One-dimensional Dirac oscillator in a thermal bath

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We analyze the one-dimensional Dirac oscillator in a thermal bath. We found that the heat capacity is two times greater than the heat capacity of the one-dimensional harmonic oscillator for higher temperatures.

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The name Dirac oscillator was first introduced by Moshinsky and Szczepaniak [1] for a Dirac equation in which the momentum $p$ is replaced by $p \rightarrow p - i m \omega \beta r$ with $r$ being the position vector, $m$ the mass of the particle and $\omega$ the frequency of the oscillator. The name was given because the determination of the spectrum and eigenstate of a Dirac oscillator [1] was obtained from that of an ordinary oscillator with a very strong spin-orbit coupling term. The concept, though not the name, had been discussed previously first by Ito et al. [2] and later by Cook [3] and Ui and Takeda [4] and possibly others. Besides intrinsic mathematical interest the study of the Dirac oscillator has invoked much attention because of its various physical applications. For instance, the last three chapter of a book of Moshinsky and Smirnov [5] deal with the Dirac oscillator relativistic interactions between systems of one, two and three particles. Moreno and Zentella showed [6] that it could be the object of an exact Foldy-Wouthuysen transformation. Benitez et al. [7] found the electromagnetic potential associated with the Dirac oscillator, and showed that this exactly soluble problem has a hidden supersymmetry, which is responsible for the special properties of its energy spectrum. They also calculated the related superpotential and discussed the implications of this supersymmetry on the stability of the Dirac sea.

Rozmej and Arvieu [8] have shown a very interesting analogy between the relativistic Dirac oscillator and the Jaynes-Cummings model. They showed that the strong spin-orbit coupling of the Dirac oscillator produces the entanglement of the spin with the orbital motion similar to what is observed in the model of quantum optics.

More recently Nogami and Toyama [9] and Toyama et al [10] have studied the behaviour of wave packets of the Dirac oscillator in the Dirac representation in (1+1) dimensions. The aim of these authors was to study wave packets which could possibly be coherent. This reduction of the dimension was brought as an attempt to get rid of spin effects and to concentrate on the relativistic effects.

As a matter of fact, the one-dimensional relativistic Dirac equation has been largely used to treat physical problems where relativistic effects could play an important role. Indeed, this treatment gives enlightening information about more realistic models in solid state physics [11], particularly in semiconductor theories [12].

In spite of the great number of papers that has been recently published concerning the solution and properties of the Dirac oscillator, as far as we know no one has reported on its thermodynamics properties. In order to overcome this lack of information, in this letter we study the one-dimensional Dirac oscillator in a thermal bath.

Let us consider the energy spectrum for a one-dimensional Dirac oscillator [13]

$$E_n = \sqrt{2(n+1)\hbar \omega mc^2 + m^2 c^4},$$

(1)

where $m$ is the mass of the particle, $\omega$ is the angular frequency of the oscillator and $n$ is a positive integer number. As we can see, for $n + 1 \ll mc^2 / 2\hbar \omega$, the spectrum of the one-dimensional Dirac oscillator is approximated, up to $E_0$, to the spectrum of the one-dimensional harmonic oscillator.

The partition function of the Dirac oscillator at temperature $T$ is obtained through the Boltzmann factor,

$$Z = \sum_{n=0}^{\infty} e^{-(E_n - E_0)\beta} = e^{\sqrt{a} \beta} \sum_{n=0}^{\infty} e^{-\sqrt{a} \sqrt{bn+1} \beta},$$

(2)

were $\beta = 1/kT$, $k$ the Boltzmann constant, $a = 2\hbar \omega mc^2$, $b = m^2 c^4 / 2\hbar \omega mc^2$, and we are using a frame in the heat bath. We have subtracted the ground state energy in order to make a comparison with the non-relativistic harmonic oscillator. All thermodynamics quantities for the Dirac oscillator are obtained through the partition function [2].

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Let us first analyze the convergence of the partition function. The function \( f(x) = \exp(-\sqrt{ax + b}\beta) \) is a monotonically decreasing function and the corresponding integral
\[
I(\beta) = \int_0^\infty e^{-\sqrt{ax + b}\beta} \, dx = \frac{2}{a\beta^2} e^{-\sqrt{b}\beta} (1 + \beta\sqrt{b}),
\]
is convergent. Thus from the theorems of convergent series, this implies that the partition function is convergent.

In order to evaluate the partition function, we can use the Euler-MacLaurin formula
\[
\sum_{n=0}^\infty f(n) = \frac{1}{2} f(0) + \int_0^\infty f(x) \, dx - \sum_{p=1}^\infty \frac{1}{(2p)!} B_{2p} f^{(2p-1)}(0),
\]
where \( B_{2p} \) are the Bernoulli numbers, \( B_2 = 1/6, B_4 = -1/30, \ldots \). They are defined through the series
\[
\frac{t}{e^t - 1} = \sum_{n=0}^\infty B_n \frac{t^n}{n!}.
\]

Then the partition function can be written as
\[
Z = \frac{1}{2} + \frac{2}{a\beta^2} (1 + \beta\sqrt{b}) - e^{\sqrt{b}\beta} \sum_{p=1}^\infty \frac{B_{2p}}{(2p)!} f^{(2p-1)}(0).
\]

To compute the partition function, we need to calculate the sum in the above expression. For our case it can be done only by numerical methods. Up to \( p = 2 \) this sum can be written as:
\[
e^{\sqrt{b}\beta} \sum_{p=1}^2 \frac{B_{2p}}{(2p)!} f^{(2p-1)}(0) = \\
\left( \frac{1}{24} \frac{\beta a}{\sqrt{b}} - \frac{1}{1920} \frac{\beta a^3}{b^{5/2}} - \frac{1}{1920} \frac{\beta^2 a^3}{b^2} - \frac{1}{5760} \frac{\beta^3 a^3}{b^{3/2}} \right).
\]

Before computing numerically the partition function, let us analyze the case of higher temperatures. This correspond to \( \beta \ll 1 \). All terms in the sum of \( Z \) have a positive power in \( \beta \), which are very small compared with the term \( 2(1 + \beta\sqrt{b})/a\beta^2 \). Hence, for \( \beta \ll 1 \)
\[
Z \simeq \frac{2}{a\beta^2}.
\]

Now we can easily obtain the mean energy and the heat capacity of the Dirac oscillator for higher temperatures
\[
U = -\frac{\partial \ln Z}{\partial \beta} \simeq 2kT,
\]
\[
C \simeq \frac{\partial U}{\partial T} \simeq 2k.
\]

These results show that, for higher temperatures, the mean energy and the heat capacity for the Dirac oscillator is two times the mean energy and the heat capacity of the non-relativistic one dimensional harmonic oscillator. This is due, only to relativistic effects, since in one dimension there is no spin coupling.

Now we briefly discuss our numerical results on the calculation of the partition function \( Z \). First of all, we should mention that, using \( Z \), the curves for the three thermal functions, namely, mean energy, heat capacity and free energy, are identical to those obtained for the nonrelativistic one-dimensional harmonic oscillator, in the limit of low temperatures (0 to 400 K).

We display a comparison between the Dirac oscillator and the harmonic oscillator only, for high temperatures (Fig. 1 and Fig. 2). It is seen that the free energy and the mean energy are greater for the Dirac oscillator than for
FIG. 1: The free energy for the harmonic and the Dirac oscillator for higher temperatures and $\hbar \omega/mc^2 = 1$. Dashed line corresponds to the Dirac oscillator and the solid line the harmonic oscillator.

FIG. 2: The mean energy for the harmonic and the Dirac oscillator for higher temperatures and $\hbar \omega/mc^2 = 1$. Dashed line corresponds to the Dirac oscillator and the solid line the harmonic oscillator.

the harmonic oscillator. From Fig. 3 it is also seen that the heat capacity for the Dirac oscillator is two times the heat capacity for the harmonic oscillator, result that was anticipated by the analytical calculations presented above.

It is worthwhile to mention that numerical calculations of the partition function for high temperatures ($\beta \ll 1$), force us to handle several very small variables, which could lead to deceiving results. So, it is required a precise estimation of the involved physical quantities. Therefore, we choose the angular frequency of the oscillator as $\omega = 10^{20} \text{Hz}$ in the region of high temperatures. We used adimensional quantities in the figures. The temperature ranges from $10^8 K$ to $10^{10} K (0.01$ to $1.0 \text{MeV})$.

As an extension of this work, we are currently studying a chain of Dirac oscillators in a thermal bath. Also, construction of higher dimensional Dirac oscillator in finite temperature, despite its technical difficulties, would be very interesting and could shed light on relativistic effects in statistical and solid state physics.

Some models used the nonrelativistic harmonic oscillator potential for describing confinement of quarks in mesons and baryons [14]. More recently, some authors suggested [8, 13] that the Dirac oscillator could be a good candidate to be used as the confinement potential in heavy quark systems. On the other hand, recently ultrarelativistic heavy ion experiments have search for the quark gluon plasma, a novel phase of QCD in which quarks and gluons are deconfined [17]. We expect that studies on quark-gluon plasma models and their thermal properties could be subsidized by our results.
FIG. 3: The heat capacity for the harmonic and the Dirac oscillator for higher temperatures and $\hbar\omega/mc^2 = 1$. Dashed line corresponds to the Dirac oscillator and the solid line the harmonic oscillator.

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