An Objective Bayesian Analysis of Life’s Early Start and Our Late Arrival

David Kipping\textsuperscript{a,b}

\textsuperscript{a}Columbia University, Dept. of Astronomy, New York, NY 10027; \textsuperscript{b}Flatiron Institute, Computational Center for Astrophysics, New York, NY 10010

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Life emerged on the Earth within the first quintile of its habitable window, but a technological civilization did not blossom until its last. Efforts to infer the rate of abiogenesis, based on its early emergence, are frustrated by the selection effect that if the evolution of intelligence is a slow process, then life’s early start may simply be a prerequisite to our existence, rather than useful evidence for optimism. In this work, we interpret the chronology of these two events in a Bayesian framework, extending upon previous work by considering that the evolutionary timescale is itself an unknown that needs to be jointly inferred, rather than fiducially set. We further adopt an objective Bayesian approach, such that our results would be agreed upon even by those using wildly different priors for the rates of abiogenesis and evolution - common points of contention for this problem. It is then shown that the earliest microfossil evidence for life indicates that the rate of abiogenesis is at least 2.8 times more likely to be a typically rapid process, rather than a slow one. This modest limiting Bayes factor rises to 8.7 if we accept the more disputed evidence of \textsuperscript{13}C depleted zircon deposits (Bell et al. 2015). For intelligence evolution, it is found that a rare-intelligence scenario is slightly favored at 3:2 betting odds. Thus, if we re-run Earth’s clock, one should statistically favor life to frequently re-emerge, but intelligence may not be as inevitable.

A fundamental question to modern science concerns the prevalence of life, and intelligence, within the Universe. At the time of writing, searches for non-terrestrial life within the Solar System have not yielded any direct evidence for life (1–3), and the remote detection of chemical biomarkers on extrasolar planets remains years ahead of present observational capabilities (4–7). The search for intelligence, through the signatures of their technology, may be detectable under certain assumptions (8–11) and limited observational campaigns have been attempted (12). However, the underlying assumptions make it challenging to use these null results to directly constrain the prevalence of life or intelligence at this time.

Despite having no observational data concerning non-terrestrial life, we are in possession of stronger constraints when it comes to life on Earth. Until this situation changes, inferences concerning the existence of life elsewhere in the Universe must unfortunately rely heavily on this single data point (13). Whilst a single data point is not ideal, it is certainly not devoid of information either (14). This is even true when strong selection biases are in play, such as the fact our existence is predicated on at least one previously successful abiogenesis event.

Problems such as these lend themselves to Bayesian analysis, where the biases can be encoded into the inference framework. The 2012 seminal paper of Spiegel & Turner (13) applied this to the question of abiogenesis. In that work, the authors treat abiogenesis as a Poisson process, which holds for systems with a discrete number of successes in a finite interval. This is used to define a likelihood function that accounts for the possibility of multiple successes and the selection effect that a success is demanded for us to observe ourselves. The authors conclude that the priors ultimately dominate their posteriors, and that even choosing between three reasonable and diffuse priors leads to greatly different answers. Accordingly, it’s very difficult to use the model to say anything definitive about how difficult or easy abiogenesis really is.

Intuitively, the possibility that life typically emerges slowly seems highly improbable given its relativity quick start on the Earth (15–18). Indeed, some commentators have remarked that this fact implies that “life is not a fussy, reluctant and unlikely thing” (ref. (19), p. 501). The plausibility of the slow start scenario can be understood to be a consequence of the selection effect, which requires that life emerges fast enough for us to have sufficient time to evolve into complex (“intelligent”) organisms capable of observing ourselves. The early emergence of life on Earth then becomes consistent with a low abiogenesis rate, since worlds where life does not emerge quickly never evolve to the point of an intelligent observer.

This reveals the important role that the evolutionary timescale plays in the inference problem, since it strongly shapes the selection bias effect. In the Bayesian analysis of (13), the evolutionary timescale is not known a-priori and thus is set to three different fiducial values (1, 2 and 3.1Gyr) in order to test the sensitivity of their results to this parameter.

In this work, we extend the model to allow the evolutionary timescale to itself now become an inferred parameter. Rather than assume several fiducial values, this parameter can be learnt by conditioning upon the time it took for observers to evolve. This enables a joint posterior distribution between the rate of abiogenesis and the rate of intelligence evolution that

\textbf{Significance Statement}

Does life’s early emergence mean that it would re-appear quickly if we were to re-run Earth’s clock? If the timescale for intelligence evolution is very slow, then a quick start to life is actually necessary for our existence - and thus doesn’t necessarily mean it’s a generally quick process. Employing objective Bayesianism and a uniform-rate process assumption, we use just the chronology of life’s appearance in the fossil record, that of ourselves, and Earth’s habitability window, to infer the true underlying rates accounting for this subtle selection effect. Our results find betting odds of >3:1 that abiogenesis is indeed a rapid process versus a slow and rare scenario, but 3:2 odds that intelligence may be rare.
Accordingly, the probability density function for the achievement of at least one success by that time.

The joint distribution of success times. Following earlier work (13, 20–22), we describe abiogenesis as uniform rate (i.e., Poisson) process, defined by a rate parameter \( \lambda_L \). As with (13), we caution that this does not imply that abiogenesis is a truly single step, instantaneous event. Rather, we interpret the process to be a model which “integrates out” the likely complex and multi-step chemistry which culminates in life. Indeed, it may be that a variety of pathways can lead to abiogenesis, but this ensemble is grouped into a single process where any of them succeeding counts as a success for the ensemble. Further, it is not necessary to strictly define what is meant by “life” here, only that the success of this Poisson process ultimately led to the geological evidence for life, and that without it said evidence would not exist.

The assumption of a uniform-rate process over some time interval can at first seem problematic when one considers the stark changes to Earth’s environment over its history. However, much of this change is due to life affecting it’s environment and thus is a consequence of a success. The abiogenesis rate may indeed be different after life begins, but it’s also irrelevant since our model only cares about the first success.

With these points in mind, we can now write that in a time interval \( t_L \) the probability of obtaining at least one successful abiogenesis event \( X_L > 0 \) will be

\[
\Pr(X_L > 0; \lambda_L, t_L) = 1 - \frac{\Pr(X_L = 0; \lambda_L, t_L)}{1 - e^{-\lambda_L t_L}}.
\]

The time interval between successes for a Poisson process follows an exponential distribution. This can be demonstrated by noting that eq. (1) corresponds to the probability of obtaining at least one success over the interval of time \( t_L \), and thus can be understood as the cumulative probability distribution for the achievement of at least one success by that time. Accordingly, the probability density function the first success with respect to \( t_L \) must be given by the derivative of eq. (1) with respect to time, yielding

\[
\Pr(t_L | \lambda_L) = \lambda_L e^{-\lambda_L t_L}.
\]

We now deviate from the approach of (13) by considering a second process, labelled by “I” for “intelligence”, which can only proceed once the previous process (“L”) is successful. The inverse of this process’ rate parameter, \( \lambda_I \), describes the characteristic timescale it takes for evolution to develop from the earliest forms of life, to an “intelligent observer” which is carefully defined in the next subsection. We truncate the times, such that both processes have to occur within a finite time \( T \). Since process I can only proceed once process L has occurred, then this requires \( t_L + t_I < T \). The joint distribution of times \( t_L \) and \( t_I \) is therefore given by

\[
\Pr(t_L, t_I | \lambda_L, \lambda_I) = \begin{cases} 
\lambda_I \lambda_L e^{-\lambda_L t_L - \lambda_I t_I} & \text{if } t_L + t_I < T, \\
0 & \text{otherwise}.
\end{cases}
\]

Imposing the condition of \( t_L + t_I < T \) serves to truncate the joint distribution and thus the above is formally a proportionality because it is not yet normalized. After normalization, the expression becomes

\[
\Pr(t_L, t_I | \lambda_L, \lambda_I) = \begin{cases} 
\frac{\lambda_I \lambda_L (\lambda_L - \lambda_I) e^{-\lambda_I (1 - e^{-\lambda_I}) - \lambda_I t_L}}{\lambda_L (1 - e^{-\lambda_I}) - \lambda_I (1 - e^{-\lambda_I})} & \text{if } t_L + t_I < T, \\
0 & \text{otherwise}.
\end{cases}
\]

Accounting for observational constraints. Before we discuss how eqn. 4 can be updated to incorporate to include observational constraints, it is first useful to define exactly what we mean by “intelligence” in this work. We adopt a functional view of this term, consider that a successful event from process I is defined as some kind of transition - which occurs after abiogenesis - which is fundamentally necessary in order for analyses such as the one presented here to be possible. In other words, this type of analysis is only possible because process I succeeded, and would be impossible if it failed.

Expounding upon this, we can consider that process I results in an observer/entity/society capable of i) obtaining and dating geological evidence pertaining to the early emergence of life ii) the ability to model the future climatic conditions of their world such that the habitability window can be estimated, and iii) interpret the ramifications of this information regarding the underlying rates of abiogenesis and evolution. For the sake of brevity we’ll refer to such outcomes as intelligent observers in what follows. Formally, these three conditions are not equivalent to a technological civilization, but this author would argue that it’s difficult to imagine how these feats would be possible in absence of one.

In what follows, we will attribute process I to correspond to the emergence of human civilization and thus will further assume that no previous Earth dwelling entities/observers/societies have had the capacity to satisfy the three conditions discussed above. This assumption would be invalidated if the “Silurian hypothesis” of (23) were confirmed, which considers the possible existence of industrial civilizations pre-dating humanity (24), in which case we would certainly advocate revisiting the calculations that will be described in this paper.

The emergence of human civilization could be defined in a variety of ways. Some possible defining “moments” could be the appearance of hominids, use the evolution of homo-sapiens, complex language, the neolithic revolution, or the first radio transmissions into space. Whatever we use, this only shifts \( t_I \) around by several million years at most. Since \( t_I \) is of order of several Gyr, these disagreements have negligible impact on our final results and thus \( t_I \) will be treated as a fixed quantity.

This is not true for the first transition, since it seems to have occurred relatively quick and the uncertainty associated with it is comparable to the actual timing. Further, it is not possible to accurately date the emergence of life, since any life could (and indeed must) have begun prior to its appearance in the geological record. Accordingly, the true date for the emergence of life, \( t_L \), must pre-date the actual observation, \( t' \), i.e., \( t_L < t' \). The probability of this can be calculated through integration:
\[\Pr(t_L < t'_L, t_I | \lambda_L, \lambda_I) = \int_{t_L=0}^{t'_L} \Pr(t_L, t_I | \lambda_L, \lambda_I) dt_L. \quad [5]\]

Since \(t_I\) is defined as the time since \(t_L\), then one cannot directly measure this value either. However, we can state that its value is somewhere between \(t'_I\) and \(t'_I + t'_I\), where \(t'_I\) is the observed time difference between the emergence of intelligence and the first evidence for life. This allows us to write our final likelihood function as

\[ L = \Pr(t_L < t'_L, t_I < t'_I + t'_L | \lambda_L, \lambda_I) \]
\[ = \int_{t_I=t'_I}^{t'_L+t'_L} \Pr(t_L < t'_L, t_I | \lambda_L, \lambda_I) dt_I. \quad [6]\]

Evaluating the above yields a piecewise closed-form likelihood function which is given in the SI Appendix. The function has two sub-domains, one which applies to the interval \( T > 2t'_L + t'_L\), and one which applies to \( T < 2t'_L + t'_L\). As shown later, the former case is applicable when \( t'_L < 0.904\) Gyr and we plot this function in fig. 1 against \( \lambda_L\) and \( \lambda_I\), along with the limiting behaviors (see the SI Appendix).

It is interesting to note that the likelihood function is not monotonic and has a global maximum which can be solved for numerically. For example, using the “optimistic” data defined in the next section, it occurs at \( \lambda_I \rightarrow 0 \) and \( \lambda_L = 21.8 \) Gyr\(^{-1}\) (the full behavior is shown in the SI Appendix). However, along the \( \lambda_I \rightarrow 0 \) axis, the likelihood is almost flat beyond this peak. For example, with the same data, the likelihood is 52.7\% of the peak when one sets \( \lambda_L = \hat{\lambda}_L/10 \), but only 99.4\% of the peak when one sets \( \lambda_L = 10\hat{\lambda}_L \). Whilst maximum likelihood parameters are instructive, we turn to Bayesian inference to determine the posterior distributions and rigorously compare different model scenarios.

**Adopted values for the observational data.** To perform any inference with our likelihood function, one first needs to assign values to the observables \( t'_L\) and \( t'_I\), as well as \( T\). All three times are relative to some initial time when conditions on Earth became suitable for life to emerge, and so let us first discuss how to define this initial time.

There is of course uncertainty about the conditions on the early Earth and when they became suitable for life (25–27). The Earth is generally thought to have been impacted by a Mars-sized body, dubbed “Theia”, 4.51 Gyr ago in a cataclysmic event that formed the Moon (28). Such an impact would have been a globally sterilizing event, and indeed may have been accompanied by another sterilizing impactor, “Monta”, 40 Myr later (29). Mineralogical evidence from zircons indicates that both an atmosphere and liquid water must have been present on the Earth’s surface (4.04 ± 0.008) Gyr ago (30). In this work, we consider these to be the necessary basic requirements for abiogenesis to take place and thus adopt \( T = 4.048\) Gyr throughout. As noted by (13), a step-function like transition from uninhabitable to habitable is surely too simplistic, but our ignorance of Earth’s early history and indeed the conditions necessary for life mean that we do not at present have a well-motivated complex model to impose in its place.

The earliest evidence for life arrives soon after this time, in the form of \(^{13}\)C depleted carbon inclusions within 4.1 Gyr old zircon deposits (17). The source of this depletion remains controversial but this would yield an optimistic estimate of \( t'_L = 0.304\) Gyr. The earliest direct and undisputed evidence for life comes from microfossils discovered in 3.465 Gyr old rocks in western Australia (15, 16, 18), yielding \( t'_L = 0.939\) Gyr. We highlight that, in both the optimistic and conservative case, \( t'_L\) is much larger than the 76 Myr uncertainty as to when the Earth became habitable and thus is not a dominant source of uncertainty.

For \( t'_I\), for reasons discussed in the previous subsection we can simply attribute this time to be the modern era. We therefore adopt intelligent observing as arriving “now” such that \( t'_I = 4.404\) Gyr − \( t'_L\) in what follows.

Finally, we turn to \( T\), which defines the interval over which the Earth is expected to persist as habitable. It’s important that we refine this definition to be habitable for intelligent beings, such as ourselves. If the Earth evolves into a state where only simple microbial life is possible, then a success can no longer occur for process I. As the Sun evolves, its luminosity will increase, which in turn increases the rate of weathering of silicate rocks on the Earth (31). This increased weathering draws down carbon from the atmosphere, thus gradually depleting the atmospheric content of carbon dioxide. Once levels drop below ~10 ppm, plant C4-photosynthesis will no longer be viable (32), leading to their imminent demise. It also possible that higher temperatures and the progressive loss of Earth’s oceans could also trigger an earlier die off (33).

The end of plant life leads to a collapse in both the food chain and Earth’s oxygen productivity, upon which animal life is critically dependent. Large endotherms, such as mammals and birds, will be the first to become extinct as a result of
their higher oxygen requirements (31, 34). Thus, one can reasonably consider that the habitable window for intelligence decidedly ends once the reign of plant life comes to a close. The timing for this is predicted to be 0.9 Gyr by (33), a value which is adopted in what follows to give \( T = 5.304 \text{Gyr} \).

In summary, we set \( T = 5.304 \text{Gyr} \) but consider two values for \( t'_L \) of \( t'_L = 0.304 \text{Gyr} \) (“optimistic”) and \( t'_L = 0.939 \text{Gyr} \) (“conservative”). This in turn gives two values of \( t'_L \) of \( t'_L = 4.404 \text{Gyr} - t'_L \).

**The Non-Objective \( \lambda \) Power-Law Prior**

**The role of the prior.** Equipped with a likelihood function, one may infer the a-posteriori distribution of \( \lambda_L \) & \( \lambda_I \) using Bayes’ theorem:

\[
Pr(\lambda_L, \lambda_I|t'_L, t'_I) = \frac{Pr(t'_L, t'_I|\lambda_L, \lambda_I)Pr(\lambda_L, \lambda_I)}{\int \int Pr(t'_L, t'_I|\lambda_L, \lambda_I)Pr(\lambda_L, \lambda_I) \, d\lambda_L \, d\lambda_I}. \tag{7}
\]

In any Bayesian inference problem, the posterior is a product of the likelihood and the prior and thus is affected by both. In cases where one possesses little or no information about the target parameters in advance, such as here, the ideal prior should be minimally informative (“diffuse”) such that it doesn’t strongly influence the result (35). In objective Bayesianism, the resulting posterior should be expected to be universally be agreed upon by everyone (36) - whereas a subjective Bayesian would argue that probability corresponds to the degree of personal belief (37).

When equipped with strongly constraining data, even those using different diffuse priors will generally find convergent solutions, since the likelihood overwhelms the prior thus naturally leading to objective Bayesianism. This is certainly not the case for our problem, since it has already been established that the posterior for \( \lambda_L \) is very sensitive to the priors (13). In such a case, one should tread carefully and seek a prior which can be objectively defined, such that other parties could agree upon the choice of prior and thus the resulting posterior.

We therefore proceed by considering how to define a prior distribution which is minimally informative and also not dependent upon subjective choices of the prior distribution parameters. However, we will later show that several important inference statements can be made independent of the prior.

**Power-law in \( \lambda \).** In previous work (13, 22), the a-priori distribution for \( \lambda_L \) was assumed to be of the form \( \lambda^0_L \) - a power-law. In extending the likelihood to include \( \lambda_I \), one may similarly extend this power-law prior to encompass \( \lambda_I \) by writing that \( Pr(\lambda_L, \lambda_I) = Pr(\lambda_L)Pr(\lambda_I) \) (i.e. assuming independence), where

\[
Pr(\lambda) = \begin{cases} 
\frac{\lambda^{n-1}}{(n+1)^{\lambda}} & \text{if } n = -1, \\
\frac{\lambda^\lambda - \lambda_{min}^\lambda}{\lambda_{max}^\lambda - \lambda_{min}^\lambda} & \text{otherwise}.
\end{cases} \tag{8}
\]

For \( n = 0 \), this returns a uniform in \( \lambda \) prior, for \( n = -1 \) a uniform in \( \log \lambda \) prior and for \( n = -2 \) a uniform in \( \lambda^{-1} \) prior; the three priors considered by (13). In adopting a prior of this form, one must choose values for three shape parameters: the index, \( n \) and the prior bounds \( \lambda_{min} \) and \( \lambda_{max} \).

**Assigning the prior shape parameters.** In (13), the favored index was \( n = -1 \) on the basis that a log-uniform prior exhibits scale-invariance ignorance for \( \lambda \). For a real-valued parameter constrained only by a minimum and maximum threshold, \( n = -1 \) also corresponds to the Jeffreys prior - a standard approach to defining objective priors (35). However, in this case, the \( n = -1 \) power-law is not actually objective since one cannot objectively define a minimum and maximum threshold. Because a power-law prior does not have semi-infinite support from \( \lambda = 0 \) to \( \lambda = \infty \), then these prior bounds have to be subjectively chosen.

For these bounds, (13) set \( \lambda_{max} = 10^3 \text{Gyr}^{-1} \) somewhat arbitrarily and a range of plausible values was offered for \( \lambda_{min} \). Whilst useful as an exercise to test the sensitivity of the posterior to the prior, this approach does not enable an objectively defined solution. In an effort to objectively assign a power-law prior, we consider here imposing the condition that the prior should be *fair* and *unbiased*, which we will define in what follows.

As currently stated, the prior in eqn. (8) appears reasonably diffuse for \( n = 0 \), \(-1\) and \(-2\), and the bounds could essentially be anything. However, it is worth recalling that the Poisson model is used as a vehicle to describe the Bernoulli probability of one or more successful events occurring. Accordingly, a natural alternative parameterization for this problem is to consider the fraction of experiments in which the Poisson processes culminate in at least one success, \( f_L \) and \( f_I \). Although these terms are similar to the fractions defined in the Drake Equation (38), here an “experiment” really refers to re-running Earth’s history back and observing how the stochastic processes play out each time (rather than some other world). Since \( f = 1 - e^{-\lambda T} \), then the power-law prior in \( \lambda \) is transformed into \( f \)-space as

\[
Pr(f) = \frac{n + 1}{1 - \int - \log(1 - f))^{n+1} - (- \log(1 - f_{min}))^{n+1}. \tag{9}
\]

Recall that we seek to define a prior which is both fair and unbiased. For a Bernoulli process, such as a coin-toss, a “fair” prior can be defined as one for which the chance of a positively-loaded coin is no more or less likely than negatively-loaded one. Accordingly, in our problem, we define a fair prior as one which does not *a-priori* favor either an optimistic \( f > 1/2 \) or pessimistic \( f < 1/2 \) world-view. This can be quantified by defining the prior odds ratio between the two scenarios using \( F \):

\[
F = \frac{\int_{f=1/2}^{f_{max}} Pr(f) df}{\int_{f_{min}}^{f_{max}/2} Pr(f) df}. \tag{10}
\]

Setting \( F = 1 \), one may solve for \( f_{max} \) (which corresponds to \( \lambda_{max} \)) as a function of \( f_{min} \) (corresponding to \( \lambda_{min} \)):

\[
\lim_{F \rightarrow 1} \lambda_{max} = \begin{cases} 
\frac{(log 2)^2}{f_{max}^{(n+1)}} & \text{if } n = -1, \\
\frac{(2 log 2)^2}{f_{max}^{(n+1)}(n+1)} & \text{otherwise}.
\end{cases} \tag{11}
\]

Although this ensures a fair prior (subject to our definition), it does not necessarily ensure an unbiased one. An unbiased
prior is defined here as one for which the \textit{a-priori} expectation value of \(f\), given by \(E[f] \equiv \int f f_{\text{prior}}(f) df\), equals one-half. After imposing the \(F = 1\) constraint enabled by eqn. (11), we evaluate \(E[f]\) in the limit of \(f_{\text{min}} \to 0\) as a function of \(n\). This reveals for \(n = -1\), the expectation value converges to a half, as desired for an unbiased prior (see the SI Appendix). The only other value of \(n\) in the range \(-1 < n < 2\) that yields a fair and unbiased distribution is \(n = -0.709\), but this is shown in the SI Appendix to require an overly restrictive prior bound limit and thus is not used.

Using these two constraints, our fair and unbiased prior for \(\lambda\) takes the form \(\lambda^{-1}\), with bounds following the relationship given by eqn. (11). Unfortunately, our two constraints applied to three parameters is insufficient to uniquely define the prior - it is still necessary to choose \(\lambda_{\text{min}}\) subjectively. The posterior could still be argued to be objective if it were found to be broadly insensitive to the choice of \(\lambda_{\text{min}}\) - however this unfortunately turns out to be false, as shown in what follows.

\textbf{Resulting posteriors.} To compute marginalized posteriors, we initially tried sampling using Markov Chain Monte Carlo and nested sampling techniques, but found that the resulting posteriors were too poorly sampled in the tails. Instead, we directly integrate the posterior density in each parameter. For example, for the \(\lambda_L\) marginalized posterior, we slide along a fine grid of \(\lambda_L\) values and numerically integrate (using the Gauss-Kronrod Rule) the joint posterior density over the limits \(\lambda_L = \lambda_{\text{min}}\) to \(\lambda_L = \lambda_{\text{max}}\).

The resulting marginalized posteriors are shown in fig. (2) for two arbitrary choices of \(\lambda_{\text{min}}\): \(10^{-3}\) Gyr\(^{-1}\) and \(10^{-6}\) Gyr\(^{-1}\). As can be seen from that figure, and perhaps not surprisingly, the resulting distributions are certainly sensitive to the choice of \(\lambda_{\text{min}}\). In conclusion, we argue here that a \(\lambda\) power-law prior is simply unacceptable as an objective prior for this problem.

\textbf{The Objective Bernoulli Prior}

\textbf{Fair and unbiased priors with semi-infinite support.} Since we have no way of objectively choosing \(\lambda_{\text{min}}\), then a fair and unbiased (subject to our definitions) power-law prior in \(\lambda\) does not provide a viable path to defining an objective posterior.

Rather than prescribing a prior in \(\lambda\)-space and then evaluating its fairness and bias in \(f\)-space, we consider here simply writing down a fair and unbiased prior in \(f\)-space directly. Since \(f\) represents a fraction, it can be interpreted as the Bernoulli probability of a success over the interval \(T\). For a Bernoulli process, a so-called Haldane prior (39) of the form \(\propto (f(1 - f))^{-1}\) was argued by the objective Bayesian Jaynes to represent the least informative prior (36). Such a prior is fair and unbiased by construction, and places extreme weight on the solutions \(f = 1\) and \(f = 0\) at the expense of intermediate values. The intuition behind this is that either a very small fraction of planets will be successful or almost all of them, but is unlikely the laws of nature are tuned such that approximately half of the planets are successful.

Using the Fisher information matrix, one can define that the Jeffreys prior for a Bernoulli distribution. The solution is not the Haldane prior, but rather a softer variant of the form \(\propto (f(1 - f))^{-1/2}\). This translates to a prior in \(\lambda\) space given by

\[
\Pr(\lambda) = \frac{T}{\pi \sqrt{\pi} \exp(-\lambda)}.
\]

[12]

Both the Haldane and Jeffreys priors are fair and unbiased with respect to \(f\) and indeed any prior of the more general...
form \( (f(1-f))^n \) satisfies these conditions. However, the \( n = -1 \) case, corresponding to the Haldane prior, is improper and indeed leads to an improper posterior too, but for all other \( n > -1 \) the prior can be normalized to a finite quantity. Accordingly, the Jeffreys prior is fair, unbiased, objectively defined and proper over the interval \( f = [0,1] \). Since \( f = 0 \) corresponds to \( \lambda = 0 \) and \( f = 1 \) corresponds to \( \lambda = \infty \), this naturally yields a proper prior with semi-infinite support in \( \lambda \), something which was not possible with the power-law case discussed earlier. Together, these properties make the distribution well-suited for our problem and we argue it solves the dilemma faced in earlier work (13).

Although we consider the Jeffreys prior to be the ideal objective prior for our problem, it is instructive to consider posteriors with \( n = 0 \) (a uniform prior in \( f \)) as well, which has a \( \lambda \)-space form of

\[
\Pr(\lambda) = Te^{-\lambda T}. \tag{13}
\]

**Bayes factors independent of the \( \lambda \) prior.** Equipped with our likelihood function and prior, one may now sample/integrate the posterior probability distribution to compute marginalized distributions. Marginalization irreversibly bakes the prior into the resulting collapsed posteriors, but specific probabilistic statements can be made in a Bayesian framework without marginalizing. In particular, we consider here an exercise in Bayesian model comparison, where we seek to compare four models, \( M \), defined as the unique corners of the parameters volume:

- \( M_{00} \): \( \lambda_L \ll 1/t_L' \) and \( \lambda_I \ll 1/t_I' \)
- \( M_{01} \): \( \lambda_L \ll 1/t_L' \) and \( \lambda_I \gg 1/t_I' \)
- \( M_{10} \): \( \lambda_L \gg 1/t_L' \) and \( \lambda_I \ll 1/t_I' \)
- \( M_{11} \): \( \lambda_L \gg 1/t_L' \) and \( \lambda_I \gg 1/t_I' \)

Binarizing the parameter volume into these four camps may at first seem arbitrary - what about intermediate values? However, this partitioning is consistent with objective Bayesianism. The objective Bernoulli prior treats life/intelligence as being either very rare or very common, but unlikely to be finely-tuned such that it approaches the intermediate value of one-half - thus motivating the models above.

Conditioned upon some available data, \( D \), one may express that the odds ratio between two models as

\[
\Pr(M_1|D)/\Pr(M_2|D) = [\Pr(D|M_1)/\Pr(D|M_2)][\Pr(M_1)/\Pr(M_2)].
\]

The terms inside the first square bracket is known as the Bayes factor, which equals the odds factor under the simple assumption that no model is a-priori preferred over any other. The Bayes factor is ratio of two “evidences” given by \( Z \equiv \Pr(D|M) \), and for the four models defined above, \( Z \) can be expressed analytically and independent of the prior \( \pi(\lambda_L, \lambda_I) \). This can be seen by noting that, for example with model \( M_{00} \):

\[
Z_{00} = \Pr(t_L', t_I'| \lambda_L \ll 1/t_L', \lambda_I \ll 1/t_I') = \lim_{\lambda_L \ll 1/t_L', \lambda_I \ll 1/t_I'} \lim_{\lambda_L \ll 1/t_L', \lambda_I \ll 1/t_I'} \Pr(t_L', t_I'| \lambda_L, \lambda_I) \tag{14}
\]

Thus, the evidences of these four corner models is independent of the prior of \( \pi(\lambda_L, \lambda_I) \), meaning that even those adopting different priors would consistently agree on the Bayes factors. We note that (13) used a similar strategy and we provide an alternative explanation of this prior-free model comparison in the SI Appendix, in terms of the Savage-Dickey ratio (40).

In practice, Bayes factors for the two corners with rapid intelligence emergence \( (\lambda_I \gg 1/t_I') \) tend to zero, since this is the behavior of the likelihood function (see the SI Appendix). This can be understood by the fact that if the rate of intelligence emergence were extremely fast, then it would be incompatible with taking as long as it did here on Earth. In contrast, life could emerge much faster than \( t_L' \) because \( t_L' \) only represents the first appearance of life in the geological record, not the actual date of abiogenesis.

Since these two corners are zero, we instead compare models \( M_{10} \) to \( M_{00} \). This represents the Bayes factor between a scenario where abiogenesis is a fast versus a slow process to compare conditional upon the premise that intelligence emergence is itself a slow process. In this case, we find

\[
\frac{Z_{10}}{Z_{00}} = \left( \frac{t_L'}{t_I'} \right)^{2L\lambda_T} \frac{t_I'}{t_L'}, \quad \text{if } T > 2t_L' + t_I', \tag{15}
\]

which evaluates to \( (Z_{10}/Z_{00}) = 8.73 \) and 2.83 for the optimistic and conservative data inputs respectively. The above also reveals the dependency on \( T \) is nearly linear; if \( T \) is revised significantly up then optimism for life would also increase. We highlight that \( Z_{10} < Z_{00} \) if \( t_L' > 3.72 \) Gyr - i.e. if the earliest evidence for life were from no earlier than 680 Myr ago, we would conclude that abiogenesis was an improbable event.

If we relax the assumption that \( \lambda_I \ll 1/t_I' \) and let intelligence become faster, then the Bayes factor monotonically rises, as shown in fig. 3. This therefore means that the Bayes factor of a quick versus slow abiogenesis scenario must be greater than the limiting case of \( Z_{10}/Z_{00} \), irrespective of whatever value \( \lambda_I \) takes (or indeed whatever the prior is).

On this basis, we can conclude that even with the most conservative date for the emergence of life, a scenario where abiogenesis occurs rapidly is at least three times more likely than a slow emergence, independent of the priors and even the timescale it takes for intelligence to emerge. If the more ambiguous evidence for an earlier start to life is confirmed (17), then this would increase the odds to a factor of nine, representing relatively strong preference for a model where life would consistently emerge rapidly on the Earth, if time were replayed.

**Bayes factors after marginalization.** Thus far we have avoided using the marginalized posteriors, which has the benefit of enabling model comparison independent of the prior \( \pi(\lambda_L, \lambda_I) \). However, it also has the disadvantage that we can only compare conditional scenarios. For example, our result for \((Z_{10}/Z_{00})\) is a Bayes factor conditional upon the assumption of a slow intelligence emergence. Whilst it turns out this can be interpreted as a lower limit on a fast versus slow abiogenesis scenario, marginalization allows for a calculation which integrates over the uncertainty in \( \lambda_I \). For example, one may write that the evidence for a model where \( \lambda_L \ll 1/t_L' \) (slow abiogenesis) marginalized over \( \lambda_I \) is given by
Further understanding of these Bayes factors can be gained by evaluating the marginalized posteriors. We numerically marginalizing the posteriors in the case of the optimistic data and show the resulting distributions in fig. 4. From these, one can see that the \( f_L \to 0 \) limit drops below the prior, whereas the \( f_L \to 1 \) limit rises above it. Together, these results paint a consistent picture that the timing of life’s emergence, and that of intelligent observers, favor the hypothesis that life would likely re-emerge rapidly on the Earth were the clock to be re-ran.

And what of intelligence?. Thus far, we have calculated Bayes factors concerning fast versus slow abiogenesis rates. For the rate of emergence of intelligent observers, Bayes factors against a fast emergence scenario tend to infinity, since \( \lim_{\lambda_i \gg 1/\ell_i} L \to 0 \) (as shown in the SI Appendix). Instead, it is more useful to compare the slow intelligence scenario (\( \lambda_i \ll 1/\ell_i \)) against the ensemble of models where \( \lambda_i \) can take any value, for which the Bayes factor is

\[
\frac{Z_{L_{\text{slow}}}}{Z_{\text{ensemble}}} = \frac{\int_{\lambda_i=0}^{\infty} \left( \lim_{\lambda_L \ll 1/\ell_L} L(\lambda_L, \lambda_I) \right) \pi(\lambda_L) d\lambda_L}{\int_{\lambda_L=0}^{\infty} \left( \lim_{\lambda_i \ll 1/\ell_i} L(\lambda_L, \lambda_I) \right) \pi(\lambda_L) d\lambda_L}, \tag{17}
\]

We evaluated the above numerically using optimistic data to yield 1.638 and 1.474 using the Jeffreys and uniform priors respectively. Switching to the conservative data paints barely affects these numbers with 1.572 and 1.416 (the full range of possibilities is depicted in panel B of fig. 3). This suggests a slight preference, 3:2 betting odds, that intelligent observers would rarely re-emerge - a value which is broadly robust against the two priors considered and the range of possible \( \ell_i \) values.

Conclusions

In this work, we have attempted to build upon the seminal paper of (13) which devised a Bayesian formalism for interpreting life’s early emergence on Earth. Unlike that work, we do not treat the timescale for life to develop into intelligent observers as a fixed quantity, but rather infer it jointly as a free parameter. This important difference feeds into an overall theme of the analysis presented here - to present an objective Bayesian analysis of life’s early emergence on Earth, and our relatively late arrival within the context of Earth’s habitable window.

In this vein, we have demonstrated that the commonly used power-law prior for this problem is not objective as the results strongly depend on arbitrary choices on the prior’s domain. We show that priors on the Bernoulli probability of life/intelligence emerging naturally provide semi-infinite support and yield distribution which can be seen to be fair and unbiased for this problem. Even so, it is possible to derive numerous model comparison results which are fully independent of these priors - meaning that even those using wildly different priors would consistently agree on the results.

The early emergence of life on Earth is naively interpreted as meaning that if we reran the tape, life would generally reappear quickly. But if the timescale for intelligence evolution is long, then a quick start to life is simply a necessary byproduct of our existence - not evidence for a general rapid abiogenesis rate. Using our objective Bayesian framework, we show that the Bayes factor between a fast versus slow abiogenesis scenario, is at least a factor of three - irrespective of the
prior or the timescale for intelligence evolution. This factor is boosted to nine when we replace the earliest microfossil evidence (15, 16, 18) with the more disputed $^{13}C$ depleted zircon deposits reported by (17). These results are also supported by marginalizing over our objective priors. An additional objective result concerning abiogenesis, is that the maximum likelihood timescale for life to first appear is 190 Myr after conditions became habitable (4.21 Gyr ago) using the microfossil evidence, or even just 46 Myr using the more disputed data. It is emphasized that these results are conditioned solely upon the chronology data concerning life.

We find that the rate at which intelligent observers evolve is less well constrained. Certainly, the possibility that the rate of intelligence emergence is rapid ($<\lambda$ Gyr) is strongly excluded, which is not surprising given that it took several Gyr here on Earth. But the possibility that intelligence is extremely rare and Earth “lucked out” remains quite viable. Overall, we find a weak preference, 3:2 betting odds, that intelligence rarely emerges given our late arrival.

It is tempting to apply these numbers to potentially habitable exoplanets being discovered. However, we caution that our analysis purely concerns the Earth, treating abiogenesis as a stochastic process against a backdrop of events and conditions which might be plausibly unique to the Earth. If conditions sufficiently similar to the early conditions exist and sustain on other worlds for a Gyr or more, then our analysis would then favor the hypothesis that life is common, by a factor of $K > 3$. However, the alternative is clearly not discounted and our Bayes factor does not cross the threshold to which it would be conventionally described as “strong” ($K > 10$) or “decisive” ($K > 100$) evidence (41). Yet, future revision regarding the earliest evidence for life could plausibly trigger this.

Overall, our work supports an optimistic outlook for future searches for biosignatures (4–7). The slight preference for a rare intelligence scenario is consistent with a straightforward resolution to the Fermi paradox. However, our work says nothing about the lifetime of civilizations, and indeed the weight of evidence in favor of this scenario is sufficiently weak that searches for technosignatures should certainly be a component in observational campaigns seeking to resolve this grand mystery.

Data Availability Statement: All data used in this work is fully stated in the text of this paper.

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Supplementary Information

Full Likelihood Function and Limiting Behavior

The main text establishes that our likelihood function is defined as (see eqns. 4-6)

\[ \mathcal{L} = \Pr(t_L < t_L', t_I < t_I' + t_L'|\lambda_L, \lambda_I) \]
\[ = \int_{t_I=0}^{t_I'+t_L} \int_{t_I'=t_I}^{t_I'+t_L} \Pr(t_L < t_L', t_I'|\lambda_L, \lambda_I)dt_I, \]
\[ = \int_{t_I=0}^{t_I'+t_L} \Pr(t_L|\lambda_L, \lambda_I)dt_Ldt_I. \]  

[18]

In the main text this is evaluated to eqn. (7) for the case where \( 0 < 2t_L' + t_I' < T \). The more general result is given by

\[ \Pr(t_L < t_L', t_I < t_I' + t_L'|\lambda_L, \lambda_I) = \begin{cases} 
(\lambda_L - \lambda_I)e^{-\lambda_I t_I'}(1-e^{-\lambda_L t_L'})(1-e^{-\lambda_I (T-t_I'-t_I')}) & \text{if } T > 2t_L + t_I, \\
\lambda_L e^{-\lambda_L (T-t_I'-t_I')}-\lambda_I e^{-\lambda_I (t_I'+t_I')}-\lambda_L T-(\lambda_I-t_I')e^{-\lambda_I (T-t_I'-t_I')} & \text{if } T < 2t_L + t_I.
\end{cases} \]

[19]

It is useful to inspect the limiting behavior of the likelihood function, which are depicted in the side panels of fig. 1 of the main text.

In the limit of fast abiogenesis, \( \lambda_L \gg 1/t_L' \), one finds that

\[ \lim_{\lambda_L \gg 1/t_L'} \mathcal{L} = \frac{e^{-\lambda_I t_I'} - e^{-\lambda_I (t_I'-t_I')}}{1-e^{-\lambda_I T}}, \]

[20]

whereas in the slow limit we find

\[ \lim_{\lambda_L \ll 1/t_L'} \mathcal{L} = \begin{cases} 
\lambda_I t_I' e^{-\lambda_I t_I'}(1-e^{-\lambda_L t_L'}) & \text{if } T > 2t_L + t_I, \\
\lambda_I T(1-e^{-\lambda_L t_L'}) - \lambda_I e^{-\lambda_I (t_I'+t_I')}(1-\lambda_L (T-t_I'-t_I')) & \text{if } T < 2t_L + t_I.
\end{cases} \]

[21]

Similarly, one can define limits for slow and fast intelligence evolution as

\[ \lim_{\lambda_I \gg 1/t_I'} \mathcal{L} = 0, \]

[22]

and

\[ \lim_{\lambda_I \ll 1/t_I'} \mathcal{L} = \begin{cases} 
\lambda_L t_L'(1-e^{-\lambda_L t_L'}) & \text{if } T > 2t_L + t_I, \\
\lambda_L t_L'+e^{-\lambda_L t_L'}(1-\lambda_I (T-t_I'-t_I')) & \text{if } T < 2t_L + t_I.
\end{cases} \]

[23]

Finally, one can evaluate the likelihood function in the corners of the entire parameter volume taking limits of these results another time. Since \( \lim_{\lambda_I \gg 1/t_I'} \mathcal{L} = 0 \), then it makes no difference what value \( \lambda_L \) takes in along the \( \lambda_I \gg 1/t_I' \) axis, the result is still zero. For the \( \lambda_I \ll 1/t_I' \) \( \mathcal{L} \) case, we have two non-zero corners of

\[ \lim_{\lambda_L \gg 1/t_L'} \lim_{\lambda_I \ll 1/t_I'} \mathcal{L} = \begin{cases} 
\frac{2t_I'^2}{T} & \text{if } T > 2t_L + t_I, \\
\frac{2t_I'(t_I'+2t_L')-T^2+4t_I'^2-2t_I'^2}{2} & \text{if } T < 2t_L + t_I.
\end{cases} \]

[24]

and

\[ \lim_{\lambda_L \gg 1/t_L'} \lim_{\lambda_I \ll 1/t_I'} \mathcal{L} = \frac{2t_I'}{T}. \]

[25]

Max Likelihoods

It is straightforward to maximize the likelihood function as a function of the parameters \( \lambda_L \) and \( \lambda_I \). The maximum likelihood always lies along the \( \lambda_I \ll 1/t_I' \) axis, since the likelihood function monotonically decreases as \( \lambda_I \) increases. We therefore simply need to maximize the function \( \lim_{\lambda_I \ll 1/t_I'} \mathcal{L} \), found earlier. Rather than do this exclusively for the optimistic and conservative data, we repeat the calculation along a sliding scale to illustrate the full behavior (see fig. 5).
Imposing Fairness on the $\lambda$ Power-Law Prior

Starting from eqn. (8) of the main text, we can transform the prior into $f$, where $f \equiv 1 - e^{-\lambda T}$ using

$$\Pr(\lambda) \left( \frac{d\lambda}{df} \right) df = \Pr(f) df$$

and noting that $\lambda = -T^{-1} \log(1 - f)$, yielding

$$\Pr(f) = \frac{\log(-\log(1-f_{\text{max}})) - \log(-\log(1-f_{\text{min}}))}{\log(1-f)} \left( \frac{1-f}{-\log(1-f)} \right)^n$$

for which the more general case of $n \neq 1$ is given in eqn. (9) of the main text. In the above, the limits can be related as

$$f_{\text{min}} = 1 - e^{-\lambda_{\text{min}} T},$$

$$f_{\text{max}} = 1 - e^{-\lambda_{\text{max}} T}.$$

Let us now consider whether this prior is biased with respect to $f$ by evaluating the odds ratio of obtaining an a-priori low value of $f$ ($f < 0.5$) versus a high value ($f > 0.5$) using the definition of $F$ given by eqn. (10) of the main text. Evaluating this function yields

$$F = \begin{cases} \log \left( \frac{\log(-\log(1-f_{\text{max}}))}{\log(1-f)} \right) & \text{if } n = -1, \\ \log(2)^n - \log(-\log(1-f_{\text{max}}))^{n+1} & \text{otherwise,} \end{cases}$$

Let’s define $F \equiv \log(1-f)$ (such that $\lambda = -F/T$) and then set the above equal to unity and solve for $F_{\text{max}}$ to give

$$\lim_{F \to 1} F_{\text{max}} = \begin{cases} (\log 2)^2 - \log(-\log(1-f_{\text{min}})) & \text{if } n = -1, \\ -2 \log 2^{n+1} + F_{\text{min}}(-F_{\text{min}})^n & \text{otherwise.} \end{cases}$$

And this may now be easily related back to $\lambda$ as

$$\lim_{F \to 1} \lambda_{\text{max}} = \begin{cases} (\log 2)^2 - \log(-\log(1-f_{\text{min}})) & \text{if } n = -1, \\ T^{-1}(2 \log 2^{n+1} - (\lambda_{\text{min}} T)^{n+1})^{1/(n+1)} & \text{otherwise.} \end{cases}$$

Note, in order for $\lambda_{\text{max}} > \lambda_{\text{min}}$, this requires $\lambda_{\text{min}} < T^{-1} \log 2 \sim 0.14 \text{ Gyr}^{-1}$. 

---

**Fig. 5.** Maximum likelihood value of $1/\lambda_L$ as a function of $t_L'$ (the earliest evidence for life).
Bias in the Power-Law Prior

The a-priori expectation value of $f$ should be equal to one half for an unbiased prior. This may be evaluated through integration of eqn. \( (27) \), yielding

$$E[f] = \begin{cases} \frac{(n+1)!}{\log(\text{min}) - \log(\text{max})} \times \frac{(n+1)!}{\log(\text{min}) - \log(\text{max})}^{-n+1} \frac{\text{E}[f_{\text{min}}] - \text{E}[f_{\text{max}}]}{\log(\text{max}) - \log(\text{min})} & \text{if } n = -1, \\ \frac{\text{E}[f_{\text{min}}] - \text{E}[f_{\text{max}}]}{\log(\text{max}) - \log(\text{min})} & \text{otherwise}, \end{cases}$$

where $\text{E}[x]$ is the exponential integral function. We may now replace $\text{F}_{\text{max}}$ using eqn. \( (30) \), to impose a fair prior. After doing so, we take the limit of the expectation value as $f_{\text{min}} \to 0$ and show the result as a function of $n$ in fig. 6. This reveals that the $n = -1$ prior converges to precisely one-half as desired, whereas other indices (in general) do not.

Aside from $n = -1$ being unbiased, we also see that $n \approx -0.709$ appears unbiased prior too. This can be understood by the fact that the minimum in $\text{Pr}(f)$ occurs at exactly 1/2 when $n = -\log 2 = -0.693\ldots$. However, $\text{Pr}(f)$ is not symmetric about $f = 1/2$ when using $n = -\log 2$ and thus is not a fair prior. Accordingly, a small perturbation away from $n = -\log 2$ is needed to obtain an unbiased distribution.

Unfortunately, it is not practical to use $n = -0.709$ as an alternative fair and unbiased prior. This can be understood by considering the behavior of eqn. \( (31) \) in the limit of $\lambda_{\text{min}} \to 0$. Unless, $n = -1$, the limit is $2^{1/(n+1)} \times \log 2$, which in general is not large enough to serve as a useful upper limit on the prior. For example, with $T = 5$ Gyr and $n = -0.709$, this yields $\lambda_{\text{max}} = 1.5$ Gyr$^{-1}$, which means that the posterior can never explore solutions with more than 1.5 successes per billion years. On that basis, it is argued here that this is an overly restrictive choice of prior distribution that significantly truncates the posterior. And thus, even though an $n = -0.709$ power-law prior can be fair and unbiased, it will not be considered further.

An Alternative Explanation of Prior Independent Bayes Factors

In the main text, it is shown that for the four models which define the four corners of the two-dimensional parameter volume, objective Bayes factors can be expressed independent of the prior $\pi(\lambda_L, \lambda_I)$. We here provide an alternative explanation of this interesting feature using the Savage-Dickey ratio instead.

This can be accomplished by first considering that the extreme model $\mathcal{M}_{00}$ is embedded within the nested model of arbitrary $\{\lambda_L, \lambda_I\}$. If we compare the extreme (and nested) model to the more general case, the Bayes factor will be given by the Savage-Dickey ratio \( (40) \):

$$\frac{Z_{00}}{Z_{\text{ensemble}}} = \lim_{\lambda_L \leq 1/t'_L} \lim_{\lambda_I \leq 1/t'_I} \left( \frac{\mathcal{P}(\lambda_L, \lambda_I)}{\pi(\lambda_L, \lambda_I)} \right),$$

where $\pi$ is the prior, $\mathcal{P}$ is the posterior and we use the notation “00” to denote model $\mathcal{M}_{00}$ and “ensemble” to denote the general case comprising the ensemble of scenarios. Note that the posterior density featured above is not marginalized, but rather directly given by the closed form formula of our likelihood multiplied by the prior. Accordingly, prior will cancel out leaving

$$\frac{Z_{00}}{Z_{\text{ensemble}}} = \lim_{\lambda_L \leq 1/t'_L} \lim_{\lambda_I \leq 1/t'_I} \mathcal{L}(\lambda_L, \lambda_I),$$

thus providing a totally prior-free measure of the Bayes factor. To finish the analysis, one may now compare different extrema against one another noting that the arbitrary “ensemble” models will cancel out. For example, $\left( \frac{Z_{10}}{Z_{00}} \right) = \left( \frac{Z_{10}/Z_{\text{ensemble}}}{Z_{00}/Z_{\text{ensemble}}} \right)$.

Fig. 6. Left: The a-priori expectation value of $f$ (the probability of at least one success in a Poisson process) for priors of the form $\lambda^n$ when imposing a fairness using eqn. \( (30) \). The expectation values shown are all in the limit of $f_{\text{min}} \to 0$ and we find that the $n = -1$ (log-uniform prior) naturally recovers an unbiased result of one-half. Right: Required $\lambda_{\text{max}}$ value to impose a fair prior as a function of $\lambda_{\text{max}}$ for several highlighted choices of $n$. Only $n = -1$ provides the necessary dynamic range to handle the scope of our problem.