GENERAL MASS INSERTION EXPANSION IN FLAVOR PHYSICS

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ABSTRACT

Calculating amplitudes for the flavor changing transitions in terms of the off-diagonal elements of mass matrices, so called Mass Insertions (MI) in the theory defined in gauge basis (before mass matrix diagonalization) is the common technique in analyzing the flavor structure of the New Physics models. I present a general method allowing to expand any QFT amplitude calculated in the mass-eigenstates (physical) basis into series in MI’s, to any required order [1]. The technique is purely algebraic, translating an amplitude written in the mass eigenbasis into MI series without performing diagrammatic calculations in the gauge basis. It can be applied for all types of mass matrices - either Hermitian (scalar or vector), general complex (Dirac fermions) or complex symmetric (Majorana fermions). Proposed expansion has been also automatized in the form of publicly available specialized Mathematica package, MassToMI [2], which features I illustrate with the example of the Higgs boson decays in the MSSM.

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1 Introduction: basis choice for flavor transition calculations

Amplitudes describing flavor changing transitions within the New Physics models can be computed using two basic approaches:

- Calculations in the “interaction” or “symmetry” basis, using fields before the mass matrices diagonalization. Gauge interactions in such basis are flavor diagonal, flavor transitions originate from the off-diagonal entries of mass matrices, so-called “Mass Insertions” (MI).
  - advantage: direct dependence on original symmetry-related parameters.
  - disadvantage: tedious and error prone diagrammatic calculations with mass insertions treated as interaction vertices, combinatorial complication quickly growing with MI order. Amplitude expressed as double infinite series, in loop order and in MI order.

- Calculations in the “mass eigenstates” basis, in terms of physical fields after mass matrices diagonalization.
  - advantage: physical external states, more compact expressions, easier diagrammatic calculations, at given loop order exact formulae in terms of flavor changing parameters.
  - disadvantage: complicated non-linear dependence on initial symmetry-related parameters, various effects can be analyzed only numerically.

Both approaches can be illustrated with the following toy example. Let us consider the scalar self-energy in a model with one real scalar field $\eta$ and one complex scalar multiplet $\Phi_I$, defined by the weak-basis Lagrangian:

$$L_{\text{int}} = \left( \partial^\mu \Phi_I^\dagger \right) \left( \partial_\mu \Phi_I \right) - M^2_{IJ} \Phi^\dagger_I \Phi_J + \frac{1}{2} \left( \partial^\mu \eta \right) \left( \partial_\mu \eta \right) - \frac{1}{2} m^2_\eta \eta^2 - Y_{IK} \eta \Phi^\dagger_I \Phi_J$$

Transition to mass eigenstates basis is done by the unitary rotation $U$:

$$\Phi_I = U_{Ii} \phi_i \quad m^2 = \text{diag}(m^2_1, \ldots, m^2_N) = U^\dagger M^2 U \quad y = U^\dagger Y U$$

Then, self-energy amplitude in the interaction basis reads as (thick dots in the diagram represent off diagonal elements of $M^2$ and $B_0, C_0, D_0, \ldots$ are the standard scalar 2-, 3-, 4-, \ldots point 1-loop functions):

$$\hat{\Sigma}_{JI}(p) = -\frac{1}{(4\pi^2)} Y_{JK} Y_{LI} \left( \delta_{KL} B_0(p; M^2_K, m^2_\eta) + \hat{M}^2_{KL} C_0(0, p; M^2_K, M^2_L, m^2_\eta) + \hat{M}^2_{KN} \hat{M}^2_{NL} D_0(0, 0, p; M^2_K, M^2_N, M^2_L, m^2_\eta) \right),$$

while the mass eigenstates basis amplitude has the form:

$$\Sigma_{ji}(p) = U_{jI} \hat{\Sigma}_{JI}(p) U_{Ii} = -\frac{1}{(4\pi^2)} y_{ji} B_0(p; m^2_i, m^2_\eta) y_{li}$$
2 Mass Insertion expansion of bosonic amplitudes

As proven in ref. [1], transition from mass to interaction basis amplitudes can be done using purely algebraic techniques, without the diagrammatic calculations with Mass Insertions. Typical term in QFT mass-eigenstates amplitude involving bosonic fields can be, through the Taylor series expansion, expressed as an element of function of the Hermitian mass matrix $M^2$:

$$U_{Ii} f (m_i^2) U_{jI}^* = U_{Ii} (f_0 + f_1 m_i^2 + f_2 m_i^4 + \ldots) U_{jI}^* = (f_0 + f_1 M^2 + f_2 M^4 + \ldots)_{I,J} = f(M^2)_{I,J}$$

MI approximation requires expanding $f(M^2)_{I,J}$ into a series with following properties:

- **Series coefficients:** depend only of diagonal elements of mass matrices.
- **Expansion parameters:** only non-diagonal elements of mass matrices.

The relevant solution is given by the “Flavor Expansion Theorem” (FET), valid for any function of Hermitian matrix (assuming only proper convergence criteria, see [1] for discussion of the most important case of the Passarino-Veltman 1-loop functions). The FET can be formulated as follows. Let’s decompose a Hermitian matrix (assuming only proper convergence criteria, see [1] for discussion of the most important case of the Passarino-Veltman 1-loop functions). The FET can be formulated as follows. Let’s decompose a Hermitian matrix $M^2$ as a sum of diagonal and non-diagonal part:

$$M^2 = M_0^2 + \hat{M}^2 \quad \text{where} \quad (M_0^2)_{I,J} \equiv M_{I,J}^2, \quad \hat{M}_{I,J} \equiv \hat{M}_{I,J}^2, \quad \hat{M}_{I,I} = 0.$$  

Then, matrix element $f(M^2)_{I,J}$ is given by (no sum over $I, J$) (see ref. [1] for formal proof):

$$f(M^2)_{I,J} = \delta_{IJ} f((M_0^2)_{I,J}) + f^{[1]}((M_0^2)_{I,J}, (M_0^2)_{J,J}) \hat{M}_{I,J} + \sum_{K_1} f^{[2]}((M_0^2)_{I,J}, (M_0^2)_{J,K_1}) \hat{M}_{I,K_1} \hat{M}_{K_1,J} \hat{M}_{K_1,K_2} \hat{M}_{K_2,J} + \ldots$$

and $f^{[n]}$ are divided differences of function $f$, defined recursively as:

$$
\begin{align*}
    f^{[0]}(x) &= f(x) \\
    f^{[1]}(x_0, x_1) &= \frac{f(x_0) - f(x_1)}{x_0 - x_1} \\
    &\quad \vdots \\
    f^{[k+1]}(x_0, \ldots, x_k, x_{k+1}) &= \frac{f^{[k]}(x_0, \ldots, x_{k-1}, x_k) - f^{[k]}(x_0, \ldots, x_{k-1}, x_{k+1})}{x_k - x_{k+1}}
\end{align*}
$$

Using FET, any amplitude involving scalar or vector fields can be calculated in mass eigenstates basis and consistently expanded to any order in MI’s (see e.g. ref [3] for example of applications).

3 Mass insertion expansion of fermionic amplitudes

FET theorem holds for Hermitian matrices. Mass terms for chiral fermions depend on *general complex* mass matrix $M$:

$$-\bar{\Psi} \left( M P_L + M^\dagger P_R \right) \Psi,$$

which can be diagonalized by 2 independent unitary rotations $U, V$, applied to left and right part of the spinor field:

$$\Psi_{L_A} = U_A \psi_{Li}, \quad \Psi_{R_A} = V_A \psi_{Ri} \quad V^\dagger M U = m = \text{diag}(m_1, \ldots, m_N)$$

However, FET still applies using the following observation. General fermion propagator can be decomposed as

$$\frac{i}{\slashed{k} - MP_L - M^\dagger P_R} = (M^\dagger P_L + \slashed{k} P_L) \frac{i}{k^2 - MM^\dagger} + (MP_R + \slashed{k} P_R) \frac{i}{k^2 - MM^\dagger}$$

Therefore:
Loop functions in fermionic amplitudes depend always only on eigenvalues of the Hermitian matrices $M^\dagger M$ or $M M^\dagger$.

Only some combinations of mixing matrices can appear in fermionic amplitudes:

\[
\begin{align*}
U_{Bi} f(m_i^2) U^*_i & = f(M^\dagger M)_{BA} \\
V_{Bi} f(m_i^2) V^*_i & = f(M M^\dagger)_{BA} \\
U_{Bi} m_i f(m_i^2) U^*_i & = M^\dagger_{BC} f(M M^\dagger)_{CA} = f(M^\dagger M)_{BC} M^\dagger_{CA} \\
V_{Bi} m_i f(m_i^2) V^*_i & = M_{BC} f(M^\dagger M)_{CA} = f(M M^\dagger)_{BC} M_{CA}
\end{align*}
\]

All such combinations can be expressed using FET formula for $f(M M^\dagger)$ or $f(M^\dagger M)$, thus algebraic expansion technique works for fermions as well. The conclusion holds also for Majorana fermions – then $M = M^T$ and $U = V^*$.

4 Automatized MI expansion: MassToMI Mathematica package

FET technique has been automatized in the symbolic Mathematica package MassToMI [2], able to expand mass eigenstates amplitude to any user-defined MI order. Using MassToMI, one can:

- Calculate flavor amplitudes in the mass eigenstates basis, with the advantage of having less diagrams and more compact expressions, better suited for numerical computations.
- Expand result using FET implemented in MassToMI package, recovering direct analytic dependence on “interaction basis” parameters (better suited for understanding of various effects).

To illustrate MassToMI features, we consider as an example the flavor violating charged Higgs boson decay $H^- \to d^\dagger u^j$ in the MSSM ($I,J$ denote the quark generation indices). The sample diagram contributing to the amplitude involving chargino (Dirac) $C_n$, neutralino (Majorana) $N_j$ and down squark $D_i$ circulating in the loop is shown in the figure below:

For simplicity, again we choose just one of many terms in the amplitude, having the “typical” form ($Z,O,U,V$ denote scalar, Majorana and Dirac fermion mixing matrices; summation convention is assumed for the repeating indices):

$$A \supset Z^I_{Dz} Z^J_{Dz} O^K_{Nz} O^{Lj}_{Nz} v^M_{Cz} U^N_{Cz} m_{Cz} c_0(p,q,m^2_{Cz},m^2_{Dz},m^2_{Nz})$$

To expand the amplitude, it has to be coded using the MassToMI input syntax rules:

- $Z^I_{Dz} \to SMIX[D,I,i]$ scalar $D$ mixing matrix
- $O^K_{Nz} \to NMIX[N,K,j]$ Majorana fermion $N$ mixing matrix
- $v^M_{Cz} \to Conjugate[FMIXL[C,M,n]]$ left Dirac fermion $C$ mixing matrix
- $U^N_{Cz} \to FMIXR[C,M,n]$ right Dirac fermion $C$ mixing matrix
- $m_{Cz} \to MASS[C,n]$ physical fermion $C$ mass
- $c_0(p,q,m^2_{Cz},m^2_{Dz},m^2_{Nz}) \to LOOP[c0,\{\{C,n\},\{D,i\},\{N,j\}\},\{p,q\}]$ loop integral
Then, \textit{MassToMI} expression for the amplitude reads as:

\[
A = \text{SMIX}[D,I,i] \text{ Conjugate}[\text{SMIX}[D,J,i]] \\
\text{NMIX}[N,K,j] \text{ Conjugate}[\text{NMIX}[N,L,j]] \\
\text{Conjugate}[\text{FMIXL}[C,M,n]] \text{ FMIXR}[C,N,n] \text{ MASS}[C,n] \\
\text{LOOP}[c0,\{(C,n),(D,i),(N,j)\},\{p,q\}];
\]

Further, user must initialize control variables defining order of MI expansion:

- **FetScalarList** = \{\{D,2\}\}. Mixing matrices for scalar \(D\) are expanded up to the 2nd order in mass insertions.

- **FetFermionList** = \{\{C,1,\text{MMH}\},\{N,1,\text{MMH}\}\}. Mixing matrices for fermions \(C, N\) are expanded to 1st order in MI. Parameters \text{MMH}, \text{MMH} decide if the final result is expressed in terms of \(M^\dagger M\) or \(MM^\dagger\).

- **FetMaxOrder** = 2. Only mass insertion products of the total order \text{FetMaxOrder} or lower are kept in the final result (e.g. terms proportional to 2nd order scalar MI and 1st order fermionic MI are truncated).

Using the setup above, command **FetExpand[A]** automatically performs the MI expansion of amplitude to the required order.

\textit{MassToMI} package has been successfully tested on the realistic case of the lepton flavor violating processes in the MSSM:

- Initial mass eigenstates expressions for amplitude: few diagrams / few lines of code.

- Mathematica expansion code / execution time: 300 lines / up to few hours on standard PC.

- \textit{Intermediate expressions before cancellations:} \(\sim 50000\) MI terms! Equivalent to evaluating of hundreds of diagrams in the interaction basis.

- Final MI expanded expressions after cancellations and simplifications: few lines for the leading MI terms.

5 Conclusions

I presented “Flavor Expansion Theorem”, the general technique of purely algebraic expansion of the mass eigenstates QFT amplitudes into Mass Insertion series, valid for all types of amplitudes involving scalar, fermionic or vector fields. Using FET allows to avoid tedious and error prone direct diagrammatic calculations with MI’s as additional interaction vertices. The technique can be fully automatized with the use of specialized \textit{MassToMI} package, written in the \textit{Mathematica} symbolic language. The current \textit{MassToMI} distribution, including detailed manual with examples, can be downloaded from the address

\texttt{www.fuw.edu.pl/masstomi}

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