SPREADING-VANISHING DICHOTOMY IN INFORMATION DIFFUSION IN ONLINE SOCIAL NETWORKS WITH INTERVENTION

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ABSTRACT. In this paper, multiple information diffusion in online social networks with free boundary condition is investigated. We prove a spreading-vanishing dichotomy for the problem: i.e., either at least one piece of information lasts forever or all information suspend in finite time. The criterion for spreading and vanishing is established, it is related to the initial spreading area and the expansion capacity. We also obtain several cases of the asymptotic behavior of the information under different conditions. When the information spreads, we provide some upper and lower bounds of the spreading speed corresponding to different cases of asymptotic behavior of the information. In addition, numerical examples are given to illustrate the impacts of the initial spreading area and the expansion capacity on the free boundary, and all cases of the asymptotic behavior of the information.

1. Introduction. Online social networks have become a global platform to disseminate various information. Popular social networks such as Twitter, Facebook, MSN and WeChat have been integrated into our daily lives. Extensive investigations have been made to understand network structure, user interactions [2, 11] and the characteristics of information diffusion [13, 12, 23, 31]. Some mathematical models were used to describe the information diffusion in online social networks, see [9, 18, 21, 25, 30, 33] and references therein.

In a recent work [24], F. Wang, H. Y. Wang and K. Xu proposed a diffusive logistic model to describe single information diffusion. They studied the temporal and spatial patterns in a real data set collected from a social news aggregation site-Digg (www.digg.com), and validated the proposed diffusive logistic equation. The experiment results showed that the diffusive logistic model is able to characterize and predict the process of information propagation in online social networks.

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The spatial domain of the partial differential equation model in [24] is chosen to be a fixed interval. But in reality, information may cross the boundary of the interval. For this reason, C. X. Lei, Z. G. Lin and H. Y. Wang [14] investigated the single information diffusion model with a free boundary for online social networks. They presented several sharp thresholds for which information diffusion either lasts forever or suspends in a finite time. In the former case, they gave the asymptotic spreading speed which is determined by the corresponding elliptic equation. In fact, free boundary value problems have attracted extensive research efforts and remarkable successes have been achieved. They were used to describe spreading phenomena in many fields, such as, wound healing [5], ice melting [22], the transmission of diseases [1, 17, 32] and species interaction [6, 7, 15, 29, 26, 28].

In this paper, we investigate the multiple information diffusion process in online social networks with intervention. This type of problem arises from many practical situations. For example, a health intervention from public health agencies can use online social networks to promote behaviour that improves mental and physical health, or discourages or reframes those with health risks. A better understanding of multiple information diffusion process with intervention is significant in both theoretically and practically. Information diffusion originated from multiple sources has not been fully understood. Recently, C. Peng, K. Xu, F. Wang and H. Y. Wang [19] studied information diffusion initiated from multiple sources in online social networks by numerical simulation. In fact, there are increasing numbers of rumors in online social networks and many challenging problems remain in modeling and analysis of multiple information diffusion process. Our results may be able to shed light on effectively controlling unwanted information in online social networks.

We shall study a simple case: there are three pieces of information A, B and C sent from different sources to compete for influence on online users. We view the official information C as an intervention from the media or government to control the spread of the ordinary information A and B. For simplicity, we assume that A and B have no influence on C, A and B compete for influence on each other. The relationship among the three pieces of information is depicted in Fig. 1. Following the spatio-temporal modeling approach in [14, 24], the online users are aggregated into a number of clusters and imbedded to the $x$-axis. As a result, $x$ represents the social distance between the groups of users. Thus this competition system with intervention can be modeled by the free boundary problem

\[
\begin{aligned}
\begin{cases}
\begin{aligned}
ut = d_1 u_{xx} &= \rho_1 (v - u - b_1 v - c_1 w),
\quad &0 < x < h(t), t > 0, \\
v_t = d_2 v_{xx} &= \rho_2 (w - a_2 u - b_2 v - c_2 w),
\quad &0 < x < h(t), t > 0, \\
w_t = d_3 w_{xx} &= \rho_3 (1 - c_3 w),
\quad &0 < x < h(t), t > 0, \\
u_x(0, t) = v_x(0, t) = w_x(0, t) = 0, \\
\end{aligned}
\end{cases}
\end{aligned}
\]

where $\mu, \rho_1, \rho_2$ and $h_0$ are given positive constants. $\mu$ means the expanding capacity of the information in the new area, $x = h(t)$ is the free boundary to be determined and represents the spreading front of news, and the initial function
\[ u_0(x), v_0(x), w_0(x) \in \Sigma(h_0) \text{ for some } h_0 > 0, \text{ where} \]
\[ \Sigma(h_0) = \{ \phi \in C^2([0, h_0]) : \phi'(0) = \phi(h_0) = 0, \phi(x) > 0 \text{ in } [0, h_0] \}. \]  
(2)

Here \( u(x, t), v(x, t), w(x, t) \) represent the density of influenced users of information A, B, C at distance \( x \) and time \( t \) respectively; \( d_i (i = 1, 2, 3) \) are the diffusion rates; \( r_i (i = 1, 2, 3) \) are the intrinsic growth rates; \( 1/a_1, 1/b_2, 1/c_3 \) are the carrying capacities; \( b_1, a_2, c_1, c_2 \) are the intervention rates. All parameters are positive.

This paper is organized as follows. In Section 2, we give some preliminaries including the existence, uniqueness and bounded estimates of the solution to problem (1), and some estimates of solutions to parabolic partial differential inequalities. In Section 3, we study the asymptotic behavior of the solution and obtain a spreading-vanishing dichotomy. We establish the criterion for spreading and vanishing in Section 4. In Section 5, we provide some estimates of asymptotic spreading speed. Moreover, we give some numerical simulations to illustrate our results and all cases of the asymptotic behavior of the solution in Section 6. Finally, we conclude the paper with a brief discussion in Section 7.

2. Preliminaries. By a similar argument as in [7, 10], we have the following result.

**Lemma 2.1.** For any given \( u_0, v_0, w_0 \) satisfying (2) and any \( \alpha \in (0, 1) \), there is a \( T > 0 \) such that problem (1) admits a unique solution
\[ (u, v, w, h) \in [C^{1+\alpha, (1+\alpha)/2}(D_T)]^3 \times C^{1+\alpha/2}([0, T]), \]
where \( D_T := \{ (x, t) : x \in [0, h(t)], t \in [0, T] \} \), \( T \) only depend on \( \alpha, h_0, \|u_0\|_{C^2([0, h_0])}, \|v_0\|_{C^2([0, h_0])}, \|w_0\|_{C^2([0, h_0])}) \).

**Lemma 2.2.** Let \( (u, v, w, h) \) be a solution of problem (1) for \( t \in (0, T) \) for some \( T > 0 \), then there exists a constant \( C_1 \) independent of \( T \) such that
\[ 0 < u(x, t), v(x, t), w(x, t) \leq C_1, \text{ for } (x, t) \in [0, h(t)] \times (0, T). \]

**Proof.** Using the strong maximum principle, we obtain
\[ u, v, w > 0 \text{ for } (x, t) \in [0, h(t)] \times (0, T). \]
Lemma 2.3. Let \((u, v, w, h)\) be the solution of the problem (1) for \(t \in (0, T)\) for some \(T > 0\), then

\[0 < h'(t) \leq C_2, \quad \text{for } t \in (0, T),\]

where \(C_2\) is independent of \(T\).

Proof. According to \(u(h(t), t) = 0\) and \(u > 0\) for \((x, t) \in [0, h(t)] \times (0, T)\), then \(u_x(h(t), t) < 0\) for \(t \in (0, T)\). Similarly, we have \(v_x(h(t), t) < 0\) and \(w_x(h(t), t) < 0\) for \(t \in (0, T)\), then

\[h'(t) = -\mu[u_x(h(t), t) + \rho_1 v_x(h(t), t) + \rho_2 w_x(h(t), t)] > 0, \quad \text{for all } t \in (0, T).\]

Let

\[\Omega_{CT} := \{(x, t) : h(t) - C^{-1} < x < h(t), 0 < t < T\},\]

where \(C > 0\) is to be determined later. Define

\[\bar{u}(x, t) = C_1[2C(h(t) - x) - C^2(h(t) - x)^2].\]

Direct calculations yield that

\[
\bar{u}_t = 2C_1 Ch'(t)[1 - C(h(t) - x)] \geq 0, \\
\bar{u}_{xx} = -2C_1 C^2, \\
\bar{u}(0) = \|u_0\|_{L^\infty}.
\]

If \(C \geq \sqrt{r_1/2d_1}\), it follows that

\[
\bar{u}_t - d_1 \bar{u}_{xx} \geq 2d_1 C_1 C^2 \geq r_1 C_1 \geq r_1 u(1 - a_1 u - b_1 v - c_1 w) \leq r_1 C_1.
\]

On the other hand,

\[
\bar{u}(h(t) - C^{-1}, t) = C_1 \geq u(h(t) - C^{-1}, t), \quad \bar{u}(h(t), t) = 0 = u(h(t), t).
\]

If \(C \geq \|u_0\|_{C([0, h_0])}/C_1\), then

\[
\bar{u}(x, 0) = C_1[2C(h_0 - x) - C^2(h_0 - x)^2] \geq C_1 C(h_0 - x) \geq (h_0 - x)\|u_0\|_{C([0, h_0])},
\]

Since

\[
u_0(x) = \int_{h_0}^x u_0(y)dy \leq \|u_0\|_{C([0, h_0])}(h_0 - x),
\]

then, \(\bar{u}(x, 0) \geq \|u_0(x)\|\). By the comparison principle, we obtain \(u(x, t) \leq \bar{u}(x, t)\) in \(\Omega_{CT}\). From \(\bar{u}(h(t), t) = u(h(t), t) = 0\),

\[
u_x(h(t), t) \geq \bar{u}_x(h(t), t) = -2C_1 C, \quad \text{for all } 0 \leq t < T.
\]

Similarly, we have

\[v_x(h(t), t), w_x(h(t), t) \geq -2C_1 C, \quad \text{for all } 0 \leq t < T.\]

Combining (3) with (4), we get

\[
h'(t) = -\mu[u_x(h(t), t) + \rho_1 v_x(h(t), t) + \rho_2 w_x(h(t), t)] \leq 2\mu C_1(1 + \rho_1 + \rho_2) := C_2
\]
Theorem 2.4. The solution of problem (1) exists and is unique for all $t \in (0, +\infty)$.

Proof. By Lemma 2.1, there exists $T_{\text{max}} > 0$ such that $[0, T_{\text{max}})$ is the maximal existence time interval. Our goal is to prove that $T_{\text{max}} = +\infty$. Suppose not, then $T_{\text{max}} < +\infty$. By Lemmas 2.2 and 2.3, there exist positive constants $C_i (i = 1, 2)$ independent of $T_{\text{max}}$ such that for all $(x, t) \in [0, h(t)] \times (0, T_{\text{max}})$,

$$0 \leq u(x, t), v(x, t), w(x, t) \leq C_1,$$

and

$$0 \leq h'(t) \leq C_2, \text{ for } t \in (0, T_{\text{max}}).$$

Hence

$$h_0 \leq h(t) \leq h_0 + C_2 t, \text{ for } t \in (0, T_{\text{max}}).$$

Choose $\vartheta \in (0, T_{\text{max}})$ and $\bar{\vartheta} \in (T_{\text{max}}, +\infty)$. From the standard regularity theory, there exists $C^* > 0$ depending only on $\vartheta, \bar{\vartheta}, C_i (i = 1, 2)$ such that

$$\|u(\cdot, t), v(\cdot, t), w(\cdot, t)\|_{C^2([0, \bar{h}(t)])} \leq C^*$$

for $t \in [\vartheta, T_{\text{max}}]$. It follows from the proof of Lemma 2.1 that there exists a $\tau > 0$ depending only on $C_1, C_2$ such that the solution of problem (1) with the initial time $T_{\text{max}} - \tau/2$ can be extended uniquely to the time $T_{\text{max}} + \tau/2$, which contradicts to the definition of $T_{\text{max}}$. The proof is completed.

In order to investigate the asymptotic behavior of the solution $(u, v, w)$ to (1) in Section 3, we state the following lemmas (Propositions 2.1-2.3 in [26]).

Lemma 2.5. Let $d, \alpha$ and $\beta$ be fixed positive constants. For any given $\varepsilon, L > 0$, there exists $l_\varepsilon > \max\{L, \frac{\pi}{2} \sqrt{\frac{d}{\alpha \beta}}\}$ such that, if a continuous and non-negative function $z(x, t)$ satisfies

$$\begin{cases}
  z_t - dz_{xx} \geq \alpha z(\beta - z), & 0 < x < l_\varepsilon, t > 0, \\
  z_x(0, t) = 0, u(l_\varepsilon, t) \geq 0, & t > 0,
\end{cases}$$

and $z(x, 0) > 0$ in $(0, l_\varepsilon)$, then we have

$$\liminf_{t \to \infty} z(x, t) > \beta - \varepsilon \text{ uniformly in } [0, L].$$

Lemma 2.6. Let $d, \alpha, \beta$ and $K$ be fixed positive constants. For any given $\varepsilon, L > 0$, there exists $l_\varepsilon > \max\{L, \frac{\pi}{2} \sqrt{\frac{d}{\alpha \beta}}\}$ such that, if a continuous and non-negative function $z(x, t)$ satisfies

$$\begin{cases}
  z_t - dz_{xx} \leq \alpha z(\beta - z), & 0 < x < l_\varepsilon, t > 0, \\
  z_x(0, t) = 0, u(l_\varepsilon, t) \leq K, & t > 0,
\end{cases}$$

and $z(x, 0) > 0$ in $(0, l_\varepsilon)$, then we have

$$\limsup_{t \to \infty} z(x, t) < \beta + \varepsilon \text{ uniformly in } [0, L].$$

Lemma 2.7. Let $d, \lambda, K > 0$ and $\zeta \geq 0$ be fixed constants. For any given $\varepsilon, L > 0$, there exists $l_\varepsilon \gg 1$ such that, if a continuous and non-negative function $z(x, t)$ satisfies

$$\begin{cases}
  z_t - dz_{xx} \leq \lambda z(\zeta + z), & 0 < x < l_\varepsilon, t > 0, \\
  z_x(0, t) = 0, u(l_\varepsilon, t) \leq K, & t > 0,
\end{cases}$$

and $z(x, 0) > 0$ in $(0, l_\varepsilon)$, then we have

$$\liminf_{t \to \infty} z(x, t) > \lambda \zeta \text{ uniformly in } [0, L].$$
and $0 < z(x, 0) \leq K$ in $(0, l_z)$, then we obtain
$$\lim_{t \to \infty} \sup_{x} z(x, t) < \varepsilon \text{ uniformly in } [0, L].$$

3. Asymptotic behavior of $(u, v, w)$. It follows from Lemma 2.3 that $x = h(t)$ is monotonically increasing, there exists $h_\infty \in (0, \infty]$ such that
$$\lim_{t \to \infty} h(t) = h_\infty.$$

Lemma 3.1. Let $(u, v, w, h)$ be the solution of problem (1). If $h_\infty < \infty$, then there exists a constant $K$ such that
$$\|u(\cdot, t), v(\cdot, t), w(\cdot, t)\|_{C^1([0, h(t)])} \leq K, \forall t > 1,$$$$
$$\lim_{t \to \infty} h'(t) = 0.$$

Proof. Define a transformation
$$(x, t) \to (y, t), y = \frac{x}{h(t)}, 0 \leq y < \infty.$$ Let $U(y, t) = u(x, t), V(y, t) = v(x, t), W(y, t) = w(x, t)$ and set $F(U, V, W) = r_1 U(1 - a_1 U - b_1 V - c_1 W), G(U, V, W) = r_2 V(1 - a_2 U - b_2 V - c_2 W)$ and $H(W) = r_3 W(1 - c_3 W)$, then problem (1) becomes
$$\begin{cases}
U_t - \beta_1 U_{yy} - \gamma U_y = F(U, V, W), & 0 < y < 1, t > 0, \\
V_t - \beta_2 V_{yy} - \gamma V_y = G(U, V, W), & 0 < y < 1, t > 0, \\
W_t - \beta_3 W_{yy} - \gamma W_y = H(W), & 0 < y < 1, t > 0, \\
U_y(0, t) = V_y(0, t) = W_y(0, t) = 0, & t > 0, \\
U(1, t) = V(1, t) = W(1, t) = 0, & t > 0, \\
U(y, 0) = u_0(y), V(y, 0) = v_0(y), W(y, 0) = w_0(y), & 0 \leq y \leq 1,
\end{cases}$$
where $\beta_i(t) := \frac{d_i}{h^2(t)}$ for $i = 1, 2, 3$ and $\gamma(y, t) := \frac{\kappa(t)y}{h(t)}$.

Denote that $h_n(t) = h(t + n), U_n(y, t) = U(y, t + n), V_n(y, t) = V(y, t + n), W_n(y, t) = W(y, t + n), (\beta_1)_n(t) = \beta_1(t + n), (\beta_2)_n(t) = \beta_2(t + n), (\beta_3)_n(t) = \beta_3(t + n), \gamma_n(y, t) = \gamma(y, t + n)$, then $U_n$ satisfies
$$\begin{cases}
(U_n)_t - (\beta_1)_n(U_n)_{yy} - \gamma_n(U_n)_y = F_n(y, t), & 0 < y < 1, t > 0, \\
(U_n)_y(0, t) = U_n(1, t) = 0, & t > 0, \\
U_n(y, 0) = u(h(n)y, n), & 0 \leq y \leq 1,
\end{cases}$$
where $F_n = r_1 U_n(1 - a_1 U_n - b_1 V_n - c_1 W_n)$. By the similar way as proposition 7.1 in [28], there exists a positive constant $K$ such that $\|U_n\|_{C^{1, \infty}(\mathbb{R}^2 \setminus \{(0, 1) \times [1, 3)\})} \leq K$ for all $n \geq 0$. This implies $\|U\|_{C^{1, \infty}(\mathbb{R}^2 \setminus \{(0, 1) \times [1, 3)\})} \leq K$ for all $n \geq 0$, where $E_n = [0, 1] \times [n + 1, n + 3]$. Similarly, we get $\|V, W\|_{C^{1, \infty}(\mathbb{R}^2 \setminus \{(0, 1) \times [1, 3)\})} \leq K$ for all $n \geq 0$. Because these rectangles $E_n$ overlap and $K$ is independent of $n$, then $\|U, V, W\|_{C^{1, \infty}(\mathbb{R}^2 \setminus \{(0, 1) \times [1, 3)\})} \leq K$. Since $u_x = h^{-1}(t)U_y, v_x = h^{-1}(t)V_y$ and $w_x = h^{-1}(t)W_y$, then
$$\|u(\cdot, t), v(\cdot, t), w(\cdot, t)\|_{C^1([0, h(t)])} \leq K, \forall t > 1.$$ Due to the Stefan condition and $0 < h'(t) \leq C_2$, we derive that $\|h'\|_{C^{1, \infty}(\mathbb{R}^2 \setminus \{(0, 1) \times [1, 3)\})} \leq K$. Since $h_\infty < \infty$, then $\lim_{t \to \infty} h'(t) = 0$.}

Lemma 3.2. Let $(u, v, w, h)$ be the solution of problem (1). If $h_\infty < \infty$, then
$$\lim_{t \to \infty} \|u(\cdot, t), v(\cdot, t), w(\cdot, t)\|_{C^{1, \infty}(\mathbb{R}^2 \setminus \{(0, 1) \times [1, 3)\})} = 0.$$
Proof. We easily see that

\[
\begin{aligned}
  w_t - d_3 w_{xx} &= r_3 w(1 - c_3 w), & 0 < x < h(t), t > 0, \\
  w_x(0, t) &= w(h(t), t) = 0, & t > 0, \\
  h'(t) &
\end{aligned}
\]

\[
\begin{aligned}
  &\geq -\mu_2 w_x(h(t), t), & t > 0, \\
  w(x, 0) &= u_0(x), h(0) = h_0, & 0 < x < h_0.
\end{aligned}
\]

In view of Proposition 3.1 in [27] and Lemma 3.1, we obtain

\[
\lim_{t \to \infty} \|w(\cdot, t)\|_{C^1((0, h(t))]} = 0.
\]

For small enough \(\varepsilon\), there exists \(T \gg 1\) such that \(w(x, t) \leq \varepsilon\) for \(t \geq T\) and \(0 \leq x \leq h(t)\). Since \(0 < v(x, t) \leq C_1\), then we have that

\[
\begin{aligned}
  &u_t - d_1 u_{xx} \geq r_1 u(1 - b_1 C_1 - c_1 \varepsilon - a_1 u), & 0 < x < h(t), t \geq T, \\
  &u_x(0, t) = u(h(t), t) = 0, & t \geq T, \\
  &h'(t) \geq -\mu_4 (h(t), t), & t \geq T, \\
  &u(x, 0) = u_0(x), & 0 \leq x \leq h(T).
\end{aligned}
\]

Applying Proposition 3.1 in [27] and Lemma 3.1 again, we have

\[
\lim_{t \to \infty} \|u(\cdot, t)\|_{C^1((0, h(t))]} = 0.
\]

We conclude that \(\lim_{t \to \infty} \|v(\cdot, t)\|_{C^1((0, h(t))]} = 0\) by using the similar way. \(\square\)

**Remark 1.** \(1/a_1\) is the carrying capacity of influenced users of information A, which is the maximum possible density of influenced users of information A; \(a_2\) is the intervention rate and measures how information A affects B in spreading within the users. Define \(P_{AB} := a_2/a_1\) as the total influence of information A w.r.t. B, that is, the maximum possible effect of information A w.r.t. B; \(P_{BA} := b_1/b_2\) as the total influence of information B w.r.t. A; \(P_{CA} := c_1/c_3, P_{CB} := c_2/c_3\) as the total influence of information C w.r.t. A and B respectively.

The main objective of the following content is to study the long time behavior of \((u, v, w)\) when \(h_\infty = \infty\) in four cases, i.e., \(P_{BA} + P_{CA} \in (1, \infty), P_{AB} + P_{CB} \in (0, 1); P_{BA} + P_{CA} \in (0, 1), P_{AB} + P_{CB} \in (1, \infty); P_{CA}, P_{CB} \in (1, \infty)\) and \(P_{BA} + P_{CA}, P_{AB} + P_{CB} \in (0, 1)\).

**Lemma 3.3.** Let \((u, v, w)\) be the solution of free boundary problem (1) with \(h_\infty = \infty\). Then

(i) \(\lim \sup_{t \to \infty} u(x, t) \leq 1/a_1\), \(\lim \sup_{t \to \infty} v(x, t) \leq 1/b_2\) and \(\lim \sup_{t \to \infty} w(x, t) \leq 1/c_3\) uniformly in \([0, \infty)\);

(ii) if \(P_{BA} + P_{CA}, P_{AB} + P_{CB} \in (0, 1)\), then \(\lim \inf_{t \to \infty} u(x, t) \geq \frac{1}{a_1} (1 - \frac{b_1}{b_2} - \frac{c_1}{c_3})\) and \(\lim \inf_{t \to \infty} v(x, t) \geq \frac{1}{b_2} (1 - \frac{a_2}{a_3} - \frac{c_2}{c_3})\) locally uniformly in \([0, \infty)\);

(iii) \(\lim_{t \to \infty} w(x, t) = 1/c_3\) locally uniformly in \([0, \infty)\).

**Proof.** We first prove (i). Consider the problem

\[
\begin{aligned}
  \bar{u}_t &= r_1 \bar{u}(1 - a_1 \bar{u}), \\
  \bar{u}(0) &= \|u_0\|_{C((0, h_0))}.
\end{aligned}
\]

By the comparison principle,

\[
\begin{aligned}
  u(x, t) &\leq \bar{u}(t), \text{ for all } x \in [0, h(t)], t \geq 0.
\end{aligned}
\]

Then \(\lim \sup_{t \to \infty} u(x, t) \leq 1/a_1\) uniformly in \([0, \infty)\). Similarly, we can prove \(\lim \sup_{t \to \infty} v(x, t) \leq 1/b_2\), \(\lim \sup_{t \to \infty} w(x, t) \leq 1/c_3\) uniformly in \([0, +\infty)\).
Then we prove (ii). By our condition, we choose $0 < \varepsilon \ll 1$ such that $1 - \left(\frac{b_1}{b_2} + \frac{c_1}{c_3}\right)(1 + \varepsilon) > 0$ and $1 - \left(\frac{a_2}{a_1} + \frac{c_2}{c_3}\right)(1 + \varepsilon) > 0$. Fix

$$l > \max\left\{ \frac{d_1}{r_1} \frac{\pi}{2\sqrt{1 - \left(\frac{b_1}{b_2} + \frac{c_1}{c_3}\right)(1 + \varepsilon)}}, \frac{d_2}{r_2} \frac{\pi}{2\sqrt{1 - \left(\frac{a_2}{a_1} + \frac{c_2}{c_3}\right)(1 + \varepsilon)}}, h_0 \right\}.$$  

Due to $h_\infty = \infty$ and (i), there exists $T_l > 0$ such that $h(T_l) = l$ and $u(x, t) \leq \frac{1}{a_1}(1 + \varepsilon), v(x, t) \leq \frac{1}{b_2}(1 + \varepsilon)$ and $u(x, t) \leq \frac{1}{c_3}(1 + \varepsilon)$ for $(x, t) \in [0, l] \times [T_l, \infty)$. Consider the problem

$$\begin{cases}
\left(u^l\right)_t - d_1\left(u^l\right)_{xx} = r_1 u^l\left[1 - a_1 u^l - \left(\frac{b_1}{b_2} + \frac{c_1}{c_3}\right)(1 + \varepsilon)\right], & 0 < x < l, t > T_l, \\
u^l(0, t) = u^l(l, t) = 0, & t > T_l, \\
u^l(x, T_l) = u(x, T_l), & 0 \leq x \leq l,
\end{cases}$$

with the solution $u^l$, and the problem

$$\begin{cases}
\left(v^l\right)_t - d_2\left(v^l\right)_{xx} = r_2 v^l\left[1 - b_2 v^l - \left(\frac{a_2}{a_1} + \frac{c_2}{c_3}\right)(1 + \varepsilon)\right], & 0 < x < l, t > T_l, \\
v^l(0, t) = v^l(l, t) = 0, & t > T_l, \\
v^l(x, T_l) = v(x, T_l), & 0 \leq x \leq l,
\end{cases}$$

with the solution $v^l$. By the comparison principle,

$$u^l \leq u \text{ and } v^l \leq v \text{ for } (x, t) \in [0, l] \times [T_l, \infty).$$

It is well known that

$$u^l(x, t) \to \phi^l(x) > 0, v^l(x, t) \to \phi^l(x) > 0 \text{ in } C([0, l]) \text{ as } t \to \infty,$$

where $\phi^l(x)$ and $\phi^l(x)$ satisfy

$$\begin{cases}
d_1\phi^l_{xx} + r_1\phi^l\left[1 - \left(\frac{b_1}{b_2} + \frac{c_1}{c_3}\right)(1 + \varepsilon)\right] = 0, & 0 < x < l, \\
\phi^l(0) = \phi^l(l) = 0,
\end{cases}$$

and

$$\begin{cases}
d_2\phi^l_{xx} + r_2\phi^l\left[1 - \left(\frac{a_2}{a_1} + \frac{c_2}{c_3}\right)(1 + \varepsilon)\right] = 0, & 0 < x < l, \\
\phi^l(0) = \phi^l(l) = 0.
\end{cases}$$

So

$$\liminf_{t \to \infty} u(x, t) \geq \phi^l(x) \text{ and } \liminf_{t \to \infty} v(x, t) \geq \phi^l(x) \text{ uniformly in } [0, l].$$

It follows from [8] that

$$\phi^l(x) \to \frac{1}{a_1} - \frac{1}{a_1}\left(\frac{b_1}{b_2} + \frac{c_1}{c_3}\right)(1 + \varepsilon) \text{ and } \phi^l(x) \to \frac{1}{b_2} - \frac{1}{b_2}\left(\frac{a_2}{a_1} + \frac{c_2}{c_3}\right)(1 + \varepsilon)$$

locally uniformly in $[0, \infty)$ as $l \to \infty$.

Let $\varepsilon \to 0$, then

$$\liminf_{t \to \infty} u(x, t) \geq \frac{1}{a_1} - \frac{1}{b_2} - \frac{c_1}{c_3} \text{ and } \liminf_{t \to \infty} v(x, t) \geq \frac{1}{b_2} - \frac{a_2}{a_1} - \frac{c_2}{c_3},$$

locally uniformly in $[0, \infty)$. The proof of (iii) is similar to Lemma 3.2 in [7]. We complete the proof of this Lemma.
Theorem 3.4. Let \((u,v,w,h)\) be a solution of problem (1) with \(h_\infty = \infty\).

(i) If \(P_{BA} + P_{CA} \in (1, \infty), P_{AB} + P_{CB} \in (0, 1)\) and \(P_{BA} + P_{CA} \geq \frac{b_1}{b_2} (P_{AB} + P_{CB}) + 1\), then

\[
\lim_{t \to \infty} (u,v,w) = (0, \frac{1}{b_2}(1 - \frac{c_2}{c_3}), \frac{1}{c_3}) \text{ locally uniformly in } [0, \infty).
\]

(ii) If \(P_{BA} + P_{CA} \in (0, 1), P_{AB} + P_{CB} \in (1, \infty)\) and \(P_{AB} + P_{CB} \geq \frac{a_2}{a_1} (P_{BA} + P_{CA}) + 1\), then

\[
\lim_{t \to \infty} (u,v,w) = \left(\frac{1}{a_1}(1 - \frac{c_1}{c_3}), 0, \frac{1}{c_3}\right) \text{ locally uniformly in } [0, \infty).
\]

Proof. We first prove part (i). Define \(\bar{u}_1 = 1/a_1\). For any fixed \(L > 0\) and \(0 < \delta < 1\) with \(a_2(1/a_1 + \delta) + c_2(1/c_3 + \delta) < 1\) and \(0 < \varepsilon < 1\), let \(l_\varepsilon\) be given by Lemma 2.5 with \(d = d_2, \alpha = r_2b_2, \beta = \frac{1}{b_2}(1 - a_2(\bar{u}_1 + \delta) - c_2(\frac{1}{c_3} + \delta))\). According to Lemma 3.3 and \(h_\infty = \infty\), there exists \(T_1 > 0\) such that \(u(x,t) \leq \bar{u}_1 + \delta, 1/c_3 - \delta < w(x,t) < 1/c_3 + \delta, h(t) > l_\varepsilon\) for \(0 \leq x \leq l_\varepsilon, t \geq T_1\). Therefore, \(v\) satisfies

\[
\begin{align*}
&v_t - d_2v_{xx} \leq r_2v(1 - b_2v - a_2(\bar{u}_1 + \delta) - c_2(\frac{1}{c_3} + \delta)), \quad 0 \leq x \leq l_\varepsilon, t \geq T_1, \\
v_x(0,t) = 0, v(l_\varepsilon,t) \geq 0, \\
v(x,T_1) > 0,
\end{align*}
\]

By Lemma 2.5, we obtain

\[
\liminf_{t \to \infty} v(x,t) \geq \frac{1}{b_2}(1 - a_2(\bar{u}_1 + \delta) - c_2(\frac{1}{c_3} + \delta)) - \varepsilon \text{ uniformly on } [0, L] .
\]

By the arbitrariness of \(\varepsilon, \delta\) and \(L\), then

\[
\liminf_{t \to \infty} v(x,t) \geq \frac{1}{b_2}(1 - a_2\bar{u}_1 - \frac{c_2}{c_3}) := \underline{v}_1 \text{ locally uniformly on } [0, \infty).
\]  

(5)

If \(P_{BA} + P_{CA} \geq \frac{b_1}{b_2} (P_{AB} + P_{CB}) + 1\), then \(1 - b_1\bar{u}_1 - c_1/c_3 < 0\). Choose \(0 < \delta < 1\) such that \(1 - b_1\bar{u}_1 - c_1/c_3 < 0\). For any fixed \(L > 0\) and \(0 < \varepsilon < 1\), let \(l_\varepsilon\) be given by Lemma 2.7 with \(d = d_1, \lambda = r_1a_1, \xi = -\frac{1}{a_1}(1 - b_1\bar{u}_1 - c_1(\frac{1}{c_3} - \delta)) > 0\) and \(K = C_1\) (given in Lemma 2.2). Taking into account of (5), there exists \(T_2 > T_1\) such that \(v(x,t) > \underline{v}_1 - \delta\) for \(0 \leq x \leq l_\varepsilon, t \geq T_2\). Then \(u\) solves

\[
\begin{align*}
u_t - d_1u_{xx} &\leq r_2u(1 - a_1u - b_1(\bar{u}_1 - \delta) - c_1(\frac{1}{c_3} - \delta)), \quad 0 \leq x \leq l_\varepsilon, t \geq T_2, \\
u_x(0,t) = 0, u(l_\varepsilon,t) \leq C_1, \\
0 < u(x,T_2) \leq C_1, \\
0 \leq u(x,T_2) \leq C_1,
\end{align*}
\]

We get \(\limsup_{t \to \infty} u(x,t) < \varepsilon\) uniformly on \([0, L]\) by Lemma 2.7. Note that \(u \geq 0\) and \(\varepsilon, L\) are arbitrary, then

\[
\lim_{t \to \infty} u(x,t) = 0 \text{ locally uniformly on } [0, \infty).
\]  

(6)

When \(P_{BA} + P_{CA} = \frac{b_1}{b_2} (P_{AB} + P_{CB}) + 1\). For the fixed \(L > 0\) and \(0 < \delta, \varepsilon < 1\), let \(l_\varepsilon\) be given in Lemma 2.6 with \(d = d_1, \alpha = r_1a_1, \beta = \frac{(b_1 + c_1)}{a_1}\) and \(K = C_1\). We can choose a \(T_3 > T_1\) such that \(v(x,t) > \underline{v}_1 - \delta\) for \(0 \leq x \leq l_\varepsilon, t \geq T_3\). Then \(u\) solves

\[
\begin{align*}
u_t - d_1u_{xx} &\leq r_1u((b_1 + c_1)\delta - a_1u), \quad 0 \leq x \leq l_\varepsilon, t \geq T_3, \\
u_x(0,t) = 0, u(l_\varepsilon,t) \leq C_1, \\
u(x,T_3) > 0, \\
0 \leq x \leq l_\varepsilon.
\end{align*}
\]
Make use of Lemma 2.6, one gets
\[
\limsup_{t \to \infty} u(x, t) < \frac{(b_1 + c_1)\delta}{a_1} + \varepsilon \text{ uniformly on } [0, L].
\]
Since \( u \geq 0 \), by the arbitrariness of \( \varepsilon, \delta \) and \( L \), then (6) still holds.

For any given \( L > 0 \) and \( 0 < \delta, \varepsilon \ll 1 \), let \( l_\varepsilon \) be given in Lemma 2.5 with \( d = d_2, \alpha = r_2 b_2, \beta = \frac{1}{L^2}(1 - a_2 \delta - c_2(\frac{1}{c_3} + \delta)) \). Thanks to (6), there exists \( T_4 > T_1 \) such that \( u(x, t) < \delta \) for \( 0 \leq x \leq l_\varepsilon, t \geq T_4 \). Hence, \( v \) satisfies
\[
\begin{cases}
v_t - d_2 v_{xx} \geq r_2 v(1 - b_2 v - a_2 \delta - c_2(\frac{1}{c_3} + \delta)), & 0 \leq x \leq l_\varepsilon, t \geq T_4, \\
u_x(0, t) = 0, u(l_\varepsilon, t) \geq 0, & t \geq T_4, \\
u(x, T_4) > 0, & 0 \leq x \leq l_\varepsilon.
\end{cases}
\]
In the same way as (i), we can show that (ii) holds.

Similar to the proof of Theorem 3.4, together with Lemma 3.3(iii), we complete the proof of Theorem 3.5.
Theorem 3.6. Let \((u, v, w, h)\) be a solution of problem \((1)\) with \(h_\infty = \infty\). If \(P_{BA} + P_{CA} + P_{AB} + P_{CB} \in (0, 1)\), then
\[
\lim_{t \to \infty} (u, v, w) = \left( \frac{b_2(c_3 - c_1) - b_1(c_3 - c_2)}{c_3(a_1 b_2 - a_2 b_1)}, \frac{a_1(c_3 - c_2) - a_2(c_3 - c_1)}{c_3(a_1 b_2 - a_2 b_1)}, \frac{1}{c_3} \right)
\]
locally uniformly on \([0, \infty)\).

Proof. From the proof of Theorem 3.4, \(1 - b_1 v_1 - c_1/c_3 > 0\) if \(P_{BA} + P_{CA} + P_{AB} + P_{CB} \in (0, 1)\). Take \(L > 0\) and \(0 < \delta, \varepsilon \ll 1\), let \(l_\varepsilon\) be determined by Lemma 2.6 with \(d = d_1, \alpha = r_1 a_1, \beta = \frac{1}{a_1}(1 - b_1(v_1 - \delta) - c_1(\frac{1}{c_3} - \delta))\) and \(K = C_1\). By virtue of (5), we can find a \(T_6 > T_1\) such that \(v(x, t) > v_1 - \delta\) for \(0 \leq x \leq l_\varepsilon, t \geq T_6\). Therefore, \(w\) satisfies
\[
\left\{ \begin{array}{l}
u_1 u - d_1 u_{xx} \leq r_1 u(1 - a_1 u - b_1(v_1 - \delta) - c_1(\frac{1}{c_3} - \delta)), \quad 0 \leq x \leq l_\varepsilon, t \geq T_6, \\
v_x(0, t) = 0, v(l_\varepsilon, t) \leq C_1, \quad t \geq T_6, \\
v(x, T_6) > 0, \quad 0 \leq x \leq l_\varepsilon.
\end{array} \right.
\]
In the same way as above,
\[
\limsup_{t \to \infty} u(x, t) \leq \frac{1}{a_1}(1 - b_1 v_1 - \frac{c_1}{c_3}) := \bar{u}_2 \text{ locally uniformly on } [0, \infty). \tag{9}
\]
Repeating the above procedure, we can find two sequences \(\{\bar{u}_i\}\) and \(\{v_i\}\) such that for all \(i\)
\[
\limsup_{t \to \infty} u(x, t) \leq \bar{u}_i, \liminf_{t \to \infty} v(x, t) \geq v_i \text{ locally uniformly in } [0, \infty), \tag{10}
\]
which can be determined by the following iterative formulas:
\[
(\bar{u}_{i+1}, v_{i+1}) := \left( \frac{1}{a_1}(1 - \frac{c_1}{c_3} - b_1 v_i), \frac{1}{b_2}(1 - \frac{c_2}{c_3} - a_2 \bar{u}_{i+1}) \right).
\]
Similar to the above, there exist two sequences \(\{\bar{v}_i\}\) and \(\{\bar{v}_i\}\) such that for all \(i\)
\[
\liminf_{t \to \infty} u(x, t) \geq \bar{u}_i, \limsup_{t \to \infty} v(x, t) \leq \bar{v}_i \text{ locally uniformly in } [0, \infty). \tag{11}
\]
They are given by the following iterative formulas:
\[
(\bar{u}_{i+1}, \bar{v}_{i+1}) := \left( \frac{1}{a_1}(1 - \frac{c_1}{c_3} - b_1 \bar{v}_{i+1}), \frac{1}{b_2}(1 - \frac{c_2}{c_3} - a_2 \bar{u}_i) \right),
\]
where \(\bar{v}_1 := 1/b_2\).

Obviously,
\[
\lim_{i \to \infty} \bar{u}_i = \lim_{i \to \infty} u_i = \frac{b_2(c_3 - c_1) - b_1(c_3 - c_2)}{c_3(a_1 b_2 - a_2 b_1)}, \tag{12a}
\]
\[
\lim_{i \to \infty} \bar{v}_i = \lim_{i \to \infty} v_i = \frac{a_1(c_3 - c_2) - a_2(c_3 - c_1)}{c_3(a_1 b_2 - a_2 b_1)}. \tag{12b}
\]
Combining (10)-(12) with Lemma 3.3(iii), the proof is completed. \(\square\)

4. The criterion for spreading and vanishing.

Lemma 4.1. If \(h_\infty < \infty\), then \(h_\infty \leq h_* = \frac{d}{2} \min\{\sqrt{\frac{d}{r_i}}, i = 1, 2, 3\}\).
Proof. Due to Lemma 3.2, then \( \lim_{t \to \infty} \|u(\cdot, t), v(\cdot, t), w(\cdot, t)\|_{C^1([0,h(t)])} = 0 \) if \( h_\infty < \infty \). We assume \( h_\infty > h_0 \) to get a contradiction. If \( h_\infty > \frac{\pi}{2} \sqrt{\frac{d_1}{r_1}} \), there exists \( \varepsilon > 0 \) such that \( h_\infty > \frac{\pi}{2} \sqrt{\frac{d_1}{r_1(1-b_1c_1 \varepsilon)}} \). For such \( \varepsilon \), there exists \( T_0 > 0 \) such that

\[
l := h(T_0) > \frac{\pi}{2} \sqrt{\frac{d_1}{r_1(1-b_1c_1 \varepsilon)}} \quad \text{and} \quad v(x, t) \leq \varepsilon, w(x, t) \leq \varepsilon \quad \text{for} \quad t \geq T_0, 0 \leq x \leq h(t).
\]

Let \( z = z(x, t) \) be the unique solution of

\[
\begin{aligned}
z_t - d_1 z_{xx} &= r_1 z(1-a_1 z - b_1 \varepsilon - c_1 \varepsilon), \quad 0 < x < l, t \geq T_0, \\
z_x(0, t) &= z(l, t) = 0, \quad t \geq T_0, \\
z(x, T_0) &= u(x, T_0), \quad 0 < x < l.
\end{aligned}
\]

Applying the comparison principle, \( z(x, t) \leq u(x, t) \) for \( 0 \leq x \leq l, t \geq T_0 \). According to Proposition 3.3 in [4], since \( l > \frac{\pi}{2} \sqrt{\frac{d_1}{r_1(1-b_1c_1 \varepsilon)}} \), then

\[
\|z(\cdot, t) - Z(\cdot)\|_{C([0,l])} \to 0 \quad \text{as} \quad t \to \infty,
\]

where \( Z(x) \) is the unique solution of

\[
\begin{aligned}
-d_1 Z_{xx} &= r_1 Z(1-a_1 Z - b_1 \varepsilon - c_1 \varepsilon), \quad 0 < x < l, \\
Z_x(0) &= Z(l) = 0.
\end{aligned}
\]

Hence, \( \liminf_{t \to \infty} u(x, t) \geq \liminf_{t \to \infty} z(x, t) = Z(x) > 0 \) in \( (0, l) \). This is a contradiction. If \( h_\infty > \frac{\pi}{2} \sqrt{\frac{d_1}{r_1}} \) or \( h_\infty > \frac{\pi}{2} \sqrt{\frac{d_2}{r_1}} \), we can get a contradiction by using the similar argument.

Here, we give a comparison principle for later applications. As in [7], the following comparison lemma can be proved analogously.

**Lemma 4.2.** Let \((u, v, w, h)\) be the solution of problem (1). \((\bar{u}, \bar{v}, \bar{w}, \bar{h})\) \( \in [C^{2,1}(D^*)]^3 \times C^1([0,\infty)) \), where \( D^* := \{(x, t) : 0 \leq x \leq \bar{h}(t), t > 0\} \), admits

\[
\begin{aligned}
\bar{u}_t - d_1 \bar{u}_{xx} &\geq r_1 \bar{u}(1-a_1 \bar{u}), & (x, t) \in D^*, \\
\bar{v}_t - d_2 \bar{v}_{xx} &\geq r_2 \bar{v}(1-b_2 \bar{v}), & (x, t) \in D^*, \\
\bar{w}_t - d_3 \bar{w}_{xx} &\geq r_3 \bar{w}(1-c_3 \bar{w}), & (x, t) \in D^*, \\
\bar{u}_x(0, t) &\leq 0, \bar{v}_x(0, t) \leq 0, \bar{w}_x(0, t) \leq 0, & t > 0, \\
\bar{u}(\bar{h}(t), t) = \bar{v}(\bar{h}(t), t) = \bar{w}(\bar{h}(t), t) = 0, & t > 0, \\
\bar{h}'(t) &\geq -\mu(\bar{u}_x(\bar{h}(t), t) + \rho_1 \bar{v}_x(\bar{h}(t), t) + \rho_2 \bar{w}_x(\bar{h}(t), t)), & t > 0,
\end{aligned}
\]

if

\[
\bar{u}(x, 0) \geq u_0(x), \bar{v}(x, 0) \geq v_0(x), \bar{w}(x, 0) \geq w_0(x), x \in [0, h_0] \quad \text{and} \quad \bar{h}(0) \geq h_0,
\]

then \( \bar{h}(t) \geq h(t) \) for all \( t > 0 \), and

\[
\bar{u}(x, t) \geq u(x, t), \bar{v}(x, t) \geq v(x, t), \bar{w}(x, t) \geq w(x, t) \quad \text{for all} \quad (x, t) \in [0, h(t)] \times (0, \infty).
\]

**Lemma 4.3.** Assume that \( h_0 < h_* \), then there exists \( \mu > 0 \) depending on \( u_0, v_0 \) and \( w_0 \) such that \( h_\infty < \infty \) if \( \mu \leq \mu_\ast \).

**Proof.** Define

\[
\tilde{h}(t) := h_0(1 + \kappa - \frac{\kappa}{2} e^{-\gamma t}), t \geq 0,
\]

\[
U(y) := \cos\left(\frac{\pi}{2} y\right), 0 \leq y \leq 1,
\]

\[
\psi(x, t) := \theta e^{-\gamma t} U\left(\frac{x}{\tilde{h}(t)}\right), 0 \leq x \leq \tilde{h}(t),
\]

\[
4 \gamma \tilde{h}(t) - \frac{\pi^2}{4} \tilde{h}(t) + d_1 \tilde{h}(t) - r_1 \tilde{h}(t)^2 \geq \kappa \tilde{h}(t) - r_1 \tilde{h}(t)^2.
\]
Lemma 4.4. Assume that \( \kappa := \frac{1}{2} \left( \frac{h_0}{h_0} - 1 \right) > 0 \), \( \gamma \) and \( \theta \) are positive constants to be determined. It is easy to see that
\[
h_0(1 + \kappa) < \frac{\pi}{2} \sqrt{\frac{d_1}{r_i}}. \tag{13}
\]

Denote
\[
\gamma := \frac{1}{2} \min\{ \frac{d_i \pi^2}{4(1 + \kappa)^2 h_0^2} - r_i, i = 1, 2, 3 \}.
\]

By (13), we get \( \gamma > 0 \).

Let
\[
\bar{u}(x, t) = \bar{v}(x, t) = \bar{w}(x, t) = \psi(x, t),
\]
then
\[
\bar{u}_t - d_1 \bar{u}_{xx} - r_1 \bar{u}(1 - a_1 \bar{u}) \geq \theta U e^{-\gamma t} \left[ \frac{d_1 \pi^2}{4(1 + \kappa)^2 h_0^2} - r_1 - \gamma \right] \geq 0,
\]
\[
\bar{v}_t - d_2 \bar{v}_{xx} - r_2 \bar{v}(1 - b_2 \bar{v}) \geq \theta U e^{-\gamma t} \left[ \frac{d_2 \pi^2}{4(1 + \kappa)^2 h_0^2} - r_2 - \gamma \right] \geq 0,
\]
\[
\bar{w}_t - d_3 \bar{w}_{xx} - r_3 \bar{w}(1 - c_3 \bar{w}) \geq \theta U e^{-\gamma t} \left[ \frac{d_3 \pi^2}{4(1 + \kappa)^2 h_0^2} - r_3 - \gamma \right] \geq 0.
\]

If we choose
\[
\theta := \max\{ \|v_0\|_{L^\infty}, \|v_0\|_{L^\infty}, \|w_0\|_{L^\infty} \} / \cos\left( \frac{\pi}{2 + \kappa} \right),
\]
we have that \( \bar{u}(x, 0) \geq u_0(x), \bar{v}(x, 0) \geq v_0(x), \) and \( \bar{w}(x, 0) \geq w_0(x) \) for all \( x \in [0, h_0] \).

For
\[
\mu \leq \frac{\gamma \kappa (2 + \kappa) h_0^2}{2 \pi \theta (1 + p_1 + p_2)},
\]
then
\[
\bar{h}'(t) + \mu[\bar{u}_x(\bar{h}(t), t) + p_1 \bar{v}_x(\bar{h}(t), t) + p_2 \bar{w}_x(\bar{h}(t), t)]
\]
\[
= \kappa h_0 \gamma \frac{1}{2 \pi \theta} \left[ 1 - \frac{\bar{h}(t) \gamma \kappa h_0}{2 \pi \theta} \right] \geq \kappa h_0 \gamma \frac{1}{2 \pi \theta} \left[ 1 - \frac{\bar{h}(0) (2 + \kappa)}{2 \bar{h}(t)} \right] = \kappa h_0 \gamma \frac{1}{2 \pi \theta} \left[ 1 - \frac{\bar{h}(0)}{\bar{h}(t)} \right] \geq 0.
\]

By Lemma 4.2, we have
\( \bar{h}(t) \geq h(t) \) for all \( t \geq 0 \).

So we obtain \( h_\infty \leq h_0(1 + \kappa) \). We complete the proof. \( \square \)

**Lemma 4.4.** Assume that \( h_0 < h_\ast \), if
\[
\mu \geq \bar{\mu} := \max\{1, c_3 \|w_0\|_{L^\infty}\} \frac{d_3}{\rho_2} \frac{\pi}{2} \sqrt{\frac{d_3}{r_3}} - h_0 \left( \int_0^{h_0} w_0(x) dx \right)^{-1},
\]
then \( h_\infty = \infty \).

**Proof.** Consider the following problem
\[
\begin{cases}
\begin{aligned}
\bar{u}_t - d_1 \bar{u}_{xx} &= r_3 \bar{w}(1 - c_3 \bar{w}), & 0 < x < \bar{h}(t), t > 0, \\
\bar{w}_x(0, t) &= \bar{w}(\bar{h}(t), t) = 0, & t > 0, \\
\bar{h}'(t) &= -\mu \rho_2 \bar{w}_x(\bar{h}(t), t), & t > 0, \\
\bar{w}(x, 0) &= w_0(x), \bar{h} = h_0, & 0 < x < h_0.
\end{aligned}
\end{cases} \tag{14}
\]

By Lemma 4.2 in [27],
\( \bar{h}(t) \leq h(t), \bar{w}(x, t) \leq w(x, t) \) for \( t > 0, 0 < x < \bar{h}(t) \).
According to the proof of Lemma 3.7 in [7], we get $h(\infty) = \infty$. Then $h_\infty = \infty$. \qed

**Theorem 4.5.** Assume that $h_0 < h_\ast$, there exists $\mu^* \geq \mu_\ast > 0$ such that $h_\infty < \infty$ if $0 < \mu \leq \mu_\ast$ or $\mu = \mu^*$, and $h_\infty = \infty$ if $\mu > \mu^*$.

**Proof.** We define $\Gamma := \{\mu > 0 : h_\infty \leq h_\ast\}$. Due to Lemma 4.3, $\Gamma \neq \emptyset$. In view of Lemma 4.4, $\mu^* := \sup \Gamma \in [\mu, \bar{\mu}]$. By the definition of $\mu^*$ and Lemma 4.1, we get that $h_\infty = \infty$ if $\mu > \mu^*$.

Next, we prove that $h_\infty < \infty$ if $\mu = \mu^*$. If not, $h_\infty = \infty$. So there exists $T$ such that $h(T) > h_\ast$. Since the solution $(u, v, w, h)$ depends on $\mu$, we write $(u_\mu, v_\mu, w_\mu, h_\mu)$ instead of $(u, v, w, h)$. By the continuous dependence of $(u_\mu, v_\mu, w_\mu, h_\mu)$ on $\mu$, for small $\varepsilon > 0$, $h_\mu(T) > h_\ast$ for all $[\mu^* - \varepsilon, \mu^* + \varepsilon]$. Then $\sup \Gamma \leq \mu^* - \varepsilon$, which contradicts to the definition of $\mu^*$. Hence $\mu^* \in \Gamma$.

Denote $\Lambda := \{\nu > 0 : \nu \geq \mu_\ast\}$ such that $h_\infty \leq h_\ast$ for all $0 < \mu \leq \nu$ and $\mu_\ast := \sup \Lambda \leq \mu^*$. Using the similar way to the above, we obtain that $\mu_\ast \in \Lambda$. The proof is completed. \qed

5. Spreading speed. We first give a result of [3] as follow, which plays a significant role to the estimates of asymptotic spreading speed.

**Proposition 1.** (Proposition 2.1 in [3]) For any given $a > 0, b > 0, d > 0$ and $k \in [0, 2\sqrt{\alpha d}]$, the problem

$$
kU' = dU'' + U(a - bU) \text{ in } (0, \infty), U(0) = 0, U(\infty) = \frac{a}{b}
$$

admits a unique positive solution $U = U_k$. Moreover, $U_k'(z) > 0$ for $z \geq 0$. $U_k'(0) > U_k(0)$, $U_k(0) > U_k(z)$ for $z > 0$ and $k_1 < k_2$, and for each $\mu > 0$, there exists a unique $k_0 = k_0(a, b, d, \mu) \in (0, 2\sqrt{\alpha d})$ such that $\mu U_k'(0) = k_0$.

**Theorem 5.1.** Let $(u, v, w, h)$ be the solution of problem (1) with $h_\infty = \infty$. Then

$$
k_1 \leq \lim \inf_{t \to \infty} \frac{h(t)}{t} \leq \lim \sup_{t \to \infty} \frac{h(t)}{t} \leq k_2,
$$

where $k_1 := k_0(r_3, r_3 c_3, d_3, \mu_\rho_2)$, $k_2 := 2 \max \{\sqrt{r_i d_i}, i = 1, 2, 3\}$.

**Proof.** We first prove $\lim \inf_{t \to \infty} \frac{h(t)}{t} \geq k_1$. In the proof of Lemma 4.4,

$$
h(t) \leq h(t), w(x, t) \leq w(x, t) \text{ for } t > 0, 0 < x < h(t),
$$

where $(w, h)$ is the solution of (14). In view of Theorem 4.2 in [7],

$$
\lim_{t \to \infty} \frac{h(t)}{t} = k_0(r_3, r_3 c_3, d_3, \mu_\rho_2).
$$

Consequently,

$$
\lim \inf_{t \to \infty} \frac{h(t)}{t} \geq k_1.
$$

For any given $K \geq 2\sqrt{r_i d_i}, i = 1, 2, 3$, consider

$$
\begin{cases}
K u_1' + d_1 u_1'' + r_1 u_1(1 - a_1 u_1) = 0, & \text{in } \mathbb{R},
K u_2' + d_2 u_2'' + r_2 u_2(1 - b_2 v_2) = 0, & \text{in } \mathbb{R},
K w_1' + d_3 w_1'' + r_3 w_1(1 - c_3 w_3) = 0, & \text{in } \mathbb{R},
(u_1, v_1, w_1)(-\infty) = (1/a_1, 1/b_2, 1/c_3), (u_1, v_1, w_1)(\infty) = (0, 0, 0),
\end{cases}
$$

(15)

$$
u_1' < 0, v_1' < 0, w_1' < 0,
$$

in $\mathbb{R}$.

It follows from [12], problem (15) admits a solution

$$(u_\ast(\zeta), v_\ast(\zeta), w_\ast(\zeta))$$

where $\zeta := x - Kt$.
and

\[ u_* (0) = 1/2a_1, \; v_* (0) = 1/2b_2, \; w_* (0) = 1/2c_3. \]

Now we make some proper conditions to apply Lemma 4.2. Firstly, we choose \( \sigma > 1 \) such that

\[ \sigma u_* (x) > \|u_0\|_{L^\infty}, \sigma v_* (x) > \|v_0\|_{L^\infty} \text{ and } \sigma w_* (x) > \|w_0\|_{L^\infty} \text{ for all } x \in [0, h_0]. \]

Next, fix \( \bar{h}_0 > h_0 \) such that

\[
 u_* (h_0) < \min_{x \in [0, h_0]} [u_* (x) - u_0 (x) / \sigma],
\]

\[
 v_* (\bar{h}_0) < \min_{x \in [0, h_0]} [v_* (x) - v_0 (x) / \sigma],
\]

\[
 w_* (\bar{h}_0) < \min_{x \in [0, h_0]} [w_* (x) - w_0 (x) / \sigma],
\]

\[ -\sigma \mu (1 + \rho_1 + \rho_2) \min \{u_* (h_0), v_* (\bar{h}_0), w_* (\bar{h}_0)\} < K, \]

\[
 u_* (\bar{h}_0) \leq \frac{1}{a_1} (1 - \frac{1}{\sigma}), v_* (\bar{h}_0) \leq \frac{1}{b_2} (1 - \frac{1}{\sigma}), w_* (\bar{h}_0) \leq \frac{1}{c_3} (1 - \frac{1}{\sigma}).
\]

Let

\[
 \bar{u} (x, t) = \sigma u_* (x - K t) - \sigma u_* (\bar{h}_0),
\]

\[
 \bar{v} (x, t) = \sigma v_* (x - K t) - \sigma v_* (\bar{h}_0),
\]

\[
 \bar{w} (x, t) = \sigma w_* (x - K t) - \sigma w_* (\bar{h}_0),
\]

\[
 \bar{h} (t) := \bar{h}_0 + K t,
\]

then

\[
 \bar{u} (x, 0) \geq u_0 (x), \; \bar{v} (x, 0) \geq v_0 (x), \; \bar{w} (x, 0) \geq w_0 (x) \text{ for all } x \in [0, h_0].
\]

Direct computations yield that

\[
 \bar{u}_t - d_1 \bar{u}_{xx} - r_1 \bar{u} (1 - a_1 \bar{u}) \geq 0,
\]

\[
 \bar{v}_t - d_2 \bar{v}_{xx} - r_2 \bar{v} (1 - b_2 \bar{v}) \geq 0,
\]

\[
 \bar{w}_t - d_3 \bar{w}_{xx} - r_3 \bar{w} (1 - c_3 \bar{w}) \geq 0,
\]

and

\[
 \bar{h}' (t) = K > -\mu [\bar{u}_x (\bar{h} (t), t) + \rho_1 \bar{v}_x (\bar{h} (t), t) + \rho_2 \bar{w}_x (\bar{h} (t), t)].
\]

Due to Lemma 4.2, we obtain that \( h (t) \leq \bar{h} (t) \). Thus

\[
 \limsup_{t \to \infty} \frac{h (t)}{t} \leq \lim_{t \to \infty} \frac{\bar{h} (t)}{t} = K.
\]

By the arbitrariness of \( K \),

\[
 \limsup_{t \to \infty} \frac{h (t)}{t} \leq k_2.
\]

Hence we complete the proof. \( \square \)

**Remark 2.** The number \( k_2 = 2 \max \{ \sqrt{r_i d_i}, i = 1, 2, 3 \} \) is the minimal speed of travel wave fronts of the following problem

\[
 \begin{cases}
 u_t - d_1 u_{xx} = r_1 u (1 - a_1 u), & x \in \mathbb{R}, t > 0, \\
 v_t - d_2 v_{xx} = r_2 v (1 - a_2 u), & x \in \mathbb{R}, t > 0, \\
 w_t - d_3 w_{xx} = r_3 w (1 - c_3 w), & x \in \mathbb{R}, t > 0.
 \end{cases}
\]
When \( P_{BA} + P_{CA}, P_{AB} + P_{CB} \in (0, 1) \), denote \((u^*, v^*, w^*) := \lim_{t \to \infty} (u, v, w)\). If 
\[ a_1 u^* > b_1 v^* + c_1 w^* \quad \text{and} \quad b_2 v^* > a_2 u^* + c_2 w^*, \]
it follows from Theorem 5.8 in [16] that \( k_2 \) is also the minimal speed of travel wave fronts of

\[
\begin{align*}
&u_t - d_1 u_{xx} = r_1 u(1 - a_1 u - b_1 v - c_1 w), \quad x \in \mathbb{R}, t > 0, \\
v_t - d_2 v_{xx} = r_2 v(1 - a_2 u - b_2 v - c_2 w), \quad x \in \mathbb{R}, t > 0, \\
w_t - d_3 w_{xx} = r_3 w(1 - c_3 w), \quad x \in \mathbb{R}, t > 0.
\end{align*}
\]

**Theorem 5.2.** Let \((u, v, w, h)\) be the solution of problem (1) with \( h_\infty = \infty \).

(i) If \( P_{BA} + P_{CA}, P_{AB} + P_{CB} \in (0, 1) \), then

\[
\lim_{t \to \infty} \frac{h(t)}{t} \geq \max\{k_3, k_4\},
\]

where \( k_3 = k_0(r_1(1-P_{BA}-P_{CA}), r_1 a_1, d_1, \mu), k_4 = k_0(r_2(1-P_{BA}-P_{CB}), r_2 b_2, d_2, \mu) \).

(ii) If \( P_{BA} + P_{CA} \in (0, 1), P_{AB} + P_{CB} \in (1, \infty) \) and \( P_{AB} + P_{CB} \geq \frac{a_1^2}{a_2^2} (P_{BA} + P_{CA}) + 1 \), then

\[
\lim_{t \to \infty} \frac{h(t)}{t} \geq k_3.
\]

(iii) If \( P_{BA} + P_{CA} \in (1, \infty), P_{AB} + P_{CB} \in (0, 1) \) and \( P_{BA} + P_{CA} \geq \frac{b_1}{b_2} (P_{AB} + P_{CB}) + 1 \), then

\[
\lim_{t \to \infty} \frac{h(t)}{t} \geq k_4.
\]

**Proof.** Here we only prove part (i). Parts (ii) and (iii) can be proved by the similar way. For any \( 0 < \varepsilon \ll 1 \), consider the problem

\[
\begin{align*}
&kU'' - d_1 U''' = r_1 U(1 - a_1 U - b_1(\frac{1}{b_2} + \varepsilon) - c_1(\frac{1}{c_3} + \varepsilon)), \quad z > 0, \\
&U(0) = 0, U(\infty) = \frac{1}{a_1}(1 - b_1(\frac{1}{b_2} + \varepsilon) - c_1(\frac{1}{c_3} + \varepsilon)), \\
&\mu U'(0) = k.
\end{align*}
\]

By Proposition 1, there exists a unique \( k_0(r_1(1-b_1(\frac{1}{b_2} + \varepsilon) - c_1(\frac{1}{c_3} + \varepsilon), r_1 a_1, d_1, \mu) \)
such that (16) has a unique positive solution \( U \). In view of Lemma 3.3 and Theorem 3.6, we can find a \( T > h_0/(2k_0) \) sufficiently large such that

\[
u(x, t) = u^* - \varepsilon(b_1 + c_1)/a_1, \quad \text{for} \quad 0 \leq x \leq h(t), t \geq T, \tag{17}
\]

and

\[
v(x, t) < 1/b_2 + \varepsilon, w(x, t) < 1/c_3 + \varepsilon, \quad \text{for} \quad 0 \leq x \leq h_0, t \geq T.
\]

Define

\[
u(x, t) = U(\eta(t) - x), \quad \eta(t) = k_0(t - T) + h_0/2.
\]

Then \((\nu, \eta)\) satisfies

\[
\begin{align*}
&\nu_t - d_1 \nu_{xx} = r_1 \nu(1 - a_1 \nu - b_1(\frac{1}{b_2} + \varepsilon) - c_1(\frac{1}{c_3} + \varepsilon)), \quad 0 < x < \eta(t), t > T, \\
&\nu(\eta(t), t) = 0, \quad t \geq 0, \\
&\eta'(t) = -\mu \nu_x(\eta(t), t), \quad t \geq T, \\
&\eta(T) = h_0/2, \quad \nu(x, T) = U(h_0/2 - x), \quad 0 \leq x \leq h_0/2.
\end{align*}
\]

Since \( U'(z) > 0 \) for \( z > 0 \), one has \( \nu(x, T), \nu(0, t) < U(\infty) = \frac{1}{a_1}(1 - b_1(\frac{1}{b_2} + \varepsilon) - c_1(\frac{1}{c_3} + \varepsilon)) \). Clearly, \( \frac{1}{a_1}(1 - b_1(\frac{1}{b_2} + \varepsilon) - c_1(\frac{1}{c_3} + \varepsilon)) < u^* - \varepsilon(b_1 + c_1)/a_1 \). Together with (17),

\[
u(x, T) < u(x, T) \quad \text{in} \quad [0, \eta(T)], \quad \nu(0, t) < u(0, t) \quad \text{in} \quad [T, \infty).
\]
By the comparison principle, \((u, \eta)\) is a lower solution of (1) and then \(\eta(t) \leq h(t)\) for \(t \geq T\). It follows that
\[
limit \inf_{t \to \infty} \frac{h(t)}{t} \geq k_0 (r_1 (1 - b_1 (\frac{1}{\beta_2} + \varepsilon) - c_1 (\frac{1}{\gamma_3} + \varepsilon)), r_1 a_1, d_1, \mu).
\]
Letting \(\varepsilon \to 0\), we derive that
\[
\lim \inf_{t \to \infty} \frac{h(t)}{t} \geq k_3.
\]
Similarly,
\[
\lim \inf_{t \to \infty} \frac{h(t)}{t} \geq k_4.
\]
The proof is completed.

6. Numerical illustrations. In this section, we present some numerical simulations of problem (1). As the boundary is unknown, it is difficult to present the numerical solution of free boundary compared with the fixed boundary problem. Here, we use a similar way in [20] to deal with the problem. First, we carry out the discretization by finite differences. Second, we use the standard implicit scheme to deal with the equation (1); then we get a nonlinear algebraic system with the same number of equations and unknowns. Finally, we use the Newton-Raphson method to solve this nonlinear algebraic system.

Fix some coefficients and initial functions. Assume that
\[
d_1 = d_2 = 1, d_3 = 2, r_1 = r_3 = 1, r_2 = 0.5, \rho_1 = 1, \rho_2 = 0.5, b_1 = 0, b_2 = 2, c_1 = c_2 = 0.5, c_3 = 1,\]
\[
u_0(x) = v_0(x) = \frac{1}{2} \cos(\frac{\pi x}{2h_0}), w_0(x) = \frac{1}{4} (-x^2 + h_0^2), h_* = \frac{\pi}{2} \min\{\sqrt{\frac{d_i}{r_i}}, i = 1, 2, 3\} = \frac{\pi}{2}.
\]
Now we consider the long time behavior of solutions \((u, v, w)\).

**Example 6.1.** Fix \(h_0 = 1(< h_* = \pi/2), a_1 = 1, a_2 = 0.3, b_1 = 0.5, b_2 = 2, c_1 = c_2 = 0.5, c_3 = 1,\) then \(P_{BA} + P_{CA}, P_{AB} + P_{CB} \in (0, 1)\). The numerical solutions of problem (1) with \(\mu = 0.2\) and \(\mu = 1\) are shown in Figs. 2 and 3 respectively. From Fig. 2, we observe that the free boundary \(x = h(t)\) increases slowly, and the solutions decay to zero quickly, implying that all information vanish with small expanding capability \((\mu = 0.2)\). In Fig. 3 the free boundary \(x = h(t)\) increases faster than in Fig. 2 and the solutions stabilize to positive equilibria, that is, all information spread with large expanding capability \((\mu = 1)\). These accord with the criterion for spreading and vanishing (Theorem 4.5).
Example 6.2. Choose $h_0 = 2 (> h_\ast = \pi/2)$, the same parameters as in Example 6.1. The simulation results with $\mu = 0.3$ and $\mu = 1$ are shown in Figs. 4 and 5. It is easy to see that in Figs. 4 and 5 the free boundary increases quickly and the solutions stabilize to positive equilibria, which means that spreading always happens when $h_0 \geq h_\ast$.

Example 6.3. Let $h_0 = 1, \mu = 1, a_1 = b_2 = 1, b_1 = a_2 = 0.5, c_1 = 2, c_2 = 2.4, c_3 = 1$. From Fig. 6, we find that in (A) and (B) the solutions decay to zero quickly; in (C) the free boundary increases quickly and the solution stabilizes to a positive equilibrium. Information A, B vanish; C spreads, which conforms to Theorem 3.5.

Example 6.4. Choose $h_0 = 1, \mu = 1, a_1 = a_2 = b_2 = 1, b_1 = 2, c_1 = 0.6, c_2 = 0.4, c_3 = 1$, then $P_{BA} + P_{CA} > P_{AB} + P_{CB} > 1$. From Fig. 7, we notice that in
Example 6.5. Consider problem (1) with the following parameters $h_0 = 1, \mu = 1, a_1 = b_2 = 1, a_2 = 1.5, b_1 = 0.4, c_1 = c_2 = 0.5, c_3 = 1$, then $P_{BA} + P_{CA} < 1 < P_{AB} + P_{CB}$. It is easy to see that in Fig. 8(B) the solution decays to zero quickly; in Fig. 8(A) and (C) the free boundaries increase fast and the solutions stabilize to positive equilibria. Information B vanishes; A, C spread.

Example 6.6. Take $h_0 = 1, \mu = 1, a_1 = b_2 = 1, b_1 = a_2 = 0.6, c_1 = c_2 = 0.8, c_3 = 1$. Here $P_{BA} + P_{CA} = P_{AB} + P_{CB}$. The free boundaries increase fast and the
solutions stabilize to positive equilibria. All information spreading occurs as shown in Fig. 9.

Figure 9. u, v and w all spread.

Example 6.7. Fix \( h_0 = 1, \mu = 1, a_1 = b_2 = 1, c_3 = 1, b_1 = a_2 = 0.5, c_2 = 1.2 \) for Fig. 10 (A) and \( a_2 = 1, c_1 = 0.4, c_2 = 0.6 \) for Fig. 10 (B). We notice that in Fig. 10 (A) with increasing \( c_1 \), information A takes less time to converge to zero. If the government or media increases the intervention rate, then the information will gradually increase the asymptotic convergence rate of \( u \). In Fig. 10 (B), with increasing \( b_1 \), information A takes less time to converge to zero. This means that if the competition rate of information B to A is large, then information A is easy to vanish.

7. Discussion. In this paper, we study a competition model with intervention in online social networks where each information compete for influence. We obtain the criterion for spreading and vanishing. If the initial spreading area \([0, h_0]\) is within the critical size, i.e., \( h_0 < h_* \), information spread or vanish depending the size of the expanding capacity \( \mu \), namely, vanishing happens with small expanding capability and spreading happens with large expanding capability. Regardless of
the expanding capability, spreading always occurs if the initial spreading area is beyond the critical size.

If the system has either small initial spreading area $h_0$ and small expanding capacity $\mu$, or relatively large intervention rates $c_1, c_2$, then the ordinary information A and B vanish. If the system has either small initial spreading area and large expanding capacity, or large initial spreading area, whether the ordinary information vanishes or spreads depends on the total influence of other information w.r.t. this one.

Based on the theoretical results and the above numerical simulations, we find several kinds of the long time behavior of the information with the spreading case, i.e., all information spread; one ordinary information and official information spread, while the other ordinary information vanishes; two pieces of ordinary information vanish and official information spreads. To be concrete, when the total influence of information C w.r.t. A and B are less than 1 ($0 < P_{CA}, P_{CB} < 1$), there are three cases: A, B and C all spread if the total influence of information B and C w.r.t. A and the total influence of information A and C w.r.t. B is less than 1 ($0 < P_{BA} + P_{CA}, P_{AB} + P_{CB} < 1$) or the total influences are equal ($P_{BA} + P_{CA} = P_{AB} + P_{CB}$); A vanishes and B, C spread if the total influence of information B and C w.r.t. A is greater than 1 and the total influence of information A and C w.r.t. B ($P_{BA} + P_{CA} > P_{AB} + P_{CB} > 1$ or $P_{BA} + P_{CA} > 1 > P_{AB} + P_{CB}$); B vanishes and A, C spread if $P_{AB} + P_{CB} > P_{BA} + P_{CA} > 1$ or $P_{AB} + P_{CB} > 1 > P_{BA} + P_{CA}$. When the total influence of information C w.r.t. A and B are greater than 1 ($P_{CA}, P_{CB} > 1$), A and B vanish, C spreads.

There remain a number of open problems worth further investigation. For example, we discuss the long time behavior of the solution under some conditions. When $P_{BA} + P_{CA}, P_{AB} + P_{CB} \in (1, \infty)$, we only obtain the long time behavior of the solution with $P_{CA}, P_{CB} > 1$. For other case, we know the dynamics of information A, B and C by numerical illustrations. The corresponding theoretical result is left for the further study. Moreover, it would be much more challenging and complex if we study the diffusion model for multiple pieces of information with different boundaries instead of the same free boundary. We will discuss these problems in future.

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