Error Avoiding Quantum Codes

PAOLO ZANARDI¹, and MARIO RASETTI²

¹ ISI Foundation, Villa Gualino, Viale Settimio Severo, Torino
² Dipartimento di Fisica and Unità INFN, Politecnico di Torino,
Corso Duca degli Abruzzi 24, I-10129 Torino, Italy

Abstract The existence is proved of a class of open quantum systems that admits a linear
subspace $C$ of the space of states such that the restriction of the dynamical semigroup to
the states built over $C$ is unitary. Such subspace allows for error-avoiding (noiseless)
encoding of quantum information.

PACS numbers: 71.10.Ad , 05.30.Fk

1. Introduction

In this letter we deal with the general question, in the frame of the mathemati-
cal theory of open quantum systems, whether a subset of the state space of a given
open system $S$ within the environment $E$ exists, unaffected by the coupling of $S$ with
$E$. Such a challenging question raises with special emphasis in the area of quantum
computation (QC) [1], where it finds strong motivations. QC aims to construct com-
utational schemes, based on quantum features, more efficient (e.g., exponentially
faster) than classical algorithms [2]. Quantum computation differs from classical
computation in that, whereas in the latter a Turing-Boole state is specified at any
time by a single integer, say $n$, written in binary form, the generic state $|\psi\rangle$ of
a quantum computer is a superposition of states $|n\rangle$ in some appropriate Hilbert
space $\mathcal{H}$, each of which can be thought of as corresponding to a classical boolean
state: $|\psi\rangle = \sum_{n=0}^{2^n-1} c_n |n\rangle$. The features of $|\psi\rangle$ are described by the probability am-
plitudes $c_n$. The higher potential efficiency one may expect of quantum with respect
to classical computation is ascribable just to characteristically quantum mechanical
properties, such as interference (the phases of the $c_n$’s play a role), entanglement
(some of the quantum states of a complete system do not correspond to definite
states of its constituting parts), von Neumann state reduction (a quantum state
cannot be observed without being irreversibly disturbed), which are absent in clas-
sical computers. Moreover, quantum information processing is inherently parallel,
due to the linear structure of state space and of dynamical evolution. For example, the quantum Turing machine proposed by Deutsch consists of a unitary evolution from a single initial state encoding input data to a final state encoding the output. As in Turing’s scheme, the initial state encodes information on both the input and the "program". It is therefore clear that in the physical implementation of QC maintaining quantum coherence (namely the phase relationship between the $c_n$’s) in any computing system is an essential requirement in order to take advantage of its specific quantum mechanical features. On the other hand any real system unavoidably interacts with some environment, which typically, consists of a huge amount of uncontrollable degrees of freedom. Such interaction causes a corruption of the information stored in the system as well as errors in computational steps, that may eventually lead to wrong outputs. One of the possible approaches to overcome such difficulty, in analogy with classical computation, is to resort to redundancy in encoding information, by means of the so-called (quantum) error correcting codes (ECC). In these schemes information is encoded in linear subspaces $C$ (codes) of the system Hilbert space in such a way that “errors” induced by the interaction with the environment can be detected and corrected. Of course, detection of correctable errors has to be carried over with no gain of which-path information about the actual system state; otherwise this would result in a further source of loss of coherence. The ECC approach appears then to aim to an active stabilization of quantum states by conditionally carrying on suitable quantum operations. The typical system considered in the ECC literature is the $N$-qubit register $R$ made of $N$ replicas of a two-level system $S$ (the qubit) where each qubit of $R$ is assumed to be coupled with an independent environment. We shall prove here that, by relaxing the latter assumption, one can identify a class of open quantum systems which admit linear subspaces $C$ such that the restriction to $C$ of the dynamics is unitary. Quantum information encoded in such subspaces is therefore preserved, thus providing a strategy to maintain quantum coherence. The approach to the decoherence problem suggested by our results is, in a sense, complementary to EC, in that it consists in a passive stabilization of quantum information. For this reason, subspaces $C$ will be referred to as Error Avoiding Codes (EAC).

2. Outline

In this paper, without loss of generality, we shall describe the quantum dynamics of a (open) system $S$ in terms of marginalization of the dynamics associated to a one-parameter unitary group $\{U_t\}_{t \in \mathbb{R}}$ of transformations acting on an enlarged Hilbert space (system plus environment). Even though this description is by no means unique, we assume that the form of the generator (Hamiltonian) of the dynamical
group is dictated by physical considerations \cite{6}. The component of the Hamiltonian that induces a non-trivial mixing of the system and environment degrees of freedom will, as usual, be referred to as the interaction Hamiltonian $H_I$. In sect. 3 after defining an EAC $\mathcal{C}$ as a subspace with unitary marginal dynamics we characterize it (Lemma 3.1) by the simple property that $H_I$ restricted to $\mathcal{C}$ should be the identity on the system space. The simplest – but physically important – example is provided by the simultaneous eigenspaces (if any) of the whole set of system operators appearing in $H_I$ (Theorem 3.1). Such a condition can be implemented in a less trivial way by means of the reducible structure of the system Hilbert space considered as a representation space of a group $G$ or of a Lie algebra $A_S$ (Theorems 3.2, 3.3). In the former case $G$ is required to be a symmetry group for $H_I$, in the latter the allowed interaction operators (error generators) have to belong to $U(A_S)$. By imposing that both module-structures are present and compatible (the representatives of elements of $A_S$ are $G$-invariant) one can identify (Theorem 3.4) a whole class of EAC’s as the singlet sector; direct sum of the one-dimensional submodules of a semisimple (dynamical) Lie algebra $A_S$. In sect. 4 we consider the case of a quantum register $R$ defined as the collection of $N$ replicas of a $d$-dimensional quantum system (cell) $C$. Assuming that the error generators \{ $S^{(i)}_{\lambda}$ \} ($i = 1, \ldots, N$) of each cell are coupled in a replica-symmetric way to a common environment, one finds that the marginal dynamics of $R$ is described in terms of the $N$-fold tensor representation $\phi_N$ of the dynamical algebra $A_S$ [isomorphic to $\text{sl}(d, \mathbb{C})$] spanned by the $S^{(i)}_{\lambda}$’s. Theorem 3.4 holds because $\phi_N$ is compatible with the natural action of the symmetric group $S_N$.

3. Error Avoiding Codes

Let $H_\alpha$, $\dim H_\alpha = d_\alpha$, ($\alpha = E, S$) be finite dimensional Hilbert spaces. The quantum system associated to $H_S$ ($H_E$) will be referred to as the system (respectively, the environment). The set of non-negative hermitian operators on Hilbert space $H$ with trace one will be denoted by $S(H)$; its elements will be referred to as states. $S(H)$ is the convex hull of the set of pure states

$$S_p(H_S) \doteq \{ \rho \in S(H_S) : \rho^2 = \rho \} \cong H_S/U(1).$$

(1)

We assume the quantum system associated with $H_{SE} = H_S \otimes H_E$ to be closed, i.e. its dynamics to be generated by a hermitian operator $H_{SE} \in \text{End}(H_{SE})$. The time evolution of any state $\rho \in S(H_{SE})$ is given by $\rho \rightarrow \rho(t) \doteq U_{SE}(t)\rho U_{SE}^\dagger(t)$, where $U_{ES}(t) \doteq \exp(-i t H_{SE})$, ($t \in \mathbb{R}$) is the one-parameter unitary group generated by $H_{SE}$. The marginal dynamics on $H_S$ (conditional to the initial preparation $\rho_E \in S(H_E)$) is given by

$$E_t^{\rho_E} : S(H_S) \rightarrow S(H_E) : \rho \rightarrow \text{tr}^E \left( U_{SE}(t) \rho \otimes \rho_E U_{SE}^\dagger(t) \right).$$

(2)

3
The dynamical semigroup \( \{ \mathcal{E}_t \}_{t \geq 0} \) does not leave invariant the set of pure states. This a characteristic quantum phenomenon known as decoherence. It reflects the fact that the system-environment interaction entangles the degrees of freedom of \( S \) with those of \( E \) in such a way that, despite unitarity (which does indeed preserve purity of the overall joint state) each of the two subsystems has no longer a (pure) state of its own: the two subsystems have become inseparable\(^8\). From the point of view of quantum information this amounts to a corruption of the initial state.

For a \( d_C \)-dimensional linear subspace of \( \mathcal{H}_S \), we denotes by \( \mathcal{A}(C) \) the subalgebra of \( \text{End}(\mathcal{H}_S) \) leaving \( C \) invariant: \( \mathcal{A}(C) = \{ X \in \text{End}(\mathcal{H}_S) : X \mathcal{C} \subset \mathcal{C} \} \).

**Definition 3.1.** A linear subspace \( C \neq \{0\} \) of \( \mathcal{H}_S \), is an error avoiding code (EAC) iff

1. \( \exists H_S \in \mathcal{A}(C) \), \( H_S \), hermitian, is such that, \( \forall \rho_E \in \mathcal{S}(\mathcal{H}_E) \), \( \rho \in \mathcal{S}(C) \Rightarrow \mathcal{E}_t^{\rho_E}(\rho) = e^{-i t H_S} \rho e^{i t H_S} (\forall t \in \mathbb{R}) \).
2. \( \mathcal{C} \) is maximal (i.e. it is not a proper subspace of any space for which i) holds).

Each state in \( \mathcal{C} \) will be referred to as noiseless.

Definition 3.1 of EAC can be summarized in terms of commutativity (\( \forall t \in \mathbb{R} \)) of the following diagram:

\[
\begin{array}{cccc}
\mathcal{S}(\mathcal{H}) & \xrightarrow{\mathcal{E}_t} & \mathcal{S}(\mathcal{H}) \\
\downarrow{} & & \downarrow{} \\
\mathcal{S}_P(\mathcal{C}) & \xrightarrow{\text{Ad}(U_t)} & \mathcal{S}_P(\mathcal{C}) \\
\downarrow{} & \cong & \downarrow{} \\
\mathcal{C}/U(1) & \xrightarrow{U_t} & \mathcal{C}/U(1)
\end{array}
\]

Here \( \cong \) is the isomorphism defined in equation (1) and \( \iota \) is the canonical inclusion map.

**Remark 1.** The eigenstates of \( H_S \) in \( C \) are stationary states (i.e. \( \mathcal{E}_t(\rho) = \rho, \forall t \in \mathbb{R} \)). \( C \neq \{0\} \) means that there exists a set of initial preparations for which no information loss occurs. Since the minimal system which permits useful encoding of quantum information is a two-level system (qubit), an EAC has use in QC if \( \dim \mathcal{C} > 1 \).

The Hamiltonian \( H_{SE} \) has the form

\[
H_{SE} = H_S \otimes I_E + I_S \otimes H_E + H_I ,
\]

where \( H_S \) (\( H_E \)) is an hermitian operator on \( \mathcal{H}_S \) (respectively, \( \mathcal{H}_E \)) and \( H_I \), hermitian, acts, for an arbitrary state, in a non trivial way on both factors of the tensor product space \( \mathcal{H}_{SE} \). The following lemma states a sufficient and necessary condition for EAC’s:

**Lemma 3.1.** A linear subspace \( C \subset \mathcal{H}_S \) is an EAC iff

4
i) $H_S \in \mathcal{A}(C)$, 
ii) $H_I|_C = I_C \otimes E(C)$, $(E(C) = E^\dagger(C) \in \text{End}({\mathcal{H}}_E))$.

Proof
We first show that i) and ii) are sufficient conditions. Let $\{|\phi_k\rangle\}$ be an orthonormal set of eigenvectors of the hermitian operator $\tilde H_E \doteq H_E + E(C)$, and $\{\tilde e_k\}$ the corresponding set of eigenvalues. Any $\rho_E \in \mathcal{S}(\mathcal{H}_E)$ can written in the form $\rho_E = \sum_{k,h} R_{kh} |\phi_k\rangle \langle \phi_h|$, where $R$ is a hermitian non-negative matrix of rank $d_E$ and trace one with complex matrix elements $R_{kh}$. For $\rho \in \mathcal{S}(C)$,

$$
\mathcal{E}_t^{\rho_E}(\rho) = \text{tr}^E \left( U_{SE}(t) \rho \otimes \rho_E U_{SE}^\dagger(t) \right) = \sum_{kh} R_{kh} \text{tr}^E \left( e^{-itH_S} \rho e^{itH_S} \otimes e^{-it(\tilde e_k - \tilde e_h)} |\phi_k\rangle \langle \phi_h| \right) = e^{-itH_S} \rho e^{itH_S} \sum_k R_{kk} = e^{-itH_S} \rho e^{itH_S},
$$

in that $\text{tr}^E (|\phi_k\rangle \langle \phi_h|) = \delta_{hk}$, and $\sum_k R_{kk} = 1$.

Suppose now that $C$ is an EAC. Expanding the identity at point i) of Definition 3.1 up to the first order in $t$ one finds $[\tilde H(\rho_E), \rho] = 0, \forall \rho \in \mathcal{S}(C)$, where $\tilde H(\rho_E) \doteq \text{tr}^E (\rho_E H'_SE)$, and $H'_SE \doteq H_{SE} - H_S \otimes I_E$. From the (manifest) commutativity of $\tilde H_{SE}$ with all the states of $C$ ensues that $\tilde H_{SE}|_C = \lambda(\rho_E) I_C$. Moreover, since this property holds for all $\rho_E \in \mathcal{S}(\mathcal{H}_E)$, one has $\langle \phi_i| H'_SE | \phi_i\rangle = \lambda_i I_C, \forall |\phi_i\rangle \in \mathcal{H}_E$. It follows from this latter relation that $\langle \phi_i| H'_SE | \phi_{i'}\rangle = \lambda_{i'i'} I_C$. Therefore the spectral resolution of $H'_SE$ finally reads

$$
H'_SE = \sum_{jj',ii'} |\psi_j\rangle \otimes |\phi_i\rangle \langle \phi_i| H'_SE | \phi_{i'}\rangle \langle \phi_{i'}| \otimes \langle \phi_{i'}| = \sum_{jj'} |\psi_j\rangle \langle \psi_j| \otimes \sum_{ii'} \lambda_{i'i'} |\phi_i\rangle \langle \phi_{i'}| = I_C \otimes E,
$$

for some $E \in \text{End}(\mathcal{H}_E)$. Here $\{|\psi_j\rangle\}_{j=1}^{d_C}$ ($\{|\phi_i\rangle\}_{i=1}^{d_E}$) is a orthonormal basis of $C$ (respectively, $\mathcal{H}_E$). The r.h.s. of eq. (5) shows that $H_{SE} - H_S \otimes I_E$, restricted to $C$ acts trivially on the system Hilbert space, as was to be proven.

Remark 1. Suppose a unitary $U \in \text{End}(\mathcal{H}_S)$ exists such that $\text{Ad} U(H_{SE}) \doteq U H_{SE} U^\dagger$ satisfies the hypothesis of Lemma 3.1 with respect to subspace $C$. Then $U^\dagger C$ is an EAC.

The physical meaning of Lemma 3.1 is quite transparent: the states over $C$ do not suffer any decoherence in that they are all affected by the environment in the same way.

The general form of the interaction Hamiltonian $H_I$ is

$$
H_I = \sum_{\lambda \in \Lambda} S_\lambda \otimes E_\lambda,
$$

(6)
where $X_\lambda \in \text{End}(\mathcal{H}_X)$, $X = S, E$, and $\Lambda$ is a suitable (finite) index set. The operators \( \{S_\lambda\} \) will be referred to as error generators. Lemma 3.1 basically asserts that $\mathcal{C}$ is an EAC iff $H_S \in A(\mathcal{C})$ and the $S_\lambda$’s belong to the subalgebra $A_1(\mathcal{C}) \subset A(\mathcal{C})$ of operators with restriction to $\mathcal{C}$ proportional to the identity. Notice that $A_1(\mathcal{C})$ contains the ideal $A_0(\mathcal{C})$ of those operators in $A_1(\mathcal{C})$ which annihilate $\mathcal{C}$. If the error generators belong to $A_0(\mathcal{C})$ the dynamics on $\mathcal{C} \otimes \mathcal{H}_E$ coincides with that generated by the free Hamiltonian.

The simplest case in which Lemma 3.1 provides an EAC is described in the following THEOREM 3.1. Let \( \{S_\lambda\}_{\lambda \in \Lambda} \) and $H_S$ form a commutative family of hermitian operators. If $\mathcal{C}$ is a maximal common eigenspace of the $S_\lambda$’s, then $\mathcal{C}$ is an EAC.

\begin{proof}
Let $\sigma_\lambda$, $(\lambda \in \Lambda)$ be the set of $S_\lambda$-eigenvalues, then one has $H_{I|\mathcal{C}} = \sum_{\lambda \in \Lambda} \sigma_\lambda \mathbb{I}_C \otimes E_\lambda = \mathbb{I}_C \otimes \sum_{\lambda \in \Lambda} \sigma_\lambda E_\lambda = \mathbb{I}_C \otimes E(\mathcal{C})$. Since $\mathcal{C}$ is maximal and $H_S$ commutes with the $S_\lambda$’s, then $H_S \in A(\mathcal{C})$, and the thesis follows from Lemma 3.1.
\end{proof}

Let now $G$ be a group, $\Phi$ a unitary representation of $G$ on $H_S$. $H_S$, considered as a $G$-module, has the decomposition, in terms of irreducible $G$-submodules

\[ \mathcal{H}_S = \bigoplus_{j \in \mathcal{J}} n_j \mathcal{H}_j , \]  

(7)

where $\mathcal{J}$ is a label set for the $G$-irreps, \( \{\mathcal{H}_j\}_{j \in \mathcal{J}} \) is the set of irreducible submodules of $G$, and the integers \( \{n_j\}_{j \in \mathcal{J}} \) are the corresponding multiplicities. Suppose $\exists j_0 \in \mathcal{J}$ such that $n_{j_0}(\Phi) = 1$, and let $\mathcal{C}$ be the corresponding submodule; then

THEOREM 3.2. If the $\{S_\lambda\}$’s in equation (6) are Ad $\Phi(G)$-invariant and $H_S \in A(\mathcal{C})$, then $\mathcal{C}$ is an EAC.

\begin{proof}
Since the $S_\lambda$’s transform according to the identity representation of $G$, they can couple only submodules corresponding to equivalent representations. Therefore it follows from $n_{j_0}(\Phi) = 1$ that $S_\lambda \in A(\mathcal{C}) (\lambda \in \Lambda)$. Hence the $S_\lambda$’s commute with all operators of the $G$-irrep labelled by $j_0$, and one obtains – from Schur’s lemma – that $S_\lambda|\mathcal{C} \sim \mathbb{I}_C (\lambda \in \Lambda)$. The thesis follows from Lemma 3.1.
\end{proof}

Let us suppose now that the error generators belong to some representation $\phi: A_S \rightarrow \text{gl}(\mathcal{H}_S)$ of a Lie algebra $A_S$ (dynamical algebra). $\phi$ turns $\mathcal{H}_S$ into an $A_S$-module that has a decomposition analogous to equation (7) ($\mathcal{J}$ being now a label set for the $A_S$-irreps).

THEOREM 3.3. Let $\mathcal{C}$ be the direct sum over a maximal set of equivalent one-dimensional $A_S$-submodules. Suppose $\mathcal{C} \neq \{0\}$ and $H_S \in A(\mathcal{C})$; then $\mathcal{C}$ is an EAC.

\begin{proof}
Since $\mathcal{C}$ is spanned by $A_S$-singlets and \( \{S_\lambda\} \subset \phi(A_S) \) one has $S_\lambda|\psi = \sigma_\lambda |\psi\),
where the $\sigma_\lambda$ are c-numbers. Therefore the assumption of Lemma 3.1 holds with $E(C) = \sum_{\lambda \in \Lambda} \sigma_\lambda E_\lambda$.

**Remark 1.** When $A_S$ is semisimple, then the $\sigma_\lambda$'s are necessarily zero, and all the one-dimensional irreps are equivalent.

**Remark 2.** When $A_S$ is abelian all the irreps are one-dimensional. The subspaces corresponding to the direct sum over a maximal set of equivalent irreps are weight spaces.

**Remark 3.** Theorem 3.3 still holds if the error generators belong to $\phi(U(A_S))$, where $U(A_S)$ denotes the universal enveloping algebra of $A_S$.

The Lie-algebra representation $\phi$ is compatible (i.e. $\text{Ad } \Phi(G)$-invariant) with the action of the group $G$ iff $\Phi(g)X\Phi^*(g) = X$, $\forall g \in G$, $X \in \phi(A_S)$. In this case, when $A_S$ is semisimple, the multiplicities of the $A_S$-irreps ($G$-irreps) appearing in the decomposition of $\phi(\Phi)$ are but the dimensions of the $G$-irreps ($A_S$-irreps) entering the decomposition of $\Phi(\Phi)$. In particular this means that the subspace $C$ obtained as direct sum over the one-dimensional $A_S$-submodules of $\phi(\text{singlet sector})$ appearing in the decomposition of $\phi$ is a $G$-module which enters with multiplicity one in the decomposition of $\Phi$.

**THEOREM 3.4.** Let $C$ be the singlet sector of the $G$-compatible Lie-algebra representation $\phi$ of $A_S$. If

- i) the error generators are $\text{Ad } \Phi(G)$-invariant,
- ii) $H_S \in A(C)$,

then $C$ is an EAC.

**Proof**

The singlet sector corresponds to a $G$-irrep appearing in the $\Phi$ decomposition with multiplicity one. The thesis follows from Theorem 3.2.

**Remark 1.** $\phi(U(A_S))$ is $\text{Ad } \Phi(G)$-invariant in that it is generated by $I$ and $\phi(A_S)$.

**Remark 2.** If $[H_S, \phi(A_S)] = 0$ or $H_S \in \phi(U(A_S))$, the condition $H_S \in A(C)$ is fulfilled.

4. Quantum Registers

In this section the physically relevant notion of register is introduced, in analogy with the case of classical computation.

**DEFINITION 4.1.** A $d$-dimensional (quantum) cell $C$ is a quantum system associated to a Hilbert space $H_C \cong \mathbb{C}^d$. A (quantum) register with $N$ cells is a quantum system given by $N$ replicas of $C$. $R$ is associated with $H_R = H_C^\otimes N$.

The register self-hamiltonian will be denoted as $H_R$. The register Hilbert space $H_R$ is a natural $S_N$-module. Let $\{|\psi_j\rangle\}_{j=1}^d$ be a basis of $H_C$; one can define $\sigma \cdot \otimes_{k=1}^N|\psi_{j_k}\rangle = \otimes_{k=1}^N|\psi_{j_{\sigma(k)}}\rangle$, $\forall \sigma \in S_N$). The latter formula defines, by linear
extension, a representation $\Phi_N$ of $S_N$ on $H_R$. The operators compatible with this $S_N$-action lie in the symmetric subspace of $\text{End}(H_R) \cong \text{End}^{\otimes N} H_C$. If each cell of $R$ is coupled with the (common) environment $E$ by a $S_N$-invariant interaction, one has

$$H_I = \sum_{i=1}^{N} \sum_{\lambda \in \Lambda} S^{(i)}_\lambda \otimes E_\lambda \in \text{End}(H_R \otimes H_E),$$

where $S^{(i)}_\lambda \in \text{End}(H_R)$, $(i = 1, \ldots, N, \lambda \in \Lambda)$ acts as $S_\lambda$ in the $i$-th factor of the tensor product $H_R$, and as the identity in the other factors.

Notice that the register-environment interaction (8) involves only the coproduct operators $\Delta^{(N)}(S_\lambda) \doteq \sum_{i=1}^{N} S^{(i)}_\lambda$. If $\phi: A_S \to \text{gl}(H_C)$ is a representation of the Lie algebra $A_S$ in $H_C$, then $\Delta^{(N)} \circ \phi: A_S \to \text{gl}(H_R)$ is the $N$-fold tensor product of $\phi$ and will be denoted as $\phi_N$. An important role in physical applications is played by the case in which $A_S \cong \text{sl}(d, \mathbb{C})$ and $\phi$ is the defining representation.

**Theorem 4.1** Let the quantum register $R$ be coupled with the environment $E$ by the Hamiltonian given by equation (8), where the interaction operators $S_\lambda$ belong to the defining representation $\tilde{\phi} N$ of $\text{sl}(d, \mathbb{C})$ in $H_C$. Let $C_N$ be the singlet sector of $\tilde{\phi} N$. If $H_R \in A(C_N)$ then $C_N$ is an EAC.

**Proof**

From Theorem 3.4, letting $A_S = \text{sl}(d, \mathbb{C})$, $\mathcal{G} = S_N$, $\phi = \tilde{\phi} N$, and $\Phi = \Phi_N$. □

**Remark 1.** Remarks 1. and 2. of Theorem 3.4 imply immediately that in the latter proposition the error generators are allowed to belong to $\tilde{\phi} N(\mathcal{U}(\text{sl}(d, \mathbb{C})))$ as well. In this case the latter subspace coincides with the whole space of $S_N$-invariant operators.

5. Conclusions

In this paper we introduced the notion of Error Avoiding Quantum Code as the subspace $C$ of the Hilbert space of an open quantum system $S$ embedded in an environment $E$, in which quantum coherence is preserved. Formally this means that the dynamical (one-parameter) semigroup of $S$ restricted to initial data in $S(C)$ is given by a (one-parameter) group of unitary transformations. We proved a number of theorems which relate the existence of an EAC to the (dynamical) algebraic structure of the interaction Hamiltonian coupling $S$ and $E$. In particular we discussed the case of a quantum register symmetrically coupled with the environment. □From the broader point of view of the theory of open quantum systems, our results provide a
systematic way of building non-trivial models in which, under quite generic assumptions, the unitary evolution of a subspace is allowed, even while the remaining part of the Hilbert space gets strongly entangled with the environment.

Acknowledgements

One of the authors (P.Z.) thanks C. Calandra for providing hospitality at the University of Modena, and Elsag-Bailey for financial support.

References

[1] For reviews, see D.P. DiVincenzo, Science 270, 255 (1995); A. Ekert, and R. Josza, Revs. Mod. Phys. 68, 733, (1996)

[2] H. Bernstein, and U. Vazirani, Proc. 25-th ACM Symposium on the Theory of Computation, 1993; p. 11

[3] D. Deutsch, Proc. Royal Soc. London A 425, 73 (1989)

[4] P.W. Shor, Phys. Rev. A 52, 2493 (1995); A. Ekert, and C. Macchiavello, Phys. Rev. Lett. 77, 2585 (1996); D. Gottesman, Phys. Rev. A 52, 1862 (1996); A.R. Calderbank, E. M. Rains, P.M. Shor, and N.J. Sloane, Phys. Rev. Lett. 78, 405 (1997)

[5] K. Kraus, ”States, Effects, and Operations: Fundamental Notions of Quantum Theory”, Lecture Notes in Physics, 190, Springer, Berlin (1983)

[6] P. Zanardi, and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997)

[7] G. Lindblad, Commun. Math. Phys. 48, 119 (1976)

[8] A. Peres, ”Quantum Theory: Concepts and Methods”, Kluwer, Dordrecht (1993).