DEFINABLE RETRACTIONS OVER COMPLETE FIELDS WITH SEPARATED POWER SERIES

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Abstract. Let $K$ be a complete non-Archimedean field with separated power series, treated in the analytic Denef–Pas language. We prove the existence of definable retractions onto an arbitrary closed definable subset of $K^n$, whereby definable non-Archimedean versions of the extension theorems by Tietze–Urysohn and Dugundji follow directly. We reduce the problem to the case of a simple normal crossing divisor, relying on our closedness theorem and desingularization of terms. The latter result is established by means of the following tools: elimination of valued field quantifiers (due to Cluckers–Lipshitz–Robinson), embedded resolution of singularities by blowing up (due to Bierstone–Milman or Temkin), the technique of quasi-rational subdomains (due to Lipshitz–Robinson) and our closedness theorem.

1. Main result

Fix a complete, rank one valued field $K$ of equicharacteristic zero (not necessarily algebraically closed). Denote by $v$, $\Gamma = \Gamma_K$, $K^\circ$, $K^{\infty}$ and $\tilde{K}$ the valuation, its value group, the valuation ring, maximal ideal and residue field, respectively. The multiplicative norm corresponding to $v$ will be denoted by $|\cdot|$. The $K$-topology on $K^n$ is the one induced by the valuation $v$. The word "definable" usually means "definable with parameters".

In the paper, we shall deal with a separated Weierstrass system $\mathcal{S} := \{S_{m,n}^\circ\} = \{S_{m,n}(E, K)\}$ from the paper \cite{7} (see also \cite{4}, Ex. 4.4.(3)). Any algebraic extension $L$ of a complete field containing $K$ carries separated analytic $\mathcal{S}$-structure. It is the collection $\{\sigma_{m,n}\}$ of homomorphisms from $S_{m,n}^\circ$ to the ring of

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\(L^\circ\)-valued functions on \((L^\circ)^m \times (L^\infty)^n\), which are canonically induced by the inclusion \(K \subset L\). These fields with analytic structure are treated in an analytic Denef–Pas language \(L\). It is the two sorted, semialgebraic language \(L_{Hen}\) (with the main, valued field sort \(K\) and the auxiliary \(RV\)-sort), augmented on the valued field sort \(K\) by the multiplicative inverse \(1/x\) (with \(1/0 := 0\)) and the names of all functions of the system \(S\), together with the induced language on the auxiliary sort \(RV\) (cf. \([4\), Section 6.2\] and also \([11\), Section 2\]). Power series \(f \in S_{m,n}^\circ\) are construed via the analytic \(A\)-structure on their natural domains and as zero outside them. Note that in the equicharacteristic case, the induced language on the sort \(RV\) coincides with the semialgebraic inclusion language.

Consider a quasi-affinoid algebra \(A\), i.e. \(A = S_{m,n}/I\) for an ideal \(I\) of \(S_{m,n}\). The space \(X = \text{Max}(A)\) of the maximal ideals of \(A\) may be regarded as a scheme with the Zariski topology and as a quasi-affinoid variety \(\text{Sp}(A)\), i.e. the locally G-tringed space \((X, \mathcal{O}_X)\) with the rigid G-topology; \(\text{Max}(A)\) carries also the canonical topology induced by the absolute value. Every algebraic \(K\)-variety \(V\) can be endowed with an analytic structure of a rigid analytic variety, called the analytification of \(V\). The analytification of a projective \(K\)-variety is a quasi-compact rigid analytic variety.

In this paper, we are mainly interested in the \(K\)-rational points \(X(K)\) of rigid analytic varieties \(X\) with the canonical topology. For simplicity of notation, we shall usually write \(X\) instead of \(X(K)\) when no confusion can arise. The main purpose is the following theorem on the existence of \(L\)-definable retractions onto an arbitrary closed \(L\)-definable subset, whereby definable non-Archimedean versions of the extension theorems by Tietze–Urysohn and Dugundji follow directly.

**Theorem 1.1.** Let \(Z \subset W\) be closed subvarieties of the unit balls \((K^\circ)^N\) or \((K^\infty)^N\), of the projective space \(\mathbb{P}^n(K)\) or of the products \((K^\circ)^N \times \mathbb{P}^n(K)\) or \((K^\infty)^N \times \mathbb{P}^n(K)\), and let \(X := W \setminus Z\). Then, for each closed \(L\)-definable subset \(A\) of \(X\), there exists an \(L\)-definable retraction \(X \rightarrow A\).

We immediately obtain

**Corollary 1.2.** For each closed \(L\)-definable subset \(A\) of \(K^n\), there exists an \(L\)-definable retraction \(K^n \rightarrow A\).

Using embedded resolution of singularities, we reduce the problem of definable retractions onto a closed \(L\)-definable subset \(A\) of \((K^\circ)^N\) to that onto a simple normal crossing divisor. This is possible via our
closedness theorem (see [1] for the analytic and [3], [4] for the algebraic versions) and desingularization of $L$-terms established in Section 2. The basic tools applied in our approach are the following:

- elimination of valued field quantifiers in the Denef–Pas language $L$ (cf. [3, Theorem 4.2] and [4, Theorem 6.3.7]);
- the analytic closedness theorem (cf. [11, Theorem 1.1]) which, in particular, enables application of resolution of singularities to problems concerning the canonical topology;
- embedded resolution of singularities by blowing up for quasi-compact rigid analytic spaces (cf. [1, 15]);
- and the technique of quasi-rational subdomains (cf. [7]).

2. Desingularization of terms

Consider the ring

$$S_{m,n} := K \otimes_{K^c} S_{m,n}^c, \quad m, n \in \mathbb{N},$$

of $K$-valued series power series. For $f = \sum a_{\mu,\nu} \xi^\mu \rho^\nu \in S_{m,n}$, the Gauss norm

$$\|f\| = \sup_{\mu,\nu} |a_{\mu,\nu}| = \max_{\mu,\nu} |a_{\mu,\nu}|$$

is attained, and one has

$$S_{m,n}^c := \{ f \in S_{m,n} : \|f\| \leq 1 \} = \{ f \in S_{m,n} : |f^\sigma(a, b)| \leq 1 \text{ for all } (a, b) \in (K_{alg}^c)^m \times (K_{alg}^c)^n \}.$$

Lipshitz–Robinson [7] proved that the rings $S_{m,n}$ are regular, excellent, unique factorization domains and satisfy the Nullstellensatz. Further, they developed a theory of quasi-affinoid algebras over the rings of separated power series, including generalized rings of fractions and rational and quasi-affinoid subdomains. This was done in analogy to the classical theory of affinoid algebras over the Tate rings of strictly convergent power series (cf. [4]). We still need the following

**Proposition 2.1.** Let $A$ be a quasi-affinoid algebra, $a, b \in A$ be two elements such that, at each point $x \in \operatorname{Max} A$, one has $aA_x \subset bA_x$ or $bA_x \subset aA_x$. Then there is a partition of $X = \operatorname{Max} A$ into quasi-rational subdomains $X_i$ such that, on each $X_i$, $a$ is divisible by $b$ or $b$ is divisible by $a$.

**Proof.** Put

$$V_a := V(\operatorname{Supp} ((a, b)/(a))) = V(\operatorname{Ann} ((a, b)/(a)))$$

and

$$V_b := V(\operatorname{Supp} ((a, b)/(b))) = V(\operatorname{Ann} ((a, b)/(b))).$$
By the assumption, $V_1 \cap V_2 = \emptyset$. Take generators $f = (f_1, \ldots, f_r)$ and $g = (g_1, \ldots, g_s)$ of $\text{Ann} \left((a, b)/(a)\right)$ and $\text{Ann} \left((a, b)/(b)\right)$, respectively. It follows from the closedness theorem that there is an $\epsilon \in |K|$, $\epsilon > 0$, such that the two quasi-rational subdomains $X_a := \{x \in \text{Max} A : |f(x)| \leq \epsilon\}$ and $X_a := \{x \in \text{Max} A : |f(x)| \leq \epsilon\}$ are disjoint. But the intersection of two quasi-rational subdomains is a quasi-rational subdomain and the complement of a quasi-rational subdomain is a finite disjoint union of quasi-rational subdomains. Hence the assertion follows. □

Let $\prec$ stands for $<$, $>$ or $\leq$. We immediately obtain

**Corollary 2.2.** On each subdomain $X_i$, the subset 

$$\{x \in X_i : |a(x)| \ll |b(x)|, \ b(x) \neq 0\}$$

is the trace on the subset $\{x \in X_i : b(x) \neq 0\}$ of the quasi-rational subdomain of $X_i$ (thus being an $R$-subdomain of $X$)

$$\left\{x \in X_i : \left|\frac{a(x)}{b(x)}\right| < 1\right\} \text{ or } \left\{x \in X_i : 1 \ll \left|\frac{b(x)}{a(x)}\right|\right\},$$

according as $a$ is divisible by $b$ or $b$ is divisible by $a$ on $X_i$. □

**Remark 2.3.** Corollary 2.2 can be, of course, generalized to the case of several pairs $a_i(x)$, $b_i(x)$ satisfying the above divisibility condition.

Similarly, the following strengthening of Proposition 2.1 can be proven.

**Proposition 2.4.** Let $A$ be a quasi-affinoid algebra, $a_1, \ldots, a_k \in A$ be $k$ elements such that, at each point $x \in \text{Max} A$, the ideals $a_i A_x$ are linearly ordered by inclusion. Then there is a partition of $X = \text{Max} A$ into quasi-rational subdomains $X_i$ such that the functions $a_1, \ldots, a_k$ are linearly ordered by divisibility relation on each $X_i$. □

The desingularization of $\mathcal{L}$-terms provided in this paper will be based on resolution of singularities by blowing up for quasi-affinoid varieties or, more generally, quasi-compact rigid analytic varieties. The canonical desingularization by Bierstone–Milman [1] by finite sequences of blow-ups (multi-blowups for short) along admissible smooth centers applies directly to quasi-compact rigid analytic spaces $X$ over a complete field of characteristic zero with non trivial absolute value; and to restrictions of such spaces to their $K$-rational points $X(K)$ as well. And so does the functorial desingularization for quasi-excellent schemes by Temkin (cf. [14], Theorem 5.2.2 and [15], Theorem 1.1.13) via analytification and functoriality.
Consider a quasi-affinoid variety \( X = \text{Max} A \) and an ideal \( I \) of \( A \) generated by \((n + 1)\) analytic functions on \( X \). Then the blow-up of \( X \) along the subvariety \( V(I) \) induced by \( I \) is a quasi-compact rigid analytic subvariety of the projective variety

\[
\mathbb{P}^n(X) = X \times_K \mathbb{P}^n(K);
\]

the projective rigid analytic space \( \mathbb{P}^n(K) \) is the analytification of the projective space over \( K \) (cf. [3, Example 9.3.4/3] for a natural construction in the classical affinoid case). By convention, we regard identity map as a multi-blowup.

**Proposition 2.5.** Let \( X \) be a quasi-compact irreducible rigid analytic variety, \( f_1, \ldots, f_p, g_1 \neq 0, \ldots, g_q \neq 0 \) be analytic functions on \( X \) and \( P \) a polynomial in \( p + q \) indeterminates with coefficients from \( K \). Then there exist an admissible multi-blowup \( \sigma : \tilde{X} \to X \) and a finite partition of \( \tilde{X} \) into \( R \)-subdomains \( \tilde{X}_i \) of \( \tilde{X} \) such that, on each subset

\[
\{ x \in \tilde{X}_i : g_1(x) \cdot \ldots \cdot g_q(x) \neq 0 \},
\]

the pull-back

\[
P(f_1, \ldots, f_p, 1/g_1, \ldots, 1/g_q) \circ \sigma
\]

is the restriction of an analytic function \( \omega \) on \( \tilde{X}_i \) or its multiplicative inverse and, moreover, we have \( |\omega| \triangleleft 1 \), where \( \triangleleft \) is either \( <, > \) or \( = \).

**Proof.** Obviously, we have

\[
(2.1) \quad P(f_1, \ldots, f_p, 1/g_1, \ldots, 1/g_q) = \frac{Q(f_1, \ldots, f_p, g_1, \ldots, g_q)}{(g_1 \cdot \ldots \cdot g_q)^d}
\]

for some polynomial \( Q \) and a positive integer \( d \). Take an admissible multi-blowup \( \sigma : \tilde{X} \to X \) such that the pull-backs under \( \sigma \) of the numerator and denominator of the above fraction are simple normal crossing divisors (unless \( Q(f_1, \ldots, f_p, g_1, \ldots, g_q) \) vanishes, yet which is a trivial case) being locally (in the Zariski topology) linearly ordered with respect to divisibility relation. Then the conclusion follows directly from Corollary 2.2. \( \square \)

**Remark 2.6.** The multi-blowup \( \sigma \) is an isomorphism of (non quasi-compact) rigid analytic varieties over the complement of the zero locus of the function

\[
g_1 \cdot \ldots \cdot g_q \cdot Q(f_1, \ldots, f_p, g_1, \ldots, g_q).
\]

It is not difficult to strengthen the above proposition as follows.

**Corollary 2.7.** The conclusion of Proposition 2.5 holds in the case of several polynomials \( P \) and several tuples of analytic functions. \( \square \)
Remark 2.8. Consider a quasi-affinoid algebra \( A \) and elements \( f_1, \ldots, f_m, g_1, \ldots, g_n \in A \).

Then one has a unique homomorphism
\[
S_{m,n} \ni \phi \mapsto \phi(f_1, \ldots, f_m, g_1, \ldots, g_n) \in A\langle f\rangle[[g]]
\]
that sends \( \xi_i \mapsto f_i \) and \( \rho_j \mapsto g_j \) for \( i = 1, \ldots, m, j = 1, \ldots, n \) (cf. \[7\], Corollary 5.1.8 and Lemma 5.2.2). Similarly, given a quasi-compact rigid analytic variety \( X \), elements \( f_1, \ldots, f_m, g_1, \ldots, g_n \in \mathcal{O}_X(X) \) determine a unique homomorphism
\[
S_{m,n} \ni \phi \mapsto \phi(f_1, \ldots, f_m, g_1, \ldots, g_n) \in \mathcal{O}_X(U),
\]
where \( U := \{ x \in X : |f(x)| \leq 1, |g(x)| < 1 \} \).

Repeated application of Corollary 2.7 enables a desingularization for \( \mathcal{L} \)-terms restricted to the closed unit balls \( \mathbb{B}_N \) via an inductive process described as follows. We shall proceed with induction on the degree \( \deg t \) of \( \mathcal{L} \)-terms \( t \), i.e. the maximum number of nested superpositions that occur in \( t \). Any \( \mathcal{L} \)-term \( t \) of degree 0 in \( N \) variables is on the unit ball \( \mathbb{B}_N \subset K^N \) of the form \( P(f_1, \ldots, f_p, 1/g_1, \ldots, 1/g_q) \) from Corollary 2.7 with \( X = \mathbb{B}_N \).

\( \mathcal{L} \)-terms \( t \) of the form \( h(t_1, \ldots, t_N) \), for a function \( h \in S_{m,n} \) with \( m + n = N \) and some \( \mathcal{L} \)-terms \( t_1, \ldots, t_N \) of degree 0, are of degree 1. Application of Corollary 2.7 to the terms \( t_1, \ldots, t_N \) gives rise to a stratification of \( X_0 := \mathbb{B}_N \) into Zariski locally closed subsets \( X_{0,i} \) resulting from equations 2.2 which occur in given terms; namely one should consider all equalities or inequalities of the form:
\[
(2.2) \quad g_1 = \ldots g_k = 0, \quad g_{k+1} \cdot \ldots \cdot g_n \neq 0,
\]
and
\[
Q(f_1, \ldots, f_p, g_{k+1}, \ldots, g_q) = 0 \quad \text{or} \quad Q(f_1, \ldots, f_p, g_{k+1}, \ldots, g_q) \neq 0,
\]
where \( Q \) is a polynomial and \( d \) a positive integer such that
\[
P(f_1, \ldots, f_p, 0, \ldots, 0, 1/g_{k+1}, \ldots 1/g_q) = \frac{Q(f_1, \ldots, f_p, g_{k+1}, \ldots, g_q)}{(g_{k+1} \cdot \ldots \cdot g_q)^d}.
\]
The further data of the process on this first stage are the restrictions \( \sigma_{i_1} \) to \( \hat{X}_{0_{i_1}} = \sigma^{-1}(X_{0_{i_1}}) \) of the admissible multi-blowup \( \sigma \) of \( X_0 \) from Proposition 2.5 if the inequality
\[
Q(f_1, \ldots, f_p, g_{k+1}, \ldots, g_q) \neq 0
\]
DEFINABLE RETRACTIONS OVER SEPARATED POWER SERIES

holds on $X_{0,i}$; otherwise let $\sigma_{1,i}$ be the identity on $\tilde{X}_{0,i} = X_{0,i}$. Note that every stratum $\tilde{X}_{0,i}$ is disjoint with the exceptional divisor of the multi-blowup $\sigma$. Finally, partition each $\tilde{X}_{0,i}$ by means of $R$-subdomains from the conclusion of Proposition 2.5. Let $X_1 := \bigsqcup X_{1,i_1}$ be the disjoint union of all subsets $X_{1,i_1}$ of those partitions (all $X_{0,i}$ are taken into account), being (non quasi-compact) rigid analytic varieties, and $\sigma_1 : X_1 \to X_0$ be the induced map.

Then, on each $X_{1,i_1}$, every pull-back $t_i \circ \sigma_1$ is an analytic function or its multiplicative inverse. In view of Remark 2.8, the remaining inequalities from Proposition 2.5 ensure that the pull-back $h(t_1, \ldots, t_N) \circ \sigma_1$ is an analytic function on each $X_{1,i_1}$.

Any $L$-term $t$ of degree 1 can be expressed polynomially by terms considered above or their multiplicative inverses. As before, application of Corollary 2.7 yields the second stage of the process with the data: $\sigma_2 : X_2 \to X_1$ and $X_2 := \bigsqcup X_{2,i_1,i_2}$ the disjoint union of (non quasi-compact) rigid analytic varieties $X_{1,i_1,i_2}$ constructed as at the first stage. Then the pull-back $t \circ \sigma_1 \circ \sigma_2$ is an analytic function on $X_2$.

Given an $L$-term $t$ of degree $k$, the process described above consists of $(k+1)$ stages which yield $(k+1)$ maps $\sigma_j : X_j \to X_{j-1}, j = 1, \ldots, k+1$, induced by admissible multi-blowups, such that the pull-back

$$t \circ \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_{k+1}$$

of the term $t$ ia an analytic function on $X_k$.

Note that the maps $\sigma_j$ are not definably closed because stratifications are involved. The restrictions of multi-blowups, of which the maps $\sigma_j$ are built, are in a sense not linked together. Therefore we need another variant which, however, does not cover all points of $X_0 = \mathbb{B}_N$. Nevertheless, it enables induction with respect to the dimension $N$ of the ambient space. Namely we shall not stratify the space with respect to equalities and inequalities 2.2, but first consider a global multi-blowup $\sigma_1 : \tilde{X}_0 \to X_0$ and next partitioning from Proposition 2.5 and Corollary 2.7, applied to the polynomials $P(f_1, \ldots, f_p, 1/g_1, \ldots, 1/g_q)$.

This allows us to control the term on the complement of the zero locus of the functions

$$g_1 \cdot \ldots \cdot g_q \cdot Q(f_1, \ldots, f_p, g_1, \ldots, g_q),$$

over which the multi-blowup is an isomorphism of (non quasi-compact) rigid analytic varieties. Therefore repeated application of Corollary 2.7 enables a desingularization for $L$-terms restricted to the closed unit balls $\mathbb{B}_N$ via the following inductive process of alternate admissible multi-blowups and partitions into R-subdomains.
Proposition 2.9. Let $t$ be an $\mathcal{L}$-term of degree $k$. Then there exists a desingularization process for the term $t$ restricted to the unit ball $B_N$ which consists of the following data:

- $X_0 := B_N$, $\sigma_1 : \widetilde{X}_0 \to X_0$ an admissible multi-blowup, $\{X_{1,i_1}\}_{i_1}$ a finite partition of $\widetilde{X}_0$ into $\mathbb{R}$-subdomains;
- $X_1 := \bigsqcup X_{1,i_1}$ the disjoint union, $\sigma_2 : \widetilde{X}_1 \to X_1$ an admissible multi-blowup, $\{X_{2,i_1,i_2}\}_{i_1,i_2}$ a finite partition of $\widetilde{X}_1$ into $\mathbb{R}$-subdomains compatible with each $\sigma_2^{-1}(X_{1,i_1})$;
- $X_k := \bigsqcup X_{k,i_1,\ldots,i_k}$ the disjoint union, $\sigma_{k+1} : \widetilde{X}_k \to X_k$ an admissible multi-blowup, $\{X_{k+1,i_1,\ldots,i_{k+1}}\}_{i_1,\ldots,i_{k+1}}$ a finite partition of $\widetilde{X}_k$ into $\mathbb{R}$-subdomains compatible with each $\sigma_{k+1}^{-1}(X_{k,i_1,\ldots,i_k})$;
- analytic functions $\chi_j$, $\psi_j$ on $X_j$ and $\phi_j$ on $\widetilde{X}_j$, $\phi_j$ being simple normal crossing divisors (unless non-vanishing), $j = 0, \ldots, k$, defined recursively as follows: the functions $\chi_0 = \psi_0, \chi_1, \ldots, \chi_k$ are provided successively (they are products of factors of the form $2.3$), next
  $\phi_j := \psi_j \circ \sigma_{j+1}, \quad \psi_{j+1} := \phi_j \cdot \chi_{j+1}, \quad j = 0, 1, \ldots, k,$
and eventually $\phi_k := \psi_k \circ \sigma_{k+1}$.
- an analytic function $\omega_k$ on the complement of the zero locus $V(\phi_k)$ of $\phi_k$ such that, on each $\mathbb{R}$-subdomain $X_{k+1,i_1,\ldots,i_{k+1}}$, $\phi_k$ and $\omega_k$ or $\phi_k$ and $1/\omega_k$ are simultaneous simple normal crossing divisors, unless $\omega_k$ vanishes.

The spaces $X_j$ and $\widetilde{X}_{j-1}$, $j = 1, \ldots, k$, are homeomorphic in the canonical topology. The multi-blowups $\sigma_j$ transform $\psi_{j-1}$ to normal crossing divisors and are homeomorphisms over the complement of $V(\psi_{j-1})$ for $j = 1, \ldots, k + 1$. By abuse of notation, we shall regard $\sigma_j$ also as a map from $X_j$ onto $X_{j-1}$.

The process results in that the pull-back

$$t \circ \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_{k+1}$$

is the restriction of $\omega_k$ away from the zero locus $V(\phi_k)$ of $\phi_k$. $\square$

It is not difficult to obtain the following generalization.

Corollary 2.10. The conclusion of Proposition 2.9 holds in the case of several $\mathcal{L}$-term restricted to the unit ball $B_N$. $\square$
Remark 2.11. Resolution of singularities including, in particular, the technique of admissible blow-ups, applies to the $K$-rational points $X(K)$ of rigid analytic varieties $X$. Therefore also valid are the counterparts of the above two results for $K$-rational points, whose formulations are straightforward. For instance, the closed unit ball $B_X$ must be replaced with $(K^o)^N$.

By the closedness theorem (cf. [11, Theorem 1.1]) and the descent property below, the technique of blowing up can be applied to constructions of definable retractions.

Lemma 2.12. Let $\sigma : Y \to X$ be a continuous $L$-definable surjective map which is $L$-definably closed, and $A$ be a closed $L$-definable subset of $X$ such that $\sigma$ is bijective over $X \setminus A$. Then every $L$-definable retraction $\tilde{r} : Y \to \sigma^{-1}(A)$ descends uniquely to a unique $L$-definable retraction $r : X \to A$.

Proof. The retraction $r$ is given by the formula (cf. [12, Lemma 3.1]):

$$r(x) = \begin{cases} \sigma(\tilde{r}(\sigma^{-1}(x))) & \text{if } x \in X \setminus A, \\ x & \text{if } x \in A. \end{cases}$$

We immediately obtain the two corollaries.

Corollary 2.13. Let $\sigma : Y \to X$ be a continuous $L$-definable surjective map which is $L$-definably closed, $A$ and $S$ be closed $L$-definable subsets of $X$ and $W := \sigma^{-1}(S)$. Suppose that the restriction of $\sigma$ to $Y \setminus W$ is bijective. Then every $L$-definable retraction $\tilde{r} : Y \to \sigma^{-1}(A) \cup W$ descends uniquely to an $L$-definable retraction $r : X \to A \cup S$.

Corollary 2.14. Under the above assumptions, suppose that there is an $L$-definable retraction $\theta : S \to A \cap S$. Then there is an $L$-definable retraction $\rho : X \to A$.

Proof. Define the map $\eta : A \cup S \to A$ by putting $\eta(x) = \theta(x)$ if $x \in S$ and $\eta(x) = x$ if $x \in A$. Obviously, $\eta$ is an $L$-definable retraction. Next set $\rho := \eta \circ r$.

3. Proof of the main theorems on definable retractions

In this section, we shall deal with the rational points $X(K)$ of rigid analytic varieties $X$. For simplicity of notation, from now on we shall write it simply $X$ instead of $X(K)$.

We begin by considering definable retractions of closed definable subsets contained in the closed unit ball $X_0 = (K^o)^N$. However all
the arguments carry over, mutatis mutandi, to subsets in the open unit ball \((K^{\circ})^N\), in the projective space \(\mathbb{P}^n(K)\) or in the products \((K^{\circ})^N \times \mathbb{P}^n(K)\) or \((K^{\circ\circ})^N \times \mathbb{P}^n(K)\). We shall proceed with induction on the dimension \(N\) of the ambient space \(X_0\). To this end, we need the following

**Lemma 3.1.** Let \(Z \varsubsetneq X\) be two closed subvarieties of the product \((K^{\circ})^N \times \mathbb{P}^n(K)\) and \(A\) a closed \(\mathcal{L}\)-definable subset of \(Z\). Suppose that \(X\) is non-singular of dimension \(N\) and Theorem 1.1 holds for closed \(\mathcal{L}\)-definable subsets of every non-singular variety of this kind of dimension < \(N\). Then there exists an \(\mathcal{L}\)-definable retraction \(r : Z \to A\).

**Proof.** We shall proceed with induction on the dimension of \(Z\). First apply embedded resolution of singularities and take a multi-blowup \(\tau : V \to X\) which is an isomorphism over the complement of the singular locus \(S\) of \(Z\) and such that the pre-image \(\tau^{-1}(Z)\) is a simple normal crossing subvariety of \(V\), which is the union of the transform \(\tilde{Z}\) of \(Z\) and the exceptional divisor \(E\).

Taking the disjoint union \(Y\) of \(\tilde{Z}\) and the components of \(E\), we get a map \(\sigma : Y \to Z\) which satisfies the assumptions from Corollary 2.13 with \(W := \sigma^{-1}(S)\). Of course, \(Y\) is non-singular of dimension < \(N\). By the assumption, there is an \(\mathcal{L}\)-definable retraction \(\tilde{\rho} : Y \to \sigma^{-1}(A) \cup W\). Hence and by Corollary 2.13, there is an \(\mathcal{L}\)-definable retraction

\[\rho : Z \to A \cup S.\]

But, by the induction hypothesis, there is an \(\mathcal{L}\)-definable retraction

\[r_1 : S \to A \cap S\]

which, similarly as it was in the proof of Lemma 2.14, gives rise to an \(\mathcal{L}\)-definable retraction

\[r_2 : A \cup S \to A.\]

Then the map \(r := r_2 \circ \rho\) is the retraction we are looking for. \(\square\)

By elimination of valued field quantifiers, every \(\mathcal{L}\)-definable subset of \((K^{\circ})^N\) is a finite union of sets of the form

\[A = \{x \in (K^{\circ})^N : (rv t_1(x), \ldots, rv t_s(x)) \in B\},\]

where \(B\) is an \(\mathcal{L}\)-definable subset of \(RV(K)^s\). We are going to combine this description with the desingularization process from Proposition 2.9 and Corollary 2.10 applied to the sets of \(K\)-rational points, as indicated in Remark 2.11. Putting

\[A^\sigma := \sigma^{-1}(A), \quad t^\sigma := t \circ \sigma \quad \text{and} \quad \tau_j := \sigma_1 \circ \cdots \circ \sigma_j,\]
we get
\[ A^{\tau_{k+1}} \cap X_{k+1,i_1,\ldots,i_{k+1}} = \{ x \in X_{k+1,i_1,\ldots,i_{k+1}} : (rv \, t_{1}^{\tau_{k+1}}(x), \ldots, rv \, t_{s}^{\tau_{k+1}}(x)) \in B \} \]

and
\[
(A^{\tau_{k+1}} \cap X_{k+1,i_1,\ldots,i_{k+1}}) \setminus V(\phi) =
\{ x \in X_{k+1,i_1,\ldots,i_{k+1}} \setminus V(\phi) : (rv \, \omega_1(x), \ldots, rv \, \omega_s(x)) \in B \}.
\]

Clearly, modifying the set \( B \), we can assume that the functions \( \phi_k \) and \( \omega_1, \ldots, \omega_s \) are simultaneous normal crossing divisors on \( X_{k+1,i_1,\ldots,i_{k+1}} \). Then the set \( B \) is a finite union of the subsets
\[
(X_{k+1,i_1,\ldots,i_{k+1}} \setminus V(\phi)) \cap V_j \cap G_j,
\]
where
\[
V_j := \{ x \in X_{k+1,i_1,\ldots,i_{k+1}} : \omega_i(x) = 0 \text{ for } i \in I \}
\]
and
\[
G_j := \{ x \in X_{k+1,i_1,\ldots,i_{k+1}} \setminus V(\phi) : (rv \, \omega_1(x), \ldots, rv \, \omega_s(x)) \in B, \omega_q(x) \neq 0 \text{ for } q \in \{1, \ldots, s\} \setminus I \}
\]
with all subsets \( I \subset \{1, \ldots, s\} \). It is easy to check that \( G_j \) are clopen subsets of \( X_{k+1,i_1,\ldots,i_{k+1}} \setminus V(\phi) \).

Hence the set
\[
(A^{\tau_{k+1}} \cup V(\phi)) \cap X_{k+1,i_1,\ldots,i_{k+1}}
\]
falls under the description from Proposition 3.3 of our paper \([12]\), which treated the algebraic case of the problems under study. The proof of Theorem 3.1 (op.cit.) on the existence of definable retractions onto closed definable subsets uses Proposition 3.2 (op.cit.) on the existence of definable retractions onto Zariski closed subsets and Proposition 3.3 (op.cit.). Note that the proofs of those Proposition 3.2 and Theorem 3.1 modulo the description from Proposition 3.3 carry over to the affinoid case treated here.

Under the circumstances, almost the same proof as in the algebraic case provides an \( \mathcal{L} \)-definable retraction
\[
r_{k+1,i_1,\ldots,i_{k+1}} : X_{k+1,i_1,\ldots,i_{k+1}} \to (A^{\tau_{k+1}} \cup V(\phi)) \cap X_{k+1,i_1,\ldots,i_{k+1}}.
\]
Since the subsets \( X_{k+1,i_1,\ldots,i_{k+1}} \) indexed by \( i_{k+1} \) are a clopen covering of \( \sigma_{k+1}^{-1}(X_{k,i_1,\ldots,i_k}) \), we get by gluing also a retraction
\[
\tilde{r}_{k,i_1,\ldots,i_k} : \sigma_{k+1}^{-1}(X_{k,i_1,\ldots,i_k}) \to (A^{\tau_{k+1}} \cup V(\phi)) \cap \sigma_{k+1}^{-1}(X_{k,i_1,\ldots,i_k}).
\]
Hence and by Lemma \([2,13]\), we get a retraction
\[
r_{k,i_1,\ldots,i_k} : X_{k,i_1,\ldots,i_k} \to (A^{\tau_{k}} \cup V(\psi)) \cap X_{k,i_1,\ldots,i_k}.
\]
As before, we get a retraction
\[ \tilde{r}_{k-1,i_1,\ldots,i_{k-1}} : \sigma_k^{-1}(X_{k-1,i_1,\ldots,i_{k-1}}) \to (A^\tau_k \cup V(\psi_k)) \cap \sigma_k^{-1}(X_{k-1,i_1,\ldots,i_{k-1}}); \]
we have \( V(\psi_k) = \sigma_k^{-1}(V(\psi_{k-1})) \cup V(\chi_k). \)

We shall now proceed with induction on the dimension \( N \) of the ambient space. By Lemma \[\text{Lemma 3.1}\], we can find a retraction
\[ (\sigma_k^{-1}(V(\psi_{k-1})) \cup V(\chi_k)) \cap \sigma_k^{-1}(X_{k-1,i_1,\ldots,i_{k-1}}) \to \]
\[ \sigma_k^{-1}(V(\psi_{k-1})) \cup (A^\tau_k \cap V(\chi_k)) \cap \sigma_k^{-1}(X_{k-1,i_1,\ldots,i_{k-1}}). \]
Since \( \sigma_k \) is a homeomorphism over the complement of \( V(\psi_{k-1}) \), the above two retractions along with Lemma \[\text{Lemma 2.14}\] yield a retraction
\[ r_{k-1,i_1,\ldots,i_{k-1}} : X_{k-1,i_1,\ldots,i_{k-1}} \to (A^\tau_{k-1} \cup V(\psi_{k-1})) \cap X_{k-1,i_1,\ldots,i_{k-1}}. \]

We can continue the reasoning along this pattern to eventually achieve an \( L \)-definable retraction \( r_0 : X_0 \to A \) we are looking for.

We have thus proven Theorem \[\text{Theorem 1.2}\] on the existence of \( L \)-definable retractions onto any closed \( L \)-definable subset \( A \) of the closed unit ball \( X_0 := (K^o)^N \).

Remark 3.2. The above results, both desingularization of terms and retractions onto definable subsets of \( X_0 := (K^o)^N \), will run almost in the same way when the space \( X_0 \) is the open unit ball \( (K^{oo})^N \), the projective space \( \mathbb{P}^n(K) \) or the products \( (K^o)^N \times \mathbb{P}^n(K) \) or \( (K^{oo})^N \times \mathbb{P}^n(K) \). Moreover, definable retractions will exist when \( X_0 \) is the complement of a closed subvariety \( Z \) in these spaces. For the last assertion, one must require at the final stage of the desingularization process that \( \phi_k, \omega_k \) and also the pre-image \( \tau_k^{-1}(Z) \) be simultaneous simple normal crossing divisors; this corresponds to the passage from the projective space \( \mathbb{P}^N(K) \) to \( K^N \) indicated in [\[12\], Remark 2.10]. Next one must apply the corresponding versions of Lemma \[\text{Lemma 2.12}\] along with its two corollaries and of Lemma \[\text{Lemma 3.1}\], which take into account the subvariety \( Z \); modifications and proofs of those versions being straightforward.

The foregoing proof of a special case of Theorem \[\text{Theorem 1.1}\] along with Remark \[\text{Remark 3.2}\] establish its full version, and thus the proof is complete.

We conclude with the following comments.

Remark 3.3. The existence of definable retractions onto closed definable subsets yields a non-Archimedean definable analogue of the Dugundji theorem on the existence of a linear (and continuous) extender and, a
fortiori, a non-Archimedean definable analogue of the Tietze–Urysohn extension theorem. These issues in the algebraic case were discussed in our previous paper [12].

**Remark 3.4.** The non-Archimedean extension problems for definable functions require a different approach and other techniques in comparison with the classical non-Archimedean purely topological ones because, among others, geometry over non-locally compact Henselian valued fields suffers from lack of definable Skolem functions. On the other hand, definability in a suitable language often makes the subject under study tamer and enables application of new tools and techniques. Let us finally emphasize that our interest to problems of non-Archimedean geometry was inspired by our joint paper [6].

**Remark 3.5.** In our subsequent paper [13], we establish Theorem 1.1 in the general settings of Henselian valued fields with analytic structure. Its proof relies on the definable version of canonical desingularization developed in that paper (and carried out within a category of definable, strong analytic manifolds and maps). Note that the theory of Henselian fields with analytic structure, unifying many earlier approaches within non-Archimedean analytic geometry, was developed in the papers [3], [4], [5], [6].

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