MULTIPLECTY OF SUBHARMONIC SOLUTIONS AND PERIODIC SOLUTIONS OF A PARTICULAR TYPE OF SUPER-QUADRATIC HAMILTONIAN SYSTEMS

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Abstract. This paper considers the Hamiltonian systems with new generalized super-quadratic conditions. Using the variational principle and critical point theory, we obtain infinitely many distinct subharmonic solutions and an unbounded sequence of periodic solutions.

1. Introduction. We consider the non-autonomous Hamiltonian system

\[
\begin{aligned}
\dot{p} &= -H_p'(t, z), \\
\dot{q} &= H_q'(t, z),
\end{aligned}
\]

(1)

which also can be written as \( \dot{z} = JH_z'(t, z) \), where \( H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R}) \), \( z(t) = (p(t), q(t)) \) \( (t \in \mathbb{R}, p(t), q(t) \in \mathbb{R}^n) \), \( H_z' = \frac{\partial H}{\partial z}, H_p' = \frac{\partial H}{\partial p}, H_q' = \frac{\partial H}{\partial q} \) and \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \) with \( I_n \) being the \( n \times n \) identity matrix.

In the pioneer work [11], the author deals with the existence of periodic solutions of the Hamiltonian systems under a classical super-quadratic condition, that is, (S) there exist constants \( \bar{\theta} \in (0, \frac{1}{2}) \) and \( R > 0 \) such that

\[
\bar{\theta}H_z'(t, z) \cdot z \geq H(t, z) > 0, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n} \quad \text{with} \quad |z| \geq R.
\]

Papers [3] and [4] consider the existence of periodic solutions of the system (1) with different super-quadratic conditions, both of which generalize the super-quadratic condition (S). Our previous paper [18] verifies the existence of periodic solutions of the system (1) with a new super-quadratic condition, that is, (NS) there exist constants \( c_1, c_2, \sigma, \tau, \mu, \nu > 0, \beta > 1 \) and \( \max\{\frac{\sigma}{\tau}, \frac{\tau}{\sigma}\} < 1 + \frac{\beta - 1}{\beta} \) such that

\[
\frac{1}{\mu}H_p'(t, z) \cdot p + \frac{1}{\nu}H_q'(t, z) \cdot q - \left( \frac{1}{\mu} + \frac{1}{\nu} \right) H(t, z) \geq c_1|z|^{\beta} - c_2, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n},
\]

\[
H(t, z) \rightarrow +\infty \quad \text{as} \quad |z| \rightarrow +\infty \quad \text{uniformly in} \quad t,
\]

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which generalizes the super-quadratic conditions in papers [3] and [4]. Paper [18] also shows that the Hamiltonian function $H(t, z) = \left( |p|^2 + |q|^2 \right) \left( \ln(1 + |z|^2) \right)$ satisfies above condition (NS) but dissatisfies the super-quadratic conditions in papers [3] and [4], where $\xi, \eta, \mu, \nu > 1$ and $\frac{\mu}{\nu} < \frac{\nu}{\mu} < 1 + \frac{\nu}{\mu}$, for instance $\mu = \frac{13}{12}, \nu = \frac{24}{13}$. We would like to remind the readers that the condition $\mu, \nu > 1$ with $\frac{1}{\nu} + \frac{1}{\nu} < 1^*$ in condition (H2) in our previous [18] (that is, above condition (NS)) is unnecessary. Paper [1] also introduces a generalized super-quadratic condition, that is, 

\[(S)' \quad \text{there exist constants } R > 0 \text{ and } \alpha_i, \beta_i > 0 (i = 1, 2, \ldots, n) \text{ with} \]

$$\frac{1}{\alpha_1} + \frac{1}{\beta_1} = \frac{1}{\alpha_2} + \frac{1}{\beta_2} = \cdots = \frac{1}{\alpha_n} + \frac{1}{\beta_n} < 1$$

such that for $t \in \mathbb{R}$, $|z| \geq R$,

$$0 < H(t, z) \leq \sum_{i=1}^{n} \left( \frac{p_i}{\alpha_i} H_i'(t, z) + \frac{q_i}{\beta_i} H_{n+i}(t, z) \right),$$

where $H_j'(t, z) = \frac{\partial H}{\partial z_j}(t, z)$ ($j = 1, 2, \ldots, 2n$) and $z = (p_1, \ldots, p_n, q_1, \ldots, q_n)$. As Remark 1 in paper [1] shows, the super-quadratic condition $(S)'$ generalizes the super-quadratic condition (H2) in paper [4]. Furthermore, Theorem 1.1 and Theorem 1.2 in paper [1] generalize Theorem 0.1 in paper [4] and Theorem 0.1 in paper [5] respectively. However, the above Hamiltonian function $H$ dissatisfies $(S)'$, which can be proved by the techniques shown in paper [18]. And it is not difficult to see that the Hamiltonian function $H(t, z) = |p_1|^2 + |p_2|^2 + |p_3|^4 + |q_1|^6 + |q_2|^{15} + |q_3|^{32}$ introduced in paper [1] dissatisfies our super-quadratic condition (NS), where $z = (p_1, p_2, p_3, q_1, q_2, q_3) \in \mathbb{R}^6$.

By using $S^1$ symmetry, paper [13] obtains an unbounded sequence of periodic solutions of autonomous Hamiltonian systems under merely the super-quadratic condition (S). One goal of paper [5] is to validate the multiplicity of periodic solutions of the system (1) under the generalized super-quadratic condition in paper [4]. Paper [9] searches for infinitely many periodic solutions of the system (1) by using the monotone truncations of $H$ developed in paper [8]. Using the monotone truncations of $H$ in paper [8], paper [16] also obtains unbounded sequences of periodic solutions of the system (1) with symmetric assumption. If we put $H(t, z) = \frac{1}{2}|p|^2 + V(t, q), (t, p, q) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, then the system (1) diverts to the second-order Hamiltonian system, that is, $q + V_q'(t, q) = 0, (t, q) \in \mathbb{R} \times \mathbb{R}^n$.

Papers [15], [19] and [17] are such materials that consider multiplicity of periodic solutions of the second-order Hamiltonian systems. Two $T$-periodic solutions $x_1$ and $x_2$ are geometrically distinct, if $x_1(\mathbb{R}) \neq x_2(\mathbb{R})$.

Paper [12] considers subharmonic solutions of the system (1) under classical super-quadratic and sub-quadratic assumptions. Paper [7] uses the Maslov-type index to solve the subharmonic solutions of the system (1). Paper [2] considers periodic and subharmonic solutions of the system (1) under the generalized super-quadratic condition in paper [4]. Paper [6] considers brake subharmonic solutions by using the Galerkin approximation method and the iteration inequalities of the $L$-Maslov type index theory. To the authors’ knowledge, the subharmonic solution problem originates from paper [12] and focuses on when two solutions are geometrically distinct. A subharmonic solution of the system (1) is a $KT$-periodic solution for some $k \geq 2$, if the Hamiltonian function $H$ is $T$-periodic with respect
to \( t \). A \( k_1T \)-periodic solution \( z_1 \) and a \( k_2T \)-periodic solution \( z_2 \) are \textit{geometrically distinct}, if \( m_1 \neq m_2 \) holds for any \( m_1, m_2 \in \mathbb{Z} \), where \((m \ast z)(t) = z(t+mT)\). Books [10] and [14] contain some results of the existence and multiplicity of periodic solutions and subharmonic solutions of the system (1).

Our aim is to use the variational principle and critical point theory to verify the multiplicity of subharmonic solutions and periodic solutions of the Hamiltonian system (1) with the super-quadratic condition (NS).

We shall prepare some preliminaries in Section 2, give the multiplicity results of subharmonic solutions of the system (1) in Section 3 and give the multiplicity result of periodic solutions of the system (1) in Section 4.

Our three results are divided into two parts. Firstly, two results about subharmonic solutions are listed as following.

**Theorem 1.1.** The system (1) possesses infinitely many distinct nontrivial subharmonic solutions, if \( H \) satisfies

\( (H1) \) \( H \in C^1(R \times R^{2n}, R) \) is nonnegative and \( T \)-periodic with respect to \( t \),
\( (H2) \) there exist constants \( c_1, c_2, \sigma, \tau, \nu > 0, \beta > 1 \) and \( \max\{\frac{\sigma}{\nu}, \frac{\tau}{\nu}\} < 1 + \frac{\beta-1}{\beta} \) such that
\[
\frac{1}{\mu} H_p'(t,z) \cdot p + \frac{1}{\nu} H_q'(t,z) \cdot q - \left(\frac{1}{\mu} + \frac{1}{\nu}\right) H(t,z) \geq c_1|z|^\beta - c_2, \quad (t,z) \in R \times R^{2n},
\]

\( (H3) \) there exists a constant \( \lambda \in \left(\max\{\frac{\sigma}{\nu}, \frac{\tau}{\nu}\}, 1 + \frac{\beta-1}{\beta}\right) \) such that
\[
|H'_z(t,z)| \leq c_2(|z|^{\lambda} + 1), \quad (t,z) \in R \times R^{2n},
\]

\( (H4) \) \( \frac{H(t,z)}{|p|^{\frac{1}{\sigma}} + |q|^{\frac{1}{\tau}}} \to 0 \) as \( |z| \to 0 \) uniformly in \( t \),
\( (H5) \) \( \frac{H(t,z)}{|p|^{\frac{1}{\sigma}} + |q|^{\frac{1}{\tau}}} \to +\infty \) as \( |z| \to +\infty \) uniformly in \( t \).

Moreover, above condition (H3) has a weakened version, that is, \( (H3)' \) there exist constants \( \xi > 0 \) and \( \eta > 0 \) with \( \max\{\frac{\xi}{\eta}, \frac{\eta}{\xi}\} < 1 + \frac{\beta-1}{\beta} \) such that
\[
\frac{1}{\xi} H_p'(t,z) \cdot p + \frac{1}{\eta} H_q'(t,z) \cdot q - \left(\frac{1}{\xi} + \frac{1}{\eta}\right) H(t,z) \geq c_1|H'_z(t,z)| - c_2, \quad (t,z) \in R \times R^{2n},
\]

where \( \beta \in (1, \frac{3+\sqrt{5}}{2}) \) and \( \max\{\frac{\xi}{\eta}, \frac{\eta}{\xi}\} < 1 + \frac{\beta-1}{\beta} \).

Replacing (H3) with (H3)', due to the lack of polynomial growth control for \( H'_z \), by modifying \( H \) with a cut-off function, we have a result similar to Theorem 1.1, that is,

**Theorem 1.2.** The system (1) possesses infinitely many distinct nontrivial subharmonic solutions, if \( H \) satisfies (H1), (H2), (H3)', (H4) and (H5).

We would like to remind the readers that the Hamiltonian functions satisfying (H1)-(H5) and (H3)' do exist, see our previous paper [18].

Next, we shall use a generalized critical point theorem of even functionals to search for an unbounded sequence of periodic solutions of the system (1).

**Theorem 1.3.** If \( H \) satisfies (H1)-(H3), (H5) and (H6) \( H(t,z) = H(t,-z), \quad (t,z) \in R \times R^{2n}, \)
then the system (1) possesses a sequence of $T$-periodic solutions $\{z_k\}$ such that $\|z_k\|_{L^\infty} \to +\infty$ as $k \to +\infty$. Moreover, there exists a constant $c > 0$ such that $\|z_k\|_{L^\infty} \geq ck^{\frac{1}{n}}$.

During the proof of the above three theorems, we need two inequalities stated below.

Remark 1 (See [18]). For $z = (p, q), p, q \in \mathbb{R}^n$, suppose $\min \{\alpha_1, \alpha_2\} \geq \max \{\alpha_3, \alpha_4\} > 0$ and $\alpha_5 \geq 1$, then we have

$$|p|^{\alpha_1} + |q|^{\alpha_2} \geq \frac{1}{2}(|p|^{\alpha_3} + |q|^{\alpha_4}), \quad |z| \geq \sqrt{2}$$

and

$$2^{-\frac{n}{4}}(|p|^{\alpha_5} + |q|^{\alpha_5}) \leq |z|^{\alpha_5} \leq 2^{\frac{n}{2}}(|p|^{\alpha_5} + |q|^{\alpha_5}).$$

As is in paper [4], by making change of variables $\zeta = \frac{z}{2}$ with $\omega = \frac{T}{2}$, seeking for $T$-periodic solutions of the system (1) diverts to searching for $2\pi$-periodic solutions of the system

$$\begin{cases}
\dot{p}(\zeta) = -\omega H'_q(\omega \zeta, z), \\
\dot{q}(\zeta) = \omega H'_p(\omega \zeta, z).
\end{cases}$$

We hence-force focus attention on $2\pi$-periodic solutions of the system (1).

2. Preliminaries. We introduce some notations and conclusions which are used later.

$$E := W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n}) = \left\{ z \in L^2(S^1, \mathbb{R}^{2n}) \|z\|^2 = \pi \sum_{j \in \mathbb{Z}} |j||a_j|^2 + |a_0|^2 < +\infty \right\},$$

where $S^1 := \mathbb{R}/2\pi \mathbb{Z}$, $z(t) = \sum_{j \in \mathbb{Z}} a_j \exp (ijt), a_j \in C^2(n)$.

$$E^+ := \mathcal{S}(\mathbb{R}^{2n}), \quad E^- := \mathcal{S}(\mathbb{R}^{2n}),$$

$$E^+ := \mathcal{S}(\mathbb{R}^{2n}), \quad E^- := \mathcal{S}(\mathbb{R}^{2n}),$$

where $\{e_k\}_{1 \leq k \leq 2n}$ is the canonical basis in $\mathbb{R}^{2n}$.

$$B[z, \zeta] := \int_0^{2\pi} (-J \dot{z}, \dot{\zeta}) \, dt$$

and $A(z) := \frac{1}{2}B[z, z] = \int_0^{2\pi} p \cdot q \, dt, z = (p, q), \zeta \in C^\infty(S^1, \mathbb{R}^{2n})$, both of which can be continuously extended onto $E$. So $B$ is a bounded bilinear form.

Set $E_1 = E^+, E_2 = E^0 \bigoplus E^-$ and $L_k : E_k \to E_k, \langle L_kz, \zeta \rangle = B[z, \zeta](k = 1, 2)$, where $\langle \cdot, \cdot \rangle$ denotes the induced inner product. References [11] and [14] indicate the following conclusions. $E = E^+ \bigoplus E^0 \bigoplus E^- = E_1 \bigoplus E_2$, and $E^+, E^0$ and $E^-$ are orthogonal and $B$-orthogonal respectively. $A$ is positive on $E^+$, null on $E^0$ and negative on $E^-$. If $z = z^+ + z^0 + z^-$, then $A(z) = \frac{1}{2}(Lz, z) = A(z^+) + A(z^-)$ and $\|z\|^2 = A(z^+) + |z^0|^2 - A(z^-)$, where $Lz := L_1P_1z + L_2P_2z$ and $P_k : E \to E_k(k = 1, 2)$ is the projective operator.

We will use the following two lemmas in our proof.

Lemma 2.1 (See [14]). $E$ can be compactly embedded into $L^s(S^1, \mathbb{R}^{2n})$ $(s \geq 1)$, in particular, there exists a constant $C_s > 0$ such that $\|z\|_{L^s} \leq C_s \|z\|$ holds for $z \in E$. 

Suppose that $I \in C^1(E, \mathbb{R})$. $I$ satisfies the (PS) condition means that if a sequence $\{z_m\}$ satisfies that $\{I(z_m)\}$ is bounded and $I'(z_m) \to 0$ as $m \to +\infty$, then $\{z_m\}$ has a convergent subsequence.

Lemma 2.2 (See [4]). Let $E$ be a real Hilbert space with $E = E_1 \oplus E_2$. Suppose $I \in C^1(E, \mathbb{R})$ with $I(z) = \frac{1}{2} \langle Lz, z \rangle + b(z)$ satisfies the (PS) condition, and

(i) $I$ is a linear, bounded and self-adjoint operator,

(ii) $b'$ is compact,

(iii) $B(v) = P_2B_1^{-1}\exp(vL)B_2 : E_2 \to E_2$ is invertible for any $v \in [0, +\infty)$, where $B_k : E \to E(k = 1, 2)$ is linear, bounded and invertible,

(iv) there exists a constant $\gamma > 0$ such that

$\langle B_2(z) - B_2(0), z \rangle \geq \gamma \|z\|^2$ for every $z \neq 0$.

Then $I$ possesses a critical value $l = \inf_{h \in \Gamma} \sup_{z \in Q} I(h(1, z)) \geq \gamma$, where $\Gamma$ is defined as

$\Gamma := \{h \in C([0, 1] \times E, E)| h \text{ satisfies } (\Gamma_1) - (\Gamma_3)\},$

where

$\Gamma_1) h(0, z) = z, z \in Q,$

$\Gamma_2) h(t, z) = z, z \in Q,$

$\Gamma_3) h(t, z) = \exp(\Phi(t, z)L)z + \Psi(t, z), where \Phi \in C([0, 1] \times E, [0, +\infty))$ transforms bounded sets into bounded sets, and $\Psi$ is compact.

3. Multiplicity of subharmonic solutions. By making change of variables $\zeta = \frac{\theta}{k} (k \in \mathbb{N}^*)$, seeking for $2k\pi$-periodic solutions of the system (1) diverts to searching for $2\pi$-periodic solutions of the system

\[
\begin{align*}
\dot{p}(\zeta) &= -kH'_{\theta}(k\zeta, z), \\
\dot{q}(\zeta) &= kH'_{\theta}(k\zeta, z).
\end{align*}
\]

For $k \in \mathbb{N}^*$, set $H_k(t, z) = kH(kt, z)$ and $I_k(z) = A(z) - \int_0^{2\pi} H_k(t, z) dt, z \in E$. If $H$ satisfies (H1)-(H5), then $H_k$ also satisfies (H1)-(H5), so the proof of Lemma 3.2 in our paper [18] shows that for every $k \in \mathbb{N}^*$, there exist constants $g_k \in (0, 1)$ and $\gamma_k > 0 \text{ (both } g_k \text{ and } \gamma_k \text{ depend on } k)$ such that $I_k|_{S_k} \geq \gamma_k > 0$, where $S_k = \{(q^{-1}_k p, q^{-1}_k q) | (p, q) \in E_1\}$.

For every $k \in \mathbb{N}^*$, define $B_{1,k} : E \to E$ as $B_{1,k}(p, q) = (g_k^{-1} p, g_k^{-1} q)$, where $(p, q) \in E$, then $B_{1,k}$ is linear, bounded and invertible. Define $B_2 : E \to E$ as $B_2(p, q) = (r^{-1} p, r^{-1} q)$, where $(p, q) \in E$, and $r > 0$ is a constant determined later, then $B_2$ is also linear, bounded and invertible.

For $e = (p^+, q^+) \in E_1$ with $\|e\| = 1$, set $\tilde{E} = \text{span}\{e\} \oplus E_2$ and $W = \{z \in \tilde{E}| |t| \leq 1, \|z\| \leq 2 \text{ and } \|z\| \leq \|\tilde{z} + z_0\|\}$, then we have

Lemma 3.1 (See [18]). There exists a constant $\varepsilon_1 > 0$ such that

$\mu\{t \in [0, 2\pi] | |z(t)| \geq \varepsilon_1\} \geq \varepsilon_1, z \in W.$

Furthermore, we set

$Q = \{(r^{-1}(sp^+ + p^- + q^0), r^{\sigma^{-1}}(sq^+ + q^- + q^0)) | 0 \leq s \leq r, (p^-, q^-) \in E^-, (p^0, q^0) \in E^0 \text{ and } \|(p^-, q^- + q^0)\| \leq r\}.$
and $\partial Q$ refers to the boundary of $Q$ relative to 
\[
\hat{E} = \left\{ (r^{-1}(sp^+ + p^0), r\sigma^{-1}(sq^+ + q^0)) \mid s \in \mathbb{R}, (p^-, q^-) \in E^-, (p^0, q^0) \in E^0 \right\}.
\]

Lemma 3.2. There exists a constant $r > 1 > \sup_{k \in \mathbb{N}^*} \frac{\rho_k}{\|B_{1,k}B_2e\|}$ such that $I_k|_{\partial Q} \leq 0$ holds for every $k \in \mathbb{N}^*$, if $H$ satisfies (H1) and (H5).

Proof. The proof is similar to that of Lemma 3.3 in our previous paper [18].

Set $A_0 = \frac{\sqrt{2}}{\varepsilon_1 \min \left\{ \left( \frac{2}{\sqrt{2}} \right)^{1+\frac{\varepsilon}{2}}, \left( \frac{2}{\sqrt{2}} \right)^{1+\frac{\varepsilon}{2}} \right\}}$, where $\varepsilon_1$ is as in Lemma 3.1. Condition (H5) implies that there exists a constant $A_1 \geq \sqrt{2}$ such that
\[
H(t, z) \geq A_0 \left( |p|^{1+\frac{s}{2}} + |q|^{1+\frac{s}{2}} \right), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n} \text{ with } |z| \geq A_1.
\]

So (H1) and (3) show that
\[
H_k(t, z) = kH(kt, z) \geq A_0 \left( |p|^{1+\frac{s}{2}} + |q|^{1+\frac{s}{2}} \right), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n} \text{ with } |z| \geq A_1.
\]

Fix a constant $r \geq \frac{A_1}{\varepsilon_1} + 1$, then $0 < \rho_k < 1 < r$ and

\[
\|B_{1,k}B_2e\|^2 = \left\| \left( \frac{r}{\rho_k} \right)^{-1}, \left( \frac{r}{\rho_k} \right)^{-\sigma} \right\|^2 = \left( \frac{r}{\rho_k} \right)^{\sigma+\tau-2},
\]

so $r > 1 > \rho_k \left( \frac{2}{\sigma} \right)^{\frac{s+\tau-1}{s+\tau}} = \frac{\rho_k}{\|B_{1,k}B_2e\|}$ holds for every $k \in \mathbb{N}^*$.

For $z = (r^{-1}(sp^+ + p^0), r\sigma^{-1}(sq^+ + q^0))$, we have
\[
A(z) = \frac{1}{r} \left| \left( \frac{r}{p^0} \right)^{\sigma+\tau} \right| \left| \left( \frac{r}{p^0} \right)^{\sigma} \right|^2 = \frac{1}{r} \left| \left( \frac{r}{p^0} \right)^{\sigma+\tau} \right|^2.
\]

We will verify for every $k \in \mathbb{N}^*$, $I_k(z) \leq 0$ holds for every $z \in \partial Q$. The proof is divided into several cases.

Case 1. If $s = 0$, then (5) and (H1) imply that $I_k(z) \leq 0$.

Case 2. If $s \neq 0$, then $z \in \partial Q$ indicates that either $s = r$ and $\|p^- + p^0, q^- + q^0\| \leq r$ or $0 < s \leq r$ and $\|p^- + p^0, q^- + q^0\| = r$. Whatever the case is, we have $1 \leq \|\hat{z}\| \leq 2$, where $\hat{z} = (\bar{p}, \tilde{q}) = \frac{1}{r}(sp^+ + p^0, sq^+ + q^0)$. Next, we will consider two subcases below.

Subcase 1. If $\|\frac{1}{r}(sp^+ + p^0, sq^+ + q^0)\| < \|\frac{1}{r}(p^-, q^-)\|$, then (5) and (H1) imply that $I_k(z) \leq 0$.

Subcase 2. If Case 1 fails, we set $\Omega_{\hat{z}} = \left\{ t \in [0, 2\pi] \mid \|\hat{z}(t)\| \geq \varepsilon_1 \right\}$, then Lemma 3.1 implies that $\text{measure}(\Omega_{\hat{z}}) \geq \varepsilon_1$. For $t \in \Omega_{\hat{z}}$, $\frac{\sqrt{2}}{\varepsilon_1} \|\hat{z}(t)\| \geq \sqrt{2}$ and

\[
|z(t)| = \left| \left( r^{\tau-1}(sp^+(t) + p^-(t) + p^0), r\sigma^{-1}(sq^+(t) + q^-(t) + q^0) \right) \right|
\geq r \left| \left( \frac{r}{p^0} \right)^{\sigma+\tau} \right| \left| \left( \frac{r}{p^0} \right)^{\sigma} \right|^2 \geq r \varepsilon_1 > A_1.
\]
For \( t \in \Omega_I \), using (6), (4), Remark 1 and the choice of \( A_0 \), we have
\[
H_k(t, z(t)) \geq A_0 \left( r\pi^{-1}(q^{-1}(t)+p^{-1}(t)+p^0) + r\sigma^{-1}(q^{-1}(t)+q^0(t)+q^0(t)) \right)^{1+\frac{\sigma}{\pi}}
\]
\[
= A_0 r^{\sigma+\tau} \left( |\frac{\sqrt{2}}{\sigma} \hat{p}(t)|^{1+\frac{\sigma}{\pi}} + |\frac{\sqrt{2}}{\sigma} \hat{q}(t)|^{1+\frac{\sigma}{\pi}} \right)
\]
\[
= A_0 r^{\sigma+\tau} \left( |\frac{\sqrt{2}}{\sigma} \hat{p}(t)|^{1+\frac{\sigma}{\pi}} + |\frac{\sqrt{2}}{\sigma} \hat{q}(t)|^{1+\frac{\sigma}{\pi}} \right)
\]
\[
\geq A_0 r^{\sigma+\tau} \min \left( \frac{\sqrt{2}}{\sigma}, \left( \frac{\sqrt{2}}{\sigma} \right)^{1+\frac{\sigma}{\pi}} \right) \left( \frac{\sqrt{2}}{\sigma} \hat{p}(t) \right)^{1+\frac{\sigma}{\pi}} + \left( \frac{\sqrt{2}}{\sigma} \hat{q}(t) \right)^{1+\frac{\sigma}{\pi}}
\]
\[
\geq A_0 r^{\sigma+\tau} \min \left( \frac{\sqrt{2}}{\sigma}, \left( \frac{\sqrt{2}}{\sigma} \right)^{1+\frac{\sigma}{\pi}} \right) \cdot \frac{1}{2} \left( \left( \frac{\sqrt{2}}{\sigma} \hat{p}(t) \right)^{1+\frac{\sigma}{\pi}} + \left( \frac{\sqrt{2}}{\sigma} \hat{q}(t) \right)^{1+\frac{\sigma}{\pi}} \right)
\]
So (5) and (7) imply that
\[
I_k(z) = A(z) - \int_0^{2\pi} H_k(t, z(t)) \, dt \leq r^{\sigma+\tau} - \int_{\Omega_I} H_k(t, z(t)) \, dt \leq 0.
\]
\[\square\]

**Proof of Theorem 1.1.** We complete the proof in two steps.

**Step 1.** We will prove that the system (2) possesses a nontrivial classical 2\( \pi \)-periodic solution for every \( k \in \mathbb{N}^* \).

Since \( H \) satisfies (H1)-(H5), for every \( k \in \mathbb{N}^* \), \( H_k \) also satisfies (H1)-(H5), which leads to the following results: book [14] implies that \( I_k \) \in \( C^1(\mathbb{R}, \mathbb{R}) \) satisfies (I1) and (I2) in Lemma 2.2, where \( A(z) = \frac{1}{2} \{ Lz, z \} \) and \( b_k(z) = - \int_0^{2\pi} H_k(t, z(t)) \, dt, z \in \mathbb{E} \); Lemma 3.1 in our paper [18] indicates that \( I_k \) satisfies the (PS) condition; Lemma 3.4 in our paper [18] shows that \( I_k \) satisfies (I3) in Lemma 2.2 for \( r \) and \( g_k \) defined as in Lemma 3.2; Lemma 3.2 and the arguments before it show that \( I_k \) satisfies (I4) in Lemma 2.2. Thus Lemma 2.2 holds for \( I_k \), if \( H \) satisfies (H1)-(H5). Then there exists a critical point \( z_k \) of \( I_k \) with \( \dot{z}_k = I_k(z_k) \geq \gamma_k > 0 \). Pages 40 and 41 in book [14] indicate that \( z_k(t) \) is a nontrivial classical 2\( \pi \)-periodic solution of the system (2) for every fixed \( k \in \mathbb{N}^* \).

**Step 2.** We will prove that the system (1) possesses a distinct sequence of subharmonic solutions.

The idea comes from paper [12].

Note that \( k \dot{z}_k = A(z_k) - k \int_0^{2\pi} H_k(t, z_k) \, dt > 0 \), and \( z_1(k) \) also satisfies the system (2). If \( z_k(t) = z_1(kt), t \in [0, 2\pi], \) then
\[
l_k = \int_0^{2\pi} p_k(t) \cdot \dot{q}_k(t) \, dt - k \int_0^{2\pi} H_k(t, z_k) \, dt
\]
\[
= k \int_0^{2\pi} \left[ p_1(kt) \cdot \dot{q}_1(kt) - H(kt, z_1(kt)) \right] \, dt
\]
\[
= \int_0^{2\pi} \left[ p_1(t) \cdot \dot{q}_1(t) - H(t, z_1(t)) \right] \, dt = kl_k \rightarrow +\infty, \quad k \rightarrow +\infty.
\]
Choosing \( h = \text{Id} \in \Gamma, \) Lemma 2.2, (H1) and (5) imply that
\[
l_k = \inf_{h \in \Gamma} \sup_{z \in Q} I_k(h(1, z)) \leq \sup_{z \in Q} A(z) \leq r^{\sigma+\tau}.
\]
From (8), (9) and for a certain \( k_1 \) sufficiently large, we can deduce that
\[
z_k(t) \neq z_1(kt), \quad k \geq k_1.
\]
Reapplying this method, we see there is a sequence of 2π-periodic solutions \( z_{k,j} \) of the system
\[
\begin{aligned}
\dot{p}(s) &= -jk_1 H'_q(jk_1 s, z), \\
\dot{q}(s) &= jk_1 H'_p(jk_1 s, z),
\end{aligned}
\]
and \( z_{k,j}(t) \neq z_k(jt), j \geq k_2, \) where \( k_2 \) is large enough. So
\[
z_{k,k_2}(t) \neq z_k(k_2t) \text{ and } z_{k,k_2}(t) \neq z_1(k_1 k_2 t).
\]

Repeating the above arguments, it follows that we have a distinct sequence of subharmonic solutions of (1), i.e., \( z_1(t), z_{k_1} \left( \frac{t}{k_1} \right), z_{k_2} \left( \frac{t}{k_2} \right), \ldots. \) \( \square \)

Now we turn to the proof of Theorem 1.2. For every \( k \in \mathbb{N}^* \), if \( H \) satisfies (H1), (H2), (H3)', (H4) and (H5), then \( H_k \) also satisfies those conditions. As our paper \([18]\) demonstrates, due to the lack of growth control for \((H_1), (H_2), (H_3)\), then
\[
\text{Lemma 3.3.}
\]

\( \text{Lemma 3.5 in our paper [18] indicates that if } \chi \in C^\infty([0, +\infty), [0, 1]) \text{ defined as} \]
\[\chi(s) = \begin{cases} 1, & 0 \leq s \leq K, \\ 0, & s \geq K + 1, \end{cases} \text{ and } \chi'_{|K+1}| < 0.\]

Choosing a constant \( \lambda_0 \in (\max\{|\frac{\mu}{\nu}, \frac{\xi}{\eta}, \frac{\sigma}{\rho}, \frac{\tau}{\tau}, \frac{\zeta}{\zeta}, 1 + \frac{\beta-1}{\beta}\}, 1) \) with \( \lambda_0 \geq \beta - 1 \) and
\[
C_{k,K} \geq \max \left\{ \frac{\max_{t \in E} \left| H_k(t, z) \right|}{\left| z \right|^2}, \left( \frac{c_1}{\min\{|\frac{\mu}{\nu}, \frac{\xi}{\eta}, \frac{\sigma}{\rho}, \frac{\tau}{\tau}, \frac{\zeta}{\zeta}, 1 + \frac{\beta-1}{\beta}\}} \right)^2 \right\}
\]
where \( A_0 \) is as in Lemma 3.2, define
\[
H_{k,K}(t, z) = \chi(|z|)H_k(t, z) + (1 - \chi(|z|))C_{k,K}|z|^\lambda_{0+1}, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}.
\]

Lemma 3.5 in our paper \([18]\) indicates that if \( H \) satisfies (H1), (H2), (H3)', (H4), (H4) and (H5), then \( H_{k,K} \) also satisfies those conditions. Clearly, \( H_{k,K} \) satisfies (H5). So \( H_{k,K} \) satisfies the conditions of Theorem 1.2. Let \( I_{k,K}(z) = A(z) - \int_0^\infty H_{k,K}(t, z)dt, z \in E, \) then the proof of Lemma 3.2 in our paper \([18]\) shows that for every \( k \in \mathbb{N}^* \), there exists a number \( \theta_{k,K} \in (0, 1) \) depending on \( k \) and \( K \) such that \( I_{k,K}|S_{k,K} \geq \gamma_{k,K} > 0, \) where \( S_{k,K} = \{(q_{k,K}, p_{k,K}) \| (p, q)\| \neq \theta_{k,K}, (p, q) \in E_1\}. \) Similarly, we set \( B_{k,K}(p, q) = (\theta_{k,K} p, \theta_{k,K} q), (p, q) \in E, \) then \( B_{k,K}(p, q) \) is bounded, closed and invertible.

**Lemma 3.3.** For \( r, B_2 \) and \( Q \) defined as in Lemma 3.2, we have \( r > 1 > \sup_{k \in \mathbb{N}^*} \frac{\theta_{k,K}}{B_{k,K}^2e}, \) and \( I_{k,K}|\partial Q \leq 0 \) holds for every \( k \in \mathbb{N}^* \), if \( H \) satisfies (H1) and (H5).

**Proof:** The proof is similar to Lemma 3.2.

By the choice of \( C_{k,K}, \) (4) and the estimate
\[
|z|^{\lambda_{0+1}} \geq 2^{-\frac{\lambda_{0+1}}{\tau}} (|p|^\lambda_{0+1} + |q|^\lambda_{0+1}) \geq 2^{-\frac{\lambda_{0+1}}{\tau}} (|p|^1 + |q|^1) , \quad |z| \geq \sqrt{2},
\]
we have
\[
H_{k,K}(t, z) \geq A_0 \left( |p|^{1+\frac{\tau}{\tau}} + |q|^{1+\frac{\tau}{\tau}} \right) , \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n} \text{ with } |z| \geq A_1.
\]

Since \( r \geq \frac{A_1}{\tau+1} + 1, \) then \( r \) is independent of \( k \) and \( K, \) \( 0 < \theta_{k,K} < 1 < r \) and
\[
\|B_{k,K}^{-1} B_2e\|^2 = \left\| \left( \left( \frac{r}{\theta_{k,K}} \right)^{\tau-1} p^+, \left( \frac{r}{\theta_{k,K}} \right)^{\tau-1} q^+ \right) \right\|^2 = \left( \frac{r}{\theta_{k,K}} \right)^{\tau-2},
\]
so \( r > 1 > g_{k,K} \left( \frac{\theta_{k,K}}{r} \right)^{\sigma^* - 1} = \frac{\theta_{k,K}}{\| B_{rK} B_{rE} \|} \) holds for every \( k \in \mathbb{N}^* \).

For \( z = (r^\sigma - 1)(sp^+ + p^- + p^0, r^\sigma - 1(sq^+ + q^- + q^0)) \in \partial Q \), we will verify \( I_{k,K}(z) \leq 0 \) holds for every \( k \in \mathbb{N}^* \). The proof is divided into several cases.

**Case 1.** If \( s = 0 \), then (H1) and (5) imply that \( I_{k,K}(z) \leq 0 \).

**Case 2.** If \( s \neq 0 \), then \( z \in \partial Q \) indicates that either \( s = r \) and \( \| (p^+ + p^0, q^- + q^0) \| \leq r \) or \( 0 < s \leq r \) and \( \| (p^- + p^0, q^- + q^0) \| = r \). So whatever the case is, we have \( 1 \leq \| \tilde{z} \| \leq 2 \), where \( \tilde{z} = (\tilde{p}, \tilde{q}) = \frac{r}{s}(sp^+ + p^- + p^0, sq^+ + q^- + q^0) \). We consider two subcases below.

**Subcase 1.** If \( \| \frac{1}{s}(sp^+ + p^0, sq^+ + q^0) \| < \| \frac{1}{s}(p^-, q^-) \| \), then (H1) and (5) imply that \( I_{k,K}(z) \leq 0 \).

**Subcase 2.** If Case 1 fails, set \( \Omega_z = \{ t \in [0, 2\pi] | \| \tilde{z}(t) \| \geq \varepsilon_1 \} \), then Lemma 3.1 shows that measure(\( \Omega_z \)) \( \geq \varepsilon_1 \). For \( t \in \Omega_z \), \( \frac{\sqrt{2}}{\varepsilon_1} \geq |z(t)| \geq r\varepsilon_1 > 1 \).

So (5) and (12) show that

\[
I_{k,K}(z) = A(z) - \int_0^{2\pi} H_{k,K}(t, z) dt \leq r^{\sigma^* + r} - \int_{\Omega_z} H_{k,K}(t, z) dt \leq 0.
\]

**Proof of Theorem 1.2.** We complete the proof in two steps.

**Step 1.** We will prove that the system (2) possesses a nontrivial classical 2\( \pi \)-periodic solution for every fixed \( k \in \mathbb{N}^* \).

The proof is similar to that of Theorem 1.3 in our paper [18].

Since \( H \) satisfies (H1), (H2), (H3)', (H4) and (H5), for every \( k \in \mathbb{N}^* \), \( H_{k,K} \) also satisfies those conditions, which leads to the following results: book [14] implies that \( I_{k,K} \in C^1(E, \mathbb{R}) \) satisfies (I1) and (I2) in Lemma 2.2, where \( A(z) = \frac{1}{2} \langle L_{\tilde{z}}, z \rangle \) and \( b_{k,K}(z) = -\int_0^{2\pi} H_{k,K}(t, z(t)) dt, z \in E \); Lemma 3.1 in our paper [18] indicates that \( I_{k,K} \) satisfies the (PS) condition; Lemma 3.4 in our paper [18] shows that \( I_{k,K} \) satisfies (I3) in Lemma 2.2 for \( r \) and \( \theta_{k,K} \) defined as in Lemma 3.3; and finally Lemma 3.3 and the arguments before it show that \( I_{k,K} \) satisfies (I4) inLemma 2.2. Thus Lemma 2.2 holds for \( I_{k,K} \), if \( H \) satisfies (H1), (H2), (H3)', (H4) and (H5).
Then there exists a critical point \( z_{k,K} \) of \( I_{k,K} \) with \( l_{k,K} = I_{k,K}(z_{k,K}) \geq \gamma_{k,K} > 0 \). Pages 40 and 41 in book [14] indicate that for every fixed \( k \in \mathbb{N}^* \), \( z_{k,K} \) is a nontrivial classical 2\( \pi \)-periodic solution of the system

\[
\begin{cases}
\dot{p} = -(H_{k,K})_1'(t,z), \\
\dot{q} = (H_{k,K})_2'(t,z).
\end{cases}
\]

We claim that for every fixed \( k \in \mathbb{N}^* \), there exists a constant \( K_0 = K_0(k) > 0 \) such that \( \| z_{k,K} \|_{L^\infty} \leq K_0 \), if \( K \geq K_0 \).

In fact, following the proof of Theorem 1.3 in our paper [18], \( I_{k,K}(z_{k,K}) = \inf_{h \in \Gamma} I_{k,K}(h(1,z)) \), where \( \Gamma \) is as in Lemma 2.2. Since \( \text{Id} \in \Gamma \), for \( z \in Q \), from (6) and (H1), we have

\[
I_{k,K}(z_{k,K}) \leq I_{k,K}(z) \leq A(z) \leq r^{\sigma + \tau}.
\]

Since \( H_{k,K} \) satisfies (H2) with coefficients independent of \( K \), then

\[
\left( \frac{1}{\mu} + \frac{1}{\nu} \right) r^{\sigma + \tau} \geq \left( \frac{1}{\mu} + \frac{1}{\nu} \right) I_{k,K}(z_{k,K})
\]

\[
= \left( \frac{1}{\mu} + \frac{1}{\nu} \right) I_{k,K}(z_{k,K}) - \frac{1}{\mu} I_{k,K}(z_{k,K})(p_{k,K}, 0) - \frac{1}{\nu} I_{k,K}(z_{k,K})(0, q_{k,K})
\]

\[
= \int_0^{2\pi} \left[ \frac{1}{\mu} (H_{k,K})_1'(t,z_{k,K}) \cdot p_{k,K} + \frac{1}{\nu} (H_{k,K})_2'(t,z_{k,K}) \cdot q_{k,K}
\right.
\]

\[
- \left( \frac{1}{\mu} + \frac{1}{\nu} \right) H_{k,K}(t,z_{k,K}) \right] dt
\]

\[
\geq \int_0^{2\pi} (c_{1,k} |\dot{z}_{k,K}|^2 - c_{2,k}) dt = c_{1,k} \| z_{k,K} \|_{L^2}^2 - 2\pi c_{2,k}
\]

\[
\geq c_{1,k} \tilde{C}_{\beta} \| z_{k,K} \|_{L^1}^{\beta} - 2\pi c_{2,k},
\]

where \( \tilde{C}_{\beta} \) is independent of \( K \).

By (H3)’ and the fact \( \dot{z}_{k,K} = J(H_{k,K})_2'(t,z_{k,K}) \), we see

\[
\left( \frac{1}{\xi} + \frac{1}{\eta} \right) r^{\sigma + \tau} \geq \left( \frac{1}{\xi} + \frac{1}{\eta} \right) I_{k,K}(z_{k,K})
\]

\[
= \int_0^{2\pi} \left[ \frac{1}{\xi} (H_{k,K})_1'(t,z_{k,K}) \cdot p_{k,K} + \frac{1}{\eta} (H_{k,K})_2'(t,z_{k,K}) \cdot q_{k,K}
\right.
\]

\[
- \left( \frac{1}{\xi} + \frac{1}{\eta} \right) H_{k,K}(t,z_{k,K}) \right] dt
\]

\[
\geq \int_0^{2\pi} (c_{1,k} |(H_{k,K})_2'(t,z_{k,K})| - c_{2,k}) dt = c_{1,k} \| \dot{z}_{k,K} \|_{L^1} - 2\pi c_{2,k}
\]

The above two estimates imply that for every \( k \in \mathbb{N}^* \), there exists a constant \( C(k) > 0 \) such that \( \| z_{k,K} \|_{L^1} \leq C(k) \) and \( \| \dot{z}_{k,K} \|_{L^1} \leq C(k) \).

For \( t \in [0, 2\pi] \), we have

\[
| z_{k,K}(t) - z_{k,K}(0) | \leq | z_{k,K}(t) - z_{k,K}(0) | = \left| \int_0^t \dot{z}_{k,K}(s) ds \right| \leq \int_0^{2\pi} | \dot{z}_{k,K}(s) | ds \leq C(k).
\]
Then
\[ |z_{k,K}(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |z_{k,K}(t)| \, dt + C(k) \leq \left( \frac{1}{2\pi} + 1 \right) C(k). \] (14)

Inequalities (13) and (14) imply that
\[ |z_{k,K}(t)| \leq \left( \frac{1}{2\pi} + 2 \right) C(k), \]
hence we verify that \( \|z_K\|_{L^\infty} \leq K_0 \) holds for some \( K_0 \) independent of \( K \).

The fact \( \|z_{k,K}\|_{L^\infty} \leq K_0 \) and the definition of \( H_{k,K} \) imply \( H_{k,K}(t, z_{k,K}) = H_k(t, z_k) \) holds for any \( K > K_0 \), then \( (H_{k,K})'_z(t, z_{k,K}) = (H_k)'_z(t, z_k) \), which implies that \( z_{k,K} \) is a \( 2\pi \)-periodic solution of the system (2).

For \( K \geq K_0 \) (\( K_0 \) depends on \( k \)), set \( z_k = z_{k,K} \) and \( l_k = I_{k,K}(z_k) > 0 \).

**Step 2.** We will prove that the system (1) possesses a distinct sequence of subharmonic solutions.

The proof is the same as Step 2 of the proof for Theorem 1.1. \( \square \)

4. **Multiplicity of periodic solutions.** Now we turn to the multiplicity of \( 2\pi \)-periodic solutions of the system (1). For this aim, we introduce a theorem coming from paper [5].

Let \( E \) be a Hilbert space with \( E = X \oplus Y \), where \( X \) and \( Y \) are infinite dimensional subspaces of \( E \). Assume we have sequences of finite dimensional subspaces \( X_m \subset X \) and \( Y_m \subset Y \) such that \( E_m = X_m \oplus Y_m \) and \( E = \bigcup_{m=1}^{\infty} E_m \). For \( m \) large enough, \( T_i : E \rightarrow E \ ) (\( i = 1, 2 \)) is linear, bounded and invertible, and satisfies \( T_i(E_m) = E_m \).

For constants \( \rho, r, M > 0 \) with \( \rho \|T_i^{-1}T_2y_1\| > \rho \) and for some fixed \( y_1 \in Y \) independent of \( m \) with \( \|y_1\| = 1 \), set \( D = \{v_1 + v_2 \in V_1 \oplus V_2 | v_1 \in V_1, v_2 \in V_2, \|v_1\| \leq r, \|v_2\| \leq M \} \) and \( S_0 = \{z \in Y | \|z\| = \rho \} \) where \( V_1 \oplus V_2 = \text{span}\{y_1\} \oplus X \) and \( V_1, V_2 \) are subspaces of \( E \).

Suppose that \( I \in C^1(E, \mathbb{R}) \), \( I \) satisfies the \((PS)^*\) condition with respect to \( \{E_m\} \) means that if a sequence \( \{z_m|z_m \in E_{N_m}\} \) satisfies that \( \{I|_{E_{N_m}}(z_m)\} \) is bounded and \( (I|_{E_{N_m}})'(z_m) \rightarrow 0 \) as \( m \rightarrow +\infty \), then \( \{z_m\} \) has a convergent subsequence, where \( \{N_m\} \) is a strictly increasing subsequence of \( \mathbb{N} \).

**Lemma 4.1** (See [5]). Let \( I \in C^1(E, \mathbb{R}) \) be an even functional and satisfy the \((PS)^*\) condition on \( E \) with respect to \( \{E_m\} \). Assume there exist constants \( u \leq v \) such that for \( m \) large enough, we have
\[
\begin{align*}
(I_1) & \quad \inf_{z \in T_1(E_m \cap S_0)} I(z) \geq u, \\
(I_2) & \quad \sup_{z \in T_2(E_m \cap \partial D)} I(z) < u, \text{ where } \partial D \text{ refers to the boundary of } V_1 \oplus V_2, \\
(I_3) & \quad \sup_{z \in T_3(E_m \cap D)} I(z) \leq v, 
\end{align*}
\]
then \( I \) has a critical value \( \theta \in [u, v] \).

For \( j \in \mathbb{N}^* \), set
\[
\begin{align*}
E_j^+ &= \text{span}\{(\sin jt)e_i - (\cos jt)e_{i+n}, (\cos jt)e_i + (\sin jt)e_{i+n}, 1 \leq i \leq n\}, \\
E_j^- &= \text{span}\{(\sin jt)e_i + (\cos jt)e_{i+n}, (\cos jt)e_i - (\sin jt)e_{i+n}, 1 \leq i \leq n\}, \\
E_m &= \oplus_{1 \leq j \leq m} (E_j^+ \oplus E_j^-) \oplus E^0.
\end{align*}
\]
Lemma 4.2. I satisfies the (PS)* condition on E with respect to \{E_m\}, if H satisfies (H1)-(H3).

Proof. The proof is similar to the proof of Lemma 3.1 in our paper [18]. Let \{z_m\}_{m \in E_{N_m}} be a (PS)* sequence, that is, \{I(z_m)\} is bounded and \((I_{|E_{N_m}})'(z_m)\to 0\) as \(m \to +\infty\). In the process, we regard \(I'(z)\) as an element in \(E\) and still write \((I'(z), \zeta)\) as \(I'(z)\zeta\), where \(z, \zeta \in E\). Let \(P_m : E \to E_{N_m}\) denote the projective operator, then for \(z, \zeta \in E_{N_m}\), we have \((I_{|E_{N_m}})'(z) = P_mI'(z)\) and \((I_{|E_{N_m}})'(z) = (P_mI'(z), \zeta) = I'(z)\zeta\).

Firstly, we show that \(\{z_m\}\) is bounded. If not, we may assume that \(\|z_m\| \to +\infty\) as \(m \to +\infty\).

Condition (H2) implies that
\[
\left(1 + \frac{1}{\nu}\right)I(z_m) - \frac{1}{\nu} (I_{|E_{N_m}})'(z_m)(p_m, 0) - \frac{1}{\nu} (I_{|E_{N_m}})'(z_m)(0, q_m)
\]
\[
= \left(1 + \frac{1}{\nu}\right)I(z_m) - \frac{1}{\nu} I'(z_m)(p_m, 0) - \frac{1}{\nu} I'(z_m)(0, q_m)
\]
\[
= \int_0^{2\pi} \left[ \frac{1}{\mu} H'_1(t, z_m) \cdot p_m + \frac{1}{\nu} H'_2(t, z_m) \cdot q_m - \left(1 + \frac{1}{\nu}\right)H(t, z_m) \right] dt
\]
\[
\geq \int_0^{2\pi} (c_1|z_m|^{\beta} - c_2) dt = c_1 \int_0^{2\pi} |z_m|^\beta dt - 2\pi c_2.
\]
Dividing both sides of (15) by \(\|z_m\|\), we have
\[
\int_0^{2\pi} \frac{|z_m|^\beta dt}{\|z_m\|} \to 0, \quad m \to +\infty.
\]

Let \(\alpha = \frac{\beta - 1}{\beta(\lambda - 1)}\), then \(\lambda > 1\) and \(\lambda \alpha - 1 = \alpha - \frac{1}{\beta}\). Condition (H3) indicates that there exists a constant \(c_3 > 0\) such that
\[
|H'_1(t, z)|^\alpha \leq c_3 + c_3|z|^{\lambda \alpha}, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}.
\]
Then we obtain
\[
(I_{|E_{N_m}})'(z_m)z_m^+ = I'(z_m)z_m^+
\]
\[
= A'(z_m)z_m^+ - \int_0^{2\pi} H'_1(t, z_m) \cdot z_m^+ dt
\]
\[
\geq 2\|z_m^+\|^2 - |H'_1(t, z_m)| L^\alpha \cdot \|z_m^+\|_{\frac{\beta-1}{\beta}}
\]
\[
\geq 2\|z_m^+\|^2 - C(\alpha)|H'_1(t, z_m)| L^\alpha \cdot \|z_m^+\|_{\frac{\beta-1}{\beta}}
\]
and
\[
\int_0^{2\pi} |H'_1(t, z_m)|^\alpha dt \leq \int_0^{2\pi} (c_3 + c_3|z_m|^\lambda \alpha) dt
\]
\[
\leq 2\pi c_3 + c_3\|z_m\| L^\beta \left(\int_0^{2\pi} |z_m|^\lambda |z_m|^{\lambda \alpha - 1} dt \right)^{\frac{\beta-1}{\beta}}
\]
\[
\leq 2\pi c_3 + C(\lambda, \alpha, \beta)\|z_m\| L^\beta \cdot \|z_m\|^{\lambda \alpha - 1},
\]
where \(C(\alpha)\) and \(C(\lambda, \alpha, \beta)\) are embedding constants in Lemma 2.1.

From (18) and (16), we have
\[
\frac{|H'_1(t, z_m)| L^\alpha}{\|z_m\|} \leq \left(\frac{2\pi c_3 + C(\lambda, \alpha, \beta)\|z_m\| L^\beta \cdot \|z_m\|^{\lambda \alpha - 1}}{\|z_m\|^\alpha} \right)^{\frac{1}{\alpha}} \to 0, \quad m \to +\infty.
\]
Consequently, dividing both sides of (17) by \( \|z_m\| + \|z_m^+\| \), we have

\[
\frac{\|z_m^+\|}{\|z_m\|} \to 0, \quad m \to +\infty. \tag{19}
\]

Similarly, computing \((I|_{E_{N_m}})'(z_m)(-z_m^-)\), we have

\[
\frac{\|z_m^-\|}{\|z_m\|} \to 0, \quad m \to +\infty. \tag{20}
\]

Condition (H2) implies that there exists a constant \( c_4 > 0 \) such that

\[
\frac{1}{\mu} \frac{d}{dt} (t \cdot z) + \frac{1}{\nu} \frac{d}{dt} (t \cdot q) - \frac{1}{\mu} H(t, z) \geq c_1 |z| - c_4, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}.
\]

Then we have

\[
\frac{1}{\mu} I'(z_m) - \frac{1}{\nu} (I|_{E_{N_m}})'(z_m)(p_m, 0) - \frac{1}{\nu} (I|_{E_{N_m}})'(z_m)(0, q_m)
\geq \int_0^{2\pi} (c_1 |z_m| - c_4) \frac{dt}{\mu} \geq c_1 \int_0^{2\pi} (|z_m^0| - |z_m^+| - |z_m^-|) dt - 2\pi c_4 \tag{21}
\]

where \( C_1 \) is the embedding constant.

From (19)-(21), we see

\[
\frac{\|z_m^0\|}{\|z_m\|} \to 0, \quad m \to +\infty.
\]

Then a contradiction is obtained from

\[
1 = \frac{\|z_m\|}{\|z_m\|} \leq \frac{\|z_m^+\| + \|z_m^0\| + \|z_m^-\|}{\|z_m\|} \to 0, \quad m \to +\infty.
\]

Next, we show that \( \{z_m\} \) has a convergent subsequence. Let \( D_m : E_{N_m} \to E_{N_m}^* \) denote the duality mapping between \( E_{N_m} \) and its dual \( E_{N_m}^* \), then \((D_m z) \zeta = \int_0^{2\pi} (-J \zeta) \cdot \zeta dt, \zeta \in E_{N_m}^*\).

Set \( T_m^\pm = -P_m^\pm D_m P_m^\pm \), then \( D_m^{-1}(I|_{E_{N_m}})'(z_m) = z_m + D_m^{-1} P_m b'(z_m) \) and

\[
\begin{align*}
\ast_m^+ &= P_m^+ D_m^{-1}(I|_{E_{N_m}})'(z_m) - P_m^+ D_m^{-1} P_m b'(z_m) \\
&= P_m^+ D_m^{-1}(I|_{E_{N_m}})'(z_m) + T_m^+(z_m).
\end{align*}
\]

From the fact that \( \{z_m\} \) is bounded, \((I|_{E_{N_m}})'(z_m) \to 0 \ (m \to +\infty)\), and \( \{T_m^+(z_m)\} \) has a convergent subsequence, we conclude that \( \{z_m^\pm\} \) has a convergent subsequence. Furthermore, the fact that \( E^0 \) has finite dimension implies \( \{z_m^0\} \) also has a convergent subsequence. Thus \( \{z_m\} \) has a convergent subsequence. \( \square \)

Fix \( k \in \mathbb{N}^* \) sufficiently large from now on, for \( m \geq k \), choose arbitrary \( y_1 = (p_1, q_1) \in E_{k}^* \) with \( \|y_1\| = 1 \) and set

\[
\begin{align*}
X &= E^0 \oplus E^- \oplus \oplus_{1 \leq j \leq k-1} E_j^+, \quad Y = \oplus_{j \geq k} E_j^+, \\
X_m &= \oplus_{1 \leq j \leq m} E^- \oplus E^0 \oplus \oplus_{1 \leq j \leq k-1} E_j^+, \quad Y_m = \oplus_{k \leq j \leq m} E_j^+,
\end{align*}
\]

then \( E_m = X_m \oplus Y_m \) and \( E = X \oplus Y = \bigcup_{m=1}^{\infty} E_m. \)
Lemma 4.3. If $H$ satisfies (H3), then there exist $u_k > 0$ and $\rho_k > 0$, both of which are independent of $m$, such that
\[
\inf_{z \in T_k(E_m \cap S_{\rho_k})} I(z) \geq u_k \text{ and } u_k \to +\infty \text{ as } k \to +\infty, \text{ where } T_k = Id \text{ and } S_{\rho_k} = \{ z \in Y \|z\| = \rho_k \}.
\]

Proof. The proof comes from paper [5], but need some modifications. To the readers’ convenience, we list it here.

Paper [5] indicates that
\[
\|z\| \geq \sqrt{k}\|z\|_{L^2}, \quad z \in Y = \overline{\oplus_{j \geq k} E_j^+},
\]
so there exists an embedding constant $C(\lambda) > 0$ such that
\[
\|z\|_{L^2}^{\lambda,1} \leq \|z\|_2 \cdot \|z\|_1^{\lambda,2} \leq \frac{C(\lambda)}{\sqrt{k}} \|z\|^{\lambda,1}.
\]

Via (H3), we can find a constant $c_5 > 0$ satisfying
\[
H(t, z) \leq c_5 (1 + |z|^{\lambda,1}), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}.
\]
For $z \in E_m \cap S_{\rho_k} = Y_m \cap S_{\rho_k} \subset E^+$, (23) and (22) indicate that
\[
I(z) = A(z) - \int_0^{2\pi} H(t, z)dt \\
\geq \|z\|^2 - 2\pi c_5 (1 + \|z\|^{\lambda,1}) \\
\geq \|z\|^2 - 2\pi c_5 \left(1 + \frac{C(\lambda)}{\sqrt{k}} \|z\|^{\lambda,1}\right) \\
= \rho_k^2 \left(1 - \frac{2\pi c_5 C(\lambda)}{\sqrt{k}} \rho_k^{-1}\right) - 2\pi c_5.
\]
Let $\frac{2\pi c_5 C(\lambda)}{\sqrt{k}} \rho_k^{-1} = \frac{1}{2}$, then $\rho_k = \left(\frac{\sqrt{k}}{4\pi c_5 C(\lambda)}\right)^{\frac{1}{\lambda+1}}$. Set $u_k = \frac{1}{2}\rho_k^2 - 2\pi c_5$, then (24) implies that
\[
\inf_{z \in T_k(E_m \cap S_{\rho_k})} I(z) \geq u_k = \frac{1}{2} \left(\frac{\sqrt{k}}{4\pi c_5 C(\lambda)}\right)^{\frac{1}{\lambda+1}} - 2\pi c_5 \to +\infty, \quad k \to +\infty.
\]

\[\square\]

Lemma 4.4. For every $k \in \mathbb{N}^\ast$, there exists a number $\varepsilon_k > 0$ such that
\[
\text{measure } \{ t \in [0, 2\pi] \|z(t)\| \geq \varepsilon_k \} \geq \varepsilon_k
\]
holds for any $z \in W_k := \{ z \in E^0 \oplus E^- \oplus \oplus_{1 \leq j \leq k} E_j^+ | 1 \leq \|z\| \leq 2, \|z^-\| \leq \|z^+ + z^0\| \}$. 

Proof. The idea comes from paper [3].

For $z \in W_k$, we have $1 \leq \|z\|^2 = \|z^+ + z^0\|^2 + \|z^-\|^2 \leq 2\|z^+ + z^0\|^2$, then
\[
\|z^+ + z^0\|^2 \geq \frac{1}{2}.
\]

If Lemma 4.4 is false, then for $i \in \mathbb{N}^\ast$, there exists $z_i \in W_k$ such that
\[
\text{measure } \{ t \in [0, 2\pi] \|z_i(t)\| \geq \frac{1}{i} \} < \frac{1}{i}.
\]

Since the dimension of $\oplus_{1 \leq j \leq k} E_j^+ \oplus E^0$ is finite and $\|z_i^+ + z_i^0\|, \|z_i^-\| \leq 2$, there exists a subsequence $\{z_{i_j}\}$ such that
\[
z_{i_j}^+ + z_{i_j}^0 \to z^+ + z^0 \oplus \oplus_{1 \leq j \leq k} E_j^+ \oplus E^0 \quad \text{and} \quad z_{i_j}^- \to z^-, \quad i \to +\infty.
\]
Hence, \( z_i \to z = z^+ + z^0 + z^- \in E \) as \( i \to +\infty \), from which we get
\[
z_i \to z \text{ in } L^2(S^1, \mathbb{R}^{2n}), \quad i \to +\infty. \tag{27}
\]

From (25), we see \( \|z\| \neq 0 \), so
\[
\|z\|^2_{L^2} \neq 0. \tag{28}
\]

We claim there exists \( \delta_1 > 0 \) such that measure \( \{ t \in [0, 2\pi] \mid |z(t)| \geq \delta_1 \} > 0 \). If not, for any \( l \), measure \( \{ t \in [0, 2\pi] \mid |z(t)| \geq \frac{1}{l} \} = 0 \). A contradiction with (28) is obtained from
\[
\int_0^{2\pi} |z(t)|^2 dt \leq \frac{2\pi}{l^2} \to 0, \quad l \to +\infty.
\]
So there exists a constant \( \delta_2 > 0 \) such that
\[
\text{measure } \{ t \in [0, 2\pi] \mid |z(t)| \geq \delta_1 \} > \delta_2. \tag{29}
\]
Set \( \Omega_0 = \{ t \in [0, 2\pi] \mid |z(t)| \geq \delta_1 \} \), \( \Omega_i = \{ t \in [0, 2\pi] \mid |z(t)| < \frac{1}{l} \} \) and \( \Omega_i \) then (29) and (26) show that
\[
\text{measure}(\Omega_0 \cap \Omega_i) = \text{measure}(\Omega_0 \setminus (\Omega_0 \cap \Omega_i))
\]
\[
= \text{measure}(\Omega_0) - \text{measure}(\Omega_0 \cap \Omega_i) \geq \delta_2 - \frac{1}{l}.
\]
Let \( i \) be large enough such that \( \delta_i - \frac{1}{i} \geq \frac{1}{2}\delta_i \) holds, then we have
\[
|z_i(t) - z(t)|^2 \geq \left( \delta_i - \frac{1}{i} \right)^2 \geq \left( \frac{1}{2}\delta_i \right)^2, \quad t \in \Omega_0 \cap \Omega_i.
\]
A contradiction with (27) is obtained from
\[
\int_0^{2\pi} |z_i - z|^2 dt \geq \int_{\Omega_0 \cap \Omega_i} |z_i - z|^2 dt
\]
\[
\geq \frac{1}{2}\delta_i^2 \cdot \text{measure}(\Omega_0 \cap \Omega_i) \geq \left( \frac{1}{2}\delta_i \right)^2 \cdot \left( \frac{1}{2}\delta_2 \right).
\]

\( \square \)

Set \( V_1 = \oplus_{1 \leq j \leq k} E_j^+ \) and \( V_2 = E^0 \oplus E^- \), then \( V_1 \oplus V_2 = \text{span}\{ y_1 \} \oplus X \).

Set \( T_1(p, q) = (p, q) \) and \( T_{2,k}(p, q) = (r_k^{-1}p, r_k^{-1}q) \), where \( (p, q) \in E \) and \( r_k > 0 \) is determined later. Paper [4] indicates that \( T_{2,k}(E_m) = E_m \) (\( m \in \mathbb{N}^* \)).

**Lemma 4.5.** If \( H \) satisfies (H5), then there exist numbers \( u_k, v_k \) and \( r_k \) independent of \( m \) satisfying \( v_k \geq u_k \) and \( r_k > \frac{\|v_k\|}{\|T_{2,k}y_1\|} \) such that
\[
\sup_{z \in T_{2,k}(E_m \cap \partial D_k)} I(z) \leq 0 \quad \text{and} \quad \sup_{z \in T_{2,k}(E_m \cap D_k)} I(z) \leq v_k,
\]
where \( m \geq k, D_k = \{ z \in V_2 \oplus \oplus_{1 \leq j \leq k} E_j^+ \mid \|z^0 + z^-\| \leq r_k, \|z^+\| \leq r_k \} \).

**Proof.** Proceeding as in Lemma 3.1, for every \( k \in \mathbb{N}^* \) large enough, condition (H5) shows that there is a number \( F_k \geq 1 \) such that
\[
H(t, z) \geq G_k \left( \|p\|^{1+\frac{\sigma}{2}} + |q|^{1+\frac{\sigma}{2}} \right), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n} \text{ with } |z| \geq F_k, \tag{30}
\]
where \( G_k = \frac{\sqrt{2}}{\varepsilon_k \min \left( \left( \frac{z_k^+}{\varepsilon_k} \right)^{1+\frac{\sigma}{2}}, \left( \frac{z_k^-}{\varepsilon_k} \right)^{1+\frac{\sigma}{2}} \right) } \) with \( \varepsilon_k \) being as in Lemma 4.4.

Fix \( r_k \geq \frac{\|v_k\|}{\|T_{2,k}y_1\|} + \varepsilon_k + 1 \), then \( \varepsilon_k + 1 < r_k \) and
\[
\|T_{1}^{-1}T_{2,k}y_1\|^2 = \| (r_k^{-1}p_1^+, r_k^{-1}q_1^-) \|^2 = r_k^{\sigma + r - 2},
\]
so \( r_k > \varrho_k \left( \frac{q_k}{r_k} \right)^{\frac{1}{\theta_k}} \).

For \( z = (r_k^{-1}(p^+ + p^- + q^0), r_k^{-1}(q^+ + q^- + q^0)) \in T_{2,k} (E_m \cap \partial D_k) \), we will verify \( I(z) \leq 0 \), where \((p^+, q^+) \in \Theta_1 \leq \Theta_2 \), \((p^-, q^-) \in E^- \) and \((p^0, q^0) \in E^0 \).

\( z \in T_{2,k} (E_m \cap \partial D_k) \) indicates that either \( \| (p^+, q^+) \| = r_k \) and \( \| (p^-, q^- + q^0) \| \leq r_k \) or \( \| (p^-, q^-) \| \leq r_k \) and \( \| (p^+, q^+ + q^0) \| = r_k \). So whatever the case is, we have \( 1 \leq \| z \| \leq 2 \), where \( z = (\bar{p}, \bar{q}) = \frac{1}{r_k} (p^+ + p^- + q^+ + q^- + q^0) \). We consider two cases below.

**Case 1.** If \( \frac{1}{r_k} (p^+ + p^- + q^+ + q^0) < \frac{1}{r_k} (p^-, q^-) \), then (H1) and (5) imply that \( I(z) \leq 0 \).

**Case 2.** If Case 1 fails, we set \( \Omega_z = \{ t \in [0, 2\pi] \| \bar{z}(t) \| \geq \epsilon_k \} \), then Lemma 4.4 shows that measure(\( \Omega_z \)) \( \geq \epsilon_k \). For \( t \in \Omega_z \), \( H(t, z(t)) \) \( \geq G_k \left( \| r_k^{-1}(p^+(t) + p^-(t) + p^0) \|^{1+\frac{r}{\theta_k}} + \| r_k^{-1}(q^+(t) + q^-(t) + q^0) \|^{1+\frac{r}{\theta_k}} \right) \)

\[ \geq G_k \left( \frac{\epsilon_k}{\sqrt{2}} \left| \frac{\sqrt{2}}{\epsilon_k} \bar{p}(t) \right|^{1+\frac{r}{\theta_k}} + \frac{\sqrt{2}}{\epsilon_k} \left| \frac{\epsilon_k}{\sqrt{2}} \bar{q}(t) \right|^{1+\frac{r}{\theta_k}} \right) \]

\[ \geq G_k \min \left( \frac{\epsilon_k}{\sqrt{2}}, \frac{\sqrt{2}}{\epsilon_k} \right) \left( \left| \frac{\sqrt{2}}{\epsilon_k} \bar{p}(t) \right|^{1+\frac{r}{\theta_k}} + \left| \frac{\sqrt{2}}{\epsilon_k} \bar{q}(t) \right|^{1+\frac{r}{\theta_k}} \right) \]

From (31), (30), Remark 1 and the choice of \( G_k \), we have

\[ H(t, z(t)) \geq G_k \left( \left| \frac{\sqrt{2}}{\epsilon_k} \bar{p}(t) \right|^{1+\frac{r}{\theta_k}} + \left| \frac{\sqrt{2}}{\epsilon_k} \bar{q}(t) \right|^{1+\frac{r}{\theta_k}} \right) \]

\[ \geq G_k \min \left( \frac{\epsilon_k}{\sqrt{2}}, \frac{\sqrt{2}}{\epsilon_k} \right) \left( \left| \frac{\sqrt{2}}{\epsilon_k} \bar{p}(t) \right|^{1+\frac{r}{\theta_k}} + \left| \frac{\sqrt{2}}{\epsilon_k} \bar{q}(t) \right|^{1+\frac{r}{\theta_k}} \right) \]

\[ \geq G_k \epsilon_k^{1+\frac{r}{\theta_k}} \min \left( \frac{\epsilon_k}{\sqrt{2}}, \frac{\sqrt{2}}{\epsilon_k} \right) \left( \left| \frac{\sqrt{2}}{\epsilon_k} \bar{p}(t) \right|^{1+\frac{r}{\theta_k}} + \left| \frac{\sqrt{2}}{\epsilon_k} \bar{q}(t) \right|^{1+\frac{r}{\theta_k}} \right) \]

\[ \geq G_k \epsilon_k^{1+\frac{r}{\theta_k}} \left( \left| \frac{\sqrt{2}}{\epsilon_k} \bar{p}(t) \right|^{1+\frac{r}{\theta_k}} + \left| \frac{\sqrt{2}}{\epsilon_k} \bar{q}(t) \right|^{1+\frac{r}{\theta_k}} \right) \]

So (5) and (32) show that

\[ I(z) = A(z) - \int_0^{2\pi} H(t, z) dt \leq r_k^{\sigma^+ + \tau} - \int_{\Omega_z} H(t, z) dt \leq 0. \]

Since \( T_{2,k} (E_m \cap D_k) \) is bounded, closed and \( \dim E_m \) is finite, the proof is complete if we take

\[ v_k > \max \left\{ u_k, \sup_{z \in T_{2,k} (E_m \cap D_k)} I(z) \right\}. \]

**Proof of Theorem 1.3.** For every fixed \( k \in \mathbb{N}^* \) large enough, Lemmas 4.2, 4.3 and 4.5 show that Lemma 4.1 holds, so there exists a critical value \( \theta_k \in [u_k, v_k] \) with the corresponding critical point \( z_k \). Pages 40 and 41 in book [14] show that \( z_k \) is a nontrivial \( 2\pi \)-periodic solution of the system (1).
The equation \( \dot{z}_k = JH'_z(t, z_k) \) and (H3) indicate that
\[
A(z_k) = \frac{1}{2} \int_0^{2\pi} (-J\dot{z}_k, z_k) dt
= \frac{1}{2} \int_0^{2\pi} (H'_z(t, z_k), z_k) dt
\leq \frac{1}{2} \int_0^{2\pi} |H'_z(t, z_k)| \cdot |z_k| dt
\leq \frac{1}{2} \int_0^{2\pi} c_2 \left( |z_k|^\lambda + 1 \right) |z_k| dt
\leq \frac{1}{2} \int_0^{2\pi} \pi \left( \|z_k\|_{L^\infty}^\lambda + 1 \right) \|z_k\|_{L^\infty} dt.
\]

Since \( u_k = \frac{1}{2} \left( \frac{\pi}{4\pi c_5 C(\lambda)} \right)^{\frac{2}{\lambda+1}} - 2\pi c_5 \) and \( u_k \leq I(z_k) \leq A(z_k) \), estimate (33) shows that if \( k \) is large enough, then there exists a constant \( c > 0 \) such that
\[
\|z_k\|_{L^\infty} \geq ck^{\frac{1}{\lambda+1}} \rightarrow +\infty, \quad k \rightarrow +\infty.
\]

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