The bound-state soliton solutions of a higher-order nonlinear Schrödinger equation for inhomogeneous Heisenberg ferromagnetic system

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Abstract The inverse scattering of a higher-order nonlinear Schrödinger equation for inhomogeneous Heisenberg ferromagnetic system with zero boundary condition is calculated by an appropriate Riemann–Hilbert (RH) problem. The RH problem of reflection coefficient with multiple high-order poles is obtained. Meanwhile, the calculation formulas of bound-state (BS) solitons and multiple BS solitons corresponding to one high-order pole and multiple high-order poles are also calculated, respectively. Finally, the corresponding soliton solution model is calculated according to the corresponding formula. Simultaneously, we also obtain the BS soliton, the multiple BS soliton and the interaction between multiple BS solitons and multiple BS solitons.

Keywords A higher-order nonlinear Schrödinger equation · Heisenberg ferromagnetism system · Riemann–Hilbert problem · One high-order pole · Multiple higher-order poles · Bound-state soliton solution

1 Introduction

Nonlinear wave phenomena have very important applications in many fields, such as fluid mechanics, nuclear physics and optical fibers. Through their solutions, we can better understand these nonlinear wave phenomenon. Then, we can get the dynamic behavior of these waves and the stability of the solutions by analyzing the graphs of the solutions. It is also of great significance to study the dynamic characteristics of these nonlinear waves and the stability of the solutions [1–8].

In recent years, the Heisenberg magnetic spin chain has drawn much attention [9–13] because one-dimensional spin array can store quantum information [10]. At the same time, it is found that the Heisenberg ferromagnetic (HBF) models with different magnetic interactions are equivalent to a class of nonlinear evolution equations in semiclassical and continuous media [14–17].

Besides, an inhomogeneous HBF spin system with extended structure is also obtained [18]
where $\epsilon$ means the perturbation parameter, $h$ is the inhomogeneity of the medium, and $\overrightarrow{T}$ represents the spin vector relative to the space coordinate $t$ and the scale time $t$. It is found that Eq. (1) can degenerate into the HBF spin system at $\epsilon = h = 0$.

In addition, the soliton solutions of some nonlinear equations have been solved, such as the nonlinear Schrödinger (NLS) equation

$$i\Phi_t + \Phi^2\Phi^* + \frac{1}{2}\Phi_{xx} = 0,$$  

(2)

where the superscript $^*$ denotes the complex conjugate, and $\Phi$ is a function of $x$ and $t$. It has very significant physical significance in fluid mechanics, Bose–Einstein condensation, nonlinear optics and so on. For example, it can describe the group evolution process of quasi-monochromatic waves in weakly nonlinear dispersive media. Because of its crucial physical significance, the NLS equations have attracted extensive attention, especially after Zakharov and Shabat et al. [19,20].

In this paper, we study the higher-order NLS equation equivalent to Eq. (1) by mapping the spin vector function $\overrightarrow{T}$ to the unit tangent vector of the motion helical space curve [14], which has the following form [21]

$$i\epsilon_t + q^2q^* - 30i\epsilon_\sigma q^2(q^*)^2 - 10i\epsilon_\sigma q q^* - 10i\epsilon_q q_{xx} q^* - 10i\epsilon_q q_{xxx} q^* - 10i\epsilon_q^2 q_x^* - 20i\epsilon_q q_{s} q_{s} q^* + \frac{1}{2} q_{ss} - ihq_x - ieq = 0,$$  

(3)

where $q$ is a function of $x$ and $t$. The above equation is the compatibility condition of the following Lax pairs

$$\frac{\partial}{\partial x}\phi(x, t, \lambda) = U\phi(x, t, \lambda),$$

$$\frac{\partial}{\partial t}\phi(x, t, \lambda) = V\phi(x, t, \lambda),$$

(4)

where $\phi(x, t, \lambda)$ is a $2 \times 2$ matrix function, and $U$ and $V$ are expressed in the following form, respectively

$$U = -i\lambda\sigma_3 + Q,$$

$$V = \frac{1}{2}[-2i(h\lambda + \lambda^2 + 16i\lambda^3)\sigma_3 - i(1 + 16i\lambda^3)Q^2\sigma_3 - 12i\epsilon\lambda Q^3\sigma_3 + 8\epsilon\lambda^2(Q_x Q - Q Q_x) + 12\epsilon Q^2(Q_x Q - Q Q_x) - 4i\epsilon\lambda Q_x^2\sigma_3 + 4\epsilon\lambda(Q_x Q_x Q + Q Q_x \sigma_3) + 2hQ + 2Q + 2\epsilon(Q Q_{xxx} - Q_{xxx} Q) - 2\epsilon(Q_x Q_{xxx} - Q_{xxx} Q_x) + 32\epsilon\lambda^4 Q + 16i\lambda^2 Q^3 + 12Q^5 + 16i\lambda^3 Q_x \sigma_3 - 24i\epsilon\lambda Q^2 Q_x \sigma_3 - 12\epsilon Q^2 Q_x^* Q_x - 8\epsilon Q_x^2 Q_x^2 - 8\epsilon\lambda^2 Q_{xx}^2 - 16\epsilon Q^2 Q_{xx} - 4\epsilon Q Q_x^* Q_{xx}^* + 4i\epsilon\lambda Q_{xxx} \sigma_3 + 2\epsilon Q_{xxxx} - iQ_x \sigma_3].$$

(5)

with

$$Q = \left( \begin{array}{cc} 0 & q \\ -q^* & 0 \end{array} \right), \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

(6)

In [21], Jia and Ma et al. worked out the higher-order rogue wave solutions of Eq. (3) and analyzed their properties by the method of generalized Darboux transformation. Note that taking $\epsilon = h = 0$, then Eq. (3) can be reduced to Eq. (2). In [22], the three-parameter two-period solution family of NLS Eq. (2), the repetition, period doubling and amplification outside the conventional modulation instability band are studied in detail.

However, so far, more soliton solutions of Eq. (3) have not been found. In this paper, we extend the inverse scattering transformation (IST) method to the higher-order nonlinear Schrödinger equation for inhomogeneous HBF spin system, thus making up for the blank in this aspect. Furthermore, some correlated soliton solutions of Eq. (3) are obtained, which not only enriches the form of its solutions, but also helps us to understand some correlated dynamical behaviors of Eq. (3).

IST is one of the effective methods for solving nonlinear integrable equations. IST was first proposed by Gardner, Green and others, and IST is mainly used to study the initial value problem of the KdV Refs. [23,24]. After that, there are many works that extended IST to other Refs. [25–34]. Based on the asymptotic behavior of the Jost solution, the Gelfand–Levitan–Marchenko (GLM) integral equation is constructed, and soliton of the KdV equation is represented by solution of the GLM equation. In recent years, a new IST for focusing on the NLS equation has been proposed [35,36], which is an improvement on the standard IST and is very helpful in dealing with the equations with serious spectral singularities. For example, it is difficult
to find the rogue wave solution of the linear integrable equation with standard IST, but this new method can be used to solve it. (Reference [35] obtained rogue wave solution of the NLS equation by using new IST.)

The structure of this paper is as follows. In Sect. 2, we first construct the Riemann–Hilbert (RH) problem of Eq. (3) and then show the relationship between the solution of RH problem and the solution of Eq. (3). In Sect. 3, we derive the bound-state (BS) soliton and multiple BS soliton solutions of Eq. (3). In Sect. 4, we give some explicit soliton solutions and corresponding dynamical properties. In the last section, we give the conclusion.

2 The construction of Riemann–Hilbert problem

2.1 Spectral analysis

In Lax pair (4), if $\phi$ is a $2 \times 2$ matrix, we can obtain that the solution of Eq. (3) satisfies the symmetric condition

$$\phi(x, t, \lambda) = \sigma_0 \phi^*(x, t, \lambda^*) \sigma_0^{-1},$$

(7)

where $\sigma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then, we find that when $\phi(x, t, \lambda)$ is a solution to Eq. (3), $\sigma_0 \phi^*(x, t, \lambda) \sigma_0^{-1}$ is also a solution to Eq. (3). Simultaneously, we can also use this symmetric condition to define the scattering matrix $S(\lambda)$ and solve the RH problem.

We introduce the following hypothesis

$$\phi(x, t, \lambda) = \psi(x, t, \lambda) e^{-i[\lambda x + (h \lambda + \lambda^2 + 16 \epsilon \lambda^5) t] \sigma_3},$$

(8)

and consider both $x$-part and $t$-part of the Lax pair (4), subsequently. Thus, we obtain that the relationship of the Lax pair $\psi(x, t, \lambda)$ is

$$\begin{align*}
\psi_x + i \lambda [\sigma_3, \psi] &= Q \psi, \\
\psi_t + i (h \lambda + \lambda^2 + 16 \epsilon \lambda^5) [\sigma_3, \psi] &= \tilde{Q} \psi,
\end{align*}$$

(9)

with

$$\tilde{Q} = -i \left( \frac{1}{2} + 8 \epsilon \lambda^3 \right) Q \sigma_3 - 6 \epsilon \lambda Q^4 \sigma_3 + 4 \epsilon \lambda^2 (Q_x Q - Q Q_x) + 6 \epsilon Q^2 (Q_x Q - Q Q_x)
- 2i \epsilon \lambda Q^2 \sigma_3 + 2i \epsilon \lambda (Q^* Q_{xx} + Q Q_{xx}) \sigma_3
+ \epsilon (Q^*_x Q_{xx} - Q_x Q_{xx})
+ \epsilon (Q Q_{xxx} - Q^* Q^*_{xxx})
+ h Q + 2 \lambda Q + 16 \epsilon \lambda^4 Q + 8 \epsilon \lambda^2 Q^3 + 6 \epsilon Q^5
- \frac{i}{x} Q_x \sigma_3 - 8i \epsilon \lambda^3 Q_x \sigma_3 - 12i \epsilon \lambda Q^2 \sigma_3
- 6 \epsilon Q^* Q_x^* Q_x - 4e Q \sigma_3^2 - 4 \epsilon \lambda^2 Q_{xx}
- 8 \epsilon \lambda Q_{xxx} + 2e \lambda Q^* Q_{xx}^* + 2i \epsilon \lambda Q_{xxx} \sigma_3
+ \epsilon Q_{xxxx}.$$  

(10)

Subsequently, we will discuss the case of rapidly decay initial conditions

$$q(x, 0) = q_0(x) \quad \text{and} \quad q_0(x) \to 0 \quad \text{as} \quad x \to \infty$$

(11)

and then, asymptotic solution $\psi(x, t, \lambda)$ will satisfy

$$\psi_{\pm}(x, t, \lambda) \to I, \quad x \to \pm \infty.$$  

(12)

The Lax pair of $\psi(x, t, \lambda)$ is formally integrable, and its Jost solutions can be expressed in the following Volterra integral form

$$\psi_{\pm}(x, t, \lambda) = I + \int_{\pm \infty}^{x} e^{-i \lambda (x - y) \sigma_3} Q(y, t) \psi_{\pm}(y, t, \lambda) \, dy.$$  

(13)

**Proposition 2.1** The Jost solution $\psi_{\pm}(x, t, \lambda)$ has the following analytic properties

(I) $[\psi_-]$, $[\psi_+]$ are analytic in $\{ \lambda \mid \text{Im} \lambda > 0 \}$ and continuous in $\{ \lambda \mid \text{Im} \lambda \geq 0 \}$;

(II) $[\psi_+]$, $[\psi_-]$ are analytic in $\{ \lambda \mid \text{Im} \lambda < 0 \}$ and continuous in $\{ \lambda \mid \text{Im} \lambda \leq 0 \}$,

and it also has the following asymptotic property

(i) $[[\psi_-], [\psi_+]] \to I$ as $\lambda \to \pm \infty$ in $\{ \lambda \mid \text{Im} \lambda > 0 \}$;

(ii) $[[\psi_+], [\psi_-]] \to I$ as $\lambda \to \pm \infty$ in $\{ \lambda \mid \text{Im} \lambda < 0 \}$

where we need to explain that $[\psi_{\pm}]$, $[\psi_{\pm}]$ represents the 1-th column and the 2-th column of matrix $[\psi_{\pm}]$, respectively. In addition, the solution of Eq. (3) can be expressed as follows

$$q(x, t) = \lim_{x \to \infty} [2i \lambda (\psi_{\pm})_{12}],$$  

(14)

where $(\psi_{\pm})_{12}$ represent the 1-th row and the 2-th column of matrix $\psi_{\pm}$.

**Proof** For the proof of the above proposition, we can refer to Refs. [37, 38].

According to Abel’s theorem, any solution $\phi(x, t, \lambda)$ of (4) satisfies $\partial_t (\det \phi) = \partial_t (\det \phi) = 0$. Due to
By using the symmetric condition given in (7), we obtain
\[
\lim_{x \to \pm \infty} \phi(x, t, \lambda) = e^{-i[\lambda x + (h\lambda + \lambda^2 + 16\epsilon\lambda^3)t]\sigma_3},
\]
we have
\[
det \phi_\pm(x, t, \lambda) = 1, \quad x, t \in \mathbb{R}; \quad (15)
\]
det \(\phi_+(x, t, \lambda)\) and det \(\phi_-(x, t, \lambda)\) are linearly correlated, where \(\phi_+(x, t, \lambda)\) and \(\phi_-(x, t, \lambda)\) represent the solution of scattering problem (4), and they have different asymptotic properties. Let
\[
\phi_+(x, t, \lambda) = \phi_-(x, t, \lambda)S(\lambda), \quad (16)
\]
or
\[
\psi_+(x, t, \lambda) = \psi_-(x, t, \lambda)e^{-i[\lambda x + (h\lambda + \lambda^2 + 16\epsilon\lambda^3)t]t}\sigma_3, \quad S(\lambda) = e^{i[\lambda x + (h\lambda + \lambda^2 + 16\epsilon\lambda^3)t]t}\sigma_3, \quad (17)
\]
where \(S(\lambda)\) is often referred to as the scattering matrix. By using the symmetric condition given in (7), we obtain
\[
S(\lambda) = \begin{pmatrix}
c^s(\lambda^*) & d(\lambda) \\
d^s(\lambda^*) & c(\lambda)
\end{pmatrix}, \quad (18)
\]
c(\lambda) and \(d(\lambda)\) are often referred to as the scattering date. Moreover, on the basis of (17) and (18), we get
\[
S(\lambda) = e^{i[\lambda x + (h\lambda + \lambda^2 + 16\epsilon\lambda^3)t]t}\sigma_3\psi_-(x, t, \lambda)e^{-i[\lambda x + (h\lambda + \lambda^2 + 16\epsilon\lambda^3)t]t}\sigma_3,
\]
or
\[
c(\lambda) = det([\psi_-]_1), \quad [\psi_-]_2,
\]
d(\lambda) = \(e^{2i[\lambda x + (h\lambda + \lambda^2 + 16\epsilon\lambda^3)t]t}\sigma_3\delta([\psi_+]_2), \quad [\psi_-]_2).
\]

From the above formula, the scattered data \(c(\lambda)\) and \(d(\lambda)\) can be represented by the initial data \(q_0(x)\). However, given the initial data \(q_0(x)\), we obtain the Jost solution \(\psi_\pm(x, t, \lambda)\), and then, \(c(\lambda)\) and \(d(\lambda)\) can be calculated by Eqs. (13) and (20).

According to the analytic analysis of Jost solutions \(\psi_\pm(x, t, \lambda)\) in property 2.1, we have the following properties.

**Proposition 2.2** The scattering date \(c(\lambda)\) and \(d(\lambda)\) satisfy

\(c(\lambda)^2 + |d(\lambda)|^2 = 1\) \(\lambda \in \mathbb{R} \); \(d(\lambda) \to 0\) as \(\lambda \to \infty \); \(\lambda \to \infty \); \(d(\lambda) \to 0\) as \(\lambda \to \infty \); \(d(\lambda)\) is analytic in \(\{\lambda \mid \text{Im} \lambda > 0\}\); \(d(\lambda)\) is continuous on the real \(\lambda\).

2.2. The Riemann–Hilbert problem

Reviewing the analytical properties of \(\psi_\pm(x, t, \lambda)\) in property 2.1, we rewrite the scattering relation (17) into a piecewise meromorphic \(2 \times 2\) matrix function. \(M(x, t, \lambda)\) is defined as

\[
M(x, t, \lambda) = \begin{pmatrix}
|\psi_-|_1(x, t, \lambda) & \frac{[\psi_+]_2(x, t, \lambda)}{c(\lambda)} \\
\frac{[\psi_-]_1(x, t, \lambda)}{c(\lambda^*)} & [\psi_-]_2(x, t, \lambda)
\end{pmatrix}, \quad \text{Im} \lambda > 0,
\]

\[
M(x, t, \lambda) = \begin{pmatrix}
|\psi_-|_1(x, t, \lambda) & \frac{[\psi_+]_2(x, t, \lambda)}{c(\lambda)} \\
\frac{[\psi_-]_1(x, t, \lambda)}{c(\lambda^*)} & [\psi_-]_2(x, t, \lambda)
\end{pmatrix}, \quad \text{Im} \lambda < 0,
\]

and

\[
M_\pm(x, t, \lambda) = \lim_{\epsilon \to 0^+} M(x, t, \lambda \pm i\epsilon),
\]

\(\epsilon, \lambda \in \mathbb{R}\). According to the scattering relation (17), the limit value \(M_\pm(x, t, \lambda)\) satisfies the following conditions

\[
M_+(x, t, \lambda) = M_-(x, t, \lambda)e^{i[\lambda x + (h\lambda + \lambda^2 + 16\epsilon\lambda^3)t]t}\sigma_3
\]

\[
J(\lambda)e^{i[\lambda x + (h\lambda + \lambda^2 + 16\epsilon\lambda^3)t]t}\sigma_3, \quad \lambda \in \mathbb{R},
\]

and jump matrix \(J(\lambda)\) is

\[
J(\lambda) = \begin{pmatrix} 1 & r(\lambda) \\ r^*(\lambda^*) & 1 + |r(\lambda)|^2 \end{pmatrix},
\]

with \(r(\lambda) = \frac{d(\lambda)}{c(\lambda)}\).

**Proposition 2.3** According to the definition of \(M(x, t, \lambda)\), the RH problem satisfies the following properties

(I) \(d M = 1\);

(II) **Normalization:** \(M(x, t, \lambda) \to I\) as \(\lambda \to \infty\);

(III) **Symmetry**

\[
M^*(x, t, \lambda^*) = \sigma_0 M(x, t, \lambda)\sigma_0^{-1}.
\]

In addition, on the basis of expression (14), we can use \(M(x, t, \lambda)\) to reconstruct the new solution of Eq. (3) as follows

\[
q(x, t) = \lim_{\lambda \to \infty} 2i\lambda M_{12}(x, t, \lambda),
\]

If \(c(\lambda) \neq 0\) for all \(\lambda\), we can get that the RH problems constructed by (21) are regular and can uniquely be solved by Plemelj formula [39]. If \(c(\lambda) = 0\) at certain discrete positions, the RH problems become singular. For the singular RH problems, we can transform it into regular RH problems by using the dressing method [40]. In the abstract, we can solve the solution of singular RH problems accurately by the Plemelj formula and
dressing method \cite{41–45}. However, when the reflection coefficient has multiple higher-order poles \cite{46}, the ansatz using the dressing matrix may become complex. Therefore, in the next section of the calculation, we will solve the RH problems directly without using the dressing matrix.

3 The RH problem with multiple higher-order poles

In order to obtain the explicit solution of Eq. \eqref{3}, in this section, we will consider the reflection-less case, i.e., \( d(\lambda) = 0 (\lambda \in \mathbb{R}) \). Since \( c(\lambda) \) is analytic on the upper half plane, there are only finite zeros points and no singular point on the upper half plane. Different from other solutions of equations, \( c(\lambda) \) has both simple zero points and higher-order zeros points. Therefore, we suppose \( c(\lambda) \) and \( c^*(\lambda^*) \) have higher-order poles \( \{ \lambda_j, \text{Im}\lambda_j > 0 \}_{j=1}^N \) and \( \{ \lambda_j^*, \text{Im}\lambda_j^* < 0 \}_{j=1}^N \). That is, \( c(\lambda) \) and \( c^*(\lambda^*) \) are extended to

\[
\begin{align*}
  c(\lambda) &= (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \times \cdots \times (\lambda - \lambda_N)^{n_N} c_0(\lambda), \\
  c^*(\lambda^*) &= (\lambda - \lambda_1^*)^{n_1}(\lambda - \lambda_2^*)^{n_2} \times \cdots \times (\lambda - \lambda_N^*)^{n_N} c_0^*(\lambda^*),
\end{align*}
\]

where \( c_0(\lambda) \neq 0 \) for all \( \lambda \) (Im\( \lambda_j > 0 \)).

Assuming \( \lambda = \lambda_j \) in the scattering relation \eqref{17}, we can obtain that the Jost solution \( [\psi_+]|_2 \) is directly proportional to \( [\psi_-]|_2 \), where the relation is

\[
[\psi_+]|_2(x, t, \lambda_j) = d(\lambda_j) e^{-2i[\lambda_j t + (\lambda_j^2 + \lambda_j^2 + 16\epsilon \lambda_j^5) t]} [\psi_-]|_1(x, t, \lambda_j),
\]

and the symmetry condition \eqref{7} is also satisfied

\[
[\psi_+]|_1(x, t, \lambda_j^*) = -d^*(\lambda_j^*) e^{-2i[\lambda_j^* t + (\lambda_j^* + \lambda_j^* + 16\epsilon \lambda_j^5) t]} [\psi_-]|_2(x, t, \lambda_j^*).
\]

Combined with the definition of RH problems given in \eqref{21}, we can obtain the following residual conditions

\[
\begin{align*}
  \text{Res}_{\lambda = \lambda_j} \left( \frac{[\psi_+]|_2(x, t, \lambda)}{e^{\lambda_j t}} \right) &= \text{Res}_{\lambda = \lambda_j} \left( \frac{d(\lambda)}{e^{\lambda_j t}} e^{-2i[\lambda_j t + (\lambda_j^2 + \lambda_j^2 + 16\epsilon \lambda_j^5) t]} [\psi_-]|_1(x, t, \lambda) \right), \\
  \text{Res}_{\lambda = \lambda_j^*} \left( \frac{[\psi_+]|_2(x, t, \lambda^*)}{e^{\lambda_j^* t}} \right) &= \text{Res}_{\lambda = \lambda_j^*} \left( \frac{d^*(\lambda_j^*)}{e^{\lambda_j^* t}} e^{2i[\lambda_j^* t + (\lambda_j^* + \lambda_j^* + 16\epsilon \lambda_j^5) t]} [\psi_-]|_2(x, t, \lambda^*) \right),
\end{align*}
\]

\[
\begin{equation}
\text{Res}_{\lambda = \lambda_j} \left( \frac{[\psi_+]|_2(x, t, \lambda)}{e^{\lambda_j t}} \right) = \text{Res}_{\lambda = \lambda_j^*} \left( \frac{d(\lambda)}{e^{\lambda_j t}} e^{-2i[\lambda_j t + (\lambda_j^2 + \lambda_j^2 + 16\epsilon \lambda_j^5) t]} [\psi_-]|_1(x, t, \lambda) \right),
\end{equation}
\]

\[
\begin{equation}
\text{Res}_{\lambda = \lambda_j^*} \left( \frac{d^*(\lambda_j^*)}{e^{\lambda_j^* t}} e^{2i[\lambda_j^* t + (\lambda_j^* + \lambda_j^* + 16\epsilon \lambda_j^5) t]} [\psi_-]|_2(x, t, \lambda^*) \right),
\end{equation}
\]

which are very useful for us to solve the solutions related to simple poles. When \( c(\lambda) \) has only \( N \) single zeros, the RH problem can be solved by subtracting the residue condition and using Plemelj formula. The formula of \( N \)-order soliton solution is obtained by \eqref{26}. Among them, we can refer to \cite{37,40,47,48}. However, the residual condition is not sufficient since \( c(\lambda) \) has multiple higher-order zeros points, so we need to consider the coefficients related to the much higher negative powers of \( \lambda - \lambda_j \) and \( \lambda - \lambda_j^* \). Here, we work on the simplest case first that \( c(\lambda) \) has only one high-order zero point.

3.1 The RH problem with one higher-order pole

Let

\[
c(\lambda) = (\lambda - \lambda_0)^N c_0(\lambda),
\]

\[
\text{with } c_0(\lambda) \neq 0 \text{ for all } \lambda (\text{Im}\lambda_j > 0),
\]

Using the Laurent series expansion, we can expand \( r(\lambda) \) and \( r^*(\lambda^*) \) as follows

\[
r(\lambda) = r_0(\lambda) + \sum_{n=1}^{N} \frac{r_n}{(\lambda - \lambda_0)^n},
\]

\[
r^*(\lambda^*) = r_0^*(\lambda^*) + \sum_{n=1}^{N} \frac{r_n^*}{(\lambda - \lambda_0^*)^n},
\]

where \( r_n(n = 1, 2, \cdots, N) \) are defined by

\[
r_n = \lim_{\lambda \to \lambda_0} \frac{1}{(N - n)!} \frac{\partial^{N-n}}{\partial \lambda^{N-n}}[(\lambda - \lambda_0)^N r(\lambda)],
\]

\[
n = 1, 2, \cdots, N,
\]

and \( r_0(\lambda) \) is analytic for all \( \lambda (\text{Im}\lambda_j > 0) \). In addition, through the definition of \( M_{11}(x, t, \lambda) \) given by \eqref{21}, we conclude that \( r(\lambda) = r_0(\lambda) \) is the \( N \)-order pole of \( M_{11}(x, t, \lambda) \), and \( r(\lambda) = r_0(\lambda) \) is the \( N \)-order pole of \( M_{12}(x, t, \lambda) \). According to property 2.3, we set the normalization condition as

\[
\begin{align*}
  M_{11}(x, t, \lambda) &= 1 + \sum_{m=1}^{N} \frac{R_m(x, t)}{(\lambda - \lambda_0)^m}, \\
  M_{12}(x, t, \lambda) &= \sum_{m=1}^{N} \frac{W_m(x, t)}{(\lambda - \lambda_0)^m},
\end{align*}
\]
where $R_m(x, t)$ and $W_m(x, t)$ are unknown functions to be determined. Once $R_m(x, t)$ and $W_m(x, t) (m = 1, 2, \ldots, N)$ are obtained, solution $M(x, t, \lambda)$ of the singular RH problem is obtained. Simultaneously, the solution $q(x, t)$ of the original Eq. (3) is obtained from expression (26).

First of all, $R_m(x, t)$ and $W_m(x, t) (m = 1, 2, \ldots, N)$ are solved. On the basis of Taylor series expansion, we obtain

$$e^{-2i[\lambda x + (b+\lambda^2 + 16e^x) \lambda^s]} = \sum_{s=0}^{+\infty} \zeta_s(x, t) (\lambda - \lambda_0)^s,$$

$$e^{2i[\lambda x + (b+\lambda^2 + 16e^x) \lambda^s]} = \sum_{s=0}^{+\infty} \xi_s(x, t) (\lambda - \lambda_0)^s,$$  

(35a)

where

$$\zeta_s(x, t) = \lim_{\lambda \to \lambda_0} \frac{1}{s!} \frac{\partial^s}{\partial \lambda^s} e^{-2i[\lambda x + (b+\lambda^2 + 16e^x) \lambda^s]},$$  

(36a)

$$\mu_s(x, t) = \lim_{\lambda \to \lambda_0} \frac{1}{s!} \frac{\partial^s}{\partial \lambda^s} M_{11}(x, t, \lambda),$$  

(36b)

$$\nu_s(x, t) = \lim_{\lambda \to \lambda_0} \frac{1}{s!} \frac{\partial^s}{\partial \lambda^s} M_{12}(x, t, \lambda), \quad s = 0, 1, 2, \ldots, \quad (36c)$$

The scattering relation given in (17) is combined with the RH problem given in (21), and then, the corresponding coefficients of $(\lambda - \lambda_0)^{-m}$ and $(\lambda - \lambda_0^*)^{-m}$ are calculated, respectively. $R_m(x, t)$ and $W_m(x, t) (m = 1, 2, \ldots, N)$ can be expressed by $\mu_s(x, t)$ and $\nu_s(x, t)$ ($s = 1, 2, \ldots, N$) as follows

$$R_m(x, t) = -\sum_{n=m}^{N} \sum_{s=0}^{n-m} r_n^s s_n^{m-s} (x, t) \nu_s(x, t),$$  

(37a)

$$W_m(x, t) = \sum_{n=m}^{N} \sum_{s=0}^{n-m} r_n^s s_n^{m-s} (x, t) \mu_s(x, t),$$  

(37b)

with $m = 1, 2, \ldots, N$.

Remark 3.1 Among them, (37a) and (36b) are actually only a general expression of the residue condition (30a) and (30b). When $N = 1$, the above relation with residue condition is still valid.

In fact, $\mu_s(x, t)$ and $\nu_s(x, t) (s = 1, 2, \ldots, N)$ also be represented by $R_m(x, t)$ and $W_m(x, t) (m = 1, 2, \ldots, N)$. By reviewing the definitions of $\mu_s(x, t)$ and $\nu_s(x, t)$ given in (36b) and (36c), and substituting (34) into the definitions of $\mu_s(x, t)$ and $\nu_s(x, t)$, we have

$$\nu_s(x, t) = \sum_{m=1}^{N} \left( \frac{m + s - 1}{s} \right) \frac{(-1)^s W_m(x, t)}{(\lambda_0^* - \lambda_0)^{m+s}}, \quad s = 0, 1, 2, \ldots,$$  

(38a)

$$\mu_s(x, t) = \left\{ \begin{array}{ll}
1 + \sum_{m=1}^{N} \frac{R_m(x, t)}{(\lambda_0^* - \lambda_0)^m}, & s = 0, \\
\sum_{m=1}^{N} \left( \frac{m + s - 1}{s} \right) \frac{(-1)^s R_m(x, t)}{(\lambda_0^* - \lambda_0)^{m+s}}, & s = 1, 2, \ldots.
\end{array} \right.$$  

(38b)

Substituting (38a) and (38b) into (37a) and (37b) leads to

$$R_m(x, t) = -\sum_{n=m}^{N} \sum_{s=0}^{n-m} \sum_{l=1}^{n-s} \left( \frac{l + s - 1}{s} \right) \frac{(-1)^s r_n^s r_{n-m-s}^l (x, t) W_l(x, t)}{(\lambda_0^* - \lambda_0)^{l+s}},$$  

(39a)

$$W_m(x, t) = \sum_{n=m}^{N} \sum_{s=0}^{n-m} \sum_{l=1}^{n-s} \frac{(-1)^s r_n^s r_{n-m-s}^l (x, t) W_l(x, t)}{(\lambda_0^* - \lambda_0)^{l+s}}.$$  

(39b)

At this point, introducing the following notations

$$|R| = [R_1 \ R_2 \ \cdots \ R_N]^T,$$

$$|W| = [W_1 \ W_2 \ \cdots \ W_N]^T,$$

$$\Omega = [\Omega_{m,n}]_{N \times N} = \left[ -\sum_{n=m}^{N} \sum_{s=0}^{n-m} \left( \frac{l + s - 1}{s} \right) \frac{(-1)^s r_n^s r_{n-m-s}^l (x, t)}{(\lambda_0^* - \lambda_0)^{l+s}} \right]_{N \times N},$$  

(40a)

$$|\xi| = [\xi_1 \ \xi_2 \ \cdots \ \xi_N]^T,$$

$$\xi_m = \sum_{n=m}^{N} r_n^s f_{n-m} (x, t), \quad m, l = 1, 2, \ldots, N.$$  

(40b)
where superscript “$T$” means transposition. Thus, we can rewrite (39a) and (39b) in the following form
\[
|R| = \Omega |W|, \quad |W| = |\xi| - \Omega^2 |R|.
\] (41)

Then, by using Cramer’s rule, $|R|$ and $|W|$ are easily solved as
\[
|R| = \Omega (I + \Omega^2)^{-1} |\xi|,
\]
\[
|W| = (I + \Omega^2)^{-1} |\xi|.
\] (42)

Substituting $|R|$ and $|W|$ into expressions $M_{11}$ $(x, t, \lambda)$ and $M_{12}(x, t, \lambda)$ given by (34), we obtain
\[
M_{11}(x, t, \lambda) = 1 + \sum_{m=1}^{N} \frac{R_m(x, t)}{(\lambda - \lambda_0)^m}
\]
\[
= \frac{\det(I + \Omega^2 + |\xi| (Y(\lambda)|\Omega)}{\det(I + \Omega^2)}, \quad (43a)
\]
\[
M_{12}(x, t, \lambda) = \sum_{m=1}^{N} \frac{G_m(x, t)}{(\lambda - \lambda_0)^m}
\]
\[
= \frac{\det(I + \Omega^2 + |\xi| (Y(\lambda^*)|\Omega)}{\det(I + \Omega^2)} - 1, \quad (43b)
\]
where
\[
(Y(\lambda)) = \left[ \begin{array}{c|c|c|c|c}
1 & 1 & \cdots & 1 \\
(\lambda - \lambda_0) & (\lambda - \lambda_0^2) & \cdots & (\lambda - \lambda_0^N) \\
\end{array} \right].
\] (44)

In the symmetric property given by (25), we obtain
\[
M_{21}(x, t, \lambda) = 1 - \frac{\det(I + \Omega^2 + |\xi^*| (Y(\lambda)|\Omega)}{\det(I + \Omega^2)}, \quad (45a)
\]
\[
M_{22}(x, t, \lambda) = \frac{\det(I + \Omega^2 + |\xi^*| (Y^*(\lambda^*)|\Omega)}{\det(I + \Omega^2)} - 1. \quad (45b)
\]

On the basis of (26), we derive the following theorem.

**Theorem 3.1** Based on the rapidly decaying initial condition given by (11), the n-th-order BS soliton of Eq. (3) is
\[
q(x, t) = 2i \left[ \frac{\det(I + \Omega^2 + |\xi| (Y(\lambda)|\Omega)}{\det(I + \Omega^2)} - 1 \right], \quad (46)
\]
where $(Y(\lambda)) = [1 \ 0 \ 0 \ \cdots \ 0]_{1 \times N}.$

**Proof** From the expression of $M_{12}(x, t, \lambda)$ given in (43b), we get
\[
q(x, t) = \lim_{\lambda \to \infty} 2i\lambda M_{12}(x, t, \lambda)
\]
\[
= \lim_{\lambda \to \infty} 2i\lambda \times (Y*(\lambda^*)|\Omega)
\]
\[
= \lim_{\lambda \to \infty} 2i\lambda \times (Y*(\lambda^*)|I + \Omega^2)^{-1} |\xi|
\]
\[
= 2i \times \left( \det(I + \Omega^2 + |\xi| (Y(\lambda)) - 1 \right]. \quad (47)
\]

The above theorem is proved. $\square$

**Remark 3.2** If $N > 1$, the BS solution of the equation can be obtained. Meanwhile, when $N = 1$, we can also use expression (46) to get the first-order soliton solution.

### 3.2 RH problem with N higher-order poles

Now, we will discuss the more general case that $c(\lambda)$ has $N$ higher-order zeros points $\lambda_1, \lambda_2, \cdots, \lambda_N$, $\lambda_1, \lambda_2, \cdots, \lambda_N$ are on the upper half plane of the $\lambda$-plane, and their powers are $n_1, n_2, \cdots, n_N$, respectively. Similar to the case of the one higher-order pole discussed above, where we also use the Laurent series expansion, $r(\lambda)$ can be expressed as follows
\[
r(\lambda) = r_0(\lambda) + \frac{r_{1,1}}{(\lambda - \lambda_1)^{n_1}} + \frac{r_{1,2}}{(\lambda - \lambda_1)^2} + \cdots
\]
\[
+ \frac{r_{2,1}}{(\lambda - \lambda_2)^{n_2}} + \frac{r_{2,2}}{(\lambda - \lambda_2)^2} + \cdots
\]
\[
+ \frac{r_{N,1}}{(\lambda - \lambda_N)^{n_N}} + \frac{r_{N,2}}{(\lambda - \lambda_N)^2} + \cdots
\]
\[
+ \frac{r_{N,N}}{(\lambda - \lambda_N)^n N} = r_0(\lambda) + \sum_{j=1}^{N} \sum_{m=1}^{n_j} \frac{r_{j,m}}{(\lambda - \lambda_j)^m}, \quad (48)
\]
and
\[
r*(\lambda^*) = r_0*(\lambda^*) + \sum_{j=1}^{N} \sum_{m=1}^{n_j} \frac{r_{j,m}}{(\lambda - \lambda_j^*)^m}, \quad (49)
\]
where
\[
r_{j,m} = \lim_{\lambda \to \lambda_j} \frac{1}{(n_j - m_j)!} \frac{\lambda^{n_j - m_j}}{(\lambda - \lambda_j)^{n_j - m_j}}
\]
and \( r_0(\lambda) \) is analytic for all \((\text{Im}\lambda_j > 0)\).

By using the similar method, we obtain the following theorem.

**Theorem 3.2** If \( c(\lambda) \) has \( N \)-different high-order poles, we obtain a result similar to that of expression (46) according to the rapidly initial condition of (11). Furthermore, the multiple BS solitons of Eq. (3) are

\[
q(x, t) = 2i \left[ \frac{\det(I + \Omega^* \Omega + |\xi|^2|Y_0|)}{\det(I + \Omega^* \Omega)} - 1 \right],
\]

where

\[
|\xi| = [\xi_1 \xi_2 \cdots \xi_N]^T,
\]

\[
\langle Y_0 \rangle = [Y_0^0 Y_0^0 \cdots Y_0^N],
\]

\[
\xi_j = [\xi_{j,1} \xi_{j,2} \cdots \xi_{j,n_j}],
\]

\[
Y_j^0 = [1 0 \cdots 0]_{1 \times n_j},
\]

\[
\xi_{j,l} = \sum_{m_j=l}^{n_j} r_{j,m_j} f_{j,m_j-l}(x, t),
\]

\[
\Omega = \begin{bmatrix} [\omega_{11}] & [\omega_{12}] & \cdots & [\omega_{1N}] \\ [\omega_{21}] & [\omega_{22}] & \cdots & [\omega_{2N}] \\ \vdots & \vdots & \ddots & \vdots \\ [\omega_{N1}] & [\omega_{N2}] & \cdots & [\omega_{NN}] \end{bmatrix},
\]

and \([\omega_{j,l}](j, l = 1, 2, \cdots, N)\) are \( n_j \times n_l \) matrices with

\[
[\omega_{j,l}] = ([\omega_{j,l}]_{p,e})_{n_j \times n_l}
\]

\[
= \left( -\sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \begin{pmatrix} e + s_j - 1 \\ s_j \end{pmatrix} \right) \frac{(-1)^{s_j} r_{j,m_j} f_{j,m_j-l}(x, t)}{(\lambda_j - \lambda)^{y_j+e}}
\]

\[
(54)
\]

**Remark 3.3** When \( c_0 \) has only one higher-order zero point, we can transform the formula of \( q(x, t) \) expressed by (51) into the form of formula (46). Therefore, we get a special case that formula (46) is only (51).

From the above results, we can see that \([\omega_{j,l}]\) given by (53) is similar to \( \Omega \) given by (40b). It can also be said that a block matrix of \( \Omega \) composed of multiple higher-order poles (53) is constructed by a matrix \( \Omega \) with one high-order pole \( \Omega \) of (40b). Based on the above formula, we will find the solution of Eq. (3) in the next section.

4 The solutions of Eq. (3)

4.1 Solution related to two simple pole

Suppose that \( \lambda_1 = a_1 + ib_1 \) and \( \lambda_2 = a_2 + ib_2 \) are one simple pole points of \( c(\lambda) \), that is, \( r(\lambda) \), which can be expressed in the following form

\[
r(\lambda) = r_0(\lambda) + \frac{r_{1,1}}{\lambda - \lambda_1} + \frac{r_{2,1}}{\lambda - \lambda_2}.
\]

According to Theorem 3.2, we get that \( \Omega \) can be expressed as follows

\[
\Omega_{1,1} = -\frac{r_{1,1}^* s_{1,0}^*}{\lambda_1^* - \lambda_1}, \quad \Omega_{1,2} = -\frac{r_{1,1}^* s_{1,0}^*}{\lambda_1^* - \lambda_2},
\]

\[
\Omega_{2,1} = -\frac{s_{2,1}^* s_{2,0}^*}{\lambda_2^* - \lambda_1}, \quad \Omega_{2,2} = -\frac{s_{2,1}^* s_{2,0}^*}{\lambda_2^* - \lambda_2},
\]

(56)

\[
|\xi| = [\xi_1 \xi_2]^T
\]

are denoted as

\[
\xi_1 = r_{1,1}^* s_{1,0}^*(x, t), \quad \xi_2 = r_{2,1}^* s_{2,0}^*(x, t),
\]

and \(|Y_0| = [1 1]\). According to the expression (51), the explicit solution of Eq. (3) can be obtained. We omit it here because of its explicit expression. In Fig. 1, we mainly draw the image of zero boundary soliton with two pole points, which describes the role of two solitons in two different orbits. In Fig. 1a, we draw a three-dimensional diagram of it, and Fig. 1b describes its density map.

4.2 Solution related to one second-order pole

Suppose that \( \lambda = \lambda_0 \) is a second-order zero point of \( c(\lambda) \), that is, \( r(\lambda) \), which can be expressed in the following form

\[
r(\lambda) = r_0(\lambda) + \frac{r_1}{\lambda - \lambda_0} + \frac{r_2}{(\lambda - \lambda_0)^2}.
\]

According to Theorem 3.2, we get that \( \Omega \) can be expressed as follows

\[
\Omega_{11} = -\frac{r_{1,1}^* s_{1,0}^*}{\lambda_0^* - \lambda_0} - \frac{r_{2,1}^* s_{2,0}^*}{\lambda_0^* - \lambda_0} + \frac{r_{2,2}^* s_{2,0}^*}{(\lambda_0^* - \lambda_0)^2},
\]

\[
\Omega_{12} = -\frac{r_{1,1}^* s_{1,0}^*}{(\lambda_0^* - \lambda_0)^2} - \frac{r_{2,1}^* s_{2,0}^*}{(\lambda_0^* - \lambda_0)^2} + \frac{2r_{2,2}^* s_{2,0}^*}{(\lambda_0^* - \lambda_0)^3},
\]

\[
\Omega_{2,1} = -\frac{r_{2,1}^* s_{2,0}^*}{\lambda_0^* - \lambda_0}, \quad \Omega_{2,2} = -\frac{r_{2,2}^* s_{2,0}^*}{(\lambda_0^* - \lambda_0)^2},
\]

(59)

\[
|\xi| = [\xi_1 \xi_2]^T
\]

are denoted as
Fig. 1 (Color online) The explicit solution of Eq. (3) with \( \lambda_1 = \frac{1}{2} + \frac{\sqrt{3}i}{2} \), \( \lambda_2 = \frac{1}{2} + \frac{\sqrt{3}i}{2} \), \( r_{1,1} = 1 \), \( r_{2,1} = 1 \), \( \epsilon = 0 \), \( h = 0 \): (a) The three-dimensional plot. (b) The density map.

\[ \dot{\xi}_1 = r_1 \xi_0 + r_2 \xi_1, \quad \dot{\xi}_2 = r_2 \xi_0, \quad (60) \]

and \( \{ Y_0 \} = \{ 1 \ 0 \} \). According to the expression (51), the explicit solution of Eq. (3) can be obtained. We omit it here because of its explicit expression. In order to better express the solution of the equation, we will describe it in the form of graph. In Fig. 2, we draw an image of the second-order BS soliton, which describes the staged interaction of two solitons in two curved orbits. In Fig. 2a, we draw a three-dimensional diagram of the second-order BS soliton and Fig. 2b describes its density map.

4.3 Solution related to one simple pole and one second-order pole

Suppose that \( \lambda_1 = a_1 + ib_1 \) is a simple zero point of \( c(\lambda) \), and \( \lambda_2 = a_2 + ib_2 \) is one second-order pole points of \( c(\lambda) \), that is, \( r(\lambda) \), which can be expressed in the following form

\[ r(\lambda) = r_0(\lambda) + \frac{r_{1,1}}{\lambda - \lambda_1} + \frac{r_{1,2}}{(\lambda - \lambda_1)^2} + \frac{r_{2,1}}{\lambda - \lambda_2}, \quad (61) \]

According to Theorem 3.2, we get that \( \Gamma = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \) can be expressed as follows

\[ \begin{align*}
\omega_{11} & = -\frac{r_{1,1}^* r_{1,0}^*}{\lambda_1^* - \lambda_1} - \frac{r_{1,2}^* r_{1,1}^*}{\lambda_1^* - \lambda_1} + \frac{r_{1,2}^* r_{1,0}^*}{(\lambda_1^* - \lambda_1)^2}, \\
\omega_{12} & = -\frac{r_{1,1}^* r_{2,0}^*}{(\lambda_1^* - \lambda_1)^2} - \frac{r_{1,2}^* r_{1,1}^*}{(\lambda_1^* - \lambda_1)^2} + \frac{2r_{1,2}^* r_{1,0}^*}{(\lambda_1^* - \lambda_1)^3}, \\
\omega_{21} & = -\frac{r_{2,1}^* r_{1,0}^*}{\lambda_2^* - \lambda_1}, \\
\omega_{22} & = -\frac{r_{2,1}^* r_{2,0}^*}{(\lambda_2^* - \lambda_1)^2}.
\end{align*} \]
Fig. 3 (Color online) The interaction between one soliton and one second-order BS solution of Eq. (3) with \( r_{1,1} = 1, r_{1,2} = 2, r_{2,1} = 3, \epsilon = 1, h = 0 \): (a) The three-dimensional plot with \( \lambda_1 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \) and \( \lambda_2 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \). (b) The density plot with \( \lambda_1 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \) and \( \lambda_2 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \). (c) The three-dimensional plot with \( \lambda_1 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \) and \( \lambda_2 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \). (d) The density plot with \( \lambda_1 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \) and \( \lambda_2 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \). (e) The three-dimensional plot with \( \lambda_1 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \) and \( \lambda_2 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \). (f) The density plot with \( \lambda_1 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \) and \( \lambda_2 = \frac{1}{5} + \frac{\sqrt{3}i}{2} \).

\[
[\omega_{21}]_{12} = -\frac{r_{2,1}^{*}r_{2,0}^{*}}{(\lambda_2^2 - \lambda_1)^2}, \quad [\omega_{22}]_{11} = -\frac{r_{2,1}^{*}r_{2,0}^{*}}{\lambda_2^2 - \lambda_2},
\]

(62)

\[
|\xi\rangle = [\xi_1 \xi_2]^T \text{ are denoted as}
\]

\[
\xi_{1,1} = r_{1,1}\xi_{1,0} + r_{1,2}\xi_{1,1},
\xi_{1,2} = r_{1,2}\xi_{1,0}, \quad \xi_{2,1} = r_{2,1}\xi_{2,0}, \quad \xi_{2,2} = r_{2,2}\xi_{2,0}.
\]

According to the expression (51), the explicit solution of Eq. (3) can be obtained. We omit it here because of its explicit expression. In Fig. 3, we mainly describe the image of the interaction solution between one regular soliton and one second-order soliton. Figure 3a implies that the regular soliton passes through the center of the second-order BS soliton. After colliding with
the branches of the two solitons, the original structure will be destroyed and oscillated periodically. In Fig. 3c, the regular soliton collides with the left branch of the second-order BS soliton, and then, the regular soliton oscillates periodically with its left branch, while the right branch remains unchanged. In Fig. 3e, the regular soliton and the second-order BS soliton are separated immediately after the interaction and keep the original shape, but there will be a certain displacement change. In Fig. 3b, d and f, we describe the density plots of their interactions, respectively.

4.4 Solution related to two second-order pole

Suppose that \( \lambda_1 = a_1 + ib_1 \) and \( \lambda_2 = a_2 + ib_2 \) are two second-order pole points of \( c(\lambda) \), that is, \( r(\lambda) \), which can be expressed in the following form

\[
r(\lambda) = r_0(\lambda) + \frac{r_{1,1}}{\lambda - \lambda_1} + \frac{r_{1,2}}{(\lambda - \lambda_1)^2} + \frac{r_{2,1}}{\lambda - \lambda_2} + \frac{r_{2,2}}{(\lambda - \lambda_2)^2},
\]

(64)

According to Theorem 3.2, we get that \( \Omega = \begin{bmatrix} [\omega_{11}] & [\omega_{12}] \\ [\omega_{21}] & [\omega_{22}] \end{bmatrix} \) can be expressed as follows:

\[
[\omega_{11}]_{11} = -\frac{r_{1,1}^{*}r_{1,0}^{*}}{\lambda_{1}^{*} - \lambda_{1}} - \frac{r_{1,2}^{*}r_{1,1}^{*}}{(\lambda_{1}^{*} - \lambda_{1})^2} + \frac{r_{2,2}^{*}r_{1,0}^{*}}{(\lambda_{1}^{*} - \lambda_{1})^3},
\]

\[
[\omega_{11}]_{12} = -\frac{r_{1,1}^{*}r_{1,0}^{*}}{(\lambda_{1}^{*} - \lambda_{1})^2} - \frac{r_{1,2}^{*}r_{1,1}^{*}}{(\lambda_{1}^{*} - \lambda_{1})^2} + \frac{2r_{2,2}^{*}r_{1,0}^{*}}{(\lambda_{1}^{*} - \lambda_{1})^3},
\]

\[
[\omega_{11}]_{21} = -\frac{r_{1,2}^{*}r_{1,0}^{*}}{\lambda_{1}^{*} - \lambda_{1}}, \quad [\omega_{11}]_{22} = -\frac{r_{2,2}^{*}r_{1,0}^{*}}{(\lambda_{1}^{*} - \lambda_{1})^2},
\]

\[
[\omega_{12}]_{11} = -\frac{r_{1,1}^{*}r_{1,0}^{*}}{\lambda_{1}^{*} - \lambda_{2}} - \frac{r_{1,2}^{*}r_{1,1}^{*}}{(\lambda_{1}^{*} - \lambda_{2})^2} + \frac{r_{2,2}^{*}r_{1,0}^{*}}{(\lambda_{1}^{*} - \lambda_{2})^3},
\]

\[
[\omega_{12}]_{12} = -\frac{r_{1,1}^{*}r_{1,0}^{*}}{(\lambda_{1}^{*} - \lambda_{2})^2} - \frac{r_{1,2}^{*}r_{1,1}^{*}}{(\lambda_{1}^{*} - \lambda_{2})^2} + \frac{2r_{2,2}^{*}r_{1,0}^{*}}{(\lambda_{1}^{*} - \lambda_{2})^3},
\]

\[
[\omega_{12}]_{21} = -\frac{r_{1,2}^{*}r_{1,0}^{*}}{\lambda_{1}^{*} - \lambda_{2}}, \quad [\omega_{12}]_{22} = -\frac{r_{2,2}^{*}r_{1,0}^{*}}{(\lambda_{1}^{*} - \lambda_{2})^2},
\]

\[
[\omega_{21}]_{11} = -\frac{r_{2,1}^{*}r_{2,0}^{*}}{\lambda_{2}^{*} - \lambda_{1}} - \frac{r_{2,2}^{*}r_{2,1}^{*}}{(\lambda_{2}^{*} - \lambda_{1})^2} + \frac{r_{2,2}^{*}r_{2,0}^{*}}{(\lambda_{2}^{*} - \lambda_{1})^3},
\]

\[
[\omega_{21}]_{12} = -\frac{r_{2,1}^{*}r_{2,0}^{*}}{(\lambda_{2}^{*} - \lambda_{1})^2} - \frac{r_{2,2}^{*}r_{2,1}^{*}}{(\lambda_{2}^{*} - \lambda_{1})^2} + \frac{2r_{2,2}^{*}r_{2,0}^{*}}{(\lambda_{2}^{*} - \lambda_{1})^3},
\]

\[
[\omega_{21}]_{21} = -\frac{r_{2,2}^{*}r_{2,0}^{*}}{\lambda_{2}^{*} - \lambda_{1}} - \frac{r_{2,2}^{*}r_{2,1}^{*}}{(\lambda_{2}^{*} - \lambda_{1})^2} + \frac{r_{2,2}^{*}r_{2,0}^{*}}{(\lambda_{2}^{*} - \lambda_{1})^3},
\]

\[
[\omega_{21}]_{22} = -\frac{r_{2,2}^{*}r_{2,0}^{*}}{(\lambda_{2}^{*} - \lambda_{1})^2}.
\]
\[ [\omega_{21}]_{12} = -\frac{r_{21}^2\xi_{21}^2}{(\lambda_2^2 - \lambda_1^2)^2} - \frac{r_{22}^2\xi_{22}^2}{(\lambda_2^2 - \lambda_1^2)^2} + \frac{2r_{22}^2\xi_{22}^2}{(\lambda_2^2 - \lambda_1^2)^3}, \]

\[ [\omega_{21}]_{21} = -\frac{r_{21}^2\xi_{21}^2}{\lambda_2^2 - \lambda_1}, \quad [\omega_{21}]_{22} = -\frac{r_{22}^2\xi_{22}^2}{\lambda_2^2 - \lambda_1}, \]

\[ [\omega_{22}]_{11} = -\frac{r_{21}^2\xi_{21}^2}{\lambda_2^2 - \lambda_2} - \frac{r_{22}^2\xi_{22}^2}{\lambda_2^2 - \lambda_2} + \frac{2r_{22}^2\xi_{22}^2}{(\lambda_2^2 - \lambda_2)^2}, \]

\[ [\omega_{22}]_{12} = -\frac{r_{21}^2\xi_{21}^2}{(\lambda_2^2 - \lambda_2)^2} - \frac{r_{22}^2\xi_{22}^2}{(\lambda_2^2 - \lambda_2)^2} + \frac{2r_{22}^2\xi_{22}^2}{(\lambda_2^2 - \lambda_2)^3}, \]

\[ [\omega_{22}]_{21} = -\frac{r_{21}^2\xi_{21}^2}{\lambda_2^2 - \lambda_2}, \quad [\omega_{22}]_{22} = -\frac{r_{22}^2\xi_{22}^2}{(\lambda_2^2 - \lambda_2)^2}. \]

(65)

\[ |\xi\rangle = [\xi_1 \xi_2]^T \] are denoted as

\[ \xi_{11} = r_{11}\xi_1, 0 + r_{12}\xi_{11}, \quad \xi_{12} = r_{12}\xi_1, 0, \]

\[ \xi_{21} = r_{21}\xi_2, 0 + r_{22}\xi_{21}, \quad \xi_{22} = r_{22}\xi_2, 0. \quad (66) \]

and \( |Y_0\rangle = [1 \ 0 \ 1 \ 0] \). According to the expression (51), the explicit solution of Eq. (3) can be obtained.

We omit it here because of its explicit expression. In Fig. 4, we describe the interaction of two second-order BS solitons. In Fig. 4a, the two BS solitons evolve along two parallel orbits, and the interaction will cause the amplitude to oscillate periodically. In Fig. 4c, the two BS solitons move at the same speed and shape. At the same time, their density maps are described in Fig. 4b and d, respectively.

5 Conclusions and discussion

In this paper, based on the nonlinear Schrödinger equation, we study a higher-order nonlinear Schrödinger equation for inhomogeneous Heisenberg ferromagnetic system. Based on the spectral analysis of Lax pairs, we study the RH problem of Eq. (3) with one high-order poles and multiple higher-order poles. Meanwhile, the formulas of BS soliton corresponding to one higher-order pole and BS soliton corresponding to multiple higher-order poles are derived, respectively. Then, some soliton solutions of Eq. (3) are derived. Because one-dimensional spin array can store quantum information, and the Heisenberg ferromagnetic model with different magnetic interactions is equivalent to a class of nonlinear evolution equations in semiclassical and continuous media, some of the relations of Eq. (3) can enrich the contents of the solutions of nonlinear evolution equations.

Finally, we will note that the Wronskian method [49], the bilinear method [50] and so on can be used to derive the exact solutions of nonlinear evolution equations. Therefore, in the future work, we need to consider whether we can use these methods to solve Eq. (3).

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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