Impact of Prandtl numbers on turbulence modeling

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Abstract. A low-Reynolds number (LRN) $k-\tau$ turbulence model is devised that includes the near-wall turbulence modeling and turbulent Prandtl numbers $\sigma(k, \epsilon, t)$. A secondary source term added in the $\tau$-equation accounts substantially for the anisotropic production in near-wall regions. Hence, improved predictions of adverse pressure-gradient flows, involving separation and reattachment are obtained with a considerable reduction in the turbulent kinetic energy and length scale magnitudes. The eddy-viscosity formulation responds to both rotational and irrotational strains to better predict nonequilibrium flows. The current model employs no wall function/distance parameter to bridge the near-wall integration as required by conventional $k-\tau$ models. Model predictions are compared with direct numerical simulation (DNS) and experimental data; a good correlation is achieved.

Nomenclature

$C_f$ skin-friction coefficient
$C_{\epsilon 1}$ model closure coefficient
$C_{\epsilon 2}$ model closure coefficient
$C_{\mu}$ eddy-viscosity coefficient
$c_p$ specific heat at constant pressure
$D$ tube diameter
$e$ specific internal energy
$f_\mu$ eddy-viscosity damping function
$F, G$ flux vectors in $x$- and $y$-directions
$Nu$ local Nusselt number
$p$ static pressure
$P$ production of turbulent kinetic energy
$Pr$ Prandtl number
$Re$ Reynolds number
$q$ heat flux
$Q$ source term
$S$ invariant of mean strain-rate tensor
$t$ time
$T$ temperature
$T_h$ hybrid time scale
$U$ vector of conservative variables

$u, v$ velocity components in $x$- and $y$-directions
$u\tau$ wall-friction velocity
$u^+$ non-dimensional flow velocity
$\overline{u_i u_j}$ Reynolds stress tensor
$W^-$ invariant of mean vorticity tensor
$y$ normal distance from wall
$y^+$ non-dimensional wall distance
$\delta$ channel half-width
$\delta_{i,j}$ Kronecker’s delta
$\theta$ azimuth angle
$\mu, \mu_t$ laminar and turbulent eddy viscosities
$\nu$ kinematic viscosity
$\rho$ density
$\epsilon$ dissipation-rate
$\sigma$ turbulent Schmidt/Prandtl number

Subscript
$ref$ reference condition
$w$ wall condition
$v$ viscous part
1. Introduction

The eddy-viscosity phenomenon is established on an analogy between turbulent transport and molecular diffusion. This analogy simply states that the turbulent transport of any quantity could be formulated as the gradient of that quantity multiplied by the eddy-viscosity. Consequently, similar to the molecular diffusion, an appropriate turbulent Prandtl number is needed for each variable to obtain plausible diffusion models. Another imperfection seems to be attached with the characteristic scale based on the turbulent kinetic energy $k$ and its dissipation-rate $\tilde{\epsilon}$ which enters the model for the diffusivity, appearing to be inadequate in accommodating the near-wall viscous effects. Paradoxically, the precise situation wherein the turbulence model encounters most difficulties is the turbulent diffusion, dominating the overall balance of the flow. However, the amount of empiricism invoked in the model equations can favor such as not confronting a failure of the diffusion models. In principle, turbulent Schmidt/Prandtl numbers (having similar types of physical interpretations in the sense that they are defined as the ratio between the momentum diffusivity and the scalar diffusivity) ranging from 0.68 to 2.0 can be found for $k$ [1], the most widely used value for $\sigma_k$ is 1.0. For $\tilde{\epsilon}$, $\sigma_k = 1.3$ is the standard choice, but values as low as 0.72 can be found. The commonly used value for $\sigma_\epsilon$ is 0.9. Nevertheless, it is not precisely correct and there are significant departures from $\sigma_t = 0.9$, particularly in wake/near-wall regions [1, 2].

In particular, discussions and improvements on RANS (Reynolds-averaged Navier-Stokes) turbulence models have never been saturated. During last several decades, a lot of innovative ideas have been developed for improving the accuracy of RANS that cannot precisely predict the nonequilibrium flows [3, 4]. The eddy-viscosity model (EVM) in conjunction with an RANS approach manipulates unclosed Reynolds stresses with the mean-velocity gradient and turbulent eddy-viscosity; it has been a very practical way to model the complex turbulent physics. Therefore, it is essential for the EVM to devise an accurate turbulent eddy-viscosity formulation as far as possible. In practice, most RANS turbulence models are based on the turbulence kinetic energy $k$ and its dissipation rate $\epsilon$ (or the specific dissipation $\omega$). The equilibrium assumption used in the derivation leads to an inadaptable scheme near the wall; the eddy-viscosity has to be damped to a reasonable level in reproducing a correct near-wall asymptotic behavior of the Reynolds shear stress. This improved scheme is called a low-Reynolds-number (LRN) correction and commonly used $k-\epsilon$ models employ an LRN correction.

On the other hand, many turbulence models usually include the distance to the wall as an explicit parameter which hinders them from simulating complex flows involving multiple surfaces; the wall distance in this case becomes cumbersome to be defined accurately [5]. A remedy to this imperfection is to develop a model which implicates no explicit wall distance while integrating it toward the solid surface. To alleviate the above limitations, recent developments in the RANS community are driven toward an attempt at introducing wall-distance-free turbulence models [5–9], in order to enhance numerical robustness in complex configurations, and to improve accuracy when applied to increasingly challenging flow cases. However, such fixes are not quite successful in predicting the flow-transition, flow separation and reattachment; they often impair the model robustness, or are active outside the flow regions they are intended for [8, 9]. In fact, involving more and more physical contents with turbulence modeling reinforces the accuracy with an inevitable decrease in robustness.

A wall-distance-free LRN $k-\tilde{\epsilon}$ turbulence model is constructed. A non-linear eddy-viscosity formulation is introduced. To enhance dissipation in nonequilibrium flow regions, an extra positive source term is included in the $\tilde{\epsilon}$ transport equation. It reduces the turbulent kinetic energy and length scale magnitudes and thus, improved predictions of adverse pressure gradient flows with separation and reattachment are obtained. The wall singularity is removed with an appropriate time scale that never falls below the Kolmogorov (dissipative eddy) time scale; it represents the time scale realizability enforcement accompanied by the near-wall turbulent...
phenomena. In this way, a pseudo-dissipation rate used to remove the singularity in the dissipation equation at the wall, is avoided. A wall-distance-free eddy-viscosity damping function is devised in terms of the total kinetic energy and invariants of strain-rate and vorticity tensors. It reaches the upper limiting value of unity in the logarithmic layer. In addition, the turbulent Prandtl numbers $\sigma_{(k, \epsilon, \tilde{\epsilon})}$ are adjusted such as to provide presumably an accurate turbulent diffusion in near-wall regions.

The proposed model performance is validated against experimental and DNS data of well documented flows, consisting of fully developed channel flows, a flat plate boundary layer flow with zero pressure-gradient and heat transfer from the circular cylinder in a cross-flow, respectively. The test cases are selected such as to justify the ability of the current model to replicate the combined effects of LRN, near-wall turbulence and nonequilibrium.

2. Governing equations

The two-dimensional Reynolds-averaged Navier-Stokes (RANS) equations, including the equations for the turbulent kinetic energy $k$ and dissipation-rate $\tilde{\epsilon}$, can be written in the following form:

$$\frac{\partial U}{\partial t} + \frac{\partial (F - F_v)}{\partial x} + \frac{\partial (G - G_v)}{\partial y} = Q$$

(1)

where $U = (\rho, \rho u, \rho v, E, \rho k, \rho \tilde{\epsilon})^T$. The inviscid fluxes are:

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p + \frac{2}{3} \rho k \\ \rho uv \\ \rho \frac{k}{T} \\ \rho u \tilde{\epsilon} \\ u(E + p) \end{pmatrix}, \quad G = \begin{pmatrix} \rho v \\ \rho vu \\ \rho v^2 + p + \frac{2}{3} \rho k \\ v(E + p) \\ \rho vk \\ \rho v \tilde{\epsilon} \end{pmatrix}$$

(2)

Here $\rho$ is the density and $p$ is the pressure. The total energy is defined as

$$E = \rho e + \frac{\rho \vec{V} \cdot \vec{V}}{2} + \rho k$$

(3)

where $e$ is the specific internal energy and $\vec{V} = u\hat{i} + v\hat{j}$ is the velocity. The viscous fluxes are:

$$F_v = \begin{pmatrix} \tau_{xx} + \frac{2}{3} \rho k \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} - q_x \\ \mu_k (\partial k / \partial x) \\ \mu_e (\partial \tilde{\epsilon} / \partial x) \end{pmatrix}, \quad G_v = \begin{pmatrix} \tau_{yy} + \frac{2}{3} \rho k \\ \tau_{yx} \\ u\tau_{xy} + v\tau_{yy} - q_y \\ \mu_k (\partial k / \partial y) \\ \mu_e (\partial \tilde{\epsilon} / \partial y) \end{pmatrix}$$

(4)

and the viscous stress tensor can be given as

$$\tau_{ij} = 2 \mu \left( S_{ij} - \frac{1}{3} S_{kk} \delta_{ij} \right) - \rho \mu_t u_i u_j$$

(5)

where $\mu$ is the laminar viscosity and the Reynolds stresses $\rho u_i u_j$ are related to the mean strain-rate tensor $S_{ij}$ through the Boussinesq approximation:

$$-\rho \mu_t u_i u_j = 2 \mu T \left( S_{ij} - \frac{1}{3} S_{kk} \delta_{ij} \right) - \frac{2}{3} \rho k \delta_{ij}, \quad S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

(6)
The heat flux is calculated from

$$\vec{q} = - \left( \mu_c \frac{c_p}{P r} + \mu_T \frac{c_p}{\sigma_t} \right) \nabla T$$ (7)

where $\mu_T$ is the coefficient of turbulent viscosity, $c_p$ is the specific heat at constant pressure, $Pr$ and $\sigma_t$ represent the molecular and turbulent Prandtl numbers, respectively, and $T$ implies the temperature. Clearly, the turbulent part of the total heat flux is estimated using the Boussinesq approximation. The diffusion of turbulence is modeled as

$$\mu_k \nabla k = (\mu + \frac{\mu_T}{\sigma_k}) \nabla k, \quad \mu_\epsilon \nabla \tilde{\epsilon} = (\mu + \frac{\mu_T}{\sigma_\epsilon}) \nabla \tilde{\epsilon}$$ (8)

where $\sigma_k$ and $\sigma_\epsilon$ are the appropriate turbulent Prandtl numbers. The source term $Q$ for the $k$ and $\tilde{\epsilon}$ equations can be written as

$$Q = \left( C_{\epsilon 1} \rho P - C_{\epsilon 2} \rho \epsilon + E_\epsilon \right)$$ (9)

where $\epsilon = \tilde{\epsilon} + 2 \nu \left( \partial \sqrt{k}/\partial x_j \right)^2$. The turbulent production term $P = -\overline{u_i u_j} (\partial u_i/\partial x_j)$ and $E_\epsilon$ is a secondary source term designed to increase the level of $\tilde{\epsilon}$ in nonequilibrium flow regions. The symbolized $T_t$ is the characteristic (mixed or hybrid) time scale, having the asymptotic consistency in the near-wall region. The associated constants are: $C_{\epsilon 1} = 1.45$ and $C_{\epsilon 2} = 1.83$.

3. Near-wall turbulence modeling

In the wall-vicinity, the effect of molecular viscosity is stronger than that of the turbulent mixing, reflecting the existence of a dominant anisotropic condition. Consequently, an important phenomenon concerning the appropriateness of turbulence model is to introduce the near-wall behavior of turbulence quantities accompanied by a preferential damping of velocity fluctuations in the direction normal to the wall that resolves the influence of wall-proximity substantially.

The mixed time scale $T_t$ associated with Eq. (9) can simply be defined as

$$T_t = \sqrt{\frac{k^2}{\bar{\epsilon}^2} + C_T^2 \frac{\nu}{\epsilon}} = \frac{k}{\epsilon} \sqrt{1 + \frac{\tilde{\epsilon}}{\epsilon} C_T^2 R_T}, \quad R_T = \frac{k^2}{\nu \bar{\epsilon}}$$ (10)

where $\nu$ denotes the kinematic viscosity and $R_T$ is the turbulent Reynolds number. Equation (10) guarantees that the eddy time scale never falls below a compatibility bound deduced from the Kolmogorov time scale $C_T \sqrt{\nu/\epsilon}$. It is dominant in the immediate neighborhood of solid wall and prevents the singularities in the dissipation equation down to the wall. Alternatively, the turbulence time scale is $k/\bar{\epsilon}$ at large $R_T$ but approaches the Kolmogorov limit $C_T \sqrt{\nu/\epsilon}$ for $R_T \ll 1$. The empirical constant $C_T$ associated with the Kolmogorov time scale is estimated as follows. In the viscous sublayer $k = y^2/(C_T^2 \nu/\epsilon)$, where the basic scale is the Kolmogorov time scale. Besides, the $k$-equation reduces to $\nu \partial^2 k/\partial y^2 = \epsilon$ as the wall is approached. Combining these relations gives $C_T = \sqrt{2}$. Obviously, the inclusion of $T_t$ in the $\tilde{\epsilon}$-equation provides a near-wall asymptotic consistency without resorting to ad hoc damping functions employed in many $k-\epsilon$ models [10]. Figure 1 shows a priori evaluation of hybrid time scale $T_t$ pertaining to the $\tilde{\epsilon}$-equation using DNS data with different frictional Reynolds numbers for a developed channel flow. As can be seen, Eq. (10) yields obviously a fair $Re_\tau$-dependent characterization of the whole channel-region, exhibiting a sharp amplification in the wall-vicinity and afterwards, $T_t$ gradually increases toward the defect layer.
Since the viscous dissipation is dominant in near-wall regions, the eddy-viscosity is calculated from the relation:

$$
\mu_T = \rho k T_t \min \left[ C_\mu f_\mu, \frac{0.5}{1 + T_t \sqrt{S^2 + W^2}} \right]
$$

(11)

where $C_\mu = 0.09$ and the dynamic time scale $k/\bar{\epsilon}$ is replaced by an hybrid time scale $T_t$. It is worth mentioning that Eq. (11) ensures the realizability in turbulence modeling [11]. The damping function is formulated as:

$$
\begin{align*}
    f_\mu &= f_\lambda + \frac{\sqrt{C_\mu}}{1 + 6 R_\lambda^2}, \\
    f_\lambda &= \tanh(C_1 R_\lambda + C_2 R_\lambda^2), \\
    R_\lambda &= \sqrt{C_\mu K_T}, \\
    K_T &= \frac{\nu \eta}{V \cdot V/2 + k}
\end{align*}
$$

(12)

where $C_1 = 1 \times 10^{-2}$ and $C_2 = 4.5 \times 10^{-3}$. The parameter $\eta = \max(S, W)$, containing the invariants of strain-rate and vorticity respectively; $S = \sqrt{2S_{ij}S_{ij}}$ and $W = \sqrt{2W_{ij}W_{ij}}$. The mean vorticity tensor $W_{ij}$ is defined as

$$
W_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)
$$

(13)

The damping function $f_\mu$ is valid in the whole flow field, regarding the viscous sublayer and logarithmic layer. Close to the wall, the Reynolds stress $-uv \sim y^3$ and $k \sim y^2$. To sustain the correct cubic power-law behavior of $-uv$, the damping function requires to increase proportionally to $y$ in the near-wall region. Equation (12) implies that as $y \to 0$, $R_\lambda \sim y$ and hence $f_\mu = O(\sqrt{C_\mu})$ at the close proximity of wall (i.e., $f_\mu$ pretends to increase like $f_\mu \sim y^{-1}$). In fact, the construction of $f_\mu$ compared to the traditional one [10] augments the potentiality of $f_\mu$ to grow particularly in near-wall regions, thereby expediting the viscous dissipation. Alternatively, the adopted form of $f_\mu$ reproduces correctly the asymptotic limit, involving the distinct effects of LRN and wall-proximity. As shown by Figure 1 in comparison with DNS data [12] for fully developed turbulent channel flows, the designed function $f_\mu = 1$, away from the wall to ensure that the model is compatible to the standard $k-\epsilon$ turbulence model.

The use of $R_\lambda$ (a new parameter with no reference to the wall-distance) faces the singularity at neither the separating nor the reattaching point in contrast to the adoption of $y^+ = u_T y/\nu$,
where $u_\tau$ is the wall-friction velocity. Consequently, the model is applicable to adverse pressure-gradient flows, involving separation and reattachment.

Budgets of $k$ and $\epsilon$ from DNS data dictate that the role of turbulent diffusion in near-wall regions is substantial. Therefore, Prandtl numbers $\sigma_k$ and $\sigma_\epsilon$ are modeled, rather than assigning constant values as are applied to the commonly used practice with $\sigma_k = 1.0$, and $\sigma_\epsilon = 1.3$:

$$\sigma_\epsilon = C_T C_\mu + f_\sigma, \quad \sigma_k = \frac{\sigma_\epsilon}{1 - C_\mu f_\sigma}$$

(14)

where $f_\sigma = 2f_\lambda/(1 + f_\lambda^2)$. Distributions of $\sigma$ are displayed in Figure 2. Model coefficients $\sigma_k$ and $\sigma_\epsilon$ are developed such as to achieve sufficient diffusion in the wall-vicinity and in the core region of flow $\sigma_k/\sigma_\epsilon > 1$ to eliminate the common drawback where the turbulent diffusion of $k$ overwhelms the diffusion of $\epsilon$ with $\sigma_k < \sigma_\epsilon$ [13].

The turbulent Prandtl number correlation of Ref. [14] is adopted with a near-wall modification:

$$\sigma_t = \frac{2}{Pr (1 + \mu_T/\mu)} + 9 C_\mu$$

(15)

The formulation is consistent with the theoretical behavior over a wide range of the molecular Prandtl number $Pr$, approaching $9 C_\mu$ as the turbulent Peclet number ($= \mu_T Pr/\mu$) is sufficiently large. To this end, it must be stressed that the modification to $\sigma_t$ (rather than applying $\sigma_t = 0.9 [2]$, traditionally) facilitates the avoidance of predicting excessive heat transfer coefficients, particularly in near-wall regions.

The positive source term $E_\epsilon$ in Eq. (9) stems from the most extensive turbulent diffusion models for $k$ and $\epsilon$ equations with the TSDIA (two-scale direct-interaction approach) having the inertial-range simplification [15]. To receive positive benefits from the numerical stability and to integrate the cross-diffusion term directly to the wall, $E_\epsilon$ is devised as [16]:

$$E_\epsilon = C_\epsilon \frac{\mu_T}{T_t} \max \left[ \frac{\partial(k/\epsilon)}{\partial x_j}, 0 \right], \quad C_\epsilon = 1 + C_{\epsilon 2}$$

(16)

Evidently, the source term $E_\epsilon$ enhances the energy dissipation in nonequilibrium flows, maintaining a reduced level of the turbulent length scale from its local equilibrium value and
enabling an improved prediction of adverse pressure-gradient flows involving separation and reattachment. To this end, it must be stressed that the term $E_\tau$ is physically beneficial in the vicinity of reattachment region and hence, it can be regarded as a substitute for the well-known Yap correction [17].

The following boundary conditions for $k$ and $\tilde{\epsilon}$ are applied at solid walls:

$$k_w = 0, \quad \left( \frac{\partial \tilde{\epsilon}}{\partial y} \right)_w = 0 \quad (17)$$

Obviously, reproducing the wall-limiting behavior necessitates a fine grid system in near-wall regions.

4. Computations

To evaluate the performance of new turbulence model, computations regarding two-dimensional turbulent flows comprising a fully developed channel flow, a flat plate boundary layer flow with zero pressure-gradient and heat transfer from the circular cylinder in a cross-flow are considered. For numerical comparisons, calculations from the original Chien (OCH) model [18], modified Chien (MCH) model [16] and Gatski & Speziale (GS) model [19] (considered only for channel flow since the GS model shows an evidence of numerical instability when the flow is far from equilibrium; the reason is that it is more susceptible to the rotational strains) are included. Note that OCH, MCH and GS models assume a constant value for $\sigma$ (i.e., $\sigma_k = 1.0, \sigma_\epsilon = 1.3, \sigma_\tau = 0.9$). An artificial compressibility approach in the frame-work of a cell centered finite-volume scheme is employed to solve the flow equations [20–23]. A second-order upwinded spatial differencing is used to compute the convective terms. Roe’s [24] damping term is applied to calculate the flux on the cell face. A diagonally dominant alternating direction implicit (DDADI) time integration method [25] is applied for the iterative solution to the discretized equations. A multigrid method is utilized for the convergence-acceleration [26]. Basic implementations of artificial compressibility approach and associated features can be found in Refs. [20–23].

4.1. Channel flow

Numerical computations are performed for fully developed turbulent channel flows at $Re_\tau = 180$ and 395, respectively; turbulence quantities are attainable from DNS data [12]. Calculations are carried out in the half-width of a channel, applying periodic boundary conditions (except for the pressure) to the upstream and downstream boundaries. Computations with a $48 \times 32$
Figure 4. Shear stress profiles of channel flow.

non-uniform grid refinement for $Re_\tau = 180$ and a $48 \times 48$ for $Re_\tau = 395$ are considered to be sufficiently accurate based on grid-independent studies (not shown). The length of computational domain is $32\delta$ for both cases, where $\delta$ is the channel half-width. The first grid node near the wall is placed at $y^+ \approx 0.4$ to ensure the resolution of viscous sublayer. The results are plotted in the forms of $u^+ = u/u_\tau$ and $uv^+ = u\nu/u_\tau^2$ versus $y^+$ for comparisons.

Figure 3 depicts the comparisons of velocity profiles for various models. As can be seen, predictions of both present and MCH models almost fairly match the DNS data. OCH and GS models slightly over-estimate the mean velocity profile in the outer layer. Turbulent shear stress profiles, pertaining to all models are displayed in Figure 4. All model predictions provide reasonable estimates when compared with DNS data.

4.2. Flat-plate boundary layer flow

The performance evaluation is further examined with the experimental data of flow over a flat-plate with a high free-stream turbulence intensity. This numerical experiment, referred to as a well-documented T3B case for bypass transition is taken from "ERCOFTAC" Fluid Dynamics Database WWW Services (http://fluindigo.mech.surrey.ac.uk/) preserved by P. Voke. Measurements down to $x = 1.495$ m corresponding to $Re_x \approx 94000$, are conducted by J. Coupland at Rolls-Royce. The inlet velocity is 9.4 m/s with a zero pressure-gradient and an upstream turbulence intensity of $Tu = 6.0\%$, defined as $Tu = \sqrt{\frac{2}{3}k/U_{ref}^2}$, where $U_{ref}$ indicates the reference velocity. To predict the bypass transition, a physical constraint is set to the dissipation such that the decay of free-stream turbulence is in balance.

Computations initiate 16 cm ahead of the leading edge and symmetric/mirror boundary conditions are imposed. The length and height of the grid are 1.6 m and 0.3 m, respectively. The near-wall grid node is placed at $y^+ < 1.0$, except at the point of leading edge where $y^+ = 2.1$. The grid size is $96 \times 64$ and heavily clustered near the wall which presumably ensures a grid-independent numerical solution (not shown).

The predicted skin friction coefficients ($C_f = 2u_\tau^2/U_{ref}^2$) are presented with experimental data in Figure 5. It can be seen that the current model has good correspondence with measurements, characterizing an interesting feature that the transition commences at the right position and it is strong enough. In contrast, both OCH and MCH models with the wall distance in their eddy-damping functions provide earlier transition than that seen in the experiment; this tendency can be seen in the comparison with the observation by Savill [27]. It seems likely that agreement between numerical results and measurements is impressive toward the end of
transition (e.g., beyond $x = 0.195 \, m$). Nevertheless, the MCH model is incapable of reproducing the experimental evidences; results offered by the MCM model stays somewhat on a lower level than those of measured data.

Figure 5. Streamwise skin-friction coefficient of boundary layer flow.

Figure 6. Local Nusselt number distribution over half of tube surface.

4.3. Heat transfer from circular cylinder in cross flow
To validate the performance in complex separated and reattaching turbulent flows, the proposed model is further applied to the turbulent fluid flow and heat transfer around a circular cylinder at $Re = 3.6 \times 10^4$ in a cross flow, for which measurements are available [28]. This is discernibly a typical $Re$ for practical heat exchangers. The geometrical aspect is simple but it remains cumbersome to be modeled due to the boundary layer separation, leading to a complex flow structure. The diameter of cylindrical tube $D = 0.025 \, m$ with an inlet velocity of $U_{ref} = 22.85 \, m/s$ and an upstream turbulence intensity of $Tu = 0.5\%$. The radial length of computational domain is taken as $60D$. An $O$ type grid with a $128 \times 96$ non-uniform resolution, clustered heavily near the solid wall, is considered to be sufficient to reproduce the flow and heat transfer characteristics. External boundary (e.g., far field) conditions are employed. A constant temperature that imitates experimental boundary conditions is prescribed at the solid wall.

Variations of local Nusselt number with the azimuth angle are compared with experimental data in Figure 6. Distinguishably, the distribution reflects the characteristic feature of a minimum Nusselt Number at the separation point corresponding to $\theta \approx 85^\circ$, accompanied by an enhanced heat transfer in the wake regions. Articulately, the current model predictions match experimental data fairly well. Obviously, it accommodates better predictions with a variable $\sigma_t$, particularly in separation and wake regions; this aspect becomes important in relation to the model extensibility to complex flow modeling with heat transfer.

5. Conclusions
The proposed turbulent model has the competency to be applied to arbitrary topology in conjunction with structured or unstructured grids since it is wall-distance-free, tensorially invariant and frame-indifferent. The model is capable of properly reproducing the near-wall and LRN effects associated with viscous and non-viscous wall phenomena. The virtual significance of non-linear eddy-viscosity coefficient and functions is evident. The included secondary source
term with the dissipation equation accounts for the anisotropic production substantially, leading to a reduced level of turbulence generation in nonequilibrium flow regions. Consequently, the model is capable of evaluating the flow cases with separation and reattachment. In particular, the turbulent Prandtl numbers must be considered as variable parameters throughout the flow, since they determine where an accurate modeling of the diffusion phenomenon is needed. A comparative assessment between the predicted and experimental results dictates that above all, the present model accurately replicates the skin-friction and near-wall heat-transfer coefficients.

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