Semigroups of real functions with dense orbits

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Abstract

Let $\mathcal{F}_I = \{ f : I \to I \mid f(x) = (Ax + B)/(Cx + D); \ AD - BC \neq 0 \}$, where $I$ is an interval. For $x \in I$, let $\Omega_x$ be the orbit of $x$ under the action of the semigroup of functions generated by $f, g \in \mathcal{F}_I$. Our main result in this paper is to describe all $f, g \in \mathcal{F}_I$ such that $\Omega_x$ is dense in $I$ for all $x$.

1 Introduction

In a one-dimensional dynamical system, one is concerned with the density of orbits (hypercyclicity) and periodic points of a single map on a one-dimensional manifold such as an interval. More generally, if $G$ is a semigroup of functions on an interval $I$, then we would like to study the denseness of the $G$-orbit of $x \in I$, which is defined as $\Omega_x = \{ f(x) : f \in G \}$. If $\Omega_x$ is dense in $I$ for some $x \in I$, we say $G$ is hypercyclic.

Several authors have studied hypercyclic continuous semigroups of bounded linear operators on Banach spaces; see [3, 4, 6] and the survey article [2]. In this paper, we study the hypercyclicity of the semigroups generated by two functions from the following set of functions:

$$\mathcal{F}_I = \left\{ f : I \to I : f(x) = \frac{Ax + B}{Cx + D}; \ AD - BC \neq 0 \right\}, \quad (1.1)$$
where \( I \subset \mathbb{R} \) is an interval (possibly infinite). A single map in \( \mathcal{F} \) has a simple dynamical system: the orbit of every \( x \) is either periodic of order at most 2 or converging to a fixed point, hence our interest in semigroups generated by a pair of functions from \( \mathcal{F}_I \).

An important example of a hypercyclic pair, which is related to continued fractions, is the pair of maps:

\[
\begin{align*}
f(x) &= x + 1, \\
g(x) &= \frac{1}{x}.
\end{align*}
\]

Note that the orbit of 1 is the set of all positive rational numbers. Another example is the pair of functions \( f(x) = ax \) and \( g(x) = bx + c \) on \((0, \infty)\), where \( b > 1 > a \) and \( c > 0 \) (see [11] or [31]). In general, it is simple to construct examples of pairs of functions where every orbit of the semigroup generated by them is dense. The following theorem (which will be proved in section 2) can be used to construct such examples. In the sequel, we call a function \( f : I \to I \) length-decreasing, if \( |f(J)| < |J| \) for every nonempty subinterval \( J \subseteq I \), where \( |J| \) means the length of the interval \( J \). Also \( \text{Im}(f) \) means the image of \( f \).

**Theorem 1.1.** Let \( \{f_i : I \to I \mid i \in \Lambda \} \) be a set of length-decreasing functions on a closed finite interval \( I \), where \( \Lambda \) is some (possibly infinite) index set. Suppose that for each \( i \in \Lambda \), the global maximum and minimum values of \( f \) on \( I \) occur at the end points of \( I \). Moreover, suppose that:

\[
\bigcup_{i \in \Lambda} \text{Im}(f_i) = I. \tag{1.2}
\]

Then the orbit of every \( x \in I \) under the action of the semigroup generated by the \( f_i \)'s, \( i \in \Lambda \), is dense in \( I \).

The main result of this paper is to describe all hypercyclic pairs of functions from \( \mathcal{F}_I \). This will be achieved through Propositions 2.3, 3.5, and 4.3. The following theorem is a more compact but slightly weaker result, since it only deals with the case where neither one of the functions \( f \) or \( g \) is onto. To state the theorem, we need the following definitions. Every \( f \in \mathcal{F}_I \) has at most one attracting fixed point (an attracting fixed point of \( f \) is a point \( \theta \in I \) such that \( f^n(x) \to \theta \) for \( x \) near \( \theta \), as \( n \to \infty \), where \( f^n \) means the composition of \( f \) with itself \( n \) times). We denote this unique attracting fixed point of \( f \) by \( o(f) \), if exists (whenever we write \( o(f) \) it is implied that it exists). If \( I \) is an infinite interval, then we allow \( o(f) = \infty \) (which means
\( f^n(x) \to \infty \) for \( x \) large enough or \( o(f) = -\infty \) (defined similarly). Finally, if \( I \subset \mathbb{R} \) is an interval, then \( \partial I \) means the end points of \( I \), possibly containing the symbols \( \infty \) or \( -\infty \) if \( I \) is an infinite interval.

**Theorem 1.2.** Let \( f, g \in \mathcal{F}_I \) such that \( f \) and \( g \) are not onto, where \( I \subset \mathbb{R} \) is an interval (possibly infinite but not \( \mathbb{R} \) itself). Then the orbit of a given \( x \in I \) is dense in \( I \) if and only if \( \text{Im}(f) \cup \text{Im}(g) = I \) and one of the following occurs:

i) Both \( f \) and \( g \) are increasing and \( \{o(f), o(g)\} = \partial I \).

ii) Exactly one of \( f \) or \( g \), say \( f \), is increasing and \( o(f) \in \partial I \).

iii) Both \( f \) and \( g \) are decreasing and \( \{o(fg), o(gf)\} = \partial I \).

We divide the proof of Theorem 1.2 into three sections based on the monotonicity types of \( f \) and \( g \). In section 2, we prove Theorem 1.1 and part (i) of Theorem 1.2. Parts (ii) and (iii) of Theorem 1.2 are proved, respectively, in sections 3 and 4.

2 Case I

In this section we present the proof of Theorem 1.1 and part (i) of Theorem 1.2.

**Proof of Theorem 1.1** Let \( \overline{\Omega}_x \) denote the closure of the orbit of a given \( x \in I = [a, b] \). The proof is by contradiction, and so suppose \( \overline{\Omega}_x \neq I \) for some \( x \in I \). We first show that \( a, b \in \overline{\Omega}_x \). We choose \( i, j \in \Lambda \) so that \( a \in \text{Im}(f_i) \) and \( b \in \text{Im}(f_j) \). Since, by our assumptions, the maximum and minimum values of \( f_i \) and \( f_j \) occur at the end points, there are four cases:

Case 1. Suppose \( f_i(a) = a \) and \( f_j(b) = b \). In this case \( f^n_i(x) \to a \) and \( f_j^n(x) \to b \), since \( f_i \) and \( f_j \) are length-decreasing, and so \( a, b \in \overline{\Omega}_x \).

Case 2. Suppose \( f_i(b) = a \) and \( f_j(b) = b \). As in the previous case, \( f_j^n(x) \to b \) which implies that \( b \in \overline{\Omega}_x \). But then \( a = f_i(b) \in \overline{\Omega}_x \) as well.

Case 3. Suppose \( f_i(a) = a \) and \( f_j(a) = b \). This case is similar to Case 2.

Case 4. Suppose \( f_i(b) = a \) and \( f_j(a) = b \). We note that both \( f_i \circ f_j \) and \( f_j \circ f_i \) are length-decreasing, \( f_i \circ f_j(a) = a \), and \( f_j \circ f_i(b) = b \). We again conclude that \( a, b \in \overline{\Omega}_x \).

Next, we let \( A \) be the maximum-length interval in \( (a, b) \setminus \overline{\Omega}_x \). Such \( A \) exists, since \( I \) is finite. We have proved that \( a, b \in \overline{\Omega}_x \) and so \( f_i(a), f_i(b) \in \overline{\Omega}_x \)
for all $i \in \Lambda$. In particular, $A \subset \text{Im}(f_i)$ for some $i \in \Lambda$. Let $B$ be a maximal interval in the open set $f_i^{-1}(A)$. Clearly $B \subseteq (a,b) \setminus \bar{\Omega}_\varepsilon$ and $|B| > |A|$. This contradicts the definition of $A$, and the theorem is proved. □

From now on, for simplicity, we only work with the interval $I = [0,1]$. For any other interval $J = [a,b]$, there is a one-to-one correspondence $\hat{\theta} : \mathcal{F}_J \rightarrow \mathcal{F}_I$ defined by

$$\hat{\theta}(f) = \theta \circ f \circ \theta^{-1},$$

so that $\hat{\theta}$ maps dense orbits to dense orbits and attracting fixed points to attracting fixed points. Here, we set $\theta(x) = (x-a)/(b-a)$ if $J$ is a finite interval. If $J = [a,\infty)$, we let $\theta(x) = 1/(x-a+1)$ and if $J = (-\infty,b]$, then we let $\theta(x) = 1/(x+b+1)$.

The following lemma is the key to the proof of part (i) of Theorem [1.2]. In the sequel, by $\langle R, T \rangle$, we mean the semigroup of functions generated by $R$ and $T$. Every element $f \in \langle R, T \rangle$ is a word in $R$ and $T$, where $R$ and $T$ appear a certain number of times in the expression of $f$. Throughout this section, $R$ and $T$ are given by:

$$R(x) = \frac{(ab-c)x + c}{(ab-a-c)x + a + c}, \quad T(x) = \frac{x}{x+a}.$$  \hspace{1cm} (2.2)

**Lemma 2.1.** Suppose $a, b > 1 \geq c > 0$ and that

$$ab - a - c \geq 0.$$  \hspace{1cm} (2.3)

Let $f \in \langle R, T \rangle$ so that the number of appearances of $R$ and $T$ in $f$ are, respectively, $m$ and $n$. Suppose $f = R^m T^k$ so that $g$ is the empty word or a word that does not end in $T$, and $k \geq 0$. Then there exist $u, v$ so that

$$u + a^k b^{n-1} + ca^{k-1}b^{n-1} \geq 0; \quad u + b^m \geq 0; \quad v \geq 0,$$  \hspace{1cm} (2.4)

and

$$f'(x) = \frac{a^n b^m}{((b^m + u)x + v + ca^{k-1}b^{m-1})^2}.$$  \hspace{1cm} (2.5)

**Proof.** The proof is by induction on the length of the word $f$. The base of the induction is the case $f = R$ with $m = 1$, $n = 0$, and $k = 0$, for which we have:

$$R'(x) = \frac{b}{((b-1-a^{-1}c)x + 1 + a^{-1}c)^2}.$$
First, we consider the case $F = fR$ and $G = fT$. A simple calculation shows that

$$F'(x) = f'(R(x))R'(x) = \frac{a^n b^m}{((b^m + u)R(x) + v + ca^{k-1}b^{m-1})^2} \times \frac{a^2 b}{((ab - a - c)x + a + c)^2} \times \frac{a^n b^m}{a^{n+1} b^m}$$

where $aV = cu + (a + c)(v + ca^{k-1}b^{m-1}) \geq c(u + a^k b^{m-1} + ca^{k-1}b^{m-1}) \geq 0$ and

$$aU = -cb^m + (ab - a - c)u + (ab - a - c)v + ca^{k-1}b^{m-1} - c^2 a^{k-1}b^{m-1} \geq -cb^m - (ab - a - c)(a^k b^{m-1} + ca^{k-1}b^{m-1}) + ca^{k-1}b^{m-1} - c^2 a^{k-1}b^{m-1} \geq -cb^m - a^{k+1}b^{m-1} \geq -a(a^k b^{m-1} + ca^{k-1}b^{m-1})$$

which implies that $U + a^k b^{m-1} + ca^{k-1}b^{m-1} \geq 0$. This completes the proof of the assertion for $F = fR$. For $G = fT$, we have

$$G'(x) = f'(T(x))T'(x) = \frac{a^n b^m}{((b^m + u)T(x) + v + ca^{k-1}b^{m-1})^2} \times \frac{a}{(x + a)^2} \times \frac{a^n b^m}{a^{n+1} b^m}$$

In this case, clearly $U + b^m \geq 0$ and $V \geq 0$. Finally,

$$U + a^{k+1}b^{m-1} + ca^{k-1}b^{m-1} = u + v + ca^{k-1}b^{m-1} + a^{k+1}b^{m-1} + ca^{k-1}b^{m-1} \geq 0 \quad \text{(2.6)}$$

This completes the inductive step and the lemma follows. \qed

**Proposition 2.2.** Let $a, b \geq 1 \geq c \geq 0$. Moreover, suppose $b > 1$ if $c = 0$. Let $R$ and $T$ be maps defined by (2.2). Then the orbit of any $x \in (0, 1]$ is dense in $[0, 1]$ under the action of the semigroup generated by $R$ and $T$.

**Proof.** First, we consider the case $c = 0$, where the maps $R$ and $T$ are:

$$R(x) = \frac{bx}{(b-1)x + 1}, \quad T(x) = \frac{x}{x + a} \quad \text{(2.7)}$$
We make the change of variable \( \phi : (0, 1) \to (0, \infty) \).

\[
\phi(x) = \frac{1}{a} \left( \frac{1}{x} - 1 \right).
\]

(2.8)

Then

\[
\phi \circ R \circ \phi^{-1}(x) = \frac{x}{b}, \quad \phi \circ T \circ \phi^{-1}(x) = ax + 1.
\]

(2.9)

Now by the results in [1, 5], the orbit of any \( x \in [0, \infty) \) is dense under the action of the semigroup generated by the pair \((x/b, ax + 1)\). We conclude that the same holds for the pair \((R, T)\) on \([0, 1]\).

In the remainder of the proof, we assume \( c > 0 \). Clearly \( T^n(x) \to 0 \) for all \( x \in [0, 1] \) as \( n \to \infty \). Thus, we only need to show that the orbit of 0 is dense. Let \( \Omega \) be the orbit of 0 in \([0, 1]\). The proof is by contradiction, and so suppose \( \Omega \neq [0, 1] \). Let \( A \) be the maximum-length interval in \((0, 1) \setminus \Omega\).

Since \( R^n(0) \to 1 \) as \( n \to \infty \), we have \( 1 \in \Omega \) and \( 1/(a + 1) = T(1) \in \Omega \). It follows that either \( A \subseteq (0, 1/(a + 1)) \) or \( A \subseteq (1/(a + 1), 1) \). Suppose first that \( A \subseteq (0, 1/(a + 1)) \) and we will derive a contradiction. Let \( B = T^{-1}(A) \).

Clearly \( B \subseteq (0, 1) \setminus \Omega \) and \(|B| > |A|\), since \(|T'(x)| < 1\) for all \( x \in (0, 1) \). This contradicts our choice of \( A \). It follows that \( A \subseteq (1/(a + 1), 1) \), and so we can write \( A = R(A_1) \) for some \( A_1 \subseteq (0, 1) \setminus \Omega \). Now, we divide the proof into three cases:

**Case 1.** Suppose that \( ab - a - c \leq 0 \). In this case, we have

\[
|A| < \max_{x \in [0,1]} R'(x) \times |A_1| \leq \frac{1}{b} |A_1| \leq |A_1|,
\]

(2.10)

which is a contradiction, and so in this case the proposition holds.

**Case 2.** Suppose that \( ab - a - c > 0 \) (hence \( b > 1 \)) and \( a > 1 \). There exist sequences of nonnegative integers \( \{\alpha_i\}_{i=1}^{\infty} \) and \( \{\beta_i\}_{i=1}^{\infty} \), \( \alpha_i + \beta_i > 0 \) for all \( i \geq 1 \), so that for each \( n \) there exists \( A_n \subseteq (0, 1) \setminus \Omega \) with

\[
A = R^{\alpha_1} T^{\beta_1} \ldots R^{\alpha_n} T^{\beta_n}(A_n).
\]

(2.11)

This follows from the fact that every \( C \subseteq (0, 1) \setminus \Omega \) is included in the image of \( R \) or \( T \). Now, from Lemma 2.11, we have

\[
1 \leq \frac{|A|}{|A_n|} \leq \max_{x \in [0,1]} (R^{\alpha_1} T^{\beta_1} \ldots R^{\alpha_n} T^{\beta_n})(x) \\
\leq e^{-(2 + \sum_i \beta_i - \beta_n) b - 2 - \sum_i \alpha_i}.
\]

(2.12)
It follows that
\[
2(1 - \log a) c + \sum_{i=1}^{n} \beta_i - \beta_n \geq (\log a \ b) \left( \sum_{i=1}^{n} \alpha_i - 2 \right) .
\] (2.13)

On the other hand,
\[
1 \leq \frac{|A|}{|A_n|} \leq \max_{x \in [0,1]} (R^{\alpha_1}T^{\beta_1} \ldots R^{\alpha_n}T^{\beta_n})'(x)
\leq (\max R'(x)) \sum_{i=1}^{n} \alpha_i (\max T'(x)) \sum_{i=1}^{n} \beta_i
\leq \left( \frac{a^2 b}{(a + c)^2} \right)^{\sum_{i=1}^{n} \alpha_i} \left( \frac{1}{a} \right)^{\sum_{i=1}^{n} \beta_i},
\]
which implies that
\[
\log a \left( \frac{a^2 b}{(a + c)^2} \right)^{\sum_{i=1}^{n} \alpha_i} \geq \sum_{i=1}^{n} \beta_i .
\] (2.14)

Since \( \sum (\alpha_i + \beta_i) = \infty \), the inequalities (2.13) and (2.14) are in direct contradiction with each other as \( n \to \infty \).

Case 3. Suppose that \( ab - a - c > 0 \) but \( a = 1 \). Inequality (2.12) implies that \( \sum_{i=1}^{\infty} \alpha_i \) is finite, i.e. \( \alpha_i = 0 \) for \( i \) large enough. In particular there is a word \( h \) so that for each \( k \geq 1 \) there exists an open interval \( S_k \subseteq (0,1) \setminus \Omega \) with \( A = hT^k(S_k) \). But \( T(S_k) \subset (0,1/2) \) on which the maximum of \( T'(x) \) is \( 4/9 \). It follows that
\[
|A| \leq \max_{x \in [0,1]} h'(x) \times \left( \frac{4}{9} \right)^{k-1} \times |S_k| \to 0 ,
\] (2.15)
as \( k \to \infty \). This is clearly a contradiction, and the proof of Proposition 2.2 is completed.

Part (i) of Theorem 1.2 follows from the following proposition.

**Proposition 2.3.** Let \( f, g \in \mathcal{F} \) be both increasing. Then the orbit of a given \( x \in (0,1) \) is dense in \([0,1]\) if and only if one of the following occurs:

i) At least one of \( f \) or \( g \) is not onto and
\[
\text{Im}(f) \cup \text{Im}(g) = [0,1] \text{ and } \{ o(f), o(g) \} = \{ 0, 1 \} .
\]
ii) Or \( f(x) = ax/((a - 1)x + 1) \) and \( g(x) = bx/((b - 1)x + 1) \), where \( a > 1 > b > 0 \) or \( b > 1 > a > 0 \), and \( \log_a b \) is irrational.

**Proof.** i) Without loss of generality, suppose that \( o(f) = \{0\} \) and \( o(g) = \{1\} \) and that \( f \) is not onto, while \( g \) may be onto (if \( f \) is onto and \( g \) is not, then by the change of variable \( x \to 1 - x \), we have a pair \((\hat{f}, \hat{g})\) so that \( o(\hat{f}) = \{1\} \), \( o(\hat{g}) = \{0\} \), and \( \hat{g} \) is not onto. Then one would replace \( f \) by \( \hat{g} \) and \( g \) by \( \hat{f} \), and go on with the proof).

By a change of variable of the form
\[
\theta(x) = \frac{x}{ux + 1 - u},
\]
for some \( u \), we can assume \( f(x) = x/(x + a) \) for some \( a \geq 1 \). Now let \( g(x) = (Ax + B)/(Cx + D) \), where \( A + B = C + D > 0 \) and \( D > B \geq 0 \). It follows from \( o(g) = 1 \) that \( B + C \geq 0 \). Next, we set
\[
b = \frac{A + B}{D - B}, \quad c = \frac{aB}{D - B}.
\]

We note that \( b > 1 \geq c \geq 0 \) and \( b > 1 \) if \( c = 0 \). The claim then follows from Proposition 2.2 where \( f \) and \( g \) are give by \( R \) and \( T \) in (2.2).

ii) By the change of variable \( \phi(x) = 1/x - 1 \), we have \( \hat{f}(x) = \phi^{-1} \circ f \circ \phi(x) = x/a \) and \( \hat{g}(x) = \phi^{-1} \circ g \circ \phi(x) = x/b \). The orbits of \((x/a, x/b)\) in \((0, \infty)\) are dense if and only if \( \log_a b \) is irrational and \( a > 1 > b > 0 \) or \( b > 1 > a > 0 \).

To prove the converse, suppose \( \Omega_{x_0} \) is dense in \([0, 1]\) for some \( x_0 \). Clearly \( \text{Im}(h) \subset \text{Im}(f) \cup \text{Im}(g) \) for every \( h \in \langle f, g \rangle \). And so if \( \text{Im}(f) \cup \text{Im}(g) \neq [0, 1] \), the orbit of \( x \) cannot be dense. It follows that \( \text{Im}(f) \cup \text{Im}(g) = [0, 1] \). Then suppose that \( f \) and \( g \) are not onto and \( f(0) = 0 \). If 0 was not an attracting fixed point of \( f \), then \( f(x) \geq x \) for \( x \) near 0. In particular, there would not exist any \( x \in \Omega \) with \( x < \min\{x_0, g(0)\} \). It follows that \( o(f) = 0 \) and similarly \( o(g) = 1 \). Next, suppose \( f \) is onto, and without loss of generality, suppose that \( o(f) = 0 \) (otherwise, we can use the change of variable \( x \to 1 - x \) to arrive at this assumption). As before, we should have \( o(g) = 1 \), otherwise there is no \( x \in \Omega_{x_0} \) with \( x > \max\{x_0, g(x_0)\} \). The case where both \( f \) and \( g \) are onto was discussed in part (ii) of this proof. The proof of the Proposition is now complete. \( \square \)
3 Case II

The proof of part (ii) of Theorem 1.2 is more technical. Through a pair of lemmas, we first prove that there are sequences of positive integers \( \{\alpha_i\}_{i=1}^{\infty} \) and positive odd integers \( \{\beta_i\}_{i=1}^{\infty} \) so that for each \( k \geq 1 \) the maximum-length interval \( A \subseteq (0,1) \setminus \Omega \) can be written as

\[
A = T^\beta R^{\alpha_1} T^{\beta_1} R^{\alpha_2} T^{\beta_2} \ldots R^{\alpha_k} T^{\beta_k}(A_k),
\]

where \( \beta \in \{0,1\} \) and \( A_k \subseteq (0,1) \setminus \Omega \). Here and throughout this section, \( \Omega \) denotes the orbit of 0. We begin with a lemma that will help us to obtain upper bounds on the derivatives of some special elements in the semigroup generated by \( R \) and \( T \), where \( R \) and \( T \) throughout this section are given by

\[
R(x) = \frac{ax}{(-ab + a + c)x + ab}, \quad T(x) = \frac{a}{x + a}. \quad (3.1)
\]

Lemma 3.1. Let \( a, b, c > 0 \). Let \( R \) and \( T \) be defined by (3.1) and \( f \) be a word such that

\[
f = R^{\alpha_1} T^{\beta_1} R^{\alpha_2} T^{\beta_2} \ldots R^{\alpha_k} T^{\beta_k}, \quad (3.2)
\]

where \( \alpha_i, \beta_i > 0 \) and \( \beta_i \) is odd for all \( i \leq k \). Moreover, let

\[
s = \left\lfloor \frac{k}{2} \right\rfloor, \quad M = \sum_{i=0}^{s} \alpha_{2i+1}, \quad N = \sum_{i=1}^{s} \alpha_{2i}. \quad (3.3)
\]

If \( k \) is even, then there exist \( u, v \geq 0 \) and \( K \geq \max\{M, N\} \) and \( L \geq N \) such that

\[
f'(x) = \frac{a^k b^{M+N}}{(c a^{s-1} b^K + a^s b^N + u)x + v + a^s b^M + c^2 a^{s-1} b^{L-1})^2}. \quad (3.4)
\]

If \( k \geq 3 \) is odd, then there exist \( u, v \geq 0 \) and \( K, L \geq N \) such that

\[
f'(x) = \frac{-a^k b^{M+N}}{(a^s b^M + c^2 a^{s-1} b^L + u)x + v + a^{s+1} b^N + c a^s b^K)^2}. \quad (3.5)
\]

Proof. We first prove (3.4) by induction on \( s \). A simple induction shows that for a positive integer \( m \) and odd integer \( n \), there exist \( \gamma, \delta, \mu, \lambda \geq 0 \) so that

\[
R^m T^n(x) = \frac{a + \gamma + \delta x}{a + c b^{m-1} + b^m x + \mu + \lambda x}. \quad (3.6)
\]
and
\[(R^mT^n)'(x) = \frac{-ab^m}{(a + cb^{m-1} + b^mx + \mu + \lambda x)^2}. \tag{3.7}\]

For \(k = 2\) and \(h = R^{\alpha_1}T^{\beta_1}R^{\alpha_2}T^{\beta_2}\), it follows that:
\[h(x) = \frac{ab^{\alpha_2}x + cab^{\alpha_2-1} + C_1 + C_2x}{(c^{\alpha_1+\alpha_2-1} + ab^{\alpha_2} + u)x + v + ab^{\alpha_1} + c^2b^{\alpha_1+\alpha_2-2}} \tag{3.8}\]

where \(C_1, C_2, u, v \geq 0\), and
\[h'(x) = \frac{a^2b^{\alpha_1+\alpha_2}}{(c^{\alpha_1+\alpha_2-1} + ab^{\alpha_2} + u)x + v + ab^{\alpha_1} + c^2b^{\alpha_1+\alpha_2-2})^2}, \tag{3.9}\]

which is of the form (3.4) with \(K = \alpha_1 + \alpha_2 - 1 \geq \max\{\alpha_1, \alpha_2\}\) and \(L = \alpha_1 + \alpha_2 - 2 \geq \max\{\alpha_1, \alpha_2\} - 1\). Next, suppose (3.4) holds for \(k = 2s\) and \(f\) given by (3.2). Let \(g = fh\), where \(h = R^{\alpha_{k+1}}T^{\beta_{k+1}}R^{\alpha_{k+2}}T^{\beta_{k+2}}\), and \(\beta_{k+1}\) and \(\beta_{k+2}\) are odd. We calculate from (3.4), (3.8), and (3.9):
\[g'(x) = f'(h(x)) \ast h'(x) \tag{3.10}\]
\[= \frac{-a^kb^{M+N}h'(x)}{((ca^{s-1}b^K + a^s b^N + u)h(x) + (v + a^s b^M + c^2a^{s-1}b^{L-1}))^2}\]

After an algebraic simplification (that includes canceling the denominators of \((h(x))^2\) and \(h'(x))\), the coefficient of \(x\) in the denominator of this fraction is given by:
\[ca^s b^{K+\alpha_{k+2}} + a^{s+1}b^{N+\alpha_{k+2}} + ca^s b^{M+\alpha_{k+1}+\alpha_{k+2}+1} + U,\]
for some \(U \geq 0\). To show that (3.4) holds, we need to show that
\[\max\{K + \alpha_{k+2}, M + \alpha_{k+1} + \alpha_{k+2} - 1\} \geq \max\{M + \alpha_{k+1}, N + \alpha_{k+2}\},\]
which clearly holds. On the other hand, the constant in the denominator of the fraction is given by
\[(ca^{s-1}b^K + a^s b^N + u)cab^{\alpha_{k+2}+1} + a^{s+1}b^{M+\alpha_{k+1}} + V\]
\[\geq c^2a^s b^{K+\alpha_{k+2}+1} + a^{s+1}b^{M+\alpha_{k+1}}.\]
for some \(V \geq 0\). This completes the proof of (3.4), since \(K + \alpha_{k+2} - 1 \geq N + \alpha_{k+2} - 1\). The proof of (3.5) follows similarly by using (3.4), (3.8), and (3.9). \(\square\)
Lemma 3.2. Suppose $a, b, c > 0$ and $c \leq a/(a+c)$. Then for any interval $A \subseteq (0,1) \setminus \overline{\Omega}$ there exists a sequence $\alpha_i$ of positive integers and $\beta \in \{0,1\}$ so that for each $n \geq 1$ there exists $A_n \subseteq (0,1) \setminus \overline{\Omega}$ with

$$A = T^{\beta R^{\alpha_1} TR^{\alpha_2} T \ldots R^{\alpha_n} T (A_n)}.$$  \hfill (3.11)

Proof. Since $a/(a+c) = RT(0) \in \Omega$, we have either $A \subseteq (a/(a+c), 1)$ or $A \subseteq (0,a/(a+c))$. First suppose that $A \subseteq (0,a/(a+c))$. We set $\beta = 0$ and choose the largest $\alpha_1$ so that $B_1 = R^{-\alpha_1}(A) \subseteq [0,1]$. Such a choice of $\alpha_1$ is possible, since $R^n(x) \to 0$ uniformly for $x \in [0,1]$ as $n \to \infty$. In particular $B_1$ is not included in $Im(R) = [0, a/(a+c)]$. On the other hand, $B_1 \subseteq (0,1) \setminus \overline{\Omega}$, and so $B_1$ does not contain the point $a/(a+c)$. It follows that $B_1 \subseteq (a/(a+c), 1) \subseteq Im(T)$. Then we let $A_1 = T^{-1}(B_1) \subseteq (0, c) \subseteq (0, a/(a+c))$. One can repeat this argument and obtain the sequence $\alpha_i$. The case of $A \subseteq (a/(a+c), 1)$ is similar but $\beta = 1$ in this case. \hfill \Box

Lemma 3.3. Suppose that $a, b, c > 0$ and $c \geq a/(c+a)$. Let $A$ be the maximum-length interval in $(0,a/(a+c)) \setminus \overline{\Omega}$. Then the same conclusion of Lemma 3.2 holds with $\beta = 0$ and $A_k \subseteq (0, a/(a+c)) \setminus \overline{\Omega}$.

Proof. We construct the sequence $\{\alpha_i\}_{i=1}^{\infty}$ inductively. Let $\alpha_1$ be the maximum integer so that $B_1 = R^{-\alpha_1}(A) \subseteq [0,1]$. Such $\alpha_1$ exists, since $R^n(x) \to 0$ uniformly for $x \in [0,1]$. By our choice of $\alpha_1$, we should have $B_1 \subseteq (a/(a+c), 1) \subseteq Im(T)$ (see the proof of Lemma 3.2). Now, choose $\beta_1$ to be the maximum number so that $A_1 = T^{-\beta_1}(B_1) \subseteq [0,1]$. In particular $A_1$ is not included in $Im(T) = [a/(a+1), 1]$. Since $A_1 \cap \overline{\Omega} = \emptyset$, $A_1$ does not include $\lim_{n \to \infty} TR^n(0) = a/(a+1)$. It follows that $A_1 \subseteq [0, a/(a+1) \subseteq Im(R)$, and so $A = R^{\alpha_1} T^{\beta_1}(A_1)$ for $A_1 \subseteq (0, a/(a+c))$.

Now suppose we have constructed the sequences $\{\alpha_i\}_{i=1}^{k-1}$ of positive integers and $\{\beta_i\}_{i=1}^{k-1}$ of positive odd integers, $k \geq 2$, so that $A = F_{k-1}(A_{k-1})$, where $F_{k-1} = R^{\alpha_1} T^{\beta_1} \ldots R^{\alpha_{k-1}} T^{\beta_{k-1}} = gT^{\beta_{k-1}}$ and $A_{k-1} \subseteq (0, a/(a+c)) \setminus \overline{\Omega}$. Choose $\alpha_k$ to be the maximum positive integer such that $B_k = R^{-\alpha_k}(A_{k-1}) \subseteq [0,1]$. Similar to the base case, by our choice of $\alpha_k$, we have $B_k \subseteq Im(T)$. Then, we choose $\beta_k$ to be the maximum integer such that $A_k = T^{-\beta_k}(B_k) \subseteq [0,1]$. It follows that $A = F_k(A_k)$, and by our choice of $\beta_k$, we have $A_k \subseteq (0, a/(a+c)) \setminus \overline{\Omega}$.

It is left to show that $\beta_k$ is odd. On the contrary, suppose $\beta_k = 2l + 2$ and write $F_k = f T$ so that Lemma 3.1 is applicable to $f$. Suppose first that $k$ is even. Then $\max F'(x) \geq 1$, since $A = F(A_k)$ and $|A| \geq |A_k|$. It follows
from Lemma 3.1 that

\[
1 \leq |(f T)'(x)| = \frac{|a^k b^{M+N} T'(x)|}{((ca^{s-1}b^K + a^s b^N + u) T(x) + v + a^s b^M + c^2 a^{s-1}b^{L-1})^2}
\]

\[
\leq \frac{a^{k+1} b^{M+N}}{(ca^s b^K + a^{s+1} b^N + a^{s+1} b^M + c^2 a^{s} b^{L-1})^2}
\]

\[
\leq \frac{a}{(c+a)^2} \leq \frac{c}{c+a} < 1 ,
\]

and so we have a contradiction in this case. Next, suppose \( k \) is odd. Similarly,

\[
1 \leq |(f T)'(x)| = \frac{|a^k b^{M+N} T'(x)|}{((a^s b^M + c^2 a^{s-1} b^L + u) T(x) + v + a^{s+1} b^N + c a^s b^K)^2}
\]

\[
< \frac{a^{k+1} b^{M+N}}{(a^{s+1} b^M + c^2 a^s b^L + c a^{s+1} b^K)^2} .
\] (3.12)

There are two cases:

Case 1. Suppose \( M \geq N \). Then continuing from (3.12), we have the contradiction:

\[
1 \leq |(f T)'(x)| < \frac{a^{k+1} b^{M+N}}{(a^{s+1} b^M)^2} \leq b^{N-M} \leq 1 .
\]

Case 2. Suppose \( N \geq M \). Again continuing from (3.12), we have the contradiction:

\[
1 \leq |(f T)'(x)| \leq \frac{a^{k+1} b^{M+N}}{(c^2 a^{s} b^L + c a^{s+1} b^K)^2} < \frac{a^2}{(c^2 + c a)^2} \leq \frac{1}{c^2} \left( \frac{a}{c+a} \right)^2 \leq 1 .
\]

In either case of \( k \) odd or even, we have proved that \( \beta_k \) is odd. The proof of the lemma is now complete.

\[\square\]

**Proposition 3.4.** Let \( b \geq 1 > c \geq 0 \) and \( a > 0 \). Moreover, suppose that \( b > 1 \) if \( c = 0 \). Let \( R \) and \( T \) be maps given by (3.1). Then the orbit of any \( x \in [0, 1] \) is dense in \([0, 1]\) under the action of the semigroup generated by \( R \) and \( T \).
Proof. Since $0, 1 \in \Omega_x$ for any $x \in [0, 1]$, we only need to show that $\Omega$ is dense. On the contrary, suppose $\Omega$ is not dense and we will derive a contradiction. By Lemmas 3.2 and 3.3 we have a sequence of positive integers $\{\alpha_i\}_{i=1}^\infty$ and positive odd integers $\{\beta_i\}_{i=1}^\infty$ so that:

$$
\Lambda_k = \max_{x \in [0, 1]} \left| (T^\beta R^{\alpha_1} T^\beta \ldots R^{\alpha_k} T^\beta)'(x) \right| \geq 1. \quad (3.13)
$$

for all $k \geq 1$. If $b = 1$, then (3.4) and (3.13) are in direct contradiction for $k$ even, since the denominator in (3.4) contains $a^s b^M = a^s$ while the numerator is $a^k b^{M+N} = a^k$, and so $f'(x) < 1$ (recall that $c > 0$ if $b = 1$). Thus, suppose $b > 1$ in the remainder of the proof. Let $u = \max\{1, 1/\sqrt{a}\}$ and note that max $\left| (T^\beta)'(x) \right| \leq u^2$. Then for $k = 2s$, we conclude from (3.9) that

$$
\Lambda_{2s} \leq u^2 \prod_{i=1}^s \max \left| (R^{\alpha_{2i-1}} T^\beta_{2i-1} R^{\alpha_{2i}} T^\beta_{2i})'(x) \right|
$$

$$
\leq u^2 \prod_{i=1}^s \frac{b^{\alpha_{2i-1} - \alpha_{2i-1}}}{(1 + (c^2/a) b^{\alpha_{2i} - 2})^2}.
$$

Let $M = \sum_{i=1}^s \alpha_{2i-1}$ and $N = \sum_{i=1}^s \alpha_{2i}$. It follows that

$$
ub^{(N-M)/2} \geq \prod_{i=1}^s \left( 1 + \frac{c^2}{ab^2} b^{\alpha_{2i}} \right) \geq 1 + \frac{c^2}{ab^2} \sum_{i=1}^s b^{\alpha_{2i}}. \quad (3.14)
$$

We conclude that $N - M \geq 0$ for all $s \geq 1$ and $N - M \to \infty$ as $s \to \infty$. We will repeat this analysis for $k = 2s + 1$. In this case, we have:

$$
\Lambda_{2s+1} \leq u^2 \max \left| (R^{\alpha_1} T)'(x) \right| \prod_{i=1}^s \max \left| (R^{\alpha_{2i}} T^\beta_{2i} R^{\alpha_{2i+1}} T^\beta_{2i+1})'(x) \right|
$$

$$
\leq \frac{u^2 ab^{\alpha_1}}{(a + cb^{\alpha_1-1})^2} \prod_{i=1}^s \frac{b^{\alpha_{2i+1} - \alpha_{2i}}}{(1 + (c^2/a) b^{\alpha_{2i+1} - 2})^2}.
$$

It follows that

$$
ub^{(\alpha_{2s+1} + M - N)/2} \geq a^{1/2} \prod_{i=1}^s \left( 1 + \frac{c^2}{ab^2} b^{\alpha_{2i+1}} \right) \geq \left( \frac{c^2}{\sqrt{ab^2}} \right) \sum_{i=1}^s b^{\alpha_{2i+1}} \quad (3.15)
$$

In particular, we should have $\alpha_{2s+1} \to \infty$ as $s \to \infty$. But the same inequality (3.15) implies that

$$
b^{\alpha_{2s+1}/2} \geq \left( \frac{c^2}{u \sqrt{ab^2}} \right) b^{\alpha_{2s+1}}, \quad (3.16)
$$
which implies that $\alpha_{2s+1}$ is bounded. This is a contradiction, and so we have proved the proposition in the case of $c > 0$.

Next, we deal with the case $c = 0$ as well. Choose $n$ large enough so that
\[
\theta = \frac{-a + \sqrt{a^2 + 4ab^n}}{2b^n} = \frac{2a}{a + \sqrt{a^2 + 4ab^n}} < \frac{1}{b}, \tag{3.17}
\]
where $\theta$ is the positive fixed point of $R^n T$. It follows that $(R^n T)(1) \leq R(\theta)$. It is then readily checked that the pair $(R, (R^n T)^2)$ satisfy all of the conditions of Proposition 2.2 on the interval $[0, \theta]$, and so the orbits of $\langle R, (R^n T)^2 \rangle$ are dense on $[0, \theta]$. But clearly $R^n T[0, \theta] = [\theta, 1]$ which implies that $\Omega$ is dense. This completes the proof of the proposition. \qed

Part (ii) of Theorem 1.2 follows from the proposition below.

**Proposition 3.5.** Let $f, g \in F$ so that $f$ is increasing and $g$ is decreasing. Then the orbit of a given $x \in [0, 1]$ is dense in $[0, 1]$ if and only if $\{o(f), g(o(f))\} = \{0, 1\}$ and one of the following occurs

i) $g$ is not onto and $\text{Im}(f) \cup \text{Im}(g) = [0, 1]$.

ii) $g$ is onto, $f$ is not onto, and $\text{Im}(f) \cup \text{Im}(gf) = [0, 1]$.

**Proof.** Proof of parts (i) is straightforward and follow from Proposition 3.4. For part (ii), note that the pair $(f, gf g)$ satisfies the conditions of Proposition 2.3 since $o(gfg) = 1 - o(f)$, and so $\{o(f), o(gfg)\} = \{0, 1\}$.

To prove the converse of (ii), suppose $g$ is onto, $f$ is not onto, and the orbit of some $x \in [0, 1]$ is dense in $[0, 1]$. We will show that $\text{Im}(f) \cup \text{Im}(gf) = [0, 1]$ and $o(f) \in \{0, 1\}$. We first show that $f(0) = 0$ or $f(1) = 1$. Otherwise, the orbit of $x$ is contained in the interval $[\min U, \max U]$, where $U = \{f(0), f(1), x, gf(0), gf(1), g(x)\}$ and so it cannot be dense. Without loss of generality, suppose $f(0) = 0$ (otherwise, we can use the change of variable $x \to 1 - x$ to arrive at this assumption). Next, we show that $o(f) = 0$, since otherwise $f(y) \geq y$ for all $y$ and so the orbit of $x$ would not contain any $y < \min\{x, g(x)\}$. It is left to show that $\text{Im}(f) \cup \text{Im}(gf) = [0, 1]$.

For every $\epsilon > 0$, there should exist an element of the orbit of $x$ which is contained in the interval $(f(1), f(1) + \epsilon)$. Let $w$ be a word in $f$ and $g$ such that $w(x) \in (f(1), f(1) + \epsilon)$. Clearly $w = gw_1$ for some word $w_1$. By making $\epsilon$ smaller, we can assume $w_1$ is not the empty word. Since $g^2 = id$, we should have $w_1 = fw_2$ and so $w = gf w_2$ for some word $w_2$. In particular
Im(gf) ∩ (f(1), f(1) + ǫ) ≠ ∅ for all positive ǫ small enough. It follows that
gf(1) ≤ f(1), which implies that Im(f) ∪ Im(gf) = [0, 1].

Next, we show that if an orbit Ωx is dense, then f and g cannot be both onto. On the contrary, suppose f and g are both onto. Then g² = id and fg = gf. It follows that the elements of ⟨f, g⟩ are g, f^m, and f^mg for m ≥ 1. In particular, the only accumulation points of the orbit Ωx are 0 and 1, and so it cannot be dense. □

4 Case III

In this section, we consider the case where both f, g ∈ F are decreasing. We first have a lemma giving us upper bounds on the derivatives of the elements in ⟨R, T⟩, where R and T in this section are defined by
\[ R(x) = \frac{a(1 - x)}{(b - a - c)x + a + c}, T(x) = \frac{a}{x + a}. \] (4.1)

**Lemma 4.1.** If m + n is even, then
\[ \max_{x \in [0,1]} |(R^mT R^nT)'(x)| \leq \max \left\{ \frac{1}{(1 + a)^2}, \frac{1}{(b + c)^2} \right\}, \] (4.2)
and if m + n is odd, then
\[ \max_{x \in [0,1]} |(R^mT R^nT)'(x)| \leq \frac{ab}{(ab + bc)^2}. \] (4.3)

**Proof.** A simple induction shows that for even m there exist γ, λ, u, v ≥ 0 such that
\[ R^mT(x) = \frac{a + \gamma + \lambda x}{(a + c + x + v + ux)^2}, (R^mT)'(x) = \frac{a}{(a + c + x + v + ux)^2}. \]
And if m is odd, then there exist γ, λ, u, v ≥ 0 such that
\[ R^mT(x) = \frac{c + x + \gamma + \lambda x}{b + c + x + v + ux}, (R^mT)'(x) = \frac{b}{(b + c + x + v + ux)^2}. \]

Now suppose both m and n are even. It follows from these equations and some algebraic simplifications that (R^mT R^nT)'(x) ≤ 1/(1 + a)^2, while if m and n are both odd, we have (R^mT R^nT)'(x) ≤ 1/(b + c)^2. The inequality (4.3) follows similarly. □
Proposition 4.2. Let \( a > 0 \) and \( b \geq 1 \geq c \geq 0 \). Moreover, suppose that \( b > 1 \) if \( c = 0 \). Let \( R \) and \( T \) be the maps given by \((4.1)\). Then the orbit of any \( x \in [0,1] \) is dense in \([0,1] \) under the action of the semigroup generated by \( R \) and \( T \).

Proof. Consider the pair \((RT,T)\). We see that \( RT(0) = 0 \) and \( RT(1) = a/(a+ab+c) \). If \( RT(1) \geq T(1) \), which is equivalent to \( ab+c \leq 1 \), then the claim follows from Proposition [3.4]. Thus, for the rest of the proof, we assume that \( ab + c > 1 \). In particular, we have

\[
\frac{ab}{(ab+c)^2} < \frac{ab}{(ab+c)} \leq 1. \tag{4.4}
\]

We divide the proof into two cases:

Case 1. Suppose \( c \leq a/(a+c) \). Let \( A \) be the maximum-length interval in \((0,1)\setminus \Omega\), where \( \Omega \) is the orbit of 0. Similar to the proof of Lemma 3.2 one shows that there exist \( \beta \in \{0,1\} \) and a sequence \( \{\alpha_i\}_{i=1}^\infty \) such that for each \( k \geq 1 \) there exists \( A_k \subseteq (0,1)\setminus \Omega \) with \( A = T^\beta R^\alpha_1 T \beta R^\alpha_2 T \ldots R^\alpha_k T(A_k) \). Let

\[
\delta = \max \left\{ \frac{1}{(1+a)^2}, \frac{1}{(b+c)^2}, \frac{ab}{(ab+c)^2} \right\} < 1.
\]

The fact that \( \delta < 1 \) follows from the facts that \( a > 0 \), \( b > 1 \) if \( c = 0 \), and \((4.4)\). Now, Lemma 4.1 implies that for \( k \) large enough

\[
\max \]|(T^\beta R^\alpha_1 T^\beta R^\alpha_2 T \ldots R^\alpha_k T)'(x)| \leq \max \{1, 1/a\} \cdot \delta^k < 1, \tag{4.5}
\]

which is in contradiction with \( A \) being the maximum-length interval in \((0,1)\setminus \Omega\).

Case 2. Suppose \( c > a/(a+c) \). Let \( A \) be the maximum-length interval in \((0,a/(a+c))\setminus \Omega\). Then \( A = R^\alpha T^\beta(B) \) for some \( B \subseteq (0,a/(a+c)) \). We show that \(|(R^\alpha T^\beta)'(x)| < 1 \) for all \( \alpha, \beta > 0 \) and \( x \in [0,1] \). This will complete the proof. Since \(|(T^m)'(x)| < 1 \) and \(|(R^m)'(x)| < 1 \) for \( m > 1 \), it is sufficient to prove the claim for \( (\alpha, \beta) = (1,1), (1,2), (2,1) \). We go through the list:

\[
(RT)'(0) = 1/b < 1, \quad (RT)'(1) = (a^2b)/(a+ab+c)^2 < 1. \tag{4.6}
\]
\[(RT^2)'(0) = \frac{ab}{(a + ab + c)^2} < \frac{ab}{(ab + c)^2} < 1 \tag{4.7}\]
\[(RT^2)'(1) = \frac{ab}{(a + b + ab + c)^2} < 1 \tag{4.8}\]
\[|\!(R^2T)'(0)| = \frac{a}{(a + c)^2} < \frac{c}{a + c} < 1 \tag{4.9}\]
\[|\!(R^2T)'(1)| = \frac{a^3b^2}{(a^2b + c^2 + ab + ac + abc)^2} < \frac{a^3b^2}{(a^2b + ab)^2} < 1 \tag{4.10}\]

This completes the proof of Proposition 4.2. \(\square\)

Part (iii) of Theorem 1.2 follows from the proposition below.

**Proposition 4.3.** Let \(f, g \in F\) be both decreasing. Then the orbit of a given \(x \in (0, 1)\) is dense in \([0, 1]\) if and only if \(\{o(fg), o(gf)\} = \{0, 1\}\) and one of the following occurs

i) Neither one of \(f\) or \(g\) is onto and \(Im(f) \cup Im(g) = [0, 1]\).

ii) Exactly one of \(f\) or \(g\) is onto and \(Im(fg) \cup Im(gf) = [0, 1]\).

**Proof.** Proof of (i) is straightforward and follows from Proposition 4.2. To prove (ii), we note that the pair \((fg, gf)\) satisfies all of the conditions of Proposition 2.2. The proof of the converse of (ii) is similar to the proof of the converse of (ii) in Proposition 3.5. Finally the proof that both \(f\) and \(g\) cannot be onto (if some orbit is dense) is similar to the proof given for the same statement at the end of the proof of Proposition 3.5. \(\square\)

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17
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