One Dimensional Locally Connected S-spaces *

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Abstract

We construct, assuming Jensen’s principle ♦, a one-dimensional locally connected hereditarily separable continuum without convergent sequences.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. A continuum is any compact connected space. A nontrivial convergent sequence is a convergent ω-sequence of distinct points. As usual, dim(X) is the covering dimension of X; for details, see Engelking [7]. “HS” abbreviates “hereditarily separable”. We shall prove:

Theorem 1.1 Assuming ♦, there is a locally connected HS continuum Z such that dim(Z) = 1 and Z has no nontrivial convergent sequences.

Note that points in Z must have uncountable character, so that Z is not hereditarily Lindelöf; thus, Z is an S-space.

Spaces with some of these features are well-known from the literature. A compact F-space has no nontrivial convergent sequences. Such a space can be a continuum; for example, the Čech remainder β[0, 1)\[0, 1) is connected, although not locally connected; more generally, no infinite compact F-space can be either locally connected or HS. In [15], van Mill constructs, under the Continuum Hypothesis, a locally connected continuum with no nontrivial convergent sequences. Van Mill’s example,

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constructed as an inverse limit of Hilbert cubes, is infinite dimensional. Here, we shall replace the Hilbert cubes by one-dimensional Peano continua (i.e., connected, locally connected, compact metric spaces) to obtain a one-dimensional limit space. Our $Z = Z_{\omega_1}$ will be the limit of an inverse system $\langle Z_{\alpha} : \alpha < \omega_1 \rangle$. Each $Z_{\alpha}$ will be a copy of the Menger sponge [13] (or Menger curve) $MS$; this one-dimensional Peano continuum has homogeneity properties similar to those of the Hilbert cube. The basic properties of $MS$ are summarized in Section 2, and Theorem 1.1 is proved in Section 3.

In [15], as well as in earlier work by Fedorchuk [9] and van Douwen and Fleissner [4], one kills all possible nontrivial convergent sequences in $\omega_1$ steps. Here, we focus primarily on obtaining an S-space, modifying the construction of the original Fedorchuk S-space [8]; we follow the exposition in [5], where the lack of convergent sequences occurs only as an afterthought. This exposition can easily be modified to make $Z$ a strong S-space as well; see Section 5.

We do not know whether one can obtain $Z$ so that it satisfies Theorem 1.1 with the stronger property $\text{ind}(Z) = 1$; that is, the open $U \subseteq Z$ with $\partial U$ zero-dimensional form a base. In fact, we can easily modify our construction to ensure that $1 = \dim(Z) < \text{ind}(Z) = \infty$; this will hold because (as in [5]) we can give $Z$ the additional property that all perfect subsets are $G_\delta$ sets; see Section 6 for details.

We can show that a $Z$ satisfying Theorem 1.1 cannot have the property that the open $U \subseteq Z$ with $\partial U$ scattered form a base; see Theorem 4.12 in Section 4. This strengthening of $\text{ind}(Z) = 1$ is satisfied by some well-known Peano continua. It is also satisfied by the space produced in [10] under $\Diamond$ by an inductive construction related to the one we describe here, but the space of [10] was not locally connected, and it had nontrivial convergent sequences (in fact, it was hereditarily Lindelöf).
2 ON SPONGES

The Menger sponge $\text{MS}$ [13] is obtained by drilling holes through the cube $[0, 1]$, analogously to the way that one obtains the middle-third Cantor set by removing intervals from $[0, 1]$. The paper of Mayer, Oversteegen, and Tymchatyn [14] has a precise definition of $\text{MS}$ and discusses its basic properties. Many pictures of $\text{MS}$ are available online, if you google “Menger sponge”.

In proving theorems about $\text{MS}$, one often refers not to its definition, but to the following theorem of R. D. Anderson [1, 2] (or, see [14]), which characterizes $\text{MS}$. This theorem will be used to verify inductively that $Z_\alpha \cong \text{MS}$. The fact that $\text{MS}$ satisfies the stated conditions is easily seen from its definition, but it is not trivial to prove that they characterize $\text{MS}$.

**Theorem 2.1** $\text{MS}$ is, up to homeomorphism, the only one-dimensional Peano continuum with no locally separating points and no non-empty planar open sets.

Here, $C \subseteq X$ is locally separating iff, for some connected open $U \subseteq X$, the set $U \setminus C$ is not connected. A point $x$ is locally separating iff $\{x\}$ is. This notion is applied in the Homeomorphism Extension Theorem of Mayer, Oversteegen, and Tymchatyn [14]:

**Theorem 2.2** Let $K$ and $L$ be closed, non-locally-separating subsets of $\text{MS}$ and let $h : K \to L$ be a homeomorphism. Then $h$ extends to a homeomorphism of $\text{MS}$ onto itself.

The non-locally-separating sets have the following closure property of Kline [11] (or, see Theorem 2.2 of [14]):

**Theorem 2.3** Let $X$ be compact and locally connected, and let $K = \bigcup\{K_i : i \in \omega\}$, where $K$ and the $K_i$ are closed subsets of $X$. If $K$ is locally separating then some $K_i$ is locally separating.

For example, these results imply that in $\text{MS}$, all convergent sequences are equivalent. More precisely, points in $\text{MS}$ are not locally separating, so if $\langle x_i : i \in \omega \rangle$ converges to $x_\omega$, then $\{x_i : i \leq \omega\}$ is not locally separating. Thus, if $\langle s_i \rangle$ and $\langle t_i \rangle$ are nontrivial convergent sequences in $\text{MS}$, with limit points $s_\omega$ and $t_\omega$, respectively, then there is a homeomorphism of $\text{MS}$ onto itself that maps $s_i$ to $t_i$ for each $i \leq \omega$.

The following consequence of Theorem 2.1 was noted by Prajs [16] (see p. 657).

**Lemma 2.4** Let $J \subseteq \text{MS}$ be a non-locally-separating arc and obtain $\text{MS}/J$ by collapsing $J$ to a point. Then $\text{MS}/J \cong \text{MS}$ and the natural map $\pi : \text{MS} \to \text{MS}/J$ is monotone.
Here, a map \( f : Y \to X \) is called monotone iff each \( f^{-1}\{x\} \) is connected; so, the monotonicity in Lemma 2.4 is obvious. When \( X, Y \) are compact, monotonicity implies that \( f^{-1}(U) \) is connected whenever \( U \) is a connected open or closed subset of \( X \).

We shall use these results to show that the property of being a Menger sponge will be preserved at the limit stages of our construction:

**Lemma 2.5** Suppose that \( \gamma \) is a countable limit ordinal and \( Z_\gamma \) is an inverse limit of \( \langle Z_\alpha : \alpha < \gamma \rangle \), where all bonding maps \( \sigma_\alpha^\beta \) are monotone and each \( Z_\alpha \cong \text{MS} \). Then \( Z_\gamma \cong \text{MS} \).

**Proof.** We verify the conditions of Theorem 2.1. \( \dim(Z_\gamma) = 1 \), since this property is preserved by inverse limits of compacta, and \( Z_\gamma \) is locally connected because the \( \sigma_\alpha^\beta \) are monotone. So, we need to verify that \( Z_\gamma \) has no locally separating points and no non-empty planar open sets.

Suppose that \( q \in Z_\gamma \) is locally separating; so we have a connected neighborhood \( U \) of \( q \) with \( U \setminus \{q\} \) not connected. Shrinking \( U \), we may assume that \( U = (\sigma_\alpha^\beta)^{-1}(V) \), where \( \alpha < \gamma \) and \( V \) is open and connected in \( Z_\alpha \). Since \( Z_\alpha \cong \text{MS} \), \( \sigma_\alpha^\beta(q) \) is not locally separating, so \( V \setminus \{\sigma_\alpha^\beta(q)\} \) is connected. Then, since \( \sigma_\alpha^\beta \) is monotone, \( (\sigma_\alpha^\beta)^{-1}(V \setminus \{\sigma_\alpha^\beta(q)\}) = U \setminus (\sigma_\alpha^\beta)^{-1}\{\sigma_\alpha^\beta(q)\} \) is connected. The same argument shows that \( U \setminus (\sigma_\alpha^\beta)^{-1}\{\sigma_\alpha^\beta(q)\} \) is connected whenever \( \alpha < \beta < \gamma \). But then \( U \setminus \{q\} = \bigcup \{U \setminus (\sigma_\alpha^\beta)^{-1}\{\sigma_\alpha^\beta(q)\} : \alpha \leq \beta < \gamma\} \) is connected also.

Suppose that \( U \subseteq Z_\gamma \) is open and non-empty; we show that \( U \) is not planar. Shrinking \( U \), we may assume that \( U = (\sigma_\alpha^\beta)^{-1}(V) \), where \( \alpha < \gamma \) and \( V \) is open in \( Z_\alpha \). Since \( Z_\alpha \cong \text{MS} \), there is a \( K_5 \) set \( F \subseteq V \); that is, \( F \) consists of 5 distinct points \( p_0, p_1, p_2, p_3, p_4 \), together with arcs \( J_{i,j} \) with endpoints \( p_i, p_j \) for \( 0 \leq i < j < 5 \), where the sets \( J_{i,j} \setminus \{p_i, p_j\} \), for \( 0 \leq i < j < 5 \), are pairwise disjoint. Now \( F \) is not planar, and, one can show that \( (\sigma_\alpha^\beta)^{-1}(F) \) is not planar either. To do this, use the fact that \( \sigma_\alpha^\beta \) is monotone, so that the sets \( (\sigma_\alpha^\beta)^{-1}\{p_i\} \) and \( (\sigma_\alpha^\beta)^{-1}(J_{i,j}) \) are all continua. \( \smile \)

The following terminology was used also in the exposition in [5] of the Fedorchuk S-space:

**Definition 2.6** Let \( \mathcal{F} \) be a family of subsets of \( X \). Then \( x \in X \) is a strong limit point of \( \mathcal{F} \) iff for all neighborhoods \( U \) of \( x \), there is an \( F \in \mathcal{F} \) such that \( F \subseteq U \) and \( x \notin F \).

In practice, we shall only use this notion when the elements of \( \mathcal{F} \) are closed. If all elements of \( \mathcal{F} \) are singletons, this reduces to the usual notion of a point being a limit point of a set of points.

The map \( \sigma_\alpha^{\alpha+1} : Z_{\alpha+1} \to Z_\alpha \) will always be obtained by collapsing a non-locally-separating arc in \( Z_{\alpha+1} \) to a point. We obtain it using:
Lemma 2.7 Assume that $X \cong MS$ and that for $n \in \omega$, $\mathcal{F}_n$ is a family of non-locally-separating closed subsets of $X$. Fix $t \in X$ such that $t$ is a strong limit point of each $\mathcal{F}_n$. Then there is a $Y \cong MS$ and a monotone $\sigma : Y \to X$ such that

1. $\sigma^{-1}\{t\}$ is a non-locally-separating arc in $Y$,
2. $|\sigma^{-1}\{x\}| = 1$ for all $x \neq t$, and
3. $y$ is a strong limit point of $\{\sigma^{-1}(F) : F \in \mathcal{F}_n\}$, for each $y \in \sigma^{-1}\{t\}$ and $n \in \omega$.

Proof. First, let $\{A_n : n \in \omega\}$ partition $\omega$ into disjoint infinite sets. In $X$, choose disjoint closed $F_i \not\ni t$ for $i \in \omega$ such that $F_i \in \mathcal{F}_n$ whenever $i \in A_n$, and such that every neighborhood of $t$ contains all but finitely many of the $F_i$. Let $L = \{t\} \cup \bigcup_i F_i$. Then $L$ is closed and non-locally-separating by Theorem 2.3.

Now, in $MS$, let $J$ be any non-locally-separating arc. Choose disjoint closed non-locally separating sets $G_i$ for $i \in \omega$ such that each $G_i \cong F_i$, every neighborhood of $J$ contains all but finitely many $G_i$, each $G_i \cap J = \emptyset$, and for each $n$ and each $y \in J$: $y$ is a strong limit point of $\{G_i : i \in A_n\}$.

Let $\rho : MS \to MS/J$ be the usual projection, and let $[J]$ denote the point to which $\rho$ collapses the set $J$. Then $MS/J \cong MS$ by Lemma 2.4. In $MS/J$, let $K = \{[J]\} \cup \{\rho(G_i) : i \in \omega\}$. Let $h : K \to L$ be a homeomorphism such that $h([J]) = t$ and each $h(\rho(G_i)) = F_i$. By Theorem 2.2, $h$ extends to a homeomorphism $\tilde{h} : MS/J \to X$.

Now, let $Y = MS$ and let $\sigma = \tilde{h} \circ \rho$. ☺

The next lemma will simplify somewhat the description of our inverse limit:

Lemma 2.8 In Lemma 2.7, we may obtain $Y \subseteq X \times [0, 1]$, with $\sigma : Y \to X$ the natural projection.

Proof. Start with any $Y, \sigma, t$ satisfying Lemma 2.7, and let $J := \sigma^{-1}\{t\}$. Apply the Tietze Extension Theorem to fix $f : Y \to [0, 1]$ such that $f|J : J \to [0, 1]$ is a homeomorphism. Then $y \mapsto (\sigma(y), f(y))$ is one-to-one on $Y$, and hence $\tilde{Y} := \{(\sigma(y), f(y)) : y \in Y\} \subseteq X \times [0, 1]$ satisfies Lemma 2.8. ☺

The following additional property of our $\sigma$ will be useful:

Lemma 2.9 Let $t$ and $\sigma : Y \to X$ be as in Lemma 2.7 or 2.8. Assume that $H \subseteq X$ is closed and nowhere dense and not locally separating. Then $\sigma^{-1}(H) \subseteq Y$ is closed and nowhere dense and not locally separating.

Proof. $\sigma^{-1}(H)$ is closed and nowhere dense because $\sigma$ is continuous and irreducible. Also note that $\sigma^{-1}(H)$ is not locally separating if either $H = \{t\}$ (trivially) or $t \not\in H$ (because $\sigma$ is a homeomorphism in a neighborhood of $\sigma^{-1}(H)$).

Next, note that every closed $K \subseteq H$ is non-locally-separating in $X$: If not, let $U \subseteq X$ be connected and open with $U \setminus K$ not connected, so that $U \setminus K = W_0 \cup W_1$,
where the $W_i$ are open in $X$, non-empty, and disjoint. Then $U \setminus H = W_0 \setminus H \cup W_1 \setminus H$, but $H$ is not locally separating, so one of the $W_i \setminus H = \emptyset$, so $W_i \subseteq H$, contradicting $H$ being nowhere dense.

Now, let $H = \bigcup_{n \in \omega} K_n$, where each $K_n$ is closed and either $K_n = \{t\}$ or $t \notin K_n$. Then $\sigma^{-1}(H) = \bigcup_n \sigma^{-1}(K_n)$, which is not locally separating by Theorem 2.3.

### 3 The Inverse Limit

We shall obtain our space $Z = Z_{\omega_1}$ as an inverse limit of a sequence $(Z_\alpha : \alpha < \omega_1)$. As with many such constructions, it is somewhat simpler to view the $Z_\alpha$ concretely as subsets of cubes, so that the bonding maps are just projections. Thus, we shall have:

**Conditions 3.1** We obtain $Z_\alpha$ for $\alpha \leq \omega_1$ and $\pi_\alpha^\beta, \sigma_\alpha^\beta$ for $\alpha \leq \beta \leq \omega_1$ such that:

- **C1.** Each $Z_\alpha$ is a closed subset of $MS \times [0, 1]^\alpha$, and $Z_0 = MS$.
- **C2.** For $\alpha \leq \beta \leq \omega_1$, $\pi_\alpha^\beta : MS \times [0, 1]^\beta \rightarrow MS \times [0, 1]^\alpha$ is the natural projection.
- **C3.** $\pi_\alpha^\beta(Z_\beta) = Z_\alpha$ whenever $\alpha \leq \beta \leq \omega_1$.
- **C4.** $Z_\alpha$ is homeomorphic to $MS$ whenever $\alpha < \omega_1$.
- **C5.** The maps $\sigma_\alpha^\beta := \pi_\alpha^\beta|Z_\beta : Z_\beta \rightarrow Z_\alpha$, for $\alpha \leq \beta \leq \omega_1$, are monotone.

Using (C1,C2,C3), the construction is determined at limit ordinals; (C4) is preserved by Lemma 2.5 and (C5). It remains to explain how, given $Z_\alpha$ for $\alpha < \omega_1$, we obtain $Z_{\alpha+1} \subseteq Z_\alpha \times [0, 1]$; as usual, we identify $MS \times [0, 1]^\alpha+1$ with $MS \times [0, 1]^{\alpha} \times [0, 1]$.

We now add:

**Conditions 3.2** We have $q_\alpha^\xi$ and $t_\alpha$ for $\xi < \alpha < \omega_1$ such that:

- **C6.** Each $(q_\alpha^\xi : \xi < \alpha)$ is a sequence of points in $MS \times [0, 1]^\alpha$.
- **C7.** Whenever $(q_\xi^\xi : \xi < \omega_1)$ is any sequence of points in $MS \times [0, 1]^\omega_1$, $\{\alpha < \omega_1 : \forall \xi < \alpha [\pi_\alpha^{\omega_1}(q_\xi^\xi) = q_\xi^\alpha]\}$ is stationary.
- **C8.** Whenever $\alpha < \beta \leq \omega_1$ and $z \in Z_\alpha$; If $q_\alpha^\xi \in Z_\alpha$ for all $\xi < \alpha$ and $z$ is a limit point of $\{q_\alpha^\xi : \xi < \alpha \& q_\alpha^\xi \neq z\}$, then all points of $(\sigma_\alpha^\beta)^{-1}\{z\}$ are strong limit points of $\{\pi_\alpha^{\omega_1}(q_\xi^\xi) : \xi < \alpha\}$.
- **C9.** $t_\alpha \in Z_\alpha$, and for all $z \in Z_\alpha$: $(\sigma_\alpha^{\omega_1+1})^{-1}\{z\}$ is a singleton if $z \neq t_\alpha$ and a non-locally-separating arc if $z = t_\alpha$.
- **C10.** $t_\alpha = q_\alpha^0$ whenever $\alpha > 0$ and $q_\alpha^0 \in Z_\alpha$. 
Proof of Theorem 1.1. The fact that one may obtain (C1 – C10) has already been outlined above. (C6,C7) are possible by △, and (C10) is just a definition. (C8,C9) are obtained by induction on β. For the successor step, we must obtain $Z_{β+1}$ from $Z_β$ using Lemmas 2.7 and 2.8. Here, $X = Z_β$, $Y = Z_{β+1}$, and $t = t_β$; the $F_n$ list all sets of the form $F^β_α := \{ (σ^α_β)^{-1}\{q^ξ_α\} : ξ < α \& q^ξ_α \in Z_α \}$ such that $α ≤ β$ and $t_β$ is a strong limit point of $F^β_α$. Observe that (C8) for $(α, β + 1)$ is immediate from (C8) for $(α, β)$ except for the points of $Z_{β+1}$ in $(σ^{β+1}_β)^{-1}\{t_β\}$. Also observe that in order to apply Lemmas 2.7 and 2.8, we must check by induction on β, using Lemma 2.9, that the sets $(σ^β_α)^{-1}\{q^ξ_α\}$ are non-locally-separating (and nowhere dense) in $Z_β$.

Note that $χ(z, Z) = 81$ for all $z \in Z$; this follows from (C9,C10) and the fact, using (C7), that $\{ α < ω_1 : π^{ω_1}_α(z) = t_α \}$ is unbounded in $ω_1$.

$Z$ is HS by (C6,C7,C8,C1,C2,C3): If not, suppose that $⟨ q^ξ : ξ < ω_1 ⟩$ is left-separated in $Z$. As in [5], we get a club $C ⊂ ω_1$ such that for all $α ∈ C$,

1. The $σ^{ω_1}_α(q^ξ)$ for $ξ < α$ are all distinct, and
2. For all $η$ with $α < η < ω_1$, $σ^{ω_1}_α(q^η)$ is a limit point of $\{ σ^{ω_1}_α(q^ξ) : ξ < α \}$.

Fix $α ∈ C$ such that $∀ξ < α [σ^{ω_1}_α(q^ξ) = q^ξ_α]$. Let $z = σ^{ω_1}_α(q^α)$. Applying (C8) with $β = ω_1$, we have in $Z$: all points of $(σ^{ω_1}_α)^{-1}\{z\}$ are strong limit points of $\{ (σ^{ω_1}_α)^{-1}\{q^ξ_α\} : ξ < α \}$. In particular, $q^α$ is a limit point of $⟨ q^ξ : ξ < α ⟩$, contradicting “left-separated”.

Similarly, $Z$ has no non-trivial convergent sequences: Suppose that $q^n → q^ω$ in $Z$, where the $q^ξ$ for $ξ ≤ ω$ are distinct. Let $q^ξ = q^ω$ when $ω < ξ < ω_1$, and apply (C7) to get $α$ with $ω < α < ω_1$ such that the $σ^{ω_1}_α(q^ξ)$ for $ξ ≤ ω$ are distinct points and $∀ξ < α [σ^{ω_1}_α(q^ξ) = q^ξ_α]$. Let $z = σ^{ω_1}_α(q^α)$. Then all points of $(σ^{ω_1}_α)^{-1}\{z\}$ are strong limit points of $\{ (σ^{ω_1}_α)^{-1}\{q^ξ_α\} : ξ < α \}$ and hence also of $\{ (σ^{ω_1}_α)^{-1}\{q^ξ_α\} : n < ω \}$. So, all points of $(σ^{ω_1}_α)^{-1}\{z\}$ are limit points of $\{ q^n : n ∈ ω \}$. Since $\{ q^ω \} \not⊆ (σ^{ω_1}_α)^{-1}\{z\}$ (by $χ(q^ω, Z) = 81$), we contradict $q^n → q^ω$. ☐

4 The Almost Clopen Algebra

We show here (Theorem 4.12) that a space $Z$ satisfying Theorem 1.1 cannot have a base of open sets with scattered boundaries; equivalently (because there are no non-trivial convergent sequences) with finite boundaries. We first note that if there were such a base, we could take the basic open sets $U$ to be regular, since $\partial(int(cl(U))) ⊆ ∂U$. To simplify notation, we define:

Definition 4.1 ro($X$) denotes the algebra of regular open subsets of $X$, and acl($X$) (the almost clopen sets) denotes the family of regular open sets $U$ such that $∂U$ is finite. For $U ∈ ro(X)$, let $U^c$ denote the boolean complement $(X \setminus U)^c$. 

Note that $\text{acl}(X)$ is a boolean subalgebra of $\text{ro}(X)$: If $U \in \text{acl}(X)$ and $W = U^g$, then $\partial W = \partial U$, so $W \in \text{acl}(X)$. Also, if $U, V \in \text{acl}(X)$ and $W = U \land V = U \cap V \in \text{ro}(X)$, then $W \in \text{acl}(X)$ because $\partial(W) \subseteq \partial(U) \cup \partial(V)$.

In a locally connected space, the connected components of an open set $U$ are open; if $V$ is any such component, then $\partial V \subseteq \partial U$ (because $V$ is relatively clopen in $U$), so $V \in \text{acl}(X)$ whenever $U \in \text{acl}(X)$. Thus,

**Lemma 4.2** If $X$ is locally connected and $\text{acl}(X)$ is a local base at $p \in X$, then \{ $U \in \text{acl}(X) : p \in U \land U$ is connected \} is also a local base at $p$.

Various LOTS sums have bases of almost clopen sets. This is true, for example, for any compact hedgehog consisting of a central point plus arbitrarily many LOTS spines. The assumption of no convergent sequences, however, puts some restrictions on the space. In particular, the hedgehog fails the following lemma (taking $U$ to be $X$ and letting $s$ be the central point):

**Lemma 4.3** Assume that $X$ is compact and locally connected, and $X$ has no nontrivial convergent sequences. Fix an open $U$ with $\partial U$ finite, and fix a finite $s \subseteq U$. Then $U \setminus s$ has finitely many components.

**Proof.** Assume that $V_n$, for $n < \omega$, are different components of $U \setminus s$. Choose $x_n \in V_n$. Then the limit points of $\{x_n : n \in \omega\}$ must lie in $\partial(U \setminus s) \subseteq \partial U \cup s$. Thus, $\{x_n : n \in \omega\}$ has finitely many limit points, which is impossible if $X$ has no nontrivial convergent sequences. ☺

We now look more closely at the locally separating points; that is, the points $p \in X$ such that $U \setminus \{p\}$ is not connected for some open connected $U \ni p$.

**Definition 4.4** If $p \in U \subseteq X$, then $c(p, U)$ is the number of components of $U \setminus \{p\}$.

**Lemma 4.5** Assume that $X$ is compact and locally connected, and $p \in X$. If $U$ and $V$ are open connected subsets of $X$ with $p \in V \subseteq U$, then:

1. Every component of $V \setminus \{p\}$ is a subset of exactly one component of $U \setminus \{p\}$.
2. $c(p, V) \geq c(p, U)$.
3. If $\text{acl}(X)$ is a local base at $p$ and $X$ has no nontrivial convergent sequences, then $c(p, U)$ is finite.

**Proof.** (1) is immediate from the fact that if $W$ is a component of $V \setminus \{p\}$ then $W$ is connected and $W \subseteq U \setminus \{p\}$. For (2), use the fact that every component of $U \setminus \{p\}$ must meet $V$ because $U$ is connected, so that (1) provides a map from the components of $V \setminus \{p\}$ onto the components of $U \setminus \{p\}$. For (3), choose $V \in \text{acl}(X)$ with $p \in V \subseteq U$, and apply (2) and Lemma 4.3. ☺

The next lemma is trivial, but useful when $\partial U$ is finite.
Lemma 4.6 Suppose that $E \subseteq X$ is connected, $U \subseteq X$ is open, and $\partial U \cap E = \emptyset$. Then $E \subseteq U$ or $E \cap U = \emptyset$.

Proof. $U \cap E = \overline{U} \cap E$ is relatively clopen in $E$, so $U \cap E$ is either $E$ or $\emptyset$. ☺

Lemma 4.7 Assume that $X$ is compact and locally connected, $\text{acl}(X)$ is a local base at $p \in X$, and $X$ has no nontrivial convergent sequences. Then there is an $n \in \omega$ such that $c(p, U) \leq n$ for all open connected $U \ni p$.

Proof. If this fails, then applying Lemma 4.5, we may fix open connected $U_n \ni p$ for $n \in \omega$ such that $U_0 \supseteq U_1 \supseteq U_2 \cdots$ and $2 \leq c(p, U_0) < c(p, U_1) < \cdots$. Then, we may define a subtree $T \subseteq \omega^{<\omega}$ and open connected $W_s$ for $s \in T$ and $k_s \in \omega \setminus \{0\}$ for $s \in T$ as follows:

1. $W_0$ is the component of $p$ in $X$.
2. If $\text{lh}(s) = n$, then $k_s$ is the number of components of $U_n \setminus \{p\}$ which are subsets of $W_s$, and these components are listed as $\{W_{s^{-i}} : i < k_s\}$.
3. $s^{-i} \in T$ iff $s \in T$ and $i < k_s$.

Item (1) is a bit artificial, but it gives $T$ a root node $()$. For the levels below the root, note that $|T \cap \omega^{n+1}| = c(p, U_n)$, and the $W_s$ for $s \in T \cap \omega^{n+1}$ list the components of $U_n \setminus \{p\}$. Let $P(T) = \{f \in \omega^\omega : \forall n[f|n \in T]\}$ be the set of paths through $T$. Since every node in $T$ has at least one child, $|P(T)|$ is either $\aleph_0$ or $2^{\aleph_0}$. Note that $\text{cl}(W_{s^{-i}}) \subseteq W_s \cup \{p\}$, since if $n = \text{lh}(s) > 0$ and $q \in \text{cl}(W_{s^{-i}}) \setminus \{p\}$, then $q$ and the points of $W_{s^{-i}}$ must all lie in the same component of $U_{n-1} \setminus \{p\}$, which is $W_s$.

Let $H = \bigcap_n U_n = \bigcap_n \overline{U}_n$. Then $H$ is a connected closed $G_\delta$ containing $p$, and $H$ must be infinite, since $p$ must have uncountable character. For each $f \in P(T)$, let $K_f = \bigcap_n \text{cl}(W_{f|n}) = \{p\} \cup \bigcap_n W_{f|n}$. Then the $K_f$ are connected and infinite, since $\{p\}$ cannot be a decreasing intersection of $\omega$ infinite closed sets (or there would be a convergent sequence). Observe that $K_f \cap K_g = \{p\}$ whenever $f \neq g$. Thus, if $p \in V \in \text{acl}(X)$ then $K_f \subseteq V$ for all but finitely many $f \in P(T)$, since $K_f \subseteq V$ whenever $K_f \cap \partial V = \emptyset$ by Lemma 4.6. Now let $f_i$, for $i \in \omega$ be distinct elements of $P(T)$, and choose $q_i \in K_{f_i} \setminus \{p\}$. Then every neighborhood of $p$ contains all but finitely many $q_i$, so the $q_i$ converge to $p$, a contradiction. ☺

Definition 4.8 Assume that $X$ is compact and locally connected, $\text{acl}(X)$ is a base for $X$, and $X$ has no nontrivial convergent sequences. Then for each $p \in X$, define $c(p) \in \omega$ to be the largest $c(p, U)$ among all open connected $U \ni p$.

By a standard chaining argument:

Lemma 4.9 Assume that $X$ is compact and locally connected and $\text{acl}(X)$ is a base for $X$. Fix a connected open $U \subseteq X$ and a compact $F \subseteq U$. Then there is a connected $V \in \text{acl}(X)$ such that $F \subseteq V \subseteq \overline{V} \subseteq U$. 

Proof. Let \( G = \{ W \in \text{acl}(X) : \emptyset \neq \overline{W} \subseteq U \& W \text{ is connected} \} \). Then \( \bigcup G = U \).

View \( G \) as an undirected graph, by putting an edge between \( W_1 \) and \( W_2 \) iff \( W_1 \cap W_2 \neq \emptyset \). Then \( G \) is connected as a graph because \( U \) is connected and the components of \( G \) yield topological components of \( U \). Fix a finite \( G_0 \subseteq G \) such that \( F \subseteq \bigcup G_0 \). Then fix a finite connected \( G_1 \) with \( G_0 \subseteq G_1 \subseteq G \). Let \( V = \bigvee G_1 = \text{int}(\text{cl}(\bigcup G_1)) \).

Lemma 4.10 Assume that \( X \) is compact and locally connected, \( \text{acl}(X) \) is a base for \( X \), and \( X \) has no nontrivial convergent sequences. Then there is no sequence of open sets \( \langle U_n : n \in \omega \rangle \) such that \( \overline{U_{n+1}} \not\subseteq U_n \) for all \( n \) and \( \overline{U_n} \setminus U_{n+1} \) is connected for all even \( n \).

Proof. Given such a sequence, choose \( x_n \in \overline{U_n} \setminus U_{n+1} \), and let \( y \) be a limit point of \( \{ x_{2m} : m \in \omega \} \). Since \( \{ x_{2m} : m \in \omega \} \) cannot converge to \( y \), fix a connected \( W \in \text{acl}(X) \) and disjoint infinite \( A, B \subseteq \{ 2m : m \in \omega \} \) such that \( x_n \in W \) for all \( n \in A \) and \( x_n \notin W \) for all \( n \in B \). Since \( \partial W \) is finite, we may also assume (shrinking \( A, B \) if necessary) that \( \partial W \cap (\overline{U_n} \setminus U_{n+1}) = \emptyset \) for all \( n \in A \cup B \). Then, by Lemma 4.6, \( \overline{U_n} \setminus U_{n+1} \subseteq W \) for all \( n \in A \) and \( (\overline{U_n} \setminus U_{n+1}) \cap W = \emptyset \) for all \( n \in B \). But then, for \( n \in B \), the connected \( W \) is partitioned into the disjoint open sets \( W \cap U_{n+1}, W \setminus \overline{U_n} \), both of which are non-empty when \( n > \min(A) \).

Lemma 4.11 Assume that \( X \) is compact and locally connected, \( \text{acl}(X) \) is a base for \( X \), and \( X \) has no nontrivial convergent sequences. Then every non-isolated point in \( X \) is locally separating.

Proof. Suppose we have a non-isolated \( p \) which is not locally separating; so \( U \setminus \{ p \} \) is connected whenever \( U \) is open and connected. Then inductively construct \( U_n \) for \( n \in \omega \) such that

1. Each \( U_n \) is open and \( p \in U_n \).
2. Each \( U_{n+1} \not\subseteq U_n \).
3. \( \overline{U_n} \setminus U_{n+1} \) is connected whenever \( n \) is even.
4. Each \( U_n \in \text{acl}(X) \).
5. \( U_n \) is connected for all even \( n \).

Then (1)(2)(3) contradict Lemma 4.10.

To construct the \( U_n \): Let \( U_0 \in \text{acl}(X) \) be such that \( p \in U_0 \) and \( U_0 \) is connected and not clopen. Given \( U_n \), where \( n \) is even, we construct \( U_{n+1} \) and \( U_{n+2} \) as follows:

Say \( \partial U_n = \{ q^i : j < r \} \); of course, \( r \) and the \( q^i \) depend on \( n \). For each \( j \), choose \( V^j \in \text{acl}(X) \) be such that \( q^j \in V^j \), \( p \notin \text{cl}(V^j) \), and \( V^j \) is connected. Also make sure that the \( V^j \) are disjoint; then \( \overline{V^j} \cap \partial U_n = \{ q^j \} \). Let \( \{ W^j_i : i < c^j \} \) list the components of \( V^j \setminus \{ q^j \} \); so \( 2 \leq c^j < \omega \). Then \( W^j_i \) is connected and \( \partial U_n \cap W^j_i = \emptyset \),
so $W_i^j \subseteq U_n$ or $W_i^j \cap U_n = \emptyset$; say $W_i^j \subseteq U_n$ for $i < d^j$ and $W_i^j \cap U_n = \emptyset$ for $d^j \leq i < c^j$; so $1 \leq d^j < c^j$. Choose $y_i^j \in W_i^j$. Now $U_n$ is connected and $p$ is not locally separating, so $U_n \setminus \{p\}$ is connected. Applying Lemma 4.9, fix a connected $R \in \text{acl}(X)$ such that \{\{y_i^j : j < r & i < d^j\} \subseteq R \subseteq R \subseteq U \setminus \{p\}$. Let $S$ be the finite union $R \cup \bigcup\{W_i^j : j < r & i < d^j\}$. Then $S$ is open and connected, $p \notin \overline{S}$, and each $q^j \in \overline{S}$. Let $U_{n+1} = U_n \setminus \overline{S} = U_n \setminus \overline{S}$. Then $p \in U_{n+1} \in \text{acl}(X)$, and $\overline{U_n \setminus U_{n+1}} = \overline{S}$ is connected. Also, each $q^j \notin \overline{U_{n+1}}$ because $U_{n+1} \cap V^j = \emptyset$, so that $\overline{U_{n+1}} \subseteq U_n$.

Now, choose a connected $U_{n+2} \in \text{acl}(X)$ so that $p \in U_{n+2} \subseteq U_{n+2} \subseteq U_{n+1}$. ☺

**Theorem 4.12** If $X$ is infinite, compact, locally connected, and $\text{acl}(X)$ is a base for $X$, then $X$ has a nontrivial convergent sequence.

**Proof.** Suppose not. Fix any non-isolated $p \in X$; then $p$ is locally separating by Lemma 4.11, so $c(p) \geq 2$ (see Definition 4.8). Fix a connected $U \in \text{acl}(X)$ such that $p \in U$ and $c(p, U) = c(p)$. Let $W_i$, for $i < c(p)$ be the components of $U \setminus \{p\}$. Then $c(p, V) = c(p)$ whenever $V \in \text{acl}(X)$ and $p \in V \subseteq U$; furthermore, the components of $V \setminus \{p\}$ are the sets $W_i \cap V$ for $i < c(p)$.

Let $Y = \text{cl}(W_0)$. Then $\text{acl}(Y)$ is a base for $Y$, $Y$ is locally connected, and $Y$ has no nontrivial convergent sequences. Furthermore, $p \in Y$ and $p$ is not locally separating in $Y$, contradicting Lemma 4.11 applied to $Y$. ☺

## 5 Strong S-spaces of Various Dimensions

Call $Z$ a Fedorchuk space iff $Z$ is compact HS and crowded, and has no nontrivial convergent sequences. So, Theorem 1.1 produces, under $\heartsuit$, a one-dimensional locally connected Fedorchuk space. Using the same method, one can modify the CH construction of van Mill [15] to produce, under $\heartsuit$, an infinite dimensional locally connected Fedorchuk space; in this construction, the Hilbert cube replaces the Menger sponge MS. The $\heartsuit$ is necessary since by Eisworth [6], CH alone does not imply the existence of any Fedorchuk space.

The referee of the original version of this paper asked whether one might also produce a $k$-dimensional locally connected Fedorchuk space for each finite $k \geq 1$. One way of doing this (the referee’s suggestion) is to replace MS by Menger’s universal $k$-dimensional compactum; these spaces are described in detail in Bestvina [3]. We are not sure if this works, since the characterization of these compacta for $k > 1$ is a bit more complex than that for MS. However, we can construct our $Z$ so that the product $Z^k$ provides a $k$-dimensional example.

Let $Z$ be as constructed in our proof of Theorem 1.1. Then $\dim(Z^k) = k$ because $Z^k$ is an inverse limit of copies of MS$^k$, which has dimension $k$. Also, $Z^k$ is certainly crowded and locally connected, and has no non-trivial convergent sequences. We need to do some extra work to ensure that $Z^k$ is HS for all $k < \omega$; that is, $Z$
is a strong S-space. Then $Z^\omega$ will also be HS, but $Z^\omega$ has non-trivial convergent sequences.

The key to making our space HS was conditions (C6,C7,C8), where we used $\diamondsuit$ to capture all $\omega_1$-sequences from $Z$, ensuring that no such sequence is left-separated. But we can also use $\diamondsuit$ to capture sequences from $Z^k$, which in our construction is a subspace of $(MS \times [0,1]^\omega)^k$. To avoid confusion in our subscripts, if $y \in Y^k$, let $\mu y$, for $\mu < k$, denote coordinate $\mu$ of $y$. Call a point $y \in Y^k$ simple iff all the $\mu y$ are different, and call a $\gamma$-sequence $\langle q^\xi : \xi < \gamma \rangle$ from $Y^k$ simple iff $\mu q^\xi \neq \nu q^n$ unless $\mu = \nu$ and $\xi = \eta$. Observe that for $Z$ to be strongly HS, it is sufficient that for each $k$, there are no simple left-separated $\omega_1$-sequences in $Z^k$.

To avoid confusion about which $k$ is handled at each stage, partition $\omega_1$ into disjoint stationary sets $S_k$ for $k < \omega$ such that $\diamondsuit(S_k)$ is true for each $k$. In (C7), require that $\{ \alpha \in S_1 : \forall \xi < \alpha [\pi^{\alpha_1}_\omega(q^\xi) = q^\alpha_0] \}$ be stationary; then $Z$ is HS and has no convergent $\omega$-sequences. To make $Z^k$ HS, add the following when $2 \leq k < \omega$:

**C6**. For $\alpha \in S_k$, $\langle q^\xi_\alpha : \xi < \alpha \rangle$ is a simple sequence of points in $(MS \times [0,1]^\alpha)^k$.

**C7**. Whenever $\langle q^\xi : \xi < \omega_1 \rangle$ is any simple sequence of points in $(MS \times [0,1]^\omega)^k$,

$\{ \alpha \in S_k : \forall \xi < \alpha [\pi^{\alpha_1}_\omega(q^\xi) = q^\alpha_0] \}$ is stationary.

**C8**. Whenever $\alpha \in S_k$ and $\alpha < \beta \leq \omega_1$ and $z \in (Z_\alpha)^k$: If $q^\alpha_\beta \in (Z_\alpha)^k$ for all $\xi < \alpha$ and $z$ is a limit point of $\{ q^\xi_\alpha : \xi < \alpha \& q^\xi_\alpha \neq z \}$, then all points of $(\sigma^{\beta}_\alpha)^{-1}\{ z \}$ are strong limit points of $\{ (\sigma^{\beta}_\alpha)^{-1}\{ q^\alpha_\beta \} : \xi < \alpha \}$.

Here, $\pi^{\beta}_\alpha$ denotes the natural projection from $(MS \times [0,1]^\beta)^k$ onto $(MS \times [0,1]^\alpha)^k$, and $\sigma^{\beta}_\alpha$ denotes the natural projection from $(Z_\beta)^k$ onto $(Z_\alpha)^k$.

Then, to achieve C8, we need the following improvement on Lemma 2.7. Call a nonempty $F \subseteq X^k$ a nice closed $k$-box iff $F = \prod_{\mu < k}(\mu F)$, where each $\mu F$ is closed and not locally separating in $X$, and the $\mu F$ are pairwise disjoint; then write Sides($F$) for $\bigcup_{\mu < k}(\mu F)$. Call $\mathcal{F}$ a nice $k$-family iff $|\mathcal{F}| = \aleph_0$ and each $F \in \mathcal{F}$ is a nice closed $k$-box and Sides($F$) \cap Sides($\tilde{F}$) $= \emptyset$ whenever $F, \tilde{F}$ are distinct elements of $\mathcal{F}$. Call $\mathcal{F}$ a nice family iff $\mathcal{F}$ is a nice $k$-family for some $k$ with $0 < k < \omega$.

**Lemma 5.1** Suppose that $X \cong MS$ and $\mathfrak{F}$ is a countable set of nice families. Fix any $t \in X$. Then there is a $Y \cong MS$ and a monotone $\sigma : Y \to X$ such that

1. $\sigma^{-1}\{ t \}$ is a non-locally-separating arc in $Y$,
2. $|\sigma^{-1}\{ x \}| = 1$ for all $x \neq t$, and
3. For each $k \in \omega$ and $y \in Y^k$, if $\sigma(y)$ is a strong limit point of a $k$-family $\mathcal{F} \in \mathfrak{F}$, then $y$ is a strong limit point of $\{ \sigma^{-1}(F) : F \in \mathcal{F} \}$. Here, $\sigma$ is applied to each coordinate of $y$; likewise, $\sigma^{-1}$ operates coordinatewise.

When $k = 1$: The result is trivial when $\sigma(y) \neq t$, and Lemma 2.7 handles those $y$ for which $\sigma(y) = t$. Lemma 2.7 did not require the sets in $\mathcal{F}$ to be disjoint, but
they are disjoint when the lemma is applied to the proof that $Z$ is HS, since our $\mathcal{F}$ arises from an inverse limit of a simple sequence. When $k > 1$, we cannot assume that $y$ is simple, so we must consider the possibility that $\sigma(\mu y) = t$ for some $\mu$ and not for other $\mu$.

**Proof of Lemma 5.1.** For each nice $k$-family $\mathcal{F}$, we describe some related families as follows: Fix $r$ with $1 \leq r \leq k$, fix $Q = \{\mu_0, \ldots, \mu_{r-1}\}$ with $\mu_0 < \cdots < \mu_{r-1} < k$, and fix a $(k-r)$-tuple $\tilde{V} = \{\nu V : \mu \in k\setminus Q\}$ of basic open subsets of $X$. Let $\mathcal{F} \upharpoonright (Q, \tilde{V})$ be the family of all nice closed $r$-boxes $H$ such that for some $F \in \mathcal{F}$: $\nu H = \mu_0 F$ for $\nu < r$ and $\mu F \subseteq \nu V$ for $\mu \in k\setminus Q$. Note that $\mathcal{F} \upharpoonright (Q, \tilde{V})$ is a nice $k$-family unless it is finite. If $r = k$, then $Q = k$ and $\tilde{V}$ is the empty sequence and $\mathcal{F} \upharpoonright (Q, \tilde{V}) = \mathcal{F}$; this will handle the special case where all $\sigma(\mu y) = t$.

Call $t$ a **sidewise strong limit** of a nice $k$-family $\mathcal{F}$ iff for all open $U \ni t$, Sides($F$) $\subseteq U$ for all but finitely many $F \in \mathcal{F}$.

Observe that we may assume the following closure properties of $\mathfrak{F}$:

a. If $\mathcal{F} \in \mathfrak{F}$ and $t$ is a sidewise strong limit of some infinite $\tilde{\mathcal{F}} \subseteq \mathcal{F}$, then some such $\tilde{\mathcal{F}}$ is in $\mathfrak{F}$.

b. If $\mathcal{F} \in \mathfrak{F}$ and $Q, \tilde{V}$ are as above, then $\mathcal{F} \upharpoonright (Q, \tilde{V}) \in \mathfrak{F}$ unless $\mathcal{F} \upharpoonright (Q, \tilde{V})$ is finite.

We next restate that part of the proof of Lemma 2.7 which remains unchanged here:

In $X$, we shall choose disjoint closed non-locally-separating $D_i \notin t$ for $i \in \omega$ such that every neighborhood of $t$ contains all but finitely many of the $D_i$. Let $L = \{\{t\} \cup \bigcup_i D_i\}$. Then $L$ is closed and non-locally-separating.

In $\mathcal{MS}$, let $J$ be any non-locally-separating arc. We shall choose disjoint closed non-locally separating sets $G_i$ for $i \in \omega$ such that each $G_i \cong D_i$ and every neighborhood of $J$ contains all but finitely many $G_i$.

$\rho : \mathcal{MS} \to \mathcal{MS}/J$ is the usual projection. Then $\mathcal{MS}/J \cong \mathcal{MS}$. In $\mathcal{MS}/J$, let $K = \{[J]\} \cup \{\rho(G_i) : i \in \omega\}$. Let $h : K \to L$ be a homeomorphism such that $h([J]) = t$ and each $h(\rho(G_i)) = D_i$; then $h$ extends to a homeomorphism $\tilde{h} : \mathcal{MS}/J \to X$. Let $Y = \mathcal{MS}$ and let $\sigma = \tilde{h} \circ \rho$. This handles everything in Lemma 5.1 except for (3), which requires more about the $D_i$ and $G_i$.

In addition to the preceding requirements, choose the $D_i$ and $G_i$ so that for all basic open $0U, \ldots, k-1U \subseteq \mathcal{MS}$ which meet $J$: whenever $t$ is a sidewise strong limit of a $k$-family $\mathcal{F} \in \mathfrak{F}$, there are infinitely many $n \in \omega$ such that for some $F \in \mathcal{F}$ and all $\mu < k$: $D_{n+\mu} = \mu F$ and $G_{n+\mu} \subseteq \mu U \setminus J$.

To see that this proves Lemma 5.1: Fix any $k$-family $\mathcal{F} \in \mathfrak{F}$. Fix any $y \in Y^k$, let $x = \sigma(y) \in X^k$, and assume that $x$ is a strong limit point of $\mathcal{F}$. We need to show that $y$ is a strong limit point of $\{\sigma^{-1}(F) : F \in \mathcal{F}\}$. Assume that exactly $r$ of the coordinates of $x$ equal $t$. Since the result is trivial if $r = 0$, assume that $1 \leq r \leq k$. Let $Q = \{\mu_0, \ldots, \mu_{r-1}\}$, with $\mu_0 < \cdots < \mu_{r-1} = k$, list the subscripts $\mu$ with $\mu x = t$.

Fix basic open neighborhoods $\mu U \ni \mu y$ for $\mu < k$; when $\mu \notin Q$, assume that $\mu U \cap J = \emptyset$ and $\mu U = \sigma^{-1}(\mu V)$, where $\mu V$ is a basic open neighborhood of $\mu x$ in
This defines $\tilde{V}$. Since $x$ is a strong limit point of $\mathcal{F}$, $\mathcal{F}|(Q,\tilde{V})$ is infinite, so $\mathcal{F}|(Q,\tilde{V}) \in \mathfrak{S}$ and hence $t$ is a sidewise strong limit of some infinite $\tilde{\mathcal{F}} \subseteq \mathcal{F}|(Q,\tilde{V})$ which, by closure property (a), is in $\mathfrak{S}$. Then there are infinitely many $n$ such that for some $H \in \tilde{\mathcal{F}}$ and all $\nu < r$: $D_{n+\nu} = \nu H$ and $G_{n+\nu} \subseteq \mu \nu U$. For these $H$, there is an $F \in \mathcal{F}$ such that $\mu F \subseteq \mu V$ for $\mu \notin Q$ and each $\mu \nu F = \nu H$; then $\sigma^{-1}(\mu F) \subseteq \mu U$ for all $\mu$. There are thus infinitely many $F \in \mathcal{F}$ with $\sigma^{-1}(\mu F) \subseteq \mu U$ for all $\mu$. ☺

6 Further Remarks

We note that in constructing a locally connected compactum, the monotone bonding maps, as used also by van Mill [15], are inevitable:

Remark 6.1 Assume that $X \subseteq [0,1]^{\omega_1}$ is compact and locally connected. Define $X_\alpha = \pi^{(\omega_1)}_\alpha(X) \subseteq [0,1]^\alpha$. Then there is a club $C \subseteq \omega_1$ such that $X_\alpha$ is locally connected for all $\alpha \in C$, and such that $\sigma^{\beta}_\alpha := \pi^{(\omega_1)}_\alpha|X_\beta$ is monotone whenever $\alpha < \beta$ and $\alpha, \beta \in C \cup \{\omega_1\}$.

Proof. Let $\mathcal{B}$ be the family of all connected open $F_\sigma$ subsets of $X$. Then $\mathcal{B}$ is a base for $X$. For $\alpha < \omega_1$, let $\mathcal{B}_\alpha$ be the family of all open $U \subseteq X_\alpha$ such that $(\sigma^{\omega_1}_\alpha)^{-1}(U) \in \mathcal{B}$. Observe that each $U \in \mathcal{B}_\alpha$ is connected. Put $\alpha \in C$ iff $\mathcal{B}_\alpha$ is a base for $X_\alpha$. Then $C$ is club.

Now, it is sufficient to show that $(\sigma^{\omega_1}_\alpha)^{-1}\{x\}$ is connected whenever $\alpha \in C$ and $x \in X_\alpha$. Choose $U_n \in \mathcal{B}_\alpha$ with $x \in U_n \supseteq \overline{U_{n+1}}$ for all $n \in \omega$ and $\{x\} = \bigcap_n U_n = \bigcap_n \overline{U_n}$. Each $(\sigma^{\omega_1}_\alpha)^{-1}(U_n)$ is in $\mathcal{B}$, so it and its closure are connected, and $\text{cl}((\sigma^{\omega_1}_\alpha)^{-1}(U_{n+1})) \subseteq (\sigma^{\omega_1}_\alpha)^{-1}(\overline{U_{n+1}}) \subseteq (\sigma^{\omega_1}_\alpha)^{-1}(U_n)$, so that $(\sigma^{\omega_1}_\alpha)^{-1}\{x\}$ is the decreasing intersection of the connected closed sets $\text{cl}((\sigma^{\omega_1}_\alpha)^{-1}(U_n))$, and hence connected. ☺

We do not know if conditions (C1 – C10) in Section 3 determine $\operatorname{ind}(Z)$, but a minor addition to the construction will ensure that $Z$ does not have small transfinite inductive dimension; that is, $\text{trind}(Z) = \infty$ (and hence $\operatorname{ind}(Z) = \infty$). The transfinite inductive dimension trind is the natural generalization of ind; see [7].

Theorem 6.2 Assuming $\emptyset$, there is a locally connected HS continuum $Z$ such that $\dim(Z) = 1$, $\text{trind}(Z) = \infty$, and $Z$ has no nontrivial convergent sequences.

To do this, we make sure that all perfect subsets are $G_\delta$ sets. Observe that by local connectedness, every non-empty closed $G_\delta$ contains a non-empty connected closed $G_\delta$ subset, which in our $Z$ cannot be a singleton. So, no non-empty closed $G_\delta$ can have dimension 0.

Lemma 6.3 Assume that $X$ is compact, connected, and infinite, and all perfect subsets of $X$ are $G_\delta$ sets. Assume also that $\lambda(x,X) > \aleph_0$ for all $x \in X$, and that in $X$, every non-empty closed $G_\delta$ set contains a non-empty closed connected $G_\delta$ subset. Then $\text{trind}(X) = \infty$. 


Proof. We prove by induction on ordinals \( \alpha \) that \( \neg(\text{trind}(X) \leq \alpha) \) for all such \( X \). This is obvious for \( \alpha = 0 \). Assume \( \alpha > 0 \) and the inductive hypothesis holds for all ordinals \( \xi < \alpha \). Suppose that \( \text{trind}(X) \leq \alpha \). Then there is a regular open set \( U \) such that \( U \neq \emptyset, U \neq X \), and \( \text{trind}(\partial U) = \xi < \alpha \). Let \( V = X \setminus \overline{U} \); then \( \overline{U} \) and \( \overline{V} \) are perfect, so \( \partial U = \overline{U} \cap \overline{V} \) is a \( G_\delta \), and hence contains a non-empty closed connected \( G_\delta \) subset \( Y \). Then \( \text{trind}(Y) \leq \text{trind}(\partial U) \leq \xi \). Since \( Y \) satisfies the conditions of the lemma, this is a contradiction. ☺

By the same argument, this space is weird in the sense of [10]; that is, no perfect subset is totally disconnected.

To construct our \( Z \) so that perfect sets are \( G_\delta \), we observe first that if \( Q \subseteq MS \times [0,1]^{\omega_1} \) is perfect, then \( C := \{ \alpha < \omega_1 : \pi^{\omega_1}_\alpha(Q) \text{ is perfect} \} \) is a club. One might then use \( \langle \rangle \), as in [5], to capture perfect subsets of \( Z \), but this is not necessary, since we already know that \( Z \) is HS, and we are already capturing countable sequences. Thus, we get:

**Conditions 6.4** We have \( P_\alpha \) and \( P_\alpha \) for \( \alpha < \omega_1 \) such that:

\begin{enumerate}
  \item \( P_\alpha = \text{cl}(Z_\alpha \cap \{ q_\alpha^n : n \in \omega \}) \) whenever \( \alpha \geq \omega \) and this set is perfect; otherwise, \( P_\alpha = Z_\alpha \).
  \item \( P_\alpha = \{(\alpha^\delta)^{-1}(P_\delta) : \delta \leq \alpha \} \).
  \item \( \sigma^{\alpha+1} \upharpoonright (\sigma^{\alpha+1})^{-1}(P) : (\sigma^{\alpha+1})^{-1}(P) \rightarrow P \) is irreducible for each \( P \in P_\alpha \).
\end{enumerate}

Proof of Theorem 6.2. To obtain these conditions, note that (C13) is trivial for \( P \) unless \( t_\alpha \in P \). If \( t_\alpha \in P \), then, since \( P \) is perfect, we may choose a sequence of distinct points \( \langle p_n : n \in \omega \rangle \) from \( P \setminus \{ t_\alpha \} \) converging to \( t_\alpha \). Then, while we are accomplishing (C8), we make sure that all points of \( \{ \sigma^{\alpha+1} \}_{\alpha \in \omega} \) are (strong) limit points of the set of singletons, \( \{ (\sigma^{\alpha+1})^{-1}(p_n) : n \in \omega \} \); this implies irreducibility.

Now, we prove by induction on \( \beta \geq \alpha \) that \( \sigma^\beta \upharpoonright (\sigma^\beta)^{-1}(P) : (\sigma^\beta)^{-1}(P) \rightarrow P \) is irreducible for each \( P \in P_\alpha \). Then, if \( Q \subseteq Z \) is perfect, we use HS and (C7) to fix some \( \alpha < \omega_1 \) such that \( P_\alpha = \sigma^{\omega_1}_\alpha(Q) \) and \( P_\alpha \) is perfect. Irreducibility then implies that \( Q = (\sigma^{\omega_1}_\alpha)^{-1}(P_\alpha) \), which is a \( G_\delta \). ☺

Finally, we remark that our space \( Z \) is dissipated in the sense of [12], since in the inverse limit, only one point \( t_\alpha \) gets expanded in passing from \( Z_\alpha \) to \( Z_{\alpha+1} \); the inverse projection of every other point is a singleton. As pointed out in [12], this is also true of the original Fedorchuk S-space [8], where one point \( t_\alpha \) got expanded to a pair of points; here, and in [10] and van Mill [15], \( t_\alpha \) gets expanded to an interval.

References

[1] R. D. Anderson, A characterization of the universal curve and a proof of its homogeneity. *Ann. of Math.* (2) 67 (1958) 313-324.
REFERENCES

[2] R. D. Anderson, One-dimensional continuous curves and a homogeneity theorem. *Ann. of Math.* (2) 68 (1958) 1-16.

[3] M. Bestvina, Characterizing $k$-dimensional universal Menger compacta, *Mem. Amer. Math. Soc.* 71 (1988), no. 380, vi+110 pp.

[4] E. K. van Douwen and W. G. Fleissner, Definable forcing axiom: an alternative to Martin’s axiom, *Topology Appl.* 35 (1990), no. 2-3, 277-289.

[5] M. Džamonja and K. Kunen, Measures on compact HS spaces, *Fundamenta Mathematicae* 143 (1993) 41-54.

[6] T. Eisworth, Countable compactness, hereditary $\pi$-character, and the continuum hypothesis, *Topology Appl.* 153 (2006) 3572-3597.

[7] R. Engelking, *Theory of Dimensions Finite and Infinite*, Heldermann Verlag, Lemgo, 1995.

[8] V. V. Fedorchuk, The cardinality of hereditarily separable bicompacta (in Russian) *Dokl. Akad. Nauk SSSR* 222 (1975) 302-305. English translation: *Soviet Math. Dokl.* 16 (1975) 651-655.

[9] V. V. Fedorchuk, A compact space having the cardinality of the continuum with no convergent sequences, *Math. Proc. Camb. Phil. Soc.* 81 (1977) 177-181.

[10] J. Hart and K. Kunen, Inverse limits and function algebras, *Topology Proceedings* 30, No. 2 (2006) 501-521.

[11] J. R. Kline, Concerning the complement of a countable infinity of point sets of a certain type, *Bull. Amer. Math. Soc.* 23 (1917) 290-292.

[12] K. Kunen, Dissipated compacta, *Topology Appl.* 155 (2008) 282-303.

[13] K. Menger, *Kurventheorie*, Teubner, Leipzig, 1932.

[14] J. C. Mayer, L. G. Oversteegen, and E. D. Tymchatyn, The Menger curve. Characterization and extension of homeomorphisms of non-locally-separating closed subsets, *Dissertationes Math. (Rozprawy Mat.)* 252 (1986), 45 pp.

[15] J. van Mill, A locally connected continuum without convergent sequences, *Topology Appl.* 126 (2002), no. 1-2, 273-280.

[16] J. Prajs, A homogeneous arcwise connected non-locally-connected curve, *Amer. J. Math.* 124 (2002) 649-675.