Crossed modules
and the homotopy 2-type of a free loop space

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Abstract
The question was asked by Niranjan Ramachandran: how to describe the fundamental
groupoid of $LX$, the free loop space of a space $X$? We show how this depends on the
homotopy 2-type of $X$ by assuming $X$ to be the classifying space of a crossed module
over a group, and then describe completely a crossed module over a groupoid determining
the homotopy 2-type of $LX$; that is we describe crossed modules representing the 2-type
of each component of $LX$. The method requires detailed information on the monoidal
closed structure on the category of crossed complexes.1

1 Introduction
It is well known that for a connected $CW$-complex $X$ with fundamental group $G$ the set of
components of the free loop space $LX$ of $X$ is bijective with the set of conjugacy classes of the
group $G$, and that the fundamental groups of $LX$ fit into a family of exact sequences derived
from the fibration $LX \to X$ obtained by evaluation at the base point.

Our aim is to describe the homotopy 2-type of $LX$, the free loop space on $X$, when $X$ is
a connected $CW$-complex, in terms of the 2-type of $X$. Weak homotopy 2-types are described
by crossed modules (over groupoids), defined in [BH81b] as follows.

A crossed module $M$ is a morphism $\delta : M \to P$ of groupoids which is the identity on objects
such that $M$ is just a disjoint union of groups $M(x), x \in P_0$, together with an action of $P$ on

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$M$ written $(m, p) \mapsto m^p$, $m \in M(x), p : x \to y$ with $m^p \in M(y)$ satisfying the usual rules for an action. We find it convenient to use (non-commutative) additive notation for composition so if $p : x \to y, q : y \to z$ then $p + q : x \to z$, and $(m + n)^p = m^p + n^p, (m^p)^q = m^{p+q}, m^0 = m$. Further we have the two crossed module rules for all $p \in P, m, n \in M$:

CM1) $\delta(m^p) = -p + \delta m + p$;

CM2) $-n + m + n = m^{\delta n}$;

whenever defined. This is a crossed module of groups if $P_0$ is a singleton.

A crossed module $\mathcal{M}$ as above has a simplicial nerve $K = N^\Delta \mathcal{M}$ which in low dimensions is described as follows:

- $K_0 = P_0$;
- $K_1 = P$;
- $K_2$ consists of quadruples $\sigma = (m; c, a, b)$ where $m \in M, a, b, c \in P$ and $\delta m = -c + a + b$ is well defined;
- $K_3$ consists of quadruples $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ where $\sigma_i \in K_2$ and the $\sigma_i$ make up the faces of a 3-simplex, as shown in the following diagrams:

```
\begin{tikzpicture}
    \node (0) at (0,0) {0};
    \node (a) at (1,0) {1};
    \node (1) at (0.5,0.866) {1};
    \node (2) at (0.5,-0.866) {2};
    \node (3) at (1,2) {3};
    \node (c) at (0.5,0) {c};
    \node (d) at (0,1) {d};
    \node (e) at (1,1) {e};
    \node (m0) at (0.5,1.732) {m_0};
    \node (m1) at (0.5,-1.732) {m_1};
    \node (m2) at (1.5,1) {m_2};
    \node (m3) at (0.5,1.732-1.732) {m_3};

    \draw (0) -- (a);
    \draw (0) -- (c);
    \draw (c) -- (2);
    \draw (2) -- (a);
    \draw (2) -- (1);
    \draw (1) -- (a);
    \draw (1) -- (3);
    \draw (3) -- (a);
    \draw (a) -- (1);
    \draw (a) -- (2);
    \draw (1) -- (3);
    \draw (2) -- (3);
    \draw (0) -- (1);
    \draw (0) -- (2);
    \draw (0) -- (3);

    \node at (0.5,1.732) {\delta m_2 = -d + a + e};
\end{tikzpicture}
```

providing we have the rules

$$\mu m_0 = -e + b + f, \quad \mu m_1 = -d + c + f,$$
$$\mu m_2 = -d + a + e, \quad \mu m_3 = -c + a + b,$$

together with the rule

$$(m_3)^f - m_0 - m_2 + m_1 = 0.$$

You may like to verify that these rules are consistent.

A crossed module is the dimension 2 case of a crossed complex, the definition of which in the single vertex case goes back to Blakers in [Bla48], there called a ‘group system’, and in
the many vertex case is in [BH81b]. The definition of the nerve of a crossed complex $C$ in the one vertex case is also in [Bla48], and in the general case is in [Ash88, BH91]. An alternative description of $K$ is that $K_n$ consists of the crossed complex morphisms $\Pi\Delta^n \to M$ where $\Pi\Delta^n$ is the fundamental crossed complex of the $n$-simplex, with its skeletal filtration, and $M$ is also considered as a crossed complex trivial in dimensions $> 2$. This shows the analogy with the Dold-Kan theorem for chain complexes and simplicial abelian groups, [Dol58].

We thus define the classifying space $BM$ of $M$ to be the geometric realisation $|N\Delta M|$, a special case of the definition in [BH91]. It follows that an $a \in P(x)$ for some $x \in P_0$ determines a 1-simplex in $X = BM$ which is a loop and so a map $a' : S^1 \to BM$, i.e. $a' \in LX$.

The chief properties of $X = BM$ are that $\pi_0(X) \cong \pi_0(P)$ and for each $x \in P_0$

$$\pi_i(X, x) \cong \begin{cases} 
\text{Cok}(\delta : M(x) \to P(x)) & \text{if } i = 1, \\
\text{Ker}(\delta : M(x) \to P(x)) & \text{if } i = 2, \\
0 & \text{if } i > 2.
\end{cases}$$

Further if $Y$ is a CW-complex, then there is a crossed module $\mathcal{M}$ and a map $Y \to BM$ inducing isomorphisms of $\pi_0, \pi_1, \pi_2$. For an exposition of some basic facts on crossed modules and crossed complexes in relation to homotopy theory, see for example [Bro99]. There are other versions of the classifying space, for example the cubical version given in [BHS10], and one for crossed module of groups using the equivalence of these with groupoid objects in groups, see for example [Lod82, BS08]. However the latter have not been shown to lead to the homotopy classification Theorem 1.6 below.

Our main result is:

**Theorem 1.1** Let $\mathcal{M}$ be the crossed module of groups $\delta : M \to P$ and let $X = BM$ be the classifying space of $\mathcal{M}$. Then the components of $LX$, the free loop space on $X$, are determined by equivalence classes of elements $a \in P$ where $a, b$ are equivalent if and only if there are elements $m \in M, p \in P$ such that $b = p + a + \delta m - p$.

Further the homotopy 2-type of a component of $LX$ given by $a \in P$ is determined by the crossed module of groups $LM[a] = (\delta_a : M \to P(a))$ where

(i) $P(a)$ is the group of elements $(m, p) \in M \times P$ such that $\delta m = [a, p]$, with composition $(n, q) + (m, p) = (m + np, q + p)$;

(ii) $\delta_a(m) = (-m^a + m, \delta m)$, for $m \in M$;

(iii) the action of $P(a)$ on $M$ is given by $n^{(m, p)} = np$ for $n \in M, (m, p) \in P(a)$. 

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In particular \( \pi_1(LX, a) \) is isomorphic to \( \text{Cok}\delta \), and \( \pi_2(LX, a) \cong \pi_2(X, *)^a \), the elements of \( \pi_2(X, *) \) fixed under the action of \( \overline{a} \), the class of \( a \) in \( G = \pi_1(X, *) \).

We give a detailed proof that \( LM[a] \) is a crossed module in Appendix 2.

**Remark 1.2** The composition in (i) can be seen geometrically in the following diagram:

\[
\begin{array}{ccc}
q & n & q \\
p & m & p \\
a & & a
\end{array}
\]

\[
q + p\quad m + np\quad q + p
\]

\[
\begin{array}{ccc}
1 & 2
\end{array}
\]

The following examples are due to C.D. Wensley.

**Example 1.3** \( \delta = 0 : M \to P \), so that \( M \) is a \( P \)-module. Then \( P(a) \) is the set of \((m, p)\) s.t. \([a, p] = 0\), i.e. \( p \in C_a(P) \), and so is \( M \rtimes C_a(P) \). (\( P = G \) the fundamental group, as \( \delta = 0 \)). But \( \delta_a(m) = (-m^a + m, 0) \). So \( \pi_1(LM, a) = (M/[a, M]) \rtimes C_a(P) \).

**Example 1.4** If \( a \in Z(P) \), the center of \( P \), then \([a, p] = 0\) for all \( p \). (For example, \( P \) might be abelian.) Hence \( P(a) = \pi \rtimes P \). Then \( \pi_1(LM, a) = (\pi \rtimes P)/\{(-m^a + m, \delta m) \mid m \in M\} \).

It is not clear to me that even in this case the exact sequence splits. (??)

It is also possible to give a less explicit description of \( \pi_1(LX, a) \) as part of an exact sequence:

**Theorem 1.5** Under the circumstances of Theorem 1.1, if we set \( \pi = \text{Ker}\delta = \pi_2(X), G = \text{Cok}\delta = \pi_1(X) \), with the standard module action of \( G \) on \( \pi \), then the fundamental group \( \pi_1(LX, a) \) in the component given by \( a \in P \) is part of an exact sequence:

\[
0 \to \pi^a \to \pi \to \pi/\{\overline{a}\} \to \pi_1(LX, a) \to C_a(G) \to 1
\]

(1)

where: \( \pi/\{\alpha\} \) denotes \( \pi \) with the action of \( \alpha \) killed; and \( C_a(G) \) denotes the centraliser of the element \( \alpha \in G \).

The proof of Theorem 1.1, which will be given in Section 2, is essentially an exercise in the use of the following classification theorem [BH91, Theorem A]:

**Theorem 1.6** Let \( Y \) be a CW-complex with its skeletal filtration \( Y_* \) and let \( C \) be a crossed complex, with its classifying space written \( BC \). Then there is a natural weak homotopy equivalence

\[
B(\text{CRS}(\Pi Y_*, C)) \to (BC)^Y.
\]
In the statement of this theorem we use the internal hom \( \text{CRS}(\cdot, \cdot) \) in the category \( \text{Crs} \) of crossed complexes: this internal hom is described explicitly in [BH87], in order to set up the exponential law

\[
\text{Crs}(A \otimes B, C) \cong \text{Crs}(A, \text{CRS}(B, C))
\]

for crossed complexes \( A, B, C \), i.e. to give a monoidal closed structure on the category \( \text{Crs} \). Note that \( \text{CRS}(B, C)_0 = \text{Crs}(B, C) \), \( \text{CRS}(B, C)_1 \) gives the homotopies of morphisms, and \( \text{CRS}(B, C)_n \) for \( n \geq 2 \) gives the higher homotopies.

## 2 Proofs

We deduce Theorem 1.1 from the following Theorem.

**Theorem 2.1** Let \( X = BM \), where \( M \) is the crossed module of groups \( \delta : M \to P \). Then the homotopy 2-type of \( LX \), the free loop space of \( X \), is described by the crossed module over groupoids \( LM \) where

(i) \( (LM)_0 = P \);

(ii) \( (LM)_1 = M \times P \times P \) with source and target given by

\[
s(m, p, a) = p + a + \delta m - p, \quad t(m, p, a) = a
\]

for \( a, p \in P, m \in M \);

(iii) the composition of such triples is given by

\[
(n, q, b) + (m, p, a) = (m + n^p, q + p, a)
\]

which of course is defined under the condition that

\[
b = p + a + \delta m - p
\]

or, equivalently, \( b^p = a + \delta m \);

(iv) if \( a \in P \) then \( (LM)_2(a) \) consists of pairs \( (m, a) \) for all \( m \in M \), with addition and boundary

\[
(m, a) + (n, a) = (m + n, a), \quad \delta(m, a) = (-m^a + m, \delta m, a);
\]

(v) the action of \( (LM)_1 \) on \( (LM)_2 \) is given by: \( (n, b)^{(m,p,a)} \) is defined if and only if \( b^p = a + \delta m \) and then its value is \( (n^p, a) \).
Proof In Theorem 1.6 we set \( Y = S^1 \) with its standard cell structure \( e^0 \cup e^1 \), and can write \( \Pi Y_* \cong \mathbb{K}(\mathbb{Z}, 1) \) where the latter is the crossed complex with a base point \( z_0 \) and a free generator \( z \) in dimension 1, and otherwise trivial. Thus morphisms of crossed complexes from \( \mathbb{K}(\mathbb{Z}, 1) \), and homotopies and higher homotopies of such morphisms, are completely determined by their values on \( z_0 \) and on \( z \).

A crossed module over a group or groupoid is also regarded as a crossed complex trivial in dimensions > 2.

All the formulae required to prove Theorem 2.1 follow from those for the internal hom \( \text{CRS} \) on the category \( \text{Crs} \) given in [BH87, Proposition 3.14] or [BHS10, §7.1.vii, §9.3].

We set \( LM = \text{CRS}(\mathbb{K}(\mathbb{Z}, 1), \mathcal{M}) \).

Since \( \mathbb{K}(\mathbb{Z}, 1) \) is a free crossed complex with one generator \( z \) in dimension 1, the elements \( a \in P \) are bijective with the morphisms \( f : \mathbb{K}(\mathbb{Z}, 1) \to \mathcal{M} \), and we write this bijection as \( a \mapsto \hat{a} \), where \( a = \hat{a}(z) \). Also the homotopies and higher homotopies from \( \mathbb{K}(\mathbb{Z}, 1) \to \mathcal{M} \) are determined by their values on \( z \) and on the element \( z_0 \) of \( \mathbb{K}(\mathbb{Z}, 1) \) in dimension 0. Thus a 1-homotopy \( (h, \hat{a}) : \hat{b} \simeq \hat{a} \) is such that \( h \) lifts dimension by 1, and is given by elements \( p = h(z_0) \in P, m = h(z) \in M \) and so \( (h, \hat{a}) \) is given by a triple \( (p, m, a) \). The condition that this triple gives a homotopy \( \hat{b} \simeq \hat{a} \) translates to

\[
b = p + a + \delta m - p
\]

or, equivalently, \( a + \delta m = b^p \). It follows easily that \( \hat{b}, \hat{a} \) belong to the same component of \( LM \) if and only if \( b, a \) give conjugate elements in the quotient group \( \pi_1(\mathcal{M}) \). (The use of such general homotopies was initiated in [Whi49].)

The composition of such homotopies \( \tilde{c} \simeq \hat{b} \simeq \hat{a} \) is given by:

\[
(n, q, b) + (m, p, a) = (m + n^p, q + p, a)
\]

which of course is defined if and only if

\[
b^p = a + \delta m.
\]

A 2-homotopy \( (H, \hat{a}) \) of \( \hat{a} \) is such that \( H \) lifts dimension by 2 and so is given by an element \( H(z_0) \in M \). There are rules giving the composition, actions, and boundaries of such 1- and 2-homotopies.

In particular the action of a 1-homotopy \( (h, f^+) : f^- \simeq f^+ \) on a 2-homotopy \( (H, f^-) \) gives a 2-homotopy \( (H^h, f^+) \) where \( H^h(c) = H(c)^{h(c)} \). Here we take \( c = z_0 \) so that we obtain the action \( (n, b)^{(m, p, a)} = n^p \).

All these formulae follow from those given in [BH87, Proposition 3.14] or [BHS10, §9.3].
A 2-homotopy \((H, \hat{a})\) is given by \(a = \hat{a}(z)\) and \(m = H(z_0) \in M\). We then have to work out \(\delta_2(H)\). We find that

\[
\delta_2(H)(x) = \begin{cases} 
\delta H(z_0) & \text{if } x = z_0, \\
-H(sz)\hat{a}(z) + H(tz) + \delta H(z) & \text{if } x = z,
\end{cases}
\]

\[
= \begin{cases} 
\delta m & \text{if } x = z_0, \\
-m^a + m & \text{if } x = z.
\end{cases}
\]

This completes the proof of Theorem 2.1. □

The proof of Theorem 1.1 now follows by restricting the crossed module of groupoids given in Theorem 2.1 to \(LM(a)\), the crossed module of groups over the object \(a \in (LM)_1 = P\). Then we have an isomorphism \(\theta : LM(a) \to LM[a]\) given by \(\theta_0(a) = *, \ \theta_1(m, p, a) = (m, p), \ \theta_2(m, a) = m\).

For the next result we need the notion of fibration of crossed modules of groupoids which is a special case of fibrations of crossed complexes as defined in [How79] and applied in [Bro08].

**Theorem 2.2** In the situation of Theorem 2.1, there is a fibration \(LM \to M\) of crossed modules of groupoids. Hence if

\[
\pi = \pi_2(X) \cong \text{Ker } \delta, \quad G = \pi_1(X) \cong \text{Cok } \delta
\]

then for each \(a \in P\) there is an exact sequence

\[
0 \to \pi^a \to \pi \to \pi/\{\bar{a}\} \to \pi_1(LX, a') \to C_\alpha(G) \to 1
\]

(2)

where: \(\bar{a}\) denotes the image of \(a'\) in \(G\); \(\pi/\{\alpha\}\) denotes \(\pi\) with the action of \(\alpha\) killed; and \(C_\alpha(G)\) denotes the centraliser of the element \(\alpha \in G\).

**Proof** We define the fibration \(\psi : LM \to M\) by the inclusion \(i : \{z_0\} \to \mathbb{K}(\mathbb{Z}, 1)\) and the identification \(\text{CRS}(\{z_0\}, M) \cong M\), where here \(\{z_0\}\) denotes also the trivial crossed complex on the point \(z_0\). Then \(\psi\) is a fibration since \(i\) is a cofibration, see [BG89]. The exact description of \(\psi\) in terms given earlier is that

\[
\psi_0(a) = *, \quad a \in P, \\
\psi_1(m, p, a) = p, \quad (m, p, a) \in M \times P \times P, \\
\psi_2(n, a) = n, \quad (n, a) \in M \times P.
\]

To say that \(\psi\) is a fibration of crossed modules over groupoids is to say that: (i) it is a morphism; (ii) \((\psi_1, \psi_0)\) is a fibration of groupoids, [Bro70, And78]; and (iii) \(\psi_2\) is piecewise surjective.
Let $\mathcal{F}$ denote the fibre of $\psi$. Then
\[ \mathcal{F}_0 = P, \quad \mathcal{F}_1 = \{0\} \times M \times P, \quad \mathcal{F}_2 = \{0\} \times P. \]
The exact sequence of the fibration for a given base point $a \in \mathcal{F}_0 = P$ is
\[
0 \to \pi_2(\mathcal{F}, a) \to \pi_2(\mathcal{LM}, a) \to \pi_2(\mathcal{M}, *) \xrightarrow{\partial} \\
\to \pi_1(\mathcal{F}, a) \to \pi_1(\mathcal{LM}, a) \to \pi_1(\mathcal{M}, *) \xrightarrow{\partial} \pi_0(\mathcal{F}) \to \pi_0(\mathcal{LM}) \to *. 
\]
Under the obvious identifications, this leads to the exact sequence of Theorem 1.5. \( \square \)

**Remark 2.3** These results and methods should be related to the description in [Bro87, §6] of the homotopy type of the function space $(BG)^Y$ where $G$ is an abstract group and $Y$ is a CW-complex, and which gives a result of Gottlieb in [Got69]. \( \square \)

**Remark 2.4** Here is a methodological point. The category $\text{Crs}$ of crossed complexes is equivalent to that of $\infty$-groupoids, as in [BH81a], where these $\infty$-groupoids are now commonly called ‘strict globular $\omega$-groupoids’. However the internal hom in the latter category is bound to be more complicated than that for crossed complexes, because the cell structure of the standard $n$-globe, $n > 1$,
\[ E^n = e_+^0 \cup e_+^1 \cup \cdots \cup e_+^{n-1} \cup e_+^n \]
is more complicated than that for the standard cell for which
\[ E^n = e_0^0 \cup e_0^{n-1} \cup e_0^n, n > 1. \]
Also we obtain a precise answer using filtered spaces and strict structures, whereas the current fashion is to go for weak structures as yielding more homotopy $n$-types for $n > 2$. In fact many results on crossed complexes are obtained using cubical methods. \( \square \)

**Appendix: Verification of crossed module rules**

We now verify the crossed module rules for the structure
\[ L\mathcal{M}[a] = (M \xrightarrow{\delta_a} P(a)) \]
defined in Theorem 1.1 from a crossed module of groups $\mathcal{M} = (M \xrightarrow{\delta} P)$ and $a \in P$ as follows:
\[ P(a) = \{(m, p) \in M \times P \mid \delta m = -[a, p] = -a - p + a + p\}; \]
\[ \delta_a m = (-m^a + m, \delta m); \]
\[ (n, q) + (m, p) = (m + n^p, q + p); \]
\[ n^{(m, p)} = n^p. \]
Proposition 2.5 If $\delta : M \to P$ is a crossed module of groups, and $a \in P$, then $LM[a]$ as defined above is also a crossed module of groups.

Proof It is easy to check that $\delta(-m^a + m) = [a, \delta m]$, so that $\delta_a(m) \in P(a)$.

We next show that $\delta_a$ is a morphism:

$\delta_a(n) + \delta_a(m) = (-n^a + n, \delta n) + (-m^a + m, \delta m)$

$= (-m^a + m + (-n^a + n)\delta m, \delta n + \delta m)$

$= (-m^a - n^a + n + m, \delta n + \delta m)$

$= \delta_a(n + m)$.

Now we verify the first crossed module rule. Let $(m, p) \in P(a), n \in M$:

$-(m, p) + \delta_a n + (m, p) = (-m^{-p}, -p) + (-n^a + n, \delta n) + (m, p)$

$= (-n^a + n + (-m^{-p})\delta n, -p + \delta n) + (m, p)$

$= (-n^a - m^{-p} + n, -p + \delta n) + (m, p)$

$= (m + (-n^a - m^{-p} + n)p, -p + \delta n + p)$

$= (m - n^{a+p} - m + np, \delta(n^p))$

$= (-n^{a+p} - \delta m + np, \delta(n^p))$

$= (np + n, \delta(n^p))$

since $\delta m = [a, p]$

$= \delta_a(np)$.

Now we verify the second crossed module rule:

$m^k a^n = m(-n^a + n, \delta n)$

$= m\delta n$

$= -n + m + n$. $\square$

In effect, this illustrates that verifying the crossed complex rules for the internal hom $CRS(C, D)$ is possible but tedious, and that is it is easier to say it follows from the general construction in terms of $\omega$-groupoids and the equivalence of categories, as in [BH87]. On the other hand, this direct proof ‘proves’, in the old sense of ‘tests’, the general theory.

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