Integrals of periodic motion for classical equations of relativistic string with masses at ends

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Boundary equations for the relativistic string with masses at ends are formulated in terms of geometrical invariants of world trajectories of masses at the string ends. In the three–dimensional Minkowski space $E^3_1$, there are two invariants of that sort, the curvature $K$ and torsion $\kappa$. For these equations of motion with periodic $\kappa_i(\tau + nl) = \kappa(\tau)$, constants of motion are obtained.

I. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Classical equations of motion and boundary conditions for a system of two point masses connected by the relativistic string follow from the action function for that system [1,2]

$$S = -\gamma \int d\tau \int d\sigma \sqrt{\dot{x}^2 - \dot{x}^2} - \sum_{i=1}^2 m_i \int d\tau \sqrt{\left(\frac{dx^\mu(\tau,\sigma_i(\tau))}{d\tau}\right)^2}$$

(1)

Here the first term is the action of a massless relativistic string; $\gamma$ is the parameter of tension of the string; $m_i$ are masses of particles at the string ends; $x^\mu(\tau,\sigma)$ are coordinates of the string points in a $D$–dimensional Minkowski space with metric $(1,-1,-1,...)$; derivatives are denoted by

$$\dot{x}^\mu = \partial x^\mu(\tau,\sigma)/\partial \tau, \quad \dot{x}^\mu = \partial x^\mu(\tau,\sigma)/\partial \sigma$$

$$\frac{dx^\mu(\tau,\sigma_i(\tau))}{d\tau} = \dot{x}^\mu(\tau,\sigma_i(\tau)) + \dot{\sigma}_i(\tau)\dot{\sigma}_i(\tau),$$

where the string endpoints with masses in the plane of parameters $\tau$ and $\sigma$ are described by functions $\sigma_i(\tau)$.

As in the case of a massless string, $m_i = 0$, the action (1) is invariant with respect to a nondegenerate change of parameters $\tilde{\tau} = \tilde{\tau}(\tau,\sigma)$ and $\tilde{\sigma} = \tilde{\sigma}(\tau,\sigma)$, which allows us to take the conformally flat metric on the string surface by imposing the conditions of orthonormal gauge

$$\dot{x}^2 + \dot{x}^2 = 0, \quad \dot{x}\dot{x} = 0$$

(2)

The action (1) results in the linear equations of motion for the string coordinates [1,2]

$$\ddot{x}^\mu(\tau,\sigma) - \ddot{x}^\mu(\tau,\sigma) = 0$$

(3)

and the boundary conditions for the ends with masses

$$m_i \frac{d}{d\tau} \left[ \frac{\dot{x}^\mu + \dot{\sigma}_i\dot{x}^\mu}{\sqrt{\dot{x}^2(1 - \sigma_i^2)}} \right] = (-1)^{i+1} \gamma [\dot{x}^\mu + \dot{\sigma}\dot{x}^\mu],$$

$$i = 1, 2; \quad x^\mu = x^\mu(\tau,\sigma_i(\tau)); \quad \sigma_i = \sigma_i(\tau)$$

(4)
The general solution to equations of motion (3) is the vector-function

\[ x^\mu(\tau, \sigma) = 1/2 \left[ \psi_+^\mu(\tau + \sigma) + \psi_-^\mu(\tau - \sigma) \right], \tag{5} \]

Inserting it into the gauge conditions (2) we obtain the equations \( \dot{\psi}_+^\mu(\tau + \sigma) = 0, \ \dot{\psi}_-^\mu(\tau - \sigma) = 0 \), where \( \dot{\psi}_\pm^\mu(\tau \pm \sigma) \) are derivatives with respect to the arguments. According to (3) the vectors \( \dot{\psi}_\pm^\mu(\tau + \sigma) \) and \( \dot{\psi}_\pm^\mu(\tau - \sigma) \) should be isotropic. For further consideration, it is convenient to represent them as expansions over a constant basis in the \( D \)-dimensional Minkowski space \( E^D_{\perp} \) consisting of two isotropic vectors \( a^\mu \) and \( c^\mu \) \( (a^\mu a_\mu = 0, c^\mu c_\mu = 0, a^\mu c_\mu = 1) \) and \( D - 2 \) orthonormal space-like vectors \( b_k^\mu \) \( (r = 1, 2, 3...D - 2), b_k^\mu b_\mu = -\delta_{kl} \) orthogonal to vectors \( a^\mu \) and \( c^\mu \) \( (a^\mu b_\mu = 0, c^\mu b_\mu = 0) \) [2]. As a result, we obtain the expansion of \( \dot{\psi}_\pm^\mu \) over this basis

\[ \dot{\psi}_+^\mu = \frac{A_+}{\sqrt{\sum_{k=1}^{D-2} f_k^2}} \left[ a^\mu + \sum_{k=1}^{D-2} b_k^\mu f_k + \frac{1}{2} c^\mu \sum_{k=1}^{D-2} f_k^2 \right], \]

\[ \dot{\psi}_-^\mu = \frac{A_-}{\sqrt{\sum_{k=1}^{D-2} g_k^2}} \left[ a^\mu + \sum_{k=1}^{D-2} b_k^\mu g_k + \frac{1}{2} c^\mu \sum_{k=1}^{D-2} g_k^2 \right], \tag{6} \]

where \( \dot{\psi}_\pm^\mu = \dot{\psi}_\pm^\mu(\tau \pm \sigma), A_\pm = A_\pm(\tau \pm \sigma), f = f(\tau + \sigma), \ g = g(\tau + \sigma). \) It can easily be verified that \( \dot{\psi}_\pm^2 = 0, \) and

\[ (\psi_{\pm}^{\nu\mu} \psi_{\pm}^{\nu\mu}) = \psi_{\pm}^{\nu\mu}(\tau \pm \sigma) = -A_{\pm}^2(\tau \pm \sigma), \]

where \( A_{\pm}^2(\tau \pm \sigma) \) are two arbitrary functions, like the functions \( f_k \) and \( g_k. \) The condition of orthogonal gauge (2) does not determine the functions \( A_\pm, \) and consequently, there is a possibility to fix them by imposing further gauge conditions since expressions (5) are invariant under conformal transformations of the parameters \( \tilde{\tau} \pm \tilde{\sigma} = V_{\pm}(\tau \pm \sigma). \) We fix them by imposing two more gauge conditions

\[ (\tilde{x}^\mu(\tau, \sigma) \pm \tilde{x}^\nu(\tau, \sigma))^2 = -A^2 = \text{const}, \tag{7} \]

which in terms of the vector-functions \( \dot{\psi}_\pm^\mu \) mean that the space-like vectors \( \psi_{\pm}^{\nu\mu}(\tau \pm \sigma) \) are modulo constant,

\[ \psi_{\pm}^{\nu\mu}(\tau + \sigma) = -A_{\pm}^2(\tau \pm \sigma) = -A^2, \tag{8} \]

In this way, we have fixed the functions \( A_\pm(\tau - \sigma) \) now equal to the constant \( A. \) At the same time, this condition fixes the values of functions \( \sigma_i(\tau) \) (see ref.[2] where it is shown that \( \sigma_i(\tau) = \sigma_i = \text{const} \)), therefore, we choose \( \sigma_1(\tau) = 0 \) and \( \sigma_2(\tau) = l. \)

Further, we will consider the dynamics of a string with masses at the ends on the plane \((x, y)\), i.e. in the Minkowski space with \( D = 3. \) In this case, the expansion (5) contains only one space-like vector \( b^\mu, \) and the expression (6) takes the form

\[ \dot{\psi}_+^\mu(\tau + \sigma) = \frac{A}{f}[a^\mu + b^\mu f + 1/2c^\mu f^2] \]

\[ \dot{\psi}_-^\mu(\tau - \sigma) = \frac{A}{g}[a^\mu + b^\mu f + 1/2c^\mu g^2], \tag{9} \]
where \( \dot{f} = \dot{f}(\tau + \sigma), \dot{g} = \dot{g}(\tau - \sigma) \) are derivatives with respect to arguments.

Boundary equations (1.4), when \( \sigma_i(\tau) = \text{const.} \), \( \dot{x}^\mu(\tau, \sigma_i) \) and \( \dot{x}^\mu(\tau, \sigma_i) \), from (3) are substituted into them, and the representation (1) is taken into account, transform into two nonlinear equations for the functions \( f \) and \( g \) [2]

\[
\frac{d}{d\tau} \ln \left[ \frac{\dot{g}}{\dot{f}} \right] + 2 \frac{\dot{f} + \dot{g}}{f - g} = \gamma \frac{|f - g|}{\sqrt{\dot{f} \dot{g}}},
\]

\[
\frac{d}{d\tau} \ln \left[ \frac{\dot{g}}{f_i} \right] + 2 \frac{\dot{f_i} + \dot{g_i}}{f_i - g_i} = -\gamma \frac{|f_i - g_i|}{\sqrt{\dot{f_i} \dot{g_i}}},
\]

where \( f = f(\tau), g = g(\tau), f_i = f(\tau + l), g_i = g(\tau - l) \). The nonzero components of the metric tensor of the string surface \( \dot{x}^2(\tau, \sigma) = -\dot{x}^2(\tau, \sigma) \) are expressed via \( f \) and \( g \) as follows [2]

\[
\dot{x}^2(\tau, \sigma) = A^2 \frac{(f(\tau + \sigma) - g(\tau - \sigma))^2}{4f(\tau + \sigma)\dot{g}(\tau - \sigma)}.
\]

From (10) we obtain the boundary values for the component of metric tensor \( \dot{x}^2(\tau, \sigma_i) \), where \( \sigma_i = 0, l \),

\[
\dot{x}^2(\tau, 0) = A^2 \frac{(f - g)^2}{4\dot{g}}, \quad \dot{x}^2(\tau, l) = A^2 \frac{(f_i - g_i)^2}{4\dot{f_i} \dot{g_i}}.
\]

By using these formulas together with eq. (10) the functions \( f(\tau) \) and \( g(\tau) \) can be expressed [2] in terms of the curvatures \( K_i \) and \( \dot{x}^2(\tau, \sigma_i) \) as follows

\[
D[f] = I(\tau, \sigma_1) + Q(\tau, \sigma_1) - 2K_1 A \frac{d}{d\tau} \sqrt{\dot{x}^2(0)} = I(\tau - l, \sigma_2) + Q(\tau - l, \sigma_2) + 2K_2 A \frac{d}{d\tau} \sqrt{\dot{x}^2(l)},
\]

\[
D[g] = I(\tau, \sigma_1) + Q(\tau, \sigma_1) + 2K_1 A \frac{d}{d\tau} \sqrt{\dot{x}^2(0)} = I(\tau + l, \sigma_1) + Q(\tau + l, \sigma_2) - 2K_2 A \frac{d}{d\tau} \sqrt{\dot{x}^2(l)}.
\]

Here

\[
I(\tau, \sigma_i) = D \left[ A \int^{\tau} \frac{d\eta}{\sqrt{\dot{x}^2(\eta, \sigma_i)}} \right], \quad \sigma_1 = 0, \sigma_2 = l,
\]

\[
Q(\tau, \sigma_i) = \frac{AK_i}{2} \left( \frac{A}{K_i \dot{x}^2(\tau, \sigma_i)} - \frac{K_i \dot{x}^2(\tau, \sigma_i)}{A} \right)
\]

and \( D(f(\tau)) \) stands for the Schwarz derivative:

\[
D(f) = \frac{f''}{f'} - 3 \left( \frac{f''}{f'} \right)^2 = -2\sqrt{f'} \frac{d^2}{d\tau^2} \left( \frac{1}{\sqrt{f}} \right).
\]
The second equalities in (13) and (14) represent just the connection between \( \dot{x}^2(\tau, 0) \) and \( \dot{x}^2(\tau, l) \).

Further, from (13) and (14) it follows that the difference of the Schwartz derivatives of the functions \( f(\tau) \) and \( g(\tau) \) is given by

\[
D[f] - D[g] = -4AK_1 \frac{d}{d\tau} \sqrt{\dot{x}^2(0)},
\]

(16)

\[
D[f_0] - D[g_0] = 4AK_2 \frac{d}{d\tau} \sqrt{\dot{x}^2(l)}.
\]

(17)

Eliminating \( D[g(\tau)] \) from these equations by the change of \( \tau \) to \( \tau + l \) in the equation (17) and then eliminating \( D[f(\tau)] \) by the change of \( \tau \) to \( \tau - l \), we obtain the equations

\[
D[f_{2l}] - D[f] = 4A \frac{d}{d\tau} [K_1 \sqrt{\dot{x}^2(0)} + K_2 \sqrt{\dot{x}^2_+(l)}],
\]

(18)

\[
D[g] - D[g_{2l}] = 4A \frac{d}{d\tau} [K_1 \sqrt{\dot{x}^2(0)} + K_2 \sqrt{\dot{x}^2_- (l)}],
\]

where \( f_{2l} = f(\tau + 2l), g_{2l} = f(\tau - 2l), \dot{x}^2(0) = \dot{x}^2(\tau, 0), \dot{x}^2_+(l) = \dot{x}^2(\tau - l, l), \dot{x}^2_-(l) = \dot{x}^2(\tau + l, l) \)

The left–hand sides of (18) contain either the function \( f \) or \( g \) with shifted arguments, whereas the right–hand sides depend on \( \sqrt{\dot{x}^2(\tau, 0)} \) and \( \sqrt{\dot{x}^2(\tau \pm l, l)} \). These equations give conserved quantities when the difference of Schwarz derivatives on the left–hand sides are zero under certain conditions of periodicity to be considered in sect. 2.

**II. CONSTANTS OF MOTION FOR BOUNDARY EQUATIONS WITH PERIODIC TORSIONS**

It is a remarkable fact that the system of boundary equations (13) and (14) possesses conserved quantities when \( \dot{x}^2(\tau, \sigma_i) \) are periodic with a period multiple of \( l \): \( \dot{x}^2(\tau, \sigma_i) = \dot{x}^2(\tau + nl, \sigma_i), n = 1, 2, 3, \ldots; \) in this case, the torsions of boundary curves will also be periodic, \( \kappa_i(\tau) = \kappa_i(\tau + nl). \)

The right–hand sides of the above equations depend only on \( \dot{x}^2(\tau, \sigma) \), consequently, their left–hand sides should be periodic with the same period:

\[
D[f_{nl}] = D[f], \quad D[g_{nl}] = D[g],
\]

(19)

where \( f_{nl} = f(\tau + nl), g_{nl} = g(\tau - nl) \) In view of the property of the Schwartz derivative we have

\[
f_{nl} = \frac{a_1 f + b_1}{c_1 f + d_1} = T_1 f; \quad g_{nl} = \frac{a_2 g + b_2}{c_2 g + d_2} = T_2 g.
\]

(20)

We will prove that these two linear-fractional transformations are to be equal: \( T_1 = T_2 \). To this end, using (20) and (12), we write the condition of periodicity for \( \dot{x}^2(\tau, \sigma_i) \)
\[ \dot{x}^2(0) = \dot{x}^2_{nl}(0) = \frac{A^2[T_1 f - T_2 g]^2}{4(T_1 f)'(T_2 g)'}, \]
\[ \dot{x}^2(l) = \dot{x}^2_{nl}(l) = \frac{A^2[T_1 f - T_2 g]^2}{4(T_1 f)'(T_2 gl)'} \tag{21} \]

Since the derivatives of the linear–fractional function are given by the expressions

\[ (T_1 f)' = \frac{f'}{c_1 f + d_1}^2, \quad (T_2 g)' = \frac{g'}{c_2 g + d_2}^2, \]

the denominators in (21) coincide, and the numerators obey the equality

\[ [f - g] = (a_1 f + b_1)(c_2 g + d_2) - (c_1 f + d_1)(a_2 g + b_2) \]

and the same equality follows from the second eq. of (21) but with shifted arguments of \( f(\tau + l) \) and \( g(\tau - l) \). These equalities, provided that \( a_i d_i - b_i c_i = 1 \), hold valid under the condition \( a_1 = a_2 = a, \ b_1 = b_2 = b, \ c_1 = c_2 = c, \ d_1 = d_1 = d \). Thus the periodicity condition (21) results in that \( f \) and \( g \) are transformed as follows

\[ f_{nl} = T f, \quad g_{nl} = T g. \tag{22} \]

Now we can consider each of periods \( l, 2l, ..., nl \) separately and consequences that follow from eqs. (18) in these cases.

For the period \( l \), \( \dot{x}^2(\tau + l, \sigma_i) = \dot{x}^2(\tau, \sigma_i) \) from eq. (22) it follows that \( f(\tau + l) = T f(\tau) \) and \( g(\tau - l) = T^{-1} g(\tau) \), where \( T^{-1} \) is the inverse linear–fractional transformation, and

\[ f_{2l} = T(T f) = \frac{(a^2 + cb)f + b(a + d)}{c(a + d)f + d^2 + cb}, \]
\[ g_{2l} = T^{-1}(T^{-1} g) = \frac{(d^2 + cb)g - b(a + d)}{-c(a + d)g + a^2 + cb} \]

are also linear–fractional transformations with the determinant equal to unity when \( ad - bc = 1 \). Then, the left–hand side of eqs. (18) are zero because

\[ D[f_{2l}] = D[f], \quad D[g] = D[g_{2l}], \]

we obtain the conserved quantity for the motion with \( \dot{x}^2(\tau, \sigma_i) = \dot{x}^2(\tau + l, \sigma_i) \)

\[ K_1 \sqrt{\dot{x}^2(0)} + K_2 \sqrt{\dot{x}^2(l)} = h_1, \tag{23} \]

where \( h_1 \) is the constant of integration.

Now let us consider the case with period \( 2l \): \( \dot{x}^2(\tau + 2l, \sigma_i) = \dot{x}^2(\tau, \sigma_i) \). According to (22), \( f(\tau + 2l) = T f(\tau) \) and \( g(\tau - 2l) = T^{-1} g(\tau) \), therefore, \( D[f(\tau + 2l)] = D[f(\tau)] \) and \( D[g(\tau - 2l)] = D[g(\tau)] \), then the left-hand sides of eqs. (18) again turn out to be zero; upon integration we obtain

\[ K_1 \sqrt{\dot{x}^2(0)} + K_2 \sqrt{\dot{x}^2_{2l}(l)} = h_2. \tag{24} \]
From these examples it is not difficult to deduce the general expression for a conserved quantity for period \(nl\) that is different for even and odd \(n\).

For even \(n = 2r (r = 1, 2, \ldots)\), it is necessary, upon adding \(2ml\) to the argument \(\tau\) in eq. (18), to sum up the obtained expressions over \(m\) from zero to \(r - 1\), which gives

\[
\sum_{m=0}^{r-1} D[f_{2(1+m)l}] - \sum_{m=0}^{r-1} D[f_{2ml}] = 4A \sum_{m=0}^{r-1} \frac{d}{d\tau} \left\{ K_1 \sqrt{\dot{x}_{+2ml}^2(0)} + K_2 \sqrt{x_{+2ml}^2(l)} \right\}.
\]

The left–hand side of this equation equals zero since under the change \(1 + m = m'\) in the first sum we have

\[
\sum_{m'=1}^{r} D[f_{2m'l}] - \sum_{m=0}^{r-1} D[f_{2ml}] = D[f_{2rl}] - D[f] = 0,
\]

and hence the constant quantity is

\[
\sum_{m=0}^{r-1} \left\{ K_1 \sqrt{\dot{x}_{+2ml}^2(0)} + K_2 \sqrt{x_{+2ml}^2(l)} \right\} = h_{2r}.
\]

When \(r = 1\), from (24) we obtain (23) with period \(2l\).

For odd \(n = 2r + 1 (r = 0, 1, 2, \ldots)\), it is necessary, adding \(ml\) to the argument in (18), to sum up the equations over \(m\) from zero to \(2r\), then

\[
\sum_{m'=1}^{2r} D[f_{2m'l}] - \sum_{m=0}^{2r} D[f_{ml}] = 4A \sum_{m=0}^{2r} \frac{d}{d\tau} \left\{ K_1 \sqrt{\dot{x}_{+ml}^2(0)} + K_2 \sqrt{x_{+ml}^2(l)} \right\}.
\]

Again, the left–hand side of the equation (3.12) is zero since setting \(2 + m = m'\) in the first sum and considering that \((1 + 2k)l\) is a period, we get

\[
\sum_{m'=2}^{2+2r} D[f_{m'l}] - \sum_{m=0}^{2r} D[f_{ml}] = D[f_{(1+2r)l}] - D[f_{2(1+r)l}] - D[f] - D[f] = 0,
\]

Consequently, in this case the quantity

\[
\sum_{m=0}^{2r} \left\{ K_1 \sqrt{\dot{x}_{+ml}^2(0)} + K_2 \sqrt{x_{+ml}^2(l)} \right\} = h_{2r+1}.
\]

is constant. In (27) we considered that the last term in the sum of the second term in (26) equals \(\dot{x}^2(\tau + l + 2rl, l) = \dot{x}^2(\tau, l)\). When \(r = 0\) and \(r = 1\), we obtain (23).

So, (25) and (27) are constants of motion of the boundary equations of a relativistic string with masses at ends when masses are moving along the curves with periodic torsion \(\kappa_i(\tau + nl) = \kappa_i(\tau)\) and constant curvature \(K; = \gamma/m_i\). The work is supported by the Russian Foundation for Fundamental Research (Grant No. 97–01–00745).
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