Electromagnetic radiation in even-dimensional spacetimes

B. P. Kosyakov

March 31, 2008

Russian Federal Nuclear Center, Sarov, 607190 Nizhnii Novgorod Region, Russia
E-mail: kosyakov@vniief.ru

Abstract

The basic concepts and mathematical constructions of the Maxwell–Lorentz electrodynamics in flat spacetime of an arbitrary even dimension \( d = 2n \) are briefly reviewed. We show that the retarded field strength \( F_{\mu \nu}^{(2n)} \) due to a point charge living in a 2\( n \)-dimensional world can be algebraically expressed in terms of the retarded vector potentials \( A_{\mu}^{(2m)} \) generated by this charge as if it were accommodated in 2\( m \)-dimensional worlds nearby, \( 2 \leq m \leq n + 1 \). With this finding, the rate of radiated energy-momentum of the electromagnetic field takes a compact form.

PACS numbers: 03.50.De; 03.50.Kk.
Key words: Electrodynamics in even-dimensional spacetimes, radiation.

1 Introduction

This paper is dedicated to Professor Iosif Buchbinder in celebration of his sixtieth birthday. A marvellous feature of my friend Iosif is his ability to grasp the essence of a challenging problem in theoretical physics and interpret it quite plainly. He makes a major effort to attain the greatest possible clarity in a complex subject. The analysis of the Maxwell–Lorentz electrodynamics in even-dimensional spacetimes presented in this paper will hopefully be found to be made in the same vein.

The physics in higher spacetime dimensions is of basic current interest. String-inspired large extra-dimensional models [1], [2], [3] and braneworld scenarios [4] [5], [6], [7], [8] (for a review see [9], [10]) offer promise for a better understanding of a rich variety of high-energy phenomena which is expected to be discovered at the Large Hadron Collider at CERN, and other coming into service colliders. Our main concern in this paper is with the concept of radiation in higher-dimensional classical electrodynamics. A further refinement of this concept is needed if we are to gain a more penetrating insight into the self-interaction problem [11], [12], [13]. Recently, this problem was addressed in Refs. [14], [15], [16], [17], [18], [19]. It should be stressed that spacetime manifolds in these papers were assumed to be flat. (Conceivably it might be prematurely to embark on a study of the radiation in curved manifolds until the energy-momentum problem in general relativity is completely solved.) We also note that the idea of radiation in odd-dimensional worlds falls far short of being clear-cut because Huygens’s principle fails in odd spacetime dimensions [20], and the same is true for massive vector fields. Consequently, we will restrict our consideration to classical electrodynamics in Minkowski spacetime of even dimension \( d = 2n \).
The paper is organized as follows. Section 2 outlines the state of the art of the $2n$-dimensional Maxwell–Lorentz theory, notably the methods for solving Maxwell’s equations with the source composed of a single point charge. A central result of this section is given by equations (40)–(44) suggesting that the retarded field strength $F_{\mu\nu}^{(2n)}$ due to a point charge living in a $2n$-dimensional world can be algebraically expressed in terms of the retarded vector potentials $A_{\mu}^{(2m)}$ generated by this charge as if it were accommodated in $2m$-dimensional worlds nearby, with $m$ being within the limits $2 \leq m \leq n + 1$. It is then shown in Sec. 3 that the rate of radiated energy-momentum of the electromagnetic field in $2n$-dimensional spacetime takes a compact form, Eqs. (72) and (73). Some implications of these results are discussed in Sec. 4.

We adopt the metric of the form $\eta_{\mu\nu} = \text{diag}(1, -1, \ldots, -1)$, and follow the conventions of Ref. [12] throughout.

2 Vector potentials, prepotentials, and field strengths

Consider a single charged point particle moving along a timelike world line in flat spacetime of an arbitrary even dimension $d = 2n$, $n = 1, 2, \ldots$. With reference to the aforementioned string-inspired models and braneworld scenarios, our prime interest is with $d$ in the range from $d = 2$ to $d = 10$. The world line $z^\mu(s)$ is regarded as a smooth function of the proper time $s$. Suppose that the Maxwell–Lorentz electrodynamics is still valid. This is tantamount to stating that the field sector is given by

$$\mathcal{L} = -\frac{1}{4\Omega_{d-2}} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu, \quad (1)$$

and the retarded boundary condition is imposed on the vector potential $A_\mu$. Here, $\Omega_{d-2}$ is the area of the unit $(d-2)$-sphere, $v^\mu = \dot{z}^\mu = dz^\mu/ds$ is the $d$-velocity, and $\delta^d(R)$ is the $d$-dimensional Dirac delta-function. In what follows the value of the charge will be taken to be unit, $e = 1$.

The field equation resulting from (1) reads

$$\mathcal{E}^\mu = \partial_\nu F^{\nu\mu} + \Omega_{d-2} j^\mu = 0. \quad (3)$$

This is accompanied by the Bianchi identity

$$\mathcal{E}^{\lambda\mu\nu} = \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0. \quad (4)$$

We take the general solution to (4), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and choose the Lorenz gauge condition $\partial_\mu A^\mu = 0$ to put (3) into the form

$$\Box A^\mu = \Omega_{d-2} j^\mu. \quad (5)$$

There are two alternative procedures for integrating the wave equation (5). The Green’s function approach holds much favor (for a review see [20], [12]). The retarded Green’s function satisfying

$$\Box G_{\text{ret}}(x) = \delta^d(x) \quad (6)$$

is given by

$$G_{\text{ret}}(x) = \frac{1}{2\pi^n} \theta(x_0) \delta^{(n-1)}(x^2), \quad d = 2n. \quad (7)$$
Here, $\delta^{(n-1)}(x^2)$ is the delta-function differentiated $n-1$ times with respect to its argument. With the Green’s function \eqref{eq:7} at our disposal it becomes possible to obtain the retarded vector potential

$$A^\mu(x) = \Sigma_{d-2} \int_{-\infty}^{\infty} ds G_{ret}(R) v^\mu(s),$$  \hspace{1cm} (8)

where

$$R^\mu = x^\mu - z^\mu(s)$$  \hspace{1cm} (9)

is the null four-vector drawn from the retarded point $z^\mu(s)$ on the world line, where the signal is emitted, to the point $x^\mu$, where the signal is received.

The other procedure consists of using the ansatz of a particular form \cite{14}, \cite{12}. To illustrate, the pertinent ansätze for $d = 2, 4, 6$ are given, respectively, by

$$A^{(2)}_\mu = \alpha(\rho) R_\mu,$$  \hspace{1cm} (10)

$$A^{(4)}_\mu = f(\rho) R_\mu + g(\rho) v_\mu,$$  \hspace{1cm} (11)

$$A^{(4)}_\mu = \Omega(\rho, \lambda) R_\mu + \Phi(\rho, \lambda) v_\mu + \Psi(\rho, \lambda) a_\mu.$$  \hspace{1cm} (12)

Here, $a_\mu = \dot{v}_\mu$ is the $d$-acceleration, and $\alpha, f, g, \Omega, \Phi, \Psi$ are unknown scalar functions. The functions $\alpha, f, g$ are assumed to depend on the retarded invariant distance

$$\rho = R \cdot v,$$  \hspace{1cm} (13)

while $\Omega, \Phi, \Psi$ are taken to depend on $\rho$ and the retarded invariant variable

$$\lambda = R \cdot a - 1.$$  \hspace{1cm} (14)

Let us introduce a further null vector $c_\mu$ aligned with $R_\mu$,

$$R_\mu = \rho c_\mu.$$  \hspace{1cm} (15)

We insert any one of the ansätze \cite{10}, \cite{11}, \cite{12} in \cite{3}, perform differentiations of the retarded variables using the rules

$$\partial_\mu s = c_\mu,$$  \hspace{1cm} (16)

$$\partial_\mu \rho = v_\mu + \lambda c_\mu,$$  \hspace{1cm} (17)

$$\partial_\mu R^\lambda = \delta^\lambda_\mu - v^\lambda c_\mu,$$  \hspace{1cm} (18)

and solve the resulting ordinary differential equations (for detail see \cite{14}, \cite{12}) to obtain

$$A^{(2)}_\mu = -R_\mu,$$  \hspace{1cm} (19)

$$F^{(2)}_{\mu\nu} = c_\mu v_\nu - c_\nu v_\mu,$$  \hspace{1cm} (20)

$$A^{(4)}_\mu = \frac{v_\mu}{\rho},$$  \hspace{1cm} (21)

$$F^{(4)}_{\mu\nu} = c_\mu U^{(4)}_\nu - c_\nu U^{(4)}_\mu, \quad U^{(4)}_\mu = -\lambda \frac{v_\mu}{\rho^2} + \frac{a_\mu}{\rho},$$  \hspace{1cm} (22)

$$A^{(6)}_\mu = \frac{1}{3} \left( -\lambda \frac{v_\mu}{\rho^2} + \frac{a_\mu}{\rho^2} \right),$$  \hspace{1cm} (23)
\[ F_{\mu \nu}^{(6)} = \frac{1}{3} \left( e_\mu U_{\nu}^{(6)} - e_\nu U_{\mu}^{(6)} + \frac{a_\mu v_\nu - a_\nu v_\mu}{\rho^3} \right), \quad U_\mu^{(6)} = \left[ 3 \lambda^2 - \rho^2 (\dot{a} \cdot c) \right] \frac{v_\mu}{\rho^4} - 3 \lambda \frac{a_\mu}{\rho^4} + \frac{\dot{a}_\mu}{\rho^2}. \] (24)

Note the overall factor \( \frac{1}{3} \) in (23). The origin of this numerical factor is most easily understood if we apply Gauss’ law to the case that \( \alpha_\mu = 0 \) and \( \dot{\alpha}_\mu = 0 \). To simplify our notations as much as possible, we introduce the net vector potentials and field strengths, \( A_\mu \) and \( F_{\mu \nu} \) (as opposed to the ordinary vector potentials and field strengths, \( A_\mu \) and \( F_{\mu \nu} \), whose normalization is consistent with Gauss’ law):

\[ A_\mu^{(2p)} = N_p^{-1} A_\mu^{(2p)}, \quad F_{\mu \nu}^{(2p)} = N_p^{-1} F_{\mu \nu}^{(2p)}, \quad \] (25)

where

\[ N_p = (p - 1)!! \]. (26)

It is an easy matter to extend the sequence of the ansätze shown in (10), (11), (12) to any \( d = 2n \) with integer \( n \geq 1 \). Based on the anzatz for \( d = 2n \), we come to the anzatz for \( d = 2n + 2 \) by appending a term proportional to the \((n - 1)\)th derivative of \( v_\mu \) with respect to \( s \), and assuming that the unknown functions depend on \( \rho \), together with scalar products of \( R_\mu \) and derivatives of \( v_\mu \) up to the \((n - 1)\)th derivative inclusive.

Proceeding in these lines, we get

\[ A_\mu^{(2)} = -R_\mu, \] (27)

\[ A_\mu^{(4)} = \frac{v_\mu}{\rho}, \] (28)

\[ A_\mu^{(6)} = -\lambda \frac{v_\mu}{\rho^3} + \frac{a_\mu}{\rho^2}, \] (29)

\[ A_\mu^{(8)} = \left[ 3 \lambda^2 - \rho^2 (\dot{a} \cdot c) \right] \frac{v_\mu}{\rho^3} - 3 \lambda \frac{a_\mu}{\rho^4} + \frac{\dot{a}_\mu}{\rho^2}, \] (30)

\[ A_\mu^{(10)} = \left\{ -15 \lambda^3 + 10 \lambda \rho^2 (\dot{a} \cdot c) - \rho^2 a^2 - \rho^3 (\ddot{a} \cdot c) \right\} \frac{v_\mu}{\rho^3} + \left[ 15 \lambda^2 - 4 \rho^2 (\dot{a} \cdot c) \right] \frac{a_\mu}{\rho^5} - 6 \lambda \frac{\dot{a}_\mu}{\rho^6} + \frac{\ddot{a}_\mu}{\rho^4}, \] (31)

\[ A_\mu^{(12)} = \left\{ 105 \lambda^2 \left[ \lambda^2 - \rho^2 (\dot{a} \cdot c) \right] \right\} \frac{v_\mu}{\rho^3} + 15 \lambda \rho^2 [\rho (\dot{a} \cdot c) + a] - \frac{5}{2} \rho^3 (a^2) \right\} \frac{a_\mu}{\rho^5} + 10 \rho^4 (\dot{a} \cdot c)^2 \frac{a_\mu}{\rho^7} \] \[ + 5 \left\{ -7 \lambda^2 + 4 \rho^2 (\dot{a} \cdot c) \right\} \frac{a_\mu}{\rho^8} + 5 \left\{ -2 \lambda^2 + 2 \rho^2 (\dot{a} \cdot c) \right\} \frac{a_\mu}{\rho^9} - 10 \lambda \frac{\dot{a}_\mu}{\rho^6} + \frac{\ddot{a}_\mu}{\rho^5}. \] (32)

Another way of looking at \( A_\mu^{(2p)} \) is to invoke the notion of prepotential. The prepotential \( H_\mu \) of the vector potential \( A_\mu \) is defined as

\[ A_\mu = \Box H_\mu. \] (33)

One can check that

\[ N_{p+1} \Box A_\mu^{(2p)} = (d - 2p) N_p A_\mu^{(2p+2)}, \quad p \geq 1. \] (34)

In other words, any 2n-dimensional retarded vector potential \( A_\mu^{(2n)} \) (up to a normalization factor) is the prepotential of the (2n + 2)-dimensional retarded vector potential \( A_\mu^{(2n+2)} \). Furthermore, \( A_\mu^{(2p)} \) can be produced by acting on \( A_\mu^{(2)} \) \( p - 1 \) times with the wave operator:

\[ A_\mu^{(2p)} = Z_{d,p}^{-1} \Box^{p-1} R_\mu = -Z_{d,p}^{-1} \Box^{p-1} A_\mu^{(2)}, \] (35)
where
\[
Z_{d,p} = (d - 2) (d - 4) \cdots (d - 2p) N_p = \frac{2^p (n - 1)!}{(n - p - 1)!} N_p.
\]

All the resulting vector potentials \(A^{(2p)}_\mu\), beginning with \(p = 2\), obey the Lorenz gauge condition. To see this, we note that \(\partial_\mu R_\mu = d - 1\), and so \(\Box \partial_\mu A^{(2)}_\mu = 0\).

This technique provides a further significant advantage if we observe that the action of the wave operator amounts to the action of the first-order differential operator
\[
\frac{1}{\rho} \frac{d}{ds}.
\]

We thus have
\[
A^{(2p)}_\mu = -\left(\frac{1}{\rho} \frac{d}{ds}\right)^{p-1} A^{(2)}_\mu.
\]

Indeed, (48) derives from (7) and (8) by noting that \(dR^2/ds = -2\rho\), \(dR_\mu/ds = -v_\mu\), and so
\[
-\frac{1}{\rho} \frac{d}{ds} A^{(2)}_\mu = \frac{1}{\rho} \frac{d}{ds} R_\mu = \frac{v_\mu}{\rho} = A^{(4)}_\mu.
\]

We now take a closer look at the field strengths \(F^{(2)}_{\mu\nu}\), \(F^{(4)}_{\mu\nu}\), and \(F^{(6)}_{\mu\nu}\) shown, respectively, in (20), (22), and (24). When their structure is compared with that of the vector potentials \(A^{(2)}_\mu\), \(A^{(4)}_\mu\), \(A^{(6)}_\mu\), \(A^{(8)}_\mu\) displayed in (27)–(30), it is apparent that
\[
\mathcal{F}^{(2)} = -A^{(2)} \wedge A^{(4)},
\]
\[
\mathcal{F}^{(4)} = -A^{(2)} \wedge A^{(6)},
\]
\[
\mathcal{F}^{(6)} = -A^{(2)} \wedge A^{(8)} - A^{(4)} \wedge A^{(6)}.
\]

In addition, one can verify that
\[
\mathcal{F}^{(8)} = -A^{(2)} \wedge A^{(10)} - 2A^{(4)} \wedge A^{(8)},
\]
\[
\mathcal{F}^{(10)} = -A^{(2)} \wedge A^{(12)} - 3A^{(4)} \wedge A^{(10)} - 2A^{(6)} \wedge A^{(8)}.
\]

We come to recognize that the retarded field strength \(\mathcal{F}^{(2p)}_{\mu\nu}\) can be expressed in a very compact and elegant form in terms of retarded vector potentials \(A^{(2m)}_\mu\), \(2 \leq m \leq p + 1\). Recall, the canonical representation of a general 2-form \(\omega^{(2n)}\) in spacetime of dimension \(d = 2n\) is the sum of \(n\) exterior products of 1-forms:
\[
\omega^{(2n)} = f_1 \wedge f_2 + \cdots + f_{2n-1} \wedge f_{2n}.
\]

In particular, by (45), \(\omega^{(10)}\) is decomposed into the sum involving five terms. However, (44) shows that the retarded field strength contains only three exterior products, two less than the canonical representation.

The validity of relations (40)–(44) can be seen by inspection. To derive them in a regular way, we take (40) as the starting point. If we apply \(Z_{d,p}^{-1} \Box^{p-1}\) to the left-hand side of this equation, then, in view of (35), we obtain \(\mathcal{F}^{(2)}\). Applying \(p - 1\) times the first-order differential operator (37) to the right-hand side of (40) and taking into account Leibnitz’s rule for differentiation of the product of two functions, in view of (58), we come to the desired result.
This explains the puzzling fact that the gauge-independent quantity $F^{(2 \mu)}$ is an algebraic function of gauge-dependent quantities $A^{(2m)}$. By the construction, the vector potentials $A^{(2m)}_{\mu}$, $m \geq 1$, are subject to the Lorenz gauge condition. Therefore, such $A^{(2m)}_{\mu}$ leave room for gauge modes $\partial_{\mu} \chi$ with $\chi$ being solutions to the wave equation, $\Box \chi = 0$. In our derivation of (11)–(14), we are entitled to apply the wave operator $\Box$, rather than the first-order differential operator (37), to the right-hand side of (10). All feasible gauge modes are then killed by the action of $\Box$.

We close this section with a remark about the behavior of the retarded electromagnetic field at spatial infinity. In general, $F^{(2n)}$ can be represented as the sum of exterior products of retarded vector potentials $A^{(2p)} \wedge A^{(2n-2p+4)}$. It is easy to understand that the infrared properties of $F^{(2n)}$ are controlled by the term $A^{(2)} \wedge A^{(2n+2)}$. (In fact, a comparison of the long-distance behavior of $A^{(2)} \wedge A^{(2n+2)}$ and $A^{(4)} \wedge A^{(2n)}$ will suffice for the present purposes. Since the least falling terms of $A^{(2n+2)}$ and $A^{(2n)}$ scale, respectively, as $\rho^{-n}$ and $\rho^{1-n}$, the leading long-distance asymptotics of $A^{(2)} \wedge A^{(2n+2)}$ is given by $\rho^{1-n}$ while that of $A^{(4)} \wedge A^{(2n)}$ is given by $\rho^{-n}$.) We segregate in $A^{(2n+2)}$ the term scaling as $\rho^{-n}$ by introducing the vectors

$$b^{(2n+2)}_{\mu} = \lim_{\rho \to \infty} \rho^n A^{(2n+2)}_{\mu}$$

and

$$\tilde{A}^{(2n+2)}_{\mu} = \frac{1}{\rho^n} b^{(2n+2)}_{\mu}.$$ 

All infrared irrelevant terms are erased by this limiting procedure, so that

$$A^{(2)} \wedge \tilde{A}^{(2n+2)}$$

represents the infrared part of $F^{(2n)}$.

We write explicitly $b^{(2n+2)}_{\mu}$ for $n = 1, 2, 3, 4, 5$:

$$b^{(4)}_{\mu} = v_{\mu},$$

$$b^{(6)}_{\mu} = - (a \cdot c) v_{\mu} + a_{\mu},$$

$$b^{(8)}_{\mu} = \begin{bmatrix} 3 (a \cdot c)^2 - (\dot{a} \cdot c) \end{bmatrix} v_{\mu} - 3 (a \cdot c) a_{\mu} + \dot{a}_{\mu},$$

$$b^{(10)}_{\mu} = - \begin{bmatrix} 15 (a \cdot c)^3 - 10 (a \cdot c) (\dot{a} \cdot c) + (\ddot{a} \cdot c) \end{bmatrix} v_{\mu} + \begin{bmatrix} 15 (a \cdot c)^2 - 4 (\dot{a} \cdot c) \end{bmatrix} a_{\mu} - 6 (a \cdot c) \dot{a}_{\mu} + \ddot{a}_{\mu},$$

$$b^{(12)}_{\mu} = \begin{bmatrix} 5 \left[ 3 \cdot 7 (a \cdot c)^2 \left( (a \cdot c)^2 - (\dot{a} \cdot c) \right) + 2 \left( \ddot{a} \cdot c \right)^2 + 3 (a \cdot c) (\ddot{a} \cdot c) - (\dddot{a} \cdot c) \right] v_{\mu} - 5 \left[ 3 (a \cdot c) \left[ 7 (a \cdot c)^2 - 4 (\dot{a} \cdot c) \right] + (\ddot{a} \cdot c) \right] a_{\mu} + 5 \left[ 9 (a \cdot c)^2 - 2 (\ddot{a} \cdot c) \right] \ddot{a}_{\mu} - 10 (a \cdot c) \dddot{a}_{\mu} + \dddot{a}_{\mu}. \end{bmatrix}$$

It follows from (50)–(53) that $b^{(6)}, \ldots, b^{(12)}$ are subject to the constraint

$$R \cdot b^{(2n+2)} = 0,$$

while $b^{(4)}$ is not. To derive (54), we note that, far apart from the world line, the field appears (locally) as a plane wave moving along a null ray that points toward the propagation vector $k_{\mu}$,

$$A_{\mu} \sim \epsilon_{\mu} \phi(k \cdot x),$$

(55)
Here, $\epsilon_\mu$ is the polarization vector, $\phi$ is an arbitrary smooth function of the phase $k \cdot x$, and the prime stands for the derivative with respect to the phase. Recall that $\partial^\mu A^{(2n)}_\mu = 0$ for $n \geq 2$. In view of (55), this equation becomes

$$(k \cdot \epsilon) \phi' = 0,$$  

which implies that the polarization vector is orthogonal to the propagation vector. On the other hand, $A^{(2n+2)}_\mu$ approaches $\bar{A}^{(2n+2)}_\mu$ as $\rho \to \infty$. Now the null vector $R_\mu$ acts as the propagation vector $k_\mu$. A comparison between (48) and (56) shows that $\epsilon_\mu$ should be identified with $b^{(2n+2)}_\mu$. Since $\partial_\mu R_\mu = d - 1$, the vector potential $A^{(2)}_\mu$ does not obey the Lorenz gauge condition, and hence (54) is not the case for $b^{(4)}_\mu$.

To sum up, the polarization of the retarded electromagnetic field is an imprint of the next even dimension $d + 2$, excluding $d = 2$ which is immune from the effect of $d = 4$.

### 3 Radiation

Apart from the overall numerical factor, the metric stress-energy tensor of the electromagnetic field takes the same form in any dimension,

$$\Theta_{\mu\nu} = \frac{1}{\Omega_{d-2}} \left( F_\alpha^{\mu} F^{\alpha\nu} + \frac{\eta_{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta} \right).$$ 

Let us substitute (8) into (58). Since the result is to be integrated over $(d - 1)$-dimensional spacelike surfaces, $\Theta^{\mu\nu}$ is conveniently split into two parts, nonintegrable and integrable,

$$\Theta^{\mu\nu} = \Theta^{\mu\nu}_I + \Theta^{\mu\nu}_{II}.$$ 

Here, our concern is only with the integrable part $\Theta^{\mu\nu}_{II}$. To identify this part of the stress-energy tensor as the radiation, we check the fulfillment of the following conditions [21], [12]:

(i) $\Theta^{\mu\nu}_I$ and $\Theta^{\mu\nu}_{II}$ are dynamically independent off the world line, that is,

$$\partial_\mu \Theta^{\mu\nu}_I = 0, \quad \partial_\mu \Theta^{\mu\nu}_{II} = 0,$$ 

(ii) $\Theta^{\mu\nu}_{II}$ propagates along the future light cone $C_+$ drawn from the emission point, and

(iii) the energy-momentum flux of $\Theta^{\mu\nu}_{II}$ goes as $\rho^{2-d}$ implying that the same amount of energy-momentum flows through spheres of different radii.

It has been found in the previous section that the infrared behavior of $F^{(2n)}$ is controlled by

$$A^{(2)} \land A^{(2n+2)}. $$  

More precisely, the leading long-distance term

$$A^{(2)} \land \bar{A}^{(2n+2)} $$

where $\bar{A}^{(2n+2)}_\mu$ is defined in (47), is responsible for the infrared properties of $F^{(2n)}$.

---

1 The term ‘infrared’ is used here in reference to what can be described by means of quantities which are either regular or having integrable singularities at the world line.
With (58), it is apparent that $\Theta_{\Pi}^{\mu\nu}$ is built up solely from the term shown in (62),

$$\Theta_{\Pi}^{\mu\nu} = \frac{-1}{N^2_\pi \Omega_{2n-2}} R^\mu R^\nu \left( A^{(2n+2)} \right) = \frac{-1}{N^2_\pi \Omega_{2n-2} \rho^{2n-2} c^\mu c^\nu \left( b^{(2n+2)} \right)^2}. \quad (63)$$

Let us check that $\Theta_{\Pi}^{\mu\nu}$ given by (63) meets every condition (i)–(iii), and hence this quantity is reasonable to call the radiation

In view of (47), the scaling properties of this $\Theta_{\Pi}^{\mu\nu}$ are in agreement with (iii). Furthermore, since the surface element of the future light cone $C_+$ is

$$d\sigma^\mu = c^\mu \rho^{2n-2} d\rho \Omega_{2n-2}, \quad (64)$$

where $c^\mu$ is a null vector on $C_+$, the flux of $\Theta_{\Pi}^{\mu\nu}$ through $C_+$ vanishes, $d\sigma^\mu \Theta_{\Pi}^{\mu\nu} = 0$. Therefore, $\Theta_{\Pi}^{\mu\nu}$ propagates along $C_+$ to suit (ii).

To verify that condition (i) holds, let us note that, for $\Theta_{\Pi}^{\mu\nu}$ and $j^\mu$ written, respectively, as (58) and (2),

$$\partial_{\nu} \Theta_{\Pi}^{\mu\nu} = -F^{\mu\nu} j_{\nu}. \quad (65)$$

Off the world line, (65) becomes

$$\partial_{\mu} \Theta_{\Pi}^{\mu\nu} = 0, \quad (66)$$

and hence either of two local conservation laws (60) implies the other one. It is sufficient to verify the conservation law for the $\Theta_{\Pi}^{\mu\nu}$. We have

$$\partial_{\mu} \Theta_{\Pi}^{\mu\nu} \propto \partial_{\mu} \left[ R^\mu R^\nu \left( \tilde{A}^{(2n+2)} \right)^2 \right] = R^\nu \left[ 2n \left( \tilde{A}^{(2n+2)} \right)^2 + (R \cdot \partial) \left( \tilde{A}^{(2n+2)} \right)^2 \right]. \quad (67)$$

Here the second equation is obtained using the differentiation rule (18) and the fact that $\delta_{\mu} = 2n$. Let us take into account that $b_{\mu}^{(2n+2)}$ depends on $v^\alpha$, $a^\alpha$, ... and their scalar products with $c^\alpha$. Since

$$(R \cdot \partial) \{ c^\nu, \ v^\nu, \ a^\nu, \ a^\nu, \ldots \} = 0, \quad (R \cdot \partial) \rho = \rho, \quad (68)$$

we apply $(R \cdot \partial)$ to $\tilde{A}_{\mu}^{(2n+2)}$ defined in (47) to conclude from (67) that $\partial_{\mu} \Theta_{\Pi}^{\mu\nu} = 0$. This is just the required result.

By (54), $b_{\mu}^{(2n+2)}$ is orthogonal to a null vector $R_{\mu}$. This suggests that $b_{\mu}^{(2n+2)}$ is a linear combination of a spacelike vector and the null vector $R_{\mu}$ itself. Referring to (48), $\tilde{A}_{\mu}^{(2n+2)}$ is defined up to adding $k R_{\mu}$, where $k$ is an arbitrary constant. If we impose the Lorenz gauge condition to this additional term, then $k (n - 1) = 0$, that is, $k = 0$ for $n \neq 1$. When it is considered that $b_{\mu}^{(2n+2)}$ is spacelike, (63) shows that $\Theta_{\Pi}^{00} \geq 0$. We thus see that $\Theta_{\Pi}^{00}$ represents positive field energy flowing outward from the source.

Let us calculate the radiation rate. The radiation flux through a $(d - 2)$-dimensional sphere enclosing the source is constant for any radius of the sphere. Therefore, the terms of $\Theta_{\mu\nu}$ responsible for this flux scale as $\rho^{2-d}$. The radiated energy-momentum is defined by

$$\mathcal{P}^\mu = \int_{\Sigma} d\sigma^\nu \Theta_{\Pi}^{\mu\nu}, \quad (69)$$

where $\Sigma$ is a $(d - 1)$-dimensional spacelike hypersurface. Since $\Theta_{\Pi}^{\mu\nu}$ involves only integrable singularities, and $\partial_{\nu} \Theta_{\Pi}^{\mu\nu} = 0$, the surface of integration $\Sigma$ in (69) may be chosen arbitrarily. It

---

3Strictly speaking, the radiation is represented by (58) only when $n \geq 2$. In a world with one temporal and one spatial dimension, the radiation is absent [14], [12].
is convenient to deform \( \Sigma \) to a tubular surface \( T_ε \) of small invariant radius \( ρ = ε \) enclosing the world line. The surface element on this tube is

\[
dσ^μ = \partial^μ ρ ρ^{d-2} dΩ_{d-2} ds = (v^μ + λc^μ) c^{d-2} dΩ_{d-2} ds.
\]  
(70)

Equation (69) becomes

\[
\mathcal{P}_{d}^{(2n)} = -\frac{1}{N^2 Ω_{2n-2}} \int_{−∞}^{∞} ds \int dΩ_{2n-2} c_μ \left( b^{2n+2} \right)^2,
\]

so that the radiation rate is given by

\[
\hat{\mathcal{P}}_{d}^{(2n)} = -\frac{1}{N^2 Ω_{2n-2}} \int dΩ_{2n-2} c_μ \left( b^{2n+2} \right)^2.
\]

This can be recast as

\[
\hat{\mathcal{P}}_{d}^{(2n)} = -\frac{1}{Z^2_{d,n} Ω_{2n-2}} \int dΩ_{2n-2} c_μ \left( \lim_{ρ→∞} ρ^n R_α \right)^2.
\]

Conceivably this form of the radiation rate might find use in a wider context of gauge theories.

The solid angle integration is greatly simplified if we introduce the spacelike normalized vector \( u^μ \) orthogonal to \( v^μ \),

\[
c^μ = v^μ + u^μ,
\]

and observe that the integrands are expressions homogeneous of some degree in \( u^μ \). Consider

\[
I_{μ_1…μ_p} = \frac{1}{Ω_{d-2}} \int dΩ_{d-2} u_μ_1 \cdots u_μ_p.
\]

In the case of odd number of multiplying vectors \( u^μ \), the integrals vanish. If the number of multiplying vectors \( u^μ \) is even, then the integration are made through the use of the following formulas

\[
I_{μν} = -\left( \frac{1}{d-1} \right)^v \mbox{v} \mu ν,
\]

\[
I_{αβμν} = \frac{1}{(d-1)(d+1)} \left( \left( \frac{v}{v} α_μ v_α β v ν v_ν \right) \right),
\]

\[
I_{αβγλμν} = -\frac{1}{(d-1)(d+1)(d+3)} \left( \left( \frac{v}{v} α_μ v_α β v γ v_γ v_γ v_γ v_γ v_μ \right) + \left( \frac{v}{v} α_γ v_γ β v ν v_ν v_ν v_μ \right) + \left( \frac{v}{v} α_β v_β γ v_γ v_γ v_γ v_λ v_μ \right) + \left( \frac{v}{v} α_γ v_γ β v_β γ v_γ v_γ v_μ v_λ \right) + \left( \frac{v}{v} α_β v_β γ v_γ v_γ v_λ v_μ v_μ \right) + \left( \frac{v}{v} α_μ v_μ γ v_γ v_γ v_γ v_λ v_λ \right) + \left( \frac{v}{v} α_ν v_ν β v_β γ v_γ v_γ v_μ v_μ \right) + \left( \frac{v}{v} α_ν v_ν β v_β v_λ v_λ v_μ \right) + \left( \frac{v}{v} α_ν v_ν β v_λ v_λ v_μ v_μ \right) + \left( \frac{v}{v} α_ν v_ν β v_λ v_λ v_λ v_μ \right) \right),
\]

which are readily derived (see, e. g., [22]). Here,

\[
\mbox{v} \mu ν = η_{μν} − v_μ v_ν
\]

(79)
is the operator that projects vectors onto a hyperplane with normal $v^\mu$, The number of terms in such decompositions of $I_{\mu_1\cdots\mu_k}$ proliferates with $k$: $I_{\mu_1\cdots\mu_4}$ contains 3 monomials $v^\mu v^\nu$, $I_{\mu_1\cdots\mu_6}$ involves 3·5 monomials $v^\mu v^\nu v^\rho v^\sigma$, $I_{\mu_1\cdots\mu_8}$ comprises 3·5·7 monomials $v^\mu v^\nu v^\rho v^\sigma v^\tau$, etc. If $k \geq 6$, then calculations with $I_{\mu_1\cdots\mu_k}$ are rather tedious, so that we restrict our discussion to the dimensions $d = 4$ and $d = 6$. In these cases we need only handling $I_{\mu \nu}$ and $I_{\alpha \beta \mu \nu}$.

Using the identities

$$v^2 = 1, \quad (v \cdot a) = 0, \quad (v \cdot \dot{a}) = -a^2,$$

we find from (50) and (51) that

$$\left(b^{(6)}\right)^2 = (a \cdot u)^2 + a^2,$$

$$\left(b^{(6)}\right)^2 = \left[(\perp \dot{a})^2 + 9 (a \cdot u)^2 a^2 + 9 (a \cdot u)^4 + (\dot{a} \cdot u)^2\right] - 3 \left[(a^2)^2 (a \cdot u) + 2 (a \cdot u)^2 (\dot{a} \cdot u)\right].$$

Thus, in four and six dimensions, the radiation rate is given, respectively, by

$$\dot{\mathcal{P}}^{(4)} = -\frac{2}{3} a^2 v_\mu$$

and

$$\dot{\mathcal{P}}^{(6)} = \frac{1}{9} \frac{1}{5 \cdot 7} \left\{ 4 \left[16 (a^2)^2 - 7 \dot{a}^2\right] v_\mu - 3 \cdot 5 (a^2)^2 a_\mu + 6 a^2 (\perp \dot{a})_\mu \right\}. \quad (84)$$

### 4 Discussion and outlook

Let us summarize our discussion of the methods for obtaining retarded field configurations due to a single point charge in $2n$-dimensional Minkowski spacetime. The retarded Green’s function technique is presently accepted as the standard approach. Iwanenko and Sokolow [20] pioneered the use of this technique. The approach based on the ansätze of a particular form, such as those defined in [110], [111], and [122], was developed in Ref. [14]. This procedure for solving Maxwell’s equations (without resort to Green’s functions) is found to be of particular assistance in solving the Yang–Mills equations [12]. It seems likely that the tool of greatest practical utility involves the notion of prepotential, in particular the simplest way for calculating the retarded vector potential $A^{(2n)}_\mu$ is given by Eq. (38).

Close inspection of exact solutions to $d$-dimensional Maxwell’s equations shows that the retarded field strength $\mathcal{F}^{(2n)}_{\mu \nu}$ generated by a point charge living in a $2n$-dimensional world is expressed in terms of the retarded vector potentials $A^{(2m)}_\mu$ due to this charge in $2m$-dimensional worlds nearby, Eqs. (40)–(44). The fact that the state of the retarded electromagnetic field in a given even-dimensional manifold is entangled with those of contiguous even-dimensional manifolds may be the subject of far-reaching philosophical speculations. To illustrate, it follows from (41) that, while living in $d = 4$, a charge feels a specific impact from $d = 2$ and $d = 6$. The responsibility for this entanglement may rest with either coexistence on an equal footing of different $2p$-branes in some braneworld scenario or manifestation of contiguous ‘parallel’ realms.

A notable feature of Eqs. (40)–(44) is that the world line $z^\mu(s)$ of the charge generating these field configurations is described by different numbers of the principal curvatures $\kappa_j$ for different spacetime dimensions. To be specific, we refer to Eq. (41). The world line appearing in $A^{(2)}_\mu$ is a planar curve, specified solely by $\kappa_1$, while that appearing in $A^{(6)}_\mu$ is a curve characterized (locally) by five essential parameters $\kappa_1$, $\kappa_2$, $\kappa_3$, $\kappa_4$, $\kappa_5$. If we regard the world line $z^\mu(s)$ in $\mathbb{M}_{1,2n-1}$ as
the basic object, then both projections of this curve onto lower-dimensional spacetimes and its extensions to higher-dimensional spacetimes are rather arbitrary. Nevertheless, Eqs. (40)–(44) are invariant under variations of these mappings of the world line $z^\mu(s)$.

The advanced fields $F_{\text{adv}}$ can be also represented as the sums of exterior products of 1-forms $A_{\mu}$ similar to (40)–(44), whereas combinations $\alpha F_{\text{ret}} + \beta F_{\text{adv}}$, $\alpha \beta \neq 0$, are not. Therefore, Eqs. (40)–(44) do not hold for field configurations satisfying the St"uckelberg–Feynman boundary condition. We thus see that the remarkably simple structures displayed in Eqs. (40)–(44) are inherently classical.

Based on Eqs. (40)–(44), we put the rate of radiated energy-momentum of electromagnetic field in a compact form, Eqs. (72) and (73). Let us recall that there are two alternative concepts of radiation, proposed by Dirac and Teitelboim (for a review see [11]); the latter was entertained in Sec. 3. Although these concepts have some points in common, they are not equivalent. Accordingly, the fact [afforded by (63) and (72)] that the radiation in 2n-dimensional spacetime is an infrared phenomenon stemming from the next even dimension $d = 2n + 2$ cannot be clearly recognized until the Teitelboim’s definition of radiation is invoked.

Why is it essential to draw the stress-energy tensor for introducing the concept of radiation? It is still common to see the assertion that the degrees of freedom related to the radiation may be identified directly in $F^{(2n)}$ if one takes the piece of $F^{(2n)}$ shown in (62) as the ‘radiation field’. However, this assertion is erroneous. First, the construction $A^{(2)} \wedge A^{(2n+2)}$ which allegedly plays the role of radiation field is in no sense dynamically independent of the rest of $F^{(2n)}$. Second, the bivector $\varpi = A^{(2)} \wedge A^{(2n+2)}$ is deprived of information about the vector $A_{\mu}^{(2n+2)}$. A pictorial view of $\varpi$ is the parallelogram of the vectors $A^{(2)}_{\mu}$ and $A^{(2n+2)}_{\mu}$. The bivector $\varpi$ is independent of concrete directions and magnitudes of the constituent vectors $A^{(2)}_{\mu}$ and $A^{(2n+2)}_{\mu}$; $\varpi$ depends only on the parallelogram’s orientation and area $S = |A^{(2)} \cdot A^{(2n+2)}|$. By virtue of (54), $A^{(2n+2)}_{\mu}$ makes no contribution to $S$. It can be shown (much as was done in [12], p. 181) that the term scaling as $\rho^{1-n}$ can be eliminated by a local SL$(2, \mathbb{R})$ transformation of the plane spanned by the vectors $A^{(2)}$ and $A^{(2n+2)}$ which leaves the bivector $\varpi$ invariant. In other words, there is a reference frame in which the ‘radiation field’ (62) vanishes over all spacetime (except for the future null infinity).

The implication of this argument is that the radiation is determined not only by the retarded field $F^{(2n)}$ as such but also by the frame of reference in which $F^{(2n)}$ is measured. On the other hand, the stress-energy tensor $\Theta^{\mu\nu}$ is not invariant under such SL$(2, \mathbb{R})$ transformations. $\Theta^{\mu\nu}$ carries information about both the field $F^{(2n)}$ and the frame which is used to describe $F^{(2n)}$.

References

[1] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, “The Hierarchy problem and new dimensions at a millimeter,” Phys. Lett. B 429 (1998) 263; hep-th/9803315.

[2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, “New dimensions at a millimeter to a Fermi and superstrings at a TeV,” Phys. Lett. B 436 (1998) 257; hep-th/9804398.

[3] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, “Phenomenology, astrophysics and cosmology of theories with sub-millimeter dimensions and TeV scale quantum gravity,” Phys. Rev. D 59 (1999) 086004; hep-th/9804398.
[4] K. Akama, “Pregeometry,” in *Gauge Theory and Gravitation*, Proceedings, Nara, 1982, edited by K. Kikkawa, N. Nakanishi and H. Nariai, Lecture Notes in Physics, 176, (Springer, Berlin, 1983) pp. 267-271; hep-th/0001113.

[5] V. A. Rubakov and M. E. Shaposhnikov, “Do we live inside a domain wall?,” *Phys. Lett. B* 125 (1983) 136.

[6] M. Visser, “An exotic class of Kaluza-Klein models,” *Phys. Lett. B* 159 (1985) 22.

[7] L. Randall and R. Sundrum, “A large mass hierarchy from a small extra dimension,” *Phys. Rev. Lett.* 83 (1999) 3370; hep-ph/9905221.

[8] L. Randall and R. Sundrum, “An alternative to compactification,” *Phys. Rev. Lett.* 83 (1999) 4690; hep-th/9906064.

[9] V. A. Rubakov, “Large and infinite extra dimensions: An introduction,” *Phys. Uspekhi* 44 (2001) 871; hep-ph/0104152.

[10] R. Maartens, “Brane world gravity,” *Living Rev. Rel.* 7 (2004) 7; gr-qc/0312059.

[11] B. P. Kosyakov, “Radiation in electrodynamics and the Yang–Mills theory,” *Sov. Phys.—Uspekhi*, 35 (1992) 135.

[12] B. Kosyakov, *Introduction to the Classical Theory of Particles and Fields* (Springer, Berlin, 2007).

[13] A. Mironov and A. Morozov, “On the problem of radiation friction beyond 4 and 6 dimensions,” hep-th/0710.5676v1.

[14] B. P. Kosyakov, “Exact solutions of classical electrodynamics and the Yang–Mills–Wong theory in even-dimensional spacetime,” *Theor. Math. Phys.* 119 (1999) 493; hep-th/0207217.

[15] D. Gal’tsov, “Radiation reaction in various dimensions,” *Phys. Rev. D* 66 (2002) 025016; hep-th/0112110.

[16] P. Kazinski, S. Lyakhovich, and A. Sharapov, “Radiation reaction and renormalization in classical electrodynamics of a point particle in any dimension,” *Phys. Rev. D* 66 (2002) 025017; hep-th/0201046.

[17] Yu. Yaremko, “Radiation reaction, renormalization and conservation laws in six-dimensional classical electrodynamics,” *J. Phys. A* 37 (2004) 1079.

[18] M. Gürses and Ö. Sarioğlu, “Liénard–Wiechert potentials in even dimensions,” *J. Math. Phys.* 44 (2003) 4672; hep-th/0303078v2.

[19] A. Mironov and A. Morozov, “Radiation beyond four-dimension space-time,” hep-th/0703097v1.

[20] D. Iwanenko & A. Sokolow, *Die Klassische Feldtheorie* (Akademie, Berlin, 1953). Translated from the Russian edition 1951.

[21] C. Teitelboim, “Splitting of Maxwell tensor: Radiation reaction without advanced fields,” *Phys. Rev. D* 1 (1970) 1572.