Symmetry preserving regularization with a cutoff

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Abstract
A Lorentz and gauge symmetry preserving regularization method is proposed in 4 dimension based on momentum cutoff. We use the conditions of gauge invariance or freedom of shift of the loop-momentum to define the evaluation of the terms carrying Lorentz indices, e.g. proportional to $k_\mu k_\nu$. The remaining scalar integrals are calculated with a four dimensional momentum cutoff. The finite terms (independent of the cutoff) are unambiguous and agree with the result of dimensional regularization.

1 Introduction
Several regularization methods are known and used in quantum field theory: three and four dimensional momentum cutoff, Pauli-Villars type, dimensional regularization, lattice regularization, Schwinger’s proper time method and others directly linked to renormalization like differential renormalization. Dimensional regularization (DREG) \cite{1} is the most popular and most appreciated as it respects the gauge and Lorentz symmetries of the Lagrangian and textbooks give a detailed recipe. However DREG is not useful in all cases, for example it is not directly applicable to supersymmetric gauge theories as it modifies the number of bosons and fermions differently. DREG gets rid of (does not identify) naive quadratic divergencies, which may be important in low energy effective theories or in the Wilson’s renormalization group method. Another shortcoming is that together with (modified) minimal subtraction DREG is a “mass independent” scheme, particle thresholds and decoupling must put in the theory by hand \cite{2}. The choice of the ultraviolet regulator always depends on the problem.

In low energy effective field theories or in the Wilson renormalization group method there is an explicit cutoff, with well defined physical meaning. The cutoff gives the range of validity of the model. There are a few implementations: sharp momentum cutoff in 3 or 4 dimensions, modified operator regularization (based on Schwinger proper time method \cite{3}). In the Nambu-Jona-Lasinio model different regularizations proved to be useful calculating different physical quantities \cite{4}.

Regularization is an arbitrary algorithm that defines how to handle divergent momentum integrals. In this paper we show that with a reasonable and definite modification the loop calculations can be reduced to scalar integrals and those can be
evaluated with a sharp momentum cutoff. The results respect gauge (chiral and other) symmetries. Using a naive momentum cutoff the symmetries are badly violated. The calculation of the QED vacuum polarization function ($\Pi_{\mu\nu}(q)$) shows the problems. The Ward identity tells us that $q^2\Pi_{\mu\nu}(q) = 0$, e.g. in

$$\Pi_{\mu\nu}(q) = q_\mu q_\nu \Pi_L(q^2) - g_{\mu\nu} q^2 \Pi_T(q^2)$$

(1)

the two coefficients must be the same $\Pi(q^2)$. Usually the condition $\Pi(0) = 0$ is required to define a subtraction to keep the photon massless at 1-loop. However this condition is ambiguous when one calculates at $q^2 \neq 0$ in QED or in more general models, in the case of two different masses in the loop, it just fixes $\Pi(q^2, m_1, m_2)$ in the limit of degenerate masses at $q^2 = 0$. Ad hoc subtractions does not necessarily give satisfactory results.

There were several proposals to define symmetry preserving cutoff regularization. Usual way is to start with a regularization that respects symmetries and find the connection with momentum cutoff. In case of dimensional regularization already Veltman observed \[5\] that the naive quadratic divergencies can be identified with the poles in two dimensions ($d=2$) besides the usual logarithmic singularities in $d=4$. This idea turned out to be fruitful. Hagiwara et al. \[6\] calculated electroweak radiative corrections originating from effective dimension-six operators and later Harada and Yamawaki performed the Wilsonian renormalization group inspired matching of effective hadronic field theories \[7\]. Based on Schwinger’s proper time approach Oleszczuk proposed the operator regularization method \[8\], and showed that it can be formulated as a smooth momentum cutoff respecting gauge symmetries \[8, 9\]. A momentum cutoff is defined in the proper time approach in \[10\] with the identification under loop integrals

$$k_\mu k_\nu \to \frac{1}{d} g_{\mu\nu} k^2$$

(2)

instead \[1\] of the standard $d = 4$. The degree of the divergence determines $d$ in the result: $\Lambda^2$ goes with $d = 2$ and $\ln(\Lambda^2)$ with $d = 4$. This way the authors get correctly the divergent parts, they checked them in the QED vacuum polarization function and in the phenomenological chiral model.

Various authors formulated consistency conditions to maintain gauge invariance during the evaluation of divergent loop integrals. Finite \[11\] or infinite \[12, 13\] number of new regulator terms added to the propagators a’la Pauli-Villars, the integrals are tamed to have at most logarithmic singularities and become tractable. Differential renormalization can be modified to fulfill consistency conditions automatically, it is called constrained differential renormalization \[14\]. Another method, later proved to be equivalent with the previous one \[15\], is called implicit regularization, a recursive identity (similar to Taylor expansion) is applied and the external momentum ($q$) is moved to finite integrals. The divergent integrals contain only the loop momentum, thus universal local counter terms can cancel the potentially dangerous symmetry violating contributions \[16, 17\]. Gauge invariant regularization is implemented in exact

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1In what follows we denote the metric tensor by $g_{\mu\nu}$ both in Minkowski and Euclidean space.
renormalization group method providing a cutoff without gauge fixing in [18]. Introducing a multiplicative regulator in the d-dimensional integral, the integrals are calculable in the original dimension with the tools of DREG [19].

We show that there is a tension between naive application of the Lorentz symmetry and gauge invariance. The proper handling of the $k_\mu k_\nu$ terms in divergent loop-integrals solve the problems of momentum cutoff regularizations. We give a simple and well defined algorithm to have unambiguous finite and infinite terms, the finite terms agree with the result of DREG.

In section 2 we present how to get a momentum cutoff from DREG calculation, then we give the gauge symmetry preserving conditions emerging during the calculation of the vacuum polarization amplitude. In section 4 we discuss the condition of independence of momentum routing in loop diagrams. Section 5 shows that gauge invariance and freedom of shift in the loop momentum have the same root. Next we show that the conditions are related to vanishing integrated surface terms. In section 7 we give a definition of the new regularization method and in section 8 as an example we present the calculation of a general vacuum polarization function at 1-loop and close with conclusions.

2 Momentum cutoff via dimensional regularization

DREG is very efficient and popular, because it preserves gauge and Lorentz symmetries. Performing standard steps the integrals evaluated in $d = 4 - 2\epsilon$ dimension. Generally the loop-momentum integral Wick rotated and with a Feynman parameter ($x$) the denominators are combined, then the order of $x$ and momentum integrals are changed. Shifting the loop-momentum does not generate surface terms and it leads to spherically symmetric denominator, terms linear in the momentum are dropped and (2) is used. Singularities identified as $1/\epsilon$ poles, naive power counting shows that these are the logarithmic divergencies of the theory. In DREG quadratic or higher divergencies are set identically to zero. However Veltman noticed [5] that quadratic divergencies can be calculated in $d = 2 - 2(\epsilon - 1)$ in the limit $\epsilon \to 1$. This observation led to a cutoff regularization based on DREG.

Carefully calculating the one and two point Passarino-Veltman functions in DREG and in 4-momentum cutoff the divergencies can be matched as [6, 7]

$$4\pi\mu^2 \left( \frac{1}{\epsilon - 1} + 1 \right) = \Lambda^2,$$

$$\frac{1}{\epsilon} - \gamma_E + \ln \left( 4\pi\mu^2 \right) + 1 = \ln \Lambda^2,$$

where $\mu$ is the mass-scale of dimensional regularization. The finite part of a divergent quantity is defined as

$$f_{\text{finite}} = \lim_{\epsilon \to 0} \left[ f(\epsilon) - R(0) \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi + 1 \right) - R(1) \left( \frac{1}{\epsilon - 1} + 1 \right) \right],$$

$^2$Similar identification can be done in three dimensional integrals, too [20].
Fig. 1. 1-loop vacuum polarization diagram

where \( R(0), \ R(1) \) are the residues of the poles at \( \epsilon = 0, 1 \) respectively. Note that in the usual \( \epsilon \to 0 \) limit the left hand side (lhs) of (3) vanishes and no quadratic divergence appears in the original DREG.

The identifications above define a momentum cutoff calculation based on the symmetry preserving DREG formulae. This cutoff regularization is well defined and unique, but still relies on DREG. Let us see the main properties in the calculation of the vacuum polarization function. In \( \Pi_{\mu\nu} \) the quadratic divergence is partly coming from a \( k_\mu k_\nu \) term via \( \frac{1}{d} \cdot g_{\mu\nu} k^2 \), which is evaluated at \( d = 2 \) instead of the \( d = 4 \) in the naive cutoff calculation. The \( \Lambda^2 \) terms cancel if and only if this term is evaluated at \( d = 2 \). This is a warning that the usual \( k_\mu k_\nu \to \frac{1}{2} g_{\mu\nu} k^2 \) substitution during the naive cutoff calculation of divergent integrals might be too naive, especially as an intermediate step, the Wick rotation is legal only for finite integrals. A further finite term additional to the logarithmic singularity is coming from the well known expansion in \( \frac{1}{4-2\epsilon} \simeq \frac{1}{4} \left( \frac{1}{\epsilon} + \frac{1}{2} \right) \), and it is essential to retain gauge invariance. We stress that the shift of the loop momentum is allowed in DREG, an improved cutoff regularization should inherit it. In the next sections we derive consistency conditions for general regularizations.

3 Consistency conditions - gauge invariance

Calculation in a gauge theory ought to preserve gauge symmetries. We start with a single example, calculate the QED vacuum polarization function with massive electrons, and present the condition(s) of gauge invariance. We start generally (see Fig. 1.) with two different masses \([21]\) and restrict it to QED later.

\[
i\Pi_{\mu\nu}(q) = -(-i g)^2 \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left( \gamma_\mu \frac{k + m_a}{k^2 - m_a^2} \gamma_\nu \frac{k + q}{(k + q)^2 - m_b^2} \right).
\]

\[
\Pi_{\mu\nu} = g^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{2l_E l_{E\nu} - g_{\mu\nu} (l_E^2 + \Delta)}{(l_E^2 + \Delta)^2} \left[ 2x(1-x)q_{E\mu} q_{E\nu} + 2x(1-x)g_{\mu\nu} q_E^2 \right],
\]
where $\Delta = x(1-x)q_E^2 + (1-x)m_a^2 + xm_b^2$. In QED $m_a = m_b = m$ and $g = e$ it simplifies to $\Delta_1 = x(1-x)q_E^2 + m^2$. Having a symmetric denominator and symmetric volume of integration the terms linear in $l_{\mu\nu}$ are dropped. After changing the order of momentum- and $x$-integration the loop momentum is shifted with $x$-dependent values, $xq_E\mu$ and sum up the results during the integration. Different shifts sums up to a meaningful result only if the shift does not modify the value of the momentum integral (it will be discussed in the next section).

In QED the Ward identity tells us, that

$$q^\mu \Pi_{\mu\nu}(q) = 0.$$  \hspace{1cm} (8)

The terms proportional to $q_E$ fulfill the Ward-identity (8) and what remains is the condition of gauge invariance

$$\int_0^1 dx \int d^4 l_E \frac{l_{\mu\nu}l_{\nu\sigma}}{(l_E^2 + \Delta_1)^2} = \frac{1}{2} g_{\mu\nu} \int_0^1 dx \int d^4 l_E \frac{1}{(l_E^2 + \Delta_1)}. \hspace{1cm} (9)$$

This condition appeared already in [13, 16]. Any gauge invariant regulator should fulfill (9). It holds in dimensional regularization and in the momentum cutoff based on DREG of Section 2. In [11, 13] a similar relation defined the finite or infinite Pauli-Villars terms to maintain gauge invariance.

So far the $x$ integrals were not performed. Expanding the denominator in $q^2$, the $x$-integration can be easily done and we arrive at a condition for gauge invariance at each order of $q^2$. At order $q^{2n}$ we get (leaving out the factor $(2\pi)^4$)

$$\int d^4 l_E \frac{l_{\mu\nu}l_{\nu\sigma}}{(l_E^2 + m^2)^{n+1}} = \frac{1}{2n} g_{\mu\nu} \int d^4 l_E \frac{1}{(l_E^2 + m^2)^n}, \hspace{1cm} n = 1, 2, ... \hspace{1cm} (10)$$

The conditions (10) are valid for arbitrary $m^2$ mass, so it holds for any $m^2 = \Delta$ in two or n-point functions with arbitrary masses in the propagators. These conditions mean that in any gauge invariant regularization the two sides of (10) should give the same result. We will use this condition to define the lhs of (10) in the new improved cutoff regularization.

### 4 Consistency conditions - momentum routing

Evaluating any loops in QFT one encounters the problem of momentum routing. The choice of the internal momenta should not affect the result of the loop calculation. The simplest example is the 2-point function. In (6) there is a loop-momentum $k$, and $q$ the external momentum (see Fig. 1.) is put on one line $(k+q,k)$, but any partition of the external momentum $(k+q+p, \ k+p)$ must be as good as the original. The arbitrary shift of the loop momentum should not change the physics. This independence of the choice of the internal momentum gives a conditions. We will impose it on a very simple loop integral

$$\int d^4 k \frac{k_\mu}{k^2 - m^2} - \int d^4 k \frac{k_\mu + p_\mu}{(k+p)^2 - m^2} = 0 \hspace{1cm} (11)$$
which turns up during the calculation of the 2-point function. Expanding (11) in powers of \( p \) we get a series of condition, meaningful at \( p, p^3, p^5 \ldots \). At linear order we arrive at

\[
\int d^4k \left( \frac{p_\mu}{k^2 - m^2} - 2 \frac{k_\mu k \cdot p}{k^2 - m^2} \right) = 0, \tag{12}
\]

which is equivalent to (10) for \( n = 1 \). At order \( p^3 \) a linear combination of two conditions should vanish

\[
p_\rho p_\alpha p_\beta \int d^4k \left[ \left( \frac{4k_\alpha k_\beta}{(k^2 - m^2)^3} - \frac{g_{\alpha\beta}}{(k^2 - m^2)^2} \right) g_{\mu\rho} - 4k_\mu \left( \frac{2k_\alpha k_\beta k_\rho}{(k^2 - m^2)^4} \right) - \frac{g_{\alpha\beta}k_\rho}{(k^2 - m^2)^3} \right] = 0. \tag{13}
\]

These two conditions get separated if the freedom of the shift of the loop-momentum is considered in

\[
\int d^4k \left( \frac{g_{\mu\nu}}{(k^2 - m^2)^2} - 4 \frac{k_\mu k_\nu}{(k^2 - m^2)^3} \right) = 0, \tag{14}
\]

equivalent with (10) for \( n = 2 \). Using (14) twice the second part of the condition (13) connects 4 loop-momenta nominators to 2 \( k \)'s. Symmetrizing the indices we get

\[
\int d^4k \frac{k_\alpha k_\beta k_\mu k_\rho}{(k^2 - m^2)^4} = \frac{1}{24} \int d^4k \frac{g_{\alpha\beta}g_{\mu\rho} + g_{\alpha\mu}g_{\beta\rho} + g_{\alpha\rho}g_{\beta\mu}}{(k^2 - m^2)^2}. \tag{15}
\]

Invariance of momentum routing provides conditions for symmetry preserving regularization and these conditions are equivalent with the conditions coming from gauge invariance.

## 5 Gauge invariance and loop-momentum shift

We show at one loop level that gauge invariance of the vacuum polarization function is equivalent to invariance of a special loop integrand against shifting the loop momentum (11). Consider \( \Pi_{\mu\nu} \) defined in (6), performing the trace we get

\[
i \Pi_{\mu\nu}(q) = -g^2 \int \frac{d^4k}{(2\pi)^4} \left( \frac{k_\mu (k_\nu + q_\nu) + k_\nu (k_\mu + q_\mu) - g_{\mu\nu} (k^2 + k \cdot q - m_\alpha m_\beta)}{(k^2 - m_\alpha^2)((k + q)^2 - m_\beta^2)} \right). \tag{16}
\]

Specially in QED \( m_\alpha = m_\beta = m \) and gauge invariance requires (8), which simplifies to

\[
i q^\nu \Pi_{\mu\nu}(q) = -g^2 \int \frac{d^4k}{(2\pi)^4} \left( \frac{(k_\mu + q_\mu)}{((k + q)^2 - m^2)} - \frac{k_\mu}{(k^2 - m^2)} \right) = 0. \tag{17}
\]

This example shows that the Ward identity is fulfilled only if the shift of the loop momentum does not change the value of the integral, like in (11).
In [17] based on the general diagrammatic proof of gauge invariance it is shown that the Ward identity is fulfilled if the difference of a general n-point loop and its shifted version vanishes

\[-i \int d^4 p_1 \text{Tr} \left[ \frac{i}{p_n - m} \gamma^{\mu_n} ... \frac{i}{p_1 - m} \gamma^{\mu_1} - \frac{i}{p_n + \beta - m} \gamma^{\mu_n} ... \frac{i}{p_1 + \beta - m} \gamma^{\mu_1} \right] = 0.\]  

(18)

We interpret (17) and (18) as a necessary condition for gauge invariant regularizations.

6 Consistency conditions - vanishing surface terms

All the previous conditions are related to the volume integral of a total derivative

\[\int d^4 k \frac{\partial}{\partial k^\nu} \left( \frac{k_\mu}{(k^2 + m^2)^n} \right) = \int d^4 k \left( \frac{k_\mu k_\nu}{(k^2 + m^2)^{n+1}} - \frac{1}{2n} g_{\mu\nu} \frac{1}{(k^2 + m^2)^n} \right), \quad n = 1, 2, ...\]  

(19)

The total derivative on the lhs leads to surface terms, which vanish for finite valued integrals and should vanish for symmetry preserving regularization. In our improved regularization this will follow from new definitions. The left hand side is in connection with an infinitesimal shift of the loop momentum \(k\), it should be zero if the integral of the term in the delimiter is invariant against the shift of the loop momentum. The vanishing of this surface terms reproduces on the rhs the previous conditions (12) and (13). In (19) starting with any odd number of \(k\)’s in the nominator we end up with some conditions, three \(k\)’s for \(n = 3\) provide (15) after some algebra. Starting with even number of \(k_\mu\)'s in the nominator on the lhs in (19) we get relations between odd number of \(k_\mu\)'s in the nominators, which vanish separately.

These surface terms all vanish in DREG and give the basis of DREG respecting Lorentz and gauge symmetries. Vanishing of the surface term is inherited to any regularization, like improved momentum cutoff, if the identification (9) is understood to evaluate integrals involving even number of free Lorentz indices, e.g. nominators alike \(k_\mu k_\nu\). Vanishing of integrals with odd number of \(k\)’s in the nominator is also required by the symmetry of the integration volume.

7 Improved momentum cutoff regularization

We propose a new symmetry preserving regularization based on 4-dimensional momentum cutoff. During this improved momentum cutoff regularization method a simple sharp momentum cutoff is introduced to calculate the divergent scalar integrals in the end. The evaluation of loop-integrals starts with the usual Wick rotation, Feynman parameterization and loop-momentum shift. The only crucial modification is that the potentially symmetry violating loop integrals containing explicitly the loop momenta with free Lorentz indices are calculated with the identification

\[\frac{l_{E,\mu} l_{E,\nu}}{(l_E^2 + \Delta)^{n+1}} \rightarrow \frac{1}{2n} g_{\mu\nu} \frac{1}{(l_E^2 + \Delta)^n},\]  

(20)
under the loop integrals or with more momenta using the condition (15) or generalizations of it, like

\[ \frac{l_{E\mu}l_{E\nu}l_{E\rho}l_{E\sigma}}{(l_E^2 + \Delta)^{n+1}} \rightarrow \frac{1}{4n(n-1)} \frac{g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} + g_{\mu\nu}g_{\rho\sigma}}{(l_E^2 + \Delta)^{n-1}}. \]  

Integrals with odd number of the loop-momenta vanish identically. These identifications guarantee gauge invariance and freedom of shift in the loop momentum. Under any regularized momentum integrals the identifications (20) or generalizations like (21) are understood as a part of the regularization procedure for \( n = 1, 2, ... \).

What is the relation with the standard (textbook) \( k_\mu k_\nu \rightarrow 1/4 g_{\mu\nu} k^2 \) substitution? We have to modify it in case of divergent integrals to respect gauge symmetry, i.e. to fulfill (10). Lorentz invariance dictates that in (10) the lhs must be proportional to the only available tensor \( g_{\mu\nu} \), i.e.

\[ l_{E\mu}l_{E\nu} \rightarrow \frac{1}{d} g_{\mu\nu} l_E^2 \]  

can be used, where \( d \) is some number to determine\(^3\). Now both sides of equation (10) can be calculated with simple 4-dimensional momentum cutoff. The different powers of \( \Lambda \) can be matched on the two sides, and for \( n = 1 \) we get the following conditions (from gauge invariance) for the value of \( d \),

\[ \frac{1}{d} \Lambda^2 \rightarrow \frac{1}{2} \Lambda^2, \]  

\[ \frac{1}{d} \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) \rightarrow \frac{1}{4} \left( \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) + \frac{1}{2} \right), \]  

\[ \frac{1}{d} \rightarrow \frac{1}{4} \text{ for finite terms}. \]

We see that for finite valued integrals when the Wick-rotation is legal, the condition (10) and the rule (20) gives the usual \( k_\mu k_\nu \rightarrow 1/4 g_{\mu\nu} k^2 \) substitution, but for divergent cases we get back the identification partially found by [6, 7, 10] and others. Quadratic divergence goes with \( d = 2 \), logarithmic divergence goes with \( d = 4 \) plus a finite term (a shift), it is the +1 in equation (1). For more than 2 even number of indices generalizations of (22) should be used, for example in case of 4 indices the

\[ l_{E\mu}l_{E\nu}l_{E\rho}l_{E\sigma} \rightarrow \frac{1}{d(d+2)} \cdot (g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) l_E^4. \]

substitution works.

Fulfilling the condition (10) via the substitution (20) the results of momentum cutoff based on DREG of section 2 are completely reproduced performing the calculation in the physical dimensions \( d = 4 \) [21, 22]. The next example shows that the new regularization provides a robust framework for calculating loop integrals and respects symmetries.

\(^3\)The usual method is to calculate the trace (and get \( d=4 \)), but the trace is not well defined for divergent integrals.
8 Vacuum polarization function

As an example let us calculate the vacuum polarization function of Fig. 1. in a general gauge theory with fermion masses \( m_a, m_b \). For sake of simplicity we consider only vector couplings. Performing the trace in (6) we get (13). Now we can introduce a Feynman \( x \)-parameter, shift the loop-momentum and get (7) after dropping the linear terms. Generally we are interested in low energy observables like the precision electroweak parameters and need the first few terms in the power series of \( \Pi_{\mu\nu}(q) \). Using the rule (20) for \( n = 1 \) and expanding the denominator in \( q^2 \) the scalar loop and \( x \)-integrals can be easily calculated with a 4-dimensional momentum cutoff (\( \Lambda \)). The result in this construction is automatically transverse

\[
\Pi_{\mu\nu}(q) = \frac{g^2}{4\pi^2} \left( q^2 g_{\mu\nu} - q_\mu q_\nu \right) \left[ \Pi(0) + q^2 \Pi'(0) + \ldots \right].
\]

The terms independent of the cutoff completely agree with the results of DREG and even the logarithmic singularity can be matched with the \( 1/\epsilon \) terms using (31)

\[
\Pi(0) = \frac{1}{4} (m_a^2 + m_b^2) - \frac{1}{2} (m_a - m_b)^2 \ln \left( \frac{\Lambda^2}{m_a m_b} \right) - \frac{m_a^4 + m_b^4}{4 (m_a^2 - m_b^2)} \ln \left( \frac{m_b^2}{m_a^2} \right).
\]

The first derivative is

\[
\Pi'(0) = -\frac{2}{9} \frac{4m_a^2 m_b^2 - 3m_a m_b (m_a^2 + m_b^2)}{6 (m_a^2 - m_b^2)^2} + \frac{1}{3} \ln \left( \frac{\Lambda^2}{m_a m_b} \right) + \frac{(m_a^2 + m_b^2) (m_a^4 - 4m_a^2 m_b^2 + m_b^4) + 6m_a^3 m_b^3}{6 (m_a^2 - m_b^2)^3} \ln \left( \frac{m_b^2}{m_a^2} \right).
\]

The photon remains massless in QED, as in the limit, \( m_a = m_b \) we get \( \Pi(0) = 0 \).

In this paragraph we show that the proposed regularization is robust and gives the same result even if the calculation is organized in a different way. Introducing Feynman parameters and shifting the loop momentum can be avoided if we need only the first few terms in the Taylor expansion of \( q \). For small \( q \) the second denominator in (16) can be Taylor expanded, for simplicity we give the expanded integrand for equal masses, up to \( O(q^4) \)

\[
\Pi_{\mu\nu}(q) \simeq -g^2 \int \frac{d^4k_E}{(2\pi)^4} \left[ 2k_\mu k_\nu \left( \frac{1}{(k_E^2 + m^2)^2} - \frac{q_E^2}{(k_E^2 + m^2)^3} + 4 \frac{(k_E \cdot q_E)^2}{(k_E^2 + m^2)^4} \right) \right.
\]

\[
- \left. \frac{2 (k_E \cdot q_E \nu + k_E \cdot q_\mu k_E \cdot q_E)}{(k_E^2 + m^2)^3} - g_{\mu\nu} \left( \frac{1}{(k_E^2 + m^2)^2} - \frac{q_E^2}{(k_E^2 + m^2)^3} + 2 \frac{(k_E \cdot q_E)^2}{(k_E^2 + m^2)^4} \right) \right].
\]

Taking into account that \( k_E \cdot q_E = k_{Ea} q_{Ea} \), (20) and (21) can be used and the remaining scalar integrals can be easily calculated. The result agrees with (28) and (29) and
the finite terms with DREG if and only if we use the correct symmetry preserving substitutions. Applying the naive $\frac{1}{2}g_{\mu\nu}k^2$ substitution in both approaches the finite terms will differ not just from each other but also from the result of DREG.

The calculation of the $\Pi_{\mu\nu}$ function at 1-loop shows that the new regularization gives a robust gauge invariant result and the finite terms agree with DREG.

9 Conclusions

We presented in this paper a new method for the reliable calculation of divergent 1-loop diagrams with four dimensional momentum cutoff. Various conditions were derived to maintain gauge symmetry, to have the freedom of momentum routing or shifting the loop-momentum. These conditions were known by several authors \cite{11, 13, 16, 17}. Our new proposal is that these conditions will be satisfied during the regularization process if terms proportional to loop-momenta with free Lorentz indices (e.g. $\sim k_\mu k_\nu$) are calculated according to the special rules (20) and (21) or generalizations thereof. In the end the scalar integrals are calculated with a simple momentum cutoff. The calculation is robust - at least at 1-loop level - as we have shown via the fermionic contribution to the vacuum polarization function. The finite terms agree with the one in DREG in all examples. The connection with DREG is more transparent if one uses alternatively the $k_\mu k_\nu \to \frac{1}{2}g_{\mu\nu}k^2$ or (26) substitution and $d$ has different values determined by the degree of divergence in each term (23, 24, 25). $d = 2$ for quadratic divergencies, for log divergent terms $d = 4$ further there is an important finite shift, and simply $d = 4$ for finite terms. The new improved momentum cutoff regularization at 1-loop gives the same results as the cutoff regularization based on DREG of section 2. This observation gives a solid basis to use the new method for complicated diagrams or at higher loops, finite terms are expected to agree with DREG. We stress that this new regularization stands without DREG as the substitutions (20), (21) and scalar integration with a cutoff are independent of DREG. The success of both regularizations based on the property that they fulfill the consistency conditions of gauge invariance and momentum shifting.

The idea to use consistency conditions has been tested in the literature by various authors, we list few examples. The infinite terms in a cutoff calculation using (2) were identified correctly in \cite{10}, the authors showed that the 1-loop QED Ward identities are fulfilled and the Goldstone theorem is recovered in the phenomenological chiral model. Constrained differential renormalization proved to be useful also in supersymmetric \cite{23} and non-Abelian gauge theories, it fulfills Slavnov-Taylor identities at one and two loops \cite{24}. Implicit regularization \cite{16, 17} requires the same conditions as we used and it was successfully applied to the Nambu-Jona-Lasinio model \cite{16} and to higher loop calculations in gauge theory. It was shown that the conditions guarantee gauge invariance generally and the Ward identities are fulfilled explicitly in QED at two-loop order \cite{17}. In an effective composite Higgs model, the Fermion Condensate Model \cite{25} oblique radiative corrections (S and T parameters) were calculated in DREG and with the improved cutoff, the finite results completely agree. The calculation involved
vacuum polarization functions with two different fermion masses and no ambiguity appeared [21, 22].

This new regularization prescription is advantageous in special loop-calculations where one wants to keep the cutoff of the model, like in effective theories, derivation of renormalization group equations, extra dimensional scenarios or in models explicitly depending on the space-time dimensions like supersymmetric theories. We argue that the method can be successfully used in higher order calculations containing terms up to quadratic divergencies in (non-Abelian) gauge theories as it allows for shifts in the loop momenta, which guarantees the ’t Hooft identity [17, 26]. This symmetry preserving method can be used also in automatized calculations (similar to [28]) as even the Veltman-Passarino functions [27] can be defined with the improved cutoff.

### A Basic integrals

In this appendix we list the basic divergent integrals calculated by the regularization proposed in this paper. In the following formulae \( m^2 \) can be any loop momentum \( k \) independent expression depending on Feynman x parameter, external momenta, etc., e.g. \( \Delta(x, q, m_a, m_b) \).

\[
\int_0^\Lambda \frac{d^4k}{i(2\pi)^4} \frac{1}{k^2 - m^2} = -\frac{1}{(4\pi)^2} \left( \Lambda^2 - m^2 \ln \left( \frac{\Lambda^2}{m^2} \right) \right) \quad (31)
\]

\[
\int_0^\Lambda \frac{d^4k}{i(2\pi)^4} \frac{k_{\mu}k_{\nu}}{(k^2 - m^2)^2} = -\frac{1}{(4\pi)^2} \frac{g_{\mu\nu}}{2} \left( \Lambda^2 - m^2 \ln \left( \frac{\Lambda^2}{m^2} \right) \right) \quad (32)
\]

\[
\int_0^\Lambda \frac{d^4k}{i(2\pi)^4} \frac{k_{\mu}k_{\nu}k_{\mu}k_{\nu}}{(k^2 - m^2)^4} = -\frac{1}{(4\pi)^2} \frac{g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}}{8} \left( \Lambda^2 - m^2 \ln \left( \frac{\Lambda^2}{m^2} \right) \right) \quad (33)
\]

\[
\int_0^\Lambda \frac{d^4k}{i(2\pi)^4} \frac{1}{(k^2 - m^2)^3} = -\frac{1}{(4\pi)^2} \frac{g_{\mu\nu}}{4} \left( \Lambda^2 - 2m^2 \ln \left( \frac{\Lambda^2}{m^2} \right) + \frac{1}{2} m^2 \right) \quad (34)
\]

\[
\int_0^\Lambda \frac{d^4k}{i(2\pi)^4} \frac{k_{\mu}k_{\nu}}{(k^2 - m^2)^2} = \frac{1}{(4\pi)^2} \ln \left( \frac{\Lambda^2}{m^2} \right) - 1 \quad (35)
\]

\[
\int_0^\Lambda \frac{d^4k}{i(2\pi)^4} \frac{k_{\mu}k_{\nu}}{(k^2 - m^2)^3} = \frac{1}{(4\pi)^2} \frac{g_{\mu\nu}}{2} \ln \left( \frac{\Lambda^2}{m^2} \right) - 1 \quad (36)
\]

\[
\int_0^\Lambda \frac{d^4k}{i(2\pi)^4} \frac{k^2k_{\mu}k_{\nu}}{(k^2 - m^2)^4} = \frac{1}{(4\pi)^2} \frac{g_{\mu\nu}}{6} \ln \left( \frac{\Lambda^2}{m^2} \right) - \frac{3}{2} \quad (37)
\]

\[
\int_0^\Lambda \frac{d^4k}{i(2\pi)^4} \frac{k_{\mu}k_{\nu}k_{\mu}k_{\sigma}}{(k^2 - m^2)^4} = \frac{1}{(4\pi)^2} \frac{g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}}{24} \left( \ln \left( \frac{\Lambda^2}{m^2} \right) - 1 \right) \quad (38)
\]

(31),(33) depend on the same function of \( \Lambda \) as (32, 33) are traced back to (31) via (20) and (21). On the other hand (34) has a different \( \Lambda \) dependence showing that (20) or (21) applies only to free indices, contraction of Lorentz indices does not commute with the integration in case of divergencies.
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