Holographic Map for Cosmological Horizons

Chang Liu and David A. Lowe

Department of Physics, Brown University, Providence, RI, 02912, USA

Abstract

We propose a holographic map between Einstein gravity coupled to matter in a de Sitter background and large $N$ quantum mechanics of a system of spins. Holography maps a spin model with a finite dimensional Hilbert space defined on a version of the stretched horizon into bulk gravitational dynamics. The full Hamiltonian of the spin model contains a non-local piece which generates chaotic dynamics, widely conjectured to be a necessary part of quantum gravity, and a local piece which recovers the perturbative spectrum in the bulk.
I. INTRODUCTION

Previous work has argued for a unitary, holographic description of black hole dynamics via certain spin models \[1, 2\] defined on the stretched horizon \[3\] of the black hole. These spin models have the common feature that non-local interactions generate chaotic dynamics, widely conjectured to be an integral part of a full quantum mechanical description of gravity \[4\]. In this paper we argue that a similar approach works for the cosmological horizon in de Sitter spacetime, given that the static patch metric has a similar form to the metric in Schwarzschild coordinates. To this end we will give an explicit prescription to map perturbative bulk fields to a quantum mechanical operator defined in the holographic spin model. This map will then allow us to construct a local Hamiltonian that reproduces the classical energy of a perturbation around de Sitter spacetime. We argue that this local Hamiltonian, together with the non-local long-range interaction necessary to generate chaotic dynamics, can potentially be a viable description of de Sitter quantum gravity.

Before we begin, we will review relevant facts of the de Sitter space-time to establish our convention of notations. We mostly follow the conventions in Ref. \[5\]. Throughout the paper we will restrict our discussion to (1 + 3)-dimensional space-time entirely, although the methodology presented can in principle be applied to higher (or lower) dimensions. Our metric signatures are always mostly positive, i.e. \((- + + \cdots )\).

Static coordinates cover only one triangular region in the Penrose diagram (see Fig. 1)

\[
ds^2 = -(1 - \frac{r^2}{\ell^2}) dt^2 + \frac{dr^2}{1 - \frac{r^2}{\ell^2}} + r^2 d\Omega^2
\]

(1)

where \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2\) is the line element on the unit 2-sphere \(S^2\) and \(\ell\) is the radius of curvature of the de Sitter spacetime.

II. MODE FUNCTIONS IN DE SITTER

A. Static patch

Our goal is to construct a spin model which reproduces the bulk spectrum in a de Sitter background. To proceed we first review some standard results concerning mode functions in static de Sitter. For simplicity, we will treat the massless minimally coupled scalar, and follow the point of view of \[6\] that with a cutoff to exclude the zero mode, the system can be
FIG. 1. Penrose diagram for de Sitter spacetime, where shaded region is covered by the static coordinates. The stretched horizon (solid curve inside the static patch) is defined as a hypersurface at fixed $r$.

quantized around the Bunch-Davies (or Euclidean) vacuum state. Had we been interested in the system including the zero-mode, this quantization preserving de Sitter isometries would be inadmissible [7]. This quantization of the massless minimally coupled scalar around the Bunch-Davies vacuum is closely related to that of perturbative gravitons, as explained in [8, 9]. The results may then be straightforwardly applied to other bulk modes once this case is understood.

We begin by considering modes in the static patch (1). The equation of motion for the free scalar field $\Phi(t, r, \theta, \phi)$ is

$$\frac{\partial^2 \Phi}{1 - r^2/\ell^2} - \frac{\partial_r [r^2(1 - r^2/\ell^2) \partial_r \Phi]}{r^2} - \frac{\partial_\theta (\sin \theta \partial_\theta \Phi)}{r^2 \sin \theta} - \frac{\partial^2 \Phi}{r^2 \sin^2 \theta} = 0.$$  

Separating variables, we have

$$\Phi_{\omega lm}(t, r, \theta, \phi) = A_{\omega l} e^{-i\omega t} f_{\omega l}(r) Y_{lm}(\theta, \phi),$$

where $f_{\omega l}(r)$ satisfies

$$(1 - r^2/\ell^2)f''_{\omega l}(r) + \left(2\frac{1 - 2r^2/\ell^2}{r}f'_{\omega l}(r) + \left(\frac{\omega^2}{1 - r^2/\ell^2} - \frac{l(l + 1)}{r^2}\right)f_{\omega l}(r) = 0.\right.$$  

We pick the set of solutions that are regular at $r = 0$ and find

$$f_{\omega l}(r) = \frac{(r/\ell)^l}{\ell}(1 - r^2/\ell^2)^{i\omega l/2} \text{}_2F_1 \left(\frac{l + i\omega \ell}{2}, \frac{l + i\omega \ell + 3}{2}; l + \frac{3}{2}; \frac{r^2}{\ell^2}\right).$$

We fix the normalization constant $A_{\omega l}$ by computing the Klein-Gordon norm. This is defined on a spacelike surface $\Sigma$ by

$$\langle f, g \rangle = -i \int_{\Sigma} d\Sigma n^\lambda (f \partial_\lambda g^* - g^* \partial_\lambda f),$$
where \( n^\lambda \) is a timelike unit vector normal to \( \Sigma \). Evaluating this on a constant \( t \) slice gives

\[
\langle f, g \rangle = -i \int (f \partial_t g^* - g^* \partial_t f) \frac{r^2 \sin \theta \, d\theta \, d\varphi}{1 - r^2 / \ell^2}.
\]

(2)

Computing the mode normalization then gives

\[
\langle \Phi_{\omega lm}, \Phi_{\omega l'm'} \rangle = A_{\omega l} A_{\omega l'}^*(\omega + \omega') \delta_{ll'} \delta_{mm'} \int_0^\ell \frac{f_{\omega l}(r) f_{\omega l'}^*(r) \, r^2 \, dr}{1 - r^2 / \ell^2}.
\]

Using the equation of motion for \( f_{\omega l}(r) \) we have

\[
[r^2(1 - r^2 / \ell^2) f_{\omega l}'(r)]^* f_{\omega l}'(r) + \left( \frac{\omega^2 r^2}{1 - r^2 / \ell^2} - l(l + 1) \right) f_{\omega l}(r) f_{\omega l}'(r) = 0
\]

and likewise

\[
[r^2(1 - r^2 / \ell^2) f_{\omega l}^*(r)]' f_{\omega l}(r) + \left( \frac{\omega^2 r^2}{1 - r^2 / \ell^2} - l(l + 1) \right) f_{\omega l}^*(r) f_{\omega l}(r) = 0.
\]

Subtracting we have

\[
\frac{(\omega^2 - \omega'^2)r^2}{1 - r^2 / \ell^2} f_{\omega l}(r) f_{\omega l}'(r) = \left[ r^2(1 - r^2 / \ell^2) f_{\omega l}'(r) \right]' f_{\omega l}(r) - \left[ r^2(1 - r^2 / \ell^2) f_{\omega l}'(r) \right]' f_{\omega l}^*(r),
\]

and integrating by parts, we have

\[
\int_0^\ell \frac{(\omega^2 - \omega'^2)r^2}{1 - r^2 / \ell^2} f_{\omega l}(r) f_{\omega l}'(r) = r^2(1 - r^2 / \ell^2) f_{\omega l}'(r) f_{\omega l}(r) - r^2(1 - r^2 / \ell^2) f_{\omega l}'(r) f_{\omega l}'(r) \bigg|_0^\ell.
\]

This gives

\[
\int_0^\ell \frac{r^2 \, dr}{1 - r^2 / \ell^2} f_{\omega l}(r) f_{\omega l}'(r) = \frac{\ell^2}{\omega^2 - \omega'^2} \lim_{r \to \ell} (1 - r^2 / \ell^2) \left[ f_{\omega l}'(r) f_{\omega l}(r) - f_{\omega l}'(r) f_{\omega l}'(r) \right].
\]

Using the hypergeometric identity near \( z = 1 \)

\[
\text{}_2\Gamma_1(a, b, c, z) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} + \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} (1 - z)^{c - a - b}
\]

we can expand \( f_{\omega l}(r) \) near \( r = \ell \) to give

\[
\ell f_{\omega l}(r) \approx \frac{\Gamma(l + \frac{3}{2}) \Gamma(-i\omega l)}{\Gamma(l - i\omega l + \frac{3}{2}) \Gamma(l - i\omega l + 3)} (1 - r^2 / \ell^2)^{\frac{i\omega l}{2}} + \frac{\Gamma(l + \frac{3}{2}) \Gamma(i\omega l)}{\Gamma(l + i\omega l + \frac{3}{2}) \Gamma(l + i\omega l + 3)} (1 - r^2 / \ell^2)^{-\frac{i\omega l}{2}}.
\]

Letting

\[
B_{\omega l} = \frac{\Gamma(l + \frac{3}{2}) \Gamma(i\omega l)}{\Gamma(l + i\omega l + \frac{3}{2}) \Gamma(l + i\omega l + 3)}
\]

we see that we have

\[
\ell f_{\omega l}(r) \approx B_{\omega l} (1 - r^2 / \ell^2)^{\frac{i\omega l}{\ell}} + B_{\omega l} (1 - r^2 / \ell^2)^{-\frac{i\omega l}{\ell}}.
\]
and
\[ f'_{\omega l}(r) \approx \frac{-i\omega}{1 - r^2/\ell^2} \left[ B^*_{\omega l}(1 - r^2/\ell^2)^{\mu/2} - B_{\omega l}(1 - r^2/\ell^2)^{-\mu/2} \right]. \]
Multiplying \( f_{\omega l}(r) \) and \( f'_{\omega l}(r) \) and dropping terms that are rapidly oscillating as \(|\omega - \omega'| > 0\) and \( r \to \ell \), we find that
\[ \int_0^\ell \frac{r^2 \, dr}{1 - r^2/\ell^2} f_{\omega l}(r) f^*_{\omega' l}(r) = \lim_{r \to \ell} 2 |B_{\omega l}|^2 \sin \left[ \frac{(\omega - \omega') \ell}{2} \log \left( \frac{1}{1 - r^2/\ell^2} \right) \right] \tag{3} \]
Using
\[ \lim_{C \to \infty} \frac{\sin C x}{x} = \pi \delta(x) \]
we have
\[ \int_0^\ell \frac{r^2 \, dr}{1 - r^2/\ell^2} f_{\omega l}(r) f^*_{\omega' l}(r) = 2\pi |B_{\omega l}|^2 \delta(\omega - \omega') \]
or
\[ \langle \Phi_{\omega l m}, \Phi_{\omega' l' m'} \rangle = 4\pi |A_{\omega l}|^2 |B_{\omega l}|^2 \delta_{\ell \ell'} \delta_{mm'} \omega \delta(\omega - \omega'). \tag{4} \]
If we normalize according to
\[ \langle \Phi_{\omega l m}, \Phi_{\omega' l' m'} \rangle = \delta_{\ell \ell'} \delta_{mm'} \omega \delta(\omega - \omega'), \]
we need to pick \( A_{\omega l} \) such that
\[ |A_{\omega l}|^2 = \frac{1}{4\pi |B_{\omega l}|^2}. \]

**B. Flat slicing**

Now let us consider the analogous problem for modes in the flat slicing. The metric takes the form
\[ ds^2 = -d\tau^2 + e^{2\tau/\ell} dx^2 = -d\tau^2 + e^{2\tau/\ell} (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2) \]
where \( \tau \in (-\infty, +\infty) \) and \( \rho \in (0, +\infty) \). The wave equation for the massless minimally coupled scalar is given by
\[ \partial^2_{\tau} \phi + \frac{3}{\ell} \partial_{\tau} \phi - e^{-2\tau/\ell} \Delta \phi = 0 \]
where \( \Delta \phi \) is the usual spatial Laplacian operator
\[ \Delta \phi = \partial^2_\rho \phi + \frac{2}{\rho} \partial_\rho \phi + \frac{\partial^2_\theta \phi}{\rho^2} + \frac{\partial_\theta \phi}{\rho \tan \theta} + \frac{\partial^2_\phi \phi}{\rho^2 \sin^2 \theta}. \]
Separating variables, we use the ansatz
\[ \phi_{klm}(\tau, \rho, \theta, \varphi) = T_k(\tau)R_{kl}(\rho)Y_{lm}(\theta, \varphi) \]
where \( T_k(\tau) \) satisfies
\[ T''(\tau) + \frac{3}{\ell} T'(\tau) + k^2 e^{-2\tau/\ell} T(\tau) = 0 \]
and \( R_{kl}(\rho) \) satisfies
\[ R''(\rho) + \frac{2}{\rho} R'(\rho) + \left( k^2 - \frac{l(l+1)}{\rho^2} \right) R(\rho) = 0 \, . \]
Assuming regularity at \( \rho = 0 \) we can solve for \( R(\rho) \)
\[ R_{kl}(\rho) = C_{kl} j_l(k \rho) \]
where \( j_l(x) \) is the spherical Bessel function of first kind \( j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x) \). We will determine the normalization constant \( C_{kl} \) later.

The equation for \( T(\tau) \) can be solved to give
\[ T_k(\tau) = c_1 e^{ikte^{-\tau/\ell}} (1 - ikte^{-\tau/\ell}) + c_2 e^{-ikte^{-\tau/\ell}} (1 + ikte^{-\tau/\ell}) \]
or, in terms of the conformal time \( \eta = -\ell e^{-\tau/\ell} \)
\[ T_k(\tau) = c_1 e^{-ik\eta} (1 + ik\eta) + c_2 e^{ik\eta} (1 - ik\eta) \, . \]
We assume Bunch-Davies vacuum and therefore pick the special solution
\[ T_k(\eta) = e^{-ik\eta} (1 + ik\eta) \]
and absorb the normalization constant into \( C_{kl} \), which we will fix now. The mode functions are normalized according to the Klein-Gordon norm
\[ \langle f, g \rangle = -i \int_{\Sigma} (f \partial_\mu g^* - g^* \partial_\mu f) n^\mu \sqrt{\gamma} \, d^3 x \, . \]
Here \( \Sigma \) is a spacelike hypersurface with unit norm \( n^\mu \) and \( \sqrt{\gamma} \) is the spatial volume element. We pick the \( \tau = 0 \) timeslice in the flat slicing since the metric at \( \tau = 0 \) is conveniently the Minkowski metric. In addition, we have \( \partial_\tau = \partial_\eta \) on the \( \tau = 0 \) timeslice. We therefore have
\[ \langle f, g \rangle = -i \int (f \partial_\tau g^* - g^* \partial_\tau f) \rho^2 \, d\rho \, \sin \theta \, d\theta \, d\varphi \, . \]
FIG. 2. Penrose diagram for de Sitter. Flat slicing modes cover the right upper half of the diagram and are matched to static patch modes on the line $u = 0$. Positive frequency modes in the Bunch-Davies vacuum are then analytic in the lower-half-complex $v$–plane.

We use this to fix the normalization factor $C_{kl}$. We have, for the $\phi_{klm}$ modes on $\tau = 0$

$$\partial_\tau \phi_{klm} = -C_{kl} k^2 \ell e^{ik\ell} j_l(k\rho) Y_{lm}(\theta, \varphi).$$

Therefore

$$\langle \phi_{klm}, \phi_{k'l'm'} \rangle = i\ell C_{kl} C_{k'l'}^{*} \delta_{ll'} \delta_{mm'} \rho^2 \int \left[ (1 - ik\ell) k^2 - (1 + ik'\ell) k'^2 \right] j_l(k\rho) j_l(k'\rho) \rho^2 d\rho.$$

Using the orthogonality of spherical Bessel functions

$$\int_0^\infty \rho^2 j_l(u\rho) j_l(v\rho) d\rho = \frac{\pi}{2u^2} \delta(u - v)$$

we have

$$\langle \phi_{klm}, \phi_{k'l'm'} \rangle = \ell^3 \pi k |C_{kl}|^2 \delta_{ll'} \delta_{mm'} \delta(k\ell - k'\ell).$$

We therefore find

$$C_{kl} = \frac{1}{\sqrt{\pi k\ell}},$$

and the full solution is therefore

$$\phi_{klm}(\tau, \rho, \theta, \varphi) = \frac{1}{\ell \sqrt{\pi k\ell}} e^{-ik\eta} (1 + ik\eta) j_l(k\rho) Y_{lm}(\theta, \varphi).$$

C. Matching modes across the cosmological horizon

Following [10], modes in the flat slicing may be viewed as modes entangled across the left and right static patches. In Kruskal coordinates in the right static patch $u = e^{x^-}/\ell$ and
\[ v = -e^{-x/\ell} \] (other choices of sign generate the other patches in static coordinates) where \( x^\pm = t \pm r^* \) and

\[ r^* = \frac{\ell}{2} \log \frac{1 + r/\ell}{1 - r/\ell} \approx \frac{\ell}{2} \log \frac{2}{1 - r/\ell} \]

we have

\[ \Phi_{\omega lm} \approx A_{\omega l}(B^*_{\omega l}2^{i\omega \ell}|v|^{\omega \ell} + B_{\omega l}2^{-i\omega \ell}|u|^{-i\omega \ell})Y_{lm}(\theta, \varphi) \]

near the cosmological horizon.

We now define a mode function \( \Phi^+_{\omega lm} \) which is non-zero on the right quadrant, and a \( \Phi^-_{\omega lm} \) which is non-zero on the left quadrant. We require that the linear combination

\[ \bar{\Phi}_{\omega lm} = c \Phi^+_{\omega lm} + d \Phi^-_{\omega lm} \]

be analytic in the lower half \( v^- \) plane on the past horizon on the right (and the past horizon on the left), i.e. the surface \( u = 0 \). On the right quadrant we can rewrite the function \((-v)^{i\omega \ell} = (e^{i\pi v})^{i\omega \ell} = e^{-\pi \omega \ell} v^{i\omega \ell} \). Therefore, the linear combination \( \Phi^+_{\omega lm} + e^{-\pi \omega \ell} \Phi^-_{\omega lm} \) is analytic in the lower-half-complex \( v^- \) plane at \( u = 0 \) (see fig. [2]) corresponding to a combination of positive frequency flat-slicing modes. The properly normalized mode is

\[ \bar{\Phi}_{\omega lm} = \frac{1}{\sqrt{2} \sinh(\pi \omega \ell)} (e^{\pi \omega \ell/2} \Phi^+_{\omega lm} + e^{-\pi \omega \ell/2} \Phi^-_{\omega lm}) \tag{5} \]

This mode is analytic in the lower-half \( v^- \) plane for either choice of sign for \( \omega \), so should be identified with a linear combination of positive \( k \) flat-slicing modes.

We can now expand a quantum field operator \( \hat{\Phi} \) in terms of normal modes

\[
\begin{align*}
\hat{\Phi} &= \int_0^\infty dk \sum_{lm} \phi_{klm} a_{klm} + \text{h.c. (flat slicing)} \\
&= \int_0^\infty d\omega \sum_{lm} \Phi^+_{\omega lm} b^+_{\omega lm} + \Phi^-_{\omega lm} b^{-\dagger}_{\omega lm} + \text{h.c. (static patches)} \\
&= \int_{-\infty}^\infty d\omega \sum_{lm} \bar{\Phi}_{\omega lm} c_{\omega lm} + \text{h.c. (entangled static patches)}.
\end{align*}
\]

We define the Bunch-Davies vacuum \( |0 \rangle \) to be annihilated by all \( a_{klm} \) with \( k > 0 \)

\[ a_{klm} |0 \rangle = 0 \]

which coincides with

\[ c_{\omega lm} |0 \rangle = 0 \]
for $-\infty < \omega < \infty$. We similarly define the static patch vacuum $|\Omega\rangle$ to be annihilated by all $b_{\omega lm}^\pm$ with $\omega > 0$

$$b_{\omega lm}^\pm |\Omega\rangle = 0.$$  

From the relation we obtained between $\Phi^\pm$ and $\bar{\Phi}$ we obtain the relations between $b_{\omega lm}^\pm$ and $c_{\omega lm}$

$$c_{\omega lm} = \begin{cases} \frac{1}{\sqrt{2 \sinh \pi \omega \ell}} \left(e^{\pi \omega \ell/2} b_{\omega lm}^+ + e^{-\pi \omega \ell/2} b_{\omega lm}^-\right) & \omega > 0 \\ \frac{1}{\sqrt{2 \sinh \pi \omega \ell}} \left(e^{-\pi \omega \ell/2} b_{\omega lm}^+ + e^{\pi \omega \ell/2} b_{\omega lm}^-\right) & \omega < 0. \end{cases}$$

As usual, the vacuum state $|0\rangle$ becomes a thermal density matrix in the right static patch when the modes $b_{\omega lm}^-$ are traced over.

We now need to compute the overlap of modes $\Phi_{\omega lm}$ with $\phi_{\omega lm}$ to construct the Bogoliubov transformation. In order to perform the integrals, we need the coordinate transformation between $(t, r)$ and $(\eta, \rho)$. We define null coordinates in the flat slicing

$$U = \frac{\eta - \rho}{2}, \quad V = \frac{\eta + \rho}{2}$$

leading to the metric. In the flat slicing using the $(\eta, \rho)$ coordinates the metric is

$$\text{d}s^2 = \frac{\ell^2}{\eta^2} (-\text{d}\eta^2 + \text{d}\rho^2 + \rho^2 \text{d}\Omega^2)$$

which in terms of $(U, V)$ coordinates becomes

$$\text{d}s^2 = \frac{\ell^2}{(U + V)^2} (-4 \text{d}U \text{d}V + (V - U)^2 \text{d}\Omega^2).$$

In the static patch, we define null coordinates $(u, v)$

$$u = e^{x^-/\ell}, \quad v = -e^{-x^+/\ell}$$

and one verifies that the metric is

$$\text{d}s^2 = \frac{\ell^2}{(1 - uv)^2} (-4 \text{d}u \text{d}v + (1 + uv)^2 \text{d}\Omega^2).$$

We see that the relation between $(u, v)$ and $(U, V)$ is simply

$$U = -\frac{\ell}{u}, \quad V = \ell v$$

which gives

$$\eta = -\frac{\ell}{u} + \ell v, \quad \rho = \frac{\ell}{u} + \ell v.$$
The flat slicing mode function therefore becomes
\[
\phi_{klm} = \frac{1}{\ell} \frac{1}{\sqrt{\pi k\ell}} e^{ik\ell (\frac{1}{u} - v)} \left[ 1 - i k\ell \left( \frac{1}{u} - v \right) \right] \frac{\sin(k\ell(v + 1/u) - l\pi/2)}{k\ell(v + 1/u)} Y_{lm}(\theta, \varphi)
\]
where we have used the behavior of spherical Bessel function at infinity
\[
\lim_{\rho \to \infty} j_l(k\rho) = \sin(k\rho - l\pi/2)
\]

Near the past horizon on the left patch, \( u \to 0 \) with \( v \) fixed, the flat slice mode becomes
\[
\phi_{klm} = \frac{1}{\ell} \frac{1}{2\sqrt{\pi k\ell}} e^{ik\ell (\frac{1}{u} - v)} \frac{\sin(k\ell(v + 1/u) - l\pi/2)}{k\ell v} Y_{lm}(\theta, \varphi).
\]

Using the identity \( \sin z = (e^{iz} - e^{-iz})/2i \) we can rewrite the flat slice mode function as
\[
\phi_{klm} = \frac{1}{\ell} \frac{1}{2\sqrt{\pi k\ell}} (i^l e^{-2ik\ell v} - i^{-l} e^{2ik\ell v}) Y_{lm}(\theta, \varphi) \sim \frac{1}{\ell} \frac{i^l}{2\sqrt{\pi k\ell}} e^{-2ik\ell v} Y_{lm}(\theta, \varphi).
\]

We have shown above that near the past horizon the static patch mode is
\[
\Phi_{\omega lm} = A_{\omega l} B_{\omega l}^* (-v)^{\omega l} 2^{\omega l} Y_{lm}(\theta, \varphi).
\]

On the past horizon the Klein-Gordon norm is
\[
\langle f, g \rangle = -i\ell^2 \int d\Omega dv \left( f \partial_v g^* - g^* \partial_v f \right) = i\ell^2 \int_{-\infty}^{0} d\Omega \left( 2 \int [g^* \partial_v f - g^* f] \bigg|_0^{-\infty} \right).
\]

Since we have
\[
\partial_v \Phi_{\omega lm} = i\omegalv^{-1} \Phi_{\omega lm}
\]
adding a small imaginary part to \( \omega \) and \( k \) to dampen the oscillation of \((-v)^{\omega l}\) and \(e^{-ikv}\) we have, since the boundary term vanishes,
\[
\langle \Phi_{\omega lm}, \phi_{klm} \rangle = \frac{(-i)^l}{\sqrt{\pi k\ell}} A_{\omega l} B_{\omega l}^* 2^{\omega l} \int_0^{+\infty} e^{-2ik\ell v} v^{\omega l-1} dv
\]
where we replaced \( v \to -v \). This can be evaluated to give
\[
\langle \Phi_{\omega lm}, \phi_{klm} \rangle = \frac{(-i)^l}{\sqrt{\pi k\ell}} A_{\omega l} B_{\omega l}^* (ik\ell)^{-i\omega l} \Gamma(i\omega\ell).
\]

By choosing the phase of \( A_{\omega l} \) appropriately such that
\[
A_{\omega l} B_{\omega l}^* = \frac{1}{\sqrt{4\pi}}
\]
we finally have
\[
\langle \Phi_{\omega lm}, \phi_{klm} \rangle = \frac{(-i)^l (ik\ell)^{-i\omega l} \Gamma(i\omega\ell)}{2\pi}.
\]

With the Bogoliubov transformation at hand, we can now map modes in the flat slicing to entangled modes in the left and right static patch. These in turn define modes in the upper quadrant of the static slicing by continuation.
III. HOLOGRAPHIC MAP

Our goal is to build a holographic version of the bulk theory that might be viewed as living on the so-called “stretched horizon” [3]. The essence of the black hole membrane paradigm is that to an external observer outside the horizon, the black hole horizon behaves more or less like a hydrodynamic membrane with properties such as resistance and viscosity. Quantum mechanically, the stretched horizon acts as a mirror [11] which scrambles and reflects information sent into it. Our viewpoint in this paper is that since the static patch of de-Sitter has essentially the same mathematical form as the Schwarzschild metric, one ought to be able to construct a similar stretched horizon theory for static de-Sitter. We further assume that the horizon entropy of de Sitter is to be matched with the logarithm of the Hilbert space dimension. Thus, the stretched horizon theory will be a finite dimensional quantum mechanical system.

Usually the stretched horizon is defined as the constant \( r \) surface such that the local temperature measured by a fiducial observer (constant \( r \)) is the Planck temperature. When redshifted down to \( r = 0 \), this will match the Hawking temperature. In other words, one usually defines \( r_* \) such that

\[
\frac{1}{2\pi\ell_p}\sqrt{1-r_*^2/\ell^2} = T_H = \frac{1}{2\pi\ell},
\]

where \( \ell_p \) is the Planck length.

As it stands, the bulk Hilbert space is infinite dimensional, labelled by the oscillators \( c_{\omega lm} \) where \( \omega \in \mathbb{R} \), and the angular momentum ranges up to infinity. Each oscillator mode creates a mode entangled across both static patches, with a stress-energy tensor non-singular on the future cosmological horizon on the right patch.

As a first step, we can discretize the sphere in coordinate space. There are many possibilities for such a discretization, and the details will not be too important for us, except to note that such a discretization will produce an effective cutoff \( l_{\text{max}} \) on the angular momentum.

Likewise, it is necessary to discretize the frequency \( \omega \) which can in turn be viewed as a radial quantum number. This discretization may then be viewed as a kind of regulator for the radial coordinate. In order to produce a useful effective field theory with such a cutoff we choose a finite set of frequencies in the range

\[
\frac{1}{\ell_p n_{\text{UV}}} > |\omega| \geq \frac{\pi}{\ell \log (\ell/\ell_p)}.
\]
For simplicity we can take the $\omega'$s to be evenly spaced in this range, with spacing $\frac{\pi}{\ell} \log \left( \frac{\ell}{\ell_p} \right)$. This corresponds to $n_{\text{rad}} = 2\ell \log(\ell/\ell_p)/\pi \ell_p n_{\text{UV}}$ radial points in the static patch. In particular, this number is conserved with time. Here $n_{\text{UV}} > 1$ is a factor introduced to parameterize the ultraviolet cutoff. We will see momentarily why the log factor appears.

An issue we immediately face is regulating the continuum mode normalization (4) to the discrete case. To do this we replace the upper limit on the radial integral in (2) by $\ell \rightarrow \ell - \epsilon$. Then the result of the integral (3) may be replaced by

$$\int_{0}^{\ell-\epsilon} \frac{r^2 dr}{1 - r^2/\ell^2} f_{\omega l}(r) f_{\omega' l}^*(r) = \frac{2|B_{\omega l}|^2}{\omega - \omega'} \sin \left[ \frac{(\omega - \omega') \ell}{2} \log \left( \frac{1}{1 - r^2/\ell^2} \right) \right] \bigg|_{r=\ell-\epsilon}.$$

Keeping in mind $\omega - \omega' = \pi n / \ell \log (\ell/\ell_p)$ for some integer $n$ we choose

$$\log \left( \frac{1}{1 - r^2/\ell^2} \right) \bigg|_{r=\ell-\epsilon} = 2 \log (\ell/\ell_p) \quad (8)$$

which fixes $r_*$ according to (6).

$$\int_{0}^{\ell-\epsilon} \frac{r^2 dr}{1 - r^2/\ell^2} f_{\omega l}(r) f_{\omega' l}^*(r) = 2 \log \left( \frac{\ell}{\ell_p} \right) |B_{\omega l}|^2 \delta_{\omega,\omega'}$$

up to rapidly oscillating terms. This unusual relation between a short distance cutoff and an infrared cutoff is typical in holographic models.

Finally, each harmonic oscillator mode $c_{\omega l m}$ produces an infinite dimensional Hilbert space. To regulate these Hilbert subspaces, we use the Holstein-Primakoff map [12] to replace $c_{\omega l m}$ by spin operators, introducing the parameter $s_{\text{max}} \gg 1$

$$s_+ = \sqrt{2s_{\text{max}}} \sqrt{1 - \frac{c_{\omega l m}^\dagger c_{\omega l m}}{2s_{\text{max}}}} c_{\omega l m}, \quad s_- = \sqrt{2s_{\text{max}}} \sqrt{1 - \frac{c_{\omega l m}^\dagger c_{\omega l m}}{2s_{\text{max}}}} c_{\omega l m}, \quad s_z = s - c_{\omega l m}^\dagger c_{\omega l m}.$$

For states near the ground state, we can approximate $\sqrt{1 - \frac{c_{\omega l m}^\dagger c_{\omega l m}}{2s_{\text{max}}}}$ by 1.

This regularization of the Hilbert space then allows us to write the energy in the scalar field at quadratic order as a spin model

$$H_0 = \sum_{\omega} \sum_{l=0}^{l_{\text{max}}} \sum_{m=-l}^{l} \omega \left( c_{\omega l m}^\dagger c_{\omega l m} + c_{\omega l m} c_{\omega l m}^\dagger \right).$$

The dimension of the Hilbert space, for large $l_{\text{max}}$ is $(2s_{\text{max}} + 1)^2 l_{\text{rad}}^2 = e^{S_{\text{BH}}}$, identified with the Bekenstein-Hawking entropy of the cosmological horizon $S_{\text{BH}} = \pi \ell^2/\ell_p^2 \equiv N$. If we follow the arguments of [13], we identify

$$l_{\text{rad}}^2 n_{\text{rad}} \log s_{\text{max}} \sim N$$
and a natural choice would be to scale $n_{\text{rad}} \sim l_{\text{max}} \sim N^{1/3}$, dropping subleading log factors for simplicity in a large $N$ limit. This leads to a short distance cutoff length of order $\ell_p N^{1/6}$ in all directions (and a choice $n_{\text{UV}} \sim N^{1/6}$). We note if our present universe was replaced by a pure de Sitter region with the same Hubble parameter, we would find $N \approx 10^{120}$ and $\ell_p N^{1/6}$ would correspond to a $\text{GeV}$ UV cutoff.

So far, we have simply regulated the scalar field theory at the level of free field theory and found a holographic dual that reproduces that. The holographic dual can be viewed as living on an $S^2$ with the discrete parameter $\omega$ labelling different variables at each point on the sphere. This construction is guaranteed to reproduce the bulk correlators of free scalar field theory with this particular regulator.

The ground state of the Hamiltonian corresponds to the Bunch-Davies vacuum state, and the Hamiltonian is diagonal in modes that are entangled between the left and right “patches”. Tracing over one set leads to an approximately thermal density matrix in the other, subject to the regulator on mode number imposed by finite $s_{\text{max}}$. The excitations of this model will lead to stress-energy tensors regular on the cosmological horizon, avoiding the firewall conundrum.

In general, we also expect to have to add perturbative interactions to this model, which will typically be suppressed by powers of $N$ relative to the quadratic term. One might hope to follow a construction paralleling HKLL [14, 15] to reproduce perturbative field theory in the bulk.

Such a theory might be satisfactory for de Sitter spacetime. Once the initial state corresponding to Bunch-Davies is specified on the past horizon of the right static patch (and its continuation onto the left static patch) it evolves according to the standard rules of quantum mechanics. The future cosmological horizon would essentially behave like a remnant, becoming entangled with the degrees of freedom in the left patch. A priori this poses no issues for the information problem, because the cosmological horizon in de Sitter is eternal.

Motivated by the physics of black hole horizons, it is interesting to explore what happens when this model is supplemented by an additional nonlocal term as studied in [1, 2, 16] which is thought to generate chaotic dynamics over sufficiently long timescales. In the black hole case, the timescale associated with quantum scrambling is linked to the timescale the horizon can retain quantum information, before emitting it to the region outside the black hole. In the de Sitter case, we view the static patch as analogous to the black hole interior.
and are mostly interested in developing the holographic map on timescales shorter than this scrambling time. We may then study the decoherence of local observables built using the holographic map described above, when supplemented by chaotic interactions.

The full Hamiltonian includes a non-local piece and a local piece, where the non-local piece is given by

$$H_{nl} = \sum_{ijkl} J_{ijkl} s_i s_j s_k s_l$$

(9)

Here the coupling $J_{ijkl}$ is drawn randomly from a Gaussian distribution with zero mean (tensor indices are suppressed). We do not have in mind averaging over this coupling, but rather work with a fixed set of $J_{ijkl}$ as needed to generate chaotic dynamics. We impose the condition that the variance of the non-local Hamiltonian $\text{var}(H_{nl}) = 1$. This forces the width of the Gaussians to scale like $1/N^2$, due to the following analysis

$$1 = \langle H_{nl}^2 \rangle \sim J^2 \left\langle \sum_{i_1 \cdots i_8} s_{i_1} \cdots s_{i_8} \right\rangle \sim J^2 N^4$$

(10)

where in the last step we have used the fact that on average $\langle s_i s_j \rangle = \delta_{ij}$. We note this unusual scaling with $N$ is designed to reproduce the Bekenstein-Hawking entropy via microstate counting for fixed $N$ as opposed to the more conventional large $N$ limit where $\langle H_{nl}^2 \rangle \sim N$, which would widen the spectrum to much larger energies.

Our proposal for the full Hamiltonian is then

$$H = H_0 + T_H H_{nl}$$

(11)

and the chaotic term may then be treated as a small perturbation for short enough time intervals, where it will shift energies at leading order by terms of order $T_H$.

One may then study how local perturbations of the thermal state decohere when this term is included. Following the analysis of [16] we expect the timescale of such decoherence to be

$$t_{dec} = \beta \log N$$

(12)

This resembles the scrambling time, however the interpretation here is somewhat different. With the scaling (10) the global scrambling time is expected to be

$$t_{scr} = \beta N^{1/2} \log N \gg t_{dec}$$

if the bounds derived in [17] happen to be saturated. However, the local decoherence time is the quantity of most relevance in deciding when the holographic map derived above breaks
down. A similar breakdown of the bulk description via effective field theory in a black hole interior was noted in [1, 2, 18].

Given that a local operator will evolve to a highly non-local operator in the time (12), rather than simply undergoing the free propagation governed by the term $H_0$, our holographic map based on the mode functions (5) will break down after this timescale. In the case of applying this to a pure de Sitter region with $\ell$ matched to our present cosmological horizon, this would imply a breakdown in the local laws of physics after a timescale of order 4000 billion years due to quantum gravity effects. It would be very interesting to devise experiments sensitive to this local decoherence. While the shifts in energy levels are tiny, of order $10^{-33} eV$ the nonlocal character of the decoherence opens the door to more sensitive experiments.

One might wonder whether such a holographic description is ruled out for primordial inflation. In that case, one can try to embed “small” de Sitter models into a much larger Hilbert space, which is needed to describe the late-time phase of cosmology. Holographic bounds with these considerations in mind were considered in [19, 20]. The decoherence times in this case can be made much longer than the timescale associated with primordial inflation.

It should also be noted that once a local basis of operators has decohered, for example in Heisenberg picture

$$c_{\omega lm}(t) = e^{iHt}c_{\omega lm}(0)e^{-iHt}$$

with $t > t_{dec}$ one may simply do a change of basis by the unitary transformation $e^{iH_{dec}t}$ to return to another local basis

$$\tilde{c}_{\omega lm}(t) = e^{-iH_{dec}t}c_{\omega lm}(t)e^{iH_{dec}t}$$

therefore, in some basis, one always retains an approximately local description of spacetime physics. This realizes the proposal of [18] in a concrete model, when adapted to de Sitter spacetime.

IV. CONCLUSIONS

Now that we have a detailed proposal for the stretched horizon theory of the de Sitter cosmological horizon, we can try to adapt the method to black holes. A key step in the
development of the holographic map was the assumption of regularity of the modes on the pole of the static patch. This eliminated the non-normalizable modes and allowed us to make a one-one map from frequency/radial quantum number space to mode functions (5). For black holes in asymptotically flat space, one would need to perform a similar restriction, which might be accomplished by placing a mirror around the black hole to prevent evaporation. In practice, as we have learned over the years, the best substitute for this procedure is simply to introduce a negative cosmological constant which has the same effect and can be handled much more precisely. Thus, we expect the present considerations will apply largely unchanged to a large black hole in anti-de Sitter spacetime which does not evaporate. In this way, we can use the present construction to derive a holographic map for the interior of such a black hole. One might then hope to derive the spin model directly from the conformal field theory description available in that case. Note here we have in mind realizing the black hole in a single conformal field theory representing, perhaps, a large black hole formed by collapse, rather than the tensor product conformal field theories describing wormholes.

Turning this argument around, we then expect the much more interesting case of the evaporating black hole in asymptotically flat space, or a small black hole in asymptotically anti de Sitter space will involve important extra ingredients. The coupling between this stretched horizon theory and some larger holographic theory describing the asymptotic region will need to be specified. Nevertheless, for timescales shorter than $t_{\text{dec}}$ we expect to be able to apply the considerations of the present paper, which is sufficient to extend the holographic map to black hole interiors.

In the case of anti-de Sitter/conformal field theory duality, it is often suggested one has control of the holographic map all the way to the stretched horizon. In that case one has a fixed local basis extending from asymptotic infinity down to the stretched horizon. The present picture implies the coupling between the exterior and the stretched horizon eventually become highly nonlocal, contaminating the exterior physics with non-local effects. Indeed, nonlocal interactions akin to (9) must emerge from the correspondence in a smooth way as one approaches the stretched horizon. This has the profound consequence that nonlocal scrambling effects might be detected outside large black holes, if sufficiently long timescales can be probed to overcome the $T_H$ suppression factor in (11). Indeed, such effects are probed in current gravitational wave experiments [21]. For example, for black
hole mergers with masses of order a solar mass, one must probe around 100 light crossing times to access the timescale [12]. As these experiments become more precise it will be very interesting to look for signs of violations of the equivalence principle. For example, one might look for anomalies in the late time ringing profile following black hole merger.

Finally, we end with a comment on an interesting numerological coincidence of this holographic model. We noted above, that if we replace our present cosmology with a de Sitter horizon with size around 14 billion light years, an unacceptably small ultraviolet cutoff emerges on bulk effectively field theory of about 1 GeV. This may simply be a signal that a more precise holographic model would produce a bulk cutoff in a much more subtle way. However, for now, let us instead explore the possibility that the current observable entropy $S \approx 10^{88}$ which arises largely from cosmic microwave background photons, might be equated with a late-time de Sitter entropy. Interestingly, this predicts the cosmic acceleration must increase versus the previous possibility, a feature also noted in the Hubble tension experiments, and the ultraviolet cutoff that emerges is the more experimentally interesting value of 100 TeV. This raises the possibility that holographic physics might appear in collider experiments at experimentally accessible scales. Unfortunately, the model also predicts the horizon size must shrink to of order $10^4$ m to reach the late time de Sitter phase, so we are presently far off from the phase, and it is not clear how much to trust the ultraviolet cutoff result. Nevertheless, the model was designed so a freely falling observer will use a cutoff with fixed proper spatial resolution, so there is reason to be optimistic.

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