A PSEUDO-TWIN PRIMES THEOREM

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Abstract. Selberg identified the “parity” barrier, that sieves alone cannot distinguish between integers having an even or odd number of factors. We give here a short and self-contained demonstration of parity breaking using bilinear forms, modeled on the Twin Primes Conjecture.

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1. Introduction

The Twin Prime Conjecture states that there are infinitely many primes $p$ such that $p + 2$ is also prime. A refined version of this conjecture is that $\pi_2(x)$, the number of prime twins lying below a level $x$, satisfies

$$\pi_2(x) \sim C \frac{x}{\log^2 x},$$

as $x \to \infty$, where $C \approx 1.32032 \ldots$ an arithmetic constant.

The best result towards the Twin Prime Conjecture is Chen’s [Che73], stating that there are infinitely many primes $p$ for which $p + 2$ is either itself prime or the product of two primes. This statement is a quintessential exhibition of the “parity” barrier identified by Selberg, that sieve methods alone cannot distinguish between sets having an even or odd number of factors. Vinogradov’s resolution [Vin37] of the ternary
Goldbach problem introduced the idea that estimating certain bilinear forms can sometimes break this barrier, and there have since been many impressive instances of this phenomenon, see e.g. [Fi98, HB01].

In this note, we aim to illustrate parity breaking in a simple, self-contained example. Consider an analogue of the Twin Prime Conjecture where instead of intersecting two copies of the primes, we intersect one copy of the primes with a set which analytically mimics the primes.

For \( x > 2 \) let

\[
iL(x) \sim x \log x
\]

denote the inverse to the logarithmic integral function,

\[
\text{Li}(x) := \int_2^x \frac{dt}{\log t}.
\]

**Definition 1.1.** Let \( \hat{\pi}(x) \) denote the number of primes \( p \leq x \) such that \( p = \lfloor iL(n) \rfloor \) for some integer \( n \).

Here \( \lfloor \cdot \rfloor \) is the floor function, returning the largest integer not exceeding its argument. Our main goal is to demonstrate

**Theorem 1.1.** As \( x \to \infty \),

\[
\hat{\pi}(x) \sim \frac{x}{\log^2 x}.
\]

Notice that the constant above is 1, that is, there is no arithmetic interference. This theorem follows also from the work of Leitmann [Lei77]; both his proof and ours essentially mimic Piatetski-Shapiro’s theorem [Pu53]. Our aim is to give a short derivation of this statement from scratch.

**Outline.** In §2 we give bounds for exponential sums of linear and bilinear type; these are used in the sequel. We devote §3 to reducing Theorem 1.1 to an estimate for exponential sums over primes. The latter are treated in §4 by Vaughan’s identity, relying on the bounds of §2 to establish Theorem 1.1.

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2. Estimates for Linear and Bilinear Sums

In this section we develop preliminary bounds of linear and bilinear type, which are used in the sequel. We require first the following two well-known estimates due originally to Weyl [Wey21] and van der Corput [vdC21, vdC22], see e.g. Theorem 2.2 and Lemma 2.5 of [GK91].

Lemma 2.1 (van der Corput). Suppose \( f \) has two continuous derivatives and for \( 0 < c < C \), we have \( c\Delta \leq f'' \leq C\Delta \) on \([N, 2N]\). Then

\[
\sum_{N < n \leq N_1 \leq 2N} e(f(n)) \ll_{C,c} N\Delta^{1/2} + \Delta^{-1/2}.
\]

This is proved by truncating Poisson summation, comparing the sum to the integral, and integrating by parts two times.

Lemma 2.2 (Weyl, van der Corput). Let \( z_k \in \mathbb{C} \) be any complex numbers, \( k = K + 1, \ldots, 2K \). Then for any \( Q \leq K \),

\[
\left| \sum_{K < k \leq 2K} z_k \right|^2 \leq \frac{K + Q}{Q} \sum_{|q| < Q} (1 - \frac{|q|}{Q}) \sum_{K < k, k+q \leq 2K} z_k \bar{z}_{k+q}.
\]

To prove this, shift the interval by \( q \) and average the contributions over \( |q| < Q \).

2.1. Estimating Type I Sums. We use Lemma 2.1 to prove

Lemma 2.3. For any integer \( h \) and \( \ell \geq 1 \),

\[
\sum_{N < n \leq N_1 \leq 2N} e(h \text{Li}(n\ell)) \ll \begin{cases} N & \text{if } h = 0 \\ (N|h|\ell) \frac{h}{2} \log(N\ell) & \text{otherwise} \end{cases}
\]

Remark 2.4. Here as throughout, the implied constant is absolute unless otherwise specified.

Proof. Let \( \Xi \) denote the sum in question. The trivial estimate is \( N \). Assume without loss of generality \( h > 0 \). Apply Lemma 2.1 with \( f(n) = h \text{Li}(n\ell) \), taking \( \Delta = \frac{h\ell}{N \log^2(N\ell)} \). Thus

\[
\Xi \ll N \left( \frac{h\ell}{N \log^2(N\ell)} \right)^{1/2} + \left( \frac{N \log^2(N\ell)}{h\ell} \right)^{1/2} \ll (Nh\ell)^{1/2} \log(N\ell),
\]

so we are done. \( \square \)
2.2. Estimating Type II Sums. We first require the following estimate.

**Lemma 2.5.** For positive integers $h, k, q$, and $L \geq 10$, let

$$S_0(q; k) := \sum_{L < \ell \leq 2L} e\left( h \left[ \text{Li}(\ell k) - \text{Li}(\ell (k + q)) \right] \right). \quad (2.6)$$

Then

$$S_0(q; k) \ll (Lhq)^{1/2},$$

where the implied constant is absolute, that is, independent of $k$.

**Proof.** We again apply Lemma 2.1, this time choosing for the function $f(x) = h(\text{Li}(xk) - \text{Li}(x(k + q)))$. Then for $L < x \leq 2L$,

$$f''(x) = h \left( \frac{-k}{x \log^2(xk)} + \frac{k + q}{x \log^2(x(k + q))} \right) = hq \frac{\log(k'x) - 2}{x \log(k'x)} \approx \frac{hq}{\ell},$$

for some $k' \in [k, k + q)$ by the Mean Value Theorem in $k$. Thus we can take $\Delta = \frac{hq}{\ell}$ and

$$S_0 \ll L \left( \frac{hq}{L} \right)^{1/2} + \left( \frac{L}{hq} \right)^{1/2} \ll (Lhq)^{1/2},$$

as desired. \qed

With this estimate in hand, we control Type II sums as follows (see also [GK91, Lemma 4.13]).

**Lemma 2.7.** Let $\alpha(\ell)$ and $\beta(k)$ be sequences of complex numbers supported in $(L, 2L]$ and $(K, 2K]$, respectively, and suppose that

$$\sum_\ell |\alpha(\ell)|^2 \ll L \log^{2A} L \quad \text{and} \quad \sum_k |\beta(k)|^2 \ll K \log^{2B} K. \quad (2.8)$$

Then

$$\sum_{L < \ell \leq 2L} \sum_{K < k \leq 2K} \alpha(\ell) \beta(k) e(h \text{Li}(\ell k)) \ll KL^{5/6}h^{1/6} \log^A L \log^B K. \quad (2.9)$$

The implied constant in (2.9) depends only on those in (2.8).

**Proof.** Let $S$ denote the sum on the left hand side. By Cauchy-Schwartz,

$$|S|^2 \ll \left( \sum_\ell |\alpha(\ell)|^2 \right) \left( \sum_\ell \left| \sum_k \beta(k) e(h \text{Li}(\ell k)) \right|^2 \right).$$
Let $Q \leq K$ be a parameter to be chosen later. Using Lemma 2.2 and (2.8), we get:

\[
|S|^2 \ll L \log^A L \frac{K + Q}{Q} \sum_{|q| < Q} \left(1 - \frac{|q|}{Q}\right) \times \sum_{\ell} \sum_{K < k, k + q \leq 2K} \beta(k) \beta(k + q) e\left(h\left[\text{Li}(\ell k) - \text{Li}(\ell(k + q))\right]\right)
\]

\[
\ll L \log^A L \frac{K}{Q} \sum_{1 \leq |q| < Q} \sum_{k} |\beta(k)\beta(k + q)||S_0(q; k)| + \frac{K^2 L^2}{Q} \log^2 A \log^2 B K,
\]

where $S_0$ is defined by (2.6).

Using Cauchy’s inequality, $|\bar{x} \bar{y}| \leq \frac{1}{2}(|x|^2 + |y|^2)$, and the fact that $|S_0(q; k)| = |S_0(-q; k + q)|$, we get

\[
|S|^2 \ll \frac{K^2 L^2}{Q} \log^2 A \log^2 B K + \frac{LK}{Q} \log^2 A \sum_{k} |\beta(k)|^2 \sum_{1 \leq q < Q} |S_0(q; k)|.
\]

From Lemma 2.5 we have the estimate:

\[
\frac{1}{Q} \sum_{1 \leq q < Q} |S_0(q; k)| \ll (LhQ)^{1/2},
\]

so we finally see that

\[
|S|^2 \ll \frac{K^2 L^2}{Q} \log^2 A \log^2 B K + L^{3/2} K^2 \log^2 A L \log^2 B KQ^{1/2} h^{1/2}.
\]

The choice $Q = \lfloor L^{1/3} h^{-1/3} \rfloor$ gives the desired result. $\blacksquare$

3. Reduction to Exponential Sums

In this section, we reduce the statement of Theorem 1.1 to a certain exponential sum over primes. We follow standard methods, see e.g. [GK91, HB83], which we include here for completeness. If $p = \lfloor iL(n) \rfloor$ then $p \leq iL(n) < p + 1, or equivalently, Li(p) \leq n < Li(p + 1)$. The existence of an integer in the interval $[Li(p), Li(p + 1))$ is indicated by the value $\lfloor Li(p + 1) \rfloor - \lfloor Li(p) \rfloor$, so we have

\[
\hat{\pi}(x) = \sum_{p \leq x} \left(\lfloor Li(p + 1) \rfloor - \lfloor Li(p) \rfloor\right).
\]

Write $[\theta] = \theta - \psi(\theta) - \frac{1}{2}$, where $\psi$ is the shifted fractional part

\[
\psi(\theta) := \{\theta\} - \frac{1}{2} \in [-\frac{1}{2}, \frac{1}{2}).
\]
So we have:
\[ \hat{\pi}(x) = \sum_{p \leq x} \left[ \text{Li}(p + 1) - \text{Li}(p) \right] + \sum_{p \leq x} \left[ \psi(\text{Li}(p)) - \psi(\text{Li}(p + 1)) \right]. \]

Since \( \text{Li}'(x) = \frac{1}{\log x} \), we use the Taylor expansion:
\[ \text{Li}(p + 1) = \text{Li}(p) + \frac{1}{\log p} + O \left( \frac{1}{p \log^2 p} \right) \]
to get:
\[ \hat{\pi}(x) = \sum_{p \leq x} \frac{1}{\log p} + \sum_{p \leq x} \left[ \psi(\text{Li}(p)) - \psi(\text{Li}(p + 1)) \right] + O(1). \]

By partial summation and a crude form of the Prime Number Theorem,
\[ \sum_{p \leq x} \frac{1}{\log p} = \int_2^x \frac{d\pi(t)}{\log t} = \frac{\pi(x)}{\log x} + O \left( \int_2^x \frac{\pi(t)}{t \log^2 t} dt \right) = \frac{x}{\log^2 x} + O \left( \frac{x}{\log^2 x} \right). \]

Therefore to prove Theorem 1.1 it suffices to show that
\[ \sum_{p \leq x} \left[ \psi(\text{Li}(p)) - \psi(\text{Li}(p + 1)) \right] \ll \frac{x}{\log^2 x}. \quad (3.1) \]

Equivalently, split the sum into dyadic segments and apply partial summation to reduce (3.1) to the statement that for any \( N < N_1 \leq 2N \),
\[ \sum_{N < n \leq N_1 \leq 2N} \Lambda(n) \left[ \psi(\text{Li}(n)) - \psi(\text{Li}(n + 1)) \right] \ll \frac{N}{\log^2 N}, \quad (3.2) \]
with \( N \ll x \). Here \( \Lambda \) is the von Mangoldt function:
\[ \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ is a \prime power} \\ 0 & \text{otherwise}. \end{cases} \]

The truncated Fourier series of \( \psi \) is
\[ \psi(\theta) = \sum_{0 < |h| \leq H} c_h e(\theta h) + O(g(\theta, H)), \quad (3.3) \]
where \( e(x) = e^{2\pi ix} \), \( c_h = \frac{1}{2\pi ih} \), and
\[ g(\theta, H) = \min \left( 1, \frac{1}{H \|	heta\|} \right). \]
Here \( \| \cdot \| \) is the distance to the nearest integer. In the above, \( H \) is a parameter which we will choose later, eventually setting

\[
H = \log^4 N. \tag{3.4}
\]

The function \( g \) has Fourier expansion

\[
g(\theta, H) = \sum_{h \in \mathbb{Z}} a_h e(\theta h),
\]

in which

\[
a_h \ll \min \left( \frac{\log 2 H}{H} \frac{H}{|h|^2} \right). \tag{3.5}
\]

Using (3.3) write the sum in (3.2) as \( \Sigma = \Sigma_1 + O(\Sigma_2) \) where

\[
\Sigma_1 := \sum_n \Lambda(n) \sum_{0 < |h| \leq H} c_h \left[ e(h \text{Li}(n)) - e(h \text{Li}(n + 1)) \right]
\]

and

\[
\Sigma_2 := \sum_{n \sim N} \Lambda(n) \left[ g(\text{Li}(n), H) + g(\text{Li}(n + 1), H) \right].
\]

We first dispose of \( \Sigma_2 \). Using positivity of \( g \), the bound (3.5), and Lemma 2.3 we have

\[
\Sigma_2 \ll \log N \sum_{n \sim N} g(\text{Li}(n), H) \ll \log N \sum_{h \in \mathbb{Z}} |a_h| \left| \sum_{n \sim N} e(\text{Li}(n) h) \right|
\]

\[
\ll \log N \left[ \frac{\log 2H}{H} N + \sum_{h \neq 0} \frac{H}{|h|^2} (N|h|)^{1/2} \log N \right]
\]

\[
\ll (\log N)^2 \left( N/H + N^{1/2} H \right).
\]

This bound is acceptable for (3.2) on setting \( H \) according to (3.4).
Next we massage $\Sigma_1$. On writing $\phi_h(x) = 1 - e(h(\text{Li}(x + 1) - \text{Li}(x)))$, we see by partial summation that

$$\Sigma_1 \ll \sum_{1 \leq h \leq H} h^{-1} \left| \sum_{N < n \leq N_1} \Lambda(n)\phi_h(n)e(h \text{Li}(n)) \right|$$

$$\ll \sum_{1 \leq h \leq H} h^{-1} \left| \phi_h(N_1) \sum_{N < n \leq N_1} \Lambda(n)e(h \text{Li}(n)) \right|$$

$$+ \int_{N}^{N_1} \sum_{1 \leq h \leq H} h^{-1} \left| \frac{\partial \phi_h(x)}{\partial x} \sum_{N < n \leq x} \Lambda(n)e(h \text{Li}(n)) \right| dx$$

$$\ll \frac{1}{\log N} \max_{N_2 \leq 2N} \sum_{1 \leq h \leq H} \left| \sum_{N < n \leq N_2} \Lambda(n)e(h \text{Li}(n)) \right|.$$ 

Here we used the bounds

$$\phi_h(x) \ll h(\text{Li}(x + 1) - \text{Li}(x)) \ll \frac{h}{\log N}$$

and

$$\frac{\partial \phi_h(x)}{\partial x} \ll h \left( \frac{1}{\log(x + 1)} - \frac{1}{\log(x)} \right) \ll \frac{h}{N \log^2 N}$$

for $N \leq x \leq 2N$. We have thus reduced Theorem 1.1 to the statement that for all $N < N_2 \leq 2N$,

$$S := \sum_{0 < h \leq H} \left| \sum_{N < n \leq N_2 \leq 2N} \Lambda(n)e(h \text{Li}(n)) \right| \ll \frac{N}{\log N}. \quad (3.6)$$

We establish this fact in the next section.

4. Proof of Theorem 1.1

Our goal in this section is to demonstrate (3.6), thereby establishing Theorem 1.1. We will actually prove more; instead of a log savings, we will save a power:

**Theorem 4.1.** For $S$ defined in (3.6) and any $\epsilon > 0$, we have

$$S \ll_{\epsilon} N^{21/22+\epsilon}, \quad \text{as} \quad N \to \infty.$$ 

Fix $u$ and $v$, parameters to be chosen later, and let $F(s) = \sum_{1 \leq n \leq v} \Lambda(n)n^{-s}$ and $M(s) = \sum_{1 \leq n \leq u} \mu(n)n^{-s}$, where $\mu$ is the Möbius function:

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{if } n \text{ is not square-free.} \end{cases}$$
The functions $F$ and $M$ are the truncated Dirichlet polynomials of the functions $-\frac{\zeta'}{\zeta}$ and $\frac{1}{\zeta}$, respectively, where $\zeta(s)$ is the Riemann zeta function. Notice, for instance, that

$$-\frac{\zeta'}{\zeta}(s) + F(s) = -\sum_{n>v} \Lambda(n)n^{-s}.$$ 

Comparing the Dirichlet coefficients on both sides of the identity

$$\frac{\zeta'}{\zeta} + F = \left(\frac{\zeta'}{\zeta} + F\right)(1 - \zeta M) + \zeta' M + \zeta FM$$

gives for $n>v$:

$$-\Lambda(n) = -\sum_{k\ell=n} \Lambda(k) \sum_{d\mid\ell, d>u} \mu(d) - \sum_{k\ell=n} \log k \mu(\ell) + \sum_{k\ell m=n} 1 \cdot \Lambda(\ell)\mu(m)$$

This formula is originally due to Vaughan [Vau77] (see also [GK91, Lemma 4.12]). Assume for now that $v\leq N$ (we will eventually set $u$ and $v$ to be slightly less than $\sqrt{N}$). Multiply the above identity by $e(h\text{Li}(n))$ and sum over $n$:

$$\sum_{N<n\leq N_2\leq 2N} \Lambda(n)e(h\text{Li}(n)) = \sum_{u\leq \ell \leq N_2/u} \sum_{N/\ell \leq k \leq N_2/\ell} \Lambda(k)a(\ell)e(h\text{Li}(k\ell))$$

$$+ \sum_{\ell \leq u} \sum_{N/\ell \leq k \leq N_2/\ell} \mu(\ell)\log k e(h\text{Li}(k\ell))$$

$$- \sum_{r \leq uv} \sum_{N/r \leq k \leq N_2/r} b(r)e(h\text{Li}(kr))$$

$$= S_1 + S_2 - S_3,$$

where

$$a(\ell) = \sum_{d\mid\ell, d>u} \mu(d), \text{ and } b(r) = \sum_{\ell m=r, \ell \leq v, m \leq u} \Lambda(\ell)\mu(m).$$

It is the bilinear nature of the above identity which we exploit, forgetting the arithmetic nature of the coefficients $a$, $b$, $\mu$, and $\Lambda$, and just treating them as arbitrary. The savings then comes from the matrix norm of $\{e(h\text{Li}(k\ell))\}_{k,\ell}$. This is achieved as follows.

Notice that $|a(\ell)|$ is at most $d(\ell)$, the number of divisors of $\ell$, and similarly $|b(r)| \leq \sum_{d\mid r} \Lambda(d) = \log r$, so we have the estimates

$$\sum_{L<\ell\leq 2L} |a(\ell)|^2 \ll L \log^3 L, \text{ and } \sum_{R<r\leq 2R} |b(r)|^2 \ll R \log^2 R.$$
It now suffices to show that $\sum_{0 < h < H} |S_i| \ll_{\epsilon} N^{21/22 + \epsilon}$ for each $i = 1, 2, 3$ by choosing $u$ and $v$ appropriately. We treat the sums of $S_i$ individually in the next three subsections.

4.1. The sum $S_2$. Let $G(x) := \sum_{k \leq x} e(h \text{Li}(k\ell))$. By Lemma 2.3, $G(x) \ll (xh\ell)^{1/2} \log(x\ell)$, so by partial integration we get

$$S_2 = \sum_{\ell \leq u} \mu(\ell) \sum_{N/\ell \leq k \leq N_2/\ell} \log k \ e(h \text{Li}(k\ell))$$

$$\ll \sum_{\ell \leq u} \int_{N/\ell}^{N_2/\ell} \log x \ dG(x)$$

$$\ll \sum_{\ell \leq u} \left( \sqrt{Nh} \log^2 N + \int_{N/\ell}^{N_2/\ell} \frac{1}{x} \sqrt{xh\ell \log(x\ell)dx} \right)$$

$$\ll \sqrt{Nh}u \log^2 N.$$ 

Thus $\sum_{1 \leq h < H} |S_2| \ll_{\epsilon} N^{21/22 + \epsilon}$ (as desired) on taking $u = N^{5/11}$ and recalling (3.4).

4.2. The sum $S_1$. Rewrite $S_1$ and split it into $\ll \log^2 N$ sums of the form:

$$S_1 = \sum_{N \leq k \leq N_2 \atop v < k, u < \ell} \alpha(k) \beta(\ell) e(h \text{Li}(k\ell))$$

$$\ll \log^2 N \sum_{L \leq \ell \leq 2L} \sum_{N \leq k \leq 2N \atop N \leq \ell \leq N_2} \alpha(k) \beta(\ell) e(h \text{Li}(k\ell)).$$

The roles of $k$ and $\ell$ are essentially symmetric (allowing $\alpha$ and $\beta$ to be either $\Lambda$ or $a$ affects only powers of log and not the final estimate) and taking $v = u$, we may arrange it so $N^{5/11} \leq K \leq N^{1/2} \leq L \leq N^{6/11}$.

Now using Lemma 2.7 we find that:

$$S_1 \ll \log^2 N \left( KL^{5/6} h^{1/6} \log^2 L \ \log^2 K \right)$$

$$\ll \log^6 N \left( N^{21/22} h^{1/6} \right).$$

Thus $\sum_{h} |S_1| \ll_{\epsilon} N^{21/22 + \epsilon}$ as desired.
4.3. The sum $S_3$. Recall $S_3$ and break it according to:

$$S_3 = \sum_{r \leq uv} b(r) \sum_{N/r \leq k \leq N_2/r} e(h \text{Li}(kr)) = \sum_{r \leq u} + \sum_{r \leq u < r \leq uv} = S_4 + S_5.$$

We treat $S_4$ exactly as $S_2$, getting $S_4 \ll (Nh)^{1/2} \log N(u \log u)$, which is clearly sufficiently small.

For $S_5$, the analysis is identical to that of $S_1$ and gives the same estimate, so we are done.

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