THE CANONICAL MODIFICATIONS BY WEIGHTED BLOW-UPS

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Abstract. In this paper we give a criterion for an isolated, hypersurface singularity of dimension $n \geq 2$ to have the canonical modification by means of a suitable weighted blow-up. Then we give a counter example to the following conjecture by Reid-Watanabe: For a 3-dimensional, isolated, non-canonical, log-canonical singularity $(X, x)$ of embedded dimension 4, there exists an embedding $(X, x) \subset (C^4, 0)$ and a weight $w = (w_0, w_1, \ldots, w_n)$, such that the $w$-blow-up gives the canonical modification of $(X, x)$.

0. Introduction

0.1. Throughout this paper all varieties are defined over the complex number field $C$. The canonical modification of a singularity $(X, x)$ is a partial resolution $\varphi : Y \to X$ such that $Y$ admits at worst canonical singularities, and the canonical divisor $K_Y$ is relatively ample with respect to $\varphi$. If a canonical modification exists, then it is unique up to isomorphisms over $X$. It is well known that it exists if the minimal model conjecture holds. For a 2-dimensional singularity $(X, x)$, the canonical modification is the RDP-resolution ([9]). For a 3-dimensional singularity $(X, x)$, the canonical modification exists by the affirmative answer ([10]) to the minimal model conjecture. For higher dimensional singularities the existence is not generally proved.

The motivation for writing this paper is the following conjecture by Miles Reid and Kimio Watanabe:

Conjecture 0.2. For a 3-dimensional, isolated, non-canonical, log-canonical singularity $(X, x)$ of embedded dimension 4, there exists an embedding $(X, x) \subset (C^4, 0)$ and a weight $w = (w_0, w_1, \ldots, w_n)$ such that the $w$-blow-up gives the canonical modification of $(X, x)$.

Primarily, both Reid and Watanabe brought up stronger versions in different ways: Reid required the statement for elliptic singularities,
not only for log-canonical singularities (Conjecture p.306, [12]); Kimio Watanabe required it for all non-canonical, log-canonical singularities defined by a non-degenerate polynomial without replacing embedding to $(\mathbb{C}^4,0)$, and he also required the weight $w$ should be in 95-weights listed by Yonemura [22] and Fletcher [1], which give the weights of quasi-homogeneous simple K3-singularities.

Tomari [15] showed an affirmative answer for log-canonical singularities of special type. Watanabe calculated many examples, and made a list of standard equations of log-canonical singularities which admit the canonical modifications by weighted blow-up with each weight of the 95’s [20].

0.3. In this paper we say that a weight is the canonical weight, if it gives the weighted blow-up which is the canonical modification. We give a criterion for an isolated, hypersurface singularity of dimension $n$ ($\geq 2$) to have the canonical weight in §2. As a consequence, for a non-canonical, log-canonical singularity $(X,x) \subset (\mathbb{C}^{n+1},0)$ defined by a non-degenerate polynomial $f$ (definition cf. [8]), a primitive vector $w = (w_0,w_1,\ldots,w_n)$ is the canonical weight if and only if $w$ is absolutely minimal (i.e. each coordinate $w_i$ is the minimal integer) in the essential cone in the dual space (Corollary 2.9). We can see many singularities for which such vectors actually exist (Corollary 2.12∼ Example 2.16). But we also observe in §4 an example for which such a vector does not exist and it turns out to be a counter example opposing the Reid-Watanabe’s conjecture. In the other sections, we prepare the formula for coefficients of divisors (in §1) and study deformations of isolated singularities (in §3).

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1. Divisors on toric varieties

1.1. Let $M$ be the free abelian group $\mathbb{Z}^r$ ($r \geq 1$) and $N$ be the dual $\text{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$. We denote $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$ by $M_\mathbb{R}$ and $N_\mathbb{R}$, respectively. Then $N_\mathbb{R} = \text{Hom}_{\mathbb{R}}(M_\mathbb{R},\mathbb{R})$. For a finite fan $\Delta$ in $N_\mathbb{R}$, we construct the toric variety $V = \text{T}_N(\Delta)$. Denote by $\Delta(1)$ the set of primitive vectors $q = (q_1,\ldots,q_r) \in \mathbb{N}$ whose rays $\mathbb{R}_{\geq 0}q$ belong to $\Delta$ as
one-dimensional cones. For $q \in \Delta(1)$, denote by $D_q$ the corresponding divisor which is denoted by $\text{orb} \mathbb{R}_{\geq 0}q$ in $\mathbb{P}^1$. Denote by $U_q$ the invariant affine open subset which contains $\text{orb} \mathbb{R}_{\geq 0}q$ as the unique closed orbit. Then $U_q = \text{Spec} \mathbb{C}[q' \cap M]$, $U_q \cap D_q = \text{orb} \mathbb{R}_{\geq 0}q = \text{Spec} \mathbb{C}[q^+ \cap M]$, and $U_q \cap D_q$ is defined in $U_q$ by the ideal $p_q$ which is generated by the elements $e \in M$ with $q(e) > 0$.

Express $\mathbb{C}[M]$ by $\mathbb{C}[x^a]_{a \in M}$, where $x^a = x_1^{a_1}x_2^{a_2} \cdots x_r^{a_r}$ for $a = (a_1, \ldots, a_r) \in M$. For convenience sake, we write $x^a \in f$, if $f = \sum_{q \in M} a_qx^q$, and $a_q \neq 0$.

**Definition 1.2.** Under the notation above, take $q \in |\Delta|$.

1. For a regular function $f$ on $T_N(\Delta)$, define:

$$q(f) := \min \{ q(a) | a \in M, x^a \in f \}$$

2. For a $\mathbb{Q}$-divisor $D$ on $T_N(\Delta)$ such that $mD$ is defined by a regular function $f$ on $T_N(\Delta)$, define:

$$q(D) := \frac{1}{m}q(f).$$

**Proposition 1.3.** Let $D \subset V$ be a $\mathbb{Q}$-principal divisor (i.e. $mD$ is defined by a regular function on $V$). Then $D$ is of the form:

$$D = D' + \sum_{q \in \Delta(1)} q(D)D_q,$$

where $D'$ is an effective divisor which does not contain any $D_q$.

**Proof.** By the definition of $q(D)$, we may assume that $D$ is defined by a regular function $f$ on $V$. Write $D = D' + \sum_{q \in \Delta(1)} m_qD_q$ such that $D'$ does not contain any $D_q$ and $m_q \geq 0$. Let $\zeta$ be the defining function of $D_q$ on $U_q$, then $f = \zeta^{m_q}h$, where $h \notin (\zeta)$. Since $q(f) = m_qq(\zeta) + q(h)$ and $q(h) = 0$, it is sufficient to prove that $q(\zeta) = 1$. And this is clear, because there is a vector $e \in M$ such that $q(e) = 1$ and $\zeta$ is a generator of the ideal $\{ g \in \mathbb{C}[q' \cap M] | q(g) > 0 \}$. \(\square\)

**1.4.** Now we consider the case that $\Delta$ consists of all faces of a simplicial cone $\sigma$ in $\mathbb{N}_\mathbb{Z}$. Let $a_1, \ldots, a_r$ be the primitive vector of the one-dimensional faces of $\sigma \cap \mathbb{N}$ and $a_1^*, \ldots, a_r^*$ be the dual system of $\{a_i\}$'s (i.e. $a_i^* \in \mathbb{M}_\mathbb{R}$ and $\langle a_i^*, a_j^* \rangle = \delta_{ij}$). Denote by $\mathbb{N}$ the subgroup of $\mathbb{N}$ generated by $\{a_i\}$. Then the morphism $\pi : T_{\mathbb{N}}(\Delta) \rightarrow T_N(\Delta) = V$ induced by $(\mathbb{N}, \Delta) \rightarrow (N, \Delta)$ is the quotient morphism of $\mathbb{C}^r$ by the finite group $N/\mathbb{N}$, and each $a_j^*$ belongs to $\overline{M} = \text{Hom}_\mathbb{Z}(\mathbb{N}, \mathbb{Z})$. Denote by $D_{a_i}$ and $\overline{D}_{a_i}$ the divisors on $T_N(\Delta)$ and $T_{\mathbb{N}}(\Delta) = \mathbb{C}^r$ respectively.
which are corresponding to $a_i$. Let $r_i$ be the ramification index of $\pi$ at $D_{a_i}$.

**Lemma 1.5.** Let $\Delta$ be as in [1.4]. Then it follows that

$$q(D_{a_i}) = r_iq(a_i^*)$$

for every $q \in |\Delta|$ and every $i$.

**Proof.** Let $D_{a_i}$ be defined by $x_i = 0$ on $U_{a_i} \subset V$. Then $x_i = y_i^{r_i}$, where the equation $y_i = 0$ defines $D_{a_i}$ on $T_N(\Delta)$. Therefore, $q(D_{a_i}) = q(x_i) = r_iq(y_i)$ and the last term equals $r_iq(a_i^*)$, because $y_i$ is the $i$-th coordinate function of $T_N(\Delta) = C^r$. \hfill $\square$

**Proposition 1.6.** Let $\Delta'$ be a finite subdivision of an arbitrary finite fan $\Delta$ and $\varphi : V' = T_N(\Delta') \to V = T_N(\Delta)$ be the corresponding birational morphism. Denote the divisor on $V'$ corresponding to $q \in \Delta'(1)$ by $D'_q$. Assume that $K_V$ is $\mathbb{Q}$-principal. Then

$$K_{V'} = \varphi^*K_V + \sum_{q \in \Delta'(1) - \Delta(1)} (q(\varphi^* (\sum_{t \in \Delta(1)} D_t)) - 1) D'_q;$$

If $\Delta$ is as in [1.4], then it follows that

$$K_{V'} = \varphi^*K_V + \sum_{q \in \Delta'(1) - \Delta(1)} (\sum_{i=1}^r r_iq(a_i^*) - 1) D'_q;$$

If moreover $\sigma$ is the positive quadrant in $N_{\mathbb{R}}$, then

$$K_{V'} = \varphi^*K_V + \sum_{q \in \Delta'(1) - \Delta(1)} (q(1) - 1) D'_q,$$

where $1 = (1, 1, \ldots, 1) \in M$.

**Proof.** For a toric variety, the canonical divisor is represented by the sum of all toric invariant divisors with coefficient $-1$. Therefore

(1) $K_{V'} = -\sum_{q \in \Delta'(1) - \Delta(1)} D'_q - \sum_{t \in \Delta(1)} D'_t$.

On the other hand, we represent $K_{V'}$ as

(2) $K_{V'} = \varphi^*K_V + \sum_{q \in \Delta'(1) - \Delta(1)} q(\varphi^* D) m_q D'_q$.

Substituting $K_V = -\sum_{t \in \Delta(1)} D_t$ into (2) and comparing (1) and (2), we obtain the value of $m_q$. For the second and the third equalities, note that $q(\varphi^* D) = q(D)$ for a $\mathbb{Q}$-principal divisor $D$ and apply [1.3]. \hfill $\square$

Now we obtain the characterization of hypersurface singularities by means of the Newton diagram. Parts of the following are stated in [21], [19] and [13].
Corollary 1.7. Let \((X, 0) \subset (\mathbb{C}^{n+1}, 0)\) be an isolated singularity defined by a polynomial \(f\). Denote by \(\Gamma_+(f)\) and \(\Gamma(f)\) Newton's diagram of \(f\) and the union of the compact faces of it, respectively. Then the following hold:

(i) if \((X, 0)\) is canonical, then \(1 = (1, 1, \ldots, 1) \in \Gamma_+(f)^{\circ}\), where \(\Gamma_+(f)^{\circ}\) is the interior of \(\Gamma_+(f)\);

(ii) if \((X, 0)\) is log-canonical, then \(1 \in \Gamma_+(f)\).

If \(f\) is non-degenerate, the following hold:

(iii) \((X, 0)\) is canonical if and only if \(1 \in \Gamma_+(f)^{\circ}\);

(iv) \((X, 0)\) is non-canonical, log-canonical if and only if \(1 \in \Gamma(f)\);

(v) \((X, 0)\) is not log-canonical if and only if \(1 \notin \Gamma_+(f)^{\circ}\).

Proof. Let \(\sigma\) be the positive quadrant in \(\mathbb{N}^n\) and \(\Delta\) be the fan consisting of all faces of \(\sigma\). For a primitive \(q \in \mathbb{N} \cap \sigma\), take the subdivision \(\Delta(q)\) of \(\Delta\) consisting of all faces of \(\sigma_i = \sum_{j \neq i} \mathbb{R}_{\geq 0} e_j + \mathbb{R}_{\geq 0}, i = 0, \ldots, n\), take the normalization \(\widetilde{X}\) of the proper transform \(\overline{X} \subset T_N(\Delta(q))\) of \(X\). For the composite \(\psi : \widetilde{X} \to X \xrightarrow{\varphi} X\), write the canonical divisor as follows:

\[
K_{\widetilde{X}} = \psi^* K_X + \sum m_i E_i,
\]

where \(E_i\)'s are the exceptional divisors of \(\psi\). On the other hand

\[
K_{T_N(\Delta(q))} + \overline{X} = \varphi^*(K_{\mathbb{C}^{n+1}} + X) + (q(1) - 1 - q(f)) D_q.
\]

For the statement (i) (resp. (ii)), it is sufficient to prove that if \(q(1) - 1 - q(f) < 0\) (resp. \(< -1\)), then \(m_i < 0\) (resp. \(m_i < -1\)) for some \(i\). Since \(T_N(\Delta(q))\) has at worst \(\mathbb{Q}\)-factorial log-terminal singularities and \(K_X\) is linearly trivial, we can apply the following lemma to a Weil divisor \(\overline{X} \subset T_N(\Delta(q))\). For the assertion of the case that \(f\) is non-degenerate, it is sufficient to prove the opposite implications in (i) and (ii). For a non-singular subdivision \(\Delta'\) of \(\Delta\), on whose toric variety the proper transform \(X'\) is non-singular and intersects transversally each orbit on \(T_N(\Delta(q))\) (for the existence of such \(\Delta'\), cf. \[8\] [17]), we have:

\[
K_{X'} = \varphi^*(K_X) + \sum_{q \in \Delta'(1) - \Delta(1)} (q(1) - 1 - q(f)) D_q|_{X'}
\]

by \[2\] [3]. If \(1 \in \Gamma_+(f)^{\circ}\), then \(q(f) < q(1)\) for all \(q \in \Delta'(1) - \Delta(1)\), which implies that \((X, 0)\) is canonical. If it is not log-canonical, then there exists \(q\) such that \(q(f) > q(1)\), which implies \(1 \notin \Gamma_+(f)^{\circ}\). \(\square\)

Lemma 1.8. Let \(Y \subset Z\) be an irreducible Weil divisor on a normal variety \(Z\). Suppose \(Z\) admit at worst \(\mathbb{Q}\)-factorial log-terminal singularities. Let \(\tau : \widetilde{Y} \to Y\) be the normalization. Then:
(i) $Y$ is a Cohen-Macaulay variety;
(ii) $\omega_Y \simeq (\omega_Z(Y) \otimes_{\mathcal{O}_Z} \mathcal{O}_Y)/\mathcal{T}$, where $\mathcal{T}$ is the torsion submodule of $\omega_Z(Y) \otimes_{\mathcal{O}_Z} \mathcal{O}_Y$;
(iii) if $\omega_Z(Y) \simeq \mathcal{O}_Z(-aD)$ $a \geq 1$ (resp. $a > 1$) for an effective divisor $D \subset Z$ such that $\phi \neq D \cap Y \neq Y$, then we have the canonical isomorphism $\omega_Y \simeq \mathcal{O}_Y(-\sum_{i=1}^k b_i E_i)$ with $b_i \geq 1$ for divisors $E_i$ ($i = 1, \ldots, k$) such that $\bigcup_{i=1}^k E_i \supset \tau^{-1}(D)$ (resp. in addition $b_i > 1$ for some $i$ such that $E_i \subset \text{Supp}(\tau^{-1}(D))$).

**Proof.** First, one can prove that every effective Weil divisor on $Z$ is a Cohen-Macaulay variety in the same way as in 0.5 of [6], because the covering constructed as in [8] has at worst rational singularities in the present case too. For the proof of (ii), take the exact sequence:

$$
\mathcal{H}_1\mathcal{H}\mathcal{L}(\mathcal{O}_Z, \omega_Z) \to \mathcal{H}_1\mathcal{H}\mathcal{L}(\mathcal{O}_Z(-Y), \omega_Z) \to \mathcal{E}_1\mathcal{L}(\mathcal{O}_{Y}, \omega_Z) \to \mathcal{E}_1\mathcal{L}(\mathcal{O}_Z, \omega_Z) = 0.
$$

Here $\mathcal{E}_1\mathcal{L}(\mathcal{O}_Y, \omega_Z) = \omega_Y$, because $Z$ is a Cohen-Macaulay variety. So $\omega_Y$ is the image of $\mathcal{H}_1\mathcal{H}\mathcal{L}(\mathcal{O}_Z(-Y), \omega_Z)$, and therefore also the image of $\omega_Z(Y) \otimes \mathcal{O}_Y$ which is isomorphic to $\omega_Y$ on general points of $Y$. Since $\omega_Y$ is torsion-free, it must be isomorphic to $(\omega_Z(Y) \otimes \mathcal{O}_Y)/\mathcal{T}$ as desired in (ii). Now, since $\tau$ is finite, we have the inclusion $\tau_\ast \omega_Y \hookrightarrow \omega_Y$. By (ii) and the assumption of (iii), $\omega_Y$ is isomorphic to the defining ideal $\mathcal{I}$ of a subscheme $aD \cap Y$ of $Y$. Therefore $\omega_Y \simeq \mathcal{O}_Y(-\sum b_i E_i)$, $b_i > 0$ for divisors $E_i$ ($i = 1, \ldots, k$) such that $\bigcup_{i=1}^k E_i \supset \tau^{-1}(D)$. Next, assume $a > 1$. For the assertion, we may replace $Y \subset Z$ with a small neighbourhood of a general point on $D \cap Y$. So we may assume that all $E_i$ are over $\text{Supp}(D|_Y)$ and $D|_Y$ is irreducible. If there is no $E_i \subset \text{Supp}(D)$ such that $b_i > 1$, then $\tau_\ast \omega_Y = \tau_\ast \mathcal{O}_Y(-\sum E_i)$ is a reduced $\mathcal{O}_Y$-ideal whose locus has the support on $D \cap Y$. On the other hand, $\mathcal{I}$ also has the locus with the support on $D \cap Y$, therefore they coincide. By this equality $\tau_\ast \omega_Y = \omega_Y$, it follows that

$$
\tau_\ast \mathcal{O}_Y = \mathcal{H}_1\mathcal{H}\mathcal{L}(\tau_\ast \omega_Y, \tau_\ast \omega_Y) \subset \mathcal{H}_1\mathcal{H}\mathcal{L}(\mathcal{O}_Y, \omega_Y) = \mathcal{O}_Y,
$$

where the left and right equalities follow from the fact that $\tilde{Y}$ and $Y$ satisfy $S_2$-condition. Now it follows that $\tau : \tilde{Y} \simeq Y$ is normal, which induces the contradiction to $a > 1$. 

**1.9.** For a normal isolated singularity $(X, x)$, we define an invariant $\kappa_\delta(X, x)$ by the growth order of the plurigenera $\delta_m$ ($m \in \mathbb{N}$) ($[13]$).

In general, n-dimensional, normal, isolated singularities $(X, x)$ are classified by the invariant $\kappa_\delta$ into $(n+1)$-classes: $\kappa(X, x) = -\infty$, 0, 1, $\ldots$, $n-2$, $n$ (skipping $n-1$ curiously) ($[1]$). For hypersurface singularities, the classes are only three: $\kappa(X, x) = -\infty$, 0, $n$ ($[13]$). A hypersurface
singularity with \( \kappa_3(X, x) = -\infty \) (resp. 0, \( n \)) is equivalent to the fact that \((X, x)\) is canonical (resp. non-canonical-log-canonical, not log-canonical) (cf. [2]). Therefore \([7]\) also gives the combinatoric characterization of non-degenerate hypersurface singularities’ classes by \( \kappa_3 \).

2. The weights which give the canonical modification

2.1. Under the notation in \([4]\), put \( r = n + 1 \) for \( n \geq 2 \) and number the elements of the basis \( \{e_i\} \) from \( i = 0 \) to \( i = n \). Let \( \sigma = \sum_{i=0}^n \mathbb{R}_{\geq 0} e_i \) be the positive quadrant in \( N_\mathbb{R} \), and \( \Delta = \langle \sigma \rangle \) be the fan consisting of all faces of \( \sigma \). Denote Newton’s diagram of a polynomial \( f \in \mathbb{C}[x_0, \ldots, x_n] \) and the union of its compact faces by \( \Gamma_+(f) \) and by \( \Gamma(f) \) respectively.

**Definition 2.2.** For a polynomial \( f \in \mathbb{C}[x_0, \ldots, x_n] \), we define the essential cone as follows:

\[
C_1(f) := \{ q \in \sigma \subset N_\mathbb{R} \mid q(f) - q(1) \geq 0 \}.
\]

**Remark 2.3.** (i) It is clear that if \( 1 \in \Gamma_+(f) \), then \( C_1(f) = \{0\} \).

(ii) If \( 1 \notin \Gamma_+(f) \), the essential cone \( C_1(f) \) is actually the cone spanned by \( \gamma_1^+, \ldots, \gamma_r^+ \), where each \( \gamma_i \) is an \( n \)-dimensional face of \( \Gamma_+(x_0 \cdots x_n + f) \) which contains 1. Let \( X \) be the divisor in \( \mathbb{C}^{n+1} = T_N(\Delta) \) defined by \( f = 0 \). If \( X \) has an isolated singularity at the origin \( 0 \in \mathbb{C}^{n+1} \), then every vector \( q \in C_1(f) \) has positive coordinates \( q_j \) for \( j = 0, 1, \ldots, n \), otherwise at least one \( \gamma_i \) is parallel to one of the coordinate axes which causes a contradiction to the isolatedness of the singularity \((X, 0)\).

(iii) In Def 3.3 of [2], we have the notion of an essential divisor of a resolution of a Gorenstein singularity. Every 1-dimensional cone in the essential cone in \([2]\) gives a component of the essential divisor in some resolution.

**Definition 2.4.** (1) Let \( C \) be a cone in \( \sigma \subset N_\mathbb{R} \). For \( p = (p_0, \ldots, p_n) \), \( q = (q_0, \ldots, q_n) \in C \), we define \( p \leq q \) if \( p_i \leq q_i \) for every \( i = 0, \ldots, n \).

We say that a primitive element \( p \in C \cap N - \{0\} \) is absolutely minimal, if \( p \leq q \) for every primitive element \( q \in C \cap N - \{0\} \).

(2) For \( p, q \in C_1(f) \), we define \( p \preceq_f q \), if \( p_i/(p(f) - p(1) + 1) \leq q_i/(q(f) - q(1) + 1) \) for every \( i = 0, \ldots, n \). We define another order \( \preceq_f \) as follows: \( p \prec_f q \) if \( p_i/(p(f)) \leq q_i/(q(f)) \) for every \( i = 0, \ldots, n \). We say that a primitive element \( p \in C_1(f) \cap N - \{0\} \) is \( f \)-minimal, if for every primitive element \( q \in C_1(f) \cap N - \{0\} \), either \( p \preceq_f q \) or \( p \prec_f q \) and \( q \) belongs to the interior of an \( n + 1 \)-dimensional cone of \( \Delta(p) \), where the fan \( \Delta(p) \) consists of all faces of \( \sigma_i = \sum_{j \neq i} \mathbb{R}_{\geq 0} e_j + \mathbb{R}_{\geq 0} p \subset N_\mathbb{R} \), \( i = 0, \ldots, n \).
2.5. For a primitive vector \( p \in \sigma \cap N - \{0\} \), we have the star-shaped decomposition \( \Delta(p) \) by adding the ray \( \mathbb{R}_{\geq 0}p \) as in the definition above. We denote the fan of all faces of \( \sigma_i \) by \( \Delta_i \). Denote the proper transform of \( X = \{ f = 0 \} \) on \( T_N(\Delta(p)) \) by \( X(p) \). The induced morphisms \( \varphi : T_N(\Delta(p)) \to T_N(\Delta) \), \( \varphi^i : X(p) \to X \) are called weighted blow-ups with weight \( p \), or simply \( p \)-blow-ups of \( \mathbb{C}^{n+1} \) and \( X \) respectively.

Let \( U_i \) be the invariant open subset \( U_{\sigma_i} \simeq T_N(\Delta_i) \) of \( T_N(\Delta(p)) \), and \( \varphi_i : U_i \to \mathbb{C}^{n+1} \) be the restriction of \( \varphi \) onto \( U_i \). Denote \( X(p) \cap U_i \) by \( X_i \).

**Proposition 2.6.** Under the notation in 2.3, let \( \psi : \tilde{U}_i \to U_i \) be the birational morphism corresponding to a finite subdivision \( \Sigma_i \) of \( \Delta_i \). Denote the proper transform of \( X_i \) by \( \tilde{X}_i \), then

\[
K_{\tilde{U}_i} + \tilde{X}_i = \psi_*(K_{U_i} + X_i) + \sum_{q \in \Sigma_i(1)-\Delta_i(1)} \frac{q}{p_i}(p(f) - p(1)+1)(f(q) - f(1)+1))D_q.
\]

**Proof.** We can assume that \( i = 0 \) without the loss of generality. Let \( \{a_j\}_{j=0}^n \) be \( \{p, e_1, \ldots, e_n\} \). By Lemma 1.3 and 1.6, it is sufficient to prove:

\[
(1) \quad \sum_{j=0}^n r_j q(a_j^*)q^*(X_0) - 1 = \frac{q_0}{p_0}(p(f) - p(1)+1)(f(q) - f(1)+1))D_q.
\]

First, we can see that \( r_j = 1 \) for every \( j \). In fact, the quotient map \( \pi : \mathbb{C}^{n+1} = T_N(\Delta_0) \to T_N(\Delta_0) = U_0 \) is defined by the action of the cyclic group generated by

\[
\begin{pmatrix}
\epsilon & 0 & \ldots & \ldots & 0 \\
0 & \epsilon^{-p_1} & 0 & \ldots & 0 \\
0 & 0 & \epsilon^{-p_2} & 0 & \\
\vdots & \vdots & \vdots & \ddots & \\
0 & 0 & 0 & \ldots & \epsilon^{-p_n}
\end{pmatrix},
\]

where \( \epsilon \) is a primitive \( p_0 \)-th root of unity. Here it is easy to check that \( \pi \) is etale in codimension one. Next, since \( a_0^* = p^* = (1/p_0, 0, \ldots, 0) \) and \( a_j^* = e_j^* = (-p_j/p_0, 0, \ldots, 0, 1, 0, \ldots, 0) \) (j-th entry is 1) for \( 1 \leq j \leq n \), one obtains \( q(\sum a_j^*) = (1 - p_1 - \ldots - p_n)q_0/p_0 + (q_1 + \ldots + q_n) \). On the other hand, since \( \varphi_0^*(X) = X_0 + p(f)D_p \) by Lemma 1.3, it follows that

\[
(2) \quad q(\psi^*X_0) = q(\psi^*(\varphi_0(X))) - q(\psi^*(p(f)D_p)) = q(f) - p(f)q(p^*) = q(f) - p(f)q_0/p_0.
\]

By substituting them into the left hand side of (1) we obtain the equality (1). \( \square \)
Lemma 2.7. Let $Y \subset Z$ be an irreducible Weil divisor on a variety $Z$. Assume that $Z$ admits at worst $\mathbb{Q}$-factorial log-terminal singularities. Let $\psi : \tilde{Y} \rightarrow Y$ be a resolution of singularities on $Y$. Assume $K_{\tilde{Y}} = \psi^*((K_Z + Y)|_Y) + \sum_i m_i E_i$ with $m_i > -1$ for all $i$, where $E_i$’s are the exceptional divisors of $\psi$.

Then $Y$ is normal, and $Y$ has at worst log-terminal singularities.

In particular, if $m_i \geq 0$ for all $i$, then $Y$ has at worst canonical singularities.

Proof. First $Y$ is a Cohen-Macaulay variety as in [1.8]. Therefore it is sufficient to prove that $\text{codim}_Y \text{Sing}(Y) \geq 2$ by Serre’s criterion. Assume that $y \in Y$ is a general point of a component of $\text{Sing}(Y)$ of codimension one. By replacing $Y$ with a small neighbourhood of $y$, we may assume that $\psi$ is the normalization.

Claim that $\psi_* \omega_{\tilde{Y}} = \omega_Y$. The inclusion $\subset$ is trivial. For the proof of the opposite inclusion, take an arbitrary $\theta \in \omega_Y$. Then $\theta^r \in \omega^r_Z(rY) \otimes \mathcal{O}_Y$ for such $r$ that $\omega^r_Z(rY)$ is invertible, because $\omega_Y = \omega_Z(Y) \otimes \mathcal{O}_Y / \mathcal{T}$ by (ii) of [1.8]. By the assumption of the lemma, one obtains:

$$\theta^r \in \omega^r_Z(rY) \otimes \mathcal{O}_Y \subset \psi_* \omega^r_{\tilde{Y}}(-\sum r m_i E_i).$$

Hence for the valuation $\nu_i$ at each $E_i$, $r\nu_i(\psi^* \theta) = \nu_i(\psi^* \theta^r) \geq r m_i > -r$. Therefore $\nu_i(\psi^* \theta) \geq 0$ for every $E_i$, which means that $\psi^* \theta \in \omega_{\tilde{Y}}$ as claimed. By the same argument as in the proof of (iii) in [1.8], it follows that $Y$ is normal. One can see also that $Y$ is $\mathbb{Q}$-Gorenstein, because $\omega^r_Y = \omega^r_Z(rY) \otimes \mathcal{O}_Y$ is invertible for $r$ above. 

Theorem 2.8. Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be an isolated singularity defined by a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$. For a primitive integral vector $p = (p_0, \ldots, p_n)$ such that $p_i > 0$ for all $i$,

(i) $p$ is the canonical weight, i.e., $p$-blow-up $\varphi : X(p) \rightarrow X$ is the canonical modification,

then

(ii) $p$ is $f$-minimal in $C_1(f) \cap N - \{0\}$.

Suppose $f$ is non-degenerate, then the converse (ii) $\Rightarrow$ (i) also holds.

Proof. We use the notation in [2.5]. If the $p$-blow-up $\varphi' : X(p) \rightarrow X$ is the canonical modification, then it follows that $p \in C_1(f)$; otherwise, $K_{X(p)} = \varphi^* K_X + (p(1) - 1 - p(f)) D_p |_{X(p)}$ with $p(1) - 1 - p(f) \geq 0$, which shows a contradiction that $(X, 0)$ itself is a canonical singularity.
Let us prove that $p$ is $f$-minimal. If there exists $q \in C_1(f) \cap N - \{0\}$ such that $q \not< f p$, then 
\[ \min_{i=0, \ldots, n} \left\{ \frac{q_i}{p_i} \right\} \left( \frac{q(f) - q(1) + 1}{p(f) - p(1) + 1} \right) | i = 0, \ldots, n < 1. \]
Let $i = 0$ attain the minimal value, then it follows that $q \in \sigma_0$, because $q$ is represented as 
\[ \frac{q_0}{p_0} p + \sum_{i=1}^n (q_i - \frac{q_0}{p_0} p_i) e_i \] and its coefficients are all non-negative. Taking the star-shaped subdivision $\Delta_0(q)$ of $\Delta_0$ by adding a ray $\mathbb{R}_{\geq 0} q$, we have a birational morphism $\psi : U_0 := T_N(\Delta_0(q)) \to U_0 = T_N(\Delta_0)$. Denote the proper transform of $X_0$ by $\tilde{X}_0$, then, by 2.6

\[ (K_{\tilde{U}_0} + \tilde{X}_0)|_{\tilde{X}_0} = \psi^*(K_{X_0}) + \frac{q_0}{p_0} (p(f) - p(1) + 1) - (q(f) - q(1) + 1)) D_q|_{\tilde{X}_0}, \]

It follows that $\tilde{X}_0$ is normal. In fact, for a resolution $\lambda : \tilde{X}_0 \to \tilde{X}_0$, denote $K_{\tilde{X}_0} = \lambda^*((K_{\tilde{U}_0} + \tilde{X}_0)|_{\tilde{X}_0})$, then $K_{\tilde{X}_0} = \lambda^* \psi^* K_{X_0} + \sum (n_i + m_i) E_i$ where $n_i$ is the coefficient of $E_i$ in $\lambda^* (\frac{q_0}{p_0} (p(f) - p(1) + 1) - (q(f) - q(1) + 1)) D_q|_{\tilde{X}_0}$ which is non-positive by the negativity of $\frac{q_0}{p_0} (p(f) - p(1) + 1) - (q(f) - q(1) + 1)$; therefore if $m_i < 0$ for some $i$, then it contradicts to the fact that $X_0$ is canonical; since $m_i \geq 0$ for all $i$, by Lemma 2.7 $\tilde{X}_0$ is normal. Now we obtain the partial resolution $\psi : \tilde{X}_0 \to X_0$ with $K_{\tilde{X}_0} = \psi^* (K_{X_0}) + \frac{q_0}{p_0} (p(f) - p(1) + 1) - (q(f) - q(1) + 1)) D_q|_{\tilde{X}_0}$.

If $D_q \cap \tilde{X}_0 \neq \phi$, the coefficient of $D_q|_{\tilde{X}_0}$ is negative by the definition of $q$, which contradicts the hypothesis that $X_0$ has at worst canonical singularities. Therefore, $D_q \cap \tilde{X}_0 = \phi$ which happens if and only if $\psi(D_q)$ is a point away from $X(p)$, because $X(p)$ is ample on $D_p$. It implies $q(\psi^* X_0) = q(f) - \frac{q_0}{p_0} p(f) = 0$ (c.f. (2) in the proof of 2.6) and \( q \) belongs to the interior of an $n + 1$-dimensional cone of $\Delta(p)$. By the minimality of $q_0/p_0$, we obtain $p \not< f q$.

Next suppose that $f$ is non-degenerate and $p$ is $f$-minimal in $C_1(f) \cap N - \{0\}$. Take a non-singular subdivision $\Delta'$ of $\Delta(p)$ such that the restriction of the corresponding morphism $\psi : T_N(\Delta') \to T_N(\Delta(p))$ onto the proper transform $X(\Delta')$ of $X(\Delta(p))$ gives a resolution of $X(\Delta(p))$ such that every intersection of $X(\Delta')$ and an orbit is transversal. Let $\Delta'_i$ be the subdivision of $\Delta_i$ which is in $\Delta'$, then $T_N(\Delta'_i)$ is covered by $T_N(\Delta'_i)'s$ and the restriction $\psi_i : T_N(\Delta'_i) \to U_i = T_N(\Delta_i)$ of $\psi$ gives a resolution $X(\Delta'_i) := X(\Delta') \cap T_N(\Delta'_i) \to X_i$ of each $X_i$. 
Represent the canonical divisor on \(X(\Delta_i')\) by
\[
K_{X(\Delta_i')} = \psi_i^*(K_{U_i} + X_i)|_{X(\Delta_i')} + \sum_{q \in \Delta'_i(1) - \Delta_i(1)} m_q(D_q \cap X(\Delta_i'))_{\text{red}}
\]
If \(D_q \cap X(\Delta_i') \neq \phi\), the both intersect generically transversally each other by the construction of \(\psi\). Therefore \(m_q = \frac{q_0}{p_i}(p(f) - p(1) + 1) - (q(f) - q(1) + 1)\). If \(q \notin C_1(f)\), then \(q(f) - q(1) + 1 \leq 0\) and therefore \(m_q > 0\). If \(q \in C_1(f)\) and \(D_q \cap X(\Delta_i') \neq \phi\), then \(p \neq q\). In fact, if \(p \prec q\), then \(q\) is in the interior of \(\Delta_i\) for some \(i\), let it be \(\Delta_0\), which implies \(\psi_0(D_q)\) is a point. We also have that \(q(f) - \frac{q_0}{p_0} p(f) \leq 0\). On the other hand, \(q(\psi_0^*X_0) = q(f) - \frac{q_0}{p_0} p(f) \geq 0\). Therefore \(q(\psi_0^*X_0) = 0\), which shows that \(X(\Delta_0') \cap D_q = \phi\). Thus it follows that \(m_q \geq 0\) by the absolute \(f\)-minimality of \(p\). Now, by Lemma 2.7, it follows that \(X_i\)'s have at worst canonical singularities.

The \(\varphi\)-ampleness of \(K_{X(p)}\) follows from the \(\varphi\)-ampleness of \(-D_p|_{X(p)}\) and \(K_{X(p)} = \varphi^*K_X + (p(1) - p(f) - 1)D_p|_{X(p)}\), where the coefficient of \(D_p\) is negative.

**Corollary 2.9.** Let \((X, 0) \subset (\mathbb{C}^{n+1}, 0)\) be an isolated, non-canonical, log-canonical singularity defined by a polynomial \(f \in \mathbb{C}[x_0, \ldots, x_n]\). For a primitive integral vector \(p = (p_0, \ldots, p_n)\) such that \(p_i > 0\) for all \(i\), if

(i) \(p\)-blow-up \(\varphi : X(p) \to X\) is the canonical modification,

then

(ii) \(p\) is absolutely minimal in \(C_1(f) \cap N - \{0\}\).

Suppose \(f\) is non-degenerate, then the converse (ii) \(\Rightarrow\) (i) holds too.

**Proof.** Since \((X, 0)\) is log-canonical, \(1 \in \Gamma_+(f)\) by [1.7]. Then \(q(f) = q(1)\), for \(q \in C_1(f)\). First, two distinct primitive \(p, q \in C_1(f)\), neither \(p \prec q\) nor \(q \prec p\) hold. In fact, if \(p \prec q\), then \(p_i/q_i(1) \leq q_i/q(1)\) for every \(i\), and moreover the equality holds for every \(i\), because, by summing all these inequalities, we obtain \(1 \leq 1\). Hence \(p\) must coincide with \(q\). On the other hand, it is clear that \(\geq f\) is equivalent to \(\geq \) and therefore

\(f\)-minimal is equivalent to absolutely minimal. \(\square\)

**Example 2.10.** (Tomari) It is possible for a singularity to have more than one canonical weights. In fact, let \(X \subset \mathbb{C}^3\) be defined by \(x_0^k + \ldots + \).
$x_1^{k+1} + x_2^{k+1} = 0 \ (k \geq 3)$, then the weights $(1, 1, 1)$ and $(k+1, k, k)$ are both canonical weights.

2.11. In the rest of this section a singularity $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ is assumed to be a non-canonical, log-canonical singularity defined by a polynomial $f$.

In some cases, one can easily see the existence of the absolutely minimal vector, therefore one also sees the existence of the canonical modification for these cases.

For 2-dimensional case, singularities as above are either $\tilde{E}_6$ or $\tilde{E}_7$ or $\tilde{E}_8$ or defined by equations of type: $x_0x_1x_2 + x_0^p + x_1^q + x_2^r = 0$ with $1/\frac{1}{p} + 1/\frac{1}{q} + 1/\frac{1}{r} < 1$ by suitable coordinates transformations. It is well known that there exist the canonical weights for singularities defined by these equations. The following corollary shows that one need not take a coordinate transformation for 2-dimensional non-canonical, log-canonical singularities to admit the canonical weight.

**Corollary 2.12.** If $n = 2$ and $f$ is non-degenerate, there exists the absolutely minimal vector in $C_1(f)$ for every $f$ as in 2.11. And the vector is either $(1, 1, 1)$ or $(3, 2, 1)$ or $(2, 1, 1)$.

**Proof.** By the direct calculation, one can find the absolutely minimal vector in $C_1(f)$ for each $f$. □

The next one was proved by Tomari under a more general situation.

**Corollary 2.13.** (13) If $1$ is in the interior of an $n$-dimensional face $\gamma$ of $\Gamma(f)$, which is equivalent to that the singularity $(X, 0)$ is of Hodge type $(0, n-1)$ (for the definition, cf. [1]), then the primitive vector $p$ generating $\gamma^\perp$ gives the canonical modification $X(p) \rightarrow X$.

**Proof.** Since $C_1(f)$ is of one dimension, the primitive vector on it is clearly absolutely minimal. It completes the proof of the non-degenerate case. If $f$ is degenerate, there may not exist toric embedded resolution. But taking a resolution $\psi: Y \rightarrow X(p)$, $\varphi\psi$ is a resolution of a log-canonical singularity of type $(0, n-1)$, which yields that $K_Y = \psi^*\varphi^*(K_X) + \sum_i m_iE_i$ with the only one negative $m_i$. By substituting $(K_{T_N(\Delta(p))})|_{X(p)} = \varphi^*(K_X) - D|_{X(p)}$ into the equality above, we can see that the pair $X(p) \subset T_N(\Delta(p))$ satisfy the conditions of 2.7. □

**Corollary 2.14.** If a non-degenerate polynomial $f$ is represented as $x_0 \cdots x_n + h(x_0, \ldots, x_n)$, where $\deg h \geq n+1$, then the blow-up by the maximal ideal of the origin is the canonical modification.
Proof. Since \( \Gamma_+(f) \) is in the domain \( \{ \mathbf{a} \in M_\mathbb{R}|a_0 + \cdots + a_n \geq n+1 \} \), it follows that \( (1,1,\ldots,1) \in C_1(f) \) and clearly this is absolutely minimal.

Corollary 2.15. If every vector \( \mathbf{a} \in \Gamma(f) \cap M \) is quasi-reduced (i.e. \( \mathbf{a} = (a_0,\ldots,a_n) \) satisfies that \( 0 \leq a_i \leq 1 \) except for at most one \( i \)), then there exists the absolutely minimal vector \( \mathbf{p} \) in \( C_1(f) \).

Proof. A positive vector \( \mathbf{q} \in N \) belongs to \( C_1(f) \), if and only if \( \mathbf{q}(\mathbf{a}) \geq \mathbf{q}(1) \) for all \( \mathbf{a} = (a_0,\ldots,a_n) \in \Gamma(f) \cap M \). These inequalities are equivalent to the inequalities of the following type: 
\[
(a_i - 1)q_i \geq \sum_{j \in \Lambda(\mathbf{a})} q_j,
\]
where \( a_i \geq 2 \) and \( \Lambda(\mathbf{a}) \) is the suitable subset of \( \{0,\ldots,n\} \) such that \( i \notin \Lambda(\mathbf{a}) \). Let \( \mathbf{p} = (p_0,\ldots,p_n) \) and \( \mathbf{q} = (q_0,\ldots,q_n) \) belong to \( C_1(f) \). Define \( \mathbf{r} = (r_0,\ldots,r_n) \) by \( r_i = \min\{p_i,q_i\} \). We show that \( \mathbf{r} \in C_1(f) \).

For \( \mathbf{a} \in \Gamma(f) \cap M \), let \( a_i \geq 2 \). We can assume that \( r_i = p_i \) by the definition of \( \mathbf{r} \). Then \( (a_i - 1)r_i = (a_i - 1)p_i \geq \sum_{j \in \Lambda(\mathbf{a})} p_j \geq \sum_{j \in \Lambda(\mathbf{a})} r_j \).

Hence \( \mathbf{r} \) also satisfies \( \mathbf{r}(\mathbf{a}) \geq \mathbf{r}(1) \) for all \( \mathbf{a} \in \Gamma(f) \cap M \).

Example 2.16. We say that \( X \) is of type \( T_\mathbf{a} \), if it is defined by \( f = x_0 \cdots x_n + \sum x_i^{a_i} \) for \( \mathbf{a} = (a_0,\ldots,a_n) \), where \( \sum 1/a_i < 1 \). Then \( f \) satisfies the condition of \([2,13]\) and therefore \( X \) has a weight which gives the canonical modification. The summary paper \([7]\) contains the table of 3-dimensional \( T_{p,q,r,s} \)-singularities \( (X,0) \) with the absolutely minimal vectors \( \mathbf{p} \). All those weights are in the weights of \( 95 \)-simple K3-singularities listed in \([22]\) which is bijective to the list of \([1]\). And therefore \( T_{p,q,r,s} \)-singularities have the same plurigenera \( \{\gamma_m\} \) with those of corresponding simple K3-singularities (cf. \([3,2]\)).

3. Deformations and the simultaneous canonical modifications

Definition 3.1. Let \( \pi : (\mathcal{X},x) \rightarrow (C,0) \) be a flat morphism over a non-singular curve \( C \). A partial resolution \( \Phi : \mathcal{Y} \rightarrow \mathcal{X} \) is called the simultaneous canonical modification, if the restriction \( \Phi_t : \mathcal{Y}_t \rightarrow \mathcal{X}_t \) is the canonical modification for every \( t \in C \), where \( \mathcal{X}_t = \pi^{-1}(t) \) and \( \mathcal{Y}_t = \Phi^{-1}(\mathcal{X}_t) \).

Proposition 3.2. Let \( (X,0) \subset (\mathbb{C}^{n+1},0) \) be an isolated, non-canonical, log-canonical singularity defined by a polynomial \( f \). Assume that \( X(\mathbf{p}) \rightarrow X \) is the canonical modification for a positive integral vector \( \mathbf{p} \). Let \( \{F_t\}_{t \in C} \) be a deformation of \( f = F_0 \) over a non-singular curve \( C \) such that \( F_t \)'s \( (t \neq 0) \) are non-degenerate and Newton’s diagrams \( \Gamma_+(F_t) \) sit
in the halfspace $1 + p^\gamma$ of $M_R$. Then the flat family $\pi : (X, 0) \to (C, 0)$ defined by \{\$F_t\}_{t \in C}$ admits the simultaneous canonical modification and $\gamma_m(X_t, 0)$ is constant in $t \in C$ for every $m \in \mathbb{N}$.

Proof. By the assumption of \{\$F_t\}_{t \in C}$, $p(1) \leq p(F_t) \leq p(f)$. Therefore $p \in C_1(F_t) \subset C_1(f)$. Since $p$ is absolutely minimal in $C_1(f)$, it is absolutely minimal in $C_1(F_t)$ for every $t \in C$, which yields that $p$ is the canonical weight for the singularities defined by $F_t = 0$. On the other hand, since $(X, 0)$ is log-canonical, $1 \in \Gamma_+(f)$ by [4.7]. Hence $p(f) = p(1)$ which implies also $p(1) = p(F_t)$. Take the morphism $\Phi := \varphi \times id_C : T_N(\Delta(p)) \times C \to \mathbb{C}^{n+1} \times C$, where $\varphi : T_N(\Delta(p)) \to \mathbb{C}^{n+1}$ is the $p$-blow-up. Denote the proper transform of $X$ in $T_N(\Delta(p)) \times C$ by $Y$, then $\Phi^*X = Y + p(F_t)(D_p \times C)$ for a general $t \in C$, where $D_p$ is the corresponding divisor to $p$ on $T_N(\Delta(p))$. Since $p(F_t)$ is constant for all $t \in C$, $Y$'s are all irreducible, and therefore these turn out to be the $p$-blow-ups of $X_t$, which shows that $Y \to X$ is the simultaneous canonical modification. By Proposition 7 of [14], $Y$ admits at worst canonical singularities, and, on the other hand, $K_{Y/X} = -(D_p \times C)|_Y$ is $\Phi$-ample, which means that $Y \to X$ is the canonical modification of $X$. Thus $\pi$ turns out to be an (FG)-deformation in terms of $\Phi$. By 1.11 of [3], it follows that $\gamma_m(X_t, 0)$ is constant for all $t \in C$. \qed

Proposition 3.3. Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be an isolated, non-canonical, log-canonical singularity defined by a polynomial $f$. Assume that $X(p) \to X$ is the canonical modification for a positive integral vector $p$. If $p / \sum_i p_i$ is the weight of a weighted-homogeneous polynomial defining an isolated singularity at the origin, then $\gamma_m(X, 0) = \gamma_m(Y, 0)$, where $(Y, 0) \subset (\mathbb{C}^{n+1}, 0)$ is defined by a non-degenerate weighted-homogeneous polynomial $g$ with the weight $p / \sum_i p_i$. Moreover there exists a flat deformation $\pi : (X, 0) \to (C, 0)$ of $(X, 0) = (X_0, 0)$ over a non-singular curve $C$ with $(X_\tau, 0) \simeq (Y, 0)$ for some $\tau \in C$ such that $\pi$ admits the simultaneous canonical modification.

Proof. Let $F_t$ be $(1 - t)f + tg$ for $t \in \mathbb{C}$. Then, taking a suitable open subset $C \subset \mathbb{C}$ with $0, 1 \in C$, it follows that $F_t$ ($t \neq 0$) defines a non-canonical, log-canonical singularity of type $\{0, n - 1\}$, because it is a small deformation of such a singularity $\{g = 0\}$ and $1 \in \Gamma_+(F_t)$ (4.4 of [3], 2.2 of $\Phi$ and [4.7]). Hence by 2.13, $p$ is the canonical weight for $F_t$ ($t \neq 0$). Since $p(F_t) = p(1)$ for all $t \in C$, we can see that the deformation $\pi : X \to C$ defined by $\{F_t\}$ admits the simultaneous canonical modification $X(p)$ in the same way as in the proof of 3.2. Therefore $\gamma_m(X, 0) = \gamma_m(X_1, 0)$. \qed
Example 3.4. (Watanabe) One can see in [20] 95-examples of deformations such as in Proposition 3.3. For example, let $X \subset \mathbb{C}^4$ be defined by $f = x_0^2 + x_1^3 + x_2^5 + x_3^{10} + x_0x_1x_2x_3 = 0$ ($s \geq 0$) and $Y \subset \mathbb{C}^4$ by $g = x_0^2 + x_1^3 + x_2^7 + x_3^{42} = 0$. Let $p_0$ be $(21, 14, 6, 1)$, then $X(p_0) \rightarrow X$ is the canonical modification and $p_0/42$ is the weight of the quasi-homogeneous polynomial $g$. One can construct a family $\{F_t\}$ connecting $f$ and $g$ as in the proof of Proposition 3.3.

4. A COUNTER EXAMPLE TO THE CONJECTURE

In this section we show a counter example to the conjecture written in the introduction. Let $f$ be the polynomial: $x_0x_1x_2x_3 + \alpha x_0^3 + \beta x_1^2x_2^2 + \gamma x_1^4 + \delta x_2^3 + \epsilon x_3^3 \in \mathbb{C}[x_0, \ldots, x_3]$, with $a_i \geq 6$ and $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}$ general. Then $f$ is non-degenerate and defines an isolated, non-canonical, log-canonical singularity $(X, 0)$ at the origin by [17]. The essential cone is as follows:

$$C_1(f) = \{q \in \sigma \mid 2q_0 - q_1 - q_2 - q_3 \geq 0, -q_0 + q_1 + q_2 - q_3 \geq 0, (a_i - 1)q_i - \sum_{j \neq i} q_j \geq 0, i = 1, 2, 3\}$$

Here $C_1(f)$ has no absolutely minimal vector. In fact, it is easy to see that $(2, 2, 1, 1)$ and $(2, 1, 2, 1)$ belong to $C_1(f)$ but neither $(2, 1, 1, 1)$ nor $(1, 1, 1, 1)$ does. This shows that under these coordinates there is no weighted blow-up which is the canonical modification of $(X, 0)$ by [28]. In the following, we prove the same statement under arbitrary coordinates.

Lemma 4.2. If $Y \rightarrow X$ is the canonical modification of $(X, 0)$, then $-K_Y^3 > 3/2$.

Proof. We use the notation in [2] and [3]. Denote the fan consisting of all faces of the positive quadrant in $\mathbb{R}^n$ by $\Delta$. Let $q$ be $(2, 1, 2, 1)$, and $\varphi : T_N(\Delta(q)) \rightarrow \mathbb{C}^4$, $\varphi' = \varphi|_{X(q)} : X(q) \rightarrow X$ be the $q$-blow-ups of $\mathbb{C}^4$ and $X$ respectively under the given coordinates. First we prove that $X(q)$ has log-terminal singularities. For any resolution $\psi : \tilde{X} \rightarrow X(q)$ of the singularities on $X(q)$, we can write $K_{\tilde{X}} = \psi^*\varphi'^*(K_X) + \sum a_iE_i$ with $a_i \geq -1$ for all exceptional divisors $E_i$, because $(X, 0)$ is log-canonical. On the other hand, by [4] and [5], $K_{T_N(\Delta(q))} + X(q) = \varphi^*(K_{C^4} + X) - D_q$, since $q(1) - 1 - q(f) = -1$. Therefore if we write: $K_{\tilde{X}} = \psi^*((K_{T_N(\Delta(q))} + X(q))|_{X(q)}) + \sum m_j E_j$, then $m_j > -1$ for every exceptional divisor $E_j$ of $\psi$. Hence, by [2] $X(q)$ has at worst log-terminal singularities. Note that there is a non-canonical singularity, because $m_j = -1/2$ for $E_j$ which corresponds to the vector $(2, 2, 1, 1)$. Next construct a flat deformation $\pi : (X', 0) \rightarrow (C, 0)$ by $(1 - t)f + t(x_0^3 + x_1^6 + x_2^3 + x_3^6)$ as in [3] so that $(X_0, 0) \simeq (X, 0)$, and
Let \( \varphi : X_q \to X \) be the restriction of \( \varphi \times \text{id}_C \) onto the proper transform \( X_q \) of \( X \) in \( T_X(\Delta(q)) \times C \); since \( q(f) = q(1) = q(F_t) \) for \( t \in C \), \( \varphi : X_q \to \text{id} \) is the \( q \)-blow-up \( X_q \) of \( X \) for every \( t \in C \) as in the proof of [13,2]; here \( X_q \) has at worst canonical singularities for \( t \neq 0 \) by [13,13] and \( X_q \) has at worst log-terminal singularities as proved above; on the other hand, it is clear that \( K_{X_q} \) and \( K_X \) are both \( \mathbb{Q} \)-Cartier divisors; hence by Proposition 7 of [14], \( \varphi \) admits at worst canonical singularities; one can easily see that \( K_{X_q} = -(D_q \times C)_{|X_q} \) which is \( \varphi \)-ample, which shows that \( X \) admits the canonical modification \( X \) (i.e. \( \pi \) is an (FG)-deformation).

Now we can apply the upper semi-continuity theorem on \( \{ \gamma_m \} \) (Theorem 1 of [13]) to our (FG)-deformation \( \pi \). Since \( -K^3/(3!) \) of the canonical modification is the coefficient of the leading term of a function \( \gamma_m \) in \( m \), it follows that \( -K^3 \geq -K^3_{X(\pi)} = \sum q_i / \Pi q_i = 3/2 \) for \( t \in C - \{ 0 \} \). Here the equality does not hold. Because if it does, \( \pi \) would admit the simultaneous canonical modification \( \Psi : Y \to X \) by Corollary 1.11 on [13]. Since the simultaneous canonical modification must be the canonical modification of \( X \) by [14] again, \( Y \) would coincide with \( X \). However \( X_q \) has a non-canonical singularity as is seen above. \( \square \)

**4.3.** Now we assume that there are coordinates \( y_0, \ldots, y_3 \) on \( \mathbb{C}^4 \) and a weight \( p = (p_0, \ldots, p_3) \) such that the \( p \)-blow-up \( X(p) \to X \) under these coordinates gives the canonical modification, and then will induce a contradiction. Let \( g(y) = 0 \) be the defining equation of \( X \) under these coordinates. By [13,8], it follows that \( p(g) = p(1) \) and therefore \( -K^3_{X(p)} = \sum p_i / \Pi p_i > 3/2 \). Now it is easy to prove that at least three of the \( p_i \)'s must be 1. Write the coordinates transformation as follows:

\[
(T_i) \quad x_i = \sum_{m \in \mathbb{Z}_{\geq 0}} a^{(i)}_m y^m \quad (a^{(i)}_m \in \mathbb{C}).
\]

We may assume that the coefficient of \( y_i \) in \( (T_i) \) is not zero for each \( i \) by reordering \( \{ y_i \} \)'s. Then \( y_0^3 \in g \) (see [13] for the notation), since \( x_0^3 \in f \) and this is the unique monomial of degree 3 in \( f \). Therefore \( p(3,0,0,0) \geq p(1) \) which means \( p_0 \geq 2 \), since \( p \) must be in \( C_1(g) \) by [13,8]. Then one obtains the fact that \( a_{0,1,0,0}^{(0)} = a_{0,0,1,0}^{(0)} = a_{0,0,0,1}^{(0)} = 0 \), otherwise \( y_i^3 \in g \), for \( i = 1, 2, 3 \) which induce \( p(0,3,0,0) \geq p(1) \) and so on, therefore \( 3 \geq p_0 + 3 \) a contradiction. One can also prove that \( a_{0,0,1,0}^{(1)} = a_{0,1,0,0}^{(2)} = 0 \) in the same way. Then it follows that \( y_1^2 y_2^2 \in g \),
because this monomial comes from the term $x_1^2x_2^2$ and is not cancelled by the contribution from other terms. Hence $p$ must satisfy $p(0, 2, 2, 0) \geq p(1)$ which is equivalent to $4 \geq p_0 + 3$, a contradiction.

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