WARPED POISSON BRACKETS ON WARPED PRODUCTS

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Abstract. In this paper, we generalize the geometry of the product pseudo-Riemannian manifold equipped with the product Poisson structure ([10]) to the geometry of a warped product of pseudo-Riemannian manifolds equipped with a warped Poisson structure. We construct three bivector fields on a product manifold and show that each of them lead under certain conditions to a Poisson structure. One of these bivector fields will be called the warped bivector field. For a warped product of pseudo-Riemannian manifolds equipped with a warped bivector field, we compute the corresponding contravariant Levi-Civita connection and the curvatures associated with.

1. Introduction. Poisson manifolds play a fundamental role in Hamiltonian dynamics, where they serve as a phase space. The geometry of Poisson structures, which began as an outgrowth of symplectic geometry, has seen rapid growth in the last three decades, and has now become a very large theory, with interaction with many domains of mathematics, including Hamiltonian dynamics, integrable systems, representation theory, quantum groups, noncommutative geometry, singularity theory . . .

The warped product provides a way to construct new pseudo-Riemannian manifolds from given ones, see [11],[2] and [1]. This construction has useful applications in general relativity, in the study of cosmological models and black holes. It generalizes the direct product in the class of pseudo-Riemannian manifolds and it is defined as follows. Let \((M_1,\tilde{g}_1)\) and \((M_2,\tilde{g}_2)\) be two pseudo-Riemannian manifolds and let \(f : M_1 \rightarrow \mathbb{R}\) be a positive smooth function on \(M_1\), the warped product of \((M_1,\tilde{g}_1)\) and \((M_2,\tilde{g}_2)\) is the product manifold \(M_1 \times M_2\) equipped with the metric tensor \(\tilde{g}_f := \sigma_1^*\tilde{g}_1 + (f \circ \sigma_1)\sigma_2^*\tilde{g}_2\), where \(\sigma_1\) and \(\sigma_2\) are the projections of \(M_1 \times M_2\) onto \(M_1\) and \(M_2\) respectively. The manifold \(M_1\) is called the base of \((M_1 \times M_2, \tilde{g}_f)\).

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and $M_2$ the fiber. The function $f$ is called the warping function.

This paper extends the study of product pseudo-Riemannian manifolds equipped with product Poisson structures, [10], to warped products equipped with warped Poisson structures. We will give an expression for the contravariant Levi-Civita connection and for its curvatures.

This paper is organized as follows. In section 2, we recall basic definitions and facts about contravariant connections, generalized Poisson brackets and pseudo-Riemannian Poisson manifolds. By analogy with the definition of the Hessian and the gradient of a smooth function in the covariant case, in section 3 we define some differential operators associated to the Levi-Civita contravariant connection. In section 4, we consider bivector fields $\Pi_1$ and $\Pi_2$ on manifolds $M_1$ and $M_2$ respectively and, for a smooth function $\phi$ on $M_1 \times M_2$ relative to $\Pi_1, \Pi_2$ and the warping function $\mu$, then, we give conditions under which $\Pi^\mu$ becomes a warped Poisson tensor; next, if $D^1$ and $D^2$ are contravariant connections with respect to $\Pi_1$ and $\Pi_2$, we define a contravariant connection $D^\mu$ with respect to the bivector field $\Pi^\mu$ as the warped product of $D^1$ and $D^2$ using the warping function $\mu$, then, we express the generalized pre-Poisson bracket associated with $D^\mu$ in terms of the generalized pre-Poisson brackets $\{\ldots\}_1$ and $\{\ldots\}_2$ associated with $D^1$ and $D^2$ respectively, it will be clear that $\{\ldots\}$ is a generalized Poisson bracket, i.e. satisfy the graded Jacobi identity, if and only if $\{\ldots\}_1$ and $\{\ldots\}_2$ are; to end this section, we give two other bivector fields on $M_1 \times M_2$ and conditions under which they become Poisson tensors. In the final section 5, we consider bivector fields $\Pi_1$ and $\Pi_2$ on pseudo-Riemannian manifolds $(M_1, \tilde{g}_1)$ and $(M_2, \tilde{g}_2)$ respectively, we compute the Levi-Civita contravariant connection $D$ associated with the pair $(g^f, \Pi^\ast)$, where $g^f$ is the cometric of the warped metric $\tilde{g}_2^f$, and we compute the curvatures of $D$; finally, if $g_1$ and $g_2$ denote the cometrics of $\tilde{g}_1$ and $\tilde{g}_2$ respectively, we conclude with some interesting relationships between the geometry of the triples $(M_1, g_1, \Pi_1)$ and $(M_2, g_2, \Pi_2)$ and that of $(M_1 \times M_2, g^f, \Pi^\ast)$.

2. Preliminaries

2.1. Poisson structures. Many fundamental definitions and results about Poisson manifolds can be found in [12] and [5].

A Poisson bracket on a manifold $M$ is a Lie bracket $\{\ldots\}$ on $C^\infty(M)$ satisfying the Leibniz identity

$$\{\varphi, \phi \psi\} = \{\varphi, \phi\} \psi + \phi \{\varphi, \psi\}, \quad \forall \varphi, \phi, \psi \in C^\infty(M).$$

A Poisson manifold is a manifold equipped with a Poisson bracket. The Leibniz identity means that, for a given function $\varphi \in C^\infty(M)$ on a Poisson manifold $M$, the map $\psi \mapsto \{\varphi, \psi\}$ is a derivation, and thus, there is a unique vector field $X_\varphi$ on $M$, called the Hamiltonian vector field of $\varphi$, such that for any $\psi \in C^\infty(M)$ we have

$$X_\varphi(\psi) = \{\varphi, \psi\}.$$

A function $\varphi \in C^\infty(M)$ is called a Casimir function if $X_\varphi \equiv 0$. It follows from the Leibniz identity that there exists a bivector field $\Pi \in \Gamma(\wedge^2TM)$ such that

$$\{\varphi, \psi\} = \Pi(d\varphi, d\psi). \quad (1)$$

If we denote by $[\ldots]_S$ the Schouten-Nijenhuis bracket, we have

$$\{\{\varphi, \phi\}, \psi\} + \{\{\phi, \psi\}, \varphi\} + \{\psi, \varphi, \phi\} = \frac{1}{2} [\Pi, \Pi]_S(d\varphi, d\phi, d\psi).$$
Therefore, the Jacobi identity for \(\{.,.\}\) is equivalent to the condition \(\Pi,\Pi\| = 0\). Conversely, if \(\Pi\) is a bivector field on a manifold \(M\) such that \(\Pi,\Pi\| = 0\), then the bracket \(\{.,.\}\) defined on \(C^\infty(M)\) by (1) is a Poisson bracket. Such a bivector field is called a Poisson tensor.

Let \(\Pi\) be a bivector field on a manifold \(M\), in a local system of coordinates \((x_1, \ldots, x_n)\) we have

\[
\Pi = \sum_{i<j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \frac{1}{2} \sum_{i,j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},
\]

where \(\Pi_{ij} = \Pi(dx_i, dx_j)\). The bivector field \(\Pi\) is a Poisson tensor if and only if it satisfies the following system of equations:

\[
\oint_{ijk} \sum_l \frac{\partial \Pi_{ij}}{\partial x_l} \Pi_{lk} = 0,
\]

for all \(i,j,k\), where \(\oint_{ijk} A_{ijk}\) means the cyclic sum \(A_{ijk} + A_{kij} + A_{jki}\).

Given a Poisson tensor \(\Pi\) (or more generally, a bivector field) on a manifold \(M\), we can associate to it a natural homomorphism called the anchor map:

\[
\sharp \Pi : T^* M \to TM
\]

defined by

\[
\beta(\sharp \Pi(\alpha)) = \Pi(\alpha, \beta)
\]

for any \(\alpha, \beta \in T^* M\). If \(\varphi\) is a function, then \(\sharp \Pi(d\varphi)\) is the Hamiltonian vector field of \(\varphi\).

Let \(x\) be a point of \(M\). The restriction of \(\sharp \Pi\) to the cotangent space \(T^*_x M\) will be denoted by \(\sharp \Pi_x\). The image \(C_x = \text{Im} \sharp \Pi_x\) of \(\sharp \Pi_x\) is called the characteristic space of \(\Pi\) at \(x\). The dimension \(\text{rank}_x \Pi = \dim C_x\) of \(C_x\) is called the rank of \(\Pi\) at \(x\) and \(\text{rank} \Pi = \max_{x \in M} \{\text{rank}_x \Pi\}\) is called the rank of \(\Pi\). When \(\text{rank}_x \Pi = \dim M\) we say that \(\Pi\) is nondegenerate at \(x\). If \(\text{rank}_x \Pi\) is a constant on \(M\), i.e. does not depend on \(x\), then \(\Pi\) is called regular.

If \(\Pi\) is nondegenerate at all \(x\) of \(M\) then the anchor map \(\sharp \Pi\) is invertible and defines a singular foliation \(\mathcal{F}\). The leaves of \(\mathcal{F}\) are symplectic immersed submanifolds of \(M\). The foliation \(\mathcal{F}\) is called the symplectic foliation associated to the Poisson structure \((M, \Pi)\).

2.2. Contravariant connections and generalized Poisson brackets. Contravariant connections associated to a Poisson structure have recently turned out to be useful in several areas of Poisson geometry. Contravariant connections were defined by Vaisman [12] and were analyzed in details by Fernandes [6]. This notion appears extensively in the context of noncommutative deformations (see [7, 8]).

Let \(M\) be a manifold. A bivector field \(\Pi\) on \(M\) induces on the space of differential 1-forms \(\Gamma(T^* M)\) a bracket \(\{.,.\}\Pi\) called the Koszul bracket:

\[
[\alpha, \beta]_\Pi = \mathcal{L}_{\sharp \Pi(\alpha)} \beta - \mathcal{L}_{\sharp \Pi(\beta)} \alpha - d(\Pi(\alpha, \beta)).
\]

If \(\Pi\) is a Poisson tensor, the bracket \(\{.,.\}\Pi\) is a Lie bracket. For this Lie bracket on \(\Gamma(T^* M)\) and for the usual Lie bracket \(\{.,.\}\) on the space of vector fields \(\Gamma(TM)\), the
anchor map $\sharp \Pi$ induces a Lie algebra homomorphism $\sharp \Pi : \Gamma(T^*M) \rightarrow \Gamma(TM)$, i.e.
\[ \sharp \Pi([\alpha, \beta]) = [\sharp \Pi(\alpha), \sharp \Pi(\beta)]. \tag{2} \]

A contravariant connection on $M$ with respect to the bivector field $\Pi$ is an $\mathbb{R}$-bilinear map
\[ D : \Gamma(T^*M) \times \Gamma(T^*M) \rightarrow \Gamma(TM), \]
\[ (\alpha, \beta) \mapsto D_{\alpha} \beta \]
such that the mapping $\alpha \mapsto D_{\alpha}$ is $C^\infty(M)$-linear, that is
\[ D_{\varphi \alpha} \beta = \varphi D_{\alpha} \beta \quad \text{for all } \varphi \in C^\infty(M), \]
and that the mapping $\beta \mapsto D_{\alpha} \beta$ is a derivation in the following sense :
\[ D_{\alpha} \varphi \beta = \varphi D_{\alpha} \beta + \sharp \Pi(\varphi) \beta, \quad \text{for all } \varphi \in C^\infty(M). \]

As in the covariant case, for a differential 1-form $\alpha$, we can use the derivation $D_{\alpha}$ to define a contravariant derivative of multivector fields of degree $r$ by
\[ (D_{\alpha}Q)(\beta_1, \ldots, \beta_r) = \sharp \Pi(\alpha)(Q(\beta_1, \ldots, \beta_r)) - \sum_{i=1}^r Q(\beta_1, \ldots, D_{\alpha} \beta_i, \ldots, \beta_r). \tag{3} \]
In particular, if $X$ is a vector field, we have
\[ (D_{\alpha}X)(\beta) = \sharp \Pi(\alpha)(\beta(X)) - (D_{\alpha} \beta)(X). \tag{4} \]
We can also define a contravariant derivative of $r$-forms by
\[ (D_{\alpha} \omega)(X_1, \ldots, X_r) = \sharp \Pi(\alpha)(\omega(X_1, \ldots, X_r)) - \sum_{i=1}^r \omega(X_1, \ldots, D_{\alpha} X_i, \ldots, X_r). \tag{5} \]

The definitions of the torsion and of the curvature of a contravariant connection $D$ are formally identical to the definitions in the covariant case (see [6]) :
\[ T(\alpha, \beta) = D_{\alpha} \beta - D_{\beta} \alpha - [\alpha, \beta]_{\Pi} \quad \text{and} \quad R(\alpha, \beta) = D_{\alpha} D_{\beta} - D_{\beta} D_{\alpha} - D_{[\alpha, \beta]_{\Pi}}. \]

We say that a connection $D$ is torsion-free (resp. flat) if its torsion $T$ (resp. its curvature $R$) vanishes identically. We say that $D$ is locally symmetric if $DR = 0$, i.e., for any $\alpha, \beta, \gamma, \delta \in \Gamma(T^*M)$, we have :
\[ (D_{\alpha} R)(\beta, \gamma) \delta := D_{\alpha}(R(\beta, \gamma) \delta) - R(D_{\alpha} \beta, \gamma) \delta - R(\beta, D_{\alpha} \gamma) \delta - R(\beta, \gamma) D_{\alpha} \delta = 0. \tag{6} \]

Let us briefly recall the definition of a generalized Poisson bracket, introduced by E. Hawkins in [8]. To a contravariant connection $D$, with respect to the bivector field $\Pi$, Hawkins associated an $\mathbb{R}$-bilinear bracket on the differential graded algebra of differential forms $\Omega^*(M)$ that generalizes the initial pre-Poisson bracket $\{ \ldots \}$ given by (1) on $C^\infty(M)$. This bracket, that we will then call a generalized pre-Poisson bracket associated with the contravariant connection $D$ and that we will still denote by $\{ \ldots \}$, is defined as follows. The bracket of a function and a differential form is given by
\[ \{ \varphi, \eta \} = D_{d \varphi} \eta, \tag{7} \]
and it is extended to two any differential forms in a way it satisfies the following properties :
\begin{itemize}
\item $\{ \ldots \}$ is antisymmetric, i.e.
\[ \{ \omega, \eta \} = -(-1)^{\deg \omega \cdot \deg \eta} \{ \eta, \omega \}, \tag{8} \]
\item the differential $d$ is a derivation with respect to $\{ \ldots \}$, i.e.
\[ d\{ \omega, \eta \} = \{ d\omega, \eta \} + (-1)^{\deg \omega} \{ \omega, d\eta \}, \tag{9} \]
\end{itemize}
When $\mathcal{D}$ is flat, Hawkins showed that there is a $(2,3)$-tensor $\mathcal{M}$ symmetric in the covariant arguments, antisymmetric in the contravariant arguments and such that the following two assertions are equivalent:

(i) the generalized pre-Poisson bracket $\{.,.\}$ satisfies the graded Jacobi identity

$$\{\{\omega,\eta\},\lambda\} - \{\omega,\{\eta,\lambda\}\} + (-1)^{\deg\omega \deg\eta}\{\eta,\{\omega,\lambda\}\} = 0$$

and (ii) the tensor $\mathcal{M}$ vanishes identically.

The tensor $\mathcal{M}$ is called the metacurvature of $(\Pi)$ at $p$. Let $(\mathcal{M},\otimes,\otimes)$ be a pseudo-Riemannian Poisson manifold. If $\mathcal{M}$ vanishes identically, the contravariant connection $\mathcal{D}$ is called metaflat and the bracket $\{.,.\}$ is called the generalized Poisson bracket associated with $\mathcal{D}$.

2.3. Pseudo-Riemannian Poisson manifolds. Let $(\tilde{M},\tilde{g})$ be a pseudo-Riemannian manifold. The metric $\tilde{g}$ defines the musical isomorphisms

$$\varphi_g : T\tilde{M} \rightarrow T^*\tilde{M}$$

$$X \mapsto \tilde{g}(X,.)$$

and its inverse $\varphi_{\tilde{g}}$. We define the metric $g$ of the metric $\tilde{g}$ by:

$$g(\alpha,\beta) = \tilde{g}(\varphi_g(\alpha),\varphi_g(\beta)).$$

For a bivector field $\Pi$ on $M$, there exists a unique contravariant connection $\mathcal{D}$ associated to the pair $(g,\Pi)$ such that the metric $g$ is parallel with respect to $\mathcal{D}$, i.e.

$$\varphi_{\Pi}(\alpha).g(\beta,\gamma) = g(\mathcal{D}_\alpha\beta,\gamma) + g(\beta,\mathcal{D}_\alpha\gamma),$$

and that $\mathcal{D}$ is torsion-free, i.e.

$$\mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha = [\alpha,\beta]_{\Pi}.$$  

(13)

The connection $\mathcal{D}$ is called the Levi-Civita contravariant connection associated with $(g,\Pi)$. It is characterized by the Koszul formula:

$$2g(\mathcal{D}_\alpha\beta,\gamma) = \varphi_{\Pi}(\alpha).g(\beta,\gamma) + \varphi_{\Pi}(\beta).g(\alpha,\gamma) - \varphi_{\Pi}(\gamma).g(\alpha,\beta)$$

$$+ g([\alpha,\beta]_{\Pi},\gamma) + g([\gamma,\alpha]_{\Pi},\beta) + g([\gamma,\beta]_{\Pi},\alpha).$$

(14)

If $\mathcal{R}$ is the curvature of $\mathcal{D}$ and if $\theta_p$ and $\eta_p$ are two non-parallel cotangent vectors at $p \in M$ then the number

$$\mathcal{K}_p(\theta_p,\eta_p) = \frac{g_p(\mathcal{R}_p(\theta_p,\eta_p)\eta_p,\theta_p)}{g_p(\theta_p,\theta_p)g_p(\eta_p,\eta_p) - g_p(\theta_p,\eta_p)^2}$$

is called the sectional contravariant curvature of $(M,g,\Pi)$ at $p$ in the direction of the plane spanned by the covectors $\theta_p$ and $\eta_p$ in $T_p^*M$. Let $\{e_1,\ldots,e_n\}$ be a local orthonormal basis of $T_p^*M$ with respect to $g$ on an open $U \subset M$. Let $\theta_p$ and $\eta_p$ be two cotangent vectors at $p \in M$. The Ricci curvature $r_p$ at $p$ and the scalar curvature $\mathcal{S}_p$ of $(M,g,\Pi)$ at $p$ are defined by

$$r_p(\theta_p,\eta_p) = \sum_{i=1}^n g_p(\mathcal{R}_p(\theta_p,e_i)e_i,\eta_p)$$

and

$$\mathcal{S}_p = \sum_{j=1}^n \sum_{i=1}^n g_p(\mathcal{R}_p(e_i,e_j)e_j,e_i).$$
With the notations above, we say that the triple $(M, g, \Pi)$ is a pseudo-Riemannian Poisson manifold if $\mathcal{D} \Pi = 0$, i.e., for any $\alpha, \beta, \gamma \in \Gamma(T^* M)$, we have
\[ z_{\Pi}(\alpha) \Pi(\beta, \gamma) - \Pi(D_\alpha \beta, \gamma) - \Pi(\beta, D_\alpha \gamma) = 0. \]
The notions of Levi-Civita contravariant connection and of pseudo-Riemannian Poisson manifold were introduced by Boucetta in [3, 4].

We say that the triple $(M, g, \Pi)$ is flat (resp. locally symmetric) if $\mathcal{R} = 0$ (resp. $\mathcal{D} \mathcal{R} = 0$). We say that the triple $(M, g, \Pi)$ is metaflat if $\mathcal{R} = 0$ and $\mathcal{M} = 0$, where $\mathcal{M}$ is the metacurvature of the Levi-Civita contravariant connection $\mathcal{D}$.

In this paper, for a pseudo-Riemannian manifold $(M, \tilde{g})$ and a bivector field $\Pi$ on $M$, we will always denote by $J \in \Gamma(M, TM \otimes T^* M)$ the field of homomorphisms defined by
\[ g(J\alpha, \beta) = \Pi(\alpha, \beta). \quad (15) \]
If $(M, g, \Pi)$ is a pseudo-Riemannian Poisson manifold and $\mathcal{D}$ the Levi-Civita contravariant connection associated with $(g, \Pi)$ then $\mathcal{D} J = 0$, i.e., for any $\alpha, \beta \in \Gamma(T^* M)$,
\[ D_\alpha (J\beta) = J D_\alpha \beta. \]

3. Some differential operators associated to the pair $(g, \Pi)$. Let $(M, \tilde{g})$ be a pseudo-Riemannian manifold, $g$ be the cometric of $\tilde{g}$ and $\Pi$ be a bivector field on $M$. In this section, we define some new differential operators on $M$.

**Definition 3.1.** Let $\mathcal{D}$ be the contravariant Levi-Civita connection associated with the pair $(g, \Pi)$. We define the contravariant Hessian $H^\gamma_{\Pi}$ of a function $\varphi \in C^\infty(M)$ with respect to the bivector field $\Pi$ by
\[ H^\gamma_{\Pi} = \mathcal{D} \mathcal{D} \varphi, \]
i.e., the contravariant Hessian $H^\gamma_{\Pi}$ of $\varphi$ is its second contravariant differential.

**Proposition 1.** The contravariant Hessian $H^\gamma_{\Pi}$ of $\varphi$ is a $(0, 2)$-tensor field and we have
\[ H^\gamma_{\Pi}(\alpha, \beta) = z_{\Pi}(\alpha)(z_{\Pi}(\beta)(\varphi)) - z_{\Pi}(D_\alpha \beta)(\varphi) = -g(D_\alpha J \varphi, \beta). \]
Moreover, when $\Pi$ is Poisson, the Hessian $H^\gamma_{\Pi}$ is symmetric.

**Proof.** We have $H^\gamma_{\Pi}(\alpha, \beta) = \mathcal{D} \mathcal{D} \varphi(\alpha, \beta) = (D_\alpha(D_\beta \varphi))(\beta)$. From (4) and since we have $\mathcal{D} \varphi = d \varphi \circ \mathcal{P}$, i.e. $D_\alpha \varphi = z_{\Pi}(\alpha)(\varphi)$, we deduce that
\[ H^\gamma_{\Pi}(\alpha, \beta) = z_{\Pi}(\alpha) (D_\beta \varphi) - D D_{\alpha \beta} \varphi = z_{\Pi}(\alpha) (z_{\Pi}(\beta)(\varphi)) - z_{\Pi}(D_\alpha \beta)(\varphi). \quad (16) \]

To prove the second equality, notice that, for any 1-form $\gamma$ on $M$, we have
\[ z_{\Pi}(\gamma)(\varphi) = \Pi(\gamma, d \varphi) = -\Pi(d \varphi, \gamma) = -g(J d \varphi, \gamma) = -g(\gamma, J d \varphi), \]
therefore, substituting in (16), taking $\gamma = \beta$ and $\gamma = D_\alpha \beta$, we get
\[ H^\gamma_{\Pi}(\alpha, \beta) = -z_{\Pi}(\alpha)(g(\beta, J d \varphi)) + g(D_\alpha \beta, J d \varphi), \]
and using (12), we get $H^\gamma_{\Pi}(\alpha, \beta) = -g(D_\alpha \beta, J d \varphi)$.

Now, assume that $\Pi$ is Poisson. By (2) and since $\mathcal{D}$ is torsion-free (13), we have
\[ z_{\Pi}(D_\alpha \beta) = z_{\Pi}(D_\beta \alpha) + [\alpha, \beta]_{\Pi} = z_{\Pi}(D_\beta \alpha) + z_{\Pi}([\alpha, \beta]_{\Pi}) = z_{\Pi}(D_\beta \alpha) + z_{\Pi}(\alpha), \]
and by substituting in (16), we get
\[ H^\gamma_{\Pi}(\alpha, \beta) = z_{\Pi}(\alpha)(z_{\Pi}(\beta)(\varphi)) - z_{\Pi}(D_\beta \alpha)(\varphi) - [z_{\Pi}(\alpha), z_{\Pi}(\beta)](\varphi) \]
\[ = z_{\Pi}(\beta)(z_{\Pi}(\alpha)(\varphi)) - z_{\Pi}(D_\beta \alpha)(\varphi), \]
therefore, $H^\gamma_{\Pi}$ is symmetric. \qed
Observe that in case $(M,g,\Pi)$ is a pseudo-Riemannian Poisson manifold, we have
\[
H^\Pi_\alpha(\alpha,\beta) = -g(JD_\alpha d\varphi,\beta) = -\Pi(D_\alpha d\varphi,\beta) = \Pi(\beta,D_\alpha d\varphi).
\]

**Proposition 2.** If for any $\varphi \in C^\infty(M)$ we put
\[
\angle_\Pi(\varphi) = \text{tr}_g(\alpha \mapsto D_\alpha Jd\varphi) \quad \text{and} \quad \triangledown_\Pi(\varphi) = \text{tr}_g(\alpha \mapsto D_\alpha d\varphi),
\]
where $\text{tr}_g$ denotes the trace with respect to $g$, then $\angle_\Pi$ is a differential operator of degree two on $C^\infty(M)$ and $\triangledown_\Pi$ is a vector field on $M$.

**Proof.** One can easily see that both $\angle_\Pi$ and $\triangledown_\Pi$ are $\mathbb{R}$-linear. Now, let $\{dx_1,\ldots, dx_n\}$ be a local, orthonormal basis of 1-forms and let us show that $\angle_\Pi$ is a differential operator of degree two on $C^\infty(M)$. If $\varphi \in C^\infty(M)$, we have
\[
\angle_\Pi(\varphi) = \sum_{i=1}^{n} g(D_{dx_i} Jd\varphi, dx_i) = \sum_{i,j} g(D_{dx_i}(\frac{\partial \varphi}{\partial x_j} Jd\varphi), dx_i).
\]
Since by $(15)$ we have $Jd\varphi_j = \sum_{k=1}^{n} \Pi_{jk} dx_k$, then
\[
D_{dx_i}(\frac{\partial \varphi}{\partial x_j} Jd\varphi_j) = \sum_{k=1}^{n} \left( \angle_\Pi(dx_i)(\frac{\partial \varphi}{\partial x_j} \Pi_{jk}) + \frac{\partial \varphi}{\partial x_j} \Pi_{jk} D_{dx_i} dx_k \right).
\]
Therefore,
\[
\angle_\Pi(\varphi) = \sum_{i,j,k=1}^{n} \Pi_{ik} \frac{\partial \varphi}{\partial x_k} \Pi_{ji} + \sum_{i,j,k=1}^{n} \frac{\partial \varphi}{\partial x_j} \Pi_{jk} \Gamma^i_{ik}.
\]
Finally, as $\Gamma^i_{ik} = 0$, we get
\[
\angle_\Pi(\varphi) = \sum_{i,j,k=1}^{n} \frac{\partial \varphi}{\partial x_k} \Pi_{ij} \frac{\partial \varphi}{\partial x_j} + \sum_{i,j,k=1}^{n} \Pi_{ij} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}.
\]
To show that $\triangledown_\Pi$ corresponds to a vector field one has to verify that it satisfies the Leibnitz rule with respect to $\varphi$, i.e. that for any $\varphi, \psi \in C^\infty(M)$, we have
\[
\triangledown_\Pi(\varphi \psi) = \varphi \sum_{i=1}^{n} g(D_{dx_i} d\psi, dx_i) + \psi \sum_{i=1}^{n} g(D_{dx_i} d\varphi, dx_i).
\]
A direct calculation using the definition of $D$ gives
\[
\triangledown_\Pi(\varphi \psi) - \varphi \triangledown_\Pi(\psi) - \psi \triangledown_\Pi(\varphi) = \sum_{i,j=1}^{n} \Pi_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \frac{\partial \psi}{\partial x_j} \frac{\partial \varphi}{\partial x_i}.
\]
It is clear that the right hand side of this formula is zero. \hfill $\square$

4. Poisson brackets on a product manifold.

4.1. Horizontal and vertical lifts. Throughout this paper $M_1$ and $M_2$ will be respectively $m_1$ and $m_2$ dimensional manifolds, $M_1 \times M_2$ the product manifold with the natural product coordinate system and $\sigma_1 : M_1 \times M_2 \to M_1$ and $\sigma_2 : M_1 \times M_2 \to M_2$ the usual projection maps.

We recall briefly how the calculus on the product manifold $M_1 \times M_2$ derives from that of $M_1$ and $M_2$ separately. For details see [11].

Let $\varphi_1$ in $C^\infty(M_1)$. The horizontal lift of $\varphi_1$ to $M_1 \times M_2$ is $\varphi_1^h = \varphi_1 \circ \sigma_1$. One can define the horizontal lifts of tangent vectors as follows. Let $p \in M_1$ and let $X_p \in T_p M_1$. For any $q \in M_2$ the horizontal lift of $X_p$ to $(p, q)$ is the unique tangent vector $X^h_{(p, q)}$ in $T_{(p, q)}(M_1 \times M_2)$ such that $d_{(p, q)} \sigma_1(X^h_{(p, q)}) = X_p$.
and \( d_{(p,q)}\sigma_2(X^h_{(p,q)}) = 0 \). We can also define the horizontal lifts of vector fields as follows. Let \( X_1 \in \Gamma(TM_1) \). The horizontal lift of \( X_1 \) to \( M_1 \times M_2 \) is the vector field \( X_1^h \in \Gamma(T(M_1 \times M_2)) \) whose value at each \((p, q)\) is the horizontal lift of the tangent vector \((X_1)_p\) to \((p, q)\). For \((p, q) \in M_1 \times M_2\), we will denote the set of the horizontal lifts to \((p, q)\) of all the tangent vectors of \( M_1 \) at \( p \) by \( L(p, q)(M_1) \). We will denote the set of the horizontal lifts of all vector fields on \( M_1 \) by \( \mathfrak{L}(M_1) \).

The vertical lift \( \varphi^v_2 \) of a function \( \varphi_2 \in C^\infty(M_2) \) to \( M_1 \times M_2 \) and the vertical lift \( X^v_2 \) of a vector field \( X_2 \in \Gamma(TM_2) \) to \( M_1 \times M_2 \) are defined in the same way using the projection \( \sigma_2 \). Note that the spaces \( \mathfrak{L}(M_1) \) of the horizontal lifts and \( \mathfrak{L}(M_2) \) of the vertical lifts are vector subspaces of \( \Gamma(TM_1 \times M_2) \) but neither is invariant under multiplication by arbitrary functions \( \varphi \in C^\infty(M_1 \times M_2) \).

We define the horizontal lift of a covariant tensor \( \omega_1 \) on \( M_1 \) to be its pullback \( \omega^h_1 \) to \( M_1 \times M_2 \) by the means of the projection map \( \sigma_1 \), i.e. \( \omega^h_1 := \sigma_1^*(\omega_1) \). In particular, for a 1-form \( \alpha_1 \) on \( M_1 \) and a vector field \( X \) on \( M_1 \times M_2 \), we have

\[
(a^h_1)(X) = \alpha_1(d\sigma_1(X)).
\]

 Explicitly, if \( u \) is a tangent vector to \( M_1 \times M_2 \) at \((p, q)\), then

\[
(a^h_1)_{(p,q)}(u) = (\alpha_1)_p(d_{(p,q)}\sigma_1)(u).
\]

Similarly, we define the vertical lift of a covariant tensor \( w_2 \) on \( M_2 \) to be its pullback \( \omega^v_2 \) to \( M_1 \times M_2 \) by the means of the projection map \( \sigma_2 \).

Observe that if \( \{dx_1, \ldots, dx_n\} \) is the local basis of the 1-forms relative to a chart \((U, \Phi)\) of \( M_1 \) and \( \{dy_1, \ldots, dy_{n_2}\} \) the local basis of the 1-forms relative to a chart \((V, \Psi)\) of \( M_2 \), then \( \{dx^1, \ldots, dx^i, \ldots, dx^{n_1}, dy^1, \ldots, dy^l, \ldots, dy^{n_2}\} \) is the local basis of the 1-forms relative to the chart \((U \times V, \Phi \times \Psi)\) of \( M_1 \times M_2 \).

Let \( Q_1 \) (resp. \( Q_2 \)) be an \( r \)-contravariant tensor on \( M_1 \) (resp. on \( M_2 \)). We define the horizontal lift \( Q^h_1 \) of \( Q_1 \) (resp. the vertical lift \( Q^v_2 \) of \( Q_2 \)) to \( M_1 \times M_2 \) by

\[
\begin{align*}
\{ Q^h_1(\alpha^h_1, \ldots, \alpha^h_v) = [Q_1(\alpha_1, \ldots, \alpha_r)]^h & \quad \text{and} \\
i_{\beta^h}Q^h_1 = 0, & \quad \forall \beta \in \Gamma(T^*M_2),
\end{align*}
\]

resp.

\[
\begin{align*}
\{ Q^v_2(\beta^v_1, \ldots, \beta^v_v) = [Q_2(\beta_1, \ldots, \beta_r)]^v & \quad \text{and} \\
i_{\alpha^v}Q^v_2 = 0, & \quad \forall \alpha \in \Gamma(T^*M_1),
\end{align*}
\]

where \( i \) denotes the inner product. The following lemma will be useful for our computations.

**Lemma 4.1.**

1. Let \( \varphi_i \in C^\infty(M_i) \), \( X_i, Y_i \in \Gamma(TM_i) \) and \( \alpha_i \in \Gamma(T^*M_i) \), \( i = 1, 2 \). Let \( \varphi = \varphi^h_1 + \varphi^v_2 \), \( X = X^h_1 + X^v_2 \) and \( \alpha, \beta \in \Gamma(T^*(M_1 \times M_2)) \). Then

   i/ For all \((i, l) \in \{(1, h), (2, v)\} \), \( i \) have

   \[
   X^i_l(\varphi) = X^i_l(\varphi^i)^l, \quad [X^i_l, Y^j_l] = [X^i_l, Y^j_l] \quad \text{and} \quad \alpha^i_l(X) = \alpha_l(X^i_l)^l.
   \]

   ii/ If for all \((i, l) \in \{(1, h), (2, v)\} \) we have \( \alpha(X^i_l) = \beta(X^i_l) \), then \( \alpha = \beta \).

2. Let \( \omega_i \) and \( \eta_i \) be \( r \)-forms on \( M_i \), \( i = 1, 2 \). Let \( \omega = \omega^h_1 + \omega^v_2 \) and \( \eta = \eta^h_1 + \eta^v_2 \). We have

\[
d\omega = (d\omega^h_1)^h + (d\omega^v_2)^v \quad \text{and} \quad \omega \wedge \eta = (\omega \wedge \eta^h_1)^h + (\omega^v_2 \wedge \eta^v_2)^v.
\]

**Proof.** See [9].
4.2. The warped Poisson tensor. Now, we construct a bivector field on a product manifold and give the conditions under which it becomes a Poisson tensor.

Let \( \Pi_1 \) and \( \Pi_2 \) be bivector fields on \( M_1 \) and \( M_2 \) respectively. Given a smooth function \( \mu \) on \( M_1 \), we define a bivector field \( \Pi^\mu \) on \( M_1 \times M_2 \) by \( \Pi^\mu = \Pi_1^h + \mu^h\Pi_2^\mu \). It is the unique bivector field such that

\[
\Pi^\mu(\alpha_1^h, \beta_1^h) = \Pi_1(\alpha_1^h, \beta_1^h) \quad \text{and} \quad \Pi^\mu(\alpha_2^\mu, \beta_2^\mu) = \mu^h\Pi_2(\alpha_2^\mu, \beta_2^\mu),
\]

for any \( \alpha_i, \beta_i \in \Gamma(T^*M_i), \ i = 1, 2 \). We call \( \Pi^\mu \) the warped bivector field relative to \( \Pi_1, \Pi_2 \) and the warping function \( \mu \).

**Proposition 3.** Let \( \alpha_i, \beta_i \in \Gamma(T^*M_i), \ i = 1, 2 \). Let \( \alpha = \alpha_1^h + \alpha_2^\mu \) and \( \beta = \beta_1^h + \beta_2^\mu \). Then

1. \( \zeta_{\Pi^\mu}(\alpha) = [\zeta_{\Pi_1}(\alpha_1)]^h + \mu^h[\zeta_{\Pi_2}(\alpha_2)]^\mu \)
2. \( \mathcal{L}_{\zeta_{\Pi^\mu}(\alpha)}\beta = (\mathcal{L}_{\zeta_{\Pi_1}(\alpha_1)}\beta_1)^h + \mu^h(\mathcal{L}_{\zeta_{\Pi_2}(\alpha_2)}\beta_2)^\mu + \Pi_2(\alpha_2, \beta_2)(d\mu)^\mu \)
3. \[ [\alpha, \beta]_{\Pi^\mu} = [\alpha_1, \beta_1]^h + \mu^h[\alpha_2, \beta_2]_{\Pi_2} + \Pi_2(\alpha_2, \beta_2)(d\mu)^\mu. \]

**Proof.**
1. By Lemma 4.1, for any \( \gamma_i \in \Gamma(T^*M_i), \ i = 1, 2 \), we have

\[
\begin{align*}
\gamma_1^h(\zeta_{\Pi^\mu}(\alpha_1^h)) &= \Pi^\mu(\alpha_1^h, \gamma_1^h) = \Pi_1(\alpha_1^h, \gamma_1^h) = (\gamma_1(\zeta_{\Pi_1}(\alpha_1^h)))^h = \gamma_1^h([\zeta_{\Pi_1}(\alpha_1^h)]^h) \\
\gamma_2^\mu(\zeta_{\Pi^\mu}(\alpha_2^\mu)) &= \Pi^\mu(\alpha_2^\mu, \gamma_2^\mu) = 0 = \gamma_2^\mu([\zeta_{\Pi_2}(\alpha_2^\mu)]^\mu).
\end{align*}
\]

and similarly

\[
\begin{align*}
\gamma_1^h(\zeta_{\Pi^\mu}(\alpha_2^\mu)) &= \Pi^\mu(\alpha_2^\mu, \gamma_1^h) = 0 = \gamma_1^h(\mu^h[\zeta_{\Pi_2}(\alpha_2^\mu)]^\mu) \\
\gamma_2^\mu(\zeta_{\Pi^\mu}(\alpha_1^h)) &= \Pi^\mu(\alpha_1^h, \gamma_2^\mu) = \mu^h\Pi_2(\alpha_2^\mu, \gamma_2^\mu) = \gamma_2^\mu(\mu^h[\zeta_{\Pi_2}(\alpha_2^\mu)]^\mu).
\end{align*}
\]

Therefore \( \zeta_{\Pi^\mu}(\alpha) = \zeta_{\Pi^\mu}(\alpha_1^h) + \zeta_{\Pi^\mu}(\alpha_2^\mu) = [\zeta_{\Pi_1}(\alpha_1)]^h + \mu^h[\zeta_{\Pi_2}(\alpha_2)]^\mu. \)

2. Using the assertion 1., the Leibniz identity and Lemma 4.1, we have

\[
\begin{align*}
\mathcal{L}_{\zeta_{\Pi^\mu}(\alpha)}\beta &= \mathcal{L}_{[\zeta_{\Pi_1}(\alpha_1)]^h + \mu^h[\zeta_{\Pi_2}(\alpha_2)]^\mu}\beta \\
&= ([\mathcal{L}_{\zeta_{\Pi_1}(\alpha_1)}\beta_1]^h + \mu^h[\mathcal{L}_{\zeta_{\Pi_2}(\alpha_2)}\beta_2]^\mu + \beta([\zeta_{\Pi_2}(\alpha_2)]^\mu)(d\mu)^\mu \\
&= ([\mathcal{L}_{\zeta_{\Pi_1}(\alpha_1)}\beta_1]^h + \mu^h[\mathcal{L}_{\zeta_{\Pi_2}(\alpha_2)}\beta_2]^\mu + \Pi_2(\alpha_2, \beta_2)(d\mu)^\mu.
\end{align*}
\]

3. It is a direct consequence of 2. \( \square \)

The following result provide a necessary and sufficient condition for the bivector field \( \Pi^\mu \) to be a Poisson tensor.

**Theorem 4.2.** Let \( (M_1, \Pi_1) \) and \( (M_2, \Pi_2) \) be two Poisson manifolds such that \( \Pi_2 \) is non trivial and let \( \mu \) be a smooth function on \( M_1 \). Then \( (M_1 \times M_2, \Pi^\mu) \) is a Poisson manifold if and only if \( \mu \) is a Casimir function.

**Proof.** A straightforward calculation using Lemma 4.1 shows that

\[
\begin{align*}
[\Pi^\mu, \Pi^\mu]_S((d\phi_1^h, (d\phi_1)^h, (d\psi_1)^h) &= ([\Pi_1, \Pi_1]_S((d\phi_1), (d\phi_1), (d\psi_1)))^h \\
[\Pi^\mu, \Pi^\mu]_S((d\phi_2)^h, (d\phi_2)^h, (d\psi_2)^h) &= (\mu^2)^h([\Pi_2, \Pi_2]_S((d\phi_2), (d\phi_2), (d\psi_2)))^h \\
[\Pi^\mu, \Pi^\mu]_S((d\phi_1^h, (d\phi_1)^h, (d\psi_2)^h) &= 0 \\
[\Pi^\mu, \Pi^\mu]_S((d\phi_1)^h, (d\phi_1)^h, (d\psi_2)^h) &= (X_\mu(\phi_1))^h\Pi_2(d\phi_2, d\psi_2)^v
\end{align*}
\]

for any \( \phi_i, \phi_i, \psi_i \in C^\infty(M_i), \ i = 1, 2 \). Since \( \Pi_1 \) and \( \Pi_2 \) are Poisson tensors, then

\[
[\Pi^\mu, \Pi^\mu]_S((d\phi_1)^h, (d\phi_1)^h, (d\psi_1)^h) = [\Pi^\mu, \Pi^\mu]_S((d\phi_2)^h, (d\phi_2)^h, (d\psi_2)^h) = 0.
\]

Therefore, \( \Pi^\mu \) is a Poisson tensor if and only if \( X_\mu = 0 \). \( \square \)
Definition 4.3. Let \((M_1, \Pi_1)\) and \((M_2, \Pi_2)\) be two Poisson manifolds such that \(\Pi_2 \neq 0\) and let \(\mu\) be a Casimir function on \((M_1, \Pi_1)\). The Poisson tensor \(\Pi^\mu\) on \(M_1 \times M_2\) is called the warped Poisson tensor relative to \(\Pi_1, \Pi_2\) and the warping function \(\mu\).

Corollary 1. (The symplectic case) Under the assumptions of Theorem 4.2. If \(\Pi_1\) and \(\Pi_2\) are nondegenerate and \(\mu\) is nonvanishing, then \((M_1 \times M_2, \Pi^\mu)\) is symplectic if and only if \(\mu\) is essentially constant (i.e. constant on each connected component).

Proof. From Theorem 4.2, \((M_1 \times M_2, \Pi^\mu)\) is a Poisson manifold if and only if \(\Pi^\mu\) is a Casimir function. Since \(\Pi_1\) is nondegenerate, the only Casimir functions on \((M_1, \Pi_1)\) are the essentially constant functions. Since \(\mu\) is nonvanishing we have rank \(\mu \Pi_2 = \text{rank} \Pi_2\) and then

\[
\text{rank} \Pi^\mu = \text{rank} \Pi_1 + \text{rank} \mu \Pi_2 = \dim M_1 + \dim M_2 = \dim(M_1 \times M_2).
\]

Therefore \((M_1 \times M_2, \Pi^\mu)\) is a Poisson manifold if and only if it is symplectic.

4.3. The warped generalized Poisson bracket. Let \(\Pi_1\) and \(\Pi_2\) be bivector fields on \(M_1\) and \(M_2\) respectively. Let \(D^1\) and \(D^2\) be contravariant connections on \((M_1, \Pi_1)\) and \((M_2, \Pi_2)\) respectively. Let \(D^\mu\) be the contravariant connection on \(M_1 \times M_2\) with respect to \(\Pi^\mu\) given by

\[
D^\mu_{\alpha_1} \beta_1^h = (D^1_{\alpha_1})^h_{\beta_1}, \quad D^\mu_{\alpha_2} \beta_2^v = \mu^h (D^2_{\alpha_2} \beta_2)^v \quad \text{and} \quad D^\mu_{\alpha_1} \beta_2^v = D^\mu_{\alpha_2} \beta_1^h = 0,
\]

for any \(\alpha_i, \beta_i \in \Gamma(T^* M_i), \ i = 1, 2\).

Proposition 4. 1. Let \(T_i\) be the torsion of \(D^i, \ i = 1, 2\), and let \(T_\mu\) be the torsion of \(D^\mu\). We have

\[
T_\mu = T_1^h + \mu^h T_2^v - (d\mu)^h \Pi_2^v.
\]

2. Let \(R_i\) be the curvature of \(D^i, \ i = 1, 2\) and let \(R_\mu\) be the curvature of \(D^\mu\). Let \(\alpha_i, \beta_i, \gamma_i \in \Gamma(T^* M_i), \ i = 1, 2\), and let \(\alpha = \alpha_1^h + \alpha_2^v, \ \beta = \beta_1^h + \beta_2^v\) and \(\gamma = \gamma_1^h + \gamma_2^v\). We have

\[
R_\mu(\alpha, \beta) \gamma = [R_1(\alpha_1, \beta_1) \gamma_1]^h + \mu^h [R_2(\alpha_2, \beta_2) \gamma_2]^v.
\]

Proof. Use the definition of \(D^\mu\) and 3. of Proposition 3.

Therefore, if \(\mu\) is nonzero, \(D^\mu\) is flat if and only if \(D^1\) and \(D^2\) are flat and, if \(d\mu = 0, \ D^\mu\) is torsion-free if and only if \(D^1\) and \(D^2\) are.

Proposition 5. Let \(\{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2\) and \(\{\cdot, \cdot\}_\mu\) be the generalized pre-Poisson brackets associated with \(D^1, D^2\) and \(D^\mu\) respectively. We have:

\[
\{\omega^h_1, \eta^h_1\}_\mu = \{\omega_1^h, \eta_1^h\}_1, \quad \{\omega^v_2, \eta^v_2\}_\mu = \mu^h \{\omega_2, \eta_2\}_2 \quad \text{and} \quad \{\omega^h_1, \eta^v_2\}_\mu = 0,
\]

for any differential forms \(\omega_i, \eta_i \in \Omega^*(M_i), \ i = 1, 2\).

Proof. Let us first, once and for all, say that all along the computations we use Lemma 4.1. Now, by (4), (5) and (7), we get for a function and a differential form:

\[
\{\varphi^h_1, \eta^h_1\}_\mu = \{\varphi_1, \eta_1\}_1, \quad \{\varphi^v_2, \eta^v_2\}_\mu = \mu^h \{\varphi_2, \eta_2\}_2 \quad \text{and} \quad \{\varphi^h_1, \eta^v_2\}_\mu = 0.
\]
By the Leibniz identity (9) and using the identities above, we get for an exact 1-form and a differential form:

\[ \{ (d\varphi)^h, \eta^h_1 \}_\mu = \{ d\varphi_1, \eta^h_1 \}_\mu, \quad \{ (d\varphi_2)^v, \eta^v_2 \}_\mu = \mu^h \{ d\varphi_2, \eta^v_2 \}_2 \]

and

\[ \{ (d\varphi)^h, \eta^v_1 \}_\mu = \{ (d\varphi_2)^v, \eta^h_1 \}_\mu = 0. \]

Using the antisymmetry (8), the product identity (10), and the identities above we get

\[ \{ (\varphi_1 d\psi^h)^h, \eta^h_1 \}_\mu = \{ \varphi_1 d\psi_1, \eta^h_1 \}_1, \quad \{ (\varphi_2 d\psi_2)^v, \eta^v_2 \}_\mu = \mu^h \{ \varphi_2 d\psi_2, \eta^v_2 \}_2 \]

and

\[ \{ (\varphi_1 d\psi^h)^h, \eta^v_1 \}_\mu = \{ (\varphi_2 d\psi_2)^v, \eta^h_1 \}_\mu = 0, \]

thus for a 1-form and any differential form we have

\[ \{ \alpha^h_1, \eta^h_1 \}_\mu = \{ \alpha_1, \eta_1 \}_1, \quad \{ \alpha^v_2, \eta^v_2 \}_\mu = \mu^h \{ \alpha_2, \eta_2 \}_2 \]

and

\[ \{ \alpha^h_1, \eta^v_2 \}_\mu = \{ \alpha^v_2, \eta^h_1 \}_\mu = 0. \]

Using again the antisymmetry (8), the product identity (10) and the identities above we get for two 1-forms and any differential form:

\[ \{ (\alpha_1 \wedge \beta_1)^h, \eta^h_1 \}_\mu = \{ \alpha_1 \wedge \beta_1, \eta_1 \}_1, \quad \{ (\alpha_2 \wedge \beta_2)^v, \eta^v_2 \}_\mu = \mu^h \{ \alpha_2 \wedge \beta_2, \eta_2 \}_2 \]

and

\[ \{ (\alpha_1 \wedge \beta_1)^h, \eta^v_2 \}_\mu = \{ (\alpha_2 \wedge \beta_2)^v, \eta^h_1 \}_\mu = 0. \]

Now, by induction we get the identities of the proposition. \( \square \)

**Corollary 2.** If \( \mu \) is nonzero, the bracket \( \{ \ldots \}_\mu \) is a generalized Poisson bracket if and only if the brackets \( \{ \ldots \}_1 \) and \( \{ \ldots \}_2 \) are.

**Proof.** Indeed, by the proposition above we can see that the bracket \( \{ \ldots \}_\mu \) satisfy the graded Jacobi identity (11) if and only if the two brackets \( \{ \ldots \}_1 \) and \( \{ \ldots \}_2 \) do. \( \square \)

### 4.4. Other remarkable Poisson tensors on a product manifold.

**Proposition 6.** Let \( \Pi_1 \) and \( \Pi_2 \) be two bivector fields on \( M_1 \) and \( M_2 \) respectively. Let \( f_1 \) and \( f_2 \) be smooth functions on \( M_1 \) and \( M_2 \) respectively and let \( X_{f_1} = \sharp_{\Pi_1}(df_1) \) and \( X_{f_2} = \sharp_{\Pi_2}(df_2) \) be the corresponding Hamiltonian fields. The bivector field \( \Pi_{f_1,f_2} = X^h_{f_1} \wedge X^v_{f_2} \) is a Poisson tensor on \( M_1 \times M_2 \).

**Proof.** Using the properties of the Schouten-Nijenhuis bracket we get

\[ [\Pi_{f_1,f_2}, \Pi_{f_1,f_2}]S = [X^h_{f_1} \wedge X^v_{f_2}, X^h_{f_1} \wedge X^v_{f_2}]S = 2\Pi_{f_1,f_2} \wedge [X^h_{f_1}, X^v_{f_2}], \]

and then, from Lemma 4.1, we deduce that \( [\Pi_{f_1,f_2}, \Pi_{f_1,f_2}]S = 0 \). \( \square \)

**Proposition 7.** Let \( (M_1, \Pi_1) \) and \( (M_2, \Pi_2) \) be two Poisson manifolds and let \( f_i, \mu_i \in C^\infty(M_i), \ i = 1, 2 \). Let \( \Pi_{f_1,f_2} \) be the Poisson tensor given in the proposition above. If \( \mu_1 \) and \( \mu_2 \) are Casimir functions, then the bivector field

\[ \Lambda = \mu^h_1 \Pi^h_1 + \mu^v_1 \Pi^v_1 + \mu^h_2 \Pi_{f_1,f_2} \]

is a Poisson tensor on \( M_1 \times M_2 \).
Proof. For any \( \varphi, \psi \in C^\infty(M_i), \) \( i = 1, 2, \) using Lemma 4.1 we can easily verify that

\[
\Pi_{f_1, f_2}(d\varphi^h_1, d\psi^h_1) = 0, \quad \Pi_{f_1, f_2}(d\varphi^v_2, d\psi^v_2) = 0
\]

and

\[
\Pi_{f_1, f_2}(d\varphi^h_1, d\varphi^v_2) = \{f_1, \varphi_1\}^h_1 \{f_2, \varphi_2\}^v_2.
\]

If for \( \varphi, \psi \in C^\infty(M_1 \times M_2) \) we put \( \{\varphi, \psi\} = \Lambda(d\varphi, d\psi) \), we deduce using the identities above that

\[
\{\varphi^h_1, \psi^h_1\} = \mu^h_1 \{\varphi_1, \psi_1\}, \quad \{\varphi^v_2, \psi^v_2\} = \mu^v_1 \{\varphi_2, \psi_2\}
\]

and

\[
\{\varphi^h_1, \varphi^v_2\} = \mu^h_1 \mu^v_2 \{f_1, \varphi_1\} \{f_2, \varphi_2\}.
\]

To prove that the bivector field \( \Lambda \) is a Poisson tensor, we need to prove that the bracket \( \{., .\}\) satisfies the Jacobi identity. Let \( \varphi_1, \phi_1, \psi_1 \in C^\infty(M_i), \) \( i = 1, 2. \) Using the above identities and the Leibniz identity we get

\[
\{\{\varphi^h_1, \phi^h_1\}, \psi^h_1\} = -\mu^h_1 \mu^v_2 \{f_1, \varphi_1\} \{f_2, \psi_1\} + (\mu^v_2)^2 \{\{\varphi_1, \phi_1\}, \psi_1\}
\]

and since \( \Pi_1 \) is Poisson, taking the cyclic sum \( \oint_{\varphi_1, \phi_1, \psi_1} \), we get

\[
\oint_{\varphi_1, \phi_1, \psi_1} \{\{\varphi^h_1, \phi^h_1\}, \psi^h_1\} = -\mu^h_1 \mu^v_2 \{f_1, \varphi_1\} \{f_2, \psi_1\}, \quad \oint_{\varphi_1, \phi_1, \psi_1} \{\phi^h_1, \psi^h_1\} = \mu^h_1 \{\varphi_1, \phi_1\} \{f_2, \psi_1\},
\]

and using the same arguments we also get

\[
\oint_{\varphi_2, \phi_2, \psi_2} \{\{\varphi^v_2, \phi^v_2\}, \psi^v_2\} = \mu^v_1 \mu^h_2 \{f_1, \varphi_1\} \{f_2, \psi_1\}, \quad \oint_{\varphi_2, \phi_2, \psi_2} \{\phi^v_2, \psi^v_2\} = \mu^v_1 \{\varphi_1, \phi_1\} \{f_2, \psi_1\}
\]

and

\[
\oint_{\varphi_1, \phi_1, \psi_1} \{\{\varphi^h_1, \phi^h_1\}, \psi^h_1\} = (\mu^h_1)^2 \{f_1, \psi_1\} + (\mu^v_2)^2 \{f_2, \psi_1\} - \{f_1, \varphi_1\} \{f_2, \psi_1\} + \{f_2, \varphi_1\} \{f_1, \psi_1\}
\]

Now, \( \mu_1 \) and \( \mu_2 \) being Casimir functions, we deduce that

\[
\oint_{\varphi_1, \phi_1, \psi_1} \{\{\varphi^h_1, \phi^h_1\}, \psi^h_1\} = \oint_{\varphi_1, \phi_1, \psi_1} \{\{\varphi^v_2, \phi^v_2\}, \psi^v_2\} = \oint_{\varphi_1, \phi_1, \psi_1} \{\{\varphi^h_1, \phi^h_1\}, \psi^h_1\}
\]

\[
= \oint_{\varphi_1, \phi_1, \psi_1} \{\{\varphi^v_2, \phi^v_2\}, \psi^h_1\} = 0.
\]

\[
\Box
\]

5. Warped bivector fields on warped products. In this section, we define the contravariant warped product in the same way the covariant warped product was defined in [2]. On a contravariant warped product equipped with a warped bivector field, we compute the Levi-Civita contravariant connection and the associated curvatures. Several proofs contain standard but long computations, and hence will be omitted.
5.1. The Levi-Civita contravariant connection on a warped product manifold equipped with a warped bivector field. Let \((M_1, g_1)\) and \((M_2, g_2)\) be two pseudo-Riemannian manifolds and let \(g_1\) and \(g_2\) be the metrics of \(g_1\) and \(g_2\) respectively. Let \(f\) be a positive smooth function on \(M_1\). The contravariant metric
\[
g^f = g_1^h + f^h g_2^v
\]
on the product manifold \(M_1 \times M_2\) is characterized by the following identities
\[
g^f(\alpha_1^h, \beta_1^h) = g_1(\alpha_1, \beta_1)^h, \quad g^f(\alpha_2^v, \beta_2^v) = f^h g_2(\alpha_2, \beta_2)^v, \quad g^f(\alpha_1^h, \alpha_2^v) = 0,
\]
for any \(\alpha_1, \beta_1 \in \Gamma(T^* M_1)\), \(i = 1, 2\). We call \((M_1 \times M_2, g^f)\) the contravariant warped product of \((M_1, g_1)\) and \((M_2, g_2)\). The following lemma shows that the contravariant tensor \(g^f\) is nothing else than the metric of the warped metric \(\tilde{g}_f\).

**Lemma 5.1.** Let \(\alpha_i, \beta_i \in \Gamma(T^* M_i)\) and \(\xi_i, \eta_i \in \Gamma(TM_i)\), \(i = 1, 2\). Let \(\alpha = \alpha_1^h + \alpha_2^v\) and \(X = X_1^h + X_2^v\). We have
1. \(\nabla_{\xi_i}^f(\alpha) = [\nabla_{\xi_i} g_1(\alpha_1)]^h + f^h [\nabla_{\xi_i} g_2(\alpha_2)]^v,
2. \(\nabla_{\xi_i}^f(X) = [\nabla_{\xi_i} X_1(\alpha_1)]^h + \frac{f}{\sqrt{g}} [\nabla_{\xi_i} X_2(\alpha_2)]^v,
3. \(g^f(X, X) = \tilde{g}_1(X_1, X_1)^h + \frac{f}{\sqrt{g}} \tilde{g}_2(X_2, X_2)^v\), where \(\tilde{g}_f\) is the metric on \(M_1 \times M_2\) whose cometric is \(g^f\).

**Proof.** The proof of the first assertion is the same as that of the first assertion in Proposition 3. For the second assertion, it suffices to put \(\nabla_{\xi_i}^f(\alpha_1) = X_1\) and \(\nabla_{\xi_i}^f(\alpha_2) = X_2\) in 1. The third assertion follows from the assertions 1 and 2.

Let \(\Pi_i\) be a bivector field on \(M_i\), \(i = 1, 2\), and let \(\mu\) be a smooth function on \(M_1\). Using the Koszul formula (14), let us compute the Levi-Civita contravariant connection \(\nabla\), associated with the pair \((g^f, \Pi)\), in terms of the Levi-Civita connections \(\nabla^1\) and \(\nabla^2\) associated with the pairs \((g_1, \Pi_1)\) and \((g_2, \Pi_2)\) respectively.

**Proposition 8.** For any \(\alpha_i, \beta_i \in \Gamma(T^* M_i)\), \(i = 1, 2\), we have
1. \(\nabla_{\alpha_i}^f \beta_1^h = (\nabla_{\alpha_i}^1 \beta_1^h)^h\),
2. \(\nabla_{\alpha_i}^f \beta_2^v = \mu^h (\nabla_{\alpha_i}^2 \beta_2^v)^v + \frac{1}{\sqrt{g}} \Pi_i(\alpha_2, \beta_2)^v (d\mu)^h + \frac{1}{\sqrt{g}} g_2(\alpha_2, \beta_2)^v (J_1 d\mu)^h\),
3. \(\nabla_{\beta_2}^f \alpha_1^h = - \frac{1}{\sqrt{g}} [g_1(J_1 d\mu, \alpha_1)^h \beta_2^v + g_1(d\mu, \alpha_1)^h (J_2 \beta_2)^v]\),
4. \(\nabla_{\beta_2}^f \alpha_2^v = \nabla_{\alpha_1}^f \beta_2^v\).

**Proof.** Let \(\alpha_i, \beta_i, \gamma_i \in \Gamma(T^* M_i)\), \(i = 1, 2\). For any \((i, l), (j, l), (k, l) \in \{(1, h), (2, v)\}\), we have
\[
2g^f(\nabla_{\alpha_i}^f \beta_j^h, \gamma_k^h) = 2\Pi_i(\alpha_i^l, \beta_j^l) g^f(\beta_j^l, \gamma_k^l) + 2\Pi_1(\alpha_2, \beta_2)^v (d\mu)^h + 2\Pi_2(\alpha_2, \beta_2)^v (J_1 d\mu)^h.
\]
\[
+ g^f(\alpha_i^l, \gamma_k^l) + g^f(\alpha_i^l, \beta_2^v) + g^f(\beta_2^v, \gamma_k^l).
\]
(18)

1. Taking \((i, l) = (j, l) = (k, l) = (1, h)\) in this formula, using Formula (17) and Proposition 3, we get
\[
2g^f(\nabla_{\alpha_1}^f \beta_1^h, \gamma_1^h) = 2g_1(\nabla_{\alpha_1}^1 \beta_1, \gamma_1)^h,
\]
and using (17) again, we get
\[
g^f(\nabla_{\alpha_1}^f \beta_1^h, \gamma_1^h) = g^f((\nabla_{\alpha_1}^1 \beta_1)^h, \gamma_1^h).
\]
Similarly, taking \((i, l) = (j, l) = (1, h)\) and \((k, l) = (2, v)\), we get \(g^f(\nabla_{\alpha_2}^f \beta_2^v, \gamma_2^v) = 0\) and then
\[
g^f(\nabla_{\alpha_2}^f \beta_2^v, \gamma_2^v) = g^f((\nabla_{\alpha_2}^1 \beta_1)^h, \gamma_2^v).
\]
The result follows.
2. Taking \((i, l) = (j, l) = (2, v)\) and \((k, l) = (1, h)\) in (18), using Formula (17) and Proposition 3, we get
\[
g^f(\mathcal{D}_{\alpha_1^v}^v \beta_2^v, \gamma_1^h) = \frac{1}{2} \left\{ g^f((\alpha_1^v, \beta_2^v)_{\Pi^v}, \gamma_1^h) - \sharp_{\Pi^v}(\gamma_1^h) \cdot g^f(\alpha_1^v, \beta_2^v) \right\}
\]
\[
= \frac{1}{2} \left\{ \Pi_2(\alpha_2, \beta_2) g_1(d\mu, \gamma_1^h) - g_2(\alpha_2, \beta_2) \Pi_1(\gamma_1, df)^h \right\}
\]
\[
= g^f \left( \frac{1}{2} \left\{ \Pi_2(\alpha_2, \beta_2) df^h + g_2(\alpha_2, \beta_2) \Pi_1(\gamma_1, df)^h \right\}, \gamma_1^h \right),
\]
and similarly, taking \((i, l) = (j, l) = (k, l) = (2, v)\) in (18), we get
\[
g^f(\mathcal{D}_{\alpha_1^v}^v \beta_2^v, \gamma_2^v) = \mu^h g^f((\mathcal{D}_{\alpha_2}^2, \beta_2^v), \gamma_2^v).
\]
3. This is analogous to the proofs of 1. and 2.

4. Since \(\mathcal{D}\) is torsion-free we have \(\mathcal{D}_{\alpha_2^v}^v \alpha_1^h = \mathcal{D}_{\alpha_1^h}^v \beta_2^v + [\alpha_1^h, \beta_2^v]_{\Pi^v}\). By Proposition 3, we have \([\alpha_1^h, \beta_2^v]_{\Pi^v} = 0\). \(\square\)

**Proposition 9.** Let \(\alpha_i, \beta_i, \gamma_i \in \Gamma(T^*M_i), i = 1, 2\). We have

1. \(\mathcal{D} \Pi^v(\alpha_1^h, \beta_1^h, \gamma_1^h) = [\mathcal{D} \Pi_1(\alpha_1, \beta_1, \gamma_1)]^h\),

2. \(\mathcal{D} \Pi^v(\alpha_2^v, \beta_2^v, \gamma_2^v) = (\mu^v)^h [\mathcal{D} \Pi_2(\alpha_2, \beta_2, \gamma_2)]^v\),

3. \(\mathcal{D} \Pi^v(\alpha_1^h, \beta_1^h, \gamma_1^h) = [\mathcal{D} \Pi_1(\alpha_1, \beta_1, \gamma_1)]^h = 0\),

4. \(\mathcal{D} \Pi^v(\alpha_1^h, \beta_2^v, \gamma_2^v) = \frac{\mu^h}{f^v} g_1(J_1 df, \alpha_1^h) \Pi_2(\beta_2, \gamma_2)^v\),

5. \(\mathcal{D} \Pi^v(\alpha_2^v, \beta_1^h, \gamma_2^v) = \frac{\mu^h}{f^v} \left[ \Pi_2(\alpha_2, \gamma_2) v_1(J_1 df, \beta_1)^h - \Pi_2(\alpha_2, J_2 \gamma_2) v_1(d\mu, \beta_1)^h \right]
\]
\[
+ \frac{1}{2} \left[ g_2(\alpha_2, \gamma_2) v_1(J_1 df, \beta_1) - g_2(\alpha_2, \gamma_1) \Pi_1(\gamma_1, df)^h \right],
\]

6. \(\mathcal{D} \Pi^v(\alpha_1^h, \beta_2^v, \gamma_1^h) = \frac{\mu^h}{f^v} \left[ \Pi_2(\alpha_2, J_2 \beta_2) v_1(d\mu, \gamma_1)^h - \Pi_2(\alpha_2, \beta_2) v_1(J_1 df, \gamma_1)^h \right]
\]
\[
+ \frac{1}{2} \left[ g_2(\alpha_2, J_2 \beta_2) v_1(d\mu, \gamma_1) - g_2(\alpha_2, \beta_2) \Pi_1(\gamma_1, df)^h \right].
\]

**Proof.** The proof uses (3), Proposition 3 and Proposition 8. \(\square\)

5.2. The curvatures of the Levi-Civita contravariant connection. In the following proposition, we express the curvature \(\mathcal{R}\) of the contravariant connection \(\mathcal{D}\) in terms of the warping functions \(f, \mu\) and the curvatures \(\mathcal{R}_1\) and \(\mathcal{R}_2\) of \(\mathcal{D}^1\) and \(\mathcal{D}^2\) respectively.

**Proposition 10.** Let \(\alpha_i, \beta_i, \gamma_i \in \Gamma(T^*M_i), i = 1, 2\) and let \(\gamma = \gamma_1^h + \gamma_2^v\). We have

1. \(\mathcal{R}(\alpha_1^h, \beta_1^h) \gamma = [\mathcal{R}(\alpha_1, \beta_1, \gamma_1)]^h + \left[ g_1(\mathcal{D}_{\alpha_1}^1 \frac{d\mu}{2f}, \alpha_1) - g_1(\mathcal{D}_{\alpha_1}^1 \frac{d\mu}{2f}, \beta_1) \right]^h (J_2 \gamma_2)^v\),

2. \(\mathcal{R}(\alpha_2^v, \beta_2^v) \gamma_1^h = \frac{1}{4(f^h)^2} \left[ g_1(J_1 df, \alpha_1) g_1(J_1 df, \gamma_1) - 2f g_1(\mathcal{D}_{\alpha_1}^1 (J_1 df), \gamma_1)^h \beta_2^v\right.
\]
\[
+ \frac{1}{4(f^h)^2} \left[ g_1(d\mu, \alpha_1) g_1(J_1 df, \gamma_1) + g_1(J_1 df, \alpha_1) g_1(d\mu, \gamma_1) - 4f^2 g_1(\mathcal{D}_{\alpha_1}^1 \frac{d\mu}{2f}, \gamma_1)^h (J_2 \beta_2)^v\right.
\]
\[
+ \frac{1}{4(f^h)^2} \left[ g_1(d\mu, \alpha_1) g_1(d\mu, \gamma_1)^h (J_2 \beta_2)^v,\right.\]
3. $R(\alpha^h_1, \beta^h_2)\gamma^v_2 = \frac{1}{4f^2} \left[ g_1(\mu_1, \alpha_1)^h \Pi_2(\beta_2, J_2\gamma_2)^v + g_1(J_1df, \alpha_1)^h \Pi_2(\beta_2, \gamma_2)^v \right] (d\mu)^h$

$+ \frac{1}{4f^2} \left[ g_1(\mu_1, \alpha_1)^h g_2(\beta_2, J_2\gamma_2)^v + g_1(J_1df, \alpha_1)^h g_2(\beta_2, \gamma_2)^v \right] (J_1df)^h$

$+ \frac{\mu^h}{2f^2} \left[ g_1(\mu_1, \alpha_1)^h (D^2_{\beta_2}(J_2\gamma_2) - J_2D^2_{\beta_2}\gamma_2)^v - g_1(J_1df, \alpha_1)^h (D^2_{\beta_2}\gamma_2)^v \right]$

$+ \frac{1}{2} \Pi_2(\beta_2, \gamma_2)^v(D^1_{\alpha_1}d\mu)^h + \frac{1}{2} g_2(\beta_2, \gamma_2)^v(D^1_{\alpha_1}(J_1df))^h - \Pi_1(\mu_1, \alpha_1)^h (D^2_{\beta_2}\gamma_2)^v,$

4. $R(\alpha^v_2, \beta^v_2)\gamma^h_2 = \frac{1}{2f^2} \Pi_2(\alpha_2, \beta_2)^v \left[ g_1(\mu_1, \gamma_1)(J_1df) - J_1(\mu_1, \gamma_1)d\mu - D^1_{\alpha_1}\gamma_1 \right]^h$

$- \frac{\mu^h}{2f^2} g_1(\mu_1, \gamma_1)^h [D^2_{\alpha_2}(J_2\beta_2) - D^2_{\beta_2}(J_2\alpha_2) - J_2[\alpha_2, \beta_2]_{\alpha_2}]^v,$

5. $R(\alpha^v_2, \beta^v_2)\gamma^h_2 = (\mu^2)^h \left[ R(\alpha_2, \beta_2)\gamma_2 \right] + \frac{\mu^h}{2} \left[ D^2\Pi_2(\alpha_2, \beta_2, \gamma_2) - D^2\Pi_2(\beta_2, \alpha_2, \gamma_2) \right]^v (d\mu)^h$

$+ \left( \frac{\|d\mu\|^2_1}{4f^2} \right)^h [J_2(\Pi_2(\alpha_2, \gamma_2)\beta_2 - \Pi_2(\beta_2, \gamma_2)\alpha_2 + 2\Pi_2(\alpha_2, \beta_2)\gamma_2)]^v$

$+ \left( \frac{\|J_1df\|_1^2}{4f^2} \right)^h [g_2(\alpha_2, \gamma_2)\beta_2 - g_2(\beta_2, \gamma_2)\alpha_2]_v$

$+ \left( \frac{g_1(\mu_1, J_1df)}{4f} \right)^h [\Pi_2(\alpha_2, \gamma_2)\beta_2 - \Pi_2(\beta_2, \gamma_2)\alpha_2 + 2\Pi_2(\alpha_2, \beta_2)\gamma_2]^v$

$+ \left( \frac{g_1(\mu_1, J_1df)}{4f} \right)^h [J_2(\alpha_2, \gamma_2)\beta_2 - g_2(\beta_2, \gamma_2)\alpha_2)]^v.$

**Proof.** Long but straightforward computations using Propositions 3 and 8. 

Now, in the following two corollaries, we express the Ricci curvature $r$ (resp. the scalar curvature $S$) of the contravariant connection $D$ in terms of the warping functions $f, \mu$ and the Ricci curvatures $r_i$ (resp. the scalar curvatures $S_i$) of $D^i$, $i = 1, 2$. For the proof, notice that if we choose $\{dx_1, dx_2, \ldots, dx_{n_1}\}$ to be a local $g_1$-orthonormal basis of the 1-forms on an open $U_1 \subseteq M_1$ and $\{dy_1, dy_2, \ldots, dy_{n_2}\}$ to be a local $g_2$-orthonormal basis of the 1-forms on an open $U_2 \subseteq M_2$, then

$$\left\{dx_1^h, \ldots, dx_{n_1}^h, \frac{1}{\sqrt{f^h}} dy_1^v, \ldots, \frac{1}{\sqrt{f^h}} dy_{n_2}^v \right\}$$

is a local $g^f$-orthonormal basis of the 1-forms on the open $U_1 \times U_2$ of $M_1 \times M_2$.

**Corollary 3.** For any $\alpha_i, \beta_i \in \Gamma(T^*M_i)$, $i = 1, 2$, we have

1. $r(\alpha^h_1, \beta^h_1) = r_1(\alpha_1, \beta_1)^h + \left( \frac{\|J_2\|^2_2}{4f^2} \right)^v \left( g_1(\mu_1, \alpha_1)^h g_1(\mu_1, \beta_1) \right)^h$

$+ \frac{n_2}{4f^2} \left[ g_1(J_1df, \alpha_1)^h g_1(J_1df, \beta_1) + 2f g_1(D^1_{\alpha_1}(J_1df), \beta_1) \right]^h$

where

$$\|J_2\|^2_2 = \sum_{i=1}^{n_2} \|J_2dy_i\|^2_2 = \sum_{i=1}^{n_2} g_2(J_2dy_i, J_2dy_i).$$
2. \( r(\alpha^h_1, \beta^v_2) = \frac{\mu^h}{2f^h} g_1(d\mu, \alpha_1)^h \left[ \sum_{i=1}^{n_2} g_2(J_2D^2_{dy_i}\beta_2, dy_i) \right] \)&&\vspace{0.5cm}
+ \left( \Pi_1(d\mu, \alpha_1) + \frac{\mu}{2f} g_1(J_1df, \alpha_1) \right)^h \left[ \sum_{i=1}^{n_2} g_2(D^2_{dy_i}\beta_2, dy_i) \right],
\vspace{0.5cm}
\vspace{0.5cm}
3. \( r(\alpha^v_2, \beta^v_2) = (\mu^2)^h r_2(\alpha_2, \beta_2)^v - \left( \frac{||d\mu||^2_1}{2f} \right)^h \Pi_2(\alpha_2, J_2\beta_2)^v \)&&\vspace{0.5cm}
- \left( \frac{(n_2 - 2)||J_1df||^2_1}{4f} \right)^h g_2(\alpha_2, \beta_2)^v - \left( \frac{n_2 g_1(d\mu, J_1df)}{4f} \right)^h \Pi_2(\alpha_2, \beta_2)^v \vspace{0.5cm}
+ \frac{1}{2}(\langle \Pi_1, \mu \rangle)^h \Pi_2(\alpha_2, \beta_2)^v + \frac{1}{2}(\langle \Pi_1, f \rangle) g_2(\alpha_2, \beta_2)^v.
\vspace{0.5cm}
Proof. It is a consequence of the foregoing proposition. □
\vspace{0.5cm}
Corollary 4. We have
\( S = S^h_1 + \left( \frac{\mu^2}{f} \right)^h S^v_2 - \left( \frac{||d\mu||^2_1}{4f^2} \right)^h (||J_2||^2_2)^v \vspace{0.5cm}
- \left( \frac{n_2(n_2 - 3)||J_1df||^2_2}{4f^2} - \frac{n_2}{f} \langle \Pi_1, f \rangle \right)^h. \)
\vspace{0.5cm}
Proof. The statement follows directly from Corollary 3. □
\vspace{0.5cm}
Finally, to end this section, in the following corollary we express the sectional contravariant curvature \( K \) of the contravariant connection \( D \) in terms of the warping functions \( f, \mu \) and the sectional contravariant curvatures \( K_1 \) and \( K_2 \) of \( D^1 \) and \( D^2 \) respectively.
\vspace{0.5cm}
Corollary 5. For any \( \alpha_i, \beta_i \in \Gamma(T^*M_i), i = 1, 2 \), we have
\vspace{0.5cm}
1. \( K(\alpha^h_1, \beta^v_1) = K_1(\alpha_1, \beta_1)^h \)
\vspace{0.5cm}
2. \( K(\alpha^h_2, \beta^v_2) = \left( \frac{g_1(d\mu, \alpha_1)}{4f^2||\alpha_1||^2_1} \right)^h \left( \frac{||J_2\beta_2||^2_2}{||\beta_2||^2_2} \right)^v \vspace{0.5cm}
+ \left( \frac{g_1(J_1df, \alpha_1)}{4f^2||\alpha_1||^2_1} + \frac{g_1(D^1_\alpha, J_1df, \alpha_1)}{2f||\alpha_1||^2_1} \right)^h, \)
\vspace{0.5cm}
3. \( K(\alpha^v_2, \beta^v_2) = \left( \frac{\mu^2}{f} \right)^h K_2(\alpha_2, \beta_2)(alpha, \beta_2)\vspace{0.5cm}
- \left( \frac{||J_1df||^2_2}{4f^2} \right)^h \left( \frac{g_1(d\mu, J_1df)}{2f^2} \right)^h \left( \frac{\Pi_2(\alpha_2, \beta_2) g_2(\alpha_2, \beta_2)}{||\alpha_2||^2_2||\beta_2||^2_2 g_2^2(\alpha_2, \beta_2)} \right)^v. \)
\vspace{0.5cm}
Proof. This is a consequence of Proposition 10. □
\vspace{0.5cm}
5.3. Geometric consequences. Finally, we will conclude with some geometric consequences.
\vspace{0.5cm}
Theorem 5.2. If \( f \) is a Casimir function and \( \mu \) a nonzero essentially constant function, the triple \( (M_1 \times M_2, g^1, \Pi^1) \) is a pseudo-Riemannian Poisson manifold if and only if \( (M_1, g_1, \Pi_1) \) and \( (M_2, g_2, \Pi_2) \) are.
\vspace{0.5cm}
Proof. First, observe that \( f \) is a Casimir function if and only if \( J_1df = 0 \). Now, by Proposition 9, we have
\vspace{0.5cm}
\( D^1\Pi^1(\alpha^h_1, \beta^h_1, \gamma^h_1) = [D^1\Pi_1(\alpha_1, \beta_1, \gamma_1)]^h, \)
\vspace{0.5cm}
\( D^2\Pi^2(\alpha^v_2, \beta^v_2, \gamma^v_2) = (\mu^2)^h[D^2\Pi_2(\alpha_2, \beta_2, \gamma_2)]^v. \)
Proof. Assume that \( \alpha \) is a constant, we deduce from Corollary (DR) that the triple \((M_2 \times g_2, \Pi^\nu)\) is Ricci flat if and only if \( \mu \) is Casimir.

This shows the equivalence. \( \square \)

**Theorem 5.3.** Under the same assumptions of Th. 5.2, the triple \((M_1 \times M_2, g^f, \Pi^\nu)\) is locally symmetric if and only if \((M_1, g_1, \Pi_1)\) and \((M_2, g_2, \Pi_2)\) are.

**Proof.** Since \( f \) is Casimir and \( \mu \) is essentially constant, by Proposition 10, for any \( \alpha_i, \beta_i, \gamma_i \in \Gamma (T^* M_i), i = 1, 2 \), we have

\[
\mathcal{R} (\alpha^h_1, \beta^h_1) \gamma^h_1 = [\mathcal{R}_1 (\alpha_1, \beta_1) \gamma_1]^h, \quad \mathcal{R} (\alpha^v_2, \beta^v_2) \gamma^v_2 = (\mu^2)^h [\mathcal{R}_2 (\alpha_2, \beta_2) \gamma_2]^v
\]

and

\[
\mathcal{R} (\alpha^h_1, \beta^v_2) \gamma^h_1 = \mathcal{R} (\alpha^v_2, \beta^h_1) \gamma^h_1 = \mathcal{R} (\alpha^v_2, \beta^v_2) \gamma^v_2 = \mathcal{R} (\alpha^h_1, \beta^v_2) \gamma^h_1 = 0.
\]

and by Proposition 8, for any \( \alpha_i, \beta_i \in \Gamma (T^* M_i), i = 1, 2 \), we have

\[
\mathcal{D}_{\alpha^h_1} \beta^h_1 = (\mathcal{D}_{\alpha^h_1} \beta_1)^h, \quad \mathcal{D}_{\alpha^v_2} \beta^v_2 = \mu^h (\mathcal{D}_{\alpha^v_2} \beta_1)^v,
\]

and

\[
\mathcal{D}_{\alpha^h_1} \beta^v_2 = \mathcal{D}_{\alpha^v_2} \beta^h_1 = 0.
\]

Therefore, by (6), for \( \alpha_i, \beta_i, \gamma_i, \delta_i \in \Gamma (T^* M_i), i = 1, 2 \), if we set \( \alpha = \alpha^h_1 + \alpha^v_2 \), \( \beta = \beta^h_1 + \beta^v_2 \), \( \gamma = \gamma^h_1 + \gamma^v_2 \) and \( \delta = \delta^h_1 + \delta^v_2 \), we have

\[
(\mathcal{D}_\alpha \mathcal{R}) (\beta, \gamma) \delta = [(\mathcal{D}_{\alpha^h_1} \mathcal{R}_1) (\beta_1, \gamma_1) \delta_1]^h + (\mu^3)^h [(\mathcal{D}_{\alpha^v_2} \mathcal{R}_2) (\beta_2, \gamma_2) \delta_2]^v.
\]

Hence \( \mathcal{D} \mathcal{R} = 0 \) if and only if \( \mathcal{D}^1 \mathcal{R}_1 = \mathcal{D}^2 \mathcal{R}_2 = 0. \) \( \square \)

**Theorem 5.4.** Under the same assumptions of Th. 5.2, the triple \((M_1 \times M_2, g^f, \Pi^\nu)\) is flat if and only if \((M_1, g_1, \Pi_1)\) and \((M_2, g_2, \Pi_2)\) are flat.

**Proof.** In the proof of the theorem above we see that \( \mathcal{R} = 0 \) if and only if \( \mathcal{R}_1 = \mathcal{R}_2 = 0. \) \( \square \)

**Theorem 5.5.** Under the same assumptions of Th. 5.2, the triple \((M_1 \times M_2, g^f, \Pi^\nu)\) is Ricci flat if and only if \((M_1, g_1, \Pi_1)\) and \((M_2, g_2, \Pi_2)\) are Ricci flat.

**Proof.** By Corollary 3, for any \( \alpha_i, \beta_i \in \Gamma (T^* M_i), i = 1, 2 \) we have

\[
r(\alpha^h_1, \beta^h_1) = r_1 (\alpha_1, \beta_1)^h, \quad r(\alpha^v_2, \beta^v_2) = (\mu^2)^h r_2 (\alpha_2, \beta_2)^v
\]

and

\[
r(\alpha^h_1, \beta^v_2) = r(\alpha^v_2, \beta^h_1) = 0.
\]

Therefore, \( r = 0 \) if and only if \( r_1 = r_2 = 0. \) \( \square \)

Also, under the same assumptions, i.e. \( f \) Casimir and \( \mu \) nonzero essentially constant, we deduce from Corollary 5 that \( K = 0 \) if and only if \( K_1 = K_2 = 0. \)

**Proposition 11.** Assume that \( \mu \) is a nonzero essentially constant function. If \((M_1 \times M_2, g^f, \Pi^\nu)\) has a constant sectional curvature \( k \), then both \((M_1, g_1, \Pi_1)\) and \((M_2, g_2, \Pi_2)\) have a constant sectional curvature,

\[
K_1 = k \quad \text{and} \quad K_2 = \frac{f}{\mu^2} k + \frac{1}{4\mu^2} \|J_1 df\|_1^2.
\]

Furthermore, if \( f \) is Casimir then it is constant.
Proof. By 1. of Corollary 5, for any $\alpha_1, \beta_1 \in \Gamma(T^*M_1)$, we have
$$K_1(\alpha_1, \beta_1) = K(\alpha_1^h, \beta_1^h) = k,$$
hence $K_1 = k$. Since $\mu$ is essentially constant, by 3. of the same corollary, for any $\alpha_2, \beta_2 \in \Gamma(T^*M_2)$, we have
$$K(\alpha_2, \beta_2) = \left( \frac{\mu^2}{f} \right)^h K_2(\alpha_2, \beta_2) = \left( \frac{\|J_1df\|_1^2}{4f^2} \right)^h,$$
hence
$$K_2(\alpha_2, \beta_2) = \left( \frac{f}{\mu^2} k + \frac{\|J_1df\|_1^2}{4\mu^2 f} \right)^h,$$
and the proposition follows.

**Theorem 5.6.** Under the same assumptions of Th. 5.2, the triple $(M_1 \times M_2, g^f, \Pi^\mu)$ is metaflat if and only if $(M_1, g_1, \Pi_1)$ and $(M_2, g_2, \Pi_2)$ are metaflat.

Proof. Since $f$ is Casimir and $\mu$ is essentially constant, by Proposition 8 we see that the contravariant Levi-Civita connection $\mathcal{D}$ is nothing else than the connection $\mathcal{D}^\mu$ defined in §4.3, therefore, the generalized pre-Poisson bracket associated with $\mathcal{D}$ is precisely the bracket $\{\cdot, \cdot\}_{\mu}$ associated with $\mathcal{D}^\mu$. Now, by Theorem 5.4 and since the vanishing of the metacurvature is equivalent to the graded Jacobi identity (11), one can deduce from Corollary 2 that $\mathcal{D}$ is metaflat if and only if $\mathcal{D}^1$ and $\mathcal{D}^2$ are metaflat. 

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