Virtually free pro-$p$ groups whose torsion elements have finite centralizer

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Abstract

A finitely generated virtually free pro-$p$ group with finite centralizers of its torsion elements is the free pro-$p$ product of finite $p$-groups and a free pro-$p$ factor.

1. Introduction

The objective of this paper is to give a complete description of a finitely generated virtually free pro-$p$ group whose torsion elements have finite centralizers. Our main result is the following theorem.

Theorem 1. Let $G$ be a finitely generated virtually free pro-$p$ group such that the centralizer of every torsion element in $G$ is finite. Then $G$ is a free pro-$p$ product of subgroups which are finite or free pro-$p$.

This is a rather surprising result from a group-theoretic point of view, since the analogous statement does not hold for abstract groups (or for profinite groups): an easy counter-example is given in Section 5. However, from a Galois-theoretic point of view it is not so surprising. Indeed, the finite centralizer condition for torsion elements arises naturally in the study of absolute Galois groups. In particular, Haran [2] (see also Efrat [1] for a different proof) proved the above theorem for the case when $G$ is an extension of a free pro-2 group by a group of order 2.

The proof of Theorem 1 explores a connection between $p$-adic representations of finite $p$-groups and virtually free pro-$p$ groups, which gives a new approach to studying virtually free pro-$p$ groups. This connection enables us to use the following beautiful result.

Theorem 2 (Weiss [10]). Let $G$ be a finite $p$-group, let $N$ be a normal subgroup of $G$ and let $M$ be a finitely generated $\mathbb{Z}_p[G]$-module. Suppose that $M$ is a free $N$-module and that $M^N$ is a permutation lattice for $G/N$. Then $M$ is a permutation lattice for $G$.

Here, $M^N$ denotes the fixed submodule for $N$, and a permutation lattice for $G$ is a direct sum of $G$-modules, each of the form $\mathbb{Z}_p[G/H]$ for some subgroup $H$ of $G$.

The connection to the representation theory cannot be used in a straightforward way, however. Indeed, if one factors out the commutator subgroup of a free open normal subgroup $F$ then the $G/F$-module obtained would, in general, not satisfy the hypothesis of Weiss’s theorem.

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In order to make the representation theory work, we use pro-$p$ HNN-extensions to embed $G$ into a rather special virtually free pro-$p$ group $\tilde{G}$, in which, after factoring out the commutator of a free open normal subgroup, the hypotheses of Weiss’s theorem are satisfied. With the aid of this we prove Theorem 1 for $\tilde{G}$ and apply the Kurosh subgroup theorem to deduce the result for $G$.

We use the notation for profinite and pro-$p$ groups from [7].

2. Preliminary results

**Theorem 3** [3, Theorem 2.6]. Let $G$ be a group of order $p$ and let $M$ be a finitely generated $\mathbb{Z}_p[G]$-module, free as a $\mathbb{Z}_p$-module. Then we can write

$$M = M_1 \oplus M_p \oplus M_{p-1}$$

in such a way that $M_p$ is a free $G$-module, $M_1$ is a trivial $G$-module and $1 + c + \ldots + c^{p-1}$ induces the zero map on $M_{p-1}$, where $G = \langle c \rangle$.

Let $G$ be a finite $p$-group. A permutation lattice for $G$ is a finite direct sum of $G$-modules, each of the form $\mathbb{Z}_p[G/H]$ for some subgroup $H$ of $G$. A permutation lattice will also be called a $G$-permutational module.

If $G$ is of order $p$, then Theorem 3 implies that $M$ is a permutation lattice if and only if $M_{p-1} = \{0\}$ in the decomposition for $M$ if and only if $M/(g-1)M$ is torsion-free for $1 \neq g \in G$.

**Lemma 4.** For any finite group $N$, integral domain $R$ and finitely generated free $R[N]$-module $M$, the map $\phi : M \to M^N$ defined by $\phi(m) := \sum_{n \in N} nm$ is an epimorphism with kernel $JM$, where $J$ is the augmentation ideal in $R[N]$.

**Proof.** The map is well defined, since $\sum_{n \in N} n$ belongs to the centre of $R[N]$, and for the same reason $JM$ is contained in the kernel of $\phi$. Consider $M = R[N] \otimes_R L$ with $L$ a free $R$-module; then, when $m = \sum_{n \in N} n \otimes l(n)$ with $l(n) \in L$ belongs to $M^N$, it follows that all $l(n)$ are equal so that $m = \phi(1_N \otimes l(n))$. Hence $\phi$ is an epimorphism. If

$$m = \sum_{x \in N} x \otimes l(x) \in \ker(\phi),$$

then modulo $JM$ it is of the form $\sum_{x \in N} 1_N \otimes l(x)$, and therefore $\sum_{x \in N} l(x) = 0$ must hold; that is, $m \in JM$.

**Remark 5.** When applying Theorem 2, in light of Lemma 4, we shall usually check the hypothesis for $M_N = M/JM$ instead of $M^N$.

We shall need the following connection between free decompositions and $\mathbb{Z}_p$-representations for free pro-$p$ by $C_p$ groups.

**Lemma 6.** Let $G$ be a split extension of a free pro-$p$ group $F$ of finite rank by a group of order $p$. Then the following hold.

(i) The extension $G$ has a free decomposition $G = (\coprod_{a \in A} C_a \times H_a) \amalg H$, with $C_a \cong C_p$ and $H_a$ and $H$ free pro-$p$ contained in $F$ (see [9]).
(ii) Set $M := F/[F, F]$. Fix $a_0 \in A$ and a generator $c$ of $C_{a_0}$. Then conjugation by $c$ induces an action of $C_{a_0}$ upon $M$. The latter module decomposes in the form

$$M = M_1 \oplus M_p \oplus M_{p-1}$$

such that $M_p$ is a free \langle$c$\rangle-module, $1 + c + \ldots + c^{p-1}$ induces the zero map on $M_{p-1}$, and $c$ acts trivially on $M_1$.

Moreover, the ranks of the three $G/F$-modules satisfy

$$\text{rank}(M_p) = \text{rank}(H), \quad \text{rank}(M_{p-1}) = |A| - 1 \quad \text{and} \quad \text{rank}(M_1) = \sum_{a \in A} \text{rank}(H_a).$$

In particular, $M$ is a $G/F$-permutational module if and only if $|A| = 1$.

**Proof.** Part (i) is [9, Theorem 1.1]. For the proof of (ii), first choose $a_0 \in A$; then for each $a \in A$ choose a generator $c_a$ of $C_a$ with $c_a c_{a_0}^{-1} \in F$, and put $c_{a_0} := c$. We claim that

$$F = \left( \prod_{a \in A} H_a \right) \oplus \left( \prod_{j=0}^{p-1} H^{c_j} \right) \oplus \left( \prod_{a \in A \setminus \{a_0\}} \prod_{j=0}^{p-2} (c_ac_{a_0}^{-1}) c_j \right).$$

Indeed, consider the epimorphism $\phi : G \to C_p$, with $H$ and all $H_a$ in the kernel and sending each generator $c_a$ of $C_a$ to the generator of $C_p$. Set $F_0 := \langle H^{c_j}, H_a, (c_ac_{a_0}^{-1}) c_j \rangle \ a \in A, j = 0, \ldots, p - 1 \rangle$ and observe that it is contained in $\ker \phi$. Then $F_0$ contains the normal closure in $G$ of $\langle H, H_a \mid a \in A \rangle$ and is itself a normal subgroup of $G$ since

$$(c_ac_{a_0}^{-1}) c_j \cdots (c_ac_{a_0}^{-1}) c_2 \cdots (c_ac_{a_0}^{-1}) c_1 = \ldots = c^{-(p-1)} c_{a_0} c = 1.$$  

Then $\langle F_0, c \rangle = G$ and as $F_0 \leq F$, we conclude that $F = F_0 = \ker \phi$. This shows that

$$\text{rank}(F) \leq \sum_{a \in A} \text{rank} H_a + p \text{rank}(H) + (p - 1)(|A| - 1).$$

On the other hand, one can use the pro-$p$-version of the Kurosh subgroup theorem, [7, Theorem 9.1.9] applied to $F$ as an open subgroup of $G$, to see that

$$F = \left( \prod_{a \in A} H_a \right) \oplus \left( \prod_{j=0}^{p-1} H^{c_j} \right) \oplus U$$

with $U$ a free pro-$p$ subgroup of $F$ having

$$\text{rank}(U) = 1 + |A|p - |F\setminus G/H| - \sum_{a \in A} |F\setminus G/(H_a \times C_a)| = 1 + |A|p - p - |A| = (|A| - 1)(p - 1).$$

This shows the validity of the claimed free decomposition of $F$.

Factoring out $[F, F]$ yields the desired decomposition — the images of the three free factors. Finally, $M_{p-1}$ appears as follows: writing $f_a := c_ac_{a_0}^{-1}$, the above calculation yields the equality

$$f_a^{c_{a_0}^{-1}} f_a^{c_{a_0}^{-2}} \cdots f_a^{c_{a_0}^{-c}} = 1$$

for every $c \in A$ which, in additive notation, reads $(c^{p-1} + c^{p-2} + \ldots + c + 1)f_a = 0$. 

**Corollary 7.** If for each $a \in A$, a basis $B_a$ of $H_a$ is given and $B$ is any basis of $H$, then $\bigcup_{a \in A} B_a [F, F]/[F, F]$ is a basis of $M_1$ and $B[F, F]/[F, F]$ a basis of the $G/F$-module $M_p$. A basis of $M_{p-1}$ is given by $\{c_ac_{a_0}^{-1} \mid a \in A, a \neq a_0\} [F, F]/[F, F]$.

The next lemma follows from [6, Corollary 3]. We include a proof for the convenience of the reader.
Lemma 8. Every finitely generated virtually free pro-$p$ group has, up to conjugation, only a finite number of finite subgroups.

Proof. Suppose that the lemma is false and that $G$ is a counter-example possessing a normal free pro-$p$ subgroup $F$ of minimal possible index. When $H$ is a maximal open subgroup of $G$ with $F \leq H$ then, as $|H : F| < |G : F|$, the proper subgroup $H$ satisfies the conclusion of the lemma, and so there are, up to conjugation, only finitely many finite subgroups of $G$ contained in $H$. Hence, in order to be a counter-example, $G$ must be of the form $G = F \times K$ for a finite subgroup $K$ of $G$ and, as $G$ contains only finitely many such subgroups $H$, the proof is complete if we can show that up to conjugation there are only finitely many finite subgroups $L \cong K$ in $G$. Let $t$ be a central element of order $p$ in $K$ and consider that $G_1 := F \times \langle t \rangle$. Certainly, $G_1$ is finitely generated. Hence, as a consequence of Lemma 6(i) in conjunction with Theorem 11 (see below), $G_1$ satisfies the conclusion of the lemma, and so $G > G_1$. Next consider any finite subgroup $L \cong K$ of $G$ containing some torsion element $s \in G_1$. Then $G = F \times K = F \times L$ and $[s,L] \in L \cap F = \{1\}$, which show that $L$ is contained in $C_G(s)$. By [9, Theorem 1.2], we see that $C_F(s)$ is a free factor of $F$, and therefore, since $F$ is finitely generated, $C_F(s)$ is finitely generated as well, and so is $C_G(s)$. Let bars denote images modulo the normal subgroup $\langle s \rangle$ of $C_G(s)$. Then $|\bar{C_G(s)} : \bar{C_F(s)}| < |G : F|$ so that $\bar{C_G(s)}$ contains only finitely many conjugacy classes of maximal finite subgroups. Therefore, $C_G(s)$ also contains only finitely many conjugacy classes of maximal finite subgroups. Since the centralizers of conjugate elements are conjugate, $G$ can, up to conjugation, contain only finitely many maximal finite subgroups, which is a contradiction. 

Lemma 9. Let $G$ be a virtually free pro-$p$ group, let $F$ be a normal open free pro-$p$ subgroup of $G$ and let $C_F(t) = \{1\}$ for every torsion element $t \in G$. Then any pair of distinct maximal finite subgroups $A, B$ of $G$ have trivial intersection.

Proof. Suppose that the lemma is false. Then one can choose maximal finite subgroups $A$ and $B \neq A$ such that $1 \neq C := A \cap B$ is of maximal possible cardinality. Then $C$ is a finite normal subgroup of $L := \langle N_A(C), N_B(C) \rangle$, so the latter is itself finite, since $N_G(C)$ must be finite (by [7, Lemma 9.2.8] a finite normal subgroup of a pro-$p$ group intersects the centre nontrivially). On the other hand, one must have $L \cap A = C$ due to the maximality assumption on the cardinality of pairwise intersections of maximal finite subgroups. Since $C < A$, one arrives at the contradiction $C < N_A(C) \leq L \cap C = C$.

We shall also frequently use the following results about virtually free pro-$p$ groups and free pro-$p$ products; assertion (ii) follows immediately from (i).

Proposition 10 [11, Proposition 1.7]. Let $G$ be a virtually free pro-$p$ group and let $N$ be a normal subgroup generated by torsion. Then:

(i) $\text{Tor}(G)$ maps onto $\text{Tor}(G/N)$ under the canonical epimorphism $G \to G/N$;
(ii) $G/\langle \text{Tor}(G) \rangle$ is free pro-$p$.

Theorem 11 [7, Theorems 9.1.12 and 9.5.1]. Let $G = \prod_{i=1}^n G_i$ be a free profinite (pro-$p$) product. Then $G_i \cap G_j^g = 1$ for either $i \neq j$ or $g \notin G_j$. Every finite subgroup of $G$ is conjugate to a subgroup of a free factor.
3. **HNN-embedding**

We introduce a notion of a pro-

$p$ HNN-group as a generalization of pro-

$p$ HNN-extension in the sense of [8, p. 97]. It can also be defined as a sequence of pro-

$p$ HNN-extensions. In the definition that follows, $i$ belongs to a finite set $I$ of indices.

**Definition 12.** Let $G$ be a pro-

$p$ group and let $A_i, B_i$ be subgroups of $G$ with isomorphisms $\phi_i : A_i \to B_i$. The pro-

$p$ HNN-group is then a pro-

$p$ group $\text{HNN}(G, A_i, \phi_i, z_i)$ having presentation

$$\text{HNN}(G, A_i, \phi_i, z_i) = \langle G, z_i \mid \text{rel}(G), \forall a_i \in A_i : a_i^{z_i} = \phi_i(a_i) \rangle.$$ 

The group $G$ is called the base group, $A_i, B_i$ are called associated subgroups and $z_i$ are called the stable letters.

For the rest of this section, let $G$ be a finitely generated virtually free pro-

$p$ group, and fix an open free pro-

$p$ normal subgroup $F$ of $G$ of minimal index. Also suppose that $C_F(t) = \{1\}$ for every torsion element $t \in G$. Let $K := G/F$ and form $G_0 := G \amalg K$. Let $\psi : G \to K$ denote the canonical projection. It extends to an epimorphism $\psi_0 : G_0 \to K$, by sending $g \in G$ to $gF/F \in K$ and each $k \in K$ identically to $k$, and using the universal property of the free pro-

$p$ product. Note that the kernel of $\psi_0$, say $L$, is an open subgroup of $G_0$ and, as $L \cap G = F$ and $L \cap K = \{1\}$, as a consequence of the pro-

$p$ version of the Kurosh subgroup theorem, [7, Theorem 9.1.9], $L$ is free pro-

$p$. Let $I$ be the set of all $G$-conjugacy classes of maximal finite subgroups of $G$, and observe that in light of Lemma 8 the set $I$ is finite. Fix, for every $i \in I$, a finite subgroup $K_i$ of $G$ in the $G$-conjugacy class $i$. We define a pro-

$p$ HNN-group by considering first $\tilde{G}_0 := \text{HNN}(G_0, K_i, \phi_i, z_i)$ associated to $\tilde{G}_0$ generated by all elements of the form $k_i^{z_i} \psi(k_i)^{-1}$, with $k_i \in K_i$ and $i \in I$. Finally, set

$$\tilde{G} := \tilde{G}_0/R,$$

and since all $K_i$ are finite, by [7, Proposition 9.4.3] it is a proper HNN-group

$$\text{HNN}(G_0, K_i, \phi_i, z_i),$$

where $\phi_i := \psi|_{K_i}$, $G_0$ is the base group, $K_i$ are the associated subgroups, and the $z_i$ form a set of stable letters in the sense of Definition 12.

Let us show that $\tilde{G}$ is virtually free pro-

$p$. The above epimorphism $\psi_0 : G_0 \to K$ extends to $\tilde{G} \to K$ by the universal property of the HNN-extension, so $\tilde{G}$ is a semidirect product $\tilde{F} \rtimes K$ of its kernel $\tilde{F}$ with $K$. By [5, Lemma 10], every open torsion-free subgroup of $\tilde{G}$ must be free pro-

$p$, so $\tilde{F}$ is free pro-

$p$.

The objective of the section is to show that the centralizers of torsion elements in $\tilde{G}$ are finite.

**Lemma 13.** Let $\tilde{G} = \text{HNN}(G_0, K_i, \phi_i, z_i)$, and let $\tilde{F}$ be as explained. Then $C_{\tilde{F}}(t) = 1$ for every torsion element $t \in \tilde{G}$.

**Proof.** There is a standard pro-

$p$ tree $S := S(\tilde{G})$ associated to $\tilde{G} := \text{HNN}(G_0, K_i, \phi_i, z_i)$ on which $\tilde{G}$ acts naturally such that the vertex stabilizers are conjugates of $G_0$ and each edge stabilizer is a conjugate of some $K_i$ (cf. [8] and [12, §3]).

Claim: Let $e_1, e_2$ be two edges of $S$ with a common vertex $v$. Then the intersection of the stabilizers $\tilde{G}_{e_1} \cap \tilde{G}_{e_2}$ is trivial.
By translating $e_1, e_2, v$, if necessary, we may assume that $G_0$ is the stabilizer of $v$. Then, up to orientation, we have two cases.

(1) The edges $e_1$ and $e_2$ have $v$ as a common initial vertex. Then $\tilde{G}_{e_1} = K_i^g$ and $\tilde{G}_{e_2} = K_j^g$ with $g, g' \in G_0$ and either $i \neq j$ or $g \not\in K_i g'$. Suppose that $K_i^g \cap K_j^g \neq \{1\}$. Then, since $G_0 = G \sqcup K$, we may apply Theorem 11, in order to deduce the existence of $g_0 \in G_0$ with $K_i^{g_0} \cap K_j^{g_0} \subseteq G$. Now we apply Lemma 9, in order to deduce the contradiction $i = j$ and $gg_0 \in K_i g' g_0$. Therefore, we have $K_i^g \cap K_j^g = \{1\}$, as needed.

(2) The edge $e_1$ has initial vertex $v$ and the latter is the terminal vertex of $e_2$. Then $\tilde{G}_{e_1} = K_i^g$ and $\tilde{G}_{e_2} = K_j^g$ for $g, g' \in G_0$ so they intersect trivially by the definition of $G_0$ and Theorem 11. Therefore, the claim holds.

Now choose a torsion element $t \in \tilde{G}$ and $f \in \tilde{F}$ with $t^f = t$. Let $e \in E(S)$ be an edge stabilized by $t$. Then $fe$ is also stabilized by $t$ and, by Lemma 9, in order to deduce the contradiction $i = j$ and $gg_0 \in K_i g' g_0$. Therefore, we have $K_i^g \cap K_j^g = \{1\}$, as needed.

Now choose a torsion element $t \in \tilde{G}$ and $f \in \tilde{F}$ with $t^f = t$. Let $e \in E(S)$ be an edge stabilized by $t$. Then $fe$ is also stabilized by $t$ and, by Lemma 9, in order to deduce the contradiction $i = j$ and $gg_0 \in K_i g' g_0$. Therefore, we have $K_i^g \cap K_j^g = \{1\}$, as needed.

4. Proof of the main result

**Proposition 14.** Let $G$ be a semidirect product of a free pro-$p$ group $F$ of finite rank with a $p$-group $K$ such that every finite subgroup is conjugate to a subgroup of $K$. Suppose that $C_F(t) = \{1\}$ holds for every torsion element $t \in G$. Then $G = K \sqcup F_0$ for a free pro-$p$ factor $F_0$.

**Proof.** Suppose that the proposition is false. Then there is a counter-example with $K$ having minimal order. When $K \cong C_p$, by Lemma 6(ii) $G = \left( \prod_{i \in I} C_i \right) \sqcup H$ with $I$ a finite set, all $C_i$ of order $p$ and $H$ free pro-$p$. By the assumptions and Theorem 11, there is a single conjugacy class of finite subgroups; that is, $|I| = 1$ so that $G$ would not be a counter-example. Therefore, $K$ is of order at least $p^2$.

Let $H$ be any maximal subgroup of $K$. Then $F \rtimes H$ satisfies the premises of the proposition, and hence $F \rtimes H$ is the form $H \sqcup F_1$ for some free factor $F_1$. Let bars denote images modulo $(H)_G$. As $(H)_G = (\text{Tor}(FH))$, by Proposition 10(ii), $F$ is free pro-$p$. Lemma 6(i) shows that $\tilde{G} \cong \prod_{i \in I} (C_i \times \text{C}_{\tilde{F}}(C_i)) \sqcup F_0$ with $I$ finite and $F_0$ a free factor of $F$. Now by Proposition 10(i) $\text{Tor}(\tilde{G}) = \text{Tor}(\tilde{G})$, and therefore every torsion element in $\tilde{G}$ can be lifted to a conjugate of an element in $K$. Hence $I$ consists of a single element so that

$$\tilde{G} = (\tilde{K} \times \text{C}_{\tilde{F}}(\tilde{K})) \sqcup F_0.$$  \hspace{1cm} (4.1)

In what follows, we shall use Lemma 6(ii) a couple of times. Consider $M := F/F'$ as a $K$-module and let $J$ denote the augmentation ideal of $\mathbb{Z}_p[H]$. Since $F \rtimes H = H \sqcup F_1 = \left( \prod_{h \in H} F_1^h \right) \rtimes H$, it follows that $H$ acts by permuting the free factors $F_1^h$ so that $M$ is a free $H$-module. Passing in equation (4.1) to the quotient modulo the commutator subgroup of $\tilde{F} = (C_{\tilde{F}}(\tilde{K}), F_0)\tilde{G}$, and using Lemma 6, one can see that $M/JM$ is a $\tilde{K}$-permutation lattice. Then an application of Theorem 2 together with Remark 5 shows that $M$ itself is a $K$-permutation lattice.

Let us show that $M$ is a free $K$-module. Suppose that this is not the case. Then there exist a non-trivial subgroup $S \leq K$ and a direct summand $M_0 = \mathbb{Z}_p[K/S]$ of $M$ as a $K$-module. Such a direct summand would be a direct summand of the free $H$-module $M$; we can conclude that $H \cap S = \{1\}$ so that $S$ is of order $p$. Let us show that $G_1 := F \times S$ satisfies the premises of the proposition. Certainly, $C_{G_1}(t) = \{1\}$ for every torsion element $t \in G_1$. Choose $x \in \text{Tor}(G_1)$. There are a $k \in K$ and an $f \in F$ with $x = kf$. Since $k \in (FS)\cap K$, we deduce that $k \in S$. Thus, there is a single conjugacy class of finite subgroups in $G_1$. But then, considering the natural homomorphism from $F \times S$ to $M \times S$ and observing the minimality assumption on
Proof of Theorem 1. Lemma 8 shows that \( G \) can have only a finite number of conjugacy classes of maximal finite subgroups. Therefore, one can form \( \tilde{G} \) as described before Lemma 13, in order to embed \( G \), such that \( \tilde{G} \) is both finitely generated and has finite centralizers of its finite subgroups and, moreover, has a single conjugacy class of maximal finite subgroups. By Proposition 14, the group \( \tilde{G} \) is of the form \( \tilde{G} = K \amalg F_0 \), where \( K \) and \( F_0 \) are free \( \mathbb{Z}_p \)-modules of the same rank \( r \), and hence \( \tilde{G} \) is an isomorphism. Therefore \( \ker \phi \) is contained in \([F, F] \). In particular, since the group is finitely generated, one has \( F \cong F \), since both groups are free \( \mathbb{Z}_p \). Since \( K \cap \ker \phi = \{1\} \), we conclude that \( \phi \) is an isomorphism, as claimed.

5. An example

We give an example of a virtually free profinite group that satisfies the centralizer condition of the main theorem but does not satisfy its conclusion. Note that the same example is valid for abstract groups.

Lemma 15. Let \( A \cong B = S_3 \) be the symmetric group on a 3-element set and let \( C := C_2 \). Form the amalgamated free profinite product \( G = A \amalg_C B \), where \( C \) identifies with the given 2-Sylow subgroups in \( A \) and \( B \), respectively. Then for every torsion element \( t \in G \) its centralizer is finite. However, \( G \) cannot be decomposed as a free profinite product with some finite factor.

Proof. It is easy to see that \( G \) can be presented in the form \( G = N \rtimes C_2 \), with \( N \cong C_3 \amalg C_3 \) and \( C_2 = \langle \alpha \rangle \) acting by inverting the generators of the two factors. Then the structure of \( N \), in light of Theorem 11, shows that no element of order 3 can have an infinite centralizer. For establishing the first statement of the lemma, it will suffice to show that all involutions in \( G \) are conjugate and that \( \alpha \) acts without fixed points upon \( N = \langle a, b \rangle \), where \( a, b \) are generators of cyclic free factors of order 3. As \( G \) is the fundamental group of the graph of groups

\[
\begin{array}{ccc}
A & \overset{C}{\longrightarrow} & B
\end{array}
\]

[12, Theorem 5.6, p. 938] shows that every involution is conjugate to an involution in one of the vertex groups \( A \) or \( B \). As \( A \) and \( B \) both have a single conjugacy class of involutions and the latter contains \( C \), the first observation holds. Since, by [7, Theorem 9.1.6], \( N' \) is freely generated by the commutators \([a^i, b^j] \) with \( i, j \in \{1, 2\} \), one can see that \( \alpha \) permutes them without fixed points, so that \( N' \rtimes \langle \alpha \rangle \) is isomorphic to \( F(x, y) \amalg C_2 \) with \( F(x, y) \) a free profinite group. Thus \( \alpha \) has no fixed points in \( N' \) and, since \( n \in N \) implies that \( n^3 \in N' \), none in \( N \).
Suppose that $G = L \amalg K$ with $L$ finite. Then, by Theorem 11, without loss of generality we can assume that $A \leq L$. The theorem just cited [12, p. 938] shows that $A$ is a maximal finite subgroup of $G$ so that $A = L$. Since the quotient modulo the normal closure of $L$ in $G$ is isomorphic to $K$ on the one hand and trivial by construction, we find $K = \{1\}$, which is a contradiction. Therefore, $G$ has no finite free factor.

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