A NOTE ON PRODUCTS OF STOCHASTIC OBJECTS

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Abstract. In recent study of partial differential equations (PDEs) with random initial data and singular stochastic PDEs with random forcing, it is essential to study the regularity property of various stochastic objects. These stochastic objects are often given as products of simpler stochastic objects. As pointed out in Hairer (2014), by using a multiple stochastic integral representation, one may use Jensen’s inequality to reduce an estimate on the product to those on simpler stochastic objects. In this note, we present a simple argument of the same estimate, based on Cauchy-Schwarz’ inequality (without any reference to multiple stochastic integrals). We present an example on computing the regularity property of stochastic objects in the study of the dispersion-generalized nonlinear wave equations, and prove their local well-posedness with rough random initial data.

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1. INTRODUCTION

Over the last decade, we have seen a significant progress in the theoretical study of singular stochastic partial differential equations (PDEs), in particular in the parabolic setting [17, 13], where we now have a satisfactory understanding of how to give a meaning to solutions to a certain class of classically ill-posed equations. See [7] for a survey of the subject. Consider an equation of the form:

\[ \partial_t u = Lu + \mathcal{N}(u) + \xi, \quad (1.1) \]

where \( L \) denotes a linear operator in the spatial variable, \( \mathcal{N}(u) \) denotes a nonlinearity, and \( \xi = \xi(x, t) \) denotes a random forcing. In constructing a solution to (1.1), we usually start out by constructing basic stochastic objects such as the stochastic convolution:

\[ Z(t) = \int_0^t e^{L(t-\tau)} \xi(d\tau) \]

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and the second order term:

\[ \mathcal{I}(N(Z))(t) = \int_0^t e^{L(t-\tau)}N(Z)(\tau) d\tau \]

(usually with a renormalization on the nonlinearity \( N(Z) \)). Then, we expand the unknown \( u \) in terms of (explicitly known) basic stochastic objects and study the equation for the remainder term [3, 8, 17, 13]. For example, with the second order expansion

\[ u = Z + \mathcal{I}(N(Z)) + v, \]

we study the equation for the remainder \( v = u - Z - \mathcal{I}(N(Z)) \):

\[ \partial_t v = Lv + N(Z + \mathcal{I}(N(Z)) + v) - N(Z). \tag{1.2} \]

In the case of a power-type nonlinearity \( N(u) = u^k \), the nonlinearity in (1.2) becomes

\[ N(Z + \mathcal{I}(N(Z)) + v) - N(Z) = \sum_{k=k_1+k_2+k_3} C_{k_1,k_2,k_3} Z^{k_1}(\mathcal{I}(Z^k))^{k_2} v^{k_3} \]

(once again with proper renormalizations), leading us to study the regularity property of the stochastic objects of the form:

\[ Z^{k_1}(\mathcal{I}(Z^k))^{k_2}. \tag{1.3} \]

In studying the regularity property of stochastic objects, the following lemma (Lemma 1.1) provides a convenient criterion. Before stating the lemma, let us first introduce some notations. See [18, 21] for basic definitions. Let \((H, B, \mu)\) be an abstract Wiener space. Namely, \(\mu\) is a Gaussian measure on a separable Banach space \(B\) over reals with \(H \subset B\) as its Cameron-Martin space. Given a complete orthonormal system \(\{e_j\}_{j \in \mathbb{N}} \subset B^*\) of \(H^* = H\), we define a polynomial chaos of order \(k\) to be an element of the form

\[ \prod_{j=1}^{\infty} H^{k_j}, \]

where \(H^{k_j}\)'s are pairwise orthogonal. In the following, we use \(H_k\) to denote the orthogonal projection onto \(H_k\). For \(k \in \mathbb{N}\), we also set

\[ H_{\leq k} = \bigoplus_{j=0}^{k} H_j. \]

For those who are not familiar with the subject, it is useful to think of \(H_{\leq k}\) as the collection of (possibly infinite) linear combinations of polynomials of degree at most \(k\) in Gaussian random variables. We say that a stochastic process \(X : \mathbb{R}_+ \to \mathcal{D}'(\mathbb{T}^d)\) is spatially homogeneous if \(\{X(\cdot,t)\}_{t \in \mathbb{R}_+}\) and \(\{X(x_0 + \cdot, t)\}_{t \in \mathbb{R}_+}\) have the same law for any \(x_0 \in \mathbb{T}^d\). Given \(h \in \mathbb{R}\), we define the difference operator \(\delta_h\) by setting

\[ \delta_h X(t) = X(t + h) - X(t). \tag{1.4} \]
Lemma 1.1. Let \( \{X_N\}_{N \in \mathbb{N}} \) and \( X \) be spatially homogeneous stochastic processes : \( \mathbb{R}^d \to \mathcal{D}'(\mathbb{T}^d) \). Suppose that there exists \( k \in \mathbb{N} \) such that \( X_N(x, t) \) and \( X(x, t) \) belong to \( \mathcal{H}_{\le k} \) for each \( x \in \mathbb{T}^d \) and \( t \in \mathbb{R}_+ \).

(i) Let \( t \in \mathbb{R}_+ \). If there exists \( s_0 \in \mathbb{R} \) such that
\[
\mathbb{E} \left[ |\hat{X}(n, t)|^2 \right] \lesssim \langle n \rangle^{-d-2s_0}
\]
for any \( n \in \mathbb{Z}^d \), then we have \( X(t) \in W^{s, \infty}(\mathbb{T}^d) \cap C^s(\mathbb{T}^d) \), \( s < s_0 \), almost surely.

(ii) Suppose that \( X_N, N \in \mathbb{N} \), satisfies (1.5). Furthermore, if there exists \( \gamma > 0 \) such that
\[
\mathbb{E} \left[ |\hat{X}_N(n, t) - \hat{X}_M(n, t)|^2 \right] \lesssim \langle n \rangle^{-d-2s_0}
\]
for any \( n \in \mathbb{Z}^d \) and \( M > N \geq 1 \), then \( X_N(t) \) is a Cauchy sequence in \( W^{s, \infty}(\mathbb{T}^d) \cap C^s(\mathbb{T}^d) \), \( s < s_0 \), almost surely, thus converging to some limit in \( W^{s, \infty}(\mathbb{T}^d) \cap C^s(\mathbb{T}^d) \).

(iii) Let \( T > 0 \) and suppose that (i) holds on \( [0, T] \). If there exist \( \sigma_1, \sigma_2 > 0 \) such that
\[
\mathbb{E} \left[ |\delta_h \hat{X}(n, t) - \delta_h \hat{X}_M(n, t)|^2 \right] \lesssim \langle n \rangle^{-d-2s_0} |h|^{\sigma_2}
\]
for any \( n \in \mathbb{Z}^d \), \( t \in [0, T] \), and \( h \in [-1, 1] \) (with \( h \geq -t \) such that \( t + h \geq 0 \)), then we have \( X \in C([0, T]; W^{s, \infty}(\mathbb{T}^d) \cap C^s(\mathbb{T}^d)) \), \( s < s_0 - \frac{\sigma_1}{2} \), almost surely.

(iv) Let \( T > 0 \) and suppose that (ii) holds on \( [0, T] \). Furthermore, if there exists \( \gamma > 0 \) such that
\[
\mathbb{E} \left[ |\delta_h \hat{X}_N(n, t) - \delta_h \hat{X}_M(n, t)|^2 \right] \lesssim \langle n \rangle^{-d-2s_0} |h|^{\sigma_2}
\]
for any \( n \in \mathbb{Z}^d \), \( t \in [0, T] \), \( h \in [-1, 1] \), and \( M > N \geq 1 \), then \( X_N \) is a Cauchy sequence in \( C([0, T]; W^{s, \infty}(\mathbb{T}^d) \cap C^s(\mathbb{T}^d)) \), \( s < s_0 - \frac{\sigma_1}{2} \), almost surely, thus converging to some process in \( C([0, T]; W^{s, \infty}(\mathbb{T}^d) \cap C^s(\mathbb{T}^d)) \).

Here, \( W^{s,p}(\mathbb{T}^d) \) denotes the \( L^p \)-based Sobolev space and \( C^s(\mathbb{T}^d) \) denotes the Hölder-Besov space \( C^s(\mathbb{T}^d) = B_{2, \infty}^s(\mathbb{T}^d) \). In the case of the heat equation, we have \( \sigma_1 = 2\sigma_2 \), while we have \( \sigma_1 = \sigma_2 \) for the wave equation. Lemma 1.1 follows from a straightforward application of the Wiener chaos estimate (Lemma 2.1). See [19, 26] for the proof.

In the study of dispersive PDEs with random initial data and/or stochastic forcing, we have also witnessed a rapid progress over the recent years [6, 34, 14, 1, 27, 15, 22, 4, 29, 30, 24, 31, 10, 37, 33, 25]. See [2] for a (recent but already somewhat outdated) survey on the subject. In the dispersive setting once again, we need to study the regularity properties of (explicitly given) stochastic objects, for example of the form (1.3), where \( Z \) may be a random linear solution, a stochastic convolution, or their sum. Lemma 1.1 turns out to be a useful tool in this setting as well. We should, however, point out that in the dispersive setting, a multilinear dispersive smoothing property often plays an important role and computation of regularities of stochastic objects becomes much more complicated than that in the parabolic

1 Strictly speaking, since \( X_N(t) \) and \( X(t) \) are distributions, we need to test them against test functions on \( \mathbb{T}^d \) to say that they belong to \( \mathcal{H}_{\le k} \). Namely, we require that \( \langle X_N(t), \varphi \rangle, \langle X(t), \varphi \rangle \in \mathcal{H}_{\le k} \) for any \( t \in \mathbb{R}_+ \) and \( \varphi \in C^\infty(\mathbb{T}^d) \), where \( \langle \cdot, \cdot \rangle \) denotes the \( \mathcal{D}'(\mathbb{T}^d)-C^\infty(\mathbb{T}^d) \) duality. Alternatively, we can require that the Fourier coefficients \( \hat{X}_N(n, t) \) and \( \hat{X}(n, t) \) belong to \( \mathcal{H}_{\le k} \) for any \( t \in \mathbb{R}_+ \) and \( n \in \mathbb{Z}^d \).

2 We point out that Lemma 1.1 does not play an important role in studying non-polynomial nonlinearities; see [28, 29, 30].
Setting: Let $A_{j,k}$, $j,k \in \mathbb{N}$, be given by that $(i,j)$ of a pairing from [5, Definition 4.30].

As seen in [13], stochastic objects are often given as a (renormalized) product of simpler stochastic objects. For example, the term $Z^{k_1}(I(Z^k))^{k_2}$ in [13] is given as the product of $Z^{k_1}$ and $(I(Z^k))^{k_2}$, where the latter is further written as the $k_2$-fold product of $I(Z^k)$. In this case, we hope to study the regularity of $Z^{k_1}(I(Z^k))^{k_2}$ via Lemma [11] using the (already studied) regularity property (essentially the second moments of the Fourier coefficients) of $Z^{k_1}$ and $(I(Z^k))^{k_2}$. In computing the second moment of the Fourier coefficient of $Z^{k_1}(I(Z^k))^{k_2}$, we need to take into account the interaction between $Z^{k_1}$ and $(I(Z^k))^{k_2}$, which can be in general quite cumbersome, when $k$, $k_1$, and $k_2$ are large. By using a multiple stochastic integral representation, however, we can invoke Jensen’s inequality to ignore such interaction between $Z^{k_1}$ and $(I(Z^k))^{k_2}$ in estimating the highest order contribution (belonging to $H_{k_1+k_2}$) from $Z^{k_1}(I(Z^k))^{k_2}$. This idea first appeared in [17, Section 10]. See also [19, Section 3] and [33, Appendix B].

Our main goal in this note is to present this argument without going through a multiple stochastic integral representation so that it is more accessible to those in analysis and PDEs who are not necessarily familiar with stochastic analysis (in particular multiple stochastic integrals). See Proposition [13, (i)]. Furthermore, by imposing more structures on stochastic objects (see Assumption B below), we establish a general product estimate.

Before we state out main result, let us introduce some notations. We first recall the notion of a pairing from [5, Definition 4.30].

**Definition 1.2.** Let $J \geq 1$. We call a relation $\mathcal{P} \subset \{1, \ldots, J\}^2$ a pairing if

(i) $\mathcal{P}$ is anti-reflexive i.e. $(j,j) \notin \mathcal{P}$ for all $1 \leq j \leq J$,

(ii) $\mathcal{P}$ is symmetric, i.e. $(i,j) \in \mathcal{P}$ if and only if $(j,i) \in \mathcal{P}$,

(iii) $\mathcal{P}$ is univalent, i.e. for each $1 \leq i \leq J$, $(i,j) \in \mathcal{P}$ for at most one $1 \leq j \leq J$.

If $(i,j) \in \mathcal{P}$, the tuple $(i,j)$ is called a pair. If $1 \leq j \leq J$ is contained in a pair, we say that $j$ is paired. With a slight abuse of notation, we also write $j \in \mathcal{P}$ if $j$ is paired. If $j$ is not paired, we also say that $j$ is unpaired and write $j \notin \mathcal{P}$. Furthermore, given a partition $\mathcal{A} = \{A_\ell\}_{\ell=1}^L$ of $\{1, \ldots, J\}$, we say that $\mathcal{P}$ respects $\mathcal{A}$ if $i,j \in A_\ell$ for some $1 \leq \ell \leq L$ implies that $(i,j) \notin \mathcal{P}$. Namely, $\mathcal{P}$ does not pair elements of the same set $A_\ell \in \mathcal{A}$. We say that $(n_1, \ldots, n_J) \in (Z^d)^J$ is admissible with respect to $\mathcal{P}$ if $(i,j) \in \mathcal{P}$ implies that $n_i + n_j = 0$.

**Setting:** Let $F_j$, $j = 1, \ldots, J$, be stochastic processes: $\mathbb{R}_+ \to \mathcal{D}'(T^d)$ such that there exists $k_j \in \mathbb{N}$ such that $F_j(t) \in H_{k_j}$ for any $t \in \mathbb{R}_+$. Suppose that the Fourier coefficients of $F_j(t)$ are given by

$$\hat{F}_j(n,t) = \sum_{n=n_1+\cdots+n_{k_j}} \hat{f}_j(n_1, \ldots, n_{k_j}, t)$$

\hfill (1.8)
for any \( n \in \mathbb{Z}^d \), where \( \mathfrak{F}_j(n_1, \ldots, n_{k_j}, t) \) satisfies

\[
\mathbb{E} \left[ \mathfrak{F}_j(n_1, \ldots, n_{k_j}, t_1) \mathfrak{F}_j(m_1, \ldots, m_{k_j}, t_2) \right] = 0 \tag{1.9}
\]

for any \( t_1, t_2 \in \mathbb{R}_+ \), unless \( \{n_1, \ldots, n_{k_j}\} = \{m_1, \ldots, m_{k_j}\} \), namely \( m_\ell = n_{\sigma(\ell)} \) for some \( \sigma \in S_{k_j} \), where \( S_{k_j} \) is the symmetric group on \( k_j \) elements.

Next, we state two separate additional assumptions. Assumption A is used in Part (i) of Proposition 1.3 where we estimate the highest order contribution (belonging to \( \mathcal{H}_{K_j} = \mathcal{H}_{k_1 + \cdots + k_J} \)) of the product of \( J \) homogeneous stochastic objects without using Jensen’s inequality. Assumption B is used in Part (ii) of Proposition 1.3 where we estimate a general product by imposing a further structure on \( \mathfrak{F}_j \).

**Assumption A:** Set \( K_0 = 0 \) and

\[
K_j = k_1 + \cdots + k_j
\tag{1.10}
\]

for \( j = 1, \ldots, J \). Recall that \( \pi_k \) denotes the orthogonal projection onto \( \mathcal{H}_k \). We assume that, if the covariance

\[
\mathbb{E} \left[ \pi_{K_j} \left( \prod_{j=1}^J \mathfrak{F}_j(n_{K_j-1+1}, \ldots, n_{K_j}, t_1) \right) \pi_{K_j} \left( \prod_{j=1}^J \mathfrak{F}_j(m_{K_j-1+1}, \ldots, m_{K_j}, t_2) \right) \right] = 0 \tag{1.11}
\]

does not vanish, then we have \( \{n_1, \ldots, n_{K_j}\} = \{m_1, \ldots, m_{K_j}\} \) (counting multiplicity) such that

\[
\sum_{\sigma \in S_{K_j}} \mathbb{E} \left[ \pi_{K_j} \left( \prod_{j=1}^J \mathfrak{F}_j(n_{K_j-1+1}, \ldots, n_{K_j}, t_1) \right) \times \pi_{K_j} \left( \prod_{j=1}^J \mathfrak{F}_j(m_{\sigma(K_j-1+1)}, \ldots, m_{\sigma(K_j)}, t_2) \right) \right] \tag{1.12}
\]

for any \( t_1, t_2 \in \mathbb{R}_+ \).

Recalling that \( F_j \in \mathcal{H}_{k_j} \), we see that the product \( \prod_{j=1}^J F_j \) belongs to \( \mathcal{H}_{\leq K_j} \). Assumption A is introduced for estimating the highest order contribution belonging to \( \mathcal{H}_{K_j} \). See Proposition 1.3(i). We also point out that Assumption A is a very natural one to impose. For example, it is easy to verify Assumption A if \( \mathfrak{F}_j \) is a renormalized product of Gaussian random variables, namely,

\[
\mathfrak{F}_j(n_1, \ldots, n_{k_j}) = \pi_{k_j} \left( \prod_{i=1}^{k_j} g_{n_i} \right),
\]

where \( \{g_n\}_{n \in \mathbb{Z}^d} \) is a family of independent standard complex-valued Gaussian random variables conditioned that \( g_{-n} = \overline{g_n} \), \( n \in \mathbb{Z}^d \). Let us also point out that Assumption A is satisfied under Assumption B (in particular, \( \mathfrak{F}_j \); \( \mathfrak{F}_j \); \( \mathfrak{F}_j \)) that we introduce below. See also Section 3 for a concrete example of the dispersion-generalized nonlinear wave equations with Gaussian random initial data, satisfying Assumption A (and also Assumption B).
Assumption B: In order to fully describe the structure of a general product, let us give a more concrete representation of \( \mathfrak{F}_j(n_1, \ldots, n_{k_j}, t) \) in case of a deterministic PDE with Gaussian random initial data. Given a PDE, let \( \mathcal{I} \) denote the Duhamel integral operator:

\[
\mathcal{I}(f)(t) := \int_0^t S(t - \tau) f(\tau) d\tau
\]

\[
= \sum_{n \in \mathbb{Z}^d} \mathcal{I}_n(\hat{f}(n, \cdot))(t) e^{in \cdot x} := \sum_{n \in \mathbb{Z}^d} \left( \int_0^t S_n(t - \tau) \hat{f}(n, \tau) d\tau \right) e^{in \cdot x},
\]

where \( S(t) \) is the linear propagator for the given PDE and \( S_n(t) \) is defined by \( S_n(t) \hat{f}(n) = \mathcal{F}_x(S(t)f)(n) \). Then, we assume that \( \mathfrak{F}_j(n_1, \ldots, n_{k_j}, t) \) is of the form:

\[
\mathfrak{F}_j(n_1, \ldots, n_{k_j}, t) = C_j(n_1, \ldots, n_{k_j}, t) \cdot D_j(g_{n_1}, \ldots, g_{n_{k_j}})(t),
\]

where \( C_j : (\mathbb{Z}^d)^{k_j} \times \mathbb{R}_+ \to \mathbb{C} \) is a deterministic function and \( D_j(g_{n_1}, \ldots, g_{n_{k_j}})(t) \) denotes a (Wick) renormalized product of Gaussian processes \( g_{n_i}(t) \), possibly with the Duhamel integral operator \( \mathcal{I} \), namely

\[
D_j(g_{n_1}, \ldots, g_{n_{k_j}})(t) = \pi_{k_j} \left( \prod_{i=1}^{k_j} g_{n_i}(t) \right)
\]

or

\[
D_j(g_{n_1}, \ldots, g_{n_{k_j}})(t) = \mathcal{I}_{n_1+\ldots+n_{k_j}} \left( \pi_{k_j} \left( \prod_{i=1}^{k_j} g_{n_i}(t) \right) \right)(t)
\]

\[
= \int_0^t S_{n_1+\ldots+n_{k_j}}(t - \tau) \pi_{k_j} \left( \prod_{i=1}^{k_j} g_{n_i}(\tau) \right) d\tau.
\]

Here, \( \pi_{k_j} \) denotes the orthogonal projection onto \( \mathcal{H}_{k_j} \) and \( \{ g_n(t) \}_{n \in \mathbb{Z}^d} \) is a family of independent Gaussian processes conditioned that \( g_{\cdot n}(t) = g_n(t) \). In case of a PDE with the linear part \( \partial_t u - Lu \) (i.e. with the first order derivative in time) such as the KdV equation, \( g_n(t) \) is simply a standard complex-valued Gaussian random variable (i.e. independent of time \( t \)). In case of a PDE with the linear part \( \partial_t^2 u + Lu \) (i.e. with the second order derivative in time), \( g_n(t) \) is indeed time-dependent. For example, for the dispersion-generalized wave equation with the linear part \( \partial_t^2 u + (1 - \Delta)^\alpha u \), we have

\[
g_n(t) = \cos(t \langle n \rangle^\alpha) g_n + \sin(t \langle n \rangle^\alpha) h_n,
\]

where \( \langle n \rangle = \sqrt{1 + |n|^2} \) and the series \( \{ g_n, h_n \}_{n \in \mathbb{Z}^d} \) is a family of independent standard complex-valued Gaussian random variables conditioned that \( g_{\cdot n} = \overline{g_n} \) and \( h_{\cdot n} = h_n \), \( n \in \mathbb{Z}^d \). See \( \text{[3.3]} \) below. Note that \( \text{[1.13]} \) is clearly satisfied under \( \text{[1.14]} \) and \( \text{[1.15]} \) (or \( \text{[1.15]} \)).

Let \( N_j = \{ K_j - 1, \ldots, K_j \} \). Given an integer \( \ell = 0, 1, \ldots, \lceil \frac{K_j}{2} \rceil \), we define \( \Pi_\ell \) to be the collection of pairings \( \mathcal{P} \) on \( \{ 1, \ldots, K_j \} \) such that

\[\text{[3.3]}\]
(i) $\mathcal{P}$ respects the partition 
$$\mathcal{A} = \{N_1, \ldots, N_J\} = \{\{1, \ldots, K_1\}, \{K_1 + 1, \ldots, K_2\}, \ldots, \{K_{J-1} + 1, \ldots, K_J\}\},$$

(ii) $|\mathcal{P}| = 2\ell$ (when we view $\mathcal{P}$ as a subset of $\{1, \ldots, n_{K_J}\}$).

With these notations, let us rewrite
$$\pi_{K_j-2\ell} \left( \prod_{j=1}^J 3_j(n_{K_{j-1}+1}, \ldots, n_{K_j}, t) \right), \quad (1.17)$$

where $\pi_{K_j-2\ell}$ denotes the orthogonal projection onto $\mathcal{H}_{K_j-2\ell}$. Given a partition $A$ and $B$ of $\{1, \ldots, J\}$, suppose that $3_j$, $j \in A$, is given by (1.13) and (1.14), while $3_j$, $j \in B$, is given by (1.13) and (1.15). Then, we have
$$\pi_{K_j-2\ell} \left( \prod_{j=1}^J 3_j(n_{K_{j-1}+1}, \ldots, n_{K_j}, t) \right) = \sum_{\mathcal{P} \in \Pi_j} 1_{\{n_1, \ldots, n_{K_j}\}} \cdot \Theta_{\mathcal{P}}(n_1, \ldots, n_{K_j}, t). \quad (1.18)$$

Here, with $n_j = n_{K_{j-1}+1} + \cdots + n_{K_j}$, the expression $\Theta_{\mathcal{P}}$ is given by
$$\Theta_{\mathcal{P}}(n_1, \ldots, n_{K_j}, t) = \left( \sum_{j=1}^J C_j(n_{K_{j-1}+1}, \ldots, n_{K_j}, t) \right) \times \left\{ \int_{[0,\ell][\mathcal{B}]} \prod_{j \in B} S_{n_j}(t - \tau_j) \cdot Q_{\mathcal{P}}(\{n_i, i \in \mathcal{P}\}, \{\tau_j, j \in B\}, t) \right. $$
$$\times \pi_{K_j-2\ell} \left( \left( \prod_{j \in B} \left( \prod_{i \in N_j \setminus \mathcal{P}} \mathcal{g}_{n_i}(\tau_j) \right) \right) \left( \prod_{i \in \bigcup_{j \in \mathcal{A}} N_j \setminus \mathcal{P}} \mathcal{g}_{n_i}(t) \right) \right) \prod_{j \in B} d\tau_j \right\} \quad (1.19)$$

for some deterministic function $Q_{\mathcal{P}} = Q_{\mathcal{P}}(\{n_i, i \in \mathcal{P}\}, \{\tau_j, j \in B\}, t)$. The expression (1.18)-(1.19) follows from (1.17) with (1.13), (1.14), and (1.15) and contracting the frequencies $n_i$, $i \in \mathcal{P}$. For simplicity, let us consider the following example. Let $g$ be a mean-zero real-valued Gaussian random variable with variance $\sigma$ i.e. $g \sim \mathcal{N}\left(0, \sigma^2\right)$, and consider $3_j = \prod_{j=1}^J 3_j$ with $3_j = g$, $j = 1, \ldots, J$, and the partition $\mathcal{A} = \{\{1\}, \{2\}, \ldots, \{J\}\}$. Then, as a corollary to Wick’s theorem (see (3.5) (1.19b)), we have
$$3 = g^J = \sum_{\ell=0}^{[J/2]} \frac{J!}{2^\ell (J-2\ell)! \ell!} H_{J-2\ell}(g; \sigma) \cdot \sigma^\ell \quad (1.20)$$

where $H_k(x; \sigma) = \sigma^{1/2} H_k(\sigma^{-1/2} x)$; see (3.5) below. Here, we used the fact that the number of $\ell$ pairings is given by
$$\sum_{\mathcal{P} \in \Pi_j} 1 = \frac{J!}{2^\ell (J-2\ell)! \ell!}.$$ 

Compare (1.20) with (1.17), (1.18), and (1.19). The power $\sigma^\ell$ in (1.20) represents the contribution of the contracted terms, which appears in $Q_{\mathcal{P}}$ in (1.19).

We now state the main product estimates.
Proposition 1.3. Let $F_j$, $j = 1, \ldots, J$, be stochastic processes, satisfying \([1.8]\) and \([1.9]\).

(i) Suppose that, for each $j = 1, \ldots, J$, the term $\mathcal{F}_j(n_1, \ldots, n_k, t)$ is symmetric in $n_1, \ldots, n_k$ for any $t \in \mathbb{R}_+$ and satisfy Assumption A (namely, \([1.12]\)). Then, the following estimates hold:

$$
\mathbb{E}\left[ |\pi_{K_j} F_x \left( \prod_{j=1}^J F_j \right)(n,t) |^2 \right] \lesssim \sum_{n=n_1+\cdots+n_J} \prod_{j=1}^J \mathbb{E}\left[ |F_j(n_j,t) |^2 \right]
$$

and

$$
\mathbb{E}\left[ \delta_h \pi_{K_j} F_x \left( \prod_{j=1}^J F_j \right)(n,t) |^2 \right] \\
\lesssim \sum_{\ell=1}^J \sum_{n=n_1+\cdots+n_J} \mathbb{E}\left[ |\delta_h F_{\ell}(n_{\ell},t) |^2 \right] \\
\times \left( \prod_{j=\ell+1}^{J} \mathbb{E}\left[ |F_j(n_j,t+h) |^2 \right] \right) \left( \prod_{j=\ell}^{J} \mathbb{E}\left[ |F_j(n_j,t) |^2 \right] \right)
$$

for any $n \in \mathbb{Z}^d$, $t \in \mathbb{R}_+$, and $h \in \mathbb{R}$ (with $h \geq -t$ such that $t+h \geq 0$), where $\delta_h$ is the difference operator defined in \([1.4]\).

(ii) Fix a partition $A$ and $B$ of $\{1, \ldots, J\}$. Suppose that $\mathcal{F}_j$, $j \in A$, is given by \([1.13]\) and \([1.14]\), while $\mathcal{F}_j$, $j \in B$, is given by \([1.13]\) and \([1.15]\). Then, we have

$$
\mathbb{E}\left[ |F_x \left( \prod_{j=1}^J F_j \right)(n,t) |^2 \right] = \sum_{\ell=0}^{[K_j/2]} \mathbb{E}\left[ |\pi_{K_j-2\ell} F_x \left( \prod_{j=1}^J F_j \right)(n,t) |^2 \right]
$$

$$
\lesssim \sum_{\ell=0}^{[K_j/2]} \sum_{P \in \Pi_{\ell} \{n_i\}_{i \in P}} \mathbb{E}\left[ \sum_{\{n_i\}_{i \in P}} 1_{n=n_1+\cdots+n_{K_j}} 1_{(n_1, \ldots, n_{K_j}, t) \text{ admissible w.r.t. } P} \mathcal{P}(n_1, \ldots, n_{K_j}, t) \right]^2,
$$

where $\mathcal{P}$ is as in \([1.13]\). In particular, from \([1.23]\) and \([1.19]\), we have

$$
\mathbb{E}\left[ |F_x \left( \prod_{j=1}^J F_j \right)(n,t) |^2 \right] = \sum_{\ell=0}^{[K_j/2]} \mathbb{E}\left[ |\pi_{K_j-2\ell} F_x \left( \prod_{j=1}^J F_j \right)(n) |^2 \right]
$$

$$
\lesssim \sum_{\ell=0}^{[K_j/2]} \sum_{P \in \Pi_{\ell} \{n_i\}_{i \in P}} \mathbb{E}\left[ \sum_{\{n_i\}_{i \in P}} 1_{n=n_1+\cdots+n_{K_j}} 1_{(n_1, \ldots, n_{K_j}) \text{ admissible w.r.t. } P} \\
\times \left( \prod_{j=1}^J C_j(n_{K_j-1+1}, \ldots, n_{K_j}, t) \right) \\
\times \left\{ \int_{[0,t]} \prod_{j \in B} S_{\eta_j}(t-\tau_j) : Q_P(\{n_i, i \in P\}, \{\tau_j, j \in B\}, t) \right\} \\
\times \pi_{K_j-2\ell} \left( \left( \prod_{j \in B} \left( \prod_{i \in N_j \setminus P} g_{\eta_j}(\tau_j) \right) \left( \prod_{i \in \bigcup_{j \in A} N_j \setminus P} g_{\eta_i}(t) \right) \right) \right) \prod_{j \in B} d\tau_j \right)^2. \quad (1.24)
$$
As in Part (i), a similar bound also holds for the time difference \( \delta_h F_x \left( \prod_{j=1}^J F_j \right)(n,t) \).

For simplicity, we stated Proposition 1.3 for a usual product but a slight modification covers the case of paraproducts and resonant products. With a multiple stochastic integral representation, Proposition 1.3 (i) follows from Jensen’s inequality; see Appendix B in [33]. In Section 2, we present a proof without a multiple stochastic integral representation.

Proposition 1.3 (ii) is in the spirit of Lemma 4.1 in [9]. In the dispersive setting, we typically apply Proposition 1.3 and then apply a counting argument (to the right-hand side of (1.24)) to exhibit multilinear dispersive smoothing. In the non-dispersive setting, for example in the case of the nonlinear Schrödinger equation with the Grushin Laplacian studied in [12], there is no such multilinear dispersive smoothing and thus we can simply apply Proposition 1.3 to estimate products of stochastic objects. In treating a nonlinearity of a very high degree, in particular in a weakly dispersive setting, Proposition 1.3 (without a counting argument) provides a sufficiently good control on products of stochastic objects.

Remark 1.4. (i) Proposition 1.3 (i) with Lemma 1.1 allows us to immediately determine the regularity of the Wick powers:

\[
Z_k = \pi_k(Z_k) \quad \text{of a basic stochastic object (such as a random linear solution or a stochastic convolution). See the proof of Proposition 3.1 (i) for such an application of Proposition 1.3 (i).}
\]

(ii) By using a symmetrization, we can always satisfy the symmetry assumption in Proposition 1.3 (i). More precisely, given \( F_j \) in (1.8), we have

\[
\hat{F}_j(n,t) = \sum_{n=n_1+\cdots+n_{k_j}} \hat{\xi}_j(n_1, \ldots, n_{k_j}, t) = \frac{1}{k_j!} \sum_{\sigma \in S_{k_j}} \sum_{n=n_1+\cdots+n_{k_j}} \hat{\xi}_j(n_{\sigma(1)}, \ldots, n_{\sigma(k_j)}, t) = \sum_{n=n_1+\cdots+n_{k_j}} \text{Sym}(\hat{\xi}_j)(n_1, \ldots, n_{k_j}, t),
\]

where \( \text{Sym}(\hat{\xi}_j) \) is the symmetrization of \( \hat{\xi}_j \) defined by

\[
\text{Sym}(\hat{\xi}_j)(n_1, \ldots, n_{k_j}, t) = \frac{1}{k_j!} \sum_{\sigma \in S_{k_j}} \hat{\xi}_j(n_{\sigma(1)}, \ldots, n_{\sigma(k_j)}, t).
\]

Hence, without loss of generality, we may assume that \( \hat{\xi}_j \) is symmetric. Under the symmetry assumption, the condition (1.9) reduces to

\[
E \left[ \hat{F}_j(n_1, t_1) \hat{F}_j(n_2, t_2) \right] \sim \sum_{n=n_1+\cdots+n_{k_j}} E \left[ \hat{\xi}_j(n_1, \ldots, n_{k_j}, t_1) \hat{\xi}(n_1, \ldots, n_{k_j}, t_2) \right].
\]

(iii) In practice, in applying Proposition 1.3 (ii) to a product, we may assume that \( A = \{1\} \) and \( B = \{2, \ldots, J\} \) since, otherwise, i.e. if \( |A| \geq 2 \), then the product \( F_j F_\ell \), \( j, \ell \in A \) is divergent in the singular setting, requiring a further renormalization. Even in the case \( |A| \geq 2 \), Proposition 1.3 (ii) is useful in estimating non-standard products such as paraproducts (without using the deterministic paraproduct estimate).

In Proposition 3.1 (iii), we apply Proposition 1.3 (ii) with \( A = \{1\} \) to estimate products of stochastic objects for the dispersion-generalized nonlinear wave equations. Our proof shows that as long as the lower order terms, belonging to \( \mathcal{H}_{K_j - 2\ell}, \ell = 1, \ldots, \left[ \frac{K_j}{2} \right] \), are convergent, the regularity of the resulting stochastic term is determined by the contribution
from the highest order term belonging to $\mathcal{H}_{K,J}$, which is estimated by Proposition 1.3(i). See Remark 3.3.

(iv) While the bounds (1.23) and (1.24) look complicated, they are in their simplest form and easy to use. See the proof of Proposition 3.1(iii) below. Without such estimates, a concrete combinatorial argument can lead to a computational nightmare, even for a product with a relatively small number of terms. See, for example, Appendix A in [31] (which is more in the context of Proposition 1.3(i)).

We first present a proof of Proposition 1.3 in Section 2. In Section 3, we then consider the dispersion-generalized nonlinear wave equations as an example, for which we construct stochastic objects, using Proposition 1.3, and prove local well-posedness (Theorem 3.4). We point out that even with Proposition 1.3 at hand, there is some nontrivial combinatorial difficulty that we need to overcome; see the proof of Proposition 3.1.

2. PROOF OF PROPOSITION 1.3

We first recall the Wiener chaos estimate ([35, Theorem I.22]) which follows as a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup due to Nelson [20]. See also [39, Proposition 2.4].

Lemma 2.1. Let $k \in \mathbb{N}$. Then, we have

$$
\|X\|_{L^p(\Omega)} \leq (p - 1)^{\frac{1}{2}} \|X\|_{L^2(\Omega)}
$$

for any $p \geq 2$ and any $X \in \mathcal{H}_{\leq k}$.

We now present the proof of Proposition 1.3.

Proof of Proposition 1.3. (i) We first carry out analysis for fixed $t \in \mathbb{R}_+$ and thus suppress the $t$-dependence for notational simplicity. Recall the definition of $K_j$ from (1.10). By (1.8), (1.12), Cauchy-Schwarz’ inequality (in $\omega$), and Cauchy’s inequality (together with the Pythagoras theorem to remove $\pi_{K_j}$), we have

$$
\mathbb{E}\left[\pi_{K_j} F_x \left( \prod_{j=1}^{J} F_j \right) (n) \right]^2 \leq \sum_{n=n_1+\cdots+n_{K,J}} \sum_{n=m_1+\cdots+m_{K,J}} \mathbb{E}\left[\pi_{K_j} \left( \prod_{j=1}^{J} \tilde{F}_j (n_{K_j-1+1}, \ldots, n_{K_j}) \right) \right] \times \pi_{K_j} \left( \prod_{j=1}^{J} \tilde{F}_j (m_{K_j-1+1}, \ldots, m_{K_j}) \right)
$$

$$
= \sum_{\sigma \in S_{K_j}} \sum_{n=n_1+\cdots+n_{K,J}} \mathbb{E}\left[\pi_{K_j} \left( \prod_{j=1}^{J} \tilde{F}_j (n_{K_j-1+1}, \ldots, n_{K_j}) \right) \right] \times \pi_{K_j} \left( \prod_{j=1}^{J} \tilde{F}_j (n_{\sigma(K_j-1)}, \ldots, n_{\sigma(K_j)}) \right)
$$
and the Wiener chaos estimate (Lemma 2.1), we have
for any \( n \in \mathbb{Z}^d \), where the last step follows since the action of the permutation \( \sigma \) amounts to relabeling indices. Then, from (2.1), Hölder’s inequality, Minkowski’s integral inequality, and the Wiener chaos estimate (Lemma 2.1), we have

\[
\text{RHS of (2.1)} = \sum_{n=m_1 + \cdots + m_j} \mathbb{E} \left[ \prod_{j=1}^J \left| \mathbb{1}_{m_j=n_{K_j+1}+\cdots+n_{K_j}} \right|^2 \right] 
\]

\[
\leq \sum_{n=m_1 + \cdots + m_j} \prod_{j=1}^J \left| \mathbb{1}_{m_j=n_{K_j+1}+\cdots+n_{K_j}} \right|^2 \times \mathbb{E} \left[ \left| \mathfrak{A}_j(n_{K_{j-1}+1}, \ldots, n_{K_j}) \right|^2 \right].
\]

(2.2)

Then, from (1.22), we obtain (1.21).

By a computation analogous to (2.1) with (1.4), we have

\[
\mathbb{E} \left[ \left| \delta_h \pi_{K_j} \mathcal{F}_x \left( \prod_{j=1}^J F_j \right)(n, t) \right|^2 \right]
\]

\[
\lesssim \sum_{n=n_1 + \cdots + n_{K_j}} \mathbb{E} \left[ \left| \delta_h \left( \prod_{j=1}^J \mathfrak{A}_j(n_{K_{j-1}+1}, \ldots, n_{K_j}, t) \right) \right|^2 \right]
\]

\[
\lesssim \sum_{\ell=1}^{J} \sum_{n=n_1 + \cdots + n_{K_j}} \mathbb{E} \left[ \delta_h \mathfrak{A}_\ell(n_{K_{\ell-1}+1}, \ldots, n_{K_\ell}, t) \times \left( \prod_{j=1}^{\ell-1} \mathfrak{A}_j(n_{K_{j-1}+1}, \ldots, n_{K_j}, t+h) \right) \left( \prod_{j=\ell+1}^J \mathfrak{A}_j(n_{K_{j-1}+1}, \ldots, n_{K_j}, t) \right) \right]^2.
\]

(2.3)

Then, by proceeding as in (2.2) to bound each term on the right-hand side of (2.3), we obtain (1.22).
(ii) We only prove the bound (1.23). As in Part (i), we suppress the \( t \)-dependence. By the orthogonality of \( H \) with \([1.8], \(1.13\), \(1.14\), and \(1.15\), we have

\[
\mathbb{E} \left[ |\mathcal{F}_x \left( \prod_{j=1}^{J} F_j \right) (n)|^2 \right] = \sum_{\ell=0}^{[K_{J}/2]} \mathbb{E} \left[ |\pi_{K_{J}-2\ell} \mathcal{F}_x \left( \prod_{j=1}^{J} F_j \right) (n)|^2 \right]
\]

\[
= \sum_{\ell=0}^{[K_{J}/2]} \sum_{n_1+\ldots+n_{K_{J}}=n} \sum_{m_1+\ldots+m_{K_{J}}=m} \mathbb{E} \left[ \pi_{K_{J}-2\ell} \left( \prod_{j=1}^{J} \mathcal{F}_j (n_{K_{J}-1+j}, \ldots, n_{K_{J}}) \right) \right. \times \left. \pi_{K_{J}-2\ell} \left( \prod_{j=1}^{J} \mathcal{F}_j (m_{K_{J}-1+j}, \ldots, m_{K_{J}}) \right) \right] .
\]

Given \( \mathcal{P} \in \Pi_\ell \) for some \( \ell = 0, 1, \ldots, \left[ \frac{K_{J}}{2} \right] \), let \( S_{K_{J}-2\ell}(\mathcal{P}) \) denote the symmetric group on the unpaired indices \( \{ i \in \{ 1, \ldots, K_{J} \} : i \notin \mathcal{P} \} \). Then, by \([1.18]\) and Cauchy's inequality, we have

\[
\sum_{n_1+\ldots+n_{K_{J}}=n} \sum_{m_1+\ldots+m_{K_{J}}=m} \mathbb{E} \left[ \pi_{K_{J}-2\ell} \left( \prod_{j=1}^{J} \mathcal{F}_j (n_{K_{J}-1+j}, \ldots, n_{K_{J}}) \right) \right. \times \left. \pi_{K_{J}-2\ell} \left( \prod_{j=1}^{J} \mathcal{F}_j (m_{K_{J}-1+j}, \ldots, m_{K_{J}}) \right) \right] \]

\[
\leq \sum_{\mathcal{P} \in \Pi_\ell} \sum_{\sigma \in S_{K_{J}-2\ell}(\mathcal{P})} \mathbb{E} \left[ \sum_{\{ n_{i} \}, i \notin \mathcal{P}} 1_{(n_{1}, \ldots, n_{K_{J}})}(\mathcal{P}) \mathcal{F}_j(n_{1}, \ldots, n_{K_{J}}) \right. \times \left. \sum_{\{ m_{i} \}, i \notin \mathcal{P}} 1_{(m_{1}, \ldots, m_{K_{J}})}(\mathcal{P}) \mathcal{F}_j(m_{1}, \ldots, m_{K_{J}}) \right] \]

\[
\leq \sum_{\mathcal{P} \in \Pi_\ell} \sum_{\{ n_{i} \}, i \notin \mathcal{P}} \mathbb{E} \left[ \sum_{\{ m_{i} \}, i \notin \mathcal{P}} 1_{(n_{1}, \ldots, n_{K_{J}})}(\mathcal{P}) \mathcal{F}_j(n_{1}, \ldots, n_{K_{J}}) \right. \times \left. \sum_{\{ m_{i} \}, i \notin \mathcal{P}} 1_{(m_{1}, \ldots, m_{K_{J}})}(\mathcal{P}) \mathcal{F}_j(m_{1}, \ldots, m_{K_{J}}) \right] .
\]

Here, the first inequality (i.e. the second step) in (2.5) follows from \([1.19]\) and the definition of \( \{ g_{n} \}_{n \in \mathbb{Z}^d} \). More precisely, noting that \( \sum_{j=1}^{J} |\mathcal{N}_{j} \setminus \mathcal{P}| = K_{J} - 2\ell \), it follows from the definition
of \{g_n\}_{n \in \mathbb{Z}^d} and Wick renormalized products that
\[
\mathbb{E} \left[ \pi_{K_j - 2\ell} \left( \prod_{j \in B} \left( \prod_{i \in N_j \setminus \mathcal{P}} g_{n_i}(t) \right) \right) \left( \prod_{i \in \bigcup_{j \in A} N_j \setminus \mathcal{P}} g_{n_i}(t) \right) \right] 
\times \pi_{K_j - 2\ell} \left( \prod_{j \in B} \left( \prod_{i \in N_j \setminus \mathcal{P}} g_{m_i}(t) \right) \right) \left( \prod_{i \in \bigcup_{j \in A} N_j \setminus \mathcal{P}} g_{m_i}(t) \right) = 0
\]

unless \{n_i\}_{i \notin \mathcal{P}} = \{m_i\}_{i \notin \mathcal{P}}.

Therefore, the bound (1.23) follows from (2.4) and (2.5). \qed

3. Example: dispersion-generalized nonlinear wave equations

3.1. Dispersion-generalized nonlinear wave equation with random initial data. For \(\alpha > 0\), consider the following dispersion-generalized nonlinear wave equation (NLW) on \(\mathbb{T}^d\):
\[
\begin{aligned}
\partial_t^2 u + (1 - \Delta)^\alpha u + u^k &= 0 \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1)
\end{aligned}
\tag{3.1}
\]

with the random initial data of the form
\[
u_0^\omega = \sum_{n \in \mathbb{Z}^d} g_n(\omega) e^{in \cdot x} \quad \text{and} \quad u_1^\omega = \sum_{n \in \mathbb{Z}^d} h_n(\omega) e^{in \cdot x}
\tag{3.2}
\]

for \(\beta \in \mathbb{R}\), where the series \(\{g_n, h_n\}_{n \in \mathbb{Z}^d}\) is a family of independent standard complex-valued Gaussian random variables conditioned that \(g_{-n} = \overline{g_n}\) and \(h_{-n} = \overline{h_n}, n \in \mathbb{Z}^d\). It is easy to see that \(u_0^\omega\) in (3.2) belongs to \(W^{s,p}(\mathbb{T}^d) \setminus W^{\beta - \frac{d}{2}, p}(\mathbb{T}^d)\) for any \(s < \beta - \frac{d}{2}\) and \(1 \leq p \leq \infty\), almost surely. In the following, we consider the singular case \(\beta \leq \frac{d}{2}\) such that \(u_0^\omega\) is merely a distribution.

Let \(Z\) denote the random linear solution given by
\[
Z(t) = \cos(t(\nabla)^\alpha)u_0^\omega + \frac{\sin(t(\nabla)^\alpha)}{(\nabla)^\alpha}u_1^\omega \\
= \sum_{n \in \mathbb{Z}^d} \frac{\cos(t(n)^\alpha)g_n(\omega) + \sin(t(n)^\alpha)h_n(\omega)}{(n)^\beta} e^{in \cdot x},
\tag{3.3}
\]

where \(\langle \nabla \rangle^\alpha = (1 - \Delta)^{\frac{\alpha}{2}}\). The case \(\beta = \alpha\) is of particular importance since the distribution of \((u_0^\omega, u_1^\omega)\) in (3.2) corresponds to the base Gaussian measure of the Gibbs measure for the dispersion-generalized NLW (3.1). See, for example, \cite{32, 33, 11} when \(d = 2\) and \(\frac{1}{2} < \alpha = \beta \leq 1\).

Given \(N \in \mathbb{N}\), we set \(Z_N = P_N Z\), where \(P_N\) denotes the frequency cutoff onto the spatial frequencies \(\{|n| \leq N\}\). Then, for each fixed \(x \in \mathbb{T}^d\) and \(t \geq 0\), a direct computation with \(\mathbb{E}\) shows that \(Z_N(x, t)\) is a mean-zero real-valued Gaussian random variable with variance
\[
\sigma_N = \mathbb{E} [(Z_N(x, t))^2] = \sum_{|n| \leq N} \frac{1}{(n)^{2\beta}} \rightarrow \infty,
\]
as \(N \rightarrow \infty\), since \(\beta \leq \frac{d}{2}\). This essentially shows that \(\{Z_N(t)\}_{N \in \mathbb{N}}\) is almost surely unbounded in \(W^{0,p}(\mathbb{T}^d)\) for any \(1 \leq p \leq \infty\). We now introduce the Wick renormalization:
\[
Z^\ell_N(x, t) \overset{\text{def}}{=} H_\ell(Z_N(x, t); \sigma_N),
\tag{3.4}
\]
where $H_{\ell}(x,\sigma)$ is the Hermite polynomial of degree $\ell$ with a variance parameter $\sigma$, defined by the generating function:

$$G(t, x; \sigma) = e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x; \sigma). \quad (3.5)$$

Next, we define the second order term $\mathcal{I}(\cdot; Z_N^k \cdot)$, where $\mathcal{I}$ denotes the Duhamel integral operator given by

$$\mathcal{I}(F)(t) = \int_0^t \sin((t - \tau)\langle \nabla \rangle^{\alpha}) F(\tau) d\tau.$$

As in [14] [16] [27] [23], we consider the renormalized version of (3.1) with the truncated initial data $(\mathcal{P}_N u_0^0, \mathcal{P}_N u_1^0) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$. Let $u_N$ be the solution to (3.1) with the truncated initial data $(u_N, \partial_t u_N)|_{t=0} = (\mathcal{P}_N u_0^0, \mathcal{P}_N u_1^0)$. Then, by writing $u_N$ in the first order expansion $u_N = Z_N + w_N$, we have

$$u_N^k = \sum_{\ell=0}^{k} \binom{k}{\ell} Z_N^\ell w_N^{k-\ell}.$$

In the singular setting, namely, $\beta \leq \frac{d}{2}$, $Z = \lim_{N \to \infty} Z_N$ is only a distribution and thus the limit of $Z_N^\ell$ does not exist. In order to overcome this issue, we consider the following renormalized nonlinearity:

$$:u_N^k := H_k(Z_N + w_N; \sigma_N) = \sum_{\ell=0}^{k} \binom{k}{\ell} :Z_N^\ell :w_N^{k-\ell},$$

where the right-hand side is well defined as long as $w_N$ has sufficient regularity. In terms of the remainder term $w_N = u_N - Z_N$, the renormalized equation reads as follows:

$$\partial_t^2 w_N + (1 - \Delta)^{\alpha} w_N + \sum_{\ell=0}^{k} \binom{k}{\ell} :Z_N^\ell :w_N^{k-\ell} = 0 \quad (3.6)$$

with the zero initial data $(w_N, \partial_t w_N)|_{t=0} = (0, 0)$. The spatial regularity of $Z_N^\ell$ (in the limiting sense as $N \to \infty$) is given by $\ell(\beta - \frac{d}{2}) - \epsilon$ for any $\epsilon > 0$ (see Proposition 3.1 (i) below) and in particular, when $\beta < \frac{d}{2}$, the worst term in the nonlinearity is given by the highest order term $:Z^k :$. In the next step, we consider the second order expansion and remove this worst term.

Write $u_N$ in the second order expansion

$$u_N = Z_N - \mathcal{I}(\cdot; Z_N^k \cdot) + v_N,$$

where $w_N$ satisfies (3.6). Then, by noting that

$$(\partial_t + (1 - \Delta)^{\alpha}) \mathcal{I}(\cdot; Z_N^k \cdot) = :Z_N^k :,$$ 

we can reduce (3.6) to the following renormalized equation for the new remainder term $v_N$:

$$\partial_t^2 v_N + (1 - \Delta)^{\alpha} v_N$$

$$+ \sum_{k_1=0}^{k-1} \sum_{k_2, k_3=0}^{k} 1_{k_1+k_2+k_3} \frac{k!}{k_1!k_2!k_3!} :Z_N^{k_1} : (\mathcal{I}(\cdot; Z_N^k \cdot))^{k_2} v_N^{k_3} = 0 \quad (3.9)$$
with the zero initial data \((v_N, \partial_t v_N)|_{t=0} = (0, 0)\). Note that, thanks to the second order expansion (in particular, (3.8)), the worst term \(Z_N^k\) in the equation (3.6) for the remainder term \(w_N\) of the first order expansion is eliminated. See also Subsection 1.2 in [27]. In order to study the well-posedness issue of (3.9), we first need to study the regularity property of the stochastic objects

\[
:Z_N^{k_1} : (\mathcal{I}(Z_N^k))^{k_2}
\]  

(3.10)

for \(k_1 = 0, \ldots, k - 1\) and \(k_2 = 0, \ldots, k\). In the next subsection, we study the regularity property of these stochastic objects by using Proposition 1.3 and Lemma 1.1. See Proposition 3.1.

In Subsection 3.3 we then establish local well-posedness of (3.9); see Theorem 3.4 below.

3.2. Regularities of stochastic objects. Our main goal in this subsection is to prove the following proposition on the regularity property of the stochastic terms in (3.10), appearing in the equation (3.9).

**Proposition 3.1.** Let \(\alpha > 0\), \(\beta \leq \frac{d}{2}\), and \(k \in \mathbb{N}\).

(i) Given \(\ell \in \mathbb{N}\), let \(\beta > \frac{4}{2\ell} d\). Then, for \(s < \ell(\beta - \frac{d}{2})\), \(\{Z_N^\ell : \mathcal{I}(Z_N^k)\}_{N \in \mathbb{N}}\) is a Cauchy sequence in \(C(\mathbb{R}_+; W^{s, \infty}(\mathbb{T}^d))\), almost surely. In particular, denoting the limit by \(Z^\ell\), we have

\[
:Z^\ell : C(\mathbb{R}_+; W^{\ell(\beta - \frac{d}{2}) - \varepsilon, \infty}(\mathbb{T}^d))
\]

for any \(\varepsilon > 0\), almost surely.

(ii) Assume \(\beta > \frac{k - 1}{2k} d\). Then, for \(s < k(\beta - \frac{d}{2}) + \alpha\), \(\{\mathcal{I}(Z_N^k)\}_{N \in \mathbb{N}}\) is a Cauchy sequence in \(C(\mathbb{R}_+; W^{s, \infty}(\mathbb{T}^d))\), almost surely. In particular, denoting the limit by \(\mathcal{I}(Z^k)\), we have

\[
\mathcal{I}(Z^k) : C(\mathbb{R}_+; W^{k(\beta - \frac{d}{2}) + \alpha, \infty}(\mathbb{T}^d))
\]

for any \(\varepsilon > 0\), almost surely.

Furthermore, suppose that \(k(\beta - \frac{d}{2}) + \alpha > 0\). Let \(k_2 \in \mathbb{N}\). Then, for \(s < k(\beta - \frac{d}{2}) + \alpha\), \(\{(\mathcal{I}(Z_N^k))^k_2\}_{N \in \mathbb{N}}\) is a Cauchy sequence in \(C(\mathbb{R}_+; W^{s, \infty}(\mathbb{T}^d))\), almost surely. In particular, denoting the limit by \((\mathcal{I}(Z^k))^{k_2}\), we have

\[
(\mathcal{I}(Z^k))^{k_2} : C(\mathbb{R}_+; W^{k(\beta - \frac{d}{2}) + \alpha - \varepsilon, \infty}(\mathbb{T}^d))
\]

for any \(\varepsilon > 0\), almost surely.

(iii) Fix integers \(0 \leq k_1 \leq k - 1\) and \(0 \leq k_2 \leq k\). Suppose that \(\beta > \frac{k_1 - 1}{2k} d\) and \(k(\beta - \frac{d}{2}) + \alpha > 0\), which are the conditions from (i) and (ii). Furthermore, we assume that

\[
\beta > \frac{d}{2} - \frac{\alpha}{2k}.
\]  

(3.11)

Given \(N \in \mathbb{N}\), define \(Y_N\) by

\[
Y_N = :Z_N^{k_1} : (\mathcal{I}(Z_N^k))^{k_2}.
\]

Then, for \(s < k_1(\beta - \frac{d}{2})\), \(\{Y_N\}_{N \in \mathbb{N}}\) is a Cauchy sequence in \(C(\mathbb{R}_+; W^{s, \infty}(\mathbb{T}^d))\), almost surely. In particular, denoting the limit by \(Y = :Z^{k_1} : (\mathcal{I}(Z^k))^{k_2}\), we have

\[
Y = :Z^{k_1} : (\mathcal{I}(Z^k))^{k_2} : C(\mathbb{R}_+; W^{k_1(\beta - \frac{d}{2}) - \varepsilon, \infty}(\mathbb{T}^d))
\]

for any \(\varepsilon > 0\), almost surely.
By putting all the conditions together, the range of admissible $\beta$ is given by
\[
\max \left( \frac{d}{2} - \frac{d}{2k} - \frac{\alpha}{2k} \right) < \beta \leq \frac{d}{2},
\]
which gets smaller and smaller as $\alpha \to 0$ and $k \to \infty$.

It is possible to exploit multilinear dispersive smoothing to improve Part (ii) (and hence Part (iii)) of Proposition 3.1 (especially when $\alpha > 0$ is not small). See for example [15, 5, 33]. Note that such multilinear dispersive analysis depends sensitively on the values of $k$ and $\alpha$ and moreover the gain from multilinear dispersive smoothing becomes small for small values of $\alpha$ (i.e. in a weakly dispersive case). We do not pursue this direction in this note.

Before proceeding to the proof of Proposition 3.1, we first recall a calculus lemma.

**Lemma 3.2.** (i) Let $d \geq 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy
\[
\alpha + \beta > d \quad \text{and} \quad \alpha, \beta < d.
\]
Then, we have
\[
\sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha-\beta}
\]
for any $n \in \mathbb{Z}^d$.

(ii) Let $d \geq 1$ and $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy
\[
\alpha + 2\beta > d, \quad \alpha + 2\gamma > d, \quad \text{and} \quad \alpha, 2\beta, 2\gamma < d.
\]
Then, we have
\[
\sum_{n \in \mathbb{Z}^d} \frac{1}{\langle n \rangle^\alpha \langle n+a \rangle^\beta \langle n+b \rangle^\gamma} \lesssim \langle a \rangle^{\frac{d}{2}-\frac{\alpha}{2}-\beta} \langle b \rangle^{\frac{d}{2}-\frac{\alpha}{2}-\gamma}
\]
for any $a, b \in \mathbb{R}^d$.

Lemma 3.2 follows from elementary computations. See, for example, Lemma 4.1 in [19] for the proof of (i). Then, (ii) is a straightforward consequence of (i) and Cauchy-Schwarz’ inequality.

**Proof of Proposition 3.7.** (i) Recalling that $Z_N^\ell := \pi_\ell(Z_N^\ell)$, Proposition 1.3(i) yields
\[
\mathbb{E}\left[ |\hat{Z}_N^\ell(n,t)|^2 \right] \lesssim \sum_{n=n_1+\ldots+n_\ell} \prod_{j=1}^\ell \mathbb{E}\left[ |\hat{Z}_N(n_j,t)|^2 \right]
\]
\[
\lesssim \sum_{n=n_1+\ldots+n_\ell} \prod_{|n_j| \leq N} \frac{1}{\langle n_j \rangle^2 \beta},
\]
Then, by iteratively carrying out the summations via Lemma 3.2(i), we obtain
\[
\mathbb{E}\left[ |\hat{Z}_N^\ell(n,t)|^2 \right] \lesssim \langle n \rangle^{(\ell-1)d-2\ell \beta + \varepsilon_0} = \langle n \rangle^{-d+2\ell(\beta-\frac{d}{2}) + \varepsilon_0}
\] (3.12)
with any small $\varepsilon_0 > 0$ when $\beta = \frac{d}{2}$ and $\varepsilon_0 = 0$ when $\beta < \frac{d}{2}$, provided that $\beta > \frac{\ell-1}{2\ell} \cdot d$. A slight modification of the computation above yields
\[
\mathbb{E}\left[ |\hat{Z}_N^\ell(n,t) - :Z_M^\ell(n,t):|^2 \right] \lesssim N^{-\gamma} \langle n \rangle^{-d-2\ell(\beta-\frac{d}{2}) + \gamma + \varepsilon_0}
\]
for some small $\gamma > 0$ and any $M \geq N \geq 1$. Hence, from Lemma 1.1, we see that $Z_N^k(t)$ converges to some limit in $W^{s,\infty}(\mathbb{T}^d)$ for $s < \ell(\beta - \frac{d}{2})$, almost surely. A similar computation with the mean value theorem gives difference estimates (1.6) and (1.7), from which we conclude that $Z_N^k(t)$ converges to some limit in $C(\mathbb{R}_+;W^{s,\infty}(\mathbb{T}^d))$ for $s < \ell(\beta - \frac{d}{2})$, almost surely.

In the remaining part of the proof, we only establish uniform (in $N$) bounds for fixed $t \in \mathbb{R}_+$ on relevant stochastic objects. A slight modification yields the continuity-in-time and convergence claims. For simplicity of notation, we drop the subscript $N$. Due to the presence of the time integration in the Duhamel integral operator, various estimates depend on the time $t$. Since such $t$-dependence plays no important role, we hide the $t$-dependence in implicit constants, appearing in the estimates.

(ii) The first claim follows easily from Part (i) with $\ell = k$ and the gain of $\alpha$-derivative under the Duhamel integral operator $I$. In particular, from (3.12), we have

$$
\mathbb{E} \left[ |I(\hat{Z}^k)(n, t)|^2 \right] \lesssim \langle n \rangle^{-d-2(k(\beta - \frac{d}{2}) + \alpha) + \varepsilon_0}$$

with any small $\varepsilon_0 > 0$ when $\beta = \frac{d}{2}$ and $\varepsilon_0 = 0$ when $\beta < \frac{d}{2}$, provided that $\beta > \frac{k-1}{2k}d$. By Lemma 1.1, we conclude that $I(\hat{Z}^k)(t) \in W^{s,\infty}(\mathbb{T}^d)$ for any $s < k(\beta - \frac{d}{2}) + \alpha$, almost surely.

Now, suppose that $k(\beta - \frac{d}{2}) + \alpha > 0$. Then, there exist $s > 0$ and $\varepsilon > 0$ with $0 < s < s + \varepsilon < k(\beta - \frac{d}{2}) + \alpha$ such that $I(\hat{Z}^k)(t) \in W^{s+\varepsilon,\infty}(\mathbb{T}^d)$, almost surely. By Sobolev’s inequality (with $\varepsilon r > d$) and the fractional Leibniz rule (see Lemma 3.5(i) below), we have

$$
\| (I(\hat{Z}^k))^k(t) \|_{W^{s,\infty}} \lesssim \| (I(\hat{Z}^k))^k(t) \|_{W^{s+\varepsilon,\infty}} \lesssim \| I(\hat{Z}^k)(t) \|^k_{W^{s+\varepsilon,\infty}}
$$

almost surely. This proves the second claim.

(iii) From (3.3) with (3.3), we have $Z^{k_1}(t) \in \mathcal{H}_{k_1}$ and $I(\hat{Z}^k)(t) \in \mathcal{H}_k$. Thus, we have

$$
Y(t) := Z^{k_1}_1(t) (I(\hat{Z}^k))^k(t) \in \mathcal{H}_{k_1+kk_2}.
$$

We first estimate the contribution of $Y(t)$ belonging to $\mathcal{H}_{k_1+kk_2}$. By Proposition 1.3(ii) with (3.12), (3.13), and Lemma 3.2(i), we have

$$
\mathbb{E} \left[ \pi_{k_1+kk_2} \hat{Y}(n, t)^2 \right] \lesssim \sum_{n=n_1+\ldots+n_{k_2+1}} \mathbb{E} \left[ |\hat{Z}^{k_1}(n_1, t)|^2 \right] \prod_{j=2}^{k_2+1} \mathbb{E} \left[ |I(\hat{Z}^k)(n_j, t)|^2 \right]
$$

$$
\lesssim \sum_{n=n_1+\ldots+n_{k_2+1}} \langle n_1 \rangle^{-d-2k_1(\beta - \frac{d}{2}) + \varepsilon_0} \prod_{j=2}^{k_2+1} \langle n_j \rangle^{-d-2(k(\beta - \frac{d}{2}) + \alpha) + \varepsilon_0}
$$

for any small $\varepsilon_1 > 0$, provided that $\beta > \frac{k_1-1}{2k_1}d$ and $k(\beta - \frac{d}{2}) + \alpha > 0$. Hence, by Lemma 1.1, we have $\pi_{k_1+kk_2}Y(t) \in W^{s+\varepsilon,\infty}(\mathbb{T}^d)$ for any $s < k_1(\beta - \frac{d}{2})$.

Next, we use Proposition 1.3(ii) to estimate the contribution of $Y(t)$ belonging to the lower order homogeneous Wiener chaoses. In the following, we first describe various parameters and objects, appearing in Proposition 1.3(ii), adapted to our current setting. Fix $\ell = 1, \ldots, \left[ \frac{k_1+kk_2}{2} \right]$. With $J = k_2 + 1$, set

$$
K_0 = 0, \quad K_1 = k_1, \quad \text{and} \quad K_j = k_1 + (j-1)k, \quad j = 2, \ldots, J,
$$
and set \( N_j = \{K_{j-1} + 1, \ldots, K_j\} \). In particular, we have \( K_j = k_1 + kk_2 \). We set \( A = \{1\} \) and \( B = \{2, \ldots, k_2 + 1\} \). In view of \((3.3)\), define \( g_n(t) \) as in \((1.15)\):

\[
g_n(t) = \cos(t \langle n \rangle^\alpha) g_n + \sin(t \langle n \rangle^\alpha) h_n.
\]

We also set

\[
S_n(t) = \frac{\sin(t \langle n \rangle^\alpha)}{\langle n \rangle^\alpha} \quad \text{and} \quad C_j(n_{K_j-1} + 1, \ldots, n_{K_j}) = \prod_{i \in N_j} \frac{1}{\langle n_i \rangle^\beta},
\]

where

\[
n_j = \sum_{i \in N_j} n_i = n_{K_j-1} + \cdots + n_{K_j}.
\]

In this setting, the deterministic function \( Q \) in \((1.19) \) and \((1.24) \) satisfies

\[
|Q(\{n_i, i \in \mathcal{P}\}, \{\tau_j, j \in B\}, t)| \lesssim 1,
\]

uniformly in \( \mathcal{P} \in \Pi_{\ell_1}, \{n_i, i \in \mathcal{P}\}, \{\tau_j, j \in B\}, t \in \mathbb{R}_+ \), since \( Q \) is given by (the product of) the second moment of products of Gaussians \( g_n \) and \( h_n \), multiplied by the trigonometric functions \( \cos(\tau \langle n \rangle^\alpha) \) and \( \sin(\tau \langle n \rangle^\alpha) \) with \( \tau = t \) or \( \tau_j \). Then, by Proposition \((1.3) (ii)\) with \( m_j = m_{K_j-1} + \cdots + m_{K_j} \), we have

\[
\mathbb{E} \left[ |\pi_{k_1+kk_2-2\ell} \hat{Y}(n,t)|^2 \right] \lesssim \sup_{\mathcal{P} \in \Pi_{\ell_1}} \sum_{\{n_i, i \in \mathcal{P}\}} \sum_{\{m_i, i \in \mathcal{P}\}} \sum_{n = n_1 + \cdots + n_{K_J}} \sum_{m = m_1 + \cdots + m_{K_J}} \prod_{i \in \mathcal{P}} \frac{1}{\langle n_i \rangle^\alpha} \prod_{i \not \in \mathcal{P}} \frac{1}{\langle m_i \rangle^\alpha} \prod_{\mathcal{P}} \frac{1}{\langle n_i \rangle^\beta} \prod_{\mathcal{P}} \frac{1}{\langle m_i \rangle^\beta} \tag{3.15}
\]

Note that the condition \( n_i = m_i, i \notin \mathcal{P} \), in \((3.15)\) follows from computing the expectation of

\[
\pi_{K_j-2\ell} \left( \left( \prod_{j \in B} \left( \frac{1}{\langle n_i \rangle^\alpha} \prod_{i \in \mathcal{P}} g_{n_i} (\tau_j) \right) \right) \left( \prod_{i \in \mathcal{P}} g_{n_i} (t) \right) \right) \prod_{j \in B} \left( \frac{1}{\langle m_i \rangle^\alpha} \prod_{i \in \mathcal{P}} g_{m_i} (\tau_j) \right) \prod_{i \in \mathcal{P}} g_{m_i} (t) \right)
\]

(which comes from \((1.24)\) ), for example by using Proposition \((1.3)(i)\) (note that \( \sum_{j = 1}^J |N_j \setminus \mathcal{P}| = K_j - 2\ell \)).

In order to estimate \((3.15)\), we first need to carry out summations over \( \{n_i\}_{i \in \mathcal{P}} \) and \( \{m_i\}_{i \in \mathcal{P}} \), which can be a priori divergent. The outside summation over \( \{n_i\}_{i \in \mathcal{P}} \) behaves better, thanks to the squared power \( \langle n_i \rangle^{-2\beta} \) (as compared to \( \langle n_i \rangle^{-\beta} \) for \( \{n_i\}_{i \in \mathcal{P}} \)). What comes in rescue in summing over the paired frequencies, say in \( \{n_i\}_{i \in \mathcal{P}} \), is the gain of derivatives \( \langle n_i \rangle^{-\alpha} \) coming from the Duhamel integral operator.

Following Definition \((1.2)\) we write \( \mathcal{P} \) as a disjoint union \( \mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \) with

\[
\mathcal{P}_1 \subset \bigcup_{j \in B} N_j = \bigcup_{j = 2}^{k_2 + 1} N_j
\]
such that an element \( i \in \mathcal{P}_1 \) is paired with exactly one element \( i_0 \in \mathcal{P}_2 \). Note that the choice of such \( \mathcal{P}_1 \) (and thus \( \mathcal{P}_2 \)) is not unique but that any such partition \( \mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \) works equally. Given \( 2 \leq j \leq k_2 + 1 \), we define the following sets:

\[
\mathcal{P}_1(N_j) \overset{\text{def}}{=} \{ i \in \mathcal{P}_1 : (i, i_0) \in \mathcal{P} \text{ for some } i_0 \in \mathcal{P}_2 \cap N_j \},
\]

\[
\mathcal{P}_2(N_j) \overset{\text{def}}{=} \{ i \in \mathcal{P}_2 : (i, i_0) \in \mathcal{P} \text{ for some } i \in \mathcal{P}_1 \cap N_j \}.
\]

With these notations (and recalling that \( \mathcal{P} \) is admissible), we have

\[
\text{(3.16)} = \sup_{\mathcal{P} \in \Pi} \sum_{\{n_i\}_{i \in \mathcal{P}}} \left( \prod_{i \in \mathcal{P}} \frac{1}{(n_i)^{2\beta}} \right) 1_{n_1 = \sum_{i \in \mathcal{P}} n_i} \left\{ S(\{n_i\}_{i \notin \mathcal{P}}) \right\}^2.
\]

where \( S(\{n_i\}_{i \notin \mathcal{P}}) \) is defined by

\[
S(\{n_i\}_{i \notin \mathcal{P}}) = \sum_{\{n_i\}_{i \in \mathcal{P}_1}} \left( \prod_{i \in \mathcal{P}_1} \frac{1}{(n_i)^{2\beta}} \right) \prod_{j=2}^{k_2+1} \frac{1}{\langle n_j \rangle^\alpha}.
\]

Here, in view of \( \text{(3.14)} \), \( \text{(3.18)} \), and the definition of admissibility in Definition 1.2, \( \bar{n}_j \) is given by

\[
\bar{n}_j = \sum_{i \in \mathcal{P}_1 \cap N_j} n_i + \sum_{i \in \mathcal{P}_2 \cap N_j} n_i + \sum_{i \in \mathcal{P}_1} n_i
\]

\[
= \sum_{i \in \mathcal{P}_1 \cap N_j} n_i + \sum_{i \in \mathcal{P}_2 \cap N_j} n_i - \sum_{j_0=2}^{k_2} \sum_{j_0 \neq j} \sum_{i \in \mathcal{P}_1(N_j \cap N_{j_0})} n_i
\]

for \( 2 \leq j \leq k_2 + 1 \). The following simple observation plays a crucial role. First, recall that, in our current setting, we have \( A = \{1\} \). Namely, the frequency \( n_i, i \in N_j \), does not involve the action of the Duhamel integral operators if and only if \( j = 1 \). Then, in view of the definition \( \text{(3.10)} \), we see that if \( i \in \mathcal{P}_1(N_1) \), then \( i_0 \in \mathcal{P}_2 \cap N_1 \) and thus the frequency \( n_{i_0} = -n_i \) does not appear in \( \bar{n}_j \) for any \( j = 2, \ldots, k_2 + 1 \). Hence, if \( i \in \mathcal{P}_1(N_1) \), we conclude that the frequency \( n_i \) appears in exactly one \( \bar{n}_j \). On the other hand, if \( i \in \mathcal{P}_1(N_{j_0}) \) for some \( 2 \leq j_0 \leq k_2 + 1 \), then \( \pm n_i \) appears in \( \bar{n}_j \) for exactly two values of \( j = 2, \ldots, k_2 + 1 \).

In estimating the right-hand side of \( \text{(3.17)} \), for simplicity of the presentation, we drop the supremum in \( \mathcal{P} \in \Pi \) with the understanding that the estimates are uniform in \( \mathcal{P} \in \Pi \). In the following, \( \varepsilon > 0 \) denotes a sufficiently small positive constant, which may be different line by line.

• Step 1: In this step, we sum over the variables \( \{n_i\}_{i \in \mathcal{P}_1(N_1)} \). By Lemma 3.2 (i), we have

\[
S(\{n_i\}_{i \notin \mathcal{P}}) \lesssim \sum_{\{n_i\}_{i \in \mathcal{P}_1\setminus \mathcal{P}_1(N_1)}} \left( \prod_{i \in \mathcal{P}_1\setminus \mathcal{P}_1(N_1)} \frac{1}{(n_i)^{2\beta}} \right) \prod_{j=2}^{k_2+1} \frac{1}{\langle n_j^{(1)} \rangle^\alpha_j},
\]
provided that $\beta > \frac{d}{2} - \frac{\alpha}{2k}$, where $n_j^{(1)}$ and $\alpha_j^{(1)}$, $j = 2, \ldots, k_2 + 1$, are given by

$$n_j^{(1)} = \sum_{i \in \mathcal{P} \cap N_j} n_i + \sum_{i \in (\mathcal{P}_1 \setminus \mathcal{P}_1(N_1)) \cap N_j} n_i - \sum_{j_0 = 2}^{k_2 + 1} \sum_{i \in \mathcal{P}_1(N_1) \cap N_{j_0}} n_i,$$

$$\alpha_j^{(1)} = \min(\alpha, d) - 2|\mathcal{P}_1(N_1) \cap N_j| \left(\frac{d}{2} - \beta\right) - \varepsilon.$$

**• Step 2:** Next, we sum over $\{n_i\}$ for $i \in (\mathcal{P}_1 \cap N_2) \setminus \mathcal{P}_1(N_1)$. Given $i \in (\mathcal{P}_1 \cap N_2) \setminus \mathcal{P}_1(N_1)$, we have $i \in \mathcal{P}_1(N_{j_0})$ for some $j_0 = 3, \ldots, k_2 + 1$, and hence the frequency $n_i$ (with the ± sign) appears in exactly $n_j^{(1)}$ and $n_j^{(1)}$ on the right-hand side of (3.19). Then, by applying Lemma 3.2(ii), we have

$$S(\{n_i\}_{i \notin \mathcal{P}}) \lesssim \sum_{\{n_i\}_{i \in \mathcal{P}_1 \setminus (\mathcal{P}_1(N_1) \cup N_2)}} \left( \prod_{i \in \mathcal{P}_1 \setminus (\mathcal{P}_1(N_1) \cup N_2)} \frac{1}{\langle n_i \rangle^{2\beta}} \right) \prod_{j=2}^{k_2+1} \frac{1}{\langle n_j^{(2)} \rangle^\alpha_j^{(2)}},$$

provided that $\beta > \frac{d}{2} - \frac{\alpha}{2k}$, where $n_j^{(2)}$ and $\alpha_j^{(2)}$ are given by

$$n_j^{(2)} = \sum_{i \in \mathcal{P} \cap N_j} n_i - \sum_{j_0 = 3}^{k_2 + 1} \sum_{i \in \mathcal{P}_1(N_j) \cap N_{j_0}} n_i,$$

$$\alpha_j^{(2)} = \min(\alpha, d) - \left(2|\mathcal{P}_1(N_1) \cap N_j| + |\mathcal{P}_1 \setminus \mathcal{P}_1(N_1)|\right) \left(\frac{d}{2} - \beta\right) - \varepsilon,$$

when $j = 2$, and

$$n_j^{(2)} = \sum_{i \in \mathcal{P} \cap N_j} n_i + \sum_{i \in (\mathcal{P}_1 \setminus \mathcal{P}_1(N_1)) \cap N_j} n_i - \sum_{j_0 = 2}^{k_2 + 1} \sum_{i \in \mathcal{P}_1(N_1) \cap N_{j_0}} n_i,$$

$$\alpha_j^{(2)} = \min(\alpha, d) - \left(2|\mathcal{P}_1(N_1) \cap N_j| + |\mathcal{P}_2 \cap N_j|\right) \left(\frac{d}{2} - \beta\right) - \varepsilon,$$

when $3 \leq j \leq k_2 + 1$.

**• Step 3:** We iteratively sum over $\{n_i\}$ for $i \in N_j$ and $j = 3, \ldots, k_2 + 1$ and obtain

$$S(\{n_i\}_{i \in \mathcal{P}_j}) \lesssim \prod_{j=2}^{k_2+1} \frac{1}{\langle n_j^{(k_2+1)} \rangle^\alpha_j^{(k_2+1)}},$$

provided that $\beta > \frac{d}{2} - \frac{\alpha}{2k}$, where $n_j^{(k_2+1)}$ and $\alpha_j^{(k_2+1)}$, $j = 2, \ldots, k_2 + 1$, are given by

$$n_j^{(k_2+1)} = \sum_{i \in \mathcal{P} \cap N_j} n_i,$$

$$\alpha_j^{(k_2+1)} = \min(\alpha, d) - \left(|\mathcal{P}_1(N_1) \cap N_j| + \mathcal{P} \cap N_j\right) \left(\frac{d}{2} - \beta\right) - \varepsilon.$$

(3.20)
Here, we used the following fact, which follows from (3.16):

\[ |P_1 \cap N_j| + \sum_{j_0=2, j_0 \neq j}^{k_2+1} |P_2(N_{j_0}) \cap N_j| = |P \cap N_j| \]

for each \( j = 2, \ldots, k_2 + 1 \).

By letting \( m = \sum_{i \in P^c \cap N_1} n_i \), it follows from \( n = \sum_{i \in P} n_i \) and (3.21) that

\[ n - m = \sum_{j=2}^{k_2+1} n_j^{(k_2+1)}. \]

Then, by combining (3.17) and (3.20) and applying Lemma 3.2(i), we have

\[
(3.17) \lesssim \sum_{m \in \mathbb{Z}^d} \sum_{\{n_i\} \in P^c \cap N_1 \atop \sum_{i \in P^c \cap N_1} n_i = m} \prod_{n_i} \frac{1}{(n_i)^{2\beta}} \\
\times \sum_{\{n_j\}^{k_2+1}_{2 \leq j \leq k_2+1} \atop \sum_{j=2}^{k_2+1} n_j^{(k_2+1)} = n - m} \prod_{j=2}^{k_2+1} \frac{1}{(n_j^{(k_2+1)})^{2\alpha_{j} + \beta_j}} \\
\lesssim \sum_{m \in \mathbb{Z}^d} \frac{1}{\langle m \rangle_{\beta_1}} \sum_{\{n_j\}^{k_2+1}_{2 \leq j \leq k_2+1} \atop \sum_{j=2}^{k_2+1} n_j^{(k_2+1)} = n - m} \prod_{j=2}^{k_2+1} \frac{1}{(n_j^{(k_2+1)})^{2\alpha_{j} + \beta_j}}, 
\]

(3.22)

provided that \( \beta > \frac{d}{2} - \frac{\alpha}{2k} \), where \( \beta_j \) is defined by

\[
\beta_j = d - 2|P^c \cap N_j| \left( \frac{d}{2} - \beta \right) - \varepsilon. 
\]

Note that, for \( 2 \leq j \leq k_2 + 1 \), we have \( |N_j| = k \) and thus

\[ 2\alpha_j^{(k_2+1)} + \beta_j = d + 2 \min(\alpha, d) - 2(|P_1(N_1) \cap N_j| + |N_j|) \left( \frac{d}{2} - \beta \right) - \varepsilon \]

\[ > d + 2 \min(\alpha, d) - 4k \left( \frac{d}{2} - \beta \right) > d, \]

where the last inequality follows from the assumption (3.11) and \( \beta > \frac{k-1}{2d} d \). Therefore, by applying Lemma 3.2(i) with (3.23) and \( |N_1| = k_1 \), we obtain

\[
(3.22) \lesssim \sum_{m \in \mathbb{Z}^d} \frac{1}{\langle m \rangle_{\beta_1} \langle n-m \rangle^{d-\varepsilon}} \\
\lesssim \langle n \rangle^{-d-2k_1(\beta-\frac{d}{2})+\varepsilon},
\]

provided that \( \beta > \frac{k_1-1}{2k_1} d \). Therefore, the desired regularity follows from Lemma 1.1. This concludes the proof of Proposition 1.3.
Remark 3.3. In the proof of Proposition 3.1 (iii), we first treated the contribution from the highest homogeneous Wiener chaos $H_{k_1+k_2}$, and then treated the contribution from the lower order terms, which required more careful analysis. Note that the analysis of the lower order terms yielded an extra condition (3.11). We also point out that the regularity of the resulting stochastic term $Y = \lim_{N \to \infty} Y_N$ is the same as that coming from the contribution from the highest homogeneous Wiener chaos.

3.3. Local well-posedness of the dispersion-generalized NLW. In this subsection, we prove local well-posedness of the renormalized dispersion-generalized nonlinear wave equation (3.9), using Proposition 3.1 and simple product estimates. By writing (3.9) in the Duhamel formulation (with the zero initial data) and dropping the subscript $N$, we have

\[
v(t) = \Phi(v)(t) = - \sum_{k_1=0}^{k-1} \sum_{k_2,k_3=0}^{k} 1_{k=k_1+k_2+k_3} \frac{k!}{k_1!k_2!k_3!} \times \int_0^t \sin((t-\tau)\langle \nabla \rangle^\alpha) \left( :Z^{k_1}:(I(:Z^{k_2}:)^{k_2})^{k_3}\right)(\tau)d\tau.
\]

(3.24)

Recall that thanks to the second order expansion, we do not have the term with $k_1 = k$. In view of Proposition 3.1, we first need to impose the condition

\[
\max\left(\frac{d}{2} - \frac{d}{2k}, \frac{d}{2k} - \frac{\alpha}{k}\right) < \beta \leq \frac{d}{2}
\]

(3.25)

to guarantee existence of the stochastic terms $:Z^{k_1}:(I(:Z^{k_2}:)^{k_2})$.

In the following, we prove local well-posedness of (3.24) by viewing

\[
\Xi = \{ :Z^{k_1}:(I(:Z^{k_2}:)^{k_2} ) : k_1 = 0, \ldots, k-1, k_2 = 0, \ldots, k \}
\]

as the collection of given deterministic distributions in $C([0,T];W^{s_{k_1},k_2,\infty}(\mathbb{T}^d))$ with (i) $s_{k_1,k_2} < k_1(\beta - \frac{d}{2})$ for $k_1 \geq 1$ and (ii) some small $s_{0,k_2} > 0$ when $k_1 = 0$. Then, we set

\[
\|\Xi\|_Z = \sum_{k_1=0}^{k-1} \sum_{k_2=0}^{k} \| :Z^{k_1}:(I(:Z^{k_2}:)^{k_2})\|_{C([0,1];W^{s_{k_1,k_2},\infty}(\mathbb{T}^d))}.
\]

We now prove local well-posedness of the equation (3.24).

**Theorem 3.4.** Fix an integer $k \geq 2$. Let $\alpha > 0$ and $\beta \leq \frac{d}{2}$, satisfying (3.25) and

\[
\alpha > (k-1)\left(\frac{d}{2} - \frac{k-1}{k}\beta\right).
\]

(3.26)

Furthermore, when $k \geq 3$, we assume that

\[
\alpha > (k-1)\left(\frac{d}{2} - \frac{k^2 - 3k + 3}{k-1}\beta\right),
\]

(3.27)

which is a stronger assumption than (3.26). Then, there exists a unique local-in-time solution $v$ to (3.24) in the class $C([0,T];H^\sigma(\mathbb{T}^d))$ for some $\sigma > 0$ and $T = T(\|\Xi\|_Z) > 0$. Moreover, the map $\Xi \in Z \mapsto v \in C([0,T];H^\sigma(\mathbb{T}^d))$ is continuous.
In view of the second order expansion (3.7) and Proposition 3.1, Theorem 3.4 yields almost sure local well-posedness of the renormalized dispersion-generalized NLW (3.1) with the random initial data \((u_0^\omega, u_1^\omega)\) in (3.2) in the following sense. Given \(N \in \mathbb{N}\), let \(u_N\) be the solution to the renormalized dispersion-generalized NLW with the truncated random initial data:

\[
\begin{aligned}
\partial_t^2 u_N + (1 - \Delta)\alpha u_N + H_k(u_N; \sigma_N) = 0 \\
(u_N, \partial_t u_N)|_{t=0} = (\mathbf{P}_N u_0^\omega, \mathbf{P}_N u_1^\omega)
\end{aligned}
\]

We have \(u_N = Z_N - \mathcal{I}(Z_N^k) + v_N\), where \(v_N\) solves (3.9). Then, \(u_N\) converges to some non-trivial process \(u = Z - \mathcal{I}(Z_N^k) + v\) in \(C([0, T]); H^\sigma(\mathbb{T}^d))\) almost surely, where \(T_\omega\) is an almost surely positive random time, \(Z\) is the random linear solution in (3.3), and \(v\) satisfies the nonlinear equation (3.24). Here, \(\mathcal{I}(Z_N^k)\) is the limit of \(\mathcal{I}(Z_N^k)\) whose existence is guaranteed in Proposition 3.1.

When \(d = 2\) and \(\alpha = \beta = 1\), Thomann and the first author \(^{32}\) proved Theorem 3.4 (corresponding to the Gibbs measure problem for the standard nonlinear wave equations on \(\mathbb{T}^3\)). See also \(^{13, 23, 26}\). When \(d = 2\) and \(\beta = 1\), Theorem 3.4 extends the result in \(^{32}\) to (i) \(\alpha > \alpha_1(k) = \frac{k-1}{2}\) when \(k = 2\) and (ii) \(\alpha > \alpha_2(k) = \frac{k+1}{k-1}\) when \(k \geq 3\). Note that, for \(j = 1, 2\), we have \(\alpha_j(k) \to 1\) as \(k \to \infty\). See also \(^{38, 11}\) for \(\alpha < 1\) and \(d = 2\). In the following, we prove Theorem 3.4 by using the product estimates (Lemma 3.5) and Sobolev’s inequality. It is possible to improve Theorem 3.4 by using the Strichartz estimates but we do not pursue this issue here, since our main focus in this section is to present a simple application of Proposition 1.3 (and Proposition 3.1). See \(^{11}\) for a local well-posedness argument with the Strichartz estimates when \(d = 2\).

Before proceeding to the proof of Theorem 3.4, we first recall the following product estimates. See \(^{13}\) for the proof. Note that while Lemma 3.5(ii) was shown only for \(\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{s}{d}\) in \(^{13}\), the general case \(\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}\) follows from the inclusion \(L^{r_1}(\mathbb{T}^d) \subset L^{r_2}(\mathbb{T}^d)\) for \(r_1 \geq r_2\).

**Lemma 3.5.** Let \(0 \leq s \leq 1\).

(i) Suppose that \(1 < p_j, q_j, r < \infty, \frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}, j = 1, 2\). Then, we have

\[
\|\langle \nabla \rangle^s (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \left( \|f\|_{L^{p_1}(\mathbb{T}^d)} \|\langle \nabla \rangle^s g\|_{L^{q_1}(\mathbb{T}^d)} + \|\langle \nabla \rangle^s f\|_{L^{p_2}(\mathbb{T}^d)} \|g\|_{L^{q_2}(\mathbb{T}^d)} \right).
\]

(ii) Suppose that \(1 < p, q, r < \infty\) satisfy the scaling condition: \(\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}\). Then, we have

\[
\|\langle \nabla \rangle^{-s} (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \|\langle \nabla \rangle^{-s} f\|_{L^{p}(\mathbb{T}^d)} \|\langle \nabla \rangle^s g\|_{L^{q}(\mathbb{T}^d)}.
\]

We now present the proof of Theorem 3.4. In the following, we use the short-hand notation \(C_T B_x\) for \((C([0, T]; B(\mathbb{T}^d))\).

**Proof of Theorem 3.4.** We show that \(\Phi\) in (3.24) is a contraction in a ball \(B \subset C([0, T]; H^\sigma(\mathbb{T}^d))\) of radius 1 centered at the origin.

Let us first treat the stochastic terms. The worst term on the right-hand side of (3.24) is \(Z^k_{-1}: \mathcal{I}(Z^k)\) with regularity \((k - 1)(\beta - \frac{d}{2})\). In view of the gain of \(\alpha\)-regularity from

\footnote{In the following, we only discuss spatial regularities. We also use \(a - \varepsilon\) to denote \(a - \varepsilon\) for arbitrarily small \(\varepsilon > 0\).}
the Duhamel integral operator, this implies that $v$ is expected to have regularity

$$
\sigma < \alpha + (k - 1) \left( \beta - \frac{d}{2} \right).
$$

(3.28)

Note that, thanks to the hypothesis (3.26), we can choose small $\sigma > 0$, satisfying (3.28).

Under the condition (3.28), we have

$$
\left\| \int_0^t \frac{\sin((t - \tau)(\nabla)^\alpha)}{(\nabla)^\alpha} \left( \frac{Z^{k_1} : (I(Z^{k_1})})^{k-k_1}(\tau)d\tau \right) \right\|_{C_T H^s}\leq T \left\| Z^{k_1} : (I(Z^{k_1})})^{k-k_1}\right\|_{C_T H^{s-\alpha}} < \infty
$$

(3.29)

for any $k_1 = 0, 1, \ldots, k - 1$.

On the other hand, by Sobolev’s inequality, Hölder’s inequality, and Sobolev’s inequality once again, we estimate the contribution from $k_1 = k_2 = 0$ and $k_3 = k$ as

$$
\left\| \int_0^t \frac{\sin((t - \tau)(\nabla)^\alpha)}{(\nabla)^\alpha} v^k(\tau)d\tau \right\|_{C_T H^s} \leq T \left\| v^k \right\|_{C_T H^{s-\sigma}} \lesssim T \left\| v^k \right\|_{C_T L^{p^*}_t} \lesssim T \left\| v^k \right\|_{C_T H^s}.
$$

(3.30)

The second inequality in (3.30) holds under the condition

$$
\frac{\alpha - \sigma}{\beta} \geq \frac{1}{p} - \frac{1}{2}
$$

with some $p \leq 2$, while the third inequality holds under the condition

$$
\frac{\sigma}{\beta} \geq \frac{1}{2} - \frac{1}{kp}.
$$

(3.31) (3.32)

Next, we consider a general term $Z^{k_1} : (I(Z^{k_1}))^{k_2} v^{k_3}$ with $1 \leq k_3 \leq k - 1$. Noting that the regularity of $Z^{k_1} : (I(Z^{k_1}))^{k_2}$ depends only on $k_1$ (and gets worse as $k_1$ gets larger), it suffices to consider $Z^{k_1} : v^{k-k_1}$ with $1 \leq k_1 \leq k - 1$, since when $k_2 > 0$, the value of $k_1$ or $k_3$ is smaller, which makes it easier to handle the term.

By Sobolev’s inequality, Lemma (3.5)(ii), the fractional Leibniz rule (Lemma (3.5)(i)), Hölder’s inequality, and Sobolev’s inequality, we have

$$
\left\| \int_0^t \frac{\sin((t - \tau)(\nabla)^\alpha)}{(\nabla)^\alpha} \left( Z^{k_1} : v^{k-k_1}(\tau)d\tau \right) \right\|_{C_T H^s} \lesssim T \left\| Z^{k_1} : v^{k-k_1} \right\|_{C_T H^{s-\alpha}}
$$

(3.33)

where we omit the step with the fractional Leibniz rule when $k_1 = k - 1$. We now list the conditions needed for each step in (3.33). At the second inequality in (3.33), we used
Sobolev’s inequality with
\[ \alpha - \sigma > k_1 \left( \frac{d}{2} - \beta \right) \quad \text{and} \quad \alpha - \sigma + k_1 \left( \beta - \frac{d}{2} \right) > \frac{d}{q} - \frac{d}{2} \]  
(3.34)
(note that the first condition in (3.34) is subsumed by (3.28)). At the third inequality in (3.33), we applied Lemma 3.5(ii) with
\[ k_1 \left( \frac{d}{2} - \beta \right) + \frac{d}{q} \geq \frac{d}{q_0}. \]  
(3.35)
The fourth inequality in (3.33) follows from the fractional Leibniz rule (Lemma 3.5(i)) with
\[ \frac{1}{q_0} = \frac{1}{r_0} + \frac{k - k_1 - 1}{r}. \]  
(3.36)
The fifth inequality in (3.33) follows from Sobolev’s inequality with
\[ \sigma - k_1 \left( \frac{d}{2} - \beta \right) > \max \left( \frac{d}{2} - \frac{d}{r_0}, 0 \right). \]  
(3.37)
At the last inequality in (3.33), we used Sobolev’s inequality with
\[ \frac{\sigma}{d} > \frac{1}{2} - \frac{1}{r}. \]  
(3.38)
We point out that, by taking \( \sigma \) sufficiently close to \( \alpha > 0 \) in (3.28), the second condition in (3.37): \( \sigma - k_1 \left( \frac{d}{2} - \beta \right) > 0 \) is satisfied; see also (3.41) and (3.42) below.

Assuming all the conditions, it follows from (3.29), (3.30), and (3.33) (which can be applied to a general product \( Z^{k_1} : (I; Z^{k_2})^I \) with \( 1 \leq k_3 \leq k - 1 \)) that the map \( \Phi \) in (3.24) is bounded on the ball \( B \subset C([0, T]; H^{\sigma}(T^d)) \) by choosing \( T = T(||\Xi||_Z) > 0 \) sufficiently small. A similar computation yields a difference estimate on \( \Phi(v_1) - \Phi(v_2) \), from which we conclude that the map \( \Phi \) in (3.24) is a contraction on the ball \( B \subset C([0, T]; H^{\sigma}(T^d)) \) by choosing sufficiently small \( T = T(||\Xi||_Z) > 0 \).

We conclude the proof by summarizing the conditions and show that there exists \( \sigma > 0 \) satisfying all the conditions. From (3.31) and (3.32), we have
\[ \alpha \geq (k - 1) \left( \frac{d}{2} - \sigma \right). \]  
(3.39)
From (3.31), (3.35), (3.36), (3.37), and (3.38) (with \( k_1 = 1 \))
\[ \alpha - \sigma \geq \frac{d}{q} - \frac{d}{2} + k_1 \left( \frac{d}{2} - \beta \right) \geq \frac{d}{q_0} - \frac{d}{2} \]
\[ = \frac{d}{r_0} + \frac{(k - k_1 - 1)d}{r} - \frac{d}{2} \]
\[ > -\sigma + k_1 \left( \frac{d}{2} - \beta \right) + (k - k_1 - 1) \left( \frac{d}{2} - \sigma \right) \]
and thus we have
\[ \alpha > k_1 (\sigma - \beta) + (k - 1) \left( \frac{d}{2} - \sigma \right) \]  
(3.40)
for any \( k_1 = 1, \ldots, k - 1 \).
From (3.28) and (3.39), we obtain
\[
\frac{d}{2} - \frac{\alpha}{k-1} \leq \sigma < \alpha + (k-1)\left(\beta - \frac{d}{2}\right).
\] (3.41)

The hypothesis (3.26) guarantees existence of \(\sigma > 0\) satisfying (3.41). Note that the condition (3.26) guarantees that the right-hand side of (3.41) is positive.

Next, we verify existence of \(\sigma > 0\), satisfying (3.40) for \(k_1 = 1, \ldots, k-1\), under (3.28). For this purpose, we only need to consider the following two extreme cases: \(k_1 = k-1\) and \(k_1 = 1\). From (3.28) and (3.40) with \(k_1 = k-1\), we obtain
\[
\alpha > (k-1)\left(\frac{d}{2} - \beta\right),
\]
which is guaranteed by the hypothesis (3.26). Next, we consider the consequence of (3.28) and (3.40) with \(k_1 = 1\). When \(k = 2\), it follows from (3.28) and (3.40) with \(k_1 = 1\) that
\[
\alpha > \frac{d}{2} - \beta = (k-1)\left(\frac{d}{2} - \beta\right),
\]
which is once again guaranteed by the hypothesis (3.26).

Finally, let \(k \geq 3\). Then, from (3.28) and (3.40) with \(k_1 = 1\), we have
\[
\frac{k-1}{k-2} \frac{d}{2} - \frac{\beta}{k-2} - \frac{\alpha}{k-2} \leq \sigma < \alpha + (k-1)\left(\beta - \frac{d}{2}\right),
\] (3.42)

The hypothesis (3.27) guarantees existence of \(\sigma > 0\) satisfying (3.42). This concludes the proof of Theorem 3.4.

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