Monomial Bases for Free Post-Lie Algebras

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Abstract. Many attempts have been made to describe bases for the post-Lie algebra, using many types of Lie bases. Here, we try to describe special kind of post-Lie bases using those bases described in the pre-Lie (respectively Lie) contexts, founded before in [8, 9]. In addition, we try to understand the effect of the second operation of the post-Lie structure on the form of these bases, using some cases of the generating set, and translate it in terms of rooted trees.

1. Introduction

First appearance of post-Lie algebras was in 2007, introduced by Valette in [1]. After that, many authors have been written in this context with diverse fields, for example: H. Munthe-Kass, D. Manchon, K. Ebrahimi-Fard, A. Lundervold, C. Curry, B. Owren and others more [2, 5, 7]. Many algebraic properties for Lie, pre-Lie, Rota Baxter, and others algebras have been extended to the post-Lie in algebraic, differential, geometrical and numerical domains. In this work, we study some post-Lie properties and relate it with these satisfied by the other algebras. Bases for free post-Lie algebras are described in several ways. H. Munthe-Kass and A. Lundervold are presented post-Lie bases with tree version (in planar case) [5], using a strategy similar to that one used by F. Chapoton and M. Livernet in [4].

This paper contains two main sections. Section 2 consists of two subsections 2.1, 2.2. Some preliminaries on post-Lie structures and some of their properties have been studied. Using preceding work in Lie (respectively pre-Lie) algebras [8, 9], some connection are founded between these algebras with the post-Lie by generalizing certain identities with nice conditions. A monomial of generators $r$’s in a post-Lie algebra is a parenthesize of these elements combined with the two binary post-Lie operations, for example, a monomial of a post-Lie element $r$ (with respect to two operations $[\ldots],[\ldots],[\ldots])$ is:

$$r \triangleright r, r \triangleright (r \triangleright r), (r \triangleright r) \triangleright r, [r, r \triangleright r]$$

Finally, (Monomial) post-Lie bases for $\mathcal{L}(E)$’s algebras are calculated, using special cases of the generating set $E$. 

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2. Post-Lie Algebras

Let $\mathcal{L}$ be a vector space endowed with two bilinear operations $[,] , \triangleright : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$, such that $[,]$ is a Lie bracket and the product $\triangleright$ satisfies:

\[
x \triangleright [y,z] = [x \triangleright y,z] + [y,x \triangleright z],
\]
\[
[x,y] \triangleright z = a_\circ(x,y,z) - a_\circ(y,x,z),
\]
for any $x,y,z \in \mathcal{L}$, where $a_\circ(x,y,z) = x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z$. A triple $(\mathcal{L},[,],\triangleright)$ is called post-Lie algebra. One can verify that every Lie algebra has a natural post-Lie structure, by setting the second product $\triangleright$ to be the Lie bracket itself. In above definition, if $[,]$ is commutative then the post-Lie identities are reduced to pre-Lie property. Next, some post-Lie properties have been presented by H. Munthe-Kass and A. Lundervold.

**Proposition 1.** [5] The post-Lie $[,],\triangleright$ operations introduced new products of $\mathcal{L}$ as:

\[
[x,y] := x \triangleright y - y \triangleright x + [x,y],
\]
\[
x \bullet y := x \triangleright y + [x,y], \quad \text{(for all } x,y \in \mathcal{L})
\]

Then the bracket $[,]$ is Lie, and the triple $(\mathcal{L},[,],\bullet)$ is post-Lie.

**Proof.** See [Propositions 2.5, 2.6, 5].

2.1. Free post-Lie Algebras

A tree presentation in the pre-Lie case has been introduced by Chapoton and Livernet in [4]. Munthe-Kass and Lundervold, in [5], followed a similar tree strategy with the post-Lie (in the planar case), we review here their joint work.

Define a magma to be a set $E$ endowed with * binary operation, which has not any property. For any (non-empty) set $E$, the free magma over $E$ collects words obtained by combining letters, in $E$, each with other by the concatenation. A concrete presentation of the free magma over a set $E$ that ones defined by the rooted trees: take the set $T_{pl}^E$ of all $E$-decorated planar rooted trees, and let $\&$ be the (left) Butcher product defined on $T_{pl}^E$ as:

\[
\sigma \& \tau = B_+(\sigma \tau_1 \tau_2 \ldots \tau_k),
\]

for each $\sigma,\tau_2,\ldots,\tau_k \in T_{pl}^E$, such that $\tau = B_+(\tau_1 \tau_2 \ldots \tau_k)$, where $B_+$ is the operator which grafts a monomial $\tau_1 \tau_2 \ldots \tau_k$ on a choosing vertex called the root, decorated by some $r$ in $E$ to obtain a new tree. For example (in the undecorated context):

\[
\bullet \& \circ \circ = \circ \circ, \quad \bullet \& \circ \circ = \circ \circ.
\]
Denote by $\mathcal{T}_{\mathcal{P}l}E$ the linear span of the set $T_{\mathcal{P}l}E$. This space has two structures of magmatic algebras constructed by $\triangleleft$ and $\searrow$, where $\searrow$ is defined by:

$$\sigma \searrow \tau = \sum_{v \text{ vertex of } \tau} \sigma \searrow_v \tau,$$

where $\sigma \searrow_v \tau$ is the tree produced by the left attracting of the tree $\sigma$, on a vertex $v$ in $\tau$ [6, 9]. Below, undecorated example:

Call $\mathcal{L}(\mathcal{T}_{\mathcal{P}l}E)$ the free Lie algebra spanned by $\mathcal{T}_{\mathcal{P}l}E$, more details exist in [3]. The left Butcher and left grafting products by $\triangleleft$, $\searrow$ supply $\mathcal{L}(\mathcal{T}_{\mathcal{P}l}E)$ by two structures of free post-Lie algebras, as in following theorem and its corollary.

**Theorem 2.** [5] $\mathcal{L}(\mathcal{T}_{\mathcal{P}l}E)$ equipped with the extension of the left grafting $\searrow$ defined on $\mathcal{T}_{\mathcal{P}l}E$ by:

$$\begin{align*}
\sigma \searrow [\tau, \tau'] &= [\sigma \searrow \tau, \tau'] + [\tau, \sigma \searrow \tau'] \\
[\sigma, \tau] \searrow \tau' &= a_{\searrow}(\sigma, \tau, \tau') - a_{\searrow}(\tau, \sigma, \tau'),
\end{align*}$$

for all $\sigma, \tau, \tau' \in \mathcal{T}_{\mathcal{P}l}E$, is the free post-Lie algebra.

**Proof.** See [Proposition 3.1, Theorem 3.2, 5].

**Corollary 3.** The left Butcher grafting $\triangleleft$ can be extended to post-Lie product on $\mathcal{L}(\mathcal{T}_{\mathcal{P}l}E)$.

From the (unpublished) work for Dominique Manchon and Ebrahimi-Fard Kuruch, detailed in [8, 9], about the magmatic algebra isomorphism $\Psi$ defined between the two magmatic algebras $(\mathcal{T}_{\mathcal{P}l}E, \triangleleft)$ and $(\mathcal{T}_{\mathcal{P}l}E, \searrow)$ by:

$$\Psi : (\mathcal{T}_{\mathcal{P}l}E, \triangleleft) \rightarrow (\mathcal{T}_{\mathcal{P}l}E, \searrow)$$

$$\Psi(\bullet) = \bullet,$$

and for any $\sigma, \tau \in \mathcal{T}_{\mathcal{P}l}E$, $\Psi(\sigma \triangleleft \tau) = \Psi(\sigma) \searrow \Psi(\tau)$

(9)

Now, we can find an extension of $\Psi$ mapping, as in the following proposition.

**Proposition 4.** There is an isomorphism between the corresponding free post-Lie algebras $(\mathcal{L}(\mathcal{T}_{\mathcal{P}l}E), [\ldots,], \triangleleft)$ and $(\mathcal{L}(\mathcal{T}_{\mathcal{P}l}E), [\ldots,], \searrow)$.

**Proof.** Using the isomorphism $\Psi$ defined in (9) above, we can define an extension of $\Psi$, call it $\tilde{\Psi}$, described in Figure 1 below, such that for any $\sigma_1, \sigma_2 \in \mathcal{T}_{\mathcal{P}l}E$, and for $\sigma = \sigma_1 \triangleleft \sigma_2, \tau \in \mathcal{L}(\mathcal{T}_{\mathcal{P}l}E)$, $\tilde{\Psi}$ satisfies:
Figure 1. Post-Lie Algebra Isomorphism

\[ \Psi([\sigma, \tau]) = [\Psi(\sigma), \Psi(\tau)], \]

\[ \Psi(\sigma) = \Psi(\sigma_1) \triangleright \Psi(\sigma_2). \]

One can note that \( \Psi \) is isomorphism of post-Lie algebras.

2.2. From free Lie to free post-Lie algebras

Let \( \mathcal{L}(E) \) be the free Lie algebra graded by a generating set \( E \) as:

\[ E = \bigcup_{n \in \mathbb{N}} E_n, \]

where \( E_n \) is the subset of all elements \( r_1^{(n)}, r_2^{(n)}, \ldots, r_{d_n}^{(n)} \) of \( E \) of degree \( n \), such that its cardinal number is \( \#E_n = d_n, \forall n \in \mathbb{N} \). Then \( \mathcal{L}(E) \) is written as:

\[ \mathcal{L}(E) := \bigoplus_{n \in \mathbb{N}} \mathcal{L}_n, \quad (10) \]

\( \mathcal{L}_n \) is the homogeneous component of \( \mathcal{L} \) which contains all elements of degree \( n \), such that \( E_n \) is a subset of \( \mathcal{L}_n \). Supplying the free Lie algebra \( \mathcal{L}(E) \) by a binary operation \( \triangleright实地 \) that satisfies the post-Lie properties indicated in (2) and (3).

Here, we present important results that it linked between the freeness properties of the Lie and post-Lie algebras respectively.

**Lemma 5.** *There is a post-Lie homomorphism between \( (\mathcal{L}(T^{pl}_E), [\ldots], \triangleright), (\mathcal{L}(E), [\ldots], \triangleright) \).*

**Proof.** Since the magmatic algebras \( (T^{pl}_E, \circ), (T^{pl}_E, \triangleright) \) respectively have isometric structures of free post-Lie algebras described in Proposition 4 above, and by the freeness property of post-Lie algebras explained (for more details about the freeness property of post-Lie algebras see [Theorem 3.2, 5]) by the Figure 2 below, we define the mapping \( \Psi \) by:
Figure 2. Free Post-Lie Algebra Homomorphism

where $\tilde{E}$ is the set of all generators $\bullet$ in $T_{pl}^E$ decorated by the elements of $E$, and $\tilde{\Phi}$ is defined by:

$$\tilde{\Phi}(\bullet) = \Phi(\bullet) = r, \forall r \in E, \text{ where } |\bullet| = |r|,$$

$$\tilde{\Phi}([\sigma, \tau]) = [\Phi(\sigma), \Phi(\tau)], \forall \sigma, \tau \in L(T_{pl}^E),$$

$$\tilde{\Phi}(\sigma_1 \lessdot (\sigma_2 \lessdot \cdots \lessdot (\sigma_k \lessdot \bullet) \cdots)) = x_1 \triangleright (x_2 \triangleright \cdots \triangleright (x_k \triangleright r) \cdots),$$

for $\Phi(\sigma_i) = x_i$, and $|\sigma_i| = |x_i|$, $\forall i = 1, 2, \ldots, k$, where $\Phi$ is the Lie isomorphism explained in Proposition 6 below. By construction, $\tilde{\Phi}$ is a post-Lie homomorphism.

Proposition 6. $\tilde{\Phi}$ is an extension of the Lie isomorphism between the two free Lie algebras $(L(T_{pl}^E), [\cdot, \cdot])$ and $(L(E), [\cdot, \cdot])$.

Proof. The free Lie algebra $L(T_{pl}^E)$ constructed by the planar rooted trees by taking two-sided ideal $I$ of $T_{pl}^E$ spanned by the Jacobi property, for the left grafting $\lessdot$, and all forms below:

$$|\sigma| \sigma \lessdot \tau + |\tau| \tau \lessdot \sigma, \text{ for all } \sigma, \tau \in T_{pl}^E,$$

$L(T_{pl}^E) = T_{pl}^E/I$ has a structure of free Lie algebra, see [Proposition 4.2, 8]. This free Lie algebra is isomorphic uniquely to $L(E)$, by $\Phi$ showed in Figure 3 below (more details about this isomorphism are explained in [Corollary 4.7, 8]):

Figure 3. Free Lie Algebra Isomorphism

$\tilde{\Phi}$ is an extension of $\Phi$, by its construction.

Theorem 7. The mapping $\tilde{\Phi}$, described in Lemma 5 above, is a unique post-Lie isomorphism.
Proof. Indeed, any element \( \sigma \) in \( \mathcal{L}(E) \) is written, in a unique way, as a parenthesize of some generators of \( a \)'s of \( E \) with the operation \( \triangleright \), and corresponding to this expression of \( \sigma \) there is an element \( \sigma \) (a tree or linear combination of trees) is written, in the same way of \( \sigma \), as parenthesize of the generators \( r \)'s with the left grafting \( \triangleleft \), such that \( \Phi(\sigma) = \sigma \), hence \( \Phi \) is a surjective. Moreover, since \( \ker \Phi = \{ \sigma \in \mathcal{T}_p^E \} \), \( \Phi(\sigma) = 0 \}, \) where \( \phi \) is the empty tree. Indeed, for any \( \sigma \in \mathcal{T}_p^E \), if \( \Phi(\sigma) = 0 \), then \( |\sigma| = |0| = 0 \) and then \( \sigma = \phi \). Thus \( \Phi \) is an injective mapping. The uniqueness of \( \Phi \) is induced by that one satisfied by \( \Phi \) in Proposition 6 above.

Remark 8. The composition \( \Phi \circ \tilde{\Psi} \), in Figure 2, is also a post-Lie algebra isomorphism.

3. Monomial Bases For Free Post-Lie Algebras

The free post-Lie algebra \( \mathcal{L}(\mathcal{T}_p^E), [\ldots], \triangleright \) has a special kind of bases called Lyndon basis. In the case of one generator \( = \{ \bullet \} \), this basis is described as [5]:

\[
\mathcal{B}_{\triangleright} = \left\{ \bullet, \hat{\bullet}, \hat{\hat{\bullet}}, \ldots \right\}
\]

Hence, using the isomorphism \( \tilde{\Psi} \) described in Figures 1 and 2, one can define a basis for the free post-Lie algebra \( \mathcal{L}(\mathcal{T}_p^E), [\ldots], \triangleright \), by considering the image of the basis above by \( \tilde{\Psi} \). The first elements are:

\[
\mathcal{B}_{\triangleleft} = \left\{ \bullet, \hat{\bullet}, \hat{\hat{\bullet}}, \ldots \right\}
\]

These two bases, above, are considered as monomial bases for the free post-Lie algebras \( \mathcal{L}(\mathcal{T}_p^E) \), in the case of one generator \( \bullet \), with respect to the graftings \( \triangleright \), \( \triangleleft \) respectively.

Here, we calculate monomial bases for the free post-Lie algebra \( \mathcal{L}(E) \), using the isomorphism \( \Phi \) described in Theorem 7 above, as in the following cases:

1. If \( E = \{ r \} \) is a singleton set, then the monomial basis for \( \mathcal{L}(E) \) is:

\[
\mathcal{B}_{\triangleright} = \{ r, r \triangleright r, (r \triangleright r) \triangleright r, r \triangleright (r \triangleright r), (r \triangleright r \triangleright r), (r \triangleright r) \triangleright (r \triangleright r), \ldots \}
\]

In this case, the dimension to each homogeneous subspace \( \mathcal{L}_n \) is calculated by the following formula [Proposition 3.3, 5]:

\[
\dim(\mathcal{L}_n) = \frac{1}{2n} \sum_{d|n} \mu \left( \frac{n}{d} \right) \binom{2d}{d} n,
\]  

(11)
where $\mu$ is the Möbius function. The numerical sequence of the dimensions of $\mathcal{L}_n$, for $n = 1, 2, 3, 4, \ldots$, is the sequence A022553: $1, 1, 3, 8, 25, 75, 245, \ldots$ [10].

2. If $E = \{r_n : n \in N\}$, such that $|r_n| = n, \forall n \in N$, and $r_n$'s are ordered as $r_1 < r_2 < \cdots < r_n < \cdots$. Then the first four elements of the monomial basis for $\mathcal{L}(E)$ are:

$$B_{\triangleright, 1} = \{r_1\},$$
$$B_{\triangleright, 2} = \{r_2, r_1 \triangleright r_1\},$$
$$B_{\triangleright, 3} = \{r_3, [r_1, r_2], r_1 \triangleright r_2, r_2 \triangleright r_1, [r_1, r_1 \triangleright r_1], r_1 \triangleright (r_1 \triangleright r_2), (r_1 \triangleright r_1) \triangleright r_1\},$$
$$B_{\triangleright, 4} = \{r_4, [r_1, r_3], r_1 \triangleright r_3, r_3 \triangleright r_2, r_2 \triangleright r_1, [r_1, r_1 \triangleright r_2], [r_1, r_2 \triangleright r_1], [r_2, r_1 \triangleright r_1], [r_2, r_1 \triangleright r_2], (r_1 \triangleright r_2) \triangleright r_1, (r_2 \triangleright r_1) \triangleright r_1, [r_1, r_1 \triangleright r_1], [r_1, r_1 \triangleright r_2], [r_1, r_2 \triangleright r_1], (r_1 \triangleright r_1) \triangleright r_1, (r_1 \triangleright r_2) \triangleright r_1, (r_2 \triangleright r_1) \triangleright r_1, (r_1 \triangleright r_2) \triangleright (r_1 \triangleright r_1) \triangleright r_1, (r_2 \triangleright r_1) \triangleright (r_1 \triangleright r_1) \triangleright r_1\},$$

3. In general, if $E = \bigcup_{n \in N} E_n$, where $E_n$ is the subset of all elements $r_1^{(n)}, r_2^{(n)}, \ldots, r_d^{(n)}$ of $E$ of degree $n$, such that its cardinal number is $\#E_n = d_n, \forall n \in N$. Then the monomial bases for the homogeneous components $\mathcal{L}_n$, up to order four, of the free post-Lie algebra $\mathcal{L}(E)$ are:

$$B_{\triangleright, 1} = \{r_i^{(1)} \in E_1, \forall i = 1, 2, \ldots, d_1\} = E_1,$$
$$B_{\triangleright, 2} = E_2 \cup \left\{[r_i^{(1)}, r_j^{(1)}] | r_i^{(1)}, r_j^{(1)} \in E_1, \forall i, j = 1, 2, \ldots, d_1, \text{ such that } i \not\leq j\right\} \cup \left\{r_i^{(1)} \triangleright r_j^{(1)} | r_i^{(1)}, r_j^{(1)} \in E_1, \forall i, j = 1, 2, \ldots, d_1\right\},$$
$$B_{\triangleright, 3} = E_3 \cup \left\{[r_i^{(1)}, r_j^{(2)}] | r_i^{(1)} \in E_1, r_j^{(2)} \in E_2, \forall i = 1, 2, \ldots, d_2, \forall j = 1, 2, \ldots, d_2\right\} \cup \left\{[r_i^{(1)}, r_j^{(1)} \triangleright r_k^{(1)}], (r_i^{(1)} \triangleright r_j^{(1)}) \triangleright r_k^{(1)}, r_i^{(1)} \triangleright (r_j^{(1)} \triangleright r_k^{(1)}) | r_i^{(1)}, r_j^{(1)}, r_k^{(1)} \in E_1, \forall i, j, k = 1, 2, \ldots, d_1\right\} \cup \left\{[r_i^{(1)}, r_j^{(1)}], r_k^{(1)}] | \forall r_i^{(1)}, r_j^{(1)}, r_k^{(1)} \in E_1, \forall i, j, k = 1, 2, \ldots, d_1, \text{ such that } i \not\leq j\right\}.$$
The dimensions of the homogeneous component $\mathcal{L}_n$ subspaces, up to order four, described in (3) above, are:

$$\dim(\mathcal{L}_1) = d_1,$$
$$\dim(\mathcal{L}_2) = d_2 + \frac{3}{2} d_1^2 - \frac{1}{2} d_1,$$
$$\dim(\mathcal{L}_3) = d_3 + \frac{7}{2} d_1 d_2 + 3 d_1 d_2 - \frac{1}{2} d_1^2,$$
$$\dim(\mathcal{L}_4) = d_4 + \frac{77}{8} d_1^4 + 3 d_1 d_3 + d_2^2 + \frac{21}{2} d_1 d_2^2 d_2 - \frac{5}{4} d_1^3 - \frac{5}{8} d_2^2 - \frac{1}{2} d_1^2 d_2 + \frac{1}{4} d_1.$$
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