Sliderule-like property of Wigner’s little groups and cyclic S-matrices for multilayer optics

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Abstract

It is noted that two-by-two “S” matrices in multilayer optics can be represented by the $Sp(2)$ group whose algebraic property is the same as the group of Lorentz transformations applicable to two space-like and one time-like dimensions. It is noted also that Wigner’s little groups have a sliderule-like property which allows us to perform multiplications by additions. It is shown that these two mathematical properties lead to a cyclic representation of the S-matrix for multilayer optics, as in the case of $ABCD$ matrices for laser cavities. It is therefore possible to write the $N$-layer S-matrix as a multiplication of the $N$ single-layer S-matrices resulting in the same mathematical expression with one of the parameters multiplied by $N$. In addition, it is noted, as in the case of lens optics, multilayer optics can serve as an analogue computer for the contraction of Wigner’s little groups for internal space-time symmetries of relativistic particles.

42.25.Gy, 42.15.Dp, 02.20.Rt, 11.30.Cp

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I. INTRODUCTION

In our previous paper on multilayer optics [1], it was shown that the complex two-by-two S-matrix formalism is equivalent to a two-by-two real matrix representation of the $Sp(2)$ group, which shares the same algebraic property as the Lorentz group applicable to two space-like and one time-like dimensions. This group has three independent parameters. It was shown furthermore that, under certain conditions, one of the off-diagonal elements vanishes, and the three remaining elements can be computed analytically. We called this the Iwasawa effect [1]. In this paper, we remove those "certain conditions" and achieve the same kind of simplification for all possible multilayer cases.

Indeed, the group $Sp(2)$ plays the central role in both quantum and classical optics, including multilayer optics [2]. It consists of two-by-two real matrices whose determinant is one. Each matrix contains at most three independent parameters. It is thus a simple matter to multiply two or three matrices. However, for multiplication of a large number of matrices presents a new problem. The product of those many matrices will also be one two-by-two matrix with a unit determinant, but how can we calculate their elements?

For example, let us look at laser cavities. It consists of a chain of $N$ identical two-lens systems, where $N$ is the number of cycles the light beam performs. The resulting $ABCD$ matrix can be written as a multiplication of $N$ identical matrices, but the resulting matrix has the same mathematical form as that for the single cycle [3].

Can we then expect a similar cyclic property in multilayer optics? We have shown in Ref. [1] that the $N$-dependence can be made quite transparent if the multilayer S-matrix [4] is reduced to the Iwasawa form. In this paper, we present the cyclic property for the most general form of multilayers, without the restriction we imposed in our previous paper [1]. We shall show that the core of the S-matrix takes the form

$$
\begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}, \quad
\begin{pmatrix}
\cosh \beta & \sinh \beta \\
\sinh \beta & \cosh \beta
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 \\
\gamma & 1
\end{pmatrix}, \quad \text{or} \quad
\begin{pmatrix}
1 & \gamma \\
0 & 1
\end{pmatrix}.
\tag{1}
$$

These matrices form the core of Wigner’s little groups applicable to the internal space-time symmetries of relativistic particles [5,6]. We note here that these matrices have the following interesting property.

We cannot write $(\cos \alpha_1 \times \cos \alpha_2) = \cos(\alpha_1 + \alpha_2)$ because it is wrong. However, in the two-by-two matrix form,

$$
\begin{pmatrix}
\cos \alpha_1 & -\sin \alpha_1 \\
\sin \alpha_1 & \cos \alpha_1
\end{pmatrix} \begin{pmatrix}
\cos \alpha_2 & -\sin \alpha_2 \\
\sin \alpha_2 & \cos \alpha_2
\end{pmatrix} = \begin{pmatrix}
\cos(\alpha_1 + \alpha_2) & -\sin(\alpha_1 + \alpha_2) \\
\sin(\alpha_1 + \alpha_2) & \cos(\alpha_1 + \alpha_2)
\end{pmatrix},
\tag{2}
$$

and similar expressions for the remaining matrices in Eq.(1). We call this the sliderule property of Wigner’s little groups.

If they are cycled $N$ times, they take the form

$$
\begin{pmatrix}
\cos(N\alpha) & -\sin(N\alpha) \\
\sin(N\alpha) & \cos(N\alpha)
\end{pmatrix}, \quad
\begin{pmatrix}
\cosh(N\beta) & \sinh(N\beta) \\
\sinh(\beta) & \cosh(\beta)
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
N\gamma & 1
\end{pmatrix}, \quad \text{and} \quad
\begin{pmatrix}
1 & N\gamma \\
0 & 1
\end{pmatrix},
\tag{3}
$$
respectively. This mathematical instrumentation works for laser cavity optics [3]. The question is whether this is applicable to multilayer optics.

The purpose of this paper is to show that the answer to the above question is YES. We note first that the S-matrix consists of N cycles. Each cycle consists two phase-shift matrices, one boundary matrix and its inverse, and this cycle does not take any of the forms given in Eq.(1) if we start the cycle from the boundary. In this paper, we show that it is possible to obtain the core in the form of Eq.(1) if we start the cycle from somewhere within one of the media between the two boundaries.

Throughout this paper, we avoid group theoretical languages and rely on explicit two-by-two matrices with real elements. However, in so doing, we are going through an important group theoretical aspect which became known to us only recently, namely on contractions of Wigner’s little groups. This aspect was discussed in detail in a recent paper on lens optics [7]. Thus, we shall borrow some of the mathematical identities from that paper.

In addition, in the present paper, we observe that Wigner’s little group has sliderule properties which allow us to convert multiplications into additions. This property was noted for one of the little groups in the paper of Han et al. In this paper, we shall show that all three of the little groups have the same sliderule property, using Eq.(2).

In Sec. II, we formulate the problem in terms of the S-matrix method widely used in multilayer optics [4,8,9], and show that the complex S-matrices can be transformed to real matrices by a conjugate transformation, and thus to the algebra of the $Sp(2)$ group which is by-now a familiar mathematical language in optics. In Sec. III, we import from the literature mathematical identities useful for the purpose of the present paper. They are derivable from Wigner’s little groups and their contractions. In Sec. IV, using the cyclic property of Eq.(3), it is shown possible to write the multilayer S-matrix as a multiplication of the $N$ single-layer S-matrices resulting in the same mathematical expression with one of the parameters multiplied by N. In Sec. V, it is pointed out that the mathematical identities presented in this paper can be tested experimentally. We discuss the condition under which the system can achieve the Iwasawa effect [1].

## II. FORMULATION OF THE PROBLEM

It was noted our previous paper that one cycle in $N$-layer optics starts with the boundary matrix of the form [10]

$$B(\eta) = \begin{pmatrix} \cosh(\eta/2) & \sinh(\eta/2) \\ \sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix},$$

(4)

which describes the transition from medium 2 to medium 1, taking into account both the transmission and reflection of the beam. As the beam goes through the medium 1, the beam undergoes the phase shift represented by the matrix

$$P(\phi_1) = \begin{pmatrix} e^{-i\phi_1/2} & 0 \\ 0 & e^{i\phi_1/2} \end{pmatrix}.$$

(5)

When the wave hits the surface of the second medium, the corresponding matrix is
\[ B(-\eta) = \begin{pmatrix} \cosh(\eta/2) & -\sinh(\eta/2) \\ -\sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix}, \]

which is the inverse of the matrix given in Eq.(4). Within the second medium, we write the phase-shift matrix as

\[ P(\phi_2) = \begin{pmatrix} e^{-i\phi_2/2} & 0 \\ 0 & e^{i\phi_2/2} \end{pmatrix}. \]

Then, when the wave hits the first medium from the second, we have to go back to Eq.(4). Thus, one cycle consists of

\[ M_1 = \begin{pmatrix} \cosh(\eta/2) & \sinh(\eta/2) \\ \sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix} \begin{pmatrix} e^{-i\phi_1/2} & 0 \\ 0 & e^{i\phi_1/2} \end{pmatrix} \begin{pmatrix} \cosh(\eta/2) & -\sinh(\eta/2) \\ -\sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix} \]
\[ \times \begin{pmatrix} e^{-i\phi_2/2} & 0 \\ 0 & e^{i\phi_2/2} \end{pmatrix}. \]

This matrix contains complex numbers, but we are interested in carrying out calculations with real matrices. This can be done if we make the following conjugate transformation [1]

Let us next consider the matrix

\[ C = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} & e^{i\pi/4} \\ -e^{-i\pi/4} & e^{-i\pi/4} \end{pmatrix}. \]

Then we have shown in our previous paper that

\[ M_2 = C M_1 C^{-1}, \]

with

\[ M_2 = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} \cos(\phi_1/2) & -\sin(\phi_1/2) \\ \sin(\phi_1/2) & \cos(\phi_1/2) \end{pmatrix} \begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix} \]
\[ \times \begin{pmatrix} \cos(\phi_2/2) & -\sin(\phi_2/2) \\ \sin(\phi_2/2) & \cos(\phi_2/2) \end{pmatrix}. \]

The conjugate transformation of Eq.(10) changes the boundary matrix \( B(\eta) \) of Eq.(4) to a squeeze matrix

\[ S(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \]

and the phase-shift matrices \( P(\phi_1) \) of Eq.(5) and Eq.(7) to rotation matrices

\[ R(\phi_i) = \begin{pmatrix} \cos(\phi_i/2) & -\sin(\phi_i/2) \\ \sin(\phi_i/2) & \cos(\phi_i/2) \end{pmatrix}, \]

with \( i = 1, 2 \).

Indeed, the matrices \( M_1 \) and \( M_2 \) can be written as
\[ M_1 = B(\eta)P(\phi_1)B(-\eta)P(\phi_2), \]
\[ M_2 = S(\eta)R(\phi_1)S(-\eta)R(\phi_2). \]  
(14)

The matrix \( M_2 \) can be obtained from \( M_1 \) by the conjugate transformation in Eq.(10). Conversely, \( M_1 \) can be obtained from \( M_2 \) through the inverse conjugate transformation:
\[ M_1 = C^{-1} M_2 C. \]  
(15)

In addition, the conjugate transformations have the following properties.
\[ (M_2)^N = C (M_1)^N C^{-1}, \quad (M_1)^N = C^{-1} (M_2)^N C. \]  
(16)

Thus, we can study \( M_2 \) in order to study \( M_1 \). The advantage of \( M_2 \) is that it consists of real matrices. The group of these matrices is called \( Sp(2) \) which is like (isomorphic) the Lorentz group applicable to three space and one time dimensions. This group contains very rich group theoretical contents including those of Wigner’s little groups. We intend to study \( M_2 \) in terms of those little groups.

The problem is that \( M_2 \) takes a simple form, and \((M_2)^2\) is manageable, but we cannot predict what form \((M_2)^N\) takes. In this paper, we shall construct the core matrix of the form of Eq.(1) for multilayer optics. Then, as we can see in Eq.(3), the chain effect is straight-forward. We shall calculate \((M_2)^N\) first and then \((M_1)^N\).

**III. MATHEMATICAL IDENTITIES FROM THE LORENTZ GROUP**

Wigner’s little groups were formulated for internal space-time symmetries of relativistic particles [5,6]. However, they produced many mathematical identities useful in other branches of physics, including classical layer optics which depends heavily on two-by-two matrices. The correspondence between the two-by-two and four-by-four representations of the Lorentz group has been repeatedly discussed in the literature [1,3,7]. In the two-by-two representation, we write the rotation matrix around the \( y \) axis as
\[ \left( \begin{array}{cc} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{array} \right), \]  
(17)
and the boost matrices along the \( z \) and \( x \) axes as
\[ \left( \begin{array}{cc} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{array} \right), \quad \left( \begin{array}{cc} \cosh \lambda & \sinh \lambda \\ \sinh \lambda & \cosh \lambda \end{array} \right), \]  
(18)
respectively. We shall use only these three matrices in this paper.

We use the following identity which Baskal and Kim introduced recently in their paper on lens optics and group contractions [7,11].
\[ \left( \begin{array}{cc} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{array} \right) \left( \begin{array}{cc} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{array} \right) \left( \begin{array}{cc} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{array} \right) \]
\[ = \left( \begin{array}{cc} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{array} \right) \left( \begin{array}{cc} \cosh \lambda & \sinh \lambda \\ \sinh \lambda & \cosh \lambda \end{array} \right) \left( \begin{array}{cc} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{array} \right), \]  
(19)
with
\[
\cos(\phi/2) = \cosh \lambda \cos \theta,
\]
\[
e^{2\eta} = \frac{\cosh \lambda \sin \theta + \sinh \lambda}{\cosh \lambda \sin \theta - \sinh \lambda}.
\]  

(20)

The left-hand side of the above expression is one rotation matrix sandwiched by one boost matrix and its inverse, while the right-hand side consists of one boost matrix sandwiched between two identical rotation matrices.

The left-hand side of Eq.(19) is the same as the first three matrices of the core matrix \(M_2\) given in Eq.(11). However, the fourth matrix is a rotation matrix. Since one-rotation matrix multiplied by another rotation matrix is still a rotation matrix, the core matrix \(M_2\) is one boost matrix sandwiched between two different rotation matrices. Thus, the problem is to find a transformation which will make those two rotation matrices the same, and go back to the form of the left-hand side of Eq.(19). We shall come back to this problem in Sec. IV.

If we complete the matrix multiplications of both side, the result is
\[
\begin{pmatrix}
\cos(\phi/2) & -e^n \sin(\phi/2) \\
e^{-n} \sin(\phi/2) & \cos(\phi/2)
\end{pmatrix}
= \begin{pmatrix}
\cosh \lambda \cos \theta & -\left(\cosh \lambda \sin \theta + \sinh \lambda\right) \\
\cosh \lambda \sin \theta - \sinh \lambda & \cosh \lambda \cos \theta
\end{pmatrix}.
\]

(21)

Then we can write \(\phi\) and \(\eta\) in terms of \(\lambda\) and \(\theta\) as given in Eq.(20). The parameters \(\lambda\) and \(\theta\) can be written in terms of \(\phi\) and \(\eta\) as
\[
\cosh \lambda = (\cosh \eta) \sqrt{1 - \cos^2(\phi/2) \tanh^2 \eta},
\]
\[
\cos \theta = \frac{\cos(\phi/2)}{(\cosh \eta) \sqrt{1 - \cos^2(\phi/2) \tanh^2 \eta}}.
\]

(22)

The above relation is valid only for \((\cosh \lambda \sin \theta/2 - \sinh \lambda)\) is positive. If it is negative, the left-hand side of the above expression should be
\[
\begin{pmatrix}
e^{n/2} & 0 \\
0 & e^{-n/2}
\end{pmatrix}
\begin{pmatrix}
\cosh(\chi/2) & -\sinh(\chi/2) \\
-\sinh(\chi/2) & \cosh(\chi/2)
\end{pmatrix}
\begin{pmatrix}
e^{-n/2} & 0 \\
0 & e^{n/2}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\cosh(\chi/2) & -e^n \sinh(\chi/2) \\
-e^{-n} \sinh(\chi/2) & \cosh(\chi/2)
\end{pmatrix},
\]

(23)

with
\[
\cosh(\chi/2) = \cosh \lambda \cos \theta,
\]
\[
e^{2\eta} = \frac{\cosh \lambda \sin \theta + \sinh \lambda}{\sinh \lambda - \cosh \lambda \sin \theta}.
\]

(24)

Conversely, \(\lambda\) and \(\theta\) can be written in terms of \(\chi\) and \(\eta\) as
cosh \lambda = (\cosh \eta) \sqrt{\cosh^2(\chi/2) - \tanh^2 \eta}, \\
\cos \theta = \frac{\cosh(\chi/2) \cosh(\eta)}{(\cosh \eta) \sqrt{\cosh^2(\chi/2) - \tanh^2 \eta}}. \tag{25}

An interesting case is when \sinh \lambda - \cosh \lambda \sin \theta becomes zero, and \eta becomes very large. If we insist that

\[ e^\eta \sin(\phi/2) = u, \tag{26} \]

remain finite, then \phi/2 must become very small. On the right-hand side,

\[ u = 2 \sinh \lambda, \text{ with } \sin \theta = \tanh \lambda. \tag{27} \]

The net result is that both sides take the form

\[ \begin{pmatrix} 1 & -2 \sinh \lambda \\ 0 & 1 \end{pmatrix}. \tag{28} \]

In their recent paper [7], Kim and Baskal studied in detail the transition from Eq.(21) to Eq.(23) through Eq.(28), and showed that the one-lens camera goes through this transition as we try to focus the image. Mathematically, the system goes through group contraction processes. In the present paper, we show that the same contraction process can be achieved in multilayer optics.

IV. CYCLIC REPRESENTATION OF THE S MATRIX

It was noted in Sec. II that each cycle consists of

\[ (SR_1S^{-1}R_2), \tag{29} \]

with

\[ R_1 = R(\phi_1), \quad R_2 = R(\phi_2), \tag{30} \]

of Eq.(13) respectively. The squeeze matrix \( S \) is given in Eq.(12). For the layer consisting of \( N \) cycles, let us consider the chain

\[ M_2^N = (SR_1S^{-1}R_2) \left( SR_1S^{-1}R_2 \right) \left( SR_1S^{-1}R_2 \right) \ldots \left( SR_1S^{-1}R_2 \right). \tag{31} \]

According to Eq.(19), we can now write \( SR_1S^{-1} \) in the above expression as

\[ SR_1S^{-1} = R_3 X R_3, \tag{32} \]

with

\[ R_3 = \begin{pmatrix} \cos(\phi_3/2) & -\sin(\phi_3/2) \\ \sin(\phi_3/2) & \cos(\phi_3/2) \end{pmatrix}, \quad X = \begin{pmatrix} \cosh \lambda & \sinh \lambda \\ \sinh \lambda & \cosh \lambda \end{pmatrix}, \tag{33} \]
and
\[
cosh \lambda = (cosh \eta)\sqrt{1 - \cos^2(\phi_1/2) \tanh^2 \eta},
\]
\[
\cos \phi_3 = \frac{\cos(\phi_1/2)}{(cosh \eta)\sqrt{1 - \cos^2(\phi_1/2) \tanh^2 \eta}}.
\] (34)

The parameters \( \lambda \) and \( \phi_3 \) are determined from \( \eta \) and \( \phi_1 \) which are the input parameters from the optical properties of the media.

The chain of Eq.(31) becomes
\[
M_2^N = (R_3X R_3 R_2) (R_3X R_3 R_2) (R_3X R_3 R_2) \ldots (R_3X R_3 R_2).
\] (35)

Let us next introduce the rotation matrix \( R(\alpha) \) as
\[
R(\alpha) = (R_2)^{1/2} R_3,
\] (36)

with
\[
\alpha = \phi_3 + \frac{1}{2} \phi_2,
\] (37)

where \( \phi_2 \) is an input parameter. Since \( \phi_3 \) is determined by \( \eta \) and \( \phi_1 \), the rotation angle \( \alpha \) is determined by the three input parameters, namely \( \eta \), \( \phi_1 \), and \( \phi_2 \).

In terms of \( R = R(\alpha) \), the chain of Eq.(35) becomes
\[
M_2^N = R_3 R^{-1} (RX R)(RX R)(RX R) \ldots (RX R) R^{-1} R_3 R_2.
\] (38)

Since \( R_3 R^{-1} = R_2^{-1/2} \) and \( R^{-1} R_3 R_2 = R_2^{1/2} \) from Eq.(37),
\[
M_2^N = (R_2)^{-1/2} (RX R)(RX R)(RX R) \ldots (RX R)(R_2)^{1/2}.
\] (39)

According to Eq.(19) and Eq.(21), we can now write \( RX R \) as
\[
RX R = \begin{pmatrix}
\cosh \lambda \cos \alpha & -(\cosh \lambda \sin \alpha + \sinh \lambda) \\
\cosh \lambda \sin \alpha - \sinh \lambda & \cosh \lambda \cos \alpha
\end{pmatrix}.
\] (40)

According to the formulas given in Sec. III, especially Eq.(19), \( RX R \) can also be written as
\[
RX R = Z A Z^{-1},
\] (41)

with
\[
Z = \begin{pmatrix}
e^{\xi/2} & 0 \\
0 & e^{-\xi/2}
\end{pmatrix}.
\] (42)

Now the two-by-two matrix \( A \) can take one of the following forms.

If the off-diagonal elements of the matrix of Eq.(40) has opposite signs, the \( A \) matrix becomes
\[ A = \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}, \]  

with

\[ \cos(\phi/2) = \cosh \lambda \cos \alpha, \]

\[ e^{2\xi} = \frac{\cosh \lambda \sin \alpha + \sinh \alpha}{\cosh \lambda \sin \alpha - \sinh \lambda}, \]  

If, on the other hand, the off-diagonal elements of the matrix \( RXR \) have the same sign, the matrix \( A \) should be written as

\[ A = \begin{pmatrix} \cosh(\chi/2) & -\sinh(\chi/2) \\ -\sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}, \]  

with

\[ \cosh(\chi/2) = \cosh \lambda \cos \alpha, \]

\[ e^{2\xi} = \frac{\cosh \lambda \sin \alpha + \sinh \lambda}{\sinh \lambda - \cosh \lambda \sin \alpha}. \]  

We note from Eq.(43) and Eq.(45) that the matrix \( A \) takes circular or hyperbolic forms depending on the sign of the lower-left element of Eq.(40) which is

\[ \sinh \lambda - (\sin \alpha) \cosh \lambda, \]  

and note that this expression can become from a positive to negative number continuously as the parameters \( \lambda \) and \( \alpha \) vary. These two parameters are determined from the reflection and transmission properties of the media.

While expression of Eq.(47) makes the continuous transition, it has to go through zero. If it vanishes,

\[ RXR = \begin{pmatrix} 1 & -2 \sinh \lambda \\ 0 & 1 \end{pmatrix}. \]

The transition of \( A \) from Eq.(43) to Eq.(45) through this process has been discussed in detail in Ref. [7] in connection with the contraction of Wigner’s little groups.

As we noted in Sec. II, the matrix \( A \) has the desired cyclic property. Thus,

\[ M_2^N = (R_2)^{-1/2} \left[ (ZAZ^{-1}) (ZAZ^{-1}) (ZAZ^{-1}) \ldots (ZAZ^{-1}) \right] (R_2)^{1/2}. \]

Consequently

\[ M_2^N = (R_2)^{-1/2} \left[ Z A^N Z^{-1} \right] (R_2)^{1/2}. \]

If \( A \) takes the form of Eq.(43),

\[ A^N = \begin{pmatrix} \cos(N\phi/2) & -\sin(N\phi/2) \\ \sin(N\phi/2) & \cos(N\phi/2) \end{pmatrix}. \]
For $A$ given in Eq.(45),

$$A^N = \begin{pmatrix} \cosh(N\chi/2) & -\sinh(N\chi/2) \\ -\sinh(N\chi/2) & \cosh(N\chi/2) \end{pmatrix}. \tag{52}$$

As Eq.(48),

$$(RXR)^N = \begin{pmatrix} 1 & -2N \sinh \lambda \\ 0 & 1 \end{pmatrix}. \tag{53}$$

Then the calculation of $(M_2)^N$ for the $N$-layer case is straightforward. We can now compute the matrix $(M_1)^N$ using the conjugate transformation of Eq.(16). Let us write our result in two-by-two matrices:

$$M_2^N = \left[\begin{array}{cc} \cos(\phi_2/4) & -\sin(\phi_2/4) \\ \sin(\phi_2/4) & \cos(\phi_2/4) \end{array}\right] \left[\begin{array}{cc} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{array}\right] \left[\begin{array}{cc} \cos(N\phi/2) & -\sin(N\phi/2) \\ \sin(N\phi/2) & \cos(N\phi/2) \end{array}\right]$$

$$\times \left[\begin{array}{cc} e^{-\xi/2} & 0 \\ 0 & e^{\xi/2} \end{array}\right] \left[\begin{array}{cc} \cos(\phi_2/4) & \sin(\phi_2/4) \\ -\sin(\phi_2/4) & \cos(\phi_2/4) \end{array}\right], \tag{54}$$

for $A$ of Eq.(43). For $A$ of Eq.(45),

$$M_2^N = \left[\begin{array}{cc} \cos(\phi_2/4) & -\sin(\phi_2/4) \\ \sin(\phi_2/4) & \cos(\phi_2/4) \end{array}\right] \left[\begin{array}{cc} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{array}\right] \left[\begin{array}{cc} \cosh(N\chi/2) & -\sinh(N\chi/2) \\ -\sinh(N\chi/2) & \cosh(N\chi/2) \end{array}\right]$$

$$\times \left[\begin{array}{cc} e^{-\xi/2} & 0 \\ 0 & e^{\xi/2} \end{array}\right] \left[\begin{array}{cc} \cos(\phi_2/4) & \sin(\phi_2/4) \\ -\sin(\phi_2/4) & \cos(\phi_2/4) \end{array}\right]. \tag{55}$$

If the lower-left element given in Eq.(47) vanishes, we have to go back to Eq.(39) and Eq.(48), and write

$$M_2^N = \begin{pmatrix} \cos(\phi_2/4) & -\sin(\phi_2/4) \\ \sin(\phi_2/4) & \cos(\phi_2/4) \end{pmatrix} \begin{pmatrix} 1 & -2N \sinh \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\phi_2/4) & \sin(\phi_2/4) \\ -\sin(\phi_2/4) & \cos(\phi_2/4) \end{pmatrix}. \tag{56}$$

As we noted in Sec. II, we use $M_2$ and $M_2^N$ for mathematical convenience. In the real world, we have to use $M_1$ and $M_1^N$. It is not difficult to write this expression using the conjugate transformation of Eq.(15). It can be written as

$$M_1^N = \left[\begin{array}{cc} e^{-i\phi_2/4} & 0 \\ 0 & e^{i\phi_2/4} \end{array}\right] \left[\begin{array}{cc} \cosh(\xi/2) & \sinh(\xi/2) \\ \sinh(\xi/2) & \cosh(\xi/2) \end{array}\right] \left[\begin{array}{cc} e^{-iN\phi/2} & 0 \\ 0 & e^{iN\phi/2} \end{array}\right]$$

$$\times \left[\begin{array}{cc} \cosh(\xi/2) & -\sinh(\xi/2) \\ -\sinh(\xi/2) & \cosh(\xi/2) \end{array}\right] \left[\begin{array}{cc} e^{i\phi_2/4} & 0 \\ 0 & e^{-i\phi_2/4} \end{array}\right]. \tag{57}$$

if $A$ takes the form of Eq.(43) with a positive value of Eq.(47). If it takes the form of Eq.(45) with a negative value of Eq.(47),

$$M_1^N = \left[\begin{array}{cc} e^{-i\phi_2/4} & 0 \\ 0 & e^{i\phi_2/4} \end{array}\right] \left[\begin{array}{cc} \cosh(\xi/2) & \sinh(\xi/2) \\ \sinh(\xi/2) & \cosh(\xi/2) \end{array}\right] \left[\begin{array}{cc} \cosh(N\chi/2) & i\sinh(N\chi/2) \\ -i\sinh(N\chi/2) & \cosh(N\chi/2) \end{array}\right]$$

$$\times \left[\begin{array}{cc} \cosh(\xi/2) & -\sinh(\xi/2) \\ -\sinh(\xi/2) & \cosh(\xi/2) \end{array}\right] \left[\begin{array}{cc} e^{i\phi_2/4} & 0 \\ 0 & e^{-i\phi_2/4} \end{array}\right]. \tag{58}$$
If the expression of Eq.(47) vanishes,

\[
M_1^N = \begin{pmatrix} e^{-i\phi_2/4} & 0 \\ 0 & e^{i\phi_2/4} \end{pmatrix} \begin{pmatrix} 1 - iN \sinh \lambda & iN \sinh \lambda \\ -iN \sinh \lambda & 1 + iN \sinh \lambda \end{pmatrix} \begin{pmatrix} e^{i\phi_2/4} & 0 \\ 0 & e^{-i\phi_2/4} \end{pmatrix}.
\] (59)

This is not yet the S-matrix. The first and the last layers have boundaries with air or the third medium. It is straightforward to take these boundary conditions into consideration. This procedure was discussed in detail in our previous paper [1].

V. EXPERIMENTAL POSSIBILITIES

The variables for the S-matrix given in Secs. III and IV are determined by the optical parameters, namely the two phase-shifts and one reflection/transmission coefficient. The combinations of these three variables will determine the form of the S-matrix, which may take three different forms.

We note first that the N-dependence of the S-matrix comes from the form of A matrix or the RXR matrix of Eq.(40). If the optical parameters are in such a way that the A matrix takes the form of Eq.(43), the elements of the $A^N$ matrix of Eq.(51) are bounded and oscillating functions of N. If A takes the form of Eq.(45), the $A^N$ matrix becomes Eq.(52). The elements of this matrix are not bounded as N becomes large. Thus, in the real world, N-layers can have two different types depending on the form of A.

In addition, the optical layers can satisfy the condition that the expression of Eq.(47) be zero:

\[
\sinh \lambda - (\sin \alpha) \cosh \lambda = 0.
\] (60)

Then the RXR matrix takes the form of Eq.(48), and the N dependence is linear. This case can be tested as the optical parameters are varied from positive values of Eq.(47) to a positive value through zero. This condition does not depend on N. We have discussed a similar case in our previous paper [1].

In their recent paper [7], Baskal and Kim noted the same transition process for one-lens optics. They noted that the camera focusing mechanism corresponds to contraction of Wigner little groups. It is interesting to note that the same contraction mechanism exists in N-layer optics.

CONCLUDING REMARKS

Based on Wigner’s little groups, we have developed an algebraic method which allows us to study the cyclic properties of two-by-two S-matrices for multilayer optics. Starting from the single-layer S-matrix, it is possible to write the N-layer matrix by multiplying one of the parameters by N. The N-dependence is therefore transparent.

This is possible because the core matrices of the Wigner’s little groups have a sliderule property which allows us to perform multiplications by additions, as noted in Eq.(2). This property is an important element in computer designs.
As was noted in Ref. [7], the transition from Eq.(43) to Eq.(20) corresponds to camera focusing in one-lens optics. From the mathematical point of view, it corresponds to the contraction and expansion of the little groups. From the geometrical point of view, this corresponds to transformation from a circle to hyperbola. It is interesting to note that we can perform these operations also in multilayer optics. Indeed, as in the case of lens optics [7], multilayer optics can serve as an analogue computer for group contractions.

The correspondence between the Lorentz group $O(3,1)$ and $SL(2,c)$, the group of two-by-two unimodular matrices, is well known. Since most of the matrices in ray optics are two-by-two, the Lorentz group is becoming the major language in this field. Ray optics is the backbone of future technology, and optical devices such as polarizers, lenses, interferometers, multilayers, all speak the language of the Lorentz group. Thus, it is possible for the Lorentz group to play computational roles in future generations of computers.

It is a prevailing view in physics, especially in optics, that group theory is only for studying symmetries and not useful for computational purposes. Indeed, we do not need group theory to carry out matrix multiplications given in this paper, and we started only with three matrices given in Eq.(17) and Eq.(18). However, are going through some important theorems in group theory while going through the simple matrix algebras given in this paper. We choose not to elaborate on this point.
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