Numerical methods for checking the regularity of subdivision schemes

Maria Charina

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Abstract

In this paper, motivated by applications in computer graphics and animation, we study the numerical methods for checking $C^k$-regularity of vector multivariate subdivision schemes with dilation $2I$. These numerical methods arise from the joint spectral radius and restricted spectral radius approaches, which were shown in [4] to characterize $W^k_p$-regularity of subdivision in terms of the same quantity. Namely, the $(k, p)$-joint spectral radius and the $(k, p)$-restricted spectral radius are equal. We show that the corresponding numerical methods in the univariate scalar and vector cases even yield the same upper estimate for the $(k, \infty)$-joint spectral radius for a certain choice of a matrix norm. The difference between the two approaches becomes apparent in the multivariate case and we confirm that they indeed offer different numerical schemes for estimating the regularity of subdivision. We illustrate our results with several examples.

Keywords: vector multivariate subdivision schemes, joint spectral radius, restricted spectral radius

1 Introduction

Subdivision schemes are recursive algorithms that starting with a coarser mesh in $\mathbb{R}^s$ determine the coordinates of the finer vertices $c^{(r+1)}$ by local averages of the coarser ones

$$c^{(r+1)} = S_A c^{(r)}, \quad r \geq 0.$$  (1)

The subdivision operator $S_A$ is linear and describes the local averaging rules. The locality of the subdivision and algorithmic simplicity of the subdivision recursion ensure that (1) is fast, efficient, and easy to implement. These features make subdivision popular in computer graphics and animation, see [3, 7, 13, 25] and the references therein.

We restrict our study to the shift–invariant setting and study the multivariate vector schemes defined by the subdivision operator

$$S_A c^{(r)}(\beta) = \sum_{\beta \in \mathbb{Z}^s} A(\beta - 2\beta) c^{(r)}(\beta), \quad r \geq 0,$$

mapping the space $\ell^n(\mathbb{Z}^s)$ of vector–sequences indexed by $\mathbb{Z}^s$ into itself. The associated subdivision mask $A = (A(\alpha))_{\alpha \in [0, N]_s}$, $N \in \mathbb{N}$, is a finitely supported matrix-valued sequence.

A challenging task is to provide a characterization of the regularity of the limits of subdivision recursion and to develop numerical methods for checking their regularity. Two prominent methods that characterize the $W^k_p$-regularity, $1 \leq p \leq \infty$, $k \in \mathbb{N}$, of subdivision in the shift-invariant setting are the so-called joint spectral radius (JSR) approach [8, 19] and restricted spectral radius (RSR) [3, 5] approach. Generally speaking, the joint spectral radius approach characterizes the regularity of subdivision limits in terms of the joint spectral radius of a finite set

$$\mathcal{A} = \{ A_\varepsilon |_{V_k} : \varepsilon \in \{0, 1\}^s \}$$

of square matrices $A_\varepsilon$ derived from the mask $A$ and restricted to a common finite dimensional invariant subspace $V_k$. The sequences in $V_k$ annihilate certain polynomial eigensequences of $S_A$.
and, thus, the process of restricting of $A_k$ to $V_k$ mimics the computation of discrete derivatives $\nabla^k$. The essence of the restricted spectral radius approach can be formulated as follows: if the operator $S_A$ admits the factorization

$$\nabla^k S_A = S_{B_k} \nabla^k, \quad k \geq 1,$$

then the restricted spectral properties of the associated difference subdivision schemes $S_{B_k}$ characterize the regularity of the underlying subdivision. It had been believed until recently that the JSR and RSR approaches are intrinsically different. In [4, 6], we showed that these two approaches characterize the $W^p_k-$regularity, $1 \leq p \leq \infty$, of subdivision in terms of the same quantity $(k,p)-$JSR and differ by the numerical methods they yield for its estimation. Motivated by applications in computer graphics and animation we compare in this paper only the numerical methods for approximation of $(k, \infty)-$JSR. The problem of computing the $(k, \infty)-$JSR is NP-hard, i.e., there is no polynomial-time algorithm for its approximation, see [29]. The brute force Branch–and-Bound algorithm dating back to [10] is available. Its approximation of $(k, \infty)-$JSR is based on the estimate

$$\max_{A_{\epsilon_J} \mid V_k \in A \mid V_k} \left( \rho \left( \prod_{j=1}^{r} A_{\epsilon_j} \mid V_k \right) \right)^{1/r} \leq \rho_{\infty}(A) \leq \max_{A_{\epsilon_J} \mid V_k \in A \mid V_k} \left( \prod_{j=1}^{r} \right)^{1/r},$$

$r \in \mathbb{N}$, and allows us to approximate $(k, \infty)-$JSR with an arbitrary precision. The estimate in (3) is independent of the choice of the matrix norm $\| \cdot \|$. The Branch–and-Bound algorithm is computationally expensive and its rate of convergence is not known. See [16, 18] for a successful attempt to reduce the computational complexity of the Branch–and-Bound algorithm. The restricted spectral radius approach leads to linear programming, see [4], that due to the usual properties of the spectral radii yields another upper bound for $(k, \infty)-$JSR. The computations in [24] show that the lower bound in (3) is not always reliable.

The main goal of this paper is to compare the numerical methods arising from the joint and restricted spectral radius approaches for approximation the $(k, \infty)-$JSR. In section 3.1 we show that if we choose the matrix norm $\| \cdot \|$ in (3) to be the infinity matrix norm, then the upper bound in (3) and the one obtained using the optimization approach in [4, Section 4.6] coincide in the univariate scalar and vector cases. Note that the flexibility of the joint spectral radius approach is that the estimate in (3) is independent of the choice of the matrix norm $\| \cdot \|$. This in some cases leads to a sharper upper estimate for $(k, \infty)-$JSR, see example 3.5 or even allows for exact computations of the $(k, \infty)-$JSR, see [16, 18]. The results of section 3.1 also show how one can easily determine the entries of the matrices $A_{\epsilon_j} \mid V_k$ from the entries of the corresponding difference masks $B_k$. In section 3.2 we show that in the multivariate scalar or vector cases the Branch–and-Bound algorithm and the optimization method presented in [4, Section 4.6] are intrinsically different and yield different upper bounds for $(k, \infty)-$JSR regardless of the choice of the matrix norm in (3). In Section 3.3 we compare the method for estimation of the $(k, \infty)-$JSR from [24] and the one based on the properties of the difference masks on an example of a divergent scheme. To summarize, examples illustrate that it is impossible to prefer one of the numerical methods discussed in this paper over the other and they can be used according to one’s personal preference.

## 2 Notation and Background

An element $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ is a multi–index whose length is given by $|\mu| := \mu_1 + \cdots + \mu_s$ and $\mu! := \mu_1! \cdots \mu_s!$. For $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$ and $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ define

$$\alpha^\mu := \alpha_1^{\mu_1} \cdots \alpha_s^{\mu_s}.$$  

Denote by $\epsilon_\ell$, $\ell = 1, \ldots, s$, the $\ell-$th standard unit vector of $\mathbb{R}^s$ and by $e_j$, $j = 1, \ldots, n$, the $j-$th standard unit vector of $\mathbb{R}^n$, respectively. Let $\ell_{\infty}^{m \times k}(\mathbb{Z}^s)$ denote the linear space of all sequences of $n \times k$ real matrices indexed by $\mathbb{Z}^s$. In addition, let $\ell_\infty^{m \times d}(\mathbb{Z}^s)$ denote the Banach space of sequences of $n \times d$ real matrices indexed by $\mathbb{Z}^s$ with finite $\infty$-norm defined as

$$\|C\|_\infty := \sup_{\alpha \in \mathbb{Z}^s} |C(\alpha)|_\infty,$$  

(4)
where $|C(\alpha)|_\infty$ is the $\infty$-operator norm on $\mathbb{R}^n$ if $d > 1$ and the $\infty$-vector norm if $d = 1$. Moreover, let $\ell_{0}^{n \times d}(\mathbb{Z}^s) \subseteq \ell_{\infty}^{n \times d}(\mathbb{Z}^s)$ be the space of finitely supported matrix valued sequences. Specific examples of such scalar and vector sequences is the scalar delta sequence $\delta \in \ell_0(\mathbb{Z}^s)$ and the vector sequence $\delta e_j^T \in \ell_0^{1 \times n}(\mathbb{Z}^s)$ defined by

$$
\delta(\alpha) := \begin{cases} 
1, & \alpha = 0, \\
0, & \alpha \in \mathbb{Z}^s \setminus \{0\}
\end{cases} \quad \text{and} \quad \delta e_j^T(\alpha) := \begin{cases} 
e_j^T, & \alpha = 0, \\
0, & \alpha \in \mathbb{Z}^s \setminus \{0\}.
\end{cases} (5)
$$

Let $A \in \ell_{0}^{n \times n}(\mathbb{Z}^s)$ be a finitely supported matrix sequence shifted so that $\text{supp} A \subseteq [0, N]^s$ for some $N \in \mathbb{N}$. The subdivision operator $S_A : \ell^n(\mathbb{Z}^s) \to \ell^n(\mathbb{Z}^s)$ associated with the mask $A \in \ell_{0}^{n \times n}(\mathbb{Z}^s)$ is defined by

$$
S_A c(\alpha) = \sum_{\beta \in \mathbb{Z}^s} A(\alpha - 2\beta) c(\beta), \quad \alpha \in \mathbb{Z}^s. \quad (6)
$$

The subdivision scheme then corresponds to a repeated application of $S_A$ to an initial vector sequence $c \in \ell^n(\mathbb{Z}^s)$ yielding

$$
c(0) := c, \quad c(r+1) := S_A c(r), \quad r \geq 0. \quad (7)
$$

The dimension of the following subspace of $\mathbb{R}^n$

$$
\mathcal{E}_A := \left\{ v \in \mathbb{R}^n : \sum_{\alpha \in \mathbb{Z}^s} A(\alpha - 2\alpha)v = v, \ v \in \{0, 1\}^s \right\} \quad (8)
$$
determines the structure of the difference operators $\nabla^k$ we define next. Let $m := \dim(\mathcal{E}_A)$. For $C \in \ell^{n \times d}(\mathbb{Z}^s)$ and $D \in \ell^{d \times n}(\mathbb{Z}^s)$ we define the $j$-th column of $\nabla \ell^C$, $\nabla \ell^C : \ell^{n \times d}(\mathbb{Z}^s) \to \ell^{n \times d}(\mathbb{Z}^s)$ as

$$
[\nabla \ell^C]_{\cdot,j} := \begin{bmatrix}
-C_{1,j} + C_{1,j}(\cdot - \epsilon) \\
-\cdots \\
-C_{m,j} + C_{m,j}(\cdot - \epsilon) \\
-C_{m+1,j} \\
\cdots \\
C_{n,j}
\end{bmatrix}, \quad 1 \leq \ell \leq s, \quad 1 \leq j \leq d, \quad (9)
$$

and $\nabla \ell D := (\nabla \ell^C)^T$, respectively. Let $k \in \mathbb{N}$. For our analysis we make use of the notion of the $k$-th difference operator, the discrete analog of a derivative. The $k$-th difference operator $\nabla^k : \ell^{n \times d}(\mathbb{Z}^s) \to \ell^{n \times N_{s,k} \times d}(\mathbb{Z}^s)$, $N_{s,k} = \begin{pmatrix} s + k - 1 \\
-1 \\
\end{pmatrix}$, is defined by

$$
\nabla^k := \begin{bmatrix}
\nabla^\mu_1 \\
\vdots \\
\nabla^\mu_s I_m \\
0 \\
I_{n-m}
\end{bmatrix}_{|\mu| = k}. \quad (10)
$$

We say that the subdivision scheme $S_A$ is $C^k$-convergent, if for any starting sequence $c \in \ell_{0}^{n \times k}(\mathbb{Z}^s)$ there exists a vector–valued function $f_{c} \in (C^{k}(\mathbb{R}^s))^n$ such that for any compactly supported stable test function $g \in C^{k}(\mathbb{R}^s)$

$$
\lim_{r \to \infty} \max_{|\mu| \leq k} \|D^\mu f_{c} - g I_{n} \ast (2^{|\mu| r \nabla^\mu_1 \cdots \nabla^\mu_s S_A c}\nabla^r)\|_\infty = 0. \quad (11)
$$

For more details on the properties of test functions see [9].

Another useful tool for studying subdivision schemes is the Laurent polynomial formalism. For a finite matrix sequence $A \in \ell_{0}^{n \times k}(\mathbb{Z}^s)$ we define the associated symbol as the Laurent polynomial

$$
A^*(z) := 2^{-s} \sum_{\alpha \in \mathbb{Z}^s} A(\alpha) z^\alpha, \quad z \in (\mathbb{C} \setminus \{0\})^s, \quad (12)
$$

where, in the usual multi-index notation, $z^\alpha := z_1^\alpha_1 \cdots z_s^\alpha_s$. 

In this section we show how the entries of the matrices $A$, $U$, and $B$, $V$. As the subspace $k$, $V$ by a finite set of square matrices annihilate the polynomial sequences in any $k$, bounds in (3) and (19) on the $(k, p)$-joint spectral radius (JSR) see [8, 19]. Generally speaking, the joint spectral radius approach characterizes the $W^k_p$-regularity of subdivision limits in terms of $(k, p)$-joint spectral radius (JSR) of a finite set of linear operators. These are derived from the linear operators $A_\varepsilon : \ell_0^{1 \times n}(Z^*) \rightarrow \ell_0^{1 \times n}(Z^*)$ satisfying

$$A_\varepsilon v = \sum_{\alpha \in \mathbb{Z}^n} v(\alpha)A(\varepsilon + 2 \cdot -\alpha), \quad v \in \ell_0^{1 \times n}(Z^*), \quad \varepsilon \in \{0, 1\}^s,$$

and then restricted to the invariant subspace $V_k \subset \ell_0^{1 \times n}(Z^*)$. The elements of the finite dimensional subspace

$$V_k := \{ v \in \ell_0^{1 \times n}(\mathbb{Z}^*): \sum_{\beta \in \mathbb{Z}^n} v(\beta)u(-\beta) = 0 \text{ for all } u \in U_k\}, \quad (13)$$

annihilate the polynomial sequences in $U_k \subset \ell^n(\mathbb{Z}^*)$ reproduced by one step of subdivision recursion in [7]. For details on the structure of $U_k$ see [4]. Thus, the process of restricting to the subspace $V_k$ imitates the computation of the partial derivatives of subdivision limits function, see [2] Section 2.3. As the subspace $V_k$ is finite dimensional we replace the finite set of operators $A|_{V_k} := \{ A_\varepsilon|_{V_k}: \varepsilon \in \{0, 1\}^s \}$ by a finite set of square matrices

$$A_\varepsilon := [A^\varepsilon(\varepsilon + 2 \cdot -\beta)]_{\alpha, \beta \in \{0, N\}^s}, \quad \varepsilon \in \{0, 1\}^s, \quad (15)$$

and use them for estimations of the $(k, \infty)$-joint spectral radius

$$\rho_\infty(A|_{V_k}) := \lim_{r \rightarrow \infty} \max_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s \in \{0, 1\}^s} \left( \prod_{j=1}^s A_{\varepsilon_j} |_{V_k} \right)^{1/r}. \quad (16)$$

For a difference scheme $S_{B_k}$, $k \geq 1$, satisfying (8) define the restricted $(k, \infty)$-norm

$$\| S_{B_k}|_{V_k} \|_{\infty} := \sup \left\{ \frac{\| S_{B_k} \nabla^k c \|_{\infty}}{\| \nabla^k c \|_{\infty}} : c \in \ell_\infty^n(Z^*), \nabla^k c \neq 0 \right\}. \quad (17)$$

and the restricted $(k, \infty)$-spectral radius ((k, \infty)-RSR)

$$\rho_\infty(S_{B_k}|_{V_k}) := \lim_{r \rightarrow \infty} \| S_{B_k}^r |_{V_k} \|_{\infty}^{1/r}. \quad (18)$$

For the results on existence of $S_{B_k}$ see e.g. [3, 4]. To estimate the $(k, \infty)$-RSR we use the standard properties of spectral radii

$$\rho_\infty(S_{B_k}|_{V_k}) = \inf_{r \in \mathbb{N}} \| S_{B_k}^r |_{V_k} \|_{\infty}^{1/r}$$

and get the following estimate

$$\rho_\infty(S_{B_k}|_{V_k}) \leq \| S_{B_k}^r |_{V_k} \|_{\infty}^{1/r} \quad (19)$$

for any $r \in \mathbb{N}$. The restricted norm $\| S_{B_k}^r |_{V_k} \|_{\infty}$ can be computed using linear programming in [4] Section 4.6. The main result of [3] shows that $(k, p)$-JSR and $(k, p)$-RSR are equal.

### 3 Methods for approximating $(k, \infty)$-JSR

In this section we show how the entries of the matrices $A_\varepsilon|_{V_k}$ and their products depend on the entries of the difference masks $B_k^r$, $r \geq 1$ and $1 \leq k < N$. In particular, we show that the upper bounds in (3) and (19) on the $(k, \infty)$-joint spectral radius coincide in scalar and vector univariate cases for the special choice of the matrix norm, i.e.

$$\| S_{B_k}^r |_{V_k} \|_{\infty} = \| S_{B_k}^r \|_{\infty} = \max_{A_\varepsilon|_{V_k} \in \mathbb{A}|_{V_k}} \left( \prod_{j=1}^s A_{\varepsilon_j} |_{V_k} \right). \quad (20)$$
Varying the matrix norm in (3) can lead to a better or a worse upper estimates of the joint spectral radius, see Example 3.5. In the scalar or vector multivariate case, the identity

$$\|S_{B_k}\|_\infty = \max_{A_{\varepsilon_j} |V_k \in A |V_k} \left\| \prod_{j=1}^{n} A_{\varepsilon_j} |V_k \right\|_\infty$$

does not hold in general, see subsection 3.2. Therefore, the JSR and RSR approaches lead to intrinsically different numerical methods for checking regularity of subdivision.

An important observation that connects the entries of the matrices $A_{\varepsilon_j} |V_k$ with the ones of the difference masks $B_k$, $k \geq 1$, states the following.

**Proposition 3.1.** For $k, r \geq 1$, $\varepsilon_j \in \{0,1\}^s$ and $\beta \in \mathbb{Z}^s$ we have

$$A_{\varepsilon_r} \ldots A_{\varepsilon_1} \nabla^k \delta I_n(\cdot - \beta) = \sum_{\beta \in \mathbb{Z}^s} B_k^{(r)}(\varepsilon_1 + \cdots + 2^{r-1}\varepsilon_r + 2^r \beta - \beta) \nabla^k \delta I_n(\cdot - \beta).$$

**Proof.** Let $r \geq 1$ and $\beta \in \mathbb{Z}^s$. By [17] Lemma 2.2 we have

$$\nabla^k S_{A_k} \delta I_n(\alpha - \beta) = A_{\varepsilon_r} \ldots A_{\varepsilon_1} \nabla^k \delta I_n(\gamma - \beta),$$

where $\alpha = \varepsilon_1 + \cdots + 2^{r-1}\varepsilon_r + 2^r \gamma$, $\gamma \in \mathbb{Z}^s$. Note that the application of $A_{\varepsilon_r}$ to $\nabla^k \delta I_n$ means that the operator $A_{\varepsilon_r}$ is applied to each row sequence of $\nabla^k \delta I_n$ separately. Next using the identity (2) we obtain

$$S_{B_k}^{(r)} \nabla^k \delta I_n(\alpha - \beta) = A_{\varepsilon_r} \ldots A_{\varepsilon_1} \nabla^k \delta I_n(\gamma - \beta), \quad \gamma \in \mathbb{Z}^s.$$  

The definition of $S_{B_k}^{(r)}$ and the definition of the standard convolution operator $\ast$ yield that the elements of the sequence $A_{\varepsilon_r} \ldots A_{\varepsilon_1} \nabla^k \delta I_n(\cdot - \beta)$ have the following representation

$$A_{\varepsilon_r} \ldots A_{\varepsilon_1} \nabla^k \delta I_n(\gamma - \beta) = \left( \nabla^k \delta I_n \ast B_k^{(r)}(\varepsilon_1 + \cdots + 2^{r-1}\varepsilon_r + 2^r \cdot - \beta) \right)(\gamma), \quad \gamma \in \mathbb{Z}^s,$$

in terms of the elements of the subsequence $B_k^{(r)}(\varepsilon_1 + \cdots + 2^{r-1}\varepsilon_r + 2^r \cdot - \beta)$ of $B_k^{(r)}$. Thus, the claim follows.



### 3.1 Univariate scalar and vector cases

Let us see how to use the result of Proposition 3.1 to compare the upper bound in (3) and (19). For that we need the following auxiliary lemma.

**Lemma 3.2.** For $r \geq 1$, $1 \leq k \leq N$, and $S_{B_k}^{(r)} : \ell^\infty(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})$ we have

$$\|S_{B_k}^{(r)}|_{\ell^\infty} = \|S_{B_k}\|_{\infty}.$$

**Proof.** We consider the vector case first. Due to [12] Proposition 6.10, the non-restricted operator $(k, \infty)$--norm $\|S_{B_k}^{(r)}\|_\infty$ is defined as usual by

$$\|S_{B_k}^{(r)}\|_\infty = \max \left\{ \|S_{B_k}^{(r)} \nabla^k c\|_\infty : c \in \ell^\infty(\mathbb{Z}), \|\nabla^k c\|_\infty = 1 \right\},$$

and can be computed as follows

$$\|S_{B_k}^{(r)}\|_\infty = \max_{\alpha \in \{0,2^r-1\}} \left\{ \left\| \sum_{\beta \in \mathbb{Z}} |B_k^{(r)}(\alpha - 2^r \beta)| \right\|_\infty \right\}. \quad (21)$$

The absolute value of the matrices $B_k^{(r)}$ in (21) means that we take the absolute values of each matrix entries. Note that due to the compact support of $A$ and the definition of $\nabla^k$, we have supp$(B_k^{(r)}) \subset [0, N+k]$ for all $r \geq 1$. Thus, (21) is equivalent to the following linear programming: for $i = 1, \ldots, n$ and $\alpha \in \{0,2^r-1\}$

$$\max \sum_{\beta \in [-N-k,0]} \sum_{j=1}^{n} (B_k^{(r)})_{ij} (\alpha - 2^r \beta) d_j(\beta)$$

$$\max \sum_{\beta \in [-N-k,0]} \sum_{j=1}^{n} (B_k^{(r)})_{ij} (\alpha - 2^r \beta) d_j(\beta) \quad (22)$$

$$-1 \leq d_j(\beta) \leq 1, \quad \beta \in [-N-k,0], \quad j = 1, \ldots, n.$$
For fixed $i$ and $\alpha$, the solution $d^{(i,\alpha)} \in \ell^a_\infty([-N-k,0])$ of this optimization problem is given by
\[
d_{j}^{(i,\alpha)}(\beta) = \text{sgn}(B_k^{(r)})_{ij} (\alpha - 2^r \beta), \quad \beta \in [-N-k,0],
\]
We get the maximizing sequence $d \in \ell^a_\infty([-N-k,0])$ for all $i$ and $\alpha$ choosing $d = d^{(i,\alpha)}$ such that the number determine by linear programming in (22) for this $d$ is maximal.

On the other hand, from [4] the computation of the restricted $(k,\infty)-$norm is equivalent to the following linear programming: for $i = 1, \ldots, n$ and $\alpha \in \{0, 2^r - 1\}$
\[
\max_{\beta \in [-N-k,0]} \sum_{j=1}^{n} (B_k^{(r)})_{ij} (\alpha - 2^r \beta) \nabla^k c_j(\beta)
-1 \leq \nabla^k c_j(\beta) \leq 1, \quad \beta \in [-N-k,0], \quad j = 1, \ldots, n.
\]
For fixed $i$ and $\alpha$, the solution $c^{(i,\alpha)} \in \ell^a_\infty([-N-2k,0])$ of this optimization problem is given for each $j$ by the linear system of equations $\nabla^k c_j^{(i,\alpha)}(\beta) = \text{sgn}(B_k^{(r)})_{ij} (\alpha - 2^r \beta), \beta \in [-N-k,0]$. The solutions $c_j$ exist and are unique, due to the invertibility of the corresponding matrix of this system, which is bi-diagonal with $-1$ and $1$ on the main and upper diagonals, respectively. We get the maximizing sequence $c \in \ell^a_\infty([-N-2k,0])$ for all $i$ and $\alpha$ choosing $c = c^{(i,\alpha)}$ such that the number determine by linear programming in (23) for this $c$ is maximal. Therefore, (23) is satisfied.

In the scalar univariate case the proof is analogous, one should only take into account that the support of $B_k$ decreases with $k$ and replace above $[0, N + k]$ by $[0, N - k]$ and $[-N - k, 0]$ by $[-N + k, 0]$, respectively.

Combining Proposition 3.1 and Lemma 3.2 we obtain the following result.

**Theorem 3.3.** For $r \geq 1$ and $1 \leq k < N$ we have
\[
\max_{\epsilon_j \in \{0,1\}} \left\| \prod_{j=1}^{r} A_{\epsilon_j} |V_k| \right\|_{\infty} = \| S_{B_k}^{r} \|_{\infty}.
\]

**Proof.** We start with the scalar case. Due to [4], Lemma 4.4, 4.9], we have
\[
V_k = \text{span} \left\{ \nabla^k \delta(\cdot - \beta) : \beta \in [0, N - k] \right\}.
\]
Thus, due to the fact that the spanning set above is also a basis for $V_k$ and by Proposition 3.1, we get for any $r \geq 1$ and $\epsilon_j \in \{0,1\}$
\[
A_{\epsilon_r} \ldots A_{\epsilon_1} |V_k| = [B_k^{(r)}(\epsilon_1 + \cdots + 2^{r-1} \epsilon_r + 2^r \beta - \beta)]_{\beta = \epsilon_1, \ldots, \epsilon_r, \epsilon_r + 1, \ldots, \alpha \in [0, N - k]}.\]
Therefore,
\[
\max_{\epsilon_j \in \{0,1\}} \left\| \prod_{j=1}^{r} A_{\epsilon_j} |V_k| \right\|_{\infty} = \| S_{B_k}^{r} \|_{\infty}
\]
and the claim follows by Lemma 3.2.

In the vector case, define as in [4], Lemma 4.9] the spanning set of $V_k$ to be
\[
\left\{ (\nabla^k \delta_{\epsilon_j})^T (\cdot - \beta) : \beta \in [0, N - k], \quad j = 1, \ldots, m \right\} \cup \left\{ \delta_{\epsilon_j}^T (\cdot - \beta) : \beta \in [0, N], \quad j = m + 1, \ldots, n \right\}.
\]
This set is also a basis for $V_k$. The corresponding matrix representations of $A_{\epsilon} |V_k|$ are obtain as follows. First consider the set $\tilde{V}_k$ spanned by
\[
\left\{ (\nabla^k \delta_{\epsilon_j})^T (\cdot - \beta) : \beta \in [0, N], \quad j = 1, \ldots, m \right\} \cup \left\{ \delta_{\epsilon_j}^T (\cdot - \beta) : \beta \in [0, N], \quad j = m + 1, \ldots, n \right\}.
\]
By Proposition 3.1, we get for any $r \geq 1$ and $\epsilon_j \in \{0,1\}$
\[
A_{\epsilon_r} \ldots A_{\epsilon_1} |\tilde{V}_k| = [B_k^{(r)}(\epsilon_1 + \cdots + 2^{r-1} \epsilon_r + 2^r \beta - \beta)]_{\beta = \epsilon_1, \ldots, \epsilon_r, \epsilon_r + 1, \ldots, \alpha \in [0, N]}.
\]
As the spanning set of $\tilde{V}_k$ is not a basis for $V_k$ we need to remove the rows and columns of $A|\tilde{V}_k$ that correspond to

$$\left\{(\nabla^k \delta_{\beta j})^T (\cdot - \beta) : \beta \in [N - k + 1, N], j = 1, \ldots, m\right\}$$

Then the claim follows as in the scalar case.

**Remark 3.4.** (i) The size of the matrices $A|\tilde{V}_k$ in (3) is of importance for numerical computations of $(k, \infty)$-JSR and is determined by the dimension of $V_k$. For a method that allows us to reduce the sizes of these matrices see [27]. In the scalar case, the restriction of $A$ to $V_k$ in (25) is equivalent to removing trivial cycles of $A^*(z)$.

(ii) Note that the choice of the matrix norm on the left hand-side of (24) is crucial. For any other matrix norm the statement of the Theorem 3.3 is not true in general, see examples below. This illustrates one of the advantages of the numerical method arising from the JSR approach and is exploited in [16], where the authors approximate the so-called extremal matrix norm that allows for exact computations of the $(k, \infty)$-JSR.

Let us next illustrate the result of Theorem 3.3 on some examples.

**Example 3.5.** Consider the 4—point scheme with the mask given by

$$A^*(z) = \frac{1}{32} (-1 + 9z^2 + 16z^3 + 9z^4 - z^6) = \frac{1}{32} (1 + z)^2 (-1 + 2z + 6z^2 + 2z^3 - z^4).$$

The corresponding subdivision scheme is $C^1$ [18]. Due to

$$A^*(1) = 1 \quad \text{and} \quad A^*(-1) = 0,$$

by [4, Corollary 3.9] we can compute the difference masks $B_1$ and $B_2$ satisfying the equivalent formulation of (2)

$$(z - 1)^k A^*(z) = B_k^*(z)(z^2 - 1)^k, \quad k = 1, 2,$$

and given by

$$B_1^*(z) = \frac{1}{32} (1 + z) B_2^*(z), \quad B_2^*(z) = \frac{1}{32} (-1 + 2z + 6z^2 + 2z^3 - z^4).$$

By [4, Lemma 2.4] we have

$$V_1 = span \left\{ \nabla \delta (\cdot - \beta) : 0 \leq \beta \leq 5 \right\}$$

and

$$V_2 = span \left\{ \nabla^2 \delta (\cdot - \beta) : 0 \leq \beta \leq 4 \right\}.$$ 

The corresponding matrix representations of $A|\tilde{V}_k$, $k = 1, 2$, are

$$A_0|V_1 \equiv \left[ B_1 (0 + 2\beta - \beta) \right]_{\beta, \beta \in \{0, 5\}} = \frac{1}{16} \begin{bmatrix} -1 & 8 & 1 & 0 & 0 & 0 & 0 & 1 & 8 & -1 & 0 & 0 \\ 0 & 1 & 8 & -1 & 0 & 0 & 0 & 0 & 1 & 8 & 1 & 0 & 0 \\ \ldots & 0 & 0 & 0 & 1 & 8 & -1 & 0 \\ \end{bmatrix},$$

$$A_1|V_1 \equiv \left[ B_1 (1 + 2\beta - \beta) \right]_{\beta, \beta \in \{0, 5\}} = \frac{1}{16} \begin{bmatrix} 1 & 8 & -1 & 0 & 0 & 0 & 0 & 1 & 8 & 1 & 0 & 0 & 0 \\ -1 & 8 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & -1 & 0 & 0 \\ \ldots & 0 & 0 & 1 & 8 & -1 & 0 & 0 \\ \end{bmatrix},$$

and

$$A_0|V_2 \equiv \left[ B_2 (0 + 2\beta - \beta) \right]_{\beta, \beta \in \{0, 4\}} = \frac{1}{16} \begin{bmatrix} -1 & 6 & -1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & -1 & 0 \\ \ldots & 0 & 0 & -1 & 6 & -1 & 0 \end{bmatrix},$$

...
Note that in the vector case, due to the definition of the difference operator \( \tilde{\nabla} \) we first, as in the proof of Theorem 3.3, define 

\[
A_1|v_2 = \begin{bmatrix}
B_2(1 + 2\tilde{\beta} - \beta)
\end{bmatrix}_{\tilde{\beta}, \beta \in \{0, 4\}} = \frac{1}{16} \begin{bmatrix}
2 & 2 & 0 & 0 & 0 \\
-1 & 6 & -1 & 0 & 0 \\
0 & 2 & 2 & 0 & 0 \\
\ldots \\
0 & 0 & 2 & 2 & 0
\end{bmatrix}.
\]

Hence, on the one hand we get

\[
\|S_{B_1}\|_\infty = \max \left\{ \|A_0\|_{v_1}, \|A_1\|_{v_1} \right\} = \frac{5}{8}
\]

and

\[
\|S_{B_2}\|_\infty = \max \left\{ \|A_0\|_{v_2}, \|A_1\|_{v_2} \right\} = \frac{1}{2}.
\]

On the other hand

\[
\|S_{B_1}\|_\infty < \max \left\{ \|A_0\|_{v_1}, \|A_1\|_{v_1} \right\} = 0.7195 \ldots
\]

and

\[
\|S_{B_2}\|_\infty > \max \left\{ \|A_0\|_{v_2}, \|A_1\|_{v_2} \right\} = 0.467 \ldots
\]

The second example is of a vector univariate subdivision scheme.

**Example 3.6.** Let the mask \( A \in \ell_0^{2 \times 2}(\mathbb{Z}) \) be given by its symbol

\[
A^*(z) = \begin{bmatrix}
1/2 & 1/4 \\
0 & 1/4
\end{bmatrix} + \begin{bmatrix}
1/2 & 3/4 \\
0 & 1/2
\end{bmatrix} z + \begin{bmatrix}
1/2 & 1/4 \\
0 & 1/4
\end{bmatrix} z^2.
\]

Due to

\[
A^*(1) \cdot (1, 0)^T = (1, 0)^T \quad \text{and} \quad A^*(-1) \cdot (1, 0)^T = (0, 0)^T,
\]

by [4] Corollary 3.9 there exist the difference mask \( B_1 \) satisfying the equivalent formulation of [2]

\[
\begin{bmatrix}
z - 1 & 0 \\
0 & 1
\end{bmatrix} A^*(z) = B^*(z) \begin{bmatrix}
z^2 - 1 & 0 \\
0 & 1
\end{bmatrix}
\]

and given by

\[
B^*_1(z) = \begin{bmatrix}
1/2 & -1/4 \\
0 & 1/4
\end{bmatrix} + \begin{bmatrix}
1/2 & -1/2 \\
0 & 1/2
\end{bmatrix} z + \begin{bmatrix}
0 & 1/2 \\
0 & 1/4
\end{bmatrix} z^2 + \begin{bmatrix}
0 & 1/4 \\
0 & 0
\end{bmatrix} z^3.
\]

Note that in the vector case, due to the definition of the difference operator \( \tilde{\nabla} \), the support of the difference mask \( B_1 \) is larger than that of \( A \). To compare the entries of \( A_1|v_1 \) to the ones of \( B_1 \), we first, as in the proof of Theorem \( \ref{thm:vector-univariate} \) define \( \tilde{V}_1 \) to be the set spanned by

\[
\left\{ (\nabla \delta e_1)^T : \beta \in [0, 2] \right\} \cup \left\{ \delta e^T_2(-\beta) : \beta \in [0, 2] \right\}.
\]

Then we get

\[
A_1|\tilde{V}_1 = \left[ B_1(\varepsilon + 2\tilde{\beta} - \beta) \right]_{\tilde{\beta}, \beta \in \{0, 2\}}; \quad \varepsilon \in \{0, 1\}
\]

from which we obtain \( 5 \times 5 \) square matrices \( A_1|\tilde{V}_1, \varepsilon \in \{0, 1\} \), by removing one but last rows and one but last columns of the corresponding matrices \( A_1|v_1 \). Note that such a choice of \( A_1|\tilde{V}_1, \varepsilon \in \{0, 1\} \), depends on the structure of the basis of \( V_1 \) given by

\[
\left\{ (\nabla \delta e_1)^T : \beta \in [0, 1] \right\} \cup \left\{ \delta e^T_2(-\beta) : \beta \in [0, 2] \right\}.
\]

And we finally get

\[
\|S_{B_1}\|_\infty = \max \left\{ \|A_0\|_{v_1}, \|A_1\|_{v_1} \right\} = \frac{5}{4}.
\]
3.2 Multivariate scalar and vector cases

The result of Theorem 3.3 does not hold in general in the multivariate case. One of the reasons for that is that the spanning set of $V_k$ in [4, Lemma 4.9] is not a basis for $V_k$, although the representation from Proposition 3.1 is still valid. Another reason is that the result of Lemma 3.2 does not hold in general in the multivariate case, see [4, Example 5.2]. The following example illustrates that the numerical methods arising from the RSR and JSR approaches are indeed two different methods for estimating the $(k, \infty)\text{--}\text{JSR}$.

**Example 3.7.** Consider the scalar multivariate scheme $S_A : \ell(\mathbb{Z}^2) \to \ell(\mathbb{Z}^2)$ given by its symbol

$$A^*(z_1, z_2) = \frac{1}{8} (1 + z_1) (1 + z_2) (1 + z_1 z_2).$$

Due to $A^*(1, 1) = 1$ and $A^*(e) = 1$, $e \in \{1, -1\}^2 \setminus \{1\}$, the first difference scheme satisfying the equivalent formulation of [2]

$$\begin{bmatrix} z_1 - 1 \\ z_2 - 1 \end{bmatrix} A^*(z) = B^*(z) \begin{bmatrix} z_2^2 - 1 \\ z_2 - 1 \end{bmatrix}$$

exists and is given, e.g. by the mask

$$B_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

As all the entries of $B_1$ are non-negative, we apply [4, Remark 4.13] and get

$$\max_{\|\nabla c\|_\infty = 1} \|S_B \nabla c\|_\infty = \max_{\|c\|_\infty = 1} \|S_B c\|_\infty = 1/2.$$

Note that, using [14, Proposition 2.9], we can even compute $\rho_\infty (S_B, \nabla)$. To do that we view $S_B$ as a scheme consisting of two scalar first difference schemes $S_{\tilde{B}_1}$ and $S_{\tilde{B}_1}$ given by the masks

$$\tilde{B}_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

with the entry at bold at the position $(0, 0)$. Then by [14, Proposition 2.9] we get

$$\rho_\infty (S_B, \nabla) = \max \left\{ \rho \left( \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} (\alpha - 2\beta) \right)_{\alpha, \beta \in \Omega}, \rho \left( \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \right)_{\alpha, \beta \in \Omega} \right\}.$$

where $\Omega = [-2, 2]^2 \cap \mathbb{Z}^2$ by [14, p. 345] is any associated good set. A simple computation yields

$$\rho_\infty (S_B, \nabla) = \frac{1}{2} = \max_{\|c\|_\infty = 1} \|S_B c\|_\infty.$$

Let us compare $\rho_\infty (S_B, \nabla)$ with the upper estimate for $(0, \infty)\text{--}\text{JSR}$ produced by [2]. We fix a basis of $V_1$ to be the following set

$$\{ \nabla_1 \delta (\cdot - \beta) : \beta \in \{0, 1\} \times \{0, 2\} \} \cup \{ \nabla_2 \delta (\cdot - \beta) : \beta = (1, 0), (1, 1) \}.$$
Note that using Proposition 3.2 we get

\[ A_{(1,0)} \nabla_2 \delta(- (1, 0)) = \frac{1}{2} \nabla_2 \delta. \]

and

\[ \frac{1}{2} \nabla_2 \delta = \frac{1}{2} \nabla_1 \delta - \frac{1}{2} \nabla_1 \delta(- (0, 1)) + \frac{1}{2} \nabla_2 \delta(- (1, 0)), \]

where \( \nabla_2 \delta \) is not one of the above basis elements of \( V_1 \). Thus, the corresponding row of \( A_{(1,0)}|_{V_1} \)
does not consist of the entries of the difference mask \( B_1 \) as it happens in the univariate case. For all other possible choices of a basis of \( V_1 \) we get analogous structure for at least one of the rows of some of \( A_{s}|_{V_1} \). Therefore, we get

\[ \max_{\epsilon_j \in \{0,1\}^s} \left\| A_{\epsilon_j}|_{V_1} \right\|_\infty = \frac{3}{2} > \| S_{B_1} \|_\infty = \| S_{B_1}|_{V_1} \|_\infty = JSR_\infty. \]

and also

\[ \max_{\epsilon_j \in \{0,1\}^s} \left\| A_{\epsilon_j}|_{V_1} \right\|_2 = 1.188 \ldots > \| S_{B_1}|_{V_1} \|_\infty. \]

Possible extensions of the result in [27] to the multivariate case is currently under investigation. Such an extension will not only allow us to reduce the size of \( A_{\epsilon_j}|_{V_k} \), but also may lead to different \( V_k \) whose structure allows for better comparison of the RSR and JSR approaches.

### 3.3 Divergence of subdivision schemes

This scalar bivariate example is taken from [17]. The mask is given by its symbol

\[ A^*(z_1, z_2) = \frac{1}{4} \left( \frac{1}{4} + z_1 + \frac{3}{4} z_1^2 + \frac{3}{4} z_2 + z_1 z_2 + \frac{1}{4} z_1^2 z_2 \right). \]

Note that the mask satisfies \( A^*(1,1) = 1 \) and \( A^*(1,-1) = A^*(-1,1) = A^*(-1,-1) = 0 \). It has been shown in [1] that

\[ \rho_\infty(A|_{V}) = \sup \left\{ |\lambda|^{1/r} : r > 0, \, \lambda \in \sigma(A_{\epsilon_j}|_{V_1} \cdots A_{\epsilon_j}|_{V_1}), \, \epsilon_j \in \{0,1\}^s \right\}, \tag{26} \]

where \( \sigma(M) \) denotes the spectrum, i.e., the set of all eigenvalues, of the matrix \( M \). In our case, one of the eigenvalues of \( A_{(0,0)}|_{V_1} \) is equal to one, see [17] page 1192 for details, and we get

\[ \rho_\infty(A|_{V_1}) \geq 1. \]

This shows that the scheme is not uniformly convergent [3]. To get the same conclusion using the restricted spectral radius approach, we need to show that there exists \( R \in \mathbb{N} \) such that for all \( r > R \) we have \( \| S_{B_1}|_{V_1} \|_\infty \geq 1 \), which is a tedious task. Another possibility to show that the scheme is not uniformly convergent is to show that \( S_{B_1} \) does not converge to a zero limit function. To do that determine first the difference scheme \( S_{B_1} \) satisfying

\[ \begin{bmatrix} z_1 - 1 \\ z_2 - 1 \end{bmatrix} A^*(z_1, z_2) = B^*(z_1, z_2) \begin{bmatrix} z_1^2 - 1 \\ z_2^2 - 1 \end{bmatrix} \]

with the matrix–valued symbol \( B^*(z_1, z_1) = (b_{i,j}^*)_{i,j=1,2} \) with the entries

\[ b_{11}^*(z_1, z_2) = \frac{1}{16} (1 + 3z_1 + 3z_2 + z_1 z_2), \]
\[ b_{12}^*(z_1, z_2) = 0, \,
\[ b_{21}^*(z_1, z_2) = \frac{1}{16} (2z_2 - 2), \]
\[ b_{22}^*(z_1, z_2) = \frac{1}{16} (z_1^2 + 4z_1 + 3). \]

The symbol satisfies

\[ B_1^*(1,1) = \frac{1}{4} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \]
and does not indicate that the associated scheme is not zero convergent. The problem is though that not all of the eigenvalues of the sub-symbols
\[ B_{1,\varepsilon}(z_1, z_2) = \sum_{\alpha \in \mathbb{Z}^2} B_1(\varepsilon - 2\alpha) z_\alpha, \quad \varepsilon \in \{0, 1\}^2, \]
at \((1, 1)\) are less than 1, e.g.
\[ B_{1,(0,0)}(1, 1) = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}. \]
This violates a necessary condition for the convergence of subdivision, see [9]. Thus, the spectral properties of the difference scheme \( S_{B_1} \) also indicates that the scheme is not uniformly convergent.

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Maria Charina
Fakultät für Mathematik
TU Dortmund
Vogelpothsweg 87
D–44227 Dortmund
maria.charina@uni-dortmund.de