ON RANKS OF POLYNOMIALS.

DAVID KAZHDAN AND TAMAR ZIEGLER

Dedicated to A. Kirillov on the occasion of his 80th birthday

Abstract. Let $V$ be a vector space over a field $k, P : V \to k, d \geq 3$. We show the existence of a function $C(r, d)$ such that $\text{rank}(P) \leq C(r, d)$ for any field $k, \text{char}(k) > d$, a finite-dimensional $k$-vector space $V$ and a polynomial $P : V \to k$ of degree $d$ such that $\text{rank}(\partial P/\partial t) \leq r$ for all $t \in V - 0$. Our proof of this theorem is based on the application of results on Gowers norms for finite fields $k$. We don’t know a direct proof even in the case when $k = \mathbb{C}$.

1. Introduction

We fix $d \geq 0$ and restrict our attention to field $k$ such that $d! \neq 0$ (in other words when either $k$ is a field of characteristic zero or $\text{char}(k) > d$). For any $k$-valued function $f$ on $V$ and $t \in V - 0$ we define $\Delta_t f(x) := f(x + t) - f(x)$. For any $t \in V$ we denote by $P \to \partial P/\partial t$ the differentiation of the ring of polynomial function on $V$ such that $\partial P/\partial t = P(t)$ if $P : V \to k$ is a linear function.

Definition 1.1 (Algebraic rank filtration). Let $k$ be a field.

a) For a homogeneous polynomial $P$ on a finite-dimensional $k$-vector space $V$, of degree $d \geq 2$ we define the rank $r(P)$ of $P$ as the minimal number $r$ such that it is possible to write $P$ in the form

$$P = \sum_{i=1}^{r} L_i R_i,$$

where $L_i, R_i \in \overline{k}[V^\vee]$ are homogeneous polynomials of positive degrees.

b) For any polynomial $P$ on $V, t \in V - 0$ we define $P_t := \partial P/\partial t$.

c) If $P : V \to k$ is any polynomial of degree $d$ we define $r(P) = r(P_d)$ where $P_d : V \to k$ the homogeneous part of $P$ of the degree $d$.

Theorem 1.2 (Main). Let $d, r > 0$. There exists a function $C(r, d)$ such that $r(P) \leq C(r, d)$ for any polynomial $P$ of degree $d$ on a finite-dimensional $k$-vector space $V$ such that $r(\Delta_t P) \leq r$ for all $t \in V$.

Remark 1.3. For any $k$-valued function $f$ on $V$ and $t \in V - 0$ we define $\Delta_t f(x) := f(x + t) - f(x)$. If $f$ is a polynomial of degree $d$ then for generic $t \in V$ both $f_t$ and $\Delta_t(f)$ are polynomials of degree $d - 1$ and the difference $h f_t - \Delta_t(f)$ is

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of degree $\leq d - 2$. Therefore $r(P_t) = r(\Delta P)$ for any polynomial $P, t \in V$ and generic $t \in V$. In other words, the condition $r(P_t) \leq r, t \in V - 0$ is equivalent to the condition $r(\Delta_t(P)) \leq r, t \in V$. This is the

We grateful for Professor Jan Draisma who informed us about Theorem 1.8 of [DES] where a stronger result is proven for the case of cubic polynomial.

2. The case of finite fields

In this section we assume that $k$ is a finite field of characteristic $>d$.

Definition 2.1. (1) We denote by $e_p : \mathbb{F}_p \to \mathbb{C}^\ast$ the additive character $e_p(x) := \exp(\frac{2\pi ix}{p})$. For any finite field $\mathbb{F}_q, q = p^l$ we define $\psi := e_p(tr_{\mathbb{F}_q/\mathbb{F}_p}(x))$ where $tr_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map.

(2) For a function $f$ on a finite set $X$ we define $E_{x \in X} f(x) = \frac{1}{|X|} \sum_{x \in X} f(x)$.

We use $X \ll L Y$ to denote the estimate $|X| \leq C(L)|Y|$, where the constant $C$ depends only on $L$.

(3) Let $k = \mathbb{F}_q$ be a finite field, $V$ be a finite dimensional $k$-vector space.

Given a function $F : V \to k$ the $m$-th Gowers norm of $F$ is defined by

$\|\psi(F)\|_U^m = E_{v,v_1,\ldots,v_m \in V} \psi(\Delta_{v_m} \ldots \Delta_{v_1} F(v))$.

These were introduced by Gowers in [G], and were shown to be norms for $m > 1$.

(4) For a homogeneous polynomial $P$ on $V$ of degree $d$ we define

\[ \hat{P}(x_1, \ldots, x_d) = \Delta_{x_d} \ldots \Delta_{x_1} P(x). \]

This is a multilinear homogeneous form in $x_1, \ldots, x_d \in V$ such that

\[ P(x) = \frac{1}{d!} \hat{P}(x, \ldots, x). \]

Claim 2.2. There exists a function $s(r, d)$ such that for any finite field $k$, a finite-dimensional $k$-vector space $V$ and a multilinear homogeneous polynomial $P : V^d \to k$ of degree $d \geq 2$ and rank $\leq r$ we have $E_{x \in V} \psi(P(x)) \geq s(r, d)$.

This Claim is Lemma 2.2 in [KZ].

Lemma 2.3. There exists a function $b(r, d)$ such that for any finite field $k$, a finite-dimensional $k$-vector space $V$ and a polynomial $P : V \to k$ of degree $d$ and rank $\leq r$ we have $\|\psi(P)\|_U > b(r, d)$. 

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Proof.
\[ \| \psi(P) \|_{U_d}^2 = E_{h_1,\ldots,h_d,x \in V} \psi(\Delta_{h_d} \ldots \Delta_{h_1} P) \]

Since \( P \) is of degree \( d \) the function \( G(h_1,\ldots,h_d) = \Delta_{h_d} \ldots \Delta_{h_1} P(x) \) is a constant equal to \( \tilde{P}(h_1,\ldots,h_d) \).

Since \( P \) is of rank \( < r \), it follows from 2.2 that
\[ E_{h_1,\ldots,h_d} G(h_1,\ldots,h_d) \geq c(r,d) \]

□

Theorem 2.4. There exists a function \( C(r,d) \) such that \( r(P) \leq C(r,d) \) for any polynomial \( P : V \to k \) of degree \( d \) such that \( r(\Delta_t(P)) \leq r \) for all \( t \in V \).

Proof. Let \( b(r,d-1) \) be as in the Lemma 2.3. We have
\[ \| \psi(P) \|_{U_d}^2 = E_t \| \psi(P(x+t) - P(x)) \|_{U_{d-1}}^2 = E_t \| \psi(P_t) \|_{U_{d-1}}^{d-1} \geq b^{d-1}(r,d-1) > 0 \]
for some constant since \( P_t \) is of rank \( < r \).

Therefore (see [BL]) there exists \( C(r,d) \) such that \( r(P) \leq C(r,d) \).

□

3. PROOF OF THE MAIN THEOREM

We start with the statement of the following well known result. Let \( S,T \) be finite sets and let \((*)\) be a system of equations of the form \( A_s(x_i) = 0 \), \( s \in S, B_t(x_i) \neq 0, t \in T \) where \( A_s, B_t \in \mathbb{Z}[x_1,\ldots,x_n] \). For any field \( k \) we denote by \( \mathbb{Z}(k) \subset k^{n} \) the subset of solutions of the system \((*)\).

Claim 3.1. Assume that \( \mathbb{Z}(\overline{\mathbb{F}}_q) = \emptyset \) for all \( q = p^l, p \geq d \). Then \( \mathbb{Z}(k) = \emptyset \) if \( k \) is either a field of characteristic \( \geq d \) or a field of characteristic zero.

Proof. It is sufficient to consider the case when the field \( k \) is algebraically closed. Consider first the case when \( k \) is a field of positive characteristic \( p \). By the assumption \( \mathbb{Z}(\overline{\mathbb{F}}_q) = \emptyset \) where \( \overline{\mathbb{F}}_p \) is the algebraic closure of \( \mathbb{F}_p \). Then the result follows from Corollary 3.2.3 of [M].

To deal with the case of fields of characteristic 0 we choose a non-primitive ultrafilter \( D \) on the set of prime numbers and define \( K = \prod \overline{\mathbb{F}}_p / D \) (see the section 2.5 in [M]). Then \( K \) is an algebraically closed field of characteristic 0 and the Loe’s theorem imply that \( Z(K) = \emptyset \). As before Corollary 3.2.3 of [M] implies that \( Z(k) = \emptyset \) for all algebraically closed fields of characteristic zero.

Since some people are not familiar with Model theory we indicate another way to prove that the conditions of Claim 3.1 imply that \( Z(\mathbb{C}) = \emptyset \). To simplify the arguments we assume that polynomials \( A_s \) and \( B_t \) are homogeneous.

As follows from the Hilbert’s Nullstellensatz theorem it is sufficient to show that \( Z(\mathbb{Q}) = \emptyset \). We show that an assumption
\[ \exists(x_1,\ldots,x_n) \in Z(\mathbb{Q}) \]
leads to a contradiction.
Let $K \subset \mathbb{Q}$ be the field generated by elements $x_i, 1 \leq i \leq n$ and $\mathcal{O} \subset K$ be the ring of integers. Since polynomials $A_s$ and $B_t$ are homogeneous we have $(cx_1, \ldots, cx_n) \in Z(\mathbb{Q})$ for any $c \in K^*$. So we can assume that $x_i \in \mathcal{O}, 1 \leq i \leq n$.

Let $N_{K/\mathbb{Q}} : K^* \to Q^*$ be the norm map. Choose a prime number $p \geq d$ such that $N_{K/\mathbb{Q}}(B_t(x_1, \ldots, x_n))$ are prime to $p$ for all $t \in T$. Let $I \subset \mathcal{O}$ be any maximal ideal containing $p\mathcal{O}$. Then $(\bar{x}_1, \ldots, \bar{x}_n) \in (\mathcal{O}/I)^n$ is a point in $Z(\mathcal{O}/I)$. But by the assumption $Z(\mathcal{O}/I) = \emptyset$. This contradiction proves the claim.

Now we can prove the Theorem 1.2

Proof. Fix $r$ and $d$ and for any $N \geq 1$ consider the algebraic variety $X_N$ of polynomials $P(x_1, \ldots, x_N) = \sum_{i \in I} c_i x^i$ where $I = \{i_1, \ldots, i_N \geq 0 | \sum_j i_j = d\}$, $x^i := x_1^{i_1} \ldots x_N^{i_N}$.

The condition $r(P) > C(r, d)$ can be written as a system $\{A_s(c_i) = 0\}, 1 \leq s \leq A$ and $\{B_t(c_i) \neq 0\}, 1 \leq t \leq B$, where $A_s, B_t$ are homogeneous polynomials in $c_i$ with integer coefficients.

On the other hand it follows from Theorem 3.2 in [M] (or from the theorem of Chevalley) that the condition

$$\forall t : r(P_t) \leq r$$

can be written as a system equations of $c_i$ the form $\{C_u(c_i) = 0\}, 1 \leq u \leq C$ and $\{D_v(c_i) \neq 0\}, 1 \leq v \leq D$, where $C_u, D_v$ are homogeneous polynomials in $c_i$ with integer coefficients.

Let $W$ be the vector space with coordinates $c_i$ and $Z_N \subset W$ be the algebraic variety defined by the system of algebraic equalities

$$\{A_s(c_i) = C_u(c_i) = 0\},$$

and inequalities

$$\{B_t(c_i) \neq 0, D_v(c_i) \neq 0\}.$$  

By definition, for any field $k$ the points $c_i \in Z_N(k)$ are in bijection with polynomials $P = \sum c_i x^i$ of degree and rank $> C(r, d)$ such that $r(\partial P/\partial t) \leq r$.

So Theorem 2.4 implies that $Z_N(F_q) = \emptyset$ for any $q = p^t, p > d$. Therefore by Claim 3.1 we have $Z_N(k) = \emptyset$ for any $N$ and any field $k$ of characteristic 0 or of characteristic $>d$.

Theorem 1.2 is proven. 

\[\square\]

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Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram The Hebrew University of Jerusalem, Jerusalem, 91904, Israel
E-mail address: david.kazhdan@mail.huji.ac.il

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram The Hebrew University of Jerusalem, Jerusalem, 91904, Israel
E-mail address: tamarz@math.huji.ac.il