The Differentiation Lemma and the Reynolds Transport Theorem for Submanifolds with Corners

Maik Reddiger∗ Bill Poirier†

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Abstract

The Reynolds Transport Theorem, colloquially known as ‘differentiation under the integral sign’, is a central tool of applied mathematics, finding application in a variety of disciplines such as fluid dynamics, quantum mechanics, and statistical physics. In this work we state and prove generalizations thereof to submanifolds with corners evolving in a manifold via the flow of a smooth time-independent or time-dependent vector field. Thereby we close a practically important gap in the mathematical literature, as related works require various ‘boundedness conditions’ on domain or integrand that are cumbersome to satisfy in common modeling situations. By considering manifolds with corners, a generalization of manifolds and manifolds with boundary, this work constitutes a step towards a unified treatment of classical integral theorems for the ‘unbounded case’ for which the boundary of the evolving set can exhibit some irregularity.

Keywords: Differentiation under the integral sign - Manifolds with corners - Integral conservation laws - Reynolds Transport Theorem

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∗Department of Physics and Astronomy, and Department of Chemistry and Biochemistry, Texas Tech University, Box 41061, Lubbock, Texas 79409-1061, USA. maik.reddiger@ttu.edu +1-806-742-3067
†Department of Chemistry and Biochemistry, and Department of Physics and Astronomy, Texas Tech University, Box 41061, Lubbock, Texas 79409-1061, USA. bill.poirier@ttu.edu +1-806-834-3099
1 Introduction

Subject In this article we derive and rigorously prove two differential-geometric generalizations of the Reynolds Transport Theorem\(^1\),

\[
\frac{d}{dt} \int_{S_t} \rho \, d^3x = \int_{S_t} \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right) d^3x, \tag{1}
\]
as well as a related version of the Differentiation Lemma (cf. Prop. 6.28 in Ref. \[1\]).

The theorem is of central importance in fluid dynamics, quantum mechanics, and many other branches of physics\(^2\) as it relates the conservation of the integral on the left throughout time to the validity of the continuity equation (see e.g. §12 in Ref. \[3\], and §14.1 in Ref. \[12\]). As the name suggests, identity (1) is generally accredited to O. Reynolds\(^3\) [16].

With the slight restriction that the integrand is assumed to be sufficiently regular, the generalizations of (1) presented here are targeted to apply to most cases of practical interest to the applied mathematician, or mathematical/theoretical physicist. In those cases one usually prefers to work with real analytic functions (e.g. Gaussians), as those tend to make calculations easier. Such functions cannot have compact support unless they vanish entirely (cf. p. 46 in Ref. \[17\]), so one requires a variant of the Transport Theorem that allows both for integrands without compact support and unbounded domains.

In the global setting such a Transport Theorem has not been previously established in the literature, though, as we shall elaborate upon below, various other avenues for generalization have been pursued (cf. \[34, 41, 42, 52, 45, 47\]). Addressing this gap is the primary aim of this work.

Roughly speaking, we establish rigorous generalizations for the case of unbounded, curved domains, which lie in an ambient manifold and are smooth up to a countable number of edges and corners—both for the time-dependent and time-independent case\(^4\). In more rigorous terms, the generalizations apply to the integral of a smooth

\(^1\)This is the formulation in three spatial dimensions. See Ex. 2 below for definitions.

\(^2\)For its importance in fluid dynamics see p. 206 in Ref. \[2\], p. 78 sq. in Ref. \[3\], and §II.6 in Ref. \[4\]. Applications to quantum mechanics can be found in Ref. \[5\], §5.1 in Ref. \[6\], §1.2.1 in Ref. \[7\], and §14.8.1 in Ref. \[8\]. For its relation to other branches physics we refer to p. 413, p. 441 & §9.3.4 in Ref. \[10\], and §6.1 in Ref. \[11\].

\(^3\)In §81 Truesdell and Toupin \[13\] also cite Jaumann (cf. §383 in Ref. \[14\]) and Spielrein \[15\] (cf. §29 in Ref. \[15\]). They write that Spielrein first supplied a proof.

\(^4\)In the mathematical literature ‘time-dependent vector fields’ are vector fields depending (smoothly) on a single parameter. When computing its ‘integral curves’ one sets the parameter of the vector field equal to the parameter of the curve, which justifies the terminology (cf. Def.
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$k$-form $\alpha_t$ over a smooth $k$-submanifold $S_t$ with corners (both depending smoothly on a real parameter $t$) of a smooth $n$-manifold $Q$ ‘without corners’ ($1 \leq k \leq n < \infty$), where $S_t$ is an image of the time-dependent flow of some time-dependent vector field $X$ on $Q$. The ‘time-independent’ case then follows as a special case. That $S_t$ may be ‘unbounded’ means that we do not assume $\alpha_t$ to have compact support on $S_t$, contrary to many similar statements in the literature. Rather, $\alpha_t$ needs to satisfy a less stringent absolute convergence condition and a suitable boundedness condition relating to its parametric derivative.

This work was motivated by the study of the continuity equation in the general theory of relativity and relativistic quantum theory (cf. Refs. [19, 6, 20, 21]). The equation has been an important – though not directly apparent – subject of interest in recent articles on the foundations of (general-)relativistic quantum theory [22, 23].

Prior work According to our research, the differential-geometric generalization of Eq. (1), as given by

$$\frac{d}{dt} \int_{S_t} \alpha_t = \int_{S_t} \left( \frac{\partial}{\partial t} + \mathcal{L}_X \right) \alpha_t,$$

first appeared in an article by Flanders in a slightly adapted form (cf. Eq. 7.2 in Ref. [24]). In his article [24], Flanders bemoaned the rarity of the Leibniz rule (see e.g. Ref. [25]) and its relatives in the calculus textbooks of his times. A decade later, Betounes (cf. Ref. [29], in particular Cor. 1) also published an article containing Eq. (2), seemingly unaware of Flanders’ work. It is notable that Betounes also knew of the importance of the identity (for parameter-independent $\alpha$) for the general theory of relativity, since in a later work he reformulated it in terms of ‘metric’ geometric structures on a special class of submanifolds of a pseudo-Riemannian manifold [30].

Recently, Niven et al. [31] considered a multi-parameter generalization of Eq. (2).

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1 We stress that this differs from the terminology in physics: First, the parameter need not correspond to any actual time in applications. Second, ‘time-dependent’ descriptions in physics can be time-independent in the mathematical sense (see e.g. Ex. 2.ii below).

2 Formal definitions and examples are given in Sec. 2. Further elementary results are provided in Appx. A.

3 As the example $\int_{-\infty}^{\infty} dx e^{-x^2} = \int_{-\pi/2}^{\pi/2} dy e^{-\tan^2 y} \cos^2 y$ with $x = \tan y$ illustrates, the treatment of ‘improper’ integrals requires that one has to allow integrals over open domains.

4 $\mathcal{L}_X$ denotes the Lie derivative along $X$ (cf. §3.3 in Ref. [18], and p. 227 sqq. & p. 372 sqq. in Ref. [17]).

5 He cites Kaplan [26] as well as Loomis and Sternberg [27] as notable exceptions [24, 28].

6 To the relativist, a common special case of interest is the one for which the ‘ambient manifold’ is Lorentzian and the submanifold is spacelike. For the lightlike case other approaches are needed, see e.g. Duggal and Sahin’s book [31].

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to smooth compact submanifolds with boundary of a smooth ambient manifold (see also Ref. [35]).

By now, Eq. (2) has found its way into the textbooks under various more or less restrictive conditions (see e.g. Refs. [36, 37, 38]).

Apart from the aforementioned differential-geometric accounts, in the modern research literature one encounters functional-analytic approaches to proving (2). Here the integral is viewed as a linear functional acting on a suitable space of test functions or test differential forms. The pioneer of this approach was Schwartz himself [39, 40], the founder of the theory of distributions.

The power of the functional-analytic perspective for the problem has recently been demonstrated by Harrison [41] within the theory of differential chains. Given an open subset $U$ of $\mathbb{R}^n$, a differential $k$-chain is a linear functional on the space of differential $k$-forms, whose coefficient functions are differentiable up to some order and the highest-order derivatives are Lipschitz continuous (cf. Prop. 3.1 and Thm. 3.6 in Ref. [41]). Such a $k$-chain can then be understood as the integral over a domain, if the pairing with an arbitrary $k$-form yields the same value as the corresponding (Riemann) integral. This includes integrals over bounded, open subsets of $U$, finite unions of affine $k$-cells, and even highly irregular domains such as fractals (cf. Sec. 4.1, 4.2, and 4.3, respectively, in Ref. [41]).

Harrison used this functional-analytic ansatz to prove a version of Eq. (2) for differential chains whose time evolution in $U$ is governed by the flow of a differentiable vector field (cf. Sec. 4 and Thm. 12.4 in Ref. [41]). Also resting on Harrison’s ‘Generalized Leibniz Integral Rule’ (Thm. 12.3 in Ref. [41]), Seguin and Fried [42] considered the more general case for which the chain is not merely ‘convecting’ in the prior sense, but ‘regularly evolving’—thus allowing for topological changes like ‘tearing’ and ‘piercing’[11]. Along with Hinz, they elaborated further on their results in Ref. [45], taking an application-oriented perspective and considering a number of explicit examples (cf. §6 in Ref. [45]). Using (parameter-dependent) de Rham currents[12] instead of differential chains, Falach and Segev [47] also considered Eq. (2) for irregular domains of integration in the smooth manifold setting.

In retrospect, the initial treatments [29, 24] of formula (2) suffered from a lack of rigor regarding the regularity assumptions on $S_0$ (resp. $S_t$), which meant that the

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10See also Sec. 2 in Ref. [45] for a brief introduction to the theory of differential chains. Note that footnote 3 therein is erroneous, i.e. the support of a chain need not be compact.

11In this respect, Seguin’s work [52] on a generalization of [1] to non-smooth domains of finite perimeter should also be mentioned, in which he combined the idea of proof via the divergence theorem from Gurtin et al. [43] with tools of geometric measure theory [44].

12This generalization of the distribution concept to the space of compactly supported, smooth $k$-forms was named after G. de Rham (cf. Ref. [40], and §5.1 in Ref. [47]).
applicability of the identity was not fully specified. In particular, classical versions of Stokes’ Theorem require either compact domains or compact support of the integrand (cf. Thm. 4.2.14 in Ref. [18], and Thm. 16.11, Thm. 16.25 & Ex. 16.16 in Ref. [17]). The close connection to Stokes’ Theorem is one of the reasons why textbook treatments also make various compactness assumptions (cf. §4.3 in Ref. [37], Thm. 7.1.12 in Ref. [36], p. 419 in Ref. [27], Thm. XII.2.11 in Ref. [38], and Prop. 3.5 in Ref. [51]). Yet, due to the ubiquity of ‘improper integrals’ in applied mathematics and theoretical physics, these Transport Theorems do not directly apply to a class of problems of significant practical relevance. Harrison (cf. §4 in Ref. [41]) as well as Seguin and Fried (cf. §2.4 in Ref. [42]) also only explicitly consider cases for which the domain is bounded. The formalism of de Rham currents in Falach’s and Segev’s work [47] explicitly calls for integrands with compact support.

**Contribution of this work** The aim of this work is twofold: First, we consider mathematically rigorous, differential-geometric versions of the differentiation lemma and the transport theorems for which neither compactness of the domain of integration nor of the support of the integrand is required (or any other ‘boundedness condition’ such as finite ‘volume’). From an application-oriented perspective, this is a serious gap in the mathematical literature, that needed to be addressed. Second, in this version we also wish to allow for the ‘manifold’ to have some type of ‘boundary’ with at least some degree of ‘irregularity’. Manifolds with corners satisfy the latter requirement and, while they are neither the most general nor the most convenient spaces to work with, the results here provide simple-to-use and rigorous generalizations in the aforementioned sense.

Nonetheless, we do wish to note that, if the space of interest is a subset of a manifold and its boundary is a set of (Lebesgue-)measure zero, then for the purpose of integration one may replace the set by its interior. The latter is then an open sub-

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13 In classical versions of Stokes’ theorem for manifolds with boundary or manifolds with corners, this assumption is a crucial step in proving the theorem. While there exist functional-analytic approaches that weaken this assumption, one still requires certain boundedness conditions on the domain or integrand for those generalizations. We refer to Refs. [38, 49] as well as Thm. 8.9 in Ref. [41] for such generalizations.

14 In the book by Abraham, Ratiu, and Marsden [36], the assumption is implicit due to the use of Thm. 7.1.7.

15 Though this assumption is not required in the theory of differential chains, it is nevertheless unable to handle such domains in general. An example is provided by the 1-form $dx \in \mathcal{B}_{\infty}^1(\mathbb{R}) := \bigcap \mathcal{B}_{\gamma=0}^1(\mathbb{R})$ (cf. Sec. 3 in Ref. [41]), which shows that there can be no ‘chain representative’ for $\mathbb{R}$—even if Def. 4.1 in Ref. [41] is generalized to the Lebesgue integral.

16 The manifold boundary of a manifold with corners has measure zero. See Def. A.2.3 in Appx. A.
manifold ‘of same measure’ and thus the generalization of the theorems to ‘ordinary’ manifolds would suffice.

In this respect, we emphasize that the three main theorems of this work (Lem. 1, Thm. 1, and Cor. 1) remain valid, if manifolds with corners are replaced by ‘ordinary’ manifolds or manifolds with boundary. Readers only interested in those cases are invited to skip the parts of the article focusing on manifolds with corners and are advised to refer directly to the respective theorems.

Still, the main advantage of considering manifolds with corners in stating the theorems is that it allows for a unified treatment, independent of whether Stokes’ theorem is applicable in the particular case of interest or not. It is the goal of attaining such a unified treatment for even more general spaces that may justify future generalizations of this work.

Structure We begin by reviewing the allowed domains of integration (i.e. manifolds with corners) for the purposes of this work by giving a brief definition along with several examples and useful propositions. After ‘having set the stage’, we prove the corresponding Differentiation Lemma (Lem. 1; see also Prop. 6.28 in Ref. [1]). This allows us to prove the generalization of the Reynolds Transport Theorem for the ‘time-dependent’ case (Thm. 1), and obtain the time-independent case as a corollary (Cor. 1). We note the close relation of the latter to the Poincaré-Cartan Theorem. The article ends with applications of the theorems to two main examples. For the convenience of the reader we also included an appendix discussing some elementary results on manifolds with corners (Appx. A) as well as integral curves and flows thereon (Appx. B).

Notation $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\} \supset \mathbb{N}$. $\mathbb{Z}$ is the set of integers. By definition, an interval is a connected subset of $\mathbb{R}$ with non-empty interior. The interval $(a, b) \subseteq \mathbb{R}$ is open, $[a, b]$ is closed. If not stated otherwise, mappings and manifolds (with corners) are assumed to be smooth. For a manifold $\mathcal{Q}$ (with corners), $T\mathcal{Q}$ denotes the tangent bundle and $T^*\mathcal{Q}$ the cotangent bundle (i.e. the respective ‘total space’). If $\varphi$ is a (smooth) map, then $\text{dom} \varphi$ is its domain, $\varphi|_{\mathcal{U}}$ the mapping restricted to the domain $\mathcal{U}$, $\varphi_*$ is the pushforward/total derivative, and $\varphi^*$ the pullback mapping. $\Omega^k(\mathcal{Q})$ is the (vector) space of smooth $k$-forms on $\mathcal{Q}$, which are the smooth sections of $\bigwedge^k T^*\mathcal{Q}$. $d$ denotes the exterior derivative, $X\cdot$ is the contraction, and $\mathcal{L}_X$ the Lie derivative with respect to a (tangent) vector (field) $X$. For convenience, we identify smooth sections of the trivial bundle $\mathcal{Q} \times \mathbb{R}$ with smooth mappings $f \in C^\infty(\mathcal{Q}, \mathbb{R})$. A dot over a letter usually denotes the derivative with respect to the parameter. We also use dots as placeholders, i.e. a function...
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\( \varphi: q \mapsto \varphi(q) \) may also be written as ‘\( \varphi(\cdot) \)’. On \( \mathbb{R}^3 \) (and \( \mathbb{R}^4 \) by ‘including time’) we employ the ordinary notation for the vector calculus operators and write \( d^3x \) for \( dx^1 \wedge dx^2 \wedge dx^3 \). If some notation is unclear, the reader is advised to consult Ref. [18].

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There exist several competing – though formally equivalent – definitions of ‘manifolds with corners’: In each instance, one considers a second countable, Hausdorff space that is locally homeomorphic to the ‘model space’—which is in turn used to define ‘local charts’, etc. ‘Ordinary’ manifolds of dimension \( n \in \mathbb{N} \) employ the ‘model space’ \( \mathbb{R}^n \). For \( n \)-manifolds with boundary it is commonly \( [0, \infty) \times \mathbb{R}^{n-1} \). Generalizing therefrom, most authors use \( [0, \infty)^k \times \mathbb{R}^{n-k} \) with \( k \in \{0, \ldots, n\} \) as a ‘model space’ for manifolds with corners (cf. Rem. 3.3 in Ref. [56]). This choice is due to Douady and Hérault [58]. Since \( [0, \infty)^k \times \mathbb{R}^{n-k} \) is homeomorphic to the (relatively) open subset \( [0, \infty)^k \times (0, \infty)^{n-k} \) in \( [0, \infty)^n \), Lee [17] uses \( [0, \infty)^n \) instead. However, both choices exhibit the drawback that there is some arbitrariness involved in the choice of ‘boundary’ in \( \mathbb{R}^n \). In applying the theory, one is thus enticed to introduce local ‘coordinate transformations’ for the mere purpose of ‘fitting the definition’. Michor’s definition of manifolds with corners alleviates this problem to some degree (cf. Chap. 2 in Ref. [57]). His definition is therefore the one we use in this article.

Definition 1

i) Let \( n, k \) be positive integers such that \( k \leq n \). Let \( \varphi^1, \ldots, \varphi^k \) be \( k \) linearly independent, linear functionals on \( \mathbb{R}^n \). A set

\[
C^n(\varphi^1, \ldots, \varphi^k) = \{ x \in \mathbb{R}^n | \forall i \in \{1, \ldots, k\}: \varphi^i(x) \geq 0 \},
\]

equipped with the subspace topology, is called a quadrant (in \( \mathbb{R}^n \)). For convenience, we set \( \mathbb{R}^0 = C^0 = \{0\} \).

ii) Let \( n, m \in \mathbb{N} \), and let \( \xi \) be a map from a (relatively) open subset \( V \) of a quadrant in \( \mathbb{R}^n \) to a (relatively) open subset \( W \) of a quadrant in \( \mathbb{R}^m \). The map \( \xi \) is smooth, if there exists a smooth extension \( \tilde{\xi}: \tilde{V} \to \mathbb{R}^m \) of \( \xi \) to an open subset \( \tilde{V} \) of \( \mathbb{R}^n \). We extend this terminology to \( m \) or \( n \) being equal to zero, in which case the map \( \xi \) is always smooth (as a constant map).

iii) Let \( n \) be a positive integer. A (smooth) \( n \)-manifold with corners is a second countable, Hausdorff topological space \( Q \) with a (smooth) atlas \( \mathcal{A} \) (with cor-
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ners), defined as follows. Given a countable index set $I$, formally set

$$\mathcal{A} = \{(U_\gamma, \kappa_\gamma) | \gamma \in I \}.$$  \hfill (3b)

By definition, each $\kappa_\gamma$ is a homeomorphism from an open $U_\gamma \subseteq \mathcal{Q}$ to a (relatively) open subset of a quadrant in $\mathbb{R}^n$. Furthermore, for any $U_\gamma \cap U_\delta \neq \emptyset$ the map $\kappa_\delta \circ \kappa_\gamma^{-1}$ is smooth in the sense of ii) above.

iv) Given a smooth manifold with corners $\mathcal{Q}$ with atlas $\mathcal{A}$, an element $(U, \kappa) \in \mathcal{A}$ is called a (local) chart with corners/corner chart on $\mathcal{Q}$. \hfill \Diamond

Manifolds and manifolds with boundary, defined as usual, are trivially manifolds with corners, making all results in this article applicable to those important special cases.

As in the case of ‘ordinary’ manifolds, one can define ‘smooth structure with corners’, ‘smoothly compatible charts with corners’, introduce partitions of unity, etc. As their definitions for manifolds is standard and the generalization to manifolds with corners is straightforward, we shall not formally discuss those. More generally, we only discuss generalizations of standard differential geometric concepts to manifolds with corners, if the analogy is non-trivial. We again emphasize that, unless stated otherwise, all manifolds (with corners) and mappings in this work are assumed to be smooth.

To support the reader in gaining some intuition regarding manifolds with corners, we consider a few further examples. These also exhibit some important techniques that one can use to show that a given set is canonically a manifold with corners—or can be turned into one by defining an appropriate topology and charts with corners.

Example 1 (Manifolds with corners)

i) The interval $[0,1]$ is a manifold with corners. We define two corner charts covering $[0,1]$ as follows: The first is the set $[0,1) = \mathcal{C}^1(1) \cap (-1,1)$ together with the identity. For the second one, consider

$$(-1,0] = \mathcal{C}^1(-1) \cap (-1,1)$$  \hfill (4a)

and observe that the map $\xi: x \mapsto x - 1: (0,1] \to (-1,0]$ is a homeomorphism. Then the tuple $((0,1], \xi)$ defines a smoothly compatible corner chart.

Note that $\xi$ is orientation-preserving. More generally, it is straightforward to show that an orientation-preserving atlas exists on any manifold with corners. That this is true even in the one-dimensional case is another advantage of Michor’s definition above (cf. Prop. 15.6 in Ref. \cite{M}).
ii) The Cartesian product of finitely many manifolds with corners is (canonically) a manifold with corners. Its dimension is equal to the sum of the dimensions of each factor. Both statements can be inferred from the following argument regarding the chart codomains of two manifolds with corners:

Let \( n_1, n_2 \in \mathbb{N} \) and let \( V_1 \subseteq \mathbb{R}^{n_1}, V_2 \subseteq \mathbb{R}^{n_2} \) be open. Consider

\[
(C^{n_1}(\varphi_1^1, \ldots, \varphi_1^{k_1}) \cap V_1) \times (C^{n_2}(\varphi_2^1, \ldots, \varphi_2^{k_2}) \cap V_2).
\]

Denote by \( \text{pr}_1 \) and \( \text{pr}_2 \) the projection of \( \mathbb{R}^{n_1+n_2} \) onto the first \( n_1 \) and the last \( n_2 \) components, respectively. Then the above set equals

\[
C^{n_1+n_2}(\varphi_1^1 \circ \text{pr}_1, \ldots, \varphi_1^{k_1} \circ \text{pr}_1, \varphi_2^1 \circ \text{pr}_2, \ldots, \varphi_2^{k_2} \circ \text{pr}_2) \cap (V_1 \times V_2).
\]

iii) By ii) and iii) above, the unit \( n \)-cube \([0,1]^n\) is (canonically) a manifold with corners.

iv) Given a point \( q \) in a manifold with corners \( Q \), we follow the analogue theory for manifolds in defining the tangent space \( T_qQ \) at \( q \) to be the set of derivations at \( q \) (cf. Appx. B).

Accordingly, we take the tangent bundle \( TQ \) of \( Q \) to be the disjoint union of tangent spaces. \( TQ \) is canonically a manifold with corners:

Let \( U \) be open in \( \mathbb{R}^n \) such that

\[
C^n(\varphi^1, \ldots, \varphi^k) \cap U
\]

is the codomain of a corner chart on \( Q \). Let \( \text{pr}: \mathbb{R}^{2n} \to \mathbb{R}^n \) be the projection onto the first \( n \) components. We construct a corner chart on \( TQ \) by taking the respective chart codomain to be

\[
C^{2n}(\varphi^1 \circ \text{pr}, \ldots, \varphi^k \circ \text{pr}) \cap (U \times \mathbb{R}^n).
\]

The rest of the proof is analogous to the proof of the corresponding statement for manifolds (cf. Prop. 3.18 in Ref. [17] and Prop. 2.1.1 in Ref. [18]).

v) Let \( N, Q \) be smooth manifolds with corners and let \( \varphi: N \to Q \) be a continuous mapping. By definition, \( \varphi \) is smooth if each ‘local representative’ of \( \varphi \) is smooth in the sense of Def. [11ii]. Such a \( \varphi \) is an immersion, if \((\varphi_*)_q\) is injective at each \( q \in N \).[17] If \( \varphi \) is an injective immersion, we define the tuple \((N, \varphi)\) to be a smooth submanifold of \( Q \) (with corners).

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[17] By a continuity argument, if \( q \) is a corner point, then \((\varphi_*)_q\) is independent of the local representative of \( \varphi \) and its chosen extension.
In that case the image \( \varphi(N) \), if equipped with the coinduced topology\(^{18}\) is also canonically a smooth manifold with corners. Moreover, if \( \iota \) is the inclusion of \( \varphi(N) \) into \( Q \), \( (\varphi(N), \iota) \) is a smooth submanifold of \( Q \) with corners. \( (N, \varphi) \) and \( (\varphi(N), \iota) \) are said to be equivalent submanifolds with corners (cf. Rem. 1.6.2.1 in Ref. \([18]\)).

As in the case of manifolds, this justifies the identification of submanifolds with corners as subsets of their ambient space.

vi) The unbounded set

\[
S_0 = \left\{ \bar{x} \in \mathbb{R}^3 \mid x^2 - x^1 \leq \sqrt{2} \sin \left( \frac{x^2 + x^1}{\sqrt{2}} \right), \quad x^3 \in \left[ -\frac{H}{2}, \frac{H}{2} \right] \right\}
\]  

(4f)

is an infinite sheet of height \( H \in (0, \infty) \), diagonally cut along a sine curve at an angle of \( \pi/4 \). We refer to the first panel in Figure [1] below. \( S_0 \) is canonically a 3-manifold with corners: First set \( y^3 = x^3 \) and rotate

\[
\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}
\]

(4g)

to find \( y^2 \leq \sin(y^1) \). Now set

\[
y^1 = z^2, \quad y^2 = \sin(z^2) - z^1 \quad \text{and} \quad y^3 = Hz^3/2
\]

(4h)

for \( \bar{z} \in N \coloneqq [0, \infty) \times \mathbb{R} \times [-1,1] \). By ii) and iii), \( N \) is a manifold with corners. If we view \( N \) as a submanifold with corners of \( \mathbb{R}^3 \) equivalent to \( S_0 \), then vi) yields the assertion. Furthermore, since the (extended) mappings \( \bar{z} \mapsto \bar{y}, \bar{y} \mapsto \bar{x} \) are homeomorphisms of \( \mathbb{R}^3, \) \( N \) carries the subspace topology. Thus \( S_0 \) is (smoothly) embedded in \( \mathbb{R}^3 \). In this sense the choice of smooth structure (with corners) is canonical.

vii) Every geometric \( k \)-simplex (with \( k \in \mathbb{N}_0 \)) is canonically a smooth manifold with corners (cf. p. 467 sq. in Ref. \([17]\)).

viii) Consider a square base pyramid of height and length \( L \) (with \( L \in (0, \infty) \)):

\[
P_0 := \left\{ \bar{x} \in \mathbb{R}^3 \mid x^3 \in [0, L], \quad \text{and} \quad |x^1|, |x^2| \leq \frac{L}{2} \left( 1 - \frac{x^3}{L} \right) \right\}
\]

(4i)

\(^{18}\) \( \varphi \) need not be a topological embedding, as the coinduced topology on \( \varphi(N) \) may be finer than the subspace topology. See Example 4.19 and 4.20 in Ref. \([17]\).
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Due to its apex, \( P_0 \) is not a manifold with corners—at least not canonically. Nonetheless, we can turn \( P_0 \) into a manifold with corners by setting

\[
P_0^1 := \{ \vec{x} \in P_0 | x^2 > x^1 \} \quad \text{and} \quad P_0^2 := \{ \vec{x} \in P_0 | x^2 \leq x^1 \},
\]

which corresponds to a cut along the diagonal. By \((\text{vii})\) \( P_0^2 \) is a manifold with corners. As an open subset of a manifold with corners, \( P_0^1 \) is a manifold with corners. Since the intersection of \( P_0^1 \) and \( P_0^2 \) is empty and both are 3-manifolds with corners, their union \( P_0 \) is a 3-manifold with corners.

Clearly, the ‘cost’ of turning \( P_0 \) into a manifold with corners was to ‘add another face’ and to ‘give up’ embeddedness into \( \mathbb{R}^3 \).

ix) More generally, if \( Q \) is an \( n \)-manifold with corners and a subset \( \mathcal{N} \) consists of a countable union of mutually disjoint submanifolds with corners of same dimension \( k \leq n \), then \( \mathcal{N} \) is a \( k \)-(sub)manifold with corners. To show this one employs the fact that the countable union of disjoint second-countable spaces is second-countable. As example \((\text{viii})\) shows, \( \mathcal{N} \) need not carry the subspace topology.

x) Continuing with \((\text{viii})\) for any \( \vec{k} \in \mathbb{Z}^3 \) we define by translation

\[
P_{\vec{k}} = P_0 + 2L \vec{k}.
\]

Then the union \( \mathcal{P} := \bigcup_{\vec{k} \in \mathbb{Z}^3} P_{\vec{k}} \) is an infinite lattice of mutually disjoint pyramids. Comparing with Ex. \((\text{viii})\), \( \mathcal{P} \) is not canonically a manifold with corners. If we equip \( P_0 \) with the ‘non-canonical’ topology and smooth structure (with corners) from \((\text{viii})\) however, then, by \((\text{ix})\), \( \mathcal{P} \) is a manifold with corners.

As for the purpose of this article manifolds with corners are considered domains of integration, this is an example where the ‘unboundedness’ comes from having countably many components. In practice, this yields a series of integrals over the individual components.

xi) The set of corner points of a manifold with corners \( Q \) – its manifold boundary \( \partial Q \) – is in general not a manifold with corners (cf. Appx. A). Michor [57] has remedied this problem by separately considering the corners/boundaries of a fixed ‘dimension’ or ‘index’. We refer the interested reader to Def. A.1 and Prop. A.1 in Appx. A.

\[^{19}\text{The countable union of countably many sets is countable (cf. Ex. 2.19 in Ref. [59]), so this follows from the definition of second-countability (cf. Def. 6.1 in Ref. [59]).}\]
Since we concern ourselves with integration theory in this article, it shall be noted here that Michor has formulated a version of Stokes’ Theorem for manifolds with corners in terms of the boundary of index 1, see Prop. 3.5 in Ref. [51]. Lee has also proven Stokes’ Theorem for his definition of manifolds with corners in terms of the manifold boundary (cf. Thm. 16.25 in Ref. [17]).

We refer the reader to Appx. A for further elementary results on manifolds with corners. An introduction to the subject may also be found on p. 415 sqq. in Ref. [17] and Chap. 2 in Ref. [57]. Refs. [54, 55, 51] and the French appendix by Douady and Hérault in Ref. [58] provide further reading.

3 The Differentiation Lemma

Before we can state the theorems of interest, we need a natural definition of the integral over a generic manifold with corners: As it is needed for our intended generalizations of the Differentiation Lemma and the Transport Theorem, such a definition needs to allow for the integration of ‘integrable’ differential forms without compact support over open domains.

To take account of these points we adapted the definition from Rudolph and Schmidt (cf. Def. 4.2.6 in Ref. [18]). For an analogous definition of integrals of ‘integrable’ differential forms over arbitrary oriented manifolds (without boundary) by Choquet-Bruhat et al. see p. 202 sqq. in Ref. [60].

Definition 2 (Integral on manifolds with corners)

Let \( S \) be a (smooth) oriented \( k \)-manifold with corners, let \( A \) as in (3b) be a smooth, countable, locally finite atlas (with corners) for \( S \), and let \( \{ \rho_\gamma | \gamma \in I \} \) be a (smooth) partition of unity subordinate to \( A \) (cf. p. 417 sq. in Ref. [17]). Further, define

\[
\text{sgn}: I \to \{-1, +1\} : \quad \gamma \mapsto \text{sgn}_\gamma := \begin{cases} +1, & \kappa_\gamma \text{ is orientation-preserving} \\ -1, & \kappa_\gamma \text{ is orientation-reversing} \end{cases}
\]  

We make the following definitions:
The Differentiation Lemma

i) If $\alpha$ is a (smooth) density\footnote{The definition of densities on manifolds with corners is analogous to the one on ‘ordinary’ manifolds. See p. 427 sqq. in Ref. [17] for an elaboration of the theory on manifolds with boundary.} on $S$, then the integral of $\alpha$ over $S$ is

$$\int_S \alpha = \sum_{\gamma \in I} \int_{\kappa_\gamma(U_\gamma)} (\kappa_\gamma^{-1})^* (\rho_\gamma \alpha),$$

provided the series converges absolutely.

ii) If $\alpha$ is a (smooth) $k$-form on $S$, then the integral of $\alpha$ over $S$ is

$$\int_S \alpha = \sum_{\gamma \in I} \text{sgn}_\gamma \int_{\kappa_\gamma(U_\gamma)} (\kappa_\gamma^{-1})^* (\rho_\gamma \alpha),$$

provided the integral $\int_S |\alpha|$ of the (positive) density $|\alpha|$ exists\footnote{By definition, $|\alpha|_q(X_1, \ldots, X_k) = |\alpha_q(X_1, \ldots, X_k)|$ for all $q \in S$ and $X_1, \ldots, X_k \in T_qS$.}

In either case $\alpha$ is called integrable (over $S$). The integrals over each $\kappa_\gamma(U_\gamma) \subseteq \mathbb{R}^k$ are taken in the sense of Lebesgue\footnote{In fact the Lebesgue-Borel measure is sufficient here (see Thm. 1.55 in Ref. [1]).}

This definition is independent of the choice of atlas and partition of unity\footnote{Observe that $\rho_\gamma \alpha$ is compactly supported on $U_\gamma$. One may then adapt the reasoning by Lee (cf. Prop. 16.5 in Ref. [17]).}

In particular, as the resulting series converges absolutely, the total integral is independent of ‘the order of summation’ (i.e. the sequence of partial sums). Integrals over submanifolds (with corners) are defined as usual via pullback (cf. Def. 4.2.7 in Ref. [18]). In practice, one may ‘chop up’ the domain of integration to get countably many (convergent) integrals over subsets of $\mathbb{R}^k$. That is – roughly speaking and for the purpose of ‘practical integration’ – one does not need to worry much about the technicalities resulting from working with manifolds with corners\footnote{Since the manifold boundary $\partial S$ has measure zero, we can exclude it and integrate over the interior $\interior S$ (cf. Def. A.2 and Prop. A.2 in Appx. A). Moreover, one can add and exclude sets of measure zero to make the integration more convenient (see e.g. Ex. I.viii)).}

Remark 1

Alternatively, it is possible to define the integral for differential forms without compact support, if a definition for the compact case over a manifold (with corners) has been given. Though Def. 2 is adequate for the case considered here, analogous reasoning may make it possible to extend results for the compact case to the non-compact one. We shall sketch this in the following.

\footnote{The definition of densities on manifolds with corners is analogous to the one on ‘ordinary’ manifolds. See p. 427 sqq. in Ref. [17] for an elaboration of the theory on manifolds with boundary.}

\footnote{By definition, $|\alpha|_q(X_1, \ldots, X_k) = |\alpha_q(X_1, \ldots, X_k)|$ for all $q \in S$ and $X_1, \ldots, X_k \in T_qS$.}

\footnote{In fact the Lebesgue-Borel measure is sufficient here (see Thm. 1.55 in Ref. [1]).}

\footnote{Observe that $\rho_\gamma \alpha$ is compactly supported on $U_\gamma$. One may then adapt the reasoning by Lee (cf. Prop. 16.5 in Ref. [17]).}

\footnote{Since the manifold boundary $\partial S$ has measure zero, we can exclude it and integrate over the interior $\interior S$ (cf. Def. A.2 and Prop. A.2 in Appx. A). Moreover, one can add and exclude sets of measure zero to make the integration more convenient (see e.g. Ex. I.viii)).}
3 The Differentiation Lemma

Let $\mathcal{S}$ be a smooth, oriented manifold with corners and let $\alpha$ be a (smooth) top-degree form. As a topological manifold with boundary, $\mathcal{S}$ is $\sigma$-compact, i.e. it has a countable cover of compact sets $\mathcal{K} = \{K_\gamma | \gamma \in I\}$. One may now choose a partition of unity $\{\rho_\gamma | \gamma \in I\}$ subordinate to this cover and set

$$\int_\mathcal{S} \alpha := \sum_{\gamma \in I} \int_\mathcal{S} \rho_\gamma \alpha,$$

provided the series converges absolutely.

Again by an argument analogous to the one of Prop. 16.5 in Ref. [17], this definition is independent of the choice of cover and partition of unity: Let $\{\rho'|_\delta | \delta \in I'\}$ be a second partition of unity subordinate to $\{K'_\delta | \delta \in I'\}$, then we may write

$$\sum_{\gamma \in I} \int_\mathcal{S} \rho_\gamma \alpha = \sum_{\gamma \in I} \int_\mathcal{S} \sum_{\delta \in I'} \rho'_\delta \rho_\gamma \alpha = \sum_{\delta \in I'} \int_\mathcal{S} \sum_{\gamma \in I} \rho_\gamma \rho'_\delta \alpha = \sum_{\delta \in I'} \int_\mathcal{S} \rho'_\delta \alpha,$$

due to the absolute convergence condition.

To prove a differentiation lemma in this setting (cf. Prop. 6.28 in Ref. [1]), we make use of the following concept.

Definition 3 (Bounded differential form)
Let $\mathcal{S}$ be a (smooth) $k$-manifold with corners, let $\alpha \in \Omega^k(\mathcal{S})$ and let $\beta$ be a (smooth, positive) density on $\mathcal{S}$. We say that $\alpha$ is bounded by $\beta$, if for all $q \in \mathcal{S}$ and for all $X_1, \ldots, X_k \in T_q\mathcal{S}$ we have

$$|\alpha|_q (X_1, \ldots, X_k) \leq \beta_q (X_1, \ldots, X_k).$$

The essential idea is that any $k$-form restricted to a $k$-submanifold (with corners) is a top-degree form. Then, by taking its absolute value, we can draw upon the one-dimensional definition of boundedness to carry it over to this case.

With an adequate notion of boundedness at our disposal, proving the lemma is straightforward.

Lemma 1 (Differentiation Lemma)
Let $\mathcal{S}$ be a smooth, oriented manifold with corners of dimension $k \in \mathbb{N}$, and let $\mathcal{I} \subseteq \mathbb{R}$ be an interval. Further, let

$$\alpha : \mathcal{I} \rightarrow \Omega^k(\mathcal{S}) : t \mapsto \alpha_t,$$

be a smooth one-parameter family of $k$-forms

If $\alpha: \mathcal{I} \times \mathcal{S} \rightarrow \bigwedge^k T^*\mathcal{S}$ is smooth as a map between manifolds with corners.
3 The Differentiation Lemma

i) the integral $\int_S \alpha_t$ exists for all $t \in I$, and

ii) there exists a (t-independent) integrable density $\beta$ on $S$ such that

$$\dot{\alpha} := \frac{\partial}{\partial t} \alpha$$

is bounded by $\beta$,

then $\int_S \dot{\alpha}$ exists and

$$\frac{d}{dt} \int_S \alpha = \int_S \dot{\alpha}.$$  \hfill (9d)

**Proof** The lemma is essentially a corollary of Prop. 6.28 in Klenke’s book [1]. Note that its proof does not rely on the openness of the interval for the parameter.

Choose $A$ and $\rho$ as in Def. 2. For each $\gamma \in I$ there exist smooth functions $f_\gamma$ on $I \times \kappa_\gamma (U_\gamma)$ and $h_\gamma$ on $\kappa_\gamma (U_\gamma)$ such that

$$(\kappa_\gamma^{-1})^* \alpha = f_\gamma \, d\kappa^1 \ldots d\kappa^k,$$

and

$$(\kappa_\gamma^{-1})^* \beta = h_\gamma \, d\kappa^1 \ldots d\kappa^k.$$  \hfill (10a)

Dropping the index $\gamma$ for ease of notation, we find

$$\int_U |\rho \alpha| = \int_{\kappa(U)} (\kappa^{-1})^* |\rho \alpha|$$

$$= \int_{\kappa(U)} |(\kappa^{-1})^* \rho \, (\kappa^{-1})^* \alpha|$$

$$= \int_{\kappa(U)} |(\rho \circ \kappa^{-1})| \, |f| \, d\kappa^1 \ldots d\kappa^k.$$  \hfill (10d)

Consult Prop. 16.38b in Ref. [17] for the second step. But $|\rho| \leq 1$, so

$$\int_U |\rho \alpha| \leq \int_U |\alpha| \leq \int_S |\alpha|.$$  \hfill (10e)

---

26 Note that $\dot{\alpha}$ is well defined via

$$(\dot{\alpha})_q (X_1, \ldots, X_k) := \frac{\partial}{\partial t} (\alpha_t)_q (X_1, \ldots, X_k)$$

for any $t \in I$, $q \in S$, and $X_1, \ldots, X_k \in T_q S$ (cf. p. 416 in Ref. [61], and Rem. 4.1.10.1 in Ref. [18]).

27 Notationally, we treat $f_\gamma$ like a function on $\kappa_\gamma (U_\gamma)$. 

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3 The Differentiation Lemma

and thus \((\rho \circ \kappa^{-1}) f\) is integrable over \(\kappa(U)\). An analogous argument for \(\beta\) shows that \((\rho \circ \kappa^{-1}) h\) is integrable as well.

The assumption that \(\dot{\alpha}\) is bounded by \(\beta\) implies that for each \(\gamma \in I\) we have \(\left| \dot{f}_\gamma \right| \leq h_\gamma\) (with \(\dot{f} := \partial f / \partial t\)). Consider now the expression

\[
\int_S \left| \dot{\alpha} \right| = \sum_{\gamma \in I} \int_{\kappa_\gamma(U_\gamma)} (\kappa_\gamma^{-1})^* |\rho_\gamma \dot{\alpha}| 
\]

\[
= \sum_{\gamma \in I} \int_{\kappa_\gamma(U_\gamma)} (\rho_\gamma \circ \kappa_\gamma^{-1}) \left| \dot{f}_\gamma \right| \, d\kappa^1 \ldots d\kappa^k 
\]

\[
\leq \sum_{\gamma \in I} \int_{\kappa_\gamma(U_\gamma)} (\rho_\gamma \circ \kappa_\gamma^{-1}) \left| h_\gamma \right| \, d\kappa^1 \ldots d\kappa^k 
\]

\[
= \int_S \beta. \tag{10i}
\]

It follows that \(\int_S \dot{\alpha}\) exists.

To obtain (9d), we need to apply the differentiation lemma (cf. Prop. 6.28 in Ref. [1]) twice.

First consider

\[
\int_{\kappa(U)} (\rho \circ \kappa^{-1}) \left( \dot{f} \right) \, d\kappa^1 \ldots d\kappa^k. \tag{10j}
\]

Using the lemma, this equals

\[
\frac{d}{dt} \int_{\kappa(U)} (\rho \circ \kappa^{-1}) \left( \dot{f} \right) \, d\kappa^1 \ldots d\kappa^k. \tag{10k}
\]

Therefore, we find that

\[
\int_S \dot{\alpha} = \sum_{\gamma \in I} \frac{d}{dt} \left( \text{sgn}_\gamma \int_{\kappa_\gamma(U_\gamma)} (\kappa_\gamma^{-1})^* (\rho_\gamma \dot{\alpha}) \right) \tag{10l}
\]

\[
= \sum_{\gamma \in I} \dot{g}_\gamma, \tag{10m}
\]

with \(g: (t, \gamma) \mapsto g_\gamma(t)\) denoting the function in parentheses in Eq. (10l) above.

To get the derivative out of the sum, consider the counting measure (cf. Ex. 1.30vii in Ref. [1])

\[
\#: 2^I \to [0, \infty]: J \mapsto \#J := \sum_{\gamma \in J} 1, \tag{10n}
\]

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4 The time-dependent Transport Theorem

where $2^I$ is the power set of $I$. Then we have

$$\int_I g \, d\# = \sum_{\gamma \in I} g_\gamma.$$  \hspace{1cm} (10o)

Thus we have reformulated the series in measure theoretic terms. As for every $\gamma \in I$ the function $g_\gamma$ is smooth,

$$\sum_{\gamma \in I} |g_\gamma| = \sum_{\gamma \in I} \left| \int_{U_\gamma} \rho_\gamma \alpha \right| \leq \int_S |\alpha|, \text{ and } |\dot{g}_\gamma| \leq \int_{U_\gamma} \rho_\gamma |\beta|,$$  \hspace{1cm} (10p)

the differentiation lemma indeed yields (9d).

For further properties of 1-parameter-families of differential forms, see Rem. 4.1.10.1 in Rudolph and Schmidt’s book [18].

4 The time-dependent Transport Theorem

We shall first state and prove the Transport Theorem for the time-dependent case, since the time-independent case can then be shown to follow as a corollary.

For the reader’s convenience, we briefly recall some facts on time-dependent vector fields on ‘ordinary’ manifolds. A more in-depth treatment thereof may be found in §3.4 in Ref. [18] and p. 236 sqq. in Ref. [17]. Do note, however, that the definition we employ here is slightly more general and arguably closer to the practical situation, as we do not assert a product structure on the domain of the vector field.

**Definition 4 (Time-dependent vector fields)**

Let $Q$ be a manifold of dimension $n \in \mathbb{N}$.

i) A flow domain on $Q$ is an open subset $U$ of $\mathbb{R} \times Q$, such that for every $q \in Q$
the set

$$\mathcal{I}_q = \{ t \in \mathbb{R} | (t, q) \in U \}$$  \hspace{1cm} (11a)

is a nonempty, open interval.

ii) Given a flow domain $U$, a (smooth) time-dependent vector field $X$ (on $Q$) is a smooth map

$$X : U \rightarrow TQ : (t, q) \mapsto (X_t)_q,$$  \hspace{1cm} (11b)

such that for every $(t, q) \in U$ the vector $(X_t)_q$ lies in $T_q Q$.

$X$ is assumed to be smooth as a map from the open submanifold $U$ of $\mathbb{R} \times Q$ to the manifold $TQ$.  

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iii) For every such \( X \) there exists a smooth map \( \Psi \) with domain \( \text{dom} \Psi \), open in \( \mathbb{R} \times U \), such that the (maximal) flow of the (time-independent) vector field

\[
\frac{\partial}{\partial t} + X, \tag{11c}
\]

on \( U \) is given by

\[
(t, t_0, q) \mapsto (t_0 + t, \Psi_t(t_0, q)). \tag{11d}
\]

The smooth map

\[
\Phi: \text{dom} \Phi \rightarrow Q : (t, t_0, q) \mapsto \Phi_{t,t_0}(q) := \Psi_{t-t_0}(t_0, q) \tag{11e}
\]

with (open) domain

\[
\text{dom} \Phi = \{(t, t_0, q) \in \mathbb{R} \times U | (t - t_0, (t_0, q)) \in \text{dom} \Psi\} \tag{11f}
\]

is called the (maximal) time-dependent flow of \( X \).

Instead of the group property, time-dependent flows \( \Phi \) satisfy the following ‘semigroup identity’

\[
\Phi_{t_3,t_2}(\Phi_{t_2,t_1}(q)) = \Phi_{t_3,t_1}(q) \tag{12}
\]

for \((t_2, t_1, q)\) and \((t_3, t_2, \Phi_{t_2,t_1}(q))\) in \( \text{dom} \Phi \).

It is also worthwhile to contemplate the fact that one essentially employs a ‘spacetime’ view to define time-dependent flows—that is, the time-dependent case is paradoxically defined via the time-independent one.

**Theorem 1 (Time-dependent Transport Theorem)**

Let \( Q \) be a smooth manifold of dimension \( n \in \mathbb{N} \), let \( X \) be a smooth, time-dependent vector field on \( Q \) with domain \( U \subseteq \mathbb{R} \times Q \) and time-dependent flow \( \Phi \). Further, let \((S_0, \iota_0)\) be a smooth, oriented \( k \)-submanifold of \( Q \) with corners for \( k \in \mathbb{N} \) and \( k \leq n \). Assume there exists an interval \( \mathcal{I} \subseteq \mathbb{R} \) such that the map

\[
\iota: \mathcal{I} \times S_0 \rightarrow Q : (t, q) \mapsto \iota_t(q) = (\Phi_{t,0} \circ \iota_0)(q) \tag{13a}
\]

is well-defined.

Then the following holds:

1) For each \( t \in \mathcal{I} \) the tuple \((S_0, \iota_t)\) is a smooth, oriented \( k \)-submanifold of \( Q \) with corners. The image \( S_t := \iota_t(S_0) \), together with the inclusion and topology coinduced by \( \iota_t \), is an oriented submanifold of \( Q \) with corners equivalent to \((S_0, \iota_t)\).
\section{The time-dependent Transport Theorem}

2) Let
\[ \alpha: U \to \bigwedge^k T^* Q : \quad (t,q) \mapsto (\alpha_t)_q \]  
be smooth and satisfy \((\alpha_t)_q \in \bigwedge^k T^*_q Q\) for all \((t,q) \in U\). If for all \(t \in I\)

i) the integral \(\int_{S_t} \alpha_t \equiv \int_{S_0} \iota_t^* \alpha_t\) exists, and

ii) the \(k\)-form
\[ \frac{\partial}{\partial t} (\iota_t^* \alpha_t) \]  
is bounded by a \((t\text{-independent})\) integrable density \(\beta\) on \(S_0\),

then we have
\[ \frac{d}{dt} \int_{S_t} \alpha_t = \int_{S_t} \left( \frac{\partial}{\partial t} \Phi_t^* \alpha_t + \mathcal{L}_X \right) \alpha_t. \]  
(13d)

\textbf{Proof}

1) For every \(t \in I\) the mapping
\[ \Phi_{t,0}: \text{dom} \Phi_{t,0} \to Q : \quad q \mapsto \Phi_{t,0}(q) \]  
is injective, smooth and has full rank (cf. Rem. 3.4.5.1 in Ref. [18]). Thus those properties carry over to its restriction to \(\iota_0(S_0)\). Then, as \(\iota_0\) is a smooth, injective immersion, \(\iota_t\) is a smooth, injective immersion. So \((S_0,\iota_t)\) is a smooth submanifold of \(Q\). Recalling Ex. 1.v) above and that as a manifold \(Q\) is a manifold with corners, the image \(S_t\) yields an equivalent submanifold.

The orientation on \(S_t\) is obtained by pushforward via \((\iota_t)_*\).

2) First observe that \(\iota: I \times S_0 \to Q\) is smooth as a map between manifolds with corners, so that all of its derivatives here are well-defined.

Now reformulate:
\[ \frac{d}{dt} \int_{S_t} \alpha_t = \frac{d}{dt} \int_{S_0} \iota_t^* \alpha_t = \frac{d}{dt} \int_{S_0} \iota_0^* \Phi_{t,0} \alpha_t. \]  
(14b)

Using the definition [9b] of the parametric derivative above, one easily shows that
\[ \frac{\partial}{\partial t} \iota_0^* \Phi_{t,0} \alpha_t = \iota_0^* \left( \frac{\partial}{\partial t} \Phi_{t,0}^* \alpha_t \right). \]  
(14c)
Hence, Lem. 1 leads us to consider\(^{29}\)

\[
\frac{\partial}{\partial t} \Phi_{t,0}^* \alpha_t = \frac{\partial}{\partial t'} \Phi_{t',0}^* \alpha_{t'}
\]
(14d)

\[
= \frac{\partial}{\partial t'} \Phi_{t',0}^* \alpha_t + \frac{\partial}{\partial t'} \Phi_{t,0}^* \alpha_{t'}.
\]
(14e)

By definition of \(\Phi\), we have

\[
\mathcal{L}_{X_t} \alpha_t = \frac{\partial}{\partial s} \bigg|_0 (\Psi_s(t, \cdot))^* \alpha_t = \frac{\partial}{\partial s} \bigg|_0 \Phi_{s+t,t}^* \alpha_t.
\]
(14f)

So, the first term in (14e) is

\[
\frac{\partial}{\partial t'} \Phi_{t',0}^* \alpha_t = \frac{\partial}{\partial s} \bigg|_0 \Phi_{s+t,t}^* \alpha_t
\]
(14g)

\[
= \frac{\partial}{\partial s} \bigg|_0 (\Phi_{s+t,t} \circ \Phi_{t,0})^* \alpha_t
\]
(14h)

\[
= \Phi_{t,0}^* \left( \frac{\partial}{\partial s} \bigg|_0 \Phi_{s+t,t} \alpha_t \right),
\]
(14i)

which finally yields

\[
\frac{\partial}{\partial t} \Phi_{t,0}^* \alpha_t = \Phi_{t,0}^* \left( \mathcal{L}_{X_t} \alpha_t + \dot{\alpha}_t \right).
\]
(14j)

Applying first Lem. 1 on (14b), and then (14j) yields the assertion. \(\blacksquare\)

Remark 2

i) Consider the situation above with \(\dim \mathcal{S}_0 = \dim \mathcal{Q}\), in which case \(\iota_0\) is a diffeomorphism onto its image. If \(\alpha_t\) is nowhere vanishing on \(\mathcal{S}_t\) for each \(t \in \mathcal{I}\), then it is a volume form on it (by choosing the corresponding orientation). In that case

\[
\mathcal{L}_{X_t} \alpha_t = \text{div}_t \left( X_t \right) \alpha_t,
\]
(15a)

\[^{29}\text{The full proof of the second equality employs the definition of the parametric derivative (9b) and the fact that for } C^1 \text{ functions } g: \mathbb{R} \to \mathbb{R}^m \text{ and } f: \mathbb{R}^{1+m} \to \mathbb{R}^n: (t, x) \mapsto f(t, x) \text{ we have}
\]

\[
\frac{\partial}{\partial t} f(t, g(t)) = \frac{\partial}{\partial t'} f(t', g(t)) + \frac{\partial}{\partial u} f(t, g(t')).
\]
where \( \text{div}_t (X_t) \) denotes the divergence of \( X_t \) induced by \( \alpha_t \). Then we find that for every \( t \in I \)

\[
\frac{d}{dt} \int_{S_t} \alpha_t = \int_{S_t} \left( \frac{\partial \alpha_t}{\partial t} + \text{div}_t (X_t) \alpha_t \right) .
\] (15b)

As shown in Ex. 2 below, (15b) is a ‘time-dependent’ generalization of Reynolds Transport Theorem.

ii) The reader may wonder why we consider the transport theorem for a submanifold with corners evolving in an ‘ambient manifold’ ‘without corners’ instead of allowing the ‘ambient manifold’ to be a manifold with corners as well.

To simplify the discussion, we shall only discuss this question for the time-independent case here (cf. Cor. 1 below). The discussion can be generalized to the time-dependent case, Thm. 1 above, in a straightforward manner.

We begin by noting that the generalization of Cor. 1 to the case that \( Q \) is a smooth manifold with corners is nontrivial, since general maximal flows on manifolds with corners are ‘ill-behaved’ in several respects. The interested reader is referred to the discussion in Appx. B.

Still, we do conjecture that the generalization holds: By assumption, we may restrict the maximal flow \( \Phi \) (cf. Def. 3) to the set

\[
I \times \iota_0 (S_0) \subseteq \text{dom } \Phi ,
\] (15c)

which is canonically a smooth manifold with corners. By a somewhat involved argument one can show that the restriction of \( \Phi \) admits smooth local representatives, so that one only needs to show continuity to obtain smoothness. The remaining argument from the proof of Thm. 1 may then be carried over.

In practical situations, the manifold with corners \( Q \) is commonly obtained from restricting an ‘ordinary’ manifold to \( Q' \). Indeed, Douady and Hérault [58] have shown that every manifold with corners can be obtained this way. If in addition the vector field \( X \) on \( Q \) is the restriction of a smooth vector field \( X' \) on \( Q' \) – which is also how one commonly obtains \( X \) – then the flow \( \Phi \) of \( X \) is the restriction of the smooth flow \( \Phi' \) of \( X' \), and hence the restriction

\[30\text{This equation is independent of the chosen orientation. Locally } \text{div} X = \partial_i (f X^i) / f \text{ with } f := \left| \alpha_1 \ldots k \right| \neq 0.\]

\[31\text{See Thm. } A.2 \text{ below and the references given thereafter.}\]
of $\Phi$ to the domain in Eq. (15c) is smooth. In this case, an appropriate generalization of Cor. 1 does hold. For Thm. 1 the situation is similar.

5 The time-independent Transport Theorem

From a relativistic physics perspective, the view of time as a ‘global parameter’ is rather unnatural. Furthermore, even within Newtonian (continuum) mechanics the ‘spacetime view’ is often conceptually more coherent (see e.g. Ex. 2 below). In this respect, we regard the following special case of Thm. 1 as a physically more appropriate generalization of Reynolds Transport Theorem to the setting of manifolds with corners. Hence we omit the words ‘time-independent’.

Corollary 1 (Transport Theorem)

Let $Q$ be a smooth manifold of dimension $n \in \mathbb{N}$, let $X$ be a smooth (time-independent) vector field on $Q$ with flow $\Phi$. Further, let $(S_0, \iota_0)$ be a smooth, oriented $k$-submanifold of $Q$ with corners for $k \in \mathbb{N}$ and $k \leq n$. Assume there exists an interval $\mathcal{I} \subseteq \mathbb{R}$ such that the map

$$\iota : \mathcal{I} \times S_0 \rightarrow Q : \ (t, q) \mapsto \iota_t(q) = (\Phi_t \circ \iota_0)(q) \quad (16a)$$

is well-defined.

Then the following holds:

1) For each $t \in \mathcal{I}$ the tuple $(S_t, \iota_t)$ is a smooth $k$-submanifold of $Q$ with corners. The image $S_t := \iota_t(S_0)$, together with the inclusion and topology coinduced by $\iota_t$, is an oriented submanifold of $Q$ with corners equivalent to $(S_0, \iota_t)$.

2) Let $\alpha$ be a smooth $k$-form on $Q$. If for all $t \in \mathcal{I}$

i) the integral $\int_{S_t} \alpha \equiv \int_{S_0} \iota_t^* \alpha$ exists, and

ii) the $k$-form

$$\frac{\partial}{\partial t} (\iota_t^* \alpha) \quad (16b)$$

is bounded by a ($t$-independent) integrable density $\beta$ on $S_0$,

then we have

$$\frac{d}{dt} \int_{S_t} \alpha = \int_{S_t} \mathcal{L}_X \alpha \quad . \quad (16c)$$

$^{32}$Smoothness of $\Phi$ is more subtle, we refer the reader to Ex. B.2 in Appx. B.

$^{33}$See also Cor. 6.27 and p. 45 sq. in Ref. 17.
5 The time-independent Transport Theorem

**Proof** For $t \in \mathbb{R}$ set $\alpha_t := \alpha$ and apply Thm. 1. ■

**Remark 3 (Poincaré-Cartan invariants)**

Cor. 1 is closely related to the theory of Poincaré-Cartan invariants. These derive their name from the Poincaré-Cartan Theorem, frequently encountered in the study of Hamiltonian systems (see p. 182 sqq. in Ref. [18], §44 in Ref. [62], and Appx. 4 in Ref. [63] for a modern treatment, Refs. [64, 65] for the original works in French). Given a vector field $X$ and a $k$-form $\alpha$, integrable on $S_t$ for all $t \in I$ (as in Cor. 1), one distinguishes three kinds of invariants:

i) $\alpha$ is *invariant (on $Q$)*, if $L_X \alpha$ vanishes on $Q$.

Then, by Cor. 1, $\int_{S_t} \alpha$ is conserved [34].

ii) $\alpha$ is *absolutely invariant (on $Q$)*, if $L_{fX} \alpha$ vanishes on $Q$ for all $f \in C^\infty(Q, \mathbb{R})$.

Note that this is equivalent to the vanishing of both $X \cdot \alpha$ and $X \cdot d\alpha$ [35].

Now, for given $f$ let $\Phi^{fX}$ be the flow of $fX$, and set

$$S_t^f := (\Phi^{fX}_t \circ \iota_0)(S_0),$$

(17a)

provided it exists for $t$ on some interval $I' \subseteq \mathbb{R}$. Then, as in i) above, we find that the quantity $\int_{S_t^f} \alpha$ is both conserved and independent of $f$.

iii) $\alpha$ is *relatively invariant (on $Q$)*, if $X \cdot d\alpha$ is exact on $Q$.

Consider the setting of Cor. 1. let $\gamma$ be the smooth form such that

$$X \cdot d\alpha = d\gamma,$$

(17b)

and assume $S_0$ is an $n$-manifold with corners with compact 1-boundary $\partial^1 S_t$. Since $S_0$ and $S_t$ are diffeomorphic, so are their boundaries. Thus, $\partial^1 S_t$ is compact, and we have

$$\partial^k S_t = \iota_t (\partial^k S_0)$$

(17c)

---

[34] Of course, one needs to show the existence of $\beta$. This is obtained from $\Phi^t_* \alpha = \alpha$ (cf. Eq. 3.3.3 in Ref. [18], Prop. 9.41 in Ref. [17]), so $\beta = 0$. This identity also yields the conservation of the integral by itself.

[35] Observe that $L_{fX} \alpha = df \wedge (X \cdot \alpha) + f L_X \alpha$ (cf. p. 182 in Ref. [18]). Choose $f = 1$ to get $L_X \alpha = 0$. Then choose coordinates $\kappa$ around any $q \in Q$ to find $(d\kappa^i \wedge (X \cdot \alpha))^i = 0$ for all $i$, implying $X \cdot \alpha = 0$ on $Q$. Finally, Cartan’s formula (cf. Prop. 4.18 in Ref. [18], and Thm. 14.35 in Ref. [17]) yields both the forward and reverse implication.
for all admissible \( t \) and \( k \in \{1, \ldots, n\} \) (cf. Ex. 1[xi]). Then, by Cor. 1, Stokes’ Theorem (cf. Prop. 3.5 in Ref. [51]), and Cartan’s formula, we find

\[
\frac{d}{dt} \int_{\partial S_t} \alpha = \int_{\partial S_t} d(\gamma + X \cdot \alpha) = \int_{\partial(\partial S_t)} (\gamma + X \cdot \alpha) = 0. \tag{17d}
\]

Hence, \( \int_{\partial S_t} \alpha \) is conserved. This constitutes a generalization of Kelvin’s circulation theorem.

Under certain conditions, the Poincaré-Cartan theorem gives a one-to-one correspondence between conservation of the integrals in [i][ii][iii] and the validity of the respective geometric differential equations. ♦

6 Applications

To support the claim that both Thm. 1 and Cor. 1 are generalizations of the Reynolds Transport Theorem, we show that the special case is indeed implied.

Example 2 (Reynolds Transport Theorem)

i) In this approach, we consider the time \( t \) in Newtonian (continuum) mechanics as a parameter. It is therefore an example for Thm. 1.

Consider \( Q = \mathbb{R}^3 \) equipped with the Euclidean metric and standard coordinates \( \vec{x} \). Let \( t \mapsto \rho(t,.) \) be a smooth 1-parameter family of real-valued, nowhere vanishing functions on \( \mathbb{R}^3 \), and let \( \vec{v} \) be a smooth time-dependent vector field with parameter values on the same interval \( \mathcal{I} \) around 0 and time-dependent flow \( \vec{\Phi}_t \) (see Def. 4). Choose a smooth 3-submanifold \( S_0 \) of \( \mathbb{R}^3 \) with corners (given as a subset), e.g. (4f) from Ex. 1[vi]. By assumption \( S_t = \vec{\Phi}_{t,0}(S_0) \) exists for every \( t \in \mathcal{I} \). A possible ‘temporal evolution’ of \( S_0 \) is shown in Figure 1. By Thm. 1, each \( S_t \) is a smooth 3-submanifold of \( Q \) with corners. So by appropriate restrictions in domain

\[
\alpha_t := \rho(t,.) \, dx^1 \wedge dx^2 \wedge dx^3 = \rho(t,.) \, d^3x \tag{18a}
\]

yields a smooth, nowhere-vanishing 3-form on \( S_t \) (identifying it as a subset of \( \mathbb{R}^3 \)). In order to apply identity (15b), \( \rho(t,.) \) needs to be integrable on \( S_t \) for all \( t \) and we need to satisfy condition 2)[ii] of Thm. 1. The latter is equivalent to the real valued function

\[
(t, \vec{x}) \mapsto \frac{\partial}{\partial t} \left( \rho \left( t, \vec{\Phi}_{t,0}(\vec{x}) \right) \det \left( \frac{\partial \vec{\Phi}_{t,0}}{\partial \vec{x}} (\vec{x}) \right) \right) \tag{18b}
\]
being bounded by some (smooth) $t$-independent, integrable function $h$ on $S_0$. Then (15b) yields

$$\frac{d}{dt} \int_{S_t} \rho(t, \vec{x}) \, d^3x = \int_{S_t} \left( \frac{\partial \rho}{\partial t} + \left( \frac{1}{\rho} \nabla \cdot (\rho \vec{v}) \right) \rho \right) (t, \vec{x}) \, d^3x = \int_{S_t} \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right) (t, \vec{x}) \, d^3x$$

(18c)

This is the Reynolds Transport Theorem for nowhere vanishing $\rho$.

By employing (13d) instead of (15b), one can arrive at this result without the artificial restriction on $\rho$. The calculation is analogous to the one in (18j)-(18l) below.

ii) We also show how to obtain the Transport Theorem from the ‘time-independent’ Cor. 1 by employing the concept of a Newtonian spacetime (see §2 in Ref. [6]).

\[
\begin{align*}
 t &= 0 & t &= 1\Delta t \\
 t &= 2\Delta t & t &= 3\Delta t
\end{align*}
\]

Figure 1: A portion of $S_t$ obtained from (4f) at four times $t$. This (time-independent) flow was obtained from the Lorenz equations, which are known for exhibiting chaotic behavior (cf. §2.3 in Ref. [66], and Ref. [67]). Nonetheless, $S_t$ is a smooth manifold with corners at each $t$ and (18d) can be used to formulate conservation laws on it (e.g. conservation of mass).
6 Applications

So let \( \mathbb{R}^4 \), equipped with the appropriate geometric structures and standard coordinates \((t, \vec{x})\), be our ‘spacetime’. Let \( \rho \) be a smooth real-valued function and \( \vec{v} \) be a smooth vector field on \( \mathbb{R}^4 \). We would like \( \vec{v} \) to be a Newtonian observer vector field (cf. Def. 2.3 & Rem. 2.4 in Ref. [6]), i.e.

\[
v = \frac{\partial}{\partial t} + \vec{v}
\]

(18e)

with \( \vec{v} \) tangent to the hypersurfaces of constant \( t \) (i.e. \( \vec{v} \) is ‘spatial’). If we again take \( S_0 \subseteq \mathbb{R}^3 \) to be a smooth 3-submanifold of \( \mathbb{R}^3 \) with corners, then

\[
S'_0 := \{0\} \times S_0
\]

(18f)

is a 3-submanifold of \( \mathbb{R}^4 \) with corners. The values of the flow \( \Phi \) of \( v \) can be written as

\[
\Phi_s (t, \vec{x}) = \left( t + s, \Phi_s (t, \vec{x}) \right).
\]

(18g)

Since we are only interested in the evolution starting from \( t = 0 \), we set \( \Phi_s (0, \vec{x}) \equiv \Phi_s (\vec{x}) \). Then we may define the ‘temporal evolution’ of \( S_0 \) via

\[
S'_t := \Phi_t (S'_0) = \{t\} \times \Phi_t (S_0) = \{t\} \times S_t,
\]

(18h)

whenever \( S_t \) exists for given \( t \in \mathbb{R} \). We would like to integrate the form

\[
\alpha := \rho \, dx^1 \wedge dx^2 \wedge dx^3
\]

(18i)

over it. One easily checks that the assumptions on \( \alpha \) demanded by Cor. [1] are the same as in the ‘time-dependent’ case above with \( \Phi_{t,0} \) replaced by \( \Phi_t \).

Finally, we employ Cartan’s formula and observe that the integrands with \( dt \)-terms vanish to find

\[
\frac{d}{dt} \int_{S_t} \rho \, d^3x = \int_{S_t} L_v \alpha
\]

\[
= \int_{S_t} \left( v (\rho) \, d^3x + \rho \, d (v \cdot d^3x) \right)
\]

\[
= \int_{S_t} \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right) \, d^3x.
\]

(18j)

(18k)

(18l)

This is to support our claim that even within Newtonian (continuum) mechanics, taking a ‘spacetime-view’ as opposed to a ‘time-as-a-parameter-view’ is often conceptually more coherent. Moreover, employing the ‘Newtonian
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spacetime’ concept allows one to choose domains of integration which are not ‘constituted of simultaneous events’. We conclude this article with a physical example from the general theory of relativity for the application of Lem. and Cor. Though the example explicitly discusses how mass conservation is achieved or violated in a curved spacetime, the mathematical theory is essentially analogous for the conservation of other scalar quantities obtained from corresponding ‘scalar densities’, such as charge and probability. The spacetime under consideration describes a linearly polarized gravitational sandwich plane wave. Such mathematical models of free gravitational radiation have been studied by Bondi, Pirani, and Robinson [68, 69]. They are of physical relevance, if the wave is sufficiently far away from the source [69], and the effect of other masses on the overall spacetime geometry is negligible.

Example 3 (Gravitational plane wave)

Consider the smooth manifold \( \mathbb{R}^4 \) with standard coordinates \((t, x, y, z) = (t, \vec{x})\) and smooth Lorentzian metric \( g \) with values

\[
g_{(t, \vec{x})} = dt \otimes dt - dx \otimes dx - dy \otimes dy - dz \otimes dz - \left( (t^2 - x^2) (\beta'(t - x))^2 + 2 \frac{y^2 - z^2}{t - x} \beta'(t - x) \right) d(t - x) \otimes d(t - x) + \beta'(t - x) (yd\,dy - zd\,dz) \otimes d(t - x) + \beta'(t - x) d(t - x) \otimes (yd\,dy - zd\,dz)
\]

Here \( \beta' \) is the derivative of an arbitrary smooth function \( \beta: \mathbb{R} \to \mathbb{R} \) for which \( \beta'(0) \) vanishes—e.g. the shifted bump function of width \( \sigma \)

\[
u \mapsto \beta(u) = \begin{cases} e^{-\left(1-\frac{(u-u_0)^2}{\sigma^2}\right)^{-1}}, & |u - u_0| < \frac{\sigma}{2} \\ 0, & \text{else} \end{cases}
\]

for \( 0 < \sigma/2 < u_0 \). Since \( g \) reduces to the standard Minkowski metric whenever the expression \( \beta'(t - x) \) is zero and our choice of \( \beta' \) has connected compact support, the gravitational wave separates the spacetime into two connected flat open sets for which

\[
t - x < u_0 - \sigma/2 \quad \text{and} \quad t - x > u_0 + \sigma/2,
\]

36Appropriate care must be taken here in the choice of integrand. 37In the literature one sometimes finds the claim that plane wave spacetimes cannot be covered by a global chart. This gives an explicit counterexample. 38Roughly speaking, a spacetime is a (smooth) Lorentzian manifold, which is both time- and space-oriented in a way that respects the metric. We refer to §2.2.3 in Ref. 70 and p. 240 sqq. in Ref.
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respectively. That is, the two flat regions enclose the curved one like a sandwich, thus the terminology “sandwich wave” (cf. p. 523 in Ref. [69]). As the metric is Ricci-flat (cf. Eq. 2.8’ and 3.2 in Ref. [69]), it is indeed a solution of the vacuum Einstein equation. A slice of constant \( y \) and \( z \) containing the curved region is indicated in Fig. 2.

To our model we add a mass density \( \rho \), which is a smooth, positive scalar field, as well as a smooth, future-directed timelike vector field \( X \), whose flow \( \Phi \) governs the motion of the mass\(^{39} \). Such a model is appropriate for modeling a gas or a fluid macroscopically. Given an ‘initial value set’ \( S_0 \subset \mathbb{R}^4 \) and denoting by \( \mu \) the volume form induced by \( g \) (cf. Eq. 2.7’ and 2.8’ in Ref. [69]), the mass contained in \( S_r = \Phi_r(S_0) \) at parameter time \( r \) is then defined as

\[
M(r) := \int_{S_r} \rho X \cdot \mu
\]  

(cf. p. 69 sqq. in Ref. [9], Ref. [32], and Sec. 3.4 in Ref. [33]).

First we define the vector field \( X \) indirectly via its flow. The auxiliary function \( \phi \) is given by

\[
\phi(u) = \frac{1}{2} \int_{u_0 - \frac{u}{2}}^u v(\beta(v))^2 \, dv
\]

for \( u \in \mathbb{R} \) (cf. Eq. 2.8’ in Ref. [69]). Using the shorthand notation

\[
\phi_r := \phi \left( \frac{(t-x)}{1-(t-x)r} \right) \quad \text{and} \quad \beta_r := \beta \left( \frac{(t-x)}{1-(t-x)r} \right),
\]

\(^{39}\) for rigorous definitions as well as to §3.1 in Ref. [70] for a physical justification. Formally, one may use \( X \) below to define a time-orientation on the spacetime—it defines one everywhere except for \( t = x \), where the choice is canonical. Given the time-orientation, equip \( \mathbb{R}^4 \) with the ‘ordinary’ standard orientation. Together with the existence of a global timelike vector field this defines a space-orientation.

\(^{39}\) One may also require \( g(X,X) = 1 \) (in natural units) to assure the integral curves of \( X \) are parametrized with respect to proper time, so that \( X \) is a ‘velocity vector field’.

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the values of \( \Phi \) are as follows

\[
\Phi_r(t, \vec{x}) = \begin{pmatrix}
\frac{1}{2} \left( e^{2 \phi_r} + 1 + y^2 e^{2(\beta_r - \beta_0)} + z^2 e^{-2(\beta_r - \beta_0)} \right) \frac{(t-x)}{(t-x)^2} \right) \frac{(t-x)}{1 - (t-x)r} \\
+ \frac{1}{2} e^{2(\phi_r - \phi_0)} \left( t + x - \frac{y^2 + z^2}{(t-x)} \right) - \frac{1}{2} e^{2 \phi_r} (t-x) \\
+ \frac{1}{2} e^{2(\phi_r - \phi_0)} \left( t + x - \frac{y^2 + z^2}{(t-x)} \right) - \frac{1}{2} e^{2 \phi_r} (t-x) \\
\end{pmatrix}
\]

\[ \text{(19h)} \]

Here \( r \in (-\infty, (t-x)^{-1}) \) for \( (t-x) > 0 \), \( r \in ((t-x)^{-1}, \infty) \) for \( (t-x) < 0 \), and \( r \in \mathbb{R} \) for the limit \( (t-x) \to 0 \). The vector field \( X \) corresponding to \( \Phi \) is smooth on all of \( \mathbb{R}^4 \) and, except for \( t = x \), future-directed timelike. Modulo this set and up to normalization of \( X \), it hence provides a reasonable model of physical motion on the spacetime. The values of the vector field on a two-dimensional slice are again indicated in Fig. 2.

Second, we consider the unbounded ‘initial value set’

\[
S_0 := \left\{ (0, x, y, z) \in \mathbb{R}^4 \mid -u_0 + \frac{\sigma}{2} \leq x < 0 \quad \text{and} \quad y^2 + z^2 \geq R^2 \right\}.
\]

\[ \text{(19i)} \]

This is a half-open, three-dimensional, infinite slab with a cylindrical hole of radius \( R > 0 \). As the product of two manifolds with boundary (cf. Ex. 1.ii)), \( S_0 \) is a smooth manifold with corners. It carries a canonical orientation. As long as the parameter time \( r \) lies within \((-\infty, (u_0 - \sigma/2)^{-1})\), the set \( S_r = \Phi_r(S_0) \) is well-defined, and by Cor. 1.1) each \( S_r \) is a smooth oriented 3-submanifold of \( \mathbb{R}^4 \) with corners (cf. Fig. 2).

Third, we directly define the integrand \( \alpha \) on the right hand side of (19e). If we use \( a_0, b_0 > 0 \) as scaling constants, omit the arguments \( (t-x) \) of \( \phi, \beta \) and \( \beta' \) for brevity, and set the factor

\[
\omega(t, \vec{x}) := \frac{a_0}{2} e^{-2\phi} \frac{\left( \frac{(x^2-t^2)e^{2\phi}}{t-x} + (t-x) - \frac{u_0 - \sigma}{2} \right)^2}{e^{4\phi_0} \left( z^2 e^{2\beta} + y^2 e^{-2\beta} \right)} ,
\]

\[ \text{(19j)} \]

then the values \( \alpha(t, \vec{x}) \) are given by
\[ \omega(t, x) \left( ((e^{2\phi} + 1 + (t^2 - x^2)\beta^2)(t - x)^2 + 2(t - x)(y^2 - z^2)\beta' + y^2 + z^2) \right) \]
\[ dx \wedge dy \wedge dz + \left( (e^{2\phi} + 1 + (t^2 - x^2)\beta^2)(t - x)^2 + 2(t - x)(y^2 - z^2)\beta' + y^2 + z^2 \right) \]
\[ dt \wedge dz \wedge dy + 2 \left( (t - x) + (t - x)^2\beta' \right) y \ dt \wedge dx \wedge dz \]
\[ + 2 \left( (t - x) - (t - x)^2\beta' \right) z \ dt \wedge dy \wedge dx \]. \quad (19k) \]

Figure 2: This graphic depicts a typical slice of constant \( y \) and \( z \) in the spacetime. In the diagonal, orange region the metric is non-flat, in the remaining regions the (tangent) light cone at each point lies at angles \( \pi/4 \) and \( 3\pi/4 \) on the graphic. The arrows indicate the vector field \( X \). The colored, horizontal line is \( S_0 \), which evolves along the flow of \( X \) at ten different parameter values \( r \) here. The brightness indicates the values of the density \( \rho \) (associated with \( \alpha \) in (19k)), with brighter colors implying higher values. Observe that the evolution along the flow of \( X \) changes the ‘causal character’ of the hypersurfaces, i.e. \( S_r \) does not stay spacelike.
The proof that the integral converges is straightforward, as $\beta$ and $\phi$ are zero on $S_0$.

We proceed by showing how Lem. 1 and Cor. 1 are of use for calculating the rate of mass change $\dot{M}(r)$ in $S_r$.

To compute the integral directly, recall that $\int_{S_r} \alpha = \int_{S_0} (\Phi_r \circ \iota_0)^* \alpha$. Taking this approach, we would determine $(\Phi_r(0, .))^* \alpha$, integrate directly over the respective region $\{19i\}$ in $\mathbb{R}^3$ and employ Lem. 1. This is laborious, but straightforward.

There is, however, a simpler approach in this case. Considering $\{16c\}$ above, we compute $\mathcal{L}_X \alpha$ via Cartan’s formula. After some labor, we find that both $X \cdot \alpha$ and $d\alpha$ vanish (cf. $\{19i\}$ and Eq. 2.8’ in Ref. [69]). Hence $\mathcal{L}_X \alpha = 0$ and thus $\Phi^* \alpha \equiv \alpha$ without having to compute the left hand side directly. Therefore, the mass is conserved in $S_r$:

$$M(r) = \int_{S_r} \alpha \equiv \int_{S_0} \alpha = M(0). \quad (19l)$$

So we found that the left hand side of Eq. (16c) vanishes without needing to check the assumptions of Cor. 1. We again refer to Fig. 2 for an illustration of how the mass gets distributed in this example.

In the more general case, where $\mathcal{L}_X \alpha \neq 0$, Cor. 1 provides an alternative for calculating $\dot{M}$ to directly computing and deriving the integral: One first computes $\mathcal{L}_X \alpha$, and then the rate is found via

$$\dot{M}(r) = \int_{S_0} (\Phi_r \circ \iota_0)^* \mathcal{L}_X \alpha, \quad (19m)$$

provided the assumptions of Cor. 1 hold true. The assumptions to check are the same as if one were to directly apply Lem. 1 to the equation above.

As a final remark, we note that this example was constructed using the coordinates $(\tau, \xi, \eta, \zeta)$ as defined in Eq. 3.1 in Ref. [69] for $(t - x) \neq 0$. In these coordinates we have

$$X_{(\tau, \xi, \eta, \zeta)} = (\tau - \xi)^2 \frac{\partial}{\partial \tau} \quad \text{and} \quad \rho(\tau, \xi, \eta, \zeta) = a_0 \frac{e^{-\frac{1}{2}((\eta - \xi) / \xi_0)^2 - 2\phi(\tau - \xi)}}{(\eta^2 + \zeta^2)^2(\tau - \xi)^4}. \quad (19n)$$

Here mass conservation is trivial, since $\alpha$ is independent of $\tau$. Generally speaking, in the case of mass conservation $\mathcal{L}_X \alpha = 0$, the Straightening Lemma (cf. Prop. 3.2.17 in Ref. [18]) implies that the (local) existence of such a coordinate system on the spacetime is generic. In practice, if the flow $\Phi$ is known, such a coordinate system can be easily constructed by restricting the flow to a (coordinate-)hypersurface nowhere tangent to $X$ and applying the flowout theorem (cf. Prop. 9.20.d in Ref. [17]).

Further examples of the application of Thm. 1 and Cor. 1 can be found in the articles by Flanders [24] and Betounes [29].
Appendix A: Elementary results on manifolds with corners

To keep the article mostly self-contained, we provide some elementary definitions and results on manifolds with corners here. Since Michor’s concept of a manifold with corners (cf. Def. [1]) has not been explored much in the literature, some of the results here are original.

Definition A.1
Let $Q$ be a manifold with corners of dimension $n \in \mathbb{N}$.

i) A point $p \in Q$ is called a corner point of index $j \in \{1, \ldots n\}$, if there exists a corner chart $(U, \kappa)$ around $p$ with codomain

$$\kappa(U) = C^n(\varphi^1, \ldots, \varphi^k) \cap \bar{U}, \quad \bar{U} \text{ open in } \mathbb{R}^n$$

such that $\varphi^i(\kappa(p)) = 0$ for exactly $j$ indices $i$.

ii) Let $(U, \kappa)$ be a corner chart on $Q$ with $\kappa(U)$ as in Eq. (A.1a) above. A linear functional $\varphi^i$ is called redundant (for $(U, \kappa)$), if $\kappa(U) \cap \ker \varphi^i = \emptyset$. A quadrant $C^n(\varphi^1, \ldots, \varphi^k)$ is called a minimal quadrant (for $(U, \kappa)$), if no $\varphi^i$ is redundant.

iii) Let $(U, \kappa)$ be a corner chart as before and let the respective quadrant be minimal. Further, let $I \subset \{1, \ldots, k\}$ be an index set containing $j$ elements,

$$j = \# I \in \{1, \ldots, k\} \subset \mathbb{N}.$$  \hspace{1cm} (A.1b)

Denote the complement of $I$ in $\{1, \ldots, k\}$ by $I^c$.

Then each

$$V_I = \{x \in \kappa(U) \mid \forall i \in I: \varphi^i(x) = 0 \quad \text{and} \quad \nexists i \in I^c: \varphi^i(x) = 0\}$$  \hspace{1cm} (A.1c)

is called a $j$-slice (of $(U, \kappa)$).

Note that, since the kernel of a linear functional uniquely defines the functional up to a nonzero factor, minimal quadrants are unique up to (strictly positive) factors of the $\varphi^i$'s. It is therefore sensible to speak of $j$-slices independent of a particular choice of quadrant, even if their label $I$ in general depends on this choice. For a given choice of minimal quadrant, each $j$-slice $V_I$ is contained in $\kappa(U)$, and it is a nonempty, relatively open subset of the $(n - j)$-dimensional linear subspace $\bigcap_{i \in I} \ker \varphi^i$ of $\mathbb{R}^n$.

As shown by the example below, the minimal quadrants of two corner charts on the same chart domain need not employ the same number of linear functionals.
Appendix A

Example A.1
Let $\kappa_1, \kappa_2$ be two coordinate maps on an open subset $U$ of a 2-manifold with corners. Denote by $\{e^1, e^2\}$ the standard dual basis of $\mathbb{R}^2$ and by $B_\varepsilon(x)$ the open ball of radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^2$. Set $\tilde{U}_1 = B_\varepsilon(0, 2\varepsilon) \cup B_\varepsilon(0, -2\varepsilon)$ and $\kappa_1(U) = C^2(\varepsilon^1) \cap \tilde{U}_1$. Similarly, define $\tilde{U}_2 = B_\varepsilon(0, 2\varepsilon) \cup B_\varepsilon(2\varepsilon, 0)$ and $\kappa_2(U) = C^2(\varepsilon^1, \varepsilon^1) \cap \tilde{U}_2$. If for $x \in \kappa_1(U)$ we have

$$
(\kappa_2 \circ \kappa_1^{-1})(x) = \begin{cases} 
  x & x^1 > 0 \\
  (-x^2, x^1) & x^1 < 0
\end{cases}
$$

\hspace{1cm} (A.2)

then the transition map $\kappa_2 \circ \kappa_1^{-1}: \kappa_1(U) \to \kappa_2(U)$ and its inverse are smooth. However, $C^2(\varepsilon^1)$ is a minimal quadrant for $(U, \kappa_1)$, while $C^2(\varepsilon^1, \varepsilon^1)$ is a minimal quadrant for $(U, \kappa_2)$. \hfill \Diamond

The following important, albeit technical, theorem provides general results on changing corner charts. It was inspired by Prop. 16.20 in Lee’s book \cite{Lee}. 

Theorem A.1
Let $(U_1, \kappa'_1)$ and $(U_2, \kappa'_2)$ be two corner charts with $U = U_1 \cap U_2 \neq \emptyset$. Restrict $\kappa'_1$ and $\kappa'_2$ in domain and codomain to obtain new corner charts $(U, \kappa_1)$ and $(U, \kappa_2)$, respectively. Set

$$
\kappa_1(U) = C^n(\varphi_1^1, \ldots, \varphi_{k^1}^1) \cap \tilde{U}_1 
$$

\hspace{1cm} (A.3a)

Figure 3: The gray shaded regions indicate the respective codomains of the maps $\kappa_1$ and $\kappa_2$ in Ex. A.1.

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\[ \kappa_2(U) = C^n(\varphi_2^1, \ldots, \varphi_2^k) \cap \tilde{U}_2 \]  

(A.3b)

with \( \tilde{U}_1, \tilde{U}_2 \) open in \( \mathbb{R}^n \), and let the respective quadrants be minimal.

Then the following holds:

i) Let \( i \in \{1, \ldots, k_1\} \), and let \( V'_{1,i} \) be a (path-)connected component of the 1-slice

\[ V_{1,i} = \left\{ x \in \kappa_1(U) \mid \varphi'_{1,i}(x) = 0 \text{ only for } i' = i \right\} \quad (A.3c) \]

of \( (U, \kappa_1) \). Choose \( x \in V'_{1,i} \) and set \( y = (\kappa_2 \circ \kappa_1^{-1})(x) \). Then there exists a \( j \in \{1, \ldots, k_2\} \) and a \( c \in \mathbb{R}_+ \) such that the linear functional

\[ \lambda: \mathbb{R}^n \to \mathbb{R}: w \mapsto \lambda(w) = \frac{d}{dt} \bigg|_0 \left( \varphi^i_{1} \circ \kappa_1 \circ \kappa_2^{-1} \right)(y + tw) , \quad (A.3d) \]

satisfies \( \lambda = c \varphi^j_2 \). Up to the factor \( c \), \( \lambda \) is independent of the choice of \( x \in V'_{1,i} \).

ii) For each \( j \in \{1, \ldots, k_2\} \) choose an arbitrary \( y_j \) in the 1-slice \( V_{2,j} \subset \kappa_2(U) \). Then for every such \( j \) there exists a unique index \( i_j \) such that \( (\kappa_1 \circ \kappa_2^{-1})(y_j) \in V_{1,i_j} \).

If we further define

\[ \lambda^j: \mathbb{R}^n \to \mathbb{R}: w \mapsto \lambda^j(w) = \frac{d}{dt} \bigg|_0 \left( \varphi^i_{1} \circ \kappa_1 \circ \kappa_2^{-1} \right)(y_j + tw) , \quad (A.3e) \]

then the quadrant \( C^n(\lambda^1, \ldots, \lambda^{k_2}) \) is minimal for \( (U, \kappa_2) \).

iii) If a \( p \in U \) is a corner point of index \( j \) with respect to \( (U_1, \kappa'_1) \), then it is a corner point of index \( j \) with respect to \( (U_2, \kappa'_2) \).

iv) For each connected component \( V'_{1,i} \) of a \( j \)-slice of \( (U, \kappa_1) \), there exists a unique connected component \( V'_{2,j} \) of a \( j \)-slice of \( (U, \kappa_2) \) such that

\[ (\kappa_2 \circ \kappa_1^{-1})(V'_{1,i}) = V'_{2,j} . \quad (A.3f) \]



Points i) and ii) establish a general relationship between the \( \varphi_1 \) and \( \varphi_2 \) functionals under change of coordinates. Point iii) means that one can speak of corner points and their index without referring to a specific chart. Point iv) is a general characterization of how a transition map maps the quadrant boundary.

We shall employ the following lemma to undergird the proof of the above theorem.
Lemma A.1
Consider the situation in Thm. A.1. Denote by $\partial$ the topological boundary operator in $\mathbb{R}^n$. Then the following hold:

\begin{enumerate}
\item The differential of the map $\alpha = \kappa_2 \circ \kappa_1^{-1}$ has full rank on $\kappa_1(U)$.
\item \[
\alpha(\partial^c\{\varphi_1^1, \ldots, \varphi_{k_1}^1\} \cap \tilde{U}_1) = \partial^c\{\varphi_2^1, \ldots, \varphi_{k_2}^2\} \cap \tilde{U}_2.
\]
\end{enumerate}

\textbf{Proof (of Lem. A.1)}
As it is customary for differential geometry in $\mathbb{R}^n$, we identify vectors in the tangent space $T_x \mathbb{R}^n$ of a point $x \in \mathbb{R}^n$ with vectors in $\mathbb{R}^n$ itself—and vice versa.

\begin{enumerate}
\item For $x$ in the interior of $\kappa_1(U)$ in $\tilde{U}_1$ this is trivial: Restrict $\alpha$ to this interior and recall that $\alpha$ is open so that the respective image is open in $\mathbb{R}^n$. As the restriction of $\alpha$ is bijective and smooth in both directions, it is a diffeomorphism between opens of $\mathbb{R}^n$ with $x$ contained in the domain.
\end{enumerate}

The case that $x$ is an element of $\partial(\kappa_1(U)) \cap \tilde{U}_1$ is therefore the one of interest.

Extend $\alpha$ and $\alpha^{-1}$ to smooth maps $\xi$ and $\zeta$ on open subsets $\tilde{U}_1'$ and $\tilde{U}_2'$ of $\mathbb{R}^n$, respectively. Then the following set is open in $\tilde{U}_1'$ and in $\mathbb{R}^n$ and contains $\kappa_1(U)$:

\[
\tilde{U}_2'' = (\tilde{U}_1 \cap \tilde{U}_1') \cap \zeta^{-1}(\tilde{U}_2').
\]

As $\xi(\tilde{U}_1'') \subseteq \tilde{U}_2'$, we restrict $\xi$ to $\tilde{U}_1''$ in domain and $\tilde{U}_2'$ in codomain, using the same letter for the new map hereafter. Then the composition $\zeta \circ \xi$ is well-defined and smooth.

Since $x$ is an element of $\partial(\kappa_1(U)) \cap \tilde{U}_1$, there exists an index set $I$ with $\#I = j$ such that $x$ is in the $j$-slice

\[
V_{1,I} = \{y \in \kappa_1(U) | \forall i \in I: \varphi_i^1(y) = 0 \text{ and } \#i \in I^c: \varphi_i^1(y) = 0\}.
\]

Furthermore, we may choose a $v \in \mathbb{R}^n$ with $\varphi_i^1(v) < 0$ for all $i \in I$ and define the curve

\[
\gamma: (t_0, 0] \to \kappa_1(U): t \mapsto \gamma(t) = tv + x
\]
for some $t_0 < 0$. By choosing a basis in $\mathbb{R}^n$ in which the $\varphi_1$s are standard covectors and $I = \{1, \ldots, j\}$, one shows that there always exist $n$ linearly independent such vectors $v$.

Observe now that $\zeta \circ \xi$ is the identity on $\kappa_1(U) \subseteq \tilde{U}_1''$ and that its derivatives are continuous on $\tilde{U}_1''$. We thus find that $\frac{\partial}{\partial t} \mid_{t=0} (\zeta \circ \xi) (x + tv) = v$ for all $v$ as above. As we may choose $n$ linearly independent $v$, we conclude that

$$((\zeta \circ \xi)_*)_x = (\xi_*)_x \circ (\zeta_*)_x = ((\alpha^{-1})_*)_x \circ (\alpha_*)_x$$

is the identity in $T_x \tilde{U}_1''$. So $(\alpha_*)_y$ has full rank, indeed.

ii) Again, for $x \in V_I$ choose $v \in \mathbb{R}^n$ such that $\varphi_i^1(v) < 0$ for all $i \in I$. Define $\gamma$ as in Eq. (A.5d) above. Then the curve $\alpha \circ \gamma$ is smooth.

Aiming for a contradiction, assume $\alpha(x) = (\alpha \circ \gamma)(0)$ does not lie in $\partial (\kappa_2(U)) \cap \tilde{U}_2$, i.e. the right hand side of Eq. (A.4). Then $\alpha(x)$ lies in the interior of $\kappa_2(U)$ in $\tilde{U}_2$. Moreover, by point i) and $v \neq 0$, the tangent vector of $\alpha \circ \gamma$ at 0 is nonzero. We can therefore extend $\alpha \circ \gamma$ via a straight line to a $C^1$-curve $\gamma': (t_0, t_1) \to \kappa_2(U)$ for some $t_1 > 0$. Yet then $\alpha^{-1} \circ \gamma'$ is a $C^1$-extension of $\gamma$ in $\kappa_1(U)$ in positive $t$-direction—which is impossible. 

We shall now return to the proof of Thm. A.1 above.

**Proof (of Thm. A.1)** We carry over the terminology of corner points from Def. A.1.i) to chart codomains by viewing the latter as manifolds with corners equipped with the global identity chart.

i) First consider the case $n = 1$. As $x = 0$ is the only possible choice, the statement is true, but vacuous.

So let $n > 1$ from hereon.

That $\lambda$ is a well-defined, linear functional follows from the chain rule on $\kappa_2(U)$:

For all $w \in \mathbb{R}^n$ we have

$$\lambda(w) = \varphi_1^1 \left( (\alpha^{-1})_* w \right).$$

As noted above, for each index $i$ the 1-slice $V_{1,i}$ is a nonempty subset of the plane $\ker \varphi_i^1$, relatively open with respect to the topology on $\mathbb{R}^n$, and contained

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$^{40}$Since $\gamma$ is an integral curve of the constant vector field $v$, existence of such a $t_0$ is a consequence of Prop. B.1.ii) below applied to the manifold with corners $U$ equipped with the identity chart.
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in $\kappa_1(U)$. Since $n > 1$, for any $x \in V_{1,i}$ and $v \in \ker \varphi^i_1$ there exists an open interval $I$ with the property that the curve $\gamma: t \mapsto \gamma(t) = x + tv$ lies in $V_{1,i}$ for all $t \in I$.

Now consider the curve $\alpha \circ \gamma$ in $\kappa_2(U)$. For every $v \in \ker \varphi^i_1$ the curve $\alpha \circ \gamma$ is tangent to the subspace $W = (\alpha)_x(\ker \varphi^i_1)$ of $T_{\alpha(x)}\mathcal{U}_2$ at $\alpha(x)$. Since $(\alpha)_x$ has full rank (cf. Lem. [A.1ii]), $W$ is $(n-1)$-dimensional. However, by Lem. [A.1ii] and the fact that $V_{1,i} \subseteq \partial \mathcal{C}^n(\varphi^1_1, \ldots, \varphi_{k_1}^1) \cap \mathcal{U}_1$, the curve $\alpha \circ \gamma$ lies in $\partial \mathcal{C}^n(\varphi^1_2, \ldots, \varphi_{k_2}^2) \cap \mathcal{U}_2$. Thus $W$ is tangent to $\partial \mathcal{C}^n(\varphi^1_2, \ldots, \varphi_{k_2}^2)$ in the sense that for some $j \in \{1, \ldots, k_2\}$ the space $W$ is a linear subspace of $\ker \varphi^j_2$. After comparing dimensions, we find $W = \ker \varphi^j_2$.

Recalling Eq. (A.6a) above, it follows

$$
\ker \lambda = \ker \left( \varphi^i_1 \circ (\alpha^{-1})_x \right) = (\alpha)_x \ker \varphi^i_1 = \ker \varphi^j_2. 
$$

(A.6b)

Since the kernel of a linear functional determines the functional itself uniquely up to a nonzero factor, different choices of $x$ in Eq. (A.3d) can only change this factor. Thus $\lambda = c \varphi^j_2$ for some nonzero $c \in \mathbb{R}$, indeed. Furthermore, $\varphi^j_2((\alpha)_x v) > 0$ whenever $\varphi^i_1(v) > 0$, hence $c > 0$.

It remains to show that $j$ is independent of the choice of $x \in V'_1$.

First we show that for every $x \in V'_1$ there exists a (unique) $j$ such that $\alpha(x) \in V_{2,j}$: Due to Lem. [A.1ii] there exists a nonempty $I$ such that $\alpha(x) \in V_{2,I}$ (cf. Eq. (A.1c)). Let $j$ be the index for which $(\alpha)_x(\ker \varphi^1_1) = \ker \varphi^j_2$, as shown above. Aiming for a contradiction, assume there exist $j' \in I$ with $j' \neq j$. Then take a $w \in \ker \varphi^j_2$ with $\varphi^{j'}_2(w) < 0$. As shown, $\alpha \circ \gamma$ for $v = ((\alpha)_x)^{-1}w$ is defined on an open interval and smooth. Yet any such curve with tangent vector $w$ at $x$ leaves $\mathcal{C}^n(\varphi^1_2, \ldots, \varphi_{k_2}^2)$—contradiction. Thus $I = \{j\}$.

To finish the proof, we observe that, since $V'_1$ is path-connected, so is $\alpha(V'_1)$. On the other hand, we have shown that

$$
\alpha(V'_1) \subseteq \bigcup_{j \in \{1, \ldots, k_2\}} V_{2,j}. 
$$

(A.6c)

The right hand side of Eq. (A.6c) is a topological $(n-1)$-manifold and its connected components are the connected components of each $V_{2,j}$. As connectedness is equivalent to path-connectedness for a topological manifold, there exists a single $j$ such that $\alpha(V'_1) \subseteq V_{2,j}$.
Appendix A

ii) For \( n = 1 \), the statement is again trivial—\( y = 0 \) and modulo a positive factor, there is only one such \( \lambda \) to choose from.

For \( n > 1 \), we first recall that in \( \text{[i]} \) we have shown that for every \( x \in V_{1,i} \) there exists a (unique) \( j \) such that \( \alpha(x) \in V_{2,j} \). An analogous statement thus holds in the reverse direction. The remaining statement follows from \( \text{[i]} \).

iii) As before, restrict \( \kappa'_1 \) and \( \kappa'_2 \) to \( \kappa_1 \) and \( \kappa_2 \), respectively.

The case \( j = 1 \) (with \( n > 0 \)) we have already shown in the proof of \( \text{[i]} \).

Now proceed with \( j = 2 \). For \( n = 2 \), \( \alpha(x) \) must indeed be 0, since, by Lem. A.1.ii), \( \alpha(x) \) has to be a corner point, and, by the prior result, it cannot have index 1.

So consider \( n > 2 \).

For \( j = 2 \) and \( x \in V_{1,I} \) with \( \#I = 2 \), we have \( I = \{i_1, i_2\} \) and \( \varphi_1^{i_1}(x) = \varphi_1^{i_2}(x) = 0 \). Again by Lem. A.1.ii), \( \alpha(x) \) has at least index 1. But \( \alpha(x) \) having index 1 is again impossible by the prior result. Therefore, \( \alpha(x) \) has at least index 2.

We now argue in analogy to the proof of \( \text{[i]} \) above: Since \( V_{1,I} \) is open in \( \ker \varphi_1^{i_1} \cap \ker \varphi_1^{i_2} \) and non-empty, for each \( v \in \ker \varphi_1^{i_1} \cap \ker \varphi_1^{i_2} \) there exists an open interval \( I \) such that the curve \( \gamma: t \mapsto \gamma(t) = x + tv \) with \( \text{dom} \gamma = I \) lies in \( V_{1,I} \).

Thus for each \( v \) the smooth curve \( \alpha \circ \gamma \) is tangent to the \((n-2)\)-dimensional subspace \( W = (\alpha_x)_x (\ker \varphi_1^{i_1} \cap \ker \varphi_1^{i_2}) \) of \( T_{\alpha(x)} \mathbb{R}^n \). By applying the previous argument to \((\alpha \circ \gamma)(t)\), we find that for all \( t \in I \) the point \((\alpha \circ \gamma)(t)\) has at least index 2. Therefore, there exist distinct \( j_1, j_2 \) such that \( W \) is a linear subspace of \( \ker \varphi_2^{j_1} \cap \ker \varphi_2^{j_2} \). Again comparing dimensions, \( W = \ker \varphi_2^{j_1} \cap \ker \varphi_2^{j_2} \).

Because \( \alpha(x) \) has at least index 2, there exists a \( J \) with \( \#J \geq 2 \) such that \( \alpha(x) \in V_{2,J} \). Moreover, \( j_1, j_2 \in J \). With the goal of producing a contradiction, assume there exists a third \( j_3 \in J \). Choose \( w \in W \) with \( \varphi_2^{j_3}(w) < 0 \). Again taking \( v = ((\alpha_x)_x)^{-1}w \) for the curve \( \gamma \), the smooth curve \( \alpha \circ \gamma \) must leave the boundary. Contradiction. Hence \( J = \{j_1, j_2\} \) and \( \alpha(x) \) has index 2.

To obtain the assertion for arbitrary \( j \), repeat the argument inductively.

iv) Since \( \alpha \) is continuous, the image \( \alpha(V'_1) \) is connected. Due to \( \text{[iii]} \), \( \alpha(V'_1) \) is contained in the union of all \( V_{2,J} \) with \( \#J = j \). But the \( V_{2,J} \) are mutually disconnected, so there exists a \( J \) and a connected component \( V'_{2,J} \) of \( V_{2,J} \) such that \( \alpha(V'_1) \subseteq V'_{2,J} \). Reversing the argument, we get \( \alpha^{-1}(V'_{2,J}) \subseteq V'_1 \). Thus \( \alpha(V'_1) \) and \( V'_{2,J} \) are one and the same set. ■
Appendix A

Having established local results, we now draw our attention to global ones. In this context, we refer the reader back to Ex. [IV] for a definition of submanifolds with corners.

**Theorem A.2 (Douady and Hérald [58])**
For every manifold with corners \( Q \) there exists a manifold \( \tilde{Q} \) ‘without corners’ and a map \( \iota \) such that \((Q, \iota)\) is a submanifold (with corners) of \( \tilde{Q} \).

See Prop. 3.1 in the French appendix of Ref. [58] for the original proof using \([0, \infty)^k \times \mathbb{R}^{n-k}\) as a model space. See §2.7 in Ref. [57] for a proof in English.

**Definition A.2**
Let \( Q \) be an manifold with corners of dimension \( n \in \mathbb{N} \).

i) The \( j \)-boundary \( \partial^j Q \) of \( Q \) (or equivalently, the boundary of index \( j \) in \( Q \)) is the set of corner points of index \( j \) in \( Q \).

ii) The (manifold) boundary of \( Q \) is

\[
\partial Q = \bigcup_{j \in \{1, \ldots, n\}} \partial^j Q. \tag{A.7}
\]

iii) The (manifold) interior of \( Q \) is \( \dot{Q} = Q \setminus \partial Q \). A point \( q \in \dot{Q} \) is called an interior point of \( Q \).

Thm. A.1[iii] assures that the \( j \)-boundary \( \partial^j Q \) in Def. A.2[i] is well-defined.

**Proposition A.1 (Michor [57])**
The \( j \)-boundary \( \partial^j Q \) of an \( n \)-manifold with corners \( Q \) is an \((n-j)\)-dimensional, embedded submanifold with corners of \( Q \) with empty boundary.

A detailed proof seems to be missing in the literature and is thus given below.

**Proof** Since \( \partial^j Q \) carries the subspace topology, it is a second-countable, Hausdorff topological space, topologically embedded in \( Q \). If \( A = \{(U_\gamma, \kappa_\gamma) | \gamma \in I\} \) is an atlas on \( Q \), then we can construct an atlas on \( \partial^j(Q) \) as follows: Consider \( I' \subseteq I \) such that for all \( \gamma \in I' \) we have \( \partial^j Q \cap U_\gamma \neq \emptyset \). Define \( U_{\gamma,i} \) to be the \( i \)th connected component of \( \partial^j Q \cap U_\gamma \), denoting the set of such \( i \) as \( I'_\gamma \subseteq \mathbb{N} \). Choose a minimal quadrant \( C^n(\varphi_1^1, \ldots, \varphi_1^n) \) for \( (U_\gamma, \kappa_\gamma) \), and complete the respective functionals to a basis \( \{\varphi_1^1, \ldots, \varphi_1^n\} \) of \( (\mathbb{R}^n)^* \). For \( l \in \{1, \ldots, n\} \) and each component \( i \) define the functions

\[
\kappa_{\gamma,i}^l = \varphi_1^l \circ \kappa_\gamma|_{U_{\gamma,i}}. \tag{A.8}
\]
Appendix B

We define a coordinate map $\kappa_{\gamma,i}$ by gathering only those $\kappa_{\gamma,i}^l$ that are nonzero. For given $\gamma$ and $i$, there are precisely $(n-j)$ such $l$s. We obtain homeomorphisms $\kappa_{\gamma,i}$ from $U_{\gamma,i}$ to their image in $\kappa_{\gamma}(U)$. Set $A' = \{(U_{\gamma,i}, \kappa_{\gamma,i})| \gamma \in I', i \in I'\}$.

Smoothness of the transition functions on $\partial^j Q$ is trivial: Consider the components of the transition functions on $Q$ with respect to the $e_{\gamma,i}$s, and then recall Thm. A.1.iv). Finally, $\partial(\partial^j Q) = \emptyset$ by definition of $\partial^j Q$. ■

Note again that, in general, the boundary $\partial Q$ of a manifold with corners $Q$ is not a manifold with corners.

**Proposition A.2**

Let $Q$ be a manifold with corners of dimension $n \in \mathbb{N}$.

i) The boundary $\partial Q$ is closed and has measure zero in $Q$.

ii) The interior $\hat{Q}$ is an open submanifold of $Q$. ◊

**Proof**

i) Let $A = \{(U_{\gamma}, \kappa_{\gamma}|\gamma \in I)\}$ be an atlas for $Q$. Then for each $\gamma$, the set $\kappa_{\gamma}(U_{\gamma} \cap \partial Q)$ has measure zero in $\kappa_{\gamma}(U_{\gamma})$. Thus, by definition, $\partial Q$ has measure zero in $Q$. Define $U'_{\gamma} = U_{\gamma} \setminus \partial Q$. $\kappa_{\gamma}(U'_{\gamma})$ is open in $\kappa_{\gamma}(U_{\gamma})$, hence $U'_{\gamma}$ is open in $Q$. Taking the union over $\gamma \in I$, $\hat{Q}$ is open in $Q$. Thus its complement $\partial Q$ is closed.

ii) As shown in i), $\hat{Q}$ is open in $Q$, so we only need to show that it is a manifold. Arguing as in Prop. A.1 above, $\hat{Q}$ is second-countable and Hausdorff. An atlas is obtained from an atlas $A$ as above, by restricting $\kappa_{\gamma}$ to $U'_{\gamma}$ in domain and to its respective image. Smoothness of the transition mappings is trivial. ■

Appendix B: Integral curves and flows on manifolds with corners

Although many differential-geometric constructions and results relating to manifolds easily carry over to manifolds with corners, there are some instances where the existence of ‘corner points’ complicates matters significantly. An example thereof is the theory of vector fields and flows on manifolds with corners.

The purpose of this appendix is to show some elementary results therein and to provide the mathematical reader with insight into the kind of ‘pathologies’ that can occur, if one tries to generalize flows to ‘spaces with boundaries’ and one does not
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put any additional restrictions on the vector fields involved (see references in Rem. 1 below). Those ‘pathologies’ are likely to occur in more general such spaces, so that their study in this setting may contribute to their understanding in a more general one.

We begin our discussion by formally defining the tangent space $T_q Q$ at a point $q$ of a manifold with corners $Q$ as the vector space of derivations at $q$—following the analogue theory for manifolds. Then tangent vectors are elements of $T_q Q$. Due to the continuity of partial derivatives in the respective corner charts, derivations at $q$ are well-defined even if $q$ is a corner point. As for manifolds, the tangent bundle is taken to be the disjoint union of all tangent spaces. It is canonically a manifold with corners (cf. Ex. [1][iv]).

We shall classify tangent vectors at corner points in a way that is convenient for our subsequent study of integral curves of vector fields. As in the proof of Thm. [A.1] we employ the canonical identification between tangent vectors in $\mathbb{R}^n$ and vectors in $\mathbb{R}^n$ itself here.

**Definition B.1**

Let $Q$ be a smooth $n$-manifold with corners with $n \in \mathbb{N}$, and let $q$ be a corner point of index $j$. Further, let $(U, \kappa)$ be a corner chart around $q$, let $C^n(\varphi^1, \ldots, \varphi^k)$ be a minimal quadrant for $(U, \kappa)$, and assume that $\kappa(q)$ is contained in the $j$-slice $V_I$ (cf. Def. [A.1][iii]).

A tangent vector $X$ at $q$ is called

i) *tangent to* $\partial Q$, if the coordinate representative of $X$ is tangent to $V_I$,

ii) *inwards-pointing*, if $\varphi^i(X) > 0$ for all $i \in I$,

iii) *outwards-pointing*, if $X$ is neither tangent to $\partial Q$ nor inwards-pointing.

One uses Thm. [A.1] to show that the above definitions are independent of the particular choice of corner chart.

Again following the analogue theory for manifolds, a vector field $X$ on a manifold with corners $Q$ is defined to be a smooth map $X: Q \to TQ: q \mapsto X_q$ such that $X_q$ is in the fiber over $q$.

**Remark 4**

The ‘pathologies’ of flows exhibited here largely follow from considering vector fields $X$ whose vectors $X_q$ at a corner point $q \in Q$ may be outwards-pointing in the sense of Def. [B.1]. This is the reason why additional assumptions are usually placed on vector fields on manifolds with boundary/manifolds with corners in the literature (cf. Sec. 4 in Appx. of Ref. [58], p. 222 sqq. in Ref. [17], and Sec. 2.6 in Ref. [57]).
We give two mathematical motivations for also allowing outwards-pointing $X_q$:

First, such general vector fields naturally arise as the restriction of a vector field $X'$ in an ‘ambient manifold’ $Q' \supset Q$ to $Q$ for the case that $X'$ is tangent to $Q$. One may thus wish to consider the restriction $X$ to $Q$ and its flow on $Q$ without having to refer to $Q'$ (or make use of pullback bundles).

Second, if one defines the tangent bundle of a manifold with corners as we did here – and as it is common in the literature – then allowing only a restricted class of sections thereof may be viewed as ‘mathematically unnatural’. Of course, one may take the alternative view that the tangent space $T_q Q$ should be an ‘infinitesimal approximation’ to the manifold $Q$ with corners also at a corner point $q$, in which case one would conclude that only nonoutwards-pointing vectors $X_q$ ought to be allowed—thus removing the ‘unnaturalness’. Yet that would imply that $T_q Q$ is not a vector space any more. Thus the tangent bundle $T Q$ would not be a ‘vector bundle’ in any meaningful sense of the word, which would in turn lead to problems regarding addition of vector fields and covector fields.

As opposed to their analogues on manifolds, maximal integral curves of vector fields on manifolds with corners can have a variety of different domains. We shall first give a rigorous definition and then a more detailed discussion.

**Definition B.2**

Let $Q$ be a smooth manifold with corners of dimension at least 1. Let $X$ be a smooth vector field on $Q$.

For $q \in Q$, an integral curve $\gamma$ of $X$ at $q$ is a curve $\gamma$ in $Q$, defined on an (open, half-open, or closed) interval $I$, satisfying the integral curve equation

$$\forall t \in I: \quad \dot{\gamma}_t = X_{\gamma(t)}$$

with initial condition $\gamma(0) = q$. The integral curve $\gamma$ is called a maximal, if there does not exist an integral curve $\gamma': I' \to Q$ of $X$ at $q$ such that $I \subset I'$.

Clearly, $X$ can be restricted to a vector field $\check{X}$ on the interior $\check{Q}$. So for $q \in \check{Q}$, the respective maximal integral curves $\gamma$ of $X$ and $\check{\gamma}$ of $\check{X}$ at $q$ coincide on a connected open interval around 0. Beyond this interval, the behavior of $\gamma$ depends on the values of $X$ on the boundary $\partial Q$. To obtain a general description of possible integral curves on $Q$, it is therefore necessary to study their behavior near the boundary $\partial Q$.

**Proposition B.1**

Let $X$ be a vector field on a manifold with corners $Q$ of dimension at least 1, and let $q$ be corner point.
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i) If $X_q$ is inwards-pointing, then there exists a unique maximal integral curve $\gamma$ at $q$ with domain $[0, t_f)$ or $[0, t_f]$ for some $t_f > 0$, or $[0, \infty)$.

ii) If $-X_q$ is inwards-pointing, then there exists a unique maximal integral curve $\gamma$ at $q$ with domain $(t_i, 0]$ or $[t_i, 0]$ for some $t_i < 0$, or $(-\infty, 0]$.

iii) If both $X_q$ and $-X_q$ are outwards-pointing, then no integral curve exists. ♦

Proof

i) As in Def. B.1 above, let $(U, \kappa)$ be a corner chart around $q$ and let $q$ be of index $j$ with $\kappa(q) \in V_I$. Extend the local representative of $X$ to a smooth vector field $\tilde{X}$ over the open set $\tilde{U}$ in $\mathbb{R}^n$. Let $\tilde{\gamma}$ be the integral curve of $\tilde{X}$ starting at $\kappa(q)$.

The mappings $\varphi^i$, considered as linear functional fields over $\tilde{U}$, are continuous. Thus $\varphi^i(\tilde{X})$ is a continuous map from $\tilde{U}$ to $\mathbb{R}$. Set

$$W = \bigcap_{i \in \{1, \ldots, j\}} \left( \varphi^i(\tilde{X})^{-1}((0, \infty)) \right) \cap \left\{ x \in \mathbb{R}^n | \forall l \in I^c: \varphi^l(x) > 0 \right\}. \quad (B.2)$$

$W$ is open in $\tilde{U}$. By assumption, $\kappa(q)$ lies in $W$, so $W \neq \emptyset$. The set $\tilde{\gamma}^{-1}(W)$ contains an open interval $\tilde{I}$ with $0 \in \tilde{I}$.

Due to the integral curve equation, the $i$th components $\tilde{\gamma}^i = \varphi^i \circ \tilde{\gamma}$ are strictly increasing in $\tilde{I} \subseteq \tilde{\gamma}^{-1}(W)$. Since $\kappa(q) \in V_I$, $\tilde{\gamma}^i(0) = 0$ for all $i \in I$. Thus, $\tilde{I} = \tilde{I} \cap [0, \infty)$ is a half-open interval with $\tilde{\gamma}(t) \in \kappa(U)$ for all $t \in \tilde{I}$ and $\tilde{\gamma}(t) \notin \kappa(U)$ for negative $t \in \tilde{I} \setminus \tilde{I} \neq \emptyset$. Restricting $\tilde{\gamma}$ to $\tilde{I}$, we obtain an integral curve of $X$ in the corner chart that is inextendible to negative $t$.

To complete the proof, we require the maximal integral curve $\gamma$ of $X$ at $q$: There is at least one such curve, since $\gamma$ coincides with $\kappa^{-1} \circ \tilde{\gamma}$ over $\tilde{I}$. Uniqueness is shown in analogy to the proof of Thm. 9.12.a in Ref. [17]—which works for general intervals, not just open ones.

Observe now that $\gamma$ is inextendible to strictly negative $t$, since this is the case for $\tilde{\gamma}$. Thus the above intervals are the only possible ones.

ii) Apply [1] to $-X$ and invert $\gamma$ at $t = 0$.

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41 One may construct examples to show that each case can indeed be realized.
iii) Consider $\tilde{X}$ with integral curve $\tilde{\gamma}$, as in (i). Obviously, this situation can only occur for $\dim Q$ and $j$ greater than 1. By an argument similar to the one in (i) applied to two different $i, i' \in I$, one shows that in a sufficiently small open neighborhood of 0 in $\text{dom} \tilde{\gamma}$, we have $\tilde{\gamma}(t) \in \kappa(U)$ only for $t = 0$. As, by Def. B.2, the set \{0\} is not an admissible domain, no integral curve exists. ■

If $X_q$ is tangent to $\partial Q$, statements about the possible maximal integral curve domains are vacuous. As can be proven by an explicit construction of examples, either no integral curve exists or a maximal one exists on an open, half-open, or closed interval.

Regarding the notion of smoothness for integral curves $\gamma$ of $X$, observe that the set $I = \text{dom} \gamma$ in Def. B.2 is also a manifold with corners. Recalling the definition of smoothness between manifolds (cf. Chap. 2 in Ref. \cite{17} and Sec. 1.3 in Ref. \cite{18}), we naturally define a map between two manifolds with corners to be smooth if and only if it is continuous and each of its coordinate representatives is smooth (cf. Ex. 1.v)).

Lemma B.1
Integral curves of smooth vector fields on manifolds with corners are smooth. ♦

**Proof** Given $t_0 \in I = \text{dom} \gamma$, take a corner chart $(U, \kappa)$ around $\gamma(t_0)$. Extend the local representative of $X$ to a smooth vector field $\tilde{X}$ on an open subset $\tilde{U}$ in $\mathbb{R}^n$ covering $\kappa(U)$. Define $\tilde{\gamma}'$ by taking the integral curve $\tilde{\gamma}: \tilde{I} \rightarrow \tilde{U}$ of $\tilde{X}$ at $\kappa(\gamma(t_0))$ and setting $\tilde{\gamma}'(t) = \tilde{\gamma}(t - t_0)$. In the neighborhood $\tilde{I}$ of $t_0$ in $I$ the curve $\tilde{\gamma}'$ is a smooth extension of the restriction of $\kappa \circ \gamma$ to $I \cap \tilde{I}$. Continuity of $\gamma$ at $t_0$ then follows from continuity of $\kappa^{-1}$ and $\tilde{\gamma}'$:

$$\lim_{t \rightarrow t_0} \gamma(t) = \lim_{t \rightarrow t_0} \left( \kappa^{-1} \circ \tilde{\gamma}' \right)(t) = \gamma(t_0).$$

(B.3)

In order to move on to our discussion of flows on manifolds with corners, we need to first establish a mathematically sensible definition.

Definition B.3
Let $Q$ be a smooth manifold with corners, let $X$ be a smooth vector field on $Q$. If there exists an integral curve at $q \in Q$, denote by

$$\Phi(q): \text{dom} (\Phi(q)) \rightarrow Q: t \mapsto \Phi_t(q)$$

(B.4a)

\[\text{42Due to the ‘Gluing Lemma’ (cf. Cor. 2.8 in Ref. \cite{17}), this is sufficient for smoothness in the sense of Def. } \text{1.16} \]
the maximal integral curve at \( q \).

Then the *maximal flow of \( X \)* is the map

\[
\Phi : \text{dom} \Phi \to \mathbb{Q} : (t, q) \mapsto \Phi_t(q)
\]

(B.4b)

with domain \( \text{dom} \Phi \subseteq \mathbb{R} \times \mathbb{Q} \).

One problem one faces in defining (maximal) flows on manifolds with corners is that there exist points \( q \in \mathbb{Q} \) for which no integral curves exist. The above definition simply excludes such \( q \) from the domain.

Though Lem. B.1 implies that any maximal flow \( (t, q) \mapsto \Phi_t(q) \) (of a smooth vector field) on \( \mathbb{Q} \) is smooth in the \( t \) variable and one expects this to be the case for the \('q\) variable' as well, the fact that its domain \( \text{dom} \Phi \) is in general not \( \mathbb{R} \times \mathbb{Q} \) means there is no directly available notion of smoothness. One might expect \( \text{dom} \Phi \) to be a manifold with corners. If that were the case, the above notion of smoothness could be employed. Yet the following example shows that \( \text{dom} \Phi \) is generically not a manifold with corners.

**Example B.1**

Define \( \mathbb{Q} \) as

\[
\mathbb{Q} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1\},
\]

and define one corner chart on \( U_0 = \mathbb{Q} \) using the identity on \( \mathbb{R}^2 \). Two more corner charts are obtained from the equation

\[
(x, y) = ((1 + \rho) \cos \phi, (1 + \rho) \sin \phi),
\]

for \( \rho \geq 0 \) and \( \phi \) in \((0, 2\pi)\) and \((-\pi, \pi)\), respectively. Equipped with those three charts, \( \mathbb{Q} \) is a manifold with boundary and thus a manifold with corners.

Consider now the flow \( \Phi \) of \( \partial / \partial x \) on \( \mathbb{Q} \):

\[
\Phi_t(x, y) = (t + x, y).
\]

(B.5c)

For \( |y| \geq 1 \), \( \Phi \) is always defined. For \( |y| < 1 \) and \( x < 0 \), we have \( t \leq -x - \sqrt{1 - y^2} \). Similarly, for \( |y| < 1 \) and \( x > 0 \), we have \( t \geq -x + \sqrt{1 - y^2} \).

It is worth looking at the boundary of \( \text{dom} \Phi \) in \( \mathbb{R} \times \mathbb{Q} \) in coordinates \((t, \rho, \phi)\): Restricting ourselves to the set \( \mathbb{R} \times [0, \infty) \times (0, \pi) \) in the chart codomain and after some algebra and trigonometry, we may express the \( \rho \) coordinate of the boundary in terms of \((t, \phi)\). The graph of this function \((t, \phi) \mapsto \rho(t, \phi)\) is depicted in Fig. 4. On an algebraic level, we define the sets

\[
V_1 = ([0, \infty) \times (0, \pi/2)) \cup ((\infty, 0] \times (\pi/2, \pi))
\]

(B.5d)
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\[ V_2 = \{ (t, \phi) \in \mathbb{R} \times (0, \pi) \mid \text{either } \phi \in (0, \pi/2) \text{ and } t \in [-\cot \phi, 0), \text{ or } \phi \in (\pi/2, \pi) \text{ and } t \in (0, -\cot \phi) \} \]  

and

\[ V_3 = \{ (t, \phi) \in \mathbb{R} \times (0, \pi) \mid \text{either } \phi \in (0, \pi/2) \text{ and } t < -\cot \phi, \text{ or } \phi \in (\pi/2, \pi) \text{ and } t > -\cot \phi \} \]

Figure 4: This graphic shows a part of the boundary of the flow domain in Ex. B.1 around the point \((0, 1, 0) \in \text{dom } \Phi \) in coordinates \((t, \rho, \phi)\). Here the boundary can be expressed in terms of the graph of the function \(\rho: (t, \phi) \mapsto \rho(t, \phi)\) (cf. Eq. (B.5g)). The function \(\rho\) is smooth on the interior of the subsets \(V_1, V_2, \) and \(V_3\) of \(\text{dom } \rho = \mathbb{R} \times (0, \pi)\) (cf. Eqs. (B.5d) to (B.5f)), yet fails to be smooth at their boundaries in \(\text{dom } \rho\) (parts thereof shown in white). At the point \((0, \pi/2)\), which corresponds to the point \((0, 1, 0)\) on \(\mathbb{R} \times Q\), those boundaries intersect. As there are six smooth lines meeting at a point on which \(\rho\) is not smooth and only three are allowed on the boundary of a smooth manifold with corners, \(\text{dom } \Phi\) cannot be a manifold with corners.
so that we may write
\[
\rho(t, \phi) = \begin{cases} 
0, & (t, \phi) \in V_1 \\
-t \cos \phi - 1 + \sqrt{1 - t^2 \sin^2 \phi}, & (t, \phi) \in V_2 \\
-1 + \sqrt{1 + \cot^2 \phi}, & (t, \phi) \in V_3
\end{cases}
\] (B.5g)

The function \( \rho \) is smooth everywhere, except on the lines \( t = 0, \phi = \pi/2 \), and \( t = -\cot \phi \). At \( t = 0 \) the differential of \( \rho \) is discontinuous. At \( \phi = \pi/2 \) as well as at \( t = -\cot \phi \) the differential of \( \rho \) is continuous, yet its Hessian is not. Since all of these lines intersect at \( (0, \pi/2) \) – which corresponds to the point \( (0, 1, 0) \) in \( \text{dom } \Phi \) – \( \text{dom } \Phi \) is not (canonically) a smooth manifold with corners.

The situation is similar for the point \( (0, -1, 0) \) in \( \text{dom } \Phi \).

Example B.2

Consider the plane \( \mathbb{R}^2 \) and let \( \mathcal{Q} \) be the subset obtained by excluding the interior of the discs at \( (\pm 1, \pm 1) \) of radius 1. As in Ex. B.1 above, we construct a chart on \( U_0 = \mathring{\mathcal{Q}} \) using the identity on \( \mathbb{R}^2 \). Analogously, four more corner charts on open subsets of \( \mathcal{Q} \) are obtained by setting
\[
(x \mp 1, y \mp 1) = ((1 + \rho) \cos \phi, (1 + \rho) \sin \phi).
\] (B.6)

We again obtain a manifold with boundary and thus a manifold with corners.

As in Ex. B.1 above, we look at the flow \( \Phi \) of \( X = \partial/\partial x \) on \( \mathcal{Q} \) with values given by Eq. (B.5c) above. Fig. 5 depicts the respective streamline plot.

We observe that for any fixed \( x < -1 \) and \( y = 0 \) the integral curve \( t \mapsto \Phi_t(x, y) \) is defined on \( \mathbb{R} \), yet for any other \( y \in (-1, 1) \) the curve terminates at some \( t > 0 \). Thus, even if one were to extend the definition of smoothness to domains of flows that are not manifolds with corners, it is not possible to define, for instance, the derivative \( (\partial \Phi_t/\partial y)(-3, 0) \) for some \( t > 4 \), despite the fact that \( (t, -3, 0) \) is contained in \( \text{dom } \Phi \). While one could define the derivative in terms of the flow of a smooth extension of \( X \) to \( \mathbb{R}^2 \), there are infinitely many such extensions and the value of the derivative depends on that choice. Thus, there cannot be any sensible notion of smoothness on the entirety of \( \text{dom } \Phi \).

Summing up, (maximal) flows of general vector fields on manifolds with corners, as considered here, are ill-behaved in three respects: First, an integral curve may
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not exist at every point (cf. Prop. B.1). Second, the maximal domain of a flow on a manifold with corners is in general not a manifold with corners (cf. Ex. B.1), which in turn implies that the canonical notion of smoothness in this setting is not sufficient. Third, even on manifolds with boundary there may exist points in the maximal domain of a flow at which its differential cannot be defined in any sensible manner (cf. Ex. B.2).

As long as one does not restrict the behavior of vector fields at the boundary (cf. Rem. 4), the first problem cannot be alleviated, even if one were to consider generalizations of manifolds with corners. The second two problems can in principle be dealt with in this manner, provided one also restricts the flow domain appropriately. Such a treatment is, however, beyond the scope of this article.

Figure 5: This graphic shows a streamline plot of the vector field $\partial/\partial x$ on a part of the manifold with corners $Q$ in Ex. B.2. If one takes, for instance, the starting point $(x, y) = (-3, 0)$, then the respective integral curve is defined on the entirety of $\mathbb{R}$. Yet if one chooses any other $y$ with $|y| < 1$, then the integral curves terminate at some finite $t > 0$. Thus, for sufficiently large $t$ the derivative $(\partial\Phi_t/\partial y)(-3, 0)$ cannot be defined in any sensible manner—despite the fact that $(t, -3, 0)$ is in the domain of $\Phi$.
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Statements and Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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