A review of matrix scaling and Sinkhorn’s normal form for matrices and positive maps

Martin Idel*

Zentrum Mathematik, M5, Technische Universität München, 85748 Garching

Abstract

Given a nonnegative matrix $A$, can you find diagonal matrices $D_1$, $D_2$ such that $D_1AD_2$ is doubly stochastic? The answer to this question is known as Sinkhorn’s theorem. It has been proved with a wide variety of methods, each presenting a variety of possible generalisations. Recently, generalisations such as to positive maps between matrix algebras have become more and more interesting for applications. This text gives a review of over 70 years of matrix scaling. The focus lies on the mathematical landscape surrounding the problem and its solution as well as the generalisation to positive maps and contains hardly any nontrivial unpublished results.

Contents

1. Introduction 3
2. Notation and Preliminaries 4
3. Different approaches to equivalence scaling 5
   3.1. Historical remarks .................................................. 6
   3.2. The logarithmic barrier function .............................. 8
   3.3. Nonlinear Perron-Frobenius theory .......................... 12
   3.4. Entropy optimisation ............................................. 14
   3.5. Convex programming and dual problems ................. 17
   3.6. Topological (non-constructive) approaches ............... 20
   3.7. Other ideas .......................................................... 21
       3.7.1. Geometric proofs ............................................ 22
       3.7.2. Other direct convergence proofs ........................ 23

*martin.idel@tum.de
1. Introduction

It is very common for important and accessible results in mathematics to be discovered several times. Different communities adhere to different notations and rarely read papers in other communities also because the reward does not justify the effort. In addition, even within the same community, people might not be aware of important results - either because they are published in obscure journals, they are poorly presented in written or oral form or simply because the mathematician did not notice them in the surrounding sea of information. This is a problem not unique to mathematics but instead inherent in all disciplines with epistemological goals.

The scaling of matrices is such a problem that has constantly attracted attention in various fields of pure and applied mathematics\textsuperscript{1}. Recently, generalisations have been studied also in physics to explore possibilities in quantum mechanics where it turns out that a good knowledge of the vast literature on the problem can help a lot in formulating approaches. This review tries to tell the mathematical story of matrix scaling, including algorithms and pointers to applications.

As a motivation, consider the following problem: Imagine you take a poll, where you ask a subset of the population of your country what version (if any) of a certain product they buy. You distinguish several groups in the population (for instance by age, gender, etc.) and you distinguish several types of product (for instance different brands of toothbrushes). From the sales statistics, you know the number of each product sold in the country and from the country statistics you know the number of people in different groups. Given the answers of a random sample of the population, how can you extrapolate results?

Central to a solution is the following innocuous theorem:

\textbf{Theorem 1.1} (Sinkhorn’s theorem, weak form Sinkhorn 1964). \textit{Given a matrix $A$ with positive entries, one can find matrices $D_1, D_2$ such that $D_1 A D_2$ is doubly stochastic.}

The literature on Sinkhorn’s theorem and its generalisations is vast. As we will see, there are some natural ways to attack this problem, which further explains why the different communities were often not aware of the efforts of their peers in other fields.

One of the main motivations for this review was a generalisation of Sinkhorn’s theorem to the noncommutative setting of positive maps on matrix algebras:

\textsuperscript{1}The term “Babylonian confusion” to describe the history of this problem was first used in Krupp 1979
Theorem 1.2 (Weak form of Gurvits 2003’s generalisation to positive maps). Given a map $\mathcal{E} : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ which maps positive semidefinite matrices to positive definite matrices one can find invertible matrices $X, Y$ such that the map $\mathcal{E}'(\cdot) := Y\mathcal{E}(X \cdot X^\dagger)Y^\dagger$ is doubly stochastic, i.e.

$$\mathcal{E}'(1) = 1, \quad \mathcal{E}'^*(1) = 1$$

with the adjoint matrices $X^\dagger$ and the adjoint map $\mathcal{E}^*$.

Some results and approaches can be translated to this noncommutative setting, but many questions remain open and the noncommutativity of the problem makes progress difficult.

Very recently, a new generalisation of Sinkhorn’s theorem to a noncommutative setting has appeared in Benoist and Nechita 2016.

The goal of this review is therefore threefold:

- Trace the historical developments of the problem and give credit to the many people who contributed to the problem.
- Illuminate the many approaches and connections between the approaches to prove Sinkhorn’s theorem and its generalisations.
- Sketch the generalisation to positive maps and its history and highlight the questions that are yet unanswered and might be attacked using the knowledge from the classical version.

In addition, I will try to give a sketch of the algorithmic developments and pointers to the literature for applications. I will probably have forgotten and/or misrepresented contributions; comments to improve the review are therefore very welcome.

2. Notation and Preliminaries

Most of the concepts and notations discussed in this short section are well-known and can be found in many books. I encourage the reader to refer to this section only if some notation seems unclear.

We will mostly consider matrices $A \in \mathbb{R}^{n \times m}$. Such matrices are called nonnegative (positive) if they have only nonnegative (positive) entries. We denote by $\mathbb{R}_+^n$ ($\mathbb{R}_+^n$) all vectors with only positive entries (nonnegative entries) and for any such $x \in \mathbb{R}_+^n$, $\text{diag}(x)$ defines the diagonal matrix with $x$ on its main diagonal, while $1/x \in \mathbb{R}_+^n$ defines the vector with entries $1/x_i$ for all $i$.

An important concept for nonnegative matrices is the pattern. The support or pattern of a matrix $A$ is the set of entries where $A_{ij} > 0$. A subpattern of the pattern of $A$ is then a pattern with fewer entries than the pattern of $A$. We write $B \prec A$ if $B$ is a subpattern of $A$, i.e. for every $B_{ij} > 0$ we have $A_{ij} > 0$. 

4
Finally, let us introduce irreducibility and decomposability. Details and connections to other notions for nonnegative matrices are explained in Appendix A. If $A$ is nonnegative, then $A$ is fully indecomposable if and only if there do not exist permutations $P, Q$ such that

$$PAQ = \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix}$$

where neither $A_1$ nor $A_2$ contain a zero row or column and $A_3 \neq 0$. The matrix is irreducible, if no permutation $P$ can be found such that already $PAP^T$ is of form (1). In particular, this implies that all fully indecomposable matrices are at least irreducible.

For positive vectors, we will not use the notation $x > 0$ to avoid confusion with the positive definite case: Especially in the second part of this review, we will be dealing mostly with positive (semi)definite matrices $A \in \mathbb{R}^{n \times n}$, which are symmetric matrices with only positive (nonnegative) eigenvalues and should not be confused with positive matrices. We also introduce the partial order $\geq$ for positive semidefinite matrices, where $A \geq B$ if and only if $A - B$ is positive semidefinite and $A > B$ if $A - B$ is positive definite.

When talking about positive maps, we will also adopt the notation that $\mathcal{M}_{n,m}$ denotes the complex $n \times m$ matrices, while the shorter $\mathcal{M}_n$ is used for complex $n \times n$ square matrices.

### 3. Different approaches to equivalence scaling

This section explores the historical development and current form of the mathematical landscape surrounding the following extension to Theorem 1.1:

**Theorem 3.1.** Let $A \in \mathbb{R}^{m \times n}$ be a matrix with nonnegative entries. Then for any vectors $r \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ with nonnegative numbers there exist diagonal matrices $D_1$ and $D_2$ such that

$$D_1AD_2e = r$$

$$D_2A^TD_1e = c$$

if and only if there exists a matrix $B$ with $Be = r$ and $B^Te = c$ and the same pattern as $A$. Here, $e = (1, \ldots, 1)^T$ which means that $r$ contains the row sums of the scaled matrix and $c$ contains the column sums.

Furthermore, if the matrix has only positive entries, $D_1$ and $D_2$ are unique up to a constant factor.

In Section 4, we give maximal formulations of this theorem. Some immediate questions emerge, such as: How to compute $D_1, D_2$ and the scaled matrix? Can this be generalised to arrays of higher dimension? All of these questions and many more have been answered in the literature.
Given a matrix \( A \in \mathbb{R}^{n \times n} \) with positive entries and the task to scale \( A \) to given row sums \( r \) and column sums \( c \), one is very naturally lead to the following approximation algorithm:

**Algorithm 3.2 (RAS method).** Given \( A \in \mathbb{R}^{m \times n} \), do:

- Multiply each row \( j \) of \( A \) with \( r_j / \left( \sum_i A_{ij} \right) \) to obtain \( A^{(1)} \) with row sums \( r \).
- Multiply each column \( j \) of \( A^{(1)} \) with \( c_j / \left( \sum_i A_{ji} \right) \) to obtain \( A^{(2)} \) with column sums \( c \).
- If the row sums of \( A^{(2)} \) are very far from \( r \), repeat steps one and two.

If the algorithm converges, the limit \( B \) will be the scaled matrix. However, there is a priori no guarantee that \( D_1, D_2 \) exist, in which case we can only ask for approximate scaling, i.e. matrices \( D_1, D_2 \) such that \( D_1 A D_2 \approx B \).

### 3.1. Historical remarks

The iterative algorithm 3.2 is extremely natural and it is therefore not surprising that it was rediscovered several times. It is at least known as *Kruithof’s projection method* (Krupp 1979) or *Kruithof double-factor model* (especially in the transportation community; Visick et al. 1980), the *Furness (iteration) procedure* (Robillard and Stewart 1974), *iterative proportional fitting procedure* (IPFP) (Ruschendorf 1995), the *Sinkhorn-Knopp algorithm* (Knight 2008), the *biproportional fitting procedure* (in the case of \( r = c = e \); Bacharach 1970) or the *RAS method* (especially in economics and accounting; Fofana, Lemelin, and Cockburn 2002). Sometimes, it is also referred to simply as *matrix scaling* (Rote and Zachariasen 2007), which is mostly used as the term for scalings of the form \( DAD^{-1} \), or *matrix balancing*, which is mostly used for scalings to equal row and column sums. The algorithm is a special case of a number of other algorithms such as Bregman’s balancing method (cf. Lamond and Stewart 1981) as we will see later on.

When was interest sparked in the RAS method and diagonal equivalence? The earliest claimed appearance of the model dates back to at least the 30s and Kruithof’s use of the method in telephone forecasting (Kruithof 1937). At a similar time, according to Bregman 1967, the Soviet architect Sheleikhovskii considered the method. Sinkhorn 1964 claims that when he started to evaluate the method, it had already been proposed and in use. His example is the unpublished report Welch unknown. Bacharach 1970 acknowledges Deming and Stephan 1940 in transportation science, who popularised the RAS method in the English speaking communities.

None of these approaches seem to have been thoroughly justified. Bacharach notes that Deming and Stephan only propose an ad-hoc justification for using their method to study their problem, which turned out to be wrong (cf. Stephan 1942). He further claims that the first well-founded approach to use the RAS - this time in economics - was given by Richard Stone, who also coined the name “RAS model” (Bacharach cites
Stone 1962, although the name RAS must have occurred earlier as it already occurs in Thionet 1961 without attribution and explanation). However, one can argue that the first justified approach occurred earlier: Schrödinger 1931 had already posed a question regarding models of Brownian motion when given a priori estimates, which led to a similar problem. His approach was justified, albeit the ultimate justification in terms of large deviation theory needed to wait for the development of modern probability theory (cf. Georgiou and Pavon 2015). The problem boils down to solving a continuous analogue of Sinkhorn’s theorem, which leads to the matrix problem using discrete distributions (essentially similar to Hobby and Pyke 1965) and was first attacked in Fortet 1940 using a fixed point approach similar to Algorithm 3.13.

However, none of the original papers provided a convergence proof with the possible exception of Fortet 1940\(^2\). As noted by Fienberg 1970, after Deming and Stephan provided their account, their community started to develop the ideas, but a proof was still lacking (Smith 1947; El-Badry and Stephan 1955; Friedlander 1961).

Summarising the last paragraphs, the RAS method was discovered independently for different reasons in the 30s to 40s, although none of the authors provided a proof (with the possible exception of Fortet). A more theoretical analysis developed in the 60s after Stone’s results in economics (e.g. Stone 1962) and Sinkhorn 1964 in statistics and algebra. Since then, a large number of papers has been published analysing or applying Theorem 3.1. Every decade since the sixties contains papers where proving the theorem or an extension thereof is among the main results (examples are Sinkhorn 1964; Macgill 1977; Pretzel 1980; Borobia and Cantó 1998; Pukelsheim and Simeone 2009; Georgiou and Pavon 2015).

Many authors are aware of at least some other attempts, but only a few try to give an overview.\(^3\) The situation is further complicated by the fact that the technical answer to the question of scalability is tightly linked with the question of patterns, which has a rich history in itself, probably starting with Fréchet (overview of a long line of work in Fréchet 1960).

The last point is particularly interesting: In fact, one could summarise matrix scaling as follows: Given a nonnegative matrix \(A\) it is scalable to a matrix \(B\) fulfilling some constraints (mostly linear but some nonlinear constraints are allowed), a matrix is scalable with diagonal matrices (in different ways, mostly \(D_1AD_2\) where \(D_1\) and \(D_2\) need not be independent) if and only if there exists a matrix \(C\) with the same pattern as \(A\) fulfilling the constraints.

---

\(^2\)The notation and writing is very difficult to read today, so I am not entirely sure whether the proof is correct and captures the case we are interested in.

\(^3\)This suggests once again that the problem had a very complicated history which also makes it difficult to find out whether a problem has already been solved in the past. Several authors have attempted more complete historical overviews such as Fienberg 1970; Macgill 1977; Schneider and Zenios 1990; Brown, Chase, and Pittenger 1993; Kalantari and Khachiyan 1996; Kalantari et al. 2008; Pukelsheim and Simeone 2009. In Rothblum and Schneider 1989, the authors claims that a colleague collected more than 400 papers on the topic of matrix scaling.
Today, proofs and generalisations of Theorem 3.1 and similar questions about scaling matrices in the form $DAD$ or $DAD^{-1}$ form a knot of largely interconnected techniques. We will now try to give an overview of these results and highlight their connections. A graphical overview is presented in Figure 3.1.

![Graphical diagram](image)

Figure 1: Connected approaches to prove Theorem 3.1 and their relationships. Red arrows and text denote natural algorithms and their connections.

### 3.2. The logarithmic barrier function

Potentials and barrier functions have been important in the study of matrix scaling since at least the unpublished results of Gorman 1963. Here, we largely follow Kalantari and Khachiyan 1996, who give a very lucid account about the interconnections between...
different barrier function formulations for $g$.

Let $A \in \mathbb{R}^{n \times n}$ be a matrix with nonnegative entries and $r, c \in \mathbb{R}_+^n$. Define the logarithmic barrier function

$$g(x, y) = y^T Ax - \sum_{i=1}^n c_i \ln x_i - \sum_{i=1}^n r_i \ln y_i$$

(2)

If we take partial derivatives, we obtain

$$\partial_{y_i} g(x, y) = Ax - r_i / y_i$$
$$\partial_{x_i} g(x, y) = y^T A - c_i / x_i$$

(3)

which implies that for any stationary point we have

$$\sum_j A_{ij} x_j y_i = r_i$$
$$\sum_j A_{ji} x_i y_j = c_i$$

and setting $D_1 = \text{diag}(y)$ and $D_2 = \text{diag}(x)$ solves the scaling problem. Conversely, any scaling gives a stationary point of the logarithmic barrier function. In summary:

**Lemma 3.3.** Given $A \in \mathbb{R}^{n \times n}$ nonnegative and two vectors $r, c \in \mathbb{R}_+^n$, then the matrix can be diagonally scaled to a matrix $B$ with row sums $r$ and column sums $c$ if and only if the corresponding logarithmic barrier function (2) has a stationary point.

According to Macgill 1977, this observation was first made by Gorman 1963 who also gave the first complete and correct proof. However, the paper only circulated privately. Gorman apparently did not consider this scaling function directly but used an approach similar or identical to the ones considered in convex geometry described in Section 3.5.

The potential barrier function can also be seen from the perspective of Lagrangian multipliers:

**Lemma 3.4** (Marshall and Olkin 1968). Given $A \in \mathbb{R}^{n \times n}$ nonnegative and two vectors $r, c \in \mathbb{R}_+^n$, then the matrix can be diagonally scaled to a matrix $B$ with row sums $r$ and column sums $c$ if and only if on the region

$$\Omega := \left\{(x, y) \mid \prod_{i=1}^m x_i^{y_i} = \prod_{i=1}^m y_i^{x_i} = 1, x_i > 0, y_i > 0 \right\}$$

(4)

the function $y^T Ax$ is bounded away from zero and is unbounded whenever $\|x\|_\infty + \|y\|_\infty \to \infty$. The function $y^T Ax$ then attains a minimum defining $D_1$ and $D_2$.

This was used to prove our Theorem 3.1 in Marshall and Olkin 1968. We observe:

**Observation 3.5.** Lemma 3.4 and 3.3 are equivalent: The logarithmic barrier function is the Lagrange function of the optimisation problem in Lemma 3.4.
Now consider $g(x, y)$ for a fixed $x$. Since $(-\ln)$ is a convex function and $x^TAy$ is linear in $y$, $g$ is convex in $y$. The same holds for a fixed $y$, i.e. $g$ is convex in both direction. It is then natural to consider the coordinate descent algorithm (for an introduction and overview see Wright 2015):

**Algorithm 3.6.** Given a nonnegative matrix $A$, take a starting point for $g$, e.g. $x_0 = y_0 = e$ and iterate:

1. For fixed $y_n$, find $x_{n+1}$ by searching for the minimum of $g(x, y_n)$.
2. For fixed $x_{n+1}$, find $y_{n+1}$ by searching for the minimum of $g(x_{n+1}, y)$.
3. Repeat until convergence.

It is possible to solve $\min_x g(x, y)$ or $\min_y g(x, y)$ analytically:

$$x_{n+1} = p / (Ay_n), \quad y_{n+1} = q / (Ax_{n+1}).$$

This leads to the following observation:

**Observation 3.7** (Kalantari and Khachiyan 1996). Algorithm 3.6 and 3.2 are the same.

**Proof.** Define $D_n^{(1)} := \text{diag}(y_n)$ and $D_n^{(2)} := \text{diag}(x_n)$. Then we have $D_{n+1}^{(1)}AD_n^{(2)}e = r$ and $e^TD_n^{(1)}AD_n^{(2)} = c^T$, which implies that we perform successive row- and column normalisations as in the RAS method. \hfill \qed

Using the fact that the algorithm is a coordinate descend method, one can obtain a convergence proof including a discussion of convergence speed of this algorithm and a dual algorithm (Luo and Tseng 1992). See also Observation 3.17 for a discussion of coordinate ascent methods.

However, $g$ is not jointly convex. For a purely (jointly) convex reformulation, consider the minimum for $t$ along any line $g(tx, ty)$, where $g$ is convex. If we define

$$k(x, y) := \min_{t > 0} g(tx, ty)$$

minimising $k(x, y)$ is still equivalent to minimising $g(x, y)$. The corresponding $k$ will be homogeneous and the domain for minimisation will in fact be compact.

**Observation 3.8** (Kalantari and Khachiyan 1996). We obtain:

$$k(x, y) = \min_{t > 0} \left( t^2 y^T A x - 2n \ln t - \sum_{i=1}^n c_i \ln x_i - \sum_{i=1}^n r_i \ln y_i \right)$$

$$= \ln \left( \frac{(y^T A x)^n}{\prod_{i=1}^n x_i^{c_i} \prod_{j=1}^n y_j^{r_j}} \right) + n - n \ln(n)$$

hence minimising $g$ is equivalent to minimising $k$. 

10
This proves the following lemma:

**Lemma 3.9.** Given a nonnegative matrix $A \in \mathbb{R}^{n \times m}$, it can be scaled to a matrix with row sums $r$ and column sums $c$ if and only if the minimum of $k(x, y)$ exists and is positive. The corresponding minima $(x, y)$ define the diagonal matrices to achieve the scaling.

The function $k$ is also similar to Karmakar’s potential function for linear programming and Algorithm 3.2 is the coordinate descent method for this function (Kalantari and Khachiyan 1996; Kalantari 1996).

Setting $y(x) = (Ax)^{-1}$, we arrive at another formulation of the problem. In the doubly stochastic case, this formulation is due to Djoković 1970; London 1971 and was later adapted to arbitrary column and row sums in Sinkhorn 1974:

**Lemma 3.10.** Let $A \in \mathbb{R}^{n \times n}$ be a nonnegative matrix. There exists a scaling to a matrix with row sums $r$ and column sums $c$ iff the infimum

$$\inf \left\{ \prod_{i=1}^{n} \left( \sum_{j=1}^{n} A_{ij} x_j \right)^{r_i} \prod_{i=1}^{n} x_i^{c_i} = 1 \right\}$$

is attained on $x, y \in \mathbb{R}_+$.\(^4\)

**Observation 3.11.** Note that the infimum is attained iff the infimum

$$\inf \left\{ \sum_{i=1}^{n} r_i \ln \left( \sum_{j=1}^{n} A_{ij} x_j \right) \left| \sum_{i=1}^{n} c_i \ln x_i = 0 \right. \right\}$$

is attained. This is the formulation in Lemma 3.4.

Finally, let us sketch a proof using potential methods.

**Sketch of proof of Theorem 3.1 (Potential version).** We sketch a proof for arbitrary row and column sums based on the short proof of Djoković 1970 for doubly stochastic scaling: First assume that $A \in \mathbb{R}^{m \times n}$ is a positive matrix. Starting with equation (8) we define the function

$$f(x_1, \ldots, x_n) := \frac{\prod_{i=1}^{m} \left( \sum_{j=1}^{n} A_{ij} x_j \right)^{r_i}}{\prod_{i=1}^{n} x_i^{c_i}}$$

on the set of $x_i$ with $x_i > 0$ and $\sum_i x_i = 1$. Consider an arbitrary point $b$ on the boundary (i.e. $b_i = 0$ for at least one $i \in 1, \ldots, n$). For $x_i \to b_i$, since $\prod_i x_i = 0$ and $\sum_j A_{ij} x_j \neq 0$,

---

\(^4\)later studied in Krupp 1979, who used an entropic approach for the generalised problem and in Berger and Kelley 1979, who used a direct convergence approach reminiscent of Sinkhorn and others.
always, we have that \(f(x_1, \ldots, x_n) \to \infty\). Hence the function takes its minimum in the interior. At the minimum, the partial derivatives must vanish and we obtain:

\[
0 = \frac{\partial}{\partial x_l} f(x_1, \ldots, x_n) = m \prod_{i=1}^n \left( \frac{\sum_{j=1}^n A_{ij} x_j}{x_i^{r_i}} \right) \left( \sum_{k=1}^m \left( \frac{x_k^{c_k}}{\sum_{p=1}^m A_{kj} x_j} \right)^{r_k} \right) - \sum_{k=1}^m A_{kl} r_k \left( \frac{\sum_{p=1}^m A_{kj} x_j}{x_i^{c_i+1}} \delta_{kl} \right).
\]

If we take all conditions for \(l = 1, \ldots, n\), then this is equivalent to the condition

\[
A^T \left( \frac{r}{(Ax)} \right) = \frac{c}{x}
\]

which boils down to equations (3).

The more technical part for nonnegative matrices is a more careful analysis of what happens for nonnegative matrices that are not positive. For doubly stochastic matrices, we can use the fact that fully indecomposable matrices have a positive diagonal, which implies once again that \(\prod_i \sum_j A_{ij} b_j \neq 0\). A similar argument can be made for arbitrary patterns, but we leave it out in this sketch.

3.3. Nonlinear Perron-Frobenius theory

Another early approach uses nonlinear Perron-Frobenius theory which is essentially a very general approach to tackle fixed point problems for (sub)homogeneous maps on cones. A short overview is given in appendix B. The basic idea is given by:

**Lemma 3.12** (Brualdi, Parter, and Schneider 1966). Given a nonnegative matrix \(A \in \mathbb{R}^{n \times n}\), there exists a scaling of \(A\) to a matrix with row sums \(r\) and column sums \(c\) if and only if the following map has a fixed point \(x > 0\):

\[
T : \mathbb{R}^n \to \mathbb{R}^n
\]

\[
T(x) = c / (A^T (r / (Ax)))
\]

This also suggests another simple algorithm:

**Algorithm 3.13.** Given a nonnegative matrix \(A\). Let \(x_0 = e\). Iterate until convergence:

\[
x_{n+1} = T(x_n).
\]
The development of this idea that started with Menon 1967 and was used to provide
a full proof of Theorem 3.1 in Brualdi, Parter, and Schneider 1966 for doubly stochastic
matrices. Menon and Schneider 1969 consider arbitrary row- and column sums and give
a complete study of the spectrum of the Menon-operator. Some contraction properties
were used to give a direct proof of convergence of the RAS algorithm in Berger and
Kelley 1979. The connection to Hilbert’s projective metric, and therefore to “Nonlinear
Perron-Frobenius theory” (cf. Lemmens and Nussbaum 2012), became clear later on
and allowed to give upper bounds on the convergence speed of the RAS (Franklin and
Lorenz 1989; Georgiou and Pavon 2015).

However, Menon was not the first to define the operator $T$: Looking closely at the
arguments given in Fortet 1940, one can see the continuous version of $T$, which lead to
an independent rediscovery of $T$ and its connection to the Hilbert metric in Georgiou
and Pavon 2015. Probably, Menon was not even the first to define the discrete version
of the operator and to note that the existence of a fixed point can be seen by invoking
Brouwer’s fixed point theorem. This dates back to Thionet 1963; Thionet 1964, building
on work about matrix patterns (Thionet 1961). According to Caussinus 1965, Thionet
1964 was also the first paper to conjecture the necessary and sufficient conditions for
scalability\(^5\). The ideas where rediscovered another time in Balinski and Demange 1989b,
where the authors used the fixed point argument to prove that a scaling exists and
fulfils their axiomatic approach.

Let us connect the approach to Section 3.2. First note that the algorithm is nothing
else but a slight variation of the RAS method:

**Observation 3.14.** Setting $y_{n+1} := r/(Ax_n)$ and $x_{n+1} := c/(ATy_{n+1})$ we can immedi-
ately see that one iteration of Algorithm 3.13 is one complete iteration of the RAS
method 3.2.

The connection with the logarithmic barrier method is also very close:

**Observation 3.15.** Any fixed point of the Menon operator defines a stationary point of
the logarithmic barrier function (2) and vice versa.

**Proof.** Let $A$ be a nonnegative matrix. The derivative conditions for the stationary
points of (2) are given in equation (3), which are equivalent to:

\[
Ax = r/y \quad y^TA = c/x \tag{11}
\]

This implies immediately that $x = p/(AT(q/Ax))$, hence $x$ is a fixed point of $T$.
Similarly, any positive fixed point immediately gives a scaling as a minimum of the
logarithmic barrier function. \(\square\)

This also proves Lemma 3.12.

\(^5\)He also notes that the early history around Deming and Stephan 1940 is a little bit curious, since the
authors claim to have a convergence proof but never publish it.
Sketch of proof of Theorem 3.1 (Nonlinear Perron-Frobenius theory version). Let us first assume $A$ has only positive entries. Then $T$ sends all vectors $x \in \mathbb{R}_{+0}^n$ to $\mathbb{R}_+^n$ hence using Brouwer’s fixed point theorem $T$ has a positive fixed point. Note that in order to apply Brouwer’s fixed point theorem, we need to have a compact set. To achieve this, consider the operator $\tilde{T}(x) = T(x)/\sum_{i=1}^n T(x)_i$.

For general nonnegative matrices $A$, one can extend $T$ to be a map from $x \in \mathbb{R}_{+0}^n$ into itself (see also Appendix B) either by a general argument (see Theorem B.8) or by defining $\infty \cdot 0 = 0$ and $\infty \cdot c = \infty$ for all positive $c$. One can easily see that $T$ will not send any entry to $\infty$.

However, it is not immediately clear when the fixed point is positive if $A$ contains zero-entries. This is the main technical difficulty for a complete proof. Brualdi, Parter, and Schneider 1966 show that if $A$ is fully indecomposable, $T(x)$ has at least $k + 1$ entries which are nonzero if $x$ has exactly $k$ entries which are nonzero, which immediately proves that the fixed point must be positive.

Upon closer observation, the map is contractive under Hilbert’s metric and Banach’s fixed point theorem immediately provides existence and uniqueness of the scaled matrix. The fixed point itself provides the diagonal of $D_2$.

3.4. Entropy optimisation

Another approach, which underlies many justifications for applications, considers entropy minimisations under linear constraints. An overview of entropy minimisation and its relation to diagonal equivalence can be found in Brown, Chase, and Pittenger 1993, a broader overview about the relation of the RAS algorithm to entropy scaling with a focus on economic settings can be found in McDougall 1999.

To formulate the problem, we define the Kullback-Leibler divergence, $I$-divergence or relative entropy, which was first described in Kullback and Leibler 1951 (see also Kullback 1959) for two vectors $x, y \in \mathbb{R}_{+0}^n$:

$$D(x \parallel y) := \sum_{j=1}^n x_j \ln \left( \frac{x_j}{y_j} \right)$$

where we use the convention that the summand is zero if $x_j = y_j = 0$ and infinity if $x_j > 0, y_j = 0$. The relative entropy is nonnegative and zero if and only if $x = y$ and it is therefore similar to a distance measure. Given a set, what is the smallest “distance” of a point to this set in relative entropy? This is known as $I$-projection (cf. Csiszár 1975).

Let $A$ be a nonnegative matrix and define

$$\Pi_1 := \{ B | Be = r \}$$
$$\Pi_2 := \{ B | e^T B = c^T \}.$$

We ask for the $I$-projection of $A$ onto the set $\Pi_1 \cap \Pi_2$, i.e. we want to find $A^*$ such that

$$D(A^* \parallel A) = \inf_{B \in \Pi_1 \cap \Pi_2} D(B \parallel A).$$  (13)
The connection to scaling was probably first used in Brown 1959, where the RAS method is used to improve an estimate for positive probability distributions of dimensions $2 \times 2 \times \ldots \times 2$ in the relative entropy measure (Brown cites Lewis 1959 as a justification for his approach, where the relative entropy is justified as a "closeness" measure). According to Fienberg 1970, this approach was later generalised to all multidimensional tables in Bishop 1967 based on some duality of optimisation by Good 1965. Another early use of relative entropy occurs in Uribe, Leeuw, and Theil 1966 (see also Theil 1967), where it was noted without proof that the results were the same as the RAS.

A very natural approach to obtain $A^*$ would be to try an iterative I-projection:

**Algorithm 3.16.** Let $A$ be nonnegative.

- Let $A^{(0)} = A$.
- If $n$ is even, find $A^{(n+1)}$ such that
  \[ D(A^{(n+1)} \| A^{(n)}) := \inf_{B \in \Pi_1} D(B \| A). \]
- If $n$ is odd, find $A^{(n+1)}$ such that
  \[ D(A^{(n+1)} \| A^{(n)}) := \inf_{B \in \Pi_2} D(B \| A). \]
- Repeat the steps until convergence.

**Observation 3.17** (cf. Csiszár 1975; Csiszár 1989). The algorithms 3.16 and 3.2 are the same.

**Proof.** This was first shown in Ireland and Kullback 1968. We give a short argument based on Lagrangian multipliers restricted to column normalisation. Given $A \in \mathbb{R}^{n \times n}$, the Lagrangian for the problem is

\[ L(B, \lambda) := D(B \| A) + \lambda_j \left( \sum_i A_{ij} - q_j \right). \]

Partial derivatives $\partial_{B_{ij}} L = 0$ and $\partial_{\lambda_j} L = 0$ lead to the system of equations:

\[ \ln \left( \frac{B_{ij}}{A_{ij}} \right) + 1 + \lambda_j = 0 \quad i, j = 1, \ldots, n \]
\[ \sum_i A_{ij} - c_j = 0 \quad j = 1, \ldots, n. \]

A solution is easily seen to be

\[ B_{ij} = A_{ij} \frac{c_j}{\sum_k A_{kj}} \quad i, j = 1, \ldots, n. \]

The latter is the column renormalisation as in the RAS (Alg. 3.2). \hfill \Box

\(^6\)Both references were not available to me.
This implies that if the iterated I-projection converges to the I-projection of (13), then matrix scalability solves equation (13). This was supposedly proved in Ireland and Kullback 1968\(^7\) and Kullback 1968, but the proofs contain an error as pointed out in Csiszár 1975 (see also Brown, Chase, and Pittenger 1993). A corrected proof appeared in Csiszár 1975, however for some of the theorems it is not immediately clear whether more assumptions are needed as noted in Borwein, Lewis, and Nussbaum 1994.

In addition, the proof in Aaronson 2005 for positive matrices proves that the RAS converges using relative entropy as a “progress measure”. He shows that it decreases under RAS steps to a unique stationary point. Another direct proof appeared in Franklin and Lorenz 1989.

At this point, let us make the following observation:

**Observation 3.18** (Cottle, Duvall, and Zikan 1986). The RAS method can also be seen as the coordinate ascent method to the dual problem of entropy minimisation.

This is justified as follows: When deriving the I-projections of each step of the algorithm, we set up the Lagrangian

\[
L(B, \lambda) := D(B \parallel A) + \lambda_j \left( \sum_i A_{ij} - c_j \right)
\]

and calculate its solution. This consists in explicitly solving the resulting equations for the Lagrangian multipliers \( \lambda_j \). In this sense, the algorithm is not really a primal problem. This is also consistent with the nomenclature above: In Section 3.5 we see that the dual problem of entropy minimisation is a convex program that is basically just the (negative) logarithmic barrier function above. Since the RAS is the coordinate descent algorithm of this problem, it is the coordinate ascent method of the dual problem of entropy minimisation.

In other word, the justification of this observation is due to:

**Observation 3.19** (Georgiou and Pavon 2015; Gurvits 2004). Given a matrix \( A \in \mathbb{R}^{n \times n} \) with nonnegative entries. Suppose there exist positive diagonal matrices such that \( D_1 A D_2 \) has row sums \( r \) and column sums \( c \), then

\[
- \ln \left( \inf \left\{ \prod_{i=1}^n r_i \sum_{j=1}^n A_{ij} x_j \left| \prod_{i=1}^n x_i = 1 \right. \right\} \right) = \inf \{ D(B \parallel A) \left| B e = r, B^T e = c \right. \} \tag{14}
\]

and in particular, the minimum is the scaled matrix.

The proof of this observation will essentially follow from the results in Section 3.5. Let us finish this section by giving another proof sketch of Sinkhorn’s theorem:

---

\(^7\)In Fienberg 1970, it is pointed out that a simplified version of this proof appeared in Dempster 1969, which however is unavailable to me.
Sketch of proof of Theorem 3.1 (Entropic version). We sketch the proof given in Csiszár 1975 restricted to our scenario, which is similar to the proof in Darroch and Ratcliff 1972 (see also Csiszár 1989 for a comment on the connection). We prove convergence of Algorithm 3.16, essentially by showing that the relative entropy of two successive iterations decreases to zero.

Given a nonnegative matrix $A$, assume that there exists a matrix $B \prec A$ with required row- and column sums. Otherwise, the relative entropy will always be infinite and the problem has no solution.

The crucial observation is that if $A'$ is the I-projection of $A$ onto $\Pi$, then for any $B \in \Pi$ we have

$$D(B\|A) = D(B\|A') + D(A'\|A).$$ (15)

This “Pythagorean identity” usually only holds with $\geq$. The equality case is a special case of the “minimum discrimination principle” (Kullback 1959; Kullback and Khairat 1966) and it is proven for constraints $\Pi_i$ in Csiszár 1975. This equality leads to a very useful transitivity result (see also Ku and Kullback 1968) stating that if $A$ has I-projection $B$ on $\Pi_i$ for some $i$ and I-projection $B'$ on $\Pi$, then $B$ has I-projection $B'$ on $\Pi$. This is not necessarily true in the general case.

Let $A'$ be the I-projection of $A$ onto $\Pi$. Denoting by $A^{(n)}$ the repeated I-projection as defined in Algorithm 3.16, repeated application of equation (15) shows

$$D(A'\|A) = D(A'\|A^{(n)}) + \sum_{i=1}^{n} D(A^{(n)}\|A^{(n-1)})$$

Therefore, the sequence $A^{(n)}$ lies in a bounded set and hence contains a convergent subsequence by compactness. However, we also have that $D(A^{(n)}\|A^{(n-1)}) \to 0$ for $n \to \infty$, which implies $\|A^{(n)} - A^{(n-1)}\|_\infty \to 0$ for $n \to \infty$, hence $A^{(n)}$ converges to some matrix $A''$. Clearly, $A'' \in \Pi$, since $A^{(2n)} \in \Pi_1$ and $A^{(2n+1)} \in \Pi_2$ for every $n \in \mathbb{N}$. Using the transitivity of the I-projection, $A''$ is the I-projection of $A^{(n)}$ for all $n$ and equation (15) holds in the form

$$D(A''\|A^{(n)}) = D(A''\|A') + D(A'\|A^{(n)})$$

Since the first and last term converge to zero, $D(A''\|A') = 0$ and the I-projection $A'$ is indeed the limit of Algorithm 3.16. □

A similar proof can be found in Brown, Chase, and Pittenger 1993.

3.5. Convex programming and dual problems

Recall the logarithmic barrier function $g$ in equation (2) and that it is not jointly convex. However, it is very beneficial to make $g$ convex for several reasons:
1. Convex programming is efficient in the complexity theoretic sense (Boyd and Vandenberghe 2004).

2. The duality theory for convex programming is very well developed and can lead to new algorithms (see littleO 2014 for a heuristic introduction and Rockafellar 1997; Boyd and Vandenberghe 2004 for a more careful analysis).

3. Uniqueness proofs can become simpler: A convex function has a unique minimum iff it is strictly convex at the minimum.

To obtain a convex program, one simply needs to substitute $x = (e^{\xi_1}, e^{\xi_2}, \ldots, e^{\xi_n})$ and $y = (e^{\eta_1}, e^{\eta_2}, \ldots, e^{\eta_n})$ into $g$ to obtain (Macgill 1977; Kalantari and Khachiyan 1996):

**Lemma 3.20.** Given a nonnegative matrix $A \in \mathbb{R}^{n \times n}$, one can find diagonal matrices to scale $A$ to a matrix with row-sum $r$ and column sum $c$ if and only if the function

$$f(\xi, \eta) := \sum_{ij} A_{ij} e^{\eta_i + \xi_j} - \sum_{i} r_i \xi_i - \sum_{j} c_j \eta_j$$

attains its minimum on $\xi, \eta \in \mathbb{R}^{n}_{\geq 0}$.

A proof based on this approach can be found in Bachem and Korte 1979.\footnote{In Bacharach 1970 it is also noted that the function is used in the approach by Gorman 1963 later to be simplified by Bingen 1965. Both papers are unavailable to me.}

---

8.\footnote{In Bacharach 1970 it is also noted that the function is used in the approach by Gorman 1963 later to be simplified by Bingen 1965. Both papers are unavailable to me.}

We have already seen:

**Observation 3.21.** The convex programming formulation in Lemma 3.20 is equivalent to the logarithmic barrier function approach in Lemma 3.3.

Likewise, it can be shown:

**Observation 3.22.** The convex programming formulation 3.20 is the Wolfe dual (Macgill 1977; Krupp 1979) or Lagrangian dual (Balakrishnan, Hwang, and Tomlin 2004) of the entropy minimisation approach.

**Proof.** The entropy minimisation problem was given as:

$$\inf_{B_{ij}} \sum_{ij} B_{ij} \ln(B_{ij} / A_{ij})$$

s.t. $\sum_{i} B_{ij} = p_j$ $\sum_{j} B_{ij} = q_i$

This implies that the Wolfe dual is given by

$$\sup_{B_{ij}} \sum_{ij} B_{ij} \ln(B_{ij} / A_{ij}) + \sum_{j} u_j \left( \sum_{i} B_{ij} - p_j \right) + \sum_{i} v_i \left( \sum_{j} B_{ij} - q_i \right)$$
s.t. \( \ln(A_{ij}/B_{ij}) + 1 + u_j + v_i = 0 \quad \forall i, j \)
\( u, v \geq 0 \)

The constrained can be rewritten as
\[ B_{ij} = A_{ij} \exp(-1 - u_j - v_i) \]
and inserting this into the Wolfe dual function (see e.g. Bot and Grad 2010) we obtain:
\[
\sup_{B_{ij}} \sum_{ij} B_{ij} \ln \left( \frac{B_{ij}}{A_{ij}} \right) + \sum_j u_j \left( \sum_i B_{ij} - p_j \right) + \sum_i v_i \left( \sum_j B_{ij} - q_i \right)
\]
\[ = \sup_{u,v} - \left( \sum_{ij} A_{ij} \exp(-1 - u_j - v_i) - \sum_j u_j p_j - \sum_i v_i q_i \right) \]
which is (up to the constant \( \sum_{ij} A_{ij} / e \)) the optimisation problem in 3.20. The calculation for the Lagrangian dual is similar (see Balakrishnan, Hwang, and Tomlin 2004).

Another connection to the barrier function is to use geometric programming:

**Observation 3.23.** Minimisation of the logarithmic barrier function \( g \) is equivalent to
\[
\min \ y^T Ax \\
\text{s.t. } \prod_{i=1}^n x_i^\xi = 1, \prod_{i=1}^n y_i^{\eta_i} = 1
\]
This is in standard form of a geometric program, which implies that a substitution \( \xi = \ln(x), \eta = \ln(y) \) gives a convex program (Boyd and Vandenberghe 2004, Section 4.5.3).

This observation was made in Rothblum and Schneider 1989, which also gives necessary and sufficient conditions for a matrix to be scalable or approximately scalable.

As described in Kalantari and Khachiyan 1996, one can also reduce the problem to an unconstrained optimisation problem for only a single variable by taking the formulation of Lemma 3.10 and substituting \( x = \exp(\xi) \) as above to obtain the minimising function

**Lemma 3.24.** Given a nonnegative matrix \( A \in \mathbb{R}^{n \times n} \), one can find diagonal matrices to scale \( A \) to a matrix with row sums \( r \) and column sums \( c \) if and only if the function
\[
f(\xi) = \sum_{j=1}^n r_j \ln \left( \sum_{i=1}^n A_{ij} e^{\xi_i} \right) - \sum_{i=1}^n c_i \xi_i.
\]
attains its minimum on \( \xi \in \mathbb{R}_{n,0}^n \).

Finally, let us return to entropy minimisation: Relative entropy is jointly convex and therefore a convex program. In fact, relative entropy is a special case of a broader class of functions called Bregman divergences which we will sketch in Section 6.5.

A proof of Theorem 3.1 using convex programming is often similar to the approach in Section 3.2. The advantage is that any critical point is automatically a minimum and one does not need to consider the boundary.
3.6. Topological (non-constructive) approaches

In the proof of Theorem 3.1 in Section 3.3, the result was achieved by Brouwer’s fixed point theorem, but it is only one of many topological proofs.

For every nonnegative matrix $A$ with a given pattern, we want to decide whether a scaling with prespecified row- and column sums exists. Assume that we also know the set of possible row- and column sums for a given pattern. In a sense, we therefore just have to prove that the map $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ defined via $(D_1, D_2) \mapsto D_1AD_2$ hits all row- and column sums, or else: we need to see that the map $\phi' : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+ \times \mathbb{R}^n_+$

$$(D_1, D_2) \mapsto (p, q) : p_i = \sum_j (D_1AD_2)_{ij}, q_j = \sum_i (D_1AD_2)_{ij}$$

is onto. This is somewhat problematic, because the spaces involved are not compact, but by normalising both diagonal matrices and row- and column-sums, one can consider the map as a map from a compact space into itself. This approach was taken in Bapat 1982 for positive matrices (based on his thesis) and the map was shown to be surjective using a topological theorem, which Bapat claims is sometimes known as Kronecker’s index theorem.\footnote{I could not find any other instance of where the theorem is given that name. The theorem simply states that for any map $f : D^{n+1} \rightarrow D^{n+1}$, if $f$ maps $\partial D^{n+1}$ into itself and is of nonzero degree, then it must be surjective.}

The case for general nonnegative matrices could only be covered by combining the approach with Raghavan 1984 (see Bapat and Raghavan 1989).

Raghavan 1984 uses yet another fixed point theorem (Kakutani’s fixed point theorem of set-valued maps). Defining the set $K$ of all matrices in $\mathbb{R}^{n \times m}$ with prescribed marginals and zero (sub)pattern of the a priori matrix $A$, he considers the map

$\phi(H) = \{Z | Z \in K, \max_{Z'} \langle C(H), Z' \rangle = \langle C(H), Z \rangle \}$

where $C(H)_{ij} = \log(A_{ij}/H_{ij})$ (if $A_{ij} > 0$, and 0 else), we take all matrices as vectors in $\mathbb{R}^{nm}$ and the usual scalar product. The fixed point theorem then implies that there exists $H$ such that

$$\max_{Z' \in K} \langle C(H), Z' \rangle = \langle C(H), H \rangle$$

and using the dual of this maximisation, one can show that it scales the matrix.

**Observation 3.25.** There is a simple connection to entropy minimalization, since $\langle C(H), H \rangle = D(H\|A)$.

However, we can also take the converse road: Instead of exploring the possibilities for every $A$, we can start with the set of matrices with prespecified row- and column sums.
and matrix pattern \( \mathcal{X} \) (call the set \( \mathcal{M}(p, q, \mathcal{X}) \)) and map it to the set of all nonnegative matrices of pattern \( \mathcal{X} \) (call it \( \mathcal{M}(\mathcal{X}) \)) by diagonal equivalence, i.e. consider the map:

\[
\psi : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathcal{M}(p, q, \mathcal{X}) \to \mathcal{M}(\mathcal{X}) \\
(D_1, D_2, B) \mapsto D_1 BD_2 
\] (20)

Again, it would be enough to show surjectivity. As such, it cannot be injective, because we can obviously shift a scalar from \( D_1 \) to \( D_2 \), hence we would at least have to restrict the first coordinate of \( D_1 \) to be 1. The resulting map \( \psi' \) is indeed a homeomorphism as shown in an overlooked paper of Tverberg 1976.

Another topological proof has recently been proposed in Friedland 2016. To describe the approach, note that the following two statements are equivalent:

1. There exist \( D_1, D_2 \) such that \( D_1 A D_2 \) has row sums \( r \) and column sums \( c \).

2. There exist \( D_1', D_2' \) such that \( D_1' A D_2' \) is a stochastic matrix with \( D_1' A D_2' c = r \).

The proof is trivial, in fact \( D_1' = D_1 \) and \( D_2' = D_2 \, \text{diag} (1/c) \). In Friedland 2016, the author therefore restricts to stochastic matrices. To do this, he defines the following map:

\[
\Phi_A : \mathbb{R}^n_+ \to \mathbb{R}^{n \times n}_+; \quad \Phi_A(x) = \text{diag}(x) A / \text{diag}(A^T x) 
\] (21)

A quick calculation shows that \( \Phi_A^T e = e \), hence the matrix is always stochastic. Hence given a nonnegative matrix \( A \) and row sums \( r \) and columns sums \( c \), the question of scalability is equivalent to the question whether there exists an \( x \in \mathbb{R}^n \) such that \( \Phi_A(x)c = r \).

For positive matrices \( A \) and any \( c \in \mathbb{R}^n_+ \), Friedland 2016 now proves scalability by proving that the map \( \Phi_{A,c} : x \to \Phi_A(x)c \) is continuous as a set from \( \mathbb{R}^n_+ \cap \{ v \mid \sum_i v_i = 1 \} \) onto itself and a diffeomorphism from \( \mathbb{R}^n_+ \cap \{ v \mid \sum_i v_i = 1 \} \) onto itself. The result is achieved using degree theory similar to Bapat and Raghavan 1989.

### 3.7. Other ideas

A very general approach to prove Theorem 3.1 was provided in Letac 1974, where matrix theorems are derived as a consequence of the following theorem:

**Theorem 3.26** (Letac 1974). Let \( X \) be a finite set, \((\mu(x))_{x \in X}\) strictly positive numbers and \( \mathcal{H} \) a fixed linear subspace of \( \mathbb{R}^X \). Then there exists a unique (nonlinear) map from \( \mathbb{R}^X \to \mathcal{H} \) denoted \( f \mapsto h_f \) such that

\[
\sum_{x \in X} [\exp(f(x)) - \exp(h_f(x))] g(x) \mu(x) = 0
\]

for all \( g \) in \( \mathcal{H} \).
Sinkhorn’s theorem follows as an easy corollary:

**Sketch of proof of Theorem 3.1, Letac 1974.** First, let \( X \subset \{1, \ldots, m\} \times \{1, \ldots, n\} \), then we first define the following maps:

\[
a : (\mathbb{R}^m, \mathbb{R}^n) \to \mathbb{R}^{m \times n}, \quad (\xi, \eta) \to (\xi_i + \eta_j)_{ij}
\]

\[
\pi : (\mathbb{R}^{m \times n}) \to \mathbb{R}^X, \quad (A_{ij})_{i=1,j=1}^{m,j=n} \to (A_{ij})_{(i,j) \in X}
\]

The second is just the natural projection from \( \mathbb{R}^{m \times n} \) to \( \mathbb{R}^X \).

Now let \( A \in \mathbb{R}^{m \times n} \) be a nonnegative matrix and let \( X := \{ (i,j) | A_{ij} > 0 \} \) be its pattern. We already know that the pattern is a necessary condition for scalability, hence we know that there exists a \( B \in \mathbb{R}^{m \times n} \) with row sums \( r \) and column sums \( c \). Given the pattern, we define \( \mathcal{H} = \text{ran}(\pi \circ a) \) the range of the composition of \( \pi \) and \( a \).

Now let \( F \in \mathbb{R}^X \) be the matrix with entries \( F_{ij} := \log(B_{ij}/A_{ij}) \) and apply the theorem to \( F \), i.e. there exists a unique matrix \( H \in \mathcal{H} \) such that

\[
\sum_{ij} \exp(F_{ij})A_{ij}G_{ij} = \sum_{ij} \exp(H_{ij})A_{ij}G_{ij} \quad \forall G \in \mathcal{H}
\]

But since \( \exp(F_{ij})A_{ij} = B_{ij} \) and \( H_{ij} = \xi_i + \eta_j \) for some \( \xi \in \mathbb{R}^m, \eta \in \mathbb{R}^n \) by definition of \( \mathcal{H} \), we have

\[
\sum_{ij} B_{ij}G_{ij} = \sum_{ij} \exp(\xi_i)A_{ij} \exp(\eta_j)G_{ij} \quad \forall G \in \mathcal{H}
\]

which implies that \( \exp(\xi_i)A_{ij} \exp(\eta_j) \) has row sums \( r \) by taking \( G = \pi \circ a(e_i, 0) \) and column sums \( c \) by taking \( G = \pi \circ a(0, e_j) \) for the unit vectors \( e_i \in \mathbb{R}^m, e_j \in \mathbb{R}^n \).

Clearly, the choice of \( (\xi, \eta) \) is unique up to \( \ker(\pi \circ a) \), which can be made explicit and leads to the usual conditions.

**3.7.1. Geometric proofs**

In principle, we have already two geometric interpretations of the RAS: First, the RAS is akin to iterated I-projections and second, the RAS is the application of a contractive mapping on a cone with a projective metric. Two other “geometric” proofs are known:

Fienberg 1970 shows that the RAS is a contractive mapping in the Euclidean metric using that the RAS preserves cross-ratios of a matrix. Given a matrix, the products

\[
\alpha_{ijkl} := \frac{A_{ij}A_{kl}}{A_{il}A_{kj}} \quad (22)
\]

remain invariant. This was first observed in Mosteller 1968, where it was used to justify the use of the RAS in statistical settings (see Section 8). Fienberg then follows that if one associates any positive matrix to a point of the simplex

\[
S_{rc} = \{ (A_{11}, \ldots, A_{1c}, \ldots, A_{r1}, \ldots, A_{rc}) | \sum_{ij} A_{ij} = 1 \}
\]

22
by normalising the matrix, then any point reachable by diagonal equivalence scaling lies on a certain type of manifold inside the simplex. Using some structural knowledge of these manifolds he then shows that each full cycle of the RAS corresponds to a contraction mapping with respect to the Euclidean metric. The result is general enough to cover multidimensional tables, but in this simplicity handles only positive matrices.

There is an interesting connection: While the cross-ratios within the matrix remain constant, Hilbert’s metric is also closely connected to cross-ratios. In fact, the contraction ratio is connected to the largest cross-ratio within the matrix and it is not finite if the matrix contains zeros. In that case, the matrix does not easily define a contraction in Hilbert’s metric. The same holds true for Fienberg’s proof.

Borobia and Cantó 1998 consider the column space of scaled matrices $A S$ and notes that $R A S$ is doubly stochastic if the columns are included in the convex hull of the columns of $R^{-1}$ and the barycentre of the sets of the two columns coincide. The observation of the barycentre then leads them to a proof involving Brouwer’s fixed point theorem once again. By some continuity argument, the proof can be extended to nonnegative matrices.

3.7.2. Other direct convergence proofs

Many papers contain direct convergence proofs, not least the original approach in Sinkhorn 1964 and the proof of the full result Sinkhorn and Knopp 1967 (another proof based on this approach is given in Pretzel 1980). The idea is to show that some seemingly unrelated quantity always converges. Often, this quantity turns out to be very much related to some potential barrier function or entropy and we already cited the approach in the corresponding section.

One different proof is the short convergence proof of Macgill 1977 establishing that $\sum_j A_{ij}^{(n)} / \sum_j A_{ij}^{(n-1)} \rightarrow 1$ and similarly $\sum_i A_{ij}^{(n)} / \sum_i A_{ij}^{(n-1)} \rightarrow 1$ for every $i, j$. This proof is in some sense derived from Bacharach’s approach (Bacharach 1965; Bacharach 1970, see also Seneta 2006) and is very straightforward. In parallel to Bacharach’s earlier work Bacharach 1965 but not cited in his later Bacharach 1970, Caussinus 1965 proved the convergence of the RAS method in the general case of multidimensional matrices via the same idea which he attributes to Thionet 1964 (see the appendix of Caussinus 1965).

A second direct proof of convergence in Sinkhorn 1967, uses a norm difference as convergence measure. More precisely, he considers the map

$$\phi(x, y) = \max \left( r_i^{-1} \sum_j x_i A_{ij} y_j \right) - \min \left( r_i^{-1} \sum_j x_i A_{ij} y_j \right)$$

10Macgill also mentions yet another work that contains a proof of Theorem 3.1, namely Herrmann 1973, however no details are given beyond the fact that it contains also approximate scaling.
on the set of all \((x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^m\) with some boundedness condition on their entries and proves that \(\phi(x, y) = 0\) is achieved by two positive vectors.

A third proof of direct and approximate scaling is given in Pukelsheim and Simeone 2009 by combining the approach of Bacharach with a simple \(L^1\)-error function borrowed from Balinski and Demange 1989b.

4. Equivalence scaling

Let us now collect maximal results. A similar but scarcely referenced collection of results was provided in Krupp 1979. We follow the cleaner presentation style of Rothblum and Schneider 1989. Starting with equivalence scaling, we have:

**Theorem 4.1.** Let \(A \in \mathbb{R}^{n \times m}\) be a nonnegative matrix and \(r \in \mathbb{R}_+^n, c \in \mathbb{R}_+^m\). Then the following are equivalent:

1. There exist positive diagonal matrices \(D_1, D_2\) such that \(D_1AD_2\) has row sums \(r\) and column sums \(c\).

2. There exists a matrix \(B\) with row sums \(r\) and column sums \(c\) with the same pattern as \(A\) (Menon 1968; Brualdi 1968).

3. There exists no pair of vectors \((u, v) \in \mathbb{R}^n \times \mathbb{R}^m\) such that (Rothblum and Schneider 1989)

   \[
   u_i + v_j \geq 0 \quad \forall (i, j) \in \text{supp}(A)
   \]

   \[
   r^T u + c^T v \leq 0
   \]

   either \(u_i + v_j > 0\) for some \((i, j) \in \text{supp}(A)\) or \(r^T u + c^T v < 0\)

4. For every \(I \subset \{1, \ldots, m\}, J \subset \{1, \ldots, n\}\) such that \(A_{IJ} = 0\) we have that

   \[
   \sum_{i \in I} r_i \geq \sum_{j \in J} c_j
   \]

   and equality holds if and only if \(A_{IJ} = 0\) (Menon and Schneider 1969).

5. The RAS method converges and the product of the diagonal matrices of the iteration also converges to positive diagonal matrices (Sinkhorn and Knopp 1967).

The equivalence of the first two items was essentially established in the proof sketches in section (3). The equivalence to the fourth item follows from the characterisation of matrix patterns (see appendix A) and the third follows from studying the geometric program 3.23.

For doubly stochastic scaling, using the classification of doubly stochastic patterns, we then know that scalability is equivalent to having total support (cf. Csima and Datta
The scaling matrices $D_1, D_2$ are unique up to scalar multiplication if and only if $A$ is fully indecomposable.

For approximate equivalence scaling, the results are similar. The only difference is that certain elements of $A$ can become zero in the limit (which implies that elements of $D_i$ must become zero and others infinite, hence the diagonal matrices cannot exist):

**Theorem 4.2.** Let $A \in \mathbb{R}^{n \times m}$ be a nonnegative matrix and $r \in \mathbb{R}^n_+, c \in \mathbb{R}^m_+$. Then the following are equivalent:

1. For every $\varepsilon > 0$ there exist diagonal matrices $D_1, D_2$ such that $B = D_1AD_2$ satisfies
   \[ \|Be - r\| < \varepsilon, \|B^Te - c\| < \varepsilon \]

2. There exists a matrix $A' \prec A$ such that $A'$ is scalable to a matrix $B$ with row sums $r$ and column sums $c$.

3. There exists a matrix $B \prec A$ with row sums $r$ and column sums $c$ (Schneider and Saunders 1980).

4. There exists no pair of vectors $(u, v) \in \mathbb{R}^{n \times m}$ such that (Rothblum and Schneider 1989)
   \[ u_i + v_j \geq 0 \quad \forall (i, j) \in \text{supp}(A) \]
   \[ r^T u + c^T v < 0 \]

5. For every $I \subset \{1, \ldots, m\}, J \subset \{1, \ldots, n\}$ such that $A_{I,J} = 0$ we have that
   \[ \sum_{i \in I} r_i \geq \sum_{j \in J} c_j \]

6. The RAS method converges (Sinkhorn and Knopp 1967 for the d.s. case).

For doubly stochastic scaling, using the classification of doubly stochastic patterns, we have that approximate scalability is equivalent to $A$ having support. Using Schneider and Saunders 1980, this is a trivial consequence of the fact that a matrix has total support if and only if it has doubly stochastic pattern and Proposition A.5\(^{11}\).

The uniqueness conditions are also simple enough to state:

**Theorem 4.3.** Let $A \in \mathbb{R}^{n \times m}$ be a nonnegative matrix and $r \in \mathbb{R}^n_+, c \in \mathbb{R}^m_+$. Then, $A$ has at most one scaling.

Furthermore, if there exist no permutations $P, Q$ such that $PAQ$ is a direct sum of block matrices, then $D_1, D_2$ are unique up to scalar multiples. Otherwise, the scaled matrices $D_1, D_2$ are only unique up to a scalar multiple in each block.

---

\(^{11}\)One recent observation of this is in Bradley 2010. The observation has however already been made before such as in Achilles 1993
For doubly-stochastic scaling, this result already appears in Brualdi, Parter, and Schneider 1966. For the case of general marginals, it occurs in Menon 1968 and for general matrices and marginals in Hershkowitz, Rothblum, and Schneider 1988; Menon and Schneider 1969. The tools can also be applied to prove that the approximately scaled matrix is unique.

Let us now have a closer look at the difference between approximate scaling and equivalence scaling. What can be said about the convergence of the RAS?

**Theorem 4.4** (Pretzel 1980, Theorem 1). *Let $A \in \mathbb{R}^{n \times n}$ be a matrix that is approximately scalable to a matrix with row sums $r$ and column sums $c$. Let $B$ be a matrix with row sums $r$ and column sums $c$ with maximal subpattern of $A$ (i.e. the number of entries $(i, j)$ such that $B_{ij} = 0$ and $A_{ij} > 0$ is minimal).

Then $A$ converges to a matrix $C \lessdot B$ and the same result holds for $A'$ with $A'_{ij} = A_{ij}$ if $B_{ij} > 0$ and $A'_{ij} = 0$ else.*

The continuity of the scaling can also be studied:

**Theorem 4.5.** *Let $A$ be nonnegative and $r, c$ be prescribed row- and column sums. Then the limit of the Sinkhorn iteration procedure is a continuous function of $A$ on the space of matrices with $r, c$-pattern.

When the scaling matrices are unique up to a scalar multiple, this also implies that the scaling is continuous in $D_1, D_2$.*

The first proof of this result limited to the doubly-stochastic case was given in Sinkhorn 1972. The full result follows directly from the homeomorphism properties of the map (20) from Tverberg 1976. A discussion is also presented in Krupp 1979. Furthermore, the continuity can be achieved using arguments of Section 9.3.

Finally, let us mention another characterisation of equivalence scaling using *transportation graphs*.

Following Schneider and Zenios 1990, let $A \in \mathbb{R}^{m \times n}$ be a nonnegative matrix. Let $M = \{1, \ldots, m\}, N = \{1, \ldots, n\}$ and consider the bipartite graph with the bipartition given by the vertices $M$ and $N$ and the edges defined via $E = \{(i, j) : A_{ij} > 0\}$, directed from $i \in M$ to $j \in N$. Now we define a source $S_1$ that connects to each vertex in $M$, where the edges have capacity $r_i$ (corresponding to the edge from $S_1$ to $i \in M$) and we define a sink $S_2$ that is connected from every vertex in $N$, where the edges have capacities $c_j$ (see Fig. 2 for an example).

Then it is easy to see that the matrix is approximately scalable if and only if the maximum flow of this network is equal to $\sum r_i$. The flows along the edges $E$ then define a matrix with the wanted pattern. The matrix is exactly scalable if and only if the maximum flow of this network is equal to $\sum r_i$ and every edge contains flow.
\[
A = \begin{pmatrix}
0 & 3 & 4 & 1 \\
2 & 1 & 0 & 0 \\
2 & 0 & 0 & 1
\end{pmatrix}
\]

Figure 2: An easy example of the transportation graph for row sums \( r \) and column sums \( c \) corresponding to the pattern of the matrix \( A \). This example is similar to an example in Schneider and Zenios 1990

5. Other scalings

The problem of equivalence scaling is closely connected to different forms of scalings, the most prominent ones asking for a diagonal matrix \( D \) such that \( DAD \) is row-stochastic or such that \( DAD^{-1} \) has equal row and column-sums.

Many modern approaches to matrix equivalence scaling are general enough to cover most of those different scalings (see Section 6).

5.1. Matrix balancing

Given a matrix \( A \), does there exist a matrix \( D \) such that \( DAD^{-1} \) has equal column- and row sums? Clearly, this is a special case of \( D_1 AD_2 \) scaling with a different set of constraints. We have the following characterisation:

**Theorem 5.1.** Let \( A \in \mathbb{R}^{n \times n} \) be a nonnegative matrix. Then the following are equivalent:

1. There exists a diagonal matrix \( D \) such that \( B = DAD^{-1} \) fulfills \( \sum_{i=1}^{n} B_{ij} = \sum_{i=1}^{n} B_{ji} \).
2. \( A \) is completely reducible or equivalently, a direct sum of irreducible matrices (Hartfiel 1971).
3. There exists \( B \) with the same pattern as \( A \) and \( \sum_{i=1}^{n} B_{ij} = \sum_{i=1}^{n} B_{ji} \) (Letac 1974).

The scaling of \( A \) is unique and \( D \) is unique up to scalars for each irreducible block of \( A \).

The problem was first considered in Osborne 1960 in the context of preconditioning matrices (see Section 8) by proposing an algorithm and proving its convergence (and uniqueness). Grad 1971, building on Osborne’s results, considers the matrix balancing method and provides an algorithm and convergence proof for completely reducible matrices. Unaware of the effort of Osborne and Grad, but considering “the analogue of [Sinkhorn’s] result in terms of irreducible matrices” Hartfiel 1971 proves essentially the same result. His approach is based on a progress measure which is basically
the maximum difference of the row- and column sums. Letac 1974 provided an interpretation in terms of patterns. The same was later proved in Schneider and Saunders 1980; Golitschek, Rothblum, and Schneider 1983; Eaves et al. 1985 by yet different means.

Similar to the RAS method, one can propose a simple iterative approximation algorithm:

**Algorithm 5.2** (Schneider and Zenios 1990). Let \( A \in \mathbb{R}^{n \times n} \) be nonnegative. Let \( A^0 := A \). For \( k = 0, 1, \ldots \) we define the steps

1. For \( i = 1, \ldots, n \), let \( u_i = \sum_{j=1}^{n} A_{ij}^k \) be the row sum and similarly \( v_i \) be the column sum. Then define \( p \) as the minimum index such that \( |u_p - v_p| \) is maximal among \( |u_i - v_i| \).

2. Define \( \alpha_k \) such that \( \alpha_k u_p = 1 / \alpha_k v_p \).

3. Let \( D = \text{diag}(1, \ldots, 1, \alpha_k, 1, \ldots, 1) \) with \( \alpha_k \) at the \( p \)-th position. Then define \( A^{k+1} = DA^kD^{-1} \) and iterate.

According to Schneider and Zenios 1990, this algorithm is also similar to the proposed scheme in Osborne 1960. At any step, the \( p \)-th row is already correctly scaled, while all other rows change their scaling a bit. Note that unlike in the RAS method, the selection of the row and column to be scaled are done using norm differences. Given the results of Brown, Chase, and Pittenger 1993 that the RAS converges regardless of the order of column and row sum normalisations, a similar condition might also accelerate RAS convergence.

We have the following observation:

**Proposition 5.3** (e.g. Schneider and Zenios 1990). The algorithm converges to a balanced matrix \( B \). This matrix is also the unique minimiser of the function

\[
\sum_{i,j=1}^{n} \left( B_{ij} \ln \left( \frac{B_{ij}}{A_{ij}} \right) - B_{ij} \right)
\]  

subject to the balancing conditions.

**Sketch of proof.** The fact that the balanced matrix minimises the entropy functional can be seen by direct calculation (the minimiser must be a scaling of the original matrix and the balancing conditions ensure that the scaling is of the form \( DAD^{-1} \)).

A proof is similar to observation 3.17: Each step of the algorithm is an I-projection onto the set of matrices with only one row/column balancing constraint. Since the conditions are linear, the repeated projection will converge.

It remains to see that the order of the projections does not matter as long as all directions are chosen arbitrarily often. 

28
As with equivalence scaling, a graph version of this problem exists, this time using transshipment graphs. A nice description can be found in Schneider and Zenios 1990 (see also Figure 3): Given a nonnegative matrix $A \in \mathbb{R}^{n \times n}$, let $V = \{1, \ldots, n\}$ and define the set of edges of the transshipment graph $(V, E)$ by $E = \{(i, j) | A_{ij} > 0, i \neq j\}$. We can then add weights $A_{ij}$ to any edge $(i, j)$. A matrix is then balanced, if and only if the incoming flow at each vertex equals the outgoing flow.

![Figure 3: An easy example of the transshipment graph corresponding to the pattern of the matrix $A$ similar to the example in Schneider and Zenios 1990.](image)

5.2. DAD scaling

Another closely related problem is the question, whether given a nonnegative matrix $A$, there exists a single diagonal matrix $D$ such that $DAD$ has prespecified row- or column sums. A short but quite good overview is given in Johnson and Reams 2009.

Symmetric nonnegative matrices Let us first focus on the case where $A$ is symmetric. It seems natural that this follows directly from Sinkhorn’s theorem: If $D_1AD_2$ has equal row-sums and $A$ is symmetric, so does $D_2AD_1$. By uniqueness of $D_i$ up to scaling, this implies that one can choose $D_1 = D_2$. This was noted for example in Sinkhorn 1964.

The first discussion of the case of symmetric $A$ can be traced back to the announcements Marcus and Newman 1961; Maxfield and Minc 1962\(^{12}\). A first proof for the case of positive matrices and doubly stochastic scaling was given in Sinkhorn 1964. Shortly later, Brualdi, Parter, and Schneider 1966 consider the case of doubly stochastic scaling for nonnegative matrices with positive main diagonal, while Csima and Datta 1972 shows that a doubly stochastic scaling exists if and only if there exists a symmetric doubly stochastic matrix with the same zero pattern if and only if the matrix has total support. This was extended in Brualdi 1974 to cover the case of arbitrary row sums giving the following theorem:

**Theorem 5.4 (Brualdi 1974).** Let $A \in \mathbb{R}^{n \times n}$ be a symmetric nonnegative matrix. Then the following are equivalent:

\(^{12}\)This is covered in many papers, for instance Marshall and Olkin 1968.
1. There exists a diagonal matrix $D$ with positive entries such that $DAD$ has row sums given by $r \in \mathbb{R}^n_+$.

2. There exists a symmetric nonnegative matrix $B$ with the same pattern as $A$ and row sums $r$.

3. For all partitions $\{I, J, K\}$ of $\{1, \ldots, n\}$ such that $A(J \cup K, K) = 0$, $\sum_{i \in I} r_i \geq \sum_{i \in K} r_i$ with equality if and only if $A(I, I \cup J) = 0$.

Furthermore, the scaling is unique.

The equivalence of 2. and 3. is given in Brualdi 1968. 1. follows from 2. using Sinkhorn’s theorem and the reverse direction is proved via contradiction. Using the uniqueness in Sinkhorn’s theorem then provides uniqueness for the scaling.

Note that the following observation gives a very simple proof of Theorem 3.1:

**Observation 5.5.** Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $r \in \mathbb{R}^m_+, c \in \mathbb{R}^n_+$ be two prescribed vectors. Then $A$ has an equivalence scaling if and only if the following symmetric matrix

$$A' = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$$

has a row-sum symmetric scaling to $(r')^T = (r^T, c^T)$.

**Proof.** First assume that there exist $D_1, D_2$ positive diagonal such that $D_1 AD_2$ fulfills $D_1 AD_2 e = r$, $D_2 A^T D_1 e = c$.

Then setting $D' := \text{diag}(D_1, D_2)$ we have

$$D'A'D' = \begin{pmatrix} 0 & D_1 AD_2 \\ D_2 A^T D_1 & 0 \end{pmatrix}$$

and clearly $D'A'D'e = (r^T, c^T)^T$.

Conversely, if $A'$ has a row-sum symmetric scaling $D'$, by an analogous argument $A$ will have an equivalence scaling with row sums $r$ and column sums $c$.

This was already known in the 70s, maybe even earlier; explicit formulations include Rothblum, Schneider, and Schneider 1994; Knight 2008; Knight and Ruiz 2012. Note that the observation can easily be extended to not just row- and column sums, but all $p$-norms for $0 < p \leq \infty$ as considered in Section 6.7. It can also be extended to approximate scalings with the same proof. This implies:

**Observation 5.6.** Results from symmetric scaling for symmetric nonnegative matrices $A$ can always be translated to cover equivalence scaling for arbitrary nonnegative matrices.

The other direction is not true, since clearly not all symmetric matrices are of the special form (24). However, it can still be beneficial to study equivalence scaling on its own, as many algorithms (e.g. the RAS) do not preserve symmetry.
Arbitrary symmetric matrices  Theorem 5.4 can be generalised to cover matrices that are not necessarily nonnegative:

**Theorem 5.7.** Let $A \in \mathbb{R}^{n \times n}$ be symmetric and $\lambda \in \mathbb{R}^n_+$ prescribed column sums. Then:

1. If $A$ is positive semidefinite, then $A$ is scalable if and only if $A$ is strictly copositive (Kalantari 1990; Kalantari 1996).

2. Any principal submatrix of $A$ (including $A$) is scalable if and only if $A$ is strictly copositive (Johnson and Reams 2009).

In general, at least one of the following two propositions is true (Kalantari 1996):

1. The following set is not empty:

$$\{ x \in \mathbb{R}^n | x^T Ax = 0, x \geq 0, x \neq 0 \} \quad (25)$$

2. For all $\lambda \in \mathbb{R}^n_+$ with $\lambda > 0$ there exists a positive diagonal matrix $D$ such that $DADe = \lambda$. In other words, for any set of prescribed row sums, there exists a scaling.

More general conditions for scalability of arbitrary symmetric $A$ can be found in Johnson and Reams 2009. We make a number of remarks concerning the results:

1. Another necessary condition for scalability (the matrix must be diluted) is provided in Livne and Golub 2004.

2. The question of equivalent conditions for the scalability of matrices remains open. However, these conditions might not have a very useful description, since scalability of arbitrary symmetric matrices is NP-hard (Khachiyan 1996\textsuperscript{13}).

3. The second result implies in particular that if a matrix is strictly copositive, it is scalable, which was first proved in Marshall and Olkin 1968. Note that positive definite matrices are in particular strictly copositive, which means that this result encompasses the claimed proofs of scalability of completely positive matrices in Maxfield and Minc 1962. An elementary proof for matrices with strictly positive entries has recently appeared in Johnson and Reams 2009 based on an iterative procedure.

4. For doubly stochastic scaling, the alternative conditions of Kalantari 1996 can also be derived using linear programming duality and/or the hyperplane separation theorem using extremely general methods of duality in self-concordant cones (Kalantari 1998; Kalantari 1999; Kalantari 2005).

\textsuperscript{13}This was conjectured also in Johnson and Reams 2009, who noted that deciding whether a matrix is (strictly) copositive is NP-complete according to Murty and Kabadi 1987. The authors seemed to have been unaware of the paper by Khachiyan. The alternative in Theorem 5.7 is also not very useful computationally, because deciding the emptiness of the set (25) is also NP-hard (Kalantari 1990, according to Kalantari 1996).
5. Scaling of the special class of Euclidean predistance matrices has been considered in Johnson, Masson, and Trosset 2005. It turns out that all such matrices are scalable.

6. Note that the equivalence conditions for positive semidefinite matrices can be strengthened. If a matrix is scalable and positive semidefinite,

\[ \mu := \min \{ x^T Ax \mid x \geq 0, \|x\|_2 = 1 \} \]

can be bounded in terms of the matrix dimension (cf. Khachiyan and Kalantari 1992, where it is also noted that the scaling problem is related to linear programming).

Uniqueness of matrix scaling has also been studied:

**Proposition 5.8.** Let \( A \in \mathbb{R}^{n \times n} \) be symmetric and \( \lambda \in \mathbb{R}^n_+ \) prescribed row sums. Then

1. If \( A \) has two or more distinct scalings, then there exists a matrix \( D \) such that \( DAD \) has eigenvalues \( +1 \) and \( -1 \) (Johnson and Reams 2009).

2. For scalable positive definite matrices \( A \) there exist \( 2^n \) diagonal matrices \( D \) such that \( DADe = \lambda \), one for each sign pattern of \( D \) (O’Leary 2003). In particular, scaling by positive diagonal matrices is unique.

3. If \( A \) is positive semidefinite, then if \( A \) is scalable to row sums \( r \), the positive diagonal matrix is unique (Marshall and Olkin 1968).

For the scaling of positive semidefinite matrices, upper and lower bounds on \( \|D\| \) were derived in Khachiyan and Kalantari 1992; O’Leary 2003.

Johnson and Reams 2009 also note that for nonnegative matrices uniqueness holds in particular if \( A \) is primitive (including the case of positive matrices already covered in Sinkhorn 1964) or if \( A \) is irreducible and there does not exist a permutation \( P \) such that

\[ PAP^T = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}. \]

It is also very simple to give an algorithm of RAS type for this problem, using the observation that a \( DAD \) scaling to row sums \( \lambda \) exists if and only if \( ADe = r/(De) \). This implies that any scaling is a fixed point of the map \( T_{\text{sym}} : \mathbb{R}^n \to \mathbb{R}^n \) with \( T_{\text{sym}}(x) = r/(Ax) \).

**Algorithm 5.9** (Knight 2008). Let \( A \in \mathbb{R}^{n \times n} \) be nonnegative and symmetric. For the algorithm, set \( x_0 = e \) and iterate

\[ x_{n+1} = T_{\text{sym}}(x_n) \quad (26) \]
Nonsymmetric matrices If we do not restrict to symmetric matrices we can only hope to scale $A$ to a matrix with given row-sums. The only notable result seems to be:

Proposition 5.10. (Sinkhorn 1966) Let $A \in \mathbb{R}^{n \times n}$ be a positive matrix. Then there exists $D$ such that $DAD$ is stochastic.

The theorem can be extended to cover arbitrary row sums. The first proof occurred in Sinkhorn 1966. Likewise, the proof in Johnson and Reams 2009 does not need symmetry of $A$.

5.3. Matrix Apportionment

Another scaling problem which is interesting particularly for its applications, is asking for an equivalence scaling, but with the added constraint that the resulting matrix have integer entries. This is important for instance when attributing votes to seats in a parliament and has been applied as early as 1997 (Balinski and González 1997, see also Pukelsheim and Schuhmacher 2004 for one of many explicit accounts for actual changes).

This problem, which is often called matrix apportionment has first been studied in Balinski and Demange 1989a; Balinski and Demange 1989b. Algorithms akin to the RAS method exist and others based on network flows can be obtained from Rote and Zachariasen 2007; an overview and many references can be found in Pukelsheim and Simeone 2009.

5.4. More general matrix scalings

This review has so far largely been concerned with nonnegative matrix scaling, with the exception of symmetric $DAD$ scaling. This is understandable, as most of the applications concern nonnegative matrices. However, in view of completeness, let us mention a few of the (mostly quite recent) other cases of matrix scaling.

Arbitrary equivalence scaling While arbitrary $D_1AD_2$ scaling is interesting for real symmetric matrices, scalings of general real matrices have never sparked a similar amount of interest. It is merely known that the question whether or not a matrix is scalable is NP-hard (Khachiyan 1996) - a question that has also been considered for matrices over the algebraic numbers in Kalantari and Emamy-K 1997. Since the problem of nonnegative matrix scaling turns out to be equivalent to the existence of matrices with given pattern, it seems natural to ask whether the $(+, -, 0)$-pattern of matrices with prescribed row- and column sums play a similar role. For positive diagonal scaling the sign pattern of the matrix cannot change and it is a necessary condition for scalability, which is not sufficient as shown in Johnson, Lewis, and Yau 2001. Nevertheless, the authors achieve a characterisation of general matrix patterns (generalising Brualdi 1968, see also Johnson and Stanford 2000; Eischen et al. 2002).
Complex matrices  Let us first start with the definition

Definition 5.11. Let $A \in \mathbb{C}^{n \times m}$ be a complex matrix, then $A$ is doubly quasistochastic if all sums and columns sum to one.

Note that in case all entries are nonnegative the matrix is doubly stochastic. For the rest of this section, let us restrict to square matrices. Quasistochasticity is interesting, because if $A$ is quasistochastic, then $F_n^* AF_n e_1 = e_1$, where $e_1 = (1, 0, \ldots, 0)^T$ and $F_n$ is the $n \times n$ discrete Fourier transformation. This is true since $F_n e_1 = e$ and $e$ is an eigenvector of $A$ by quasistochasticity. A doubly quasistochastic matrix $A$ therefore satisfies that $F_n^* AF_n$ has $e_1$ as its first row and column. Repeating diagonal scalings and Fourier transform can then lead to new matrix decompositions.

The natural generalisation of $DAD$ scaling would be $D^* AD$-scalings for positive semidefinite matrices. These were first studied in Pereira 2003 and later in Pereira and Boneng 2014. Observing that the proof of Marshall and Olkin 1968 extends to complex entries, the authors obtain already part of the following partial results:

Theorem 5.12 (Pereira and Boneng 2014). Let $A \in \mathbb{C}^{n \times n}$ be positive definite. Then there exist diagonal matrices $D_1, D_2$ such that $D_1 AD_2$ is doubly quasistochastic. Neither $D_1, D_2$ nor the scaled matrices are necessarily unique. However, there exists at most one scaling with positive matrices $D_1, D_2$.

The authors suggested that such scalings can be applied to generate highly entangled symmetric states. They furthermore conjectured that the number of such scalings would be upper-bounded, but this was disproved recently in Hutchinson 2016 by giving counterexamples for $n \geq 4$, which have infinitely many scalings. For $n = 3$, there exist at most four scalings. An RAS type algorithm can be obtained from the fact that an equivalent version of Observation 3.14 also holds in the complex case.

Unitary matrices  For the subclass of unitaries, we proved the following theorem:

Theorem 5.13 (Idel and Wolf 2015). For every unitary matrix $U \in U(n)$ there exist diagonal unitary matrices $D_1, D_2$ such that $D_1 UD_2$ is doubly quasistochastic. Neither $D_1, D_2$ nor $D_1 UD_2$ are generally unique, in fact in some cases there may even be a continuous group of scalings.

An algorithm how to obtain $D_1, D_2$ similar to the RAS method is given and studied in De Vos and De Baerdemacker 2014a, however its convergence is unknown.

The theorem was conjectured in De Vos and De Baerdemacker 2014a and used later (De Vos and De Baerdemacker 2014b; Idel and Wolf 2015) to prove that any unitary matrix can be considered as a product of diagonal unitary matrices and Fourier transforms on principal submatrices. Recently, it has also been applied to prove an analogue of the famous Birkhoff theorem for doubly-stochastic matrices (De Vos and De Baerdemacker 2016).
The proof of Theorem (5.13) boils down to noticing that a scaling exists if and only if there exists a vector $x$ with $Ux = y$ and $|x_i| = |y_i| = 1$ for all $i = 1, \ldots, n$. This is a problem of symplectic topology in disguise and can be solved using a theorem in Biran, Entov, and Polterovich 2004. When we published the theorem in Idel and Wolf 2015 we were unaware of the fact that this proof had in principle already been found, since the equation $Ux = y$ with $|x_i| = |y_i| = 1$, which defines so called biunimodular vectors (see for instance Führ and Rzeszotnik 2015), also pops up in several other places. In this context, essentially the same proof was described in Lisi 2011. A first formal publication containing this proof was probably Korzekwa, Jennings, and Rudolph 2014 applying it to error-disturbance relations in quantum mechanics.

6. Generalised approaches

All of the approaches above can be generalised to some extend. Many can then incorporate also different scalings. With an eye towards matrix equivalence, we will attempt to see the different ways of generalisations and what can be gained. A quick summary can be found in Table 6.2.

6.1. Direct multidimensional scaling

Especially in transportation planning, equivalence scaling of arrays with three indices has been important from the beginning. Except for nonlinear Perron-Frobenius theory, the approaches can be readily generalised to this case. As already pointed out, Brown 1959 was the first to consider multidimensional scaling. According to Evans and Kirby 1974 (see also Evans 1970), Furness pointed out iterative scaling as a possible solution to certain transportation planning problems in the unpublished paper Furness 1962. Evans and Kirby themselves proved convergence in a limited scenario by extending the convex programming approach of equation (16) and proofs have been provided or pointed out in several other papers such as Fienberg 1970; Krupp 1979. The case of approximate multidimensional scaling is discussed in Brown, Chase, and Pittenger 1993.

For multidimensional exact or approximate scaling, the convergence results of Pretzel 1980 reflected in Theorem 4.4 still hold. In addition, the order in which we normalise any of the indices of the multidimensional array is irrelevant:

**Theorem 6.1** (Brown, Chase, and Pittenger 1993 and comment in Brown 1959). Let $A$ be an array with $m$ indices (or dimensions) and let $i_k$ be the dimension of the array that is scaled in the $k$-th step. If each element of $\{1, \ldots, m\}$ appears in the sequence $\{i_1, i_2, \ldots\}$ infinitely often, then the scaling converges to the limit of the cyclic RAS method, the I-projection of $A$.  

35
| Type                  | Base case                                      | $D_1AD_2$ | $DAD$ | $DAD^{-1}$ | multi-dim. | continuous | additional generalisation |
|----------------------|-----------------------------------------------|-----------|-------|------------|------------|------------|--------------------------|
| Algorithmic approaches | RAS-type algorithms with row or column norm constraints (also max-sum) | ✓         | ✓     |            |            |            |                          |
| Axiomatic approach   | $D_1AD_2$ scaling with inequality constraints | ✓         | -     |            |            |            |                          |
| Convex optimisation | $D_1AD_2, DAD$ scaling                        | ✓         | ✓     |            |            |            | also copositive matrices, complex scaling |
| Entropies            | minimising relative entropy minimisation + (non)linear constraints | ✓         | (✓)   | (✓)        | ✓          |            | special case of Bregman divergences; cross entropies; justifications |
| Letac’s approach     | no scaling: existence of some function        | ✓         | ✓     | ✓          |            |            | completely different applications |
| Log linear models    | scaling of prob. distributions $w_j = x_i \prod_j d_{ij}^{C_{ij}}$ given $C, x$ with constraints $Cw = b$ | ✓         | ✓     | ✓          | ✓          |            | Different scalings defined via $C$ |
| N-L Perron-Frobenius Theory | fixed points of homogeneous maps on cones | ✓         | -     |            | ✓          |            | Maps in different vector spaces (such as positive maps); infinite matrices |
| Truncated matrix scaling | $X = \Lambda DAD^{-1}$ balancing with $L \leq X \leq U$ entrywise | ✓         | ✓     | ✓          |            |            |                          |

Table 1: This table gives an overview about possible approaches to matrix scaling, their application to various different scalings discussed in Section 6 and additional possible applications.
6.2. Log-linear models and matrices as vectors

Most of the ideas above use matrices as matrices, as sets of numbers with two indices. One can likewise consider just vectors of numbers and define columns and rows by defining partitions of the vectors. This approach has the advantage that the generalisation to multidimensional matrices is immediate. It was probably pioneered by Darroch and Ratcliff 1972, although Lamond and Stewart 1981 credit Murchland, who circulated his results later (Murchland 1977; Murchland 1978\(^{14}\)). The approach was then taken on in Bapat and Raghavan 1989 (see also Bapat and Raghavan 1997, Chapter 6 for an overview and a more lucid presentation of their ideas). While Darroch and Ratcliff 1972 used an entropic approach, Bapat and Raghavan 1989 is based on a combination of optimisation and topological approaches as discussed in Section 3.6. The same theorem is also proved in Franklin and Lorenz 1989 in a very elementary fashion and in Rothblum 1989 using optimisation techniques.

The original goal of Darroch and Ratcliff 1972 was not to study matrix scaling but rather obtaining probability distributions using so called log-linear models. Given a positive (sub)probability distribution \(\pi\) over some finite index set \(I\), a log-linear model is a probability distribution \(p\) such that

\[
p_i = \pi_i D \prod_{s=1}^d D_s^{C_{si}}
\]

which satisfies some constraints \(\sum_{i \in I} C_{si} p_i = k_s\). Here, \(D\) and \(D_s\) have to be determined while \(C\) is given from the problem. The name derives from the fact that the solution is an exponential family of probability distributions.

Depending on the choice of \(C\), one can write matrix balancing, equivalence scaling or \(DAD\) scaling as finding a log-linear model.

To achieve equivalence scaling with row-sums \(r\) and column sums \(s\), consider for simplicity the case of a \(2 \times 3\) matrix. Then \(C\) and \(b\) are given by

\[
C = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}, \quad b = \begin{pmatrix}
 r_1 \\
r_2 \\
s_1 \\
s_2 \\
s_3
\end{pmatrix}
\]

and we define \(y_1 = A_{11}, y_2 = A_{12}, \ldots, y_5 = A_{22}, y_6 = A_{23}\) (example from Bapat and Raghavan 1989; Rothblum 1989).

To achieve matrix balancing with row-sums equaling column sums, consider for simplicity the case \(3 \times 3\), then \(C\) is given by

\[
C = \begin{pmatrix}
0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0
\end{pmatrix}
\]

\(^{14}\)The papers were not available to me
and \( b = 0 \) and we order \( x, y \) again as before (example from Rothblum 1989).

We have the following theorem:

**Theorem 6.2** (Bapat and Raghavan 1989). Let \( C \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m_+ \). Let \( K = \{ v | Cv = b, v \geq 0 \} \) be bounded. Let \( x \in \mathbb{R}^n_+ \). Then there exists a \( w \in K \) such that for some \( D \in \mathbb{R}^n_+ \) we have

\[
w_j = x_j \prod_{i=1}^m D_{ij}, \quad j = 1, \ldots, n
\]

if and only if there exists a vector \( y \in \mathbb{R}^n_+ \) with \( y \in K \) and the same zero pattern as \( x \).

Note that this is a major generalisation of scaling as the matrix \( C \) can contain any real numbers.

The limiting factor of the theorem is the boundedness of \( K \). While the constraints in the case of matrix equivalence are bounded, the constraint set defined by (28) is not necessarily bounded. Rothblum 1989 applies a completely different proof which only works for positive matrices. However we can still apply Theorem 6.2: \( K \) is unbounded, because the matrix entries can become unbounded since we only want equal row and column sums but do not specify them further. We fix that by using

\[
\tilde{C} = \begin{pmatrix}
0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

and \( b_4 = 1 \). The last row just implies that the sum of all matrix entries should be one which makes \( K \) a bounded set. A simple calculation then shows that this is equivalent to searching for a diagonal matrix \( D \) and a scalar \( d \) such that \( dDAD^{-1} \) has equal row- and column sums and the sum of all matrix entries is one. Clearly, this is equivalent to matrix balancing and we can apply Theorem 6.2.

The connection to entropy minimisation is simple:

**Lemma 6.3** (Darroch and Ratcliff 1972, Lemma 2). Given a positive (sub)probability distribution \( \pi \), if a positive probability distribution \( p \) satisfying (27) and the linear constraints exists, then it minimises relative entropy \( \sum_i p_i \log(p_i/\pi_i) \) subject to the linear constraints.

**Proof.** The proof in Darroch and Ratcliff 1972 is a straightforward calculation and follows directly from Kullback and Khairat 1966. If \( q \) is a probability distribution satisfying the linear constraints, then

\[
D(p||\pi) = \sum_{i \in I} p_i (\log \xi + \sum_{s=1}^d C_{si} \log \xi_s) \\
= \log \xi \left( \sum_{i \in I} p_i \right) + \sum_{s=1}^d \log \xi_s \left( \sum_{i \in I} C_{si} p_i \right)
\]
which implies the lemma by the nonnegativity of relative entropy.

6.3. Continuous Approaches

Nonnegative matrices were always tied to joint probability distributions. Obviously, there is no reason to only study discrete probability distributions. The first such generalisation was obtained in Hobby and Pyke 1965. Also the basic theorems of Kullback 1968 and Csiszár 1975 are more general than counting measures (although both have problems with parts of their arguments, see Borwein, Lewis, and Nussbaum 1994).

As pointed out in Borwein, Lewis, and Nussbaum 1994, there are essentially two approaches to continuous versions, the entropy maximisation approach studied by Kullback and later Csiszár, and the approach via fixed point theorems or contractive ratios (one can see this as a precursor to nonlinear Perron-Frobenius theory) studied in, for instance, Fortet 1940; Nussbaum 1987; Nussbaum 1993. The natural continuous extension of the DAD theorem for symmetric matrices was studied in Nowosad 1966; Karlin and Nirenberg 1967 (via fixed points or iterative contractions). The most general results in Borwein, Lewis, and Nussbaum 1994 combine these two approaches. To give a flavour of their results, we cite

**Theorem 6.4** (Borwein, Lewis, and Nussbaum 1994 Theorem 3.1). Given a finite measure spaces \( \mu(s,t) = k(s,t) \, ds \, dt \) and marginal distributions \( \alpha(s), \beta(t) \in L^1(dt/ds) \), consider the following minimisation problem:

\[
\begin{align*}
\min & \int_\mathcal{S} \int_\mathcal{T} [u(x,y) \log(u(x,y)) - u(x,y)] k(s,t) \, ds \, dt \\
\text{s.t.} & \quad \int_\mathcal{T} u(s,t) k(s,t) \, dt = \alpha(s) \quad \text{a.e.} \\
& \quad \int_\mathcal{S} u(s,t) k(s,t) \, ds = \beta(t) \quad \text{a.e.}
\end{align*}
\]

where \( u \in L^1(dt,ds) \). Furthermore, we require \( \int_\mathcal{S} \alpha(s) \, ds = \int_\mathcal{T} \beta(t) \, dt \). Then the minimisation problem has a unique optimal solution. If there exists a \( u_0 \) which fulfils the constraints and there exist \( x_0 \in L^\infty \) and \( y_0 \in L^\infty \) such that \( \log u_0(s,t) = x(s) + y(t) \) almost everywhere, then \( u_0 \) is the unique solution. Conversely, if there exists a feasible solution \( u \) with \( u \geq 0 \) almost everywhere, then the unique optimal solution satisfies \( u_0 > 0 \) almost everywhere and there exist sequences \( x_n \in L^\infty \) and \( y_n \in L^\infty \) such that

\[
\lim_{n \to \infty} (x_n(s) + y_n(t)) = \log u_0(s,t) \quad \text{a.e.}
\]

39
This in fact also covers the approximate scaling case. Note that the results also extend to more than two marginals.

6.4. Infinite matrices

Instead of continuous functions, we can also consider infinite matrices. Results usually posit that row and column sums should be finite in some norm.

The first such result was obtained in Netanyahu and Reichaw 1969, which proves Theorem 3.1 in the case where the column and row sums are uniformly bounded in the \( l^1 \)-norm and the matrix entries are uniformly bounded. The proof is reminiscent of Brualdi, Parter, and Schneider 1966 and Nonlinear Perron-Frobenius theory, using a fixed point argument involving Schauder’s fixed point theorem.

Another approach was presented in Berger and Kelley 1979, where matrices that are infinite in one direction are studied (rows or columns are finite in a \( l^p \)-norm). Once again, the matrix entries must be bounded uniformly (in this case, in a \( l^p \)-norm) and convergence of the iterative algorithm to a unique solution is proved in certain topologies.

6.5. Generalised entropy approaches

As we saw in Section 3.4, we can write matrix scaling as the problem

\[
\min_P D(P\|Q) \quad \text{s.t. } P \in \Pi
\]

where \( \Pi \) is an intersection of linear constraints. This approach can be generalised in two ways: First, one could consider other functions than relative entropy but related to it or second, one can consider more general sets \( \Pi \).

Relative entropy is a special case of Bregman divergences. These were originally introduced in Bregman 1967 and later named in Censor and Lent 1981. The idea is to study distance measures derived from functions \( \phi : S \in \mathbb{R}^n \rightarrow \mathbb{R} \), which are defined on a closed convex set \( S \), continuously differentiable and strictly convex. Then

\[
\Delta_\phi(x, y) = \phi(x) - \phi(y) - \langle \nabla \phi(y), x - y \rangle
\]

behaves similarly to a metric, although it is not necessarily symmetric and obeys no triangle inequality. If one takes \( \phi(x) = \sum_i (x_i \log(x_i) - x_i) \) (negative entropy modulo the linear term), then \( \Delta_\phi(x, y) = \sum_i (x_i \ln(x_i/y_i) - x_i + y_i) \). This example was already studied in Bregman 1967 giving in addition an iterative algorithm to find the projections onto the minimum Bregman distance given linear constraints, which is a variant of the RAS method (see also Lamond and Stewart 1981).

Another way to generalise \( D \) is the basic observation underlying McDougall 1999: Relative entropy for matrices is equivalent to the sum of cross-entropies between matrix columns, where a cross-entropy of the column \( j \) of the matrices \( A, B \) is just

40
D_j(A\|B) := \sum_i B_{ij} \log(B_{ij}/A_{ij}). Instead of taking the sum of all cross-entropies, it might be justified to take weighted sums of cross-entropies. This is relevant in economic settings and, aside from McDougall 1999, was studied in e.g. Golan and Judge 1996; Golan, Judge, and Miller 1997.\footnote{References corrected but taken from McDougall 1999 as they were unavailable to me} On the other hand, we can work with relaxed constraints. This was covered in Brown, Chase, and Pittenger 1993: The extension of linear families of probability distributions is still covered by Csizsár 1975, while finding the I-projection for closed, convex but nonlinear constraints requires different means such as Dykstra’s iterative fitting procedure (cf. Dykstra 1985).

6.6. Row and column sum inequalities scaling

Instead of wishing for matrices to have prespecified row and column sums, it might be interesting to consider cases where only lower and upper bounds on the row and column sums and the matrix entries are considered. If we denote the set of all nonnegative matrices with row sums between $r^- \in \mathbb{R}^n_+$ and $r^+ \in \mathbb{R}^n_+$(and column sums between $c^- \in \mathbb{R}^n_+$ and $c^+ \in \mathbb{R}^n_+$ and total sum of its entries $h$ by $R(r^-,r^+,c^-,c^+,h)$, then we can ask the question, whether for a given nonnegative $A \in \mathbb{R}^{n \times n}$, there exists $\delta > 0$ and $D_1,D_2$ diagonal matrices such that

\[
B := \delta D_1 AD_2 \in R(r^-,r^+,c^-,c^+,h)
\]

such that if $(D_1)_{ii} > 1$, then $\sum_j B_{ij} = r^-$ and $(D_1)_{ii} < 1$, then $\sum_j B_{ij} = r^+$ and the same conditions for $D_2$ and $c$. This problem was studied in Balinski and Demange 1989a; Balinski and Demange 1989b, where they call such a matrix a fair share matrix. The main purpose of the approach is described in Section 8.2. Using the arguments from nonlinear Perron-Frobenius theory (Section 3.3), they prove

**Theorem 6.5** (Balinski and Demange 1989a). Let $A$ be a nonnegative matrix. There exists a unique fair share matrix for $A$ if and only if there exists a matrix $B \in R(r^-,r^+,c^-,c^+,h)$ with the same pattern as $A$.

A different generalisation is called truncated matrix scaling. It is studied in Schneider 1989; Schneider 1990 and can also account for equivalence scaling. While the proofs use a combination of optimisation techniques for an optimisation problem defined via entropy functionals, the motivation and interpretation uses graphs and transportation problems for graphs.

The problem considers matrix balancing and not equivalence scaling as its basic problem and then explains the connection. A problem consists of an ordered triple $(A,L,U)$ of nonnegative matrices in $\mathbb{R}^{n \times n}$ satisfying

1. $0 \leq L \leq U \leq \infty$,  

References corrected but taken from McDougall 1999 as they were unavailable to me
2. There is a matrix $X$ whose pattern is a subpattern of $A$ such that $X$ is balanced and $L \leq X \leq U$.

3. $U_{ij} > 0$ whenever $A_{ij} > 0$.

and asks for a diagonal matrix $D$ and a nonnegative matrix $\Lambda$ such that

1. $X = \Lambda D A D^{-1}$ is balanced and $L \leq X \leq U$,

2. $X$ and $\Lambda$ satisfy

\[
\begin{cases}
A_{ij} > 1 \Rightarrow X_{ij} = L_{ij} \\
A_{ij} < 1 \Rightarrow X_{ij} = U_{ij}
\end{cases}
\]

The conditions above are consistency conditions which are trivially necessary for the existence of $D, \Lambda$. One can easily see that for $L = 0$ and $U = \infty$ the problem is equivalent to matrix scaling, because $\Lambda = 1$. Note the similarity of the treatment of inequalities to the ideas of Balinski and Demange.

The connection to equivalence scaling is simple (cf. Schneider 1989): Given a nonnegative matrix $A$ and row and column sums $r, c$, we start with the graph of Figure 2: we join the two vertices $S_1$ and $S_2$ into one vertex (call it $S$), keeping everything else fixed. We label the edges between the nodes with the corresponding matrix entries $A_{ij}$. $A'$ is now the matrix corresponding to the graph.

Now we copy the graph twice and erase the weights of the edges and instead label the first graph by $l_{(i,j)}$ and the second by $u_{(i,j)}$ where

\[
l_{(i,j)} := \begin{cases}
0 & \text{if } A_{ij} > 0 \\
r_i & \text{if } j = 0 \\
c_j & \text{if } i = 0
\end{cases} \quad u_{(i,j)} := \begin{cases}
\infty & \text{if } A_{ij} > 0 \\
r_i & \text{if } j = 0 \\
c_j & \text{if } i = 0
\end{cases}
\]

Now $L'$ ($U'$) is the matrix corresponding to the graph with labels $l_{(i,j)}$ ($u_{(i,j)}$). Finally, $(A', L', U')$ is the triple for truncated matrix scaling.

The main result of the paper then includes:

**Theorem 6.6** (Schneider 1989 Theorem 14 (part of it)). Let $(A, L, U)$ be a triple in $\mathbb{R}^{n \times n}$ fulfilling the consistency conditions 1.-3. above. Then the truncated matrix scaling has a solution satisfying the conditions 1. and 2. above if and only if there exists a balanced matrix $X$ such that

- $L \leq X \leq U$,
- $X_{ij} > 0$ iff $A_{ij} > 0$ always.

Once again, the answer is dominated by the pattern of the matrix and the conditions boil down to the usual conditions for similarity scaling. In a sense, this gives another explanation as to why both problems, equivalence scaling and matrix similarity, need pattern conditions for feasibility: They are both similar graph-related problems.
6.7. Row and column norm scaling

Instead of asking the question whether one can scale a matrix to prescribed row- and column sums, one can ask for a scaling to prescribed row- and column norms.

For the ∞-norm, this is discussed in Rothblum, Schneider, and Schneider 1994. Their proof relies on an algorithm for symmetric DAD scaling using Observation 5.5. In fact, an algorithm for the problem had already been studied for the symmetric case in Bunch 1971.\(^{16}\)

**Theorem 6.7** (Rothblum, Schneider, and Schneider 1994). Let \(A \in \mathbb{R}^{m \times n}\) be a nonnegative matrix and \(r \in \mathbb{R}^m_+, c \in \mathbb{R}^n_+\) be prescribed row and column maxima. Then the following are equivalent:

1. There exist diagonal matrices \(D_1\) and \(D_2\) such that \(D_1AD_2\) has prescribed row and column maxima \(r\) and \(c\).

2. There exists a matrix \(B\) with row and column maxima \(r\) and \(c\) with the same pattern as \(A\).

3. There exists a matrix \(B\) with row and column maxima \(r\) and \(c\) with some subpattern of \(A\).

4. The vectors \(r\) and \(c\) fulfil

\[
\max_{i=1,\ldots,m} r_i = \max_{j=1,\ldots,n} c_j \quad (30)
\]

\[
\max_{i \in I} r_i \leq \max_{j \in J^c} c_j \quad (31)
\]

\[
\max_{j \in J} c_j \leq \max_{i \in I^c} r_i \quad (32)
\]

for every subsets \(I \subset \{1,\ldots,m\}\) and \(J \subset \{1,\ldots,n\}\) such that \(A_{IJ} = 0\).

There are two further technical conditions given in Rothblum, Schneider, and Schneider 1994 as well as an algorithm that converges to the solution.

The usual equivalence scaling now corresponds to 1-norm scaling. For \(p\)-norms of row and columns with \(0 < p < \infty\), it is shown that this problem reduces to 1-norm scaling in Rothblum, Schneider, and Schneider 1994. If \(A^{(p)}\) denotes the entrywise power, we have:

**Theorem 6.8** (Rothblum, Schneider, and Schneider 1994). Let \(A \in \mathbb{R}^{m \times n}\) be a nonnegative matrix and \(r \in \mathbb{R}^m, c \in \mathbb{R}^n\). Then the following are equivalent:

1. There exist matrices \(D_1\) and \(D_2\) such that \(D_1AD_2 = B\) has prescribed row and column \(p\)-norms \(r\) and \(c\).

2. There exist matrices \(D_1\) and \(D_2\) such that \(D_1^{(p)}A^{(p)}D_2^{(p)}\) has prescribed row and column sums \(r^{(p)}\) and \(c^{(p)}\).\(^{16}\)

\(^{16}\)Reference from Knight 2008 among others. I could not obtain the reference.
3. There exists a matrix $B$ with the same pattern as $A$ and row and column sums given by $r^{(p)}$ and $c^{(p)}$.

Hence the answer again reduces to a question of patterns. Likewise, the $\varepsilon$-scalability can immediately be transferred. Much weaker results were obtained in Livne and Golub 2004, where the problem of 2-norm scaling was studied for arbitrary (not necessarily nonnegative) matrices. A (fast) algorithm is also derived in Knight and Ruiz 2012.

6.8. Row and column product scaling

At this point, one might wonder what happens when replacing the row- and column sums by row- and column products. This has been treated in Rothblum and Zenios 1992, however it is not connected to entropy or maximum likelihood estimation, but instead to least square estimations, which is why we will not discuss the techniques here. However, this is interesting in light of the original justification of the RAS method in transportation planning by Deming and Stephan 1940. The results are simple:

**Theorem 6.9** (Rothblum and Zenios 1992). Let $A$ be a nonnegative matrix. Then the following are equivalent:

1. There exist positive diagonal matrices $D_1$ and $D_2$ such that $D_1AD_2$ has row and column products $r_p$ and $c_p$.

2. There exists a matrix $B$ with the same zero pattern as $A$ and row and column products $r_p$ and $c_p$.

The scalded matrix $D_1AD_2$ is unique.

Furthermore, if $A$ has no zero rows or columns, there always exists a matrix $D$ such that $DAD^{-1}$ has equal row and column products.

Note that in the case of matrix balancing to equal row and column products, the result is also the same: This is possible if and only if a balanced matrix with the same pattern exists which is always the case (cf. Rothblum and Zenios 1992, Theorem 5.2).

7. Algorithms and Convergence complexity

After the basic existence problems of matrix scaling were solved in the 60s to 80s, the focus shifted to algorithms and complexity theory in the 90s. The story is equally convoluted, not least because algorithmic complexity is difficult and not always well-defined in itself: One can decide to study worst case or average convergence speed, count algorithm steps or computational operations. Given the RAS method and the fact that it is a coordinate descent method for an intrinsically convex optimisation problem, which is amenable to a host of other techniques, the choice of a relevant class of algorithms is already not unique.
Since this review is geared more towards the mathematical aspects of the problem, our focus will lie on exact complexity results instead of proofs by example. Papers focussed on numerical aspects appeared as early as the late 70s, early 80s with Robillard and Stewart 1974 average convergence considerations, Bachem and Korte 1979 and Parlett and Landis 1982. A small overview about many of the recent developments can be found in Knight and Ruiz 2012.

7.1. Scalability tests

Most algorithms explicitly require that the matrix $A$ be scalable (or positive). This means that we first need to check for scalability.

**Proposition 7.1.** Let $r \in \mathbb{R}^m_+$, $c \in \mathbb{R}^n_+$ be two positive vectors with $\sum_i r_i = \sum_j c_j$. Let $A$ be a nonnegative matrix, then one can check whether $A$ is approximately scalable in polynomial time $O(pq \log(q^2 / p))$ with $q = \min\{m, n\}$ and $p$ the number of nonzero elements in $A$.

If $r \in \mathbb{Q}^m_+$, $c \in \mathbb{Q}^n_+$, then one can check for exact scalability in polynomial time of the same order.

The fact that approximate scalability can be efficiently checked was probably first seen in Linial, Samorodnitsky, and Wigderson 2000. A complete and well-readable proof giving explicit bounds appeared in Balakrishnan, Hwang, and Tomlin 2004, exact scalability can be found in Kalantari et al. 2008.

**Sketch of Proof.** We first follow the proof in Balakrishnan, Hwang, and Tomlin 2004, which uses the transportation graph described in Figure 2.

The matrix is approximately scalable iff the maximum flow of this network is equal to $\sum_i r_i$. The flows along the edges $E$ then define a matrix with the wanted pattern. Such a network flow problem can be solved in time $O(pq \log(q^2 / p))$ with $q = \min\{m, n\}$ and $p$ the number of nonzero elements in $A$ (Ahuja et al. 1994).

In order to check for exact scalability, one has to check whether there exists a solution where each edge has a positive amount of flow (otherwise the entry would have to be reduced to zero). We can check for a solution to the maximum flow problem with minimum flow through each edge bigger than a prespecified value $\epsilon$ with the same costs as solving a maximum flow problem twice. Clearly, this does not help as, we would have to check scalability for any $\epsilon > 0$.

However (following Kalantari et al. 2008) if $r, c$ have only rational entries, we can find a number $h$ such that $hr, hc$ have only integer values. In this case, the flow problem has a solution iff there exists a matrix $B$ with column sum $hc$ and row sum $hr$ where each positive entry fulfils $B \geq 1/|E|$, where $|E|$ denotes the number of edges in $E$.

This implies that it suffices to check for a solution with capacities $hr, hc$ and minimum flow through each edge having prespecified value $1/(2|E|)$. □

For positive semidefinite matrices, scalability can also be checked easily:
Proposition 7.2 (Khachiyan 1996). Let \( A \in \mathbb{R}^{n \times n} \) be positive semidefinite. Then \( A \) is scalable if and only if \( Ax = 0 \) and \( e^T x = 1 \) has no solution \( x \geq 0 \). This can be tested by a linear program.

Proof. The formulation is already nearly in canonical form. We maximize \( e^T x \) subject to the equality constraints \( Ax = 0 \) and \( x \geq 0 \).

For arbitrary matrices scalability is mostly NP-hard (see Section 5.4).

7.2. The RAS algorithm

The RAS algorithm, being the natural algorithm to compute approximate scaling, is also the most studied algorithm. For the case of doubly stochastic matrices, it has long been known (cf. Sinkhorn 1967) that for positive matrices, the RAS converges linearly (sometimes called geometrically) in the \( l_\infty \) norm. Krupp 1979 gave a simple argument that the iteration will get better at any step. This can also be inferred from the fact that the RAS method is iterated I-projection onto a convex set using Csiszár 1975. Later, Franklin and Lorenz 1989 showed that the convergence is also linear in Hilbert’s projective metric, while Soules 1991 showed linear convergence for all exactly scalable matrices basically in arbitrary vector norms, albeit without explicit bounds. Conversely, it was shown that only scalable matrices can have linear convergence meaning that the RAS converges sublinear for matrices with support that is not total (Achilles 1993). We have the following best bounds:

Theorem 7.3 (Knight 2008, Theorem 4.5). Let \( A \in \mathbb{R}^{n \times n} \) be a fully indecomposable matrix and denote by \( D_1, D_2 \) the diagonal matrices such that \( D_1 A D_2 \) is doubly stochastic. Let \( D_1^k \) and \( D_2^k \) be the diagonal matrices after the \( k \)-th step of the Sinkhorn iteration, there exists a \( K \in \mathbb{N} \) such that for all \( k \geq K \), in an appropriate matrix norm

\[
\|D_1^{k+1} \oplus D_2^{k+1} - D_1 \oplus D_2\| \leq \sigma_2 \|D_1^k \oplus D_2^k - D_1 \oplus D_2\| \tag{33}
\]

where \( \sigma_2 \) is the second largest singular value of \( D_1 A D_2 \).

The proof of this theorem crucially relies on the fact that matrices with a doubly stochastic pattern are direct sums of primitive matrices (modulo row and column permutations). Hence it cannot easily be extended to matrices with arbitrary row and column sums if those matrix patterns allow for non-primitive matrices. The approach in Franklin and Lorenz 1989 for positive matrices can also be extended to arbitrary row and column sums.

We observe that the occurrence of the second singular value should not come as a big surprise: Given a stochastic matrix \( A \), \( A^k \) converges to a fixed matrix and the convergence is dominated also by the gap between the largest singular value 1 and the second largest singular value of \( A \).

For practical purposes, one then needs to work out how many operations are needed to obtain a given accuracy of the solution. The first such bounds can be derived
from the bounds in Franklin and Lorenz 1989. The main study of these questions was conducted in the early 90s and 2000s, starting with Kalantari and Khachiyan 1993. Let \( A \in \mathbb{R}^{n \times n \times \ldots \times n} \) with \( d \) copies of \( \mathbb{R}^n \) be a positive multidimensional matrix which can be scaled to doubly stochastic form, then they proved that the RAS takes at most

\[
O \left( \frac{1}{\varepsilon} + \frac{\ln(n)}{\sqrt{d}} \right) \, d^{3/2} \sqrt{n} \ln \left( \frac{V}{\nu} \right)
\]  

steps, where all matrix entries are in the interval \((\nu, V]\) and the maximal error is upper-bounded by \(\varepsilon\). (Kalantari and Khachiyan 1993, Theorem 1). They also derive a bound for a randomised version of the RAS, where at each step, the direction of descent is selected randomly and once in a while, the whole error function is computed randomly. The expected runtime is then slightly lower.

In the case of positive matrix scaling, Kalantari et al. 2008 give better bounds covering also the case of inequality constraints as in Balinski and Demange 1989a. In particular, let \( A \in \mathbb{R}^{n \times m} \) be a positive matrix with \( \nu \leq A_{ij} \leq V \), let \( N = \max\{n, m\} \), let \( \rho = \max\{r_i, c_j\} \) and \( h = \sum_{ij} A_{ij} \). Then the number of iterations needed to scale \( A \) to accuracy \( \varepsilon \) is of order

\[
O \left( \frac{1}{\varepsilon} + \ln(hN) \right) \rho \sqrt{N} \left( \ln(\rho) + \ln \left( \frac{V}{\nu} \right) \right).
\]

The two results (specific bounds and asymptotic linear behaviour for convergence speed) imply that the RAS method has generally good convergence properties if the matrix is positive.

A fully polynomial time algorithm (i.e. without a factor involving the size of the matrix entries) for general marginals was given in Linial, Samorodnitsky, and Wigderson 2000 based on the RAS method with preprocessing. However, the algorithm scales with \( O(n^7 \log(1/\varepsilon)) \) for the general \((r,c)\)-scaling and (using a different algorithm closer to the RAS) with \( O((n/\varepsilon)^2) \) for doubly-stochastic scaling.

In summary, the RAS method, while not fully polynomial by itself, can be tweaked in various ways to allow for fully polynomial algorithms. In addition, it has the advantage of being parallelisable as demonstrated in Zenios and Iu 1990; Zenios 1990. However, the scaling behaviour is not particularly fast in specific examples (see for instance Balakrishnan, Hwang, and Tomlin 2004; Knight and Ruiz 2012), in particular it doesn’t seem to be very good at handling sparse matrices.

### 7.3. Newton methods

One of the first alternative algorithms to the RAS methods was provided in Marshall and Olkin 1968 as a minimisation of \( x^T Ay \) using a modified Newton method as described in Goldstein and Price 1967 (it is not related how the equality constraints are introduced into the problem. This can be done using a \( C^2 \)-penalty function).
Newton methods were also developed to solve the scaling problem for positive semidefinite matrices. They can either be seen as Newton’s method applied to $x^T Ax$ for symmetric $A$ (cf. Khachiyan and Kalantari 1992) or as Newton’s method applied to the Sinkhorn iteration equation $x_{k+1} := e/(Ax_k)$ (cf. Knight and Ruiz 2012). Yet a different method was considered in Fürer 2004.

Kalantari 2005 shows that their algorithm converges in $O\left(\sqrt{n \ln\left(n/(\mu\epsilon)\right)}\right)$ Newton iteration steps, where $\mu := \inf\{x^T Ax | x \geq 0\}$, if the matrix is scalable.

7.4. Convex programming

As noted in section 3.5, the convex programming formulation of the problem makes it amenable to a host of (polynomial time) techniques such as the ellipsoid method or interior point algorithms.

In the case of nonnegative matrices $A \in \mathbb{R}^{n \times n}$ with doubly stochastic marginals, a good bound was found in Kalantari and Khachiyan 1996, with operations of order $O(n^4 \ln(n/\epsilon) \ln(1/\nu))$. The bound uses ellipsoid methods. Later, the bounds were extended to cover generalised marginals in Nemirovski and Rothblum 1999 (also including the generalisation discussed in Rothblum 1989) specifically using ellipsoid methods for the convex optimisation formulation of equation (16). The first instance of an interior point algorithm was probably formulated in Balakrishnan, Hwang, and Tomlin 2004 applied to the entropy formulation. The authors find a strongly polynomial algorithm which scales better than Linial, Samorodnitsky, and Wigderson 2000 with $O(n^6 \log(n/\epsilon))^{17}$. A different ansatz for an algorithm was used in Schneider 1990, where the author uses the duality in convex programming and a coordinate ascent algorithm for the dual problem of his truncated matrix scaling. This algorithm will then be some form of generalisation of the RAS method.

7.5. Other ideas

We give a short primer of other algorithms considered in the literature:

1. The first paper to develop new algorithms with a focus on speed and not only concepts was Parlett and Landis 1982, where a bunch of slightly different and optimised algorithms is derived.

2. An algorithm which is somewhat related to convex algorithms is considered in Kalantari 1996. It is a total gradient based, steepest descent algorithm for the homogeneous log-barrier potential.

It seems that the authors were unaware of Kalantari et al. and Nemirovski and Rothblum 1999.
3. In Rote and Zachariasen 2007, using an algorithm for the matrix apportionment problem involving network flows and using ideas of Karzanov and McCormick 1997, they provide an algorithm where the number of iterations scales with 

\[ O(n^3 \log n (\log(1/\varepsilon) + \log(n) + \log \log(V/v))). \]  

(35)

Once again, \( A_{ij} \in [v, V] \) for all \( i, j \).

4. With ever larger matrices, it is sometimes infeasible to access each element of the matrix on its own, because the matrix is not stored in that form or processed somewhere else. This makes it interesting to consider algorithms that do not need access to all elements, such as the RAS method for nonnegative matrices. Algorithms that are “matrix free” in this sense were developed in Bradley 2010; Bradley and Murray 2011 for doubly stochastic scaling of positive semidefinite matrices.

5. Finally, let us mention that algorithms were also developed for infinity norm scaling (cf. Bunch 1971; Ruiz 2001; Knight, Ruiz, and Uçar 2014) and other norm scaling (cf. Ruiz 2001).

7.6. Comparison of the algorithms

A first comparison of several algorithms was performed in Schneider and Zenios 1990, however, the comparison is not really in terms of speed (for instance, all algorithms were implemented on different programming platforms), but in terms of useability.

While there have been many papers claiming superior convergence speed for their algorithm, the most comprehensive analysis has probably been achieved in Knight and Ruiz 2012, which is limited to doubly-stochastic scalings. In the paper, the authors compare the RAS method, a Gauss-Seidel implementation of the ideas of Livne and Golub 2004, and the fastest algorithm in Parlett and Landis 1982 with their own Newton-method algorithm. The test matrices are mostly large sparse matrices and the new algorithm is usually the fastest and most robust algorithm. The authors also claim that their Newton-based implementation is superior to Khachiyan and Kalantari 1992 and Fürer 2004. They also suggest that the algorithm should outperform the convex optimisation based algorithms, albeit a direct comparison to the most recent algorithm in Balakrishnan, Hwang, and Tomlin 2004 is missing, who only showed that their algorithm clearly outperforms the RAS method. Bradley and Murray 2011 also mention that their purely matrix free algorithm will outperform explicit methods such as those in Knight and Ruiz 2012 if accessing single elements in the matrix is actually slow.

In general, matrix scaling can today be done on a routine basis even for very large matrices.
8. Applications of Sinkhorn’s theorem

The following problem can be encountered in many areas of applied mathematics (see also Schneider and Zenios 1990):

**Problem 1.** Let \( A \in \mathbb{R}^{m \times n} \) be a nonnegative matrix. Find a matrix \( B \) which is close to \( A \) and which fulfills a set of linear inequalities, for instance

\[
\sum_{j=1}^{n} A_{ij} = r_i, \quad \sum_{i=1}^{m} A_{ij} = c_j.
\]

We could also ask for balanced marginals or any other type of marginals. The problem is certainly not well-posed. What does “close” mean? This part of the review will try to give an overview why “close” means equivalence scaling in many applications. We will limit our attention mostly to the mathematical justification of matrix scalings, but I will try to give pointers to other literature.

This implies that we only consider “nearness” leading to equivalence scaling or matrix balancing as the result. In the literature, other nearest matrices have also been considered such as addition of small matrices (e.g. Bachem and Korte 1980). Schneider and Zenios 1990 describe network flow algorithms that allow for a wider variety of applications.

Matrix scaling has many different real world applications, which implies that it also needs different justifications. While statistical justifications exist, many applications argue with the simplicity of the method and the fact that it performs well in practice. These are valid arguments, but they are unsatisfactory from a mathematical point of view. In the two following subsections we collect mathematically rigorous (or partly rigorous) justifications and their history.

8.1. Statistical justifications

As seen, matrix scaling solves entropy minimisation with marginal constraints. This is one of the most powerful entries for justifications of equivalence scaling as the right model, since relative entropy has strong statistical justifications, mostly in the form of maximum entropy or minimum discrimination information - see Jaynes 1957 or later in Kullback and Khairat 1966 for a justification in physics, or Kullback 1959 and Gokhale and Kullback 1978 for a justification in statistics.

Another justification closely connected to entropy minimisation is maximum likelihood models. For instance, if given a set of distributions \( Q \) and an empirical i.i.d. sample \( P \), then the maximum likelihood for \( P \) being a sample of \( Q \) is given by the minimal relative entropy (Csiszár 1989; Darroch and Ratcliff 1972). Max-Likelihood justifications for applications in contingency tables are given in Fienberg 1970; Good 1963.

A different class of justifications for the validity of the matrix scaling approach are arguments showing that matrix scaling conserves certain form of interactions within
the matrix. For instance, matrix scaling conserves cross products (Mosteller 1968) and so-called $k$-cycles (Berger and Kelley 1979, $k$-cycles are certain products of matrix and inverse matrix entries). Both can be desirable for modeling reasons.

Finally, let us mention that the original justification (matrix scaling is a least-square type optimisation) made in Deming and Stephan 1940 turned out to be wrong very quickly and was superseded by real least-square methods in Stephan 1942 and later in Friedlander 1961 or Carey, Hendrickson, and Siddharthan 1981. Those however are not the same as equivalence scaling (see Section 6.8).

8.2. Axiomatic justification

Another justification for matrix scaling, which is particularly useful for application in elections is given in Balinski and Demange 1989b. Instead of considering just any matrix “close” to the original estimate, we want this matrix to fulfil a set of axioms.

Let $A$ be a nonnegative matrix, $r_+, c_+$ ($r_-, c_-$) be upper (lower) bounds to the row and column sums and $h > 0$ be a scalar. As in Section 6.6, we denote the set of all matrices $B$ fulfilling the bounds $q := (r_-, r_+, c_-, c_+, h)$ with $h = \sum_{ij} B_{ij}$ by $R(q)$. For any matrix $A$ and any set of bounds $q$, we search for a method $F(A, q)$ to allocate one out of potentially many matrices $A'$ fulfilling $q$ and the following axioms:

Axiom 1 Exactness: If $r_+ = c_+ = 0$ and $r_- = c_- = \infty$ then $A' = (h/\sum_{ij} A_{ij}) A$

Axiom 2 Relevance: If $q'$ is another set of bounds such that $R(q') \subset R(q)$ and there exists a possible $A' \in R(q')$, then $F(A, q') \subset F(A, q) \cap R(q')$.

Axiom 3 Uniformity: For any matrix $A'$ with bounds $q$, if we construct a new matrix $A''$ by exchanging any submatrix $A'_{I \times J}$ by another submatrix $B_{I \times J}$ which fulfils the same row and column sums minus the part of these bound allocated in $A'_{(I \times J) \setminus c}$, then $A'' \in F(A, q)$.

Axiom 4 Monotonicity: If we have two matrices $A, B$ with $A_{ij} \leq B_{ij}$ for all $(i, j)$, then it also holds that $A'_{ij} \leq B'_{ij}$ for all possible allocations.

Axiom 5 Homogeneity: Suppose $r_- = r_+$ and $c_- = c_+$. Then, if two rows of $A$ are proportional and are constrained to the same row sum, then the corresponding rows in $A'$ are always equal.

Then Balinski and Demange 1989b show that equivalence scaling (the fair share matrix of Section 6.6) is the unique allocation method $F(A, q)$ for all nonnegative matrices $A$ where $R(q)$ contains a matrix with the same pattern as $A$.

8.3. A primer on applications

We will only sketch applications here since a complete list and discussion is probably infeasible.
Transportation planning A natural problem in geography is connected to predicting flows in a traffic network. If one considers for example a network of streets in a city at rush hour and a number of workers that want to get home, it is important to know how the traffic will be routed through the network. This is to a large degree a problem of physical modeling and a number of methods have been developed in the last century (for a recent introduction and overview see Ortúzar and Willumsen 2011\textsuperscript{18}).

For our purposes, the most interesting question results from estimating trip distribution patterns from prior or incomplete data. In a simplified model, one could consider only origin and destination nodes (e.g. home quarters and work areas), given by a nonnegative matrix $A$. While the matrix is known for one year, it might be necessary to predict the changes given that the amount of trips to and from one destination change.

Several papers have treated a justification of the RAS method in this case. For instance, Evans 1970 argues that the method provides a unique outcome and it is easier to handle and to compute than other methods (Detroit method, growth factor method,...). A discussion of trip distribution with respect to Problem 1 can be found in Schneider and Zenios 1990.

Contingency table analysis In many situations ranging from biology to economics, contingency tables need to be estimated from sample data. Contingency tables list the frequency distributions of events in surveys, experiments, etc. They are highly useful to map several variables and study their relations.

As a specific example, suppose a small census in Germany tries to estimate migration between the states. While the number of citizens is recorded, which means that the total net migration is known, it is not known where each individual migrant came from. From a small survey among migrants, how can one estimate the true table with correct marginals in the best possible way? If one does a maximum likelihood estimation, the result is once again matrix scaling (cf. Fienberg 1970; Plane 1982).

Social accounting matrices Social accounting matrices, or SAMs, are an old tool developed in Stone 1962 and later popularised in Pyatt and Thorbecke 1976 to represent the national account of a country. To date, it is an important aspect of national and international accounting (as a random example see Klose, Opitz, and Schwarz 2004 from the German national institute of statistics. An introduction to social accounting can also be found in Pyatt and Round 1985 and, from a short mathematical perspective, in Schneider and Zenios 1990).

The idea is to represent income and outcome of a national economy in a matrix. Often, good growth estimates are known for the row and column sums and certain estimates are known for individual cells. The account estimates are then often not

\textsuperscript{18}The authors also discuss the RAS method in chapter 5. I am however not convinced by their claim that Bregman provided the best analysis of the mathematics of the problem.
balanced, which can be achieved using matrix balancing or matrix scaling. Justifications can be imported from statistics, most notably maximum likelihood.

**Schrödinger bridges** In Schrödinger 1931, the author considered the following setup: Suppose we have a Brownian motion and a model which we are very confident about. In an experiment we observe its density at two times $t_0, t_1$. Now suppose they differ significantly from the model predictions. How can we reconcile these observations by updating our model without discarding it completely?

This problem has been studied in a whole line of papers since then from Fortet 1940 to Georgiou and Pavon 2015. The minimum relative entropy approach can be justified using large deviations (see Ruschendorf 1995).

**Decreasing condition numbers** Given a system of linear equations $Ax = b$ with nonsingular $A$, solving it relies on the Gaussian elimination procedure, which is known to be numerically unstable for matrices with bad condition number $\kappa(A) := \|A\|_\infty \|A^{-1}\|_\infty$. In order to increase the stability, we have to modify $A$, for example by multiplying with diagonal matrices $D_1, D_2$ and considering $D_1 AD_2$. Given that linear systems are ubiquitous in numerical analysis, it is of paramount importance to know how best to precondition a matrix in order to minimise calculation errors (see for instance the survey Benzi 2002). The answer to this question is problem dependent. Particular problems, where equivalence scaling is helpful to go include integral controllability tests based on steady-state information and the selection of sensors and actuators using dynamic information (see Braatz and Morari 1994).

One of the first papers to consider minimisation of $\kappa$ using diagonal scaling was Osborne 1960, who focused on matrix balancing instead of equivalence scaling (see also Livne and Golub 2004 and Chen and Demmel 2000 for sparse matrices). Since the condition number contains the maximum norm, it might be best to require balanced maximum rows and columns instead of balanced row sums as observed in Bauer 1963; Curtis and Reid 1972. This works particularly well for sparse matrices. Equivalence scaling has been studied as early as Householder 2006 and later in Olschowka and Neumaier 1996. If we use other $p$-norms in the definition of $\kappa$, a convex programming solution for minimising $\kappa$ using equivalence scaling is provided in Braatz and Morari 1994.

Note that unlike in all applications studied so far, preconditioning a matrix is useful not only for nonnegative matrices. This is one reason why matrix balancing was studied for copositive and not simply nonnegative matrices. A very different measure of the “goodness” of scaling which might also be of numerical relevance was studied in Rothblum and Schneider 1980, where the authors solved the problem of matrix balancing of a matrix such that the ratio between the biggest and smallest element of the scaled matrix becomes minimal.

---

19Reference from Braatz and Morari 1994
Elections  A very important application of equivalence scaling can be found in Voting: Given election results in a federal election, how can one best distribute the seats among the parties within the states such that each party and each state is represented according to the outcome of the election? Note that here, we need to adjust for natural numbers, which requires rounding (cf. Maier, Zachariassen, and Zachariasen 2010).

Early methods based on a discretised RAS method were developed in Balinski and Demange 1989a. The problem is very intricate in itself, because the justifications rely on what is perceived as “fair” and any method that is fair in some instances is unfair in others (see for instance the discussions in Balinski and González 1997; Pukelsheim and Schuhmacher 2004; for an overview, see Niemeyer and Niemeyer 2008).

Other applications  Various other applications of equivalence scaling and matrix balancing exist such as:

1. A Sudoku Solver based on a stochastic algorithm based on the RAS was developed in Moon, Gunther, and Kupin 2009.

2. An algorithm to rank web-pages was developed in Knight 2008. The RAS allows to derive an algorithm similar in scope to the HITS algorithm (Kleinberg 1999).

3. The RAS method is analysed as a relaxed clustering algorithm in data mining (Wang, Li, and König 2010). However, it turns out that methods based on other Bregman-divergences are more favourable.

4. Given a (discretised) quantum mechanical time evolution, can we construct a local hidden variable model of its evolution corresponding to a deterministic stochastic transition matrix (Aaronson 2005)?

5. Given a Markov chain with a doubly stochastic transition matrix and given an estimate of the transition matrix, the best estimate of the real transition matrix is given by a scaled matrix (Sinkhorn 1964).

6. Regularising optimal transportation by an entropy penalty term such that it can be computed using the RAS, which is already much faster than optimal transport algorithms (Cuturi 2013).

9. Scalings for positive maps

We have already seen that Sinkhorn scaling is interesting for classical Schrödinger bridges as well as for scaling transition maps of Markov processes, etc. From a physics perspective, all these applications are classical physics, transforming classical states (probability distributions) to classical states.

In quantum mechanics, the basic objects are quantum states. For finite dimensional systems (such as spin systems), these quantum states are positive semidefinite matrices
with unit trace. A quantum operation then maps states to states, i.e. it needs to be positive: If $A \geq 0$, then $T(A) \geq 0$. In fact, this is not all that is required for quantum operations, but one actually needs $T$ to be completely positive (for an overview about quantum operations and quantum channels, see Nielsen and Chuang 2000; Wolf 2012).

(Completely) Positive trace-preserving maps are then the natural generalisation of stochastic matrices. This raises the question whether concepts as irreducibility and a Perron-Frobenius theorem exist also for quantum channels and indeed they do. A Perron-Frobenius analogue was probably first described in Schrader 2000, while the analogue for full indecomposability was first used in Gurvits 2004.

Let us define the concepts:

**Definition 9.1.** A positive map $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ with $\mathcal{M}_d = \mathbb{C}^{d \times d}$ is called irreducible (as in Evans and Høegh-Krohn 1978; Farenick 1996) if for any nonzero orthogonal projection $P$ such that

$$\mathcal{E}(P \mathcal{M}_d P) \subseteq P \mathcal{M}_d P$$

we have $P = 1$.

Likewise, it is called fully indecomposable if for any two nonzero orthogonal projections $P, Q$ with the same rank such that

$$\mathcal{E}(P \mathcal{M}_d P) \subseteq Q \mathcal{M}_d Q$$

we have $P = Q = 1$.

Finally, a map is called positivity improving (the analogue to positive matrices) if for all $A \geq 0$, $\mathcal{E}(A) > 0$.

A lot of different characterisations have been found (see Appendix C).

Furthermore, let us define:

**Definition 9.2.** Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a positive map. Then $\mathcal{E}$ is called rank non-decreasing if for all $A \geq 0$

$$\text{rank}(\mathcal{E}(A)) \geq \text{rank}(A).$$

It is called rank increasing, if the $\geq$ sign in equation (38) is replaced by a $>$.

The connections of Definitions 9.1 and 9.2 are explained in Appendix C.

Let us now define what we mean by scaling a positive map:

**Definition 9.3.** Let $\mathcal{E} : \mathcal{M}_n \rightarrow \mathcal{M}_n$ be a positive, linear map. We say that $\mathcal{E}$ is scalable to a doubly stochastic map, if there exist $X, Y \in \mathcal{M}_d$ such that

$$\mathcal{E}'(\cdot) := Y^\dagger \mathcal{E}(X \cdot X^\dagger) Y$$

(39)
is doubly stochastic (i.e. \( \mathcal{E}'(1) = \mathcal{E}''(1) = 1 \)).

We call a positive map \( \varepsilon \)-doubly stochastic, if

\[
DS(\mathcal{E}) := \text{tr}((\mathcal{E}(1) - 1)^2) + \text{tr}((\mathcal{E}^*(1) - 1)^2) \leq \varepsilon^2
\]

We call \( \mathcal{E} \) \( \varepsilon \)-scalable if there exists a scaling as in equation (39) to an \( \varepsilon \)-doubly-stochastic map \( \mathcal{E}' \).

The error function \( DS \), which is similar to an \( L^2 \)-error function for matrices, will serve twofold: first, it defines approximate scalability (which can alternatively be defined by convergence of the RAS) and second, it defines a progress measure for convergence similar to error functions as considered in Balinski and Demange 1989b.

We can now state the full analogue of equivalence scaling to doubly stochastic form:

**Theorem 9.4.** Given a positive map \( \mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d \), it is scalable to a doubly stochastic map iff there exist some matrices \( X \) and \( Y \) such that \( Y\mathcal{E}(X \cdot X^\dagger)Y^\dagger \) is a direct sum of fully indecomposable maps.

The scaling matrices are unique iff \( \mathcal{E} \) is fully indecomposable.

The fact that fully indecomposable matrices are uniquely scalable was first proved in Gurvits 2003. His work built on earlier work in Gurvits and Samorodnitsky 2002; Gurvits 2002 (see also Gurvits 2004), based on a generalisation of the convex approach in equation (16) and the London-Djokovic approach in equation (8).

Recently, the problem was considered with the hope to apply it for unital quantum channels in Idel 2013 (which has never been formally published) and shortly afterwards in Georgiou and Pavon 2015 while trying to define and study “quantum” Schrödinger bridges. Both approaches use nonlinear Perron-Frobenius theory and thereby a generalisation of equation (11) to get a result. The approaches derived from classical results are discussed in Section 9.1.

Even earlier than Gurvits, a very limited version of the theorem was proven in Kent, Linden, and Massar 1999 (with subsequent generalisations) using an approach that does not derive from any of the classical approaches but instead uses the Choi-Jamiolkowski isomorphism. This is described in Section 9.2.

The extension of the theorem to necessary and sufficient conditions has as far as I know not been formally published\(^{20}\).

Furthermore, we can state a analogue of approximate scaling, which to date has only been considered in Gurvits 2004:

**Theorem 9.5.** Let \( \mathcal{E} : \mathcal{M}_n \to \mathcal{M}_n \) be a positive, linear map. Then \( \mathcal{E} \) is approximately scalable (i.e. \( \varepsilon \)-scalable for any \( \varepsilon > 0 \)) if and only if \( \mathcal{E} \) is rank non-increasing.

An overview about different approaches and how they derive from existing approaches can be found in Figure 9.

\(^{20}\)Gurvits actually claims a proof for the fact that a positive map is uniquely scalable iff it is fully indecomposable, but I did not understand how the only if part follows.
Figure 4: Approaches to positive map scalings and their connections. The classical approaches of Figure 3.1 are depicted in grey, while positive map approaches derived from classical approaches are overlayed in black.
9.1. Operator Sinkhorn theorem from classical approaches

We will now study how the theorems above were derived extending classical approaches to positive maps starting with an analogue of the RAS method for positive maps:

Algorithm 9.6. Let \( \mathcal{E} : \mathcal{M}_n \to \mathcal{M}_n \) be a positive, linear map.

1. Start with \( \mathcal{E}_0 := \mathcal{E} \).

2. For each \( i = 0, \ldots, n \) define:

\[
\mathcal{E}_{2i+1}(\cdot) := \mathcal{E}_{2i}(\cdot) \left( \cdot \right)^{1/2} \mathcal{E}_{2i}(\cdot)^{1/2} \\
\mathcal{E}_{2i+2}(\cdot) := \mathcal{E}_{2i+1}(\mathcal{E}_i^*(\cdot))^{1/2} \cdot \mathcal{E}_i^*(\cdot)^{1/2} 
\]

3. Iterate till convergence

By construction, we iterate between trace-preserving (even) and unital (odd) maps.

9.1.1. Potential Theory and Convex programming

As stated, an approach along the lines of the London-Djokovic approach of equation (8) is found in Gurvits 2003; Gurvits 2004 (with methods of Gurvits 2002; Gurvits and Samorodnitsky 2000; Gurvits and Samorodnitsky 2002). Since the complete proofs are lengthy and scattered over several papers, we provide full proofs in Appendix D for the benefit of the reader. In this section, we only sketch the path of the proofs.

Recall that a matrix scaling exists iff the following is positive and the minimum is attained:

\[
c(A) := \inf \left\{ \prod_{i=1}^n \sum_{j=1}^n A_{ij} x_j \left| \prod_{i=1}^n x_i = 1 \right. \right\} 
\]

(43)

Exchanging products with determinants and sums with traces, we obtain the following definition:

Definition 9.7. Let \( \mathcal{E} : \mathcal{M}_n \to \mathcal{M}_n \) be a positive, linear map. Then define the capacity via

\[
\text{Cap}(\mathcal{E}) := \inf \{ \det(\mathcal{E}(X)) | X > 0, \det(X) = 1 \} 
\]

(44)

We will start with covering approximate scaling:

Approximate scalability The capacity is the right functional to study scaling:

Lemma 9.8 (Gurvits 2004). Let \( \mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d \) be a positive map. If \( \text{Cap}(\mathcal{E}) > 0 \) then the RAS method of Algorithm 9.6 converges and \( \mathcal{E} \) is \( \varepsilon \)-scalable for any \( \varepsilon > 0 \).
The proof of this lemma uses the following observation: For any $C_1, C_2 > 0$ we have
\[ \text{Cap}(C_1 \mathcal{E}(C_2^\dagger \cdot C_2) C_1^\dagger) = \det(C_1 C_1^\dagger) \det(C_2 C_2^\dagger) \text{Cap}(\mathcal{E}). \]

Then, a quick calculation shows that Algorithm 9.6 only decreases Cap using this equality. If $\text{Cap}(\mathcal{E}) \neq 0$, one can then show that $\text{DS}(\mathcal{E}_i) \rightarrow 0$ for $i \rightarrow \infty$.

Next, we need to see when the capacity is actually positive. To do this, for every unitary $U$ we need to define the tuple
\[ A_{\mathcal{E}, U} := (\mathcal{E}(u_1 u_1^\dagger), \ldots, \mathcal{E}(u_n u_n^\dagger)), \]
where $u_i$ is the $i$-th column of $U$. This is done to connect the capacity with so called mixed discriminants (see also C), which are needed for the proof. In fact, we have:

**Lemma 9.9.** Let $\mathcal{E} : \mathcal{M}_n \rightarrow \mathcal{M}_n$ be a positive, linear map and $U \in U(n)$ a fixed unitary. Then defining
\[ \text{Cap}(A_{\mathcal{E}, U}) := \inf \left\{ \det \left( \sum_i \mathcal{E}(u_i u_i^\dagger) \gamma_i \right) \mid \gamma_i > 0, \prod_{i=1}^n \gamma_i = 1 \right\} \]
where $u_i$ are once again the rows of $U$, we have the following properties:

1. Using the mixed discriminant $M$ defined in Appendix C, we have
\[ M(A_{\mathcal{E}, U}) \leq \text{Cap}(A_{\mathcal{E}, U}) \leq \frac{n^n}{n!} M(A_{\mathcal{E}, U}) \]

2. $\inf_{U \in U(n)} \text{Cap}(A_{\mathcal{E}, U}) = \text{Cap}(\mathcal{E})$

Most of the proof is very technical and found in Gurvits and Samorodnitsky 2002. Some parts are explained in Appendix D.

Finally, this proves most of Theorem 9.5: We know that $\mathcal{E}$ is rank non-decreasing if and only if $\text{Cap}(\mathcal{E})$ is positive. In that case, the RAS algorithm converges in which case the map is approximately scalable. For the other direction, one can use a simple contradiction argument: Any map close to a doubly stochastic map must be rank non-decreasing and as scaling does not change this property of a map, any approximately scalable map must be rank non-decreasing.

**Exact scalability** The capacity is also the correct generalisation for exact scaling:

**Lemma 9.10** (Gurvits 2004). Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a positive map. Then $\mathcal{E}$ is scalable to a doubly-stochastic map if and only if $\text{Cap}(\mathcal{E}) > 0$ and the capacity can be achieved.
The proof of this Lemma following Gurvits 2004 is given in Appendix D. The direction “Capacity is achieved ⇒ the map is scalable”, is proved by taking the Lagrangian and showing that at the minimum we have that
\[ \nabla \ln(\det(\mathcal{E}(X))) = \mathcal{E}^*(\mathcal{E}(C)^{-1}) \]
which implies scalability by Lemma 9.14. The converse direction is given by a direct calculation.

In order to prove that a map can be scaled to doubly stochastic form, one then needs to connect this lemma to full indecomposability of matrices. The proof is done using an argument involving strict convexity. Like the original London-Djokovic potential (8), the capacity is not a convex function, but one can make a substitution similar to Formulation (16) by considering the following function for any tuple \((A_i)_i\) of positive definite matrices:

\[ f_A(\xi_1, \ldots, \xi_n) := \ln \det(e^{\xi_1}A_1 + \ldots + e^{\xi_n}A_n). \tag{46} \]

Then we have:

**Lemma 9.11.** Let \(\mathcal{E} : \mathcal{M}_n \to \mathcal{M}_n\) be a positive, linear map and given \(U \in U(n)\), let \(A = A_{\mathcal{E},U}\). Then

1. \(f_A\) is convex on \(\mathbb{R}^n\).
2. If \(\mathcal{E}\) is fully indecomposable, then \(f_A\) is strictly convex on \(\{\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n | \sum_i \xi_i = 0\}\).

The proof is technical and uses mixed discriminants as well as results about them from Bapat 1989, which is why we only discuss it in the appendices. This is then used to prove

**Lemma 9.12.** Let \(\mathcal{E} : \mathcal{M}_n \to \mathcal{M}_n\) be a positive, linear map. If \(\mathcal{E}\) is fully indecomposable, there exists a unique scaling of \(\mathcal{E}\) to a doubly stochastic map.

The idea is somewhat similar to the approximate Sinkhorn theorem: Since fully indecomposable maps are in particular rank non-decreasing, we know that the capacity is positive. For any \(X > 0\), which is diagonalised by \(U\), using the tuple \(A_{\mathcal{E},U}\), one can then see that \(\det(\mathcal{E}(X)) = f_A(\log \lambda)\) with the eigenvalues \(\lambda\) of \(X\). Showing that the infimum must lie inside a compact set then finishes the proof, since Lemma 9.11 implies existence and uniqueness of the minimum as \(f_A\) is strictly convex.

Using Lemma C.7, we can see then see that up to a unitary, every doubly stochastic map is a direct sum of fully indecomposable maps (much like doubly stochastic matrices are up to permutations a direct sum of fully indecomposable matrices, see Proposition A.5). Hence, any map which is a direct sum of fully indecomposable maps up to some scaling is clearly scalable to doubly stochastic maps. Sadly, the condition seems not very useful and the question remains open, whether one can simplify this condition.
However, that does in fact answer the question of unique scaling: Since the scaling for direct sums is not unique (we can always interchange summands), a map is uniquely scalable if and only if it is fully indecomposable.

9.1.2. Nonlinear Perron-Frobenius theory

The main idea of the alternative proofs in my Master’s Thesis (Idel 2013) and the paper Georgiou and Pavon 2015 is to extend the Menon operator to positive semidefinite matrices:

**Definition 9.13.** Let $E : M_n \to M_n$ be a positive map, such that $E(A), E^*(A) > 0$ for all $A > 0$. Let $D$ denote matrix inversion, then we define the following nonlinear map:

$$T_{\text{pos}} : \{A \in M_n | A > 0\} \to \{A \in M_n | A > 0\}$$

$$T_{\text{pos}}(\cdot) := D \circ E^* \circ D \circ E(\cdot)$$

This map is well-defined and after normalisation, it sends positive definite matrices of trace one onto itself.

We can then reformulate the existence problem into a fixed point problem:

**Lemma 9.14.** Let $E : M_n \to M_n$ be a positive map such that $E(A), E^*(A) > 0$ for all $A > 0$. Then there exist invertible $X, Y \in M_n$ such that $Y^{-1}E(X(\cdot)X^\dagger)Y^{-\dagger}$ is a doubly stochastic map if and only if $T_{\text{pos}}$ has an eigenvector (a fixed point after normalisation) in the set of positive definite trace one matrices. Furthermore, $X, Y$ can be chosen such that $X, Y > 0$.

**Proof.** Let $\rho > 0$ be the positive definite eigenvector of $G$. Then define $0 < \sigma := E(\rho)$. Since $\rho$ is an eigenvector, one immediately sees that $E^*(\sigma^{-1}) = \lambda \rho^{-1}$ with $\lambda = \text{tr}(E^*(\sigma^{-1}))^{-1}$. Now define $X := \sqrt{\rho}$ and $Y := \sqrt{\sigma}$ (i.e. $XX^\dagger = \rho$ and $YY^\dagger = \sigma$), then $X, Y$ are positive definite and if we define the map:

$$E' : M_n \to M_n$$

$$E'(\cdot) := Y^{-1}E(X(\cdot)X^\dagger)Y^{-\dagger}$$

then a quick calculation shows $E'(1) = 1$ and $E'^*(1) = 1$:

$$E'(1) = Y^{-1}E(X(1)X^\dagger)Y^{-\dagger} = Y^{-1}E(\rho)Y^{-\dagger}$$

$$= Y^{-1}\sigma Y^{-\dagger} = Y^{-1}YY^\dagger Y^{-\dagger} = 1$$

On the other hand, a similar calculation shows

$$E'^*(1) = X^\dagger E^*(Y^{-\dagger}1Y^{-1})X = \lambda 1$$

but since $E'$ was shown to be unital, $E'^*$ is trace-preserving and $\lambda = 1$ to begin with. Conversely, given $X, Y$ as in the lemma, $XX^\dagger$ would be a fixed point of the Menon-operator.
Note that this completes the proof of Lemma 9.10. We only have to see that the conditions at the minimum are met if and only if the Menon operator has a fixed point. This also provides the connection between the Menon operator and Gurvits’ approach: As in the classical case, the conditions for a fixed point of the Menon operator are given by the Lagrange conditions of the London-Djokovic potential.

We observe that the Menon-operator, if it is defined, is a continuous, homogeneous positive map. This lets us give a proof of a weak form of the Operator Sinkhorn Theorem:

**Proposition 9.15** (Idel 2013). Given a positive trace-preserving map \( \mathcal{E} : \mathcal{M}_n \rightarrow \mathcal{M}_n \) such that there exists an \( \varepsilon > 0 \) such that for all matrices \( \rho \geq \varepsilon \mathbb{1} \) with unit trace it holds that \( \mathcal{E}(\rho) \geq \frac{\varepsilon n}{1+(n-1)\varepsilon} \mathbb{1} \), then we can find \( X, Y > 0 \) such that \( Y^{-1} \mathcal{E}(X(\cdot)X^\dagger)Y^{-\dagger} \) is a doubly stochastic map.

**Proof.** Let \( T_{\text{normpos}}(\cdot) := T_{\text{pos}}(\cdot) / \text{tr}(T_{\text{pos}}(\cdot)) \) is the normalised operator. Now assume that for all \( \rho \geq \varepsilon \mathbb{1} \) with \( \text{tr}(\rho) = 1 \), it holds \( \mathcal{E}(\rho) \geq \delta \mathbb{1} \). In particular, if we call \( \lambda_{\text{max}} \) the maximal eigenvalue of \( \mathcal{E}(\rho) \), then \( \lambda_{\text{max}} \leq 1 - (n-1)\delta \). Hence we have:

\[
\begin{align*}
\delta \mathbb{1} & \leq \mathcal{E}(\rho) \\
& \leq (1 - (n-1)\delta) \mathbb{1} \\
& \Rightarrow \frac{1}{\delta} \mathbb{1} \geq \mathcal{D}(\mathcal{E}(\rho)) \geq \frac{1}{1 - (n-1)\delta} \mathbb{1} \\
& \Rightarrow \delta \mathbb{1} \leq \mathcal{D}(\mathcal{E}^*(\mathcal{D}(\mathcal{E}(\rho)))) \leq (1 - (n-1)\delta) \mathbb{1}
\end{align*}
\]

where we used the unitality of \( \mathcal{E}^* \) in the third step. This implies

\[
T_{\text{normpos}}(\rho) \geq \frac{\delta}{1 - (n-1)\delta} \frac{\mathbb{1}}{n}
\]

Now we want \( \frac{\delta}{1 - (n-1)\delta} \geq \varepsilon n \), in which case the compact set of matrices \( \{ \rho > 0 | \text{tr}(\rho) = 1, \rho \geq \varepsilon \mathbb{1} / n \} \) is mapped into itself, hence by Brouwer’s fixed point theorem, we obtain a positive definite fixed point of \( \mathcal{G} \). A quick calculation shows that this implies \( \delta > n\varepsilon / (1 + (n-1)n\varepsilon) \).

Since a positive map \( \mathcal{E} : \mathcal{M}_n \rightarrow \mathcal{M}_n \) can always be converted into a trace-preserving map \( \mathcal{E}' \) by setting \( \rho := \mathcal{E}^*(\mathbb{1}) \) and \( \mathcal{E}'(\cdot) := \mathcal{E}(\sqrt{\rho^{-\dagger} \cdot \rho^{-\dagger}}) \), the assumption that \( \mathcal{E} \) be trace-preserving is not really necessary. As a direct corollary, we obtain a similar result, which might be easier to use:

**Corollary 9.16** (Idel 2013; Georgiou and Pavon 2015). Let \( \mathcal{E} : \mathcal{M}_n \rightarrow \mathcal{M}_n \) be a trace-preserving and positivity improving map, then there exist maps \( X, Y > 0 \) such that \( Y^{-1} \mathcal{E}(X(\cdot)X^\dagger)Y^{-\dagger} \) is a doubly stochastic map.
As in the classical matrix case in Brualdi, Parter, and Schneider 1966, one idea to obtain necessary and sufficient criteria is to extend the map $T_{\text{normpos}}$ to positive semidefinite matrices for all cases and then prove that there is a fixed point of the map inside the cone of positive definite matrices. However, we run into additional problems, since the cone of positive semidefinite matrices is not polyhedral (cf. Lemmens and Nussbaum 2012: there is no and cannot exist an equivalent version of theorem B.8; this does not preclude that an extension exists, but such a result must depend on the operator in question). Moreover, even if a continuous extension may be possible by using perturbation theory (for instance), the hardest part is to prove the existence of a fixed point inside the cone.

9.1.3. Other approaches and generalised scaling

We have seen that at least two classical approaches for proving that a matrix can be scaled to a doubly stochastic matrix can be extended to the quantum case without too much trouble (the proofs however might be more difficult): nonlinear Perron-Frobenius theory and the barrier function approach. In a sense, we have also seen convex programming approaches. An immediate question is whether one can extend the entropy approach. This was also asked in Gurvits 2004 and it is a major open question in Georgiou and Pavon 2015, since the motivation of Schrödinger bridges heavily relies on relative entropy minimisation. The answer is not clear since something like a quantum relative entropy is only used only on the level of matrices and a justification via the Choi-Jamiolkowski isomorphism is not immediate (see Section 9.2):

$$D(\rho\|\sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma).$$ 

Another question is how to extend the theorems from the doubly-stochastic map to cover arbitrary marginals, i.e. we want to scale a positive map $\mathcal{E}$ such that

$$\mathcal{E}(\rho) = \sigma, \quad \mathcal{E}^\ast(1) = 1$$

with some prespecified $\rho, \sigma$. For Gurvits’ approach based on equation (8) this is not really straightforward, since it is unclear how to take appropriate powers of $P, Q$. For the approach via nonlinear Perron-Frobenius theory, this can be done to some degree:

**Theorem 9.17** (Georgiou and Pavon 2015). Given a positivity improving map $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ and two matrices $V, W > 0$ with $\text{tr}(V) = \text{tr}(W)$, there exist matrices $X, Y \in \mathcal{M}_d$ and a constant $\lambda > 0$ such that $\mathcal{E}'(\cdot) := Y\mathcal{E}(X \cdot X^\dagger)Y^\dagger$ fulfills

$$\mathcal{E}'(V) = W \quad \mathcal{E}'^\ast(1) = 1$$

**Sketch of proof.** The proof is a variation of the methods for the case $V = W = 1$. We consider the following Menon-type operator, which was essentially defined in Georgiou
and Pavon 2015:

\[ T_{\mathcal{E},V,W} := D_1 \circ \mathcal{E}^* \circ D_2 \circ \mathcal{E} \]

where

\[ D_1(\rho) = \rho^{-1/2}V^{-1}\rho^{-1/2} \]
\[ D_2(\rho) = (W^{1/2}(W^{-1/2}\rho^{-1/2}W^{-1/2})^{1/2}W^{1/2})^2 \]

**Step 1:** Let \( \mathcal{E} \) be positivity improving, then \( T_{\mathcal{E},V,W} : \mathcal{C}^d \rightarrow \mathcal{C}^d \) is a well-defined, continuous, and homogeneous map. It is well-defined, since \( \mathcal{E} \) maps \( \mathcal{C}^d \rightarrow \mathcal{C}^d \) and \( D_1 \) and \( D_2 \) send \( \mathcal{C}^d \rightarrow \mathcal{C}^d \) if \( V, W \in \mathcal{C}^d \). It is homogeneous, because \( \mathcal{E} \) is linear and \( D_i(\lambda \rho) = \lambda^{-1}D_i(\rho) \) for \( i = 1, 2 \). Finally, \( D_1 \) is continuous as taking the square root of a positive definite matrix is continuous and matrix multiplication and inversion of positive definite matrices is continuous. Likewise, \( D_2 \) is continuous and thus \( T_{\mathcal{E},V,W} \) as composition of continuous maps.

**Step 2:** We now claim that a scaling of \( \mathcal{E} \) with as in the theorem with \( X, Y > 0 \) exists iff \( T_{\mathcal{E},V,W} \) has an eigenvector. This was observed in Georgiou and Pavon 2015 and is a straightforward but lengthy calculation.

**Step 3:** Finally, we can prove the existence of \( X, Y > 0 \) such that a scaling exists by invoking Brouwer’s fixed point theorem. The map

\[ \tilde{T}(\cdot) : \mathcal{C}_1^d \rightarrow \mathcal{C}_1^d \tilde{T}(\cdot) := T_{\mathcal{E},V,W}(\cdot) / \text{tr}(T_{\mathcal{E},V,W}(\cdot)) \]

is a continuous, well-defined map, hence it has a fixed point. This is necessarily an eigenvector of \( T_{\mathcal{E},V,W} \), hence defines a scaling. \( \square \)

The problem with this proof is that this Menon operator is no longer clearly a contraction mapping, hence uniqueness and convergence speed of the algorithm are not clear. Also, the obvious algorithm derived from this proof differs from the usual RAS algorithm. It is not clear how to remedy this or extend one of the other approaches. Also, this map has even worse prospect of being generalised to positive and not necessarily positivity improving maps. In any case, it is not immediately clear what the right necessary and sufficient conditions are. In the case of matrices, patterns were the important concept, but what is a pattern supposed to be for positive maps? One can always choose a basis and represent the map as matrix, but this is very much map-dependent and it is not clear what the correct interpretation will be.

Nevertheless, partial results for uniqueness have been achieved in Friedland 2016: The author proves that for positivity preserving maps \( \mathcal{E} \), there exists a ball around 1 such that if \( V, W \) lie inside this ball, there exists a unique scaling of \( \mathcal{E} \) to a trace-preserving positive map with \( \mathcal{E}(V) = W \).
9.2. Operator Sinkhorn theorem via state-channel duality

Another formulation of the operator Sinkhorn theorem is given by the Choi-Jamiolkowski isomorphism. It states that given any positive map $E : \mathcal{M}_n \rightarrow \mathcal{M}_n$, we have that

$$\tau_E := (\text{id} \otimes E)(\omega)$$

is a block-positive matrix (i.e. $\langle \phi_1 | (\tau_E | \phi_1) \rangle \geq 0$ for all $|\phi_1\rangle, |\phi_2\rangle \in \mathcal{M}_n$). Here $\omega := 1/n \sum_{i,j=1}^n |ii\rangle \langle jj|$ is the so-called maximally entangled state. If $E$ is completely positive, i.e. $E \otimes \text{id}_n$ is a positive map, then $\tau_E$ is a positive semi-definite matrix. Now consider $X_1, X_2 \geq 0$ and $E' := X^\dagger_1 E (X_1 \cdot X^\dagger_1) X_2$. We have

$$\tau_{E'} = (X^\dagger_1 \otimes X^\dagger_2) \tau_E (X_1 \otimes X_2)^\dagger$$

where we use $(1 \otimes X_1) \sum |ii\rangle = (X^\dagger_1 \otimes 1) \sum |ii\rangle$ and therefore

$$\tau_{E'} = (1 \otimes X^\dagger_1)(\text{id} \otimes E)((1 \otimes X_1) \omega (1 \otimes X_1)^\dagger)(1 \otimes X_2) \quad (51)$$

Therefore, the task can be reformulated: Given a block positive matrix $\tau$, find $X_1, X_2 \in \mathcal{M}_d$ such that

$$\tau' := (X_1 \otimes X_2) \tau (X_1 \otimes X_2)^\dagger$$

fulfils $\text{tr}_2(\tau) = \text{tr}_1(\tau) = 1/d$, where $\text{tr}_i$ denotes the partial trace over the $i$-th system in $\mathcal{M}_d \otimes \mathcal{M}_d \equiv \mathcal{M}_{d^2}$. For $\tau \geq 0$ these operations are called (local) filtering operations. Often (c.f. Gittsovich et al. 2008; Wolf 2012), one asks for $X_1, X_2 \in \text{SL}(d)$ and the resulting trace being merely proportional to the identity, but this is of course just a normalisation.

We can then state an equivalent version of Sinkhorn scaling for positive map:

Proposition 9.18 (Kent, Linden, and Massar 1999; Leinaas, Myrheim, and Ovrum 2006; Verstraete, Dehaene, and De Moor 2001). Let $\rho \in \mathcal{M}_d \otimes \mathcal{M}_d$ be a positive definite density matrix. Then there exist matrices $X_1, X_2 \in \mathcal{M}_d$ such that

$$(X_1 \otimes X_2) \rho (X_1 \otimes X_2)^\dagger = \frac{1}{d^2} \mathbb{1} + \sum_{k=1}^{d^2-1} \xi_k J^1_k \otimes J^2_k$$

where $\{J^1_k\}_k \subset \mathcal{M}_d$ and $\{J^2_k\}_k \subset \mathcal{M}_d$ form a basis of the traceless complex matrices and $\xi \in \mathbb{C}$ for the first and second tensor factor in $\mathcal{M}_d \otimes \mathcal{M}_d$ respectively.

Proof. Note that a positive definite $\rho$ corresponds to a completely positive map, which maps positive semidefinite matrices to positive definite ones. In particular, the corresponding map is fully indecomposable. Hence by Theorem 9.4, there exists a scaling to a doubly stochastic map, which again corresponds to a positive definite $\tilde{\rho} \in \mathcal{M}_d \otimes \mathcal{M}_d$
such that $\operatorname{tr}_1(\tilde{\rho}) = \operatorname{tr}_2(\tilde{\rho}) = 1/d$. By construction, $\{1/d, J_1^1\}_k$ and $\{1/d, J_2^2\}_k$ form an orthonormal basis of $\mathcal{M}_d$, hence we can express $\tilde{\rho}$ as

$$\tilde{\rho} = \frac{1}{d^2} \mathbb{1} + \sum_{k=1}^{k^2-1} \xi_k J_1^1 \otimes J_2^2 + \sum_{k=1}^{k^2-1} \chi_k \frac{1}{d} \otimes J_1^1 + \chi_k J_2^2 \otimes \frac{1}{d}$$

with $\xi_k, \chi_k^1, \chi_k^2 \in \mathbb{C}$ for all $k$. Then

$$\operatorname{tr}_1(\tilde{\rho}) = \frac{1}{d} \mathbb{1} + \sum_{k=1}^{k^2-1} \xi_k \operatorname{tr}(J_1^1 \otimes J_2^2) + \sum_{k=1}^{k^2-1} \left( \chi_k \frac{1}{d} \otimes J_1^1 + \chi_k \operatorname{tr}(J_2^2) \otimes \frac{1}{d} \right)$$

$$= \frac{1}{d} \mathbb{1} + \sum_{k=1}^{k^2-1} \chi_k J_1^1 \frac{1}{d} \mathbb{1} = \frac{1}{d} \mathbb{1}$$

But then, since the $J_1^1$ are linearly independent, $\chi_k^1 = 0$ for all $k$. Likewise, $\chi_k^2 = 0$ for all $k$ and we have the required normal form. \qed

The proposition has direct proofs and extensions to more than two parties (see for instance Verstraete, Dehaene, and De Moor 2002; Verstraete, Dehaene, and De Moor 2003; Wolf 2012). Here, it only uses the sufficient part of the criterion for scalability of positive maps, hence we can strengthen it to include parts of all block-positive matrices. Since, however, not all completely positive maps are fully indecomposable (e.g. the map $\mathcal{E} : \rho \mapsto |\psi\rangle \langle \psi|$ for some vector $|\psi\rangle \in \mathbb{C}^d$ is not), it certainly does not extend to all states.

### 9.3. Convergence speed and stability results

Gurvits’ proof already gives an estimate for the convergence speed of the scheme (see Theorem 4.7.3. in Gurvits 2004). Let us give an alternative proof using Hilbert’s metric which is equivalent to the classical proof in Franklin and Lorenz 1989 and reminiscent of the convergence proof in Georgiou and Pavon 2015. Throughout the proof, we use several results from Appendix B, in particular the definition of Hilbert’s projective metric $d_H$ on the cone of positive semidefinite matrices and the definition of the contraction ratio $\gamma$ in equation (68). To proceed, we define a metric on the space of positive maps that are scalable:

**Definition 9.19.** Let $\mathcal{E}, \mathcal{T} : \mathcal{M}_n \to \mathcal{M}_n$ be two positive maps such that $\mathcal{T}(\cdot) = \mathcal{Y} \mathcal{E}(X \cdot X^\dagger) \mathcal{Y}^\dagger$ for some positive matrices $X, Y$. Then

$$\Delta(\mathcal{E}, \mathcal{T}) = d_H(X, \mathbb{1}) + d_H(Y, \mathbb{1})$$

defines a metric on the space of positive maps (two maps that cannot be scaled to each other have infinite distance).
A proof that this constitutes a metric may be found in Lemmens and Nussbaum 2012, Chapter 2. Recall the Sinkhorn iteration as defined in equations (41)-(42). For convenience we use a slightly different notation:

\[ E^{(i)} := E_{2i} \quad i > 0 \]
\[ E^{(i)} := E_{2i+1} \quad i \geq 0 \]
\[ \rho^{(i)} := E^{(i-1)}(1) \quad i > 0 \]
\[ \sigma^{(i)} := E^{(i)}(1) \]
\[ E^{(0)} := E. \]

Then:

**Proposition 9.20.** Let \( E : \mathcal{M}_n \to \mathcal{M}_n \) be a positivity improving, trace preserving map. Let \( T := Y^{-1}E(X \cdot X^t)Y \) be the unique doubly stochastic scaling limit.

\[ \Delta(E^{(k)}), T) \leq \frac{\gamma^k}{1 - \gamma} (d_H(\rho^{(1)}, e) + d_H(\sigma^{(1)}, e)) \quad (55) \]
\[ \Delta(E^{(k)'}, T) \leq \frac{\gamma^k}{1 - \gamma} (d_H(\rho^{(1)}, e) + d_H(\sigma^{(1)}, e)) \quad (56) \]

where \( \gamma^{1/2} = \gamma^{1/2}(E) \) is the contraction ratio of equation (68). In particular, this implies via proposition B.6 (implying that here, \( \gamma < 1 \)) that the convergence is geometric.

**Proof.** The proof is similar to the classical one in Franklin and Lorenz 1989. First recall the definition of \( \Delta \) from Appendix B:

\[ \Delta(E) := \sup \{ d_H(E(\rho), E(\sigma)) | \rho, \sigma \geq 0 \} \]

Then \( \Delta > 0 \) but finite, since \( E \) is a positivity improving map and the maximum is attained. This is true, because it suffices to consider \( d_H(E(\rho), E(\sigma)) \) on the compact set \( \{ A \geq 0 | \| A \|_\infty = 1 \} \) using Proposition B.6 (iii).

We first make the following observations:

\[ d_H(\rho, \sigma) = d_H(\sigma^{-1/2} \rho \sigma^{-1/2}, 1) \quad \forall \rho, \sigma > 0 \quad (57) \]
\[ d_H(E(\rho), E(\sigma)) \leq \gamma^{1/2}(E) d_H(E(\rho), E(\sigma)) \quad \forall \rho, \sigma > 0, E : \mathcal{M}_n \to \mathcal{M}_n \quad (58) \]

Equation (58) follows from the definition of \( \gamma^{1/2} \). Equation (57) follows from the definition of \( M \) and \( m \) in the definition of the Hilbert metric and the fact that taking noncommutative inverses does not change positivity.

Let us now focus on \( \gamma(E) \). Let \( X, Y \in \mathcal{M}_n \) be invertible, then

\[ \Delta = \sup \{ d_H(Y^t E(X \rho X^t)Y, Y^t E(X \sigma X^t)Y) | \rho, \sigma \geq 0 \} \]
\[ = \sup \{ d_H(E(X \rho X^t), E(X \sigma X^t)) | \rho, \sigma \geq 0 \} \]

67
where we used (58) and then $X$ being invertible. In particular, this implies that for every $\gamma^{1/2}(\mathcal{E}^{(i)})$ we have a universal upper bound
\[
\gamma^{1/2}(\mathcal{E}^{(i)}) < \tanh(\Delta/4)
\] (59)

Since $\Delta > 0$ but finite, this implies that we can upper bound each $\gamma(\mathcal{E}^{(i)})$ and $\gamma(\mathcal{E}'^{(i)})$ by some $\gamma < 1$. The rest is basically an iteration.

Consider $d_H(\rho^{(2)}, \mathbb{1})$. By definition, $\rho^{(2)} = \mathcal{E}^{(1)}(\mathbb{1}) = \mathcal{E}'^{(1)}(\sigma^{(1)})^{-1}$, and since all $\mathcal{E}^{(i)}$ are unital:
\[
d_H(\rho^{(2)}, \mathbb{1}) = d_H(\mathcal{E}^{(1)}, \mathcal{E}'^{(1)}(\sigma^{(1)})^{-1}) \leq \gamma^{1/2}(\mathcal{E}'^{(1)})d_H(\mathbb{1}, \sigma^{(1)})
\] (60)

where we used (58) and then (57). Similarly, since $\sigma^{(1)} = \mathcal{E}'^{(0)*}(\mathbb{1}) = \mathcal{E}^{(0)*}(\rho^{(1)})^{-1}$ and $\mathcal{E}^{(0)*}(\mathbb{1}) = \mathbb{1}$ by construction, we obtain:
\[
d_H(\mathbb{1}, \sigma^{(1)}) = d_H(\mathcal{E}'^{(0)*}(\mathbb{1}), \mathcal{E}^{(0)*}(\rho^{(1)})^{-1}) \leq \gamma^{1/2}(\mathcal{E}^{(0)*})d_H(\rho^{(1)}, \mathbb{1})
\] (61)

Combining (60) and (61) we obtain:
\[
d_H(\rho^{(2)}, \mathbb{1}) \leq \gamma d_H(\rho^{(1)}, \mathbb{1})
\] (62)

Similarly,
\[
d_H(\sigma^{(2)}, \mathbb{1}) \leq \gamma d_H(\sigma^{(1)}, \mathbb{1})
\] (63)

These are the key observations. Now using the definition of $\Delta(\cdot, \cdot)$ we obtain:
\[
\Delta(\mathcal{E}^{(k)}, \mathcal{E}^{(k+1)}) = d_H((\rho^{(k)})^{-1}, \mathbb{1}) + d_H((\sigma^{(k)})^{-1}, \mathbb{1}) \\
= \gamma^{k-1}(d_H(\rho^{(1)}, \mathbb{1}) + d_H(\sigma^{(1)}, \mathbb{1}))
\]

Hence we have by the triangle inequality
\[
\Delta(\mathcal{E}^{(0)}, \mathcal{E}^{(k+1)}) \leq \sum_{l=0}^{k-1} \gamma^l(d_H(\rho^{(1)}, \mathbb{1}) + d_H(\sigma^{(1)}, \mathbb{1}))
\]

and therefore, if $\mathcal{T}$ denotes the limit of the Sinkhorn iteration, using the geometric series
\[
\Delta(\mathcal{E}^{(0)}, \mathcal{T}) \leq \frac{1}{1 - \gamma}(d_H(\rho^{(1)}, \mathbb{1}) + d_H(\sigma^{(1)}, \mathbb{1}))
\] (64)
\[
\Delta(\mathcal{E}^{(k)}, \mathcal{T}) \leq \frac{\gamma^k}{1 - \gamma}(d_H(\rho^{(1)}, \mathbb{1}) + d_H(\sigma^{(1)}, \mathbb{1}))
\] (65)

The other inequality for the maps $\mathcal{E}'$ follows from symmetric arguments. □
Note that in contrast to the classical case in Franklin and Lorenz 1989, because of
the noncommutativity in equation (57), a simple extension to the general scaling of
positivity improving maps seems not possible.

Next, we wish to generalise also the stability results. It seems natural that this should
follow from the contraction results above:

**Corollary 9.21.** Let \( \mathcal{E} : \mathcal{M}_n \rightarrow \mathcal{M}_n \) be positivity improving, then the scaling is continuous
in \( \mathcal{E} \).

**Proof.** Let \( \mathcal{E} \) be a positivity improving map and \( \mathcal{E}' = \mathcal{E} + \delta \mathcal{T} \) be a perturbation which
is again positivity preserving, where \( \mathcal{T} \) is a positive map with \( \| \mathcal{T} \| = 1 \) (for instance in
the operator norm).

Then let \( X, Y \) be such that they scale \( \mathcal{E} \) to a doubly stochastic map. This implies that
\[
Y\mathcal{E}'(XX^\dagger)Y^\dagger = 1 + \delta Y\mathcal{T}(XX^\dagger)Y^\dagger
\]
\[
X^\dagger\mathcal{E}'^*(Y^\dagger Y)X = 1 + \delta X^\dagger\mathcal{T}^*(Y^\dagger Y)X
\]
and the marginals are also close to \( 1 \). In fact, for any \( \epsilon > 0 \) we can find \( \delta > 0 \) such that
\[
d_H(Y\mathcal{E}'(XX^\dagger)Y^\dagger, 1) + d_H(X^\dagger\mathcal{E}'^*(Y^\dagger Y)X, 1), < \epsilon
\]
But then, by equation (64), we have that if \( \mathcal{E}' \) is the scaling of \( Y\mathcal{E}'(XX^\dagger)Y^\dagger \) to a doubly
stochastic map, then
\[
\Delta(Y\mathcal{E}'(XX^\dagger)Y^\dagger, \mathcal{E}') \leq \frac{1}{1 - \gamma}(d_H(\rho^{(1)}, 1) + d_H(\sigma^{(1)}, 1)) < \frac{1}{1 - \gamma} \epsilon
\]
Using the triangle inequality and the fact that \( \Delta(\mathcal{E}, \mathcal{E}') < C\epsilon \) for some constant \( C \)
finishes the proof.

As noted, both theorems can be extended to cover all exactly scalable positive maps
using Gurvits’ Theorem 4.7.3. of Gurvits 2004. Given the result for classical matrices,
its seems natural that the convergence speed for rank non-decreasing but not exactly
scalable matrices should not be geometric.

**9.4. Applications of the Operator Sinkhorn Theorem**

Let us finally mention applications of the operator version of Sinkhorn’s theorem. The
state-version of the theorem, since it can be seen as a normal form for states under local
operations, has been applied in the study of states under LOCC operations (see for
instance Kent, Linden, and Massar 1999; Leinaas, Myrheim, and Ovrum 2006).

The approximate operator version was developed to obtain polynomial-time algo-
rithms (Sinkhorn scaling) for a problem known as “Edmond’s problem”. It asks the
following question (Gurvits 2004): Given a linear subspace \( A \) of \( \mathcal{M}_n \), does there exist a
nonsingular matrix in \( V \)? The question can be asked also over different number fields
and in different contexts. It is particularly interesting, because it is related to rational identity testing over non-commutative variables as studied in Garg et al. 2015; Ivanyos, Qiao, and Subrahmanyam 2015. For further input we refer the reader to the extended and well-written review of the applications in Garg et al. 2015.

Finally, the exact scalability of fully indecomposable positive maps provided bounds on the mixed discriminant of matrix tuples, which is interesting to provide permanent bounds (Gurvits and Samorodnitsky 2000).

Acknowledgement I was first acquainted with the topic of matrix scaling through my Master’s thesis supervisor Michael Wolf. We also worked together on unitary scaling and I thank him for his valuable input. I am supported by the Studienstiftung des deutschen Volkes.

References

Aaronson, Scott (Mar. 2005). “Quantum computing and hidden variables”. In: Phys. Rev. A 71 (3). doi: 10.1103/PhysRevA.71.032325.

Achilles, Eva (1993). “Implications of convergence rates in Sinkhorn balancing”. In: Linear Algebra and its Applications 187, pp. 109 –112. doi: 10.1016/0024-3795(93)90131-7.

Ahuja, Ravindra K. et al. (1994). “Improved Algorithms for Bipartite Network Flow”. In: SIAM Journal on Computing 23.5, pp. 906–933. doi: 10.1137/S0097539791199334.

Bacharach, Michael (1965). “Estimating Nonnegative Matrices from Marginal Data”. In: International Economic Review 6.3, pp. 294–310. issn: 00206598, 14682354. — (1970). Biproportional Matrices and Input-Output Change. Cambridge University Press.

Bachem, Achim and Bernhard Korte (1979). “On the RAS-algorithm”. In: Computing 23.2, pp. 189–198. doi: 10.1007/BF02252097. — (1980). “Minimum norm problems over transportation polytopes”. In: Linear Algebra and its Applications 31, pp. 103 –118. doi: 10.1016/0024-3795(80)90211-6.

Balakrishnan, H., Inseok Hwang, and C. J. Tomlin (Dec. 2004). “Polynomial approximation algorithms for belief matrix maintenance in identity management”. In: Decision and Control, 2004. CDC. 43rd IEEE Conference on. Vol. 5, pp. 4874–4879. doi: 10.1109/CDC.2004.1429569.

Balinski, Michel and Gabrielle Demange (1989a). “Algorithms for proportional matrices in reals and integers”. In: Mathematical Programming 45.1-3, pp. 193–210. doi: 10.1007/BF01589103. — (1989b). “An axiomatic approach to proportionality between matrices”. In: Mathematics of Operations Research 14.4, pp. 700–719. doi: 10.1287/moor.14.4.700.

Balinski, Michel and Victoriano Ramírez González (1997). “Mexican electoral law: 1996 version”. In: Electoral Studies 16.3, pp. 329–340. doi: 10.1016/S0261-3794(97)00025-8.

Bapat, Ravindra B. (1982). “D1AD2 theorems for multidimensional matrices”. In: Linear Algebra and its Applications 48.0, pp. 437 –442. doi: 10.1016/0024-3795(82)90125-2. — (1989). “Mixed discriminants of positive semidefinite matrices”. In: Linear Algebra and its Applications 126.0, pp. 107 –124. doi: 10.1016/0024-3795(89)90009-8.
Bapat, Ravindra B. and T. E. S. Raghavan (1989). “An extension of a theorem of Darroch and Ratcliff in loglinear models and its application to scaling multidimensional matrices”. In: *Linear Algebra and its Applications* 114 - 115.0. Special Issue Dedicated to Alan J. Hoffman, pp. 705 –715. doi: 10.1016/0024-3795(89)90489-8.

— (1997). *Nonnegative Matrices and Applications*. Cambridge Books Online. Cambridge University Press. isbn: 9780511529979.

Bauer, F. L. (1963). “Optimally scaled matrices”. In: *Numerische Mathematik* 5.1, pp. 73–87. doi: 10.1007/BF01385880.

— (1965). “An elementary proof of the Hopf inequality for positive operators”. In: *Numerische Mathematik* 7.4, pp. 331–337. doi: 10.1007/BF01436527.

Benoist, Tristan and Ion Nechita (2016). *On bipartite unitary matrices generating subalgebra-preserving quantum operations*. arXiv:1608.05811v1 [quant-ph].

Benzi, Michele (2002). “Preconditioning Techniques for Large Linear Systems: A Survey”. In: *Journal of Computational Physics* 182.2, pp. 418 –477. doi: 10.1006/jcph.2002.7176.

Berger, Marc A. and C. T. Kelley (1979). “A variational equivalent to diagonal scaling”. In: *Journal of Mathematical Analysis and Applications* 72.1, pp. 291 –304. doi: 10.1016/0022-247X(79)90290-7.

Bhatia, Rajendra (1996). *Matrix Analysis*. Springer. isbn: 978-1-4612-0653-8.

Bingen, F (1965). “Simplification de la Démonstration d’un Théorème de MWM Gorman”. In: *Université Libre de Bruxelles, duplicated*.

Biran, Paul, Michael Entov, and Leonid Polterovich (2004). “Calabi quasimorphisms for the symplectic ball”. In: *Communications in Contemporary Mathematics* 06.05, pp. 793–802. doi: 10.1142/S0219199704001525.

Birkhoff, Garrett (1957). “Extensions of Jentzsch’s Theorem”. In: *Transactions of the American Mathematical Society* 85.1, pp. 219–227. doi: 10.1090/S0002-9947-1957-0087058-6.

Bishop, Y. M. M. (1967). “Multidimensional contingency tables: cell estimates”. PhD thesis. Harvard University.

Borobia, Alberto and Rafael Cantó (1998). “Matrix scaling: A geometric proof of Sinkhorn’s theorem”. In: *Linear Algebra and its Applications* 268.0, pp. 1 –8. doi: 10.1016/S0024-3796(97)00010-4.

Borwein, Jonathan M., Adrian Stephen Lewis, and Roger D. Nussbaum (1994). “Entropy minimization, DAD problems, and doubly stochastic kernels”. In: *Journal of Functional Analysis* 123.2, pp. 264–307. doi: 10.1006/jfan.1994.1089.

Bot, Radu Ioan and Sorin-Mihai Grad (2010). “Wolfe duality and Mond-Weir duality via perturbations”. In: *Nonlinear Analysis: Theory, Methods & Applications* 73.2, pp. 374 –384. doi: 10.1016/j.na.2010.03.026.

Boyd, Stephen and Lieven Vandenberghe (2004). *Convex Optimization*. New York, NY, USA: Cambridge University Press. isbn: 0521833787.

Braatz, Richard D. and Manfred Morari (1994). “Minimizing the Euclidean Condition Number”. In: *SIAM Journal on Control and Optimization* 32.6, pp. 1763–1768. doi: 10.1137/S0363012992238680.

Bradley, Andrew M. (2010). “Algorithms for the Equilibration of Matrices and Their Application to Limited-Memory Quasi-Newton Methods”. PhD thesis. Stanford ICME.

Bradley, Andrew M. and Walter Murray (2011). *Matrix-free approximate equilibration*. arXiv preprint arXiv:1110.2805.
Bregman, L. M. (1967). “The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming”. In: USSR Computational Mathematics and Mathematical Physics 7.3, pp. 200–217. doi: 10.1016/0041-5553(67)90040-7.

Brown, David T. (1959). “A note on approximations to discrete probability distributions”. In: Information and Control 2.4, pp. 386–392. doi: 10.1016/S0019-9958(59)80016-4.

Brown, Jack B., Phillip J. Chase, and Arthur O. Pittenger (1993). “Order independence and factor convergence in iterative scaling”. In: Linear Algebra and its Applications 190.0, pp. 1–38. doi: 10.1016/0024-3795(93)90218-D.

Brualdi, Richard A. (1968). “Convex sets of non-negative matrices”. In: Canadian Journal of Mathematics 20, pp. 144–157. doi: 10.4153/CJM-1968-016-9.

— (1974). “The DAD theorem for arbitrary row sums”. In: Proc. Amer. Math. Soc. 45, pp. 189–194. doi: 10.1090/S0002-9939-1974-0354737-8.

Brualdi, Richard A., Seymour V. Parter, and Hans Schneider (1966). “The diagonal equivalence of a nonnegative matrix to a stochastic matrix”. In: Journal of Mathematical Analysis and Applications 16.1, pp. 31–50. doi: 10.1016/0022-247X(66)90184-3.

Bunch, James R. (Oct. 1971). “Equilibration of Symmetric Matrices in the Max-Norm”. In: J. ACM 18.4, pp. 566–572. doi: 10.1145/321662.321670.

Carey, Malachy, Chris Hendrickson, and Krishnaswami Siddharthan (1981). “A Method for Direct Estimation of Origin/Destination Trip Matrices”. In: Transportation Science 15.1, pp. 32–49. doi: 10.1287/trsc.15.1.32.

Caussinus, Henri (1965). “Contribution à l’analyse statistique des tableaux de corrélation”. In: Annales de la Faculté des sciences de Toulouse : Mathématiques 29, pp. 77–183.

Censor, Yair and Arnold Lent (1981). “An iterative row-action method for interval convex programming”. In: Journal of Optimization Theory and Applications 34.3, pp. 321–353. doi: 10.1007/BF00934676.

Chen, Tzu-Yi and James W. Demmel (2000). “Balancing sparse matrices for computing eigenvalues”. In: Linear Algebra and its Applications 309.1, pp. 261–287. doi: 10.1016/S0024-3796(00)00014-8.

Cottle, Richard W., Steven G. Duvall, and Karel Zikan (1986). “A Lagrangean relaxation algorithm for the constrained matrix problem”. In: Naval Research Logistics Quarterly 33.1, pp. 55–76. doi: 10.1002/nav.3800330106.

Csima, J. and B. N. Datta (1972). “The DAD theorem for symmetric non-negative matrices”. In: Journal of Combinatorial Theory, Series A 12.1, pp. 147–152. doi: 10.1016/0097-3165(72)90090-8.

Csizsár, Imre (1975). “I-divergence geometry of probability distributions and minimization problems”. In: The Annals of Probability, pp. 146–158.

— (1989). “A geometric interpretation of Darroch and Ratcliff’s generalized iterative scaling”. In: The Annals of Statistics, pp. 1409–1413.

Curtis, A. R. and J. K. Reid (1972). “On the Automatic Scaling of Matrices for Gaussian Elimination”. In: IMA Journal of Applied Mathematics 10.1, pp. 118–124. doi: 10.1093/imamat/10.1.118.

Cuturi, Marco (2013). “Sinkhorn distances: Lightspeed computation of optimal transport”. In: Advances in Neural Information Processing Systems, pp. 2292–2300.

Darroch, J. N. and D. Ratcliff (Oct. 1972). “Generalized Iterative Scaling for Log-Linear Models”. In: The Annals of Mathematical Statistics 43.5, pp. 1470–1480. doi: 10.1214/aoms/1177692379.
Franklin, Joel and Jens Lorenz (1989). “On the scaling of multidimensional matrices”. In: Linear Algebra and its Applications 114 - 115.0. Special Issue Dedicated to Alan J. Hoffman, pp. 717 –735. doi: 10.1016/0024-3795(89)90490-4.

Fréchet, M. (1960). “Sur les tableaux dont les marges et des bornes sont données”. In: Revue de l’Institut International de Statistique / Review of the International Statistical Institute 28.1/2, pp. 10–32. doi: 10.2307/1401846.

Friedland, Shmuel (2016). On Schrödinger’s bridge problem. arXiv:1608.05862v1 [math-ph].

Friedlander, D. (1961). “A Technique for Estimating a Contingency Table, Given the Marginal Totals and Some Supplementary Data”. In: Journal of the Royal Statistical Society. Series A (General) 124.3, pp. 412–420. doi: 10.2307/2343244.

Frobenius, Ferdinand Georg (1912). “Über Matrizen aus nicht negativen Elementen”. In: Sitzungsbericht Königl. Preuss. Akad. Wiss. Pp. 456–477. doi: 10.3931/e-rara-18865.

Führ, Hartmut and Ziemowit Rzeszotnik (2015). “On biunimodular vectors for unitary matrices”. In: Linear Algebra and its Applications 484.0, pp. 86 –129. doi: 10.1016/j.laa.2015.06.019.

Furrer, Martin (2004). “Quadratic convergence for scaling of matrices”. In: ALENEX/ANALC, pp. 216–223.

Furness, K. P . (1962). “Trip forecasting”. Paper presented at a seminar on the use of computers in traffic planning. London, unpublished.

Garg, Ankit et al. (2015). A deterministic polynomial time algorithm for non-commutative rational identity testing with applications. arXiv:1511.03730v2 [cs.CC].

Georgiou, Tryphon T. and Michele Pavon (2015). “Positive contraction mappings for classical and quantum Schrödinger systems”. In: Journal of Mathematical Physics 56.3, 033301. doi: 10.1063/1.4915289.

Gittsovich, O. et al. (Nov. 2008). “Unifying several separability conditions using the covariance matrix criterion”. In: Phys. Rev. A 78 (5), p. 052319. doi: 10.1103/PhysRevA.78.052319.

Gokhale, D. V. and Solomon Kullback (1978). “The minimum discrimination information approach in analyzing categorical data”. In: Communications in Statistics - Theory and Methods 7.10, pp. 987–1005. doi: 10.1080/03610927808827687.

Golan, Amos and George Judge (1996). Econometric Methodology, Part I: Recovering information in the case of underdetermined problems and incomplete economic data”. In: Journal of Statistical Planning and Inference 49.1, pp. 127 –136. doi: 10.1016/0378-3758(95)00033-X.

Golan, Amos, George Judge, and Douglas Miller (1997). Maximum entropy econometrics: Robust estimation with limited data. Chichester (United Kingdom) John Wiley and Sons.

Goldstein, A. A. and J. F. Price (1967). “An effective algorithm for minimization”. In: Numerische Mathematik 10.3, pp. 184–189. doi: 10.1007/BF02162162.

Golitschek, Manfred, Uriel G. Rothblum, and Hans Schneider (1983). “A conforming decomposition theorem, a piecewise linear theorem of the alternative, and scalings of matrices satisfying lower and upper bounds”. In: Mathematical Programming 27.3, pp. 291–306. doi: 10.1007/BF02591905.

Good, I. J. (1963). “Maximum Entropy for Hypothesis Formulation, Especially for Multidimensional Contingency Tables”. In: The Annals of Mathematical Statistics 34.3, pp. 911–934. doi: 10.1214/aoms/1177704014.

— (1965). “The estimation of probabilities: An essay on modern Bayesian methods”. MIT Research Monograph No. 30.

Gorman, W. M. (1963). “Estimating trends in Leontief matrices: a note on Mr. Bacharach’s paper”. In: Nuffield College, Oxford.
Grad, J. (1971). “Matrix Balancing”. In: The Computer Journal 14.3, pp. 280–284. doi: 10.1093/comjnl/14.3.280.

Gurvits, Leonid (2002). Quantum Matching Theory (with new complexity theoretic, combinatorial and topological insights on the nature of the Quantum Entanglement). arXiv:0201022 [quant-ph].
— (2003). “Classical Deterministic Complexity of Edmonds’ Problem and Quantum Entanglement”. In: Proceedings of the Thirty-fifth Annual ACM Symposium on Theory of Computing. STOC ’03. New York, NY: ACM, pp. 10–19. isbn: 1-58113-674-9. doi: 10.1145/780542.780545.
— (2004). “Classical complexity and quantum entanglement”. In: Journal of Computer and System Sciences 69.3. Special Issue on [STOC] 2003, pp. 448 –484. doi: 10.1016/j.jcss.2004.06.003.

Gurvits, Leonid and Alex Samorodnitsky (2000). “A Deterministic Polynomial-time Algorithm for Approximating Mixed Discriminant and Mixed Volume”. In: Proceedings of the Thirty-second Annual ACM Symposium on Theory of Computing. STOC ’00. New York, NY: ACM, pp. 48–57. isbn: 1-58113-184-4. doi: 10.1145/335305.335311.

Harrington, D. (1973). “Some Results on Uniqueness and Existence of Constrained Matrix Problems”. PhD thesis. Faculty of Commerce, University of Birmingham.

Hershkowitz, Daniel, Uriel G. Rothblum, and Hans Schneider (1988). “Classifications of Non-negative Matrices Using Diagonal Equivalence”. In: SIAM Journal on Matrix Analysis and Applications 9.4, pp. 455–460. doi: 10.1137/0609038.

Hetyei, G. (1964). “2 x l-es téglapalapkok lefedhető idomokról”. In: Pécsi Tanárképző Főisk. Tud. Közl, pp. 351–368.

Hobby, Charles and Ronald Pyke (1965). “Doubly stochastic operators obtained from positive operators”. In: Pacific Journal of Mathematics 15.1, pp. 153–157. doi: 10.2140/pjm.1965.15.153.

Householder, Alston S. (2006). The Theory of Matrices in Numerical Analysis. First published in 1964. Dover Publications. isbn: 978-0486449722.

Hutchinson, George (2016). “On the cardinality of complex matrix scalings”. In: Special Matrices 4.1, pp. 141–150. doi: 10.1515/spma-2016-0014.

Idel, Martin (2013). “On the structure of positive maps”. MA thesis. Ludwig-Maximilian Universität München, Technische Universität München.

Idel, Martin and Michael M. Wolf (2015). “Sinkhorn normal form for unitary matrices”. In: Linear Algebra and its Applications 471, pp. 76 –84. doi: 10.1016/j.laa.2014.12.031.

Ireland, C. T. and Solomon Kullback (1968). “Contingency tables with given marginals”. In: Biometrika 55, pp. 179–189. doi: 10.1093/biomet/55.1.179.

Ivanyos, Gábor, Youming Qiao, and K. V. Subrahmanym (2015). Non-commutative Edmonds’ problem and matrix semi-invariants. arXiv:1508.00690v2 [cs.DS].

Jaynes, Edwin T. (1957). “Information theory and statistical mechanics”. In: Physical review 106.4, p. 620. doi: 10.1103/PhysRev.106.620.

Johnson, Charles R., Suzanne A. Lewis, and Donald Y. Yau (2001). “Possible line sums for a qualitative matrix”. In: Linear Algebra and its Applications 327.1, pp. 53 –60. doi: 10.1016/S0024-3795(00)00306-2.
Johnson, Charles R., Robert D. Masson, and Michael W. Trosset (2005). “On the diagonal scaling of Euclidean distance matrices to doubly stochastic matrices”. In: Linear Algebra and its Applications 397, pp. 253 –264. doi: 10.1016/j.laa.2004.10.023.

Johnson, Charles R. and Robert Reams (2009). “Scaling of symmetric matrices by positive diagonal congruence”. In: Linear and Multilinear Algebra 57.2, pp. 123–140. doi: 10.1080/03081080600872327.

Johnson, Charles R. and David P. Stanford (2000). “Patterns that allow given row and column sums”. In: Linear Algebra and its Applications 311.1, pp. 97 –105. doi: 10.1016/S0024-3795(00)00071-9.

Kalantari, Bahman (1990). “Canonical problems for quadratic programming and projective methods for their solution”. In: Proc. AMS Conference on Mathematical Problems Arising from Linear Programming, 1988, in Contemp. Math. Pp. 243–263. isbn: 1-58113-184-4.

— (1996). “A theorem of the alternative for multihomogeneous functions and its relationship to diagonal scaling of matrices”. In: Linear Algebra and its Applications 236.0, pp. 1 –24. doi: 10.1016/0024-3795(94)00162-6.

— (1998). Scaling dualities and self-concordant homogeneous programming in finite dimensional spaces. Tech. rep. DIMACS Technical Report 98-37. Department of Computer Science, Rutgers University.

— (1999). Scaling dualities and self-concordant homogeneous programming in finite dimensional spaces. Tech. rep. Technical Report LCSR-TR-359, Department of Computer Science. Department of Computer Science, Rutgers University.

— (2005). Matrix scaling dualities in convex programming. Tech. rep. Department of Computer Science, Rutgers University.

Kalantari, Bahman and M.R. Emamy-K (1997). “On linear programming and matrix scaling over the algebraic numbers”. In: Linear Algebra and its Applications 262, pp. 283 –306. doi: 10.1016/S0024-3795(97)80036-5.

Kalantari, Bahman and Leonid Khachiyan (1993). “On the rate of convergence of deterministic and randomized RAS matrix scaling algorithms”. In: Operations Research Letters 14.5, pp. 237 –244. doi: 10.1016/0167-6377(93)90087-W.

— (1996). “On the complexity of nonnegative-matrix scaling”. In: Linear Algebra and its Applications 240.0, pp. 87 –103. doi: 10.1016/0024-3795(94)00188-X.

Kalantari, Bahman et al. (2008). “On the complexity of general matrix scaling and entropy minimization via the RAS algorithm”. In: Mathematical Programming 112.2, pp. 371–401. doi: 10.1007/s10107-006-0021-4.

Karlin, S. and L. Nirenberg (1967). “On a Theorem of P. Nowosad”. In: Journal of Mathematical Analysis and Applications 17.1, pp. 61 –67. doi: 10.1016/0022-247X(67)90165-5.

Karzanov, Alexander V. and S. Thomas McCormick (1997). “Polynomial Methods for Separable Convex Optimization in Unimodular Linear Spaces with Applications”. In: SIAM Journal on Computing 26.4, pp. 1245–1275. doi: 10.1137/S0097539794263695.

Kent, Adrian, Noah Linden, and Serge Massar (Sept. 1999). “Optimal Entanglement Enhancement for Mixed States”. In: Phys. Rev. Lett. 83 (13), pp. 2656–2659. doi: 10.1103/PhysRevLett.83.2656.

Khachiyan, Leonid (1996). “Diagonal matrix scaling is NP-hard”. In: Linear Algebra and its Applications 234.0, pp. 173 –179. doi: 10.1016/0024-3795(94)00099-9.

Khachiyan, Leonid and Bahman Kalantari (1992). “Diagonal Matrix Scaling and Linear Programming”. In: SIAM Journal on Optimization 2.4, pp. 668–672. doi: 10.1137/0802034.
Kleinberg, Jon M. (Dec. 1999). “Hubs, Authorities, and Communities”. In: ACM Comput. Surv. 31.4es. doi: 10.1145/345966.345982.

Klose, Manfred, Alexander Opitz, and Norbert Schwarz (June 2004). “Sozialrechnungsmatrix für Deutschland”. In: Wirtschaft und Statistik, pp. 605–620.

Knight, Philip A. (2008). “The Sinkhorn-Knopp Algorithm: Convergence and Applications”. In: SIAM J. Matrix Anal. Appl. 30.1, pp. 261–275. doi: 10.1137/060659624.

Knight, Philip A. and Daniel Ruiz (2012). “A fast algorithm for matrix balancing”. In: IMA Journal of Numerical Analysis. doi: 10.1093/imajnum/drs019.

Knight, Philip A., Daniel Ruiz, and Bora Uçar (2014). “A Symmetry Preserving Algorithm for Matrix Scaling”. In: SIAM Journal on Matrix Analysis and Applications 35.3, pp. 931–955. doi: 10.1137/110825753.

Korzekwa, Kamil, David Jennings, and Terry Rudolph (May 2014). “Operational constraints on state-dependent formulations of quantum error-disturbance trade-off relations”. In: Phys. Rev. A 89 (5), p. 052108. doi: 10.1103/PhysRevA.89.052108.

Kruithof, R. (1937). “Telefoonverkeersrekening”. In: De Ingenieur 52, E15–E25.

Krupp, R. S. (1979). “Properties of Kruithof’s Projection Method”. In: The Bell System Technical Journal 58.2, pp. 517–538. doi: 10.1002/j.1538-7305.1979.tb02231.x.

Ku, Harry H. and Solomon Kullback (1968). “Interaction in multidimensional contingency tables: an information theoretic approach”. In: J. Res. Nat. Bur. Standards 72, pp. 159–199.

Kullback, Solomon (1959). Information Theory and Statistics. John Wiley & Sons. — (1958). “Probability Densities with Given Marginals”. In: The Annals of Mathematical Statistics 39.4, pp. 1236–1243. doi: 10.1214/aoms/1177704350.

Kullback, Solomon and M. A. Khairat (Feb. 1966). “A Note on Minimum Discrimination Information”. In: Ann. Math. Statist. 37.1, pp. 279–280. doi: 10.1214/aoms/1177699619.

Kullback, Solomon and R. A. Leibler (Mar. 1951). “On Information and Sufficiency”. In: Ann. Math. Statist. 22.1, pp. 79–86. doi: 10.1214/aoms/1177729694.

Lamond, B. and N. F. Stewart (1981). “Bregman’s balancing method”. In: Transportation Research Part B: Methodological 15.4, pp. 239–248. doi: 10.1016/0191-2615(81)90010-2.

Leinaas, Jon Magne, Jan Myrheim, and Eirik Ovrum (July 2006). “Geometrical aspects of entanglement”. In: Phys. Rev. A 74 (1), p. 012313. doi: 10.1103/PhysRevA.74.012313.

Lemmens, Bas and Roger Nussbaum (2012). Nonlinear Perron-Frobenius Theory. Cambridge University Press.

Letac, Gerard (1974). “A Unified Treatment of some Theorems on Positive Matrices”. In: Proc. Amer. Math. Soc. 43.1, pp. 11–17. doi: 10.1090/S0002-9939-1974-0338037-8.

Lewis, P. M. (1959). “Approximating probability distributions to reduce storage requirements”. In: Information and Control 2.3, pp. 214–225. doi: 10.1016/S0019-9958(59)90207-4.

Linial, Nathan, Alex Samorodnitsky, and Avi Wigderson (2000). “A Deterministic Strongly Polynomial Algorithm for Matrix Scaling and Approximate Permanents”. In: Combinatorica 20.4, pp. 545–568. doi: 10.1007/s004930070007.

Lisi, Sam (2011). Given two basis sets for a finite Hilbert space, does an unbiased vector exist? Mathematics Stack Exchange. (version: 2011-04-05). url: http://math.stackexchange.com/q/29819.

littleO, (http://math.stackexchange.com/users/40119/littleo) (2014). Please explain the intuition behind the dual problem in optimization. Mathematics Stack Exchange. url: http://math.stackexchange.com/q/624633.
Livne, Oren E. and Gene H. Golub (2004). “Scaling by Binormalization”. In: *Numerical Algorithms* 35.1, pp. 97–120. doi: 10.1023/B:NUMA.0000016606.32820.69.

London, David (1971). “On matrices with a doubly stochastic pattern”. In: *Journal of Mathematical Analysis and Applications* 34.3, pp. 648–652. doi: 10.1016/0022-247X(71)90104-1.

Lovász, László and M. D. Plummer (2009). *Matching Theory*. AMS Chelsea Publishing Series. AMS Chelsea Pub. isbn: 9780821847596.

Luo, Z. Q. and P. Tseng (1992). “On the convergence of the coordinate descent method for convex differentiable minimization”. In: *Journal of Optimization Theory and Applications* 72.1, pp. 7–35. doi: 10.1007/BF00939948.

Macgill, Sally M. (1977). “Theoretical properties of biproportional matrix adjustments”. In: *Environment and Planning A* 9.6, pp. 687–701. doi: 10.1068/a090687.

Maier, Sebastian, Petur Zachariassen, and Martin Zachariasen (2010). “Divisor-based biproportional apportionment in electoral systems: A real-life benchmark study”. In: *Management Science* 56.2, pp. 373–387. doi: 10.1287/mnsc.1090.1118.

Marcus, M. and M. Newman (1961). “The permanent of a symmetric matrix”. In: *Notices of the A.M.S.* 8.

Marshall, Albert W. and Ingram Olkin (1968). “Scaling of matrices to achieve specified row and column sums”. In: *Numerische Mathematik* 12.1, pp. 83–90. doi: 10.1007/BF02170999.

Maxfield, J. and H. Minc (1962). “A doubly stochastic matrix equivalent to a given matrix”. In: *Notices of the A.M.S.* 9.

McDougall, Robert (1999). *Entropy Theory and the RAS are friends*. GTAP Working Paper No. 06 (300).

Menon, M. V. (1967). “Reduction of a Matrix with Positive Elements to a Doubly Stochastic Matrix”. In: *Proc. Amer. Math. Soc.* 18.2, pp. 244–247. doi: 10.1090/S0002-9939-1967-0215873-6.

— (1968). “Matrix links, an extremization problem, and the reduction of a non-negative matrix to one with prescribed row and column sums”. In: *Canad. J. Math.* 20, pp. 225–232. doi: 10.4153/CJM-1968-021-9.

Menon, M. V. and Hans Schneider (1969). “The spectrum of a nonlinear operator associated with a matrix”. In: *Linear Algebra and its Applications* 2.3, pp. 321–334. doi: 10.1016/0024-3795(69)90034-2.

Moon, Todd K., Jacob H. Gunther, and Joseph J. Kupin (2009). “Sinkhorn solves sudoku”. In: *Information Theory, IEEE Transactions on* 55.4, pp. 1741–1746. doi: 10.1109/TIT.2009.2013004.

Mosteller, Frederick (1968). “Association and Estimation in Contingency Tables”. In: *Journal of the American Statistical Association* 63.321, pp. 1–28. doi: 10.1080/01621459.1968.11009219.

Murchland, J. D. (1977). “The multiproportional problem”. Manuscript JDM-263, draft 1, University College London Transport Studies Group.

— (1978). “Applications, history and properties of bi- and multi-proportional models”. Unpublished Seminar at London School of Economics, London.

Murty, Katta G. and Santosh N. Kabadi (1987). “Some NP-complete problems in quadratic and nonlinear programming”. In: *Mathematical Programming* 39.2, pp. 117–129. doi: 10.1007/BF02592948.

Nemirovski, Arkadi and Uriel Rothblum (1999). “On complexity of matrix scaling”. In: *Linear Algebra and its Applications* 302-303.0, pp. 435–460. doi: 10.1016/S0024-3795(99)00212-8.
Netanyahu, E. and M. Reichaw (1969). “A Theorem on Infinite Positive Matrices”. In: Proceedings of the American Mathematical Society 20.1, pp. 13–15. doi: 10.1090/S0002-9939-1969-0236203-1.

Nielsen, Michael and Isaac Chuang (2000). Quantum Computation and Quantum Information. Cambridge University Press.

Niemeyer, Horst F. and Alice C. Niemeyer (2008). “Apportionment methods”. In: Mathematical Social Sciences 56.2, pp. 240 –253. doi: 10.1016/j.mathsocsci.2008.03.003.

Nowosad, P. (1966). “On the integral equation $K f = 1/f$ arising in a problem in communication”. In: Journal of Mathematical Analysis and Applications 14.3, pp. 484 –492. doi: 10.1016/0022-247X(66)90008-4.

Nussbaum, Roger D. (1987). “Iterated Nonlinear Maps and Hilbert’s Projective Metric: A Summary”. In: Dynamics of Infinite Dimensional Systems. Ed. by Shui-Nee Chow and Jack K. Hale. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 231–248. isbn: 978-3-642-86458-2. doi: 10.1007/978-3-642-86458-2_23.

— (1993). “Entropy Minimization, Hilbert’s Projective Metric, and Scaling Integral Kernels”. In: Journal of Functional Analysis 115.1, pp. 45 –99. doi: 10.1006/jfan.1993.1080.

O’Leary, Dianne P. (2003). “Scaling symmetric positive definite matrices to prescribed row sums”. In: Linear Algebra and its Applications 370, pp. 185 –191. doi: 10.1016/S0024-3795(03)00387-2.

Olschowka, Markus and Arnold Neumaier (1996). “A new pivoting strategy for Gaussian elimination”. In: Linear Algebra and its Applications 240, pp. 131 –151. doi: 10.1016/0024-3795(94)00192-8.

Osborne, E. E. (1960). “On Pre-Conditioning of Matrices”. In: J. ACM 7.4, pp. 338–345. doi: 10.1145/321043.321048.

Panov, A. (1985). “On mixed discriminants connected with positive semidefinite quadratic forms”. In: Soviet Mathematics - Doklady 31.

Parlett, B. N. and T. L. Landis (1982). “Methods for scaling to doubly stochastic form”. In: Linear Algebra and its Applications 48.0, pp. 53 –79. doi: 10.1016/0024-3795(82)90099-4.

Pereira, Rajesh (2003). “Differentiators and the geometry of polynomials”. In: Journal of Mathematical Analysis and Applications 285.1, pp. 336 –348. doi: 10.1016/S0022-247X(03)00465-7.

Pereira, Rajesh and Joanna Boneng (2014). “The theory and applications of complex matrix scalings”. In: Special Matrices 2.1. doi: 10.2478/spma-2014-0007.

Perron, Oskar (1907). “Zur Theorie der Matrices”. In: Mathematische Annalen 64.2, pp. 248–263. doi: 10.1007/BDK0149896.

Plane, David A. (1982). “An information theoretic approach to estimation of migration flows”. In: Journal of Regional Science 22.4, pp. 441–456. doi: 10.1111/j.1467-9787.1982.tb00769.x.

Pretzel, Oliver (1980). “Convergence of the iterative scaling procedure for non-negative matrices”. In: J. London Math. Soc. 21.2, pp. 379,384. doi: 10.1112/jlms/s2-21.2.379.

Pukelsheim, F. and C. Schuhmacher (2004). “Das neue Zürcher Zuteilungsverfahren für Parlamentswahlen”. In: Aktuelle Juristische Praxis, Pratique Juridique Actuelle 13, pp. 505–522.

Pukelsheim, Friedrich and Bruno Simeone (2009). On the iterative proportional fitting procedure: Structure of accumulation points and L1-error analysis. Preprint. url: http://www.dss.uniroma1.it/sites/default/files/vecchie-pubblicazioni/RT_7_2009_Pukelsheim.pdf.

79
Pyatt, Graham and Jeffery I. Round (1985). *Social accounting matrices: a basis for planning*. Washington DC. The World Bank. url: http://documents.worldbank.org/curated/en/1985/09/439689/social-accounting-matrices-basis-planning.

Pyatt, Graham and E. Thorbecke (1976). *Planning Techniques for a better Future (A WEP study)*. International Labour Office. isbn: 978-9221015529.

Raghavan, T. E. S. (1984). “On pairs of multidimensional matrices”. In: *Linear Algebra and its Applications* 62.0, pp. 263–268. doi: 10.1016/0024-3795(84)90101-0.

Robillard, Pierre and Neil F. Stewart (1974). “Iterative numerical methods for trip distribution problems”. In: *Transportation Research* 8.6, pp. 575–582. doi: 10.1016/0041-1647(74)90034-3.

Rockafellar, R. T. (1997). *Convex Analysis*. Convex Analysis. Princeton University Press. isbn: 9780691015866.

Rote, Günter and Martin Zachariasen (2007). “Matrix scaling by network flow”. In: *Symposium on Discrete Algorithms: Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*. Vol. 7. 09, pp. 848–854. isbn: 9780898716245.

Rothblum, Uriel G. (1989). “Generalized scalings satisfying linear equations”. In: *Linear Algebra and its Applications* 114, pp. 765–783. doi: 10.1016/0024-3795(89)90492-8.

Rothblum, Uriel G. and Hans Schneider (1980). “Characterizations of optimal scalings of matrices”. In: *Mathematical Programming* 19.1, pp. 121–136. doi: 10.1007/BF01581636.

— (1989). “Scalings of matrices which have prespecified row sums and column sums via optimization”. In: *Linear Algebra and its Applications* 114 - 115.0. Special Issue Dedicated to Alan J. Hoffman, pp. 737 –764. doi: 10.1016/0024-3795(89)90491-6.

Rothblum, Uriel G., Hans Schneider, and Michael H. Schneider (1994). “Scaling Matrices to Prescribed Row and Column Maxima”. In: *SIAM Journal on Matrix Analysis and Applications* 15.1, pp. 1–14. doi: 10.1137/S0895479891222088.

Rothblum, Uriel G. and Stavros A. Zenios (1992). “Scalings of matrices satisfying line-product constraints and generalizations”. In: *Linear Algebra and its Applications* 175, pp. 159 –175. doi: 10.1016/0024-3795(92)90307-V.

Ruiz, Daniel (2001). *A scaling algorithm to equilibrate both rows and columns norms in matrices*. Tech. rep. RAL-TR-2001-034. Computational Science and Engineering Department, Atlas Centre, Rutherford Appleton Laboratory.

Ruschendorf, Ludger (1995). “Convergence of the iterative proportional fitting procedure”. In: *The Annals of Statistics*, pp. 1160–1174.

Samelson, Hans (1957). “On the Perron-Frobenius theorem”. In: *Michigan Math. J. 4.1*, pp. 57–59. doi: 10.1307/mmj/1028990177.

Sanz, Mikel et al. (2010). “A Quantum Version of Wielandt’s inequality”. In: *IEEE Transactions on Information Theory* 56 (9). doi: 10.1109/TIT.2010.2054552.

Schneider, H. and B. D. Saunders (1980). “Applications of the Gordan-Stiemke Theorem in Combinatorial Matrix Theory”. In: *Combinatorics* 79. Ed. by Peter L. Hammer. Vol. 9. Annals of Discrete Mathematics. Elsevier, pp. 247 –. doi: 10.1016/S0167-5060(08)70073-6.

Schneider, Hans (1977). “The concepts of irreducibility and full indecomposability of a matrix in the works of Frobenius, König and Markov”. In: *Linear Algebra and its Applications* 18.2, pp. 139–162. doi: 10.1016/0024-3795(77)90070-2.

Schneider, Michael H. (1989). “Matrix scaling, entropy minimization, and conjugate duality. I. existence conditions”. In: *Linear Algebra and its Applications* 114-115.0. Special Issue Dedicated to Alan J. Hoffman, pp. 785–813. doi: 10.1016/0024-3795(89)90493-X.
Schneider, Michael H. (1990). “Matrix scaling, entropy minimization, and conjugate duality (II): The dual problem”. In: *Mathematical Programming* 48.1-3, pp. 103–124. doi: 10.1007/BF01582253.

Schneider, Michael H. and Stavros A. Zenios (1990). “A comparative study of algorithms for matrix balancing”. In: *Operations Research* 38.3, pp. 439–455. doi: 10.1287/opre.38.3.439.

Schrader, R. (2000). “Perron-Frobenius Theory for positive maps on trace ideals”. In: *Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects*. Ed. by Robert Longo. American Mathematical Society. ISBN: 978-0-8218-2814-4.

Schrödinger, Erwin (1931). Sonderausgabe a. d. Sitz.-Ber. d. Preuß. Akad. d. Wiss., Phys.-math. Klasse. Verlag W. de Gruyter, Berlin.

Seneta, E. (2006). *Nonnegative Matrices and Markov Chains*. Springer. ISBN: 978-0-387-32792-1.

Sinkhorn, Richard (1964). “A Relationship Between Arbitrary Positive Matrices and Doubly Stochastic Matrices”. In: *Annals of Mathematical Statistics* 35.2, pp. 876–879. doi: 10.1214/aoms/1177730591.

— (1966). “A relationship between arbitrary positive matrices and stochastic matrices”. In: *Canad. J. Math.* 18, pp. 303–306. doi: 10.4153/CJM-1966-033-9.

— (1967). “Diagonal Equivalence to Matrices with Prescribed Row and Column Sums”. In: *The American Mathematical Monthly* 74.4, pp. 402–405. doi: 10.2307/2314570.

— (1972). “Continuous Dependence of $A$ in the $D_1AD_2$ Theorems”. In: *Proceedings of the American Mathematical Society* 32.2, pp. 395–398. doi: 10.1090/S0002-9939-1972-0297792-4.

— (1974). “Diagonal Equivalence to Matrices with Prescribed Row and Column Sums. II”. In: *Proceedings of the American Mathematical Society* 45.2, pp. 195–198. doi: 10.2307/2040061.

Sinkhorn, Richard and Paul Knopp (1967). “Concerning Nonnegative Matrices and Doubly Stochastic Matrices”. In: *Pacific Journal of Mathematics* 21.2, pp. 343–348. doi: 10.2140/pjm.1967.21.343.

Smith, John H. (June 1947). “Estimation of Linear Functions of Cell Proportions”. In: *The Annals of Mathematical Statistics* 18.2, pp. 231–254. doi: 10.1214/aoms/1177730440.

Soules, George W. (1991). “The rate of convergence of Sinkhorn balancing”. In: *Linear Algebra and its Applications* 150.0, pp. 3–40. doi: 10.1016/0024-3795(91)90157-R.

Stephan, Frederick F. (1942). “An Iterative Method of Adjusting Sample Frequency Tables When Expected Marginal Totals are Known”. In: *The Annals of Mathematical Statistics* 13.2, pp. 166–178. doi: doi:10.1214/aoms/1177731604.

Stone, R. (1962). “Multiple classifications in social accounting”. In: *Bulletin de l’institut International de Statistique* 39.3, pp. 215–33.

Theil, H. (1967). *Economics and information theory*. Studies in mathematical and managerial economics. North-Holland Pub. Co.

Thionet, Pierre (1961). “Sur le remplissage d’un tableau à double entrée”. In: *Journal de la société française de statistique* 102, pp. 331–345.

— (1963). “Sur certaines variantes des projections du tableau d’échanges inter-industriels”. In: *Bull. Inst. Int. Stat.* 40, pp. 119–132. ISSN: 0373-0441.

— (1964). “Note sur le remplissage d’un tableau à double entrée”. In: *Journal de la société française de statistique* 105, pp. 228–247.

Tverberg, Helge (1976). “On Sinkhorn’s representation of nonnegative matrices”. In: *Journal of Mathematical Analysis and Applications* 54.3, pp. 674–677. doi: 10.1016/0022-247X(76)90186-4.
A. Preliminaries on matrices

Sinkhorn’s theorem is closely related to irreducibility and notions connected to it. Therefore, we first recall the following characterisation of irreducible matrices.

**Proposition A.1.** Let $A \in \mathbb{R}^{n \times n}$ be nonnegative. The following are equivalent:

1. $A$ is irreducible.

2. The digraph associated to $A$ is strongly connected.

3. For each $i$ and $j$ there exists a $k$ such that $(A^k)_{ij} > 0$.

4. For any partition $I \cup J$ of $\{1, \ldots, n\}$, there exists a $j \in J$ and an $i \in I$ such that $A_{ij} \neq 0$.

An overview about these and similar properties can be found in Schneider 1977. Graph related properties are proven in Brualdi 1968.

Let us now describe a graph for any matrix: Given a nonnegative matrix $A \in \mathbb{R}^{n \times n}$ and any partition $I \cup J = \{1, \ldots, 2n\}$, let $I \cup J = V$ and $E = \{(i,j)|A_{ij} > 0\} \subseteq I \times J$ be the vertices and edges of the (bipartite) Graph $G_A := (V,E)$. 
Definition A.2. A bipartite graph $G = (V, E)$ has a perfect matching, if it contains a subgraph where the degree of any vertex is exactly one, i.e. any vertex is matched with exactly one other vertex.

Note that this definition is not dependent on the size of the entries of $A$, it only depends on whether an entry is positive or zero.

Proposition A.3 (Brualdi, Parter, and Schneider 1966 Lemma 2.3). Let $A \in \mathbb{R}^{m \times n}$ be nonnegative. The following are equivalent:

1. $A$ is fully indecomposable,
2. $PAQ$ is fully indecomposable for all permutations $P, Q$,
3. There exist permutations $P, Q$ such that $PAQ$ is irreducible and has a positive main diagonal.
4. For any $(i, j) \in E$ the edge set of the bipartite graph $G_A$ for $A$, there exists a perfect matching in $G_A$ containing this edge.

Sketch of proof. The equivalence (1) ⇔ (2) is obvious and (1) ⇔ (3) is done in Brualdi, Parter, and Schneider 1966. The direction ⇒ follows from the Frobenius-König theorem (cf. Bhatia 1996, Chapter 2). The converse direction follows from a short contradiction proof.

Finally, (1) ⇔ (4) follows essentially from Theorem 4.1.1 in Lovász and Plummer 2009, which was first observed in Hetyei 1964\textsuperscript{21}.

Since multiplication of positive diagonal matrices from the right and from the left does not change the pattern of a matrix, having a matrix that has the required row and column sums is an easy necessary condition for scalability.

Let us now consider the special case of doubly stochastic matrices in more detail. We define:

Definition A.4. Let $A \in \mathbb{R}^{n \times n}$ be a nonnegative matrix. Then $A$ has total support if it is nonzero and for every $A_{ij} > 0$ there exists a permutation $\sigma$ such that $\sigma(i) = j$ and $\prod_{k=1}^{n} A_{\sigma(k)k} \neq 0$. In other words, $A$ has total support if any nonzero element lies on a positive diagonal (Sinkhorn 1967).

Furthermore, $A$ has support, if there exists a positive diagonal, i.e. there exists an $A_{ij}$ such that for some permutation $\sigma$ with $\sigma(i) = j$ we have $\prod_{k=1}^{n} A_{\sigma(k)k} \neq 0$.

Proposition A.5. Let $A \in \mathbb{R}^{n \times n}$ be a nonnegative matrix. The following are equivalent:

1. After independent permutations of rows and columns, $A$ is a direct sum of fully indecomposable matrices.

\textsuperscript{21}In Hungarian. Reference taken from Lovász and Plummer 2009
2. $A$ has a doubly-stochastic pattern.

3. $A$ has total support

Furthermore, $A$ has support if and only if there exists a matrix $B$ with a subpattern of $A$ with total support.

**Sketch of Proof.** For $1 \iff 2$ we follow the proof in Brualdi, Parter, and Schneider 1966.

Let $A$ be doubly stochastic. Since we can permute rows and columns independently, we can assume that $A$ is of the form

$$
A := \begin{pmatrix}
A_1 & 0 & \ldots & 0 \\
A_{21} & A_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{k1} & A_{k2} & \ldots & A_k
\end{pmatrix}
$$

for some $k \in \mathbb{N}$. All $A_i$ are either $1 \times 1$ zero-matrices or fully indecomposable (otherwise iterate). Since $A$ is doubly stochastic one can quickly see that $A_{ij} = 0$ for all $i < j$. Furthermore, no $A_i$ can be zero, because this would then result in a zero-row. Hence, $A$ can be decomposed as a direct sum of fully indecomposable maps.

For $2 \iff 3$ see Sinkhorn 1967.

Finally, consider a matrix $B$ with total support. Clearly, if it is a submatrix of some other matrix $A$, then $A$ will have support, since any element $A_{ij} > 0$ which is contained in $B$ will lie on a nonzero diagonal. Conversely, if $A$ has support, setting any element $A_{ij}$ which does not lie on a positive diagonal to zero produces a matrix that has total support.

\[ \square \]

**B. Introduction to Nonlinear Perron-Frobenius theory**

The basic result underlying Perron-Frobenius theory is an old theorem from Perron 1907 and Frobenius 1912 stating:

**Theorem B.1.** Let $A \in \mathbb{R}^{n \times n}$ be a nonnegative, irreducible matrix with spectral radius $\rho(A)$. Then $\rho(A) > 0$ is a nondegenerate positive eigenvalue of $A$ with a one-dimensional eigenspace consisting of a vector $x$ with only positive components.

The theorem was later interpreted geometrically in Birkhoff 1957; Samelson 1957, where the authors noted that it follows using Hilbert’s projective metric and contraction principles. From then on, it was slowly extended to (not necessarily linear) operators, which lies at the heart of nonlinear Perron-Frobenius theory. The connection to matrix scaling became clear int the 60s to 80s and parts of each theory have developed alongside each other since. Probably the best current reference on the topic is Lemmens and Nussbaum 2012. In the following, we sketch some of the most important ideas surrounding the theory:
Recall that given a topological vector space $V$, a cone is a set $C \subset V$ such that for all $v \in C$, $\alpha v \in C$ for all $\alpha > 0$. A convex cone is a cone that contains all convex combinations. By definition, it is equivalent to say that $C$ is a convex cone if and only if $\alpha v + \beta w \in C$ for all $v, w \in C$ and all $\alpha, \beta > 0$. A cone is called solid, if it contains an interior point in the topology of the vector space and closed if it is closed in the given topology. It is called polyhedral, if it is the intersection of finitely many closed half-spaces (cf. Rockafellar 1997).

An easy way to construct a convex, solid cone is by using a partial order $\geq$. Then the set $C$ defined via $v \in C \iff v \geq 0$ is a closed convex cone (see Rockafellar 1997 for the connection between ordered vector spaces and convex cones). Given two cones $C, K$ which are defined by a partial order, we call a map $T : C \to K$ order-preserving or monotonic, if for $v \geq w$ we have $T(v) \geq T(w)$. We call it strongly order-preserving, if for every $v \geq w \in V$ we have $T(v) - T(w) \in \mathcal{V}' > 0$. Last but not least, we call $T$ homogeneous, if $T(\alpha v) = \alpha T(v)$ for all $\alpha \in \mathbb{R}$.

As is the case with classical Perron-Frobenius theory, the spectral radius is the crucial notion. For general maps between cones, there are several definitions, which turn out to be the same for most purposes, hence we restrict to one such notion:

**Definition B.2** (Lemmens and Nussbaum 2012, Chapter 5.2). Let $K$ be a solid closed cone in a finite dimensional vector space with a fixed norm $\| \cdot \|$ and $f : K \to K$ a continuous homogeneous map. Define the cone spectral radius as

$$r_K(f) := \sup \{ \limsup_{m \to \infty} \| f^m(x) \|^{1/m} \mid 0 \neq x \in K \}$$

The first crucial observation is that the spectral radius is actually attained for homogeneous order-preserving maps (which is not the case for general maps):

**Theorem B.3** (Lemmens and Nussbaum 2012, Cor. 5.4.2). Let $K$ be a solid closed cone in a finite dimensional vector space $V$. If $f : K \to K$ is a continuous, homogeneous, order-preserving map, then there exists $x \in K \setminus \{0\}$ with $f(x) = r_K(f)x$.

Note that the theorem does not tell us whether the eigenvector lies inside the cone or on its boundary. The next and very powerful theorem does also not settle existence, but if existence is known, then it assures uniqueness and convergence:

**Theorem B.4** (Lemmens and Nussbaum 2012, Thm. 6.5.1). Let $K$ be a solid closed cone in a finite dimensional vector space $V$ and let $\varphi \in K^*$ the dual cone. If $f : \text{int}(K) \to \text{int}(K)$ is a homogeneous strongly order-preserving map and there exists a $u \in \text{int}(K)$ with $\varphi(u) = 1$ such that $f(u) = ru$, then

$$\lim_{k \to \infty} \frac{f^k(x)}{\varphi(f^k(x))} = u$$

(66)

for all $x \in \text{int}(K)$.
This theorem is essentially due to the fact that the map is contractive in what is known as Hilbert’s projective metric of the cones. Once the attractiveness is established, uniqueness follows immediately, since if we had another fixed point \( v \in \text{int}(K) \), then \( f^k(v) = v \) would imply a contradiction to equation 66. Since the attractiveness can be used to estimate convergence speeds (see Franklin and Lorenz 1989), we include the relevant proposition due to Birkhoff (Birkhoff 1957).

**Definition B.5** (Lemmens and Nussbaum 2012, Chapter 2.1). Let \( K \subset V \) be a closed, convex, solid cone in some vector space, then for any \( x, y \in K \) such that \( x \leq \alpha y \) and \( y \leq \beta x \) for some \( \alpha, \beta > 0 \), define

\[
M(x/y; K) := \inf \{ \beta > 0 | x \leq \beta y \} \\
m(x/y; K) := \sup \{ \alpha > 0 | \alpha y \leq x \}
\]

Then we can define Hilbert’s projective metric as

\[
d_H(x, y, K) := \ln \left( \frac{M(x/y)}{m(x/y)} \right)
\]

We will leave out \( K \), when it is clear from the context. Furthermore, we set \( d_H(0, 0) = 0 \) and \( d_H(x, y) = \infty \) if \( d_H \) is otherwise not well-defined.

We have the following properties:

**Proposition B.6** (Lemmens and Nussbaum 2012 Proposition 2.1.1). Let \( K \subset V \) be a closed, convex, solid cone in some vector space \( V \). Then \( d_H \) satisfies:

(i) \( d_H(x, y) \geq 0 \) and \( d_H(x, y) = d_H(y, x) \) for all \( x, y \in K \).

(ii) \( d_H(x, z) \leq d_H(x, y) + d_H(y, z) \) for all \( x, y, z \in K \) such that the quantities are well-defined.

(iii) \( d_H(\alpha x, \beta y) = d_H(x, y) \) for all \( x, y \in K \) and \( \alpha, \beta > 0 \).

Note that the first two properties show that \( d_H \) is indeed a metric and the third property shows why it is called a projective metric.

**Proposition B.7** (Birkhoff 1957). Let \( K \subset V \) be a bounded, closed, convex and solid cone in some vector space \( V \). Let \( \mathcal{E} : K \to K \) be a linear map, then

\[
\gamma^{1/2}(\mathcal{E}) := \sup \left\{ \frac{d_H(\mathcal{E}(x), \mathcal{E}(y))}{d_H(x, y)} \middle| x, y \in K \right\} \leq \tanh(\Delta/4)
\]

where \( \Delta := \max \{ d_H(\mathcal{E}(x), \mathcal{E}(y)) | x, y \in K \} \).

Furthermore, \( \gamma^{1/2}(\mathcal{E} \circ \mathcal{F}) \leq \gamma^{2}(\mathcal{E}) \gamma^{2}(\mathcal{F}) \) and \( \gamma^{1/2}(\mathcal{E}^*) = \gamma^{2}(\mathcal{E}) \).

86
A proof can be found in Birkhoff 1957; Bauer 1965. The result can be extended to much more general scenarios, see also Eveson and Nussbaum 1995 and references therein. One can actually show that equality holds, i.e. \( \tanh(\Delta/4) \) is also attained, but this is not important here.

With all this machinery, we still need to prove existence of a fixed point in the interior of \( \mathcal{P} \). The general theory for proving that a fixed point lies in the interior is weak and we generally have to prove it “by hand”.

For, the Menon operator we have an additional problem: It is at first only well-defined on the interior of the cone of positive semidefinite matrices. A natural question is whether it can be extended to cover the closed cone as well. For matrices, this is covered by the following theorem:

**Theorem B.8** (Lemmens and Nussbaum 2012, theorem 5.1.5). Let \( \mathcal{C}, \mathcal{K} \) be cones and \( S : \mathcal{C} \to \mathcal{K} \) be an order-preserving, homogeneous map. If \( \mathcal{C} \) is solid and polyhedral, then there exists a continuous, order-preserving, homogeneous extension \( \overline{S} : \overline{\mathcal{C}} \to \overline{\mathcal{K}} \).

### C. Preliminaries on Positive Maps

In this section, let \( \mathcal{C}^n \subset \mathcal{M}_n \) be the cone of positive definite matrices (elements will also be written as \( A > 0 \)) with its closure \( \overline{\mathcal{C}^n} \), the cone of positive semidefinite matrices (elements will also be written as \( A \geq 0 \)). Likewise, a subscript \( 1 \) at any of the cones denotes the bounded subset of unit trace matrices. Positive maps on cones are elements form a cone themselves, the dual cone, which will be denoted by \( (\mathcal{C}^n)^* \) and \( (\overline{\mathcal{C}^n})^* \).

Let us start with irreducible maps. Many different characterizations exist, which we recall for the reader’s convenience:

**Proposition C.1**. For a positive, linear map \( T : \mathcal{M}_d \to \mathcal{M}_d \) the following properties are equivalent:

1. \( T \) is irreducible,
2. if \( P \in \mathcal{M}_d \) is a Hermitian projector such that \( T(P\mathcal{M}_dP) \subset P\mathcal{M}_dP \) then \( P \in \{0,1\} \),
3. for every nonzero \( A \geq 0 \) we have \( (\text{id} + T)^{d-1}(A) > 0 \),
4. for every nonzero \( A \geq 0 \) and every strictly positive \( t \in \mathbb{R} \) we have \( \exp(tT)(A) > 0 \),
5. There does not exist a nontrivial orthogonal projection \( P \) s.th. \( \text{tr}(T(P)(1 - P)) = 0 \).

Most of these properties are well-known. A proof can be found in Wolf 2012.

As with matrices in the original Sinkhorn theorem, irreducibility is not the right characterization to work with, since given an irreducible map \( \mathcal{E} \) and two \( X, Y > 0 \), \( Y\mathcal{E}(X,X^\dagger)Y^\dagger \) is not necessarily irreducible. Before giving a characterization of fully indecomposable maps, we will study rank non-decreasing maps.
Definition C.2 (Gurvits 2004). To every positive map $\mathcal{E} : \mathcal{M}_n \to \mathcal{M}_n$ and any unitary $U \in U(n)$, we associate the decoherence operator $\mathcal{E}_U$ via:

$$\mathcal{E}_U(X) := \sum_i \mathcal{E}(u_i u_i^\dagger) \text{tr}(X u_i u_i^\dagger)$$

(69)

where $u_i$ is the $i$-th row of $U$. Furthermore, we associate to every decoherence operator the tuple

$$A_{\mathcal{E},U} := (\mathcal{E}(u_1 u_1^\dagger), \ldots, \mathcal{E}(u_n u_n^\dagger))$$

(70)

This will be important during the proof of the Sinkhorn scaling, because every map $\mathcal{E}$ will be associated to the mixed discriminants of its decoherence operators:

Definition C.3. Let $(A_i)_i$ be an $n$-tuple with $A_i \in \mathcal{M}_n$, then

$$M(A_1, \ldots, A_n) := \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \det(x_1 A_1 + \ldots + x_n A_n)|_{x_1, \ldots, x_n = 0}$$

(71)

is called the mixed discriminant.

Then we have the following characterization of rank non-decreasing maps, which is essentially due to Gurvits 2004:

Proposition C.4. Let $\mathcal{E} : \mathcal{M}_n \to \mathcal{M}_n$ be a positive, linear map. Then the following expressions are equivalent:

(i) $\mathcal{E}$ is rank non-decreasing.

(ii) $\mathcal{E}_U$ is rank non-decreasing for any unitary $U \in U(n)$.

(iii) For any $U \in U(n)$, if $(A_i)_i := A_{\mathcal{E},U}$, then

$$\text{rank} \left( \sum_{i \in S} A_i \right) \geq |S| \quad \forall S \subseteq \{1, \ldots, n\}$$

(iv) For any $U \in U(n)$, $M(A_{\mathcal{E},U}) > 0$.

(v) $\mathcal{E}'(\cdot) := Y^\dagger \mathcal{E}(X \cdot X^\dagger) Y$ is rank non-decreasing for any $X, Y$ of full rank.

The proofs that (i), (ii), (iii) and (v) are equivalent are essentially the same as for the fully indecomposable case in C.6. It remains to show the equivalence of (v) with (i). This was done in Panov 1985.

We can define what will turn out as a measure of being indecomposable for a tuple of matrices:
Definition C.5. Let $A := (A_i)_i$ be an $n$-tuple of matrices $A_i \in \mathcal{M}_n$ and denote by $A^{ij}$ the tuple where $A_i$ is substituted by $A_j$. Then define:

$$M(A) := \min_{i \neq j} M(A^{ij})$$

(72)

the minimal mixed discriminant.

For fully decomposable maps, we have the following characterization (part of which is already present in Gurvits 2004):

Proposition C.6. Let $\mathcal{E} : \mathcal{M}_n \rightarrow \mathcal{M}_n$ be a positive, linear map. Then the following expressions are equivalent:

(i) $\mathcal{E}$ is fully indecomposable

(ii) $\mathcal{E}^*$ is fully indecomposable

(iii) For all singular, but nonzero $A \geq 0$, rank$(\mathcal{E}(A)) >$ rank $A$.

(iv) Property (iii) holds for $\mathcal{E}^*(X \cdot X^\dagger)Y^\dagger$ for every $X, Y > 0$.

(v) There do not exist nontrivial orthogonal projections $P, Q$ of the same rank such that $\text{tr}(\mathcal{E}(P)(1 - Q)) = 0$.

(vi) $\mathcal{E}_U$ is fully indecomposable for all $U \in U(n)$.

(vii) For any $U \in U(n)$, if $(A_i)_i := A_{\mathcal{E}_U}$, then

$$\text{rank} \left( \sum_{i \in S} A_i \right) > |S| \quad \forall S \subset \{1, \ldots, n\}, 0 < |S| < n$$

(viii) $\overline{M}(A_{\mathcal{E}_U}) > 0$ for all $U \in U(n)$.

Furthermore, when this is satisfied, $\mathcal{E}$ and via (v) also $\mathcal{E}^*$ map the open cone $\mathbb{C}^n$ into itself. Note also that the properties (vi) to (viii) are also equivalent for any fixed unitary.

Proof. (i) $\rightarrow$ (iii): let $\mathcal{E}$ be fully indecomposable and assume there was a nonzero $A \geq 0$, with rank$(\mathcal{E}(A)) \leq$ rank $A$. Since the kernels are vector spaces, this implies we can find a unitary matrix $U$ transforming the basis such that ker$(\mathcal{E}(A)) \supseteq U \cdot \ker A$. Thus:

$$\text{ker}(\mathcal{E}(A)) \supseteq U \cdot \ker A$$

$$\iff \text{ker}(\mathcal{E}(A)) \supseteq \ker UAU^\dagger$$

$$\iff \text{ker}(U^\dagger \mathcal{E}(A) U) \supseteq \ker A$$

The latter implies supp$(U^\dagger \mathcal{E}(A) U) \subseteq$ supp $A$. Let $P$ be the projection onto the support of $A$. By assumption, $P \neq \{0, 1\}$ since $A$ is nonzero and singular. For any positive
matrix $B$ with $BP = B$, we have $\text{supp } B \subseteq \text{supp } A$. Hence, there exists a constant $r > 0$ such that $A \geq rB$. Then $\mathcal{E}(A) \geq r\mathcal{E}(B)$ and $\text{supp}(\mathcal{E}(B)) \subseteq \text{supp}(\mathcal{E}(A))$. But this implies via linearity $\text{supp}(Q\mathcal{E}(M_n)Q) \subseteq \text{supp}(P\mathcal{E}(M_n)P)$, where $Q := UPU^\dagger$ is an orthogonal projection.

(iii) $\leftrightarrow$ (iv): Given (iii), the claim follows immediately from the fact that since $X, Y > 0$, the matrix ranks are not changed. For any nonzero and singular $A$ we have $\text{rank}(A) = \text{rank}(XAX^\dagger)$. By assumption, for any nonzero, singular $A \geq 0$ we have $\text{rank}(\mathcal{E}(A)) > \text{rank}(A)$, $\text{rank}(\mathcal{E}(XAX^\dagger)) > \text{rank}(XAX^\dagger)$ and hence

$$\text{rank}(Y\mathcal{E}(XAX^\dagger)Y^\dagger) > \text{rank}(A)$$

(iii) $\implies$ (i): Note that given $P, Q$ of the same rank such that $\mathcal{E}(P, M_nP) \subseteq Q, M_nQ$, we have in particular $\mathcal{E}(P) = QAQ$ for some $A \in M_n$. Since $Q$ is of the same rank as $P$, $\text{rank}(\mathcal{E}(P)) = \text{rank}(QAQ) \leq \text{rank } P$, which is a contradiction.

(v) $\implies$ (i): Note that if $\mathcal{E}(P, M_nP) \subseteq Q, M_nQ$, then in particular $\text{tr}(\mathcal{E}(P)(1 - Q)) = 0$ since $(1 - Q)$ is the orthogonal complement of $Q$.

(i) $\implies$ (v): Let $A \geq 0$. By positivity of $\mathcal{E}$, we have

$$0 \leq \text{tr}(\mathcal{E}(PAP)(1 - Q)) = \text{tr}(APE^\dagger(1 - Q)P)$$

$$\leq \|A\|_\infty \text{tr}(P1P^\dagger(1 - Q)) = \|A\|_\infty \text{tr}(\mathcal{E}(P)(1 - Q)) = 0$$

Hence in particular $\text{supp}(\mathcal{E}(PAP)) \subseteq \text{supp}(Q)$ and $\mathcal{E}(P, M_nP) \subseteq Q, M_nQ$.

(i) $\leftrightarrow$ (ii): This equivalence follows directly from (iv) by expressing $Q$ and $P$ in terms of the projections onto the orthogonal complements.

The remaining equivalences (i) $\leftrightarrow$ (vi),(vii),(viii) can be found in Gurvits 2004 (with proofs scattered throughout the earlier papers by the same author). We just repeat them here using our notation for the reader’s convenience.

(iii) $\implies$ (vi): By definition, $\mathcal{E}_U = \mathcal{E} \circ \mathcal{U}$ where $\mathcal{U}(X) = \sum_i \text{tr}(Xu_iu_i^\dagger)$. Obviously, $\mathcal{U}$ is doubly stochastic, hence rank non-decreasing. Therefore, if (iii) holds for $\mathcal{E}$, it must also hold for $\mathcal{E} \circ \mathcal{U}$.

(vi) $\implies$ (iii): Let $\mathcal{E}_U$ be fully indecomposable for all unitaries $\mathcal{U}$ and assume that $\text{rank}(\mathcal{E}(X)) \leq \text{rank}(X)$ for some $X \geq 0$ with $0 < \text{rank}(X) < n$. Let $\mathcal{U}$ be the unitary that diagonalizes $X$, then $\mathcal{E}_U(X) = \mathcal{E}(X)$, hence $\mathcal{E}_U$ is not fully indecomposable. This is a contradiction.

(vi) $\Leftrightarrow$ (vii): Let $\mathcal{T}_U$ be fully indecomposable. Then

$$\text{rank}(X) < \text{rank}(\mathcal{E}_U(X))$$

$$= \text{rank} \left( \sum_{i=1}^n \mathcal{E}(u_iu_i^\dagger) \text{tr}(Xu_iu_i^\dagger) \right)$$

$$= \text{rank} \left( \sum_{1 \leq i < n, \text{tr}(Xu_iu_i^\dagger) \neq 0} \mathcal{E}(u_iu_i^\dagger) \right)$$
Note that if \( S := \{i | \text{tr}(Xu_iu_i^\dagger)\} \), then \( \text{rank}(X) \leq |S| \) and hence follows the claim. For the other direction, we can use the same idea.

(vii) \( \leftrightarrow \) (viii): Let \( A := A_E, U \) for all unitary \( U \) fulfill (vii). Define \( A^{ij} \) as the tuple where the \( i \)-th element is replaced by the \( j \)-th. Note that

\[
\text{rank} \left( \sum_{k \in S} A_{ik} \right) \geq \text{rank} \left( \sum_{k \in S \setminus \{j\}} A_{ik} \right) \geq |S|
\]

for any \( S \subset \{1, \ldots, n\} \), where the last inequality follows from the fact that \( E \) is fully indecomposable by assumption. Hence, from the proposition C.4 we know that the mixed discriminant of \( A^{ij} \) cannot vanish, i.e. \( M(A^{ij}) > 0 \). Minimizing over \( i \neq j \) and the compact \( U(n) \) gives \( \tilde{M}(A_E, U) > 0 \).

Conversely, let \( A := A_E, U \) not fulfill (vii) for some unitary \( U \), i.e. for some \( S \subset \{1, \ldots, n\} \) with \( 0 < S < n \) we have \( \text{rank} \left( \sum_{k \in S} A_{ik} \right) \leq |S| \). Let \( i \in S, j \notin S \), then for the tuple \( A^{(ij)} \) as before, we have:

\[
\text{rank} \left( \sum_{k \in S \cup \{j\}} A_{ik} \right) = \text{rank} \left( \sum_{k \in S} A_{ik} \right) < |S| + 1 = |S \cup \{j\}|
\]

But then, by proposition C.4, \( M(A^{ij}) = 0 \) and hence also \( \tilde{M} = 0 \).

This proposition shows in particular that any fully indecomposable map is primitive: For any unit trace \( \rho \geq 0, \mathcal{E}^d(\rho) > 0 \). Note that the converse might not be true. By the characterization of primitive maps (Sanz et al. 2010, Theorem 6.7), this implies that each fully indecomposable map has only one fixed point.

**Lemma C.7.** If \( T \) is a doubly-stochastic positive linear map, then there exists a unitary matrix \( U \) such that \( UT(\cdot)U^\dagger \) admits a set of orthogonal projections \( \{P_i\} \), such that \( \sum_i P_i = 1 \), \( P_iP_j = \delta_{ij}P_i \) and \( UT(P_iM_dP_i)U^\dagger \subseteq P_iM_dP_i \). Furthermore, the restriction of \( UT(\cdot)U^\dagger \) to \( P_iM_dP_i \) is fully indecomposable for every \( i \).

**Proof.** Note that for an arbitrary unitary \( U > 0 \) the maps \( T(U(\cdot)U^\dagger) \) and \( UT(\cdot)U^\dagger \) are still doubly-stochastic. Let \( P, Q \) be a nontrivial Hermitian projector decomposing \( T \), i.e. \( \text{tr}(T(P)(1 - Q)) = 0 \) by proposition C.6 (if no such projector exists, we are finished). Then we have:

\[
0 = \text{tr}(T(P)(1 - Q)) = \text{tr}(PT^*(1 - Q)) = \text{tr}(P) - \text{tr}(PT^*(Q)) = \text{tr}(Q) - \text{tr}(QT(P)) = \text{tr}(Q(T(1 - P))
\]

where we used that \( T \) is doubly-stochastic in the second and last equality and in between we only used the cyclicity and linearity of the trace as well as the fact that \( P \)
and \( Q \) have equal rank and thus their traces equal. This means that if \( P \) reduces \( T \), then also \( 1 - Q \) reduces \( T \), i.e.

\[
T(P, M_n P) \subset Q, M_n Q
\]

\[
\Rightarrow \ T((1 - P), M_n (1 - P)) \subset (1 - Q), M_n (1 - Q)
\]

Since the two projections \( P, Q \) are of the same rank, there exists a unitary matrix \( U \) such that \( Q = UPU^\dagger \). This implies that \( T'(\cdot) = UT(\cdot)U^\dagger \) is reducible by \( P \) and \( (1 - P) \), which implies that \( T' \) is a direct sum of maps defined on \( P, M_n P \) and \( (1 - P), M_n (1 - P) \).

We obtain these maps by setting

\[
T'_1 := T'(P \cdot P)|_{P, M_n P}
\]

\[
T'_2 := T((1 - P) \cdot (1 - P))|_{(1 - P), M_n (1 - P)}
\]

By construction, \( P, (1 - P) \) are the identities on the respective subspaces and the maps are therefore doubly stochastic, i.e. \( T'_1(1, P, M_n P) = T(P) = P = 1, P, M_n P \) (and for \( T'_2 \) equivalently).

If the restricted maps are not fully indecomposable, we can iterate the procedure, thereby going over to \( T''(\cdot) = (U_1 \oplus U_2)T'(\cdot)(U_1 \oplus U_2) \) and so forth, which will terminate after finitely many steps, since the ranks of the projections involved have to decrease, thus giving a map \( \tilde{T}(\cdot) = \tilde{U}T(\cdot)\tilde{U}^\dagger \), which admits the stated decomposition.

\[\square\]

D. Gurvits' proof of scaling and approximate scaling

This appendix provides the details of Gurvits' approach, hence it does not contain original material. For easier readability, we repeat all Lemmata.

D.1. Approximate scalability

We need a way to study scalability:

**Definition D.1.** Let \( C_1, C_2 \in M_n \) and \( E : M_n \to M_n \) a positive, linear map. Then we define a **locally scalable functional** to be a map \( \varphi \in \mathcal{C}^d_+ \) such that

\[
\varphi(C_1 E(C_2^\dagger \cdot C_2) C_1^\dagger) = \det(C_1 C_1^\dagger) \det(C_2 C_2^\dagger) \varphi(E)
\]  

(73)

A locally scalable functional will be called **bounded**, if \( |\varphi(E)| \leq f(\text{tr}(E(1))) \) for some function \( f \).

Locally bounded functionals are the right tools to study scalability:

**Proposition D.2.** Let \( E : M_n \to M_n \) be a positive, linear map. Given a bounded locally scalable functional \( \varphi \) such that \( \varphi(E) \neq 0 \), the Sinkhorn-iteration procedure converges:

\[
DS(E_n) \to 0 \quad n \to \infty
\]  

(74)
Proof. We follow Gurvits 2004. Recall the definitions of the Sinkhorn iteration in equations (41)-(42). Because of property (73), we have

\[
\phi(\mathcal{E}_{i+1}) = a(i) \phi(\mathcal{E}_i)
\]

\[
a(i) = \begin{cases} 
\det(\mathcal{E}_i^*(1))^{-1} & \text{if } i \text{ odd} \\
\det(\mathcal{E}_i(1))^{-1} & \text{if } i \text{ even}
\end{cases}
\]

Let \( i \) be even. Note that \( \mathcal{E}_i \) is trace-preserving for \( i \) even, hence \( \text{tr}(\mathcal{E}_i(1)) = n \). Let \( s_j^{(i)} \) be the singular values of \( \mathcal{E}_i(1) \) and observe:

\[
|\det(\mathcal{E}_i(1))| = \prod_{j=1}^{n} s_j^{(i)} \leq \frac{1}{n} \sum_{j=1}^{n} s_j^{(i)} = \frac{1}{n} \text{tr}(\mathcal{E}_i(1)) = 1
\]

(75)

using the arithmetic-geometric mean inequality (AGM). Similarly, for \( i \) odd, \( \mathcal{E}_i \) is unital, hence \( \text{tr}(\mathcal{E}_i^*(1)) = n \) and we can use the AGM inequality again to obtain that \( a(i) \geq 0 \) for all \( i \geq 0 \) and therefore

\[
|\phi(\mathcal{E}_{i+1})| \geq |\phi(\mathcal{E}_i)|
\]

and thus, as \( \phi \) was assumed to be bounded, \( |\phi(\mathcal{E}_i)| \) converges to some value \( c \leq f(\text{tr}(\mathcal{E}(1))) \).

It remains to prove that for \( |\phi(\mathcal{E})| \neq 0 \), \( \text{DS}(\mathcal{E}_i) \) converges to zero for \( i \to \infty \). The idea is of course that if \( |\phi(\mathcal{E})| \neq 0 \), then \( |a(i)| \) converges to one and thus \( \mathcal{E}_i(1) \) and \( \mathcal{E}_i^*(1) \) converge to \( 1 \).

To make this more formal, since \( |\phi(\mathcal{E})| \) converges, for all \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for all \( d \geq N \):

\[
||\phi(\mathcal{E}_d)| - |\phi(\mathcal{E}_{d+1})| \leq \varepsilon
\]

\[
\iff |\phi(\mathcal{E}_d)| - \frac{1}{|a(i)|} |\phi(\mathcal{E}_d)| \leq \varepsilon
\]

\[
\iff |a(i)| \geq \frac{1}{1 + \varepsilon |\phi(\mathcal{E}_d)|^{-1}} \geq \frac{1}{1 + \varepsilon |\phi(\mathcal{E})|^{-1}}
\]

where we used that \( |\phi(\mathcal{E}_d)| \) increases monotonically in the last inequality.

Let us now only consider \( i \) even. Then we have just seen that

\[
\frac{1}{1 + \varepsilon |\phi(\mathcal{E})|^{-1}} \leq \det(T_i(1)) \leq 1
\]

hence, for \( i \geq N \) even, we have:

\[
\text{DS}(\mathcal{E}_i) = \text{tr}((\mathcal{E}_i(1) - 1)^2) = \sum_{j=1}^{n} (s_j^{(i)} - 1)^2
\]
where the $s_{ij}^{(i)}$ are the singular values of $E_i(1)$. If we can upper bound the last quantity by $\tilde{\varepsilon}(\varepsilon)$, we are done. This is an exercise in using logarithms:

Since $E_i$ is trace-preserving as $i$ is even, $s_{ij}^{(i)} \leq d$ for all $i$. If we set $\alpha := \frac{(n-1)-\ln(n)}{(n-1)^2}$, then by strict concavity of the logarithm,

$$\ln(x) \leq (x-1) - \alpha(x-1)^2 \quad x \leq d$$

since $\ln(d) = (x-1) - \alpha(x-1)^2$ and $\ln(1) = 0$. But then:

$$0 \leq \sum_{j=1}^{n}(s_{ij}^{(i)} - 1)^2 \leq \sum_{j=1}^{n}\left(\frac{s_{ij}^{(i)} - 1}{\alpha} - \frac{\ln(s_{ij}^{(i)})}{\alpha}\right)$$

$$= -\frac{n}{\alpha} \ln\left(\prod_{i=1}^{n} s_{ij}^{(i)}\right) \leq -\frac{1}{\alpha} \ln(1 - \varepsilon)$$

$$\leq \frac{\varepsilon}{\alpha}$$

where we used that $\sum_{j=1}^{n}s_{ij}^{(i)} = \text{tr}(E_i(1)) = n$. But $\frac{\varepsilon}{\alpha} \to 0$ for $i \to \infty$.

Exchanging $E_i$ with $E_i^*$ gives the same reasoning for odd $i$. In total, we get that for any $\varepsilon > 0$ exists an $N \in \mathbb{N}$ such that for all $d \geq n$

$$0 \leq DS(E_d) \leq \frac{\varepsilon}{\alpha}$$

hence $DS(E_i) \to 0$ for $i \to \infty$. □

**Lemma D.3.** Cap is a bounded locally scalable functional.

**Proof.** Note that for

$$\inf\{|\det(C_i^*E_iC_1C_i^*)C_2|X > 0, \det(X) = 1\}$$

$$= \inf\{\det(C_i^*C_2) \det(C_2) \det(E_iC_1C_i^*)|X > 0, \det(X) = 1\}$$

$$= \det(C_i^*C_2) \inf\{|\det(E_iC_1C_i^*)|X > 0, \det(X) = 1\}$$

$$= \det(C_i^*C_2) \inf\{|\det(E_iX)|X > 0, \det(X) = \det(C_i) \det(C_i^*) | \det(X) = 1\}$$

$$= \det(C_i^*C_2) \inf\{|\det(E_iX)|X > 0, \det(X) = 1\}$$

hence Cap is a locally scalable functional. Via the AGM inequality, we have

$$0 \leq \text{Cap}(E) \leq \det(E(1)) \leq \left(\frac{\text{tr}(E(1))}{n}\right)^{\frac{1}{n}}$$

hence Cap is bounded. □
This gives a proof of Lemma 9.8.

**Lemma D.4** (Lemma 9.9 of the main text). Let $\mathcal{E} : \mathcal{M}_n \to \mathcal{M}_n$ be a positive, linear map and $U \in U(n)$ a fixed unitary. Then defining

$$\text{Cap}(A_{\mathcal{E},U}) := \inf \left\{ \det \left( \sum_i \mathcal{E}(u_i u_i^\dagger) \gamma_i \right) \mid \gamma_i > 0, \prod_i \gamma_i = 1 \right\}$$

where $u_i$ are once again the rows of $U$, we have the following properties:

1. Using the mixed discriminant $M$, we have
   $$M(A_{\mathcal{E},U}) \leq \text{Cap}(A_{\mathcal{E},U}) \leq \frac{n^n}{n!} M(A_{\mathcal{E},U})$$

2. $\inf_{U \in U(n)} \text{Cap}(A_{\mathcal{E},U}) = \text{Cap}(\mathcal{E})$

**Proof.** The first part of the lemma is one of the main results of Gurvits and Samorodnitsky 2002. Since the proof is long due to many technicalities, we leave it out here.

The second part gives the relation between the two capacities. Let $\{X_d\}_d$ with $\det(X_d) = 1$ and $X_d > 0$ be such that $\det(\mathcal{E}(X_d)) \to \text{Cap}(\mathcal{E})$, $d \to \infty$. Then there exist unitaries $U_d \in U(n)$ such that $U_d X_d U_d^\dagger$ is diagonal with diagonal entries $\lambda_i^{(d)}$. By construction,

$$\det(\sum_{1 \leq i \leq n} \mathcal{E}((u_d)_i (u_d)_i^\dagger) \lambda_i^{(d)}) = \det(\mathcal{E}(X_d))$$

where $(u_d)$ are again the columns of $U_d$. Hence

$$\inf_{U \in U(n)} \text{Cap}(A_{\mathcal{E},U}) \leq \text{Cap}(\mathcal{E})$$

Likewise, we can construct a sequence of $U_d$ such that $\text{Cap}(A_{\mathcal{E},U_d})$ converges to the infimum and we can construct a sequence $(\gamma_{(k)}^{(d)})_k$ with $(\gamma_{(k)}^{(d)})_i > 0$ for each $U_d$ such that

$$\det(\sum_{i=1}^n \mathcal{E}((u_d)_i (u_d)_i^\dagger) \gamma_{(k)}^{(d)}_{(i)}) \to \text{Cap}(A_{\mathcal{E},U_d}) \quad \text{for } k \to \infty$$

Taking the diagonal sequence $\gamma_{(d)}^{(d)}$ we obtain a sequence converging to $\inf \text{Cap}(A_{\mathcal{E},U})$. Finally, define $X_k = U_k \text{diag}(\gamma_{(k)}^{(d)}) U_k^\dagger$, then $X_k > 0$ and

$$\det(\mathcal{E}(X_k)) = \det(\sum_{i=1}^n \mathcal{E}((u_d)_i (u_d)_i^\dagger) \gamma_{(k)}^{(d)}_{(i)})$$

and hence

$$\text{Cap}(\mathcal{E}) \leq \inf_{U \in U(n)} \text{Cap}(A_{\mathcal{E},U})$$

after taking the limit $k \to \infty$. \qed
Finally, we can write down the Operator Sinkhorn theorem (Theorem 9.5 in the main text):

**Theorem D.5** (Approximate Operator Sinkhorn Theorem, Gurvits 2004 Theorem 4.6). Let $\mathcal{E} : \mathcal{M}_n \to \mathcal{M}_n$ be a positive, linear map. Then $\mathcal{E}$ is $\varepsilon$-scalable for all $\varepsilon > 0$ iff $\mathcal{E}$ is rank non-decreasing.

**Proof.** We mostly need to combine the lemmas. By lemma D.3, the capacity is a bounded, locally scalable functional, which implies by proposition D.2 that $DS(\mathcal{E}_i)$ converges, if $\text{Cap}(\mathcal{E}) > 0$. Now, by lemma 9.9,

$$\text{Cap}(\mathcal{E}) = \inf\{\text{Cap}(A_{\mathcal{E},U}) | U \in U(n)\}$$

Since $U(n)$ is compact, it suffices to show that for every $U$, $\text{Cap}(A_{\mathcal{E},U}) > 0$. Again, by lemma 9.9,

$$\text{Cap}(A_{\mathcal{E},U}) \geq M(A_{\mathcal{E},U})$$

but $M(A_{\mathcal{E},U}) > 0$ for every $U$ if and only if $\mathcal{E}$ is rank non-decreasing by proposition C.4. Hence, $DS(\mathcal{E}_i)$ converges for rank non-decreasing maps.

Now suppose that $\mathcal{E}$ is a positive map such that in the Sinkhorn iteration, $DS(\mathcal{E}_i)$ converges. Then, for some $i \in \mathbb{N}$, $DS(\mathcal{E}_i) < \frac{1}{n}$. We claim that then $\mathcal{E}_i$ is rank non-decreasing and by consequence, also $\mathcal{E}$ is rank non-decreasing via proposition C.4.

To see this, assume $\mathcal{E}(\mathbf{1}) = 1$ and $\mathcal{E}^*(\mathbf{1}) = 1 + \mathbf{E}$, where $\mathbf{E}$ is Hermitian and $\text{tr}(\mathbf{E}^2) \leq 1/n$. We can do this, because this is exactly what $\mathcal{E}_i$ looks like for $i$ big enough such that $DS(\mathcal{E}_i) < \frac{1}{n}$ and $i$ is odd. Given a matrix $U \in U(n)$ and the corresponding $A := A_{\mathcal{E},U}$, we have that

$$\sum_{i=1}^{n} A_i = \mathcal{E}(\mathbf{1}) = 1$$

Likewise, for every $i$:

$$\text{tr}(A_i) = \text{tr}(A_{i,\mathbf{1}}) = \text{tr}(u_i u_i^\dagger T^*(\mathbf{1})) = 1 + \text{tr}(u_i u_i^\dagger E) =: 1 + \delta_i \quad (76)$$

But by assumption,

$$\sum_{i=1}^{n} |\delta_i|^2 \leq \sum_{i,j=1}^{n} |\text{tr}(u_i u_j^\dagger E)|^2$$

$$= \sum_{i,j=1}^{n} \langle u_i |E| u_j \rangle \langle u_j |E^\dagger| u_i \rangle$$

$$= \text{tr}(E^2) \leq \frac{1}{n} \quad (77)$$
Now, suppose that $E$ is not rank non-decreasing. Then, by proposition A.5 (vii), there is a $U$ such that $A_{E,U}$ fulfills
\[ \text{rank} \left( \sum_{i=1}^{k} A_i \right) < k \]
for some $0 < k < n$. Note that, since $E$ is positive, $A_i \geq 0$, hence $H := \sum_{i=1}^{k} A_i$ fulfills $0 \leq H \leq 1$. As rank$(H) \leq k - 1$, we have tr$(H) \leq k - 1$. From equation (76), we obtain
\[ \text{tr}(H) = \sum_{i=1}^{k} A_i = k + \sum_{i=1}^{k} \delta_i \]
Using equation (77), by the Cauchy Schwarz inequality,
\[ \sum_{i=1}^{k} |\delta_i| \leq \sqrt{k/n} < 1 \]
which contradicts tr$(H) \leq k - 1$, hence $E$ must be rank non-decreasing.

D.2. Exact scalability

**Lemma D.6** (Lemma 9.10 of the main text). Let $E : M_d \to M_d$ be a positive map. Then $E$ is scalable to a doubly-stochastic map if and only if Cap$(E) > 0$ and the capacity can be achieved.

**Proof.** Suppose there exists $C > 0$ with det$(E(C)) = \text{Cap}(E)$. The Lagrangian of the capacity is
\[ \mathcal{L}(X) := \ln(\text{det}(E(X))) + \lambda \ln(\text{det}(X)) \]
with the Lagrangian multiplier $\lambda \in \mathbb{R}$. Therefore, the minimum fulfills
\[ \nabla \ln(\text{det}(E(X)))|_{X=C} = (-\lambda) \nabla \ln(\text{det}(X))|_{X=C} \quad (78) \]
We claim that the conditions are equivalent to
\[ E^*((E(C))^{-1})^{-1} = C^{-1} \quad (79) \]
Let $E_{ij}$ be the usual matrix unit, then
\[ (\nabla \ln(\text{det}(E(X)))|_{X=C})_{jk} = \frac{\partial}{\partial E_{jk}} \ln \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} E(C)_{\sigma(i)} \right) \]
\[ = \frac{1}{\text{det}(E(C))} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \frac{\partial}{\partial E_{jk}} \prod_{i=1}^{n} E(C)_{\sigma(i)}. \]
Noting that
\[ \partial E_{jk} \mathcal{E}(C)_{(i)} = \text{tr}(E_{(i)}^t \partial \mathcal{E}(C) \partial E_{jk}) = \text{tr}(\mathcal{E}^*(E_{(i)})E_{jk}) = \mathcal{E}^*(E_{(i)}E_{jk}) \]
we have
\[
(\nabla \ln (\det(\mathcal{E}(X))))_{X=C} = \frac{1}{\det(\mathcal{E}(C))} \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) \sum_{l=1}^n \mathcal{E}^*(E_{(l)}E_{jk}) \prod_{i \neq l} \mathcal{E}(C)_{(i)}
\]
\[
= \mathcal{E}^*\left( \frac{1}{\det(\mathcal{E}(C))} \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) \sum_{l=1}^n E_{(l)} \prod_{i \neq l} \mathcal{E}(C)_{(i)} \right)_{jk}
\]
\[
= \mathcal{E}^*\left( \frac{1}{\det(\mathcal{E}(C))} \sum_{m,n=1}^n \left( \sum_{\sigma(m) = n \in \mathcal{S}_n} \text{sgn}(\sigma)(-1)^{n-m} \prod_{i \neq m} \mathcal{E}(C)_{(i)} \right) E_{mn} \right)_{jk}
\]
where in the last step we use Cramer’s rule. For \( \mathcal{E} = \text{id} \) we obtain the right hand side of equation (79) from equation (78), hence follows the claim. It now follows from Lemma 9.14 that any \( C \) fulfilling Equation (79) defines a scaling.

Conversely, suppose \( \tilde{\mathcal{E}}(\cdot) = C_1 \mathcal{E}(C_2^t C_2)C_1^t \) is a doubly stochastic map. Since \( \tilde{\mathcal{E}} \) is unital, the eigenvalues of \( \tilde{\mathcal{E}}(X) \) are majorized by the eigenvalues of \( X \) (cf. Wolf 2012 Theorem 8.8). Majorization stays invariant under strictly increasing functions (cf. Bhatia 1996, Chapter 1), hence we have \( (\lambda_i \text{ being the eigenvalues of } X \text{ and } \lambda_i^\tilde{\mathcal{E}} \text{ the eigenvalues of } \tilde{\mathcal{E}}(X)) \):

\[
\sum_i -\ln(\lambda_i^\tilde{\mathcal{E}}) \leq \sum_i -\ln(\lambda_i)
\]
which is equivalent to \( \det(\tilde{\mathcal{E}}(X)) \geq \det(X) \). Hence, a doubly stochastic map is in particular determinant increasing. Obviously, equality is attained at \( X = 1 \). But then:

\[
\det(\tilde{\mathcal{E}}(X)) = |\det(C_1)|^{-2} |\det(C_2)|^{-2} \det(\tilde{\mathcal{E}}(X)) \geq |\det(C_1)|^{-2} |\det(C_2)|^{-2} \det(X).
\]
A quick calculation shows that \( X = C_2^t C_2 / \det(C_2^t C_2)^{1/n} \) attains equality in Equation (81). This then necessarily minimises the capacity.

**Lemma D.7** (Lemma 9.11 of the main text). Let \( \mathcal{E} : \mathcal{M}_n \to \mathcal{M}_n \) be a positive, linear map and given \( U \in U(n) \), let \( A = A_{\mathcal{E},U} \). Then

1. \( f_A \) is convex on \( \mathbb{R}^n \).

2. If \( \mathcal{E} \) is fully indecomposable, then \( f_A \) is strictly convex on \( \{ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n | \sum_i \xi_i = 0 \} \).

98
Proof. We follow the proof of Gurvits and Samorodnitsky 2002. Given a tuple $A$ of positive definite matrices, one can show (Bapat 1989):
\[
\det(e^{\xi_1}A_1 + \ldots + e^{\xi_n}A_n) = \sum_{r \in P_n} t_r e^{(\xi,r)}
\] (82)
where $(\cdot, \cdot)$ denotes the general inner product, $P_n$ is the set of $n$-tuples of integers $r_i \geq 0$ such that $\sum r_i = n$ and
\[
t_r := \frac{1}{r_1! \ldots r_n!} M(A_{r_1}, \ldots, A_{r_n})
\] (83)
This implies that we can rewrite $f_A$:
\[
f_A(\xi_1, \ldots, \xi_n) = \ln \det(e^{\xi_1}A_1 + \ldots + e^{\xi_n}) = \ln \left( \sum_{r \in P_n} t_r e^{(\xi,r)} \right)
\]
It is well known that for positive matrices this is a convex function, but let us follow the proof of Gurvits and Samorodnitsky 2002 here.

Let $f := \ln g$. We need to prove that $\nabla^2 f$, the Hessian, is positive (semi)definite. By definition, $\nabla^2 f = \frac{1}{g^2}(g(\nabla^2 g) - (\nabla g)(\nabla g)^{tr})$, hence it is sufficient that $g(\nabla^2 g) \geq (\nabla g)(\nabla g)^{tr}$.

Note that for any $v \in \mathbb{R}^n$ we have $\nabla e^{(\xi,v)} = e^{(\xi,v)} \cdot v$ and $\nabla^2 e^{(\xi,v)} = e^{(\xi,v)} vv^{tr}$, where $vv^{tr}$ is positive definite. Therefore:
\[
g(\nabla^2 g) - (\nabla g)(\nabla g)^{tr} = \sum_{r \in P_n} t_r e^{(\xi,r)} \cdot \sum_{s \in P_n} t_s e^{(\xi,s)} ss^{tr} - \sum_{r,s \in P_n} t_r t_s e^{(\xi,r+s)} rs^{tr}
\]
\[
= \frac{1}{2} \sum_{r,s \in P_n} t_r t_s e^{(\xi,r+s)} (r-s)(r-s)^{tr} \geq 0
\]
hence the Hessian of $f$ is positive semi-definite and therefore $f$ is convex.

Now, assume that $E$ is fully indecomposable, hence the tuple $A := A_{E,U}$ fulfills proposition A.5 (vii) and (viii) for all $U \in U(n)$. In particular, if $A^{ij}$ is the tuple $A$ with the $j$-th entry being replaced by the $i$-th. entry. Then $M(A^{ij}) > 0$ in particular. Note that $M(A^{ij}) = 2t_{ij}$ by equation (83), where
\[
(r_{ij})_k := \begin{cases} 2 & k = i \\ 0 & k = j \\ 1 & \text{else} \end{cases}
\]
Then,
\[
\nabla^2 f \geq \frac{1}{g^2} \sum_{r,s \in P_n} t_r t_s e^{(\xi,r+s)} (r-s)(r-s)^{tr}
\]
\[
\begin{align*}
&\geq \frac{1}{8S^2} \sum_{i \neq j, k \neq l} M(A_{ij})M(A_{kl})\xi(r_{ij} + s_{kl})(r_{ij} - r_{kl})(r_{ij} - r_{kl})^t \\
&\geq \frac{cM^2}{8S^2} \sum_{i \neq j, k \neq l} (r_{ij} - r_{kl})(r_{ij} - r_{kl})^t
\end{align*}
\]

where \(c := \min_{i \neq j, k \neq l} \xi(r_{ij} - r_{kl})\) and \(M := \min_{i \neq j} M(A_{ij}) > 0\) by proposition A.5 (viii).

We only need to consider \(\sum_{i \neq j, k \neq l} (r_{ij} - r_{kl})(r_{ij} - r_{kl})^t =: S\) and show that this is a positive definite matrix on the hyperplane \(H\). Using the usual matrix units \(E_{mn}\) we can write:

\[
S := \sum_{i \neq j, k \neq l} (E_{ii} + E_{jj} + E_{kk} + E_{ll} + 2(E_{il} + E_{jk} - E_{ik} - E_{jl} - E_{kl}))
\]

We find that \(S_{ii} = (n - 1)(n - 2)(n - 3)\), since only the first four summands contribute to the diagonal terms. For the off-diagonal terms, note that in \(2(E_{ii} + E_{jk} - E_{ik} - E_{jl} - E_{kl})\), all unordered combinations of \(i, j, k, l\) occur, twice with a positive sign and four times with a negative. Hence we obtain \((n - 2) \cdot (n - 3)\) terms with either \(E_{ij}\) or \(E_{ji}\) that are not cancelled by other terms and therefore \(S_{ij} = -(n - 2)(n - 3)\). In short:

\[
S = (n - 1)(n - 2)(n - 3) \left( \begin{array}{cccc}
1 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\
\frac{1}{n-1} & 1 & \cdots & \frac{1}{n-1} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{1}{n-1} & \cdots & \frac{1}{n-1} & 1
\end{array} \right)
\]

Note that the image of \(S\) is just the hyperplane \(H\) and it is easy to see that \(S\) is a multiple of the projection onto the hyperplane \(S\). Therefore, \(\nabla^2 f\) is strictly convex on \(H\).

Finally, we obtain the theorem:

**Lemma D.8** (Lemma 9.12 of the main text). Let \(E : \mathcal{M}_n \to \mathcal{M}_n\) be a positive, linear map. If \(E\) is fully indecomposable, there exists a unique scaling of \(E\) to a doubly stochastic map.

**Proof.** Recall that one can show (Bapat 1989):

\[
\det(e^{\xi_1 A_1} + \cdots + e^{\xi_n A_n}) = \sum_{r \in P_n} t_r e^{\xi r} \tag{84}
\]

where \((\cdot, \cdot)\) denotes the general inner product, \(P_n\) is the set of \(n\)-tuples of integers \(r_i \geq 0\) such that \(\sum_i r_i = n\) and \(t_r := \frac{1}{r_1! \cdots r_n!} M(A_{1r_1}, \ldots, A_{nr_n})\).

Suppose \(X \geq 0\), \(\det(X) = 1\) and \(E\) is fully indecomposable. Let \(U \in \mathcal{U}(n)\) diagonalize \(X\) with eigenvalues \(\gamma_i = e^{\psi_i}\). Assume the eigenvalues are ordered \(\gamma_1 \geq \cdots \geq \gamma_n\).
Observe that then $\det(\mathcal{E}(X)) \leq \det(\mathcal{E}(\mathbb{1}))$ is equivalent to say that $f_A(\xi) \leq f_A(0)$, where $A = A_{\mathcal{E},U}$. We know:

$$\det(A_1 + \ldots + A_n) \geq \det(\gamma_1 A_1 + \ldots + \gamma_n A_n)$$

$$= \sum_{r \in R_n} t_r e^{(\xi,r)} \geq \frac{1}{2} \sum_{i \neq j} M_{ij} e^{(\xi,r_{ij})}$$

where we use that certainly for all $i \neq j \in \{1, \ldots, n\}$, $r_{ij} := r_k = 1$ for all $k \neq i,j$ and $r_i = 2, r_j = 0$ is a valid $n$-tuple where the coefficient $t_r = \frac{1}{2} M^{ij}$. Since all the terms in the sum of equation 84 are positive, we can just leave out all other $r$. By definition, $M(\mathcal{E}) \leq M^{ij}$ for every $A$, hence:

$$\det(A_1 + \ldots + A_n) \geq \frac{1}{2} M(\mathcal{E}) \sum_{i \neq j} e^{(\xi,r_{ij})}$$

$$\geq \frac{1}{2} M(\mathcal{E}) e^{\max_{i \neq j}(\xi_i - \xi_j)}$$

$$\geq \frac{1}{2} \frac{\gamma_1}{\gamma_n}$$

where we used that $(\xi, r_{ij}) = \sum_{k \neq j} \xi_k + \xi_i = \xi_i - \xi_j$ since $\sum_i \xi_i = 0$. Since $\det(A_1 + \ldots + \ldots) = \det(\mathcal{E}(\mathbb{1}))$, we have

$$\frac{\gamma_1}{\gamma_n} \leq \frac{2 \det(\mathcal{T}(\mathbb{1}))}{M(A)} \leq \frac{2 \det(\mathcal{E}(\mathbb{1}))}{M} < \infty$$

from the lemma above. But then, the infimum must be attained on the compact subset $\{\det(X) = 1 | \gamma_1 \leq \frac{2 \det(\mathcal{E}(X))}{M}\}$. Therefore, also for the capacity $\text{Cap}(\mathcal{E})$ the infimum can be considered on a compact subset of $\{\det(X) = 1\}$ and is then attained. Uniqueness is ensured by the strict convexity of $f_A$. \hfill \square