Market Fluctuations II: multiplicative and percolation models, size effects and predictions

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Abstract

We present a set of models of the main stylized facts of market price fluctuations. These models comprise dynamical evolution with threshold dynamics and Langevin price equation with multiplicative noise, percolation models to describe the interaction between traders and hierarchical cascade models to unravel the possible correlation across time scales, including the log-periodic signatures associated to financial crashes. The main empirical knowledge is summarized and some key empirical tests are presented.
1 Stylized facts of financial time series

The attraction of physicists to finance and to the study of stock markets is grounded on several factors.

• Physics and finance are both fundamentally based on the theory of random walks (and their generalizations to higher dimensions) and on the collective behavior of large numbers of correlated variables. Finance thus offers another fascinating playground for the application of concepts and methods developed in the Natural Sciences which have traditionally focused their attention on a description and understanding of the surrounding inanimate world at all possible scales.

• Stock markets offer maybe one of the simplest real life experimental system of co-evolving competing learning agents and can thus be thought of as a proxy for studying biological evolution [1, 2, 3, 4].

• It is tempting to believe that the technical abilities developed in the Physical Sciences could help to “beat the market”: predicting a complex time series like the market price evolution shown in figure 1 is an exciting intellectual challenge as well as potentially rewarding financially.

In this short review, we present a series of models of stock markets that each provide a particular window of understanding. The different models do not play the same role. In its broadest sense, recall that a model (usually formulated using the language of mathematics) is a mathematical representation of a condition, process, concept, etc, in which the variables are defined to represent inputs, outputs, and intrinsic states and equations or inequalities are used to describe interactions of the variables and constraints on the problem. In theoretical physics, models take a narrower meaning, such as in the Ising, Potts,..., percolation models. In economy and finance, the term model is usually used in the broadest sense. Here, we will use both types of models: The microscopic threshold models of the interplay between supply and demand discussed in section 2 and the percolation models of section 3 fall in the second category. The cascade of correlations across scales [5] briefly summarized in section 4 belong to this first class of models. The log-periodic signatures preceding crashes also discussed in section 4 relies on both types of models. This diversity of models reflects our burgeoning understanding of this field which has not yet fully matured.

The first most striking observation of a stock market is that prices variations seem to fluctuate randomly, leading to a price trajectory as a function of time which looks superficially similar to a random walk with markovian increments, as shown in the upper left panel (a) of figure 1. This view was first expoused by Bachelier in his 1900 thesis [6] and later formalized rigorously by Samuelson [7]. This fundamental thesis in finance is called the
efficient market hypothesis and states in a nutshell that price variations are essentially random as a result of the incessant activity of traders who attempt to profit from small price differences (so-called arbitrage opportunities); the mechanism is that their investment strategies produce feedbacks on the prices that become random as a consequence. One important domain of research consists in determining the detailed mechanisms by which this feedback operates dynamically and statistically. Correlatively, the search of deviations from this efficient market state may lead to significant understanding on the way the markets function.

At first glance, the concept that price variations are uncorrelated is confirmed by looking at the two-point correlation function of the price increments. For liquid markets such as the Standards and Poor’s (SP&500), it is found significantly different from zero with statistical confidence only for very short times of the order of a few minutes, as shown in panel (a’) of figure 1. On the other hand, the correlations of the amplitude of the variations, called the volatility, are very long-ranged. This can be visualized qualitatively by looking at panel (b) of figure 1 which constructs a random walk by successive addition of the logarithm of a measure of the amplitude of the price variation obtained through a wavelet transform (see [5] for details). One observes long periods of persistences which can be compared with the random walk in panel (c) obtained by first reshuffling the price variations and performing the same analysis as for panel (b). Panel (b’) and (c’) show the correlation functions of the volatilities corresponding respectively to panels (b) and (c). For the real SP&500 time series, one observes an extremely slow approximately power law decay, which provides a measure of the well-documented clustering or persistence of volatility.

In addition to the almost complete absence of correlation of price increments and the long-range correlation of volatilities, the last striking stylized fact is the “fat tail” nature of the distribution of price variations or of returns. The qualification “fat tail” is used to stress that the distributions of price variations decay usually more slowly than a Gaussian, which is taken as the reference that would be valid under the random walk hypothesis. Exponentially truncated Lévy laws [8, 9, 10] with exponent around $\alpha \approx 1.5$ (see definition (15)) for the 6-year period 1984-1989 and power laws with exponents $\alpha \approx 3$ [11, 12], superposition of Gaussian motivated by an analogy with turbulence [13, 14] or stretched exponentials [14] have been proposed to describe the empirical distribution of price returns in organized markets.

In addition to these three well-documented stylized facts, many other studies have been performed to test for the possible existence of dependence between successive price variations at many different time scales that go beyond the method of correlation functions. There is indeed increasing evidences that even the most competitive markets are not completely free from correlations (i.e. are not strictly “efficient”) [1]. In particular, a set of studies in the academic finance literature have reported anomalous earnings
Figure 1: 
(a) Time evolution of $\ln P(t)$, where $P(t)$ is the S&P 500 US index, sampled with a time resolution $\delta t = 5$ min in the period October 1991-February 1995. The data have been preprocessed in order to remove "parasitic" daily oscillatory effects: if $m_i$ and $\sigma_i$ are respectively the mean and the r.m.s. of the signal within the i-th 5-min. interval of a day, the value of the signal $x(i)$ has been replaced by $m + \sigma (x(i) - m_i)/\sigma_i$. 
(b) The corresponding "centered log-volatility walk", $v_\alpha(t) = \sum_{i=0}^t \tilde{\omega}_\alpha(i)$, as computed with the derivative of the Haar function as analyzing wavelet for a scale $a = 4$ ($\approx 20$ min). 
(c) $v_\alpha(t)$ computed after having randomly shuffled the increments of the signal in (a). 
(a') The 5 min ($a = 1$) return correlation coefficient $C_{11}^r(\Delta t)$ versus $\Delta t$. 
(b') The correlation coefficient $C_{1}^{\omega}(\Delta t)$ of the log-volatility of the S&P 500 at scale $a = 4$ ($\approx 20$ min); the solid line corresponds to a fit of the data using Eq. (23) with $\lambda^2 \simeq 0.015$ and $T \simeq 3$ months. 
(c') same as in (b') but for the randomly shuffled S&P 500 signal. In (a'-c') the dashed lines delimit the 95% confidence interval. Taken from [5].
which support technical analysis strategies \cite{13,16,17,18} (see \cite{19} for a different view). A recent study of 60 technical indicators on 878 stocks over a 12-year period \cite{20} finds that the trading signals from technical indicators do on average contain information that may be of value in trading, even if they generally underperform (without taking due consideration to risk-adjustments of the returns) a buy-and-hold strategy in a rising market by being relatively rarely invested.

In this review, we are going to present several models that propose to explain some of these empirical observations. Microscopic models of unbalance between supply and demand discussed in the next section provide an understanding for the possible existence of different market phases, such as random, bubble-like and cyclic as well as a simple mechanism for “fat tails” based on multiplicative noise. As we will summarize in section 3, the percolation models give probably the simplest possible mechanisms for the observation of “fat tails” and long-range volatility clustering while being compatible with the absence of correlations of price variations. They also teach us that finite size effects are probably important, i.e. the number $N$ of traders that count on the market is perhaps no more than a few hundreds to a few thousands. This is because most of the complexity observed in these and other similar models disappear when the limit of large $N$ is taken. If correct, this suggests that a suitable modeling of the stock market belongs to the most difficult intermediate asymptotics between a few degrees of freedom and the thermodynamic limit. As a consequence, it should take into account effects of discreteness. This might be one of the ingredients at the basis of the observation of the remarkable log-periodic signatures preceding large crashes discussed in section 4. The accumulate evidence now comprises more than 35 five crashes and to our knowledge, no major financial crash preceded by an extended bubble has occurred in the past two decades without exhibiting log-periodic signature. The exception is the East-European stock markets which seem to be following a completely different logic than their larger Western counterparts and their indices does not resemble those of the other markets. In particular, we find that they do not follow neither power law accelerations nor log-periodic patterns though large crashes certainly occurs.

\section{Fluctuations of demand and supply in open markets}

\subsection{Optimization of supply faced to an uncertain demand}

Contrary to the common sense in economics, demand and supply do not balance in reality. You can find that all shelves are always full of commodities in any department store in developed countries implying that supply is in
excess. On the other hand, people are sometimes making queues in front of a popular bakery shop and fresh baked croissants are sold out immediately, which clearly shows that demand is in excess. Such excess-supply or excess-demand states can be shown to be maximal profit strategies if we take into account the fluctuations of demand as follows.

Let us define the variables needed to describe the bakery’s strategy:

1. $x$, the selling price of a croissant;
2. $y$, its production cost;
3. $s$, the production number of croissant per a day;
4. $n$, the number of croissants requested by customers per day;
5. $d$, the demand which is the averaged value of $n$.

We assume that $n$ fluctuates in the interval $[d - \delta, d + \delta]$ uniformly for simplicity, and we also assume that the remainders will be thrown away at the end of each day. The problem is what is the best $s$ which maximizes the total profit. Obviously if $s = d + \delta$ then the bakery does not miss any customer’s request and the gross sale is maximal. However, there is a possibility that it will have many unsold croissants when $n$ is small and in that case the production cost of the remainders may cause a big loss. At the other extreme, if $s = d - \delta$, the bakery sells all its croissant and has no loss but on the other hand misses good selling opportunities. Therefore, there should be an optimal value of $s$ between these two extremes that maximizes the expectation of the total profit.

Let us denote the expectation of the total profit by $L(s)$, then we have the following evaluation,

$$L(s) = \langle x \min(n, s) - ys \rangle = (x - y)(d - \delta y/x) - \frac{x}{4\delta} \{s - d - \delta(1 - 2y/x)\}^2$$ \hspace{1cm} (1)

The maximal value of $L$ is given by the following value of $s$;

$$s^* = d + \delta(1 - 2y/x) \ .$$ \hspace{1cm} (2)

From this equation, it is clear that in the case where the sale price is not very high, here $x < 2y$, the optimal producing $s^*$ is smaller that the average demand $d$. This corresponds to the excess-demand state which the popular bakery shop follows. On the other hand, if the sale price $x$ is higher than twice the production price $y$, then the best strategy is to keep the excess-supply state just like all department stores actually do ($s^* > d$). It should be noted that the balanced state of $s^* = d$ is the best strategy only when $x = 2y$. This is the reason why almost all commodities in our daily life are out of the
balance of demand and supply. The coefficient 2 is of course modified if we assume a different probability density for the fluctuation of demand \( n \). The key point in this discussion is the fluctuation of demand that is inevitable in any free economy society, and the best strategy taking such effect into account proves that the balance of demand and supply should almost always be broken to earn largest income on average.

A similar result is obtained if the bakery follows a different strategy, i.e. strives to minimize its probability of loss: the probability of losing is the same as the probability that \( x \min(n, s) - ys \) be negative. In the interesting regime where \( x > y \), this probability is the same as the probability for the total sale \( xn \) to be less than the total production cost \( ys \). This leads to a probability to lose equal to

\[
\text{Prob}_{\text{loss}} = \frac{y}{2x\delta} [s - s^{**}],
\]

where

\[
s^{**} = (d - \delta) \frac{x}{y}.
\]

We see that the production \( s^{**} \) that gives no loss with certainty is larger than the average demand \( d \) only if the sale price \( x \) is larger than \( \frac{d}{x\delta} y \).

### 2.2 Consequence for the bid-ask spread in liquid markets

In an open market such as stock markets or foreign currency exchange markets, the situation is very different because there are speculative dealers who try to earn money by changing their position from a seller to a buyer or vice versa rather frequently. By this effect, the demand and supply can not be regarded as independent functions and furthermore we need to introduce a dynamic model to describe the pricing process correctly. Willing to buy from a market maker for instance, you will buy a stock at the ‘ask’ price \( p_{\text{ask}} \equiv x \) and resell it at the ‘bid’ price \( p_{\text{bid}} \equiv y \). The spread \( \delta p_{\text{spread}} = p_{\text{ask}} - p_{\text{bid}} = x - y \) is usually small. Indeed, the relevant situation for a liquid market is that the ‘ask’ price \( p_{\text{ask}} \equiv x \) is only slightly larger than the ‘bid’ price \( p_{\text{bid}} \equiv y \):

\[
x = y(1 + \epsilon), \quad \text{with } \epsilon << 1.
\]

Expanding (2) for small \( \epsilon \) gives

\[
\frac{s^* - (d - \delta)}{\delta} \approx \frac{\delta p_{\text{spread}}}{p_{\text{bid}}},
\]

i.e. the relative over-supply with respect to the minimum possible value \( d - \delta \) is essentially equal to the relative spread. The implication of this result (6) is the following: reading (3) from right to left, we find that a market maker will
be tempted to increase the spread between bid and ask if he has difficulty in getting rid of excess inventory, but this will be a smaller effect, the larger are the fluctuations of the demands, i.e. the possibility of selling in future occasions.

### 2.3 Microscopic model of market with threshold dynamics

We assume that every dealer in an open market has two prices in mind, the selling and buying prices. For each dealer, the buying price is always lower than the selling price, and the difference of these prices may represent his greediness. A dealer’s action is rather simple, namely, if the market price is higher than the selling price in mind she will sell, and if the market price is lower than the buying price in mind he will buy. Let us assume the simplest case that there are only two dealers, $A$ and $B$, and let their prices in minds be, $p_b(A)$, $p_s(A)$, $p_b(B)$ and $p_s(B)$, where the subscripts $b$ and $s$ represent the buying and selling prices and the capital letters specify the dealers, $A$ and $B$. When these prices are changed continuously, a trade occurs suddenly when either of the following two conditions is realized [21]:

$$p_b(A) \geq p_s(B) \quad \text{or} \quad p_b(B) \geq p_s(A).$$

(7)

Note that the occurrence of a trade is characterized by a nonlinear function such as a step function.

As the greediness of the dealers always require $p_b(A) < p_s(A)$ and $p_b(B) < p_s(B)$, there is no possibility of realizing the two conditions of Eq.(7) simultaneously, namely, the transaction is microscopically one-sided or irreversible. After the trade, these dealers renew their prices in their mind so that the trade condition does not hold any more.

Due to the nonlinear and irreversible nature of trades, dynamic models of dealers generally behave chaotically even if the dynamics is deterministic. There is a nonlinear effect that enhances any microscopic difference, but the estimated maximum Lyapunov exponent is 0 implying that the system is at the edge of chaos [21].

There are two extreme cases in this type of deterministic dealer models: one is the large asset limit and the other is the small asset limit.

- In the case of large asset limit, dealers are assumed to have an infinite amount of asset and all dealers can keep their positions, namely, a buyer can be always a buyer and a seller can be always a seller. In this limit, it is shown that there is a kind of phase transition behavior between excess-demand and excess-supply states as a function of the number ratio of buyers to sellers. In the excess-demand state, there are more buyers than sellers and the prices fluctuate with a linear upgrade trend [23]. In the excess-supply phase, the situation is just opposite. At the
critical point, that is realized when the numbers of buyers and sellers are the same, there is no trend and the power spectrum of the price fluctuations follow an inverse square law implying that the fluctuations are quite similar to the Brownian motion.

- In the small asset limit, each dealer changes position alternatively between a buyer and a seller, namely, after the dealer bought a stock, he tries to sell the stock. As all dealers change their positions alternatively, the numbers of demand and supply automatically balances and the system always shows critical behaviors, namely, the price fluctuations are similar to the Brownian motion even though the dynamics is deterministic [31]. This result indicates that the existence of speculative dealers who frequently change their positions is essential for the market to follow a random walk scale-free behavior. Note that this kind of stationary self-organized criticality must be distinguished from the critical behavior describing large crashes as described in section 4. The two phenomena are not mutually excluding as shown for instance in ref. [22].

As dealers in any open market are sensitive to the market price changes, it is important to introduce a response effect in the dealer model to explain the fat-tail distribution of price changes as reported e.g. by Mantegna and Stanley [3]. When dealers change their buying and selling prices in mind based on their own strategy independent of market price changes, the resulting price change distribution does not have long tails of power law. However, by adding the term that uniformly shifts all the dealers prices in mind proportional to the latest market price change, the distribution of market price changes become a power law in general [31].

2.4 Derivation of Langevin market dynamics with multiplicative noise

The reason for the fat-tail distributions can be theoretically explained by introducing a Langevin type stochastic equation with multiplicative noise:

\[ \Delta P(t + \Delta t) = B(t)\Delta P(t) + F(t) \]  

(8)

Here, \( P(t) \) represents the market price at time step \( t \) and \( \Delta P(t) \equiv P(t) - P(t - \Delta t) \) is the price change where \( \Delta t \) is the unit time interval. The effect of dealers’ response on the market price change is given by \( B(t) \), which is regarded as a random variable. The random additive term \( F(t) \) is due to the chaotic behaviors inherent in the dealer model.

In the low asset limit, it can be shown that the market price changes of the deterministic dealer model nicely approximated by the multiplicative stochastic process described by Eq. (8) [31]. We now present a more direct
derivation of Eq.(8) by considering the dealers' dynamics in a macroscopic way [32]. Let \( p_b(j,t) \) and \( p_s(j,t) \) be the \( j \)-th dealer's buying and selling prices at time \( t \), then the total balance of demand and supply in the market is described by the following function called the cumulative demand, \( I(P,t) \):

\[
I(P,t) = \sum_j \Theta(p_b(j,t) - P) - \Theta(P - p_s(j,t)),
\]

where \( \Theta(x) \) is the step function which is 0 for \( x < 0 \) and is 1 for \( x > 0 \). When \( P \) is such that \( I(P,t) > 0 \), the number of buyers is larger than that of sellers at the price. Therefore, the balanced price at time \( t \), \( P^*(t) \), is given by the equation, \( I(P^*(t), t) = 0 \). It is a natural assumption for an open market that the price change in a unit time is proportional to \( I(P(t), t) \) when the market price is \( P(t) \); therefore, we have the following equation:

\[
P(t + \Delta t) - P(t) \propto I(p(t), t).
\]

As the buying and selling prices are not announced openly, no one knows the value of \( P^*(t) \). Traders can only estimate it from the past market price data \( \{P(t - \Delta t), P(t - 2\Delta t), \ldots\} \). Taking into account the effect that each dealer thinks in a different way, we can write down the time evolution equation of \( P^*(t) \) as follows:

\[
P^*(t + \Delta t) = P^*(t) + F(t) + W(P(t), P(t - \Delta t), \ldots),
\]

Here, \( F(t) \) represents a random variable showing the statistical fluctuation of dealers’ expectation, and \( W(P(t), P(t - \Delta t), \ldots) \) is the averaged dealers’ response function. Considering the simplest non-trivial case, we have the following set of linear equations.

\[
P(t + \Delta t) = P(t) + A(t)(P^*(t) - P(t))
\]

\[
P^*(t + \Delta t) = P^*(t) + F(t) + B(t)(P(t) - P(t - \Delta t)).
\]

Here, \( A(t) \) is given by the inverse of the slope of \( I(P(t),t) \) at \( P = P^*(t) \) which is proportional to the inverse of the price elasticity coefficient in economics, and \( B(t) \) shows the dealers’ mean response to the latest market price change, and both of these coefficients can be random variables. If we can assume that \( P(t) \) and \( P^*(t) \) are always very close, the set of Eqs.(12) and (13) become identical to Eq.(8). Namely, if the market price always follows the motion of the balanced price and if the dealers’ responses to the latest price change averaged over all the dealers fluctuates randomly for different time, then the market price fluctuation is well-approximated by the Langevin type equation, Eq.(8).

It is well known that such a stochastic process (8) generally produces large fluctuations following power law distributions when \( B(t) \) takes larger values than unity with finite probability [24, 25, 26, 27, 28, 29, 30]. A important
condition to get a power law distribution is that the multiplicative noise \( B(t) \) must sometimes take values larger than one, corresponding to intermittent amplifications. This is not enough: the presence of the additive term \( F(t) \) (which can be constant or stochastic) is needed to ensure a “reinjection” to finite values, susceptible to the intermittent amplifications. It was thus shown [29] that (8) is only one among many convergent \( \langle \ln B(t) \rangle < 0 \) multiplicative processes with repulsion from the origin (due to the \( F(t) \) term in (8)) of the form

\[
x(t+1) = e^{H(x(t), \{b(t), f(t), \ldots\})} B(t) x(t) ,
\]

such that \( H \to 0 \) for large \( x(t) \) (leading to a pure multiplicative process for large \( x(t) \)) and \( H \to \infty \) for \( x(t) \to 0 \) (repulsion from the origin). \( H \) must obey some additional constraint such a monotonicity which ensures that no measure is concentrated over a finite interval. All these processes share the same power law probability density function (pdf)

\[
P(x) = Cx^{-1-\alpha}
\]

for large \( x \) with \( \alpha \) solution of

\[
\langle B(t)^\alpha \rangle = 1 .
\]

The fundamental reason for the existence of the powerlaw pdf (15) is that \( \ln x(t) \) undergoes a random walk with drift to the left and which is repelled from \(-\infty\). A simple Boltzmann argument [29] gives an exponential stationary concentration profile, leading to the power law pdf in the \( x(t) \) variable.

These results were proved for the process (8) by Kesten [25] using renewal theory and was then revisited by several authors in the differing contexts of ARCH processes in econometry [26] and 1D random-field Ising models [27] using Mellin transforms, and more recently using extremal properties of the \( G \)-harmonic functions on non-compact groups [28] and the Wiener-Hopf technique [29].

In the case that \( B(t) \) depends on \( \Delta P(t) \) especially when it does not take a large value if the magnitude of price change exceeds a threshold value, exponential cutoffs appear in the tails of distribution of price changes resulting in a more realistic distributions [31].

There are cases where the behaviors of the set of equations, Eqs.(12) and (13), deviate from that of Eq.(8). For example, in the special case that \( B(t) \) is larger than 1 and \( A(t) \) is smaller than 1 for a certain time interval then both \( P(t) \) and \( P^*(t) \) grow nearly exponentially and the difference of these values also grow exponentially. This case corresponds to the phenomenon called a bubble [33]. We can also find an oscillatory behavior of market price when \( A(t) > 1 \). Namely, the set of price equations derived theoretically can show typical behaviors of second order difference equation for different parameter combinations, as also proposed in Refs.[34, 3]. Real data analysis based on this formulation is now under intensive study.
3 Percolation Models

3.1 Basic percolation model of market price dynamics in 2 to infinite dimensions

Besides the Levy-Levy-Solomon model \cite{35}, the Cont-Bouchaud model \cite{36} seems to be the one investigated by the largest number of different authors. It uses the well-known percolation model and applies its cluster concept to groups of investors acting together. This percolation model thus, similar to the random-field Ising markets \cite{37}, applies physics knowledge collected over decades, instead of inventing new models for market fluctuations.

In percolation theory \cite{38}, every site of a large lattice is occupied randomly with probability $p$ and empty with probability $1 - p$; a cluster is a group of neighbouring occupied sites. For $p$ above some percolation threshold $p_c$, an infinite cluster appears spanning the lattice from one side to the opposite side. The average number $n_s(p)$ of clusters containing $s$ sites each varies for large $s$ right at the percolation threshold as a power law:

$$n_s \propto s^{-\tau}$$

with an exponent $\tau$ increasing from about 2.05 in two dimensions to $5/2$ in six and more dimensions. Close to $p_c$ a scaling law for large $s$ holds:

$$n_s = s^{-\tau} f ((p - p_c)s^\sigma)$$

with $\sigma \simeq 0.5$. For $p < p_c$, the cluster numbers decay asymptotically with a simple exponential, while above $p_c$ they follow a stretched exponential with $\log(n_s) \propto -s^{1-1/d}$ in $d$ dimensions.

Quite similar results are obtained if we switch from this site percolation problem to bond percolation, where all sites are occupied but the bonds between nearest neighbours are occupied with probability $p$; then clusters are groups of sites connected by occupied bonds.

For dimensionality $d > 6$, the critical exponents like $\tau$ are those of the Bethe-lattice or mean-field approximation, invented by Flory in 1941, where no closed loops are possible and for which analytic solutions are possible: $\tau = 5/2$, $\sigma = 1/2$, $f = \text{Gaussian}$. In three dimensions, most percolation results are only estimated numerically. Infinite-range bond percolation is also called random graph theory; then every site can be connected with all other sites, each with probability $p$. This infinite-range bond percolation limit was selected by Cont and Bouchaud \cite{36} in order to give exact solutions, while the later simulations concentrated on two- or three-dimensional site percolation, with nearest neighbours only forming the clusters.

For market applications, the occupied sites are identified with investors, and the percolation clusters are groups of investors acting together. Thus at each iteration, every cluster has three choices: all investors belonging to
the cluster buy (probability \(a/2\)); all of them sell (also probability \(a/2\)); and none of them acts at this time step (probability \(1-a\)). Thus the activity \(a\) measures the time with which we identify one iteration: if this time step is one second, \(a\) will be very low since very few investors act every second; if the time step is one year, \(a\) will be closer to its maximum value \(1/2\). All investors trade the same amount, and have an infinite supply of money and stocks to spend. Summation over all active clusters gives the difference between supply and demand and drives the price \(P(t)\):

\[
R(t) = \left[ P(t + 1) - P(t) \right] / P(t) \propto \sum_{\text{buy}} n_s s - \sum_{\text{sell}} n_s s
\]

In this way, the Cont-Bouchaud model has for a given lattice very few free parameters: the occupation probability \(p\) and the activity \(a\). Moreover, algorithms to find the clusters in a randomly occupied lattice are known since decades [38], and thus a computer simulation is quite simple if one has already a working (Hoshen-Kopelman) algorithm to find clusters.

Without any simulation [36], one can predict the results for very small \(a\). If for a lattice of \(N = L^d\) sites, we have \(a\) of order \(1/N\), then typically no cluster, or only one, is active during one iteration. The price change then is either zero or \(\pm\) the size \(s\) of the cluster. The distribution of absolute returns \(|R|\) thus is identical to the cluster size distribution \(n_s\), apart from a large contribution at \(R = 0\). In particular, right at the percolation threshold \(p = p_c\) we have a distribution \(\pi(|R|)\) of returns obeying a power law

\[
\pi(|R|) \propto 1/R^\tau
\]

for not too small \(|R|\), similar to Mandelbrot’s Lévy-stable Pareto distributions [39]. The probability to have a jump of at least \(|R|\) then decays asymptotically as \(1/|R|^\alpha\) with \(\alpha = \tau - 1\) between 1 and 3/2. The volatility or variance of the return distribution is thus infinite at the percolation threshold, apart from finite-size and finite-time corrections; the same holds for skewness and kurtosis.

For larger activities, but still \(a \ll 1\), scaling holds [40]: if we normalize height and width of the return distribution to unity, the curves for various activities \(a\) at \(p = p_c\) overlap, and thus still give the above power law. This scaling is no longer valid for large \(a \simeq 1/3\) where the curves become more like a Gaussian.

This model thus reproduces some stylized facts of real markets, when inflation effects are subtracted: i) The average return \(\langle R \rangle\) is zero. ii) There is no correlation between two successive returns or two successive volatilities, since all active clusters decide randomly and without memory whether to buy or to sell, and since the occupied sites are distributed randomly. iii) At the percolation threshold, a simple asymptotic power law holds for small activities (short times) and becomes more Gaussian for large \(a\) (long times).
This latter crossover to Gaussians, seen also in some analyses of real markets \[12, 41\], is not seen if we replace the percolation model by a Lévy walk for the price changes \[42\], where the return is a sum of steps distributed with the same power law exponent \( \tau \) as the above percolation clusters. In this simplification, the power law remains valid also for large \( a \) without a crossover to Gaussians. Note that in percolation, as opposed to Lévy walks, the clusters are correlated by the sum rule \( \sum_{s} n_{s} s = pN \).

### 3.2 Improvements of the percolation model

#### 3.2.1 Clustering by diffusion

Another advantage of this percolation model compared with Lévy walks is volatility clustering. While \( \langle R(t)R(t+1) \rangle \) in real markets decays rapidly to zero (but see \[4\] for different information), the autocorrelations of the absolute returns \[43\] \( \langle |R(t)R(t+1)| \rangle \) decay slowly as shown in panel (b') of figure \[4\] (see also e.g. Fig.2 of \[44\]: a turbulent day on the stock market is often followed by another turbulent day, though the sign of change for the next day is less predictable. We simulate this volatility clustering by letting the investors diffuse slowly on the lattice; thus in the above picture, a small fraction of the investors move to another neighbourhood of the city where they get a different advice from a different expert. Now the autocorrelation functions decay smoothly, with unexplained size effects \[45\].

#### 3.2.2 Feedback from the last price

So far the model assumes the investors or their advisors to be complete idiots: They decide randomly whether to buy or to sell, without regard to any economic facts. Such an assumption is acceptable for the author from Cologne since the local stock market is in Düsseldorf, not Cologne. However, the discussions of log-periodic oscillations earlier in this review made clear that not everything should be regarded as random. The simplest way to include some economic reason is the assumption that prudent investors prefer to sell if the price is high and to buy if it is low. Thus the probabilities to buy or to sell are no longer \( a/2 \) but are changed by an amount proportional to the difference between the actual price and the initial price; the latter one is regarded as the fundamental or just price. Surprisingly, simulations \[46\] show that the distribution \( \pi(R) \) is barely changed; as expected the price itself is now stabilized to values close to the fundamental price. Little changes if we allow the fundamental price to undergo Gaussian fluctuations as in \[43\].

The distribution of the wealths of the investors can be investigated only if one gives each investor a finite initial capital, and adds to it the profits and subtracts the losses made by the random decisions to buy and sell. Bankrupt investors are removed from the market. Simulations \[47\] give reasonable...
return distributions, but in disagreement with reality [48] no clear power laws with universal exponents.

3.2.3 How to get the correct empirical exponent $\alpha \approx 3$?

The above power law $\pi \propto 1/R^\tau$ with $2 < \tau \leq 2.5$ may have been sufficient some time ago [39, 49] for which Lévy stable distributions requiring $\tau = \alpha + 1 < 3$ could be qualified, but today’s more accurate statistics shows fat tails decaying faster with $\tau > 3$, though slower than a Gaussian. They may be such power laws multiplied with an exponential function, also called truncated Lévy distributions [8, 10], or stretched exponentials [14], or most likely power laws with an exponent near $\alpha = \tau - 1 = 3$ [12, 11].

Several ways were invented to correct this exponent and get $\alpha \simeq 3$. One may work with $p$ slightly above $p_c$, where an effective power law with $\alpha = 3$ can be seen over many orders of magnitude [45]. (In this case, as is traditional for percolation studies, one omits the contribution from the infinite cluster.) Or one integrates over all $p$ between zero and the percolation threshold, thus avoiding the question how investors work at $p = p_c$ without ever having read a percolation book [38]; now $\alpha = \tau - 1 + \sigma \simeq 1.5$ to 2 [50]. Much better agreement with the desired $\alpha \simeq 3$ is obtained if we follow Zhang [4] and take the price change $R$ not linear in the difference between supply and demand, as assumed above, but proportional to the square root of this difference. Then $\alpha = 2(\tau + \sigma - 1)$ is about 2.9 in two dimensions, just as desired. Numerically [50], this power law could be observed over five orders of magnitude, similar to reality [12]. Changing the activity $a$ proportionally to the last known price change breaks the up-down symmetry for price movements; now sharp peaks in the price, with high activity, are followed by calmer periods with low prices and low activity [11].

Fig. 2 shows price change versus time, both in arbitrary units, for 0.001 as the lower limit for the activity in the model of [51]. Clearly, we see sharp peaks but not equally sharp holes (the downward trend also indicates the survival probability of the first author if the Nikkei index fails to obey the prediction of Fig. 1 below). Fig. 3 shows the desired slow decay of the autocorrelation function for the volatility of this market model, and for the same simulation Fig. 4 gives the histogram of price changes.

In the opposite direction, Focardi et al [52] assume the price change to be quadratic in the difference between supply and demand when the market gets into a crash. Using also other modifications of the infinite-range Cont-Bouchaud model, their simulations show exponentially growing prices followed, at irregular time intervals, by rapid crashes.

3.2.4 Log-periodicity and finite size effects

None of these models has the ingredients which seem needed for log-periodic oscillations before or after crashes, see Sec.4.2. Percolation can give such
Figure 2: Single run of price versus time over 3,000 iterations, where the activity is between 0.001 and 0.5. Such a simulation takes less than a minute on a workstation. The units for the price change and the time are arbitrary. The choice of parameters exaggerates on purpose the asymmetry between flat valleys and sharp peaks.
Figure 3: Autocorrelations for the volatility, averaged over 4800 simulations similar to figure 2, requiring $10^2$ hours simulation time on a Cray-T3E.
Figure 4: Histogram for positive price changes on double-logarithmic scales. The straight line has slope $-(1 + \alpha) = -4$ \cite{74}. Negative price changes show the same behaviour.
oscillations if we let particles diffuse on the occupied sites of the infinite cluster for \( p > p_c \), and if there is one preferred and fixed direction for this diffusion ("bias")\(^{[5]}\). Now the rms displacement of the diffusors varies approximately as a power \( t^k \) of time, and the effective exponent \( k(t) \) approaches unity with oscillations \( \propto \sin(\lambda \ln(t)) \). However, here the percolation clusters remain fixed while some additional probing particle diffuses through the disordered medium; in the above algorithm to produce volatility clustering, the investors themselves diffuse and there is no additional probing particle. Thus these log-periodic oscillations of diffusive percolation have not yet been related by a simulation to the Cont-Bouchaud percolation model for markets.

The Cont-Bouchaud percolation model is particularly suited to look at effects of finite lattice sizes, since size effects at such critical points have been studied since decades. In most of the other microscopic models \(^{[54]}\), the "thermodynamic limit" \( N \rightarrow \infty \) means that the fluctuations die out or become nearly periodic. Real markets, according to these models, are dominated by the \( 10^2 \) most important players and not by millions of small investors. Also for the present Cont-Bouchaud model, the behaviour becomes unrealistic (Gaussian \( \pi(R) \)) in this limit if \( p < p_c \). Right at \( p = p_c \), however, the lattice is no longer self-averaging, and the simulated return distributions keep the same shape for \( N = 10^3 \) to \( 10^6 \).

Of course, the extreme tails are always dominated by size effects: No investor can own more than 100 percent of the market, and no cluster can contain more than the \( N = L^d \) lattice sites of the model. However, this trivial limit is relevant mainly above \( p_c \); at the percolation threshold, the largest cluster is a fractal and contains on average \( \propto L^D \) sites, where the fractal dimension \( D = d/(\tau - 1) \) is smaller than \( d \). Investigations of the distribution of sizes for the largest critical cluster have only begun \(^{[53]}\).

4 Critical crashes

4.1 Multiplicative cascades on the stock market

The analogy between finance and hydrodynamic turbulence developed by Ghashghaie et al. \(^{[13]}\) implicitly assumes that price fluctuations can be described by a multiplicative cascade along which the return \( r \) at a given time scale \( a < T \), is given by:

\[
 r_a(t) \equiv \ln P(t+a) - \ln P(t) = \sigma_a(t)u(t) ,
\]

where \( u(t) \) is some scale independent random variable, \( T \) is some coarse “integral” time scale and \( \sigma_a(t) \) is a positive quantity that can be multiplicatively decomposed, for any decreasing sequence of scales \( \{a_i\}_{i=0}^{n} \) with \( a_0 = T \) and \( a_n = a \), as \(^{[13]}\)

\[
 \sigma_a = \prod_{i=0}^{n-1} W_{a_{i+1},a_i} \sigma_T .
\]

18
Equation (21) together with (22) writes that the logarithm of the price is a multiplicative process. But, this is different from a standard multiplicative processes due to the tree-like structure of the correlations that are added by the hierarchical construction of the multiplicands. We use \( \omega_a(t) \equiv \ln \sigma_a(t) \) as a natural variable.

If one supposes that \( W_{a_{i+1},a_i} \) depends only on the scale ratio \( a_{i+1}/a_i \) and are i.i.d. variables with log-normal distribution of mean \(-H \ln 2\) and variance \( \lambda^2 \ln 2 \), one can show [5] that the correlation function of the volatility field \( \omega_a(t) \) averaged over a period of length \( T \) is given by

\[
\Gamma^\omega_a(\Delta t) = \lambda^2 \left( \log_2 \frac{T}{\Delta t} - 2 + 2 \frac{\Delta t}{T} \right) + \lambda_T^2 ,
\]

(23)

for \( a \leq \Delta t \leq T \) (\( \langle \cdot \rangle \) means mathematical expectation and \( \lambda_T^2 \) is the variance of \( \omega_T \)). For \( \lambda^2 \approx 0.015 \) that can be obtained independently from the fit of the pdf’s, Eq. (23) provides a very good fit of the data (Fig 1(b’)) for the slow decay of the correlation coefficient with only one adjustable parameter \( T \approx 3 \) months. Let us note that \( C^\omega_a(\Delta t) \) can be equally well fitted by a power law \( \Delta t^{-\alpha} \) with \( \alpha \approx 0.2 \). In view of the small value of \( \alpha \), this is undistinguishable from a logarithmic decay. Moreover, Eq. (23) predicts that the correlation function \( \Gamma^\omega_a(\Delta t) \) should not depend of the scale \( a \) provided \( \Delta t > a \) in agreement with data [5].

Another very informative quantity is the cross-correlation function of the volatility measured at different time scales:

\[
C^\omega_{a_1,a_2}(\Delta t) \equiv \var(\omega_{a_1})^{-1} \var(\omega_{a_2})^{-1} \langle \omega_{a_1}(t) \omega_{a_2}(t+\Delta t) \rangle .
\]

(24)

It is found that \( C^\omega_{a_1,a_2}(\Delta t) > C^\omega_{a_1,a_2}(-\Delta t) \) if \( a_1 > a_2 \) and \( \Delta t > 0 \). From the near-Gaussian properties of \( \omega_a(t) \), the mean mutual information of the variables \( \omega_a(t + \Delta t) \) and \( \omega_{a_1}(t) \) reads:

\[
I_a(\Delta t, \Delta a) = -0.5 \log_2 \left( 1 - (C^\omega_{a,a+\Delta a}(\Delta t))^2 \right) .
\]

(25)

Since the process is causal, this quantity can be interpreted as the information contained in \( \omega_{a+\Delta a}(t) \) that propagates to \( \omega_a(t + \Delta t) \). The remarkable observation [5] is the appearance of a non-symmetric propagation cone of information showing that the volatility at large scales influences in the future the volatility at shorter scales. This clearly demonstrates of the pertinence of the notion of a cascade in market dynamics.

### 4.2 Log-periodicity for “foreshocks”

A hierarchical cascade process as just described implies the existence of a discrete scale invariance if the branching ratio and scale factor along the tree are not fluctuating too much [56]. This possibility is actually born out by the data under the frame of log-periodic oscillations.
As alluded to in the section on percolation models, log-periodicity refers to the context of the accelerating oscillations that have been documented in stock market prices prior and also sometimes following major crashes. The formula typifying this behavior is the time-to-failure equation

$$I(t) = p_c + B(t_c - t)^m \left[ 1 + C \cos \left( \frac{2\pi \log(t_c - t)}{\log \lambda} + \Psi \right) \right],$$

where $I$ is the price when the crash is a correction for a bubble developing above some fundamental value (it is the logarithm of the price if the crash drop is proportional to the total price), $t_c$ is the critical time at which the crash is the most probable, $m$ is a critical exponent, and $\Psi$ is a phase in the cosine that can be get rid of by a change of time units. $\lambda$ is the preferred scale factor of the accelerating oscillations giving the ratio between the successive shrinking periods. This expression reflects a discrete scale invariance of the price around the critical time, i.e. the price exhibits self-similarity only under magnification around $t_c$ that are integer powers of $\lambda$. Figure (5) shows three cases illustrating the behavior of market prices prior to large crashes [57].

Since our initial proposition of the existence of log-periodicity preceding stock market crashes [58, 59, 60], several works have extended the empirical investigation [61, 62, 63, 64, 65, 66, 67]. A recent compilation of many crashes [57, 68, 69, 70] provides increasing evidence of the relevance of log-periodicity and of the application of the concept of criticality to financial crashes. The events that have been found to qualify comprise:

- the Oct. 1929, the Oct. 1987, the Hong-Kong Oct. 1987, the Aug. 1998 crashes, which are global market events,
- the 1985 foreign exchange event on the US dollar,
- the correction of the US dollar against the Canadian dollar and the Japanese Yen starting in Aug. 1998,
- the bubble on the Russian market and its ensuing collapse in 1997-98,
- twenty-two significant bubbles followed by large crashes or by severe corrections in the Argentinian, Brazilian, Chilean, Mexican, Peruvian, Venezuelan, Hong-Kong, Indonesian, Korean, Malaysian, Philippine and Thai stock markets [70].

In all these cases, it has been found that log-periodic power laws adequately describe speculative bubbles on the western as well as on the emerging markets with very few exceptions.

The underlying mechanism which has been proposed [57, 58, 59] is that bubble develops by a slow build-up of long-range time correlations reflecting
Figure 5: The S&P 500 US index prior to the October 1987 crash on Wall Street and the US $ against deutschmark (DEM) and Swiss franc (CHF) prior to the collapse mid-85. The fit to the S&P 500 is equation \([26]\) with \(p_c \approx 412, B \approx -165, BC \approx 12.2, m \approx 0.33, t_c \approx 87.74, \Psi \approx 2.0, \lambda \approx 2.3\). The fits to the DM and CHF currencies against the US dollar gives \(p_c \approx 3.88, B \approx -1.2, BC \approx 0.08, m \approx 0.28, t_c \approx 85.20, \Psi \approx -1.2, \lambda \approx 2.8\) and \(p_c \approx 3.1, B \approx -0.86, BC \approx 0.05, m \approx 0.36, t_c \approx 85.19, \Psi \approx -0.59, \lambda \approx 3.3\), respectively. Reproduced from \([57]\).
those between traders leading eventually to a collapse of the stock market in one critical instant. This build-up manifest itself as an over-all power law acceleration in the price decorated by “log-periodic” precursors. This mechanism can be analysed in an expectation model of bubbles and crashes which is essentially controlled by a crash hazard rate becoming critical due to a collective imitative/herding behavior of traders \[57, 68, 69\]. A remarkable *universality* is found for all events, with approximately the same value of the fundamental scaling ratio \(\lambda\) characterising the log-periodic signatures.

To test for the statistical significance of these analyses, extensive statistical tests have been performed \[69, 71\] to show that the reported “log-periodic” structures essentially never occurred in \(\approx 10^5\) years of synthetic trading following a “classical” time-series model, the GARCH(1,1) model with student-t statistics (which has a power law tail with exponent \(\alpha = 4\)), often used as a benchmark in academic circles as well as by practitioners. Thus, the null hypothesis that log-periodicity could result simply from random fluctuations is strongly rejected.

### 4.3 Logperiodicity for “aftershocks”

Log-periodic oscillations decorating an overall acceleration of the market have their symmetric counterparts after crashes. It has been found \[72\] that imitation between traders and their herding behaviour not only lead to speculative bubbles with accelerating over-valuations of financial markets possibly followed by crashes, but also to “anti-bubbles” with decelerating market de-valuations following all-time highs. The mechanism underlying this scenario assumes that the demand decreases slowly with barriers that progressively quench in, leading to a power law decay of the market price decorated by decelerating log-periodic oscillations. This mechanism is actually very similar to that operating in the random walk of a brownian particle diffusing in a random lattice above percolation in a biased field \[53\].

The strongest signal has been found on the Japanese Nikkei stock index from 1990 to present and on the Gold future prices after 1980, both after their all-time highs. Figure \[8\] shows the Nikkei index representing the Japanese market from the beginning of 1990 to present. The data from 1 Jan 1990 to 31 Dec. 1998 has been fitted (the ticked line) by an extension of (26) using the next order terms in the expansion of a renormalization group equation \[72\]. This fit has been used to issue a forecast in early January 1999 for the recovery of the Nikkei in 1999 \[72\]. The forecast, performed at a time when the Nikkei was at its lowest, has correctly captured the change of regime and the overall upward trend since the beginning of this year. This prediction has first been released in january 1999 on the Los Alamos server at [http://xxx.lanl.gov/abs/cond-mat/9901268](http://xxx.lanl.gov/abs/cond-mat/9901268). The detailed publication for IJMPD \[72\] was mentionned already with its prediction in a wide-circulation journal which appeared in May 1999 \[73\]. One of the authors would not
Figure 6: In [72], the Nikkei was fitted from 1 Jan 1990 to 31 Dec. 1998 with an extended log-periodic formula and its extrapolation predicted that the Japanese stock market should increase as the year 2000 was approached. In this figure, the value of the Nikkei is represented as the solid line after the last point used in the analysis (31 Dec. 1998) until 21 Sep. 1999. and can be compared with our prediction (the ticked line). The dots after Dec. 1989 until 31 Dec. 1998 represent the data used in the prediction. This figure is as in [72] except for the solid line starting 3rd Jan. 1999 which represents the realized Nikkei prices since the prediction was issued.

survive a Nikkei drop since another author relied on the Nikkei prediction and invested in Japan.

A set of secondary western stock market indices (London, Sydney, Auckland, Paris, Madrid, Milan, Zurich) as well as the Hong-Kong stock market have also been shown to exhibit well-correlated log-periodic power law anti-bubbles over a period 6-15 months triggered by a rash of crises on emerging markets in the early 1994 [70]. As the US market declined by no more than 10% during the beginning of that period and quickly recovered, this suggests that these smaller stock western markets can “phase lock” (in a weak sense) not only because of the over-arching influence of Wall Street but also independently of the current trends on Wall Street due to other influences.
5 Conclusion

This review has attempted to present results that may advance our understanding of the working of stock markets. First, we proved that the demand and supply should be deviating from the balanced point in general cases when there are fluctuations in demand. In the case of an open market in which prices can change instantly following the unbalance of demand and supply, the speculative actions of dealers can be modeled numerically by models with threshold dynamics. The resulting market price fluctuations are characterized by a fat tail distribution when the dealers’ response to latest price change is positive. We have also shown that simple Langevin equation with multiplicative noise account for the threshold-type dynamics of traders and rationalize the “fat tail” nature of distribution of returns. By solving the set of macroscopic market price equations, we have shown that there are three types in price changes: 1) stationary fluctuations, 2) bubble behavior, and 3) oscillatory phase. We believe that price fluctuations in open markets should be better understood by considering such dynamical effect that has been neglected in ordinary approach of financial technology. We have also presented models inspired by percolation that are probably the simplest microscopic models capturing the effect of imitation/clustering of traders in groups of various sizes. Improvements of the model provide reasonable agreement with the empirical value of the exponent \( \alpha \) of the distribution of price variations. Clustering of volatility can also emerge rather naturally by feedback effect of the price on the activity of the traders. The initial main weakness of the model, namely the fact that the connectivity had to be tuned to its critical value, has also been cured by allowing it to become a dynamical variable.

The review ends up by summarizing the evidence for critical behavior associated with the formation of speculative bubbles in large stock markets and their associated log-periodicity, corresponding to accelerated oscillations up to the time of crashes. Whether this will allow to prevent future crashes remains to be seen.

The overall picture that emerges is quite interesting: a mixture of more or less stationary self-similar statistical time series, maybe self-organized critical, with cascades of correlations across time scales and once in a while a (truncated) divergence reflecting probably the crowd effect between traders culminating in a critical point with rather specific log-periodic signatures. According to the different models that we have presented, a crash has probably an endogenous origin and is constructed progressively by the market as a whole. In this sense, this could be termed a systemic instability. Further study might clarify what could be the regulations and informations that should be released to stabilize the market and prevent these systemic instabilities.

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