Ng, S.-L. and Paterson, Maura B. (2019) Functional repair codes: a view from projective geometry. Designs, Codes and Cryptography 87 (164), pp. 2701-2722. ISSN 0925-1022.

Downloaded from:

Usage Guidelines:
Please refer to usage guidelines at contact lib-eprints@bbk.ac.uk. or alternatively
Functional repair codes: a view from projective geometry

Siaw-Lynn Ng∗and Maura B. Paterson†

June 10, 2019

Abstract

Storage codes are used to ensure reliable storage of data in distributed systems; functional repair codes have the additional property that individual storage nodes that fail may be repaired efficiently, preserving the ability to recover original data and to further repair failed nodes. In this paper we show that the existing predominant coding theoretic and vector space models of repair codes can be given a unified treatment in a projective geometric framework, which permits a natural treatment of results such as the cutset bound. We find that many of the constructions proposed in the literature can be seen to arise from well-studied geometric objects, and that this perspective provides opportunities for generalisations and new constructions that can lead to greater flexibility in trade-offs between various desirable properties.

We use this framework to explore the notion of strictly functional repair codes, for which there exist nodes that cannot be replaced exactly, and discuss how strict functionality can arise. We also consider the issue that the view of a repair code as a collection of sets of vector/projective subspaces is recursive in nature and makes it hard to discern when a collection of nodes forms a repair code. We provide another view using directed graphs that gives us non-recursive criteria for determining whether a family of collections of subspaces constitutes a functional, exact, or strictly functional repair code, which may be of use in searching for new codes with desirable properties.

1 Introduction

The growth of data and an increasing reliance on digital information have led to much research into ensuring that data can be stored reliably. One predominant solution is the use of storage codes for distributed storage systems: a database is coded and stored in multiple nodes (servers) in such a way that if a number of nodes fail, the data can still be recovered from the functioning nodes. One technique used in practice (for example, RAID [15], Total Recall [2]) is that of erasure coding: for instance, MDS codes such as the Reed-Solomon code [12] can be used to ensure that any number of node failures up to a certain threshold does not impede the recovery of the entire database. However, many distributed storage systems also require additional resilience properties. In particular, we may want to mitigate node failures: if a node should fail, we would like to repair it using information in some of the functioning nodes so that the recovery property of the system still holds. Clearly one could do that by simply recovering the entire database and re-encoding it.

∗Information Security Group, Royal Holloway, University of London, Egham, Surrey, TW20 0EX, U.K. S.Ng@rhul.ac.uk
†Department of Economics, Mathematics and Statistics, Birkbeck, University of London, Malet St, London WC1E 7HX, U.K.m.paterson@bbk.ac.uk
This involves a sometimes unacceptable overhead in storage and communication. Much work has been done to minimise the amount of data to be stored and the amount of data to be transmitted for repair. Using techniques from network coding, Dimakis et al. [3] showed that one could significantly reduce the amount of data to be communicated for repair and showed that there is a trade-off between storage and repair efficiency. Since then considerable attention has been devoted to modeling and constructing efficient repair codes. Here we consider two strands of this work.

In [16], Rashmi et al. proposed a product-matrix framework for repair codes. This is an essentially coding theoretic approach, where the database is treated as messages that are encoded using a generator matrix. The resulting codewords are then stored in individual nodes. Using this framework, repair codes can be constructed with parameters that sit on various points on the storage-repair trade-off curve. On the other hand, in [9], Hollmann and Poh viewed a repair code as a collection of sets of subspaces of a vector space. Recovery corresponds to generating the vector space while repair corresponds to generating a subspace. In this paper (Section 2.4) we explore the relationship between these two models and motivate the interpretation of the vector space model in terms of projective geometry. The connection between repair codes and projective spaces was noted by Etzion and Storme [4], who noted that “the use of subspaces in Galois geometries for distributed storage codes is relatively new and provides new challenges for future research to those who are working in both areas.” We will see that many constructions arise naturally from looking at repair codes from a projective geometric point of view (Section 3) and these include the constructions in [9, 17]. We also frame the cut-set bound of Dimakis et al. in terms of projective spaces, and show it has a straightforward proof in this model (Section 2.3).

There are broadly speaking two types of repair. In exact repair, if a node fails then the new node contains the same information that the failed node stored. (We clarify this notion in the relevant setting in Definition 2.4.) In functional repair, the new node does not necessarily contain the same symbols as the failed node, but the set of nodes after repair should remain a repair code: one should still be able to recover the original database, and future repair should be possible. We will call a functional repair code (FRC) that does not admit exact repair a strictly functional repair code. These definitions will be made more precise in Section 2. The focus of this paper is functional repair. In Section 3 we see that there are repair codes that can be both functional and exact, but that in [9] there is a construction that is strictly functional. This appears to be the only example in the literature so far. In Sections 3.2.2 and 4 we examine this structure from a projective geometry point of view and explore possibilities for generalisation. We give another example of a strictly functional repair code which arises from a familiar structure in projective planes (Section 3.1.1).

The strictly functional construction from [9] is also motivated by the following: the view of a repair code as a collection of sets of vector/projective subspaces is recursive in nature: one must be able to derive a new subspace from an “admissible” set, and the new subspace, together with all but one of the subspaces from the “admissible” set must again be “admissible”. This models the repair property, insisting that future repairs must be possible. However, this recursive nature makes it hard to distinguish when a collection of sets is admissible: it is hard to discern the “global view” of the whole set of nodes from the “local view” of individual node repairs. The construction of [9] is described using symmetry to bypass the recursiveness of the definition. Here we explore another view using directed graphs, discussing exact and functional repairs in terms of the properties of these graphs in Section 5.

We will make these aims more precise when we introduce notation. We would like to note that constructing new efficient storage codes is not the primary focus of this work, even though
many objects in projective geometry appears to offer good repair as well as flexibility in terms of resilience and trade-offs between locality and repair. We intend rather to clarify the definition and properties of functional repair codes, and to consider their possible relationship with other combinatorial objects.

2 Definitions and basic properties

An \((m; n, k, r, \alpha, \beta)\)-functional repair code (FRC) stores \(m\) information symbols from some finite alphabet \(\mathbb{F}\), encoded across \(n\) storage nodes. Each storage node can hold \(\alpha\) symbols. The following properties hold:

(I) (Recovery)

The original information can be recovered from the data stored on any set of \(k\) nodes (a recovery set).

(II) (Repair)

If a storage node fails then a newcomer node contacts some set of \(r\) surviving nodes (the repair set) and downloads \(\beta\) symbols from each of these \(r\) nodes. From these symbols the newcomer node constructs and stores \(\alpha\) symbols in such a way that (I) holds and (II) holds if another node fails.

We note that there is a dichotomy in the definition of the repair set: in some work (for example, \([3, 16]\)) it is stipulated that the repair set is any set of \(r\) surviving nodes, while in others (for example, \([9, 21]\)) it is only required that there exists some \(r\) nodes to form the repair set. The same is true for recovery sets, although with the exception of the example from Section 3.1.4, we will focus on the more desirable case where every set of \(k\) nodes is a recovery set. We will continue this discussion after Definition 2.2.

2.1 Performance measures for FRCs

The commonly-studied measures of efficiency of an FRC are the storage rate \(R_s = \frac{m}{n\alpha}\) (the number of message symbols divided by the total number of stored symbols) and the repair rate \(R_r = \frac{\alpha}{r\beta}\) (the number of symbols required for the repaired node divided by the number of symbols requested in order to facilitate repair). The value \(r\beta\) is called the repair bandwidth. Another performance metric is locality - the number of nodes to be contacted for repair, given by \(r\).

Other performance metrics that we will not describe formally include availability, which is the number of disjoint repair sets for a node. Recent interest in this includes \([19]\) where fractional repetition codes are used to construct codes with high availability and nodes are partitioned into clusters, each cluster providing a set of helper nodes to repair a failed node, and \([25]\), where codes with different repair bandwidth for repair within clusters and across clusters are proposed.

The ability to repair multiple failures is also obviously of interest, and this may also be studied under different models, for example, \([29, 30]\) study centralised repair (where repair is carried out in one location) and cooperative repair (where failed nodes may communicate) for multiple failures.

Much existing literature seeks to construct codes that optimise one or more of these measures \([3, 16, 23]\). This is not the primary motivation of this paper, although we will examine the trade-offs that arise from the various possible construction choices we discuss. We will see that most
geometrical constructions seem to have good repair rates but less than ideal storage rates; some of them offer a trade-off between repair rate and locality.

2.2 Vector space and geometric characterisations of FRCs

In [9] and various subsequent work, a functional repair code is viewed as a collection of sets of subspaces of an $m$-dimensional vector space over a finite field $\mathbb{F}_q$. The underlying storage codes work as follows:

- For $i$ with $0 \leq i \leq n - 1$, the $i^{th}$ node is assigned a vector space represented by a specified basis $\{v^i_0, v^i_1, \ldots, v^i_{t-1}\}$.
- To store a message $x = (x_0, x_1, \ldots, x_{m-1}) \in \mathbb{F}_q^m$, each node $i$ with $0 \leq i \leq n - 1$ stores the $\alpha$ scalar values $\{x \cdot v^i_0, x \cdot v^i_1, \ldots, x \cdot v^i_{t-1}\}$.
- If the vector $(1, 0, \ldots, 0)$ is in the span of a set of vectors $\{u_0, \ldots, u_{t-1}\}$, then the values $\{x \cdot u_0, \ldots, x \cdot u_{t-1}\}$ can be used to recover $x_0$. For, if $(1, 0, \ldots, 0) = \sum_{i=0}^{t-1} a_i u_i$ for $a_i \in \mathbb{F}_q$, then $x_0 = \sum_{i=0}^{t-1} a_i x \cdot u_i$. If the vectors $\{u_0, \ldots, u_{t-1}\}$ span $\mathbb{F}_q^m$ then the entire message $x$ can similarly be recovered from these values.

The properties of the storage code are hence determined by the relationship between the subspaces that correspond to the nodes, in particular, the spans and the intersections of these subspaces. The projective space $\text{PG}(m-1, q)$ provides a very natural setting for studying spans and intersections in $\mathbb{F}_q^m$. It can make the relationship between spaces easier to visualise and, furthermore, many classical geometric structures in $\text{PG}(m-1, q)$ have well-understood span/intersection properties that can be useful in constructing storage codes. The basic results we rely on are as follows: the points of projective space $\text{PG}(m-1, q)$ are the elements of the set $\mathbb{F}_q^m \setminus \{(0, 0, 0, \ldots, 0)\}$ under the equivalence relation $P \equiv \lambda P$ for any $\lambda \in \mathbb{F}_q^*$. There are $q^{m-1} + q^{m-2} + \cdots + q + 1$ such points. The span of a set of $t$ points consists of all points that can be written as a linear combination of those $t$ points. If the $t$ points are linearly independent, then their span is a $(t-1)$-dimensional subspace of $\text{PG}(m-1, q)$. The intersection of two subspaces of $\text{PG}(m-1, q)$ is itself a subspace of $\text{PG}(m-1, q)$. The key result that we will need to investigate the properties of FRCs in this setting is the fact that if $\Pi$ and $\Sigma$ are subspaces of dimensions $t_1$ and $t_2$ respectively, then the dimension of the space spanned by $\Pi$ and $\Sigma$ (denoted $(\Pi, \Sigma)$) is equal to $t_1 + t_2$ minus the dimension of their intersection. (Note that for the sake of this formula we take the dimension of the empty set to be $-1$.)

In what follows we will translate the vector-space definitions of [9, Definitions 3.1, 3.2] into the language of projective spaces. We will see that this provides new insight into existing constructions of repair codes, such as [9, 17], as well as suggesting useful frameworks for new construction of such codes.

**Definition 2.1** ($r, \beta$-repair). Let $\Sigma = \text{PG}(m-1, q)$ be an $(m-1)$-dimensional projective space over the finite field $\mathbb{F}_q$. We say that we can obtain a subspace $U'$ of $\Sigma$ from a set $\mathcal{U}$ of subspaces of $\Sigma$ by $(r, \beta)$-repair if there is an $r$-subset $\{U_{i_1}, \ldots, U_{i_r}\}$ in $\mathcal{U}$ such that there exists a $(\beta-1)$-dimensional subspace $W_{i_j} \subset U_{i_j}$ for each $i_j$ such that $U' \subset \langle W_{i_1}, \ldots, W_{i_r} \rangle$.

**Definition 2.2** (Functional repair codes). Let $\Sigma = \text{PG}(m-1, q)$ and let $\mathcal{A}$ be a collection of $(n-1)$-sets $\mathcal{U}$ of $(\alpha-1)$-dimensional subspaces of $\Sigma$ such that:
(A) (Recovery)

For each set \( U \in \mathcal{A} \) we have that any \( k \)-subset \( \{U_{i_1}, \ldots, U_{i_k}\} \) of the subspaces in \( U \) span \( \Sigma \).

(B) (Repair)

Given any \((n - 1)\)-set \( U = \{U_1, \ldots, U_{n-1}\} \) in \( \mathcal{A} \), there exists an \((\alpha - 1)\)-dimensional subspace \( U_n \subset \Sigma \) that can be obtained from \( U \) by \((r, \beta)\)-repair, such that for every \( i = 1, \ldots, n-1 \), \( U \cup \{U_n\} \setminus \{U_i\} \) is again in \( \mathcal{A} \).

We will call \((\Sigma = \text{PG}(m - 1, q), \mathcal{A})\) an \((m; n, k, r, \alpha, \beta)\)-functional repair code (or \((m; n, k, r, \alpha, \beta)\)-FRC for convenience).

In Definition 2.2 there is no stipulation on the size of \( \mathcal{A} \), nor on the number of \((\alpha - 1)\)-dimensional subspaces in an \((m; n, k, r, \alpha, \beta)\)-FRC. Let \( \mathcal{N} \) be the number of distinct \((\alpha - 1)\)-dimensional subspaces used in \( \mathcal{A} \). We will consider bounds on the value of \( \mathcal{N} \) in Section 5.

Here \( \mathcal{A} \) corresponds to all possible sets of \( n - 1 \) subspaces that belong to the nodes remaining after a single node has failed. The repair property ensures that there is always a suitable subspace that can be constructed by \((r, \beta)\)-repair from these nodes in order to construct a replacement for the node that has failed. Here we require that arbitrary \( k \)-sets of nodes are recovery sets, but we only require that there exists some repair set (although in many of the constructions we describe in Section 3, repair can be effected by arbitrary sets. We will clarify each case as we go along.)

To avoid triviality, it is standard to assume that \( m, n \geq 2 \), that \( 1 \leq k < n \), that \( k \leq r \leq n - 1 \), that \( 1 \leq \alpha \leq m - 1 \), and that \( 1 \leq \beta \leq \alpha \), and we shall make these assumptions throughout, with the exception of the generalisations of Construction 3.9, where there is interest in minimising \( r \) (the minimum locality case).

**Definition 2.3.** Let \((\Sigma = \text{PG}(m - 1, q), \mathcal{A})\) be an \((m; n, k, r, \alpha, \beta)\)-FRC. An \( n \)-set \( \{U_1, \ldots, U_n\} \) of \((\alpha - 1)\)-dimensional subspaces of \( \Sigma \) with the property that \( \{U_1, \ldots, U_n\} \setminus \{U_j\} \in \mathcal{A} \) for all \( j \in \{1, \ldots, n\} \) is said to be repairable.

It is the repairable sets corresponding to \((\Sigma, \mathcal{A})\) that can be used as storage codes; if any node fails, the repair property then ensures that the resulting \((n - 1)\)-set permits a new repairable set to be obtained through \((r, \beta)\)-repair. Now we define exact and strictly functional repairs:

**Definition 2.4** (Exact repair). Let \((\Sigma = \text{PG}(m - 1, q, \mathcal{A})\) be an \((m; n, k, r, \alpha, \beta)\)-FRC. We say that \((\Sigma, \mathcal{A})\) is an exact repair code if for any repairable set \( \{U_1, \ldots, U_n\} \) we have the additional property that \( U_i \) can be obtained by \((r, \beta)\)-repair from \( \{U_1, \ldots, U_n\} \setminus \{U_i\} \) for any \( U_i \in \{U_1, \ldots, U_n\} \).

We observe that if \((\Sigma, \mathcal{A})\) is an exact repair code, then for any repairable set \( R = \{U_1, U_2, \ldots, U_n\} \), the collection \( \mathcal{A}' = \{R \setminus \{U_i\} | 1 \leq i \leq n\} \) has the property that \((\Sigma, \mathcal{A}')\) is itself an exact repair code.

**Definition 2.5** (Strictly functional repair). Let \((\Sigma = \text{PG}(m - 1, q), \mathcal{A})\) be an \((m; n, k, r, \alpha, \beta)\)-FRC. We say that \((\Sigma, \mathcal{A})\) is a strictly functional repair code if there exists a repairable set \( \{U_1, \ldots, U_n\} \) for which there is a \( U_i \in \{U_1, \ldots, U_n\} \) that cannot be obtained from \( \{U_1, \ldots, U_n\} \setminus \{U_i\} \) by \((r, \beta)\)-repair.

In other words, \((\Sigma, \mathcal{A})\) is a strictly functional repair code if there is some subspace in a repairable set such that exact repair from the remaining \( n - 1 \) subspaces of the set is not possible. For these definitions we are focussing on the subspaces stored by the nodes, rather than explicitly referring to bases for these spaces. This is due to the fact that the elements stored by a node allow them
to recover any desired element in the corresponding space, and this ability does not depend on the choice of basis used to describe the space. We note that in [17], the term functional repair is used in a scenario in which the failed node and the repaired node correspond to different bases of the same space. However, this would satisfy Definition 2.4 for exact repair, and hence would not represent a strictly functional repair code according to our usage of terminology in this paper. We will later discuss two examples of codes that do satisfy our stronger definition of strictly functional repair: one from [9] (Section 4) and a new example that arises almost immediately from phrasing the definition in terms of projective geometry (Section 3.1.1).

2.3 Geometric interpretation of the cut-set bound

In [3], the cut-set bound of network coding is used to establish an upper bound on the number of information symbols \( m \) that can be stored in an \((m; n, k, r, \alpha, \beta)\)-FRC. Here we interpret this bound in terms of finite projective geometry for the case \( n = r + 1, \beta = 1 \).

**Theorem 2.6.** Let \( (\Sigma = \text{PG}(m - 1, q), \mathcal{A}) \) be an \((m; r + 1, k, r, \alpha, 1)\)-FRC. Then

\[
m \leq \sum_{i=1}^{k} \min(\alpha, (r - k) + i).
\]

**Proof.** Each node \( i \) corresponds to a subspace \( U_i \) of \( \Sigma \) of dimension \( \alpha - 1 \), and any \( k \) of them span \( \text{PG}(m - 1, q) \). In particular, the spaces corresponding to the first \( k \) nodes span \( \Sigma \), i.e. \( \langle U_1, U_2, \ldots, U_k \rangle = \Sigma \). This implies that \( m - 1 \) is at most \( k\alpha - 1 \).

Consider a repair of node 1. The repair property implies it is possible to choose one point \( P_j \) from each node \( j \) with \( 2 \leq j \leq r + 1 \) such that there is an \((\alpha - 1)\)-dimensional subspace \( U'_1 \) contained in their span with \{\(U'_1, U_2, \ldots, U_{r+1}\)\} repairable. Since we require \( \langle U'_1, U_2, \ldots, U_k \rangle = \Sigma \), it follows that \( \langle U_2, U_3, \ldots, U_k, P_{k+1}^1, P_{k+2}^1, \ldots, P_{r+1}^1 \rangle = \Sigma \). This implies that \( m - 1 \) is at most \((k - 1)\alpha - 1 + (r + 1 - k)\).

We now consider a repair of node 2. There exists a point \( P_j \) in each node with \( j \neq 2 \) (including \( P_2 \) in \( U'_1 \)) such that there is a \((\alpha - 1)\)-dimensional subspace \( U'_2 \) contained in their span with \{\(U'_1, U'_2, \ldots, U_{r+1}\)\} repairable, and \( \langle U_3, \ldots, U_k, P_{k+1}^1, P_{k+2}^1, \ldots, P_{r+1}^1, P_{k+1}^2, P_{k+2}^2, \ldots, P_{r+1}^2 \rangle = \Sigma \). This implies that \( m - 1 \) is at most \((k - 2)\alpha - 1 + (r + 1 - k) + (r + 2 - k)\).

We can repeat this process, continuing to replace each \( U_i \) in the set by a collection of repair points whose inclusion ensures that the replacement \( U'_i \) will be contained in the relevant span. After repair of node \( i \) we have the result that \( m - 1 \) is at most \((k - i)\alpha - 1 + \sum_{j=1}^{i} (r + j - k)\). The bound on \( m - 1 \) is lowered at each step until either we reach a point at which the number of additional points we have to add \((r + i - k)\) is greater than \( \alpha \), or we have replaced all of \( U_1, \ldots, U_k \) with the relevant repair sets of points. At this point we stop, and we have \( m - 1 \leq \left( \sum_{j=1}^{k} \min(\alpha, r + j - k) \right) - 1 \), so \( m \leq \sum_{j=1}^{k} \min(\alpha, r + j - k) = \sum_{i=0}^{k-1} \min(\alpha, r - i) \).

Generalising to \( \beta > 1 \) is entirely straightforward: in each step of the proof we take \( \beta \) points per node rather than 1 point. This approach would also work in the \( n > r + 1 \) case if we make the assumption that any set of \( r \) nodes can be used for repair. This is the assumption made in [3, 16].
2.4 The product-matrix model

The other widely used model of \((m; n, k, r, \alpha, \beta)\)-FRC is the product-matrix model [16] mentioned in the Introduction. In this model, the \(m\) information symbols are formatted into an \(r \times \alpha\) message matrix, and the encoding process involves multiplication by an \(n \times r\) encoding matrix. The resulting \(n \times \alpha\) matrix gives the symbols stored on each of the \(n\) nodes: row \(i\) of the matrix denotes the \(\alpha\) symbols stored in node \(i\). This can be viewed as an instantiation of the vector space model of [9]: if the entries in the \(i^{th}\) row of the encoding matrix are \(E_{i1}, E_{i2}, \ldots, E_{ir}\), then the \(i^{th}\) node corresponds to the subspace spanned by the vectors \(v_0, v_1, \ldots, v_{\alpha-1}\), where \(v_j\) has the values \(E_{i1}, E_{i2}, \ldots, E_{ir}\) in positions \(jr + 1\) through \(jr + r\) and 0 in the remaining positions. If a length \(m\) message is obtained by concatenating the columns of the message matrix, then the resulting symbols stored by each node according to this vector space scheme are precisely those that would be stored using the product-matrix model.

2.5 Subpacketisation/vectorisation

We now consider a well-known example of an FRC that can be generated using the product-matrix model with \(\alpha = 1\), together with the application of a technique proposed by Shanmugam et al. for improving the repair bandwidth [24]. We will see that this example can be described very naturally in the projective geometry setting.

Example 2.7 (Scalar MDS code). A file \(x_0 \ldots x_{m-1}\) consisting of \(m\) symbols belonging to the field \(\mathbb{F}_p^s\), \(p\) a prime power and \(s > 1\), is stored across \(n\) storage nodes using an \([n, m]\)-MDS code over \(\mathbb{F}_p^s\). (This is referred to as a scalar MDS code.) Each storage node stores exactly \(\alpha = 1\) symbol of \(\mathbb{F}_p^s\). If a storage node should fail, a repair would involve contacting \(r = m\) nodes, each contributing \(\beta = 1\) symbol. Altogether it would take \(r\beta = m\) symbols to repair one symbol.

Following the approach of Definition 2.2, the scalar MDS code construction translates to a collection of \(n\) points \(P_0, \ldots, P_{n-1}\) in \(\Sigma = PG(m-1, p^s)\), every \(m\) of which span \(\Sigma\); this is precisely an \(n\)-arc in \(PG(m-1, p^s)\). Any failed node can only be obtained by a \((m, 1)\)-repair, since any given point of the arc is not contained in the space spanned by \(m - 1\) further points of the arc. This is an \((m; n, m, 1, 1)\)-FRC with storage rate \(R_s = \frac{m}{n}\) and repair rate \(R_r = \frac{1}{m}\).

In [24] Shanmugam et al. proposed a “vectorisation” of MDS codes over fields of prime power in order to obtain a better repair bandwidth. “Vectorisation” or “subpacketisation” involves treating each symbol \(x_i \in \mathbb{F}_p^s\) as \(s\) symbols of \(\mathbb{F}_p\). As a consequence, instead of having to downloading all the symbols in each node, one may be able to effect repair by downloading fewer symbols (from perhaps more nodes), resulting in a reduction of repair bandwidth.

To explore the vectorisation process more explicitly, let \(f(x) = a_0 + a_1 x + \cdots + a_{s-1} x^{s-1} + x^s\) be a primitive polynomial of degree \(s\) over \(\mathbb{F}_p\) and let \(\zeta\) be a root of \(f(x)\). Then every element \(b \in \mathbb{F}_p^s\) can be written as \(b = b_0 + b_1 \zeta + \cdots + b_{s-1} \zeta^{s-1}\), \(b_i \in \mathbb{F}_p\). Using this correspondence, \(b \in \mathbb{F}_p^s\) can be viewed as \((b_0, b_1, \ldots, b_{s-1}) \in \mathbb{F}_p^s\). This is the basis of the technique of field reduction used to construct Desarguesian spreads of \(PG(sm-1, p)\) from the points of \(PG(m-1, p^s)\) ([7, Section 4]). A point \((x_0, x_1, \ldots, x_{m-1})\) in \(PG(m-1, p^s)\), with \(x_i \in \mathbb{F}_p^s\) viewed as \((x_0, x_1, \ldots, x_{s-1}) \in \mathbb{F}_p^s\), can be written as the point \((x_0^0, x_1^0, \ldots, x_0^{s-1}, x_1^0, x_1^1, \ldots, x_1^{s-1}, \ldots, x_0^{m-1}, x_1^{m-1}, \ldots, x_{s-1}^{m-1})\) in \(PG(sm-1, p)\). Now, take a point \((p_0, p_1, \ldots, p_{m-1}) \in PG(m-1, p^s)\) and all its multiples \(\{(p_0 \zeta^i, p_1 \zeta^i, \ldots, p_{m-1} \zeta^i) \mid i = 0, \ldots, p^s-2\}\). Then the corresponding points of this set in \(PG(sm-1, p)\) form an \((s-1)\)-dimensional
subspace. The set of all such \((s - 1)\)-dimensional subspaces partitions \(\text{PG}(m - 1, p^s)\) and is a Desarguesian spread.

(The “vectorisation” process in [24] uses another map: each \(b \in \mathbb{F}_{p^s}\) can be treated as a linear transformation \(x \mapsto bx\) in \(\mathbb{F}_{p^s}\), so \(b\) can be described as an \(s \times s\) matrix acting on the basis of \(\mathbb{F}_{p^s}\) over \(\mathbb{F}_p\). Each element of the MDS code is thus replaced by its corresponding \(s \times s\) matrix. This process is equivalent to the field reduction construction of Desarguesian spreads described above.)

The “vectorised” functional repair code is now an \((sm; n, m, r, \leq m, s, \beta)\)-FRC for some \(r\) and \(\beta\) and storage rate \(R_s = \frac{m}{n}\), repair rate \(R_r = \frac{s}{r^2}\). It corresponds to a set of \(n\) \((s - 1)\)-dimensional subspaces of \(\text{PG}(sm - 1, p)\), and we can see that with more room to manoeuvre we may be able to repair one subspace without having to use entire subspaces.

We give a small example to illustrate this principle:

**Example 2.8.** Take \(s = 3, k = 3, n = 5\), we have a 5-arc in \(\text{PG}(2, 8)\) (taking primitive element \(\zeta^3 = \zeta + 1\)): \{\((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (1, \zeta, \zeta^2)\}\).

This is an \((m = 3; n = 5, k = 3, r = 3, \alpha = 3, \beta = 1)\)-FRC with \(R_s = \frac{2}{5}, R_r = \frac{1}{3}\). “Vectorisation” gives 5 planes \(U_1, \ldots, U_5\) in \(\text{PG}(8, 2)\):

\[
\begin{align*}
U_1 &= \langle (100 000 000), (010 000 000), (001 000 000) \rangle, \\
U_2 &= \langle (000 100 000), (000 010 000), (000 001 000) \rangle, \\
U_3 &= \langle (000 000 100), (000 000 010), (000 000 001) \rangle, \\
U_4 &= \langle (100 100 100), (010 010 100), (001 001 001) \rangle, \\
U_5 &= \langle (100 001 010), (010 101 011), (001 010 101) \rangle.
\end{align*}
\]

This is now an \((m = 9; n = 5, k = 3, r = 5, \alpha = 3, \beta = 2)\)-FRC. If \(U_1\) fails, one could repair \(U_1\) by downloading the following points:

- \(R_{21} = (000 110 000), R_{22} = (000 011 000)\) from \(U_2\),
- \(R_{31} = (000 000 110), R_{32} = (000 000 011)\) from \(U_3\),
- \(R_{41} = (110 110 110), R_{42} = (011 011 011)\) from \(U_4\),
- \(R_{51} = (010 101 011)\) from \(U_5\) (and another one if we must have symmetry).

Then we can get \((010 000 000) = R_{51} + R_{21} + R_{22} + R_{32}, (110 000 000) = R_{41} + R_{21} + R_{31}, \) and \((011 000 000) = R_{42} + R_{22} + R_{32}\). This gives us \(U_1\).

In the scalar version, to repair one point (9 bits of information) we need to use three points (27 bits). The repair rate is therefore \(1/3\). In the “vectorised” version, to repair one subspace (27 bits) we need to use 8 points (72 bits). The repair rate is thus \(3/8 > 1/3\). (Or \(3/7\) if we don’t mind lopsidedness.)

The motivation in [24] is to obtain a better repair rate, which the example illustrated. In addition, we see that this process has a natural counterpart in projective geometry that is also intuitive.

Much work has been done further along these lines with some variations. For instance, [1] studies the lower bound for \(\alpha\) (the “sub-packetisation”) in MSR codes that allow “repair-by-transfer”, that is, symbols from the remaining functioning nodes are downloaded directly without computation
during repair, and [26] provides further examples of codes reaching the lower bound for $\alpha$ for different values of locality. Meanwhile, [5] studies trading off repair bandwidth for better sub-packetisation, and [18] also provides constructions for MSR codes achieving the lower bound for $\alpha$ for “repair-by-transfer”.

3 Projective geometric constructions of functional repair codes

We will examine some existing constructions and also some constructions that arise naturally from looking at functional repair codes from a projective geometric point of view. The construction of a vector space/projective geometric functional repair code involves choosing both the dimensions of the spaces corresponding to the nodes, and selecting which subspaces of these dimensions to use. The properties of the code are determined entirely by the manner in which the various spaces intersect.

Broadly speaking, assigning low-dimension subspaces over a given field to nodes is efficient from a storage perspective, while assigning larger spaces over the same field can allow the repair bandwidth to be reduced. When spaces of dimension greater than one are used, there is the potential for the spaces assigned to distinct nodes to have a non-trivial intersection. In what follows we will consider separately constructions with intersecting subspaces and those with non-intersecting subspaces. Both cases are potentially of interest: non-intersecting spaces are efficient in the sense of avoiding direct redundancy, however there is an upper bound to how large spaces can be without intersecting, and redundancy may be desirable for facilitating recovery and/or repair.

3.1 Constructions using intersecting subspaces.

We begin by considering the simplest possible case for intersecting subspaces, that of lines in a plane, then use the results obtained to suggest useful constructions in higher dimensions.

3.1.1 Dual arcs

A neat construction of an exact repair code can be obtained from three lines in a plane:

**Example 3.1** (Three lines in a plane.). Any three non-concurrent lines in a plane will give an exact repair code: let $l_1$, $l_2$, $l_3$ be three non-concurrent lines in $\text{PG}(2, q)$, and let $A$ be the collection of the sets of pairs of distinct lines $\{l_i, l_j\} \subseteq \{l_1, l_2, l_3\}$. Then $A$ is an $(m = 3; n = 3, k = 2, r = 2, \alpha = 2, \beta = 1)$-FRC. Here the storage rate $R_s = 1/2$ and the repair rate is $R_r = 1$.

This example tolerates a single node failure. In order to protect against additional failures we may desire schemes permitting more nodes. We can generalise the idea of Example 3.1 to a larger set of lines: a *dual arc* in a projective plane of order $q$ is a set of at most $q + 1$ lines, no three concurrent.

**Theorem 3.2** (Dual arcs in a plane). Let $\mathcal{L}$ be a dual arc with $n$ lines in $\Sigma = \text{PG}(2, q)$, $3 \leq n \leq q+1$. Let $A$ be the collection of pairs of distinct lines of $\mathcal{L}$. Then $(\Sigma, A)$ is a $(3; n, 2, 2, 2, 1)$-FRC that can tolerate up to $n - 2$ node failures (if $n > 3$), with storage rate $R_s = 3/2n \leq 1/2$ and repair rate $1$.

**Proof.** Any subset of three nodes in $\mathcal{L}$ can be considered to be an exact repair code, as seen in Example 3.1. Thus, provided two nodes survive, any failed node can be recovered by exact repair. \qed
This approach leads naturally to a generalisation to higher dimensional spaces:

**Example 3.3** (Planes in PG(3, q)). Consider a dual arc in PG(3, q): a set of q + 1 planes, any 4 meeting trivially. (So 2 planes meet in a line, 3 planes meet in a point.)

Take 3 of the planes \( \pi_1, \pi_2, \pi_3 \). If \( \pi_3 \) fails, repair to \( \pi_3' \) using lines \( l_1 \in \pi_1 \setminus \pi_2, l_2 \in \pi_2 \setminus \pi_1 \). This gives a \((m = 4; 3 \leq n \leq q + 1, k = 2, r = 2, \alpha = 3, \beta = 2)\)-FRC with \( R_s = 4/3n \leq 4/9, R_r = 3/4 \).

On the other hand, we could take 4 planes, for example, \( \pi_0: x_0 = 0, \pi_1: x_1 = 0, \pi_2: x_2 = 0, \pi_3: x_3 = 0 \). If \( \pi_3 \) fails, it can be repaired by \((4,1)\)-repair, using \( P_0 = (0, 1, 0, 0) \in \pi_0, P_1 = (0, 0, 1, 0) \in \pi_1, \) and \( P_2 = (1, 0, 0, 0) \in \pi_2 \). This gives an \((m = 4; n = 4, k = 2, r = 3, \alpha = 3, \beta = 1)\)-FRC, with \( R_s = 4/3n = 1/3 \) and better repair rate, \( R_r = 1 \).

There are two important features in the simple construction of Example 3.3: the ability to trade off locality and repair bandwidth without having to make a decision during the set up, and the ability to repair multiple failures. Before we discuss this in more detail, we give the general construction:

**Construction 3.4.** Take a dual arc in PG\((m - 1, q)\): a set of \( q + 1 \) hyperplanes, any \( m \) of which meet trivially. We may take the set of hyperplanes in a dual normal rational curve \( \{H_t = [1, t, t^2, \ldots, t^{m-1}] : t \in \mathbb{F}_q \} \cup \{H_\infty = [0,0,\ldots,0,1]\} \), where \([z_0, z_1, \ldots, z_{m-1}]\) denotes the set of points

\[ \{(x_0, x_1, \ldots, x_{m-1}) \mid z_0x_0 + z_1x_1 + \cdots + z_{m-1}x_{m-1} = 0\}. \]

However, to make the description of the trade-off clearer, we will take an \( m \)-subset of these hyperplanes and coordinatise them as follows, writing \( e_i \) to denote the point with a 1 in position \( i \) and 0 everywhere else: \( H_i: x_i = 0 \), that is, \( H_i = \{e_j, j \in \{0, \ldots, m - 1\} \setminus \{i\}\} \).

This gives an \((m; n = m, k = 2, r = \lceil \frac{m-1}{\beta} \rceil, \alpha = m - 1, \beta)\)-FRC with \( R_s = \frac{m}{nm} \) and \( R_r = \frac{m-1}{m-1+\delta} \), where \( \delta = 0 \) if \( \beta | m - 1 \). Otherwise \( \delta = \beta - \Delta \) where \( \Delta = m - 1 \mod \beta \). Here \( \beta \geq 1 \) and \( r \geq 2 \). Indeed, if we choose \( m \) odd, and \( \beta = (m - 1)/2 \), then we achieve both minimum locality and optimum repair bandwidth.

For simplicity we describe what happens if \( H_0 \) fails. An \((r, \beta)\)-repair can be performed, with \( r = \lceil \frac{m-1}{\beta} \rceil \), with each of the active \( H_i \) contributing \( \beta \) points as follows:

\[
\begin{align*}
H_1 & \rightarrow e_2, \ldots, e_{\beta+1}, \\
H_2 & \rightarrow e_{\beta+2}, \ldots, e_{2\beta+1}, \\
& \vdots \\
H_i & \rightarrow e_{(i-1)\beta+2}, \ldots, e_{i\beta+1}, \\
& \vdots \\
H_{\lceil \frac{m-1}{\beta} \rceil} & \rightarrow e_{\lceil \frac{m-1}{\beta} \rceil+2}, \ldots, e_{m-1}, e_1.
\end{align*}
\]

Clearly at any repair one could choose the locality \( r \) to suit the circumstances. In [17] a construction was given that also allows such a trade-off - one can choose between minimum bandwidth repair or low locality repair, by assigning the subspaces accordingly, but this assignment has to be determined at set up. Construction 3.4 allows the trade-off to be performed at each repair according to the network conditions.
Construction 3.4 also tolerates multiple node failures: we can choose \( m \leq n \leq q + 1 \), and any failure of up to \( n - 2 \) nodes still allows recovery and repair. It also gives high availability. For example, when we consider the special case of Theorem 3.2 using dual arcs in planes, we see that any line can be repaired using any pair of lines, so that many sets of nodes can be used to repair a failed node.

Note also that if we start with \( n < q + 1 \), additional nodes can be created by accessing information from existing nodes using the repair process. This may be useful if resilience requirements change during the lifetime of the storage system.

3.1.2 Concurrent lines and strictly functional repair

The use of dual arcs in constructing functional repair codes is appealing due to the high availability that results. However Example 3.1 also prompts another question: what happens if we allow sets of nodes that correspond to concurrent lines? In Theorem 3.2 and Construction 3.4, the spaces assigned to nodes correspond to hyperplanes forming a dual arc in the underlying space. This enables us to control the way the spaces corresponding to sets of nodes intersect: any \( t \) of them intersect in a space of dimension \( m - 1 - t \). However, we may wish to allow more general patterns of intersection (for example, in order to permit more than \( q + 1 \) nodes). To explore this, we return to the case of lines in the plane, and consider collections of lines that include sets of three concurrent lines. The following example shows this takes us into the realm of strictly functional repair codes:

**Example 3.5** (A strictly functional repair code.). Let \( l_1, l_2, l_3, l_4 \) be four lines of \( \Sigma = \text{PG}(2, q) \), \( q > 3 \), such that \( l_1, l_2, l_3 \) are concurrent at a point \( P \), and \( l_4 \) does not pass through \( P \). (See Figure 1.) Let \( \mathcal{A} \) be the collection of pairs of lines \( \{l_i, l_j\}, i, j \in \{1, 2, 3, 4\}, i \neq j \). Then \((\Sigma, \mathcal{A})\) is an \((m = 3; n = 3, k = 2, r = 2, \alpha = 2, \beta = 1)\)-FRC which is a strictly functional repair code.

This is because there is a set \( \{l_1, l_2, l_3\} \) with \( \{l_1, l_2\}, \{l_1, l_3\}, \{l_2, l_3\} \in \mathcal{A} \) but \( l_3 \) cannot be obtained from \( \{l_1, l_2\} \) by \((2, 1)\)-repair.

As far as we are aware, this appears to be the only other example of a strictly functional repair code in the literature, apart from an example due to [9] that we will discuss in Section 3.2.2.

3.1.3 Grassmann varieties

The constructions we discussed in Section 3.1.1 all involve subspaces that are hyperplanes of the ambient space. This represents one extreme point of the possible trade-off between low repair bandwidth and flexibility of repair at the cost of high storage. Using smaller dimensional spaces both reduces the storage overhead, and allows for greater flexibility in terms of the size of pairwise
intersections between the spaces. In this environment where greater flexibility is possible, this implies that the spaces must be chosen carefully to achieve the desired intersection properties. Here we consider an example of a construction from [17]. It uses subspace codes constructed from Grassmann varieties in vector spaces. We will describe it from the point of view of projective geometry, in order to see how known properties of Grassman varieties make it possible to choose collections of subsets with suitable intersections. We note that codes arising from Grassman varieties have received considerable attention for applications such as network coding (see [4] for a survey of such work.)

Let $b \geq 2$ and $t \leq b$ be integers. Consider $\Pi_t$, a $t$-dimensional projective subspace of $\mathbb{P}(b,q)$. Let the points $X_0, \ldots, X_t$ be a basis for $\Pi_t$. Write $X_i = (x^i_0, x^i_1, \ldots, x^i_b)$ and let $M_{\Pi_t}$ be the $(t+1) \times (b+1)$ matrix

$$
M_{\Pi_t} = \begin{pmatrix}
X_0 \\
X_1 \\
\vdots \\
X_t \\
\end{pmatrix} = \begin{pmatrix}
x^0_0 & x^0_1 & \cdots & x^0_b \\
x^1_0 & x^1_1 & \cdots & x^1_b \\
\vdots & \vdots & \ddots & \vdots \\
x^t_0 & x^t_1 & \cdots & x^t_b \\
\end{pmatrix}.
$$

Write $M_{\Pi_t}(i_0, \ldots, i_t)$ to denote the $(t+1) \times (t+1)$ submatrix of $M_{\Pi_t}$ consisting of columns $i_0, \ldots, i_t$. Let $\mathcal{V}$ be the set of $\binom{b+1}{t+1}$ subsets $\{i_0, \ldots, i_t\}$ of $\{0, 1, \ldots, b\}$, ordered in some way. Let $\phi(M_{\Pi_t}(i_0, \ldots, i_t))$ be defined as $\det(M_{\Pi_t}(i_0, \ldots, i_t))$. Then $\phi(M_{\Pi_t})$ is defined as a point in $\mathbb{P}(B, q)$, where $B = \binom{b+1}{t+1} - 1$, and the $j$th position of $\phi(M_{\Pi_t})$ is $\phi(M_{\Pi_t}(i_0, \ldots, i_t))$ with $\{i_0, \ldots, i_t\}$ in the given order in $\mathcal{V}$.

For example, take $t = 1$, $b = 3$. Suppose $\Pi_1$ is a line in $\mathbb{P}(3,q)$ with basis points $(x_0, x_1, x_2, x_3)$, $(y_0, y_1, y_2, y_3)$, and

$$
M_{\Pi_1} = \begin{pmatrix}
x_0 & x_1 & x_2 & x_3 \\
y_0 & y_1 & y_2 & y_3 \\
\end{pmatrix}.
$$

Then $\phi(M_{\Pi_1})$ is a point in $\mathbb{P}(5,q)$ given by

$$(x_0 y_1 - x_1 y_0, x_0 y_2 - x_2 y_0, x_0 y_3 - x_3 y_0, x_1 y_2 - x_2 y_1, x_1 y_3 - x_3 y_1, x_2 y_3 - x_3 y_2).$$

We call these Grassmann coordinates (or Plücker coordinates, when $t = 1$). The set of points in $\mathbb{P}(B, q)$ corresponding to all the $t$-dimensional subspaces of $\mathbb{P}(b,q)$ is called the Grassmannian, or the Grassmann variety of the $t$-spaces of $\mathbb{P}(b,q)$. We will concentrate on the case $t = 1$ here and refer the reader to [8, Chapter 24] for more details and for the general case.

For $t = 1$, the lines of $\mathbb{P}(b,q)$ are mapped to points of $\mathbb{P}(B, q)$, $B = \binom{b+1}{2} - 1$. The $q^2 + q + 1$ lines lying on a plane in $\mathbb{P}(b,q)$ are mapped to a plane in $\mathbb{P}(B,q)$ - the collection of such planes in $\mathbb{P}(B,q)$ are called the Greek spaces. The $q^{t-1} + q^{t-2} + \cdots + b+1$ lines through a point in $\mathbb{P}(b,q)$ are mapped to a $(b - 1)$-dimensional subspace in $\mathbb{P}(B,q)$ - the collection of such subspaces are called the Latin spaces. Two Latin (Greek) spaces meet in at most one point, and a Latin and a Greek space meet in either a line or the empty set. If there are three distinct Latin (Greek) spaces $\pi, \pi', \pi''$ such that their pairwise intersections are distinct points, then any other Latin (Greek) space $\tilde{\pi}$ having distinct points in common with $\pi$ and $\pi'$ will also have a point in common with $\pi''$. These properties allow the construction of the functional repair codes described in [17].

**Construction 3.6** (Grassman variety construction [17]). The storage nodes $V_0, \ldots, V_{n-1}$ are associated with points $P_0, \ldots, P_{n-1}$ in $\mathbb{P}(b,q)$. Each point $P_i$ can be associated with a collection
of lines through that point, which, in turn, gives a \((b-1)\)-dimensional subspace \(M_i\) in \(PG(B,q)\). The recovery and repair properties then depend on how the points \(P_t\) are chosen: every \(b\) of the \(M_i\) should span \(PG(B,q)\), and if an \(M_i\) should fail, one should be able to obtain it by some \((r,\beta)\)-repair. In [17], it is shown that this can be a \((b,1)\)-repair or a \((c,b)\)-repair for any \(c|b\). This gives an \((m = B+1; n,k = b, r = b, \alpha = b, \beta = 1)\)-FRC (or an \((m = B+1; n,k = b, r = c, \alpha = b, \beta = b)\)-FRC for any \(c|b\), where \(B = \binom{t+1}{t} - 1, t \leq b\).

Consider the example with \(t = 1, b = 3\). Take \(n \geq 4\) points in \(PG(3,q)\) such that no 4 points lie in a plane (an \(n\)-arc). The corresponding Grassmannian would then consist of \(n\) planes in \(PG(5,q)\) with the property that every pair of planes meet in a point, and for any plane, the points of intersection with the other \(n-1\) planes form an \((n-1)-\text{arc on the plane}\. It is then clear that any three planes would span \(PG(5,q)\), while any plane can be obtained by \((3,1)\)-repair. This gives a repair rate of 1, and a storage rate of \(\frac{2}{n} \leq \frac{1}{2}\).

### 3.1.4 Segre varieties

Another class of varieties having subspaces with specific intersection properties are the Segre varieties. These can also be used to construct functional repair codes with intersecting subspaces. It gives storage rate \(R_s = \frac{1}{2}\), and has some restrictive recovery properties, but may still be of some interest.

A Segre variety \(SV_{s,t}\) in \(PG((s+1)(t+1) - 1, q)\) is defined as follows:

Let \(S_t\) be a \(t\)-dimensional projective space \(PG(t,q)\) and \(S_s\) be an \(s\)-dimensional projective space \(PG(s,q)\). Then

\[
SV_{s,t} = \{ (y_0z_0, y_0z_1, \ldots, y_0z_s; y_1z_0, y_1z_1, \ldots, y_1z_s; \ldots; y_tz_0, y_tz_1, \ldots, y_tz_s) | (y_0, y_1, \ldots, y_t) \in S_t, (z_0, z_1, \ldots, z_s) \in S_s \}.
\]

\(SV_{s,t}\) consists of two opposite systems of subspaces \(\Sigma_1, \Sigma_2\): \(\Sigma_1\) consists of \(q^s + q^{s-1} + \cdots + q + 1\) mutually skew \(t\)-dimensional subspaces, and \(\Sigma_2\) consists of \(q^t + q^{t-1} + \cdots + q + 1\) mutually skew \(s\)-dimensional subspaces. Each subspace in \(\Sigma_1\) meets a subspace in \(\Sigma_2\) in exactly one point.

**Example 3.7.** Suppose \(s = t = 1\). Then \(SV_{1,1}\) is a hyperbolic quadric in \(PG(3,q)\) which consists of \((q+1)^2\) points lying on \(2(q+1)\) lines. These lines form the two opposite systems of subspaces, each consisting of \(q+1\) mutually skew lines. If we take two lines from each system, then if one line fails it can always be repaired by \((2,1)\)-repair from the two lines from the opposite system. For recovery, however, we must have \(k = 2\) lines from the same system. The collection of 3-subsets of these \(4\) lines gives an \((m = 4; n = 4, k = 2, r = 2, \alpha = 2, \beta = 1)\)-FRC, with \(R_s = \frac{1}{2}\) and \(R_r = 1\).

If we compare this construction to a construction in which we only take lines from a single system, we can view this as providing the trade-off of obtaining more convenient repair at the cost of adding more nodes to the system in such a way that only certain 2-sets of nodes allow for recovery. (Note that any 3-set of nodes would suffice for recovery, however.) This example illustrates the importance of the assumption of arbitrary recovery and repair sets in the cut-set bound: Theorem 2.6 says that \(m \leq 3\) for \((k, r, \alpha, \beta) = (2, 2, 2, 1)\). Here we achieve \(m = 4\), but the pairs of lines that constitute a recovery set are more restrictive.

This can be generalised to \(SV_{t,t}, t \geq 1\): take \(t+1\) \(t\)-dimensional subspace from \(\Sigma_1\), and \(t+1\) \(t\)-dimensional subspaces from \(\Sigma_2\). Any one subspace may be obtained by \((t+1,1)\)-repair from the \(t+1\) subspaces in the opposite system. For recovery, we must have \(k = t+1\) subspaces from the
same system. The collection of \((2t+1)\)-subsets of these \(2t+2\) subspaces gives an \((m = (t + 1)^2; n = 2(t + 1), k = t + 1, r = t + 1, \alpha = t + 1, \beta = 1)\)-FRC (again, with the possibility of adding more nodes by the repair process), with \(R_s = \frac{1}{2}\) and \(R_r = 1\).

### 3.2 Constructions using non-intersecting subspaces.

#### 3.2.1 Spreads and partial spreads

Another natural object to look at when one considers projective space constructions is spreads and partial spreads.

In [9, Example 2.1] an \((m = 4; n = 4, k = 2, r = 3, \alpha = 2, \beta = 1)\)-FRC is constructed using four mutually skew lines in PG\((3, 2)\). Here we show that the construction works over \(\mathbb{F}_q\) for any \(q \geq 2\). We describe this construction as elements from a spread in PG\((3, q)\), \(q \geq 2\).

**Theorem 3.8.** Let \(S\) be a regular spread in PG\((3, q)\). Let \(l_1, l_2, l_3\) be three lines of \(S\) and let \(R\) be the (unique) regulus containing them. Let \(l_4 \in S \setminus R\). Then \(l_4\) can be obtained from \(l_1, l_2, l_3\) by \((3, 1)\)-repair, \(\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}\).

**Proof.** It is clear that any three of \(l_1, \ldots, l_4\) are contained in a regulus that does not contain the fourth line, so without loss of generality it suffices to prove that \(l_4\) can be obtained from \(l_1, l_2, l_3\) by \((3, 1)\)-repair.

Let \(Q_1\) be any point on \(l_4\). Let \(l_5\) be the transversal through \(Q_1\) to \(l_2, l_3\) - this line exists and is unique. Let \(P_2 = l_5 \cap l_2\) and \(P_3 = l_5 \cap l_3\).

Now consider \(\{l_1, l_3, l_4\}\). There is a unique regulus containing them but not \(l_2\). Let \(l_6\) be the transversal to them through \(P_3\). Let \(P_1 = l_6 \cap l_1\) and \(Q_2 = l_6 \cap l_4\). (We know that \(Q_1 \neq Q_2\) since otherwise \(l_5 = l_6\) and \(l_6\) meets all four lines, which means all four lines are in a regulus.)

Now consider the space spanned by \(P_1, P_2, Q_1, Q_2, \pi = \langle P_1, P_2, Q_1, Q_2 \rangle\). Since \(P_2Q_1 \cap P_1Q_2 = P_3\), \(\pi\) is a plane. So \(P_1P_2\) and \(l_4\) are both lines in \(\pi\) and therefore \(P_1P_2\) meets \(l_4\) in a point \(Q_3\). Hence \(l_4 \subseteq \langle P_1 \in l_1, P_2 \in l_2, P_3 \in l_3 \rangle\) and thus is obtained from \(l_1, l_2, l_3\) by \((3, 1)\)-repair. \(\square\)

**Construction 3.9.** [9, Example 2.1] The collection of pairs of distinct lines from \(\{l_1, l_2, l_3, l_4\}\) forms an \((m = 4; n = 4, k = 2, r = 3, \alpha = 2, \beta = 1)\)-FRC which has \(R_s = \frac{1}{2}\) and \(R_r = \frac{2}{3}\).

For example, we may choose \(l_1, l_2, l_3\) to be

\[
\begin{align*}
l_1 &= \langle (1, 0, 0, 0), (0, 0, 1, 1) \rangle, \\
l_2 &= \langle (0, 1, 0, 0), (1, 0, 0, 1) \rangle, \\
l_3 &= \langle (0, 0, 1, 0), (1, 1, 0, 0) \rangle.
\end{align*}
\]

These are lines on the quadric/regulus \(x_0x_2 - x_0x_3 - x_1x_2 - x_2x_3 + x_3^2 = 0\). (The other lines of the regulus are \(\langle (1, 0, y, 1), (1, y, 0, 0) \rangle\). We can take \(l_4\) to be \(\langle (0, 0, 0, 1), (0, 1, 1, 0) \rangle\), which does not belong to this regulus.

A natural generalisation of such a construction would be to take planes in spreads in PG\((5, q)\). Indeed, in Section 2.4 a construction is given using elements of an \((s - 1)\)-spread in PG\((sm - 1, q)\). In [13, 14], regular \(t\)-spreads in PG\((m - 1, q)\) are used to give \((m; k \leq n \leq \frac{2m - 1}{t + 1}, k = 2, r = 2, \alpha = t + 1, \beta = \alpha)\)-FRC, with the aim of minimising \(r\). These functional repair codes have the additional property of allowing repairs of multiple node failures simultaneously. This follows from the property
of regular spreads, where one can always choose two spread elements that span a subspace that contains a third given element.

These elements are subsets of a system of subspaces in a Segre varieties. Hence it is also natural to consider the generalisation to subspaces on a Segre varieties. In contrast to the constructions in Section 3.1.4 where elements are taken from both systems of subspaces of a Segre variety, here we only take subspaces from one system of subspaces, and these are mutually skew. Consider again an $\mathcal{SV}_{s,t}$ as described in Section 3.1.4. For every point in $S_t$, there is a corresponding $s$-dimensional subspace belonging to $\Sigma_2$ in $\mathcal{SV}_{s,t}$. Take a $t'$-dimensional subspace $V'$ of $\text{PG}(t,q)$, $t' \leq t$, and consider $\Sigma'$, the $s$-dimensional subspaces contained in $\mathcal{SV}_{s,t}$ corresponding to the points of $V'$. Then, any subspace $W$ in $\Sigma'$ can be obtained by $(2, s+1)$-repair from two other subspaces in $\Sigma'$: suppose $W$ corresponds to the point $P \in V'$, pick a point $P' \in V'$ and another point $P'' \in V'$ collinear with $P$ and $P'$. Then the subspaces in $\Sigma'$ corresponding to $P'$ and $P''$ will span a subspace containing $W$. Let $n = \frac{q^{t+1} - 1}{q-1}$. The collection of $(n-1)$-subsets of $s$-dimensional subspaces from $\Sigma'$ gives an $(m = (s+1)(t+1); n, k = t + 1, r = 2, \alpha = s + 1, \beta = \alpha)$-$\text{FRC}$.

### 3.2.2 Focal spreads

Let $\Sigma_{2t-1} = \text{PG}(2t-1, q)$, $t > 1$, and let $S_t$ be a $(t-1)$-spread in $\Sigma_{2t-1}$. Let $L$ be an element of $S_t$. Let $\Sigma_{t+d-1}, t > d$, be a $(t+d-1)$-dimensional subspace of $\Sigma_{2t-1}$ that contains $L$. Then $\{L\} \cup \{M' = M \cap \Sigma_{t+d-1} | M \in S_t \setminus \{L\}\}$ is a focal spread consisting of the focus $L$, and the $(d-1)$-dimensional subspaces $M'$ partitioning the points of $\Sigma_{t+d-1}$ not in $L$. Focal spreads are described in greater details in [11].

In [9] an $(m = 5; n = 4, k = 3, r = 3, \alpha = 2, \beta = 1)$-$\text{FRC}$ was constructed using focal spreads with $t = 3$, $d = 2$: a 2-spread in $\text{PG}(5, 2)$, intersected by a 4-space, the focus being a plane, and there are 8 lines partitioning the points not in the plane. The storage code consists of the collection of 3-subsets of these 8 lines.

This can clearly be generalised. For example, using $t = 4$, $d = 2$, we have the storage code being 16 lines partitioning the set of points of a 5-dimensional space that are not contained in the focus, which is a 3-dimensional space. A computer search shows that a line cannot be obtained by $(3,1)$-repair but can be obtained by $(4,1)$-repair, making this an $(m = 6; n = 16; k = 3, r = 4, \alpha = 2, \beta = 1)$-$\text{FRC}$.

However, the example in [9] turns out to be strictly functional, while our generalisation allows both functional and exact repair. Indeed, this appears to be the only strictly functional repair code that is known (apart from Example 3.5). In the next section we prove this property and examine the structure further.

## 4 Anatomy of a strictly functional repair code

In [9, Example 2.2 and Section VI], an $(m = 5; n = 4, k = 3, r = 3, \alpha = 2, \beta = 1)$-$\text{FRC}$ was given which turns out to be a strictly functional repair code. This is constructed using focal spreads and is described in Section 3.2.2. Here we prove that it is strictly functional, and consider whether it can be generalised.

Firstly we write the $(m = 5; n = 4, k = 3, r = 3, \alpha = 2, \beta = 1)$-$\text{FRC}$ according to Definition 2.2:

**Definition 4.1.** Let $\Sigma = \text{PG}(4, q)$ and let $\mathcal{A}$ be a set of 3-tuples $\mathcal{U}$ of lines such that
(a) (Recovery) For every \( \mathcal{U} \in \mathcal{A} \), the 3 lines in \( \mathcal{U} \) span \( \text{PG}(4, q) \).

(b) (Repair) For each \( \mathcal{U} = \{U_1, U_2, U_3\} \) there is a point \( P_i \) on \( U_i \), \( i = 1, 2, 3 \), such that there is another line \( U_4 \subseteq \langle P_1, P_2, P_3 \rangle \), and \( \mathcal{U}' = \mathcal{U} \cup \{U_4\} \setminus \{U_i\}, i = 1, 2, 3 \), again belongs to \( \mathcal{A} \).

We will give a brief description of this construction in terms of projective spaces. We will describe the lines using the correspondence between \( \text{PG}(1, 2^3) \) and the spread in \( \text{PG}(5, 2) \) in the manner described in Section 2.4.

Write \( \mathbb{F}_8 \) as \( \{0, \zeta^i : i = 0, \ldots, 6, \zeta^3 = \zeta + 1\} \). If \( a = a_0 + a_1\zeta + a_2\zeta^2 \) and \( b = b_0 + b_1\zeta + b_2\zeta^2 \) then \( (a, b) \in \text{PG}(1, 2^3) \) can be thought of as a point \( (a_0, a_1, a_2, b_0, b_1, b_2) \) in \( \text{PG}(5, 2) \). The point \( (a, b) \in \text{PG}(1, 2^3) \) thus gives a plane \( \Pi_{(a,b)} \) in \( \text{PG}(5, 2) \) consisting of the points \( \{(ax, bx) : x \in \mathbb{F}_8\} \).

So the point \((1,0) \in \text{PG}(1, 2^3) \) corresponds to the plane

\[
\Pi_{(1,0)} = \langle (1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0) \rangle.
\]

The point \((a,1), a \in \mathbb{F}_8, \) corresponds to the plane

\[
\Pi_{(a,1)} = \langle (a_0, a_1, a_2, 1, 0, 0), (a_2, a_0 + a_2, a_1, 0, 1, 0), (a_1, a_1 + a_2, a_0 + a_2, 0, 0, 1) \rangle.
\]

We can take the plane in the focal spread as the plane \( \Pi_{(1,0)} \), and the lines \( l_a \) as the intersection of the hyperplane \( x_5 = 0 \) with the planes \( \Pi_{(a,1)}, a \in \mathbb{F}_8 \). Treating the hyperplane \( x_5 = 0 \) as \( \text{PG}(4, q) \), we may write

\[
l_a = \{(a_0, a_1, a_2, 1, 0), (a_2, a_0 + a_2, a_1, 0, 1), (a_1, a_1 + a_2, a_0 + a_2, 0, 0, 1)\}.
\]

Let \( \mathcal{L} = \{l_a : a \in \mathbb{F}_8\} \). The functional repair code consists of the collection of all 3-subsets of \( \mathcal{L} \). It is not hard to show that any set of 3 lines \( l_a, l_b, l_c \) from \( \mathcal{L} \) will allow exactly one line \( l_d \in \mathcal{L} \) by \((3, 1)\)-repair, and this line satisfies \( d^2 = ab + ac + bc \). It is also not hard to see that the following two conditions ([9, Example 2.2]) are satisfied by the lines of \( \mathcal{L} \):

(L1) Any 3 lines span \( \text{PG}(4, q) \).

(L2) Any pair of lines are skew.

This construction works for \( q > 2 \), in the sense that such a construction for focal spread works over \( q > 2 \), and also a line can be obtained by \((3, 1)\)-repair from any three lines (Theorem 4.3). However, it is not clear that there is a nice relationship between \( a, b, c \) and \( d \), as in the case for \( q = 2 \). For example, for the case \( q = 3 \):

Take \( x^3 - x + 1 = 0 \) over \( \mathbb{F}_3 \) to get \( \mathbb{F}_{3^3} = \{0, \alpha^i | \alpha^3 = \alpha - 1\} \). The point \((a,1) \) on \( \text{PG}(1, 3^3) \) with \( a = a_0 + a_1\alpha + a_2\alpha^2 \) gives the plane

\[
\langle (a_0, a_1, a_2, 1, 0, 0), (-a_2, a_0 + a_2, a_1, 0, 1, 0), (-a_1, a_1 - a_2, a_0 + a_2, 0, 0, 1) \rangle
\]

in \( \text{PG}(5, 3) \). Intersecting with \( x_5 = 0 \) gives lines

\[
l_a = \langle (a_0, a_1, a_2, 1, 0), (-a_2, a_0 + a_2, a_1, 0, 1) \rangle.
\]

We can construct \( l_{a_12} \) by \((3, 1)\)-repair from \( l_0, l_1 \) and \( l_\alpha \), but it is not clear what the relationship between \( a, b, c, d \) is.
4.1 The focal spread construction is strictly functional

The repair process described above corresponds to functional repair. In this section we show that this is necessary: this FRC does not admit exact repair. We begin with a geometric lemma that we will use in the proof of this fact.

**Lemma 4.2.** Let \( \ell_1, \ell_2, \ell_3 \) be lines in \( \text{PG}(4, q) \) that satisfy (L1) and (L2). Then there is a unique line \( m \) with \( m \cap \ell_i \neq \emptyset \) for \( i = 1, 2, 3 \).

**Proof.** By (L2) we know that \( \ell_1 \) and \( \ell_2 \) span a hyperplane \( \Pi \subset \text{PG}(4, q) \). By (L1) we know that \( \ell_3 \) intersects \( \Pi \) in a unique point \( P_3 \). Consider the plane \( \sigma = \langle P_3, \ell_2 \rangle \). Since \( \ell_1 \) and \( \ell_2 \) span \( \Pi \), it follows that \( \sigma \) intersects \( \ell_1 \) in a unique point \( P_1 \). The line \( m = \langle P_1, P_3 \rangle \neq \ell_2 \) lies in \( \sigma \), as does \( \ell_2 \), and hence these two lines intersect in a unique point \( P_2 \). Thus the line \( m \) intersects each of the lines \( \ell_1, \ell_2 \) and \( \ell_3 \), and it is unique by construction. \( \square \)

**Theorem 4.3.** Let \( \ell_1, \ell_2, \ell_3, \ell_4 \) be lines in \( \text{PG}(4, q) \) that satisfy (L1) and (L2). Then at most one of the lines can be obtained by exact (3,1)-repair from the remaining three lines.

**Proof.** Suppose (without loss of generality) that \( \ell_4 \) can be obtained by (3,1)-repair from \( \{\ell_1, \ell_2, \ell_3\} \). Then there exist points \( P_1 \in \ell_1, P_2 \in \ell_2 \) and \( P_3 \in \ell_3 \) such that \( \ell_4 \subset \langle P_1, P_2, P_3 \rangle \). We note that it is not the case that \( \ell_4 = \langle P_1, P_2, P_3 \rangle \), for this would imply that \( \ell_4 = \langle P_1, P_2 \rangle \), in which case \( \ell_4 \) would be contained in \( \langle \ell_1, \ell_2 \rangle \), in violation of (L1). Hence \( \ell_4 \subset \langle P_1, P_2, P_3 \rangle \). The line \( \langle P_1, P_2 \rangle \) therefore intersects \( \ell_4 \) in a unique point, and hence by Lemma 4.2 is the unique line \( m_{124} \) meeting \( \ell_1, \ell_2 \) and \( \ell_4 \). Similarly, \( \langle P_1, P_3 \rangle \) is the unique line \( m_{134} \) meeting \( \ell_1, \ell_3 \) and \( \ell_4 \).

Suppose now that some other line (say, \( \ell_1 \)) can be obtained by (3,1)-repair from the remaining lines (i.e. \( \{\ell_2, \ell_3, \ell_4\} \)). See Figure 2. Repeating the above argument we observe that there are points \( Q_2 \in \ell_2, Q_3 \in \ell_3 \) and \( Q_4 \in \ell_4 \) such that \( \ell_1 \subset \langle Q_2, Q_3, Q_4 \rangle \). However, in this case the line \( \langle Q_2, Q_4 \rangle \) meets \( \ell_1 \) in a point, which implies \( \langle Q_2, Q_4 \rangle = m_{124} \) (by Lemma 4.2), and so \( Q_2 = P_2 \). Similarly, \( \langle Q_3, Q_4 \rangle \) meets \( \ell_1 \) in a point, so \( \langle Q_3, Q_4 \rangle = m_{134} \), and so \( Q_3 = P_3 \). But now we have that \( Q_2, Q_3, Q_4 \in \langle P_1, P_2, P_3 \rangle \), and hence \( \ell_1 \subset \langle P_1, P_2, P_3 \rangle \). This contradicts the fact that \( \ell_1 \) and \( \ell_4 \) are not coplanar, by (L2). \( \square \)
This shows that this focal spread construction is strictly functional: one can always construct a fourth line \( l_4 = m \) from any three lines \( l_1, l_2, l_3 \), and if one of \( l_1, l_2 \) or \( l_3 \) fails, it cannot be repaired exactly from the three remaining lines.

### 4.2 A simpler description

In our examples and constructions, we could enumerate a set of subspaces, and simply state that a collection of subsets of these subspaces constitute a functional repair code, bypassing the recursive nature of the definition (Definition 2.2). However, such a description is not always useful, or easy to arrive at. Firstly, we would in general like to find small codes. For example, Theorem 3.2 allows \( \mathcal{L} \) to be the set of all lines of a dual arc, but we see in Example 3.1 that 3 lines suffices. Hollmann and Poh [9, Theorem 5.1] give a method of starting with a possible set of subspaces \( \mathcal{U} = \{ U_1, \ldots, U_{n-1} \} \) and another subspace \( U_n \) constructed by \((r, \beta)\)-repair from \( \mathcal{U} \), and obtaining a functional repair code from it using the image under a group action. In Section 5 we model this process of building a functional repair code using digraphs.

Secondly, this kind of description does not always convey the complications of the repair process. For example, the focal spread construction of Section 4 admits a straightforward description similar to that of Theorem 3.2:

Let \( \mathcal{L} \) be a set of lines in \( \Sigma = \text{PG}(4, q) \) satisfying conditions (L1), (L2):

(L1) Any 3 lines span \( \text{PG}(4, q) \).

(L2) Any pair of lines are skew.

Let \( \mathcal{A} \) be a collection of 3-subsets of \( \mathcal{L} \). Then \((\Sigma, \mathcal{A})\) is a functional repair code.

If we wanted to construct a set of such lines, how would we start? Because \( \mathcal{L} \) is a strictly functional repair code (Theorem 4.3), given a 3-subset \( \{ l_1, l_2, l_3 \} \) in \( \mathcal{A} \), we obtain an \( l_4 \) by \((3, 1)\)-repair, but the 3-subset containing \( l_4 \), say, \( \{ l_2, l_3, l_4 \} \) will give an \( l_5 \neq l_1 \) by \((3, 1)\)-repair. This motivates the following steps in the construction:

Let \( \mathcal{L} \) be a set of three lines satisfying (L1), (L2) to start with.

1. Take any 3 lines of \( \mathcal{L} \). Use \((3, 1)\)-repair to get a fourth line.
2. Add this fourth line to \( \mathcal{L} \) if it is not already in it.
3. Repeat until no new lines are constructed.

Take \( \mathcal{A} \) to be the 3-subsets of \( \mathcal{L} \). Then \( \mathcal{A} \) is a functional repair code à la Definition 4.1.

This motivates a clearer modelling of the repair properties. We examine this in the next section.

### 5 A non-recursive repair condition via digraphs

We write this with \( m = 5, n = 4, k = 3, r = 3, \alpha = 2, \beta = 1 \), for simplicity, but it can easily be written more generally.

We can think of the repair condition (Definition 2.2(B)) of an \((m; n,k,r,\alpha,\beta)\)-FRC \((\Sigma, \mathcal{A})\) as a bipartite digraph \( \mathcal{G}(\mathcal{A}) = (V(\mathcal{A}) \cup V'(\mathcal{A}), \mathcal{E} \cup \mathcal{E}') \) as follows:
Let $\mathcal{V}(A)$ be a set of vertices corresponding to the sets $U$ of 3 lines in $A$ - each set $U \in A$ is a vertex in $\mathcal{V}(A)$. By the repair condition, one could obtain a fourth line $U'$ by $(r, \beta)$-repair from any set $U$ of 3 lines. Let $\mathcal{V}'(A)$ be another set of vertices corresponding to these sets $U \cup \{U\}', U \in A$, of four lines. The set of vertices of $\mathcal{G}(A)$ will be the (disjoint) union of these two sets of vertices.

The (directed) edges of $\mathcal{G}(A)$ are as follows: There is an edge from $V = \{U_1, U_2, U_3\} \in \mathcal{V}(A)$ to $V' = \{U_1, U_2, U_3, U_4\} \in \mathcal{V}'(A)$ if and only if $U_4$ is obtained by $(r, \beta)$-repair from $\{U_1, U_2, U_3\}$. We denote this set of edges by $E$. In addition, there is an edge from $V' = \{U_1, U_2, U_3, U_4\} \in \mathcal{V}'(A)$ to $V \in \mathcal{V}(A)$ if and only if $V = V' \setminus \{U_i\}, i \in \{1, 2, 3, 4\}$. We denote this set of edges by $E'$. The set of edges of $\mathcal{G}(A)$ will be the (disjoint) union of these two sets of edges.

Clearly there are edges only between $\mathcal{V}(A)$ and $\mathcal{V}'(A)$ and $\mathcal{G}(A)$ is a bipartite digraph. An edge from $\mathcal{V}(A)$ to $\mathcal{V}'(A)$ signifies a repair while an edge from $\mathcal{V}'(A)$ to $\mathcal{V}(A)$ signifies a node failure. Figure 3 gives a small example of what the node failures and repairs might look like.

Since each node may fail, there must be four out-edges from each vertex in $\mathcal{V}'(A)$, and since every three nodes must be able to repair a fourth node, there must be at least one out-edge from each vertex in $\mathcal{V}(A)$.

**Definition 5.1.** Let $\mathcal{G} = (V_1 \cup V_2, E)$ be a bipartite digraph with parts $V_1$, $V_2$. We say that $\mathcal{G}$ satisfies the repair condition if all vertices in $V_1$ has outdegree at least 1 and all vertices in $V_2$ has outdegree $n$.

This view of a functional repair code immediately gives us some idea on the number of subspaces we need and the size of $A$, as well as the characterisation of exact repair.

**Lemma 5.2.**

$$|\mathcal{V}(A)| \leq \binom{N}{n-1}, \quad |\mathcal{V}'(A)| \leq \binom{N}{n}.$$  

As a consequence, $N \geq n$.

**Lemma 5.3.**

$$|E(A)| \geq |\mathcal{V}(A)|, \quad |E'(A)| = n|\mathcal{V}'(A)|.$$  

This leads to the characterisation:

**Lemma 5.4.** A functional repair code $(\Sigma, A)$ is an exact repair code if and only if $\mathcal{G}(A)$ is a complete bipartite digraph (with an in-edge and out-edge between each pair of vertices from different parts) with $|\mathcal{V}(A)| = n$, $|\mathcal{V}'(A)| = 1$.  

![Figure 3: $\mathcal{G}(A)$ with $n = 4$, $k = 3$, $r = 3$.](image)
A functional repair code admits exact repair if it has a subgraph that satisfies the condition in Lemma 5.4, while a strictly functional repair code would satisfy the condition that there exists $V' \in \mathcal{V}(A)$, $V \in \mathcal{V}(A)$, such that $(V', V) \in \mathcal{E}'(A)$ but $(V, V') \notin \mathcal{E}(A)$.

We illustrate this with the strictly functional repair code of Example 3.5. Figure 4 is the digraph corresponding to the example. The dotted lines represent repairs. The node $\{l_1, l_2, l_3\}$ and the dashed lines show that if any of $l_1$, $l_2$ or $l_3$ failed, they cannot be repaired from the remaining lines. And if all nodes containing $l_1$ are removed, we have an exact repair code consisting of three non-concurrent lines.

Note that we are only encoding the repair process. We say nothing about $m$, $q$, $r$, $k$, $\beta$ and $\alpha$. If a bipartite digraph satisfies the repair condition it still doesn’t say if it can be realised by any parameters. We call the digraph $G$ realisable if there is $\langle m, q, r, k, \beta, \alpha \rangle$ such that there is an $(m; n, k, r, \alpha, \beta)$-FRC $(\text{PG}(m - 1, q), A)$ with $G(A) \equiv G$.

6 Further work

The construction of Theorem 3.2 does not require the projective plane to be Desarguesian. This leads to the question of whether one could construct more functional repair codes from designs, if linearity is not required. This approach may be useful for functional repair code requiring repair-by-transfer ([20, 1, 22]), where the nodes contributing information for repair do not perform any computations. There has also been studies of locally repairable codes via matroid theory ([27, 28]) which may also be of interest for functional repair codes.

Construction 3.4 gives a functional repair code that is flexible in terms of locality and availability for node repairs. There are some recent work ([23]) in symbol locality and availability: not necessarily repairing whole nodes but only some symbols in a node. It would be interesting to see how this translate into projective geometry.

The focal spread construction in Section 4 gives the only known example of a strictly functional repair code. However, it is not clear whether a generalisation to larger fields or to higher dimensions would retain this property. Indeed, it is not even clear whether one could still have a succinct description of the repair process. This indicates that there is still much to understand about this interesting structure. It is also not clear whether the distilling of the properties of functional repair from this focal spread construction into a non-recursive definition (Section 4.2) may be generalised. Again, this indicates that further study of this structure may be profitable.

The view of a functional repair code as a digraph allows some characterisation of exact repair codes. However, as yet it is not clear when a digraph with the right properties are actually realisable as a functional repair code. Another aspect to consider is: given a digraph, is it always possible to
“complete” it so that it satisfies the repair condition or are there cases where this is impossible?

References

[1] B. S. Babu and P. Vijay Kumar. A tight lower bound on the sub-packetization level of optimal-access MSR and MDS codes. 2018 IEEE International Symposium on Information Theory (ISIT), Vail, Colorado, USA. June 2018. Available at https://arxiv.org/abs/1710.05876.

[2] R. Bhagwan, K. Tati, Y. C. Cheng, S. Savage and G. M. Voelker. Total Recall: System support for automated availability management. In Proceedings of the 1st conference on Symposium on Networked Systems Design and Implementation (NSDI’04), Vol. 1. USENIX Association, Berkeley, CA, USA, pp. 25–25.

[3] A. G. Dimakis, P. B. Godfrey, Y. Wu, M. J. Wainwright and K. Ramchandran. Network coding for distributed storage systems. IEEE Transactions on Information Theory, 56(9):4539–4551, Sept. 2010.

[4] T. Etzion and L. Storme. Galois geometries and coding theory. Des. Codes Cryptography, 78(1): 311-350, 2016.

[5] V. Guruswami, S. V. Lokam and S. V. M. Jayaraman. ϵ-MSR codes: Contacting fewer code blocks for exact repair. 2018 IEEE International Symposium on Information Theory (ISIT), Vail, Colorado, USA. June 2018. Available at https://arxiv.org/abs/1807.01166.

[6] J. W. P. Hirschfeld. Finite projective spaces of three dimensions. Oxford University Press. 1985.

[7] J. W. P. Hirschfeld. Projective geometries over finite fields (2nd ed). Oxford University Press. 1998.

[8] J. W. P. Hirschfeld and J. A. Thas. General galois geometries. Oxford University Press. 1991.

[9] H. D. L. Hollmann and W. Poh. Characterizations and construction methods for linear functional-repair storage codes. 2013 IEEE International Symposium on Information Theory (ISIT), Istanbul, Turkey, July 2013, pp. 336–340. Available at http://arxiv.org/abs/1511.02924.

[10] H. Hou, H. Li and K. W. Shum. General self-repairing codes for distributed storage systems. IEEE International Conference on Communications (ICC), Budapest, 2013, pp. 4358–4362.

[11] V. Jha and N. L. Johnson. Vector space partitions and designs. Part I - Basic Theory. Note di Matematica 29(2):165-189, 2009.

[12] F. J. MacWilliams and N. J. A. Sloane The theory of error-correcting codes. North Holland. 1983.

[13] M. Y. Nam and H. Y. Song. Binary locally repairable codes with minimum distance at least six based on partial t-spreads. IEEE Communications Letters 21(8):1683–1686, Aug. 2017.

[14] F. Oggier and A. Datta. Self-repairing codes for distributed storage - A projective geometric construction. IEEE Information Theory Workshop, Paraty, 2011, pp. 30–34.

21
[15] D. A. Patterson, G. Gibson and R. H. Katz. A case for Redundant Arrays of Inexpensive Disks (RAID). In Proc. ACM SIGMOD international conference on management of data, Chicago, USA, Jun. 1988, pp. 109–116.

[16] K. V. Rashmi, N. B. Shah and P. Vijay Kumar. Optimal exact-regenerating codes for distributed storage at the MSR and MBR points via a product-matrix construction. IEEE Transactions on Information Theory, 57(8):5227–5239, 2011.

[17] N. Raviv and T. Etzion. Distributed storage systems based on intersecting subspace codes. 2015 IEEE International Symposium on Information Theory (ISIT), Hong Kong, 2015, pp. 1462–1466.

[18] N. Raviv, N. Silberstein and T. Etzion. Constructions of high-rate minimum storage regenerating codes over small fields. IEEE Transactions on Information Theory, 63(4):2015–2038, April 2017.

[19] S. Sahraei and M. Gastpar. Increasing Availability in Distributed Storage Systems via Clustering. 2018 IEEE International Symposium on Information Theory (ISIT), Vail, Colorado, USA. June 2018. Available at https://arxiv.org/abs/1710.02653.

[20] K. W. Shum and Y. Hu, Functional-repair-by-transfer regenerating codes. 2012 IEEE International Symposium on Information Theory, Cambridge, MA, 2012, pp. 1192–1196.

[21] N. Silberstein. Fractional repetition and erasure batch codes. In Coding Theory and Applications, edited by R. Pinto, P. Rocha Malonek, P. Vettori. CIM Series in Mathematical Sciences, vol 3. Springer, Cham, 2015.

[22] N. Silberstein and T. Etzion. Optimal fractional repetition codes and fractional repetition batch codes. 2015 IEEE International Symposium on Information Theory, Hong Kong, 2015, pp. 2046–2050.

[23] N. Silberstein, T. Etzion and M. Schwartz. Locality and availability of array codes constructed from subspaces. 2017 IEEE International Symposium on Information Theory (ISIT), Aachen, 2017, pp. 829–833.

[24] K. Shanmugam, D. S. Papailiopoulos, A. G. Dimakis and G. Caire. A repair framework for scalar MDS codes. IEEE Journal on Selected Areas in Communications, 32(5):998–1007, May 2014.

[25] J. Sohn, B. Choi and J. Moon. A class of MSR codes for clustered distributed storage. 2018 IEEE International Symposium on Information Theory (ISIT), Vail, Colorado, USA. June 2018. Available at https://arxiv.org/abs/1801.02014.

[26] M. Vajha, B. S. Babu and P. Vijay Kumar. Explicit MSR codes with optimal access, optimal sub-packetization and small field size for \( d = k + 1, k + 2, k + 3 \). 2018 IEEE International Symposium on Information Theory (ISIT), Vail, Colorado, USA. June 2018. Available at https://arxiv.org/abs/1804.00598.

[27] T. Westerbäck, T. Ernvall and C. Hollanti. Almost affine locally repairable codes and matroid theory. 2014 IEEE Information Theory Workshop (ITW 2014), Hobart, TAS, 2014, pp. 621–625.
[28] T. Westerbäck, R. Freij-Hollanti, T. Ernvall and C. Hollanti. On the combinatorics of locally repairable codes via matroid theory. IEEE Transactions on Information Theory, vol. 62, no. 10, pp. 5296–5315, Oct. 2016.

[29] M. Ye and A. Barg. Cooperative repair: Constructions of optimal MDS codes for all admissible parameters. 2018 IEEE International Symposium on Information Theory (ISIT), Vail, Colorado, USA. June 2018. Available at https://arxiv.org/abs/1801.09665.

[30] M. Zorgui and Z. Wang. On the achievability region of regenerating codes for multiple erasures. 2018 IEEE International Symposium on Information Theory (ISIT), Vail, Colorado, USA. June 2018. Available at https://arxiv.org/abs/1802.00104.