The spectrum of quantum-group-invariant transfer matrices

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Abstract

Integrable open quantum spin-chain transfer matrices constructed from trigonometric R-matrices associated to affine Lie algebras \( \hat{g} \), and from certain K-matrices (reflection matrices) depending on a discrete parameter \( p \), were recently considered in \texttt{arXiv:1802.04864} and \texttt{arXiv:1805.10144}. It was shown there that these transfer matrices have quantum group symmetry corresponding to removing the \( p^{th} \) node from the \( \hat{g} \) Dynkin diagram. Here we determine the spectrum of these transfer matrices by using analytical Bethe ansatz, and we determine the dependence of the corresponding Bethe equations on \( p \). We propose formulas for the Dynkin labels of the Bethe states in terms of the numbers of Bethe roots of each type. We also briefly study how duality transformations are implemented on the Bethe ansatz solutions.

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1 Introduction

Several infinite families of integrable open quantum spin chains with quantum group (QG) symmetry have recently been identified [1, 2]. The transfer matrices for these models are constructed [3] from trigonometric R-matrices, which are associated to non-exceptional affine Lie algebras $\hat{g}$ [4, 5, 6, 7], and from certain K-matrices (also known as reflection matrices, or boundary S-matrices) depending on a discrete parameter $p$ [8, 9, 10, 11, 12, 13, 2]. These transfer matrices have QG symmetry corresponding to removing the $p^{th}$ node from the $\hat{g}$ Dynkin diagram, as summarized in Tables 2 and 3.

The main aim of this paper is to determine the spectrum of these transfer matrices. This work can be regarded as a generalization of the well-known work by Reshetikhin [14], who solved the corresponding problem for closed spin chains with periodic boundary conditions, which however do not have QG symmetry. Following [14], we use analytical Bethe ansatz to determine the eigenvalues of the transfer matrices and the corresponding Bethe equations. We expect that the Bethe states are highest/lowest weights of the quantum groups, which leads to formulas for the Dynkin labels of the Bethe states in terms of the numbers of Bethe roots of each type. Analogous formulas have been known for integrable closed spin chains constructed from rational R-matrices, which have classical (Lie group) symmetries, see e.g. [15]. From knowledge of the Dynkin labels of an irreducible representation, one can determine its dimension, which in turn helps determine the degeneracy of the corresponding transfer-matrix eigenvalue.

The K-matrices that we consider here, which do not depend on continuous boundary parameters, presumably correspond to conformal boundary conditions. Hence, these models may have interesting applications to boundary critical phenomena, see e.g. [16, 17].

The outline of this paper is as follows. The construction of the transfer matrix and key results from [1, 2] are briefly reviewed in Sec. 2. Expressions for the eigenvalues of the transfer matrix and corresponding Bethe equations are obtained in Sec. 3. Formulas for the Dynkin labels of the Bethe states (in terms of the numbers of Bethe roots of each type) are obtained and illustrated with some examples in Sec. 4. We briefly study how duality transformations are implemented on the Bethe ansatz solutions in Sec. 5. Some interesting open problems are listed in Sec. 6. A connection between “bonus” symmetry and singular solutions of the Bethe equations is noted in Appendix A, and some additional cases are considered in Appendix B.

2 Review of previous results

We briefly review here the construction of the transfer matrix (whose main ingredients are the R-matrix and the K-matrices) and its symmetries.
2.1 R-matrix

The R-matrix $R(u)$, which encodes the bulk interactions, is a solution of the Yang-Baxter equation

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v), \quad (2.1)$$

where $R_{12} = R \otimes I$, $R_{23} = I \otimes R$ and $R_{13} = \mathcal{P}_{12} R_{23} \mathcal{P}_{12}$; moreover, $I$ is the identity matrix and $\mathcal{P}$ is the permutation matrix. We consider here the trigonometric R-matrices given by Jimbo [6] (except for $A_{2n-1}^{(2)}$, in which case we consider instead Kuniba’s R-matrix [7]), corresponding to the following non-exceptional affine Lie algebras

$$\hat{g} = \{ A_{2n-1}^{(2)}, A_{2n}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)} \}. \quad (2.2)$$

We use the specific expressions for the R-matrices given in Appendix A of [1] and (for $D_{n+1}^{(2)}$) Appendix A of [21], where the anisotropy parameter is denoted by $\eta$. We emphasize that, as in [1, 2], we consider here exclusively generic values of $\eta$. Various useful parameters related to these R-matrices are collected in Table 1. In particular, $d$ is the dimension of the vector space at each site of the spin chain; hence, the R-matrix is a $d^2 \times d^2$ matrix. Also, $\delta = 0$ ($\delta = 2$) for the untwisted (twisted) cases, respectively.

$$\begin{array}{cccccccc}
\hat{g} & A_{2n-1}^{(2)} & A_{2n}^{(2)} & B_n^{(1)} & C_n^{(1)} & D_n^{(1)} & D_{n+1}^{(2)} \\
\hline
d & 2n & 2n + 1 & 2n + 1 & 2n & 2n & 2n + 2 \\
k & 2n & 2n + 1 & 2n - 1 & 2n + 2 & 2n - 2 & 2n \\
\rho & -2\kappa \eta - i\pi & -2\kappa \eta - i\pi & -2\kappa \eta & -2\kappa \eta & -2\kappa \eta & -\kappa \eta \\
\omega & \kappa + 2 & \kappa - 2 & \kappa + 2 & \kappa - 2 & \kappa + 2 & \kappa \\
\bar{\omega} & \kappa - 2 & \kappa + 2 & \kappa - 2 & \kappa + 2 & \kappa - 2 & \kappa \\
\delta & 2 & 2 & 0 & 0 & 0 & 2 \\
\xi & 1 & 0 & 0 & 1 & 0 & 0 \\
\xi' & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{array}$$

Table 1: Parameters related to the R-matrices.

2.2 K-matrices

The right $(K^R(u))$ and left $(K^L(u))$ K-matrices, which encode the boundary conditions on the right and left ends of the spin chain, respectively, are solutions of the boundary Yang-Baxter equations [22, 23, 3, 24]

$$R_{12}(u-v) K_1^R(u) R_{21}(u+v) K_2^R(v) = K_2^R(v) R_{21}(u+v) K_1^R(u) R_{12}(u-v), \quad (2.3)$$

\footnote{We do not consider here the case $A_n^{(1)}$, since this R-matrix does not have crossing symmetry for $n > 1$. This case has been studied in a similar context in [18, 19, 20].}
and

\[ R_{12}(-u + v) K_1^{L_1}(u) M_1^{-1} R_{21}(-u - v - 2\rho) M_1 K_2^{L_2}(v) = K_2^{L_2}(v) M_1 R_{12}(-u - v - 2\rho) M_1^{-1} K_1^{L_1}(u) R_{21}(-u + v), \]

respectively. The crossing parameter \( \rho \) is given in Table 1, and the matrix \( M \) can be found in [1] and (for \( D^{(2)}_{n+1} \)) [21].

For all the cases except \( D^{(2)}_{n+1} \), we take for the right K-matrices the following \( d \times d \) diagonal matrices [9, 11, 12, 13]

\[ K^R(u, p) = \text{diag} \left( e^{-u}, \ldots, e^{-u}, \frac{\gamma e^u + 1}{\gamma + e^u}, \ldots, e^u, \ldots, e^u \right), \]

where \( p \) is a discrete parameter taking the values

\[ p = 0, \ldots, n \quad \text{for} \quad A^{(2)}_{2n} , C^{(1)}_n , \]
\[ p = 0, \ldots, n , \quad p \neq 1 , \quad \text{for} \quad A^{(2)}_{2n-1} , B^{(1)}_n , \]
\[ p = 0, \ldots, n , \quad p \neq 1 , n - 1 , \quad \text{for} \quad D^{(1)}_n . \]

Moreover, \( \gamma \) is defined by

\[ \gamma = \begin{cases} 
\gamma_0 e^{(4p-2)\eta + \frac{1}{2}\rho} & \text{for} \quad A^{(2)}_{2n-1} , B^{(1)}_n , D^{(1)}_n \\
\gamma_0 e^{(4p+2)\eta + \frac{1}{2}\rho} & \text{for} \quad A^{(2)}_{2n} , C^{(1)}_n 
\end{cases} , \]

where \( \gamma_0 \) is another discrete parameter

\[ \gamma_0 = \pm 1 . \]

It is convenient to define the corresponding parameter

\[ \varepsilon = \frac{1 - \gamma_0}{2} , \]

which therefore can take the values \( \varepsilon = 0, 1 \).

Note that in (2.6) (as well as in [1]) the following cases are excluded:

\[ A^{(2)}_{2n-1} \quad \text{with} \quad p = 1 , \]
\[ B^{(1)}_n \quad \text{with} \quad p = 1 , \]
\[ D^{(1)}_n \quad \text{with} \quad p = 1 , n - 1 . \]

For these cases, to which we henceforth refer as “special” cases, we take instead the following right K-matrices:

\[ K^R(u, 1) = \text{diag}(e^{-2u}, 1, \ldots, 1, e^{2u}) \]
for $A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}$ with $p = 1$; and

$$K^R(u, n-1) = \text{diag}(e^{-u}, \ldots, e^{-u}, e^u, \ldots, e^u)$$

(2.12)

for $D_n^{(1)}$ with $p = n-1$. We choose these K-matrices because they lead to QG symmetry, as explained in Sec. 2.4.

For the case $D_{n+1}^{(2)}$, the right K-matrix is given by the $d \times d$ block-diagonal matrix

$$K^R(u, p) = \begin{pmatrix}
    k_-(u) I_{p \times p} & g(u) I_{(n-p) \times (n-p)} & & & \\
    & k_1(u) & k_2(u) & & \\
    & k_2(u) & k_1(u) & & \\
    & & & g(u) I_{(n-p) \times (n-p)} & \\
    & & & & k_+(u) I_{p \times p}
\end{pmatrix},$$

(2.13)

where

$$k_\pm(u) = e^{\pm 2u},$$

$$g(u) = \frac{\cosh(u-n-2p) \eta + \frac{i \pi}{2} \varepsilon}{\cosh(u+n-2p) \eta - \frac{i \pi}{2} \varepsilon},$$

$$k_1(u) = \frac{\cosh(u) \cosh((n-2p) \eta + \frac{i \pi}{2} \varepsilon)}{\cosh(u+n-2p) \eta + \frac{i \pi}{2} \varepsilon},$$

$$k_2(u) = -\frac{\sinh(u) \sinh((n-2p) \eta + \frac{i \pi}{2} \varepsilon)}{\cosh(u+n-2p) \eta + \frac{i \pi}{2} \varepsilon}. $$

(2.14)

and $\varepsilon$ is, again, a discrete parameter that can take the values $\varepsilon = 0, 1$.

Finally, for the left K-matrices, we take

$$K^L(u, p) = K^R(-u-p, p) M,$$

(2.15)

which is a solution of left boundary Yang-Baxter equation (2.4), and corresponds to imposing the “same” boundary conditions on the two ends.

### 2.3 Transfer matrix

The transfer matrix for an integrable open spin chain with $N$ sites is given by

$$t(u, p) = \text{tr}_a K^L_a(u, p) T_a(u) K^R_a(u, p) \hat{T}_a(u),$$

(2.16)

$^2$The $D_{n+1}^{(2)}$ K-matrices $K^R(u, n)$ (i.e. with $p = n$) with $\varepsilon = 0$ and $\varepsilon = 1$ are proportional to the $D_{n+1}^{(2)}$ K-matrices $K^-(u)$ in [21] for the cases I and II, respectively; explicitly, $K^L_{21, II}(u) = -2e^{2u+n\eta} \cosh(u-n\eta + \frac{i \pi}{2} \varepsilon) K^R(u, n)$. 
where the single-row monodromy matrices are defined by
\[
T_a(u) = R_{aN}(u) R_{aN-1}(u) \cdots R_{a1}(u), \\
\hat{T}_a(u) = R_{1a}(u) \cdots R_{N-1a}(u) R_{Na}(u),
\] (2.17)
and the trace in (2.16) is over the “auxiliary” space, which is denoted by \(a\). The transfer matrix satisfies the fundamental commutativity property
\[
[t(u, p), t(v, p)] = 0 \text{ for all } u, v,
\] (2.18)
and contains the Hamiltonian \(\mathcal{H}(p) \sim t'(0, p)\) as well as higher local conserved quantities. The transfer matrix is also crossing invariant
\[
t(u, p) = t(-u - \rho, p),
\] (2.19)
where the crossing parameter \(\rho\) is given in Table 1.

### 2.4 Symmetries of the transfer matrix

It has been shown in [1, 2] that the transfer matrices (2.16) constructed using the K-matrices (2.5) and (2.13) have the QG symmetries in Table 2. For \(0 < p < n\), the QG symmetries are given by a tensor product of two factors, to which we refer as the “left” and “right” factors. For \(p = 0\), the “right” factors are absent; while for \(p = n\), the “left” factors are absent. That is,
\[
\begin{align*}
\Delta_N(H^{(l)}_i(p)), t(u, p) & = 0, & i = 1, \ldots, n - p, \\
\Delta_N(H^{(r)}_i(p)), t(u, p) & = 0, & i = 1, \ldots, p,
\end{align*}
\] (2.20)
where $H_i^{(l)}(p), E_{i}^{\pm (l)}(p)$ are generators of the “left” algebra $g^{(l)}$; $H_i^{(r)}(p), E_{i}^{\pm (r)}(p)$ are generators of the “right” algebra $g^{(r)}$; and $\Delta_N$ denotes the $N$-fold coproduct.

It can be shown in a similar way that the transfer matrices for the “special” cases (2.10), which are constructed using the K-matrices (2.11)-(2.12), have the QG symmetries in Table 3.

| $\hat{g}$                  | QG symmetry       | Representation at each site |
|----------------------------|-------------------|----------------------------|
| $A_{2(n-1)}^{(2)}(p=1)$   | $U_q(C_n)$        | $2n$                       |
| $B_n^{(1)}(p=1)$          | $U_q(B_n)$        | $2n+1$                     |
| $D_n^{(1)}(n>1, p=1, n-1)$| $U_q(D_n)$        | $2n$                       |

Table 3: QG symmetries of the transfer matrix $t(u,p)$ for the “special” cases (2.10).

The QG symmetries displayed in Tables 2 and 3 correspond to removing the $p^{th}$ node from the $\hat{g}$ Dynkin diagram, as can be seen in Fig. 1.

For the cases $C_n^{(1)}, D_n^{(1)}$ and $D_{n+1}^{(2)}$ (i.e., the last three rows of Table 2), the transfer matrices have a $p \leftrightarrow n-p$ duality symmetry

$$U t(u,p) U^{-1} = f(u,p) t(u,n-p) ,$$

see [1, 2] for explicit expressions for the quantum-space operator $U$ and the scalar factor $f(u,p)$. In particular, for $p = \frac{n}{2}$ (n even), the transfer matrix is self-dual

$$[U, t(u, \frac{n}{2})] = 0 ,$$

since $f(u, \frac{n}{2}) = 1$. For $p = \frac{n}{2}$ (n even) and $\varepsilon = 1$, there is an additional (“bonus”) symmetry, which leads to even higher degeneracies for the transfer-matrix eigenvalues [1, 2].

The cases $A_{2(n-1)}, B_n^{(1)}$ and $D_n^{(1)}$ (for which the “right” factor in Table 2 is $U_q(D_p)$) have a “right” $Z_2$ symmetry

$$[Z^{(r)}, t(u,p)] = 0 ;$$

and the case $D_n^{(1)}$ (for which the “left” factor in Table 2 is $U_q(D_{n-p})$) also has a “left” $Z_2$ symmetry

$$[Z^{(l)}, t(u,p)] = 0 ,$$

see [1] for explicit expressions for the quantum-space operators $Z^{(r)}$ and $Z^{(l)}$.

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3The explicit form of $\Delta_N$ for $N = 2$ can be found in [1, 2].
4The symmetries for the “special” cases with $p = 1$ are the same as for $p = 0$, while the symmetry for $D_{n}^{(1)}$ with $p = n - 1$ is the same as for $p = n$. (See Table 2) These observations can be readily understood from the Dynkin diagrams, see Fig. 1.
Figure 1: Subalgebras of affine Lie algebras corresponding to removing the \( p^{th} \) node from the extended Dynkin diagram.
We now proceed to determine the spectrum of the transfer matrix (2.16) for all the cases in Tables 2 and 3. The results hold for both values $\varepsilon = 0, 1$ except for the case $D_{n+1}^{(2)}$, where we consider only $\varepsilon = 0$. The results for some of these cases have already been known:

- For $p = 0$:
  - $A_{2n}^{(2)}$ [25, 26, 27, 28]
  - $A_{2n-1}^{(2)}$ [29]
  - $B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ [29, 27]

- For $p = n$:
  - $A_{2n-1}^{(2)}$ [26, 28]
  - $A_{2n-1}^{(2)}, D_{n+1}^{(2)}$ [21]

- For $0 < p < n$:
  - $A_{2n}^{(2)}$ [26]

The eigenvalues of the transfer matrix are determined in Secs. 3.1, 3.2, and the corresponding Bethe equations are obtained in Sec. 3.3.

### 3.1 Eigenvalues of the transfer matrix

The transfer matrix and Cartan generators can be diagonalized simultaneously

$$
t(u, p) |\Lambda^{(m_1, \ldots, m_n)}(u, p)\rangle = \Lambda^{(m_1, \ldots, m_n)}(u, p) |\Lambda^{(m_1, \ldots, m_n)}\rangle,
\Delta_N(H_i^{(l)}(p)) |\Lambda^{(m_1, \ldots, m_n)}\rangle = h_i^{(l)} |\Lambda^{(m_1, \ldots, m_n)}\rangle, \quad i = 1, \ldots, n - p,
\Delta_N(H_i^{(r)}(p)) |\Lambda^{(m_1, \ldots, m_n)}\rangle = h_i^{(r)} |\Lambda^{(m_1, \ldots, m_n)}\rangle, \quad i = 1, \ldots, p,
$$

(3.1)

as follows from (2.18) and (2.20). We focus here on determining the transfer matrix eigenvalues $\Lambda^{(m_1, \ldots, m_n)}(u, p)$; the eigenvalues of the Cartan generators $h_i^{(l)}, h_i^{(r)}$ are determined in Sec. 4.1.

We take the analytical Bethe ansatz approach, whereby the eigenvalues of the transfer matrix are obtained by “dressing” the reference-state eigenvalue. The “dressing” is assumed to be “doubled” with respect to the corresponding closed chain. Hence, the main difficulty is to determine the reference-state eigenvalue. For the reference state corresponding to the cases in Table 2 we choose

$$
|0, p\rangle = v_p^{\otimes N}, \quad v_p = \begin{cases} 
e_1 & \text{for } p = 0 \\ e_p & \text{for } p = 1, \ldots, n \end{cases},
$$

(3.2)

5 For the special cases in Table 3 we choose (see again footnote 4) the reference state $|0, 0\rangle$ for $A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}$ with $p = 1$; and the reference state $|0, n\rangle$ for $D_n^{(1)}$ with $p = n - 1$. 

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where \( e_i \) are \( d \)-dimensional elementary basis vectors \((e_i)_j = \delta_{i,j}\). Like the usual reference state \( \psi^i \otimes N \), the state \( (\ref{3.2}) \) is an eigenstate of the transfer matrix with no Bethe roots \((m_1 = \ldots = m_n = 0)\)

\[
t(a,b) | 0 , p \rangle = \Lambda^{(0,\ldots,0)}(a,b) | 0 , p \rangle .
\]

However, in addition, this state is a highest weight of the “left” algebra

\[
\Delta_N(E_i^{+;1}(p)) | 0 , p \rangle = 0 , \quad i = 1, \ldots, n - p ,
\]

and a lowest weight of the “right” algebra

\[
\Delta_N(E_i^{-;1}(p)) | 0 , p \rangle = 0 . \quad i = 1, \ldots, p .
\]

In view of the crossing invariance (2.19) and the known results for \( p = 0 \) \cite{25,29} and for \( D_{n+1}^{(2)} \) \cite{21}, we propose that the eigenvalues of the transfer matrix for general values of \( p \) are given by the T-Q equation

\[
\Lambda^{(m_1,\ldots,m_n)}(u,p) = \phi(u,p) \left\{ A(u) z_0(u) y_0(u,p) c(u)^{2N} + \tilde{A}(u) \tilde{z}_0(u) \tilde{y}_0(u,p) \tilde{c}(u)^{2N} + \sum_{l=1}^{n-1} \left[ z_l(u) y_l(u,p) B_l(u) + \tilde{z}_l(u) \tilde{y}_l(u,p) \tilde{B}_l(u) \right] + w_1(u) y_n(u,p) B_n(u) + w_2 \left[ z_n(u) y_n(u,p) B_n(u) + \tilde{z}_n(u) \tilde{y}_n(u,p) \tilde{B}_n(u) \right] \right\} b(u)^{2N} . \tag{3.6}
\]

The overall factor \( \phi(u,p) \) is given by

\[
\phi(u,p) = \begin{cases} 
(-1)^\xi \left( \frac{\gamma^{u+1}_{-u+p}}{e^{u_+}} \right) \left( \frac{\gamma^{-u+p+1}_{-u-p}}{e^{u_-}} \right) & \text{for } A_{2n}^{(2)}, C_n^{(1)}, A_{2n-1}^{(2)} (p \neq 1) , B_n^{(1)} (p \neq 1 , n - 1) , \\
(-1)^\xi & \text{for } A_{2n-1}^{(2)} (p = 1) , B_n^{(1)} (p = 1) , D_n^{(1)} (p = 1 , n - 1) , \\
\frac{\cosh(u-(n-2)p)\eta}{\cosh(u+(n-2)p)\eta} & \text{for } D_{n+1}^{(2)} 
\end{cases}
\]

where \( \gamma \) is defined in (2.7), and the parameters \( \xi \) and \( \rho \) are given in Table 1. The tilde denotes crossing \( \tilde{A}(u) = A(-u - \rho) \), etc. The functions \( A(u) \) and \( B_l(u) \) for \( \tilde{g} = A_{2n}^{(2)} \),
Moreover, for the values of $l$ not included above:

\[ A_{2n-1}^{(2)} : \quad B_{n-1}(u) = \frac{Q^{[n-1]}(u - 2(n + 1)\eta) Q^{[n]}(u - 2(n - 2)\eta)}{Q^{[n-1]}(u - 2(n - 1)\eta) Q^{[n]}(u - 2n\eta)} \times \frac{Q^{[n]}(u - 2(n - 2)\eta + i\pi)}{Q^{[n]}(u - 2n\eta + i\pi)}, \]  

\[ A_{2n}^{(2)} : \quad B_{n}(u) = \frac{Q^{[n]}(u - 2(n + 2)\eta) Q^{[n]}(u - 2(n - 1)\eta + i\pi)}{Q^{[n]}(u - 2n\eta) Q^{[n]}(u - 2(n + 1)\eta + i\pi)}, \]  

\[ B_{n}^{(1)} : \quad B_{n}(u) = \frac{Q^{[n]}(u - 2(n + 2)\eta) Q^{[n]}(u - 2(n + 1)\eta)}{Q^{[n]}(u - 2n\eta) Q^{[n]}(u - 2n\eta)}, \]  

\[ C_{n}^{(1)} : \quad B_{n-1}(u) = \frac{Q^{[n-1]}(u - 2(n + 1)\eta) Q^{[n]}(u - 2(n - 3)\eta)}{Q^{[n-1]}(u - 2(n - 1)\eta) Q^{[n]}(u - 2(n + 1)\eta)}, \]  

\[ D_{n}^{(1)} : \quad B_{n-2}(u) = \frac{Q^{[n-2]}(u - 2n\eta) Q^{[n-1]}(u - 2(n - 3)\eta)}{Q^{[n-2]}(u - 2(n - 2)\eta) Q^{[n-1]}(u - 2(n - 1)\eta) Q^{[n]}(u - 2(n - 1)\eta)}, \]  

\[ B_{n-1}(u) = \frac{Q^{[n-1]}(u - 2(n - 3)\eta) Q^{[n]}(u - 2(n + 1)\eta)}{Q^{[n-1]}(u - 2(n - 1)\eta) Q^{[n]}(u - 2(n - 1)\eta)}. \]  

For the values of $n$ not included above:

\[ A_{1}^{(2)} : \quad A(u) = \frac{Q^{[1]}(u + 2\eta) Q^{[1]}(u + 2\eta + i\pi)}{Q^{[1]}(u - 2\eta) Q^{[1]}(u - 2\eta + i\pi)}, \]  

\[ C_{1}^{(1)} : \quad A(u) = \frac{Q^{[1]}(u + 4\eta)}{Q^{[1]}(u - 4\eta)}, \]  

\[ D_{2}^{(1)} : \quad A(u) = \frac{Q^{[1]}(u + 2\eta) Q^{[2]}(u + 2\eta)}{Q^{[1]}(u - 2\eta) Q^{[2]}(u - 2\eta)}, \]  

\[ B_{1}(u) = \frac{Q^{[1]}(u - 6\eta) Q^{[2]}(u + 2\eta)}{Q^{[1]}(u - 2\eta) Q^{[2]}(u - 2\eta)}. \]
Finally, for $D_{n+1}^{(2)}$:

$$A(u) = \frac{Q^{[1]}(u + \eta) Q^{[1]}(u + \eta + i\pi)}{Q^{[1]}(u - \eta) Q^{[1]}(u - \eta + i\pi)},$$

$$B_l(u) = \frac{Q^{[l]}(u - (l + 2)\eta) Q^{[l]}(u - (l + 2)\eta + i\pi)}{Q^{[l]}(u - l\eta) Q^{[l]}(u - l\eta + i\pi)}$$

$$\times \frac{Q^{[l+1]}(u - (l - 1)\eta) Q^{[l+1]}(u - (l - 1)\eta + i\pi)}{Q^{[l+1]}(u - (l + 1)\eta) Q^{[l+1]}(u - (l + 1)\eta + i\pi)}, \quad l = 1, ..., n - 1,$$

$$B_n(u) = \frac{Q^{[n]}(u - (n + 2)\eta) Q^{[n]}(u - (n - 2)\eta + i\pi)}{Q^{[n]}(u - n\eta) Q^{[n]}(u - n\eta + i\pi)}. \quad \text{(3.17)}$$

In the above equations (3.8) - (3.17) for the functions $A(u)$ and $B_l(u)$, the functions $Q^{[l]}(u)$ are given by

$$Q^{[l]}(u) = \prod_{j=1}^{m_l} \sinh \left( \frac{1}{2} (u - u^{[l]}_j) \right) \sinh \left( \frac{1}{2} (u + u^{[l]}_j) \right), \quad Q^{[l]}(-u) = Q^{[l]}(u), \quad \text{(3.18)}$$

where the zeros $u^{[l]}_j$ (and their number $m_l$) are still to be determined. Note that these expressions for $A(u)$ and $B_l(u)$ are “doubled” with respect to those in [14] for the corresponding closed chains.

The functions $c(u)$ and $b(u)$ are given by

$$c(u) = \begin{cases} 
2 \sinh \left( \frac{u}{2} - 2\eta \right) \cosh \left( \frac{u}{2} - \kappa\eta \right) & \text{for } A_{2n}^{(2)}, A_{2n-1}^{(2)}, \\
2 \sinh \left( \frac{u}{2} - 2\eta \right) \sinh \left( \frac{u}{2} - \kappa\eta \right) & \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, \\
4 \sinh (u - 2\eta) \sinh (u - \kappa\eta) & \text{for } D_n^{(2)}, 
\end{cases} \quad \text{(3.19)}$$

and

$$b(u) = \begin{cases} 
2 \sinh \left( \frac{u}{2} \right) \cosh \left( \frac{u}{2} - \kappa\eta \right) & \text{for } A_{2n}^{(2)}, A_{2n-1}^{(2)}, \\
2 \sinh \left( \frac{u}{2} \right) \sinh \left( \frac{u}{2} - \kappa\eta \right) & \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, \\
4 \sinh (u) \sinh (u - \kappa\eta) & \text{for } D_n^{(2)}, 
\end{cases} \quad \text{(3.20)}$$

For all $\hat{g}$ except $D_{n+1}^{(2)}$, the functions $z_l(u)$ are given by

$$z_l(u) = \frac{\sinh u \sinh (u - 2\kappa\eta) \cosh \left( u - \omega\eta + (2 - \delta) \frac{i\pi}{4} \right)}{\sinh(u - 2l\eta) \sinh \left( u - 2(l + 1)\eta\right) \cosh \left( u - \kappa\eta + (2 - \delta) \frac{i\pi}{4} \right)}, \quad \text{(3.21)}$$

where $\omega$ and $\delta$ are given in Table [11]. For $D_{n+1}^{(2)}$

$$z_l(u) = \begin{cases} 
\frac{\cosh (u - (n - 1)\eta) \sinh(2u - 4n\eta) \sinh(u - (n + 1)\eta)}{\sinh(u - n\eta) \cosh(u - n\eta) \sinh(2u - 2n\eta) \sinh(2u - 2(l + 1)\eta)} & l = 0, ..., n - 1, \\
z_{n-1}(u) & l = n, 
\end{cases} \quad \text{(3.22)}$$
Finally, the quantities \( w_1(u) \) and \( w_2 \) are defined as
\[
\begin{align*}
  w_1(u) &= \begin{cases} 
    \frac{\sinh u \sinh(u-2\kappa)}{\sinh(u-(\kappa+1)\eta)\sinh(u-(\kappa-1)\eta)} & \text{for } A_{2n}^{(2)}, B_n^{(1)} , \\
    0 & \text{for } A_{2n-1}^{(2)}, C_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)} ,
  \end{cases} \\
  w_2 &= \begin{cases} 
    1 & \text{for } D_{n+1}^{(2)} \\
    0 & \text{otherwise}.
  \end{cases}
\end{align*}
\] (3.23)

In the expression (3.6) for the transfer-matrix eigenvalue, only the functions \( y_l(u, p) \) remain to be specified. For \( y_l(u, p) = 1 \), the expression (3.6) reduces (apart from the overall factor) to the transfer-matrix eigenvalue for the case \( p = 0 \) for all the cases except \( D_{n+1}^{(2)} \). The functions \( y_l(u, p) \) for general values of \( p \) are determined in the following section.

### 3.2 Determining \( y_l(u, p) \)

We now proceed to determine the functions \( y_0(u, p), \ldots, y_n(u, p) \) for general values of \( p \). We emphasize that these are the only functions (besides the overall factor \( \phi(u, p) \)) through the quantity \( \gamma \) in the expression (3.6) for the transfer-matrix eigenvalue with explicit dependence on \( p \).

For the special cases in Table 3, the functions \( y_l(u, p) \) are simply given by
\[
y_l(u, p) = 1, \quad l = 0, \ldots, n, \quad (3.24)
\]
i.e., the same as for the case \( p = 0 \). We therefore focus our attention in the remainder of this section on the cases in Table 2.

We make the ansatz
\[
y_l(u, p) = \begin{cases} 
  F(u) & \text{for } 0 \leq l \leq p-1 \\
  G(u) & \text{for } p \leq l \leq n
  \end{cases}, \quad (3.25)
\]
and
\[
\tilde{G}(u) \equiv G(-u - \rho) = G(u), \quad (3.26)
\]
which guarantees that the only Bethe equation with an extra factor (in comparison with the case \( p = 0 \)) is the equation for the \( p \)th Bethe roots \( \{u_j^{[p]}\} \), as discussed further in Sec. 3.3.

The explicit form of \( F(u) \) and \( G(u) \) are
\[
F(u) = \frac{\cosh \left( \frac{u}{2} - \frac{(\omega - 4p)\eta}{2} + (\delta - 4\varepsilon)\frac{i\pi}{8} \right)}{\cosh \left( \frac{u}{2} - \frac{(\omega - 4p)\eta}{2} - (\delta - 4\varepsilon)\frac{i\pi}{8} \right)} G(u), \quad (3.27)
\]

\[
G(u) = \frac{\cosh \left( \frac{u}{2} - \frac{(\omega - 4p)\eta}{2} - (\delta - 4\varepsilon)\frac{i\pi}{8} \right)}{\cosh \left( \frac{u}{2} - \frac{(\omega - 4p)\eta}{2} + (\delta - 4\varepsilon)\frac{i\pi}{8} \right)} \frac{\cosh \left( \frac{u}{2} - \frac{(\omega - 4p)\eta}{2} - (\delta - 4\varepsilon)\frac{i\pi}{8} \right)}{\cosh \left( \frac{u}{2} - \frac{(\omega - 4p)\eta}{2} + (\delta - 4\varepsilon)\frac{i\pi}{8} \right)} \frac{\cosh \left( \frac{u}{2} + \frac{(\omega - 4p)\eta}{2} + (\delta - 4\varepsilon)\frac{i\pi}{8} \right)}{\cosh \left( \frac{u}{2} + \frac{(\omega - 4p)\eta}{2} - (\delta - 4\varepsilon)\frac{i\pi}{8} \right)} G(u), \quad (3.28)
\]
for $\hat{g} = A_{2n}^{(2)}, A_{2n-1}^{(2)}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$. Note that $\omega, \tilde{\omega}, \delta$ are given in Table I. Moreover,

$$G(u) = \frac{\cosh^2(u - n\eta)}{\cosh(u - (n - 2p)\eta) \cosh(u - (n + 2p)\eta)},$$  \hspace{1cm} (3.29)

$$F(u) = \frac{\cosh^2(u + (n - 2p)\eta)}{\cosh^2(u - n\eta)} G(u),$$  \hspace{1cm} (3.30)

for $\hat{g} = D^{(2)}_{n+1}$. Note that $G(u) = 1$ for $p = 0$ in all cases. The rest of this section is dedicated to explaining how the above expressions can be obtained, starting with $F(u)$.

According to (3.25), $y_0(u, p)$ is equal to $F(u)$ for any value of $p$ except $p = 0$. We can use this fact to determine $F(u)$ by arranging to kill all the terms in (3.6) except the one with $y_0(u, p)$, which can be accomplished by judiciously introducing inhomogeneities. Indeed, it is well known that arbitrary inhomogeneities $\{\theta_i\}$ can be introduced in the transfer matrix $t(u, p; \{\theta_i\})$ while maintaining the commutativity property

$$[t(u, p; \{\theta_i\}), t(v, p; \{\theta_i\})] = 0.$$  \hspace{1cm} (3.31)

By appropriately choosing the inhomogeneities, all the terms in (3.6) except the first one can be made to vanish. A similar procedure has been used in e.g. [21, 30].

As an example, let us consider the case $A^{(2)}_{2n}$. The only effect on the eigenvalue (3.6) of introducing inhomogeneities $\{\theta_i\}$ in the transfer matrix is to modify the expressions for $c(u)$, $\tilde{c}(u)$ and $b(u)$ (3.19), (3.20) as follows:

$$c(u)^{2N} = \left[2 \sinh \left(\frac{u}{2} - 2\eta\right) \cosh \left(\frac{u}{2} - \kappa\eta\right)\right]^{2N}$$

$$\mapsto \prod_{i=1}^{N} \left[2 \sinh \left(\frac{u + \theta_i}{2} - 2\eta\right) \cosh \left(\frac{u + \theta_i}{2} - \kappa\eta\right)\right] \left[2 \sinh \left(\frac{u - \theta_i}{2} - 2\eta\right) \cosh \left(\frac{u - \theta_i}{2} - \kappa\eta\right)\right],$$

$$\tilde{c}(u)^{2N} = \left[2 \sinh \left(\frac{u}{2} \cosh \left(\frac{u}{2} - (\kappa - 2)\eta\right)\right]\right]^{2N}$$

$$\mapsto \prod_{i=1}^{N} \left[2 \sinh \left(\frac{u + \theta_i}{2} \cosh \left(\frac{u + \theta_i}{2} - (\kappa - 2)\eta\right)\right]\right] \left[2 \sinh \left(\frac{u - \theta_i}{2} \cosh \left(\frac{u - \theta_i}{2} - (\kappa - 2)\eta\right)\right]\right],$$

$$b(u)^{2N} = \left[2 \sinh \left(\frac{u}{2} \cosh \left(\frac{u}{2} - \kappa\eta\right)\right]\right]^{2N}$$

$$\mapsto \prod_{i=1}^{N} \left[2 \sinh \left(\frac{u + \theta_i}{2} \cosh \left(\frac{u + \theta_i}{2} - \kappa\eta\right)\right]\right] \left[2 \sinh \left(\frac{u - \theta_i}{2} \cosh \left(\frac{u - \theta_i}{2} - \kappa\eta\right)\right]\right].$$

By choosing $\theta_i = u$, the modified expressions for $\tilde{c}(u)$ and $b(u)$ (but not $c(u)$) evidently become zero; hence, the only term in (3.6) that survives is the first term, which is proportional to $y_0(u, p) = F(u)$. On the other hand, by acting with the transfer matrix $t(u, p; \{\theta_i = u\})$
with $N = 1$ and $n = p = 1$ on the reference state (3.2), we explicitly obtain the corresponding eigenvalue. Comparing these two results, keeping in mind that the reference state is the Bethe state with no Bethe roots and therefore $A(u) = 1$, we can solve for $F(u)$. By repeating this procedure for $n = 2$ and $p = 1, 2$, we infer the general result (3.28), which can then be easily checked in a similar way for higher values of $n, p, N$.

In order to determine $G(u)$, we return to the homogeneous case $\theta_i = 0$, so that all the functions $y_0(u, p), \ldots, y_n(u, p)$ again appear in (3.6). Using (3.6), the ansätze (3.25) and (3.26), and the result (3.28) for $F(u)$, we obtain an expression for the reference-state eigenvalue ($A(u) = B_l(u) = 1$) in terms of $G(u)$. We also calculate this eigenvalue explicitly by acting with $t(u, p)$ (with $N = 1$) on the reference state (3.2). By comparing both expressions, we can solve for $G(u)$. We again use the results for small values of $n$ and $p$ to infer the general result (3.27). Having obtained both $F(u)$ and $G(u)$ for general values of $n$ and $p$, the reference-state eigenvalue can be easily checked for higher values of $n, p, N$.

Using the same procedure for the other $\hat{g}$ in Table 2, we arrive at the results (3.27) - (3.30). As already noted, for the special cases in Table 3 we have $y_l(u, p) = 1$ (3.24).

### 3.3 Bethe equations

The expression (3.6) for the transfer-matrix eigenvalues is in terms of the zeros $u_j^{[l]}$ of the functions $Q_j^{[l]}(u)$, which are still to be determined. In principle, these zeros can be determined by solving corresponding Bethe equations, which we now present. We find that these Bethe equations are the same as for the case $p = 0$ [25, 29], except for the presence of an extra factor $\Phi_{l,p,n}(u)$ (3.57), (3.64) that is different from 1 only if $l = p$. The only dependence on $p$ in the Bethe equations is in this factor.

#### 3.3.1 For $\hat{g} = A^{(2)}_{2n}, A^{(2)}_{2n-1}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$

We determine the Bethe equations from the requirement that the expression (3.6) for the transfer-matrix eigenvalues have vanishing residues at the poles. In this way, we obtain the following Bethe equations for all the cases in Tables 2 and 3 except for $D^{(2)}_{n+1}$:

$$
\Phi_{1,p,n}(u_k^{[1]}) = \frac{Q_k^{[1]}(u_k^{[1]} + 4\eta)}{Q_k^{[1]}(u_k^{[1]} - 4\eta)} \frac{Q_k^{[2]}(u_k^{[1]} - 2\eta)}{Q_k^{[2]}(u_k^{[1]} + 2\eta)}, \quad k = 1, \ldots, m_1,
$$

(3.32)
Moreover, for the values of $l$ where $A_{l,p,n}^{(2)} \Phi_2(n,n) = 1$, for $D_n^{(1)} (n > 2)$

$l = 1, \ldots, n - 3$ for $D_n^{(1)} (n > 2)$

$l = 1, \ldots, n - 2$ for $C_n^{(1)} (n > 1), A_{2n-1}^{(2)} (n > 1)$

$l = 1, \ldots, n - 1$ for $A_{2n}^{(2)}, B_n^{(1)}$,

where $Q^{[l]}(u)$ is given by (3.18), and $Q^{[l]}_k(u)$ is defined by

$$Q^{[l]}_k(u) = \prod_{j=1,j \neq k}^{m} \sinh \left( \frac{1}{2} (u - u_j^{[l]}) \right) \sinh \left( \frac{1}{2} (u + u_j^{[l]}) \right).$$

Moreover, for the values of $l$ not included above:

$$A_{2n-1}^{(2)}: \quad \Phi_{n-1,p,n}^{[n-1]}(u_k^{[n-1]}) = \frac{Q^{[n-1]}(u_k^{[n-1]} - 2\eta) Q^{[n-1]}_k(u_k^{[n-1]} + 4\eta)}{Q^{[n-1]}(u_k^{[n-1]} + 2\eta) Q^{[n-1]}_k(u_k^{[n-1]} - 4\eta)} \times \frac{Q^{[n]}(u_k^{[n-1]} - 2\eta) Q^{[n]}_k(u_k^{[n-1]} - 2\eta + i\pi)}{Q^{[n]}(u_k^{[n-1]} + 2\eta) Q^{[n]}_k(u_k^{[n-1]} + 2\eta + i\pi)},$$

$$\Phi_{n,p,n}^{[n]}(u_k^{[n]}) = \frac{Q^{[n]}(u_k^{[n]} - 2\eta) Q^{[n]}_k(u_k^{[n]} - 2\eta + i\pi)}{Q^{[n]}(u_k^{[n]} + 2\eta) Q^{[n]}_k(u_k^{[n]} + 2\eta + i\pi)} \times \frac{Q^{[n]}_k(u_k^{[n]} + 4\eta) Q^{[n]}_k(u_k^{[n]} + 4\eta + i\pi)}{Q^{[n]}_k(u_k^{[n]} - 4\eta) Q^{[n]}_k(u_k^{[n]} - 4\eta + i\pi)},$$

$$A_{2n}^{(2)}: \quad \Phi_{n,p,n}^{[n]}(u_k^{[n]}) = \frac{Q^{[n-1]}(u_k^{[n]} - 2\eta) Q^{[n]}_k(u_k^{[n]} + 4\eta) Q^{[n]}_k(u_k^{[n]} - 2\eta + i\pi)}{Q^{[n-1]}(u_k^{[n]} + 2\eta) Q^{[n]}_k(u_k^{[n]} - 4\eta) Q^{[n]}_k(u_k^{[n]} + 2\eta + i\pi)};$$

$$B_n^{(1)}: \quad \Phi_{n,p,n}^{[n]}(u_k^{[n]}) = \frac{Q^{[n-1]}(u_k^{[n]} - 2\eta) Q^{[n]}_k(u_k^{[n]} + 2\eta)}{Q^{[n-1]}(u_k^{[n]} + 2\eta) Q^{[n]}_k(u_k^{[n]} - 2\eta)}.$$
\[
C_{n}^{(1)} : \Phi_{n-1,p,n}(u_{k}^{[n-1]}) = \frac{Q^{[n-2]}(u_{k}^{[n-1]} - 2\eta) \times Q_{k}^{[n-2]}(u_{k}^{[n-1]} + 4\eta) \times Q^{[n]}(u_{k}^{[n-1]} - 4\eta)}{Q^{[n-2]}(u_{k}^{[n-1]} + 2\eta) \times Q_{k}^{[n-2]}(u_{k}^{[n-1]} - 4\eta) \times Q^{[n]}(u_{k}^{[n-1]} + 4\eta)}, \tag{3.39}
\]

\[
\Phi_{n,p,n}(u_{k}^{[n]}) = \frac{Q^{[n-1]}(u_{k}^{[n]} - 4\eta) \times Q_{k}^{[n]}(u_{k}^{[n]} + 8\eta)}{Q^{[n-1]}(u_{k}^{[n]} + 4\eta) \times Q_{k}^{[n]}(u_{k}^{[n]} - 8\eta)}, \tag{3.40}
\]

\[
D_{n}^{(1)} : \Phi_{n-2,p,n}(u_{k}^{[n-2]}) = \frac{Q^{[n-3]}(u_{k}^{[n-2]} - 2\eta) \times Q_{k}^{[n-2]}(u_{k}^{[n-2]} + 4\eta)}{Q^{[n-3]}(u_{k}^{[n-2]} + 2\eta) \times Q_{k}^{[n-2]}(u_{k}^{[n-2]} - 4\eta)}, \tag{3.41}
\]

\[
\times \frac{Q^{[n-1]}(u_{k}^{[n-2]} - 2\eta) \times Q_{k}^{[n]}(u_{k}^{[n-2]} - 2\eta)}{Q^{[n-1]}(u_{k}^{[n-2]} + 2\eta) \times Q_{k}^{[n]}(u_{k}^{[n-2]} + 2\eta)}, \tag{3.41}
\]

\[
\Phi_{n-1,p,n}(u_{k}^{[n-1]}) = \frac{Q^{[n-2]}(u_{k}^{[n-1]} - 2\eta) \times Q_{k}^{[n-1]}(u_{k}^{[n-1]} + 4\eta)}{Q^{[n-2]}(u_{k}^{[n-1]} + 2\eta) \times Q_{k}^{[n-1]}(u_{k}^{[n-1]} - 4\eta)}, \tag{3.42}
\]

\[
\Phi_{n,p,n}(u_{k}^{[n]}) = \frac{Q^{[n-2]}(u_{k}^{[n]} - 2\eta) \times Q_{k}^{[n]}(u_{k}^{[n]} + 4\eta)}{Q^{[n-2]}(u_{k}^{[n]} + 2\eta) \times Q_{k}^{[n]}(u_{k}^{[n]} - 4\eta)}. \tag{3.43}
\]

The Bethe equations for values of \( n \) not included above:

\[
A_{1}^{(2)} : \quad \Phi_{1,p,1}(u_{k}^{[1]}) = \frac{Q_{k}^{[1]}(u_{k}^{[1]} + 4\eta) \times Q_{k}^{[1]}(u_{k}^{[1]} + 4\eta + i\pi)}{Q_{k}^{[1]}(u_{k}^{[1]} - 4\eta) \times Q_{k}^{[1]}(u_{k}^{[1]} - 4\eta + i\pi)}, \tag{3.44}
\]

\[
A_{3}^{(2)} : \quad \Phi_{1,p,2}(u_{k}^{[1]}) = \frac{Q_{k}^{[1]}(u_{k}^{[1]} + 4\eta) \times Q_{k}^{[2]}(u_{k}^{[1]} + 2\eta) \times Q_{k}^{[2]}(u_{k}^{[1]} - 2\eta + i\pi)}{Q_{k}^{[1]}(u_{k}^{[1]} - 4\eta) \times Q_{k}^{[2]}(u_{k}^{[1]} + 2\eta) \times Q_{k}^{[2]}(u_{k}^{[1]} - 2\eta + i\pi)}, \tag{3.45}
\]

\[
A_{2}^{(2)} : \quad \Phi_{1,p,1}(u_{k}^{[1]}) = \frac{Q_{k}^{[1]}(u_{k}^{[1]} + 4\eta) \times Q_{k}^{[1]}(u_{k}^{[1]} - 2\eta + i\pi)}{Q_{k}^{[1]}(u_{k}^{[1]} + 4\eta) \times Q_{k}^{[1]}(u_{k}^{[1]} - 2\eta + i\pi)}, \tag{3.46}
\]

\[\text{16}\]
\[
B_1^{(1)}(u) = \frac{\sinh\left(\frac{u^{[1]}_k}{2} + \eta\right)}{\sinh\left(\frac{u^{[1]}_k}{2} - \eta\right)}^{2N} \Phi_{1,p,1}(u^{[1]}_k) = \frac{Q^{[1]}_k(u^{[1]}_k + 2\eta)}{Q^{[1]}_k(u^{[1]}_k - 2\eta)}, \quad (3.47)
\]

\[
C_1^{(1)}(u) = \frac{\sinh\left(\frac{u^{[1]}_k}{2} + 2\eta\right)}{\sinh\left(\frac{u^{[1]}_k}{2} - 2\eta\right)}^{2N} \Phi_{1,p,1}(u^{[1]}_k) = \frac{Q^{[1]}_k(u^{[1]}_k + 8\eta)}{Q^{[1]}_k(u^{[1]}_k - 8\eta)}, \quad (3.48)
\]

\[
C_2^{(1)}(u) = \frac{\sinh\left(\frac{u^{[1]}_k}{2} + \eta\right)}{\sinh\left(\frac{u^{[1]}_k}{2} - \eta\right)}^{2N} \Phi_{1,p,2}(u^{[1]}_k) = \frac{Q^{[1]}_k(u^{[1]}_k + 4\eta)}{Q^{[1]}_k(u^{[1]}_k - 4\eta)} \frac{Q^{[2]}(u^{[1]}_k - 4\eta)}{Q^{[2]}(u^{[1]}_k + 4\eta)}, \quad (3.49)
\]

\[
\Phi_{2,p,2}(u^{[2]}_k) = \frac{Q^{[1]}_k(u^{[2]}_k - 4\eta)}{Q^{[1]}_k(u^{[2]}_k + 4\eta)} \frac{Q^{[2]}(u^{[2]}_k + 8\eta)}{Q^{[2]}(u^{[2]}_k - 8\eta)}, \quad (3.50)
\]

\[
D_1^{(1)}(u) = \frac{\sinh\left(\frac{u^{[1]}_k}{2} + \eta\right)}{\sinh\left(\frac{u^{[1]}_k}{2} - \eta\right)}^{2N} \Phi_{1,p,2}(u^{[1]}_k) = \frac{Q^{[1]}_k(u^{[1]}_k + 4\eta)}{Q^{[1]}_k(u^{[1]}_k - 4\eta)}, \quad (3.51)
\]

\[
\Phi_{2,p,2}(u^{[2]}_k) = \frac{Q^{[1]}_k(u^{[2]}_k + 4\eta)}{Q^{[2]}(u^{[2]}_k - 4\eta)}, \quad (3.52)
\]

\[
D_3^{(1)}(u) = \frac{\sinh\left(\frac{u^{[1]}_k}{2} + \eta\right)}{\sinh\left(\frac{u^{[1]}_k}{2} - \eta\right)}^{2N} \Phi_{1,p,3}(u^{[1]}_k) = \frac{Q^{[1]}_k(u^{[1]}_k + 4\eta)}{Q^{[1]}_k(u^{[1]}_k - 4\eta)} \frac{Q^{[2]}(u^{[1]}_k - 2\eta)}{Q^{[2]}(u^{[1]}_k + 2\eta)} \times \frac{Q^{[3]}(u^{[1]}_k - 2\eta)}{Q^{[3]}(u^{[1]}_k + 2\eta)}, \quad (3.53)
\]

\[
\Phi_{2,p,3}(u^{[2]}_k) = \frac{Q^{[1]}_k(u^{[2]}_k - 2\eta)}{Q^{[1]}_k(u^{[2]}_k + 2\eta)} \frac{Q^{[2]}(u^{[2]}_k + 4\eta)}{Q^{[2]}(u^{[2]}_k - 4\eta)}, \quad (3.54)
\]

\[
\Phi_{3,p,3}(u^{[3]}_k) = \frac{Q^{[1]}_k(u^{[3]}_k - 2\eta)}{Q^{[1]}_k(u^{[3]}_k + 2\eta)} \frac{Q^{[3]}(u^{[3]}_k + 4\eta)}{Q^{[3]}(u^{[3]}_k - 4\eta)}. \quad (3.55)
\]

The \(u^{[1]}_k \leftrightarrow u^{[2]}_k\) symmetry of the Bethe equations (3.51), (3.52) is a reflection of the \(U_q(D_2)\) symmetry (see again Table 2) and the fact \(D_2 = A_1 \otimes A_1\).

The important factor \(\Phi_{t,p,n}(u)\) in the Bethe equations for most of the cases in Table 2 is
given by\(^6\)

\[
\Phi_{l,p,n}(u) = \frac{y_l(u + 2l\eta, p)}{y_{l-1}(u + 2l\eta, p)} = \begin{cases} 
\frac{G(u+2p\eta)}{F(u+2p\eta)} & \text{for } l = p, \\
1 & \text{for } l \neq p,
\end{cases}
\] (3.56)

where the second equality follows from (3.25). Using the expressions for $G(u)$ (3.24) and $F(u)$ (3.28), we conclude that $\Phi_{l,p,n}(u)$ is given by

\[
\Phi_{l,p,n}(u) = \begin{cases} 
\left[\frac{\cosh\left(\frac{1}{2} - \delta_{l,p} \left(\frac{u-2}{2} + \frac{u}{2}(\delta - 4\epsilon)\right)\right)}{\cosh\left(\frac{1}{2} + \delta_{l,p} \left(\frac{u-2}{2} + \frac{u}{2}(\delta - 4\epsilon)\right)\right)} \right]^2 & \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n}^{(1)} \\
\left[\frac{\sinh\left(\frac{1}{2} - \delta_{l,p} \left(\frac{u-2}{2} + \frac{u}{2}(\delta - 4\epsilon)\right)\right)}{\sinh\left(\frac{1}{2} + \delta_{l,p} \left(\frac{u-2}{2} + \frac{u}{2}(\delta - 4\epsilon)\right)\right)} \right]^2 & \text{for } A_{2n-1}^{(2)} \text{ with } l < n, \\
\left[\frac{\sinh\left(\frac{1}{2} - \delta_{l,p} \left(\frac{u-2}{2} + \frac{u}{2}(\delta - 4\epsilon)\right)\right)}{\sinh\left(\frac{1}{2} + \delta_{l,p} \left(\frac{u-2}{2} + \frac{u}{2}(\delta - 4\epsilon)\right)\right)} \right]^2 & \text{for } A_{2n-1}^{(2)} \text{ with } l = n.
\end{cases}
\] (3.57)

Note that $\Phi_{l,p,n}(u)$ is different from 1 only if $l = p$. That is, the Bethe equations are the same as for the case $p = 0$ [25, 29], except for an extra factor in the equation for the $p^{th}$ Bethe roots \{\(u^{[p]}\}\).

The factor $\Phi_{l,p,n}(u)$ for all the special cases in Table 3 is simply given by

\[
\Phi_{l,p,n}(u) = 1,
\] (3.58)

as follows from [3,24].

For $p = n$, the Bethe equations for $A_{2n}^{(2)}$ with $\epsilon = 1$ reduce to those found in [28]; and (again for $p = n$) the Bethe equations for $A_{2n-1}^{(2)}$ with $\epsilon = 0$ reduce to those found in [21]. We have numerically verified the completeness of all the above Bethe ansatz solutions for small values of $n$ and $N$ (for all $p = 0, \ldots, n$ and $\epsilon = 0, 1$), along the lines in [28, 21].

### 3.3.2 For $\hat{g} = D_{n+1}^{(2)}$

We emphasize that, for $D_{n+1}^{(2)}$, we consider only the case $\epsilon = 0$. We obtain the following Bethe equations:

For $n = 1$ with $p = 0, 1$:

\[
\left[\frac{\sinh(u^{[1]}_k + \eta)}{\sinh(u^{[1]}_k - \eta)} \right]^{2N} = \frac{Q^{[1]}_k (u^{[1]}_k + 2\eta)}{Q^{[1]}_k (u^{[1]}_k - 2\eta)}, \quad k = 1, \ldots, m_1.
\] (3.59)

\(^6\)The exceptions are as follows:

- $A_{2n-1}^{(2)}, p = n$: $\Phi_{l,p,n}(u) = \begin{cases} 
\frac{\tilde{y}_{n-1}(u+2n\eta, p)}{\tilde{y}_{n-1}(u+2n\eta, p)} & \text{for } l = n, \\
1 & \text{for } l \neq n.
\end{cases}$

- $C_n^{(1)}, p = n$: $\Phi_{l,p,n}(u) = \begin{cases} 
\frac{\tilde{y}_{n-1}(u+(n+1)\eta, p)}{\tilde{y}_{n-1}(u+(n+1)\eta, p)} & \text{for } l = n, \\
1 & \text{for } l \neq n.
\end{cases}$

- $D_n^{(1)}, p = n$: $\Phi_{l,p,n}(u) = \begin{cases} 
\frac{\tilde{y}_{n-1}(u+(n-1)\eta, p)}{\tilde{y}_{n-1}(u+(n-1)\eta, p)} & \text{for } l = n, \\
1 & \text{for } l \neq n.
\end{cases}$
For $n > 1$ with $p = 0, \ldots, n$:

$$
\left[ \frac{\sinh(u_k^{[1]} + \eta)}{\sinh(u_k^{[1]} - \eta)} \right]^{2N} \Phi_{1,p,n}(u_k^{[1]}) = \frac{Q_k^{[1]}(u_k^{[1]} + 2\eta)}{Q_k^{[1]}(u_k^{[1]} - 2\eta)} \frac{Q_k^{[1]}(u_k^{[1]} + 2\eta + i\pi)}{Q_k^{[1]}(u_k^{[1]} - 2\eta + i\pi)} \\
\times \frac{Q_k^{[2]}(u_k^{[1]} - \eta)}{Q_k^{[2]}(u_k^{[1]} + \eta)} \frac{Q_k^{[2]}(u_k^{[1]} - \eta + i\pi)}{Q_k^{[2]}(u_k^{[1]} + \eta + i\pi)},
$$

$k = 1, \ldots, m_1$,

$$
\Phi_{l,p,n}(u_k^{[l]}) = \frac{Q_k^{[l-1]}(u_k^{[l]} - \eta)}{Q_k^{[l-1]}(u_k^{[l]} + \eta)} \frac{Q_k^{[l-1]}(u_k^{[l]} - \eta + i\pi)}{Q_k^{[l-1]}(u_k^{[l]} + \eta + i\pi)} \\
\times \frac{Q_k^{[l]}(u_k^{[l]} + 2\eta)}{Q_k^{[l]}(u_k^{[l]} - 2\eta)} \frac{Q_k^{[l]}(u_k^{[l]} + 2\eta + i\pi)}{Q_k^{[l]}(u_k^{[l]} - 2\eta + i\pi)} \\
\times \frac{Q_k^{[l+1]}(u_k^{[l]} - \eta)}{Q_k^{[l+1]}(u_k^{[l]} + \eta)} \frac{Q_k^{[l+1]}(u_k^{[l]} - \eta + i\pi)}{Q_k^{[l+1]}(u_k^{[l]} + \eta + i\pi)},
$$

$k = 1, \ldots, m_l$, $l = 2, \ldots, n - 1$,

$$
\Phi_{n,p,n}(u_k^{[n]}) = \frac{Q_k^{[n-1]}(u_k^{[n]} - \eta)}{Q_k^{[n-1]}(u_k^{[n]} + \eta)} \frac{Q_k^{[n-1]}(u_k^{[n]} - \eta + i\pi)}{Q_k^{[n-1]}(u_k^{[n]} + \eta + i\pi)} \frac{Q_k^{[n]}(u_k^{[n]} + 2\eta)}{Q_k^{[n]}(u_k^{[n]} - 2\eta)},
$$

$k = 1, \ldots, m_n$.

The factor $\Phi_{l,p,n}(u)$ in the above Bethe equations is given by

$$
\Phi_{l,p,n}(u) = \frac{y_l(u + ln, p)}{y_l-1(u + ln, p)} = \begin{cases} 
\frac{G(u + ln)}{F(u + ln)} & \text{for } l = p \\
1 & \text{for } l \neq p
\end{cases}.
$$

Using the results for $G(u)$ [3.29] and $F(u)$ [3.30], we obtain

$$
\Phi_{l,p,n}(u) = \left[ \frac{\cosh (u - \delta_{l,p} (n - p)\eta)}{\cosh (u + \delta_{l,p} (n - p)\eta)} \right]^2.
$$

As for [3.57], this factor $\Phi_{l,p,n}(u)$ is different from 1 only if $l = p$.

For $p = n$, these Bethe equations reduce to the one found in [21]. We have numerically verified the completeness of the above Bethe ansatz solutions for small values of $n$ and $N$ (for all $p = 0, \ldots, n$) along the lines in [21].
3.3.3 Towards a universal formula for the Bethe equations

Let us denote in this subsection the affine Lie algebras \( \hat{g} \) in Tables 2 and 3 by \( g^{(t)} \), where \( g \) is a (non-affine) Lie algebra with rank \( r \), and \( t = 1 \) (untwisted) or \( t = 2 \) (twisted). The above formulas for the \( g^{(t)} \) Bethe equations can be rewritten in a more compact form in terms of representation-theoretic quantities following \cite{[4]}

\[
\prod_{s=0}^{t-1} \frac{\sinh \left( \frac{u_k^i}{2} + (\lambda_1, \theta^s \alpha_l) \eta + \frac{i\pi s}{2} \right)}{\sinh \left( \frac{u_k^i}{2} - (\lambda_1, \theta^s \alpha_l) \eta + \frac{i\pi s}{2} \right)}
\]

\[
= \prod_{s=0}^{t-1} \prod_{r=1}^{n} \prod_{j=1}^{m^r} \frac{\sinh \left( \frac{1}{2} \left( u_k^i - u_j^l \right) \right) + (\alpha_i, \theta^s \alpha_l) \eta + \frac{i\pi s}{2}}{\sinh \left( \frac{1}{2} \left( u_k^i - u_j^l \right) \right) - (\alpha_i, \theta^s \alpha_l) \eta + \frac{i\pi s}{2}} \sinh \left( \frac{1}{2} \left( u_k^i + u_j^l \right) \right) + (\alpha_i, \theta^s \alpha_l) \eta + \frac{i\pi s}{2}]
\]

\[
\cdot \prod_{s=0}^{t-1} \prod_{r=1}^{n} \prod_{j=1}^{m^r} \frac{\sinh \left( \frac{1}{2} \left( u_k^i + u_j^l \right) \right) + (\alpha_i, \theta^s \alpha_l) \eta + \frac{i\pi s}{2}}{\sinh \left( \frac{1}{2} \left( u_k^i + u_j^l \right) \right) - (\alpha_i, \theta^s \alpha_l) \eta + \frac{i\pi s}{2}},
\]

where the product over \( j \) has the restriction \((j, l') \neq (k, l)\). The simple roots \( \alpha_i \) of \( g \) are given in the orthogonal basis by

\[
\alpha_i = e_i - e_{i+1}, \quad i = 1, \ldots, r - 1,
\]

\[
\alpha_r = \begin{cases} 
  e_r - e_{r+1} & \text{for } A_r \\
  e_r & \text{for } B_r \\
  2e_r & \text{for } C_r \\
  e_{r-1} + e_r & \text{for } D_r
\end{cases}
\]

where \( e_i \) are \( r \)-dimensional elementary basis vectors (except for \( A_r \), in which case the dimension is \( r + 1 \)). The notation \((\ast, \ast)\) denotes the ordinary scalar product, and \( \lambda_1 \) is the first fundamental weight of \( g \), with \((\lambda_1, \alpha_i) = \delta_{i,1}\). For the twisted cases \( g^{(2)} \), the order-2 automorphisms \( \theta \) of \( g \) are given by

\[
\theta \alpha_i = \alpha_{2n-i}, \quad i = 1, \ldots, 2n - 1 \quad \text{for } A^{(2)}_{2n-1},
\]

\[
\theta \alpha_i = \alpha_{2n+1-i}, \quad i = 1, \ldots, 2n \quad \text{for } A^{(2)}_{2n},
\]

\[
\theta \alpha_i = \alpha_i, \quad i = 1, \ldots, n - 1, \quad \theta \alpha_n = \alpha_{n+1} \quad \text{for } D^{(2)}_{n+1}.
\]

The factor \( \Phi_{t,p,n}(u) \) in (3.65) is understood to be the appropriate one for \( g^{(t)} \), see (3.57), (3.58), (3.64). It would be interesting to also have a universal expression for this factor.

\footnote{The notation \( g^{(t)} \) introduced here for affine Lie algebras should not be confused with the “left” and “right” algebras \( g^{(l)} \) and \( g^{(r)} \) introduced in Sec. 2.4.}

\footnote{For \( D^{(2)}_{n+1} \), a rescaling \( \eta \rightarrow \frac{\eta}{2} \) in (3.65) is necessary in order to match with the Bethe equations as written in Sec. 3.3.2.}

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4 Dynkin labels of the Bethe states

In this section we obtain formulas for the Dynkin labels of the Bethe states in terms of the numbers of Bethe roots of each type. Since the Dynkin labels of an irrep determine its dimension, these formulas help determine the degeneracies of the transfer-matrix eigenvalues.

4.1 Eigenvalues of the Cartan generators

We now argue that the eigenvalues of the Cartan generators for the Bethe states (3.1) are given in terms of the cardinalities of the Bethe roots of each type by

\[ h_i^{(l)} = m_{p+i-1} - m_{p+i} - \xi \delta_{i,n-p} m_n - \xi' \delta_{i,n-p-1} m_n, \quad i = 1, \ldots, n - p, \]

\[ h_i^{(r)} = m_i - m_{i-1} + \xi \delta_{i,n} m_n + \xi' \delta_{i,n-1} m_n, \quad i = 1, \ldots, p, \]

where \( \xi \) and \( \xi' \) are given in Table 1.

The first step is to compute the asymptotic behavior of \( \Lambda^{(m_1, \ldots, m_n)}(u, p) \) by computing the expectation value

\[ \langle \Lambda^{(m_1, \ldots, m_n)} | t(u, p) | \Lambda^{(m_1, \ldots, m_n)} \rangle \]

for \( u \to \infty \). The main idea is to perform a gauge transformation to the “unitary” gauge \[1, 2\], so that the asymptotic limit of the monodromy matrices in \( t(u, p) \) become expressed in terms of the QG generators. We assume that the Bethe states \( |\Lambda^{(m_1, \ldots, m_n)}\rangle \) are highest-weight states of the “left” algebra

\[ \Delta_N(E_i^{+(l)}(p)) |\Lambda^{(m_1, \ldots, m_n)}\rangle = 0, \quad i = 1, \ldots, n - p, \]

and lowest-weight states of the “right” algebra

\[ \Delta_N(E_i^{-(r)}(p)) |\Lambda^{(m_1, \ldots, m_n)}\rangle = 0, \quad i = 1, \ldots, p, \]

as is the reference state \[3,4\], \[3,5\]. We eventually obtain

\[
\Lambda^{(m_1, \ldots, m_n)}(u, p) \sim \sigma(u) e^{-2\kappa \eta N} \left\{ d - 2n + \sum_{j=1}^{p} \left[ f^{(r)}(r) e^{4\eta(-j + h_{p+1-j}^{(r)})} + \frac{1}{f^{(l)}} e^{-4\eta(-j + h_{p+1-j}^{(r)})} \right] \right. \\
+ \sum_{j=p+1}^{n} \left[ f^{(l)}(l) e^{-4\eta(n-j + h_{j-p}^{(l)})} + \frac{1}{f^{(l)}} e^{4\eta(n-j + h_{j-p}^{(l)})} \right] \right\} \text{ for } u \to \infty, \quad (4.4)
\]

where

\[
\sigma(u) = \begin{cases} 
2^{-2N} e^{2N_u} & \text{for } A_{2n-1}^{(2)}, A_{2n}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, \\
e^{4N_u} & \text{for } D_{n+1}^{(2)} 
\end{cases}
\]

(4.5)
and

\[ f^{(r)} = \begin{cases} -1 & \text{for } A^{(2)}_{2n}, C^{(1)}_n, \\ e^{4\eta} & \text{for } A^{(2)}_{2n-1}, B^{(1)}_n, D^{(1)}_n, \\ e^{2\eta} & \text{for } D^{(2)}_{n+1} \end{cases} \]

\[ f^{(l)} = \begin{cases} e^{-2\eta} & \text{for } A^{(2)}_{2n}, B^{(1)}_n, D^{(2)}_{n+1}, \\ -e^{-4\eta} & \text{for } A^{(2)}_{2n-1}, C^{(1)}_n, \\ 1 & \text{for } D^{(1)}_n. \end{cases} \] (4.6)

Note that the result (4.3) is in terms of the eigenvalues of the Cartan generators for the Bethe states.

The second step is to compute again the asymptotic behavior of \( \Lambda^{(m_1,\ldots,m_n)}(u, p) \), but now using instead the T-Q equation (3.6). We obtain in this way

\[ \Lambda^{(m_1,\ldots,m_n)}(u, p) \sim \sigma(u) e^{-2\kappa \eta N} \left( d - 2n + \sum_{l=0}^{n-1} \left[ g_l e^{4\eta(l-n+m_{l+1}-m_l+\xi\delta_{l,n-1}m_{n}+\xi'\delta_{l,n-2}m_{n})} 
\right. \right. \\
\left. \left. + \frac{1}{g_l} e^{-4\eta(l-n+m_{l+1}-m_l+\xi\delta_{l,n-1}m_{n}+\xi'\delta_{l,n-2}m_{n})} \right] \right) \] for \( u \to \infty \), (4.7)

where

\[ g_l = \begin{cases} f^{(r)} e^{4\eta(n-p)} & \text{for } g^{(l)} = B_{n-p}, i.e., \text{for } A^{(2)}_{2n}, B^{(1)}_n, D^{(2)}_{n+1}, \\ f^{(l)} e^{4\eta} & \text{for } g^{(l)} = C_{n-p}, i.e., \text{for } A^{(2)}_{2n-1}, C^{(1)}_n, \\ 1 & \text{for } g^{(l)} = D_{n-p}, i.e., \text{for } D^{(1)}_n. \end{cases} \] (4.8)

\( \sigma(u) \) is given by (4.5), and \( f^{(r)}, f^{(l)} \) are given by (4.6). Moreover, we define \( m_0 \) as

\[ m_0 = N. \] (4.9)

Note that the result (4.7) is in terms of the cardinalities of the Bethe roots of each type.

Finally, by comparing (4.4) and (4.7), we obtain the desired result (4.1).

### 4.2 Formulas for the Dynkin labels

The “left” Dynkin labels are expressed in terms of the eigenvalues of the “left” Cartan generators by (see, e.g. [28, 21])

\[ a_i^{(l)} = h_i^{(l)} - h_{i+1}^{(l)}, \quad i = 1, \ldots, n - p - 1, \]

\[ a_{n-p}^{(l)} = \begin{cases} 2h_{n-p}^{(l)} & \text{for } g^{(l)} = B_{n-p}, i.e., \text{for } A^{(2)}_{2n}, B^{(1)}_n, D^{(2)}_{n+1}, \\ h_{n-p}^{(l)} & \text{for } g^{(l)} = C_{n-p}, i.e., \text{for } A^{(2)}_{2n-1}, C^{(1)}_n, \\ h_{n-p-1}^{(l)} + h_{n-p}^{(l)} & \text{for } g^{(l)} = D_{n-p}, i.e., \text{for } D^{(1)}_n. \end{cases} \] (4.10)
Similarly, the “right” Dynkin labels are expressed in terms of the eigenvalues of the “right” Cartan generators by
\[
\begin{align*}
a_i^{(r)} &= -h_i^{(r)} + h_{i+1}^{(r)} & i = 1, \ldots, p - 1, \\
a_p^{(r)} &= \begin{cases} 
-2h_p^{(r)} & \text{for } g^{(r)} = B_p \text{ i.e., for } D_{n+1}^{(2)} \\
-h_p^{(r)} & \text{for } g^{(r)} = C_p \text{ i.e., for } A_{2n}^{(2)}, C_n^{(1)} \\
-h_{p-1}^{(r)} - h_p^{(r)} & \text{for } g^{(r)} = D_p \text{ i.e., for } A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}
\end{cases}
\end{align*}
\]

We introduce extra minus signs in (4.11) (in comparison with corresponding formulas in (4.10)) since the Bethe states are lowest weights of the “right” algebra \((\hat{g})\). The algebras \(g^{(l)}\) and \(g^{(r)}\) for the various affine algebras \(\hat{g}\) are given in Table 2.

Finally, using the results (4.1) for the eigenvalues of the Cartan generators in terms of the cardinalities of the Bethe roots of each type, we obtain formulas for the Dynkin labels in terms of the cardinalities of the Bethe roots. Explicitly, for the “left” Dynkin labels \((p = 0, 1, \ldots, n - 1)\):
\[
a_i^{(l)} = m_{p+i-1} - 2m_{p+i} + m_{p+i+1},
\]
\[
i = 1, \ldots, n - p - 1 \text{ for } A_{2n}^{(2)}, B_n^{(1)}, D_{n+1}^{(2)},
\]
\[
i = 1, \ldots, n - p - 2 \text{ for } A_{2n-1}^{(2)}, C_n^{(1)},
\]
\[
i = 1, \ldots, n - p - 3 \text{ for } D_n^{(1)}.
\]

Moreover, for the values of \(i\) not included above:
\[
A_{2n}^{(2)}, B_n^{(1)}, D_{n+1}^{(2)}: \quad a_{n-p}^{(l)} = 2m_{n-1} - 2m_n,
\]
\[
A_{2n-1}^{(2)}, C_n^{(1)}: \quad a_{n-p-1}^{(l)} = m_{n-2} - 2m_{n-1} + 2m_n,
\]
\[
a_{n-p}^{(l)} = m_{n-1} - 2m_n,
\]
\[
D_n^{(1)}: \quad a_{n-p-2}^{(l)} = m_{n-3} - 2m_{n-2} + m_{n-1} + m_n,
\]
\[
a_{n-p-1}^{(l)} = m_{n-2} - 2m_{n-1},
\]
\[
a_{n-p}^{(l)} = m_{n-2} - 2m_n.
\]

For the “right” Dynkin labels \((p = 1, \ldots, n)\):
\[
a_i^{(r)} = m_{i-1} - 2m_i + m_{i+1},
\]
\[
i = 1, \ldots, p - 1 \text{ for } A_{2n}^{(2)}, B_n^{(1)}, D_{n+1}^{(2)},
\]
\[
i = 1, \ldots, p - 2 \text{ for } A_{2n-1}^{(2)}, C_n^{(1)},
\]
\[
i = 1, \ldots, p - 3 \text{ for } D_n^{(1)}.
\]
Moreover, for the values of $i$ not included above:

\[
A^{(2)}_{2n} : \quad a_p^{(r)} = m_{p-1} - m_{p}, \quad (4.17)
\]

\[
B^{(1)}_n : \quad a_p^{(r)} = m_{p-2} - m_{p}, \quad (4.18)
\]

\[
A^{(2)}_{2n-1} : \quad a_{p-1}^{(r)} = m_{p-2} - 2m_{p-1} + (1 + \delta_{p,n})m_{p},
\]
\[
a_p^{(r)} = m_{p-2} - (1 + \delta_{p,n})m_{p}, \quad (4.19)
\]

\[
C^{(1)}_n : \quad a_{p-1}^{(r)} = m_{p-2} - 2m_{p-1} + (1 + \delta_{p,n})m_{p},
\]
\[
a_p^{(r)} = m_{p-1} - (1 + \delta_{p,n})m_{p}, \quad (4.20)
\]

\[
D^{(1)}_n : \quad a_{p-2}^{(r)} = m_{p-3} - 2m_{p-2} + m_{p-1} + \delta_{p,n}m_{p},
\]
\[
a_{p-1}^{(r)} = m_{p-2} - 2m_{p-1} + m_p + (\delta_{p,n-1} - \delta_{p,n})m_{n},
\]
\[
a_p^{(r)} = m_{p-2} - (\delta_{p,n-1} + \delta_{p,n})m_{p}, \quad (4.21)
\]

\[
D^{(2)}_{n+1} : \quad a_p^{(r)} = 2m_{p-1} - 2m_{p}. \quad (4.22)
\]

We remind the reader that $m_0$ is defined in (4.9).

For the cases of overlap with previous results (namely, $A^{(2)}_{2n}$ with $p = 0, n$ \cite{28}; $A^{(2)}_{2n-1}$ with $p = 0, n$ \cite{21}; and $D^{(2)}_{n+1}$ with $p = n$ \cite{21}), the results match.

### 4.3 Examples

We now illustrate the results of Sec. 4.2 with two simple examples.

#### 4.3.1 $A^{(2)}_{2n}$ with $n = 3$

As a first example, we consider the case $A^{(2)}_{2n}$ with $n = 3$, two sites ($N = 2$), and either $\varepsilon = 0$ or $\varepsilon = 1$. The four possibilities $p = 0, 1, 2, 3$ are summarized in Table 4. By solving the Bethe equations (see Sec. 3.3.1) with a generic value of anisotropy $\eta$, we obtain solutions (not shown\footnote{For the cases $p = 0$ and $p = n$, such solutions can be found in tables in \cite{28}.}) with the values of $m_1, m_2, m_3$ displayed in the table. The corresponding Dynkin labels obtained using the formulas from Sec. 4.2, are also displayed in the table. Finally, the irreducible representations of the “left” and “right” algebras corresponding to these Dynkin labels (obtained e.g. using LieART \cite{31}) are shown in the final column. By explicit diagonalization of the transfer matrix, we confirm that the degeneracies of the eigenvalues exactly match with the dimensions of the corresponding irreps.
Table 4: Numbers of Bethe roots and Dynkin labels for $A_{2n}^{(2)}$ with $n = 3, N = 2$.

| $p$ | $U_q(B_3)$ | $U_q(B_2) \otimes U_q(C_1)$ | $U_q(B_1) \otimes U_q(C_2)$ | $U_q(C_3)$ |
|-----|------------|-----------------------------|-----------------------------|------------|
| $0$ | 0 0 0 2 0 0 | 0 0 0 0 0 2 (1,3)           | 0 0 0 2 0 0 (1,10)          | 0 0 0 2 0 0 21 |
|     | 1 0 0 1 0 1 | 1 0 0 1 0 1 (1,5)           | 1 0 0 1 0 1 (5,1)           | 1 0 0 1 0 1 14 |
|     | 2 2 2 0 0 0 | 2 1 0 1 0 1 (14,1)          | 2 1 0 1 0 1 (10,1)          | 2 2 2 0 0 0 21 |
| $1$ | 1 0 0 0 0 0 | 1 0 0 0 0 0 (3,1)           | 1 0 0 0 0 0 (3,1)           | 1 0 0 0 0 0 21 |
|     | 2 0 0 2 0 0 | 2 0 0 2 0 0 (1,10)          | 2 0 0 2 0 0 (1,10)          | 2 0 0 2 0 0 21 |
| $2$ | 1 1 0 1 0 1 | 1 1 0 2 1 0 (10,1)          | 1 1 0 2 1 0 (10,1)          | 1 1 0 1 0 1 14 |
|     | 2 2 2 0 0 0 | 2 2 2 0 0 0 (1,10)          | 2 2 2 0 0 0 (1,10)          | 2 2 2 0 0 0 21 |
|     | 2 2 2 0 0 0 | 2 2 2 0 0 0 (1,10)          | 2 2 2 0 0 0 (1,10)          | 2 2 2 0 0 0 21 |
| $3$ | 1 0 0 0 1 0 | 1 1 1 1 1 0 (2,6)           | 1 1 1 1 1 0 (2,6)           | 1 0 0 0 1 0 14 |
|     | 2 2 2 0 0 0 | 2 2 2 0 0 0 (1,10)          | 2 2 2 0 0 0 (1,10)          | 2 2 2 0 0 0 21 |

4.3.2 $D_n^{(1)}$ with $n = 4$

As a second example, we consider the case $D_n^{(1)}$ with $n = 4$, two sites ($N = 2$), and with $\varepsilon = 0$. The three cases $p = 0, 2, 4$ are summarized in Table 5 (We omit the “special” cases $p = 1, 3$, whose results are the same as for $p = 0, 4$, respectively, see Table 3.) By solving the Bethe equations (see Sec. 3.3.1) with a generic value of anisotropy $\eta$, we obtain solutions (not shown) with the values of $m_1, m_2, m_3, m_4$ displayed in the table. The corresponding Dynkin labels obtained using the formulas from Sec. 4.2 are also displayed in the table. Finally, the irreps of the “left” and “right” algebras corresponding to these Dynkin labels are shown in the final column.

Notice that the values of $m$’s and Dynkin labels for $p = 0$ and $p = 4$ in Table 5 are exactly the same, which is due to the $p \leftrightarrow n - p$ duality (2.21).

The degeneracy pattern is particularly interesting for the case $p = 2$ in Table 5. Indeed, by explicitly diagonalizing the transfer matrix for this case\textsuperscript{10}, we find the following degeneracies

$$\{1, 1, 12, 16, 16, 18\}.$$ (4.23)

That is, one eigenvalue is repeated 18 times; two distinct eigenvalues are each repeated 16 times.

\textsuperscript{10}We emphasize that we restrict to generic values of $\eta$. 

25
times; etc. What is happening is that the irreps \((1, 9), (9, 1)\) are degenerate, thereby giving rise to the 18-fold degeneracy, due to the self-duality \((2.22)\). Moreover, the irreps \((1, 3), (3, 1), (1, \bar{3}), (\bar{3}, 1)\) are all degenerate, thereby giving rise to the 12-fold degeneracy, due to the self-duality \((2.22)\) and \(Z_2\) symmetries \((2.23), (2.24)\).

For eigenvalues corresponding to more than one irrep, it is enough to solve the Bethe equations corresponding to just one of those irreps, such as the irrep with the minimal values of \(m\)'s. Hence, for the example \((4.24)\), it is enough to consider the reference state \((m_1 = m_2 = m_3 = m_4 = 0)\). For the example \((4.25)\), it is enough to consider the state with \(m_1 = 1, m_2 = m_3 = m_4 = 0\). Note that a non-minimal set \(\{m_1, m_2, \ldots, m_n\}\) generally does not form a monotonic decreasing sequence, i.e. does not satisfy \(m_1 \geq m_2 \geq \ldots \geq m_n\).

---

| \(p = 0\) & \(U_q(D_4)\) | \(m_1\) | \(m_2\) | \(m_3\) | \(m_4\) | \(a_1^{(l)}\) | \(a_2^{(l)}\) | \(a_3^{(l)}\) | \(a_4^{(l)}\) | Irreps. |
|---|---|---|---|---|---|---|---|---|---|
| 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 | 35 |
| 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 | 28 |
| 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 | 1 |

| \(p = 2\) & \(U_q(D_2) \otimes U_q(D_2)\) | \(m_1\) | \(m_2\) | \(m_3\) | \(m_4\) | \(a_1^{(l)}\) | \(a_2^{(l)}\) | \(a_1^{(r)}\) | \(a_2^{(r)}\) | Irreps. |
|---|---|---|---|---|---|---|---|---|---|
| 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 | \(\{1, 9\}\) |
| 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 | \(\{9, 1\}\) |
| 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 | \(\{1, 3\}\) |
| 2 & 2 & 0 & 1 & 2 & 0 & 0 & 0 | \(\{3, 1\}\) |
| 2 & 2 & 1 & 0 & 0 & 2 & 0 & 0 | \(\{3, 1\}\) |
| 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 | \(\{2, 1, 1\}\) |

| \(p = 4\) & \(U_q(D_4)\) | \(m_1\) | \(m_2\) | \(m_3\) | \(m_4\) | \(a_1^{(r)}\) | \(a_2^{(r)}\) | \(a_3^{(r)}\) | \(a_4^{(r)}\) | Irreps. |
|---|---|---|---|---|---|---|---|---|---|
| 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 | 35 |
| 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 | 28 |
| 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 | 1 |

Table 5: Numbers of Bethe roots and Dynkin labels for \(D_n^{(1)}\) with \(n = 4, N = 2\).

\[ (1, 9), (9, 1) \]  \hspace{1cm} (4.24)

\[ (1, 3), (3, 1), (1, \bar{3}), (\bar{3}, 1) \]  \hspace{1cm} (4.25)

---

\footnote{11 It can happen that an eigenvalue corresponding to a single irrep is described by more than one set of Bethe roots, and therefore by more than one set of \(m\)'s; and for some cases with \(\frac{n}{2} \leq p < n\), the set of \(m\)'s corresponding to the Dynkin labels for the irrep may not be minimal. For example, for \(C_4^{(1)}\) with \(p = 3\) and \(N = 2\), the transfer matrix has an eigenvalue with degeneracy 12 and Dynkin labels \((a_1^{(l)}, a_1^{(r)}, a_2^{(r)}, a_3^{(r)}) = (1, 1, 0, 0)\), which according to the formulas in section 4.2 corresponds to \((m_1, m_2, m_3, m_4) = (1, 1, 1, 0)\).}
For $\varepsilon = 1$ (and still $n = 4, p = 2$), the transfer matrix has an additional “bonus" symmetry \[1\]. Consequently, the two irreps $(4,4)$ in Table 5 become degenerate (giving rise to a 32-fold degeneracy), and the two irreps $(1,1)$ become degenerate (giving rise to a 2-fold degeneracy). Interestingly, these levels have the singular (exceptional) Bethe roots $u^{(1)} = 2\eta, u^{(2)} = 4\eta$; and for the 2-fold degenerate level, these Bethe roots are repeated. This phenomenon is discussed further in Appendix A.

5 Duality and the Bethe ansatz

For the cases $C_n^{(1)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$, the $p \leftrightarrow n-p$ duality property of the transfer matrix (2.21) is reflected in the Bethe ansatz solution. For concreteness, we restrict our attention here to the case $C_n^{(1)}$, for which

$$f(u, p) = -\phi(u, p), \quad (5.1)$$

where $\phi(u, p)$ is given by (3.7).

The duality property of the transfer matrix (2.21) implies that corresponding eigenvalues satisfy

$$\Lambda(u, p) = f(u, p) \Lambda(u, n-p). \quad (5.2)$$

Let us define the rescaled eigenvalue $\lambda(u, p)$ such that

$$\Lambda(u, p) = \phi(u, p) \lambda(u, p). \quad (5.3)$$

In terms of $\lambda(u, p)$, the duality relation (5.2) takes the form

$$\lambda(u, p) = \frac{1}{f(u, p)} \lambda(u, n-p), \quad (5.4)$$

as follows from (5.1), (5.3) and $f(u, n-p) = 1/f(u, p)$.

Let us now try to understand how the duality relation (5.4) emerges from the Bethe ansatz solution (3.6), which in terms of $\lambda(u, p)$ (5.3) reads

$$\lambda(u, p) = A(u) z_0(u) y_0(u, p) c(u)^{2N} + A(u) \tilde{z}_0(u) \tilde{y}_0(u, p) \tilde{c}(u)^{2N} + \{ \sum_{l=1}^{n-1} \left[ z_l(u) y_l(u, p) B_l(u) + \tilde{z}_l(u) \tilde{y}_l(u, p) \tilde{B}_l(u) \right] \} b(u)^{2N}. \quad (5.5)$$

For the self-dual case $p = n/2$, the relation (5.4) is obvious, since $f(u, n/2) = 1$. For the case $p = 0$, we note the identity

$$\frac{y_l(u, 0)}{y_l(u, n)} = \frac{1}{f(u, 0)}, \quad l = 0, 1, \ldots, n-1. \quad (5.6)$$

Indeed, one can solve the Bethe equations (3.32), (3.33), (3.39), (3.40) and find such a solution for this eigenvalue. However, this set of $m$’s is not minimal, as one can find another solution of these Bethe equations for this eigenvalue with only $(m_1, m_2, m_3, m_4) = (1, 1, 0, 0)$. Another example is $D_4^{(2)}$ with $p = 2$ and $N = 2$, for which there is an eigenvalue with degeneracy 3 and Dynkin labels $(a_1^{(l)}, a_1^{(r)}, a_2^{(r)}) = (2, 0, 0)$, corresponding to $(m_1, m_2, m_3) = (2, 2, 1)$; but by solving the Bethe equations we can also find it with $(m_1, m_2, m_3) = (2, 1, 1)$.

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Since $A(u)$ and $\{B_i(u)\}$ for $p = 0$ are the same as for $p = n$ (the Bethe equations for $p = 0$ are the same as for $p = n$), it follows from (5.5) and (5.6) that

$$\lambda(u, 0) = \frac{1}{f(u, 0)} \lambda(u, n),$$

in agreement with (5.4).

To derive the duality relation (5.4) from the Bethe ansatz solution for $0 < p < n/2$ requires more effort. For simplicity, let us consider as an example the case $n = 3$ with $p = 1$, which is related by duality to $p = 2$. The rescaled eigenvalue is given by (5.5)

$$\lambda(u, p) = z_0(u) y_0(u, p) \frac{Q^{[1]}(u + 2\eta)Q^{[2]}(u - 2\eta)}{Q^{[1]}(u - 2\eta)Q^{[2]}(u - 4\eta)} \left[2 \sinh(\frac{u}{2} - 2\eta) \sinh(\frac{u}{2} - 8\eta)\right]^{2N} \left[2 \sinh(\frac{u}{2} - 8\eta)\right]^{2N} + \ldots,$$

where the crossed terms (indicated by the ellipsis) have not been explicitly written. Let us define the barred Q-functions

$$\bar{Q}^{[l]}(u) = \prod_{j=1}^{\bar{m}_1} \sinh(\frac{1}{2}(u - \bar{\omega}_j^{[l]})) \sinh(\frac{1}{2}(u + \bar{\omega}_j^{[l]})),$$

$$\bar{Q}^{[l]}(-u) = \bar{Q}^{[l]}(u),$$

(in terms of unbarred ones $Q^{[l]}(u)$) as follows:

$$S(u) - S(-u) = \mathcal{C} \sinh^{2N}(\frac{u}{2}) \sinh(u) Q^{[2]}(u),$$

$$S(u) = \chi(u + 2\eta) Q^{[1]}(u + 2\eta) \bar{Q}^{[1]}(u - 2\eta),$$

$$\bar{Q}^{[2]}(u) = Q^{[2]}(u),$$

$$\bar{Q}^{[3]}(u) = Q^{[3]}(u),$$

where

$$\chi(u) = 1 + \cosh(u), \quad \mathcal{C} = 2 \sinh(2\eta(1 + 2m_1 - m_2 - N)),$$

and

$$\bar{m}_1 = N - m_1 + m_2, \quad \bar{m}_2 = m_2, \quad \bar{m}_3 = m_3.$$  

(The above results for $\bar{m}_1$ and $\mathcal{C}$ follow from the asymptotic limit $u \to \infty$ of (5.10).) We show below that, if $Q^{[l]}(u)$ are the Q-functions for $p = 1$, then $\bar{Q}^{[l]}(u)$ are the Q-functions for $p = 2$. 

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5.1 Duality of the Bethe equations

We first show that (5.10)–(5.13) map the $p = 1$ Bethe equations:

$$\left[ \frac{\sinh \left( \frac{u_1^2}{2} + \eta \right)}{\sinh \left( \frac{u_1^2}{2} - \eta \right)} \right]^{2N} \left[ \frac{\cosh \left( \frac{u_1^2}{2} - 2\eta \right)}{\cosh \left( \frac{u_1^2}{2} + 2\eta \right)} \right]^2 = \frac{Q^{[1]}_k (u_k^1 + 4\eta) Q^{[2]}_k (u_k^1 - 2\eta)}{Q^{[1]}_k (u_k^1 - 4\eta) Q^{[2]}_k (u_k^1 + 2\eta)},$$

(5.16)

$$1 = \frac{Q^{[1]}_k (u_k^2 - 2\eta) Q^{[2]}_k (u_k^2 + 4\eta) Q^{[3]}_k (u_k^2 - 2\eta)}{Q^{[1]}_k (u_k^2 + 2\eta) Q^{[2]}_k (u_k^2 - 4\eta) Q^{[3]}_k (u_k^2 + 2\eta)},$$

(5.17)

$$1 = \frac{Q^{[2]}_k (u_k^3 - 4\eta) Q^{[3]}_k (u_k^3 + 8\eta)}{Q^{[2]}_k (u_k^3 + 4\eta) Q^{[3]}_k (u_k^3 - 8\eta)},$$

(5.18)

to the $p = 2$ Bethe equations:

$$\left[ \frac{\sinh \left( \frac{\bar{u}_1^2}{2} + \eta \right)}{\sinh \left( \frac{\bar{u}_1^2}{2} - \eta \right)} \right]^{2N} = \frac{\bar{Q}^{[1]}_k (\bar{u}_k^1 + 4\eta) \bar{Q}^{[2]}_k (\bar{u}_k^1 - 2\eta)}{\bar{Q}^{[1]}_k (\bar{u}_k^1 - 4\eta) \bar{Q}^{[2]}_k (\bar{u}_k^1 + 2\eta)},$$

(5.19)

$$\left[ \frac{\cosh \left( \frac{\bar{u}_1^2}{2} - \eta \right)}{\cosh \left( \frac{\bar{u}_1^2}{2} + \eta \right)} \right]^2 = \frac{\bar{Q}^{[1]}_k (\bar{u}_k^2 - 2\eta) \bar{Q}^{[2]}_k (\bar{u}_k^2 + 4\eta) \bar{Q}^{[3]}_k (\bar{u}_k^2 - 2\eta)}{\bar{Q}^{[1]}_k (\bar{u}_k^2 + 2\eta) \bar{Q}^{[2]}_k (\bar{u}_k^2 - 4\eta) \bar{Q}^{[3]}_k (\bar{u}_k^2 + 2\eta)},$$

(5.20)

$$1 = \frac{\bar{Q}^{[2]}_k (\bar{u}_k^3 - 4\eta) \bar{Q}^{[3]}_k (\bar{u}_k^3 + 8\eta)}{\bar{Q}^{[2]}_k (\bar{u}_k^3 + 4\eta) \bar{Q}^{[3]}_k (\bar{u}_k^3 - 8\eta)},$$

(5.21)

Evidently, it follows from (5.12) and (5.13) that (5.18) implies (5.21).

Setting $u = u_k^1$ in (5.10), remembering that $Q^{[2]}(u_k^2) = 0$, we obtain the relation

$$\frac{Q^{[1]}_k (u_k^2 - 2\eta)}{Q^{[1]}_k (u_k^2 + 2\eta)} = \frac{\chi (u_k^2 + 2\eta)}{\chi (u_k^2 - 2\eta)} \frac{\bar{Q}^{[1]}_k (\bar{u}_k^2 - 2\eta)}{\bar{Q}^{[1]}_k (\bar{u}_k^2 + 2\eta)}.$$

(5.22)

With the help of this relation, it follows that (5.17) implies (5.20).

Setting $u = \pm u_k^1 + 2\eta$ in (5.10), noting that therefore $Q^{[1]}(u - 2\eta) = 0$ and $S(-u) = 0,$
we obtain the pair of relations
\[
\chi(u^{[1]}_n) Q^{[1]}(u^{[1]}_n) Q^{[2]}(u^{[1]}_n) = c \sinh^{2N}(\frac{u^{[1]}_n}{2}) \sinh(u^{[1]}_n) Q^{[2]}(u^{[1]}_n) + \eta \sinh(u^{[1]}_n) Q^{[2]}(u^{[1]}_n) = \eta \sinh(u^{[1]}_n) Q^{[2]}(u^{[1]}_n) - \eta \sinh(u^{[1]}_n - 2\eta) Q^{[2]}(u^{[1]}_n - 2\eta). \tag{5.23}
\]

Forming the ratio of these relations, we arrive at the Bethe equation (5.16). Similarly, setting \( u = \pm u^{[1]}_n - 2\eta \) in (5.10), we obtain the Bethe equation (5.19).

### 5.2 Duality of the transfer-matrix eigenvalues

In order to relate the transfer-matrix eigenvalues for \( p = 1 \) and \( p = 2 \), we observe from (5.10) that
\[
c = \frac{S(u) - S(-u)}{\sinh^{2N}(\frac{u}{2}) \sinh(u) Q^{[2]}(u)} = \frac{S(u - 4\eta) - S(-u + 4\eta)}{\sinh^{2N}(\frac{u}{2} - 2\eta) \sinh(u - 4\eta) Q^{[2]}(u - 4\eta)} \tag{5.24}
\]

where the second equality follows from shifting \( u \mapsto u - 4\eta \). Making use of (5.11) and (5.12), and rearranging terms, we obtain the relation
\[
\sinh^{2N}(\frac{u}{2} - 2\eta) \sinh(u - 4\eta) \chi(u + 2\eta) \frac{Q^{[1]}(u + 2\eta)}{Q^{[1]}(u - 2\eta)}
\]
\[
+ \sinh^{2N}(\frac{u}{2}) \sinh(u) \chi(u + 2\eta) \frac{Q^{[1]}(u + 6\eta)}{Q^{[1]}(u - 2\eta) Q^{[2]}(u - 4\eta)}
\]
\[
= \sinh^{2N}(\frac{u}{2} - 2\eta) \sinh(u - 4\eta) \chi(u - 2\eta) \frac{Q^{[1]}(u - 2\eta)}{Q^{[1]}(u - 2\eta)} \frac{Q^{[2]}(u)}{Q^{[2]}(u - 4\eta)}
\]
\[
+ \sinh^{2N}(\frac{u}{2}) \sinh(u) \chi(u - 2\eta) \frac{Q^{[1]}(u - 6\eta)}{Q^{[1]}(u - 2\eta) Q^{[2]}(u - 4\eta)} \tag{5.25}
\]

This relation implies that
\[
z_0(u) y_0(u, 1) \frac{Q^{[1]}(u + 2\eta)}{Q^{[1]}(u - 2\eta)} \left[2 \sinh(\frac{u}{2} - 2\eta) \sinh(\frac{u}{2} - 2\eta) \right]^{2N}
\]
\[
+ z_1(u) y_1(u, 1) \frac{Q^{[1]}(u + 6\eta)}{Q^{[1]}(u - 2\eta) Q^{[2]}(u - 4\eta)} \left[2 \sinh(\frac{u}{2} - 2\eta) \sinh(\frac{u}{2} - 2\eta) \right]^{2N}
\]
\[
= \frac{1}{f(u, 1)} \left\{ z_0(u) y_0(u, 2) \frac{Q^{[1]}(u + 2\eta)}{Q^{[1]}(u - 2\eta)} \left[2 \sinh(\frac{u}{2} - 2\eta) \sinh(\frac{u}{2} - 2\eta) \right]^{2N}
\]
\[
+ z_1(u) y_1(u, 2) \frac{Q^{[1]}(u + 6\eta)}{Q^{[1]}(u - 2\eta) Q^{[2]}(u - 4\eta)} \left[2 \sinh(\frac{u}{2} - 2\eta) \sinh(\frac{u}{2} - 2\eta) \right]^{2N} \right\} . \tag{5.26}
\]
Finally, in view of also (5.8), (5.12), (5.13) and the identity
\[ \frac{y_2(u, 1)}{y_2(u, 2)} = \frac{1}{f(u, 1)}, \] (5.27)
we conclude that the duality relation (5.4) is indeed satisfied by the Bethe ansatz solution for \( n = 3, p = 1 \).

### 5.3 Duality of the Dynkin labels

It is interesting to see if the formulas in Sec. 4.2 for the Dynkin labels are compatible with duality. For the case \( n = 3, p = 1 \), where the QG symmetry is \( U_q(C_2) \otimes U_q(C_1) \), the Dynkin labels are given by
\[ a_1^{(l)} = m_1 - 2m_2 + 2m_3, \]
\[ a_2^{(l)} = m_2 - 2m_3, \]
\[ a_1^{(r)} = N - m_1. \] (5.28)

On the other hand, for the dual case \( n = 3, p = 2 \), where the QG symmetry is \( U_q(C_1) \otimes U_q(C_2) \), the Dynkin labels are given by
\[ \bar{a}_1^{(l)} = \bar{m}_2 - 2\bar{m}_3, \]
\[ \bar{a}_1^{(r)} = N - 2\bar{m}_1 + \bar{m}_2, \]
\[ \bar{a}_2^{(r)} = \bar{m}_1 - \bar{m}_2, \] (5.29)

where we again use a bar to denote quantities for the \( p = 2 \) case. If a transfer-matrix eigenvalue (\( \Lambda(u, 1) \) or equivalently its dual \( \Lambda(u, 2) \)) forms a single irreducible representation of the QG, then we expect that the corresponding Dynkin labels (5.28) and (5.29) should be related by the duality relations\[12\]
\[ \bar{a}_1^{(l)} = a_1^{(r)}, \]
\[ \bar{a}_i^{(r)} = a_i^{(l)}, \quad i = 1, 2. \] (5.30)

Making use of the relation (5.15) between \( \{m_i\} \) and \( \{\bar{m}_i\} \), we find that the relations (5.30) are indeed satisfied, provided that the \( m \)'s satisfy the constraint
\[ N = m_1 + m_2 - 2m_3 \quad \text{or equivalently} \quad \bar{m}_1 - 2\bar{m}_2 + 2\bar{m}_3 = 0. \] (5.31)

Some simple examples for \( N = 2 \) are displayed in Table 6.

\[12\] For general values of \( n \) and \( p \), we expect the duality relations
\[ \bar{a}_i^{(l)} = a_i^{(r)}, \quad i = 1, \ldots, p, \]
\[ \bar{a}_i^{(r)} = a_i^{(l)}, \quad i = 1, \ldots, n - p, \]
where the unbarred and barred quantities correspond to \( p \) and \( n - p \), respectively.
Table 6: Numbers of Bethe roots, which satisfy the constraint (5.31), and the corresponding Dynkin labels for $C_n^{(1)}$ with $n = 3, N = 2$ and $p = 1, 2$.

Interestingly, not all transfer-matrix eigenvalues have Bethe roots that satisfy the constraint (5.31). (A simple example is the reference-state eigenvalue, for which $m_1 = m_2 = m_3 = 0$.) Such transfer-matrix eigenvalues correspond to reducible representations of the QG (i.e., they correspond to a direct sum of two or more irreps). Indeed, it was noted in [1] (see Sec. 6.4.2) that for $C_n^{(1)}$ with odd $n$ and $p = n+1$, there are additional degeneracies in the spectrum, which may be due to some yet unknown discrete symmetry.

5.4 Further remarks

We have seen that, for the case $C_n^{(1)}$ with $n = 3$, the relations (5.10)-(5.13) implement the duality transformation $p = 1 \leftrightarrow p = 2$ on the Bethe ansatz solution. Note that the Bethe roots corresponding to transfer-matrix eigenvalues related by this duality satisfy $u_k[2] = \bar{u}_k[2]$ and $u_k[3] = \bar{u}_k[3]$; i.e. only the type-1 Bethe roots $(u_k[1], \bar{u}_k[1])$ are different. We expect that, for $C_n^{(1)}$ with other values of $n$, as well as for $D_n^{(1)}$ and $D_{n+1}^{(2)}$, generalizations of the relations (5.10)-(5.13) can be found to implement the duality transformations $p \leftrightarrow n - p$ on the Bethe ansatz solutions. For supersymmetric (graded) integrable spin chains, a different type of “duality” transformation can be defined, which can be implemented on the corresponding Bethe ansatz solutions by relations somewhat analogous to (5.10)-(5.13), see e.g. [32, 33] and references therein.

6 Discussion

We have proposed Bethe ansatz solutions for several infinite families of integrable open quantum spin chains with QG symmetry that were identified in [1, 2]. In particular, we have found that the Bethe equations take the simple form (3.65), where the factor $\Phi_{l,p,n}(u)$, which is different from 1 only if $l = p$, is given by (3.57), (3.58), (3.64). We have also proposed formulas for the Dynkin labels of the Bethe states in terms of the numbers of Bethe roots of each type, see Eqs. (4.12) - (4.22). Finally, we have initiated an investigation of how the duality transformations (2.21) are implemented on the Bethe ansatz solutions, see (5.10)-(5.13).
We mention here a few of the many interesting problems that remain to be addressed. It would be desirable to use nested algebraic Bethe ansatz (see e.g. [26, 27]) to rederive the Bethe ansatz solutions, to obtain the Bethe states, and to prove the highest/lowest weight conjectures ([1,2], [1,3]). However, the latter computation would require using the reference state (3.2), which would in turn require a set of creation operators different from those used in [26, 27].

It would be interesting to find a Bethe ansatz solution for the case $D^{(2)}_{n+1}$ with $\varepsilon = 1$ (we considered in Sec. 3.3.2 only $\varepsilon = 0$), to find a completely universal form of the Bethe equations for the QG-invariant models considered here (see Sec. 3.3.3), to further investigate how Bethe ansatz solutions transform under duality (we focused in Sec. 5 primarily on the case $C^{(1)}_3$), and to understand the connection between bonus symmetry and singular solutions of the Bethe equations (see Appendix A). It would also be interesting to investigate the rational limit of these models, and to compare with results in the literature e.g. [34].

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A Bonus symmetry and singular solutions

For the cases $C^{(1)}_n, D^{(1)}_n, D^{(2)}_{n+1}$ with $p = \frac{n}{2}$ (n even) and $\varepsilon = 1$, the transfer matrix has a “bonus” symmetry (i.e., a symmetry in addition to self-duality), leading to higher degeneracies in comparison with $\varepsilon = 0$ [1, 2]. We observe here that the solutions of the Bethe equations corresponding to such degenerate levels are singular (exceptional).

As an example, we consider the case $C^{(1)}_2$ with $n = 2, p = 1$. From the $U_q(C_1) \otimes U_q(C_1)$ symmetry of the transfer matrix, we expect (for generic values of $\eta$) the following Hilbert space decompositions

\begin{equation}
N = 2 : \quad [(2,1) \oplus (1,2)]^{\otimes 2} = 2(1,1) \oplus 2(2,2) \oplus (3,1) \oplus (1,3), \quad (A.1)
\end{equation}

\begin{equation}
N = 3 : \quad [(2,1) \oplus (1,2)]^{\otimes 3} = 5(2,1) \oplus 5(1,2) \oplus 3(3,2) \oplus 3(2,3) \oplus (4,1) \oplus (1,4). \quad (A.2)
\end{equation}

However, by diagonalizing the transfer matrix directly, we observe the following degeneracy
patterns

\[ N = 2 : \quad \{1, 1, 4, 4, 6\} \quad \text{when } \varepsilon = 0, \quad (A.3) \]
\[ \{2, 8, 6\} \quad \text{when } \varepsilon = 1, \quad (A.4) \]

\[ N = 3 : \quad \{4, 4, 4, 4, 8, 12, 12, 12\} \quad \text{when } \varepsilon = 0, \quad (A.5) \]
\[ \{4, 8, 8, 12, 24\} \quad \text{when } \varepsilon = 1. \quad (A.6) \]

Let us first consider the case \( N = 2 \). Comparing the decomposition (A.1) with the degeneracies for \( \varepsilon = 0 \) (A.3), we see that they do not completely match: the \((3, 1)\) and \((1, 3)\) are degenerate (thereby giving rise to the 6-fold degeneracy) due to the self-duality (2.22). However, the degeneracies for \( \varepsilon = 1 \) (A.4) are even higher: the two \((2, 2)\) are degenerate (thereby giving rise to the 8-fold degeneracy) and the two \((1, 1)\) are degenerate (thereby giving rise to the 2-fold degeneracy) due to the “bonus” symmetry.

The key new point is that, among the Bethe roots corresponding to the levels with 8-fold degeneracy and 2-fold degeneracy, is the exact Bethe root \( u^1 = 2 \eta \) (which is repeated for the 2-fold degenerate level), for which the Bethe equations have a zero or pole.

The bonus symmetry is also present for \( N = 3 \), see (A.2), (A.5), (A.6). The levels that are degenerate due to the bonus symmetry (namely, the level with 24-fold degeneracy, and two levels with 8-fold degeneracy) again contain the singular solution \( u^1 = 2 \eta \), which is repeated for the 8-fold degenerate levels.

For all the examples that we have checked (another example is noted in Sec. 4.3.2), singular solutions occur if and only if the states are affected by the bonus symmetry. However, a general understanding of this phenomenon is still lacking.

**B Bethe ansatz solutions for some additional cases**

In the main part of this paper, we do not consider the K-matrices (2.5) for the cases \( A_{2n-1}^{(2)} \) and \( B_{n}^{(1)} \) with \( p = 1 \), and \( D_{n}^{(1)} \) with \( p = 1, n - 1 \), as emphasized in (2.10). These K-matrices are excluded because the corresponding transfer matrices do not have QG symmetry corresponding to removing one node from the Dynkin diagram. (This is the reason why we consider instead the K-matrices (2.11) and (2.12) for these cases.) Nevertheless, the transfer matrices for these cases are integrable, and we have also determined their spectra. We briefly note here the Bethe ansatz solutions for these cases.

For these cases (i.e., for the transfer matrices constructed using the K-matrices (2.5) for \( A_{2n-1}^{(2)} \) and \( B_{n}^{(1)} \) with \( p = 1 \), and for \( D_{n}^{(1)} \) with \( p = 1, n - 1 \)), the transfer matrix eigenvalues are in fact given by (3.6), where the functions \( y_l(u, p) \) are given by (3.25), (3.27), (3.28). Hence, the Bethe equations for \( A_{2n-1}^{(2)}, B_{n}^{(1)} \) and \( D_{n}^{(1)} \) with \( p = 1 \) are again those in Sec. 3.3.1, with the functions \( \Phi_{l,p,n} \) given by (3.57).

For \( D_{n}^{(1)} (n > 3) \) with \( p = n - 1 \), the Bethe equations for \( l \leq n - 2 \) are the ones given in...
but the Bethe equations for \( l = n - 1, n \) are given by

\[
\begin{bmatrix}
\cosh \left( \frac{u_{[n-1]}^{[n-1]}}{2} + \eta + \frac{i \pi \varepsilon}{2} \right) \\
\cosh \left( \frac{u_{[n-1]}^{[n-1]}}{2} - \eta + \frac{i \pi \varepsilon}{2} \right)
\end{bmatrix}^2 = \frac{Q^{[n-2]} \left( u_{[k]}^{[n-1]} - 2 \eta \right) Q_{[k]}^{[n-1]} \left( u_{[k]}^{[n-1]} + 4 \eta \right)}{Q^{[n-2]} \left( u_{[k]}^{[n-1]} + 2 \eta \right) Q_{[k]}^{[n-1]} \left( u_{[k]}^{[n-1]} - 4 \eta \right)}, \tag{B.1}
\]

\[
\begin{bmatrix}
\cosh \left( \frac{u_{[n]}^{[n]} + \eta + \frac{i \pi \varepsilon}{2}}{2} \right) \\
\cosh \left( \frac{u_{[n]}^{[n]} - \eta + \frac{i \pi \varepsilon}{2}}{2} \right)
\end{bmatrix}^2 = \frac{Q^{[n-2]} \left( u_{[k]}^{[n]} - 2 \eta \right) Q_{[k]}^{[n]} \left( u_{[k]}^{[n]} + 4 \eta \right)}{Q^{[n-2]} \left( u_{[k]}^{[n]} + 2 \eta \right) Q_{[k]}^{[n]} \left( u_{[k]}^{[n]} - 4 \eta \right)}. \tag{B.2}
\]

instead of by (3.42) and (3.43). In contrast with the QG-invariant case, the LHS of (B.2) has a nontrivial (\( \neq 1 \)) factor, even though \( l = n \neq p \).

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