SOME REMARKS ON THE SYMPLECTIC GROUP $\text{Sp}(2g, \mathbb{Z})$.

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Abstract

Let $G = \text{Sp}(2g, \mathbb{Z})$ be the symplectic group over the integers. Given $m \in \mathbb{N}$, it is natural to ask if there exists a non-trivial matrix $A \in G$ such that $A^m = I$, where $I$ is the identity matrix in $G$. In this paper, we determine the possible values of $m \in \mathbb{N}$ for which the above problem has a solution. We also show that there is an upper bound on the maximal order of an element in $G$. As an illustration, we apply our results to the group $\text{Sp}(4, \mathbb{Z})$ and determine the possible orders of elements in it. Finally, we use a presentation of $\text{Sp}(4, \mathbb{Z})$ to identify some finite order elements and do explicit computations using the presentation to verify their orders.

1. Introduction

Given a group $G$ and a positive integer $m \in \mathbb{N}$, it is natural to ask if there exists $k \neq e \in G$ such that $k^m = e$ where $e$ is the identity element in $G$. In this paper, we address this question in the case of the symplectic group $\text{Sp}(2g, \mathbb{Z})$.

One of the principal reasons for focusing on the group $\text{Sp}(2g, \mathbb{Z})$ is the following. It plays an important role in geometry and number theory and comes up in many interesting situations. For example, in geometry it plays a significant role in the study of certain type of surfaces due to its connections with the mapping class group. It also comes up in number theory in the study of Siegel modular forms.

Before we state the main results, we first recall the definition of $\text{Sp}(2g, \mathbb{Z})$ and fix some notation.

The group $\text{Sp}(2g, \mathbb{Z})$ is the group of all $2g \times 2g$ matrices with integral entries satisfying

$$A^\top JA = I$$

where $A^\top$ is the transpose of the matrix $A$ and $J = \begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}$.

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Throughout we write \( m = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \), where \( p_i \) is a prime and \( \alpha_i > 0 \) for all \( i \in \{1, 2, \ldots, k\} \). We also assume that the primes \( p_i \) are such that \( p_i < p_{i+1} \) for \( 1 \leq i < k+1 \). Also for \( A \in G \) we let \( o(A) \) denote the order of \( A \).

We now state the main results of this paper.

**Theorem 1.1.** Let \( A = \{m \in \mathbb{N} \mid p_i \leq 2g + 1 \text{ for some } i\} \). For \( A \in G \), we have \( A^m = I \) if and only if \( m \in A \).

**Theorem 1.2.** Let \( A \in G \) be such that \( o(A) = m \). Then \( m \leq \max\{30, M\} \) where \( M = \max\{2(2g)^{\frac{\alpha}{\log 2}}, (2g)^{\frac{(g+1)}{\alpha}}\} \) with \( \alpha = \frac{\log 2}{\log 3} \).

The paper is organized as follows. In section 2, we recall some important results that we need in the sequel. Section 3 contains the proofs of the main results (Theorem 1.1, Theorem 1.2) of this paper. In section 4 we explicitly identify some finite order elements in \( G \) when \( g = 2 \).

### 2. Preliminaries

Throughout this section we let \( \phi \) denote the Euler’s phi function. For the sake of completeness, we recall the definition of \( \phi \) and record a few properties we need. For a detailed account of the properties of the Euler’s \( \phi \) function, we refer the reader to [3].

For \( n \in \mathbb{N} \), \( \phi(n) \) is defined to be the number of positive integers less than or equal to \( n \) and relatively prime to \( n \). The function \( \phi \) is a multiplicative function. i.e., for \( m, n \in \mathbb{N} \) which are relatively prime, we have \( \phi(mn) = \phi(m)\phi(n) \). Using the fact that every positive integer \( n > 1 \), can be expressed in a unique way as

\[
n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k} = \prod_{i=1}^{k} p_i^{\alpha_i}
\]

where \( p_1 < p_2 < \cdots < p_k \) are primes and \( \alpha_i \)'s are positive integers and the fact that \( \phi \) is multiplicative, it is clear that \( \phi(n) = \phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2}) \cdots \phi(p_k^{\alpha_k}) \).

It is therefore useful to know the value of \( \phi \) for prime powers. It is easy to see that

\[
\phi(p^\alpha) = p^{\alpha-1}(p - 1),
\]

where \( p \) is a prime and \( \alpha \) is a positive integer.

We now state the results we shall use to prove the main theorems in this paper. We refer the reader [4] and [2] for a more detailed account of these results.

**Theorem 2.1** (Shapiro). Let \( m \in \mathbb{N} \) be such that \( m \notin \{1, 2, 3, 4, 6, 10, 12, 18, 30\} \) and \( \alpha = \frac{\log 2}{\log 3} \). Then \( \phi(m) > m^\alpha \).
Theorem 2.2 (Bürgisser). Let \( m = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \), where the primes \( p_i \) satisfy \( p_i < p_{i+1} \) for \( 1 \leq i < k \) and where \( \alpha_i \geq 1 \) for \( 1 \leq i \leq k \). There exists a matrix \( A \in \text{Sp}(2g, \mathbb{Z}) \) of order \( m \) if and only if

\[
\sum_{i=2}^{k} \phi(p_i^{\alpha_i}) \leq 2g, \text{ if } m \equiv 2(\text{mod}4). \\
\sum_{i=1}^{k} \phi(p_i^{\alpha_i}) \leq 2g, \text{ if } m \not\equiv 2(\text{mod}4).
\]

3. Main Results

Throughout this section we take \( m \in \mathbb{N} \) to be as in Theorem 2.2. We let \( A = \{m \in \mathbb{N} \mid p_i \leq 2g+1 \text{ for some } i\} \) and \( B = \mathbb{N} \setminus A \). Before we prove the main theorem, we record a lemma we need.

Lemma 3.1. Let \( A \in G \) such that \( o(A) = m \). Then \( p_i \leq 2g+1, \forall i \in \{1,2,\ldots,k\} \).

Proof. Suppose \( p_i > 2g+1 \) for some \( i \in \{1,2,\ldots,k\} \). This would imply that \( \phi(p_i^{\alpha_i}) = p_i^{\alpha_i-1}(p_i-1) > 2g \) and we get a contradiction to Theorem 2.2. \( \square \)

Theorem 3.2. Let \( m \in \mathbb{N} \). Then \( A^m = I \) if and only if \( m \in A \).

Proof. Suppose \( m \in A \). Choose \( p_i \) such that \( p_i \leq 2g+1 \). Clearly, \( \phi(p_i) \leq 2g \) and it follows from Theorem 2.2 that we have \( A \in G \) such that \( o(A) = p_i \). Let \( n = p_1^{\alpha_1} \cdots p_i^{\alpha_i-1} \cdots p_k^{\alpha_k} \). Now \( A^m = (A^{p_i})^n = I \).

Suppose there exists \( A \in G \) such that \( A^m = I \). We show that \( m \notin B \). Since \( A^m = I \), it follows that \( o(A)|m \). Let \( o(A) = n = q_1^{r_1} \cdots q_\ell^{r_\ell} \). By Lemma 3.1 we know that each \( q_i \leq 2g+1 \). The result now follows from the following simple observation. Since \( q_i|n \) and \( n|m \), we have \( q_i|m \) for each \( i \in \{1,2,\ldots,\ell\} \). This is indeed not possible if \( m \in B \). \( \square \)

3.1. An upper bound for the order. We show that the maximal order in \( G \) is always bounded. First, we introduce some notation.

Let \( m \in \mathbb{N} \), be as in Theorem 2.1 and let \( n = p_2^{\alpha_2} \cdots p_k^{\alpha_k} \). Suppose that there exists \( A \in G \) such that \( o(A) = m \). Consider the following sums:

\[
S_1 = \sum_{i=1}^{k} \phi(p_i^{\alpha_i}) \quad \text{if } m \not\equiv 2(\text{mod}4) \quad \text{and}
\]
\[
S_2 = \sum_{i=2}^{k} \phi(p_i^{\alpha_i}) \quad \text{if } m \equiv 2(\text{mod}4).
\]

In the following lemmas, We show that \( S_1 \) and \( S_2 \) are bounded below.
Lemma 3.3. $S_1 = \sum_{i=1}^{k} \phi(p_i^{\alpha_i}) > m^{\frac{\alpha}{g+1}}$.

Proof. We know that $p_i \leq 2g + 1$, $1 \leq i \leq k$. From this it follows that $k \leq g + 1$. Now consider the sum $S_1$. We have

\[
\phi(p_1^{\alpha_1}) + \cdots + \phi(p_k^{\alpha_k}) \geq k\left(\phi(p_1^{\alpha_1})\cdots\phi(p_k^{\alpha_k})\right)^{\frac{1}{k}} = k\left(\phi(m)\right)^{\frac{1}{k}} > m^{\frac{\alpha}{g+1}}.
\]

Lemma 3.4. $S_2 = \sum_{i=2}^{k} \phi(p_i^{\alpha_i}) > n^{\frac{\alpha}{g}}$.

Proof. Since $m \equiv 2 \pmod{4}$, it follows that $m = 2n = 2(p_2^{\alpha_2} \cdots p_k^{\alpha_k})$. Clearly, $n \notin \{1, 2, 3, 4, 6, 10, 12, 18, 30\}$ and the inequality in theorem 2.1 applies. Applying a similar argument as in lemma 3.3 to $n$ gives us the desired lower bound for $S_2$.

Theorem 3.5. Let $A \in G$ be such that $o(A) = m$. Then $m \leq \max\{30, M\}$ where $M = \max\{2(2g)^{\frac{\alpha}{g}}, (2g)^{\frac{(g+1)\alpha}{g}}\}$ with $\alpha = \frac{\log 2}{\log 3}$.

Proof. Suppose that $m \not\equiv 2 \pmod{4}$. By lemma 3.3, we have $S_1 > m^{\frac{\alpha}{g+1}}$. If $m > (2g)^{\frac{(g+1)\alpha}{g}}$, then we have $S_1 > 2g$. This is clearly not possible. Thus it follows that $m \leq (2g)^{\frac{(g+1)\alpha}{g}}$.

Similarly, we see that if $m \equiv 2 \pmod{4}$, then lemma 3.4 applies and we have $S_2 > n^{\frac{\alpha}{g}}$. If $m > 2(2g)^{\frac{\alpha}{g}}$, then $S_2 > 2g$. As this is not possible, it follows that $m \leq 2(2g)^{\frac{\alpha}{g}}$.

Taking $M = \max\{2(2g)^{\frac{\alpha}{g}}, (2g)^{\frac{(g+1)\alpha}{g}}\}$, we obtain $m \leq \max\{30, M\}$.

4. Finite order elements in $\text{Sp}(4, \mathbb{Z})$

Using Lemma 3.3, Theorem 3.5 and Theorem 2.2, it is easy to see that the possible orders of elements in $\text{Sp}(4, \mathbb{Z})$ are precisely 2, 3, 4, 5, 6, 8, 10 and 12. Since this is computational, we leave the details to the reader. In this section, we explicitly identify matrices in $\text{Sp}(4, \mathbb{Z})$ of these orders. The main tool we use is Bender’s presentation of $\text{Sp}(4, \mathbb{Z})$. Throughout this section we use the same notation as in [1] in all our computations.

In [1], Bender gives a presentation of $\text{Sp}(4, \mathbb{Z})$ using two generators and eight defining relations. We recall his result below.
Theorem 4.1 (Bender). The group $\text{Sp}(4,\mathbb{Z})$ is generated by the two elements

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

subject to the following eight relations:

a) $K^2 = I$,

b) $L^{12} = I$,

c) $(KL^7KL^5K)L = L(KL^5KL^7K)$,

d) $(L^2KL^4)(KL^5KL^7K) = (KL^5KL^7K)(L^2KL^4)$,

e) $(L^2KL^3)(KL^5KL^7K) = (KL^5KL^7K)(L^2KL^3)$,

f) $(L^2(KL^5KL^7K))^2 = ((KL^5KL^7K)L^2)^2$,

g) $L(L^6(KL^5KL^7K))^2 = (L^6(KL^5KL^7K))^2L$,

h) $(KL^5)^5 = (L^6(KL^5KL^7K))^2$.

Before we proceed further, we quickly recall some notation from [1] that we frequently use. We write the exponent $m$ for the word $L^m$, ($m \in \mathbb{N}$) and $H$ for the word $KL^5KL^7K$. For example, in our new notation, the word $H = KL^5KL^7K$ will be written as $H = K5K7K$. We also note that $H^2 = I$ and let $w_\alpha = H6$, $w_\beta = 9H6H$ and $x_\alpha = 5K1$.

It is clear from the presentation that we have elements of orders 2, 3, 4, 6 and 12 ($o(K) = 2$, $o(L^4) = 3$, $o(L^3) = 4$, $o(L^2) = 6$ and $o(L) = 12$).

Consider the word $K5$. We first show that $(K5)^{10} = I$. Using (h) it follows that $(K5)^5 = (6H)^2$. It is enough to show that $(6H)^4 = I$. Indeed,

$$(6H)^4 = (6H)^2(6H)^2 \\
\overset{(i)}{=} (H6)^2(H6)^2 \\
= H6(H6)^2H6 \\
\overset{(i)}{=} H6(H6)^2H6 \\
= H66H6H6H6 \\
= I.$$ 

Since $(K5)^{10} = I$, it follows that $o((K5)^2)$ is either 1 or 5. We will prove that $o((K5)^2) = 5$ by showing that $(K5)^2 \neq I$. Suppose $(K5)^2 = I$. A simple computation shows that $K5 = 5K$ and we get a contradiction.
Indeed,

\[ K^5 \overset{(h)}{=} (6H)^2 \]
\[ \overset{(i)}{=} (HK)^3 \]
\[ K5K = (HK)^3K \]
\[ \overset{(i)}{=} K(HK)^3 \]
\[ = K(K5) \]
\[ = 5 \]
\[ K5 = 5K \]

The result follows from the computation below.

\[(4.1) \quad I = (K5)^2 = (K5)(K5) = (K5)(5K) = K10K = 10. \]

Since \((K5)^{10} = I\), it follows that \(o(K5)\) is either 1, 2, 5 or 10. We will show that \(o(K5) = 10\). It is enough to show that \((K5)^5 \neq I\) (we know that \(K5 \neq I\) and \((K5)^2 \neq I\)). Suppose \((K5)^5 = I\). Using (h) and (i) it follows that

\[(K5)^5 = (6H)^2 = (HK)^3 = w^2 \alpha \]

Since \(w^2 \alpha = 1\), we see that

\[(4.2) \quad 6H = H6. \]

Using (4.2), we have

\[(4.3) \quad w_\beta = 9H6H = 9HH6 = 3. \]

Before we proceed further, we do a computation which is essential. To be more precise, we show that

\[(4.4) \quad 3 = w_\beta = 8K7KHK. \]

Indeed,

\[ 3 = w_\beta \]
\[ = 9H6H \]
\[ \overset{(k)}{=} (8K7\overline{5}K7)(\overline{5}K7K) \]
\[ = 8K7(\overline{5}K7)^2K \]
\[ = 8K7(HK)^2K \]
\[ = 8K7KHK \quad [\text{since } (HK)^3 = 1 \text{ by assumption}] \]

From (4.4), it follows that

\[(4.5) \quad w^2_\beta = 6 = 11K7KHK. \]
We will use (4.5) to show that \( H = K \) and ultimately \( K^5 = 5K \) giving us a contradiction. Before we continue we make the following observation.

We show that

\[
(KH)^2K = KHK.
\]

This follows from \((i)\) and the fact that \((HK)^3 = I\). Indeed,

\[
(KH)(KH)^2K = (KH)^3K \overset{(i)}{=} K(HK)^3 = K.
\]

Consider (4.5). We have

\[
6 = 11K7KHK \overset{(2)}{=} H6KHKHK = H6(KH)^2K \overset{(3)}{=} 6HKHK = 6(HK)^2 = 6KH
\]

The above computation shows that \( KH = 1 \) and \( H = K \). Since we have \( H = K \), it follows that \( K5K7 = 1 \) and \( K^5 = 5K \). The result follows from (4.1).

Consider the word \( 9H \). Clearly, \( 9H \neq I \). We will show that \((9H)^4 \neq I \) and \((9H)^8 = I \).

Using (9) in [1], we have

\[
(9H)^4 = (w_\alpha w_\beta)^4 = (w_\alpha w_\beta)(w_\beta w_\alpha)^2(w_\alpha w_\beta) = w_\alpha w_\beta^2 w_\alpha w_\beta^2 w_\alpha w_\beta \overset{(7)}{=} w_\beta^3 w_\alpha w_\beta w_\alpha \overset{(8)}{=} w_\beta^3 w_\alpha w_\beta \overset{(10)}{=} w_\alpha w_\beta
\]

The result now follows from a simple observation. We have

\[
(9H)^4 = w_\alpha^2 = (6H)^2 \overset{(i)}{=} (H6)^2 \overset{(h)}{=} (K5)^5 \neq I
\]

and

\[
(9H)^8 = (K5)^{10} = I.
\]
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REFERENCES

1. Peter Bender, *Eine Präsentation der symplektischen Gruppe Sp(4, \mathbb{Z}) mit 2 Erzeugenden und 8 definierenden Relationen*, J. Algebra 65 (1980), no. 2, 328–331. MR 585724 (81j:20062)
2. B. Bürgisser, *Elements of finite order in symplectic groups*, Arch. Math. (Basel) 39 (1982), no. 6, 501–509. MR 690470 (85b:20062)
3. David M. Burton, *Elementary number theory*, Allyn and Bacon Inc., Boston, Mass., 1980, Revised printing. MR 567137 (81c:10001b)
4. Harold Shapiro, *An arithmetic function arising from the \phi function*, Amer. Math. Monthly 50 (1943), 18–30. MR 0007755 (4,188c)

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