A Morita type proof of the replica-symmetric formula for SK

Erwin Bolthausen, University of Zurich

Abstract
We give a proof of the replica symmetric formula for the free energy of the Sherrington-Kirkpatrick model in high temperature which is based on the TAP formula. This is achieved by showing that the conditional annealed free energy equals the quenched one, where the conditioning is given by an appropriate $\sigma$-field with respect to which the TAP solutions are measurable.

1 Introduction
We consider the standard Sherrington-Kirkpatrick model with an external field having the random Hamiltonian

$$H_{\beta,h}(\sigma) := \beta \frac{1}{\sqrt{2}} \sum_{i,j=1}^{N} g_{ij}^{(N)} \sigma_i \sigma_j + h \sum_{i=1}^{N} \sigma_i$$  \hspace{1cm} (1.1)

where $\beta > 0$ and $h \in \mathbb{R}$ are real parameters, $\sigma = (\sigma_i) \in \Sigma_N := \{-1,1\}^N$, and $g_{ij}^{(N)}$ for $i,j$ are i.i.d. centered Gaussians with variance $1/N$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The random partition function is

$$Z_{N,\beta,h} := 2^{-N} \sum_{\sigma} \exp \left[ H_{\beta,h}(\sigma) \right],$$

and the Gibbs distribution is

$$\text{GIBBS}_{N,\beta,h}(\sigma) := \frac{2^{-N}}{Z_{N,\beta,h}} \exp \left[ H_{\beta,h}(\sigma) \right].$$  \hspace{1cm} (1.2)

It is known that

$$f(\beta,h) := \lim_{N \to \infty} \frac{1}{N} \log Z_{N,\beta,h} = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_{N,\beta,h}$$

exists, is non-random, and is given by the Parisi variational formula (see [6], [14], [10]). Furthermore, for small $\beta$, $f(\beta,h)$ is given by the replica-symmetric formula, originally proposed by Sherrington and Kirkpatrick (12):

Theorem 1 There exists $\beta_0 > 0$ such that for all $h, \beta$ with $\beta \leq \beta_0$

$$f(\beta,h) = \text{RS}(\beta,h) := \inf_{q \geq 0} \left[ \int \log \cosh (h + \beta \sqrt{q} x) \phi(dx) + \frac{\beta^2 (1-q)^2}{4} \right].$$

Here, $\phi$ is the standard Gaussian distribution.
For \( h \neq 0 \), the infimum is uniquely attained at \( q = q(\beta, h) \) which satisfies

\[
q = \int \tanh^2 (h + \beta \sqrt{q} x) \phi (dx).
\]

This equation has a unique solution for \( h \neq 0 \), and for \( h = 0 \) if \( \beta \leq 1 \). For \( \beta > 1 \) (and \( h = 0 \)), there are two solutions, one being 0, and a positive one, which is the relevant for the minimization (see [13]). We will assume \( h > 0 \), and \( q \) will exclusively be used for this number.

\( f(\beta, h) = \text{RS}(\beta, h) \) is believed to be true under the de Almeida-Thouless condition (AT-condition for short)

\[
\beta^2 \int \frac{\phi (dx)}{\cosh^4 (h + \beta \sqrt{q} x)} \leq 1,
\]

but this is still an open problem. At \( h = 0 \), the AT-condition is \( \beta \leq 1 \), and in this regime, \( f(\beta, 0) = \text{RS}(\beta, 0) = \beta^2 / 4 \) is known since long and can easily be proved by a second moment method. In fact, in this case, the free energy equals the annealed free energy

\[
f(\beta, 0) = f_{\text{ann}}(\beta, 0) = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} Z_{N,\beta,0} = \beta^2 / 4.
\]

It is however easy to see that for \( h \neq 0 \), and any \( \beta > 0 \), neither \( f(\beta, h) \) nor \( \text{RS}(\beta, h) \) equals \( f_{\text{ann}}(\beta, h) \).

The aim of this note is to prove that \( f(\beta, h) = \text{RS}(\beta, h) \) can, for small \( \beta \), be proved by a conditional “quenched=annealed” argument, via a second moment method. Roughly speaking, we prove that there is a sub-\( \sigma \)-field \( G \subset \mathcal{F} \) such that \( f(\beta, h) = \lim_{N \to \infty} N^{-1} \log \mathbb{E} (Z_N | G) = \text{RS}(\beta, h) \) almost surely, and where we can estimate the conditional second moment by the square of the first one. A key point is the connection of \( G \) (it will actually be a sequence of \( \sigma \)-fields) with the Thouless-Anderson-Palmer equation, introduced in [15], and in particular with the recursive construction given in [3]. The reason the method works is that the conditionally annealed Gibbs measure is essentially a Curie-Weiss type model, centered at the solution of the TAP equation, and as such it can be analyzed as a classical mean-field model.

The method is closely related to arguments used for the first time by Morita in [9]. In fact, Morita invented the method to derive the quenched free energy by a partial annealing, fixing part of the Hamilton which is handled in a “quenched way”, but where this quenched part can be analyzed much easier than for the full Hamiltonian. This is exactly what we do here by the conditioning.

Unfortunately, the argument does not seem to work in the full AT-region. This is partly due to the fact that the second moment method does not work up to the correct critical line. There are however also other difficulties.

Therefore, the result we prove is not new at all, and in fact, the proof is quite longer than existing proofs. However, we believe the method is of interest, and can be used quite broadly for other models.

A related approach has recently been developed independently by Jian Ding and Nike Sun [5] for the lower bound of the memory capacity of a version of the perceptron at zero temperature up to the predicted critical value for the validity of the replica symmetric solution.
Our proof given does not use any of the results on the SK-model obtained previously, except for very simple ones, like the proof of the uniqueness of $q$ for $h \neq 0$, and on some simple computations from [3]. The core of the argument given here does not use the result from [3], but it is motivated by the construction given there.

**Basic assumptions and notations:** We always assume $h \neq 0$, as there is nothing new in the argument for $h = 0$ (but see the comments at the end of the paper). For convenience, we assume $h > 0$. We usually drop the $N$ in $g_i^{(N)}$, but the reader should keep in mind that essentially any formula we write depends on the size parameter $N$. We also often drop the parameters $\beta, h$ in the notation. If we write “for $\beta$ small enough”, we mean that there exists $\beta_0 > 0$ such that the statement holds for $\beta \leq \beta_0$ and for all $h > 0$. We will not be specific about $\beta_0$.

We typically use boldface letters, like $x$, for vectors in $\mathbb{R}^N$, occasionally random vectors, with components $x_1, \ldots, x_N$. If $f : \mathbb{R} \to \mathbb{R}$, we write $f(x) \in \mathbb{R}^N$ for the vector with components $f(x_i)$.

In $\mathbb{R}^N$, we will use the inner product

$$\langle x, y \rangle := \frac{1}{N} \sum_{i=1}^{N} x_i y_i,$$

and the norm $\|x\| := \sqrt{\langle x, x \rangle}$. We will also use the shorthand $\text{Th}(x) := \tanh( h + \beta x )$.

We use $Z, Z', Z_1$ etc. for generic standard Gaussian random variables. If several of them appear in one formula, then they are assumed to be independent.

We write $\mathbb{E}$ for the expectation with respect to them. “Gaussian” always means centered Gaussian unless stated otherwise. We hope the reader will not confuse these $Z$’s with the partition functions, but it should always be clear from the context what is what.

The Gibbs expectation under (1.2) is usually written as $\langle \cdot \rangle$. $C$ is used as a generic positive constant which may change from line to line.

If $a, b \in \mathbb{R}^N$, we write $a \otimes b$ for the matrix

$$(a \otimes b)_{ij} := \frac{a_i b_j}{N}.$$  

Remark that if $a, b, c \in \mathbb{R}^N$, then $(a \otimes b) c = (b, c) a$.

If $A$ is matrix, we write $A^T$ for the transposed, and if $A$ is square

$$\bar{A} := \frac{1}{\sqrt{2}} (A + A^T).$$

**Outline of the argument:** We end the introduction with a quick outline of the main idea. The Gibbs means $\mathbf{m} := \langle \sigma \rangle$ are random variables. These random variables satisfy (in the $N \to \infty$ limit) the so-called TAP equations. The basic idea is to write the partition function $Z_N$ in terms of an average over an appropriately tilted coin-tossing measure

$$p(\sigma) = \prod_{i=1}^{N} p_i(\sigma_i)$$
where
\[ p_i(\sigma_i) = \frac{2^{-N}e^{h_i} \sigma_i}{\cosh (h_i)}, \quad \sigma_i = \pm 1, \]
where \( h \) satisfies \( m = \tanh (h) \), i.e. the expectation of \( \sigma_i \) under \( p_i \) is exactly \( m_i \) where \( m \) satisfies (approximately) the TAP equations
\[ m = Th (\bar{g}m - \beta (1 - q)m). \]

Then
\[ Z_N = \sum_\sigma 2^{-N} \exp [H(\sigma)] = \prod_{i=1}^{N} \cosh (h_i) \sum_\sigma p(\sigma) \exp [H(\sigma) - N \langle h, \sigma \rangle]. \]

\[ \frac{1}{N} \log Z = \frac{1}{N} \sum_{i=1}^{N} \log \cosh (h_i) + \frac{1}{N} \log \sum_\sigma p(\sigma) \exp [H(\sigma) - N \langle h, \sigma \rangle]. \]

The a.s.-limit of the first part will be easy to evaluate, and gives
\[ E \log \cosh (h + \beta \sqrt{q}Z) \]
which is the first part of the replica symmetric formula. For the second part, we apply a variant of the second moment method, but it is quite delicate, as the measures \( p \) depend on the random variables \( g_{ij} \). Therefore, we construct a sub-\( \sigma \)-field \( \mathcal{G} \) which has the property that \( m \) is \( \mathcal{G} \)-m.b. Then one has
\[ E \left( \sum_\sigma p(\sigma) \exp [H(\sigma) - N \langle h, \sigma \rangle] \middle| \mathcal{G} \right) = \sum_\sigma p(\sigma) E \left( \exp [H(\sigma) - N \langle h, \sigma \rangle] \middle| \mathcal{G} \right), \]
and it will turn out that \( E \left( \exp [H(\sigma) - N \langle h, \sigma \rangle] \middle| \mathcal{G} \right) \approx \exp \left[ N \beta^2 (1 - q)^2 / 4 \right] \) for small \( \beta \). Furthermore, one can estimate the conditional second moment. The implementation of this idea requires not one \( \sigma \)-field \( \mathcal{G} \), but a sequence \( \{\mathcal{G}_k\} \).

2 The recursive modification of the interaction matrix

We will not explicitly use the TAP equations, but the reader should keep in mind the rough outline of the argument given above. In spirit, we will heavily rely on the construction in [3], but we will not use in a substantial way the results of this paper. For the purpose here, it is simpler to work directly with random variables which are approximations of the iterative scheme in [3] which constructed approximations for the TAP equations through
\[ m^{(k+1)} := Th \left( \bar{g}m^{(k)} - \beta (1 - q)m^{(k-1)} \right) \quad (2.1) \]
with some initialization. We proved in [3] that these random variables defined through this iteration have a representation which makes it possible to prove the convergence in the full high temperature region. We directly use here this representation without using the iterative scheme above. There is a further
slight, but technically convenient, modification to the approach in [3]. There, we took the symmetrized matrix $g = (g_{ij})$ which has i.i.d. Gaussian entries for $i < j$ with variance $1/N$, and $g_{ii} = 0$. Fixing the diagonal to be 0 is of course of no relevance as the diagonal part cancels out in the Gibbs distribution. We then did construct a sequence $g$ of modifications $g^{(k)}$, and a sequence $F_k$ of sub-$\sigma$-fields, whose behavior is the crucial part of the analysis. In particular, the $g^{(k)}$ are conditionally Gaussian, given $F_{k-2}$, and conditionally independent of $F_{k-1}$. Of crucial importance for the analysis in [3] and also for the analysis here is the behavior of the conditional covariances. Unfortunately, the estimates for these in [3] were quite complicated, and we need them here still a bit more precise.

It turns out that these computations are simpler by sticking to $g_{ij}$ which are independent for $i, j \leq N$. The symmetrized matrix is then $\bar{g} = \frac{(g + g^T)}{\sqrt{2}}$. This looks being a trivial rewriting, but we will define the $\sigma$-fields $G_k$ here in terms of $g$, and therefore, they are different from the $F_k$ used in [3]. The main advantage is that the construction of the $g^{(k)}$ is explicit for all $k$, and the conditional covariances we need are totally explicit as well, which simplifies the computations considerably.

We construct sequence $\{\gamma_k\}_{k \geq 1}$, $\{\rho_k\}_{k \geq 1}$ of real numbers, and sequences of random matrices $g^{(k)}$ together with sequences of random vectors $\phi^{(k)} \in \mathbb{R}^N$, $k \geq 1$. Define

$$\gamma_1 = E \tanh (h + \beta \sqrt{q} Z), \quad \rho_1 := \sqrt{q} \gamma_1,$$

and recursively

$$\rho_k := \psi (\rho_{k-1}), \quad \gamma_k := \frac{\rho_k - \sum_{j=1}^{k-1} \gamma_j^2}{\sqrt{q} - \sum_{j=1}^{k-1} \gamma_j^2}$$

where $\psi : [0, q] \to (0, q]$ is defined by

$$\psi (t) := E \Th \left( \sqrt{t} Z + \sqrt{q - t} Z' \right) \Th \left( \sqrt{t} Z + \sqrt{q - t} Z'' \right).$$

Remark that $\psi (q) = q$, and $\psi (0) = \gamma_1^2$. The following easy result was proved in [3].

**Lemma 2**

a) $\{\rho_k\}$ is an increasing sequence of positive numbers. $\lim_{k \to \infty} \rho_k = q$ holds if and only if (1.4) is satisfied. If (1.4) holds with the strict inequality, then the convergence of $\{\rho_k\}$ is exponentially fast.

b) $\Gamma_{k-1}^2 := \sum_{j=1}^{k-1} \gamma_j^2 < \rho_k < q$ holds for all $k$, and $\sum_{j=1}^{\infty} \gamma_j^2 = q$ holds if and only if (1.4) is satisfied.

Next, we define the recursions for $g^{(k)}$, $\phi^{(k)}$. It is convenient to also introduce vectors $h^{(k)}$, $m^{(k)}$ which are directly related to the $\phi$’s. (The $m^{(k)}$ are the approximate solutions of the TAP equations). For $k = 1$:

$$g^{(1)} := g, \quad \phi^{(1)} := 1, \quad m^{(1)} := \sqrt{\gamma_1}.$$
Assume that \( g^{(s)} \), \( \phi^{(s)} \), \( m^{(s)} \) are defined for \( s \leq k \). Set
\[ \xi^{(s)} := g^{(s)} \phi^{(s)}, \quad \eta^{(s)} := g^{(s)T} \phi^{(s)}, \] (2.2)
\[ \zeta^{(s)} := \frac{\xi^{(s)} + \eta^{(s)}}{\sqrt{2}} = g^{(s)} \phi^{(s)}, \]
and we write
\[ G_k := \sigma \left( \zeta^{(m)}, \eta^{(m)} : m \leq k \right). \] (2.3)
We will write \( E_k \) for the conditional expectation with respect to to \( G_k \). Remark that \( \langle \phi^{(k)}, \zeta^{(k)} \rangle = \langle \eta^{(k)}, \phi^{(k)} \rangle \).

Put first
\[ h^{(k+1)} := h1 + \beta \sum_{s=1}^{k-1} \gamma_s \zeta^{(s)} + \beta \sqrt{q - \Gamma_{k-1}^2} \zeta^{(k)}, \] (2.4)
\[ m^{(k+1)} := \tan \left( h^{(k+1)} \right). \] (2.5)
We haven’t defined \( h^{(1)} \), but we could put it \( \tan^{-1} \left( \sqrt{q} \right) \).

We next define
\[ \phi^{(k+1)} := \frac{m^{(k+1)} - \sum_{s=1}^{k} \langle m^{(k+1)}, \phi^{(s)} \rangle \phi^{(s)}}{\| m^{(k+1)} - \sum_{s=1}^{k} \langle m^{(k+1)}, \phi^{(s)} \rangle \phi^{(s)} \|}. \] (2.6)
This requires that the denominator is \( \neq 0 \) which is true with probability 1 (Lemma 5), assuming \( N > k \). Finally
\[ g^{(k+1)} := g^{(k)} - \rho^{(k)}, \] (2.7)
with
\[ \rho^{(k)} := \xi^{(k)} \otimes \phi^{(k)} + \phi^{(k)} \otimes \eta^{(k)} - \left\langle \phi^{(k)}, \xi^{(k)} \right\rangle \left( \phi^{(k)} \otimes \phi^{(k)} \right). \] (2.8)

**Lemma 3**

a) \( \| \phi^{(k)} \| = 1 \) for all \( k \), and \( \langle \phi^{(k)}, \phi^{(t)} \rangle = 0 \) for \( k \neq t \).

b) For \( s < k \), one has \( g^{(k)} \phi^{(s)} = 0 \), and \( g^{(k)T} \phi^{(s)} = 0 \).

c) \( m^{(k)} \) and \( \phi^{(k)} \) are \( \mathcal{G}_{k-1} \)-m.b. for all \( k \geq 1 \).

**Proof.** a) is evident by the definition.

b) We use induction on \( k \). For \( k = 1 \), there is nothing to prove. For \( k = 2 \), one just has to check that \( g^{(2)} \mathbf{1} = 0, g^{(2)T} \mathbf{1} = 0 \), which are straightforward. So, we assume \( k \geq 3 \). If \( s = k - 1 \), using \( \langle \phi^{(k-1)}, \phi^{(k-1)} \rangle = 1 \)
\[ g^{(k)} \phi^{(k-1)} := g^{(k-1)} \phi^{(k-1)} - \rho^{(k-1)} \phi^{(k-1)} = \xi^{(k-1)} - \xi^{(k-1)} - \left\langle \eta^{(k-1)}, \phi^{(k-1)} \right\rangle \phi^{(k-1)} + \left\langle \eta^{(k-1)}, \phi^{(k-1)} \right\rangle \phi^{(k-1)} = 0. \]

If \( s \leq k - 2 \), we have by induction
\[ g^{(k)} \phi^{(s)} = -\rho^{(k-1)} \phi^{(s)}, \]
and using $\langle \phi^{(k-1)}, \phi^{(s)} \rangle = 0$, and again induction, we have
\[
\rho^{(k-1)} \phi^{(s)} = \xi^{(k-1)} \langle \phi^{(k-1)}, \phi^{(s)} \rangle + \phi^{(k-1)} \langle \phi^{(k-1)}, \phi^{(s)} \rangle - \langle \phi^{(k-1)}, \phi^{(k-1)} \rangle \langle \phi^{(k-1)}, \phi^{(s)} \rangle \phi^{(k-1)} = 0.
\]
$\phi^{(s)} g^{(k)} = 0$ is proved similarly.

c) It suffices to check that for $m^{(k)}$. As $\zeta^{(s)}$ is $G$-m.b. for $s \leq k - 1$, the claim follows. 

The motivation for the construction of $g^{(k)}$ in the form given in (2.7) is the following

**Proposition 4**

a) Conditionally on $G_{k-2}$, $g^{(k)}$ and $g^{(k-1)}$ are Gaussian. The conditional covariances of $g^{(k)}$ given $G_{k-2}$ are given by
\[
E_{k-2} \left( g^{(k)}_{ij} g^{(k)}_{st} \right) = \frac{1}{N} \left[ \delta_{is} - \alpha_{is}^{(k-1)} \right] \left[ \delta_{jt} - \alpha_{jt}^{(k-1)} \right], \tag{2.9}
\]
with the abbreviation
\[
\alpha_{ij}^{(m)} := \frac{1}{N} \sum_{r=1}^{m} \phi^{(r)}_i \phi^{(r)}_j.
\]

(By Lemma 3 c), $\alpha^{(k-1)}$ is $G_{k-2}$-m.b.)

b) Conditionally on $G_{k-2}$, $g^{(k)}$ is independent of $G_{k-1}$.

c) The variables $\zeta^{(k)}$ are conditionally Gaussian, given $G_{k-1}$ with covariances
\[
E_{k-1} \zeta^{(k)} \zeta^{(k)} = \delta_{ij} + \frac{1}{N} \phi^{(k)}_i \phi^{(k)}_j - \alpha^{(k-1)}_{ij}. \tag{2.10}
\]

**Proof.** We use the following induction scheme to prove a) and b):

(i) We assume that the statements a), b) are correct for $k$.

(ii) b) implies trivially that $g^{(k)}$ is Gaussian conditionally on $G_{k-1}$. So, this part of a) for $k+1$ is already settled.

(iii) As $\phi^{(k)}$ is $G_{k-1}$-m.b., it follows that $\zeta^{(k)}$, $\eta^{(k)}$ are Gaussian, conditionally on $G_{k-1}$, simply because they are linear combinations of the $g^{(k)}_{ij}$ with coefficients which are $G_{k-1}$-m.b.

(iv) From the form of $\rho^{(k)}$, it then follows that it is also Gaussian, conditionally on $G_{k-1}$, and therefore, $g^{(k+1)}$ is Gaussian, conditionally on $G_{k-1}$.

(v) The rest is just a covariance check: In order to prove that $g^{(k+1)}$ is independent of $G_k = \sigma \left( G_{k-1}, \zeta^{(k)}, \eta^{(k)} \right)$, we have to check that the conditional covariances between $g^{(k+1)}$ and $\zeta^{(k)}$ given $G_{k-1}$, and between $g^{(k+1)}$ and $\eta^{(k)}$, vanish, which in fact heavily uses (2.9) for $k$. Finally we have to boost this formula to $k+1$. 7
We first have to compute the conditional covariances among the $\zeta^{(k)}$'s and $\eta^{(k)}$'s.

\[
E_{k-1}\xi_i^{(k)}\xi_j^{(k)} = \sum_{s,t} \phi_s^{(k)} \phi_t^{(k)} E_{k-1}g_{is}^{(k)}g_{jt}^{(k)}
\]
\[
= \frac{1}{N} \sum_{s,t} \phi_s^{(k)} \phi_t^{(k)} [\delta_{ij} - \alpha_{ij}^{(k-1)}] [\delta_{st} - \alpha_{st}^{(k-1)}]
\]
\[
= \delta_{ij} - \alpha_{ij}^{(k-1)},
\]
and symmetrically the same for $E_{k-1}\eta_i^{(k)}\eta_j^{(k)}$.

\[
E_{k-1}\xi_i^{(k)}\eta_j^{(k)} = \sum_{s,t} \phi_s^{(k)} \phi_t^{(k)} E_{k-1}g_{is}^{(k)}g_{jt}^{(k)}
\]
\[
= \sum_{s,t} \phi_s^{(k)} \phi_t^{(k)} \frac{1}{N} [\delta_{it} - \alpha_{it}^{(k-1)}] [\delta_{sj} - \alpha_{sj}^{(k-1)}]
\]
\[
= \frac{1}{N} \phi_i^{(k)} \phi_j^{(k)}
\]

Let’s next check that the covariances between $g^{(k+1)}$ and $\xi^{(k)}$ vanish:

\[
E_{k-1}g_{ij}^{(k+1)}\xi_s^{(k)} = E_{k-1}g_{ij}^{(k)}\xi_s^{(k)} - E_{k-1}\rho_{ij}^{(k)}\xi_s^{(k)}
\]
\[
E_{k-1}g_{ij}^{(k)}\xi_s^{(k)} = E_{k-1}g_{ij}^{(k)} \sum_t g_{st}^{(k)} \phi_t^{(k)}
\]
\[
= \sum_t \phi_t^{(k)} E_{k-1}g_{ij}^{(k)} g_{st}^{(k)} = \sum_t \phi_t^{(k)} E_{k-1}g_{ij}^{(k)} g_{st}^{(k)}
\]
\[
= \frac{1}{N} \left[ \delta_{is} - \alpha_{is}^{(k-1)} \right] \sum_t \phi_t^{(k)} [\delta_{jt} - \alpha_{jt}^{(k-1)}]
\]
\[
= \frac{1}{N} \left[ \delta_{is} - \alpha_{is}^{(k-1)} \right] \phi_j^{(k)}
\]

\[
E_{k-1}\rho_{ij}^{(k)}\xi_s^{(k)} = \frac{1}{N} \phi_j^{(k)} E_{k-1} \left( \xi_s^{(k)} \phi_t^{(k)} \right) + \frac{1}{N} \phi_i^{(k)} E_{k-1} \left( \xi_s^{(k)} \eta_j^{(k)} \right)
\]
\[
= \frac{1}{N} \phi_j^{(k)} \frac{1}{N^2} \sum_u \phi_u^{(k)} E_{k-1} \xi_u^{(k)} \xi_s^{(k)}
\]
\[
= \frac{1}{N} \left( \delta_{is} - \alpha_{is}^{(k-1)} \right) \phi_j^{(k)} + \frac{1}{N} \phi_i^{(k)} \left( \frac{1}{N} \phi_s^{(k)} \phi_j^{(k)} \right)
\]
\[
- \phi_i^{(k)} \phi_j^{(k)} \frac{1}{N^2} \sum_u \phi_u^{(k)} \left( \delta_{us} - \alpha_{us}^{(k-1)} \right)
\]
\[
= \frac{1}{N} \left( \delta_{is} - \alpha_{is}^{(k-1)} \right) \phi_j^{(k)}.
\]

Therefore, $E_{k-1}g_{ij}^{(k)}\xi_s^{(k)} = 0$, and similarly (and symmetrically) $E_{k-1}g_{ij}^{(k)}\eta_s^{(k)} = 0$ for all $i, j, s$. So, this proves that the $\mathcal{G}_{k-1}$-conditional covariances between $g^{(k+1)}$ and $(\xi^{(k)}, \eta^{(k)})$ vanish which implies that $g^{(k+1)}$ is conditionally independent of $\mathcal{G}_k$ given $\mathcal{G}_{k-1}$, as everything is conditionally Gaussian.
As a consequence, we also have

\[ E_{k-1}\left(\rho^{(k)}_{ij}, g_{st}^{(k)}\right) = 0, \ \forall i, j, s, t. \] (2.13)

To finish the induction, it remains to prove the validity of (2.9) with \( k \) replaced by \( k+1 \). Using (2.13), one has

\[ E_{k-1}\left(g_{ij}^{(k+1)}, g_{st}^{(k+1)}\right) = E_{k-1}\left(\left[g_{ij}^{(k)} - \rho^{(k)}_{ij}\right]\left[g_{st}^{(k)} - \rho^{(k)}_{st}\right]\right) \]

\[ = E_{k-2}\left(g_{ij}^{(k)}, g_{st}^{(k)}\right) + E_{k-1}\left(\rho^{(k)}_{ij}, \rho^{(k)}_{st}\right). \] (2.14)

Plugging that into (2.14), and using (2.9) for \( k \), one gets it for \( k+1 \). So, we have proved a) and b). c) follows from (2.11) and (2.12).

**Lemma 5** For all \( k \), and \( N > k \)

\[
\left\| m^{(k+1)} - \sum_{s=1}^{k} \left\langle m^{(k+1)}, \phi^{(s)} \right\rangle \phi^{(s)} \right\| > 0, \ \mathbb{P} - \text{a.s.}
\]

**Proof.** We use induction on \( k \). For \( k = 0 \), there is nothing to prove, and \( k = 1 \) is evident, so we assume \( k \geq 2 \), and that \( \phi^{(s)}, s \leq k \) is well-defined, and we can use the covariance computation in Proposition 4 c). We prove that

\[
\mathbb{P}_{k-1}\left(\left\| m^{(k+1)} - \sum_{s=1}^{k} \left\langle m^{(k+1)}, \phi^{(s)} \right\rangle \phi^{(s)} \right\| > 0\right) = 1, \ \mathbb{P} - \text{a.s.}
\]

In the expression (2.3) of \( m^{(k+1)} \), all the entries are \( G_{k-1} \)-m.b. except \( \zeta^{(k)} \), and \( q - \Gamma_{k-1}^{2} > 0 \). Therefore, conditionally on \( G_{k-1} \), we have

\[ m_{i}^{(k+1)} = \tanh\left(x_{i} + \alpha \zeta_{i}^{(k)}\right) \]

with \( x_{i} \in \mathbb{R}, \ \alpha > 0 \). From (2.10), the conditional distribution of \( \zeta^{(k)} \) is Gaussian with a covariance matrix of rank \( N - k \). From that, it is immediate that \( \mathbb{P}\)-a.s. there exists \( i \leq N \) with \( m_{i}^{(k+1)} \) having a non-degenerate conditional distribution under \( \mathbb{P}_{k-1} \). This implies the claim. ■

For the formulation of the next result, we introduce the following notation. If \( X_{N}, Y_{N} \) are two sequences of random variables, depending possibly on other parameters like \( \beta, h, k \) etc., we write

\[ X_{N} \stackrel{d}{=} Y_{N} \]
if there exists a constant \( C > 0 \), depending possibly on these other parameters, but not on \( N \), with
\[
\mathbb{P} (|X_N - Y_N| \geq t) \leq C \exp \left[ -C t^2 N \right].
\]
\( X_N \simeq Y_N \) in particular implies \( \|X_N - Y_N\|_p \to 0 \) for every \( p \geq 1 \) as \( N \to \infty \).

**Proposition 6**

a) For any \( j < k \), one has
\[
\left\langle m^{(k)}_i, \phi^{(j)} \right\rangle \simeq \gamma_j.
\]

b) For any \( k \)
\[
\|m^{(k)}\|^2 \simeq q.
\]

and for \( j < k \)
\[
\left\langle m^{(k)}_i, m^{(j)}_i \right\rangle \simeq \rho_j.
\]

**Proof.** This was proved in \( [3] \). The \( m^{(k)}_i \) there were defined through the iteration (2.1), and we proved that these random variables can be approximated by the ones essentially given by (2.5). However, we have here a slightly different version, as our \( G_k \) are not the same as the \( F_k \) in \( [3] \). Therefore, we give a sketch of the proof here again.

a) is a simple consequence of b), see \( [3] \), Lemma 2.7.

So, we prove b). We first prove (2.16). (2.17) will be proved by a small modification of the argument. \( k = 1 \) is trivial, and
\[
m^{(2)}_i = \text{Th} \left( \sqrt{q} \zeta^{(1)}_i \right),
\]
and then (2.16) follows from the LLN and the fixed point equation for \( q \). So, we assume \( k \geq 3 \). We have
\[
m^{(k)}_i = \text{Th} \left( \sum_{s=1}^{k-2} \gamma_s \zeta^{(s)}_i + \sqrt{q - \Gamma_k^{2s-2}} \zeta^{(k-1)}_i \right)
\]
We observe that \( \text{Th} (x + \cdot) \) is Lipschitz continuous with \( \|\text{Th} (x + \cdot)\|_{lip} = \max (1, \beta) \) for any \( x \in \mathbb{R} \). We consider now the conditional distribution of \( m^{(k)}_i \) with respect to \( G_{k-2} \). The Lipschitz norm of \( x \mapsto \text{Th} \left( \sum_{s=1}^{k-2} \gamma_s \zeta^{(s)}_i + \sqrt{q - \Gamma_k^{2s-2}} \right) \) is \( \max \left( 1, \beta \sqrt{q - \Gamma_k^{2k-2}} \right) \). As \( \text{Th} \) is bounded by 1, we have that the Lipschitz norm of \( x \mapsto \text{Th}^2 \left( \sum_{s=1}^{k-2} \gamma_s \zeta^{(s)}_i + \sqrt{q - \Gamma_k^{2k-2}} x \right) \) is bounded by \( 2 \max \left( 1, \beta \sqrt{q - \Gamma_k^{2k-2}} \right) \).

Applying Lemma 12 and the conditional covariances of \( \zeta^{(k-1)} \) given in Proposition 4 above, we obtain
\[
\mathbb{P}_{k-2} \left( \frac{1}{N} \sum_{i=1}^{N} \left[ m^{(k)}_i - E \text{Th}^2 \left( \sum_{s=1}^{k-2} \gamma_s \zeta^{(s)}_i + \sqrt{q - \Gamma_k^{2s-2}} Z_{k-1} \right) \right] \geq t \right) \leq C \exp \left[ -C t^2 N \right],
\]
where \( C \) depends on \( k, \beta, h \), but is non-random, as the bound in Lemma 12 depends only on the the Lipschitz constant, and the other parameters.
We proceed in this way, replacing \( \zeta^{(s)} \), \( s \leq k-2 \) successively by \( Z_{k-2}, Z_{k-3}, \ldots, Z_1 \), condition first on \( G_{k-3} \), etc. This finally leads to

\[
\left\| \mathbf{m}^{(k)} \right\|^2 \asymp E \, \text{Th}^2 \left( \sum_{s=1}^{k-2} \gamma_s Z_s + \sqrt{q - \Gamma_{k-2}^2 Z_{k-1}} \right) = q.
\]

(2.17) follows by a straightforward modification: The case \( j = 1 \) is trivial, and so we assume \( j \geq 2 \). As \( j < k \), the conditioning on \( G_{k-2} \) fixes \( \mathbf{m}^{(j)} \), and we therefore get in the first step

\[
P_{k-2} \left( \left| \frac{1}{N} \sum_{i=1}^{N} m_i^{(j)} m_i^{(k)} - \frac{1}{N} \sum_{i=1}^{N} m_i^{(j)} \right| \geq \varepsilon \right) \leq C \exp \left[ -C \varepsilon^2 N \right].
\]

This replacement, we do up to replacing \( \zeta^{(j)} \) which is \( G_{j-1} \)-m.b. whereas \( \mathbf{m}^{(j)} \) is \( G_{j-2} \)-m.b. We therefore obtain

\[
\frac{1}{N} \sum_{i=1}^{N} m_i^{(j)} m_i^{(k)} \simeq \frac{1}{N} \sum_{i=1}^{N} m_i^{(j)} E \text{Th} \left( \sum_{s=1}^{j-1} \gamma_s \zeta_i^{(s)} + \sum_{s=j}^{k-2} \gamma_s Z_s + \sqrt{q - \Gamma_{j-1}^2 Z_{j-1}} \right).
\]

Performing this conditioning argument now with respect to \( G_{j-2} \), we get first

\[
\frac{1}{N} \sum_{i=1}^{N} m_i^{(j)} m_i^{(k)} \simeq \frac{1}{N} \sum_{i=1}^{N} E \left[ \text{Th} \left( \sum_{s=1}^{j-2} \gamma_s \zeta_i^{(s)} + \sqrt{q - \Gamma_{j-2}^2 Z_{j-1}} \right) \right] \times \text{Th} \left( \sum_{s=1}^{j-2} \gamma_s \zeta_i^{(s)} + \gamma_{j-1} Z_{j-1} + \sqrt{q - \Gamma_{j-1}^2 Z_j} \right),
\]

and now in the same way as for (2.16)

\[
\frac{1}{N} \sum_{i=1}^{N} m_i^{(j)} m_i^{(k)} \simeq E \left[ \text{Th} \left( \sum_{s=1}^{j-2} \gamma_s Z_s + \sqrt{q - \Gamma_{j-2}^2 Z_{j-1}} \right) \right] \times \text{Th} \left( \sum_{s=1}^{j-2} \gamma_s Z_s + \gamma_{j-1} Z_{j-1} + \sqrt{q - \Gamma_{j-1}^2 Z_j} \right).
\]

A simple computation, as in [3] in the evaluation of (5.12) there, shows that the right hand side equals \( \psi (\rho_{j-1}) = \rho_j \). \( \blacksquare \)

**Remark 7** The argument given here is considerably simpler than the one in [3]. On one hand, this is due to the fact that we don’t consider here the random variables given by the iteration (2.4). Also the explicit representation of the conditional covariances of the \( \zeta \) is very helpful.

### 3 Estimates for the first and second conditional moments

The two basic results are:

11
Proposition 8 If $h > 0$ and $\beta$ is small enough then
\[
\lim_{k \to \infty} \limsup_{N \to \infty} E \left| \frac{1}{N} \log E_k (Z_N) - RS (\beta, h) \right| = 0. \tag{3.1}
\]

Proposition 9 Under the same conditions as in Proposition 8
\[
\lim_{k \to \infty} \limsup_{N \to \infty} E \left| \frac{1}{N} \log E_k (Z_N^2) - 2 RS (\beta, h) \right| \leq 0. \tag{3.2}
\]

Remark 10 The requirement on $\beta$ is rather unsatisfactory. I believe that at least Proposition 8 is correct in the full AT-region (1.4). Actually, only the very last argument given in the proof in the next section requires an unspecified “small $\beta$” argument. The problem is coming from using the Schwarz inequality and the Hölder-inequality in the proof, but I haven’t found a better estimate.

The propositions are proved in the next section. We give now the proof of Theorem 1 based on these propositions.

We will use that, actually for all $\beta, h$, the free energy is self-averaging:
\[
\lim_{N \to \infty} \frac{1}{N} \log Z_N = \lim_{N \to \infty} \frac{1}{N} E \log Z_N, \tag{3.3}
\]
assuming the limit on the right hand side exists, which is the result in [6]. This is a simple consequence of the Gaussian isoperimetric inequality, a fact which is well known since long. In fact writing $J_{ij} := \sqrt{N} g_{ij}$ which are standard Gaussians, we have
\[
\left| \frac{1}{N} \log Z_N (J) - \frac{1}{N} \log Z_N (J') \right| \leq \frac{\beta}{\sqrt{2N}} \| J - J' \|
\]
where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^{N(N-1)/2}$. Therefore
\[
P \left( \left| \frac{1}{N} \log Z_N - \frac{1}{N} E \log Z_N \right| \geq t \right) \leq \exp \left[ -t^2 N / \beta^2 \right].
\]

By Jensen’s inequality
\[
\limsup_{N \to \infty} \frac{1}{N} E \log Z_N \leq \limsup_{N \to \infty} \frac{1}{N} E \log E_k (Z_N)
\]
for all $k$. Therefore, by Proposition 8
\[
\limsup_{N \to \infty} \frac{1}{N} E \log Z_N \leq RS (\beta, h). \tag{3.4}
\]

For the estimate in the other direction, we rely on a second moment argument. For $k, N \in \mathbb{N}$, set $A_{k, N} := \{ Z_N \geq \frac{1}{2} E_k (Z_N) \}$
\[
E_k (Z_N) = E_k (Z_N; A_{k, N}^c) + E_k (Z_N; A_{k, N})
\leq \frac{1}{2} E_k (Z_N) + \frac{1}{N} E_k (Z_N^2) \mathbb{P}_k (A_{k, N})
\]
and therefore
\[
\mathbb{P}_k (A_{k, N}) \geq \frac{E_k (Z_N)^2}{4 E_k (Z_N^2)}. \tag{3.5}
\]
Using Proposition 9 for an arbitrary \( \varepsilon > 0 \) there exists \( k_0(\varepsilon) \) such that for \( k \geq k_0(\varepsilon) \) we find \( N_0(\varepsilon, k) \) with
\[
P \left( \frac{E_k(Z_N)^2}{4E_k(Z_N^2)} \geq e^{-\varepsilon N} \right) \geq \frac{1}{2}, \quad N \geq N_0.
\]
and therefore, by (3.3), and the definition of \( A_{k,N} \),
\[
P_k \left( \frac{1}{N} \log Z_N - \log \mathbb{E}_k(Z_N) - \log \frac{2}{N} \right) \geq e^{-\varepsilon N} \geq \frac{1}{2}.
\]
By Proposition 8 we find for any \( \varepsilon' > 0 \), a \( c(\varepsilon') > 0 \) and a \( k_0'((\varepsilon')) \in \mathbb{N} \) such that for \( k \geq k_0'(\varepsilon') \), we find \( N_0'(\varepsilon', k) \) such that for \( N \geq N_0' \), we have
\[
P \left( \frac{1}{N} \log \mathbb{E}_k(Z_N) \geq \text{RS} (\beta, h) - \varepsilon' \right) \geq e^{-\varepsilon N} \geq \frac{1}{4},
\]
and \( N^{-1} \log 2 \leq \varepsilon'/2 \). Therefore, for \( k \geq \max (k_0(\varepsilon), k_0'(\varepsilon')) \), \( N \geq \max (N_0', N_0) \)
\[
P \left( \frac{1}{N} \log Z_N \geq \text{RS} (\beta, h) - \varepsilon' \right) \geq e^{-\varepsilon N} \geq \frac{1}{4},
\]
implying by the Markov inequality
\[
P \left( \frac{1}{N} \log Z_N \geq \text{RS} (\beta, h) - \varepsilon' \right) \geq \frac{1}{4} e^{-\varepsilon N}.
\] (3.6)
By Gaussian isoperimetry, we have for any \( \eta > 0 \) and large enough \( N \)
\[
P \left( \left| \frac{1}{N} \log Z_N - \frac{1}{N} \mathbb{E} \log Z_N \right| \leq \eta \right) \geq 1 - \exp \left( -\eta^2 N/\beta^2 \right).
\]
If we choose \( \varepsilon < \eta^2/\beta^2 \), it follows that for \( N \) large enough one has
\[
\frac{1}{N} \mathbb{E} \log Z_N \geq \text{RS} (\beta, h) - \varepsilon' - \eta
\]
and as \( \eta \) and \( \varepsilon' \) are arbitrary, we get
\[
\liminf_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_N \geq \text{RS} (\beta, h).
\]
Together with (3.4), this proves
\[
\liminf_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_N = \text{RS} (\beta, h).
\]

4 Proofs of the propositions

Proof of Proposition 8

\[
\mathbb{E}_k(Z_N) = \sum_{\sigma} 2^{-N} \exp \left[ h \sum_i \sigma_i \right] \mathbb{E}_k \left( \exp \left[ \frac{\beta N}{\sqrt{2}} \langle g_\sigma, \sigma \rangle \right] \right)
\]
\[
= \sum_{\sigma} 2^{-N} \exp \left[ h \sum_i \sigma_i + \frac{\beta N}{\sqrt{2}} \sum_{s=1}^k w \langle \rho^{(s)} \sigma, \sigma \rangle \right]
\]
\[
\times \mathbb{E}_k \left( \exp \left[ \frac{\beta N}{\sqrt{2}} \langle g^{(k+1)} \sigma, \sigma \rangle \right] \right).
\]
$g^{(k+1)}$ is Gaussian conditionally on $\mathcal{G}_k$, and therefore

$$
\mathbb{E}_k \left( \exp \left[ \frac{\beta N}{\sqrt{2}} \langle g^{(k+1)} \sigma, \sigma \rangle \right] \right) = \exp \left[ \frac{\beta^2 N^2}{4} \mathbb{E}_k \left( \langle g^{(k+1)} \sigma, \sigma \rangle \right)^2 \right]
$$

According to Proposition 4 a), b)

$$
\mathbb{E}_k \left( \langle g^{(k+1)} \sigma, \sigma \rangle \right)^2 = \frac{1}{N} \left( 1 - \sum_{r=1}^k \langle \phi^{(r)} \sigma, \sigma \rangle \right)^2.
$$

Therefore

$$
\mathbb{E}_k (Z_N) = \sum_{\sigma} 2^{-N} \exp \left[ h \sum_i \sigma_i + \frac{\beta N}{\sqrt{2}} \sum_{s=1}^k \langle \rho^{(s)} \sigma, \sigma \rangle \right.
\left. + \frac{\beta^2 N}{4} \left( 1 - \sum_{r=1}^k \langle \phi^{(r)} \sigma, \sigma \rangle \right)^2 \right].
$$

With $h^{(k+1)}$ and $m^{(k+1)}$ defined in (2.4), (2.5), which are $\mathcal{F}_k$-m.b., we put

$$
p^{(k)} (\sigma) := 2^{-N} \exp \left[ \frac{N \langle h^{(k+1)} \sigma \rangle}{\prod_{i=1}^N \cosh (h_i^{(k+1)})} \right],
$$

which is the product measure of tilted coin tossing, the $\sigma_i$ having mean $m_i^{(k+1)}$. Then,

$$
\mathbb{E}_k (Z_N) = \exp \left[ \sum_{i=1}^N \log \cosh (h_i^{(k+1)}) \right] \sum_{\sigma} p^{(k)} (\sigma) \exp \left[ N \beta F_{N,k} (\sigma) \right], \quad (4.1)
$$

where with $\gamma_s$

$$
F_{N,k} (\sigma) := \sum_{s=1}^k \langle 2^{-1/2} \rho^{(s)} \sigma, \sigma \rangle - \sum_{s=1}^{k-1} \gamma_s \langle \zeta^{(s)} \sigma, \sigma \rangle
\left. - \sqrt{q} - \Gamma_{k-1} \langle \zeta^{(k)} \sigma, \sigma \rangle + \frac{\beta}{4} \left( 1 - \sum_{r=1}^k \langle \phi^{(r)} \sigma, \sigma \rangle \right)^2 \right].
$$

Up to here, this is an exact computation.

The first part on the right hand side of (4.1) does not depend on $\sigma$, and by Lemma 14 we get for any $k$ :

$$
\lim_{N \to \infty} \mathbb{E}_k \left( \frac{1}{N} \sum_{i=1}^N \log \cosh (h_i^{(k+1)}) \right) = E \log \cosh (h + \beta \sqrt{q} Z) = 0
$$

and therefore, we only have to prove that with

$$
Z (F_{N,k}) := \sum_{\sigma} p^{(k)} (\sigma) \exp \left[ N \beta F_{N,k} (\sigma) \right]
$$

we have

$$
\lim_{k \to \infty} \lim_{N \to \infty} \mathbb{E}_k \left( \frac{1}{N} \log Z (F_{N,k}) - \frac{\beta^2 (1-q)}{4} \right) = 0. \quad (4.3)
$$
We will perform a number of approximations which are negligible in the $N \to \infty$, $k \to \infty$, in this order. More precisely, consider a random function

$$F_{N,k}'(\sigma) = F_{N,k}(\sigma) + \Delta_{N,k}(\sigma)$$

with the property that

$$\lim_{k \to \infty} \limsup_{N \to \infty} E \sup_{\sigma} |\Delta_{N,k}(\sigma)| = 0,$$

then

$$\lim_{k \to \infty} \limsup_{N \to \infty} E \left| \frac{1}{N} \log Z(F_{N,k}) - \frac{1}{N} \log Z(F_{N,k}') \right| = 0. \tag{4.5}$$

For instance, taking $\Delta_{N,k}(\sigma) := \gamma_k \langle \zeta^{(k)}, \sigma \rangle$, we have $\sup_{\sigma} |\Delta_{N,k}(\sigma)| \leq \gamma_k \|\zeta^{(k)}\|$, and using the covariance structure of $\zeta^{(k)}$ in Proposition 4 c), we have $\sup_{k} E \|\zeta^{(k)}\| \leq 1$. As $\gamma_k \to 0$ for $k \to \infty$, (4.4) is satisfied. By the same reasoning, we can neglect $\sqrt{q - \Gamma_{k-1}} \langle \zeta^{(k)}, \sigma \rangle$ under the AT-condition (4). Therefore, we can replace $F_{N,k}$ by

$$F_{N,k}'(\sigma) := \sum_{s=1}^{k} \left( 2^{-1/2} \rho^{(s)}(\sigma, \sigma) \right) - \sum_{s=1}^{k} \gamma_s \langle \zeta^{(s)}, \sigma \rangle$$

$$+ \frac{\beta}{2} \left( 1 - \sum_{r=1}^{k} \langle \phi^{(r)}, \sigma \rangle^2 \right)^2,$$

and get (4.5).

We do a further approximation for the first summand. Plugging in the first two summands of the definition of $\rho^{(s)}$ (2.3), the contribution to $\langle 2^{-1/2} \rho^{(s)}(\sigma, \sigma) \rangle$ is exactly $\langle \phi^{(s)}, \sigma \rangle \langle \zeta^{(s)}, \sigma \rangle$. The third term gives $\langle \phi^{(s)}, \xi^{(s)} \rangle \langle \phi^{(s)}, \sigma \rangle^2$, and we claim that we can neglect that. Indeed

$$\sup_{\sigma} \left| \langle \phi^{(s)}, \xi^{(s)} \rangle \langle \phi^{(s)}, \sigma \rangle^2 \right| \leq \left| \langle \phi^{(s)}, \xi^{(s)} \rangle \right|,$$

and using Lemma 11 we see that

$$\limsup_{N \to \infty} E \sup_{\sigma} \left| \langle \phi^{(s)}, \xi^{(s)} \rangle \langle \phi^{(s)}, \sigma \rangle^2 \right| = 0$$

for all $s$. Therefore, we can indeed neglect this part. We now center the $\sigma$ by putting

$$\sigma^{(k)} := \sigma - m^{(k+1)}.$$

Then

$$\sum_{s=1}^{k} \langle \phi^{(s)}, \sigma \rangle \langle \zeta^{(s)}, \sigma \rangle = \sum_{s=1}^{k} \langle \phi^{(s)}, \sigma^{(k)} + m^{(k+1)} \rangle \langle \zeta^{(s)}, \sigma^{(k)} + m^{(k+1)} \rangle$$

$$= \sum_{s=1}^{k} \langle \phi^{(s)}, \sigma^{(k)} \rangle \langle \zeta^{(s)}, \sigma^{(k)} \rangle + \sum_{s=1}^{k} \langle \phi^{(s)}, m^{(k+1)} \rangle \langle \zeta^{(s)}, \sigma^{(k)} \rangle$$

$$+ \sum_{s=1}^{k} \langle \phi^{(s)}, \sigma^{(k)} \rangle \langle \zeta^{(s)}, m^{(k+1)} \rangle + \sum_{s=1}^{k} \langle \phi^{(s)}, m^{(k+1)} \rangle \langle \zeta^{(s)}, m^{(k+1)} \rangle.$$
We claim that we can replace the second summand on the right hand side by \( \sum_{s=1}^{k} \gamma_s \langle \zeta(s), \check{\sigma}(k) \rangle \). Indeed
\[
\left| \langle \zeta(s), \check{\sigma}(k) \rangle \left( \langle \phi(s), \mathbf{m}^{(k+1)} \rangle - \gamma_s \right) \right| \leq \| \zeta(s) \| \| \phi(s) \| \mathbf{m}^{(k+1)} - \gamma_s \|
\]
and
\[
\mathbb{E} \left( \| \zeta(s) \| \| \phi(s), \mathbf{m}^{(k+1)} - \gamma_s \| \right) \leq \sqrt{\mathbb{E} \| \zeta(s) \|^2 \mathbb{E} \| \phi(s), \mathbf{m}^{(k+1)} - \gamma_s \|^2}
\]
which converges to 0 for \( N \to \infty \), by Proposition 6 a). In a similar way, using Lemma 16, we can replace
\[
\sum_{s=1}^{k} \langle \phi(s), \check{\sigma}(k) \rangle \langle \zeta(s), \mathbf{m}^{(k+1)} \rangle
\]
by
\[
\beta (1 - q) \sum_{s=1}^{k} \gamma_s \langle \phi(s), \check{\sigma}(k) \rangle.
\]
In the end, we replace \( F''_{N,k} \) by
\[
F''_{N,k} (\sigma) := \sum_{s=1}^{k} \langle \phi(s), \check{\sigma}(k) \rangle \langle \zeta(s), \check{\sigma}(k) \rangle + \beta (1 - q) \sum_{s=1}^{k} \gamma_s \langle \phi(s), \check{\sigma}(k) \rangle + \frac{\beta}{4} \left( 1 - \sum_{r=1}^{k} \langle \phi(r), \sigma \rangle^2 \right)^2,
\]
achieving
\[
\lim_{k \to \infty} \lim_{N \to \infty} \mathbb{E} \left( \frac{1}{N} \log Z (F''_{N,k}) - \frac{1}{N} \log Z (F''_{N,k}) \right) = 0. \tag{4.6}
\]
where we have made repeated use of Proposition 6 and Lemma 16 and \( \sum_{s=1}^{k} \gamma_s^2 \to q \), as \( k \to \infty \), under the AT-condition. Using (4.5), it therefore remains to prove
\[
\lim_{k \to \infty} \lim_{N \to \infty} \mathbb{E} \left( \frac{1}{N} \log Z (F''_{N,k}) - \frac{\beta^2 (1 - q)}{4} \right) = 0.
\]

The most “dangerous” part in \( \log Z (F''_{N,k}) \) is the presence of \( \sum_{s=1}^{k} \gamma_s \langle \phi^{(s)}, \check{\sigma}(k) \rangle \), but fortunately, it cancels in leading order when centering the third part.
\[
\sum_{r=1}^{k} \langle \phi^{(r)}, \sigma \rangle^2 = \sum_{r=1}^{k} \left( \langle \phi^{(r)}, \check{\sigma}(k) \rangle + \langle \phi^{(r)}, \mathbf{m}^{(k+1)} \rangle \right)^2.
\]
For the same reason as repeatedly use above, we may replace \( \langle \phi^{(r)}, \mathbf{m}^{(k+1)} \rangle \) by \( \gamma_r \) (in the \( N \to \infty \), \( k \to \infty \) limit), and replace \( \sum_{r=1}^{k} \gamma_r^2 \) by \( q \) under the AT-condition. By these approximations, we replace the right hand side of the expression above by
\[
\sum_{r=1}^{k} \left( \langle \phi^{(r)}, \check{\sigma}(k) \rangle + \gamma_r \right)^2 \approx q + 2Y_k + S_k^2,
\]
where
\[ Y_k := \sum_{r=1}^{k} \gamma_r \left\langle \phi^{(r)}, \hat{\sigma}^{(k)} \right\rangle, \]
\[ S_k^2 := \sum_{r=1}^{k} \left\langle \phi^{(r)}, \hat{\sigma}^{(k)} \right\rangle^2. \]

Therefore, with these approximations, we have
\[
\beta (1 - q) Y_k + \beta \left( 1 - \frac{\beta}{4} \right) \left( 1 - q - 2 Y_k - S_k^2 \right)^2.
\]

The first summand is exactly what we want, and we “only” have to check that the rest does not harm. In other words, putting
\[ F''''_{N,k} (\sigma) := \sum_{s=1}^{k} \left\langle \phi^{(s)}, \hat{\sigma}^{(k)} \right\rangle \left\langle \zeta^{(s)}, \hat{\sigma}^{(k)} \right\rangle + \beta Y_k^2 + \frac{\beta^4}{4} S_k^4 - \frac{\beta}{2} (1 - q) S_k^2 + \beta Y_k S_k^2, \]
we have
\[ \lim_{k \to \infty} \lim_{N \to \infty} \mathbb{E} \left| \frac{1}{N} \log Z \left( F''''_{N,k} \right) - \frac{1}{N} \log Z \left( \frac{\beta (1 - q)^2}{4} + F''''_{N,k} \right) \right| = 0, \quad (4.7) \]
and using (4.5), (4.6), and (4.7), it remains to prove
\[ \lim_{k \to \infty} \lim_{N \to \infty} \mathbb{E} \left| \frac{1}{N} \log Z \left( F''''_{N,k} \right) \right| = 0. \quad (4.8) \]

This is a somewhat complicated Curie-Weiss type computation. An important point is that \( F''''_{N,k} \) contains only summands which are at least quadratic in the \( \hat{\sigma}^{(k)} \). If there would be a linear term, (4.8) would for any \( \beta > 0 \) not be true, of course. I strongly believe that (4.8) is correct under the AT-condition (1.4), but a prove eludes me. The reader should also be aware, that we haven’t lost anything in the AT-region. In other words, if for a parameter \( (\beta, h) \) satisfying (1.4), (4.8) is not true, then for these \( (\beta, h) \), (3.1) is not correct.

First remark that
\[ \frac{1}{N} \log Z \left( F''''_{N,k} \right) \geq \beta \sum_{\sigma} P^{(k)} (\sigma) F''''_{N,k} (\sigma) \]
and \( \mathbb{E} \left| \sum_{\sigma} P^{(k)} (\sigma) F''''_{N,k} (\sigma) \right| = O \left( N^{-1/2} \right) \) through the independence of the components under \( P^{(k)} (\sigma) \) and the centering.

It remains to prove the upper bound. We use some rather crude and certainly not optimal bounds.
\[ \sum_{s=1}^{k} \left\langle \phi^{(s)}, \hat{\sigma}^{(k)} \right\rangle \left\langle \zeta^{(s)}, \hat{\sigma}^{(k)} \right\rangle \leq \sqrt{S_k^2 \sum_{s=1}^{k} \left\langle \zeta^{(s)}, \hat{\sigma}^{(k)} \right\rangle^2} \leq \frac{1}{2} S_k^2 + \frac{1}{2} \sum_{s=1}^{k} \left\langle \zeta^{(s)}, \hat{\sigma}^{(k)} \right\rangle^2. \]
Also
\[ |Y_k| \leq \sum_{s=1}^{k} \gamma_s^2 \| \hat{\sigma}^{(k)} \| \leq q \| \hat{\sigma}^{(k)} \| \leq 2q, \]
\[ S_k^2 \leq \| \hat{\sigma}^{(k)} \| \leq 2. \]

Using these crude estimates, and the Hölder inequality, one sees that it satisfies to prove
\[
\limsup_{N \to \infty} \frac{1}{N} \mathbb{E} \log \sum_{\sigma} p(\sigma) \exp \left[ \lambda N S_k^2 \right] \leq 0, \quad (4.9)
\]
\[
\limsup_{N \to \infty} \frac{1}{N} \mathbb{E} \log \sum_{\sigma} p(\sigma) \exp \left[ \lambda N Y_k^2 \right] \leq 0, \quad (4.10)
\]
\[
\limsup_{N \to \infty} \frac{1}{N} \mathbb{E} \log \sum_{\sigma} p^{(k)}(\sigma) \exp \left[ \lambda N \sum_{s=1}^{k} \langle \zeta^{(s)}, \hat{\sigma} \rangle \right] \leq 0, \quad (4.11)
\]
for small enough \( \lambda > 0 \), where “small enough” does not depend on \( k \). This latter requirement looks somewhat dangerous, but here it helps that the \( \phi^{(s)} \) are orthogonal with respect to our inner product on \( \mathbb{R}^N \), and the \( \zeta^{(s)} \) are approximately so. We start with (4.9)
\[
\sum_{\sigma} p(\sigma) \exp \left[ \lambda N S_k^2 \right] = \sum_{\sigma} p^{(k)}(\sigma) \exp \left[ \lambda N \sum_{s=1}^{k} \langle \phi^{(s)}, \hat{\sigma}^{(k)} \rangle^2 \right]
\]
\[ = E \sum_{\sigma} p^{(k)}(\sigma) \exp \left[ \sum_{i=1}^{N} \left( \sum_{s=1}^{k} Z_s \sqrt{\frac{2\lambda}{N}} \phi^{(s)} \right) \hat{\sigma}_i \right]
\]
\[ \leq E \exp \left[ \sum_{i=1}^{N} \chi_i \left( \sum_{s=1}^{k} Z_s \sqrt{\frac{2\lambda}{N}} \phi^{(s)} \right) \right]. \]

where
\[ \chi_i (x) := \log \cosh (h_i + x) - \log \cosh (h_i) - x m_i. \]

By Lemma 12 we have \( \chi_i (x) \leq x^2 / 2 \), so, using also the fact that the \( \phi^{(s)} \) are orthonormal, one has that the above is
\[ \leq E \exp \left[ \frac{\lambda}{N} \sum_{i=1}^{N} \left( \sum_{s=1}^{k} Z_s \phi^{(s)} \right)^2 \right] = (E \exp [\lambda Z])^k \]

which finite for \( \lambda < 1/2 \). Therefore, we have for this part a deterministic upper bound and therefore (4.9) follows.
We next prove (4.10).

\[
\sum_\sigma p(\sigma) \exp [\lambda N Y_k^2] = E \sum_\sigma p^{(k)}(\sigma) \exp \left[ \sqrt{\frac{2}{N}} \sum_{s=1}^{k} \sum_{r=1}^{N} \gamma_r \sigma_i^{(r)} \sigma_i^{(k+1)} \right]
\]

\[
= E \sum_\sigma p^{(k)}(\sigma) \exp \left[ \sum_{i=1}^{N} \phi_i^{(k+1)} \right] \sqrt{\frac{2}{N}} \sum_{r=1}^{k} \gamma_r \phi_i^{(r)} \]

\[
\leq E \exp \left[ \frac{\lambda}{N} Z^2 \sum_{i=1}^{N} \left( \sum_{r=1}^{k} \gamma_r \phi_i^{(r)} \right)^2 \right]
\]

\[
= E \exp \left[ \sum_{s=1}^{k} \sum_{r=1}^{N} \gamma_r \phi_i^{(r)} \right] \leq E \exp \left[ \lambda q Z^2 \right] < \infty
\]

for \( \lambda q < 1/2 \).

(4.11) is slightly more complicated. We start in the same way as above and reach

\[
\sum_\sigma p^{(k)}(\sigma) \exp \left[ \lambda N \sum_{s=1}^{k} \gamma_s \phi_i^{(s)} \right] = E \exp \left[ \sum_{i=1}^{N} \chi_i \left( \sum_{s=1}^{k} Z_s \sqrt{\frac{2}{N}} \gamma_s^{(s)} \right) \right]
\]

(4.12)

Fix and \( \varepsilon > 0 \), and consider the event

\[
A_{k,N} := \bigcup_{s: s \leq k} \left\{ \| \zeta^{(s)} \| > 1 + \varepsilon \right\} \cup \bigcup_{s, t: s, t \leq k} \left\{ \| \zeta^{(s)} \zeta^{(t)} \| > \frac{2 \varepsilon}{k} \right\}
\]

On \( A_{k,N}^c \), we estimate the rhs of (4.12) by

\[
(E \exp [\lambda (1 + 2 \varepsilon) Z])^k
\]

which is finite if \( \lambda (1 + 2 \varepsilon) < 1/2 \). On the other hand

\[
\frac{1}{N} \log \sum_\sigma p(\sigma) \exp \left[ \lambda N \sum_{s=1}^{k} \gamma_s \phi_i^{(s)} \right] \leq 2 \lambda \sum_{s=1}^{k} \| \zeta^{(s)} \|^2
\]

and by Lemma 15

\[
\lim_{N \to \infty} \mathbb{E} \left( 1_{A_{N,k}} \sum_{s=1}^{k} \| \zeta^{(s)} \|^2 \right) \leq \lim_{N \to \infty} \sqrt{\mathbb{P}(A_{N,k})} \sqrt{\mathbb{E} \left[ \sum_{s=1}^{k} \| \zeta^{(s)} \|^2 \right]} = 0
\]

for all \( k \). Therefore,

\[
\lim_{N \to \infty} \mathbb{E} \frac{1}{N} \log \sum_\sigma p^{(k)}(\sigma) \exp \left[ \lambda N \sum_{s=1}^{k} \gamma_s \phi_i^{(s)} \right] = 0
\]

**Proof of Proposition 9.** This is parallel, and we will be brief. A similar
computation as in the previous proof leads to

\[
E_k(Z_N^2) = \sum_{\sigma, \tau} 2^{-2N} \exp \left[ h \sum \langle \sigma_i + \tau_i \rangle + \frac{\beta N}{\sqrt{2}} \sum_{s=1}^{k} \langle \rho^{(s)} \sigma, \sigma \rangle + \langle \rho^{(s)} \tau, \tau \rangle \right] \\
\times \exp \left[ \frac{\beta^2 N}{4} \left( \left( 1 - \sum_{r=1}^{k} \langle \phi^{(r)}, \sigma \rangle \right)^2 \right) \right] \\
+ \left( 1 - \sum_{r=1}^{k} \langle \phi^{(r)}, \tau \rangle \right)^2 \\
+ \left[ \langle \sigma, \tau \rangle - \sum_{r=1}^{k} \langle \sigma, \phi^{(r)} \rangle \langle \tau, \phi^{(r)} \rangle \right]^2 \right].
\]

The only difference between \( E_k(Z_N^2) \) and \( (E_k Z_N)^2 \) come from the presence of the last cross term in the expression above. We therefore only have to check that after the centering of \( \sigma \) around \( m^{(k+1)} \), and switching to \( p^{(k)}(\sigma) \), \( p^{(k)}(\tau) \), this cross term does not cause problems for \( \beta \) small. Writing \( \sigma = \hat{\sigma}^{(k)} + m^{(k+1)} \) and multiplying out, the only contribution in \( \langle \sigma, \tau \rangle - \sum_{r=1}^{k} \langle \sigma, \phi^{(r)} \rangle \langle \tau, \phi^{(r)} \rangle \) which is not linear or quadratic in \( (\hat{\sigma}^{(k)} \hat{\tau}^{(k)}) \) is

\[
\| m^{(k+1)} \|^2 - \sum_{r=1}^{k} \langle m^{(k+1)}, \phi^{(r)} \rangle \langle m^{(k+1)}, \phi^{(r)} \rangle.
\]

But

\[
\lim_{k \to \infty} \lim_{N \to \infty} \mathbb{E} \left[ \| m^{(k+1)} \|^2 - \sum_{r=1}^{k} \langle m^{(k+1)}, \phi^{(r)} \rangle \langle m^{(k+1)}, \phi^{(r)} \rangle \right] = 0,
\]

so what remains after this (asymptotic) cancellation are terms which are linear or quadratic in \( (\hat{\sigma}^{(k)} \hat{\tau}^{(k)}) \). Therefore after squaring this expression, they are quadratic or of higher order. Writing \( G_{N,k}(\sigma, \tau) \) for this, we see in the same way as in the proof of Proposition 8 that

\[
E_k(Z_N^2) \leq \exp \left[ 2N \mathbb{R} S(\beta, h) \sum_{\sigma, \tau} p^{(k)}(\sigma) p^{(k)}(\tau) \times \exp \left[ N \beta \left( F_{N,k}^{(\sigma)}(\sigma) + F_{N,k}^{(\tau)}(\tau) + G_{N,k}(\sigma, \tau) \right) \right] \right]
\]

and with the same argument as before, one sees that for small enough \( \beta \)

\[
\lim_{k \to \infty} \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \sum_{\sigma, \tau} p^{(k)}(\sigma) p^{(k)}(\tau) \times \exp \left[ N \beta \left( F_{N,k}^{(\sigma)}(\sigma) + F_{N,k}^{(\tau)}(\tau) + G_{N,k}(\sigma, \tau) \right) \right] \leq 0.
\]

\section{Technical lemmas}

\textbf{Lemma 11} \( \langle \phi^{(m)}, \xi^{(m)} \rangle \) is (unconditionally) Gaussian with variance \( 1/N \).

\textbf{Proof.}

\[
\langle \phi^{(m)}, \xi^{(m)} \rangle = \frac{1}{N} \sum_{i,j} \phi_i^{(m)} g_{ij}^{(m)} \phi_j^{(m)}
\]

20
\( \phi^{(m)} \) is \( F_{m-1} \)-m.b., and \( g^{(m)} \) is conditionally Gaussian given \( F_{m-1} \) with covariances given by (2.9). Computing the conditional variance, using this expression, yields
\[
\mathbb{E}_{m-1} \left( \frac{1}{N} \sum_{i,j} \phi^{(m)}_i \phi^{(m)}_j g^{(m)}_i g^{(m)}_j \right)^2 = \frac{1}{N}
\]
This proves the claim.

Below, we denote by \( \chi_n(x) \), \( x \geq 0 \), the density of the \( \chi^2 \)-distribution of degree \( n \), i.e.
\[
\chi_n(x) := \frac{x^{n/2-1}e^{-x/2}}{2^{n/2}\Gamma(n/2)},
\]
\( \Gamma \) here gamma function, and
\[
\Xi_n(x) := \int_{x}^{\infty} \chi_n(y) dy.
\]
For fixed \( n \), \( \Xi_n(x) \) is exponentially decaying for \( x \to \infty \).

**Lemma 12** Let \( y^{(1)}, \ldots, y^{(k)} \) be orthonormal vectors in \( \mathbb{R}^N \), and \( X \) be a Gaussian random variable with covariances
\[
\mathbb{E} X_i X_j = \delta_{ij} - \frac{1}{N} \sum_{s=1}^{k_1} y^{(s)}_i y^{(s)}_j + \frac{1}{N} \sum_{s=k_1+1}^{k} y^{(s)}_i y^{(s)}_j
\]
with \( 1 \leq k_1 \leq k \), \( k_2 := k - k_1 \).

a) If \( f_i : \mathbb{R} \to \mathbb{R} \) are Lipshitz continuous functions with
\[
\lambda := \sup_i \|f_i\|_{lip} < \infty
\]
then
\[
\mathbb{P} \left( \left| \frac{1}{N} \sum_{i=1}^{N} [f_i(X_i) - Ef_i(Z)] \right| \geq t \right) \leq \Xi_{k_1} \left( \frac{t^2 N}{9k_1 \lambda^2} \right) + \Xi_{k_2} \left( \frac{t^2 N}{9k_2 \lambda^2} \right) + \exp \left[ -t^2 N/\lambda^2 \right].
\]

b) \[
\mathbb{E} \|X\|^2 = n - k + 2k_2,
\]
and
\[
\mathbb{P} \left( \|X\|^2 \geq t \right) = \int_{0}^{t N/2} \Xi_{N-k} (Nt - 2x) \chi_{k_2} (x) dx
\]

**Proof.** a) Choose i.i.d. standard Gaussian variables \( U_1, \ldots, U_N \), and \( Z_1, \ldots, Z_k \). Then \( Y \) with
\[
Y_i := X_i + \frac{1}{\sqrt{N}} \sum_{s=1}^{k_1} y^{(s)}_i Z_s
\]
has the same distribution as $Y'$ given by

$$Y'_i := U_i + \frac{1}{\sqrt{N}} \sum_{s=k_1+1}^{k} y^{(s)}_s Z_s.$$  

Then

$$\text{P} \left( \left| \frac{1}{N} \sum_{i=1}^{N} \left[ f_i (X_i) - Ef_i (Z) \right] \right| \geq t \right) \leq \text{P} \left( \left| \frac{1}{N} \sum_{i=1}^{N} \left[ f_i (X_i) - f_i (Y_i) \right] \right| \geq t/3 \right) + P \left( \left| \frac{1}{N} \sum_{i=1}^{N} \left[ f_i (Y'_i) - f_i (U_i) \right] \right| \geq t/3 \right) + P \left( \left| \frac{1}{N} \sum_{i=1}^{N} \left[ f_i (U_i) - Ef_i (Z) \right] \right| \geq t/3 \right).$$

Estimating the first two parts using the orthonormality of the $y^{(s)}$, and the last summand using Gaussian isoperimetry, leads to the desired bound.

b) The covariance matrix $(\mathbb{E} X_i X_i)_{i,j}$ has spectrum $\{0, 1, 2\}$ with multiplicities $k_1, N - k, k_2$. From that, the estimate follows. ■

**Lemma 13** If (1.4) is satisfied, then $\lim_{k \to \infty} \lim_{N \to \infty} \mathbb{E} \|g^{(k)} m^{(k)}\|^2 = 0$.

**Proof.** As $m^{(k)}$ is $G_{k-1}$-m.b., $g^{(k)} m^{(k)}$ is conditionally Gaussian with covariances, using Proposition 4,

$$\mathbb{E}_{k-1} \left( \left[ \left( g^{(k)} m^{(k)} \right)_i \left( g^{(k)} m^{(k)} \right)_s \right] \right) = \left[ \delta_{is} - \alpha_{is}^{(k-1)} \right] \left[ \left\| m^{(k)} \right\|^2 - \sum_{m=1}^{k-1} \left( m^{(k)}, \phi^{(m)} \right)^2 \right].$$

(5.1)

Using Lemma 12 one gets

$$\mathbb{E}_{k-1} \left\| g^{(k)} m^{(k)} \right\|^2 = \left\| m^{(k)} \right\|^2 - \sum_{m=1}^{k-1} \left( m^{(k)}, \phi^{(m)} \right)^2.$$

$$\mathbb{E} \left\| g^{(k)} m^{(k)} \right\|^2 = \mathbb{E} \left\| m^{(k)} \right\|^2 - \sum_{m=1}^{k-1} \left( m^{(k)}, \phi^{(m)} \right)^2.$$

By Proposition 3 the rhs converges, as $N \to \infty$, to $q = \sum_{m=1}^{k-1} s_m^2$, which converges to 0, as $k \to \infty$, by (1.4) and Lemma 2 ■

**Lemma 14** For any function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $|f(x)| \leq C (1 + |x|)$ for some $C$, and with $\|f\|_{lip} < \infty$, and any $k \geq 2$, one has

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f \left( h^{(k)}_i \right) = Ef (h + \beta \sqrt{N} Z)$$

in $L_1$.  

22
Proof. For $k = 2$, this is immediate from the definition of $h^{(2)}$ and Lemma 12 a). So, we assume $k \geq 3$. Conditionally on $G_{k-2}$, $g^{(k-1)}m^{(k-1)}$ is Gaussian with the covariances given in (5.1). For abbreviation, write

$$Y_i^{(t)} := h + \beta \sum_{s=1}^t \gamma_s \zeta_i^{(s)}$$

As $\|m^{(k)}\|^2$ and $\langle m^{(k)}, \phi^{(m)} \rangle^2$ are bounded (by 1), it follows from Lemma 12 a) that

$$\frac{1}{N} \sum_{i=1}^N \left[ f(h_i^{(k)}) - E_k f(h_i^{(k)}) \right] \to 0$$
in $L_1$, as $N \to \infty$, and using Proposition 6, one gets

$$\frac{1}{N} \sum_{i=1}^N \left[ E_k f(Y_i^{k-2}) + \sqrt{q - \sum_{s=1}^{k-2} \gamma_s^2 Z} \right] \to 0.$$

Next, in the same way, one obtains

$$\frac{1}{N} \sum_{i=1}^N \left[ E f(Y_i^{(k-2)} + \sqrt{q - \sum_{s=1}^{k-2} \gamma_s^2 Z}) \right] \to 0.$$

Going on in the same way, and observing that

$$\sum_{s=1}^{k-2} \gamma_s Z_s + \sqrt{q - \sum_{j=1}^{k-2} \gamma_j^2 Z}$$
is identical in law as $\sqrt{q}Z$, one gets

$$\lim_{N \to \infty} \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N f(h_i^{(k)}) - Ef(h + \beta \sqrt{q}Z) \right| = 0.$$

Lemma 15 a) For any $k$

$$\sup_N \mathbb{E} \|\zeta^{(k)}\|^4 < \infty.$$

b) For any $k$ and $\varepsilon > 0$

$$\lim_{N \to \infty} \mathbb{P} \left( \|\zeta^{(k)}\|^2 \geq 1 + \varepsilon \right) = 0.$$

c) For $s \neq k$, and $\varepsilon > 0$

$$\lim_{N \to \infty} \mathbb{P} \left( \|\zeta^{(k)}, \zeta^{(s)}\| \geq \varepsilon \right) = 0.$$
\textbf{Proof.} By the conditional covariances of $\zeta^{(k)}$ given in Proposition 3 c), $\zeta^{(k)}$ has, conditionally on $G_{k-1}$, the covariance structure of $X$ in Lemma 12 a) and b) of the present lemma are then immediate from b) of Lemma 12.

For c), we assume $s < k$. Then $\langle \zeta^{(k)}, \zeta^{(s)} \rangle$ is Gaussian, conditioned on $F_{k-1}$, with conditional variance

$$E_{k-1} \left( \zeta^{(k)}, \zeta^{(s)} \right)^2 = s_N^2 (k, s) := \frac{1}{N} \left[ \left\langle \zeta^{(s)} \right\rangle^2 + \left\langle \zeta^{(s)}, \phi^{(k)} \right\rangle^2 - \sum_{u=1}^{k-1} \left\langle \zeta^{(s)}, \phi^{(u)} \right\rangle^2 \right],$$

and therefore

$$P \left( \left| \left\langle \zeta^{(k)}, \zeta^{(s)} \right\rangle \right| \geq \varepsilon \right) = 2 \Phi \left( 1 - \Phi \left( \frac{\varepsilon}{s_N (k, s)} \right) \right),$$

$\Phi$ being the distribution function of the standard Gaussian distribution. Estimating $s_N^2 \leq 2 \left\| \zeta^{(s)} \right\|^2 / N$, and using again Lemma 12 b) proves that the rhs goes to 0 for $N \to \infty$.  

\textbf{Lemma 16} For any $n \geq 2$

$$\lim_{N \to \infty} \left\langle \zeta^{(n-1)}, \mathbf{m}^{(n)} \right\rangle = \beta (1 - q) \sqrt{q - \sum_{j=1}^{n-2} \gamma_j^2},$$

and for $1 \leq m \leq n - 2$

$$\lim_{N \to \infty} \left\langle \zeta^{(m)}, \mathbf{m}^{(n)} \right\rangle = \beta \gamma_m (1 - q)$$

in $L_2 (P)$.

\textbf{Proof.} This is very similar to the proof of Lemma 13 and we will be brief. The case $n = 2$ is straightforward, and we assume $n \geq 3$

$$\mathbf{m}^{(n)} = \text{Th} \left( \sum_{j=1}^{n-2} \gamma_j \zeta^{(j)} + g^{(n-1)} \mathbf{m}^{(n-1)} \right).$$

Using Lemma 3 we have

$$g^{(n-1)} \mathbf{m}^{(n-1)} = \left\| \mathbf{m}^{(n-1)} - \sum_{j=1}^{n-2} \left\langle \mathbf{m}^{(n-1)}, \phi^{(j)} \right\rangle \phi^{(j)} \right\| \zeta^{(n-1)}$$

and by Proposition 3 we can replace (in the $N \to \infty$ limit) the above norm by $\sqrt{q - \sum_{j=1}^{n-2} \gamma_j^2}$. Therefore $\langle \zeta^{(n-1)}, \mathbf{m}^{(n)} \rangle$ behaves in the $N \to \infty$ limit similarly to

$$\left\langle \zeta^{(n-1)}, \text{Th} \left( \sum_{j=1}^{n-1} \gamma_j \zeta^{(j)} + \sqrt{q - \sum_{j=1}^{n-2} \gamma_j^2} \zeta^{(n-1)} \right) \right\rangle.$$ 

Arguing in the same way as in the proof of Lemma 13 one sees that this converges to

$$EZ \text{Th} \left( \sum_{j=1}^{n-1} \gamma_j Z_j + \sqrt{q - \sum_{j=1}^{n-2} \gamma_j^2} Z \right)$$

$$= \beta \sqrt{q - \sum_{j=1}^{n-2} \gamma_j^2} \left( 1 - E \text{Th}^2 \left( \sum_{j=1}^{n-1} \gamma_j Z_j + \sqrt{q - \sum_{j=1}^{n-2} \gamma_j^2} Z \right) \right)$$

$$= \beta (1 - q) \sqrt{q - \sum_{j=1}^{n-2} \gamma_j^2},$$

24
the first equality by Gaussian partial integration. The case where $m \leq n - 2$ is going by the same argument, but where we get from partial integration $\gamma_m$ instead of $\sqrt{q - \sum_{j=1}^{n-2} \gamma_j^2}$. □

6 Comments

There are a number of issues and open problems we shortly want to comment on.

On the first moment evaluation: The key idea proposed here is to derive the free energy by a conditionally annealed argument, where the $\sigma$-field for the conditioning is chosen such that the solutions of the TAP equations are measurable. This can reasonably only be done by an approximating sequence $m^{(k)}$ for the TAP equations, where for fixed $k$ one lets first $N \to \infty$, and afterwards $k \to \infty$. For finite $N$, the TAP equations are not exactly valid, and we wouldn’t know how to characterize $\langle \sigma_i \rangle$ for finite $N$ without knowing the Gibbs measure already precisely. Therefore, it would be natural just to condition with respect to $\sigma(m^{(k)})$, and try to prove the corresponding versions of Proposition 8 and 9. We however didn’t see how to do this, and therefore, we took the $\sigma$-fields, generated by $\zeta(s)$, $s \leq k$, with respect to which $m^{(k)}$ is measurable. This choice may well be “too large”, in particular as the $\zeta(s)$ depend on the starting version of $m^{(1)}$ which we took just as $\sqrt{q}$. On the other hand, taking $\sigma$-fields which are larger than necessary should not do any harm for proving Proposition 8 except that the computations may become unnecessarily complicated. Anyway, assuming that the replica symmetric solution is valid in the full AT-region, it looks to me that (3.1) should be correct in the full AT-region. This belief is based on the hope that the Morita type argument could give the evaluation in the full high-temperature region. This hope is also substantiated by the recent work by Jian Ding and Nike Sun [5] who, for the Ising perceptron, obtained a one-sided (and partly computer assisted) result in the full replica symmetric region, based on a method which is related to ours.

Even if our conjecture is correct, there remains the issue how to prove it, and in particular, whether our choice of the $\sigma$-fields is the best one. As remarked before, there is nothing lost till (4.8): The region for $(\beta, h)$ where (4.8) is correct is exactly the region where (3.1) is correct. (4.8) is a standard large deviation problem with a Hamiltonian which is of ordinary mean-field type. In principle, it is not difficult to write down a variational formula for

$$
\lim_{N \to \infty} \frac{1}{N} \log \sum_{\sigma} p^{(k)}(\sigma) \exp \left[ N \beta F_{N,k}^{m^{(k)}}(\sigma) \right]
$$

or its $\mathbb{E}$-expectation, and then try to evaluate the $k \to \infty$ limit. I have not been able to do that in the full high temperature region, but, it doesn’t appear being impossible. It would be interesting to clarify this point. It is possible that the above limit is 0 even beyond the AT-line, but of course the AT-condition was used to prove that (4.8) is equivalent to (3.1).

The second moment: Regardless what the outcome for the first moment is, I wouldn’t expect that the plain vanilla second moment estimate used here would work in the full high temperature regime. This disbelief is based on a simple computation for the following toy model: Take $\sigma_i$ i.i.d. $\{-1,1\}$-valued with
mean \( m \in (-1, 1) \), \( m \neq 0 \), and consider the spin glass model with partition function

\[
Z_{N,\beta, m} := \sum_\sigma p(\sigma) \exp \left[ \frac{\beta}{\sqrt{2}} \sum_{i,j} g_{ij} \hat{\sigma}_i \hat{\sigma}_j - \frac{\beta^2 N}{4} \left( \frac{1}{N} \sum_i \hat{\sigma}_i^2 \right)^2 \right],
\]

where \( \hat{\sigma}_i := \sigma_i - m \). If \( m = 0 \), then the second summand in the exponent is \( \beta^2 N/4 \), and we have the standard SK-model at \( h = 0 \). Of course, \( \mathbb{E} Z_{N,\beta, h} = 0 \), and the question is whether

\[
\lim_{N \to \infty} \frac{1}{N} \log Z_{N,\beta, m} = 0. \tag{6.1}
\]

This is certainly correct for small enough \( \beta \), as can for small \( \beta \), easily be proved by a second moment computation. Indeed,

\[
\mathbb{E} Z^2_{N,\beta, m} = \sum_\eta \bar{p}(\eta) \exp \left[ \frac{N \beta^2}{2} \left( \frac{N^{-1} \sum_i \eta_i}{2} \right)^2 \right],
\]

where the \( \eta_i \) under \( \bar{p} \) are i.i.d., and have the distribution of \( (\sigma_i - m)(\sigma'_i - m) \) where \( \sigma_i, \sigma'_i \) are independent with distribution \( p \). Therefore

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} Z^2_{N,\beta, m} = \sup_x \left( \beta^2 x^2/2 - J(x) \right),
\]

\( J \) being the standard rate function for \( \bar{p} \). It is easily checked that the right hand side is 0 for small \( \beta \), and as the second derivative of \( J \) at 0 is \( (1 - m^2)^{-2}/2 \), one would expect that this is true as long as \( \beta^2 (1 - m^2)^2 \leq 1 \). That looks to be the right de Almeida-Thouless condition. However, one easily checks that \( \sup_x \left( \beta^2 x^2/2 - J(x) \right) > 0 \) for \( \beta \) sufficiently close but smaller than \( (1 - m^2)^{-1} \), for any choice of \( m \neq 0 \), a fact which is due to the non-vanishing third derivative of \( J \) at 0. Therefore, (6.1) cannot be proved with a simple second moment computation up to the “natural” AT-condition. Actually, I don’t know if (6.1) is correct under \( \beta^2 (1 - m^2)^2 \leq 1 \). (If not already known, it could be a level-2-problem in Talagrand’s difficulty scale).

The computation in this toy case suggests that a simple second moment estimate, in our asymmetric situation when \( h \neq 0 \), is not sufficient to cover the full high temperature regime.

**Gibbs distributions:** It is suggestive to conjecture that the Gibbs distribution (1.2), in high temperature, is somehow close to the conditional annealed measure, i.e. the measure on \( \Sigma_N \) defined by

\[
\mathbb{E}_k \exp \left[ H_{\beta, h} (\sigma) \right] / \sum_\sigma \mathbb{E}_k \exp \left[ H_{\beta, h} (\sigma) \right]
\]

which, according to the analysis given in this paper, is a kind of complicated random Curie-Weiss type model, with the centering of the \( \sigma \) given at the solution of the TAP-equation. If correct, this would suggest that the finite \( N \) high temperature Gibbs distributed can be approximated by random Curie-Weiss models, with however infinitely (if \( k \to \infty \)) many random quadratic interaction terms.
**Low temperature:** A main problem is to extend the method to low temperature. There are many results in the physics literature about the validity of the TAP equations in low temperature, see [7], [11], but the plain iteration method in [3] is certainly not able to catch such solutions. However, it has recently been shown by Marc Mézard that a similar iterative scheme for Hopfield model converges in the retrieval phase of the model (see [8]). The approximate validity of the TAP equations in generic $p$-spin models has recently been shown in [1]. See also the results of [2] and [4] on the TAP variational problem, and [13] on the $p$-spin spherical model where the TAP equations in the full temperature regime are discussed. These results (except [2]) depend on already having a rather detailed picture of the Gibbs distribution, whereas the attempt here is to present a new viewpoint.

**References**

[1] Auffinger, A., and Jagannath, A.: *Thouless-Anderson-Palmer equations for conditional Gibbs measures in the generic p-spin glass model.* arXiv:1612.06359, to appear in Ann. Prob.

[2] Belius, D., and Kistler, N.: *The TAP-Plefka variational principle for the spherical SK model.* arXiv:1802.05782

[3] Bolthausen, E.: *An iterative construction of solutions of the TAP equations for the Sherrington-Kirkpatrick model.* Comm. Math. Phys., **325**, 333–366 (2014).

[4] Chen, W.-K., and Panchenko, D.: *On the TAP free energy in the mixed p-spin models.* Comm. Math. Phys. **362**, 219–252 (2018).

[5] Ding, J., and Sun, N.: *Capacity lower bound for the Ising perceptron.* to appear in arXiv.

[6] Guerra, F. and Toninelli, F. L.: *The thermodynamic limit in mean field spin glass models.* Comm. Math. Phys. **230**, 71-79 (2002).

[7] Mézard, M., Parisi, G., and Virasoro, M.A.: *Spin glass theory and beyond.* World Scientific LN in Physics, Vol 9. World Scientific 1987.

[8] Mézard, M.: *Mean-field message-passing equations in the Hopfield model and its generalizations.* Phys. Rev. E **95**, 22117-22132 (2017).

[9] Morita,T.: *Statistical mechanics of quenched solid solutions with application to magnetically dilute alloys.* J. Math. Phys. **5**, 1401-1405 (1966).

[10] Panchenko, D.: *The Sherrington-Kirkpatrick model.* Springer, New York, 2013.

[11] Plefka, T.: *Convergence condition of the TAP equation for the infinite-ranged Ising spin glass model.* J. Phys. A: Math. Gen. **15**, 1971–1978 (1982).

[12] Sherrington, D., and Kirkpatrick, S.: *Solvable model of a spin-glass.* Phys. Rev. Lett. **35**, 1792–1795 (1975).
[13] Subag, E.: *Free energy landscapes in spherical spin glasses.* arXiv:1804.10576

[14] Talagrand, M.: *Mean field models for spin glasses. Volume I&II.* Springer, Berlin, 2011.

[15] Thouless, D.J., Anderson, P.W., and Palmer, R.G.: *Solution of “solvable model in spin glasses”.* Philosophical Magazin **35**, 593-601 (1977).