EXISTENCE AND A PRIORI ESTIMATES OF SOLUTIONS FOR QUASILINEAR SINGULAR ELLIPTIC SYSTEMS WITH VARIABLE EXPONENTS

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Abstract. This article sets forth results on the existence, a priori estimates and boundedness of positive solutions of a singular quasilinear systems of elliptic equations involving variable exponents. The approach is based on Schauder’s fixed point Theorem. A Moser iteration procedure is also obtained for singular cooperative systems involving variable exponents establishing a priori estimates and boundedness of solutions.

1. Introduction

In the present paper we focus on the system of quasilinear elliptic equations

\[
\begin{aligned}
-\Delta_{p(x)} u &= f(u, v) \quad \text{in } \Omega \\
-\Delta_{q(x)} v &= g(u, v) \quad \text{in } \Omega \\
u, v > 0 &\quad \text{in } \Omega \\
u, v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

\((P)\)

on a bounded domain \(\Omega\) in \(\mathbb{R}^N\) \((N \geq 2)\) with Lipschitz boundary \(\partial \Omega\), which exhibits a singularity at zero. Here \(\Delta_{p(x)}\) (resp. \(\Delta_{q(x)}\)) stands for the \(p(x)\)-Laplacian (resp. \(q(x)\)-Laplacian) differential operator on \(W^{1,p(x)}_0(\Omega)\) (resp. \(W^{1,q(x)}_0(\Omega)\)) with \(p, q : \Omega \to [1, \infty)\),

\[1 < p^- \leq p^+ < N \quad \text{and} \quad 1 < q^- \leq q^+ < N,\]

which satisfy the log-Hölder continuous condition, i.e., there is constants \(C_1, C_2 > 0\) such that

\[|p(x_1) - p(y_1)| \leq \frac{C_1}{\ln |x_1 - y_1|} \quad \text{and} \quad |q(x_2) - q(y_2)| \leq \frac{C_2}{\ln |x_2 - y_2|},\]

for every \(x_i, y_i \in \Omega\) with \(|x_i - y_i| < 1/2, i = 1, 2\).

Throughout this paper, we denote by \(p^*\) and \(q^*\) the Sobolev critical exponents

\[p^*(x) = \frac{Np(x)}{N - p(x)} \quad \text{and} \quad q^*(x) = \frac{Nq(x)}{N - q(x)}\]

and we set

\[s^- = \inf_{x \in \Omega} s(x) \quad \text{and} \quad s^+ = \sup_{x \in \Omega} s(x).\]

A solution \((u, v) \in W^{1,p(x)}_0(\Omega) \times W^{1,q(x)}_0(\Omega)\) of problem \((P)\) is understood in the weak sense, that is, it satisfies

\[
\begin{aligned}
\int_{\Omega} |\nabla u|^{p(x) - 2} \nabla u \nabla \varphi \, dx &= \int_{\Omega} f(u, v) \varphi \, dx \\
\int_{\Omega} |\nabla v|^{q(x) - 2} \nabla v \nabla \psi \, dx &= \int_{\Omega} g(u, v) \psi \, dx,
\end{aligned}
\]

\((1.3)\)

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for all \((\varphi, \psi) \in W_{0}^{1,p(x)}(\Omega) \times W_{0}^{1,q(x)}(\Omega)\).

Nonlinear boundary value problems involving \(p(x)\)-Laplacian operator are mathematically challenging and important for applications. Their study is stimulated by their applications in physical phenomena related to electrorheological fluids and image restorations, see for instance [11, 21]. When \(p(x) \equiv p\) and \(q(x) \equiv q\) are constant functions, \(\Delta_{p(x)}\) and \(\Delta_{q(x)}\) coincide with the well-known \(p\)-Laplacian and \(q\)-Laplacian operators. However, it is worth pointing out that \(p(x)\)-Laplacian operator possesses more complicated nonlinearity than \(p\)-Laplacian since it is inhomogeneous and in general, it has no first eigenvalue, that is, the infimum of the eigenvalues of \(p(x)\)-Laplacian equals 0 (see, e.g., [14, 20]). This point constitute a serious technical difficulty in the study of problem \((P)\), for which topological methods are difficult to apply. Another serious difficulty encountered in studying system \((P)\) is that the nonlinearities \(f(u,v)\) and \(g(u,v)\) can exhibit singularities when the variables \(u\) and \(v\) approach zero. Specifically, we assume that \(f,g : (0, +\infty) \times (0, +\infty) \to (0, +\infty)\), are continuous functions satisfying the conditions:

\[
\text{(H.}f\text{)}:
\]

\[
f(s_1, s_2) \leq m_1 (1 + s_1^{\alpha_1(x)}) (1 + s_2^{\beta_1(x)}) \quad \text{for all } s_1, s_2 > 0,
\]

with a constant \(m_1 > 0\) and continuous functions \(\alpha_1, \beta_1 : \Omega \to \mathbb{R}^*\).

\[
\text{(H.}g\text{)}:
\]

\[
g(s_1, s_2) \leq m_2 (1 + s_1^{\alpha_2(x)}) (1 + s_2^{\beta_2(x)}) \quad \text{for all } s_1, s_2 > 0,
\]

with a constant \(m_2 > 0\) and continuous functions \(\alpha_2, \beta_2 : \Omega \to \mathbb{R}^*\).

We explicitly observe that under assumptions (H.f) and (H.g) and depending on the sign of the variable exponents \(\alpha_i(\cdot)\) and \(\beta_i(\cdot)\), \(i = 1, 2\), system \((P)\) presents two types of complementary structures:

\[
(1.4) \quad \alpha_2^- , \beta_1^- > 0 \quad \text{(cooperative structure),}
\]

\[
(1.5) \quad \alpha_2^+ , \beta_1^+ < 0 \quad \text{(competitive structure).}
\]

If \((1.4)\) holds, we assume

\[
\text{H}(f,g)_1:\n\]

\[
\sigma := \min\{\inf_{s_1, s_2 > 0} f(s_1, s_2), \inf_{s_1, s_2 > 0} g(s_1, s_2)\} > 0.
\]

This assumption is useful in the subsequent estimates keeping the values of \(f(s_1, s_2)\) and \(g(s_1, s_2)\) above zero. In the case of competitive system \((P)\), in addition of \((1.5)\), we assume

\[
\text{H}(f,g)_2: \text{For all constant } M \geq 0 \text{ it hold}
\]

\[
\lim_{s_1 \to 0} \frac{f(s_1, s_2)}{s_1^{\alpha_1(x)}} = +\infty \quad \text{for all } s_2 \in (0, M)
\]

and

\[
\lim_{s_2 \to 0} \frac{g(s_1, s_2)}{s_2^{\beta_1(x)}} = +\infty \quad \text{for all } s_1 \in (0, M).
\]
This type of problem is rare in the literature. Actually, according to our knowledge, the only class of singular problems incorporated in statement (P) patterns the system for \( f(u, v) = u^{\alpha_1(x)}v^{\beta_1(x)} \) and \( g(u, v) = u^{\alpha_2(x)}v^{\beta_2(x)} \) was studied recently by Alves & Moussaoui [3]. The authors obtained the existence of solutions through new theorems involving sub and supersolutions for singular systems with variable exponents by dealing with cooperative and competitive structures. However, when the exponent variable functions \( p(\cdot), q(\cdot), \alpha_1(\cdot) \) and \( \beta_1(\cdot), \beta_2(\cdot), \) \( i = 1, 2, \) are reduced to be constants, problem (P) have been thoroughly investigated, we refer to [12] for system (P) with cooperative structure, while we quote [17, 18] for the study of competitive structure in (P). Furthermore, in the constant exponent context, the singular problem (P) arise in several physical situations such as fluid mechanics, pseudoplastics flow, chemical heterogeneous catalysts, non-Newtonian fluids, biological pattern formation, for more details about this subject, we cite the papers of Fulks & Maybe [12], Callegari & Nashman [7, 8] and the references therein.

Our goal is to establish the existence and regularity of (positive) solutions for problem (P) by processing the cases (1.4) and (1.5) related to the structure of (P). Our main results are stated as follows.

**Theorem 1.1.** Let assumptions (H.f), (H.g), H(f,g)_1 and (1.4) hold with
\[
\begin{align*}
\beta_1(x) &\leq \frac{q^*(x)}{p^*(x)}(p^*(x) - 1), \quad \alpha_2(x) \leq \frac{p^*(x)}{q^*(x)}(q^*(x) - 1) \\
\text{and} \\
\left\{ 
- \frac{1}{N} < \alpha_1^- \leq \alpha_1^+ < 0 \right. \\
\left. - \frac{1}{N} < \beta_2^- \leq \beta_2^+ < 0.
\right.
\end{align*}
\]

Then, problem (P) possesses at least one (positive) solution in \( C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \) satisfying
\[
\text{(1.8)} \quad u(x), v(x) \geq c_0d(x),
\]
where \( d(x) := d(x, \partial \Omega) \) and \( c_0 \) is a positive constant.

**Theorem 1.2.** Under assumptions (H.f), (H.g), H(f,g)_2 and (1.5) with
\[
\text{(1.9)} \quad \max\left\{ -\frac{1}{N}, -\alpha_1^- \right\} < \beta_1^- \leq \beta_1^+ < 0 < \alpha_1^- \leq \alpha_1^+ < p^* - 1
\]
and
\[
\text{(1.10)} \quad \max\left\{ -\frac{1}{N}, -\beta_2^- \right\} < \alpha_2^- \leq \alpha_2^+ < 0 < \beta_2^- \leq \beta_2^+ < q^* - 1,
\]
problem (P) possesses at least one (positive) solution \((u, v)\) in \( C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \) satisfying (1.8).

The main technical difficulty consists in the presence of \( p(x)\)-Laplacian and \( q(x)\)-Laplacian operators in the principle parts of equations in (P) on the one hand and, on the other the presence of singular terms through variable exponents that can occur under hypotheses (H.f) and (H.g). Under cooperative structure (1.4), by adapting Moser iterations procedure to problem (P), together with an adequate truncation, we prove a priori estimates for an arbitrary solution of (P). In particular, it provides that all solution \((u, v)\) of (P) are bounded in \( L^\infty(\Omega) \times L^\infty(\Omega) \). Taking advantage of this boundedness and applying Schauder’s fixed point Theorem we obtain the existence of a solution of problem (P). To the best of our knowledge,
it is for the first time when Moser iterations method is applied for problems with variable exponents.

For system $\mathcal{P}$ subjected to competitive structure (1.5), we develop some comparison arguments which provide a priori estimates on solutions of $\mathcal{P}$. In turn, these estimates enable us to obtain our main result by applying the Schauder’s fixed point theorem. It is worth noting that besides our method is different from that used by Alves & Moussaoui [3], our assumptions, precisely $H(f,g)_1$, (1.9) and (1.10), are not satisfied by hypotheses considered there.

We indicate simple examples showing the applicability of Theorems 1.1 and 1.2. Related to system $\mathcal{P}$ under assumptions above, we can handle singular cooperative systems of the form

$$
\begin{cases}
-\Delta_{p(x)} u = (u^{\alpha_1(x)} + 1)(v^{\beta_1(x)} + 1) & \text{in } \Omega \\
-\Delta_{q(x)} v = (u^{\alpha_2(x)} + 1)(v^{\beta_2(x)} + 1) & \text{in } \Omega \\
u, v > 0 & \text{in } \Omega \\
u, v = 0 & \text{on } \partial \Omega,
\end{cases}
$$

and singular competitive systems of type

$$
\begin{cases}
-\Delta_{p(x)} u = u^{\alpha_1(x)} + v^{\beta_1(x)} & \text{in } \Omega \\
-\Delta_{q(x)} v = u^{\alpha_2(x)} + v^{\beta_2(x)} & \text{in } \Omega \\
u, v > 0 & \text{in } \Omega \\
u, v = 0 & \text{on } \partial \Omega,
\end{cases}
$$

with variable exponents $\alpha_1, \alpha_2, \beta_1, \beta_2$ as in hypotheses (1.6), (1.7) and (1.9), (1.10), respectively.

The rest of this article is organized as follows. Section 2 deals with a priori estimates and regularity of solutions of cooperative system $\mathcal{P}$, whereas Section 3 presents comparison properties of competitive system $\mathcal{P}$. Sections 4 and 5 contain the proof of Theorems 1.1 and 1.2.

### 2. A priori estimates and regularity

Let $L^{p(x)}(\Omega)$ be the generalized Lebesgue space that consists of all measurable real-valued functions $u$ satisfying

$$
\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} \, dx < +\infty,
$$

endowed with the Luxemburg norm

$$
\|u\|_{p(x)} = \inf \{ \tau > 0 : \rho_{p(x)}(\frac{u}{\tau}) \leq 1 \}.
$$

The variable exponent Sobolev space $W^{1,p(x)}_0(\Omega)$ is defined by

$$
W^{1,p(x)}_0(\Omega) = \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \}.
$$

The norm $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}$ makes $W^{1,p(x)}_0(\Omega)$ a Banach space. On the basis of (1.2), the following embedding

$$
W^{1,p(x)}_0(\Omega) \hookrightarrow L^{r(x)}(\Omega)
$$

is continuous with $1 < r(x) \leq p^*(x)$ (see [9, Corollary 5.3]).
Lemma 2.1. (i) For any $u \in L^{p(x)}(\Omega)$ we have
\[
\|u\|_{p(x)}^p - \rho_{p(x)}(u) \leq \|u\|_{p(x)}^p \quad \text{if} \quad \|u\|_{p(x)} > 1,
\]
\[
\|u\|_{p(x)}^p - \rho_{p(x)}(u) \leq \|u\|_{p(x)}^p \quad \text{if} \quad \|u\|_{p(x)} \leq 1.
\]
(ii) For $u \in L^{p(x)}(\Omega) \setminus \{0\}$ we have
\[
\|u\|_{p(x)} = a \quad \text{if and only if} \quad \rho_{p(x)}\left(\frac{u}{a}\right) = 1.
\]

The next result provides a priori estimates for an arbitrary solution of \([P]\) subjected to cooperative structure.

**Theorem 2.2.** Assume that \([L_4]\) and the growth conditions \((H.f)\) and \((H.g)\) hold with 
\[
\begin{align*}
\alpha^+_1 < 0 < \beta_1(x) \leq \frac{q^-_1(x)}{p^+_1(x)}(p^+(x) - 1) \\
\beta^+_2 < 0 < \alpha_2(x) \leq \frac{q^-_2(x)}{p^+_2(x)}(q^+(x) - 1)
\end{align*}
\]
in $\Omega$. Then there exist positive constants $C = C(m_1, \beta_1, N, \Omega, p, q)$ and $C' = C'(m_2, \alpha_2, N, \Omega, p, q)$ such that every solution $(u, v) \in W^{1,p(x)}_0(\Omega) \times W^{1,q(x)}_0(\Omega)$ of \([P]\) satisfies the estimate
\[
\|u\|_{\infty} \leq C \max(1, \|u\|_{p^+(x)}^{p^+ / p^-}(1 + \max(1, \|u\|_{q^+(x)}^{\beta^+_{2}(x)})))^{\frac{1}{p^- - p^+}}.
\]
\[
\|v\|_{\infty} \leq C' \max(1, \|v\|_{q^+(x)}^{q^+ / q^-}(1 + \max(1, \|u\|_{p^+(x)}^{\beta^+_{1}(x)})))^{\frac{1}{q^- - q^+}}.
\]
In particular, problem \([P]\) has only bounded solutions.

**Proof.** Let $\phi : \mathbb{R} \rightarrow [0, 1]$ be a $C^1$ cut-off function such that
\[
\phi(s) = \begin{cases} 0 & \text{if} \ s \leq 0, \\ 1 & \text{if} \ s \geq 1 \end{cases} \quad \text{and} \quad \phi'(s) \geq 0 \text{ in } [0, 1].
\]

Given $\delta > 0$, we define $\phi_\delta(t) = \phi(t - \frac{1}{\delta})$ for all $t \in \mathbb{R}$. It follows that 
\[
\phi_\delta \circ z \in W^{1,p(x)}_0(\Omega) \quad \text{and} \quad \nabla (\phi_\delta \circ z) = (\phi_\delta' \circ z) \nabla z, \quad \text{for } z \in W^{1,p(x)}_0(\Omega).
\]

Let $(u, v) \in W^{1,p(x)}_0(\Omega) \times W^{1,q(x)}_0(\Omega)$ be a weak solution of \([P]\). Acting in the first equation in \([1.3]\) with the test function $\varphi = (\phi_\delta \circ u)\varphi$ with $\varphi \in W^{1,p(x)}_0(\Omega)$ and $\varphi \geq 0$ in $\Omega$, we obtain
\[
\int_{\Omega} \nabla u|^{p(x) - 2} \nabla u \nabla ((\phi_\delta \circ u)\varphi) \, dx = \int_{\Omega} f(u, v)(\phi_\delta \circ u)\varphi \, dx.
\]
Hence, by \([3.13]\), we get
\[
\int_{\Omega} \nabla u|^{p(x)}(\phi_\delta' \circ u)\varphi \, dx + \int_{\Omega} \nabla u|^{p(x) - 2} \nabla u \nabla \varphi \, dx \leq \int_{\Omega} f(u, v)(\phi_\delta \circ u)\varphi \, dx.
\]

Since $\phi_\delta' \circ u \geq 0$, it follows that
\[
\int_{\Omega} \nabla u|^{p(x) - 2} \nabla u \nabla \varphi \, dx \leq \int_{\Omega} f(u, v)(\phi_\delta \circ u)\varphi \, dx.
\]
Letting $\delta \to 0$, we achieve
\[
\int_{\{u > 1\}} \nabla u|^{p(x) - 2} \nabla u \nabla \varphi \, dx \leq \int_{\{u > 1\}} f(u, v)\varphi \, dx,
\]
for all $\varphi \in W^{1,p(x)}_0(\Omega)$ with $\varphi \geq 0$ in $\Omega$. Repeating the same argument with the second equation in \([P]\), we get
\[
\int_{\{v > 1\}} \nabla v|^{q(x) - 2} \nabla v \nabla \psi \, dx \leq \int_{\{v > 1\}} g(u, v)\psi \, dx,
\]
for all $\psi \in W^{1,q(x)}_0(\Omega)$ with $\psi \geq 0$ in $\Omega$.

Given $M > 0$, define
\[
u_M(x) = \min\{u(x), M\}, \quad \mu_M(x) = \min\{v(x), M\}.
\]

Observe that $h(s) = k^+ s^{p^+ - 1}$ is a $C^1$ function, $h(0) = 0$ and there is a constant $L > 0$ such that $|h'(s)| \leq L$ for all $0 \leq s \leq M$. By proceeding analogously to the proof of [5], Proposition XI.5, page 155], it follows that $u^{k^+ P^+ + 1}_M \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega)$.

Similarly we get $v^{k^+ q^+ + 1}_M \in W^{1,q(x)}_0(\Omega) \cap L^\infty(\Omega)$.

Step 1. Estimation of the left-hand side in (2.10) and (2.11)

In what follows denote by $(s - 1)^+ := \max\{s, 1\}$ for $s \geq 0$.

First, observe that
\[
|\nabla u_M|^{p(x)} u_M^{k^+ p^+} = \frac{1}{(k^+ p^+ + 1)^{p(x)}} |\nabla (u_M)^{p^+ + 1}|^{p(x)} \geq \frac{1}{(k^+ p^+ + 1)^{p(x)}} |\nabla (u_M)^{k^+ + 1}|^{p(x)}.
\]

Then
\[
(2.12)
\]
\[
(2.13)
\]
On the other hand, using (2.2) and through the mean value theorem, there exists $x_0 \in \Omega$ such that
\[
(2.14)
\]
which implies
\[
(2.15)
\]
Similarly, following the same argument as above leads to

\[
\int_{\Omega} \nabla ((u_M - 1)^{k_1^{-} + 1})^p(x) \, dx = \|((u_M - 1)^{k_1^{-} + 1})^{p(x)}\|_{1, p(x)},
\]

where \(C_1 = C_1(p, N, \Omega)\) is a positive constant and

\[
p^* = \begin{cases} 
p^+ & \text{if } \|((u_M - 1)^{k_1^{-} + 1})^{p^*(x)}\| > 1 \\
p^- & \text{if } \|((u_M - 1)^{k_1^{-} + 1})^{p^*(x)}\| \leq 1.
\end{cases}
\]

Similarly, following the same argument as above leads to

\[
k_1^{-q^+} \int_{\{v > 1\}} \nabla (v_M^{k_1^{+}})^q(x) \, dx \geq C_2 \frac{k_1^{-q^+} \int_{\{v > 1\}} \nabla (v_M^{k_1^{+}})^{q^+}}{(k_1^{+})^{q^+}} ((v_M - 1)^{k_1^{+}})^{q^+} \|((v_M - 1)^{k_1^{+}})^{q^+})\|_{(k_1^{+})^{q^+}}(x),
\]

with positive constants \(C_2 = C_2(q, N, \Omega)\) and

\[
q^* = \begin{cases} 
q^+ & \text{if } \|((v_M - 1)^{q^+})\|_{(k_1^{+})^{q^+}}(x) > 1 \\
q^- & \text{if } \|((v_M - 1)^{q^+})\|_{(k_1^{+})^{q^+}}(x) \leq 1.
\end{cases}
\]

**Step 2. Estimation of the right-hand side in (2.10) and (2.11).**

Using (2.7), (H.f), (2.9), (2.3), (2.1) together with Hölder’s inequality and [4 Proposition 2.3], we get

\[
\int_{\{u > 1\}} f(u, v) u_M^{k_1^{+} p^+} \, dx \leq \int_{\{u > 1\}} f(u, v) u_M^{k_1^{+} p^+} \, dx \\
\leq 2m_1 \int_{\{u > 1\}} (1 + v^{\beta_1(x)}) u_M^{k_1^{+} p^+} \, dx \\
= 2m_1 \int_{\{u > 1\}} (u - 1)^{k_1^{+} p^+} \, dx + 2m_1 \int_{\{u > 1\}} v^{\beta_1(x)} ((u - 1)^{k_1^{+} p^+} \, dx \\
\leq \tilde{C}_2 \left( \|(u - 1)^{k_1^{+} p^+} + \|(u - 1)^{k_1^{+} p^+} \|_{p^+} \|v^{\beta_1(x)}\|_{p^+} \|v^{\beta_1(x)}\|_{p^+} \right) \\
\leq \tilde{C}_2 \left( \|(u - 1)^{k_1^{+} p^+} + \|(u - 1)^{k_1^{+} p^+} \|_{p^+} \|v^{\beta_1(x)}\|_{p^+} \|v^{\beta_1(x)}\|_{p^+} \right) \\
\leq C_2 \left( u^{-1} (u - 1)^{k_1^{+} p^+} (1 + v^2)^{\beta^-_1(x)} \right),
\]

with a positive constant \(C_2 = C_2(m_1, \beta_1, N, \Omega, p, q)\) and

\[
\beta^-_1 = \begin{cases} \beta^+_1 & \text{if } \|v\|_{q^+} > 1 \\
\beta^+_1 & \text{if } \|v\|_{q^+} \leq 1.
\end{cases}
\]

Similarly, by (2.8), (H.g), (2.9), (2.3), (2.1), combined with Hölder’s inequality and [4 Proposition 2.3], one has

\[
\int_{\{v > 1\}} g(u, v) v_M^{k_1^{+} q^+} \, dx \leq C_2 (1 + \|v^\beta_2(x)\| \|v\|_{q^+}(x)^{k_1^{+} q^+} + 1),
\]
where the positive constant $C^*_2 = C^*_2(m_2, \alpha_2, N, \Omega, p, q)$ and

$$
\alpha^2 = \begin{cases} 
\frac{\alpha_2}{\alpha_2} & \text{if } \|u\|_{p^*(x)} > 1 \\
\frac{\alpha_2}{\alpha_2} & \text{if } \|u\|_{p^*(x)} \leq 1.
\end{cases}
$$

(2.24)

**Step 3. Moser iteration procedure and passage to the limit.**

We note that if $\|(u - 1)^+\|_{p^*(x)}, \|(v - 1)^+\|_{q^*(x)} > 1$, then there hold

$$
\|(u - 1)^+\|_{k^+ + 1} \leq \|(u - 1)^+\|_{p^*(x)} \quad \text{and} \quad \|(v - 1)^+\|_{q^*(x)} \leq \|(v - 1)^+\|_{q(q^*(x))}
$$

(2.25)

because $p^+, q^+ > 1$. Then, it follows from (2.17) - (2.25) that

$$
\|(u - 1)^+\|_{(k^+ + 1)p^*(x)} \leq C\left(\frac{k^+ + 1}{(k^+ + 1)p^+}\right) \frac{p^+}{(k^+ + 1)p^+} \|(u - 1)^+\|_{p^*(x)} \left(1 + \|v\|_{q^*(x)}\right) \frac{1}{(v - 1)^+}
$$

(2.26)

and

$$
\|(v - 1)^+\|_{(k^+ + 1)q^*(x)} \leq C\left(\frac{k^+ + 1}{(k^+ + 1)q^+}\right) \frac{q^+}{(k^+ + 1)q^+} \|(v - 1)^+\|_{q^*(x)} \left(1 + \|u\|_{p^*(x)}\right) \frac{1}{(v - 1)^+}
$$

(2.27)

with a constant $C = C(m_1, \alpha_2, \beta_1, N, \Omega, p, q) > 0$.

Inductively, we construct the sequences $\{k_n\}_{n \geq 1}$ and $\{\overline{k}_n\}_{n \geq 1}$ by defining

$$
k_n(x) + 1 = (k_{n-1}(x) + 1)p^*(x) = \left(\frac{p^*(x)}{p(x)}\right)^n,
$$

$$
\overline{k}_n(x) + 1 = (\overline{k}_{n-1}(x) + 1)q^*(x) = \left(\frac{q^*(x)}{q(x)}\right)^n,
$$

(2.28)

for all $n \geq 2$ starting with (2.20). If we have for infinitely many $n$ that

$$
\|(u - 1)^+\|_{(k_n + 1)p^*(x)} \leq 1 \quad \text{and} \quad \|(v - 1)^+\|_{(\overline{k}_n + 1)q^*(x)} \leq 1,
$$

then letting $n \to \infty$ we get $\|u\|_{\infty} \leq 1$ and $\|v\|_{\infty} \leq 1$, and we are done. If not, it suffices to consider the case

$$
\|(u - 1)^+\|_{(k_n + 1)p^*(x)} > 1 \quad \text{and} \quad \|(v - 1)^+\|_{(\overline{k}_n + 1)q^*(x)} > 1
$$

for all $n$ because otherwise the proof reduces to special case of Moser iteration procedure for an elliptic equation. In this case, we argue as for obtaining (2.26) and (2.27). Namely, proceeding by induction through (2.28) and then letting $M \to \infty$ we arrive at

$$
\|(u - 1)^+\|_{(k_n + 1)p^*(x)} \leq C^{k_n + 1} \left(\frac{k_n + 1}{(k_n + 1)p^+}\right) \frac{p^+}{(k_n + 1)p^+} \|(u - 1)^+\|_{(k_n + 1)p^*(x)} \left(1 + \|v\|_{q^*(x)}\right) \frac{1}{(k_n + 1)(p - 1)^+}
$$

(2.29)
and
\[(2.30)\]
\[\|(v - 1)^+\|_{(\mathcal{F}^{-1}_n + 1)p^*(x)} \leq C_2^{1/n-1} \left(\frac{\mathcal{F}^{-1}_n}{(\mathcal{F}^{p^+}_n + 1)})^{q^+} \right) \|(v - 1)^+\|_{(\mathcal{F}^{q^-}_n + 1)p^*(x)} (1 + \|u\|_{p^*(x)}^{\beta_1}) \left(\frac{1}{(n-1)q^-} \right),\]
with positive constants \(C_1 = C_1(N, \Omega, m_1, p, \beta_1)\) and \(C_2 = C_2(N, \Omega, m_2, q, \alpha_2)\). It turns out from (2.29) that
\[\|(u - 1)^+\|_{(\mathcal{F}^{-1}_n + 1)p^*(x)} \leq C_1 \sum_{i=1}^n \frac{1}{\mathcal{F}^{-1}_i + 1} \left(\frac{1}{\mathcal{F}^{p^+}_i} \right) \|(u - 1)^+\|_{p^*(x)} (1 + \|v\|_{q^+(x)}^{\beta_1})^{-\frac{1}{p^+}} \left(1 + \sum_{i=1}^n \frac{1}{\mathcal{F}^{-1}_i + 1} \right).\]
Furthermore, since \(\lim_{z \to \infty} \left(\frac{z}{\mathcal{F}^p(z+1)}\right) = 1\), there is a positive constant \(C_0\)
for which one has
\[(2.31)\]
\[\|(u - 1)^+\|_{(\mathcal{F}^{-1}_n + 1)p^*} \leq C_1 \sum_{i=1}^n \frac{1}{\mathcal{F}^{p^+}_i} \left(\frac{1}{\mathcal{F}^{p^+}_i} \right) \|(u - 1)^+\|_{p^*(x)} (1 + \|v\|_{q^+(x)}^{\beta_1})^{-\frac{1}{p^+}} \left(1 + \sum_{i=1}^n \frac{1}{\mathcal{F}^{-1}_i + 1} \right).\]
Similarly, we obtain
\[(2.32)\]
\[\|(v - 1)^+\|_{(\mathcal{F}^{-1}_n + 1)p^*(x)} \leq C_2 \sum_{i=1}^n \frac{1}{\mathcal{F}^{p^+}_i} \left(\frac{1}{\mathcal{F}^{p^+}_i} \right) \|(v - 1)^+\|_{q^*(x)} (1 + \|u\|_{p^*(x)}^{\alpha_2})^{-\frac{1}{q^+}} \left(1 + \sum_{i=1}^n \frac{1}{\mathcal{F}^{-1}_i + 1} \right).\]
Moreover, (2.28) guarantees the convergence of the series in (2.31) and (2.32), for instance
\[1 + \sum_{i=1}^{n-1} \frac{1}{\mathcal{F}^{-1}_i + 1} \to \frac{1}{(p^-)^{q^+} - p^-} + \frac{1}{(p^+)^{q^-} - p^+} \to \infty \] \(n \to \infty\) in (2.31) and (2.32) we derive the estimates (4.7) and (2.5). This completes the proof. \(\square\)

Next result is a consequence of Theorem 2.2.

**Proposition 2.1.** Under the assumptions of Theorem 1.1, every solutions \((u, v)\) of (2.1) is bounded in \(C^{1,\gamma}(\Omega) \times C^{1,\gamma}(\Omega)\) and there is a constant \(R > 0\) such that
\[\|u\|_{C^{1,\gamma}(\Omega)}, \|v\|_{C^{1,\gamma}(\Omega)} < R.\]
Moreover, it holds
\[(2.33)\]
\[u(x), v(x) \geq c_0 d(x),\]
with some constant \(c_0 > 0\).

**Proof.** We first show (2.33). Recalling the constant \(\sigma > 0\) in \(H(f, g)\), let \(z_1, z_2\) the only positive solutions of
\[(2.34)\]
\[\begin{cases}
-\Delta_{p^+} z_1 = \sigma \text{ in } \Omega \\
z_1 = 0 \text{ on } \partial \Omega
\end{cases} \quad \text{and} \quad \begin{cases}
-\Delta_{q^+} z_2 = \sigma \text{ in } \Omega \\
z_2 = 0 \text{ on } \partial \Omega,
\end{cases}\]
which are known to satisfy
\[(2.35) \quad z_1(x) \geq c_2 d(x) \quad \text{and} \quad z_2(x) \geq c'_2 d(x) \quad \text{in} \ \Omega,
\]
for certain positive constants \(c_2\) and \(c'_2\) (see, e.g., [3]). Then, from (P), (2.34) and
\(H(f,g)\), it follows that
\[
\begin{cases}
-\Delta p(x) u \geq -\Delta p(x) z_1 & \text{in} \ \Omega \\
u = z_1 & \text{on} \ \partial \Omega
\end{cases}
\]
and
\[
\begin{cases}
-\Delta q(x) v \geq -\Delta q(x) z_2 & \text{in} \ \Omega \\
v = z_2 & \text{on} \ \partial \Omega
\end{cases}
\]
Therefore, the weak comparison principle leads to (2.33).

By virtue of (H.\(f\)), (H.\(g\)), (2.33), (1.4), (1.5) and (1.7), on account of Theorem 2.2, one has
\[(2.36) \quad f(u,v) \leq C_0 d(x)^{\alpha - 1} \quad \text{and} \quad f(u,v) \leq C'_0 d(x)^{\beta - 2}\]
in \(\Omega\), for some positive constants \(C_0\) and \(C'_0\). Then, the \(C^{1,\alpha}\)-boundedness of \(u\) and \(v\) follows from [3, Lemma 2]. The proof is completed. \(\square\)

3. Comparison properties

In this section, we assume that (1.9) and (1.10) hold. For a fixed \(\delta > 0\) small, define \(\overline{\nu}\) and \(\underline{\nu}\) in \(C^{1,\gamma}(\Omega)\), for certain \(\gamma \in (0,1)\), as the unique weak solutions of the problems
\[
(3.1) \quad -\Delta p(x) \overline{\nu} = \lambda \begin{cases}
1 & \text{in} \ \Omega \setminus \overline{\Omega}_\delta \\
\nu^{-\alpha_1(x)} & \text{in} \ \Omega_\delta
\end{cases}, \ \overline{\nu} > 0 \text{ in} \ \Omega, \ \overline{\nu} = 0 \text{ on} \ \partial \Omega
\]
\[
(3.2) \quad -\Delta q(x) \underline{\nu} = \lambda \begin{cases}
1 & \text{in} \ \Omega \setminus \underline{\Omega}_\delta \\
\nu^{-\beta_2(x)} & \text{in} \ \Omega_\delta
\end{cases}, \ \underline{\nu} > 0 \text{ in} \ \Omega, \ \underline{\nu} = 0 \text{ on} \ \partial \Omega.
\]
where \(\lambda > 1\) is a constant and
\[
\Omega_\delta = \{x \in \Omega : d(x, \partial \Omega) < \delta\}.
\]
Combining the results in [3] Lemmas 1 and 3 and [11], it is readily seen that for \(\lambda > 1\) large \(\overline{\nu}\) and \(\underline{\nu}\) verify
\[
(3.3) \quad \min\{\delta, d(x)\} \leq \overline{\nu}(x) \leq c_1 \lambda^{\frac{1}{\alpha - 1}} \quad \text{in} \ \Omega,
\]
and
\[
(3.4) \quad \min\{\delta, d(x)\} \leq \underline{\nu}(x) \leq c_2 \lambda^{\frac{1}{\beta - 1}} \quad \text{in} \ \Omega,
\]
for some positive constant \(c_1, c_2\) independent of \(\lambda\) and for \(\delta > 0\) small. Moreover, similar arguments explored in the proof of [23] Theorem 4.4 produce constants \(c_0, c'_0 > 0\) such that
\[
(3.5) \quad \overline{\nu}(x) \leq c_0 d(x)^{\theta_1} \quad \text{and} \quad \overline{\nu}(x) \leq c'_0 d(x)^{\theta_2}\]
in \(\Omega_\delta\), for some constants \(\theta_1, \theta_2 \in (0,1)\), which assumed to satisfy the estimates
\[
(3.6) \quad \theta_1 \geq -\frac{\beta_1^{-1}}{\alpha_1} \quad \text{and} \quad \theta_2 \geq -\frac{\alpha_2^{-1}}{\beta_2}.
\]
Notice that \(\theta_1\) and \(\theta_2\) exist since \(-\beta_1^{-1} < \alpha_1^{-1}\) and \(-\alpha_2^{-1} < \beta_2^{-1}\) (see (H.\(f\)) and (H.\(g\))).

Now, let consider the functions \(\overline{u}\) and \(\underline{u}\) defined by
\[
(3.7) \quad -\Delta p(x) \overline{u} = \lambda^{-1} \begin{cases}
1 & \text{in} \ \Omega \setminus \overline{\Omega}_\delta \\
-1 & \text{in} \ \Omega_\delta
\end{cases}, \ \overline{u} = 0 \text{ on} \ \partial \Omega
\]
and
\begin{equation}
\Delta_g(u) = \lambda^{-1} \begin{cases} 
1 & \text{in } \Omega \setminus \overline{\Omega}_\delta \\
-1 & \text{in } \Omega_\delta 
\end{cases}, \quad u = 0 \text{ on } \partial \Omega.
\end{equation}

where \( \Omega_\delta \) is given by
\begin{equation}
\Omega_\delta = \{ x \in \Omega : d(x, \partial \Omega) < \delta \},
\end{equation}
with a fixed \( \delta > 0 \) sufficiently small. Combining [11] Lemma 2.1 and [13] Theorem 1.1 with [8] Lemma 3, we get
\begin{equation}
c_3 d(x) \leq u(x) \leq c_4 \lambda^{\frac{1}{p-1}} \quad \text{and} \quad c_3' d(x) \leq g(x) \leq c_4' \lambda^{\frac{1}{p-1}} \text{ in } \Omega,
\end{equation}
where \( c_3, c_4, c_3', \text{ and } c_4' \) are positive constants. Obviously, from (3.1), (3.2) (3.7) and (3.8), we have \((u, v) \leq \overline{\Omega}, \overline{\Omega}) \) for \( \lambda > 0 \) large.

The following result allows us to achieve useful comparison properties.

**Proposition 3.1.** Assume that \((H.f), (H.g) \) and \( H(f, g)_2 \) hold. Then, for \( \lambda > 0 \) large enough, we have
\begin{equation}
- \Delta_{p(x)} u \leq f(u, \overline{u}), \quad - \Delta_{q(x)} u \leq g(u, \overline{u}) \text{ in } \Omega,
\end{equation}
\begin{equation}
- \Delta_{p(x)} v \geq f(v, \overline{v}), \quad - \Delta_{q(x)} v \geq g(u, \overline{v}) \text{ in } \Omega.
\end{equation}

**Proof.** For all \( \lambda > 0 \) one has
\begin{equation}
- \lambda^{-1} \lambda^{-1} \leq 0 < 1 \quad \text{and} \quad - \lambda^{-1} \lambda^{-1} \leq 0 < 1 \text{ in } \Omega_\delta.
\end{equation}
By (3.10), it follows that
\begin{equation}
\lambda^{-1} \lambda^{-1} \leq \lambda^{-1} (c_3 d(x))^{-1} \leq \lambda^{-1} (c_3 \delta)^{-1} \leq 1 \text{ in } \Omega \setminus \overline{\Omega}_\delta,
\end{equation}
and
\begin{equation}
\lambda^{-1} \lambda^{-1} \leq \lambda^{-1} (c_3' d(x))^{-1} \leq \lambda^{-1} (c_3' \delta)^{-1} \leq 1 \text{ in } \Omega \setminus \overline{\Omega}_\delta,
\end{equation}
provided that \( \lambda \) is sufficiently large. Another hand, by \( H(f, g)_2 \) there exist constants \( \rho, \overline{\rho} > 0 \) such that
\begin{equation}
f(s_1, s_2) \geq s_1^{p^{-1}}, \text{ for all } 0 < s_1 < \rho, \text{ for all } 0 < s_2 < \lambda^{\frac{1}{p-1}},
\end{equation}
and
\begin{equation}
g(s_1, s_2) \geq s_2^{q^{-1}}, \text{ for all } 0 < s_1 < \lambda^{\frac{1}{q-1}}, \text{ for all } 0 < s_2 < \overline{\rho}.
\end{equation}
Then, for \( \lambda > 0 \) sufficiently large so that
\[
\max \{ c_4 \lambda^{\frac{1}{p-1}}, c_4' \lambda^{\frac{1}{p-1}} \} < \min \{ \rho, \overline{\rho} \},
\]
combining (3.13) - (3.17) together, we infer that (3.11) holds true.

Next, we show (3.12). By \((H.f), (H.g), (3.10), (2.13) \) and (3.14), it follows that
\begin{equation}
f(\overline{u}, \overline{v}) \leq M_1 (1 + \overline{u}^{\alpha_1(x)})(1 + \overline{u}^{\beta_1(x)})
\leq M_1 (1 + c_4 \lambda^{\frac{1}{p-1}} \lambda^{\frac{1}{p-1}})(1 + (c_3 d(x))^{\beta_1(x)}) \leq \lambda \text{ in } \Omega \setminus \overline{\Omega}_\delta
\end{equation}
and
\begin{equation}
g(\overline{u}, \overline{v}) \leq M_2 (1 + \overline{u}^{\beta_2(x)})(1 + \overline{v}^{\beta_2(x)})
\leq M_2 (1 + (c_3 d(x))^{\alpha_2(x)})(1 + (c_3' \delta)^{\beta_2(x)} \lambda^{\frac{1}{p-1}}) \leq \lambda \text{ in } \Omega \setminus \overline{\Omega}_\delta.
\end{equation}
provided that $\lambda > 0$ is large enough. Now we deal with the corresponding estimates on $\Omega_\delta$. From (H.f), (H.g), (3.10), (2.13), (3.4) and (1.3), we get

\begin{align}
\text{(3.20)} \quad & P^\alpha(x)f(u, w) \leq M_1(P^\alpha(x) + P^{2\alpha}(x))(1 + u^{3\alpha}(x)) \\
& \leq M_1 \left( (c_0 d(x)^{\beta_1} + (c_0 d(x)^{\beta_1})^{2\alpha}(x)) (1 + (c_1 d(x))^{\beta_1}(x)) \right) \\
& \leq M_1 \max \left\{ (c_0)^{\alpha}(x), (c_0)^{2\alpha}(x) \right\} (d(x)^{\beta_1} + d(x)^{2\beta_1} + (c_1 d(x))^{\beta_1}) \leq \lambda \quad \text{in } \Omega_\delta
\end{align}

and similarly

\begin{align}
\text{(3.21)} \quad & P^{\beta_2}(x)g(u, v) \leq M_2(1 + u^{\alpha_2}(x))(P^{\beta_2}(x) + P^{2\beta_2}(x)) \\
& \leq M_2 \left( (c_0 d(x)^{\beta_2} + (c_0 d(x)^{\beta_2})^{2\beta_2}(x)) (c_1 d(x)^{\beta_2} + (c_1 d(x)^{\beta_2})^{2\beta_2}(x)) \leq \lambda \quad \text{in } \Omega_\delta
\end{align}

provided that $\lambda > 0$ is sufficiently large. Consequently, (3.18), (3.19), (3.20) and (3.21) allow to infer that (3.12) holds. This ends the proof. \hfill \Box

4. PROOF OF THEOREM 1.1

For every $z_1, z_2 \in C^0_0(\Omega)$, let us state the auxiliary problem

\begin{align}
(P_z) \quad & \begin{cases}
-\Delta_p u = f(z_1, z_2) & \text{in } \Omega, \\
-\Delta_q v = g(z_1, z_2) & \text{in } \Omega, \\
u, v = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align}

where

\begin{align}
(4.1) \quad & f(z_1, z_2) = f(\tilde{z}_1, \tilde{z}_2) \quad \text{and} \quad g(z_1, z_2) = g(\tilde{z}_1, \tilde{z}_2),
\end{align}

with

\begin{align}
(4.2) \quad & \tilde{z}_i = \min \left\{ \max \{z_i, c_0 d(x)\}, R \right\} \quad \text{for } i = 1, 2.
\end{align}

On account of (4.2) it follows that $c_0 d(x) \leq \tilde{z}_i \leq R$ for $i = 1, 2$.

The next result establishes an a priori estimate for system (Pz). In addition, it shows that solutions $(u, v)$ of problem (Pz) cannot occur outside the rectangle $[c_0 d(x), L_R] \times [c_0 d(x), L_R]$, with a constant $L_R > 0$ defined below.

**Proposition 4.1.** Assume (H.f), (H.g) and (1.4) hold. Then all solutions $(u, v)$ of (Pz) belong to $C^{1, \gamma} (\Omega) \times C^{1, \gamma} (\Omega)$ for some $\gamma \in (0, 1)$ and there is a positive constant $L_R$, depending on $R$, such that

\begin{align}
(4.3) \quad & \|u\|_{C^{1, \gamma}(\Omega)}, \|v\|_{C^{1, \gamma}(\Omega)} < L_R.
\end{align}

Moreover, it holds

\begin{align}
(4.4) \quad & u(x), v(x) \geq c_0 d(x) \quad \text{in } \Omega.
\end{align}

**Proof.** First, we prove the boundedness for solutions of (Pz) in $L^\infty(\Omega) \times L^\infty(\Omega)$. To this end, we adapt the argument which proves [3] Lemma 2. For each $k \in \mathbb{N}$, set

\begin{align}
U_{k,R} = \{ x \in \Omega : u(x) > kR \} \quad \text{and} \quad V_{k,R} = \{ x \in \Omega : v(x) > kR \},
\end{align}

where the constant $R > 0$ is given by Proposition 2.1. Since $u, v \in L^1(\Omega)$, we have

\begin{align}
(4.5) \quad & |U_{k,R}|, |V_{k,R}| \to 0 \quad \text{as } k \to +\infty.
\end{align}
Using \((u - kR)^+\) and \((v - kR)^+\) as a test function in \((P_2)\), we get

\[
\begin{align*}
\int_{U_{k, R}} |\nabla u|^p dx &= \int_{U_{k, R}} f_+(\tilde{z}_1, \tilde{z}_2)(u - kR)^+ dx \\
\int_{V_{k, R}} |\nabla v|^q dx &= \int_{V_{k, R}} g_+(\tilde{z}_1, \tilde{z}_2)(v - kR)^+ dx.
\end{align*}
\]

By (H.f) and (4.2) observe that

\[
\int_{\Omega} |f_+(\tilde{z}_1, \tilde{z}_2)|^N dx \leq C_1 \int_{\Omega} (1 + \tilde{z}_1^{N\alpha_1(x)})(1 + \tilde{z}_2^{N\beta_1(x)}) dx \leq C_1 \int_{\Omega} (1 + d(x)^{N\alpha_1}) dx.
\]

Since \(N\alpha_1 > -1\) (see (1.7)), Lemma in page 726] guarantees that

\[
\int_{\Omega} d(x)^{N\alpha_1} dx < \infty.
\]

Then, it follows that \(f_+(\tilde{z}_1, \tilde{z}_2) \in L^N(\Omega)\) and therefore

\[
\|f_+(\tilde{z}_1, \tilde{z}_2)\|_{L^N(U_{k, R})} \to 0 \quad \text{as} \; k \to +\infty.
\]

Similarly, we obtain

\[
\|g_+(\tilde{z}_1, \tilde{z}_2)\|_{L^N(V_{k, R})} \to 0 \quad \text{as} \; k \to +\infty.
\]

Now, proceeding analogously to the proof of [9 Lemma 2] provides a constant \(k_0 \geq 1\) such that

\[|u(x)|, |v(x)| \leq k_0 R \quad \text{a.e in } \Omega.\]

Consider now functions \(w_1\) and \(w_2\) defined by

\[
\begin{align*}
-\Delta w_1 &= \tilde{f}(z_1, z_2) \quad \text{in } \Omega \\
\quad w_1 &= 0 \quad \text{on } \partial \Omega \quad \text{and} \\
-\Delta w_2 &= \tilde{g}(z_1, z_2) \quad \text{in } \Omega \\
\quad w_2 &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

On account of (4.1), (H.f), (H.g), (4.2), (1.4) and (1.7), one has

\[
\tilde{f}(z_1, z_2) \leq C_2 d(x)^{\alpha^-} \quad \text{and} \quad \tilde{g}(z_1, z_2) \leq C_2' d(x)^{\beta^-} \quad \text{in } \Omega,
\]

for some positive constants \(C_2\) and \(C_2'\). On the basis of (1.7) and Thanks to [10 Lemma in page 726], the right-hand side of problems in (4.9) belongs to \(H^{-1}(\Omega)\). Consequently, the Minty-Browder theorem (see [5 Theorem V.15]) implies the existence and uniqueness of \(w_1\) and \(w_2\) in (4.9). Moreover, bearing in mind (1.7) and (4.10), the regularity theory found in [15] Lemma 3.1 implies that \(w_1\) and \(w_2\) are bounded in \(C^{1,\gamma}(\Omega)\), for certain \(\gamma \in (0, 1)\).

Therefore, subtracting (4.9) from (P_2) yields

\[-\text{div}(\|\nabla u\|^p - \nabla u - \nabla w_1) = 0 \quad \text{and} \quad -\text{div}(\|\nabla v\|^q - \nabla v - \nabla w_2) = 0,
\]

and the \(C^{1,\alpha}\)-boundedness of \(u\) and \(v\) follows from [10] Theorem 1.2].

Summarizing, we have obtained that solutions \((u, v)\) of \((P_2)\) belong to \(C^{1,\gamma}(\Omega) \times C^{1,\gamma}(\Omega)\), for certain \(\gamma \in (0, 1)\), and there exists a constant \(L_R > 0\) such that (1.3) holds. Furthermore, a quite similar argument showing the second part of Proposition 2.1 leads to (1.4). This completes the proof. \(\square\)

Next we prove the existence result for cooperative system \((P_2)\).
Proof of Theorem 1.1. Denote by
\[ B(0, L_R) = \{(u, v) \in C^1(\Omega) \times C^1(\Omega) : \|u\|_{C^1(\Omega)} + \|v\|_{C^1(\Omega)} < L_R \}\]
and
\[ \mathcal{O} = \{(u, v) \in B(0, L_R) : u(x), v(x) \geq c_0d(x) \quad \text{in } \Omega \}. \]
Let us introduce the operator \( P : \mathcal{O} \to C^1(\Omega) \times C^1(\Omega) \) by \( P(z_1, z_2) = (u, v) \), where \((u, v)\) is the solution of problem \((P)\). Bearing in mind \((4.10)\) and \((1.7)\), the Minty-Browder theorem together with \([3, \text{Lemma 2}]\) guarantee that problem \((P)\) has a unique solution \((u, v)\) in \( C^{1,\gamma}(\Omega) \times C^{1,\gamma}(\Omega) \), for certain \( \gamma \in (0,1) \). This ensures the operator \( P \) is well defined. Moreover, analysis similar to that in the proof of Theorem 3 in \([3]\) imply that \( P \) is continuous and compact operator. On the other hand, according to Proposition 4.1, it follows that \( \mathcal{O} \) is invariant by \( P \), that is, \( P(\mathcal{O}) \subset \mathcal{O} \). Therefore we are in a position to apply Schauder’s fixed point Theorem to the set \( \mathcal{O} \) and the map \( P : \mathcal{O} \to \mathcal{O} \). This ensures the existence of \((u, v) \in \mathcal{O}\) satisfying \( P(u, v) = (u, v) \), that is, \((u, v) \in C^1(\Omega) \times C^1(\Omega) \) is a solution of problem
\[
\begin{align*}
-\Delta_p(u) &= f(y_1, y_2) \quad \text{in } \Omega, \\
-\Delta_q(v) &= g(y_1, y_2) \quad \text{in } \Omega, \\
u, v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
Finally, thank’s to proposition \([2, 4]\) it turns out that \((u, v) \in C^1(\Omega) \times C^1(\Omega) \) is a (positive) solution of problem \((P)\). \(\square\)

5. PROOF OF THEOREM 1.2

The proof is based on Schauder’s fixed point Theorem. Using the functions \((u, v)\) and \((\mu, \nu)\) given in \([3.1], [3.2], [3.7]\) and \([3.8]\) let introduce the set
\[ \mathcal{K} = \{(y_1, y_2) \in C(\Omega) \times C(\Omega) : u \leq y_1 \leq \mu \text{ and } \nu \leq y_2 \leq \nu \in \Omega \}, \]
which is closed, bounded and convex in \( C(\Omega) \times C(\Omega) \). Then we define the operator \( T : \mathcal{K} \to C(\Omega) \times C(\Omega) \) by \( T(y_1, y_2) = (u, v) \), where \((u, v)\) is required to satisfy
\[
\begin{align*}
-\Delta_p(x)u &= f(y_1, y_2) \quad \text{in } \Omega, \\
-\Delta_q(x)v &= g(y_1, y_2) \quad \text{in } \Omega, \\
u, v &= 0 \quad \text{on } \partial \Omega. \tag{P_y}
\end{align*}
\]
For \((y_1, y_2) \in \mathcal{K}\), we derive from \((H.f), (H.g), [2,13], [3.9], [3.10], \text{and } [3.11]\) the estimates
\[ f(y_1, y_2) \leq m_1(1 + \mu^{\alpha_1}(x))(1 + \nu^{\beta_1}(x)) \leq C_1d(x)^{\beta_1}(x) \quad \text{in } \Omega \]
and
\[ g(y_1, y_2) \leq m_2(1 + \mu^{\alpha_2}(x))(1 + \nu^{\beta_2}(x)) \leq C_2d(x)^{\alpha_2}(x) \quad \text{in } \Omega, \]
with positive constants \( C_1, C_2 \). We point out that estimates \((5.1)\) and \((5.2)\) combined with \([1.9]\) and \([1.10]\) enable us to deduce that \( f(y_1, y_2) \in W^{-1,p}(\Omega) \) and \( g(y_1, y_2) \in W^{-1,q}(\Omega) \). Then the unique solvability of \((u, v) \in \mathcal{P}_y \) is readily derived from Minty-Browder theorem (see, e.g., [5]). Hence, the operator \( T \) is well defined.

Using the regularity theory up to the boundary (see \([3, \text{Lemma 2}]\)), it follows that \((u, v) \in C^{1,\beta}(\Omega) \times C^{1,\beta}(\Omega) \), with some \( \beta \in (0,1) \), and there is a constant \( M > 0 \) such that \( \|u\|_{C^{1,\beta}(\Omega)} \leq M \), whenever \((u, v) = T(y_1, y_2) \). \(\square\)
(y_1, y_2) \in \mathcal{K}$. Then, analysis similar to that in the proof of [3] Theorem 3 imply that $\mathcal{T}$ is continuous and compact operator.

The next step in the proof is to show that $\mathcal{T}(\mathcal{K}) \subset \mathcal{K}$. Let $(y_1, y_2) \in \mathcal{K}$ and denote $(u, v) = \mathcal{T}(y_1, y_2)$. Using the definitions of $\mathcal{K}$ and $\mathcal{T}$, on the basis of Proposition 3.1 (H$_f$) and (H$_g$), it follows that

$$-\Delta_{p(x)} u(x) = f(y_1(x), y_2(x)) \leq f(\tilde{u}(x), \tilde{v}(x)) \leq -\Delta_{p(x)} \tilde{u}(x) \text{ in } \Omega,$$

and similarly

$$-\Delta_{q(x)} v(x) = g(y_1(x), y_2(x)) \leq g(\tilde{u}(x), \tilde{v}(x)) \leq -\Delta_{q(x)} \tilde{v}(x) \text{ in } \Omega.$$

Proceeding in the same way, via Proposition 3.1 and hypotheses (H$_f$), (H$_g$), leads to

$$-\Delta_{p(x)} u(x) = f(y_1(x), y_2(x)) \geq f(\tilde{u}(x), \tilde{v}(x)) \geq -\Delta_{p(x)} \tilde{u}(x) \text{ in } \Omega,$$

and similarly

$$-\Delta_{q(x)} v(x) = g(y_1(x), y_2(x)) \geq g(\tilde{u}(x), \tilde{v}(x)) \geq -\Delta_{q(x)} \tilde{v}(x) \text{ in } \Omega.$$

Then from the strict monotonicity of the operators $-\Delta_{p(x)}$ and $-\Delta_{q(x)}$ we get that $(u, v) \in \mathcal{K}$, which establishes that $\mathcal{T}(\mathcal{K}) \subset \mathcal{K}$. Therefore we are in a position to apply Schauder’s fixed point Theorem to the set $\mathcal{K}$ and the map $\mathcal{T}: \mathcal{K} \to \mathcal{K}$. This ensures the existence of $(u, v) \in \mathcal{K}$ satisfying $(u, v) = \mathcal{T}(u, v)$. Moreover, because the solution $(u, v) \in \mathcal{K}$ and (H$_f$), (H$_g$), [13] and [14] are fulfilled, we conclude from [3] Lemma 2 that $(u, v) \in C^1(\bar{\Omega}) \times C^1(\bar{\Omega})$. This ends the proof.

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