The work of Jorge Ize regarding the $n$-body problem

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Abstract

In this paper we present a summary of the last works of Jorge Ize regarding the global bifurcation of periodic solutions from the equilibria of a satellite attracted by $n$ primary bodies. We present results on the global bifurcation of periodic solutions for the primary bodies from the Maxwell’s ring, in the plane and in space, where $n$ identical masses on a regular polygon and one central mass are turning in a plane at a constant speed. The symmetries of the problem are used in order to find the irreducible representations, and with the help of the orthogonal degree theory, all the symmetries of the bifurcating branches. The results presented in this paper were done during the Ph.D. of the author under the direction of Jorge Ize (see [16–20]). This paper is dedicated to his memory.

This paper is devoted to present the results of the author in collaboration to Jorge Ize regarding the movement of a satellite attracted by $n$ primary bodies. In particular, when the primary bodies form the polygonal relative equilibrium corresponding to $n$ identical masses arranged on a regular polygon with one mass in the centre. This model was posed by Maxwell in order to explain the stability of Saturn’s rings.

For this polygonal relative equilibrium, we give also a description of bifurcation of planar and spatial periodic solutions. According to the value of the central mass, there are up to $2n$ branches of planar periodic solutions, with different symmetries, and up to $n$ additional branches, with non trivial vertical components, if some non resonance condition is satisfied. The linearization of the system is degenerated due to rotational symmetries. These facts imply that the classical bifurcation results for periodic solutions may not be applied directly. The proof is carried on with the use of a topological degree for maps that commute with symmetries and are orthogonal to the infinitesimal generators for these symmetries.

We also expose the global bifurcation of periodic solutions for a satellite attracted by $n$ primary bodies. These solutions will form a continuum in the plane of the primaries and other solutions outside the plane. A particular attention is given to the case where $n+1$ primaries form the Maxwell’s Saturn ring.

In order to explain the results, we give a short description of the steps to prove the bifurcation theorem. The ideas we follow are from the book [24].
where general bifurcation theorems are proven. In addition, in [15] there is a systematic application to Hamiltonian systems. The results exposed here for the \(n\)-body problem and the satellite are from the papers [16], [17], and [19].

1 Orthogonal degree

A Hilbert space \(V\) is a \(\Gamma\)-representation if there is a morphism of groups

\[ \rho : \Gamma \to GL(V). \]

The action of the group over a point generates one orbit denoted by \(\Gamma x\). A set \(\Omega \subset V\) is \(\Gamma\)-invariant if it is made of orbits, this is \(\Gamma x \subset \Omega\) for all \(x \in \Omega\).

The isotropy group of a point \(x\) is defined by and the fixed point space of the subgroup \(H\) is

\[ X^H = \{ x \in X : hx = x, \forall h \in H \}. \]

A space \(V\) is an irreducible representation when \(V\) does not have \(\Gamma\)-invariant proper subspaces. The irreducible representations of the action of a compact abelian Lie group are always two dimensional, and as such, equivalent to the complex space.

A function \(f : \Omega \to W\) is \(\Gamma\)-equivariant if

\[ f(\gamma x) = \gamma' f(x), \]

and \(\Gamma\)-invariant if the action in the range is trivial, \(f(\gamma x) = f(x)\).

**Proposition 1** A differentiable \(\Gamma\)-equivariant function at \(x\) satisfies

\[ df(\gamma x)\gamma = \gamma' df(x) \]

for all \(\gamma \in \Gamma\). In particular, the derivative \(f'(x)\) is a \(\Gamma_x\)-equivariant map. Moreover, the gradient of a \(\Gamma\)-invariant functional is a \(\Gamma\)-equivariant map when the action is orthogonal.

**Proof.** The first statement follows from the uniqueness of the derivative, and from the equality

\[ df(\gamma x)\gamma y + o(y) = f(\gamma(y + x)) - f(\gamma x) = \gamma'[f(y + x) - f(x)] = \gamma' df(x)y + o(y). \]

The second statement is a consequence of

\[ \gamma^T \nabla f(\gamma x) = [Df(\gamma x)\gamma]^T = Df(x)^T = \nabla f(x). \]
Let \( \gamma \) be an element of a torus \( \gamma = (\varphi_1, \ldots, \varphi_n) \in T^n \), with \( \varphi_j \in (-\pi, \pi) \). The \( j \)-th generator of the torus \( T^n \) is the vector fields tangent to the orbit
\[
A_j x = \frac{\partial}{\partial \varphi_j} (\gamma x)|_{\gamma=0}.
\]

The gradient of a \( \Gamma \)-invariant function is \( \Gamma \)-equivariant by the previous proposition. Moreover, this kind of gradient is orthogonal to the generators because
\[
\langle \nabla f(x), A_j x \rangle = \frac{\partial}{\partial \varphi_j} f(\gamma x)|_{\gamma=0} = 0.
\]

A general \( \Gamma \)-equivariant map is called \( \Gamma \)-orthogonal if it satisfy
\[
\langle f(x), A_j x \rangle = 0 \text{ for all } x \in \Omega.
\]

The following definition of \( \Gamma \)-orthogonal degree for compact abelian Lie groups is due to J. Ize and A. Vignoli, see \[23\]. Let \( \Gamma \) be a compact abelian Lie group, an \( \Omega \) a \( \Gamma \)-invariant domain of \( V \). Let \( f_0 \) an \( f_1 \) two \( \Gamma \)-orthogonal maps which are non-zero on the boundary \( \partial \Omega \). It is said that two maps \( f_0 \) and \( f_1 \) are \( \Gamma \)-orthogonal homotopic when there is a continuous deformation
\[
f_t : \Omega \times [0, 1] \to E,
\]
where the map \( f_t \) is \( \Gamma \)-orthogonal and non-zero in the boundary \( \partial \Omega \) for each step \( t \).

The ball \( B = \{ x \in V : \| x \| \leq r \} \) is \( \Gamma \)-invariant when the representation in \( V \) is an isometry. In this case, let us define \( \mathcal{C} \) as the set of \( \Gamma \)-orthogonal maps of the form
\[
f : \partial([0, 1] \times B) \to \mathbb{R} \times V - \{0\}.
\]
Since the boundary of \( [0, 1] \times B_r \) is isomorphic to the sphere \( S^V \), and since the set \( \mathbb{R} \times V - \{0\} \) is \( \Gamma \)-homotopic to \( S^V \), then the map \( f \) may be thought from \( S^V \) into \( S^V \).

Since the \( \Gamma \)-orthogonal homotopy forms an equivalent relation in \( \mathcal{C} \), then one define \( \Pi_{\perp}\left[S^V\right] \) as the set of equivalent classes of \( \mathcal{C} \) and
\[
[f]_{\perp} \in \Pi_{\perp}\left[S^V\right]
\]
as the equivalent class of \( f \).

Shrinking the top \( \{0\} \times \partial B \) and the bottom \( \{1\} \times B \) to the point \( (1, 0) \), one may prove that all homotopy classes \( [f]_{\perp} \) have one function such that \( f(t, x) = (1, 0) \) for \( t \in \{0, 1\} \). With these functions one may define the sum of homotopy classes as \( [f]_{\perp} + [g]_{\perp} = [f \oplus g]_{\perp} \) with
\[
f \oplus g = \begin{cases} f(2t, x) & \text{for } t \in [0, 1/2], \\
g(2t - 1, x) & \text{for } t \in [1/2, 1]. \end{cases}
\]

With this sum, the set \( \Pi_{\perp}\left[S^V\right] \) has a group structure. The identity is the map \( [(1, 0)]_{\perp} \), and the inverse of some class \( [f]_{\perp} \) is the class \( [f(1 - t, x)]_{\perp} \).
Moreover, one may prove that the group $\Pi_{\perp} [S^V]$ is abelian when $V^\Gamma$ is non trivial.

To define the $\Gamma$-orthogonal degree of a map $f : \Omega \rightarrow V$, it is necessary to extend the function $f$ to a ball, $\overline{f} : \Omega \subset B \rightarrow V$. Also, one needs a Urysohn $\Gamma$-invariant map with value 0 in $\Omega$, and value 1 in $B \setminus N$, where $N$ is a small neighborhood of $\Omega$. The existence of the Urysohn map $\varphi$ and the extension $\overline{f}$ follows from the $\Gamma$-orthogonal extension theorem of Borsuk. The proof of this theorem is only for actions of compact abelian Lie groups on finite spaces, see [24].

Figure 1: Degree definition.

**Definition 2**  The $\Gamma$-orthogonal degree of $f$ is defined as the homotopy class

$$
\deg_{\perp}(f; \Omega) = [(2t + 2\varphi - 1, \overline{f})]_{\perp} \in \Pi_{\perp} [S^V].
$$

When the domain is a ball, the degree is just the homotopy class of the suspension $\deg_{\perp}(f; B) = [(2t - 1, f)]_{\perp}$, and this definition is equivalent to the Brouwer degree if the action of the group is trivial.

In [24] it is proven that for each isotropy group of $\Gamma$, $H \in \text{Iso}(\Gamma)$, the group $\Pi_{\perp} [S^V]$ has a copy of a group isomorphic to the group $\mathbb{Z}$, this is

$$
\Pi_{\perp} [S^V] = \bigoplus_{H \in \text{Iso}(\Gamma)} \mathbb{Z}.
$$

Moreover, the degree of the map $f$ is

$$
\deg_{\perp}(f; \Omega) = \sum_{H \in \text{Iso}(\Gamma)} d_H [F_H]_{\perp},
$$

where $[F_H]_{\perp}$ is the generator of one $\mathbb{Z}$ corresponding to each isotropy group $H \in \text{Iso}(\Gamma)$, and $d_H$ is just an integer.

The orthogonal degree has the known properties of a degree: existence, excision and $\Gamma$-orthogonal homotopy invariance. In this case, the existence property means that the map $f$ must have a zero in $\Omega \cap V^H$ if $d_H \neq 0$. 

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Remark 3  For a $k$-dimensional orbit, with a tangent space generated by $k$ of the infinitesimal generators of the group, one uses a Poincaré section for the map augmented with $k$ Lagrange-like multipliers for the generators. (See the construction in [24], section 4.3). For instance, for the action of $SO(2)$, the study of zeros of the equivariant map $F(x)$, orthogonal to the generator $Ax$, is equivalent to the study of the zeros of $F(x) + \lambda Ax$; if $x$ is not fixed by the group, i.e., if $Ax$ is not 0, for which $\lambda$ is 0. In this way, one has added an artificial parameter. This trick has been used very often and, in the context of a topological degree argument, was called “orthogonal degree” by Rybicki in [29]. See also [10] and [22] for the case of gradients. The general case of the action of abelian groups was treated in [23]. The complete study of the orthogonal degree theory is given in [24], Chapters 2 and 4. From the theoretical point of view, the theory has to be extended to the action of non-abelian groups and to abstract infinite dimensional spaces.

2 Satellite

The restricted $n$-body problem is the study of the movement of a satellite attracted by $n$ primary bodies which are rotating, at a constant angular speed, around an axis. Since the mass of the satellite is small, one assumes that the satellite does not perturb the trajectories of the primaries, which follow the trajectories of relative equilibrium and, as such, are in a plane.

Let $q(t) \in \mathbb{R}^3$ be the position of the satellite without mass, and let $(a_j, 0)$ be the position of a primary body with mass $m_j$. Let $J$ be the standard symplectic matrix in $\mathbb{R}^2$. In rotating coordinates $q(t) = (e^{\omega t} u(t), z(t))$, $u, z \in \mathbb{R}^2$, Newton’s equations describing the movement of the satellite, with angular speed $\omega = 1$, are

\begin{equation}
\ddot{u} + 2J\dot{u} - u = - \sum_{j=1}^{n} m_j \frac{u - a_j}{\|(u, z) - (a_j, 0)\|_3},
\end{equation}

\begin{equation}
\ddot{z} = - \sum_{j=1}^{n} m_j \frac{z}{\|(u, z) - (a_j, 0)\|_3}.
\end{equation}

One may ask for existence of bifurcation of periodic solutions starting from the equilibria of the satellite. These solutions will form a continuum in the plane of the primaries and there are other global branches outside of that plane. The proof is based on the use of the orthogonal degree.

2.1 The orthogonal bifurcation map

Let $H^2_{2\pi}(\mathbb{R}^n)$ be the Sobolev space of $2\pi$-periodic functions. Define the collision points set as $\Psi = \{a_1, ..., a_n\}$, and the collision-free paths as

$H^2_{2\pi}(\mathbb{R}^3 \setminus \Psi) = \{x \in H^2_{2\pi}(\mathbb{R}^3) : x(t) \neq a_j\}$. 

Changing variables from $t$ to $t/\nu$, the $2\pi/\nu$-periodic solutions are zeros of the map

$$f : H^{2}_{2\pi}(\mathbb{R}^{3}\setminus \Psi) \times \mathbb{R}^{+} \to L^{2}_{2\pi}$$

$$f(x, \nu) = -\nu^{2}\ddot{x} - 2\nu \text{diag}(J, 0) \dot{x} + \nabla V(x).$$

where $V$ is the potential

$$V(u, z) = |u|^{2}/2 - \sum_{j=1}^{n} \frac{m_{j}}{\|(u, z) - (a_{j}, 0)\|}. $$

In view of the definitions, the collision-free $2\pi$-periodic solutions are zeros of the bifurcation operator $f(x, \nu)$. Furthermore, the operator $f$ is well defined and continuous.

Define the actions of the group $\mathbb{Z}_{2} \times S^{1}$ on $H^{2}_{2\pi}(\mathbb{R}^{3}\setminus \Psi)$ as

$$\rho(\kappa)x = \text{diag}(1, 1, -1)x$$

$$\rho(\varphi)x = x(t + \varphi).$$

Since the equation of the satellite is invariant by this reflection, and since the equation is autonomous, then $f$ is $\mathbb{Z}_{2} \times S^{1}$-equivariant. The generator of the group $S^{1}$ in the space $H^{2}_{2\pi}$ is

$$Ax = \frac{d}{d\varphi}(\rho(\varphi)x)_{\varphi=0} = \dot{x}.$$ 

Moreover, the map $f$ is $\mathbb{Z}_{2} \times S^{1}$-orthogonal because it satisfies the orthogonal condition

$$\langle f(x), \dot{x} \rangle_{L^{2}_{2\pi}} = \int_{0}^{2\pi} \left( -\nu^{2} |\dot{x}|^{2}/2 + V(x) \right)' dt = 0.$$

**Remark 4** For periodic and non-periodic solutions of the equations, the conservation of energy is written as

$$E = -\nu^{2} |\dot{x}|^{2}/2 + V(x) = \text{cte.}$$

Thus, one may think that the orthogonal condition is equivalent to conservation of energy.

The Fourier transform of the bifurcation map is

$$f(x) = \sum_{l \in \mathbb{Z}} (l^{2}\nu^{2}x_{l} - 2il\nu \text{diag}(J, 0)x_{l} + g_{l}) e^{ilt},$$

where $x_{l}$ and $g_{l}$ are the Fourier modes of $x$ and $\nabla V(x)$ respectively.

Since the matrix

$$l^{2}\nu^{2}I - 2il\nu \text{diag}(J, 0)$$

is invertible for all $l$'s, except a finite number. One may perform a global Lyapunov-Schmidt reduction using the global implicit function theorem for non-collision paths. In this way, one gets the reduced map $f_{1}(x_{1}, x_{2}(x_{1}, \nu), \nu)$, where
$x_1$ corresponds to a finite number of modes and $x_2$ to the complement. Moreover, the reduced map is a $\Gamma$-orthogonal map, see [24] or [15] for details. Furthermore, for bifurcation without resonances one may reduce the map to the principal Fourier mode $l = 1$.

For isolated orbits $\Gamma x_0$, the degree is calculated in terms of the linearization at $x_0$. Close to an equilibrium $x_0$ one has that $\nabla V(x_0 + h) = D^2 V(x_0)h + o(h)$, then the linearization of the reduced map is

$$f'_1(x_0, \nu)x_1 = \sum_{\text{finite } l's} M(l\nu)x_te^{ilt} \text{ with } M(\nu) = \nu^2 I - 2i\nu \text{diag}(J, 0) + D^2 V(x_0).$$

So the linearization of the reduced map is a diagonal matrix with blocks $M(l\nu)$ for a finite number of $l$'s. For bifurcation without resonances, it has only the block $M(\nu)$, for the 1-th Fourier mode.

### 2.2 Symmetries

The action of the element $(\kappa, \varphi) \in \mathbb{Z}_2 \times S^1$ satisfy

$$\rho(\kappa, \varphi)x = \rho(\kappa)x(t + \varphi) = \sum_{l} \rho(\kappa)e^{ilt}x_te^{ilt},$$

thus the action of the group is inherited on the Fourier modes as

$$\rho(\kappa, \varphi)x_l = \rho(\kappa)e^{ilt}x_l.$$

Since all the equilibria are planar, the isotropy subgroup of any equilibrium is $\mathbb{Z}_2 \times S^1$, this means that all equilibria are fixed by the action of $\mathbb{Z}_2 \times S^1$. When one apply orthogonal degree to the bifurcation problem, one need to know the irreducible representations of the action of $\Gamma x_0 = \mathbb{Z}_2 \times S^1$.

#### 2.2.1 Planar symmetries

In order to simplify the exposition, only the symmetries of the group $\mathbb{Z}_2 \times S^1$ for the 1-th mode will be studied. This correspond to the case without resonances. The space $\mathbb{C}^3$ corresponding to the 1-th mode has two spaces of similar irreducible representations: $V_0 = \mathbb{C}^2 \times \{0\}$ and $V_1 = \{0\} \times \mathbb{C}$. This is, the group $\mathbb{Z}_2$ acts as $\rho(\kappa) = I$ on $V_0$, and as $\rho(\kappa) = -1$ on $V_1$. Consequently, the action of the group $\mathbb{Z}_2 \times S^1$ in $V_0$ for the 1-th mode is

$$\rho(\kappa, \varphi)x = e^{i\varphi}x.$$

Since $(\kappa, 0)$ is the only element that fix the points of $V_0$, the isotropy subgroup of the points in $V_0$ is generated by $(\kappa, 0)$,

$$\mathbb{Z}_2 = \langle (\kappa, 0) \rangle.$$
Solutions $x = (u, z)$ to the equation (1) with isotropy group $\mathbb{Z}_2$ satisfy
\[ x(t) = \rho(\kappa)x(t) = \text{diag}(1, 1, -1)x(t). \]
Therefore, solutions to the equation (1) with symmetry $\mathbb{Z}_2$ are just planar solutions, i.e. $z(t) = 0$.

2.2.2 Spatial symmetries

In $V_1$ the action of the group $\mathbb{Z}_2 \times S^1$ is
\[(\kappa, \varphi)x = -e^{i\varphi}x.\]
Since $(\kappa, \pi)$ is the only element that fix the points of $V_1$, thus the isotropy subgroup for $V_1$ is generated by $(\kappa, \pi)$,
\[\tilde{\mathbb{Z}}_2 = \langle (\kappa, \pi) \rangle.\]

Solutions $x = (u, z)$ to the equation (1) with isotropy group $\tilde{\mathbb{Z}}_2$ satisfy
\[ x(t) = \rho(\kappa, \pi)x(t) = \text{diag}(1, 1, -1)x(t + \pi), \]
this is
\[ u(t) = u(t + \pi) \text{ and } z(t) = -z(t + \pi). \]

Solutions to the equation (1) with these symmetries follows twice the planar $\pi$-periodic curve $u$, one time with the spatial coordinate $z$ and a second time with $-z$. Consequently, there is at least one $t_0$ where $z(t_0) = z(t_0 + \pi) = 0$. For instance, if only one of these zeros exists, then the solution looks like a spatial eight near the equilibrium. For this reason, these solutions will be called eight-solutions.

2.3 Bifurcation theorem

For bifurcation without resonances, one may reduce the bifurcation study to the 1-th Fourier mode. In this case, the $\mathbb{Z}_2 \times S^1$-orthogonal degree of the reduced map complemented by the right function is
\[ \eta_{\mathbb{Z}_2}(\nu_0)[F_{\mathbb{Z}_2}] + \eta_{\tilde{\mathbb{Z}}_2}(\nu_0)[F_{\tilde{\mathbb{Z}}_2}], \]
where $[F_{\mathbb{Z}_2}]$ and $[F_{\tilde{\mathbb{Z}}_2}]$ are generators of one $Z$ in the homotopy group $\Pi_\perp$. The numbers $\eta_{\nu_0}$ correspond to the change of Morse index of the block $M(\nu)$ in the space $V_0$, for $\mathbb{Z}_2$, and in the space $V_1$, for $\tilde{\mathbb{Z}}_2$.

From the existence property of the degree, one has a zeros of the bifurcation map when $\eta(\nu_0) \neq 0$, this is, there is periodic solutions near $(x_0, \nu_0)$ with isotropy group $\mathbb{Z}_2$, if $\eta_{\mathbb{Z}_2}(\nu_0) \neq 0$, and with isotropy group $\tilde{\mathbb{Z}}_2$, if $\eta_{\tilde{\mathbb{Z}}_2}(\nu_0) \neq 0$. For resonances one may have more generators of $\Pi_\perp$ corresponding to bifurcation of harmonic periods of the principal one.

What remains is to analyze the Morse index in the subspaces $V_0$ and $V_1$. This is done in [16], where one arrives at the following conclusion.
Theorem 5 Let $T$ and $D$ be the trace and determinant of the Hessian of the potential in the plane, $V$, at the equilibrium $x_0$. If $D < 0$, there is one global bifurcation of planar periodic solutions from $x_0$. If $0 < D < (2 - T/2)^2$, there are two global bifurcations of planar periodic solutions.

Theorem 6 Every equilibrium $x_0$ has a global bifurcation of periodic eight solutions $$u(t) = u(t + \pi) \text{ and } z(t) = -z(t + \pi).$$
Moreover, the local branch is truly spatial, $z(t) \neq 0$, provided that some non-resonant condition between the periods of the spatial and the planar solutions is satisfy.

By global branch, one means that there is a continuum of solutions starting at the equilibrium, where the continuum goes to infinity in the norm of the solution or in the period, or goes to collision, or otherwise goes to other relative equilibria in such a way that the sum of the jumps in the orthogonal degrees is zero.

2.3.1 A Morse potential

One may easily prove that all equilibria for the satellite are planar. Moreover, provided that the potential in the plane $V$ is a Morse function, there are at least one global minimum and $n$ saddle points, see [16]. For example, in the classical restricted three body problem, case $n = 2$, there are two minimums where the satellite form an equilateral triangle with the primaries, and three saddle points where the satellite is collinear with the two primaries.

Theorem 7 Provided that the potential in the plane $V$ is a Morse function, each one of the $n$ saddle points has one global bifurcation of planar periodic solutions, and one global bifurcation of periodic eight solutions.

Theorem 8 The minimum point satisfy one of the following options: (a) it has two global bifurcations of planar periodic solutions and one bifurcation of periodic eight solutions, or (b) it has only one bifurcation of spatial periodic eight solutions.

2.3.2 The Maxwell’s Saturn ring

One may apply these results when the primaries form the Maxwell’s Saturn ring, see Proposition [14] This is a classical model for Saturn and one ring around it. In this case one has the following theorem.

Theorem 9 The potential has two $\mathbb{Z}_n$-orbits of saddle points (r1) and (r2), when $n \geq 2$, and one more $\mathbb{Z}_n$-orbit of saddle points when $n \geq 3$ and $\mu$ is near from zero. Furthermore, each saddle point has one global bifurcation of planar periodic solutions and one global bifurcation of periodic eight-solutions.
Theorem 10 The potential has one $\mathbb{Z}_n$-orbit of minimum points (r3) for $n \geq 2$. Moreover, provided $\mu$ is big enough, each minimum point has two global bifurcations of planar periodic solutions, and one global bifurcation of periodic eight solutions. On the other hand, if $\mu$ is small and $n \geq 3$, there is another $\mathbb{Z}_n$-orbit of minimum points with only one bifurcation of spatial periodic eight-solutions.

Remark 11 The equilibria of the $\mathbb{Z}_n$-orbit of minimum points (r3) are linearly stable if $\mu$ is big enough. This is proven in the paper [2]. The existence of the two extra orbits of equilibria for $\mu$ small was pointed out in the paper [1]. The stability and this fact is proven also in the paper [10]. The orthogonal degree has been used to prove bifurcation in the restricted three body problem also in the paper [25].

Remark 12 The degree arguments, coupled with group representation ideas, give global information, i.e., an indication of where the bifurcation branches could go. Also, since the results are valid for problems which are deformation of the original problem, the method does not require high order computations and they may be applied in some degenerate cases (for instance it is not necessary that the bifurcation parameter crosses a critical value with non-zero speed; it is enough that it crosses it eventually). An immediate drawback of this approach is that topological methods do not provide a detailed information on the local behavior of the bifurcating branch, such as stability or the existence of other type of solutions, like KAM tori. Other methods, such as normal forms or special coordinates, should be used for these purposes but they only provide local information near the critical point. In a similar way, the degree arguments give only partial results on resonances and other tools should be used.

3 The $n$-body problem

Let $q_j(t) \in \mathbb{R}^3$ be the position of the $j$-th body with mass $m_j$, for $j \in \{0, 1, ..., n\}$. Let $J$ be the standard symplectic matrix in $\mathbb{R}^2$. Newton’s equations of the $n$
bodies, in rotating coordinates \(q_j(t) = (e^{\sqrt{iJ}u_j(t)}, z_j(t))\), are

\[
m_j \ddot{u}_j + 2m_j \sqrt{J} \dot{u}_j = \omega m_j u_j - \sum_{i=0(i \neq j)}^n m_i m_j \frac{u_j - u_i}{\|(u_j, z_j) - (u_i, z_i)\|^3} \tag{2}
\]

\[
m_j \ddot{z}_j = - \sum_{i=0(i \neq j)}^n m_i m_j \frac{z_j - z_i}{\|(u_j, z_j) - (u_i, z_i)\|^3}.
\]

Relative equilibria of the \(n\)-body problem correspond to equilibria in these rotating coordinates. Since all relative equilibria are planar, the positions \((a_j, 0)\) correspond to a relative equilibrium if they satisfy the relations

\[
\omega a_j = \sum_{i=0(i \neq j)}^n m_i \frac{a_j - a_i}{\|a_j - a_i\|^3}. \tag{3}
\]

**Remark 13** Actually, identifying the plane and the complex plane, solutions of (3) may give also homographic solutions of the form \(q_j = qa_j\), where the function \(q(t) \in \mathbb{C}\) satisfy the Kepler equation. In these general solutions, the bodies may move in ellipses, parabolas or hyperbolas, instead of circular orbits. One may have also solutions with total collapse or growing like \(q(t) = (9\omega/2)^{1/3}t^{2/3}\).

**Proposition 14** Set the position of the bodies as:

\[a_0 = 0\]

\[a_j = e^{ij\zeta}\]

with mass 1 for \(j \in \{1, \ldots, n\}\), where \(\zeta = 2\pi/n\). The \(a_j\)'s correspond to a relative equilibrium when \(\omega = \mu + s_1\), where

\[
s_1 = \sum_{j=1}^{n-1} \frac{1 - e^{ij\zeta}}{\|1 - e^{ij\zeta}\|^3} = \frac{1}{4} \sum_{j=1}^{n-1} \frac{1}{\sin(j\zeta/2)}.
\]

**Proof.** For \(j = 0\) the equality is \(\omega a_j = \mu \sum_{j=0}^{n-1} e^{ij\zeta} = 0\). For \(j \neq 0\) the equality is

\[
\sum_{i=1(i \neq j)}^n \frac{a_j - a_i}{\|a_j - a_i\|^3} + \mu a_j = (\mu + s_1)a_j = \omega a_j,
\]

Therefore, the \(a_j\)'s form a relative equilibrium for the frequency \(\omega = \mu + s_1\). This relative equilibrium was studied by Maxwell as a simplified model of Saturn and its rings.

In the paper [17], Proposition 23, one finds that for each \(k \in \{1, \ldots, n - 1\}\), there is one mass \(\mu_k\) with one global bifurcation of relative equilibria. Let \(h\) be the maximum common divisor of \(k\) and \(n\), the bifurcation branch from \(\mu_k\) has solutions where \(n\) bodies are arranged as \(n/h\) regular polygons of \(h\) sides. See the example for \(n = 6\).

**Remark 15** The \(n\)-body problem has been the object of many papers, with different techniques and different purposes. For the stability of the polygonal equilibrium, or the bifurcation of relative equilibria from it, one shall mention: [33], [30], [27], [21], among others.
3.1 The orthogonal bifurcation map

Changing variables from \( t \) to \( t/\nu \), the \( 2\pi/\nu \)-periodic solutions of equation (2) are zeros of the bifurcation map \( f \) defined in the spaces

\[
f : H^2_{2\pi}(\mathbb{R}^{3(n+1)} \setminus \Psi) \times \mathbb{R}^+ \to L^2_{2\pi},
\]

where \( \Psi = \{ x \in \mathbb{R}^{3(n+1)} : x_i = x_j \} \) is the collision set, corresponding to two or more of the bodies colliding, and \( H^2_{2\pi}(\mathbb{R}^{3(n+1)} \setminus \Psi) \) is the open subset, consisting of the collision-free periodic (and continuous) functions of the Sobolev space \( H^2_{2\pi}(\mathbb{R}^{3(n+1)}) \).

Define the action of \((\kappa, \theta)\in \mathbb{Z}_2 \times SO(2)\) in \( \mathbb{R}^{3(n+1)} \) as

\[
\rho(\kappa)(u_j, z_j) = (u_j, -z_j),
\]

\[
\rho(\theta)(u_j, z_j) = (e^{-J\theta}u_j, z_j),
\]

where the group \( \mathbb{Z}_2 \) reflects the \( z \)-axis, and where \( SO(2) \) rotates the \((x, y)\)-plane.

Since Newton’s equations are invariant by isometries, the group \( \mathbb{Z}_2 \times SO(2) \) represents the inherited isometries in rotating coordinates and the map \( f \) is \( \mathbb{Z}_2 \times SO(2) \)-equivariant.

Let \( S_n \) be the group of permutations of the numbers \( \{1, \ldots, n\} \). Define the action of an element \( \gamma \in S_n \) in \( x \in \mathbb{R}^{3(n+1)} \) as \( \rho(\gamma)x_0 = x_0 \) for \( j = 0 \), and for \( j \in \{1, \ldots, n\} \) as

\[
\rho(\gamma)x_j = x_{\gamma(j)}.
\]

Since the action of \( S_n \) permutes the \( n \) bodies with equal mass, then the map \( f \) is \( S_n \)-equivariant.

The map \( f \) is \( S^1 \)-equivariant with the action \( \rho(\varphi)x(t) = x(t + \varphi) \), because the equations are autonomous. As the orthogonal degree is defined only for abelian groups, the map \( f \) will be considered only as \( \Gamma \times S^1 \)-equivariant, where \( \Gamma \) is the abelian group

\[
\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_n \times SO(2),
\]
and $\mathbb{Z}_n$ is the subgroup of $S_n$ generated by $\zeta(j) = j + 1$.

The element $\kappa \in \mathbb{Z}_2$ always leaves an equilibrium fixed because all equilibria are planar, see [15] for a proof. Let $\tilde{\mathbb{Z}}_n$ be the subgroup of $\Gamma$ generated by

$$(\zeta, \zeta) \in \mathbb{Z}_n \times SO(2),$$

where $\zeta = 2\pi/n \in SO(2)$. The actions of $(\zeta, \zeta)$ send the point $x_0$ to $e^{-J\zeta}x_0$, and it sends $x_j$ to $e^{-J\zeta}x_{j+1}$ for the other $j$'s. One may easily verify that the $a_j$'s are fixed by the action of $(\zeta, \zeta)$, thus the isotropy group of $a$ is the group $\Gamma_a \times S^1$ with

$$\Gamma_a = \mathbb{Z}_2 \times \tilde{\mathbb{Z}}_n.$$

In each component, the infinitesimal generator of the action of $S^1$ is given by $A_0x_j = \dot{x}_j$, and the infinitesimal generators of $SO(2)$ is given by

$$A_1x_j = \frac{\partial}{\partial \theta}|_{\theta=0}(e^{-J\theta}u_j, z_j) = diag(-J, 0)x_j.$$

Thus, the equalities $\langle f(x), \dot{x} \rangle_{L^2} = 0$ and $\langle f(x), A_1x \rangle_{L^2} = 0$ follow as the proof of conservation of energy and angular momentum for Newton’s equations, see [19] for a proof. Thus the map $f$ is a $\Gamma \times S^1$-orthogonal map.

**Remark 16** The orbit of the polygonal equilibrium $a$ consists of all the rotations in the $(x, y)$-plane. Since $f = 0$ on the orbit $\Gamma a$, deriving the map $f$ along a parametrization of this orbit one gets that the generator $A_1a$ is tangent to the orbit, and must be in the kernel of $f'(a)$. This is a well known fact where symmetries imply degeneracies.

### 3.2 Symmetries

#### 3.2.1 Planar symmetries

In the paper [17], it is proven that there are $n$ subspaces $W_k$ for the similar irreducible representations of $\tilde{\mathbb{Z}}_n$, where the action of $\kappa \in \mathbb{Z}_2$ is $\rho(\kappa) = I$, and the action of $(\zeta, \zeta) \in \tilde{\mathbb{Z}}_n$ is given by

$$\rho(\zeta, \zeta, \varphi) = e^{ik\zeta}.$$ 

Moreover, since the action of $S^1$ on the fundamental Fourier mode is given by

$$\rho(\varphi) = e^{i\varphi},$$

the isotropy subgroup of $\Gamma_a \times S^1$ in the space $W_k$ is generated by $\kappa \in \mathbb{Z}_2$ and $(\zeta, \zeta, -k\zeta) \in \tilde{\mathbb{Z}}_n \times S^1$. This is, the points of $W_k$ are fixed by the group

$$\tilde{\mathbb{Z}}_n(k) \times \mathbb{Z}_2 = ((\zeta, \zeta, -k\zeta)) \times (\kappa).$$

As for the satellite, solutions with isotropy group $\mathbb{Z}_2$ must satisfy $z_j(t) = 0$, and solutions with isotropy group $\tilde{\mathbb{Z}}_n(k)$ satisfy the symmetries

$$u_j(t) = \rho(\zeta, \zeta, -k\zeta)u_j(t) = e^{-ik\zeta}u_{\zeta(j)}(t - k\zeta).$$
In this case, for the central body one has the symmetry

\[ u_0(t) = e^{i\zeta} u_0(t + jk\zeta). \]

Using the notation \( u_j = u_{j+k\zeta} \) for \( j \in \{1, \ldots, n\} \), one has that \( \zeta(j) = j + 1 \), then the \( n \) bodies with equal mass satisfy

\[ u_{j+1}(t) = e^{i\zeta} u_1(t + jk\zeta). \]

Thus, each one of the \( n \) bodies with equal mass follows the same planar curve, but with different phase and with some rotation in the \((x, y)\)-plane.

\[ \text{Figure 4: For } n = 5. \]

**Remark 17** In fixed coordinates, the solutions are \( q_j(t) = e^{i\sqrt{\omega} t} u_j(\nu t) \). Thus in fixed coordinates the solutions are in general quasi-periodic solutions. In particular, when the central body has mass zero, we are considering the \( n \)-body problem with equal masses. In this case, one has for \( j \in \{1, \ldots, n\} \) that

\[ q_{j+1}(t) = e^{ij\zeta \Omega} q_1(t + jk\zeta) \]

with \( \Omega = 1 - k\sqrt{\omega}/\nu \). If \( \Omega \in n\mathbb{Z} \), then solutions with isotropy group \( \mathbb{Z}_n(k) \) satisfy

\[ q_{j+1}(t) = q_1(t + jk\zeta). \]

These solutions where all the bodies follow the same path are known as choreographies, see [8].

### 3.2.2 Spatial symmetries

In the paper [19] it is proven that there are \( n \) subspaces \( W_k \) for the similar irreducible representations of \( \mathbb{Z}_n \), where the action of \( \kappa \in \mathbb{Z}_2 \) is given by \( \rho(\kappa) = -I \), and the action of the element \( (\zeta, \zeta) \in \mathbb{Z} \) is

\[ \rho(\zeta, \zeta, \varphi) = e^{ik\zeta}. \]
Since the action of $S^1$ on the fundamental mode is $\rho(\varphi) = e^{i\varphi}$, then the elements $(\zeta, \zeta, -k\zeta) \in \tilde{\mathbb{Z}}_n \times S^1$ and $(\kappa, \pi) \in \mathbb{Z}_2 \times S^1$ act trivially on $W_k$. Thus, the isotropy group of $W_k$ is generated by $(\zeta, \zeta, -k\zeta)$ and $(\kappa, \pi)$.

As we saw for the satellite, solutions with isotropy group $\tilde{\mathbb{Z}}_2$ satisfy

$$u_j(t) = u_j(t + \pi) \text{ and } z_j(t) = -z_j(t + \pi),$$

thus the projection of this solution on the $(x, y)$-plane follows twice the $\pi$-periodic curve $u(t)$, one time with the spatial coordinate $z(t)$ and a second time with $-z(t)$. Thus solution looks like a spatial eight near the equilibrium.

Since the group $\tilde{\mathbb{Z}}_n(k)$ is generated by $(\zeta, \zeta, -k\zeta)$, the solutions satisfy also the symmetries

$$u_j(t) = e^{-i\kappa}u_{\zeta(j)}(t - k\zeta),$$

$$z_j(t) = z_{\zeta(j)}(t - k\zeta).$$

**Remark 18** To see one example, suppose that $n = 2m$ and choose $k = m$. In this case the central body remains at the center. Moreover, the $n$ bodies with equal masses satisfy

$$u_{j+1}(t) = e^{ij\kappa}u_1(t + j\pi) = e^{ij\kappa}u_1(t)$$

and

$$z_{j+1}(t) = z_1(t + j\pi) = (-1)^jz_1(t).$$

Thus, there are two $m$-polygons which oscillate vertically, one with $z_1(t)$ and the other with $-z_1(t)$. Furthermore, the projection of the two $m$-polygons in the plane is always a $2m$-polygon. These solutions are known as Hip-Hop orbits.

See [19] for a general description of the symmetries.

### 3.3 Bifurcation theorem

The linearization of the system at the polygonal equilibrium is a $3(n + 1) \times 3(n + 1)$ matrix, which is non invertible due to the rotational symmetry. In [19] one finds a change of variables that organize the spaces $W_k$’s of similar irreducible representation of $\Gamma_n \times S^1$, and also simplify the analysis of the spectrum.

This is, the arrange of the subspaces of similar irreducible representations gives a decomposition of the linearization in $2n$ blocks, $n$ of them for the spatial coordinates, given in [19], and $n$ of them for the planar coordinates, given in [17].

Applying orthogonal degree to the reduced bifurcation map, one finds that the degree has one component for each one of these $2n$ blocks, when there are no resonances. In the case of the satellite there were only two components. Each component has one number $\eta(\nu)$ which is the change of Morse index of
the corresponding block. By the existence property of the degree, there is one bifurcation branch starting from \((a, \nu_0)\) each time \(\eta(\nu_0) \neq 0\), with the symmetries of the corresponding block.

In this way one get the following theorems, see [19] for details.

**Theorem 19** For \(n \geq 3\) and each \(k \in \{2, \ldots, n-2\}\), the polygonal equilibrium has a global bifurcation of planar periodic solutions with symmetries \(\tilde{Z}_n(k)\), if \(\mu \in (-s_1, \mu_k)\), and two global bifurcations if \(\mu \in (m_+, \infty)\).

For \(n \geq 7\) and each \(k \in \{1, n-1\}\), the polygonal equilibrium has two global bifurcation branches of planar periodic solutions with symmetries \(\tilde{Z}_n(k)\) when \(\mu > m_+\).

By global branch, one means that there is a continuum of solutions starting at the ring configuration, and the continuum goes to infinity in the norm of the solution or in the period, or goes to collision, or otherwise goes to other relative equilibria in such a way that the sum of the jumps in the orthogonal degrees is zero.

**Theorem 20** The polygonal equilibrium has a global bifurcation of periodic solutions with symmetries \(\tilde{Z}_n(k) \times \tilde{Z}_2\) for each \(k \in \{1, \ldots, n\}\). Except for a possible finite number of \(\mu\)'s and frequencies, for \(\mu\) positive, bounded and different from \(\mu_k\), due to resonances, these solutions are truly spatial, this is \(z_j(t) \neq 0\) for some \(j\)-th body.

**Remark 21** Here, only the generic cases were exposed for simplicity. In [19] all cases of bifurcation from the polygonal equilibrium were studied for \(n \geq 2\). In the paper [19], there is also a theorem for the general \(n\)-body problem, where it is proven that any relative equilibrium has one bifurcation of spatial like eight solutions, and that generically there are \(n-1\) of these bifurcations.

The planar bifurcation for \(k = n\) consists of solutions with \(u_0(t) = 0\) and \(u_j(t) = e^{ij\xi} u_n(t)\). This branch was constructed in an explicit way in [28], by reducing the problem to a 6-dimensional dynamical system and a normal form argument.

The spatial bifurcation for \(k = n\) is made of solutions where the ring moves as a whole and the central body makes the contrary movement in order to stabilize the forces, that is \(z_j(t) = z_1(t)\) for \(j \in \{1, \ldots, n\}\) and \(z_0 = -nz_j\). This solution was called an oscillating ring in [28].

The spatial bifurcation for \(k = n/2\) has the symmetries of the well known Hip-Hop orbits. This kind of solutions appears first in the paper [11] without the central body. Later on, in [28] for a big central body in order to explain the pulsation of the Saturn ring, where they are called kink solutions. Finally, there is a proof in [4] when there is no central body.

**Remark 22** The same group of symmetries of the polygonal equilibrium for the \(n\)-body problem is present also in the papers: In [19] for charges instead of bodies. In [18] for vortices and traveling waves in almost parallel filaments,
and in [20] for a periodic lattice of coupled nonlinear Schrödinger oscillators. Although there are many similarities with the $n$ body problem, in particular in the change of variables, these results are of a quite different nature.

As we saw before, due to the rotational symmetry, the linearization at any equilibrium has at least one dimensional kernel. In order to find bifurcation of relative equilibria, in the paper [17], one get rid of this degeneracy looking for solutions in fixed-point subspaces of some reflection, where one is able to use ordinary degree or another method.

For bifurcation of periodic solutions, the polygonal equilibrium is fixed only by the action of

$$\tilde{\kappa}x_j(t) = \text{diag}(1,-1,1)x_{n-j}(-t),$$

which is a coupling between the reflection on the plane, a reversal of time, and a permutation of bodies. This is the only reflection able to get ride of the degeneracy.

However, when one restricts the problem to the fixed-point subspace of $\tilde{\kappa}$, one may proves bifurcation of periodic solutions only for the symmetries $k = n$ and $k = n/2$. For the remaining $k$’s, the linearization on the fixed-point subspace of $\tilde{\kappa}$ is a complex matrix with non-negative determinant as a real matrix. One could also use the gradient structure and apply the results for bifurcation based on Conley index. Actually, analytical studies with normal forms of high order and additional hypotheses of non-resonance are proposed in [7] for these cases. However, this approach do not provide the proof of the existence of a global continuum, something which follows from the application of the orthogonal degree. This fact implies that one may not use a classical degree argument or other simple analytical proofs to find the solutions presented here.

Remark 23 Variational techniques have been quite successful in treating the existence of closed solutions. In particular, [14], [12] and [13], classify all the possible groups which give periodic solutions which are minimizers of the action without collisions. Thus, the issue is different from ours, since one has the proof of the existence of a solution in the large, with a specific symmetry. For choreographies, following the seminal paper [8], with no central mass, there are studies with more than 3 bodies in [6] and [5], for instance. In the case of hip-hop solutions, these methods were successful in [9] and [32]. One of the advantages of the orthogonal degree is that it applies to problems which are not necessarily variational, but present conserved quantities.

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