Some Generalized Multi-sum Chu-Vandermonde Identities

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Abstract
Some generalized multi-sum Chu-Vandermonde identities are presented and proved, generalizing some known multi-sum Chu-Vandermonde identities from literature and adding some quadratic and cubic examples of these identities. Some other closely related identities are provided. The identities are proved with the integral representation method using complex residues, a method also used in some earlier papers.

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1 Generalized Multi-sum Chu-Vandermonde Identities

Let \((a_i)_{i=1}^m\) and \((c_i)_{i=1}^m\) be two sequences each of \(m\) nonnegative integers, and let \((x_i)_{i=1}^m\) and \((y_i)_{i=1}^m\) be two sequences each of \(m\) complex numbers, and let \(\overline{w}\) be the complex conjugate of \(w\). Let the following definitions be given.

\[ A_{p,q} = \sum_{i=1}^{m} x_i^p a_i^q \]  \hspace{1cm} (1.1)

\[ A_p = A_{p,1} = \sum_{i=1}^{m} x_i^p a_i \]  \hspace{1cm} (1.2)

\[ A^*_{p,q} = \sum_{i=1}^{m} x_i^p y_i^q a_i \]  \hspace{1cm} (1.3)

\[ A^*_p = A^{*}_{0,p} = \sum_{i=1}^{m} y_i^p a_i \]  \hspace{1cm} (1.4)

\[ A_{\text{abs}} = \sum_{i=1}^{m} |x_i|^2 a_i \]  \hspace{1cm} (1.5)

\[ C_{p,q} = \sum_{i=1}^{m} x_i^p c_i^q \]  \hspace{1cm} (1.6)
\[ C_p = C_{p,1} = \sum_{i=1}^{m} x_i^p c_i \quad (1.7) \]

\[ C_{p,q}^* = \sum_{i=1}^{m} x_i^p y_i^q c_i \quad (1.8) \]

\[ C^*_p = C^*_{0,p} = \sum_{i=1}^{m} y_i^p c_i \quad (1.9) \]

\[ C_{\text{abs}} = \sum_{i=1}^{m} |x_i|^2 c_i \quad (1.10) \]

\[ S_{p,q} = \sum_{i=1}^{m} x_i a_i^p c_i^q \quad (1.11) \]

The following generalized multi-sum Chu-Vandermonde identities are proved.

For integer \( m \geq 1 \) and \( n \geq 0 \):

\[ \sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{c_i}{k_i} = \binom{A_0}{n} \prod_{i=1}^{m} \binom{a_i}{c_i} \quad (1.12) \]

\[ \sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{c_i}{k_i} \sum_{i=1}^{m} x_i k_i \]

\[ = \frac{(A_0 - C_0)}{(n - C_0)} \prod_{i=1}^{m} \binom{a_i}{c_i} \left( \frac{(n - C_0)A_1 + (A_0 - n)C_1}{A_0 - C_0} \right) \sum_{i=1}^{m} x_i k_i \]

\[ . \binom{A_0}{n} \binom{A_0 - C_0}{n - C_0} \binom{A_1}{A_0 - C_0} \binom{C_1}{n - C_0 - 1} \binom{A_0}{(n - C_0 - 1) \binom{A_0}{n}} \binom{(A_0 - n)(A_0 - C_0)}{n - C_0} \binom{(A_0 - n) \binom{A_0}{n} + (A_0 - n) \binom{A_0}{n} - 1}{C_1} \]

\[ \frac{(n - C_0) \binom{A_0}{n} + (A_0 - n) \binom{A_0}{n} - 1}{C_1} \]

\[ = \frac{(A_0 - C_0)}{(n - C_0)} \prod_{i=1}^{m} \binom{a_i}{c_i} \frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)} \]

\[ \cdot \{ (n - C_0) \binom{A_0}{n} + (A_0 - n) \binom{A_0}{n} + A_0 \overline{C_1} + A_1 \overline{C_1} \} \]

\[ + (A_0 - n) \binom{A_0}{n} + (A_0 - n - 1) |C_1|^2 \} \]
\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} a_i \right] \left( \begin{array}{c}
C_k \\
k_i
\end{array} \right) \frac{1}{\sum_{i=1}^{m} x_i k_i} = (A_0 - C_0) \cdot \left\{ (n - C_0)(n - C_0 - 1)A_{1,2} + (A_0 - n)(A_1 - C_1 + 2S_{12}) \right\} \tag{1.16}
\]

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} a_i \right] \left( \begin{array}{c}
C_k \\
k_i
\end{array} \right) \frac{1}{\sum_{i=1}^{m} x_i k_i^2} = (A_0 - C_0) \cdot \left\{ (n - C_0)(n - C_0 - 1)A_{1,2} + (A_0 - n)(A_1 - C_1 + 2S_{12}) \right\} \tag{1.17}
\]

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} a_i \right] \left( \begin{array}{c}
C_k \\
k_i
\end{array} \right) \frac{1}{\sum_{i=1}^{m} x_i k_i^3} = (A_0 - C_0) \cdot \left\{ (n - C_0)(n - C_0 - 1)A_{1,2} + (A_0 - n)(A_1 - C_1 + 2S_{12}) \right\} \tag{1.18}
\]

Taking all \( c_i = 0 \) and therefore all \( C_{p,q} = C_p = C_{p,q}^* = C_p^* = C_{abs} = 0 \), identities (1.12), (1.13) and (1.15) can be found in [8].
2 Unrestricted Multi-sum Identities

When the multiple sum is without the restriction on the $k_i$-indices, other closely related identities result which are given below.

For integer $m \geq 1$:

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} = 2^{A_1-C_0} \prod_{i=1}^{m} \binom{a_i}{c_i} \tag{2.1}
\]

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \sum_{i=1}^{m} x_i k_i = 2^{A_0-C_0-1} \prod_{i=1}^{m} \binom{a_i}{c_i} \left( A_1 + C_1 \right) \tag{2.2}
\]

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \left( \sum_{i=1}^{m} x_i k_i \right)^2 = 2^{A_0-C_0-2} \prod_{i=1}^{m} \binom{a_i}{c_i} \left[ A_2 - C_2 + (A_1 + C_1)^2 \right] \tag{2.3}
\]

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \sum_{i=1}^{m} x_i k_i^2 = 2^{A_0-C_0-2} \prod_{i=1}^{m} \binom{a_i}{c_i} \left( A_{abs} - C_{abs} + |A_1 + C_1|^2 \right) \tag{2.4}
\]

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \left( \sum_{i=1}^{m} x_i k_i \right) \left( \sum_{i=1}^{m} y_i k_i \right) = 2^{A_0-C_0-2} \prod_{i=1}^{m} \binom{a_i}{c_i} \left[ A_{1,1}^* - C_{1,1}^* + (A_1 + C_1)(A_1^* + C_1^*) \right] \tag{2.5}
\]

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \sum_{i=1}^{m} x_i k_i^2 = 2^{A_0-C_0-2} \prod_{i=1}^{m} \binom{a_i}{c_i} \left( A_1 - C_1 + A_{1,2} + C_{1,2} + 2S_{1,1} \right) \tag{2.6}
\]

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \sum_{i=1}^{m} x_i k_i^3 = 2^{A_0-C_0-3} \prod_{i=1}^{m} \binom{a_i}{c_i} \left( A_1 + C_1 \right) \left[ (A_1 + C_1)^2 + 3(A_2 - C_2) \right] \tag{2.7}
\]

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \sum_{i=1}^{m} x_i k_i^3 = 2^{A_0-C_0-3} \prod_{i=1}^{m} \binom{a_i}{c_i} \left[ A_{1,3} + C_{1,3} + 3(A_{1,2} - C_{1,2} + S_{1,2} + S_{2,1}) \right] \tag{2.8}
\]
3 Proof of the Multi-sum Chu-Vandermonde Identities

For the proof of the multi-sum identities, the order of summation is changed:

\[
\sum_{k_1=0}^{m} \cdots \sum_{k_m=0}^{s} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \sum_{i_1=0}^{m} \cdots \sum_{i_s=0}^{s} \prod_{j=1}^{s} w_{i_j} k_{i_j}^{p_j}
\]

\[
= \sum_{i_1=0}^{m} \cdots \sum_{i_s=0}^{s} \prod_{j=1}^{s} w_{i_j} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \prod_{j=1}^{s} k_{i_j}^{p_j}
\]

(3.1)

The powers \( p_j \) in the right side must be constant in the second multiple summation over the \( k_i \), which means that the indices \( i_j \) must be mutually unequal. A product of sums in the summand is therefore splitted into multiple sums over mutually unequal indices:

\[
(\sum_{i=1}^{m} x_i k_i)(\sum_{i=1}^{m} y_i k_i) = \sum_{p=1}^{m} \sum_{q=1}^{m} x_p y_q k_p k_q + \sum_{p=1}^{m} x_p y_p k_p^2
\]

(3.2)

\[
(\sum_{i=1}^{m} x_i k_i)(\sum_{i=1}^{m} y_i k_i)(\sum_{i=1}^{m} z_i k_i) = \sum_{p=1}^{m} \sum_{q=1}^{m} \sum_{r=1}^{m} x_p y_q z_r k_p k_q k_r
\]

\[
+ \sum_{p=1}^{m} \sum_{q=1}^{m} (x_p y_q z_q + y_p x_q z_q + z_p x_q y_q) k_p k_q^2 + \sum_{p=1}^{m} x_p y_p z_p k_p^3
\]

(3.3)

For the multi-sum identities up to cubic order, in the right side of (3.1) the power combinations \( 1, k_p, k_p k_q, k_p k_q k_r, k_p k_q^2 \) and \( k_p^3 \) are needed, where as mentioned \( p \neq q \neq r \) and \( p \neq r \), and the second multiple summation over the \( k_i \) is evaluated below with the integral representation method using complex residues as in earlier papers [8] [9]:

\[
\binom{n}{k} = \text{Res}_w \frac{(1 + w)^n}{w^k + 1}
\]

(3.4)

When \( f(w) \) does not have a pole at \( w = w_p \), then the following is proved in section 5:

\[
\text{Res}_{w=w_p} \frac{f(w)}{(w - w_p)^k} = \frac{1}{(k - 1)!} D_{w}^{k-1} f(w) |_{w=w_p}
\]

(3.5)

where \( D_{w}^{n} f(w) |_{w=w_p} \) is the \( n \)-th derivative of \( f(w) \) at \( w = w_p \).

**Theorem 3.1.** For integer \( m \geq 1 \):

\[
\sum_{k_1=0}^{m} \cdots \sum_{k_m=0}^{s} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} = \binom{A_0 - C_0}{n - C_0} \prod_{i=1}^{m} \binom{a_i}{c_i}
\]

(3.6)
Proof. The trinomial revision identity \[5, 10\] is applied:
\[
\binom{a_i}{k_i} \binom{k_i}{c_i} = \binom{a_i - c_i}{k_i - c_i} \binom{a_i}{c_i}
\] (3.7)

Because this expression is zero when \(c_i > a_i\), in the multiple summation it can be assumed that \(0 \leq c_i \leq a_i\) and that \(c_i \leq k_i \leq a_i\). The restriction on the \(k_i\)-indices in the multiple sum can therefore be realized by setting \(k_m\) equal to:
\[
k_m = n - \sum_{i=1}^{m-1} k_i
\] (3.8)

reducing an \(m\)-fold sum to an \((m - 1)\)-fold sum. The infinite geometric series \[9\] is used:
\[
\sum_{k=0}^{\infty} w^k = \frac{1}{1 - w}
\] (3.9)

The upper bounds \(a_i\) can be replaced by \(\infty\) because the summand is zero when any \(k_i > a_i\).
\[
\sum_{k_i=0}^{a_i} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i}
\]
\[
= \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] \sum_{k_i=0}^{a_i} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \binom{a_i - c_i}{k_i - c_i}
\]
\[
= \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] \sum_{k_i=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \prod_{i=1}^{m} \text{Res}_{w_i} \frac{(1 + w_m)^{a_m - c_m}}{w_m^{n - c_m - \sum_{i=1}^{m-1} k_i + 1}} \prod_{i=1}^{m-1} \frac{(1 + w_i)^{a_i - c_i}}{w_i^{k_i - c_i + 1}}
\]
\[
= \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] \text{Res}_{w_i} \frac{1}{w_m^{1 - w_m}} \left[ \prod_{i=1}^{m} \frac{(1 + w_i)^{a_i - c_i}}{w_i^{c_i}} \right] \prod_{k_i=0}^{\infty} \sum_{i=1}^{\infty} \prod_{i=1}^{m-1} \frac{w_m}{w_i}
\]
\[
= \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] \text{Res}_{w_i} \frac{(1 + w_m)^{a_m - c_m}}{w_m^{n - c_m + 1}} \prod_{i=1}^{m-1} \text{Res}_{w_i} \frac{(1 + w_i)^{a_i - c_i}}{w_i^{c_i}}
\]

(3.10)

The residue of \(w_i\) is evaluated with \(3.5\), using that \(c_i \geq 0\) and \(a_i - c_i \geq 0\):
\[
= \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] \text{Res}_{w_m} \frac{(1 + w_m)^{a_m - c_m}}{w_m^{n - c_m + 1}} \prod_{i=1}^{m-1} \text{Res}_{w_i} \frac{(1 + w_i)^{a_i - c_i}}{w_i^{c_i}}
\]
\[
= \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] \text{Res}_{w_m} \frac{(1 + w_m)^{\sum_{i=1}^{m}(a_i - c_i)}}{w_m^{n - \sum_{i=1}^{m} c_i + 1}} = \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] \left( \sum_{i=1}^{m}(a_i - c_i) \right)
\]

(3.11)
Theorem 3.2. For integer $m \geq 1$ and $1 \leq p \leq m$:

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \left( \frac{a_i}{k_i} \right) \left( \frac{k_i}{c_i} \right) \cdot k_p = \left( A_0 - C_0 \right) \left( \frac{n - C_0}{n - C_0} \right) \left( \frac{A_0 - n c_p}{A_0 - C_0} \right)
\]

(3.12)

Proof. The following modified infinite geometric series [9] is used:

\[
\sum_{k=0}^{\infty} k w^k = \frac{w}{(1-w)^2}
\]

(3.13)

The theorem is first proved for $1 \leq p < m$:

\[
\left[ \prod_{i=1}^{m} \text{Res}_{w_i} \right] \left[ \frac{1}{w_i^m} \right] \left[ \prod_{i=1}^{m} (1 + w_i)^{a_i - c_i} w_i^{c_i - 1} \right] \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} k_p \prod_{i=1}^{m} \left( \frac{w_i}{w_i^m} \right)^{k_i}
\]

\[
= \left[ \prod_{i=1}^{m} \text{Res}_{w_i} \right] \left[ \frac{1}{w_i^m} \right] \left[ \prod_{i=1}^{m} (1 + w_i)^{a_i - c_i} w_i^{c_i - 1} \right] \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} k_p \prod_{i=1}^{m} \left( \frac{w_i}{w_i^m} \right)^{k_i}
\]

\[
= \text{Res}_{w_m} \frac{(1 + w_m)^{a_m - c_m}}{w_m} \text{Res}_{w_p} \frac{(1 + w_p)^{a_p - c_p} w_p^{c_p}}{(w_p - w_m)^2} \prod_{i=1}^{m-1} \text{Res}_{w_i} \frac{(1 + w_i)^{a_i - c_i} w_i^{c_i}}{w_i - w_m}
\]

(3.14)

The residue of $w_p$ is evaluated with [34] using the product rule for the derivative:

\[
\text{Res}_{w_p} \frac{(1 + w_p)^{a_p - c_p} w_p^{c_p}}{(w_p - w_m)^2} = (a_p - c_p)(1 + w_m)^{a_p - c_p - 1} w_m^{c_p} + c_p(1 + w_m)^{a_p - c_p} w_m^{c_p - 1}
\]

(3.15)

and evaluating the residue of $w_m$ and using the absorption identity [35 [100]:

\[
\text{Res}_{w_m} \left[ (a_p - c_p) \left( 1 + w_m \right)^{\sum_{i=1}^{m} (a_i - c_i) - 1} \right] + c_p \left( 1 + w_m \right)^{\sum_{i=1}^{m} (a_i - c_i) + 1}
\]

\[
= \left( \frac{A_0 - C_0}{n - C_0 - 1} \right) (a_p - c_p) + \left( \frac{A_0 - C_0}{n - C_0} \right) c_p
\]

\[
= \left( \frac{A_0 - C_0}{n - C_0} \right) (a_p - c_p) (n - C_0) + c_p
\]

(3.16)

The theorem is now proved for $1 \leq p < m$, but because the indices of the $k_i$ in the left side of the theorem can be interchanged, the theorem is true for $1 \leq p \leq m$. □
Theorem 3.3. For integer $m \geq 1$:

$$
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \frac{(a_i)}{(k_i)} \right] \sum_{i=1}^{m} x_i k_i 
= \left( A_0 - C_0 \right) \left[ \prod_{i=1}^{m} \frac{(a_i)}{(c_i)} \right] \frac{(n-C_0)A_1 + (A_0 - n)C_1}{A_0 - C_0}
$$

(3.17)

Proof. Changing the order of summation as in (3.1) and using theorem 3.2:

$$
\left( A_0 - C_0 \right) \left[ \prod_{i=1}^{m} \frac{(a_i)}{(c_i)} \right] \sum_{i=1}^{m} x_i (n-C_0)a_i + (A_0 - n)c_i
= \left( A_0 - C_0 \right) \left[ \prod_{i=1}^{m} \frac{(a_i)}{(c_i)} \right] \frac{(n-C_0)A_1 + (A_0 - n)C_1}{A_0 - C_0}
$$

(3.18)

Theorem 3.4. For integer $m \geq 1$ and $1 \leq p \leq m$:

$$
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \frac{(a_i)}{(k_i)} \right] k_p = \left( A_0 - C_0 \right) \left[ \prod_{i=1}^{m} \frac{(a_i)}{(c_i)} \right] \frac{(n-C_0)(n-C_0-1)a_1^2 + (A_0 - n)(a_p - c_p + 2a_p c_p) + (A_0 - n)(A_0 - n - 1)c_p^2}{(A_0 - C_0)(A_0 - C_0 - 1)}
$$

(3.19)

Proof. The following modified infinite geometric series [8] is used:

$$
\sum_{k=0}^{\infty} k^2 w^k = \frac{w}{(1-w)^3} + \frac{2w^2}{(1-w)^3}
$$

(3.20)

The first term gives a result identical to theorem 3.2, and for the second term:

$$
\left[ \prod_{i=1}^{m} \text{Res}_{w_i} \right] \frac{1}{w^m} \left[ \prod_{i=1}^{m} (1 + w_i)^a_i - c_i w_i^{a_i-1} \right] \frac{2(\frac{w_m}{w_p})^2}{(1 - \frac{w_m}{w_p})^3} \prod_{i=1}^{m-1} \frac{1}{1 - \frac{w_m}{w_i}}

= \text{Res}_{w_m} \frac{(1 + w_m)^a_m - c_m}{w_m^{a_m-c_m} - 1} \text{Res}_{w_p} \frac{2(1 + w_p)^{a_p - c_p} w_p^{c_p}}{(w_p - w_m)^3} \prod_{i=1}^{m-1} (1 + w_m)^{a_i - c_i} w_i^{c_i}
$$

(3.21)

The residue of $w_p$ is evaluated with (3.5) using the product rule for the second derivative:

$$
\text{Res}_{w_p} \frac{2(1 + w_p)^{a_p - c_p} w_p^{c_p}}{(w_p - w_m)^3} = (a_p - c_p)(a_p - c_p - 1)(1 + w_m)^{a_p - c_p - 2} w_p^{c_p}
+ 2(a_p - c_p)c_p(1 + w_m)^{a_p - c_p - 1} w_m^{c_p - 1} + c_p(c_p - 1)(1 + w_m)^{a_p - c_p} w_m^{c_p - 2}
$$

(3.22)
Theorem 3.5. For integer $m \geq 2$, $1 \leq p \leq m$, $1 \leq q \leq m$ and $p \neq q$:

$$\sum_{k_1=0}^{a_1-1} \cdots \sum_{k_m=0}^{a_m-1} \left[ \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{c_i}{k_i} \right] k_p k_q = \binom{A_0 - C_0}{n - C_0} \binom{A_0 - C_0}{n - C_0}$$

This result simplifies to the theorem.

Proof. The proof is similar to theorem 3.2 but with two quadratic residues instead of one.

$$\text{Res}_{w_m} \left( \frac{(1 + w_m)^{a_m - c_m}}{w_m^{a_m - c_m - 1}} \right) \text{Res}_{w_p} \left( (1 + w_p)^{a_p - c_p} w_p^{c_p} \right) \text{Res}_{w_q} \left( (1 + w_q)^{a_q - c_q} w_q^{c_q} \right)$$

\[
\prod_{i=1}^{m-1} (1 + w_m)^{a_i - c_i} w_m^{c_i}
\]

\[
(3.26)
\]
The two last residues are given by (3.15), and multiplying these out gives:

\[
[(a_p - c_p)(1 + w_m)^{a_p - c_p - 1} w_m^{a_p - c_p - 1} + c_p(1 + w_m)^{a_p - c_p} w_m^{a_p - c_p - 1}]
\cdot [(a_q - c_q)(1 + w_m)^{a_q - c_q - 1} w_m^{a_q - c_q} + c_q(1 + w_m)^{a_q - c_q} w_m^{a_q - c_q - 1}]
= (a_p - c_p)(a_q - c_q)(1 + w_m)^{a_p - c_p + a_q - c_q - 2} w_m^{a_p + c_q}
+ [(a_p - c_p)c_p + (a_q - c_q)c_q](1 + w_m)^{a_p - c_p + a_q - c_q - 1} w_m^{a_p + c_q} - 1
+ c_p c_q(1 + w_m)^{a_p - c_p + a_q - c_q} w_m^{a_p + c_q - 2}
\tag{3.27}
\]

This yields the following:

\[
\begin{align*}
&\left(\frac{A_0 - C_0 - 2}{n - C_0 - 2}\right)(a_p - c_p)(a_q - c_q) + \left(\frac{A_0 - C_0 - 1}{n - C_0 - 1}\right)[(a_p - c_p)c_q + (a_q - c_q)c_p] \\
&+ \left(\frac{A_0 - C_0}{n - C_0}\right)c_p c_q
\end{align*}
\tag{3.28}
\]

and using again the absorption identity:

\[
\begin{align*}
&\left(\frac{A_0 - C_0}{n - C_0}\right)\frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)} \\
&\cdot [(n - C_0)(n - C_0 - 1)(a_p - c_p)(a_q - c_q) + (A_0 - C_0 - 1)((a_p - c_p)c_q + (a_q - c_q)c_p)] \\
&+ (A_0 - C_0)(A_0 - C_0 - 1)c_p c_q
\end{align*}
\tag{3.29}
\]

This result simplifies to the theorem.

\[\square\]

**Theorem 3.6.** For integer \(m \geq 1\):

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \left( \begin{array}{c} a_i \\ c_i \end{array} \right) \left( \begin{array}{c} k_i \\ c_i \end{array} \right) \right] \sum_{i=1}^{m} x_i k_i \sum_{i=1}^{m} y_i k_i
= \left(\frac{A_0 - C_0}{n - C_0}\right)\prod_{i=1}^{m} \left( \begin{array}{c} a_i \\ c_i \end{array} \right)\frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)} \\
\cdot [(n - C_0)(n - C_0 - 1)A_1^* A_1^* + (A_0 - n)(A_1^* - C_1^* + A_1^* C_1^* + A_1^* C_1^*)] \\
+ (A_0 - n)(A_0 - n - 1)C_1^* C_1^*
\tag{3.30}
\]

**Proof.** The product in the summand becomes:

\[
\left( \sum_{i=1}^{m} x_i k_i \right) \left( \sum_{i=1}^{m} y_i k_i \right) = \sum_{i=1}^{m} \sum_{j=1}^{m} x_i y_i k_i k_j + \sum_{i=1}^{m} x_i y_i k_i^2
\tag{3.31}
\]
Theorem 3.8. For integer \( m \geq 1 \):

\[
\sum_{i=1}^{m} \sum_{j=1, j \neq i}^{m} x_i y_j \left\{ (n - C_0)[(n - C_0 - 1)a_i a_j + (A_0 - n)(a_i c_j + c_a a_j)] + (A_0 - n)(A_0 - n - 1)c_i c_j \right\} + \sum_{i=1}^{m} x_i y_i \left\{ (n - C_0)[(n - C_0 - 1)a_i a_j + (A_0 - n)(a_i c_j + c_a a_j)] + (A_0 - n)(A_0 - n - 1)c_i c_j \right\} = \sum_{i=1}^{m} x_i y_i (n - C_0)\]

\[
\sum_{i=1}^{m} x_i y_i (n - C_0) (n - C_0) (A_1 A_1^* + (A_0 - n)(A_2 - C_2 + 2A_1 C_1)) + (A_0 - n)(A_0 - n - 1)C_1^2
\]

\[
(A_0 - C_0)(A_0 - C_0 - 1)
\]

(3.32)

Proof. In theorem 3.6 taking \( y_i = x_i \), then \( A_1^* = A_1, C_1^* = C_1, A_{1,1}^* = A_2 \) and \( C_{1,1}^* = C_2 \), giving this theorem. \( \square \)

Theorem 3.7. For integer \( m \geq 1 \):

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \frac{a_i}{k_i} \right] \left[ \prod_{i=1}^{m} \frac{a_i}{c_i} \right] \sum_{i=1}^{m} x_i k_i^2 = \left( \frac{A_0 - C_0}{n - C_0} \right) \frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)} \cdot \{(n - C_0)[(n - C_0 - 1)|A_1|^2 + (A_0 - n)(A_{abs} - C_{abs} + A_1 C_1 + A_1 C_1)] + (A_0 - n)(A_0 - n - 1)|C_1|^2\}
\]

(3.33)

Proof. In theorem 3.8 taking \( y_i = x_i \), then \( A_1^* = A_1, C_1^* = C_1, A_{1,1}^* = A_2 \) and \( C_{1,1}^* = C_2 \), giving this theorem. \( \square \)
Proof. For this theorem the following is used.

\[
|\sum_{i=1}^{m} x_i k_i|^2 = \text{Re}^2(\sum_{i=1}^{m} x_i k_i) + \text{Im}^2(\sum_{i=1}^{m} x_i k_i) = (\sum_{i=0}^{m} \text{Re}(x_i)k_i)^2 + (\sum_{i=0}^{m} \text{Im}(x_i)k_i)^2 \tag{3.35}
\]

Both of these terms are evaluated by theorem 3.7 replacing \(x_i\) by \(\text{Re}(x_i)\) and then by \(\text{Im}(x_i)\), and then the results are added, using the following.

\[
(\sum_{i=1}^{m} \text{Re}(x_i)a_i)^2 + (\sum_{i=1}^{m} \text{Im}(x_i)a_i)^2 = \text{Re}^2(\sum_{i=1}^{m} x_ia_i) + \text{Im}^2(\sum_{i=1}^{m} x_ia_i) = |A_1|^2 \tag{3.36}
\]

\[
(\sum_{i=1}^{m} \text{Re}(x_i)c_i)^2 + (\sum_{i=1}^{m} \text{Im}(x_i)c_i)^2 = \text{Re}^2(\sum_{i=1}^{m} x_ic_i) + \text{Im}^2(\sum_{i=1}^{m} x_ic_i) = |C_1|^2 \tag{3.37}
\]

\[
\sum_{i=1}^{m} \text{Re}^2(x_i)a_i + \sum_{i=1}^{m} \text{Im}^2(x_i)a_i = \sum_{i=1}^{m}[\text{Re}^2(x_i) + \text{Im}^2(x_i)]a_i = \sum_{i=1}^{m} |x_i|^2 a_i = A_{\text{abs}} \tag{3.38}
\]

\[
\sum_{i=1}^{m} \text{Re}^2(x_i)c_i + \sum_{i=1}^{m} \text{Im}^2(x_i)c_i = \sum_{i=1}^{m}[\text{Re}^2(x_i) + \text{Im}^2(x_i)]c_i = \sum_{i=1}^{m} |x_i|^2 c_i = C_{\text{abs}} \tag{3.39}
\]

and using for complex \(a, c\):

\[
2[\text{Re}(a)\text{Re}(c) + \text{Im}(a)\text{Im}(c)] = a\bar{c} + c\bar{a} \tag{3.40}
\]

and the theorem is proved.

\[\Box\]

Theorem 3.9.

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_n=n}^{a_n} \left[ \prod_{i=1}^{m} \left( a_i \right) \left( k_i \right) \right] \sum_{i=1}^{m} x_i k_i = \left( A_0 - C_0 \right) \left( n - C_0 - 1 \right) A_{1,2} + \left( A_0 - n \right) \left( A_1 - C_1 + 2S_{1,1} \right) + \left( A_0 - n \right) \left( A_0 - n - 1 \right) C_{1,2} \]

\[
\frac{(n - C_0)(n - C_0 - 1)A_{1,2} + (A_0 - n)(A_1 - C_1 + 2S_{1,1}) + (A_0 - n)(A_0 - n - 1)C_{1,2}}{(A_0 - C_0)(A_0 - C_0 - 1)} \tag{3.41}
\]

Proof. Changing the order of summation as in (3.41) and using theorem 3.4.

\[
\sum_{i=1}^{m} \left\{ (n - C_0)(n - C_0 - 1)a_i^2 + (A_0 - n)(a_i - c_i + 2a_ic_i) + (A_0 - n)(A_0 - n - 1)c_i^2 \right\}
\]

\[
= (n - C_0)(n - C_0 - 1)A_{1,2} + (A_0 - n)(A_1 - C_1 + 2S_{1,1}) + (A_0 - n)(A_0 - n - 1)C_{1,2} \tag{3.42}
\]

\[\Box\]
Theorem 3.10. For integer $m \geq 1$ and $1 \leq p \leq m$:

$$\sum_{k_1=0}^{a_m} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \left( \frac{a_i}{k_i} \right) k_i^3$$

The residue of $w_p$ is evaluated with (3.5) using the product rule for the third derivative:

$$\text{Res}_{w_p} \left( \frac{6(1 + w_p)^{a_p-c_p}w_p^{c_p}}{(w_p - w_m)^4} \right) \prod_{i=1}^{m-1} (1 + w_m)^{a_i - c_i} w_m^{c_i}$$

Using the absorption identity [5, 10] as in theorem 3.4 and adding as mentioned the result
Theorem 3.11. For integer \( m \geq 2, 1 \leq p \leq m, 1 \leq q \leq m \) and \( p \neq q \):

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \right] k_p k_q^2
\]

\[
= \left( \frac{A_0 - C_0}{n - C_0} \right) \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] \frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)(A_0 - C_0 - 2)}
\]

\[
\cdot \frac{(n - C_0)(n - C_0 - 1)[(n - C_0 - 2)(a_p a_q^2 + (A_0 - n)(c_p a_q^2 + a_p a_q - a_p c_q + 2a_p a_q c_q)]}
\]

\[
+ (A_0 - n)(A_0 - n - 1)[(A_0 - n - 2)c_p c_q^2 + (n - C_0)(a_p c_q^2 - c_p c_q + c_p a_q + 2c_p a_q c_q)]
\]

This result simplifies to the theorem. \( \square \)

Theorem 3.11. For integer \( m \geq 2, 1 \leq p \leq m, 1 \leq q \leq m \) and \( p \neq q \):

\[
\left( \frac{A_0 - C_0}{n - C_0} \right) \left[ \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \right] k_p k_q^2
\]

\[
= \left( \frac{A_0 - C_0}{n - C_0} \right) \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] \frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)(A_0 - C_0 - 2)}
\]

\[
\cdot \frac{(n - C_0)(n - C_0 - 1)[(n - C_0 - 2)(a_p a_q^2 + (A_0 - n)(c_p a_q^2 + a_p a_q - a_p c_q + 2a_p a_q c_q)]}
\]

\[
+ (A_0 - n)(A_0 - n - 1)[(A_0 - n - 2)c_p c_q^2 + (n - C_0)(a_p c_q^2 - c_p c_q + c_p a_q + 2c_p a_q c_q)]
\]

Proof. The proof is similar to theorem 3.5 but with two quadratic and one cubic residues instead of two quadratic residues.

\[
\text{Res}_{w_m} \frac{(1 + w_m)^{a_m-c_m}}{w_m^{e_m-2}} \cdot \frac{(1 + w_p)^{a_p-c_p} w_p^c}{w_p - w_m^j} \cdot \frac{(1 + w_q)^{a_q-c_q} w_q^c}{w_q - w_m^j}
\]

\[
\cdot \left[ \text{Res}_{w_q} \frac{(1 + w_q)^{a_q-c_q} w_q^c}{(w_q - w_m^j)^2} + \text{Res}_{w_q} \frac{2(1 + w_q)^{a_q-c_q} w_q^c}{(w_q - w_m^j)^3} \right] \prod_{i=1}^{m-1} (1 + w_m^{a_i-c_i}) w_m^{c_i}
\]

The first product gives a result identical to theorem 3.5 and for the second product the
Theorem 3.12. For integer $m \geq 3$, $1 \leq p \leq m$, $1 \leq q \leq m$, $1 \leq r \leq m$, $p \neq q \neq r$ and $p \neq r$:

$$\sum_{k_1=0}^{a_1} \ldots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \left( \frac{a_i}{k_i} \right) \left( \binom{k_i}{c_i} \right) \right] k_p k_q k_r$$

$$= \left( \frac{A_0 - C_0}{n - C_0} \right) \left[ \prod_{i=1}^{m} \left( \frac{a_i}{c_i} \right) \right] \left( \frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)(A_0 - C_0 - 2)} \right)$$

$$\cdot \{(n - C_0)(n - C_0 - 1)(a_p - c_p)(a_q - c_q)[(n - C_0 - 2)(a_p - c_p)(a_q - c_q - 1)
+ (A_0 - C_0 - 2)(c_p(a_q - c_q - 1) + 2(a_p - c_p)c_q)]
+ (A_0 - C_0 - 1)(A_0 - C_0 - 2)c_q[(A_0 - C_0)c_p(c_q - 1)
+ (n - C_0)((a_p - c_p)(c_q - 1) + 2c_p(a_q - c_q))]
+ (A_0 - C_0 - 2)(A_0 - n)(A_0 - n - 1)c_pc_q
+ (n - C_0)((n - C_0 - 1)a_p a_q + (A_0 - n)(a_p c_q + c_p a_q))]\}$$

This result simplifies to the theorem. 

Theorem 3.12. For integer $m \geq 3$, $1 \leq p \leq m$, $1 \leq q \leq m$, $1 \leq r \leq m$, $p \neq q \neq r$ and $p \neq r$:

$$\sum_{k_1=0}^{a_1} \ldots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \left( \frac{a_i}{k_i} \right) \left( \binom{k_i}{c_i} \right) \right] k_p k_q k_r$$

$$= \left( \frac{A_0 - C_0}{n - C_0} \right) \left[ \prod_{i=1}^{m} \left( \frac{a_i}{c_i} \right) \right] \left( \frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)(A_0 - C_0 - 2)} \right)$$

$$\cdot \{(n - C_0)(n - C_0 - 1)(a_p - c_p)(a_q - c_q)[(n - C_0 - 2)(a_p - c_p)(a_q - c_q - 1)
+ (A_0 - C_0 - 2)(c_p(a_q - c_q - 1) + 2(a_p - c_p)c_q)]
+ (A_0 - C_0 - 1)(A_0 - C_0 - 2)c_q[(A_0 - C_0)c_p(c_q - 1)
+ (n - C_0)((a_p - c_p)(c_q - 1) + 2c_p(a_q - c_q))]
+ (A_0 - C_0 - 2)(A_0 - n)(A_0 - n - 1)c_pc_q
+ (n - C_0)((n - C_0 - 1)a_p a_q + (A_0 - n)(a_p c_q + c_p a_q))]\}$$

Proof. The proof is similar to theorem 3.10 but with three quadratic residues instead of
two.

\[
\text{Res}_{w_m} \left( \frac{(1 + w_m)^{a_{m-cm}}}{w_m^{c_m-2}} \right) \text{Res}_{w_p} \left( \frac{(1 + w_p)^{a_{p-cp}}}{(w_p - w_m)^2} \right) \text{Res}_{w_q} \left( \frac{(1 + w_q)^{a_{q-cq}}}{(w_q - w_m)^2} \right)
\]

\[
\cdot \frac{\text{Res}_{w_r} \left( \frac{(1 + w_r)^{a_{r-cr}}}{(w_r - w_m)^2} \right)}{(1 + w_m)^{a_{i-c_i}} w_m^{c_i}} \prod_{i \neq j \neq k \neq r}^{m-1}
\]

(3.53)

The three last residues are given by (3.15), and multiplying these out gives:

\[
[(a_p - c_p)(1 + w_m)^{a_{p-cp}} w_m^{c_p} + c_p(1 + w_m)^{a_{p-cp}} w_m^{c_p-1}]
\]

\[
\cdot [(a_q - c_q)(1 + w_m)^{a_{q-cq}} w_m^{c_q} + c_q(1 + w_m)^{a_{q-cq}} w_m^{c_q-1}]
\]

\[
\cdot [(a_r - c_r)(1 + w_m)^{a_{r-cr}} w_m^{c_r} + c_r(1 + w_m)^{a_{r-cr}} w_m^{c_r-1}]
\]

\[
= (a_p - c_p)(a_q - c_q)(a_r - c_r)(1 + w_m)^{a_{p-cp} + a_{q-cq} + a_{r-cr} - 3} w_m^{c_p + c_q + c_r} + [(a_p - c_p)(a_q - c_q)c_r + (a_p - c_p)c_q(a_r - c_r) + c_p(a_q - c_q)(a_r - c_r)]
\]

\[
\cdot (1 + w_m)^{a_{p-cp} + a_{q-cq} + a_{r-cr} - 2} w_m^{c_p + c_q + c_r - 1}
\]

\[
+ [(a_p - c_p)c_q c_r + c_p(a_q - c_q)c_r + c_p c_q(a_r - c_r)]
\]

\[
\cdot (1 + w_m)^{a_{p-cp} + a_{q-cq} + a_{r-cr} - 1} w_m^{c_p + c_q + c_r - 2}
\]

\[
+ c_p c_q c_r (1 + w_m)^{a_{p-cp} + a_{q-cq} + a_{r-cr} - w_m^{c_p + c_q + c_r - 3}}
\]

(3.54)

Using the absorption identity [8][10] as in theorem [3.11],

\[
\frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)(A_0 - C_0 - 2)}
\]

\[
\cdot \{(n - C_0)(n - C_0 - 1)[(n - C_0 - 2)(a_p - c_p)(a_q - c_q)(a_r - c_r) + (A_0 - C_0 - 2)]
\]

\[
\cdot (a_p - c_p)(a_q - c_q)c_r + (a_p - c_p)c_q(a_r - c_r) + c_p(a_q - c_q)(a_r - c_r)]
\]

\[
+ (A_0 - C_0 - 1)(A_0 - C_0 - 2)[(A_0 - C_0)c_p c_q c_r
\]

\[
+ (n - C_0)(a_p - c_p)c_q c_r + c_p(a_q - c_q)c_r + c_p c_q(a_r - c_r)]
\}

(3.55)

This result simplifies to the theorem.

\[
\text{Theorem 3.13. For integer } m \geq 1:
\]

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \frac{a_i}{k_i} \frac{k_i}{c_i} \right] \left[ \prod_{i=1}^{m} x_i k_i \right]^3
\]

\[
= \frac{A_0 - C_0}{n - C_0} \left[ \prod_{i=1}^{m} \frac{a_i}{c_i} \right] \left[ \frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)(A_0 - C_0 - 2)} \right]
\]

\[
\cdot \{(n - C_0)(n - C_0 - 1)[(n - C_0 - 2)A_1^3
\]

\[
+ (A_0 - n)(A_3 - A_3 + 3A_1(A_2 - C_2 + A_1 C_1))
\]

\[
+ (A_0 - n)(A_0 - n - 1)[(A_0 - n - 2)C_1^3
\]

\[
+ (n - C_0)(A_3 - C_3 + 3C_1(A_2 - C_2 + A_1 C_1))]\}
\]

(3.56)
Proof. The product of the sum in the summand is expressed in sums with unequal indices:

\[
\left( \sum_{i=1}^{m} x_i k_i \right)^3 = \sum_{p=1}^{m} \sum_{q=1}^{m} \sum_{r=1}^{m} x_p x_q x_r k_p k_q k_r + 3 \sum_{p=1}^{m} \sum_{q=1}^{m} x_p x_q^2 k_p k_q^2 + \sum_{p=1}^{m} x_p^3 k_p^3 \tag{3.57}
\]

Changing the order of summation, the multiple sum over the products of \(k_p, k_q\) and \(k_r\) is substituted from theorems 3.12, 3.11 and 3.10

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} x_i x_j x_k \{ (n - C_0)(n - C_0 - 1) \} [(n - C_0 - 2)a_i a_j a_k + 3(A_0 - n)a_i a_j c_k]
\]

\[
+ (A_0 - n)(A_0 - n - 1) [(A_0 - n - 2)c_i c_j c_k + 3(n - C_0)c_i c_j a_k]
\]

\[
+ 3 \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j^2 \{ (n - C_0)(n - C_0 - 1) \} [(n - C_0 - 2)a_i a_j^2]
\]

\[
+ (A_0 - n)(c_i a_j^2 + a_i a_j - a_i c_j + 2a_i a_j c_j)]
\]

\[
+ (A_0 - n)(A_0 - n - 1) [(A_0 - n - 2) c_i c_j^2
\]

\[
+ (n - C_0)(a_i c_j^2 - c_i c_j + c_i a_j + 2c_i a_j c_j)]
\]

\[
+ \sum_{i=1}^{m} x_i^3 \{ (n - C_0)(n - C_0 - 1)(n - C_0 - 2)a_i^3 + (n - C_0)(n - C_0 - 2) a_i^2 c_i + (n - C_0)(n - C_0 - 2) c_i^2 + (n - C_0)(n - C_0 - 2) c_i a_i + (n - C_0)(n - C_0 - 2) a_i c_i]
\]

\[
+ (A_0 - n)(A_0 - n - 1) [(A_0 - n - 2) c_i c_j^3 + (n - C_0)(n - C_0 - 2) c_i^2 + (n - C_0)(n - C_0 - 2) c_i a_i + (n - C_0)(n - C_0 - 2) a_i c_i]
\]

The expressions of \(A_1^3, C_1^3, A_1^2 C_1, A_1 C_1^2\) as sums over unequal indices are given by equation 3.58:

\[
(n - C_0)(n - C_0 - 1) [(n - C_0 - 2) A_1^3 + (3(A_0 - n)) A_1^3 C_1]
\]

\[
+ (A_0 - n)(A_0 - n - 1) [(A_0 - n - 2) C_1^3 + (3(n - C_0)) A_1 C_1^2]
\]

\[
+ 3 \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j^2 (n - C_0)(n - C_0 - 1)(A_0 - n)(a_i a_j - a_i c_j)
\]

\[
+ (A_0 - n)(A_0 - n - 1)(n - C_0)(c_i a_j - c_i c_j]
\]

\[
+ \sum_{i=1}^{m} x_i^3 ((n - C_0)(n - C_0 - 1)(A_0 - n)(3a_i^2 c_i - 3a_i c_i - a_i + c_i)
\]

\[
+ (A_0 - n)(A_0 - n - 1)(n - C_0)(3a_i c_i^2 + 3a_i c_i + a_i - c_i)]
\]
The remaining sums can be expressed in $A_1$, $C_1$, $A_2$, $C_2$, $A_3$ and $C_3$:

$$(n - C_0)(n - C_0 - 1)(n - C_0 - 2)A_1^3 + 3(A_0 - n)A_1^2 C_1$$
$$+ (A_0 - n)(A_0 - n - 1)(A_0 - n - 2)C_1^2 + 3(n - C_0 - 1)A_1 C_2$$
$$+ 3(n - C_0)(A_0 - n)C_1 A_2 + (A_0 - n - 1)(C_1 A_2 - C_1 C_2)$$
$$+ (n - C_0)(A_0 - n - 1)(C_3 - A_3)$$

This result simplifies to the theorem.

**Theorem 3.14.** For integer $m \geq 1$:

$$\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \frac{a_i}{k_i} \right] \prod_{i=1}^{m} x_i k_i^3$$

$$= \left( \frac{A_0 - C_0}{n - C_0} \right) \left[ \prod_{i=1}^{m} \frac{a_i}{c_i} \right] \frac{1}{(A_0 - C_0)(A_0 - C_0 - 1)(A_0 - C_0 - 2)}$$

$$\cdot \{(n - C_0)(n - C_0 - 1)[(n - C_0 - 2)A_{1,3}$$
$$+ (A_0 - n)(C_1 - A_1 + 3(S_{2,1} + A_{1,2} - S_{1,1}))$$
$$+ (A_0 - n)(A_0 - n - 1)[(A_0 - n - 2)C_{1,3}$$
$$+ (n - C_0)(A_1 - C_1 + 3(S_{1,2} - C_{1,2} + S_{1,1}))]} \}$$

**Proof.** Changing the order of summation as in (3.61) and using theorem 3.10:

$$\sum_{i=1}^{m} x_i \{(n - C_0)(n - C_0 - 1)(n - C_0 - 2)a_i^3 + (A_0 - n)(3a_i^2 c_i + 3a_i^2 - 3a_i c_i - a_i + c_i)$$
$$+ (A_0 - n)(A_0 - n - 1)[(A_0 - n - 2)c_i^3 + (n - C_0)(3a_i^2 c_i^2 - 3c_i^2 + 3a_i c_i + a_i - c_i)]\}$$

$$= (n - C_0)(n - C_0 - 1)[(n - C_0 - 2)A_{1,3} + (A_0 - n)(3S_{2,1} + A_{1,2} - 3S_{1,1} - A_1 + C_1)]$$
$$+ (A_0 - n)(A_0 - n - 1)[(A_0 - n - 2)C_{1,3} + (n - C_0)(3S_{1,2} - 3C_{1,2} + 3S_{1,1} + A_1 - C_1)]$$

which results in the theorem.

**4 Proof of the Unrestricted Multi-sum Identities**

**Theorem 4.1.** For integer $m \geq 1$:

$$\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \frac{a_i}{k_i} \left( \frac{k_i}{c_i} \right) = 2A_0 - C_0 \prod_{i=1}^{m} \frac{a_i}{c_i}$$

**Proof.** The multiple sum of the product on the left side is a product of sums:

$$\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \prod_{i=1}^{m} \frac{a_i}{k_i} \left( \frac{k_i}{c_i} \right) = \prod_{i=1}^{m} \sum_{k_i=0}^{a_i} \frac{a_i}{k_i} \left( \frac{k_i}{c_i} \right)$$

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The trinomial revision identity is applied as in theorem 3.1:

\[
\binom{a_i}{k_i} = \binom{a_i - c_i}{k_i - c_i} (a_i)_{c_i}
\] (4.3)

Because this expression is zero when \( c_i > a_i \), in the individual sums it can be assumed that \( 0 \leq c_i \leq a_i \). Then each sum can be evaluated as follows, where the upper bound \( a \) can be replaced by \( \infty \) because the summand is zero when \( k > a \):

\[
\sum_{k=0}^{a} \frac{(a - c)}{k - c} = \sum_{k=0}^{\infty} \text{Res}_w (1 + w)^{a-c} w^{k-c+1} \sum_{k=0}^{\infty} \binom{1}{w}^k
\]

\[
= \text{Res}_w (1 + w)^{a-c} w^{c-1} \frac{1}{1 - 1/w} = \text{Res}_w \frac{(1 + w)^{a-c} w^c}{w - 1} = 2^{a-c}
\] (4.4)

The product of these individual sums gives the theorem.

**Theorem 4.2.** For integer \( m \geq 1 \) and \( 1 \leq p \leq m \):

\[
\prod_{i=1}^{m} \sum_{k=0}^{a_i} \frac{(a_i - c_i)}{k_i - c_i} (a_i)_{c_i}
\]

\[
\sum_{k_i=0}^{\infty} \binom{a_i}{k_i} (k_i)_{c_i} = 2^{A_0 - C_0 - 1} \prod_{i=1}^{m} \binom{a_i}{c_i} (a_p + c_p)
\] (4.5)

**Proof.** The following modified infinite geometric series is used:

\[
\sum_{k=0}^{\infty} k w^k = \frac{w}{(1 - w)^2}
\] (4.6)

One of the individual sums of the product becomes:

\[
\sum_{k=0}^{a} \frac{(a - c)}{k - c} = \sum_{k=0}^{a} k \text{Res}_w (1 + w)^{a-c} w^{k-c+1} \sum_{k=0}^{\infty} \binom{1}{w}^k
\]

\[
= \text{Res}_w (1 + w)^{a-c} w^{c-1} \frac{1}{w(1 - 1/w)} = \text{Res}_w \frac{(1 + w)^{a-c} w^c}{(w - 1)^2} = (a - c) 2^{a-c-1} + c 2^{a-c}
\] (4.7)

The other individual sums of the product remain identical, so the product becomes the theorem.

**Theorem 4.3.** For integer \( m \geq 1 \):

\[
\prod_{i=1}^{m} \sum_{k_i=0}^{a_i} \binom{a_i}{k_i} (k_i)_{c_i} = 2^{A_0 - C_0 - 1} \prod_{i=1}^{m} \binom{a_i}{c_i} (A_1 + C_1)
\] (4.8)

**Proof.** Changing the order of summation and applying theorem 4.2:

\[
\sum_{i=1}^{m} x_i (a_i + c_i) = A_1 + C_1
\] (4.9)
Theorem 4.4. For integer $m \geq 1$ and $1 \leq p \leq m$:

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \left( \frac{a_i}{k_i} \right) \left( \frac{k_i}{c_i} \right) \right] k_p^2 = 2^{A_0-C_0-2} \left[ \prod_{i=1}^{m} \left( \frac{a_i}{c_i} \right) \right] [(a_p + c_p)^2 + a_p - c_p] \tag{4.10}
\]

Proof. The following modified infinite geometric series \[ is used:

\[
\sum_{k=0}^{\infty} k^2 w^k = \frac{w}{(1-w)^2} + \frac{2w^2}{(1-w)^3} \tag{4.11}
\]

One of the individual sums of the product becomes:

\[
\sum_{k=0}^{a} \left( \frac{a-c}{k-c} \right) k^2 = \text{Res}_w(1+w)^{a-c} w^{c-1} \sum_{k=0}^{\infty} k^2 \left( \frac{1}{w} \right)^k
\]

\[
= \text{Res}_w(1+w)^{a-c} w^{c-1} \left[ \frac{1}{w(1-1/w)^2} + \frac{2}{w^2(1-1/w)^3} \right]
\]

\[
= \text{Res}_w(1+w)^{a-c} w^{c-1} \left[ \frac{1}{(w-1)^2} + \frac{2}{(w-1)^3} \right]
\]

\[
= 2^{a-c-1}(a+c) + (a-c)(a-c+1)2^{a-c-2} + 2(a-c)c2^{a-c-1} + c(c-1)2^{a-c}
\]

\[
= 2^{a-c-2}[2(a+c) + (a-c)(a-c+1) + 4(a-c)c + 4c(c-1)]
\]

The other individual sums of the product remain identical, so the product becomes the theorem.

\[ \Box \]

Theorem 4.5. For integer $m \geq 1$, $1 \leq p \leq m$, $1 \leq q \leq m$ and $p \neq q$:

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \left( \frac{a_i}{k_i} \right) \left( \frac{k_i}{c_i} \right) \right] k_p k_q = 2^{A_0-C_0-2} \left[ \prod_{i=1}^{m} \left( \frac{a_i}{c_i} \right) \right] (a_p + c_p)(a_q + c_q) \tag{4.13}
\]

Proof. This theorem follows from applying theorem \[ twice.

\[ \Box \]

Theorem 4.6. For integer $m \geq 1$:

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \left( \frac{a_i}{k_i} \right) \left( \frac{k_i}{c_i} \right) \right] \left[ \sum_{i=1}^{m} x_i k_i \right] = 2^{A_0-C_0-2} \left[ \prod_{i=1}^{m} \left( \frac{a_i}{c_i} \right) \right] \left[ A_{1,1}^* - C_{1,1}^* + (A_1 + C_1)(A_1^* + C_1^*) \right] \tag{4.14}
\]

Proof. The product in the summand becomes:

\[
\left( \sum_{i=1}^{m} x_i k_i \right) \left( \sum_{i=1}^{m} y_i k_i \right) = \sum_{i=1}^{m} \sum_{j=1}^{m} x_i y_j k_i k_j + \sum_{i=1}^{m} x_i y_i k_i^2 \tag{4.15}
\]
Changing the order of summation and using theorem 4.4 and 4.5:

\[
\sum_{i=1}^{m} \sum_{j=1, j \neq i}^{m} x_i y_j (a_i + c_i)(a_j + c_j) + \sum_{i=1}^{m} x_i y_i [(a_i + c_i)^2 + a_i - c_i] = \sum_{i=1}^{m} \sum_{j=1}^{m} x_i y_j (a_i + c_i)(a_j + c_j) + \sum_{i=1}^{m} x_i y_i (a_i - c_i) = A_{1,1}^* - C_1^* + (A_1 + C_1)(A_1^* + C_1^*)
\]

\[ (4.16) \]

**Theorem 4.7.** For integer \( m \geq 1 \):

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \right] \sum_{i=1}^{m} x_i k_i^2 = 2^{A_0 - C_0 - 2} \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] [A_2 - C_2 + (A_1 + C_1)^2]
\]

\[ (4.17) \]

**Proof.** In the previous theorem taking \( y_i = x_i \), then \( A_1^* = A_1, C_1^* = C_1, A_{1,1}^* = A_2 \) and \( C_{1,1}^* = C_2 \) gives this theorem.

**Theorem 4.8.** For integer \( m \geq 1 \):

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \right] \sum_{i=1}^{m} |x_i k_i|^2 = 2^{A_0 - C_0 - 2} \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] (A_{\text{abs}} - C_{\text{abs}} + |A_1 + C_1|^2)
\]

\[ (4.18) \]

**Proof.** Using the previous theorem and using the same method as in theorem 3.8 and using:

\[
[\text{Re}(A_1) + \text{Re}(C_1)]^2 + [\text{Im}(A_1) + \text{Im}(C_1)]^2 = \text{Re}^2(A_1 + C_1) + \text{Im}^2(A_1 + C_1) = |A_1 + C_1|^2
\]

\[ (4.19) \]

gives the theorem.

**Theorem 4.9.** For integer \( m \geq 1 \):

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \right] \sum_{i=1}^{m} x_i k_i^2 = 2^{A_0 - C_0 - 2} \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] (A_1 - C_1 + A_{1,2} + C_{1,2} + 2S_{1,1})
\]

\[ (4.20) \]
Theorem 4.10. For integer \( m \geq 1 \) and \( 1 \leq p \leq m \):

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \left( \frac{a_i}{k_i} \right)^{(k_i)} \prod_{i=1}^{m} \left( \frac{a_i}{c_i} \right)^{(c_i)} \right] k_p^3 = 2^{A_0-C_0-3} \sum_{k=0}^{\infty} \frac{1}{w(k+1)} \prod_{i=1}^{m} \left( \frac{a_i}{c_i} \right)^{(c_i)} (a_p + c_p) \left( (a_p + c_p)^2 + 3(a_p - c_p) \right)
\]

(4.22)

Proof. The following modified infinite geometric series \( \sum \) is used:

\[
\sum_{k=0}^{\infty} k^p w^k = \frac{w}{(1-w)^2} + \frac{6w^2}{(1-w)^3} + \frac{6w^3}{(1-w)^4}
\]

(4.23)

One of the individual sums of the product becomes:

\[
\sum_{k=0}^{a} \left( \frac{a}{k-c} \right)^{k^p} = \text{Res}_w(1+w)^{a-c}w^{c-1} \sum_{k=0}^{\infty} \left( \frac{1}{w} \right)^k
\]

\[
= \text{Res}_w(1+w)^{a-c}w^{c-1} \left[ \frac{1}{w(1-w)^2} + \frac{6}{w^2(1-w)^3} + \frac{6}{w^3(1-w)^4} \right]
\]

\[
= \text{Res}_w(1+w)^{a-c}w^{c-1} \left[ \frac{1}{w(1-w)^2} + \frac{6}{w^2(1-w)^3} + \frac{6}{w^3(1-w)^4} \right]
\]

(4.24)

The other individual sums of the product remain identical, so the product becomes the theorem.

Theorem 4.11. For integer \( m \geq 1 \), \( 1 \leq p \leq m \), \( 1 \leq q \leq m \) and \( p \neq q \):

\[
\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \left( \frac{a_i}{k_i} \right)^{(k_i)} \prod_{i=1}^{m} \left( \frac{a_i}{c_i} \right)^{(c_i)} \right] k_p k_q^2 = 2^{A_0-C_0-3} \sum_{k=0}^{\infty} \frac{1}{w(k+1)} \prod_{i=1}^{m} \left( \frac{a_i}{c_i} \right)^{(c_i)} (a_p + c_p) \left( (a_p + c_p)^2 + a_q - c_q \right)
\]

(4.25)

Proof. This theorem follows from applying theorem 4.12 and 4.14.

\[ \square \]
Theorem 4.12. For integer $m \geq 3$, $1 \leq p \leq m$, $1 \leq q \leq m$, $1 \leq r \leq m$, $p \neq q \neq r$ and $p \neq r$:

$$\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \right] \sum_{i=1}^{m} x_i k_i k_r = 2^{A_0-C_0-3} \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] (a_p + c_p)(a_q + c_q)(a_r + c_r)$$

(4.26)

Proof. This theorem follows from applying theorem 4.11 three times.

Theorem 4.13. For integer $m \geq 1$:

$$\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \right] (\sum_{i=1}^{m} x_i k_i)^3$$

$$= 2^{A_0-C_0-3} \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] (A_1 + C_1)(A_1 + C_1)^2 + 3(A_2 - C_2)$$

(4.27)

Proof. Changing the order of summation and using theorem 4.12

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \left[ x_i x_j x_k (a_i + c_i) (a_j + c_j) (a_k + c_k) \right]$$

$$+ 3 \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \left[ x_i x_j^2 (a_i + c_i) (a_j + c_j)^2 + a_j - c_j \right]$$

$$+ \sum_{i=1}^{m} x_i^3 (a_i + c_i) [(a_i + c_i)^2 + 3(a_i - c_i)]$$

$$= (A_1 + C_1)^3 + 3 \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j^2 (a_i + c_i) (a_j - c_j) + 3 \sum_{i=1}^{m} x_i^3 (a_i + c_i) (a_i - c_i)$$

$$= (A_1 + C_1)^3 + 3(A_1 + C_1)(A_2 - C_2) = (A_1 + C_1)(A_1 + C_1)^2 + 3(A_2 - C_2)$$

(4.28)

Theorem 4.14. For integer $m \geq 1$:

$$\sum_{k_1=0}^{a_1} \cdots \sum_{k_m=0}^{a_m} \left[ \prod_{i=1}^{m} \binom{a_i}{k_i} \binom{k_i}{c_i} \right] \sum_{i=1}^{m} x_i k_i^3$$

$$= 2^{A_0-C_0-3} \left[ \prod_{i=1}^{m} \binom{a_i}{c_i} \right] [A_{1,3} + C_{1,3} + 3(A_{1,2} - C_{1,2} + S_{1,2} + S_{2,1})]$$

(4.29)
Proof. Changing the order of summation and using theorem \[4.10\]

\[
\sum_{i=1}^{m} x_i(a_i + c_i) [(a_i + c_i)^2 + 3(a_i - c_i)]
\]

\[
= \sum_{i=1}^{m} x_i(a_i^3 + c_i^3 + 3a_i^2 - 3c_i^2 + 3a_i c_i^2 + 3a_i^2 c_i)
\]

\[
= A_{1,3} + C_{1,3} + 3(A_{1,2} - C_{1,2} + S_{1,2} + S_{2,1})
\]  \hfill \Box

5 A Proof of Residue Identity 3.5

Theorem 5.1. When a complex function \(f(z)\) has a pole of order \(m\) at \(z = z_p\), then the residue of this function at \(z = z_p\) is \([7]\):

\[
\text{Res}_{z=z_p} f(z) = \frac{1}{(m-1)!} D_z^{m-1} [(z-z_p)^m f(z)]|_{z=z_p}
\]  \hfill (5.1)

where \(D_z^m f(z)|_{z=z_p}\) is the \(n\)-th derivative of \(f(z)\) at \(z = z_p\).

Proof. When a complex function \(f(z)\) has a pole of order \(m\) at \(z = z_p\), then the residue of \(f(z)\) at \(z = z_p\) can be defined as the coefficient \(a_{-1}\) in the Laurent series expansion of \(f(z)\) at \(z = z_p\), where \(a_{-m} \neq 0\) \([7, 12]\):

\[
f(z) = \sum_{k=-m}^{\infty} a_k(z-z_p)^k
\]  \hfill (5.2)

From this follows:

\[(z-z_p)^m f(z) = \sum_{k=-m}^{\infty} a_k(z-z_p)^{m+k} = \sum_{k=0}^{\infty} a_{k-m}(z-z_p)^k
\]  \hfill (5.3)

For integer \(k \geq 0\):

\[
D_z^n (z-z_p)^k = \begin{cases} (z-z_p)^{k-n} \prod_{j=0}^{n-1} (k-j) & \text{if } k \geq n \\ 0 & \text{if } k < n \end{cases}
\]  \hfill (5.4)

The \(n\)-th derivative of \((z-z_p)^m f(z)\) at \(z = z_p\) becomes, by differentiating the infinite power series term by term \([9]\), and using \(0^0 = 1\) \([5]\), where \(\delta_{k,n}\) is the Kronecker delta:

\[
\frac{1}{n!} D_z^n \sum_{k=0}^{\infty} a_{k-m}(z-z_p)^k|_{z=z_p} = \frac{1}{n!} \sum_{k=0}^{\infty} a_{k-m} D_z^n (z-z_p)^k|_{z=z_p}
\]

\[
= \frac{1}{n!} \sum_{k=n}^{\infty} a_{k-m}(z-z_p)^{k-n} \prod_{j=0}^{n-1} (k-j)|_{z=z_p} = \frac{1}{n!} \sum_{k=n}^{\infty} a_{k-m} \delta_{k,n} \prod_{j=0}^{n-1} (k-j)
\]  \hfill (5.5)

\[
= \frac{1}{n!} \sum_{k=n}^{\infty} a_{k-m}\delta_{k,n} \prod_{j=0}^{n-1} (k-j) = \frac{1}{n!} a_{n-m} \prod_{j=0}^{n-1} (n-j) = \frac{1}{n!} a_{n-m} n! = a_{n-m}
\]
Taking \( n = m - 1 \) the theorem is proved.  

**Theorem 5.2.** When a complex function \( f(z) \) does not have a pole at \( z = z_p \), then:

\[
\text{Res}_{z=z_p} \frac{f(z)}{(z-z_p)^m} = \frac{1}{(m-1)!} D_z^{m-1} f(z)|_{z=z_p} \quad (5.6)
\]

**Proof.** Because \( f(z) \) does not have a pole at \( z = z_p \), the power series expansion of \( f(z) \) at \( z = z_p \) is the Taylor series expansion:

\[
f(z) = \sum_{k=0}^{\infty} a_k (z-z_p)^k \quad (5.7)
\]

and therefore:

\[
\frac{f(z)}{(z-z_p)^m} = \sum_{k=0}^{\infty} a_k (z-z_p)^{k-m} = \sum_{k=-m}^{\infty} a_{m+k} (z-z_p)^k = \sum_{k=-m}^{\infty} b_k (z-z_p)^k \quad (5.8)
\]

where \( b_k = a_{m+k} \). The residue of this function at \( z = z_p \) is \( b_{-1} = a_{m-1} \), and using the same method as in the previous theorem:

\[
\frac{1}{n!} D_z^n f(z)|_{z=z_p} = \frac{1}{n!} D_z^n \sum_{k=0}^{\infty} a_k (z-z_p)^k|_{z=z_p} = a_n \quad (5.9)
\]

Taking \( n = m - 1 \) the theorem is proved.  

When in the last theorem \( a_0 \neq 0 \), the last two theorems are equivalent.

**References**

[1] L.V. Ahlfors, *Complex Analysis*, McGraw-Hill, 1979.

[2] G.P. Egorychev, *Integral Representation and the Computation of Combinatorial Sums*, Translations of Mathematical Monographs, 59, Amer. Math. Soc., 1984.

[3] H.W. Gould, *Combinatorial Identities*, rev. ed., Morgantown, 1972.

[4] H.W. Gould, H.M. Srivastava, Some Combinatorial Identities Associated with the Vandermonde Convolution, *Appl. Math. Comput.* 84 (1997) 97-102.

[5] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics, A Foundation for Computer Science*, 2nd ed., Addison-Wesley, 1994.

[6] K. Knopp, *Theory and Application of Infinite Series*, Dover Publications, 1990.

[7] S.G. Krantz, *A Guide to Complex Variables*, The Mathematical Association of America, 2008.
[8] M.J. Kronenburg, A Generalization of the Chu-Vandermonde Convolution and some Harmonic Number Identities, arXiv:1701.02768 [math.CO]

[9] M.J. Kronenburg, Some Weighted Generalized Fibonacci Number Summation Identities, Part 1, arXiv:1903.01407 [math.NT]

[10] M.J. Kronenburg, The Binomial Coefficient for Negative Arguments, arXiv:1105.3689 [math.CO]

[11] R. Meštrović, Several Generalizations and Variations of Chu-Vandermonde Identity, arXiv:1807.10604 [math.CO]

[12] T. Rowland, E.W. Weisstein, Complex Residue. From Mathworld - A Wolfram Web Resource, https://mathworld.wolfram.com/ComplexResidue.html