SYZ MIRROR SYMMETRY FOR HYPERTORIC VARIETIES

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Abstract. We construct a Lagrangian torus fibration on a smooth hypertoric variety and a corresponding SYZ mirror variety using $T$-duality and generating functions of open Gromov-Witten invariants. The variety is singular in general. We construct a resolution using the wall and chamber structure of the SYZ base.

1. Introduction

Mirror symmetry has made powerful and striking predictions in enumerative geometry. It has led to groundbreaking results in algebraic and differential geometry, number theory, gauge theory and other branches of mathematics.

Strominger-Yau-Zaslow [SYZ96] proposed that mirror symmetry can be understood as torus duality. It conjectured a geometric construction of mirror manifolds and a canonical transformation to derive the homological mirror symmetry conjecture [Kon95].

There have been a lot of breakthroughs in SYZ mirror symmetry. The Gross-Siebert program [GS11] gave a purely algebraic method to reconstruct the mirror manifolds. The family Floer theory initiated by Fukaya [Fuk02] and further developed by Tu [Tu14] and Abouzaid [Abo, Abo17] provides a canonical functor which realizes the SYZ mirror transformation. Moreover, based on the works of Fukaya-Oh-Ohta-Ono [FOOO10, FOOO09], Seidel [Sei11] and Akaho-Joyce [AJ10], deformation and moduli theory of Lagrangian immersions are being developed by Cho-Hong-Lau [CHL17, CHL, HL] which enhance and generalize the SYZ program.

In view of the recent developments in SYZ, mirror symmetry can be understood via a local-to-global approach. First we need to understand SYZ transformation for local geometries around singular Lagrangians. Second we need to glue the local mirrors using Floer-theoretical methods.

Toric Calabi-Yau manifolds and their mirrors provide a rich source of local models. Wall-crossing and SYZ mirror construction have been understood due to the works of Auroux [Aur07, Aur09], Chan-Lau-Leung [CLL12], Abouzaid-Auroux-Katzarkov [AAK16] and Chan-Cho-Lau-Tseng [CCLT16]. Using the local models, geometric transitions have been studied by Castano-Bernard and Matessi [CnBM14] and other groups [Lau14, CPU16, KLa, KLb, Lau].

In this paper we study SYZ for the hyper-Kähler analog of toric manifolds. Analogous to toric manifolds, they are obtained as hyper-Kähler quotients of $T^*\mathbb{C}^n$. Typical examples of hypertoric manifolds include $T^*\mathbb{P}^n$ and crepant resolutions of $A_n$ singularities. We expect that they should provide useful local models to understand mirror symmetry for holomorphic symplectic manifolds.

First we construct Lagrangian fibrations on hypertoric manifolds (Section 3). It uses the techniques of Gross [Gro11a] and Goldstein [Gol01] by symplectic reduction, and Abouzaid-Auroux-Katzarkov [AAK16] by Moser argument. The Lagrangian fibrations have codimension-one amoeba-like discriminant loci.

Second we analyze the walls over which the Lagrangian torus fibers bound holomorphic discs of Maslov index 0 (Section 4.2). The walls divide the base of a Lagrangian fibration into chambers.

Third we find all the holomorphic discs of Maslov index 2 bounded by a fiber in each chamber (Section 4.4). As a result we obtain the generating functions of open Gromov-Witten invariants which are countings of these holomorphic discs.

Finally we construct a mirror variety as the ring of generating functions associated to boundary divisors. We compactify the manifold in order to have sufficiently many boundary divisors (Section 4.6).
By construction the mirror variety we obtain is affine. Unfortunately, it is singular in general, and a resolution is necessary to better understand the geometry. We glue up a resolution using local charts coming from the wall and chamber structure of the SYZ base (Section 4.8). The gluing can be explained using Floer-theoretical techniques as in [Sei, PT, HL].

The variety admits another resolution by a multiplicative hypertoric variety (Section 4.9). It may be related to study equivariant Floer theory. Further, we believe hypertoric varieties are useful to understand mirror symmetry for cotangent bundles of flag manifolds. Toric degenerations of flag manifolds were used to construct their mirrors by Nishinou-Nohara-Ueda [NNU10, NU14]. It is reasonable to expect that mirrors of the total spaces of cotangent bundles of flag manifolds are given by deformations of that of (singular) hypertoric varieties.

The SYZ construction can also be carried out in the reverse direction. This direction would be useful to understand the hypertoric category $O$ (which is an analogue of BGG category $O$ for hypertoric varieties) developed by Braden-Licata-Proudfoot-Webster [BLPW12]. It may be related to the SYZ perspective in this paper, we need to study equivariant Floer theory.

Below we describe some important questions that we wish to understand in future works.

Closed-string equivariant mirror symmetry for hypertoric manifolds was found by McBreen and Shenfeld [MS13]. They derived a presentation of the $T^*\mathbb{C}^n$-equivariant quantum cohomology of a hypertoric manifold and relate it with the Gauss-Manin connection of the mirror moduli. To understand the equivariant quantum cohomology from the SYZ perspective in this paper, we need to equip the hypertoric variety with a generic character.

Below we describe some important questions that we wish to understand in future works.

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2. Review of hypertoric varieties

In this section, we review the construction and basic properties of a hypertoric variety. We refer to [BD00], [HS02], [Pro04] for more detailed introductions. All material in this section, except Proposition 2.13, come from the existing literature.

2.1 Hypertoric varieties. Let $\mathfrak{t}^n$ and $\mathfrak{t}^d$ be real vector spaces of dimension $n$ and $d$, respectively. Let $\mathbb{Z}^n \subset \mathfrak{t}^n$ and $\mathbb{Z}^d \subset \mathfrak{t}^d$ be the integer lattices. Let $\{e_1, \ldots, e_n\} \subset \mathfrak{t}^n$ be a basis that spans $\mathbb{Z}^n$ over $\mathbb{Z}$, and let $\{e_1, \ldots, e_n\} \subset (\mathfrak{t}^d)^*$ be the dual basis. Given a collection $u = \{u_i\}_{i=1}^n$ of $n$ primitive integer vectors in $\mathbb{Z}^n$ that span $\mathfrak{t}^d$ over $\mathbb{Z}$, we define $\pi : \mathfrak{t}^n \to \mathfrak{t}^d$ by $\pi(e_i) = u_i$. We have the following exact sequences:

$$0 \longrightarrow \mathfrak{k} \mathbb{Z} \longrightarrow \mathfrak{t}^n \mathbb{R} \longrightarrow \mathfrak{t}^d \longrightarrow 0,$$

$$0 \leftarrow (\mathfrak{k})^* \mathbb{R} \longrightarrow (\mathfrak{t}^n)^* \mathbb{R} \longrightarrow (\mathfrak{t}^d)^* \mathbb{R} \longrightarrow 0,$$

where $\mathfrak{k} = \ker \pi$, and (2.2) is the dual sequence of (2.1). Exponentiating (2.1) gives an exact sequence of real tori

$$0 \longrightarrow K \longrightarrow T^n \longrightarrow T^d \longrightarrow 0.$$

Let $(z, \bar{w}) = (z_1, w_1, \ldots, z_n, w_n)$ be the standard coordinates on $T^*\mathbb{C}^n$. We equip $T^*\mathbb{C}^n$ with the Kähler form $\omega_R$ and holomorphic symplectic form $\omega_C$ defined by

$$\omega_R = \sqrt{-1} \sum_{i=1}^n (dz_i \wedge d\bar{z}_i + dw_i \wedge d\bar{w}_i), \quad \omega_C = \sum_{i=1}^n dz_i \wedge dw_i.$$
This endows $T^*\mathbb{C}^n$ with its standard hyperkähler structure. Let $\tilde{t} = (t_1, \ldots, t_n) \in T^n$ act on $T^*\mathbb{C}^n$ by

$$\tilde{t} \cdot (\vec{z}, \vec{w}) = (t_1 z_1, t_1^{-1} w_1, \ldots, t_n z_n, t_n^{-1} w_n).$$

The hyperkähler moment map for the restriction to $K$ of the $T^n$-action on $T^*\mathbb{C}^n$ is given by

$$\mu_K : T^*\mathbb{C}^n \to (\mathfrak{t})^* \oplus (\mathfrak{t}_C)^*,$$

$$\mu_K(\vec{z}, \vec{w}) = \frac{1}{2} \sum_{i=1}^n (|z_i|^2 - |w_i|^2) e_i, \quad \mu_C(\vec{z}, \vec{w}) = \sum_{i=1}^n (z_i w_i) \lambda e_i.$$

**Definition 2.1.** Given a collection of primitive integer vectors $u$ and parameters $\lambda = (\lambda_K, \lambda_C) \in (\mathfrak{t})^* \oplus (\mathfrak{t}_C)^*$, the hyperkähler quotient

$$\mathcal{M}_{u, \lambda} = (\mu_{\lambda_K}^{-1}(\lambda_K) \cap \mu_C^{-1}(\lambda_C)) / K$$

is called a hypertoric variety.

Alternatively, $\mathcal{M}_{u, \lambda}$ can be constructed as the GIT quotient

$$\mathcal{M}_{u, \lambda} = \mu_C^{-1}(\lambda_C) / \lambda_K K_C = \text{Proj} \left( \bigoplus_{k=0}^{\infty} \mathbb{C}[\mu_C^{-1}(\lambda_C)]^k_{\lambda_K} \right),$$

where $K_C \hookrightarrow (\mathbb{C}^*)^n$ is defined by the complexification of the sequence, and $\lambda \in (\mathfrak{t})^*$ is understood as a character $\lambda_K : K_C \to \mathbb{C}^*$.

The quotient torus $T^d = T^n / K$ acts on $\mathcal{M}_{u, \lambda}$ with the hyperkähler moment map $(\bar{\mu}_K, \bar{\mu}_C) : \mathcal{M}_{u, \lambda} \to (\mathfrak{t}^d)^* \oplus (\mathfrak{t}_C^d)^*$,

$$(\bar{\mu}_K(\vec{z}, \vec{w}) = \frac{1}{2} \sum_{i=1}^n (|z_i|^2 - |w_i|^2) e_i \oplus \sum_{i=1}^n (z_i w_i + \hat{\lambda}_C, e_i) e_i \in \text{Ker}(\mathfrak{t}^*) \oplus \text{Ker}(\mathfrak{t}_C^*) = (\mathfrak{t}^d)^* \oplus (\mathfrak{t}_C^d)^*,$$

where $(\lambda_{R,1}, \ldots, \lambda_{R,n}, \hat{\lambda}_{C,1}, \ldots, \hat{\lambda}_{C,n}) \in (\mathfrak{t}^n)^* \oplus (\mathfrak{t}_C^n)^*$ is a lift of $\lambda$. Note that this map is always surjective.

**Example 2.2.** Let $\{u_1, \ldots, u_d\} \subset \mathfrak{t}^d$ be a primitive integer basis. Define $\pi : \mathfrak{t}^{d+1} \to \mathfrak{t}^d$ by $\pi(e_i) = u_i$ for $i = 1, \ldots, d$, and $\pi(e_{d+1}) = u_{d+1} := \sum_{j=1}^{d} (-u_j)$. Then $K \hookrightarrow T^{d+1}$ is the diagonal sub-torus. If we set $\lambda_K \in (\mathfrak{t})^*$ to be a regular value, and $\lambda_C = 0$, then the hypertoric variety $\mathcal{M}_{u, \lambda}$ is $T^*\mathbb{P}^d$ (with the standard complex structure).

**Example 2.3.** Let $u_1 \in \mathfrak{t}^1$ be a primitive integer vector. Define $\pi : \mathfrak{t}^n \to \mathfrak{t}^1$ by $\pi(e_i) = u_1$ for $i = 1, \ldots, n$. $K \hookrightarrow T^n$ is then the subtorus

$$K = \{(t_1, \ldots, t_n) \in T^n | \prod_{i=1}^n t_i = 1\}.$$

For $\lambda_K$ a regular value and $\lambda_C = 0$, the hypertoric variety $\mathcal{M}_{u, \lambda}$ is $\tilde{\mathbb{A}}_{n-1}$, the crepant resolution of $\mathbb{A}_{n-1}$ singularity $\mathbb{C}^2 / \mathbb{Z}_n$.

### 2.2 Hyperplane arrangements.

Let $\mathcal{M}_{u, \lambda}$ be a hypertoric variety. Denote by $\mathcal{H}_R = \{H_{R,i}\}_{i=1}^n$ and $\mathcal{H}_C = \{H_{C,i}\}_{i=1}^n$ the collections of hyperplanes

$$H_{R,i} = \{ s \in (\mathfrak{t}^d)^* | \langle s, u_i \rangle - \hat{\lambda}_{R,i} = 0 \},$$

and

$$H_{C,i} = \{ v \in (\mathfrak{t}_C^d)^* | \langle v, u_i \rangle - \hat{\lambda}_{C,i} = 0 \}.$$

$\mathcal{H}_R$ and $\mathcal{H}_C$ are called the associated hyperplane arrangements of $\mathcal{M}_{u, \lambda}$. The hyperplane arrangements $\mathcal{H}_R$ and $\mathcal{H}_C$ are independent of the choice of the lift of $\lambda$ up to a translation and determine $\mathcal{M}_{u, \lambda}$ up to a canonical isomorphism.

The following definition is important for smoothness of hypertoric varieties.

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1 A usual convention is setting $\lambda_C = 0$ in the definition. In this paper we work with a generic complex structure and don’t make this assumption.
**Definition 2.4.** A hyperplane arrangement \( \mathcal{H}_R \) (resp. \( \mathcal{H}_C \)) is called simple if every subset of \( k \) hyperplanes with nonempty intersection intersects in codimension \( k \). \( \mathcal{H}_R \) (resp. \( \mathcal{H}_C \)) is called unimodular if every collection of \( d \) linearly independent vectors \( \{u_{i_1}, \ldots, u_{i_d}\} \) spans \( \mathbb{R}^d \) over \( \mathbb{Z} \).

**Remark 2.5.** The holomorphic moment map \( \bar{\mu}_C : \mathcal{M}_{u,\lambda} \to (\mathbb{C}^*)^d \) is a holomorphic \((\mathbb{C}^*)^d\) fibration with discriminant loci the union of complex hyperplanes in \( \mathcal{H}_C \). The geometry of this fibration will be useful for our mirror construction later on.

Figure 1 shows some examples of hyperplane arrangements.

![Figure 1](example.jpg)

**Figure 1.** Examples of hyperplane arrangements. The left corresponds to \( A_n \) resolutions. The middle corresponds to \( T^*\mathbb{P}^2 \). The right corresponds to a hypertoric variety which contains both \( T^*\mathbb{F}_1 \) and \( T^*\mathbb{P}^2 \)

### 2.3 Geometry and topology of hypertoric varieties

Let \( A = \{A_i\}_{i=1}^n \) be the collection of affine subspaces \( A_i = H_{R,i} \times H_{C,i} \) in \((\mathbb{C}^*)^d \). We have the following necessary and sufficient conditions for \( \mathcal{M}_{u,\lambda} \) to be an orbifold or smooth manifold:

**Theorem 2.6.** [BD00, Theorem 3.2,3.3] \( \mathcal{M}_{u,\lambda} \) is an orbifold with at worst Abelian quotient singularities if and only if every \( d+1 \) distinct elements in \( A \) have empty intersection. It is a smooth manifold if and only if, in addition, whenever \( d \) distinct elements \( A_{i_1}, \ldots, A_{i_d} \) have nonempty intersection, the set \( \{u_{i_1}, \ldots, u_{i_d}\} \) spans \( \mathbb{R}^d \) over \( \mathbb{Z} \).

**Corollary 2.7.** For \( \lambda_C = 0 \), \( \mathcal{M}_{u,\lambda} \) is a smooth manifold if and only if \( \mathcal{H}_R \) is both simple and unimodular.

In this paper we shall always work with smooth hypertoric varieties.

**Remark 2.8.** The expression for the SYZ mirror in Theorem 4.27 still makes sense even when the hyperplane arrangements are not simple nor unimodular. We speculate that it is useful for the study of hypertoric degenerations.

For a generic choice of \( \lambda_C \), \( \mathcal{M}_{u,\lambda} \) is simply an affine variety.

**Theorem 2.9.** [BD00, Theorem 5.1] Let \( \mathcal{M}_{u,\lambda} \) be a hypertoric orbifold, and suppose \( \mathcal{H}_C \) is simple. Then, \( \mathcal{M}_{u,\lambda} \) equipped with the complex structure inherited from \( T^*\mathbb{C}^n \) is biholomorphic to affine variety \( \text{Spec}(\mathbb{C}[W]^{K_C}) \), where \( W \subset T^*\mathbb{C}^n \times \mathbb{C}^d \) is defined by the equations

\[
\tilde{z}_iw_i = \langle v, u_i \rangle - \tilde{\lambda}_{C,i}, \quad i = 1, \ldots, n,
\]

and \( K_C \) acts on \( T^*\mathbb{C}^n \times \mathbb{C}^d \) by \( \bar{\Gamma} : (\bar{z}, \bar{w}, \bar{v}) = (t_1z_1, t_1^{-1}w_1, \ldots, t_nz_n, t_n^{-1}w_n, v) \).

In general it is difficult to write down an explicit hyperkähler metric. For a hypertoric variety, the Kähler metric is descended from the standard metric on \( T^*\mathbb{C}^n \) and has a simple expression.

**Theorem 2.10.** [BD00, Theorem 8.3] Let \( s_i = |z_i|^2 - |w_i|^2, v_i = z_iw_i, \) and \( r_i = \sqrt{s_i^2 + 4v_i} \). Then, on the open dense subset of \( \mathcal{M}_{u,\lambda} \) where the \( T^d \)-action is free, the induced Kähler form \( \omega \) is given by

\[
\omega = \frac{1}{4d^c(2\bar{\mu}_R, \bar{\mu}_C)^*} \left( \sum_{i=1}^n (r_i + 2\bar{\lambda}_{R,i} \ln(s_i + r_i)) \right),
\]

where \( d^c = \sqrt{-1}(\bar{\partial} - \partial) \).
2.4 Circuits and primitive curve classes. The SYZ mirrors that we are going to construct depend on Kähler parameters, which are recording the symplectic areas of curve classes in $H_2(\mathcal{M}_{u,\lambda},\mathbb{Z})$. The following definition is crucial to understand curve classes in hypertoric varieties.

**Definition 2.11.** [MS13] A circuit $S \subseteq \{1, \ldots, n\}$ in $\mathcal{H}_R$ is a minimal subset satisfying

$$\bigcap_{i \in S} H_{R,i} = \emptyset.$$ 

A circuit $S$ admits a unique splitting $S = S^+ \bigsqcup S^-(\text{up to switching } S^+ \text{ and } S^-)$, which is characterized by the equation

$$\sum_{i \in S^+} u_i - \sum_{i \in S^-} u_i = 0 \in \mathbb{R}^d.$$ 

For each circuit $S$, we fix the splitting $S = S^+ \bigsqcup S^-$ such that if we set

$$\beta_S = \sum_{i \in S^+} e_i - \sum_{i \in S^-} e_i,$$

then $\hat{\lambda}_R(\beta_S) > 0$ for any lift $\hat{\lambda}_R \in (t^d)^* \lambda_{\mathbb{R}}$.

$\beta_S$ is a primitive class in $t_{\mathbb{Z}} = H_2(\mathcal{M}_{u,\lambda},\mathbb{Z})$. It can be understood as a union of discs emanated from the hyperplanes indexed by $S$. We denote by $q^{\beta_S}$ the Kähler parameter associated to $\beta_S$.

2.5 Cotangent bundles of toric varieties in a hypertoric variety. Let $\mathcal{H}_R$ be the real hyperplane arrangement of $\mathcal{M}_{u,\lambda}$. Let $\Delta$ be a convex, possibly unbounded polytope in $(t^d)^*$ with its interior being a chamber in the complement of $\mathcal{H}_R$. We will assume $\Delta$ is simple, which is the case when $\mathcal{H}_R$ is simple. We further assume $\lambda_C = 0$, so that $\mathcal{M}_{u,\lambda}$ is equipped with its canonical complex structure. Then, the cotangent bundle $T^*X_\Delta$ of the toric variety $X_\Delta$ naturally embeds into $\mathcal{M}_{u,\lambda}$ as an open dense subset.

**Theorem 2.12.** [BD00] Theorem 7.1 $T^*X_\Delta$ with its canonical holomorphic-symplectic structure is $T^d$-equivariantly isomorphic to an open dense subset $U_\Delta$ of $\mathcal{M}_{u,\lambda}$. The hyperkähler metric of $\mathcal{M}_{u,\lambda}$ restricted to the zero section of $T^*X_\Delta$ is the Kähler metric on $X_\Delta$ determined by $\Delta$.

We give an explicit construction of $T^*X_\Delta$ in $\mathcal{M}_{u,\lambda}$ following [BD00]. Let $\{H_{R,i}^+\}_{i=1}^n$ and $\{H_{R,i}^-\}_{i=1}^n$ be the half-spaces

$$H_{R,i}^+ = \{s \in (t^d)^* | \langle s, u_i \rangle - \hat{\lambda}_{R,i} \geq 0\},$$

$$H_{R,i}^- = \{s \in (t^d)^* | \langle s, u_i \rangle - \hat{\lambda}_{R,i} \leq 0\}.$$ 

Let $\sigma : \{1, \ldots, n\} \to \{+, -\}$ be a sign vector, and let $\Delta$ be given by

$$\Delta = \bigcap_{i=1}^n H_{R,i}^{\sigma(i)}.$$ 

Each face $F$ of $\Delta$ is given by intersection of hyperplanes $\bigcap_{i \in I} H_{R,i}$, for some $I \subseteq \{1, \ldots, n\}$. For $F$ a face of $\Delta$, we define $Y_F \subseteq T^*\mathbb{C}^n$ to be

$$Y_F = \{(z, w) \in T^*\mathbb{C}^n | z_i = 0 \iff i \in I, \forall i \text{ with } \sigma(i) = +; w_i = 0 \iff i \in I, \forall i \text{ with } \sigma(i) = -\}.$$ 

Note that if $F$ is the codimension-0 face, we have $I = \emptyset$, and by definition

$$Y_F = \{(z, w) \in T^*\mathbb{C}^n | z_i \neq 0, \forall i \text{ with } \sigma(i) = +; w_i \neq 0, \forall i \text{ with } \sigma(i) = -\}.$$ 

We define $Y_\Delta \subset T^*\mathbb{C}^n$ to be the union

$$Y_\Delta = \bigcup_F Y_F,$$

where $F$ runs over all faces of $\Delta$. Notice that $Y_\Delta$ is $T^n$-invariant. $T^*X_\Delta \subset \mathcal{M}_{u,\lambda}$ is then obtained by restricting the hyperkähler quotient construction to $Y_\Delta$:

$$T^*X_\Delta = (Y_\Delta \cap \mu_R^{-1}(\lambda_R) \cap \mu_C^{-1}(0)) / K.$$
We also provide a description of the complement of $T^*\mathcal{X}_\Delta$ in $\mathcal{M}_{u,\lambda}$. Let $\mathcal{J}$ be the collection of all subsets $J \subset \{1, \ldots, n\}$ such that the intersection $\bigcap_{j \in J} H_j$ is nonempty, and is not a face of $\Delta$. Denote by $\Delta_J$ the polytope
\[ \Delta_J = \bigcap_{j \in J} H_{\mathbb{R},j}^{-\sigma(j)}. \]
Notice that $\Delta_J$ is a nonadjacent to $\Delta$.

**Proposition 2.13.** $\mathcal{M}_{u,\lambda}\setminus T^*\mathcal{X}_\Delta$ is the union $\bigcup_{J \in \mathcal{J}} X_J$, where $X_J = (\bar{\mu}_{\mathbb{R}}, \bar{\mu}_{\mathbb{C}})^{-1}\left(\Delta_J \times \bigcap_{j \in J} H_{\mathbb{C},j}\right)$.

**Proof.** Let $J \in \mathcal{J}$, and denote by $Y_J \subset T^*\mathbb{C}^n \setminus Y_\Delta$ the subset
\[ Y_J = \{(z, w) \in T^*\mathbb{C}^n | z_j = 0 \iff j \in J, \forall j \text{ with } \sigma(j) = +; w_j = 0 \iff j \in J, \forall j \text{ with } \sigma(j) = -\}. \]
Restricting the hyperkähler quotient construction to $Y_J$ gives $X_J = (Y_J \cap \mu_{\mathbb{R}}^{-1}(\lambda_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(0)) / K \subset \mathcal{M}_{u,\lambda}\setminus T^*\mathcal{X}_\Delta$.

By construction, we have $\mathcal{M}_{u,\lambda}\setminus T^*\mathcal{X}_\Delta = \bigcup_{J \in \mathcal{J}} X_J$. The image of $X_J$ under the hyperkähler moment map $(\bar{\mu}_{\mathbb{R}}, \bar{\mu}_{\mathbb{C}})$ is $\Delta_J \times \bigcap_{j \in J} H_{\mathbb{C},j}$. It is disjoint from the image of $T^*\mathcal{X}_\Delta$ since if we have $[\vec{z}, \vec{w}] \in T^*\mathcal{X}_\Delta$ with $\bar{\mu}_{\mathbb{C}}([\vec{z}, \vec{w}]) \in \bigcap_{j \in J} H_{\mathbb{C},j}$, then we must have $z_j = 0, w_j \neq 0$ and $\sigma(j) = -$ or $z_j \neq 0, w_j = 0$ and $\sigma(j) = +$ for some $j \in J$, but then $\bar{\mu}_{\mathbb{R}}([z, w]) \notin H_{\mathbb{R},j}^{-\sigma(j)} \supset \Delta_J$. \[ \square \]

**Figure 2.** A hypertoric manifold that contains both $T^*\mathbb{P}^2$ and $T^*\mathbb{F}_1$. The shaded region (which are closed) on the left (resp. right) corresponds to the image of the complement of $T^*\mathbb{P}^2$ (resp. $T^*\mathbb{F}_1$) under $\bar{\mu}_{\mathbb{R}}$.

In addition to $\mathcal{M}_{u,\lambda}$ being smooth, we shall assume $\mathcal{H}_{\mathbb{C}}$ is simple for the rest of this paper. When $\mathcal{H}_{\mathbb{C}}$ is simple, under the unimodularity assumption, $\mathcal{M}_{u,\lambda}$ is smooth for all choices of $\lambda_{\mathbb{R}}$ by Theorem 2.6. We do not assume $\mathcal{H}_{\mathbb{R}}$ is simple.

### 3. Lagrangian Torus Fibrations on Hypertoric Varieties

In this section, we construct piecewise smooth Lagrangian torus fibrations on hypertoric varieties. It was first suggested by Joyce in [Joy04] that special Lagrangian fibrations should in general be piecewise smooth. In [AAK16], Auroux, Abouzaid and Katzarkov constructed piecewise smooth Lagrangian torus fibrations on the anticanonical divisor complement $X^0$ of the blowup $X$ of $V \times \mathbb{C}$ along $H \times \{0\}$, where $V$ is a toric variety and $H \subset V$ is a hypersurface, by pulling back Lagrangian torus fibrations on the symplectic reductions of $X^0$ (which are isomorphic to the open dense torus orbit $V^0 \subset V$) and assembling them together. This construction is similar to those previously considered by Gross [Gro01b, Gro01a] and Castaño-Bernard and Matessi [CnBM10, CnBM09].
additional technical input in [AAK16] was the use of Moser’s trick to interpolate between the reduced (possibly singular) Kähler forms and the torus-invariant Kähler forms on $V^0$.

3.1 Lagrangian torus fibration on reduced spaces. Denote by $\tilde{s} = (s_1, \ldots, s_n)$ the standard coordinates on $(t^n)^*$ rescaled by a factor of 2, and $\tilde{v} = (v_1, \ldots, v_n)$ the standard complex-coordinates on $(t^n)^*$. We first construct Lagrangian torus fibrations on the symplectic reductions $X_{\tilde{s}}$ of $\mathfrak{M}_{u, \lambda}$ at level $\tilde{s} \in (t^n)^* \subset (t^n)^*$. $X_{\tilde{s}}$ can be constructed as

$$X_{\tilde{s}} = \tilde{\mu}_R^{-1} \left( \frac{\tilde{s}}{2} \right) / (T^n / K).$$

For simplicity, we will assume from now on that the vectors $u_1, \ldots, u_d$ are linearly independent, and $u_l = \sum_{i=1}^d a_{il} u_i$ for $l = d + 1, \ldots, n$. Note that $a_{il}$ are integers, since $\{u_1, \ldots, u_d\}$ spans $t^n$ over $\mathbb{Z}$. We also fix a lift $\tilde{\lambda} = (\tilde{\lambda}_{R, 1}, \ldots, \tilde{\lambda}_{R, n}, \tilde{\lambda}_{C, 1}, \ldots, \tilde{\lambda}_{C, n}) \in (t^n)^* \oplus (t^n)^*$ of $\lambda$ such that $\tilde{\lambda}_{R, i} = 0$ and $\tilde{\lambda}_{C, i} = 0$ for $i = 1, \ldots, d$. We can then identify the $\tilde{\mu}_C : \mathfrak{M}_{u, \lambda} \to (t^n)^*$ with the map $(z_1 w_1, \ldots, z_d w_d) : \mathfrak{M}_{u, \lambda} \to \mathbb{C}^d$ via the projection to the first $d$ components. The restriction of $\tilde{\mu}_C$ to $\tilde{\mu}_R^{-1} \left( \frac{\tilde{s}}{2} \right)$ then descends to a biholomorphism

$$X_{\tilde{s}} \to (t^n)^*.$$

We can therefore identify the reduced spaces $X_{\tilde{s}}$ with $(t^n)^*$ equipped with complex-coordinates $(v_1, \ldots, v_n)$. We will abuse notations and write

$$s_i = |z_i|^2 - |w_i|^2, \quad v_i = z_i w_i, \quad i = 1, \ldots, n,$

and set

$$r_i = \sqrt{s_i^2 + 4v_i \bar{v}_i}.$$

These can be viewed as functions on $X_{\tilde{s}}$. In particular, $s_i$ are constant functions. The Kähler potential of the reduced Kähler form on $X_{\tilde{s}}$ has a simple expression in term of $s_i$ and $r_i$:

**Lemma 3.1.** The Kähler potentials $K_{\text{red}, \tilde{s}}$ for the reduced Kähler forms $\omega_{\text{red}, \tilde{s}}$ on $X_{\tilde{s}}$ are given by

$$K_{\text{red}, \tilde{s}} = \frac{1}{4} \sum_{i=1}^n \left( r_i - s_i \ln(s_i + r_i) \right).$$

**Proof.** Consider the action of $T^n$ and its complexification $(\mathbb{C}^*)^n$ restricted to the invariant subvariety $W = \mu_c^{-1}(\lambda_c) \subset T^* \mathbb{C}^n$. As we have noted, $X_{\tilde{s}}$ can be obtained either as a symplectic reduction or a GIT quotient of $W$:

$$X_{\tilde{s}} = (\tilde{\mu}_R|W)^{-1} \left( \frac{\tilde{s}}{2} \right) / T^n = W / \tilde{\omega}(\mathbb{C}^*)^n,$$

where $\tilde{\mu}_R$ is the moment map of the $T^n$-action on $T^* \mathbb{C}^n$ with respect $\omega_R$. For any $(\tilde{z}, \tilde{w}) \in W$, there exist a unique element $\tilde{t}(\tilde{z}, \tilde{w}) \in \exp(it^n)$ such that $\tilde{t}(\tilde{z}, \tilde{w}) \cdot (\tilde{z}, \tilde{w}) \in (\tilde{\mu}_R|W)^{-1} \left( \frac{\tilde{s}}{2} \right)$. Denote by $q : W \to (\tilde{\mu}_R|W)^{-1} \left( \frac{\tilde{s}}{2} \right)$ the map $q(\tilde{z}, \tilde{w}) = \tilde{t}(\tilde{z}, \tilde{w}) \cdot (\tilde{z}, \tilde{w})$, and by $p : (\tilde{\mu}_R|W)^{-1} \left( \frac{\tilde{s}}{2} \right) \to X_{\tilde{s}}$ the quotient map. Let $\tilde{\omega}$ be the pull-back of $\omega_{\text{red}, \tilde{s}}$ on $W$ via $p \circ q$. Let $\chi_{\tilde{s}} : (\mathbb{C}^*)^n \to \mathbb{C}^*$ the character given by $\frac{\tilde{s}}{2}$. By [BG97 Theorem 7], we have $\tilde{\omega} = dd^c \tilde{K}$, for a $T^n$-invariant function $\tilde{K}$ on $W$ defined as

$$\tilde{K}(\tilde{z}, \tilde{w}) = K_0(\tilde{t}(\tilde{z}, \tilde{w}) \cdot (\tilde{z}, \tilde{w})) + \frac{1}{4\pi} \ln |\chi_{\tilde{s}}(\tilde{t}(\tilde{z}, \tilde{w}))|^2,$$

where $K_0$ is the Kähler potential $\frac{1}{4} \sum_{i=1}^n |z_i|^2 + |w_i|^2$ restricted to $W$. We have

$$K_0(\tilde{t}(\tilde{z}, \tilde{w}) \cdot (\tilde{z}, \tilde{w})) = \frac{1}{4} \sum_{i=1}^n \sqrt{s_i^2 + 4v_i \bar{v}_i} = \sum_{i=1}^n r_i,$$

whereas

$$|\chi_{\tilde{s}}(\tilde{t}(\tilde{z}, \tilde{w}))|^2 = \prod_{i=1}^n |\tilde{t}(\tilde{z}, \tilde{w})|^{-2s_i}.$$
\( \tilde{r}_{(\vec{z}, \vec{w}), i} \) is determined by
\[
| (\tilde{r}_{(\vec{z}, \vec{w}), i} z_i) |^2 - | \tilde{r}_{(\vec{z}, \vec{w}), i} w_i |^2 = s_i.
\]
This means
\[
| \tilde{r}_{(\vec{z}, \vec{w}), i} |^2 = \frac{s_i + \sqrt{s_i^2 + 4|z_i|^2|w_i|^2}}{2|z_i|^2}.
\]
Thus,
\[
\frac{1}{4\pi} \ln | \tilde{z}^n (\tilde{r}_{(\vec{z}, \vec{w}), i}) |^2 = \frac{1}{4} \sum_{i=1}^{n} -s_i \ln(s_i + \sqrt{s_i^2 + 4|z_i|^2|w_i|^2}) + s_i \ln(2|z_i|^2).
\]
Notice that \( s_i \) in the above expressions are constants. Denote by \( \iota : (\hat{\mu}_R | W)^{-1} (\frac{\pi}{2}) \to W \) the inclusion map. Since \( \partial \bar{\partial} \ln(2|z_i|^2) = 0 \), the terms \( \frac{1}{2} s_i \ln(2|z_i|^2) \) do not contribute to \( \tilde{\omega} \). Thus, we have
\[
p^* \omega_{red, \vec{s}} = \iota^* \tilde{\omega} = \iota^* dd^c \left( \tilde{K} (\vec{z}, \vec{w}) - \frac{1}{4} \sum_{i=1}^{n} s_i \ln(2|z_i|^2) \right) = \iota^* dd^c (p \circ q)^* K_{red, \vec{s}} = p^* dd^c K_{red, \vec{s}}.
\]

\[\square\]

**Remark 3.2.** If we view
\[
F = \frac{1}{4} \sum_{i=1}^{n} (r_i - s_i \ln(s_i + r_i))
\]
as a function on \((t^1)^* \oplus (t^2)^*\), it is then the Legendre transform of the Kähler potential \( \frac{1}{4} \sum_{i=1}^{n} |z_i|^2 + |w_i|^2 \) on \( T^* \mathbb{C}^n \). In [BD00], (2.4) was obtained as the Legendre transform of \( F \) restricted to the subspace \((t^3)^* \oplus (t^4)^*\). We can alternatively derive (3.1) as the Legendre transform of \( F \) further restricted to \( (\frac{\pi}{2}) \times (t^4)^* \).

The reduced Kähler forms \( \omega_{red, \vec{s}} \) are not invariant under the standard \( T^d \)-action on \( X_{\vec{s}} \cong \mathbb{C}^d \). It is singular along the hyperplane \( H_{\mathbb{C}, i} \), when \( \vec{s} \in H_{R, i} \). These obstacles to constructing Lagrangian torus fibrations on the reduced spaces are also encountered in [AAK16]. We will use their strategy to construct Lagrangian torus fibrations on \( X_{\vec{s}} \).

We first introduce an explicit family of smoothing \( \omega_{sm, \vec{s}} \) of \( \omega_{red, \vec{s}} \):
\[
\omega_{sm, \vec{s}} = dd^c K_{sm, \vec{s}} = \frac{1}{4} dd^c \left( \sum_{i=1}^{n} \sqrt{s_i^2 + 4v_i \bar{v}_i + \kappa^2} - s_i \ln \left( s_i + \sqrt{s_i^2 + 4v_i \bar{v}_i + \kappa^2} \right) \right)
\]
where \( \kappa > 0 \) is an arbitrarily small constant. \( \omega_{sm, \vec{s}} \) is Kähler by construction. Since \( H^2(X_{\vec{s}}, \mathbb{R}) = 0 \), we have \( [\omega_{sm, \vec{s}}] = [\omega_{red, \vec{s}}] \). Write \( v_l = \sum_{i=1}^{d} a_{li} v_i + b_l \) for \( l = d+1, \ldots, n \), where \( b_l \) are determined by \( \lambda_{\mathbb{C}} \). Notice that the terms
\[
\frac{1}{4} dd^c \left( \sum_{i=d+1}^{n} \sqrt{s_i^2 + 4v_i \bar{v}_i + \kappa^2} - s_i \ln \left( s_i + \sqrt{s_i^2 + 4v_i \bar{v}_i + \kappa^2} \right) \right)
\]
in (3.3) are not invariant under the standard \( T^d \)-action. To remedy this, we isotope \( \omega_{sm, \vec{s}} \) to the family of \( T^d \)-invariant Kähler form \( \omega_{avg, \vec{s}} \) defined by averaging \( \omega_{sm, \vec{s}} \) over the \( T^d \)-action,
\[
\omega_{avg, \vec{s}} = \frac{1}{(2\pi)^d} \int_{g \in T^d} g^* \omega_{sm, \vec{s}} dg.
\]
Since the pullback of \( \omega_{inv, \vec{s}} \) to each \( T^d \)-orbit is the exterior derivative of a \( T^d \)-invariant 1-form, it must vanish. This means the \( T^d \)-orbits in \( X_{\vec{s}} \) are Lagrangian with respect to \( \omega_{avg, \vec{s}} \).

We now prove the following lemma.

**Lemma 3.3.** There exists a family of homeomorphisms \( (\phi_{\vec{s}})_{\vec{s} \in \{t_l\}} \) of \( X_{\vec{s}} \) such that:

(1) \( \phi_{\vec{s}} \) is a diffeomorphism of \( X_{\vec{s}} \) if \( \vec{s} \notin H_{R, i} \) for all \( i \). It is a diffeomorphism away from \( H_{\mathbb{C}, i} \) if \( \vec{s} \in H_{R, i} \) for some \( i \);
(2) \( \phi \) intertwines the reduced (possibly singular) Kähler form \( \omega_{\text{red}, \bar{s}} \) and the \( T^d \)-invariant Kähler form \( \omega_{\text{avg}, \bar{s}} \);

(3) \( \phi \) depends on \( \bar{s} \) continuously, and smoothly away from \( \bigcup_{i=1}^n H_{\bar{R}^d,i} \).

**Proof.** We construct \( \phi \) as the composition of \( \phi_{\text{sm}, \bar{s}} \) and \( \phi_{\text{avg}, \bar{s}} \) such that \( \phi_{\text{sm}, \bar{s}} \) takes \( \omega_{\text{red}, \bar{s}} \) to \( \omega_{\text{sm}, \bar{s}} \), and \( \phi_{\text{avg}, \bar{s}} \) takes \( \omega_{\text{sm}, \bar{s}} \) to \( \omega_{\text{avg}, \bar{s}} \), each satisfying all desired properties.

**Step 1.** We interpolate between \( \omega_{\text{red}, \bar{s}} \) and \( \omega_{\text{sm}, \bar{s}} \) via the family of Kähler forms \( \omega_{t, \bar{s}}, t \in [0, \kappa] \),

\[
(3.4) \quad \omega_{t, \bar{s}} = dd^c K_{t, \bar{s}} = \frac{1}{4} dd^c \left( \sum_{i=1}^n r_{t,i} - s_i \ln(s_i + r_{t,i}) \right),
\]

where \( r_{t,i} = \sqrt{s_i^2 + 4v_t \bar{v}_i + t^2} \). We use Moser’s trick and look for the vector field \( V_{t, \bar{s}} \) satisfying

\[
\mathcal{L}_{V_{t, \bar{s}}} \omega_{t, \bar{s}} + \frac{d}{dt} \omega_{t, \bar{s}} = \mathcal{L}_{V_{t, \bar{s}}} \omega_{t, \bar{s}} + dd^c \left( \frac{dK_{t, \bar{s}}}{dt} \right) = 0.
\]

By Cartan’s formula, we have

\[
\frac{d}{dt} \omega_{t, \bar{s}} = -dd^c \left( \frac{dK_{t, \bar{s}}}{dt} \right),
\]

from which we deduce

\[
\nu_{V_{t, \bar{s}}} \omega_{t, \bar{s}} = -a_{t, \bar{s}} = -d^c \left( \frac{dK_{t, \bar{s}}}{dt} \right) = -\frac{1}{4} d^c \left( \sum_{i=1}^n \frac{t}{s_i + r_{t,i}} \right).
\]

We have

\[
\omega_{t, \bar{s}} = \sum_{1 \leq i, j \leq d} \omega_{t, \bar{s}, ij} dv_i \wedge d\bar{v}_j
\]

\[
= \sum_{1 \leq i, j \leq d} \frac{\sqrt{-1}}{2} \left( \delta_{ij} \left( \frac{r_{t,i}^2 + s_i r_{t,i} - 2|v_t|^2}{(s_i + r_{t,i})^2 r_{t,i}} \right) + \sum_{l=d+1}^n a_{l,i} a_{l,j} \left( \frac{r_{t,l}^2 + s_l r_{t,l} - 2|v_l|^2}{(s_l + r_{t,l})^2 r_{t,l}} \right) \right) dv_i \wedge d\bar{v}_j,
\]

and

\[
-a_{t, \bar{s}} = -\sum_{i=1}^d a_{t, \bar{s}, i} dv_i + \bar{a}_{t, \bar{s}, i} d\bar{v}_i
\]

\[
= \sum_{i=1}^d -\frac{t\sqrt{-1}}{2} \left( \frac{v_i}{(s_i + r_{t,i})^2 r_{t,i}} + \sum_{l=d+1}^n \frac{a_{l,i} v_i}{(s_l + r_{t,l})^2 r_{t,l}} \right) d\bar{v}_i + \frac{t\sqrt{-1}}{2} \left( \frac{\bar{v}_i}{(s_i + r_{t,i})^2 r_{t,i}} + \sum_{l=d+1}^n \frac{a_{l,i} \bar{v}_i}{(s_l + r_{t,l})^2 r_{t,l}} \right) dv_i.
\]

Denote by \( A = (A_{l,j})_{1 \leq i, j \leq d} \) the matrix with entries \( A_{l,j} = \omega_{t, \bar{s}, ij} \), and let \( A^{-1} = (A^{ij}) \) be its inverse. The vector field \( V_{t, \bar{s}} \) is given by

\[
V_{t, \bar{s}} = \sum_{j=1}^d f_{t, \bar{s}, j} \frac{\partial}{\partial v_j} + g_{t, \bar{s}, j} \frac{\partial}{\partial \bar{v}_j} = \sum_{j=1}^d -\left( \sum_{l=1}^d \frac{A_{l,j} a_{t, \bar{s}, l}}{A^{lj}} \right) \frac{\partial}{\partial v_j} - \left( \sum_{l=1}^d \frac{A^{lj} a_{t, \bar{s}, l}}{A_{l,j}} \right) \frac{\partial}{\partial \bar{v}_j}.
\]

\( V_{t, \bar{s}} \) is smooth except when \( t = 0 \) and \( \bar{s} \in H_{\bar{R}^d,i} \) for some \( i \), in which case it is singular along \( H_{\bar{C},i} \).

We will show that the flow of \( V_{t, \bar{s}} \) is well-defined even for \( t = 0 \), and \( V_{t, \bar{s}} \) is complete.

Let \( I \subset \{1, \ldots, n\} \) be a multi-index such that \( \bigcap_{k \in I} H_{\bar{C},k} \neq \emptyset \). Let \( \bar{s} \in (t^d)^* \) be a point such that \( \bar{s}_k \in H_{\bar{R}^d,k} \) if and only if \( k \in I \), and let \( \bar{v}_0 \in \bigcap_{k \in I} H_{\bar{C},k} \). We now analyze the singularities of the functions \( f_{t, \bar{s}, j} \) (the analysis for \( g_{t, \bar{s}, j} \) is identical and thus omitted) by considering the following limits:

\[
\lim_{(t, \bar{v}) \to (0, \bar{v}_0)} f_{t, \bar{s}, j} = -\lim_{(t, \bar{v}) \to (0, \bar{v}_0)} \sum_{l=1}^n A_{l,j} a_{t, \bar{s}, l} = -\lim_{(t, \bar{v}) \to (0, \bar{v}_0)} \sum_{l=1}^n \frac{C_{l,j}}{\det A} a_{l,j},
\]

where \( C_{l,j} \) is the \((j, i)\)-cofactor of \( A \). Since \( \bigcap_{k \in I} H_{\bar{C},k} \neq \emptyset \) and the hyperplane arrangement \( H_{\bar{C}} \) is simple, the vectors \( \{u_k\}_{k \in I} \) are linearly independent. Thus we can assume \( J \subset \{1, \ldots, d\} \) by
rearranging indices (notice that the coefficients in \( u_l = \sum_{i=1}^{d} a_{li} u_i \), \( l = d + 1, \ldots, n \), will change accordingly but this does not affect our argument).

Let \( A_I \) be the matrix obtained from \( A \) by removing the \( k \)th row and column for \( k \in I \). Denote by \( A_I,ij \) the matrix obtained from \( A \) by removing the \( j \)th row, \( i \)th column, and \( k \)th row and column for \( k \in I \). Note that \( \det A_I \neq 0 \), and \( \det A_{I,ij} \) is nonsingular. As \((t, \vec{v}) \to (0, \vec{v}_0)\), \( \det A \) is dominated by the term

\[
\prod_{k \in I} \frac{r_{i,k}^2 + s_k r_{i,k} - 2|v_k|^2}{(s_k + r_{i,k})^2 r_{i,k}} \times \det A_I.
\]

If \( i \notin I \), \( C_{ji} \) is dominated by the term

\[
\prod_{k \notin I \setminus \{j\}} \frac{r_{i,k}^2 + s_k r_{i,k} - 2|v_k|^2}{(s_k + r_{i,k})^2 r_{i,k}} \times \det A_{I,ij},
\]

As \((t, \vec{v}) \to (0, \vec{v}_0)\), \( C_{ji} \) blows up of at most the same order as \( \det A \), while \( a_{t,\vec{s},i} \) converges since \( i \notin I \). In this case, the term \( A^{ji} a_{t,\vec{s},i} \) converges. If \( i \in I \), \( C_{ji} \) is dominated by the term

\[
\prod_{k \in I \setminus \{j\}} \frac{r_{i,k}^2 + s_k r_{i,k} - 2|v_k|^2}{(s_k + r_{i,k})^2 r_{i,k}} \times \det A_{I,ij}.
\]

Then, \( A^{ji} \) vanishes of order at least 1, while \( a_{t,\vec{s},i} \) blows up of order 1 as \((t, \vec{v}) \to (0, \vec{v}_0)\), in which case the term \( A^{ji} a_{t,\vec{s},i} \) again converges. This shows \( V_{t,\vec{s}} \) has only removable singularities.

On the other hand, taking the dual vector field of \( a_{t,\vec{s}} \) gives

\[ V_{t,\vec{s}} = \nabla^t,\vec{s} \left( \frac{dK_{t,\vec{s}}}{dt} \right). \]

Let \( \| \cdot \| \) is the norm with respect to the Kähler metric determined by \( \omega_{t,\vec{s}} \). We have

\[
\| V_{t,\vec{s}} \|^2 = \| a_{t,\vec{s}} \|^2 = \| a_{t,\vec{s}} \otimes a_{t,\vec{s}} \| \leq \| \omega_{t,\vec{s}} \| = 1,
\]

for outside of a ball \( B(0, R) \) of sufficiently large radius. We define an auxiliary complete metric (see [Gli97] Theorem 1.3) \( g \) by

\[
(3.5) \quad g = \sum_{i=1}^{d} dv_i dv_i + \sum_{i \neq j} (\vec{v}_i \vec{v}_j) d\theta - \frac{d}{(\vec{v} \cdot \vec{v})^2}.
\]

The norm of \( V_{t,\vec{s}} \) is uniformly bounded with respect to \( g \). We can therefore define \( \phi_{sm,\vec{s}} \) to be the time-\( \kappa \) flow generated by \( V_{t,\vec{s}} \).

**Step 2.** We interpolate between \( \omega_{sm,\vec{s}} \) and \( \omega_{avg,\vec{s}} \) via the family of Kähler forms \( \omega_{t,\vec{s}}, t \in [0, 1] \), defined by

\[
\omega_{t,\vec{s}} = t \omega_{avg,\vec{s}} + (1 - t) \omega_{sm,\vec{s}} = d\bar{d}^c (tK_{avg,\vec{s}} + (1 - t)K_{sm,\vec{s}}),
\]

where

\[
K_{avg,\vec{s}} = \frac{1}{(2\pi)^d} \int_{T^d} g^* K_{sm,\vec{s}} dg.
\]

We again use Moser’s trick and look for the vector field \( V_{t,\vec{s}} \) satisfying

\[
\mathcal{L}_{V_{t,\vec{s}}} \omega_{t,\vec{s}} + \frac{d}{dt} \omega_{t,\vec{s}} = 0.
\]

By Cartan’s formula, we have

\[
dt V_{t,\vec{s}} = \omega_{t,\vec{s}} - \omega_{avg,\vec{s}},
\]

from which we deduce

\[ t V_{t,\vec{s}} = -a_{t,\vec{s}} = -d^c (K_{avg,\vec{s}} - K_{sm,\vec{s}}). \]

Taking the dual vector field, we get

\[ V_{t,\vec{s}} = \nabla^t,\vec{s} (K_{avg,\vec{s}} - K_{sm,\vec{s}}). \]
where $\nabla^{t,\bar{t}}$ is the gradient with respect to the Kähler metric determined by $\omega'_t, \bar{\omega}_t$. Write $K = K_{avg, \bar{s}} - K_{sm, \bar{s}}$. We have

$$
\|V_t^t\|' = 2 |K|^{1/2} \left\| \nabla^{t,\bar{t}} K^{1/2} \right\|' = 2 |K|^{1/2} \left\| dK^{1/2} \wedge d^* K^{1/2} \right\|',
$$

where $\|\cdot\|$' is the norm with respect to the Kähler metric determined by $\omega'_t, \bar{\omega}_t$. To estimate the norm of $V_t, \bar{\omega}_t$, we write down the relevant terms explicitly and find

$$K = \frac{1}{4} \sum_{l=d+1}^{d+n} \frac{1}{(2\pi)^d} \int_{g \in T^d} g^* \left( \frac{a_{lj} v_l}{s_l + r_{k,l}} \right) \left( \frac{a_{lj} \bar{v}_l}{s_l + r_{k,l}} \right) dg - \left( \frac{a_{lj} v_l}{s_l + r_{k,l}} \right) \left( \frac{a_{lj} \bar{v}_l}{s_l + r_{k,l}} \right) dv_l \wedge d\bar{v}_l,
$$

and

$$\omega'_t, \bar{\omega}_t = \sqrt{-1} \left( \delta_{lj} \left( \frac{r^2_{k,l} + s_l r_{k,l} + 2 |v_l|^2}{(s_l + r_{k,l})^2 r_{k,l}} \right) + \sum_{l=d+1}^{d+n} \frac{t}{(2\pi)^d} \int_{g \in T^d} g^* \left( \frac{a_{lj} \bar{v}_l}{r^2_{k,l} + s_l r_{k,l} + 2 |v_l|^2} \right) \left( \frac{a_{lj} v_l}{(s_l + r_{k,l})^2 r_{k,l}} \right) dg \right)
$$

outside of a ball $B(0, R)$ of sufficiently large radius. This means

$$\|V_t^t\|' \leq |\bar{v}|^{1/2} \left\| \omega'_t, \bar{\omega}_t \right\|' = |\bar{v}|^{1/2},
$$

outside of $B(0, R)$. Denote by $\|\cdot\|_g$ the norm with respect to the metric $g$ defined in \([3.5]\). We can find a constant $C > 0$ such that

$$C \|\bar{x}\|' \geq |\bar{v}| \|\bar{x}\|_g,
$$

for all $\bar{x} \in T_{\bar{v}} \mathbb{C}^d$, and $\bar{v} \in \mathbb{C}^d \setminus B(0, R)$. This means

$$\|V_t, \bar{\omega}_t\|_g \leq \frac{2C|K|^{1/2}}{|\bar{v}|^{1/2}},
$$

outside of $B(0, R)$. Thus, $\|V_t, \bar{\omega}_t\|_g$ is uniformly bounded. Moreover, the time-1 flow $\phi_{avg, \bar{s}}$ generated by $V_t, \bar{\omega}_t$ intertwines $\omega_{sm, \bar{s}}$ and $\omega_{avg, \bar{s}}$, as desired. 

Denote by $T$ the tropical semi-field $T = \mathbb{R} \cup \{ -\infty \}$. Let $c = (c_1, \ldots, c_d) \in \mathbb{C}^d$ is a point away from the hyperplanes in $H_C$.

**Definition 3.4.** Let $\text{Log}_t : \mathbb{C}^d \to T^d$ be the map defined by

$$\text{Log}_t(v_1, \ldots, v_d) = (\text{Log}_t(|v_1 - c_1|), \ldots, \text{Log}_t(|v_d - c_d|)),
$$

where $t \gg 0$ is a constant. Denote by $\pi_{\bar{s}} : X_{\bar{s}} \to T^d$ the composition $\pi_{\bar{s}} = \text{Log}_t \circ \phi_{\bar{s}}$. $\pi_{\bar{s}}$ is our preferred Lagrangian torus fibration on $X_{\bar{s}}$. 

3.2 Lagrangian torus fibration on $\mathfrak{M}_{u,\lambda}$ and discriminant loci.

**Definition 3.5.** We denote by $\pi: \mathfrak{M}_{u,\lambda} \to B = \mathbb{R}^d \times \mathbb{T}^d$ the map which sends a point $x \in \tilde{\mu}^{-1}(\xi)$ to $\pi(x) = (\tilde{s}, \tilde{\pi}(\tilde{x}))$, where $\tilde{x} \in X_{\xi}$ is the $T^d$-orbit of $x$. $\pi$ is a piecewise smooth Lagrangian torus fibration and generic fibers of $\pi$ are smooth Lagrangian tori $T^2_d$.

Let $b = (\tilde{s}, \tilde{\tau}) = (s_1, \ldots, s_d, \tau_1, \ldots, \tau_d) \in B$. When $\tau_i = -\infty$ for some $i$, the fiber $\pi^{-1}(b)$ is a torus $T^k, d \leq k < 2d$. If $\tilde{s} \in H_{R,i}$ and $\tilde{\tau} \in \pi_{\tilde{s}}(H_{C,i})$ for some $i$, the fiber $\pi^{-1}(b)$ is a pinched torus (i.e. a product of immersed $S^2$ and torus) of dimension $2d$. These are the only singular fibers of $\pi$. The image of all singular fibers under $\pi$ is the set $\Sigma = \partial B \cup \bigcup_{i=1}^n \{ (\tilde{s}, \tilde{\tau}) | \tilde{s} \in H_{R,i} \text{ and } \tilde{\tau} \in \pi_{\tilde{s}}(H_{C,i}) \}$, which we call the discriminant loci of $\pi$ (see Fig 3). Let $B^0 = B \setminus \Sigma$. $\pi$ restricts to a $T^{2d}$-bundle over $B^0$, and induces an integral affine structure on $B^0$.

**Figure 3.** Lagrangian fibrations on $T^*\mathbb{P}^1$ and $T^*\mathbb{P}^2$, where the base are $\mathbb{R} \times \mathbb{T}$ and $\mathbb{R}^2 \times \mathbb{T}^2$, respectively. The complex hyperplanes are taken to be in general positions, and their tropical image of shown as amoebas.

4. SYZ Mirror Construction for Hypertoric Varieties

In this section, we carry out the SYZ mirror construction for smooth hypertoric varieties. We begin by reviewing the SYZ construction.

4.1 The SYZ mirror construction. Let $\pi: X \to B$ be a proper Lagrangian torus fibration of a compact Kähler manifold $(X, \omega)$ of dimension $d$ such that the base $B$ is a compact manifold with corners, and the preimage of each codimension-one facet of $B$ is a smooth irreducible divisor denoted as $D_i$ for $1 \leq i \leq m$.

We assume that the regular Lagrangian fibers of $\mu$ are special with respect to a nowhere-vanishing meromorphic top-form $\Omega$ on $X$ whose pole divisor is the boundary divisor $D := \sum_{i=1}^m D_i$ (and hence $D$ is an anti-canonical divisor). We denote by $B^0 \subset B$ the complement of the discriminant locus of $\pi$, and we assume that $B^0$ is connected.\footnote{When the discriminant locus has codimension-two, $B^0$ is automatically connected. Although the Lagrangian fibrations on hypertoric varieties that we constructed have codimension-one discriminant loci, $B^0$ is still connected.}

We denote by $L_b$ a fiber of $\pi$ at $b \in B^0$.

**Lemma 4.1** (Maslov index of disc classes [Aur07, Lemma 3.1]). For a disc class $\beta \in \pi_2(X, F_b)$ where $b \in B^0$, the Maslov index of $\beta$ is $\mu(\beta) = 2D \cdot \beta$.  

---

[Figure 3]
Definition 4.2 (Wall [CLL12]). The wall $W$ of a Lagrangian fibration $\pi : X \to B$ is the set of point $b \in B^0$ such that $L_b := \pi^{-1}(b)$ bounds non-constant holomorphic discs with Maslov index $0$.

The complement of $W \subset B^0$ consists of several connected components, which we call chambers. Over different chambers the Lagrangian fibers behave differently in a Floer-theoretic sense. Away from the wall $W$, the one-pointed open Gromov–Witten invariants are well-defined using the machinery of Fukaya–Oh–Ohta–Ono [FOOO09].

Definition 4.3 (Open Gromov–Witten invariants [FOOO09]). For $b \in B^0 \setminus W$ and $\beta \in \pi_2(X, L_b)$, let $M_1(L_b, \beta)$ be the moduli space of stable discs with one boundary marked point of class $\beta$, and $[M_1(L_b, \beta)]^{vir}$ be the virtual fundamental class of $M_1(\beta)$. The open Gromov–Witten invariant associated to $\beta$ is $n_\beta := \int_{[M_1(L_b, \beta)]^{vir}} \ev^*[pt]$, where $\ev : M_1(L_b, \beta) \to L_b$ is the evaluation map at the boundary marked point and $[pt]$ is the Poincaré dual of the point class of $L_b$.

We will restrict to disc classes which are transversal to the boundary divisor $D$ when we construct the mirror space (while for the mirror superpotential we need to consider all disc classes).

Definition 4.4 (Transversal disc class). A disc class $\beta \in \pi_2(X, L_b)$ for $b \in B^0$ is said to be transversal to the boundary divisor $D$, which is denoted as $\beta \cap D$, if it is represented by a map $u$ with $\text{Im}(u) \cap D$ being a finite set of points and the intersections are transversal.

Due to dimension reason, the open Gromov–Witten invariant $n_\beta$ is nonzero only when the Maslov index $\mu(\beta) = 2$. When $\beta$ is transversal to $D$ or when $X$ is semi-Fano, namely $c_1(\alpha) = D \cdot \alpha \geq 0$ for all holomorphic sphere classes $\alpha$, the number $n_\beta$ is invariant under small deformation of complex structure and under Lagrangian isotopy in which all Lagrangian submanifolds in the isotopy do not intersect $D$ nor bound non-constant holomorphic discs with Maslov index $\mu(\beta) < 2$.

The SYZ mirror construction can be realized as follows [CLL12]. First, the semi-flat mirror $X_0^\vee$ is defined as the space of pairs $(L_b, \nabla)$ where $b \in B^0$ and $\nabla$ is a flat $U(1)$-connection on the trivial complex line bundle over $L_b$ up to gauge. There is a natural map $\pi^\vee : X_0^\vee \to B^0$ given by forgetting the second coordinate. The semi-flat mirror $X_0^\vee$ has a canonical complex structure [Leu05] and the functions $e^{-\tilde{\beta}} \omega \Hol_{\nabla}(\tilde{\beta})$ on $X_0^\vee$ for disc classes $\beta \in \pi_2(X, L_b)$ are called semi-flat complex coordinates. Here $\Hol_{\nabla}(\tilde{\beta})$ denotes the holonomy of the flat $U(1)$-connection $\nabla$ along the loop $\tilde{\beta} \in \pi_1(L_b)$.

Then the generating functions of transversal open Gromov–Witten invariants are defined by

$$I_i(L_b, \nabla) := \sum_{\beta \in \pi_2(X, L_b) \setminus \beta : \partial D_i = \beta \cap D} n_\beta e^{-\tilde{\beta}} \omega \Hol_{\nabla}(\tilde{\beta}),$$

for $1 \leq i \leq m$, $(L_b, \nabla) \in (\pi^\vee)^{-1}(B^0 \setminus H)$. They serve as quantum corrected complex coordinates. The function $I_i$ can be written in terms of the semi-flat complex coordinates, and hence they generate a subring $\mathbb{C}[I_1, \ldots, I_m]$ in the function ring $\mathbb{C}$ of $(\pi^\vee)^{-1}(B^0 \setminus W)$.

Definition 4.5. An SYZ mirror of $X$ is the pair $(X^\vee, W)$ where $X^\vee := \text{Spec} (\mathbb{C}[I_1, \ldots, I_m])$ and $W := \sum_{\beta \in \pi_2(X, L_b)} n_\beta e^{-\tilde{\beta}} \omega \Hol_{\nabla}(\tilde{\beta})$.

Moreover, $X^\vee$ is called to be an SYZ mirror of $X - D$.

Remark 4.6. In general the mirror space $X^\vee$ defined in this way, which only uses the generating functions of stable discs emanated from boundary divisors, is always affine and can be singular. The reason is that it ignores the detailed chamber structure in the base and only take the chambers adjacent to the boundary divisors into account.

Indeed for most hypertoric varieties this is the case. A resolution is necessary, and this will be carried out in Section 4.8. The choice of a resolution does not affect the derived category.

---

3 In general we need to use the Novikov ring instead of $\mathbb{C}$ since $I_i$ could be a formal Laurent series. In the cases that we study later, $I_i$ are Laurent polynomials whose coefficients are convergent, and hence the Novikov ring is not necessary.
Remark 4.7. Note that \( W \) is a sum over all disc classes which are not necessarily transversal. If \( X \) is semi-Fano, then every stable holomorphic disc class of Maslov index 2 is of the form \( \beta + \alpha \) where \( \beta \) is transversal with \( \mu(\beta) = 2 \), and \( \alpha \in H_2(X) \) with \( c_1(\alpha) = 0 \). Hence it takes the form \( W = \sum_{a_i} a_i \mathcal{L} \), where \( a_i \) are certain series in Kähler parameters. If \( X \) is not semi-Fano, then some algebraic manipulation is necessary to write \( W \) as a series in \( \mathcal{L} \) over the Novikov ring. In this paper we deal with \( X - D \) and hence do not seriously concern about \( W \).

4.2 Maslov index 0 holomorphic discs and walls. Let \( c = (c_1, \ldots, c_d) \in (\mathbb{C}^d)^* \) be as in Definition 3.4. Denote by \( D_i \) the divisor \( D_{-i} = \{ [z, w] \in \mathcal{M}_{u,\lambda} | z_i w_i = c_i \} \). Let \( D_\mathcal{R} = D_{-1} + \ldots + D_{-d} \). We will assume the isotopies \( \phi_0 \) of Lemma 3.3 preserves \( D_\mathcal{R} \). This can be achieved by modifying \( \phi_0 \) using the construction in [AAK16] Lemma B.2.

Proposition 4.8. Let \( L_b \) be a fiber of \( \pi \) over \( b \in B^0 \). For any disc class \( \beta \in \pi_2(\mathcal{M}_{u,\lambda}, L_b) \), the Maslov index \( \mu(\beta) \) is twice the algebraic intersection number \( \beta \cdot D_\mathcal{R} \).

Proof. Let \( \Omega \) be the meromorphic volume form on \( \mathcal{M}_{u,\lambda} \) with generically simple poles along \( D_\mathcal{R} \) defined by

\[
\Omega = \frac{\prod_{i=1}^{d} dz_i \wedge dw_i}{\prod_{i=1}^{d} z_i w_i - c_i}.
\]

Let \( b = (\tilde{s}, \tilde{\tau}) \). If \( \tilde{s} \notin H_{R,i} \) for all \( i \), the residual \( T^0/K \)-action on the level set \( \tilde{\mu}^{-1}_R(\tilde{s}) \) containing \( L_b \) is free, and hence \( \tilde{\mu}^{-1}_R(\tilde{s}) \) is a trivial \( T^d \)-bundle over \( \mathbb{C}^d \). From Lemma 3.3 we have a one parameter family \( (\tilde{\phi}_{\tilde{s},t})_{t \in [0,1]} \) of homeomorphisms taking the projection \( \tilde{\mu}^{-1}_C(L_b) \subset \mathbb{C}^d \) of \( L_b \) to a standard product torus centered at the point \( c \). We lift \( (\tilde{\phi}_{\tilde{s},t})_{t \in [0,1]} \) to \( \mu^{-1}_R(\tilde{s}) \) by defining it to be fiber-wise constant. Then we extend it to a one parameter family of homeomorphisms of \( (\tilde{\Phi}_{\tilde{s},t})_{t \in [0,1]} \) of \( \mathcal{M}_{u,\lambda} \). If \( \tilde{s} \in H_{R,i} \) for some \( i \), we cannot write \( L_b \) to a nearby smooth fiber \( L_{s'} \) contained in a level set \( \tilde{\mu}^{-1}_R(\tilde{s}) \) with \( s' \notin H_{R,i} \) for all \( i \). We define \( (\tilde{\Phi}_{s,t})_{t \in [0,1]} \) by pre-composing \( (\tilde{\Phi}_{s',t})_{t \in [0,1]} \) with this isotopy. Since phase function \( \arg(\Omega)_{s,t} \) is trivial, \( \arg(\Omega)_{s,t} \) lifts to a real-valued function. It is then a well known fact that \( \mu(\beta) = 2\beta \cdot D_\mathcal{R} \).

\[
\boxdot
\]

Proposition 4.9. The walls of the Lagrangian torus fibration \( \pi : \mathcal{M}_{u,\lambda} \to B \) are the sets \( \{W_i\}_{i=1}^N \subset B^0 \) defined by

\[
W_i = \{ (\tilde{s}, \tilde{\tau}) \in B^0 | \tilde{\tau} \in \pi_2(H_{C,i}) \}.
\]

Proof. Let \( L_b \) be a smooth fiber of \( \pi \) over \( b = (\tilde{s}, \tilde{\tau}) \). \( L_b \) is contained in the level set \( \tilde{\mu}^{-1}_R(\tilde{s}) \). Let \( u : (D^3, \partial D^3) \to (\mathcal{M}_{u,\lambda}, L_b) \) be a holomorphic disc with boundary in \( L_b \) representing \( \beta \in \pi_2(\mathcal{M}_{u,\lambda}, L_b) \), with \( \mu(\beta) = 0 \). Denote by \( L_{red} \) the projection of \( L_b \) to \( \mathbb{C}^d \) via \( \mu_C \). \( L_{red} \) is a Lagrangian torus with respect to \( \omega_{red} \), and its projection to the \( i \)-th component is a loop around \( c_i \).

The holomorphic disc \( \tilde{\mu}_C \circ u : (D^3, \partial D^3) \to (\mathbb{C}^d, L_{red}) \) is contained in \( \mathbb{C}^d \setminus \{ c \} \) by Proposition 4.8. By maximal principle, \( \tilde{\mu}_C \circ u \) is necessarily constant. This means the image of \( u \) is contained in a fiber \( \tilde{\mu}_C^{-1}(v_0) \) for some \( v_0 \in \mathbb{C}^d \).

If \( b \notin W_i \) for all \( i \), then \( v_0 \) is away from all hyperplanes in \( \mathcal{H}_C \). This means \( \tilde{\mu}_C^{-1}(v_0) = (\mathbb{C}^\times)^d \), while \( \tilde{\mu}_C^{-1}(v_0) \cap L_b \) is a product torus in \( \mathbb{C}^d \). Maximal principle then implies that \( u \) is necessarily constant. On the other hand, suppose \( b \in \bigcap_{i=1}^N W_i \), then, for \( \{ j_1, \ldots, j_m \} \subset \{ i_1, \ldots, i_N \} \) such that \( \bigcap_{i=1}^m H_{C,j_i} \neq \emptyset \), we can have \( v_0 \in \bigcap_{i=1}^m H_{C,j_i} \). In this case, \( \tilde{\mu}_C^{-1}(v_0) \subset (\mathbb{C} \cup \mathbb{C})^m \subset (\mathbb{C}^\times)^{d-m} \), where \( \mathbb{C} \cup \mathbb{C} \) denotes the union of two affine lines intersecting transversely at the origin. \( \tilde{\mu}_C^{-1}(v_0) \cap L_b \) is a product torus in \( (\mathbb{C} \cup \mathbb{C})^m \subset (\mathbb{C}^\times)^{d-m} \) such that each \( \mathbb{C} \cup \mathbb{C} \) contains one of its \( 3^1 \)-component in one of the irreducible components (depending on the signs of components of \( \tilde{s} \)). It is then easy to see...
that $\mu_{C}^{-1}(v_{0}) \cap L_{b}$ bounds exactly $m$ non-constant holomorphic discs (and all their multiple covers) of Maslov index 0. □

**Remark 4.10.** The construction of $(\phi_{C})_{t \in (0, \ell)}^{\ast}$ in Lemma 3.3 gives us an one-parameter family of homeomorphisms of $B^{0}$ taking each $W_{i}$ to $(\mathbb{R}^{d} \setminus H_{R,i}) \times \text{Log}_{t}(H_{C,i})$, where $\text{Log}_{t}(H_{C,i}) \subset \mathbb{T}^{d}$ is a amoeba that retracts to a tropical hyperplane in $\mathbb{T}^{d}$ as $t \to \infty$. Since we only need the wall and chamber structure on $B^{0}$ for the mirror construction, which is purely combinatorial, we will simply illustrate each $W_{i}$ as a tropical hyperplane in $\mathbb{T}^{d}$ (see Fig 4.3).

### 4.3 Chambers and simply connected affine charts

Let $H$ be a tropical hyperplane in $\mathbb{T}^{d}$ defined by the tropical polynomial $\max\{\tau_{1}, \ldots, \tau_{m}, a\}$. $H$ divides $\mathbb{T}^{d}$ into tropical chambers each of which a monomial of the defining equation attains maximum. We label the chamber where the constant $a$ attains maximum by 0, and the chamber where the monomial $\tau_{i}$ attains maximum by $i$. Using this convention, we can label the (closed) chambers given by a simple arrangement of tropical hyperplanes $\{H_{i}\}_{i=1}^{n}$ by $n$-tuples $h := (h_{1}, \ldots, h_{n})$, where $h_{i} \in \{0, \ldots, d\}$ indicates the position of the chamber relative to $H_{i}$.

Now, let $\mathcal{H} = \{H_{i}\}_{i=1}^{n}$ be the arrangement of tropical hyperplanes $H_{i}$, where $H_{i}$ is the tropical limit of the projection of the wall $W_{i}$ to $\mathbb{T}^{d}$. We can choose $\lambda_{C}$ to be sufficiently generic such that for $l = d+1, \ldots, n$, $|b_{l}|$ (in the expression $v_{l} = \sum_{i=1}^{d} a_{i} v_{i} + b_{l}$) are distinct powers of $t$, making $\mathcal{H}$ simple (i.e. every subset of $k$ tropical hyperplanes with nonempty intersection intersects in codimension $k$). We will denote by $C_{h}$ both the tropical chambers and their preimages in $B^{0}$. This shall not cause any confusion. Note that the wall and chamber structure depends on the choice of $\lambda_{C}$.

![Figure 4: Tropical hyperplane arrangement and chambers](image)

Let $\sigma$ be a sign vector. $B^{0}$ is covered by simply connected affine charts $B^{0}_{\sigma}$ defined by

$$B^{0}_{\sigma} = \{(\vec{s}, \vec{\tau}) \in B^{0} | \vec{s} \in H_{R,i}^{\sigma_{i}} \text{ if } \vec{\tau} \in H_{i}; \vec{s} \in \mathbb{R}^{d} \text{ if } \vec{\tau} \notin H_{i}, \forall i \}.$$
The fibration \( \pi \) trivializes over each \( B^0_\sigma \).

### 4.4 Effective disc classes of Maslov index 2

Let \( b \in B^0_\sigma \cap C_\eta \). We now classify the effective disc classes \( \beta \in \pi_2(\mathcal{M}_{u,\lambda}, L_b) \) of Maslov index 2.

**Proposition 4.11.** The effective disc classes \( \beta \in \pi_2(\mathcal{M}_{u,\lambda}, L_b) \) of Maslov index 2 are of the following form:

\[
\beta = \beta_{-j} + \delta_1 \alpha_{j_1} + \ldots + \delta_N \alpha_{j_N}, \quad j = 1, \ldots, d,
\]

where \( \delta_i \in \{0, 1\} \), and \( j_1, \ldots, j_N \in \{1, \ldots, n\} \) are the set of indices such that \( h_{j_i} = j \), i.e. the images of holomorphic discs of class \( \beta \) cross the walls \( W_{j_1}, \ldots, W_{j_N} \).

**Proof.** Let \( u : (D^2, \partial D^2) \to (\mathcal{M}_{u,\lambda}, L_b) \) be a holomorphic disc of Maslov index 2. Denote by \( Z_i \) and \( W_i \) the irreducible components

\[
Z_i = \{ [z, w] \in \mathcal{M}_{u,\lambda} | |z_i| = 0 \},
\]

and

\[
W_i = \{ [z, w] \in \mathcal{M}_{u,\lambda} | |w_i| = 0 \},
\]

of the subvariety \( [z, w] \in \mathcal{M}_{u,\lambda} | z_i w_i = 0 \).

For \( i = 1, \ldots, n \), by Proposition 4.8 and positivity of intersection, \( u \) intersects exactly one \( D_{-j} \) with multiplicity 1. Thus, by a winding number argument, \( u \) cannot intersect both \( Z_i \) and \( W_i \).

For each sign vector \( \sigma' \), we define an open subset \( U_{\sigma'} \subset \mathcal{M}_{u,\lambda} \) by

\[
U_{\sigma'} = \{ [z, w] \in \mathcal{M}_{u,\lambda} | z_i \neq 0 \text{ if } \sigma'(i) = + \text{ and } w_i \neq 0 \text{ if } \sigma'(i) = -, \text{ for } i = 1, \ldots, n \}.
\]

We have \( L_b \subset U_{\sigma'} \) for all \( U_{\sigma'} \), and \( u(D^2) \subset U_{\sigma'} \) for exactly one \( \sigma' \). Note that each \( U_{\sigma'} \) is biholomorphic to the trivial \( (\mathbb{C}^*)^d \)-bundle over \( \mathbb{C}^d \). Let \( (v_1, \ldots, v_d) \in \mathbb{C}^d \) be the complex coordinates on \( U_{\sigma'} \), where \( v_i = z_i w_i \) are the base coordinates and \( v_i \) are the fiber coordinates. Assume \( u(D^2) \subset U_{\sigma'} \) and write \( u : (D^2, \partial D^2) \to (U_{\sigma'}, L_b) \) as

\[
u(\zeta) = (v_1(\zeta), \ldots, v_d(\zeta), v_1(\zeta), \ldots, v_d(\zeta)).
\]

By maximal principle, only the \( v_j \)-component of \( u \) is nonconstant. The \( v_j \)-component of \( u \) is unique up to reparametrization. This means all holomorphic discs \( u \) of Maslov index 2 with \( u(D^2) \subset U_{\sigma'} \), for a fixed \( \sigma' \) represent the same disc class in \( \pi_2(\mathcal{M}_{u,\lambda}, L_b) \). We denote the disc class contained in \( U_{\sigma'} \) by \( \beta_{-j} + \delta_1 \alpha_{j_1} + \ldots + \delta_N \alpha_{j_N} \), where \( \delta_i = 1 \) if \( h_{j_i} = j \) and \( \sigma'(j_i) \neq \sigma(j_i) \), and \( \delta_i = 0 \) otherwise.

Note that the argument above has already shown that these disc classes are effective.

To see that these disc classes are distinct, one can simply consider their intersection numbers with the cycles \([Z_i]\) and \([W_i]\).

\[ \square \]

**Remark 4.12.** The disc classes \( \beta_{-j}, \ldots, \beta_{-d} \in \pi_2(\mathcal{M}_{u,\lambda}, L_b) \) are given by vanishing cycles \( \gamma_{\sigma,1}, \ldots, \gamma_{\sigma,d} \in H_1(L_b) \) such that \( \gamma_{\sigma,i} \) shrinks to a point when parallel transported to a singular fiber contained in \( D_{-i} \). When \( b \in W_i \) for some \( i \), \( \alpha_1, \ldots, \alpha_n \) are in fact disc classes given by vanishing cycles \( \gamma_{\sigma,d+1}, \ldots, \gamma_{\sigma,d+n} \in H_1(L_b) \) such that \( \gamma_{\sigma,d+i} \) shrinks to a point over the discriminant locus \( H_b \times H_1 \). Notice that if \( b \in W_i \) for some \( i \), \( \alpha_i \) is the Maslov index 0 disc class described in Proposition 4.9 and if \( b \notin W_i \), \( \alpha_i \) is no longer effective.

**Remark 4.13.** Suppose \( \beta \) and \( \beta + \alpha_i \) are disc classes in \( \pi_2(\mathcal{M}_{u,\lambda}, L_b) \). Starting with a holomorphic representative \( u \) of \( \beta \) (resp. \( \beta + \alpha_i \)), one can construct a holomorphic representative \( u' \) of \( \beta + \alpha_i \) (resp \( \beta \)) by multiplying components of \( u \) by a Blaschke factor and its reciprocal in appropriate local coordinates.

### 4.5 Regularity and open Gromov-Witten invariants

We now prove regularity of the disc classes in \( \pi_2(\mathcal{M}_{u,\lambda}, L_b) \) and compute relevant open Gromov-Witten invariants necessary for the mirror construction. Our strategies of proofs are similar to that of Lemma 7 and Corollary 8 in \cite{Aur15}.

Let \( u : (D^2, \partial D^2) \to (\mathcal{M}_{u,\lambda}, L_b) \) be a holomorphic disc. Denote by \( (E, F) \) the sheaf of holomorphic sections of \( E = u^* T \mathcal{M}_{u,\lambda} \) with boundary values in \( F = (u|_{\partial D^2})^* TL_b \). Denote by \( A^0(E, F) \) the sheaf of smooth sections of \( E \) with boundary values in \( F \), and \( A^{(0,1)}(E) \) the sheaf of smooth \( E \)-valued \((0,1)\) forms.
Lemma 4.14. [CO06, Lemma 6.2] The sequence
\[
0 \to (E, F) \to A^0(E, F) \to \hat{A}^{(0,1)}(E) \to 0
\]
defines a fine resolution of \((E, F)\).

Proposition 4.15. The holomorphic discs representing classes in \(\{4.2\}\) are Fredholm regular, i.e. its linearization \(\hat{\partial}\) is surjective.

Proof. Let \(u : (D^2, \partial D^2) \to (\mathbb{R}u_\lambda, L_b)\) be a holomorphic disc of Maslov index 2. Denote by \(u_{\text{red}}\) the composition \(\mu_C \circ u : (D^2, \partial D^2) \to (\mathbb{C}^d, L_{\text{red}})\), where \(L_{\text{red}} = \mu_C(L_b)\). Let \(L_R\) and \(L_C\) be the real and complex spans of the vector fields generating the \(T^n/K\)-action. As noted in the proof of Proposition 4.11, both \(L_b\) and the image of \(u\) are contained in \(U_{\sigma'}\) (see \(\{4.3\}\)) for some sign vector \(\sigma'\).

The \(T^n/K\)-action is free on \(U_{\sigma'}\), and thus we have the following short exact sequences:
\[
0 \to L_C \to T\mathbb{R}u_\lambda \to \mu_C^* T\mathbb{C}^d \to 0,
\]
\[
0 \to L_R \to TL_b \to \mu_C^* TL_{\text{red}} \to 0,
\]
in \(U_{\sigma'}\). Pulling back the exact sequences above along \(u\), we find that \(E\) admits a trivial holomorphic subbundle \(u^* L_C\), with a trivial real subbundle \((u_\partial^* D^2)^* L_R \subset F\) on the boundary. Since the \(\hat{\partial}\)-operator for complex-valued functions on the unit disc with trivial real boundary condition on the boundary circle is surjective, the surjectivity of \(\hat{\partial}\) on sections of \(E\) with boundary conditions \(F\) is equivalent to the surjectivity of \(\hat{\partial}\) on sections of \(E\) with boundary conditions \(F/(u_\partial^* D^2)^* L_R = (u_{\text{red}}^* D^2)^* TL_{\text{red}}\). Since only the \(j\)-th component of \(u_{\text{red}}\) is non-constant, the surjectivity of \(\hat{\partial}\) reduces to a one-dimensional Riemann-Hilbert problem which then follows from Theorem II and III in \([Oh95]\).

\[\square\]

Proposition 4.16. With the notations as in Proposition \(\{4.11\}\), we have
\[
n_\beta = \begin{cases} 
 1 & \text{for } \beta = \beta_j + \delta_1 \alpha_{j_1} + \ldots + \delta_n \alpha_{j_n}, \quad j = 1, \ldots, n; \\
 0 & \text{otherwise.}
\end{cases}
\]

Proof. Due to dimension reason, we have \(n_\beta = 0\) for \(\mu(\beta) \neq 2\). Suppose \(\beta\) is an effective disc class with \(\mu(\beta) = 2\), intersecting the divisor \(D_{-j}\). Denote by \(p\) be the unique boundary marked point on the unit disc. Let \(L_{\text{red}} = \mu_C(L_b) \subset \mathbb{C}^d\), and let \(\beta = (\mu_C)_{\beta} \in \pi_2(\mathbb{C}^d, L_{\text{red}})\). Denote by \(D_{i}\) the divisor \(\{(v_1, \ldots, v_d) \in \mathbb{C}^d | v_i = c_i\}\). We have \(\beta \cdot D_{-j} = 1\), and \(\beta \cdot D_i = 0\) for \(i \neq j\). We first consider the moduli space \(M_1(L_{\text{red}}, \beta)\). By maximal principle, for any \([u] \in M_1(L_{\text{red}}, \beta)\), all but the \(j\)-th component of \(u\) are constant, and the \(j\)-th component of \(\hat{u}\) is unique up to automorphisms of \(D^2\) fixing \(p\). Thus, for each \(q \in L_{\text{red}}\), there exists a unique \([\hat{u}] \in M_1(L_{\text{red}}, \beta)\) with \(\hat{u}(p) = q\). Moreover, the map \(ev : M_1(L_{\text{red}}, \beta) \to L_{\text{red}}\) given by evaluation at the boundary marked point is a diffeomorphism. Now, consider the projection \(M_1(L_b, \beta) \to M_1(L_{\text{red}}, \beta)\) given by post-composition with \(\mu_C\). We will show momentarily that for any given \([\hat{u}] \in M_1(L_{\text{red}}, \beta)\), and a lift \(\tilde{q} \in L_b\) of \(q\), there exists a unique \([u] \in M_1(L_b, \beta)\) with \(\mu_C \circ u = \hat{u}\) and \(u(p) = \tilde{q}\). Any holomorphic disc in \(M_1(L_b, \beta)\) has its image contained in \(U_{\sigma'}\) (see \(\{4.3\}\)) for some sign vector \(\sigma'\). \(U_{\sigma'}\) is biholomorphic to the trivial \((\mathbb{C}^x)^d\)-bundle over \(\mathbb{C}^d\). Denote by \((v_1, \ldots, v_d, v_1, \ldots, v_d)\) the complex coordinates on this open set with \(v_1, \ldots, v_d\) being the base coordinates and \(v_1, \ldots, v_d\) the fiber coordinates. In this coordinate, write \(\hat{q} = (\hat{q}_1, \ldots, \hat{q}_{2d})\). We define the lift of \(\hat{u}\) to be the holomorphic disc \(u : (D^2, \partial D^2) \to (U_{\sigma'}, L_b)\) defined by
\[
u(\zeta) = (\hat{u}(\zeta), \hat{q}_{d+1}, \ldots, \hat{q}_{2d}).
\]
We have a free \(T^d\)-action on \(M_1(L_b, \beta)\) given by composing holomorphic discs \([u] \in M_1(L_b, \beta)\) with the \(T^d\)-action on \(\mathbb{R}u_\lambda\). The orbits of this action are exactly the fibers of \(M_1(L_b, \beta) \to M_1(L_{\text{red}}, \beta)\). Therefore, \(M_1(L_b, \beta) \to M_1(L_{\text{red}}, \beta)\) is a \(T^d\)-bundle. Since the evaluation map \(ev : M_1(L_b, \beta) \to L_{\text{red}}\) is \(T^d\)-equivariant, it is again a diffeomorphism, i.e. it is of degree \(\pm 1\).

As for the orientations of \(M_1(L_b, \beta)\), recall that a spin structure on \(L_b\) determines an orientation on \(M_1(L_b, \beta)\) (see \([FOOO09]\, \text{Chapter 8}\)). Since \(L_{\text{red}}\) is isotopic to the standard product torus in \(\mathbb{C}^d\) via \(\phi_\alpha\), we can choose the standard spin structure on \(L_{\text{red}}\) such that \(ev : M_1(L_{\text{red}}, \beta) \to L_{\text{red}}\)
is orientation-preserving. We choose the spin structure on $L_b$ to be standard along the $T^d$-orbits and consistent under the splitting (4.6) with the spin structure previously chosen on $L_{\text{red}}$. Then, with the induced orientation on $\mathcal{M}_1(L_b, \beta)$, the evaluation map $\text{ev} : \mathcal{M}_1(L_b, \beta) \rightarrow L_b$ is orientation-preserving, i.e. it is of degree 1.

\begin{proposition}
$L_b$ is weakly unobstructed.
\end{proposition}

\begin{proof}
Due to degree reason, only stable holomorphic discs of Maslov index less than or equal to 2 can contribute to $m^b_{0\beta}$. In our case there is no stable discs with negative Maslov index. Thus the only discs with Maslov index less than 2 are the constant ones, which are not stable since there is only one output marking. For an effective disc class $\beta$ of Maslov index 2, the evaluation map at the boundary marked point gives a homeomorphism $\mathcal{M}_1(L_b, \beta) \rightarrow L_b$. Hence $m^b_{0\beta}$, which is the sum over $\beta$ of $\text{ev}_* [\mathcal{M}_1(L_b, \beta)]$ weighted by $T^{-\frac{1}{2}\omega}$ (where $T$ is the formal Novikov parameter), is proportional to the fundamental class of $L_b$.
\end{proof}

4.6 Partial compactifications of hypertoric varieties. Our idea of constructing the mirror $\mathfrak{M}^\rho_{u,\lambda}$ is to construct coordinate functions of $\mathfrak{M}^\rho_{u,\lambda}$ by counting holomorphic discs emanating from boundary divisors of $\mathfrak{M}_{u,\lambda}$. The problem is that in our situation, $B$ has only $d$ codimension-one boundary, while we need $2d$ coordinate functions. To resolve this, one may consider counting holomorphic cylinders, which requires the extra work of defining rigorously the corresponding Gromov-Witten invariants. Another way is to consider a partial compactification of $\mathfrak{M}_{u,\lambda}$ by adding divisors at infinity and count the additional holomorphic Maslov index 2 discs emanated from these divisors. We will use the second approach in this paper. This method was used in [CLL12], [AAK16] to construct mirrors of Calabi-Yau toric varieties, and blow-ups of toric varieties along a hypersurface.

\begin{figure}[h]
\centering
\includegraphics{wall-crossing.png}
\caption{Wall-crossing.}
\end{figure}
Recall from Remark 2.5 that the holomorphic moment map $\bar{\mu}_C : \mathcal{M}_{u,\lambda} \to \mathbb{C}^d$ is a holomorphic $(\mathbb{C}^\times)^d$ fibration. We partially compactify $\mathcal{M}_{u,\lambda}$ by extending $\bar{\mu}_C$ to a holomorphic $(\mathbb{C}^\times)^d$ fibration over $(\mathbb{P}^1)^d$.

Let $([\zeta_1 : \hat{\zeta}_1], \ldots, [\zeta_n : \hat{\zeta}_n])$ be the homogeneous coordinates on $(\mathbb{P}^1)^n$. We embed $\mathbb{C}^d$ into $(\mathbb{P}^1)^n$ via the map $(v_1, \ldots, v_d) \mapsto ([v_1 : 1], \ldots, [v_d : 1], [v_{d+1} : 1], \ldots, [v_n : 1])$, where $v_l = \sum_{k=1}^d a_{lk} v_k + b_l$ for $l = d + 1, \ldots, n$. Its closure $\overline{\mathbb{C}^d}$ in $(\mathbb{P}^1)^n$ is defined by the following homogeneous polynomials

$$f_l = \zeta_1 \cdots \zeta_d \hat{\zeta}_l - \sum_{i=1}^d a_{li} \zeta_1 \cdots \zeta_i \hat{\zeta}_i - b_l \zeta_1 \cdots \zeta_i \hat{\zeta}_i \quad l = d + 1, \ldots, n,$$

and is biholomorphic to $(\mathbb{P}^1)^d$. The hyperplanes $\{\hat{H}_{C,i}\}_{i=1}^n$ extends naturally to divisors $\{\hat{H}_{C,i}\}_{i=1}^n$ on $\overline{\mathbb{C}^d}$ defined by

$$\hat{H}_{C,i} = \{([\zeta_1 : \hat{\zeta}_1], \ldots, [\zeta_n : \hat{\zeta}_n]) \in \overline{\mathbb{C}^d} | \zeta_i = 0\}.$$

Let $W$ be total space of the rank $2n$ complex vector bundle on $(\mathbb{P}^1)^n$ defined by

$$W = \mathcal{O}(\hat{H}_{C,1}) \oplus \mathcal{O}_1 \oplus \ldots \oplus \mathcal{O}(\hat{H}_{C,n}) \oplus \mathcal{O}_n \to (\mathbb{P}^1)^n,$$

where $\mathcal{O}_i = \mathcal{O}$ are trivial complex line bundles. Denote by $w_i$ the fiber coordinate of $\mathcal{O}_i$, $z_i$ the local coordinate of the $\mathcal{O}(\hat{H}_{C,i})$ over $U_i = \{\zeta_i \neq 0\}$, and $\hat{z}_i$ the local coordinate of $\mathcal{O}(\hat{H}_{C,i})$ over $\hat{U}_i = \{\zeta_i \neq 0\}$. The gluing between $\mathcal{O}(\hat{H}_{C,i})|_{U_i}$ and $\mathcal{O}(\hat{H}_{C,i})|_{\hat{U}_i}$ is given by $z_i \hat{z}_i = \zeta_i \hat{\zeta}_i$. Let $V \subset W$ the subvariety defined by the ideal $(f_{d+1}, \ldots, f_n)$ and global sections $(g_1, \ldots, g_n)$ of $\mathcal{O}(\hat{H}_{C,i})$ defined by $g_i = z_i \hat{z}_i w_i - \zeta_i$.

We now define a $(\mathbb{C}^\times)^n$-action on $W$. For $\bar{t} = (t_1, \ldots, t_n) \in (\mathbb{C}^\times)^n$, let $\bar{t}$ act on $\mathcal{O}(\hat{H}_{C,i})$ via multiplication by $t_i$ and on $\mathcal{O}_i$ via multiplication by $t_i^{-1}$. Let $\bar{t}$ act trivially on the base $(\mathbb{P}^1)^n$. $V$ is then a $(\mathbb{C}^\times)^n$-invariant subvariety of $W$. Let $K_C \subset (\mathbb{C}^\times)^n$, and $\lambda_{\mathbb{R}} : K_C \to \mathbb{C}^\times$ be the same as in Definition 2.1. Then, the GIT quotient

$$\overline{\mathcal{M}}_{u,\lambda} = V//_{\lambda_{\mathbb{R}}} K_C$$

is a partial compactification of $\mathcal{M}_{u,\lambda}$. The embedding $\mathcal{M}_{u,\lambda} \hookrightarrow \overline{\mathcal{M}}_{u,\lambda}$ is holomorphic and $(\mathbb{C}^\times)^n/K_C$-invariant.

Alternatively, we can construct $\overline{\mathcal{M}}_{u,\lambda}$ via symplectic reduction. Notice that the subbundles $\mathcal{O}(\hat{H}_{C,i}) \to (\mathbb{P}^1)^n$ of $W$ are the pullbacks of $\mathcal{O}(1) \to \mathbb{P}^1$ via the projections $(\mathbb{P}^1)^n \to \mathbb{P}^1$ to the $i$th component. The sum of pullbacks of Fubini-Study form then defines a Kähler form on the total space of the subbundle $\bigoplus_{i=1}^n \mathcal{O}(\hat{H}_{C,i})$. Combined with the standard symplectic pairing on the fibers of $\mathcal{O}_i$, we have a $T^n$-invariant Kähler form $\omega_W$ on $W$. We can construct $\overline{\mathcal{M}}_{u,\lambda}$ as the symplectic reduction of $V$ at level $\lambda_{\mathbb{R}}$ with respect to the action of the maximal torus $K \subset K_C$ and the restriction of $\omega_W$ to $V$. This equips $\overline{\mathcal{M}}_{u,\lambda}$ with a $T^n/K$-invariant Kähler form $\overline{\omega}$.

We can construct a Lagrangian torus fibration

$$\overline{\pi} : \overline{\mathcal{M}}_{u,\lambda} \to \overline{B} = \mathbb{R}^d \times (\mathbb{R} \cup \{\pm \infty\})^d$$

using symplectic reductions as in Section 3. The reduced spaces are biholomorphic to $(\mathbb{P}^1)^d$. Since the reduced spaces are now compact, the construction of $\overline{\pi}$ is simply applications of Moser's trick, and hence omitted. The discriminant loci $\Sigma$ of $\overline{\pi}$ is the union $\Sigma$ and the new boundaries of $\overline{B}$ at infinity. Notice that we have $\overline{B} \setminus \Sigma = B^0 \subset B$.

We now state the results analogous to Propositions 4.8, 4.9, 4.11, 4.15, 4.17 and 4.16 in order to define the additional generating functions. We will be brief since the proofs are nearly identical to the previous ones.

Denote by $D_{\pm}$ the divisor $D_{\pm} = \{(\zeta_i = 0) \cap V//_{\lambda_{\mathbb{R}}} K_C\}$. Let $D_+ := \sum_{i=1}^d D_{1+i}$, and $D := D_- + D_+$. We assume the isotopy obtained from Moser’s trick leaves $D$ invariant.

**Proposition 4.18.** Let $L_b$ be the fiber of $\overline{\pi}$ over $b \in B^0$. For any disc class $\beta \in \pi_2(\overline{\mathcal{M}}_{u,\lambda}, L_b)$, the Maslov index $\mu(\beta)$ is equal to twice the algebraic intersection number $\beta \cdot D$. 
Proof. We first extend the meromorphic volume form $\Omega$ (see Proposition 4.8) on $\mathcal{M}_{u,\lambda}$ to a meromorphic volume form on $\overline{\mathcal{M}}_{u,\lambda}$ with additional (generically) simple poles along $D$. Consider the form

$$\tilde{\Omega} = \frac{\wedge_{i=1}^{n} d\log \xi_i \wedge d\log w_i}{\prod_{i=1}^{n} 1 - \frac{c_i}{\xi_i}}$$

defined on $U = \{ \tilde{\xi_i} \neq 0, \forall i \} \subset W$. Its restriction to $U \cap V$ descends to $\Omega$ on $\mathcal{M}_{u,\lambda}$. Let $I \subset \{1, \ldots, n\}$, and set $U_I = \{ \tilde{\xi_i} \neq 0, \forall i \in I \}$, $\tilde{U_I} = \{ \xi_i \neq 0, \forall i \in I \}$. Let $I_- \bigsqcup I_+$ be a splitting of $\{1, \ldots, n\}$. We extend $\tilde{\Omega}$ to $W$ by defining it to be

$$(-1)^{\text{sgn}(I_- \cdot I_+)} \left( \wedge_{i \in I_-} d\log \xi_i \wedge d\log w_i \right) \left( \wedge_{j \in I_+} -d\log \tilde{\xi}_j \wedge d\log w_j \right) \left( \prod_{i \in I_-} 1 - \frac{c_i}{\xi_i} \right) \left( \prod_{j \in I_+} 1 - \frac{c_j}{\tilde{\xi}_j} \right)$$
on $U_I \cap \tilde{U}_I$, where $\text{sgn}(I_- \cdot I_+)$ is the sign of the concatenation of $I_-$ and $I_+$ as a permutation. Note that the expression above is simply given by rewriting $\tilde{\Omega}$ under change of coordinates. We denote the extension of $\tilde{\Omega}$ to $W$ by $\bar{\Omega}$. $\bar{\Omega}$ is $(\mathbb{C}^*)^d$-invariant, hence its restriction to $V$ descends to a meromorphic volume form on $\overline{\mathcal{M}}_{u,\lambda}$, which is the extension of $\Omega$. With $\bar{\Omega}$ constructed, the proof then follows from Proposition 4.8.

The restriction of the projection $W \rightarrow (\mathbb{P}^1)^n$ to $V$ descends to a holomorphic $(\mathbb{C}^*)^d$-fibration $\rho : \mathcal{M}_{u,\lambda} \rightarrow (\mathbb{P}^1)^d$, extending $\bar{\mu}_{\mathcal{C}} : \mathcal{M}_{u,\lambda} \rightarrow \mathbb{C}^d$. We denote by $\rho_0 = \bar{\mu}_{\mathcal{C}} : \overline{\mathcal{M}}_{u,\lambda} \times D_+ \rightarrow \mathbb{C}^d$, and $\rho_{\mathcal{C}} : \overline{\mathcal{M}}_{u,\lambda} \times D_- \rightarrow \mathbb{C}^d$ the restrictions of $\rho$ to the respective domains.

**Proposition 4.19.** The walls of the Lagrangian torus fibration $\tilde{\pi} : \overline{\mathcal{M}}_{u,\lambda} \rightarrow \mathcal{B}$ are the sets $\{W_i\}_{i=1}^{3}$ defined in Proposition 4.8.

Proof. Since $\tilde{\mathcal{B}}$, $\tilde{\Sigma} = \mathcal{B}^0$, any fiber over $\mathcal{B}^0$ is contained in $\mathcal{M}_{u,\lambda}$. By Proposition 4.18 any Maslov index 0 holomorphic disc $u : (D^2, \partial D^2) \rightarrow (\overline{\mathcal{M}}_{u,\lambda}, L_b)$ is contained in $\mathcal{M}_{u,\lambda}$. Composing $u$ with $\rho_0$ reduces this to Proposition 4.9.

We again denote by $\mathcal{C}_{\mathcal{B}}$ and $\mathcal{B}_{\mathcal{B}}^0$ the chambers and simply connected affine charts on $\mathcal{B}^0 \subset \mathcal{B}$, respectively. Let’s fix a reference point $b \in \mathcal{B}_{\mathcal{B}}^0$ and assume $b \in \mathcal{C}_{\mathcal{B}}$. We now classify the effective disc classes $\beta \in \pi_2(\overline{\mathcal{M}}_{u,\lambda}, L_b)$ with $\mu(\beta) = 2$.

We express any vector $\hat{v}$ in the basis $\{u_1, \ldots, u_d\}$, and denote the corresponding coefficients by $v^{(\beta)}$.

**Proposition 4.20.** Denote by $\beta_{+1}, \ldots, \beta_{+d} \in \pi_2(\overline{\mathcal{M}}_{u,\lambda}, L_b)$ the disc classes given by the cycles $\gamma_{\sigma,1}, \ldots, \gamma_{\tau,d} \in H_1(L_b, \mathbb{Z})$ (see Remark 4.12) vanishing on $D_{+1}, \ldots, D_{+d}$. The effective disc classes $\beta = \pi_2(\overline{\mathcal{M}}_{u,\lambda}, L_b)$ with $\mu(\beta) = 2$ are of the form

$$\beta = \beta_{+j} + \delta_1 \alpha_{j_1} + \ldots + \delta_n \alpha_{j_n}, \quad j = 1, \ldots, d,$$

where $\delta_i \in \{0, 1\}$, and $j_1, \ldots, j_n \in \{1, \ldots, n\}$ is the set of indices such that $h_{j_i} = j$ if $\beta$ has the $\beta_{-j}$ component, and it is the set indices such that $u_{i}^{(j)} \neq 0$, and $h_{j_1} \neq j$ if $\beta$ has the $\beta_{+j}$ component. In other words, the images of holomorphic discs of class $\beta$ in $\mathcal{B}$ cross the walls $W_{j_1}, \ldots, W_{j_n}$.

Proof. Let $u : (D^2, \partial D^2) \rightarrow (\overline{\mathcal{M}}_{u,\lambda}, L_b)$ be a holomorphic disc of Maslov index 2. By Proposition 4.18 $u$ intersects either $D_-$ or $D_+$ with multiplicity 1. In either case, we can classify the potential disc classes of $u$ and show that they are effective by using local charts as in Proposition 4.11.

**Proposition 4.21.** The holomorphic discs representing classes in (4.7) are Fredholm regular.

Proof. Let $u : (D^2, \partial D^2) \rightarrow (\overline{\mathcal{M}}_{u,\lambda}, L_b)$ be a holomorphic disc representing $\beta$ in (4.7). By the same argument as in 4.13, regularity of $u$ is equivalent to regularity of $\rho_0 \circ u$ if $\beta \cdot D_- = 1$ and of $\rho_{\mathcal{C}} \circ u$ if $\beta \cdot D_+ = 1$, which is then a one-dimensional Riemann-Hilbert problem and follows from Theorem II and III of [Oh85].
Proposition 4.22. With the notations as in Proposition 4.20 we have 
\[ n_\beta = \begin{cases} 
1 & \text{for } \beta = \beta_{z_j} + \delta_1 \alpha_{j_1} + \ldots + \delta_N \alpha_{j_N}, \ j = 1, \ldots, n; \\
0 & \text{otherwise}. 
\end{cases} \]

Proof. The proof is identical to that of Proposition 4.16 except we have $T^d$-bundles $\mathcal{M}_1(L_0, \beta) \to \mathcal{M}_1(p_0(L_b), (p_0)_*(\beta))$ and $\mathcal{M}_1(L_0, \beta) \to \mathcal{M}_1(p_1(L_b), (p_1)_*(\beta))$ depending on whether $\beta \cdot D_+ = 1$ or $\beta \cdot D_- = 1$.

Similar to Proposition 4.17, we have Proposition 4.23. $L_b$ is weakly unobstructed.

4.7 Generating functions of open Gromov-Witten invariants and wall-crossing. Denote by $\mathcal{M}_0^\gamma$ the semi-flat mirror of $\mathcal{M}_{u, \lambda}$ (see Section 4.1). The semi-flat complex coordinates on $\mathcal{M}_0^\gamma$ is defined as follows.

For each simply connected affine chart $B_{\sigma}^0$, we have an open subset $\mathcal{M}_{0, \sigma}^\gamma = (\pi^\gamma)^{-1}(B_{\sigma}^0) \subset \mathcal{M}_0^\gamma$. For each $\sigma$, we fix a reference point $b_\sigma \in B_{\sigma}^0$. Let $\{\gamma_{\sigma, 1}, \ldots, \gamma_{\sigma, 2d}\} \subset H_1(L_{b_\sigma})$ be the cycles described in Remark 4.12 and note that it is a basis for $H_1(L_{b_\sigma})$.

Definition 4.24. The semi-flat complex coordinates on $\mathcal{M}_0^\gamma$ is defined locally on the charts $\mathcal{M}_{0, \sigma}^\gamma$ by 
\[ Z_{\sigma,i}(L_b, \nabla) = e^{-\frac{1}{\hbar} \gamma_{\sigma,i}(b) \bar{\omega}} \text{Hol}_\mathcal{V}(\gamma_{\sigma,i}(b)), \ i = 1, \ldots, 2d, \]
where $\gamma_{\sigma,i}(b) \in H_1(L_b, \mathbb{Z})$ is the parallel transport of $\gamma_{\sigma,i}$, and $\Gamma_{\sigma,i}(b)$ is the cylinder given by parallel transporting $\gamma_{\sigma,i}$.

The transition map between the charts $\mathcal{M}_{0, \sigma}^\gamma$ and $\mathcal{M}_{0, \sigma'}^\gamma$ is given by the integral affine transformation between $B_{\sigma}^0$ and $B_{\sigma'}^0$. In particular, we have $C_{\sigma, \sigma'}Z_{\sigma,i} = Z_{\sigma', i}$ for $i = d + 1, \ldots, 2d$, where $C_{\sigma, \sigma'} = e^{-\frac{1}{\hbar} \gamma_{\sigma'} \cdot \bar{\omega}}$, where $\Gamma_{\sigma,i}$ is the cylinder oriented from $b_{\sigma}$ to $b_{\sigma'}$ with boundary cycles $\gamma_{\sigma,d+i} \in H_1(L_{b_\sigma})$ and $\gamma_{\sigma', d+i} \in H_1(L_{b_{\sigma'}})$.

Definition 4.25. The generating functions $u_i$ (resp. $v_i$) for discs emanated from boundary divisors $D_{-i}$ (resp. $D_{+i}$), $i = 1, \ldots, d$, are given by 
\[ u_i(L_b, \nabla) = \sum_{\beta \in \pi_1(X(L_b))} \sum_{\beta \cdot D_{-i} = 1, \beta \cdot D = 0} n_\beta e^{-\frac{1}{\hbar} \beta \cdot \bar{\omega}} \text{Hol}_\mathcal{V}(\beta). \]
\[ v_i(L_b, \nabla) = \sum_{\beta \in \pi_1(X(L_b))} \sum_{\beta \cdot D_{+i} = 1, \beta \cdot D = 0} n_\beta e^{-\frac{1}{\hbar} \beta \cdot \bar{\omega}} \text{Hol}_\mathcal{V}(\beta). \]

Let $C_{\sigma,i} = e^{-\frac{1}{\hbar} \gamma_{-i} \cdot \bar{\omega}}$ and $C_{\sigma,d+i} = e^{-\frac{1}{\hbar} \gamma_{i} \cdot \bar{\omega}}$ for $i = 1, \ldots, d$, where $\gamma_{-i}, \gamma_i \in H_2(\mathcal{M}_{u, \lambda}, L_{b_\sigma})$ are as described in Remark 4.12. Note that we have $C_{\sigma,d+i}Z_{\sigma,d+i} = C_{\sigma', d+i}Z_{\sigma', d+i}$ on $\mathcal{M}_{0, \sigma}^\gamma \cap \mathcal{M}_{0, \sigma'}^\gamma$ for any pair $\sigma, \sigma'$ of sign vectors. Thus, $Z_{d+i} := C_{\sigma,d+i}Z_{\sigma,i}$ are global holomorphic functions on $\mathcal{M}_0^\gamma$.

Let $S_i$ be the circuits corresponding to the relation $u_i = \sum_{l=1}^d a_{l,i} u_l$ for $l = d + 1, \ldots, n$. We have Kähler parameters $q^{\beta_{S_l}}$ associated to the primitive curve classes $\beta_{S_l}$. For $j = 1, \ldots, d$, denote by $j$ be the collection of all $k \in \{1, \ldots, n\}$ such that $u_k^{(j)} \neq 0$. Let 
\[ Z_i := Z_{d+i}, \]
for $i = 1, \ldots, d$, and let 
\[ Z_i := q^{\beta_{S_i}} \prod_{l=1}^d Z_{d+i}^{u_l^{(j)}} \]
for $l = d + 1, \ldots, n$. The generating functions can be expressed locally in term of semi-flat flat coordinates as follows.
Proposition 4.26. On the open subset \((\pi^\vee)^{-1}(B_0^\vee \cap C_h) \subset \mathcal{M}_0^\vee\), we have
\[ u_j = C_{\sigma,j}Z_{\sigma,j}(1 + Z_{j_1})\ldots(1 + Z_{j_N}), \]
where for each \(j \in \{1, \ldots, d\}\), \(j_1, \ldots, j_N\) are the set of indices such that \(h_{j_i} = j\), and
\[ v_j = e^{v_{j_1}} C_{\sigma,j}^{-1} \prod_{k \in \hat{j} \setminus \{j_1, \ldots, j_N\}} (1 + Z_k). \]
where \(\mathbb{P}_j^1\) is the holomorphic curve obtained from gluing \(\beta_{-j}\) and \(\beta_{+j}\) in \(H_2(\mathcal{M}_u, L_u)\).

Proof. This follows from Propositions 4.20, 4.22 and Definitions 4.24, 4.25. See also [CLL12, Proposition 4.39].

4.8 SYZ mirror and its resolution. Set \(q_{\beta_S} = e^{-\frac{i}{2} \beta_S \bar{\omega}}\). By Definition 4.5, an SYZ mirror is given by \(\text{Spec}(R)\) where \(R\) is the ring of generating functions \(u_i, v_i, Z_i, Z_i^{-1}\) for \(i = 1, \ldots, d\). By combining the above propositions, we obtain the following.

Theorem 4.27. An SYZ mirror of \(\mathcal{M}_{u,\lambda} - D_-\) is
\[ \mathcal{M}_{u,\lambda}^\vee = \left\{ (u_1, v_1, \ldots, u_d, v_d, Z_1, \ldots, Z_d) \in \mathbb{C}^{2d} \times (\mathbb{C}^\times)^d | u_i, v_i = \prod_{k \in \hat{j}} (1 + Z_k), i = 1, \ldots, d \right\}. \]

For simplicity we have rescaled the variables \(u_i\) so that the Kähler parameters \(e^{-\frac{i}{2} \beta_S j} \bar{\omega}\) for \(j = 1, \ldots, d\) do not appear in the above expression.

Example 4.28. An SYZ mirror of \(T^*\mathbb{P}^2\) is the subvariety of \(\mathbb{C}^4 \times (\mathbb{C}^\times)^2\) given by
\[ u_1 v_1 = (1 + Z_1)(1 + q^{\beta_{S_3}} Z_2^{-1} Z_2^{-1}); \]
\[ u_2 v_2 = (1 + Z_2)(1 + q^{\beta_{S_3}} Z_2^{-1} Z_2^{-1}). \]
Note that this subvariety is singular at the one-dimensional loci \(\{Z_1 = -1, Z_2 = q^{\beta_{S_3}}, u_1 = v_1 = 0\}\) and \(\{Z_2 = -1, Z_1 = q^{\beta_{S_3}}, u_2 = v_2 = 0\}\).

In general \(\mathcal{M}_{u,\lambda}^\vee\) is singular. The wall and chamber structure of the Lagrangian torus fibration explained in Section 4.3 gives a resolution of \(\mathcal{M}_{u,\lambda}^\vee\), provided that \(\mathcal{M}_{u,\lambda}\) is smooth. In the following we construct this resolution. The construction can be justified by Lagrangian Floer theory of immersed Lagrangians which is partly explained in [HL]. We will study more about Lagrangian Floer theory in future work. In the following we glue up the resolution from local charts by hand.

Step 1. First we glue the charts corresponding to smooth torus fibers by wall-crossing functions. Recall that we have a collection of tropical hyperplanes which divide the base into chambers (see Figure 4.3 and Section 4.3 for the labels). For each chamber \(C_h\), we define a chart \(U_h \cong (\mathbb{C}^\times)^d \times (\mathbb{C}^\times)^d\) by
\[ U_h = \left\{ \left( u_i^{(h)}, v_i^{(h)}, \ldots, u_d^{(h)}, v_d^{(h)}, Z_1, \ldots, Z_d \right) \in (\mathbb{C}^\times)^{2d} \times (\mathbb{C}^\times)^d | u_i^{(h)} v_i^{(h)} = 1, i = 1, \ldots, d \right\}. \]

Consider a pair of chambers \(C_h\) and \(C_{h'}\) where \(h = (h_1, \ldots, h_n)\) and \(h' = (h'_1, \ldots, h'_n)\). For \(j = 1, \ldots, d\), let \(J_{j, h, h'}\) be the set of all indices \(k \in \{1, \ldots, n\}\) such that \(h_k \neq h'_k\) and either \(h_k = j\) or \(h'_k = j\). These indices label the hyperplanes which give walls in between the two chambers involving the \(j\)-th direction.

Let \(U_{h, h'} \subset U_h\) be the open subset defined by
\[ U_{h, h'} = \left\{ \left( u_i^{(h)}, v_i^{(h)}, \ldots, u_d^{(h)}, v_d^{(h)}, Z_1, \ldots, Z_d \right) \in U_h | 1 + Z_k \neq 0 \text{ for all } k \in \bigcup_{j=1}^d J_{j,h, h'} \right\}. \]

Let \(\delta_j^{(h, h')} = \delta_j^{(h', h)} = 0\) if \(J_{j, h, h'} = \emptyset\). Let \(\delta_j^{(h, h')} = 1\) and \(\delta_j^{(h', h)} = 0\) if there exist (and hence for all) \(k \in J_{j, h, h'}\) such that \(h'_k = j\). Let \(\delta_j^{(h, h')} = 0\) and \(\delta_j^{(h', h)} = 1\) if there exist \(k \in J_{j, h, h'}\) such that
We define the 0-dimensional strata of \( H_\sigma \) if the \( k \)-dimensional face \( \sigma \) of \( \Delta \) corresponds to a \( d-k \)-dimensional tropical stratum \( H_\sigma \) of \( H \). The 0-dimensional faces(vertices) of \( \Delta \) corresponds to the tropical chambers adjacent to \( H \). We will denote by \( |\sigma| \) the dimension of \( \sigma \), and note that \( \sigma \) is itself a simplex. Let \( \Delta_1, \ldots, \Delta_n \) be the dual simplexes of the tropical hyperplanes \( H_1, \ldots, H_n \).

We will abuse notation and denote by \( H \) both the tropical hyperplane arrangement \( H = \{ H_i \}_{i=1}^n \) and the union \( H = \bigcup_{i=1}^n H_i \). We define a stratification of \( H \) as follows. Let \( \sigma = \{ \sigma_1, \ldots, \sigma_{\nu} \} \) be a collection such that \( \sigma_{ji} \) is a face of \( \Delta_j \), and set \( |\sigma| := \sum_{\sigma_j \in \sigma} |\sigma_j| \). Let \( \mathcal{H}(\sigma) \subset H \) be the set

\[
\mathcal{H}(\sigma) = \bigcap_{\sigma_j \in \sigma} H_j(\sigma_j).
\]

We define the 0-dimensional strata of \( H \) to be points of the form \( H(\sigma) \) where \( |\sigma| = d \). We then define the \( l \)-dimensional strata of \( H \) for \( l \geq 1 \) to be the connected components of \( \mathcal{H}(\sigma) \setminus \mathcal{H}_{l-1} \), where \( |\sigma| = d-l \), and \( \mathcal{H}_{l-1} \) denotes the union of the \((l-1)\)-dimensional strata \( \Theta \) of \( H \).

Let \( \{ e_1, \ldots, e_d \} \) be the standard basis on \( (\mathbb{R}^d)^* \), and denote by \( \langle , \rangle \) the standard pairing between \( \mathbb{R}^d \) and \( (\mathbb{R}^d)^* \). For each \( \sigma \) with \( 1 \leq |\sigma| \leq d \), we associate to it a collection of primitive and linearly independent vectors \( \{ \vec{a}^\sigma_1, \ldots, \vec{a}^\sigma_{|\sigma|} \} \) parallel to \( \mathcal{H}(\sigma) \). For each \( \sigma_j \in \sigma \), we associate to it a collection of primitive vectors \( \{ \vec{a}^{\sigma_{ji}}_1, \ldots, \vec{a}^{\sigma_{ji}}_{|\sigma_{ji}|+1} \} \) such that \( \vec{a}^{\sigma_{ji}}_k \) is normal to the \( k \)-th facet of \( \sigma_{ji} \) (notice that the number of facets of \( \sigma_{ji} \) is \( |\sigma_{ji}| + 1 \), and parallel to \( \bigcap_{\sigma_{ji} \in \Theta, j \neq ji} H_j(\sigma_j) \). In particular, we can choose \( \{ \vec{a}^{\sigma_{ji}}_1, \ldots, \vec{a}^{\sigma_{ji}}_{|\sigma_{ji}|+1} \} \) such that \( \vec{a}^{\sigma_{ji}}_{|\sigma_{ji}|+1} = \sum_{k=1}^{|\sigma_{ji}|} (\vec{a}^{\sigma_{ji}}_k) \).

For each \( l \)-dimensional stratum \( \Theta \), there exists a unique collection \( \sigma = \{ \sigma_1, \ldots, \sigma_{\nu} \} \) such that \( |\sigma| = d-l \), and \( \Theta \subset \mathcal{H}(\sigma) \). We will call a stratum \( \Theta \) admissible if \( \bigcap_{j=1}^{\nu} H_{\sigma, ji} \neq \emptyset \).

Now, we associate to each admissible stratum \( \Theta \) a chart \( U_{\Theta} \) defined by

\[
U_{\Theta} = \left\{ \left( (\vec{y}^{(\Theta, \sigma_{ji})}_1, \ldots, \vec{y}^{(\Theta, \sigma_{ji})}_{|\sigma_{ji}|+1}), (\vec{x}^{(\Theta, \sigma_{ji})}_1, \ldots, \vec{x}^{(\Theta, \sigma_{ji})}_{d}) \mid (\vec{y}^{(\Theta)}_1, \ldots, \vec{y}^{(\Theta)}_{|\sigma|_{\nu}}), (\vec{Z}_1, \ldots, \vec{Z}_d) \in \left( \prod_{i=1}^{\nu} \mathbb{C}^{|\sigma_i|_{\nu}+1} \right) \times \mathbb{C}^l \times \mathbb{C}^d \right. \right\}.
\]

We glue \( U_{\Theta} \) to the resulting space from Step 1. For any tropical chamber \( C_\Theta \) adjacent to \( \Theta \), we define an open embedding \( \psi_{\Theta, \Theta} : U_\Theta \to U_\Theta \) by

\[
y^{(\Theta)}_{k} = \prod_{i=1}^{d} (u^{(h)}_i)_{<e_i, \vec{a}^{\sigma_{ji}}_k>} \quad k = 1, \ldots, l;
\]

\[
x^{(\Theta, \sigma_{ji})}_i = \prod_{m=1}^{d} (u^{(h)}_m)_{<e_m, \vec{a}^{\sigma_{ji}}_i>} \quad i = 1, \ldots, \nu;
\]

if the \( k \)-th facet of \( \sigma_{ji} \) is adjacent to the vertex of \( \sigma_{ji} \) corresponding to the chamber \( C_h \), and

\[
x^{(\Theta, \sigma_{ji})}_i = (1 + \vec{Z}_{ji}) \prod_{m=1}^{d} (u^{(h)}_m)_{<e_m, \vec{a}^{\sigma_{ji}}_i>} \quad i = 1, \ldots, \nu,
\]

otherwise. The variables \( Z_i \) are identified trivially.
We denote the smooth variety obtained from this gluing by $\mathcal{M}_{u,\lambda}$. We have $H^0(\mathcal{M}_{u,\lambda}, O_{\mathcal{M}_{u,\lambda}}) = R$.

Thus, the resolution $\mathcal{M}_{u,\lambda} \rightarrow \mathcal{M}_{u,\lambda}$ is the affinization map. Figure 6 shows an example of the above gluing procedure.

Remark 4.29. $\mathcal{M}_{u,\lambda}$ is equipped with the holomorphic symplectic form

$$\sum_{i=1}^{d} \frac{du_i \wedge dv_i}{Z_i}.$$  

With respect to this holomorphic symplectic form, the projection $(Z_1, \ldots, Z_d) : \mathcal{M}_{u,\lambda} \rightarrow (\mathbb{C}^\times)^d$ is a holomorphic Lagrangian $(\mathbb{C}^\times)^d$ fibration with discriminant loci the sets

$$\{(Z_1, \ldots, Z_d) \in (\mathbb{C}^\times)^d | Z_i = -1\}, \quad i = 1, \ldots, n.$$  

Remark 4.30. The combinatorics of matroids is intimately related to the geometry and topology of hypertoric varieties (See [Hau05]). Our construction of the resolved SYZ mirrors associates to each tropical oriented matroid (See [AD09]) a holomorphic symplectic manifold. We expect that the geometry and topology of these manifolds are related to combinatorics of tropical oriented matroids.

4.9 Multiplicative hypertoric varieties. It was observed in [MS13] that certain period integrals defined on the target space of the multiplicative moment map (for the general theory of Lie-group valued moment maps, see [AM98]) of a multiplicative hypertoric variety are solutions to the quantum differential equation of the equivariant quantum cohomology of the corresponding hypertoric variety. In this subsection, we show that smooth multiplicative hypertoric varieties provide alternative resolutions to our SYZ mirrors.

Let’s first review the construction of multiplicative hypertoric varieties. Let

$$T^* \mathbb{C}^{n,\circ} = \{(z, w) \in T^* \mathbb{C}^n | 1 + z_i w_i \neq 0, i = 1, \ldots, n\}.$$  

We equip $T^* \mathbb{C}^{n,\circ}$ with the holomorphic symplectic form

$$\omega^\circ = \sum_{i=1}^{n} \frac{dz_i \wedge dw_i}{1 + z_i w_i}.$$
Let $\mathbf{f} = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ act on $T^*\mathbb{C}^{n,0}$ by
$$\mathbf{f} \cdot (\mathbf{z}, \mathbf{w}) = (t_1z_1, t_1^{-1}w_1, \ldots, t_nz_n, t_n^{-1}w_n).$$
This action comes with a $(\mathbb{C}^*)^n$-valued moment map $\mu : T^*\mathbb{C}^{n,0} \to (\mathbb{C}^*)^n$ given by
$$\mu(\mathbf{z}, \mathbf{w}) = \left( (1 + z_1w_1), \ldots, (1 + z_nw_n) \right).$$
Let $K_C \subset (\mathbb{C}^*)^n$ be the subtorus defined by the collection of vectors $\mathbf{u}$ as in Section 2.1. Let
$$\left( \begin{array}{c} \mathbf{u}^*_1 \\ \vdots \\ \mathbf{u}^*_d \end{array} \right) \in \mathbb{Z}^{d \times n}$$
be the matrix associated to $\mathbf{u}^*: (\mathbb{C}^*)^n \to \mathbb{C}$. The multiplicative moment map $\mu : T^*\mathbb{C}^{n,0} \to K_C$ of the $(\mathbb{C}^*)^n$-action on $T^*\mathbb{C}^{n,0}$ restricted to $K_C$ is given by
$$\mu(\mathbf{z}, \mathbf{w}) = \left( \prod_{j=1}^n (1 + z_jw_j)^{\mathbf{u}^*_j}, \ldots, \prod_{j=1}^n (1 + z_jw_j)^{(n-d)_{\mathbf{u}^*_j}} \right).$$
Let $\eta = (\eta_1, \ldots, \eta_{n-d}) \in K_C$, and let $\chi : K_C \to \mathbb{C}^\times$ be a character. We define multiplicative hypertoric variety to be the GIT quotient
$$X_{u,\chi,\eta} = \mu^{-1}(\eta)/\chi K_C,$$
or equivalently,
$$X_{u,\chi,\eta} = \text{Proj} \left( \bigoplus_{k \geq 0} \mathcal{O}(\mu^{-1}(\eta))^k \right).$$
Set $q = \prod_{j=1}^{d+1} (-1)^{\sigma_{d+1}} q^{\beta} \in K_C$, where $\sigma_i$ is the parity of $\sum_{j=1}^d a_{ii}$, and $a_{ii}$ are coefficients in $u_i = \sum_{j=1}^d a_{ii}u_{ji}$. Consider the multiplicative hypertoric variety $X_{u,0,q}$. We have
$$X_{u,0,q} = \text{Spec} \left( \mathbb{C}[\mu^{-1}(q)]^{K_C} \right),$$
where $\mathbb{C}[\mu^{-1}(q)]^{K_C}$ denotes the $K_C$-invariant subring of $\mathbb{C}[\mu^{-1}(q)]$.

Let $\Pi = (\pi_{\mathbf{u}^*_i})_{1 \leq i \leq n, 1 \leq i \leq d}$ be the matrix associated to the map $\pi^*: (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ with respect to the ordered basis $u_1, \ldots, u_d$. $(\pi_{\mathbf{u}^*_i})_{1 \leq i \leq n}$ is the identity $d \times d$ matrix. Since $\Pi$ is totally unimodular, the remaining entries take values in $\{-1,0,1\}$. The columns of $\Pi$ correspond to $K_C$-invariant polynomials $z_i = \prod_{j=1}^n x_{ij}^{\pi_{\mathbf{u}^*_i}}$ and $u_i = \prod_{j=1}^n y_{ij}^{\pi_{\mathbf{u}^*_i}}$, where $x_{ij} = z_j$, $y_{ij} = w_j$ if $\pi_{\mathbf{u}^*_i} \geq 0$, and $x_{ij} = w_j$, $y_{ij} = z_j$ if $\pi_{\mathbf{u}^*_i} < 0$. Denote by $S$ the multiplicative system generated by $z_i$, $w_i$, and $z_iw_i$ for $i = 1, \ldots, d$.

**Lemma 4.31.** $S^{-1}\mathbb{C}[\mu^{-1}(q)]^{K_C}$ is generated by $z_i^{\pm 1}$, $w_i^{\pm 1}$, and $(z_iw_i)^{\pm 1}$ for $i = 1, \ldots, d$.

**Proof.** Let $f = \prod_{i=1}^{n} z_i^{a_i} \prod_{i=1}^{n} w_i^{b_i}$ be an arbitrary nonconstant Laurent monomial in $S^{-1}\mathbb{C}[\mu^{-1}(q)]$. If $f$ is not divisible by either $z_i^{\pm 1}$ nor $w_i^{\pm 1}$ for $i = 1, \ldots, d$, then the vector $\langle a_1 - b_1, \ldots, a_n - b_n \rangle$ is not in the kernel of $\mathbf{u}^*: (\mathbb{C}^*)^n \to K_C$ unless it is the zero vector. In the first case, $f$ is not $K_C$-invariant, while in the second case, $f$ is a product of $(z_iw_i)^{\pm 1}$.

**Proposition 4.32.** For a generic choice of $\chi$, $X_{u,\chi,q}$ is a resolution of $\mathfrak{M}^\chi_{u,\lambda}$.

**Proof.** We have a ring homomorphism $\varphi : R \to \mathbb{C}[\mu^{-1}(q)]^{K_C}$ given by
$$\varphi(u_i) = (-1)^{\text{sgn}(\sum_{j=1}^d |\pi_{\mathbf{u}^*_i}|)} z_i, \quad \varphi(v_i) = w_i, \quad \varphi(Z_i) = -1 - z_iw_i, \text{ for } i = 1, \ldots, d.$$ Denote by $R'$ the ring obtained by localizing $R$ at the multiplicative system generated by $u_i$, $v_i$, and $1 + Z_i$ for $i = 1, \ldots, d$. The induced map $\varphi_* : X_{u,0,q} \to \mathfrak{M}^\chi_{u,\lambda}$ is birational since $\varphi$ descends to a ring isomorphism $R' \cong S^{-1}\mathbb{C}[\mu^{-1}(q)]^{K_C}$. When the Kähler parameters of $\mathfrak{M}^\chi_{u,\lambda}$ are generic (i.e. $\mathcal{H}_R$ is simple), $X_{u,\chi,q}$ is smooth and is independent of $\chi$, and therefore we have a resolution of $\mathfrak{M}^\chi_{u,\lambda}$ by $X_{u,\chi,q}$. On the other hand, if the Kähler parameters are not generic, $X_{u,0,q}$ is singular. However, the affinization map $X_{u,\chi,q} \to X_{u,0,q} = \text{Spec}(R'(X_{u,\chi,q}, \mathcal{O}_{X_{u,\chi,q}}))$ is a resolution. In this case, the composition $X_{u,\chi,q} \to X_{u,0,q} \to \mathfrak{M}^\chi_{u,\lambda}$ is a resolution.
[GS11] M. Gross and B. Siebert, From real affine geometry to complex geometry, Ann. of Math. (2) 174 (2011), no. 3, 1301–1428.

[Hau05] T. Hausel, Quaternionic geometry of matroids, Cent. Eur. J. Math. 3 (2005), no. 1, 26–38.

[HL] H. Hong and S.-C. Lau, Moduli of Lagrangian immersions with formal deformations, preprint, arXiv:1801.07100.

[HS02] T. Hausel and B. Sturmfels, Toric hyperKähler varieties, Doc. Math. 7 (2002), 495–534.

[Joy04] D. Joyce, Singularities of special Lagrangian submanifolds, Different faces of geometry, Int. Math. Ser. (N. Y.), vol. 3, Kluwer/Plenum, New York, 2004, pp. 163–198.

[KLa] A. Kanazawa and S.-C. Lau, Geometric transitions and SYZ mirror symmetry, preprint, arXiv:1503.03829.

[KLb] A. Kanazawa and S.-C. Lau, Local Calabi-Yau manifolds of affine type A and open Yau-Zaslow formula via SYZ mirror symmetry, preprint, arXiv:1605.00342.

[Kon95] M. Kontsevich, Homological algebra of mirror symmetry, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120–139.

[Lau] S.-C. Lau, Affine elliptic surfaces with type-A singularities and orbiconifolds, preprint, arXiv:1802.08891.

[Lau14] S.-C. Lau, Open Gromov-Witten invariants and SYZ under local conifold transitions, J. Lond. Math. Soc. (2) 90 (2014), no. 2, 413–435.

[Lau05] N.C. Leung, Mirror symmetry without corrections, Comm. Anal. Geom. 13 (2005), no. 2, 287–331.

[MS13] M. B. McBreen and D. K. Shenfeld, Quantum cohomology of hypertoric varieties, Lett. Math. Phys. 103 (2013), no. 11, 1273–1291.

[NNU10] T. Nishinou, Y. Nohara, and K. Ueda, Toric degenerations of Gelfand-Cetlin systems and potential functions, Adv. Math. 224 (2010), no. 2, 648–706.

[NU14] Y. Nohara and K. Ueda, Toric degenerations of integrable systems on Grassmannians and polygon spaces, Nagoya Math. J. 214 (2014), 125–168.

[NZ09] D. Nadler and E. Zaslow, Constructible sheaves and the Fukaya category, J. Amer. Math. Soc. 22 (2009), no. 1, 233–286.

[Oh95] Y.-G. Oh, Riemann-Hilbert problem and application to the perturbation theory of analytic discs, Kyungpook Math. J. 35 (1995), no. 1, 39–75.

[Pro04] N. J. Proudfoot, Hyperkahler analogues of Kahler quotients, ProQuest LLC, Ann Arbor, MI, 2004, Thesis (Ph.D.)–University of California, Berkeley.

[PT] J. Pascaleff and D. Tonkonog, The wall-crossing formula and lagrangian mutations, preprint, arxiv:1711.03209.

[Sei] P. Seidel, Lectures on categorical dynamics and symplectic topology, preprint, http://math.mit.edu/~seidel/937/lecture-notes.pdf.

[Sei11] P. Seidel, Homological mirror symmetry for the genus two curve, J. Algebraic Geom. 20 (2011), no. 4, 727–769.

[SYZ96] A. Strominger, S.-T. Yau, and E. Zaslow, Mirror symmetry is T-duality, Nuclear Phys. B 479 (1996), no. 1-2, 243–259.

[Tu14] J. Tu, On the reconstruction problem in mirror symmetry, Adv. Math. 256 (2014), 449–478.

[Tu15] J. Tu, Homological mirror symmetry and Fourier-Mukai transform, Int. Math. Res. Not. IMRN (2015), no. 3, 579–630.

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