THE TRISPECTRUM OF THE 4 YEAR COBE-DMR DATA

M. Kunz\textsuperscript{1}, A.J. Banday\textsuperscript{2}, P.G. Castro\textsuperscript{1}, P.G. Ferreira\textsuperscript{1}, K.M. Górski\textsuperscript{3,4}

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ABSTRACT

We propose an estimator for the trispectrum of a scalar random field on a sphere, discuss its geometrical and statistical properties, and outline its implementation. By estimating the trispectrum of the 4 year COBE-DMR data (in HEALPix pixelization) we find new evidence of a non-Gaussian signal associated with a known systematic effect. We find that by removing data from the sky maps for those periods of time perturbed by this effect, the amplitudes of the trispectrum coefficients become completely consistent with predictions for a Gaussian sky. These results reinforce the importance of statistical methods based in harmonic space for quantifying non-Gaussianity.

\textit{Subject headings:} cosmic microwave background — cosmology: observations

1. INTRODUCTION

The Cosmic Microwave Background (CMB) is the cleanest window on the origin of structure in the very early universe. A complete description of the statistical properties of cosmological fluctuations at a redshift \( z \approx 1000 \) affords us an essential insight into those processes which may have seeded the formation of galaxies. In a Gaussian theory of structure formation, such as the currently favored model of Inflation, the power spectrum contains all the possible information about the fluctuations. Any higher order moment can subsequently be described in terms of it. However, if the theory is non-Gaussian (as expected for structure formation theories due to local effects from primordial phase transitions or more generally from non-linear processes), then there will be deviations from the simple Gaussian expressions for the higher order moments. Such behavior can serve as a powerful discriminator between different models of structure formation.

Most analyses of CMB data to-date have focused on the angular power spectrum and its sensitivity to various parameters of cosmological theories. Some work has been done on the estimation of the three point correlation function and its analogue in spherical harmonic space, with intriguing results (Heavens 1998, Ferreira, Magueijo and Górski 1998, Magueijo 2000, Banday, Zaroubi and Górski 2000). It is the purpose of this letter to propose a method for estimating the four point spectrum, the \textit{trispectrum}, and to apply it to the COBE 4 year DMR data. This work complements the recent work of Hu (2001) where some of the properties of the angular trispectrum of the CMB are discussed.

The outline of this letter is as follows. In Section \ref{sec:estimator}, we construct a set of orthonormal estimators and describe their properties for a Gaussian random field. In Section \ref{sec:results}, we apply the estimators to the COBE 4 year DMR data. We show that we detect the non-Gaussian signal found in Ferreira, Magueijo & Górski (1998) and that it can be explained by the arguments presented in Banday, Zaroubi & Górski (2000), and in particular that this is a manifestation of a known systematic effect. We therefore conclude that the COBE 4 year data is consistent with a Gaussian cosmological signal. In Section \ref{sec:discussion} we summarize our results.

2. THE ESTIMATOR

In this section we wish to construct a set of quantities for estimating the trispectrum of a random field on the sphere. The temperature anisotropy in a given direction on the celestial sphere, \( T(\mathbf{n}) \), can be expanded in terms of spherical harmonic functions, \( Y_{\ell m}(\mathbf{n}) \):

\begin{equation}
T(\mathbf{n}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\mathbf{n})
\end{equation}

For any theory of structure formation, the \( a_{\ell m} \) coefficients are a set of random variables; we shall restrict ourselves to theories which are statistically homogeneous and isotropic. In this case we can define the power spectrum \( C_\ell \) of the temperature anisotropies by \( \langle a_{\ell m} a_{\ell' m'}^\ast \rangle = C_\ell \delta_{\ell \ell'} \delta_{mm'} \).

We now seek to construct a set of tensors that are geometrically independent, describe their statistical properties for a Gaussian random field and then discuss the practical issue of their implementation. Given a set of \( a_{\ell m} \) we wish to find the index structure of the set of four point correlators such that (1) they are rotationally invariant (2) they form a complete basis (preferably orthonormal) of the whole space of admissible four-point correlators and (3) they satisfy the appropriate symmetries under interchanges of \( m \)- and \( \ell \)-values. We shall restrict ourselves to the case in which \( \ell_1 = \ell_2 = \ell_4 = \ell_4 = \ell \). Furthermore, throughout this section we keep \( \ell \) fixed. We determine the tensor \( T \) such that

\begin{equation}
\langle a_{\ell m_1} a_{\ell m_2} a_{\ell m_3} a_{\ell m_4} \rangle = \sum_{a=0}^n T_{\ell a} T_{m_1 m_2 m_3 m_4}^{a \ast} \label{eq:trispectrum}
\end{equation}

where \( n=\text{int}(\ell/3) \) (due to reflection, permutation and rotational symmetry). The \( T_{\ell a} \) values are then the compo-

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1 Astrophysics, University of Oxford, Keble Road, Oxford OX1 3RH, UK
2 Max Planck Institut fuer Astrophysik, Karl-Schwarzschildstr. 1, Postfach 13 17, D-85741 Garching, Germany
3 European Southern Observatory, Garching, Germany
4 Warsaw University Observatory, Aleje Ujazdowskie 4, 00-478 Warszawa, Poland.
nents of the trispectrum which we wish to estimate. The explicit form of the $T$ are

$$T_{a1m2m3m4} = \sum_{\alpha=0}^{2} L_{\alpha} T_{a1m2m3m4} \quad (3)$$

$$\tilde{T}_{a1m2m3m4} = \sum_{M=-2\alpha}^{2\alpha} (-1)^{M} \left( \begin{array}{ccc} \ell & \ell & 2\alpha \\ m_1 & m_2 & M \end{array} \right) \times$$

$$\left( \begin{array}{ccc} 2\alpha & \ell & -M \\ -M & m_3 & m_4 \end{array} \right) + \text{inequiv. permutations} \quad (4)$$

where the matrices in parentheses are the Wigner 3-J symbols. The $T_{\alpha\ell}$ are not orthogonal and satisfy

$$\tilde{T}_{a1m2m3m4} \tilde{T}_{\alpha1m2m3m4} = \frac{3}{4\alpha + 1} \delta_{\alpha\beta} + 6 \left\{ \begin{array}{ccc} \ell & \ell & 2\alpha \\ \ell & \ell & 2\beta \end{array} \right\}$$

(whence summation over the $m_i$ is assumed) which has rank $n + 1$. The matrix $L_{\alpha}$ in (3) is a rectangular matrix (with a triangular sub-block) with $n + 1$ columns and $\ell + 1$ rows. It is constructed through a Gram-Schmidt procedure by subtracting for each $\alpha$ (starting from $\alpha = 0$) the projection onto all $a' < a$ and then normalizing the result. The $\alpha = 0$ (and hence $a = \alpha = 0$) tensor is proportional to the Gaussian contribution. This can be easily seen given that for $\alpha = 0$ the Wigner 3J symbols are simply Kronecker $\delta$ symbols in the corresponding indices. The remaining $a > 0$ terms contain therefore no Gaussian signal and quantify the non-Gaussian part of the trispectrum.

The $T$ are orthonormal and can be used to construct an estimator for $T_a$ from a realization of $a_{\ell m}$:

$$\tilde{T}_{\ell a} = T_{m1m2m3m4} a_{\ell m1} a_{\ell m2} a_{\ell m3} a_{\ell m4} \quad (5)$$

For a Gaussian random field we expect $\sigma^2[\tilde{T}_{\ell a}] \gg \sigma^2[T_{\ell a}]$ for $a > 0$, where $\sigma^2[A]$ denotes the variance of the random variable $A$ and $T_{\ell a}$ is simply the square of the minimum variance estimator of the $C_{\ell}$. One finds that $\langle \tilde{T}_{\ell a} \rangle = 0$ and $\sigma^2[\tilde{T}_{\ell a}] = 24C_{\ell}^4$ for all $a > 0$.

To show that the $\tilde{T}_{\ell a}$ constitute a family of minimum variance estimators we construct a linear combination of the estimators:

$$T_{m1m2m3m4} = \sum_{a=0}^{n} c_a T_{m1m2m3m4} \quad (6)$$

and minimize the function

$$\sigma^2[c_a, \lambda] = \langle (T_{m1m2m3m4} a_{\ell m1} a_{\ell m2} a_{\ell m3} a_{\ell m4})^2 \rangle$$

$$- \langle (T_{m1m2m3m4} a_{\ell m1} a_{\ell m2} a_{\ell m3} a_{\ell m4}) \rangle^2$$

$$- \lambda C_{\ell}^2 \langle T_{m1m2m3m4} T_{m1m2m3m4} - 1 \rangle \quad (7)$$

where summation over all $m_i$ is implied. The last term, a Lagrange multiplier, ensures that $T$ is normalized. We solve $\partial_{c_a} \sigma^2[c_a, \lambda] = 0$ to find a set of two equations

$$(24I + 72A_{\ell})^{ab} c_b + \lambda c_b = 0 \quad \text{and} \quad c^2 = 1 \quad (8)$$

where

$$A_{\ell}^{ab} = T_{m1m2m3m4} T_{m1m2m3m4}$$

This is an eigenvector equation where, for a given eigenvector $c$, the eigenvalue $\lambda$ will give the expected variance of the estimator. Of the $n + 1$ eigenvalues, one is large and has an eigenvector proportional to $\tilde{T}_{\ell a}$. The remaining eigenvalues have an amplitude $\lambda = 24$ and each eigenvector is a $\tilde{T}_{\ell a}$ for $a > 0$.

Note that we can relate our parameterization to the one proposed in Hu (2001); If we reexpress equation (3) as

$$\langle a_{\ell m1} a_{\ell m2} a_{\ell m3} a_{\ell m4} \rangle = \sum_{a=0}^{\ell} \tilde{T}_{\ell a} \tilde{T}_{\ell a} \quad (9)$$

where $T_{\ell a} = L_{\alpha}^{a} \tilde{T}_{\ell a}$ then $Q_{\ell a}^{\ell}$ as defined in equation 15 of Hu (2001) can be written as

$$Q_{\ell a}^{\ell}(2\alpha) = \tilde{T}_{\ell a} + 2(4\alpha + 1) \sum_{\beta} \left\{ \begin{array}{ccc} \ell & \ell & 2\alpha \\ \ell & \ell & 2\beta \end{array} \right\} \tilde{T}_{\ell \beta} \quad (10)$$

The numerical implementation of these estimators is more involved than for the bispectrum. If we omit the numerous symmetries, we have to consider for each $\ell$ a set of up to $8\ell^3$ Wigner 3J symbols (compared to just one for the bispectrum). There are reasonably fast ways for constructing the Wigner 3J symbols (Schulten and Gordon 1976) but the number of operations per estimator scales as $O(\ell^6)$. For repeated computations of the estimators (eg. in Monte Carlo studies), this can partially be avoided by storing the precomputed estimators in a lookup table, with the amount of memory required scaling as $O(\ell^4)$.

Clearly, to be able to estimate the trispectrum on small angular scales, approximate methods must be developed to make the procedure computationally feasible. However, the ability to constrain non-Gaussianity on large angular scales is in any case more important physically for two reasons; the ratio of the non-Gaussian to the Gaussian signal will in general be higher for lower moments, and the signal to noise is better for low $l$. To understand these points, let us assume a source for non-Gaussianity which leads to approximately scale invariant moments of the gravitational potential on arbitrary scales. i.e. $\langle \Phi(R)^N \rangle$ is constant for any $R$, where $\Phi(R)$ is the gravitational potential within a ball of radius $R$ and $\langle \cdots \rangle$ denotes the ensemble average. This might be expected from a primordial source with no preferred scale such as inflation (Komatsu & Spergel 2000) or from an active source where the only scale is set by the horizon today (Durrer et al. 2000). Current observations of the CMB certainly favor such scale-invariant descriptions of the potential. One then expects the moment of order $N$ of the $a_{\ell m}$ to scale as $\ell^{2(1-N)}$. This signal will be competing against the fluctuations due to the disconnected (or Gaussian) part, which is proportional to $N!\ell^{-(2N+1)/2}$, the former therefore dominating for $N > 2$. Since the power spectrum for white noise has constant amplitude, the signal to noise as a function of scale will have the same form as the scale invariant power spectrum itself, therefore being larger for smaller $l$, i.e. larger angular scales.
As an application of the formalism described in Section 2, we estimate the trispectrum of the coadded 53 and 90 GHz COBE-DMR 4 year sky maps in HEALPix format (Górski et al 1999). The resolution of the maps is $N_{side} = 64$ or 49152 pixels. We do not extend our analysis beyond $\ell_{\text{max}} = 20$ since the signal to noise is poor for higher $\ell$. Hence the maximal number of independent non-Gaussian estimators for the trispectrum is $\operatorname{int}(\ell_{\text{max}}/3) = 6$. We set the pixels in the extended Galactic cut (Banday et al 1997) to zero and subtract the residual monopole and dipole of the resulting map. After convolving the maps with spherical harmonics to extract a set of $a_{l,m}$’s for $\ell \leq 20$ we then apply equation (3). To validate our software, we have estimated the bispectrum of the COBE-DMR 4 year sky data repixelized in the HEALPix format (for convenience denoted by EC) and reproduced the results of Ferreira, Magueijo & Górski (1998), and in particular the strong non-Gaussian signal present at $\ell = 16$. When an equivalent map, from which that part of the DMR time stream contaminated by the ‘eclipse effect’ is removed (denoted NEC), is subsequently analyzed we also reproduce the results of Banday, Zaroubi & Górski 2000, namely that the non-Gaussian signal is no longer detected. For our subsequent analysis we will present the trispectra of both the EC and NEC data.

One of our primary concerns is to compare our results with the assumption that the CMB sky measured by COBE-DMR is Gaussian. To do so, we generate 10000 full-sky maps at the same resolution using a scale invariant power spectrum normalized to $Q_{\ell_{\text{rms}} - PS} = 18 \mu K$ (Górski et al 1998). We convolve each map with the DMR beam and add uncorrelated pixel noise with rms amplitude $\sigma_n = 15.95 \mu K/\sqrt{N_{\text{obs}}}$ (where $N_{\text{obs}}$ is the number of times a given pixel was observed); we then subject the synthetic map to the same procedure as the original data.

Figure 1 shows the trispectra of the DMR data together with Gaussian 95% confidence limits. Instead of the “raw” estimator $\hat{T}$ we prefer to use the normalized trispectrum, $\tau_{a}^{(a)} = \hat{T}_{a}/C_{a}^{2}$ for $a > 1$ (where $\hat{C}_{a} = \frac{1}{2\ell+1} \sum_{m} |a_{\ell,m}|^{2}$), thus effectively removing the dependence on the power spectrum. This prevents fluctuations in the power spectrum from introducing spurious signals and from masking real non-Gaussianities. Figure 1 shows that in this case, most values fall within the 95% confidence lines and demonstrate the scatter expected for a Gaussian random field.

Of particular interest is the value of the normalized $\tau^{(3)}$ at $\ell = 16$ in figure 1. One finds that 99.9% of the Gaussian models in the EC case have a smaller $\tau^{(3)}$ than the measured one. This is clearly a manifestation of the non-Gaussianity found in Ferreira, Magueijo & Górski (1998) which is highly localized in $\ell$ space. However, if we estimate $\tau^{(3)}$ for the NEC we find that it falls comfortably within the 95% confidence limits. This leads us to believe that this detection of non-Gaussianity results from the ‘eclipse effect’, consistent with the hypothesis of Banday, Zaroubi & Górski (1999).

Let us now construct a goodness of fit for our statistic. In Ferreira, Magueijo & Górski (1998), a modified $\chi^{2}$ was constructed which took into account the non-Gaussian distribution of each method: as above, the distribution of each estimator for a Gaussian sky was constructed and used as an approximate likelihood function to evaluate the goodness of fit. One shortcoming of such a method

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5 The ‘eclipse effect’ was an orbitally modulated signal which took place for approximately two months every year around the June solstice when the COBE spacecraft repeatedly flew through the Earth’s shadow.
was that correlations between the estimates for different \(\ell\)s were discarded. To include them, we use the Gaussian ensemble of data sets to derive the expectation values \(<\tilde{\mathcal{E}}\rangle\) and the covariance matrix \(\mathcal{C}\) for both the power spectrum, \(\mathcal{C}_\ell\), and all seven trispectrum estimators, \(\tau^{(0)}_\ell\) to \(\tau^{(6)}_\ell\). We proceed to calculate the \(\chi^2\) value for the estimator \(\tilde{\mathcal{E}}\) and the data set \(\mathcal{D}\),

\[
\chi^2[\tilde{\mathcal{E}}, \mathcal{D}] = \sum_{\ell,\ell'} \left(\langle \tilde{\mathcal{E}}\rangle_{\mathcal{G}} - \tilde{\mathcal{E}}\mathcal{D}\rangle_{\ell}\right) C^{-1}_{\ell\ell'} \left(\langle \tilde{\mathcal{E}}\rangle_{\mathcal{G}} - \tilde{\mathcal{E}}\mathcal{D}\rangle_{\ell}\right),
\]

using as data sets the EC data and the NEC data. Finally we use another 10000 Gaussian realizations to estimate the expected distribution of the \(\chi^2\) for both the EC and the NEC data.

For all normalized non-Gaussian trispectrum estimators (\(\tau^{(1)}\) to \(\tau^{(6)}\)) we find that 94\% of the Gaussian models have a smaller \(\chi^2\) than the EC data as can be seen in figure 2. As expected the main contribution to the \(\chi^2\) for the EC data stems from \(\tau^{(3)}\) at \(\ell = 16\); indeed, this is the only normalized trispectrum estimator which exhibits any significant non-Gaussianity, in this case at about 99.9\%. If we use the NEC data, the detection vanishes. In this case, 60\% of all Gaussian models have a lower \(\chi^2\) when computed over all six trispectrum estimators (83\% for \(\tau^{(3)}\) alone). Hence the NEC data is compatible with Gaussianity.

4. DISCUSSION

In this paper, we have derived an estimator for the trispectrum of a scalar random field on the sphere. Application of this estimator, normalized by the power spectrum (a procedure adopted in Ferreira, Magueijo & Górski, 1998 for the bispectrum, see also Komatsu et al 2002 for a detailed discussion), to the COBE-DMR data provides evidence for non-Gaussianity at the 94\% confidence level. As in the case of the bispectrum, the signal is mainly present in the \(\ell = 16\) multipole (and the \(\tau^{(3)}\) estimator here). However, when data is excluded to correct for the `eclipse effect’, the non-Gaussian behavior is removed, allowing us to conclude that the non-Gaussianity present in the uncorrected sky maps is not cosmological in origin.

The detection of a signal that is so strongly localized in \(\ell\) space provides convincing support to our contention that the trispectrum is an important and sensitive probe of non-Gaussianity in the frequency (scale) domain. It affords complementary information to the bispectrum since it is an even moment, and, despite the higher computational effort required, has the obvious advantage in that it can probe all values of \(\ell\), not just the even ones.

Interestingly enough, from a theoretical perspective, there may be some possible sources of non-Gaussianity for which the trispectrum provides a far more sensitive test than the bispectrum. In many cases a given moment of the \(a_{\ell m}\)s can be expressed as the projection of a cosmological field. If that field is vector-like in nature (as in the case of the Doppler effect or the Ostriker-Vishniac effect and its non-linear extensions), any odd moment may suffer from the Sunyaev-Kaiser cancellation, where the integral of a given wavenumber, \(k\), over a smoothly varying projection function with width \(\sigma\) tends to suppress the moment by a factor of order \(1/(\sigma k)^2\) (Sunyaev 1978, Kaiser 1985, Scannapieco 2000). For even moments one can always construct a scalar component which will not be subject to this cancellation. Such a tool will be of great use in the analysis of the data sets from the MAP and Planck Surveyor satellites.

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Note added in proof – E. Komatsu investigates the trispectrum of the COBE DMR data in his Ph. D. thesis. His conclusions agree with ours, namely that the COBE data is consistent with Gaussian initial fluctuations.