To help focus ideas regarding possible routes to the breakdown of Lorentz invariance, it is extremely useful to explore concrete physical models that exhibit similar phenomena. In particular, acoustics in Bose–Einstein condensates has the interesting property that at low-momentum the phonon dispersion relation can be written in a “relativistic” form exhibiting an approximate “Lorentz invariance”. Indeed all of low-momentum phonon physics in this system can be reformulated in terms of relativistic curved-space quantum field theory. In contrast, high-momentum phonon physics probes regions where the dispersion relation departs from the relativistic form and thus violates Lorentz invariance. This model provides a road-map of at least one route to broken Lorentz invariance. Since the underlying theory is manifestly physical this type of breaking automatically avoids unphysical features such as causality violations. This model hints at the type of dispersion relation that might be expected at ultra-high energies, close to the Planck scale, where quantum gravity effects are suspected to possibly break ordinary Lorentz invariance.
have at least one concrete physical model of Lorentz symmetry breaking at hand which is guaranteed to be physically consistent. Even if the symmetry breaking believed to arise from quantum gravity does not follow this particular pattern, exhibiting such a model unambiguously settles an important matter of principle. Additionally, the type of dispersion relation arising in this model (and many other related analog models\(^3\)) gives an important hint as to what type of modified dispersion relation to look for at Planck energies.

## 2 From BEC to Lorentzian geometry

Bose–Einstein condensates are described by the nonlinear Schrödinger equation (Gross–Pitaevskii equation):

\[
- \imath \hbar \partial_t \psi(t, \vec{x}) = \frac{\hbar^2}{2m} \nabla^2 \psi(t, \vec{x}) + \lambda \|\psi\|^2 \psi(t, \vec{x}).
\]  

(1)

(We have suppressed the externally applied trapping potential for algebraic simplicity. For technical details see Barceló et al.\(^1\) That reference also contains an extensive background bibliography.) Now use the Madelung representation to put the Schrödinger equation in “hydrodynamic” form:

\[
\psi = \sqrt{\rho} \exp(-i\theta m/\hbar).
\]  

(2)

Take real and imaginary parts: The imaginary part is a continuity equation for an irrotational flow of velocity \(\vec{v} = \nabla \theta\) and the real part is a Hamilton–Jacobi equation (its gradient leads to the Euler equation). Specifically:

\[
\partial_t \rho + \nabla \cdot (\rho \nabla \theta) = 0.
\]  

(3)

\[
\frac{\partial}{\partial t} \theta + \frac{1}{2} (\nabla \theta)^2 + \frac{\lambda \rho}{m} - \frac{\hbar^2}{2m^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = 0.
\]  

(4)

That is, the nonlinear Schrödinger equation is completely equivalent to irrotational inviscid hydrodynamics with an enthalpy

\[
h = \int \frac{dp}{\rho} = \frac{\lambda \rho}{m},
\]  

(5)

plus a peculiar self-interaction:

\[
V_Q = -\frac{\hbar^2}{2m^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = 0.
\]  

(6)
The equation of state for this Madelung fluid is \( p = \frac{\lambda \rho^2}{2m} \), so that \( \frac{dp}{d\rho} = \frac{\lambda \rho}{m} \). To now extract a Lorentzian geometry, linearize around some background. In the low-momentum limit it is safe to neglect \( V_Q \). It is a by now standard result that the phonon is described by a massless minimally-coupled scalar that satisfies the d’Alembertian equation in the effective (inverse) background-dependent metric

\[
g_{\mu\nu}(t, \vec{x}) \equiv \frac{\rho_0}{c_s} \begin{pmatrix} -1 & : & -v_0 \\ : & \ddots & \vdots \\ -v_0 & : & (c_s^2 \mathbf{I} - v_0 \otimes v_0) \end{pmatrix}. \tag{7} \]

Here

\[
c_s^2 = \frac{\lambda \rho_0}{m}; \quad v_0 = \nabla \theta_0. \tag{8} \]

It cannot be overemphasized that low-momentum phonon physics is completely equivalent to quantum field theory in curved spacetime. That is, everything that theorists have learned about curved space QFT can be carried over to this acoustic system, and conversely acoustic experiments can in principle be used to experimentally investigate curved space QFT. In particular, it is expected that acoustic black holes (dumb holes) will form when the condensate flow goes supersonic, and that they will emit a thermal bath of (acoustic) Hawking radiation at a temperature related to the physical acceleration of the condensate as it crosses the acoustic horizon.

For completeness we mention that the metric is

\[
g_{\mu\nu}(t, \vec{x}) \equiv \frac{\rho_0}{c_s} \begin{pmatrix} -(c_s^2 - v_0^2) & : & -\vec{v}_0 \\ : & \ddots & \vdots \\ -\vec{v}_0 & : & \mathbf{I} \end{pmatrix}. \tag{9} \]

so the space-time interval can be written

\[
ds^2 = \frac{\rho_0}{c_s} \left[ -c_s^2 dt^2 + ||d\vec{x} - \vec{v}_0 dt||^2 \right]. \tag{10} \]

### 3 Characteristics

But there is a bit of a puzzle here: We started with the nonlinear Schrödinger equation. That equation is parabolic, so we know that the characteristics move at infinite speed. How did we get a hyperbolic d’Alembertian equation with a finite propagation speed? The subtlety resides in neglecting the higher-derivative term \( V_Q \). To see this, keep \( V_Q \), and go to the eikonal approximation.
One obtains the dispersion relation

\[
(\omega - \vec{v}_0 \cdot \hat{k})^2 = c_s^2 k^2 + \left(\frac{\hbar}{2m} k^2\right)^2.
\]  

(11)

This is the curved-space generalization of the well-known Bogolubov dispersion relation. Equivalently

\[
\omega = \vec{v}_0 \cdot \hat{k} + \sqrt{c_s^2 k^2 + \left(\frac{\hbar}{2m} k^2\right)^2}.
\]  

(12)

The group velocity is

\[
\vec{v}_g = \frac{\partial \omega}{\partial \vec{k}} = \vec{v}_0 + \frac{\left(c_s^2 + \frac{\hbar^2}{2m^2} k^2\right)}{\sqrt{c_s^2 k^2 + \left(\frac{\hbar}{2m} k^2\right)^2}} \hat{k},
\]  

(13)

while for the phase velocity

\[
\vec{v}_p = \frac{\omega \hat{k}}{||k||} = (\vec{v}_0 \cdot \hat{k}) \hat{k} + \sqrt{c_s^2 k^2 + \frac{\hbar^2 k^2}{4m^2}} \hat{k}.
\]  

(14)

Both group and phase velocities have the appropriate relativistic limit at low momentum, but then grow without bound at high momentum, leading to an infinite signal velocity and the recovery of the parabolic nature of the differential equation at high momentum. \((k \gg k_c \equiv m c_s/\hbar);\) equivalently in terms of the acoustic Compton wavelength \(\lambda \ll \lambda_C \equiv \hbar/(m c_s).\)

It is amusing, and perhaps somewhat surprising, to notice that while at low momentum the Bogolubov dispersion relation is (approximately) relativistic, at large momenta \((k \gg k_c)\) the dispersion relation again takes on Newtonian form, complete with “rest mass” contribution

\[
\omega(k) = \frac{\hbar k^2}{2m} + \vec{v}_0 \cdot \hat{k} + \frac{m c_s^2}{\hbar} + O(k^{-2}).
\]  

(15)

4 Discussion

The key message to take from this article is that BECs are examples of physically realizable systems with “broken” Lorentz invariance. The way to preserve causality in this system is clear — there is an intrinsic preferred frame in this model and one should quantize by applying equal time commutators in the inertial frame in which the original nonlinear Schrödinger equation is written.
down. In particular, field commutators need not generically vanish identically outside the lightcone. In the low-momentum limit, where the physics decouples from this preferred frame, the commutators are approximately zero outside the lightcone; this is good enough for an approximate Lorentz symmetry to arise. (In the preferred frame, at equal times commutators are exactly zero at disjoint points.) This is not the only way of dealing with causality in theories with broken Lorentz invariance, but it is at least one route that is guaranteed (by its very construction) to be internally consistent.

It is also useful to remember that BECs are not the only interesting systems exhibiting broken Lorentz symmetry, in fact condensed matter physics is littered with “analog models” for low-energy Lorentz invariance. For instance we mention acoustics in the presence of viscosity and Lattice phonons. Finding an approximate Lorentzian geometry is really just a matter of isolating a particular degree of freedom that is approximately decoupled from the rest of the physics and doing a low-momentum field-theory normal-modes analysis. In all of the condensed matter systems we are aware of, the modifications to the high-energy dispersion relation show up as even powers of energy and/or momentum. Ultimately this is due to the fact that the condensed matter systems considered so far explicitly conserve parity and time reversal invariance. In contrast some of the models based on quantum gravity hint at cubic deviations from a quadratic dispersion relation — this can often be traced back to some underlying assumption of an intrinsic violation of parity or time reversal invariance.

In summary: BECs in particular, and condensed matter systems in general, provide useful explicit models of an approximate low-energy Lorentz invariance that is broken by higher-energy “fundamental” physics. They provide useful templates for comparison with (and contrast to) the models of Lorentz symmetry breaking currently emerging from various quantum gravity scenarios.

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