Meson correlation functions are studied in the model with four-fermion interaction Lagrangian. We demonstrate that despite the singular character of system mean energy and corresponding quark condensate found, the meson observables are finite, quite well identified and compatible with experimental energy scale. It allows the similar model Hamiltonians to be used for describing the nonequilibrium features of quark/hadron systems which reveal themselves in studying ultrarelativistic heavy ion collisions. The analytical results for meson correlation functions in the Keldysh model are given.

The intensive experimental study of ultrarelativistic heavy ion collisions pushes forward a great interest in developing theoretical and phenomenological description of nonequilibrium processes in quark/hadron matter. This stage preceding a thermalization and chemical equilibration is of crucial importance in governing the collision process development but very complicated for reliable theoretical interpretation. The analysis of experimental data available leads to the conclusion that at the initial moment of clashing, system with great number of degrees of freedom appears and its constituents are strongly interacting. The characteristic time of nonequilibrium stage at the RHIC experiments is roughly estimated, for example, as 1 fm/c [1] and the energy density reached exceeds 15 GeV/fm$^3$ which is much higher than the corresponding quantity for the nuclear matter and its value for the bag model. Clearly, such estimates urge (and allow) to speculate on the description of processes at the quark level with the mechanism of dynamical mass generation included.

In such a context the study of equilibrated states and phase diagram of quark/hadron matter is based on the Nambu–Jona-Lasinio model (NJL) [2] in which the adequate picture of spontaneous chiral symmetry breaking is properly incorporated and low energy meson physics is well understood [3], [4]. However, the NJL model does not accommodate the gluon degrees of freedom and is not directly applicable for analyzing the nonequilibrium processes. Thus, searching the related models which are free of such shortages and share the attractive features of NJL in the low energy region is a topical and practical task. In Ref. [5] the effective Hamiltonians with four-fermion interaction in the form of the product of two spatially separated currents mediated by a formfactor have been considered. As in the NJL model it is supposed that the ground state of the system is formed by the quark–anti-quark pairs with vacuum quantum numbers and oppositely directed momenta and in the framework of the Bogolyubov–Hartree–Fock approximation the description of quarks as the quasiparticles which is appropriate in the broad momentum range of momenta has been developed. A comparative analysis of the models with the formfactor behaving as the δ-function in the coordinate space (the NJL model) and the similar formfactor behaviour but in the momentum space (the model allied to the Keldysh model which is well known in the condensed matter physics) [6] teaches that these dissimilar models lead to the equivalent quasi-particles when the dynamical quark masses are comparable. Actually, it turns out the parameters characterizing the quasiparticles are developed by the common dynamical mechanism which is practically insensitive to the formfactor species.
One unexpected feature of these models is a discontinuity of mean energy functional considered as a function of current quark mass what results in some difficulties at fitting the quark condensate beyond the chiral limit. Speaking literally the quark condensate and the mean energy of quark ensemble are infinite. Fortunately, neither are physically observable quantities and in order to make a reliable conclusion about the models we should study the meson correlation functions, for example.

In this note we explore the meson observables in the Keldysh model which resembles a toy model but in some aspects elucidated below, turns out quite instructive. The meson characteristics despite the singular character of the ground state are finite and fairly adequate to correspond to the energy scale of existing experimental data. Certainly, such a conclusion does not concern all the features which is quite understandable within so a simple model. For example, the \( \pi \)-meson mass is slightly underestimated when the tuning parameters leading to the dynamical quark mass compatible with the NJL result are used and the pion decay constant \( f_\pi \) disappears (see below).

1 Model Lagrangian

We take the model Lagrangian density (discussed in Ref. [5]) in the form of a product of two quark currents localized at the space coordinates \( x \) and \( y \) which are bound by the formfactor \( F_{\mu\nu} \), i.e.

\[
\mathcal{L} = \bar{q} (i\gamma_\mu \partial_\mu + im) q - \bar{q} t^a \gamma_\mu q \int d\mathbf{y} F_{\mu\nu}(\mathbf{x} - \mathbf{y}) \bar{q}' t^a \gamma_\nu q',
\]

where \( q = q(x, t), \bar{q} = \bar{q}(\mathbf{x}, t) \), \( q' = q(y, t), \bar{q}' = \bar{q}(\mathbf{y}, t) \) are the (anti-)quark fields, \( t^a = \lambda^a/2 \) are the generators of \( SU(N_c) \) colour gauge group and \( m \) is a current quark mass. The Lagrangian density is given in the context of the Euclidean field theory and \( \gamma_\mu \) are the Hermitian Dirac matrices, \( \mu, \nu = 1, 2, 3, 4 \). The effective Hamiltonian corresponding to the Eq. (1) results from the averaging procedure when the quark behaviour is strongly affected by intensive stochastic gluon field. The (anti-)instanton ensemble was considered as such a background. As a general form the formfactor in Eq. (1) can be presented by a sum of two components \( F_{\mu\nu}(\mathbf{x} - \mathbf{y}) = G F(x - y) \delta_{\mu\nu} + J_{\mu\nu}(x - y) \), where the second term is spanned on the relative distance vector. In the first component we single out the constant \( G \) characterizing the strength of four-fermion interaction. In particular cases when the formfactor \( F \) has the \( \delta \)-function shape in coordinate space we come to the NJL model. With the formfactor behaving as \( F(p) = (2\pi)^3 \delta(p) \) the model is similar to the Keldysh model. In order to simplify the consideration we ignore the contribution of the correlator corresponding to \( J_{\mu\nu}(p) \). The prejudiced analysis of the system behaviour beyond the chiral limit performed in Ref.[5] demonstrates that the four-fermion interaction develops a singularity. The mean energy of the ensemble goes to infinity and the quark condensate demonstrates the singular behaviour as well. In the leading order in the \( N_c \)-expansion we obtain for the generators of colour group \( \sum_{a=1}^{N_c^2-1} t_a \delta_{ij} t_k = 1/2 \delta_{ij} \delta_{kl} \) and utilizing the Fierz transformation \( \gamma_\mu \otimes \gamma_\nu = 1 \otimes 1 + i\gamma_5 \otimes i\gamma_5 - \frac{1}{2} \gamma_\mu \otimes \gamma_\nu - \frac{1}{2} \gamma_\mu \gamma_5 \otimes \gamma_\nu \gamma_5 \), we have for the scalar term \( \langle \bar{q} q' \rangle \bar{q}' q \) in the mean field approximation the following effective Lagrangian density

\[
\mathcal{L} \approx \bar{q} (i\gamma_\mu \partial_\mu + im) q - \int d\mathbf{y} G F(x - y) \langle \bar{q} q' \rangle \bar{q}' q.
\]

where the angle brackets denote the corresponding average. It is interesting to notice the interaction term is composed with the colourless quark operators with \( x \) and \( y \) coordinates interchanged. The selfconsistency condition allows us to extract the dynamical quark mass as

\[
M(p) = 2N_c \int \frac{d\mathbf{q}}{(2\pi)^3} G F(p - q) \frac{m + M(q)}{[q^2 + (m + M(q))^2]^{1/2}}.
\]

At the spontaneous breaking of chiral symmetry takes place at this stage we have for the formfactor behaving as \( F(x) = \delta(x) \) the well known gap equation

\[
M = 2N_c G \int^{\Lambda_{NJL}} \frac{d\mathbf{q}}{(2\pi)^3} \frac{m + M}{[q^2 + (m + M)^2]^{1/2}},
\]
where $\Lambda_{NJL}$ is the cut off parameter. For the Keldysh model in the mean field approximation it looks like

$$M(p) = 2N_c G \frac{m + M(p)}{[p^2 + (m + M(p))^2]^{1/2}}. \quad (4)$$

Transforming this solution into the function $p(M)$ we have for the quark mass in the chiral limit

$$M(p) = [(2N_c G)^2 - p^2]^{1/2}.$$ 

In the further analysis bearing in mind the Fierz transformation we consider a more general form of the Lagrangian with two constants included. They characterize the interacting strength for the scalar and pseudoscalar channels — $G$ and for the vector and the axial-vector ones — $G_V$. Clearly, the scalar and pseudo-scalar coupling constants are identical in the chiral limit $[2]$.

2 Borzonization

Here we introduce the meson fields adapting well known bosonization procedure for the scalar channel (similar relations are valid for the other channels). It is convenient to introduce the auxiliary variables

$$Q_s(x, y; t) = F^{1/2}(x - y) \frac{1}{2} \left[ \bar{q}(x; t)q(y; t) + \bar{q}(y; t)q(x; t) \right],$$

$$Q_a(x, y; t) = F^{1/2}(x - y) \frac{1}{2} \left[ \bar{q}(x; t)q(y; t) - \bar{q}(y; t)q(x; t) \right], \quad (5)$$

for the symmetric and anti-symmetric combinations of quarks. Since the formfactor is a symmetric function with respect to an interchange of coordinates $x \to y$ then interaction contribution in the scalar sector which is just the point of our interest can be written as

$$\mathcal{V}_{int}^S = \frac{G}{2} F(x - y) \frac{1}{2} \left[ \bar{q}(x; t)q(y; t) \bar{q}(y; t)q(x; t) + \bar{q}(y; t)q(x; t) \bar{q}(x; t)q(y; t) \right] = \frac{G}{2} (Q_s^2 - Q_a^2).$$

It is easy to see that now the standard procedure of bosonization may be realized with the Gaussian integration which concerns auxiliary meson fields $\sigma_s(x, y; t)$ and $\sigma_a(x, y; t)$. Indeed, the integration is performed over the combinations including the meson and quark fields as

$$\frac{i\sigma_s}{(2G)^{1/2}} + \left(\frac{G}{2}\right)^{1/2} Q_a.$$ 

Then the interaction term may be presented in the form including the meson fields as

$$\mathcal{V}_{int}^S = \frac{G}{2} (Q_s^2 - Q_a^2) \to$$

$$\to - \frac{\sigma_s^2 + \sigma_a^2}{2G} + \frac{\sigma_s + i\sigma_a}{2} F^{1/2}(x - y) \bar{q}(x; t)q(y; t) + \frac{\sigma_s - i\sigma_a}{2} F^{1/2}(x - y) \bar{q}(y; t)q(x; t). \quad (6)$$

and integrating over the quark fields we obtain the effective theory operating with the mesons only. We do not show the detailed calculations here because they follow the standard procedure with one minor distinction which is a doubling of meson fields. Because of the same reason we present the succinct exposition of calculating the equation for dynamical quark mass and extracting meson correlator behaviour. We remind only that the first variation of the effective action allows us to determine the dynamical quark mass

$$\frac{\sigma_s^{(1)}}{G} + \langle Q_s \rangle = 0, \quad \frac{\sigma_a^{(1)}}{G} + i \langle Q_a \rangle = 0, \quad (7)$$

$\text{3}$
Here we imply that the meson fields with the primes are dependent on the coordinates \( k \), where the following notations are introduced
\[
\tilde{\sigma}(k) = \frac{1}{4} \tilde{\sigma}(x) \cdot q(x) q(x) \cdot \sigma, \quad \tilde{\sigma}(x) = -\frac{G}{2} F^{1/2}(x - y) \langle \sigma | \bar{q}(x) q(y) + \bar{q}(y) q(x) | \sigma \rangle. \tag{8}
\]
Certainly, we are interested in the real solutions and should consider only the case when the average contribution of anti-symmetric quark combination becomes trivial \( (Q_a) = 0 \).

For the quadratic terms of effective meson Lagrangian in the scalar channel we have
\[
\frac{\sigma^2 + \sigma^2_2}{2G} + \frac{1}{2} \left( \frac{\sigma^2 + \bar{\sigma}^2}{2} F^{1/2}(x - y) q(y; t) + \frac{\sigma^2 - \bar{\sigma}^2}{2} \bar{q}(y; t) F^{1/2}(x - y) q(x; t) \right)
\cdot \left( \tilde{\sigma}^{\prime} + i \tilde{\sigma}^a \right) F^{1/2}(x' - y') q(y'; t') + \frac{\sigma^2 - \bar{\sigma}^2}{2} \tilde{q}(y'; t') \tilde{F}^{1/2}(x' - y') q(x'; t'),
\]
here we imply that the meson fields with the primes are dependent on the coordinates \( x' \) and \( y' \) and \( t' \). Then the pairing of quark fields with utilizing the corresponding Green functions leads in the momentum representation (the integrations over the corresponding ‘internal’ variables are dropped) to the equation
\[
\sigma_{\alpha}(p, q; p_4) K_{\alpha, \beta}(p, q; p, q'; p_4) \sigma_{\beta}(p', q'; p_4) = \frac{\sigma_{\alpha}(p, q; p_4) \sigma_{\alpha}(-p - q - p_4) + \sigma_{\alpha}(p, q; p_4) \sigma_{\alpha}(-p, q; p_4)}{2G} \left[ \frac{\sigma^2 + \bar{\sigma}^2}{2} F^{1/2}(k - p) F^{1/2}(k + q') \tilde{\sigma}^{\prime} + \tilde{\sigma}^a \right] + \frac{\sigma^2 - \bar{\sigma}^2}{2} F^{1/2}(k - p) F^{1/2}(k + p') \tilde{\sigma}^{\prime} - \tilde{\sigma}^a + \frac{\sigma^2 - \bar{\sigma}^2}{2} F^{1/2}(k - q) F^{1/2}(k + q') \tilde{\sigma}^{\prime} + \tilde{\sigma}^a + \frac{\sigma^2 - \bar{\sigma}^2}{2} F^{1/2}(k - q) F^{1/2}(k + p') \tilde{\sigma}^{\prime} - \tilde{\sigma}^a \right], \tag{9}
\]
where the following notations are introduced \( \sigma_{\alpha,a} = \sigma_{\alpha,a}(p, q; p_4); \tilde{\sigma}_{\alpha,a} = \sigma_{\alpha,a}(p', q'; -p_4); q' = -p - q - p', \alpha, \beta = s, a \).

Apparently, there is no special need to investigate the meson correlation functions in so general form. We can obtain quite enough information on the solutions analyzing some particular cases. First we consider the conditions when the formfactors becomes identical \( F^{1/2}(k - q) = F^{1/2}(k - p - q - p') \), \( F^{1/2}(k - q) = F^{1/2}(k + p') \). These allow to conclude that the momenta of quarks coincide \( p = q \), i.e. there is no a relative motion of quarks in such a situation. Then it is easy to understand that the contribution of antisymmetric fields becomes degenerate in this configuration and the remaining symmetric part corresponds explicitly to the standard bosonization procedure. Finally, we have for the meson correlators in scalar and pseudoscalar channels
\[
K^{\sigma, \pi} = -\frac{1}{2G} + 2N_c \int \frac{dk}{(2\pi)^4} F(k - p) k_4 (k_4 - p_4) + k(k - 2p) \mp (m + M(k))(m + M(k - 2p)) \frac{1}{[k^2 + (m + M(k))^2][k_4 + (k - 2p)^2 + (m + M(k - 2p))^2]^{3/2}}, \tag{10}
\]
\( k^2 = k_4^2 + k^2 \). In particular, for the Keldysh model it reads
\[
K^{\sigma, \pi} = -\frac{1}{2G} + 2N_c \int \frac{dk_4}{2\pi} k_4 (k_4 - p_4) - p^2 \mp (m + M(p))^2 \left[ \frac{1}{[k_4^2 + E^2(p)][(k_4 - p_4)^2 + E^2(p)]} \right], \tag{10}
\]
where the notation \( E^2(p) = p^2 + (m + M(p))^2 \) for the quark energy is used. To simplify the presentation of formulae we omit the energy \( E \) dependence on the momentum. Due to the fact that

\[1\] In the momentum representation, respectively, we have \( M(p) = G \int \frac{dq}{(2\pi)^4} F(p - q) i \text{ Tr } S(q) \), where \( S(q) \) is the quark Green function.
only the integration over \( k_4 \) is essential we say about the one-dimensional model for mesons in this paper.

The denominator of the \( \pi \)-meson in Eq. (10) can be written in more convenient form as

\[
\frac{1}{[k_4^2 + E^2][(k_4 - p_4)^2 + E^2]} = \frac{1}{2 k_4 (k_4 - p_4) + p_4^2 + 2 E^2} \left[ \frac{1}{k_4^2 + E^2} + \frac{1}{(k_4 - p_4)^2 + E^2} \right].
\]  (11)

We see the correlation function of the \( \pi \)-meson in Eq. (10) is expressed by three integrals

\[
K^\pi = -\frac{1}{2G} + 2N_c \int \frac{dk_4}{2\pi} \frac{1}{2} \left( \frac{1}{k_4^2 + E^2} + \frac{1}{(k_4 - p_4)^2 + E^2} \right) - N_c \int \frac{dk_4}{2\pi} \frac{p_4^2 + 4 p^2}{[k_4^2 + E^2][(k_4 - p_4)^2 + E^2]}.
\]

Calculating them we have finally the following result

\[
K^\pi = -\frac{1}{2G} + \frac{N_c}{2E} + N_c \left\{ \begin{array}{ll}
\frac{p_4^2 + 4 p^2}{E (p_4 + 2iE)}, & \text{Im} p_4 > iE, \\
\frac{p_4^2 + 4 p^2}{2E} - \frac{p_4^2 + 4 p^2}{E (p_4 + 4iE)}, & \text{Im} p_4 < E, \\
\frac{p_4^2 + 4 p^2}{2 p_4 E (p_4 - 2iE)}, & \text{Im} p_4 < -iE,
\end{array} \right.
\]

where \( p = |p| \). In the Euclidean domain we have for the real values of energy \( p_4 \) that

\[
K^\pi = -\frac{N_c}{E \left( p_4^2 + 4E^2 \right)} \left( \frac{E}{G} p_4^2 + \frac{E - \tilde{G}}{G} 4E^2 + 4p^2 \right),
\]  (12)

where \( \tilde{G} = 2N_cG \). Then we find that the meson correlation function resembles a screening factor. In order to investigate the pseudo-euclidean situation we continue the \( p_4 \) variable to the imaginary axis. Introducing the notation \( p_4 = iP_0 \) we have

\[
K^\pi = N_c \left\{ \begin{array}{ll}
\frac{1}{E \left( P_0^2 - 4E^2 \right)} \left( -\frac{E}{G} P_0^2 + \frac{E - \tilde{G}}{G} 4E^2 + 4p^2 \right), & P_0 < E, \\
\frac{1}{P_0 E \left( P_0 + 2E \right)} \left( -\frac{E}{G} P_0^2 + \frac{G - 2E}{G} E P_0 + 2p^2 \right), & P_0 > E,
\end{array} \right.
\]

Comparing this expression at \( P_0 < E \) with Eq. (12) we make certain that the transition from the Euclidean variables \( p_4 \) to pseudo-euclidean ones do not change its form. The branch \( \text{Im} p_4 < -iE \) is not be considered because of the symmetry reason.

Now we continue with searching the \( \pi \)-meson dispersion law which is defined by the zeros of correlation function \( K^\pi = 0 \). The results for scalar and pseudo-scalar mesons are presented in Fig. 1 and for the vector and axial vector mesons are shown in Fig. 2. For the branch \( P_0 < E \) the dispersion can be received from the following equation

\[
P_0^2 = 4 \left( E - \tilde{G} \right) E + \tilde{G} (2p)^2.
\]  (13)

It can be obtained from Eq. (1) for the quark energy

\[
E = \tilde{G} \frac{m + M}{M}.
\]  (14)

In particular, for zero quark momentum we have \( M(0) = \tilde{G} \) for induced quark mass and for the \( \pi \)-meson energy we receive

\[
P_0^2 = 4 \left( E - \tilde{G} \right) E = 4 m \left( m + \tilde{G} \right),
\]  (15)
MeV. When tuning the model parameters as was proposed in Ref. [5] we get the following parameters for the NJL model which meet the constraint 

\[ p \approx 76 \text{ MeV} \quad \text{for the zero quark momentum but the quark energy at low momenta is} \quad E \approx m + M \approx 286 \text{ MeV} \]

When \( P_0 > E \) the interesting branch looks like \( P_0 = N_c G - E + \left[(E - N_c G)^2 + \frac{4N_c G E^2}{E}\right]^{1/2} \) but the analysis shows this branch does not satisfy the constraint \( P_0 > E \).

Turning now to the scalar channel we present the integrand in the convenient form as (10)

\[ K^\sigma = -\frac{1}{2G} + 2N_c \int \frac{dk_4}{2\pi} \left[ \frac{1}{k_4^2 + E^2} + \frac{1}{(k_4 - p_4)^2 + E^2} \right] - N_c \int \frac{dk_4}{2\pi} \frac{p_4^2 + 4E^2}{(k_4 + p_4)^2} \]

Calculating the integrals we come to the following result

\[ K^\sigma = -\frac{1}{2G} + \frac{N_c}{2E} + N_c \begin{cases} \frac{p_4 - 2iE}{2p_4 E}, & \text{Im} p_4 > iE, \\ \frac{1}{2E}, & \text{Im} p_4 < E, \\ \frac{p_4 + 2iE}{2p_4 E}, & \text{Im} p_4 < -iE. \end{cases} \]

In the pseudo-euclidean regime for the branch \( P_0 < E \) the \( \sigma \)-meson correlation function is degenerated \( K^\sigma = -\frac{1}{2G} \) and for \( P_0 > E \) we have \( K^\sigma = -\frac{P_0 + \tilde{G}}{2GP_0} \), Comparing to the quark energy of Eq. (14) we conclude that at \( P_0 > E \) there are not the interesting zeros in \( K^\sigma \).

We have considered the configuration when the relative momentum of quark and anti-quark equals zero and below we address the quark and anti-quark system with zero total momentum \( p + q = 0 \), see. Eq. (9). For the outgoing quark momenta two configurations are possible: a) \( p' = p \) and b) \( p' = -p \). In the a)-situation we obtain for the correlation functions in scalar and pseudo-scalar channels

\[ K^\pi_{\alpha, \beta} = \frac{\pi_\alpha \pi_\beta}{2} \quad \text{and} \quad K^\sigma_{\alpha, \beta} = \frac{\sigma_\alpha \sigma_\beta}{2} \]

where \( \alpha, \beta = s, a \). Then for the \( \pi \)-meson we have

\[ K^\pi_s \begin{cases} \frac{-P_0 + \tilde{G} - 2E}{2G(P_0 + 2E)}, & P_0 > E \\ \frac{P_0^2 - 4E(E - \tilde{G})}{2G(4E^2 - P_0^2)}, & P_0 < E \end{cases} \]

The dispersion law for the \( K^\pi_s \) correlation function at \( P_0 < E \) is extracted from the following equation

\[ P_0^2 = 4 \left( E - \tilde{G} \right) E \]

which is in a full agreement with Eq. (13) if the total momentum of quark anti-quark pair is taken to develop value \( 2p \to 0 \) (see. Fig. 1, curve 2). For the \( K^\sigma_a \) correlation function there is no solution meeting the constraint \( P_0 < E \) and there is no any solution for the \( K^\pi_s \) correlation function at \( P_0 > E \) as well as for the branch \( K^\pi_s \).
In the scalar channel one can obtain

\[
K_s^\sigma = \begin{cases} 
  -EP_0^2 + EP_0(\bar{G} - 2E) + 2\bar{G}(m + M)^2 \\
  2GEP_0(P_0 + 2E) 
\end{cases}, \\
K_a^\sigma = \begin{cases} 
  -EP_0^2 - EP_0(\bar{G} + 2E) - 2\bar{G}(m + M)^2 \\
  2GEP_0(P_0 + 2E)
\end{cases},
\]

The upper ratios are written for \( P_0 > E \) and the lower ones for \( P_0 < E \). The dispersion law for the branch \( K_s^\sigma \) at \( P_0 < E \) is determined by the solution of the following equation

\[
P_0^2 = 4 \left( E - \bar{G} \right) E + 4 \frac{\bar{G}}{E} (m + M)^2,
\]

see the dashed curve in Fig. 1. We did not manage to find the appropriate solution of dispersion equation for the correlation function \( K_a^\sigma \) in this case as the condition \( P_0 < E \) is invalid. Analysis of the correlation function roots at \( P_0 > E \) gives the same message that the suitable solutions are absent. We omit the discussion of the configuration b) because the results already given demonstrate how rich and complicated the analysis of solution branches could be. We would like to mention only that the presence of bound state even for the quarks with comparatively large momenta looks improbable.

![Figure 1: The \( \sigma \)-meson (dashed curve) and \( \pi \)-meson (curves 1 and 2) energies in MeV as the functions of momenta. The curve 1 corresponds to the configuration of zero relative quark momentum. The curve 2 describes the quark and anti-quark system with zero total momentum. The dots demonstrate the quark energy \( E \).](image)

In order to calculate the pion decay constant it is necessary to calculate a loop integral which is similar to Eq. (10) in which one of the vertices responsible for the weak interaction of the quarks does not contain the formfactor \( F^{1/2}(k - p) \) relevant for the meson fields. Then for the Keldysh model the integral with weak singularity develops formally the zero value what leads to the pion decay constant \( f_\pi \) equal to zero.

One can consider the vector and axial-vector meson channels in the similar way if the corresponding substitutions \( \bar{q}q \rightarrow \bar{q}\gamma_\mu q \), \( \sigma \rightarrow V_\mu \), \( \bar{q}\gamma_5 q \rightarrow \bar{q}\gamma_5 \gamma_\mu q \), \( \pi \rightarrow A_\mu \) are done in the relevant formulae. However, we omit those calculations here and do use the method of the equations for vertex functions (Bethe–Salpeter equation) to analyze the correlation functions quantitatively.
It was demonstrated above that the Gaussian integration should be performed with the symmetric and anti-symmetric combinations of auxiliary meson fields and the anti-symmetric fields should contain an imaginary unit factor. Clearly, the corresponding analysis of the functional integral should be performed with the symmetric vertex functions some special diagrams as follows.

It should be taken into account from the beginning that each line of these graphs has to be depicted as the doubled one because we consider the nonlocal meson fields. The first diagram of this set describes the initial interaction \( \Gamma_0(p, q) = (2\pi)^3 \Gamma F(p - q) \) where the matrix \( \Gamma \) denotes the interaction channel, \( i\gamma_5, \gamma_\mu, \gamma_5\gamma_\mu \). If we single out the combination \( \bar{q}_\alpha(x; t) \tilde{\Gamma}_{\alpha\beta}(x, y; t) q_\beta(y; t) \) then the following equation can be calculated for the series sum

\[
-(2\pi)^3 G \Gamma F(p - q) + \tilde{\Gamma}(-p, -q; r_4) =
\]

\[
= - \int \frac{dk_4 dk}{2\pi} G \Gamma F(-k - q) \text{Tr} S(k; k_4) \Gamma S(k - p + q; k_4 - r_4) \tilde{\Gamma}(k - p + q, k; k_4 - r_4),
\]

where \( r_4 = p_4 - q_4 \) and \( p_4, q_4 \) are the corresponding components of quark and anti-quark momenta. We search for the solution for the vertex function in the Keldysh model, for example, in the form

\[
\tilde{\Gamma}(p, q; r_4) = (2\pi)^3 \Gamma \delta(p - q) V(p; r_4)
\]

assuming that for the imaginary values of \( r_4 \) the solution for the vertex function possesses the pole singularity. Picking out the singular contributions we can obtain approximately that

\[
-G + V = G \Pi V
\]

where the polarization operator in the Keldysh model can be represented in the form

\[
\Pi = \int \frac{dk_4}{2\pi} \text{Tr} S(-k; k_4) \Gamma S(-p; k_4 - r_4) \Gamma.
\]

Therefore, we have for the vertex function

\[
V = \frac{G}{1 - G \Pi^s}
\]

and its denominator zeros determine the pole positions. Discussing the intermediate calculaions we show here the results for pseudo-scalar and scalar channels at \( |r_4| < E \) as

\[
1 - G \Pi^s = \frac{4E}{G} (E - \tilde{G}) - R_0^2,
\]

\[
1 - G \Pi^s = \frac{(4E^2 - R_0^2) E + 4[(m + M)^2 - E^2]}{G E (4E^2 - R_0^2)},
\]

where \( R_0 = P_0 - Q_0, P_0 = ip_4, Q_0 = iq_4 \). In the pseudo-scalar channel the \( \pi \)-meson dispersion law coincides explicitly with Eq. (17) and for the \( \sigma \)-meson we have Eq. (18).

For vector and axial-vector channels in the Keldysh model we have

\[
\Pi_{44}^{V,A} = -4N_c \int \frac{dk_4}{2\pi} \frac{k_4(k_4 - r_4) - p^2 \pm (m + M)^2}{[k_4^2 + E^2][(k_4 - r_4)^2 + E^2]},
\]

\[
\Pi_{4i}^{V,A} = 4N_c \int \frac{dk_4}{2\pi} \frac{(2k_4 - r_4) p_i}{[k_4^2 + E^2][(k_4 - r_4)^2 + E^2]},
\]

\[
\Pi_{0i}^{V,A} = 4N_c \int \frac{dk_4}{2\pi} \frac{k_4(k_4 - r_4) \delta_{ij} - 2 p_ip_j + \delta_{ij} p^2 \pm \delta_{ij} (m + M)^2}{[k_4^2 + E^2][(k_4 - r_4)^2 + E^2]},
\]

\[
\Pi_{0i}^{V,A} = 4N_c \int \frac{dk_4}{2\pi} \frac{k_4(k_4 - r_4) \delta_{ij} - 2 p_ip_j + \delta_{ij} p^2 \pm \delta_{ij} (m + M)^2}{[k_4^2 + E^2][(k_4 - r_4)^2 + E^2]},
\]

\[
\Pi_{0i}^{V,A} = 4N_c \int \frac{dk_4}{2\pi} \frac{k_4(k_4 - r_4) \delta_{ij} - 2 p_ip_j + \delta_{ij} p^2 \pm \delta_{ij} (m + M)^2}{[k_4^2 + E^2][(k_4 - r_4)^2 + E^2]},
\]

\[
\Pi_{0i}^{V,A} = 4N_c \int \frac{dk_4}{2\pi} \frac{k_4(k_4 - r_4) \delta_{ij} - 2 p_ip_j + \delta_{ij} p^2 \pm \delta_{ij} (m + M)^2}{[k_4^2 + E^2][(k_4 - r_4)^2 + E^2]},
\]
$\Pi_{ij}^{V,\lambda} = \Pi_{ij}^{\nu,\lambda}$. It is easy to see that these results coincide with the corresponding meson correlation function obtained above. Calculating the integrals in vector and axial-vector channels for $Im \, r_4 > iE$ we have:

$$
\Pi_{44}^{V} = -2N_c \frac{i}{r_4}, \quad \Pi_{44}^{\lambda} = \frac{2N_c}{E} \frac{i \, r_4 E - 2p^2}{r_4(r_4 + 2iE)}, \\
\Pi_{ii}^{V} = -2N_c \frac{p_i}{r_4}, \quad \Pi_{ii}^{\lambda} = \Pi_{ii}^{V}, \\
\Pi_{ij}^{V} = 2N_c \frac{i \, r_4 E \, \delta_{ij} - 2p_i p_j}{r_4(r_4 + 2iE)}, \quad \Pi_{ij}^{\lambda} = \frac{2N_c}{E} \frac{i \, r_4 E - 2(m + M)^2 \delta_{ij} - 2p_i p_j}{r_4(r_4 + 2iE)).}
$$

At $|r_4| < E$ we obtain

$$
\Pi_{44}^{V} = 0, \quad \Pi_{44}^{\lambda} = -8N_c \frac{(m + M)^2}{E^{2}} \frac{r_4^2 + 4E^2}{}, \quad \Pi_{ii}^{V} = 0, \quad \Pi_{ii}^{\lambda} = \Pi_{ii}^{V}, \\
\Pi_{ij}^{V} = 8N_c \frac{E^2 \, \delta_{ij} - p_i p_j}{E (r_4^2 + 4E^2)}, \quad \Pi_{ij}^{\lambda} = \frac{8N_c}{E} \frac{p^2 \delta_{ij} - p_i p_j}{r_4^2 + 4E^2}.
$$

Now we diagonalize the correlation functions using the fact that corresponding quadratic forms determine simply the Lagrangian of free vector and axial-vector mesons

$$
K^{V,\lambda} = C_{44}^{V,\lambda} V_4^2 + 2 C_{4i}^{V,\lambda} V_4 \tilde{V}_i + \tilde{V}_i \tilde{C}_{ij}^{V,\lambda} \tilde{V}_j,
$$

where we imply the summation over the indices which are repeated. It is valid by definition that $C_{\mu \nu}^{V,\lambda} = \delta_{\mu \nu} - G_\nu \Pi_{i}^{\nu,\lambda}$. If we redefine the space components of (axial-)vector fields by substituting $\tilde{V}_i = V_i + \alpha \, p_i V_4$, where $\alpha^{V,\lambda} = -\frac{C_{4i}^{V,\lambda} p_i}{p_i C_{ij}^{V,\lambda} p_j}$, and exclude the mixed components $V_4 V_i$ from quadratic form (23) we get that the fourth component of vector field enters the quadratic form with coefficient $C_{44}^{V,\lambda} = \tilde{C}_{44}^{V,\lambda} - \frac{(C_{4i}^{V,\lambda} p_i)^2}{p_i C_{ij}^{V,\lambda} p_j}$. The components of the tensor $C_{ij}^{V,\lambda}$ remain unchanged. The numerical analysis demonstrates the acceptable solution for the dispersion of the fourth component exists only for the axial-vector field at $R_0 > E$. However, we do not discuss this solution in the present paper. The spatial components of vector fields are searched as having two different forms for the transversal $V_i = (\delta_{ij} - \frac{p_i p_j}{p^2}) \, v_j^\perp$ and longitudinal $V_i = p_i \, v^\parallel$ components. First we consider the case of $R_0 > E$. The dispersion law for the transversal component of vector field $v^\perp$ has the form $R_0 = \tilde{G}_\nu - 2E$, where $\tilde{G}_\nu = 2N_c G_\nu$. Therefore the nontrivial solution is possible when the condition $\tilde{G}_\nu > 3E$ is satisfied. For clarity we take the constant as $\tilde{G}_\nu = 1.5 \tilde{G}$ which corresponds at the low quark momentum to the value $\tilde{G}_\nu \approx 1.5E$ and therefore for the fitting parameter set selected this solution branch does not manifest itself. The dispersion of longitudinal component $v^\parallel$ in this case is defined as $R_0 = N_c G_\nu - E + \left[ (E - N_c G_\nu)^2 + \frac{4N_c G_\nu v^2}{E} \right]^{1/2}$. The branches of meson observables in the vector and axial-vector channels are depicted in Fig. 2. The branch discussed above is shown by the solid curve and denoted by 1. For the transversal component of axial-vector meson $a^\perp$ we obtain $R_0 = N_c G_\nu - E + \left[ (E - N_c G_\nu)^2 + \frac{4N_c G_\nu (m + M)^2}{E} \right]^{1/2}$. This curve is depicted in Fig. 2 by dashed line and marked by 2. The dispersion of the longitudinal component $a^\parallel$ is calculated from $R_0 = \tilde{G}_\nu$. This branch appears when the condition $\tilde{G}_\nu > E$ is satisfied. In Fig. 2 it corresponds to the dashed straight line 3. We see at low momentum the longitudinal $a^\parallel$ and transversal $a^\perp$ components practically coincides.

Now we address the situation of $R_0 < E$. The dispersion of the transversal component of the vector meson $v^\perp$ is defined by $R_0^2 = 4E \left( E - \tilde{G}_\nu \right)$. This curve is shown in Fig. 2 by solid line 4.
For the longitudinal components $v^\parallel$ we have $R_0^2 = 4 \left( E^2 - \tilde{G}_V \frac{(m + M)^2}{E} \right)$, and the solid curve 5 shows its behaviour. For the transversal component of the axial-vector meson $a^\perp$ we obtain $R_0^2 = 4E^2 - 8N_c G_V \frac{p^2}{E}$. The corresponding curve is presented as the dashed line 6. Its right-hand component is practically degenerated with the curve 4 because the induced quark mass goes to zero at large momenta and the restoration of chiral symmetry takes place. The longitudinal component of axial-vector field $a^\parallel$ is degenerated.

The $\pi$-meson energy in the NJL model for the tuning parameter values considered coincides with experimental data and looks like $E_{\pi}^{NJL} = 140$ MeV. In Ref. [5] it was supposed that for the Keldysh model the dynamical quark mass in the low momentum region is equal to the dynamical quark mass of the NJL model. This assumption has led to the almost identical quasiparticles for both models. But the $\pi$-meson energy turned out rather underestimated $E_{\pi}^{K} = 76$ MeV although the relative scale of mesons for different channels was maintained. The axial-vector meson had gotten heavier than the vector meson and the meson of pseudoscalar channel was the lightest one, indeed.

Calculating the correlation functions here we did not exploit the convenient trick of shifting the integration variable to make the integrals symmetric and were keeping the integration contour fixed. The one-dimensional character of the model provides us with the obvious possibility to study the dependence of the correlation functions on the integration contour shape as well. In particular, it is interesting to trace the turn of integration contour to the imaginary axis. Then two quark poles could also be treated as the "Wigner" phase and the continuation of correlation functions will be different from what has been done in this note.

## 4 Correlation functions in the Minkowski space

The model developed allows us to study easily the meson correlation functions in the Minkowski space as well and to compare them to what we obtained above. In fact, the task is technically related to computing the following integral (dependent on the Euclidean variables) within the fixed
Our concern here is the particular situation when the parameter \( p \) is pure imaginary \( p = iP \). Now we should calculate the similar integral with another fixed contour which corresponds to the Minkowski space (turned to 90 degrees regarding the Euclidean integration) contour

\[
I = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{i}{(q^2 - E^2 + i\varepsilon)[(q - p)^2 - E^2 + i\varepsilon]} = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{i}{[2(E - i\varepsilon)]^2} \left[ \frac{2(E - i\varepsilon)}{p[p - 2(E - i\varepsilon)]} q - E + i\varepsilon + \frac{2(E - i\varepsilon)}{p[p + 2(E - i\varepsilon)]} q - p - E + i\varepsilon \right] + \left[ \frac{-2(E - i\varepsilon)}{p[p + 2(E - i\varepsilon)]} q + E - i\varepsilon + \frac{-2(E - i\varepsilon)}{p[p - 2(E - i\varepsilon)]} q - p + E - i\varepsilon \right],
\]

and its calculation gives

\[
J = \frac{1}{4E^2} \left\{ \begin{array}{ll}
\frac{2E}{p[p - 2(E - i\varepsilon)]}, & \text{Im } p > i\varepsilon, \\
\frac{4E}{p^2 - 4(E - i\varepsilon)^2}, & \text{Im } p < \varepsilon, \\
\frac{2E}{p[p + 2(E - i\varepsilon)]}, & \text{Im } p < -i\varepsilon.
\end{array} \right.
\]

Compared to the Euclidean configuration the pole contributions to the integral in the Minkowski space are interchanged, i.e. the poles of \( e \)-type become the poles of \( m \)-type. Thus, the result obtained for the Minkowski space will be valid for the Euclidean configuration if the contribution of the \( m_2 \)-pole at \( \text{Im } p > iE \) is omitted and the contribution of \( e_2 \)-pole at \( \text{Im } p < -iE \) is added. One should not change anything at \( |\text{Im } p| < E \). The figure shows one of possible integration contours corresponding to the situation \( \text{Im } p > iE \). Similarly at \( \text{Im } p < -iE \) the contour should be deformed in order to have the contribution of \( e_2 \)-pole. The detailed analysis make possible to formulate the general rule for reproducing the proper result in the Minkowski space. The integration contour should be deformed in such a way to have the contributions of the \( e_1 \) and \( e_2 \)-poles only, i.e. the integration contour looks like being squeezed in between the poles \( e_1 \) and \( m_2 \).
In order to calculate the integrals $I$ and $J$ in $x$-representation we rewrite the integral $I$ (using the well-known identity and implying the analytical continuation of all auxiliary functions in the parameter $p$) as

$$I = \int_{-\infty}^{\infty} dq \frac{1}{2\pi (q^2 + E^2)[(q-p)^2 + E^2]} = \int_0^{1/2} dx \int_{-\infty}^{\infty} dq \frac{1}{2\pi \{x(q^2 + E^2) + (1-x)((q-p)^2 + E^2)\}^{3/2}} = \frac{1}{4p^2} \int_{-1/2}^{1/2} d\xi \frac{1}{\left[\frac{1}{4p^2} + \frac{E^2}{p^2} - \xi^2\right]^{3/2}} = \frac{1}{E(p^2 + 4E^2)},$$

where $\xi = x - 1/2$. Such an integral treatment makes transparent that the result above corresponds to the calculation of $I$ in the fixed contour for the branch when $|\text{Im} \, p| < E$. Changing the variables of integration as $E^2 \to -E^2$ we are able to reproduce the corresponding result for $J$. Thus, we may conclude that calculating in the $x$-representation fully reproduces the result for the Minkowski space being analytically continued.

Considering the meson correlation functions in the Minkowski space we are interested in the situation when the external parameters are real what corresponds to the constraint $|\text{Im} \, p| < \varepsilon$ for the $J$ integral. It is a pretty simple task to obtain the final results for the corresponding dispersion laws using the results of calculations for the Euclidean space. Below we show as an example the results for the (anti- )quark total momentum equal zero

$$P^2_\pi = 4E(E - \tilde{G}), \quad P^2_\sigma = 4E^2 - 4\frac{\tilde{G}}{E} \, p^2,$$

$$P^2_{\nu\perp} = 4E(E - \tilde{G}_\nu), \quad P^2_{\lambda\perp} = 4E^2 - 4\frac{\tilde{G}_\nu}{E} \, p^2,$$

$$P^2_{\nu\parallel} = 4E^2 - 4\frac{\tilde{G}_\nu}{E} \, (m + M)^2,$$

the axial vector field correlator $a_{\parallel}$ becomes degenerate. At $\tilde{G} = \tilde{G}_\nu$ $\pi$-meson becomes degenerate with the vector meson and $\sigma$-meson with the axial vector meson. It is clear the realistic relations between the meson masses correspond to the situation when $\tilde{G}_\nu < \tilde{G}$. In order to give another example we take $\tilde{G}_\nu = \tilde{G}/2$ (in addition to the Euclidean consideration in which the constant was
Figure 3: The meson energies obtained by using the correlation functions in the Minkowski space as the functions of quark momentum. The double quark energy is depicted by dots. The dispersion law for the longitudinal component of vector meson field is depicted by stars and the dashed line with the symbol $v_\perp$ corresponds to the transverse component. The dashed line with the symbol $a_\perp$ shows the dispersion law for the transverse component of axial vector meson. The solid line with the symbol $\sigma$ is devoted for the scalar meson whereas the similar line with the symbol $\pi$ corresponds to $\pi$-meson.

taken as $\tilde{G}_v = 1.5 \tilde{G}$. The meson energies as quark momentum functions are shown in Fig. 3 in the Minkowski space. Curiously, the bound states of quark and anti-quark do exist at any quark momentum for the present configuration.

5 Conclusion

In this note we demonstrate that despite the singular behaviour of mean energy and quark condensate which was observed in Ref. [5] the meson observables are finite, well identified and compatible with the experimental energy scale. The number of effective degrees of freedom which define the quasi-particle picture in the NJL and Keldysh model are comparable. The Keldysh model being as simple as the NJL model looks like a suitable candidate for describing the nonequilibrium processes in the (anti-)quark ensembles. Due to the one-dimensional character of the Keldysh model the analytical continuation from the Euclidean region of meson observables to the pseudo-euclidean one is easily performed and controled. The amazing feature of our consideration is that the bound states are revealed at any quark momenta in Minkowsky space. Eventually we conclude that if the quasi-particles in the different models are similar the meson observables are also alike.

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