AN EXAMPLE OF NON-HOMOEO MORPHIC CONJUGATE VARIETIES

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ABSTRACT. We give examples of smooth quasi-projective varieties over complex numbers, in the context of connected Shimura varieties, which are not homeomorphic to a conjugate of itself by an automorphism of the complex numbers.

1. INTRODUCTION

Let $X$ be a quasi-projective variety defined over $\mathbb{C}$. Suppose $\sigma$ is an automorphism of $\mathbb{C}$. Denote by $X^\sigma := X \times_{\sigma} \mathbb{C}$, the conjugate of $X$ by the automorphism $\sigma$ of $\mathbb{C}$, obtained by applying the automorphism $\sigma$ to the coefficients of the polynomials defining $X$. It is known that the varieties $X$ and $X^\sigma$ have the same Betti numbers.

In [Se], Serre gave an example such that the topological spaces $X(\mathbb{C})$ and $X^\sigma(\mathbb{C})$ are not homeomorphic.

Recently, Milne and Suh [MS], gave further examples in the context of connected Shimura varieties. Their method is to find a conjugate such that the reductive group underlying the Shimura datum is different, and then apply the super-rigidity results of Margulis.

Our examples are in the same context as that of Milne and Suh, but we work with Shimura’s construction of canonical models (SH). Shimura’s construction allows us to identify the adelic congruence subgroup defining the conjugate variety as a conjugate by an element of the adjoint group. We then appeal to Mostow rigidity and the failure of strong approximation (or non-triviality of class number) for the adjoint group to get at the desired examples. In our example, the congruent lattices defining the variety and it’s conjugate are commensurable. Earlier in [R], we observed using Shimura’s construction coupled with the theorems of Labesse and Langlands on the multiplicity of cusp forms for $SL(1, D)$, that a Galois twist of these spaces attached to $SL(1, D)$ over the reflex field preserves the spectrum of the Laplacian; this provides examples of locally symmetric spaces attached to a quaternion division algebra over a number field which are isospectral but not isometric.

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Let $F$ be a totally real number field of degree at least two. Let $D$ be an indefinite quaternion algebra defined over $F$. We assume that $D$ is split at exactly one real place, say $\tau_1$ of $F$. This assumption allows us to assume that the reflex field of $(F, \tau_1)$ to be $F$ itself. Let $V$ be a vector space of rank $n \geq 2$ over $D$, equipped with a hermitian inner product with respect to the standard involution on $D$. We assume that the inner product is definite on the spaces $V \otimes_\tau \mathbb{R}$, for all real embeddings $\tau$ of $F$ different from $\tau_1$. In particular, since we have assumed that the degree of $F$ is at least two, the form $h$ is anisotropic. Let $G$ be the group of unitary similitudes of $h$. We consider $G$ as an algebraic group defined over $\mathbb{Q}$, and let $G_d$ the derived group of $G$. We let $\text{PG}$ denote the projective group attached to $G$, the group obtained by taking the quotient of $G$ modulo its centre. Under our assumptions, it follows that

$$G_d(\mathbb{R}) \simeq \text{Sp}(2n, \mathbb{R}) \times \text{a compact group}, \quad n \geq 2.$$ 

Let $K_\infty$ be a maximal compact subgroup of $G_d(\mathbb{R})$ and let $X = G_d(\mathbb{R})/K_\infty$ be the non-compact symmetric space associated to $G$. By our assumptions, $X$ is isomorphic to the Siegel upper half space $\mathbb{H}_n$ of dimension $n$. Denote by $\mathbf{A}$ the adele ring of $F$, and by $\mathbf{A}_f$ the subring of finite adeles. Let $K$ be a compact open subgroup of $G(\mathbf{A}_f)$, and let $K_d = K \cap G_d(\mathbf{A}_f)$. Denote by

$$\Gamma_K = G(\mathbb{R})K \cap G(\mathbb{Q}) \quad \text{and} \quad \Gamma_{d,K} = G_d(\mathbb{R})K_d \cap G_d(\mathbb{Q}),$$

the corresponding arithmetic lattices in $G(\mathbb{R})$ and $G_d(\mathbb{R})$ respectively. We assume that $K$ is such that $\Gamma_{d,K}$ is torsion-free, and the natural inclusion $\Gamma_{d,K} \subset \Gamma_K$ is an isomorphism modulo the centre of $\Gamma_K$.

By a theorem of Baily-Borel, the quotient space $X_K = \Gamma_K \backslash X$ is a connected, smooth, projective variety. The fundamental group $\overline{\Gamma}_K$ of the variety $X_K$ can be identified with the projection of $\Gamma_K$ to $\text{PG}(\mathbb{R})$, and also with the lattice $\Gamma_{d,K}$ contained in $G_d(\mathbb{R})$.

For an element $x \in G(\mathbf{A}_f)$, denote by $K^x$ the conjugate lattice $x^{-1}Kx$, and by $\overline{x}$ its image in $\text{PG}(\mathbf{A}_f)$. Further, let $N(\overline{K})$ denote the normalizer of $\overline{K}$ in $\text{PG}(\mathbf{A}_f)$, where $\overline{K}$ is the image of $K_d$ in $\text{PG}(\mathbf{A}_f)$. The desired example is provided by the following theorem:

**Theorem 1.** With notation and assumptions as above, suppose $x$ is an element in $G(\mathbf{A}_f)$ such that $\overline{x}$ does not belong to the set $N(\overline{K})\text{PG}(\mathbb{Q})$. Then $X_K$ and $X_{K^x}$ are conjugate by an automorphism $\sigma$ of $\mathbb{C}$, but the respective fundamental groups $\overline{\Gamma}_K$ and $\overline{\Gamma}_{K^x}$ are not isomorphic. In particular, $X_K$ and $X_{K^x}$ are not homeomorphic.

**Remark.** The hypothesis can be seen to hold from two different but related aspects of the arithmetic of algebraic groups. The normalizer $N(\overline{K})$ is a compact open subgroup of $\text{PG}(\mathbf{A}_f)$. By the failure of strong approximation for the adjoint group $\text{PG}$ (see [PR, Proposition 7.13]), the rational points $\text{PG}(\mathbb{Q})$ are not dense in $\text{PG}(\mathbf{A}_f)$. Hence, the hypothesis that $\overline{x}$ does not belong to the double coset $N(\overline{K})\text{PG}(\mathbb{Q})$ is satisfied provided $N(\overline{K})$ is small enough.
On the other hand, the adjoint group $PG$ is not simply connected, hence has a non-trivial fundamental group. The results of Section 8.2 of [PR], show that the class group of $G$ is non-trivial for suitably chosen congruence lattices in $PG(A_f)$. This allows us to work with large congruence lattices in $PG(A_f)$.

Proof. We first show that the varieties $X_K$ and $X_{K^x}$ are conjugate by an automorphism of $\mathbb{C}$. For this, we recall Shimura’s theory of canonical models [Sh]. Let $\nu: G \to G_m$ be the reduced norm. By class field theory, the subgroup $F^*\nu(K)$ of the idele group $A^*$ defines an abelian extension $F_K$ of $F$. The reciprocity morphism of class field, 

$$\text{rec}: A^*/F^* \to \text{Gal}(F^{ab}/F),$$

defines an element $\sigma(x) \in \text{Gal}(F^{ab}/F)$ by the prescription

$$\sigma(x) = \text{rec}(\nu(x)^{-1}).$$

As a consequence of the main theorem of canonical models in [Sh, Theorem 2.5, page 159, Section 2.6], the variety $X_K$ has a model defined over the field $F_K$, and

$$X_{K^\sigma(x)} \simeq X_{K^x}.$$  
(2.1)

Thus the varieties $X_K$ and $X_{K^x}$ are conjugate.

Suppose on the contrary, that $X_K$ and $X_{K^x}$ have isomorphic fundamental groups. Since these spaces are Eilenberg-Maclane spaces, there exists a homotopy equivalence

$$\phi: X_K \to X_{K^x}.$$ 

Since the lattices are irreducible in $PG(\mathbb{R})$ and the real rank of $PG$ is at least two, by Mostow rigidity [Mo], the spaces $X_K$ and $X_{K^x}$ are isometric.

Hence, there exists $\overline{g} \in PG(\mathbb{R})$ such that

$$\overline{g}^{-1}\Gamma_{K^x}\overline{g} = \Gamma_K.$$ 

Since the lattices $\Gamma_K$ and $\Gamma_{K^x}$ are arithmetic and commensurable, it follows by a theorem of Borel ([Bo]), that $\overline{g} \in PG(Q)$. Hence there is an element $g \in G(Q)$ satisfying,

$$g^{-1}\Gamma_{d,K}g = \Gamma_{d,K}.$$ 

Consider now $G_d(Q)$ embedded diagonally in $G_d(A_f)$. By the strong approximation theorem for $G_d$, the closure of $\Gamma_{d,K}$ in $G_d(A_f)$ can be identified with $K_d$. Further, the closure of $\Gamma_{d,K^x}$ in $G_d(A_f)$ can be identified with $g^{-1}K_d^xg$, where we now consider $g \in G(Q)$ as diagonally embedded in $G(A_f)$. Hence, we have

$$g^{-1}K_d^xg = K_d.$$ 

Projecting to $PG$, we obtain

$$\overline{g}^{-1}\overline{\tau}^{-1}\overline{K\tau g} = \overline{K},$$

where $\overline{K}$ denotes the image of $K_d$ in $PG(A_f)$. This implies that $\overline{\tau} \in N(\overline{K})PG(Q)$, contradicting our choice of $\overline{\tau}$. $\square$

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REFERENCES

[Bo] A. Borel, *Density and maximality of arithmetic subgroups*, J. Reine Angew. Math. **224** (1966) 78–89.

[MS] J. S. Milne and J. Suh, *Nonhomeomorphic conjugates of connected Shimura varieties*, Amer. J. Math. **132**, no. 3 (2010), 731-750.

[Mo] G. D. Mostow, *Strong rigidity of locally symmetric spaces* Annals of Mathematics Studies, No. 78, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, (1973).

[PR] V. P. Platonov and A. R. Rapinchuk, *Algebraic groups and number theory*, Translated from the 1991 Russian original by Rachel Rowen. Pure and Applied Mathematics, 139. Academic Press, Inc., Boston, MA, (1994).

[R] C. S. Rajan, On isospectral arithmetical spaces, Amer. J. Math. **129** (2007), no. 3, 791–806.

[Se] J.-P. Serre, *Exemples de variétés projectives conjuguées nonhoéomorphes*, C. R. Acad. Sci. Paris 258 (1964) 4194-4196.

[Sh] G. Shimura, *On canonical models of arithmetic quotients of bounded symmetric domains*, Annals of Mathematics, 91, (1970), 144-222.

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