Effective Actions and Gauge Field Stability

Amit Giveon\footnote{e-mail address: giveon@vms.huji.ac.il}

Racah Institute of Physics, The Hebrew University
Jerusalem, 91904, ISRAEL

and

Martin Roček\footnote{e-mail address: rocek@insti.physics.sunysb.edu}

Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, NY 11794-3840 USA

ABSTRACT

By studying an effective action description of the coupling of charged gauge fields in $N = 2$ $SU(n)$ supersymmetric Yang-Mills theories, we can describe regions of moduli space where one or more of these fields becomes unphysical. We discuss subtleties in the structure of the moduli space for $SU(3)$. 
1 Introduction and summary

The low energy effective action for the massless fields in $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory was found by Seiberg and Witten in [1, 2]. These results gave new insight into the dynamics of strongly coupled theories. In particular, in [1] it was shown that there exists a curve in moduli space across which certain BPS-bound saturated states may cease to exist. In [3], it was noted that the naive $SU(2)$ covariantization of the effective action of [1] gives a striking signal of this phenomenon: as one crosses the curve described in [1], the norm of some states (the charged gauge field supermultiplets) appears to change sign. If the theory is to remain unitary, the massive states described by this covariantized effective action must become unphysical.

In a series of papers [4, 5, 6, 7, 8], the methods of [1, 2] were extended from the case of $SU(2)$ to general groups. In this paper, we consider what we can learn by covariantizing the effective action and studying where various charged gauge bosons become unphysical. We give some general results for the Coulomb phase of $N = 2$ $SU(n)$ gauge theories with or without matter hypermultiplets (the generalization to arbitrary simple groups should be straightforward). We then focus on the case of pure $N = 2$ $SU(3)$ gauge theory. We find that the moduli space is divided into regions where one or more gauge bosons destabilize.

In more detail, for a general group $G$ with a basis of generators $\{T_i, T_\alpha\}$, where $\{T_i\}$ generate the Cartan subalgebra, we write an adjoint representation $N = 1$ chiral superfield $\phi$ as

$$
\phi^{ab} = A^i(T_i)^{ab} + A^{\alpha}(T_\alpha)^{ab} .
$$

(1.1)

Here, $A^i$ describe Abelian superfields, and $A^{\alpha}$ describe charged superfields. The low energy effective action for an $N = 2$ supersymmetric Yang-Mills theory in the Coulomb phase is given in terms of a single holomorphic function $\mathcal{F}(A^i)$ [3]:

$$
\frac{1}{4\pi} Im \left[ \int d^4 \theta \frac{\partial^2 \mathcal{F}(\{A^k\})}{\partial A^i} \bar{A}^i + \frac{1}{2} \int d^2 \theta \frac{\partial^2 \mathcal{F}(\{A^k\})}{\partial A^i \partial A^j} W^{a i} W^j_a \right] .
$$

(1.2)
In terms of $\mathcal{F}$, the gauge coupling constants of the theory are given by the imaginary part of $\tau_{ij}$, where

$$
\tau_{ij} = \left\langle \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} \rightangle ,
$$

and

$$
a_i = \langle A^i \rangle , \quad 0 = \langle A^\alpha \rangle .
$$

The expectation values $a_i$ vary as one moves through the moduli space.

We define a gauge-invariant function $\mathcal{F}(\phi^{ab})$ by the condition that it reduces to $\mathcal{F}(a_i)$:

$$
\mathcal{F}(\langle \phi^{ab} \rangle) = \mathcal{F}(a_i) ;
$$

for $G = SU(2)$, $\mathcal{F}(\phi) = \mathcal{F}(\sqrt{\frac{1}{2} Tr \phi^2})$. Then a manifestly gauge invariant $N = 2$ supersymmetric action which reduces to (1.2) at low energies is

$$
\frac{1}{4\pi} \text{Im} \left[ \int d^4 \theta \frac{\partial \mathcal{F}(\phi)}{\partial \phi^{ab}} (e^V)_{ab,cd} \phi^{cd} + \frac{1}{2} \int d^2 \theta \frac{\partial^2 \mathcal{F}(\phi)}{\partial \phi^{ab} \partial \phi^{cd}} W^{\alpha ab} W_{\alpha}^{cd} \right] ,
$$

where $V$ is the gauge superfield (see, e.g., [11]). In this work, we focus on the quadratic terms in (1.6); their coefficient is

$$
\left\langle \frac{\partial^2 \mathcal{F}}{\partial \phi^{ab} \partial \phi^{cd}} \rightangle ,
$$

where here, and subsequently,

$$
\langle f(\phi) \rangle \equiv f(\phi)|_{\phi = \langle \phi \rangle} .
$$

In section 2, we compute the explicit form of the generalized gauge couplings (1.7) for arbitrary $\mathcal{F}$ in the case of $G = SU(n)$, and find it has a

1This is not a low momentum expansion of the 1PI generating functional even at the one-loop level [10].

2For general groups, $\mathcal{F}(\phi)$ is a complicated function of the group invariants; we will not need its explicit form.
simple decomposition as a sum of projectors: The terms in the Cartan subalgebra have $\tau_{ij}$ as coefficients; by construction, these have positive norm everywhere in moduli space. The remaining terms have coefficients with imaginary parts that may vanish and change sign in certain regions of moduli space. As described above, we interpret this phenomenon as a signal that certain states are destabilizing and disappearing from the spectrum. We compare this result with the BPS mass formula, and find complete consistency: As in the $SU(2)$ case, when a gauge boson destabilizes, its mass becomes degenerate with a monopole-dyon pair with the same total quantum numbers.

In section 3, we analyze the case of $SU(3)$ in more detail. We find a puzzle, and offer a resolution: certain points in the moduli space should be blown up into $S^2$'s.

2 $SU(n)$: The calculation

We begin by defining our notation: We will work in the fundamental $n \times n$ matrix representation of $SU(n)$, and therefore denote a basis for all $n \times n$ matrices by

$$(E^{ij})_{ab} = \delta^i_a \delta^j_b ;$$

we give a special name to the diagonal matrices

$$(e_i)_{ab} = (E^{ii})_{ab} , \quad i = 1, ..., n ,$$

and choose a basis $H_i$ of the Cartan subalgebra of $SU(n)$ as well as a dual basis $H^*_i$:

$$H_i = e_i - e_n , \quad Tr(H^*_i H_j) = \delta_{ij} \Rightarrow H^*_i = e_i - \frac{1}{n} I , \quad i = 1, ..., n - 1 .$$

3M. Douglas suggests that, while this may be true mathematically, it isn’t relevant physically; see Section 3.

4The vectors $\alpha_i \equiv diag(H_i)$ defined by the diagonal matrices $H_i$, are a basis for the root lattice. Similarly, $\alpha^*_i \equiv diag(H^*_i)$ is a basis for the weight lattice. We do not choose a basis of simple roots corresponding to $H_i = e_i - e_{i+1}$ as used in, e.g., [1], because this leads to $H^*_i = \sum_j e_j - \frac{i}{n} I$, which we find less convenient than (2.3).
Here I is the $n \times n$ identity matrix. We parametrize the classical expectation values of the field $\phi$ by eigenvalues $a_i$:

$$
\langle \phi \rangle = \sum_{1}^{n-1} a_i H_i = \sum_{1}^{n} a_i e_i, \quad a_n = -\sum_{1}^{n-1} a_i. \quad (2.4)
$$

We now compute the (generalized) coupling matrix for the extended low energy effective action defined above in terms of the function $F$:

$$
\left\langle \frac{\partial^2 F}{\partial \phi^{ab} \partial \phi^{cd}} \right\rangle = \tau_{ij} \left\langle \frac{\partial a_i}{\partial \phi^{ab}} \right\rangle \left\langle \frac{\partial a_j}{\partial \phi^{cd}} \right\rangle + a_{Di} \left\langle \frac{\partial^2 a_i}{\partial \phi^{ab} \partial \phi^{cd}} \right\rangle, \quad (2.5)
$$

where

$$
\tau_{ij} = \left\langle \frac{\partial^2 F}{\partial a_i \partial a_j} \right\rangle \quad \text{and} \quad a_{Di} = \left\langle \frac{\partial F}{\partial a_i} \right\rangle. \quad (2.6)
$$

We need to find

$$(a'_{\scriptscriptstyle i})_{ab} = \left\langle \frac{\partial a_i}{\partial \phi^{ab}} \right\rangle \quad \text{and} \quad (a''_{\scriptscriptstyle i})_{ab,cd} = \left\langle \frac{\partial^2 a_i}{\partial \phi^{ab} \partial \phi^{cd}} \right\rangle. \quad (2.7)
$$

We do this by differentiating the invariants

$$
u_k = \frac{1}{k} Tr(\phi^k) = \frac{1}{k} \sum_{1}^{n} (a_i)^k, \quad (2.8)
$$

and solving the resulting linear equations. Since the $\phi^{ab}$ are traceless, differentiation acts as

$$
\frac{\partial \phi^{ab}}{\partial \phi^{cd}} = \delta_c^a \delta_d^b - \frac{1}{n} \delta^{ab} \delta_{cd}. \quad (2.9)
$$

Using (2.9), we differentiate $u_k$ (2.8), and find

$$
\frac{\partial u_k}{\partial \phi^{ab}} = (\phi^{k-1})_{ba} - \frac{1}{n} Tr(\phi^{k-1}) \delta_{ab} = \sum_{1}^{n} (a_i)^{k-1} \frac{\partial a_i}{\partial \phi^{ab}}. \quad (2.10)
$$

Taking the expectation value, and using $\langle \phi^k \rangle = \sum (a_i)^k e_i$ (which follows from $e_i e_j = \delta_{ij} e_j$), we find

$$
\sum_{1}^{n} (a_i)^{k-1} a'_i = \sum_{1}^{n} (a_i)^{k-1} e_i - \frac{1}{n} (\sum_{1}^{n} (a_i)^{k-1} I = \sum_{1}^{n} (a_i)^{k-1} H'_i. \quad (2.11)
$$
This is clearly solved by
\[ \mathbf{a}'_i = H_i^* ; \]  
(2.12)

for consistency, we must check that \( \mathbf{a}'_n = -\sum_{i=1}^{n-1} \mathbf{a}'_i \), and this is indeed the case. Thus the \( \mathbf{a}'_i \) are constant diagonal matrices that span the Cartan subalgebra, and we have found that, just as for \( SU(2) \) [3], the term in the coupling matrix (2.5) proportional to \( \tau_{ij} \) is projected onto the \( U(1)^{n-1} \) subgroup of \( SU(n) \), i.e., onto massless fields.

We now turn to the computation of \( \mathbf{a}'' \). We differentiate (2.10) again, and find:

\[
\left\langle \frac{\partial^2 u_k}{\partial \phi^{ab} \partial \phi^{cd}} \right\rangle = \sum_{l=0}^{k-2} \left\langle (\phi^l)_{bc} (\phi^{k-l-2})_{da} \right\rangle
- \frac{k-1}{n} \left( \left\langle (\phi^{k-2})_{ba} \right\rangle \delta_{cd} + \left\langle (\phi^{k-2})_{dc} \right\rangle \delta_{ab} \right)
+ \frac{k-1}{n^2} \left( Tr(\phi^{k-2}) \right) \delta_{ab} \delta_{cd}
\]

\[
= \sum_1^n (a_i)^{k-1}(a''_{i})_{ab,cd} + (k-1) \sum_1^n (a_i)^{k-2}(a'_{i})_{ab}(a'_{i})_{cd}.
\]

(2.13)

From (2.4) and (2.12), we find

\[
\sum_{l=0}^{k-2} \left( \sum_{i=1}^n (a_i)^l(e_i)_{bc} \sum_{j=1}^n (a_j)^{k-l-2}(e_j)_{da} \right)
- \frac{k-1}{n} \left( \sum_1^n (a_i)^{k-2}(e_i)_{ba} \delta_{cd} + \sum_1^n (a_i)^{k-2}(e_i)_{dc} \delta_{ab} \right)
+ \frac{k-1}{n^2} \sum_1^n (a_i)^{k-2} \delta_{ab} \delta_{cd}
\]

\[
= \sum_1^n (a_i)^{k-1}(a''_{i})_{ab,cd} + (k-1) \sum_1^n (a_i)^{k-2}(H_i^*)_{ab}(H_i^*)_{cd}.
\]

(2.14)
Substituting the explicit form of $e_i$ (2.2) and $H^*_i$ (2.3), this simplifies to
\[ \sum_{l=0}^{k-2} \sum_{i,j=1}^{n} (a_i)^l (a_j)^{k-l-2} \delta_b^i \delta_c^j \delta_d^l = (k-1) \sum_{i,j=1}^{n} (a_i)^{k-2} \delta_b^i \delta_c^j \delta_d^l = \sum_{i=1}^{n} (a_i)^{k-1} (a''_i)_{ab,cd}, \] (2.15)

which can be further simplified to give:
\[ \sum_{l=0}^{k-2} \sum_{i,j=1}^{n} (a_i)^l (a_j)^{k-l} (E^{ij})_{ba} (E^{ji})_{dc} = \sum_{i=1}^{n} (a_i)^{k-1} (a''_i)_{ab,cd}. \] (2.16)

To solve this, we observe that
\[ \sum_{l=0}^{k-2} (a_i)^l (a_j)^{k-l-2} = \frac{(a_i)^{k-1} - (a_j)^{k-1}}{a_i - a_j}, \] (2.17)

which allows us to rewrite (2.16) as
\[ \sum_{i=1}^{n} \sum_{j \neq i=1}^{n} \frac{(a_i)^{k-1} - (a_j)^{k-1}}{a_i - a_j} (E^{ij})_{ba} (E^{ji})_{dc} = \sum_{i=1}^{n} (a_i)^{k-1} (a''_i)_{ab,cd}. \] (2.18)

This is clearly solved by
\[ (a''_i)_{ab,cd} = \sum_{j \neq i}^{n} \frac{(E^{ij})_{ba} (E^{ji})_{dc} + (E^{ij})_{ba} (E^{ji})_{dc}}{a_i - a_j} ; \] (2.19)

as before, we can easily check the consistency condition $a''_i = -\sum_{i=1}^{n-1} a''_i$. Note that the $a''_i$ project onto $SU(n)/U(1)^{n-1}$ (for $i \neq j$, $E^{ij}$ are traceless, and hence are generators of $SU(n)$ outside the Cartan subalgebra), again as in the $SU(2)$ case [3]. Thus our final expression for the coupling constant matrix (2.5) is
\[ \langle \frac{\partial^2 F}{\partial \phi^{ab} \partial \phi^{cd}} \rangle = \sum_{i,j=1}^{n-1} \tau_{ij} (H^*_i)_{ab} (H^*_j)_{cd} + \sum_{i=1}^{n-1} \sum_{j \neq i}^{n-1} \frac{a_{Di} - a_{Dj}}{a_i - a_j} (E^{ij})_{ba} (E^{ji})_{dc} \]
\[ + \sum_{i=1}^{n-1} \frac{a_{Di}}{a_i - a_n} ((E^{in})_{ba} (E^{ni})_{dc} + (E^{ni})_{ba} (E^{in})_{dc}) . \] (2.20)
This is our main result. It is valid whenever the low energy physics is described by a function $\mathcal{F}$, i.e., for the Coulomb phase of an $N = 2$ $SU(n)$ gauge theory with or without matter multiplets. What is the physical consequence of the computation? By construction, $\tau_{ij}$, the coupling that we have found in the Cartan subalgebra, has an imaginary part that is positive everywhere in moduli space. On the other hand, the ratios $(a_{Di} - a_{Dj})/(a_i - a_j)$ and $a_{Di}/(a_i - a_n)$ can have vanishing and even negative imaginary parts. The real codimension 1 surfaces where one of these ratios becomes real split moduli space into regions where the number of physical charged gauge field supermultiplets is different.

We now consider the relation of our results in (2.20) to the BPS mass formula. In our notation, the electric charges $q$ are vectors on the root lattice $q = \sum_{i=1}^{n-1} q_i \alpha_i \equiv \text{diag}(Q)$ where $Q = \sum_{i=1}^{n-1} q_i H_i$, and by the Dirac quantization condition, the magnetic charges $g$ are vectors on the weight lattice $g = \sum_{i=1}^{n-1} g_i \alpha_i^* \equiv \text{diag}(G)$, $G = \sum_{i=1}^{n-1} g_i H_i^*$. Here $q_i, g_i$ are integers, and the Dirac quantization condition reads $\sum q_i g_i = Tr(QG) = \text{integer}$. In particular, the charged gauge bosons $W_{ij}$ have vanishing magnetic charges $g_{k}(W_{ij}) = 0$ and electric charges

$$q_k(W_{in}) = \delta_{ik}, \quad q_k(W_{ij}) = \delta_{ik} - \delta_{jk}, \quad i, j, k = 1...n - 1,$$  

(2.21)

where we define

$$W_{ab} \equiv \sum_{i=1}^{n-1} W_i^{(H_i)_{ab}} + \sum_{i \neq j=1}^{n} W_{ij}^{(E_{ij})_{ab}}.$$  

(2.22)

The eigenvalues $a_{Di}$ defined in (2.8) parametrize the expectation values of the dual field $\phi_D$:

$$\langle \phi_D \rangle = \sum_{i=1}^{n-1} a_{Di} H_i = \sum_{i=1}^{n} a_{Di} e_i, \quad a_{Dn} = - \sum_{i=1}^{n-1} a_{Di}.$$  

(2.23)

The mass $M_{q,g}$ of BPS saturated states is given in terms of the central charge $Z_{q,g}$ by

$$M_{q,g} = \sqrt{2} |Z_{q,g}|,$$

$$Z_{q,g} = Tr(Q \langle \phi \rangle + G \langle \phi_D \rangle) = \sum_{i=1}^{n-1} (q_i (a_i - a_n) + g_i a_{Di}).$$  

(2.24)
Comparing the central charge (2.24) with our generalized gauge coupling matrix (2.20), we see that when the imaginary parts of the coefficients of the charged gauge fields vanish, their mass becomes degenerate with bound states of dyons (at threshold, i.e., with no binding energy). This is an essential consistency check.\footnote{In the presence of massive hypermultiplets, (2.24) receives corrections \cite{4}, which do not affect the charged gauge boson masses.}

3 \textit{SU}(3): The Physics

For pure \(N = 2\ \textit{SU}(3)\) gauge theory, the holomorphic function \(\mathcal{F}(a_1, a_2)\) has been described in detail \cite{5, 7}; we can use this to extract more explicit information about the different regions of moduli space. The terms in the effective action with the gauge field-strength multiplets \(W_\alpha\) are (from (1.6), (2.20)):

\begin{equation}
\frac{1}{4\pi} \text{Im} \left[ \int d^2\theta \frac{1}{2} \tau_{ij} W_\alpha^i W_\alpha^j + \frac{a_{D1}}{a_1 - a_2} W_\alpha^{a12} W_\alpha^{21} + \frac{a_{D1}}{a_1 - a_3} W_\alpha^{a13} W_\alpha^{31} + \frac{a_{D2}}{a_2 - a_3} W_\alpha^{a23} W_\alpha^{32} \right],
\end{equation}

where we have used (2.22) with \(n = 3\). Clearly, depending on the phases of the three ratios in (3.1), one, two or three charged gauge bosons (with their \(\text{CPT}\)-conjugates) may destabilize. The authors of \cite{4, 5, 7} postulate that the massless gauge coupling matrix \(\tau_{ij}\) is defined as the period matrix of the genus 2 hyperelliptic curve

\begin{equation}
y^2 = (x^3 - ux - v)^2 - 1,
\end{equation}

where \(u, v\) are coordinates on the quantum moduli space that correspond to \(u_2, u_3\), respectively, in the semiclassical domain, and where, without loss of generality, we have chosen the dynamically generated scale \(\Lambda = 1\). Then \(a_i\)
and $a_{Di}$ can be calculated as contour integrals:

$$I_C = \frac{1}{2\pi i} \oint_C \frac{x(3x^2 - u)dx}{y(x)},$$  \hspace{1cm} (3.3)$$

where the contour $C$ runs around various homology cycles on the genus 2 surface corresponding to $a_i, a_{Di}$. This turns out to mean that the integral (3.3) is evaluated between various roots of $[y(x)]^2 = 0$. Different choices of contours and cuts give different $Sp(4, \mathbb{Z})$ sections; given an explicit section on which the charged gauge superfields have a local description, we could draw a map of walls in moduli space across which gauge bosons destabilize and disappear. This requires a careful analysis of the contours and cuts, which we leave to the future. However, using the results of [4, 5, 7], we can describe some of the phenomena we should find, as well as a puzzle and a possible resolution.

This quantum moduli space of $SU(3)$ admits a natural $\mathbb{Z}_3 \times \mathbb{Z}_2$ action. When the roots of $y^2 = 0$ fall into three pairs with separations much larger than the scale (in our conventions, 1), one is in a semiclassical or weakly coupled region of the moduli space. When precisely two roots degenerate, then one of three $SU(2)$ subgroups becomes strongly coupled (an "$SU(2)$ vacuum"); there are six ways that this can happen, and they are rotated into each other by $\mathbb{Z}_3 \times \mathbb{Z}_2$. There are also five special points where the whole $SU(3)$ is strongly coupled: At three, two pairs of roots simultaneously degenerate to two distinct points; these are $\mathbb{Z}_2$ invariant and rotate into each other under $\mathbb{Z}_3$ ("$SU(3)$ vacua"). At the remaining two, three roots all degenerate to a single point; these are $\mathbb{Z}_3$ invariant and are interchanged by $\mathbb{Z}_2$ ("$\mathbb{Z}_3$ vacua"). The $SU(2)$ vacua are characterized by the existence of one massless dyon, the $SU(3)$ vacua are characterized by the existence of two mutually local massless dyons, and the $\mathbb{Z}_3$ vacua are characterized by the existence of three mutually nonlocal massless dyons.

Far from the strongly coupled $SU(3)$ region, each $SU(2)$ vacuum should reproduce the results of [4]. That is, we expect to find a curve passing through (or close to) the two paired $SU(2)$ vacua on which a charged gauge field destabilizes, and $\mathbb{Z}_3$ to act by permuting the $SU(2)$ vacua, and correspondingly, the disappearing gauge fields. As one moves toward stronger
SU(3) coupling, this curve sweeps out a cylinder $(S_1 \times \mathbb{C})$; the six cylinders corresponding to the six asymptotic SU(2) vacua must meet in some way in the strongly coupled region.

It is straightforward to see how they meet at the SU(3) vacua: At these vacua, $v = 0$ and $u = 3r^2 \theta^j$, where

$$\theta = e^{\frac{2\pi i}{3}}, \quad r = 2^{-\frac{1}{3}}, \quad \text{(3.4)}$$

and $j = 0, 1, 2$ labels the three different SU(3) vacua. Without loss of generality, we may choose $j = 0$ (the other choices are found simply by a $\mathbb{Z}_3$ rotation). Then the six roots of $y^2 = 0$ are $-2r, -r, -r, r, r, 2r$. All possible integrals (3.3) are real linear combinations of the integrals

$$I_1 = \int_{-2r}^{-r}, \quad I_2 = \int_{-r}^{r}, \quad I_3 = \int_{r}^{2r}. \quad \text{(3.5)}$$

However, a glance at (3.3) shows that $I_1 = -I_3, I_2 = 0$. Thus all the quantities $a_i, a_{Di}$ are relatively real at the SU(3) vacuum for any Sp(4, $\mathbb{Z}$) section, and we can conclude that, at the SU(3) vacua, all three charged bosons simultaneously destabilize. This is consistent with [7], where $a_i, a_{Di}$ are explicitly calculated near an SU(3) vacuum for some choice of Sp(4, $\mathbb{Z}$) section.

Our puzzle arises at the $\mathbb{Z}_3$ vacua. On general principles, if the $\mathbb{Z}_3$ rotates the various charged gauge fields into each other, as it does in the semiclassical regions and along the SU(2) vacua, at a $\mathbb{Z}_3$ invariant point, either zero or three charged gauge fields may destabilize. For a broad class of Sp(4, $\mathbb{Z}$) sections, including those of [5, 7], it appears that at least one charged gauge field destabilizes; however, as we show below, because mutually nonlocal dyons are becoming massless at the $\mathbb{Z}_3$ vacua, all three charged gauge fields cannot simultaneously destabilize. At the $\mathbb{Z}_3$ vacua, $u = 0, v = \pm 1$; without loss of generality, we may take $v = 1$. Then the six roots are $0, 0, 0, 1/r, \theta/r, \theta^2/r$ (recall (3.4)). By comparing to the semiclassical limit, it is clear that for any Sp(4, $\mathbb{Z}$) section on which the charged gauge bosons are local fields, the $a_i$ are integrals from 0 to the root $\theta^{i-1}/r$. Then looking at (3.3), we see that $a_2 = \theta a_1$. However, whenever $a_1$ and $a_2$ are not relatively real, it follows that all three charged bosons destabilize only if $a_{Di} = c(a_i - a_3)$ for some real $c$. 

11
c. The mass formula (2.24) then implies that only mutually local dyons can become massless simultaneously, which does not occur at the $\mathbb{Z}_3$ vacua [6].

A possible resolution of this puzzle seems to be suggested by the work of [5]: They study the vicinity of the $\mathbb{Z}_3$ vacua, and find a modular parameter $\rho$ that survives at the $\mathbb{Z}_3$ vacuum. The gauge couplings of the massless $U(1)$ fields depend on this parameter. This suggests that at the $\mathbb{Z}_3$ vacua, the coordinates $u, v$ are not good coordinates, and each $\mathbb{Z}_3$ vacuum should be blown up into an $S^2$ (with coordinate $\rho$). This would resolve our puzzle: depending on where on the $S^2$ one sits (which $\rho$), different charged gauge fields destabilize. However, [5] do not make this interpretation; they argue that, although blowing up the $\mathbb{Z}_3$ points looks more natural mathematically, it is not what is seen physically: at the singularity, the theory becomes conformally invariant, and the measurable couplings are the $\mathbb{Z}_3$ symmetric ones, not the $\rho$ dependent ones. With this interpretation, it is not clear how to interpret the multiplets of charged gauge bosons near the $\mathbb{Z}_3$ vacua.

We close by noting that a similar analysis could be performed for the Coulomb phase of the $SU(n)$ theory, with and without matter hypermultiplets.

Acknowledgments

We are happy to thank Mike Douglas for an illuminating discussion about the physics of the $\mathbb{Z}_3$ point. This work is supported in part by the BSF (the American-Israel Bi-National Science Foundation). AG thanks the ITP at Stony Brook and MR thanks the Racah Institute for their respective hospitality. The work of AG is supported in part by the BRF (the Basic Research Foundation) and by an Alon Fellowship. The work of MR is supported in part by NSF Grant No. PHY 93 09888.

6Indeed, the coordinate transformation $\delta u, \delta v \to \rho, \epsilon$ of $\mathbb{Z}_3$, $\delta u = 3\rho \epsilon^2, \delta v = 2\epsilon^3$, is precisely such a blowup.

7We thank M. Douglas for a discussion on this interpretation.
References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087.

[2] N. Seiberg and E. Witten, Nucl. Phys. B431 (1994) 484, hep-th/9408099.

[3] U. Lindström and M. Rocek, Phys. Lett. B355 (1995) 492, hep-th/9503012.

[4] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. B344 (1995) 169, hep-th/9411048;
P. Argyres and A. Faraggi, Phys. Rev. Lett. 73 (1995) 3931, hep-th/9411057;
M. Douglas and S. Shenker, Nucl. Phys. B447 (1995) 271, hep-th/9503163.

[5] P. Argyres and M. Douglas, preprint IASSNS-HEP-95/31, RU-95-28, hep-th/9505062.

[6] A. Hanany and Y. Oz, preprint TAUP-2248-95, WIS-95/19/May-PH, hep-th/9505075;
P. Argyres, M. Plesser and A. Shapere, preprint IASSNS-HEP-95/32, UK-HEP/95-06, hep-th/9505100;
J. Minahan and D. Nemeschansky, preprint USC-95/019, CERN-TH.95-167, hep-th/9507032.

[7] A. Klemm, W. Lerche and S. Theisen, preprint CERN-TH/95-104, LMU-TPW 95-7, hep-th/9505150.

[8] U. Danielsson and B. Sundborg, preprint USITP-95-06, UUITP-4/95, hep-th/9504102;
A. Brandhuber and K. Landsteiner, preprint CERN-TH/95-180, hep-th/9507008.
[9] S. J. Gates, Jr., Nucl. Phys. B238 (1984) 349;
    B. de Wit, P. G. Lauwers, R. Philippe, S.-Q. Su and A. Van Proeyen,
    Phys. Lett. 134 (1984) 37;
    M. Günaydin, G. Sierra and P. K. Townsend, Nucl. Phys. B242 (1984)
    244.

[10] M. T. Grisaru and M. Roček, unpublished results.

[11] S. J. Gates, Jr., M. T. Grisaru, M. Roček and W. Siegel, *Superspace*,
    Benjamin/Cummings Pub. Co. (1983).