Lagrangian perfect fluids and black hole mechanics

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Abstract

The first law of black hole mechanics (in the form derived by Wald), is expressed in terms of integrals over surfaces, at the horizon and spatial infinity, of a stationary, axisymmetric black hole, in a diffeomorphism invariant Lagrangian theory of gravity. The original statement of the first law given by Bardeen, Carter and Hawking for an Einstein-perfect fluid system contained, in addition, volume integrals of the fluid fields, over a spacelike slice stretching between these two surfaces. One would expect that Wald’s methods, applied to a Lagrangian Einstein-perfect fluid formulation, would convert these terms to surface integrals. However, because the fields appearing in the Lagrangian of a gravitating perfect fluid are typically nonstationary, (even in a stationary black hole-perfect fluid spacetime) a direct application of these methods generally yields restricted results. We therefore first approach the problem of incorporating general nonstationary matter fields into Wald’s analysis, and derive a first law-like relation for an arbitrary Lagrangian metric theory of gravity coupled to arbitrary Lagrangian matter fields, requiring only that the metric field be stationary. This relation includes a volume integral of matter fields over a spacelike slice between the black hole horizon and spatial infinity, and reduces to the first law originally derived by Bardeen, Carter and Hawking when the theory is general relativity coupled to a perfect fluid. We then turn to consider a specific Lagrangian formulation for an isentropic perfect fluid given by Carter, and directly apply Wald’s analysis, assuming that both the metric and fluid fields are stationary and axisymmetric in the black hole spacetime. The first law we derive contains only surface integrals at the black hole horizon and spatial infinity, but the assumptions of stationarity and axisymmetry of the fluid fields make this relation much more restrictive in its allowed fluid configurations and perturbations than that given by Bardeen, Carter and Hawking. In the Appendix, we use the symplectic structure of the Einstein-perfect fluid system to derive a conserved current for perturbations of this system: this current reduces to one derived ab initio for this system by Chandrasekhar and Ferrari.

1 Introduction

The first law of black hole mechanics as stated by Bardeen, Carter and Hawking relates small changes in the mass of a stationary, axisymmetric black hole to small changes in its horizon surface
area, angular momentum and the properties of a stationary perfect fluid that might surround it: one first fixes a stationary axisymmetric Einstein-perfect fluid black hole solution with stationary killing field $\xi^a$ (with asymptotically unit norm) and axial killing field $\varphi^a$ (with closed orbits). One then defines $\delta$ to be an infinitesimal perturbation to a nearby stationary axisymmetric solution; then the first law in \cite{1} is

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J_H - \int_\Sigma \mu' |v| \delta N_{abc} + \int_\Sigma \Omega \delta J_{abc} + \int_\Sigma T |v| \delta S_{abc},$$

where the spacetime is characterised by an ADM mass, $M$, and the black hole by its horizon surface area, $A$, surface gravity, $\kappa$, angular velocity, $\Omega_H$, and angular momentum $J_H$ (measured at the horizon). The fields associated to the perfect fluid are its four velocity, $U^a$ (which here is taken to be of the form $U^a = v^a / |v|$, where $v^a = \xi^a + \Omega \varphi^a$, for some (generally non-constant) $\Omega$), the chemical potential $\mu'$, the temperature $T$, stress-energy $T_{ab}$, and number and entropy densities $n$ and $S$. The three-forms $N_{abc} = n U^d \varepsilon_{abcd}$, $J_{abc} = T^{de} \varphi^e \varepsilon_{abcd}$, and $S_{abc} = S U^d \varepsilon_{abcd}$ represent the fluid number density, angular momentum density and entropy density on a spacelike 3-surface, $\Sigma$, that has boundaries at the black hole horizon and the two-sphere at spatial infinity. We have also set $\varepsilon_{abcd}$ to be the canonical volume element on spacetime.

Considerable effort has been spent on weakening the assumptions made in \cite{1} on the background fields and their perturbations. For instance, consider an arbitrary diffeomorphism invariant Lagrangian theory with both metric and matter fields, and let the theory possess stationary, axisymmetric black hole solutions, which are asymptotically flat, and have a bifurcate killing horizon (for an explanation of these terms see \cite{2, 3}). Then it was shown \cite{2, 4}, providing the metric and matter fields appearing in the Lagrangian were stationary and axisymmetric in the black hole background, that there existed a first law of black hole mechanics in a form only involving surface integrals on the sphere at spatial infinity and the bifurcation sphere of the black hole horizon. Namely, given the Lagrangian for the theory, one could algorithmically define integrals $E$ and $J$ over the sphere at spatial infinity, and $S$ over the bifurcation sphere, satisfying the following identity:

$$\delta E = \frac{\kappa}{2\pi} \delta S + \Omega_H \delta J.$$

(Here $\delta$ denotes a perturbation from the background black hole solution to any nearby solution.) The quantity $E$ was interpreted as the canonical energy of the black hole system, $J$ as the canonical angular momentum and $S$ as the black hole entropy.

We might therefore expect that the volume integrals in \cite{1} involving the fluid can be converted to surface integrals in the form \cite{2}, by choosing a suitable variational form for the Einstein-perfect fluid system and using the methods of \cite{4}. In fact, we are unable to reproduce the first law \cite{1} in a form only containing surface integrals, using these methods; the difficulty is that at least one of the fields appearing in each of the Lagrangian formulations for a perfect fluid (that we are aware of) is generally non-stationary, even when the fluid four velocity, number density, entropy, and functions of these fields (which we refer to collectively as the physical fields), are stationary. Since the methods of \cite{4} require that all fields appearing in the Lagrangian (which we refer to henceforth as the dynamical fields) are stationary and axisymmetric in the black hole background, the allowed background solutions for the perfect fluid in the resulting first law are restricted.

This paper gives two results in response to this problem: we first relax all explicit symmetry assumptions on matter fields appearing in the Lagrangian, and find the consequence for the first law given in
In section (2) we consider an arbitrary Lagrangian theory of gravity coupled to arbitrary matter fields, assuming only that the metric is stationary and axisymmetric in the black hole background, but making no such assumptions about the matter dynamical fields. We then modify the methods of [4] to generate a perturbative relation, but instead of attempting to express the matter contribution to the first law via surface integrals, we leave it instead as a volume integral over a hypersurface, $\Sigma$, joining the bifurcation sphere to the sphere at spatial infinity. In restricted cases (which we explain later) we can motivate an independent measurement of the “vacuum” black hole mass, $M_g$. In these cases we can also define quantities which resemble the “vacuum” black hole entropy, $S_g$, and angular momentum, $J_g H$, and having done so our perturbative relation takes the form

$$\delta M_g = \frac{\kappa}{2\pi} \delta S_g + \Omega_H \delta J_g H, + \int_{\Sigma} \left[ \frac{1}{2} \mathbf{\xi} \cdot \mathbf{\epsilon} \right] T^{ab} \delta g_{ab} - \delta (\mathbf{\epsilon} \cdot \mathbf{T} \cdot \mathbf{\xi}),$$

(3)

where $T^{ab}$ is the stress-energy of the matter fields. We will see that this relation defines a black hole entropy, $S_g$, which is in general not the black hole entropy defined in [4]; however, in special cases the interpretation of $S_g$ as black hole entropy can be appropriate (for instance, as we show in section (4), this relation reduces to [4] when the gravitational theory is chosen to be general relativity, and the matter source is chosen to be a perfect fluid). Our result differs from a similar relation presented by Schutz and Sorkin [4], in that they conjectured, but did not explicitly include the black hole entropy and angular momentum boundary terms, and so did not explicitly generalise the full form of [4]. In addition, as we shall explain, the definition of our “Noether current” (involved in the intermediate calculations) is both less ambiguous than that presented by Schutz and Sorkin [4] and more general than the definition given by Sorkin [8]. The range of theories in which our methods are well defined is therefore larger than those addressed by their methods.

In section (3) we define a gravitating perfect fluid and review some variational principles for it: Schutz’s “velocity-potential” formulation [1], which uses the dynamical fields $(\phi, \alpha, \beta, \theta, \sigma)$ to define the product of the (physical) specific inertial mass and four velocity $\mu U_a \equiv \nabla \Phi + \alpha \nabla \alpha + \beta \nabla \sigma$, and Carter’s more recent “axionic vorticity” formulation [2] for an isentropic perfect fluid, which uses a dynamical field $b_{ab}$ to define the number current $N_{abc}$ (given in [1]) via $N_{abc} \equiv 3 \nabla_{[a} b_{bc]}$, and the dynamical fields $\chi^\pm$ to define the fluid vorticity via $2 \nabla_{[a} \mu U_b] \equiv 2 \nabla_{[a} \chi^+ \nabla^b] \chi^-$. In section (4) we present two forms of the first law for the Einstein-perfect fluid system. The first form is derived from the relation (2) and is the same as (4), with the exception that $\delta$ is now allowed to be a perturbation from the (stationary axisymmetric) background to an arbitrary nearby solution. (Note that this form of the first law contains volume integrals.) It is of also interest to know if we can construct any form of the first law with perfect fluids only involving surface integrals; in fact, by directly applying the methods of [4] for a metric theory of gravity coupled to a perfect fluid described using Carter’s variational principle, (with the potential $b_{ab}$ for $N_{abc}$, and $\chi^\pm$ for $\omega_{ab}$) we can derive a first law of the form

$$\delta M + \mu_\infty \delta \int_{S_\infty} b_{qr} - \mu_\infty \delta \int_{H} b_{qr} = \frac{\kappa}{8\pi} \delta A - \Omega_H \delta J_H + \int_{H} X_{qr} - \int_{S_\infty} X_{qr},$$

(4)

where $M$ is the ADM mass, $A$ is the black hole surface area, $J_H$ is the black hole angular momentum appearing in [4], $X_{qr}$ is the two-form $2 \xi^p b_{pq} \left[ \delta (\mu U_r) - \nabla_r \chi^- \delta \chi^+ + \nabla_r \chi^+ \delta \chi^- \right]$, and we have written $S_\infty$ and $H$ for the sphere at spatial infinity and the bifurcation sphere, respectively. We will see that
this first law is more restrictive than (1), but it is the only non-trivial rule of the type (2) involving a perfect fluid that we can currently construct.

In the Appendix we evaluate the symplectic form of the Einstein-perfect fluid system, using the variational formulation given by Schutz [9] for the perfect fluid. The symplectic form is dual to a generally conserved current, quadratic in the field perturbations [11]. We find (in parallel with Burnett and Wald’s calculation for the Einstein-Maxwell system [12]) that this conserved current reduces to a current previously derived \textit{ab initio} by Chandrasekhar and Ferrari [13] for the polar perturbations of a static axisymmetric black hole.

2 A perturbative relation for black hole mechanics with non-stationary matter fields

In this section we give a perturbative relation that resembles the first law of black hole mechanics, for an arbitrary theory of gravity with a diffeomorphism invariant Lagrangian. We assume the theory possesses black hole solutions in which the metric is stationary and axisymmetric, but place no restrictions on the other fields appearing in the Lagrangian (we refer to these fields collectively as the dynamical fields). The motivation for this is, as we have indicated, that variational formulations for gravitating Einstein-perfect fluid systems have fluid dynamical fields which are nonstationary even when the fluid’s physical fields (the four-velocity, number density and entropy) are stationary and axisymmetric. We first make some necessary definitions related to the the symplectic structure of a diffeomorphism invariant Lagrangian theory. These are explained in detail in [4]; here we merely state (and, in one case, refine) the relevant definitions and results. In the following we often use bold face type to denote differential forms on spacetime, suppressing their indices when convenient.

2.1 Some Preliminaries

All theories we consider arise from a Lagrangian, which is taken to be a diffeomorphism invariant four-form on spacetime, dependent on the metric, $g_{ab}$, and some arbitrary set of matter fields, $\psi$. (We collectively refer to all the dynamical fields by $\phi$.) By this we mean that the Lagrangian has the functional dependence

$$\mathcal{L} = \mathcal{L}(g_{ab}, R_{abcd}, \nabla R_{abcd}, \ldots, (\nabla)^p R_{abcd}, \psi, \nabla \psi, \ldots, (\nabla)^q \psi),$$

(here multiple derivatives appearing in the above expression are assumed to be symmetrised - see [4] for further discussion about this dependence). In particular we require that every field appearing in the Lagrangian give rise to an equation of motion (there are no “background” fields). The variation of the Lagrangian defines these equations, $\mathbf{E} = 0$, along with the symplectic potential $\Theta$, by

$$\delta \mathcal{L} = \mathbf{E} \delta \phi + d\Theta(\phi, \delta \phi).$$

(Here $\Theta(\phi, \delta \phi)$ is a linear differential operator in the field variations $\delta \phi$. Because the Lagrangian is only defined up to the addition of an exact form, $\mathcal{L} \rightarrow \mathcal{L} + d\mu$, the symplectic potential is only defined up to the following terms: $\Theta(\phi, \delta \phi) \rightarrow \Theta(\phi, \delta \phi) + dY(\phi, \delta \phi) + \delta \mu(\phi)$, where $Y$ and $\mu$ are covariant
forms with the same type of functional dependence as $\Theta$ and $L$, respectively. These ambiguities were discussed in [4].

Now fix a smooth vector field, $\xi^a$, on spacetime. Then the Noether current $J[\xi]$ associated to $\xi^a$ is a three-form defined by

$$J[\xi] \equiv \Theta(\phi, L_\xi \phi) - \xi \cdot L$$

(7)

where the centred dot denotes contraction of the vector into the first index of the form. This Noether current can be seen [4] to obey the following identity:

$$dJ[\xi] = -EL_\xi \phi,$$

(8)

which we now use to further elucidate its structure. (Although they appear in a different context, the calculations below have the same flavour as those in the Appendix of [4].)

**Lemma 1:** Fix $L$ to be the Lagrangian of a diffeomorphism invariant theory of gravity and matter fields, with equation of motion $E = 0$ as given in (6). Without loss of generality, label each dynamical field $\phi$ by $i$, and give each field $u_i$ upper, and $d_i$ lower indices: also label the equations of motion for each field similarly, so that the equation of motion term in (6) becomes

$$E(\delta \phi) = \epsilon E_{i1 \cdots b_{ui}} \delta_{a1 \cdots a_{di}} \phi_{b1 \cdots b_{ui}}.$$

(9)

Then for any smooth field $\xi^a$ there exists a two-form, $Q[\xi]$, called the Noether charge associated to $\xi^a$ (which is local in the dynamical fields and $\xi^a$), such that the Noether current $J[\xi]$, defined in (7), can be written

$$J[\xi] = -(\epsilon \cdot E \cdot \phi \cdot \xi) + dQ[\xi],$$

(10)

where we define the three-form

$$(\epsilon \cdot E \cdot \phi \cdot \xi)_{abc} \equiv \epsilon_{abc} \sum_i E_{\phi i1 \cdots b_{ui}} \delta_{a1 \cdots a_{di}} (\delta_{b1 \cdots b_{ui}} \phi_{e1 \cdots e_{di}} \delta^{e_{di}}_{a1 \cdots a_{di}}) \xi^p.$$

(11)

**Proof:**

For clarity, we first consider the case where the metric is the only dynamical field: $\phi \rightarrow g_{ab}$. Then setting the metric field equations $E_{ab} = \epsilon E_{ab}$, Eq. (5) reads

$$dJ[\xi] = -2\epsilon E_{gab} \nabla_a \xi_b - 2\epsilon \nabla_a (E_{gab} \xi_b) + 2\epsilon \nabla_a (E_{gab}) \xi_b.$$

(12)

Therefore setting $(\epsilon \cdot E_g \cdot \xi)_{abc} \equiv \epsilon_{abc} E^{de}_{g} \xi_{de}$, we have

$$d(J[\xi] + 2\epsilon \cdot E_g \cdot \xi) = 2\epsilon \nabla_a (E_{gab}) \xi_b,$$

(13)

which shows that the right side of (13) is both linear in $\xi^a$, and exact for all $\xi^a$. The results of [5] now imply that the right side must vanish identically - and so $\nabla_a E_{gab} = 0$. This in turn implies that the left side of (13) must be an identically closed three-form, which (using the results of [5] again) implies the existence of a two-form, $Q[\xi]$, local in the dynamical fields and $\xi^a$, such that

$$J[\xi] + 2\epsilon \cdot E_g \cdot \xi = dQ[\xi].$$

(14)
We define $Q[\xi]$, the Noether charge associated to $\xi^a$, as any two form which is local in the dynamical fields and $\xi^a$, and satisfies this relation.

We can also perform this analysis for $L$ with the general dependence in Eq. (3). With the labels for each field and its equation of motion given in Eq. (3), the first equation in Eq. (13) becomes

$$dJ[\xi] = -\varepsilon \sum_{i} E_{\phi b_1...b_{u_i}} a_1...a_{d_i} L_{\xi} \phi_{b_1...b_{u_i} a_1...a_{d_i}},$$  

(15)

which, through a similar manipulation to (12) leads to the structure for $J[\xi]$ and the definition of the Noether charge, $Q[\xi]$, in (11). □

The Noether charge was defined in (4) only when $E = 0$, via $J[\xi] = dQ[\xi]$. This left open the definition of $Q[\xi]$ when $E \neq 0$. In the Appendix of (3), however, it was shown that $Q[\xi]$ could be defined when $E \neq 0$, such that there existed forms $C_a$ with $J[\xi] - dQ[\xi] = C_a \xi^a$, and where the $C_a$ vanished when $E = 0$. At that time it was not known whether $Q[\xi]$ was uniquely defined this way, nor was the explicit form of $C_a$ specified. We have given this explicit form in (11). Moreover, the above analysis uniquely defines the Noether charge via (11), without imposing the field equations, up to the following ambiguities (which were discussed in detail in (3)): The ambiguity in $\Theta$ described after Eq. (3) means that $J[\xi]$ is only defined up to the following terms: $J[\xi] \to J[\xi] + d(Y(\phi, L_{\xi} \phi) - \xi \cdot \mu)$, and so the ambiguity in $Q[\xi]$ is $Q[\xi] \to Q[\xi] + Y(\phi, L_{\xi} \phi) - \xi \cdot \mu$. These ambiguities will not affect the results stated in the following sections.

We now define the symplectic current, $\omega(\phi, \delta_1 \phi, \delta_2 \phi)$, (a three-form on spacetime) by:

$$\omega(\phi, \delta_1 \phi, \delta_2 \phi) \equiv \delta_2 \Theta(\phi, \delta_1 \phi) - \delta_1 \Theta(\phi, \delta_2 \phi).$$  

(16)

Note that $\omega$ is a function of an unperturbed set of fields, $\phi$, and is bilinear and skew in pairs of variations ($\delta_1 \phi, \delta_2 \phi$). It can be shown (see (11)) that this three-form is closed when $\phi$ is a solution of the field equations and $\delta_1 \phi$ and $\delta_2 \phi$ are solutions of the linearised equations of motion (In the Appendix we examine this closed form - it is dual to a conserved vector field, which we evaluate for perturbations of an Einstein-perfect fluid system). Moreover, if we let $\xi^a$ be a smooth vector field, set $\delta_1 \phi = L_{\xi} \phi$ and let $\delta_2 \phi = \delta \phi$ be a variation to a nearby solution (with $\delta \xi^a = 0$), then $\omega(\phi, L_{\xi} \phi, \delta \phi)$ can be shown (3) to be exact:

$$\omega(\phi, L_{\xi} \phi, \delta \phi) = d[\delta Q[\xi] - \xi \cdot \Theta(\phi, \delta \phi)].$$  

(17)

Now fix a black hole spacetime with a stationary and axisymmetric metric, for the theory given by the Lagrangian in (3); let the stationary killing field with unit norm at spatial infinity be $\xi^a$ and the axial killing field (with closed orbits) be $\varphi^a$. Let the black hole have a bifurcate killing horizon, with bifurcation sphere $\mathcal{H}$, and let it be asymptotically flat, with the two-sphere at spatial infinity $S^\infty$. Let $\Sigma$ be a three-surface with these two boundaries, and set $\delta \phi$ to be an arbitrary perturbation of the background which satisfies the linearised equations. Then the first law of black hole mechanics as stated in (3) is an interpretation of the identity

$$\int_{\Sigma} \omega(\phi, L_{\xi} \phi, \delta \phi) = \int_{S^\infty} \delta Q[\xi] - \xi \cdot \Theta(\phi, \delta \phi) - \int_{\mathcal{H}} \delta Q[\xi] - \xi \cdot \Theta(\phi, \delta \phi)$$  

(18)

(which arises from integrating (17) over $\Sigma$). When $\xi^a$ Lie derives all the dynamical fields in the background, the left side of (18) vanishes, and one is left with a relation between surface integrals on the boundaries of $\Sigma$, which can be shown to be of the form (3). In section (4) we present an explicit Lagrangian for the Einstein-perfect fluid system, and, assuming that all dynamical fields are stationary and axisymmetric, compute the surface terms arising from this Lagrangian.
2.2 The perturbative identity

Having stated these necessary definitions we turn to construct our perturbative identity. We start by decomposing the Lagrangian $L$ into a part $L_g$, depending on the metric, $g_{ab}$, (which is assumed to be stationary and axisymmetric in the black hole background), and a part $L_m$, dependent on both the metric and a set of matter fields, $\psi$, (on which we place no restrictions):

$$ L = L_g(g_{ab}, R_{abcd}, \ldots, (\nabla)^p R_{abcd}) + L_m(\psi, \nabla \psi, \ldots, (\nabla)^q \psi, g_{ab}, R_{abcd}, \ldots, (\nabla)^p R_{abcd}). $$ (19)

Since this breakup only requires that $L_g$ be independent of any matter fields, it is very non-unique, and in general we have no method of controlling the ambiguity

$$ L_g \rightarrow L_g + \lambda, \quad L_m \rightarrow L_m - \lambda. $$ (20)

where $\lambda = \lambda(g_{ab}, R_{abcd}, \ldots, (\nabla)^s R_{abcd})$.

The variation of the Lagrangian yields equations of motion for the metric, $E^a_b = 0$, and matter fields, $E^m = 0$, via

$$ \delta L = E^a_b \delta g_{ab} + E^m \delta \psi + d\Theta(\phi, \delta \phi). $$ (21)

For convenience we set $E^a_b = \epsilon E^a_b$ and $E^m = \epsilon E^m$. As discussed above we can compute $J[\xi]$ (defined by (7)), and define $Q[\xi]$, for the theory described by (19): it must have the form given in (10):

$$ J[\xi] = -2\epsilon \cdot E^a_b \cdot \xi - \epsilon \cdot E^m \cdot \psi \cdot \xi + dQ[\xi] $$ (22)

(the factor of two between the terms with equations of motion here is purely a matter of convention).

We can also use the individual Lagrangians $L_g$ and $L_m$ to define the stress-energy tensor $T^a_b$, and symplectic potentials $\Theta_g(g, \delta g)$ and $\Theta_m(\phi, \delta \phi)$:

$$ \delta L_g = E^a_b \delta g_{ab} + d\Theta_g(g, \delta g) \quad \delta L_m = E^m \delta \psi + \epsilon \frac{1}{2} T^a_b \delta g_{ab} + d\Theta_m(\phi, \delta \phi). $$ (23)

Clearly $E^a_b = E'_g \delta g_{ab} + \epsilon \frac{1}{2} T^a_b$, and up to the ambiguities present in the symplectic potentials, we also have $\Theta = \Theta_g + \Theta_m$. Similarly, if we define the Noether currents for the individual Lagrangians by

$$ J_g[\xi] = \Theta_g(g, L_g \xi) - \xi \cdot L_g, \quad J_m[\xi] = \Theta_m(\phi, L_\phi \xi) - \xi \cdot L_m, $$ (24)

then it follows that

$$ J[\xi] = J_g[\xi] + J_m[\xi]. $$ (25)

Now we impose the structure (11) on each of $J_g$ and $J_m$, in the process defining $Q_g$ and $Q_m$, which are the Noether charges in the theories arising from these Lagrangians:

$$ J_g[\xi] = -2\epsilon \cdot E'_g \cdot \xi + dQ_g[\xi] \quad J_m[\xi] = -\epsilon \cdot E_m \cdot \psi \cdot \xi - \epsilon \cdot T \cdot \xi + dQ_m[\xi]. $$ (26)
Finally we substitute (24) into the right side of (25) and (22) into the left side, obtaining
\[
-2\epsilon \cdot E_g \cdot \xi - \epsilon \cdot E_m \cdot \psi \cdot \xi + dQ[\xi] = -2\epsilon \cdot E_g' \cdot \xi - \epsilon \cdot T \cdot \xi - \epsilon \cdot E_m \cdot \psi \cdot \xi + dQ_g[\xi] + dQ_m[\xi]. \quad (27)
\]
All the terms involving equations of motion and stress-energy tensors can be seen to cancel, and the resulting identity implies
\[
Q[\xi] = Q_g[\xi] + Q_m[\xi] + dZ,
\]
(28)
(where \(Z\) is some arbitrary covariant one-form). We therefore have a relation (independent of any field equations) between the Noether charge, \(Q\), of the full theory given by \(L\), and that of the “pure gravity” theory \(Q_g\), arising from \(L_g\). We are now ready to state the identity:

**Lemma 2:** Fix \(L, L_g\) (the “vacuum” Lagrangian) and \(L_m\) (the “matter” Lagrangian) to be diffeomorphism invariant Lagrangians related as given in (19) with the functional dependence shown there. Fix a smooth vector field \(\xi^a\), let \(\Theta_g\) be defined by (23), let \(Q_g[\xi]\) be the Noether charge defined by (23) for the theory described by \(L_g\), and let \(T^{ab}\) be the stress-energy tensor of the matter fields defined by (23). Now consider an asymptotically flat, stationary, axisymmetric black hole solution with bifurcate Killing horizon, in the theory described by \(L\), with stationary Killing field \(\xi^a\) (with unit norm at the sphere \(S^\infty\), at spatial infinity), and axial Killing field \(\varphi^a\) (with closed orbits), so that \(\xi^a\) and \(\varphi^a\) Lie derive the metric but not necessarily the matter fields. Let the horizon Killing field (which vanishes on the bifurcation sphere \(\mathcal{H}\)) be given by \(\chi^a = \xi^a + \Omega_H \varphi^a\), where \(\Omega_H\) is a constant. Then for \(\delta\) a perturbation to an arbitrary nearby solution, such that \(\delta\xi^a = 0\),
\[
\int_{S_\infty} \delta Q_g[\xi] - \xi \cdot \Theta_g = \int_{\mathcal{H}} \delta Q_g[\chi] - \Omega_H \int_{\mathcal{H}} \delta Q_g[\varphi] + \int_S \frac{1}{2} \xi \cdot \epsilon \cdot T^{ab} \delta g_{ab} - \delta(\epsilon \cdot T \cdot \xi). \quad (29)
\]

**Proof:**
We evaluate the expression (28) for the theory (19), where the background solution is a black hole with the symmetry and structure described above (24), demanding that the metric be stationary and axisymmetric in the background spacetime, but placing no restrictions on the matter fields. In this case the integrand on the left side of (28) is generally nonvanishing. Assuming that the field equations hold in background for the matter fields, \(E_m = 0\), and that \(\delta \psi\) is a solution to the linearised matter equations of motion off this background (\(\delta E_m = 0\)), we find the left side of (28) is
\[
\omega(\phi, L_\xi \phi, \delta \phi) = \delta \Theta_g(g, L_\xi g) - L_\xi \Theta_g(g, \delta g) + \delta \Theta_m(\phi, L_\xi \phi) - L_\xi \Theta_m(\phi, \delta \phi) = \delta \Theta_m(\phi, L_\xi \phi) - L_\xi \Theta_m(\phi, \delta \phi) = \delta(\delta Q_m[\xi] - \epsilon \cdot T \cdot \xi + \xi \cdot L_m) - L_\xi \Theta_m(\phi, \delta \phi) = \delta dQ_m[\xi] - \epsilon \cdot T \cdot \xi + \xi \cdot L_m - \epsilon \cdot d\Theta_m(\phi, \delta \phi) - d(\xi \cdot \Theta_m(\phi, \delta \phi)) = d(\delta Q_m[\xi] - \xi \cdot \Theta_m(\phi, \delta \phi)) - \delta(\epsilon \cdot T \cdot \xi) + \frac{1}{2} \xi \cdot \epsilon \cdot T^{ab} \delta g_{ab}, \quad (30)
\]
where we used the stationarity of \(g_{ab}\) in the second line, the expression (26) for \(J_m\) in the third, the Lie derivative identity \(L_\xi \lambda = \xi \cdot d\lambda + d(\xi \cdot \lambda)\) (which holds for an arbitrary form \(\lambda\)) in the fourth line, and the definition (23) of \(\Theta_m\) and the stress-energy \(T^{ab}\) in the fifth line. Now also assuming \(E_g = \delta E_g = 0\), and substituting (30) into the left side of (13) yields
\[
\int_S \delta dQ_m[\xi] - \xi \cdot \Theta_m + \frac{1}{2} \xi \cdot \epsilon \cdot T^{ab} \delta g_{ab} - \delta(\epsilon \cdot T \cdot \xi) = \int_{S_\infty} \delta Q[\xi] - \xi \cdot \Theta(\phi, \delta \phi) - \int_{\mathcal{H}} \delta Q[\xi] - \xi \cdot \Theta(\phi, \delta \phi), \quad (31)
\]
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and so, cancelling the boundary terms \( \delta Q_m[\xi] - \xi \cdot \Theta_m \) from both sides (and using (28)) we get

\[
\int \frac{1}{2} \xi \cdot \epsilon \ T^{ab} \delta g_{ab} - \delta(\epsilon \cdot T \cdot \xi) = \int_{S_{\infty}} \delta Q_g[\xi] - \xi \cdot \Theta_g(g, \delta g) - \int_{\mathcal{H}} \delta Q_g[\xi] - \xi \cdot \Theta_g(g, \delta g). \tag{32}
\]

Now writing \( \xi^a \) in terms of \( \chi^a \) and \( \phi^a \) at the boundary \( \mathcal{H} \) (and discarding terms which vanish as a result of the vanishing of \( \chi^a \) at \( \mathcal{H} \), or which vanish because \( \phi^a \) is tangent to \( \mathcal{H} \)) we get

\[
\int \frac{1}{2} \xi \cdot \epsilon \ T^{ab} \delta g_{ab} - \delta(\epsilon \cdot T \cdot \xi) = \int_{S_{\infty}} \delta Q_g[\xi] - \xi \cdot \Theta_g(g, \delta g) - \int_{\mathcal{H}} \delta Q_g[\chi] + \Omega_H \int_{\mathcal{H}} \delta Q_g[\varphi], \tag{33}
\]

which is what we wished to show. □

The identity (29) has physical significance when we can interpret the surface integrals appearing there as (variations of) the energy, entropy and angular momentum of the black hole. When is this possible? If the theory had no matter fields then we could choose \( L_\text{m} \) to vanish, and the terms involving \( T^{ab} \) in (29) would vanish (we could also choose other breakups of \( L \), and we’ll return to this shortly). In this case we’d have \( L_g = L, \Theta_g = \Theta, \) and \( Q_g = Q \). If in addition there existed a three-form \( B \), (local in the dynamical fields, the flat metric \( \eta^{ab} \), and its associated derivative, \( \partial \), at spatial infinity) such that at spatial infinity, \( \xi \cdot \Theta(\phi, \delta \phi) = \xi \cdot \delta B \), then (29) can be written

\[
\delta E = \frac{\kappa}{2\pi} \delta S + \Omega_H \delta J_H, \tag{34}
\]

where the varied quantities in (34) are defined below, and have well-known physical interpretations [4]. These are (i) The canonical energy of the system, which we define as

\[
E \equiv \int_{S_{\infty}} Q[\xi] - \xi \cdot B, \tag{35}
\]

(ii) The entropy \( S \) of the black hole; by taking the functional derivative of the Lagrangian with respect to the Riemann tensor (treated as an independent field) we know (setting \( \epsilon_{ab} \) to be the binormal to the bifurcation sphere) that

\[
\delta \int_\mathcal{H} Q[\chi] = \frac{\kappa}{2\pi} \delta S, \tag{36}
\]

where

\[
S \equiv -2\pi \int_\mathcal{H} \frac{\delta L}{\delta R_{abcd}} \epsilon_{ab} \epsilon_{cd}, \tag{37}
\]

and \( \kappa \) is the surface gravity of the background black hole horizon. (iii) The angular momentum of the system measured at the black hole, defined by

\[
J_H \equiv -\int_\mathcal{H} Q[\varphi]. \tag{38}
\]

In fact, the angular momentum can be measured either at the black hole horizon or at spatial infinity; since the metric is axisymmetric with axial killing field \( \varphi^a \), it can be seen from (7) that \( J[\varphi] \) vanishes, when pulled back to a slice to which \( \varphi^a \) is tangent. This ensures (integrating the relation \( J[\varphi] = dQ[\varphi] \) over \( \Sigma \)) that, for the background solution,

\[
\int_{S_{\infty}} Q[\varphi] = \int_\mathcal{H} Q[\varphi]. \tag{39}
\]
In addition, by considering the identity

$$\int_{\Sigma} \omega(\phi, \delta \phi, \mathcal{L}_\phi \phi) = \int_{\partial \Sigma} \delta \mathbf{Q}[\phi] - \phi \cdot \Theta,$$

(40)

we see that when $\phi^a$ Lie derives all the dynamical fields, the left side of this equation vanishes. Since $\phi^a$ is tangent to the two-spheres $\mathcal{H}$ and $S^\infty$, the pullback of the second term on the right side vanishes. It follows that

$$\delta \int_{S^\infty} \mathbf{Q}[\phi] = \delta \int_{\mathcal{H}} \mathbf{Q}[\phi],$$

(41)

Therefore, in spacetimes which have axisymmetric background configurations, the angular momentum measured at the black hole is equivalent to the canonical angular momentum $\mathcal{J}$, measured at spatial infinity

$$\mathcal{J} \equiv - \int_{S^\infty} \mathbf{Q}[\phi],$$

(42)

both when $\phi$ is an axial Killing field, (in the background solution) and for arbitrary solutions which are perturbations, $\delta \phi$, of the axisymmetric solution. This calculation also shows that the definition of $\mathcal{J}_H$ is gauge independent, for arbitrary perturbations of an axisymmetric solution. This is because $\delta \mathcal{J}_H = 0$ when we choose $\delta \phi$ to be pure gauge, which we see by first setting $\hat{\delta} \phi \equiv \mathcal{L}_v \phi$ for some smooth $v^a$, and then replacing $\delta \phi$ with a gauge transform $\hat{\delta}' \phi$ which coincides with $\hat{\delta} \phi$ in a neighbourhood of the bifurcation sphere, but vanishes in a neighbourhood of spatial infinity. Then we have for every $\delta \phi$ (using (41)),

$$\hat{\delta} \mathcal{J}_H = \hat{\delta} \int_{\mathcal{H}} \mathbf{Q}[\phi] = \hat{\delta}' \int_{\mathcal{H}} \mathbf{Q}[\phi] = \hat{\delta}' \int_{S^\infty} \mathbf{Q}[\phi] = 0.$$

(43)

So we have that when $T^{ab}$ vanishes (along with $\mathbf{L}_m$), the interpretation of the terms in (29) is straightforward, and one obtains a formula (34) which (bearing in mind the equivalence of $\mathcal{J}$ and $\mathcal{J}_H$) is the formula (2).

What if the set of fields $\psi$ is non-empty? In general, the ambiguity (20) in breaking $\mathbf{L}$ into $\mathbf{L}_g$ and $\mathbf{L}_m$ stops us from meaningfully interpreting the surface terms in (29) as perturbations of mass, entropy and angular momentum: even if the overall theory is fixed, every choice of $\mathbf{L}_g$ generates a different relation, with different choices of $\mathbf{Q}_g$ etc. We therefore seek more restrictive assumptions under which we might successfully identify the surface terms in (29). One approach is to fix a particular choice of $\mathbf{L}_g$ and think of it as specifying an independent theory. We assume there exists a form $\mathbf{B}_g$ such that at spatial infinity, $\delta (\xi \cdot \mathbf{B}_g) = \xi \cdot \Theta_g$, and consider the functional $M_g$ defined by

$$M_g \equiv \int_{S^\infty} \mathbf{Q}_g[\xi] - \xi \cdot \mathbf{B}_g.$$

(44)

If we now require that the stress-energy of the matter distribution falls off sufficiently rapidly at spatial infinity, such that (near spatial infinity) the metric for any solution of the $\mathbf{L}$-theory approaches a metric solution of the $\mathbf{L}_g$-theory, and $M_g$ yields the same result on both metrics, then it makes sense to define the mass of the system as $M_g$. We note that if we can also find a form $\mathbf{B}(\phi)$ for the full theory, such that at spatial infinity $\delta (\xi \cdot \mathbf{B}) = \Theta(\phi, \delta \phi)$, then we can also define a canonical energy, $\mathcal{E}$, for the full theory given by (35), and in general $\mathcal{E} \neq M_g$. 

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Therefore, when the stress-energy of the matter distribution falls off sufficiently rapidly, we can interpret the left side of (29) as the variation of the mass of the system. The surface terms on the right side of (29) are (variations of) the functionals that would measure the entropy and angular momentum of a stationary black hole in the $L_g$-theory. We might therefore be tempted to interpret them as the black hole entropy and angular momentum; indeed, since $Q_g$ is the Noether charge of the $L_g$ theory, we know from [4] that one can define a quantity, $S_g$, by

$$ S_g \equiv -2\pi \int_{\mathcal{H}} \frac{\delta L_g}{\delta R_{abcd}} \epsilon_{ab} \epsilon_{cd}, \quad (45) $$

such that

$$ \delta \int_{\mathcal{H}} Q_g[\chi] = \frac{\kappa}{2\pi} \delta S_g. \quad (46) $$

One might also define a quantity, $J_{gH}$, by

$$ J_{gH} \equiv -\int_{\mathcal{H}} Q_g[\varphi]. \quad (47) $$

Although we made no assumptions about the axisymmetry of the matter fields, we can show, providing the support of $T^{ab}$ does not intersect some neighbourhood, $U$, of the bifurcation sphere, that $J_{gH}$ is also well-defined (gauge independent) for arbitrary perturbations of the axisymmetric solution. This follows by evaluating the left side of (40), using the fact that the calculation (30) also holds when $\xi^a$ is replaced by $\varphi^a$. Taking $\varphi^a$ to be tangent to the spatial slice, Eq. (40) then becomes

$$ \int_{S_{\infty}} \delta Q_g[\varphi] + \int_{\Sigma} \delta(\epsilon \cdot T \cdot \varphi) = \int_{\mathcal{H}} \delta Q_g[\varphi] \quad (48) $$

Now, as before, let the perturbation in this equation be gauge, $\delta \phi = \hat{\delta} \phi$. Then we again can replace the perturbation on the right side with an equivalent gauge change, which vanishes outside $U$, and so intersects neither the support of $T^{ab}$ nor spatial infinity. Then we have the left side of (48) vanishes, and so

$$ \hat{\delta} J_{gH} = -\int_{\mathcal{H}} \delta Q_g[\varphi] = 0. \quad (49) $$

Therefore $J_{gH}$ is defined for arbitrary perturbations of an axisymmetric solution.

Now having defined $M_g$, $S_g$ and $J_{gH}$, we could write out (29) in the form

$$ \delta M_g = \frac{\kappa}{2\pi} \delta S_g + \Omega_H \delta J_{gH} + \int_{\Sigma} \frac{1}{2} \xi \cdot \epsilon \cdot T^{ab} \delta g_{ab} - \delta(\epsilon \cdot T \cdot \xi), \quad (50) $$

where $\delta$ is a perturbation to an arbitrary nearby solution. However, we caution the reader that the identification of black hole entropy with $S_g$ in general gives results in conflict with those in [4]; consider a theory of gravitation with a scalar field, for which the matter Lagrangian couples to the spacetime curvature, and which displays stationary black hole configurations in which the scalar field has sufficiently rapid spatial falloff. We can therefore write out (50) and interpret the black hole entropy as $S_g$. From the results of [4] we expect the entropy of the black hole to include contributions from the scalar field; equation (50), however, defines a black hole entropy $S_g$ with only metric contributions, with the entropy contribution of the scalar field somehow distributed in the volume integral of its stress-energy. These two points of view are contradictory; therefore, while there are clearly special cases (for instance, the Einstein-perfect fluid system) in which we can identify $S_g$ as the black hole entropy, and terms in the volume integral as (variations of) the matter entropy, in general we regard the
We note parenthetically that we can write out an alternative form of (50) by replacing the stationary killing field $\xi^a$ in (13) with the horizon killing field $\chi^a$. (The analysis up to (12) is unchanged except for the substitution $\xi^a \rightarrow \chi^a$.) Then expanding $\chi^a = \xi^a + \Omega_H \varphi^a$ at spatial infinity and on the slice $\Sigma$, (but not at $\mathcal{H}$) and making the definitions discussed above gives the identity

$$\delta M_g = \frac{\kappa}{2\pi} \delta S_g + \Omega_H \delta J_{g,\infty} + \int_{\Sigma} \frac{1}{2} \xi \cdot \epsilon T^{ab} \delta g_{ab} - \delta (\epsilon \cdot T \cdot \xi) - \Omega_H \int_{\Sigma} \delta (\epsilon \cdot T \cdot \varphi)$$  \hspace{1cm} (51)$$

where $J_{g,\infty} \equiv - \int_{S^\infty} Q_g[\varphi]$, is the system angular momentum measured at spatial infinity. Therefore the cost we have incurred for the transfer of the angular momentum integral to spatial infinity is the appearance of an extra term in the volume integral.

A relation of the form (50), was first given by Schutz and Sorkin [7], in the case where $L_g$ was fixed to be the Lagrangian for general relativity, $L_m$ was any matter Lagrangian, and there was no black hole boundary $\mathcal{H}$, for the hypersurface $\Sigma$. The relation stated in [7] is correct, but we comment here on the ambiguity of the “Noether operators” used by Schutz and Sorkin to derive it: In its initial definition [6] the Noether operator for a Lagrangian $L$ and a smooth vector field $\xi^a$ was defined to be any (not necessarily covariant) three form $J^3[\xi]$ satisfying the relation

$$L_\xi L = E L_\xi \phi + d(J^3[\xi] + \xi \cdot L),$$  \hspace{1cm} (52)$$

for every smooth field vector field $\xi^a$. This definition leaves $J^3[\xi]$ ambiguous by an arbitrary exact three-form which is a linear differential operator in $\xi^a$. Since we know from [11] that $J^3 = dQ[\xi]$ when the field equations hold, this ambiguity would permit $J^3 = 0$ as a valid Noether operator (which, following Schutz and Sorkin’s methods, would yield a correct but trivial relation). On the other hand, our definition of the Noether current admits a limited set of ambiguities (stated after (13)), which cannot be used to annihilate the Noether charge, and in particular do not change the content of the first law.

Sorkin introduced an augmented definition of the Noether operator in [3], requiring that for a variation of the dynamical fields given by $\delta \phi = f L_\xi \phi$, where $f$ is any function, the Noether operator $J^S$ be defined by

$$\delta L = E f L_\xi \phi + d(f J^S[\xi] + f \xi \cdot L).$$  \hspace{1cm} (53)$$

Providing one can find a $J^S$ which satisfies this relation, it is easy to see that one cannot add a term to $J^S$ which is both exact and linear in $f$, for arbitrary $f$. For a theory with a first order Lagrangian, finding such a $J^S$ is always possible: in [3] a first-order (noncovariant) Lagrangian for Einstein-Maxwell theory was used to yield an unambiguous Noether operator. It is not clear, however, that any general Lagrangian theory has a first order Lagrangian formulation, so in general, Sorkin’s definition may not even yield a Noether operator. In contrast, all of our Noether currents $J[\xi]$ defined above can be computed for Lagrangian theories of arbitrary derivative order, and are manifestly covariant, requiring no additional background fields (apart from the symmetry field $\xi^a$) for their definition. For these reasons, we feel that whilst our relation (50) and that in [6] coincide for an Einstein-matter system without the black hole, (51) is defined more generally.

We finally remark that we could have carried out the entire analysis leading up to (29) allowing the Lagrangian $L_g$ to depend on a set of stationary axisymmetric fields, $s_i$, including the metric, and the
Lagrangian $L_m$ to depend on $s_i$ and a distinct set of fields, $\psi$, which didn’t appear in $L_g$, to obtain a relation very similar to (29). The resulting perturbative identity has the terms $Q_g$ and $\Theta_g$ in (29) replaced with the Noether charge and symplectic potential in the theory described by $L_g$ (which now depends on both the metric and the other matter fields in the set $s_i$), and the volume term is now given by

$$\int \Sigma \frac{1}{2} \xi \cdot \epsilon \ T_{s_i} \delta s_i - \delta(\epsilon \cdot T_s \cdot s \cdot \xi)$$

where the first term in the volume integral is defined by the variation of $L_m$:

$$\delta L_m = E_m \delta \psi + \frac{1}{2} T_{s_i} \delta s_i + d \Theta_m (\phi, \delta \phi).$$

Giving each $s_i$ field $u_i$ upper, and $d_i$ lower indices, in the manner

$$s_i \rightarrow s_i^{b_1 \cdots b_{u_i} a_1 \cdots a_{d_i}},$$

the second term in the volume integral is defined by

$$(\epsilon \cdot T_s \cdot s \cdot \xi)_{abc} \equiv \epsilon_{eabc} \sum_i [T_{s_i b_1 \cdots b_{u_i} a_1 \cdots a_{d_i}} (-s_i e \cdots b_{u_i} a_1 \cdots a_{d_i} \delta_{p}^{b_1} \cdots - s_i b_1 \cdots e a_1 \cdots a_{d_i} \delta_{p}^{b_{u_i}} + s_i b_1 \cdots b_{u_i} p \cdots a_{d_i} \delta_{a_1}^{e} \cdots + s_i b_1 \cdots b_{u_i} p \delta_{a_1}^{e} \cdots \delta_{a_1}^{e} \delta_{a_1}^{e} \xi_p)].$$

(57)

3 A review of perfect fluids, and three variational formulations.

In this section we recall the definition, the relevant properties, and three variational principles for a self-gravitating perfect fluid: one given by Schutz [14], (which we use in the Appendix to derive a conserved current for perturbations of Einstein-perfect fluid systems), the “axionic vorticity” formulation given by Carter [10] for an isentropic perfect fluid (which we use in the next section, to derive a first law), and a “convective” approach also described by Carter [10]. Our aim is to gather the results we need for the calculations of the following sections; detailed treatments of these variational principles can be found in [3, 10, 14, 15].

From the viewpoint of black hole mechanics, we would like a stationary axisymmetric black hole configuration to be represented by a Lagrangian theory in which all the fields appearing in the Lagrangian (the dynamical fields) are also stationary and axisymmetric. Having stated these formulations, however, we will see that they all have fluid configurations in which the physical fields (the fluid four velocity, number density, entropy and functions of these fields) are stationary and axisymmetric, but in which the dynamical fields possibly share neither of these symmetries. The question as to whether a variational principle exists that always represents (physically) stationary axisymmetric configurations with dynamical fields that also have these properties is (as far as we are aware) open.

By a perfect fluid on a fixed spacetime background [14, 15] we mean a system described by five scalar fields, $(n, s, \rho, p, T)$, on spacetime and one (unit, timelike) vector field $U^a$, such that $\rho = \rho(n, s)$ is a fixed function, and the following equations hold on the fields: the first law of thermodynamics,

$$d\rho(n, s) = \frac{(p + \rho)}{n}dn + nTds,$$

(58)
and the equations of motion,
\[ \nabla_a (nU^a) = 0, \quad \text{and} \quad \nabla_a T^{ab} = 0, \] 
where \( T^{ab} \) is defined by
\[ T^{ab} \equiv (p + \rho)U^aU^b + pg^{ab}. \]
The fields \( n, \rho, s, p, T \) and \( U^a \) have physical interpretations as the number density, energy density, entropy per particle (specific entropy), pressure, temperature, and four velocity of the fluid, respectively.

We note that (59) can be given a useful alternative form, by first defining the specific inertial mass \( \mu \):
\[ \mu \equiv \frac{p + \rho}{n} \]
which along with (58) implies
\[ dp = nd\mu - nTd\sigma. \]
By using these relations in the second equation of (59) we get (see [14]) an equivalent pair of equations of motion -
\[ \nabla_a (nU^a) = 0, \quad \text{and} \quad nU^a \omega_{ab} = nT \nabla_b \sigma, \]
where the fluid vorticity two-form, \( \omega_{ab} \), is defined as
\[ \omega_{ab} = 2\nabla_{[a}(\mu U_{b]}). \]
If desired, one can define the entropy per unit volume, \( S \) (entropy density), by \( S = ns \). Substituting this definition of \( S \) into (58) and defining the chemical potential, \( \mu' \), by
\[ \mu' \equiv \frac{p + \rho - TS}{n}, \]
then gives the relation
\[ d\rho(n, S) = \mu'dn + TdS. \]

We now specify three variational formulations for this perfect fluid, over a fixed spacetime background (coupling the theories to gravitation amounts to adding the appropriate metric Lagrangian, which we do later). First, we state the “velocity-potential” representation of Schutz [9]: here the dynamical fields of the fluid are given by scalars \( \Phi, \alpha, \beta, \theta, \) and \( \sigma \). One now defines a function \( m \) which depends on these fields via the relation
\[ m^2 = - (\nabla_a \Phi + \alpha \nabla_a \beta + \theta \nabla_a \sigma)(\nabla^a \Phi + \alpha \nabla^a \beta + \theta \nabla^a \sigma), \]
and the fluid Lagrangian is given by
\[ L_f \equiv \epsilon P(m, \sigma). \]
where \( P(m, \sigma) \) is some fixed function. One can verify [9, 14] that we recover (58), and also that the equations of motion for the fields \( \Phi, \alpha, \beta, \theta, \sigma \) arising from this Lagrangian reduce to (63), provided one defines the physical fields in terms of the dynamical fields in these equations by:
\[
\begin{align*}
P & \rightarrow p \\
m & \rightarrow \mu \\
\sigma & \rightarrow s \\
(\partial P/\partial m)_\sigma & \rightarrow n \\
(\partial P/\partial \sigma)_m & \rightarrow -nT \\
\nabla_a \Phi + \alpha \nabla_a \beta + \theta \nabla_a \sigma & \rightarrow \mu U_a.
\end{align*}
\]
Conversely, given any configuration of the physical fields \((n, \rho, s, p, T, U^a)\) satisfying (58) and (59), it can be shown (see [9]) that there exist functions \((P, m)\) and (non-unique) dynamical fields \((\Phi, \alpha, \beta, \theta, \sigma)\) related to the physical fields by (69), which satisfy the equations of motion arising from Lagrangian (68).

Next, Carter’s variational formulation [10] for an isentropic perfect fluid, (by which we mean that the fluid has an everywhere constant specific entropy \(s\)), defines the dynamical fields to be a two-form and two scalars, \(b_{ab}\) and \(\chi^\pm\). The fluid Lagrangian is given in terms of these fields by

\[
L_f = \left[ -r(\nu) - \frac{1}{2} \epsilon^{abcd} b_{ab} \nabla_c \chi^+ \nabla_d \chi^- \right] e. 
\]

where the function \(r(\nu)\) is fixed, and the function \(\nu\) is defined in term of the potentials \(b_{ab}\) by the relation

\[
\nu^2 \equiv \frac{3}{2} (\nabla_{[a} b_{bc]})(\nabla^{[a} b^{bc]}). 
\]

As shown in [10], if one defines the physical fields as follows:

\[
\begin{align*}
 r & \rightarrow \rho \\
 \nu & \rightarrow n \\
 \nu(\partial r/\partial \nu) - r & \rightarrow p, \\
 3\nabla_{[c} b_{ab]} & \rightarrow N_{abc},
\end{align*}
\]

where the number-density three-form \(N_{abc}\) is given by

\[
N_{abc} = n \epsilon_{abc} U^d, 
\]

then we recover (58), and the field equations for \(b_{ab}\) and \(\chi^\pm\) yield the second equation in (63) in the case \(\nabla_a s = 0\), as well as the relation

\[
\omega_{ab} = 2\nabla_{[a} \chi^+ \nabla_{b]} \chi^-. 
\]

Given relation (72) between \(b_{ab}\) and \(N_{abc}\), one sees that the first equation in (63) is satisfied vacuously, since it can be rewritten as

\[
\nabla_{[a} N_{bcd]} = 0, 
\]

but the definition of \(N_{abc}\) shows \(dN = dd b = 0\) automatically.

A third type of variational formulation given by Carter [10], and treated in more detail by Brown [14], (which is the equivalent diffeomorphism invariant version of the formalisms specified by Taub [16], or Hawking and Ellis [17]), has dynamical fields \(X^A\) for \(A = 1, 2, 3\). In this formalism one must specify two functions- \(r(\nu, \sigma)\), and \(\sigma(X)\), where \(\nu\) is defined in terms of the \(X^A\) by

\[
\nu^2 \equiv 6[N_{ABC}(X) \nabla_a X^A \nabla_b X^B \nabla_c X^C][N_{DEF}(X) \nabla^a X^D \nabla^b X^E \nabla^c X^F],
\]

and \(N_{ABC}(X)\) is a fixed three-form on the three-dimensional manifold which has \(X^A\) as coordinate fields. The Lagrangian is then given by

\[
L_f = -\epsilon r(\nu, \sigma). 
\]
The equations resulting from this Lagrangian for the fields \( X^A \) are seen to reduce to the second equation in (63) after one has set
\[
\begin{align*}
 r & \rightarrow \rho \\
 \nu & \rightarrow \nu \\
 \sigma & \rightarrow s
\end{align*}
\]
\[
\nu(\partial r/\partial \nu)_{\sigma} - r \rightarrow p \\
(\partial r/\partial \sigma)_{m} \rightarrow nT
\]
\[
N_{ABC}(X)\nabla_a X^A \nabla_b X^B \nabla_c X^C \rightarrow N_{abc},
\] (78)
where \( N_{abc} \) is defined from (73). (This relation between the physical \( N_{abc} \) and the dynamical fields also ensures that \( N_{abc} \) is automatically conserved.) The \( X^A \) are interpreted as coordinates on a “base manifold”, obtained by treating the spacetime as a bundle with fibres given by the integral curves of the four-velocity. We will not use this formulation for two reasons: firstly, the assignment of the entropy, \( s \), as a fixed function of the \( X^A \) only allows us to perturb it by diffeomorphisms of the base manifold (for this reason we use Schutz’s formalism for the calculation in the Appendix). Secondly, it is unclear that there are any solutions in which the \( X^A \) are globally well-defined axisymmetric fields on spacetime (for this reason, in section 4, we use the formulation due to Carter with Lagrangian (70)).

In order to write the first law in form (2), only involving surface integrals, we must assume that all the dynamical fields are stationary and axisymmetric in the background solution. Now even if a fluid configuration has stationary and axisymmetric physical fields (the fluid number density, entropy and functions of these fields), the dynamical fields (the fields appearing in the Lagrangian) corresponding to these physical fields may not possess these symmetries. Therefore, the requirement of stationarity and axisymmetry on the dynamical fields may restrict the choice of background configurations. In fact, for Schutz’s formulation, we see from the definition of the four velocity (69) that physical fluid configurations with an everywhere causal four-velocity (including those which are stationary and axisymmetric) must include at least one nonstationary dynamical field. There are therefore no physically interesting fluid configurations in which all the dynamical fields in this formulation are stationary.

On the other hand, for Carter’s formulation, it is evident that there must be some physically stationary fluid configurations with stationary dynamical fields; (for instance, a static spherically symmetric fluid distribution could have the field \( b_{ab} \) given by \( b \sim f(r)^2 \epsilon \) and \( \chi^\pm = 0 \), where \( \epsilon \) is the volume element on the spheres of symmetry). However, we will see in the next section (in the discussion above (95)) that a stationary, axisymmetric, circular flow (in a spacetime which also has these symmetries) must be vortex-free, if \( \chi^\pm \) are restricted to be stationary and axisymmetric. That is, the assumption of stationarity and axisymmetry on the vorticity potentials \( \chi^\pm \) restricts the allowed stationary axisymmetric configurations a fluid can adopt. We make no attempt here to enumerate the set of physically stationary and axisymmetric configurations which also have these symmetries in the dynamical fields (or indeed, in the case of black hole spacetimes, to investigate whether this set is non-empty). Rather, in the following section we will assume the potentials are stationary and axisymmetric, and write out the resulting first law involving only surface terms, looking for any non-trivial modifications arising from the fluid fields.

We are unaware of a variational formulation for a perfect fluid which represents all stationary axisymmetric fluid configurations with stationary axisymmetric dynamical fields. If it exists, then the following argument by Schutz and Sorkin shows that certain compactly supported perturbations of
the physical fields must correspond to non-compactly supported perturbations of the dynamical fields. Since the calculation given in (30) does not depend on the fulfillment of the field equations for $g_{ab}$, it is still valid if we consider the fields $\psi$ to be the dynamical fields for a perfect fluid over a fixed spacetime background, and we let $\delta\psi$ be a perturbation to a nearby solution of the perfect fluid equations, with $\delta g_{ab} = 0$. Now consider a formulation for a perfect fluid where, for a general configuration in which all the physical fields (and the metric of the spacetime background) are stationary, all the dynamical fields are also stationary. Then the left side of (30) vanishes, and integrating the right side over a spatial slice $\Sigma$, we are left with

$$\int_{\Sigma} \delta (\epsilon \cdot T \cdot \xi) = \int_{\partial \Sigma} \delta Q_m[\xi] - \xi \cdot \Theta_m(\phi, \delta \psi).$$

(79)

This implies that for perturbations of the physical fields for which the corresponding perturbations of the dynamical fields are compact, we must have

$$\int_{\Sigma} \delta (\epsilon \cdot T \cdot \xi) = 0,$$

(80)

which, for a perfect fluid, is clearly false for a general stationary background. This implies that if a variational formulation is to have dynamical fields which are always stationary when the physical fields are stationary, then perturbations of the physical fields which yield a non-zero result on the left side of (79) must correspond to spatially non-compact perturbations of the dynamical fields. This requirement rules out the existence of a variational principle in which the physical fields are the dynamical fields. However, the existence of a variational principle for a perfect fluid in which all configurations with stationary and axisymmetric physical fields are represented by dynamical fields with these symmetries is still an open question.

4 First laws of black hole mechanics with perfect fluids

We now present two forms of the first law of black hole mechanics which incorporate perfect fluids. The first form is a special case of the perturbative identity (50), where $L_g$ is the usual Hilbert Lagrangian for general relativity, and $L_m$ is any Lagrangian for a perfect fluid. This form of the first law allows non-stationary dynamical fields, at the cost of having volume integrals in the interior of the spacetime. We then compute a second form of the first law only involving surface integrals for both metric and fluid fields, using Carter’s variational formulation presented above, and the methods of [4].

4.1 The first law with volume integrals

We now write out the perturbative relation (50), setting $L_g = 1/16\pi R$, and $L_m$ to be any perfect fluid Lagrangian which allows all possible perturbations of the physical fields of the perfect fluid off an arbitrary background. (From the comments below Eq. (69) it is evident that Schutz’s variational formulation, with Lagrangian (68) satisfies this criterion.) As stated in Lemma 2, we assume the metric of the background spacetime is asymptotically flat, stationary and axisymmetric with a stationary killing field $\xi^a$ and axial killing field $\phi^a$. We also assume the existence of a bifurcate killing horizon, with horizon killing field $\chi^a = \xi^a + \Omega_H \varphi^a$, where $\Omega_H$ is the angular velocity of the horizon.
Lemma 3: Let \( L \), given by
\[
L = L_g - e(r) + \frac{1}{2} \epsilon^{abcd} b_{ab} \nabla_{[c} \chi^+ \nabla_{d]} \chi^-,
\]

When all these substitutions are inserted into (50), it reduces to
\[
\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J_H - \int \mu' |v| \delta N_{abc} + \int \Omega \delta J_{abc} + \int T |v| \delta S_{abc},
\]
which is identical to (1), except that \( \delta \) now represents an arbitrary perturbation (not necessarily stationary or axisymmetric) of the background. In this sense, (82) is a generalisation of (1).

### 4.2 A (restricted) first law with surface integrals

In the previous section we observed that the variational formulations we presented were constrained in the stationary axisymmetric fluid configurations they could represent, given the requirement that their dynamical fields obeyed these symmetries. One might therefore suspect that any form of the first law involving only surface integrals could not include non-trivial fluid contributions. Indeed, if we add Schutz’s Lagrangian (68) to the Lagrangian of an arbitrary metric theory of gravity, and construct a first law using the analysis of (1) then we find no additional contributions to this first law from the fluid fields, providing the fluid’s number density decays sufficiently rapidly at spatial infinity, and does not intersect the black hole horizon. It is possible, however, to convert some of the volume integrals in (1) into surface integrals, by choosing Carter’s variational formulation (7). We do so below, finding a first law for an arbitrary metric theory of gravity coupled to an isentropic perfect fluid, in which the background configuration for the perfect fluid as well as the allowed perturbations of the physical fields are restricted. (Note that the gravitational contributions to such a first law have been considered in detail in (1). We are interested in the fluid contributions.) We finally verify that this first law reduces to (1) when the assumptions made in the two derivations overlap. Our first law is the following result:

\[
\delta(T_a^b \xi^b \epsilon_{apqr}) = v^b \delta T_a^b \epsilon_{apqr} - \Omega \delta(T_a^b \epsilon_{apqr})
\]
\[
= v^b (\mu' n + TS) v_a (-v \cdot v)^{-1/2} U^b \epsilon_{apqr} + p \epsilon_{apqr} - \Omega \delta(T_a^b \epsilon_{apqr})
\]
\[
= (p + \rho) v^a \delta(v_a (-v \cdot v)^{-1/2}) U^b \epsilon_{apqr} + \frac{1}{2} \rho g^{cd} \delta g_{cd} \xi^a \epsilon_{apqr}
\]
\[
- \mu' (-v \cdot v)^{1/2} \delta(n U^b \epsilon_{apqr}) - T(v \cdot v)^{1/2} \delta(S U^b \epsilon_{apqr})
\]
\[
- (n \mu'_* + S \delta T) \xi^a \epsilon_{apqr} + v^a \epsilon_{apqr} \delta p - \Omega \delta(T^b_{a \nu} \epsilon_{apqr})
\]
\[
= \xi^a \epsilon_{apqr} \frac{1}{2} T^{cd} \delta g_{cd} + \mu' (-v \cdot v)^{1/2} \delta(n U^b \epsilon_{apqr}) - T(v \cdot v)^{1/2} \delta(S U^b \epsilon_{apqr})
\]
\[
- \Omega \delta(T^b_{a \nu} \epsilon_{apqr}),
\]

(81)
be the Lagrangian for an isentropic perfect fluid coupled to an arbitrary metric theory of gravity, where \( L_g = \varepsilon L_g(g_{ab}, R_{abcd}, \nabla R_{abcd}, \ldots, (\nabla^n)R_{abcd}) \), and the perfect fluid formulation, with dynamical fields \((b_{ab}, \chi^\pm)\), is summarised below (70). Fix an asymptotically flat black hole solution with bifurcate Killing horizon, with the spacetime structure and the Killing fields described in Lemma (2), with the additional assumptions that all the dynamical fields, (not just the metric) in this theory are stationary and axisymmetric, and that all the dynamical fields are globally defined. Let \( \delta \phi \) be a perturbation of the dynamical fields, from such a solution to an arbitrary nearby solution, with \( \delta \xi^a = 0 \). With these assumptions the following identity is the first law of black hole mechanics for this system:

\[
\delta M_g = \frac{\kappa}{2\pi} \delta S + \Omega_H \delta J_H + \int_H \mu_\infty \delta b_{qr} - \int S_\infty \mu_\infty \delta b_{qr} + \int X_{qr} - \int S_\infty X_{qr} \tag{84}
\]

Here we define the mass of the system, \( M_g \), as

\[
M_g \equiv \int_{S_\infty} Q_g[\xi^a] - \xi \cdot B_g
\]

and the entropy, \( S \), and angular momentum, \( J_H \), of the system by

\[
S \equiv -2\pi \int_H \frac{\delta L_g}{\delta g_{ab}} \epsilon_{ab} \epsilon_{cd}
\]

\[
J_H \equiv -\int_H Q_g[\varphi], \tag{85}
\]

where \( \kappa \) is the surface gravity of the black hole, the two-form \( Q_g[\xi] \) was defined in (26), and the three-form \( B_g \) is such that, at spatial infinity, \( \delta (\xi \cdot B_g) = \xi \cdot \Theta_g \), with \( \Theta_g \) given by (23). Finally, the two-form \( X_{qr} \) is defined by

\[
X_{qr} \equiv 2\xi^p b_{pq}[\delta (\mu U_r)] - \nabla_r \chi^- \delta \chi^+ + \nabla_r \chi^+ \delta \chi^- \tag{86}
\]

Proof:
The first law of black hole mechanics in \cite{4} is essentially given by the right side of (18), when the left side vanishes because of the assumed symmetries of the background fields. We therefore compute the quantities appearing in the right side of (18): Varying the dynamical fields in \( L \) (and performing the substitutions (72) where applicable) yields the equations of motion and the symplectic potential \( \Theta \):

\[
\delta L = \varepsilon \left( \frac{\delta L_g}{\delta g_{ab}} + \frac{1}{2} T^{ab} \right) \delta g_{ab} + \frac{1}{6} \varepsilon N_{abc} \epsilon^{abcd} (\nabla d \chi^+ \delta \chi^- - \nabla d \chi^- \delta \chi^+) + \varepsilon (\nabla c \left( \frac{\mu}{2n} N^{abc} \right) - \frac{1}{4} \epsilon^{abcd} \omega_{cd}) \delta b_{ab} + d \Theta, \tag{87}
\]

with the stress-energy tensor

\[
T^{ab} = \frac{\mu}{2n} N^{a\overline{cd}} N^{b\overline{cd}} - \rho g^{ab}, \tag{88}
\]

and the symplectic potential

\[
\Theta_{pqr} (\phi, \delta \phi) = \Theta_{g_{pqr}} (g, \delta g) - \frac{\mu}{2n} N^{abc} \delta b_{bc} \epsilon_{apqr} + \frac{1}{2} \epsilon_{apqr} b_{cd} \epsilon^{abcd} (\nabla b \chi^+ \delta \chi^- - \nabla b \chi^- \delta \chi^+). \tag{89}
\]

It can be verified that the equations of motion for the fluid fields reduce to (83) using the definitions (72, 73). The stress-energy tensor (89) is also seen to reduce to the usual form (18) by expanding its first term:

\[
\frac{\mu}{2n} N^{a\overline{cd}} N^{b\overline{cd}} = \frac{\mu}{2n} n U^e \epsilon^{a\overline{cde}} \epsilon^{bf} n U_f = \mu n (g^{ab} + U^a U^b). \tag{90}
\]
The Noether current associated to \( \xi^a \) is
\[
J_{pqr}[\xi] = J_{pqr}[\xi] - \left( \frac{\mu}{2n} N^{def} N_{efc} \xi^e - \rho \xi^d \right) \epsilon_{dpqr} - \nabla_b \left( \frac{\mu}{2n} N^{def} \xi^e b_{ec} \epsilon_{dpqr} \right).
\] (91)

Therefore, the integrand on the right side of (18) evaluates to
\[
(\delta Q[\xi] - \xi \cdot \Theta)_{qr} = \delta Q_{gqr}[\xi] - \delta \left( \frac{\mu}{2n} N^{abc} b_{ec} \xi^e \epsilon_{abqr} \right) - \xi^p \Theta_{gqr} - \frac{\mu}{2n} N^{abc} \delta b_{ec} \epsilon_{apqr} \\
+ \epsilon_{apqr} \frac{1}{2} \epsilon^{abcd}\partial_b(\nabla_b \chi^d - \nabla_b \chi^+ \delta \chi^-) \\
= \delta Q_{gqr}[\xi] - \xi^p \Theta_{gqr} - \xi^p U_{p\mu} \delta b_{qr} \\
- \frac{1}{2} b_{qr} \xi^p (\nabla_p \chi^- \delta \chi^+ - \nabla_p \chi^+ \delta \chi^-) + X_{qr}
\] (92)
where we define the two-form \( X_{qr} \) by (87), and we used the identification \( N_{abc} \equiv \epsilon_{abcd} n U^d \) to obtain the second line of (92).

When the background solution is a black hole with the structure and symmetries specified in the statement of the Lemma, the fourth term in the second equation of (92) vanishes because the dynamical fields are stationary: \( \xi \cdot \nabla \chi^\pm = 0 \). Now given the definition of vorticity (64) and its relation to the potentials (74), it is evident that (locally) there exists some function \( f \) such that \( \mu U_a \) can be rewritten
\[
\mu U_a = \nabla_a f + \chi^+ \nabla_a \chi^-.
\] (93)

Let \( t \) be a function such that \( \xi^a dt_a = 1 \). Then the requirements that the four-velocity be causal, stationary and axisymmetric, along with the assumed stationarity and axisymmetry of \( \chi^\pm \) force \( f \) to be a sum of terms, one of which is strictly linear in \( t \) (we define the constant of proportionality to be \( -\mu_\infty \)). For the same reason the \( \varphi \)-dependence of \( f \) must be also linear, but this dependence can be ruled out because the occurrence of such a term would force \( U^a \) to be acausal near spatial infinity. We therefore have that the form of \( f \) is
\[
f = -\mu_\infty t + g.
\] (94)

where \( \xi \cdot \nabla g = \varphi \cdot \nabla g = 0 \). Therefore we see that the assumption of stationarity and axisymmetry on the dynamical fields (taking the four-velocity to be everywhere causal) has restricted us to a very narrow range of allowed background four-velocities; for instance, we must have \( \varphi^a U_a = 0 \). Moreover, when the vacuum theory is general relativity, with the flow assumed to be circular (tangent to the \( \xi - \varphi \) subspaces), there is only one possible solution: for this theory the subspaces orthogonal to \( \xi^a \) and \( \varphi^a \) are integrable, and the resulting submanifolds can be endowed with coordinates \( (x^1, x^2) \), such that the metric is “block diagonal” with no “cross-terms” between the subspace spanned by \( \xi^a, \varphi^a \) and its orthogonal complement (see Chapter 7 of [3]). Now the assumption of circular flow forces \( g = 0 \) and \( \chi^+ d\chi^- = 0 \), leaving us with only
\[
\mu U_a = -\mu_\infty dt_a.
\] (95)

In any case, using just the form of \( f \) in (94), we see
\[
\xi^a \mu U_a = -\xi^a \mu_\infty dt_a = -\mu_\infty,
\] (96)
and the boundary term (92) reduces to
\[
\delta Q_{qrf}[\xi] - \xi^p \Theta_{pqr} = \delta Q_{gqr}[\xi] - \xi^p \Theta_{gqr} + \mu_\infty \delta b_{qr} + X_{qr},
\] (97)
We now assume the existence of a form \( B_g \) such that at spatial infinity \( \xi \cdot \Theta_g = \delta(\xi \cdot B_g) \), and write out the first law of black hole mechanics by substituting (97) into the surface integrals on the right side of (18), observing that the left side of (18) vanishes due to the symmetries assumed on the dynamical fields. If we expand \( \xi^a = \chi^a - \Omega_H \varphi^a \) at the bifurcation sphere for the first two terms of (97), then we obtain (84) which is what we wished to show. ✷

The results of [4] predicted that the first law (84) would only contain surface integrals, and we see this is indeed the case. Note, however, that the assumptions made about the symmetry of the dynamical fields restricted the allowed background fluid configurations for the fluid fields. Moreover, by perturbing the local form of \( \mu U_a \) in (93) we see that the restriction to stationary and axisymmetric \( \chi^\pm \) in background also prevents us from achieving all possible perturbations of \( \mu U_a \), by perturbing only the dynamical fields \( b_{ab} \) and \( \chi^\pm \). Finally both the background and the perturbed configurations must be restricted such that the integral \( \int_{\Sigma} \mu v |\delta N|_{abc} \) converges. (This, along with the following result relating this term to the fluid angular momentum will guarantee the convergence of the corresponding boundary term at the bifurcation sphere).

We finally show that (84) reduces to (1) when the assumptions made in the two derivations overlap. From our discussion in the last section we know that \( M_g, S \) and \( J_H \) reduce to their values for general relativity given in (1), when \( L_g = (1/16\pi)R \). We start by considering the fluid contribution in our first law (84) from the integral

\[
\delta \int_{\Sigma} \mu v |\delta N|_{abc} = \delta \int_{\Sigma} \mu v |\delta N|_{pqr},
\]

where the last line follows because the fluid flow in [1] is assumed to be tangent to the subspaces spanned by \( \xi^a \) and \( \varphi^a \): so taking the velocity to be \( U^a = v^a/|v| \) where \( v^a = \xi^a + \Omega H \varphi^a \), we see from the discussion above (95) that \( \mu = -\mu U \cdot U = \mu_{\infty} v \cdot dt/|v| = \mu_{\infty}/|v| \), and so \( \mu |v| = \mu_{\infty} \). Our first law now takes the form

\[
\delta M = \frac{\kappa}{8\pi} \delta A + \Omega H \delta J_H - \int_{\Sigma} \mu |v| |\delta N|_{abc} + \int_{\Sigma} X_{q\bar{r}} - \int_{S^\infty} X_{q\bar{r}} \tag{99}
\]

We now concentrate on the original form of the first law in (1) and show that it agrees with (99). By repeating the calculation (81) using the relation (58) instead of (66) along with the assumption \( \delta s = 0 \) (as befits an isentropic fluid), we find the form of (1) for an isentropic fluid -

\[
\delta M = \frac{\kappa}{8\pi} \delta A + \Omega H \delta J_H - \int_{\Sigma} \mu |v| |\delta N|_{abc} + \int_{\Sigma} \Omega \delta J_{abc}. \tag{100}
\]

Next, we demonstrate that the pullback to \( \Sigma \) of the angular momentum density given in (100) reduces to the exterior derivative of the two form \( X_{q\bar{r}} \) defined in (87), given the assumption that the dynamical fields are stationary and axisymmetric, i.e.,

\[
\Omega \delta J_{pqr} = -(dX)_{pqr}, \tag{101}
\]

where both sides are assumed pulled back to \( \Sigma \). To do this we compute the exterior derivative of (87), finding

\[
(dX)_{pqr} = 3\xi^e N_{[p} \delta(\mu_{U^r]) - \nabla^r \delta \chi^+ \nabla^r \chi^+ \delta \chi^-), \tag{102}
\]
where we have assumed $\mathcal{L}_\xi b_{ab} = 0$. Pulling this form back to $\Sigma$ by contracting with $1/6 \epsilon^{spqr} n_s$ (where $n_s$ is the unit normal to $\Sigma$) yields

$$dX = -3 \epsilon (2 nn_e \delta(\mu U_r) U^r/|v|) + 2 \epsilon (\nabla U^r) n_e (\nabla r \chi^- \delta \chi^+ - \nabla r \chi^+ \delta \chi^-),$$

(103)

where $3 \epsilon$ is the volume form induced on $\Sigma$: $3 \epsilon^{bcd} \equiv n^a \epsilon_{abcd}$. Now using the axisymmetry of the $\chi^{\pm}$, and writing $U^a$ as $U^a = v^a/|v|$ with angular velocity $\Omega$ as given above in (99), we have (using $\delta \xi^a = \delta \phi^a = 0$)

$$dX = -3 \epsilon 2 nn_e \Omega \delta(\mu U_r) \varphi^r/|v|$$

$$= 3 \epsilon (p + \rho) \Omega (n_e \xi^e) \delta(\mu U_r) \varphi^r/|v|$$

$$= -3 \epsilon \Omega (p + \rho) \delta(\mu U_r) \varphi^r$$

$$= \Omega \delta(3 \epsilon n_a T^a b \varphi^b)$$

$$= -\Omega \delta J$$

(104)

where $J$ is the pullback of $J_{abc}$ to $\Sigma$. Therefore (99) now matches (100) and so the first law (84) now agrees with the first law given in (1).

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### Appendix: The Chandrasekhar-Ferrari conserved current

The symplectic form $\omega(\phi, \delta_1 \phi, \delta_2 \phi)$ defined in (11) is closed when $\delta_{1,2} \phi$ satisfy the linearised equations. Its dual $\omega^a(\phi, \delta_1 \phi, \delta_2 \phi)$, defined by $\omega_{abc} = \omega^d \epsilon_{dabc}$, is therefore a covariantly conserved current for the Einstein-perfect fluid system. Chandrasekhar and Ferrari [13] have, from first principles, also derived a conserved current, $\mathcal{E}^a(\phi, \delta \phi)$, for the Einstein-perfect fluid system. Their current is quadratic in the (complex) perturbations $\delta \phi$, and is restricted to the case where $\phi$ is a static axisymmetric solution, and $\delta \phi$ is a “polar” (even parity) perturbation with harmonic time dependence (we will define this below). We now show the equivalence of the $\omega^a(\phi, \delta \phi, \delta \phi^*)$ and $\mathcal{E}^a$ for the Einstein-perfect fluid system. This calculation is the analogue for the Einstein-perfect fluid system of the calculation by Burnett and Wald [12] for the Einstein-Maxwell system.

We start by choosing the Lagrangian for the Einstein-perfect fluid system to be

$$L_{pqrs} = \epsilon_{pqrs} [-\frac{1}{4} R + P(m, \sigma)],$$

(105)

where we have set the constant in front of the Ricci scalar to give the field equations in [13], and used Schutz’s velocity-potential representation, with Lagrangian (68). The symplectic potential, $\Theta$, arising
from this Lagrangian is (after substituting \((69)\) where applicable),

\[
\Theta_{pqr} = -\frac{1}{4} \epsilon_{apqr} (\nabla^b \gamma^a - \nabla^a \gamma) - nU^a \epsilon_{apqr} (\delta \Phi + \alpha \delta \beta + \theta \delta s),
\]

where \(\gamma_{ab} \equiv \delta g_{ab}\) and \(\gamma \equiv g^{ab} \gamma_{ab}\). The resulting presymplectic form is (from \([11]\))

\[
\omega_{pqr}(\phi, \delta_1 \phi, \delta_2 \phi) = \frac{1}{8} \epsilon_{apqr} [(\gamma_{2}^{cd} - g^{cd} \gamma_{2}) \nabla^a \gamma_{1cd} - (2 \gamma_{2}^{cd} - \gamma_{2} g^{cd}) \nabla c \gamma_{1d}^a + \gamma_{2}^{cd} \nabla d \gamma_{1}^a] - \delta_2 (\epsilon_{apqr} nU^a) (\delta_1 \Phi + \alpha \delta_1 \beta + \theta \delta_1 s) - \epsilon_{apqr} nU^a (\delta_2 \alpha \delta_1 \beta + \delta_2 \theta \delta_1 s) - (1 \leftrightarrow 2).
\]

This form is dual to a generally conserved current: it can be shown \([11]\) that for \(\omega^a\) defined above, we have (for perturbations \(\delta_1 \phi\) and \(\delta_2 \phi\) satisfying the linearised field equations),

\[
\nabla_a \omega^a = 0.
\]

We now relate this conserved current to the current presented in \([13]\), by fixing a coordinate system with derivative operator \(\partial_a\), and writing the volume element \(\epsilon\) in terms of the coordinate volume element \(\epsilon\) of this system –

\[
\epsilon_{abcd} = \sqrt{-g} \epsilon_{abcd}
\]

– then the vector field \(w^a\) defined by \(\omega_{pqr} = w^a \epsilon_{apqr}\) is conserved in the sense \(\partial_a w^a = 0\). If we follow Chandrasekhar and Ferrari \([13]\) and specialise to the case where the background spacetime is static (with static killing field \(t^a\)) and axisymmetric (with axial killing field \(\varphi^a\)), and the perturbations are time and angle-dependent only “harmonically” (that is, there are constants \(\sigma\) and \(\omega\) such that

\[
\mathcal{L}_t \delta \eta = i \sigma \delta \eta
\]

\[
\mathcal{L}_\omega \delta \eta = i \omega \delta \eta,
\]

for all the dynamical fields \(\eta\) then (following \([12]\)) it’s easy to see that for complex \(\delta \phi\), \(w^a(\phi, \delta \phi, \delta \phi^*)\) and \(w^3(\phi, \delta \phi, \delta \phi^*)\) are independently conserved: \(\partial_a w^a = 0\). We can therefore restrict our attention to the vector components \((w^2, w^3)\). Moreover, \((110)\) allows us to substitute the variations of the fluid potentials \(\delta(\Phi, \beta, s)\) for variations of their time derivatives: we do this and (recalling \((69)\)) find

\[
w^a = w^a_{gr} - \delta_2 (\sqrt{-g} nU^a) \frac{1}{i \sigma} \delta_1 (\mu t b U_b) - \sqrt{-g} \frac{1}{i \sigma} nU^a [\delta_2 (t \cdot \nabla \alpha) \delta_1 (t \cdot \nabla \beta) + \delta_2 (t \cdot \nabla \theta) \delta_1 (t \cdot \nabla s)] - (1 \leftrightarrow 2),
\]

where we labelled the contribution from the first two lines of \((107)\) by \(w^a_{gr}\).

Our aim is now to show the equality of \((w^2(\phi, \delta \phi, \delta \phi^*), w^3(\phi, \delta \phi, \delta \phi^*))\) and \((E^2, E^3)\). To do this we first specialise the background and perturbations in \(w^a\) to those used by Chandrasekhar and Ferrari. In the coordinates given in \([13]\) the metric is written

\[
g_{ab} = -e^{2\nu} dt_a dt_b + e^{2\psi} d\varphi_a d\varphi_b + e^{2\mu_2} dx_a^2 dx_b^2 + e^{2\mu_3} dx_3^2 dx_3^2,
\]

and the nonvanishing (polar) metric perturbations are taken to be

\[
\begin{align*}
\gamma_{1tt} &= -2e^{2\nu} \delta \nu \\
\gamma_{1\varphi \varphi} &= 2e^{2\psi} \delta \psi \\
\gamma_{122} &= 2e^{2\mu_2} \delta \mu_2 \\
\gamma_{133} &= 2e^{2\mu_3} \delta \mu_3,
\end{align*}
\]
We then set $\gamma_{2ab} = \gamma^{*}_{ab}$, where the perturbed functions, $\delta\nu$, etc. are complex, but the unperturbed functions are real.

A direct substitution of these perturbations into $w_{gr}^{2}$ yields the result already known from (113):

$$w_{gr}^{2} = -\frac{1}{2} e^{\nu + \psi + \mu_{2} + \mu_{3}} [\delta \nu_{2, \delta} (\psi + \mu_{3})^* + \delta \mu_{3, \delta} (\nu + \psi)^* + \nu_{2, \delta} (\psi + \mu_{3})^* \delta (\nu - \mu_{2}) + \psi_{2, \delta} (\nu + \mu_{3})^* \delta (\psi - \mu_{2}) + \delta (\mu_{3} - \mu_{2})] - c.c. \quad (114)$$

We now turn to the fluid contributions $w_{f}^{2}$ to the conserved current, defined by $w_{f}^{2} = w^{a} - w_{gr}^{a}$. We set the four-velocity of the background to be $U^{a} = e^{-\nu} t^{a}$ and (following (13)) and denote the perturbations of the orthonormal frame components of $U^{a}$ by $i\sigma \xi^{a}$: $\delta U_{a} = i\sigma \xi_{a}$. We then find the ‘2’-component of $w_{f}^{2}$ given by

$$w_{f}^{2} = -dx_{a}^{2} \frac{1}{i\sigma} \delta_{2}(e^{\nu + \psi + \mu_{2} + \mu_{3}} nU_{0}) \delta_{1}(\mu t\xi_{e}) - (1 \leftrightarrow 2) = \frac{1}{i\sigma} e^{2\nu + \psi + \mu_{2}} \delta_{2}(nU_{2})(\delta_{1} \mu + \mu \delta_{1} \nu - \mu \delta_{1} U_{0}) - (1 \leftrightarrow 2) = -e^{2\nu + \psi + \mu_{2}} (n \delta \mu + n \mu \delta \nu) \xi^{2} - c.c., \quad (115)$$

where we have set $\delta_{1} = \delta$ and $\delta_{2} = \delta^{*}$, and used the result (see (13)) that $\delta U_{0} = 0$. We can also put $n \delta \mu = \delta p + nT \delta s$, and bearing in mind that $\delta s$ must also have harmonic time dependence, we can write

$$i\sigma n T \delta s = T \delta (U \cdot \nabla s) - \delta (U \cdot \nabla s). \quad (116)$$

Referring to (13) we see that the first term on the right side of (116) vanishes whenever the perturbation satisfies the linearised equations. Since the background is vortex-free, we see that the second term also vanishes as a consequence of (13). Adding the resulting fluid contribution to the gravitational terms (114) yields

$$w^{2} = -\frac{1}{2} e^{\nu + \psi - \mu_{2} - \mu_{3}} [\delta \nu_{2, \delta} (\psi + \mu_{3})^* + \delta \mu_{3, \delta} (\nu + \psi)^* + \nu_{2, \delta} (\psi + \mu_{3})^* \delta (\nu - \mu_{2}) + \psi_{2, \delta} (\nu + \mu_{3})^* \delta (\psi - \mu_{2}) + \delta (\mu_{3} - \mu_{2})] - e^{2\nu + \psi + \mu_{3}} (\delta p + (p + \rho) \delta \nu) \xi^{2} - c.c. \quad (117)$$

Now using the appropriate linear combinations of the linearised Einstein constraint –

$$\delta (\psi + \mu_{3} + \nu_{2}) + \psi_{2, \delta} (\psi + \mu_{3} + \nu_{2}) + \delta (\mu_{3} - \mu_{2}) - \nu_{2, \delta} (\psi + \mu_{3} + \nu_{2}) = 2 e^{\nu + \mu_{2}} (p + \rho) \xi^{2} \quad (118)$$
- to replace the third, fourth and fifth lines of (117) we get

\[ w^2 = -\frac{1}{2}e^{\psi+\mu_3} \{ \delta \nu_2 \delta (\psi + \mu_3) + \delta \mu^* (\delta \psi + \delta \mu_3) \cdot 2 
- [\delta \psi, \delta \psi^*] \cdot 2 - [\delta \mu_3, \delta \mu_3^*] \cdot 2 
+ 2e^{\mu+\mu_2} (5 \psi + \mu_3 - \mu_2)^* - \delta \rho)^\xi_2 \} - c.c. \] (119)

where, we define \([A,A^*]_i \equiv A_A A^* - AA^*_i\). This is seen to agree (up to an overall constant) with \(E^2\) of the conserved current in \([13]\). A similar calculation for \(w^3\) yields \(E^3\) (which is obtained from \(E^2\) by interchanging \(2 \leftrightarrow 3\)), and so we find \((w^2,w^3) = (E^2,E^3)\), and our symplectic current \(w^a\) for the Einstein-perfect fluid system agrees with the Chandrasekhar-Ferrari current for this system.

We make two final comments. Firstly, from the comment following Eq.(69), we know that every configuration of the physical fields of a perfect fluid has a corresponding equivalence class of configurations of the dynamical fields, and as a consequence, every perturbation of the physical fields has a corresponding perturbation of the dynamical fields. Now, two distinct perturbations of the physical fields off the same background (physical field) configuration will each select a corresponding perturbation of the dynamical fields. The background dynamical field configuration for each of these perturbations will certainly lie within the equivalence class corresponding to the given background physical field configuration: however, in general, these background dynamical field configurations will be distinct elements of this equivalence class. In using symplectic methods to derive \(E^a\) we have implicitly restricted ourselves to those pairs of perturbations of the physical fields where the corresponding pairs of dynamical field perturbations \((\delta_1 \phi, \delta_2 \phi)\) have identical background configurations. In fact, as we have seen above, the resulting conserved current agrees with the Chandrasekhar-Ferrari current for all pairs of perturbations of the physical fields, not just those restricted in this way.

Secondly, we notice from (111) that as long as the \(U^a\) of the background solution lies in a plane tangent to the subspace spanned by \(t^a\) and \(\phi^a\), the last term in (111) vanishes for the components of interest. This in turn yields a conserved current \((w^2,w^3)\) which only depends on perturbations of the physical fields, without the explicit appearance of the fluid potentials, for any stationary background configuration in which the fluid velocity is tangent to the \(t-\phi\) subspaces. Of course, we know that \(\omega^a\) is a conserved current off any background; this observation suggests only that a current similar in style to that presented by Chandrasekhar and Ferrari also exists for a background with a fluid in circular motion, as well as the static case considered in [13].

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