A Re-examination of the isometric embedding approach to General

Relativity

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Abstract

We consider gravitational field equations which are Einstein equations written in terms of embedding coordinates in some higher dimensional Minkowski space. Our main focus is to address some tricky issues relating to the Cauchy problem and possible non-equivalence with the intrinsic Einstein theory. The well known theory introduced by Regge and Teitelboim in 9+1 dimensions is cast in Cauchy-Kowalevskaya form and therefore local existence and uniqueness results follow for analytic initial data.

In seeking a weakening of the regularity conditions for initial data, we are led naturally to propose a 13+1 dimensional theory. By imposing an appropriate conserved initial value constraint we are able, in the neighbourhood of a generic (free) embedding, to obtain a system of nonlinear hyperbolic differential equations. The questions of long time or global existence and uniqueness are formidable, but we offer arguments to suggest that the situation is not hopeless if the theory is modified in an appropriate way. We also present a modification of the perturbation method of Günther to weighted Sobolev spaces, appropriate to noncompact initial data surfaces with asymptotic fall-off conditions.

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I. INTRODUCTION

The isometric embedding of a riemannian or pseudo-riemannian manifold into an ambient flat space is a mathematical subject much explored over the last century. It also crops up occasionally in various guises in the context of gravitational physics. Manifolds were originally conceived of as embedded surfaces but after seminal work of the nineteenth century geometers the intrinsic viewpoint emerged. Full equivalence between the two viewpoints was established by Whitney in the case of abstract differentiable manifolds and by Nash \cite{1} for manifolds with a Riemannian metric. The latter equivalence has been extended to pseudo-riemannian manifolds by various people \cite{2,3,4}. Therefore, in principle, any physics which can be formulated in terms of curved manifolds, can alternatively be reformulated in terms of curved submanifolds of flat space(-time) and vice versa.

In Ref. \cite{6} Regge and Teitelboim considered the Hamiltonian formulation of General Relativity (GR), observing a similarity between the constraint algebra of GR and that of the bosonic string. (In strings, the auxiliary flat space-time structure plays a key role in determining a clever choice of non-local operators.) This motivated a study of embedded GR as a possible route to canonical quantisation via the Dirac procedure. The constraint algebra has been subsequently worked out in detail \cite{7,8} but the question of quantisation remains unclear. Similarly in Ref. \cite{9} a 10-vector formulation was proposed, effectively introducing 10 vielbeins to describe four dimensional space-time, leading to a system very similar to the embedded one. This was motivated by analogy with the Weinberg-Salaam theory of massive vector field, which can be cured of non-renormalisability by the Higgs mechanism. Generally, we may say that the currently well understood approaches to quantisation make some use of the properties of Minkowski space and particularly the Lorentz group. As such one hopes that by writing gravity as some kind of field theory living in Minkowski space, the quantum theory may be made more tractable. Some works along these lines are Refs. \cite{10,11}. The possibility to relate the Lorentz symmetry of the embedding space with the internal symmetries of particle physics was explored in refs \cite{12}.

Perhaps a less ambitious hope is that the embedding point of view may provide new methods to prove results in mathematical relativity. It may also be that an embedded theory can shed light on certain physical notions which do not sit well with the intrinsic tensorial formulation of gravity, such as a local notion of gravitational energy or the nonlinear completion of massive gravity.
In this article we will not venture into such possible applications, but will discuss some basic problems of existence and uniqueness which potentially plague any embedded theory of gravity. These kind of problems were known from the beginning, see e.g. Refs [6], [13]. By reinvestigating the mathematical foundations of the theory in some detail we are able to clarify to some extent the situation. The point of view adopted here has some similarities with that of Ref. [14] in that we emphasise the so called free embedding as the central idea. Our concern here is with the physical theory, but we find it worthwhile to get a little into the nuts and bolts of the various embedding theorems along the way. Since this is rather a large and complicated subject, and quite removed from standard methods in Relativity, we shall aim to consider in detail only those ideas which we will need, and to introduce them in a pedagogical way. For more detailed reviews see Refs. [4, 15–17].

A. Einstein-Hilbert action and embedding variables

We shall consider an $n$-dimensional Lorentzian space-time $(\mathcal{M}^n, g_{\mu\nu})$ and embeddings of (subsets of) $\mathcal{M}$ into Minkowski space $\mathbb{E}^{N-1,1}$. If at some point $p \in \mathcal{M}$ there exists a neighbourhood $U \subset \mathcal{M}$ of $p$ with local coordinates $(\xi^\mu)$ and a map $X: U \to \mathbb{E}^{N-1,1}$ such that

$$\xi^\mu \to X^A(\xi), \quad \eta_{AB} \frac{\partial X^A}{\partial \xi^\mu} \frac{\partial X^B}{\partial \xi^\nu} = g_{\mu\nu},$$

and $X(U)$ is a submanifold, then we say that $\mathcal{M}$ admits a local isometric embedding $X$ at $p$. Naturally the case of most interest in physics is $n = 4$. The choice of $N$ is more a matter of circumspection. For a given metric, we will have 10 equations, 4 of which may be considered redundant due to coordinate transformations. Since transformations $X \to X'$ which map the sub-manifold $X(U)$ to itself are also redundant, giving also 4 redundant degrees of freedom, we might expect $N \geq 10$ is necessary to avoid fatally over-determining the equations. The same counting of degrees of freedom applies if, instead of embedding a fixed metric, we are solving Einstein’s equations for $X$. More careful arguments for choice of $N$ can be made and will be given in what follows.

Following Ref. [6] we can attempt to describe gravity with $X(\xi)$ as the fundamental variable.
instead of a fundamental metric field $g$. Beginning with the Einstein-Hilbert action

$$I[X, \phi_{\text{matter}}] = \int \sqrt{-g} R d^4 \xi + \int \mathcal{L}_{\text{matter}},$$

and assuming that the matter Lagrangian depends on $X$ only implicitly through $g_{\mu \nu}$, the Euler Lagrange equation $\delta I / \delta X^A = 0$ gives

$$X^A_{;\mu \nu} (G^{\mu \nu} - T^{\mu \nu}) = 0.$$  

There are three issues preventing us from immediately concluding that this is equivalent to the metric formulation of General Relativity:

**Issue i)** If the second fundamental form $X^A_{;\mu \nu}$ is not invertible in the appropriate sense, there are more solutions besides $G^{\mu \nu} = T^{\mu \nu}$.

Even if we suppose $G^{\mu \nu} = T^{\mu \nu}$ holds:

**Issue ii)** Degrees of freedom may be missing, due to the impossibility of embedding certain spacetimes.

**Issue iii)** There may be extra degrees of freedom due to the possibility of isometric bending.

Due to well known theorems, issue ii) can be resolved by choosing $N$ large enough. For local embeddings $N = 10$ is sufficient for analytic metrics or $N = 14$ for smooth metrics.

Issue iii) is really two issues: firstly there may be extra physical degrees of freedom, which is not necessarily undesirable; secondly there may be nonphysical degrees of freedom with ill defined evolution. To illustrate what can go wrong consider the following: let $N = 5$ and choose for initial data the surface $X^0 = X^5 = 0$, and a time-like normal vector field along which we are to develop our space-time, which we shall take to be $\partial / \partial X^0$. Thus we specify the intrinsic metric to be flat and extrinsic curvature to be zero. Let $T^{\mu \nu} = 0$. From the point of view of intrinsic geometry, the unique maximal development of this data is Minkowski space $\mathbb{E}^{3,1}$. From the embedding point of view however there is not uniqueness. One possible development is the hypersurface $X^5 = 0$, but any hypersurface $X^5 = f(X^0)$ satisfying $f'(0) = 0$, and $|f'| \leq C$ with $C < 1$ will also be a solution. Any complete surface of this form is globally isometric to $\mathbb{E}^{3,1}$.

One might suspect from this example that nonuniqueness is generic and that it becomes worse whenever it is necessary to make the distinction clear, we will use $g$ to denote an induced metric, which is implicitly a function of $\partial X$ via (1) and $g$ to denote a fundamental metric field. Likewise we distinguish between curvatures $R_{\mu \nu \kappa \lambda}$ and $\mathcal{R}_{\mu \nu \kappa \lambda}$ and Einstein tensors $G_{\mu \nu}$ and $\mathcal{G}_{\mu \nu}$.

Suppose $X_t$ is a one-parameter family of small deformations of $X$ such that $X_0 = X$ and $X_t(M) \neq X(M)$ for $t > 0$ and the induced metrics are $g_t$ and $g$ respectively. If $(\mathcal{M}, g)$ is isometric to $(\mathcal{M}, g_t)$ for some interval $[0, a]$ of $t$ we shall say that $X_t$ is an isometric bending of $X$.  

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in higher dimensions. But the reality turns out to be quite the contrary. The nonuniqueness here is in fact a consequence of the special way in which the initial surface is embedded, which is unavoidable in low dimensions but becomes non-generic in higher dimensions. If we choose \( N = 10 \) then we find that a generic choice of (analytic) embedding and normal vector field constitutes initial data for a well defined local evolution problem. This shall be shown in detail in section II. For higher \( N \), nonuniqueness will reemerge, but there is a possibility to eliminate it for \( N = 14 \) with a judiciously imposed constraint, which will be discussed in section IV.

To guarantee the existence of global embeddings in principle requires larger \( N \), in which case problems of non-uniqueness may become formidable. We pick up on this matter briefly in section VC.

The type of embedding mentioned above is basically what is called a free embedding, defined in section I C. If \( N = 10 \) the space-like sections will generically be freely embedded. It is well known that, if we are in an open set of the function space where this property holds, this partially resolves issue i), subject to imposing the constraints \( G_{0\mu} = T_{0\mu} \) by hand. If \( N \geq 14 \) the space-time is generically freely embedded and issue i) is resolved if we assume we are in a neigbourhood of such a generic solution. However there is a subtlety due to the fact that genericity in the sense of space-time free condition is not a condition purely on initial data, since it depends on second time derivatives of the embedding [18]. We return to this point in section IV where we discuss the 14 dimensional theory, which is of necessity a background dependent theory.

B. Intrinsic and extrinsic geometry

Notation: For brevity, we shall use a vector space notation both for position vectors in \( \mathbb{R}^{N-1,1} \) and tangent vectors, without distinction. Let \( p \) be a point in \( \mathcal{M} \) with coordinates \( (\xi^\mu) \). Its image, \( X(p) \in \mathcal{E} \), is represented by a position vector \( X = (X^A) \). The tangent space \( T_{X(p)}\mathcal{E} \) is decomposed into components tangential and orthogonal to \( X(\mathcal{M}) \): \( T_X(\mathcal{E}) = T^\parallel_X \oplus T^\perp_X \). The vectors \( X_{,\mu} \) form a basis of \( T^\parallel_X \). The dot product \( V \cdot W \) denotes \( \eta_{AB} V^A W^B \).

Basic geometrical quantities are the metric \( g_{\mu\nu} = X_{,\mu} \cdot X_{,\nu} \), Christoffel symbol (of the first kind) \( \Gamma_{\mu\rho\sigma} = X_{,\mu} \cdot X_{,\rho\sigma} \) and second fundamental form \( X_{,\mu\nu} \). Here \( ; \) denotes covariant differentiation with respect to \( g \). The Riemann curvature tensor is obtained from the Gauss formula:

\[
R_{\mu\nu\rho\sigma} = X_{,\mu\rho} \cdot X_{,\nu\sigma} - X_{,\mu\sigma} \cdot X_{,\nu\rho} - X_{,\mu\sigma} \cdot X_{,\nu\rho} + X_{,\mu\rho} \cdot X_{,\nu\sigma}
\]  

\( (4) \)
It is important to keep in mind that the second fundamental form is normal-vector valued:

\[ \mathbf{X}_{\mu
u} \cdot \mathbf{X}_{\sigma} = 0. \]

In fact \( \mathbf{X}_{\mu
u} \) is the projection of \( \mathbf{X}_{\mu
u} \) onto \( T_{\mathbf{X}}^\perp \).

C. Free embeddings

Let \( M \) be of dimension \( n \) and \( \mathbb{E} \) be some flat space of dimension \( N \) and unspecified signature. Consider a given embedding \( \hat{\mathbf{X}} : \mathcal{M} \rightarrow \mathbb{E} \) and some perturbed embedding \( \mathbf{X} \) such that \( \mathbf{X} = \hat{\mathbf{X}} + \mathbf{U} \). The induced metric will be perturbed \( g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu} \), with

\[
    h_{\mu\nu} = 2 \hat{\mathbf{X}}_{(\mu} \cdot \mathbf{U}_{\nu)} + \mathbf{U}_{\mu} \cdot \mathbf{U}_{\nu} \\
    = 2(\hat{\mathbf{X}}_{(\mu} \cdot \mathbf{U})_{|\nu}) - 2 \hat{\mathbf{X}}_{[\mu} \cdot \mathbf{U} + \mathbf{U}_{\mu} \cdot \mathbf{U}_{\nu} \tag{5}
\]

where here "|" denotes the covariant derivative using some arbitrary background, not necessarily the physical background \( \hat{g} \). In order to motivate the free embedding, we neglect for now the nonlinear part, considering the following system for \( \mathbf{U} \):

\[
    \hat{\mathbf{X}}_{\mu} \cdot \mathbf{U} = 0, \\
    \hat{\mathbf{X}}_{\mu\nu} \cdot \mathbf{U} = -\frac{1}{2} h_{\mu\nu}.
\]

At the linearised level the tangential components of \( \mathbf{U} \) generate automorphisms of \( \mathbf{X}(\mathcal{M}) \) and therefore the first condition can be regarded as a gauge fixing. Representing \((\hat{\mathbf{X}}^A_{\mu}, \hat{\mathbf{X}}^A_{\mu\nu})\) by \( \hat{\mathbf{X}}^A_{\Lambda} \) with index \( \Lambda = 1, \ldots, \frac{1}{2}n(n+3) \), this can be written in matrix form:

\[
    U^A \eta_{AB} \hat{X}^B_{\Lambda} = H_{\Lambda}, \quad H_{\Lambda} := (0, -h_{\mu\nu}/2)
\]

For \( N \geq \frac{1}{2}n(n+3) \) we find that the linear system can be solved for \( \mathbf{U} \) in terms of \( h_{\mu\nu} \) provided that \( \det (\hat{\mathbf{X}}_{\Omega} \cdot \hat{\mathbf{X}}_{\Lambda}) \neq 0 \). If furthermore \( N = \frac{1}{2}n(n+3) \) then the solution is unique: \( \hat{\mathbf{X}}^A_B := (\hat{\mathbf{X}}^A_{\mu}, \hat{\mathbf{X}}^B_{[\mu\nu]}) \) is an invertible square matrix and the solution is \( U^A = (\hat{\mathbf{X}}^{-1})^A_B H^B \).

**Definition I.1.** An embedding \( \mathbf{X} : \mathcal{U} \rightarrow \mathbb{E} \) is said to be free at \( p \) if \( \det (\mathbf{X}_{\Omega} \cdot \mathbf{X}_{\Lambda}) \neq 0 \). If the condition holds for all \( p \in \mathcal{U} \) then \( \mathbf{X} \) is a free embedding.

For brevity, a free isometric embedding will be denoted FIE.
**Definition I.2.** For \( p \in U \) and an embedding \( X : U \to \mathbb{E} \), the vector subspace \( \mathcal{O}_{X(p)} := \text{span}(X_\mu, X_{\rho\sigma}) \in T_{X(p)}\mathbb{E} \) is called the osculating space.

**Remark I.3.** An equivalent definition of free embedding is that: i) \( \mathcal{O}_{X(p)} \) is not a null surface of \( T_{X(p)}\mathbb{E} \) and; ii) the \((X_\mu, X_{\rho\sigma})\) are a set of linearly independent vectors, i.e. \( \dim \mathcal{O}_{X(p)} = n(n+3)/2 \). For an embedding of a Lorentzian manifold into \( \mathbb{E}^{N-1,1} \), condition i) is automatic. But in the next section we will consider embedding of a Riemannian initial value surface into \( \mathbb{E}^{N-1,1} \), where condition i) is nontrivial.

**Remark I.4.** A free embedding requires \( N \geq n(n+3)/2 \) dimensions. If \( X \) is free and \( N = n(n+3)/2 \), then \((X_\mu, X_{\rho\sigma})\) form a basis of \( T_{X(p)}\mathbb{E} \).

We can now see why free embeddings are relevant to issue ii) of section A. In 13+1 dimensions, the perturbations \( U \) about a free embedding are in 1-to-1 correspondence with \( h_{\mu\nu} \) at the linearised level. This is also true for finite perturbations if they are suitably small. Therefore there is the possibility to establish an equivalence between the degrees of freedom by gauge fixing the isometric bending in the same way one gauge fixes \( h \) in the intrinsic theory. Although it is not completely satisfactory to restrict the function space to free embeddings, they are generic in a certain sense. Point-wise, a set of \( n(n+3)/2 \) vectors in \( N \geq n(n+3)/2 \) dimensions will generically be linearly independent. This remains true, in the sense of open sets on the function space, for vector fields if they are continuous and bounded. We may therefore adopt a background field approach, fixing a background embedding and studying close-by solutions. The existence of an appropriate background is guaranteed by the theorem:

**Theorem I.5** (Greene). Any Lorentzian manifold \((\mathcal{M}_4, g)\) with \( g \in C^m, \ m \geq 3 \) or \( C^\infty \) admits local free isometric embedding in \( \mathbb{E}^{13,1} \).

The following weaker definition is also of some importance:

**Definition I.6.** Let \( U \) be Lorentzian and \( X : U \to \mathbb{E}^{N-1,1} \) be an embedding. If there exists a foliation of \( U \) by space-like surfaces with adapted frame \((e_a, e_0)\) of \( TU \) such that the vectors \((X_a, X_0, X_{ab})\) are linearly independent and span a non-null vector subspace of \( T\mathbb{E} \), then we say that \( X \) is spatially free.

A spatially free embedding requires \( N \geq n(n+1)/2 \). The terminology is our own, but the idea is not new. This condition, or rather the analogous one for Riemannian geometry
is important in establishing the Janet-Cartan-Burstin theorem. In the context of the Regge-Teitelboim theory, it has been used widely in the literature and its importance explicitly emphasised in Ref. [14].

The Lorentzian version of the Janet-Cartan-Burstin theorem is the following:

**Theorem 1.7** (Friedman [20]). Any Lorentzian manifold $(\mathcal{M}_4, g)$ with $g$ analytic, admits an analytic local isometric embedding in $E^{9,1}$.

Theorems I.5 and I.7 are special cases ($n = 4$) of the original theorems which hold for any $n$ and for more general pseudo-riemannian signature. For smooth metrics, Theorem I.5 is the best we can do. It is not known whether the dimension can be reduced by dropping the requirement that the embedding be free. In particular, we wish to stress that a smooth version of Theorem I.7 is not available.

### II. ANALYTIC EVOLUTION PROBLEM FOR THE 9+1 DIMENSIONAL THEORY

The problem of embedding a given space-time into $E^{9,1}$ can be formulated as a Cauchy-Kowalewskaya system. Likewise for the evolution of the metric under Einstein’s equations. Therefore it stands to reason that Einstein’s equations expressed in embedding variables are also of that form. However, we find it worthwhile to briefly go through the details, not least because it brings out the role of what we call admissible embeddings of the initial data surface. Basically, for a given choice of extrinsic curvature $K_{ab}$, not every free embedding of the initial data surface allows us to extend the embedding into a space-time neighbourhood. The condition of admissibility is a Lorentzian analogue of the one discussed in Ref. [19] but the signature change makes for a quite different conclusion- certain embeddings, i.e. those for which the osculating space is space-like, are admissible for any choice of $K_{ab}$.

The theory introduced in Ref. [6] and developed extensively in the literature describes (local) time evolution of a 3-membrane embedded in $E^{9,1}$. The Euler Lagrange variation of the Einstein-Hilbert action gives:

$$X_{;ab}(G^{ab} - T^{ab}) + 2X_{;0a}(G^{0a} - T^{0a}) + X_{;00}(G^{00} - T^{00}) = 0.$$  \hspace{2cm} (6)

Since we have only 10 embedding dimensions the $X_{;\mu\nu}$ cannot be all linearly independent, so as things stand we may not deduce the Einstein equations from the action principle. But we may proceed as follows:
Let $\Sigma$ be an analytic 3-manifold and $Y : \Sigma \to \mathbb{E}^{9,1}$ be an analytic free embedding which induces a riemannian metric $\gamma$. Let $p$ be a point in $\Sigma$. Since $Y$ is free at $p$, we can take as basis for the tangent space $T_{Y(p)}\Sigma$ the set of vectors $\{Y_a, Y_{ab}, E_\perp\}$, where $E_\perp$ is orthogonal to $O_Y$. By remark 3 it follows that $E_\perp$ is not null and the $Y_{ab}$ are linearly independent, which implies in turn that $P_{abcd} := Y_{[ab} \cdot Y_{cd]}$ possesses a unique inverse $P^{abcd}$ such that $P_{abcd} P^{cdef} = \frac{1}{2} (\delta^a_c \delta^b_f + \delta^a_f \delta^b_c)$.

We assume that $\Sigma$ is a submanifold of an analytic 4-manifold $M$. For the metric theory, one gives as initial data $(\Sigma, \gamma_{ab}, K_{ab})$, subject to the constraints $G_{\mu 0} = T_{\mu 0}$. For the Regge-Teitelboim theory on the other hand, we will need to specify $Y$ and some unit time-like vector field $V$ over $\Sigma$, and solve the equations to find an embedding $X : M \to \mathbb{E}^{9,1}$ satisfying $X|_\Sigma = Y$ and $X_{;0}|_\Sigma = V$. Suppose we are given $Y$ and $V$. Then

$$K_{ab} = V \cdot Y_{|a}$$

uniquely determines $K$. However, if we wish to prescribe $\gamma$ and $K$, and are given some isometric embedding $Y$ of $(\Sigma, \gamma)$, we need to face the issue of whether or not an appropriate vector field $V$ exists.

**Definition II.1.** Given analytic $(\Sigma, \gamma, K)$, a free analytic embedding $Y$ is admissible if $\gamma_{ab} := Y_{|a} \cdot Y_{|b}$ and $\exists$ an analytic vector field $V \in T\Sigma$ which satisfies at each point on $\Sigma$:

i) $V \cdot Y_{|a} = 0$;

ii) $V \cdot Y_{|ab} = K_{ab}$;

iii) $V \cdot V = -1$;

iv) $V$ is linearly independent of $O_Y$.

Note that if such a $V$ exists, it is unique. Conditions i) and ii) require that $V$ is decomposed as: $V = P^{abcd} K_{ab} Y_{|cd} + V_\perp E_\perp$. Condition iv) requires that $V_\perp$ be nonvanishing. Its value will be determined by solving condition iii), i.e. $-1 = P^{abcd} K_{ab} K_{cd} + E_\perp \cdot E_\perp (V_\perp)^2$ for $V_\perp$ if a solution exists. If the osculating space is space-like (i.e. $E_\perp \cdot E_\perp = -1$) then there is always a nonvanishing solution. If the osculating space is space-timelike, a solution exists if $P^{abcd} K_{ab} K_{cd} \leq -1$ on $\Sigma$ (in terms of a local condition: if at a point $p \in \Sigma$ we have the strict inequality $P^{abcd} K_{ab} K_{cd} < -1$ then we can find an analytic vector field $V_\perp$ so that the embedding is admissible in some open neighbourhood of $p$.).

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3 In this section ; and $|$ denotes the 4 dimensional covariant derivative w.r.t $g$ and 3-dimensional covariant derivative w.r.t. $\gamma$ respectively.
**Remark II.2.** If \((\Sigma, \gamma) \to \mathbb{E}^9\) is free, then the embedding \(Y\) induced by identifying \(\mathbb{E}^9\) with a constant time surface of \(\mathbb{E}^9\) will also be free, with space-like osculating space. Therefore \(Y\) will be admissible for any \(K_{ab}\). Since \((\Sigma, \gamma)\) always admits local analytic FIE into \(\mathbb{E}^9\) \([20]\), a local admissible embedding always exists.

Returning to (6), and working in an adapted frame such that \(e_0\) is orthogonal to \(\Sigma\), we impose

\[
(G_{\mu 0} - T_{\mu 0} )|_\Sigma = 0
\]

by hand (but see \([7]\) where a modified action principle implementing this was proposed). Then (6) reduces to

\[
G_{\mu \nu}(\partial X, \partial \partial X) = T_{\mu \nu}.
\]  

Equation (8) can be put into (second order) Cauchy-Kowalevskaya form by further specialising to a Gaussian normal coordinate basis. The embedding equations \(g = g\) become \(X_{a} \cdot X_{b} = g_{ab}\), \(X_{a} \cdot X_{0} = g_{a0} = 0\) and \(X_{0} \cdot X_{0} = g_{00} = -1\). Following a standard procedure \([17, 19]\), we can differentiate to obtain the equivalent second order system:

\[
X_{,00} \cdot X_{,0} = 0, \quad X_{,00} \cdot X_{,a} = 0, \quad X_{,00} \cdot X_{,ab} = -\frac{1}{2} g_{ab,00} + X_{,0a} \cdot X_{,0b},
\]

with initial value constraints

\[
X_{,a} \cdot X_{,b} = g_{ab}, \quad X_{,0} \cdot X_{,0} = -1, \quad X_{,0} \cdot X_{,a} = 0, \quad X_{,0} \cdot X_{,ab} = -\frac{1}{2} g_{ab,0}.
\]

If \(X\) is spatially free then the matrix \(M^{A}_{B} := (X_{,0}^{A}, X_{,a}^{A}, X_{,(ab)}^{A})\) is invertible. If \(g\) were a given background field then the system can immediately be put in Cauchy-Kowalevskaya form. In order to deal with (8) we have a bit more work to do, but it is also a standard construction \([21]\). In Gaussian normal coordinates the Einstein equations take the form

\[
\frac{1}{2} g_{ab,00} = S_{ab} + \mathcal{F}_{ab},
\]

\[
\mathcal{R}_{0\mu} = S_{0\mu},
\]

where \(\mathcal{F}_{ab}\) and \(\mathcal{R}_{0\mu}\) are first order in time derivatives and \(S_{\mu \nu} - \frac{1}{2} g_{\mu \nu} S_{\sigma} = T_{\mu \nu}\). (One can consider a combined Einstein-matter system but for simplicity, the stress tensor will be regarded as a given background field.) The combined system

\[
X_{,00} \cdot X_{,0} = 0, \quad X_{,00} \cdot X_{,a} = 0, \quad X_{,00} \cdot X_{,ab} = -S_{ab} - F_{ab} + X_{,0a} \cdot X_{,0b},
\]

\[
\frac{1}{2} g_{ab,00} = S_{ab} + \mathcal{F}_{ab},
\]
along with initial value constraints (10) and (12), is equivalent to (8). We can write (13) as:

\[ X^A_{,\alpha} = (M^{-1})^A_B F^B \]  

(15)

Where \( F^B := (0, 0, -S_{ab} - F_{ab} + X_{,\alpha} \cdot X_{,\beta}) \) and \( M^{-1} \) is an analytic function of \( X_{,\alpha}, X_{,\beta}, X_{,\alpha} \).

**Theorem II.3.** Given analytic \((\Sigma, \gamma, K)\) satisfying the Hamiltonian and momentum constraints and an admissible embedding \( Y : \Sigma \to \mathbb{E}^{9,1} \) then \( \exists \) a neighbourhood \( \mathcal{N} \) of \( \Sigma \) in \( \mathcal{M} \), homeomorphic to \( \Sigma \times (-\tau, \tau) \), a nondegenerate Lorentzian metric \( g_{\mu\nu} \) on \( \mathcal{N} \) and an analytic isometric embedding \( X : (\mathcal{N}, g) \to \mathbb{E}^{9,1} \) such that: \( g_{\mu\nu} \) satisfies the Einstein equations \( G_{\mu\nu} = T_{\mu\nu} \) on \( \mathcal{N} \); \( X(\Sigma) = Y(\Sigma) \); the pullback of \( g \) onto \( \Sigma \) is \( \gamma \); the extrinsic curvature of \( \Sigma \) is equal to \( K \). The solution \( X \) is unique up to diffeomorphisms of \( \mathcal{N} \) which reduce to the identity on \( \Sigma \).

For any \( p \in \Sigma \) we can introduce a gaussian normal coordinate neighbourhood \( \mathcal{U} \) of \( p \) in \( \mathcal{M} \). Since \( Y \) is admissible, we can find a unique \( V \) such that (11) is satisfied for \( X(\Sigma) = Y(\Sigma) \) and \( X_{,\alpha}|_\Sigma = V \). The local result follows by applying the Cauchy-Kowalevskaya theorem to the above equation system. This implies local geometrical uniqueness up to diffeomorphisms which reduce to the identity on \( \mathcal{U} \cap \Sigma \). Therefore the result can be be stated in the above tensorial form, whereupon the global-in-space result follows in standard fashion by considering an appropriate coordinate atlas on \( \Sigma \) and patching together solutions. \( \square \)

Given \((\Sigma, \gamma, K)\), by remark II.2 we can always find an admissible embedding in the neighbourhood of any \( p \in \Sigma \). However, without some assumptions on the geometry and topology, a global admissible embedding may not exist. This motivates the search for global admissible embeddings \( M^3 \to \mathbb{E}^{9,1} \) where \( M^3 \) is a case of physical interest. This is left as an open problem.

The fact that we obtain uniqueness does not mean that there is no isometric bending. It means that there is no residual isometric bending left over once we have fixed the initial surface. The choice of \( Y \) itself is not unique - from naive counting of components we expect 4 degrees of isometric bending. The appearance of extra independent initial data whose evolution is well defined, suggests that we have more than the 2 physical degrees of freedom. We could simply declare these extra degrees of freedom to be pure gauge but it is not clear from our analysis why this should be done. On the other hand it is natural from the Hamiltonian point of view - isometric bending is an invariance of the action and so will not have a corresponding conjugate canonical momentum. This manifests itself in terms of additional constraints on the canonical variables. This approach is in fact shown in [8] to be consistent and reproduce the degrees of
freedom of GR. In that reference, four (linear combinations of) the constraints were isolated as
generators of isometric bending and were shown to form the ideal of the constraint algebra.

It does not seem to be possible to relax the analyticity assumption in the above results without
some other modifications to the theory. Even for \( C^\infty \) initial data, no local existence result is
available. This is why we are reliant on the Cauchy-Kowalevskaya theorem to establish existence
and uniqueness. This leads to what seems to be a fundamental theoretical weakness of the 9+1
dimensional theory. This problem is very similar to one raised in Ref. \[22\] regarding the use
of the Campbell-Magaard theorem in physical applications. Basically, the embedding theorem
of Friedman, combined with the well-posedness of the intrinsic Einstein equation system, is not
strong enough to deduce the well-posedness of the embedded Einstein system. This is because the
CK theorem does not guarantee continuous dependence on the data. Furthermore, the domain of
dependence property does not strictly speaking make sense for analytic data. From the physical
point of view one might be content with an approximate local domain of dependence result for
analytic fields, whereby two different sets of analytic data which agree closely in some small
enough region \( U \subset \Sigma \) will agree closely regarding the development \( X(D^+(U)) \). But in order
to be able to deduce this we would need continuous dependence. Therefore it is unclear how to
proceed with the 9+1 dimensional theory.

Let us consider two possible approaches to smooth data:

1) A natural way to relax the regularity and attempt to obtain continuous dependence is
to formulate the problem in terms of a (probably highly nonlinear) system of wave equations.
This will require an approach which is more democratic with respect to space and time which
ultimately requires the background to be free rather than merely spatially free. For this it is
necessary to go to 13+1 dimensions.

2) At first glance one might think that Theorem \[1.3\] can be strengthened to the smooth case
by approximating the initial data by analytic functions, and then applying the implicit function
theorem to obtain smooth solutions with the original data. The problem with this is that the
implicit function theorem can only be invoked if the spacetime is freely embedded. If we follow
the above construction which led to Theorem \[1.3\] in 13+1 dimensions, the analytic solution is
not unique, and it turns out it can always be modified to a free embedding as described in Ref.
Ref.\[19\].

The two approaches are slightly different but complementary. The second enables us to
immediately conclude local existence of solutions in 13+1 dimensions, but without giving any
uniqueness result. Indeed there is extra gauge freedom to be fixed, as can be seen more clearly in the first approach. We will focus on the first approach, which will be elaborated on in section IV. In any case, all roads seem to lead us to $N \geq 14$.

III. THE PERTURBATION PROBLEM FOR ISOMETRIC EMBEDDINGS OF INITIAL DATA SURFACES

This section is somewhat of an aside, and can be omitted by the reader wishing to proceed directly to the 13+1 dimensional theory. It is included here since some of the ideas introduced provide some context and inspiration for the methods used in the next section.

A question of considerable interest regarding isometric embeddings is the following: given $(\mathcal{U}, \hat{g})$ which admits a FIE into $\mathbb{R}^N$, do all nearby metrics also admit such an embedding? Well known proofs exist in which the concept of “nearby” is defined in terms of $C^n$ or Hölder norm. Günther[23] introduced a useful method to effectively reduce the problem to an elliptic one. This method is powerful because it can be used to solve compactly supported metric perturbations in terms of compactly supported perturbations of $X$. This in turn gave a reduction of the dimensions for Nash’s theorem[24]. Our interests are somewhat different so we will here give only a simplistic introduction which is sufficient for our purposes. There are two quite different reasons for our interest in this method, both of which arise when we abandon the notion of analytic gravitational field. Firstly, it becomes natural to formulate the field equations as a hyperbolic system. In so doing, the method of Günther provides a guide for how to proceed by analogy. Our second reason relates to the embedding of the initial data surface. We would like to be sure that all close-by data can be achieved by perturbing it, so as to account for all the degrees of freedom.

In Ref. [23] Holder spaces were used and by a trick the invertibility of the Laplacian minus a constant on compact manifolds or with trivial boundary conditions was exploited to obtain the perturbation result. However, for cases of physical interest, such as asymptotically flat manifolds, we are interested in special classes of non-compact manifolds with some fall-off conditions at infinity. So it would be desirable to have a perturbation result more directly applicable to these manifolds by working with weighted Sobolev spaces. As a preliminary step in this direction we consider Riemannian 3-manifolds with trivial topology with perturbations decaying appropriately at infinity.

The case of nontrivial topology should in principle be tractable, with the appropriate generalisations of the weighted sobolev space and (Lichnerowicz) laplacian.
A perturbation of an isometric embedding is described by (5). Note that the equation remains valid if \( \| \) is a covariant derivative with respect to any background metric, not necessarily the metric we wish to perturb. The most appropriate choice is the background metric \( e \) used to define the norm on the function space. Here we shall restrict ourselves to manifolds with topology \( \mathbb{R}^3 \) so we take \( e \) to be the standard cartesian metric. Hence:

\[
h_{ab} = 2(\mathbf{\hat{X}}_{(a} \cdot \mathbf{U})_{,b}) - 2\mathbf{\hat{X}}_{ab} \cdot \mathbf{U} + U_{,a} \cdot U_{,b},
\]  

(16)
is valid globally.

In section I C we saw that the perturbation result at the linearised level follows immediately upon setting \( \mathbf{\hat{X}}_{,\mu} \cdot \mathbf{U} = 0 \). Now keeping the nonlinear terms, one obtains

\[
\mathbf{\hat{X}}_{,a} \cdot \mathbf{U} = 0
\]
\[
\mathbf{\hat{X}}_{,ab} \cdot \mathbf{U} = -\frac{1}{2} h_{ab} + U_{,a} \cdot U_{,b}.
\]

The existence of solutions for small \( h \) and \( \mathbf{U} \) is not obvious, since the nonlinear terms contain derivatives. The result can be obtained by using a Nash-Moser type iteration scheme which introduces smoothing operators to restore differentiability at each step (see e.g. the appendix of ref [3]). The approach of Ref. [23] dispenses with such technicalities by introducing a more complicated gauge fixing of \( \mathbf{\hat{X}}_{,\mu} \cdot \mathbf{U} \) so as to obtain an elliptic system. We introduce the Laplacian \( \Delta = \delta^{ab} \partial_a \partial_b \) and consider instead the system

\[
\Delta(\mathbf{\hat{X}}_{,a} \cdot \mathbf{U}) = -\Delta \mathbf{U} \cdot U_{,a},
\]  

(17)
\[
\Delta(\mathbf{\hat{X}}_{,ab} \cdot \mathbf{U}) = -\frac{1}{2} \Delta h_{ab} + U_{,ac} \cdot U_{,b}^c - U_{,ab} \cdot \Delta U.
\]  

(18)

By applying \( \Delta \) to (16) and imposing (17) we obtain (18). Supposing that \( \Delta^{-1} \) is well defined we can write

\[
\mathbf{U} = M^{-1} \mathbf{H} + M^{-1} \Delta^{-1} Q(\mathbf{U}),
\]  

(19)

where

\[
M = (\hat{X}_{,a}^A, \hat{X}_{,ab}^A), \quad \mathbf{H} = \begin{pmatrix} 0 \\ -\frac{1}{2} h_{ab} \end{pmatrix},
\]

\[
Q(\mathbf{U}) = \begin{pmatrix} -\Delta \mathbf{U} \cdot U_{,a} \\ U_{,ac} \cdot U_{,b}^c - U_{,ab} \cdot \Delta U \end{pmatrix}.
\]
If \( N = 9 \) then \( M^{-1} \) is unique. If \( N > 9 \) we can choose the unique \( M^{-1} \) such that \( M^{-1}H \) lies within the osculating space \( O_X \). A solution of (19) will solve (16). The existence and uniqueness (modulo the choice of \( M^{-1} \)) of such a solution can be established by contraction mapping arguments.

Here we revisit the proof using weighted Sobolev spaces \( H_{p,\alpha} \). Notation and basic properties are in appendix A. We will assume that an explicit FIE, \( \hat{X} \), of \( (\mathbb{R}^3, \gamma) \) into \( \mathbb{E}^N \) is known which has reasonable asymptotic behaviour.\(^5\) Then we obtain:

**Theorem III.1.** Let \( \alpha, \beta > 0, q \geq p \geq 4 \). Given a FIE \( \hat{X} \) for \( (\mathbb{R}^3, \gamma) \) such that \( M^{-1} = K + L \) where \( K \) is a constant matrix and \( L \in H_{q,\beta} \), then there exists some constant \( C \) such that for any metric \( \gamma + h \) such that \( M^{-1}H \in H_{p,\alpha} \) and satisfies

\[
\|M^{-1}H\|_{p,\alpha} (|K| + \|L\|_{q,\beta}) \leq C,
\]  

there exists a FIE \( X \) for \( (\mathbb{R}^3, \gamma + h) \) such that \( X = \hat{X} + U \) with \( U \in H_{p,\alpha} \) satisfying

\[
\|U\|_{p,\alpha} \leq \sqrt{\|M^{-1}H\|_{p,\alpha} (|K| + \|L\|_{q,\beta})}.
\]  

With the above regularity assumptions the proof goes through much the same as given in e.g. \([17]\), and so we relegate it to appendix B. If we further assume \( h \in H_{p,\alpha} \) we obtain the following corollary, which admits a more intuitive interpretation.

**Corollary III.2.** Under the additional hypothesis \( h \in H_{p,q} \), Theorem III.1 holds with (20) and (21) replaced by

\[
\|h\|_{p,\alpha} \leq \frac{C}{(|K| + \|L\|_{q,\beta})^2}
\] 

and

\[
\|U\|_{p,\alpha} \leq D \sqrt{\|h\|_{p,\alpha}}
\] 

respectively, for some constants \( C \) and \( D \).

Roughly speaking, \( |K| + \|L\|_{q,\beta} \) is a measure of how close the vectors \( (\hat{X}_{,a}, \hat{X}_{,ab}) \) are to being linearly dependent. So from (22) we can say heuristically that the “more free” we make the  

---

\(^5\) Here reasonable means that \( |K| + \|L\|_{q,\beta} \) is finite. We include the constant term \( K \), since this seems to be important in finding examples, e.g. by some modification of the simple Veronese type embedding \( \xi \to X(\xi) = (\xi_1, \xi_2, \xi_3, (\xi_1)^2, (\xi_2)^2, (\xi_3)^2, \xi_1\xi_2, \xi_2\xi_3, \xi_3\xi_1) \).
background embedding, the larger the perturbations for which we are able to exactly solve the system.

It is important to note that we have only assumed some fall-off conditions for the metric perturbation \( h \), not for the background \( \gamma \). This includes as a special case all small perturbations within the class \( \gamma, h \in H_{p,\alpha}, p \geq 4, \alpha > 1/2 \) of asymptotically flat metrics on \( \mathbb{R}^3 \) with well defined ADM total energy. Such metrics are \( C^2 \) and have an asymptotic fall-off faster than \( r^{-\alpha} \).

**IV. PROPOSAL: 13+1 DIMENSIONAL THEORY**

Here we will argue that it is natural to generalise the Regge-Teitelboim theory to 13+1 dimensions, being the minimal dimension in which a local free embedding of space-time exists. If we consider the theory in the neighbourhood of some freely embedded space-time, there are two interesting features. The Euler-Lagrange equations (3) are equivalent to the Einstein equations in terms of the embedding coordinates. Secondly, with a bit of work, we can obtain a system of nonlinear wave equations for the perturbed embedding coordinates \( U \). The first feature is subject to the following caveat [18]: Given initial data such that \( X_{;\alpha}, X_{;0}, X_{;ab}, X_{;a0} \) are close to our background embedding, we can not deduce a priori from (3) that \( X_{;00} \) is close to the background, and therefore we can not guarantee that \( X \) is free. We may impose the Hamiltonian constraint \( G_{00} = T_{00} \) by hand, whereupon (3) do reduce to the Einstein equations. These are linear in \( X_{;00} \), but we can not solve algebraically for \( X_{;00} \) because they only determine the projection of \( X_{;00} \) onto \( \text{span}(X_{;\alpha}, X_{;0}, X_{;ab}, X_{;a0}) \) leaving one component undetermined. So the issue still remains that we have no a priori bound on \( X_{;00} \). In view of this, we will defer the question of an action principle for the future, and simply look for a set of equations which will: reproduce the Einstein equations in some gauge fixing; completely determine the evolution of \( X \); ensure that \( X \) remains close to some freely embedded background, at least for small initial data and for some finite time. Therefore we will necessarily have a background-dependent formulation. As a first step, we write the Einstein equations as \( R_{\mu\nu} = S_{\mu\nu} \) i.e.

\[
X_{;\mu\nu} \cdot X_{;\rho}^{\rho} - X_{;\mu\rho} \cdot X_{;\rho}^{\rho} = S_{\mu\nu}.
\]

(24)

This equation system does not belong to any of the well known types and little can be said about it in its current form.

We shall expand the space-time metric about a background, which we shall assume here to be flat, \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \). (If we take some other background the analysis is basically the same in
terms of obtaining a hyperbolic system but the equations are considerably more complicated.)

Let \( \hat{X} : (U, \eta) \to \mathbb{R}^{13,1} \) be a FIE in a neighbourhood \( U \) of \( p \in \mathcal{M} \). The osculating space spans the tangent space, with vectors \( \hat{X}_\mu \) spanning \( T_\parallel \) and vectors \( \hat{X}_{\mu\nu} \) spanning \( T_\perp \). A nearby solution \( X \) will be a FIE of \( (U, g) \) such that \( X = \hat{X} + U \). Let us make use of the notation:

\[
\| U_\mu \coloneqq \hat{X}_\mu \cdot U, \quad ^\parallel U_{\mu\nu} \coloneqq \hat{X}_{\mu\nu} \cdot U. \tag{25}
\]

With these definitions, \( U \) can be decomposed uniquely as:

\[
U = ^\parallel U_{\mu\nu} (P^{-1})^{\mu\nu\rho\sigma} \hat{X}_{\rho\sigma} + \| U_\mu g^{\mu\nu} \hat{X}_{\nu}, \tag{26}
\]

where \( P_{\mu\nu\rho\sigma} \coloneqq \hat{X}_{\mu\nu} \cdot \hat{X}_{\rho\sigma} \) and \( (P^{-1})^{\mu\nu\rho\sigma} \) is the unique inverse \( (P^{-1})^{\mu\nu\rho\sigma} P_{\rho\sigma\kappa\lambda} = \frac{1}{2} (\delta_{\kappa}^\mu \delta_{\lambda}^\sigma + \delta_{\lambda}^\mu \delta_{\kappa}^\sigma). \)

In the metric theory of GR, a standard method\[21\] to obtain a hyperbolic equation system is to consider the reduced Einstein equations: \((\Gamma_\mu := g^{\rho\sigma} \Gamma_{\mu\rho\sigma})\)

\[
R_{\mu\nu} - \Gamma_{(\mu,\nu)} = S_{\mu\nu} \tag{27}
\]

along with initial value constraints

\[
\Gamma_\mu |_{\Sigma} = 0 \quad G_{0\mu} |_{\Sigma} = 0. \tag{28}
\]

The constraints are conserved and therefore this system is equivalent to the Einstein equations under the de Donder gauge fixing of the coordinate freedom. The point being that \( R_{\mu\nu} = -\frac{1}{2} \Box g_{\mu\nu} + \Gamma_{(\mu,\nu)} + \cdots \), where \( \Box \coloneqq g^{\mu\nu} \partial_\mu \partial_\nu \) and the ellipsis means we are concerned only with the principal part. Therefore (27) is hyperbolic about any non-degenerate \( g \).

Simply following the above approach will not quite serve for the embedding theory. The problem can be seen from (5) - \( h_{\mu\nu} \) is first derivative in \( U \). This means third derivatives of \( U \) will show up in the reduced Einstein equations. In order to get around this, we propose to split the de Donder condition into two separate gauge fixing conditions, something which is possible precisely because we have enlarged the embedding space to 14 dimensions. We hope that what follows will convince the reader that this is also a natural and useful thing to do.

Forgetting for a moment the Einstein equations, if we wished to cast the embedding problem for a pre-prescribed metric \( g = \eta + h \) as a hyperbolic system we might perform a Lorentzian version of (17) and (18), introducing the gauge fixing \( \Box U_\mu = -U_{\mu} \cdot \Box U \) so as to obtain \( -\frac{1}{2} \Box h_{\mu\nu} = \Box ^\parallel U_{\mu\nu} + \cdots \). However, when we come to consider the Einstein equations we wish instead to
obtain $-\frac{1}{2} \square h_{\mu\nu} + \Gamma_{(\mu,\nu)} = \square^2 U_{\mu\nu} + \cdots$. It turns out that the relevant condition is: (this will be shown in detail below, see (34))

$$\square U_{\mu} + U_{\mu} \cdot \square U - \Gamma_{\mu} = g^{\rho\sigma} \left( \square U_{\rho\sigma,\mu} - 2 \square U_{\rho\mu,\sigma} \right) = 0 .$$

This is the appropriate modification of (17), the novelty being that this condition is first derivative in $U$. Define $\Psi_{\mu} := g^{\rho\sigma} \left( \square U_{\rho\sigma,\mu} - 2 \square U_{\rho\mu,\sigma} \right)$. The candidate Reduced Einstein equations for our system will be

$$R_{\mu\nu} - \Psi_{(\mu,\nu)} = S_{\mu\nu} ,$$

together with initial value constraints $\Psi_{\mu}|_\Sigma = 0$, $G_{0\mu}|_\Sigma = 0$. Now in order to get a hyperbolic system we need to give some wave equation $\square \|U_{\mu} = \cdots$. In the method of Gunther, the right hand side of (17) was fixed by the need to reduce (5) to second order elliptic form (18). In our case, the tangential perturbations may be given arbitrary dynamics and it is the isometric bending degrees of freedom contained in the normal perturbations which must be fixed in a specific way in order to obtain a hyperbolic system. This difference is a consequence of the geometrical nature of the Einstein equation. We shall see that our choice of isometric bending gauge condition, namely $\Psi_{\mu} = 0$, can be implemented by initial value constraint which is conserved by the evolution equations and that, for technical reasons, a preferred canonical choice of wave equation for $\|U$ is indeed $\Gamma_{\mu} - \Psi_{\mu} = 0$.

### A. The nonlinear theory

We now turn to consider (24) in more detail. We recall that $X = \dot{\hat{X}} + U$ and that both the covariant derivatives and the inverse metric used to raise indices depend on the physical metric. Therefore (24) is highly nonlinear in $U$.

Introduce the noncovariant derivative operator $\square := g^{\mu\nu} \partial_\mu \partial_\nu$. The Ricci tensor may be re-expressed as:

$$R_{\mu\nu} - \Gamma_{(\mu,\nu)} = -\frac{1}{2} \square h_{\mu\nu} + \Gamma_{\sigma\nu}^\rho \Gamma_{\rho \mu}^{\sigma} - \Gamma_{\rho}^\rho \Gamma_{\mu\nu}^{\rho} + 2 \Gamma_{(\mu}^{\rho\sigma} \Gamma_{\nu)\rho\sigma} .$$

Define:

$$\Psi_{\sigma\mu\nu} := -\frac{1}{2} U_{\sigma\mu,\nu} - \frac{1}{2} U_{\sigma\nu,\mu} + \frac{1}{2} U_{\mu\nu,\sigma} , \quad \Psi_{\sigma} := g^{\mu\nu} \Psi_{\sigma\mu\nu} .$$

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Then we have the following useful identities for the Christoffel symbols:

\[ \Gamma_{\sigma \mu \nu} = \parallel U_{\sigma, \mu \nu} + U_{\sigma} \cdot U_{\mu \nu} + \Psi_{\sigma \mu \nu}, \]

\[ \Gamma_{\sigma} = \Box \parallel U_{\sigma} + U_{\sigma} \cdot \Box U + \Psi_{\sigma}. \] (31) (32)

Applying \(-\frac{1}{2} \Box\) to (3) we have:

\[-\frac{1}{2} \Box h_{\mu \nu} = \Box U_{\mu \nu} + \Box U \cdot U_{\mu \nu} - U_{\mu \rho} \cdot U_{\nu \sigma} g^{\rho \sigma} - \Gamma_{(\mu, \nu)} + \Psi_{(\mu, \nu)} - 2 \Gamma_{(\mu}^{\rho} \Gamma_{\nu)_{\rho \sigma}} + 2 \Gamma_{(\mu}^{\rho} \Psi_{\nu)_{\rho \sigma}} \]

(33)

Combining (29) and (33) we obtain:

\[ R_{\mu \nu} - \Psi_{(\mu, \nu)} = \Box U_{\mu \nu} = \Box U \cdot U_{\mu \nu} - U_{\mu \rho} \cdot U_{\nu \sigma} g^{\rho \sigma} - 2 \Gamma_{(\mu}^{\rho} \Psi_{\nu)_{\rho \sigma}} + \Gamma_{\rho}^{\sigma} \Gamma_{\sigma \rho \mu} - \Gamma_{\rho}^{\rho \mu} . \] (34)

We introduce the equation system:

\[ R_{\mu \nu} - \Psi_{(\mu, \nu)} = S_{\mu \nu}, \]

\[ \Gamma_{\mu} - \Psi_{\mu} = 0, \] (35) (36)

which, explicitly reads:

\[ \Box U_{\mu \nu} = - \Box U \cdot U_{\mu \nu} + U_{\mu \rho} \cdot U_{\nu \sigma} g^{\rho \sigma} - 2 \Gamma_{(\mu}^{\rho} \Psi_{\nu)_{\rho \sigma}} + \Gamma_{\rho}^{\sigma} \Gamma_{\sigma \rho \mu} + S_{\mu \nu} \]

\[ \Box U_{\sigma} = - U_{\sigma} \cdot \Box U , \] (37) (38)

and the initial value constraints:

\[ \Psi_{\mu}|_{\Sigma} = 0, \] (39)

\[ n^\nu(G_{\mu \nu} - T_{\mu \nu})|_{\Sigma} = 0, \] (40)

where \( n^\mu \) is a time-like normal vector orthogonal to \( \Sigma \). By going to an orthonormal basis \( e_{\bar{\mu}} \) such that \( e_{\bar{0}} \) is normal to \( \Sigma \) we observe that \( G_{\bar{0} \bar{0}} = \sum_{i,j=1}^{3} (X_{\bar{i} \bar{i}} \cdot X_{\bar{j} \bar{j}} - X_{\bar{i} \bar{j}} \cdot X_{\bar{j} \bar{i}}) \) and \( G_{\bar{0} \bar{a}} = \sum_{i=1}^{3} (X_{\bar{i} \bar{a}} \cdot X_{\bar{i} \bar{i}} - X_{\bar{0} \bar{i}} \cdot X_{\bar{a} \bar{i}}) \) do not depend on \( X_{\bar{i} \bar{0}} \). Substituting the initial value constraints into (35) we obtain \( n^\nu(G_{\mu \nu} - T_{\mu \nu})|_{\Sigma} = 0 \). Taking the covariant divergence of (35), using the Bianchi identity and \( T_{\mu \nu} = 0 \) we obtain

\[ \Box \Psi_{\mu} = 2 \Gamma_{\mu}^{\rho \sigma} \Psi_{(\rho, \sigma)} + 2 \Gamma^{\nu} \Psi_{(\mu, \nu)}. \] (41)

Therefore (39) is conserved and any solution of our system satisfies \( G_{\mu \nu} = T_{\mu \nu}, \Gamma_{\mu} = \Psi_{\mu} = 0. \)
Equations \((37)\) and \((38)\) constitute a nonlinear hyperbolic equation system for \(U\). We note that the right hand side of \((38)\) is somewhat arbitrary and could instead be chosen to be zero. The choice we have made is so as to recover \(\Gamma^{\mu} = 0\). Regardless of this choice the equation system is far from being quasilinear. The nonlinear terms in \((37)\) contain second derivatives of \(U\) explicitly and also implicitly in the \(\Gamma\Gamma\) terms. We will now show that, whilst the former seem to be an essential feature, the latter can be effectively smuggled into the initial conditions.

**B. Reformulating the nonlinear theory by including an auxiliary metric field**

Let us introduce now a metric \(g\) as an auxiliary variable, a priori independent of \(U\). We first define

\[
\Phi_{\mu\nu} := -2\perp U_{\mu\nu} + 2\perp U_{(\mu,\nu)} + U_{,\mu} \cdot U_{,\nu} + \eta_{\mu\nu} - g_{\mu\nu} = g_{\mu\nu} - g_{\mu\nu},
\]

\[
\Psi_{\mu} := g^{\sigma\rho}(-2\perp U_{\sigma\mu,\nu} + \perp U_{\mu\nu,\sigma})
\]

so that for an isometric embedding \(\Phi_{\mu\nu} = 0\) and \(\Psi_{\mu} = \Psi_{\mu}\). Consider now the equation system:

\[
g^{\rho\sigma} \perp U_{\mu\nu,\rho\sigma} = -g^{\rho\sigma} U_{,\rho\sigma} \cdot U_{,\mu\nu} + U_{,\rho\nu} \cdot U_{,\nu\sigma} g^{\rho\sigma} - 2\perp (\mu \Psi_{\nu})_{\rho\sigma} - \perp \Gamma_{\gamma}^{\rho \gamma \sigma} \Gamma^{\sigma}_{\mu \nu} + \Gamma_{\rho \gamma}^{\sigma} + \Gamma^{\rho}_{\gamma} + S_{\mu\nu},
\]

\[
g^{\rho\sigma} U_{,\mu\nu,\rho\sigma} = -g_{,\sigma} \cdot g^{\rho\sigma} U_{,\rho\sigma},
\]

\[
-\frac{1}{2} g^{\rho\sigma} g_{\mu\nu,\rho\sigma} = -\Gamma_{\gamma}^{\rho \gamma \sigma} \Gamma^{\sigma}_{\mu \nu} + \Gamma_{\rho \gamma}^{\sigma} + \Gamma_{\rho \gamma}^{\sigma} - 2\perp (\mu \Gamma_{\nu})_{\rho\sigma} + S_{\mu\nu},
\]

with initial conditions:

\[
n^{\nu}(G_{\mu\nu} - T_{\mu\nu})|_{\Sigma} = 0,
\]

\[
\Psi_{\mu}|_{\Sigma} = 0,
\]

\[
\Phi_{\mu\nu}|_{\Sigma} = 0,
\]

\[
n^{\rho}\Phi_{\mu\nu,\rho}|_{\Sigma} = 0.
\]

In \((44)-(50)\) we regard \(\Gamma\) with indices in various positions and \(G_{\mu\nu}\) to be functions of \(g\) and its derivatives rather than functions of \(U\).

Equation \((50)\) is not as it stands an equation for initial data. But, going again to an adapted frame, we find that the second time derivatives appear in the combination \(\perp U_{\mu|00} + U_{|\mu} \cdot U_{|00}\). Therefore we can use evolution equation \((45)\) to eliminate them.

Taking the difference between \((44)\) and \((46)\) and using \((45)\) we obtain:

\[
g^{\rho\sigma} \Phi_{\mu\nu,\rho\sigma} = \Gamma^{\rho \sigma}_{\mu} \left[ \Phi_{\nu,\rho,\sigma} + \Phi_{\nu,\sigma,\rho} - \Phi_{\rho,\sigma,\nu} \right] + \Gamma^{\rho \sigma}_{\nu} \left[ \Phi_{\mu,\rho,\sigma} + \Phi_{\mu,\sigma,\rho} - \Phi_{\rho,\sigma,\mu} \right].
\]
Therefore the constraint for $\Phi_{\mu\nu}$ is conserved, our embedding is isometric, equations (46) and (44) become equivalent and our system reduces to the previous one (37-40). Conservation of the remaining constraints then follows and solutions of the system satisfy:

$$R_{\mu\nu} = S_{\mu\nu}, \quad \Gamma_\mu = \Psi_\mu = 0, \quad g_{\mu\nu} = g_{\mu\nu}.$$  

(52)

By introducing $g_{\mu\nu}$ as an auxiliary field we obtained a kind of hybrid between the standard formulation of GR in De Donder gauge by Choquet-Bruhat, and a hyperbolic version of Gunther’s perturbation method. In this formulation, the nonlinear term on the right hand side of equation (45) is not arbitrary. It is in just the right form to allow us to conserve the initial condition $g = g$. The evolution equations are of the form:

$$g^{\sigma\rho}(\hat{X}_{,\mu\nu} \cdot U)_{,\rho\sigma} = g^{\sigma\rho}(U_{,\mu\nu} \cdot U_{,\rho\sigma} - U_{,\mu\rho} \cdot U_{,\nu\sigma}) + \cdots, \quad \text{(53)}$$

$$g^{\sigma\rho}(\hat{X}_{,\mu\nu} \cdot U)_{,\rho\sigma} = -g^{\sigma\rho}U_{,\mu} \cdot U_{,\rho\sigma}, \quad \text{(54)}$$

$$\frac{1}{2}g^{\sigma\rho}g_{,\mu\nu,\rho\sigma} = \cdots. \quad \text{(55)}$$

where the ellipsis denotes terms of at most first order in derivative. Since $\hat{X}$ is free, (53) and (54) can be expressed in the form:

$$g^{\sigma\rho}\partial_{\mu}\partial_{\rho}U^A = (\hat{M}^{-1})_A^B Q^B(g, \partial g; U, \partial U, \partial U). \quad \text{(56)}$$

where $\hat{M} = (\partial \partial \hat{X}, \partial \hat{X})$ understood as a 14 by 14 matrix. Second derivatives in $Q$ occur only in terms which are quadratic order in $U$ or derivatives. $Q$ also depends on the background embedding up to its fourth derivatives.

Equations (53, 56) constitute a second order fully non-linear differential equation system which is hyperbolic about $U = 0$. Returning to the caveat mentioned at the beginning of the section, note that (56) is linear in $U_{,00}$ and can be algebraically solved for it, allowing us to get a bound on it in terms of the initial data. Therefore, for small enough initial data, the perturbed embedding is guaranteed to be free, at least for some period of time. To see that the evolution must be well defined for some finite time period, we note that it is possible to eliminate $U_{,00}$ from the right hand side of (56). Then, by introducing new variables equal to spatial derivatives of the original ones, one can construct a new system, which is quasilinear. The principal symbol will be modified, but as long as all fields remains small, the equations will remain hyperbolic. This will be discussed more fully in a follow up article. Another matter which deserves further study is the question of solving the initial data, which basically amounts to the perturbation problem for the initial surface subject to the isometric bending constraints $\Psi_\mu = 0$. 

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C. Comments on the linearised theory

By considering the action for the linearised theory, we see another explanation for why we obtained a wave equation by imposing (39). The massless Fierz-Pauli action is:

\[ I^{(\text{lin})} = \frac{1}{4} \int_M \left( 2h_{\mu\nu,\rho} h^{\mu\rho,\nu} - 2h_{\mu\nu} h^{\rho,\mu} + h_{\mu}^{\mu,\nu} h^{\rho,\nu} - h_{\mu\nu,\rho} h^{\mu\nu,\rho} \right) \]  

(57)

In terms of the embedding coordinates, we have

\[ I^{(\text{lin})} = - \int_M 2\perp U_{\mu,\rho} \perp U^{\mu,\rho,\nu} - 2\perp U_{\mu,\nu} \perp U^{\rho,\mu} + \perp U_{\mu}^{\mu,\nu} \perp U^{\rho,\nu} - \perp U_{\mu,\rho} \perp U^{\mu,\rho} + \partial_\mu \Omega^\mu \]  

(58)

with

\[ \Omega^\mu := 2U_{\rho,\sigma}^{\mu} \perp U^{\rho,\sigma} - 2U_{\rho,\sigma}^{\mu} \perp U^{\rho,\sigma} - U_{\rho,\sigma}^{\mu} \perp U^{\rho,\sigma} + U_{\rho,\sigma}^{\mu} \perp U^{\rho,\sigma} - U_{\rho,\sigma}^{\mu} \perp U^{\rho,\sigma} \]

Viewed from this point of view it is trivial to establish equivalence with the metric theory - it is simply a matter of replacing \( h \) with \( \perp U \). However, the interpretation is somewhat different. In the metric theory the gauge invariance \( h_{\mu\nu} \rightarrow h_{\mu\nu} + v_{\mu,\nu} + v_{\nu,\mu} \) is associated with infinitesimal coordinate transformations. In the embedding theory the automorphisms of the surface are associated with \( \perp U \rightarrow \perp U^f \) which is manifested in the fact that \( \perp U \) drops out of the linearised action, whereas the transformations \( \perp U_{\mu,\rho} \rightarrow \perp U_{\mu,\rho} + v_{\mu,\rho} + v_{\rho,\mu} \) are geometrically nontrivial from the extrinsic point of view. They have the status of pure gauge only because the action is an intrinsic invariant. In view of this, we are at liberty to modify the action to promote the isometric bending to physical degrees of freedom without breaking intrinsic diffeomorphism invariance. It would be interesting to pursue some modification such as massive gravity via this route.

V. FURTHER DISCUSSION

A. Equivalence with GR at the linearised level?

In ref. [13] it was pointed out that, for typical examples of embeddings of interest, the degrees of freedom of linearised GR tend to show up only at quadratic order in the embedded theory. For example, consider the embedding of the Schwarzschild space-time into 5+1 dimensions due to Fronsdal[28]. A perturbation of the Fronsdal embedding has some non-trivial content at the linearised level. Adding extra dimensions (in straightforward fashion to create a product space) does not add anything at linear level[29]. However, degrees of freedom are certainly missing.
Why? As was pointed out in section I C, the linearised embedded theory fails to encompass the linearised metric theory to the extent to which the osculating space is degenerate. Adding trivial extra dimensions will not remedy the problem. Instead we must find an alternative embedding which is free. Now local free embeddings are generic in high enough dimension. In \( N \geq 14 \) (or 10 for analytic data) the set of embeddings for which the full linearised theory does not show up is therefore expected to be of measure zero. The problem is that this measure zero set consists of all the concrete examples that are well known to Relativists. So one must be careful of intuition based on specific examples until appropriate examples are found.

Let us consider now how such examples may be constructed. One way is to take our favourite known embedding of \( \mathcal{M} \to \mathbb{R}^{1,N} \) and compose it with a free embedding of \( \mathbb{R}^{N-1,1} \to \mathbb{E}^{N-1,1} \) for some \( N \). This has the advantage that we only ever need to find one explicit FIE. The disadvantage is that this is not very economical at all with dimensions. For example, for the Fronsdal embedding, we have \( R=6 \) which will require \( N = 27 \) even for a local FIE. The other approach is to look directly for a free embedding in each case of interest. There does not seem to be any examples in the literature, although some spatially free embeddings were given in Ref. [14].

### B. The 13+1 dimensional theory

The study of (55-56) is equivalent to the study of the Einstein equations, for an embedded surface close to a given freely embedded background, as an initial value problem. This approach is useful for identifying appropriate gauge fixing and perhaps for identifying the degrees of freedom. It also allows us to confirm the intuitive picture of gravitational waves as ripples propagating along the world-sheet of the embedded manifold. However, it may not be the most appropriate method for addressing long time existence and uniqueness of solutions to the evolution equations. To attack directly such highly nonlinear equations may be a very hard problem (although it seems some results are known in certain cases[30]). Alternatively we may abandon the idea of a wave equation for \( U \). Indeed the problem can be broken down into two steps: first one evolves the initial data for the intrinsic metric using the Einstein equation, then one embeds the resulting manifold, according to some initial constraints on the embedding. For the second step the complication arises because we have introduced the lorentzian physical metric into (53) and (54). It may be better to adapt existing methods which in one way or another introduce an auxiliary euclidean metric. If we are sufficiently close to the background, then the second step is a 4
dimensional perturbation problem similar to that discussed in chapter III. This would give a combined hyperbolic/elliptic system. A metric perturbation will be realisable as a perturbation of $\hat{X}$ when some condition like (20) or (22) is satisfied. Heuristically
\[ \|h\| \lesssim \|\hat{M}^{-1}\|^{-2}. \]

Since the Einstein system is well posed, by making the initial data small enough, we may ensure that the condition is met up to arbitrarily large time. To ensure that, with finite initial data, it is satisfied for all time would require some strong stability result for the background spacetime.

More generally we might hope for some version of cosmic censorship, whereby an initially free embedding will generically remain: 1) an immersion; 2) free, except perhaps inside some horizons. We might also add a stronger condition: 1') embedded. Roughly speaking 1) is a poor man's cosmic censorship which says generically there will be no naked singularities where the metric becomes degenerate or singular; 2) is an additional requirement to allow us to develop the embedding; 1') says that $X(M)$ does not self-intersect. These are properties we would like solutions to have, but at the moment we have no arguments in favour of such a conjecture. It is not even easy to make a precise statement of it. Any statement in terms of inextendibility of developments is problematic. For example, even if an embedding can be developed out to infinity of $\mathbb{E}^{13,1}$, it is possible that as an intrinsic manifold, $M$ is extendible. This can happen if $X(M)$ asymptotically approaches a null surface of $\mathbb{E}$.

C. Global embeddings. $N > 14$?

In order to consider the global problem, and accomodate a general background, we will presumably need to go to higher than $13 + 1$ dimensions. The current lower bound on $N$ for embedding any Riemannian $n$-manifold is $N_\ast = \min(\frac{1}{2}n(n + 5), \frac{1}{2}n(n + 3) + 5)$ [24]. For globally hyperbolic Lorentzian manifolds it is $N_\ast + 1$ [5]. The way in which $N_\ast$ is obtained is a long story which we are not able to summarise here, so we will simply give some comments. The $\frac{1}{2}n(n + 5)$ comes purely from differential topology- basically the generalisation of Whitney’s theorem to free embeddings. Since we are restricting our interests to globally hyperbolic manifolds diffeomorphic to $\mathbb{R} \times \Sigma$, this number might conceivably be reduced. The $\frac{1}{2}n(n + 3) + 5$ comes from requiring a 5 dimensional

\footnote{This is only roughly speaking because there are certain types of purely extrinsic singularity that can occur in higher dimensions whereby the map $dX$ is not injective but the metric $dX \cdot dX$ is well defined e.g. the “embedding” of a flat two-plane into Minkowski space $(x,y) \rightarrow (r,x,y,r)$.}
orthogonal compliment to the osculating space with which to make coarser adjustments to the induced metric without loosing control of $M^{-1}$, so as to apply the perturbation result. Depending upon whether this last number can be improved upon, we may need up to 19+1 dimensions for our embedding space. Even if $\mathcal{M}_4$ is flat, it is not clear whether a global free embedding into 13+1 dimensions exists.

Let $\hat{X} : \mathcal{M} \rightarrow \mathbb{E}^{N-1,1}$ be free. Introduce an orthonormal basis $e_I$, $I = 15, \ldots, N$, of the orthogonal compliment to the osculating space. Then a perturbation $U$ can be decomposed as

$$U = U_{\mu\nu}(P^{-1})^{\mu\rho\sigma} \hat{X}_{,\rho\sigma} + U_{\mu} g^{\mu\nu} \hat{X}_{,\nu} + Y^I e_I.$$  \hspace{1cm} (59)

At the linearised level $Y$ will drop out of the Einstein equations. At the nonlinear level it will still apply, so that $Y$ will contribute to the nonlinear self interaction terms, but will not have any propagator. In order to recover a hyperbolic system we may then proceed to treat $Y$ in the same way as $\parallel U$, by giving some dynamics

$$\Box Y^I = \ldots$$  \hspace{1cm} (60)

which can presumably be interpreted as a gauge fixing. Certainly, locally $Y$, like $\parallel U$, can be gauged away. But over long timescales, $U$ may grow too large for the perturbation result to apply, in which case $Y$ may need to be nonzero in order for a solution to exist. In a sense, a global change in $Y$ is more like changing the background rather than a component of the perturbation. At the present, it is not clear whether a full global description of the gravitational field can be recovered in this way and if so what the appropriate form of (60) should be.

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Appendix A: Weighted Sobolev spaces

Let $e$ be the Euclidean metric on $\mathbb{R}^3$ and $\mu_e$ be the standard volume element. We shall be interested in quantities which decay faster than $r^\delta$ so let us introduce the quantity $\sigma := \sqrt{1 + r^2}$
which is asymptotically \( \sigma \sim r \) but has the advantage of being positive definite.

**Definition A.1.** \( H_{s,\delta} = W^{2,\delta-3/2}_{s,\delta} \), \( s \in \mathbb{N}, \delta \in \mathbb{R} \) is the space of functions or tensor fields \( f \) over \( \mathbb{R}^n \) with square-integrable weak derivatives of order up to \( s \) such that

\[
\|f\|_{s,\delta} := \left( \sum_{|\alpha| \leq s} \int_{\mathbb{R}^3} \sigma^{2(|\alpha|+\delta-3/2)} |\partial^\alpha f|^2 \mu_e \right)^{1/2} < \infty \tag{A1}
\]

where \( |f| \) denotes the pointwise norm of tensor \( f \) in the metric \( e \).

We shall need the following elementary properties which follow directly from the definition:

- \( \|Kf\|_{s,\delta} = |K|\|f\|_{s,\delta} \) for \( K \) constant;
- \( \|f, \alpha\|_{s-1,\delta+1} \leq \|f\|_{s,\delta} \); \( \|\sum_c f, ac, f, bc\| \leq C_1\|f, ab\|\|f, cd\| \);
- \( \|\Delta f\| \leq C_2\|f, ab\| \) for some constants \( C_1, C_2 \). We shall also need the following two well known properties:

**Theorem A.2** (Continuous multiplication rule). If \( s_1, s_2 \geq s \), \( s_1 + s_2 > s + 3/2 \), \( \delta_1 + \delta_2 > \delta \) then \( f \in H_{s_1,\delta_1}, g \in H_{s_2,\delta_2} \Rightarrow f \otimes g \in H_{s,\delta} \) and, for some constant \( C \):

\[
\|f \otimes g\|_{s,\delta} \leq C\|f\|_{s_1,\delta_1}\|g\|_{s_2,\delta_2}.
\]

**Theorem A.3** (Invertibility and open mapping property of Laplacian). If \( s \geq 2, 0 < \delta < 1 \) then \( \Delta : H_{s,\delta} \rightarrow H_{s-2,\delta+2} \) is an isomorphism and for some constant \( C \):

\[
\|f\|_{s,\delta} \leq C\|\Delta f\|_{s-2,\delta+2}.
\]

**Appendix B: Proof of theorem III.1**

Let \( \hat{X} \) be a FIE of \((\mathbb{R}^3, \gamma)\) into \( \mathbb{E}^N \) satisfying \( M^{-1} = K + L \) where \( K \) is a constant matrix. Assume and \( L \in H_{q,\beta} \) and \( M^{-1}H \in H_{p,\alpha} \). We prove the existence of a FIE \( X \) for \((\mathbb{R}^3, \gamma + h)\) under the conditions of theorem [III.1].

Let \( U = \lambda W \) where

\[
\lambda := \sqrt{\frac{\|M^{-1}H\|_{p,\alpha}}{|K| + \|L\|_{q,\beta}}} \leq D\sqrt{\|h\|_{p,\alpha}}. \tag{B1}
\]

Then [19] becomes

\[
W = \mathcal{O}(W), \tag{B2}
\]

\[
\mathcal{O}(W) := \frac{1}{\lambda} M^{-1}H + \lambda M^{-1}\Delta^{-1}Q(W). \tag{B3}
\]

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Assume $W_m \in H_{p,\alpha}$ and consider the iteration $W_{m+1} = O(W_m)$ with $W_0 = \frac{1}{X} M^{-1} H$ and prove convergence to a solution by the contraction mapping theorem. Suppose

Assume $V \in \mathcal{F} := \{V \in H_{p,\alpha} : \|V\|_{p,\alpha} \leq 1\}$. Then

\[
\|O(W)\|_{p,\alpha} \leq \frac{1}{\lambda} \left\| M^{-1} H \right\|_{p,\alpha} + \lambda |K| \left\| \Delta^{-1} Q(W) \right\|_{p,\alpha} + \lambda \left\| L \Delta^{-1} Q(W) \right\|_{p,\alpha} \tag{B4}
\]

\[
\leq \sqrt{\left\| M^{-1} H \right\|_{p,\alpha} (|K| + \left\| L \right\|_{q,\beta}) \left( 1 + C_1 \left\| \Delta^{-1} Q(W) \right\|_{p,\alpha} \right)}, \tag{B5}
\]

\[
\leq \sqrt{\left\| M^{-1} H \right\|_{p,\alpha} (|K| + \left\| L \right\|_{q,\beta}) \left( 1 + C_2 \left\| Q(W) \right\|_{p-2,\alpha+2} \right)}, \tag{B6}
\]

\[
\leq \sqrt{\left\| M^{-1} H \right\|_{p,\alpha} (|K| + \left\| L \right\|_{q,\beta}) \left( 1 + C_3 \left\| W'' \right\|_{p-2,\alpha+2} \left( \left\| W'' \right\|_{p-2,\alpha+2} + \left\| W' \right\|_{p-1,\alpha+1} \right) \right)}, \tag{B7}
\]

\[
\leq C_5 \sqrt{\left\| M^{-1} H \right\|_{p,\alpha} (|K| + \left\| L \right\|_{q,\beta})}, \tag{B8}
\]

\[
\leq C_5 \sqrt{\left\| M^{-1} H \right\|_{p,\alpha} (|K| + \left\| L \right\|_{q,\beta})}. \tag{B9}
\]

The above inequalities are applications of Theorems A.2 and A.3 which are valid provided we require certain conditions in $p, q, \alpha, \beta$: (B7) requires $p \geq 4$; (B5) requires $\alpha, \beta > 0$, $q \geq p$. These are sufficient to ensure all other inequalities hold.

Therefore $O : \mathcal{F} \to \mathcal{F}$ provided that:

\[
\left\| M^{-1} H \right\|_{p,\alpha} (|K| + \left\| L \right\|_{q,\beta}) \leq \frac{1}{C_5^2}. \tag{B10}
\]

Now consider $W_1, W_2 \in \mathcal{F}$.

\[
\|O(W_2) - O(W_1)\|_{p,\alpha} = \lambda \left\| M^{-1} \Delta^{-1} Q(W_2) - M^{-1} \Delta^{-1} Q(W_1) \right\|_{p,\alpha} \tag{B11}
\]

\[
\leq \lambda C_6 (|K| + \left\| L \right\|_{q,\beta}) \left\| Q(W_2) - Q(W_1) \right\|_{p-2,\alpha+2}, \tag{B12}
\]

\[
\leq \lambda C_7 (|K| + \left\| L \right\|_{q,\beta}) \left( \left\| W'' + W'' \left\|_{p-2,\alpha+2} \left\| W_2 - W_1 \right\|_{p-2,\alpha+2} + \left\| W_2 - W_1 \right\|_{p-1,\alpha+1} \right) + \left\| W'' + W'' \left\|_{p-2,\alpha+2} \left\| W_2 - W_1 \right\|_{p-1,\alpha+1} \right) \tag{B13}
\]

\[
\leq \lambda C_8 (|K| + \left\| L \right\|_{q,\beta}) \left\| W_2 + W_1 \right\|_{p,\alpha} \left\| W_2 - W_1 \right\|_{p,\alpha} \tag{B14}
\]

\[
\leq C_9 \sqrt{\left\| M^{-1} H \right\|_{p,\alpha} (|K| + \left\| L \right\|_{q,\beta}) \left\| W_2 - W_1 \right\|_{p,\alpha}}. \tag{B15}
\]

Therefore $O : \mathcal{F} \to \mathcal{F}$ is a contraction mapping if

\[
\left\| M^{-1} H \right\|_{p,\alpha} (|K| + \left\| L \right\|_{q,\beta}) < \max \left\{ \frac{1}{C_5^2}, \frac{1}{C_9^2} \right\}. \tag{B16}
\]
Applying the contraction mapping theorem we obtain the desired result. 

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