GLOBAL SOLUTIONS AND SELF-SIMILAR SOLUTIONS OF THE COUPLED SYSTEM OF SEMILINEAR WAVE EQUATIONS IN THREE SPACE DIMENSIONS

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Abstract. In this paper, we treat the coupled system of wave equations whose nonlinearities are $|u|^{p_1}|v|^{q_1}$ and propagation speeds may be different from each other. We study the lower bounds of $p_1$ and $q_1$ to assure the global existence of a class of small amplitude solutions which includes self-similar solutions. The exponent of self-similar solutions plays crucial role to find the lower bounds. Moreover, we prove that the discrepancy of propagation speeds allow us to bring them down. Conversely, if such conditions for the global existence do not hold, then no self-similar solution exists even for small initial data.

1. Introduction. In the present paper, we treat the coupled system of semilinear wave equations:

$$\begin{align*}
(\partial_t^2 - \Delta) u(x, t) &= |u(x, t)|^{p_1}|v(x, t)|^{q_1}, & x \in \mathbb{R}^n, t \geq 0, \\
(\partial_t^2 - s^2 \Delta) v(x, t) &= |u(x, t)|^{p_2}|v(x, t)|^{q_2}, & x \in \mathbb{R}^n, t \geq 0,
\end{align*}$$

$$u(x, 0) = \epsilon f_1(x), \quad \partial_t u(x, 0) = \epsilon g_1(x), \quad x \in \mathbb{R}^n,$$

$$v(x, 0) = \epsilon f_2(x), \quad \partial_t v(x, 0) = \epsilon g_2(x), \quad x \in \mathbb{R}^n.$$  \hspace{1cm} (1.1) (1.2) (1.3) (1.4)

where $s > 0$, $p_1, q_1, p_2, q_2 \geq 0$ and $\epsilon > 0$. The problem was studied by D. Del Santo, V. Georgiev and E. Mitidieri [8] when $s = 1$ and $p_1 = q_2 = 0$, and the following quantity, determined by $p_2, q_1$ and the spatial dimensions $n \geq 2$, was introduced:

$$\Lambda \equiv \Lambda(p_2, q_1, n) = \max\left\{\frac{q_1 + 2 + p_2^{-1}}{p_2q_1 - 1}, \frac{p_2 + 2 + q_1^{-1}}{p_2q_1 - 1}\right\} - \frac{n - 1}{2},$$  \hspace{1cm} (1.5)

which plays the same role as the so called Fujita-type critical exponent. Namely, it was shown in [8] that when $\Lambda < 0$ (with some additional conditions on $p_2, q_1$), then the small data global existence holds (see also [7]), and when $\Lambda > 0$, the small data blow-up occurs. The blow-up result was also obtained by K. Deng [10]...
independently. As for the critical case \( \Lambda = 0 \), the small data blow-up holds. (See [1], [9] and [19]). Here, we say that the small data global existence holds for (1.1)–(1.4) if for any \( f_j, g_j \in C^\infty_0(\mathbb{R}^n) \) \( (j = 1, 2) \) there exists a constant \( \epsilon_0 = \epsilon_0(p_2, g_1, f_j, g_j) > 0 \) such that (1.1)–(1.4) has a global classical (resp. weak) solution for \( n = 2, 3 \) (resp. for \( n \geq 4 \)), provided \( 0 < \epsilon \leq \epsilon_0 \). Otherwise, we say that the small data blow-up occurs for (1.1)–(1.4). In that sense \( \Lambda \) characterizes the behavior of the solution well. However it seems to be difficult to understand from where \( \Lambda \) comes and how to generalize the definition of it for the general case.

Before we proceed further, we briefly recall the known results concerning the Cauchy problem for the single equation:

\[
(\partial_t^2 - \Delta) u(x, t) = |u(x, t)|^p, \quad x \in \mathbb{R}^n, \quad t \geq 0. \tag{1.6}
\]

For simplicity, we take \( n = 3 \). (For the other space dimensional case, see [12], [13], [30], [23], [32], [11], and references cited therein).

Let \( u_0(x, t) \) be the solution of the homogeneous wave equation with compactly supported initial data. Then, as is well known, \( u_0(x, t) \) is supported in \( \{(x, t) \mid |x| \leq t + R\} \) with \( R \) the diameter of a ball containing the support of the data, and

\[
|u_0(x, t)| \leq C(1 + |x| + t)^{-1}.
\]

Therefore, the first iterate \( u_1(x, t) \), i.e., the solution of \( (\partial_t^2 - \Delta) u_1(x, t) = |u_0(x, t)|^p \) with zero initial data, satisfies

\[
|u_1(x, t)| \leq C(1 + |x| + t)^{-1}(1 + ||x| - t|)^{-(p-2)}. \tag{1.7}
\]

Employing those estimates and applying the successive iteration, F. John [16] was able to show that \( p_0(3) = 1 + \sqrt{2} \) is the critical exponent of the problem in the above sense. (See also [29] for the critical case). This exponent had already been conjectured by W. Strauss in [31]. Namely, the exponent \( p_0(n) \) is given by the positive root of

\[
(n-1)p^2 - (n+1)p - 2 = 0. \tag{1.8}
\]

Moreover, F. Asakura [3] proved the small data global existence for a wider class of the initial data. Roughly speaking, we have only to assume that the Cauchy data \((f, g)\) decay faster than \( (|x|^{-m_0}, |x|^{-m_0-1}) \) as \(|x| \to \infty \). Here \( m_0 = 2/(p - 1) \).

To understand the number \( m_0 \), it is convenient to consider the self-similar solution \( u(x, t) \), which is a solution of (1.6) verifying \( u(x, t) = \lambda^{m_0}u(\lambda x, \lambda t) \) for any \( \lambda > 0 \) and \((x, t)\). The existence of the self-similar solution of (1.6) is proved in [28] for \( p > 2 + \sqrt{5} \), in [25] for \( p > (\sqrt{15} + 4)/3 \) and in [26] and [14] for \( p > 1 + \sqrt{2} \). Pecher [26] also showed that even for small data no self-similar solutions in general exist if \( p \leq 1 + \sqrt{2} \). If \( u(x, t) \) is a self-similar solution, we easily see that

\[
u(x, t) = t^{-m_0}u(x/t, 1)\tag{1.9}.
\]

Now, heuristically, we may say that the condition \( p > p_0(3) \) for the small data global existence comes from

\[
m_0 < 1 + (p - 2),
\]
in view of both (1.9), which is connected with the most slowly decaying data, and (1.7), which is the estimate for the compactly supported data. For general \( n \geq 2 \), replacing the right hand side as

\[
m_0 < \frac{n - 1}{2} + p^*, \quad p^* := \frac{n - 1}{2}p - \frac{n + 1}{2}, \tag{1.10}
\]
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we can recover the condition \( p > p_0(n) \). We remark that this type of observation
holds true for the Schrödinger and heat equations. (For the existence of self-similar
solutions for those equations and related topics, see Cazenave-Weissler [4],[5] and
Ribaud-Youssfi [27]).

Turning to our problem, we need to study the self-similar solution \((u(x,t),v(x,t))\)
of (1.1)–(1.4), which is a solution of (1.1)–(1.4) satisfying

\[ u(x,t) = \lambda^{m_1} u(\lambda x, \lambda t), \quad v(x,t) = \lambda^{m_2} v(\lambda x, \lambda t) \]

for any \( \lambda > 0 \) and \((x,t)\), where

\[ m_1 = \frac{2((q_2 - 1) - q_1)}{(p_1 - 1)(q_2 - 1) - q_1 p_2}, \quad m_2 = \frac{2((p_1 - 1) - p_2)}{(p_1 - 1)(q_2 - 1) - q_1 p_2}. \]

Here we assumed \((p_1 - 1)(q_2 - 1) - q_1 p_2 \neq 0\). Then the analogue to (1.10) is

\[ m_1 < \frac{n - 1}{2} + (p_1 + q_1)^*, \quad m_2 < \frac{n - 1}{2} + (p_2 + q_2)^*. \quad (1.11) \]

Actually, if we put \( p_1 = q_2 = 0 \) in (1.11), then we get \( \Lambda < 0 \) as we desired.

Next we turn our attention to the role of propagation speeds. The small data
global existence for systems of nonlinear wave equations with different propagation
speeds has been well developed when the nonlinear terms depend only on the deriva-
tives of unknown functions but not on unknown functions themselves in the works
of [18], [2], [15], and [33]. Those result suggest that the discrepancy of propagation
speeds may relax the condition (1.11) to prove the small data global existence also
for our problem (1.1)–(1.4).

In this spirit, the works of [20], [21] and [6] deal with (1.1)–(1.4) with \( s \neq 1 \)
and \( p_2 = q_1 = 0 \), and found that \( \Lambda \) still works as the Fujita-type critical exponent.
However, when \( s \neq 1 \) and either \( p_1 > 0 \) and \( p_2 > 0 \) or \( q_1 > 0 \) and \( q_2 > 0 \), it was
shown in [22] that something different happens for some peculiar values of \( p_j, q_j \)
\((j = 1, 2)\). Here we try to formulate the condition in general. Since \( m_1 \) and \( m_2 \)
satisfy \( m_1 + 2 = m_1 p_1 + m_2 q_1, \quad m_2 + 2 = m_1 p_2 + m_2 q_2, \quad (1.12) \)
we see that (1.11) is equivalent to

\[ (m_1 - \frac{n - 1}{2}) p_j + (m_2 - \frac{n - 1}{2}) q_j < 1 \quad \text{for} \quad j = 1, 2. \quad (1.13) \]

We suppose that the following weaker version of (1.13) is sufficient to prove the small
data global existence:

\[ \max \{(m_1 - \frac{n - 1}{2}) p_j, \quad (m_2 - \frac{n - 1}{2}) q_j\} < 1, \quad \text{for} \quad j = 1, 2, \quad (1.14) \]

and will realize the conjecture for the case \( n = 3 \) in this paper. Notice that if
\( p_2 = q_1 = 0 \), then (1.14) coincides with (1.13), hence (1.11).

Remark: Since we have strict inequality in (1.11) and (1.14), we are able to
show a kind of the stability of self-similar solutions in Theorem 3.3 below when
\( n = 3 \), as in [28], [25], [26] and [14]. Besides, at least in the formal level, we could
derive such conditions as in (1.11) and (1.14), even though the number of unknown
functions increase.

This paper is organized as follows. In section 2, following Pecher's argument
in [26], we prepare several estimates which is connected with pointwise estimates
for the solution. For that reason, we restrict ourselves to the case \( n = 3 \) in what
follows. The proof of estimates for the solution of inhomogeneous wave equation is simplified (see Proposition 2.2 and its proof). In section 3, we prove the small data global existence for a class of data which includes homogeneous data $f_j$ and $g_j$ of degree $-m_j$ and $-m_j - 1$, respectively. Hence the existence of self-similar solutions is shown, and the solutions are not necessarily radially symmetric. Moreover, we show the existence of asymptotically self-similar solutions. In section 4, we show that even for small data no self-similar solutions in general exist if $(p_1, q_1) \in \Omega_s$ or $(p_2, q_2) \in \Omega_s$, where we have set for $s = 1$: 

$$\Omega_s = \{(p, q) \in [0, \infty) \times [0, \infty) | (m_1 - 1)p + (m_2 - 1)q \geq 1\},$$

(1.15)

and for $s \neq 1$:

$$\Omega_s = \{(p, q) \in [0, \infty) \times [0, \infty) | (m_1 - 1)p \geq 1 \text{ or } (m_2 - 1)q \geq 1\}.$$ 

(1.16)

This result suggests that the conditions (1.11) for $s = 1$ and (1.14) for $s \neq 1$ with $n = 3$ are sharp for the existence of self-similar solutions.

We conclude this introduction by giving some notations. For $s > 0$, we define the operators as follows:

$$K_s(f, g)(x, t) = \frac{1}{4\pi s^2t} \int_{|x-y|=st} g(y) dS_y + \partial_t \left( \frac{1}{4\pi s^2t} \int_{|x-y|=st} f(y) dS_y \right),$$

(1.17)

and

$$L_s(F)(x, t) = \frac{1}{4\pi s^2t} \int_0^t \int_{|x-y|=s(t-t')} F(y, t') dS_y dt'.$$

(1.18)

We define the space $V_1$, $V_s$ and $V_{1,s}$ as follows:

$$V_1 = \{ u \in C^0(\mathbb{R}^3 \times \mathbb{R}^+) \setminus \{|x| = t\}) | \|u\|_{V_1} < \infty \},$$

(1.19)

$$\|u\|_{V_1} = \sup_{|x| \neq t} \{|x| + t| |x| - t|^{m_1-1}|u(x, t)|\}.$$

and

$$V_s = \{ v \in C^0(\mathbb{R}^3 \times \mathbb{R}^+) \setminus \{|x| = st\}) | \|v\|_{V_s} < \infty \},$$

(1.20)

$$\|v\|_{V_s} = \sup_{|x| \neq st} \{|(x+t)| |x| - st|^{m_2-1}|v(x, t)|\},$$

and

$$V_{1,s} = \{ (u, v) \in V_1 \times V_s | \|(u, v)\|_{V_{1,s}} := \|u\|_{V_1} + \|v\|_{V_s} < \infty \},$$

where we have set $\mathbb{R}^3 = \mathbb{R}^+ \setminus \{0\}$ and $\mathbb{R}^+ = (0, \infty)$.

2. Preliminary estimates. In this section, we mention preliminary estimates.

**Proposition 2.1.** If $f \in C^1(\mathbb{R}^3)$, $g \in C^0(\mathbb{R}^3)$, and

$$|f(x)| \leq |x|^{-k}, \quad |g(x)| + |\nabla f(x)| \leq |x|^{-k-1},$$

where $k > 1$, then the following estimate holds for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$ with $|x| \neq st$:

$$|K_s(f, g)(x, t)| \leq C_0 \left( |x|^{-k} \right) |x| - st|^{-k+1},$$

where $C_0$ is a constant depending only on $k$ and $s$. If $f \in C^1(\mathbb{R}^3)$, $g \in C^0(\mathbb{R}^3)$, and

$$|f(x)| \leq (1 + |x|)^{-k}, \quad |g(x)| + |\nabla f(x)| \leq (1 + |x|)^{-k-1},$$

where $k > 1$, then the following estimate holds for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$:

$$|K_s(f, g)(x, t)| \leq C_0 \left( 1 + |x| + t \right)^{-1} (1 + |x| - st)^{-k+1}.$$
Proposition 2.2. Let \((x,t) \in \mathbb{R}^3 \times \mathbb{R}^+\) and let \(s, \kappa_1, \kappa_2 > 0\) and \(\mu > 0\). Assume that \(\mu_1 > 0, \mu_2 > 0\) if \(\kappa_1 \neq \kappa_2\) and that \(\mu_1 + \mu_2 > 1\) if \(\kappa_1 = \kappa_2\). If

\[
F(y, t') = (|y| + t')^{-1-\mu-\mu_1-\mu_2} ||y| - \kappa_1 t'|^{\mu_1-1} ||y| - \kappa_2 t'|^{\mu_2-1},
\]

then we have

\[
|L_s(F)(x,t)| \leq C(|x| + t)^{-1} ||x| - st|^{-\mu},
\]

where \(C\) is a constant depending only on \(s, \kappa_1, \kappa_2, \mu, \mu_1\) and \(\mu_2\). If

\[
G(y, t') = (1 + |y| + t')^{-1-\mu-\mu_1-\mu_2}(1 + ||y| - \kappa_1 t'|^{\mu_1-1}(1 + ||y| - \kappa_2 t'|^{\mu_2-1},
\]

then we have

\[
|L_s(G)(x,t)| \leq C(1 + |x| + t)^{-1}(1 + ||x| - st|)^{-\mu}.
\]

This proposition is the estimate for the nonlinear terms. Before carrying out a proof of Proposition 2.2, we prepare several lemmas. First one is a fundamental identity concerning the spherical mean. For the proof of Lemma 2.3, see, e.g., [24].

Lemma 2.3. Let \(F \in C^0([0, \infty)), x \in \mathbb{R}^3\) and \(t > 0\). Then we have

\[
\frac{1}{t} \int_{|y - x| = t} F(|y|) dS_y = 2\pi \int_{|x| - t}^{x + t} |y| F(|y|) dy.
\]

Lemma 2.4. Let \(\mu > 0\) and let \(r, t > 0\) with \(t \neq r\). Then we have

\[
\int_{|t - r|}^{t + r} \xi^{-\mu-1} d\xi \leq \frac{Cr}{(r + t)|r - t|^\mu},
\]

\[
\int_{|t - r|}^{t + r} (1 + \xi)^{-\mu-1} d\xi \leq \frac{Cr}{(1 + r + t)(1 + |r - t|)^\mu},
\]

where \(C\) is a constant depending only on \(\mu\).

Proof: We prove only (2.5), since the other can be handled analogously. If \(t \geq 2r\), then \(t - r \geq (r + t)/3\), hence

\[
\int_{t - r}^{t + r} \xi^{-\mu-1} d\xi \leq \frac{2r}{(t - r)^{\mu+1}} \leq \frac{6r}{(r + t)(r - t)^\mu}.
\]

If \(0 < t \leq 2r\), then we have \(r/(r + t) \geq 1/3\). Therefore we obtain (2.5) by a direct calculation. This completes the proof.

Lemma 2.5. Let \(\xi > 0\) and \(-1 < \sigma_j < 1\) (\(j = 1, 2\)). Assume that \(\mu_1 > 0, \mu_2 > 0\) if \(\sigma_1 \neq \sigma_2\) and that \(\mu_1 + \mu_2 > 1\) if \(\sigma_1 = \sigma_2\). Then we have

\[
\int_{-\xi}^{\xi} |\eta + \sigma_1 \xi|^{\mu_1-1} |\eta + \sigma_2 \xi|^{\mu_2-1} d\eta \leq C \xi^{\mu_1+\mu_2-1},
\]

where \(C\) is a constant depending only on \(\mu_1, \mu_2, \sigma_1\) and \(\sigma_2\), but not on \(\xi\).

Proof: By changing the variable, we see that (2.7) is equivalent to

\[
\int_{-1}^{1} |\eta + \sigma_1 |^{\mu_1-1} |\eta + \sigma_2 |^{\mu_2-1} d\eta \leq C.
\]

It is clear that (2.8) holds for \(\sigma_1 = \sigma_2\) and \(\mu_1 + \mu_2 > 1\). When \(\sigma_1 \neq \sigma_2\), by the assumptions \(\mu_1 > 0\) and \(\mu_2 > 0\), we can derive (2.8). This completes the proof.
End of the proof of Proposition 2.2: We prove only (2.2), since we can get (2.4) less hard. Since \( L_s(F)(x, t) = s^{-2}L_1(F)(x, st) \) with \( F(y, t') = F(y, t'/s) \), it suffices to show (2.2) with \( s = 1 \) for any \( \kappa_1 \) and \( \kappa_2 \). We slightly modify \( F(y, t') \) by replacing \( |y - \kappa_1 t'| \) by \( \varepsilon + |y - \kappa_1 t'| \) (\( \varepsilon > 0 \)) if \( \mu_j - 1 < 0 \) (\( j = 1 \) or \( j = 2 \)), so that \( F \in C^0(\mathbb{R}^3 \times \mathbb{R}_+) \). We denote it by \( F_{\varepsilon} \) to indicate the modification. Then it follows from Lemma 2.3 that

\[
\frac{1}{t-t'} \int_{|x-y|=t-t'} F_{\varepsilon}(y, t')dS_y = \frac{2\pi}{|x|} \int_{|x|-t+t'} ^{|x|+t-t'} \frac{|y|F_{\varepsilon}(y, t')dy}{|y|},
\]

for \( x \in \mathbb{R}^3 \) and \( t' < t \).

Next we introduce new variables:

\[
\xi = |y| + t', \quad \eta = |y| - t'.
\]

Then there is a constant \( C \) independent of \( \varepsilon \) such that

\[
F_{\varepsilon}(y, t') \leq C \xi^{-1-\mu_1-\mu_2}(\xi + \sigma_1\xi^{|m_1-1|} + \sigma_2\xi^{|m_2-1|}),
\]

where \( \sigma_j = (1 - \kappa_j)/(1 + \kappa_j) \) (\( j = 1, 2 \)), which satisfies \(-1 < \sigma_j < 1 \). Besides, \( \sigma_1 = \sigma_2 \) if and only if \( \kappa_1 = \kappa_2 \). In addition, if \( 0 < t' < t \) and \(|x| - t + t' < |y| < |x| + t - t'\), then we have \(-\xi < \eta < \xi, |t|-|x| < \xi < t + |x|\). Therefore, it follows from (1.18), (2.9) and Lemma 2.5 that

\[
|L_1(F_{\varepsilon})(x, t)| \leq \frac{1}{2|x|} \int_0^t \left( \int_{|x|-t+tt'} |y|F_{\varepsilon}(y, t')dy \right)dt' \quad (2.10)
\]

\[
\leq C \frac{1}{|x|} \int_{|x|-|t|t} ^{|x|+|t|t} \xi^{-\mu_1-\mu_2} \left( \int_{-\xi} ^{\xi} (\eta + \sigma_1\eta^{|m_1-1|} + \sigma_2\eta^{|m_2-1|}d\eta \right) d\xi
\]

\[
\leq C \frac{1}{|x|} \int_{|x|-|t|t} ^{|x|+|t|t} \xi^{-\mu_1} d\xi,
\]

which is bounded by \( C(|x| + t)^{-1}||x| - t|^{-\mu} \), by (2.5). Letting \( \varepsilon \to 0 \), we obtain (2.2) with \( s = 1 \) by Beppo Levi’s theorem. This completes the proof. \( \square \)

3. Main results. In this section, we first prove the existence of global solutions to the Cauchy problem (1.1)–(1.4) including self-similar solutions. More precisely, we consider the corresponding integral equation:

\[
u = \epsilon K_1(f_1, g_1) + L_1(|u|^{p_1}|v|^{q_1}), \quad (3.1)
\]

\[
u = \epsilon K_1(f_2, g_2) + L_2(|u|^{p_2}|v|^{q_2}). \quad (3.2)
\]

Throughout this section, we assume that

\[
p_1, p_2, q_1, q_2 \in \{0\} \cup [1, \infty) \quad (3.3)
\]

so that \(|u|^{p_1}|v|^{q_1} \) and \(|u|^{p_2}|v|^{q_2} \) satisfy the Lipschitz condition. Besides, we require that

\[
m_1 = \frac{2((q_1 - 1) - q_1)}{(p_1 - 1)(q_1 - 1) - q_1 p_2} > 1, \quad m_2 = \frac{2((p_1 - 1) - p_2)}{(p_1 - 1)(q_1 - 1) - q_1 p_2} > 1 \quad (3.4)
\]

with \((p_1 - 1)(q_2 - 1) - q_1 p_2 \neq 0\). If \( m_1 \leq 1 \) or \( m_2 \leq 1 \), we have to modify the function space \( V_1 \) or \( V_3 \) defined by (1.19) and (1.20) as in [3]. But we do not go further in this direction, since we are interested in values of \((p_j, q_j) \) which are close to \( \Omega_s \) in the \( p-q \) plane. Actually, it is clear that \((p_j, q_j) \notin \Omega_s \) if \( m_1 \leq 1 \) or \( m_2 \leq 1 \).
In addition, we introduce a set from where we take the Cauchy data:
\[ W_k = \{ (f, g) \in C^1(\mathbb{R}^2) \times C^0(\mathbb{R}^2) \mid |f(x)| \leq |x|^{-\kappa}, \ |g(x)| + |\nabla f(x)| \leq |x|^{-\kappa-1} \}. \] (3.5)

Then we have the following.

**Theorem 3.1.** Let \( p_j, q_j \ (j = 1, 2) \) satisfy (3.3) and (3.4). Suppose that for \( s = 1 \):
\[ (m_1 - 1)p_j + (m_2 - 1)q_j < 1 \quad \text{for} \quad j = 1, 2, \] (3.6)
and for \( s \neq 1 \):
\[ (m_1 - 1)p_j < 1, \quad \text{and} \quad (m_2 - 1)q_j < 1 \quad \text{for} \quad j = 1, 2. \] (3.7)
Assume \((f_j, g_j) \in W_{m_j} \) for \( j = 1, 2 \). If \( \epsilon > 0 \) is sufficiently small, then there exists a unique global solution of (3.1)–(3.2) in the space
\[ X = \{(u, v) \in V_{1,s} \mid \|(u, v)\|_{V_{1,s}} \leq 4C_0\epsilon \}, \]
where \( C_0 \) is the number in Proposition 2.1.

**Remark:** For instance, if \( p_1 + q_1 = p_2 + q_2 =: \alpha + 1 \), then (3.4) and (3.6) become \( m_1 = m_2 = 2/\alpha > 1 \) and \((2/\alpha - 1)(\alpha + 1) < 1, \ i.e., \ \alpha > \sqrt{2} \). While (3.7) is equivalent to \((2/\alpha - 1) p_j < 1 \) and \((2/\alpha - 1) q_j < 1 \), which are always fulfilled when \( \alpha > 1 \) and (3.3) hold. Therefore, we have a significant difference according to the discrepancy of the propagation speeds.

**Proof:** We have only to prove that \( M \) defined by
\[ M(u, v) = (\epsilon K_1(f_1, g_1), \epsilon K_s(f_2, g_2)) + (L_1(|u|^{p_1}|v|^q_1), L_s(|u|^{p_2}|v|^{q_2}) \]
is a contraction in \( X \). Since \((f_j, g_j) \in W_{m_j} \) with (3.4), it follows from Proposition 2.1 that
\[ \|(K_1(f_1, g_1), K_s(f_2, g_2))\|_{V_{1,s}} \leq 2C_0. \] (3.8)
Let \( h(y, t') = (|y| + t')^{-(p_1 + q_1)}|y| - t'^{-(m_1 - 1)p_1}|y| - s|t'|^{-(m_2 - 1)q_1} \). Then we have
\[ |L_1(|u|^{p_1}|v|^{q_1})(x, t)| \leq \|u\|_{P_1}^{p_1} \|v\|_{V_1}^{q_1} |L_1(h)(x, t)|. \]
Setting \( \mu = m_1 - 1 > 0, \mu_1 = -(m_1 - 1)p_1 + 1 \) and \( \mu_2 = -(m_2 - 1)q_1 + 1 \), we have \( p_1 + q_1 = 1 + \mu_1 + \mu_2 \) by (1.12). By the assumptions (3.6) and (3.7), we are able to apply Proposition 2.2 and to get
\[ |L_1(h)(x, t)| \leq C(|x| + t)^{-1} |x| - t|^{m_1 + 1}, \] (3.9)
hence
\[ \|L_1(|u|^{p_1}|v|^{q_1})\|_{V_1} \leq C\|(u, v)\|_{V_{1,s}}^{p_1 + q_1}. \] (3.10)
In the same manner, we obtain
\[ \|L_s(|u|^{p_2}|v|^{q_2})\|_{V_1} \leq C\|(u, v)\|_{V_{1,s}}^{p_2 + q_2}. \] (3.11)
By collecting (3.8), (3.10) and (3.11), we arrive at
\[ \|M(u, v)\|_{V_{1,s}} \leq 2C_0\epsilon + C\|(u, v)\|_{V_{1,s}}^{p_1 + q_1} + C\|(u, v)\|_{V_{1,s}}^{p_2 + q_2}, \]
hence \( M \) maps \( X \) into itself, if we take \( \epsilon \) sufficiently small.

Because
\[
\begin{align*}
|u|^{p_1}|v|^{q_1} - |u'|^{p_1}|v'|^{q_1} & \leq |u|^{p_1}|v - v'| \left( |v|^{q_1 - 1} + |v'|^{q_1 - 1} \right) \\
& \quad + |v'|^{q_1}|u - u'| \left( |u|^{p_1 - 1} + |u'|^{p_1 - 1} \right),
\end{align*}

(3.12)
we get
\[
\|L_1(|u|^{p_1}|v|^{q_1}) - L_1(|u'|^{p_1}|v'|^{q_1})\|_{V_1} \leq C\|\|u\|^{p_1+n-1} + \|u'\|^{p_1+n-1}\|\|L_1(u-u', v-v')\|_{V_{1,s}},
\]
by using (3.9). Thus, taking \(\epsilon\) sufficiently small, we see that \(M\) is a contraction in \(X\). The rest of the proof is standard, so we omit the details.

**Corollary 3.2.** Let \(p_j, q_j\) \((j = 1, 2)\) satisfy the assumptions of Theorem 3.1, and let
\[
\phi_j(x) = F_j\left(\frac{x}{|x|}\right)|x|^{-m_j}, \quad \psi_j(x) = G_j\left(\frac{x}{|x|}\right)|x|^{-m_j-1},
\]
where \(F_j \in C^1(S^2)\), \(G_j \in C^0(S^2)\) for \(j=1,2\). Then for sufficiently small \(\epsilon\), the integral equation (3.1)-(3.2) with \(f_j = \phi_j, g_j = \psi_j\) has a unique global self-similar solution in \(X\).

**Proof:** Since \(\phi_j(x), \psi_j(x) \in W_{m_j}\) (recalling (3.5)), we obtain the global solution \((U(x,t), V(x,t)) \in X\) of (3.1)-(3.2) with initial data \((\phi_j, \psi_j)\). Then \((\lambda^{m_1}U(\lambda x, \lambda t), \lambda^{m_2}V(\lambda x, \lambda t))\) automatically satisfies the same equation with the same data. Therefore, by the uniqueness, \((U(x,t), V(x,t))\) should be the desired self-similar solution.

We next consider the existence of a class of solutions which asymptotically behave like a self-similar solution.

**Theorem 3.3.** Assume that \(p_j, q_j\) \((j = 1, 2)\) satisfy the assumptions of Theorem 3.1. We take a positive number \(\delta > 0\) such that for \(s = 1\):
\[
(m_1 - 1)p_j + (m_2 - 1)q_j + \delta < 1 \quad \text{for} \quad j = 1, 2,
\]
and for \(s \neq 1\):
\[
(m_1 - 1)p_j + \delta < 1, \quad \text{and} \quad (m_2 - 1)q_j + \delta < 1 \quad \text{for} \quad j = 1, 2.
\]
Let \(\phi_j, \psi_j\) be the functions in (3.14) and \((U, V)\) be the self-similar solution of (3.1)-(3.2) with initial data \((\phi_j, \psi_j)\) (see Corollary 3.2). Assume \((f_j, g_j) \in W_{m_j}\) satisfies \((f_j - \phi_j, g_j - \psi_j) \in W_{m_j+s}\). If \(\epsilon > 0\) is sufficiently small, then there exists a unique solution \((u,v)\) in \(X\) of (3.1)-(3.2) with initial data \((f_j, g_j)\). Moreover, we have
\[
(|x| + t)||x| - t|^{m_1-1}|u(x,t) - U(x,t)| \leq C\epsilon(1 + ||x| - t|)^{-\delta},
\]
\[
(|x| + t)||x| - st|^{m_2-1}|v(x,t) - V(x,t)| \leq C\epsilon(1 + ||x| - st|)^{-\delta}.
\]

**Remark:** The existence of \((f_j, g_j)\) described in the theorem can be seen as follows: Let \(\chi \in C^1(R^3_+)\) and \(\tilde{\chi} \in C^0(R^3_+)\) be cut functions such that \(\chi(x) = \tilde{\chi}(x) = 1\) for \(|x| \geq 1\), and \(\chi(x) = \tilde{\chi}(x) = 0\) for \(|x| \leq 1/2\). For \((f'_j, g'_j) \in W_{m_j} \cap W_{m_j+s}\), if we set
\[
f_j = \chi\phi_j + f'_j, \quad g_j = \tilde{\chi}\psi_j + g'_j,
\]
then \((f_j, g_j)\) has the desired properties, modulo some universal constants.

**Proof:** Because \((f_j, g_j) \in W_{m_j}\), we have the existence and uniqueness of \((u,v)\) in \(X\) from Theorem 3.1. Since \((U,V)\) is also in \(X\), it is easy to see that (3.17) and (3.18) holds for \(\delta = 0\). Therefore, we have only to prove for \(\delta > 0\) satisfying (3.15)
or (3.16):

\[ \|u - U \|_{V_1} := \sup_{|x| \neq 0} \{(|x| + t)|x| - t|^{m_1 + \delta - 1}|u(x, t) - U(x, t)|\} \leq 2C_0 \varepsilon. \]  
\[ (3.19) \]

\[ \|v - V\|_{V_2} := \sup_{|x| \neq 0} \{(|x| + t)\|x| - st|^{m_2 + \delta - 1}|v(x, t) - V(x, t)|\} \leq 2C_0 \varepsilon. \]

If we set \( \Phi_j := f_j - \phi_j, \) \( \Psi := g_j - \psi_j, \) then we get

\[ |u - U| \leq |K_1(\Phi_1, \Psi_1)| + |L_1(|u|^{p_1}|v|^{q_1}) - L_1(|U|^{p_1}|V|^{q_1})|, \]
\[ (3.21) \]

\[ |v - V| \leq |K_2(\Phi_2, \Psi_2)| + |L_2(|u|^{p_2}|v|^{q_2}) - L_2(|U|^{p_2}|V|^{q_2})| \]  
\[ (3.22) \]

from the integral equation (3.1)–(3.2). Since \( (\Phi_j, \Psi_j) \in W_{m_j, \delta}, \) we have

\[ |K_1(\Phi_1, \Psi_1)| \leq C_0 \varepsilon(|x| + t)^{-1}\|x| - t|^{-m_1 - \delta + 1}, \]
\[ (3.23) \]

\[ |K_2(\Phi_2, \Psi_2)| \leq C_0 \varepsilon(|x| + t)^{-1}\|x| - st|^{-m_2 - \delta + 1}. \]  
\[ (3.24) \]

Then, in the same manner as we derived (3.13), we have

\[ |L_1(|u|^{p_1}|v|^{q_1}) - L_1(|U|^{p_1}|V|^{q_1})| \leq C|L_1(h_1)|\|u\|^{p_1}_V|v|^{q_1}_V - \|u\|^{p_1}_V|V|^{q_1}_V| \]
\[ + C|L_1(h_2)|\|u\|^{p_1}_V - \|u\|^{p_1}_V|V|^{q_1}_V| |u - U|\|V\|_V. \]

Proposition 2.2 with \( \mu = m_1 - 1 > 0, \) \( \mu_1 = -(m_1 - 1)p_1 - \delta + 1 \) and \( \mu_2 = -(m_2 - 1)q_1 + 1, \) and with \( \mu = m_1 - 1 > 0, \) \( \mu_1 = -(m_1 - 1)p_1 + 1 \) and \( \mu_2 = -(m_2 - 1)q_1 - \delta + 1 \) leads to

\[ |L_1(h_i)(x, t)| \leq C(|x| + t)^{-1}\|x| - t|^{-m_1 - \delta + 1} \quad \text{for} \quad i = 1, 2. \]

Therefore, we have

\[ \|L_1(|u|^{p_1}|v|^{q_1}) - L_1(|U|^{p_1}|V|^{q_1})|_{V_1} \leq C_1 \varepsilon^{p_1 + q_1}|v - V|_{V_2} + \|u - U\|_{V_1}, \]
\[ (3.25) \]

where we have used the fact that \( \|u\|_{V_1}, \|v\|_{V_2}, \|U\|_{V_1}, \|V\|_{V_2} \leq 4C_0 \varepsilon. \) In the same manner, we have

\[ \|L_1(|u|^{p_1}|v|^{q_1}) - L_1(|U|^{p_1}|V|^{q_1})|_{V_2} \leq C_1 \varepsilon^{p_2 + q_2}|v - V|_{V_2} + \|u - U\|_{V_1}. \]  
\[ (3.26) \]

If we choose \( \varepsilon \) such that \( C_1 \varepsilon^{p_2 + q_2} \leq 1/4, \) then (3.19) and (3.20) follows from (3.21)–(3.26). This completes the proof. \( \Box \)

Finally we consider the problem (1.1)–(1.4) for smooth initial data at the origin. The proof of the following result is analogous to that of Theorem 3.1, so is omitted.

**Theorem 3.4.** Assume that \( p_j, q_j \) \( (j = 1, 2) \) satisfy the assumptions of Theorem 3.1 and that \( f_j(x) \in C^1(\mathbb{R}^3), \) \( g_j(x) \in C^0(\mathbb{R}^3) \) satisfy

\[ |f_j(x)| \leq (1 + |x|)^{-m_1}, \quad |g_j(x)| + |\nabla f_j(x)| \leq (1 + |x|)^{-m_1 - 1}, \]
\[ (3.27) \]

for \( j = 1, 2. \) If \( \varepsilon > 0 \) is sufficiently small, then there exists a unique global solution of (3.1)–(3.2) in the space

\[ \tilde{X} = \{(u, v) \in V_{1, s} \| (u, v)\|_{V_{1, s}} \leq 4C_0 \varepsilon, \}. \]
where $C_0$ is the number in Proposition 2.1, and

$$V_{1,s} = \{(u,v) \in C^0(\mathbb{R}^3 \times \mathbb{R}) \times C^0(\mathbb{R}^3 \times \mathbb{R}) | \|u,v\|_{V_{1,s}} < \infty\},$$

$$\|u,v\|_{V_{1,s}} = \sup\{(1 + |x| + t)(1 + |x| - t)^{m_1-1}\|u(x,t)\| + \sup\{(1 + |x| + t)(1 + |x| - st)^{m_2-1}\|v(x,t)\|\}.$$ 

4. A counterexample. The following example shows that even for small data no self-similar solution in general exists if $(p_1, q_1) \in \Omega_s$ or $(p_2, q_2) \in \Omega_s$, where $\Omega_s$ is defined by (1.15) and (1.16). We consider the Cauchy problem:

$$(\partial_t^2 - \Delta) u(x,t) = |u(x,t)|^{p_1} v(x,t)^{q_1}, \quad (4.1)$$

$$(\partial_t^2 - s^2 \Delta) v(x,t) = |u(x,t)|^{p_2} v(x,t)^{q_2}, \quad (4.2)$$

$$u(x,0) = 0, \quad \partial_t u(x,0) = \epsilon |x|^{-m_1-1}, \quad (4.3)$$

$$v(x,0) = 0, \quad \partial_t v(x,0) = \epsilon |x|^{-m_2-1}, \quad (4.4)$$

via its corresponding integral equation:

$$u = \epsilon K_1(0, |x|^{-m_1-1}) + L_1(|u|^{p_1}|u|^{q_1}), \quad (4.5)$$

$$v = \epsilon K_2(0, |x|^{-m_2-1}) + L_2(|v|^{p_2}|v|^{q_2}). \quad (4.6)$$

We assume that $(u,v)$ is a local solution of integral equation (4.5)–(4.6) which is measurable in $(x,t) \in \mathbb{R}^3 \times (0,T)$. By symmetry, we have only to treat the case $(p_1, q_1) \in \Omega_s$. Besides, we assume $s \neq 1$ in the following, since the case of $s = 1$ can be handled analogously.

From the positivity of $L_s(F)$, we have

$$u(x,t) \geq u_0(|x|, t) := \epsilon K_1(0, |x|^{-m_1-1}), \quad v(x,t) \geq v_0(|x|, t) := \epsilon K_2(0, |x|^{-m_2-1}). \quad (4.7)$$

Since $m_j > 1$, it follows from (1.17) and Lemma 2.3 that

$$u_0(|x|, t) = C_j \epsilon \frac{1}{|x|} (|x| - t)^{1-m_1} - (x + t)^{1-m_1}, \quad (4.8)$$

$$v_0(|x|, t) = C_j \epsilon \frac{1}{|x|} (|x| - st)^{1-m_2} - (x + st)^{1-m_2}, \quad (4.9)$$

where $C_j$ are the positive constants depending only on $s, m_1$ and $m_2$.

First suppose $(m_2 - 1)q_2 \geq 1$. We fix $(x,t) \in \mathbb{R}^3 \times (0,T)$ so that $t/2 < |x| < t$. We choose a positive number $\delta_0$ such that

$$D_\delta = \{ (\lambda, t') | (t - |x|)/(s + 1) < t' < 3(t - |x|)/(s + 3), \quad t - |x| - t' < \lambda < st' - \delta \}$$

is not the empty set for $\delta = \delta_0$. Then we define a characteristic function $H_\delta$ on $D_\delta$ for $\delta$ with $0 < \delta \leq \delta_0$, namely $H_\delta(\lambda, t') = 1$ if $(\lambda, t') \in D_\delta$, and $H_\delta(\lambda, t') = 0$ otherwise. Notice that if $(|y|, t') \in D_\delta$, then we have $|y| > st'/3$, hence

$$\frac{1}{s} < \frac{t'}{|y|} < \frac{3}{s}, \quad |y| + st' \geq 2(st' - |y|), \quad |y| \geq s(t - |x|)/3(s + 1).$$

Therefore, we get

$$v_0(|y|, t') \geq C_\delta |y|^{-m_1} \left( |1 - \frac{t'}{|y|}|^{1-m_1} - (1 + \frac{t'}{|y|})^{1-m_1} \right) \quad (4.10)$$

from (4.8), and

$$v_0(|y|, t') \geq C_\delta (t - |x|)^{-m_1} (st' - |y|)^{1-m_2} \quad (4.11)$$
from (4.9). It follows from (4.5), (4.7), (4.10) and (4.11) that

\[
(4.14)
\]

\[
(4.15)
\]

\[
(4.16)
\]

by Lemma 2.3. For \((|y|, t') \in D_\delta,\) we have \(|x| - t + t' = t - |x| - t' < st' - \delta < |x| + t - t',\) since \(t - |x| < (s + 1)t' < 3(s + 1)(t - |x|)/(s + 3) < |x| + t.\) Thus we conclude

\[
(4.17)
\]

Because \((1 - m_2)q_1 \leq -1,\) the inner integral goes to \(\infty\) as \(\delta \to 0,\) which leads to a contradiction to the assumed local existence of \(u.\)

Next suppose \((m_1 - 1)p \geq 1.\) We consider the region \(t/2 < |x| < t\) and put \(s = 1\) in the definition of \(D_\delta.\) Then, for \((|y|, t') \in D_\delta,\) we get

\[
(4.18)
\]

from (4.8) and (4.9). as before. Therefore, the similar argument to the previous case yields the contradiction of the same type. This completes the proof. \(\square\)

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