Gravitational recoil in nonspinning black-hole binaries: The span of test-mass results

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We consider binary systems of coalescing, nonspinning, black holes of masses \(m_1\) and \(m_2\) and show
that the gravitational recoil velocity for any mass ratio can be obtained accurately by extrapolating
the waveform of the test-mass limit case. The waveform obtained in the limit \(m_1/m_2 \ll 1\) via a
perturbative approach is extrapolated in \(\nu = m_1 m_2 / (m_1 + m_2)^2\) multipole by multipole using
the corresponding, analytically known, leading-in-\(\nu\) behavior. The final kick velocity computed from this
\(\nu\)-flexed waveform is written as \(v(\nu)/c = 0.04457/\nu^2 \sqrt{1-4\nu} (1 - 2.07106/\nu + 3.93472/\nu^2 - 4.78404/\nu^3 +
2.5204/\nu^4)\) and is compatible with the outcome of numerical relativity simulations.

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I. INTRODUCTION

Interference between the multipoles of the gravitational waves (GW) emitted from coalescing black-hole
binaries of masses \(m_1\) and \(m_2\) carries away linear momentum and thus imparts a recoil to the final merged black
hole. The accurate calculation of this recoil velocity, also referred as kick, has been the topic of analytical and nu-
merical studies in recent years [1–11]. In particular, after assessing the properties of the kick velocity for nonspinning
black-hole binaries, numerical relativity (NR) went to investigate the effect the black-hole spins have on the
final kick. The most interesting and astrophysically relevant result is that high recoil velocities, of about a
few thousands of km/s, can be reached for nonaligned spin configurations [9, 10].

When one black hole is much more massive than the other, \(M \equiv m_2 \gg m \equiv m_1\ (m/M \equiv 1/q \ll 1)\), the
kick is obtained from the GW emission computed using black hole perturbation theory [12, 13]. When the
larger black hole is nonspinning, Ref. [12] used Regge-Wheeler-Zerilli (RWZ) perturbation theory [14] to calcu-
late the GW emission from the transition from inspiral to plunge of a point-source particle subject to leading-
order (LO) analytical (effective-one-body), resummed radiation reaction force. When the larger black hole is spin-
ing, [13] solved the Teukolsky equation with a point-source particle term subject to a numerical, adiabatic,
radiation reaction force. In the nonspinning case, both studies essentially agreed on the value of the final re-
coil velocity: Ref. [13] got \(v/\sqrt{c(m/M)}^2 = 0.044\), using up to \(\ell = 6\) multipoles, while Ref. [12] estimated \(v/\sqrt{c(m/M)}^2 = 0.0446\) using multipoles up to \(\ell = 8\). Reference [13] studied whether the perturbative result can be accurately extrapolated to any mass ratio using the \(\nu\)-scaling corresponding to the LO multipolar contribution [15]

\[
v(\nu)/c = 0.044/\nu^2 \sqrt{1-4\nu}, \tag{1}
\]

where \(\nu = m_1 m_2 / M^2\), with \(M = m_1 + m_2\), is the symmetric mass ratio. It was found that this scaling is rather
inaccurate when \(\nu \sim 0.2\), as it predicts values that are larger by \(\sim 50\%\) than the NR results.

In this paper we show that extrapolating in \(\nu\) the test-mass waveform multipole by multipole up to multipole order \(\ell = 8\) and then computing the recoil from this \(\nu\)-flexed waveform, allows one to get an improved version of the LO scaling that is compatible with the NR results of Refs. [5, 6, 11].

II. EXTRAPOLATING IN \(\nu\) TEST-MASS RESULTS

Let us start by pointing out a systematic flaw in assuming the LO scaling (1). The RWZ-normalized multipolar
decomposition of the waveform is (for equatorial motion)

\[
h_+ - i h_\times = \frac{1}{r} \sum_{\ell=2}^{\ell_{\text{max}}} \sum_{m=-\ell}^{\ell} \sqrt{\frac{(\ell+2)}{(\ell-2)!}} i^\ell \Psi_{\ell m}^{(\epsilon)} Y_{\ell m}(\theta, \phi),
\]

where \(\epsilon = 0, 1\) is the parity of \(\ell + m\). The functions \(\Psi_{\ell m}^{(\epsilon)} = \Psi_{\ell m}^{(\epsilon)}(t, \nu)\), (e.g., computed from a NR simulation), are normalized as in Ref. [12]. In the perturbative context (\(\nu \rightarrow 0\)), they are a solution of the Zerilli (\(\epsilon = 0\) and Regge-Wheeler (\(\epsilon = 1\)) equations with a point-source particle term [12, 16]. The GW linear momentum flux in the equatorial plane is

\[
\mathcal{F}_x + i \mathcal{F}_y = \frac{1}{8\pi} \sum_{\ell=2}^{\ell_{\text{max}}} \sum_{m=-\ell}^{\ell} \left[ a_{\ell m} \Psi_{\ell m}^{(0)} \Phi_{\ell m+1}^{(1)} + b_{\ell m} \sum_{\epsilon=0,1} \Psi_{\ell m}^{(\epsilon)} \Phi_{\ell m+1}^{(\epsilon)} \right], \tag{2}
\]

where the numerical coefficients \((a_{\ell m}, b_{\ell m}) > 0\) are given in Eqs. (16)-(17) of [12], and \(\Psi_{\ell m} = (-1)\ell m \Psi_{\ell -m}\). The (complex) recoil velocity at time \(t\) is obtained as

\[
v_x + iv_y = -\frac{1}{M} \int_{-\infty}^{t} (\mathcal{F}_x + i \mathcal{F}_y) dt'. \tag{3}
\]

For each multipole, the leading-in-\(\nu\) (completely explicit) dependence is [17] \(\Psi_{\ell m}^{(\epsilon)} \propto \nu c_{\ell+\epsilon}(\nu)\), where \(c_{\ell+\epsilon}(\nu) \equiv
TABLE I. Final recoil velocity: comparing the (multipolar) $\nu$-extrapolated RWZ result, $v^{(\nu)}_{\text{end}}$, the leading-order extrapolation, Eq. (1), $v^{\text{RWZLO}}_{\text{end}}$, and the NR values of [11]. As a conservative error estimate, the $v^{\text{RWZ}}_{\text{end}}$ can be larger by 1 to 2\%.

| $q$ | $\nu$ | $v^{\text{NR}}_{\text{end}}$ [km/s] | $v^{\text{RWZ}}_{\text{end}}$ [km/s] | $v^{\text{RWZLO}}_{\text{end}}$ [km/s] |
|-----|------|-----------------|-----------------|-----------------|
| 2   | 0.2  | 148 $\pm$ 2    | 151.3           | 219.9           |
| 3   | 0.1875 | 174 $\pm$ 6    | 169.5           | 234.8           |
| 4   | 0.1600 | 157 $\pm$ 2    | 154.2           | 205.2           |
| 6   | 0.1224 | 114 $\pm$ 6    | 114.1           | 143.1           |

$X^{\nu+e-1}_i + (-)^m X^{\nu+e-1}_1$, with $X_i = m_i/M$ so that $X_1 + X_2 = 1$ and $X_1X_2 = \nu$. The convention we adopt here is $X_2 > X_1$, i.e., $X_2 - X_1 = \sqrt{1 - 4\nu}$, so that $c_{\nu+e}(0) = 1$. The explicit $\nu$-dependence in Eq. (2) comes as sum of products of $c_{\nu+e}(\nu)$. Defining individual rescaled fluxes as

$$F^{(\nu)}_x + iF^{(\nu)}_y = \nu^2 \sqrt{1 - 4\nu} \left\{ \tilde{F}_{223-3} + \tilde{F}_{223-2} + \tilde{F}_{222-2} + \cdots + (1 - 3\nu) \tilde{F}_{334-4} + \cdots + (1 - 3\nu)(1 - 2\nu)\tilde{F}_{445-5} + \cdots \right\},$$

(4)

where we wrote just a few terms to indicate that the explicit (leading) $\nu$-dependence of the flux is more complicated than just the LO one. Let us consider now the $\nu \to 0$ gravitational waveform $\Psi^{(\nu)}_{\ell m}(t; 0)$ obtained solving the RWZ equations with a point-particle source subject to leading-order, resummed, analytical radiation reaction force. The mass ratio is $m/M = 10^{-3}$. This waveform was computed in Ref. [18] using the hyperboloidal layer approach [19], which allowed us to: i) extract waves at $\ell \to \infty$; ii) obtain high-resolution data (the numerical error is not an issue). The quasicircular inspiral starts at $r_0 = 7M$. The recoil velocity obtained from Eq. (2) with $\ell_{\text{max}} = 7$ is $v(0)/[c(m/M)^2] = 0.04457$, consistent with [12].

To extrapolate in $\nu$ the multipolar waveform, we take $\Psi^{(\nu)}_{\ell m}(\ell; 0) \equiv \nu \Psi^{(\nu)}_{\ell m}(\ell; 0)/(m/M)$, multiply it by the corresponding leading-order $\nu$ dependence, so to get the $\nu$-dependent function (addressed as RWZ, in the following) $\Psi^{(\nu)}_{\ell m}(\ell; 0) \equiv \nu \Psi^{(\nu)}_{\ell m}(\ell; 0)$. The notation $0_\nu$ is a reminder that only the leading order $\nu$ dependence of each multipole is included and so $\Psi^{(\nu)}_{\ell m}(\ell; 0) \neq \Psi^{(\nu)}_{\ell m}(\ell; 0)$. Using $\Psi^{(\nu)}_{\ell m}(\ell; 0)$ in Eq. (2) we get the linear momentum flux versus time and then the kick velocity via Eq. (3). Since the waveform starts at time $t_0 > -\infty$, the boundary condition $\nu v_0 \equiv \int_{t_0}^{\infty} (F^{(\nu)}_x + iF^{(\nu)}_y) dt$ in Eq. (3) is fixed as the center of the velocity hodograph during the inspiral [12].

Table I compares the final kick velocity $v \equiv |v_x + iv_y|$ obtained from the RWZ, waveform with the most recent NR calculations [11], using the SpEC [22] code, with $q = (2, 3, 4, 6)$ (and retaining only multipoles with $\ell \leq 6$). The extrapolated values are very close to the NR ones, in two cases within their error bars. By contrast, the last column of the table highlights how inaccurately the leading-order scaling is. The uncertainty on the RWZ, values has essentially two sources: (i) the fact that $m/M \ll 1$, but always $m/M \neq 0$ and (ii) the effect of multipoles selected by the condition $\ell_{\text{max}} > 7$. In Table III of Ref. [12] it was shown that changing $m/M = 10^{-3}$ to $m/M = 10^{-4}$ was increasing the final kick by $\sim 0.5$%. In
addition, we checked that the relative difference between taking $\ell_{\text{max}} = 6$ \((v(0)(M/m)^2 = 0.04383)\) and $\ell_{\text{max}} = 7$ \((v(0)(M/m)^2 = 0.04457)\) is as large as \(\sim 1.7\%\) when \(m/M = 10^{-3}\), but becomes as small as \(10^{-3}\) for $q = 6$ and \(10^{-4}\) for $q = 2$. As a conservative error estimate, the extrapolated values of Table I can be larger by $1$ to $2\%$.

Figure 1 compares $v(\nu)$ with $0 \leq \nu \leq 0.25$ (solid curve, red online) with available fits obtained from the comprehensive numerical study of Refs. [5, 6]. We also show the data of Ref. [11]. The data of Refs. [5, 6] are represented by two different fits: $v_{\text{NR}} = 1.20 \times 10^\nu \sqrt{1 - 4\nu(1 - 0.93\nu)}$ (dashed, blue online), proposed in Ref. [5] without including the $q = 10$ data of [6], and $v_{\text{NR}} = 0.0436\nu^2 \sqrt{1 - 4\nu(1 - 1.3012\nu)}$, with $c = 299792.458$ km/s (dot-dashed) done in [11] including the $q = 10$ data. The maximum value of the RWZ$_\nu$ curve is $v_{\text{max}} = 170.164$ km/s (at $\nu = 0.194$), quite close to $v_{\text{max}} = 175.2 \pm 11$ km/s computed in [5]. A more precise quantitative information is given by (bottom panel of Fig. 1) the normalized quantity $\tilde{f} = v(\nu)/v(0)^2 \sqrt{1 - 4\nu}$ obtained from the extrapolated $v(\nu)$ (solid line). For completeness, we also exhibit the raw NR data of Refs. [5, 6, 11] as well as those of Refs. [20, 21] for the challenging values $q = 15$ and $q = 100$, the highest simulated so far. Note that for these $q$’s the recoil velocity is systematically underestimated since the multipoles with $\ell > 4$ were neglected in Refs. [20, 21]. Notably, if the extrapolation is done retaining only the multipoles with $\ell \leq 4$, the RWZ$_\nu$ result for $q = 15$ and $q = 100$ (red circles in the bottom panel of Fig. 1) is compatible with the NR points. The complete RWZ$_\nu\tilde{f}(\nu)$ curve is accurately fitted $(\Delta \tilde{f} \equiv \tilde{f} - \tilde{f}_{\text{RWZ}_\nu} \sim 10^{-5})$ by the quartic trend $\tilde{f}(\nu) = 1 - 2.07106\nu + 3.93472\nu^2 - 4.78404\nu^3 + 2.52040\nu^4$. A cubic trend yields instead $\tilde{f}(\nu) = 1 - 2.06407\nu + 3.76663\nu^2 - 3.60498\nu^3$ with $\Delta \tilde{f} \sim 10^{-4}$, undistinguishable on the scale of Fig. 1. Note that the (less accurate) quadratic trend was instead suggested in both Ref. [1] using the effective-one-body formalism and Ref. [2] using the close-limit approximation. It would be interesting to extract $\tilde{f}(\nu)$ accurately from ad hoc NR simulations.

*Time evolution of kick velocity.* We investigate now if the $\nu$-extrapolation is able to reproduce the structure of the well-known (post-merger) local maximum of $v(t)$, predicted and analytically explained in [1] (see also [23]) and now known as “antikick” [3, 24]. Since this information is not given in [11], we have to compute $v_{\text{NR}}(t)$ from the (limited) number of NR $(\ell, m)$ waveform multipoles of [11] to which we have access. For both NR and RWZ$_\nu$ we use $\Psi^{(P)}_\nu$ with $m = \ell$ up to $\ell = 6$ plus (2.1) and (3.2). Table II lists the final and maximum velocity obtained from NR (boldface) and RWZ$_\nu$ data (cf. with Table I), together with the magnitude of the antikick, $\Delta \hat{v} \equiv \max(\hat{v}) - \hat{v}_{\text{end}}$, with $\hat{v} \equiv v(t)/(c\nu^2 \sqrt{1 - 4\nu})$. Even with a limited number of multipoles, the $\nu$-extrapolated $v_{\text{end}}$ is accurate; by contrast, the extrapolated antikick is much smaller than the corresponding NR one. The table is complemented by the main panel of Fig. 2, where we contrast the $q = 2$ $\hat{v}(t)$ for both NR and RWZ$_\nu$ data (the original $\nu \to 0$ curve is also added for completeness). Note that $\hat{v}(t)$ is plotted versus $t \equiv t - t_{\text{max}}$, where $t_{\text{max}}$ corresponds to the maximum of $F_P \equiv |F^P_x + iF^P_y|$. The vertical line indicates the NR merger. Inset: corresponding analytical approximations, Eq. (5), to $\hat{v}(t)$. The nonextrapolated $\nu \to 0$ curves are also shown for completeness.

![Figure 2](image-url)
TABLE III. Characterization of max($\mathcal{F}_p$) for the NR (boldface) and RWZ$_\nu$ waveforms (with a restricted sample of dominant multipoles). Here is $\tilde{\mathcal{F}}_p^\text{max} \equiv \mathcal{F}_p^\text{max}/\nu^4 \times 10^3$. The analytical estimate $v^\text{end}$ of the final recoil velocity (last two columns) is obtained from Eq. (6).

| $q$ | $\tilde{\mathcal{F}}_p^\text{max}$ | $\tau_{\text{max}}$ | $Q^\text{max}$ | $\epsilon_{\text{max}}$ | $v^\text{end}$ [km/s] | $v^\text{end}$ |
|-----|-------------------------------|-----------------|----------------|-----------------|-----------------|----------------|
| 2   | 3.009                         | 7.505           | 1.770          | 0.011           | 174.85          | 0.0354         |
|     | 1.463                         | 7.780           | 1.298          | -0.486          | 202.57          | 0.0410         |
| 3   | 4.222                         | 7.485           | 1.666          | -0.028          | 208.47          | 0.0396         |
|     | 2.330                         | 7.823           | 1.319          | -0.465          | 224.30          | 0.0426         |
| 4   | 4.816                         | 7.526           | 1.607          | -0.065          | 192.39          | 0.0418         |
|     | 2.930                         | 7.858           | 1.335          | -0.447          | 201.621         | 0.0438         |
| 6   | 5.347                         | 7.689           | 1.552          | -0.136          | 141.29          | 0.0440         |
|     | 3.730                         | 7.905           | 1.356          | -0.422          | 146.07          | 0.0455         |
| $\infty$ | 6.499                  | 8.043           | 1.418          | -0.330          | 146.07          | 0.0455         |

nonadiabatic character of the evolution of the momentum flux, this integral is dominated by what happens near max[$\mathcal{F}_p(t)$]. Expanding around $\tau_{\text{max}}$ one gets [1]

$$v_x + iv_y \simeq i\mathcal{F}_p^\text{max} e^{i\tau_{\text{max}}} \sqrt{\frac{\pi}{2\alpha}} e^{-\beta^2/(2\alpha)} \text{erfc}(z), \tag{5}$$

with $z = -\sqrt{\alpha/2} (\bar{t} - \beta/\alpha)$, where $\alpha \equiv 1/\tau_{\text{max}}^2 (1 - i\epsilon_{\text{max}})$ and $\beta = iQ/\tau_{\text{max}}$. Here $\tau_{\text{max}}^2 \equiv -\mathcal{F}_p^\text{max}/(d^2\mathcal{F}_p/d\tau^2)_{\text{max}}$ is the characteristic time scale associated to the resonance peak of $\mathcal{F}_p$: $Q \equiv \omega_{\text{max}}\tau_{\text{max}}$, where $\omega \equiv \dot{\varphi}$ can be interpreted as the “quality factor” associated to the same peak, and $\epsilon_{\text{max}} \equiv \omega_{\text{max}}\tau_{\text{max}}^2$. When $\bar{t} \gg \tau_{\text{max}}$, the integrated recoil is analytically expected to be [1]

$$v^\text{end}_\lambda \simeq \sqrt{2\pi \mathcal{F}_p^\text{max}} \frac{\tau_{\text{max}}}{(1 + \tau_{\text{max}}^2)^{1/4}} e^{-Q^2/[2(1 + \epsilon_{\text{max}}^2)]}. \tag{6}$$

All relevant information to numerically evaluate Eqs. (5)-(6) for NR (boldface) and RWZ$_\nu$ data is listed in Table III. Several observations can be made. First, the presence of the antikick is qualitatively explained by the behavior of the complementary error function $\text{erfc}(z)$, Eq. (5), when $z$ is complex. Since $\epsilon_{\text{max}}$ is small, one sees that $\Im(z)$ is essentially given by $Q$ [1]. When $Q > 0$ the usual, monotonic, behavior of $\text{erfc}(z)$ is modified so that a local peak (the antikick) appears (see inset of Fig. 2). In particular, when $Q$ is small one finds small or negligible antikicks; when $Q$ is larger the antikicks are larger. Second, looking at the values of Table III one sees that, from the quantitative point of view the analytical result leads to estimates of $v^\text{end}_\lambda$ that are always systematically larger than the exact one, from $\sim 25\%$ ($q = 2$) to $\sim 38\%$ ($q = \infty$). Third, focusing on the RWZ$_\nu$ data, from Table III one sees that the values of $\tau_{\text{max}}$ and $Q$ do not vary much with the extrapolation with respect to the test-mass ones, contrary to $\mathcal{F}_p^\text{max}$, which is then the main responsible of getting $v^\text{end}_\lambda$ smaller than in the $\nu \rightarrow 0$ case. This gives a qualitative, analytical, consistency check of Table I and Fig. 1. In addition, from Table III one sees that $Q$ is always larger in the NR case than in the RWZ$_\nu$ one, which explains qualitatively Table II. The reason for this is that the extrapolation acts only on the waveform modulus, and not on its phase (and frequency). As $Q = \omega_{\text{max}}\tau_{\text{max}}$, in the RWZ$_\nu$ case $\omega_{\text{max}}$ is still driven by the underlying, less bound, dynamics of a particle on Schwarzschild spacetime, which, during late plunge and merger, spans frequencies that are smaller than the corresponding (more bound) NR ones. Similarly one explains the dependence of $\Delta \dot{v}$ on $q$.

IV. CONCLUSIONS

In the context of coalescing, nonspinning, black-hole binaries, we have found a simple way to correct the leading-order $\nu$-extrapolation of the recoil velocity in the test-mass limit, Eq. (1) (obtained via a perturbative approach) that is fully compatible with state-of-the-art numerical relativity simulations. Our approach is based on extrapolating in $\nu$ the test-mass waveform multipole by multipole using the corresponding leading-in-$\nu$ behavior before computing the recoil. An analogous $\nu$-extrapolation to get the final recoil velocity can be applied to the the waveform generated by a (spinning) particle plunging on a Kerr black hole. In this case, the subtlety is to separately extrapolate in $\nu$ the spin-dependent and the spin-independent part of the waveform because of their different, leading-order, $\nu$-dependence. The accuracy of the procedure will be discussed in future work.

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