Keldysh Nonlinear Sigma Model for a Free-Fermion Gas under Unconditional Continuous Measurements

Qinghong Yang\textsuperscript{1}*, Yi Zuo\textsuperscript{2}, and Dong E. Liu\textsuperscript{1,3,4†}

\textsuperscript{1}State Key Laboratory of Low Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing, 100084, China
\textsuperscript{2}Beijing National Laboratory for Condensed Matter Physics, and Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
\textsuperscript{3}Beijing Academy of Quantum Information Sciences, Beijing 100193, China and
\textsuperscript{4}Frontier Science Center for Quantum Information, Beijing 100184, China

(Dated: July 8, 2022)

We analytically analyze the quantum dynamics of a free-fermion gas subject to unconditional continuous projective measurements. By mapping the Lindblad master equation to the functional Keldysh field theory, we observe that the Keldysh Lindblad partition function is resemble to that in the Keldysh treatment of the disordered fermionic systems. Based on this observation, we obtain an effective theory termed the Keldysh nonlinear sigma model to describe the low-energy physics. Two types of diffusion correlators, similar with those in the disordered fermionic systems, are derived. In addition, up to the one-loop level, we obtain a Drude-form conductivity where the elastic scattering time is replaced by the inverse measurement strength.

Introduction.—The entanglement entropy, as a characteristic measure of quantum correlations, has been intensively studied in many fields, such as quantum information and quantum computation [1], condensed matter physics [2], and high energy physics [3, 4]. Subsystem entanglement entropies follow significantly distinct scaling laws for different dynamical phenomena in quantum-many body systems. By adjusting the system parameters, different scaling laws can be mutually converted. One typical example is the transition between the phase obeying the eigenstate thermalization hypothesis (ETH) [5, 6] and the many-body localized (MBL) phase [7–10]. When quantum many-body systems obey ETH, the entanglement entropy of subsystems presents a volume-law scaling. By increasing the disorder strength, the systems will enter the MBL phase where the subsystem entanglement entropy obeys the area law instead [11–13].

An alternative way to obtain the entanglement transition from the volume law to the area law has been proposed by using projective measurements [14]. Intuitively, one can imagine that local projective measurements will collapse a highly entangled many-body state, and thus enough measurements will convert the volume-law entangled state to an area-law one. In recent years, people have studied this phenomenon, both numerically and analytically, in a wide variety of models, ranging from random unitary circuits subjected to random local measurements [14–17] to quantum many-body systems under continuous projective measurements [18–21].

In the disorder-induced entanglement transition case, when a system is in the MBL phase, degrees of freedom of the system are indeed being localized, which is a manifestation of the area-law entanglement entropy. This localization effect can be easily demonstrated experimentally through the DC conductivity [9, 22–24]. Specifically, the DC conductivity is zero in the MBL regime [9], while in the ETH regime, the conductivity is nonzero. This motivates us to dig into the recently proposed measurement-induced transition counterpart [14–21]. Since in this scenario, the dynamics will be hindered by continuous projective measurements and the subsystem entanglement entropy also has an area-law scaling, one may ask that is there also a localization effect and what is the behavior of the conductivity.

In this work, we analytically study these problems in a free-fermion gas under unconditional continuous projective measurements. In the literature, people mostly focus on the quantum trajectory dynamics conditioned on measurement outcomes [20, 25, 26], and in order to reveal the entanglement transition, the subsystem entanglement entropy is firstly calculated for each quantum trajectory and then averaged over all trajectories. In contrast to their calculations, our theoretical scheme directly capture the unconditional dynamics generated by the full Lindblad master equation [26]. Note that if the quantity is a linear function of the system’s state described by the density matrix, the conditional and the unconditional approaches will give the same result. For examples, many physical observables including the conductivity are linear functions of the state, while the entanglement entropy is not.

We modify the Keldysh field theory mapping [27] to capture the Lindblad master equation for open fermionic systems. We find that the dissipator in the Lindblad master equation will present in a four-fermion interaction form in the Keldysh Lindblad partition function; and very surprisingly, the four-fermion interaction term resembles the four-fermion term in the Keldysh treatment of the disordered fermionic problem [28–30]. Inspired by this observation, we derive a low-energy effective bosonic theory, which we termed as the modified Keldysh nonlin-
ear sigma model, to describe the low-energy physics of the free-fermion gas under projective measurements. Up to the one-loop level, we obtain the Drude-form conductivity in which the inverse measurement strength plays the role of the elastic scattering time, and this result shows a clear slow-down effect due to the projective measurements. Intuitively, external projective measurements will not intrinsically change eigenstates of a system, and thus if the Hamiltonian generating unitary dynamics satisfies ETH, one may imagine that there does not exist a localization phenomenon, but instead a slow-down effect may occur due to the quantum Zeno effect [31].

Setup.— We consider a $d$-dimension spinless free-fermion gas, whose Hamiltonian reads

$$H = \int d\mathbf{x} c^\dagger(\mathbf{x}) \left( -\frac{1}{2m} \nabla^2 - \epsilon_F \right) c(\mathbf{x}),$$

(1)

where $c$ ($c^\dagger$) is the annihilation (creation) operator of fermions, $m$ is the mass of fermions, and $\epsilon_F$ is the Fermi energy which equals to the chemical potential. This free-fermion gas is subject to continuous projective measurements, in which the projective operations $\mathbf{F}$ occur due to the quantum Zeno effect [31].

In order to preserve the normalization condition, at least in the first order, the bare Green’s function of free fermions should be in its full form, that is

$$\hat{G}(\mathbf{k}; t, t') = \begin{bmatrix} G^K_0(\mathbf{k}; t, t') & G^K_0(\mathbf{k}; t, t') \\ G^A_0(\mathbf{k}; t, t') & i \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \delta_{t, t'} \end{bmatrix},$$

(4)

where $G^K_0$ are the three typical bare Green’s functions used in the standard Keldysh field theory [28, 33], and $\delta_{t, t'}$ is the Kronecker delta symbol, which comes from the discrete time version $\delta_{j, j'}$ with $j, j'$ standing for the $j$th time slice and $j'$th time slice. Note that in the standard functional Keldysh partition function derived from the Hamiltonian of a closed system, the extra term $\propto \delta_{t, t'}$ also exists. However, one usually omits it. One argument is that the $t = t'$ line is a manifold of measure zero and omitting it is inconsequential for most purposes [28]. We emphasize that this $\delta_{t, t'}$ term cannot be directly omitted in open system problems due to the normalization condition mentioned above.

In the Keldysh Lindblad partition function Eq. (3), we add two extra terms $\bar{\psi}_a(x) \psi_a(x) \bar{\psi}_b(x) \psi_b(x)$ with $a = b$, which are null due to the property of Grassmann numbers $-\bar{\psi}_a^2 = \psi_a^2 = 0$. After adding these two terms, one finds that the four-fermion term in the partition function is similar to the four-fermion term after doing the disorder averaging in the Keldysh treatment of the disordered fermionic problem [28–30] (also see SI [32] for a brief introduction to the disordered fermionic system). This inspires us to deal with this problem following the approach people used to treat the disordered fermionic system. However, there are also some differences between these two problems. For example, the four-fermion term in Eq. (3) only depends on one time variable, while in the disordered fermionic problem, the four-fermion term depends on two time variables. In addition, there is no time-reversal symmetry in our case due to the nature of open quantum systems, while the time-reversal symmetry is present in the free-fermion gas with disorders.

Modified Keldysh nonlinear sigma model.— Just as the treatment used in the disordered fermionic system
and many other problems, we will derive a low-energy effective theory for our problem in the following. To this end, we employ the Hubbard-Stratonovich (HS) transformation [28, 33] by introducing a matrix-valued bosonic field $Q$ to decouple the four-fermion term, where $Q$ is defined as

$$\hat{Q} = \int dx \begin{bmatrix} Q_{11}^{11}(x) & Q_{12}^{12}(x) \\ Q_{21}^{12}(x) & Q_{22}^{22}(x) \end{bmatrix} |x\rangle \langle x|,$$

and it is Hermitian in the Keldysh space, i.e., $Q_{ab}(x) = [Q_{ba}(x)]^\dagger$. Note that due to the fact that the four-fermion term depends on two time variables in the disordered fermionic case, the matrix HS field is not diagonal in the time basis. The HS transformation and Gaussian integral lead the partition function Eq. (3) to an effective bosonic theory (see SI [32] for details):

$$Z = \int D[\hat{Q}] \exp \left\{ -\frac{\pi \nu}{2} \text{tr} \left[ \hat{Q}^2 + \left( \frac{1}{2\pi \nu} \hat{\tau}_1 \right)^2 \right] + \text{tr} \ln \left( -i\hat{G}_0^{-1} + \pi \nu \hat{Q} \right) \right\},$$

where $\text{tr}$ stands for the trace over the Keldysh space as well as time and spatial integrations, $\nu$ is density of states (DOS) in the vicinity of the Fermi surface and $G_0^{-1}$ is the inverse of $G + (i/2)\delta_{t,t'}\hat{\tau}_1$ (see Eq. (4)). In the procedure of replacing $\hat{G}^{-1}$ with $G_0^{-1}$, we have employed the argument that the $t = t'$ line is a manifold of measure zero to higher-order ($\geq 2$) terms of $\gamma$. As mentioned in the previous, the time-reversal symmetry is absent in our case, thus we just decouple the four-fermion term in the density channel. In contrast, in the disordered fermionic case, one can also decouple the four-fermion term in the Cooper channel, and this procedure results in Cooperons, which accounts for the weak localization effect in the one-loop level [28, 29].

To proceed, we need to find the saddle point configuration of the action in Eq. (5), which contributes most to the functional integral. Taking the variation over $\hat{Q}(x)$, one gets the saddle point equation:

$$\gamma \pi^2 \nu^2 \hat{Q}(x) = \gamma \pi \nu \left( -i\hat{G}_0^{-1} + \pi \nu \hat{Q} \right)^{-1}(x,x).$$

One can check that the constant configuration $\hat{\Lambda} = \frac{1}{\pi \nu} \hat{\tau}_3$ satisfies the saddle point equation when $\gamma$ satisfies $\gamma \ll \epsilon_F$. Note that this condition also validates the procedure of replacing $\hat{G}^{-1}$ with $G_0^{-1}$ in Eq. (5). Fluctuations around the saddle point can be classified into two classes: the massive and the massless modes. In the low-energy physics regime, the dynamics is mostly contributed by the massless modes. Thus, we here focus on fluctuations of the $\hat{Q}$-matrix along the massless “direction”, and they can be generated through the similarity transformation: $\hat{Q} = \hat{R}^{-1}\hat{\Lambda}\hat{R}$. In the spacetime basis, $\hat{Q}(x) = \hat{R}^{-1}(x)\hat{\Lambda}\hat{R}(x)$, and $\hat{Q}(x)$ satisfies the nonlinear constraint: $\hat{Q}^2(x) = \left( \frac{1}{\pi \nu} \right)^2 \hat{\tau}_0$.

In order to derive an effective low-energy theory for our problem, one can employ the gradient expansion, that is, we expand the $\text{tr} \ln$ term in Eq. (5) in powers of $\partial_t\hat{R}^{-1}$ and $\nabla \hat{R}^{-1}$. Keeping terms up to the first order of $\partial_t\hat{R}^{-1}$ and the second order of $\nabla \hat{R}^{-1}$, one arrives at the modified Keldysh nonlinear sigma model (see SI [32] for details):

$$iS[\hat{Q}] = \pi \nu \text{tr} \left[ \partial_t \hat{Q} - \frac{1}{4} \pi \nu D \text{tr} \left( \nabla \hat{Q} \right)^2 \right],$$

where we just keep those non-constant terms in the action. Here, $\hat{Q}$ is redefined as $\hat{Q} = \hat{U}^{-1}\hat{R}^{-1}\hat{\tau}_3\hat{R}\hat{U}$, where $\hat{U}$ encodes the statistical information and is defined as

$$\hat{U}^{-1} = \hat{U} = \sum_{\epsilon} \begin{bmatrix} 1 & F_{\epsilon} \\ 0 & -1 \end{bmatrix} |\epsilon\rangle \langle \epsilon|$$

with $F_{\epsilon} = \tanh(\beta \epsilon/2)$ relating to the Fermi-Dirac distribution. In Eq. (7), the constant $D$ is defined as $D = v_F^2/(\gamma d)$ with $v_F$ being the Fermi velocity, and is named as the modified diffusive constant. Comparing with the traditional diffusive constant in the disordered fermionic systems, one finds that the inverse measurement strength $1/\gamma$ plays the role of the elastic scattering time. Intuitively, this makes sense, as the elastic scattering time represents the mean time within which a fermion hits the disorder, or in other words, is measured by the disorder. Associating with the fact that the disordered fermionic system is also described by a similar nonlinear sigma model, we know that the effect of the projective measurements has some similarities with that of disorder. Indeed, in the following, we will show that up to the one-loop level, the conductivity is presented in the already familiar Drude form [28, 33].

**Gaussian fluctuation and modified diffusion.**— Having derived the saddle point and the low-energy effective theory for our problem, we are now in a position to draw the consequences from our low-energy effective theory. To this end, we write the similarity transformation matrix $\hat{R}$ through its generator $\hat{W}$ as $\hat{R} = \exp(\hat{W}/2)$. In the spacetime basis, we have $\hat{R}(x) = \exp(\hat{W}(x)/2)$. To generate a non-trivial transformation for $\hat{\tau}_3$, the generator $\hat{W}(x)$ should be an off-diagonal matrix in the Keldysh space, and can be expressed as

$$\hat{W}(x) = \begin{bmatrix} 0 & d^{12}(x) \\ d^{21}(x) & 0 \end{bmatrix},$$

where $\{\hat{W}(x), \hat{\tau}_3\} = 0$, and $d^{12}$ and $d^{21}$ are two independent fields. Substituting Eq. (8) into the nonlinear sigma model Eq. (7), and expanding the action in powers of $d^{12}$ and $d^{21}$, up to the second order, one obtains the Gaussian
action

\[ iS \left[ d_{12}^{a}, d_{21}^{a} \right] = 2\pi \nu \int dx \, d_{21}^{a}(x) \left( \partial_{t} - \frac{1}{4} D \nabla^{2} \right) d_{12}^{a}(x). \]  

(9)

With the help of the Fourier transformation, one finds that this Gaussian action will generate two types of correlations — \( \langle d_{12}^{a}(x) d_{21}^{a}(x) \rangle \) and \( \langle d_{21}^{a}(x) d_{12}^{a}(x) \rangle \), which are defined as

\[ \langle d_{12}^{a}(x) d_{21}^{a}(x) \rangle = \frac{1}{2\pi \nu} \frac{1}{D^{2} \kappa^{2} - i\epsilon}, \]

\[ \langle d_{21}^{a}(x) d_{12}^{a}(x) \rangle = \frac{1}{2\pi \nu} \frac{1}{D^{2} \kappa^{2} + i\epsilon}, \]  

(10)

where \( D' = (1/4)D \), and \( \langle \cdot \rangle \) stands for taking expectation values with weight \( \exp(iS[d_{12}^{a}, d_{21}^{a}]) \). We name these two correlators in Eq. (10) modified diffusions, as they are similar with those diffusions in the disorder fermionic systems [28, 29]. The modified diffusions play the role of bare Green’s functions and serve as the starting point to consider higher-order interaction effects and other phenomena underneath [30, 34].

**Linear response: the conductivity.**— Although the evolution according to the Lindblad master equation with Hermitian jump operators will result in featureless steady state [19, 35], due to the projection nature of the quantum jump operator \( n(x) \) considering here, one can imagine that the continuous projective measurements will have some impacts on the linear response. Here, we consider the most common linear response function in the condensed matter theory: the conductivity. To this end, we introduce the vector potential \( A(x) \), to which the current \( j \) couples, through the action \( S_{A} = -\int dx \, \dot{\psi}_{a}(x) \nu_{F} A^{a}(x) \dot{\psi}_{b}(x) \) [28, 29], where \( a, b \in \{1, 2\} \), \( \alpha \in \{0, 1\} \), and \( A^{0} \) stands for the classical component of the vector potential while \( A^{1} \) for the quantum component after the Keldysh transformation. Since the vector potential is classical, the quantum component \( A^{1} \) is actually zero. In the Keldysh field theory, it is preserved to generate observables by appropriate variations and is set to zero in the end. Following the procedures of deriving Eq. (7), one can get the Keldysh nonlinear sigma model in the presence of the vector potential:

\[ iS \left[ \hat{Q}, A \right] = \pi \nu \text{tr} \left[ \partial_{t} \hat{Q} \right] - \frac{1}{4} \pi \nu D \text{tr} \left[ \left( \partial \hat{Q} \right)^{2} \right], \]  

(11)

where we have assumed that the vector potential is small enough such that it does not alter the previous saddle point, \( \partial \hat{Q} = \nabla \hat{Q} + i \left[ A^{0} \hat{\gamma}_{0}, \hat{Q} \right] \), and \( \hat{Q} \) is also defined as \( \hat{Q} = \hat{U}^{-1} \hat{R}^{-1} \hat{\gamma}_{3} \hat{R} \hat{U} \).

The longitudinal AC conductivity can be derived through \( \sigma(q, \omega) = (-i/\omega) K^{R}(q, \omega) \), where \( K^{R}(q, \omega) \) is the retarded current-current response function, and is defined as

\[ K^{R}(q, \omega) = \frac{e^{2}}{2i} \frac{\delta^{2} Z[A]}{\delta A^{0}(q, \omega) \delta A^{1}(-q, -\omega)} |_{A=0} \]  

(12)

with \( Z[A] \) now being \( Z[A] = \int D[\hat{Q}] \exp\{iS[\hat{Q}, A] \} \). To calculate the retarded current-current response function, one may expand \( Z[A] \) in powers of \( A \) and keep terms up to the second order of \( A \). Then, one finds that, up to the one-loop level of the nonlinear sigma model (Eq. (9)), the longitudinal DC conductivity for the spatially-uniform potential, reads

\[ \sigma(q \to 0, \omega \to 0) = e^{2} \nu D. \]  

(13)

Thus, we reproduce the conductivity of the Drude form in a monitored free-fermion gas. Note that for a purely free-fermion gas, the conductivity is infinite, but in a monitored free-fermion gas, the conductivity is finite and is inversely proportional to the measurement strength \( \gamma \). Thus, the larger \( \gamma \) is, the smaller the conductivity. This slow-down effect of the projective measurements on the transport property is the manifestation of the quantum Zeno effect.

**Conclusions and discussions.**— In this paper, we have derived an effective low-energy theory for a free-fermion gas under unconditional continuous projective measurements. Up to the one-loop level, we obtain a Drude-form conductivity which is inversely proportional to the measurement strength \( \gamma \), and this shows that the projective measurements cause a slow-down effect on the free-fermion gas. Interestingly, the projective measurements manifest in a form that is comparable to that of the disorders in the framework of the Keldysh field theory. Nevertheless, the original Lindblad master equation formalism does not explicitly show this connection between the monitored system and the disordered system. Note that in the disordered fermionic system case, the weak localization effect exists in the one-loop level due to the time reversal symmetry, while in our case, we do not see the weak localization effect in the one-loop level. Although our effective theory works when the measurement strength is smaller than the Fermi energy, it can still cover a part of the area-law entanglement regime, as Ref. [20] shows that the critical measurement strength \( \gamma_{c} \) of the entanglement transition is around \( 0.31E_{K} \) with \( E_{K} \) being the kinetic energy, where the typical kinetic energy in our case is the Fermi energy.

The measurement of local particle number can be implemented via homodyne detection for ultracold fermions in optical lattices or Rydberg atom arrays [36], or by a local coupling to a projectively measured photon bath [25, 37]. Unlike measuring the entanglement entropy which needs huge overheads, it is more easier to measure the conductivity experimentally. Therefore, in addition to the theoretical and conceptual progress, we believe our work has potential experimental consequences.

In this work, we just focus on the one-loop level calculation, the effect of higher-order terms representing interactions is still unclear and will be our following focus. The Keldysh field theory approach also enables us to
study the physics of the monitored system in the presence of fermion interactions, such as the information scrambling [30, 34]. This will also be considered in the future.

Acknowledgments.— Authors thank Jing-Yuan Chen and Sebastian Diehl for helpful discussions on the functional Keldysh field theory from Lindblad master equations, and Jing-Yuan Chen and Yunxiang Liao for helpful discussion on the Keldysh nonlinear sigma model in the disordered fermionic systems. The work is supported by NSF-China (Grant No. 11974198) and the startup grant from State Key Laboratory of Low-Dimensional Quantum Physics of Tsinghua University.

∗ yqh19@mails.tsinghua.edu.cn
† dongeliu@mail.tsinghua.edu.cn

[1] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information, 1st ed. (Cambridge University Press, 2004).
[2] N. Laflorencie, Physics Reports 646, 1 (2016).
[3] P. Calabrese and J. Cardy, Journal of statistical mechanics: theory and experiment 2004, P06002 (2004).
[4] S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 181602 (2006).
[5] J. M. Deutsch, Phys. Rev. A 43, 2046 (1991).
[6] M. Srednicki, Phys. Rev. E 50, 888 (1994).
[7] L. Fleishman and P. W. Anderson, Phys. Rev. B 21, 2366 (1980).
[8] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Phys. Rev. Lett. 95, 206603 (2005).
[9] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Annals of Physics 321, 1126 (2006).
[10] V. Oganesyan and D. A. Huse, Phys. Rev. B 75, 155111 (2007).
[11] A. Pal and D. A. Huse, Phys. Rev. B 82, 174411 (2010).
[12] B. Bauer and C. Nayak, Journal of Statistical Mechanics: Theory and Experiment 2013, P09005 (2013).
[13] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Annals of Physics 321, 1126 (2006).
[14] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Phys. Rev. X 7, 031009 (2019).
[15] M. J. Gullans and D. A. Huse, Phys. Rev. X 10, 041020 (2020).
[16] C.-M. Jian, Y.-Z. You, R. Vasseur, and A. W. W. Ludwig, Phys. Rev. B 101, 104302 (2020).
[17] X. Cao, A. Tilloy, and A. D. Luca, SciPost Phys. 7, 24 (2019).
[18] Y. Fuji and Y. Ashida, Phys. Rev. B 102, 054302 (2020).
[19] O. Alberton, M. Buchhold, and S. Diehl, Phys. Rev. Lett. 126, 170602 (2021).
[20] T. Maimbourg, D. M. Basko, M. Holzmann, and A. Rosso, Phys. Rev. Lett. 126, 120603 (2021).
[21] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Phys. Rev. B 76, 052203 (2007).
[22] S. Gopalakrishnan, M. Müller, V. Khemani, M. Knap, E. Demler, and D. A. Huse, Phys. Rev. B 92, 104202 (2015).
[23] R. Nandkishore and D. A. Huse, Annual Review of Condensed Matter Physics 6, 15 (2015).
[24] K. Jacobs and D. A. Steck, Contemporary Physics 47, 279 (2006).
[25] H.-P. Breuer, F. Petruccione, et al., The theory of open quantum systems (Oxford University Press on Demand, 2002).
[26] L. M. Sieberer, M. Buchhold, and S. Diehl, Reports on Progress in Physics 79, 096001 (2016).
[27] A. Kamenev, Field Theory of Non-Equilibrium Systems (Cambridge University Press, 2011).
[28] M. L. Horbach and G. Schön, Annalen der Physik 505, 51 (1993).
[29] A. A. Patel, D. Chowdhury, S. Sachdev, and B. Swingle, Phys. Rev. X 7, 031047 (2017).
[30] Y. Minoguchi, A. Altland, and S. Diehl, Phys. Rev. X 11, 041004 (2021).
[31] D. Yang, C. Laflamme, D. V. Vasilyev, M. A. Baranov, and P. Zoller, Phys. Rev. Lett. 120, 133601 (2018).
[32] H. M. Wiseman and G. J. Milburn, Phys. Rev. A 47, 1652 (1993).
Supplementary Information for
“Keldysh Nonlinear Sigma Model for a Free-Fermion Gas under Unconditional Continuous Measurements”

Qinghong Yang\textsuperscript{1}, Yi Zuo\textsuperscript{2}, and Dong E. Liu\textsuperscript{1,2,3}
\textsuperscript{1}State Key Laboratory of Low Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing 100084, China
\textsuperscript{2}Beijing National Laboratory for Condensed Matter Physics, and Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
\textsuperscript{3}Beijing Academy of Quantum Information Sciences, Beijing 100193, China and Beijing National Laboratory for Condensed Matter Physics, Beijing 100190, China
\textsuperscript{3}Frontier Science Center for Quantum Information, Beijing 100184, China

In this supplementary information, we will show some details about how to map the Lindblad master equation to the Keldysh field theory in Sec. I. We will also briefly introduce the Keldysh treatment of the disordered fermionic system, and discuss the similarity of our problem with the disordered fermionic system in Sec. II. The derivation of the effective bosonic theory and the Keldysh nonlinear sigma model will be discussed in more detail in Sec. III.

I. FROM LINDBLAD MASTER EQUATION TO KELDYSH FIELD THEORY

In order to introduce the mapping between the Lindblad master equation and the Keldysh field theory, we consider a trivial one-site case. The detailed procedure can be found in Ref. [1], and here we focus mostly on the differences: 1) We introduce a method to make the continuum limit mathematically rigorous; 2) We show that in order to preserve the normalization condition, one should retain the $t = t'$ contribution in the bare Green’s function.

The Hamiltonian of the trivial one-site model reads $H = \mu c^\dagger c$, where $\mu$ is the on-site energy and can be regarded as the chemical potential. The projective quantum jump operator is the particle number operator $c^\dagger c$. Thus, the Lindblad master equation describing the evolution under the Hamiltonian and the unconditional continuous projective measurements can be expressed as

$$\dot{\rho} = -i[H, \rho] + \gamma \left( c^\dagger c \rho c - \frac{1}{2} \{c^\dagger c, \rho\} \right).$$

This equation can be formally expressed as $\rho_{t_f} = \lim_{N \to \infty} (1 + \delta t \cdot \mathcal{L})^N \rho_0$, where we have divide the time interval into $N$ slices, and $\mathcal{L}$ is Liouvillian superoperator, which is defined as

$$\mathcal{L}(\rho) = -i[H, \rho] + \gamma \left( c^\dagger c \rho c^\dagger - \frac{1}{2} \{c^\dagger c, \rho(t)\} \right).$$

Based on the recursion equation $\rho_{n+1} = (1 + \delta t \cdot \mathcal{L}) \rho_n$, one can get the final state $\rho_{t_f}$. In order to get the path integral based on the fermionic coherent state, we should first expand the density matrix in the fermionic coherent basis. Thus, we have

$$\rho_n = \int d\bar{\psi}_{+, n} d\psi_{+, n} d\bar{\psi}_{-, n} d\psi_{-, n} e^{-\bar{\psi}_{+, n} \psi_{+, n}} e^{-\bar{\psi}_{-, n} \psi_{-, n}} \langle \psi_{+, n} | \rho_n | - \psi_{-, n} \rangle \langle - \psi_{-, n} | \psi_{+, n} \rangle,$$

where $|\psi\rangle$ is the fermionic coherent state, and $\bar{\psi}_+, \bar{\psi}_-$ are independent Grassmann numbers. We also have

$$\langle \psi_{+, n+1} | \rho_{n+1} | - \psi_{-, n+1} \rangle$$

$$= \int d\bar{\psi}_{+, n} d\psi_{+, n} d\bar{\psi}_{-, n} d\psi_{-, n} e^{\left( \bar{\psi}_{+, n+1} - \bar{\psi}_{+, n} \right) \psi_{+, n} + \bar{\psi}_{-, n} (\psi_{-, n+1} - \psi_{-, n})} \langle \psi_{+, n} | \rho_n | - \psi_{-, n} \rangle$$

$$+ \delta t \int d\bar{\psi}_{+, n} d\psi_{+, n} d\bar{\psi}_{-, n} d\psi_{-, n} e^{-\bar{\psi}_{+, n} \psi_{+, n}} e^{-\bar{\psi}_{-, n} \psi_{-, n}} \langle \psi_{+, n+1} | \mathcal{L}(\psi_{+, n}) (- \psi_{-, n}) \rangle - \psi_{-, n+1} \rangle \langle - \psi_{-, n} | \psi_{+, n} | \rho_n | - \psi_{-, n} \rangle.$$
coherent basis, and we have
\[
\text{tr}(\rho_{n+1}) = \int d\bar{\psi}_{+,n} d\psi_{+,n} e^{\bar{\psi}_{-,n+1} \psi_{+,n+1} + e^{-\bar{\psi}_{-,n+1} \psi_{+,n+1}}} e^{-\bar{\psi}_{-,n} \psi_{-,n+1} + e^{-\bar{\psi}_{-,n} \psi_{-,n+1}}} e^{\bar{\psi}_{+,n+1} \psi_{+,n} + e^{-\bar{\psi}_{+,n+1} \psi_{+,n}}} \psi_{+,n} e^{\bar{\psi}_{-,n} (\psi_{-,n+1} - \psi_{-,n})} \\
\times \langle \psi_{+,n} | \rho_n | -\psi_{-,n} \rangle \\
+ \int d\bar{\psi}_{+,n} d\psi_{+,n} d\bar{\psi}_{-,n} d\psi_{-,n} e^{\bar{\psi}_{-,n+1} \psi_{+,n+1} + e^{-\bar{\psi}_{-,n+1} \psi_{+,n+1}}} e^{-\bar{\psi}_{-,n} \psi_{-,n+1} + e^{-\bar{\psi}_{-,n} \psi_{-,n+1}}} e^{\bar{\psi}_{+,n+1} \psi_{+,n} + e^{-\bar{\psi}_{+,n+1} \psi_{+,n}}} \psi_{+,n} e^{\bar{\psi}_{-,n} (\psi_{-,n+1} - \psi_{-,n})} \\
\times \delta_t \left\{ -i \left[ H (\bar{\psi}_{+,n+1}, \psi_{+,n}) - H (\bar{\psi}_{-,n}, \psi_{-,n-1}) \right] + \gamma \bar{\psi}_{+,n+1} \psi_{+,n} \psi_{-,n+1} - \frac{1}{2} \gamma \left( \bar{\psi}_{+,n+1} \psi_{+,n} + \psi_{-,n} \psi_{-,n+1} \right) \right\} \\
\times \langle \psi_{+,n} | \rho_n | -\psi_{-,n} \rangle.
\]
(5)

In Ref. [1], in the continuum limit, \( \bar{\psi}_{+,n+1} \psi_{+,n} \psi_{-,n+1}, \bar{\psi}_{+,n+1} \psi_{+,n}, \) \( \psi_{-,n} \psi_{+,n} \) are directly set to \( \psi_+ (t) \psi_+ (t) \psi_- (t) \psi_- (t), \psi_+ (t) \psi_+ (t), \) and \( \psi_- (t) \psi_- (t), \) respectively. Here, in order to make the continuum limit rigorous, we make those Grassmann numbers of the dissipation part be at the time argument following the procedure:

\[
\int d\bar{\psi}_{+,n} d\psi_{+,n} d\bar{\psi}_{-,n} d\psi_{-,n} e^{\bar{\psi}_{-,n} (\psi_{+,n+1} - \psi_{-,n})} \psi_{-,n} \psi_{+,n+1} \\
= \int d\bar{\psi}_{+,n} d\psi_{+,n} d\bar{\psi}_{-,n} d\psi_{-,n} e^{\bar{\psi}_{-,n} (\psi_{+,n+1} - \psi_{-,n})} \psi_{-,n} (\psi_{+,n+1} - \psi_{-,n}) \\
= \int d\bar{\psi}_{+,n} d\psi_{+,n} d\bar{\psi}_{-,n} d\psi_{-,n} \left[ \frac{\delta}{\delta \psi_{-,n}} e^{\bar{\psi}_{-,n} (\psi_{+,n+1} - \psi_{-,n})} \right] \psi_{-,n} + \int d\bar{\psi}_{+,n} d\psi_{+,n} d\bar{\psi}_{-,n} d\psi_{-,n} e^{\bar{\psi}_{-,n} (\psi_{+,n+1} - \psi_{-,n})} \psi_{-,n} \psi_{-,n} \\
= \int d\bar{\psi}_{+,n} d\psi_{+,n} d\bar{\psi}_{-,n} d\psi_{-,n} e^{\bar{\psi}_{-,n} (\psi_{+,n+1} - \psi_{-,n})} (\psi_{-,n} \psi_{-,n} + 1).
\]

(6)

\( \bar{\psi}_{+,n+1} \psi_{+,n} \psi_{-,n} \) and \( \psi_{+,n} \psi_{+,n+1} \psi_{+,n} \) can be treated in the same way.

Therefore, we have
\[
Z = \frac{1}{\text{tr} (\rho_0)} \int d\bar{\psi}_{+,0} d\psi_{+,0} d\bar{\psi}_{-,0} d\psi_{-,0} \left[ \begin{array}{cccc} -1 & 0 & 0 & -\rho \\ h_- & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & h_+ & -1 \end{array} \right] \left[ \begin{array}{c} \psi_{+,0} \\ \psi_{+,1} \\ \psi_{-,1} \\ \psi_{-,0} \end{array} \right] + \gamma \delta_t \left[ \begin{array}{c} \psi_{+,0} \psi_{+,0} \psi_{-,0} \psi_{-,0} \\ \psi_{+,1} \psi_{+,1} \\ \psi_{-,1} \psi_{-,1} \\ \psi_{-,0} \psi_{-,0} \end{array} \right] + \frac{1}{2} \gamma \left( \bar{\psi}_{+,0} \psi_{+,0} + \psi_{-,0} \psi_{-,0} \right) \\
\times \exp \left\{ -i G^{-1} \right\}.
\]
(7)

where we choose \( N = 1 \) for simplicity, \( h_\mp = 1 \mp i \mu \delta_t, \rho = \langle \psi_{+,0} | \rho_0 | -\psi_{-,0} \rangle, \) and the initial state is chosen to be an exponential form, such as a thermal state. One finds that after doing the treatment shown in Eq. (6), the sign before the factor \( 1/2 \) in the dissipation term is changed (see Eq. (6) and Eq. (7)). Note that the dissipation part (the second term of the second line in Eq. (7)) depends on the same time argument, thus one can directly take the continuum limit and this procedure is mathematically rigorous now. The Keldysh-Larkin-Ovchinnikov (KLO) transformation [2] leads Eq. (7) to
\[
Z = \frac{1}{\text{tr} (\rho_0)} \int d\bar{\psi}_{1,0} d\psi_{1,0} d\bar{\psi}_{2,0} d\psi_{2,0} \exp \left\{ -i \bar{\Psi} \left( -i G^{-1} \right) \Psi + \gamma \delta_t \left[ -\bar{\psi}_{1,0} \psi_{1,0} \psi_{2,0} \psi_{2,0} + \frac{1}{2} (\bar{\psi}_{1,0} \psi_{2,0} + \psi_{2,0} \psi_{2,0}) \right] \right\},
\]
(8)

where \( \bar{\Psi} = [\bar{\psi}_{1,0} \bar{\psi}_{2,0} \bar{\psi}_{2,0} \bar{\psi}_{2,0}], \Psi = [\psi_{1,0} \psi_{1,0} \psi_{2,0} \psi_{2,0}]^T, \psi_{a,j} \) is the Grassmann number after the KLO transformation with \( a \in \{1, 2\} \) being the Keldysh indices and \( j \in \{0, 1\} \) being the discrete time indices, and
\[
-i G^{-1} = -\frac{1}{2} \left[ \begin{array}{cccc} -\rho & -h_- & h_+ & -2 + \rho \\ h_- & -1 & -3 & h_- \\ h_- & -1 & 1 & h_- \\ -2 - \rho & h_+ & -h_+ & \rho \end{array} \right].
\]
(9)
The bare Green’s function in the discrete time version reads
\[
i\hat{G} = \left( -i\hat{G}^{-1} \right)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1-\rho}{1+\rho} & \frac{1-\rho}{1+\rho} \\ \frac{1}{2} & 0 & \frac{1-\rho}{1+\rho} & \frac{1-\rho}{1+\rho} \\ 0 & \frac{1}{2} & \frac{1-\rho}{1+\rho} & \frac{1-\rho}{1+\rho} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

In the continuum limit, Eq. (8) can be expressed as
\[
i\hat{G}(t, t') = \left[ i G^R_0(t, t') \right] + i G^K_0(t, t') + \frac{1}{2} \left[ 0 \ 1 \right] \delta_{t, t'},
\]
where \(\delta_{t, t'}\) should be interpreted as the Kronecker symbol. In the standard Keldysh field theory [2], people usually omit the term proportional to \(\delta_{t, t'}\) in Eq. (11), and only keep the first term of Eq. (11). In this formalism, we need to preserve the normalization condition \(Z = 1\), which can be checked by expanding Eq. (8) in powers of \(\gamma\), and treat each order with the help of Wick’s theorem. In our problem here, one will find that this \(\delta_{t, t'}\) term has to be kept so as to preserve the normalization, as one will encounter the equal-time correlation: \(\langle \bar{\psi}_2(t)\psi_1(t) \rangle\).

Generalizing to the model considered in the main text, one can obtain the Keldysh Lindblad partition function Eq. (3) of the main text.

II. THE KELDYSH TREATMENT OF THE DISORDERED FERMIONIC SYSTEM AND THE SIMILARITY WITH OUR PROBLEM

The Keldysh treatment of the disordered fermionic system can be found in Ref. [2, 3], and here we just quote some discussions connected with our problem.

In the traditional studying of the disordered fermionic system or the weak localization effect, one usually assume a static and spatial-dependent disorder potential \(V_{dis}(x)\) through the disorder action
\[
S_{dis}[V_{dis}] = \int dx V_{dis}(x) \bar{\psi}_a(x) \gamma^a \psi_b(x),
\]
where the configuration of \(V_{dis}(x)\) satisfies the Gaussian distribution and thus the disorder averaging takes the form
\[
\langle \cdots \rangle = \int \mathcal{D}[V_{dis}] \exp \left\{ -\pi \nu \tau_{el} \int dx V^2_{dis}(x) \right\} \cdots,
\]
where \(\tau_{el}\) is the elastic scattering time. Performing the disorder averaging for \(\exp i S_{dis}\), one can get
\[
\langle e^{iS_{dis}} \rangle_{dis} = \int \mathcal{D}[V_{dis}] \exp \left\{ -\int dx \pi \nu \tau_{el} V^2_{dis}(x) + i V_{dis}(x) \int dt \bar{\psi}_a(x, t) \gamma^a \psi_b(x, t) \right\},
\]
\[
= \exp \left\{ -\frac{1}{4\pi \nu \tau_{el}} \int dx \int dt dt' \bar{\psi}_a(x, t) \psi_a(x, t) \bar{\psi}_b(x, t') \psi_b(x, t') \right\}.
\]

And then, the partition function after disorder averaging reads
\[
Z = \int \mathcal{D}[\psi] \exp \left\{ iS_0 - \frac{1}{4\pi \nu \tau_{el}} \int dx \int dt dt' \bar{\psi}_a(x, t) \psi_a(x, t) \bar{\psi}_b(x, t') \psi_b(x, t') \right\},
\]
where \(S_0\) is the free-fermion action. Note that the disorder averaging introduce a four-fermion term into the action.

For convenience, we also put the Keldysh Lindblad partition function of our problem here:
\[
Z = \int \mathcal{D}[\psi] \exp \left\{ iS_0 - \frac{\gamma}{2} \int dx dt [\bar{\psi}_a(x, t) \psi_a(x, t) \bar{\psi}_b(x, t) \psi_b(x, t) - \bar{\psi}_a(x, t) \gamma_1^a \psi_b(x, t)] \right\}.
\]

Comparing these two equations, Eq. (15) and Eq. (16), one can observe that the two four-fermion terms are in a similar form. Thus, in some sense, these two different problems are unified in the framework of the functional Keldysh field theory.
III. BOSONIC EFFECTIVE THEORY AND THE KELDYSH NONLINEAR SIGMA MODEL

A. Bosonic Effective Theory

Following the procedure introduced in Sec. I, one can get the Keldysh Lindblad partition function for our problem:

\[
Z = \int \mathcal{D}[\hat{\psi}] \exp \left\{ i S_0 - \frac{\gamma}{2} \int dx \left[ \bar{\psi}_a(x) \psi_a(x) \bar{\psi}_b(x) \psi_b(x) - \bar{\psi}_a(x) \hat{Q}^{ab} \psi_b(x) \right] \right\},
\]

where \(S_0\) is the free-fermion action. For the four-fermion term in the dissipation part, we introduce an auxiliary matrix field \(\hat{Q}\) to decouple it with the help of the identity

\[
i = \int \mathcal{D} [\hat{Q}] \exp \left[ -\frac{\gamma}{2} (\pi \nu)^2 \text{tr} \left( \hat{Q}^2 \right) \right] = \int \mathcal{D} [\hat{Q}] \exp \left[ -\frac{\gamma}{2} (\pi \nu)^2 \int dx \hat{Q}^{ab}(x) \hat{Q}^{ba}(x) \right].
\]

And one arrives at

\[
Z = \int \mathcal{D} [\hat{Q}] \mathcal{D}[\psi] \exp \left\{ i S_0 - \frac{\gamma}{2} (\pi \nu)^2 \text{tr} \left( \hat{Q}^2 \right) - \gamma \int dx \left[ \pi \nu \bar{\psi}_a(x) \hat{Q}^{ab}(x) \psi_b(x) - \frac{1}{2} \bar{\psi}_a(x) \hat{Q}^{ab} \psi_b(x) \right] \right\}.
\]

Using the Gaussian integration, one arrives at the effective bosonic theory depending only on \(\hat{Q}\):

\[
Z = \int \mathcal{D} [\hat{Q}'] \exp \left\{ -\frac{\gamma}{2} (\pi \nu)^2 \text{tr} \left[ \left( \hat{Q}' + \frac{1}{2 \pi \nu} \hat{\tau}_1 \right)^2 \right] + \text{tr} \ln \left[ -i \hat{G}^{-1} + \gamma \pi \nu \hat{Q}' \right] \right\},
\]

where we have let \(\hat{Q}' = \hat{Q} - \frac{1}{2 \pi \nu} \hat{\tau}_1\), and \(\hat{Q}'\) is still Hermitian. In the following, we will relabel \(\hat{Q}'\) as \(\hat{Q}\) again. For higher orders (\(\geq 2\)) of the expansion in powers of \(\gamma\), we can replace \(\hat{G}\) with \(\hat{G}_0\) (the first term in Eq. (11)) due to the fact that the \(t = t'\) line is only a manifold of measure zero [2]. Then, one gets the Keldysh Lindblad Partition function shown in Eq. (5) of the main text:

\[
Z = \int \mathcal{D} \left[ \hat{Q} \right] \exp \left\{ -\frac{\gamma}{2} (\pi \nu)^2 \text{tr} \left[ \left( \hat{Q} + \frac{1}{2 \pi \nu} \hat{\tau}_1 \right)^2 \right] + \text{tr} \ln \left[ -i \hat{G}_0^{-1} + \gamma \pi \nu \hat{Q} \right] \right\}.
\]

Note that for \(\gamma \ll \epsilon_F\) (this condition is also used to deriving the saddle point), this result is also valid up to the first order of \(\gamma\).

B. Keldysh Nonlinear Sigma Model

Taking the variation over \(\hat{Q}(x)\), one gets the saddle point equation of the action in Eq. (21), and one can check that the constant configuration \(\hat{\Lambda} = \frac{1}{2 \pi \nu} \hat{\tau}_3\) satisfies the saddle point equation. For the low-energy physics, we just focus on the massless fluctuation, which can be generated by \(\hat{\Lambda} = \hat{R}^{-1}(x) \hat{\Lambda} \hat{R}(x)\). Note that now \(\hat{Q}(x)\) is constrained by \(\hat{Q}^2(x) = (\frac{1}{2 \pi \nu})^2 \hat{\tau}_0\). And then one finds that only the \(\text{tr} \ln\) term in Eq. (21) will contribute to the dynamics, while other terms only contribute some constants. Thus, in the following, we can just focus on the \(\text{tr} \ln\) term.

Note that the bare Green’s function \(\hat{G}_0\) can be expressed as \(\hat{G}_0 = \hat{U}^{-1} \hat{G}_{od} \hat{U}\), where

\[
\hat{U}^{-1} = \hat{U} = \sum_{\epsilon} \begin{pmatrix} F_\epsilon & \rho_{\epsilon} \\ \rho_{\epsilon} & 0 \end{pmatrix} |\epsilon\rangle \langle \epsilon|, \quad \hat{G}_{od} = \sum_{k, \epsilon} \begin{pmatrix} 0 & G^R_0(k, \epsilon) \\ G'_0(k, \epsilon) & 0 \end{pmatrix}, \quad |k, \epsilon\rangle |k, \epsilon|,
\]

and \(F_\epsilon = 1 - 2n_F(\epsilon)\) with \(n_F(\epsilon)\) being the Fermi-Dirac distribution function. Thus, the statistical information is encoded in the matrix \(\hat{U}\). We would like to obtain an effective low-energy theory depending only on \(\hat{Q}\) to describe the physics of our problem. To this end, we first make a similarity transformation to encode the statistical information in \(\hat{Q}\) instead. Note that due to the cyclic property of the trace operation, this similarity does not change the theory.
And then the tr ln term in Eq. (21) now becomes \( \text{tr} \ln \left[ -i \hat{G}_{0d}^{-1} + (\gamma/2) \hat{Q} \right] \), where \( \hat{Q} \) is redefined as \( \hat{Q} = \hat{U}^{-1} \hat{R}^{-1} \hat{\tau}_3 \hat{R} \hat{U} \), and in the spacetime basis, \( \hat{G}_{0d}^{-1} = i \partial_t + \frac{\nabla^2}{2m} + \epsilon_F + i 0 \hat{\tau}_3 \). Therefore, we have

\[
iS[\hat{Q}] = \text{tr} \ln \left[ -i \hat{G}_{0d}^{-1} + (\gamma/2) \hat{U}^{-1} \hat{R}^{-1} \hat{\Lambda} \hat{R} \hat{U} \right]
= \text{tr} \ln \left\{ -i \hat{R} \left( i \partial_t + \frac{\nabla^2}{2m} + \epsilon_F \right) \hat{R}^{-1} - \hat{U}^{-1} i 0 \hat{\tau}_3 \hat{U} \right\} + \frac{\gamma}{2} \hat{\tau}_3 \}
\approx \text{tr} \ln \left[ \hat{G}^{-1} + i \hat{U}^{-1} \hat{R} \left( \partial_t \hat{R}^{-1} \right) \hat{U} + i \hat{U}^{-1} \hat{R} \left( v_F \cdot \nabla \hat{R}^{-1} \right) \hat{U} \right],
\]

where \( \hat{G}^{-1} = i \partial_t + \frac{\nabla^2}{2m} + \epsilon_F + i 2 \hat{U} \hat{\tau}_3 \hat{U} \), and \( v_F \cdot \nabla \) comes from the linearization of the dispersion relation near the Fermi energy: \( k^2/(2m) - \epsilon_F \approx v_F \cdot k \rightarrow -iv_F \cdot k \). Note that the saddle point configuration \( \propto \hat{\tau}_3 \) plays the role of the self-energy. In the energy-momentum basis, we have

\[
\hat{G}(k, \epsilon) = \hat{U}_\epsilon \begin{bmatrix} 1 & 0 \\ \frac{-\epsilon - i \gamma/2}{\xi_k + i \gamma/2} & \frac{1}{\epsilon - \xi_k - i \gamma/2} \end{bmatrix} \hat{U}_\epsilon,
\]

where \( \xi_k = k^2/(2m) - \epsilon_F \). Expanding the tr ln term in powers of \( \partial_t \hat{R}^{-1} \) and \( \nabla \hat{R}^{-1} \), one will arrives at the modified Keldysh nonlinear sigma model

\[
iS[\hat{Q}] = \pi \nu \text{tr} \left[ \partial_t \hat{Q} \right] - \frac{1}{4} \pi \nu \text{D} \text{tr} \left[ \left( \nabla \hat{Q} \right)^2 \right],
\]

where \( \partial_t \hat{Q} \equiv \partial_t (\hat{U} \hat{R}^{-1} \hat{\tau}_3 \hat{R} \hat{U}) \). A similar calculation can be found in Chapter 11 of Ref. [2]. The Keldysh nonlinear sigma model in the presence of the vector potential can be derived from those similar calculations.

[1] L. M. Sieberer, M. Buchhold, and S. Diehl, Reports on Progress in Physics 79, 096001 (2016).
[2] A. Kamenev, Field Theory of Non-Equilibrium Systems (Cambridge University Press, 2011).
[3] M. L. Horbach and G. Schönh, Annalen der Physik 505, 51 (1993).