LOWER BOUNDS FOR MAHLER MEASURE THAT DEPEND ON THE NUMBER OF MONOMIALS

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Abstract. We prove a new lower bound for the Mahler measure of a polynomial in one and in several variables that depends on the complex coefficients, and the number of monomials. In one variable our result generalizes a classical inequality of Mahler. In \( M \) variables our result depends on \( \mathbb{Z}^M \) as an ordered group, and in general our lower bound depends on the choice of ordering.

1. Introduction

Let \( P(z) \) be a polynomial in \( \mathbb{C}[z] \) that is not identically zero. We assume to begin with that \( P \) has degree \( N \), and that \( P \) factors into linear factors in \( \mathbb{C}[z] \) as

\[
P(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_N z^N = c_N \prod_{n=1}^{N} (z - \alpha_n).
\]

If \( e : \mathbb{R}/\mathbb{Z} \to \mathbb{T} \) is the continuous isomorphism given by \( e(t) = e^{2\pi it} \), then the Mahler measure of \( P \) is the positive real number

\[
\mathfrak{M}(P) = \exp \left( \int_{\mathbb{R}/\mathbb{Z}} \log |P(e(t))| \, dt \right) = |c_N| \prod_{n=1}^{N} \max\{|1, |\alpha_n||}. \tag{1.2}
\]

The equality on the right of (1.2) follows from Jensen’s formula. If \( P_1(z) \) and \( P_2(z) \) are both nonzero polynomials in \( \mathbb{C}[z] \), then it is immediate from (1.2) that

\[
\mathfrak{M}(P_1P_2) = \mathfrak{M}(P_1)\mathfrak{M}(P_2).
\]

Mahler measure plays an important role in number theory and in algebraic dynamics, as discussed in [6], [12], [14, Chapter 5], and [16]. Here we restrict our attention to the problem of proving a lower bound for \( \mathfrak{M}(P) \) when the polynomial \( P(z) \) has complex coefficients. We establish an analogous result for polynomials in several variables.

For \( P(z) \) of degree \( N \) and given by (1.1), there is a well known lower bound due to Mahler which asserts that

\[
|c_n| \leq \binom{N}{n} \mathfrak{M}(P), \quad \text{for each } n = 0, 1, 2, \ldots, N. \tag{1.3}
\]

The inequality (1.3) is implicit in [9], and is stated explicitly in [11, section 2], (see also the proof in [11, Theorem 1.6.7]). If

\[
P(z) = (z \pm 1)^N,
\]
then there is equality in (1.3) for each $n = 0, 1, 2, \ldots, N$.

We now assume that $P(z)$ is a polynomial in $\mathbb{C}[z]$ that is not identically zero, and we assume that $P(z)$ is given by

$$P(z) = c_0 z^{m_0} + c_1 z^{m_1} + c_2 z^{m_2} + \cdots + c_N z^{m_N},$$

where $N$ is a nonnegative integer, and $m_0, m_1, m_2, \ldots, m_N$, are nonnegative integers such that

$$m_0 < m_1 < m_2 < \cdots < m_N.$$ 

We wish to establish a lower bound for $M(P)$ which depends on the coefficients and on the number of monomials, but which does not depend on the degree of $P$. Such a result was recently proved by Dobrowolski and Smyth [5]. We use a similar argument, but we obtain a sharper result that includes Mahler’s inequality (1.3) as a special case.

**Theorem 1.1.** Let $P(z)$ be a polynomial in $\mathbb{C}[z]$ that is not identically zero, and is given by (1.4). Then we have

$$|c_n| \leq \left( \frac{N}{n} \right) \mathfrak{M}(P), \quad \text{for each } n = 0, 1, 2, \ldots, N.$$

Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be a trigonometric polynomial, not identically zero, and a sum of at most $N + 1$ distinct characters. Then we can write $f$ as

$$f(t) = \sum_{n=0}^{N} c_n e(m_n t),$$

where $c_0, c_1, c_2, \ldots, c_N$, are complex coefficients, and $m_0, m_1, m_2, \ldots, m_N$, are integers such that

$$m_0 < m_1 < m_2 < \cdots < m_N.$$ 

As $f$ is not identically zero, the Mahler measure of $f$ is the positive number

$$\mathfrak{M}(f) = \exp \left( \int_{\mathbb{R}/\mathbb{Z}} \log |f(t)| \, dt \right).$$

It is trivial that $f(t)$ and $e(-m_0 t) f(t)$ have the same Mahler measure. Thus we get the following alternative formulation of Theorem 1.1.

**Corollary 1.1.** Let $f(t)$ be a trigonometric polynomial with complex coefficients that is not identically zero, and is given by (1.7). Then we have

$$|c_n| \leq \left( \frac{N}{n} \right) \mathfrak{M}(f), \quad \text{for each } n = 0, 1, 2, \ldots, N.$$

For positive integers $M$ we will prove an extension of Corollary 1.1 to trigonometric polynomials

$$F : (\mathbb{R}/\mathbb{Z})^M \to \mathbb{C},$$

that are not identically zero. The Fourier transform of $F$ is the function

$$\hat{F} : \mathbb{Z}^M \to \mathbb{C},$$

defined at each lattice point $k$ in $\mathbb{Z}^M$ by

$$\hat{F}(k) = \int_{(\mathbb{R}/\mathbb{Z})^M} F(x) e(-k^T x) \, dx.$$
In the integral on the right of (1.10) we write $dx$ for integration with respect to a Haar measure on the Borel subsets of $(\mathbb{R}/\mathbb{Z})^M$ normalized so that $(\mathbb{R}/\mathbb{Z})^M$ has measure 1. We write $k$ for a (column) vector in $\mathbb{Z}^M$, $k^T$ for the transpose of $k$, $x$ for a (column) vector in $(\mathbb{R}/\mathbb{Z})^M$, and therefore

$$k^T x = k_1 x_1 + k_2 x_2 + \cdots + k_N x_N.$$ 

As $F$ is not identically zero, the Mahler measure of $F$ is the positive real number

$$\mathcal{M}(F) = \exp\left( \int_{(\mathbb{R}/\mathbb{Z})^M} \log|F(x)| \, dx \right).$$

We assume that $S \subseteq \mathbb{Z}^M$ is a nonempty, finite set that contains the support of $\hat{F}$. That is, we assume that

(1.11) \hspace{1cm} \{k \in \mathbb{Z}^M : \hat{F}(k) \neq 0\} \subseteq S,

and therefore $F$ has the representation

(1.12) \hspace{1cm} F(x) = \sum_{k \in S} \hat{F}(k) e^{(k^T x)}.

Basic results in this setting can be found in Rudin [13, Sections 8.3 and 8.4].

If $\alpha = (\alpha_m)$ is a (column) vector in $\mathbb{R}^M$, we write

$$\varphi_\alpha : \mathbb{Z}^M \to \mathbb{R}$$

for the homomorphism given by

(1.13) \hspace{1cm} \varphi_\alpha(k) = k^T \alpha = k_1 \alpha_1 + k_2 \alpha_2 + \cdots + k_M \alpha_M.

It is easy to verify that $\varphi_\alpha$ is an injective homomorphism if and only if the coordinates $\alpha_1, \alpha_2, \ldots, \alpha_M$, are $\mathbb{Q}$-linearly independent real numbers.

Let the nonempty, finite set $S \subseteq \mathbb{Z}^M$ have cardinality $N + 1$, where $0 \leq N$. If $\varphi_\alpha$ is an injective homomorphism, then the set

$$\{\varphi_\alpha(k) : k \in S\}$$

consists of exactly $N + 1$ real numbers. It follows that the set $S$ can be indexed so that

(1.14) \hspace{1cm} S = \{k_0, k_1, k_2, \ldots, k_N\},

and

(1.15) \hspace{1cm} \varphi_\alpha(k_0) < \varphi_\alpha(k_1) < \varphi_\alpha(k_2) < \cdots < \varphi_\alpha(k_N).

By using a limiting argument introduced in a paper of Boyd [2], we will prove the following generalization of (1.8).

**Theorem 1.2.** Let $F : (\mathbb{R}/\mathbb{Z})^M \to \mathbb{C}$ be a trigonometric polynomial that is not identically zero, and is given by (1.12). Let $\varphi_\alpha : \mathbb{Z}^M \to \mathbb{R}$ be an injective homomorphism, and assume that the finite set $S$, which contains the support of $\hat{F}$, is indexed so that (1.14) and (1.13) hold. Then we have

(1.16) \hspace{1cm} |\hat{F}(k_n)| \leq \left(\frac{N}{n}\right) \mathcal{M}(F), \quad \text{for each } n = 0, 1, 2, \ldots, N.
Let \( F \) and \( \varphi_{\alpha} : \mathbb{Z}^M \to \mathbb{R} \) be as in the statement of Theorem 1.2 and then let \( \varphi_{\beta} : \mathbb{Z}^M \to \mathbb{R} \) be a second injective homomorphism. It follows that \( \mathcal{S} \) can be indexed so that (1.14) and (1.15) hold, and \( \mathcal{S} \) can also be indexed so that

\[
\mathcal{S} = \{ \ell_0, \ell_1, \ell_2, \ldots, \ell_N \},
\]

and

\[
\varphi_{\beta}(\ell_0) < \varphi_{\beta}(\ell_1) < \varphi_{\beta}(\ell_2) < \cdots < \varphi_{\beta}(\ell_N).
\]

In general the indexing (1.14) is distinct from the indexing (1.17). Therefore the system of inequalities

\[
|\hat{F}(k_n)| \leq \left( \frac{N}{n} \right) \mathcal{M}(F), \quad \text{for each } n = 0, 1, 2, \ldots, N,
\]

and

\[
|\hat{F}(\ell_n)| \leq \left( \frac{N}{n} \right) \mathcal{M}(F), \quad \text{for each } n = 0, 1, 2, \ldots, N,
\]

which follow from Theorem 1.2, are different, and in general neither system of inequalities implies the other.

2. Proof of Theorem 1.1

It follows from (1.2) that the polynomial \( P(z) \), and the polynomial \( z^{-m_0}P(z) \), have the same Mahler measure. Hence we may assume without loss of generality that the exponents \( m_0, m_1, m_2, \ldots, m_N \), in the representation (1.4) satisfy the more restrictive condition

\[
0 = m_0 < m_1 < m_2 < \cdots < m_N.
\]

If \( N = 0 \) then (1.6) is trivial. If \( N = 1 \), then

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1,
\]

and using Jensen’s formula we find that

\[
\mathcal{M}(c_0 + c_1z^{m_1}) = \max\{|c_0|, |c_1|\}.
\]

Therefore the inequality (1.6) holds if \( N = 1 \). Throughout the remainder of the proof we assume that \( 2 \leq N \), and we argue by induction on \( N \). Thus we assume that the inequality (1.6) holds for polynomials that can be expressed as a sum of strictly less than \( N + 1 \) monomials.

Besides the polynomial

\[
P(z) = c_0z^{m_0} + c_1z^{m_1} + c_2z^{m_2} + \cdots + c_Nz^{m_N},
\]

we will work with the polynomial

\[
Q(z) = z^{m_N}P(z^{-1}) = c_0z^{m_N-m_0} + c_1z^{m_N-m_1} + c_2z^{m_N-m_2} + \cdots + c_N.
\]

It follows from (1.2) that

\[
\mathcal{M}(Q) = \exp\left( \int_{\mathbb{R}/\mathbb{Z}} \log|e(m_N t) P(e(-t))| \, dt \right) = \mathcal{M}(P).
\]

Next we apply an inequality of Mahler \cite{10} to conclude that both

\[
\mathcal{M}(P') \leq m_N \mathcal{M}(P), \quad \text{and} \quad \mathcal{M}(Q') \leq m_N \mathcal{M}(Q).
\]
Because
\[ P'(z) = \sum_{n=1}^{N} c_n m_n z^{m_n-1} \]
is a sum of strictly less than \( N + 1 \) monomials, we can apply the inductive hypothesis to \( P' \). It follows that
\[ |c_n|m_n \leq \binom{N-1}{n-1} \mathfrak{M}(P') \leq m_N \binom{N-1}{n-1} \mathfrak{M}(P) \]
for each \( n = 1, 2, \ldots, N \). As
\[ m_0 = 0, \quad \text{and} \quad \binom{N-1}{-1} = 0, \]
it is trivial that (2.6) also holds at \( n = 0 \).

In a similar manner,
\[ Q'(z) = \sum_{n=0}^{N-1} c_n (m_N - m_n) z^{m_N - m_n - 1} \]
is a sum of strictly less than \( N + 1 \) monomials. We apply the inductive hypothesis to \( Q' \), and get the inequality
\[ |c_n|(m_N - m_n) \leq \binom{N-1}{N-1-n} \mathfrak{M}(Q') \leq m_N \binom{N-1}{n} \mathfrak{M}(Q) \]
for each \( n = 0, 1, 2, \ldots, N - 1 \). In this case we have
\[ (m_N - m_N) = 0, \quad \text{and} \quad \binom{N-1}{N} = 0, \]
and therefore (2.7) also holds at \( n = N \).

To complete the proof we use the identity (2.4), and we apply the inequality (2.6), and the inequality (2.7). In this way we obtain the bound
\[ |c_n|m_N = |c_n|m_n + |c_n|(m_N - m_n) \leq m_N \binom{N-1}{n-1} \mathfrak{M}(P) + m_N \binom{N-1}{n} \mathfrak{M}(P) = m_N \binom{N}{n} \mathfrak{M}(P). \]
This verifies (1.6).

3. Archimedean orderings in the group \( \mathbb{Z}^M \)

In this section we consider \( \mathbb{Z}^M \) as an ordered group. To avoid degenerate situations, we assume throughout this section that \( 2 \leq M \).

Let \( \alpha \) belong to \( \mathbb{R}^M \), and let \( \varphi_{\alpha} : \mathbb{Z}^M \to \mathbb{R} \) be the homomorphism defined by (1.13). We assume that the coordinates \( \alpha_1, \alpha_2, \ldots, \alpha_M \), are \( \mathbb{Q} \)-linearly independent so that \( \varphi_{\alpha} \) is an injective homomorphism. It follows, as in [13 Theorem 8.1.2 (c)], that \( \varphi_{\alpha} \) induces an archimedean ordering in the group \( \mathbb{Z}^M \). That is, if \( k \) and \( \ell \) are distinct points in \( \mathbb{Z}^M \) we write \( k < \ell \) if and only if
\[ \varphi_{\alpha}(k) = k^T \alpha < \varphi_{\alpha}(\ell) = \ell^T \alpha \]
in \( \mathbb{R} \). Therefore \( (\mathbb{Z}^M, <) \) is an ordered group, and the order is archimedean. If \( \mathcal{S} \subseteq \mathbb{Z}^M \) is a nonempty, finite subset of cardinality \( N + 1 \), then the elements of \( \mathcal{S} \) can be indexed so that
\[
\mathcal{S} = \{ k_0, k_1, k_2, \ldots, k_N \}
\]
and
\[
k_0^T \alpha < k_1^T \alpha < k_2^T \alpha < \cdots < k_N^T \alpha.
\]
A more general discussion of ordered groups is given in [13, Chapter 8]. Here we require only the indexing (3.1) that is induced in the finite subset \( \mathcal{S} \) by the injective homomorphism \( \phi_\alpha \).

If \( b = (b_m) \) is a (column) vector in \( \mathbb{Z}^M \), we define the norm
\[
\|b\|_\infty = \max \{ |b_m| : 1 \leq m \leq M \}.
\]
And if \( \mathcal{S} \subseteq \mathbb{Z}^M \) is a nonempty, finite subset we write
\[
\|\mathcal{S}\|_\infty = \max \{ \|k\|_\infty : k \in \mathcal{S} \}.
\]
Following Boyd [2], we define the function \( \nu : \mathbb{Z}^M \setminus \{0\} \to \{1, 2, 3, \ldots\} \) by
\[
\nu(a) = \min \{ \|b\|_\infty : b \in \mathbb{Z}^M, b \neq 0, \text{ and } b^T a = 0 \}.
\]
It is known (see [2]) that the function \( a \mapsto \nu(a) \) is unbounded, and a stronger conclusion follows from our Lemma 3.2. Moreover, if \( \nu(a) \) is sufficiently large, then the map \( k \mapsto k^T a \) restricted to points \( k \) in the finite subset \( \mathcal{S} \) takes distinct integer values, and therefore induces an ordering in \( \mathcal{S} \). This follows immediately from the triangle inequality for the norm (3.3), and was noted in [2]. As this result will be important in our proof of Theorem 1.2, we prove it here as a separate lemma.

**Lemma 3.1.** [D. Boyd] Let \( \mathcal{S} \subseteq \mathbb{Z}^M \) be a nonempty, finite subset with cardinality \( |\mathcal{S}| = N + 1 \), and let \( a \neq 0 \) be a point in \( \mathbb{Z}^M \) such that
\[
2\|\mathcal{S}\|_\infty < \nu(a).
\]
Then
\[
\{ k^T a : k \in \mathcal{S} \}
\]
is a collection of \( N + 1 \) distinct integers.

**Proof.** If \( N = 0 \) the result is trivial. Assume that \( 1 \leq N \), and let \( k \) and \( \ell \) be distinct points in \( \mathcal{S} \). If
\[
k^T a = \ell^T a,
\]
then
\[
(k - \ell)^T a = 0.
\]
It follows that
\[
\nu(a) \leq \|k - \ell\|_\infty \leq \|k\|_\infty + \|\ell\|_\infty \leq 2\|\mathcal{S}\|_\infty,
\]
and this contradicts the hypothesis (3.5). We conclude that (3.6) contains \( N + 1 \) distinct integers. \( \square \)
Let $\varphi_\alpha : \mathbb{Z}^M \to \mathbb{R}$ be an injective homomorphism, and let $\mathcal{S} \subseteq \mathbb{Z}^M$ be a nonempty, finite subset of cardinality $N + 1$. We assume that the elements of $\mathcal{S}$ are indexed so that both (3.1) and (3.2) hold. If $a \neq 0$ in $\mathbb{Z}^M$ satisfies (3.5), then it may happen that the indexing (3.1) also satisfies the system of inequalities

$$k_0^T a < k_1^T a < k_2^T a < \cdots < k_N^T a.$$  

We write $\mathcal{B}(\alpha, \mathcal{S})$ for the collection of such lattice points $a$. That is, we define

$$\mathcal{B}(\alpha, \mathcal{S}) = \{a \in \mathbb{Z}^M : 2\|\mathcal{S}\|_\infty < \nu(a)$$

and $k_0^T a < k_1^T a < k_2^T a < \cdots < k_N^T a$. (3.7)

The following lemma establishes a crucial property of $\mathcal{B}(\alpha, \mathcal{S})$.

**Lemma 3.2.** Let the subset $\mathcal{B}(\alpha, \mathcal{S})$ be defined by (3.7). Then $\mathcal{B}(\alpha, \mathcal{S})$ is an infinite set, and the function $\nu$ restricted to $\mathcal{B}(\alpha, \mathcal{S})$, is unbounded on $\mathcal{B}(\alpha, \mathcal{S})$.

**Proof.** By hypothesis

$$\eta = \eta(\alpha, \mathcal{S}) = \min \{k_n^T \alpha - k_{n-1}^T \alpha : 1 \leq n \leq N\}$$

is a positive constant that depends on $\alpha$ and $\mathcal{S}$.

By Dirichlet’s theorem in Diophantine approximation (see [3] or [15]), for each positive integer $Q$ there exists an integer $q$ such that $1 \leq q \leq Q$, and

$$(3.9) \quad \max \{|q\alpha_m| : m = 1, 2, \ldots, M\} \leq (Q + 1)^{-\frac{1}{d}} \leq (q + 1)^{-\frac{1}{d}},$$

where $\|\|_\|$ on the left of (3.9) is the distance to the nearest integer function. Let $Q$ be the collection of positive integers $q$ such that

$$(3.10) \quad \max \{|q\alpha_m| : m = 1, 2, \ldots, M\} \leq (q + 1)^{-\frac{1}{d}}.$$

Because $2 \leq M$, at least one of the coordinates $\alpha_m$ is irrational, and it follows from (3.10) that $Q$ is an infinite set.

For each positive integer $q$ in $Q$, we select integers $b_{1q}, b_{2q}, \ldots, b_{Mq}$, so that

$$(3.11) \quad |q\alpha_m| = |q\alpha_m - b_{mq}|, \quad \text{for } m = 1, 2, \ldots, M.$$  

Then (3.10) can be written as

$$\max \{|q\alpha_m - b_{mq}| : m = 1, 2, \ldots, M\} \leq (q + 1)^{-\frac{1}{d}}.$$  

Let $b_q = (b_{mq})$ be the corresponding lattice point in $\mathbb{Z}^M$, so that $q \mapsto b_q$ is a map from $Q$ into $\mathbb{Z}^M$. It follows using (3.8) and (3.12), that for each index $n$ we have

$$q\eta \leq qk_n^T \alpha - qk_{n-1}^T \alpha$$

$$= k_n^T b_q - k_{n-1}^T b_q + (k_n - k_{n-1})^T (q\alpha - b_q)$$

$$\leq k_n^T b_q - k_{n-1}^T b_q + 2\|\mathcal{S}\|_\infty \sum_{m=1}^{M} |q\alpha_m - b_{mq}|$$

$$\leq k_n^T b_q - k_{n-1}^T b_q + 2\|\mathcal{S}\|_\infty M (q + 1)^{-\frac{1}{d}}.$$  

Therefore for each sufficiently large integer $q$ in $Q$, the lattice point $b_q$ satisfies the system of inequalities

$$k_0^T b_q < k_1^T b_q < k_2^T b_q < \cdots < k_N^T b_q.$$  

We conclude that for a sufficiently large integer $L$ we have

$$\{b_q : L \leq q \text{ and } q \in Q\} \subseteq \mathcal{B}(\alpha, \mathcal{S}).$$  

(3.13)
This shows that \( \mathcal{B}(\alpha, \mathcal{S}) \) is an infinite set.

To complete the proof we will show that the function \( \nu \) is unbounded on the infinite collection of lattice points

\[
\{ b_q : L \leq q \text{ and } q \in \mathbb{Q} \}.
\]

If \( \nu \) is bounded on (3.14), then there exists a positive integer \( B \) such that

\[
\nu(b_q) \leq B
\]

for all points \( b_q \) in the set (3.14). Let \( C_B \) be the finite set

\[
C_B = \{ c \in \mathbb{Z}^M : 1 \leq \| c \|_\infty \leq B \}.
\]

Because \( \alpha_1, \alpha_2, \ldots, \alpha_M \), are \( \mathbb{Q} \)-linearly independent, and \( C_B \) is a finite set of nonzero lattice points, we have

\[
0 < \delta_B = \min \left\{ \left| \sum_{m=1}^{M} c_m \alpha_m \right| : c \in C_B \right\}.
\]

By our assumption (3.15), for each point \( b_q \) in (3.14) there exists a point \( c_q = (c_{mq}) \) in \( C_B \), such that

\[
c_q^T b_q = \sum_{m=1}^{M} c_{mq} b_{mq} = 0.
\]

Using (3.12) and (3.17), we find that

\[
q \delta_B \leq q \left| \sum_{m=1}^{M} c_{mq} \alpha_m \right|
\]

\[
= \left| \sum_{m=1}^{M} c_{mq} (q \alpha_m - b_{mq}) \right|
\]

\[
\leq \left( \sum_{m=1}^{M} |c_{mq}| \right) \max \{ |q \alpha_m - b_{mq}| : m = 1, 2, \ldots, M \}
\]

\[
\leq M B (q + 1)^{-\frac{1}{M}}.
\]

But (3.18) is impossible when \( q \) is sufficiently large, and the contradiction implies that the assumption (3.15) is false. We have shown that \( \nu \) is unbounded on the set (3.14). In view of (3.13), the function \( \nu \) is unbounded on \( \mathcal{B}(\alpha, \mathcal{S}) \). \( \square \)

4. Proof of Theorem 1.2

If \( M = 1 \) then the inequality (1.16) follows from Corollary 1.1. Therefore we assume that \( 2 \leq M \).

Let \( \varphi_\alpha : \mathbb{Z}^M \to \mathbb{R} \) be an injective homomorphism, and let the set \( \mathcal{S} \) be indexed so that (1.14) and (1.15) hold. It follows from Lemma 3.2 that the collection of lattice points \( \mathcal{B}(\alpha, \mathcal{S}) \) defined by (3.7), is an infinite set, and the function \( \nu \) defined by (3.4) is unbounded on \( \mathcal{B}(\alpha, \mathcal{S}) \).

Let \( a \) be a lattice point in \( \mathcal{B}(\alpha, \mathcal{S}) \). If \( F : (\mathbb{R}/\mathbb{Z})^M \to \mathbb{C} \) is given by (1.12), we define an associated trigonometric polynomial \( F_a : \mathbb{R}/\mathbb{Z} \to \mathbb{C} \) in one variable by

\[
F_a(t) = \sum_{k \in \mathcal{S}} \hat{F}(k) e(k^T a t) = \sum_{n=0}^{N} \hat{F}(k_n) e(k_n^T a t),
\]
where the equality on the right of (4.1) uses the indexing (1.14) induced by \( \varphi_\alpha \). The hypothesis (1.15) implies that the integer exponents on the right of (4.1) satisfy the system of inequalities
\[
(4.2) \quad k_1^T a < k_1^T a < k_2^T a < \cdots < k_N^T a.
\]
Then it follows from (1.8), (4.1), and (4.2), that
\[
(4.3) \quad |\hat{F}(k_n)| \leq \frac{\binom{N}{n}}{n} \mathcal{M}(F_{a}), \quad \text{for each } n = 0, 1, 2, \ldots, N.
\]
We have proved that the system of inequalities (4.3) holds for each lattice point \( a \) in \( B(\alpha, \mathcal{S}) \).

To complete the proof we appeal to an inequality of Boyd [2, Lemma 2], which asserts that if \( b \) is a parameter in \( \mathbb{Z}^M \) then
\[
(4.4) \quad \limsup_{\nu(b) \to \infty} \mathcal{M}(F_b) \leq \mathcal{M}(F).
\]
More precisely, if \( b_1, b_2, b_3, \ldots \), is a sequence of points in \( \mathbb{Z}^M \) such that
\[
(4.5) \quad \lim_{j \to \infty} \nu(b_j) = \infty,
\]
then
\[
(4.6) \quad \limsup_{j \to \infty} \mathcal{M}(F_{b_j}) \leq \mathcal{M}(F).
\]
Because \( \nu \) is unbounded on \( B(\alpha, \mathcal{S}) \), there exists a sequence \( b_1, b_2, b_3, \ldots \), contained in \( B(\alpha, \mathcal{S}) \) that satisfies (4.5). Hence the sequence \( b_1, b_2, b_3, \ldots \), in \( B(\alpha, \mathcal{S}) \) also satisfies (4.6) From (4.3) we have
\[
(4.7) \quad |\hat{F}(k_n)| \leq \frac{\binom{N}{n}}{n} \mathcal{M}(F_{b_j}), \quad \text{for each } n = 0, 1, 2, \ldots, N, \text{ and for each } j = 1, 2, 3, \ldots.
\]
The inequality (1.16) plainly follows from (4.6) and (4.7). This completes the proof of Theorem 1.2.

Boyd conjectured in [2] that (4.4) could be improved to
\[
(4.8) \quad \lim_{\nu(b) \to \infty} \mathcal{M}(F_b) = \mathcal{M}(F).
\]
The proposed identity (4.8) was later verified by Lawton [8] (see also [4] and [7]). Here we have used Boyd’s inequality (4.4) because it is simpler to prove than (4.8), and the more precise result (4.8) does not effect the inequality (1.16).

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