QUASILINEAR NONLOCAL ELLIPTIC PROBLEMS WITH VARIABLE SINGULAR EXPONENT

PRASHANTA GARAIN
Department of Mathematics and Systems Analysis, Aalto University, Espoo - 02150, Finland.

TUHINA MUKHERJEE
Department of Mathematics, National Institute of Technology Warangal, Warangal-506004, India.

(Communicated by Changfeng Gui)

Abstract. In this article, we provide existence results to the following nonlocal equation
\[
\begin{aligned}
(-\Delta)^s_p u &= g(x,u), \quad u > 0 \text{ in } \Omega, \\
\end{aligned}
\] (P_λ)
\[
\begin{aligned}
u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]
where \((-\Delta)^s_p\) is the fractional p-Laplacian operator. Here \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain, \(s \in (0, 1), p > 1\) and \(N > sp\). We establish existence of at least one weak solution for \((P_\lambda)\) when \(g(x,u) = f(x)u^{-q(x)}\) and existence of at least two weak solutions when \(g(x,u) = \lambda u^{-q(x)} + u^r\) for a suitable range of \(\lambda > 0\). Here \(r \in (p-1, p^*_s - 1)\) where \(p^*_s\) is the critical Sobolev exponent and \(0 < q \in C^1(\overline{\Omega})\).

1. Introduction. In this article, we are interested in the existence result of the following singular problem
\[
\begin{aligned}
(-\Delta)^s_p u &= g(x,u) \text{ in } \Omega; \quad u > 0 \text{ in } \Omega; \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\] (1.1)
where \((-\Delta)^s_p\) is the fractional p-Laplacian operator defined by
\[
\begin{aligned}
(-\Delta)^s_p u(x) := 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} dy, \quad x \in \mathbb{R}^N.
\end{aligned}
\]
Here \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain, \(s \in (0, 1), p > 1\) and \(N > sp\). We consider the following two type of nonlinearities:

Case (I) \(g(x,u) = f(x)u^{-q(x)}\), and
Case (II) \(g(x,u) = \lambda f(x)u^{-q(x)} + u^r\).

Throughout the paper, we assume \(\lambda > 0\), \(q \in C^1(\overline{\Omega})\) and \(f \in L^m(\Omega)\) \((m > 1\) is appropriately chosen in each case (I) and (II)) both being positive and \(p-1 < r < p^*_s - 1\). The problem (1.1) is singular in nature due to the fact that the nonlinearity \(g(x,t)\) in both Case (I) and Case (II) blows up as \(t \to 0^+\). The novelty in our work relies on the fact that the exponent \(q\) is variable and we provide sufficient conditions

2020 Mathematics Subject Classification. Primary: 35J35, 35J60; Secondary: 35J92.

Key words and phrases. Fractional p-Laplacian, singular nonlinearity, multiple weak solutions, variational method, variable exponent.

* Corresponding author.
on $q$ to assure the existence of at least one solution in Case (I) and at least two solutions in Case (II). Additionally, we also establish a regularity result in Case (I).

Before proceeding to state our main results, let us briefly mention the state of the art concerning singular elliptic problems both in the local and nonlocal case.

Under the hypothesis that $q(x) \equiv q$, singular problems with constant exponent has been investigated widely both in the local and nonlocal case. Starting point of study of singular problems is the pioneering work of Crandall et al. [13]. Let us briefly discuss the local case first. Consider the following $p$-Laplace equation

$$-\Delta_p u = \lambda \frac{f(x)}{u^q} + \mu u^r \text{ in } \Omega; \ u > 0 \text{ in } \Omega; \ u = 0 \text{ on } \partial \Omega. \quad (1.2)$$

For $p = 2$, $\mu = 0$ and $f(x) \equiv 1$, existence of a classical solution to the problem (1.2) is proved in [13] for any $q > 0$. Later for certain restricted range of $q$, the existence of weak solution was proved in [25]. This restriction on $q$ was removed in [8] to obtain existence of at least one weak solution. Indeed the authors in [8] proved the existence of solution in $H_0^1(\Omega)$ for $0 < q \leq 1$ and in $H_{loc}^1(\Omega)$ for $q > 1$. The case of $p \in (1, \infty)$ was settled in [11], where existence of weak solution in $W_0^{1,p}(\Omega)$ was proved for $0 < q \leq 1$ and in $W_{loc}^{1,p}(\Omega)$ for $q > 1$.

On the other hand, for $p = 2$, $f(x) \equiv \mu = 1$ and $1 < r \leq 2^* = \frac{2N}{N-2}$, multiplicity of weak solutions was established using Nehari manifold and sub super solution techniques in [21, 22] for $0 < q < 1$. Whereas the case of any $q > 0$ was settled in [4, 23]. In the nonlinear case that is for $p \in (1, \infty)$, authors in [20] answered the question of existence, multiplicity and regularity of weak solutions in the case of $0 < q < 1$ which was further extended to the case of $q \geq 1$ in [6] deducing the multiplicity of weak solutions. Recently for the weighted $p$-Laplace operator with Muckenhoupt class of weights existence and multiplicity is deduced in [16, 17].

In the nonlocal case, after immense activeness in the study of elliptic problems involving the fractional Laplace operator by researchers, it was a natural question to study the nonlocal singular problems. In this context, authors in [7] studied the following problem

$$(-\Delta)^s u = \lambda \frac{f(x)}{u^\gamma} + Mu^p, \ u > 0 \text{ in } \Omega, \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

where $N > 2s$, $M \geq 0$, $0 < s < 1$, $\gamma$, $\lambda > 0$, $1 < p < 2^*_s - 1$, $2^*_s = 2N/(N - 2s)$, $f \in L^m(\Omega)$ for $m \geq 1$ is a non-negative function and

$$(-\Delta)^s u(x) = -\frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{N+2s}} \, dy, \text{ for all } x \in \mathbb{R}^N.$$

Here authors studied the existence of distributional solutions for small $\lambda$ using the uniform estimates of $\{u_n\}$ which are solutions of the regularized problems with singular term $u^{-\gamma}$ replaced by $(u + \frac{1}{n})^{-\gamma}$. This was extended for the $p$-fractional Laplace operator by Canino et al. in [10]. In the critical case for $0 < q < 1$, the question of existence and multiplicity of weak solutions to nonlocal singular problems has been answered in [19, 27, 28] whereas $q \geq 1$ case has been dealt in [18]. We also refer a recent article [5] related to nonlocal singular problem with exponential nonlinearity. Moreover, we refer readers to [1, 2] concerning the existence and multiplicity results for the fractional $p$-Laplacian problems.
Naturally, now a question arises that what is the result when \( q \) is a function depending on \( x \). In this direction, for the following model problem
\[
- \Delta u = \frac{f(x)}{u^q(x)} \quad \text{in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega
\]  
(1.3)
existence of at least one weak solution was proved in [12]. Motivated by this and the current interest in the study of nonlocal problems involving fractional Laplacian, we study the singular problem (1.1) where the singular exponent is a function of \( x \). The case (I) is treated using the solutions to an approximated nonsingular formulation of (1.1) whereas case (II) has been dealt using the critical point theory and Mountain pass Lemma. We have also established that every weak solution is bounded which contributes to the regularity part. This article is divided into 5 sections- Section 2 contains the preliminaries and main results. We prove the existence result for (1.1) in case (I) in Section 3. The regularity of such weak solutions has been proved in Section 4. Finally, the existence and multiplicity of weak solution to (1.1) in case (II) have been established in Section 5.

2. Preliminaries and main results. To study the concerned problem (1.1) and give it a variational characterization, for any \( s \in (0, 1) \) and \( p \in (1, \infty) \), we consider the fractional Sobolev spaces
\[
W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy < +\infty \right\}
\]
endowed with the norm \( \| \cdot \|_{L^p(\mathbb{R}^N)} + [\cdot]_{s,p} \). Now for \( \Omega \subset \mathbb{R}^N \) which is smooth and bounded domain, we define the spaces
\[
W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy < +\infty \right\}
\]
endowed with the norm
\[
\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left\| \frac{u(x) - u(y)}{|x-y|^{N+sp}} \right\|_{L^p(\Omega \times \Omega)}.
\]
Moreover, we define the space
\[
W^{s,p}_0(\Omega) := \{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}
\]
endowed with the norm \( [\cdot]_{s,p} \) which we denote by \( \| \cdot \|_{W^{s,p}_0(\Omega)} \). Both the spaces \( W^{s,p}(\Omega) \) and \( W^{s,p}_0(\Omega) \) are reflexive Banach spaces. We have the following useful embedding result.

**Lemma 2.1** (Embedding). *The inclusion map*
\[
W^{s,p}_0(\Omega) \hookrightarrow L^r(\Omega)
\]
*is continuous for* \( 1 \leq r \leq p^*_s = \frac{Np}{N-sp} \), *provided* \( N > sp \). *Moreover, the above embedding is compact except for* \( r = p^*_s \).

For more detailed discussion on such spaces, we refer the interested reader to the article [15].

**Definition 2.2.** We say that \( u \in W^{s,p}_0(\Omega) \) is a weak solution of (1.1) if \( u > 0 \) in \( \Omega \) and for all \( \phi \in C_c^\infty(\Omega) \), one has
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x-y|^{N+sp}} \, dx \, dy = \int_{\Omega} g(x,u)\phi \, dx. \quad (2.1)
\]
Moreover we say a function \( u \in W_{0}^{s,p}(\Omega) \) is a subsolution (or supersolution) of (1.1) if
\[
\int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} |x-y|^{N+sp} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y)) \, dx \, dy \leq (or \geq) \int_{\Omega} g(x,u)\phi \, dx \tag{2.2}
\]
for every \( 0 \leq \phi \in C_{c}^{\infty}(\Omega) \) respectively. Therefore every weak solution is both a sub and supersolution. We shall use the following notations throughout the article.

\[
X = W_{0}^{s,p}(\Omega), \quad X_{+} = \{ u \in X : u \geq 0 \text{ a.e. in } \Omega \}, \quad v^{+}(x) = \max\{v(x),0\}, \quad v^{-}(x) = \max\{-v(x),0\}, \quad |S| = \text{Lebesgue measure of } S \text{ and } t' = \frac{1}{t^2-1} \text{ for } t > 1.
\]

We have the following property of weak solutions.

**Lemma 2.3.** If \( u \in X \) is a weak solution of (1.1), then the equality (2.1) holds for every \( \phi \in X \).

**Proof.** Following the proof of [22, Lemma A.1], we get for any \( \phi \in X_{+}, \) there exists a sequence \( \{v_{n}\} \subset W_{0}^{s,p}(\Omega) \) such that each \( v_{n} \) has a compact support in \( \Omega, \) \( 0 \leq v_{1} \leq v_{2} \leq \ldots \) and \( \{v_{n}\} \) converges strongly to \( v \) in \( W_{0}^{s,p}(\Omega) \). Now arguing similarly as in [22, Lemma 9] we get the result. \( \square \)

For a given \( \delta > 0, \) we denote by
\[
\Omega_{\delta} := \{ x \in \Omega : \text{dist}(x,\partial \Omega) < \delta \}.
\]

Our main results in this paper reads as follows:

**Theorem 2.4.** Assume that there exists a \( \delta > 0 \) such that \( q(x) \leq 1 \) in \( \Omega_{\delta} \) and let \( f \in L^{m}(\Omega) \) for \( m = (p_{s}^{\ast})' \). Then the problem (1.1) admits a weak solution \( u \in X \) in Case (I).

**Theorem 2.5.** Assume that there exists a \( q^{\ast} > 1 \) and \( \delta > 0 \) such that \( ||q||_{L^{\infty}(\Omega)} \leq q^{\ast} \). Then the problem (1.1) admits a weak solution \( u \in W_{0}^{s,p}(\Omega) \) such that \( u = \frac{u^{n}}{u + \frac{1}{n}q(x)} \in X, \) provided \( f \in L^{m}(\Omega) \) for \( m = (q^{\ast} + p - 1)p_{s}^{\ast})' \) in Case (I).

**Theorem 2.6.** Let \( 0 < q(x) < 1 \) for \( x \in \Omega \). Then there exists a \( \Lambda > 0 \) such that for all \( \lambda \in (0,\Lambda) \), the problem (1.1) admits at least two distinct weak solutions in \( X \) in Case (II).

3. **Existence results in case (I).** To prove our main results we make use of the following approximated problem:
\[
(-\Delta)_{s}^{u} = \frac{f_{n}(x)}{(u + \frac{1}{n})^{q}(x)} \quad \text{in } \Omega, \quad u > 0 \in \Omega, \quad u = 0 \in \mathbb{R}^{N} \setminus \Omega, \tag{3.1}
\]
where \( f_{n}(x) = \min\{f(x),n\} \) for \( n \in \mathbb{N} \) and \( q \in C^{1}(\bar{\Omega}) \) is positive.

**Lemma 3.1.** For each \( n \geq 1, \) there exists a weak solution \( u_{n} \in X \cap L^{\infty}(\Omega) \) of the problem (3.1).

**Proof.** Since \( \frac{f_{n}(x)}{(u^{+} + \frac{1}{n})^{q}(x)} \in L^{\infty}(\Omega) \) for each fixed \( n \geq 1, \) by [10, Lemma 2.1] there exists a unique solution \( w \in X \) to the problem
\[
(-\Delta)_{s}^{u} w = \frac{f_{n}(x)}{(u^{+} + \frac{1}{n})^{q}(x)} \quad \text{in } \Omega, \quad w > 0 \in \Omega, \quad w = 0 \in \mathbb{R}^{N} \setminus \Omega. \tag{3.2}
\]
Therefore we can define the operator \( S : X \to X \) by \( S(u) = w \) where \( w \) satisfies (3.2). Since \( q \in C^{1}(\Omega) \) is positive, proceeding as in the proof [10, Proposition 2.3],
it follows that the operator $S$ is both continuous and compact. Choosing $w$ as a test function in (3.2) and using the embedding result, Lemma 2.1 we get
\[
\|w\|_X \leq C n_{\frac{q(\Omega)(p+1)-1}{q-1}},
\]
for some constant $C = C(p, s, N, \Omega)$ (independent of $u$). Therefore, by Schauder fixed point theorem we get the existence of a solution $u_n$ of the problem (3.1). Since the R.H.S of (3.2) belong to $L^\infty(\Omega)$, by [10, Lemma 2.2], we get $u_n \in L^\infty(\Omega)$. \qed

**Lemma 3.2.** The sequence $\{u_n\}_{n \in \mathbb{N}}$ found in Lemma 3.1 satisfies
\[
u_n(x) \leq u_{n+1}(x), \text{ for a.e. } x \in \Omega,
\]
and
\[
u_n(x) \geq l > 0, \text{ for a.e. } x \in K \Subset \Omega \text{ and for all } n \in \mathbb{N}
\]
where $l = l(K) > 0$ is a constant.

**Proof.** We observe that
\[
\int_{\Omega} \left\{ \frac{f_n(x)}{(u_n + \frac{1}{n})^q(x)} - \frac{f_{n+1}(x)}{(u_{n+1} + \frac{1}{n+1})^q(x)} \right\} (u_n - u_{n+1})^+ dx 
\]
\[
\leq \int_{\Omega} f_{n+1}(x) \frac{(u_{n+1} + \frac{1}{n+1})^q(x) - (u_n + \frac{1}{n})^q(x)}{(u_n + \frac{1}{n})^q(x)(u_{n+1} + \frac{1}{n+1})^q(x)} (u_n - u_{n+1})^+ dx \leq 0.
\]
Now following the arguments of [10, Lemma 2.4], the lemma follows. \qed

Next, we present the proof of our first main theorem.

**Proof of Theorem 2.4:** By Lemma 3.1, choosing $u_n$ as a test function in (3.1), we get
\[
\int_{\mathbb{R}^2N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} dxdy \leq \int_{\Omega} \frac{f_n(x)u_n}{(u_n + \frac{1}{n})^q(x)} dx.
\]
Now denote by $\omega_\delta = \Omega \setminus \Omega_\delta$, then by Lemma 3.2, we get $u_n \geq l > 0$ in $\omega_\delta$. Observe that
\[
\int_{\Omega} \frac{f_n(x)u_n}{(u_n + \frac{1}{n})^q(x)} dx = \int_{\Omega_\delta} \frac{f_n(x)u_n}{(u_n + \frac{1}{n})^q(x)} dx + \int_{\omega_\delta} \frac{f_n(x)u_n}{(u_n + \frac{1}{n})^q(x)} dx
\]
\[
\leq \int_{\Omega_\delta} f(x)u_n^{1-q(x)} dx + \int_{\omega_\delta} \frac{f(x)}{q(x)} u_n dx
\]
\[
\leq \int_{\Omega_\delta \cap \{u_n \leq l\}} f(x) dx + \int_{\Omega_\delta \cap \{u_n \geq l\}} f(x)u_n dx + \int_{\omega_\delta} \frac{f(x)}{q(x)} u_n dx
\]
\[
\leq \|f\|_{L^1(\Omega)} + (1 + \|l^{-q(x)}\|_{L^\infty(\Omega)}) \int_{\Omega} f(x)u_n dx.
\]
Using Hölder’s inequality and Lemma 2.1 we obtain
\[
\|u_n\|_X^p \leq \|f\|_{L^1(\Omega)} + C \|f\|_{L^\infty(\Omega)} \|u_n\|_X,
\]
which implies that the sequence $\{u_n\}$ is uniformly bounded in $X$. Thus up to a subsequence, by Lemma 2.1, we get $u_n \rightharpoonup u$ weakly in $X$, $u_n \to u$ strongly in $L^r(\Omega)$ for $1 \leq r < p^*_s$ and $u_n \to u$ pointwise a.e. in $\Omega$ as $n \to \infty$. Since $u_n$ is a weak solutions of (3.1), we have
\[
\int_{\mathbb{R}^2N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dxdy = \int_{\Omega} \frac{f_n(x)}{(u_n + \frac{1}{n})^q(x)} \phi(x) dx \quad (3.3)
\]
for all $\phi \in C_0^\infty(\Omega)$. Let $\phi \in C_0^\infty(\Omega)$ with $\text{supp}(\phi) = K$, then by Lemma 3.2 there exists $l > 0$ (independent of $n$) such that $u_n(x) \geq l > 0$ for a.e. $x \in K$. Therefore, since
\[
\left| \frac{f_n(x)\phi}{(u_n + \frac{1}{n}q(x))} \right| \leq |l^{-q(x)}f(x)\phi(x)| \in L^1(\Omega),
\]
by the Lebesgue dominated convergence theorem, we have
\[
\lim_{n \to \infty} \int_\Omega \frac{f_n(x)\phi}{(u_n + \frac{1}{n}q(x))} \phi \, dx = \int_\Omega \frac{f(x)}{u^{q(x)}} \phi \, dx.
\]
Now the convergence of L.H.S of (3.3) follows proceeding similarly as in the proof of [10, Theorem 3.2]. Therefore we can pass to the limit as $n \to \infty$ to conclude that
\[
\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} \, dx dy = \int_\Omega \frac{f(x)}{u^{q(x)}} \phi \, dx
\]
for each $\phi \in C_0^\infty(\Omega)$. Hence $u \in X$ is a weak solution of (1.1).

Finally, we prove our second main result below.

**Proof of Theorem 2.5:** Denote by $\Phi(t) := t^{\frac{q^*-p+1}{p}}$, for $t \geq 0$. Since $u_n \in L^\infty(\Omega)$ for each $n$ and $\Phi$ is Lipschitz, we have $\Phi(u_n) \in \mathcal{W}^{s,p}_0(\Omega)$. Also since $u_n$ solves (3.1) and the R.H.S of (3.1) belong to $L^\infty(\Omega)$, we can use [10, Proposition 3.3] to get
\[
\int_{\mathbb{R}^N} \frac{\Phi(u_n)(x) - \Phi(u_n(y))}{|x - y|^{N+sp}} \, dx dy \leq \int_\Omega \frac{f_n(x)}{(u_n + \frac{1}{n})^{q(x)}} |\Phi'(u_n)|^{p-1} \Phi(u_n) \, dx.
\]
Note that
\[
\int_\Omega \frac{f_n(x)}{(u_n + \frac{1}{n})^{q(x)}} |\Phi'(u_n)|^{p-1} \Phi(u_n) \, dx
\]
\[
\leq c \int_\Omega \frac{f_n(x)}{(u_n + \frac{1}{n})^{q(x)}} u_n^{q^*-s(x)} \, dx
\]
\[
\leq c \int_\Omega f(x) u_n^{q^*-q(x)} \, dx + \int_{\Omega} \frac{f(x)}{u_n^{q(x)}} u_n^{q^*} \, dx
\]
\[
\leq \|f\|_{L^1(\Omega)} + \left(1 + \|l^{-q(x)}\|_{L^\infty(\Omega)}\right) \int_\Omega f(x) \Phi(u_n) \frac{u_n^{q^*-p+1}}{X^{q^*-p+1}} \, dx
\]
\[
\leq \|f\|_{L^1(\Omega)} + \left(1 + \|l^{-q(x)}\|_{L^\infty(\Omega)}\right) \|f\|_{L^\infty(\Omega)} \|\Phi(u_n)\|_{X^{q^*-p+1}}^{q^*-p+1},
\]
which gives
\[
\|u_n\|_{\mathcal{W}^{s,p}(\mathbb{R}^N)} \leq C,
\]
for some constant $C$ independent of $n$. Therefore
\[
\sup_{n \in \mathbb{N}} \|u_n\|_{\mathcal{W}^{s,p}(\mathbb{R}^N)} \leq C.
\]
By Lemma 3.2, since $u_n$ is monotone increasing, we can define the pointwise limit $u$ of $u_n$ as $n \to \infty$. Therefore by Fatou's lemma
\[
\left| u^{\frac{q^*-p+1}{p}} \right|_{W^{s,p}(\mathbb{R}^N)} \leq \liminf_{n \to \infty} \left| u_n^{\frac{q^*-p+1}{p}} \right|_{W^{s,p}(\mathbb{R}^N)} \leq C.
\]
Hence $u^{\frac{q^*-p+1}{p}} \in W^{s,p}_0(\Omega)$ and since $q^* > 1$ we have $u \in L^p(\Omega)$. Moreover, by Lemma 3.2 for every $K \subseteq \Omega$, there exists a constant $l(K) > 0$ such that $u(x) \geq l(K) > 0$ for a.e. $x \in \Omega$. Now the fact that $u \in W^{s,p}_{loc}(\Omega)$ and is a weak solution
of (1.1) follows from the lines of proof of [10, Theorem 3.6] while realising that the role of $\gamma$ there is played by $q^*$ here. \hfill \Box

4. **Regularity results in case (I).** Following is a local regularity result which follows directly as a consequence of Lemma 3.2.

**Theorem 4.1.** Every weak solution of (1.1) in Case (I) as obtained through Theorem 2.4 and Theorem 2.5 belongs to $C^\alpha_{loc}(\Omega)$ for some $\alpha \in (0,1)$.

**Proof.** Let $u$ denotes a weak solution to (1.1) obtained in Theorem 2.4 and Theorem 2.5 then using Lemma 3.2 and $u$ being a pointwise limit of $\{u_n\}$ we obtain that there exists a $l_K > 0$ such that

$$u(x) \geq l_K$$

for a.e. $x \in K \subset \subset \Omega$.

This implies $u^{-q(x)}(x) \leq C_K$ for a.e. $K \subset \subset \Omega$ for some positive constant $C(K)$ depending on $K$. Therefore using [24, Corollary 4.2] we conclude that $u \in L^\infty_{loc}(\Omega)$. Furthermore, from [24, Corollary 5.5] it follows that $u \in C^\alpha_{loc}(\Omega)$ for some $\alpha \in (0,1)$.

While restricting $N > sp(p+1)$, we can get that solutions of (1.1) belong to $L^\infty(\Omega)$. Before proving this result, we recall the following Lemmas.

**Lemma 4.2.** [9, Lemma A.1] Let $1 < p < \infty$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a $C^1$ convex function. If $a, b \in \mathbb{R}$ and $A, B > 0$, then

$$|f(a) - f(b)|^p - 2(f(a) - f(b))(A - B) \leq |a - b|^p - 2(a - b)(Af'(a)|^p - 2f'(a) - B|f'(b)|^p - 2f'(b)).$$

**Lemma 4.3.** [9, Lemma A.2] Let $1 < p < \infty$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function, then we have

$$|G(a) - G(b)|^p \leq |a - b|^p - 2(a - b)(g(a) - g(b))$$

where $G(t) = \int_0^t g'(\tau)^{\frac{p}{2}} d\tau$, for $t \in \mathbb{R}$.

Now below follows the regularity result.

**Theorem 4.4.** Let $u$ be a positive weak solution of (1.1) in Case (I) with $f \in L^{p_\omega}(\Omega)$ and $N > sp(p+1)$. Then $u \in L^\infty(\Omega)$.

**Proof.** The proof here is adopted from Brasco and Parini [9]. Let $\epsilon > 0$ be very small and define

$$z_\epsilon(t) = (\epsilon^2 + t^2)^{\frac{1}{2}}$$

which is smooth, convex and Lipschitz. By taking the choices

$$a = u(x), \ b = u(y), \ A = \psi(x), \ B = \psi(y)$$

in Lemma 4.2 and by Lemma 2.3 choosing $\varphi = \psi|z_\epsilon'(u)|^{p-2}z_\epsilon'(u)$ for $0 < \psi \in C^\infty_c(\Omega)$ as a test function in (1.1) we get

$$\int_{\mathbb{R}^N} \frac{|z_\epsilon(u(x)) - z_\epsilon(u(y))|^{p-2}(z_\epsilon(u(x)) - z_\epsilon(u(y)))(\psi(x) - \psi(y))}{|x - y|^{N+sp}} dxdy$$

$$\leq \int_{\Omega} |f(x)u^{-q(x)}||z_\epsilon'(u)|^{p-1}\psi \ dx.$$
As \( t \to 0 \), \( z_\epsilon(t) \to |t| \) and we have \( |z'_\epsilon(t)| \leq 1 \). So using Fatou’s Lemma, we let \( \epsilon \to 0 \) in the above inequality to get

\[
\int_{\mathbb{R}^N} \frac{|u(x)| - |u(y)|}{|x-y|^{N+p}} \cdot |(u(x) - |u(y)|)(\psi(x) - \psi(y))| \, dx \, dy \leq \int_{\Omega} f(x) u^{-q(x)} \psi \, dx, \tag{4.1}
\]

for every \( 0 < \psi \in C^\infty_0(\Omega) \). The inequality (4.1) still holds for \( 0 \leq \psi \in X \) (similar proof as of [28, Lemma 6.1]). Now, define \( u_K = \min\{(u-1)^+, K\} \in X \), for \( K > 0 \).

For \( \beta > 0 \) and \( \rho > 0 \), we take \( \psi = (u_K + \rho) - \rho \beta \) as test function in (4.1) and get

\[
\int_{\mathbb{R}^N} \frac{|u(x)| - |u(y)|}{|x-y|^{N+p}} \cdot |(u(x) - |u(y)|)((u_K(x) + \rho) - (u_K(y) + \rho)\beta)\| \, dx \, dy \\
\leq \int_{\Omega} \left| f(x) u^{-q(x)} \right| \left| (u_K + \rho) - \rho \beta \right| \, dx.
\]

Then, by using Lemma 4.3 with the function \( g(u) = (u_K + \rho)^\beta \) we get

\[
\int_{\mathbb{R}^N} \frac{|(u_K(x) + \rho)^\beta - (u_K(y) + \rho)^\beta|}{|x-y|^{N+p}} \, dx \, dy \\
\leq \frac{(\beta + p - 1)^p}{\beta^p} \int_{\mathbb{R}^N} \frac{|U(x,y)|^p}{|x-y|^{N+p}} \, dx \, dy \\
\leq \int_{\Omega} \frac{(\beta + p - 1)^p}{\beta^p} \left| f(x) u^{-q(x)} \right| \left| (u_K + \rho)^\beta - \rho \beta \right| \, dx,
\]

where \( U(x, y) = |u(x)| - |u(y)| \). Now, using the support of \( u_K \) we have

\[
\int_{\Omega} \left| f(x) u^{-q(x)} \right| \left| (u_K + \rho)^\beta - \rho \beta \right| \, dx = \int_{\{u > 1\}} \left| f(x) u^{-q(x)} \right| \left| (u_K + \rho)^\beta - \rho \beta \right| \, dx \\
\leq \int_{\{u > 1\}} |f(x)| \left| (u_K + \rho)^\beta - \rho \beta \right| \, dx \\
\leq \|f\|_{L^r(\Omega)} \|u_K + \rho\|_{L^{r'}(\Omega)} \|u_K + \rho\|_{L^r(\Omega)},
\]

where \( r = p^* \) and \( r' = \frac{p^*_r}{p^*_r - 1} \). By using Sobolev inequality given in [26, Theorem 1], we get

\[
\int_{\Omega} \frac{|(u(x) + \rho)^\beta - (u_K(y) + \rho)^\beta|}{|x-y|^{N+p}} \, dx \, dy \\
\geq \frac{1}{T_{p,s}} \left\| (u_K + \rho)^\beta - \rho \beta \right\|_{L^{p'}(\Omega)}^p \\
\geq \frac{1}{T_{p,s}} \left( \frac{(\beta + p - 1)^p}{\beta^p} \right) \left\| f \right\|_{L^r(\Omega)} \left\| (u_K + \rho)^\beta \right\|_{L^{r'}(\Omega)}^2 \left| \Omega \right|^{1/p'},
\]

where \( T_{p,s} \) is a nonnegative constant and the last inequality follows from triangle inequality and \( (u_K + \rho)^{\beta + p - 1} \geq \rho^{p - 1}(u_K + \rho)^\beta \). Using all these estimates, we now have

\[
\left\| (u_K + \rho)^\beta \right\|_{L^{p'}(\Omega)}^p \\
\leq C \left( \frac{(\beta + p - 1)^p}{\beta^p} \right) \left\| f \right\|_{L^r(\Omega)} \left\| (u_K + \rho)^\beta \right\|_{L^{r'}(\Omega)} \left| \Omega \right|^{1/p'},
\]
Hence in particular, we say that
\[
\left(\frac{\beta + p - 1}{p}\right)^{\frac{1}{\beta}} \geq 1.
\]
Using this we can also check that
\[
\rho^{|\Omega|^{\frac{1}{p}} \geq \beta\frac{1}{\beta} \left(\frac{\beta + p - 1}{p}\right)^{\frac{1}{\beta}} \|u_K + \rho\|^2_{\nu}.
\]
Hence we have
\[
\left\|u_K + \rho\right\|^p_{L^{p}(\Omega)} \leq \frac{C}{\beta} \left(\frac{\beta + p - 1}{p}\right)^{\frac{1}{\beta}} \|u_K + \rho\|^p_{L^{\nu}(\Omega)} \left(\frac{T_{p,s}}{\rho^{p-1}} + \|\Omega\|^{1 - \frac{1}{p} - \frac{\nu}{p}}\right),
\]
for \(C = C(p) > 0\) is constant. We now choose
\[
\rho = \left(T_{p,s}\|f\|_{L^{\nu}(\Omega)}\right)^{\frac{1}{p-1}} \|\Omega\|^{\frac{1}{p-1} \left(1 - \frac{1}{p} - \frac{\nu}{p}\right)}
\]
and let \(\beta \geq 1\) be such that
\[
\frac{1}{\beta} \left(\frac{\beta + p - 1}{p}\right)^{\frac{1}{\beta}} \leq \beta^{p-1}.
\]
In addition, if we let \(\tau = \beta^{\nu}\) and \(\nu = \frac{\nu'}{p'} > 1\) since we have assumed \(N > sp(p + 1)\), then the above inequality reduces to
\[
\left\|u_K + \rho\right\|^p_{L^{\nu}(\Omega)} \leq \left(C\|\Omega\|^{1 - \frac{1}{p} - \frac{\nu}{p}}\right)^{\frac{1}{\beta}} \left(\frac{\tau}{\rho}\right)^{\frac{p-1}{p}} \|u_K + \rho\|_{L^{\nu}(\Omega)}.
\]
Now, we iterate (4.3) using \(\tau_0 = r\) and let
\[
\tau_{m+1} = \nu^{\tau_m} = \nu^{m+1}\nu',
\]
which gives
\[
\left\|u_K + \rho\right\|^p_{L^{\nu}(\Omega)} \leq \left(C\|\Omega\|^{1 - \frac{1}{p} - \frac{\nu}{p}}\right)^{\frac{1}{\beta}} \left(\frac{\tau}{\rho}\right)^{\frac{p-1}{p}} \|u_K + \rho\|_{L^{\nu}(\Omega)}.
\]
Since \(\nu > 1\),
\[
\sum_{i=0}^{\infty} \tau_i = \sum_{i=0}^{\infty} \frac{1}{\nu_i} = \nu - 1
\]
and
\[
\prod_{i=0}^{\infty} \left(\frac{T_i}{T_{i+1}}\right)^{\frac{1}{T_{i+1}}} = \nu^{\frac{1}{\nu - 1}}.
\]
Taking limit as \(m \to 0\) in (4.4), we finally get
\[
\left\|u_K\right\|^p_{L^{\infty}(\Omega)} \leq \left(C\nu^{\frac{1}{\nu - 1}}\right)^{\frac{p-1}{p}} \left(\|\Omega\|^{1 - \frac{1}{p} - \frac{\nu}{p}}\right)^{\frac{p-1}{p}} \|u_K + \rho\|_{L^{\nu}(\Omega)}.
\]
Since \(u_K \leq (u - 1)^{+}\), using the triangle inequality in the above inequality we get,
\[
\left\|u_K\right\|^p_{L^{\infty}(\Omega)} \leq \left(C\nu^{\frac{1}{\nu - 1}}\right)^{\frac{p-1}{p}} \left(\|\Omega\|^{1 - \frac{1}{p} - \frac{\nu}{p}}\right)^{\frac{p-1}{p}} \left(\|u - 1\|^1\right)^{\frac{p-1}{p}} \|u_K + \rho\|_{L^{\nu}(\Omega)}.
\]
for some constant \(C = C(p) > 0\). If we now let \(K \to \infty\), we get
\[
\left\|u - 1\right\|^1_{L^{\infty}(\Omega)} \leq \left(C\nu^{\frac{1}{\nu - 1}}\right)^{\frac{p-1}{p}} \left(\|\Omega\|^{1 - \frac{1}{p} - \frac{\nu}{p}}\right)^{\frac{p-1}{p}} \left(\|u - 1\|^1\right)^{\frac{p-1}{p}} \|u_K + \rho\|_{L^{\nu}(\Omega)}.
\]
Hence in particular, we say that \(u \in L^{\infty}(\Omega)\).
5. Multiplicity result in case (II). This section is devoted to prove our second main result that is Theorem 2.6 using the method of approximation. We follow [3] here. Let us denote the energy functional $I_\lambda : X \to \mathbb{R} \cup \{\pm \infty\}$ corresponding to the problem (1.1) for Case (II) by

$$I_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy - \lambda \int_{\Omega} \frac{(u^+)^{1-q(x)}(x)}{1-q(x)} \, dx - \frac{1}{r+1} \int_{\Omega} (u^+)^{r+1} \, dx.$$ 

Now for $\epsilon > 0$, we consider the following approximated problem

$$\begin{cases}
(-\Delta)_\rho^s u = \frac{\lambda}{(u^+ + \epsilon)^{q(x)}} + (u^+)^r & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases} \quad (P_\lambda, \epsilon)$$

for which the corresponding energy functional is given by

$$I_{\lambda, \epsilon}(u) = \frac{\|u\|^p}{p} - \lambda \int_{\Omega} \frac{[(u^+ + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)}]}{1-q(x)} \, dx - \frac{1}{r+1} \int_{\Omega} (u^+)^{r+1} \, dx.$$ 

It is easy to verify that $I_{\lambda, \epsilon} \in C^1(\mathbb{R}^N)$, $I_{\lambda, \epsilon}(0) = 0$ and $I_{\lambda, \epsilon}(v) \leq I_{0, \epsilon}(v)$ for all $0 \leq v \in X$. From [30], we have the existence of the first nonnegative eigenfunction $e_1 \in X \cap L^\infty(\Omega)$ corresponding to the first eigenvalue $\lambda_1$ satisfying the equation

$$(-\Delta)_\rho^s v = \lambda_1 |v|^{p-2} v \text{ in } \Omega, \quad v = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \quad (5.1)$$

W.l.o.g. we may assume that $\|e_1\|_{L^\infty(\Omega)} = 1$. Our next Lemma states that $I_{\lambda, \epsilon}$ satisfies the Mountain Pass Geometry.

**Lemma 5.1.** There exists $R > 0$, $\rho > 0$ and $\Lambda > 0$ depending on $R$ such that $\inf_{\|v\| \leq R} I_{\lambda, \epsilon}(v) < 0$ and $\inf_{\|v\| = R} I_{\lambda, \epsilon}(v) \geq \rho$, for $\lambda \in (0, \Lambda)$. Moreover there exists $T > R$ such that $I_{\lambda, \epsilon}(Te_1) < -1$, for $\lambda \in (0, \Lambda)$.

**Proof.** We fix $l = |\Omega|^{1/(\frac{p}{p+1})}$. Then using H"{o}lder’s inequality and Lemma 2.1 we get that

$$\int_{\Omega} (v^+)^{r+1} \, dx \leq \left( \int_{\Omega} |v|^l \right)^{\frac{\frac{r+1}{p}}{\frac{l}{p}}} |\Omega|^{\frac{1}{l}(\frac{p}{p+1})} \leq C \|v\|^{r+1},$$

for some positive constant $C$ independent of $v$. We now observe that

$$\lim_{t \to 0} \frac{I_{\lambda, \epsilon}(te_1)}{t} = -\lambda \int_{\Omega} \epsilon^{-q(x)} e_1 \, dx < 0,$$

which implies that it is possible to choose $k \in (0, 1)$ sufficiently small and to set $\|v\| = R := k(\frac{1}{p+1})^{-\frac{1}{p+1}}$ such that $\inf_{\|v\| \leq R} I_{\lambda, \epsilon}(v) < 0$. Moreover, since $R < (\frac{\rho}{pC})^{\frac{1}{p+1}}$ we obtain

$$I_{0, \epsilon}(v) \geq \frac{R^p}{p} - \frac{C \rho^{r+1}}{r+1} := 2\rho \text{ (say)} > 0.$$ 

We define

$$\Lambda := \sup_{\|v\| = R} \frac{\rho}{\left( \frac{1}{1-q(x)} \int_{\Omega} |v|^{1-q(x)} \, dx \right)^{\frac{1}{p+1}}},$$

which is a positive constant and since $\rho, R$ depends on $k, r, p, |\Omega|, C$ so does $\Lambda$. We know that

$$(v^+ + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)} \leq (v^+)^{1-q(x)}, \quad (5.2)$$
which gives
\[ I_{\lambda,\epsilon}(v) \geq \frac{\|v\|^p}{p} - \frac{1}{r+1} \int_\Omega (v^+)^{r+1} \, dx - \frac{\lambda}{1-q(x)} \int_\Omega (v^+)^{1-q(x)} \, dx \]
\[ = I_{0,\epsilon}(v) - \frac{\lambda}{1-q(x)} \int_\Omega (v^+)^{1-q(x)} \, dx. \]
Therefore
\[ \inf_{\|v\|=R} I_{\lambda,\epsilon}(v) \geq \inf_{\|v\|=R} I_{0,\epsilon}(v) - \lambda \sup_{\|v\|=R} \left( \frac{1}{1-q(x)} \int_\Omega |v|^{1-q(x)} \, dx \right) \]
\[ \geq 2\rho - \lambda \sup_{\|v\|=R} \left( \frac{1}{1-q(x)} \int_\Omega |v|^{1-q(x)} \, dx \right) \geq \rho, \]
if \( \lambda \in (0, \Lambda) \). Lastly, it is easy to see that \( I_{0,\epsilon}(te_1) \to -\infty \) as \( t \to +\infty \), which implies that we can choose \( T > R \) such that \( I_{0,\epsilon}(Te_1) < -1 \). Hence
\[ I_{\lambda,\epsilon}(Te_1) \leq I_{0,\epsilon}(Te_1) < -1, \]
which completes the proof. \( \square \)

As a consequence of Lemma 5.1, we have
\[ \inf_{\|v\|=R} I_{\lambda,\epsilon}(v) \geq \rho \max\{I_{\lambda,\epsilon}(Te_1), I_{\lambda,\epsilon}(0)\} = 0. \]

Our next Lemma ensures that \( I_{\lambda,\epsilon} \) satisfies the Palais Smale \((PS)_c\) condition.

**Proposition 6.** \( I_{\lambda,\epsilon} \) satisfies the \((PS)_c\) condition, for any \( c \in \mathbb{R} \), that is if \( \{u_k\} \subset X \) is a sequence satisfying
\[ I_{\lambda,\epsilon}(u_k) \to c \text{ and } I'_{\lambda,\epsilon}(u_k) \to 0 \]
as \( k \to \infty \), then \( \{u_k\} \) contains a strongly convergent subsequence in \( X \).

**Proof.** Let \( \{u_k\} \subset X \) satisfies (6.1) then we claim that \( \{u_k\} \) must be bounded in \( X \). To see this using (5.2), we obtain
\[ I_{\lambda,\epsilon}(u_k) - \frac{1}{r+1} I'_{\lambda,\epsilon}(u_k) u_k \]
\[ = \left( \frac{1}{p} - \frac{1}{r+1} \right) \|u_k\|^p - \lambda \int_\Omega \frac{(u_k^+ + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)}}{1-q(x)} \, dx \]
\[ + \frac{\lambda}{r+1} \int_\Omega (u_k^+ + \epsilon)^{-q(x)} u_k \, dx \]
\[ \geq \left( \frac{1}{p} - \frac{1}{r+1} \right) \|u_k\|^p - \lambda \int_\Omega \frac{(u_k^+)^{1-q(x)}}{1-q(x)} \, dx + \frac{\lambda}{r+1} \int_\Omega (u_k^+ + \epsilon)^{-q(x)} u_k \, dx \]
\[ \geq \left( \frac{1}{p} - \frac{1}{r+1} \right) \|u_k\|^p - \lambda \int_\Omega \frac{(u_k^+)^{1-q(x)}}{1-q(x)} \, dx - \frac{\lambda C}{\epsilon(r+1)} \|u_k\|, \]
for some positive constant \( C \) (independent of \( k \)), where we have used the embedding result, Lemma 2.1 and the fact \( 0 < q(x) < 1 \) in \( \Omega \). Due to the same reasoning, we obtain
\[ - \int_\Omega \frac{(u_k^+)^{1-q(x)}}{1-q(x)} \, dx \geq - \int_\Omega \frac{|u_k|^{1-q(x)}}{1-q(x)} \, dx \]
\[ \geq \frac{-1}{1-\|q\|_{L^\infty(\Omega)}} \left( \int_{\Omega \cap \{|u_k| \geq 1\}} |u_k|^{1-q(x)} \, dx + \int_{\Omega \cap \{|u_k| < 1\}} |u_k|^{1-q(x)} \, dx \right) \]
\[ \geq \frac{-1}{1 - \|u\|_{L^\infty(\Omega)}} \left( \int_{\Omega \cap \{|u_k| \geq 1\}} |u_k| dx + \int_{\Omega \cap \{|u_k| < 1\}} |u_k|^{1-\|u\|_{L^\infty(\Omega)}} dx \right) \]
\[ \geq -C(\|u_k\| + \|u_k\|^{1-\|u\|_{L^\infty(\Omega)}}), \tag{6.3} \]
for some positive constant $C$ independent of $k$. Thus inserting (6.3) into (6.2), for some positive constant $C_1$ (independent of $k$), we get
\[ I_{\lambda, \epsilon}(u_k) - \frac{1}{r+1} I'_{\lambda, \epsilon}(u_k) u_k \geq C_1 \|u_k\|^p - C \left( \|u_k\| + \|u_k\|^{1-\|u\|_{L^\infty(\Omega)}} \right). \tag{6.4} \]
Also from (6.1) it follows that for $k$ large enough
\[ \left| I_{\lambda, \epsilon}(u_k) - \frac{1}{r+1} I'_{\lambda, \epsilon}(u_k) u_k \right| \leq c + o(\|u_k\|). \tag{6.5} \]
Combining (6.4) and (6.5), our claim follows since $p > 1$. By reflexivity of $X$, there exists $u_0 \in X$ such that up to a subsequence, $u_k \rightarrow u_0$ weakly in $X$ as $k \rightarrow \infty$.

Claim: $u_k \rightarrow u_0$ strongly in $X$ as $k \rightarrow \infty$. For convenience, we denote by
\[ A(v, \phi) := \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(|\phi(x) - \phi(y)|)}{|x - y|^{N+p}} \ dx \ dy. \]
Then by (6.1), we have that
\[ \lim_{k \rightarrow \infty} (A(u_k, u_0) - \lambda \int_{\Omega} (u_k^+ + \epsilon)^{-q(x)} u_0 \ dx - \int_{\Omega} (u_k^+)^r u_0 \ dx) = 0 \]
and
\[ \lim_{k \rightarrow \infty} (A(u_k, u_k) - \lambda \int_{\Omega} (u_k^+ + \epsilon)^{-q(x)} u_k \ dx - \int_{\Omega} (u_k^+)^r u_k) = 0. \]
Now setting $U_k(x, y) = u_k(x) - u_k(y)$ and $U_0(x, y) = u_0(x) - u_0(y)$ we get
\[ \lim_{k \rightarrow \infty} \int_{\mathbb{R}^{2N}} \left( |U_k(x, y)|^{p-2} U_k(x, y) - |U_0(x, y)|^{p-2} U_0(x, y) \right) (U_k(x, y) - U_0(x, y)) \ dx \ dy \]
\[ = \lim_{k \rightarrow \infty} \left( \lambda \int_{\Omega} (u_k^+ + \epsilon)^{-q(x)} u_k \ dx + \int_{\Omega} (u_k^+)^r u_k \ dx - \lambda \int_{\Omega} (u_k^+ + \epsilon)^{-q(x)} u_0 \ dx \right. \]
\[ - \int_{\Omega} (u_k^+)^r u_0 \ dx \left. \right) - \lim_{k \rightarrow \infty} (A(u_0, u_k) - A(u_0, u_0)). \tag{6.6} \]
From weak convergence of $\{u_k\}$ in $X$, we get
\[ \lim_{k \rightarrow \infty} (A(u_0, u_k) - A(u_0, u_0)) = 0. \tag{6.7} \]
Indeed, $u_k \rightarrow u_0$ weakly in $X$ implies that
\[ \frac{u_k(x) - u_k(y)}{|x - y|^{N+p}} \overset{\text{weakly}}{\longrightarrow} \frac{u_0(x) - u_0(y)}{|x - y|^{N+p}} \]
weakly in $L^p(\mathbb{R}^{2N})$. Now it is easy to see that
\[ \frac{|u_0(x) - u_0(y)|^{p-2}(u_0(x) - u_0(y))}{|x - y|^{N+p}} \in L^p(\mathbb{R}^{2N}) \]
which proves our statement. Also $|(u_k^+ + \epsilon)^{-q(x)} u_0| \leq \epsilon^{-q(x)} u_0$ and $\int_\Omega |\epsilon^{-q(x)} u_0| dx \leq (1 + \epsilon^{-\|u\|_{L^\infty(\Omega)}}) \int_\Omega |u_0| \ dx < +\infty$. Thus Lebesgue Dominated convergence theorem gives that
\[ \lim_{k \rightarrow \infty} \int_\Omega (u_k^+ + \epsilon)^{-q(x)} u_0 \ dx = \int_\Omega (u_0^+ + \epsilon)^{-q(x)} u_0 \ dx. \tag{6.8} \]
Since $u_k \to u_0$ a.e. in $\Omega$ and for any measurable subset $E$ of $\Omega$ we have
\[
\int_E |(u_k^+ + \varepsilon)^{-q(x)} u_k| \, dx \\
\leq \int_E \varepsilon^{-q(x)} u_k \, dx \leq (1 + \varepsilon^{-||q||_{L^\infty(\Omega)}}) ||u_k||_{L^{p'}(\Omega)} \, E^\frac{p_q'-1}{p_q'} \leq C(\varepsilon) ||E||^\frac{p_q'-1}{p_q'},
\]
so from Vitali convergence theorem, it follows that
\[
\lim_{k \to \infty} \lambda \int_\Omega (u_k^+ + \varepsilon)^{-q(x)} u_k \, dx = \lambda \int_\Omega (u_0^+ + \varepsilon)^{-q(x)} u_0 \, dx. \tag{6.9}
\]
Similarly, we have
\[
\int_E |(u_k^+)^r u_0| \, dx \leq ||u_0||_{L^{p'}(\Omega)} \left( \int_E (u_k^+)^{r p_q'} \, dx \right)^\frac{1}{p_q'} \leq C_3 ||E||^\alpha
\]
and
\[
\int_E |(u_k^+)^r u_k| \, dx \leq ||u_k||_{L^{p'}(\Omega)} \left( \int_E (u_k^+)^{r p_q'} \, dx \right)^\frac{1}{p_q'} \leq C_4 ||E||^\beta
\]
for some positive constants $C_3, C_4, \alpha$ and $\beta$. Therefore Vitali convergence theorem gives
\[
\lim_{k \to \infty} \int_\Omega (u_k^+)^r u_0 \, dx = \int_\Omega (u_0^+)^r u_0 \, dx, \tag{6.10}
\]
and
\[
\lim_{k \to \infty} \int_\Omega (u_k^+)^r u_k \, dx = \int_\Omega (u_0^+)^r u_0 \, dx. \tag{6.11}
\]
Using (6.7), (6.8), (6.9), (6.10) and (6.11) into (6.6), we obtain
\[
\lim_{k \to \infty} \int_{\mathbb{R}^{2N}} \frac{(|U_k(x, y)|^{p-2} U_k(x, y) - |U_0(x, y)|^{p-2} U_0(x, y)) (U_k(x, y) - U_0(x, y))}{|x-y|^{N+sp}} \, dx \, dy = 0.
\]
From Hölder’s inequality, we get that
\[
\lim_{k \to \infty} \int_{\mathbb{R}^{2N}} \frac{(|U_k(x, y)|^{p-2} U_k(x, y) - |U_0(x, y)|^{p-2} U_0(x, y)) (U_k(x, y) - U_0(x, y))}{|x-y|^{N+sp}} \, dx \, dy \geq (||u_k||^{p-1} - ||u_0||^{p-1}) (||u_k|| - ||u_0||),
\]
which proves our claim. \qed

From Lemma 5.1, Proposition 6 and Mountain Pass Lemma, for every $\lambda \in (0, \Lambda)$, there exists a $\zeta_\varepsilon \in X$ such that $I_{\lambda, \varepsilon}'(\zeta_\varepsilon) = 0$ and
\[
I_{\lambda, \varepsilon}(\zeta_\varepsilon) = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_{\lambda, \varepsilon}(\gamma(t)) \geq \rho > 0,
\]
where $\Gamma = \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = T e_1 \}$. Using (6.3) together with Vitali convergence theorem, if $u_k \rightharpoonup u_0$ weakly in $X$, then we have
\[
\lim_{k \to \infty} \int_\Omega \frac{(u_k + \varepsilon)^{1-q(x)} - (u_0 + \varepsilon)^{1-q(x)}}{1 - q(x)} \, dx = \int_\Omega \frac{(u_0 + \varepsilon)^{1-q(x)} - (u_0)^{1-q(x)}}{1 - q(x)} \, dx.
\]
Therefore, $I_{\lambda, \varepsilon}$ is weakly lower semicontinuous. Furthermore, as a consequence of Lemma 5.1, since for every $\lambda \in (0, \Lambda)$ we have $\inf_{||v|| \leq R} I_{\lambda, \varepsilon}(v) < 0$, so there exists nonzero $\nu_\varepsilon \in X$ such that $||\nu_\varepsilon|| \leq R$ and
\[
\inf_{||v|| \leq R} I_{\lambda, \varepsilon}(v) = I_{\lambda, \varepsilon}(\nu_\varepsilon) < 0 < \rho \leq I_{\lambda, \varepsilon}(\zeta_\varepsilon). \tag{6.12}
\]
Thus, \( \zeta_e \) and \( \nu_e \) are two different non trivial critical points of \( P_{\lambda,e} \), provided \( \lambda \in (0, A) \).

**Lemma 6.1.** The critical points \( \zeta_e \) and \( \nu_e \) of \( P_{\lambda,e} \) are nonnegative in \( \Omega \).

*Proof.* Testing \( (P_{\lambda,e}) \) with \( \min \{ \zeta_e, 0 \} \) and \( \min \{ \nu_e, 0 \} \), it is easy to verify that \( \zeta_e, \nu_e \geq 0 \) since the R.H.S. of \( (P_{\lambda,e}) \) remains a nonnegative quantity.

**Lemma 6.2.** There exists a constant \( \Theta > 0 \) (independent of \( \epsilon \)) such that \( \| \nu_e \| \leq \Theta \) where \( \nu_e = \zeta_e \) or \( \nu_e \).

*Proof.* The result trivially holds if \( \nu_e = \zeta_e \) so we deal with the case \( \nu_e = \zeta_e \). Recalling the terms from Lemma 5.1, we define \( A = \max_{\gamma \in \Gamma} I_{\gamma,0}(t T e_1) \) then

\[
A \geq \max_{\gamma \in \Gamma} I_{\lambda,0}(t T e_1) \geq \inf_{\gamma \in \Gamma} \max_{\gamma \in \Gamma} I_{\lambda,\gamma}(\gamma(t)) = I_{\lambda,\gamma}(\zeta_e) \geq \rho > 0 \geq I_{\lambda,\gamma}(\nu_e).
\]

Therefore

\[
\frac{1}{p} \| \zeta_e \|^{p} - \lambda \int_{\Omega} \left( \frac{(\zeta_e + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)}}{1-q(x)} \right) dx - \frac{1}{r+1} \int_{\Omega} \zeta_e^{r+1} dx \leq A. \tag{6.13}
\]

Choosing \( \phi = -\frac{\zeta_e}{p+1} \) as a test function in \( (P_{\lambda,e}) \) we obtain

\[
-\frac{1}{r+1} \| \zeta_e \|^{p} + \frac{\lambda}{r+1} \int_{\Omega} \frac{\zeta_e}{(\zeta_e + \epsilon)^{q(x)}} dx + \frac{1}{r+1} \int_{\Omega} \zeta_e^{r+1} dx = 0. \tag{6.14}
\]

Adding (6.13) and (6.14) we get

\[
\left( \frac{1}{p} - \frac{1}{r+1} \right) \| \zeta_e \|^{p} \leq \lambda \int_{\Omega} \left( \frac{(\zeta_e + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)}}{1-q(x)} \right) dx - \frac{\lambda}{r+1} \int_{\Omega} \frac{\zeta_e}{(\zeta_e + \epsilon)^{q(x)}} dx + A
\]

\[
\leq \lambda \int_{\Omega} \left( \frac{(\zeta_e + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)}}{1-q(x)} \right) dx + A \leq C(\| \zeta_e \| + \| \zeta_e \|^{1-\| \zeta_e \|}) + A,
\]

for some positive constant \( C \) being independent of \( \epsilon \), where the last inequality is deduced using the estimate (6.3), Hölder inequality along Lemma 2.1. Therefore, since \( r+1 > p \), the sequence \( \{ \zeta_e \} \) is uniformly bounded in \( X \) with respect to \( \epsilon \). This completes the proof.

Now as a resultant of Lemma 6.1 and Lemma 6.2, upto a subsequence we get that \( \zeta_e \to \zeta_0 \) and \( \nu_e \to \nu_0 \) weakly in \( X \) as \( \epsilon \to 0^+ \), for some nonnegative \( \zeta_0, \nu_0 \in X \).

In the sequel, we establish that \( \zeta_0 \neq \nu_0 \) and forms a weak solution to our problem (1.1). For convenience, we denote by \( \nu_0 \) either \( \zeta_0 \) or \( \nu_0 \).

**Lemma 6.3.** \( \nu_0 \in X \) is a weak solution to the problem (1.1).

*Proof.* We observe that for any \( \epsilon \in (0, 1) \) and \( t \geq 0 \),

\[
\frac{\lambda}{(t + \epsilon)^{q(x)}} + t^r \geq \frac{\lambda}{(t + 1)^{q(x)}} + t^r \geq \min \{ 1, \frac{\lambda}{2} \}.
\]

As a consequence we get

\[
(-\Delta)_p^s \nu_e = \frac{\lambda}{(\nu_e + \epsilon)^{q(x)}} + \nu_e^r \geq \min \{ 1, \frac{\lambda}{2} \} := C, \text{ say.}
\]

Consequently, by [10, Lemma 2.1] if \( \xi \in X \) satisfies

\[
(-\Delta)_p^s \xi = C \text{ in } \Omega, \xi > 0 \text{ in } \Omega,
\]
we get
\[
\int_{\mathbb{R}^N} \frac{|v_\epsilon(x) - v_\epsilon(y)|^{p-2}(v_\epsilon(x) - v_\epsilon(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} \, dx \, dy
\geq \int_{\mathbb{R}^N} \frac{|\xi(x) - \xi(y)|^{p-2}(\xi(x) - \xi(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} \, dx \, dy
\tag{6.15}
\]
for every nonnegative \( \phi \in X \). Therefore choosing \( \phi = (\xi - v_\epsilon)^+ \in X \) as a test function in (6.15), proceeding the idea of the proof of [10, Lemma 2.4], we obtain
\[
v_\epsilon \geq \xi \text{ in } \Omega.
\]
Now by [14, Theorem 1.2] and [29, Proposition 1.1], we obtain
\[
v_\epsilon \geq c_K > 0
\tag{6.16}
\]
for every \( K \in \Omega \). This gives \( v_\epsilon \geq c_K > 0 \) for every \( K \in \Omega \), \( v_0 > 0 \) in \( \Omega \) and
\[
0 \leq \left| \frac{\lambda \phi}{(v_\epsilon + \epsilon)^{q(x)}} \right| \leq \lambda \|c_K \|_{L^\infty(\Omega)}, \text{ for every } \phi \in C_c^\infty(\Omega).
\]
By the Lebesgue dominated theorem, we have
\[
\lim_{\epsilon \to 0^+} \int_{\Omega} \frac{\lambda}{(v_\epsilon + \epsilon)^{q(x)}} \phi \, dx = \int_{\Omega} \frac{\lambda}{v_0^{q(x)}} \phi \, dx.
\tag{6.17}
\]
Moreover, by the compact embedding from Lemma 2.1, we have
\[
\lim_{\epsilon \to 0^+} \int_{\Omega} v_\epsilon^r \phi \, dx = \int_{\Omega} v_0^r \phi \, dx, \quad \forall \phi \in C_c^\infty(\Omega).
\tag{6.18}
\]
Moreover, using the weak convergence of \( v_\epsilon \) we get
\[
\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} \frac{|v_\epsilon(x) - v_\epsilon(y)|^{p-2}(v_\epsilon(x) - v_\epsilon(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} \, dx \, dy
\geq \int_{\mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^{p-2}(v_0(x) - v_0(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} \, dx \, dy, \quad \forall \phi \in C_c^\infty(\Omega).
\tag{6.19}
\]
Using (6.17), (6.18) and (6.19), we get
\[
\int_{\mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^{p-2}(v_0(x) - v_0(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} \, dx \, dy
= \lambda \int_{\Omega} \frac{\phi}{v_0^{q(x)}} \, dx + \int_{\Omega} v_0^r \phi \, dx,
\forall \phi \in C_c^\infty(\Omega).
\]
This completes the proof. \( \square \)

**Proof of Theorem 2.6:** Using Lemma 6.3 we get that \( \zeta_0 \) and \( v_0 \) are two positive weak solution of (1.1) for \( \lambda \in (0, \Lambda) \). Now we are going to prove that \( \zeta_0 \neq v_0 \). Choosing \( \phi = v_\epsilon \in X \) as a test function in (P\(_{\lambda, \epsilon}\)) we get
\[
\int_{\mathbb{R}^N} \frac{|v_\epsilon(x) - v_\epsilon(y)|^{p-2}(v_\epsilon(x) - v_\epsilon(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} \, dx \, dy
= \lambda \int_{\Omega} \frac{v_\epsilon}{(v_\epsilon + \epsilon)^{q(x)}} \, dx + \int_{\Omega} v_\epsilon^{r+1} \, dx.
\]
Since \( r + 1 < p_\epsilon^* \), using Lemma 2.1 we obtain
\[
\lim_{\epsilon \to 0^+} \int_{\Omega} (v_\epsilon)^{r+1} \, dx = \int_{\Omega} v_0^{r+1} \, dx.
\]
Moreover, since
\[
0 \leq \frac{v_\epsilon}{(v_\epsilon + \epsilon)^q(x)} \leq v_\epsilon^{1-q(x)},
\]
using (6.3) together with Vitali convergence theorem, we get
\[
\lambda \lim_{\epsilon \to 0^+} \int_\Omega \frac{v_\epsilon}{(v_\epsilon + \epsilon)^q(x)} dx = \lambda \int_\Omega v_0^{1-q(x)} dx.
\]
Therefore for every \( \phi \in X \), we have
\[
\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} \frac{|v_\epsilon(x) - v_\epsilon(y)|^{p-2}(v_\epsilon(x) - v_\epsilon(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dxdy = \lambda \int_\Omega v_0^{1-q(x)} dx + \int_\Omega v_0^{p+1} dx.
\]
Using Lemma 2.3 we can choose \( \phi = v_0 \) as a test function in (1.1) to deduce that
\[
||v_0||^p = \lambda \int_\Omega v_0^{1-q(x)} dx + \int_\Omega v_0^{p+1} dx.
\]
Hence we obtain
\[
\lim_{\epsilon \to 0} ||v_\epsilon||^p dx = ||v_0||^p,
\]
which gives the strong convergence of \( v_\epsilon \) to \( v_0 \) in \( X \). Now by the Vitali convergence theorem, we get
\[
\lim_{\epsilon \to 0} \int_\Omega [(v_\epsilon + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)}] dx = \int_\Omega v_0^{1-q(x)} dx,
\]
which together with the strong convergence of \( v_\epsilon \) implies \( \lim_{\epsilon \to 0} I_{\lambda,\epsilon}(v_\epsilon) = I_{\lambda}(v_0) \).
Hence from (6.12) we get \( \zeta_0 \neq v_0 \). \( \square \)

REFERENCES

[1] V. Ambrosio, Nontrivial solutions for a fractional \( p \)-Laplacian problem via Rabier theorem, Complex Var. Elliptic Equ., 62 (2017), 838–847.
[2] V. Ambrosio and T. Isernia, Multiplicity and concentration results for some nonlinear Schrödinger equations with the fractional \( p \)-Laplacian, Discrete Contin. Dyn. Syst., 38 (2018), 5855–5881.
[3] D. Arcoya and L. Boccardo, Multiplicity of solutions for a Dirichlet problem with a singular and a supercritical nonlinearities, Differ. Integral Equ., 26 (2013), 119–128.
[4] D. Arcoya and L. M. Mérida, Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearity, Nonlinear Anal., 95 (2014), 281–291.
[5] R. Arora, J. Giacomoni, D. Goel, and K. Sreenadh, Positive solutions of 1-D half-laplacian equation with singular and exponential nonlinearity, Asymptotic Anal., 118 (2020), 1–34.
[6] K. Bal and P. Garain, Multiplicity of Solution for a Quasilinear Equation with Singular Nonlinearity, Mediterr. J. Math., 17 (2020), 1–20.
[7] B. Barrios, I. D. Bonis, M. Medina and I. Peral, Semilinear problems for the fractional laplacian with a singular nonlinearity, Open Math., 13 (2015), 390–407.
[8] L. Boccardo, A Dirichlet problem with singular and supercritical nonlinearities, Nonlinear Anal., 75 (2012), 4436–4440.
[9] L. Brasco and E. Parini, The second eigenvalue of the fractional \( p \)-laplacian, Adv. Calc. Var., 9 (2016), 323–355.
[10] A. Canino, L. Montoro, B. Sciunzi and M. Squassina, Nonlocal problems with singular nonlinearity, Bull. Sci. Math., 141 (2017), 223–250.
[11] A. Canino, B. Sciunzi and A. Trombetta, Existence and uniqueness for \( p \)-Laplace equations involving singular nonlinearities, Nonlinear Differ. Equ. Appl., 23 (2016), 1–18.
[12] J. Carmona and P. J. M. Aparicio, A singular semilinear elliptic equation with a variable exponent, Adv. Nonlinear Stud., 16 (2016), 491–498.
[13] M. G. Crandall, P. H. Rabinowitz and L. Tartar, On a Dirichlet problem with a singular nonlinearity, Commun. Partial. Differ. Equ., 2 (1977), 193–222.
[14] L. M. Del Pezzo and A. Quaas, A Hopf’s lemma and a strong minimum principle for the fractional $p$-Laplacian, *J. Differ. Equ.*, **263** (2017), 765–778.

[15] E. D. Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573.

[16] P. Garain and T. Mukherjee, On a class of weighted $p$-Laplace equation with singular nonlinearity, *Mediterr. J. Math.*, **17** (2020), 110.

[17] P. Garain, On a degenerate singular elliptic problem, Preprint, arXiv:1803.02102.

[18] J. Giacomoni, T. Mukherjee and K. Sreenadh, Positive solutions of fractional elliptic equation with critical and singular nonlinearity, *Adv. Nonlinear Anal.*, **6** (2017), 327–354.

[19] J. Giacomoni, T. Mukherjee, and K. Sreenadh, A global multiplicity result for a very singular critical nonlocal equation, *Topol. Methods Nonlinear Anal.*, **54** (2019), 345–370.

[20] J. Giacomoni, I. Schindler and P. Takáč, Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **6** (2007), 117–158.

[21] Y. Haitao, Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem *J. Differ. Equ.*, **189** (2003), 487–512.

[22] N. Hirano, C. Saccon and N. Shioji, Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities, *Adv. Differ. Equ.*, **9** (2004), 197–220.

[23] N. Hirano, C. Saccon and N. Shioji, Brezis-nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem, *J. Differ. Equ.*, **245** (2008), 1997–2037.

[24] A. Iannizzotto, S. Mosconi and M. Squassina, Global hölder regularity for the fractional $p$-laplacian, *Rev. Mat. Iberoam.*, **32** (2016), 1353–1392.

[25] A. C. Lazer and P. J. McKenna, On a singular nonlinear elliptic boundary-value problem, *Proc. Amer. Math. Soc.*, **111** (1991), 721–730.

[26] V. Maz’ya and T. Shaposhnikova, On the bourgain, brezis, and mironescu theorem concerning limiting embeddings of fractional sobolev spaces, *J. Func. Anal.*, **195** (2002), 230–238.

[27] T. Mukherjee and K. Sreenadh, Fractional elliptic equations with critical growth and singular nonlinearity, *Electron. J. Differ. Equ.*, **23** (2016), 54.

[28] T. Mukherjee and K. Sreenadh, On Dirichlet problem for fractional $p$-Laplacian with singular non-linearity, *Adv. Nonlinear Anal.*, **8** (2019), 52–72.

[29] X. R. Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, *J. Math. Pure. Appl.*, **101** (2014), 275–302.

[30] R. Servadei and E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension, *Commun. Pure Appl. Anal.*, **12** (2013), 2445–2464.

Received February 2020; revised April 2020.

E-mail address: pgarain92@gmail.com

E-mail address: tulimukh@gmail.com