Restriction theorem for the Fourier–Hermite transform and solution of the Hermite–Schrödinger equation

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Abstract
In this article, we prove a restriction theorem for the Fourier–Hermite transform and obtain a Strichartz estimate for the system of orthonormal functions for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ as application. Further, we show the optimal behavior of the constant in the Strichartz estimate as limit of a large number of functions.

Keywords Restriction theorem · Strichartz inequality · Schrödinger equations · Hermite operator · Hartree equation

Mathematics Subject Classification 35Q41 · 47B10 · 35P10 · 35B65

1 Introduction

A long-standing but persistent classical topic in harmonic analysis is the so-called restriction problem. Originally emerged by the works of Stein in the late 1960s, the restriction problem is a key problem for understanding the general oscillatory integral operators. The restriction problem and its applications are crucial from the point of view of their credible implementation in many areas of mathematical analysis, geometric measure theory, combinatorics, harmonic analysis, number theory, including the Bochner–Riesz conjecture, Kakeya conjecture, the estimation...
of solutions to the wave, Schrödinger, and Helmholtz equations, and the local smoothing conjecture for PDEs [23].

For a Schwartz class function $f$ on $\mathbb{R}^n$, the Fourier transform and the inverse Fourier transform of $f$ are defined as

$$\mathcal{F}(f)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{i\xi \cdot x} \, dx, \quad \xi \in \mathbb{R}^n,$$

and

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi) e^{-i\xi \cdot x} \, d\xi, \quad x \in \mathbb{R}^n,$$

respectively.

Given a surface $S$ embedded in $\mathbb{R}^n$ with $n \geq 2$, the classical restriction problem is the following:

**Problem A** For which exponents $1 \leq p \leq 2$, the Fourier transform of a function $f \in L^p(\mathbb{R}^n)$ belongs to $L^q(S)$, $1 \leq q \leq \infty$, where $S$ is endowed with its $(n-1)$-dimensional Lebesgue measure $d\sigma$?

A model case of the restriction problem which is often considered in the literature is the case $q = 2$ (see [20, 22, 25]). For example, the celebrated Stein–Tomas Theorem (see [20, 25, 26]) gives an affirmative answer to Fourier restriction problem for compact surfaces with non-zero Gaussian curvature if and only if $1 \leq p \leq \frac{2(n+1)}{n+3}$. For quadratic surfaces, Strichartz [22] gave a complete solution to Fourier restriction problem, when $S$ is a quadratic surface given by $S = \{ x \in \mathbb{R}^n : R(x) = r \}$, where $R(x)$ is a polynomial of degree two with real coefficients and $r$ is a real constant. For a more detailed study on the history of the restriction problem, we refer to the excellent survey of Tao [23]. The Stein–Tomas Theorem is further generalized to a system of orthonormal functions with respect to the Fourier transform by Frank–Lewin–Lieb–Seiringer [9] and Frank–Sabin [8] and the corresponding Strichartz bounds to the Schrödinger equations up to the end point are obtained.

The main aim of this article is to investigate the validity of Problem A for the Fourier–Hermite transform and obtain a Strichartz type estimate for a system of orthonormal functions for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$.

Let $f \in L^1(\mathbb{R}^n)$. The Hermite transform of $f$ is defined by

$$\hat{f}(\mu) = \int_{\mathbb{R}^n} f(x) \Phi_\mu(x) \, dx, \quad \mu \in \mathbb{N}_0^n,$$

where $\mathbb{N}_0$ denotes the set of all non-negative integers and $\Phi_\mu$’s are the normalized $n$-dimensional Hermite functions (defined in Sect. 2). If $f \in L^2(\mathbb{R}^n)$ then $\{ \hat{f}(\mu) \} \in \ell^2(\mathbb{N}_0^n)$ and the Plancherel formula is of the form
The inverse Hermite transform is given by

\[ f(x) = \sum_{k \in \mathbb{N}_0^n} \hat{f}(\mu) \Phi_\mu(x), \]

i.e., the orthonormal basis expansion of \( f \) with respect to \( \{\Phi_\mu(x)\} \). Given a discrete surface \( S \) in \( \mathbb{N}_0^n \times \mathbb{Z} \), we define the restriction operator \( \mathcal{R}_S f := \{\hat{f}(\mu, v)\}_{(\mu, v) \in S} \) and the operator dual to \( \mathcal{R}_S \) (called the extension operator) as

\[ \mathcal{E}_S(\{\hat{f}(\mu, v)\}) := \sum_{(\mu, v) \in S} \hat{f}(\mu, v) \Phi_\mu(\cdot) e^{-i(\cdot)v}, \]  

where the Fourier–Hermite transform of \( f \) is given by

\[ \hat{f}(\mu, v) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^n} \int_{(-\pi, \pi)} f(t, x) \Phi_\mu(x) e^{ivt} \, dx \, dt. \]  

We consider the following problem:

**Problem 1** For which exponents \( 1 \leq p \leq 2 \), the sequence of Fourier–Hermite transforms of a function \( f \in L^p((-\pi, \pi) \times \mathbb{R}^n) \) belongs to \( \ell^2(S) \)?

This question can be reframed to the boundedness of the operator \( \mathcal{E}_S \) from \( \ell^2(S) \) to \( L^{p'}((-\pi, \pi) \times \mathbb{R}^n) \), where \( p' \) is the conjugate exponent of \( p \), i.e., \( \frac{1}{p} + \frac{1}{p'} = 1 \). Since \( \mathcal{E}_S \) is bounded from \( \ell^2(S) \) to \( L^{p'}((-\pi, \pi) \times \mathbb{R}^n) \) if and only if \( T_S := \mathcal{E}_S(\mathcal{E}_S)^* \) is bounded from \( L^p((-\pi, \pi) \times \mathbb{R}^n) \) to \( L^{p'}((-\pi, \pi) \times \mathbb{R}^n) \), Problem 1 can be re-written as follows:

**Problem 2** For which exponents \( 1 \leq p \leq 2 \), the operator \( T_S := \mathcal{E}_S(\mathcal{E}_S)^* \) is bounded from \( L^p((-\pi, \pi) \times \mathbb{R}^n) \) to \( L^{p'}((-\pi, \pi) \times \mathbb{R}^n) \)?

Note that Hölder’s inequality implies that the operator \( T_S = \mathcal{E}_S(\mathcal{E}_S)^* \) is bounded from \( L^p((-\pi, \pi) \times \mathbb{R}^n) \) to \( L^{p'}((-\pi, \pi) \times \mathbb{R}^n) \) if and only if for any \( W_1, W_2 \in L^{2p}((-\pi, \pi) \times \mathbb{R}^n) \), the operator \( W_1 T_S W_2 \) (composition of the multiplication operator associated with \( W_1, T_S \) and the multiplication operator associated with \( W_2 \)) is bounded on \( L^2((-\pi, \pi) \times \mathbb{R}^n) \) with

\[ \|W_1 T_S W_2\|_{L^2((-\pi,\pi)\times\mathbb{R}^n)} \leq C \|W_1\|_{L^{2p}((-\pi,\pi)\times\mathbb{R}^n)} \|W_2\|_{L^{2p}((-\pi,\pi)\times\mathbb{R}^n)}, \]

for some \( C > 0 \).

We introduce an analytic family of operators \( (T_c) \) defined on the strip \( a \leq \text{Re} \ z \leq b \) in the complex plane such that \( T_S = T_c \) for some \( c \in (a, b) \) and show that the operator \( W_1 T_S W_2 \) belongs to a Schatten class with
\[ \|W TW_2\|_{L^2((-\pi,\pi))^n} \leq C\|W_1\|_{L^2((-\pi,\pi))^n}\|W_2\|_{L^2((-\pi,\pi))^n}, \]

for some \( C > 0 \) and some \( \alpha > 0 \), which is more general result \( L^p - L^{p'} \) boundedness of \( T_S \).

Another motivation to consider Problem 1 is due to the connection to Frame theory: the Fourier–Hermite restriction and extension operators seem to be what is called analysis and synthesis operator in Gabor Analysis [5, 11]. More precisely, the extension operator as defined by (1) is not only a synthesis operator, it is already the frame operator for the union of modulated ONBs consisting of Hermite functions. The question about the boundedness of the extension operator defined by (1) is, therefore, a question about the boundedness of the frame operator of a degenerated multi-window Gabor system. The degeneracy stems from the fact that no translations are used. The multi-window Gabor system is built from the eigenfunctions of a Daubechies localization operator [6] (see also [7]).

Although, the Strichartz inequality for the system of orthonormal functions for the Hermite operator has been proved in [4] using the classical Strichartz estimates for the free Schrödinger propagator for orthonormal systems [8, 9] and the link between the Schrödinger kernel and the Mehler kernel associated with the Hermite semigroup [18], it is important to note that this result can also be obtained independently as a direct application of the Fourier–Hermite restriction theorem. To the best of our knowledge, the study on restriction theorem with respect to the Fourier–Hermite transform has not been considered in the literature so far. However, we prove the restriction theorem for the Fourier–Hermite transform and obtain the full range Strichartz estimate for the system of orthonormal functions for the Hermite operator \( H = -\Delta + |x|^2 \) on \( \mathbb{R}^n \) as an application. We also show that the constant obtained in the Strichartz inequality is optimal in terms of the limit of a large number of functions. In addition, we discuss the global well-posedness results in Schatten spaces for the non-linear Hermite–Hartree equation as an application to our main result.

The schema of the paper is as follows: in Sect. 2, we discuss the spectral theory of the Hermite operator and the kernel estimates for the Hermite semigroup. In Sect. 3, we obtain the duality principle in terms of Schatten bounds of the operator \( W e^{-iH}(e^{-iH})^*W \) and give an affirmative answer to Problem 2 when \( p = \frac{2\nu_0}{1+\nu_0} \) for some \( \lambda_0 > 1 \). In Sect. 4, we obtain the Strichartz estimate for \( 1 \leq q < \frac{n+1}{n-1} \) for the system of orthonormal functions associated with the Hermite operator as the restriction of the Hermite–Fourier transform to the discrete surface \( S = \{(\mu, v) \in \mathbb{N}_0^n \times \mathbb{Z} : v = 2|\mu| + n\} \). In Sect. 5, we prove the optimality of Schatten exponent and we obtain the global well-posedness result for the non-linear Hermite–Hartree equation in Schatten spaces.

2 Preliminary

In this section, we discuss some basic definitions and provide necessary background information about the Hermite semigroup.
2.1 Hermite operator and the spectral theory

Let \(\mathbb{N}_0\) be the set of all non-negative integers. Let \(H_k\) denote the Hermite polynomial on \(\mathbb{R}\), defined by

\[
H_k(x) = (-1)^k \frac{d^k}{dx^k} (e^{-x^2}) e^{x^2}, \quad k \in \mathbb{N}_0,
\]

and \(h_k\) denote the normalized Hermite functions on \(\mathbb{R}\) defined by

\[
h_k(x) = (2^k \sqrt{\pi k!} )^{-1} H_k(x) e^{-x^2}, \quad k \in \mathbb{N}_0.
\]

The higher dimensional Hermite functions denoted by \(U_a\) are obtained by taking tensor product of one dimensional Hermite functions. Thus, for any multi-index \(a \in \mathbb{N}_0^n\) and \(x \in \mathbb{R}^n\), we define

\[
U_a(x) = \prod_{j=1}^n h_{a_j}(x_j).
\]

The family \(\{U_a\}\) forms an orthonormal basis for \(L^2(\mathbb{R}^n)\). They are eigenfunctions of the Hermite operator \(H = -\Delta + |x|^2\) corresponding to eigenvalues \((2|a| + n)\), where \(|a| = \sum_{j=1}^n a_j\).

Given \(f \in L^2(\mathbb{R}^n)\), we have the Hermite expansion

\[
f = \sum_{a \in \mathbb{N}_0^n} (f, \Phi_a) \Phi_a = \sum_{k=0}^\infty \sum_{|a|=k} (f, \Phi_a) \Phi_a = \sum_{k=0}^\infty P_k f,
\]

where \(P_k\) denotes the orthogonal projection of \(L^2(\mathbb{R}^n)\) onto the eigenspace spanned by \(\{\Phi_a : |a| = k\}\). The operator \(H\) defines a semigroup called the Hermite semigroup \(e^{-tH}, t > 0\), defined by

\[
e^{-tH} f = \sum_{k=0}^\infty e^{-(2k+n)t} P_k f
\]

for \(f \in L^2(\mathbb{R}^n)\). On a dense subspace, say the space of all Schwartz functions, the above expression can be written as

\[
e^{-tH} f(x) = \int_{\mathbb{R}^n} f(y) K_t(x,y) dy,
\]

where the kernel \(K_t(x,y)\) is given by the expansion

\[
K_t(x,y) = \sum_{a \in \mathbb{N}_0^n} e^{-(2|a| + n)t} \Phi_a(x) \Phi_a(y).
\]

For \(z' = r + it, r > 0, t \in \mathbb{R}\), the kernel of the operator \(e^{-z'H}\) is given by

\[
K_{z'}(x,y) = \sum_{k=0}^\infty e^{-z'(2k+n)} \sum_{|a|=k} \Phi_a(x) \Phi_a(y).
\]

Using Mehler’s formula [12], the kernel of the operator \(e^{-z'H}\) can be obtained as
\[ K_{x'}(x, y) = \frac{1}{(2\pi \sinh 2z')^2} e^{\frac{i}{2}(\coth 2z'(\lvert x \rvert^2 + \lvert y \rvert^2) + \frac{2z'}{\sinh 2z'})}. \]

For \( t \in \mathbb{R} \setminus \left( \frac{q}{2} \right) \mathbb{Z} \), letting \( r \to 0 \), the kernel of the operator \( e^{-itH} \) can be written as
\[ K_{it}(x, y) = \frac{e^{-i2\pi r}}{(2\pi \sin 2t)^2} e^{\left( \cot 2t\lvert x \rvert^2 + \lvert y \rvert^2 \right) - \frac{2r}{\sin 2t}}. \quad (3) \]

For \( t \in \mathbb{R} \setminus \left( \frac{q}{2} \right) \mathbb{Z} \), we have
\[ K_{it}(x, y) = K_{it}(y, x) \quad \text{and} \quad K_{i(t+\frac{q}{2})}(x, y) = e^{-ir\frac{q}{2}} K_{it}(-x, y). \quad (4) \]

For real valued functions \( f \) the \( L^p(\mathbb{R}^n) \) norm of \( e^{-itH}f \) is even and \( \frac{q}{2} \)-periodic as a function of \( t \).

We refer to [12, 14, 24] for a detailed study on the kernel associated with the operator \( e^{-itH} \).

### 2.2 Schatten class and the duality principle

Let \( \mathcal{H} \) be a complex and separable Hilbert space in which the inner product is denoted by \( (\cdot, \cdot)_{\mathcal{H}} \). Let \( T : \mathcal{H} \to \mathcal{H} \) be a compact operator and let \( T^* \) denote the adjoint of \( T \). For \( 1 \leq r < \infty \), the Schatten space \( \mathcal{G}^r(\mathcal{H}) \) is defined as the space of all compact operators \( T \) on \( \mathcal{H} \) such that
\[ \sum_{n=1}^{\infty} (s_n(T))^r < \infty, \]
where \( s_n(T) \) denotes the singular values of \( T \), i.e., the eigenvalues of \( |T| = \sqrt{T^*T} \) counted according to multiplicity. For \( T \in \mathcal{G}^r(\mathcal{H}) \), the Schatten \( r \)-norm is defined by
\[ \|T\|_{\mathcal{G}^r} = \left( \sum_{n=1}^{\infty} (s_n(T))^r \right)^{\frac{1}{r}}. \]

An operator belonging to the class \( \mathcal{G}^1(\mathcal{H}) \) is known as Trace class operator. In addition, an operator belongs to \( \mathcal{G}^2(\mathcal{H}) \) is known as Hilbert–Schmidt operator.

### 3 The restriction theorem

In this section, we set a platform to prove the restriction theorem with respect to the Fourier–Hermite transform for a given discrete surface \( S \subset \mathbb{N}_0^n \times \mathbb{Z} \). Recall that, a family of operators \( \{T_z\} \) on \( \mathbb{R}^n \) defined on a strip \( a \leq \text{Re}z \leq b \) with \( a < b \), in the complex plane is analytic in the sense of Stein [19] if for all simple functions \( f, g \) on \( \mathbb{R}^n \), the map \( z \mapsto \langle g, T_z f \rangle \) is analytic in \( a < \text{Re}z < b \), continuous in \( a \leq \text{Re}z \leq b \), and if \( \sup_{a \leq z \leq b} \lvert \langle g, T_{z+is} f \rangle \rvert \leq C(s) \), where \( C(s) \) with at most a (double) exponential growth in \( s \).

\[ \text{Birkhäuser} \]
The following proposition assures an affirmative answer to Problem 2 under certain assumptions.

In order to obtain the Strichartz inequality for the system of orthonormal functions we need the duality principle lemma in our context. We refer to Proposition 1 and Lemma 3 of [9] with appropriate modifications to obtain the following two results:

**Proposition 3.1** Let \((T_z)\) be an analytic family of operators on \((-\pi, \pi) \times \mathbb{R}^n\) in the sense of Stein defined on the strip \(-\lambda_0 \leq \text{Re} z \leq 0\) for some \(\lambda_0 > 1\). Assume that we have the following bounds:

\[
\begin{align*}
\|T_{iz}\|_{L^2((-\pi, \pi)\times \mathbb{R}^n) \to L^2((-\pi, \pi)\times \mathbb{R}^n)} & \leq M_0 e^{a|s|}, \\
\|T_{-\lambda_0 + iz}\|_{L^2((-\pi, \pi)\times \mathbb{R}^n) \to L^\infty((-\pi, \pi)\times \mathbb{R}^n)} & \leq M_1 e^{b|s|},
\end{align*}
\]

(5)

for all \(s \in \mathbb{R}\), for some \(a, b, M_0, M_1 \geq 0\). Then, for all \(W_1, W_2 \in L^{2\lambda_0}((-\pi, \pi) \times \mathbb{R}^n)\), the operator \(W_1 T_{-1} W_2\) belongs to \(\mathcal{G}^{2\lambda_0}(L^2((-\pi, \pi) \times \mathbb{R}^n))\) and we have the estimate

\[
\|W_1 T_{-1} W_2\|_{\mathcal{G}^{2\lambda_0}(L^2((-\pi, \pi) \times \mathbb{R}^n))} \leq M_0^{1-\frac{1}{2\lambda_0}} M_1^{\frac{1}{2\lambda_0}} \|W_1\|_{L^{2\lambda_0}((-\pi, \pi) \times \mathbb{R}^n)} \|W_2\|_{L^{2\lambda_0}((-\pi, \pi) \times \mathbb{R}^n)},
\]

(6)

where the general mixed norm \(\|W\|_{L^p_t L^q_x((-\pi, \pi) \times \mathbb{R}^n)}\) is defined as

\[
\|W\|_{L^p_t L^q_x((-\pi, \pi) \times \mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^2} |W(t,x)|^q dx \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}
\]

for \(1 \leq p, q < \infty\).

**Lemma 3.2** (Duality principle) Let \(p, q \geq 1\) and \(\alpha \geq 1\). Let \(A\) be a bounded linear operator from \(L^2(\mathbb{R}^n)\) to \(L^{2\alpha}_t L^{2\alpha}_x((-\pi, \pi) \times \mathbb{R}^n)\). Then, the following statements are equivalent.

1. There is a constant \(C > 0\) such that

\[
\|W W^* A^* W\|_{\mathcal{G}^{\alpha}(L^2((-\pi, \pi) \times \mathbb{R}^n))} \leq C \|W\|_{L^{2\alpha}_t L^{2\alpha}_x((-\pi, \pi) \times \mathbb{R}^n)}^2
\]

(7)

for all \(W \in L^{2\alpha}_t L^{2\alpha}_x((-\pi, \pi) \times \mathbb{R}^n)\).

2. For any orthonormal system \((f_j)_{j \in J}\) in \(L^2(\mathbb{R}^n)\) and any sequence \((n_j)_{j \in J} \subset \mathbb{C}\), there is a constant \(C' > 0\) such that

\[
\left\| \sum_{j \in J} n_j |Af_j|^2 \right\|_{L^{2\alpha}_t L^{2\alpha}_x((-\pi, \pi) \times \mathbb{R}^n)} \leq C' \left( \sum_{j \in J} |n_j|^{2\alpha'} \right)^{1/\alpha'}.
\]

(8)
Note that Lemma 3.2 and Proposition 3.1 are also valid in the domain \((-\pi, \pi) \times \mathbb{R}^n\).

Let \(S\) be the discrete surface \(S = \{(\mu, v) \in \mathbb{N}_0^d \times \mathbb{Z} : R(\mu, v) = 0\}\), where \(R(\mu, v)\) is a polynomial of degree one, with respect to the counting measure.

For \(-1 < \text{Re} \ z \leq 0\), consider the analytic family of generalized functions

\[
G_z(\mu, v) = \psi(z)R(\mu, v)^z_+,
\]

where \(\psi(z)\) is an appropriate analytic function with a simple zero at \(z = -1\) with exponential growth at infinity when \(\text{Re}(z) = 0\) and

\[
R(\mu, v)^z_+ = \begin{cases} \frac{R(\mu, v)}{z} & \text{for } R(\mu, v) > 0, \\ 0 & \text{for } R(\mu, v) \leq 0. \end{cases}
\]

Restricting the Schwartz class function \(\phi\) (defined on \(\mathbb{Z}^{n+1}\)) to \(\mathbb{N}_0^d \times \mathbb{Z}\), we have

\[
\langle G_z, \phi \rangle := \psi(z) \sum_{k \in \mathbb{Z}} k^z_+ \sum_{\{(\mu, v) : R(\mu, v) = k\}} \phi(\mu, v),
\]

where \(k^z_+\) is defined as in (10). Using one dimensional analysis of \(k^z_+\) (see [10]), we have

\[
\lim_{z \to -1} \langle G_z, \phi \rangle = \sum_{(\mu, v) \in S} \phi(\mu, v).
\]

For \(-1 < \text{Re} \ z \leq 0\), define the analytic family of operators \(T_z\) (acting on Schwartz class functions on \((-\pi, \pi) \times \mathbb{R}^n\) by

\[
T_z g(t, x) = \sum_{\mu, v} \hat{g}(\mu, v)G_z(\mu, v)\Phi_\mu(x)e^{-i\mu t},
\]

where \(\hat{g}(\mu, v)\) is defined in (2). Then, \((T_z)\) is an analytic in the sense of Stein defined on the strip \(-\lambda_0 \leq \text{Re} z \leq 0\) for some \(\lambda_0 > 1\) and

\[
T_z g(t, x) = \int_{\mathbb{R}^n} (K_z(x, y, \cdot) * g(y, \cdot))(t) \, dy,
\]

where \(K_z(x, y, t) = (2\pi)^{-\frac{1}{2}} \sum_{\mu, v} \Phi_\mu(x)\Phi_\mu(y)G_z(\mu, v)e^{-i\mu t}\). When \(\text{Re}(z) = 0\), we have

\[
\|T_is\|_{L^2((-\pi, \pi) \times \mathbb{R}^n)} \to L^2((-\pi, \pi) \times \mathbb{R}^n) = \|G_is\|_{L^\infty((-\pi, \pi) \times \mathbb{R}^n)} \leq |\psi(is)|. \tag{12}
\]

Again an application of Hölder and Young inequalities in (11) gives

\[
|T_z g(t, x)| \leq \sup_{t \in (-\pi, \pi), \ y \in \mathbb{R}^n} |K_z(x, y, t)| \|g\|_{L^1((-\pi, \pi) \times \mathbb{R}^n),} \quad \forall x \in \mathbb{R}^n, \tag{13}
\]

for \(g \in L^1((-\pi, \pi) \times \mathbb{R}^n)\). Denoting \(T_S = E_S E^*_S\), we obtain the following Schatten bound (see (14) below) for the operator \(W_1 T_S W_2\), where \(W_1, W_2 \in L^2_{\ell_2, \ell_2}((-\pi, \pi) \times \mathbb{R}^n)\).
Theorem 3.3 (Fourier–Hermite restriction theorem) Let $n \geq 1$ and let $S \subset \mathbb{N}_0^n \times \mathbb{Z}$ be a discrete surface. Suppose that for each $x, y \in \mathbb{R}$, $|K_z(x, y, t)|$ is bounded and has at most exponential growth at infinity when $z = -\lambda_0 + i$ is for some $\lambda_0 > 1$. Then, $T_S$ is bounded from $L^p((-\pi, \pi) \times \mathbb{R}^n)$ to $L^p((-\pi, \pi) \times \mathbb{R}^n)$ for $p = \frac{2\lambda_0}{1+\lambda_0}$.

**Proof** It is enough to show

$$\|W_1 T_S W_2\|_{L^{2\lambda_0}(L^2((-\pi, \pi) \times \mathbb{R}^n))} \leq C\|W_1\|_{L^{2\lambda_0}_{x,y}(L^2((-\pi, \pi) \times \mathbb{R}^n))} \|W_2\|_{L^{2\lambda_0}_{x,y}(L^2((-\pi, \pi) \times \mathbb{R}^n))}$$

for all $W_1, W_2 \in L^{2\lambda_0}_{x,y}((-\pi, \pi) \times \mathbb{R}^n)$ by Lemma 3.2. By our assumption, together with (12), (13) and Proposition 3.1, we get (14).

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4 Strichartz inequality for system of orthonormal functions

Consider the Schrödinger equation associated with the Hermite operator $H = -\Delta + |x|^2$:

$$i\partial_t u(t, x) = Hu(t, x) \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n$$

$$u(x, 0) = f(x).$$

If $f \in L^2(\mathbb{R}^n)$, the solution of the initial value problem (15) is given by $u(t, x) = e^{-itH}f(x)$. The solution to the initial value problem (15) can be realized as the extension operator of some function $f$ on $(-\pi, \pi) \times \mathbb{R}^n$. To estimate the solution to the initial value problem (15) is equivalent to obtain the Schatten bound (7) with $A = e^{-itH}$.

Let $S$ be the discrete surface $S = \{(\mu, v) \in \mathbb{N}_0^n \times \mathbb{Z} : v = 2|\mu| + n\}$ with respect to the counting measure. Then, for all $f \in \ell^1(S)$ and for all $(t, x) \in [-\pi, \pi] \times \mathbb{R}^n$, the extension operator can be written as

$$E_Sf(t, x) = \sum_{\mu, v \in S} \hat{f}(\mu, v) \Phi_\mu(x)e^{-ivt},$$

where $\hat{f}(\mu, v)$ is defined in (2). Choosing

$$\hat{f}(\mu, v) = \begin{cases} \hat{u}(\mu) & \text{if } v = 2|\mu| + n, \\ 0 & \text{otherwise,} \end{cases}$$

for some $u : \mathbb{R}^n \to \mathbb{C}$ in (16), we get...
\[ E_S f(t, x) = \sum_{\mu, \nu \in S} f(\mu, \nu) \Phi_\mu(x) e^{-i\nu y} \]
\[ = \int_{\mathbb{R}^n} \left( \sum_\mu \Phi_\mu(x) \Phi_\mu(y) e^{-i(2|\mu| + n)} \right) u(y) \, dy \]
\[ = e^{-iH} u(x). \]

Restricting Schwartz class functions \( \phi \) to \( \mathbb{N}_0^n \times \mathbb{Z} \), using the discrete Taylor series expansion (see [15]), we have
\[ \langle k^z_+, \phi \rangle = \sum_{k \in \mathbb{N}_0^n \times \mathbb{Z}} k^z_+ \phi(k) \] (17)
\[ = \sum_{k \in \mathbb{N}_0^n \times \mathbb{Z}} k^z_+ \left[ \phi(k) - \sum_{|x| < M} \frac{1}{M} k^x \Delta^x \phi(0) \right] + \sum_{|x| < M} \frac{1}{M} \Delta^x \phi(0) \sum_{k \in \mathbb{N}_0^n \times \mathbb{Z}} k^x k^z_+. \] (18)

The above formula is valid for \( z \neq -1, -2, \ldots \), regularizing (17). Notice that (18) shows that \( \langle k^z_+, \phi \rangle \) has simple poles at \( z = -1, -2, \ldots \) as a function of \( z \). Setting \( \psi(z) = \frac{1}{\Gamma(z+1)} \) and \( R(\mu, v) = v - (2|\mu| + n) \) in (9), we get
\[ G_\psi(\mu, v) = \frac{1}{\Gamma(z+1)} (v - (2|\mu| + n))^z_+ \]
and
\[ \lim_{z \to -1} \langle G_\psi, \phi \rangle = \lim_{z \to -1} \frac{1}{\Gamma(z+1)} \sum_{k \in \mathbb{Z}} k^z_+ \sum_{(\mu, v) \in \mathbb{N}_0^n} \phi(\mu, v) \]
\[ = \sum_{(\mu, v) \in \mathbb{N}_0^n} \phi(\mu, v). \]

Thus, \( G_{-1} = \delta_S \) and
\[ T_z g(t, x) = \int_{\mathbb{R}^n} (K_z(x, y, \cdot) * g(y, \cdot))(t) \, dy, \] (19)
where
\[ K_z(x, y, t) = \frac{(2\pi)^{-\frac{n}{2}}}{\Gamma(z+1)} \sum_\mu \Phi_\mu(x) \Phi_\mu(y) e^{-i(2|\mu| + n)} \sum_{k=0}^{\infty} k^z_+ e^{-itk}. \] (20)

We use the notation \( x \sim y \) if there exists constants \( C_1, C_2 > 0 \) such that \( C_1 x \leq y \leq C_2 x \). With this convention, we estimate \( \sum_{k=0}^{\infty} k^z_+ e^{-itk} \) in the following proposition.

**Proposition 4.1** Let \(-1 < \text{Re } z < 0\). Then, the series \( \sum_{k=0}^{\infty} k^z_+ e^{-itk} \) is the Fourier series of an integrable function on \([-\pi, \pi]\) which is of class \( C^\infty \) on \([-\pi, \pi] \setminus \{0\} \).
Near the origin this function has the same singularity as the function whose values are \( \Gamma(z + 1)(it)^{-z-1} \), i.e.,

\[
\sum_{k=0}^{\infty} k^z e^{-ikt} \sim \Gamma(z + 1)(it)^{-z-1} + b(t).
\]  

where \( b \in C^\infty[-\pi, \pi] \).

**Proof** For \( \tau > 0 \), we calculate the inverse Fourier transform of \( u^\tau_+ e^{-\tau u} \).

\[
\mathcal{F}^{-1}[u^\tau_+ e^{-\tau u}](x) = (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} u^\tau_+ e^{-\tau u} e^{-iux} du = (2\pi)^{-\frac{1}{2}} \int_0^\infty u^\tau e^{-iux} du,
\]

where \( s = x - i\tau \) so that \(-\pi < \arg s < 0\). Then, \( u^\tau_+ e^{-\tau u} \) converges to \( u^\tau_+ \) in the sense of distributions as \( \tau \to 0 \). In addition, the inverse Fourier transform of \( u^\tau_+ e^{-\tau u} \) converges to the inverse Fourier transform of \( u^\tau_+ \). Using the change of variable \( isu = \xi \) and proceeding as in page 170 of [10], we get

\[
\mathcal{F}^{-1}[u^\tau_+ e^{-\tau u}](x) = (2\pi)^{-\frac{1}{2}} \frac{1}{(is)^{\frac{1}{z}+1}} \int_L \xi^\tau e^{-\xi} d\xi = (2\pi)^{-\frac{1}{2}} \frac{\Gamma(z + 1)}{(is)^{\frac{1}{z}+1}},
\]

where the contour \( L \) of the integral is a ray from origin to infinity whose angle with respect to the real axis is given by \( \arg \xi = \arg s + \frac{\pi}{2} \). Letting \( \tau \to 0 \) in (22), we have

\[
\mathcal{F}^{-1}[u^\tau_+](x) = (2\pi)^{-\frac{1}{2}} \frac{\Gamma(z + 1)}{(ix)^{\frac{1}{z}+1}}.
\]

By analytic continuation (23) is valid for all \( z \neq -1 \).

We use the idea given in Theorem 2.17 of [21] to prove (21). To make the paper self contained, we will only indicate the main steps. Let us consider a function \( \eta \in C^\infty(\mathbb{R}) \) such that \( \eta(x) = 1 \) if \( |x| \geq 1 \), and vanishes in a neighborhood of the origin. Let \( F(x) = \eta(x)x^\tau_+ \) for \( x \in \mathbb{R} \). Writing \( F(x) = x^\tau_+ + (\eta(x) - 1)x^\tau_+ \), using (23) and denoting \( \hat{f} \) to be the inverse Fourier transform of \( F \) in the sense of distributions, we have \( f(x) = \Gamma(z + 1)(ix)^{-z-1} + b_1(x) \), where \( b_1 \) is the inverse Fourier transform of the integrable function \( (\eta(x) - 1)x^\tau_+ \) whose support is bounded. Moreover, \( b_1 \in C^\infty(\mathbb{R}) \) and \( f \in L^1(\mathbb{R}) \).

Applying Poisson summation formula (see page 250 of [21]) to the function \( f \) and using the fact \( \hat{\hat{f}} = F \), we get.
\[
\sum_{k=0}^{\infty} k^z e^{-itk} = \sum_{k \in \mathbb{Z}} F(k)e^{-itk}
\]
\[
\sim \sum_{k \in \mathbb{Z}} f(2k\pi + t)
\]
\[
= f(t) + \sum_{|k| > 0} f(2k\pi + t)
\]
\[
= \Gamma(z + 1)(it)^{-z-1} + b_1(t) + \sum_{|k| > 0} f(2k\pi + t)
\]
\[
= \Gamma(z + 1)(it)^{-z-1} + b(t),
\]
where \( b(t) = b_1(t) + \sum_{|k| > 0} f(2k\pi + t) \in C^\infty[-\pi, \pi]. \)

Now, we are in a position to prove the following Strichartz inequality for the diagonal case.

**Theorem 4.2** (Diagonal case) Let \( n \geq 1 \). Then, for any (possibly infinite) system \((u_j)\) of orthonormal functions in \( L^2(\mathbb{R}^n) \) and any coefficients \( (n_j) \subset \mathbb{C} \), we have

\[
\left\| \sum_{j} n_j |e^{-itH}u_j|^2 \right\|_{L_{x,t}^{n+\frac{1}{2}}((-\pi,\pi) \times \mathbb{R}^n)} \leq C \left( \sum_{j} |n_j|^{\frac{n+1}{n+2}} \right)^{\frac{n+2}{n+1}},
\]

where \( C > 0 \) is independent of \( n \) and \( q \).

**Proof** To prove (24), it is enough to show

\[
\left\| W_1 T_s W_2 \right\|_{C^{n+2}(L^2((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n))} \leq C \left\| W_1 \right\|_{L_{x,t}^{n+2}((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n)} \left\| W_2 \right\|_{L_{x,t}^{n+2}((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n)}
\]

for all \( W_1, W_2 \in L_{x,t}^{n+2}((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n) \), where \( S = \{(\mu, v) \in \mathbb{N}_0^n \times \mathbb{Z} : v = 2|\mu| + n\} \).

Applying Lemma 3.2 to (25) gives

\[
\left\| \sum_{j} n_j |e^{-itH}u_j|^2 \right\|_{L_{x,t}^{n+\frac{1}{2}}((-\pi,\pi) \times \mathbb{R}^n)} \leq C \left( \sum_{j} |n_j|^{\frac{n+1}{n+2}} \right)^{\frac{n+2}{n+1}}.
\]

Using the kernel properties \((4)\) of the semigroup \( e^{-itH} \) the range of \( t \) can be extended to \((-\pi, \pi)\). We show that the family of operators \((T_z)\) (defined in (19)) satisfies (5). When \( Re(z) = 0 \), we have

\[
\left\| T_{iz} \right\|_{L^2((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n) \rightarrow L^2((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n)} = \left\| G_{iz} \right\|_{L^\infty((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n)} \leq \left| \frac{1}{\Gamma(1 + is)} \right| \leq Ce^{\pi|s|/2}.
\]

When \( z = -\lambda_0 + is \), (13) gives \( T_z \) is bounded from \( L^1((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n) \) to
$L^\infty((-rac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}^n)$ if and only if $|K_z(x,y,t)|$ is bounded for each $x,y \in \mathbb{R}^n$. But by (20), Proposition 4.1 and (3), we get

$$|K_z(x,y,t)| \sim \frac{C}{|t|^{\Re(z+1+\frac{p}{2})}} e^{ibz}.$$  \hspace{1cm} (28)

Therefore, for each $x,y \in \mathbb{R}$, $|K_z(x,y,t)|$ is bounded if and only if $\Re(z) = - \frac{n+2}{2}$. The conclusion of the theorem follows by choosing $\lambda_0 = \frac{n+2}{2}$ by Proposition 3.1 and the identity $T_S = T_{-1}$.

To obtain the Strichartz inequality for the general case, we need to observe the following.

**Theorem 4.3** Let $S$ be the discrete surface $S = \{(\mu, v) \in \mathbb{N}_0^n \times \mathbb{Z} : v = 2|\mu| + n\}$ with respect to the counting measure. Then, for all exponents $p, q \geq 1$ satisfying

$$\frac{2}{p} + \frac{n}{q} = 1, \quad q > n + 1$$

we have

$$\|W_1 T_S W_2\|_{\mathcal{L}^p(\mathbb{L}^2((-\pi, \pi) \times \mathbb{R}^n))} \leq C \|W_1\|_{L^{q_0}_x L^{q_1}_t((-\pi, \pi) \times \mathbb{R}^n)} \|W_2\|_{L^{q_1}_x L^{q_2}_t((-\pi, \pi) \times \mathbb{R}^n)}$$

\hspace{1cm} (29)

with $C > 0$ independent of $W_1, W_2$.

**Proof** The operator $T_{-\lambda_0 + i \delta}$ is an integral operator with kernel $K_{-\lambda_0 + i \delta}(x,x', t-t')$ defined in (19). An application of Hardy–Littlewood–Sobolev inequality (see page 39 in [1]) along with (27) and (28) yields

$$\left\| W_1^{\lambda_0 - i \delta} T_{-\lambda_0 + i \delta} W_2^{\lambda_0 - i \delta} \right\|_{L^2}^2 \leq \int_{(-\frac{\pi}{4}, \frac{\pi}{4})} \int_{\mathbb{R}^{2n}} W_1(t,x)^{2\lambda_0} |K_{-\lambda_0 + i \delta}(x,t,x', t')|^2 W_2(t',x')^{2\lambda_0} dxdx' dt'dt' \leq C_1 \int_{(-\frac{\pi}{4}, \frac{\pi}{4})} \int_{\mathbb{R}^{2n}} \left| t - t' \right|^{n+2-2\lambda_0} dxdx' dt'dt' \leq C_1 e^{\pi|x|} \int_{(-\frac{\pi}{4}, \frac{\pi}{4})} \int_{(-\frac{\pi}{4}, \frac{\pi}{4})} \frac{\|W_1(t)\|_{L^{2\lambda_0}_x L^{2q_1}_t(\mathbb{R}^n)} \|W_2(t')\|_{L^{2\lambda_0}_x L^{2q_1}_t(\mathbb{R}^n)}}{|t - t'|^{n+2-2\lambda_0}} dt'dt' \leq C_1 e^{\pi|x|} \left\| W_1 \right\|_{L^{\frac{2\lambda_0}{q_0}}_x L^{\frac{2q_1}{q_0}}_t((-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}^n)} \left\| W_2 \right\|_{L^{\frac{2\lambda_0}{q_0}}_x L^{\frac{2q_1}{q_0}}_t((-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}^n)}$$

provided we have $0 \leq n + 2 - 2\lambda_0 < 1$, that is $\frac{n+1}{2} < \lambda_0 \leq \frac{n+2}{2}$. By Theorem 2.9 of [17], we have
\section{Optimality of the Schatten exponent}

In this section, we show that the power \( \frac{p(q+1)}{2q} \) on the right hand side in (30) is optimal. The inequality (30) can also be written in terms of the operator

\[ \gamma_0 := \sum_j n_j |u_j\rangle \langle u_j| \]  

on \( L^2(\mathbb{R}^n) \), where the Dirac’s notation \( |u\rangle \langle v| \) stands for the rank-one operator \( f \mapsto \langle v,f \rangle u \). For such \( \gamma_0 \), let
\[
\gamma(t) := e^{-iH\gamma_0}e^{iH} = \sum_j n_j |e^{-iH}u_j\rangle \langle e^{-iH}u_j|.
\]

Then, the density of the operator \(\gamma(t)\) is given by
\[
\rho_{\gamma(t)} := \sum_j n_j |e^{-iH}u_j|^2.
\]

With these notations (30) can be re-written as
\[
\left\| \rho_{e^{-iH\gamma_0}e^{iH}} \right\|_{L^r_k((-\pi,\pi) \times \mathbb{R}^n)} \leq C_{n,q} \left\| \gamma_0 \right\|_{\mathcal{G}^r}^{2q},
\]

where \(\left\| \gamma_0 \right\|_{\mathcal{G}^r}^{2q} = \left( \sum_j |n_j|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}}\).

**Proposition 5.1** (Optimality of the Schatten exponent) Assume that \(n, p, q \geq 1\) satisfy \(\frac{2}{p} + \frac{n}{q} = n\). Then, we have
\[
\sup_{\gamma_0 \in \mathcal{G}^r} \left\| \rho_{e^{-iH\gamma_0}e^{iH}} \right\|_{L^r_k((-\pi,\pi) \times \mathbb{R}^n)} = +\infty
\]
for all \(r > \frac{2q}{q+1}\).

**Proof** Depending on the positive parameters \(\beta, L\) and \(\mu\), we construct the family of operators
\[
\gamma_0 = \frac{1}{(2\pi)^n} \int e^{-\frac{x^2}{2} - \frac{z^2}{2\pi}} \left| F_{x,\xi} \right| dx d\xi,
\]
where \(F_{x,\xi}(z) = (2\pi\beta)^{-\frac{n}{2}} e^{-(x-z)^2} e^{i\xi \cdot z}\). The functions \(F_{x,\xi}\) are normalized and satisfy
\[
\int e^{-(x-z)^2} \left| F_{x,\xi} \right| dx = 1.
\]

By Mehler’s formula, we get
\[
e^{iH} F_{x,\xi}(z) = \left( -2\pi i \sin 2t \right)^{-\frac{n}{2}} (2\pi\beta)^{-\frac{n}{2}} \int e^{-4\cot 2t(y^2+\frac{1}{\sin 2t})} e^{\frac{(y-z)^2}{4\beta^2}} e^{i\xi \cdot y} dy.
\]

Therefore,
\[
\left| e^{iH} F_{x,\xi}(z) \right| = \left( \frac{2\beta}{\pi(4\beta^2 \cos^2 2t + \sin^2 2t)} \right)^{\frac{n}{2}} e^{-\frac{\beta(z-x \cos 2t + i \sin 2t)^2}{4\beta^2 \cos^2 2t + \sin^2 2t}}
\]
and
\[ \rho_{\gamma(t)}(z) := \rho_{e^{i\alpha}e^{-i\theta}(z)} = \int \int \frac{\mathrm{d}x \mathrm{d}y}{2\pi n}(2\pi)^n |e^{iH_{x,y}(z)}|^2 \]

\[ = \left( \frac{2\pi\beta\mu L^2}{(4\beta^2 + 2\beta L^2) \cos^2 2t + (1 + 2\mu\beta) \sin^2 2t} \right)^{\frac{q}{2}} e^{\frac{2\beta}{L^2} + (1 + 2\mu\beta) \sin^2 2t}. \]

Therefore,

\[ \| \rho_{\gamma(t)} \|_{L^q([R^n])}^{q} = \left( \frac{\pi}{q} \right)^{\frac{q}{2}} (\mu L^2)^{\frac{q}{2}} \beta \left( \frac{4\beta^2 + 2\beta L^2}{\cos^2 2t + (1 + 2\mu\beta) \sin^2 2t} \right)^{\frac{n(q-1)}{2}}. \]

Using the fact that \( n(q-1)p = 2q \), we have

\[ \| \rho_{\gamma(t)} \|_{L^p([-\pi,\pi] \times R^n)}^p = \left( \frac{\pi}{q} \right)^{\frac{q}{2}} (\mu L^2)^{\frac{q}{2}} \int_{[-\pi,\pi]} \beta \left( \frac{4\beta^2 + 2\beta L^2}{\cos^2 2t + (1 + 2\mu\beta) \sin^2 2t} \right)^{\frac{n(q-1)}{2}} \mathrm{d}t = \sqrt{2\pi} \left( \frac{\pi}{q} \right)^{\frac{q}{2}} (\mu L^2)^{\frac{q}{2}} \frac{\beta}{\sqrt{2\beta^2 + \beta L^2 \sqrt{1 + 2\mu\beta}}}. \]

Thus,

\[ \| \rho_{\gamma(t)} \|_{L^p([-\pi,\pi] \times R^n)} = A_{n,p}(\mu L^2)^{\frac{q}{2}}(L^2)^{-\frac{q}{2p}} \mu^{-\frac{1}{p}} \left( \frac{2\beta}{L^2} + 1 \right)^{\frac{1}{2p}} \left( \frac{1}{n} + 2 \right)^{\frac{1}{2p}} = A_{n,p}(\mu L^2)^{\frac{q}{2} - \frac{1}{2p}} \left( \frac{2\beta}{L^2} + 1 \right)^{\frac{1}{2p}} \left( \frac{1}{n} + 2 \right)^{\frac{1}{2p}}. \]

Using the fact that \( \frac{n}{q} \left( 1 + \frac{1}{q} \right) = \frac{q}{2} - \frac{1}{2p} \) and choosing \( 1/\mu < \beta < L^2 \), we obtain

\[ \| \rho_{\gamma(t)} \|_{L^p([-\pi,\pi] \times R^n)} \geq A_{n,p} 2^{-\frac{1}{2p}} N^{\frac{1+q}{2p}}, \]

where
\[ N = \int_{\mathbb{R}^n} \gamma_0(z, \bar{z}) dz \]
\[ = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{dx d\bar{x}}{(2\pi)^n} e^{-\frac{x^2 + \bar{x}^2}{r}} \left| F_{x,\bar{x}}(z) \right|^2 dz \]
\[ = \int_{\mathbb{R}^n} \frac{dx d\bar{x}}{(2\pi)^n} e^{-\frac{x^2 + \bar{x}^2}{r}} \]
\[ = A_n L^n \mu^2. \]

An application of Berezin–Lieb inequality [3, 13] gives that
\[ \text{Tr} \gamma_0 \leq \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{dx d\bar{x}}{(2\pi)^n} e^{-\frac{x^2 + \bar{x}^2}{r}} = r^{-n} N, \]
where \( r \geq 1 \) and \( N = \frac{(\mu^2)^\frac{n}{2}}{2^n}. \) Therefore,
\[ \frac{\| \rho e^{-itH} \rho e^{itH} \|_{L^q_t L^\infty_x([0, T] \times \mathbb{R}^n)}}{\| \gamma_0 \|_{G^q}} \geq \frac{A_n 2^{-\frac{3q}{4}}} {r^{-\frac{n}{4}}} N^{\frac{1}{2q} + \frac{1}{4q}}. \]

Using the similar idea as in Theorem 14 of [9], the following global well posedness for the Hermite–Hartree equation in Schatten spaces can be obtained as an application of Theorem 4.4.

**Theorem 5.2** (Global well posedness for the Hermite–Hartree equation) Let \( w \in L^q([\mathbb{R}^n]). \) Under the assumptions of Theorem 4.4, for any \( \gamma_0 \in G^{\frac{2q}{q+1}}, \) there exists a unique \( \gamma \in C^0_t([0, T], G^{\frac{2q}{q+1}}) \) (see [16]) satisfying \( \rho_\gamma \in L^q_t L^q_x([0, T] \times \mathbb{R}^n) \) and
\[ i\partial_t \gamma = [H + w \ast \rho_\gamma, \gamma], \quad \gamma|_{t=0} = \gamma_0. \]

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