Discrete integrable equations on face-centred cubics: consistency and Lax pairs of corner equations

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A new set of discrete integrable equations, called face-centred quad equations, was recently obtained using new types of interaction-round-a-face solutions of the classical Yang–Baxter equation. These equations satisfy a new formulation of multidimensional consistency, known as consistency-around-a-face-centred-cube (CAFCC), which requires consistency of an overdetermined system of 14 five-point equations on the face-centred cubic unit cell. In this paper, a new formulation of CAFCC is introduced where so-called type C equations are centred at faces of the face-centred cubic unit cell, whereas previously they were only centred at corners. This allows type C equations to be regarded as independent multidimensionally consistent integrable systems on higher dimensional lattices and is used to establish their Lax pairs.

1. Introduction

The property of multidimensional consistency has been at the forefront of studies of integrability of partial difference equations [1] for the past two decades. For partial difference equations defined on faces of a square lattice, commonly known as quad equations, multidimensional consistency arises as a consequence of the consistency of an overdetermined system of six equations on the faces of a cube, a property commonly known as consistency-around-a-cube (CAC) [2–5]. Furthermore, by comparing two different evolutions on the cube one may derive Lax pairs for CAC equations [3,6–8], through which the equations themselves are essentially reinterpreted as their own Lax matrices.
The author has recently introduced the concept of face-centred quad equations [9] as a new type of multidimensionally consistent equation on the square lattice. The analogue of CAC for face-centred quad equations is the property of consistency-around-a-face-centred-cube (CAFCC), which requires consistency of an overdetermined system of 14 equations on the face-centred cube (face-centred cubic unit cell). Similarly to CAC equations, a comparison of two different directions of evolution on the face-centred cube was shown to lead to Lax pairs [10], through which the equations themselves are reinterpreted as their own Lax matrices. Besides multidimensional consistency and Lax pairs, the CAFCC equations also possess vanishing algebraic entropy [11] (as do CAC equations), which is another important property associated with integrability of lattice equations [12–14].

In the original formulation of CAFCC [9], there arose three types of equations which were denoted as one of type A, B or C. The type C equations were centred at corners of the face-centred cube and provided an intermediate evolution equation between type A and B equations that were centred at faces of the face-centred cube. This means that CAFCC in its original formulation can only imply the consistency of systems of either type A or type B equations in higher dimensional lattices. Furthermore, the method that was introduced for deriving Lax pairs [10] only applies to CAFCC equations appearing on faces of the face-centred cube, and thus Lax pairs have only been obtained for type A and type B equations, but not for type C equations.

The purpose of this paper is to establish a new formulation of CAFCC where the type C equations are centred at faces of the face-centred cube. The motivation behind the search for this new formulation of CAFCC arises from the recent discovery of the vanishing algebraic entropy for type C equations in appropriate arrangements in the square lattice [11], suggesting that they are integrable systems which should also have consistent extensions into higher dimensional lattices, similarly to type A and B equations. The new formulation of CAFCC in this paper gives the multidimensional consistency of systems of type C equations in combination with type A equations, and furthermore allows for the derivation of their Lax pairs.

This paper is organized as follows. In §2, an overview of the concept of face-centred quad equations is given, and the main result of this section is the new formulation of CAFCC involving only type A and type C equations, which is given in §2a. In §3, it is shown how the property of CAFCC leads to Lax pairs of type C equations, and explicit examples are presented. The type A and type C equations obtained from [9] are given in appendix A, and in appendix B a new form of the classical Yang–Baxter equation is given that is related to the property of CAFCC of §2a.

### 2. Face-centred quad equations and consistency-around-a-face-centred-cube

The author has recently introduced the concept of face-centred quad equations and their consistency as CAFCC [9]. A face-centred quad equation may be written as

\[ A(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0, \quad (2.1) \]

where \( A \) is a multivariate polynomial of five variables \( x, x_a, x_b, x_c, x_d \), with linear dependence on each of the four variables \( x_a, x_b, x_c, x_d \), but no restriction on the degree of \( x \). This equation is associated with a square as shown in figure 1a, where the variables \( x_a, x_b, x_c, x_d \) are at corners and the variable \( x \) is centred at the face. The multilinearity in the four corner variables \( x_a, x_b, x_c, x_d \) is a natural requirement to have a well-defined evolution in the square lattice. The parameters \( \alpha \) and \( \beta \) each have two components,

\[ \alpha = (\alpha_1, \alpha_2) \quad \text{and} \quad \beta = (\beta_1, \beta_2), \quad (2.2) \]

and are associated with horizontal and vertical lines, as indicated in figure 1.
A general polynomial form of a face-centred quad equation is given by

\[
A(x; x_{a}, x_{b}, x_{c}, x_{d}; \alpha, \beta) = \kappa_{1}x_{a}x_{b}x_{c}x_{d} + \kappa_{2}x_{a}x_{b}x_{c} + \kappa_{3}x_{a}x_{b}x_{d} + \kappa_{4}x_{a}x_{c}x_{d} + \kappa_{5}x_{b}x_{c}x_{d} + \kappa_{6}x_{a}x_{b} + \kappa_{7}x_{a}x_{c} + \kappa_{8}x_{a}x_{d} + \kappa_{9}x_{b}x_{c} + \kappa_{10}x_{b}x_{d} + \kappa_{11}x_{c}x_{d} + \kappa_{12}x_{a} + \kappa_{13}x_{b} + \kappa_{14}x_{c} + \kappa_{15}x_{d} + \kappa_{16} = 0,
\]

(2.3)

where the coefficients \( \kappa_{i} = \kappa_{i}(x, \alpha, \beta) \) \( (i = 1, \ldots, 16) \) depend on the face variable \( x \) and the four components \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) of the parameters \( \alpha, \beta \). The multilinear expression (2.3) resembles the general expression for regular quad equations, except in the latter case the coefficients would depend only on two one-component parameters and no variables.

Originally, three types of face-centred quad equations were distinguished as type A, B or C. The type A equations satisfy the following three symmetries:

\[
A(x; x_{a}, x_{b}, x_{c}, x_{d}; \alpha, \beta) = -A(x; x_{b}, x_{a}, x_{d}, x_{c}; \alpha, \beta),
\]

(2.4)

\[
A(x; x_{a}, x_{b}, x_{c}, x_{d}; \alpha, \beta) = -A(x; x_{c}, x_{d}, x_{a}, x_{b}; \alpha, \beta)
\]

(2.5)

\[
A(x; x_{a}, x_{b}, x_{c}, x_{d}; \alpha, \beta) = -A(x; x_{d}, x_{b}, x_{c}, x_{a}; \beta, \alpha),
\]

(2.6)

where \( \hat{\alpha} \) and \( \hat{\beta} \) represent \( \alpha \) and \( \beta \) with the components exchanged

\[
\hat{\alpha} = (\alpha_2, \alpha_1) \quad \text{and} \quad \hat{\beta} = (\beta_2, \beta_1).
\]

(2.7)

Type B equations only satisfy (2.4) and (2.5), while type C equations only satisfy (2.4). The symmetries (2.4)–(2.6) are natural analogues for face-centred quad equations of the square symmetries that are satisfied by regular quad equations.

For the purposes of this paper, it will be useful to distinguish the three different types of face-centred quad equations by using different types of edges, as is shown in figure 1. For face-centred quad equations that were obtained in [9], this graphical representation is related to the existence of ‘four-leg’ forms of the equations, which arose as a step in the derivation of equations (2.3) that came from the classical Yang–Baxter equation (however, a generic face-centred quad equation (2.3) does not have this property). Specifically, the face-centred quad equations derived in [9] may be written in the following forms:

\[
\text{type A:} \quad \frac{a(x; x_{a}; \alpha_{2}, \beta_{1})a(x; x_{d}; \alpha_{1}, \beta_{2})}{a(x; x_{d}; \alpha_{1}, \beta_{1})} = 1,
\]

(2.8)

\[
\text{type B:} \quad \frac{b(x; x_{a}; \alpha_{2}, \beta_{1})b(x; x_{d}; \alpha_{1}, \beta_{2})}{b(x; x_{d}; \alpha_{1}, \beta_{1})} = 1
\]

(2.9)

and

\[
\text{type C:} \quad \frac{c(x; x_{a}; \alpha_{2}, \beta_{1})c(x; x_{d}; \alpha_{1}, \beta_{2})}{c(x; x_{d}; \alpha_{1}, \beta_{1})} = 1.
\]

(2.10)

The function \( a(x, y; \alpha, \beta) \) satisfies \( a(x, y; \alpha, \beta)a(x, y; \beta, \alpha) = 1 \), and the three functions \( a(x, y; \alpha, \beta) \), \( b(x, y; \alpha, \beta) \) and \( c(x, y; \alpha, \beta) \) are each fractional linear functions of a corner variable \( y \). The function

\[
\begin{align*}
A & (x; x_{a}, x_{b}, x_{c}, x_{d}; \alpha, \beta) \\
B & (x; x_{a}, x_{b}, x_{c}, x_{d}; \alpha, \beta) \\
C & (x; x_{a}, x_{b}, x_{c}, x_{d}; \alpha, \beta)
\end{align*}
\]
Figure 2. Two different type C equations (2.12) distinguished by the orientations of directed edges. The convention used in the following is that $C$ will be assigned to faces of the face-centred cube, and $\tilde{C}$ will be assigned to corners (see figure 3a,b).

$a(x; y; \alpha, \beta)$ is associated with the solid edges of both type A and type C equations in figure 1, and the functions $b(x; y; \alpha, \beta)$ and $c(x; y; \alpha, \beta)$ are associated with the solid double edges of type B and type C equations, respectively. The reason that the edges of both type A and type B equations appear for type C equations is that the latter originally arose in CAFCC as an intermediate equation between the ‘face’ vertices of both type A and type B equations. There are type B and type C equations for which the functions $b(x; y; \alpha, \beta)$ and $c(x; y; \alpha, \beta)$ are the same, but to satisfy CAFCC [9] it requires for some cases that the functions are different, which is the reason that they are denoted as different functions. Examples of the functions $a(x; y; \alpha, \beta)$ and $c(x; y; \alpha, \beta)$ are given in table 2 in appendix A for type A and type C equations that will be used in this paper.

Starting from a four-leg form, the multilinear form (2.3) of a face-centred quad equation is typically recovered by multiplying both sides of one of (2.8)–(2.10) by its denominator, and cancelling common terms appearing on both sides. On the other hand, a generic multilinear face-centred quad equation (2.3) does not possess a four-leg form. The explicit multilinear face-centred quad equations and associated four-leg forms for type A and type C equations that were derived in [9] are given in appendix A. The expressions for the type A equations in the four-leg form (2.8) are related to certain expressions for discrete Toda equations that have been known for a long time [3,15,16], while the type B and C equations in the four-leg form (2.9) and (2.10) (as well as multilinear form) appear to be new equations that arose from the discovery of CAFCC [9].

(a) A new formulation of consistency-around-a-face-centred-cube

Type A equations have been shown to satisfy CAFCC either by themselves or in combination with type B and type C equations [9]. In either case, both type A and type B equations only appeared centred at faces of the face-centred cube, and type C equations only appeared centred at corners. Here, a new formulation of CAFCC will be given that involves both type A and type C equations that are centred at both faces and corners.

Let two different type A equations, and two different type C equations, be denoted as

$$A(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0, \quad \tilde{A}(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0 \quad (2.11)$$

and

$$C(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0, \quad \tilde{C}(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0. \quad (2.12)$$

For convenience, the two different type C equations (2.12) will be distinguished graphically by different orientations of directed edges, as shown in figure 2.

Two halves of the face-centred cube are shown in figure 3a,b. This presentation is used to more clearly show how edges and parameters are arranged, while it also reflects the form of Yang–Baxter equation that is associated with CAFCC (see appendix B). As usual, the three parameters $\alpha, \beta, \gamma$, are associated with three orthogonal lattice directions, and the components of $\gamma$ are exchanged on two faces which meet orthogonally.
Figure 3. (a,b) Labelling of vertices and edges for CAFFC involving type C equations centred at both faces and corners. (c,d) Comparison with a diagram for CAFFC from [9] that involves type A and type B equations on faces, and a type C equation on corners. The directed edges for this diagram have only been included for the purpose of comparison and are unnecessary as the diagram involves only one type of type B equation and one type of type C equation.

There are 14 equations on the face-centred cube shown in figure 3a,b, made up of the six equations

$$C(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0, \quad C(z_w; y_a, y_c, x_c; \alpha, \beta) = 0,$$

$$A(z_n; y_a, y_b, x_a, x_b; \gamma, \beta) = 0$$

and

$$C(y; y_a, y_b, y_c, y_d; \alpha, \beta) = 0, \quad C(z_s; y_b, x_b, x_d; \alpha, \gamma, \beta) = 0,$$

$$\overline{A}(z_o; y_c, y_d, x_a, x_c; \gamma, \beta) = 0$$

on faces and the eight equations

$$A(x_a; z_w, y_a, x, z_n; (\beta_1, \gamma_2), (\alpha_2, \gamma_1)) = 0,$$

$$A(x_b; z_e, y_b, x, z_n; (\beta_2, \gamma_2), (\alpha_2, \gamma_1)) = 0,$$

$$\overline{C}(x_a; z_s, x_c, z_w; (\alpha_1, \gamma_1), (\beta_1, \gamma_2)) = 0,$$

$$\overline{C}(x_d; y_d, z_s, x_c; (\alpha_1, \gamma_1), (\beta_1, \gamma_2)) = 0,$$

$$A(y_a; z_w, x_a, y, z_n; (\beta_1, \gamma_1), (\alpha_2, \gamma_2)) = 0,$$

$$A(y_b; z_e, x_b, y, z_n; (\beta_2, \gamma_1), (\alpha_2, \gamma_2)) = 0$$

and

$$\overline{C}(y_c; x_c, z_a, z_w; (\alpha_1, \gamma_2), (\gamma_1, \beta_1)) = 0,$$

$$\overline{C}(y_d; x_d, z_s, z_o; (\alpha_1, \gamma_2), (\gamma_1, \beta_2)) = 0$$

(2.13)
on corners. An important difference from the original formulation of CAFCC is that there are now type C equations in (2.13) that are centred at faces of the face-centred cube.

(i) Consistency-around-a-face-centred-cube algorithm

The six components of the parameters $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$ are fixed, while

$$x, x_a, x_b, x_c, z_n, z_w$$

are chosen as initial variables. There remain a total of eight undetermined variables on vertices of the face-centred cube, and each of the 14 equations (2.13) and (2.14) to be satisfied.

For the initial conditions (2.15), the six steps below may be used to determine whether equations (2.11) and (2.12) satisfy CAFCC.

(i) The two equations

$$A(x_a; z_w, y_n, x, z_n; (\beta_1, \gamma_2), (\alpha_2, \gamma_1)) = 0$$

and

$$C(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0$$

are used to uniquely determine the variables $y_a$ and $x_d$, respectively.

(ii) The three equations

$$C(z_w; y_a, x_a, y_c, x_c; \alpha, \gamma) = 0,$$

$$A(z_n; y_n, y_b, x_n; \gamma, \beta) = 0$$

and

$$A(y_a; z_w, x_a, y, z_n; (\beta_1, \gamma_1), (\alpha_2, \gamma_2)) = 0$$

are used to uniquely determine the three variables $y_c$, $y_b$ and $y$, respectively.

(iii) The two equations

$$A(x_b; z_c, y_b, x, z_n; (\beta_2, \gamma_2), (\alpha_2, \gamma_1)) = 0$$

and

$$A(y_b; z_c, x_b, y, z_n; (\beta_2, \gamma_1), (\alpha_2, \gamma_2)) = 0$$

must agree for the solution of the variable $z_e$.

(iv) The two equations

$$\overline{C}(y_c; x_c, z_c, z_w, y; (\alpha_1, \gamma_2), (\gamma_1, \beta_1)) = 0$$

and

$$\overline{C}(x_c; z_s, y_c, x, z_w; (\alpha_1, \gamma_1), (\beta_1, \gamma_2)) = 0$$

must agree for the solution of the variable $z_s$.

(v) The four equations

$$C(z_e; y_b, y_n, y_c, x; \alpha, \gamma) = 0,$$

$$\overline{A}(z_s; y_c, y_d, x_c, x_d; \gamma, \beta) = 0,$$

$$C(y; y_f, y_b, y_c, y_d; \alpha, \beta) = 0$$

and

$$\overline{C}(x_d; y_d, z_s, z_e, x; (\alpha_1, \gamma_1), (\gamma_1, \beta_2)) = 0$$

must agree for the solution of the final variable $y_d$.

(vi) The remaining equation

$$\overline{C}(y_d; x_d, z_s, z_e, y; (\alpha_1, \gamma_2), (\gamma_1, \beta_2)) = 0$$

must be satisfied by the variables obtained in the previous steps.

If following steps (i) and (ii) the remaining steps (iii)–(vi) can be satisfied, then equations (2.11) and (2.12) satisfy the property of CAFCC.

As for the original formulation of CAFCC [9], the above property of CAFCC may be derived using solutions of a particular form of an interaction-round-a-face (IRF)-type classical Yang–Baxter equation. Further details on this connection are given in appendix B.
follows.

Specific pairs of type C equations that have vanishing algebraic entropy in the arrangement of figure 4 have vanishing algebraic entropy in the arrangement of figure 4. Same as A (or ‘same as C’) indicates that the same equation that is listed for A (C) should be used for A (C).

(b) Integrable type C lattice, consistency-around-a-face-centred-cube and consistency across adjacent cubes

Because type C equations possess the least symmetry in comparison with type A and type B equations, only particular arrangements of type C equations in the square lattice were found previously to possess vanishing algebraic entropy as required for integrability. At present, a general set of conditions for when an arrangement of equations will have vanishing entropy are not known, but figure 4 shows one example of an arrangement that was found [11] to give vanishing algebraic entropy for a pair of type C equations, as indicated in figure 2.

Specific pairs of type C equations that have vanishing algebraic entropy in the arrangement of figure 4 are listed in the last two columns of table 1 (where the expressions for the equations are given in appendix A). These equations satisfy CAFC in combination with type A equations as follows.
Figure 5. (a) Four CAFCC cubes. The northwest, northeast, southwest and southeast cubes are, respectively, denoted by $cF^{(nw)}$, $cF^{(ne)}$, $cF^{(sw)}$ and $cF^{(se)}$. (b) A seven-point equation on white vertices of $cF^{(sw)}$ and another CAFCC cube. (c) A nine-point equation on white vertices of $cF^{(ne)}$, $cF^{(sw)}$ and another CAFCC cube.

Theorem 2.1. Let the four equations $C$ in (2.13) and the four equations $\overline{C}$ in (2.14) be chosen from one of the pairs of equations $C$ and $\overline{C}$ that are listed in the rows of table 1. Then the choices of the equations $A$ and $\overline{A}$ in (2.13) and (2.14) can be uniquely fixed by requiring their respective leg functions $a(x; y; \alpha, \beta)$ from (2.8) to match the leg functions of $C$ and $\overline{C}$ from (2.10), as indicated in table 2 of appendix A. Each of the resulting 10 combinations of equations listed in the rows of table 1 satisfy the property of CAFCC.

Theorem 2.1 may be checked by direct computation. Theorem 2.1 gives the desired multidimensional consistency property for type $C$ equations that appear on faces of the face-centred cube, which allows them to be considered as integrable systems in higher dimensional lattices and may be used to derive their Lax pairs, which will be shown in the following section.

One of the consequences of the property of CAFCC is that because the evolution of face-centred quad equations involves equations that overlap, when considering consistency on adjacent face-centred cubes in higher dimensional lattices, there must be more instances of consistent equations than those just defined on the individual cubes. This also applies (but was not considered) to the original formulation of CAFCC [9].

This can be illustrated for the simplest case of extending the equations into a third dimension, with the example of the four CAFCC cubes in figure 5a. These four individual cubes will be denoted as $cF^{(nw)}$, $cF^{(ne)}$, $cF^{(sw)}$ and $cF^{(se)}$, for the northwest, northeast, southwest and southeast cubes, respectively. The types of equations on the pair of cubes $(cF^{(sw)}, cF^{(se)})$, and on the pair of cubes $(cF^{(nw)}, cF^{(ne)})$, are related by the exchanges $A \leftrightarrow \overline{A}$ and $C \leftrightarrow \overline{C}$. Since each cube involves 14 equations, there are a total of 56 equations on the individual cubes (but four pairs of these equations that appear on coinciding faces are equivalent), and these equations involve 38 different variables (associated with distinct vertices).

The variables on each of the four cubes will be labelled according to figure 3a,b and distinguished by the superscripts $(nw), (ne), (sw), (se)$ for the respective cubes. An initial value problem for the 56 equations can be posed by choosing 10 initial variables as

$$x^{(nw)}_a, x^{(nw)}_b, x^{(nw)}_c = x^{(nw)}_d, x^{(sw)}_a, x^{(sw)}_b, x^{(sw)}_c, x^{(ne)}_a, x^{(ne)}_b, x^{(ne)}_c. \quad (2.22)$$

Proposition 2.2. With the assignment of variables and equations on the four cubes according to (2.13), (2.14) and figure 3a,b, and with the initial conditions (2.22), there is an overdetermined system of 56 equations for 28 unknown variables on the four cubes $cF^{(nw)}$, $cF^{(ne)}$, $cF^{(sw)}$, $cF^{(se)}$. If the four equations (2.12) and (2.11) satisfy CAFCC then this overdetermined system of equations is consistent.

Besides the 56 equations on the individual cubes, there are additional equations that will be satisfied which involve variables on two or more different cubes. However, these equations are
and two type C equations:

\[
A(y^{(nw)}_b, z^{(nw)}_n, y^{(ne)}_n, y^{(ne)}_c; (\alpha_2, \gamma_2), \beta) = 0, \\
A(x^{(nw)}_b, z^{(nw)}_n, x^{(ne)}_n, x^{(ne)}_c; (\alpha_2, \gamma_1), \beta) = 0, \\
\tilde{C}(y^{(nw)}_d, z^{(nw)}_n, y^{(ne)}_n, y^{(ne)}_c; (\alpha_1, \gamma_2), \beta) = 0, \\
\tilde{C}(x^{(nw)}_d, b^{(nw)}_n, z^{(ne)}_n, x^{(ne)}_n; (\alpha_1, \gamma_1), \beta) = 0,
\] (2.23)

respectively, centred at the four common variables \(y^{(nw)}_b, z^{(nw)}_n, y^{(ne)}_n, y^{(ne)}_c\) of the pair of cubes \(cF^{(nw)}, cF^{(ne)}\). There are similar sets of four equations on three other pairs of cubes \((cF^{(nw)}, cF^{(sw)}), (cF^{(ne)}, cF^{(se)})\) and \((cF^{(sw)}, cF^{(se)})\), where each pair shares a face. As mentioned above, such equations are satisfied as a consequence of the CAFCC equations on the individual cubes. For example, the third equation of (2.23) is a consequence of the second type C equations:

\[
\begin{align*}
\tilde{C}(y^{(nw)}_d, x^{(nw)}_d, z^{(nw)}_n, z^{(ne)}_c, y^{(ne)}_n, y^{(ne)}_c; (\alpha_1, \gamma_1), (\gamma_1, \beta_2)) = 0, \\
\tilde{C}(y^{(ne)}_c, x^{(ne)}_c, z^{(ne)}_n, z^{(ne)}_c, y^{(ne)}_n; (\alpha_1, \gamma_2), (\gamma_1, \beta_1)) = 0,
\end{align*}
\] (2.24)

which are centred at the common vertex \(y^{(nw)}_d = y^{(ne)}_c\) of the pair of cubes \((cF^{(nw)}, cF^{(ne)})\). This may be seen by rewriting the above two equations into their four-leg forms (2.10) and then dividing the second equation by the first. Similarly, the other equations in (2.23) are consequences of two equations centred at the other three common vertices of the pair of cubes \((cF^{(nw)}, cF^{(ne)})\).

As another example, there are the following two equations:

\[
\begin{align*}
\tilde{C}(y^{(nw)}_d, y^{(se)}_n, y^{(sw)}_n, y^{(sw)}_c, y^{(ne)}_n, y^{(ne)}_c; \alpha, \beta) = 0, \\
\tilde{C}(x^{(nw)}_d, z^{(se)}_b, z^{(sw)}_b, z^{(sw)}_c, z^{(ne)}_b, z^{(ne)}_c; \alpha, \beta) = 0,
\end{align*}
\] (2.25)

which involve distinct variables of all four cubes in figure 5a. Once again these are satisfied as a consequence of CAFCC equations which are centred at a common vertex of the four cubes. For example, the first equation in (2.25) is a consequence of the two type C equations (2.24) and the two type A equations on \(cF^{(sw)}\) and \(cF^{(se)}\) which are each centred at the same vertex \(y^{(nw)}_d = y^{(ne)}_c = y^{(sw)}_c = y^{(se)}_a\).

This section considered only examples of CAFCC equations on multiple cubes that involve five different vertices; however, by using different arrangements of cubes there can also appear equations that involve from six to nine vertices (e.g. see the examples in figure 5). It is not clear what the equations involving six to nine vertices would be (although for systems of only type A four-leg equations these should be related to Adler’s discrete Toda equations on planar graphs [15]), but such equations could also turn out to possess nice properties associated with discrete integrability, which warrants further investigation.

3. Lax matrices

Using the original formulation of CAFCC [9], it has been shown how to derive the Lax matrices for type A and type B CAFCC equations [10]. This method relied on the fact that CAFCC was formulated with type A and type B equations centred at faces of the face-centred cube. Importantly, the new formulation of CAFCC given in §2 has type C equations centred at faces of the face-centred cube, which will allow for the derivation of their Lax matrices. Lax matrices derived from both type A and type C CAFCC equations from [10] will be required to construct the Lax pairs of type C equations, and for convenience the method used to obtain these Lax matrices is included below.
(i) Type A Lax matrix

Because of the symmetries (2.4)–(2.6), for the type A equation it does not matter which two variables are chosen for constructing the matrix equation. Here, the two variables $x_c$ and $x_d$ will be chosen to be written as

$$x_c = \frac{f}{g}$$  and  $$x_d = \frac{f_L}{g_L}.$$  \hspace{2cm} (3.1)

Then because of linearity in both $x_c$ and $x_d$, the equation $A(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0$ may be solved uniquely for the variable $x_d = \frac{f_L}{g_L}$ in the following form:

$$x_d = \frac{f_L}{g_L} = \frac{L_{11}(x; x_a, x_b; \alpha, \beta)f + L_{12}(x; x_a, x_b; \alpha, \beta)g}{L_{21}(x; x_a, x_b; \alpha, \beta)f + L_{22}(x; x_a, x_b; \alpha, \beta)g}. \hspace{2cm} (3.2)$$

The above equation can then be rewritten in a matrix form

$$\psi_L = L_A(x; x_a, x_b; \alpha, \beta)\psi,$$  \hspace{2cm} (3.3)

where

$$\psi = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \psi_L = \begin{pmatrix} f_L \\ g_L \end{pmatrix}$$  \hspace{2cm} (3.4)

and $L_A(x; x_a, x_b; \alpha, \beta)$ is the $2 \times 2$ matrix

$$L_A(x; x_a, x_b; \alpha, \beta) = D L_A \begin{pmatrix} L_{11}(x; x_a, x_b; \alpha, \beta) & L_{12}(x; x_a, x_b; \alpha, \beta) \\ L_{21}(x; x_a, x_b; \alpha, \beta) & L_{22}(x; x_a, x_b; \alpha, \beta) \end{pmatrix}. \hspace{2cm} (3.5)$$

The $D_{L_a}$ is an as yet unspecified normalization factor that arises from separating the numerator and denominator of (3.2). The matrix $L_A(x; x_a, x_b; \alpha, \beta)$ is one of the desired Lax matrices that will be used for the Lax pair of type C equations. Since it is obtained from a type A equation it will be referred to as a type A Lax matrix. The reinterpretation of the type A equation in figure 1c as a matrix equation for the vectors $\psi$ and $\psi_L$ is shown in figure 6. Note that, because of the symmetry (2.4), the matrix $L_A(x; x_a, x_b; \alpha, \beta)$ is proportional to the inverse of $L_A(x; x_a, x_b; \alpha, \beta)$.

(ii) Type C Lax matrix

Since they have less symmetry, the expression for the Lax matrix derived from type C equations will depend on which choices of variables are used to construct the matrix equation. Let the variables $x_a$ and $x_c$ be written as

$$x_a = \frac{f_L}{g_L}$$  and  $$x_c = \frac{f}{g}.$$  \hspace{2cm} (3.6)

Then because of linearity in both $x_a$ and $x_c$, the equation $C(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0$ may be solved uniquely for the variable $x_a = \frac{f_L}{g_L}$ in the following form:

$$x_a = \frac{f_L}{g_L} = \frac{L_{11}(x; x_b, x_d; \alpha, \beta)f + L_{12}(x; x_b, x_d; \alpha, \beta)g}{L_{21}(x; x_b, x_d; \alpha, \beta)f + L_{22}(x; x_b, x_d; \alpha, \beta)g}. \hspace{2cm} (3.7)$$
The other evolution equations \([10]\) may also be used in the Lax equation, and thus for convenience it will be denoted separately by \(\psi\).

The reinterpretation of the type C equation in figure 1 as a matrix equation for the Lax pair of type C equations. Since it is obtained from a type C equation it will be referred to as a type C Lax matrix (although it is not the only Lax matrix that can be obtained from type C equations). The compatibility equation for the Lax matrices (3.5) and (3.10) that reproduces a type C equation \((a)\) Lax compatibility equation for type C consistency-around-a-face-centred-cube.

The above equation can then be rewritten in a matrix form

\[
\psi_L = L_C(x; x_b, x_d; \alpha, \beta) \psi,
\]

where

\[
\psi = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \psi_L = \begin{pmatrix} f_L \\ g_L \end{pmatrix}
\]

and the \(L_C(x; x_b, x_d; \alpha, \beta)\) is a \(2 \times 2\) matrix given by

\[
L_C(x; x_b, x_d; \alpha, \beta) = D_L_C \begin{pmatrix} L_{11}(x; x_b, x_d; \alpha, \beta) & L_{12}(x; x_b, x_d; \alpha, \beta) \\ L_{21}(x; x_b, x_d; \alpha, \beta) & L_{22}(x; x_b, x_d; \alpha, \beta) \end{pmatrix}.
\]

The \(D_L_C\) is an as yet unspecified normalization factor that arises from separating the numerator and denominator of (3.7).

The type C equation from figure 1 is reinterpreted as equation (3.8) for the matrix (3.10). Both of these diagrams are equivalent, because of the symmetry (2.4).

\[
\text{Figure 7. The type C equation from figure 1 is reinterpreted as equation (3.8) for the matrix (3.10). Both of these diagrams are equivalent, because of the symmetry (2.4).}
\]

The compatibility equation for the Lax matrices (3.5) and (3.10) that reproduces a type C equation can be derived from the property of CAFCC given in \(\S2\). The main idea from \([10]\) is to take two different evolutions between two vertices at the centres of opposite faces of the face-centred cube. In terms of figure 3\(a, b\), the two type A equations centred at \(x_a\) and \(y_a\) will be used for one evolution \(x \to z_n \to y\), and the two type C equations centred at \(x_c\) and \(y_c\) will be used for the other evolution \(x \to z_s \to y\). The consistency of these two different evolutions then provides a compatibility equation for the type C equation centred at \(z_w\).

Specifically, the following four equations from (2.14)

\[
\begin{align*}
A(x_a; z_w, x_a, z_n; (\beta_1, \gamma_2), (\alpha_2, \gamma_1)) &= 0, & \overline{C}(x_a; z_s, y_c, x_a, z_n; (\alpha_1, \gamma_1), (\beta_1, \gamma_2)) &= 0, \\
A(y_a; z_w, y_c, z_n; (\beta_1, \gamma_1), (\alpha_2, \gamma_2)) &= 0, & \overline{C}(y_c; x_a, z_n, z_w, y_a; (\alpha_1, \gamma_2), (\gamma_1, \beta_1)) &= 0
\end{align*}
\]

(a) Lax compatibility equation for type C consistency-around-a-face-centred-cube equations

The compatibility equation for the Lax matrices (3.5) and (3.10) that reproduces a type C equation can be derived from the property of CAFCC given in \(\S2\). The main idea from \([10]\) is to take two different evolutions between two vertices at the centres of opposite faces of the face-centred cube. In terms of figure 3\(a, b\), the two type A equations centred at \(x_a\) and \(y_a\) will be used for one evolution \(x \to z_n \to y\), and the two type C equations centred at \(x_c\) and \(y_c\) will be used for the other evolution \(x \to z_s \to y\). The consistency of these two different evolutions then provides a compatibility equation for the type C equation centred at \(z_w\).
Figure 8. Type A and type C equations (3.12) centred at $x_a, y_a, x_c, y_c$ on the CAFCC cube of figure 3a,b are reinterpreted as four matrices $L_1, L_3, L_2, L_4$ in (3.13) using figures 6 and 7. The compatibility condition $\psi_{13} \doteq \psi_{24}$ implies $L_4 L_2 - L_3 L_1 \doteq 0$, where $\doteq$ indicates equality on solutions of equation (3.16) centred at $z_w$, which according to figure 1 is type C.

are used to define matrices for the transitions $x \to z_n$, $x \to z_s$, $z_n \to y$, $z_s \to y$, respectively. These matrices are given in terms of (3.5), (3.10) and (3.11) as

$$L_1 = L_A(x_a; z_w, y_a; (\beta_1, \gamma_2), (\alpha_2, \gamma_1)), \quad L_2 = L_C(x_c; y_c, z_w; (\alpha_1, \gamma_1), (\beta_1, \gamma_2)), \quad L_3 = L_A(y_a; x_a, z_w; (\beta_1, \gamma_1), (\gamma_2, \alpha_2)), \quad L_4 = \hat{L}_C(y_c; x_c, z_w; (\alpha_1, \gamma_2), (\beta_1, \gamma_1)).$$

(3.13)

The arrangement of equations (3.12) and associated matrices (3.13) is shown schematically in figure 8. Note that, according to figure 1, the equation that is centred at $z_w$ as a result of this arrangement is a type C equation.

Next denoting

$$\psi_{13} = L_3 \psi_1 = L_3 L_1 \psi \quad \text{and} \quad \psi_{24} = L_4 \psi_2 = L_4 L_2 \psi,$$

(3.14)

the compatibility condition is

$$\psi_{13} \doteq \psi_{24}.$$  

(3.15)

The $\doteq$ denotes equality on solutions of the equation centred at $z_w$ in figure 8, which is

$$C(z_w; y_a, x_a, y_c, x_c; \alpha, \gamma) = 0.$$  

(3.16)

This corresponds to an equation in (2.13) that appears on the face of the CAFCC cube in the top diagram of figure 3. The compatibility condition (3.15) is equivalently written in terms of Lax matrices (3.13) as

$$L_4 L_2 - L_3 L_1 \doteq 0.$$  

(3.17)

This is the desired equation which reinterprets the CAFCC property of the equations (3.12) as a compatibility condition for the Lax matrices (3.13) satisfied on the solution of a type C CAFCC equation. Analogously to CAC equations and their Lax pairs, $\beta_1$ is identified as the spectral parameter that comes from the additional third parameter associated with the CAFCC cube of figure 3a,b. A comparison of the Lax pairs arising from CAC and CAFCC is shown in figure 9.
Figure 9. Lax matrix equations for regular quad equations satisfying CAC (a), and face-centred quad equations satisfying CAFCC (b–d). In (a), the Lax compatibility equation would be satisfied on solutions of a quad equation \(Q(x'_a, x'_b, x'_c, x'_d; \alpha, \beta) = 0\), and \(\gamma\) would be identified as the spectral parameter. In (b), the Lax compatibility equation \((3.17)\) is satisfied on solutions of the type C equation \((3.16)\) (using the variable labelling of figure 8), and \(\beta_1\) is identified as the spectral parameter. In (c,d), the Lax equations are satisfied on solutions of type A and B equations \([10]\).

(i) Compatibility equation for \(C3(\delta_3; \delta_1; \delta_2)\)

The type A equation \(A3(\delta)\) is given in (A 1). The Lax matrix obtained from this equation may be written as \([10, \S 4.1.1]\)

\[
L_A(x; x_a, x_b; \alpha, \beta) = \frac{1}{D_{L_A}} \sum_{i=0}^{2} (L^{x^i} + \delta \Delta^{x^i}) x^i,
\]

where

\[
L_{x^2} = M(x_a, x_b), \quad L_{x^0} = x_a x_b PM\left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\right) P,
\]

\[
L_{x^1} = 4 \left( \begin{array}{cc}
\frac{\alpha_1}{\alpha_2} - \frac{\alpha_3}{\alpha_4} & \frac{\alpha_2}{\alpha_3} - \frac{\alpha_1}{\alpha_4} \frac{\beta_1}{\beta_2} - \frac{\beta_2}{\beta_1} \\
\frac{\alpha_1}{\alpha_2} & \frac{\alpha_2}{\alpha_1}
\end{array} \right) x_a + \left( \frac{\alpha_1}{\alpha_2} - \frac{\alpha_3}{\alpha_4} \right) x_b,
\]

\[
\Delta_{x^2} = 0, \quad \Delta_{x^0} = \left( \begin{array}{cc}
0 & \left( \frac{\alpha_1}{\alpha_2} - \frac{\alpha_3}{\alpha_4} \right) \left( \frac{\beta_1}{\beta_2} - \frac{\beta_2}{\beta_1} \right)
\end{array} \right) x_a + \left( \frac{\alpha_1}{\alpha_2} - \frac{\alpha_3}{\alpha_4} \right) x_b,
\]

\[
\Delta_{x^0} = \frac{(\alpha_1^2 - \beta_1^2)(\alpha_2^2 - \beta_2^2)(\alpha_3^2 - \beta_3^2)(\alpha_4^2 - \beta_4^2)}{(\alpha_1 \alpha_2 \beta_1 \beta_2)^2} \left( \begin{array}{cc}
\frac{\alpha_1 \alpha_2 \beta_1 \beta_2}{\alpha_1^2 - \alpha_2^2} & \frac{\alpha_1 \alpha_2 \beta_1 \beta_2}{\beta_1^2 - \beta_2^2}
\end{array} \right)
\]

\[
P = \left( \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} \right), \quad M(x_a, x_b) = 4 \left( \begin{array}{cc}
\frac{\beta_1}{\beta_2} - \frac{\beta_2}{\beta_1} & \left( \frac{\alpha_1}{\alpha_2} - \frac{\alpha_3}{\alpha_4} \right) x_a + \left( \frac{\beta_1}{\beta_2} - \frac{\beta_2}{\beta_1} \right) x_b
\end{array} \right) \left( \begin{array}{cc}
\frac{\beta_1}{\beta_2} - \frac{\beta_2}{\beta_1} & \left( \frac{\alpha_1}{\alpha_2} - \frac{\alpha_3}{\alpha_4} \right) x_a + \left( \frac{\beta_1}{\beta_2} - \frac{\beta_2}{\beta_1} \right) x_b
\end{array} \right). \]


The type C equation $C_3(\delta_1; \delta_2; \delta_3)$ is given in (A 3). The Lax matrix obtained from this equation may be written as [10, §4.2.1]

$$L_C(x; x_b, x_d; \alpha, \beta) = \frac{1}{D_{L_C}} \sum_{i=0}^{2} (L_x^i + \delta_1 \Delta_x) x^i,$$

(3.24)

where

$$L_{x^2} = \begin{pmatrix} -\beta_2 & \beta_1 x_d \\ 0 & 0 \end{pmatrix}, \quad L_{x^1} = -\begin{pmatrix} -\alpha_2 x_b & \frac{\beta_1 \beta_2}{\alpha_2} x_b x_d \cline{1-2} \beta_1 & -\alpha_2 x_d \end{pmatrix}, \quad L_x^0 = x_b \begin{pmatrix} 0 & 0 \\ \beta_1 & -\beta_2 x_d \end{pmatrix},$$

and

$$\Delta_{x^2} = \frac{2 \delta_3 \beta_1 \beta_2}{\alpha_1} \begin{pmatrix} 0 & (\frac{\beta_1}{\alpha_2} - \frac{\alpha_2}{\beta_1}) x_b \\ 0 & 0 \end{pmatrix}, \quad \Delta_{x^1} = \frac{2 \beta_1 \beta_2}{\alpha_1} \left( \begin{pmatrix} \frac{\beta_1}{\alpha_2} - \frac{\alpha_2}{\beta_1} \\ 0 \end{pmatrix} \right) \left( \begin{pmatrix} \delta_2 x_d \\ 0 \end{pmatrix} \right),$$

$$\Delta_x^0 = \left( \begin{pmatrix} \delta_2 \left( \frac{\beta_1}{\alpha_2} - \frac{\alpha_2}{\beta_1} \right) x_b \\ 2 \delta_2 \left( \frac{\beta_1}{\alpha_2} - \frac{\alpha_2}{\beta_1} \right) \right) \begin{pmatrix} \frac{\beta_1}{\alpha_2} - \frac{\alpha_2}{\beta_1} \\ \alpha_1 \left( \frac{\beta_1}{\alpha_2} - \frac{\alpha_2}{\beta_1} \right) \end{pmatrix}.$$

(3.25)

(3.26)

(3.27)

**Proposition 3.1.** For the case $\delta = 0$ of the Lax matrix (3.18) defined with (3.19)–(3.22), and the three cases $(\delta_1, \delta_2, \delta_3) = (0, 0, 0), (1, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0)$ of the Lax matrix (3.24) defined with (3.25)–(3.27), and the normalizations chosen as

$$D_{L_4} = 4(\alpha_1^2 - \beta_1^2)(\beta_1 x_a - \alpha_2 x_b)(\beta_2 x - \alpha_2 x_b)((\alpha_1 \alpha_2 \beta_1 \beta_2)^{-1},$$

$$D_{L_C} = (\alpha_2)^{-1},$$

the Lax equation (3.17) is satisfied on solutions of $C_3(\delta_1; \delta_2; \delta_3)(z; x_b, x_d; \alpha, \gamma) = 0.$

**Proof.** Using the definitions (3.13), the Lax equation of proposition 3.1 for the three cases $(\delta_1, \delta_2, \delta_3) = (0, 0, 0), (1, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0)$ may be written as

$$L_4 L_2 - L_3 L_1 = \frac{C_3(\delta_1; \delta_2; \delta_3)(z; x_b, x_d; \alpha, \gamma) \left( \begin{pmatrix} \gamma_1 \\ \beta_1^2 \right) \right) \otimes \left( \begin{pmatrix} -\gamma_2 \\ \beta_1 \beta_2 \end{pmatrix} \right)}{(\beta_1^2 - \gamma_2^2)(\gamma_2 x_a - \alpha_2 z_{\gamma})(\alpha_2 z_{\gamma} - \gamma_1 z_{\gamma})(\gamma_c - \delta_1(\gamma_1 z_{\gamma}) - \delta_2(\gamma_1 z_{\gamma}/\alpha_1)).}

(3.29)

(ii) Compatibility equation for $C_2(\delta_1; \delta_2; \delta_3)$ and $C_1(\delta_1)$

In the following, $\theta_{ij}$ are defined as differences of parameters as follows:

$$\theta_{ij} = \theta_i - \theta_j, \quad i, j \in \{1, 2, 3, 4\}, \quad (\theta_1, \theta_2, \theta_3, \theta_4) = (\alpha_1, \alpha_2, \beta_1, \beta_2).$$

(3.30)

The type A equation $A_2(\delta_1; \delta_2)$ is given in (A 2). The parameters have been denoted as $\delta_1$ and $\delta_2$ to avoid confusion with the $\delta_i$ parameters for the type C equation (which may take different values in the construction of the Lax equation). The Lax matrix obtained from this equation is
where

\[
\begin{align*}
(\Delta_{x^1})_{12} &= \theta_{12}\theta_{34}(\theta_{13} + \theta_{24})(x_a + x_b - \theta_{13}^2 - \theta_{24}^2 - \theta_{12}\theta_{34})\hat{\delta}_2 \\
&\quad - 2\hat{\delta}_2\theta_{13}\theta_{14}(x_a\theta_{24} + x_b\theta_{32}), \\
(\Delta_{x^0})_{12} &= \theta_{13}\theta_{14}(x_b\theta_{23}(\theta_{12}\theta_{43} - \theta_{13}^2)\hat{\delta}_2 - x_a\theta_{24}(\theta_{12}\theta_{34} - \theta_{14}^2)\hat{\delta}_2 \\
&\quad - \hat{\delta}_2\theta_{34}(x_a x_b - (\theta_{13} + \theta_{24})\theta_{12}\theta_{23}\theta_{24})) \\
\text{and} \\
(\Delta_{x^0})_{22} &= \theta_{13}\theta_{23}\theta_{24}(x_a + \theta_{12}\theta_{34} - \theta_{23}^2\hat{\delta}_2 + \hat{\delta}_2\theta_{13}(x_a\theta_{14}\theta_{24} - x_b\theta_{12}\theta_{23}).
\end{align*}
\]

The type C equation C2(δ,δ2,δ3) is given in (A 4). The Lax matrix obtained from this equation is (3.24) with [10, §4.2.2]

\[
L_{x^2} = \begin{pmatrix}
\theta_{31} & \theta_{23}x_a - \theta_{24}x_b \\
0 & \theta_{41}
\end{pmatrix}, \\
L_{x^1} = \begin{pmatrix}
x_a\theta_{12} + x_b(\theta_{13} + \theta_{24}) & \theta_{34}x_a x_b \\
\theta_{34} & x_b\theta_{12} + x_a(\theta_{13} + \theta_{24})
\end{pmatrix}, \\
L_{x^0} = x_a x_b \begin{pmatrix}
\frac{1}{x_a}\theta_{41} - \frac{1}{x_b}\theta_{24} & 0 \\
0 & \theta_{31}
\end{pmatrix}, \\
\Delta_{x^2} = \begin{pmatrix}
0 & \hat{\delta}_2\theta_{12}\theta_{34}(\theta_{13} + \theta_{24}) \\
0 & 0
\end{pmatrix}, \\
(\Delta_{x^1})_{12} = \hat{\delta}_2(2\theta_{13}\theta_{24}^2 + \theta_{12}\theta_{34}(\theta_{12} - \theta_{34})) \\
\Delta_{x^0} = \begin{pmatrix}
\theta_{14}\theta_{23}\theta_{24}(x_b - \theta_{12}\theta_{34} - \theta_{24}^2\hat{\delta}_2 - \hat{\delta}_2\theta_{14}(x_a\theta_{12}\theta_{24} - x_b\theta_{12}\theta_{23}) & (\Delta_{x^0})_{12} \\
\hat{\delta}_2\theta_{23}\theta_{24}\theta_{43} & (\Delta_{x^0})_{22}\end{pmatrix},
\]
where

\[
\begin{align*}
(\Delta x_{i})_{12} &= (\theta_{43} + \delta_{3}x_{b})(\theta_{43} + \theta_{41})^{2}(-1)^{\delta_{3}} + 2\delta_{3}\theta_{12}^{2} + (\theta_{11}^{1+\delta_{3}} + \theta_{11}^{1+\delta_{3}})(\theta_{32} + \theta_{42})^{2} \\
&+ 2\delta_{2}(\theta_{23} - x_{d}) + \theta_{24}(x_{b} - x_{d}x_{23})) - \delta_{3}x_{b}(x_{d} + 2\theta_{12}^{2} - (\theta_{23} + \theta_{24})), \\
(\Delta x_{i})_{22} &= 2(\delta_{2} + \delta_{3}x_{d} - 2\delta_{3}x_{b} - (\theta_{23} + \theta_{24})(\theta_{32} + \theta_{34})(\theta_{31} + \theta_{34} + 1)^{\delta_{3}}), \\
(\Delta x_{i})_{11} &= \theta_{32}(\theta_{24}(\theta_{34}x_{12} + \theta_{14} - 2x_{d}) + \theta_{24}^{2} - x_{b})^{2} + x_{b}(2x_{d} - (\theta_{31} + \theta_{34})^{2})^{2}x_{b}, \\
(\Delta x_{i})_{12} &= \theta_{32}(\theta_{24}(\theta_{23}\theta_{24} - 2\theta_{31}\theta_{14})^{2}(\theta_{31} + \theta_{34} - x_{d}(2\theta_{13}\theta_{43} + \theta_{23}\theta_{24})^{2}(-1)^{\delta_{3}}) \\
&+ x_{b}(\theta_{31} + \theta_{34})^{1+\delta_{3}} - 2\theta_{24}\theta_{34} - (\theta_{23} + \delta_{3})2\theta_{13}\theta_{14} + x_{d}(\theta_{21} + \theta_{31})^{2}(-1)^{\delta_{3}}), \\
(\Delta x_{i})_{22} &= \theta_{24}(-\theta_{31}^{1+\delta_{3}} + \theta_{31}^{1+\delta_{3}}) - x_{d}(2\theta_{31} - \theta_{24})^{2}(-1)^{\delta_{3}} - \delta_{2}\theta_{23}\theta_{34})x_{b} - (2\delta_{2} + \theta_{23})x_{d} - \delta_{3}(x_{d} - \theta_{34}(\theta_{31} + \theta_{14}))x_{b}.
\end{align*}
\]

(3.40)

**Proposition 3.2.** For the \((\delta_{1}, \delta_{2}) = (1, 0)\) case of the Lax matrix (3.18) defined with (3.31)–(3.34), the three cases \((\delta_{1}, \delta_{2}, \delta_{3}) = (0, 0, 0), (1, 0, 0), (1, 1, 0)\) of the Lax matrix (3.24) defined with (3.36)–(3.39) and the normalizations chosen as

\[
\begin{align*}
D_{L_{A}} &= i(\alpha_{1} - \beta_{1})(\beta_{1} + x_{a} - \alpha_{2} - x)(\beta_{2} + x - \alpha_{2} - x_{b}) \\
D_{L_{C}} &= 1,
\end{align*}
\]

(3.41)

and the Lax equation (3.17) is satisfied on solutions of \(C_{2}(\delta_{1}; \delta_{2}; \delta_{3}) (z_{w}, y_{a}, x_{a}, y_{c}, x_{c}; \alpha, \gamma) = 0\) if \((\delta_{1}, \delta_{2}, \delta_{3}) = (1, 0, 0)\) or \((1, 1, 0)\) and on solutions of \(C_{1}(z_{w}, y_{a}, x_{a}, y_{c}, x_{c}; \alpha, \gamma) = 0\) if \((\delta_{1}, \delta_{2}, \delta_{3}) = (0, 0, 0)\).

**Proof.** Using the definitions (3.13), the Lax equation of proposition 3.2 for the three cases \((\delta_{1}, \delta_{2}, \delta_{3}) = (0, 0, 0), (1, 0, 0), (1, 1, 0)\) may be written as

\[
L_{4}L_{2} - L_{3}L_{1} = \frac{(-C/2(\beta_{1} - \gamma_{2}))}{(\gamma_{2} + x_{d} - \alpha_{2} - z_{w})(\gamma_{2} + y_{a} - \gamma_{1} - z_{w})^{(\alpha_{1} - \gamma_{1} - z_{w})^{1+\delta_{2}}}},
\]

(3.42)

where

\[
C = \begin{cases} 
C_{2}(\delta_{1}; \delta_{2}; \delta_{3})(z_{w}, y_{a}, x_{a}, y_{c}, x_{c}; \alpha, \gamma), & (\delta_{1}, \delta_{2}, \delta_{3}) = (1, 0, 0) \text{ or } (1, 1, 0), \\
-C_{1}(z_{w}, y_{a}, x_{a}, y_{c}, x_{c}; \alpha, \gamma), & (\delta_{1}, \delta_{2}, \delta_{3}) = (0, 0, 0).
\end{cases}
\]

(3.43)

\[\text{(iii) Compatibility equation for } C_{2}(0; 0; 0) \text{ and } C_{1}(0)\]

(iii) Compatibility equation for \(C_{2}(0; 0; 0)\) and \(C_{1}(0)\)

The Lax matrix obtained from the type C equation \(C_{1}(\delta)\) given in (A5) is (this matrix is slightly different from [10, §4.2.3] because of a slightly different expression for \(C_{1}(\delta=0)\) given in (A5), while the \(\delta = 1\) case is new)

\[
L_{C} = \frac{(-1)^{\delta}}{D_{L_{C}}} \begin{pmatrix} 
(\alpha_{2} - \beta_{1})(x_{c} - \delta_{3}x_{b}) & L_{12} \\
(\alpha_{2} - \beta_{1})(x_{c} - \delta_{3}x_{b}) & 2(\alpha_{2} - \beta_{2})(-\frac{z_{w}}{2})^{3} + (x_{b} - x_{d})x_{d}
\end{pmatrix},
\]

(3.44)

where

\[
L_{12} = 2((\beta_{1} - \beta_{2})x + (\alpha_{2} - \beta_{1})x_{b})(-\frac{z_{w}}{2})^{3} + (\delta(\alpha_{2} - \beta_{1})(\alpha_{2} - \beta_{2}) - x(x - x_{b}))x_{d}.
\]

(3.45)
Proposition 3.3. For the $(\hat{\delta}_1, \hat{\delta}_2) = (0, 0)$ case of the Lax matrix (3.18) defined with (3.31)–(3.34), the Lax matrix (3.44) and with the normalizations chosen as

\[ DL_A = (\alpha_1 - \beta_1)(x - x_a)(x - x_b) \]
\[ DL_C = x - x_b, \]

the Lax equation (3.17) is satisfied on solutions of $C_2(0;0;0)(z_w; y_d, x_a, y_c; x_c; \alpha, \gamma) = 0$ if $\delta = 1$ in (3.44) and on solutions of $C_1(0)(z_w; y_d, x_a, y_c; x_c; \alpha, \gamma) = 0$ if $\delta = 0$ in (3.44).

Proof. Using the definitions (3.13), the Lax equation of proposition 3.3 may be written as

\[ L_4 L_2 - L_3 L_1 = \tilde{C} \left( \frac{1}{2(\beta_1 - \gamma_2)(x_a - z_w)(y_d - z_w)} \begin{pmatrix} z_w \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -z_w \end{pmatrix} \right), \]

where

\[ \tilde{C} = \begin{cases} C_1(0)(z_w; y_d, x_a, y_c; x_c; \alpha, \gamma), & \delta = 0, \\ \frac{1}{z_w} C_2(0;0;0)(z_w; y_d, x_a, y_c; x_c; \alpha, \gamma), & \delta = 1. \end{cases} \]

4. Conclusion

CAFCC has recently been introduced as a new form of multidimensional consistency for integrability of systems of five-point face-centred quad equations in the square lattice [9]. This paper introduces a new formulation of CAFCC which involves so-called type C equations that are centred at faces of the face-centred cube, whereas previously they were only centred at corners. This is essential for consistently extending systems of type C equations into higher dimensional lattices as well as for deriving their Lax pairs, as has been done in §3. The establishing of the Lax pairs and multidimensional consistency of type C equations in this paper, along with the previous establishing of their vanishing algebraic entropy [11], shows that systems of type C CAFCC equations (as well as type A and B equations) satisfy the important properties that are associated with integrability of lattice equations.

An important problem will be to determine whether there are other equations that satisfy CAFCC, for which the results here (and [10]) would imply their Lax pairs. This might be done by establishing a classification result similar to [2], which has not yet been explored for multilinear face-centred quad equations of the form (2.3). It is also worth investigating potential integrability of the different equations that arise on adjacent cubes in higher dimensional lattices, as illustrated in §2b. Besides providing new examples of integrable systems, this could also potentially lead to other interesting forms of consistency connected to some different forms of the Yang–Baxter equation. It would also be interesting if there is some relation between CAFCC and the recently proposed consistency-around-a-cuboctahedron property [17], which also involves some face-centred cubic structure, but applies to regular quad equations rather than face-centred quad equations. This could also inspire new reductions to discrete Painlevé equations [18].

Data accessibility. Mathematica code is available as electronic supplementary material [19].

Conflict of interest declaration. I declare I have no competing interests.

Funding. I received no funding for this study.

Acknowledgements. The author would like to thank Giorgio Gubbiotti for fruitful discussions, and also the anonymous referees for several helpful suggestions that led to improvement of the manuscript.
Appendix A. Type A and type C consistency-around-a-face-centred-cube equations

Here, the definition (3.30) is used, i.e. $\theta_{ij} = \theta_i - \theta_j$, where $(\theta_1, \theta_2, \theta_3, \theta_4) = (\alpha_1, \alpha_2, \beta_1, \beta_2)$.

$$A3(\delta) = \frac{\delta}{4} \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \left( \frac{\beta_1 - \beta_2}{\beta_1} \right) \left( \frac{\beta_1 \beta_2}{\alpha_1 \beta_2} \right) x$$

$$+ \left( \frac{\delta}{4} \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \left( \frac{\beta_1 - \beta_2}{\beta_1} \right) \left( \frac{\beta_1 \beta_2}{\alpha_1 \beta_2} \right) \right) x$$

$$+ \left( \frac{\delta}{4} \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \left( \frac{\beta_1 - \beta_2}{\beta_1} \right) \left( \frac{\beta_1 \beta_2}{\alpha_1 \beta_2} \right) \right) x$$

$$+ \left( \frac{\delta}{4} \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \left( \frac{\beta_1 - \beta_2}{\beta_1} \right) \left( \frac{\beta_1 \beta_2}{\alpha_1 \beta_2} \right) \right) x$$

$$= 0.$$  \hspace{1cm} (A 1)

$$A2(\theta_1; \theta_2; \theta_3; \theta_4) = \left( \theta_1 + \theta_2 \right) x_1 x_2 x_3 x_4 + \theta_2 x_1 x_2 x_3 x_4$$

$$+ \theta_3 x_1 x_2 x_3 x_4 + \theta_4 x_1 x_2 x_3 x_4 \right) x$$

$$+ \theta_2 \left( x_1 x_2 x_3 x_4 + \theta_4 \right) x$$

$$+ \theta_3 \left( x_1 x_2 x_3 x_4 + \theta_4 \right) x$$

$$+ \theta_4 \left( x_1 x_2 x_3 x_4 + \theta_4 \right) x = 0.$$  \hspace{1cm} (A 2)

$$C2(\theta_1; \theta_2; \theta_3; \theta_4) = \left( \alpha_2 \left( \beta_1 x_2 - \beta_2 x_4 \right) - \delta_3 \left( \alpha_2 \left( \beta_1 x_2 - \beta_2 x_4 \right) + \beta_1 \beta_2 \left( \beta_1 x_2 - \beta_2 x_4 \right) \right) \right) \alpha_1^{-1} x^2$$

$$+ \alpha_2 x_1 x_3 \left( \beta_1 x_2 - \beta_2 x_4 \right) + \delta_1 \alpha_1 \left( \beta_1 x_2 - \beta_2 x_4 + \alpha_2 \left( x_1 x_3 - \delta_4 \right) \right) \left( \alpha_2 \left( x_1 x_3 - \delta_4 \right) \right)$$

$$+ \beta_1 \beta_2 \left( x_1 x_3 - \delta_4 \right) \left( \alpha_2 \left( x_1 x_3 - \delta_4 \right) \right) \left( \alpha_2 \left( x_1 x_3 - \delta_4 \right) \right)$$

$$+ \delta_2 \left( \alpha_2 \left( x_1 x_3 - \delta_4 \right) \right) \left( \alpha_2 \left( x_1 x_3 - \delta_4 \right) \right) \left( \alpha_2 \left( x_1 x_3 - \delta_4 \right) \right)$$

$$= 0.$$  \hspace{1cm} (A 3)

$$C1(\delta) = \left( \frac{\delta}{4} \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \left( \frac{\beta_1 - \beta_2}{\beta_1} \right) \left( \frac{\beta_1 \beta_2}{\alpha_1 \beta_2} \right) \right) x$$

$$+ \left( \frac{\delta}{4} \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \left( \frac{\beta_1 - \beta_2}{\beta_1} \right) \left( \frac{\beta_1 \beta_2}{\alpha_1 \beta_2} \right) \right) x$$

$$+ \left( \frac{\delta}{4} \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \left( \frac{\beta_1 - \beta_2}{\beta_1} \right) \left( \frac{\beta_1 \beta_2}{\alpha_1 \beta_2} \right) \right) x$$

$$+ \left( \frac{\delta}{4} \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \left( \frac{\beta_1 - \beta_2}{\beta_1} \right) \left( \frac{\beta_1 \beta_2}{\alpha_1 \beta_2} \right) \right) x$$

$$= 0.$$  \hspace{1cm} (A 4)

(a) Four-leg expressions

In table 2, the abbreviation $add_0$ indicates an additive form of one of equations (2.8) or (2.10), given, respectively, by

$$a(x; x_a; \alpha_1, \alpha_2, \beta_1) - a(x_a; x_a; \alpha_1, \alpha_2, \beta_1) - a(x_a; x_a; \alpha_1, \alpha_2, \beta_1) = 0.$$  \hspace{1cm} (A 6)

and

$$a(x; x_a; \alpha_1, \alpha_2, \beta_1) + c(x; x_a; \alpha_1, \alpha_2, \beta_1) - a(x_a; x_a; \alpha_1, \alpha_2, \beta_1) = 0.$$  \hspace{1cm} (A 7)
is represented by the difference of rapidity variables \( \alpha \).

Table 2. Left: A list of \( a(x; y; \alpha, \beta) \) in (2.8) for type A equations (A 1) and (A 2). Right: A list of \( c(x; y; \alpha, \beta) \) in (2.10) for type C equations (A 3), (A 4) and (A 5). For \( C_{3}(\delta_{1}; \delta_{2}; \delta_{3}) \), \( C_{2}(\delta_{1}; \delta_{2}) \) and \( C(\delta_{1}) \), the \( a(x; y; \alpha, \beta) \) are, respectively, given by \( A_{3}(\delta_{1}; \delta_{2}; \delta_{3}) \) and \( A_{2}(\delta_{1}; \delta_{2}) \) \( A_{2}(\delta_{1}; \delta_{2}) \). Here \( x \equiv x + \sqrt{x^2 - 1} \).

| type A | \( a(x; y; \alpha, \beta) \) | type C | \( c(x; y; \alpha, \beta) \) |
|--------|-----------------|--------|-----------------|
| \( A_{3}(\delta_{1}; \delta_{2}; \delta_{3}) \) | \( (\alpha^2 + \beta^2 x^2 - 2\alpha\beta xy)/(\beta^2 + \alpha^2 x^2 - 2\alpha\beta xy) \) | \( C_{3}(\delta_{1}; \delta_{2}; \delta_{3}; \delta_{4}; \delta_{5}; \delta_{6}; \delta_{7}; \delta_{8}) \) | \( (\alpha - \beta xy)/(\alpha x - \beta y) \) |
| \( A_{3}(\delta_{1}; \delta_{2}; \delta_{3}) \) | \( (\beta x - \alpha y)/(\alpha x - \beta y) \) | \( C_{3}(\delta_{1}; \delta_{2}; \delta_{3}; \delta_{4}; \delta_{5}; \delta_{6}; \delta_{7}; \delta_{8}) \) | \( (\alpha^2/\beta) + \beta x^2 - 2\alpha xy \) |
| \( A_{2}(\delta_{1}; \delta_{2}) \) | \( \left( \pm \sqrt{x + \alpha - \beta \pm \beta} \right)^2 - y)/(\pm \sqrt{x - \alpha + \beta \pm \beta} \left)^2 - y \) | \( C_{2}(\delta_{1}; \delta_{2}; \delta_{3}; \delta_{4}; \delta_{5}; \delta_{6}; \delta_{7}; \delta_{8}) \) | \( (\mp \sqrt{x + y - \alpha + \beta} \mp \beta)/(\mp \sqrt{x + y + \alpha - \beta} \mp \beta) \) |
| \( A_{2}(\delta_{1}; \delta_{2}) \) | \( -x + y + \alpha - \beta)/(x - y + \alpha - \beta) \) | \( C_{2}(\delta_{1}; \delta_{2}; \delta_{3}; \delta_{4}; \delta_{5}; \delta_{6}; \delta_{7}; \delta_{8}) \) | \( x + y - \alpha + \beta \) |
| \( A_{2}(\delta_{1}; \delta_{2}) \) | \( (\alpha - \beta)/(x - y) \) | \( C_{1}(\delta_{1}; \delta_{2}; \delta_{3}; \delta_{4}; \delta_{5}; \delta_{6}; \delta_{7}; \delta_{8}) \) | \( y \) |

Appendix B. Classical Yang–Baxter equation and consistency-around-a-face-centred-cube

The original formulation of CAFCC was derived from new types of IRF forms of the classical Yang–Baxter equation (CYBE), and explicit solutions of the latter were constructed from solutions of the classical star–triangle relations. For the new formulation of CAFCC given in §2a, there is a similar connection to a different form of the CYBE.

The relevant CYBE may be defined in terms of six complex-valued functions

\[
\begin{align*}
\mathcal{L}_{\alpha}(x_{i}, x_{j}), & \quad \hat{\mathcal{L}}_{\alpha}(x_{i}, x_{j}), & \quad \mathcal{T}_{\alpha}(x_{i}, x_{j}), & \quad \hat{\mathcal{T}}_{\alpha}(x_{i}, x_{j}), & \quad A_{\alpha}(x_{i}, x_{j}), & \quad \bar{A}_{\alpha}(x_{i}, x_{j}), & \quad B(\alpha, x_{i}, x_{j}), & \quad C(\alpha, x_{i}, x_{j}),
\end{align*}
\]

which each depend on two complex variables \( x_{i}, x_{j} \) and a complex parameter \( \alpha \). These six functions are associated with the edges and vertices shown in figure 10, where the parameter \( \alpha \) is represented by the difference of rapidity variables \( u \) and \( v \) associated with directed edges. It is assumed that the partial derivatives also satisfy the following relations:

\[
\frac{\partial D_{\alpha}(x_{i}, x_{j})}{\partial x_{i}} = \frac{\partial D_{\alpha}(x_{i}, x_{j})}{\partial x_{i}} \quad \text{(mod 2\pi i)},
\]

\[
\frac{\partial D_{\alpha}(x_{j}, x_{i})}{\partial x_{j}} = -\frac{\partial D_{\alpha}(x_{j}, x_{i})}{\partial x_{j}} \quad \text{(mod 2\pi i)},
\]

where \( D \) represents one of \( \mathcal{L}, \mathcal{T}, \hat{\mathcal{L}}, \hat{\mathcal{T}} \) from (B 1), and

\[
\frac{\partial F_{\alpha}(x_{i}, x_{j})}{\partial x_{i}} = \frac{\partial F_{\alpha}(x_{i}, x_{j})}{\partial x_{i}} \quad \text{(mod 2\pi i)},
\]

\[
\frac{\partial G_{\alpha}(x_{i}, x_{j})}{\partial x_{j}} = -\frac{\partial G_{\alpha}(x_{i}, x_{j})}{\partial x_{j}} \quad \text{(mod 2\pi i)},
\]

where \( \eta \) is some constant, \( F \) is one of \( \mathcal{L}, \mathcal{T}, \hat{\mathcal{L}}, \hat{\mathcal{T}}, \mathcal{A}, \bar{\mathcal{A}} \) from (B 1) and \( (G, H) \) represents one of the three pairs \( (\mathcal{L}, \mathcal{T}), (\hat{\mathcal{L}}, \hat{\mathcal{T}}), (\mathcal{A}, \bar{\mathcal{A}}) \). The relations (B 2) and (B 3) reduce the number of different types of equations that need to be considered for CAFCC [9].

Let \( B_{L} \) and \( B_{R} \) denote the graphs that appear on the left- and right-hand sides of the equality of figure 11, with respective sets of vertices \( V(B_{L}) = \{a, b, c, d, e, f, h, i, j, k\} \) and \( V(B_{R}) = \{a, b, c, d, e, f, l, m, n, o\} \), and sets of edges \( E(B_{L}) \) and \( E(B_{R}) \) which connect two vertices. Then the
Functions associated with edges for solutions of the classical Yang–Baxter equation.

Figure 10. Functions associated with edges for solutions of the classical Yang–Baxter equation.

Figure 11. Expression for the classical Yang–Baxter equation corresponding to (B 4) and (B 5).

CYBE is defined by

\[ \sum_{(ij) \in E(B_k)} K_{aij}(x_i, x_j) - \sum_{(ij) \in E(B_k)} K_{aij}(x_i, x_j) = 2\pi i \sum_{i \in V(B_L) \cup V(B_R)} k_i x_i + C(\alpha_{ij}), \]  

for some integers \( k_i \in \mathbb{Z} \), where \( C(\alpha_{ij}) \) is a constant with respect to the variables \( x_i \) (\( i \in V(B_L) \cup V(B_R) \)), \( K_{aij}(x_i, x_j) \) is the function associated with an edge \( (ij) \) according to the assignment of figure 10, and the variables \( x_i \) (\( i \in V(B_L) \cup V(B_R) \)) are subject to the following eight constraints:

\[ \frac{\partial}{\partial x_j} \sum_{(ij) \in E(B_k)} K_{aij}(x_i, x_j) = 2\pi i k_j, \quad j \in \{ h, i, j, k, l, m, n, o \}. \]

The CAFCC property may be thought of as a reinterpretation of the classical Yang–Baxter equation of (B 4). First, there are a further six partial derivatives of the CYBE in addition to (B 5), which are given by the equations

\[ \frac{\partial}{\partial x_j} \left( \sum_{(ij) \in E(B_k)} K_{aij}(x_i, x_j) - \sum_{(ij) \in E(B_k)} K_{aij}(x_i, x_j) \right) = 2\pi i k_j, \quad j \in V(B_L) \cap V(B_R). \]

Introduce the following four functions defined in terms of (B 1):

\[ a(x; x_a, x_b, x_c, x_d; \alpha, \beta) = e^{\frac{\alpha}{\gamma}} \left( \mathcal{L}_{x_a \rightarrow x_b}(y, y) + \mathcal{Z}_{x_a \rightarrow x_c}(y, y) + \mathcal{Z}_{x_a \rightarrow x_d}(y, y) + \mathcal{L}_{x_a \rightarrow x_b}(y, y) \right), \]

\[ \bar{a}(x; x_a, x_b, x_c, x_d; \alpha, \beta) = e^{\frac{\alpha}{\gamma}} \left( \mathcal{L}^*_{x_a \rightarrow x_b}(y, y) + \mathcal{Z}^*_{x_a \rightarrow x_c}(y, y) + \mathcal{Z}^*_{x_a \rightarrow x_d}(y, y) + \mathcal{L}^*_{x_a \rightarrow x_b}(y, y) \right), \]

\[ c(x; x_a, x_b, x_c, x_d; \alpha, \beta) = e^{\frac{\alpha}{\gamma}} \left( \mathcal{L}_{x_a \rightarrow x_b}(y, y) + \mathcal{Z}_{x_a \rightarrow x_c}(y, y) + \mathcal{Z}_{x_a \rightarrow x_d}(y, y) + \mathcal{L}_{x_a \rightarrow x_b}(y, y) \right), \]

and

\[ \bar{c}(x; x_a, x_b, x_c, x_d; \alpha, \beta) = e^{\frac{\alpha}{\gamma}} \left( \mathcal{L}^*_{x_a \rightarrow x_b}(y, y) + \mathcal{Z}^*_{x_a \rightarrow x_c}(y, y) + \mathcal{Z}^*_{x_a \rightarrow x_d}(y, y) + \mathcal{L}^*_{x_a \rightarrow x_b}(y, y) \right). \]
The variables on the left- and right-hand sides of (B 7) for \( a \) and \( \bar{a} \) are related by
\[
x = f(y), \quad x_i = f(y_i), \ i \in \{a, b, c, d\},
\]
and the variables on the left- and right-hand sides for \( c \) and \( \bar{c} \) are related by
\[
x = f(y), \quad x_i = \begin{cases} f(y_i), & i \in \{a, b\}, \\ g(y_i), & i \in \{c, d\}, \end{cases}
\]
where \( f(y) \) and \( g(y) \) are chosen so that each of \( a, \bar{a}, c, \bar{c} \) are ratios of multilinear polynomials in the four variables \( x_a, x_b, x_c, x_d \). The relation between parameters in (B 7) is
\[
\alpha_1 = h(u_1), \quad \alpha_2 = h(u_2), \quad \beta_1 = h(v_1), \quad \beta_2 = h(v_2),
\]
where \( h(z) \) is chosen so that each of \( a, \bar{a}, c, \bar{c} \) has an algebraic dependence on \( \alpha_1, \alpha_2, \beta_1, \beta_2 \). Because of the above choices of \( f(y) \) and \( g(y) \), the four equations
\[
a(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 1, \quad \bar{a}(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 1,
\]
\[
c(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 1, \quad \bar{c}(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 1,
\]
can, respectively, be written in the equivalent forms
\[
A(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0, \quad \bar{A}(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0,
\]
\[
C(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0, \quad \bar{C}(x; x_a, x_b, x_c, x_d; \alpha, \beta) = 0,
\]
where each of \( A, \bar{A}, C, \bar{C} \) are multilinear polynomials of the form (2.3). Then analogously to what was found for the CYBE for the original formulation of CAFCC [9], if the six functions (B 1) are a solution of the CYBE (B 4) and (B 5), the four equations (B 12) will satisfy the property of CAFCC given in §2. This follows from the fact that each of the 14 partial derivatives of the CYBE (B 5) and (B 6) may be identified (mod 2 \( \pi i \)) and up to the change of variables (B 8)–(B 10)) with the 14 equations (2.13) and (2.14) of CAFCC, using the bijection between the vertices of figure 11 and figure 3a,b, given by
\[
(x, x_a, x_b, x_c, x_d) \mapsto (k, h, d, b, c), \quad (y, y_a, y_b, y_c, y_d) \mapsto (a, f, e, a, l), \quad (z_m, z_e, z_s, z_w) \mapsto (i, m, n, j).
\]

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