Abstract

In this paper, a nonlinear system aiming at reducing the signal transmission rate in a networked control system is constructed by adding nonlinear constraints to a linear feedback control system. Its stability is investigated in detail. It turns out that this nonlinear system exhibits very interesting dynamical behaviors: in addition to local stability, its trajectories may converge to a non-origin equilibrium or be periodic or just be oscillatory. Furthermore it exhibits sensitive dependence on initial conditions — a sign of chaos. Complicated bifurcation phenomena are exhibited by this system. After that, control of the chaotic system is discussed. All these are studied under scalar cases in detail. Some difficulties involved in the study of this type of systems are analyzed. Finally an example is employed to reveal the effectiveness of the scheme in the framework of networked control systems.

Keywords: stability, attractor, nonlinear constraint, chaos, bifurcation, tracking, networked control systems.
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1 Introduction

1.1 Limited information related control

In the past decade great interest has been devoted to the study of limited information related control problems. Limited information related control is defined as follows: Given a physical plant $G$ and a set of performance specifications such as tracking, design a controller $C$ based on limited information such that the resulting closed-loop system meets the prespecified performance specifications. There are generally two sources of limited information, one is signal quantization, and the other is signal transmission through various networks.

In designing a digital control system, signal quantization induced by signal converters such as A/D, D/A and computer finite word-length limitation is unavoidable. To compensate this, traditional design methods generally proceed like this: First design a controller ignoring the effect of signal quantization, then model it as external white noise and analyze its effect on the designed system. If the performance is acceptable, it is okay; otherwise, adjust controller parameters such as the sampling frequency, or do redesign (including the choice of converters) until satisfactory performance is obtained. Recently the following problems have been asked:

1. How to study the effect of signal quantization more rigorously? More precisely, how will it genuinely affect the performance of the underlying control system?

2. If there are positive answers to the above question, can one design better controllers based on this knowledge?

To address these two problems, stability, the fundamental requirement of a control system, has been studied recently in somewhat detail. Delchamps [1990] studied the problem of stabilizing an unstable linear time-invariant discrete-time system via state feedback where the state is quantized by an arbitrarily given quantizer of fixed quantization sensitivity. It turned out that there are no state feedback strategies ensuring asymptotic stability of the closed-loop system in the sense of Lyapunov. Instead, the resulting closed-loop system behaves chaotically. Fagnani & Zampieri [2003] continued this research in the context of a linear discrete-time scalar system. Based on the flow information provided by the system invoked by quantization, stabilizing methods based on the Lyapunov approach and chaotic dynamics of the system were discussed. Ishii & Francis [2003] studied the quadratic stabilization of an unstable linear time-invariant continuous-time system by designing a digital controller whose input was the quantized system state; an upper bound of sampling periods was calculated geometrically using state feedback for the system $G$ with a carefully designed quantizer of fixed quantization sensitivity, by which the trajectories of the closed-loop system would enter and stay in a region of attraction around the origin. Clearly in order to achieve asymptotic stability, quantizers with variable quantization sensitivities must be adopted. In Brockett & Liberzon [2000], for the system $G$, by choosing a quantizer $q$ with time-varying sensitivities, a linear time-invariant feedback was designed to yield global asymptotic stability. This problem was also studied in Elia & Mitter [2001] for exponential stability using
logarithmic quantizers. In Nair & Evans [2002], exponential stabilization of the system $G$ with a quantizer is studied under the framework of probability theory. More interestingly, the simultaneous effect of sampling period $T$ and quantization sensitivity was studied in Bamieh [2003], where it is shown via simulation that system performance would become unbounded as $T \to 0$ if a quantizer of fixed sensitivity was inserted into a control loop composed of a system and an unstable controller. Therefore it is fair to say that the problem—performance of quantized systems—is quite complicated as well as challenging. Much research is still required in this area.

Another representation of limited information is signals suffering from time-delays or even loss, which are ubiquitous in the networked control systems (Wong & Brockett [1997], Walsh et al. [2001], and Ray [1987]). The fast-developing secure, high speed networks (Varaiya & Walrand [1996] and Peterson & Davie [2000]) make control over networks possible. Compared to the traditional point-to-point connection, the main advantages of connecting various system components such as processes, controllers, sensors and actuators via communication networks are wire reduction, low cost and easy installation and maintenance, etc. Thanks to these merits, networked control systems have been built successfully in various fields such as automobiles (Krtolica et al. [1994], Ozguner et al. [1992]), aircrafts (Ray [1987] and Sparks [1997]), robotic controls (Malinowshi et al. [2001], Safaric et al. [1999]) and so on. In addition, in the field of distributed control, networks may provide distributed subsystems with more information so that performance can be improved (Ishii & Francis [2002]). However, networks inevitably introduce time delays and packet dropouts due to network propagation, signal computation and coding, congestion, etc., which lead to limited information for the system to be controlled as well as the controller, thus complicating the design of controllers and degrading the performance of control systems or even destabilizing them (Zhang et al. [2001]). Therefore it is very desirable to reduce time delays and packet dropouts when implementing a networked control system. For the limitation of space, for now we will concentrate on discussing a network protocol proposed by Walsh, Beldiman, Bushnell, and Hong, et al. (Walsh et al. [1999, 2001, 2002a, 2002b]) since our proposed one is in the same spirit as theirs. For a more complete review on networked control systems and more references, please refer to Zhang & Chen [2003].

1.2 Network based control

One effective way to avoid large time delays and high probability of packet dropouts is by reducing network traffic. In a series of papers published by Walsh, Beldiman, Busnell, and Hong, et al. (Walsh et al. [1999, 2001, 2002a, 2002b]), a network protocol called try-once-discard (TOD) is proposed. In that scheme, there is a network along the route from a MIMO plant to its controller. At each transmission time, each sensor node calculates the importance of its current value by comparing it with the latest one, the larger the difference is, the more important the current value is, then the most important one gets access to the network. For this scheme, based on the Lyapunov method and the perturbation theory, a minimal time within which there must have at least one network transmission to guarantee stability of networked control systems is derived.
This network protocol, TOD, essentially belongs to the category of dynamical schedulers. In comparison with static schedulers such as token rings, it allocates network resources more effectively. However, a supervisor computer, i.e., a central controller, is required to compare those differences and decide which node should get access to the network at each transmission time. It is therefore complicated and possibly difficult to implement.

1.3 A new networked control technique

Consider the feedback system in Fig. 1, where $G$ is a discrete-time system of the form:

$$
x(k+1) = Ax(k) + Bu(k),
$$
$$
y(k) = Cx(k),
$$

with the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, the output $y \in \mathbb{R}^p$ and the reference input $r \in \mathbb{R}^p$ respectively; $C$ is a stabilizing controller:

$$
x_d(k+1) = A_dx_d(k) + B_d e(k),
$$
$$
u(k) = C_d x_d(k) + D_d e(k),
$$
$$
e(k) = r(k) - y(k),
$$

with its state $x_d \in \mathbb{R}^{n_c}$. Let $\xi = \begin{bmatrix} x \\ x_d \end{bmatrix}$, then the closed-loop system from $r$ to $e$ can be modeled by

$$
\xi(k+1) = \begin{bmatrix} A - BD_d C & BC_d \\ -B_d C & A_d \end{bmatrix} \xi(k) + \begin{bmatrix} BD_d \\ B_d \end{bmatrix} r(k),
$$

$$
e(k) = \begin{bmatrix} -C & 0 \end{bmatrix} \xi(k) + r(k).
$$

Now we add nonlinear constraints on both $u$ and $y$. Specifically, consider the system in Fig. 2. The nonlinear constraint $H_1$ is defined as, for a given $\delta_1 > 0$, let $v(-1) = 0$, and for $k \geq 0$,

$$
v(k) = H_1(u_c(k), v(k-1)) = \begin{cases} u_c(k), & \text{if } \|u_c(k) - v(k-1)\|_\infty > \delta_1, \\ v(k-1), & \text{otherwise.} \end{cases}
$$

Similarly $H_2$ is defined as, for a given $\delta_2 > 0$, let $z(-1) = 0$, and for $k \geq 0$,

$$
z(k) = H_2(y_c(k), z(k-1)) = \begin{cases} y_c(k), & \text{if } \|y_c(k) - z(k-1)\|_\infty > \delta_2, \\ z(k-1), & \text{otherwise.} \end{cases}
$$
Fig. 2. A constrained feedback system

It can be shown that $\|H_1\|$, the induced norm of $H_1$, equals 2, so is $\|H_2\|$.

In a networked control system, there are normally computer networks along the routes from the controller $C$ to the system $G$ and from $G$ to $C$. These networks (usually shared by other clients) will introduce time delays into the closed-loop system. It is quite appealing to compensate this adverse effect. If we regard $H_1$ as a component of $C$ and $H_2$ of $G$, $G$ (resp. $C$) contains previous version of $u_c$ (resp. $y_c$), then there will have no signal transmission from $C$ to $G$ and (or) from $G$ to $C$ if the inequalities in Eqs. (4)-(5) are not satisfied, suggesting that we are reducing network traffic. We expect this will benefit the overall system connected by the common networks. One example will be given in Sec. 3 to illustrate this point.

Similar work is done in Otanez et al. [2003] where adjustable deadbands are proposed to reduce network traffics. In that formulation, the closed-loop system with deadbands is modeled as a perturbed system, then its exponential stability follows that of the original system [Khalil 1996]. The constraints proposed here are fixed ($\delta_1$ and $\delta_2$), we will see the stability of the system in Fig. 2 is quite complicated (e.g., only local stability can be obtained). However, the advantage of fixed deadbands is that it will reduce network traffic more effectively. Furthermore, the stability region can be scaled as large as desired. This is one advantage of our proposed scheme. Moreover, we find out that the system in Fig. 2 has rather complex dynamics — it appears chaotic. As is known chaotic behavior will in general provide more system dynamics, i.e., more information of the underlying system, therefor we hope we can achieve better control in the framework of Fig. 2. We will address this problem more rigorously in Sec. 2.2.

For the “constrained” system in Fig. 2, let $p$ denote the state of the system $G$, and $p_d$ denote the state of the controller $C$, then

$$p(k+1) = Ap(k) + Bv(k),$$
$$y_c(k) = Cp(k),$$

and

$$p_d(k+1) = A_d p_d(k) + B_d e_c(k),$$
$$u_c(k) = C_d p_d(k) + D_d e_c(k),$$
$$e_c(k) = r(k) - z(k).$$
Let $\eta = \begin{bmatrix} p \\ p_d \end{bmatrix}$, then the closed-loop system from $r$ to $e$ is

$$
\eta(k+1) = \begin{bmatrix} A & 0 \\ 0 & A_d \end{bmatrix} \eta(k) + \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \begin{bmatrix} v(k) \\ -z(k) \end{bmatrix} + \begin{bmatrix} 0 \\ B_d \end{bmatrix} r(k),
$$

(6)

where $v$ and $z$ are given in Eqs. (4)-(5).

To test whether the scheme adopted here is useful in the framework of networked control systems, we have to address at least the following two concerns:

- The stability of the system in Fig. 2. Since stability is fundamental to any control system, the first question about this system is its stability. In this paper, the Lyapunov stability is studied in detail:

1. Given that both $G$ and $C$ are stable, the system is locally exponentially stable (Lemma 1).
2. However, the behavior of the state trajectory $(p, p_d)$, starting outside the stability region, is hard to predict. A scalar case is studied in detail to illustrate various dynamics the system can exhibit (Sec. 2.1): Its trajectory may converge to an equilibrium which is not necessarily the origin (Proposition 1, Corollary 1), or be periodic (Theorem 3, Theorem 4), or aperiodic (Theorem 1, Theorem 2), which can either be quasiperiodic or exhibit sensitive dependence on initial conditions—a sign of chaos, advocating novel control method—chaotic control.
3. For higher-order cases, a positively invariant set is constructed (Theorem 5).
4. Finally, it is proved that the set of all initial points $\eta(0)$ whose closed-loop trajectories tend to an equilibrium as $k \to \infty$ has Lebesgue measure zero if either $G$ or $C$ is unstable (Theorem 6).

- This research is mainly devoted to the study of networked control systems (NCSs), hence it is natural and necessary to analyze its effectiveness in the framework of networked control systems. An example is used to illustrate the efficacy of our scheme (Sec. 3).

The outline of this paper as follows. Sec. 2 is devoted to the study of stability. An example is constructed to show the effectiveness of our scheme in Sec. 3. Some concluding remarks are in Sec. 4.

2 Stability

In this section, we discuss the stability of the system in Eq. (6). Firstly a sufficient condition ensuring local exponential stability is derived. Secondly concentrated mainly on scalar cases, the intriguing behavior of the dynamics of the system is studied in detail. It appears that the system behaves chaotically. Finally it is proven that the Lebesgue measure of the set of trajectories converging to a certain equilibrium is zero if either the system $G$ or the controller $C$ is unstable.
Letting $r = 0$, the system in Eq. (3) becomes

$$\eta(k + 1) = \begin{bmatrix} A & 0 \\ 0 & A_d \end{bmatrix} \eta(k) + \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \begin{bmatrix} v(k) \\ -z(k) \end{bmatrix},$$

$$\begin{bmatrix} u_c(k) \\ y_c(k) \end{bmatrix} = \begin{bmatrix} 0 & C_d \\ C & 0 \end{bmatrix} \eta(k) + \begin{bmatrix} 0 & D_d \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v(k) \\ -z(k) \end{bmatrix},$$

$$\begin{bmatrix} v(k) \\ -z(k) \end{bmatrix} = \begin{bmatrix} H_1(u_c(k), v(k - 1)) \\ -H_2(y_c(k), z(k - 1)) \end{bmatrix}, \quad k \geq 0.$$  \hspace{1cm} (7)

Then, we have the following result regarding local stability.

**Lemma 1** If both the system $G$ and the controller $C$ are stable, then the origin is locally exponentially stable.

**Proof:** Define

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A_d \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & C_d \\ C & 0 \end{bmatrix}.$$  

Since both $G$ and $C$ are stable, $\rho(\tilde{A}) < 1$ where $\rho(M)$ is the spectral radius of a square matrix $M$. Then for any given $\varepsilon > 0$ satisfying $\rho(\tilde{A}) + \varepsilon < 1$, there exists a matrix norm $\|\cdot\|_*$ such that $\|\tilde{A}\|_* \leq \rho(\tilde{A}) + \varepsilon$ [Huang, 1984]. Furthermore, this matrix norm satisfies $\|MN\|_* \leq \|M\|_* \|N\|_*$ for any two matrices $M$ and $N$ of dimension $n + n_c$. Therefore, for a vector $x$ of dimension $n + n_c$, one can define a vector norm $|x|_*$ such that $|Mx|_* \leq \|M\|_* |x|_*$. One way to define such a norm is the following: Let $O$ denote the zero vector of dimension $n + n_c$, define

$$|x|_* := \left\| \begin{bmatrix} x, O, \ldots, O \end{bmatrix}_{n+n_c-1} \right\|_*,$$

then

$$|Mx|_* = \| [Mx, O, \ldots, O] \|_* \leq \|M\|_* \| [x, O, \ldots, O] \|_* = \|M\|_* |x|_*.$$  

For a vector $\omega$ of dimension $\nu < n + n_c$, denote by $O$ the zero vector of dimension $n + n_c - \nu$, define $|\omega|_* := \left\| \begin{bmatrix} \omega' \\ O' \end{bmatrix} \right\|_*$, then $|\cdot|_*$ is a norm on the vector space $\mathbb{R}^{\nu\times 1}$. We treat a matrix of dimension less than $n + n_c$ in the similar way.

Let $\|\cdot\|_1$ be the induced matrix norm of the vector norm $\|\cdot\|_\infty$, then there exist positive constants $c_1$ and $c_2$ such that $c_1 \|M\|_* \leq \|M\|_1 \leq c_2 \|M\|_*$ for any matrix $M \in \mathbb{R}^{n+n_c}$. Let $\delta := \min\{\delta_1, \delta_2\}$, then $\|M\|_1 \leq \delta$ if $\|M\|_* \leq \delta/c_2$. Hence, in the sequel we concentrate on the matrix norm $\|\cdot\|_*$ and the upper bound $\delta/c_2$. Now we are ready to derive the local stability of the system in Eq. (7). We claim that the stability region contains a ball centered at the origin with radius

$$rd := \min \left\{ \frac{\delta_1}{c_2 \| \begin{bmatrix} 0 & C_d \end{bmatrix} \|_*}, \frac{\delta_2}{c_2 \| \begin{bmatrix} C & 0 \end{bmatrix} \|_*} \right\}$$  \hspace{1cm} (8)

(denoted $\mathcal{B}(0, rd)$).
Suppose $|\eta(0)|_* \leq r_d$, by Eq. (7),

$$|y_c(0)|_* \leq \| \begin{bmatrix} C & 0 \end{bmatrix} \|_* |\eta(0)|_* \leq \frac{\delta_2}{c_2},$$
then

$$\|y_c(0)\|_\infty \leq \delta_2,$$

hence

$$z(0) = H_2(y_c(0), z(k-1)) = z(-1) = 0.$$

Therefore

$$|u_c(0)|_* \leq \| \begin{bmatrix} 0 & C_d \end{bmatrix} \|_* |\eta(0)|_* \leq \frac{\delta_1}{c_2},$$

which means

$$\|u_c(0)\|_\infty \leq \delta_1,$$

and

$$v(0) = H_1(u_c(0), v(k-1)) = v(-1) = 0.$$

Then

$$\eta(1) = \tilde{A} \eta(0).$$

Similarly,

$$|y_c(1)|_* \leq \| \begin{bmatrix} C & 0 \end{bmatrix} \|_* |\eta(1)|_* = \| \begin{bmatrix} C & 0 \end{bmatrix} \|_{\tilde{A}} \|\tilde{A}\|_* |\eta(0)|_* \leq \frac{\delta_2}{c_2},$$

$$\|y_c(1)\|_\infty \leq \delta_2,$$

$$H_2(y_c(1), z(0)) = z(0) = 0.$$

Moreover,

$$|u_c(1)|_* \leq \| \begin{bmatrix} 0 & C_d \end{bmatrix} \|_* |\eta(1)|_* = \| \begin{bmatrix} 0 & C_d \end{bmatrix} \|_{\tilde{A}} \|\tilde{A}\|_* |\eta(0)|_* \leq \frac{\delta_1}{c_2},$$

which means

$$\|u_c(1)\|_\infty \leq \delta_1,$$

and

$$v(1) = H_1(u_c(1), v(0)) = v(0) = 0.$$

Then

$$\eta(2) = \tilde{A} \eta(1) = \tilde{A}^2 \eta(0).$$
implying there is no updating for the inputs to $G$ and $C$. Following this process, we see

$$\eta(k) = \tilde{A}^k \eta(0)$$

converges to zero as $k$ tends to $\infty$. ■

**Remark 1:** Though this system is locally exponentially stable, it is hard to find the exact stability region except for a scalar system controlled by a static feedback. However, even in this scalar case, very complex dynamics can be exposed by the system. This is the topic of the next subsection.

### 2.1 Scalar case

In this part, the definitions of such concepts as (positively) invariant sets, topological transitivity, structural stability, invariant sets and $\omega$–limit sets, etc., are adopted from Robinson [1995] or Robinson [2004] unless otherwise specified.

To get a flavor of the complexity that the system in Fig. 2 may exhibit, we first study a simple one-dimensional system:

\[
x(k+1) = ax(k) + bv(k), \tag{9}
\]

\[
u_c(k) = x(k),
\]

with $v(-1) \in \mathbb{R}$ without loss of generality, and for $k \geq 0$,

\[
v(k) = H_1(u_c(k), v(k-1)) = \begin{cases} u_c(k), & \text{if } |u_c(k) - v(k-1)| > \delta_1, \\ v(k-1), & \text{else}, \end{cases}
\]

where $\delta_1 = 0.01$. The system in Eq. (9) is a static state feedback system with feedback gain equal to 1. Note that in this example there is no constraint on the output of the system $G$. Now let $a = 9/10$ and $b = -3/10$. By choosing different initial values $(v(-1), x(0))$, Figs. 3–4 are obtained. In these two figures, the horizontal axis

![Fig. 3. Two trajectories converging to two different fixed points](image)

stands for the iteration time $k$, and the vertical axis denotes the value of $x$. It is clear from these two figures...
that different initial conditions give rise to significantly different types of trajectories: the first converging to the origin and the second converging to a non-origin point and the last just oscillating. Furthermore, the system in Eq. (9) is actually able to exhibit “chaotic” behavior, i.e., sensitive dependence on initial conditions. Fig. 5 reveals this phenomenon clearly. Is the trajectory in the lower part of Fig. 5 aperiodic? Fig. 6 is its spectrum produced using the function “pmtm” in Matlab. One can see that this trajectory contains a broad band of frequencies.

Next let $a = 1$ and $b = -3/10$, and we get Figs. 7-8 where the horizontal axis denotes $v(k - 1)$ and the vertical axis stands for $x(k)$. The first two (in Fig. 7) are eventually periodic orbits of different periods, the third one (in Fig. 8) is aperiodic.

The complicated behavior of the system in Fig. 2 is due to its nonlinearity. To some extent, invariant sets provide some measure of how complex the dynamics of a system is. According to the above examples, the
invariant sets of the system in Eq. (9) contain not only the origin, non-origin fixed points (Fig. 3), but also periodic (Fig. 7) and aperiodic orbits (Fig. 8). Furthermore, it may contain a strange attractor if chaos is indeed present in the system. In the rest of this subsection, we will analyze the dynamics of this system. We always assume that $|a + b| < 1$ which guarantees the boundedness of trajectories of the system.

### 2.1.1 Case 1: $|a| < 1$

For convenience, define

$$\xi(k) := \begin{bmatrix} v(k-1) \\ x(k) \end{bmatrix},$$

then the system can be written as

$$\xi(k + 1) = \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} \xi(k) + s_k \begin{bmatrix} -1 & 1 \\ -b & b \end{bmatrix} \xi(k)$$

$$:= (A + s_k B) \xi(k) := F(\xi(k)) \, \forall k \geq 0,$$ (10)

and

$$s_k = 1 \text{ if } |x(k) - v(k-1)| > \delta;$$

$$s_k = 0 \text{ if } |x(k) - v(k-1)| \leq \delta.$$ (11)

Based on this representation, the fixed points of the system are the line segment:

$$x = \frac{b}{1 - a} v_-, \quad \text{(12)}$$

within the region:

$$|x - v_-| \leq \delta.$$ (13)
Fig. 7. Two periodic orbits

(Note \( v_- \) indicates that \( v \) is one step behind \( x \).) For the local stability of fixed points, we have the following result.

**Proposition 1** For the system in Eq. (10) with \(|a| < 1\), a local stability region, denoted by \( R_{\text{loc}} \subset \mathbb{R}^2 \), of its fixed points is the region encircled by

\[
|x - v_-| = \delta, \tag{14}
\]

and

\[
|v_-| = \frac{1 - |a|}{1 - (a + b)} \delta. \tag{15}
\]

**Proof:** Given an initial point \((v(-1), x(0)) \in R_{\text{loc}}\), we have

\[
x(1) = ax(0) + bv(-1).
\]

In general,

\[
x(k) = a^k x(0) + \sum_{i=0}^{k-1} a^i b v(-1), \tag{16}
\]

provided that

\[
|x(k) - v(-1)| \leq \delta, \forall k > 0. \tag{17}
\]

Now we show that Eq. (17) indeed holds.

Since

\[
x(k) - v(-1) = a^k x(0) + \sum_{i=0}^{k-1} a^i b v(-1) - v(-1)
\]

\[
= a^k (x(0) - v(-1)) + \left( 1 - a^k \right) \frac{b + a - 1}{1 - a} v(-1),
\]

13
one has

\[ |x(k) - v(-1)| \leq a^k |x(0) - v(-1)| + \left(1 - a^k\right) \frac{1 - (a + b)}{1 - a} |v(-1)|. \]

If \(0 \leq a < 1\), then

\[
|x(k) - v(-1)| \leq a^k \delta + \left(1 - a^k\right) \frac{1 - (a + b)}{1 - a} \frac{1 - a}{1 - (a + b)} \delta
= \delta.
\]

If \(-1 < a < 0\) and \(a^k > 0\), then

\[
|x(k) - v(-1)| \leq a^k \delta + \left(1 - a^k\right) \frac{1 - (a + b)}{1 - a} \frac{1 + a}{1 - (a + b)} \delta
= \left(a^k + \left(1 - a^k\right) \frac{1 + a}{1 - a}\right) \delta
\leq \left(a^k + \left(1 - a^k\right)\right) \delta = \delta.
\]

If \(-1 < a < 0\) and \(a^k < 0\), then

\[
|x(k) - v(-1)| \leq -a^k \delta + \left(1 - a^k\right) \frac{1 - (a + b)}{1 - a} \frac{1 + a}{1 - (a + b)} \delta
= \left(-a^k + \left(1 - a^k\right) \frac{1 + a}{1 - a}\right) \delta
= \frac{1 + a - 2a^k}{1 - a} \delta.
\]

Therefore it suffices to show that

\[
\frac{1 + a - 2a^k}{1 - a} \leq 1.
\]
However, it is equivalent to

\[ a \leq a^k, \]

which holds for \(-1 < a < 0\) and \(a^k < 0\). By taking limit in Eq. (16) with respect to \(k\), \((v(k-1), x(k))\) converges to a fixed point defined by Eqs. (12)-(13). The proof is completed.

Having identified a local stability region, next we will study the following problem: Can the actual stability region of the fixed points be larger than the region given in Proposition 1? We will see that this problem is actually a difficult one in that it heavily depends on system parameters. Before doing so, we first concentrate on the “one-dimensional case”, i.e., the dynamics of \(x\), to get the globally attracting region of \(x\) of the system in Eq. (9). We have the following result.

**Proposition 2** The globally attracting region of \(x\) is given by

\[ |x| \leq \frac{|b|}{1 - |a + b|} \delta. \]  

Furthermore, it is positively invariant.

**Proof:** According to Eq. (9),

\[
\begin{align*}
x(1) &= ax(0) + bv(0) = (a + b)x(0) + b(v(0) - x(0)), \\
x(2) &= ax(1) + bv(1) = (a + b)^2x(0) + (a + b)b(v(0) - x(0)) + b(v(1) - x(1)), \\
&\vdots \\
x(k) &= (a + b)^kx(0) + \sum_{i=0}^{k-1} (a + b)^i b(v(k-1-i) - x(k-1-i)),
\end{align*}
\]

hence

\[ |x(n)| \leq |a + b|^n |x(0)| + \frac{1 - |a + b|^n}{1 - |a + b|} |b| \delta, \quad \forall n \geq 1. \]  

(19)

By taking limit on both sides, one gets Eq. (18). Moreover, if

\[ |x(0)| \leq \frac{|b|}{1 - |a + b|} \delta, \]

then

\[ |x(n)| \leq \frac{|b|}{1 - |a + b|} \delta, \quad \forall n \geq 1, \]

which means that the region given by Eq. (18) is positively invariant.

Based on this observation, we are ready to derive a positive invariant set for the system in Eq. (10).
then region defined by

\[ |x| \leq \frac{|b|}{1 - |a + b|} \delta \]

and

\[ |v_-| \leq \frac{|b|}{1 - |a + b|} \delta \]

is a positively invariant set. Otherwise, the region defined by

\[ |x| \leq \frac{1 - |a|}{1 - (a + b)} \delta \]

and

\[ |v_-| \leq \frac{1 - |a|}{1 - (a + b)} \delta \]

is globally attracting, which indicates that the fixed points given by Eqs. (12)-(13) are the only invariant set of the system (For convenience, we call such a system a generic system).

**Proof:** It readily follows from Proposition 1 and Proposition 2. ■

The following result is an immediate consequence of Theorem 1.

**Corollary 1** If the system in Eq. (10) satisfies either of

- \( a > 0 \) and \( b > 0 \);
- \( a < 0 \) and \( b < 0 \),

then it is a generic system.

**Proof:** Suppose \( a > 0 \) and \( b > 0 \). Then

\[
\frac{|b|}{1 - |a + b|} = \frac{b}{1 - (a + b)} \leq \frac{1 - a}{1 - (a + b)} = \frac{1 - |a|}{1 - (a + b)}.
\]

Hence the system is generic. On the other hand, given \( a < 0 \) and \( b < 0 \),

\[
\frac{|b|}{1 - |a + b|} = \frac{-b}{1 + (a + b)}, \quad \frac{1 - |a|}{1 - (a + b)} = \frac{1 + a}{1 - (a + b)}.
\]

Since

\[
\frac{-b}{1 + (a + b)} \leq \frac{1 + a}{1 - (a + b)}
\]

is equivalent to

\[ a^2 \leq 1 + b^2, \]

which says

\[ \frac{|b|}{1 - |a + b|} \leq \frac{1 - |a|}{1 - (a + b)}. \]
i.e., the system is generic.

Theorem 1 tells us that, in order to have complex dynamics,

\[
\frac{|b|}{1 - |a + b|} > \frac{1 - |a|}{1 - (a + b)}
\]

must be satisfied. However, this is not a sufficient condition. For the case when

\[
a = \frac{9}{10}, \quad b = -\frac{3}{10},
\]

(which satisfies Eq. (20)), we have already known that the system exhibits complicated dynamics (see Figs. 3-6). However, for the case when

\[
a = \frac{3}{10}, \quad b = -\frac{9}{10},
\]

which also satisfies Eq. (20), there is no complex dynamic behavior, i.e., the system is generic. The following argument provides a sufficient proof for this specific system.

Given \((v (-1), x (0))\) satisfying

\[
|x (0) - v (-1)| > \delta,
\]

one has

\[
x (1) = (a + b) x (0), \quad v (0) = x (0).
\]

Suppose

\[
|x (1) - v (0)| > \delta,
\]

then

\[
|x (0)| > \frac{\delta}{1 - (a + b)}, \tag{21}
\]

and

\[
x (2) = (a + b) x (1) = (a + b)^2 x (0), \quad v (1) = x (1) = (a + b) x (0).
\]

If

\[
|x (2) - v (1)| \leq \delta, \tag{22}
\]

and

\[
|v (1)| \leq \frac{1 - |a|}{1 - (a + b)} \delta,
\]

then the trajectory will converge to some fixed point. Meanwhile,

\[
|x (0)| \leq \frac{1 - |a|}{1 - (a + b)} \frac{1}{|a + b|} \delta. \tag{23}
\]
Note that Eq. (22) holds given Eq. (21). Therefore, only
\[
\frac{\delta}{1 - (a + b)} \leq \frac{1 - |a|}{1 - (a + b)} \frac{1}{|a + b|} \delta 
\] (24)
is required. Moreover, Eq. (24) is equivalent to
\[-b \leq 1.\] (25)

Systems with
\[a = 3/10, b = -9/10,\] (26)
and
\[a = 8/10, b = -9/10,\] (27)
both satisfy Eq. (25). However, for a sufficiently large time \(k\), any trajectory \((v(k - 1), x(k))\) governed by Eq. (26) will satisfy
\[|x(k) - v(k - 1)| > \delta,
\]
and
\[|x(k)| > \frac{\delta}{1 - (a + b)}.
\]
consequently, it will converge to a fixed point. However, any trajectory \((v(k - 1), x(k))\) governed by Eq. (27) violates these two conditions, predicting complex dynamics, see Fig. 11 below.

We have already analyzed three cases:

- \(a > 0\) and \(b > 0\);
- \(a < 0\) and \(b < 0\);
- \(a > 0\) and \(b < 0\).

What about the case when \(a < 0\) and \(b > 0\)? Next we will prove that such a system is generic. It is easy to see that the transition matrix of the system in Eq. (9) is some combination of \((a + b)^k\) and \((a^m + \sum_{i=0}^{m-1} a^i b)\) with scalar multiplication as the involved operation, where \(k \geq 0\) and \(m > 1\), since \(|a + b| < 1\), if
\[\left| a^m + \sum_{i=0}^{m-1} a^i b \right| < 1 \] (28)
for all \(m > 1\), the state \(x\) will tend to the origin unless it reaches a fixed point. In this case, the system is generic. Via simple manipulation, Eq. (28) is equivalent to
\[0 \leq (1 - a^m) \frac{1 - (a + b)}{1 - a} \leq 2.\] (29)
Given \(a < 0\) and \(b > 0\), define
\[f(m) := (1 - a^m) \frac{1 - (a + b)}{1 - a}, \quad \forall m > 1,\]
then
\[ f(m) \geq 0, \quad \forall m > 1, \]
and
\[ f(3) = \max_{m > 1} f(m). \]
However,
\[
\begin{align*}
f(3) - 2 &= (1 - a^3) \frac{1 - (a + b)}{1 - a} - 2 \\
&= (1 + a + a^2) (1 - (a + b)) - 2 \\
&\leq 0.
\end{align*}
\]
hence, Eq. (29) (then Eq. (28)) holds for all \(m > 1\), which means the system is generic.

In the rest of this part, we will concentrate on a specific system and study its complexity. Consider the system in Fig. 2, where

\[
x(k + 1) = ax(k) + bv(k),
\]
\[
u_c(k) = x(k),
\]
with \(a = 0.9\), \(b = -0.3\), \(v(-1) \in \mathbb{R}\), and for \(k \geq 0\),
\[
v(k) = H_1(u_c(k), v(k - 1)) = \begin{cases} u_c(k), & \text{if } |u_c(k) - v(k - 1)| > 0.01, \\
v(k - 1), & \text{else.} \end{cases}
\]

Fig. 9. Diagram for the case of \(a = 0.9\) and \(b = -0.3\)
We make the following definitions (Fig. 9):

\[
L_{Hb^+} := \left\{ \left( v_-, \frac{-b}{1 - |a + b|} \delta \right) : |v_-| \leq \frac{-b}{1 - |a + b|} \delta \right\},
\]

\[
L_{Hb^-} := \left\{ \left( v_-, \frac{b}{1 - |a + b|} \delta \right) : |v_-| \leq \frac{-b}{1 - |a + b|} \delta \right\},
\]

\[
L_{Vb^+} := \left\{ \left( \frac{-b}{1 - |a + b|}, x \right) : |x| \leq \frac{-b}{1 - |a + b|} \delta \right\},
\]

\[
L_{Vb^-} := \left\{ \left( \frac{b}{1 - |a + b|}, x \right) : |x| \leq \frac{-b}{1 - |a + b|} \delta \right\},
\]

\[
I_b := \left\{ (v_-, x) : |v_-| \leq \frac{-b}{1 - |a + b|} \delta, |x| \leq \frac{-b}{1 - |a + b|} \delta \right\}.
\]

\[
L_{a^+} := \left\{ \left( \frac{1 - |a|}{1 - (a + b)}, x \right) : |x| \leq \frac{-b}{1 - (a + b)} \delta \right\},
\]

\[
L_{a^-} := \left\{ \left( -\frac{1 - |a|}{1 - (a + b)}, x \right) : |x| \leq \frac{-b}{1 - (a + b)} \delta \right\},
\]

\[
I_a := \left\{ (v_-, x) \in I_b : |v_-| \leq \frac{1 - |a|}{1 - (a + b)} \delta, |x| \leq \frac{-b}{1 - |a + b|} \delta \right\}.
\]

\[
L_\delta^+ := \left\{ (v_-, x) \in I_b : x - v_- = \delta \right\},
\]

\[
L_\delta^- := \left\{ (v_-, x) \in I_b : x - v_- = -\delta \right\},
\]

\[
L_{18^+} := \left\{ (v_-, x) \in I_b : x = (a + b) v_-, v_- > \frac{1 - |a|}{1 - (a + b)} \delta \right\},
\]

\[
L_{18^-} := \left\{ (v_-, x) \in I_b : x = (a + b) v_-, v_- < -\frac{1 - |a|}{1 - (a + b)} \delta \right\}.
\]

Clearly, \( L_\delta \) is the set of equilibria, \( I_a \) is a local stability region of \( L_\delta \), and \( I_b \) is a globally attracting region and is also positively invariant. Denote the two endpoints of \( L_\delta \) by \( E^+ \) and \( E^- \), i.e., \( E^+ = \left( -\frac{1 - |a|}{1 - (a + b)}, 1 - |a| \delta + \delta \right) \) and \( E^- = \left( \frac{1 - |a|}{1 - (a + b)}, \frac{1 - |a|}{1 - (a + b)} \delta - \delta \right) \). Define

\[
E_s := L_\delta \setminus \{ E^+, E^- \} := \left\{ (v_-, x) \in L_\delta : (v_-, x) \notin \{ E^+, E^- \} \right\}.
\]

Then each point in \( E_s \) is stable in the sense of Lyapunov, however it is not asymptotically. As for the stability of \( E^+ \) (resp. \( E^- \)), each trajectory starting from a point in \( I_b \) on \( v_- = \frac{1 - |a|}{1 - (a + b)} \delta \) (resp. \( v_- = -\frac{1 - |a|}{1 - (a + b)} \delta \)) will converge to \( E^+ \) (resp. \( E^- \)). How about trajectories starting from points in \( I_b \)? It turns out that they never converge to \( E^+ \) (or \( E^- \)); therefore the two equilibria \( E^+ \) (resp. \( E^- \)) are not stable. To wit, we need more preparations.

For convenience, we regard the system in Eq. (30) as a map, i.e., adopt the notation defined in Eq. (10):

\[
\xi (k + 1) = F (\xi (k)).
\]

20
Given a set \( \Omega \subset I_b \), define

\[
\text{pre}^n (\Omega) := \{(v_-, x) \in I_b : F^n ((v_-, x)) \subset I_b\}, \quad \forall n \geq 0, \tag{31}
\]

where \( F^0 ((v_-, x)) = (v_-, x) \), iteratively \( F^n ((v_-, x)) = F^{n-1} ((v_-, x)) \) for \( n \geq 1 \).

Then

\[
L_{\delta_-} \setminus \left\{ \left( -\frac{1-|a|}{1-(a+b)} \delta, -\frac{1-|a|}{1-(a+b)} \delta - \delta \right) \right\} \subset \text{pre}^2 (L_{\delta_-}),
\]

\[
L_{\delta_-} \setminus \left\{ \left( \frac{1-|a|}{1-(a+b)} \delta, \frac{1-|a|}{1-(a+b)} \delta - \delta \right) \right\} \subset \text{pre}^2 (L_{\delta_-}).
\]

Based on this observation, we have

\[
F (I_b \setminus \{LV_a^- \cup LV_a^+\}) \subset I_b \setminus \{LV_a^- \cup LV_a^+\},
\]

i.e., \( I_b \setminus \{LV_a^- \cup LV_a^+\} \) is positively invariant. As a result, trajectories starting from points in \( I_b \setminus \{LV_a^- \cup LV_a^+\} \), no matter how close to \( E^+ \) (resp. \( E^- \)) they are, will not converge to \( E^+ \) (resp. \( E^- \)), indicating that neither \( E^+ \) nor \( E^- \) is locally stable.

Moreover, for a given set \( \Omega \subset I_b \setminus \{LV_a^- \cup LV_a^+\} \), define

\[
\text{Img}^n (\Omega) := \{ F^n (\Omega) \},
\]

\[
\Psi (\Omega) := \cup_{n=0}^{\infty} \text{Img}^n (\Omega),
\]

then it is easy to verify that

\[
F (\Psi (L_{\delta+})) \subset \Psi (L_{\delta+}),
\]

\[
F (\Psi (L_{\delta-})) \subset \Psi (L_{\delta-}),
\]

which furthermore imply all trajectories starting within \( I_b \setminus \{LV_a^- \cup LV_a^+\} \) will eventually move along the line segment \( \Psi (L_{\delta+}) = \Psi (L_{\delta-}) \). For a point \( \xi \in I_b \setminus \{LV_a^- \cup LV_a^+\} \), let \( \omega (x) \) denote its \( \omega \)-limit set, define

\[
\omega (I_b \setminus \{LV_a^- \cup LV_a^+\}) := \{ \cup_{\xi \in I_b \setminus \{LV_a^- \cup LV_a^+\}} \omega (\xi) \},
\]

then

\[
\omega (I_b \setminus \{LV_a^- \cup LV_a^+\}) \subset \Psi (L_{\delta+}).
\]

Obviously

\[
\omega (I_a) = L_o.
\]

Thus we get a characterization of the \( \omega \)-limit sets of the system in Eq. (30). However, we have to admit that this characterization is somewhat crude because all trajectories starting within \( I_b \setminus \{LV_a^- \cup LV_a^+\} \) will eventually move along merely a part of each line segment in \( \Psi (L_{\delta+}) \) instead of the whole line segment. Fig.
10 given later will visualize this observation. Now the problem of finding the exact \( \omega \)-limit set of the system is still under our study. Nevertheless, adopting the argument on pp. 24 in Robinson [1995], it is easy, though not straightforward due to the nature of the map \( F \), to show that this \( \omega \)-limit set is indeed invariant. By extensive simulation, we find that this \( \omega \)-limit set is also topological transitive, however, up to now we have not been able to build solid theoretic background to support it.

Based on the above analysis, it is fair to say that the dynamics of the system in Eq. (30) is remarkably complicated: It indeed exhibits the feature of sensitive dependence on initial conditions, this sensitivity locates only on \( \cup_{n=0}^{\infty} (\text{pre}^n (L_{\delta_-}) \cup \text{pre}^n (L_{\delta_+})) \), a subset of \( \cup_{n=0}^{\infty} (\text{pre}^n (L_{1\delta_-} \cup L_{1\delta_+})) \). Hence it is weakly chaotic. Next we will calculate its generalized topological entropy in the spirit of Kopf [2000] and Galatolo [2003].

Denote by \( I_{\text{binv}+} \) the region encircled by the lines \( L_{V a+}, L_{V b+}, L_{1\delta+}, \text{Img}^1(L_{\delta-}) \). Similarly denote the region encircled by the lines \( L_{V a-}, L_{V b-}, L_{1\delta-}, \text{Img}^1(L_{\delta+}) \) by \( I_{\text{binv}-} \), based on the above analysis, we have the following claim:

**Claim 1:** The steady state of the system will settle in the region \( I_{\text{binv}+} \cup I_{\text{binv}-} \).

This claim is a straightforward application of the foregoing analysis, however it plays an important role in the calculation of the topological entropy of the system.

For the definition of topological entropy for piecewise monotone transformations with discontinuities, please refer to Kopf [2000]. Now we will give a construction in order to compute the topological entropy for our system, which is clearly piecewise monotone (under some metric defined on the system rather than under the usual Euclidean metric; however, this is not essential) with discontinuities.

Define

\[
P_{\text{imF}}(0) := \{L_{\delta-}, L_{\delta+}\},
\]

\[
P_{\text{imF}}(1) := \{L : L \cap P_{\text{imF}}(0) = \phi, F(L) \subset P_{\text{imF}}(0)\},
\]

\[
\vdots
\]

\[
P_{\text{imF}}(m) := \{L : L \cap (\cup_{i=0}^{m-1} P_{\text{imF}}(0)) = \phi, F^m(L) \subset P_{\text{imF}}(0)\}, \quad m \geq 1,
\]

(32)

where \( \phi \) stands for the empty set. Note that the elements in each \( P_{\text{imF}}(m) \) are line segments.

Denote by \( \#(P_{\text{imF}}(m)) \) the number of elements in \( P_{\text{imF}}(m) \).

Before calculating the topological entropy, we need to pay a bit more attention to the mapping \( F \). Clearly, according to Fig. 9 there exists a positive integer \( M \) such that

\[
\text{pre}^{M+l}(L_{\delta+}) \cap L_{1\delta-} \neq \phi,
\]

\[
\text{pre}^{M+l}(L_{\delta-}) \cap L_{1\delta+} \neq \phi, \quad \forall l \geq 1.
\]

(33)

In fact, each set of intersections contains exactly one element (one line segment). For each given integer \( n > 0 \), define

\[
\#F(n) := \sum_{m=0}^{n} \#(P_{\text{imF}}(m)),
\]

22
and define the topological entropy of $F$ as

$$\mathcal{H}(F) := \lim_{n \to \infty} \frac{\log \#F(n)}{n},$$

which is well-defined (see the proof below). Then we have

**Theorem 2** For the system in Eq. (30), the following statements hold:

- For $m \leq M$, 
  $$\#(\text{Pim}_{F}(m)) = 2.$$ 
  (35)

- For $m > M$, 
  $$\#(\text{Pim}_{F}(m)) = 2 + 2 \cdot (m - M),$$ 
  (36)
  and
  $$\mathcal{H}(F) = 0.$$ 
  (37)

**Proof:** Eq. (35) is self-evident, Eq. (36) follows from Claim 1 restricting $\#(\text{Pim}_{F}(m))$ on $I_{\text{inv}+} \cup I_{\text{inv}-}$ for $m > M$ and the analysis above. Then for sufficiently large $n (n \geq M)$,

$$\# F(n) = 2(M + 1) + 2\frac{(n-M)(n-M+1)}{2},$$

thus

$$\mathcal{H}(F) := \lim_{n \to \infty} \frac{\log \#F(n)}{n} \quad \quad = \quad \quad \lim_{n \to \infty} \frac{\log \left(2(M + 1) + (n-M)(n-M+1)\right)}{n} \quad \quad = \quad \quad 0.$$ 

**Remark 2:** In light of this result, from the perspective of topological entropy, our system is a weakly chaotic system.

The above discussion is mainly for the case of $|b| < a$. For example, given $a = 0.9$ and $b = -0.3$, Fig. 10 plots a trajectory at large time instants, i.e., its asymptotic behavior. Now consider the case when $a = 0.8$ and $b = -0.9$, hence $|b| < a$, and we also draw its asymptotic behavior in Fig. 11 from the same initial point. We observe that their asymptotic behavior is different. The reason is still unclear up to now.
Fig. 10. A trajectory at large time instants for $|b| < a$

### 2.1.2 Case 2: $a = 1$

Consider the system

\[
\begin{align*}
    x(k+1) &= x(k) + bv(k), \\
    u_c(k) &= x(k),
\end{align*}
\]

where $|1 + b| < 1$. Let $v(-1) \in \mathbb{R}$, and for $k \geq 0$,

\[
v(k) = H_1(u_c(k), v(k-1)) = \begin{cases} 
    u_c(k), & \text{if } |u_c(k) - v(k-1)| > 0.01, \\
    v(k-1), & \text{otherwise}. 
\end{cases}
\]

Figs. 7-8 show that the dynamics of the system in Eq. (38) can be fairly complicated. In the rest of this subsection we mainly study the problem when the system will have periodic orbits. From now on, we assume $-1 \leq b < 0$.

For this case, $L_{V_{a-}}$, $L_{V_{a+}}$ and $L_o$ in Fig. 9 now become one line segment

\[
L := \{(0, x) : |x| \leq \delta\}.
\]

Define

\[
\Gamma_{in} := \{(v_-, x) \in I_b : |x - v_-| \leq \delta\},
\]

and

\[
\Gamma_{ex} := I_b \setminus \Gamma_{in},
\]

the following is a necessary condition for the existence of periodic orbits.
**Theorem 3** If the system in Eq. (38) has periodic orbits, then there exist an even integer \( n > 0 \) and integers \( K_i > 0 \) such that

\[
\prod_{i=1}^{n} (1 + K_i b) = 1. \tag{39}
\]

Without loss of generality, we here prove the case of \( n = 2 \). The following Lemma is used in the proof of Theorem 3:

**Lemma 2** Suppose \( \xi_0 \in \Gamma_{ex} \) is a point on a periodic orbit at time \( K_0 \), it will be inside \( \Gamma_{in} \) at \( K_0 + 1 \).

**Proof:** Let \( \xi_0 = \begin{bmatrix} v \vline x \end{bmatrix} \). Then

\[
\xi_1 = \begin{bmatrix} v_1 \\ x_1 \end{bmatrix},
\]

where,

\[
x_1 = (a + b)x,
\]

\[
v_1 = x.
\]

Hence

\[
|x_1 - v_1| = |1 - (a + b)||x| = |b||x|.
\]

Since \( \begin{bmatrix} v \vline x \end{bmatrix} \) is the steady state (a periodic point),

\[
|x| \leq \frac{|b| \delta}{1 - (a + b)} = \delta.
\]
One obtains
\[ |x_1 - v_1| = |1 - (a + b)|x| \leq |b|\delta \leq \delta, \]
i.e., \( \xi_1 \in \Gamma_{in} \).

Note that Lemma 2 is not trivial because there are systems such as the case when \( a = 3/10 \) and \( b = -9/10 \) violating this property.

**Proof of Theorem 3:** Without loss of generality, suppose the periodic orbit begins with \( \xi_0 \in \Gamma_{ex} \), and by Lemma 2,
\[ \xi_1 = A_1 \xi_0 \in \Gamma_{in}, \tag{40} \]
where
\[ A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 + b \end{bmatrix}. \]
Assume after a time \( K_1 \) the state
\[ \xi_2 = (A_1 + A_2)^{K_1-1} \xi_1 \]
is in \( \Gamma_{ex} \), where
\[ A_2 = \begin{bmatrix} 1 & -1 \\ b & -b \end{bmatrix}. \]
Then
\[ \xi_3 = A_1 \xi_2 \in \Gamma_{in}. \]
After a time \( K_2 \) the state
\[ \xi_4 = (A_1 + A_2)^{K_2-1} \xi_3 \]
returns to \( \xi_0 \). Then
\[ (A_1 + A_2)^{K_2-1} A_1 (A_1 + A_2)^{K_1-1} A_1 \xi_0 = \xi_0. \tag{41} \]
By straightforward algebraic computations, one gets
\[ d\! \! \! / \! \! \! / x = x, \tag{42} \]
\[ d\! \! \! / \! \! \! / x = v_-, \]
where
\[ d = 1 + K_2b, \ c = 1 + K_1b. \]
Since \( \xi_0 \) is outside the sector,
\[ |x - v_-| > \delta. \]
According to Eq. (42), note that \( c \neq 0, d \neq 0 \), then
\[ cd = 1. \tag{43} \]
Given \( a = 1 \),
\[
K_1 + K_2 = K_1K_2 (-b),
\]
which is equivalent to Eq. (39) for \( n = 2 \).
\[\Box\]

**Remark 3:** Theorem 3 provides a necessary condition for having periodic orbits. Interestingly for the case of \( n = 2 \), extensive experiments imply that there are periodic orbits of period \( K_1 + K_2 \) if \( K_1 \) and \( K_2 \) satisfy Eq. (39), and there are no periodic orbits if there are no such \( K_1 \) and \( K_2 \) that satisfy Eq. (39). Based on the observation, Theorem 3 is not severely conservative.

Following the above analysis, we immediately have

**Corollary 2** Suppose \( a \) is rational and \( b \) is irrational in Eq. (10), then there are no periodic orbits.

**Proof:** Following the proof above, it suffices to show that
\[
\prod_{i=1}^{n} \left( a^{K_i} + \sum_{j=0}^{K_i-1} a^j b \right) = 1
\]
has no positive integer solutions for any given even number \( n > 0 \). This can be easily verified.

\[\Box\]

Though the above result is simple, its significance can not be underestimated: If a system has periodic orbits, then it is not structurally stable.

Now suppose a system has periodic orbits, how to find them? And how to determine their periods? we first consider an example.

**Example 1:** In the case when \( a = 1 \) and \( b = -0.3 \), there are two periodic solutions. One is of period 24 corresponding to \( K_1 = 4 \) and \( K_2 = 20 \), the other is of period 15 corresponding to \( K_1 = 5 \) and \( K_2 = 10 \). Observe that
\[
-b = 0.3 = \frac{3}{2 \cdot 5} = \frac{p}{q_1 \cdot q_2},
\]
where \( p = 3 \), \( q_1 = 2 \), \( q_2 = 5 \). Interestingly
\begin{align*}
4 & = \frac{q_2 + 1}{p}q_1, \quad 20 = \frac{q_2 + 1}{p}q_1q_2, \\
5 & = \frac{q_1 + 1}{p}q_2, \quad 10 = \frac{q_1 + 1}{p}q_1q_2.
\end{align*}
Based on this observation, we propose a necessary condition for Theorem 3 for the case when \( n = 2 \).

**Theorem 4** Given
\[
a = 1 \quad \text{and} \quad b = -\frac{p}{q},
\]
suppose positive integers \( p \) and \( q \) satisfy \( 1 < p < q \), \( p \neq 2 \) (which is the trivial case), and \( \gcd(p, q) = 1 \), i.e., the greatest common divisor of \( p \) and \( q \) is 1. Define
\[
\Delta := \{ q_i : q_i \text{ is a prime number, } q_i | q \}.
\]

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Then if $p \mid (q_i + 1)$,

$$
\left( \frac{q_i + 1}{p}, \frac{q_i + 1}{q} \right)
$$

is a solution of Eq. (37).

**Proof:** Obviously, given $p \mid (q_i + 1)$, $\left( \frac{q_i + 1}{p}, \frac{q_i + 1}{q} \right)$ is a solution of Eq. (39). Now we show how the set $\Delta$ is constructed in the above way. Given $q_i \in \Delta$, If there are two positive numbers $m$ and $n$ satisfying

$$
\frac{1}{m} + \frac{1}{n} = \frac{p}{q_i}, \tag{47}
$$

then $(m \frac{q}{q_i}, mq)$ is a solution to Eq. (39). Hence we need only to pay attention to solutions to Eq. (47). Suppose $(m, n)$ is a solution of Eq. (47), then either gcd $(m, q_i) = 1$ or gcd $(n, q_i) = 1$ (otherwise, $p = 2$ or does not exist). For convenience, we always assume gcd $(m, q_i) = 1$. According to Eq. (47),

$$\frac{m + n}{nm} = \frac{p}{q_i},$$

i.e.,

$$(m + n) q_i = pmn.$$

Then $q_i \mid pmn$. Since gcd $(p, q_i) = \gcd (m, q_i) = 1$, $q_i \mid n$. Let

$$n = kq_i, \tag{48}$$

which leads to

$$\frac{1}{m} + \frac{1}{kq_i} = \frac{p}{q_i}.$$ 

Consequently,

$$m (pk - 1) = kq_i,$$

hence $m \mid kq_i$. Since gcd $(m, q_i) = 1$, $m \mid k$. In light of Eq. (48), we set

$$n = ml.$$

Substituting it into Eq. (47), one has

$$\frac{1}{m} + \frac{1}{ml} = \frac{p}{q_i},$$

equivalently,

$$mpl = q_i (l + 1),$$

which means $q_i \mid mpl$. i.e., $q_i \mid l$. Similarly, $l \mid q_i (l + 1)$, hence $l = q_i$. Therefore,

$$m = \frac{q_i + 1}{p}.$$

If $p \mid (q_i + 1)$, then

$$\left( \frac{q_i + 1}{p}, \frac{q_i + 1}{q} \right)$$

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solves Eq. (47), and

\[ \left( \frac{q_i + 1}{p}, \frac{q_i + 1}{p} \right) \]

is a solution of Eq. (39).

The above theorem provides a construction for the solutions to Eq. (39). However, this is somewhat inadequate. For example, for \( a = 1 \) and \( b = -\frac{3}{7} \), the set \( \Delta \) is empty. There are no positive integers satisfying Eq. (39) for \( n = 2 \) either. This is good for us. However for \( a = 1 \) and \( b = -\frac{3}{20} \cdot 5 \cdot 11 \), we have the following observations (Table 1):

| \( (v_-, x_0) \) | \( \left( \frac{7}{1000}, 0 \right) \) | \( \left( \frac{1}{1000}, 0 \right) \) | \( \left( \frac{1}{2000}, 0 \right) \) |
|---|---|---|---|
| Periods | 165 | 147 | 243 |
| \( (v_-, x_0) \) | \( \left( \frac{1}{2000}, \frac{1}{2000} \right) \) | \( \left( \frac{1}{2000}, \frac{8000}{2000} \right) \) | \( \left( \frac{1}{3000}, 0 \right) \) |
| Periods | 264 | 480 | 243 |

Periods 165, 264 and 480 can be obtained based on Theorem 4, however, others can not. Actually there are more periodic and aperiodic orbits (Table 2):

| \( (v_-, x_0) \) | \( \left( \frac{\delta}{100}, \frac{(a+b)\delta}{100} \right) \) | \( \left( \frac{\delta}{10}, \frac{(a+b)\delta}{10} \right) \) | \( \left( \frac{5\delta}{10}, \frac{5(a+b)\delta}{10} \right) \) | \( \left( \frac{10\delta}{10}, \frac{2(a+b)\delta}{10} \right) \) |
|---|---|---|---|
| Periods | 4107 | 264 | 243 | 165 |
| \( (v_-, x_0) \) | \( \left( \frac{5\delta}{10}, \frac{4(a+b)\delta}{10} \right) \) | \( \left( \frac{8\delta}{20}, \frac{9(a+b)\delta}{20} \right) \) | \( \left( \frac{5\delta}{10}, \frac{5(a+b)\delta}{10} \right) \) | \( \left( \frac{10\delta}{10}, \frac{6(a+b)\delta}{10} \right) \) |
| Periods | aperiodic | aperiodic | 147 | aperiodic |
| \( (v_-, x_0) \) | \( \left( \frac{5\delta}{10}, \frac{7(a+b)\delta}{10} \right) \) | \( \left( \frac{8\delta}{10}, \frac{8(a+b)\delta}{10} \right) \) | \( \left( \frac{9\delta}{10}, \frac{9(a+b)\delta}{10} \right) \) | \( \left( \frac{10\delta}{10}, \frac{10(a+b)\delta}{10} \right) \) |
| Periods | 165 | 165 | 243 | 4107 |

Furthermore, we observed that there are at least 55 solutions to Eq. (39) for \( n = 4 \).

The foregoing analysis tells us:

- There may exist periodic orbits of very large periods.
- There are always aperiodic orbits.

Inspired by the proof of Corollary 2, especially by Eq. (45), now we attempt to construct systems with \( |a| < 1 \) that have periodic orbits. First choose \( n = 2, a = 9/10 \), choose \( K_1 = 15 \) and \( K_2 = 7 \). Then
solves Eq. (45). Surprisingly the trajectory starting from

\[(v_-, x_0) = \left( \frac{-b \cdot \delta}{1 - |a + b|} - \frac{1}{10^4}, 0 \right)\]

will become a periodic orbit of period 22\((= K_1 + K_2)\) after some iterations, i.e., it is an eventually periodic orbit. It can be shown this periodic orbit is locally stable. However, a trajectory starting outside the stability region, say, from

\[(v_-, x_0) = \left( \frac{-b \cdot \delta}{1 - |a + b|} - \frac{1}{10^4}, \frac{1}{10^4} \right)\]

is aperiodic. For the case when \(|a| > 1\), suppose \(a = 11/10\), choose \(K_1 = 7\) and \(K_2 = 5\). Then

\[b = -\frac{2138428376721}{5792012767210}\]

solves Eq. (45). And the trajectory starting from

\[(v_-, x_0) = \left( \frac{\delta}{10^5}, \frac{(a + b) \cdot \delta}{10^5} \right)\]

will become a periodic orbit of period 12\((= K_1 + K_2)\) after some iterations, i.e., it is an eventually periodic orbit. It can be shown this periodic orbit is also locally stable and there are aperiodic orbits too.

**Remark 4:** From this construction, one finds out that most systems with \(|a| < 1\) or \(|a| > 1\) will be unlikely to have periodic orbits.

### 2.1.3 Case 3: \(|a| > 1\)

This case is analogous to that of \(|a| < 1\) except that all the fixed points are unstable. The Fig. 12 is one trajectory at sufficiently large time instants.

The complex dynamics exhibited by our system is due to its nonlinearity. This is different from a quantized system. The complicated behavior of an unstable quantized scalar system is extensively studied in Delchamps [1988, 1989, 1990], Fagnani & Zampieri [2003], etc. In Delchamps [1990], it is mentioned that given that the system parameter \(a\) is stable, a quantized system may have many fixed points as well as many periodic orbits which are all asymptotically stable. However, for our constrained systems, most of them will not possess periodic orbits. For the systems with \(a = 1\), periodic orbits are locally stable, this is not the case of for a quantized system (Delchamps [1990]). Given that \(a\) is unstable, the ergodicity of the quantized system is studied in Delchamps [1990]. In essence, related results there depend heavily on the affine representation of the system by which the system is piecewise expanding, i.e., the absolute value of derivative of the piecewise affine map in each interval is greater than 1. Based on this crucial property, the main theorem (Theorem 1) in Lasota & Yorke [1973] and then that of Li & Yorke [1978] are employed to show there exists a unique invariant
measure under the affine mapping and which is also ergodic with respect to that mapping. Therefore ergodicity is established for scalar unstable quantized systems. However, this is not the case for our system. Though the system is piecewise linear, it is singular with respect to the Lebesgue measure and furthermore, the derivative of the system in the region Γ_{in} is (a + b), whose absolutely value is strictly less than 1. Hence the results in Lasota & Yorke [1973] and Li & Yorke [1978] are not applicable here. By extensive experiments, we strongly believe that the system indeed has the property of ergodicity, however, the problem still remains open.

To appreciate what qualitative behavior of a higher dimensional system can have, we give the following example.

**Example 2:** Suppose the system $G$ in Fig. 2 is given by

\[
\begin{align*}
    x(k+1) &= 2x(k) + 3v(k), \\
    y_c(k) &= x(k),
\end{align*}
\]

and the controller $C$ is given by

\[
\begin{align*}
    x_d(k+1) &= -2x_d(k) + 1.5e_c(k), \\
    u_c(k) &= x_d(k), \\
    e_c(k) &= r(k) - z(k),
\end{align*}
\]

where $v(k)$ and $z(k)$ are outputs of Eqs. (4)-(5) respectively. We set $r \equiv 0$. We call the resulting system $\Sigma_\circ$. It is easy to see that the closed-loop system without the constraints $H_1$ and $H_2$ is asymptotically stable. Under $H_1$ and $H_2$, three figures, Figs. 13–15, are drawn. The first is for $(v(k-1), x(k))$, the second for $(z(k-1), x_d(k))$ and the last for $(x(k), x_d(k))$. 

Fig. 12. A trajectory at large time instants
Next we analyze the chaotic behavior of system $\Sigma_o$ using nonlinear data analysis. First we show sensitive dependence on initial conditions. Choose an initial condition

$$[v(-1), x(0), z(-1), x_d(0)] = [-1/1000, 1/1000, 2/1000, -1/1000],$$

set the iteration number to be 600000, then we get trajectory of $x$; perturb the initial condition above slightly to $[-1/1000, 1/1000 + 1/10^{13}, 2/1000, -1/1000]$, under the same iteration, we get another trajectory of $x$, the following plot (Fig.16) is the difference between these two $x$ of the last 1200 points of the iteration: From this figure, one can clearly see sensitive dependence on initial conditions. In general, the spectra of a chaotic orbit will be continuous. Here we draw the spectrum of $x$ starting from $[-1/1000, 1/1000, 2/1000, -1/1000]$ (Fig. 17): What about the Lyapunov exponents? based on the last 10000 point of $x$, using the software “Chaos Data Analyzer”, choosing parameters $D = 3$, $n = 3$ and $A = 10^{-4}$, we get the largest Lyapunov exponent $0.407 \pm 0.027$, indicating the trajectory is indeed a chaotic one.

Now we look at the dynamics of Example 2 geometrically. For a given dynamical system, generally complicated manifold structure will lead to complex dynamics. we now indicate that the manifold structure of system $\Sigma_o$ is indeed very complicated. To simplify the discussion, suppose there is no constraint $H_2$ in Fig. 2, i.e., $v(k) \equiv u_c(k)$. The fixed points of the system $\Sigma_o$ is given by

$$\left\{ (x, x_d, z_-) : x = \frac{3}{2}z_-, x_d = -\frac{1}{2}z_- , |z_-| \leq 2\delta \right\}$$

Define

$$\mathbf{x} := \begin{bmatrix} x & x_d & z_- \end{bmatrix}'.$$
Then the system under new coordinates is

\[
\Sigma_{n1} : \tilde{x}(k+1) = \begin{bmatrix}
\frac{1}{2} & 3 & -\frac{3}{2} \\
-1 & -2 & -\frac{3}{2} \\
1 & 0 & \frac{3}{2}
\end{bmatrix} \tilde{x}(k)
\]

under

\[
|\tilde{x} - \tilde{z}_-| > \delta,
\]

and

\[
\Sigma_{n2} : \tilde{x}(k+1) = \begin{bmatrix}
2 & 3 & 0 \\
0 & -2 & 0 \\
0 & 0 & 1
\end{bmatrix} \tilde{x}(k)
\]

under

\[
|\tilde{x} - \tilde{z}_-| \leq \delta.
\]

For convenience, we denote this system by \(\Sigma_n\). It is easy to see that the fixed points of \(\Sigma_n\) are

\[
\{(0, 0, \tilde{z}_-) : |\tilde{z}_-| \leq 2\delta\}.
\]

Some comments are appropriate here:

- The subsystem \(\Sigma_{n1}\) is a stable system.
If Eq. (51) is satisfied, a trajectory (governed by $\Sigma_{n2}$) will move on a surface $\tilde{z}_- = \gamma$ for some $\gamma \in [-2\delta, 2\delta]$. We call such a surface $\Omega_{\gamma}$. The point $(0, 0, \gamma)$ is the origin of $\Sigma_{n2}$ on $\Omega_{\gamma}$. Furthermore, the line

$$\Gamma_{s,\gamma} : \tilde{x} = 0, \quad \tilde{z}_- = \gamma$$

(53)

is the stable manifold of $\Sigma_{n2}$ and similarly, the line

$$\Gamma_{u,\gamma} : \tilde{x}_d = 0, \quad \tilde{z}_- = \gamma$$

(54)

is the unstable manifold of $\Sigma_{n2}$.

Suppose a trajectory $\Gamma$ of the system $\Sigma_n$ starts from a point $p$ and is governed by $\Sigma_{n2}$, if $p \in \Gamma_{u,\gamma}$ (or in general $p \notin \Gamma_{s,\gamma}$) on some surface $\Omega_{\gamma}$, then the trajectory will contract along $\tilde{x}_d$–axis and stretch along $\tilde{x}$–axis. Due to the Eq. (51), after some time, $\Gamma$ will move according to the stable subsystem $\Sigma_{n1}$. At this moment, $\Gamma$ will leave the surface $\Omega_{\gamma}$, and move toward the origin $(0, 0, 0)$. Due to the Eq. (50), after some time, it will move again on some surface $\Omega_{\gamma'}$ for some $\gamma' \in [-2\delta, 2\delta]$. If it is not exactly on the line $\Gamma_{s,\gamma'}$, it will once more contract along $\tilde{x}_d$–axis and stretch along $\tilde{x}$–axis and repeat the above behavior. So normally a trajectory never settles done, indicating its intriguing behavior.

### 2.2 Chaotic control?

The complex dynamical behavior of the system in Fig. 2 has been studied in detail in the foregoing sections, compared to the standard control scheme such as that in Fig. 2, whose dynamics can only be either converging
to the origin, or being periodic or unbounded trajectories, the scheme adopted in Fig. 2 provides much more dynamical properties. Of course this means that a control engineer has more flexibility at his/her disposal. This is particularly attracting from the viewpoint of multi-purpose control. We believe this is the main merit this control scheme can provide. In this subsection, we will study the following problem: Given a control performance specification, can we achieve it by possibly adjusting the system parameters? We will discuss two control specifications:

(1) The system has one unique fixed point.

(2) A periodic orbit is desirable.

For item (1), without loss of generality, assume that the desirable unique fixed point is the origin. If the parameter $a$ in the system in Eq. (9) satisfies $|a| < 1$, then we can achieve asymptotic stability with respect to the origin by adjusting the nonlinear block $H_1$, though the system itself has no such property. According to Fig. 9, we need merely to let the value $v(k-1)$ stored in $H_1$ be 0 when $|x(k)| < \delta$ (This feature is illustrated in Figs. 10-11). Then the trajectory will move along the $x-$axis toward the origin, i.e., the asymptotic stability of the origin is achieved. If the parameter $a$ in the system in Eq. (9) satisfies $|a| \geq 1$, we can not expect asymptotic stability of the origin because it itself is unstable. However, we can keep the trajectory arbitrarily close to the origin at large time instants, by adopting the following scheme: Suppose it is desirable to keep the trajectory within the distance $\epsilon$ around the origin, then choose $\delta$ small enough so that $x(k^*)$ satisfies $|x(k^*)| < \epsilon/|a|^2$ at some time instant $k^*$ (this can be realized, see Fig. 12). Next let $v(k^*-1) = 0$ when $|x(k^*)| < \epsilon/|a|$. If $|a| = 1$, then the trajectory will stay at $(0, x(k^*))$ forever. The goal is achieved. On the other hand, assume $a > 1$. If $x(k^*) > 0$, we first let the trajectory move along the $x-$axis until we get
Fig. 17. Sensitive dependence on initial conditions

\[ x(k^* + 1) < \frac{\epsilon}{|a|}, \text{ then choose } v(k^*) > 0. \text{ In this way } (x(k^* + 1), v(k^*)) \text{ is below the line segment of fixed}
\]
\[ \text{point (then } (x(k^* + 1), v(k^*)) \text{ will move downward at the next step) such that}
\]
\[ x(k^* + 2) > 0,
\]
and
\[ x(k^* + 2) = ax(k^* + 1) + bv(k^*) < \frac{\epsilon}{|a|^2}. \]

(Note this is guaranteed by the property of the vector field of the system.) Then let \( v(k^* + 1) = 0 \), and repeat
the above procedure. Similarly if \( x(k^*) < 0 \), all we need to do is to choose suitable \( v(k^*) < 0 \) such that
\( (x(k^* + 1), v(k^*)) \) is above the line segment of fixed points, then follow the above procedure. In this way we
\[ \text{can keep the trajectory within the distance } \epsilon \text{ of the origin. Based on the above analysis, we observe that the}
\]
\[ \text{instability of the parameter } a \text{ poses a difficulty for implementing our scheme, there are more discussions from}
\]
\[ \text{the perspective of higher dimensional systems (e.g. Theorem 6 below). The foregoing discussion is reminiscent}
\]
of Proposition 2.2 in Delchamps [1990], however our scheme is better since the \( K_1 \) in that paper can be \( \infty \) here. Moreover, our algorithm is simpler too.

For the item (2), suppose it is desired that the system operates on a periodic orbit \( \Gamma \) of periodic \( T \). If
\( a = 1 \), according to Theorem 4, by suitably choosing \( b \), a periodic orbit of period \( T \) can be built. If \( a \neq 1 \), then
\[ \text{following the discussion at the end of Sec. 2.1.2, it is also possible to construct a periodic orbit of period } T. \]
Then the real question is: Can we really find an initial condition which produce or converge to the desirable
\[ \text{periodic orbit } \Gamma? \text{ If } \Gamma \text{ is within a strange attractor, then from almost all initial points, trajectories will be within}
\]
an arbitrarily small neighborhood of \( \Gamma \) at some time \( k; \) this is the property of a stranger attractor. So we can
\[ \text{just pick up such an initial condition, let the system run automatically first, and apply control similarly to the}
\]
exists a Lyapunov function $v$ for this system. For simplicity, let $D$ now we return to our analysis of higher dimensional system in Fig. 2. We will find a positively invariant set especially suitable for multi-purpose controller design. However in order to make the proposed scheme more useful, a thorough study of this scheme has to be conducted.

We have to acknowledge that the preceding analysis is naive, nevertheless, it illustrates that by using the trajectories of the system, i.e., some extra information in addition to system parameters, we can achieve better control in some sense. For chaotic control, interested readers may refer to Schuster [1999]. These will be our future research directions. Here we still adhere to classic control theory.

### 2.3 Stability analysis of higher-dimensional systems

Now we return to our analysis of higher dimensional system in Fig. 2. We will find a positively invariant set for this system. For simplicity, let $D_d = 0$. Define

$$
\tilde{A} := \begin{bmatrix}
A & BC_d \\
-B_dC & A_d
\end{bmatrix}, \quad \tilde{B} := \begin{bmatrix}
B & 0 \\
0 & -B_d
\end{bmatrix}, \quad \tilde{B} := \tilde{B}\tilde{C} = \begin{bmatrix}
0 & BC_d \\
-B_dC & 0
\end{bmatrix}.
$$

Since the controller $C$ is stabilizing, the closed-loop system in Fig. 1 is asymptotically stable. Then there exists a Lyapunov function $v(\xi(k)) = \xi'(k)P\xi(k)$ with $P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} > 0$ such that

$$
\Delta v(\xi(k)) = \xi'(k + 1)P\xi(k + 1) - \xi'(k)P\xi(k)
\begin{equation}
= \xi'(k) \left( \tilde{A}'P\tilde{A} - P \right) \xi(k)
= -\|\xi(k)\|^2_2 \quad \text{for all } \xi(k).
\end{equation}
$$

Correspondingly, define $v_c(\eta(k)) = \eta'(k)P\eta(k)$, then

$$
\begin{align*}
\Delta v_c(\eta(k)) &= \eta'(k + 1)P\eta(k + 1) - \eta'(k)P\eta(k) \\
&= \eta'(k) \left( \tilde{A}'P\tilde{A} - P \right) \eta(k) + 2\eta'(k)\tilde{A}'P\tilde{B} \left[ \begin{array}{c}
H_1(u_c(k), v(k-1)) \\
H_2(y_c(k), z(k-1))
\end{array} \right] - \left[ \begin{array}{c}
u_c(k) \\
y_c(k)
\end{array} \right]
\begin{align*}
&+ \left( \begin{bmatrix}
H_1(u_c(k), v(k-1)) \\
H_2(y_c(k), z(k-1))
\end{bmatrix} - \left[ \begin{array}{c}
u_c(k) \\
y_c(k)
\end{array} \right] \right)' \tilde{B}' P\tilde{B} \left[ \begin{array}{c}
H_1(u_c(k), v(k-1)) \\
H_2(y_c(k), z(k-1))
\end{array} \right] - \left[ \begin{array}{c}
u_c(k) \\
y_c(k)
\end{array} \right] \cdot \gamma \cdot \delta + (\gamma \cdot \delta)^2 \cdot \|P\tilde{B}\|_\infty \\cdot \|\tilde{B}' P\tilde{B}\|_\infty,
\end{align*}
$$

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where the positive constant $\gamma = \sqrt{m + p}$ and $\delta = \max \{\delta_1, \delta_2\}$. Hence $\triangle v_c(\eta(k)) < 0$ if
\[
\|\eta(k)\|_2 > \gamma \cdot \delta \|A'PB\|_\infty + \gamma \cdot \delta \sqrt{\|A'PB\|_\infty^2 + \|B'PB\|_\infty},
\]
For convenience, define
\[
\begin{align*}
    r_1 &:= \gamma \cdot \delta \|A'PB\|_\infty + \gamma \cdot \delta \sqrt{\|A'PB\|_\infty^2 + \|B'PB\|_\infty}, \\
    r_2 &:= \|A\|_\infty r_1 + \|B\|_\infty \delta,
\end{align*}
\]
then we have
\[
\begin{align*}
    \eta' &= \left[ \frac{I}{-C(I - A)^{-1}B - C_d(I - A_d)^{-1}B_d + D_d} \right], \\
    \Phi := -\left[ \frac{I}{-C(I - A)^{-1}B - C_d(I - A_d)^{-1}B_d + D_d} \right],
\end{align*}
\]
then we have:

**Theorem 5** The set $\Omega$ defined by
\[
\Omega := \left\{ \eta \left| \eta'(k)P\eta(k) \leq \max \{\sigma(P)r_1^2, \sigma(P)r_2^2\} \right. \right\}
\]
is a positively invariant set, where $\sigma(P)$ is the largest singular value of $P$.

**Proof:** We need only to show that for each $\eta(0) \in \Omega$, $\eta(k) \in \Omega$ for all $k \geq 1$. Suppose for some integer $k_0 > 0$, we have $\|\eta(k_0)\|_2 \leq r_1$, and $\|\eta(k_0 + 1)\|_2 > r_1$, then $\triangle v_c(\eta(k_0 + 1)) < 0$, which means $\eta'(k_0 + 2)P\eta(k_0 + 2) < \eta'(k_0 + 1)P\eta(k_0 + 1)$. Furthermore, the trajectory will eventually fall into the set $\left\{ \eta \left| \eta'(k)P\eta(k) \leq \sigma(P)r_2^2 \right. \right\}$. Therefore it suffices to show $\eta(k_0 + 1) \in \Omega$. Since
\[
\|\eta(k_0 + 1)\|_2 \leq \|A\|_\infty r_1 + \|B\|_\infty \delta,
\]
one has
\[
\eta'(k_0 + 1)P\eta(k_0 + 1) \leq \sigma(P)r_2^2,
\]
which gives $\eta(k_0 + 1) \in \Omega$. $\square$

The preceding result ascertains the existence of a positively invariant set for the system in Fig. 2, the system behavior inside this invariant set may be very complex. The next result gives an upper bound for all equilibria of the system in Eq. (7).

Defining
\[
\Phi := -\left[ \frac{I}{-C(I - A)^{-1}B - C_d(I - A_d)^{-1}B_d + D_d} \right],
\]
then we have:

**Corollary 3** For the system in Eq. (7), supposing both $G$ and $C$ are stable, if the matrix
\[
\left( C_d(I - A_d)^{-1}B_d - D_d \right)C(I - A)^{-1}B \text{ has no eigenvalue at } (-1, 0), \text{ then } \left\| \left( I - \tilde{A} \right)^{-1}\tilde{B} \right\|_1 \cdot \|\Phi^{-1}\|_1 \bar{\delta} \text{ is an upper bound for all equilibria of this system.}
Proof: Suppose $\bar{x}$ is an equilibrium of the system in Eq. (7), then there are an integer $K > 0$ and some vector $\varpi$ such that

$$\eta(k+1) = A\eta(k) + \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \varpi \tag{56}$$

for all $k > K$. Letting $k \to \infty$, we get

$$\bar{x} = \bar{A}\bar{x} + \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \varpi,$$

then

$$\bar{x} = (I - \bar{A})^{-1} \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \varpi,$$

and

$$\|C\bar{x} + \begin{bmatrix} 0 & D_d \\ 0 & 0 \end{bmatrix} \varpi - \varpi\|_\infty = \|\Phi \varpi\|_\infty.$$

Because the matrix $\left(C_d(I - A_d)^{-1}B_d - D_d\right)C(I - A)^{-1}B$ has no eigenvalue at $(-1, 0)$, $\Phi$ is invertible. Furthermore, since $\|C\bar{x} + \begin{bmatrix} 0 & D_d \\ 0 & 0 \end{bmatrix} \varpi - \varpi\|_\infty \leq \tilde{\delta}$, $\|\varpi\|_\infty \leq \|\Phi^{-1}\|_1 \|\Phi \varpi\|_\infty \leq \tilde{\delta}$, we have

$$\|\bar{x}\|_\infty \leq \|(I - \bar{A})^{-1}\tilde{B}\|_1 \cdot \|\Phi^{-1}\|_1 \tilde{\delta}. \tag{57}$$

Because $\bar{x}$ is arbitrarily chosen, the result follows.

In particular, assume we have a scalar system with a static state feedback:

$$\begin{align*}
x(k+1) &= ax(k) + bv(k), \\
u(k) &= -fx(k), \\
v(k) &= H_1(u(k), v(k-1))
\end{align*} \tag{58}$$

where $|a - bf| < 1$. Then following the above procedure, $|\bar{x}| \leq (\delta |bf|) / (1 - |a - bf|)$ where $\bar{x}$ can be any equilibrium.

An upper bound has been found for all equilibria. Will any of these equilibria be stable if either $G$ or $C$ is unstable? We have a result reminiscent of that in Delchamps [1990]

**Theorem 6** Assume either $G$ or $C$ is unstable, and $\bar{A}$ is invertible, then the set of all initial points $\eta_0$ whose closed-loop trajectories tend to an equilibrium as $k \to \infty$ has Lebesgue measure zero.

**Proof:** Denote this set by $U$. Let $E^s$ be the generalized stable eigenspace of Eq. (7), Then the Lebesgue measure of $E^s$ is zero since Eq. (7) is unstable. Suppose $\eta(0) \in U$, following the process in the proof of Corollary 1, there exist $K > 0$ and some vector $\varpi$ such that

$$\eta(K) = \begin{bmatrix} A - BD_dC & BC_d \\ -B_dC & A_d \end{bmatrix} \varpi, \tag{59}$$

39
and Eq. (56) holds for all \( k > K \). Since \( \tilde{A} \) is unstable, \( \eta(k) \in E^s \) for all \( k \geq K \). Furthermore, the invertibility of \( \tilde{A} \) implies that \( \bar{w} \) is uniquely determined by \( \eta(K) \). Due to the uniqueness of the state trajectory the system in Eq. (7), note also that this system is essentially a system with unit time delay, the trajectory starting from \( (\eta(-1) = 0, \eta(0)) \) is identical to that starting from \((\bar{w}, \eta(K))\). Define a mapping \( F \) as

\[
F : U \to E^s, \\
\eta(0) \longmapsto \eta(K),
\]

where \( \eta(0) \) and \( \eta(K) \) satisfying Eqs. (59) and (56), then \( F \) is injective. Therefore the Lebesgue measure of \( U \) is zero.

\[\square\]

### 3 An Example

In this section, one example will be used to illustrate the effectiveness of the scheme proposed in this paper. In this example, the networked control system consists of two subsystems, (each composed of a system and its controller), the outputs of the controlled systems will be sent respectively to controllers via a network. For the ease of notation, we denote the two systems, their controllers and their outputs by \( G_1 \), \( G_2 \), \( C_1 \), \( C_2 \), \( y_1 \) and \( y_2 \) respectively. Here two transmission methods will be compared: one is just letting the outputs transmitted sequentially, i.e., the communication order is \([y_1(0), y_2(0), y_1(1), y_2(1), \ldots] \). Another method is adding the nonlinear constraint \( H_2 \) to the subsystem composed of \( G_1 \) and \( C_1 \), if the difference between the two adjacent signals are greater than \( \delta_2 = 0.01 \), then this subsystem gets access to the network; otherwise the other gets access. Here, we will compare the tracking errors produced under these two schemes respectively. For convenience, we call the first method the regular static scheduler and the second the modified static scheduler.

The controlled system \( G_1 \) is:

\[
x_1(k+1) = \begin{bmatrix}
1.0017 & 0.1000 & 0.0250 & 0.0009 \\
0.0500 & 1.0000 & 0.5000 & 0.0259 \\
0.2000 & -0.0003 & 1.0000 & 0.1052 \\
-0.0034 & -0.2103 & -0.0517 & 1.1034
\end{bmatrix} x_1(k) + \\
\begin{bmatrix}
0.0050 \\
0.0991 \\
-0.0052 \\
-0.1155
\end{bmatrix} w(k) + \\
\begin{bmatrix}
-0.0050 & -0.0000 \\
-0.1000 & -0.0001 \\
0.0000 & -0.0005 \\
0.0103 & -0.0105
\end{bmatrix} u_1(k),
\]

\[
z_1(k) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0
\end{bmatrix} x_1(k) + \begin{bmatrix}
-1 \\
0
\end{bmatrix} w(k),
\]

\[
y_1(k) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} x_1(k),
\]
and $G_2$ is:

$$x_2(k + 1) = \begin{bmatrix}
1.0000 & 0.0100 & 0.0002 & 0.0000 \\
0.0050 & 1.0000 & 0.0500 & 0.0003 \\
0.0200 & -0.0000 & 1.0000 & 0.0101 \\
-0.0000 & -0.0201 & -0.0005 & 1.0100
\end{bmatrix}
\begin{bmatrix}
x_2(k)
\end{bmatrix}
+ \begin{bmatrix}
0.0000 \\
0.0100 \\
-0.0001 \\
-0.0102
\end{bmatrix} w(k) + \begin{bmatrix}
-0.0000 \\
-0.0100 \\
0.0000 \\
0.0001
\end{bmatrix} u_2(k),$$

$$z_2(k) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
x_2(k)
\end{bmatrix}
+ \begin{bmatrix}
-1
\end{bmatrix} w(k),$$

$$y_2(k) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_2(k)
\end{bmatrix},$$

where $w$ is a unit step. $z_1$ and $z_2$ are tracking errors. Controllers $C_1$ and $C_2$ can be obtained using the technique in Chen & Francis [1995]. Denote the first element of $y_1$ by $y_{11}$ and that of $y_2$ by $y_{21}$; the second element of $y_1$ by $y_{12}$ and that of $y_2$ by $y_{22}$, then the subsystem with variables $x_1, x_2, z_1, z_2, y_{11}, y_{21}$ is $G_1$ controlled by $C_1$ and the subsystem with variables $x_1, x_2, z_1, z_2, y_{12}, y_{22}$ is $G_2$ controlled by $C_2$. The simulation results are in Figs. 18–19.

![Fig. 18. The first elements of $z_1$ and $z_2$](image)

From these two figures, one finds that the tracking errors approach zero faster under the modified static scheduler than under the regular one. Note that both systems are unstable. If one of the two systems is stable, one can expect better convergence rate. In essence, our scheme is based on the following principle: Allocate access to the network to the systems with faster dynamics first, then take care of the systems of slower dynamics. In this way, we hope we can improve system performance. Interestingly, a similar idea is explored
in Hristu & Morgansen [1999].

4 Conclusion

In this paper, a new networked control technique is proposed and its effectiveness is illustrated via simulations. The complicated dynamics of this type of systems is studied both numerically and theoretically. A simulation shows that the scheme proposed here has possible application in networked control systems. There are several problems guiding our further research: 1) Continuity of state trajectories with respect to the initial points under space partition induced by the discontinuities of the system. 2) How to find a precise characterization of the attracting set for our system, and is it topologically transitive (i.e., is it a chaotic attractor)? Topological transivity, an indispensable feature of a chaotic attractor, is closely related to ergodicity of a map. As discussed in Sec. 2.1.3, the proof of topological transitivity or ergodicity is difficult for our system from the point of view of measure theory due to the singularity of the map and its violation of conditions in Lasota & Yorke [1973]. However, this investigation is unavoidable should one want to find the chaotic attractor inherited in the system studied. 3) For different system parameters, different aperiodic orbits can be obtained, what are the differences among these orbits? In particular, given two aperiodic orbits, one generated from a system having no periodic orbits and the other generated by a system having periodic orbits, is there any essential difference between them? 4) In Sec. 2.1.2, periodic orbits are constructed for some originally stable ($|a| < 1$) and originally unstable ($|a| > 1$) systems. However given a system, how to determine if there are periodic orbits, and if so, how to find all of them is still an unsolved problem. 5) How to effectively design controllers based on chaotic control? Obviously the solution of this problem depends on the forgoing ones. 6) How to incorporate properly the scheme proposed in this paper into the framework of networked control systems? The simulation in Sec. 3
is naive, more research is required here to make the proposed scheme practical.

5 Acknowledgement

The first author is grateful to discussions with Dr. Michael Li. This work was partially supported by NSERC. The authors are also grateful to the anonymous reviewers and the Editor for their resourceful comments and constructive suggestions.
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