A Noether-Lefschetz theorem for vector bundles

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Abstract

In this note we use the monodromy argument to prove a Noether-Lefschetz theorem for vector bundles.

1 Introduction

Let \( X \) be a smooth complex projective manifold of dimension \( n \) and let \( E \) be a very ample vector bundle on \( X \) of rank \( r \). This means that the tautological quotient line bundle \( L \) on the bundle \( Y = \mathbb{P}(E^*) \) of hyperplanes in \( E \) is very ample. For almost all \( s \in H^0(X, E) \) the zero-locus \( Z \) is smooth, irreducible and of dimension \( n - r \). In [8, prop. 1.16] Sommese proved that \( H^i(X, Z; \mathbb{Z}) \) vanishes for \( i < n - r + 1 \) and is torsion free for \( i = n - r + 1 \). Assume that \( n - r \) is even, say \( n - r = 2p \). Let \( \text{Alg} \subset H^{n-r}(Z) \) be the space of algebraic classes and let \( \text{Im} = \text{Im}(H^{n-r}(X) \hookrightarrow H^{n-r}(Z)) \). (We always take coefficients in \( \mathbb{C} \) unless other coefficients are mentioned explicitly (cf. Remark 3).) In this note we prove the following Noether-Lefschetz theorem for this situation.

**Theorem 1** If \( E \) is very ample and \( s \) is general, then either \( \text{Alg} \subset \text{Im} \) or \( \text{Alg} + \text{Im} = H^{n-r}(Z) \).

(With “general” we shall always mean general in the usual Noether-Lefschetz sense.) The following theorem, which generalizes the Noether-Lefschetz theorem for complete intersections in projective space (see [2, pp. 328–329]) is an immediate corollary.

**Theorem 2** If \( h^{\alpha \beta}(X) < h^{\alpha \beta}(Z) \) for some pair \((\alpha, \beta)\) with \( \alpha + \beta = n - r \) and \( \alpha \neq \beta \), then every algebraic class on \( Z \) is induced from \( X \).

**Remark 3** Notice that the unique pre-images of algebraic classes are themselves Hodge classes, i.e. lie in \( H^{p,q}(X) \cap H^{n-r}(X; \mathbb{Z}) \). This follows from the fact that the cokernel of \( H^{n-r}(X, \mathbb{Z}) \to H^{n-r}(Z, \mathbb{Z}) \) is torsion free.
It is not difficult to show that after replacing $E$ with $E \otimes L^k$, where $k \gg 0$ and $L$ is an ample line bundle, the assumption of theorem 2 is satisfied. (E.g. the geometric genus of $X$ goes to infinity as $k$ goes to infinity.) In [9] we used the notion of Castelnuovo-Mumford regularity (cf. [7, p. 99]) to make the positivity assumption on $E$ more precise if $X = \mathbb{P}^n$. Notations are as in theorem 1.

Hdg is defined to be the space of Hodge classes on $Z$ of codimension $p$, i.e. 
$$Hdg = H^{p,p}(Z) \cap H^{n-r}(Z,\mathbb{Z}).$$

Theorem 4 If $E$ is a $(-3)$-regular vector bundle of rank $r$ on $X = \mathbb{P}^n$ and $Z$ is the zero-locus of a general global section of $E$, then $Hdg \subset \text{Im}$, unless $(X, E) = (\mathbb{P}^3, \mathcal{O}(3))$. If $\dim Z = 2$, then it suffices that $E$ be $(-2)$-regular unless $(X, E) = ((\mathbb{P}^3, \mathcal{O}(2)), (\mathbb{P}^3, \mathcal{O}(3))$ or $(\mathbb{P}^4, \mathcal{O}(2) \oplus \mathcal{O}(2))$.

(Notice that $(-3)$-regularity $\Rightarrow$ $(-1)$-regularity $\Rightarrow$ very ampleness.) For the case $\dim Z = 2$ theorem 4 is due to Ein [3, thm. 3.3]. The advantage of theorem 4 is that it applies to Hodge rather than algebraic classes on $Z$. For example, it implies that if all Hodge classes of codimension $n-r$ on $\mathbb{P}^n$ are algebraic, then the same holds for $Z$. The advantage of theorem 4 is that the positivity condition on $E$ is more geometric: the cohomological conditions from [9] are replaced with the condition that $E$ be very ample plus a Hodge number inequality (cf. theorem 2).

In other words, for very ample vector bundles, the Noether-Lefschetz property holds as soon as this is allowed by the Hodge numbers. However, this Hodge number inequality condition is of course a cohomological condition on $E$ in disguise.

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2 Proof of the main result

Let $V = H^0(X, E)$, let $\mathbb{P}(V)$ be the set of lines in $V$, let $N = \dim \mathbb{P}(V) = h^0(X, E) - 1$ and set $X' = \mathbb{P}(V) \times X$. Set $E' = p_1^* \mathcal{O}(1) \otimes p_2^* E$, where $p_i$ are the projections. $E'$ has a canonical section $s'$. Let $Z$ be the zero locus of $s'$. The restriction $p: Z \to \mathbb{P}(V)$ of $p_1$ to $Z$ is the universal family of zero loci of sections in $E$. We leave the proof of the following easy lemma to the reader.

Lemma 5 If $E$ is very ample, then it is generated by its sections. If $E$ is generated by its sections, then $Z$ is smooth, irreducible and of dimension $N + n - r$.

Let $\Delta \subset \mathbb{P}(V)$ be the discriminant of $p$, i.r.

$$\Delta = p\{z \in Z : \text{rk}_z p \leq N - 1\}$$
$$= \{[s] \in \mathbb{P}(V) : p^{-1}(s) \text{ is not smooth of dimension } n - r\}.$$
Fix a point \([s_0] \in \mathbb{P}(V) \setminus \Delta\) and let \(Z \subset X\) be the corresponding smooth fibre of \(p\). Let \(\Gamma\) the image of the monodromy representation \(\pi_1(\mathbb{P}(V) \setminus \Delta) \to \text{Aut}(H^{n-r}(Z))\).

Let \(\text{Im}^\perp\) be the orthogonal complement of \(\text{Im}\) with respect to the intersection form on \(H^{n-r}(Z)\). Since for general \(s \in H^0(X, E)\), \(\text{Alg}\) is a \(\Gamma\)-module (cf. [5, p. 141]), theorem [4] from the following proposition.

**Proposition 6** (*Second Lefschetz Theorem*)

1. \(H^{n-r}(Z) = \text{Im} \oplus \text{Im}^\perp\)
2. \(\text{Im} = H^{n-r}(Z)\Gamma\)
3. \(\text{Im}^\perp\) is an irreducible \(\Gamma\)-module

**Proof:**

1. Arguing as in the proof of [4, thm. 6.1 (i)] one shows that if \(Z\) is submanifold of a compact Kähler manifold \(X\) such that \(H^i(X, Z) = 0\) for \(i \leq m = \dim Z\), then the restriction of the intersection form to \(\text{Im}(H^m(X) \to H^m(Z))\) is non-degenerate.

2. The inclusion \(\text{Im} \subset H^{n-r}(Z)\Gamma\) is trivial. To prove that \(H^{n-r}(Z)\Gamma \subset \text{Im}\), we argue as in [4, thm. 6.1 (iii)]. Consider the commutative diagram

\[
\begin{array}{ccc}
H^{n-r}(\mathbb{P}(V) \times X) & \longrightarrow & H^{n-r}(Z) \\
\downarrow & & \downarrow \\
H^{n-r}(X) & \longrightarrow & H^{n-r}(Z)\Gamma.
\end{array}
\]

By [4, théorème 4.1.1 (ii)] the map \(H^{n-r}(Z) \to H^{n-r}(Z)\Gamma\) is surjective. By [4, prop. 1.16] the map \(H^{n-r}(\mathbb{P}(V) \times X) \to H^{n-r}(Z)\) is surjective.

3. Since the monodromy respects the intersection form, \(I^\perp\) is a \(\Gamma\)-module.

The standard argument using Lefschetz pencils and the theory of vanishing cycles reduces the problem of irreducibility to proposition [6] below (cf. [4, pp. 46–48]).

\(\blacksquare\)

**Proposition 7**

1. The discriminant \(\Delta\) is an irreducible, closed, proper subvariety of \(\mathbb{P}(V)\).

2. Let \(G \subset \mathbb{P}(V)\) be a general line. Then \(Z_G := p^{-1}(G)\) is smooth, irreducible of dimension \(n-r+1\) and the restricted family \(p_G: Z_G \to G\) is a holomorphic Morse function, i.e. all critical points are non-degenerate and no two lie in the same fibre (cf. [4, p. 34]). \(g \in G\) is a critical value of \(p_G\) if and only if it is a critical value of \(p\).
PROOF: The statements about $\mathcal{Z}_C$ follow from Bertini. The remaining assertions are well-known if $\text{rk} E = 1$ (cf. p. 19]). In particular, they are true for $(Y, L)$, where $Y$ is the hyperplane bundle $\mathbb{P}(E)$ of $E$ and $L$ is the tautological quotient line bundle $\mathcal{O}_Y(1)$. The following proposition reduces the general case $(X, E)$ to this line bundle case $(Y, L)$, thus finishing the proof.

Before we state the last proposition, notice that the natural map $s \mapsto \bar{s}: H^0(X, E) \to H^0(Y, L)$, where $\bar{s}(x, h) := s(x) \in E(x)/h = L(x, h)$ for $(x, h) \in Y$, is an isomorphism. Indeed, the map is clearly injective and $h^0(Y, L) = h^0(X, \pi_* L) = h^0(X, E)$. For $s \in H^0(X, E)$ we denote by $Z_X(s)$ the zero-locus of $s$ in $X$ and by $Z_Y(\bar{s})$ the zero-locus of $\bar{s}$ in $Y$.

**Proposition 8** For $s \in H^0(X, E) \setminus \{0\}$, $Z = Z_X(s)$ is singular if and only if $W = Z_Y(\bar{s})$ is singular. More precisely, if $x \in \text{Sing} Z$, then there exists a $y \in \text{Sing} W$ with $\pi(y) = x$ and conversely, if $(x, h) \in \text{Sing} Z$, then $x \in \text{Sing} W$. Finally, if $(x, h)$ is a non-degenerate quadratic singularity, then so is $x$.

**Proof:** This is a calculation in local coordinates. Let $x_0 \in Z$, i.e. $s(x_0) = 0$. After choosing local coordinates $x_1, \ldots, x_n$ on $X$ and a local trivialization of $E$ near $x_0$ we may regard $s$ to be a function in $x_1, \ldots, x_n$. Then $x_0 \in \text{Sing} Z$ if and only if $\left\{ \frac{\partial s}{\partial x_j}(x_0) \right\}_{j=1}^n$ does not span $\mathbb{C}^r$. Let $h_0 \subset \mathbb{C}^r$ be a hyperplane containing span $\left\{ \frac{\partial s}{\partial x_j}(x_0) \right\}_{j=1}^n$. We claim that $y_0 = (x_0, h_0) \in \text{Sing} W$. We may assume that the local trivialization of $E$ has been chosen in such a way that $h_0$ is given by $z_r = 0$, where $z_1, \ldots, z_r$ are coordinates on $\mathbb{C}^r$. Let $s = (f_1, \ldots, f_r)$. Local coordinates on $Y$ near $y_0$ are provided by the local coordinates $x_1, \ldots, x_n$ on $X$ near $x_0$ together with $(y_1, \ldots, y_{r-1}) \in \mathbb{C}^{r-1}$: we let $(y_1, \ldots, y_{r-1}) \in \mathbb{C}^{r-1}$ correspond to the hyperplane $\sum_{i=1}^r y_i z_i = 0$, where $y_r := 1$. The point $y_0$ has coordinates $(x_0, 0)$. In these local coordinates $\bar{s}(x, y) = \sum_{i=1}^r y_i f_i(x)$. It now suffices to calculate $\frac{\partial \bar{s}}{\partial y_j}(x_0, 0) = 0$ for $k = 1, \ldots, n$ and $\frac{\partial \bar{s}}{\partial y_0}(x_0, 0) = f_j(x_0) = 0$ for $j = 1, \ldots, r-1$. The converse is proven similarly.

Let $y_0 = (x_0, h_0) \in \text{Sing} W$. We may again assume that $h_0$ is given by $z_r = 0$. The Hessian of $\bar{s}$ in $y_0$ is of the form $\begin{pmatrix} h & d' \\ d & 0 \end{pmatrix}$, where the $n \times n$ matrix $h$ is the Hessian of $f_r$ and the $(r-1) \times n$ matrix $d$ is the Jacobian of $f' := (f_1, \ldots, f_{r-1})$ in $x_0$. Let $Z' = \{ x \in X : f'(x) = 0 \}$. We have to check that the Hessian of $f_r|_{Z'}$ in $0$ is non-degenerate. Since we assume that the Hessian of $\bar{s}$ has maximal rank in $y_0$, so has $d$. Thus, after a change of coordinates, we may assume that $f_i(x) = x_i$ for $i < r$. Then $\bar{s}(x, y) = \sum_{i=1}^{r-1} x_i y_i + f_r(x)$, hence the Hessian of $\bar{s}$ in $y_0$ is

$$
\begin{pmatrix}
* & * & E_{r-1} \\
* & H & 0 \\
E_{r-1} & 0 & 0
\end{pmatrix},
$$

where $H$ is the Hessian of $f_r|_{Z'}$ in $x_0$. It follows that $H$ is non-degenerate. \(\square\)
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