CONVERGENCES AND THE INTERMEDIATE VALUE PROPERTY IN FERMAT REALS

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Abstract. This paper contains two topics of Fermat reals, as suggested by the title. In the first part, we study the $\omega$-topology, the order topology and the Euclidean topology on Fermat reals, and their convergence properties, with emphasis on the relationship with the convergence of sequences of ordinary smooth functions. We show that the Euclidean topology is best for this relationship with respect to pointwise convergence, and Lebesgue dominated convergence does not hold, among all additive Hausdorff topologies on Fermat reals. In the second part, we study the intermediate value property of quasi-standard smooth functions on Fermat reals, together with some easy applications. The paper is written in the language of Fermat reals, and the idea could be extended to other similar situations.

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1. Introduction

The idea of using infinitesimals in geometry and analysis, even from its birth, was on the one hand very intuitive and computable, and hence led to great development of mathematics and physics, and on the other hand very controversial for its rigor. It was A.-L. Cauchy who made the definition of limit rigorous using the epsilon-delta language. Since then, infinitesimals gradually left the mainstream of mathematics, but its idea was still used while doing research. The renaissance of infinitesimals happened when they were made rigorous, together with many applications in other fields of mathematics (see for example Non-Standard Analysis [R] and Synthetic Differential Geometry [K]).

Among all the existing infinitesimal theories, the theory of Fermat reals introduced by P. Giordano in [G2] has the properties that the theory is compatible with classical logic, all infinitesimals are nilpotent, and the ring \( \ast \mathbb{R} \) of Fermat reals is well-ordered. The whole theory is a mixture of algebra and analysis: the model of infinitesimals are polynomial-like function (called little-oh polynomials) modulo certain degree, the functions (called quasi-standard smooth functions) are locally extensions of ordinary smooth functions with parameters, and the calculations are given by Taylor’s expansion at standard point together with the nilpotency of infinitesimals (and hence a finite sum); see Section 2 for a quick review of the basics of Fermat reals.

In the current paper, we continue developing calculus of Fermat reals (see [GW] for the integral calculus). More precisely, we study two questions: (1) Does Lebesgue dominated convergence hold in Fermat reals? (2) Does every quasi-standard smooth function (of one variable) have the intermediate value property?

To settle the first question, we first study three natural topologies on Fermat reals (the \( \omega \)-topology, the order topology and the Euclidean topology) and their properties of convergence (pointwise and uniform), with emphasis on the relationship with the convergence of ordinary smooth functions. Then we show that the Euclidean topology is best for pointwise convergence (Theorem 27), and by a similar method that the Lebesgue dominated convergence does not hold (Theorem 32), for any additive Hausdorff topologies on the Fermat reals.

For the second question, the general answer is no (Remark 34 (3)). So the real interesting question is, which quasi-standard smooth functions have the intermediate value property. We study this in depth from simple to general, together with (counter-)examples and some applications. We show in Corollary 38 that the extension (without parameter) of ordinary smooth functions with no flat point have the intermediate value property, and the general case is solved in Subsection 6.4 (especially Proposition 41) by a similar method. The proof of Corollary 38 contains three ingredients: the slice image theorem (Theorem 33), the slice monotonicity (Observation (4) in Subsection 6.2) and some real analysis (the proof of Corollary 38). And the slice image theorem (Theorem 33) is indeed an algorithm, whose finite termination is proved with the usage of an unusual method (a mixture of real and symbolic computations).

Although this paper is written in the language of Fermat reals, many examples and some methods of proof can be applied to other similar situations.

I would like to thank P. Giordano for raising the question of Lebesgue dominated convergence in the collaboration of [GW] together with some comments on the first draft of Section 6, and also to G. Sinnamon for providing Example 37 and some discussion of Example 39.
2. Basics on Fermat reals

Fermat reals were introduced by P. Giordano in [G1, G2, G3, GK]. Let us review the basic theory here; see these references for detailed proof of these results.

Let \( U \) be an open subset of \( \mathbb{R}^n \). We define \( U_0[t] \), the little-oh polynomials on \( U \), to be the set of functions \( x: [0, \epsilon) \to U \) for some (not fixed) \( \epsilon \in \mathbb{R}_{>0} \) with the property that

\[
\|x(t) - r - \sum_{i=1}^{k} \alpha_i t^{a_i}\| = o(t) \quad \text{i.e.,} \quad \lim_{t \to 0^+} \frac{\|x(t) - r - \sum_{i=1}^{k} \alpha_i t^{a_i}\|}{t} = 0
\]

for some \( r \in U, \ k \in \mathbb{N}, \ \alpha_i \in \mathbb{R}^n \) and \( a_i > 0 \). Two little-oh polynomials \( x \) and \( y \) are called equivalent if \( x(0) = y(0) \) and \( x(t) - y(t) = o(t) \). This is an equivalence relation on \( U_0[t] \), and the quotient set is denoted by \( *U \). As a consequence, every element in \( *U \) has a unique representing little-oh polynomial of the form

\[
y(t) = \circ y + \sum_{i=1}^{l} \beta_i t^{b_i}
\]

for some \( \circ y := y(0) \in U, \ l \in \mathbb{N}, \ \beta_i \in (\mathbb{R}^n \setminus \{0\}) \) and \( 0 < b_1 < b_2 < \cdots < b_l \leq 1 \), defined on \( [0, \delta) \) for some maximum \( \delta \in \mathbb{R}_{>0} \cup \{\infty\} \). We call this the decomposition of the element \([y] \), \( \circ y \) the standard part, and we define \( \omega([y]) := \frac{1}{\delta} \) the order of \([y] \). For convenience, we sometimes use a similar form of \( y(t) \) as \eqref{eq:decomposition} but allowing \( \beta_i = 0 \), and we call such a form a quasi-decomposition of \([y] \). From now on, we write elements in \( *U \) by \( y \) instead of \([y] \) whenever there is no confusion.

Given a finite set of open subsets \( \{U_i\}_{i \in I} \) of Euclidean spaces, \( \prod_{i \in I} *U_i \) naturally bijects \( \prod_{i \in I} *U_i \). Therefore, we do not distinguish \( *U \) and \( (\mathbb{R}^n) \), and write it as \( \mathbb{R}^n \). We can also identify \( *U \) as a subset of \( \mathbb{R}^n \) by \( *U = \{x \in \mathbb{R}^n \mid \circ x \in U\} \) when \( U \) is an open subset of \( \mathbb{R}^n \).

There is a canonical injective map \( i_U: U \to *U \) defined by \( i_U(u)(t) = u \). So \( *U \) is an extension of \( U \), and for \( x \in *U \), we call \( \delta x := x - \circ x \) the infinitesimal part of \( x \). The meaning is clear when \( U = \mathbb{R} \): we can give a well ordering on \( \mathbb{R} \) by \( x \leq y \) if \( x = \circ x + \sum_{i=1}^{n} \alpha_i t^{a_i} \) and \( y = \circ y + \sum_{i=1}^{n} \beta_i t^{b_i} \), both in the quasi-standard form, with \( (\circ x, \alpha_1, \ldots, \alpha_n) \leq (\circ y, \beta_1, \ldots, \beta_n) \) in the dictionary order, and then \( D_\infty := \{x \in \mathbb{R} \mid \circ x = 0\} = \{x \in \mathbb{R} \mid -r < x < r \text{ for all } r \in \mathbb{R}_{>0}\} \). Moreover, every infinitesimal part \( \delta x \) of \( x \in *U \) is nilpotent, i.e., there exists some \( m = m(x) \in \mathbb{N} \) such that \( (\delta x)^m = 0 \).

Using this ordering, we can define intervals on \( \mathbb{R} \), e.g. \( (0, 1) := \{x \in \mathbb{R} \mid 0 < x < 1\} \). Instead, the usual intervals on \( \mathbb{R} \) will be denoted, e.g. \( (0, 1)_{\mathbb{R}} = (0, 1) \cap \mathbb{R} \).

On \( \mathbb{R}^n \), define \( \tau := \{\bigcup_{i \in I} U_i \mid U_i \text{ is an open subset of } \mathbb{R}^n\} \). Then \( \tau \) is a topology on \( \mathbb{R}^n \), called the Fermat topology, since \( \bigcap \tau = \mathbb{R}^n \). Without specification, for every subset \( A \) of \( \mathbb{R}^n \), we always equip it with the sub-topology of the Fermat topology of \( \mathbb{R}^n \).

Let \( f: U \to V \) be a smooth map between open subsets of Euclidean spaces. Then \( \circ f: \circ U \to \circ V \) by \( \circ f(x) = f \circ x \) is a well-defined map extending \( f \) (called the Fermat extension of \( f \)), i.e., \( \circ f(u) = f(u) \) whenever \( u \in U \). The calculation of \( \circ f(x) = \circ f(\circ x + \delta x) \) can be done by Taylor’s expansion of \( f \) at the point \( \circ x \), using the nilpotency of \( \delta x \). More precisely, if the \( (m+1)^{th} \) power of each component of \( \delta x \) is 0 for some \( m \in \mathbb{N} \), then we have

\[
\circ f(x) = \circ f(\circ x + \delta x) = \sum_{\substack{i \in \mathbb{N}^n, |i| \leq m}} \frac{1}{i!} \frac{\partial^{\|i\|}}{\partial x^i} f(\circ x) \cdot (\delta x)^i.
\]

\(^{1}\) It is a commutative unital ring under pointwise addition and pointwise multiplication, called the ring of Fermat reals.
Therefore, for any open subset \( W \) of \( V \), we have \( \ast(f^{-1}(W)) = (\ast f)^{-1}(\ast W) \), i.e., \( \ast f \) is continuous with respect to the Fermat topology.

Note that when \( U \neq \emptyset \) and \( \dim(V) > 0 \), not every constant map \( \ast U \to \ast V \) is of the form \( \ast f \) for some smooth function \( f : U \to V \), since otherwise \( \ast f(u) \in V \subset \ast V \) for every \( u \in U \subset \ast U \). We introduce the following definition:

**Definition 1.** Let \( A \subseteq \ast \mathbb{R}^n \) and \( B \subseteq \ast \mathbb{R}^m \) be arbitrary subsets. A function \( f : A \to B \) is called **quasi-standard smooth** if for every \( a \in A \), there exists an open neighborhood \( U \) of \( \circ a \) in \( \mathbb{R}^n \), an open subset \( V \) of some Euclidean space, a smooth map \( \alpha : V \times U \to \mathbb{R}^m \) and some fixed point \( v \in \ast V \), such that for every \( x \in A \cap \ast U \), we have

\[
\ast f(x) = \ast \alpha(v, x).
\]

In particular, every constant map \( A \to B \) and \( i_U : U \to \ast U \) are quasi-standard smooth. Moreover, every quasi-standard smooth map is continuous with respect to the Fermat topology.

### 3. Quasi-standard smooth functions revisited

In this section, we give another characterization of quasi-standard smooth functions.

**Proposition 2.** Let \( A \) be a subset of \( \ast \mathbb{R}^n \). Then \( f : A \to \ast \mathbb{R} \) is a quasi-standard smooth function if and only if for every \( a \in A \), there exist an open neighborhood \( U \) of \( \circ a \) in \( \mathbb{R}^n \), some \( m = m(f, U) \in \mathbb{N} \), a finite number of ordinary smooth functions \( \{\alpha_i : U \to \mathbb{R}\}_{i=0}^m \), and \( a_1, \ldots, a_m \in \mathbb{R} \) with \( 0 < a_1 < a_2 < \ldots < a_m \leq 1 \) such that

\[
\ast f(x) = \ast \alpha_0(x) + \sum_{i=1}^m \ast \alpha_i(x) \cdot t^{a_i} \quad \forall x \in A \cap \ast U.
\]

**Proof.** \((\Rightarrow)\) Let \( \ast \alpha(p, -) : A \cap \ast U \to \ast \mathbb{R} \) be a local expression of \( f \) near \( a \in A \), where \( p \) is a fixed parameter. Then the result follows from rearranging the terms according to the decomposition of \( \delta p, (\delta p)^2, \ldots, (\delta p)^k \) for \( k = \omega(\delta p) \), after Taylor’s expansion of \( \ast \alpha(p, -) = \ast \alpha(\circ p + \delta p, -) \) with respect to \( \circ p \). Here we have also used the fact that total Taylor’s expansion of a smooth function with several variables (for nilpotent infinitesimals) is the same as Taylor’s expansion by one variable after another.

\((\Leftarrow)\) This is clear. \(\square\)

The key point of the above proposition is, if we further assume that \( \circ A := \{\circ x \mid x \in A\} \) is an open subset of \( \mathbb{R}^n \) and \( \circ A \subseteq A \), then the expression in (3.1) is unique, while the expression in [G1, Theorem 12.1.9] is not in general. Here is the proof. Assume that we have two expressions:

\[
f(x) = \ast \alpha_0(x) + \sum_{i=1}^m \ast \alpha_i(x) \cdot t^{a_i} = \ast \beta_0(x) + \sum_{j=1}^l \ast \beta_j(x) \cdot t^{b_j}
\]

for all \( x \in A \cap \ast U \) with \( m, l \in \mathbb{N} \), \( \alpha_i : U \to \mathbb{R} \) and \( \beta_j : U \to \mathbb{R} \) ordinary smooth functions, and \( 0 < a_1 < a_2 < \ldots < a_m \leq 1, 0 < b_1 < b_2 < \ldots < b_l \leq 1 \). We may assume that \( U \subseteq \circ A \subset \circ \mathbb{R}^n \). For every \( x \in \circ A \subseteq A \), we can conclude that \( m = l \), \( \{a_1, \ldots, a_m\} = \{b_1, \ldots, b_m\} \), and \( \alpha_i(x) = \beta_i(x) \) for \( i = 0, 1, \ldots, m \) by the uniqueness of decomposition of elements in \( \ast \mathbb{R} \). Hence, \( \ast \alpha_i(x) = \ast \beta_i(x) \) for each \( i \).
Corollary 3. Let $A$ be a subset of $\mathbb{R}^n$ such that \(^o\!A\) is an open subset of $\mathbb{R}^n$ and \(^o\!A \subseteq A$. Then $f : A \to \mathbb{R}$ is a quasi-standard smooth function if and only if for every precompact subset $K$ of $A$ in the Fermat topology (i.e., the closure of \(^o\!K\) is compact in $\mathbb{R}^n$), there exist $m = m(f, K) \in \mathbb{N}$, a finite number of ordinary smooth functions $\{\alpha_i : U \to \mathbb{R}\}_{i=0}^m$ with $U$ an open neighborhood of \(^o\!K\) in $\mathbb{R}^n$, and $a_1, \ldots, a_m \in \mathbb{R}$ with $0 < a_1 < a_2 < \ldots < a_m \leq 1$ such that

$$f(x) = \alpha_0(x) + \sum_{i=1}^m \alpha_i(x) \cdot t^{a_i} \quad \forall x \in K.$$ 

Proof. This is straightforward from the above discussion together with Proposition 2. \qed

4. Topologies and convergences in Fermat reals

The main focus of the first part of this paper is to discuss convergences in Fermat reals. To define convergences, we need a topology on Fermat reals, and in order to make limit unique, we need the topology to be Hausdorff. Since the Fermat topology is not Hausdorff, we will introduce and study new Hausdorff topologies: the $\omega$-topology, the order topology, and the Euclidean topology. Note that the $\omega$-topology was first introduced in [GK]. We also explore the properties of convergences with respect to these topologies, together with comparisons to the convergences of ordinary smooth functions.

We first fix some notations:

Definition 4. A topology on a group is called additive if the group operations are continuous with respect to this topology. In other words, the group with this topology is a topological group.

A topology on Fermat reals is an additive Hausdorff topology on $\mathbb{R}$, which then induces the product topology on $\mathbb{R}^n$ for each $n \in \mathbb{N}$.

Note that $\mathbb{R}$ with coordinate-wise addition and the induced topology is a topological group, since topological groups are closed under finite products.

Definition 5. Let $\tau$ be a topology on Fermat reals. A sequence $(f_n : U \to \mathbb{R})_{n \in \mathbb{N}}$ of quasi-standard smooth functions from $U \subseteq \mathbb{R}^k$ is called pointwise convergent in $\tau$ if for each $x \in U$, $\lim_{n \to \infty} f_n(x)$ exists in $\tau$. In other words, there exists a function (not necessarily quasi-standard smooth; see Example 30) $f : U \to \mathbb{R}$ with the property that for every $x \in U$, for any $\tau$-open neighborhood $T$ of $f(x)$, there exists $N = N(x) \in \mathbb{N}$, such that for any $n > N$, we have $f_n(x) \in T$.

Note that we do not need additivity of the topology to define pointwise convergence, but we need it for uniform convergence:

Definition 6. Let $\tau$ be a topology on Fermat reals. A sequence $(f_n : U \to \mathbb{R})_{n \in \mathbb{N}}$ of quasi-standard smooth functions from $U \subseteq \mathbb{R}^k$ is called uniformly convergent in $\tau$ if there exists a function (not necessarily quasi-standard smooth) $f : U \to \mathbb{R}$ with the property that for any $\tau$-open neighborhood $T$ of $0 \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $f_n(x) - f(x) \in T$ for every $x \in U$.

Note that the convergence (both pointwise and uniform) of a sequence of quasi-standard smooth functions only depends on the topology of the codomain.

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2 Set theoretically, $\mathbb{R} = \mathbb{R} \times D_\infty$, i.e., the Cartesian product of standard part and infinitesimal part. The $\omega$-topology essentially relates to the standard part (i.e., the points in a small neighborhood only differ from the standard part), and the order topology essentially relates to the infinitesimal part.

3 As a convention, from now on, whenever there is no adjective in front of the word “topology” for Fermat reals, we mean the topology in this sense; otherwise, it has the usual meaning.
4.1. The $\omega$-topology. The $\omega$-topology on Fermat reals was first introduced in [GK]. We review some basics of the $\omega$-topology here without any details. The $\omega$-topology on $\mathbb{R}^n$ is induced by a complete metric $d_\omega: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $d_\omega(x, y) = ||x - y|| + \sum_{i=1}^n \omega(x_i - y_i)$, where $x_i, y_i \in \mathbb{R}$ are the $i^{th}$-coordinates of $x, y \in \mathbb{R}^n$ respectively. It has a base consisting of all balls $B_s(x; d_\omega)$ for $x \in \mathbb{R}^n$ and $s \in (0, 1]_\mathbb{R}$, where $B_s(x; d_\omega)$ is simply $\{x + r \mid r \in \mathbb{R}, ||r|| < s\}$. It is clear that the $\omega$-topology on $\mathbb{R}^n$ defined this way coincides with the product topology of the $\omega$-topology on $\mathbb{R}$, and the restriction of the $\omega$-topology to $\mathbb{R}^n$ is the standard topology. Moreover, the $\omega$-topology is strictly finer than the Fermat topology, and it does not behave well with quasi-standard smooth functions (see [GW, Section 3]). In other words, $\mathbb{R}$ with the $\omega$-topology is a topological group, but not a topological $\mathbb{R}$-vector space.

Compared to convergences of sequences of ordinary smooth functions, convergences in the $\omega$-topology is very restrictive; see the following three results.

**Lemma 7.** Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$. If it converges in the $\omega$-topology, then there exist a convergent sequence $(b_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ and $N \in \mathbb{N}$ such that for all $n > N$, we have $a_n = a_N + b_n$.

The converse of the above lemma is trivially true.

**Proof.** Write $a$ for the limit of the sequence $(a_n)_{n \in \mathbb{N}}$ in the $\omega$-topology. Since the $\omega$-open neighborhood $B_1(a; d_\omega)$ of $a$ is the set $\{a + r \mid r \in \mathbb{R}, ||r|| < 1\}$, we know that there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have $a_n \in B_1(a; d_\omega)$. In other words, $a_n - a_N \in \mathbb{R}$ for $n \geq N$. We set

$$b_n = \begin{cases} a_n - a_N, & \text{if } n > N \\ 0, & \text{otherwise.} \end{cases}$$

The rest of the proof is easy. \hfill $\square$

**Proposition 8.** Let $(f_n : U \to \mathbb{R})$ be a sequence of ordinary smooth functions from an open connected subset $U \subseteq \mathbb{R}$. If the sequence $(\star f_n : \star U \to \star \mathbb{R})_{n \in \mathbb{N}}$ converges uniformly in the $\omega$-topology, then there exists $N \in \mathbb{N}$ and a convergent sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ such that for every $n > N$, we have $f_n = f_N + a_n$.

The converse of the above proposition is trivially true.

**Proof.** By the previous lemma and the definition of uniform convergence, we know that there exists $N \in \mathbb{N}$ such that for any $n > N$ and any $u \in \star U$, $\star f_n(u)$ and $\star f_N(u)$ only differ from the standard part. In particular, this implies that for any $n > N$ and any $x \in U$, we have $f'_n(x) = f'_N(x)$. By the constant function theorem, we know that $f_n - f_N$ is constant for every $n > N$, since the domain $U$ of these functions is connected. Let $x_0 \in U$ be any point, and define

$$a_n = \begin{cases} f_n(x_0) - f_N(x_0), & \text{if } n > N \\ 0, & \text{otherwise.} \end{cases}$$

The rest of the proof is easy. \hfill $\square$

More generally, we have

**Theorem 9.** Let $U$ be a connected open subset of $\mathbb{R}^n$, and let $A$ be a subset of $\star \mathbb{R}^n$ such that $\star U \subseteq A \subseteq \overline{\star U}$, where $\overline{\star U}$ denotes the closure of $\star U$ in $\star \mathbb{R}^n$ with respect to the $\omega$-topology. If $(f_m : A \to \star \mathbb{R})_{m \in \mathbb{N}}$ is a uniformly convergent sequence of quasi-standard smooth functions in the $\omega$-topology, then there exist a convergent sequence $(a_m)_{m \in \mathbb{N}}$ in $\mathbb{R}$ and $N \in \mathbb{N}$ such that for every $m > N$, we have $f_m = f_N + a_m$. In particular, the limit function is also quasi-standard smooth.
The converse of the above theorem is trivially true.

Proof. Note that \( \overline{U} = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus U) \), where \( \overline{U} \) is the closure of \( U \) in \( \mathbb{R}^n \). By the uniqueness theorem ([GW, Theorem 5]), it is enough to prove the statement for \( A = \overline{U} \).

Let \( f \) be the limit of the sequence \( (f_m : A \to \mathbb{R})_{m \in \mathbb{N}} \) of quasi-standard smooth functions. By the definition of uniform convergence and Lemma 7, we know that there exists \( N \in \mathbb{N} \) such that for any \( m \geq N \) and any \( x \in \overline{U} \), \( f_m(x) \) and \( f(x) \) only differ from the standard part. Hence, \( f_m(x) \) and \( f_N(x) \) only differ from the standard part. In other words, we have a quasi-standard smooth function \( g_m : \overline{U} \to \mathbb{R} \) defined by \( x \mapsto f_m(x) - f_N(x) \). Since \( \overline{U} \) is connected in \( \mathbb{R}^n \), \( \overline{U} \) is connected in \( \mathbb{R}^n \) with respect to the Fermat topology. By [W, Proposition 24], we know that \( g_m \) is constant. The rest of the proof is easy. \( \square \)

4.2. The order topology. Since \( \mathbb{R} \) is a totally ordered ring, we have the order topology on \( \mathbb{R} \).

Note that the order topology on \( \mathbb{R} \) is a Hausdorff topology, which has a base consisting of open intervals \((a, b) \) for \( a, b \in \mathbb{R} \). Hence, the restriction of the order topology to \( \mathbb{R} \) is the discrete topology (for example, \((-t, t) \cap \mathbb{R} = \{0\})

Proposition 10. \( \mathbb{R} \) with the order topology is a topological group, but not a topological \( \mathbb{R} \)-vector space. In other words, the order topology does not behave well with quasi-standard smooth functions.

Proof. Note that the preimage of the order open subset \((t^{1/2} - t, t^{1/2} + t)\) of the scalar multiplication map \( \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) contains \((1, t^{1/2}) \in \mathbb{R} \times \mathbb{R} \), but none of its open neighborhoods. The rest is straightforward. \( \square \)

Lemma 11. The order topology on \( \mathbb{R}^n \) is first countable.

Proof. It is enough to prove this for \( n = 1 \). By the previous proposition, it is enough to show that \( 0 \in \mathbb{R} \) has a countable neighborhood base in the order topology, which can in fact be chosen as \((-\frac{1}{n}, \frac{1}{n})\) for \( n \in \mathbb{Z}^>0 \). \( \square \)

It is clear that the order topology on \( \mathbb{R}^n \) contains the Fermat topology, but it is not comparable with the \( \omega \)-topology.

Compared to convergences of sequences of ordinary smooth functions, convergences in the order topology is also very restrictive; see the following two results.

Lemma 12. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence in \( \mathbb{R} \). If it converges in the order topology, then there exists \( N \in \mathbb{N} \) such that for all \( n > N \), \( a_n - a_N = b_n t \) for some convergent sequence \((b_n)_{n \in \mathbb{N}}\) in \( \mathbb{R} \). In other words, the coefficient of \( t^i \) in the decomposition of \( a_n \) are fixed for all \( 0 < i < 1 \) and all \( n > N \).

The converse of the above lemma is trivially true.

Proof. Assume that the sequence \((a_n)_{n \in \mathbb{N}}\) converges to \( a \in \mathbb{R} \) in the order topology. The result follows directly by considering the order open neighborhoods \((a - \frac{1}{m} t, a + \frac{1}{m} t)\) of \( a \) for \( m \in \mathbb{Z}^>0 \). \( \square \)

Proposition 13. Let \((f_n : U \to \mathbb{R})_{n \in \mathbb{N}}\) be a sequence of ordinary smooth functions from an open subset \( U \subseteq \mathbb{R}^m \). If the sequence \((f'_n : \mathbb{R} \to \mathbb{R})_{n \in \mathbb{N}}\) converges uniformly in the order topology, then there exists \( N \in \mathbb{N} \) such that for every \( n > N \), we have \( f_n = f_N \).

The converse of the above proposition is trivially true.
Proof. By the previous lemma and the definition of uniform convergence, we know that there exists \( N \in \mathbb{N} \) such that for any \( n > N \) and any \( u \in \mathbb{R} \), \( f_n(u) \) and \( f_N(u) \) only differ from the coefficients of \( t \) in their decomposition. In particular, \( f_n = f_N \) for all \( n > N \), by taking the standard part. \( \square \)

As [GW, Theorem 5], a similar uniqueness theorem holds for the order topology:

**Theorem 14.** Let \( A \) be an arbitrary subset of \( \mathbb{R}^n \), and let \( f, g : A \to \mathbb{R} \) be quasi-standard smooth functions. If \( f(x) = g(x) \) for all \( x \) in a dense subset \( B \) of \( A \) in the order topology, then \( f = g \).

**Proof.** Without loss of generality, we may assume that \( g = 0 \). By definition of quasi-standard smooth function, assume that \( f(x) = \alpha(x) \) for \( x \in A \cap \mathbb{R}^n \), where \( U \) is an open subset of a Euclidean space, \( \alpha : W \times U \to \mathbb{R} \) is an ordinary smooth function, and \( p \in W \) is a fixed parameter. By definition of the order topology on Fermat reals, for any \( x_0 \in (A \setminus B) \cap U \), there exists a sequence \( (a_i)_{i \in \mathbb{Z}^+} \) in \( \mathbb{R}^n \) converging to 0, such that \( x_i := x_0 + a_i t \). By Taylor's formula with nilpotent increments, one checks that

\[
 f(x_0) = f(x_0) - f(x_i) = - \sum_{j=1}^{n} \frac{\partial \alpha}{\partial x_j}(p, 0) t_a + \sum_{j=1}^{n} \frac{\partial \alpha}{\partial x_j}(p, x_0) \cdot a_{ij} t
\]

for every \( i \in \mathbb{Z}^+ \), where \( a_{ij} \) is the \( j \)-th-component of \( a_i \), which implies that \( f(x_0) = 0 \), since \( a_{ij} \to 0 \) as \( i \to \infty \). \( \square \)

### 4.3. The Euclidean topology.

Note that as an \( \mathbb{R} \)-vector space, \( \mathbb{R}^n \) can be viewed as a linear subspace of \( \mathbb{R}^n \times (\mathbb{R}^n)^{(0,1)}_\mathbb{R} \), consisting of elements \( (x, y) \in \mathbb{R}^n \times (\mathbb{R}^n)^{(0,1)}_\mathbb{R} \) such that \( y_i = 0 \in \mathbb{R}^n \) for all except finitely many \( i \in (0,1] \). So we can write elements in \( \mathbb{R}^n \) as \( x = \alpha + \sum_{i \in (0,1]} \alpha_i \cdot t^i \) (called the quasi-decomposition of \( x \in \mathbb{R}^n \)), and remembering that all \( \alpha_i = 0 \in \mathbb{R}^n \) except for finitely many \( i \in (0,1] \). The notion of quasi-decomposition here is consistent with that in Section 2.

**Definition 15.** Let \( x = \alpha + \sum_{i \in (0,1]} \alpha_i \cdot t^i \) and \( y = \beta + \sum_{i \in (0,1]} \beta_i \cdot t^i \) be quasi-decomposition of two elements in \( \mathbb{R}^n \). Recall that only finitely many \( \alpha_i \) and \( \beta_i \)'s are non-zero. We define

\[
 \langle x, y \rangle = \langle \alpha, \beta \rangle_\mathbb{R} + \sum_{i \in (0,1]} \langle \alpha_i, \beta_i \rangle_\mathbb{R}
\]

where \( \langle -, - \rangle_\mathbb{R} \) denotes the standard inner product on \( \mathbb{R}^n \).

**Lemma 16.** \( \langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) defined above is an inner product on the infinite-dimensional \( \mathbb{R} \)-vector space \( \mathbb{R}^n \). It makes \( \mathbb{R}^n \) into a topological \( \mathbb{R} \)-vector space, but not a Hilbert space.

We call this inner product the Euclidean inner product on \( \mathbb{R}^n \), and the topology induced by it the Euclidean topology. It is clear that the Euclidean topology is strictly finer than the Fermat topology, but strictly coarser than both the \( \omega \)- and the order topologies. Moreover, the restriction of the Euclidean inner product and the Euclidean topology to \( \mathbb{R}^n \) is the standard inner product and the standard topology.

**Proof.** It is straightforward to show that \( \langle -, - \rangle \) is an inner product on \( \mathbb{R}^n \), and \( \mathbb{R}^n \) is a topological \( \mathbb{R} \)-vector space in the Euclidean topology. To see that the Euclidean topology is not complete, let \( n = 1 \) and take \( a_m = \sum_{i=1}^{m} \frac{1}{i} t^{i/1} \in \mathbb{R} \). It is easy to see that the sequence \( (a_m)_{m \in \mathbb{N}} \) is Cauchy, but it has no limit in \( \mathbb{R} \) with respect to the Euclidean topology. \( \square \)

The Euclidean topology also does not behave well with quasi-standard smooth functions:
Remark 17. Let \( \| \cdot \| \) be the norm induced by the Euclidean inner product \( \langle \cdot , \cdot \rangle \) on \( \star \mathbb{R} \). In general, there is no definite inequality relating \( \| xy \| \) and \( \| x \| \cdot \| y \| \) for \( x, y \in \star \mathbb{R} \). For example,

(i) \[ \| t \cdot t \| = \| t \|^{2} < 1 = \| t \| \cdot \| t \| ; \]
(ii) \[ \| (1 + t^{1/2}) \cdot t^{1/2} \| = \sqrt{2} = \| 1 + t^{1/2} \| \cdot \| t^{1/2} \| ; \]
(iii) \[ \| (1 + t^{1/2})^{2} \| = \sqrt{6} > 2 = \| 1 + t^{1/2} \|^{2} . \]

As a consequence, one can show that the multiplication map \( \star \mathbb{R} \times \star \mathbb{R} \to \star \mathbb{R} \) is not continuous when \( \star \mathbb{R} \) is equipped with the Euclidean topology. This is because for any \( \delta \in \mathbb{R}_{> 0} \), there exist \( x_{n}, y_{n} \in B_{\delta}(0; (-, -)) \) such that \( \lim_{n \to \infty} \| x_{n} \cdot y_{n} \| = \infty \). In fact, one can take \( x_{n} = y_{n} = \frac{\delta}{\sqrt{n+1}} \sum_{i=1}^{n} t^{i/2} \).

Theorem 18. The Cauchy completion of \( \star \mathbb{R} \) with respect to the Euclidean inner product is the linear subspace \( V \) of \( \mathbb{R} \times [0, 1] \) consisting of elements of the form \( x = \sum_{i \in [0, 1]} \alpha_{i} t^{i} \) with all \( \alpha_{i} = 0 \) except for countably many \( i \in (0, 1]_{\mathbb{R}} \), and \( \sum_{i \in (0, 1]} \alpha_{i}^{2} < \infty \). The Euclidean inner product on \( \star \mathbb{R} \) extends canonically to an inner product on \( V \), which makes \( V \) a (non-separable) Hilbert space.

Proof. This is straightforward. \( \square \)

Remark 19. We no longer have the nilpotency of infinitesimals and the dictionary order in \( V \).

In order to state the next theorem, we need the following definition:

Definition 20. A sequence \( (f_{n} : U \to \mathbb{R})_{n \in \mathbb{N}} \) of ordinary smooth functions defined on an open subset \( U \) of \( \mathbb{R}^{m} \) is called pointwise Taylor convergent, if for every \( k \in \mathbb{N}^{m} \), the sequence \( (\frac{d^{k}}{d x^{n}} f_{n})_{n \in \mathbb{N}} \) is pointwise convergent.

Example 21. Let \( f_{n} : (-1, 1)_{\mathbb{R}} \to \mathbb{R} \) be defined by \( f_{n}(x) = x^{n} \). Then for any \( k \in \mathbb{N} \), we have

\[ f_{n}^{(k)}(x) = \begin{cases} 0, & \text{if } n < k \\ \frac{n!}{(n-k)!} x^{n-k}, & \text{otherwise.} \end{cases} \]

So for any \( x \in (-1, 1)_{\mathbb{R}} \) and any \( k \in \mathbb{N} \), \( \lim_{n \to \infty} f_{n}^{(k)}(x) = 0 \), and hence the sequence \( (f_{n})_{n \in \mathbb{N}} \) is pointwise Taylor convergent.

Note that the ordinary smooth function \( f_{n} \) above can be defined on the whole \( \mathbb{R} \), and the sequence \( (f_{n})_{n \in \mathbb{N}} \) is pointwise convergent on \( (-1, 1)_{\mathbb{R}} \), but the sequence of the Fermat extension \( (\star f_{n})_{n \in \mathbb{N}} \) is not pointwise convergent in the Euclidean topology at \( 1 + \epsilon \) for any non-zero infinitesimal \( \epsilon \).

Here is an example of a pointwise convergent sequence which is not pointwise Taylor convergent:

Example 22. Let \( f_{n} : \mathbb{R} \to \mathbb{R} \) be defined by \( f_{n}(x) = \frac{1}{n} \sin(nx) \). Then \( f_{n}'(x) = \cos(nx) \). So for any \( x \in \mathbb{R} \), \( \lim_{n \to \infty} f_{n}(x) = 0 \), but \( \lim_{n \to \infty} f_{n}'(x) \) does not exist in general. Hence, the sequence \( (f_{n})_{n \in \mathbb{N}} \) is pointwise convergent, but not pointwise Taylor convergent.

Theorem 23. Let \( (f_{n} : U \to \mathbb{R})_{n \in \mathbb{N}} \) be a sequence of ordinary smooth functions defined on an open subset \( U \) of \( \mathbb{R}^{m} \). Then \( (f_{n})_{n \in \mathbb{N}} \) is pointwise Taylor convergent if and only if the sequence of the Fermat extension \( (\star f_{n} : U \to \star \mathbb{R})_{n \in \mathbb{N}} \) is pointwise convergent in the Euclidean topology.

A similar statement does not hold if we change the Euclidean topology to the \( \omega \)-topology or the order topology; see Example 21, Lemmas 7 and 12.
Proof. For any $x \in \bullet U$, write $x = \circ x + \delta x$ for the standard and the infinitesimal parts of $x$, and write $k$ for the integer part of $\omega(x)$. Then we have

$$\bullet f_n(x) = \sum_{i \in \mathbb{N}^m, |i| \leq k} \frac{1}{i!} \partial^{|i|} f_{\circ}(x) \cdot (\delta x)^i.$$  

($\Leftarrow$) For $y \in \bullet U$, write $\bullet f^w_n(y)$ for the coefficient of $t^w$ of the quasi-decomposition of $\bullet f_n(y)$. By the definition of the Euclidean topology, we know that the fact that $\lim_{n \to \infty} \bullet f_n(y)$ exists implies that $\lim_{n \to \infty} \circ \bullet f_n(y)$ and $\lim_{n \to \infty} \bullet f^w_n(y)$ exist for each $w \in \{0, 1\}$. The conclusion then follows from the facts that for any $x \in U$, $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \bullet f_n(x)$, and that $\lim_{n \to \infty} \frac{\partial^{|i|} f_{\circ}}{i!}(x)$ is $j!$ times the coefficient of $t$ of the quasi-decomposition of $\lim_{n \to \infty} \bullet f_n(x + \sum_{i=1}^m t^{|i|} \cdot e_i)$ for each $j \in \mathbb{N}^m \setminus \{0\}$, where $e_1, \ldots, e_m$ is the standard basis for $\mathbb{R}^m$, and $(a_1, \ldots, a_m) = (a_1(j), \ldots, a_m(j)) \in \mathbb{R}^m$ are suitably chosen such that $\sum_{i=1}^m i j a_i = 1$, and for any $s \in \mathbb{N}^m$ with $s \neq j$, we have $\sum_{i=1}^m s i a_i \neq 1$. The existence of such $a_i$ is guaranteed by the following remark.

($\Rightarrow$) Since the sequence $(f_n)_{n \in \mathbb{N}}$ of ordinary smooth functions is pointwise Taylor convergent, $\lim_{n \to \infty} \circ \bullet f_n(x)$ exists for each $i \in \mathbb{N}^m$, and hence $\lim_{n \to \infty} \bullet f_n(x)$ exists. \qed

Remark 24. In this case for any fixed $j \in \mathbb{N}^m \setminus \{0\}$, there exists $a \in \mathbb{R}^m$ such that for any $s \in \mathbb{N}^m$ with $s \neq j$, we have $\sum_{i=1}^m s_i a_i \neq \sum_{i=1}^m j_i a_i$. We prove this by induction on $m$. It is clearly true for $m = 1$. Assume that we have proved the statement for $m = k$, and now we consider $m = k + 1$. Note that we have a continuous function $\phi : \mathbb{R}_0^k \times \mathbb{R}_0^{k+1} \to \mathbb{R}$ defined by $(b_1, \ldots, b_{k+1}, u_1, \ldots, u_{k+1}) \mapsto \sum_{i=1}^{k+1} u_i b_i$, and $\phi((1, \ldots, 1) \times \mathbb{N}^{k+1}) \subseteq \mathbb{N}$. Our strategy is to perturb $(1, \ldots, 1)$ a bit to get the required $a$. So we are left to show that there exists $c \in \mathbb{R}_0^{k+1}$ such that $\sum_{i=1}^{k+1} c_i s_i \neq \sum_{i=1}^{k+1} c_i j_i$ for every $s \in \mathbb{N}^{k+1}$ with $|s| = |j|$ and $s \neq j$. Then $a$ defined by $a_i = 1 + \frac{c_i}{s_i}$ for some large enough $N \in \mathbb{N}$ is what we are looking for, by the continuity of the map $\phi$ and the discreteness of $\mathbb{N}$ in $\mathbb{R}$. Note that $|s| = |j|$ and $s \neq j$ imply that $(s_1, \ldots, s_k) \neq (j_1, \ldots, j_k)$. Now we split into two cases: $(j_1, \ldots, j_k) \in \mathbb{N}^k \setminus \{0\}$ and $(j_1, \ldots, j_k) = (0, \ldots, 0)$. For the first case, we get the conclusion by the induction hypothesis and setting $c_{k+1} = 0$. For the second case, we get the conclusion by setting $(c_1, \ldots, c_k, c_{k+1}) = (0, \ldots, 0, 1)$.

Uniform convergence in the Euclidean topology for sequences of the Fermat extension of ordinary smooth functions is also trivial:

Proposition 25. Let $(f_n : U \to \mathbb{R})_{n \in \mathbb{N}}$ be a sequence of ordinary smooth functions defined on a connected open subset $U$ of $\mathbb{R}^m$. If the sequence of the Fermat extension $(\bullet f_n)_{n \in \mathbb{N}}$ converges uniformly in the Euclidean topology, then there exist a convergent sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ and $N \in \mathbb{N}$ such that for every $n > N$, we have $f_n = f_N + a_n$.

The converse of the above proposition is trivially true.

Proof. Since $(\bullet f_n)_{n \in \mathbb{N}}$ converges uniformly in the Euclidean topology, for every open neighborhood $T$ of $0 \in \bullet \mathbb{R}$, there exists $N \in \mathbb{N}$ such that for every $n, l > N$, we have $\bullet f_n(x) - \bullet f_l(x) \in T$ for every $x \in \bullet U$. This could not hold if $\frac{\partial (f_n - f_l)}{\partial x_j}(x_0) \neq 0$ for some $j = 1, 2, \ldots, m$ and some $x_0 \in U$. The rest of the proof is straightforward. \qed

\footnote{\textsuperscript{4}As a warning, the converse of this statement is not necessarily true, unless the sequence $(\bullet f_n(y))_{n \in \mathbb{N}}$ is uniformly bounded, in the sense that there exists a finite subset $A$ of $[0, 1] \setminus A$ and for every large enough $n$.}
5. Some general results for Fermat reals

In this section, we prove some results which hold for any topology on Fermat reals. And recall from Definition 4 that by a topology on Fermat reals, we always mean an additive Hausdorff topology on \(\mathbf{\bullet} \mathbb{R}\), which then induces the product topology on \(\mathbf{\bullet} \mathbb{R}^n\).

5.1. The Euclidean topology is best for pointwise convergence. From the previous section, we know that the pointwise convergence of the sequence of the Fermat extension of ordinary smooth functions in any natural topology on Fermat reals always imposes extra conditions on the sequence of the original ordinary smooth functions. In fact, this is a common phenomenon:

\textbf{Theorem 26.} There is no topology \(\tau\) (Definition 4) on Fermat reals such that for every pointwise convergent sequence of ordinary smooth functions \((f_n : U \to \mathbb{R})_{n \in \mathbb{Z}^+}\), where \(U\) is an open subset of \(\mathbb{R}^m\), the sequence of the Fermat extension \((\mathbf{\bullet} f_n)_{n \in \mathbb{Z}^+}\) is pointwise convergent in \(\tau\).

\textbf{Proof.} We prove below that every additive topology \(\tau\) satisfying the above condition for \(U = \mathbb{R}\) is not Hausdorff.

Let \(f_n(x) = \frac{1}{n} \sin(nx)\). Then \((f_n)_{n \in \mathbb{Z}^+}\) pointwise converges to the constant function with value 0. By assumption, the sequence \((\mathbf{\bullet} f_n)_{n \in \mathbb{Z}^+}\) pointwise converges in \(\tau\). So, for any \(x \in D := \{x \in \mathbf{\bullet} \mathbb{R} \mid x^2 = 0\}\), we have

\[\mathbf{\bullet} f_n(\frac{n}{2} + x) = \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{2}\right) \cdot x.\]

Therefore, \(2x \in T\) for any \(\tau\)-open neighborhood \(T\) of 0 in \(\mathbf{\bullet} \mathbb{R}\). In other words, every \(\tau\)-open neighborhood of 0 in \(\mathbf{\bullet} \mathbb{R}\) contains \(D\), which is an ideal of the commutative unital ring \(\mathbf{\bullet} \mathbb{R}\) ([GK, Theorem 23]). Therefore, \(\tau\) cannot be Hausdorff.

Moreover, we have:

\textbf{Theorem 27.} The Euclidean topology is a best topology (Definition 4) for pointwise convergence in the sense of Theorem 23.

\textbf{Proof.} We prove below that for any \(k \in \mathbb{N}\), if an additive topology \(\tau\) on Fermat reals has the property that for every sequence of ordinary smooth functions \((f_n : \mathbb{R} \to \mathbb{R})_{n \in \mathbb{Z}^+}\) such that \((f_n^{(i)})_{n \in \mathbb{Z}^+}\) is pointwise convergent for each \(i = 0, 1, \ldots, k\), the sequence of the Fermat extension \((\mathbf{\bullet} f_n)_{n \in \mathbb{Z}^+}\) is pointwise convergent in \(\tau\), then \(\tau\) cannot be Hausdorff.

It is easy to see that for any \(c \in \mathbb{R}\), there is an ordinary smooth function \(g : \mathbb{R} \to \mathbb{R}\) with the properties that \(g(x + 2) = g(x)\) and \(g(x) = g(1 - x)\) for each \(x \in \mathbb{R}\), \(g(0) = g'(0) = \cdots = g^{(k)}(0) = 0\) and \(g^{(k+1)}(0) = c\). From these properties, one derives that

\[g^{(i)}(m) = \begin{cases} g^{(i)}(0), & \text{if } m \text{ is even} \\ (-1)^{m}g^{(i)}(0), & \text{if } m \text{ is odd} \end{cases}\]

for any \(i \in \mathbb{N}\) and any \(m \in \mathbb{Z}\). Now we define \(f_n(x) = \frac{1}{n^{k+1}} g(nx)\). Then \(f_n^{(i)}(x) = \frac{1}{n^{k+1+i}} g^{(i)}(nx)\) for any \(i \in \mathbb{N}\). So for any \(x \in \mathbb{R}\), \(\lim_{n \to \infty} f_n^{(i)}(x) = 0\) for each \(i = 0, 1, \ldots, k\), and \(f_n^{(k+1)}(x) = g^{(k+1)}(nx)\).

For any \(a \in D_{k+1} := \{x \in \mathbf{\bullet} \mathbb{R} \mid x^{k+2} = 0\}\), we have

\[
\mathbf{\bullet} f_n(1 + a) = f_n(1) + f'_n(1) \cdot a + \frac{f''_n(1)}{2!} \cdot a^2 + \cdots + \frac{f^{(k)}(1)}{k!} \cdot a^k + \frac{f^{(k+1)}(1)}{(k + 1)!} \cdot a^{k+1}
\]

\[= \frac{g^{(k+1)}(n)}{(k + 1)!} \cdot a^{k+1}.
\]
By assumption, the sequence \((f_n(1 + a))_{n \in \mathbb{Z} > 0}\) converges in \(\tau\). So we can conclude that every \(\tau\)-open neighborhood of \(0\) in \(*\mathbb{R}\) contains \(A := \{x^{k + 1} \mid x \in D_{k + 1}\}\). Fix any \(u, v \in A\) with \(u \neq v\). Since \(A \cap (u - v + A) \neq \emptyset\), we know that the additive topology \(\tau\) is not Hausdorff. \(\square\)

Finally, we discuss the uniqueness of the pointwise convergence in the Euclidean topology in the following sense:

Let \(X\) be a set, and let \(\tau_1, \tau_2\) be two topologies on \(X\). We say that \(\tau_1\) and \(\tau_2\) are strongly convergence equivalent if a sequence \((x_n)_{n \in \mathbb{N}}\) in \(X\) converges to \(x \in X\) in \(\tau_1\) if and only if it converges to \(x\) in \(\tau_2\). We have:

**Theorem 28.** Let \(X\) be a set, and let \(\tau_1, \tau_2\) be first-countable topologies on \(X\). Then \(\tau_1\) and \(\tau_2\) are strongly convergence equivalent if and only if \(\tau_1 = \tau_2\).

**Proof.** It is enough to show that for any \(\tau_1\)-open subset \(A\) of \(X\) and any \(a \in A\), there exists a \(\tau_2\)-open neighborhood \(B\) of \(a\) such that \(B \subseteq A\). Assume that this is not true, i.e., there exist a \(\tau_1\)-open subset \(A\) of \(X\) and \(a \in A\) such that for any \(\tau_2\)-open neighborhood \(B\) of \(a\), \(B \setminus A \neq \emptyset\). Since \(\tau_2\) is first-countable, there exists a countable \(\tau_2\)-neighborhood basis \(\{B_i\}_{i \in \mathbb{N}}\) of \(a\) such that \(\cdots \subseteq B_1 \subseteq B_0\). Pick \(b_i \in B_i \setminus A\). Then the sequence \((b_n)_{n \in \mathbb{N}}\) is a sequence which converges to \(a\) in \(\tau_2\). Since \(\tau_1\) and \(\tau_2\) are strongly convergence equivalent, this sequence also converges to \(a\) in \(\tau_1\), which implies that \(b_i \in A\) for \(i\) large enough, i.e., we reach a contradiction. \(\square\)

Note that all of the \(\omega\)-topology, the order topology and the Euclidean topology are first-countable.

In particular of the above theorem, we have:

**Corollary 29.** The pointwise convergence in the Euclidean topology is unique among those in all first-countable topologies on Fermat reals.

5.2. **Impossibility of Lebesgue dominated convergence.** As expected, the pointwise limit of a sequence of quasi-standard smooth functions may not be quasi-standard smooth in any topology:

**Example 30.** For \(i = 2, 3, 4, \ldots\), let \(\sigma_i : \mathbb{R} \rightarrow \mathbb{R}\) be an ordinary smooth function with the properties that \(\sigma_i(x) = 1\), \(\text{Im}(\sigma_i) = [0, 1]_{\mathbb{R}}\), \(\text{Supp}(\sigma_i) \subseteq \left[\frac{1}{2}(\frac{1}{i + 1} + \frac{1}{i}), \frac{1}{2}(\frac{1}{i - 1} + \frac{1}{i})\right]_{\mathbb{R}}\), and \(\int_0^1 \sigma_i(x)dx = 1^5\). Then \((f_n : *\mathbb{R} \rightarrow *\mathbb{R})_{n \in \mathbb{Z} > 0}\) defined by

\[
  f_n(x) = \sum_{i=1}^n \sigma_{i+1}(x) \cdot t^{i/i}
\]

is a sequence of quasi-standard smooth functions, which pointwise converges to the function \(f : *\mathbb{R} \rightarrow *\mathbb{R}\) with

\[
  f(x) = \begin{cases} \sigma_1(x) \cdot t^{1/i}, & \text{if } x \in \left(\frac{1}{2}(\frac{1}{i + 1} + \frac{1}{i}), \frac{1}{2}(\frac{1}{i - 1} + \frac{1}{i})\right) \\ 0, & \text{else} \end{cases}
\]

in any topology (not necessarily additive Hausdorff). By Proposition 2, \(f\) is not quasi-standard smooth (around \(0\)).

**Remark 31.** By the results we developed in the previous section, a similar statement to the classical Lebesgue dominated convergence theorem does not hold for quasi-standard smooth functions in

\(^5\)The last condition is a normalization, which will only be used in the remark following this example.
either of the $\omega$-topology, the order topology or the Euclidean topology. This is because, in the above example, $|f_n(x)| \leq 1$, but the limit of
\[
\int_0^1 f_n(x) dx = \sum_{i=1}^n t^{1/i} \cdot \int_0^1 \sigma_{i+1}(x) dx = \sum_{i=1}^n t^{1/i} \cdot \int_0^1 \sigma_{i+1}(x) dx = \sum_{i=1}^n t^{1/i}
\]
as $n \to \infty$ does not exist in the $\omega$-topology (Lemma 7) or the order topology (Lemma 12) or the Euclidean topology (Definition 15); for integration of quasi-standard smooth functions, see [GW].

More generally, we have:

**Theorem 32.** The similar statement to the classical Lebesgue dominated convergence theorem does not hold for $\bullet \mathbb{R}$ with any topology (Definition 4).

*Proof.* We prove below that if $\tau$ is an additive topology on $\bullet \mathbb{R}$ such that for every pointwise convergent sequence $(f_n : \bullet \mathbb{R} \to \bullet \mathbb{R})_{n \in \mathbb{Z}^+}$ in $\tau$ with $|f_n| \leq g$ for some quasi-standard smooth function $g : \bullet \mathbb{R} \to \bullet \mathbb{R}$, we have the existence of $\lim_{n \to \infty} \int_0^1 f_n(x) dx$ in $\bullet \mathbb{R}$ with respect to $\tau$, then $\tau$ cannot be Hausdorff.

In fact, we can take $g$ to be the constant function with value 1. Observe that in Example 30, we can change $t^{1/i}$ to any $\delta_i \in D_\infty$, so that we still have a pointwise convergent sequence $(f_n^{(\delta_i)} : n \in \mathbb{N})_{n \in \mathbb{N}}$ in any topology, and we have
\[
\int_0^1 f_n^{(\delta_i)}(x) dx = \sum_{i=1}^n \delta_i =: b_n.
\]
By assumption, the sequence $(b_n)_{n \in \mathbb{Z}^+}$ converges in $\tau$, i.e., for any $\tau$-open neighborhood $T$ of 0 $\in \bullet \mathbb{R}$, there exists $N \in \mathbb{Z}^+$ such that for every $n, m > N$, $b_n - b_m \in T$. In particular, $\delta_i \in T$ for $l$ large enough. So we get the following information of $\tau$: every $\tau$-open neighborhood of 0 $\in \bullet \mathbb{R}$ contains all but finitely many points of $D_\infty$. Therefore, $\tau$ cannot be Hausdorff. \hfill $\Box$

6. **Intermediate value property**

Recall that the Fermat reals $\bullet \mathbb{R}$ has a total ordering, which is an extension of the usual ordering on $\mathbb{R}$, and which makes $\bullet \mathbb{R}$ an ordered commutative ring. In this section, we investigate which quasi-standard smooth functions $f : \bullet U \to \bullet \mathbb{R}$, with $U$ an open connected subset of $\mathbb{R}$, has the intermediate value property, i.e., if $a, b \in \bullet U$ with $a < b$, then for any $y \in \bullet \mathbb{R}$ between $f(a)$ and $f(b)$, there exists some $c \in \bullet \mathbb{R}$ between $a$ and $b$ such that $f(c) = y$.

Since quasi-standard smooth functions are locally restrictions (in the sense of Fermat topology) of the Fermat extension of ordinary smooth functions, we first discuss the question for Fermat extension of ordinary smooth functions, which is answered in Corollary 38. We subsequently use the idea of the proof of this corollary to get a general criteria (Proposition 41), together with some applications. In order to reach these results, we will need some preparations.

6.1. **The slice image theorem.** Here is the slice image theorem for one variable, as one of the key ingredients for solving the intermediate value property of Fermat extension of ordinary smooth functions:

**Theorem 33.** Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, and let $a \in \mathbb{R}$.

(i) If $f^{(n)}(a) = 0$ for all $n \in \mathbb{Z}^+$, then

$$\bullet f(a + D_\infty) = \{ f(a) \};$$
(ii) Assume that there exists some \( n \in \mathbb{Z}^{> 0} \) such that \( f^{(n)}(a) \neq 0 \). Denote by \( m \) the smallest of such \( n \).

(a) If \( m \) is odd, then

\[
\star f(a + D_\infty) = f(a) + D_\infty;
\]

(b) If \( m \) is even, then

\[
\star f(a + D_\infty) = f(a) + sgn(f^{(m)}(a)) \cdot D^\infty_\infty,
\]

where \( D^\infty_\infty := \{ h \in D_\infty | h \geq 0 \} \).

Proof. (1) This is clear from Taylor’s expansion

\[
(6.1) \quad \star f(a + x) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \ldots
\]

for any \( x \in D_\infty \). As usual, the sum in (6.1) is finite since \( x \) is nilpotent.

(2a) The inclusion \( \star f(a + D_\infty) \subseteq f(a) + D_\infty \) follows from (6.1). Since \( m \) is the smallest positive integer such that \( f^{(m)}(a) \neq 0 \), we have \( m \geq 1 \), and

\[
\star f(a + x) = f(a) + \frac{f''(a)}{m!}x^m + \frac{f'''(a)}{(m + 1)!}x^{m+1} + \ldots
\]

for any \( x \in D_\infty \). This is a finite sum since \( x \) is nilpotent. We introduce the following terminologies: given a quasi-decomposition \( z = \circ z + \sum_{i=1}^k z_it^{k_i} \in \circ \mathbb{R} \), we say that the leading term of \( z = \circ z \) is \( z_1t^{k_1} \) with degree \( k_1 \) and that the second leading term is \( z_2t^{k_2} \) with degree \( k_2 \). We will do a certain mixture of real and symbolic calculations below. More precisely, first of all, we can think of doing symbolic computations in the following algorithm using this Taylor’s expansion for all \( m \geq 1 \), and

Step 0: Let \( G_0 = w \) and let \( F_0 = f(a) \).

Step 1: Let \( G_1 = G_0 - F_0 = w - f(a) \), let \( \eta_1 = \frac{a}{m} \) and let \( F_1 = \star f(a + c_1t^{b_1}) - f(a) \), where \( c_1 \) and \( b_1 \) are chosen so that the leading terms of \( F_1 \) and \( G_1 \) are equal. This is possible since we can set \( b_1 = \frac{a}{m} \), and \( c_1 \) is the solution of the equation \( f^{(m)}(a)x^m = m!c_1 \) since \( m \) is odd by assumption. Note that the second leading term of \( F_1 \) has degree \( (m + 1)b_1 \).

Step 2: Let \( G_2 = G_1 - F_1 = w - \star f(a + c_1t^{b_1}) \), let \( \eta_2 = \text{leading}(G_2) - (m - 1)b_1 \) and let \( F_2 = \star f(a + c_1t^{b_1} + c_2t^{b_2}) - \star f(a + c_1t^{b_1}) \), where \( c_1 \) and \( b_1 \) are determined in Step 1, and \( c_2 \) and \( b_2 \) are chosen so that the leading terms of \( F_2 \) and \( G_2 \) are equal. This is possible since \( b_2 > b_1 \) from the

---

\(^6\)After having a global picture of the idea of this proof, one could instead think of a refined proof of using the decomposition of \( c \) and real computations, and one will realize how complicated it becomes.
requirement and Step 1, and the leading term of $F_2$ is $\frac{f(m)(a)}{(m-1)!}c_2t^{(m-1)b_1+b_2}$. Moreover, since the degree of the leading term of $G_2$ is either $a_2$ or the degree of the second leading term of $F_1$, we get

$$(m - 1)b_1 + b_2 = \min\{a_2, (m + 1)b_1\} \leq (m + 1)b_1,$$

i.e., $b_2 \leq 2b_1$. Hence, the second leading term of $F_2$ has degree $(m - 2)b_1 + 2b_2$.

\[ \cdots \]

Step $r$: Let $G_r = G_{r-1} - F_{r-1} = w - \frac{f(a + \sum_{i=1}^{r-1} c_it^{b_i})}{(m-1)!} \cdot r!$, let $\eta_r = \text{leading}(G_r) - (m - 1)b_1$ and let $F_r = \frac{f(a + \sum_{i=1}^{r-1} c_it^{b_i})}{(m-1)!} - \frac{f(a + \sum_{i=1}^{r-1} c_it^{b_i})}{(m-1)!}$, where $(c_1, b_1), \ldots, (c_{r-1}, b_{r-1})$ are determined in Step 1, \ldots, Step (r-1), and $c_r$ and $b_r$ are chosen so that the leading terms of $F_r$ and $G_r$ are equal. This is possible since $b_r > b_{r-1}$ from the requirement and Step (r-1), and the leading term of $F_r$ is $\frac{f(m)(a)}{(m-1)!}c_1t^{(m-1)b_1+b_r}$. Note that the second leading term of $F_r$ has degree $(m - 2)b_1 + b_2 + b_r$ since $b_2 \leq 2b_1$.

\[ \cdots \]

Now we show that this procedure terminates in finitely many steps, i.e., $G_s = o(t)$ for some $s \in \mathbb{N}$. Note that $G_r$ measures the closeness of $a + \sum_{i=1}^{r-1} c_it^{b_i}$ to the solution of the equation $\frac{f(a)}{(m-1)!}x = w$ at Step (r-1), and $F_r$ measures the new extra terms created at Step $r$. From the above analysis, we know that at Step $r$, there are only finitely many terms in $G_r$ which has degree less than the degree of the second leading term of $F_r$, and the new extra terms created in later on steps all have degree greater than the degree of the second leading term of $F_r$. This is to say that for any fixed $r$, after finitely many steps, the degree of the second leading term of $F_r$ becomes the degree of the leading term of $G_l$ for some $l > r$. So for any $r \in \mathbb{Z}^{>0}$, there exists $l > r$ such that $(m - 1)b_1 + b_l = (m - 2)b_1 + b_2 + b_r$, i.e., $b_1 = b_2 + (b_r - b_1)$. This is to say that after finitely many steps, the terms of the terms in $c$ will raise at least a fixed positive constant, although we do not know explicitly how much $b_i$ increases at each step. Therefore, the degree of some term in $c$ will be greater than 1 after finitely many steps, which implies the termination of the algorithm after finitely many steps.

(2b) The proof is similar to (2a), except in Step 1 when solving the equation $f(m)(a)x^m = m!u_1$, since $m$ is even by assumption. In this case, we can only solve this equation in $\mathbb{R}$ when $f(m)(a)$ and $u_1$ have the same sign.

\begin{remark}
\begin{enumerate}
\item As a warning, the proof of (2a) of the above theorem does not mean that in that case the restriction map $\frac{f(a)}{(m-1)!} : a + D_\infty \to \frac{f(a)}{(m-1)!} + D_\infty$ is injective. Instead, the algorithm in the proof gives the simplest solution (called the fundamental solution) to the equation $\frac{f(a)}{(m-1)!}x = w$ with real part $a$, in the sense that every solution with real part $a$ is of this form
\end{enumerate}
\end{remark}

\begin{footnotes}
\footnote{It is easy to check that if $b_1 \geq b_2$, then the degree of the leading term of $F_2$ is $mb_2$. So we have $b_1 \geq b_2 = \frac{\text{leading}(G_2)}{\text{leading}(G_1)} \cdot \frac{a_1}{m}$, contradicting that $b_1 = \frac{a_1}{m}$ in the first step.}
\footnote{It is easy to check that the degree of the leading term of $F_r$ is $mb_r$ if $b_1 \geq b_r$ or $(m - 1)b_1 + b_r$ otherwise. The conclusion then follows easily.}
\footnote{To see this, note that if exists, we can order the terms in $G_r$ by its degree which has degree strictly between the degree of the leading term of $F_r$ (which is $(m - 1)b_1 + b_r$) and the degree of the second leading term of $F_r$ (which is $(m - 2)b_1 + b_2 + b_r$), say they are $\{x_1, x_2, \ldots, x_s\}$ with $\deg(x_1) < \deg(x_2) < \cdots < \deg(x_s)$. Then $x_1$ becomes the leading term of $G_{r+1}$, and the second leading term of $F_{r+1}$ is $(m - 2)b_1 + b_2 + b_{r+1}$ which is strictly greater than the second leading term of $F_r$. One can then see that $x_1$ becomes the leading term of $G_{r+i}$, and the second leading term of $F_{r+i} = (m - 2)b_1 + b_2 + b_{r+i}$ which is strictly greater than the second leading term of $F_{r+i}$, for each $i = 1, 2, \cdots, s$.}
\end{footnotes}
possibly plus some more terms. This can be proved using (2) below. For example, by the algorithm, the equation \( x^3 = t \) has a solution \( x = t^{1/3} \). In fact, \( x = t^{1/3} + y \) is a solution of this equation as long as \( y \in D_2 \). In other words, the polynomial equation \( x^3 = t \) has uncountably many solutions in \( \bullet \mathbb{R} \).

(2) Under the assumption of (2a), we further assume that \( y \in \bullet \mathbb{R} \) is already a solution to the equation \( \bullet f(x) = w \). Here is the procedure to get all the solutions of this equation with real part \( \circ y \). First, we can refine \( y = \circ y + \sum_{i=1}^{\infty} \alpha_i t^{n_i} \) as its decomposition to get the fundamental solution: if \( m \) is the smallest positive integer such that \( f^{(m)}(\circ y) \neq 0 \), then \( \hat{y} = \circ y + \sum_i \alpha_i t^{\alpha_i} \) with the sum indexed by all \( i = 1, 2, \ldots, n \) with \( (m-1)a_1 + a_i \leq 1 \), is the fundamental solution. Then we get all the solutions: \( \hat{y} + z \) for \( z \in D_{\infty} \) such that the degree of \( z \) is greater than

\[
\begin{cases}
1 - (m-1)a_1, & \text{if } ma_1 \leq 1 \\
1/m, & \text{otherwise}.
\end{cases}
\]

We can get a similar result under the assumption of (2b), noting that there are two fundamental solutions in that case when \( w \) is not real.

(3) Not every smooth function \( \bullet \mathbb{R} \to \bullet \mathbb{R} \) has the intermediate value property. For example, let \( f : \mathbb{R} \to \mathbb{R} \) be defined by

\[
f(x) = \begin{cases} e^{-1/x}, & \text{if } x > 0 \\
0, & \text{else}.
\end{cases}
\]

Then \( f \) is smooth, and \( \bullet (0, \infty) \cup \{0\} = \text{Im}(\bullet f) \).

(4) As a refinement of (2b) of the above theorem, we have

\[
\bullet f(a + D_{\infty}^{>0}) = f(a) + \text{sgn}(f^{(m)}(a)) \cdot D_{\infty}^{>0} = \bullet f(a + D_{\infty}^{>0})
\]

under the same assumption.

So we can determine the images of the Fermat extension of elementary functions:

**Example 35.**

1. Let \( n \in \mathbb{Z}^{>0} \) and let \( f : \mathbb{R} \to \mathbb{R} \) be the function \( x \mapsto x^n \). Then

\[
\text{Im}(\bullet f) = \begin{cases} \bullet \mathbb{R}, & \text{if } n \text{ is odd} \\
\bullet \mathbb{R}^{>0}, & \text{if } n \text{ is even}.
\end{cases}
\]

2. Let \( f : \mathbb{R} \to \mathbb{R} \) be \( x \mapsto a^x \) with \( 0 < a < 1 \) or \( a > 1 \). Then \( \bullet (0, \infty) = \text{Im}(\bullet f) \).

3. Let \( f : (0, \infty) \to \mathbb{R} \) be \( x \mapsto \log_a x \) with \( 0 < a < 1 \) or \( a > 1 \). Then \( \text{Im}(\bullet f) = \bullet \mathbb{R} \).

4. Let \( f : \mathbb{R} \to \mathbb{R} \) be either \( x \mapsto \sin x \) or \( x \mapsto \cos x \). Then \( \text{Im}(\bullet f) = [-1, 1] \).

5. Let \( f : \mathbb{R} \to \mathbb{R} \) be either \( x \mapsto \tan x \) or \( x \mapsto \cot x \). Then \( \text{Im}(\bullet f) = \bullet \mathbb{R} \).

6.2. **Monotonicity.** Here are some important observations, with the last one another important ingredient for solving the intermediate value property for Fermat extension of ordinary smooth functions:

1. There is no smooth function \( f : \mathbb{R} \to \mathbb{R} \) such that there exists a point \( a \in \mathbb{R} \) with the properties that for every solution \( b \) of the equation \( f(x) = a \), the smallest \( m \in \mathbb{Z}^{>0} \) such that \( f^{(m)}(b) \neq 0 \) exists and is even, say it is \( m_b \), and all these \( f^{(m)}(b) \)'s are not of the same sign.

To prove this, one observes that \( f(x) = a \) can only have finitely many solutions on any
closed interval by the evenness assumption. Clearly it is impossible to connect the image of \( f \) if \( f(x) = a \) has two consecutive solutions \( b \) and \( c \), say \( b < c \), (i.e., there is no solution in the interval \( (b, c) \)), with \( f^{(m)}(b) \cdot f^{(m)}(c) < 0 \) and \( m_b, m_c \) both even.

(2) (Boundary) Let \( f : \mathbb{R} \to \mathbb{R} \) be a smooth function. If \( b \in \text{Im}(f) \) is minimum, then \( \text{Im}(f) \cap (b + D_{\infty}) \) is either \( \{b\} \) or \( b + D_{\infty} \). Dually, if \( c \in \text{Im}(f) \) is maximum, then \( \text{Im}(f) \cap (c + D_{\infty}) \) is either \( \{c\} \) or \( c + D_{\infty} \).

These follow easily from Taylor’s expansion of \( f \). This leads to the monotonicity discussed below.

(3) (Global monotonicity) Let \( U \) be an open subset of \( \mathbb{R} \), and let \( f : U \to \mathbb{R} \) be a smooth function. If \( f'(x) > 0 \) for all \( x \in U \), then \( f^* : U \to \mathbb{R} \) is strictly increasing, i.e., if \( x, y \in U \) with \( x < y \), then \( f(x) < f(y) \). More generally, if \( f \) has the property that for any \( u \in U \), there exists some \( m = m(u) \in \mathbb{Z}^{>0} \) such that \( f^{(m)}(u) \neq 0 \), \( m_u := \) the smallest such \( m \) is odd, and \( f^{(m_u)}(u) > 0 \), then \( f \) is increasing, i.e., for any \( x, y \in U \) with \( x < y \), we have \( f(x) \leq f(y) \).

(A typical such example is \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x^3 \).) Dually, if \( f'(x) < 0 \) for all \( x \in U \), then \( f^* : U \to \mathbb{R} \) is strictly decreasing, i.e., if \( x, y \in U \) with \( x < y \), then \( f(x) > f(y) \). More generally, if \( f \) has the property that for any \( u \in U \), there exists some \( m = m(u) \in \mathbb{Z}^{>0} \) such that \( f^{(m)}(u) \neq 0 \), \( m_u := \) the smallest such \( m \) is odd and \( f^{(m_u)}(u) < 0 \), then \( f \) is decreasing, i.e., for any \( x, y \in U \) with \( x < y \), we have \( f(x) \geq f(y) \).

Let us sketch the proof of the statements in the first paragraph, since the others can be proved similarly. It is easy to show that under these assumptions, the function \( f \) is increasing, i.e., if \( u, u' \in U \) with \( u < u' \), then \( f(u) < f(u') \). So we only need to prove the slice version of the statements, which will be in the next observation.

(4) (Slice monotonicity) Let \( U \) be an open subset of \( \mathbb{R} \), let \( a \in U \) be a fixed point, and let \( f : U \to \mathbb{R} \) be a smooth function. Assume that there exists \( n \in \mathbb{Z}^{>0} \) such that \( f^{(n)}(a) \neq 0 \). Let \( m \) be the smallest such \( n \).

Assume that \( m \) is odd. If \( f^{(m)}(a) > 0 \) (resp. \( f^{(m)}(a) < 0 \)), then \( f^* \mid_{a + D_{\infty}} : a + D_{\infty} \to f(a) + D_{\infty} \) is increasing (resp. decreasing).

Assume that \( m \) is even. If \( f^{(m)}(a) > 0 \) (resp. \( f^{(m)}(a) < 0 \)), then \( f^* \mid_{a + D_{\infty}} : a + D_{\infty} \to f(a) + \text{sgn}(f^{(m)}(a)) \cdot D_{\infty} \) is increasing (resp. decreasing), and \( f^* \mid_{a + D_{\infty}} : a + D_{\infty} \to f(a) + \text{sgn}(f^{(m)}(a)) \cdot D_{\infty} \) is decreasing (resp. increasing).

We prove the case when \( m \) is odd and \( f^{(m)}(a) > 0 \), since the others can be proved similarly. For any \( x_1, x_2 \in a + D_{\infty} \) with \( x_1 < x_2 \), write \( x_1 = a + \sum_{i=1}^{m} \alpha_i t_i \) and \( x_2 = a + \sum_{i=1}^{m} \beta_i t_i \) for their quasi-decompositions with at least one of \( \alpha_1 \) and \( \beta_1 \) non-zero, and assume that \( k \in \{1, 2, \ldots, n\} \) is the smallest integer such that \( \alpha_k < \beta_k \). Let \( x = a + \sum_{i=1}^{k-1} \alpha_i t_i \). Note that Taylor’s expansions of both \( f(x_1) \) and \( f(x_2) \) have \( f(x) \) in common. And the leading terms of \( f(x_1) - f(x) \) and \( f(x_2) - f(x) \) are

\[
\begin{align*}
\left( \sum_{i=1}^{(m-1)/2} \alpha_i t_i \right) & \alpha_k t^{(m-1)} a_1 a_k + \sum_{i=(m-1)/2+1}^{(m-1)/2} \alpha_i t^{(m-1)} a_1 a_k, \\
\left( \sum_{i=1}^{(m-1)/2} \alpha_i t_i \right) & \alpha_k t^{(m-1)} a_1 a_k + \sum_{i=(m-1)/2+1}^{(m-1)/2} \beta_i t^{(m-1)} a_1 a_k,
\end{align*}
\]

if \( k > 1 \),

\[
\begin{align*}
\sum_{i=1}^{(m-1)/2} \alpha_i t_i & \alpha_k t^{(m-1)} a_1 a_k + \sum_{i=(m-1)/2+1}^{(m-1)/2} \alpha_i t^{(m-1)} a_1 a_k, \\
\sum_{i=1}^{(m-1)/2} \alpha_i t_i & \alpha_k t^{(m-1)} a_1 a_k + \sum_{i=(m-1)/2+1}^{(m-1)/2} \beta_i t^{(m-1)} a_1 a_k,
\end{align*}
\]

if \( k = 1 \),

respectively. Since \( m \) is odd and \( \alpha_k < \beta_k \), in both cases we have \( f(x_1) \leq f(x_2) \), or more precisely, \( f(x_1) < f(x_2) \) if the degree of that leading term in the above expression is less than or equal to 1, and \( f(x_1) = f(x_2) \) if the degree is greater than 1.

Here is an application to transferring monotonicity:
Proposition 36. Let \( f : U \to \mathbb{R} \) be a smooth map from an open subset of \( \mathbb{R} \). Then \( f \) is increasing (resp. decreasing) if and only if \( \overset{\bullet}{f} \) is increasing (resp. decreasing).

Proof. The “if” part is clear. For the “only if” part, we are left to deal with monotonicity within \( f^{-1}(a) \) for every fixed \( a \in \mathbb{R} \). Note that if \( f^{-1}(a) \neq \emptyset \), since \( f \) is monotone, \( f^{-1}(a) \) is the intersection of a closed interval (possibly a single point) in \( \mathbb{R} \) with \( U \). Now we have two cases: (1) if the interior of \( f^{-1}(a) \) is non-empty, then every point in \( f^{-1}(a) \) is a flat point of \( f \); (2) if \( f^{-1}(a) \) contains a single point, denoted by \( b \), then either \( b \) is a flat point of \( f \), or there exists some \( m \in \mathbb{Z}^{>0} \) such that \( f^{(m)}(b) \neq 0 \) and the smallest such \( m \) is odd. So one can apply Observation (3), the global monotonicity, to conclude the result. \( \square \)

Here are some more complicated examples, which are complimentary to Observation (1) above:

Example 37.

(1) Let \( f : \mathbb{R} \to \mathbb{R} \) be

\[
 f(x) = \begin{cases} 
 x(x-1)^2e^{-1/x^2}, & \text{if } x \neq 0 \\
 0, & \text{else.}
 \end{cases}
\]

Then \( f \) is smooth and surjective, \( f(x) = 0 \) has two solutions \( x = 0 \) and \( x = 1 \), \( f^{(m)}(0) = 0 \) for all \( n \in \mathbb{N} \), the smallest \( m \in \mathbb{N} \) such that \( f^{(m)}(1) \neq 0 \) is 2, and \( f''(1) > 0 \). Therefore, 0 is in the interior of \( \text{Im}(f) \), and \( \text{Im}(\overset{\bullet}{f}) \cap D_\infty = D_{\geq 0} \). This example shows that if the ordinary smooth function has a flat point, then it is possible that the image of its Fermat extension has “holes”, but not always (see the following one).

(2) Let \( f : \mathbb{R} \to \mathbb{R} \) be

\[
 f(x) = \begin{cases} 
 x(x-1)^2(x+1)^2e^{-1/x^2}, & \text{if } x \neq 0 \\
 0, & \text{else.}
 \end{cases}
\]

Then \( f \) is smooth and surjective, \( f(x) = 0 \) has three solutions \( x = 0 \), \( x = 1 \) and \( x = -1 \), \( f^{(n)}(0) = 0 \) for all \( n \in \mathbb{N} \), the smallest \( m \in \mathbb{N} \) such that \( f^{(m)}(\pm 1) = 0 \) are both 2, \( f''(1) > 0 \) and \( f''(-1) < 0 \). Therefore, 0 is in the interior of \( \text{Im}(f) \), and \( \text{Im}(\overset{\bullet}{f}) \cap D_\infty = D_\infty \).

6.3. Fermat extension of ordinary smooth functions. Now we can prove the intermediate value property for certain Fermat extension of ordinary smooth functions:

Corollary 38. Let \( U \) be an open connected subset of \( \mathbb{R} \), and let \( f : U \to \mathbb{R} \) be a smooth function without any flat point, i.e., for any \( u \in U \), there exists \( m = m(u) \in \mathbb{Z}^{>0} \) such that \( f^{(m)}(u) \neq 0 \). Then \( \overset{\bullet}{f} \) has the intermediate value property.

Proof. Let \( a, b \in \overset{\bullet}{U} \) with \( a < b \). If \( \overset{\bullet}{f}(a) = \overset{\bullet}{f}(b) \), then we are done. Without loss of generality, we may assume that \( \overset{\bullet}{f}(a) < \overset{\bullet}{f}(b) \). For any \( x \in \mathbb{R} \) with \( \overset{\bullet}{f}(a) < x < \overset{\bullet}{f}(b) \), we need to find \( c \in \overset{\bullet}{\mathbb{R}} \) with \( a < c < b \) (since \( U \) is connected) such that \( \overset{\bullet}{f}(c) = x \). We prove this in several cases below.

Case 1: Assume \( f(\overset{\circ}{a}) < \overset{\circ}{x} < f(\overset{\circ}{b}) \). Then by the classical intermediate value theorem for \( f \), the set \( f^{-1}(\overset{\circ}{x}) \cap (\overset{\circ}{a}, \overset{\circ}{b})_R \) is non-empty. Now if there exists some \( c_0 \) in this set with the property that the smallest positive integer \( m \) such that \( f^{(m)}(c_0) \neq 0 \) is odd, or if there exist \( c_1, c_2 \) in this set with the property that the smallest positive integer \( m_i \) such that \( f^{(m_i)}(c_i) \neq 0 \) are both even and \( f^{(m_1)}(c_1)f^{(m_2)}(c_2) < 0 \), then we are done by Theorem 33. So we may assume that for every point \( c_0 \) in this set, the smallest positive integer \( m(c_0) \) such that \( f^{(m(c_0))}(c_0) \neq 0 \) is even, and all these \( f^{(m(c_0))}(c_0) \)'s are of the same sign. Without loss of generality, we may assume that they are all positive. This implies that \( \overset{\circ}{x} \) has to be the minimum of the smooth function
\[ f|_{(c_a,c_b)} : (c_a,c_b) \to \mathbb{R}, \] contradicting the assumption that \( f(c_a) < c x < f(c_b) \) at the beginning of this case. (Actually in this case, one can conclude together with Observation (1) that there always exists \( c_0 \in f^{-1}(c_a) \cap (c_a,c_b) \) with the property that the smallest positive integer \( m \) such that \( f^{(m)}(c_0) \neq 0 \) is odd.)

Case 2: Assume that \( f(c_a) < f(c_b) \) and \( c x \) is equal to one of them. Without loss of generality, we may assume \( f(c_a) = c x < f(c_b) \). If \( m \) is the smallest positive integer such that \( f^{(m)}(c_a) \neq 0 \), and \( f^{(m)}(c_a) > 0 \), then we are done by Theorem 33 together with slice monotonicity (Observation (4)). If \( f^{(m)}(c_a) < 0 \), then \( f \) is decreasing on \((c_a,c_a + \delta) \) for some \( \delta \in \mathbb{R}^{>0} \), and the claim then follows from Case 1.

Case 3: Assume that \( f(c_a) = c x = f(c_b) \). We may further assume that \( f^{(m)}(c_a) < 0 \), where \( m \) is the smallest positive integer such that \( f^{(m)}(c_a) \neq 0 \), and \( f^{(n)}(c_b) > 0 \) if \( n \) is even or \( f^{(n)}(c_b) < 0 \) if \( n \) is odd, where \( n \) is the smallest positive integer such that \( f^{(n)}(c_b) \neq 0 \), since otherwise the claim is true by slice monotonicity and Theorem 33. Under these assumptions, \( f \) is decreasing near both \( a \) and \( b \), and the claim then follows from Case 1.

However, the converse of the above corollary is not true:

**Example 39.** It is not true that if a smooth function \( f : \mathbb{R} \to \mathbb{R} \) has a flat point, then \( \cdot f \) does not have the intermediate value property. For example, let \( f \) be defined by

\[
f(x) = \begin{cases} 
  e^{-1/x^2} \cos(1/x), & \text{if } x \neq 0 \\
  0, & \text{else.}
\end{cases}
\]

Then one can check that (1) \( f \) is a smooth function; (2) \( x = 0 \) is the only flat point of \( f \). The “only” part of the second statement follows from the fact that the system of equations

\[
\begin{cases}
  f'(x) = 0 \\
  f''(x) = 0
\end{cases}
\]

has only one solution: \( x = 0 \). Note that \( f^{-1}(0) = \{0, x_k\} \subseteq \mathbb{Z} \) with \( x_k = \frac{1}{\pi + \pi k} \), and \( f'(x_k) \neq 0 \) for all \( k \). Together with Corollary 38 and the evenness of \( f \), one can show that \( f \) has the intermediate value property.

Here is an application to extrema problems:

**Proposition 40.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a smooth map, and let \( a, b \in \mathbb{R}^\bullet \) with \( a < b \). Then \( \cdot f|_{[a,b]} \) always has maximum and minimum. Moreover, if \( f \) has no flat point, then there exist \( c, d \in \mathbb{R}^\bullet \) with \( c < d \) such that \( \cdot f([a,b]) = [c,d] \).

**Proof.** The first statement follows easily from the extreme value property of \( f|_{[a,x]} \) together with Corollary (2) (boundary) and Observation (4) (slice monotonicity), and the second statement then follows from Corollary 38, the intermediate value property for \( \cdot f \).

6.4. **Quasi-standard smooth functions.** Now we turn to the intermediate value property problem for a general quasi-standard smooth function \( g : \mathbb{R}^\bullet \to \mathbb{R}^\bullet \), where \( U \) is an open connected subset of \( \mathbb{R}^\bullet \). For \( a, b \in \mathbb{R}^\bullet \) with \( a < b \), if there exist \( c_1, \ldots, c_n \in \mathbb{R}^\bullet \) with \( a < c_1 < \ldots < c_n < b \) for some \( n \in \mathbb{N} \) such that \( g \) has the intermediate value property on each of \([a,c_1],[c_1,c_2],\ldots,[c_n,b]\), then it is easy to see that \( g \) also has the intermediate value property on \([a,b]\). In other words, if each local expression of \( g \) has the intermediate value property, then so does \( g \). So we are led to study when the function \( \cdot h(v,-) : \mathbb{R}^\bullet \to \mathbb{R}^\bullet \) has the intermediate value property, where \( h : V \times U \to \mathbb{R} \) is a smooth function with \( V \) an open subset of some Euclidean space, and \( v \in \mathbb{V} \) is a fixed point.
It is slightly more complicated than the Fermat extension \( \bullet f : \bullet U \to \bullet \mathbb{R} \) we have discussed in the previous subsection, because of the parameter \( v \). For example, let \( g : \bullet \mathbb{R} \to \bullet \mathbb{R} \) be defined by \( g(y) = y^3 + xy^2 \), where \( h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a smooth function defined by \( h(x, y) = y^3 + xy^2 \). Then \( g|_{D_{\infty}} : D_{\infty} \to D_{\infty} \) is not surjective since the equation \( g(y) = -y^{31/100} \) has no solution. But \( g(0) = 0 \) and \( g(-1) = -1 + t^{1/100} \), which implies that \( g \) does not have the intermediate value property over \( \bullet \mathbb{R} \). The main problem here is that the highest degree term is \( y^3 \) or \( xy^2 \) will have the leading term for variant \( y \). After excluding such functions, we can prove the following general criteria:

**Proposition 41.** Let \( U \) be an open connected subset of \( \mathbb{R} \), and let \( g : \bullet U \to \bullet \mathbb{R} \) be a quasi-standard smooth function defined by \( g(x) = \bullet h(v, x) \), where \( h : V \times U \to \mathbb{R} \) is a smooth function, \( V \) is an open subset of some Euclidean space, and \( v = \delta v \in \bullet V \) is a fixed point with \( \delta v \neq 0 \). Assume that for each \( x \in \bullet U \) there exists \( m = m(x) \in \mathbb{Z}^>0 \) such that \( D^{(0, m)} h(v, x) \neq 0 \). Write \( m_x(=m_x) \) for the smallest such \( m \). If \( D^a h(v, x) = 0 \) for any multi-index \( a = (a_v, a_x) \) with \( 0 < a_x < m_x \) for every \( x \in \bullet U \), then \( g \) has the intermediate value property.

Note that Corollary 38 applies to the case when \( \delta v = 0 \).

**Proof.** The proof is very similar to that of Corollary 38, so we only sketch here.

The assumptions in the statement imply that if we omit all constant terms in \( g(x) = \bullet h(v, x) \), i.e., if we consider \( g(x) - \bullet g(x) \), then the term \( \frac{D^{(0, m)} h(v, x)}{m_x!} x^{m_x} \) will always provide the leading term when varying \( x \).

Step 1: One can prove the slice image theorem in this case: Under these assumptions, (1) if \( m_x \) is odd, then \( g|_{x \in D_{\infty}} : D_{\infty} \to g(v, x) + D_{\infty} \) is surjective; (2) if \( m_x \) is even, then \( g|_{x \in D_{\infty}} : D_{\infty} \to g(v, x) + \text{sgn}(D^{(0, m_x)} h(v, x)) \cdot D_{\infty} \) is surjective, or in the refined version,

\[
g(v, x) + D_{\infty} = g(v, x) + \text{sgn}(D^{(0, m_x)} h(v, x)) \cdot D_{\infty} = g(v, x) + D_{\infty}.
\]

Step 2: One can prove the slice monotonicity in this case: Under these assumptions, (1) if \( m_x \) is odd and \( D^{(0, m_x)} h(v, x) > 0 \) (resp. \( D^{(0, m_x)} h(v, x) < 0 \)), then \( g|_{x \in D_{\infty}} \) is increasing (resp. decreasing); (2) if \( m_x \) is even and \( D^{(0, m_x)} h(v, x) > 0 \) (resp. \( D^{(0, m_x)} h(v, x) < 0 \)), then \( g|_{x \in D_{\infty}} \) is decreasing (resp. increasing). Step 3: We can split into three cases depending on the order of the real parts, and prove the intermediate value property as the proof of Corollary 38.

Here is a way to apply this proposition:

**Example 42.** Let \( h : \mathbb{R}^l \times \mathbb{R} \to \mathbb{R} \) be a smooth function with variables \((x, y) \in \mathbb{R}^l \times \mathbb{R} \), and let \( a \in \mathbb{R}^l \) be a fixed point. Assume that \( h \) is a polynomial in \( y \) together with the property that there exists some \( m \in \mathbb{Z}^>0 \) such that \( \frac{\partial^m h}{\partial y^m}(a, 0) \neq 0 \). Then there exist finitely many connected Fermat open subsets \( A_i \) of \( \bullet \mathbb{R} \) such that on each \( A_i \) the smooth function \( \bullet h(v, -) \), for any fixed \( v \in a + D_{\infty}^l \), has the intermediate value property.

For example, let \( h(x, y) = y^3 + xy^2 \) and let \( v = t^{1/100} \). We know from the paragraph above Proposition 41 that \( \bullet h(v, -) \) does not have the intermediate value property over \( \bullet \mathbb{R} \). Here is the procedure to find all connected Fermat open subsets of \( \bullet \mathbb{R} \) on each of which \( \bullet h(v, -) \) has the
intermediate value property. Note that Taylor’s expansion of $h$ at $(0, c)$ is given by

$$h(v, c + y) = (c + y)^3 + (c + y)^2v$$

$$= c^3 + c^2v + 3c^2y + 2cyv + 3cy^2 + y^2v + y^3.$$

According to Proposition 41, $\bullet h(v, -)$ has the intermediate value property on each connected Fermat open subset as long as the subset does not contain $c$ with $3c^2 = 0$, i.e., $c = 0$. So we get such connected Fermat open subsets: $\bullet (-\infty, 0)$ and $\bullet (0, \infty)$.

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