Generalized derivations and general relativity

Michael Heller
Copernicus Center for Interdisciplinary Studies, Cracow, Poland

Tomasz Miller*   Leszek Pysiak   Wiesław Sasin
Department of Mathematics and Information Science,
Warsaw University of Technology
Plac Politechniki 1, 00-661 Warsaw, Poland
and Copernicus Center for Interdisciplinary Studies, Cracow, Poland

December 12, 2013

Abstract

We construct differential geometry (connection, curvature, etc.) based on generalized derivations of an algebra $\mathcal{A}$. Such a derivation, introduced by Brešar in 1991, is given by a linear mapping $u : \mathcal{A} \rightarrow \mathcal{A}$ such that there exists a usual derivation $d$ of $\mathcal{A}$ satisfying the generalized Leibniz rule $u(ab) = u(a)b + ad(b)$ for all $a, b \in \mathcal{A}$. The generalized geometry “is tested” in the case of the algebra of smooth functions on a manifold. We then apply this machinery to study generalized general relativity. We define the Einstein–Hilbert action and deduce from it Einstein’s field equations. We show that for a special class of metrics containing, besides the usual metric components, only one nonzero term, the action reduces to the O’Hanlon action that is the Brans–Dicke action with potential and with the parameter $\omega$ equal to zero. We also show that the generalized Einstein equations (with zero energy–stress tensor) are equivalent to those of the Kaluza–Klein theory satisfying a “modified cylinder condition” and having a noncompact extra dimension. This opens a possibility to consider Kaluza–Klein models with a noncompact extra dimension that

*Corresponding author. E-mail: T.Miller@mini.pw.edu.pl
remains invisible for a macroscopic observer. In our approach, this extra dimension is not an additional physical space-time dimension but appears because of the generalization of the derivation concept.

PACS Nos.: 02.40.-k, 04.50.-h, 04.50.Cd, 02.10.De

1 Introduction

In the present paper we investigate differential geometry based on generalized derivations introduced in 1991 by Brešar [1]. He originally used them in his algebraic research concerning a certain generalization of Posner’s theorem [2]. Systematic studies of algebraic properties of generalized derivations were initiated in 1998 by Hvala [3]. Since then, generalized derivations have been thoroughly studied by numerous researchers [4, 5, 6] and the concept itself was further developed to encompass e.g. higher order derivations [7] and nonassociative settings [8]. For a brief summary and further references, see Ashraf et al. [9]. However, as far as we know the geometric content of this notion has not yet been investigated. The aim of the present paper is to develop elements of differential geometry based on the concept of generalized derivations and to see how this geometry works in the context of general relativity theory.

After briefly presenting the generalization itself (section 2), we construct basic notions of differential geometry based on this generalization (section 3), and apply them to the case of algebra of smooth functions on a lorentzian manifold (section 4). Because the general case leads to rather involved calculations, we specify to the case of a simplified metric with only one additional nonzero term (section 5). We apply the generalized geometry to formulate a generalized theory of relativity (section 6). We start with a natural choice of the Einstein–Hilbert action and deduce from it the generalized Einstein equations. They involve no free parameters. A term modeling the space–time dependence on the gravitational “constant” $G$ leads to similar effects as the ones in Brans–Dicke theory [10]. In fact, we show that for a special class of metrics discussed in section 5 the action reduces to the O’Hanlon action [11], that is the Brans–Dicke action with potential and with the Brans–Dicke parameter $\omega$ equal to zero.

Even at first glance, the generalized derivation-based approach to general relativity seems to resemble the idea standing behind Kaluza–Klein–type the-
ories. We show (section 7) that indeed this observation is correct and the generalized Einstein equations (with zero energy–stress tensor) can be equivalently obtained from a Kaluza–Klein theory involving a modified version of the “cylinder condition”. However, unlike in standard gravity theories with extra dimensions, this equivalent Kaluza–Klein theory features a noncompact extra dimension. The generalized general theory of relativity may thus serve as an alternative formulation of a Kaluza–Klein theory with a single noncompact extra dimension that is not associated with any extra space–time dimension. This effectively avoids the conundrum: why is this extra dimension not physically observed?

Let us finally mention that there exist many generalizations of standard geometry, of which the most renowned is the one developed by Connes and his collaborators (see for instance the monographs [12, 13, 14]). The present work can be situated in the stream of derivation-based approaches developed by Dubois-Violette [15, 16, 17].

2 Generalized derivations

Throughout this section, \( \mathcal{A} \) denotes an (abstract) associative algebra over \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). The algebra \( \mathcal{A} \) can in general be nonunital and noncommutative. \( \mathcal{Z}(\mathcal{A}) \) denotes the center of the algebra \( \mathcal{A} \).

To begin with, let us recall that a linear mapping \( d : \mathcal{A} \rightarrow \mathcal{A} \) is called a derivation if it satisfies the Leibniz rule: \( d(ab) = d(a)b + ad(b) \) for all \( a, b \in \mathcal{A} \). The set of all derivations of \( \mathcal{A} \) is denoted \( \text{Der}(\mathcal{A}) \).

Derivations have the following four properties, indispensable for the derivation-based approach to differential geometry

(i) \( \forall d_1, d_2 \in \text{Der}(\mathcal{A}) \forall \lambda_1, \lambda_2 \in \mathbb{K} \quad \lambda_1 d_1 + \lambda_2 d_2 \in \text{Der}(\mathcal{A}) \),

(ii) \( \forall d_1, d_2 \in \text{Der}(\mathcal{A}) \quad [d_1, d_2] \in \text{Der}(\mathcal{A}) \),

(iii) \( \forall d \in \text{Der}(\mathcal{A}) \forall f \in \mathcal{Z}(\mathcal{A}) \quad fd \in \text{Der}(\mathcal{A}) \),

(iv) \( \forall d \in \text{Der}(\mathcal{A}) \forall f \in \mathcal{Z}(\mathcal{A}) \quad d(f) \in \mathcal{Z}(\mathcal{A}) \).

By (i, ii) \( \text{Der}(\mathcal{A}) \) possesses the Lie algebra structure. By (i, iii), it is also a \( \mathcal{Z}(\mathcal{A}) \)-module. Finally, (iv) states that derivations leave the center of \( \mathcal{A} \) invariant.
By *inner derivation* induced by \( a \in \mathcal{A} \) we mean a derivation \( \text{ad}_a(b) = [a, b] = ab - ba \) for any \( b \in \mathcal{A} \). The set of all inner derivations is denoted \( \text{Inn}(\mathcal{A}) \).

In their 1991 paper [1], Brešar considered what was called a *generalized inner derivation*, that is a map \( I_{a,b} : \mathcal{A} \to \mathcal{A} \) given by

\[
\forall x \in \mathcal{A} \quad I_{a,b}(x) = ax + xb.
\]

Of course, \( \text{ad}_a = I_{a,-a} \). One can also easily notice that \( I_{a,b} \) satisfies

\[
I_{a,b}(xy) = I_{a,b}(x)y + x\text{ad}_b(y)
\]

for all \( x, y \in \mathcal{A} \). This fact motivated Brešar to formulate the following definition.

A linear mapping \( u : \mathcal{A} \to \mathcal{A} \) is called a *generalized derivation* if there exists \( d \in \text{Der}(\mathcal{A}) \) such that the *generalized Leibniz rule*

\[
u(ab) = u(a)b + a\,d(b)
\]

holds for all \( a, b \in \mathcal{A} \). Derivation \( d \) in the preceding definition is called *associated* with \( u \). If such a derivation is unique, it is written as \( d_u \).

The set of all generalized derivations of \( \mathcal{A} \) is denoted \( \text{GDer}(\mathcal{A}) \).

The concept of generalized derivation covers the notion of a derivation and that of a *left multiplier*, that is, a linear map \( \mathcal{L} : \mathcal{A} \to \mathcal{A} \) satisfying \( \mathcal{L}(ab) = \mathcal{L}(a)b \) for all \( a, b \in \mathcal{A} \). In fact, one can show that any \( u \in \text{GDer}(\mathcal{A}) \) is a sum of a left multiplier and a derivation associated with \( u \). If this decomposition is unique, the left multiplier \( \mathcal{L}_u = u - d_u \) will also be called *associated* with \( u \).

Simple examples of left multipliers include the maps \( l_a \) defined as \( l_a(b) = ab \) for any \( b \in \mathcal{A} \). Left multipliers of this form we shall call *inner*. Another important example of a left multiplier is the identity map \( \text{id}_\mathcal{A} \). For algebras without left unity, \( \text{id}_\mathcal{A} \) is not inner.

One can easily prove that generalized derivations satisfy (i–iii). However, in the case of some algebras (iv) does not hold for all generalized derivations. For our later geometrical applications it is important to single out those elements of \( \text{GDer}(\mathcal{A}) \) for which (iv) holds.

By \( \text{CGDer}(\mathcal{A}) \) we shall denote the set of generalized derivations of \( \mathcal{A} \) that leave \( \mathcal{Z}(\mathcal{A}) \) invariant. One can easily check that this set satisfies all properties (i–iv) and is a proper superset of \( \text{Der}(\mathcal{A}) \), because \( \text{id}_\mathcal{A} \in \text{CGDer}(\mathcal{A}) \setminus \text{Der}(\mathcal{A}) \).
3 Generalized derivation-based differential geometry

In this section we construct elements of differential geometry based on generalized derivations; we adopt the method analogous to what is done in similar situations [18, 19, 20]. The interested reader is referred also to works by Dubois-Violette [15, 16, 17].

For the sake of readability, let us denote the \( \mathcal{Z}(\mathcal{A}) \)-module \( \text{CGDer}(\mathcal{A}) \) simply by \( V \). Then \( V^* \equiv \text{Hom}_{\mathcal{Z}(\mathcal{A})}(V, \mathcal{Z}(\mathcal{A})) \) is its dual \( \mathcal{Z}(\mathcal{A}) \)-module.

Let \( G : V \times V \to \mathcal{Z}(\mathcal{A}) \) be a symmetric, \( \mathcal{Z}(\mathcal{A}) \)-bilinear map called metric. We will also assume that \( G \) is nondegenerate, that is, that the map \( \Phi_G : V \to V^* \) given by

\[
\Phi_G(u)(v) = G(u,v)
\]

is an isomorphism of \( \mathcal{Z}(\mathcal{A}) \)-modules.

We are now ready to define the preconnection \( \nabla^* : V \times V \to V^* \) by using the Koszul formula [19]

\[
(\nabla^*_u v)(w) = \frac{1}{2} \left[ u( G(v, w)) + v( G(u, w)) - w( G(u, v)) + G(w, [u, v]) + G(v, [w, u]) - G(u, [v, w]) \right]
\]

and then the Levi–Civita connection \( \nabla : V \times V \to V \) by

\[
\nabla = \Phi_G^{-1} \circ \nabla^*.
\]

As one can show by tedious but straightforward calculations, \( \nabla \) has almost identical properties to its well-known derivation-based counterpart, the only difference lying in the generalized Leibniz rule, we thus have

1° \( \nabla_{u_1+u_2} v = \nabla_{u_1} v + \nabla_{u_2} v \),
2° \( \nabla_{fu} v = f \nabla_u v \),
3° \( \nabla_u (v_1 + v_2) = \nabla_u v_1 + \nabla_u v_2 \),
4° \( \nabla_u (fv) = d_u(f) v + f \nabla_u v \) (generalized Leibniz rule),
5° \( \nabla_u v - \nabla_v u - [u, v] = 0 \) (torsion-freeness),
6° \( w( G(u, v)) = G( \nabla_u v, v) + G(u, \nabla_v v) \) (metric compatibility)
for all \( u, u_1, u_2, v, v_1, v_2, w \in V \) and \( f \in \mathcal{Z}(A) \). Moreover, the Levi–Civita connection is the unique connection satisfying 5° and 6°.

Having defined the Levi–Civita connection, one can readily introduce the \emph{Riemann curvature map} \( R : V \times V \times V \to V \) by

\[
R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.
\]

It is not difficult to check that, although the property 4° differs from the ordinary Leibniz rule for connections, the map \( R \) is \( \mathcal{Z}(A) \)-trilinear and thus it can be called the \emph{Riemann tensor}.

\( R \) can be shown to satisfy the usual Riemann tensor identities

\[
R(u, v) = -R(v, u),
\]

\[
\mathcal{G}(R(u, v)w, z) = -\mathcal{G}(R(u, v)z, w),
\]

\[
R(u, v)w + R(v, w)u + R(w, u)v = 0,
\]

\[
\mathcal{G}(R(u, v)w, z) = \mathcal{G}(R(w, z)u, v)
\]

for all \( u, v, w, z \in V \).

By demanding the \( \mathcal{Z}(A) \)-module \( \text{CGDer}(A) \) to be (at least locally) free, one can define the Ricci 2-form and the scalar curvature using standard construction involving the notion of a trace of an operator. With this in mind, let us move to an illustrative example of a (commutative) algebra, whose generalized derivations will be shown to possess interesting physical interpretation.

\section{Generalized derivations of the algebra of smooth functions}

Let \( M \) be an \( N \)-dimensional lorentzian manifold (we assume \( N \geq 2 \)) and let us consider the algebra \( A = \mathcal{C}^\infty(M) \) of smooth real-valued functions on \( M \) with the pointwise multiplication.

The set of derivations on \( A \) is a locally free \( A \)-module and

\[
\text{Der}(A) = \text{span}_{A}(\partial_0, \partial_1, \ldots, \partial_{N-1}),
\]

where \( \partial_\mu \equiv \frac{\partial}{\partial x_\mu} \) in a fixed map \( x = (x^0, x^1, \ldots, x^{N-1}) \).
Because \( \mathcal{Z}(\mathcal{A}) = \mathcal{A} \), all generalized derivations trivially leave the center invariant, \( \text{CGDer}(\mathcal{A}) = \text{GDer}(\mathcal{A}) \). In order to find the local basis of \( \text{GDer}(\mathcal{A}) \), notice that, by the generalized Leibniz rule (1),

\[
u(f) = \nu(1 \cdot f) = \nu(1)f + d\nu(f) = \mathcal{L}\nu(1)f + d\nu(x^\mu)\partial_\mu f \tag{2}\]

for any \( \nu \in \text{GDer}(\mathcal{A}) \) and \( f \in \mathcal{A} \), where \( 1 \) denotes a constant function equal to one. Thus

\[
\text{GDer}(\mathcal{A}) = \text{span}_\mathcal{A} (\partial_0, \partial_1, \ldots, \partial_{N-1}, \text{id}_\mathcal{A}). \tag{3}
\]

Therefore, \( \dim \text{GDer}(\mathcal{A}) = 1 + \dim M \).

For the sake of brevity let us denote \( \partial_N \equiv \text{id}_\mathcal{A} \). In what follows we shall also adopt the convention that capital Latin indices \( A, B, C, \ldots \) run from 0 to \( N \), whereas Greek indices \( \mu, \nu, \alpha, \beta, \ldots \) do not cover the additional “generalized” index value \( N \). Thus, (2) can be re-expressed as

\[
u(f) = \nu^A \partial_A f
\]

with \( \nu^\mu = d\nu(x^\mu) = \nu(x^\mu) - \nu(1)x^\mu \) and \( \nu^N = \mathcal{L}\nu(1) = \nu(1) \).

Of course, coordinate transformations affect all index values but \( N \).

Setting \( g_{AB} \equiv G(\partial_A, \partial_B) \), we use the Koszul formula to express the coefficients of the Levi–Civita connection

\[
\nabla_{\partial_A} \partial_B = \Gamma^C_{AB} \partial_C \quad \text{where}
\]

\[
\Gamma^C_{AB} = \frac{1}{2} g^{CD} (\partial_A g_{BD} + \partial_B g_{AD} - \partial_D g_{AB}). \tag{4}
\]

Although the preceding expressions are identical to those known from the pseudo-riemannian geometry, the connection acts in a slightly different way because of the presence of \( \text{id}_\mathcal{A} \) in the basis

\[
\nabla_{u^A \partial_A}(v^B \partial_B) = (u^A v^B \Gamma^C_{AB} + u^\mu \partial_\mu v^C) \partial_C.
\]

Notice that among the indices used here one is Greek.

This can be written in the abstract index notation as follows:

\[
\nabla_A v^B = (\partial_A - \delta^A_N) v^B + \Gamma^B_{AC} v^C. \tag{5}
\]

Let us now consider the coefficients \( R^C_{DAB} \) of the Riemann tensor, defined by the equality

\[
R(\partial_A, \partial_B) \partial_D = R^C_{DAB} \partial_C.
\]

7
Using (5), one obtains the following formula for these coefficients:

\[
R^C_{DAB} = (\partial_A - \delta^N_A) \Gamma^C_{BD} - (\partial_B - \delta^N_B) \Gamma^C_{AD} + \Gamma^K_{BD} \Gamma^C_{AK} - \Gamma^K_{AD} \Gamma^C_{BK}.
\] (6)

Note that (6) differs from the standard result if \( A \) or \( B \) is equal to \( N \).

As for the coefficients of the Ricci 2-form \( \text{ric} \) and the scalar curvature \( r \), we have, as usual,

\[
\text{ric}_{AB} = R^C_{ACB} \quad \text{and} \quad r = g^{AB} \text{ric}_{AB}.
\] (7)

Let us now visualize the effect the introduction of generalized derivations has on Christoffel symbols, on Riemann and Ricci tensors’ coefficients and on the scalar curvature, by calculating them for a simple (but nontrivial) metric.

5 Example: a simple metric

In this section, we consider a metric that does not mix derivations \( \partial_\mu \) with the identity \( \partial_N \). To do so, let us take any \( N \)-dimensional metric \( g_{\alpha\beta} \) and let us set

\[
g_{AB} = \begin{bmatrix}
g_{\alpha\beta} & 0 & \vdots & 0 \\
0 & \ddots & \vdots & \vdots \\
0 & \ddots & 0 & 0 \\
0 & \cdots & 0 & \varepsilon \Phi^2
\end{bmatrix},
\] (8)

where \( \Phi = \Phi(x^0, x^1, \ldots, x^{N-1}) \) denotes a smooth positive function and \( \varepsilon = \pm 1 \). For clarity, we separate the parts of matrices associated with the additional “generalized” degree of freedom from the “classical” \( N \times N \) parts.

In the following, the tilde above a given object signifies that the object is obtained from the \( N \)-dimensional metric \( g_{\alpha\beta} \) according to the standard (i.e. not “generalized”) pseudo-riemannian-geometrical formulae.
One obtains the following Christoffel symbols:

\[ \Gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\lambda} \left( \partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta} \right) = \tilde{\Gamma}_{\alpha\beta}, \]

\[ \Gamma^N_{\alpha\beta} = -\frac{\epsilon}{2\Phi^2} g_{\alpha\beta}, \]

\[ \Gamma^\gamma_{N\beta} = \Gamma^\gamma_{\beta N} = \frac{1}{2} \delta^\gamma_\beta, \]

\[ \Gamma^\gamma_{NN} = -\epsilon \Phi \tilde{\nabla}_\gamma, \]

\[ \Gamma^N_{\alpha N} = \Gamma^N_{N\alpha} = \frac{\partial_\alpha \Phi}{\Phi}, \]

\[ \Gamma^N_{NN} = \frac{1}{2} \]

where \( \partial^\gamma \equiv g^{\gamma\lambda} \partial_\lambda \).

As for the non-zero Riemann tensor coefficients, one gets

\[ R^{\gamma}_{\lambda\alpha\beta} = \tilde{R}^{\gamma}_{\lambda\alpha\beta} + \frac{\epsilon}{4\Phi^2} \left( g_{\alpha\lambda} \delta^\gamma_\beta - g_{\beta\lambda} \delta^\gamma_\alpha \right), \]

\[ R^N_{\lambda\alpha\beta} = \frac{\epsilon}{2\Phi^3} \left( g_{\beta\lambda} \partial_\alpha - g_{\alpha\lambda} \partial_\beta \right) \Phi, \]

\[ R^\gamma_{N\alpha\beta} = \frac{1}{2\Phi} \left( \delta^\gamma_\alpha \partial_\beta - \delta^\gamma_\beta \partial_\alpha \right) \Phi, \]

\[ R^\gamma_{\lambda N\beta} = -R^\gamma_{\beta N\lambda} = \frac{1}{2\Phi} \left( g_{\beta\lambda} \partial_\gamma - \delta^\gamma_\beta \partial_\lambda \right) \Phi, \]

\[ R^N_{\lambda\alpha N} = -R^N_{\lambda N\alpha} = \frac{1}{\Phi} \tilde{\nabla}_\alpha \partial_\lambda \Phi, \]

\[ R^\gamma_{NN\beta} = -R^\gamma_{N\beta NN} = \epsilon \Phi \tilde{\nabla}_\beta \partial^\gamma \Phi \]

where \( \tilde{\nabla}_\alpha \) denotes the standard covariant derivative along the \( \alpha \)-th direction.

The Ricci tensor, written in matrix form, reads

\[
\text{ric}_{AB} = \begin{bmatrix}
\tilde{\text{ric}}_{\alpha\beta} - \frac{\epsilon N-1}{4\Phi^2} g_{\alpha\beta} - \frac{1}{\Phi} \tilde{\nabla}_\alpha \partial_\beta \Phi \\
\frac{N-1}{2\Phi} \partial_\beta \Phi \\
-\epsilon \Phi \tilde{\Delta} \Phi
\end{bmatrix}
\]

where \( \tilde{\Delta} \) denotes the standard Laplace–Beltrami operator

\[ \tilde{\Delta} = \tilde{\nabla}_\lambda \partial^\lambda = \partial^\lambda \partial_\lambda - g^{\mu\nu} \tilde{\Gamma}_{\lambda\mu\nu} \partial_\lambda. \]

Finally, the scalar curvature takes the following form

\[ r = \tilde{r} - \frac{\epsilon N(N-1)}{4\Phi^2} - \frac{2}{\Phi} \tilde{\Phi}. \]
Notice that the introduction of the generalized derivation $\partial_N$ leads to the appearance of additional terms depending on $\Phi$, similar to when considering an extra dimension within Kaluza–Klein theories (see section 7 for the list of references). We shall investigate this similarity more closely in section 7.

6 Action principle and generalized Einstein’s equations

We have now all the geometric notions needed to investigate the impact the generalized derivations have on Einstein’s field equations. Let us start with the following Einstein–Hilbert action:

$$S_{EH} = \frac{1}{2\kappa} \int r \sqrt{|g|} \, d^N x,$$

where $g$ denotes the determinant of the $(N + 1) \times (N + 1)$ matrix $g_{AB}$. The coefficient $\kappa$ is a physical constant equal to $\frac{8\pi G}{c^4}$, where $G$ is the gravitational constant and $c$ is the speed of light.

One should notice that the term $\sqrt{|g|} \, d^N x$ differs from the standard volume $N$-form, which would involve only the determinant of the “classical” $N \times N$ part of the matrix $g_{AB}$. Nevertheless, just as the standard volume $N$-form, the term $\sqrt{|g|} \, d^N x$ is invariant under coordinate transformations.

Let us vary thus defined $S_{EH}$ with respect to $\delta g^{AB}$

$$\delta S_{EH} = \frac{1}{2\kappa} \int \left( \mathbf{ric}_{AB} - \frac{1}{2} r g_{AB} \right) \delta g^{AB} \sqrt{|g|} \, d^N x$$

$$+ \frac{1}{2\kappa} \int \delta \mathbf{ric}_{AB} g^{AB} \sqrt{|g|} \, d^N x.$$ (10)

The integrand involving $\delta \mathbf{ric}_{AB}$ does not vanish and can be expressed via the variations of Christoffel symbols as follows:

$$\delta \mathbf{ric}_{AB} g^{AB} \sqrt{|g|} = \partial_\lambda \left[ (g^{AB} \delta \Gamma^\lambda_{AB} - g^{\lambda B} \delta \Gamma^A_{AB}) \sqrt{|g|} \right]$$

$$+ \frac{N-1}{2} \left( g^{AB} \delta \Gamma^N_{AB} - g^{NB} \delta \Gamma^A_{AB} \right) \sqrt{|g|}.$$ (11)

The first term on the right-hand side of (11) is a divergence and as such can be omitted in further considerations.
In order to express the remaining term with $\delta g^{AB}$, let us notice that
\[
g^{AB}\Gamma^{N}_{AB} - g^{NB}\Gamma^{A}_{AB} = -Ng^{NN} - \partial_\lambda g^{N\lambda} - g^{N\lambda}g^{AB}\partial_\lambda g_{AB}.
\]

By varying the preceding equality, one gets
\[
g^{AB}\delta \Gamma^{N}_{AB} - g^{NB}\delta \Gamma^{A}_{AB} = \Gamma^{A}_{AB}\delta g^{NB} - \Gamma^{N}_{AB}\delta g^{AB} - N\delta g^{NN} - \partial_\lambda\delta g^{N\lambda} - g^{N\lambda}\partial_\lambda g_{AB}\delta g^{AB} - g^{N\lambda}g^{AB}\partial_\lambda\delta g_{AB}.
\]

Inserting this expression into (11), one can further simplify it by first realizing that
\[
\Gamma^{A}_{AB}\delta g^{NB} = \frac{1}{2}g^{AC}\partial_B g_{AC}\delta g^{NB} = \frac{1}{2}g^{AC}\partial_\lambda g_{AC}\delta g^{N\lambda} + \frac{N+1}{2}\delta g^{NN}
\]
and that
\[
-\partial_\lambda\delta g^{N\lambda} \sqrt{|g|} = -\partial_\lambda \left( \delta g^{N\lambda} \sqrt{|g|} \right) + \frac{1}{2}g^{CD}\partial_\lambda g_{CD}\sqrt{|g|} \delta g^{N\lambda}.
\]

Therefore, up to divergence terms
\[
(\Gamma^{A}_{AB}\delta g^{NB} - N\delta g^{NN} - \partial_\lambda\delta g^{N\lambda} - g^{AB}\partial_\lambda g_{AB}\delta g^{N\lambda}) \sqrt{|g|} = -\frac{N-1}{2} \sqrt{|g|} \delta g^{NN}.
\]

Let us now move to the two rightmost terms in (12). Because it is true that
\[
\delta g_{AB}\partial_\lambda g^{AB} = \delta g^{AB}\partial_\lambda g_{AB},
\]
one can write that
\[
- g^{N\lambda}\partial_\lambda g_{AB}\delta g^{AB} - g^{N\lambda}g^{AB}\partial_\lambda\delta g_{AB} = -g^{N\lambda}\partial_\lambda g^{AB}\delta g_{AB} - g^{N\lambda}g^{AB}\partial_\lambda\delta g_{AB} = -g^{N\lambda}\partial_\lambda \left( g^{AB}\delta g_{AB} \right).
\]

\footnote{To prove this claim, one can use the formula for the derivative of the matrix inverse, obtaining
\[
\delta g_{AB}\partial_\lambda g^{AB} = \left( -g_{AC}\delta g^{CD}g_{DB} \right) \partial_\lambda g^{AB} = \delta g^{CD} \left( -g_{CA}\partial_\lambda g^{AB}g_{BD} \right) = \delta g^{CD}\partial_\lambda g_{CD}.
\]}

11
This means, however, that

$$
\left( - g^{N\lambda} \partial_{\lambda} g^{AB} \delta g^{AB} - g^{N\lambda} g^{AB} \partial_{\lambda} \delta g^{AB} \right) \sqrt{|g|} \\
= g^{N\lambda} \sqrt{|g|} \partial_{\lambda} \left( g_{AB} \delta g^{AB} \right) \\
= - \partial_{\lambda} \left( g^{N\lambda} \sqrt{|g|} \right) g_{AB} \delta g^{AB}
$$

(14)

up to divergence terms.

All in all, (11)–(14) imply that up to divergence terms

$$
\delta \text{ric}_{AB} g^{AB} \sqrt{|g|} = \frac{N - 1}{2} \left( - \Gamma_{AB}^{N} \sqrt{|g|} \delta g^{AB} - \frac{N - 1}{2} \sqrt{|g|} \delta g^{NN} \\
- \partial_{\lambda} \left( g^{N\lambda} \sqrt{|g|} \right) g_{AB} \delta g^{AB} \right).
$$

(15)

Therefore, the variation of (10) can finally be put into the following form:

$$
\delta S_{EH} = \frac{1}{2\kappa} \int \left( \text{ric}_{AB} - \frac{1}{2} r g_{AB} - \frac{N - 1}{2} \Pi_{AB} \right) \delta g^{AB} \sqrt{|g|} d^N x
$$

(16)

where

$$
\Pi_{AB} = \Gamma_{AB}^{N} + \frac{N - 1}{2} \delta_{A}^{N} \delta_{B}^{N} + g_{AB} \frac{1}{\sqrt{|g|}} \partial_{\lambda} \left( g^{N\lambda} \sqrt{|g|} \right) \\
= \Gamma_{AB}^{N} + \frac{N - 1}{2} \delta_{A}^{N} \delta_{B}^{N} + g_{AB} \sqrt{|g^{NN}|} \tilde{\nabla}_{\lambda} \left( \frac{g^{N\lambda}}{\sqrt{|g^{NN}|}} \right)
$$

(17)

is a symmetric tensor. Notice that it naturally involves the term proportional to the metric\(^2\). It is thus tempting to associate it with Einstein’s cosmological term \(\Lambda g_{\alpha\beta}\) with nonconstant \(\Lambda\) which could model dynamical dark energy, here being of a purely geometrical (or rather generalized-geometrical) origin.

Let now \(S_M\) denote the standard action for matter. Applying the action principle to the sum \(S_{EH} + S_M\) one gets the following generalized Einstein’s equations

$$
\text{ric}_{AB} - \frac{1}{2} r g_{AB} - \frac{N - 1}{2} \Pi_{AB} = \kappa \sqrt{|g^{NN}|} T_{AB},
$$

(18)

\(^2\)Also, an additional term of this kind is possibly implicit in \(\Gamma_{AB}^{N}\), as for the case of the metric discussed in section \(5\)
where the stress–energy tensor with indices raised $T^{\alpha\beta}$ is given as usual by

$$T^{AB} = -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta S_M}{\delta g_{AB}}.$$ 

Because we have assumed that $S_M$ is standard, that is, it does not involve metric coefficients $g_{\alpha N}, g_{N\beta}, g_{NN}$, the stress–energy tensor will be of the form

$$T^{AB} = \begin{bmatrix}
  T^{\alpha\beta} & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0
\end{bmatrix}$$

and so $T_{AB} = g_{A\alpha} g_{B\beta} T^{\alpha\beta}$. Notice the equality of the traces

$$T^A_A = g^{AB} g_{A\alpha} g_{B\beta} T^{\alpha\beta} = g_{\alpha\beta} T^{\alpha\beta} = T^\alpha_\alpha.$$ 

By calculating the trace of both sides of (18) one obtains

$$r + \frac{N}{\sqrt{|g|}} \partial_\alpha \left( \sqrt{|g|} g^{N\alpha} \right) = -\frac{2\kappa}{N-1} \sqrt{|g|} g^{NN} |T^\alpha_\alpha|.$$ 

(19)

It is appropriate to call the left-hand side of (18) the generalized Einstein tensor. What is worth noticing is that it involves no free parameters.

The term $\sqrt{|g^{NN}|}$ on the right-hand sides of (18) and (19) is in general nonconstant; it models the space–time-dependency of the gravitational constant $G$, similarly to the Brans–Dicke scalar field [10].

In fact, by considering only the metrics studied in section 5, that is, those for which $g_{\alpha N} = g_{N\alpha} = 0$ and $g_{NN} = \epsilon \Phi^2$ (where $\epsilon = \pm 1$), the theory reduces exactly to the scalar–tensor theory governed by the action

$$S_{O'H} = \frac{1}{2\kappa} \int (\Phi \tilde{r} - V[\Phi]) \sqrt{-\tilde{g}} d^N x$$ 

(20)

with the “Coulomb” potential $V[\Phi] = \epsilon^N (N-1)$.

An action of this kind was first considered by O’Hanlon [11] and is sometimes referred to as the O’Hanlon action (consult Sotiriou and Faraoni [21] for more references). It is a special case of the Brans–Dicke action with a
nonzero potential and the Brans–Dicke parameter $\omega$ equal to zero (in the Jordan frame).

One can also put \([20]\) into an equivalent, \(f(R)\)-theoretical form\(^3\) \([21, 22]\)

\[
S_{f(R)} = \frac{\sqrt{N(N-1)}}{2\kappa} \int \sqrt{|\tilde{\tau}|} \sqrt{-\tilde{g}} \, d^N x.
\]

Thus, from the point of view of \(f(R)\)-gravity theory, the introduction of generalized derivations (and considering only the narrowed class of metrics) leads to the action with \(f(R) = \sqrt{N(N-1)} |R|^{1/2}\). Actions of the form \(f(R) \propto |R|^n\) (with \(n\) not necessarily integer) have been studied by numerous authors; see Faraoni \([23]\) for a list of references. It is already known that only for \(n \approx 1\) with the level of accuracy of about \(10^{-19}\) theories of this type meet current observational data as far as Solar System physics is concerned \([23]\).

## 7 Noncompact invisible dimension

The idea of the celebrated Kaluza–Klein theory (and its modifications) is to assume that the physical space–time is a pseudo-riemannian manifold of dimension \(D > 4\), on which one considers the Einstein–Hilbert action

\[
S_{KK} = \frac{1}{2\kappa} \int \hat{\tau} \sqrt{|\hat{g}|} \, d^D x
\]

where \(\hat{\tau}\) denotes the determinant of the \(D \times D\) matrix of the metric tensor \(\hat{g}_{AB}\) and \(\hat{\tau}\) is the scalar curvature obtained from that metric with the standard pseudo-riemannian-geometric formulae.

Kaluza \([24]\) showed that in \(D = 5\) dimensions varying \([21]\) over the class of metrics satisfying the so-called “cylinder condition” (i.e. the class of metrics independent of the extra coordinate \(x_4\)) leads to a theory unifying Einstein’s theory of gravity with Maxwell’s theory of electromagnetism. Shortly after, Klein \([25]\) realized that if the extra dimension is compact and has a small

---

\(^3\)Actions considered in \(f(R)\)-theory of gravity have the general form

\[
S = \frac{1}{2\kappa} \int f(\tilde{\tau}) \sqrt{-\tilde{g}} \, d^N x.
\]
enough scale, the seemingly artificial “cylinder condition” arises naturally in the low-energy physics regime. Moreover, the previously mentioned features of the fifth dimension explain why it is not physically observed [20].

Throughout the decades, numerous authors introduced and studied various modifications and generalizations of Kaluza’s original idea, in order to incorporate other physical phenomena into a unified geometrical formalism. Because it is far beyond the scope of this work even to briefly describe them, the interested reader is referred to excellent books [27, 28, 29, 30, 31] and review papers [32, 33, 34, 35, 36, 37].

It turns out that the generalized derivation-based approach to general relativity presented in previous sections can be equivalently formulated in a Kaluza-Klein-theoretical way.

Concretely, we shall prove that generalized Einstein’s equations (18) with $T_{AB} = 0$ can be obtained from varying Kaluza–Klein action (21) involving one noncompact extra dimension $(D = N + 1)$ over the class of metrics satisfying the “modified cylinder condition”. Namely, the metrics are assumed to depend on the extra coordinate $x_N$ exponentially, that is

$$\hat{g}_{AB} = e^{x_N} g_{AB}$$

where $g_{AB}$ already does not depend on $x_N$. According to the authors’ best knowledge, this particular version of Kaluza–Klein theory has not so far been studied.

To start the proof, notice that the determinants of these two matrices satisfy the equality

$$\hat{g} = e^{(N+1)x_N} g,$$

which can be inserted into (21). We would like to do something similar with $\hat{r}$. It is crucial to realize that

$$\hat{r} = e^{-x_N r}$$

where $r$ is a scalar curvature obtained from $g_{AB}$ with the help of formulae [4, 7].

Before we prove this claim, let us introduce the symbol $\hat{\partial}_C$ to denote $\frac{\partial}{\partial x^C}$. We use the hat here so as to avoid confusion, because throughout the paper the symbol $\partial_N$ denotes the identity map $\text{id}_A$. Let us also recall that we follow
the convention that capital Latin indices run from 0 to \(N\), whereas Greek indices run from 0 to \(N - 1\).

In order to prove (23), let us first notice that

\[
\hat{\partial}_C \hat{g}_{AB} = e^{x_N} \partial_C g_{AB}. \tag{24}
\]

Indeed, by a simple computation

\[
\hat{\partial}_\lambda \hat{g}_{AB} = \hat{\partial}_\lambda (e^{x_N} g_{AB}) = e^{x_N} \hat{\partial}_\lambda g_{AB} = e^{x_N} \partial_\lambda g_{AB},
\]

where the hat can be dropped because for \(\lambda = 0, 1, \ldots, N - 1\) the meanings of the symbols \(\hat{\partial}_\lambda\) and \(\partial_\lambda\) coincide.

One also obtains

\[
\hat{\partial}_N \hat{g}_{AB} = \hat{\partial}_N (e^{x_N} g_{AB}) = \hat{\partial}_N (e^{x_N} g_{AB}) + e^{x_N} \hat{\partial}_N (g_{AB}) = e^{x_N} g_{AB} = e^{x_N} \partial_N g_{AB},
\]

which proves (24).

The next step is to show that the coefficients \(\hat{\Gamma}^C_{AB}\) of the Levi–Civita connection associated with \(\hat{g}_{AB}\) do not depend on \(x_N\) and are in fact equal to \(\Gamma^C_{AB}\) given by (4). Indeed, by (24) one has

\[
\hat{\Gamma}^C_{AB} = \frac{1}{2} \hat{g}^{CD} (\hat{\partial}_A \hat{g}_{BD} + \hat{\partial}_B \hat{g}_{AD} - \hat{\partial}_D \hat{g}_{AB})
\]

\[
= \frac{1}{2} e^{-x_N} g^{CD} (\partial_A g_{BD} + \partial_B g_{AD} - \partial_D g_{AB}) = \frac{1}{2} g^{CD} (\partial_A g_{BD} + \partial_B g_{AD} - \partial_D g_{AB}) = \Gamma^C_{AB}.
\]

Since \(\hat{\Gamma}^C_{AB}\) does not depend on \(x_N\), the standard formula for the Riemann tensor coefficients leads to a formula identical to (6). Consequently, the same concerns the Ricci tensor coefficients, therefore,

\[
\hat{R}^C_{DAB} = R^C_{DAB} \quad \text{and} \quad \hat{\text{ric}}_{AB} = \text{ric}_{AB}
\]

with the right-hand sides obtained from \(g_{AB}\) via formulae (6) and (7).

Finally, for the scalar curvature, one has

\[
\hat{r} = \hat{g}^{AB} \hat{\text{ric}}_{AB} = e^{-x_N} g^{AB} \text{ric}_{AB} = e^{-x_N} r,
\]

which proves claim (23).
With all of this in mind, let us now vary action (21) with respect to \( \delta \hat{g}_{AB} = e^{x_N} \delta g_{AB} \). One obtains

\[
\delta S_{KK} = \frac{1}{2\kappa} \int \left( \hat{\text{ric}}_{AB} - \frac{1}{2} \hat{\text{r}} \hat{g}_{AB} \right) \delta \hat{g}^{AB} \sqrt{\left| \hat{g} \right|} d^{N+1}x
\]

\[
+ \frac{1}{2\kappa} \int \delta \hat{\text{ric}}_{AB} \hat{g}^{AB} \sqrt{\left| \hat{g} \right|} d^{N+1}x. \tag{25}
\]

The integrand of the leftmost integral can be equivalently written as

\[
\left( \hat{\text{ric}}_{AB} - \frac{1}{2} \hat{\text{r}} \hat{g}_{AB} \right) \delta \hat{g}^{AB} \sqrt{\left| \hat{g} \right|} = e^{N-1\times_N} \left( \text{ric}_{AB} - \frac{1}{2} r g_{AB} \right) \delta g^{AB} \sqrt{\left| g \right|}.
\]

As for the integrand of the rightmost integral, it can be expressed as a \((N+1)\)-dimensional divergence

\[
\delta \hat{\text{ric}}_{AB} \hat{g}^{AB} \sqrt{\left| \hat{g} \right|} = \hat{\partial}_C \left[ \sqrt{\left| \hat{g} \right|} \left( \delta \hat{\Gamma}^C_{AB} \hat{g}^{AB} - \delta \hat{\Gamma}^D_{DA} \hat{g}^{AC} \right) \right].
\]

This, however, does not imply that the rightmost integral in (25) vanishes. Indeed, because the Christoffel symbols do not depend on \( x_N \), one cannot argue that their variations are zero on a boundary of a sufficiently large \((N+1)\)-dimensional domain. However, writing down the dependence on \( x_N \) explicitly, one obtains

\[
\delta \hat{\text{ric}}_{AB} \hat{g}^{AB} \sqrt{\left| \hat{g} \right|} = \hat{\partial}_C \left[ e^{N-1\times_N} \sqrt{\left| \hat{g} \right|} \left( \delta \hat{\Gamma}^C_{AB} \hat{g}^{AB} - \delta \hat{\Gamma}^D_{DA} \hat{g}^{AC} \right) \right]
\]

\[
= e^{N-1\times_N} \hat{\partial}_C \left[ \sqrt{\left| \hat{g} \right|} \left( \delta \hat{\Gamma}^C_{AB} \hat{g}^{AB} - \delta \hat{\Gamma}^D_{DA} \hat{g}^{AC} \right) \right]
\]

\[
+ \frac{N-1}{2} e^{N-1\times_N} \sqrt{\left| \hat{g} \right|} \left( \delta \hat{\Gamma}^C_{AB} \hat{g}^{AB} - \delta \hat{\Gamma}^D_{DA} \hat{g}^{AC} \right) \tag{26}
\]

In fact, this is an alternate derivation of (11).

By Fubini’s theorem and by the fact that the variations \( \delta \hat{\Gamma}^C_{AB} \) vanish on a boundary of a sufficiently large \( N \)-dimensional subset of any surface of fixed \( x_N \), one has

\[
\int e^{N-1\times_N} \hat{\partial}_C \left[ \sqrt{\left| \hat{g} \right|} \left( \delta \hat{\Gamma}^C_{AB} \hat{g}^{AB} - \delta \hat{\Gamma}^D_{DA} \hat{g}^{AC} \right) \right] d^{N+1}x = 0.
\]

In other words, one can omit the first term on the right-hand side of (26) and from now on proceed exactly as shown in the previous section, eventually
obtaining Einstein’s equations (18) with zero stress–energy tensor.

This result has an interesting interpretative aspect. Every Kaluza–Klein model involving noncompact extra dimensions has to address the question of why these additional dimensions are not observed. For instance Schmutzer’s five-dimensional Projective Unified Field Theory (PUFT) \[38, 39, 40\] assumes the additional dimension merely as a mathematical tool without direct physical meaning. On the other hand, the 5-dimensional Space–Time–Matter (STM) theory (the interested reader is referred to Overduin and Wesson \[32\] for a brief introduction and a list of references, and to Wesson \[41\] for a more detailed course) treats the fifth coordinate as physical, but not lengthlike.

Our case in principle seems to resemble Schmutzer’s in the sense that the extra dimension is nonphysical and effectively results from modifying the standard pseudo-riemannian geometry. Indeed, in terms of generalized differential geometry, space–time has an extra “generalized-differential dimension” spanned by $i\delta_A$ (see [3]), which is not associated with any extra space–time coordinate.

It is worth noting that Einstein’s equations differ here from the Ricci-flatness condition $\text{ric}_{AB} = 0$ as obtained in other Kaluza–Klein theories without higher-dimensional matter \[32\]. Indeed, setting $T_{AB} = 0$ in (18)–(19) yields

$$\text{ric}_{AB} - \frac{1}{2N} g_{AB} - \frac{N-1}{2} \left( \Gamma^N_{AB} + \frac{N-1}{2} \delta^N_A \delta^N_B \right) = 0,$$

which does not in general imply that $\text{ric}_{AB} = 0$, as one can check for example for metrics discussed in section 5.

One can regard this effect as a new realization of an induced matter (“matter-out-of-geometry”) mechanism. In Kaluza’s original work, (as well as in the STM theory \[32, 41, 42\]), a four-dimensional stress-energy tensor appears when the five-dimensional Ricci-flatness condition is projected onto the four-dimensional setting. In our case additional terms are present in Einstein’s equations (27) already before the projection. Therefore, we could say that the “modified cylinder condition” \[22\] induces a certain form of five-dimensional matter.

Let us finally remark, that the approach based on generalized derivations does not exclude other Kaluza–Klein-type approaches. In other words, in
addition to treating space–time as a $D$-dimensional manifold with $D > 4$, one can consider its generalized geometry, and interpret it physically if there is such a necessity.

References

[1] M. Brešar. Glasgow Math. J. 33(1), 89 (1991).
[2] E. Posner. Proc. Amer. Math. Soc. 8, 1093 (1957).
[3] B. Hvala. Comm. Algebra, 26(4), 1147 (1998).
[4] M.A. Quadri, M. Shadab Khan, and N. Rehman. Indian J. Pure Appl. Math. 34(9), 1393 (2003).
[5] F. Ali, and M.A. Chaudhry. Int. J. Algebra, 5(8), 397 (2011).
[6] A. Nakajima. Scientiae Mathematicae, 2(3), 345 (1999).
[7] A. Nakajima. Turk. J. Math. 24(3), 295 (1999).
[8] G.F. Leger, and E.M. Luks. J. Algebra, 228, 165 (2000).
[9] M. Ashraf, Sh. Ali, and C. Haetinger. Aligarh Bull. Maths. 25(2), 79 (2006).
[10] C. Brans, and R.H. Dicke. Phys. Rev. 124, 925 (1961).
[11] J. O’Hanlon. J. Phys. Rev. Lett. 29, 137 (1972).
[12] A. Connes. Noncommutative Geometry. Academic Press, New York. 1994.
[13] J. Madore. An Introduction to Noncommutative Differential Geometry and Its Physical Applications, 2nd ed. Cambridge University Press, Cambridge. 1999.
[14] J.M. Gracia-Bondía, J.C. Várilly, and H. Figueroa. Elements of noncommutative geometry. Birkhäuser, Boston. 2001.
[15] M. Dubois-Violette. C.R. Acad. Sci Paris, 307(1), 403 (1988).
[16] M. Dubois-Violette, R. Kerner, and J. Madore. J. Math. Phys. 31, 316 (1990).

[17] A.E.F. Djemai. Int. J. Theor. Phys. 34(6), 801 (1995).

[18] W. Sasin, and M. Heller. Acta Cosmologica, 21(2), 235 (1995).

[19] M. Heller, L. Pysiak, and W. Sasin. J. Math. Phys. 46, 122501 (2005).

[20] G.N. Parfionov, and R.R. Zapatrin. Int. J. Theor. Phys. 34, 717 (1995).

[21] T.P. Sotiriou, and V. Faraoni. Rev. Mod. Phys. 82, 451 (2010).

[22] V. Faraoni. Phys. Rev. D, 75, 067302 (2007).

[23] V. Faraoni. Phys. Rev. D, 83, 124044 (2011).

[24] T. Kaluza. Sitz. Preuss. Akad. Wiss. Phys. Math. K1, 966 (1921).

[25] O. Klein. Zeits. Phys. 37, 895 (1926).

[26] A. Einstein, and P. Bergmann. Ann. Math. 39, 683 (1938).

[27] V. De Sabbata, and E. Schmutzer (Editors). Unified field theories of more than 4 dimensions, proc. international school of cosmology and gravitation (Erice). World Scientific, Singapore. 1983.

[28] H.C. Lee (Editor). An introduction to Kaluza-Klein theories, proc. Chalk River workshop on Kaluza-Klein theories. World Scientific, Singapore. 1984.

[29] K. Kang, H. Fried, and P. Frampton (Editors). Fifth workshop on grand unification. World Scientific, Singapore. 1984.

[30] T. Piran, and S. Weinberg (Editors). Physics in higher dimensions, proc. 2nd Jerusalem winter school of theoretical physics. World Scientific, Singapore. 1986.

[31] T. Appelquist, A. Chodos, and P.G.O. Freund (Editors). Modern Kaluza-Klein theories. Addison-Wesley, Menlo Park. 1987.

[32] J.M. Overduin, and P.S. Wesson. Phys. Rept. 283, 303 (1997).
[33] D.J. Toms. *In* An introduction to Kaluza-Klein theories, proc. Chalk River workshop on Kaluza-Klein theories. *Edited by* H.C. Lee. World Scientific, Singapore. 1984. p. 185.

[34] T. Appelquist. *In* Fifth workshop on grand unification. *Edited by* K. Kang, H. Fried, and P. Frampton. World Scientific, Singapore. 1984. p. 474.

[35] M.J. Duff. *In* Physics in higher dimensions, proc. 2nd Jerusalem winter school of theoretical physics. *Edited by* T. Piran, and S. Weinberg. World Scientific, Singapore. 1986. p. 40.

[36] M.J. Duff, B.E.W. Nilsson, and C.N. Pope. Phys. Rep. 130, 1 (1986).

[37] D. Bailin, and A. Love. Rep. Prog. Phys. 50, 1087 (1987).

[38] E. Schmutzer. *In* Unified field theories of more than 4 dimensions, proc. international school of cosmology and gravitation (Erice). *Edited by* V. De Sabbata, and E. Schmutzer. World Scientific, Singapore. 1983. p. 81.

[39] E. Schmutzer. Astron. Nachr. 311, 329 (1990).

[40] E. Schmutzer. Fortschr. Phys. 43, 613 (1995).

[41] P.S. Wesson. Space-Time-Matter: Modern Kaluza-Klein Theory. World Scientific, Singapore. 1999.

[42] P.S. Wesson, and H. Liu. Int. J. Mod. Phys. D, 10, 905 (2001).