Low-energy holographic models for QCD

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Abstract

We consider the bottom-up holographic models for QCD which contain the ultraviolet (UV) cutoff. Such models are supposed to describe exclusively the low-energy sector of QCD. The introduction of UV cutoff in the soft wall model is shown to result in a model with qualitatively different predictions. The ensuing model seems to be able to incorporate the constituent quark mass. It is also demonstrated that in order to reproduce the results of the usual soft wall model for the vector and higher spin mesons in the presence of the UV cutoff one can consider the flat bulk space with a modified dilaton background.

1 Introduction

The fundamental theory of strong interactions — Quantum Chromodynamics (QCD) — is known to be highly difficult for the analytical analysis at low energies because of the strong coupling regime. Meanwhile this regime triggers the most interesting phenomena in QCD and therefore the low-energy domain is of extreme theoretical interest. For this reason, it has become a common practice to replace the low-energy QCD by some effective description. Examples of such descriptions include the Sigma-model, Nambu–Jona-Lasinio model, chiral quark model and others. They provide a simple framework for the analysis of some aspects of non-perturbative QCD and lead to relations which are qualitatively or even semiquantitatively right. In spite of much efforts, no any rigorous relation of those models to QCD has been established. Nevertheless the simplicity of theoretical setup makes the effective models very interesting and useful approach to the phenomenology of strong interactions.

Recently a qualitatively new approach emerged on the market that intends to replace QCD at low and intermediate energies and perhaps to do more than the traditional effective models. This approach was inspired by the ideas of gauge/gravity correspondence from the string theory [1,2] which suggest that the 4D strongly coupled gauge theories may have a dual semiclassical description in the 5D anti-de Sitter (AdS) space. There is no recipe
how to construct the holographic duals for the confining theories like QCD. One can try however to guess such a dual model for a limited class of physical problems. This ambitious idea received a concrete realization in the form of so-called bottom-up holographic models proposed some time ago [3,4]. They have been extensively applied to various problems more or less successfully. On the other hand, the bottom-up approach faced a certain criticism, one of worrying point is that the ensuing models are usually matched to QCD in the UV regime where QCD represents a weakly coupled theory due to its asymptotic freedom. At the same time, one expects that the corresponding dual theory (if it exists) should be then in the strongly coupled regime, hence, the applicability of semiclassical treatment for the latter becomes questionable. It is therefore interesting to construct bottom-up models which are free of this conceptual drawback. A straightforward idea consists in imposing the UV cutoff that removes the high-energy region from a tentative holographic model. In the present Letter, we undertake an exploratory consideration of two bottom-up models with the UV cutoff.

The paper is organized as follows. In Section 2, we introduce the UV cutoff to a simple soft wall model [6] and discuss the resulting model which turns out to be quite different. The main results of the soft wall model may be however reproduced if one accepts the flat bulk space after imposing the UV cutoff. This is shown in Section 3. Our conclusions are summarized in Section 4.

2 Soft wall model with the UV cutoff

A straightforward possibility in constructing the holographic models for the IR domain of QCD seems to impose the UV cutoff in the existing bottom-up holographic models. As an educative exercise let us analyze how the Soft Wall (SW) model introduced in Ref. [6] is modified after imposing the UV boundary $z_{\text{UV}}$ that corresponds to the inverse scale of the onset of the strongly coupled regime, about 1 GeV$^{-1}$. The simplest version of the SW model is defined by the action [6]

$$S = -\frac{1}{4g_5^2} \int d^4x \, dz \sqrt{g} \, e^{-N^2z^2} F_{MN} F^{MN}, \quad (1)$$

where $F_{MN} = \partial_M V_N - \partial_N V_M$, $M, N = 0, 1, 2, 3, 4$, and the metric of the AdS$_5$ space having radius $R$ is given by ($\mu = 0, 1, 2, 3$)

$$ds^2 = \frac{R^2}{z^2} (dx_\mu^2 - dz^2); \quad 0 \leq z < \infty. \quad (2)$$

1A similar issue was considered numerically in Ref. [5].
The dilaton background $e^{-\Lambda^2 z^2}$ introduces the scale $\Lambda$ that determines the slope of the linear mass spectrum of normalizable modes, $m_n^2 = 4\Lambda^2(n + 1)$, $n = 0, 1, 2, \ldots$.

Without loss of generality we identify the UV cutoff $z_{UV}$ with the radius of the AdS$_5$ space, $z_{UV} = R$. It means that in performing the integral over $z$,

$$\int_0^\infty dz = \int_0^R dz + \int_R^\infty dz,$$

we do not consider the region $0 \leq z < R$ since we do not expect the validity of the semiclassical approximation in that region. In the axial gauge, $V_z(x, z) = 0$, the equation of motion for the 4D Fourier transform $V_\mu(q, z)$ of the transverse components, $\partial_\mu V_\mu(x, z) = 0$, takes the form

$$-\partial_z \left( \frac{e^{-\Lambda^2 z^2}}{z} \partial_z V_\mu(q, z) \right) = q^2 \frac{e^{-\Lambda^2 z^2}}{z} V_\mu(q, z).$$

Letting $V_\mu(q, z) = v(q, z)V_\mu^0(q)$, we require that $v(q, R) = 1$; the source for the 4D vector current in the momentum space is then given by $V_\mu^0(q)$. The corresponding solution to the Eq. (4) bounded as $z \to \infty$ is

$$v(q, z) = \frac{U(-q^2/4\Lambda^2, 0, \Lambda^2 z^2)}{U(-q^2/4\Lambda^2, 0, \Lambda^2 R^2)},$$

where $U$ is the Tricomi confluent hypergeometric function.

Evaluating the action (1) on the solution leaves the boundary term

$$S = \frac{R}{2g_5^2} \int d^4x \left. \left( \frac{e^{-\Lambda^2 z^2}}{z} V_\mu \partial_\mu V_\mu \right) \right|_{z=R}.$$  

According to the conjecture of the AdS/CFT correspondence [2], the vector two-point correlation function,

$$\int d^4x e^{iqx} \langle J_\mu(x)J_\nu(0) \rangle = (q_\mu q_\nu - q^2 \eta_{\mu\nu}) \Pi_V(Q^2); \quad Q^2 = -q^2,$$

is given by the second derivative of the term (6) with respect to the source $V_\mu^0$:

$$\Pi_V(Q^2) = -\frac{Re^{-\Lambda^2 z^2}}{g_5^2 Q^2} v \partial_z v \bigg|_{z=R}.$$  

Using the normalization $v(q, R) = 1$ and the property $\partial_z U(a, 0, x) = -a U(1 + a, 1, x)$, we obtain finally

$$\Pi_V(Q^2) = \frac{Re^{-\Lambda^2 R^2} U(1 + Q^2/4\Lambda^2, 1, \Lambda^2 R^2)}{2g_5^2 U(Q^2/4\Lambda^2, 0, \Lambda^2 R^2)}.$$  

The expression (9) has poles \( q_n^2 = 4\Lambda^2 f_n \), \( n = 0, 1, 2, \ldots \), where \( f_n \to n + 1 \) as \( \Lambda R \to 0 \). At \( \Lambda R > 0 \), \( f_n \) tends to the equidistant behavior, \( f_{n+1} - f_n \to 1 \), at large \( n \). For instance, the choice \( \Lambda R = 1 \) leads to \( f_n \approx n + 1 \) as \( \Lambda R \to 0 \). In contrast to the SW model, the residues are vanishing as \( n \to \infty \). In the limit \( \Lambda R \to 0 \), the residues of the expression \( U(1+Q^2/4\Lambda^2,1,\Lambda^2 R^2) \) tend to \( 4\Lambda^2 \). Thus, taking into account the general factor in the relation (9), we arrive at the same residues as in the vector correlator of the SW model,

\[
\Pi_V^{(SW)}(Q^2) = \frac{R}{2g_5^2} \left[ \sum_{n=0}^{\infty} \frac{4\Lambda^2}{Q^2 + 4\Lambda^2(n+1)} + \gamma - \sum_{k=1}^{\infty} \frac{1}{k} - \log(z^2\Lambda^2) \right] \bigg|_{z \to 0} \tag{10}
\]

The expression (10) contains two infinite terms since the limit \( \Lambda R = 0 \) is implied from the very beginning. They are subtracted in the final answer while in our case we did not make any subtractions.

The introduction of the UV cutoff may solve the problem with a natural description of the Chiral Symmetry Breaking (CSB) within the SW model. We remind the reader the essence of the problem. The simplest way for incorporating the CSB consists in introducing an action quadratic in a scalar field \( X \) [3, 4],

\[
S_{CSB} = \int d^4x \, dz \sqrt{g} e^{-\Lambda^2 z^2} \left( |\partial_M X|^2 + \frac{3}{R^2} |X|^2 \right), \tag{11}
\]

where the background field \( \frac{2}{z}X(z) \) corresponds to the quark bilinear operator \( \bar{q} q \) and this correspondence dictates the mass term from the relation (26). In the full model, the usual derivative in (11) should be replaced by the covariant one to have a coupling of \( X(z) \) with the vector fields; this is not relevant for the present discussion.

According to the AdS/CFT based prescriptions, the bulk scalar field responsible for the CSB should have the following UV asymptotics [7]

\[
X(z)_{z \to 0} \sim Mz + \Sigma z^3, \tag{12}
\]

the coefficient \( M \) represents then the quark mass and \( \Sigma \) is the chiral condensate. However, the equation of motion for \( X(z) \) following from the action (11) has only one solution bounded as \( z \to \infty \), \( X(z) = zU(\frac{1}{2}, 0, \Lambda^2 z^2) \). Its expansion around \( z = 0 \) reads

\[
X(z)_{z \to 0} \sim 2\Lambda z + \left( 1 + \gamma - \log 4 + \log(\Lambda^2 z^2) \right) \Lambda^3 z^3, \tag{13}
\]

where \( \gamma \approx 0.577 \) is the Euler constant. Thus, \( \Sigma \) is proportional to \( M \). Since this is not what one expects in QCD, the given description of the CSB was
rejected in Ref. [6]. However, if we assume that the prescription (12) holds approximately also for a non-zero \( z = R \) and take into account that the model now is defined in the IR domain only, \( z \geq R \), the behavior above is exactly what one expects in the effective description of the low-energy QCD — the light quarks acquire a constituent (called also dynamical) mass, \( M \approx 320 \text{ MeV} \), that is proportional to \( \Sigma \). For example, within the Nambu–Jona-Lasinio model for the low-energy QCD [8], the relation is \( M = -2GN\Sigma + M_0 \), where \( G \) represents the four-fermion coupling and \( M_0 \) is the current quark mass.

We may attempt to exploit the relation (13) for a rough estimate of the quantity \( \Lambda R \). Multiplying (13) by a constant \( C \) and comparing with (12) one obtains

\[
C = \frac{M}{\Lambda^2 \Sigma} \quad \text{and} \quad \Sigma \simeq \frac{1}{2} \left( 1 + \gamma - \log 4 + \log (\Lambda^2 R^2) \right) M \Lambda^2.
\]

(14)

Taking \( \Lambda = 550 \text{ MeV} \) from the approximate fits for the slope \( 4\Lambda^2 \) in the vector mass spectrum and \( \Sigma = (-235 \text{ MeV})^3 \), we have the estimate \( \Lambda R \approx 0.8 \), i.e. \( R \approx \frac{1}{0.7 \text{ GeV}} \). This means that the model is defined roughly below the mass of the \( \rho \)-meson.

It is interesting to note a similarity between the expression (14) and the relation for the chiral condensate provided by the Nambu–Jona-Lasinio model regularized by the 4D momentum cutoff \( \Lambda_{\text{cut}} \) [8], \( \Sigma \simeq -MA^2_{\text{cut}} + O(M^3) \).

3 Flattened soft wall model

In building the holographic models that describe exclusively the low-energy sector of QCD we do not need to impose the AdS\(_5\) metric in the UV limit since the conformal symmetry is believed to be strongly broken.\(^2\) Phenomenologically the most satisfactory bottom-up model seems to be the SW one. An interesting question arises: Which holographic models with the UV cutoff reproduce the basic results of the SW model, say the form of the vector two-point correlator? We are going to argue that such a model can be constructed in the flat 5D space.

3.1 The model

The metric (2) can be cast into the form

\[
ds^2 = e^{-ky/R}dx^2_\mu - dy^2; \quad k = 2,
\]

(15)

\(^2\)There are, however, some suggestions in the literature (see, e.g., [9]) that the conformal symmetry is restored at very low energies.
where \( y = R \log \frac{z}{R} \) and \( -\infty < y < \infty \). The fifth coordinate \( y \) has then the physical meaning of the logarithm of energy scale. The UV bounded interval \( R \leq z < \infty \) translates into \( 0 \leq y < \infty \). Let us consider the case of the flat space, \( k = 0 \) in (15), and write an action for the free vector field with some unknown dilaton background \( f(y) \),

\[
S = -\frac{1}{4} \int d^4 x \, dy \, e^{-f(y)} F_{MN} F^{MN}.
\]

(16)

Proceeding as in Section 2, we arrive at the equation of motion for the field \( v(q, y) \),

\[
- \partial_y \left( e^{f} \partial_y v \right) = q^2 e^{-f} v.
\]

(17)

The substitution

\[
\varphi = e^{f/2} \psi,
\]

(18)
transforms the Eq. (17) into a Schrödinger like equation

\[
- \psi'' + \left( \frac{(f')^2}{4} - \frac{f''}{2} \right) \psi = q^2 \psi.
\]

(19)

Following the Ref. [6] we try the ansatz (\( R \) below is simply a constant of dimension [mass\(^{-1}\)])

\[
f = Ay^2 + B \log \frac{y}{R}
\]

(20)
and consider the exactly solvable case \( A = \Lambda^2, \ B = 1 \). The "potential" of Eq. (19) is then

\[
\frac{(f')^2}{4} - \frac{f''}{2} = \Lambda^4 y^2 + \frac{3}{4 y^2}.
\]

(21)

The further results are identical to those of the SW model but with \( z \) replaced by \( y \): The normalizable solutions form a discrete set of modes with the masses \( q_n^2 = m_n^2 \),

\[
m_n^2 = 4\Lambda^2 (n+1); \quad n = 0, 1, 2, \ldots ,
\]

(22)
and wave functions

\[
v(y) = \sqrt{\frac{2n!}{(n+1)!}} \Lambda^2 y^2 L_n(\Lambda^2 y^2),
\]

(23)
Thus we see that without the CSB the SW model and the flattened SW model with appropriately chosen background lead to identical predictions for the vector mesons. The appearance of factor $y^{-1}$ in the dilaton background may be interpreted as an "encoded" rest of conformal symmetry ($=\text{AdS metric}$). The UV limit, $y \to 0$, of the flattened SW model, however, does not correspond to the UV limit in QCD and can be taken about 1 GeV in order to describe the strongly coupled regime only. The description of the CSB requires an analogue of the prescription (12) for the flat space. This problem will be considered somewhere.

### 3.2 Higher spin fields

In what follows we will try to include the Higher Spin Fields (HSF) into the Flattened Soft Wall (FSW) model. The free massless HSF are described by symmetric double traceless tensors $\Phi_{M_1...M_J}$ \[^{10}\]. The corresponding action is invariant under the gauge transformations $\delta \Phi_{M_1...M_J} = \nabla (M_1 \xi_{M_2...M_J})$, where $\nabla$ is covariant derivative with respect to the general coordinate transformations and the gauge parameter $\xi$ represents a traceless symmetric tensor. We first remind the reader the result of incorporation of HSF into the usual SW model \[^{6}\]. The quadratic part of the action reads

$$S^{(J)} = \frac{1}{2} \int d^4x \sqrt{g} e^{-\Lambda^2 z^2} \left( \nabla_N \Phi_{M_1...M_J} \nabla^N \Phi^{M_1...M_J} + \ldots \right), \tag{24}$$

where further terms are omitted. As was argued in Ref. \[^{11}\], in the axial gauge, $\Phi_{z...} = 0$, the action for a rescaled field $\Phi = \left( \frac{z}{R} \right)^{2(1-J)} \tilde{\Phi}$ contains only the first kinetic term written in (24). The resulting equation of motion for $\tilde{\Phi}(x)$ in the SW model results in the mass spectrum \[^{6}\]

$$m_{nJ}^2 = 4\Lambda^2 (n + J), \tag{25}$$

which generalizes the spectrum for the $J = 1$ case. The spectrum (25) corresponds to the poles of Veneziano amplitude and is expected (up to some additional intercept) in the effective string description of QCD. It is interesting to note that such a spectrum seems to hold approximately in the phenomenology of light mesons \[^{12}\] (again up to a general shift). For the universality of intercept, however, one must replace $J$ by the relative angular momentum $L$ of pions produced via the strong decay of resonance under consideration \[^{13}\]. In the framework of the non-relativistic quark model, the account for the quark spin leads to the relation $J = L, L \pm 1$.

In order to include the HSF into the FSW model we first reinterpret the corresponding result of the Ref. \[^{6}\]. The rescaled field $\tilde{\Phi}_{M_1...M_J}$ corresponds to
a twist-two operator with the canonical dimension $\Delta = J + 2$ [6]. According to the AdS/CFT prescriptions [2], the masses of 5D fields propagating in the AdS$_5$ space which correspond to the $p$-form operators of dimension $\Delta$ in the equivalent 4D theory are given by

$$R^2m_5^2(p) = (\Delta - p)(\Delta + p - 4),$$

(26)

The question arises how this relation could show up in the holographic models describing the $J > 1$ mesons? It is easy to observe that the action (24) in terms of the rescaled field $\tilde{\Phi}$ can be written as

$$S^{(J)} = \frac{1}{2} \int d^4x \, dz \sqrt{g} \left( \frac{R}{z} \right)^{R^2m_5^2(J)} \, e^{-\Lambda^2z^2} \nabla_N \tilde{\Phi}_{M_1...M_J} \nabla^N \tilde{\Phi}_{M_1...M_J}.\quad (27)$$

This implies that the description of the HSF proposed in [6,11] is equivalent to the assumption that such fields couple to different 5D backgrounds. The form of these backgrounds is dictated by the relation (26) in which $p$ is replaced by $J$. Contracting the Lorentz indices in (27) for the case of pure AdS$_5$ space (2) one arrives finally at the action

$$S^{(J)} = \frac{1}{2} \int d^4x \, dz \left( \frac{R}{z} \right)^{2J-1} \, e^{-\Lambda^2z^2} (\nabla_N \tilde{\Phi})^2.\quad (28)$$

In the flat space we expect that the incorporation of the HSF should follow the same principle. It is easy to verify that postulating in the axial gauge the action (28) for the flat space (with $z$ replaced by $y$) leads to the spectrum (25) which represents the eigenvalues of the equation of motion,

$$\partial_y \left( \frac{\partial_y \tilde{\Phi}_n}{y^{2J-1}} \right) + \frac{m_n^2}{y^{2J-1}} \tilde{\Phi}_n = 0,$$

(29)

for the 4D Fourier transform $\tilde{\Phi}(q^2, y)$, $q^2_n = m_n^2$. The 5D background in the action (28) generalizes that of the previous Section to the $J > 1$ states. The parameter $R$ in (28) can be regarded just as a dimensional parameter, say $R = 1/\Lambda$.

4 Conclusions

We have considered a couple of bottom-up holographic models for QCD with the UV cutoff imposed. They are closer in spirit to the traditional effective models for the low-energy QCD which possess in a direct or indirect way
the UV cutoff showing the applicability of a model. Like in the effective field theories, the present approach neglects the running of some physical quantities with the energy scale.

In the first part, the UV cutoff was introduced to the conventional soft wall model and it was demonstrated that the ensuing model differs substantially from the original one. We also argued that the disadvantage of the original model in describing the chiral symmetry breaking — the resulting proportionality of the quark mass to the chiral condensate — may be converted into an advantage after imposing the UV cutoff as exactly this pattern holds in the low-energy effective approaches.

In the second part, we put forward a kind of soft wall model in the flat 5D space motivating this choice by the absence of conformal invariance at low energies where the model is applicable. We proposed an ansatz for modification of dilaton background that reproduces the predictions of the soft wall model both for the vector mesons (neglecting the chiral symmetry breaking) and for the higher spin mesons. The description of the chiral symmetry breaking within such a setup is left for the future.

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