Mass singularity and confinement in QED

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Infrared behaviour of the fermion propagator is examined by spectral representation. Assuming asymptotic states and using LSZ reduction formula we evaluate the the lowest order spectral function by definition. After exponentiation of it we derive the non perturbative propagator. It shows confinement and dynamical mass generation explicitly.

1. Introduction

To search the infrared behaviour of the propagator in three dimensional QED ordinary perturbative method is not appropriate because of its had infrared divergences. Assuming spectral representation and LSZ reduction formula, we evaluate the infrared behaviour or the mass singularity of the fermion propagator in three dimensional QED as in four dimension. In quenched case with bare photon mass we obtain the Coulomb energy and Dynamical mass by lowest intermediate state. After exponentiation of this leads the non-perturbative propagator. Including massless vacuum polarization the short distance part of the propagator has a logarithmic correction. In the long distance Coulomb force is screened to suppress mass generation. Confining property and renormalization constants are discussed.

2. Position space propagator

2.1. spectral function

Here we assume spectral representation of the propagator which is defined by

\[ S_F(p) = \int d^3x \exp(-ip \cdot x) \langle \Omega | \psi(x) \bar{\psi}(0) | \Omega \rangle \]

\[ = \int ds \frac{\gamma \cdot \left(p_1(s) + p_2(s)\right)}{p^2 - s + i\epsilon}. \tag{1} \]

\[ \frac{1}{\pi} \Im S_F(p) = \int ds \delta(p^2 - s) \left[ \gamma \cdot \left(p_1(s) + p_2(s)\right)\right] = \gamma \cdot p_1(p) + \rho_2(p). \tag{2} \]

\[ \rho(p) = (2\pi)^3 \sum_N \delta(p - p_N) \int d^3x \exp(ip \cdot x) \times \langle \Omega | \psi(x) | N \rangle \langle N | \bar{\psi}(0) | \Omega \rangle. \tag{3} \]

Total three-momentum of the state \( |N\rangle \) is \( p_N^0 \). The only intermediates \( N \) contain one spinor and an arbitrary number of photons. Setting \( |N\rangle = |r; k_1, ..., k_n\rangle \),

where \( r \) is the momentum of the spinor \( r^2 = m^2 \), and \( k_i \) is the momentum of \( i \)th soft photon, we have

\[ \rho(p) = \int d^4x \exp(-ip \cdot x) \int \frac{md^2r}{r^0} \times \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int \frac{d^3k}{(2\pi)^3} \theta(k_0) \delta(k^2) \sum_i \delta(p - r - \sum_{i=1}^{n} k_i) \times \langle \Omega | \psi(x) | r; k_1, ..., k_n \rangle \langle r; k_1, ..., k_n | \bar{\psi}(0) | \Omega \rangle \right). \]

Here the notations

\[ (f(k))_0 = 1, (f(k))_n = \prod_{i=1}^{n} f(k_i) \] (6)

have been introduced to denote the phase space integral of each photon. The initial sum over \( \epsilon \) is a sum over polarization of photon. To evaluate the contribution of the soft photons, we consider when only the \( n \)th photon is soft. Our main problem is the determination of the matrix element. Here we define the following matrix element

\[ T_n = \langle \Omega | \psi | r; k_1, ..., k_n \rangle \]

\[ = \langle \Omega | \psi \phi^n_{in}(k_n) | r; k_1, ..., k_{n-1} \rangle. \] (7)
We consider $T_n$ for $k_n^2 \neq 0$, we continue off the photon mass-shell by Lehmann-Symanzik-Zimmermann (LSZ) formula:

$$T_n = e^{\mu} T_{n\mu},$$

$$e^{\mu} T_{n\mu} = \frac{i}{\sqrt{Z_3}} \int d^3 y \exp(ik_n \cdot y)$$

$$\times \Box_y \langle \Omega | T\psi(x) e^{\mu} A_\mu(y) | r; k_1, ..., k_{n-1} \rangle$$

$$= -\frac{i}{\sqrt{Z_3}} \int d^3 y \exp(ik_n \cdot x)$$

$$\times \langle \Omega | T\psi(x) e^{\mu} j_\mu(y) | r; k_1, ..., k_{n-1} \rangle ,$$

provided

$$\Box x T(\psi A_\mu(x)) = T \Box x A_\mu(x)$$

$$= T \psi(-j_\mu(x) + \frac{d-1}{d} \partial^\nu (\partial \cdot A(x))),$$

$$\partial \cdot A^{(\mu)}|_{\text{phys}} > 0,$$  

(9)  

where the electromagnetic current is

$$j_\mu(x) = -\Box x \psi(x) \gamma_\mu \psi(x).$$

(11)

From the definition (8) $T_n^{\mu}$ is seen to satisfies Ward-Takahashi-identity:

$$k_{n\mu} T_{n\mu}^{\mu}(r, k_1, ..., k_n) = eT_{n-1}(r, k_1, ..., k_{n-1}), r^2 = m^2,$$

(12)

provided the equal-time commutation relations

$$\partial^\nu T(\psi j_\mu(x)) = -e\psi(x),$$

$$\partial^\nu T(\psi j_\mu(x)) = e\psi(x).$$

(13)

In the Bloch-Nordsieck approximation we have most singular contributions of photons which are emitted from external lines. In perturbation theory one photon matrix element $T_1$ is given

$$\langle \Omega | T(\psi_{in}(x), i e \int d^3 y \bar{\psi}_{in}(y) \gamma_\mu \psi_{in}(y) A^{\mu}_n(y)) | r; k \ in \rangle$$

$$= \langle \Omega | T(\gamma_{\mu} \psi_{in}(x) A^{\mu}_n(y)) | r; k \ in \rangle$$

$$= i e \int d^3 y d^3 z S_P(x - y) \gamma_\mu \delta^{(3)}(y - z)$$

$$\times \exp(i(k \cdot y + r \cdot z)) \frac{e^{\mu}(k, \lambda) U(r, s)}{(r + k)^2 - m^2} \gamma_\mu e^{\mu}(k, \lambda) \exp(i(k + r) \cdot x) U(r, s),$$

(15)

where $U(r, s)$ is a four-component free particle spinor with positive energy. $U(r, s)$ satisfies the relations

$$\langle \gamma \cdot r - m \rangle U(r, s) = 0,$$

$$\sum_s U(r, s) \bar{U}(r, s) = \frac{\gamma \cdot r + m}{2m}.$$

(16)

In this case the Ward-Takahashi-identity follows

$$k_\mu T_{1\mu}^{\mu} = -ie \frac{1}{\gamma \cdot (r + k) - m} (\gamma \cdot k) U(r, s)$$

$$= -ie U(r, s) = eT_0.$$

(17)

For general $T_n$ low-energy theorem determines the structure of non-singular terms in $k_n$. Detailed discussions are given in ref [1] and non-singular terms are irrelevant for the single particle singularity in four-dimension. Under the same assumption in three-dimension we have

$$T_n|_{k_n^2 = 0} = e \frac{\gamma \cdot \epsilon}{\gamma \cdot (r + k) - m} T_{n-1}.$$

(18)

From this relation the $n$-photon matrix element

$$\langle \Omega | \psi(x) | r; k_1, ..., k_n \rangle \langle r; k_1, ..., k_n | \bar{\psi}(0) | \Omega \rangle$$

is reduced to the product of lowest-order one-photon matrix element

$$T_n T_n = \prod_{j=1}^{n} T_1(k_j) T_1^{+}(k_j) \gamma_0.$$  

(20)

In this case the spectral function $\rho$ in (5) is given by exponentiation of one-photon matrix element, which yields an infinite ladder approximation for the propagator. Thus we obtain the spectral function and the propagator in the followings.
forms

\[ \rho(p) = \int d^3x \exp(-ip \cdot x) \int \frac{md^2r}{r^0} \times \exp(ir \cdot x) \exp(F), \]  

\[ F = \sum_{\text{one photon}} (\Omega|\psi(x)|r; k) \langle r; k|\psi(0)|\Omega \rangle \]

\[ = \int \frac{d^2k}{(2\pi)^2} \delta(k^2)\theta(k_0) \exp(ik \cdot x) \sum_{\lambda,S} T_1T_1, \]

\[ \sum_{\lambda,S} T_1T_1 = e^{2i(r + k) \cdot \gamma + m(r + k)^2 - m^2} \frac{r \cdot \gamma + m}{2m} \times \gamma^\nu (r + k) \cdot r + m \]

\[ \times \gamma^\mu (r + k)^2 - m^2 \Pi_{\mu \nu}. \]  

(22)

Here \( \Pi_{\mu \nu} \) is the polarization sum

\[ \Pi_{\mu \nu} = \sum_{\lambda} \epsilon_{\mu}(k, \lambda)\epsilon_{\nu}(k, \lambda) = -g_{\mu\nu} - (d-1) \frac{k_\mu k_\nu}{k^2}. \]

(24)

and the free photon propagator is

\[ D_{\mu \nu} = \frac{1}{k^2 + i\epsilon} [g_{\mu\nu} + (d-1) \frac{k_\mu k_\nu}{k^2}]. \]

(25)

We get

\[ F = -e^2 \int \frac{d^3k}{(2\pi)^2} \exp(ik \cdot x)\theta(k^0) \times [\delta(k^2)(\frac{m^2}{r \cdot k^2} + \frac{1}{r \cdot k}) \]

\[ + (d-1) \frac{\delta(k^2)}{k^2}], \]

(26)

The second term \( \delta(k^2)/k^2 \) equals to \( -\delta'(k^2). \) Our calculation is the same with the imaginary part of the photon propagator. To avoid infrared divergence which arises in the phase space integral we must introduce small photon mass \( \mu \) as an infrared cut-off. Therefore (22) is modified to

\[ F = -e^2 \int \frac{d^3k}{(2\pi)^2} \exp(ik \cdot x)\theta(k^0) \times [\delta(k^2 - \mu^2)(\frac{m^2}{r \cdot k^2} + \frac{1}{r \cdot k}) \]

\[ - (d-1) \frac{\partial}{\partial k^2} \delta(k^2 - \mu^2)]. \]

(27)

It is helpful to use function \( D_+(x) \) to determine \( F \)

\[ D_+(x) = \frac{1}{(2\pi)^2} \int \exp(ik \cdot x)d^3k\theta(k^0)\delta(k^2 - \mu^2) \]

\[ = \frac{1}{(2\pi)^2} \int_0^\infty J_0(kx) \frac{\pi kd}{2\sqrt{k^2 + \mu^2}} = \frac{\exp(-\mu x)}{8\pi \mu x}. \]

(28)

If we use parameter trick (exponential cut-off)

\[ \lim_{\epsilon \to 0} \int_0^\infty d\alpha \exp(i(k + i\epsilon) \cdot (x + \alpha r)) = \frac{\exp(ik \cdot x)}{(k \cdot r)^2}, \]

\[ \lim_{\epsilon \to 0} \int_0^\infty d\alpha \exp(i(k + i\epsilon) \cdot (x + \alpha r)) = - \frac{\exp(ik \cdot x)}{(k \cdot r)^2}. \]

(29)

\[ F_1 = ie^2m^2 \int_0^\infty d\alpha \exp(-\mu |x| + |x|Ei(\mu |x|)), \]

\[ F_2 = -ie^2 \int_0^\infty d\alpha D_+(x + \alpha r) \]

\[ = -e^2 \frac{\exp(-\mu |x|)}{8\pi \mu} Ei(\mu |x|), \]

\[ F_g = ie^2(d-1) \frac{\partial}{\partial \mu} D_+(x) \]

\[ = \frac{(d-1)e^2}{8\pi \mu} \exp(-\mu |x|). \]

(32)

where the function \( Ei(\mu |x|) \) is defined

\[ Ei(\mu |x|) = \int_1^\infty \frac{\exp(-\mu |x| t)}{t} dt. \]

(34)

It is understood that all terms which vanishes with \( \mu \to 0 \) are ignored. The leading non trivial contributions to \( F \) are

\[ Ei(\mu |x|) = -\gamma - \ln(\mu |x|) + O(\mu |x|), \]

(35)

\[ F_1 = \frac{e^2m^2}{8\pi \mu^2} \left( \frac{1}{\mu} + |x| (1 - \ln(\mu |x|) - \gamma) \right) + O(\mu), \]

\[ F_2 = \frac{e^2}{8\pi \sqrt{r^2}} (\ln(\mu |x|) + \gamma) + O(\mu), \]

\[ F_g = \frac{e^2}{8\pi} \left( \frac{1}{\mu} - |x| \right)(d-1) + O(\mu), \]

(36)
\[ F = \frac{e^2}{8\pi \mu}(d - 2) + \frac{\gamma e^2}{8\pi \sqrt{r^2}} \]
\[ + \frac{e^2}{8\pi \sqrt{r^2}} \ln(\mu |x|) \]
\[ - \frac{e^2}{8\pi} |x| \ln(\mu |x|) - \frac{e^2}{8\pi} |x| (d - 2 + \gamma), \quad (37) \]

where \( \gamma \) is Euler’s constant. We set \( r^2 = m^2 \) in the phase space integral;

\[ \rho(p) = F.T \int \frac{md^2r}{\sqrt{r^2 + m^2}} \exp(ir \cdot x) \exp(F(m, |x|)) \]
\[ = F.T \frac{\exp(-m |x|)}{4\pi |x|} \exp(F(m, |x|)), \quad (38) \]
\[ \overline{\rho(x)} = \frac{\exp(-m |x|)}{4\pi |x|} \exp(F(m, |x|)). \quad (39) \]

Here \( \exp(-m |x|)/4\pi |x|, |x| = \sqrt{-x^2} \) is a free scalar propagator with physical mass \( m \) and \( \exp(F) \) denotes the quantum correction for the propagator involving infinite numbers of photons in our approximation.

### 2.2. Confining property

Here we mention the confining property of the propagator \( S_F(x) \) in position space

\[ S_F(x) = \left( \frac{i\gamma \cdot \partial}{m} + 1 \right) \frac{m \exp(-m |x|)}{4\pi |x|} (\mu |x|)^{D(1-m|x|)} . \quad (40) \]

The \( S_F \) dumps strongly at large \( x \) provided

\[ \lim_{x \to \infty} (\mu |x|)^{-Dm|x|} = 0. \quad (41) \]

The profiles of the \( \overline{\rho(x)} \) for various values of \( D \geq 1 \) are shown in Fig.1. The effect of \( (\mu |x|)^{-Dm|x|} \) in position space is seen to decrease the value of the propagator at low energy and shown in Fig.2.

In section IV and V we discuss dynamical mass, the renormalization constant, and bare mass in connection of each terms in \( F \).

After angular integration of (38), we get the propagator

![Figure 1. \( \overline{\rho(x)} \) for \( D = 1, 1.5, m = \mu = \text{unit} \)](image1)

![Figure 2. \( (\mu |x|)^{-Dm|x|} \), \( D = 1, m = \mu = \text{unit} \)](image2)
\[ S_F(p) = \frac{\gamma \cdot B}{m} + 1) \rho(p), \quad (42) \]

\[ \rho(p) = \frac{m}{2\pi \sqrt{-p^2}} \int_0^\infty \, d|x| \sin(\sqrt{p^2 x^2}) \]

\[ \times \exp(A - (m_0 + B) |x|)(\mu |x|)^{-C|x| + D}, \quad (43) \]

where

\[ A = \frac{e^2}{8\pi \mu} (d - 2) + \frac{\gamma e^2}{8\pi m}, B = \frac{e^2}{8\pi} (d - 2 + \gamma), \]

\[ C = \frac{e^2}{8\pi}; D = \frac{e^2}{8\pi m}. \quad (44) \]

If we discuss the Euclidean or off-shell propagator we can omitt the linear infrared divergent part in \( A \). In this case \( m \) denotes a physical mass

\[ m = |m_0 + \frac{e^2}{8\pi} (d - 2 + \gamma)|. \quad (45) \]

3. Analysis in momentum space

To search the infrared behaviour we expand the propagator in the powers of the coupling constant \( e^2 \) and obtained the Fourier transform of \( \mathcal{F}(x) \) [2,3]. In that case it is not enough to see the structure of infrared behaviour which can be compared to the well-known four dimensional QED[7]. Instead we make Laplace transformation to dynamical mass and wave function renormalization. To see this let us think about position space propagator

\[ \rho(x) = \frac{\exp(-m_0 |x|)}{4\pi |x|} (\mu |x|)^{-C|x|} (\mu |x|)^D. \quad (49) \]

It is easy to see that the probability of particles which are separated with each other in the long distance is suppressed by the factor \((\mu |x|)^{-C|x|}\), and the Coulomb energy modifies the short distance behaviour from the bare 1/|x| to 1/|x|\(^{1-D}\). The effect of Coulomb energy for the infrared behaviour of the free particle with mass \( m \) can be seen by its Fourier transform [2,9]

\[ 4\pi \int_0^\infty \, x^2 \frac{\sin(\sqrt{p^2 x^2})}{\sqrt{p^2 x^2}} \frac{\exp(-m_0 |x|)}{4\pi |x|} (\mu |x|)^D \, d|x| \]

\[ = \mu^D \Gamma(D + 1) \sin((D + 1) \arctan(\sqrt{-p^2/m})) \]

\[ \frac{\sqrt{-p^2/m}}{\sqrt{-p^2 + m^2}^{(1+D)/2}} \]

\[ \sim \mu^D (-p^2 - m)^{-1-D} \text{ near } p^2 = -m^2. \quad (51) \]

Above formula shows the structure in momentum space is modified for both infrared and ultraviolet regions. Usually constant \( D \) represents the coefficient of the leading infrared divergence for fixed mass in four dimension. Therefore Coulomb energy in three dimension has the same effects as in four dimension but change the ultraviolet behaviour since the coupling constant \( e^2 \) is not renormalized. Now we consider the role of \( M(x) \) as the dynamical mass at low momentum. First we define Fourier transform of the scalar part of the propagator;

\[ \rho(p) = F.T \left( \frac{m \exp(-m_0 |x|)}{4\pi |x|} (\mu |x|)^{-C|x| + D} \right). \quad (52) \]

If we use Laplace transformation

\[ F(s) = \int_0^\infty \exp(-s |x|)(\mu |x|)^{-C|x|} d|x| \]
we easily see that \((m |x|)^{-C|x|}\) acts as mass changing operator \(m \rightarrow m - s\)

\[
\exp(-m |x|)(m |x|)^{-C|x|} = \int ds F(s) \exp(-(m - s) |x|).
\]

To separate the \(\mu\) dependence we define \(m^* = m(1 + D \ln(\mu/m))\). \(F(s)\) is shown in Fig.4.

We have the complete expression of the spectral function

\[
\rho(x) = \frac{m \mu^D}{4\pi |x|^{1-D}} \int_0^\infty \exp(s |x|) F(s) ds,
\]

\[
\rho(p) = m \mu^D \Gamma(D + 1) \int_0^\infty ds F(s) \times \frac{\sin((D + 1) \arctan(\sqrt{-p^2/(m - s)})\)}{\sqrt{-p^2(-p^2 + (m - s)^2(1+D)/2)}}
\]

4. Bare mass and vacuum expectation value \(\langle \bar{\psi}\psi \rangle\)

In this section we examine the renormalization constant and bare mass and study the condition of vanishing bare mass based on spectral representation. The spinor propagator in position space is expressed in the following for \(\rho_1 = \rho_2 = \rho\) which is the case in our model[2]. The equation for the renormalization constant in terms of the spectral functions read

\[
m_0 Z_2^{-1} = m \int \omega \rho_2(\omega) d\omega = \frac{1}{4} \lim_{p \to \infty} tr[p^2 S_F(p)],
\]

\[
Z_2^{-1} = \int \rho_1(\omega) d\omega = \frac{1}{4} \lim_{p \to \infty} tr[\gamma \cdot p S_F(p)].
\]

we obtain

\[
m_0 Z_2^{-1} \sim \lim_{p^2 \to \infty} \sqrt{-p^2}^D = \begin{bmatrix} 0 & 0 \ end{bmatrix} \end{bmatrix},
\]

\[
Z_2^{-1} = \lim_{p^2 \to \infty} \sqrt{-p^2}^D = \begin{bmatrix} 0 & 0 \ end{bmatrix} \end{bmatrix}.
\]

This means that propagator in the high energy limit has no part which is proportional to the free one. Usually mass is a parameter which appears in the Lagrangean. For example chiral symmetry is defined for the bare quantity. In ref[6]
the relation between bare mass and renormalized mass of the fermion propagator in QED is discussed based on renormalization group equation with the assumption of ultraviolet stagnant point and shown that the bare mass vanishes in the high energy limit even if we start from the finite bare mass in the theory. It suggests that symmetry properties can be discussed in terms of renormalized quantities. In QCD bare mass vanishes in the short distance by asymptotic freedom. And the dynamical mass vanishes too[5]. In our approximation this problem is understood that at short distance propagator in position space tends to

$$\bar{\rho}(x)_{x \to 0} \to |x|^{D-1} (\mu |x|)^{-C|x|},$$  \hspace{1cm} (60)

where we have $\bar{\rho}(0) = \text{finite at } D = 1$ case which is independent of the bare mass $m_0$. Thus we have a same effect as vanishing bare mass in four dimensional model. Of course we have a dynamical mass generation which is $m = |\frac{\nabla^2}{\mu^2}(d-2+\gamma) + M(x)|$ for $m_0 = 0$ in our approximation. There is a chiral symmetry at short distance where the bare or dynamical mass vanishes in momentum space but its breaking must be discussed in terms of the values of the order parameter. Therefore it is interesting to study the possibility of pair condensation in our approximation. The vacuum expectation value of pair condensate is evaluated as

$$\langle \bar{\psi} \psi \rangle = -tr S_F(x) = -2 \int_0^\infty \frac{p^2 dp}{2\pi^2} \frac{\Gamma(D+1)\mu^D}{(D+1)^2p^2}  \left( \frac{p^2}{(m-s)} \right)^{(D+1)/2},$$  \hspace{1cm} (61)

is finite for $D \geq 1$ for finite cut-off $\mu$. In the weak coupling limit we obtain $Z_\rho = 1, m_0 = m$ and $\langle \bar{\psi} \psi \rangle = \infty$. If we introduce chiral symmetry as a global $U(2n)$, it breaks dynamically into $SU(n) \times SU(n) \times U(1) \times U(1)$ as in QCD[9,11] for $D = 1$ for finite infrared cut-off. Our model may be applicable to relativistic model of super fluidity in three dimension. Usually we do not find the critical coupling $D = 1$ in the analysis of the Dyson-Schwinger equation in momentum space where only continuum contributions are taken into account and we do not define physical mass. Finally we notice the effects of vacuum polarization[4,10,11,12]. In the presence of vacuum polarization with $N$ massless fermion the mass shift and dynamical mass we find in the quenched case are decreased by screening at short distance. In the long distance mass shift is strongly suppressed and we have only wave renormalization as in four dimension. We have an infrared behaviour of the propagator in the Landau gauge

$$S_F(p) \sim \left( \frac{\mu}{m} \right)^{D'} \frac{p + m}{2m^2} (1 - \frac{p^2}{m^2})^{-(1+D')}, \quad D' = \frac{8}{N\pi^2}. \hspace{1cm} (62)$$

5. References

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