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Theory of noiseless phase-mixing amplification in an optomechanical system

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The investigation of the ultimate limits imposed by quantum mechanics on amplification represents an important topic both on a fundamental level and from the perspective of potential applications. We propose here a novel setup for an optomechanical amplifier, constituted by a mechanical resonator which is dispersive coupled to an optomechanical cavity asymmetrically driven around both mechanical sidebands. We show that, on general grounds, this amplifier operates in a novel regime— which we here call phase-mixing amplification. For a suitable choice of parameters, the amplifier proposed here operates as a phase-sensitive amplifier. We show that both configurations allow amplification with an added noise below the quantum limit of (phase-insensitive) amplification in a parameter range compatible with current experiments in microwave circuit optomechanics. In particular, we show that introducing phase-mixing amplification typically allows for a significant reduction of the added noise.

The amplification of a signal constitutes one of the fundamental aspects through which modern information and communication technology operates, potentially paving the way towards the full technological exploitation of quantum mechanics [1]. At the same time it also represents a fundamental tool in the exploration of the properties of the world around us: with implications ranging from the exploration of quantum-mechanical properties of macroscopic objects [2] to the detection of gravitational waves [3]. In this context, it is thus relevant both from a conceptual and the applied point of view to investigate the boundaries imposed on the amplification of a signal, e.g. what kind of input we can effectively amplify and what are the properties of the output of a given specific amplification setup.

In the context of quantum physics, a general result about the limits of amplification was derived by Haus [4] and Caves [5]. According to these results, an amplifier, in order for its behaviour to be consistent with quantum mechanics, must add a minimum amount of noise, effectively preventing the possibility of cloning a quantum state [6]. In particular, if both quadratures of the input signal are amplified by the same amount, the minimum added noise corresponds, in the large-gain limit, to half a quantum. In this article, we refer to this limit as the amplification quantum limit (AQL) for phase-insensitive amplifiers. In the recent past, a lot of experimental and theoretical effort has been devoted to the amplification of quantum signals close to the AQL, in particular in the context of circuit quantum electrodynamics [7–9], and in optomechanical setups [10–14]. In the optical regime the quantum limit was closely approached earlier [15], owing to the intrinsically low added thermal noise in the optical regime.

From the theoretical point of view, two possible options have been contemplated. One option relies on the concept of “nondeterministic noiseless linear amplification” [16], according to which, with a probability of success $p$, it is possible to improve the signal-to-noise ratio beyond the AQL, with the limiting case of $p = 0$ to attain noiseless amplification. The second idea dates back to Haus and Caves’ work, and considers a phase-sensitive amplifier. In such a device it is possible to reduce the fluctuations in the other below the AQL, at the expenses of increased fluctuations in one quadrature.

In this article, we elaborate on the second idea and report how it is possible to reach below-AQL amplification in an optomechanical device (Fig. 1) suitably driven by two strong pumping tones. The conceptual relevance of such a device lies in the fact that it allows for a faithful amplification on the level of single quanta, thus representing an ideal candidate in quantum-information processing applications, and in the detection of ultraweak signals. This amplifier design possesses other advantages with respect to previous proposals. Contrary to amplifiers based on Josephson junctions (see e.g. [7–9, 17]), whose inputs have relatively small dynamic range, the current amplifier works with comparably large inputs;

![FIG. 1. Schematic representation of the optomechanical setup discussed here. The cavity, depicted by two mirrors, is driven by two tones at frequencies and intensities $\omega_+, \alpha_+$ (blue in the figure) and $\omega_-, \alpha_-$, respectively (in red). In the figure we also indicate the input and output signals ($a_{\text{in}}, a_{\text{out}}$), and the internal and mechanical noise ($a_{\text{in}}$ and $b_{\text{in}}$).]

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FIG. 2. Phase mixing amplifier concept. (a) We assume the signal in general to contain several frequencies, so that (each) signal frequency \( \omega_s = \omega_0 - \omega \) is paired with an image frequency at \( \omega_0 + \omega \). (b) The signal from the amplifier is detected via heterodyne scheme, i.e., by mixing it with a reference at frequency \( \omega_0 \).

The most general multimode linear amplifier can be described by the input/output relations [18]

\[
a_{i\omega} = A_{\omega} a_{i\omega} + B_{\omega} a_{i\omega}^\dagger + F_{i\omega}, \tag{1}
\]

where \( a_{i\omega} \), \( a_{i\omega} \) and \( F_{i\omega} \) represent the operators associated with the input, output and added noise fields respectively and the frequency is measured with respect to a reference frequency \( \omega_0 \) [19]. Such a generic scheme is pictured in Fig. 1 for the case of our optomechanical amplifier. The relation given by Eq. (1) assumes that the heterodyne mixing specifies the reference frequency \( \omega_0 \) around which we assume the signal to have non-vanishing components. As an illustrative example, let us consider the case where the input (electric) field is characterized by a coherent signal defined around \( \omega_0 \) as

\[
\langle E \rangle \propto x_1(t) \cos \omega_0 t + x_2(t) \sin \omega_0 t \tag{2}
\]

where \( x_1(t) = \langle X_1 \rangle \), \( x_2(t) = \langle X_2 \rangle \) represent the (slowly) time-varying expectation values of quadrature fields, defined with respect to the carrier frequency \( \omega_0 \). If we assume that the quadrature fields oscillate at a (small) frequency \( \bar{\omega} \)

\[
x_1(t) = \Xi_1 \cos(\bar{\omega} t + \phi) \]
\[
x_2(t) = \Xi_2 \sin(\bar{\omega} t + \phi). \tag{3}
\]

we see that for general values of \( \Xi_1 \) and \( \Xi_2 \), the signal defined by Eq. (2) will have components at \( \omega_0 \pm \bar{\omega} \). However, for the specific choice \( \Xi_1 = \Xi_2 \) (respectively\( \Xi_1 = -\Xi_2 \)), \( \langle E \rangle \) will have components at \( \omega_s = \omega_0 - \bar{\omega} \) (respectively \( \omega_s = \omega_0 + \bar{\omega} \) only). This implies, on one hand, the dependence of the quadratures on the carrier frequency \( \omega_0 \) and on the other hand the necessity of defining signal and idler modes—oscillating at \( \omega_s \) and \( \omega_1 \), respectively—for the full characterization of a field of the form given by Eq. (2).

Going back to the general description provided by Eq. (1), we can write Eq. (1) in terms of input \( X^\dagger_{i\omega} \) and output \( Y^\dagger_{i\omega} \) quadratures as

\[
Y^\theta_{\omega} = [A_{11} \cos \theta - iA_{21} \sin \theta]X^\dagger_{\omega} + iA_{12} \cos \theta + A_{22} \sin \theta]X^\theta_{\omega} + F^\theta_{\omega}, \tag{4}
\]

where

\[
A_{11} = \left( (A_{\omega} + A_{\omega}^*) + (B_{\omega} + B_{\omega}^*) \right) / 2,
\]
\[
A_{12} = \left( (A_{\omega} - A_{\omega}^*) - (B_{\omega} - B_{\omega}^*) \right) / 2,
\]
\[
A_{21} = \left( (A_{\omega} - A_{\omega}^*) + (B_{\omega} - B_{\omega}^*) \right) / 2,
\]
\[
A_{22} = \left( (A_{\omega} + A_{\omega}^*) - (B_{\omega} + B_{\omega}^*) \right) / 2 \tag{5}
\]

and 

\[
X^\dagger_{\omega} = a_{i\omega}^\dagger + a_{i\omega}, \quad X^\theta_{\omega} = i \left( a_{i\omega}^\dagger - a_{i\omega} \right),
\]

\[
Y^\theta_{\omega} = \left( a_{i\omega}^\dagger e^{i\theta} + a_{i\omega} e^{-i\theta} \right) \quad \text{and} \quad F^\theta_{\omega} = (F^\dagger_{i\omega} e^{i\theta} + F_{i\omega} e^{-i\theta}).
\]

The phase \( \theta \) of the local oscillator represents a controllable parameter, related to the detection scheme (Fig. 2) characterising phase-sensitive amplification both in the optical and in the microwave regime [20].

Defining \( Y_1 = Y^\theta_{\omega} / 2 \) and \( Y_2 = Y^\theta_{\omega} \), we can write Eq. (4) in matrix form

\[
Y = AX + F \tag{6}
\]

with \( A = [A_{11}, iA_{21}, -iA_{21}, A_{22}], \quad Y = [Y_1, Y_2]^T, \quad X = [X_1, X_2]^T, \quad F = [F_1, F_2]^T. \)

Equation (6) constitutes a generalisation of the analysis performed by Caves [5] in the sense that we do not...
constrain the coefficients of Eq. (1) (and the corresponding equation for $a_{+\omega}$) to obey the relation $A_{\omega}^* = A_{\omega}$, $B_{\omega}^* = B_{\omega}$, as in the case discussed by Caves for multimode phase-sensitive amplifiers (see discussion before Eqs. (4.40) in Ref. [5]), for which $A_{12}$ and $A_{21}$ would be identically zero.

In order to characterise the deviation from the case of multimode phase-sensitive amplification, we write the coefficients $A_{\omega}$ and $B_{\omega}$ in terms of their symmetric and antisymmetric frequency components

$$
A_{\omega} = A_{\Delta} + A_{\Sigma},
$$
$$
B_{\omega} = B_{\Delta} + B_{\Sigma},
$$
where $A_{\Sigma} = (A_{\omega} + A_{-\omega})/2$, $A_{\Delta} = (A_{\omega} - A_{-\omega})/2$ and analogously for $B$. This decomposition allows a direct quantification of the deviation of the phase-mixing amplifier behaviour from the one of the phase-sensitive case. In the latter case, for a multimode setup, $A_{\Delta}$ and $B_{\Delta,\omega}$ are assumed to be equal to zero.

In addition to the decomposition given in Eq. (7), we exploit the gauge freedom for the input ($a_{n,\omega} \rightarrow a_{n,\omega} \exp[i\phi_{\omega}]$) and output fields ($a_{o,\omega} \rightarrow a_{o,\omega} \exp[i\phi_{\omega}]$) imposing that

$$
\phi_{\Delta}^A = \phi_{o,\omega} - \phi_{\omega},
$$
$$
\phi_{\Sigma}^A = \phi_{o,\omega} - \phi_{1-\omega} = \phi_{o,\omega} + \phi_{\omega},
$$
where $\phi_{\Delta}^A = \text{Arg}[A_{\Sigma,\omega}], \phi_{\Sigma}^A = \text{Arg}[B_{\Sigma,\omega}]$. We can write the equations of motion in a rotated frame, corresponding to the preferred quadratures for a phase-sensitive amplifier. In this frame the input/output relations for the quadratures for which

$$
\hat{Y} = \hat{A}\hat{X}
$$
are characterised by the following coefficients

$$
\hat{A}_{11} = |A_{\Sigma}| + |B_{\Sigma}| + i|A_{\Delta}| \cos \phi_1 + i|B_{\Delta}| \cos \phi_2
$$
$$
\hat{A}_{12} = i|B_{\Delta}| \cos \phi_2 - i|A_{\Delta}| \cos \phi_1
$$
$$
\hat{A}_{21} = i|A_{\Delta}| \cos \phi_1 + i|B_{\Delta}| \cos \phi_2
$$
$$
\hat{A}_{22} = |A_{\Sigma}| - |B_{\Sigma}| + i|A_{\Delta}| \sin \phi_1 - i|B_{\Delta}| \sin \phi_2
$$
with $\phi_1 = \text{Arg}[A_{\Delta}] - \text{Arg}[A_{\Sigma,\omega}]$ and $\phi_2 = \text{Arg}[B_{\Delta}] - \text{Arg}[B_{\Sigma,\omega}]$. It is thus clear that while for $A_{\Delta} = B_{\Delta} = 0$, Eq. (9) corresponds to the usual input/output relation for a phase sensitive amplifier in the preferred quadratures. However, if $A_{\Delta}, B_{\Delta} \neq 0$, the off-diagonal terms $\hat{A}_{12}$ and $\hat{A}_{21}$ are generally different from 0, and therefore, even in the frame that should correspond to the preferred quadratures each output quadrature will depend on both input quadrature.

Let us analyze the requirements for the possibility of diagonalising the matrix $A$ through a phase rotation of the input and output fields, which corresponds to an orthogonal transformation — i.e. a transformation constituted, at most of a rotation and a reflection— of the quadratures $X \rightarrow R_X X$, $Y \rightarrow R_Y Y$. This requirement is equivalent to the statement that for each real matrix $M$ there exists the singular value decomposition (SVD)

$$
M = UDV^\dagger
$$
where $D$ is a diagonal matrix and $U$ and $V$ are orthogonal matrices. However, if $M$ is a complex matrix, the SVD is in general possible only if $U$ and $V$ are unitary matrices. A unitary transformation does not necessarily map quadrature operators to quadrature operators — for instance, $a_1^\dagger$ are obtained from the (normalised) quadrature operators through a unitary transformation, but they do not represent quadrature operators. Therefore, not all matrices describing linear amplifiers can be diagonalized to the form describing preferred quadratures, because due to the residual gauge freedom in the definition of input and output phases, the only transformations allowed are those defined by orthogonal matrices modulo an overall phase factor. We designate the regime for which the preferred quadratures cannot be found as phase-mixing amplification (PMA). In this case both output quadratures depend on both input quadratures.

In addition, from the expression of the matrix elements given in Eq. (10), we note that $\hat{A}$ is a diagonal matrix for $\omega = 0$, since, in this case, $A_{\Delta}$ and $B_{\Delta}$ are identically zero. For $\omega = 0$, we thus recover the usual input/output expressions for a narrowband phase-sensitive linear amplifier

$$
Y_1 = A_{11}X_1 + F_1
$$
$$
Y_2 = A_{22}X_2 + F_2.
$$
Thus an amplifier can be PMA only if it has a finite bandwidth. Otherwise stated, following the definition given in [5] of multimode amplifier, a PMA is intrinsically a multimode device.

While we elaborate more about the noise analysis in the specific case of the optomechanical PMA in section II below, we note here that the general-PMA noise analysis is somewhat complicated by the fact that the output in each quadrature depends on both input quadratures. In general, we can write the output power spectrum as

$$
S_Y^\theta = O_1^\theta S_1 + O_2^\theta S_2 + S_P^\theta
$$
where $S_Y^\theta = \frac{1}{2} \langle Y^\dagger Y\rangle$, $S_1^\theta = \frac{1}{2} \langle (X_{1-\omega}X_{1\omega}) - 1 \rangle$, $S_2^\theta = \frac{1}{2} \langle (X_{2-\omega}X_{2\omega}) - 1 \rangle$, and analogously for the added noise. From Eqs. (5, 6), we obtain

$$
O_1^\theta = [|A_{11}|^2 \cos^2 \theta + |A_{21}|^2 \sin^2 \theta]
$$
$$
+ \sin 2\theta |A_{11}A_{21}| \sin [\phi_{21} - \phi_{11}]$$
$$
O_2^\theta = [|A_{22}|^2 \sin^2 \theta + |A_{12}|^2 \cos^2 \theta]
$$
$$
- \sin 2\theta |A_{22}A_{12}| \sin [\phi_{12} - \phi_{22}]
$$
with $\phi_{ij} = \text{Arg}[A_{ij}]$. In order to simplify our analysis, we consider here an input for which $S_1 = S_2$ (i.e. we exclude
from our noise analysis the possibility of a squeezed input state) and therefore define the PMA power gain as

$$|G^0|^2 = O_1^0 + O_2^0.$$  

(15)

This allows us to evaluate the added noise as referred to the input as

$$S^0_X = \left. \frac{S^0_Y}{|G^0|^2} \right|_{s_1,s_2=0} = \left. \frac{S^0_Y}{O_1^0 + O_2^0} \right|_{s_1,s_2=0}.$$  

(16)

We note that, In the context of microwave circuit QED, the typical approach to characterize an amplifier is send a broadband noise input signal and observe the frequency dependence of the output. In this case the input contains both the signal frequency $\omega_s$ and its image frequency $\omega_i$, and the analysis of a PMA needs to include all components of $\bar{A}$.

II. OPTOMECHANICAL PMA

We demonstrate PMA in an optomechanical cavity system, consisting of an electromagnetic (optical or microwave) cavity with resonant frequency $\omega_c$ dispersive coupled to a mechanical oscillator whose resonance frequency is $\omega_m$ (see, for example, [21]).

The Hamiltonian of the system can be written as

$$H = \omega_c a^\dagger a + \omega_m b^\dagger b + g_0 a^\dagger a \left( b^\dagger + b \right),$$  

(17)

where $a$ ($a^\dagger$) and $b$ ($b^\dagger$) represent the raising (lowering) operators associated with the electromagnetic cavity field and the mechanical oscillator, respectively, and $g_0$ is the single-photon optomechanical coupling strength. In addition to its internal dynamics, the system is coupled to an environment, which provides the possibility of driving and probing the system and, at the same time, represents a source for noise and dissipation, both for the mechanics and the cavity. Furthermore, we describe the noise/dissipation properties of the mechanical resonator through the coupling with a phononic thermal reservoir with average occupation number $n_m$, and define a characteristic linewidth $\gamma$. An analogous assumption is adopted for the cavity. In this case, however, we consider a coupling to two different baths: the external bath (characterized by the linewidth $\kappa_e$) providing both input signal and input noise, and an internal bath (linewidth $\kappa_i$), associated with the internal losses of the cavity and whose population is given by $n_e^\dagger$. We assume that the cavity is driven by two strong pumps of amplitude $\alpha_+$ and $\alpha_-$, which are detuned with respect to the cavity resonant frequency $\omega_c$, by $\omega_+ - \omega_c = \omega_m + \delta$ and $\omega_- - \omega_c = -\omega_m - \delta$, respectively (Fig. 3). A related two-tone setup has been previously considered in the context of backaction evading (BAE) measurements of the mechanical oscillator position [22–25], and in the generation of mechanical squeezing [26–28]. In both cases the frequencies of the driving tones were considered to fulfill the relation $\omega_{\pm} = \omega_c \pm \omega_m$. For equal pump amplitudes ($\alpha_+ = \alpha_-$) this leads to the BAE detection of the mechanical oscillator position, and for $\alpha_+ < \alpha_-$ to the squeezing of the mechanics below the standard quantum limit, defined as the uncertainty associated with the ground state of the mechanical oscillator.

In the presence of two strong driving tones, we can follow a standard approach and linearise the Hamiltonian given in Eq. (17). Neglecting fast oscillating terms (rotating-wave approximation) and moving to a frame rotating at $\omega_c$ and $\omega_m - \delta$ for the cavity and the mechanical field, respectively, we can write the Hamiltonian as

$$H = \delta b^\dagger b + G_+ a^\dagger b^\dagger + G_- a^\dagger b + h.c..$$  

(18)

Without loss of generality, due to the gauge freedom associated with the definition of the operators $a$ and $b$, we can assume $G_\pm$ to be real and positive. With a view to the amplification mechanism we discuss, we furthermore assume $G_- \gtrsim G_+$. The solution of the equations of motion becomes simple after expressing Eq. (18) in terms of Bogoliubov modes for the cavity field,

$$\alpha = ua + v a^\dagger,$$  

(19)

where $u = G_+/G_{BG}$, $v = G_-/G_{BG}$ and $G_{BG} = (G_+^2 - G_-^2)^{1/2}$. Writing Eq. (18) in terms of the transformed operators given by Eq. (19), leads to the following beam-splitter Hamiltonian

$$H = \delta b^\dagger b + G_{BG} \left( a^\dagger b + a b^\dagger \right),$$  

(20)

where $G_{BG} = (G_+^2 - G_-^2)^{1/2}$. The beam-splitter term $G_{BG} \left( a^\dagger b + a b^\dagger \right)$ in Eq. (20) points towards the cooling of the mechanical motion to the temperature of the Bogoliubov cavity mode. As we show below, this entails the amplification of the unrotated cavity mode $a$.

From Eq. (20) we can determine the following quantum
Langevin equations in the frequency domain for \(a\) and \(b\)
\[-i\omega a_\omega = -iG_{BG}b_\omega - \frac{\kappa}{2} a_\omega + \sqrt{\kappa}\alpha_{in}\omega + \sqrt{\kappa} a_{in}\omega^d \]
\[-i\omega b_\omega = -i\delta b_\omega - iG_{BG}a_\omega - \frac{\gamma}{2} b_\omega + \sqrt{\gamma} b_{in}\omega, \quad (21)\]
where \(\alpha_{in} = w_{in} + v_{in} a^d\). Eliminating the mechanical degrees of freedom from Eq. (21), considering the usual input-output relation \(a_{o\omega} + \alpha_{in\omega} = \sqrt{\kappa}\alpha_{o\omega}\), and transforming back to \(a_\omega\), we can obtain an input/output relation for the output field \(a_{o\omega}\) (see Appendix B)
\[a_{o\omega} = A_o a_{in\omega} + B_o a_{in\omega}^d + A_{1-\omega} a_{in\omega}^d + B_{1-\omega} a_{in\omega}^d + C_\omega b_{in\omega} + D_\omega b_{in\omega}^d. \quad (22)\]
The coefficients in Eq. (22) are given by
\[\begin{align*}
A_o &= \kappa_e \left( u^2 \chi^e - v^2 \chi^e \right) - 1 \\
A_{1-\omega} &= \sqrt{\kappa_e} \left( u^2 \chi^e - v^2 \chi^e \right) \\
B_o &= uu_{\kappa_e} \left( \chi^e - \chi^e \right) \\
B_{1-\omega} &= uu_{\kappa_e} \left( \chi^e - \chi^e \right) \\
C_o &= -iG_{-\sqrt{\gamma}\epsilon e} \chi^e \chi^m \\
D_o &= iG_{+\sqrt{\gamma}\epsilon e} \chi^e \chi^m \\
\end{align*}\]
with \(\chi = \chi^e(\omega \rightarrow -\omega)\) for both \(\chi = \chi^e, \chi^m\), and
\[\chi^e = \left[ \kappa / 2 - i\omega + G_{BG}\chi^m \right]^{-1}, \quad \chi^m = \left[ \gamma / 2 - i(\omega - \delta) \right]^{-1}. \quad (23)\]

Equations (23, 24) allow us to identify, for the optomechanical case, the parameters defined in Eq. (1). More specifically, the definitions given in (23) allow us to evaluate \(O_1, O_2, \text{ and } S_F\), therefore characterising the PMA properties of the system. We characterize them in Fig. 4. We can see that the gain is highest at a frequency \(\omega = \pm \omega_{\text{max}} \approx \pm \delta\) (better approximation in Eq. (26) below), and \(\theta \approx \pi / 2\), whereas for \(\theta \approx 0\) there is attenuation instead of gain. Moreover, at the maximum-gain frequency we find \(|G^{\pi / 2}(\omega_{\text{max}})|^2 \approx O_1^{\pi / 2}\) (Fig. 4(b)) and \(|G^{0}(\omega_{\text{max}})|^2 \approx O_0^0\) (Fig. 4(c)). From Eqs. (14,15), this implies the coefficients \(A_{11}\) and \(A_{22}\) are negligible with respect to the off-diagonal terms. This property allows us to describe the device as a phase sensitive amplifier,
\[\begin{align*}
Y_1 &\approx A_{12}X_2 + F_2 \\
Y_2 &\approx A_{21}X_1 + F_1. \quad (25)\]

If we are in a sideband resolved-like regime, i.e. if the two peaks depicted in Fig. 4 can be approximately treated as separate peaks for \(\kappa_e \approx \kappa\) and \(\gamma \approx 0\), it is possible to express the gain in terms of a Lorentzian centered around \(\omega_{\text{max}}\) and linewidth \(\gamma\), where
\[\omega_{\text{max}} = \pm \delta \left[ 1 + \frac{G_{BG}^2}{\kappa^2 / 4 + \delta^2} \right] \]
\[\gamma = \frac{G_{BG}^2\kappa}{\kappa^2 / 4 + \delta^2}. \quad (26)\]

These expressions are hence valid for \(\omega_{\text{max}} \gg \gamma\). Crucially, for the description of this optomechanical system in terms of PMA, away from the resonance defined by Eq. (26), the mixing coefficients \(A_{11}\) and \(A_{22}\) start to play a significant role (see Fig. 4), and a real-valued singular value decomposition becomes impossible. In the limit \(G_{BG} \ll \delta \ll \kappa\) the coefficients \(A_{ij}\) assume a particularly simple form
\[A_{21} = -\frac{2/\kappa (G_- + G_+)^2}{\sqrt{2} - i(\omega - \omega_{\text{max}})} \]
\[A_{12} = -\frac{2(\kappa (G_- - G_+)^2}{\sqrt{2} - i(\omega - \omega_{\text{max}})} \]
\[A_{11} = A_{22} = \left[ 1 - \frac{2G_{BG}^2/\kappa}{\sqrt{2} - i(\omega - \omega_{\text{max}})} \right]. \quad (27)\]

Equations (26, 27) allow us to evaluate an approximate expression for the gains at \(\omega = \omega_{\text{max}}\)
\[|G_1| = (u + v)^2 \]
\[|G_2| = (u - v)^2 \quad (28)\]
and therefore the value of the gain-bandwidth product
\[G_1\gamma = 16G_{BG}^2/\kappa^2. \quad (29)\]

Furthermore, since the Bogoliubov parameters \(u\) and \(v\) satisfy \(u^2 - v^2 = 1\), we can recover the condition
\[|G_1G_2| = 1 \quad (30)\]
characterizing a degenerate parametric amplifier, which can be considered as the “gold standard” of phase-sensitive amplifiers. Furthermore, it is clear from Eq. (27) that, in the limit discussed here, the frequency range around \(\omega_{\text{max}}\) for which the system can be characterized as a phase-sensitive amplifier is given by \(\gamma\). In Fig. 5 we plot the gain \(|G^\theta(\omega)|^2\) as a function of \(\theta\) for different values of \(\omega\). The crucial feature of this plot is the \(\omega\)-dependence of the gain maximum. This dependence, which plays an important role in the determination of the noise properties of the system, can be ascribed to a finite value of \(A_{11}\) and \(A_{22}\). From Eqs. (14,15), it is possible to write \(|G^\theta|^2\) as
\[|G^\theta|^2 = |A_{11}|^2 \cos^2 \theta + |A_{21}|^2 \sin^2 \theta \]
\[+ \sin 2\theta |A_{11}A_{21}| \sin (\phi_{21} - \phi_{11}) \]
\[+ |A_{22}|^2 \sin^2 \theta + |A_{12}|^2 \cos^2 \theta \]
\[+ \sin 2\theta |A_{22}A_{12}| \sin (\phi_{12} - \phi_{22}). \quad (31)\]
Moreover, since \(|A_{12}| = |A_{21}|\), we can write
\[|G^\theta|^2 = A_s + A_x + A_\Delta \cos (\theta + \phi) \quad (32)\]
with \(A_s, D = \frac{1}{2} \left( |A_{11}|^2 \pm |A_{22}|^2 \right), A_s = |A_{21}|^2 = |A_{21}|^2, A_\Delta = \sqrt{A_s} \sin (\phi_{21} - \phi_{11}) - |A_{21}| \sin (\phi_{12} - \phi_{21}), \)
\[ A_\Delta = \sqrt{A_{11}^2 + A_{22}^2}, \phi = \arctan \frac{A_{42}}{A_{42}}. \] Equation (32) expresses the frequency dependence of the maximum gain angle through the frequency dependence of the added phase factor \( \phi \).

In order to show that away from \( \omega = \omega_{\text{max}} \) the amplifier cannot be described in terms of phase-sensitive amplification, we evaluate for the optomechanical case the frequency dependence of the phases \( \text{Arg} \left[ \tilde{A}_{11} \right] \) and \( \text{Arg} \left[ \tilde{A}_{22} \right] \) – note that \( \text{Arg} \left[ \tilde{A}_{12} \right] = \text{Arg} \left[ \tilde{A}_{21} \right] = \pm \pi/2 \), see Eq. (10). The evaluation of the phase for the matrix \( \tilde{A} \) allows to verify the possibility of PMA, which has to be excluded for matrices which are either real or proportional to a real one. Since for \( \omega = 0 \), \( \tilde{A}_{21} = \tilde{A}_{12} = 0 \) and \( \tilde{A}_{11}, \tilde{A}_{22} \) are both real (see Fig. 6), the matrix \( \tilde{A} \) is, in this case, real, and therefore real-valued SVD, corresponding to a rotation to the preferred quadratures, is possible. Analogously, for \( \omega = \omega_{\text{max}} \) the phases of all four elements are equal, implying that, in this case the matrix \( \tilde{A} \) is proportional to a real matrix and thus, again it can be rotated to the preferred quadratures. For all other frequency values, since all elements of matrix \( \tilde{A} \) are non-vanishing and possess different phases, real-valued SVD is not possible and thus a rotation to the preferred quadratures is not possible.

### III. Noise Properties

We turn now to the discussion of the added noise properties of the amplifier, assuming that both the mechanical oscillator and the cavity field are subject to noise – below referred to as mechanical and internal noise (see Fig. 1). Otherwise stated, we assume that we can write the added noise as \( \mathcal{F}_\theta = \mathcal{F}_m^\theta + \mathcal{F}_m^\phi \), where

\[
\mathcal{F}_m^\theta = [A_{11}^m \cos \theta - i A_{12}^m \sin \theta] X_\omega^1 + i [A_{12}^m \cos \theta + A_{22}^m \sin \theta] X_\omega^2, 
\]

(33)

where the coefficients \( A_{ij}^m \) are defined analogously to the input/output quadratures coefficients given in Eq. (5), with the replacement \( A_\omega \rightarrow C_\omega \) and \( B_\omega \rightarrow D_\omega \), using the definitions of \( C_\omega, D_\omega \) given in Eq. (23). Furthermore, the operators \( X_\omega^1 \) and \( X_\omega^2 \) are defined as

\[
X_\omega^1 = b_{\omega}^+ + b_{\omega}, \quad X_\omega^2 = i \left( b_{\omega}^+ - b_{\omega} \right) \]

(34)

The added noise from the cavity internal bath \( \mathcal{F}_\theta^I \) is obtained the same way with the substitutions \( A_\omega \rightarrow A_f \omega \) and \( B_\omega \rightarrow B_f \omega \), and with definitions analogous to Eq. (34) for the input quadrature operators

\[
X_{\omega}^{1f} = a_{\omega}^{f+} + a_{\omega}^{f-}, \quad X_{\omega}^{2f} = i \left( a_{\omega}^{f+} - a_{\omega}^{f-} \right). 
\]

(35)

FIG. 4. (\textit{a}) Gain of the optomechanical amplifier as a function of the signal frequency evaluated with respect to \( \omega_c \). (\textit{b}) |\( G^\theta |^2 \) (green), \( O_\theta^m \) (blue), \( O_\theta^f \) (red), for \( \theta = \pi/2 \) and (c) |\( G^\theta |^2 \) (green), \( O_\theta^m \) (blue), \( O_\theta^f \) (red), \( \theta = 0 \). Parameters: \( G_+ = 0.06, G_- = 0.072, \delta = 0.04, \kappa_c = 0.99, \gamma = 4 \cdot 10^{-5} \), energies in units of the cavity linewidth \( \kappa \). (d) Detail of the frequency dependence around \( \omega_{\text{max}} \) for \( \theta = 0 \). The frequency \( \omega = 0 \) corresponds to the cavity resonant frequency.
Focusing on the $\omega = \omega_{\text{max}}$ resonance, with the same approximations as the ones used in the derivation of the gain coefficients, we have

$$A_{11}^m(\omega) = A_{12}^m(\omega) = -ie^{i\gamma\kappa_1/\kappa}(\omega - \omega_{\text{max}})$$

$$A_{22}^m(\omega) = A_{21}^m(\omega) = -ie^{i\gamma\kappa_2/\kappa}(\omega - \omega_{\text{max}})$$

(36)

For the internal cavity noise, with the same approximations considered for the calculation of the gain, we have

$$A_{21} = \frac{-2\kappa_1/\kappa^2(G_- + G_+)^2}{\omega_{\text{max}} - i(\omega - \omega_{\text{max}})}$$

$$A_{12} = \frac{-2\kappa_1/\kappa^2(G_- - G_+)^2}{\omega_{\text{max}} - i(\omega - \omega_{\text{max}})}$$

$$A_{11} = A_{22} = \frac{\kappa_1}{\kappa} \left[ 1 - \frac{2G_{BG}^2}{\kappa^2} \right] + \frac{\kappa_1}{\kappa}.$$  (37)

With the expressions given by Eq. (33), and excluding the possibility of squeezed noise, considering that for the mechanical and for the cavity internal bath the input quadratures satisfy the following relations

$$\langle X_{\omega}^1 X_{\omega}^1 \rangle = \langle X_{\omega}^2 X_{\omega}^2 \rangle = 2n_{\text{m}} + 1$$

$$\langle X_{\omega}^1 X_{\omega}^1 \rangle = \langle X_{\omega}^2 X_{\omega}^2 \rangle = 2n_{\text{c}} + 1$$  (38)

we can write the mechanical contribution to the added noise as

$$(\Delta F_{\omega}^m)^2 = \left[ |A_{11}^m| \cos^2 \theta + |A_{22}^m| \sin^2 \theta \right] (n_{\text{m}} + 1/2),$$  (39)

where $(\Delta F_{\omega}^m)^2$ is related to the variance of $F_{\omega}$ and $n_{\text{m}}$ is the thermal population of the mechanical bath as

$$(\Delta F_{\omega}^m)^2 = \frac{1}{2} \langle X_{\omega}^\theta X_{\omega}^\theta \rangle.$$  

Let us assume that $\kappa_1 \ll \kappa$, and $n_{\text{c}} \ll n_{\text{m}}$. In this case the contribution from the mechanical noise is dominant. The approximate expressions given in Eqs. (28,36) allow us to write the total added noise at $\omega = \omega_{\text{max}}$ as

$$n_{\text{add}}^\theta = \frac{(\Delta F_{\omega}^m)^2 + (\Delta F_{\omega}^m)^2}{|G|^2}$$

$$\approx \frac{\gamma\kappa}{2(G_- - G_+)^2} \frac{(G_- - G_+)^2}{(G_- + G_+)^2} \cot^2 \theta + 1$$

$$\times (n_{\text{m}} + 1/2)$$  (40)

For $G_- - G_+ \ll G_- + G_+$ this allows establishing a condition under which the quantum limit for phase-insensitive amplification is overcome by the (phase-sensitive) optomechanical amplifier discussed here, namely

$$(G_- + G_+)^2 > \gamma\kappa(n_{\text{m}} + 1/2) \Rightarrow n_{\text{add}}^\theta < 1/2$$  (41)
as long as $\theta \neq 0, \pi$. On the other hand, even if the condition given by Eq. (41) is not fulfilled, it is still possible to "beat" the quantum limit in the PMA regime, reaching $n_{\text{add}}^\theta < 1/2$ away from $\omega = \omega_{\text{max}}$. This relies on the different phase dependence of mechanical added noise and gain. Namely, the condition $A_{11} \neq 0$ allows for a shift in the location of the maximum of $G^\theta_\omega$ as a function of $\theta$. Since this phase shift is absent for the added mechanical noise (see Fig. 5(b)), the presence of an $A_{11} \neq 0$ term effectively allows for a relative shift of the phases for which gain and noise reach their maxima.

Stated otherwise, it is possible to reach amplification with noise properties below the quantum limit by shifting the input signal frequency away from $\omega_{\text{max}}$. In Figs. 7,8 we depict the added noise as a function of $\omega$ and $\theta$ for a value of the pump intensities leading to amplification with noise properties below the AQL for $\omega = \omega_{\text{max}}$. In Fig. 8(b),(c) it is possible to see that shifting away from $\omega = \omega_{\text{max}}$ leads to a reduction of the range of $\theta$ for which $n_{\text{add}}^\theta < 1/2$.

Conversely, in Figs. 9, 10, where we plot the total added noise for a pump leading to amplification with noise properties above the AQL for $\omega = \omega_{\text{max}}$: shifting away from $\omega = \omega_{\text{max}}$, leads to the possibility of reaching below AQL amplification. This is a direct consequence of the different $\theta$-dependence of the gain $G^\theta_\omega$ and the mechanical contribution to the added noise $n_{\text{add}}^\theta_{m}$.

IV. CONCLUSIONS

We have here proposed a novel regime of quantum signal amplification beyond the usual phase-insensitive/phase-sensitive amplification paradigm, which we call phase-mixing amplification. In addition, we have provided a specific example of phase-sensitive amplification in the context of optomechanics, demonstrating the possibility of below-AQL amplification and
FIG. 9. Total added noise for a drive below $\gamma_\kappa(2n_m + 1)$ ($G_+ = 0.04, G_- = 0.048$, all other parameters as in the previous figures). Grey areas correspond to regions with added noise larger than the AQL showing how the different phase dependence of gain and noise can increase the parameter range over which below-AQL amplification is possible. Our results are directly relevant for experiments.

Appendix A: PMA for a coherent field

In order to further clarify the concept of phase-mixing amplification, we provide a simple example of how a phase-mixing amplifier works for an input characterised by a coherent monochromatic signal defined around a carrier frequency $\omega_0$ as in Eqs. (2,3). The slowly time-varying quadrature fields can be analogously defined in frequency domain as

\[
x_1(\omega) = \Xi_1 \left[ e^{-i\phi_1} \delta (\bar{\omega} - \omega) + e^{i\phi_1} \delta (\bar{\omega} + \omega) \right],
\]

\[
x_2(\omega) = \Xi_2 \left[ e^{-i\phi_2} \delta (\bar{\omega} - \omega) + e^{i\phi_2} \delta (\bar{\omega} + \omega) \right],
\]

where we have set $\phi_1 = \phi$ and $\phi_2 = \phi - \pi/2$. Considering the I/O relations for the phase-mixing amplifier –Eqs. (6)– we can write the expression for the output field quadrature’s time dependence, (neglecting the noise sources) as

\[
y_\theta(t) = \left\{ \begin{array}{l}
[A_{11} (\bar{\omega}) \cos \theta - iA_{21} (\bar{\omega}) \sin \theta] e^{-i(\bar{\omega}t + \phi_1)} + \\
[A_{11} (-\bar{\omega}) \cos \theta - iA_{21} (-\bar{\omega}) \sin \theta] e^{i(\bar{\omega}t + \phi_1)} \end{array} \right\} \Xi_1 + \\
\left\{ \begin{array}{l}
[iA_{12} (\bar{\omega}) \cos \theta + A_{22} (\bar{\omega}) \sin \theta] e^{-i(\bar{\omega}t + \phi_2)} + \\
[iA_{12} (-\bar{\omega}) \cos \theta + A_{22} (-\bar{\omega}) \sin \theta] e^{i(\bar{\omega}t + \phi_2)} \end{array} \right\} \Xi_2.
\]

FIG. 10. (a) Zoom of figure 9 for $\omega \approx \delta$, dashed lines correspond to the plots in (b) and (c). (b-c) Total (green), internal (red, dashed), and mechanical (blue, dot-dashed), added noise for the same drive as in Fig 9: (b) on resonance ($\omega = \omega_{\text{max}}$), (c) off resonance ($\omega = \omega_{\text{max}} + 2\gamma_{\text{eff}}$).
Since $A_{ij}(\tilde{\omega}) = A^*_{ij}(-\tilde{\omega})$, Eq. (A2) can be written as

$$y_\theta(t) = \left[|A_{11}| \cos \theta \cos(\tilde{\omega} t + \tilde{\phi}_{11}) + |A_{21}| \sin \theta \sin(\tilde{\omega} t + \phi_{21}) \right] \Xi_1 + \left[|A_{12}| \sin \theta \cos(\tilde{\omega} t + \phi_{22}) + |A_{22}| \cos \theta \sin(\tilde{\omega} t + \phi_{22}) \right] \Xi_2,$$

where we have defined $\tilde{\phi}_{ij} = \phi_j - \phi_{ij}$, and defined $A_{11} = A_{11}(\omega)$. Equation (A3) can be written also as

$$y_\theta(t) = \left[A_1 \cos(\tilde{\omega} t + \tilde{\phi}_1) \right] \Xi_1 + A_2 \sin(\tilde{\omega} t + \tilde{\phi}_2) \Xi_2$$

(A4)

with

$$A_1 = |A_{11}| \cos \theta |^2 + |A_{21}| \sin \theta |^2 + A_{11} A_{21} \sin 2\theta \sin(\tilde{\phi}_{11} - \phi_{21})} / 1/2$$

$$A_2 = |A_{12}| \sin \theta |^2 + |A_{22}| \cos \theta |^2 - A_{12} A_{22} \sin 2\theta \sin(\tilde{\phi}_{22} - \phi_{22})} / 1/2$$

(A5)

and

$$\tilde{\phi}_1 = \arctan \left[ A_{11} \cos \theta \sin \tilde{\phi}_{11} + A_{21} \sin \theta \cos \phi_{21} \right] / A_{11} \cos \theta \sin \phi_{11} - A_{21} \sin \theta \sin \phi_{21}$$

$$\tilde{\phi}_2 = \arctan \left[ A_{12} \cos \theta \sin \tilde{\phi}_{22} + A_{22} \sin \theta \cos \phi_{22} \right] / A_{12} \cos \theta \sin \phi_{12} - A_{22} \sin \theta \sin \phi_{22}$$

(A6)

Finally, from Eq. (A6), it is possible to write

$$y_\theta(t) = A_\theta \cos(\tilde{\omega} t + \eta_\theta) \Xi$$

(A7)

with

$$A_\theta \Xi = \sqrt{A_1^2 \Xi_1^2 + A_2^2 \Xi_2^2 + 2A_1 A_2 \Xi_1 \Xi_2 \sin(\tilde{\phi}_2 - \phi_1)}$$

(A8)

and

$$\eta_\theta = \arctan \left[ A_1 \sin \tilde{\phi}_1 - A_2 \cos \tilde{\phi}_2 \right] / A_1 \cos \phi_{11} - A_2 \sin \phi_{22}$$

(A9)

which, in general, does not allow a formulation in terms independent quadratures, unless as noted in Section I, the matrix $A$ can be expressed in terms of real coefficients.

Appendix B: Derivation of the input/output equations of motion

The mechanical degrees of freedom can be eliminated from Eq. (21), leading to the following equation for the Bogoliubov mode $\alpha$

$$i \omega \alpha - G_{BG}^2 \chi m \alpha - \frac{\kappa}{2} \alpha$$

$$+ \sqrt{\kappa} \alpha_{\text{in}} - i G_{BG} \chi \sqrt{\gamma} b_{\text{in}}$$

(B1)

which can be solved to give

$$\alpha = \chi^c \sqrt{\kappa} \alpha_{\text{in}}.$$

(B2)

where

$$\sqrt{\kappa} \alpha_{\text{in}} = \sqrt{\kappa} \alpha_{\text{in}} - i G_{BG} \chi \sqrt{\gamma} b_{\text{in}}.$$

(B3)

These are used to obtain Eqs. (22,23,24).

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