\frac{\lambda}{2}\lambda+\mu+\mu'))} \), which is the same as taking

\[
\begin{align*}
S^{i,i}_{i,i}(\mu) &= \sin(\pi(\lambda-\mu)) , \\
S^{j,j}_{i,j}(\mu) &= \sin(\pi\mu) , \\
S^{i,j}_{i,j}(\mu) &= \sin(\pi\lambda)e^{(i,j)\pi\frac{i}{2}} , \quad (\text{for } i,j = 1,\ldots,3) ,
\end{align*}
\]

instead of (1.5).

First let us consider equation (1,1,2,2) which only depends on the diagonal elements \( K_1^1 \) and \( K_2^2 \). Assuming neither of the diagonal elements is identically zero, this equation can be written in the very simple form

\[
\frac{K_1^1(\mu)}{K_2^2(\mu)} = \frac{K_1^1(\mu')}{K_2^2(\mu')} ,
\]

which implies that \( K_1^1 \) and \( K_2^2 \) have the same rapidity dependence. We can also find similar equations involving \( K_3^3 \) and we thus define:

\[
\begin{align*}
K_2^2(\mu) &= D_2(\lambda)K_1^1(\mu) , \\
K_3^3(\mu) &= D_3(\lambda)K_1^1(\mu) ,
\end{align*}
\]

in which \( D_2(\lambda) \) and \( D_3(\lambda) \) are two arbitrary functions depending on the coupling constant \( \lambda \) (but not on the rapidity \( \mu \)).

Next we consider equation (1,2,1,2), which only depends on the two functions \( K_1^1 \) and \( K_2^2 \), and which can be simplified to

\[
\begin{align*}
\sin(\pi(\mu-\mu')) \left[ K_1^2(\mu)K_2^2(\mu')e^{i\pi((\frac{\mu+\mu'}{2})-\frac{\mu+\mu'}{2})} - K_2^1(\mu)K_2^1(\mu')e^{-i\pi((\frac{\mu+\mu'}{2})-\frac{\mu+\mu'}{2})} \right] &= \\
&= \sin\left(\pi(\mu+\mu'-\frac{\lambda}{2})\right) \left[ K_1^2(\mu)K_2^2(\mu')e^{i\pi\frac{\mu'+\mu}{2}} - K_2^1(\mu)K_2^1(\mu')e^{i\pi\frac{\mu+\mu'}{2}} \right] .
\end{align*}
\]

\(^h\)All the computations were done using MapleV. We present all the results here in terms of \( \mu \) and \( \lambda \). However, for technical reasons most of the computation have been performed using the variables \( x = e^{i\pi\mu/3} \) and \( q = e^{i\pi\lambda/4} \), since in this case all trigonometric equations can be written in terms of simpler rational equations.
Apart from the trivial solution in which both $K_1^2$ and $K_2^1$ are zero, there are two other possible solutions to this equation for general $\mu$ and $\mu'$, namely either
\begin{align}
K_1^2(\mu) &= e^{-\frac{2}{3}i\pi \mu} A_1(\mu), \\
K_2^1(\mu) &= e^{\frac{2}{3}i\pi \mu} A_1(\mu),
\end{align}
(A.3)
or
\begin{align}
K_1^2(\mu) &= e^{i\pi \left(\frac{\mu}{3} - \frac{1}{2}\right)} A_1(\mu), \\
K_2^1(\mu) &= \epsilon_1 e^{-i\pi \left(\frac{\mu}{3} - \frac{1}{2}\right)} A_1(\mu),
\end{align}
(A.4)
in which $\epsilon_1 = \pm 1$ and $A_1(\mu)$ is an arbitrary function of $\mu$ and $\lambda$. However, after a careful analysis of the other equations we find that the first of these solutions inevitably leads to contradictions and we are therefore left with solution (A.4). The corresponding equations for the other non-diagonal elements are (1,3,1,3) and (2,3,2,3), and we can make the following ansatz:
\begin{align}
K_3^2(\mu) &= e^{i\pi \left(\frac{\mu}{3} - \frac{1}{2}\right)} A_2(\mu), \\
K_2^3(\mu) &= \epsilon_2 e^{-i\pi \left(\frac{\mu}{3} - \frac{1}{2}\right)} A_2(\mu),
\end{align}
(A.5)
and
\begin{align}
K_3^1(\mu) &= e^{i\pi \left(\frac{\mu}{3} - \frac{1}{2}\right)} A_3(\mu), \\
K_1^3(\mu) &= \epsilon_3 e^{-i\pi \left(\frac{\mu}{3} - \frac{1}{2}\right)} A_3(\mu).
\end{align}
(A.6)
So far we have reduced the nine unknown functions $K_j^i(\mu)$ to four $\mu$ dependent functions $K_1^1(\mu)$, $A_1(\mu)$, $A_2(\mu)$, $A_3(\mu)$, two $\lambda$ dependent functions $D_2(\lambda)$, $D_3(\lambda)$, and the choices of signs $\epsilon_1, \epsilon_2, \epsilon_3$.

Putting the above ansatz back into the BYBE we find that, for example, equation (1,1,1,2) now only depends on $A_1$ and $K_1^1$ and can be simplified to
\[ K_1^1(\mu) A_1(\mu') \sin(\pi(2\mu' - \frac{\lambda}{2})) = K_1^1(\mu') A(\mu) \left[ \sin(\pi(\mu + \mu' - \frac{\lambda}{2})) + \epsilon_1 \sin(\pi(\mu - \mu')) \right]. \]

We can separate the $\mu$ and $\mu'$ dependence in this equation and write
\begin{align*}
\frac{A(\mu)}{K_1^1(\mu)} \sin(\pi(\mu - \frac{\lambda}{4})) &= \frac{A(\mu')}{K_1^1(\mu')} \sin(\pi(\mu' - \frac{\lambda}{4})) \quad \text{if } \epsilon_1 = +1, \\
\frac{A(\mu)}{K_1^1(\mu)} \cos(\pi(\mu - \frac{\lambda}{4})) &= \frac{A(\mu')}{K_1^1(\mu')} \cos(\pi(\mu' - \frac{\lambda}{4})) \quad \text{if } \epsilon_1 = -1.
\end{align*}

This can be solved by introducing yet another $\lambda$ dependent parameter $\alpha(\lambda)$:
\[ K_1^1(\mu) = \alpha(\lambda) \sin \left( \pi(\mu - \frac{\lambda}{4} + \frac{1}{4}(1 - \epsilon_1)) \right) A_1(\mu). \]
Considering equation (1.2,1.3) we then obtain
\[
\mathcal{A}_1(\mu)\mathcal{A}_3(\mu')\epsilon_3 \left[ \epsilon_1 \sin(\pi(\mu + \mu' - \frac{\lambda}{2})) + \sin(\pi(\mu - \mu')) \right] = \\
= \mathcal{A}_1(\mu')\mathcal{A}_3(\mu)\epsilon_1 \left[ \epsilon_3 \sin(\pi(\mu + \mu' - \frac{\lambda}{2})) + \sin(\pi(\mu - \mu')) \right]. \tag{A.8}
\]

However from equations (1.1,2,3) we find that we must have
\[
\epsilon_1 = \epsilon_2 = \epsilon_3 \equiv \epsilon,
\]
which simplifies (A.8) to
\[
\frac{\mathcal{A}_1(\mu)}{\mathcal{A}_3(\mu)} = \frac{\mathcal{A}_1(\mu')}{\mathcal{A}_3(\mu')}.
\]

We can find a similar relation (from equation (2.3,2.1)) for the function \( \mathcal{A}_2(\mu) \) and we thus make the ansatz
\[
\mathcal{A}_2(\mu) = g(\lambda)\mathcal{A}_1(\mu), \\
\mathcal{A}_3(\mu) = h(\lambda)\mathcal{A}_1(\mu), \tag{A.9}
\]
in which \( h(\lambda) \) and \( g(\lambda) \) are again arbitrary functions of \( \lambda \).

Now we are left with one \( \mu \) dependent function \( \mathcal{A}_1(\mu) \), five \( \lambda \) dependent parameters \( \mathcal{D}_2(\lambda), \mathcal{D}_3(\lambda), \alpha(\lambda), h(\lambda) \) and \( g(\lambda) \), and one choice of sign of \( \epsilon \). If we put all these back into the BYBE we find that only three independent non-trivial equations remain (namely (1.1,2,3), (1.2,2,3) and (1.2,3,3)), which for the case of \( \epsilon = +1 \) can be written as
\[
\begin{align*}
    h(\lambda) &= \sin(\pi \frac{\lambda}{2})\alpha(\lambda)g(\lambda), \\
    g(\lambda) &= \sin(\pi \frac{\lambda}{2})\alpha(\lambda)h(\lambda)\mathcal{D}_2(\lambda), \\
    h(\lambda)g(\lambda) &= \sin(\pi \frac{\lambda}{2})\alpha(\lambda)\mathcal{D}_3(\lambda),
\end{align*}
\]
and therefore
\[
\begin{align*}
    \alpha(\lambda) &= \frac{1}{\sin(\pi \frac{\lambda}{2})} \frac{h(\lambda)}{g(\lambda)}, \\
    \mathcal{D}_2(\lambda) &= \left( \frac{g(\lambda)}{h(\lambda)} \right)^2, \\
    \mathcal{D}_3(\lambda) &= (g(\lambda))^2.
\end{align*}
\]

The only difference in the case of \( \epsilon = -1 \) is
\[
\alpha(\lambda) = \frac{i}{\sin(\pi \frac{\lambda}{2})} \frac{h(\lambda)}{g(\lambda)}.
\]
Now, putting everything together we obtain the two non-diagonal solutions (3.7) to the BYBE as given in section 3. The reflection matrices for the reflection of antisolitons into solitons can be obtained in a completely analogous way by using equation (3.8). In all these solutions we have assumed that none of the matrix elements are identically zero. In a careful case by case study one can see that the only possibility of having a solution with any matrix elements being identically zero is the diagonal solution (3.10). We therefore expect that the three types of solutions given in section 3 are indeed the most general solutions to the BYBE (3.6).

Finally we briefly want to mention possible solutions to a slightly different BYBE which, nevertheless, is also associated to the quantum algebra $U_q(a_2^{(1)})$. Unlike previously thought, $K$-matrices of the form (3.1) are not the only possible solutions to the BYBE. There are in fact also reflection matrices which map each multiplet into itself:

$$\tilde{K}(\mu) : V_1 \rightarrow V_1.$$  \hspace{1cm} (A.10)

Reflection matrices of this type have to satisfy the following BYBE:

$$\tilde{K}_j^k(\mu') S_{i,k}^{d,m}(\mu + \mu') \tilde{K}_m^n(\mu) S_{i,n}^{p,r}(\mu - \mu') = S_{i,j}^{k,l}(\mu - \mu') \tilde{K}_l^n(\mu) S_{i,m}^{p,n}(\mu + \mu') \tilde{K}_r^n(\mu').$$ \hspace{1cm} (A.11)

The most general diagonal solutions to this equation (and its generalisation to $a_n^{(1)}$) have been found some time ago in [24]. Up to an arbitrary overall scalar factor these are

$$\tilde{K}_j^k(\mu) = 0, \quad \text{for} \quad j \neq k,$$ \hspace{1cm} (A.12)

and either

$$\tilde{K}_j^j(\mu) = e^{i\pi \frac{4j}{3}(j-1)}, \quad (j = 1, 2, 3),$$ \hspace{1cm} (A.13)

or

$$\tilde{K}_j^j(\mu) = e^{i\pi \frac{4j}{3}(j-1)} \tilde{a}(\mu),$$

$$\tilde{K}_k^k(\mu) = e^{i\pi \frac{4k}{3}(k-1)} \tilde{b}(\mu), \quad (1 \leq j < k \leq 3)$$ \hspace{1cm} (A.14)

in which

$$\tilde{a}(\mu) = e^{i\pi \mu} \sin(\pi(\mu + \xi)), \quad \tilde{b}(\mu) = e^{-i\pi \mu} \sin(\pi(\xi - \mu)),$$ \hspace{1cm} (A.15)

and $\xi$ is a free boundary parameter. Note that the difference between (A.13, A.14) and the expressions in [24] are due to the fact that in [24] the $R$-matrix in the homogeneous gradation was used whereas here we are working with the $R$-matrix in the principal gradation. This means that the solutions to the corresponding BYBE are related to each other by a simple ‘gauge’ transformation, which leads to the additional factor $e^{i\pi \frac{4j}{3}(j-1)}$ in the above expressions.

These $K$-matrices have been used in the study of open quantum spin chains (see for instance [25]). However, as mentioned earlier, because of the fact that affine Toda solitons change multiplets after reflection, we do not expect the reflection matrices (A.13, A.14) to be relevant in the study of affine Toda field theories.

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iI am grateful to Rafael Nepomechie for bringing this to my attention.
B Evaluating the boundary bootstrap relations

B.1 The soliton bootstrap equations

In this appendix we provide the details of the soliton bootstrap relations which were illustrated in figure 3 in section 5.1. We make explicit use of the FZ algebra which was defined in (2.3), (3.3) and (5.1). In order to avoid any confusion we write out explicitly all summations over indices in this appendix.

The soliton bootstrap involves the reflection of two solitons from the boundary, and we therefore write

\[ A_j(\mu_1) A_k(\mu_2) B = \sum_{m,n,p,r=1}^3 K_p^k(\mu_2) S_{j,p}^{m,n}(\mu_1 + \mu_2) K_r^n(\mu_1) \overline{A}_m(-\mu_2) \overline{A}_n(-\mu_1) B, \]  

(B.1)

which describes the reflection of two incoming solitons \( A_j, A_k \) into two outgoing solitons \( \overline{A}_m, \overline{A}_n \).

For later convenience we define

\[ H_{j,k}^{m,n} = \sum_{p,r=1}^3 K_p^k(\mu + \frac{\lambda}{2}) S_{j,p}^{m,n}(2\mu) K_r^n(\mu - \frac{\lambda}{2}). \]  

(B.2)

Using the explicit expression for the S-matrix and K-matrices we can check the following identity, which is true for all \( j \neq k \):

\[ H_{j,k}^{m,n} + \gamma_{j,k} H_{k,j}^{m,n} = \begin{cases} \gamma_{m,n} \left[ H_{j,k}^{n,m} + \gamma_{j,k} H_{k,j}^{n,m} \right], & \text{if } m \neq n \\ 0, & \text{if } m = n. \end{cases} \]  

(B.3)

These identities will turn out to be necessary conditions for the soliton bootstrap relations to be satisfied. Now let \( (i, j, k) \) be an even permutation of \( (1, 2, 3) \), and define \( \mu = \frac{\mu_1 + \mu_2}{2}, \quad \mu_1 = \mu - \frac{\lambda}{2}, \) and \( \mu_2 = \mu + \frac{\lambda}{2} \). Then using (5.1) and the above identities we can write

\[ \overline{A}_i(\mu) B = \left[ A_j(\mu_1) A_k(\mu_2) + \gamma_{j,k} A_k(\mu_1) A_j(\mu_2) \right] B \]

\[ = \sum_{m,n=1}^3 \left( H_{j,k}^{m,n} + \gamma_{j,k} H_{k,j}^{m,n} \right) \overline{A}_m(-\mu_2) \overline{A}_n(-\mu_1) B \]

\[ = \sum_{(m,n)=(2,1),(1,3),(3,2)} \left( H_{j,k}^{m,n} + \gamma_{j,k} H_{k,j}^{m,n} \right) \overline{A}_m(-\mu_2) \overline{A}_n(-\mu_1) B \]

\[ + \sum_{(m,n)=(2,1),(1,3),(3,2)} \left( H_{j,k}^{m,n} + \gamma_{j,k} H_{k,j}^{m,n} \right) \overline{A}_n(-\mu_2) \overline{A}_m(-\mu_1) B \]

\[ + \sum_{m=1}^3 \left( H_{j,k}^{m,m} + \gamma_{j,k} H_{k,j}^{m,m} \right) \overline{A}_m(-\mu_2) \overline{A}_m(-\mu_1) B \]
\[ \sum_{(m,n)=(2,1),(1,3),(3,2)} (H_{j,k}^{m,n} + \gamma_{j,k} H_{k,j}^{m,n}) (A_m(-\mu_2)A_n(-\mu_1) + \gamma_{n,m} A_n(-\mu_2)A_m(-\mu_1)) B \]
\[ = \sum_{(l,n,m)=(1,2,3),(3,1,2),(2,3,1)} (H_{j,k}^{m,n} + \gamma_{j,k} H_{k,j}^{m,n}) A_l(-\mu) B. \]

Hence, we find that the soliton bootstrap relations imply

\[ \overline{K}^l_i(\mu) = H_{j,k}^{m,n} + \gamma_{j,k} H_{k,j}^{m,n}, \quad (B.4) \]

in which \((i, j, k)\) and \((l, n, m)\) are even permutations of \((1,2,3)\). For the sake of clarity let us consider one specific example here, namely \(i = 1, \ l = 2\) and \(\epsilon = + : \)

\[ H_{2,3}^{1,3} + \gamma_{2,3} H_{3,2}^{1,3} = \sum_{p,r=1}^3 \left[ K^p_p(\mu + \frac{\lambda}{2}) S_p^r(2\mu) K^q_q(\mu - \frac{\lambda}{2}) + e^{\pi \frac{2}{3}} K^p_p(\mu + \frac{\lambda}{2}) S_p^r(2\mu) K^q_q(\mu - \frac{\lambda}{2}) \right] = \]
\[ = -g(\lambda)h(\lambda) e^{-i\pi(\frac{\lambda}{2} - \frac{\lambda}{4})} \sin \left( \frac{\pi(\mu + \frac{\lambda}{2})}{2} \right) \sin \left( \frac{\pi(\mu - \frac{\lambda}{2})}{2} \right) \cos \left( \frac{\pi(\mu - \frac{\lambda}{4})}{2} \right) \sin \left( \frac{\pi(\mu + \frac{\lambda}{4})}{2} \right) \cos \left( \frac{\pi(\mu - \frac{\lambda}{4})}{2} \right) \]
\[ \times A^+(\mu + \frac{\lambda}{2}) F(\frac{3}{2}\lambda - 2\mu) A^+(\mu - \frac{\lambda}{2}). \quad (B.5) \]

First we need to compute the overall scalar factor on the right hand side. After some lengthy calculations involving the usual Gamma function manipulations we obtain the following identities:

\[ a_1(\mu + \frac{\lambda}{2}) a_1(\mu - \frac{\lambda}{2}) F(\frac{3}{2}\lambda - 2\mu) = \frac{\sin \left( \frac{\pi(2\mu - \frac{\lambda}{2})}{2} \right)}{\sin \left( \frac{\pi(2\mu - \frac{3}{4}\lambda)}{2} \right)} \left( \frac{\lambda}{4} \right) \left( -3 \frac{\lambda}{4} \right) \left( \frac{5}{4} \lambda \right) a_1(\mu), \]

and

\[ a_0^+(\mu - \frac{\lambda}{2}) a_0^+(\mu - \frac{\lambda}{2}) = \frac{\sin \left( \frac{\pi(\mu - \frac{3}{4}\lambda)}{2} \right)}{\sin \left( \frac{\pi(\mu - \frac{3}{4}\lambda)}{2} \right) \sin \left( \frac{\pi(\mu + \frac{\lambda}{4})}{2} \right)} \left( \frac{\lambda}{4} \right) \left( -3 \frac{\lambda}{4} \right) \left( \frac{5}{4} \lambda \right) a_0^+(\mu), \]

\[ a_0^-(\mu - \frac{\lambda}{2}) a_0^-(\mu - \frac{\lambda}{2}) = \frac{\cos(\frac{\pi(\mu - \frac{3}{4}\lambda)}{2})}{\cos(\frac{\pi(\mu - \frac{3}{4}\lambda)}{2}) \cos(\frac{\pi(\mu + \frac{\lambda}{4})}{2})} \left( -\frac{\lambda}{4} \right) \left( 3 \frac{\lambda}{4} \right) \left( -\frac{5}{4} \lambda \right) a_0^-(\mu). \]

And these lead to

\[ A^+(\mu + \frac{\lambda}{2}) F(\frac{3}{2}\lambda - 2\mu) A^+(\mu - \frac{\lambda}{2}) = \frac{\sin \left( \frac{\pi}{2} \right) \sin \left( \frac{\pi(\mu - \frac{\lambda}{4})}{2} \right) \cos \left( \frac{\pi(\mu - \frac{\lambda}{4})}{2} \right)}{\sin \left( \frac{\pi(\mu + \frac{\lambda}{4})}{2} \right) \sin \left( \frac{\pi(\mu - \frac{3}{4}\lambda)}{2} \right) \cos \left( \frac{\pi(\mu - \frac{3}{4}\lambda)}{2} \right)} \]
\[ \times \left( \frac{\lambda}{4} \right) \left( -3 \frac{\lambda}{4} \right) \left( \frac{5}{4} \lambda \right) A^+(\mu), \]

\[ A^-(\mu + \frac{\lambda}{2}) F(\frac{3}{2}\lambda - 2\mu) A^-(\mu - \frac{\lambda}{2}) = \frac{\sin \left( \frac{\pi}{2} \right) \sin \left( \frac{\pi(\mu - \frac{\lambda}{4})}{2} \right) \cos \left( \frac{\pi(\mu - \frac{\lambda}{4})}{2} \right)}{\cos \left( \frac{\pi(\mu + \frac{\lambda}{4})}{2} \right) \cos \left( \frac{\pi(\mu - \frac{3}{4}\lambda)}{2} \right) \sin \left( \frac{\pi(\mu - \frac{3}{4}\lambda)}{2} \right)} A^-(\mu), \]

\[ A^d(\mu + \frac{\lambda}{2}) F(\frac{3}{2}\lambda - 2\mu) A^d(\mu - \frac{\lambda}{2}) = \frac{\sin \left( \frac{\pi(2\mu - \frac{\lambda}{4})}{2} \right)}{\sin \left( \frac{\pi(2\mu - \frac{3}{4}\lambda)}{2} \right)} A^d(\mu). \]
From this we can see that only in the case of $A$ we have to include an additional CDD-type factor $\sigma(\mu)$, which satisfies equations (4.19) and

$$\sigma(\mu + \frac{\lambda}{2})\sigma(\mu - \frac{\lambda}{2}) = \left(-\frac{\lambda}{4}\right)^2 \left(\frac{3}{4}\lambda\right)^2 \left(-\frac{5}{4}\lambda\right)^2 \sigma(\mu). \quad (B.6)$$

It is easy to check that a solution to these equations if provided by $\sigma^+(\mu)$ which was given in (5.3).

Now we can go back to the above example and we find that relation (B.5) simplifies to

$$H_{2,3}^{1,3} + \gamma_{2,3} H_{5,2}^{1,3} = -g(\lambda)h(\lambda)K_1^2(\mu).$$

Hence in order for the bootstrap to be satisfied we require that

$$g(\lambda)h(\lambda) = -1.$$

It is easily verified that this restriction is sufficient to ensure that (B.4) is true for all $i, l = 1, 2, 3$ (with $i \neq l$). We can also check that similar relations hold for the cases of $\epsilon = -$ and $\epsilon = d$, and we find that the soliton bootstrap equations are all satisfied if the free parameters in the $K$-matrices are restricted by (5.4). Therefore, we obtain the full reflection matrices (5.5) and (5.6), which were given in section 5.1.

### B.2 The breather bootstrap equations

Using the exchange relations (2.3) and (3.3) we can write the scattering process on the right hand side of figure 4 in the following form:

$$A_m(\mu_1)\overline{A}_m(\mu_2)B = \sum_{j,k,l,n=1}^3 K^j_m(\mu_2) S^{k,l}_{m,j}(\mu_1 + \mu_2) K^n_l(\mu_1) A_k(-\mu_2)\overline{A}_n(-\mu_1)B. \quad (B.7)$$

Let us therefore introduce the abbreviations

$$G_{m,n}(\mu) \equiv \sum_{j,l=1}^3 K^j_m(\mu + 3\frac{\lambda}{4} - \frac{1}{2}) S^{k,l}_{m,j}(2\mu) K^n_l(\mu - 3\frac{\lambda}{4} + \frac{1}{2}),$$

and

$$J_{m,n}(\mu) \equiv \sum_{m=1}^3 \alpha_m G_{m,n}(\mu).$$

Then we can check that the following identities hold:

$$J_{k,n}(\mu) = 0, \quad \text{(if } k \neq n\text{)}, \quad (B.8)$$

and

$$J^{1,1}(\mu) = \alpha_1 J^{2,2}(\mu) = \alpha_3 J^{3,3}(\mu). \quad (B.9)$$
Using these identities and the FZ generators for the lowest breathers (2.8), it is now straightforward to express the breather reflection amplitude in terms of the FZ algebra:

\[ B_1(\mu) B = 3 \sum_{m=1}^{\infty} \alpha_m A_m(\mu - \frac{3}{4} \lambda + \frac{1}{2}) \bar{A}_m(\mu + \frac{3}{4} \lambda - \frac{1}{2}) B \]

\[ = 3 \sum_{m=1}^{\infty} \alpha_m \sum_{k,n=1}^{\infty} C_{mn}^{k,n}(\mu) A_k(-\mu - \frac{3}{4} \lambda + \frac{1}{2}) \bar{A}_n(-\mu + \frac{3}{4} \lambda - \frac{1}{2}) B \]

\[ = 3 \sum_{k,n=1}^{\infty} J_{k,n}(\mu) A_k(-\mu - \frac{3}{4} \lambda + \frac{1}{2}) \bar{A}_n(-\mu + \frac{3}{4} \lambda - \frac{1}{2}) B \]

\[ = \alpha_1^{-1} J_{1,1}^{1,1}(\mu) \sum_{k=1}^{3} \alpha_k A_k(-\mu - \frac{3}{4} \lambda + \frac{1}{2}) \bar{A}_k(-\mu + \frac{3}{4} \lambda - \frac{1}{2}) B \]

\[ = \alpha_1^{-1} J_{1,1}^{1,1}(\mu) B_1(-\mu) B , \]

We therefore define the lowest breather reflection factor as

\[ K_B(\mu) \equiv \alpha_1^{-1} J_{1,1}^{1,1}(\mu) , \] (B.10)

which can now be computed explicitly, and we obtain

\[ \alpha_1^{-1} J_{1,1}^{1,1}(\mu) = A^\epsilon(\mu - \frac{3}{4} \lambda + \frac{1}{2}) F(2\mu) \bar{A}^\epsilon(\mu + \frac{3}{4} \lambda - \frac{1}{2}) E^\epsilon(\mu) , \]

in which

\[ E^\epsilon(\mu) = \begin{cases} 
- \frac{\cos(\pi \mu) \cos(\pi(\mu + \frac{1}{2})) \cos(\pi(\mu - \frac{3}{4} \lambda))}{\sin^2(\pi \frac{1}{2}) \cos(\pi(\mu - \frac{1}{2}))} , & \text{(if } \epsilon = +) , \\
- \frac{\sin(\pi \mu) \sin(\pi(\mu + \frac{1}{2})) \sin(\pi(\mu - \frac{3}{4} \lambda))}{\sin^2(\pi \frac{1}{2}) \sin(\pi(\mu - \frac{1}{2}))} , & \text{(if } \epsilon = -) , \\
1 , & \text{(if } \epsilon = d) . 
\end{cases} \] (B.11)

And after some straightforward computations involving the scalar factors we find

\[ A^+(\mu - \frac{3}{4} \lambda + \frac{1}{2}) F(2\mu) \bar{A}^+(\mu + \frac{3}{4} \lambda - \frac{1}{2}) = - \frac{\cos(\pi(\mu - \frac{1}{2})) \sin^2(\pi \frac{1}{2})}{\cos(\pi \mu) \cos(\pi(\mu + \frac{1}{2})) \cos(\pi(\mu - \frac{3}{4} \lambda))} \times \left( \frac{1}{2} \right) (-\lambda) \left( -2 \lambda + \frac{1}{2} \right) \left( \frac{\lambda}{2} - \frac{1}{2} \right) \left( -\frac{\lambda}{2} - \frac{1}{2} \right) , \]
\[ A^- (\mu - \frac{3}{4} \lambda + \frac{1}{2}) F(2\mu) \mathcal{A}^- (\mu + \frac{3}{4} \lambda - \frac{1}{2}) = - \frac{\sin \left( \pi (\mu - \frac{1}{2}) \right) \sin^2 \left( \pi \frac{\lambda}{2} \right)}{\sin(\pi \mu) \sin \left( \pi (\mu + \frac{1}{2}) \right) \sin \left( \pi (\mu - \frac{1}{2} \lambda) \right)} \times \left( \frac{\lambda}{2} \right) \left( -\frac{\lambda}{2} - \frac{1}{2} \right) \left( -\frac{3}{2} \lambda + \frac{1}{2} \right), \]

\[ A^d (\mu - \frac{3}{4} \lambda + \frac{1}{2}) F(2\mu) \mathcal{A}^d (\mu + \frac{3}{4} \lambda - \frac{1}{2}) = \left( -\frac{\lambda}{2} - \frac{1}{2} \right) \left( -\lambda \right) \left( -\frac{3}{2} \lambda + \frac{1}{2} \right), \]

from which we can see that all the terms in (B.11) get cancelled. Finally, in the case of \( K^+(\mu) \) we also have to take the contribution from the additional CDD factor (5.3) into account, i.e.

\[ \sigma^+(\mu - \frac{3}{4} \lambda + \frac{1}{2}) \sigma^+(\mu + \frac{3}{4} \lambda - \frac{1}{2}) = \left( \lambda - \frac{1}{2} \right) \left( 2\lambda - \frac{1}{2} \right) \left( -\frac{\lambda}{2} + \frac{1}{2} \right) \left( \frac{\lambda}{2} + \frac{1}{2} \right). \]

And, therefore, we obtain the final result, the reflection factors for the reflection of a breather \( B_1(\mu) \) from the boundary

\[ B_1(\mu) \mathcal{B} = K_B^{(+)}(\mu) B_1(-\mu) \mathcal{B}, \quad \text{(B.12)} \]

in which

\[ K_B^{(+)}(\mu) = \left( \frac{\lambda}{2} \right) \left( -\lambda \right) \left( \lambda - \frac{1}{2} \right), \]

\[ K_B^{(-)}(\mu) = -\left( \frac{\lambda}{2} \right) \left( -\lambda - \frac{1}{2} \right) \left( -\frac{3}{2} \lambda + \frac{1}{2} \right), \]

\[ K_B^{(d)}(\mu) = \left( -\frac{\lambda}{2} - \frac{1}{2} \right) \left( -\lambda \right) \left( -\frac{3}{2} \lambda + \frac{1}{2} \right). \]

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