Balanced Policy Evaluation and Learning

Nathan Kallus
School of Operations Research and Information Engineering and Cornell Tech, Cornell University
New York, New York 10011
kallus@cornell.edu

Abstract

We present a new approach to the problems of evaluating and learning personalized decision policies from observational data of past contexts, decisions, and outcomes. Only the outcome of the enacted decision is available and the historical policy is unknown. These problems arise in personalized medicine using electronic health records and in internet advertising. Existing approaches use inverse propensity weighting (or, doubly robust versions) to make historical outcome (or, residual) data look like it were generated by a new policy being evaluated or learned. But this relies on a plug-in approach that rejects data points with a decision that disagrees with the new policy, leading to high variance estimates and ineffective learning. We propose a new, balance-based approach that too makes the data look like the new policy but does so directly by finding weights that optimize for balance between the weighted data and the target policy in the given, finite sample, which is equivalent to minimizing worst-case or posterior conditional mean square error. Our policy learner proceeds as a two-level optimization problem over policies and weights. We demonstrate that this approach markedly outperforms existing ones both in evaluation and learning, which is unsurprising given the wider support of balance-based weights. We establish extensive theoretical consistency guarantees and regret bounds that support this empirical success.
1 Introduction

Using observational data with partially observed outcomes to develop new and effective personalized decision policies has received increased attention recently [1, 5, 6, 11, 21, 23, 34–36, 39]. The aim is to transform electronic health records to personalized treatment regimes, transactional records to personalized pricing strategies, and click- and “like”-streams to personalized advertising campaigns – problems of great practical significance. Many of the existing methods rely on a reduction to weighted classification via a rejection and importance sampling technique related to inverse propensity weighting and to doubly robust estimation. However, inherent in this reduction are several shortcomings that lead to reduced personalization efficacy: it involves a naive plug-in estimation of a denominator nuisance parameter leading either to high variance or scarcely-motivated stopgaps; it necessarily rejects a significant amount of observations leading to smaller datasets in effect; and it proceeds in a two-stage approach that is unnatural to the single learning task.

In this paper, we attempt to ameliorate these by using a new approach that directly optimizes for the balance that is achieved only on average or asymptotically by the rejection and importance sampling approach. We demonstrate that this new approach provides improved performance and explain why. And, we provide extensive theory to characterize the behavior of the new methods. The proofs are given in the appendix.

1.1 Setting, Notation, and Problem Description

The problem we consider is how to choose the best of \( m \) treatments based on an observation of covariates \( x \in \mathcal{X} \subseteq \mathbb{R}^d \) (also known as a context). An instance is characterized by the random variables \( X \in \mathcal{X} \) and \( Y(1), \ldots, Y(m) \in \mathbb{R} \), where \( X \) denotes the covariates and \( Y(t) \) for \( t \in [m] = \{1, \ldots, m\} \) is the outcome that would be derived from applying treatment \( t \).

We always assume that smaller outcomes are preferable, i.e., \( Y(t) \) corresponds to costs or negative rewards.

A policy is a map \( \pi : \mathcal{X} \to \Delta^m \) from observations of covariates to a probability vector in the \( m \)-simplex \( \Delta^m = \{ p \in \mathbb{R}_+^m : \sum_{t=1}^m p_t = 1 \} \). Given an observation of covariates \( x \), the policy \( \pi \) specifies that treatment \( t \) should be applied with probability \( \pi_t(x) \). There are two
key tasks of interest: policy evaluation and policy learning. In policy evaluation, we wish to evaluate the performance of a given policy based on historical data. This is also known as off-policy evaluation, highlighting the fact that the historical data was not necessarily generated by the policy in question. In policy learning, we wish to determine a policy that has good performance.

We consider doing both tasks based on data consisting of \( n \) passive, historical observations of covariate, treatment, and outcome: \( S_n = \{ (X_1, T_1, Y_1), \ldots, (X_n, T_n, Y_n) \} \), where the observed outcome \( Y_i = Y_i(T_i) \) corresponds only to the treatment \( T_i \) historically applied. We use the notation \( X_{1:n} \) to denote the data tuple \((X_1, \ldots, X_n)\). The data is assumed to be iid. That is, the data is generated by drawing from a stationary population of instances \((X, T, Y(1)), \ldots, Y(m))\) and observing a censored form of this draw given by \((X, T, Y(T))\).

From the (unknown) joint distribution of \((X, T)\) in the population, we define the (unknown) propensity function \( \varphi_t(x) = P(T = t \mid X = x) = \mathbb{E}[\delta_{T=t} \mid X = x] \), where \( \delta_{st} = \mathbb{I}[s = t] \) is the Kronecker delta. And, from the (unknown) joint distribution of \((X, Y(t))\) in the population, we define the (unknown) mean-outcome function \( \mu_t(x) = \mathbb{E}[Y(t) \mid X = x] \). We use the notation \( \varphi(x) = (\varphi_1(x), \ldots, \varphi_m(x)) \) and \( \mu(x) = (\mu_1(x), \ldots, \mu_m(x)) \).

Apart from being iid, we also assume the data satisfies unconfoundedness:

**Assumption 1.** For each \( t \in [m] \): \( Y(t) \) is independent of \( T \) given \( X \), i.e., \( Y(t) \perp T \mid X \).

This assumption is equivalent to there being a logging policy \( \varphi \) that generated the data by prescribing treatment \( T_i = t \) with probability \( \varphi_t(X_i) \) to each instance \( i \) and recording the outcome \( Y_i = Y_i(T_i) \). Therefore, especially in the case where the logging policy \( \varphi_t \) is in fact known to the user, the problem is often called learning from logged bandit feedback [34, 35].

In policy evaluation, given a policy \( \pi \), we wish to estimate its sample-average policy effect (SAPE),

\[
\text{SAPE}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{m} \pi_t(X_i) \mu_t(X_i),
\]

by an estimator \( \hat{\tau}(\pi) = \hat{\tau}(\pi; X_{1:n}, T_{1:n}, Y_{1:n}) \) that depends only on the observed data and the policy \( \pi \). The SAPE quantifies the average outcome that a policy \( \pi \) would induce in the

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1Thus, although the data is iid, the \( t \)-treated sample \( \{ i : T_i = t \} \) may differ systematically from the \( t' \)-treated sample \( \{ i : T_i = t' \} \) for \( t \neq t' \), i.e., not necessarily just by chance as in a randomized controlled trial (RCT).
sample and hence measures its risk. SAPE is strongly consistent for the population-average policy effect (PAPE):

\[
\text{PAPE}(\pi) = \mathbb{E}[\text{SAPE}(\pi)] = \mathbb{E}\left[\sum_{t=1}^{m} \pi_t(X) \mu_t(X)\right] = \mathbb{E}\left[Y(\bar{T}_\pi(X))\right],
\]

where \(\bar{T}_{\pi(x)}\) is defined as \(\pi\)’s random draw of treatment when \(X = x\), \(\bar{T}_{\pi(x)} \sim \text{Multinomial}(\pi(x))\). Moreover, if \(\pi^*\) is such that \(\pi^*_t(x) > 0 \iff t \in \arg\min_{s\in[m]} \mu_s(x)\), then \(\bar{R}(\pi) = \text{SAPE}(\pi) - \text{SAPE}(\pi^*)\) is the regret of \(\pi\) [8]. The policy evaluation task is closely related to causal effect estimation [17] where, for \(m = 2\), one is interested in estimating the sample and population average treatment effects:

\[
\text{SATE} = \frac{1}{n} \sum_{i=1}^{n} (\mu_2(X_i) - \mu_1(X_i)),
\]

\[
\text{PATE} = \mathbb{E}[\text{SATE}] = \mathbb{E}[Y(2) - Y(1)].
\]

In policy learning, we wish to find a policy \(\hat{\pi}\) that achieves small outcomes, i.e., small SAPE and PAPE. The optimal policy \(\pi^*\) minimizes both \(\text{SAPE}(\pi)\) and \(\text{PAPE}(\pi)\) over all functions \(X \rightarrow \Delta^m\).

### 1.2 Existing Approaches and Related Work

The so-called “direct” approach fits regression estimates \(\hat{\mu}_t\) of \(\mu_t\) on each dataset \(\{(X_i, Y_i) : T_i = t\}, t \in [m]\). Given these estimates, it estimates SAPE in a plug-in fashion:

\[
\hat{\tau}_{\text{direct}}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{m} \pi_t(X_i) \hat{\mu}_t(X_i).
\]

A policy is learned either by \(\hat{\pi}_{\text{direct}}(x) = \arg\min_{t\in[m]} \hat{\mu}_t(x)\) or by minimizing \(\hat{\tau}_{\text{direct}}(\pi)\) over \(\pi \in \Pi\) [26]. However, direct approaches may not generalize as well as weighting-based approaches [5].

Weighting-based approaches seek weights based on covariate and treatment data \(W(\pi) = W(\pi; X_{1:n}, T_{1:n})\) that make the outcome data, when reweighted, look as though it were generated by the policy being evaluated or learned, giving rise to estimators that have the form

\[
\hat{\tau}_W = \frac{1}{n} \sum_{i=1}^{n} W_i Y_i.
\]

Bottou et al. [6], e.g., propose to use inverse propensity weighting (IPW). Noting that \([15, 16] \text{SAPE}(\pi) = \mathbb{E}[\frac{1}{n} \sum_{i=1}^{n} Y_i \times \pi_{T_i}(X_i)/\varphi_{T_i}(X_i) \mid X_{1:n}]\), one first fits a probabilistic clas-
classification model $\hat{\varphi}$ to $\{(X_i, T_i): i \in [n]\}$ and then estimates SAPE in an alternate but also plug-in fashion:

$$\hat{\tau}_{\text{IPW}}(\pi) = \hat{\tau}_{W_{\text{IPW}}(\pi)}, \quad W_i^{\text{IPW}}(\pi) = \pi_T(X_i)/\hat{\varphi}_T(X_i)$$

For a deterministic policy, $\pi(x) \in \{0, 1\}^m$, this can be interpreted as a rejection and importance sampling approach [23, 34]: reject samples where the observed treatment does not match $\pi$’s recommendation and up-weight those that do by the inverse (estimated) propensity. For deterministic policies $\pi_t(x) \in \{0, 1\}$, we have that $\pi_T(X) = \delta_{T, \hat{T}_{\text{pi}(X)}}$ is the complement of 0-1 loss in predicting $t$. By scaling and constant shifts, one can reduce minimizing $\hat{\tau}_{\text{IPW}}(\pi)$ over policies $\pi \in \Pi$ to minimizing a weighted classification loss over classifiers $\pi \in \Pi$, providing a reduction to classification [5, 39].

Given both $\hat{\mu}(x)$ and $\hat{\varphi}(x)$ estimates, Dudík et al. [11] propose a weighting-based approach that combines the direct and IPW approaches by adapting the doubly robust (DR) estimator [9, 27, 28, 31]:

$$\hat{\tau}_{\text{DR}}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{m} \pi_t(X_i) \hat{\mu}_t(X_i) + \frac{1}{n} \sum_{i=1}^{n} (Y_i - \hat{\mu}_T(X_i)) \pi_T(X_i)/\hat{\varphi}_T(X_i).$$

$\hat{\tau}_{\text{DR}}(\pi)$ can be understood either as debiasing the direct estimator by via the reweighted residuals $\hat{e}_i = Y_i - \hat{\mu}_T(X_i)$ or as denoising the IPW estimator by subtracting the conditional mean from $Y_i$. As its bias is multiplicative in the biases of the regression and propensity estimates, the estimator is consistent so long as one of the estimates is consistent. For policy learning, [1, 11] minimize this estimator via weighted classification. Athey and Wager [1] provide a tight and favorable analysis of the corresponding uniform consistency (and hence regret) of the DR approach to policy learning.

Based on the fact that $1 = E[\pi_T(X)/\varphi_T(X)]$, a normalized IPW (NIPW) estimator is given by normalizing the weights so they sum to $n$, a common practice in causal effect estimation [2, 24]:

$$\hat{\tau}_{\text{NIPW}}(\pi) = \hat{\tau}_{W_{\text{NIPW}}(\pi)}, \quad W_i^{\text{NIPW}}(\pi) = W_i^{\text{IPW}}(\pi) \bigg/ \sum_{i'=1}^{n} W_i^{\text{IPW}}(\pi).$$

Both IPW approaches are subject to considerable variance because the plugged-in propensities are in the denominator so that small errors can have outsize effects on the total estimate.
As a stopgap measure is to clip the propensities \([12, 18]\) resulting in the clipped IPW (CIPW) estimator:

\[
\hat{\tau}^{M-CIPW}(\pi) = \hat{\tau}_{M-CIPW(\pi)}, \quad W_i^{M-CIPW}(\pi) = \pi_{T_i}(X_i)/\max\{M, \hat{\varphi}_{T_i}(X_i)\}.
\]

While effective in reducing variance, the practice remains ad-hoc, loses the unbiasedness of IPW (with true propensities), and requires the tuning of \(M\). For policy learning, Swaminathan and Joachims [35] propose to minimize over \(\pi \in \Pi\) the \(M\)-CIPW estimator plus a regularization term of the sample variance of the estimator, which they term POEM. The sample variance scales with the level of overlap between \(\pi\) and \(T_{1:n}\), i.e., the prevalence of \(\pi_{T_i}(X_i) > 0\). Indeed, when the policy class \(\Pi\) is very flexible relative to \(n\) and if outcomes are nonnegative, then the anti-logging policy \(\pi_{T_i}(X_i) = 0\) minimizes any of the above estimates. POEM avoids learning the anti-logging policy by regularizing overlap, reducing variance but limiting novelty of \(\pi\). A refinement, SNPOEM [36] uses a normalized and clipped IPW (NCIPW) estimator (and regularizes variance):

\[
\hat{\tau}^{M-NCIPW}(\pi) = \hat{\tau}_{M-NCIPW(\pi)} , \quad W_i^{M-NCIPW}(\pi) = W_i^{M-CIPW}(\pi)/\sum_{i'=1}^{n} W_{i'}^{M-CIPW}(\pi).
\]

Kallus [21] proposed a recursive partitioning approach to policy learning, the Personalization Tree (PT) and Personalization Forest (PF), that eschews a two-stage procedure that first estimates propensities and then plugs these in but still uses within-partition IPW with dynamically estimated propensities.

### 1.3 A Balance-Based Approach

**Shortcomings in existing approaches.** All of the above weighting-based approaches seek to reweight the historical data so that they look as though they were generated by the policy being evaluated or learned. Similarly, the DR approach seeks to make the historical residuals look like those that would be generated under the policy in question so to remove bias from the estimated regression model of the direct approach. However, the way these methods achieve this through various forms and versions of inverse propensity weighting, has three critical shortcomings:
By taking a simple plug-in approach for a nuisance parameter (propensities) that appears in the denominator, existing weighting-based methods are either subject to very high variance or must rely on scarcely-motivated stopgap measures such as clipping.

In the case of deterministic policies (such as an optimal policy), existing methods all have weights that are multiples of $\pi_T(X_i)$, which means that one necessarily throws away every data point $T_i$ that does not agree with the new policy recommendation $\tilde{T}_{\pi(X_i)}$. This means that one is essentially only using a much smaller dataset than is available, leading again to higher variance.

The existing weighting-based methods all proceed in two stages: first estimate propensities and then plug these in to a derived estimator (when the logging policy is unknown). On the one hand, this raises model specification concerns, and on the other, is unsatisfactory when the task at hand is not inherently two-staged – we wish only to evaluate or learn policies, not to learn propensities.

A new approach. We propose a balance-based approach that, like the existing weighting-based methods, also reweights the historical data to make it look as though they were generated by the policy being evaluated or learned and potentially denoises outcomes in a doubly robust fashion, but rather than doing so circuitously via a plug-in approach, we do it directly by finding weights that optimize for balance between the weighted data and the target policy in the given, finite sample.

In particular, we formalize balance as a discrepancy between the reweighted historical covariate distribution and that induced by the target policy and prove that it is directly related to the worst-case conditional mean square error (CMSE) of any weighting-based estimator.

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2 It is important to note that this is a problem that is unique to policy evaluation and learning – in causal effect estimation, the IPW estimator for SATE, $\frac{1}{n} \sum_{i=1}^{n} (\delta_{T_i,2} - \delta_{T_i,1}) Y_i / \hat{\pi}_{T_i}(X_i)$ (and all other extensions or DR versions), has nonzero weights on all of the data points.

3 For $m = 2$, [1, 5] minimize estimates of the form $\frac{1}{2} (\tilde{\tau}(\pi) - \tilde{\tau}(1(\cdot) - \pi))$ with $\tilde{\tau}(\pi) = \hat{\tau}_{IPW}(\pi)$ or $= \hat{\tau}_{DR}(\pi)$. This evaluates $\tau$ relative to the uniformly random policy and the resulting total weighted sums over $Y_i$ or $\hat{\varepsilon}_i$ have nonzero weights whether $\pi_{T_i}(X_i) = 0$ or not. While a useful approach for reduction to weighted classification [5] or invoking semi-parametric theory [1], it only works for $m = 2$, has no effect on learning (except algorithmically) as the correction is constant in $\pi$, and, for evaluation, is not an estimator for SAPE.
estimator. Given a policy \( \pi \), we then propose to choose (policy-dependent) weights \( W^*(\pi) \) that optimize the worst-case CMSE and therefore achieve excellent balance while controlling for variance. For evaluation, we use these optimal weights to evaluate the performance of \( \pi \) by the estimator \( \hat{\tau}_{W^*(\pi)} \) as well as a doubly robust version. For learning, we propose a bilevel optimization problem: minimize over \( \pi \in \Pi \), the estimated risk \( \hat{\tau}_{W^*(\pi)} \) (or a doubly robust version thereof and potentially plus a regularization term), given by the weights \( W^*(\pi) \) that in turn minimize the estimation error.

## 2 Balanced Evaluation

We begin by presenting the approach in the context of evaluation. Given a policy \( \pi \), consider any weights \( W = W(\pi; X_{1:n}, T_{1:n}) \) that are based on the covariate and treatment data. Given these weights we can consider both a simple weighted estimator as well as a \( W \)-weighted doubly robust estimator given a regression estimate \( \mu \):

\[
\hat{\tau}_W = \frac{1}{n} \sum_{i=1}^{n} W_i Y_i, \quad \hat{\tau}_{W,\hat{\mu}} = \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{m} \pi_t(X_i) \hat{\mu}_t(X_i) + \frac{1}{n} \sum_{i=1}^{n} W_i (Y_i - \hat{\mu}_T(X_i)).
\]

We can measure the risk of either such estimator as the conditional mean square error (CMSE), conditioned on all of the data upon which the chosen weights depend:

\[
\text{CMSE}(\hat{\tau}, \pi) = \mathbb{E}[(\hat{\tau} - \text{SAPE}(\pi))^2 | X_{1:n}, T_{1:n}].
\]

Minimal CMSE is the target of choosing weights for weighting-based policy evaluation. Our first result decomposes the CMSE of any weighting-based policy evaluation estimator into its conditional bias and variance under the unconfoundedness assumption:

**Theorem 1.** Let \( \epsilon_i = Y_i - \mu_{T_i}(X_i) \) and \( \Sigma = \text{diag}(\mathbb{E}[\epsilon_i^2 | X_1, T_1], \ldots, \mathbb{E}[\epsilon_n^2 | X_n, T_n]) \). Define

\[
B_t(W; \pi_t; f_t) = \frac{1}{n} \sum_{i=1}^{n} (W_i \delta_{t,i} - \pi_t(X_i)) f_t(X_i) \quad \text{and} \quad B(W, \pi; f) = \sum_{t=1}^{m} B_t(W; \pi_t; f_t)
\]

Then we have that:

\[
\hat{\tau}_{W,\hat{\mu}} - \text{SAPE}(\pi) = B(W, \pi; \mu) + \frac{1}{n} \sum_{i=1}^{n} W_i \epsilon_i.
\]

Moreover, under Asn. 1:

\[
\text{CMSE}(\hat{\tau}_{W,\hat{\mu}}, \pi) = B^2(W, \pi; \mu) + \frac{1}{n} W^T \Sigma W.
\]

**Corollary 2.** Let \( \hat{\mu} \) be given such that \( \hat{\mu} \perp Y_{1:n} | X_{1:n}, T_{1:n} \) (e.g., trained on a split sample). Then we have that:

\[
\hat{\tau}_{W,\hat{\mu}} - \text{SAPE}(\pi) = B(W, \pi; \mu - \hat{\mu}) + \frac{1}{n} \sum_{i=1}^{n} W_i \epsilon_i.
\]

Moreover, under Asn. 1:

\[
\text{CMSE}(\hat{\tau}_{W,\hat{\mu}}, \pi) = B^2(W, \pi; \mu - \hat{\mu}) + \frac{1}{n} W^T \Sigma W.
\]
In Thm. 1 and Cor. 2, $B(W, \pi; \mu)$ and $B(W, \pi; \mu - \hat{\mu})$ are precisely the conditional bias in evaluating $\pi$ for $\hat{\pi}_W$ and $\hat{\pi}_{W, \hat{\mu}}$, respectively, and $\frac{1}{n} W^T \Sigma W$ the conditional variance for both. In particular, $B_t(W, \pi_t; \mu_t)$ or $B_t(W, \pi_t; \mu_t - \hat{\mu}_t)$ is the conditional bias in evaluating the effect on the instances where $\pi$ assigns $t$. Note that for any function $f_t$, $B_t(W, \pi_t; f_t)$ corresponds to the discrepancy between the $f_t(X)$-moments of the measure $\nu_{t, \pi}(A) = \frac{1}{n} \sum_{i=1}^{n} \pi_t(X_i) \mathbb{I}[X_i \in A]$ on $\mathcal{X}$ and the measure $\nu_{t, W}(A) = \frac{1}{n} \sum_{i=1}^{n} W_i \delta_{t, i} \mathbb{I}[X_i \in A]$. The sum $B(W, \pi; f)$ corresponds to the sum of moment discrepancies over the components of $f = (f_1, \ldots, f_m)$ between these measures. The moment discrepancy of interest is that of $f = \mu$ or $f = \mu - \hat{\mu}$, but neither of these are known.

Balanced policy evaluation seeks weights $W$ to minimize a combination of imbalance, given by the worst-case value of $B(W, \pi; f)$ over functions $f$, and variance, given by the norm of weights $W^T \Lambda W$ for a specified positive semidefinite (PSD) matrix $\Lambda$. This general approach is in the spirit of that introduced by [20] for the case of causal effect estimation. Here, we focus on functions $f$ in the unit ball of a direct product of reproducing kernel Hilbert spaces (RKHS):

$$
\| f \|_{\mathcal{K}_{1:m, \gamma_{1:m}}} = \left( \sum_{i=1}^{m} \| f_t \|_{\mathcal{K}_i}^p / \gamma_i^p \right)^{1/p},
$$

where $\| \cdot \|_{\mathcal{K}_i}$ is the norm of the RKHS given by the PSD kernel $\mathcal{K}_i(\cdot, \cdot) : \mathcal{X}^2 \rightarrow \mathbb{R}$, i.e., the unique completion of span($\mathcal{K}_i(x, \cdot) : x \in \mathcal{X}$) endowed with $\langle \mathcal{K}_i(x, \cdot), \mathcal{K}_i(x', \cdot) \rangle = \mathcal{K}_i(x, x')$ [see...
Table 1: Policy evaluation performance in Ex. 1

| Weights W | Vanilla $\hat{\tau}_W$ | Doubly robust $\hat{\tau}_{W,\hat{\mu}}$ | $\|W\|_0$ |
|-----------|------------------------|----------------------------------|--------|
| IPW, $\varphi$ | 2.209 | 4.196 | 13.6 ± 2.9 |
| IPW, $\varphi$ | 0.568 | 0.428 | 13.6 ± 2.9 |
| .05-CIPW, $\varphi$ | 0.581 | 0.520 | 13.6 ± 2.9 |
| .05-CIPW, $\varphi$ | 0.568 | 0.428 | 13.6 ± 2.9 |
| NIPW, $\varphi$ | 0.519 | 0.754 | 13.6 ± 2.9 |
| NIPW, $\varphi$ | 0.463 | 0.692 | 13.6 ± 2.9 |
| .05-NCIPW, $\varphi$ | 0.485 | 0.724 | 13.6 ± 2.9 |
| .05-NCIPW, $\varphi$ | 0.463 | 0.692 | 13.6 ± 2.9 |
| Balanced eval | 0.280 | 0.251 | 90.7 ± 3.2 |

32. (We say $\|f\|_{K_i} = \infty$ if $f$ is not in the RKHS.) One common kernel is the Mahalanobis RBF kernel: $K_n(x, x') = \exp(-(x - x')^T \hat{S}^{-1}(x - x')/s^2)$ where $\hat{S}$ is the sample covariance of $X_1, \ldots, X_n$ and $s$ is a parameter. Any choice of $\| \cdot \|$ gives rise to our worst-case CMSE objective:

$$
\mathcal{E}^2(W, \pi; \|\cdot\|, \Lambda) = \sup_{f: \|f\| \leq 1} B^2(W, \pi; f) + \frac{1}{n^2} W^T \Lambda W.
$$

The next result characterizes this objective in terms of both worst-case and posterior risk analysis.

**Lemma 1.** (1) $\mathcal{E}^2(W, \pi; \|\cdot\|_{p,K_{1:n},\gamma_{1:n}}, \Lambda) = (\sum_{t=1}^n \gamma_t^q \mathfrak{B}_t^q(W, \pi_t; \|\cdot\|_{K_t}))^{2/q} + \frac{1}{n^2} W^T \Lambda W$, where $\mathfrak{B}_t^q(W, \pi_t; \|\cdot\|_{K_t}) = \sum_{i,j=1}^n (W_i \delta_{i,t} - \pi_t(X_i))(W_j \delta_{j,t} - \pi_t(X_j))K_t(X_i, X_j)$ and $1/p + 1/q = 1$.

(2) CMSE bound: If $\ker(\Lambda) \subseteq \ker(\Sigma)$ then $\text{CMSE}(\hat{\tau}_W, \pi) \leq \max\{\|\mu\|^2, \|A^1 \Sigma\|_2^2\} \mathcal{E}^2(W, \pi; \|\cdot\|, \Lambda)$ and $\text{CMSE}(\hat{\tau}_{W,\hat{\mu}}, \pi) \leq \max\{\|\mu - \hat{\mu}\|^2, \|A^1 \Sigma\|_2^2\} \mathcal{E}^2(W, \pi; \|\cdot\|, \Lambda)$, where $A^1$ is the pseudo inverse and $\|A\|_2$ is the spectral (operator) norm.

(3) Posterior CMSE: If $p = 2$ and $\mu_t$ has a Gaussian process prior [38] with constant mean $c$ and covariance $\gamma_t K_t$ then $\text{CMSE}(\hat{\tau}_W, \pi) = \mathcal{E}^2(W, \pi; \|\cdot\|, \Sigma)$, where the CMSE marginalizes over $\mu$. Similarly, if $\mu_t$’s process has instead mean $\hat{\mu}_t$ then the equation instead holds for $\hat{\tau}_{W,\hat{\mu}}$. 10
Point (3) in Lemma 1 suggests that, in practice, model selection of $\gamma_{1:m}$, $\Lambda$, and kernel hyperparameters such as $s$ or even $S$, can be done by the marginal likelihood method [see 38, Ch. 5].

**Balanced policy evaluation** is given by either the estimator $\hat{\tau}_{W^{\ast} (\pi; ||\cdot||, \Lambda)}$ or $\hat{\tau}_{W^{\ast} (\pi; ||\cdot||, \Lambda), \hat{d}}$ where $W^{\ast} (\pi) = W^{\ast} (\pi; ||\cdot||, \Lambda)$ is the minimizer of $\mathcal{E}^2 (W; \pi; ||\cdot||, \Lambda)$ over the space of all weights $W$ that sum to $n$, $W = \{ W \in \mathbb{R}_+^n : \sum_{i=1}^n W_i = n \} = n \Delta^n$. Specifically,

$$W^{\ast} (\pi; ||\cdot||, \Lambda) \in \arg \min_{W \in \mathcal{W}} \mathcal{E}^2 (W; \pi; ||\cdot||, \Lambda).$$

When $||\cdot|| = ||\cdot||_{p, \kappa_{1:m}, \gamma_{1:m}}$, this problem is a quadratic program for $p = 2$ and a second-order cone program for $p = 1, \infty$. Both are efficiently solvable [7]. In practice, we solve these using Gurobi 7.0.

In Lemma 1, $\mathcal{B}_i (W; \pi_i; ||\cdot||_{K_i})$ measures the *imbalance* between $\nu_{i, \pi}$ and $\nu_{i, W}$ as the *worst-case* discrepancy in means over functions in the unit ball of an RKHS. In fact, as a distributional distance metric, it is the maximum mean discrepancy (MMD) used, for example, for testing whether two samples come from the same distribution [14]. Thus, minimizing $\mathcal{E}^2 (W; \pi; ||\cdot||_{p, \kappa_{1:m}, \gamma_{1:m}}, \Lambda)$ is simply seeking the weights $W$ that balance $\nu_{i, \pi}$ and $\nu_{i, W}$ subject to variance regularization in $W$.

**Example 1.** We demonstrate balanced evaluation with a mixture of $m = 5$ Gaussians: $X \mid T \sim \mathcal{N}(\overline{X}_T, I_2 \times 2)$, $\overline{X}_t = (0, 0)$, $\overline{X}_t = (\text{Re}, \text{Im}) (e^{i2\pi(t-2)/(m-1)})$ for $t = 2, \ldots, m$, and $T \sim \text{Multinomial}(1/5, \ldots, 1/5)$. Fix a draw of $X_{1:n}, T_{1:n}$ with $n = 100$ shown in Fig. 1a (numpy seed 0). Color denotes $T_i$ and size denotes $\varphi_{T_i}(X_i)$. The centers $\overline{X}_t$ are marked by a colored number. Next, we let $\mu_t (x) = \exp (1 - 1/||x - \chi_t||_2)$ where $\chi_t = (\text{Re}, \text{Im}) (e^{-i2\pi t/m} / \sqrt{2})$ for $t \in [m]$, $\epsilon_t \sim \mathcal{N}(0, \sigma)$, and $\sigma = 1$. Fig. 1b plots $\mu_t (x)$. Fig. 1c shows the corresponding optimal policy $\pi^\ast$.

Next we consider evaluating $\pi^\ast$. Fixing $X_{1:n}$, as in Fig. 1a, we have $\text{SAPE} (\pi^\ast) = 0.852$. With $X_{1:n}$ fixed, we draw 1000 replications of $T_{1:n}, Y_{1:n}$ from their conditional distribution. For each replication, we fit $\hat{\varphi}$ by estimating the (well-specified) Gaussian mixture by maximum likelihood and fit $\hat{\mu}$ using $m$ separate gradient-boosted tree models (sklearn defaults). We consider evaluating $\pi^\ast$ either using the vanilla estimator $\hat{\tau}_W$ or the doubly robust estimator $\hat{\tau}_{W, \hat{d}}$ for $W$ either chosen in the 4 different standard ways laid out in Sec. 1.2, using either
the true $\varphi$ or the estimated $\hat{\varphi}$, or chosen by the balanced evaluation approach using untuned parameters (rather than fit by marginal likelihood) using the standard ($s = 1$) Mahalanobis RBF kernel for $K_t$, $\|f\|^2 = \sum_{i=1}^m \|f_i\|_{K_t}^2$, and $\Lambda = I$. (Note that this misspecifies the outcome model, $\|\mu\|_{K_t} = \infty$.) We tabulate the results in Tab. 1.

We note a few observations on the standard approaches: vanilla IPW with true $\varphi$ has zero bias but large SD (standard deviation) and hence RMSE (root mean square error); a DR approach improves on a vanilla IPW with $\hat{\varphi}$ by reducing bias; clipping and normalizing IPW reduces SD. The balanced evaluation approach achieves the best RMSE by a clear margin, with the vanilla estimator beating all standard vanilla and DR estimators and the DR estimator providing a further improvement by nearly eliminating bias (but increasing SD). The marked success of the balanced approach is unsurprising when considering the support $\|W\|_0 = \sum_{i=1}^n [W_i > 0]$ of the weights. All standard approaches use weights that are multiples of $\pi_t(X_i)$, limiting support to the overlap between $\pi$ and $T_{1:n}$, which hovers around 10–16 over replications. The balanced approach uses weights that have significantly wider support, around 88–94. In light of this, the success of the balanced approach is expected.

2.1 Consistent Evaluation

Next we consider the question of consistent evaluation: under what conditions can we guarantee that $\hat{\varphi}(\pi) - \text{SAP}(\pi)$ and $\hat{\varphi}(\pi)\hat{\mu} - \text{SAP}(\pi)$ converge to zero and at what rates. We use the notation $Z_n = \alpha_p(a_n)$ or $Z_n/a_n \to_p 0$ for random variable $Z_n$ and positive numbers $a_n$ to mean $|Z_n|/a_n$ converges to 0 in probability, i.e., $\forall \epsilon \; \mathbb{P}(|Z_n| \geq a_n\epsilon) \to 0$. We use $Z_n = O_p(a_n)$ to mean that $|Z_n|/a_n$ is stochastically bounded, i.e., $\forall \epsilon \; \exists M : \mathbb{P}(|Z_n| > Ma_n) < \epsilon$.

One key requirement for consistent evaluation is a weak form of overlap between the historical data and the target policy to be evaluated using this data:

**Assumption 2** (Weak overlap). $\mathbb{P}(\varphi_t(X) > 0 \lor \pi_t(X) = 0) = 1$ for all $t \in [m]$ and $\mathbb{E}[\pi^2_t(X)/\varphi^2_t(X)] < \infty$.

Another key requirement is specification. If the mean-outcome function is well-specified in that it is in the RKHS product used to compute $W^*(\pi)$ then convergence at rate $1/\sqrt{n}$ is guaranteed. Otherwise, for a doubly robust estimator, if the regression estimate is well-specified then consistency is still guaranteed. In lieu of specification, consistency is also
guaranteed if the RKHS product consists of $C_0$-universal kernels, defined below, such as the RBF kernel [33].

**Definition 1.** A PSD kernel $K$ on a Hausdorff $X$ (e.g., $\mathbb{R}^d$) is $C_0$-universal if, for any continuous function $g: X \rightarrow \mathbb{R}$ with compact support (i.e., for some $C$ compact, $\{x : g(x) \neq 0\} \subseteq C$) and $\eta > 0$, there exists $m, \alpha_1, x_1, \ldots, \alpha_m, x_m$ such that $\sup_{x \in X} |\sum_{j=1}^{m} \alpha_j K(x_j, x) - g(x)| \leq \eta$.

**Theorem 3.** Fix $\pi$ and let $W_n^*(\pi) = W_n^*(\pi; \|f\|_{p, K_{1:m}, \gamma_{n,1:m}, \Lambda_n})$ with $0 < \kappa I \preceq \Lambda_n \preceq \bar{\kappa} I$, $0 < \gamma \leq \gamma_{n,t} \leq \bar{\gamma} \forall t \in [m]$ for each $n$. Suppose Asns. 1 and 2 hold, $\text{Var}(Y | X)$ a.s. bounded, $\mathbb{E}[\sqrt{K_t(X, X)}] < \infty$, and $\mathbb{E}[K_t(X, X) \pi_t^2(X)/\varphi_t^2(X)] < \infty$. Then the following two results hold:

(a) If $\|\mu_t\|_{K_t} < \infty$ for all $t \in [m]$:
\[
\hat{\tau}_{W_n^*(\pi)} - \text{SAPE}(\pi) = O_p(1/\sqrt{n}).
\]

(b) If $K_t$ is $C_0$-universal for all $t \in [m]$:
\[
\hat{\tau}_{W_n^*(\pi)} - \text{SAPE}(\pi) = o_p(1).
\]

**Corollary 4.** Suppose the assumptions of Thm. 3 hold and. Then

(a) If $\|\hat{\mu}_n - \mu_t\|_{K_t} = o_p(1) \forall t \in [m]$:
\[
\hat{\tau}_{W_n^*(\pi), \hat{\mu}_n} - \text{SAPE}(\pi) = (\frac{1}{n} \sum_{i=1}^{n} W_{ni}^2 \text{Var}(Y_i | X_i))^{1/2} + o_p(1/\sqrt{n}).
\]

(b) If $\|\hat{\mu}_n(X) - \mu(X)\|_2 = O_p(r(n))$, $r(n) = \Omega(1/\sqrt{n})$:
\[
\hat{\tau}_{W_n^*(\pi), \hat{\mu}_n} - \text{SAPE}(\pi) = O_p(r(n)).
\]

---

4The latter two requirements are trivial for bounded kernels like the RBF kernel, which is bounded 1.
(c) If $\mu_t, \mu_t k K_t < \infty$, $\hat{\mu}_ntkK_t = O_p(1)$ for all $t \in [m]$: $\hat{\tau}_{\pi,\hat{\mu}_n} - \text{SAPE}(\pi) = O_p(1/\sqrt{n})$.

(d) If $K_t$ is $C_0$-universal for all $t \in [m]$: $\hat{\tau}_{\pi,\hat{\mu}_n} - \text{SAPE}(\pi) = o(1)$.

Cor. 4(a) is the case where both the balancing weights and the regression function are well-specified, in which case we can show that the multiplicative bias disappears faster than $o_p(1/\sqrt{n})$, leaving us only with the irreducible residual variance, leading to an efficient evaluation of policy. The other cases concern the “doubly robust” nature of the balanced DR estimator, where Cor. 4(b) requires only that the regression be well-specified and Cor. 4(c)-(d) require only the balancing weights to be well-specified or arise from a $C_0$-universal kernel. Finally, note that, as $\text{SAPE}(\pi) - \text{PAPE}(\pi) = O_p(1/\sqrt{n})$, all results above also hold for convergence to PAPE in both Thm. 3 and Cor. 4.

3 Balanced Learning

Next we consider a balanced approach to policy learning. Given a policy class $\Pi \subset [X \rightarrow \Delta^m]$, we let the balanced policy learner yield the policy $\pi \in \Pi$ that minimizes the balanced policy evaluation using either a vanilla or DR estimator plus a potential regularization term in the worst-case/posterior CMSE of the evaluation. We formulate this as a bilevel optimization problem:

\[
\hat{\pi}^\text{balanced} = \arg\min_{\pi} \left\{ \hat{\tau}_W + \lambda \mathcal{E}(W, \pi; \| \cdot \|, \Lambda) : \pi \in \Pi, W \in \arg\min_{W \in W} \mathcal{E}^2(W, \pi; \| \cdot \|, \Lambda) \right\}
\] (1)

\[
\hat{\pi}^\text{balanced-DR} = \arg\min_{\pi} \left\{ \hat{\tau}_{W,\hat{\mu}} + \lambda \mathcal{E}(W, \pi; \| \cdot \|, \Lambda) : \pi \in \Pi, W \in \arg\min_{W \in W} \mathcal{E}^2(W, \pi; \| \cdot \|, \Lambda) \right\}
\] (2)

The regularization term regularizes both the balance (i.e., worst-case/posterior bias) that is achievable for $\pi$ and the variance in evaluating $\pi$. We include this regularizer for completeness and motivated by the results of [35] (which regularize variance), but find that it not necessary to include it in practice.

3.1 Optimizing the Balanced Policy Learner

Unlike [1, 5, 11, 34, 39], our (nonconvex) policy optimization problem does not reduce to weighted classification precisely because our weights are not multiplies of $\pi T_i(X_i)$ (but
therefore our weights also lead to better performance). Instead, like [35], we use gradient
descent. For that, we need to be able to differentiate our bilevel optimization problem. We
focus on $p = 2$ for brevity.

**Theorem 5.** Let $\| \cdot \| = \| \cdot \|_{2,K_{1:m},\gamma_{1:m}}$. There is $W^*(\pi) \in \arg\min_{W \in \mathcal{W}} \mathcal{E}^2(W, \pi; \| \cdot \|, \Lambda)$ such that

\[
\nabla_{\pi_t(X_1), \ldots, \pi_t(X_n)} \tilde{W}^*(\pi) = \frac{1}{n} Y_{1:n}^{T} \tilde{H}(I - (A + (I - A)\tilde{H})^{-1}(I - A)\tilde{H})J_t
\]

\[
\nabla_{\pi_t(X_1), \ldots, \pi_t(X_n)} \tilde{W}^*(\pi), \tilde{\pi} = \frac{1}{n} \tilde{\epsilon}_{1:n}^{T} \tilde{H}(I - (A + (I - A)\tilde{H})^{-1}(I - A)\tilde{H})J_t + \frac{1}{n} \tilde{\mu}_{t}(X_{1:n})
\]

\[
\nabla_{\pi_t(X_1), \ldots, \pi_t(X_n)} \mathcal{E}(W^*(\pi), \pi; \| \cdot \|, \Lambda) = -D_{t}/\mathcal{E}(W^*(\pi), \pi; \| \cdot \|, \Lambda)
\]

where $\tilde{H} = -F(F^T HF)^{-1}F^T$, $F_{ij} = \delta_{ij} - \delta_{in}$ for $i \in [n], j \in [n-1]$, $A_{ij} = \delta_{ij}I[W^*_i(\pi) > 0]$, $D_{ti} = \gamma^2 \sum_{j=1}^{n} \mathcal{K}_t(X_i, X_j)(W_j \delta_{T_t, i} - \pi_t(X_j))$, $H_{ij} = 2 \sum_{t=1}^{m} \gamma^2 \delta_{T_t, i} \delta_{T_t, j} \mathcal{K}_t(X_i, X_j) + 2\Lambda$, and $J_{tij} = -2\gamma^2 \delta_{T_t, i} \mathcal{K}_t(X_i, X_j)$.

To leverage this result, we use a parameterized policy class such as $\Pi_{\text{logit}} = \{ \pi_t(x; \beta_t) \propto \exp(\beta_0 + \beta^T_t x) \}$ (or kernelized versions thereof), apply chain rule to differentiate objective in
the parameters $\beta$, and use BFGS [13] with random starts. The logistic parametrization allows
us to smooth the problem even while the solution ends up being deterministic (extreme $\beta$).

This approach requires solving a quadratic program for each objective gradient evaluation.
While this can be made faster by using the previous solution as warm start, it is still
computationally intensive, especially as the bilevel problem is nonconvex and both it and each
quadratic program solved in “batch” mode. This is a limitation of the current optimization
algorithm that we hope to improve on in the future using specialized methods for bilevel
optimization [4, 25, 30].

**Example 2.** We return to Ex. 1 and consider policy learning. We use the fixed draw shown
in Fig. 1a and set $\sigma$ to 0. We consider a variety of policy learners and plot the policies
in Fig. 2 along with their population regret PAPE($\pi$) – PAPE($\pi^*$). The policy learners
we consider are: minimizing standard IPW and DR evaluations over $\Pi_{\text{logit}}$ with $\hat{\varphi}, \hat{\mu}$ as in
Ex. 1 (versions with combinations of normalized, clipped, and/or true $\varphi$, not shown, all have
regret 0.26–0.5), the direct method with Gaussian process regression gradient boosted trees
(both sklearn defaults), weighted SVM classification using IPW and DR weights (details in
appendix), SNPOEM [36], PF [21], and our balanced policy learner (1) with parameters as
in Ex. 1, $\Pi = \Pi_{\text{logit}}, \lambda = \Lambda = 0$ (the DR version (2), not shown, has regret .08).
3.2 Uniform Consistency and Regret Bounds

Next, we establish consistency results uniformly over policy classes. This allows us to bound the regret of the balanced policy learner. We define the sample and population regret, respectively, as

\[ R(\hat{\pi}) = \text{PAPE}(\hat{\pi}) - \min_{\pi \in \Pi} \text{PAPE}(\pi), \quad \hat{R}(\hat{\pi}) = \text{SAPE}(\hat{\pi}) - \min_{\pi \in \Pi} \text{SAPE}(\pi) \]

A key requirement for these to converge is that the best-in-class policy is learnable. We quantify that using Rademacher complexity [3] and later extend our results to VC dimension. Let us define

\[ \hat{R}_n(\mathcal{F}) = \frac{1}{2^n} \sum_{\rho \in \{-1, +1\}^n} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \rho_i f(X_i), \quad R_n(\mathcal{F}) = \mathbb{E}[\hat{R}_n(\mathcal{F})]. \]

E.g., for linear policies \( \hat{R}_n(\mathcal{F}) = O(1/\sqrt{n}) \) [19]. If \( \mathcal{F} \subseteq [\mathcal{X} \to \mathbb{R}^m] \) let \( \mathcal{F}_\ell = \{(f(\cdot))_\ell : f \in \mathcal{F}\} \) and set \( R_n(\mathcal{F}) = \sum_{\ell=1}^m R_n(\mathcal{F}_\ell) \) and same for \( \hat{R}_n(\mathcal{F}) \). We also strengthen the overlap assumption.

**Assumption 3** (Strong overlap). \( \exists \alpha \geq 1 \) such that \( \mathbb{P}(\varphi_t(X) \geq 1/\alpha) = 1 \ \forall t \in [m] \).

**Theorem 6.** Fix \( \Pi \subseteq [\mathcal{X} \to \Delta^m] \) and let \( W_\pi^*(\pi) = W_\pi^*(\pi; \|f\|_{\mu, K_{1, m}, \gamma_{n, 1, m}, \Lambda_n}) \) with \( 0 \leq \xi I \leq \Lambda_n \leq \pi I \leq \pi^* I \), \( 0 < \gamma \leq \gamma_{n, t} \leq \gamma \ \forall t \in [m] \) for each \( n \) and \( \pi \in \Pi \). Suppose Assns. 1 and 3 hold, \( |\epsilon_t| \leq B \) a.s. bounded, and \( \sqrt{K_t(x, x)} \leq \Gamma \ \forall t \in [m] \) for \( \Gamma \geq 1 \). Then the following two results hold:

(a) If \( \|\mu_t\|_{K_t} < \infty, \ \forall t \in [m] \) then for \( n \) sufficiently large (\( n \geq 2\log(4m/\nu)/(1/(2\alpha) - \mathbb{R}_n(\Pi)^{2}) \)), we have that, with probability at least \( 1 - \nu \) and \( 1 - 5\nu/4, \) respectively,

\[
\sup_{\pi \in \Pi} |\hat{\gamma}_W^*(\pi) - \text{SAPE}(\pi)| \leq 8\alpha\Gamma\gamma m(\|\mu\| + \sqrt{2\log(4m/\nu)}k^{-1}B)\mathbb{R}_n(\Pi)
+ \frac{2\alpha\bar{\gamma}\|\mu\| + 12\alpha\Gamma^2\gamma m\|\mu\|}{\sqrt{n}}
+ \frac{(2\alpha\bar{\gamma}k^{-1}B + 12\alpha\Gamma^2\gamma mk^{-1}B + 3\alpha\Gamma\gamma m\|\mu\|)\sqrt{2\log(4m/\nu)}}{\sqrt{n}}
+ \frac{6\alpha\Gamma\gamma mk^{-1}B\log(4m/\nu)}{\sqrt{n}},
\]

(b) \( \sup_{\pi \in \Pi} |\hat{\gamma}_W^*(\pi) - \text{SAPE}(\pi)| \leq 8\alpha\Gamma\gamma m(\|\mu\| + \sqrt{2\log(4m/\nu)}k^{-1}B)\hat{R}_n(\Pi) \)
\[
\begin{align*}
&+ \frac{2\alpha\pi \|\mu\| + 12\alpha \Gamma^2 \pi m \|\mu\|}{\sqrt{n}} \\
&+ \frac{(2\alpha \pi \mu^{-1}B + 12\alpha \Gamma^2 \pi m \mu^{-1}B + 35\alpha \pi m \|\mu\|) \sqrt{2 \log(4m/\nu)}}{\sqrt{n}} \\
&+ \frac{38\alpha \pi \mu^{-1}B \log(4m/\nu)}{\sqrt{n}}.
\end{align*}
\]

(b) If \(K_t\) is \(C_0\)-universal for all \(t \in [m]\) and either \(\mathcal{R}_n(\Pi) = o(1)\) or \(\hat{\mathcal{R}}_n(\Pi) = o_p(1)\) then

\[
\sup_{\pi \in \Pi} |\hat{\pi}_W(\pi) - \text{SAPE}(\pi)| = o_p(1).
\]

**Corollary 7.** Suppose the assumptions of Thm. 6 hold. If \(\hat{\pi}_n^{\text{balanced}}\) is as in (1) then:

(a) If \(\|\mu_t\|_{K_t} < \infty\) for all \(t \in [m]\):

\[
R_{\Pi}(\hat{\pi}_n^{\text{balanced}}) = O_p(\mathcal{R}_n(\Pi) + 1/\sqrt{n}).
\]

(b) If \(K_t\) is \(C_0\)-universal for all \(t \in [m]\):

\[
R_{\Pi}(\hat{\pi}_n^{\text{balanced}}) = o_p(1).
\]

If \(\hat{\pi}_n^{\text{balanced-DR}}\) is as in (2) then:

(c) If \(\|\hat{\mu}_n - \mu_t\|_{K_t} = o_p(1)\) for all \(t \in [m]\): \(R_{\Pi}(\hat{\pi}_n^{\text{balanced-DR}}) = O_p(\mathcal{R}_n(\Pi) + 1/\sqrt{n})\). case (a).

(d) If \(\|\hat{\mu}_n(X) - \mu(X)\|_{X} = O_p(r(n))\):

\[
R_{\Pi}(\hat{\pi}_n^{\text{balanced-DR}}) = O_p(r(n) + \mathcal{R}_n(\Pi) + 1/\sqrt{n}).
\]

(e) If \(\|\mu_t\|_{K_t} < \infty, \|\hat{\mu}_n - \mu_t\|_{K_t} = O_p(1)\) for all \(t \in [m]\):

\[
R_{\Pi}(\hat{\pi}_n^{\text{balanced-DR}}) = O_p(\mathcal{R}_n(\Pi) + 1/\sqrt{n}).
\]

(f) If \(K_t\) is \(C_0\)-universal for all \(t \in [m]\):

\[
R_{\Pi}(\hat{\pi}_n^{\text{balanced}}) = o_p(1).
\]

All the same results hold when replacing \(\mathcal{R}_n(\Pi)\) with \(\hat{\mathcal{R}}_n(\Pi)\) and/or replacing \(R_{\Pi}\) with \(\hat{R}_{\Pi}\).

The constants in case (c) are strictly smaller than those in (a) by virtue of the multiplicative bias of the DR estimator. However, an exact analysis and comparison of the constants is beyond our scope.

In some cases it is easier to deal with VC (Vapnik-Chervonenkis) dimension [37] rather than Sauer’s and Massart’s lemmas:

**Lemma 2.** Suppose \(\Pi \subseteq \{X \rightarrow \{0, 1\}^m\}\) is a class of deterministic policies and \(\Pi_t\) has VC dimension \(h_t\). Let \(h = (\sum_{t=1}^m \sqrt{h_t})^2\). Then \(\hat{\mathcal{R}}_n(\Pi) \leq (\sqrt{2m} + \sqrt{h \log n})/\sqrt{n}\).
4 Conclusion

Considering the policy evaluation and learning problems using observational or logged data, we presented a new method that is based on finding optimal balancing weights that make the data look like the target policy and that is aimed at ameliorating the shortcomings of existing methods, which included having to deal with near-zero propensities, using too few positive weights, and using an awkward two-stage procedure. The new approach showed promising signs of fixing these issues in some numerical examples. However, the new learning method is more computationally intensive than existing approaches, solving a QP at each gradient step. Therefore, in future work, we plan to explore faster algorithms that can implement the balanced policy learner, perhaps using alternating descent, and use these to investigate comparative numerics in much larger datasets.

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A Omitted Proofs

Proof of Thm. 1. Noting that $Y_i = Y_i(T_i) = \sum_{t=1}^m \delta_{T_t} \mu_t(X_i) + \epsilon_i$, let us rewrite $\hat{\tau}_W$ as

$$\hat{\tau}_W = \frac{1}{n} \sum_{t=1}^m \sum_{i=1}^n W_i \delta_{T_t} \mu_t(X_i) + \frac{1}{n} \sum_{i=1}^n W_i \epsilon_i.$$  

Recalling that $\text{SAPE} (\pi) = \frac{1}{n} \sum_{i=1}^n \sum_{t=1}^m \pi_t(X_i) \mu_t(X_i)$ immediately yields the first result. To obtain the second result note that $\text{SAPE}(\pi)$ is measurable with respect to $X_{1:n}, T_{1:n}$ so that

$$\text{CMSE}(\hat{\tau}_W, \pi) = (\mathbb{E}[\hat{\tau}_W | X_{1:n}, T_{1:n}] - \text{SAPE}(\pi))^2 + \text{Var}(\hat{\tau}_W | X_{1:n}, T_{1:n}).$$

By Asn. 1

$$\mathbb{E}[\delta_{T_t} \epsilon_i | X_{1:n}, T_{1:n}] = \delta_{T_t} \mathbb{E}[\epsilon_i | X_i] = \delta_{T_t} (\mathbb{E}[Y_i(t) | X_i] - \mu_t(X_i)) = 0.$$ 

Therefore,

$$\mathbb{E}[\hat{\tau}_W | X_{1:n}, T_{1:n}] = \frac{1}{n} \sum_{t=1}^m \sum_{i=1}^n W_i \delta_{T_t} \mu_t(X_i),$$

giving the first term of $\text{CMSE}(\hat{\tau}_W, \pi)$. Moreover, since

$$\mathbb{E}[\epsilon_i \epsilon_i' | X_{1:n}, T_{1:n}] = \delta_{ii'} \sigma_{T_t}^2,$$

we have

$$\text{Var}(\hat{\tau}_W | X_{1:n}, T_{1:n}) = \mathbb{E}[(\hat{\tau}_W - \mathbb{E}[\hat{\tau}_W | X_{1:n}, T_{1:n}])^2 | X_{1:n}, T_{1:n}] = \frac{1}{n^2} \mathbb{E}(\sum_{i=1}^n W_i \epsilon_i^2 | X_{1:n}, T_{1:n}) = \frac{1}{n^2} \sum_{i=1}^n W_i^2 \sigma_{T_t}^2,$$

giving the second term. \qed

Proof of Cor. 2. This follows from Thm. 1 after noting that $\hat{\tau}_{W, \hat{\mu}} = \hat{\tau}_W - B(W, \pi; \hat{\mu})$ and that

$$B_t(W, \pi; \hat{\mu}_t) - B_t(W, \pi; \hat{\mu}_t) = B_t(W, \pi; \mu_t - \hat{\mu}_t).$$ \qed

Proof of Lemma 1. For (1), we have

$$\mathcal{E}^2(W; \pi; \| \cdot \|_{p, \kappa_{1:m}, \gamma_{1:m}}, \Lambda) = \sup_{\|v\|_p \leq 1, \|f\|_{K_t} \leq \gamma_t} (\sum_{i=1}^m B_t(W, \pi_i; f_i))^2 + \frac{1}{n^2} W^T \Lambda W$$

$$= \sup_{\|v\|_p \leq 1} (\sum_{i=1}^{m} \sup_{\|f\|_{K_t} \leq \gamma_t} B_t(W, \pi_i; f_i))^2 + \frac{1}{n^2} W^T \Lambda W$$

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\[
= \sup_{\|v\| \leq 1} \left( \sum_{t=1}^{m} v_t \gamma_t \mathfrak{B}_t(W, \pi_t; \| \kappa_i \|) \right)^2 + \frac{1}{n^2} W^T \Lambda W \\
= \left( \sum_{t=1}^{m} \gamma_t^q \mathfrak{B}_t^q(W, \pi_t; \| \kappa_i \|)^2 \right)^{2/q} + \frac{1}{n^2} W^T \Lambda W.
\]

For (2), we have
\[
\sum_{t=1}^{m} B_t(W, \pi_t; f_t) \leq \sum_{t=1}^{m} \| \mu_t \| \kappa_i \mathfrak{B}_t(W, \pi_t; \| \kappa_i \|) \\
\leq \left( \sum_{t=1}^{m} \| \mu_t \|_{K_i}^p / \gamma_t^p \right)^{1/p} \left( \sum_{t=1}^{m} \gamma_t^q \mathfrak{B}_t^q(W, \pi_t; \| \kappa_i \|)^1 \right)^{1/q}
\]
and \( W^T \Sigma W \leq \| \Lambda^1 \Sigma \|_2^2 W^T \Lambda W \).

For (3), since \( B_t(W, \pi_t; c) = 0 \) for a constant function \( c \), WLOG we can assume that \( \mu_t \)'s constant prior is zero. Letting \( z_{ti} = (W_t \delta_{ti} - \pi_t(X_i)) \), since \( \mathbb{E}[\mu_t(X_i)\mu_s(X_j) \mid X_{1:n}, T_{1:n}] = \delta_{ts} \kappa_t(X_i, X_j) \), we then have
\[
\text{CMSE}(\hat{\tau}_t, \pi) = \mathbb{E}[(\sum_{t=1}^{m} B_t(W, \pi_t; \mu_t))^2 \mid X_{1:n}, T_{1:n}] + \frac{1}{n^2} W^T \Sigma W \\
= \sum_{t,s=1}^{m} \sum_{i,j=1}^{m} z_{ti} z_{sj} \mathbb{E}[\mu_t(X_i)\mu_s(X_j) \mid X_{1:n}, T_{1:n}] + \frac{1}{n^2} W^T \Sigma W \\
= \sum_{t=1}^{m} \sum_{i,j=1}^{m} z_{ti} z_{tj} \kappa_t(X_i, X_j) + \frac{1}{n^2} W^T \Sigma W.
\]
And similarly for \( \text{CMSE}(\hat{\tau}_w, \pi) \).

**Proof of Thm. 3.** Let \( Z = \frac{1}{n} \sum_{i=1}^{n} \pi_T(X_i) / \varphi_T(X_i) \) and \( \bar{W}_t(\pi) = \frac{1}{Z} \pi_T(X_i) / \varphi_T(X_i) \) and note that \( \bar{W} \in \mathcal{W} \). Moreover, note that
\[
\mathfrak{B}_t(\bar{W}, \pi_t; \| \kappa_i \|) = \frac{1}{Z} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\delta_{T_{ij}}}{\varphi_T(X_i)} - Z \right) \pi_t(X_i) E_X \| \kappa_i \right) \\
\leq \frac{1}{Z} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\delta_{T_{ij}}}{\varphi_T(X_i)} \right) - 1 \right) \pi_t(X_i) E_X \| \kappa_i + \frac{1}{Z} \left( \frac{1}{n} \sum_{i=1}^{n} \left( Z - 1 \right) \pi_t(X_i) E_X \| \kappa_i ight) \\
\leq \frac{1}{Z} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\delta_{T_{ij}}}{\varphi_T(X_i)} \right) - 1 \right) \pi_t(X_i) E_X \| \kappa_i + \frac{|Z - 1|}{n} \sum_{i=1}^{n} \sqrt{\kappa_t(X_i, X_i)}.
\]

Let \( \xi_i = \left( \frac{\delta_{T_{ij}}}{\varphi_T(X_i)} - 1 \right) \pi_t(X_i) E_X \), and note that \( \mathbb{E}[\xi_i] = \mathbb{E}[\mathbb{E}[\delta_{T_{ij}} / \varphi_T(X_i) \mid X_i] - 1] \pi_t(X_i) E_X] = 0 \)
and that \( \xi_1, \xi_2, \ldots \) are iid. Therefore, letting \( \xi_1, \xi_2, \ldots \) be iid replicates of \( \xi_1, \xi_2, \ldots \) (ghost
sample) and letting \( \rho_i \) be iid Rademacher random variables independent of all else, we have

\[
\mathbb{E}[\left\| \frac{1}{n} \sum_{i=1}^{n} \xi_i \right\|_{K_t}^2] = \frac{1}{n^2} \mathbb{E}[\left\| \sum_{i=1}^{n} (\mathbb{E}[\xi_i] - \xi_i) \right\|_{K_t}^2] \\
= \frac{1}{n^2} \mathbb{E}[\left\| \sum_{i=1}^{n} \rho_i (\xi_i' - \xi_i) \right\|_{K_t}^2] \leq 4 \frac{1}{n^2} \mathbb{E}[\left\| \sum_{i=1}^{n} \rho_i \xi_i \right\|_{K_t}^2].
\]

Note that \( \|\xi_1 - \xi_2\|_{K_t}^2 + \|\xi_1 + \xi_2\|_{K_t}^2 = 2\|\xi_1\|_{K_t}^2 + 2\|\xi_2\|_{K_t}^2 + 2 \langle \xi_1, \xi_2 \rangle = 2\|\xi_1\|_{K_t}^2 + 2\|\xi_2\|_{K_t}^2. \)

By induction, \( \sum_{\rho_i \in \{-1, 1\}} \| \sum_{i=1}^{n} \rho_i \xi_i \|_{K_t}^2 = 2^n \sum_{i=1}^{n} \| \xi_i \|_{K_t}^2. \) Since

\[
\mathbb{E}[\| \xi_i \|_{K_t}^2] \leq 2 \mathbb{E}\left[ \frac{\pi_t^2(X)}{\varphi_t^2(X)} \mathcal{K}_t(X, X) \right] + 2 \mathbb{E}[\pi_t^2(X) \mathcal{K}_t(X, X)] \leq 4 \mathbb{E}\left[ \frac{\pi_t^2(X)}{\varphi_t^2(X)} \mathcal{K}_t(X, X) \right] < \infty,
\]

we get \( \mathbb{E}[\frac{1}{n} \sum_{i=1}^{n} \xi_i \|_{K_t}^2] = O(1/n) \) and therefore \( \frac{1}{n} \sum_{i=1}^{n} \xi_i \|_{K_t}^2 = O_p(1/n) \) by Markov’s inequality. Moreover, as \( \mathbb{E}[\pi_t(X)/\varphi_t(X)] = \mathbb{E}[\sum_{l=1}^{m} \epsilon_l \mathcal{K}_l \mathcal{K}_l(X, X)] = 1 \) and \( \mathbb{E}[\pi_t^2(X)/\varphi_t^2(X)] < \infty, \) by Chebyshev’s inequality, \( \mathbb{E}[(Z - 1)^2] = O(1/n) \) so that \( (Z - 1)^2 = O_p(1/n) \) by Markov’s inequality. Similarly, as \( \mathbb{E}[(Z_2)^2] = O(1/n) \), we have

\[
\mathbb{E}^2(W_n^*, \pi; \| \cdot \|_{\mathcal{K}_{1:m}, \gamma_{n,1:m}, \Lambda_n}) \leq \mathbb{E}^2(W, \pi; \| \cdot \|_{\mathcal{K}_{1:m}, \gamma_{n,1:m}, \Lambda_n}) \\
\leq \gamma^2 \left( \sum_{l=1}^{m} \mathfrak{B}_t^2(W, \pi_l; \| \cdot \|_{\mathcal{K}_l}) \right)^{2/q} + \bar{\sigma}^2 \| \bar{W} \|_2^2 = O_p(1/n)
\]

Therefore,

\[
\mathfrak{K}_t^2(W_n^*, \pi_l; \| \cdot \|_{\mathcal{K}_l}) \leq \gamma^{-2} \mathbb{E}^2(W_n^*, \pi; \| \cdot \|_{1:m, \gamma_{n,1:m}, \Lambda_n}) = O_p(1/n),
\]

\[
\frac{1}{n^2} \| W_n^* \|_2^2 \leq \frac{1}{n^2} \mathbb{E}^2(W_n^*, \pi; \| \cdot \|_{1:m, \gamma_{n,1:m}, \Lambda_n}) = O_p(1/n).
\]

Now consider case (a). By assumption \( \| \Sigma \|_2 \leq \bar{\sigma}^2 < \infty \) for all \( n \). Then we have

\[
\text{CMSE}(\hat{\gamma}_{W_n}^*, \pi) \leq \frac{m}{n} \sum_{l=1}^{m} \| \mu_l \|_{K}^2 \mathfrak{K}_t^2(W_n^*, \pi_l; \| \cdot \|_{\mathcal{K}_l}) + \frac{\bar{\sigma}^2}{n^2} \| W_n^* \|_2^2 = O_p(1/n).
\]

Letting \( D_n = \sqrt{n} | \hat{\gamma}_{W_n} - \text{SAPE}(\pi) | \) and \( \mathcal{G} \) be the sigma algebra of \( X_1, T_1, X_2, T_2, \ldots, \) Jensen’s inequality yields \( \mathbb{E}[D_n | \mathcal{G}] = O_p(1) \) from the above. We proceed to show that \( D_n = O_p(1) \), yielding the first result. Let \( \nu > 0 \) be given. Then \( \mathbb{E}[D_n | \mathcal{G}] = O_p(1) \) says that there exist
\(N, M\) such that \(\mathbb{P}(E[D_n | G] > M) \leq \nu/2\) for all \(n \geq N\). Let \(M_0 = \max\{M, 2/\nu\}\) and observe that, for all \(n \geq N\),

\[
\mathbb{P}(D_n > M_0^2) = \mathbb{P}(D_n > M_0^2, E[D_n | G] > M_0) + \mathbb{P}(D_n > M_0^2, E[D_n | G] \leq M_0) \\
= \mathbb{P}(D_n > M_0^2, E[D_n | G] > M_0) + \mathbb{E}[\mathbb{P}(D_n > M_0 | G) \mathbb{I}[E[D_n | G] \leq M_0]] \\
\leq \nu/2 + \mathbb{E}\left[\frac{|E[D_n | G]|}{M_0^2}\right] \mathbb{I}[E[D_n | G] \leq M_0] \leq \nu/2 + 1/M_0 \leq \nu
\]

Now consider case (b). We first show that \(B_t(W_n^*, \pi; \mu_t) = o_p(1)\). Fix \(t \in [m]\) and \(\eta > 0, \nu > 0\). Because \(\mathfrak{B}_t(W_n^*, \pi_i; \| \kappa_i \|) = O_p(n^{-1/4}) = o_p(n^{-1/4})\) and \(\|W_n^*\|_2 = O_p(\sqrt{n})\), there are \(M, N\) such that for all \(n \geq N\) both \(\mathbb{P}(n^{1/4}\mathfrak{B}_t(W_n^*, \pi_i; \| \kappa_i \|) > \sqrt{\eta}) \leq \nu/3\) and \(\mathbb{P}(n^{-1/2}\|W_n^*\|_2 > M\sqrt{\eta}) \leq \nu/3\). Next, fix \(\tau = \sqrt{n\eta/3}/M\). By existence of second moment, there is \(g_0 = \sum^t \beta_i I_{S_i}\) with \((\mathbb{E}[(\mu_t(X) - g_0(X))^2])^{1/2} \leq \tau/2\) where \(I_{S}(x)\) are the simple functions \(I_{S}(x) = \mathbb{I}[x \in S]\) for \(S\) measurable. Let \(i = 1, \ldots, \ell\). Let \(U_i \supset S_i\) open and \(E_i \subset S_i\) compact be such that \(\mathbb{P}(X \in U_i \setminus E_i) \leq \tau^2/(4\ell |\beta_i|)^2\).

By Urysohn’s lemma [29], there exists a continuous function \(h_i\) with support \(C_i \subset U_i\) compact, \(0 \leq h_i \leq 1\), and \(h_i(x) = 1 \forall x \in E_i\). Therefore, \((\mathbb{E}[(I_{S_i}(X) - h_i)^2])^{1/2} = \mathbb{E}[(I_{S_i}(X) - h_i)^2]^{1/2} \leq \mathbb{P}(X \in U_i \setminus E_i))^{1/2} \leq \tau/(4\ell |\beta_i|)\). By \(C_0\)-universality, \(\exists g_i = \sum^n \alpha_j K(x_j, \cdot)\) such that \(\sup_{x \in X} |h_i(x) - g_i(x)| < \tau/(4\ell |\beta_i|)\). Because \(\mathbb{E}[(h_i - g_i)^2] \leq \sup_{x \in X} |h_i(x) - g_i(x)|^2\), we have \(\sqrt{\mathbb{E}[(I_{S_i}(X) - g_i)^2]} \leq \tau/(2\ell |\beta_i|)\). Let \(\tilde{\mu}_t = \sum^t \beta_i g_i\). Then \((\mathbb{E}[(\mu_t(X) - \tilde{\mu}_t)^2])^{1/2} \leq \tau/2 + \sum_{i=1}^\ell |\beta_i| \tau/(2\ell |\beta_i|) = \tau\). Let \(\delta_n = \sqrt{\frac{1}{n} \sum^n (\mu_t(X_i) - \tilde{\mu}_t(X_i))^2}\) so that \(\mathbb{E}\delta_n^2 \leq \tau^2\). Now, because we have

\[
B_t(W_n^*, \pi_i; \mu_t) = B_t(W_n^*, \pi; \tilde{\mu}_t) + B_t(W_n^*, \pi_i; \mu_t - \tilde{\mu}_t) \\
\leq \|\tilde{\mu}_t\|_{C_i} \mathfrak{B}_t(W_n^*, \pi_i; \| \kappa_i \|) + \sqrt{\frac{1}{n} \sum^n (W_n^* \delta_t - \pi_t(X_i))^2} \delta_n \\
\leq \|\tilde{\mu}_t\|_{C_i} \mathfrak{B}_t(W_n^*, \pi_i; \| \kappa_i \|) + (n^{-1/2}\|W_n^*\|_2 + 1)\delta_n,
\]

letting \(N' = \max\{N, 2\|\delta_t\|_{C_i}/\nu^2\}\), we must then have, for all \(n \geq N'\), by union bound and by Markov’s inequality, that

\[
\mathbb{P}(B_t(W_n^*, \pi; \mu_t) > \eta) \leq \mathbb{P}(n^{-1/4}\|\tilde{\mu}_t\|_{C_i} > \sqrt{\eta}) + \mathbb{P}(n^{1/4}\mathfrak{B}_t(W_n^*, \pi_i; \| \kappa_i \|) > \sqrt{\eta}) \\
+ \mathbb{P}(n^{-1/2}\|W_n^*\|_2 > M\sqrt{\eta}) \leq \nu/3 + \mathbb{P}(\delta_n > \sqrt{\eta}/M)
\]

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\[ \leq 0 + \nu/3 + \nu/3 + \nu/3 = \nu. \]

Following the same logic as in case (a), we get \( \text{CMSE}(\hat{\tau}_W \, \tau_n, \pi) = o_p(1) \), so letting \( D_n = |\hat{\tau}_W \, \tau_n - \text{SAPE}(\pi)| \) and \( \mathcal{G} \) be as before, we have \( \mathbb{E}[D_n | \mathcal{G}] = o_p(1) \) by Jensen’s inequality. Let \( \eta > 0, \nu > 0 \) be given. Let \( N \) be such that \( \mathbb{P}(\mathbb{E}[D_n | \mathcal{G}] > \nu \eta / 2) \leq \nu / 2 \). Then for all \( n \geq N \):

\[
\mathbb{P}(D_n > \eta) = \mathbb{P}(D_n > \eta, \mathbb{E}[D_n | \mathcal{G}] > \nu \eta / 2) + \mathbb{P}(D_n > \eta, \mathbb{E}[D_n | \mathcal{G}] \leq \nu \eta / 2)
\]

\[
= \mathbb{P}(D_n > \eta, \mathbb{E}[D_n | \mathcal{G}] > \nu \eta / 2) + \mathbb{E}[\mathbb{P}(D_n > \eta | \mathcal{G}) | \mathbb{E}[D_n | \mathcal{G}] \leq \nu \eta / 2]
\]

\[
\leq \nu / 2 + \mathbb{E}\left[ \frac{\mathbb{E}[D_n | \mathcal{G}]}{\eta} \right] \mathbb{E}[\mathbb{E}[D_n | \mathcal{G}] \leq \eta \nu / 2] \leq \nu / 2 + \nu / 2 \leq \nu,
\]

showing that \( D_n = o_p(1) \) and completing the proof.  \( \square \)

**Proof of Cor. 4.** Case (a) follows directly from the proof of Thm. 3 noting that the bias term now disappears at rate \( o_p(1)O_p(1/\sqrt{n}) = o_p(1/\sqrt{n}) \). For Case (b), observe that by Cauchy-Schwartz and Slutsky’s theorem \( |B_t(W, \tau_i; \mu - \hat{\mu}_n)| \leq (n^{-1/2}\|W_n\|_2 + 1)(\frac{1}{n} \sum_{i=1}^{n} (\hat{\mu}_n(X_i) - \mu(X_i))^2)^{1/2} = O_p(r_n^0) \). For cases in cases (c) and (d) we treat \( B_t(W, \tau_i; \mu - \hat{\mu}_n) \) as in the proof of Thm. 3 noting that \( \|\mu_i - \hat{\mu}_{nt}\|_{K_i} \leq \|\mu_i\|_{K_i} + \|\hat{\mu}_{nt}\|_{K_i} \) and that, in case (c), \( \|\hat{\mu}_{nt}\|_{K_i} = O_p(1) \) implies by Markov’s inequality that \( \|\hat{\mu}_{nt}\|_{K_i} = O_p(1) \). The rest follows as in the proof of Thm. 3.  \( \square \)

**Proof of Thm. 5.** First note that because our problem is a quadratic program, the KKT conditions are necessary and sufficient and we can always choose an optimizer where strict complementary slackness holds.

Ignore previous definitions of some symbols, consider any linearly constrained parametric nonlinear optimization problem in standard form: \( z(x) \in \arg\min_{y \geq 0, By = b} f(x, y) \) where \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \), and \( b \in \mathbb{R}^l \). KKT says there exist \( \mu(x) \in \mathbb{R}^m \), \( \lambda(x) \in \mathbb{R}^l \) such that

- (a) \( \nabla_y f(x, z(x)) = 0 \),
- (b) \( Bz(x) = b \),
- (c) \( z(x) \geq 0 \),
- (d) \( \mu(x) \geq 0 \), and
- (e) \( \mu(x) \odot z(x) = 0 \), where \( \odot \) is the Hadamard product. Suppose strict complementary slackness holds in that (f) \( \mu(x) + z(x) > 0 \). By (a), we have that

\[
\nabla_{xy} f(x, z(x)) + \nabla_{yy} f(x, z(x)) \nabla_y z(x) = \nabla_x \mu(x) + B^T \nabla_x \lambda(x),
\]

and hence, letting \( H = \nabla_{yy} f(x, z(x)) \) and \( J = \nabla_{xy} f(x, z(x)) \),

\[
\nabla z(x) = H^{-1}(\nabla_x \mu(x) + B^T \nabla_x \lambda(x) - J).
\]
By (b), we have that \( B \nabla z(x) = 0 \) so that
\[
BH^{-1} \nabla_x \mu(x) + BH^{-1} B^T \nabla \lambda = BH^{-1} J,
\]
and hence if the columns of \( F \) form a basis for the null space of \( B \) and \( \tilde{H} = -F(F^T HF)^{-1} F^T \),
\[
\nabla_x z(x) = (H^{-1} B^T (AH^{-1} A^T)^{-1} AH^{-1} - H^{-1})(J - \nabla_x \mu(x)) = \tilde{H}(J - \nabla_x \mu(x)).
\]
By (e), we have that
\[
z_i(x) \nabla_x \mu_i(x) + \mu_i(x) \nabla_x z_i(x) = 0,
\]
and then by (f), letting \( A = \text{diag}(\mathbb{I}[z_1(x) > 0, \ldots, z_m(x) > 0]) \) we have
\[
A \nabla_x \mu(x) = 0, \quad (I - A) \nabla_x z(x) = 0,
\]
and therefore
\[
A \nabla_x \mu(x) - (I - A) \tilde{H}(J - \nabla_x \mu(x)) = 0
\]
yielding finally that
\[
\nabla_x z(x) = \tilde{H}(I - (A + (I - A) \tilde{H})^{-1}(I - A) \tilde{H}) J.
\]
The rest of the theorem is then begged by applying this result and using chain rule. \( \square \)

**Proof of Thm. 6.** Let \( Z(\pi) = \frac{1}{n} \sum_{i=1}^{n} \pi_{T_i}(X_i)/\varphi_{T_i}(X_i) \) and \( \tilde{W}_i(\pi) = \frac{1}{Z(\pi)} \pi_{T_i}(X_i)/\varphi_{T_i}(X_i) \) and note that \( \tilde{W} \in \mathcal{W} \). Moreover, note that
\[
\sup_{\pi \in \Pi} \mathcal{B}_1(\tilde{W}_i, \pi; \| \cdot \|_{K_1}) = \sup_{\pi \in \Pi, \| f_i \|_{K_1} \leq 1} \frac{1}{Z(\pi)} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\delta \pi_{T_i}}{\varphi_{T_i}(X_i)} - Z(\pi) \right) \pi_{T_i}(X_i) f_i(X_i)
\]< (\sup_{\pi \in \Pi} Z(\pi))^{-1} \sup_{\pi \in \Pi, \| f_i \|_{K_1} \leq 1} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\delta \pi_{T_i}}{\varphi_{T_i}(X_i)} - 1 \right) \pi_{T_i}(X_i) f_i(X_i) + \Gamma \sup_{\pi \in \Pi} |1 - Z(\pi)|^{-1}.
\]
We first treat the random variable
\[
\Xi_t(X_{1:n}, T_{1:n}) = \sup_{\pi_t \in \Pi_t, \| f_t \|_{K_1} \leq 1} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\delta \pi_t}{\varphi_{T_i}(X_i)} - 1 \right) \pi_t(X_i) f_t(X_i).
\]
Fix \( x_{1:n}, t_{1:n}, x'_{1:n}, t'_{1:n} \) such that \( x'_i = x_i, t'_i = t_i \forall i \neq i' \) and note that
\[
\Xi_t(x_{1:n}, t_{1:n}) - \Xi_t(x'_{1:n}, t'_{1:n}) \leq \sup_{\pi_t \in \Pi_t, \| f_t \|_{K_1} \leq 1} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\delta \pi_t}{\varphi_{T_i}(x_i)} - 1 \right) \pi_t(x_i) f_t(x_i) \right)
\]

By McDiarmid’s inequality, \( P(\Xi(X_{1:n}, T_{1:n}) \geq \Xi(\Xi(X_{1:n}, T_{1:n}) + \eta) \leq e^{-n\eta^2\alpha^2\Gamma^{-2}/2} \). Let

\[ 
\xi_i(\tau_i, f_i) = (\frac{\delta_{\tau_i,t}}{\varphi_t(x_i)} - 1)\pi_i(x_i) f_i(x_i) + \delta_{\tau_i,t} \varphi_t(x_i) \]

and note that for all \( \tau_i, f_i \) we have \( E[\xi_i(\tau_i, f_i)] = E[(\frac{\delta_{\tau_i,t}}{\varphi_t(X_i)} - 1)\pi_i(X_i) f_i(X_i)] = 0 \) and that \( \xi_1(\cdot, \cdot), \xi_2(\cdot, \cdot), \ldots \) are iid. Therefore, letting \( \xi_1^*(\cdot, \cdot), \xi_2^*(\cdot, \cdot), \ldots \) be iid replicates of \( \xi_1(\cdot, \cdot), \xi_2(\cdot, \cdot), \ldots \) (ghost sample) and letting \( \rho_i \) be iid Rademacher random variables independent of all else, we have

\[
E[\Xi(X_{1:n}, T_{1:n})] = E[\sup_{\pi_i \in \Pi, \|\tau_i\|_{\mathcal{K}_i} \leq 1} \frac{1}{n} \sum_{i=1}^{n} (E[\xi_i(\tau_i, f_i)] - \xi_i(\pi_i, f_i))]
\leq E[\sup_{\pi_i \in \Pi, \|\tau_i\|_{\mathcal{K}_i} \leq 1} \frac{1}{n} \sum_{i=1}^{n} (\xi_i(\tau_i, f_i) - \xi_i(\pi_i, f_i))]
= E[\sup_{\pi_i \in \Pi, \|\tau_i\|_{\mathcal{K}_i} \leq 1} \frac{1}{n} \sum_{i=1}^{n} \rho_i \xi_i(\tau_i, f_i) - \xi_i(\pi_i, f_i))]
\leq 2 E[\sup_{\pi_i \in \Pi, \|\tau_i\|_{\mathcal{K}_i} \leq 1} \frac{1}{n} \sum_{i=1}^{n} \rho_i \xi_i(\tau_i, f_i)].
\]

Note that by bounded kernel we have \( \|\mathcal{K}_i(x, \cdot)\|_{\mathcal{K}_i} = \sqrt{\mathcal{K}_i(x, x)} \leq \Gamma \) and therefore

\[ 
\sup_{\|\tau_i\|_{\mathcal{K}_i} \leq 1, x \in \mathcal{X}} f_i(x) = \sup_{\|\tau_i\|_{\mathcal{K}_i} \leq 1, x \in \mathcal{X}} (f_i, \mathcal{K}_i(x, \cdot)) \leq \sup_{\|\tau_i\|_{\mathcal{K}_i} \leq \Gamma, \|\tau_i\|_{\mathcal{K}_i} \leq \Gamma} (f_i, g) = \Gamma.
\]

As before, \( \|\xi_1 - \xi_2\|_{\mathcal{K}_i}^2 + \|\xi_1 + \xi_2\|_{\mathcal{K}_i}^2 = 2\|\xi_1\|_{\mathcal{K}_i}^2 + 2\|\xi_2\|_{\mathcal{K}_i}^2 \) implies by induction that \( \sum_{i=1}^{n} \rho_i \xi_i \|\xi_i\|_{\mathcal{K}_i}^2 = 2^n \sum_{i=1}^{n} \|\xi_i\|_{\mathcal{K}_i}^2 \). Hence,

\[
E[\frac{1}{n} \sum_{i=1}^{n} \rho_i E_{X_i}], \|\xi_i\|_{\mathcal{K}_i}^2 \leq (\frac{1}{n} \sum_{i=1}^{n} \rho_i E_{X_i}], \|\xi_i\|_{\mathcal{K}_i}^1 \leq \frac{E[\frac{1}{n} \sum_{i=1}^{n} \rho_i E_{X_i}], \|\xi_i\|_{\mathcal{K}_i}^2]^{1/2}}{\Gamma} \leq \frac{\Gamma}{\sqrt{n}}.
\]

Note that \( |\frac{\delta_{\tau_i,t}}{\varphi_t(X_i)} - 1| \leq \alpha \), that \( x^2 \) is \( 2b \)-Lipschitz on \([-b, b]\), and that \( ab = \frac{1}{2}((a+b)^2 - a^2 - b^2) \).

Therefore, by the Rademacher comparison lemma [22, Thm. 4.12], we have

\[
E[\Xi(X_{1:n}, T_{1:n})] \leq 2\alpha E[\sup_{\pi_i \in \Pi, \|\tau_i\|_{\mathcal{K}_i} \leq 1} \frac{1}{n} \sum_{i=1}^{n} \rho_i \pi_i(X_i) f_i(X_i)]
\leq \alpha E[\sup_{\pi_i \in \Pi, \|\tau_i\|_{\mathcal{K}_i} \leq 1} \frac{1}{n} \sum_{i=1}^{n} \rho_i \pi_i(X_i) + f_i(X_i))^2]
\]

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With a symmetric argument, letting 

\[
\omega_t(\pi_i) = (\delta_{T_i,t}/\varphi_{T_i,t} - 1)\pi_i(X_i) \quad \text{and} \quad \Omega_t(X_{1:n}, T_{1:n}) = \sup_{\pi_i \in \Pi} \frac{1}{n} \sum_{i=1}^{n} \omega_t(\pi_i).
\]

Note that \(\sup_{\pi \in \Omega}(Z(\pi) - 1) \leq \sum_{i=1}^{m} \Omega_t(X_{1:n}, T_{1:n})\). Fix \(x_{1:n}, t_{1:n}, x_{1:n}', t_{1:n}'\) such that \(x_i' = x_i, t_i' = t_i \forall i \neq i'\) and note that

\[
\Omega_t(x_{1:n}, t_{1:n}) - \Omega_t(x_{1:n}', t_{1:n}') \leq \frac{1}{n} \sup_{\pi_i \in \Pi} \left( \frac{\delta_{T_i,t}}{\varphi_{T_i,t} - 1} \pi_i(x_i') - \frac{\delta_{T_i,t}}{\varphi_{T_i,t}} \pi_i(x_i) \right) \leq \frac{2}{n} \alpha
\]

By McDiarmid’s inequality, \(\mathbb{P}(\Omega_t(X_{1:n}, T_{1:n}) \geq \mathbb{E}[\Omega_t(X_{1:n}, T_{1:n})] + \eta) \leq e^{-\eta^2/n^2/2}\). Note that \(\mathbb{E}[\omega_t(\pi_t)] = 0\) for all \(\pi_t\) and that \(\omega_1(\cdot), \omega_2(\cdot), \ldots\) are iid. Using the same argument as before, letting \(\rho_i\) be iid Rademacher random variables independent of all else, we have

\[
\mathbb{E}[\Omega_t(X_{1:n}, T_{1:n})] \leq 2\mathbb{E}[\sup_{\pi_i \in \Pi} \frac{1}{n} \sum_{i=1}^{n} \rho_i \omega_t(\pi_i)] \leq 2\alpha \mathfrak{R}_n(\Pi).
\]

With a symmetric argument, letting \(\delta = 3m\nu/(3m + 2)\), with probability at least \(1 - 2\delta/3\), we have \(\sup_{\pi \in \Pi} |1 - Z(\pi)| \leq 2\alpha \mathfrak{R}_n(\Pi) + \alpha \sqrt{2 \log(3m/\delta)/n} \leq 2\alpha \mathfrak{R}_n(\Pi) + \alpha \sqrt{2 \log(4m/\nu)/n} \leq 1/2\).

Since \(\|\tilde{W}\|_2 \leq \sqrt{n} \alpha / Z(\pi)\), we get that, with probability at least \(1 - \delta\), both \(\sup_{\pi \in \Pi} \|\tilde{W}\|_2 \leq 2\alpha \sqrt{n}\) and for all \(t \in [m]\)

\[
\sup_{\pi \in \Pi} \mathfrak{B}_t(\tilde{W}, \pi, \| \cdot \|_{K_t}) \leq \alpha \Gamma(12 \mathfrak{R}_n(\Pi_i) + 2 \mathfrak{M}_n(\Pi) + 12\Gamma/\sqrt{n} + 3\sqrt{2 \log(3m/\delta)/n}).
\]

Therefore, with probability at least \(1 - \delta\), using twice that \(\ell_1\) is the biggest \(p\)-norm,

\[
\mathcal{E} = \sup_{\pi \in \Pi} \mathcal{E}(W_{n, \pi}, \pi; \| \cdot \|_{p,K_{1:n,1:m}}, \Lambda_n) \leq \sup_{\pi \in \Pi} \mathcal{E}(\tilde{W}, \pi; \| \cdot \|_{p,K_{1:n,1:m}}, \Lambda_n)
\]

\[
\leq \sum_{t=1}^{m} \gamma_t \sup_{\pi \in \Pi} \mathfrak{B}_t(\tilde{W}, \pi; \| \cdot \|_{K_t}) + \frac{\pi}{n} \sup_{\pi \in \Pi} \|\tilde{W}\|_2
\]

\[
\leq 8\alpha \Gamma m \mathfrak{R}_n(\Pi) + \frac{2\alpha \Gamma + 12\alpha \Gamma^2 m + 3\alpha \Gamma \gamma m \sqrt{2 \log(3m/\delta)} \sqrt{n}}{n}.
\]
Consider case (a). Note that \( \sup_{\pi \in \Pi} \sum_{t=1}^m |B_t(W^*_n, \pi; \mu_t)| \leq \|\mu\|_E \) and \( \sup_{\pi \in \Pi} \|W^*_n\|_2 \leq \log^{-1} E \). Since \( E[\sum_{t=1}^m W_t \epsilon_t | X_{1:n}, T_{1:n}] = 0, \epsilon_t \in [-B, B] \) and \( W_t \epsilon'_t - W_t \epsilon''_t \leq 2BW_t \) for \( \epsilon'_t, \epsilon''_t \in [-B, B] \), by McDiarmid’s inequality (conditional on \( X_{1:n}, T_{1:n} \)), we have that with probability at least \( 1 - \delta' \), \( |\sum_{t=1}^m W^*_n \epsilon_t| \leq \|W^*_n\|_2 B \sqrt{2 \log(2/\delta')} \). Therefore, letting \( \delta' = 2\nu/(3m + 2) \) so that \( 3m/\delta = 2/\delta' = (3m + 2)/\nu \leq 4m/\nu \), with probability at least \( 1 - \nu \), we have

\[
\sup_{\pi \in \Pi} |\pi_{W^*_n} - \text{SAPE}(\pi)| \leq 8\alpha \Gamma m(\|\mu\| + \sqrt{2 \log(4m/\nu) m^{-1} B}) \mathcal{R}_n(\Pi)
+ \frac{2\alpha \pi \|\mu\| + 12\alpha \Gamma^2 \pi m \|\mu\|}{\sqrt{n}}
+ \frac{(2\alpha \pi \log^{-1} B + 12\alpha \Gamma^3 \pi m \log^{-1} B + 3\alpha \Gamma \pi m \|\mu\|) \sqrt{2 \log(4m/\nu)}}{\sqrt{n}}
+ \frac{6\alpha \Gamma \pi \log^{-1} B \log(4m/\nu)}{\sqrt{n}}.
\]

This gives the first result in case (a). The second is given by noting that, by McDiarmid’s inequality, with probability at least \( 1 - \nu/(4m) \), \( \mathcal{R}_n(\Pi_t) \leq \hat{\mathcal{R}}_n(\Pi_t) + 4 \sqrt{2 \log(4m/\nu)} \). Case (b) is given by following a similar argument as in the proof of Thm. 3(b).

**Proof of Cor. 7.** These results follow directly from the proof of Thm. 6, the convergence in particular of \( E^2(W^*(\pi), \pi; \| \cdot \|, \Lambda) \), the decomposition of the DR estimator in Thm. 1, and a standard Rademacher complexity argument concentrating \( \text{SAPE}(\pi) \) uniformly around \( \text{PAPE}(\pi) \).

**Proof of Lemma 2.** Let \( \mathcal{G}_n(\Pi_t) = \sup_{x_1, \ldots, x_n \in X} |\{ (\pi_1(x_1), \ldots, \pi_t(x_n)) : \pi \in \Pi_t \}| \) be the growth function. By Sauer’s lemma, \( \mathcal{G}_n(\Pi_t) \leq en^{ht} \). By Massart’s lemma, \( \hat{\mathcal{R}}_n(\Pi_t) \leq \sqrt{2 \log \mathcal{G}_n(\Pi_t)/n} \leq \sqrt{2 + h_t \log n}/\sqrt{n} \). By Jensen’s inequality, \( \hat{\mathcal{R}}_n(\Pi) = \sum_{t=1}^m \hat{\mathcal{R}}_n(\Pi_t) \leq (\sqrt{2m} + \sum_{t=1}^m \sqrt{h_t \log n})/\sqrt{n} \), completing the proof.

### B IPW and DR weight SVM details

To reduce training a deterministic linear policy using IPW evaluation to weighted SVM classification, we add multiples of \( \sum_{i=1}^n \pi_{T_i}(X_i)/\hat{\phi}_{T_i}(X_i) \) (1 in expectation) and note that

\[
\frac{1}{B}(\hat{\pi}_{\text{IPW}}(\pi) - C \sum_{i=1}^n \frac{\pi_{T_i}(X_i)}{\hat{\phi}_{T_i}(X_i)} - \sum_{i=1}^n \frac{Y_i - C}{\hat{\phi}_{T_i}(X_i)}) = \sum_{i=1}^n \frac{C - Y_i}{B \hat{\phi}_{T_i}(X_i)} (1 - \pi_{T_i}(X_i))
\]
\[
= \sum_{i=1}^{n} \frac{C - Y_i}{B \phi_{T_i}(X_i)} \mathbb{I}[T_i \neq \tilde{T}_{\pi}(X_i)].
\]

Choosing \( C \) sufficiently large so that all coefficients are nonnegative and choosing \( B \) so that all coefficients are in \([0, 1]\), we replace the indicators \( \mathbb{I}[T_i \neq \tilde{T}_{\pi}(X_i)] \) with their convex envelope hinges to come up with a weighted version of Crammer and Singer [10]'s multiclass SVM.

For the DR version, we replace \( \pi(X_i) \) with \( \mathbb{I}[t = \tilde{T}_{\pi}(X_i)] \) and we do the above with but using \( \hat{\epsilon}_i \) and also add multiples of \( \tau^\text{direct}(1(\cdot)) = \sum_{i=1}^{n} \sum_{t=1}^{m} \mu_t(X_i) \) to make all indicators be 0-1 loss and have nonnegative coefficients. Replacing indicators with hinge functions, we get a weighted multiclass SVM with different weights for each observation and each error type.