Existence condition and phase transition of Reissner-Nordström-de Sitter black hole

Meng-Sen Ma\textsuperscript{a,b,\ast}, Hui-Hua Zhao\textsuperscript{a,b,\dagger}, Li-Chun Zhang\textsuperscript{b}, Ren Zhao\textsuperscript{a,b,\ddagger}

\textsuperscript{a} Department of Physics, Shanxi Datong University, Datong 037009, China
\textsuperscript{b} Institute of Theoretical Physics, Shanxi Datong University, Datong 037009, China

After introducing the connection between the black hole horizon and the cosmological horizon, we discuss the thermodynamic properties of Reissner-Nordstrom-de Sitter (RN-dS) spacetime. We present the condition under which RN-dS black hole can exist. Employing Ehrenfest’ classification we conclude that the phase transition of RN-dS black hole is the second-order one. The position of the phase transition point is irrelevant to the electric charge of the system. It only depends on the ratio of the black hole horizon and the cosmological horizon.

\textbf{keywords:} Reissner-Nordstrom-de Sitter black hole, critical phenomena, the second order phase transition

\begin{center}
\textbf{Contents}
\end{center}

| I. Introduction | 1 |
| II. The effective thermodynamic quantities of RN-dS spacetime | 3 |
| III. P-V criticality in RN-dS black hole spacetime | 4 |
| IV. The second-order phase transition of RN-dS spacetime | 6 |
| V. Conclusions and Discussion | 8 |
| Acknowledgments | 9 |
| References | 9 |

\section*{I. INTRODUCTION}

Black holes supply an ideal arena to research and test kinds of quantum gravity\cite{1}. On the one hand, black holes are the solutions of classical gravity (GR), thus are classical systems. On the other hand, black holes can also be a kind of quantum system, in which thermodynamics plays an important role\cite{2,3}. The entropy, temperature and the holographic properties of black holes are essentially related to quantum mechanics. Although the statistical explanation of the thermodynamic states of black holes is lacked yet, the relevant studies on the properties

\* Email: mengsenma@gmail.com
\dagger Email:kietemap@126.com
\ddagger Email: zhao2969@sina.com
of black hole thermodynamics still received a lot of attentions, such as Hawking-Page phase transition\cite{8}, the critical phenomena. More interestingly, it is found for some black holes there exist similar phase transition and critical behaviors to the van der Waals-Maxwell system\cite{9, 10}.

Recently, the idea of including the variation of the cosmological constant $\Lambda$ in the first law of black hole thermodynamics has attained increasing attention\cite{11–31}. Matching the thermodynamic quantities with the ones in usual thermodynamic system, the critical behavior of black holes can be investigated and the phase diagram like the van der Waals vapor-liquid system can be obtained. This helps to further understand black hole entropy, temperature, heat capacity, et.al, and it is also very important to improve the self-consistent geometric theory of thermodynamics of black hole.

The black holes mentioned above must be stable in the appropriate range of parameters. Only in this case the corresponding relationship between the black hole and the usual thermodynamic system can be true. The black holes, like Schwarzschild black hole, are thermodynamic unstable. To investigate their thermodynamic properties, some methods are proposed, for example black branes thermodynamics\cite{32–36}. The black holes in de Sitter space usually possess not only the black hole horizon, but the cosmological horizon. The horizons all have thermal radiation, thus different temperatures. Therefore black holes in de Sitter spacetime are thermodynamic unstable. The thermodynamic quantities on the black hole horizon and the cosmological horizon all satisfy the first law of thermodynamics, moreover the corresponding entropies both fulfill the area formula\cite{16, 37, 38}. In recent years the studies on the thermodynamic properties of de Sitter space have aroused wide concern\cite{16, 37–41}. In the era of inflation the universe lies in a quasi-de Sitter space. The cosmological constant corresponds to vacuum energy and is usually considered as a candidate of dark energy. The accelerating universe will evolve into another de Sitter phase. In order to construct the entire history of evolution of the universe, we should have a clear perspective to the classical and quantum properties of de Sitter space\cite{16, 38, 42, 43}. Firstly, we anticipate the entropy should satisfy the Nernst theorem\cite{39, 40, 44}. Secondly, after introducing the connection between the black hole horizon and the cosmological horizon, we want to know whether the thermodynamic quantities in de Sitter space fulfill the conditions of thermodynamic stability, whether in de Sitter space there exists similar phase transition and critical phenomena like in AdS space? Hence constructing a self-consistent relation between the thermodynamic quantities in de Sitter space is worth studying.

The thermodynamic quantities corresponding to the black hole horizon and the cosmological horizon are all functions of the mass $M$, electric charge $Q$ and the cosmological constant $\Lambda$. Therefore the two pairs of thermodynamic quantities are not independent each other. Taking their relations into account is very important to study the thermodynamic properties in de Sitter space. In \cite{45} we set the position $r_c$ of the cosmological horizon to be invariant and studied the phase transition of RN-dS black hole. Based on the results in \cite{45}, in this paper we investigate the critical behaviors of the effective thermodynamic quantities of RN-dS spacetime when the cosmological horizon is variable. According to Ehrenfest’s classification for phase transition of thermodynamic system, we find that what happens in RN-dS black hole is the second-order phase transition.

The paper is arranged as follows: In Sec.2 we first review the RN-dS spacetime and give the thermodynamic quantities corresponding to the two horizons. After considering the connection between the two horizons we introduce the effective temperature, effective pressure and effective potential. In Sec. 3 phase transition in charged dS black hole spacetime is investigated. We
discuss the relation between the effective pressure and the effective volume in RN-dS spacetime and analyze its critical phenomena. We will analyze the nature of the phase transition using Ehrenfest’s equations in Sec. 4. Finally, the paper ends with a brief conclusion. (we use the units $G = \hbar = k_B = c = 1$)

II. THE EFFECTIVE THERMODYNAMIC QUANTITIES OF RN-DS SPACETIME

The line element of the RN-dS black hole is given by

$$ds^2 = -f(r)dt^2 + f^{-1}dr^2 + r^2d\Omega^2,$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2,$$ 

The above geometry possesses three horizons: the black hole Cauchy horizon locates at $r = r_-$, the black hole event horizon (BEH) locates at $r = r_+$ and the cosmological horizon (CEH) locates at $r = r_c$, where $r_c > r_+ > r_-$. The only real, positive zeroes of $f(r) = 0$. After considering the connection between the black hole horizon and the cosmological horizon, the thermodynamic relation in RN-dS spacetime is provide in [45]

$$dM = T_{eff}dS - P_{eff}dV + \varphi_{eff}dQ,$$  

Where the effective temperature is

$$T_{eff} = \frac{(1 + x - 2x^2 + x^3 + x^4)}{4\pi r_c x(1 + x)(1 + x + x^2)} - \frac{Q^2}{4\pi r_c^3 x^3 (x + 1)(x^2 + x + 1)} (1 + x + x^2 - 2x^3 + x^4 + x^5 + x^6)$$

$$= \frac{B_1}{4\pi r_c x(1 + x)} - \frac{Q^2 B_2}{4\pi r_c^3 x^3 (1 + x)}$$

(2.4)

The effective pressure is

$$P_{eff} = \frac{(1 - x)(1 + 3x + 3x^2 + 3x^3 + x^4)}{8\pi r_c^2 x(1 + x)(1 + x + x^2)^2} - \frac{Q^2 (1 + 2x + 3x^2 - 3x^5 - 2x^6 - x^7)}{8\pi r_c^4 x^3 (1 + x)(1 + x + x^2)^2}$$

$$= \frac{B_3}{8\pi r_c^2 x(1 + x)} - \frac{Q^2 B_4}{8\pi r_c^4 x^3 (1 + x)}$$

(2.5)

And the effective electric potential

$$\varphi_{eff} = \frac{Q r_c^4 + 2r_c^2 r_+ + 2r_c^2 r_+ + 2r_c r_3 + r_+^4}{r_c r_+ (r_c + r_+)(r_c^2 + r_+^2 + r_c r_+)} = \frac{Q}{r_c x(1 + x + x^2)}$$

(2.6)

The thermodynamic volume in RN-dS spacetime is [32, 43]

$$V = \frac{4\pi}{3} (r_c^3 - r_+^3).$$

(2.7)

The entropy of RN-dS system is [48]

$$S = S_+ + S_c.$$  

where

$$B_1 = \frac{1 + x - 2x^2 + x^3 + x^4}{1 + x + x^2}, \quad B_2 = \frac{1 + x + x^2 - 2x^3 + x^4 + x^5 + x^6}{1 + x + x^2},$$

$$B_3 = \frac{1 + 2x - 2x^4 - x^5}{(1 + x + x^2)^2}, \quad B_4 = \frac{1 + 2x + 3x^2 - 3x^5 - 2x^6 - x^7}{(1 + x + x^2)^2}. $$

(2.9)
and \( x := r_+/r_c, \) \( 0 < x < 1. \) \( S_+ \) and \( S_c \) are the entropies which correspond to the black hole horizon and the cosmological horizon respectively. When \( Q = 0, \) the thermodynamic equation (2.3) will return back to the known result in [39]. When the relation \( r_c^2 x^2 B_1 > Q^2 B_2 \) is fulfilled, \( T_{\text{eff}} > 0 \) is always satisfied.

The effective quantities defined in (2.4), (2.5) and (2.6) and the thermodynamic equation (2.3) are self-consistent. When \( x \to 1, \) namely the two horizons tend to coincide,

\[
P_{\text{eff}} \to 0, \quad T_{\text{eff}} \to \frac{1}{12\pi r_c^3} (r_c^2 - 2Q^2),
\]

(2.10)

In this state, because

\[
Q^2 = r_+ r_c \left( 1 - \frac{r_c^2 + r_c r_+ + r_+^2}{3} \right), \quad 2M = (r_c + r_+) \left( 1 - \frac{r_c^2 + r_+^2}{3} \Lambda \right)
\]

\[
2M = \frac{(r_c + r_+)}{r_+^2 + r_c r_+ + r_+^2} \left( r_c r_+ + Q^2 \frac{r_c^2 + r_+^2}{r_c r_+} \right),
\]

(2.11)

thus one can obtain

\[
Q^2 = r_c^2 \left( 1 - r_+^2 \Lambda \right), \quad M = r_c \left( 1 - \frac{2}{3} r_+^2 \Lambda \right),
\]

(2.12)

Due to \( Q^2 \geq 0, \) \( M \geq 0, \) so \( r_+^2 \Lambda \leq 1. \) When \( M^2 \geq Q^2 \) is satisfied, \( r_+^2 \Lambda > \frac{3}{4} \) can be derived according to (2.12), therefore \( T_{\text{eff}} \to \frac{1}{12\pi r_c^3} (2r_c^2 \Lambda - 1) > 0, \) which fulfills the requirement of the thermodynamic equilibrium stability. If do not considering the connection between the two horizons and taking them into account as independent thermodynamic systems, due to the different radiation temperature for the two horizons, the spacetime is not stable.

Another problem to perceive the two horizons as independent thermodynamic systems is when the two horizons coincide, \( \kappa_{++/c} = 0 \) which means the temperature from the both horizons is zero. In this case the areas of the both horizons are obviously not zero, namely the entropies correspond to the black hole horizon and the cosmological horizon are not zero. Therefore Nernst theorem cannot be satisfied [49, 50].

When the two horizons coincide, from (2.7), the thermodynamic volume \( V \to 0 \) in the RN-dS spacetime. Hence the thermodynamic system transits from volume distribution to area distribution. The pressure of the “thermodynamic brane” is zero and volume tends to zero, but the temperature \( T_{\text{eff}} \) of the “thermodynamic brane” is not zero, the entropy \( S \to 2\pi r_c^2 \). This may explain the problem that extremal de Sitter black holes do not satisfy Nernst theorem.

### III. P-V CRITICALITY IN RN-DS BLACK HOLE SPACETIME

The investigation of phase transition in thermodynamic system has been an active subject. Recently by treating black holes as the thermodynamic systems many works on the phase transition of black holes has been done [15–26, 22, 23, 32, 36, 51–63]. Moreover by comparing with the Van der Waals system the critical behaviors of black hole system are also be studied [15–23, 32, 36]. However, due to the existence of two horizons in de Sitter spacetime, two different thermodynamic systems for the two horizons should be built. There is few research on the phase transition of this kind of non-equilibrium system. Based on Sec. 2, we will investigate the phase transition of RN-dS black holes. First we compare the effective thermodynamic quantities in RN-dS black hole with the Van der Waals equation and discuss the relation between pressure
and volume at constant temperature. Then we will analyze the nature of the phase transition using Ehrenfest’s equations.

Comparing with the Van der Waals equation

\[ (P + \frac{a}{v^2})(v - b) = kT, \]  

(3.1)

Here, \( v = V/N \) is the specific volume of the fluid, \( P \) its pressure, \( T \) its temperature, and \( k \) is the Boltzmann constant. From (3.1) one can plot the \( P - v \) curves at constant \( T \). The critical temperature, critical pressure and the critical specific volume can be determined according to the first and second derivatives.

To compare with Van der Waals equation we set \( P_{\text{eff}} \to P, \ v \to v \) and discuss the phase transition and critical phenomena at constant \( Q \).

Substituting (2.4) into (2.5), one can derive

\[ P_{\text{eff}} = T_{\text{eff}} \frac{B_1}{2r_cB_2} + \frac{B_2B_3 - B_1B_4}{8\pi r_c^2x(1 + x)B_2}. \]  

(3.2)

\( x \) is a dimensionless parameter. Employing (3.2) and the thermodynamic volume (2.7) in the RN-dS spacetime, by dimensional analysis\(^{15}\) we can set the specific volume to be

\[ v = r_c(1 - x). \]  

(3.3)

The critical point occur when the two equations set up at the same time:

\[ \left( \frac{\partial P_{\text{eff}}}{\partial v} \right)_{T_{\text{eff}}} = 0, \quad \left( \frac{\partial^2 P_{\text{eff}}}{\partial v^2} \right)_{T_{\text{eff}}} = 0, \]  

(3.4)

\[ \left( \frac{\partial P_{\text{eff}}}{\partial v} \right)_{T_{\text{eff}}} = \left( \frac{\partial (T_{\text{eff}}, P_{\text{eff}})}{\partial (x, r_c)} \right)_{T_{\text{eff}}} = \left( \frac{\partial(T_{\text{eff}}, P_{\text{eff}})}{\partial (x, r_c)} \right)_{T_{\text{eff}}} = \left( \frac{\partial P_{\text{eff}}}{\partial r_c} \right)_{T_{\text{eff}}} \left( \frac{\partial T_{\text{eff}}}{\partial r_c} \right)_{T_{\text{eff}}} + (1 - x) \left( \frac{\partial T_{\text{eff}}}{\partial x} \right)_{T_{\text{eff}}} \right) \]  

(3.5)

letting

\[ \left( \frac{\partial P_{\text{eff}}}{\partial r_c} \right)_{T_{\text{eff}}} \left( \frac{\partial T_{\text{eff}}}{\partial r_c} \right)_{T_{\text{eff}}} + (1 - x) \left( \frac{\partial T_{\text{eff}}}{\partial x} \right)_{T_{\text{eff}}} = f(x, r_c), \]  

(3.6)

From which we can derive

\[ \left( \frac{\partial^2 P_{\text{eff}}}{\partial v^2} \right)_{T_{\text{eff}}} = \left( \frac{\partial f}{\partial r_c} \right)_{T_{\text{eff}}} \left( \frac{\partial T_{\text{eff}}}{\partial r_c} \right)_{T_{\text{eff}}} + (1 - x) \left( \frac{\partial T_{\text{eff}}}{\partial x} \right)_{T_{\text{eff}}} = 0. \]  

(3.7)

One can choose different \( Q \) and calculate the critical values of the effective thermodynamic quantities numerically. We give them in Table I. In Fig.1 we plot the \( P_{\text{eff}} - v \) curves when \( Q = 1, 3, 10 \) respectively.

From the above calculation, we find that the value of \( x \) is \( x^c = 0.732216 \) at the critical point in the RN-dS black hole, which is independent of the electric charge. The critical temperature is inversely proportional to the electric charge, \( T_{\text{eff}}^c = \frac{0.00281475}{Q} \). The critical cosmological horizon \( r_c^c \) is proportional to the charge, \( r_c^c = 10.5186Q \), and the critical specific volume \( v^c \) is also related to the electric charge, \( v^c = 0.938907Q \).

From Fig.1, when \( T_{\text{eff}} > T_{\text{eff}}^c \), RN-dS system satisfies the stability condition \( \left( \frac{\partial P}{\partial v} \right)_{T_{\text{eff}}} < 0 \); when \( T_{\text{eff}} < T_{\text{eff}}^c \), for larger value of \( v \) RN-dS system satisfies the stability condition \( \left( \frac{\partial P}{\partial v} \right)_{T_{\text{eff}}} < 0 \), however, for smaller value of \( v \) RN-dS system does not satisfy the stability condition. Thus those states may not exist in nature.
TABLE I: Numerical solutions for $x^c$, $r_c^c$, $v^c$, $T_c^{\text{eff}}$ and $P_c^{\text{eff}}$ for given values of $Q = 1, 3, 10$ respectively.

$$
\begin{array}{cccccc}
Q & x^c & r_c^c & v^c & T_c^{\text{eff}} & P_c^{\text{eff}} \\
1 & 0.732216 & 3.5062 & 0.938907 & 0.00801475 & 0.00060544 \\
3 & 0.732216 & 10.5186 & 2.81672 & 0.00267158 & 0.0000672711 \\
10 & 0.732216 & 35.062 & 9.38907 & 0.000801475 & 6.0554 \times 10^{-6} \\
\end{array}
$$

For Van der Waals system there is no latent heat and at the critical point the liquid-gas structure do not change suddenly. Therefore this kind of phase transition belongs to the continuous phase transition according to Ehrenfest’s classification. Below we will discuss the behaviors of RN-dS system near the phase transition point.

When the chemical potential and its first derivative is continuous, whereas the second derivative of chemical potential is discontinuous, this kind of phase transition is called the second-order phase transition. We can calculate the specific heat of RN-dS system at constant pressure $C_P$, the expansion coefficient $\beta$ and the compressibility $\kappa$

$$
C_P = T_{\text{eff}} \left( \frac{\partial S}{\partial T_{\text{eff}}} \right)_{P_{\text{eff}}} = -T_{\text{eff}} \frac{\partial^2 G}{\partial T_{\text{eff}}^2}
$$

$$
= \pi r_c T_{\text{eff}} \left( \frac{T_{\text{eff}}}{r_c} \left( \frac{\partial P_{\text{eff}}}{\partial r_c} \right)_x - (1 + x^2) \left( \frac{\partial P_{\text{eff}}}{\partial x} \right)_{r_c} \right),
$$

(4.1)

$$
\beta = \frac{1}{v} \left( \frac{\partial v}{\partial T_{\text{eff}}} \right)_{P_{\text{eff}}} = \frac{1}{v} \frac{\partial^2 \mu}{\partial T_{\text{eff}} \partial P_{\text{eff}}}
$$

$$
= -\frac{1}{v} \left( \frac{T_{\text{eff}}}{r_c} \left( \frac{\partial P_{\text{eff}}}{\partial r_c} \right)_x + (1 - x) \left( \frac{\partial P_{\text{eff}}}{\partial x} \right)_{r_c} \right),
$$

(4.2)
\[ \kappa = -\frac{1}{v} \left( \frac{\partial v}{\partial P_{\text{eff}}} \right)_{T_{\text{eff}}} = -\frac{1}{v} \frac{\partial^{2} \mu}{\partial P_{\text{eff}}^{2}} \]

\[ = -\frac{1}{v} \left( \frac{r_{c} \left( \frac{\partial T_{\text{eff}}}{\partial r_{c}} \right)_{x} + (1 - x) \left( \frac{\partial T_{\text{eff}}}{\partial x} \right)_{r_{c}}}{r_{c} \left( \frac{\partial P_{\text{eff}}}{\partial r_{c}} \right)_{x} - \left( \frac{\partial P_{\text{eff}}}{\partial x} \right)_{r_{c}} \left( \frac{\partial T_{\text{eff}}}{\partial r_{c}} \right)_{x}} \right). \quad (4.3) \]

From (2.8), the entropy is

\[ S = \pi r_{c}^{2}(1 + x^2). \quad (4.4) \]

Below we will give the $\kappa - x$, $\beta - x$, $C_{P} - x$, $S - T$, $G - T$ curves for the fixed values of $Q = 1, 3, 10$.

![FIG. 2: $\kappa - x$ curves for RN-dS black hole corresponding to the critical effective temperature $T_{\text{eff}}^c = 0.00801475$, $T_{\text{eff}}^c = 0.00267158$ and $T_{\text{eff}}^c = 0.000801475$ respectively.](image1)

![FIG. 3: $\beta - x$ curves for RN-dS black hole corresponding to the critical effective pressure $P_{\text{eff}}^c = 0.00060544$, $P_{\text{eff}}^c = 0.0000672711$ and $P_{\text{eff}}^c = 6.0554 \times 10^{-6}$ respectively.](image2)

From the above figures, it can be found that the specific heat at constant pressure, the expansion coefficient $\beta$ and the compressibility $\kappa$ exist infinite peak. While the Gibbs function $G$ and the entropy $S$ are both continuous at the critical point. According to Ehrenfest, the phase transition of the RN-dS black hole should be the second-order one.
FIG. 4: $C_P - x$ curves for RN-dS black hole corresponding to the critical effective pressure $P_{eff}^c = 0.00060544$, $p_{eff}^c = 0.0000672711$ and $P_{eff}^c = 6.0554 \times 10^{-6}$ respectively.

FIG. 5: $S - T$ curves for RN-dS black hole corresponding to the critical effective pressure $P_{eff}^c = 0.00060544$, $p_{eff}^c = 0.0000672711$ and $P_{eff}^c = 6.0554 \times 10^{-6}$ respectively.

V. CONCLUSIONS AND DISCUSSION

After introducing the connection between the thermodynamic quantities corresponding to the black hole horizon and the cosmological horizon, we give the effective thermodynamic quantities of the RN-dS system, (2.21), (2.22) and (2.26). When describing the RN-dS system by the effective thermodynamic quantities, it will exhibit a similar phase transition to Van der Waals equation. In Sec.3 it shows that the position $x$ of the phase transition point in RN-dS system is irrelevant to the electric charge of the system. This indicates that for fixed charge when the ratio of the black hole horizon and the cosmological horizon is $x^c$, the second –order phase transition will occur. From Fig.1, when the effective temperature $T_{eff} < T_{eff}^c$, the system lies at a non-equilibrium state because of $(\frac{\partial P_{eff}}{\partial v})_{T_{eff}} > 0$ for some values of $v$. These states turn up at the small value of $v = r_c(1 - x)$, namely at the large value of $x > x^c$. This means that when the two horizons are close to each other, the system is in non-equilibrium state. Therefore the state in which the two horizons of RN-dS approach does not exist. Only the states of RN-dS black holes with $x < x^c$ can exist.

In Sec. 4 we analyzed the phase transition of RN-dS system. It shows that at the critical point the specific heat at constant pressure, the expansion coefficient $\beta$ and the compressibility $\kappa$ of the RN-dS system exist infinite peak, while the entropy and the Gibbs potential $G$ are
FIG. 6: $G - T$ curves for RN-dS black hole corresponding to the critical effective pressure $P_{\text{eff}}^{c} = 0.00060544$, $P_{\text{eff}}^{c} = 0.0000672711$ and $P_{\text{eff}}^{c} = 6.0554 \times 10^{-6}$ respectively. For fixed charge $Q$, the Gibbs free energy can be expressed as $G = M - T_{\text{eff}}S - P_{\text{eff}}V$.\[1109.2433; 1208.6251;1306.4516 1209.17071203.2279]\]

continuous. Therefore for the phase transition of the RN-dS system no latent heat and no specific volume changes suddenly, it belongs to the second-order phase transition.

To understand black hole and cosmological singularities, or distinguish all kinds of inflation models, or study the physics at the Planck scale, specially to investigate the nature of the dark energy which accounts for about 68.3% of the substance of the universe, a complete quantum theory of gravity is needed. Black holes refer to gravity, quantum mechanics and thermodynamics, in particular black holes in de Sitter space combine black holes with cosmology. When considering the connection between the black hole horizon with the cosmological horizon, it is possible to study the non-equilibrium gravitational system, like RN-dS black hole. We are looking forward to the research on the thermodynamic properties of de Sitter space, such as phase transition and critical phenomena can supply more information about quantum gravity and help to understand the classic and quantum properties of de Sitter space.

**Acknowledgments**

This work is supported by NSFC under Grant Nos.(11175109;11075098;11247261;11205097).

[1] J. X. Lu, “The Thermodynamical Phase Structure of black branes in String/M Theory” (in Chinese). Sci Sin- Phys Mech Astron, 2012, 42: 1099C1111, doi:10.1360/132012-610
[2] J. D. Bekenstein, Black holes and the second law, Lett. Nuovo Cimento 4 737 (1972).
[3] J. D. Bekenstein, Generalized second law of thermodynamics in black hole physics, Phys. Rev. D 9 3292 (1974).
[4] J. D. Bekenstein, “Extraction of Energy and Charge from a Black Hole”, Phys. Rev. D 7, 949 (1973).
[5] J. M. Bardeen, B. Carter, S. W. Hawking, “The Four laws of black hole mechanics”, Commun. Math. Phys. 31, 161 (1973).
[6] S.W.Hawking, Black Hole Explosions, Nature, 248 30 (1974).
[7] S. W. Hawking, Particle Creation by Black Holes, Commun. Math. Phys. 43 199 (1975).
[8] S. Hawking and D. N. Page, “Thermodynamics of black holes in anti-de Sitter space”, Commun. Math. Phys. 87, 577 (1983).

[9] A. Chamblin, R. Emparan, C. Johnson, and R. Myers, “Charged AdS black holes and catastrophic holography”, Phys.Rev. D60 (1999) 064018, hep-th/9902170.

[10] A. Chamblin, R. Emparan, C. Johnson, and R. Myers, “Holography, thermodynamics and fluctuations of charged AdS black holes”, Phys.Rev. D60 (1999) 104026, hep-th/9904197.

[11] B. P. Dolan, “The cosmological constant and black-hole thermodynamic potentials”, Class.Quant.Grav.28:125020,2011;

[12] B. P. Dolan, “Compressibility of rotating black holes”, Phys. Rev. D84 (2011) 127503, arXiv:1109.0198

[13] B. P. Dolan, D. Kastor, D. Kubiznak, R. B. Mann, J. Traschen, “Thermodynamic Volumes and Isoperimetric Inequalities for de Sitter Black Holes”, arXiv:1301.5926

[14] B. P. Dolan, “Where is the PdV term in the first law of black hole thermodynamics?”, arXiv:1209.1272[gr-qc].

[15] M. Cvetic, G. Gibbons, D. Kubiznak, and C. Pope, “Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume”, Phys.Rev. D84 (2011) 024037, arXiv:1012.2888.

[16] D. Kubiznak, and R. B. Mann, ”P-V criticality of charged AdS black holes”, JHEP 2012(2012): 1-25.

[17] S. Gunasekaran, D. Kubiznak, R. B. Mann, “Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization”, Journal of High Energy Physics, 2012, 2012(11): 1-43.

[18] Shao-Wen Wei, Yu-Xiao Liu, “Critical phenomena and thermodynamic geometry of charged Gauss-Bonnet AdS black holes”, Phys. Rev. D 87, 044014 (2013), arXiv:1209.1707

[19] Rong-Gen Cai, Li-Min Cao, Li Li, and Run-Qiu Yang, “P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space”, JHEP, 2013, 2013:5, arXiv:1306.6233[gr-qc].

[20] S. H. Hendi and M. H. Vahidinia, “Extended phase space thermodynamics and P-V criticality of black holes with nonlinear source”, Phys. Rev. D 88, 084045 (2013)arXiv:1212.6128[hep-th];

[21] A. Belhaj, M. Chabab, H. E. Moumni and M. B. Sedra, “Critical Behaviors of 3D Black Holes with a Scalar Hair”, arXiv:1306.2518 [hep-th].

[22] R. Zhao, H. -H. Zhao, M. -S. Ma and L. -C. Zhang, “On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes”, European Physical Journal C, 2013, 73:2645, arXiv:1305.3725 [gr-qc];

[23] M. Bagher, J. Poshteh, B. Mirza, Z.Sherkatghanad, “Phase transition, critical behavior, and critical exponents of Myers-Perry black holes”, Phys. Rev. D 88, 024005 (2013) arXiv:1306.4516

[24] E. Spallucci, A. Smailagic, “Maxwell’s equal-area law for charged Anti-de Sitter black holes”,Physics Letters B 723(2013)436C441.

[25] M. Eune, W. Kim, “Entropy and temperatures of Nariai black hole”, Physics Letters B723(2013)177C181.

[26] R Zhao, M S Ma, H F Li, L C Zhang, “On Thermodynamics of Charged and Rotating Asymptotically AdS black strings”, Advances in High Energy PhysicsVolume 2013, Article ID 371084

[27] J X Mo, W B Liu, “Ehrenfest scheme for PV criticality in the extended phase space of black holes”, Physics Letters B, 727(2013)336C339.

[28] N. Altamirano, D. Kubiznak, and R. B. Mann “ Reentrant phase transitions in rotating antiCde Sitter black holes ”, Phys. Rev. D 88, 101502(R) (2013).
[29] De-Cheng Zou, Shao-Jun Zhang, Bin Wang, “Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics”, arXiv:1311.7299.

[30] Wei Xu, Hao Xu, Liu Zhao, “Gauss-Bonnet coupling constant as a free thermodynamical variable and the associated criticality”, arXiv:1311.3053.

[31] M. Cadoni, G. DAppollonio, P. Pani, “Phase transitions between Reissner-Nordstrom and dilatonic black holes in 4D AdS spacetime”, JHEP, 2010(3), 1-27.

[32] S. Vaidya, “Phase transitions and critical behavior for charged black holes”. Class Quantum Gravity, 2003, 20: 3827C3838

[33] A. P. Lundgren, “Charged black hole in a canonical ensemble”. Phys Rev D, 2008, 77: 044014

[34] Lu J X, Roy S, Xiao Z. “Phase transitions and critical behavior of black branes in canonical ensemble”, JHEP, 2011, 1: 133

[35] Lu J X, Roy S, Xiao Z. “The enriched phase structure of black branes in canonical ensemble”, Nucl Phys B, 2012, 854: 913C925

[36] Lu J X, Roy S, Xiao Z. “Phase structure of black branes in grand canonical ensemble”, JHEP, 2011, 5: 091 J. X. Lu, Ran Wei, J. F Xu, “The phase structure of black system in canonical ensemble”, JHEP12(2012)012 arXiv:1210.0708

[37] R.G. Cai, “Cardy-Verlinde formula and thermodynamics of black holes in de Sitter spaces”, Nucl. Phys. B 628 (2002)375

[38] Y. Sekiwa, “Thermodynamics of de Sitter black holes: Thermal cosmological constant”, Phys. Rev. D 73(2006)084009; hep-th/0602269

[39] M. Urano, A, Tomimatsu, “Mechanical First Law of Black Hole Spacetimes with Cosmological Constant and Its Application to Schwarzschild-de Sitter Spacetime”, Class.Quant.Grav.26(2009)105010 arXiv:0903.4230gr-qc

[40] L. C. Zhang, H. F. Li and R. Zhao, SCIENCE CHINA, Physics, Mechanics & Astronomy, 54(2011)1384.

[41] Y. S. Myung, “Thermodynamics of the Schwarzschild-de Sitter black hole: Thermal stability of the Nariai black hole”, Phys. Rev. D 77(2008)104007

[42] R. G. Cai, Physics 34(2005) 555( in Chinese)

[43] S. Bhattacharya, A. Lahiri, “Mass function and particle creation in Schwarzschild-de Sitter spacetime”, arXiv:1301.4532 [gr-qc]

[44] R.G. Cai, J. Y. Ji, K. S. Soh, “Action and entropy of black holes in spacetimes with cosmological constant”, Classical. Quantum. Grav. 15 (1998) 2783; arXiv:gr-qc/9708062

[45] R Zhao, M S Ma, H H Zhao, L C Zhang,“On the critical phenomena and Thermodynamics of the Reissner-Nordstrom-de Sitter black hole”, to be published.

[46] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, “Rotating black holes in higher dimensions with a cosmological constant”, Phys. Rev. Lett. 93 (2004) 171102, hep-th/0409155.

[47] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, “The general Kerr-de Sitter metrics in all dimensions”, J.Geom. Phys. 53 (2005) 49C73, hep-th/0404008

[48] R. Zhao, L. C. Zhang, H. F. Li. “Hawking radiation of Reisser-Nordstrom-de Sitter space”, Gen. Relat. Grav. 42(2010)975 ; R. Zhao, L. C. Zhang, H. F. Li, et al. “Hawking radiation of high-dimensional rotating black hole”, Eur.Phys. J. C. 65(2010)289

[49] R. Zhao, L. C. Zhang, Y. Q. Wu, “The Nernst Theorem and the Entropy of the ReissnerCNordstrom Black Hole”, Gen. Relat. Grav. 32(2000)1639;
[50] Li-Qin Mi, “Extremal black hole entropy satisfying the Nernst theorem”, Astrophysics and Space Science. 343(2013)599
[51] A. Sahay, T. Sarkar, and G. Sengupta, Thermodynamic Geometry and Phase Transitions in Kerr-Newman-AdS Black Holes, JHEP 1004, 118 (2010), [arXiv:1002.2538[hep-th]].
[52] A. Sahay, T. Sarkar, and G. Sengupta, On the Thermodynamic Geometry and Critical Phenomena of AdS Black Holes, JHEP 1007, 082 (2010), [arXiv:1004.1625[hep-th]].
[53] A. Sahay, T. Sarkar, and G. Sengupta, On The Phase Structure and Thermodynamic Geometry of R-Charged Black Holes, JHEP 1011, 125, (2010), [arXiv:1009.2236[hep-th]].
[54] D. Kastor, S. Ray, and J. Traschen, Enthalpy and the Mechanics of AdS Black Holes, Class. Quant. Grav. 26, 195011 (2009), [arXiv:0904.2765[hep-th]].
[55] R. Banerjee, S. K. Modak, S. Samanta, Second Order Phase Transition and Thermodynamic Geometry in Kerr-AdS Black Hole, Phys.Rev.D84:064024,2011 [arXiv:1005.4832].
[56] R. Banerjee, D. Roychowdhury, Critical behavior of Born Infeld AdS black holes in higher dimensions, Phys. Rev. D 85, 104043 (2012), [arXiv:1203.0118].
[57] R. Banerjee, D. Roychowdhury, Critical phenomena in Born-Infeld AdS black holes, Phys. Rev. D 85, 044040 (2012), [arXiv:1111.0147].
[58] R. Banerjee, D. Roychowdhury, Thermodynamics of phase transition in higher dimensional AdS black holes, JHEP 11(2011)004, [arXiv:1109.2433].
[59] R. Banerjee, S. Ghosh, D. Roychowdhury, New type of phase transition in Reissner Nordstrom - AdS black hole and its thermodynamic geometry, Phys. Lett. B696: 156-162, 2011, [arXiv:1008.2644].
[60] R. Banerjee, S. K. Modak, D. Roychowdhury, A unified picture of phase transition: from liquid-vapour systems to AdS black holes, JHEP 1210:125,2012, [arXiv:1106.3877].
[61] R. Banerjee, S. K. Modak, S. Samanta, Glassy Phase Transition and Stability in Black Holes, Eur.Phys.J.C70:317-328,2010, [arXiv:1002.0466].
[62] A. Lala, D. Roychowdhury, ”Ehrenfests scheme and thermodynamic geometry in Born-Infeld AdS black holes,” Phys Rev D 86, 084027 (2012);
[63] B. R. Majhi, D. Roychowdhury, Phase transition and scaling behavior of topological charged black holes in Horava-Lifshitz gravity, Class.Quant.Grav. 29 (2012) 245012, [arXiv:1205.0146] [gr-qc].
[64] C. Niu, Y. Tian, X. N. Wu, “Critical phenomena and thermodynamic geometry of Reissner-Nordström-anti-de Sitter black holes”, Physical Review D, 2012, 85(2): 024017.