Abstract

Siegel’s action is generalized to the $D = 2(p + 1)$ ($p$ even) dimensional space-time. The investigation of self-duality of chiral $p$-forms is extended to the momentum frame, using Siegel’s action of chiral bosons in two space-time dimensions and its generalization in higher dimensions as examples. The whole procedure of the investigation is realized in the momentum space which relates to the configuration space through the Fourier transformation of fields. These actions correspond to non-local Lagrangians in the momentum frame. The self-duality of them with respect to dualization of chiral fields is uncovered. The relationship between two self-dual tensors in momentum space, whose similar form appears in configuration space, plays an important role in the calculation, that is, its application realizes solving algebraically an integral equation.

PACS number(s): 11.10.Kk
1 Introduction

Many models of chiral bosons and/or their generalizations to higher (than two) space-time dimensions, i.e., chiral $p$-forms have been proposed [1-12]. Among them, some [1-7] are non-manifestly space-time covariant, while the others [8-12] manifestly space-time covariant. Moreover, these chiral $p$-form models have close relationship among one another, especially various dualities that have been demonstrated [12-14] in detail. Incidentally, the self-duality that exists beyond chiral $p$-form actions has also been uncovered [15].

As the investigations of duality symmetries mentioned above are limited only in configuration space, it may be interesting to extend these investigations to the momentum space that relates to the configuration space through the Fourier transformation of fields. It is quite natural to have this idea because the Fourier transformation plays an important role in field theory. Perhaps motivated similarly, the duality in harmonic oscillators obtained by Fourier decomposition was discussed [16] by considering several simple models, such as the free scalar, Maxwell and Kalb-Ramond theories, as examples. However, it is quite unsatisfactory to leave a variety of attractive chiral $p$-form models unnoticed. In this note we re-investigate in momentum space the duality symmetries of various chiral $p$-forms that exist in the configuration space, and we do find something non-trivial. The non-triviality we mention here means duality investigation of non-local Lagrangians and algebraic solution of integral equations.

We choose Siegel’s action as our example. To this end, we have to generalize the action to the $D = 2(p + 1)$ dimensional space-time. The main reason to make this choice is that after one makes the Fourier transformation to fields its formulation is non-trivial because of its cubic Lagrange-multiplier term as shown below. Starting from this formulation, the whole procedure of investigation is realized in the momentum space. As a result, the self-duality of Siegel’s action with respect to dualization of chiral fields is uncovered in the momentum frame. In the next section we deal with the special case in $D = 2$ space-time dimensions, and in section 3 we turn to the general case in $D = 2(p + 1)$ dimensions. Finally, section 4 is devoted to a conclusion.

\footnote{Note that the space-time is twice odd dimensional, i.e., $p$ is even throughout this paper.}
The metric notation we use throughout this note is

\[
\begin{align*}
g_{00} = -g_{11} = \cdots = -g_{D-1,D-1} = 1, \\
e^{012\cdots D-1} = 1.
\end{align*}
\]  

(1)

Greek letters stand for indices \((\mu, \nu, \sigma, \cdots = 0, 1, 2, \cdots, D-1)\) in both the configuration and momentum spaces.

## 2 Self-duality of the chiral 0-form action in \(D=2\) momentum frame

We begin with Siegel’s action \([8]\) in \(D=2\) space-time dimensions,

\[
S_c = \int d^2x \left\{ -\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{1}{2} \lambda_{\mu\nu}(x) [\partial^\mu \phi(x) - \epsilon^{\mu\sigma} \partial_\sigma \phi(x)] [\partial^\nu \phi(x) - \epsilon^{\nu\rho} \partial_\rho \phi(x)] \right\},
\]

(2)

where \(\phi(x)\) is a scalar field, and \(\lambda_{\mu\nu}(x)\) a symmetric auxiliary second-rank tensor field.

Substituting the Fourier transformations of \(\phi(x)\) and \(\lambda_{\mu\nu}(x)\),

\[
\begin{align*}
\phi(x) &= \frac{1}{2\pi} \int d^2k e^{-ik\cdot x} \phi(k), \\
\lambda_{\mu\nu}(x) &= \frac{1}{2\pi} \int d^2k e^{-ik\cdot x} \lambda_{\mu\nu}(k),
\end{align*}
\]

(3)

into eq.(2), one arrives at the Siegel action in the momentum frame spanned by \(k_\mu\),

\[
S_m = -\frac{1}{2} \int d^2k (-ik_\mu)(ik^\mu) \phi(k)\phi(-k)
+ \frac{1}{4\pi} \int d^2kd^2k' \lambda_{\mu\nu}(-k-k')(ik^\mu - \epsilon^{\mu\sigma} i k_\sigma)(ik'^\nu - \epsilon^{\nu\rho} i k'_\rho) \phi(k)\phi(k').
\]

(4)

Note that \(S_m\) contains quartic integrals over the momentum space because of the cubic Lagrange-multiplier term in \(S_c\), which shows that \(S_m\) is non-trivial. This non-triviality can also be understood as the non-locality of the corresponding Lagrangian. Therefore, we have to envisage the duality investigation of non-local Lagrangians.

We investigate the duality property of \(S_m\) with respect to dualization of the field \(\phi(k)\).

By introducing two independent vector fields in the momentum space, \(F_\mu(k)\) and \(G_\mu(k)\), we construct a new action to replace \(S_m\),

\[
S'_m = -\frac{1}{2} \int d^2k F_\mu(k) F^\mu(-k)
\]

3
\[
+ \frac{1}{4\pi} \int d^2kd^2k'\lambda_{\mu\nu}(-k-k') [F^\mu(k) - \epsilon^{\mu\sigma} F_\sigma(k)] [F^\nu(k') - \epsilon^{\nu\rho} F_\rho(k')]
+ \int d^2kG^\mu(-k) [F_\mu(k) + ik_\mu\phi(k)],
\]

(5)

where the third term is nothing but the \(k\)-space formulation of \(\int d^2 x G^\mu(x) [F_\mu(x) - \partial_\mu\phi(x)]\).

Variation of \(S'_m\) with respect to \(G^\mu(-k)\) gives

\[
F_\mu(k) = -ik_\mu\phi(k),
\]

(6)

which, when substituted into \(S'_m\), yields the classical equivalence between the two actions, \(S_m\) and \(S'_m\). Furthermore, variation of \(S'_m\) with respect to \(F^\mu(k)\) leads to the expression of \(G^\mu(k)\) in terms of \(F^\mu(k)\),

\[
G^\mu(k) = F^\mu(k) - \frac{1}{2\pi} \int d^2k'(g^{\mu\nu} + \epsilon^{\mu\nu})\lambda_{\nu\sigma}(-k-k') [F^\sigma(k') - \epsilon^{\nu\rho} F_\rho(k')].
\]

(7)

Note that eq.(7) is an integral equation other than an algebraic one that happens in configuration space, which is induced by the non-locality of the corresponding Lagrangian of eq.(4). At first sight it seems difficult to solve \(F^\mu(k)\) from eq.(7). In fact, one can deal with this problem in terms of an algebraic method shown in the following. In order to avoid solving this integral equation, one defines two self-dual tensors as

\[
\mathcal{F}^\mu(k) \equiv F^\mu(k) - \epsilon^{\mu\nu} F_\nu(k),
\]

\[
\mathcal{G}^\mu(k) \equiv G^\mu(k) - \epsilon^{\mu\nu} G_\nu(k),
\]

(8)

and establishes their relationship by using eq.(7),

\[
\mathcal{F}^\mu(k) = \mathcal{G}^\mu(k).
\]

(9)

We should emphasize that a similar relationship exists in configuration space for various chiral \(p\)-form actions as pointed out in Ref.[14]. With the aid of eq.(9), one can easily obtain algebraically from eq.(7) \(F^\mu(k)\) expressed in terms of \(G^\mu(k)\),

\[
F^\mu(k) = G^\mu(k) + \frac{1}{2\pi} \int d^2k'(g^{\mu\nu} + \epsilon^{\mu\nu})\lambda_{\nu\sigma}(-k-k') [G^\sigma(k') - \epsilon^{\nu\rho} G_\rho(k')].
\]

(10)

We can check from eq.(7) that when the self-duality condition is satisfied, i.e., \(\mathcal{F}^\mu(k) = 0\), which is also called an “on mass shell” condition, \(F^\mu(k)\) and \(G^\mu(k)\) relate with a duality,
\( G^\mu(k) = \epsilon^{\mu\nu} F^\nu(k) \). Substituting eq.(10) into eq.(5), we obtain the dual action of \( S_m \),

\[
S_{m}^{\text{dual}} = \frac{1}{2} \int d^2 k G_\mu(k) G^\mu(-k) + \frac{1}{4\pi} \int d^2 k d^2 k' \lambda_{\mu\nu}(k - k') [G^\mu(k) - \epsilon^{\mu\sigma} G_\sigma(k)] [G^\nu(k') - \epsilon^{\nu\rho} G_\rho(k')] + \int d^2 k \phi(k) [i k_\mu G^\mu(-k)]. \tag{11}
\]

Variation of eq.(11) with respect to \( \phi(k) \) gives \( i k_\mu G^\mu(-k) = 0 \) or \( -i k_\mu G^\mu(k) = 0 \), whose solution has to be

\[
G^\mu(k) = \epsilon^{\mu\nu} (-ik_\nu) \psi(k) \equiv \epsilon^{\mu\nu} F^\nu[\psi(k)], \tag{12}
\]

where \( \psi(k) \) is an arbitrary scalar field in momentum space. Substituting eq.(12) into eq.(11), one finally obtains the dual action in terms of \( \psi(k) \),

\[
S_{m}^{\text{dual}} = -\frac{1}{2} \int d^2 k (-i k_\mu) (ik^\mu) \psi(k) \psi(-k) + \frac{1}{4\pi} \int d^2 k d^2 k' \lambda_{\mu\nu}(k - k') (ik^\mu - \epsilon^{\mu\sigma} ik_\sigma) (ik'^\nu - \epsilon^{\nu\rho} ik'_\rho) \psi(k) \psi(k'). \tag{13}
\]

This action has the same form as the original one, eq.(4), only with the replacement of \( \phi(k) \) by \( \psi(k) \). As analysed above, \( \phi(k) \) and \( \psi(k) \) coincide with each other up to a constant when the self-duality condition is imposed. Therefore, the \( k \)-space formulation of Siegel’s action is self-dual with respect to \( \phi(k) - \psi(k) \) dualization expressed by eq.(6) and eq.(12).

## 3 Self-duality of the chiral p-form action in \( D=2(p+1) \) momentum frame

By introducing a real \( p \)-form field, \( A_{\mu_1...\mu_p}(x) \), we generalize the Siegel action to the \( D = 2(p + 1) \) dimensional space-time,

\[
S_c = \int d^D x \left\{ -\frac{1}{2(p+1)!} \partial_{[\mu_1} A_{\mu_2...\mu_{p+1}]}(x) \partial^{\mu_1} A^{\mu_2...\mu_{p+1}}(x) + \frac{1}{2} \lambda^\mu(x) \left[ \partial_{[\mu A_{\mu_1...\mu_p]}(x) - \frac{1}{(p+1)!} \epsilon_{\mu_1...\mu_p\nu_1...\nu_{p+1}} \partial^{[\nu_1} A^{\nu_2...\nu_{p+1}]}(x) \right] \times \left[ \partial^{[\nu A_{\mu_1...\mu_p]}(x) - \frac{1}{(p+1)!} \epsilon_{\nu_1...\nu_p\sigma_1...\sigma_{p+1}} \partial_{[\sigma_1 A_{\sigma_2...\sigma_{p+1}]}(x) \right] \right\}. \tag{14}
\]

It can be verified that \( A_{\mu_1...\mu_p}(x) \) indeed describes a chiral \( p \)-form by following Ref.[8] in which only the \( D = 2 \) and \( D = 6 \) cases are dealt with. Moreover, the quantization of
the theory in higher dimensions \((D = 10, 14, \cdots)\) can be discussed by following Siegel's approach to the case \(D = 6\). As pointed out [8] that the generalization is straightforward, a kind of light-cone gauge should be imposed and then all degrees of freedom associated with the auxiliary tensor field are eliminated for the cases of dimensions higher than \(D = 6\).

As done in the above section, substituting the Fourier transformations,

\[
A_{\mu_1 \cdots \mu_p}(x) = \frac{1}{(2\pi)^{D/2}} \int d^D k e^{-ik \cdot x} A_{\mu_1 \cdots \mu_p}(k),
\]

\[
\lambda_{\mu \nu}(x) = \frac{1}{(2\pi)^{D/2}} \int d^D k e^{-ik \cdot x} \lambda_{\mu \nu}(k),
\]

(15)

into eq.(14), one arrives at Siegel's action in the \(D = 2(p + 1)\) momentum space spanned by \(k_\mu\),

\[
S_m = -\frac{1}{2(p+1)!} \int d^D k \left[ -ik_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}(k) \right] \left[ ik_{[\nu_1} A^{\nu_2 \cdots \nu_{p+1}]}(k) \right] \left[ ik_{[\sigma_1} A^{\sigma_2 \cdots \sigma_{p+1}]}(k) \right]
\]

+ \[
\frac{1}{2(2\pi)^{D/2}} \int d^D k d^D k' \lambda_{\mu \nu}(-k - k') \times \left[ i k'_{[\mu_1} A_{\nu_2 \cdots \nu_{p+1}]}(k') \right] \left[ i k'_{[\sigma_1} A_{[\sigma_2} A^{\sigma_3 \cdots \sigma_{p+1}]}(k') \right] \times \left[ i k'_{[\sigma_1} A_{\sigma_2 \cdots \sigma_{p+1}]}(k') \right].
\]

(16)

Here we note that eq.(16) includes 2D momentum integrals. This shows the non-locality of the corresponding Lagrangian as was seen in the \(D = 2\) case.

In order to discuss the duality of \(S_m\), we introduce two \((p + 1)\)-form fields in the momentum space, \(F_{\mu_1 \cdots \mu_{p+1}}(k)\) and \(G_{\mu_1 \cdots \mu_{p+1}}(k)\), and replace eq.(16) by the following action,

\[
S'_m = -\frac{1}{2(p+1)!} \int d^D k F_{\mu_1 \cdots \mu_{p+1}}(k) F^{\mu_1 \cdots \mu_{p+1}}(-k)
\]

+ \[
\frac{1}{2(2\pi)^{D/2}} \int d^D k d^D k' \lambda_{\mu \nu}(-k - k') \times \left[ F_{\mu_1 \cdots \mu_{p+1}}(k) \right] \left[ F^{\nu_1 \cdots \nu_{p+1}}(k) \right]
\]

\times \[
\left[ F^{\nu_1 \cdots \nu_{p+1}}(k') \right] \left[ F_{\sigma_1 \cdots \sigma_{p+1}}(k') \right] \left[ G_{\sigma_1 \cdots \sigma_{p+1}}(k') \right] \left[ G^{\mu_1 \cdots \mu_{p+1}}(-k) \right] \left[ F_{\mu_1 \cdots \mu_{p+1}}(k) + i k_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}(k) \right],
\]

(17)

where \(F_{\mu_1 \cdots \mu_{p+1}}(k)\) and \(G_{\mu_1 \cdots \mu_{p+1}}(k)\) act, at present, as independent auxiliary fields. Similar to the \(D = 2\) case, the third term can be obtained by substituting the Fourier trans-
expressed in terms of variation of eq.(17) with respect to $G$ which yields the equivalence between the actions, eq.(16) and eq.(17). On the other hand, Hodge dual, dual tensor which is relevant to $G$ equation, as was done in the last section, one defines another field strength difference/self-formations of the fields, i.e., eq.(15), into the following term,

$$
\frac{1}{(p+1)!} \int d^D x G^{\mu_1 \cdots \mu_{p+1}}(x) \left[ F_{\mu_1 \cdots \mu_{p+1}}(x) - \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}(x) \right].
$$

Variation of eq.(17) with respect to $G^{\mu_1 \cdots \mu_{p+1}}(-k)$ gives

$$
F_{\mu_1 \cdots \mu_{p+1}}(k) = -ik [\mu_1 A_{\mu_2 \cdots \mu_{p+1}}](k),
\tag{18}
$$

which yields the equivalence between the actions, eq.(16) and eq.(17). On the other hand, variation of eq.(17) with respect to $F_{\mu_1 \cdots \mu_{p+1}}(k)$ leads to the expression of $G^{\mu_1 \cdots \mu_{p+1}}(k)$ in terms of $F^{\mu_1 \cdots \mu_{p+1}}(k)$,

$$
G^{\mu_1 \cdots \mu_{p+1}}(k) = F^{\mu_1 \cdots \mu_{p+1}}(k) - \frac{1}{(2\pi)^{D/2}} \int d^D k' \lambda_{\mu \nu}(k - k')
\times \left[ g^{\mu[1} F^{\mu_2 \cdots \mu_{p+1}]\nu}(k') + \epsilon^{\mu_1 \cdots \mu_{p+1}\nu_1 \cdots \nu_{p+1}} F^{\nu_1 \cdots \nu_{p+1}}(k') \right],
\tag{19}
$$

where $F^{\mu_1 \cdots \mu_{p+1}}(k)$ is defined as the difference of the field strength $F^{\mu_1 \cdots \mu_{p+1}}(k)$ and its Hodge dual,

$$
F^{\mu_1 \cdots \mu_{p+1}}(k) \equiv F^{\mu_1 \cdots \mu_{p+1}}(k) - \frac{1}{(p+1)!} \epsilon^{\mu_1 \cdots \mu_{p+1}\nu_1 \cdots \nu_{p+1}} F_{\nu_1 \cdots \nu_{p+1}}(k),
\tag{20}
$$

which is also called the self-dual tensor. Note that eq.(19) is an integral equation. In order to obtain algebraically $F^{\mu_1 \cdots \mu_{p+1}}(k)$ in terms of $G^{\mu_1 \cdots \mu_{p+1}}(k)$ without solving the integral equation, as was done in the last section, one defines another field strength difference/self-dual tensor which is relevant to $G^{\mu_1 \cdots \mu_{p+1}}(k)$,

$$
G^{\mu_1 \cdots \mu_{p+1}}(k) \equiv G^{\mu_1 \cdots \mu_{p+1}}(k) - \frac{1}{(p+1)!} \epsilon^{\mu_1 \cdots \mu_{p+1}\nu_1 \cdots \nu_{p+1}} G_{\nu_1 \cdots \nu_{p+1}}(k),
\tag{21}
$$

and then establishes the relationship between these two self-dual tensors by using eq.(19),

$$
F^{\mu_1 \cdots \mu_{p+1}}(k) = G^{\mu_1 \cdots \mu_{p+1}}(k),
\tag{22}
$$

whose similar form, as mentioned above, exists in various chiral $p$-forms in configuration space [14]. With eq.(22), one can algebraically invert eq.(19) and obtain $F^{\mu_1 \cdots \mu_{p+1}}(k)$ expressed in terms of $G^{\mu_1 \cdots \mu_{p+1}}(k)$,

$$
F^{\mu_1 \cdots \mu_{p+1}}(k) = G^{\mu_1 \cdots \mu_{p+1}}(k) + \frac{1}{(2\pi)^{D/2}} \int d^D k' \lambda_{\mu \nu}(k - k')
\times \left[ g^{\mu[1} G^{\mu_2 \cdots \mu_{p+1}]\nu}(k') + \epsilon^{\mu_1 \cdots \mu_{p+1}\mu_1 \cdots \mu_{p+1}} G^{\nu_1 \cdots \nu_{p+1}}(k') \right].
\tag{23}
$$
We can verify from eq.(19) that when the self-duality condition is satisfied, i.e.,

$$ F^{\mu_1 \cdots \mu_{p+1}}(k) = 0, $$

$F^{\mu_1 \cdots \mu_{p+1}}(k)$ and $G^{\mu_1 \cdots \mu_{p+1}}(k)$ relate with a duality,

$$ G^{\mu_1 \cdots \mu_{p+1}}(k) = \frac{1}{(p+1)!} \epsilon^{\mu_1 \cdots \mu_{p+1} \nu_1 \cdots \nu_{p+1}} F_{\nu_1 \cdots \nu_{p+1}}(k). $$

Now substituting eq.(23) into the action, eq.(17), and making tedious calculations, we obtain the dual Siegel action in the $D = 2(p+1)$ momentum space,

$$ S_{m}^{\text{dual}} = \frac{1}{2(p+1)!} \int d^Dk G^{\mu_1 \cdots \mu_{p+1}}(k) G^{\mu_1 \cdots \mu_{p+1}}(-k) $n + \frac{1}{2(2\pi)^{D/2}} \int d^Dk d^Dk' \lambda_{\mu \nu}(k-k') G^{\mu \mu_1 \cdots \mu_p}(k) G^{\nu \mu_1 \cdots \mu_p}(k') $n + \int d^Dk A_{\mu_1 \cdots \mu_p}(k) [ik_{\mu} G^{\mu_1 \cdots \mu_p}(-k)] . \quad (24) $$

Variation of eq.(24) with respect to $A_{\mu_1 \cdots \mu_p}(k)$ gives $ik_{\mu} G^{\mu_1 \cdots \mu_p}(-k) = 0$, whose solution has to be

$$ G^{\mu_1 \cdots \mu_{p+1}}(k) = \frac{1}{(p+1)!} \epsilon^{\mu_1 \cdots \mu_{p+1} \nu_1 \cdots \nu_{p+1}} [-ik_{[\nu_1} B_{\nu_2 \cdots \nu_{p+1}]}(k)] $n \equiv \frac{1}{(p+1)!} \epsilon^{\mu_1 \cdots \mu_{p+1} \nu_1 \cdots \nu_{p+1}} F_{\nu_1 \cdots \nu_{p+1}}[B(k)], \quad (25) $$

where $B_{\nu_1 \cdots \nu_p}(k)$ is an arbitrary $p$-form field. When eq.(25) is substituted into the dual action, eq.(24), one finally obtains the result that the dual action is the same as the original action, eq.(16), only with the replacement of $A_{\nu_1 \cdots \nu_p}(k)$ by $B_{\nu_1 \cdots \nu_p}(k)$. Consequently, the $k$-space formulation of Siegel’s action in $D = 2(p+1)$ dimensions is self-dual with respect to $A_{\nu_1 \cdots \nu_p}(k) - B_{\nu_1 \cdots \nu_p}(k)$ dualization given by eq.(18) and eq.(25).

4 Conclusion

In this note we have generalized Siegel’s model to the $D = 2(p+1)$ dimensional space-time, and have extended for this model duality investigations from the configuration frame to the momentum frame and hence uncovered its self-duality with respect to dualization of chiral fields in the momentum space. The characteristic that does not exist in configuration space is duality investigation of non-local Lagrangians and algebraic solution
of integral equations. Here we emphasize that the introduction of two self-dual tensors and the establishment of their relationship are crucial in realizing the whole procedure of investigation.

The PST action [11,13] is not polynomial, however, the non-polynomiality can be formally eliminated in terms of a re-definition of auxiliary fields. Here we take the two dimensional case as our example. (Generalization to higher dimensions is straightforward. See Ref.[12].) The PST action reads

$$S = \int d^2x \left\{ -\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{1}{2} \frac{\partial_\eta a(x) \partial^\eta a(x)}{2 \partial_\eta a(x) \partial^\eta a(x)} \partial_\mu a(x) [\partial^\mu \phi(x) - \epsilon^{\mu \sigma} \partial_\sigma \phi(x)] \partial_\nu a(x) [\partial^\nu \phi(x) - \epsilon^{\nu \rho} \partial_\rho \phi(x)] \right\},$$

where $a(x)$ is an auxiliary scalar field. If we introduce an auxiliary tensor field $\lambda_{\mu \nu}(x)$ which is defined by

$$\lambda_{\mu \nu}(x) \equiv \frac{\partial_\mu a(x) \partial_\nu a(x)}{\partial_\eta a(x) \partial^\eta a(x)},$$

the PST action appears then in the form of the Siegel action, eq.(2). In this sense, we may say that the duality investigation of the PST action in the momentum space is the same as that of Siegel’s action formally. We would like to thank the referee for pointing out this solely formal equivalence. However, our treatment is valid for any chiral $p$-form action in which polynomial or specifically Lagrange-multiplier auxiliary fields are introduced. This validity has nothing to do with gauge symmetries of these actions, that is, our treatment works whether the actions contain first-class or second-class constraints. The property appears quite naturally when one applies the whole procedure of our duality investigation to a gauge invariant (first-class constraint) model, for instance, to the McClain-Wu-Yu action [10]. As to the direct duality investigation of the PST action in the momentum space, difficulties exist not due to the first-class constraints, but the non-polynomiality of the auxiliary fields. To deal with the problem requires us to develop an appropriate technique of the Fourier transformation. This topic is beyond the context of the present note, and we shall consider it later in a separate work.

Finally, we make a few remarks upon the non-locality of Siegel’s Lagrangian in the momentum frame.
(i) As we know, the non-locality originates from the non-linearity which is imposed by the manifest Lorentz invariance of Siegel’s action. Therefore, we conclude that the manifest Lorentz invariance of actions in configuration space, which is a quite commonly required property of actions, sometimes produces the non-locality of corresponding Lagrangians in momentum space.

(ii) In Ref.[4] a non-local Lagrangian of a chiral 0-form was proposed and then a local fermionic Lagrangian was proved to be equivalent to the non-local bosonic one in terms of chiral bosonization in configuration space. In our case we have obtained the non-local Lagrangian eq.(4) in momentum space. We may ask whether or not its local and/or equivalent fermionic formulation exists. In addition, we may need to develop a technique of chiral bosonization in momentum space.

(iii) A problem which is closely related to (ii) is the quantization of non-local Lagrangians in momentum space. For instance, we may generalize Dirac’s method [17] which has been treated as a standard approach of quantization. To this end, we have to deal with several basic concepts, such as the definition of canonical momenta, the classification of first and second class constraints, and the construction of Dirac brackets, and so on. This is now under consideration.

Acknowledgments

Y.-G. Miao acknowledges supports by an Alexander von Humboldt fellowship, by the National Natural Science Foundation of China under grant No.19705007, and by the Ministry of Education of China under the special project for scholars returned from abroad. D.K. Park acknowledges support from the Basic Research Program of the Korea Science and Engineering Foundation (Grant No. 2001-1-11200-001-2).
References

[1] D. Zwanziger, Phys. Rev. D 3, 880 (1971).

[2] S. Deser and C. Teitelboim, Phys. Rev. D 13, 1592 (1976); S. Deser, J. Phys. A 15, 1053 (1982).

[3] N. Marcus and J. H. Schwarz, Phys. Lett. 115B, 111 (1982).

[4] R. Floreanini and R. Jackiw, Phys. Rev. Lett. 59, 1873 (1987).

[5] M. Henneaux and C. Teitelboim, Phys. Lett. B 206, 650 (1988).

[6] A. Tseytlin, Phys. Lett. B 242, 163 (1990); Nucl. Phys. B350, 395 (1991).

[7] J. H. Schwarz and A. Sen, Nucl. Phys. B411, 35 (1994).

[8] W. Siegel, Nucl. Phys. B238, 307 (1984).

[9] P. P. Srivastava, Phys. Rev. Lett. 63, 2791 (1989).

[10] B. McClain, Y.-S. Wu and F. Yu, Nucl. Phys. B343, 689 (1990).

[11] P. Pasti, D. Sorokin and M. Tonin, Phys. Lett. B 352, 59 (1995); Phys. Rev. D 52, R4277 (1995); Phys. Rev. D 55, 6292 (1997).

[12] Y.-G. Miao, H. J. W. Müller-Kirsten and D. K. Park, Nucl. Phys. B612, 215 (2001).

[13] A. Maznytsia, C. R. Preitschopf and D. Sorokin, Nucl. Phys. B539, 438 (1999); “Dual actions for chiral bosons,” hep-th/9808049.

[14] Y.-G. Miao and H. J. W. Müller-Kirsten, Phys. Rev. D 62, 045014 (2000).

[15] Y.-G. Miao, R. Manvelyan and H. J. W. Müller-Kirsten, Phys. Lett. B 482, 264 (2000).

[16] R. Banerjee and B. Chakraborty, J. Phys. A 32, 4441 (1999).

[17] P. A. M. Dirac, Lectures on Quantum Mechanics (Yeshiva Univ. Press, New York, 1964).