Solvable model for quantum gravity?

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Abstract

We study a type of geometric theory with a non-dynamical 1-form field. Its dynamical variables are an $su(2)$ gauge field and a triad of $su(2)$ valued 1-forms. Hamiltonian decomposition reveals that the theory has a true Hamiltonian, together with spatial diffeomorphism and Gauss law constraints, which generate the only local symmetries. Although perturbatively non-renormalizable, the model provides a test bed for the non-perturbative quantization techniques of loop quantum gravity.

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1. Introduction

It is well known that the perturbative approach to finding a theory of quantum gravity faces the problem of non-renormalizability. This is partly solved in string theory, but at the expense of introducing extra dimensions. There exist however, non-renormalizable field theory models for which the quantum theory is known to exist non-perturbatively.

One example of this is the Gross–Neveu model in three dimensions which is a theory with a four-fermion interaction; the interaction coupling constant has negative mass dimension indicating power counting non-renormalizability. The model exists non-perturbatively in the ultraviolet regime [1]. Another is Einstein gravity in three dimensions [2, 3]. But this is a theory with no propagating degrees of freedom, and it is still not clear how to construct a consistent quantum theory of 3D gravity [4]. In any case, we do not know how to proceed from lower to higher dimensional theories, and so this case too may be considered special.

There is so far only one example of a four dimensional diffeomorphism invariant theory that is not renormalizable, but which exists as a non-perturbative quantum theory [5, 6]. The catch is that although the model has local degrees of freedom, its dynamics is trivial. It nevertheless shows that perturbative non-renormalizability is not necessarily a sound criteria for discarding a theory.

The question of whether this could be the case for quantum gravity in four dimensions has been one motivation for seeking a non-perturbative canonical formulation. However, no such
program has yet been completed due to the problem of dealing with the Hamiltonian constraint in the Dirac quantization approach, which leads to the Wheeler–DeWitt equation. Nevertheless there are recent indications that this can be circumvented by deparametrizing the system by using matter fields to fix time and space gauges. One such approach uses a pressureless dust to fix only a time gauge, thereby eliminating the Hamiltonian constraint problem [7] and replacing it by a true Hamiltonian with only spatial diffeomorphism symmetry.

Motivated by such approaches we present a new type of theory with dynamical metric and a fixed 1-form field\(^1\). The theory is such that it has a built in time that does not arise via a gauge fixing as in the aforementioned approaches. Its canonical decomposition reveals that there is a true Hamiltonian together with spatial diffeomorphism and Gauss constraints, which generate the only gauge symmetry. The theory can be coupled to matter in natural way. Its quantization can be carried out using the methods of loop quantum gravity (LQG). It therefore provides an example of a non-renormalizable geometric theory whose quantum theory exists non-perturbatively.

In the following we describe the theory and its canonical formulation, and then outline a non-perturbative quantization scheme using the background independent techniques developed in the LQG program.

2. The model

The fields in the theory are an \(su(2)\) gauge field \(A^i_\mu\), a dreibein \(e^i_\mu\), scalar field \(\phi\), and a fixed non-dynamical 1-form field \(\zeta_\alpha\) which gives the 2-form \(\omega = d\zeta\) (\(i, j, k\ldots\) are \(su(2)\) indices, and \(\alpha, \beta\ldots\) are world indices.) The dreibein fields \(e^i_\mu\) define a degenerate 4-metric, and give rise to the tensor density

\[
\tilde{u}^\alpha = \frac{1}{3!} \tilde{\eta}^{\alpha\beta\mu\nu} e^j_\beta e^k_\mu \epsilon_{ijk}. \tag{1}
\]

where \(\tilde{\eta}^{\alpha\beta\mu\nu}\) is the Levi–Civita symbol (independent of \(e^i_\alpha\)) and \(\epsilon_{ijk}\) is the \(su(2)\) structure constant. Using this we define a scalar density and vector field by

\[
\tilde{u} = \tilde{u}^\alpha \zeta_\alpha, \quad \tilde{u}^\alpha = \frac{\tilde{u}}{\tilde{u}}, \tag{2}
\]

and a co-triad by

\[
e^i_\alpha = \frac{1}{2\tilde{u}} \tilde{\eta}^{\alpha\beta\mu\nu} \zeta_\beta e^j_\mu \epsilon_{ijk}. \tag{3}
\]

The scalar density \(\tilde{u}\) would vanish if \(\zeta_\alpha\) were a linear combination of the \(e^i_\alpha\), so we assume this is not the case. These definitions give the relations\(^2\)

\[
u^\alpha \zeta_\alpha = 1, \quad \nu^\alpha e^i_\alpha = 0, \quad \zeta_\alpha e^i_\alpha = 0 \tag{4}
\]

\[
e^i_\alpha e^j_\alpha = \delta^i_j, \quad e^i_\alpha e^j_\beta = \delta^i_\beta. \tag{5}
\]

We note finally that a non-degenerate Euclidean or Lorentzian signature 4-metric may be defined by

\[
g_{\alpha\beta} = \pm \zeta_\alpha \zeta_\beta + e^i_\alpha e^i_\beta. \tag{6}
\]

\(^1\) We note that there is a model with a dynamical scalar field [8] to which the methods of this paper may be applied; in a particular time gauge, this model has an interesting physical Hamiltonian with a diffeomorphism constraint.

\(^2\) With these relations we note that the 2-form \(\omega\) is invertible (because \(u^\cdot \omega = \zeta_\alpha \neq 0\), and \(\nu^\cdot \omega \neq 0\)), therefore it is a symplectic form. However this fact is not needed in our subsequent development of the model.
We are now ready to define the action for the model which contains the field $\zeta_a$ as a fixed ‘background’ structure. The action is

$$S = S_G + S_\Lambda + S_\phi$$

$$= \frac{1}{12} \int_M \tilde{\eta}^{\mu \nu} \epsilon^{ijk} e_{\mu}^{a} F_{\nu}^{ij}(A) + \Lambda \int_M \bar{u}$$

$$+ \int_M \bar{u} \left( -6 u^a u^b \partial_a \phi \partial_b \phi + e^a_i e^b_j \partial_a \phi \partial_b \phi \right).$$

The first term is the action of the model introduced in [5], where $F(A)$ is the curvature of the gauge field $A$. Its canonical theory has an identically vanishing Hamiltonian constraint, so it is a theory with three local degrees of freedom and no dynamics. The fixed 1-form field $\zeta_a$ makes it possible to introduce the cosmological constant term and coupling to matter in the manner displayed. The coupling constant $\Lambda$ is a fundamental length scale obtained by assigning the usual canonical dimension to the connection, i.e. $A$ has mass dimension one, and $e$ is dimensionless. This assignment makes the theory power counting non-renormalizable just as in Einstein gravity, since changing the gauge algebra from $so(3, 1)$ to $su(2)$ does not affect this counting.

2.1. Hamiltonian theory

To construct the Hamiltonian theory let us introduce the embedding variable $X^a (t, x^a)$ which provides a smooth map

$$X : \mathbb{R} \times \Sigma \longrightarrow M$$

where $\Sigma$ is a three manifold. The inverse map gives the functions $x^a (X)$ and $t (X)$. The 3+1 split of the first term in the action is obtained [5] by substituting into the action the decompositions

$$\tilde{\eta}^{a b c} = \tilde{\eta}^{a b c} X_a X_b X_c,$$

where the time deformation vector field $\dot{X}^a$ decomposes as

$$\dot{X}^a = u^a + N^a = u^a + X^a_x N^a.$$  

We also use the spatial projections of the fields defined by

$$e^a_i = e^a_i X^a, \quad A^a_\mu = A^a_\mu X^a, \quad \tilde{e}^a_i = \tilde{\eta}^{b c} e^a_i e^b_j e^c_k, \quad e^a_i = \tilde{e}^a_i / \tilde{e},$$

where $\tilde{e} = \tilde{\eta}^{a b c} e^a_i e^b_j e^c_k$. These are the decompositions needed to arrive at the canonical form of the first part of the action, which is

$$S_G = \int_M d^3 x d t \left[ 2 \tilde{\eta}^{a b c} (A^a_\mu - N^a \partial_\mu A^b_\nu \partial_\nu) - \Lambda \delta^a_{a'} e^a_i \right]$$

where $N^a = e^a_i (e^a_\mu \dot{X}^\mu)$ and $\Lambda^i = A^a_\mu \partial_\mu$. This identifies the fundamental Poisson brackets for the geometric variables:

$$\{ A^a_\mu (x), \tilde{e}^b_j (x') \} = \delta^a_i \delta_{\mu j} \delta^b_a \delta_{ij} \delta_{\mu j}.$$  

To obtain the canonical decomposition of $S_\Lambda$ and $S_\phi$ we note first that

$$\bar{u} = \dot{X}^a \zeta_a \tilde{e} = (1 + X^a_x \zeta_a N^a) \tilde{e},$$

$$e^a_i = X^a_x e^a_i + \dot{X}^a (t, \phi^a_\mu).$$

Now the identity $e^a_i \zeta_a = 0$ applied to the last equation gives

$$0 = X^a_x \zeta_a e^a_i + \dot{X}^a (t, \phi^a_\mu).$$

(16)
Thus if we choose the foliation \( X^a(t, x^a) \) such that \( X^a_\alpha \xi_a = 0 \) (i.e. adapted to the fixed field \( \xi_a \)) we have

\[
\tilde{u} = \tilde{e}, \quad e^{ai} = X^a_\alpha e^{ai}. \tag{17}
\]

Substituting these together with (11) into the action gives

\[
S_{\Lambda} + S_{\phi} = \int d^3x \, dt \left[ \Lambda \tilde{e}^a + \frac{P^2_\phi}{2\tilde{e}} + \tilde{e}^{ai} \epsilon_b \delta_\phi \partial_\phi \phi - N a P_\phi \partial_\phi \phi \right]. \tag{18}
\]

The Hamiltonian decomposition of the full action then shows that the phase space variables are the canonical pairs \((e^{ai}, A^\alpha_a)\) and \((\phi, P_\phi)\) with the Hamiltonian

\[
H = \int d^3x \left[ \Lambda \tilde{e}^a + \frac{P^2_\phi}{4\tilde{e}} + \tilde{e}^{ai} \epsilon_b \delta_\phi \partial_\phi \phi \right]. \tag{19}
\]

The theory has two sets of first class constraints that generate SU(2) gauge transformations and spatial diffeomorphisms. Thus the theory has four local configuration degrees of freedom of which three are geometric and one is matter. The remarkable feature is that true dynamics is obtained by introducing a fixed 1-form field which may be interpreted as providing a symplectic structure on the manifold. Thus the presence of this structure may be viewed as providing a time variable, while maintaining full general covariance of the action.

The Hamiltonian equations of motion provide a view of the dynamics. Evolution is a combination of gauge (Gauss and spatial diffeomorphisms) and true motion via \( H \). We note first that the 3-geometry does not evolve:

\[
\dot{\tilde{e}}^a = \{ \tilde{e}^a, H \} = 0, \tag{20}
\]

but its conjugate connection does

\[
\dot{A}^i_a = \{ A^i_a, H \} = e^a_a \left( \Lambda - \frac{P^2_\phi}{4\tilde{e}} - \epsilon^b \epsilon^c \partial_\delta \phi \partial_\phi \phi \right) + 2\partial_\phi \phi \partial^i \phi \partial_\phi \phi. \tag{21}
\]

The scalar field equations are the usual ones for a field on a curved space-time given by the metric (6).

The geometrical phase space variables in our theory are identical to those of the Ashtekar–Barbero canonical formulation of general relativity. There the connection \( A^\alpha_a \) is a sum of the (spatial) metric connection and the extrinsic curvature. Thus the comparison allows us to interpret the canonical equations of motion of our model as evolving the extrinsic curvature, but not the spatial metric. With this in mind, the model may be viewed as evolving both the matter field and the 4-geometry (through the connection \( A^\alpha_a \)).

3. Quantization

The similarity of the geometrical part of the phase space with that of general relativity in the connection-triad variables serves as a starting point for quantization. We use an extension of the spin network Hilbert space used in LQG to include scalar matter degrees of freedom \([9]\). The starting point of the LQG approach is the set of phase space functions

\[
U_{\gamma} [A] = P \exp \int \gamma A^i_a \tau^i_a dx^a, \quad F^i_s = \int_S \tilde{e}^i_a dS^a, \tag{22}
\]

where \( \gamma \) is a loop and \( S \) a surface in a spatial hypersurface \( \Sigma \), and \( \tau^i \) are generators of SU(2). Gauge invariant versions of these were first used for quantization of BF theory \([10]\) and in \([11]\). Their Poisson bracket forms the so-called holonomy-flux algebra

\[
\{ H_{\gamma} [A], F^i_s \} = \int_{\gamma} ds \int_S d^2\sigma U_{\gamma} [A] \tau^i \delta^3(\gamma(s), S(\sigma)). \tag{23}
\]
The analogous observables for the scalar field are

$$V_k(\phi(x)) = \exp[ik\phi(x)], \quad P_j = \int_{\Sigma} f P_0 d^3x,$$

where $k \in \mathbb{R}$ and $f(x)$ is a suitable function with rapid fall-off. These satisfy

$$\{V_k(x), P_j\} = ikf(x)V_k(x).$$

### 3.1. Geometry Hilbert space

There is a well-defined path to quantization of the gravitational variables which are discussed in detail in a number of reviews [12]. Therefore we restrict attention to describing the basic guidelines. A crucial first step is the choice of Hilbert space for a connection representation $\Psi$. One considers an oriented graph $\Gamma$ with ordered edges $e_1, e_2, \ldots, e_n$, and vertices $n_1, n_2, \ldots, n_M$ embedded in the spatial surface $\Sigma$, and associates the holonomy function in the representation $j$ of SU(2), $H_j[A]$, with edge $e$. A spin network state is a function of such holonomies

$$f[A] = f(U_{e_1}^i, U_{e_2}^j, \ldots, U_{e_n}^{i\prime}).$$

These are essentially functions of SU(2) group elements, so the natural inner product is the Haar measure on (tensor product copies) of this group. A convenient orthonormal basis for this space of functions is the spin network basis; the wave function of a graph with a single vertex is the matrix

$$\rho = \frac{1}{\sqrt{2\pi}} \int \frac{d^3p}{2\pi^2} e^{-ip \cdot x}.$$

Having characterized $H_{\text{Kin}}$ in this manner, the next step is to address the requirement of invariance under spatial diffeomorphisms. We note first that there is a natural action of diffeomorphisms on the gauge invariant spin network states such as (29). This stems from the observation that such transformations ‘drag the graph around’ but do not affect the

$$\psi[A]_{1,1;1} = \langle 1|1, 1; 1, \frac{1}{2} : \sigma, \sigma \rangle = \left[U_{e_1}^{1\prime} \right]_A^B \left[U_{e_2}^{1\prime} \right]_C^D \sigma_{AC} \sigma^{BD},$$

where $\sigma^i$ are the Pauli matrices. This example also illustrates why edges must be oriented; the matrix indices (â€œACâ€) come together at one vertex and (â€œBDâ€) at the other.
combinatoric information in the spins and intertwiners [6, 12]. Formally, for \( \phi \in \text{Diff}(\Sigma) \), we have

\[
U_0[\phi][\Gamma; j_1, \ldots, j_N; I_1, \ldots, I_M] = |\phi^{-1}\Gamma; j_1, \ldots, j_N; I_1, \ldots, I_M\rangle. \tag{30}
\]

Thus for a fixed graph \( \Gamma \) the diffeomorphism invariant information is just the set of spins and intertwiners (up to some subtleties [12]). We denote this Hilbert space by \( \mathcal{H}_{\text{geom}} \), and in the following consider the case where the underlying graph is a cubic (abstract) lattice. Thus each node will be 6-valent, and we will assume that the associated non-zero spins and intertwiners form a finite set. This will aid in defining the physical Hamiltonian operator\(^3\).

### 3.2. Matter Hilbert space

The geometry Hilbert space \( \mathcal{H}_{\text{geom}} \) described above is the physical Hilbert space of the model without matter. Its extension to include matter is accomplished by associating an additional quantum number with the vertices of graphs. Given a graph \( \Gamma \) with vertices \( v_1, \ldots, v_M \), a basis for the matter Hilbert space, \( \mathcal{H}_{\text{matter}} \), is \( |k_1, \ldots, k_M\rangle \), where \( k_i \in \mathbb{R} \) are the quantum numbers associated with matter. The inner product is

\[
\langle k'_1, \ldots, k'_M | k_1, \ldots, k_M \rangle = \delta_{k'_1, k_1}, \ldots, \delta_{k'_M, k_M}. \tag{31}
\]

The classical scalar field variables \( V_{k}(\phi(x)), P_f \) defined above have the quantum realizations

\[
\hat{V}_{k}(v_i)|k_1, \ldots, k_M\rangle = |k_1, \ldots, k+ k, \ldots, k_M\rangle, \tag{32}
\]

\[
\hat{P}_f|k_1, \ldots, k_M\rangle = \sum_{i=1}^{M} k_i f(v_i)|k_1, \ldots, k_M\rangle, \tag{33}
\]

where \( v_i \) is a vertex. It is readily verified that these definitions provide a representation of the classical Poisson algebra.

### 3.3. Physical Hilbert space and Hamiltonian

The physical Hilbert space is the tensor product \( \mathcal{H}_{\text{geom}} \otimes \mathcal{H}_{\text{matter}} \), with basis

\[
|\Gamma; j; I; k\rangle = |\Gamma; j_1, \ldots, j_N; I_1, \ldots, I_M; k_1, \ldots, k_M\rangle. \tag{34}
\]

As mentioned above we assume that the geometric and matter excitations are on an infinite cubic graph. Its regularity provides a systematic way to construct the Hamiltonian operator to which we now turn.

The classical expression for the Hamiltonian (19) contains geometric terms that appear in the Hamiltonian constraint of LQG. The operator realizations of these are well studied in the literature [9]. For example the \( \tilde{e} \) term in the Hamiltonian is realized using the LQG volume operator, and its inverse is realized as a commutator of the square root of the volume and holonomy operators, a construction well known in LQG.

Turning to the matter operators, the \( P^2 \) factor is diagonal in the basis we are using. It can be localized by writing the integral for \( P_f \) as a sum over vertices of the graph, taking \( f \) to be unity, i.e.

\[
\int d^3 x \frac{P^2}{\tilde{e}} \longrightarrow \sum_{i} \frac{1}{\tilde{e}_{v_i}} P^2(v_i). \tag{35}
\]

\(^3\) The choice of cubic graph represents a restriction of the quantum theory, since in principle all graphs should be included; this choice allows a systematic construction of the Hamiltonian operator. The solution of the diffeomorphism constraint to yield \( \mathcal{H}_{\text{geom}} \) for a cubic lattice proceeds as in [6], with a finite set of excitations on the lattice.
The factors of $\partial_a \phi$ may be realized by using a ‘finite difference’ approach. We first define a local field operator as

$$\Phi_k(v_i) := \frac{1}{2i} \left( V_k(v_i) - V_{-k}(v_i) \right).$$

(36)

Using this, one way to define the operator corresponding to the matter gradient $e^a \partial_a \phi$ via a finite difference scheme. The simplest such scheme is forward Euler, where for a single direction $z_k$ on the cubic lattice we have

$$e^a \partial_a \phi(v_i) \rightarrow \tilde{F}_{z_k}(\Phi_k(v_i + z_k) - \Phi_k(v_i)).$$

(37)

where $\tilde{F}_{z_k}$ is the flux operator associated with the edge $z_k$ that connects the adjacent vertices $v_i + z_k$ and $v_i$. It is evident that there are other ways to write this operator; our purpose is to point out that the Hamiltonian can be defined using the basic operators.

4. Summary

We have developed a new type of geometric theory with a fixed 1-form field. The theory has a ‘built in’ time that does not arise via a gauge fixing as in the aforementioned approaches. Its canonical decomposition reveals a true Hamiltonian together with spatial diffeomorphism and Gauss constraints, which generate the only gauge symmetry. The theory can be coupled to matter in a natural way. The canonical connection $A^a_i$ defines an extrinsic curvature via the Ashtekar–Barbero relation $A^a_i = \Gamma^a_i(e) + \gamma K^a_i$. This provides an interpretation of the theory as giving a dynamical 4-geometry, even though the 3-geometry given by $e^a_i$ does not evolve; (we note that the pair $(e^a_i, K^a_i)$ so defined do not give the same metric as the auxiliary one in equation (6), which is not dynamical). Quantization can be carried out using the methods of LQG. The model therefore provides an example of a perturbatively non-renormalizable geometric theory that exists non-perturbatively at the quantum level.

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