On principal fibrations associated with one algebra

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To the memory of B.N. Shapukov

Abstract In this paper we study two types of fibrations associated with a 3-dimensional unital associative irreducible algebra and their basic properties. We investigate trivial principal fibrations of degenerate semi-Euclidean sphere and their semi-conformal and projective models. We use Norden normalization method for constructing second model.

1. Principal fibrations of \( G \) by its subgroups

Let us denote by \( \mathfrak{A} \) a unital associative \( n \)-dimensional algebra with multiplication \( xy \), by \( G \subset \mathfrak{A} \) the set of invertible elements. It is well known that \( G \) is a Lie group with the same multiplication rule. Let \( \mathfrak{B} \) be a unital subalgebra of algebra \( \mathfrak{A} \) and \( H \subset G \) the set of invertible elements. So \( H \) is a Lie subgroup of group \( G \). We consider the factor-space \( G/H \) of right cosets. A fibration \( (G, \pi, M = G/H) \) is a principal fibration with the structure group \( H \), where \( \pi \) is a canonical projection [II]. Here we have a fibration:

\[ H \rightarrow G \rightarrow G/H. \]

We consider all 3-dimensional unital associative irreducible algebras up to isomorphism. It is known that there exist only tree types of such algebras [2], [3]. In special choice of basic units they have the following multiplication rules:

\[
\begin{array}{c|ccc}
I) & 1 & e_1 & e_2 \\
1 & e_1 & e_2 & 0 \\
e_2 & 0 & 0 & 0 \\
\end{array}
\quad \begin{array}{c|ccc}
II) & 1 & e_1 & e_2 \\
1 & e_1 & 1 & e_2 \\
e_2 & e_2 & e_2 & 0 \\
\end{array}
\quad \begin{array}{c|ccc}
III) & 1 & e_1 & e_2 \\
1 & e_1 & 0 & 0 \\
e_2 & e_2 & 0 & 0 \\
\end{array}
\]
For these algebras N. Belova found all subalgebras with unit and principal fibrations \[4\]. Only in algebras of type II and III exist the associated conjugation
\[ x = x_0 + x^i e_i \rightarrow \overline{x} = x_0 - x^i e_i \] which has the property \( \overline{xy} = y \overline{x} \).

We consider the bilinear form
\[ (x, y) = \frac{1}{2} (x\overline{y} + y\overline{x}). \] (1)

For algebras of type II and III this form takes the real values and it determines a degenerate scalar product on them:
\[ \text{II})(x, y) = x_0 y_0 - x_1 y_1, \quad \text{III})(x, y) = x_0 y_0. \]

So these algebras have a structure of semi-Euclidean vector spaces with rank 2 and 1, respectively. In our discussion we concentrate on the algebra of type II, because it is less degenerate than type III.

This is non-Abelian algebra. Multiplication rule for algebra elements is:
\[ xy = (x_0 + x_1 e_1 + x_2 e_2)(y_0 + y_1 e_1 + y_2 e_2) = \]
\[ x_0 y_0 + x_1 y_1 + (x_0 y_1 + x_1 y_0) e_1 + (x_2(y_0 - y_1) + (x_0 + x_1)y_2) e_2. \] (2)

An inverse element of \( x \) is:
\[ x^{-1} = \frac{x_0 - x_1 e_1 - x_2 e_2}{(x_0)^2 - (x_1)^2}. \] (3)

The set of invertible elements \( G = \{ x \in A \mid (x_0)^2 - (x_1)^2 \neq 0 \} \) is a non-Abelian Lie group. Its underlying manifold is \( \mathbb{R}^3 \) without two transversal 2-planes, hence it consists from 4 connected components.

**Proposition** (N. Belova) *Any 2-dimensional subalgebra of algebra of type II with an algebra unit is isomorphic to double or dual numbers algebras.*

We consider a subalgebra \( R(e_1) \) with basis \( \{1, e_1\} \), it is an algebra of double numbers, and a subalgebra \( R(e_2) \) with basis \( \{1, e_2\} \), it is an algebra of dual numbers. The set of their invertible elements \( H_1 = \{ x_0 + x_1 e_1 \in R(e_1) \mid x_0^2 - x_1^2 \neq 0 \} \) and \( H_2 = \{ x_0 + x_2 e_2 \in R(e_2) \mid x_0 \neq 0 \} \) are Lie subgroups of the Lie group \( G \). First we take the space of right cosets by \( R(e_1) \) subalgebra and fibration by it. Then we have the following proposition:

**Proposition** (N. Belova) *Fibration \( (G, \pi, M = G/H_1) \) determined by formula
\[ \pi(x) = \frac{x_2}{x_0 - x_1} \] (4) is a trivial principal fibration over the real line \( \mathbb{R} \). The typical fiber, it is a plane without two transversal lines. The structure group is \( H_1 \).*

Therefore, the manifold of the group \( G \) is diffeomorphic to direct sum \( \mathbb{R} \times H_1 \). The equation of fibers is:
\[ u(x_0 - x_1) - x_2 = 0, \quad u \in \mathbb{R}. \] (5)
This is 1-parametric family of planes with common axis: \( x_0 - x_1 = 0, x_2 = 0 \). Of course it is necessary to remove the intersection points of these planes with planes: \( x_0 \pm x_1 = 0 \).

We consider the left multiplications \( x' = ax \) on the group \( G \). They form the 3-parametric Lie group of linear transformations:

\[
L(a) = \begin{pmatrix} a_0 & a_1 & 0 \\ a_1 & a_0 & 0 \\ a_2 & -a_2 & a_0 + a_1 \end{pmatrix},
\]

where determinant \( \det L(a) = (a_0^2 - a_1^2)(a_0 + a_1) \neq 0 \). The multiplications preserve fibers if and only if when \( a \in H_1 \). This group consists of four connected components.

The right multiplications \( x' = xb \) on the group \( G \) forms the 3-parametric Lie group of linear transformations:

\[
R(b) = \begin{pmatrix} b_0 & b_1 & 0 \\ b_1 & b_0 & 0 \\ b_2 & b_2 & b_0 - b_1 \end{pmatrix}.
\]

They preserve the fibration and induce the 2-parametric group of affine transformations on the base:

\( u' = \alpha u + \beta \),

here \( \alpha = \frac{b_0 - b_1}{b_0 + b_1} \neq 0, \beta = \pi(b) \).

The right multiplications include involutions of hyperbolic type with matrices:

\[
R(b_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ b_2 & b_2 & -1 \end{pmatrix}, \quad R^2 = \text{Id}.
\]

They induce the 1-parametric family involutions on the base:

\( u' = -u + b_2 \).

Let us consider rotations and anti-rotations of this semi-Euclidean space. Multiplications \( x' = ax \) and \( x' = xa \), where \( |a|^2 = \pm 1 \), are rotations and anti-rotations because of the relations:

\[
|x'|^2 = |a|^2|x|^2 = \pm |x|^2.
\]

Each element \( a \in \mathfrak{A}, |a|^2 = \pm 1 \), can be represented as:

\[
a = \cosh \varphi + a_0 \sinh \varphi \quad \text{or} \quad a = \sinh \varphi + a_0 \cosh \varphi,
\]

where \( a_0 = -\bar{a}_0, |a_0|^2 = -1 \). The bilinear form (1) in the algebra \( \mathfrak{A} \) takes real values, therefore it is possible to present it as: \( (x, y) = \frac{1}{2}(x \bar{y} + y \bar{x}) = \frac{1}{2}(\bar{x}y + \bar{y}x) \).

Consequently, the hyperbolic cosines or sines of angle between \( x \) and \( x' \) is equal:

\[
\frac{(x, ax)}{|x||ax|} = \frac{1}{2}(x \bar{\alpha} + ax \bar{\alpha}) = \frac{1}{2}(x \bar{x} + ax \bar{x}) = \frac{1}{2}(a + \bar{a}),
\]
\[
\frac{(x, xa)}{|x||xa|} = \frac{1}{2} \frac{(x x a + \overline{x} a x)}{|x|^2} = \frac{1}{2} \frac{(x x a + \overline{a} x x)}{|x|^2} = \frac{1}{2} (a + \overline{a}).
\]

In the last two equations we get \(\cosh \varphi\) if \(|a|^2 = 1\) and we get \(\sinh \varphi\) if \(|a|^2 = -1\). It means that this angle does not depend on \(x\).

Transformations

\[x' = axb,\]

where \(|a|^2 = \pm 1, |b|^2 = \pm 1\), are compositions of rotations or anti-rotations \(x' = ax\) and \(x' = xb\). So, there are proper rotations or anti-rotations.

Transformations

\[x' = a\overline{a}b\]

are compositions of (8) and reflection \(x' = \overline{x}\). So, there are improper rotations or anti-rotations.

**Lemma** Any proper or improper rotation and anti-rotation of this semi-Euclidean space can be represented by (8) or (9).

**Proof** These rotations and anti-rotations are compositions of odd and even numbers of reflections of planes passing through the origin. To each plane corresponds its orthonormal vector \(n\). If vectors \(x_1\) and \(n\) are collinear, then \(x_1 n = \overline{n} x_1 = -x_1\). If vectors \(x_2\) and \(n\) are orthogonal, then \(\overline{x}_2 n + \overline{n} x_2 = 0\) and \(x'_2 = -\overline{n} x_2 n = \overline{n} x_2 = x_2\). On the other hand any vector \(x\) can be represented as a sum of vectors \(x_1\) and \(x_2\). It means, that a reflection of plane is: \(x' = -n\overline{x}\). So, composition of odd and even numbers of planes reflections are transformation (8) or (9). □

Let us introduce adapted coordinates \((u, \lambda, \varphi)\) to fibration in semi-Euclidean space, here \(u\) is a basic coordinate, \(\lambda, \varphi\) are fiber coordinates. If \(|x|^2 > 0\), we denote \(\lambda = \pm \sqrt{x_0^2 - x_1^2} \neq 0\), the sign of \(\lambda\) is equal to the sign of \(x_0\). The adapted coordinates of fibration in this case are:

\[x_0 = \lambda \cosh \varphi, \quad x_1 = \lambda \sinh \varphi, \quad x_2 = u \lambda \exp \varphi,\]

where \(\lambda \in \mathbb{R}_0, \quad u, \varphi \in \mathbb{R}\).

If \(|x|^2 < 0\), then we write \(\lambda = \pm \sqrt{x_1^2 - x_0^2}\), the sign of \(\lambda\) is equal to the sign of \(x_1\):

\[x_0 = \lambda \sinh \varphi, \quad x_1 = \lambda \cosh \varphi, \quad x_2 = u \lambda \exp \varphi.\]

The structure group acts as follows:

\[u' = u, \quad \lambda' = \lambda \rho, \quad \varphi' = \varphi + \psi,\]

where the element \(a(0, \rho, \psi)\) of structure group acts on the element \(x(u, \lambda, \varphi) \in G\). This group consists of 4 connected components.

2. **The Principal Subfibration of the Fibration** \((G, \pi, M = G/H_1)\)

As we already said, the scalar product in the algebra \(\mathfrak{A}\) of type II is: \((x, y) = x_0 y_0 - x_1 y_1\). So, the algebra \(\mathfrak{A}\) is a 3-dimensional semi-Euclidean space with
rank 2. We call semi-Euclidean sphere with unit radius the set of all elements of algebra \( \mathfrak{A} \) whose square is equal to one,

\[ S^2(1) = \{ x \in \mathfrak{A} \mid x_0^2 - x_1^2 = 1 \}. \]

It looks like a hyperbolic cylinder. The set of elements with imaginary unit module \( |x|^2 = -1 \) we call semi-Euclidean sphere with imaginary unit radius \( S^2(-1) \). One of these spheres can be obtained up from another one by rotation.

We consider the subfibration of the fibration \( (G, \pi, M = G/H_1) \) to semi-Euclidean sphere \( S^2(1) \), i.e. the fibration \( \pi : S^2(1) \to M \). The fibers of new fibration are intersections of \( S^2(1) \) and planes \([13]\). The restriction of the group of double numbers \( H_1 \) to \( S^2(1) \) is a Lie subgroup \( S_1 \) of double numbers with unit module

\[ S_1 = \{ a_0 + a_1 e_1 \in H_1 \mid a_0^2 - a_1^2 = 1 \}. \]

This group consists of two connected components.

**Proposition** The fibration \( (S^2(1), \pi, M) \) is principal fibration of the group \( S^2(1) \) by the Lie subgroup \( S_1 \) to right cosets.

**Proof.** Let \( x \) and \( y \) be two sphere points from one fiber. They are also from the same fiber of the principal fibration \( (G, \pi, M) \). So, there exists a unique element \( a \in H_1 \) such that \( y = ax \). Then \( |y|^2 = |a|^2 |x|^2 \) and hence \( |a|^2 = 1 \). It means that \( a \in S_1 \). \( \Box \)

We define coordinates adapted to the fibration on semi-Euclidean sphere \( S^2(1) \). If \( x \in S^2(1) \) then by \([10]\) we get \( \lambda = \varepsilon, \varepsilon = \pm 1 \). The parametric equation of semi-Euclidean sphere in the adapted coordinates \((u, \varphi)\) is:

\[ r(u, \varphi) = \varepsilon (\cosh \varphi, \sinh \varphi, u \exp \varphi), \quad (13) \]

where \( u \) is a basis coordinate, \( \varphi \) is a fiber coordinate. Different values of \( \varepsilon \) correspond to different connected components of semi-Euclidean sphere \( S^2(1) \).

Let us define the action of the structure group \( S_1 \) on semi-Euclidean sphere. By \([12]\) and the adapted coordinates of elements \( a(0, \varepsilon_1, \psi), x(u, \varepsilon, \varphi) \in S^2(1) \) we get:

\[ u' = u, \quad \varepsilon' = \varepsilon \varepsilon_1, \quad \varphi' = \varphi + \psi. \]

This group also consists of two connected components.

The metric tensor for semi-Euclidean sphere has the matrix representation:

\[ (g_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{rank}(g_{ij}) = 1 \]

So the linear element of metric is:

\[ ds_1^2 = -d\varphi^2. \quad (14) \]
3. Semi-conformal model of the sphere fibration \((S^2(1), \pi, \mathbb{R})\)

We consider the semi-conformal model of the fibration \((S^2(1), \pi, \mathbb{R})\). We project stereographically the sphere \(S^2(1)\) from the point \(N(1,0,0) \in S^2(1)\) (pole) to the equatorial plane \(\mathbb{R}^2\) with equation \(x_0 = 0\). We write the stereographic map in the coordinate form. For this we find the equation of line which pass through the pole and an arbitrary point of the sphere \(X(x_0, x_1, x_2)\): \(\bar{x}_0 = 1 + (x_0 - 1)t, \bar{x}_1 = x_1t, \bar{x}_2 = x_2t\). When this line crosses the equatorial plane, we get the formulas of stereographic map \(f: S^2(1) \to \mathbb{R}^2\) where \(x_0 \neq 1\):

\[
x = \frac{x_1}{1 - x_0}, \quad y = \frac{x_2}{1 - x_0},
\]

(15)

here \((x, y) \in \mathbb{R}^2, (x_0, x_1, x_2)\) are coordinates on \(S^2(1)\). An inverse map \(f^{-1}: \mathbb{R}^2 \to S^2(1)\) when \(x \neq \pm 1\) is:

\[
x_0 = \frac{1 + x^2}{1 - x^2}, \quad x_1 = \frac{2x}{1 - x^2}, \quad x_2 = \frac{2y}{1 - x^2}.
\]

(16)

If we put formulas (15) into (13) then we obtain the relations between coordinates \(x, y\) and adapted coordinates \(u, \varphi\) which are on semi-Euclidean sphere:

\[
f: x = \frac{\sinh \varphi}{\varepsilon - \cosh \varphi}, \quad y = \frac{u \exp \varphi}{\varepsilon - \cosh \varphi}.
\]

Then the inverse map is:

\[
\varphi = \ln \left(\frac{\varepsilon x - 1}{x + 1}\right), \quad u = -\frac{2y}{(1 - x)^2}.
\]

(17)

If we expand \(\mathbb{R}^2\) to the semi-conformal plane \(C^2\) by infinitely distant point and ideal line crossing this point, then the stereographic map \(f\) becomes diffeomorphism \(\tilde{f}\). This infinitely distant point is the image of point \(N\). The ideal line is the image of straight line belonging to \(S^2(1)\) and crossing the pole: \(x_0 = 1, x_1 = 0\). Do not forget, that we need to exclude lines \(x \neq \pm 1\) from the the plane \(\mathbb{R}^2\).

Let us now consider the commutative diagram:

\[
\begin{array}{ccc}
S^2(1) & \xrightarrow{\tilde{f}} & C^2 \\
\pi \downarrow & & \downarrow p \\
\mathbb{R} & & \\
\end{array}
\]

The map \(p = \pi \circ \tilde{f}^{-1}: C^2 \to \mathbb{R}\) is defined by this diagram. We find the coordinate form of this map:

\[
u = -\frac{2y}{(1 - x)^2}.
\]

So, the map \(p: C^2 \to \mathbb{R}\) defines the trivial principal fibration with the base \(\mathbb{R}\) and the structure group \(S_1\).

**Proposition.** The map \(\tilde{f}: S^2(1) \to C^2\) is conformal.
Proof. The metric on $G$ induces the metric on $C^2$. In the coordinates $x, y$ it has the form:

$$ds^2 = -dx^2.$$  \hfill (18)

Let us find the metric of semi-Euclidean sphere from the metric on $C^2$. By (17) we get $d\varphi = \frac{2}{x^2-1}$. So, by (14) and (18), we find:

$$d\tilde{s}^2 = \frac{4}{(x^2-1)^2} ds^2.$$  \hfill (19)

Hence, the linear element of semi-Euclidean sphere differs from the linear element of semi-plane by conformal factor and so, the map $\tilde{f}$ is conformal. $\Box$

We find the fibers equations on $C^2$. The 1-parametric fibers family of the fibration $(S^2(1), \pi, \mathbb{R})$ in the adaptive coordinates (13) is: $u = c$, $c \in \mathbb{R}$. By (17) we get the image of this family under map $\tilde{f}$:

$$y = -c/2 \cdot (x - 1)^2.$$  \hfill (19)

The semi-conformal plane is also fibred by this 1-parametric family of curves. It is parabolas with axis $x = 1$ and top $(1, 0)$.

4. THE PROJECTIVE SEMI-CONFORMAL MODEL OF THE FIBRATION OF SPHERE $S^2(1)$

Now we construct the projective semi-conformal model of the sphere $S^2(1)$ and the principal fibration on it. We use normalization method of A.P. Norden [5]. A. P. Shirokov in his work [6] constructed conformal models of Non-Euclidean spaces with this method.

In a projective space $\mathbb{P}^n$ a hypersurface $X_{n-1}$ as an absolute is called normalized if with every point $Q \in X_{n-1}$ is associated:

1) line $P_I$ which has the point $Q$ as the only intersection with the tangent space $T_{n-1}$,

2) linear space $P_{II}$ that belongs to $T_{n-1}$, but it does not contain the point $Q$.

We call them *normals of first and second types*, $P_I$ and $P_{II}$.

In order that normalization be polar, $P_I$ and $P_{II}$ must be polar with respect to absolute $X_{n-1}$.

We expand the semi-Euclidean space $E^3$ to a projective space $P^3$. Here $kE^n_l$ denotes an $n$-dimensional semi-Euclidean space with metric tensor of rank $k$, $l$ is the number of negative inertia index in a quadric form. We consider homogeneous coordinates $(y_0 : y_1 : y_2 : y_3)$ in $P^3$, where $x_i = \frac{y_i}{y_3}$, $i = 0, 1, 2$. Thus $S^2(1)$:

$$x_0^2 - x_1^2 = 1$$

describes the hyperquadric in $P^3$:

$$y_0^2 - y_1^2 - y_2^2 = 0.$$  \hfill (20)

Here the projective basis $(E_0, E_1, E_2, E_3)$ is chosen in the following way. The vertex $E_0$ of basis is inside the hyperquadric. The other vertices $E_1, E_2, E_3$ are on its polar plane, $y_0 = 0$. The line $E_0E_3$ cross the hyperquadric at poles $N(1:$
0 : 0 : 1), \( N'(1 : 0 : 0 : -1) \). Vertices \( E_1, E_2 \) lie on the polar of the line \( E_0E_3 \). The vertex of the hyperquadric coincides with the vertex \( E_2 \).

The stereographic map of projective plane \( P^2 : y_0 = 0 \) to the hyperquadric (20) from the pole \( N(1 : 0 : 0 : 1) \) is shown on the picture. Let \( U(0 : y_1 : y_2 : y_3) \in P^2 \). If \( y_3 = 0 \), then the line \( UN \) belongs to the tangent plane \( T_N : y_0 - y_3 = 0 \) of the hyperquadric (20) at the point \( N \) and in this case the intersection point of the line \( UN \) with the hyperquadric is not uniquely determined. If \( y_3 \neq 0 \), then the intersection point of line \( UN \) with the hyperquadric is unique. So, we choose the line \( E_1E_2 : y_3 = 0 \) as the line at infinity. In the area \( y_3 \neq 0 \) we consider the Cartesian coordinates \( x_1 = \frac{y_1}{y_3}, x_2 = \frac{y_2}{y_3} \). Then the plane \( \alpha : y_0 = 0, y_3 \neq 0 \) becomes a plane with an affine structure \( A^2 \). It is possible to introduce the structure of semi-Euclidean plane \( E^2 \) with the linear element

\[ ds^2_0 = dx_1^2. \]  

(21)

The hyperquadric and the plane \( \alpha \) do not intersect or intersect in two imaginary parallel lines

\[ x_1^2 = -1. \]  

(22)

The restriction of the stereographic projection to the plane \( \alpha \) maps the point \( U(0 : x_1 : x_2 : 1) \) into the point \( X_1 \)

\[ X_1(-1 - x_1^2 : 2x_1 : 2x_2 : 1 - x_1^2). \]  

(23)

So, the Cartesian coordinates \( x_i \) can be used as the local coordinates at the hyperquadric except the point of its intersection with the tangent plane \( T_N \).

We construct an autopolar normalization of the hyperquadric. As a normal of first type we take lines with fixed center \( E_0 \) and as a normal of second type we
take their polar lines which belong to the plane \( \alpha \) and cross the vertex \( E_2 \) of the hyperquadric. The line \( E_0X_1 \) intersects the plane \( \alpha \) at the point

\[
X(0 : 2x_1 : 2x_2 : 1 - x_1^2).
\]

In this normalization the polar of the point \( X \) intersects the plane \( \alpha \) on normal \( P_{II} \). Thus for any point \( X \) in the plane \( \alpha \) there corresponds a line which does not cross this point. It means that the plane \( \alpha \) is also normalized. Normalization of \( \alpha \) is defined by an absolute quadric \((22)\).

We consider the derivative equations for this normalization. If we take normals of the first type with fixed center \( E_0 \), then the derivative equations \([5], \text{p.204}\) have the form:

\[
\begin{align*}
\partial_i X &= Y_i + l_i X, \\
\nabla_j Y_i &= l_j Y_i + p_{ji} X.
\end{align*}
\]

(24)

The points \( X, Y_i, E_0 \) define a family of projective frames. Here \( Y_i \) are generating points of normal \( P_{II} \).

We can calculate the values \((X, X), (X, Y_i)\) on the plane \( \alpha \) using the quadric form, which is in the left part of equation \((20)\). So, \((X, X) = -(1 + x_1^2)^2\).

Let us find coordinates of metric tensor on plane \( \alpha \). So, we take the Weierstrass standartization

\[
(\widetilde{X}, \widetilde{X}) = -1, \quad \widetilde{X} = \frac{X}{1 + x_1^2}.
\]

Then the coordinates of metric tensor are scalar product of partial derivatives \( g_{ij} = -(\partial_i \widetilde{X}, \partial_j \widetilde{X}):\)

\[
(g_{ij}) = \begin{pmatrix}
4 & 0 \\
0 & 0
\end{pmatrix}.
\]

So, we get the conformal model of polar normalized plane \( \alpha : y_0 = 0, y_3 \neq 0 \) with a linear element

\[
ds^2 = \frac{dx_1^2}{(1 + x_1^2)^2}.
\]

(25)

It means that this non-Euclidean plane is conformal to semi-Euclidean plane \( 1E^2 \).

The points \( X \) and \( Y_i \) are polarly conjugated, \((X, Y_i) = 0\). From this equation and the derivative equations \([24]\) we can get the non-zero connection coefficients:

\[
\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\frac{2x_1}{1 + x_1^2}, \quad \Gamma^2_{11} = \frac{2x_2}{1 + x_1^2}.
\]

The sums \( \Gamma^k_{ks} = \partial_k \ln \frac{c}{(1 + x_1^2)^2} \) \((c = \text{const})\) are gradient, so the connection is equiaffine. Curvature tensor has the following form:

\[
R^1_{121} = R^2_{211} = -\frac{4}{(1 + x_1^2)^2}.
\]
Ricci curvature tensor $R_{sk} = R_{isk}^i$ is symmetric: $R_{11} = \frac{4}{(1+x_1^2)^2}$. Metric $g_{ij}$ and curvature $R_{sk}^i$ tensors are covariantly constant in this connection: $\nabla_k g_{ij} = 0, \nabla_l R_{sk}^i = 0$.

Geodesic curves in this connection satisfy the following equations:

$$x_2 = A(x_1^2 - 1) + Bx_1$$ and $x_1 = 0$,

where $A$ and $B$ are arbitrary constants. There are parabolas and lines orthogonal to axis $Ox_1$.

Let us consider the fibration of this plane by double numbers subalgebra. We write the equations of fibers of semi-Euclidean sphere $S^2(1)$ in homogeneous coordinates:

$$\begin{cases} (y_0 - y_1)v - y_2 = 0, \\ y_0^2 - y_1^2 - y_3 = 0. \end{cases}$$ (26)

This 1-parametric family of curves fibers the hyperquadric and it defines a fibration on it. The stereographic projection of these fibers from the pole $N$ to the plane $\alpha$ is:

$$x_2 = -v/2 \cdot (x_1 + 1)^2.$$ It is 1-parametric family of parabolas.

**Remark**

We would obtain the similar results for the space of right cosets by Lie subgroup $H_2$ (it is the subgroup of invertible dual numbers) and the fibration of the group $G$ by $H_2$. However, $H_2$ is a normal divisor of the group $G$. Therefore, the spaces of right and left cosets coincide. We have the proposition:

**Proposition** The fibration $(G, \pi', M' = G/H_2)$ determined by the formula

$$\pi'(x) = \frac{x_1}{x_0}$$ (27)

is a trivial principal fibration over the real line $\mathbb{R}\\{0\}$. The typical fiber is a plane without a line. The structure group is $H_2$.

The equations of fibers are:

$$ux_0 - x_1 = 0, \quad u \in \mathbb{R}.$$ (28)

This is a 1-parametric family of planes with common axis $Ox_2$. It is necessary to remove intersection points of these planes with planes: $x_0 \pm x_1 = 0, \quad x_0 = 0$.

The left multiplications $x' = ax$ on the group $G$ preserve fibres if and only if when $a \in H_2$.

The right multiplications: $x' = xb$ on group $G$ forms the 3-parametric Lie group of linear transformations that preserve the fibration and induce a 1-parametric group of hyperbolic transformations on the base with two fixed points $u = \pm 1$:

$$u' = \frac{u + \alpha}{\alpha \cdot u + 1},$$

here $\alpha = \pi(b)$. 


The right multiplications include involutions of hyperbolic type:

\[ R(b) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ b_2 & b_2 & -1 \end{pmatrix}, R^2 = \text{Id.} \]

They induce the involution on the base:

\[ u' = \frac{1}{u}. \]

The similar properties has the fibration of semi-Euclidean sphere by subgroup

\[ S_2 = \{ x \in H_2 | x_0^2 - x_1^2 = 1 \} \]

and its models.

ACKNOWLEDGEMENT

I would like to thank Professor Jiri Vanzura for fruitful discussions and support with writing this paper. This work was supported in part by grant No. 201/05/2707 of The Czech Science Foundation and by the Council of the Czech Government MSM 6198959214.

REFERENCES

[1] B. N. Shapukov, Exercises on Lie groups and its applications, Moskow: RXD (2002).
[2] Study E., Cartan E. Nombres complexes, Encyclopedie des sciences mathematigues pures et appliquees, t.1. Vol.1. (1908) 329-468.
[3] V. V. Vishnevskii, A.P. Shirokov, V.V. Shurygin, Spaces over algebras, Izdat. Kazan. univ. (1985).
[4] N. E. Belova, various articles and works published in Kazan, for example: Mat. vseros. molod. nauc. shkoli-konf. po mat. mod., geom. i algeb. Kazan. (1998) 169-174.
[5] A. P. Norden, Spaces with affine connection, Moscow: Nauka (1976).
[6] A. P. Shirokov, Non-Euclidean spaces, Izdat. Kazan. univ. (1997).
[7] B. A. Rosenfeld, Geometry of Lie Groups, Kluwer (1991).