Low Temperature Properties of the Mermin-Ho Texture of Superfluid $^3$He-A in a Cylinder

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Abstract. A quasi-classical theory of the Mermin-Ho texture in a cylinder is presented. We obtain the self consistent order parameter and calculate the total angular momentum of the system from the mass current distribution. We show that the profile of the bending angle $\beta$ of the $l$ vector significantly changes at low temperatures $T < 0.5T_c$ in contrast to the behavior of the bulk energy gap. Accordingly, the total angular momentum also changes at low temperatures, but it seems to tend to $\frac{1}{2}Nh$ when $T \to 0K$, which agrees with the prediction by McClure and Takagi.

1. Introduction
Existence of a coreless vortex texture in the A phase of superfluid $^3$He was predicted by Mermin and Ho[1] in 1976. Recent rotating cryostat experiments[2] indicate the existence of the Mermin-Ho texture in a cylinder and call a renewed attention to this texture. Theoretical studies of the texture have been limited, however, to the GL temperature range.[1, 3] We consider the low temperature properties of the Mermin-Ho texture in a cylinder using the quasi-classical theory. In this report, we consider a cylinder with specular wall. Using cyclic polygons for the trajectories, we have succeeded in constructing a formal solution for the quasi-classical Green’s function that already satisfies the specular boundary condition. One of the interests is the total angular momentum. We obtain the self consistent order parameter in a cylinder and calculate the angular momentum of the system from the mass current distribution.

2. Quasi-classical Green’s Function
We consider a system of superfluid $^3$He-A in a long cylinder along the $z$ axis with radius $R$. We assume that the system is uniform along the $z$ axis; therefore, we consider the polar coordinates $(r, \phi)$ in the $x-y$ plane. In the Mermin-Ho texture, the $l$ vector bends continuously from axial at the center to radial at the walls with the bending angle $\beta(r)$. The order parameter takes a form

$$\Delta(\hat{p}, r) = \Delta(r) \left( \sin \theta_p (\cos \beta \cos \phi_p + i \sin \phi_p) - \sin \beta \cos \theta_p \right)$$

(1)

where $\theta_p$ is the polar angle of the Fermi momentum $p$ with respect to the $z$-axis and $\phi_p = \phi_p - \phi$ is the relative azimuthal angle of the Fermi momentum measured from the azimuthal angle $\phi$ of $r$. 

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We use the quasi-classical theory to study the properties of the system. The Eilenberger equation for the quasi-classical Green’s function should be solved along the trajectory parallel to the Fermi momentum component in the $x-y$ plane. Instead of directly solving the Eilenberger equation, we introduce an evolution operator which obeys

$$v_F \sin \theta_p \partial_s U_p(s, s') = i \begin{pmatrix} i\omega_n & \Delta(p, r) \\ -\Delta^*(p, r) & -i\omega_n \end{pmatrix} U_p(s, s'), \quad U_p(s, s) = 1 \quad (2)$$

where $s$ is the position along the trajectory (see Fig. 1) and $\omega_n$ is the Matsubara frequency. Then the quasi-classical Green’s function at position $s$ is related to that at position $s'$ by

$$\hat{G}(p, s) = U_p(s, s') \hat{G}(p, s') U_p(s', s) \quad (3)$$

In this report, we consider the specularly reflecting walls. For that boundary condition, it is convenient to use cyclic $m$-polygon for the trajectories. Connecting the vertices by diagonals, we can construct specularly reflecting closed trajectories. In Fig. 2, pentagon is shown for an example. In case of $m$-polygon with odd $m$, we obtain $m - 1$ independent closed trajectories.

Each trajectory visits all the $m$ vertices, when $m$ is an odd number. Let us put numbers from 0 to $m - 1$ on the vertices in order of the visit by the trajectory (see Fig. 2). Since the trajectory is closed, the quasi-classical Green’s function $\hat{G}_0$ at the vertex 0 is given in a form

$$\hat{G}_0(p_{01}) = i \frac{A_0 - \frac{1}{2} \text{Tr} A_0}{\sqrt{\left(\frac{1}{2} \text{Tr} A_0\right)^2 - \det A_0}}, \quad A_0 = U_{0,1} U_{1,2} \cdots U_{m-1,0}, \quad (4)$$

where $p_{01}$ is the Fermi momentum along the diagonal from the vertex 0 to 1, $U_{i,i+1}$ is the evolution operator from the vertex $i$ to $i + 1$. One needs not to solve $U_{i,i+1}$ for all the $i$, because
one can show that \( \exp[-\frac{i}{2}\phi_{p,i+1}\rho_3]U(i,i+1)\exp[\frac{i}{2}\phi_{p,i+1}\rho_3] \) is independent of \(i\), where \( \phi_{p,i+1} \) is the azimuthal angle of the momentum along the diagonal from \(i\) to \(i+1\) and \( \rho_3 \) is a Pauli matrix in the particle-hole space. This comes from the fact that the order parameter of Eq. (1) depends upon the momentum through the relative azimuthal angle \( \tilde{\phi}_p = \phi_p - \phi \). Once \( \tilde{G}_0 \) is known, the quasi-classical Green’s function at the point \( P \) on the diagonal (see Fig. 1) is given using Eq. (3). Moreover, the Green’s function on the diagonal \( \tilde{G}(\theta_p, \phi_p, r, \phi) \) can be converted to that on the \( x \)-axis (\( \phi = 0 \), see Fig. 1) through the relationship

\[
\tilde{G}(\theta_p, \phi_p - \phi, r, \phi = 0) = \exp\left[-\frac{i}{2}\phi_p \rho_3\right]\tilde{G}(\theta_p, \phi_p, r, \phi) \exp\left[\frac{i}{2}\phi_p \rho_3\right],
\]

which can be also proved from the \( \tilde{\phi}_p \) dependence of the order parameter. Choosing the polygon trajectories, we can obtain quasi-classical Green’s function on the \( x \)-axis for various direction of the Fermi momentum.

3. Numerical Results

We solve numerically the system of equations digitizing the radius to 500 points. We use \( m=361 \) polygon for the trajectories, but near the center we use \( M=15001 \) polygon to obtain sufficient number of directions of the Fermi momentum. In this report, we show the results for the cylinder with radius \( R = 30\xi \). Here, \( \xi = v_F/\pi T_c \) is the coherence length. The results for cylinders with larger radius (\( R = 50, 60\xi \)) are almost the same when plotted as functions of \( r/R \).

\[\begin{array}{c}
\text{Figure 3. Self-consistent order parameter at } T = 0.2T_c.
\text{Figure 4. The bending angle } \beta(r) \text{ scaled by } \pi/2. \text{ Temperatures are } 0.2,0.3,0.4,0.5,0.8 T_c \text{ from the top to the bottom.}
\end{array}\]

Let us consider the order parameter. The gap equation for \( \Delta(r) \cos \beta(r) \) is given by

\[
\Delta(r) \cos \beta(r) = 3g_1 N(0) \pi T \sum_{\omega_n > 0} \frac{d\Omega_p}{4\pi} \sin \theta_p \cos \phi_p \Re(\hat{G}^{12} - \hat{G}^{21})
\]

where \( g_1 \) is the \( p \)-wave pairing interaction, \( N(0) \) is the density of states at the Fermi surface and \( \hat{G}^{12}, \hat{G}^{21} \) are the off-diagonal element of the quasi-classical Green’s function \( \hat{G}(\theta_p, \phi_p, r, \phi = 0) \). The gap equation for \( \Delta(r) \) is given by changing \( \cos \phi_p \) and \( \Re(\hat{G}^{12} - \hat{G}^{21}) \) in the right hand side of Eq. (6) into \( \sin \phi_p \) and \( \Im(\hat{G}^{12} + \hat{G}^{21}) \), respectively. We have solved iteratively the set of
gap equations following the prescription given by Nagato et al.\cite{6} and obtained self-consistent \( \Delta(r), \beta(r) \). In Fig. 3, we show the order parameter at \( T = 0.2T_c \). \( \Delta(r) \) is almost constant and is nearly equal to the ABM state bulk gap, but has a hollow at the center. The hollow becomes less prominent at higher temperatures. The bending angle of the \( l \) vector is shown in Fig. 4. At low temperatures, \( \beta \) grows quite rapidly in contrast to the result\cite{3} of GL theory; \( \beta \sim \frac{2}{r R} \).

**Figure 5.** Mass current density scaled by \( n\hbar/\xi \) at T=0.2, 0.3, 0.5 and 0.8 \( T_c \) (solid lines). The current density by Cross’s formula is also plotted by dashed lines.

In the Mermin-Ho texture, mass current flows around the \( z \)-axis. We calculate the mass current density from the diagonal element of the quasi-classical Green’s function:

\[
J_y(r) = -\frac{6n}{\xi} \sum_{\omega_n>0} \frac{T}{T_c} \int \frac{d\Omega_p}{4\pi} \sin \theta_p \sin \phi_p R(\hat{G}^{11}),
\]

where \( n \) is the particle number density. The results are shown in Fig. 5 together with the current density calculated using Cross’s current formula\cite{7} for the ABM state. The present result is systematically smaller than Cross’s result in particular near the center at low temperatures. This is due to the existence of the bound states near the center. Cross’s current was derived using gradient expansion method, therefore the bound states were not taken into account. From the current density, we can calculate the total angular momentum \( L_z \). The results are shown in Fig. 6. \( L_z \) changes still below \( T \sim 0.3T_c \) in contrast to the low temperature behavior of the bulk energy gap, but it seems to tend to \( \frac{1}{2}N\hbar \) when \( T \to 0K \), which is in agreement with McClure and Takagi\cite{8}.

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