HYPERCOMPLEX QUANTUM MECHANICS

L.P. Horwitz
School of Physics
Raymond and Beverley Sackler Faculty of Exact Sciences
Tel Aviv University, Ramat Aviv 69978, Israel

and

Department of Physics
Bar Ilan University, Ramat Gan 52400, Israel

Abstract The fundamental axioms of the quantum theory do not explicitly identify the algebraic structure of the linear space for which orthogonal subspaces correspond to the propositions (equivalence classes of physical questions). The projective geometry of the weakly modular orthocomplemented lattice of propositions may be imbedded in a complex Hilbert space; this is the structure which has traditionally been used. This paper reviews some work which has been devoted to generalizing the target space of this imbedding to Hilbert modules of a more general type. In particular, detailed discussion is given of the simplest generalization of the complex Hilbert space, that of the quaternion Hilbert module.

Key Words: Octonions, Clifford Algebras, quantum theory, quaternion Hilbert modules, tensor product.
1. Introduction

In his discussion of the development of the theory of matrices in the middle of the nineteenth century, in which he remarked that “it seems almost uncanny how mathematics now prepared itself for its future service in quantum mechanics,” Max Jammer recounted how the natural generalization of the real numbers to complex numbers and quaternions played a central role. He cites Tait as attributing to Hamilton the discovery of matrices in a letter to A. Cayley, who discovered that quaternions could be represented as \(2 \times 2\) matrices over complex elements. Hamilton invented the quaternions in 1844; Tait referred to Hamilton’s “linear and vector operators,” and called Cayley’s discovery only a modification of Hamilton’s ideas. Taber, in 1890, renewed the claim that Hamilton had indeed originated the theory of matrices. Gibbs “regarded 1844 as a ‘memorable’ year in the annals of mathematics because it was the year of the appearance of Hamilton’s first paper on quaternions.”

John von Neumann, in fact, emphasized the result of Hurwitz, that there are just four normed division algebras, the real \(\mathbb{R}\), complex \(\mathbb{C}\), quaternion \(\mathbb{H}\), and octonion, or Cayley, algebra \(\mathbb{O}\), and that “nature must make use of them.”

Herman Goldstine and I set out to investigate the possibility of constructing a Hilbert space, with application to a more general form of the quantum theory, using the Cayley numbers \(\mathbb{O}\) as coefficients on the vector space (they are not commutative or associative).

In fact, the octonions arise in a natural way as a result of the attempt of Jordan, von Neumann and Wigner to set up a quantum theory in which the products of observables (represented, in general, by non-commuting self-adjoint operators on the complex Hilbert space) remain observables. They defined multiplication as the symmetric product \(A \circ B = \frac{1}{2} (AB + BA)\). This definition gives rise to a set of algebraic relations, and one may ask for the solution to the inverse problem, i.e., to ask for which algebraic structures could these relations be valid. The answer, provided by Albert, was that only the symmetric product of matrices on the real, or complex, are allowed, with one exception, the \(3 \times 3\) matrices over octonions.

The lack of associativity formed an obstacle to us in constructing the adjoint of operators on the Hilbert space. We found, however, in the construction of the representation of a vector in terms of a complete orthogonal set, that the alternative property of the octonions admits the closure of the subspace generated by (successively associated) products of the vector with octonion elements to order seven, i.e., after multiplication seven times by octonions, the subspace no longer grows. The algebra of successive multiplications is isomorphic to the Clifford algebra of order seven. We found, furthermore, that the minimal ideals of this Clifford algebra reduce products of Clifford elements into single elements, resulting in effective multiplication laws that reproduce those of the non-associative Cayley algebra. We therefore turned to the general problem of constructing Hilbert modules over Clifford algebras, in particular, that of \(C_7\), the Clifford algebra of order seven.

The automorphism group of the Clifford algebra \(C_7\) which stabilizes a minimal ideal is isomorphic to the automorphism group of the octonions, i.e., \(G_2\). In 1965, this result enabled Biedenharn and me to generalize the quark-lepton model of Günaydin and Gürsey, set in the framework of an octonionic Hilbert space, to the Hilbert module over \(C_7\).

In the following, I discuss some properties of the octonionic Hilbert space, and how
this structure leads to a Clifford Hilbert module. In the succeeding sections, I specialize to the simplest of the Clifford algebras beyond the complex, which is also a division algebra, that of the quaternions. In this work, spanning many years of attention, the historical account and scientific evaluation given in Max Jammer’s book was a strong element of encouragement. It is a pleasure to dedicate this review to him on the occasion of his eightieth birthday.

2. Octonionic and Clifford Modules

To see how the Clifford module structure emerges, consider the definition\(^8\) of the octonion Hilbert space. If \(f, g \in \mathcal{H}\), we suppose that \(fa + gb \in \mathcal{H}\), \((f, f) = \|f\|^2 \geq 0\) (\(f = 0\) in case of the equality), and the linearity property \((fa, fa) = \|f\|^2|a|^2\), where the constants \(a, b\), of the form \(\sum_{i=0}^{7} \lambda_i e_i\), for \(\{\lambda_i\}\) real, are elements of the involution \(e_i^* = -e_i, i \neq 0\), \(e_i e_j = -e_j e_i, e_i^2 = -1\) (non-zero \(i \neq j\)), \(e_1 e_2 = e_3, e_5 e_6 = e_4, e_4 e_2 = e_6, e_6 e_3 = e_5, e_6 e_7 = e_1, e_5 e_7 = e_2, e_4 e_7 = e_3\) (and cyclic), \(e_0 = 1\). These are seven (associative) quaternion subalgebras. The non-associativity follows from the relation \(e_i e_j e_k = -(e_i e_j) e_k\) for \(i, j, k\) not all in one of the quaternion subalgebras. Assuming that the space is separable, one may construct an orthonormal set of vectors \(\{\varphi_n\}\) which generate subspaces \(\{\varphi_n a_n\}\) that span the space\(^6\). Orthogonalization is carried out by a variational principle, by means of which we can construct, for any vector \(g\), a part in \(\{fa | a \in \mathbf{O}\}\), and a part orthogonal to this manifold. The orthogonal part is found by imposing the requirement that \(g - fa = h\) be minimum (in norm), and hence \(\|h + fb\| \geq \|h\|\).

With this, one obtains \(\text{Re}(h, fb) = 0\) for any \(b\). The factor \(b\) may be extracted under the real part\(^{14}\), so that \(\text{Re}((h, fb)) = 0\), or \(h, f) = 0\). But, if we expand some arbitrary vector \(g\) in terms of the orthogonal expansion \(\sum \varphi_n a_n\), in general \(\varphi_m \varphi_n a_n \neq 0, n \neq m\), so we cannot solve for the coefficients \(a_n\). One may then attempt to widen the set of subspaces to the form \((fa)b\); the orthogonalization can be carried out in the same way. But again, one cannot solve for the coefficients in \(\sum (\varphi_n a_n)b_n\).

The algebra has, however, the alternative property\(^{14}\), for which \((ab)b = ab^2\), so one may assume \((fa)a = fa^2\) for any \(f \in \mathcal{H}\). It follows that \((fa)b + (fb)a = f(ab + ba)\), so that for the octonion elements, for example, \((fe_1)e_2 + (fe_2)e_1 = 0\). Hence the growth of the size of the subspace generated by a vector \(f\) is limited by the multiplication, successively, by only seven octonion elements. The operation of successive multiplication \((\cdots ((fa)b) \cdots c)\) is necessarily associative, and the seven elements of the Cayley algebra, with unity, generate in this way the algebra of the seventh Clifford algebra \(C_7\).

The elements of the Clifford algebra \(C_7\) close on a group of 256 elements; since (we now drop the associating parentheses) \(e_1 e_2 \cdots e_7\) commutes with all elements, and its square is one, we can define the algebraic projections \(P_\pm = \frac{1}{2}(1 \pm e_1 e_2 \cdots e_7)\). Defining a new scalar product in the module over \(C_7\) so that \((f, g) = (g, f)^\dagger \in C_7, \text{Tr}(f, ga) = \text{Tr}(fa^\dagger, g)\) (\(C_7\) is isomorphic to a matrix algebra)\(^{11}\), we see that the (right-acting) algebraic projections \(P_\pm\) are Hermitian (this involution is carried to \(e_i^* = -e_i, i \neq 0\)). The norm is defined by \(\|f\|^2 = \text{Tr}(f, f)\). We remark that transition probabilities in such a theory are calculated by \(P_{\psi \chi} = \text{Tr}[(\psi, \chi)(\chi, \psi)]\), if \(\psi, \chi\) are normalized.
The Clifford projections

\[ P_0^\pm = \frac{1}{8}(1 - e_1e_2e_3)(1 - e_4e_2e_6)(1 - e_4e_5e_1)P_\pm \]

are minimal ideals in each sector of the split representation. The elements

\[ e_1e_2e_3, e_4e_2e_6, \text{ and } e_4e_5e_1 \]

commute with each other and are Hermitian. The representation of \( C_7 \) is reduced by \( P_\pm \), and the \( 8 \times 8 \) matrices in each of the two invariant subspaces are carried algebraically by the \( e_{ij}^\pm = e_iP_0^\pm e_j^\dagger, i, j = 0, \ldots, 7 \), corresponding to matrices of all zero elements but for the \( i, j \) component, which is unity. It is remarkable that, as mentioned above, for example,

\[ e_1e_2P_0^\pm = e_3P_0^\pm. \]

The remaining non-associative multiplication laws of the Cayley algebra reappear in this way as well (with a sign change in those involving \( e_7 \) in the \( \pm \) sectors).

Hence the automorphisms of \( C_7 \) that preserve \( P_0^\pm \) are the same as those of the Cayley algebra, constituting the group \( G_2 \).

We leave the generalized structure of the \( C_7 \) module for discussion elsewhere, and concentrate in the following on the properties of the simplest Clifford module beyond the complex, that of \( C_2 \), equivalent to the quaternions (generated by \( 1 \equiv e_0, e_1, e_2, \text{ and } e_3 \equiv e_1e_2 \)).

### 3. Quaternionic Quantum Mechanics

The quaternionic Hilbert space\(^{15}\) is defined by a set of elements \( f \in \mathcal{H} \); if \( f \in \mathcal{H}, g \in \mathcal{H}, fa + gb \in \mathcal{H}, \) where \( a, b \in \mathbb{H} \), the algebra of quaternions defined as real linear combinations of the \( \{e_i\} \), where \( e_i^2 = -1, e_1e_2e_3 = -1, e_i^* = -e_i, i \neq 0, \) and \( (f, g) = (g, f)^* \in \mathbb{H} \), with \( (f, f) = \|f\|^2 \geq 0 \) defining the norm. Transition probabilities are given by \( P_{\psi\chi} = |(\psi, \chi)|^2 \), which coincides with the general form given above for Clifford algebras since the quaternions are a normed division algebra as well.

The scalar product has the linearity property \( (f, ga) = (f, g)a, \forall a \in \mathcal{H} \). Such a Hilbert module can be used to represent the quantum theory\(^{15,16,17}\); this structure contains the usual complex theory and its results, but predicts new effects as well.

In order to construct some physical observables in terms of self-adjoint and anti-self-adjoint operators, let us define the action of translation by means of the representation of a state vector on the spectral resolution of the self adjoint operator of position \( X \) (the spectral theorem of von Neumann is true in the quaternion Hilbert space\(^{15,16}\)) as

\[ \langle x + \delta x|f \rangle = \langle x|T(\delta x)f \rangle, \quad (3.1) \]

where \( T(\delta x) \) is a unitary operator. Expanding to lowest order in \( \delta x \), so that

\[ T(\delta x) = 1 + \delta xS + O(\delta x^2), \quad (3.2) \]
it follows from (3.1) that
\[ \langle x | S f \rangle = \frac{\partial}{\partial x} \langle x | f \rangle, \] (3.3)
where \( S \) is the anti-self-adjoint operator
\[ S = \int |x\rangle \frac{\partial}{\partial x} \langle x | dx. \] (3.4)

If we define the quaternion linear operator (satisfying \( P(fa) = (Pa)f \))*
\[ P = \hbar E_i S \]
\[ = -\hbar \int |x\rangle e_i \frac{\partial}{\partial x} \langle x | dx, \] (3.5)
where \( E_i \) belongs to a left algebra isomorphic to \( \mathbf{H} \),
\[ E_i = \int |x\rangle e_i \langle x | dx, \] (3.6)
then one finds that
\[ [X, P] = -\hbar E_i, \] (3.7)
and, by a proof somewhat more involved than that for the complex Hilbert space\(^{16}\),
\[ \Delta X \Delta P \geq \hbar / 2. \] (3.8)

The existence of such left algebras is important (in the presence of these left algebras, the vector space is often called a bimodule) for the construction of tensor product spaces to represent many-body systems, as well as in applications to the one particle problem; I recall here a basic representation theorem\(^{16,18}\). Let us define a left acting algebra \( \{E_i\} \) isomorphic to \( \mathbf{H} \), for which the operator norm is unity, i.e., \( \sup \|E_i h\| = \|E_i\| \|h\| = \|h\| \).

Then, for any \( g, f \in \mathcal{H} \), there is a unique decomposition
\[ f = \sum_i E_i f_i, \quad g = \sum_i E_i g_i \] (3.9)
where \( E_i f_i = f_i e_i, E_i g_i = g_i e_i \), for which \( (f_i, g_j) \) is real. We call the vector valued coefficients \( \{f_i\} \) “formally real”. The operators \( \{E_i\} \) may be of the form (3.6), but there are, in principle, an infinite number of such algebras.

It is interesting to examine the structure of the Schrödinger equation in this context. As Adler\(^{17}\) has shown, the time independent Schrödinger equation implies the existence of an “optical potential” which can break time reversal invariance in a very natural way. Consider the time dependent Schrödinger equation
\[ \frac{\partial \psi}{\partial t} = -\hat{H} \psi, \] (3.10)

* There are, as well, complex linear operators \( A(fz) = (Af)z, z \in \mathbb{C}(1, e), e^2 = -1 \), but for which \( A(fa) \neq (Af)a \) in general, and operators that are real linear only.
where $\psi$ is a quaternion-valued $L^2$ function and $\tilde{H}$ is an anti-self-adjoint Hamiltonian operator. The stationary problem corresponds to

$$\tilde{H}\psi = \psi eE,$$  \hspace{1cm} (3.11)

where $E$ is real, and $e$ is an imaginary quaternion (as required by the anti-self-adjoint property of $\tilde{H}$). Multiplying (3.11) by the quaternion $q$ on the right, we obtain

$$\tilde{H}\psi q = \psi(q^{-1}eq)E,$$  \hspace{1cm} (3.12)

and by an appropriate choice of $q$, the imaginary unit $e$ can be brought, say to $e_1$, and the sign of $E$ assumed positive. The standard form of (3.11) can therefore be written as

$$\tilde{H}\psi = \psi e_1 E, \hspace{1cm} E \geq 0. \hspace{1cm} (3.13)$$

Now, every quaternion

$$a = a_0 + \sum_{i=1}^{3} a_i e_i \hspace{1cm} (\{a_i\text{real})$$

can be written as

$$a = a_{\alpha} + e_2 a_{\beta}, \hspace{1cm} (3.14)$$

where $a_{\alpha} = a_0 + a_1 e_1$, $a_{\beta} = a_2 - a_3 e_1$, so that

$$\psi(x) = \psi_{\alpha}(x) + e_2 \psi_{\beta}(x), \hspace{1cm} (3.15)$$

where

$$\psi_{\alpha}(x) = \psi_0(x) + e_1 \psi_1(x)$$

$$\psi_{\beta}(x) = \psi_2(x) - e_1 \psi_3(x). \hspace{1cm} (3.16)$$

The relations (3.15), (3.16) are the $x$-representation of

$$\psi = \psi_{\alpha} + E_2 \psi_{\beta}$$

$$\psi_{\alpha} = \psi_0 + E_1 \psi_1$$

$$\psi_{\beta} = \psi_2 + E_1 \psi_3$$

in the abstract quaternionic Hilbert space, where the $\{E_i\}$ are of the form (3.6).

If $\tilde{H}$ is quaternion linear, it also has the decompositon (in $x$-representation, as in (3.15))

$$\tilde{H} = H_{\alpha} + e_2 H_{\beta} = -\tilde{H}^\dagger \hspace{1cm} (3.18)$$

where $H_{\alpha} = e_1 H_1, H_1^\dagger = H_1$. Decomposing (3.13) by components, one finds

$$e_1 H_1 \psi_{\alpha} - H_{\beta}^* \psi_{\beta} = e_1 E \psi_{\alpha}$$

$$-e_1 H_1^* \psi_{\beta} + H_{\beta} \psi_{\alpha} = e_1 E \psi_{\beta}, \hspace{1cm} (3.19)$$
where the asterisk indicates complex conjugation \((e_1 \rightarrow -e_1)\) in the complex subalgebra \(\mathbb{C}(1, e_1)\) of \(\mathbb{H}\). Solving for \(\psi_\beta\) in the second of (3.19) and substituting into the first, we obtain the complex Schroödinger type equation

\[
H_{eff} \psi_\alpha = E \psi_\alpha,
\]

where

\[
H_{eff} = H_1 + H_\beta^* \frac{1}{E + H_1^{-1}} H_\beta \equiv H_1 + V_{opt}(e),
\]

and the effective Hamiltonian on the complex degrees of freedom contains the dynamics of the quaternionic sector in the form of an “optical potential”. Clearly, the quaternionic degrees of freedom can break time reversal invariance.

4. Spectral Properties and the Time Operator

In this section, I discuss the spectral properties of quaternion anti-self-adjoint operators, their implication for the existence of a “time operator” and the possibility of describing irreversible processes.

The definition of the spectrum of \(\tilde{H}\) in (3.13) appears to admit of only positive spectra. In fact, (3.12) indicates that the spectrum of a quaternionic anti-self-adjoint operator is a sphere; only the two points \(\pm E\) for each \(E\) will concern us now. Multiplying (3.13) by \(e_2\) on the right, one obtains

\[
\tilde{H}(\psi e_2) = \psi e_2(-e_1 E),
\]

so that if \(\psi\) is an eigenfunction with eigenvalue \(E\), \(\psi e_2\) is an eigenfunction with eigenvalue \(-E\). If we suppose an absolutely continuous spectrum for \(E\) in \((0, \infty)\), then \(\tilde{H}\) has an absolutely continuous spectrum \(^{19}\) in \((-\infty, \infty)\). Discrete spectra imbedded on this continuum (and interacting with it) may lead to resonances with interesting properties \(^{20}\), which will be discussed briefly below.

The spectral representation for an anti-self-adjoint operator is (in the Dirac form for absolutely continuous spectrum)

\[
\tilde{H} = \int_0^\infty dE |E\rangle \psi_i E \langle E|,
\]

so that in the sense of generalized eigenfunctions,

\[
\tilde{H} |E\rangle = |E\rangle \psi_1 E.
\]

We therefore define

\[
|E\rangle e_2 \equiv | - E\rangle.
\]

Then, calling the complex part (the \(\alpha\)-component) of \(\langle E|\psi\rangle\) by \(c\langle E|\psi\rangle\), we have

\[
\begin{align*}
    c\langle E|\psi\rangle &= \psi_\alpha(E) \\
    c\langle -E|\psi\rangle &= \psi_\beta(E)
\end{align*}
\]
and it follows that, with these definitions (in an obvious notation),

\[
\tilde{H} = \int_{-\infty}^{\infty} dE |E\rangle \langle e_1^n E | e_1^n \langle E|,
\]

displaying explicitly the negative spectrum. There exists an operator \(T\) such that (in units \([T] = [\tilde{H}^{-1}]\))

\[
[T, \tilde{H}] = 1,
\]

where \(T\) is the Hermitian operator

\[
T = -\int_0^\infty dE |E\rangle e_1^n \frac{\partial}{\partial E} \langle E| = \int_{-\infty}^{\infty} dE |E\rangle e_1^n \frac{\partial}{\partial E} c \langle E|.
\]

It is a straightforward consequence of the symmetry of the spectrum that the quaternionic analog of the Lee-Friedrichs model\(^{21}\), based on an unperturbed Hamiltonian with a discrete eigenvalue embedded in an absolutely continuous spectrum with perturbation that connects the continuum only to the discrete state, develops a complex pole below the real negative axis as well as the usual pole below the real positive axis. The resulting interference term in the decay law can lead to oscillations, and may conceivably be observable\(^{20}\).

Misra, Prigogine and Courbage\(^{21}\) have stressed the importance of the existence of a time operator for the description of irreversible processes. Their demonstration assumes that there is a Lyapunov operator (entropy) \(M\) for which \(\dot{M} = dM/dt \geq 0\) and commutes with \(M\). One then easily shows that the expectation value of the Hamiltonian in the state \(e^{iM_s \psi}\), for some \(\psi\) in the domain of \(M\), \(\dot{M}\) must be unbounded from below; it therefore admits a conjugate time operator. It was partly for this reason that emphasis has been placed, in the Brussels school, on developments of methods in the Liouville space, where the generator of evolution (whose prototype is the commutator with the Hamiltonian) has unbounded spectrum.

It has, moreover, been recently shown that the beautiful theory of Lax and Phillips\(^{22}\) for describing scattering and resonances in hyperbolic systems is applicable in the framework of the quantum theory as well\(^{23}\). The semigroup property of the evolution of an unstable system (exact exponential decay)\(^{24,25}\) can be achieved in this structure, in which the usual Hilbert space of the quantum theory is expanded to a direct integral of Hilbert spaces over the time axis. The existence of a time operator is implicit in this structure, which may be realized directly in the Liouville space\(^{26,27}\), in a relativistic quantum theory (in which the evolution operator is also not bounded from below)\(^{28}\), or in the quaternionic Hilbert space as described above.

5. Fock Space and Quantum Field Theory

The construction of a tensor product of quaternion modules, following the usual method for the representation of many-body systems (and the Fock space that is the
prototype for quantum field theory) has long been an obstacle in quaternionic theories, since there is an essential destruction of linearity, i.e., \( fg \otimes g \neq (f \otimes g)q \), where \( q \in \mathbb{H} \).

In his book\(^\text{17}\), Adler has described a new construction in the framework of path integrals which appears to be a very powerful way of handling the problem of the construction of a quaternionic quantum field theory, and, in fact, of more general quantum field theories. It follows, however, from the theorem stated in (3.9), that one may consider the tensor product problem in terms of that of a real Hilbert space and the vector space of a tensor product of quaternion algebras, e.g., for

\[
f \otimes g = \sum_{i,j} f_i \otimes g_j \cdot e_i \otimes e_j. \tag{5.1}
\]

The first factor is relatively simple to study, since it involves the tensor product of real Hilbert spaces. The requirements in dealing with the second factor stem from the fact that the tensor product space must be a \textit{quaternionic} Hilbert space, and hence the scalar products must be quaternion valued. Furthermore, we wish to construct a scalar product in the direct product algebra space which is totally symmetric, so that Bose-Einstein or Fermi-Dirac statistics can be achieved for the physical states by appropriate symmetrization of the \( \{f, g, \ldots\} \) entering the tensor product. This was done by Razon and me\(^\text{29}\); we found and studied the properties of the corresponding annihilation-creation operators which create and annihilate states with correct Bose-Einstein and Fermi-Dirac symmetry, but satisfy commutation and anti-commutation relations which are deformed from the usual ones, i.e.,

\[
a(f)a^\dagger(g) = \lambda a^\dagger(g)a(f) = F(f, g), \tag{5.2}
\]

where \( F(f, g) \) is a simple functional of \( f \) and \( g \), and \( \lambda \) is a real number determined by the occupation number of the quaternionic state this relation acts on (it may therefore be considered a function of the quaternion number operator). It is interesting that the scalar product of two one-particle states does not coincide with that of the original Hilbert space, i.e.,

\[
(\Psi(f), \Psi(g)) = \frac{1}{3}(2(f, g) + (g, f)), \tag{5.3}
\]

and the annihilation of a one-particle state yields

\[
a(f)\Psi(g) = \Psi_0 \frac{1}{3}((2(f, g) + (g, f)). \tag{5.4}
\]

The reason for this is that the vacuum \( \Psi_0 \) carries a non-trivial quaternionic phase, and the creation of a one-particle state therefore involves the construction of a non-trivial tensor product. The one-particle function \( \Psi(f) \) carries, as shown in ref. 29, a linearity property constructed for the scalar product of N-body functionals, but not the linearity under \( f \to fg \) associated with the original space.

6. Comments
The associative Clifford algebras appear to provide models for quantum theories generalized, in their realization, beyond the usual complex structure. The natural “phase” of the linear spaces in these theories is non-abelian, and they may provide models for describing the quantum states of the non-abelian gauge theories entering in recent attempts to describe strong and electroweak interactions, and to account for the observed particle spectrum. It appears that some fundamental structural modification should be made on the basic form and realization of the general quantum theory if progress is to be made in the description of these phenomena\textsuperscript{17}. The generalization of the idea of locality, as in the theory of strings and conformal field theories\textsuperscript{30} is a possibility that is being investigated widely; the modification of the algebraic structure of the realization of quantum theory is yet another possibility that I have discussed here.

References

1. M. Jammer, \textit{The Conceptual Development of Quantum Mechanics}, McGraw Hill, N.Y. (1966), pp. 205, 375-377.
2. Mr. Tait, in a letter to A. Cauchy \textsuperscript{1}.
3. W.R. Hamilton, Phil. Mag. 25, 10, 241, 489 (1844).
4. H. Taber, Amer. Jour. Math. 12, 337 (1890).
5. J.W. Gibbs, Nature 44, 79 (1891).
6. A. Hurwitz, Nachr. Gesell. Wiss., Göttingen, Math-Phys. Kl., 309 (1898).
7. Personal communication to H.H. Goldstine. The axiomatic foundation of the quantum theory does not restrict the structure of the Hilbert module in which the propositional system is embedded, provided that it is isomorphic to a projective geometry. See, for example, C. Piron, \textit{Mécanique quantique}, Presses polytechniques et universitaires romandes, Lausanne (1990).
8. H.H. Goldstine and L.P. Horwitz, Proc. Nat. Aca. Sci. 48, 1134 (1962); Math. Ann. 154, 1 (1964).
9. P. Jordan, J. von Neumann and E.P. Wigner, Ann. Math. N.Y. 35, 29 (1934).
10. A.A. Albert, Ann. Math. 35, 65(1934).
11. H.H. Goldstine and L.P. Horwitz, Math. Ann. 164, 291 (1966).
12. M. Günyaydin and F. Gürsey, Phys. Rev. D9, 3387 (1974).
13. L.P. Horwitz and L.C. Biedenharn, Jour. Math. Phys. 20, 269 (1979).
14. W. Freudenthal, Math. Inst. der Rijksuniversiteit te Utrecht (1951).
15. D. Finkelstein, J.M. Jauch, S. Schiminovitch and D. Speiser, Jour. Math. Phys. 3, 207 (1962); 4, 788 (1963).
16. L.P. Horwitz and L.C. Biedenharn, Ann. Phys. 157, 432 (1984).
17. S.L. Adler, \textit{Quaternionic Quantum Mechanics and Quantum Fields}, Oxford University Press, Oxford (1995).
18. A. Razon, L.P. Horwitz and L.C. Biedenharn, Jour. Math. Phys. 30, 59 (1989).
19. L.P. Horwitz, Jour. Math. Phys. 34, 3405 (1993).
20. L.P. Horwitz, Jour. Math. Phys. 35, 2743,2760 (1994).
21. T.D. Lee, Phys. Rev. 95, 1329 (1954); K.O. Friedrichs, Comm. Pure and Appl. Math. 1, 361 (1950).
22. B. Misra, I. Prigogine and M. Courbage, Proc. Nat. Aca. 76, 4768 (1979).
24. C. Flesia and C. Piron, Helv. Phys. Acta 57, 697 (1984).
25. L.P. Horwitz and C. Piron, Helv. Phys. Acta 66, 693 (1993).
26. E. Eisenberg and L.P. Horwitz, Adv. Chemical Phys., to be published.
27. E. Eisenberg and L.P. Horwitz, Phys. Rev. A 52, 70 (1995).
28. Y. Strauss and L.P. Horwitz, in preparation.
29. A. Razon and L.P. Horwitz, Acta. Appl. Math. 24, 141, 179 (1991).
30. For example, M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory I and II, Cambridge Univ. Press, Cambridge (1987).