Number of quantum measurement outcomes as a resource

Weixu Shi · Chaojing Tang

Received: 8 May 2020 / Accepted: 20 October 2020 / Published online: 3 November 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
The number of outcomes is an intrinsic property of a quantum measurement, which has the potential to improve the performance of quantum information processing. Recently, there have been fruitful results on resource theories of quantum measurements. In this paper, we investigate the number of measurement outcomes as a kind of resource. We cast the robustness of this resource as a semi-definite positive program. Its dual problem confirms that if a measurement cannot be simulated by the measurements with a smaller number of outcomes, there exists a state discrimination task where it outperforms the latter. An upper bound of this advantage is derived, which can be saturated if the number of outcomes is smaller than the dimension of the Hilbert space. We also show that the possible tasks to reveal the advantage can be more general and not restricted to state discrimination.

Keywords Quantum information · Quantum measurements · Resource theory

1 Introduction
Quantum measurements are of central interest in quantum information theory. They collapse the quantum state to the classical world which we can perceive, thus allowing us to probe the quantum system. One intrinsic property of a measurement is the number of outcomes. It plays an important role in both fundamentals and applications of quantum theory. On the fundamental side, Refs. [1–3] have shown the existence of bipartite quantum correlations produced by measurements with certain outcomes that cannot be produced by measurements with fewer outcomes in any non-signaling theory. In other words, quantum theory cannot be explained by any general probabilistic theory [4] with limited number of measurement outcomes. This implies that the number of measurement outcomes, like incompatibility of measurements [5,6],
is a property that holds even when one relaxes the underlying theory to any general probabilistic theories. On the practical side, it has been proved that measurements with more outcomes can verify more randomness in quantum random number generation [7]. Moreover, in device-independent quantum key distribution, increasing the number of measurement outcomes can bring higher key rate and lower minimal detection efficiency [8].

To avoid confusion, let us introduce the distinction between the apparent and effective number of outcomes. The former is the number of possible outcomes at the output of the measurement box. Inside the box, however, it is possible that someone mixes several measurements with fewer outcomes and then enlarges the alphabet of outcomes at the output by relabeling. We call the above classical processing simulation. For a given measurement, there may exist several ways to simulate it. We define the effective number of outcomes of a measurement as the largest number of outcomes it must involve to simulate the measurement.

There are several methods that one can certify the effective number of outcomes. At the level of quantum correlations, in a bipartite device-independent scenario, one can always obtain a smaller value on a specific inequality by quantum theory than by any non-signaling theories with any limited outcomes [1] and thus can exclude a particular number of outcomes according to the value of the inequality. In the semi-device-independent scenario where the distinguishability of the prepared states is upper bounded, it is possible to verify the quantum correlations generated by three-outcome measurements from those by two-outcome measurements via semi-definite programming (SDP), analytical characterization of the quantum correlations, and unambiguously state discrimination tasks [9]. At the level of operators, Ref. [10] has studied simulating POVM with projective measurements, which can have a number of outcomes up to the dimension of the Hilbert space. Note that although the simulability of projective measurements is limited [10], one can simulate any POVMs with two-outcome projective measurements using postselection [11]. In quantum steering, given some specific bipartite state, three-outcome measurements can retain more steerability of the state than two-outcome measurements when the state is mixed with a trivial state [12].

Since the measurements with more outcomes can generate a larger set of quantum correlations, they should be advantageous in quantum information processing tasks. It is then natural to see this property as a kind of resource. Resource theory provides a framework to quantify the resource, look into the operational meaning, and explore the possibility of application. The recent years have seen fruitful results on quantum resources [13], among which to our most interest are resource theories regarding quantum measurements. Incompatibility of measurements was first focused on and was associated operationally with state discrimination tasks [14–17]. More generally, for every resourceful measurement, there exists a state discrimination task in which it outperforms all the non-resourceful ones [14,18,19].

In this paper, the number of measurement outcomes that is invariant under classical processing is investigated as a quantum resource. We show how to compute the robustness of a given measurement via SDP. The dual problem confirms the advantage of larger number of outcomes in state discrimination tasks. Next, we give an upper bound of the advantage, which can be saturated under certain condition. Lastly, we
present a more general kind of prepare-and-measure experiment that can reveal the advantage of resourceful measurements.

2 Preliminaries

In quantum theory, measurements can be formulated as positive operator-valued measures (POVMs). A POVM with $n$ outcomes is a collection of positive semi-definite operators $M = \{M_i\}_{i=1}^n$ which satisfies $\sum_i M_i = I$. The probability of obtaining an outcome $i$ by measuring $M$ on quantum state $\rho$ is given by $p(i) = \text{tr}(\rho M_i)$. The set of all the $m$-outcome POVMs is denoted by $P_m$. A set of POVMs is called a POVM assemblage.

In this paper, a set of POVMs or a set of POVM assemblages is denoted by calligraphic symbols like $\mathcal{M}$. An element of a POVM in $\mathcal{M}$ is denoted by $M_a|x$, where $a$ and $x$ are labels for the outcomes and the POVMs.

2.1 Mathematical form of simulation

Mathematically, we say a POVM $M$ can be simulated by a set of POVMs $N = \{N_{a|x}\}_{a,x}$ if it can be written as

$$M_b = \sum_{a,x} p(x) p(b|a,x) N_{a|x}$$

$$= \sum_{a,x} p(x) D(b|a,x) N_{a|x}, \quad (1)$$

where $p(x)$ is randomness for mixing the measurements, $p(b|a,x)$ the randomness for relabeling the outcome $a$ to $b$, and $D(b|a,x)$ a deterministic function. The second equation is due to the fact that the randomness in the relabeling process can be combined into the mixing process. To see this, let us encode the relabeling strategy in $x$ by writing $x$ as strings made up of choosing $n$ outcomes from the $m$ outcomes at a time, namely $x = x^1 x^2 \ldots x^n$ with $x^i = 1, \ldots, m$ and $x^i < x^j$ if $i < j$. Let the deterministic relabeling be $D(b|a,x) = \delta_{b,x^a}$, which means that the relabeling function outputs the $a^{th}$ digit in string $x$ as the outcome. Now, suppose we have probabilistic relabeling $p(b_1|a_1,x_1) = p$ and $p(b_2|a_1,x_1) = 1 - p$. This process is equivalent to mixing with probability distribution $(p, 1 - p)$ two identical POVMs $N_{x_1}$ and $N_{x_2}$ that has labels $x^i_1 = x^i_2$ for all $i$ except $i = a_1$. In this way, not only $p(b|a,x)$ can be restricted to deterministic functions, but also the number of $n$-outcome POVMs to be considered can be fixed to $C_m^n = \frac{m(m-1)\ldots(m-n+1)}{n!}$. This is because two $n$-outcome POVMs which have the same relabeling can always be combined to a new $n$-outcome POVMs, and there are $C_m^n$ possible relabeling functions.

To rephrase the definition, a POVM is said to have effective number of outcomes $n$ if it cannot be simulated by any $(n-1)$-outcome POVMs.
2.2 Quantum resource theory

A quantum resource theory is defined by free operations and free objects [13]. The latter can be quantum states, measurements and channels. The set of free objects is also called free set, and denoted by $F$. Note that all the free sets we discuss about are convex and compact. Here, we restrict the free operations to classical processing which is mixing and relabeling. The free set $F_{nm}$ is defined to be the subset of $P_m$ which can be simulated by $n$-outcome POVMs. A POVM that is not free is deemed resourceful. The discussion in the following is based on $n < m$ because when $n \geq m$, we have trivially $F_{nm} = P_m$. Note that $F_{nm}$ is a convex and compact set [18].

One quantifier of the resource in a POVM is the robustness with respective to the free set $F_{nm}$, which is defined as

$$R_{F_{nm}}(M) = \min_{N \in P_m} \left\{ t \geq 0 \left| \frac{M_b + tN_b}{1 + t} = O_b \right. \right. \text{s.t. } O \in F_{nm} \right\}. \quad (2)$$

It characterizes the relative distance from the POVM to the surface of $F_{nm}$. It can also be interpreted as the minimal “noise” that can make the POVM not resourceful (fall into the free set). The robustness $R_{F_{nm}}(M)$ can be cast as an SDP:

$$\min_{\tilde{Q}_{a|x}} \quad 1 + R_{F_{nm}}(M) = \frac{1}{d} \sum_{a,x,b} D(b|a,x) \text{tr}(\tilde{Q}_{a|x})$$

subject to:

$$\tilde{Q}_{a|x} \geq 0 \quad \forall a, x,$$

$$\sum_{a,x} D(b|a,x) \tilde{Q}_{a|x} - M_b \geq 0 \quad \forall b,$$

$$\sum_{a} \tilde{Q}_{a|x} - \frac{1}{d} \sum_{a} \text{tr}(\tilde{Q}_{a|x})I = 0 \quad \forall x,$$ \quad (3)

where $N_b$ and $(1 + t)O_b$ have been substituted with Eq. (2) and $(1 + t)O_b = \sum_{a,x} D(b|a,x)(1 + t)p(x)Q_{a|x} = \sum_{a,x} D(b|a,x)\tilde{Q}_{a|x}$. The dual program reads

$$\max_{Y_b, Z_x} \quad \sum_{b} \text{tr}(M_bY_b)$$

subject to:

$$Z_x - \frac{1}{d} \text{tr}(Z_x)I + \sum_{b} D(b|a,x)Y_b \leq \frac{1}{d} \sum_{b} D(b|a,x)I \quad \forall a, x,$$

$$Z_x \text{ is Hermitian } \forall x,$$

$$Y_b \geq 0 \quad \forall b.$$ \quad (4)

Strong duality holds because $\tilde{Q}_{a|x} = I$ is a strictly feasible point for the primal problem. The problems (3) and (4) are instances of those in Ref. [14,18].
By choosing the state ensemble as $\mathcal{E} = \{ Y_b / \text{tr}(\sum_b Y_b) \}$ and combining the positivity inequality derived from Eq. (2), one can obtain [14,18]

$$\max_{\mathcal{E}} \frac{P_{\text{guess}}(\mathcal{E}, M)}{\max_{\mathcal{O} \in \mathcal{F}_m} P_{\text{guess}}(\mathcal{E}, \mathcal{O})} = 1 + R_{\mathcal{F}_n}(M),$$

(5)

where $P_{\text{guess}}(\mathcal{E}, T) = \sum_b \text{tr}(\hat{\rho}_b T_b)$ is the probability of guessing correctly in a state discrimination instance of ensemble $\mathcal{E} = \{ \hat{\rho}_b \}$. Note that the prior probabilities of the states are absorbed into the ensemble for compactness. This proves that for a resourceful $m$-outcome measurement, there exists a state discrimination task in which the measurement outperforms all the free ones.

Note that Eq. (5) also provides a semi-device-independent certification of the number of outcomes larger than $n$. The tested measurement can be totally unknown, while the state preparation should be perfectly known. Obtaining an experimental guessing probability larger than the computed optimal one by $\mathcal{O} \in \mathcal{F}_m$ indicates an effective number of outcomes no smaller than $n$.

3 Maximal advantage

We have seen in Sect. 2.2 that every POVM that is not in $\mathcal{F}_m^n$ can show advantage over all the members in $\mathcal{F}_m^n$ in a state discrimination task. One may next wonder what is its greatest advantage. Via see-saw [20] method on the dual problem, one can have an empirical result as well as the corresponding quantum realization. Here, we give an analytical upper bound of the maximal advantage.

Proposition 1 The maximal advantage of $\mathcal{P}_m$ over $\mathcal{F}_m^n$ in a quantum state discrimination task is upper bounded as

$$\max_{M \in \mathcal{P}_m} 1 + R_{\mathcal{F}_m}(M) \leq \frac{m}{n}.$$  

(6)

The inequality is saturated when $m \leq d$, where $d$ is the dimension of the Hilbert space of $M$.

Proof By substituting $O_b = \sum_{a,x} D(b|a, x) p(x) Q_{a|x}$ with $Q_{x} \in \mathcal{P}_n$ into $P_{\text{guess}}(\mathcal{E}, O)$, we have

$$\max_{O \in \mathcal{F}_m^n} P_{\text{guess}}(\mathcal{E}, \mathcal{O}) \Rightarrow \max_{Q_x \in \mathcal{P}_n, p(x)} \sum_b \text{tr}(\hat{\rho}_b \sum_{a,x} D(b|a, x) p(x) Q_{a|x})$$

$$= \max_{Q_x \in \mathcal{P}_n, p(x)} \sum_{a,x} p(x) \text{tr}(\hat{\rho}_{x_a} Q_{a|x})$$

$$= \max_{Q_x \in \mathcal{P}_n, p(x)} \sum_{x} p(x) q_x P_{\text{guess}}(\hat{\mathcal{E}}_x, Q_x)$$

$$= \max_{p(x)} \sum_{x} p(x) q_x P_{\text{guess}}(\hat{\mathcal{E}}_x)$$
\[\begin{align*}
\sum_{x} q_x' P_{\text{guess}}(\hat{\mathcal{E}}_x) &\geq \frac{C_{m-1}^{n}}{C_m^n} \sum_{x} q_x' P_{\text{guess}}(\hat{\mathcal{E}}_x) \\
&= \frac{n}{m} \sum_{x} q_x' P_{\text{guess}}(\hat{\mathcal{E}}_x),
\end{align*}\]

where

\[q_x = \sum_{b \in x} \text{tr}(\tilde{\rho}_b) = \sum_{a} \text{tr}(\tilde{\rho}_{xa}).\]

The coefficient \(q_x\) leaves \(\hat{\mathcal{E}}_x = \{\tilde{\rho}_{xa}/q_x\}\) a valid ensemble. The inequality is attained by letting \(p(x) = 1/|x| = 1/C_m^n\) and noticing that \(C_{m-1}^{n-1}\) is the factor which normalizes \(\{p(x)q_x\}\) to a valid probability distribution, \(\{q_x'\}\). The term \(\sum_{x} q_x' P_{\text{guess}}(\hat{\mathcal{E}}_x)\) can be interpreted as the probability of guessing correctly in a state discrimination game with information prior to the measurement, which tells from which \(n\)-state sub-ensemble the state is chosen. In this case, the measurements can be optimized according to each sub-ensemble. Apparently, this information cannot decrease the probability of guessing correctly over the whole ensemble. Hence, it holds that

\[\max_{O \in \mathcal{F}_m^n} P_{\text{guess}}(\mathcal{E}, O) \geq \frac{n}{m} \max_{M} P_{\text{guess}}(\mathcal{E}, M).\]

Note that this inequality holds for any \(\mathcal{E}\), we have

\[\max_{\mathcal{E}} \frac{\max_{M} P_{\text{guess}}(\mathcal{E}, M)}{\max_{O \in \mathcal{F}_m^n} P_{\text{guess}}(\mathcal{E}, O)} \leq \frac{m}{n}.\]

When \(m \leq d\), the bound is saturated by choosing \(\mathcal{E}\) to be \(m\) orthogonal states with uniform probability distribution. In this case, the optimal \(M\) is the set of projectors corresponding to the orthogonal states in \(\mathcal{E}\), while the optimal \(Q_x\) to the ones in \(\mathcal{E}_x\). Note that the optimal \(M\) and \(Q_x\) include one projector onto an \(m-n+1\)-dimensional space to meet the completeness rule of measurements.

The above proposition shows that increasing the dimension can no longer augment the largest advantage when the dimension is larger than the size of the ensemble. In addition, larger \(m\) may bring larger greatest advantage because \(\mathcal{P}_m \subseteq \mathcal{P}_{m'}\) if \(m' > m\). However, the increasing stops at \(m = d^2\). This is because any \(d\)-dimensional measurements can be simulated by \(d^2\)-outcome ones [21]. And since the simulation is free, it does not generate any advantage.

Little is known about the tight upper bound of advantage for the case \(m > d\). An instance for the saturation of inequality (6) is lacking. However, from the fourth line of Eq. (7) one can observe the following interesting fact, which might give some intuition about the above problem.
Observation 1 Given a state ensemble \( \mathcal{E} \), the optimal guessing probability to discriminate it using \( \mathbf{O} \in \mathcal{F}^n_m \) is equal to the largest weighted optimal guessing probability to discriminate its sub-ensembles \( \hat{\mathcal{E}}_x \), i.e.,

\[
\max_{\mathbf{O} \in \mathcal{F}^n_m} P_{\text{guess}}(\mathcal{E}, \mathbf{O}) = \max_x q_x P_{\text{guess}}(\hat{\mathcal{E}}_x),
\]

where \( q_x \) and \( \hat{\mathcal{E}}_x \) are defined as Eq. (8).

4 Generalization of tasks

Here, we make a generalization of tasks that can reveal the advantage of resourceful measurement assemblages. We show that not only state discrimination, but a type of quantum prepare-and-measure game can reveal the advantage of all the resourceful measurement assemblages.

Consider a quantum prepare-and-measure game. The state and the measurement are chosen according to \( x \) and \( y \), and the outcomes of the measurement are labeled \( b \). The value of a game is a quantity that tells how well one plays the game obeying the rules. Reusing the proving skill in Ref. [18], we have the following proposition.

**Proposition 2** Let the value of the quantum prepare-and-measure game be defined by

\[
S = \sum_{x,y,b} c_{x,y,b} p(x) p(b|x, y),
\]

where \( c_{x,y,b} \) are real coefficients. Let \( \mathcal{F} \) be a convex and compact set of POVM assemblages. Let \( f : \mathcal{M} \mapsto N \) be the linear map such that \( N_x = \sum_{b, y} c_{x,y,b} M_{b|y} \). If \( f \) is a bijection, then for any POVM assemblage \( \mathcal{M} \notin \mathcal{F} \), there exists an instance of the above game where \( \mathcal{M} \) strictly outperforms all the members in \( \mathcal{F} \) with respect to the value.

**Proof** For any POVM assemblage \( \mathcal{M} \) that is not in \( \mathcal{F} \), its image \( N = f(\mathcal{M}) \) must be outside \( f(\mathcal{F}) \). The reason is that if \( N \in f(\mathcal{F}) \), it must have an preimage in \( \mathcal{F} \), which contradicts with the premise that \( f \) is a bijection. Since the linear map \( f \) is convexity preserving, \( f(\mathcal{F}) \) is also a convex set. It follows from the separating hyperplane theorem that there exists a hyperplane described by \( \{W_i\}_i \) such that \( \sum_i \text{tr}(W_i N_i') < 0 \) for all \( N' \in f(\mathcal{F}) \), and \( N \notin f(\mathcal{F}) \) when \( \sum_i \text{tr}(W_i N_i) \geq 0 \). Define \( \tilde{W}_i = W_i + |\lambda|I \), where \( \lambda \) is the smallest eigenvalue of \( W_i \). We have \( \sum_x \text{tr}(\tilde{W}_x N_x') < \sum \text{tr}(\tilde{W}_x N_x) \). Letting \( \mathcal{E} = \{p(x)\rho_x = \tilde{W}_x / \sum_x \text{tr}(\tilde{W}_x)\} \), one can find

\[
\sum_x \text{tr}(\tilde{W}_x N_x) = \sum_{x,y,b} c_{x,y,b} p(x) \text{tr}(\rho_x M_{b|y}) = S(\mathcal{E}, \mathcal{M}),
\]

and similarly \( \sum_x \text{tr}(\tilde{W}_x N_x') = S(\mathcal{E}, \mathcal{M}') \), where \( \mathcal{M}' \in \mathcal{F} \). We can see that \( \mathcal{E} \) gives an instance of state discrimination which reveals the advantage of \( \mathcal{M} \). \( \square \)
We present two examples of such games. One is the state discrimination task with information prior to the measurement, which has been so far widely used to discuss the outperformance of resourceful measurements \cite{14-17}. It corresponds to the case \( x = (w, a) \) and \( c_{x, y, b} = \delta_{b, a} \delta_{w, y} \), which is a bijection. Another example is the following variation of the quantum random access code (QRAC) \( 2 \rightarrow 1 \). The task involves four possible preparations labeled by \( (x_0, x_1) \), where \( x_0, x_1 \in \{0, 1\} \), and two measurements \( y \in \{0, 1\} \) with outcomes \( b \in \{0, 1\} \). In the original QRAC, the value of the task is given by the probability of guessing correctly, \( S = \sum_{x_0, x_1, y} p(x_0, x_1) p(y) p(b = x_y | x_0, x_1, y) \). Here, instead of assigning a weight of 1 to each case of guessing correctly, we let the case of \( x_0 = x_1 = y = b = 1 \) weigh \(-1\). The value of the game is then

\[
S = \sum_{x_0, x_1, y} (-1)^{x_0 \cdot x_1 \cdot y \cdot 1} p(x_0, x_1) p(y) p(b = x_y | x_0, x_1, y).
\]

This definition of value is equivalent to Eq. (12) with \( c_{x_0, x_1, y, b} = \delta_{x_y, b} (-1)^{x_0 \cdot x_1 \cdot y \cdot b \cdot 1} p(y) \), which forms a bijection map between \( M_y | b \) and \( N_{x_0, x_1} = \sum_{y, b} c_{x_0, x_1, y, b} M_y | b \). Thus, the game offers an instance for every resourceful measurement to outperform all the free ones.

The maximal relative advantage of the resource can be related to the robustness of \( N_x \) with respect to \( f(\mathcal{F}) \) as in Ref. \cite{18}

\[
\max_{\mathcal{E}} \frac{S(\mathcal{E}, \mathcal{M})}{\max_{\mathcal{O} \in \mathcal{F}} S(\mathcal{E}, \mathcal{O})} = 1 + R_{f(\mathcal{F})}(N). \tag{14}
\]

It is easy to verify that \( R_{f(\mathcal{F})}(N) = R_{\mathcal{F}}(\mathcal{M}) \) combining the fact that \( f \) is a bijection and the definition of robustness (2).

Note that Proposition 2 gives a sufficient condition for a game to reveal the advantage of a resourceful measurement. More generally, the map \( f \) can be any linear one that has property that \( f(\mathcal{A} \setminus \mathcal{F}) \cap f(\mathcal{F}) = \emptyset \) when \( \mathcal{F} \subset \mathcal{A} \), and not necessarily a bijection. In other words, the map \( f \) preserves the membership relation in their images with respect to the free set. This property of \( f \) depends on the resource being investigated, and it would be interesting to find a general method to certify this. Note that in this case, we have only \( R_{f(\mathcal{F})}(N) \leq R_{\mathcal{F}}(\mathcal{M}) \). And if \( S \) as a function of probabilities is nonlinear, the proof can no longer work because of the use of separating hyperplane theorem.

\section{5 Conclusion}

In this paper, we investigated the number of measurement outcomes as a resource. We characterized the robustness via SDP. The dual problem confirms the operational meaning of robustness as advantage in state discrimination. With the duality giving us one way to interpret operationally a measure of the resource, more possibilities on other tasks are to be found. We gave an upper bound of the maximal advantage. A tighter upper bound for the case \( d < m \) remains to be found. Finally, we have shown...
that a broaden kind of prepare-and-measure experiment can be used to demonstrate the advantage of the resource. Whether there is a general way to check the “membership-preserving” property of the map would be an interesting problem.

Acknowledgements We would like to thank the reviewers of this paper and Roope Uola for their kind comments.

References

1. Kleinmann, M., Cabello, A.: Quantum correlations are stronger than all nonsignaling correlations produced by n-outcome measurements. Phys. Rev. Lett. 117(15), 150401 (2016). https://doi.org/10.1103/PhysRevLett.117.150401
2. Kleinmann, M., Vértesi, T., Cabello, A.: Proposed experiment to test fundamentally binary theories. Phys. Rev. A 96(3), 032104 (2017)
3. Hu, X.M., Liu, B.H., Guo, Y., Xiang, G.Y., Huang, Y.F., Li, C.F., Guo, G.C., Kleinmann, M., Vértesi, T., Cabello, A.: Observation of stronger-than-binary correlations with entangled photonic qutrits. Phys. Rev. Lett. 120(18), 180402 (2018)
4. Barrett, J.: Information processing in generalized probabilistic theories. Phys. Rev. A 75(3), 032304 (2007)
5. Busch, P., Heinosaari, T., Schultz, J., Stevens, N.: Comparing the degrees of incompatibility inherent in probabilistic physical theories. EPL 103(1), 10002 (2013)
6. Heinosaari, T., Schultz, J., Toigo, A., Ziman, M.: Maximally incompatible quantum observables. Phys. Lett. A 378(24), 1695 (2014)
7. Ioannou, M., Brask, J.B., Brunner, N.: Upper bound on certifiable randomness from a quantum black-box device. Phys. Rev. A 99(5), 052338 (2019)
8. Brown, P., Fawzi, H., Fawzi, O.: Computing conditional entropies for quantum correlations, arXiv:2007.12575 [quant-ph] (2020)
9. Shi, W., Cai, Y., Brask, J.B., Zbinden, H., Brunner, N.: Semi-device-independent characterization of quantum measurements under a minimum overlap assumption. Phys. Rev. A 100(4), 042108 (2019)
10. Oszmaniec, M., Guerini, L., Wittek, P., Acín, A.: Simulating positive-operator-valued measures with projective measurements. Phys. Rev. Lett. 119(19), 190501 (2017)
11. Oszmaniec, M., Maciejewski, F.B., Puchała, Z.: Simulating all quantum measurements using only projective measurements and postselection. Phys. Rev. A 100(1), 012351 (2019)
12. Nguyen, H.C., Gühne, O.: Some quantum measurements with three outcomes can reveal nonclassicality where all two-outcome measurements fail. arXiv:2001.03514 [quant-ph] (2020)
13. Chitambar, E., Gour, G.: Quantum resource theories. Rev. Mod. Phys. 91(2), 025001 (2019)
14. Uola, R., Kraft, T., Shang, J., Yu, X.D., Gühne, O.: Quantum resource theories. Phys. Rev. Lett. 122(13), 130404 (2019)
15. Skrzypczyk, P., Šupić, I., Cavalcanti, D.: All Sets of Incompatible Measurements give an Advantage in Quantum State Discrimination. Phys. Rev. Lett. 122(13), 130403 (2019)
16. Buscemi, F., Chitambar, E., Zhou, W.: Complete resource theory of quantum incompatibility as quantum programmability. Phys. Rev. Lett. 124(12), 124001 (2020)
17. Carmeli, C., Heinosaari, T., Toigo, A.: Quantum incompatibility witnesses. Phys. Rev. Lett. 122(13), 130402 (2019)
18. Oszmaniec, M., Biswas, T.: Operational relevance of resource theories of quantum measurements. Quantum 3, 133 (2019)
19. Takagi, R., Regula, B.: General resource theories in quantum mechanics and beyond: operational characterization via discrimination tasks. Phys. Rev. X 9(3), 031053 (2019)
20. Werner, R.F., Wolf, M.M.: Bell inequalities and entanglement. Quantum Inf. Comput. 1(3), 1–25 (2001)
21. D’Ariano, G.M., Presti, P.L., Perinotti, P.: Classical randomness in quantum measurements. J. Phys. A Math. General 38(26), 5979 (2005)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.