Group properties and invariant solutions of a sixth-order thin film equation in viscous fluid

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(Dated: February 24, 2012)

Using group theoretical methods, we analyze the generalization of a one-dimensional sixth-order thin film equation which arises in considering the motion of a thin film of viscous fluid driven by an overlying elastic plate. The most general Lie group classification of point symmetries, its Lie algebra, and the equivalence group are obtained. Similar reductions are performed and invariant solutions are constructed. It is found that some similarity solutions are of great physical interest such as sink and source solutions, travelling-wave solutions, waiting-time solutions, and blow-up solutions.

PACS numbers: 47.15.G-, 02.20.Sv, 02.30.Jr

I. INTRODUCTION

In the past several decades there is an increasing interest in physics and mathematics literatures in higher-order nonlinear diffusion equations because they are models of various interesting phenomena in fluid physics and have surprising mathematical structure and properties. Probably, one of the most famous examples is the fourth-order thin film equation in the form

\[ u_t = (u^m u_{xxxx})_x, \quad \alpha > 0 \]  

(1)

which was first introduced by Greenspan in 1978 [1]. This equation describes the surface-tension-dominated motion of thin viscous films for the film height \( u(t, x) \) and spreading droplets in the lubrication approximation [1]. In particular, for \( \alpha = 3 \) it describes a classical thin film of Newtonian fluid, as reviewed in [2], \( \alpha = 1 \) occurs in the dynamics of a Hele-Shaw cell [3] and \( \alpha = 2 \) arises in a study of wetting films with a free contact line between film and substrate [4]. There also exist many interesting generalizations of the famous equation (1) (see [2,6,10] and reference therein).

Apart from the fourth-order equations, another interesting higher-order diffusion model is the sixth-order nonlinear thin film equation in the form

\[ u_t = (u^m u_{xxxxx})_x, \]  

(2)

which appear in flow modeling. The case \( m = 3 \), for instance, was first introduced by King in [7] as a model of the oxidation of silicon in semiconductor devices [8] or for a moving boundary given by a beam of negligible mass on a surface of a thin film [9]. Here \( u \geq 0 \) will be treated as the thickness of a fluid film beneath an elastic plate and \( p = u_{xxx} \) as the pressure within the film [7]. The other derivatives of \( u \) can in the usual way be assigned different physical meaning, for instance \( \Gamma = -u_{xx} \) is the bending moment on the overlying plate and \( \Sigma = u_{xxxx} \) is the shearing force [10], here all such expressions are dimensionless. An equation of this type can be used to model the motion of a thin film of viscous fluid overlain by an elastic plate [8]; see also Hobart et al. [11] and Huang et al. [12] for possible applications of such modelling approaches to the wrinkling upon annealing of SiGe films bonded to Si substrates. Other plausible applications of Eq. (2), and suitable generalizations thereof, include a simple model for the influence of a crust on a solidifying melt or for a microfluidic pump (see Koch et al. [13], for instance).

Eqs. (1) and (2) are also the second and the third member of a hierarchy arising from the generalized Reynolds equation

\[ u_t = (u^m p_x)_x, \]  

(3)

under different driving forces respectively. For gravity driven flows, we have \( p = u \), giving the very widely studied porous-medium equation (see, for example, Aronson [14]). For surface-tension driven flows we have \( p = -u_{xx} \), leading to the fourth-order thin film equation (1). For elastic plate driven flows, we have \( p = u_{xxx} \), which give the sixth-order thin film equation (2) [8].

Up to now, the mathematical structure and properties of the fourth-order thin film equation (1) have been widely investigated, including (non-)uniqueness, wetting behaviour and contact line motion, in particular optimal propagation rates, waiting time or dead core phenomena and self-similar solutions (see Hulshof [15], for instance). Recent years, there are also many researches devoted to symmetry group structure and exact solutions of the fourth-order thin film equations (1) and their generalizations [16–25], or searching for special invariant finite vector spaces of solutions [26].

However, the sixth-order thin film equation (2) has been much less extensively investigated. It was only a few
researches that were devoted to qualitative mathematic properties such as the existence of weak solutions, initial boundary value problems (see Bernis Friedman [27], King et al. [28], Smith et al. [29], Evans et al. [30], Barrett et al. [31] for existing studies, the first two being analytical and the others primarily numerical), while the symmetry group properties and corresponding algebraic structure as well as explicit exact solutions of Eq. (2) still remain open. Therefore, the aim of the present work is to find such group properties, algebraic structure and exact solutions. To do this, we investigate alternatively a more general sixth-order nonlinear diffusion equation in the form

\[ u_t = (f(u) u_{xxxxx})_x, \]

(4)

than the original equation (2), where \( f(u) \) is an arbitrary smooth function depending on the geometry of the problem and \( f_u \neq 0 \) (i.e., (4) is a nonlinear equation).

We use the method of Lie groups, one of the powerful tools available to solve nonlinear PDEs, and which was discovered and applied firstly by S. Lie in the nineteenth century, but only in the last decades has it become a common tool for both mathematicians and physicists (see for examples [32–41]). The method consists of looking for the infinitesimal generators of a group of point transformations which leave the equation under study invariant. An important point of the Lie theory is that the conditions for an equation to admit a group of transformations are represented by a set of linear equations, the so-called "determining equations", which are usually completely solvable. Having once found the groups of transformations, one can obtain a number of interesting results, which include the possibility to reduce a partial differential equation with two independent variables to an ordinary differential equation with one independent variable, etc.. Solving these reduced equations, one can obtain some particular solutions for the original equations. These particular solutions are usually called "similarity solutions" or "invariant solutions" [32–35]. When the equation contains "arbitrary elements" (a variable coefficient deriving from the particular equation of state chosen to characterize the physical mechanism. "Arbitrary elements" are functions or variable parameters whose form is not strictly fixed and can be assigned freely on the grounds of physical hypotheses about the nature of the medium under consideration.), the theory gives rise to the problem of group classification of differential equations which is the core stone of modern group analysis [33, 35]. In particular, in the past several years, a numbers of novel techniques, such as algebraic methods based on subgroup analysis of the equivalence group [42, 45], compatibility and direct integration [33, 46] (also referred as the Lie-Ovsiannikov method) as well as their generalizations (eg. method of furcate split [47], additional and conditional equivalence transformations [48, 49], extended and generalized equivalence transformation group, gauging of arbitrary elements by equivalence transformations [50, 51]) have been proposed to solve group classification problem for numerous nonlinear partial differential equations. Although a great deal of classification was solved by these methods, almost all of them are limited to the equations whose order are lower than four (see [51] for details).

In this paper we extend these new techniques, specific compatibility and direct integration as well as equivalence transformation techniques, to sixth-order nonlinear diffusion equations. We first carry out group classification of Eq. (4) under the usual equivalence group. The Lie group of point symmetries of Eq. (4), as a special case of Eq. (4), and its Lie algebra are also obtained. Then similar reductions of the classification models are performed and invariant solutions are also constructed. It is found that some similarity solutions are solutions with physical interest: sink and source solutions, travelling-wave solutions, waiting-time solutions and blow-up solutions.

The rest of this paper is organized as follows: In Sec. II we derive the equivalence group and perform the group classification related to Eq. (4). In Sec. III similar reductions of classification models are carried out. Sec. IV contains examples of some specific exact solutions, including sink and source solutions, travelling-wave solutions, waiting-time solutions and blow-up solutions, while in Sec. V some concluding remarks are reported.

II. SYMMETRY CLASSIFICATION

Background and procedures of the modern Lie group theory are well described in literature [32, 53, 48, 49]. Without going into the details of the theory, we present only the results below.

Let

\[ Q = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \phi(t, x, u) \partial_u \]

be a vector field or infinitesimal operator on the space of independent and dependent variables \( t, x, u \). A local group of transformations \( G \) is a symmetry group of Eq. (4) if and only if

\[ \text{pr}^{(6)}Q(\Delta) = 0, \]

(5)

whenever \( \Delta = u_t - [f(u) u_{xxxxx}]_x = 0 \) for every generator of \( G \), where \( \text{pr}^{(6)}Q \) is the sixth-order prolongation of \( Q \).

Expanding Eq. (5) we get

\[ \phi^t = f''(u) u_x u_{xxxxx} + f'(u) u_{xxxxx} \phi^x + f'(u) u_x \phi^{xxxxx} + \phi^x(u) u_{xxxxx} + f(u) \phi^{xxxxx} \]

(6)

which must be satisfied whenever Eq. (4) is satisfied. Substituting the formulae of \( \phi^t, \phi^x, \phi^{xxxxx} \) and \( \phi^{xxxxx} \) into Eq. (6) we get an equation of \( t, x, u \) and the derivatives of \( \tau, \xi, \phi, u \). Replacing \( u_t \) by the right hand side of Eq. (4) whenever it occurs, and equating the coefficients of the various independent monomials to zero, we obtain
the space \( (3) \) and \( \frac{\partial}{\partial u} \), which together with the sixth, the eighth and the ninth equations imply that \( \phi_x = \phi = \xi_t = 0 \), so the determining equations reduce to

\[
\begin{align*}
\tau_x - \tau_u &= 0, \\
\xi_t &= \xi_u = 0, \\
\phi_x &= \phi_u = 0, \\
(\tau_t - 6\xi_x) f(u) + \phi f'(u) &= 0,
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
\xi &= ax + b, \\
\tau &= ct + d, \\
\phi &= pu + q, \\
(3c - 6a) f(u) + (pu + q)f'(u) &= 0,
\end{align*}
\]

where \( a, b, c, d, p, \) and \( q \) are arbitrary constants.

In order to make the classification as simple as possible, we next look for equivalence transformations of class \([1] \), and then solve system \([5] \) under these transformations. An equivalence transformation is a nondegenerate change of the variables \( t, x \) and \( u \) taking any equation of the form \([4] \) into an equation of the same form, generally speaking, with different \( f(u) \). The set of all equivalence transformations forms the equivalence group \( G^- \). To find the connected component of the unity of \( G^- \), we have to investigate Lie symmetries of the system that consists of Eq. \([4] \) and some additional conditions, i.e.

\[
\begin{align*}
u_t &= f_t u_x u_{xxxx} + f u_{xxxx}, \\
f_t &= 0, \\
f_x &= 0.
\end{align*}
\]

That is to say we must seek for an operator of the Lie algebra \( A^- \) of \( G^- \) in the form

\[
\begin{align*}
X &= \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x \\
&\quad + \phi(t, x, u) \partial_u + \psi(t, x, u, f) \partial f.
\end{align*}
\]

The coordinates \( \tau, \xi, \phi \) of the operator \([10] \) are sought as functions of \( t, x, u \) while the coordinates \( \psi \) are sought as functions of \( t, x, u \) and \( f \).

Applying \( \text{pr}^{(6)} X \) to Eq. \([9] \) we get the infinitesimal criterion

\[
\begin{align*}
\phi^t &= u_x u_{xxxx} \psi^u + f_u u_{xxxx} \phi^x \\
&\quad + f_u u_x \phi^{xxxxx} + u_{xxxxx} \psi^x + f \phi^{xxxxxxx}, \\
\psi^t &= 0, \\
\phi^x &= 0,
\end{align*}
\]

which must be satisfied whenever Eq. \([9] \) is satisfied. Substituting the formulae of \( \phi^t, \phi^x, \phi^{xxxxx}, \psi^t, \psi^x, \) and \( \psi^u \) into Eq. \([11] \) we get equations of \( t, x, u, f, \) and the partial derivatives of \( \tau, \xi, \phi, u, f, \) and \( \psi \). Replacing \( u, f_t \) and \( f_x \) by the right hand side of Eq. \([9] \) whenever they occur, and equating the coefficients of various independent monomials to zero, we obtain

\[
\begin{align*}
\xi_x &= 0, \quad \xi_t = 0, \quad \xi_u = 0, \\
\tau_x &= 0, \quad \tau_u = 0, \\
\phi_x &= 0, \quad \phi_t = 0, \quad \phi_u = 0, \\
\psi_x &= 0, \quad \psi_t = 0, \quad \psi_u = 0, \\
\psi &= (6\xi_x - \tau_t)f
\end{align*}
\]

which can be reduced to

\[
\begin{align*}
\tau &= c_1 t + c_1, \\
\xi &= c_2 x + c_2, \\
\phi &= c_3 u + c_3, \\
\psi &= (6c_5 - c_4)f
\end{align*}
\]

where \( c_1, c_2, \ldots, c_6 \) are arbitrary constants.

Thus the Lie algebra of \( G^- \) for class \([1] \) is

\[
A^- = \langle \partial_t, \partial_x, \partial_u, t\partial_t - f\partial_f, x\partial_x + 6f\partial_f, u\partial_u \rangle.
\]

Continuous equivalence transformations of class \([1] \) are generated by the operators from \( A^- \). In fact, \( G^- \) contains the following continuous transformations:

\[
\begin{align*}
\dot{t} &= t\varepsilon_4 + \varepsilon_1, \\
\dot{x} &= x\varepsilon_5 + \varepsilon_2, \\
\dot{u} &= u\varepsilon_6 + \varepsilon_3, \\
\dot{f} &= f\varepsilon_4^{-1}\varepsilon_5
\end{align*}
\]

where \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6 \) are arbitrary constants.

Solve the last equation of system \([5] \) under the above equivalence group \( G^- \), we can obtain three inequivalent equations of class \([1] \) with respect to the transformations from \( G^- \):

**Case 1:** \( f(u) \) is an arbitrary nonconstant function, the symmetry algebra of class \([1] \) is a three-dimensional Lie algebra which is generated by the operators

\[
\begin{align*}
Q_1 &= \partial_t, \quad Q_2 = \partial_x, \quad Q_3 = t\partial_t + \frac{1}{6} x \partial_x;
\end{align*}
\]

**Case 2:** \( f = e^{\lambda u} \mod G^- (\lambda \neq 0) \), the symmetry algebra of class \([1] \) is a four-dimensional Lie algebra which
is generated by the operators

\[ Q_1 = \frac{1}{\lambda} \partial_u - t \partial_t, \quad Q_2 = \partial_t, \]
\[ Q_3 = -x \partial_x - \frac{6}{m} u \partial_u, \quad Q_4 = \partial_x. \quad (14) \]

**Case 3:** \( f = u^m \mod G^\sim (m \neq 0) \), the symmetry algebra of class \([3]\) is a four-dimensional Lie algebra which is generated by the operators

\[ Q_1 = \frac{1}{m} u \partial_u - t \partial_t, \quad Q_2 = \partial_t, \]
\[ Q_3 = -x \partial_x - \frac{6}{m} u \partial_u, \quad Q_4 = \partial_x. \quad (15) \]

From the above results, it is easy to see that equation \([2]\) is exactly corresponding to case 3, thus possess a four-dimensional symmetry algebra.

### III. SIMILARITY REDUCTION

In order to obtain all the inequivalent reductions, we look for the one-dimensional optimal systems (see \([25]\)). These systems, similarity variables and reduced equations are listed below. In the following tables II III, each row shows the infinitesimal generators \( Q_i \) of each optimal system, as well as its similarity variable, similarity solution and reduced equation. \( \alpha \) is an arbitrary constant, while \( \lambda \) is a non-vanishing arbitrary constant. Note that in the case \( f(u) = u^m \) which corresponds to Eq. \([2]\), we only consider \( m \neq 0 \), otherwise the equation is linear.

**A. \( f(u) \) is an arbitrary nonconstant function**

In this case, the symmetry operators are Eq. \([13]\). These operators satisfy the commutation relations

\[ [Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = \frac{1}{6} Q_2 \]

and thus the corresponding symmetry algebra is a realization of the algebra \( A^3_9 (0 < |a| < 1) \) \([52]\). According to the results of Patera and Winternitz \([52]\), an optimal system of one-dimensional subalgebras is those spanned by

\[ Q_1, \quad Q_2, \quad Q_3, \quad Q_1 + \alpha Q_2. \]

Therefore, the corresponding similarity variables and reduced ODEs can be easily calculated. Such results are listed in Table I.

**B. \( f(u) = e^{\lambda u} (\lambda \neq 0) \)**

In this case, the symmetry operators are given by Eq. \([13]\), which satisfy the commutation relations

\[ [Q_1, Q_2] = Q_2, \quad [Q_3, Q_4] = Q_4 \quad (16) \]

and thus the corresponding symmetry algebra is a realization of the algebra \( 2A_2 \). According to the results of Patera and Winternitz \([52]\) again, an optimal system of one-dimensional subalgebras is those generated by

\[ Q_2, Q_3, Q_4; Q_1 + \alpha Q_3, Q_2 + \alpha Q_4, Q_2 + \alpha Q_3. \]

The corresponding similarity variables and reduced ODEs are listed in Table II.

**C. \( f(u) = u^m (m \neq 0) \)**

In this case, the symmetry operators are Eq. \([15]\). These operators share the same commutation relations Eq. \([10]\). Hence an optimal system of one-dimensional subalgebras is the same as the case \( f(u) = e^{\lambda u} \). The corresponding similarity variables and reduced ODEs are listed in Table III.

### IV. INVARIANT SOLUTIONS

Using the above reduced ODEs, we can construct some invariant solutions for the original equations \([3]\). It is easy to see that some of the similarity variables in the tables II III and III have a clear physical interpretation. Besides, for some higher order reduced ODEs, they can be further reduced by using new symmetries. Below, we discuss some facts related with some types of similarity
solutions with physical interest and obtain some particular solutions. Different types of solutions are separately analyzed.

A. Source and Sink Solutions

There are two ODEs, i.e., ODE\(15\) and ODE\(18\) among our reduced equations are related to this type of solutions. In fact, if we choose \(\alpha = \frac{1}{m+6}\) in ODE\(15\) then the similarity solution has the form
\[ u(t, x) = \frac{1}{t^{m+6}} v\left( \frac{x}{t^{m+6}} \right). \]
Thus, if \(m > -6\) it is clear that \(u(t, x) \to \delta(x)\) as \(t \to 0\) and the similarity solution is a source solution; if \(m < -6\) it is clear that \(u(t, x) \to \delta(x)\) as \(t \to +\infty\) and the similarity solution is a sink solution. Furthermore, we can also observe that, for the above choice of \(\alpha = \frac{1}{m+6}\), ODE\(15\) can be integrated once to obtain
\[ v^m v_{yyyyy} + \frac{1}{m+6} yv = k, \]
where \(k\) is an arbitrary constant. Thus, we can obtain a class of source solutions and sink solutions for the general thin-film equations \(4\) with \(f(u) = u^m\) (i.e., equation \(2\)) by solving the above fifth-order ODE. If we further choose \(k\) as zero, we have
\[ v_{yyyyy} = -\frac{1}{m+6} yv^{1-m}. \]
This equation admits the symmetry group corresponding to the infinitesimal generator \(v = y \partial_y + \frac{6}{m} v \partial_v\). Taking into account that the invariants of its first prolongation and setting
\[ x_1 = v y - \frac{6}{m}, \quad u_1 = y \frac{6}{m} (yv' - 6/m v)^{-1}, \]
this equation becomes a fourth-order ODE:
\[
\begin{align*}
-6u_1^2 &+ m^2 u_1 u_{1x} + 5m^4 (3m u_{1xx} + 2m u_{1x}^2) \\
-6u_1^2 &+ m^2 u_1 u_{1x} + 5m^4 (3m u_{1xx} + 2m u_{1x}^2) \\
-5m^3 (21m^2 u_{1x}^2 + 20m (m-3) u_{1x}^2) u_{1x} \\
&+ (7m^2 - 48m + 72) u_1^4 u_{1x} + 105m^5 u_{1x}^4 \\
&+ 150m^4 (m-3) u_{1x}^3 + 15m^3 (7m^2 - 48m) \\
&+ 72 u_1^4 u_{1x} + 105m (m-3) (5m^2 - 48m) \\
&+ 72 u_1^4 u_{1x} + [m^5 x^{-m+1}]/(m+6) \\
&+ 72 (m-2) (m-3) (2m-3) x_1 u_{1x}^9 \\
&+ 12 m (2m^3 - 50m^3 + 315m^2) \\
&- 720 m + 540) u_{1x}^9 = 0.
\end{align*}
\]
Thus, source and sink solutions can be also obtained by solving the above fourth-order ODE.

If we choose \(m = -6\) in ODE\(18\), then the similarity solution has the form
\[ u(t, x) = \frac{1}{e^{-\alpha t}} v\left( \frac{x}{e^{-\alpha t}} \right), \]
thus, if \(\alpha > 0\) it is clear that \(u(t, x) \to \delta(x)\) as \(t \to +\infty\) and the similarity solution is a sink solution; if \(\alpha < 0\) it is clear that \(u(t, x) \to \delta(x)\) as \(t \to -\infty\) and the similarity solution is a source solution. As in the previous case, for the choice of \(m = -6\), ODE\(18\) can be integrated once to obtain a fifth-order ODE
\[ v^{-6} v_{yyyyy} - \alpha v y = k, \]
where \(k\) is an arbitrary constant. Consequently, source and sink solutions can be computed by solving a fifth-order ODE.

B. Travelling-wave Solutions

This type of solution corresponds to the reductions \(10\) and \(17\) in fact, in these three reductions the similarity variables are given by \(y = x - \alpha t, u = v\), so that \(u(t, x) = v(x - \alpha t)\), thus the corresponding solutions are travelling-wave solutions. Due to the physical interest of this type of solutions, in what follows we study further symmetries of the associated ODEs and then construct some such kinds of solutions. First of all, we integrate these three equations once trivially and obtain
\[
\begin{align*}
\text{ODE}10: \quad & f(v) v_{yyyyy} + \alpha v = k, \\
\text{ODE}10': \quad & e^{\lambda} v_{yyyyy} + \alpha v = k, \\
\text{ODE}17': \quad & v^m v_{yyyyy} + \alpha v = k,
\end{align*}
\]
where \(k, k_1, k_2\) are arbitrary constants. Because ODE\(10\) and ODE\(17\) are given by ODE\(10\) for \(f(u) = e^{\lambda u}\) and \(f(u) = u^m\) respectively, so we will focus on the ODE\(10\) below. This equation is invariant under the group of translations in the \(y\)-direction, with infinitesimal generator \(\frac{\partial}{\partial y}\). Set
\[ x_1 = v, \quad u_1 = v^{-1}, \]
then the equation becomes a fourth-order ODE:
\[
\begin{align*}
f(x_1)(105u_1^4 - 105u_1 u_{1x} + 10u_1^2 u_{1x} + 105u_1^2 u_{1x} + 15u_1^2 u_{1x} + u_{1x} - u_{1x} - u_{1x} + u_{1x}) = \alpha x_1 u_1^9 = k_1 u_1^9.
\end{align*}
\]
We further suppose that
\[ v = \xi(x_1, u_1) \partial_{x_1} + \phi(x_1, u_1) \partial_{u_1} \]
is an infinitesimal generator of the last equation, then the coefficients \(\xi(x_1, u_1)\) and \(\phi(x_1, u_1)\) are satisfied with
\[
\begin{align*}
\xi &= ax_1 + b, \\
\phi &= cu_1, \\
\left[5(a + c) x_1 + (a + 5c) k\right] f(x_1) \\
+ &[-aa x_1^2 + (ak - ba) x_1 + kb] f'(x_1) = 0.
\end{align*}
\]
If \( f(u) = e^{\lambda u} \), from the above system we can obtain \( \xi = \phi = 0 \), which means that ODE (11) has no nontrivial symmetry. Thus, it cannot be reduced again. Consequently, the travelling wave solutions for Eq. (4) with \( f(u) = e^{\lambda u} \) can be computed by solving a fourth-order ODE:
\[
e^{\xi x}(105u_{11}^4 - 105u_{11}^2u_{12}x_1 + 10u_{11}^2u_{11}x_1 + 15u_{12}^2u_{11}x_1 + u_{12}^2u_{12}x_1 + u_{12}^2u_{11}x_1) + ax_1u_1 = ku_9.
\]

If \( f(u) = u^m \), then we have three cases from system (13):

(i) \( k \neq 0, \xi = 0, \phi = 0 \);
(ii) \( k = 0, m = 1, \xi = ax_1 + b, \phi = \frac{1}{k}u_1 \);
(iii) \( k = 0, m \neq 1, \xi = ax_1, \phi = \frac{m-5}{5}u_1 \).

Due to the triviality, the first case is excluded from the consideration. From the second case, we can get a rational travelling wave solution for Eq. (4) with \( f(u) = u \) in the form
\[
u(t,x) = -\frac{1}{120}(x - \alpha t)^5 + \sum_{i=0}^{4} c_i(x - \alpha t)^i.
\]

For the third case, we can set
\[
x_2 = u_1x_1, \quad u_2 = x_1^{-1}(x_1u_1' + \frac{5-m}{5}u_1)^{-1},
\]
then Eq. (17) can be reduced to:
\[
625x_3^2u_2u_{22x_2x_2} - 125[50x_2u_{22x_2} + (11m
- 25)x_2u_2^2 + 75u_2x_2^2u_{22x_2} + 9375x_2u_{22x_2} + 125[3x_2u_2(11m - 25) + 275]x_2u_2u_{22x_2}
+ 25[125(5m - 12)x_2u_2 + (46m^2 - 225m
+ 250)x_2u_2^2 + 2625]x_2u_2u_{22x_2} + (24m^4 + 875m^2
- 250m^3 - 1250m + 625m^2 + 625)x_2u_2^7
+ 10(48m^3 - 375m^2 + 875m - 625)x_2u_2^5
+ 125(38m^2 - 195m + 225)x_2u_2^5 + 13125(2m
- 5)x_2u_2^4 + 65625u_2^2 = 0.
\]

Consequently, the travelling wave solutions for Eq. (4) with \( f(u) = u^m \) (\( m \neq 1 \)) can be computed by solving a third-order ODE.

Finally, we consider a special situation when \( f(u) = u^m \) and \( k = 0 \), in which system (13) infers that \( x = -5\partial_{x_1} + u_1\partial_{u_1} \). Let
\[
x_2 = u_1e^{\frac{\phi}{\lambda}}, \quad u_2 = -(5u_1 + u)^{-1}e^{-\frac{\phi}{\lambda}},
\]
then Eq. (17) is reduced to:
\[
x_2^3u_2u_{22x_2x_2} - x_2u_2^2(10x_2u_{2x_2} + 11u_2x_2
+ 15u_2u_{2x_2} + 15x_2u_2^3u_{2x_2} + 11x_2u_2^5\)
+ 3x_2u_2u_2 + x_2u_2^2(46x_2u_2^2 + 125x_2u_2
+ 105)u_2x_2 + x_2^2(625x_2^5 + 24)u_2^2 + 96x_2u_2^6
+ 190x_2^2u_2^5 + 210x_2u_2^4 + 105u_2^3 = 0
\]

Therefore, the travelling wave solutions for equation (11) with \( f(u) = u^m \) can be computed by solving a third-order ODE too.

C. Waiting-Time Solutions

ODE (13) is a first-order equation that can be easily solved, in this way we obtain a family of waiting-time solutions for the sixth-order thin film equation (4) corresponding to \( f(u) = u^m \) (if \( m \neq 3/2, 2, 3 \) or 6). These solutions are given by
\[
u(t,x) = \begin{cases} 
\frac{6}{m} \ln \left( \frac{m(t_0 - t)}{144(t_0 - t)} \right)^{-1}, & x \geq 0, \\
0, & x < 0.
\end{cases}
\]

where \( t_0 \) being an arbitrary constant.

D. Blow-up Solutions

ODE (4) is also a first-order equation. Solving it we get for the sixth-order thin film equation (4) with \( f(u) = e^{\lambda u} \) the corresponding similarity solution
\[
u(t,x) = \frac{1}{\lambda} \ln \left( \frac{(x - x_0)^6}{144(t_0 - t)} \right)
\]
where \( t_0 \) is an arbitrary constant. This solution describes a localized blow-up at \( x = x_0 \). Note that the solution is only valid if \( \frac{(x-x_0)^6}{144(t_0 - t)} \leq 1 \), then, it ceases before \( t = t_0 \).

V. CONCLUDING REMARKS

We have carried out a detailed group-theoretical analysis for the generalized one-dimensional sixth-order thin film equation (4), which arises in considering the motion of a thin film of viscous fluid driven by an overlying elastic plate. A complete Lie point symmetry group classification for the class (4) have been performed under the continuous equivalence transformation group. Based on these, a complete list of symmetry reductions of the classification cases have been derived by making use of the optimal system of one-dimensional subalgebras of the corresponding Lie symmetry algebras. Furthermore, invariant solutions of the Eq. (4) with different functional form of \( f \) have been constructed by solving the reduced ODEs. In particular, by focusing our attention in those aspects with physical interest, we have found:

1. The thin film equation (4), for the case \( f(u) = u^m \) (which corresponds to equation (2)), \( m > -6 \) admits source solutions and \( m < -6 \) admits sink solutions. These solutions are related to the solutions
of a fourth-order ODE. If \( m = -6 \), Eq. (11) admits source and sink solutions. In this case these families of solutions are related to a fifth-order ODE.

2. The thin film equation (11) has travelling-wave solutions. In the case \( f(u) = u^m \), for \( m = 1 \) the equation admits a rational travelling-wave solutions, for \( m \neq 1 \) the problem of finding these solutions can be transformed into the problem of solving third-order ODEs. In the case \( f(u) = e^{iu} \), the travelling wave solutions can be computed by solving a fourth-order ODE. While for the case \( f(u) = u e^{-u} \), the travelling wave solutions of equation (11) can be computed by solving a third-order ODE.

3. Waiting-time solutions in the case \( f(u) = u^m \), and blow-up solutions in the case \( f(u) = e^{iu} \) are obtained in the context of symmetry reductions. However, it should be noted that these two types of solutions can also be obtained by means of variable separation. In the first case one takes \( u(t, x) = T(t)X(x) \) and in the second case \( u(t, x) = T(t) + X(x) \).

These results may lead to further applications in physics and engineering such as tests in numerical solutions of Eq. (11) and as trial functions for application of variational approach in the analysis of different perturbed versions of Eq. (11). Other topics including nonclassical symmetry, non-Lie exact solutions and physical applications of class (11) will be studied in subsequent publication.

**ACKNOWLEDGMENTS**

This work was partially supported by the National Key Basic Research Project of China under Grant No. 2010CB126600, the National Natural Science Foundation of China under Grant No. 60873070, Shanghai Leading Academic Discipline Project No. B114, the Post-doctoral Science Foundation of China under Grant Nos. 20090450067, 201104247, Shanghai Postdoctoral Science Foundation under Grant No. 09R21410600 and the Fundamental Research Funds for the Central Universities under Grant No. WM0911004.

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