A PROOF OF T. WILL WACHER’S CONJECTURE

ALEXEY KALUGIN

Abstract. In the present paper we continue the study of the relation between the cohomology of moduli stacks of smooth and proper curves $\mathcal{M}_{g,n}$ and the cohomology of ribbon graph complexes. The main results of this work are proofs of T. Willwacher’s conjecture and A. Căldăraru’s conjecture.

1. Introduction

1.1. Introduction. In this paper we study the relation between the cohomology of moduli stacks of smooth proper algebraic curves with marked points $\mathcal{M}_{g,n}$ and the cohomology of ribbon graph complexes. For a natural number $d \in \mathbb{N}$ denote by $\text{RGC}_d(\delta)$ the Kontsevich-Penner ribbon graph complex [Kon93] [Kon94] [Kon92] [Pen87] [MW15]. This is a combinatorial cochain complex with entries being ribbon graphs i.e. one dimensional CW-complexes equipped with cyclic order on a set of half-edges at each vertex and a choice of a certain certain orientation (for details see [MW15]). The differential is defined by splitting a vertex in all possible ways which preserves a cyclic order.

$$\Gamma_1 = \bullet \circ \bullet, \quad \Gamma_2 = \bullet \circ \bullet \circ \bullet, \quad \Gamma_3 = \circ \circ \circ$$

Examples of ribbon graphs.

According to the fundamental results of D. Mumford, R. C. Penner, J. Harer and W. Thurston [Mum] [Pen87] [Har88] there is a quasi-isomorphism of complexes\footnote{Here $\epsilon_n$ is a one dimensional sign local system.}:

$$\text{RGC}_{d}^{\bullet}(\delta)_{\geq 3} \cong \prod_{g \geq 0, n \geq 1, 2g + n - 2 > 0}^{\infty} C^{* + 2dg - n}(\mathcal{M}_{g,n}/\Sigma_n, \epsilon_n).$$

This result comes from introducing an orbispace of metric ribbon graphs $\mathcal{M}_{g,n}^{\text{rib}}$ and constructing a homeomorphism $\mathcal{M}_{g,n}^{\text{rib}} \cong \mathcal{M}_{g,n} \times \mathbb{R}_{>0}$. The description of the cohomology of moduli stacks of curves in terms of ribbon graphs plays a crucial role in the study of $\mathcal{M}_{g,n}$. In particular this description was used by J. Harer and D. Zagier [HZ86] to calculate the Euler characteristic of moduli stack of curves and by M. Kontsevich [Kon92] to prove E. Witten’s conjecture.
1.2. Main results. The main result of this paper is a proof of the following Theorem conjectured by T. Willwacher [Wil19]:

**Theorem 1 (T. Willwacher’s conjecture).** For every \( g \geq 0 \) and \( n \geq 1 \) such that \( 2g + n - 2 > 0 \) the following square commutes (in the derived category):

\[
\begin{array}{ccc}
\prod_{g,n} C^*_{c+2d-g-n}(\mathcal{M}_{g,n}/\Sigma_n, \epsilon_n) & \xrightarrow{\nabla_1} & \prod_{g,n} C^*_{c+2d-g-n}(\mathcal{M}_{g,n+1}/\Sigma_{n+1}, \epsilon_{n+1}) \\
\sim & \Delta_1 & \sim \\
RGC^*_d(\delta)_{\geq 3} & & RGC^*_{d+1}(\delta)_{\geq 3}
\end{array}
\]

Here \( \nabla_1 \) is the Willwacher differential introduced in [Kal22] which is defined\(^2\) as a pullback along the forgetful morphism:

\( \pi: \mathcal{M}_{g,n+1} \longrightarrow \mathcal{M}_{g,n} \)

and \( \Delta_1 \) is the so-called Bridgeland differential [Bri21] which is defined by adding an edge to a ribbon graph, splitting a boundary in two:\(^3\)

\[ \Delta_1: \begin{array}{c}
\bullet
\end{array} \longrightarrow \sum \begin{array}{c}
\square
\end{array} \]

Recall that the Merkulov-Willwacher ribbon graph complex \( RGC^*_d(\delta+\Delta_1) \) [MW15] is a version of the Kontsevich-Penner ribbon graph complex where the differential is twisted by the Bridgeland differential. This complex plays an important in the deformation theory of Lie bialgebras. From Theorem 1 and Theorem 4.4.5 from [Kal22] we compute the cohomology of the Merkulov-Willwacher ribbon graph complex:

**Theorem 2 (A. Căldăraru’s conjecture).** For every \( d \geq 0 \) the cohomology of the Merkulov-Willwacher ribbon graph complex \( RGC^*_d(\delta+\Delta_1) \) is isomorphic to the totality of shifted cohomology with compact support of moduli stacks \( \mathcal{M} \):

\[
H^*(RGC^*_d(\delta+\Delta_1)) \cong \prod_{g \geq 1} H^*_{c+2d-g-1}(\mathcal{M}_g, \mathbb{Q}).
\]

Denote by \( GC^*_d \) the Kontsevich graph complex [Kon93, Kon94, Wil15a] i.e. the combinatorial cochain complex with entries being at least two-valent non-loop graphs with a certain choice of an orientation (See [Wil15a] for the precise definitions) and a differential is defined by splitting a vertex in all possible ways.

**Theorem 3.** We have the canonical injection:

\[
H^*(GC^*_d) \longrightarrow H^*_{c+1}(RGC^*_d(\delta+\Delta_1))
\]

\(^2\)This morphism exists at the level of derived categories due to non-properness of the forgetful morphism.

\(^3\)More precisely we connect distinct corners attached to the same boundary i.e. intervals on ‘blown-up’ vertices of a ribbon graph lying between half-edges [MW15].
This result immediately follows from the M. Chan S. Galatius and S. Payne description of the weight zero part of the compactly supported cohomology of $\mathcal{M}_g$ [CGP21b] (See also [AWŽ20], [Kal22]) and Theorem 2.

1.3. Techniques. The main idea in a proof of Theorem 1 is very simple: we show that both differentials $\Delta_1$ and $\nabla_1$ are induced by the pullback along the morphism on moduli spaces of bordered surfaces which forgets a marking. A crucial role in the construction is played by moduli spaces of stable bordered surfaces in the sense of M. Liu and K. Costello [Lin20] [Cos07]. Costello’s moduli space $\mathcal{N}_{g,n}$ is a moduli space of stable bordered surfaces of genus $g$ with $n$-boundary components such that we only allows nodes on the boundary. This moduli space is a non-compact but smooth orbifold with corners. Analogous to the Deligne-Mumford moduli spaces these moduli spaces have versions with marked points. In the world of bordered surfaces we have versions of marked points depending where we allow them: on a boundary or (and) in an interior of the surface. Following op. cit we have a moduli space $\mathcal{N}_{g,n,p,q}$ of stable bordered surfaces with singularities only on a boundary and $p$ points on a boundary and $q$ points in an interior of the surface. We have morphisms:

$$\pi^{\text{real}}_p: \mathcal{N}_{g,n,p+1,q} \to \mathcal{N}_{g,n,p,q}, \quad \pi^{\text{comp}}_q: \mathcal{N}_{g,n,p,q+1} \to \mathcal{N}_{g,n,p,q}$$

which forget real or complex marking and stabilise the resulting bordered surface. Following op. cit we have a diagram of weak homotopy equivalences (the Costello homotopy equivalence):

$$v: \mathcal{N}_{g,n} \to \mathcal{N}_{g,n} \leftarrow D_{g,n}: \ u$$

Here $\mathcal{N}_{g,n}$ is a locus of smooth bordered surfaces which is an interior in $\mathcal{N}_{g,n}$ and $D_{g,n}$ is a locus of bordered surfaces with irreducible components of genus 0 (the maximally degenerated locus). Note that the chain complex which computes the homology of $D_{g,n}$ with coefficients in the sign local system can be identified with a direct summand of the Kontsevich-Penner ribbon graph complex (See [Cos07] [MW15]).

The Costello moduli space $\mathcal{N}_{g,n}$ is an open suborbifold in the Liu moduli space $\mathcal{N}_{g,n}^L$ [Lin20]. The latter moduli space consists of stable bordered surfaces of genus $g$ with $n$ (non-labelled) boundary components where we allow not only nodes on the boundary of the surface but also nodes in the interior and shrinking lengths of boundaries to zero. The moduli space $\mathcal{N}_{g,n}^L$ is a smooth and compact orbifold with corners of dimension $6g - 6 + 3n$. Our first result is the following:

**Theorem 4.** We have a diagram of weak homotopy equivalence:

$$a: \mathcal{M}_{g,n}/\Sigma_n \to \mathcal{N}_{g,n} \leftarrow \mathcal{N}_{g,n}: b.$$

Here $\mathcal{N}_{g,n}$ is open suborbifold of $\mathcal{N}_{g,n}^L$ which consists of stable bordered surfaces with the most singularities given by shrinking lengths of boundaries to zero (cusps) and $\mathcal{M}_{g,n}/\Sigma_n$ is identified with the maximal degenerated locus in $\mathcal{N}_{g,n}^L$ by filling

\[\text{This result is established with a help of } \text{CAT}(0)-\text{properties of the Weil-Petersson geometry of the moduli space } \mathcal{N}_{g,n}^L.\]
cusps with marked points. Theorem \[\text{[1]}\] allows us to define the so called boundary shrinking morphism:

\[
\rho_{\mathcal{M}_{g,n}/\Sigma_n} := a_1 \circ b_1^{-1} : C^*_c(\mathcal{M}_{g,n}/\Sigma_n, \epsilon_n) \xrightarrow{\sim} C_{c}^{*+n}(\mathcal{N}_{g,n}, \mathbb{Q})
\]

Building on Costello’s homotopy equivalence we explicitly relate the homology of \(D_{g.n}\) to the compactly supported cohomology of \(\mathcal{M}_{g,n}/\Sigma_n\):

\[
\text{RGC}_d^\delta(\delta) \xrightarrow{\alpha_n} \prod_{g,n} C_{6g-6+3n-2dg-\cdots}(\mathcal{N}_{g,n}, \epsilon_n) \xleftarrow{\Theta} \prod_{g,n} C_{c}^{*+2dg}(\mathcal{N}_{g,n}, \mathbb{Q})
\]

We defined the geometric Bridgeland differential \(\Delta^\text{bor}_1\) as the following composition: we pullback along the proper morphism which forgets the marking in the interior (induced by \(\pi^\text{comp}\)):

\[
\pi_{D_{g,n,0,1}} : D_{g,n,0,1} \longrightarrow D_{g,n}
\]

and then we take a Gysin pushforward along the inclusion to \(K_{g,n+1}^L\). The latter space is defined as the closure (in Liu’s moduli space) of the locus of nodal surfaces of the following type:

(a) All irreducible components are disks
(b) All irreducible components are disks except the unique one which is an annulus

The canonical inclusion \(D_{g,n+1} \rightarrow K_{g,n+1}^L\) is a weak-homotopy equivalence (Lemma \[\text{5.1.1}\]) and hence we define the geometric Bridgeland differential \(\Delta^\text{bor}_1\) as the following zig-zag morphism:

\[
C^*(D_{g,n}, \mathbb{Q}) \xrightarrow{\pi_{D_{g,n,0,1}^\text{bor}}} C^*(D_{g,n,0,1}, \mathbb{Q}) \xrightarrow{m_1} C^*(K_{g,n+1}^L, \mathbb{Q}) \xleftarrow{c'} C^*(D_{g,n+1}, \mathbb{Q})
\]

In order to relate the morphism \(\Delta^\text{bor}_1\) to the Bridgeland differential \(\Delta_1\) we have to introduce the intermediate operators \(\Theta_0\) and \(\Theta_1\). The morphism \(\Theta_1\) is an obvious geometric incarnation of the Bridgeland differential. Denote by \(D_{g,n,2}^1 \subset D_{g,n,0,2}\) the (non-compact) locus which consist of stable bordered surfaces where (non-labelled) markings belong to distinct corners of the same boundary. We have morphisms:

\[
\pi_{D^1_{g,n,2}} : D^1_{g,n,2} \longrightarrow D_{g,n}, \quad \xi_{D^1_{g,n,2}} : D^1_{g,n,2} \longrightarrow D_{g,n+1}
\]

This first morphism is induced by morphism \(\pi^\text{real}\) and defined by forgetting marked points and stabilising the nodal curve. The second morphism is a version of clutching morphism for bordered surfaces i.e. we connect two marked points by a node (cf. [Knu83]). Note that a clutching morphism is proper and hence the push-forward in Borel-Moore chains is well defined. We set:

\[
\Theta_1 := \xi_{D^1_{g,n,2}} \circ \pi^1_{D^1_{g,n,2}} : C_*(D_{g,n}, \epsilon_n) \longrightarrow C_{*+2}(D_{g,n+1}, \epsilon_{n+1})
\]

We prove (Lemma \[\text{5.1.8}\]) that via the identification between homology of \(D_{g,n}\) and cohomology of the Kontsevich-Penner ribbon graph complex the morphism \(\Theta_1\) correspond to the Bridgeland differential \(\Delta_1\). The second morphism \(\Theta_0\) is defined by "connecting identical corners by an edge". Denote by \(D_{g,n,2}^2 \subset D_{g,n,0,2}\) the locus
of stable bordered surfaces with markings lying on the same boundary, by \( D_{g,n,2}^0 \) we denote the closure of the complement \( D_{g,n,2}^2 \setminus D_{g,n,2}^1 \). We have morphisms:

\[
\pi_{D_{g,n,2}^0} : D_{g,n,2}^0 \longrightarrow D_{g,n}, \quad \xi_{D_{g,n,2}^0} : D_{g,n,2}^0 \longrightarrow D_{g,n+1},
\]

defined analogically to morphisms above. Note that a clutching morphism is proper hence we set:

\[
\Theta_0 := \xi_{D_{g,n,2}^0}^! \pi_{D_{g,n,2}^0}^* : C^*_{q}(D_{g,n}, \epsilon_n) \longrightarrow C^*_{q+2}(D_{g,n+1}, \epsilon_{n+1})
\]

We show (Proposition 5.1.6) that the morphism \( \Theta_0 \) is the Poincaré–Verdier dual to the geometric Bridgeland differential \( \Delta^{\text{bor}}_1 \). Finally using the gluing of sheaves and base change theorems we show (Lemma 5.1.7) that morphisms \( \Theta_0 \) and \( \Theta_1 \) coincide in the derived category. Hence we get the following:

**Theorem 5.** For \( g \geq 0, n \geq 1 \) such that \( 2g + n - 2 > 0 \), the following diagram commutes (in the derived category of vector spaces):

\[
\begin{array}{ccc}
\prod_{g,n} C^{*+2g(d-1)-n+1}(D_{g,n}, \mathbb{Q}) & \xrightarrow{\Delta^{\text{bor}}_1} & \prod_{g,n} C^{*+2g(d-1)-n+1}(D_{g,n+1}, \mathbb{Q}) \\
D \sim & & D \sim \\
\text{RGC}_{c}(d)_{\geq 3} & \xrightarrow{\Delta_1} & \text{RGC}_{c+1}(d)_{\geq 3}
\end{array}
\]

The geometric Bridgeland differential admits an obvious version for the moduli space of smooth bordered surfaces: we pullback along the (proper) morphism which forgets a marking in the interior:

\[
\pi_{N_{g,n,0,1}} : N_{g,n} \times \nabla_{g,n} N_{g,n,0,1} \longrightarrow N_{g,n}
\]

Further we take a Gysin pushforward to the moduli space of bordered surfaces \( N_{g,n+1}^L \). The latter is moduli space is defined as a subspace in Lui’s moduli space where we do not allow singularities in an interior. The canonical inclusion \( N_{g,n} \rightarrow N_{g,n}^L \) is a homotopy equivalence. Hence we define the morphism \( \nabla^{\text{bor}}_1 \) as the following zig-zag morphism:

\[
C^*_c(N_{g,n}, \mathbb{Q}) \xrightarrow{\pi_{N_{g,n,0,1}}^*} C^*_c(N_{g,n} \times \nabla_{g,n} N_{g,n,0,1}, \mathbb{Q}) \longrightarrow C^*_c(N_{g,n+1}^L, \mathbb{Q}) \xleftarrow{\sim} C^*_c(N_{g,n+1}, \mathbb{Q})
\]

It is almost obvious (Proposition 5.2.2) that under the Costello homotopy equivalence morphisms \( \Delta^{\text{bor}}_1 \) and \( \nabla^{\text{bor}}_1 \) correspond to each other. Finally the proof of T. Willwacher’s conjecture follows from the following comparison result:

**Theorem 6.** For \( g \geq 0, n \geq 1 \) such that \( 2g + n - 2 > 0 \), the following diagram commutes in the derived category of vector spaces:

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5The reason for this is that one of the possible degenerations of surfaces having an annulus as an irreducible component are given by contracting arcs with endpoints on two boundaries of an annulus, this leads to the morphism which connect identical corners by an edge.
This Theorem compares the Willwacher differential $\nabla_1$ and the differential $\nabla^{bor}$ under the homotopy equivalence from Theorem 4. More precisely we show that both differentials are induced by the forgetful morphism $\pi^{\text{comp}} : \mathcal{N}^L_{g,n,0,1} \to \mathcal{N}^L_{g,n}$ on Liu’s moduli spaces of stable bordered surfaces.

1.4. Overview. In the second section we recollect some facts from Borel-Moore homology and operations on Borel-Moore chains which we will need. In the third section we recall basic construction from ribbon graphs as well as constructions of the Merkulov-Willwacher ribbon graph complex and the Bridgeland differential. The fourth section is devoted to recalling definitions of the Liu and Costello moduli spaces of stable bordered surfaces as well as Costello’s result about the homotopy equivalence between $\mathcal{N}^L_{g,n}$ and $D_{g,n}$. In this section, we prove our Theorem 4. The fifth section is a heart of the present work, we make all necessary definitions and give proofs of Theorem 5 and Theorem 6. Further we get the proof of T. Willwacher’s conjecture and deduce A. Căldăraru’s conjecture and Theorem 3. The last Section (Appendix) is devoted to recollecting some facts about the Teichmüller space of a bordered surface and Weil-Petersson geometry of this space.

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2. Preliminaries

2.1. Notation. For a natural number $n \in \mathbb{N}_+$ we will denote by $[n]$ a finite set such that $[n] := \{1, 2, \ldots, n\}$. Let $I$ be a finite set, $\text{Aut}(I)$ we will denote the group of automorphisms of this set, in the case when $I = [n]$ we will use a notation $\Sigma_n := \text{Aut}([n])$ for a symmetric group on $n$-letters. We usually work over the field of rational number $\mathbb{Q}$. For $\mathbb{Q}$-linear representation $V$ of a finite group $G$ we will denote by $V^G$ (resp. $V_G$) the space of $G$-invariant (resp. $G$-coinvariants). Since the characteristic of a field $\mathbb{Q}$ is zero the canonical morphism $V_G \to V^G$ is an isomorphism and hence we will freely switch between invariants and coinvariants. For a finite $S$ we will denote by $\det(S) := \wedge^{\dim V(S)} V(S)$ the determinant on the free $\mathbb{Q}$-vector space $V(S)$ generated by a set $S$.

For any $g$, and $n$ we have the moduli stack $\mathcal{M}_{g,n}$ which parametrizes smooth proper algebraic curves of genus $g$ with $n$ (labelled) marked points. This is a smooth Artin stack of dimension $3g - 3 + n$ over $\mathbb{Z}$. When $2g + n - 2 > 0$ this is a smooth Deligne-Mumford stack. For $2g + n - 2 > 0$ according to [DM69] there exists a
compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$ which is defined by adding all stable nodal curves. $\overline{\mathcal{M}}_{g,n}$ is a smooth and proper Deligne-Mumford stack over $\mathbb{Z}$. The complement to a smooth locus is a divisor with normal crossings. By the abuse of notation we will identify the moduli stack with corresponding coarse moduli spaces.

2.2. Borel-Moore homology. For a locally compact topological space $X$ we will use a notation $D(X)$ for the derived category of sheaves of $\mathbb{Q}$-vector spaces on $X$. It is well known that the formalism of six functors exists for the category $D(X)$. By $\omega_X$ we denote the dualising complex on $X$ i.e. an object in the derived category of sheaves $D(X)$, defined by the rule:

$$\omega_X := a_X^! \mathbb{Q}, \quad a_X : X \to *$$

In particular when $X$ is a topological manifold (possible with a boundary) of finite dimension we have an isomorphism of sheaves $\omega_X \cong or_X[\dim X]$, here $or_X$ is the sheaf of orientations on $X$. Note that if $X$ is also an orientable manifold we have an isomorphism of sheaves $or_X \cong \mathbb{Q}_X$.

We will use a notation: $D := D(X) \to D(X)^{op}$, $D := R\text{Hom}(\ , \omega_X)$ for the Verdier duality functor defined as internal hom functor with values in the dualising sheaf. For an object $K \in D(X)$ we define the Borel-Moore chain complex $C_{BM}^\ast(X, K)$ with coefficients in $K$ by the rule:

$$C_{BM}^\ast(X, K) := R^{-\ast}\Gamma(X, D(K)).$$

When $X$ is a compact manifold the Borel-Moore chains are quasi-isomorphic to the standard chains with coefficients in a sheaf $K$:

$$C_{BM}^\ast(X, K) := R^{-\ast}\Gamma(X, D(K)) \cong R^{-\ast}\Gamma_c(X, (K)^\ast) := C_c^\ast(X, K).$$

In the case when $K$ is a constant sheaf $\mathbb{Q}_X$ we recover the usual definition of the Borel-Moore homology $C_{BM}^\ast(X, \mathbb{Q}) := R^{-\ast}\Gamma(X, \omega_X)$). When $X$ is a finite dimensional topological manifold we have the Poincaré-Verdier duality for the Borel-Moore chains:

$$D : C_{BM}^\ast(X, \mathbb{Q}) \cong C^{\dim X - \ast}(X, or_X)$$

Note that Borel-Moore chains is dual to the compactly supported chains i.e. we have a morphism:

$$C_{BM}^\ast(X, K)^\ast := R^{-\ast}\Gamma(X, D(K))^\ast \to R^\ast\Gamma_c(X, K)^\ast := C_c^\ast(X, K).$$

The latter morphism is induced by the natural transformation:

$$D^2 \to \text{Id}.$$

When $K$ is a cohomologically constructible complex the above morphism is quasi-isomorphism.

Suppose that a space $X$ is equipped with a stratification $S = \{S_\alpha\} i_\alpha : S_\alpha \to X$. Following [KS16] we can define the homological Cousin complex i.e. the complex which computes the Borel-Moore homology of $X$ with coefficients in $K$:

**Proposition 2.2.1.** The DG-vector space $C_{BM}^\ast(X, K)$ is quasi-isomorphic naturally to (the total DG-vector space arising from) the complex:

$$\bigoplus_{\text{codim } S_\alpha = 0} C_{BM}^\ast(S_\alpha, i_\alpha^* K) \to \bigoplus_{\text{codim } S_\alpha = 1} C_{BM}^\ast(S_\alpha, i_\alpha^* K) \to \ldots$$
In particular when all strata $S_\alpha$ are contractible the DG-vector space $C^{BM}_\ast(X, K)$ is quasi-isomorphic to:

$$\bigoplus_{\text{codim } S_\alpha = 0} K_\alpha \otimes \mathcal{O}_\alpha \rightarrow \bigoplus_{\text{codim } S_\alpha = 1} K_\alpha \otimes \mathcal{O}_\alpha \rightarrow \ldots$$

Here $K_\alpha := H^{BM}_{\dim S_\alpha}(S_\alpha, K)$ and $\mathcal{O}_\alpha := H^{BM}_{\dim S_\alpha}(S_\alpha, \mathbb{Q})$ is the one-dimensional space of orientations. Note that the sum over strata of codimension $m$ is placed in degree $-m$.

**Proof.** Consider the filtration of the space $X$:

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X$$

Here $X_k$ is a union of strata of dimension $k$. Denote by $Y_k := X_k \setminus X_{k-1}$ with an inclusion $j_k : Y_k \rightarrow X$. With a sheaf $D(K)$ we can associated the Postnikov tower:

$$R(j_{\dim X \ast}) j_{\dim X \ast}^! D(K) \rightarrow R(j_{\dim X-1 \ast}) j_{\dim X-1 \ast}^! D(K) \rightarrow \cdots \rightarrow R(j_0 \ast) j_0^! D(K).$$

Then applying the derived sections we get the desired result. □

### 2.3. Operations on BM-chains

**For any proper morphism** $f : X \rightarrow Y$ and any sheaf $K$ on $Y$ we define the (proper) pushforward in Borel-Moore chains:

$$(1) \quad f^* : C^{BM}_\ast(X, f^* K) \rightarrow C^{BM}_\ast(Y, K)$$

as the derived global sections the morphism of sheaves which comes from the following natural transformation of functors:

$$R(f_!) f_! D \cong R(f_!) f_!^* D \rightarrow D$$

Let $f : X \rightarrow Y$ be an arbitrary morphism of locally compact topological spaces. We define the **Gysin pullback in Borel-Moore chains**:

$$(2) \quad f^! : C^{BM}_\ast(Y, K) \rightarrow C^{BM}_\ast(X, f^! K)$$

as the derived global sections of the following morphism:

$$D(K) \rightarrow R(f_!) f_!^* (D(K)) \cong R(f_!) D(f^! K).$$

We will be mostly interested in the case when $K$ is the orientation sheaf. In this case we have:

$$f^! : C^{BM}_\ast(Y, \mathcal{O}_Y) \rightarrow C^{BM}_\ast(X, \mathcal{O}_X)$$

In particular when $j : U \rightarrow X$ is an open embedding we recover a notion of usual pullback in Bore-Moore chain [CG10].

**Remark 2.3.1.** Note that if a morphism $f$ is a weak-homotopy equivalence and both $X$ and $Y$ are topological manifolds possible with a boundary then the Gysin pullback induces the quasi-isomorphism:

$$f^! : C^{BM}_\ast(Y, \mathcal{O}_Y) \approx C^{BM}_\ast(X, \mathcal{O}_X).$$

This immediately follows from the definition and the Poincare-Verdier duality for Borel-Moore chains.

**Remark 2.3.2.** In the dual setting of the compactly supported cochains for any proper morphism $f : X \rightarrow Y$ and a sheaf $K$ on $Y$ we will use a notation:

$$f^* : C^c_\ast(Y, K) \rightarrow C^c_\ast(X, f^* K)$$
for the corresponding proper pullback dual to \(\mathcal{1}\). With an arbitrary morphism \(f: X \to Y\) we can associate the *Gysin-pushforward* in compactly supported cochains:

\[
f_! : C^*_c(X, f^! \mathcal{K}) \to C^*_c(Y, \mathcal{K})
\]

which is dual to a morphism \(\mathcal{2}\).

The Gysin pullback satisfies the following base change property:

**Proposition 2.3.3.** Consider the fibered diagram of locally compact spaces:

\[
\begin{array}{ccc}
W := Z \times_X Y & \xrightarrow{\tilde{f}} & Y \\
\downarrow{\tilde{g}} & & \downarrow{g} \\
Z & \xrightarrow{f} & X
\end{array}
\]

such that the following conditions hold:

(i) \(\dim Y - \dim X = \dim W - \dim Z\),
(ii) \(g\) is proper and oriented i.e. \(\tilde{g}^! \mathbb{Q}_X \cong \mathbb{Q}_Y\),
(iii) We have a transversality condition:

\[
\tilde{f}^* \tilde{g}^! \sim \tilde{g}^! f^*.
\]

Then we have the commutative diagram:

\[
\begin{array}{ccc}
C^*_{BM} \left( W, or_W \right) & \xrightarrow{\tilde{f}^!} & C^*_{BM} \left( Y, or_Y \right) \\
\downarrow{\tilde{g}^*} & & \downarrow{g^*} \\
C^*_{BM} \left( Z, or_Z \right) & \xrightarrow{f^!} & C^*_{BM} \left( X, or_X \right)
\end{array}
\]

**Proof.** Using the standard adjunction morphisms we have a commutative diagram (in the category of sheaves):

\[
\begin{array}{ccc}
\mathbf{R}f_* f^! g^! & \xleftarrow{\text{adj}} & \mathbf{R}g_* g^! \\
\downarrow{\text{adj}} & & \downarrow{\text{adj}} \\
\mathbf{R}f_* f^* & \xleftarrow{\text{adj}} & \text{Id}
\end{array}
\]

Now plug in a sheaf \(\mathcal{D}(or_X)\) in the diagram above we get:

\[
\begin{array}{ccc}
\mathbf{R}f_* f^* g^! \mathcal{D}(or_X) & \xleftarrow{\text{adj}} & \mathbf{R}g_* g^! \mathcal{D}(or_X) \\
\downarrow{\text{adj}} & & \downarrow{\text{adj}} \\
\mathbf{R}f_* f^* \mathcal{D}(or_X) & \xleftarrow{\text{adj}} & \mathcal{D}(or_X)
\end{array}
\]

\[\text{This morphism is defined as an adjoint to the base change isomorphism Proposition 3.1.9 [KS90].}\]
Applying the functor of derived sections, using the base change Theorem for sheaves, transversality and orientability we get a diagram:

\[
\begin{array}{ccc}
C_{\star + \dim Z - \dim X} (W, or_W) & \xrightarrow{f^!} & C_{\star} (Y, or_Y) \\
\downarrow g^* & & \downarrow g^* \\
C_{\star + \dim Z - \dim X} (Z, or_Z) & \xrightarrow{f^!} & C_{\star} (X, or_X)
\end{array}
\]

Further using the transversality we get that the diagram above commutes.

\[\square\]

**Remark 2.3.4.** Using the fact that the compactly supported cohomology are dual to the Bore-Moore chains in the assumptions of Proposition 2.3.3 we get the following commutative square:

\[
\begin{array}{ccc}
C_{\star + \dim Z - \dim X} (W, or_W) & \xrightarrow{\tilde{f}^!} & C_{\star} (Y, or_Y) \\
\downarrow \tilde{g}^* & & \downarrow g^* \\
C_{\star + \dim Z - \dim X} (Z, or_Z) & \xrightarrow{f^!} & C_{\star} (X, or_X)
\end{array}
\]

Let \(f: X \to Y\) be a morphism between stratified manifolds with contractible strata. We can explicitly describe the Gysin pullback in Borel-Moore chains (cf. [Gor13]):

**Proposition 2.3.5.** In terms of the Cousin complexes from Proposition 2.2.1 the Gysin pullback \(f^! : C_{\star} (Y, or_Y) \to C_{\star + \dim X - \dim Y} (X, or_X)\) acts by the rule:

\[
f^! : H_{BM}^{\dim Y} (S_\alpha, or_Y) \to H_{BM}^{\dim Y - (\dim - 1) (S_\alpha), or_X |_{f^{-1} (S_\alpha)}}
\]

**Proof.** Consider the adjunction morphism: \(adj : R(f)_* \to or_Y\). Applying the Verdier duality and using the fact that \(f^! or_Y \simeq or_X [\dim X - \dim Y]\) we have:

\[
\mathbb{D}(or_Y) \to R(f)_* \mathbb{D}(or_X)
\]

Consider the Cousin resolution \[2.2.1\] for the sheaf \(R(f)_* \mathbb{D}(or_X)\)

\[
R(j_{\dim Y*})_\dim Y R(f)_* \mathbb{Q}_X \to \cdots \to R(j_{0*})_0 R(f)_* \mathbb{Q}_X.
\]

Let \(S_\alpha\) be a stratum in \(Y\), \(j_\alpha : S_\alpha \to Y\), we have \(f^{-1} (S_\alpha) = S_\alpha \times_Y X\), with the corresponding fibered square:

\[
\begin{array}{ccc}
\tilde{j}_\alpha & \to & X \\
\downarrow f_\alpha & & \downarrow f \\
S_\alpha & \to & Y
\end{array}
\]

Applying the base change theorem for sheaves [KS90] we get:

\[
R(j_{\dim Y*} f_{\dim Y*})_\dim Y \mathbb{Q}_X \to \cdots \to R(j_{0*} f_{0*})_0 \mathbb{Q}_X.
\]

\[\square\]
3. Ribbon graph complexes

3.1. Ribbon graphs. By a ribbon graph \( \Gamma \) we understand a triple \((H(\Gamma), \sigma_1, \sigma_0)\) where \( H(\Gamma) \) is a finite set called the set of half edges of \( \Gamma \), \( \sigma_1: H(\Gamma) \rightarrow H(\Gamma) \) is a fixed point free involution whose set of orbits, and a permutation \( \sigma_0: H(\Gamma) \rightarrow H(\Gamma) \). Orbits of \( \sigma_1 \) are called the edges of \( \Gamma \), we will denote the set of edges of \( \Gamma \) by \( E(\Gamma) \). A set of orbits, \( V(\Gamma) := H(\Gamma)/\sigma_0 \), is called the set of vertices of the ribbon graph \( \Gamma \). We have a canonical map:
\[
p: H(\Gamma) \rightarrow V(\Gamma) := H(\Gamma)/\sigma_0
\]
For each \( v \in V(\Gamma) \) the pre-image of \( p \) will be called a set of half-edges attached to a vertex \( v \):
\[
p^{-1}(v) := H_v(\Gamma).
\]
The orbits of the permutation \( \sigma_2 := \sigma_0^{-1}\sigma_1 \) are called boundaries of the ribbon graph \( \Gamma \). The set of boundaries of \( \Gamma \) is denoted by \( B(\Gamma) \). By a genus of a ribbon graph we understand the following quantity:
\[
g(\Gamma) := 1 + \frac{1}{2}(E(\Gamma) - V(\Gamma) - B(\Gamma))
\]
The definition implies that a ribbon graph \( \Gamma \) is the same as a standard graph with a fixed cyclic structure on the set of half-edges \( H_v(\Gamma) \) at each vertex \( v \in V(\Gamma) \). The definition of a genus and boundaries of a ribbon graph is consistent with the following well-known construction: with a ribbon graph \( \Gamma \) it is always possible to associate a compact bordered surface \( \Sigma_\Gamma \) by replacing vertices with disks and edges with strips \([Kon92]\). Then a set of boundary components of a ribbon graph corresponds to the set boundaries of the surface \( \Sigma_\Gamma \) and \( g(\Sigma_\Gamma) = g(\Gamma) \).

Recall that in \([MW15]\) for a ribbon graph \( \Gamma \) a notion of a corner of a ribbon graph is defined. Representing ribbon graphs as graphs with 'blown up' vertices a corner is defined as an interval between two half-edges. Set of corners associated with a ribbon graph \( \Gamma \) will be denoted by \( C(\Gamma) \). For a ribbon graph \( \Gamma \) we have the following partition:
\[
C(\Gamma) = \coprod_{b \in B(\Gamma)} C(b)
\]
Elements from \( C(b) \) will be called corners attached to a boundary \( b \). Following \( op. cit. \) with a ribbon graph \( \Gamma \) with \( l \)-edges and \( n \)-boundaries we can associate a ribbon graph \( c_{c_1,c_2}(\Gamma) \) with \( l+1 \) edges and \( n+1 \) boundaries, here \( c_1, c_2 \) are corners which belong to the same boundary \( b \). This operation is defined by attaching a new edge \( e \) connecting corners \( c_1 \) and \( c_2 \). Pictorially that means:

3.2. Ribbon graph complexes. Let us briefly recall some of the main results of \([MW15]\). Denote by \( \text{RGra}_d \) the properad of ribbon graphs. This properad is defined by the collection of vector spaces \( \text{RGra}_d(n,m) \) which consists of directed and connected ribbon graphs with \([n]\)-labelled boundaries and \([m]\)-labelled vertices. We assume the choice of the orientation on the set of edges of a ribbon graph (See \( op. cit. \)). The composition is defined by attaching a vertex of one ribbon graph to the boundary of another. More precisely with every ribbon graph \( \Gamma \) we consider
the corresponding bordered Riemann surface $\Sigma_\Gamma$, where boundaries of the surface we represent as "out-circle" further we remove disks in $\Sigma_\Gamma$ at vertices of $\Gamma$ and represent these boundaries as "in-circle", finally we "glue" two Riemann surfaces in all possible ways by identifying "in-circle" with "out-circles", we refer to [MW15] for details (cf. [TZ07]). The Theorem 4.2.3 from [MW15] (based on [CS02]) claims that there is a natural morphism of properads:

$$s^* : \Lambda \text{LieB}_{d,d} \to \text{RGra}_d,$$

Here $\Lambda \text{LieB}_{d,d}$ is the standard properad controlling $\text{Lie } d-1$-bialgebras with a bracket and a cobracket being symmetric operations of degree $d-1$. It will be convenient to represent operations in the properad $\Lambda \text{LieB}_{d,d}$ as trees. For the generators of $\Lambda \text{LieB}_{d,d}$ it means:

- bracket = \[ \begin{array}{c} \bullet \\ \Gamma \end{array} \]
- cobracket = \[ \begin{array}{c} \Gamma \\ \bullet \end{array} \]

The morphism $s^*$ is defined on generators of $\Lambda \text{LieB}_{d,d}$ by the following rule:

$$s^* : \begin{array}{c} \bullet \\ \Gamma \end{array} \to 0.$$  

Hence using the theory of deformation complexes for properads [MV09a][MV09b] one may consider the deformation complex of this morphism of properads:

$$RGC_d^*(\delta) := \text{Def}(\Lambda \text{LieB}_{d,d} \xrightarrow{s^*} \text{RGra}_d)$$

According to [MV09a] and [MV09b] complex $RGC_d^*(\delta)$ is a DG-Lie algebra with a bracket $[-,-]$ which comes from the properadic composition. The differential in the complex (8) is defined by the rule $\delta := [-,\Gamma]$, here $\Gamma$ is the following element:

$$\Gamma = \begin{array}{c} \bullet \\ \bullet \end{array}$$

We will this object the Kontsevich-Penner ribbon graph complex. This complex can be explicitly realised as a complex with $k$-cochains $RGC_d^k$ being isomorphism classes of ribbon graphs $\Gamma$ with:

$$k = -2gd + |E(\Gamma)|$$

together with an orientation $\text{or}$ defined by a choice of an element in the vector space:

(a) $\det(V(\Gamma)) \otimes \det(B(\Gamma)) \otimes_{e \in E(\Gamma)} \det(H(e)^{\text{or}})$ for $d$ odd.

(b) $\det(E(\Gamma))$ for $d$ even.

We assume that:

$$(\Gamma, \text{or}) = -(\Gamma, \text{or}^{\text{op}})$$

Remark 3.2.1. It is evident that for different odd (resp. even) values of $d$ the Kontsevich-Penner ribbon graph complexes are isomorphic up to a shift. Moreover according to [MW15] these complexes are isomorphic for all values of $d$ (See the proof of Proposition 4.3.1)).

---

7 For $d=1$ algebras over this properad are Lie bialgebras in the sense of Drinfeld [Dri87].

8 Note that here we consider ribbon graphs of arbitrary valency.

9 Here by $H(e)$ we have denote the set of two half-edges associated with an edge $e$. 

The differential $\delta: \text{RGC}_d^k \rightarrow \text{RGC}_d^{k+1}$ is defined by the vertex splitting (which preserves the cyclic structure (See op. cit for details). We have the following decomposition:

$H^*(\text{RGC}_d(\delta)) = H^*(\text{RGC}_d(\delta)_{\geq 3}) \oplus H^*(\text{RGC}_d(\delta)_{\leq 2})$ 

Here by $\text{RGC}_d^k(\delta)_{\geq 3}$ we have denoted the ribbon graph complex which consist of at least trivalent ribbon graphs and by $\text{RGC}_d^k(\delta)_{\leq 2}$ we have denoted the ribbon graph complex which consist of ribbon graphs of valence $\leq 2$. Following the argument from [Wil15a] (Proposition 3.4) the cohomology of the second direct summand occur only in degrees $1, 5, 9, \ldots, 2k + 1$:

$H^*(\text{RGC}_d(\delta)_{\leq 2}) \cong \bigoplus_{i=k+1, j \equiv 1 \mod 4} \mathbb{Q}[j]$ 

These elements can be represented by the loop type ribbon graphs $R_k$ (ribbon graphs with $k$-edges, $k$-vertices and two boundaries). Note that the Kontsevich-Penner ribbon graph complex $\text{RGC}_d(\delta)_{\geq 3}$ can be identified with the following product:

$\bigoplus_{g \geq 0, n \geq 1} C_{c,-2g}^*(M_{g,n}^{\text{rib}}/\Sigma_n, \mathbb{Q})$

Here $M_{g,n}^{\text{rib}}$ is Kontsevich’s moduli space of ribbon graphs of genus $g$ with $n$-labelled boundaries (See [Kon93] and [Kon94]).

Analogically to the morphism we can consider the morphism $s$ defined by the rule:

$s: \bigoplus \rightarrow \bigoplus$  

We can consider the following deformation complex:

$\text{RGC}_d^*(\delta_s) := \text{Def}(\Lambda \text{LieB}_{d,d} \rightarrow \text{RGr}_{d})$

We will call this object the Merkulov-Willwacher ribbon graph complex. Like the Kontsevich-Penner ribbon graph complex the complex $\text{RGC}_d^*(\delta_s)$ is a DG-Lie algebra. The differential is defined by the following rule $\delta := [-,\Gamma]$, here $\Gamma$ is the following element:

$\Gamma = \bigoplus + \bigcirc$

The differential $\delta_s$ can be decomposed as $\delta_s = \delta + \Delta_1$, here:

$\Delta_1 := [-, \bigcirc]$ 

is the so-called Bridgeland differential. This morphism can be explicitly described by the rule (up to signs):

$\Delta_1(\Gamma) = \sum_{b \in B(\Gamma)} \sum_{c_1,c_2 \in C(b)} e_{c_1,c_2}(\Gamma)$.

The morphism $\Delta_1$ satisfies the following properties:

$\delta\Delta_1 + \Delta_1\delta = 0, \quad \Delta_1^2 = 0.$
Moreover the Bridgeland differential preserves a genus of a ribbon graph. Hence we have a natural product decomposition:

\[(10) \quad \text{RGC}_d^*(\delta + \Delta_1) := \prod_{g=0}^{\infty} B_g \text{RGC}_d^*(\delta + \Delta_1).\]

Here we have used the notation \(B_g \text{RGC}_d^*(\delta + \Delta_1)\) for the direct summand in \(\text{RGC}_d^*(\delta + \Delta_1)\), which consists of ribbon graphs with genus equal to \(g\).

**Remark 3.2.2.** A definition \((9)\) of the morphism \(\Delta_1\) was first given by T. Bridgeland [Bri21] in the beginning of 00's. According to op. cit the operation \(\Delta_1\) appeared in studying of \(N = (2, 2)\) SCFT as the dual differential to the standard vertex splitting differential \(\delta\). Namely there is an involution on ribbon graphs given by interchanging vertices with boundaries (See [MW15] and [CFL16]). Under this involution the differential \(\delta\) corresponds to the Bridgeland differential \(\Delta_1\). Later S. Merkulov and T. Willwacher [MW15] rediscovered this operation.\[10\]

4. **Moduli spaces of surfaces with boundary**

4.1. Bordered surfaces. There are two classical ways to relate moduli spaces of ribbon graphs to moduli spaces of algebraic curves. One is due to D. Mumford [Mum] [Har88] based on the technique of Jenkins-Strebel differentials [Str84], which was later popularised by M. Kontsevich [Kon92] and the second one is due to R. Penner [Pen87] based on the hyperbolic geometry. For us it will be convenient to use the third approach due to K. Costello [Cos07].

Let \(I, K\) and \(J\) be a finite sets and \(\pi: I \to K\) be a morphism of finite sets. Following M. Liu [Liu20] for \(g \geq 0\) and we denote by \(\mathcal{N}_{g,K,\pi,J}^L\) the moduli space of stable bordered surfaces of genus \(g\) with \(K\)-labelled boundaries, \(I\)-labelled marked points on the boundary such that exactly points \(\pi^{-1}(k)\) attached to \(k\)-boundary and \(J\)-labelled marked points in the interior of the surface. Here we assume that singularities can be of three types:

(a) Singularities on the boundary locally isomorphism to:
\[x^2 - y^2 = 0\] (real node of type one).

(b) Singularities on the boundary i.e. locally isomorphism to:
\[x^2 + y^2 = 0\] (real node of type two).

(c) Singularities in the interior locally isomorphism to:
\[xy = 0\] (complex singularities).

This is an orbifold with corners of dimension \(6g - 6 + 3|K| + |I| + 2|J|\). The open locus of smooth (hyperbolic) surfaces will be denoted by \(\mathcal{N}_{g,K,\pi,J}^L\). We will use notation \(\mathcal{N}_{g,|K|,|I|,|J|}^L\) for the following orbifold:

\[\mathcal{N}_{g,|K|,|I|,|J|}^L := \left( \prod_{\pi: I \to K} \mathcal{N}_{g,K,\pi,J}^L \right)/G\]

\[10\]In our work we also follow the notation from [MW15].
Here the group $G$ is defined by the following rule:

$$G := \prod_{k \in K} \mathbb{Z}_{|\pi^{-1}(k)|} \rtimes \text{Aut}(K) \times \text{Aut}(J)$$

Note that the quotient orbifold $N_{g,[K],[I],[J]}$ is non-orientable and the corresponding orientation sheaf $\mathcal{O}^{N_{g,[K],[I],[J]}}$ is defined by the following rule. We consider a one dimensional representation of a group $G$ defined by the rule (cf. [HVZ10]):

1. It acts as a trivial representation of $\text{Aut}(J)$,
2. It acts as $(-1)^{|\pi^{-1}(k)|-1}$ on generators of cyclic groups,
3. It acts as $(-1)^{|(\pi^{-1}(k)|-1)|(|\pi^{-1}(p)-1|}$ on generators of $\text{Aut}(K)$

In particular the orientation sheaf $\mathcal{O}^{N_{g,n}}$ for the orbifold $N_{g,n}$ is given by the sign representation of $\Sigma_n$ and will be denoted by $\epsilon_n$. By $N_{g,n,k,p} \subset N_{g,n,k}$, we will denote the locus of stable bordered surfaces with at most real singularities of first type, in the sense of K. Costello [Cos07]. This is an orbifold with corners of dimension $6g - 6 + 3n + k + 2p$ Since the most of the time we will be interested in surfaces without cusps and marked points on the boundary we will use the notation $N_{g,n}$ (resp. $N_{g,0}$) for $N_{g,n,0,0}$ (resp. $N_{g,0}$).

Analogically to the case of Deligne-Mumford moduli spaces we have morphisms which forgets markings. In our case there will be morphisms of two types:

(11) $\pi_{\text{real}}^{\text{real}} : N_{g,n,k+1,p}^{k+1} \to N_{g,n,k,p}$

This is a proper morphism which forgets the real marked point and stabilise the resulting curve. We also have:

(12) $\pi_{\text{comp}}^{\text{comp}} : N_{g,n,k,p+1}^{k+1} \to N_{g,n,k,p}$

This is also a proper morphism which forgets the complex marked point and stabilise the resulting curve. Let us consider some examples which we will be interested in later on. We will use the notation for the pullback of a morphism (12) along in inclusion of Costello moduli space:

(13) $\pi_{N_{g,n,0}} : N_{g,n,0,1} \to N_{g,n}$

We will also use the notation:

(14) $\pi_{N_{g,n,2,0}} : N_{g,n,2,0} \to N_{g,n}$

For the pullback of the composition of morphism (11) along in the inclusion of the Costello moduli space.

**Remark 4.1.1.** These moduli spaces of bordered surfaces in the sense of Liu are related to the moduli spaces of real algebraic curves (See [Cey07], [GM04], [Kap93], [Sep91], [Sil92]). Namely starting with the nodal bordered Riemann surface $\Sigma \in N_{g,K,\pi,J}$ we can always associate the complex algebraic curve $\Sigma_C := \Sigma \cup_{\partial \Sigma} \Sigma$, called the complex double of $\Sigma$ The complex curve $\Sigma_C$ has the complex anti-involution $i: \Sigma_C \to \Sigma_C$ and therefore defines the real algebraic curve with some additional structures (labelling of irreducible components (ovals) of $\Sigma_C(\mathbb{R})$, a choice of an orientation of $\Sigma_C(\mathbb{R})$ [FOOO09] [Cos07] (cf. Appendix).
4.2. Shrinking morphisms. Denote by $\tilde{N}_{g,n} \subset N_{g,n}'$ the locus of stable bordered surfaces without real nodes of type one and complex nodes. The moduli space $\tilde{N}_{g,n}$ is an orbifold with corners with an interior being $b: N_{g,n} \hookrightarrow \tilde{N}_{g,n}$. By the construction we have an inclusion $a: M_{g,n}/\Sigma_n \hookrightarrow \tilde{N}_{g,n}$, given by removing marked points. We have the following:

**Theorem 4.2.1.** The inclusion above induces the weak-homotopy equivalence of orbifolds:

$$a_n: M_{g,n}/\Sigma_n \xrightarrow{\sim} \tilde{N}_{g,n}.$$  

**Proof.** Let $\Sigma_{g,n}$ be a topological surface of genus $g$ and $n$-boundary components. We have the corresponding augmented Teichmüller space $B\tilde{T}_{g,n}$ (See Appendix Definition 6.2.4). We have a canonical morphism $p: B\tilde{T}_{g,n} \rightarrow N_{L,g,n}$. Denote by $\sigma \in C(D\Sigma_{g,n})$ the $n$-simplex which is given by curves lying on the real part of the surface i.e. stable under the diffeomorphism $\tau$. The Weil-Petersson metric provides us with the nearest point projection morphism (15):

$$\Pi_{B\tilde{T}(\sigma)}: B\tilde{T}_{g,n} \rightarrow B\tilde{T}(\sigma).$$

Denote by $B\tilde{T}_{g,n}$ the locus in $B\tilde{T}_{g,n}$ which consists of surfaces $(X,f)$ such that $\ell_{\alpha}(X) = 0$ iff $\sigma_k$ for any $k \geq 0$, here $\sigma_k$ we denote the $k$-simplex with curves lying in the real part of the surface $D\Sigma_{g,n}$, in particular for $k = 0$ we assume that this set is empty.

**Lemma 4.2.2.** The pullback of the nearest point projection (15) along the inclusion $B\tilde{T}(\sigma) \hookrightarrow B\tilde{T}(\sigma)$ is $B\tilde{T}_{g,n}$:

$$B\tilde{T}_{g,n} \xrightarrow{\Pi_{B\tilde{T}(\sigma)}} B\tilde{T}(\sigma).$$

**Proof.** Building on Theorem 4.18 from [Wo08] and following the scheme of Lemma 4.4 from [Fuj12] (See also [Yam13]) one can easily show that for $x \in T(\sigma_k)^\tau$ and for any natural numbers $l$ and $p$ such that $k < l < p$ then the Weil-Petersson metric satisfies the property:

$$d_{WP}(x,T(\sigma_l)^\tau) < d_{WP}(x,T(\sigma_p)^\tau).$$

Then the desired result follows.

Taking the composition of the inclusion $B\tilde{T}_{g,n} \hookrightarrow B\tilde{T}_{g,n}$ with the pullback morphism of (15) we get the morphism:

$$\Pi_{B\tilde{T}(\sigma)}: B\tilde{T}_{g,n} \rightarrow B\tilde{T}(\sigma),$$

which defines the homotopy equivalence. Denote by $MCG_{g,n,n}$ the subgroup of the mapping class group of the bordered surface which consists of elements $g$ such that $g(\gamma)$ is homotopic to $\gamma$ for any $\gamma \in \sigma$ and $g$ fixes each component of $D\Sigma_{g,n} \setminus \sigma$. We have the following:
Lemma 4.2.3. The canonical inclusion of groups $\text{MCG}_{g,n,\sigma}^B \rightarrow \text{MCG}_{g,n}^B$ is an isomorphism.

Proof. The proof easily follows from the presentation of the mapping class group in terms of Dehn twists.

From this Lemma we have that morphism (16) is a $\text{MCG}_{g,n}^B$-equivariant morphism between contractible spaces, such that the mapping class group action is properly discontinuous. Hence it induces the homotopy equivalence of the corresponding topological stack:

$$\Pi_{\mathcal{M}_{g,n}/\Sigma_n} : \mathcal{N}_{g,n} \cong [B\mathcal{T}_{g,n}/\text{MCG}_{g,n}^B] \xrightarrow{\sim} [B\mathcal{T}(\sigma)/\text{MCG}_{g,n}^B] \cong \mathcal{M}_{g,n}/\Sigma_n$$

Recall that a homotopy type of a quotient stack $[X/G]$ is given by the Borel construction $EG \times_G X$ \cite{Noo05}. Applying this we get the morphism:

$$EM\text{MCG}_{g,n,\sigma}^B \times_{EM\text{MCG}_{g,n}^B} B\mathcal{T}_{g,n} \rightarrow EM\text{MCG}_{g,n}^B \times_{EM\text{MCG}_{g,n}^B} B\mathcal{T}(\sigma)$$

The morphism (18) is a homotopy equivalence since (16) is a homotopy equivalence.

We have the following diagram of topological stacks:

$$[B\mathcal{T}(\sigma)/\text{MCG}_{g,n}^B] \rightarrow [B\tilde{\mathcal{T}}_{g,n}/\text{MCG}_{g,n}^B] \leftarrow [B\mathcal{T}_{g,n}/\text{MCG}_{g,n}^B]$$

Note that the left inverse to the morphism $a := i_\sigma$ is $\tilde{\Pi}_{B\mathcal{T}(\sigma)}$, since the composition $b \circ \tilde{\Pi}_{B\mathcal{T}(\sigma)}$ is a homotopy equivalence by two-out-of-three property of weak homotopy equivalences we get that $a$ is a homotopy equivalence as well.

Definition 4.2.4. For every $g$ and $n$ such that $n > 0$ and $2g + n - 2 > 0$ we define the boundary shrinking morphism (as a morphism in the derived category):

$$\rho_{\mathcal{M}_{g,n}/\Sigma_n} : C_c^\ast(\mathcal{M}_{g,n}/\Sigma_n, \epsilon_n) \xrightarrow{\sim} C_c^{\ast+n}(\tilde{\mathcal{N}}_{g,n}, \mathbb{Q})$$

as the zig-zag quasi-isomorphism:

$$C_c^\ast(\mathcal{M}_{g,n}/\Sigma_n, \epsilon_n) \xrightarrow{a_\ast} C_c^{\ast+n}(\tilde{\mathcal{N}}_{g,n}, \mathbb{Q}) \xleftarrow{b_!} C_c^{\ast+n}(\mathcal{N}_{g,n}, \mathbb{Q})$$

Remark 4.2.5. Let us explain the coefficients in the definition of the boundary shrinking morphism. We consider the sign local system $\epsilon_n$ on Liu moduli space of curves $\tilde{\mathcal{N}}_{g,n}$. By the construction above we have a quasi-isomorphism:

$$C_\ast(\mathcal{M}_{g,n}/\Sigma_n, \epsilon_n) \xrightarrow{a_\ast} C_\ast(\tilde{\mathcal{N}}_{g,n}, \epsilon_n)$$

Applying the Poincaré-Verdier duality to both sides and noticing that $\mathcal{M}_{g,n}/\Sigma_n$ is orientable while the orientation sheaf for Liu’s moduli space coincides with a sign local system we get the desired result.

---

\(^{11}\)Since the $\text{MCG}_{g,n}^B$-group actions are properly discontinuous the corresponding quotient stacks are topological Deligne-Mumford stack \cite{Noo05} Corollary 14.6

\(^{12}\)Recall that the homotopy type of a topological stack $X$ equipped with an atlas $f : U \rightarrow X$ is defined as homotopy type of the total space of the simplicial space: $X^\Delta := \{U \times_X \cdots \times_X U\}$. 
4.3. Costello’s homotopy equivalence. We denote by $D_{g,n,k,p} \subset \overline{N}_{g,n,k,p}$ the locus of the stable bordered Riemann surfaces with irreducible components given by disks and the number of complex marked points at each component of the normalization is required to be less or equal to one. Just like in the case of the space $\overline{N}_{g,n,k,p}$ we will use the special notation $D_{g,n}$ for $D_{g,n,0,0}$. Note that $D_{g,n,k,p}$ is the compact orbispace of the following dimension (See the proof of Proposition 4.3.1):

$$\dim \mathbb{R} D_{g,n,k,p} = 4g - 5 + 2n + k + 2p.$$ 

It is naturally stratified by saying that two nodal surfaces $\Sigma$ and $\Sigma'$ are in the same stratum if there exists an orientation preserving homeomorphism $\Sigma \to \Sigma'$.

**Proposition 4.3.1.** We have the following (well-known) quasi-isomorphism:

$$D : RGCG_{q}((k)_{\geq 3} \xrightarrow{\sim} \prod_{g \geq 0, n \geq 1, 2g + n - 2 > 0} C^{* - 2g - n + 1 + 2d g}(D_{g,n}, \mathbb{Q}),$$

defined by the Poincaré-Verdier duality.

**Proof.** Following the scheme of [Cos07] for a given nodal surface $\Sigma_{g,n} \in D_{g,n}$ we replace each irreducible component of a normalization $\overline{N}_{g,n}$ by a vertex and for each node we define an edge between the vertices (a vertex) which corresponds to the irreducible components (a component) connected by a node. A cyclic order on half-edges attached to a vertex comes from the natural orientation of irreducible components. Hence we obtain a ribbon graph $\Gamma$, with at least trivalent vertices. Stratification of $D_{g,n}$ into orbicells in defined by taking all surfaces $\Sigma_{g,n} \in D_{g,n}$ with the dual ribbon graph being $\Gamma$:

$$X(\Gamma) := \prod_{v \in V(\Gamma)} X_v / \text{Aut}(\Gamma),$$

here $X_v$ is a set of injective functions $H_v : \mathbb{R}^1 \to \mathbb{R}^1$ modulo the action of $PSL_2(\mathbb{R})$ on $S^1$ by Möbius transformations. We compute the dimension of an orbicell:

Note that equivalently we have:

$$X_v = Con_{f^{[H_v(\Gamma)]}}(S^1)/PSL_2(\mathbb{R})$$

Here by $Con_{f^p}(S^1)$ we have denote by configuration space of $p$-ordered points in the circle $S^1$. For $p \geq 3$ it is easy to see that the configuration space $Con_{f^p}(S^1)/PSL_2(\mathbb{R})$ is homeomorphic to the union of copies of $\mathbb{R}^{p-3}$ labelled by number of cyclic orders on the set $\{p\}$. Hence the dimension of the orbicell (20) is $|H(\Gamma)| - 3|V(\Gamma)| = 6g - 6 + 3|B(\Gamma)| - |E(\Gamma)|$ (cf. [Sta63]). From this we get that $\dim \mathbb{R} D_{g,n} = 6g - 6 + 3n - |E(\Gamma_{\min})|$ where $\Gamma_{\min}$ is a ribbon graph of genus $g$ with $n$ boundaries with a minimal number of edges. Hence we get $\dim \mathbb{R} D_{g,n} = 4g - 5 + 2n$.

The chains of the complex $C_{*}(D_{g,n}, \epsilon_{n})$ correspond to the ribbon graphs $\Gamma$ which are placed in degree $6g - 6 + 3|B(\Gamma)| - |E(\Gamma)|$. The orientation of the orbicell (20) in $C_{*}(D_{g,n}, \epsilon)$ can be computed as follows. We have a canonical $\text{Aut}(\Gamma)$-equivariant isomorphism:

$$\otimes_{v \in V(\Gamma)} H_0(X_v, \mathbb{Q}) \cong \text{det}(V(\Gamma)) \otimes \text{det}(H(\Gamma)).$$

Hence the orientation of the orbicell $X(\Gamma)$ is give by:

$$\det(V(\Gamma)) \otimes \det(H(\Gamma)) \otimes \det(B(\Gamma)).$$

Further following [CFL16] (See also [MW15]) we explicitly compute:
Recall that by $H_e(\Gamma)$ we have denoted a set of two half edges contained in the edge $e \in E(\Gamma)$. We can identify canonically $\det(H(\Gamma)) \cong \otimes_{e \in E(\Gamma)} \det H_e(\Gamma)$. Consider the bordered Riemann surface $\Sigma_\Gamma$ associated with a graph $\Gamma$. By gluing disks to the boundary components of $\Sigma_\Gamma$ we obtain the closed Riemann surface denoted by $\hat{\Sigma}_\Gamma$. Using the cellular decomposition which comes from a ribbon graph $\Gamma$ we have the following complex which computes singular homology of $\hat{\Sigma}_\Gamma$ with $\mathbb{Q}$-coefficients:

\begin{equation}
C_2(\hat{\Sigma}_\Gamma, \mathbb{Q}) \xrightarrow{d_2} C_1(\hat{\Sigma}_\Gamma, \mathbb{Q}) \xrightarrow{d_1} C_0(\hat{\Sigma}_\Gamma, \mathbb{Q})
\end{equation}

Here the space of 2-chains can be identified with space of boundaries $C_2(\hat{\Sigma}_\Gamma, \mathbb{Q}) = \mathbb{Q}[B(\Gamma)]$, space of 1-chains with set of half-edges $C_1(\hat{\Sigma}_\Gamma, \mathbb{Q}) = \oplus_{e \in E(\Gamma)} \mathbb{Q}[H_e(\Gamma)]$ and 0-chains with vertices $C_0(\hat{\Sigma}_\Gamma, \mathbb{Q}) = \mathbb{Q}[V(\Gamma)]$. We have the following exact sequences:

$$0 \to H_2(\hat{\Sigma}_\Gamma, \mathbb{Q}) \to C_2(\hat{\Sigma}_\Gamma, \mathbb{Q}) \to \text{Im}(d_2) \to 0,$$

$$0 \to \text{Im}(d_2) \to \text{Ker}(d_1) \to H_1(\hat{\Sigma}_\Gamma, \mathbb{Q}) \to 0,$$

$$0 \to \text{Ker}(d_1) \to C_1(\hat{\Sigma}_\Gamma, \mathbb{Q}) \to C_0(\hat{\Sigma}_\Gamma, \mathbb{Q}) \to H_0(\hat{\Sigma}_\Gamma, \mathbb{Q}) \to 0,$$

Further using Lemma 1 from [CV03] and identifications above we have the following isomorphisms:

$$\det(B(\Gamma)) \cong \det(H_2(\hat{\Sigma}_\Gamma, \mathbb{Q})) \otimes \det(\text{Im}(d_2))$$

$$\det(\text{Ker}(d_1)) \cong \det(\text{Im}(d_2)) \otimes \det(H_1(\hat{\Sigma}_\Gamma, \mathbb{Q}))$$

$$\det(\text{Ker}(d_1)) \otimes \det(V(\Gamma)) \cong \det(E(\Gamma)) \otimes \bigotimes_e \det(H_e(\Gamma)) \otimes \det(H_0(\hat{\Sigma}_\Gamma, \mathbb{Q}))$$

Note that since spaces $H_0(\hat{\Sigma}_\Gamma, \mathbb{Q})$ and $H_2(\hat{\Sigma}_\Gamma, \mathbb{Q})$ are one dimensional and have a canonical basis we can omit them from expressions above. Further vector space $H_1(\hat{\Sigma}_\Gamma, \mathbb{Q})$ can be equipped with a basis of cycles $\{\gamma_i, \delta_i\}_{i \in K}$ where $|K| = g$, hence can be also omitted. Hence we have:

\begin{equation}
\det(E(\Gamma)) \cong \det(V(\Gamma)) \otimes \det(B(\Gamma)) \otimes \bigotimes_e \det(H_e(\Gamma))
\end{equation}

Comparing the right-hand side of (23) with (21) we obtain the desired identification of orientation spaces.

Finally it is easy to see that the differential $C_*(D_{g,n}, \epsilon_n)$ is coincides with the differential $\delta$ in the Kontsevich-Penner ribbon graph complex. Hence we get:

$$C_{6g-6+3n-\bullet}(D_{g,n}, \epsilon_n) \cong C_*(M_{g,n}/\Sigma_n, \mathbb{Q})$$

Applying the Poincaré-Verdier duality we get:

$$C_*(M_{g,n}/\Sigma_n, \mathbb{Q}) = C_{6g-6+3n-\bullet}(D_{g,n}, \epsilon_n) \xrightarrow{\mathcal{D}} C^{\text{dim}D_{g,n}}_{-6g+6-3n+\bullet}(D_{g,n}, \mathbb{Q})$$

$$= C_{-2g-n+1}(D_{g,n}, \mathbb{Q}).$$

\[\square\]
Remark 4.3.2. Following [Cos07] Proposition 4.3.1 can be extended to the case of moduli spaces of nodal surfaces with markings in the interior and markings on the boundary. Namely in these cases we replace the notion of a ribbon graph with a notion of a ribbon graph with black and white vertices (resp. ribbon graphs with hairs), where white vertices (resp. hairs) corresponds to a components which have a complex marking (resp. marking on a boundary).

The main result (Theorem 2.2.2) of [Cos07] gives us a weak equivalence of orbispaces:

\[
\nu_{g,n,k,p} : \mathcal{N}_{g,n,k,p} \rightarrow \mathcal{N}_{g,n,k,p} \quad \text{and} \quad \eta_{g,n,k,p} : D_{g,n,k,p} \rightarrow u_{g,n,k,p},
\]

Further we will use a notation \(\nu\) (resp. \(u\)) for a morphism \(\nu_{g,n,k,p}\) (resp. \(u_{g,n,k,p}\)) if it will not lead to confusion. We have the following (well known):

Proposition 4.3.3. The Costello weak equivalence (24) together with the quasi-isomorphism (19) define the zig-zag quasi-isomorphism:

\[
\text{cost} : \prod_{g \geq 0, n \geq 1, 2g + n - 2 > 0} (C^*_c(M_{g,n}, \mathbb{Q}) \otimes \Sigma_n \text{sgn}_n)^{\Sigma_n} \xrightarrow{\rho_{M_{g,n}/\Sigma_n}} C^*_c(\mathcal{N}_{g,n}, \mathbb{Q}) \xrightarrow{\psi_1} C^*_c(\mathcal{N}_{g,n}, \mathbb{Q}) \xrightarrow{\nu_1} C^*_c(\mathcal{N}_{g,n}, \mathbb{Q}) \xrightarrow{D} C^*_c(M^\text{rib}_{g,n}/\Sigma_n, \mathbb{Q}).
\]

\[\square\]

5. Proof of T. Willwacher’s conjecture

5.1. The differential \(\Delta^1_{\text{tor}}\). By \(T_{g,n}\) we denote the locus in \(\mathcal{N}_{g,n}\) which consists of nodal surfaces such that all irreducible components are disks except the unique component which is an annulus (we also assume that all nodes are connected to one boundary of the annulus). By \(K^L_{g,n}\) we denote the closure of the union of \(D_{g,n}\) with \(T_{g,n}\) in Liu’s moduli space:

\[K^L_{g,n} := T_{g,n} \cup D_{g,n} \subset \mathcal{N}^L_{g,n}\]

Note that by the construction the locus \(K^L_{g,n}\) is a compact orbispace. We have the canonical inclusions:

\[m : D_{g,n,0,1} \hookrightarrow K^L_{g,n+1} \hookrightarrow D_{g,n+1} : n\]

Here the morphism \(m\) is given by removing a marked point in the interior. We have the following:

Lemma 5.1.1. The push-forward along the morphism \(n\) induces the quasi-isomorphism:

\[n_* : C_*(D_{g,n+1}, \epsilon_{n+1}) \rightarrow C_*(K^L_{g,n+1}, \epsilon_{n+1})\]

\[\text{13}\]The fact that \(v_{g,n,k,p}\) is an equivalence is obvious, since \(\mathcal{N}_{g,n,k,p}\) is an interior of \(\mathcal{N}_{g,n,k,p}\).
Proof. Consider the Gysin triangle:
\[ C_\bullet(D_{g,n+1}, \epsilon_{n+1}) \longrightarrow C_\bullet(K_{g,n+1}^L, \epsilon_{n+1}) \longrightarrow C_{BM}^\bullet(K_{g,n+1}^L \setminus D_{g,n+1}, \epsilon_{n+1}) \longrightarrow \]
in Borel-Moore chains associated with the diagram:
\[ n : D_{g,n+1} \hookrightarrow K_{g,n+1}^L \hookrightarrow K_{g,n+1}^L \setminus D_{g,n+1} \]
In order to prove the morphism \( n_* \) is acyclic it is enough to show that the complex:
\[ C_{BM}^\bullet(K_{g,n+1}^L \setminus D_{g,n+1}, \epsilon_{n+1}) \]
is acyclic. We have a sequence of inclusions:
\[ T_{g,n} \hookrightarrow K_{g,n+1}^L \setminus D_{g,n+1} \hookrightarrow D_{g,n,0,1} \]
Here the inclusion \( T_{g,n} \hookrightarrow K_{g,n+1}^L \) is open and \( D_{g,n,0,1} \) is the corresponding complement. Hence the complex \( C_{BM}^\bullet(K_{g,n+1}^L \setminus D_{g,n+1}, \epsilon_{n+1}) \) is quasi-isomorphic to the mapping cone of the following morphism:
\[ 0'' : C_{BM}^\bullet(T_{g,n+1}, \epsilon_{n+1}) \longrightarrow C_{-1}(D_{g,n,0,1}, \epsilon_n) \]
Analogous to Proposition 4.3.1 we can realise complex \( C_\bullet(D_{g,n,0,1}, \epsilon_n) \) as complex of pairs \((\Gamma, or)\). Here \( \Gamma \) is a ribbon graph with a unique white vertex \cite{Cos07}. The orientation \( or \) is given by:
\[ or = \det(V_\bullet(\Gamma)) \otimes \det(H(\Gamma)) \otimes \det(B(\Gamma)) \]
Here \( V_\bullet(\Gamma) \) is a set of black vertices. The differential is defined by splitting a vertex. Analogous to \( D_{g,n,0,1} \) we can stratify \( T_{g,n+1} \) by associating with a surfaces \( \Sigma_{g,n+1} \in T_{g,n+1} \) the corresponding dual graph \( \Gamma \) (as an element in the moduli space of real curves). The unique vertex \( v \) of genus \( \frac{13}{2} \) will be called the special vertex and denoted by \( v_{03} \). Hence the corresponding stratum is defined by the rule:
\[ S(\Gamma) := \left( \prod_{v \in V_w(\Gamma)} X_v \times N_{0,2,\pi} \right) / \text{Aut}(\Gamma) \]
Here \( \pi : H_{v_{03}} \to [2] \) is a non-surjective morphism and \( V_w(\Gamma) \) is a set of vertices with zero weights attached. Hence we can compute the dimension of a stratum \( S(\Gamma) \):
\[ | v_{03} \in V_w(\Gamma) | H_v | - 3|V_{w(0)}| + \dim H_{N_{0,2,\pi}} = |H(\Gamma)| - 3|V_{w(0)}| \]
Note that according to [DHV11] the moduli space \( N_{0,2,\pi} \) is an interior of convex polytope and it is homeomorphic to the union of \( \mathbb{R}^{H_{v_{03}}} \) over all cyclic orders on \( H_{v_{03}} \). Hence we compute:
\[ \otimes_{v \in H_0(V_\cdot(\Gamma))} H_0(\mathbb{R}^{H_{v_{03}}}) \cong \det(V_{w(0)}(\Gamma)) \otimes \det(H(\Gamma)) \]
Thus we may identify chains in \( C_{BM}^\bullet(T_{g,n+1}, \epsilon_{n+1}) \) with pairs \((\Gamma, \tilde{\omega})\) consisting of a ribbon graph \( \Gamma \) with a unique white vertex and \( \tilde{\omega} \in \det(V_\bullet(\Gamma)) \otimes \det(H(\Gamma)) \otimes \det(B(\Gamma) \sqcup V_\bullet(\Gamma)) \). Let \( \text{Rib}_{g,n,1}^k \) be a set of ribbon graphs of genus \( g \) with \( n \) boundary components and with a unique white vertex. We can explicitly realize \( C_{BM}^\bullet(K_{g,n+1}^L \setminus D_{g,n+1}, \epsilon_{n+1}) \) as the total DG-vector space of the following complex:

---

\(^{14}\) The unique genus one vertex comes from considering an annulus as a real curve.
Here morphism $\delta'''$ acts on the orientation space by the following rule: on a component corresponding to the ordering of vertices and half-edges it is identical and on $\det(B(\Gamma))$ it acts by the rule:

$$\delta''' : b' \land \cdots \land b'' \land \cdots \land b''' \mapsto (-1)^j b' \land \cdots \land b''' .$$

Here $b''$ is a unique boundary without attached nodes which is placed on $j$-slot in the sequence above. Since in each homological degree a morphism $\delta'''$ is an isomorphism the desired result follows.

\[ \square \]

Remark 5.1.2. Alternatively following the scheme of the proof of Lemma 3.0.7 from [Cos07] one can prove Lemma 5.1.1 by induction. In order to do that one has to consider the version $K^L_{g,n,l}$ of the orbispace $K^L_{g,n}$ where we allow marked points on the boundary (if a component has genus one we allow marked points only on a boundary with nodes attached). Hence we consider the morphism:

$$D_{g,n,l} \hookrightarrow K^L_{g,n,l}$$

Further we apply the construction from op. cit and get a morphism between simplicial orbispaces, with simplicies give by products of the same orbispaces of lower dimension. Passing to the total space we get the induction step.

Definition 5.1.3. We define the geometric Bridgeland differential (as the morphism in the derived category of vector spaces):

$$\Delta^\text{bor}_1 : C^*(D_{g,n}, \mathbb{Q}) \longrightarrow C^*(D_{g,n+1}, \mathbb{Q})$$

By the rule:

$$C^*(D_{g,n,0,1}, \mathbb{Q}) \xrightarrow{m!} C^*(D_{g,n+1,1}, \mathbb{Q}) \xrightarrow{\pi^*_{D_{g,n,0,1}}} C^*(D_{g,n}, \mathbb{Q}) \xrightarrow{\Delta^\text{bor}_1} C^*(D_{g,n+1,1}, \mathbb{Q})$$
Theorem 5.1.4. For \( g \geq 0, n \geq 1 \) such that \( 2g + n - 2 > 0 \), the following diagram commutes (in the derived category of vector spaces):

\[
\begin{array}{ccc}
\prod_{g,n} C^* + 2g(d-1) - n + 1(D_{g,n}, \mathbb{Q}) & \xrightarrow{\Delta_1} & \prod_{g,n} C^* + 2g(d-1) - n + 1(D_{g,n+1}, \mathbb{Q}) \\
D \sim & & D \sim \\
\Delta_1 & & \Delta_1 \\
\text{RGC}_d(\delta)_{\geq 3} & & \text{RGC}_{d+1}(\delta)_{\geq 3}
\end{array}
\]

Proof. Consider the moduli space \( D_{g,n,2,0} \) of stable bordered surfaces with irreducible components being disks and with two marked points attached to a boundary. Inside the moduli space \( D_{g,n,2,0} \) we consider the locus \( D^0_{g,n,2} \) which consists of nodal surfaces \( \Sigma \) with marked points lying on the same boundary, such that marked points belong to distinct corners \( c \) and \( c' \) only if corners \( c \) and \( c' \) lie on distinct irreducible components \( \Sigma_1 \) and \( \Sigma_2 \) of a surface \( \Sigma \) which share a common node. These surfaces (as well as the corresponding marked ribbon graphs) will be called special surfaces (resp. special ribbon graphs). We have the corresponding forgetful morphism:

\[
\pi^0_{D^0_{g,n,2}} : D^0_{g,n,2} \rightarrow D_{g,n}
\]

We have the clutching morphism for bordered surfaces defined by gluing two marked points:

\[
\xi^0_{D^0_{g,n,2}} : D^0_{g,n,2} \rightarrow D_{g,n+1}
\]

The clutching morphism \( \xi^0_{D^0_{g,n,2}} \) is an oriented i.e. \( \pi^0_{D^0_{g,n,2}} \xi^0_{D^0_{g,n,2}} \cong \xi^0_{D^0_{g,n,2}} \epsilon_{n+1} \) and proper morphisms of orbi-spaces. Hence we define the following pull-push morphism:

\[
\Theta_0 : C_*(D_{g,n}, \epsilon_n) \xrightarrow{\pi^0_{D^0_{g,n,2}} \epsilon_n} C_{BM}^{g+2}(D^0_{g,n,2}, \epsilon_{n+1} \xi^0_{D^0_{g,n,2}} \epsilon_{n+1}) \xrightarrow{\xi^0_{D^0_{g,n,2}} \epsilon_{n+1}^*} C_*(D_{g,n+1}, \epsilon_{n+1}),
\]

Lemma 5.1.5. The morphism \( \Theta_0 \) admits the following explicit description (up to signs):

\[
\Theta_0(\Gamma) = \sum_{b \in B(\Gamma)} \theta_b(\Gamma)
\]

Here \( \theta_b(\Gamma) \) is zero cohomology of the following complex:

\[
\sum_{c \in C(b)} e_{c,c}(\Gamma) \rightarrow \sum_{c \in C(b)} e'_c(\Gamma) \oplus \sum_{c \in C(b)} e''_c(\Gamma)
\]

Here we have:

\[15\text{By a corner we understand an open interval on the boundary bounded by nodes of the surface.}
\[16\text{Note that in order to be well defined it is necessary to consider marked points lying on the same boundary, otherwise the resulting surface will have genus } g + 1.\]
(i) The operation $e'_c(\Gamma)$ is defined by attaching a ribbon graph:

\[
\begin{array}{c}
\circ \\
\end{array}
\]

to a ribbon graph $\Gamma$ at a corner $c$.

(ii) The operation $e''_a(\Gamma)$ replaces an edge $a$ with the following ribbon graph:

\[
\begin{array}{c}
\circ \circ \\
\end{array}
\]

in all possible ways.

Proof. Let $(\Gamma, or)$ be a cochain in $C_*(D_{g,n}, \epsilon_n)$ i.e. a ribbon graph with an element $or$ in $\text{det}(E(\Gamma))$. We suppose that the number of edges equals to $l$. The corresponding stratum in $D_{g,n}$ will be denoted by $X(\Gamma)$. By Proposition 2.3.3 we have the following description of the Gysin pullback:

\[
\pi^!_{D_{g,n}, 2}(\Gamma, or) = H^{BM \dim \pi_{D_{g,n}, 2}^{-1}(X(\Gamma))}(\pi^{-1}_{D_{g,n}, 2}(X(\Gamma)), \pi^!_{D_{g,n}, 2} \epsilon_n)
\]

The homology above are equal to the zero cohomology of the following complex:

\[
V_0 \rightarrow V_1.
\]

Here $V_0$ consists of all possible pairs $(\Gamma_h, or)$ where $\Gamma_h$ is a marked ribbon graphs with two markings lying on the same corner or a special ribbon graph (number of edges of $\Gamma_h$ is equal to $l$). The vector space $V_1$ consists of the following marked ribbon graphs with $l + 1$-edges:

(a) Marked ribbon graphs with a unique one-valent vertex and two markings at this vertex (cf. (i)).

(b) Marked ribbon graphs with a unique two-valent vertex with two markings attached at the same corner (cf. (ii)).

(c) Marked ribbon graphs with a unique marking on a two-valent vertex and another marking on a vertex which is connected to the latter one by the unique edge (special ribbon graphs).

We will show that all entries with markings on distinct corners vanish (under the clutching morphism):

Every special ribbon graph $\Gamma_h$ admits a partner, namely suppose that $\Gamma_h$ has marked points lying on a corner $c_v$ associated with a vertex $v$ and on a corner $c_w$ associated with a vertex $w$ the edge which connects vertices $v$ and $w$ will be denoted by $e$. Then we also have a ribbon graph $\Gamma'_h$ when we remove edge $e$ and mark the corresponding half-edges and put an edge $e'$ with half-edges lying on corners $c_v$ and $c_w$. Then a sum:

\[
\xi_{D_{g,n}, 2}(\Gamma_h, or) \oplus (\Gamma'_h, or)
\]

admits an odd automorphism (the one which interchanges parallel edges) and hence vanishes.

\[\square\]

Proposition 5.1.6. The diagram below is homotopy commutative:
\[
\begin{align*}
C_{q+2}(D_{g,n,0,1}, \epsilon_n) & \xrightarrow{m_*} C_{q+2}(K_{g,n+1}^L, \epsilon_{n+1}) \\
\pi_{D_{g,n,0,1}}^j & \downarrow \quad \Theta_0 \quad \uparrow n_* \sim \\
C_* (D_{g,n}, \epsilon_n) & \xrightarrow{} C_{q+2}(D_{g,n+1}, \epsilon_{n+1})
\end{align*}
\]

**Proof.** We shall prove this Lemma by constructing an explicit homotopy

\[ h_k : C_k(D_{g,n}, \epsilon_n) \rightarrow C_{k-1}(K_{g,n+1}^L, \epsilon_{n+1}) \]

between morphisms \( n_* \circ \Theta_0 \) and \( m_* \circ \pi_{D_{g,n,0,1}}^j \):

Let \( (\Gamma, or) \) be a chain in \( C_* (D_{g,n,0,1}, \epsilon_n) \) of degree \( k \). We define the homotopy \( h := \{ h_k \} \) by the rule:

\[ h_k : (\Gamma, or) \rightarrow \rho(\pi_{D_{g,n,0,1}}^j(\Gamma, or)) \]

Here \( \rho \) is a morphism inverse to the morphism \( \delta''' \) from Lemma 5.1.1

\[ \rho : C_k(D_{g,n,0,1}, \epsilon_n) \rightarrow C_{k+1}^{BM}(T_{g,n+1}, \epsilon_{n+1}) \]

More precisely the morphism \( \rho \) is defined by the following rule: on a pair \( (\Gamma, or) \) this morphism acts identically on the first component and on the orientation it acts by the rule:

\[ \rho: b' \wedge \cdots \wedge b'' \mapsto \frac{1}{n+1} \sum_{j} (-1)^j b' \wedge \cdots \wedge v_{\emptyset} \wedge \cdots \wedge b'' \]

Consider a differential \( D \) in the complex \( C_* (K_{g,n}^L, \epsilon_n) \). This differential has the following decomposition:

\[ D = \delta + \delta' + \delta'' + \delta''' + \delta'''. \]

Here we have:

(a) \( \delta \) is the standard differential in the Kontsevich-Penner ribbon graph complex,

(b) \( \delta' \) is the differential which is defined by splitting a vertex on nodal surfaces which have genus one component,

(c) \( \delta'' \) is defined by splitting a vertex on surfaces with a marking on an irreducible component,

(d) \( \delta''' \) is defined by contracting a loop over a "non-singular" boundary of a nodal surface with a genus one component,

(e) \( \delta'''' \) is defined by contracting an arc with endpoints on distinct boundary components of a genus one irreducible component.

Hence we compute:

\[
(Dh_k + h_{k+1}\delta)(\Gamma) =
\]

\[
\delta''' \rho \pi_{D_{g,n,0,1}}^j(\Gamma) + \delta''' \rho \pi_{D_{g,n,0,1}}^j(\Gamma) + \\
\delta' \rho \pi_{D_{g,n,0,1}}^j(\Gamma) + \rho \pi_{D_{g,n,0,1}}^j \delta(\Gamma) \\
\pi_{D_{g,n,0,1}}^j(\Gamma) + \delta''' \rho \pi_{D_{g,n,0,1}}^j(\Gamma).
\]
Finally note that both morphisms $n_*$ and $m_*$ are inclusion on chains and by Lemma 5.1.5 the morphism $\delta'''(\rho(\pi^1_{D_{g,n,0,1}}))$ coincides with the morphism $\Theta_0$ (up to a sign). The last claim easily follows from the description of the Gysin pullback $\pi^1_{D_{g,n,0,1}}$.

By Proposition 2.3.5 for a chain $(\Gamma, \sigma)$ the Gysin pullback is equivalent (up to signs) to the zero cohomology of the complex:

$$W_0 \rightarrow W_1$$

Here $W_0$ consists of all possible attachment of a white marking to a ribbon graph $\Gamma$.

The vector space $W_1$ consists of possible attachments of the marked ribbon graph:

$$\circ \bullet$$

to a ribbon graph $\Gamma$ and replacement of any edge of a ribbon graph $\Gamma$ by the following marked ribbon graph:

$$\bullet \circ \bullet$$

Hence applying the differential $\delta'''$ to the image of the element above under the morphism $\rho$ we create loops in all possible ways (by connecting an identical corners) on a white vertex. □

Consider the locus $D^1_{g,n,2}$ in $D_{g,n,2}$ which consists of surfaces $\Sigma$ with marked points lying on distinct corners attached to the same boundary. This moduli space is equipped with a morphism:

$$\pi_{D^1_{g,n,2}} : D^1_{g,n,2} \rightarrow D_{g,n}$$

which forget two marked points and stabilises the resulting nodal surface. By $\xi_{D^1_{g,n,2}}$ we denote the corresponding clutching morphisms defined by gluing two marked points (cf. (28)):

$$\xi_{D^1_{g,n,2}} : D^1_{g,n,2} \rightarrow D_{g,n+1}$$

Note that the clutching morphism (30) is an oriented and proper morphism of orbispaces. We have the following:

**Lemma 5.1.7.** The following morphisms:

$$\Theta? : C_*(D_{g,n}, \epsilon_n) \rightarrow C_{*+2}(D_{g,n+1}, \epsilon_{n+1}), \quad ? = 0, 1.$$  

$$\Theta? := \xi_{D^1_{g,n,2}} \pi^1_{D^1_{g,n,2}}, \quad ? = 0, 1$$

coincide in the derived category of vector spaces.

**Proof.** Consider the following locus $\overline{N^2}_{g,n,2}$ in $\overline{N}_{g,n,2}$ which consists of all stable bordered surfaces with marking lying on the same boundary. We have the corresponding forgetful morphism $\pi_{\overline{N}^2_{g,n,2}} : \overline{N}^2_{g,n,2} \rightarrow \overline{N}_{g,n}$. We also have the corresponding clutching morphism $\xi_{\overline{N}^2_{g,n,2}} : \overline{N}^2_{g,n,2} \rightarrow \overline{N}_{g,n+1}$ which is oriented and proper. The locus of nodal surfaces with irreducible components being disks will be denoted by $D^2_{g,n,2}$. The corresponding forgetful morphism will be denoted by $\pi_{D^2_{g,n,2}} : D^2_{g,n,2} \rightarrow D_{g,n}$ and by $\xi_{D^2_{g,n,2}} : D^2_{g,n,2} \rightarrow D_{g,n+1}$ we will denote the corresponding clutching morphism. First note that we have the following equivalence:

$$\xi_{D^2_{g,n,2}} \pi^1_{D^2_{g,n,2}} = \Theta_0 - \Theta_1$$
This follows directly by applying the Mayer-Vietoris sequence and using the fact that the Gysin pullback of \( \pi_{D_{g,n+2}}^* \) to \( D_{g,n+2}^1 \cap D_{g,n+2}^0 \) vanishes. Next we will show that morphism \( \xi_{D_{g,n+2},2}^* \pi_{D_{g,n+2}}^1 \) vanishes in the derived category. Consider the following diagram:

\[
\begin{array}{ccc}
N^2_{g,n+2} & \xrightarrow{j} & \overline{N}_{g,n+1} \\
N^2_{g,n+2} & \xrightarrow{\sim} & \overline{N}_{g,n+2} \\
N^2_{g,n+2} & \xrightarrow{\sim} & \overline{N}_{g,n+2} \\
N_{g,n} & \xrightarrow{\sim} & \overline{N}_{g,n} \\
\end{array}
\]

By \( N^2_{g,n+2} \) we have denoted the fibered product:

\[
N^2_{g,n+1} := N_{g,n} \times_{\overline{N}_{g,n}} N^2_{g,n+2}
\]

Here \( N^2_{g,n+1} \) is the locus in \( \overline{N}_{g,n+1} \) which consists of stable bordered surfaces of the following type:

1. \( \Sigma \) is a smooth bordered surface of genus \( g \) with \( n+1 \) boundary components,
2. \( \Sigma \) is a nodal surface such that the normalisation \( \overline{\Sigma} \) is a smooth surface of genus \( g \) with \( n \) boundary components and two marked points.
3. \( \Sigma \) is a nodal surface such that the normalisation \( \overline{\Sigma} = \Sigma' \cup \Sigma'' \) here \( \Sigma \) is a smooth surface of genus \( g \) with \( n \) boundary components and one marked point and \( \Sigma'' \) is a disk with three marked points.

Note that \( N^2_{g,n+1} \) is an orbifold with corners such that the interior is \( N_{g,n+1} \). From this fact we get that the open morphism \( j: N^2_{g,n+1} \to \overline{N}_{g,n+1} \) is a homotopy equivalence by the two-out-of-three property of homotopy equivalences. By \( \xi_{N^2_{g,n+2}}: N^2_{g,n+2} \to N^2_{g,n+1} \) we have denoted the corresponding (proper) clutching morphism. Applying the base change theorem we get the following equality:

\[
\xi_{N^2_{g,n+2}}^* \pi_{N^2_{g,n+2}}^1 = j^! u^! \xi_{D_{g,n+2}}^* \pi_{D_{g,n+2}}^1 u^! v^! \]

We claim that the morphism \( \xi_{N^2_{g,n+2}}^* \pi_{N^2_{g,n+2}}^1 \) vanishes in the derived category. Consider the following morphism of sheaves:

\[
R(\xi_{N^2_{g,n+2}}^* \pi_{N^2_{g,n+2}}^1) : C^BM_{\bullet}(N^2_{g,n+2}, \epsilon_n) \to C^BM_{\bullet}(N_{g,n+1}, \epsilon_n)
\]

Here \( i: N_{g,n+1} \to N^2_{g,n+1} \) is the open inclusion. If one applies the global section functor to the morphism above one gets the morphism:

\[
i^! \xi_{N^2_{g,n+2}}^* : C_{BM}(N_{g,n+2}, \epsilon_n) \to C_{BM}(N_{g,n+1}, \epsilon_n)
\]
It is easy to see that the morphism (31) vanishes. Consider the following fibered product:

\[
\begin{array}{ccc}
N_{g,n+1} & \xrightarrow{i} & N^2_{g,n+1} \\
\emptyset & \xrightarrow{} & \xi N_{g,n+2}^2 \\
\end{array}
\]

By the base change Theorem for sheaves we have

\[
i^* R(\xi N_{g,n+2}^2) = 0
\]

in particular we get

\[
R(\xi N_{g,n+2}^2) \pi_{D g,n,2}^l
\]

Also one can consider the restriction to the closed complement \( f : N_{g,n+1} \setminus N_{g,n+1} \to N_{g,n+1}^2 \). In this case a sheaf \( f! \mathcal{R}(i^*) \) will be zero as well. Hence by gluing construction for sheaves the morphism (31) will be zero and hence the morphism (32) vanishes. Finally since the morphism \( i^! \) is a quasi-isomorphism we get that \( \xi N_{g,n+2}^2 \pi_{D g,n,2}^l \) vanishes. □

Finally the desired result will follow from the following:

Lemma 5.1.8. For \( g \geq 0, n \geq 1 \) such that \( 2g + n - 2 > 0 \), the following diagram commutes:

\[
\prod_{g,n} C_{\dim N_{g,n},-2g-\epsilon_n} \xrightarrow{\Theta_1} \prod_{g,n} C_{\dim N_{g,n+2,-2g-\epsilon_{n+1}}}
\]

\[
\sim \xrightarrow{\Delta_1} \sim
\]

\[
\text{RGC}^l_1(\delta)_{\geq 3} \xrightarrow{\pi_D^1} \text{RGC}^{l+1}(\delta)_{\geq 3}
\]

Proof. In order to prove the result we have to calculate the Gysin pullback along the morphism \( \pi_{D g,n,2}^l \). Let \( \Gamma \) be a ribbon graph with \( l \)-edges. By Proposition 2.3.5 we have:

\[
\pi_{D g,n,2}^l (\Gamma, \eta) = H_{\pi_{D g,n,2}^l}^{BM} (X(\Gamma), (\pi_{D g,n,2}^{-1}(X(\Gamma)), \pi_{D g,n,2}^l \epsilon_n)
\]

The vector space is equivalent to the zero cohomology of the following complex:

\[
Q_0 \longrightarrow Q_1
\]

Here \( Q_0 \) consists of marked ribbon graphs \( \Gamma_h \) (with \( l \)-edges) which are define by all possible attachments of markings to a ribbon graph \( \Gamma \) to distinct corners (at the same boundary). The vector space \( Q_1 \) consists of the following ribbon graphs (with \( l + 1 \)-edges):

(a) Marked ribbon graph with a unique marking on a two-valent vertex and another marking on a vertex which is connected to the latter one by the unique edge (very special ribbon graphs).
(b) Marked ribbon graph which has a unique two-valent vertex with a unique marking attached and another marking lying on a distinct vertex, not connected by an edge to the first vertex.
Every very special stratum is special. Hence using the same argument as in proof of Lemma 5.1.5, we can show that the image of every very special ribbon graph $\Gamma_h$ (under the clutching morphism) admits a partner and hence the image of all very special entries in $Q_1$ vanishes. Next we can show that every marked ribbon graph $\Gamma_h$ of type two from the list above admits an odd automorphism (given by interchanging two edges attached to a unique two-valent vertex and mapping a marked point to the other corner of this vertex) and hence vanishes. Thus we get the desired result.

Now combining all equivalences above and applying the Poincaré-Verdier duality we get the desired result.

5.2. The differential $\nabla_{\text{bor}}^1$. Consider the following fibered product:

$$\begin{array}{c}
\mathcal{N}_{g,n} \times \mathcal{N}_{g,n} \xrightarrow{\mathcal{v}} \mathcal{N}_{g,n,0,1} \\
\downarrow \pi_{\mathcal{N}_{g,n,0,1}} \\
\mathcal{N}_{g,n} \\
\downarrow \mathcal{v} \\
\mathcal{N}_{g,n,0,1} \xrightarrow{\mathcal{q}_{\pi}} \mathcal{N}_{g,n,0,1}
\end{array}$$

(34)

Note that by the construction the morphism

$$\pi_{\mathcal{N}_{g,n,0,1}} : \mathcal{N}_{g,n} \times_{\mathcal{N}_{g,n}} \mathcal{N}_{g,n,0,1} \rightarrow \mathcal{N}_{g,n},$$

is a proper morphism which forgets the unique marking in the interior of a surface. Denote by $\mathcal{N}_{g,n}^{\text{fL}}$ the open sub-orbifold in Lui’s moduli space which consists of stable bordered surfaces without complex nodes. We have sequence of inclusions:

$$\mathcal{N}_{g,n} \xrightarrow{u} \mathcal{N}_{g,n} \xrightarrow{q} \mathcal{N}_{g,n}^{\text{fL}}.$$

Note that since $\mathcal{N}_{g,n}$ is an interior in $\mathcal{N}_{g,n}^{\text{fL}}$, the morphism $q \circ u$ is a weak homotopy equivalence. Hence by two-out-of-three property of weak equivalences the morphism $q$ is a homotopy equivalence as well. We also have a morphism:

$$\tilde{m} : \mathcal{N}_{g,n,0,1} \rightarrow \mathcal{N}_{g,n+1}^{\text{fL}}.$$

Here $\tilde{m}$ is a morphism which is defined by replacing the unique marked point in the interior of the surface by the puncture. We have the following:

**Definition 5.2.1.** By $\nabla_{\text{bor}}^1$ we have denoted the morphism:

$$\nabla_{\text{bor}}^1 : C_c^\ast(\mathcal{N}_{g,n}, Q) \rightarrow C_c^\ast+1(\mathcal{N}_{g,n+1}, Q)$$

defined by the rule:

$$\begin{array}{c}
C_c^\ast(\mathcal{N}_{g,n,0,1} \times \mathcal{N}_{g,n}, \mathcal{N}_{g,n}, Q) \xrightarrow{\tilde{m}_\ast \circ \mathcal{v}_\ast} C_c^\ast+1(\mathcal{N}_{g,n+1}^{\text{fL}}, Q) \\
\downarrow \pi_{\mathcal{N}_{g,n,0,1}}^\ast \\
C_c^\ast(\mathcal{N}_{g,n}, Q) \xrightarrow{\nabla_{\text{bor}}^1} C_c^\ast+1(\mathcal{N}_{g,n+1}, Q)
\end{array}$$
Proposition 5.2.2. For every $g \geq 0$ and $n \geq 1$ such that $2g + n - 2 > 0$ the following square commutes (in the derived category of vector spaces):

\[
\begin{array}{c}
\prod_{g,n}^{\infty} C^*_{c + 2dg}(N_{g,n}, \mathbb{Q}) \\
\xrightarrow{\nabla^1_{bor}} \\
\prod_{g,n}^{\infty} C^*_{c + 2dg + 1}(N_{g,n+1}, \mathbb{Q})
\end{array}
\]

(35) \[ v_1^{-1} \circ u \sim v_1^{-1} \circ u \sim \]

\[
\begin{array}{c}
\Delta_1 \\
\xrightarrow{\nabla^1_{bor}} \\
\Delta_1
\end{array}
\]

\[ RGC_d(\delta)_{\geq 3} \rightarrow RGC_d^{+1}(\delta)_{\geq 3} \]

Proof. We have the following diagram:

\[
\begin{array}{c}
C^*_{c + 1}(N_{g,n+1}, \mathbb{Q}) \\
\xrightarrow{u_1} \\
C^*_{c - 2g + 1 - n}(D_{g,n+1}, \mathbb{Q})
\end{array}
\]

\[
\begin{array}{c}
q! \\
\xrightarrow{n!} \\
q!
\end{array}
\]

\[
\begin{array}{c}
C^*_{c + 1}(N_{g,n+1}^L, \mathbb{Q}) \\
\xrightarrow{\bar{u}_1} \\
C^*_{c - 2g + 2 - n}(K_{g,n+1}^L, \mathbb{Q})
\end{array}
\]

\[
\begin{array}{c}
\bar{m}_1 \\
\xrightarrow{m_1} \\
\bar{m}_1
\end{array}
\]

\[
\begin{array}{c}
C^*_{c}(N_{g,n,0,1} \times N_{g,n}, \mathbb{Q}) \\
\xrightarrow{\pi_{N_{g,n,0,1}}} \\
C^*_{c}(N_{g,n,0,1}, \mathbb{Q})
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{\pi_{N_{g,n}}^*} \\
\xrightarrow{\pi_{N_{g,n}}^*} \\
\pi_{D_{g,n,0,1}}^*
\end{array}
\]

\[
\begin{array}{c}
C^*_{c}(N_{g,n}, \mathbb{Q}) \\
\xrightarrow{v_1} \\
C^*_{c}(N_{g,n}, \mathbb{Q})
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{u_1} \\
\xrightarrow{u_1} \\
\xrightarrow{u_1}
\end{array}
\]

\[
\begin{array}{c}
\sim \\
\sim \\
\sim
\end{array}
\]

\[
\begin{array}{c}
(C^*_{c + 2dg}(N_{g,n}, \mathbb{Q})) \\
\xrightarrow{\nabla^1_{bor}} \\
(C^*_{c + 2dg + 1}(N_{g,n+1}, \mathbb{Q}))
\end{array}
\]

By the base change theorem for the compactly supported chains (Remark 2.3.4) all bottom squares of the diagram above commute. Moreover since $q_!$ is a quasi-isomorphism we get that $\bar{u}_1$ is a quasi-isomorphism as well (by two-out-of-three property of weak equivalences). Hence we get that:

\[
\nabla^1_{bor} \circ v_1^{-1} \circ u_1 = v^{-1} u_1 \circ \Delta^1_{bor}.
\]

And hence from the Theorem 5.1.4 we have:

\[
\nabla^1_{bor} \circ v_1^{-1} \circ u_1 = v^{-1} u_1 \circ \Delta^1 \circ D.
\]

\[
\square
\]

Theorem 5.2.3. The following diagram commutes (in the derived category of vector spaces):
Proof. For every $g,n$ such that $2g + n - 2 > 0$ we denote by $\mathcal{N}_L^{rt}_{g,n}$ the locus in Liu’s moduli space which has only "rational tails" as complex singularities i.e. nodal surfaces $\Sigma \in \overline{\mathcal{N}}^{L}_{g,n}$ such that the normalization of complex nodes of $\Sigma$ consists of the unique stable bordered surfaces of genus $g$ and stable bordered surfaces of genus 0. We will also use a notation $\mathcal{N}_L^{rt}_{g,n,0,1}$ for the pullback of $\pi^{comp}: \overline{\mathcal{N}}^{L}_{g,n,0,1} \to \overline{\mathcal{N}}^{L}_{g,n}$ along the canonical inclusion $\mathcal{N}_L^{rt}_{g,n} \to \overline{\mathcal{N}}^{L}_{g,n}$. Note that by the construction we have a proper morphism:

\[
\mu: \mathcal{N}_L^{rt}_{g,n,0,1} \to \mathcal{N}_L^{rt}_{g,n}
\]

Following [Kal21] we will use a notation $\mathcal{M}^{rt}_{*g,n}$ for the moduli stack $\mathcal{M}^{rt}_{g,n}/\Sigma_n$, here $\mathcal{M}^{rt}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is a moduli stack of curves with rational tails. We also have a canonical inclusion:

\[
e^{-1}_{n}: \mathcal{M}^{rt}_{*g,n} \to \mathcal{N}_L^{rt}_{g,n}
\]

Consider the pullback of the morphism (36) along the inclusion above. We have an equivalence of stacks:

\[
\mathcal{N}_L^{rt}_{g,n,0,1} \times \mathcal{N}_L^{rt}_{g,n} (\mathcal{M}^{rt}_{*g,n}) \cong \mathcal{M}^{rt}_{*g,n *g,n+1}
\]

Here by $\mathcal{M}^{rt}_{*g,n *g,n+1}$ we have denoted the following stack $(\mathcal{M}^{rt}_{g,n+1} \times [n+1])/\Sigma_{n+1}$. Following [Kal21] the pullback of the morphism (36) will be denoted by:

\[
\tilde{\mu}_{*g,n *g,n+1}: \mathcal{M}^{rt}_{*g,n *g,n+1} \to \mathcal{M}^{rt}_{*g,n}
\]

Note that we also have an inclusion:

\[
m_{rt}: \mathcal{N}_L^{rt}_{g,n,0,1} \to \mathcal{N}_L^{rt}_{g,n+1}
\]

which is defined by replacing the unique complex marked point with a puncture. We have an obvious commutativity property $m_{rt} \circ \tilde{e}_n = e_{n+1} \circ \pi_{*g,n *g,n+1}$, here by $\pi_{*g,n *g,n+1}: \mathcal{M}^{rt}_{*g,n *g,n+1} \to \mathcal{M}^{rt}_{*g,n+1}$ we have denoted the canonical projection. Following [Kal21] we have an isomorphism of sheaves:

\[
m_{rt}^! \mathcal{Q} \cong \mu^* \mathcal{Q}
\]
We can organise all data above into the commutative diagram:

\[
\begin{array}{cccc}
C_c(M_{g,n}^{\text{rt}}, \epsilon_{n+1}) & \xrightarrow{c_{n+1}!} & C_c^*(N_{g,n+1}^{\text{rt}}, \mathbb{Q}) \\
\pi_{g,n} & & \ & m_{rt}! \\
C_c^*(M_{g,n}^{\text{rt}}, \epsilon_{n,n+1}) & \xrightarrow{\bar{c}_{n!}} & C_c^*(N_{g,n,0,1}^{\text{rt}}, \mathbb{Q}) \\
\tilde{\mu}_{g,n} & & \ & \mu^* \\
C_c^*(M_{g,n}^{\text{rt}}, \epsilon_n) & \xrightarrow{c_n!} & C_c^*(N_{g,n}^{\text{rt}}, \mathbb{Q}) \\
\end{array}
\]

Note that by the construction we have an inclusion \( N_{g,n} \rightarrow N_{g,n,0,1}^{\text{rt}} \) the pullback of the moduli space \( N_{g,n}^{\text{rt}} \) is isomorphic to a moduli space [1]:

\[ N_{g,n}^{\text{rt}} \times N_{g,n}^{\text{rt}} N_{g,n} \cong N_{g,n} \times N_{g,n} N_{g,n,0,1} \]

We also have the following commutative diagram:

\[
\begin{array}{cccc}
C_c^{*+n+1}(N_{g,n+1}^{\text{rt}}, \mathbb{Q}) & \xrightarrow{g'_{n+1}!} & C_c^{*+n+1}(N_{g,n+1}^{\text{rt}}, \mathbb{Q}) \\
m_{rt}! & & \ & \bar{m}_! \circ \bar{v}_! \\
C_c^{*+n}(N_{g,n,0,1}^{\text{rt}}, \mathbb{Q}) & \xrightarrow{\bar{g}_{n!}} & C_c^{*+n}(N_{g,n} \times N_{g,n} N_{g,n,0,1}) \\
\mu^* & & \ & 
\pi_{N_{g,n,0,1}}^* \\
C_c^{*+n}(N_{g,n}^{\text{rt}}, \mathbb{Q}) & \xrightarrow{g_{n!}} & C_c^{*+n}(N_{g,n,0,1}) \\
\end{array}
\]

Consider the following:

**Lemma 5.2.4.** The morphism:

\[
j_{n!} : C_c^*(N_{g,n+1}, \mathbb{Q}) \rightarrow C_c^*(N_{g,n+1}^{\text{rt}}, \mathbb{Q})
\]

is a quasi-isomorphism.

**Proof.** Note that the interior of the moduli space \( N_{g,n+1}^{\text{rt}} \) is given by the moduli space \( N_{g,n+1}^{\text{rt}} \) of stable bordered surface without real singularities and all possible complex singularities being 'rational tails'. Following the scheme of the proof of Proposition 4.4.1 [Kal22] we can prove that the Gysin pushforward along the open inclusion \( N_{g,n} \rightarrow N_{g,n+1}^{\text{rt}} \). Hence by two-out-of-three property of weak-homotopy equivalences we get that \( j_{n!} \) is a quasi-isomorphism as well.

\[ \square \]
From diagrams above we get:
\[
g_{n+1}^{-1} \circ c_n+1 \circ \nabla^{rt} = \tilde{\mu}_1 \circ \tilde{\nu}_1 \circ \pi_{N^g,n,0,1} \circ g_n^{-1} \circ c_n!
\]
Here following [Kal21] we have denoted the composition \(\pi_{N^g,n,0,1} \circ \tilde{\mu}_1 \circ \tilde{\nu}_1\) by \(\nabla_1^{rt}\). Recall that \(\text{opcit}\) the Willwacher differential \(\nabla_1\) was defined as \(\nabla_1 := (j_1^{rt})^{-1} \nabla^{rt} j_1^{rt}\). Since we have an obvious commutativity property \(\nabla_1^{rt} \circ j_1^{rt} = 1\), we get:
\[
\rho \mathcal{M}_{g,n} \circ \nabla_1 := b^{-1}_{n+1} \circ a_{n+1} \circ (j_1^{rt})^{-1} \nabla^{rt} j_1^{rt} = 1.
\]

**Theorem 5.2.5** (T. Willwacher’s conjecture). For every \(g \geq 0\) and \(n \geq 1\) such that \(2g + n - 2 > 0\) the following square commutes (in the derived category):

\[
\begin{array}{ccc}
\Pi \mathcal{C}_c^{g+2d-2n} \mathcal{M}_{g,n} / \Sigma_n & \xrightarrow{\nabla_1} & \Pi \mathcal{C}_c^{g+2d-2n} \mathcal{M}_{g,n+1} / \Sigma_{n+1} \\
\text{cost} & \sim & \text{cost} \\
\text{RGC}_{d}(\delta)_{\geq 3} & \xrightarrow{\Delta_1} & \text{RGC}_{d}(\delta)_{\geq 3}
\end{array}
\]

**Proof.** Combining Theorems 5.2.3 with Proposition 5.2.2 we obtain the result. \(\square\)

**Theorem 5.2.6** (A. Căldăraru’s conjecture). The cohomology of the Merkulov-Willwacher ribbon graph complex \(\text{RGC}_{d}(\delta + \Delta_1)\) is isomorphic to the totality of the shifted cohomology with compact support of the moduli stacks \(\mathcal{M}_g\):

\[
\text{cald}: H^* (\text{RGC}_{d}(\delta + \Delta_1)) \cong \prod_{g \geq 1} H^*_{c} (\mathcal{M}_g, \mathbb{Q}).
\]

**Proof.** For every \(g > 0\) the genus \(g\)-part of the Merkulov-Willwacher ribbon graph complex can be identified with the total DG-vector space of the complex:

\[
(38) \quad C^*_{c}(\mathcal{M}_{g,n}^{\text{rib}}, \mathbb{Q}) \xrightarrow{\nabla_1} \ldots \xrightarrow{\nabla_1} C^*_{c}(\mathcal{M}_{g,n}^{\text{rib}}/\Sigma_n, \mathbb{Q}) \xrightarrow{\nabla_1} \ldots
\]

This follows from the decomposition (5) and the fact that the genus of ribbon graphs \(\mathcal{M}_{g,n}^{\text{rib}}\) is always zero. Consider a case when \(g = 0\). Denote by \(B_0 \text{RGC}_{d}(\delta + \Delta_1)_{\geq 3}\) the genus zero part of the Merkulov-Willwacher ribbon graph complex, which is quasi-isomorphic to the total DG-vector space of the complex:

\[
(39) \quad \ldots \xrightarrow{\nabla_1} (\mathcal{C}_c^*(\mathcal{M}_{0,n}, \mathbb{Q}) \otimes_{\Sigma_n} \text{sgn}_n)^{\Sigma_n} \xrightarrow{\nabla_1} \ldots
\]

Then it is easy to notice that (5)

\[
H^* (B_0 \text{RGC}_{d}(\delta + \Delta_1)_{\geq 3}) = 0.
\]

Hence from the result above we get that \(H^* (B_0 \text{RGC}_{d}(\delta + \Delta_1))\) is controlled by the loop classes above. Thus using the involution on the Merkulov-Willwacher ribbon graph complex we get that any class \(R_k \ k \geq 1\) must has a partner with at least

\[17\] By the Theorem 4.3, Corollary 2 from [Vas92].
\[18\] This involution is defined by replacing the vertices of a ribbon graph with boundaries and vice versa [MW15, CFL10].
trivalent vertices and \( R_1 \) must has a partner with one boundary and one edge. Finally applying Theorem 4.4.5 from [Kal22] we get the desired quasi-isomorphism.

Denote by \( \text{GC}_{2d}^2(\delta) \) the Kontsevich graph complex \([Wil15a]\). This is a combinatorial cochain complex with cochains given by pairs \((\Gamma, or_\Gamma)\) here \( \Gamma \) is a graph with at least two valent vertices and no loop edges and \( or_\Gamma \in \text{det}(E(\Gamma)) \) is an orientation. These elements are considered modulo the relation \((\Gamma, or_\Gamma) = -(\Gamma, or_\Gamma)\). We define the grading of \((\Gamma, or_\Gamma)\) by the rule \( |\Gamma| = 2d(|V(\Gamma)| - 1) + (1 - 2d)|E(\Gamma)| \). The differential \( \delta \) is defined by splitting a vertex. Note that the differential \( \delta \) preserves a genus of a graph and hence we have a decomposition of the Kontsevich graph complex into subcomplexes by fixing a genus. For \( g \geq 1 \) \( B_g \text{GC}_{2d}^2(\delta) \) we denote the genus \( g \)-part of the Kontsevich graph complex.

**Theorem 5.2.7.** We have the canonical injection:

\[
H^q(\text{GC}_{2d}^2) \longrightarrow H^{q+1}(\text{RGC}_d(\delta + \Delta_1))
\]

**Proof.** By Theorem 5.2.6 we have an equivalence between the compactly supported cohomology of \( \mathcal{M}_g \) and the cohomology of the Merkulov-Willwacher complex. For \( g \geq 2 \) by the Theorem of Chan-Galatius-Payne [CGP21b] (See also [AWŻ20], [Kal21]) we have an inclusion:

\[
H^q(B_g \text{GC}_{2d}^2) \cong W_0 H^{q+2dg}(\mathcal{M}_g, \mathbb{Q}) \hookrightarrow H^{q+2dg}(\mathcal{M}_g, \mathbb{Q})
\]

Here \( W_0 \) is a weight zero piece of the weight filtration on the compactly supported cohomology of \( \mathcal{M}_g \) \([Del71] \ [Del74]\). For \( g = 1 \) the result follows from the explicit computations of \( B_1 \text{GC}_{2d}^2(\delta) \) from [Wil15a] and the explicit computations of the compactly supported cohomology of the Artin stack \( \mathcal{M}_1 \) \([Kal22]\) (cf. [Beh03]).

**Remark 5.2.8.** Denote by \( \text{OGC}_{2d+1}^2(\delta) \) the oriented graph complex \([Wil15b]\). This complex naturally appears as a deformation complex of Lie bialgebras i.e. of a propered \( \Lambda \text{Lie}_{B, d} \) \([MW18]\). One can explicitly describe this complex as complex with cochains given by pairs \((\Gamma, or_\Gamma)\), where \( \Gamma \) is an oriented graph i.e. a graph equipped with a direction on each edge such that there are no closed circles formed by paths \([Wil15b]\) and the differential given by splitting a vertex. Applying the deformation complex to the morphism \( \mathcal{M}_1 \) and passing to the cohomology, following [MW15] we get a morphism:

\[
\text{mw}: H^*(\text{OGC}_{2d+1}) \longrightarrow H^{*+1}(\text{RGC}_d(\delta + \Delta_1))
\]

Note that it was shown in [Wil15b] that the oriented graph complex \( \text{OGC}_{2d+1}^2(\delta) \) is quasi-isomorphic to the Kontsevich graph complex \( \text{GC}_{2d}^2(\delta) \). Further in \([Ziv20]\) the explicit combinatorial quasi-isomorphism:

\[
\text{ziv}: \text{OGC}_{2d+1}^2(\delta) \longrightarrow \text{GC}_{2d}^2(\delta)
\]

was constructed. We can organise this data into the diagram:

\[
\begin{align*}
H^*(\text{GC}_{2d}) & \xrightarrow{\text{cgp}} \prod_{g=1}^{\infty} H^{*+2dg}(\mathcal{M}_g, \mathbb{Q}) \\
H^*(\text{OGC}_{2d+1}) & \xrightarrow{\text{mw}} H^{*+1}(\text{RGC}_d(\delta + \Delta_1)) \\
\text{ziv} & \sim \quad \text{cald} \sim
\end{align*}
\]
It was conjectured in \[ AWŽ20 \] that the square above is commutative. This conjecture in particular implies the original conjecture from \[ MW15 \] that the morphism \( (41) \) is injective. From Theorem 5.2.6 and the fact that \( \nabla_1 \) preserves the weight filtration (Corollary 4.4.4 in \[ Kal22 \]) the commutativity of the square \( (43) \) follows from the commutativity of the following square:

\[
\begin{array}{ccc}
H^*(H_n GC_{2d}) & \xrightarrow{cgp_n} & \prod_{g \geq 0, 2g + n - 2 > 0} H^*(\mathcal{M}_{g,n}, \mathbb{Q}) \\
\downarrow \sim & & \downarrow \sim \\
H^*(H_n OC_{2d+1}) & \xrightarrow{awz} & H^+(RGC_{d})
\end{array}
\]

By \( H_n GC_{2d}(\delta) \) we have denoted the hairy graph complex \[ CGP21a \] with labelled hairs, \( H_n OG C_{2d+1}(\delta) \) is the hairy oriented graph complex \[ AWŽ20 \] and \( RG C^*_d(\delta) \) is a ribbon graph complex which consists of ribbon graphs with \( [n] \)-labelled boundaries. Here \( az \) is a version of a morphism \( (42) \) for hairy graph complexes \[ AWŽ20 \] (See also \[ AŽ20 \]), \( cgp_n \) is a "weight zero inclusion" \[ CGP21a \] (See also \[ AWŽ20 \] and \[ Kal22 \]) and \( awz \) is a version of \( (41) \) for hairy graph complexes.

**Remark 5.2.9.** We can define a degree \( -1 \) DG-Lie algebra structure on the genus totality of \( C_c^*(\mathcal{M}_g, \mathbb{Q}) \):

\[
\{ \cdot, \cdot \} : \prod_{g=1}^{\infty} C_c^*(\mathcal{M}_g, \mathbb{Q}) \times \prod_{g=1}^{\infty} C_c^*(\mathcal{M}_g, \mathbb{Q}) \longrightarrow \prod_{g=1}^{\infty} C_c^{*+1}(\mathcal{M}_g, \mathbb{Q})
\]

By the rule:

\[
\{ \cdot, \cdot \} := Res_{\mathcal{M}_\Gamma/\text{Aut}(\Gamma)} \circ pt \circ (\pi^* \otimes \pi^*)(\cdot, \cdot), \quad \Gamma = \cdot^{g_1 \otimes g_2} \cdot
\]

Here morphisms in the definitions are:

\[
\mathcal{M}_{\Gamma} := \mathcal{M}_{g_1,1} \times \mathcal{M}_{g_2,1}
\]

\[
\begin{array}{ccc}
\mathcal{M}_{\Gamma}/\text{Aut}(\Gamma) & \xrightarrow{\pi \times \pi} & \mathcal{M}_{g_1} \times \mathcal{M}_{g_2}
\end{array}
\]

Here \( \pi \) is a forgetful (proper) morphism of stacks and \( Res_{\mathcal{M}_{\Gamma}/\text{Aut}(\Gamma)} \) is a residue morphism along the normalised open stratrum in \( \mathcal{M}_{g_1 + g_2} \). We believe that via the Theorem 5.2.6 the shifted DG-Lie algebra structure \( (45) \) coincides with a combinatorial DG-Lie algebra structure on the Merkulov-Willwacher ribbon graph complex \( (10) \).

6. **Appendix: The Teichmüller space of a bordered surface**

6.1. **Preliminaries.** Let \( (X, d) \) be a metric spaces. Recall that a geodesic \( \gamma \) joining points \( x \) and \( y \) is an isometric morphism \( \gamma : [0,1] \rightarrow X \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \), we write \([x,y] \subset X \) for the set \( \gamma([0,1]) \). We say that the metric spaces is **uniquely geodesic** if for any two points there exists a unique geodesic joining them. The important class of uniquely geodesic comes from \( CAT(0)\)-spaces.


Riemannian manifold on

Then we define the closed curves in with 0-simplicies given by homotopy classes of non trivial onc peripheral simple

ded by homotopy classes of non trivial homotopic to $f$. One can show that every $CAT(0)$-space is uniquely geodesic with geodesic varying continuously \cite{BH99} (Proposition 1.4). We say that a subset $C \subset X$ is convex if for any $x$ and $y$ in $C$ the unique geodesic $[x, y]$ joining points $x$ and $y$ lies in $C$. Recall that for a point $x \in X$ in a metric spaces $X$ and a subset $C$ in $X$ a distance $d(x, C)$ from the point $x$ to the subset $C$ is defined by the rule $d(x, C) := \inf_{y \in C} d(x, y)$.

We have the following well known result Proposition 2.4 \cite{BH99}:

**Proposition 6.1.1.** Let $X$ be $CAT(0)$-space then for every closed convex subset $C$ and $x \in X$ there exists a unique point $\pi_C(x) \in X$ such that:

$$d(x, C) = d(c, \pi_C(x))$$

The point $\pi_C(x)$ is called the nearest point in $C$ to a point $x$.

Hence for any closed and convex subset in $CAT(0)$-space one can define the nearest point projection:

$$\Pi_C : X \to C, \quad \Pi_C : x \mapsto \pi_C(x).$$

This morphism is 1-Lipschitz and hence continues. Moreover one can show (Proposition 2.4 \cite{BH99}) that the inclusion $C \to X$ is a deformation retract and in particular taking $C$ to be a point one can show that every $CAT(0)$-space is contractible.

Let $S_{g, n}$ be a closed topological surface of genus $g$ with $n$-punctures. We define the Teichmüller space $\mathcal{T}_{g, n}$ of $S_{g, n}$ by the rule: a set of pairs $(X, g)$, where $X$ is a Riemann surface and $g: \Sigma \to X$ is a diffeomorphism modulo the equivalence: $(X, g) \cong (Y, f)$ if an only if $fg^{-1}$ is isotopic to a biholomorphic morphism. One can show that the Teichmüller space $\mathcal{T}_{g, n}$ is a manifold with the a cotangent spaces defined by the rule: a fiber over a point $(X, g)$ is given by the space $\mathcal{Q}(X)$ of holomorphic quadratic differentials on $X$ with at most simple poles at the punctures. Recall that every Riemann surface $X$ has a hyperbolic structure with a hyperbolic metric $dh^2$. We define the Petersson Hermitian pairing i.e. a structure of the Riemannian manifold on $\mathcal{T}_{g, n}$ by the rule:

$$\langle \psi, \phi \rangle := \int_{\mathcal{T}_{g, n}} \phi \bar{\psi} (dh^2)^{-1}$$

Then we define the Weil-Petersson metric $d_{WP}(x, y)$ on $\mathcal{T}_{g, n}$ as the dual to the Hermitian pairing above \cite{Ah61}. The important fact that this metric is Kähler and uniquely geodesic.

Denote by $C(S_{g, n})$ the complex of curves in $S_{g, n}$. This is a simplicial complex with 0-simplicies given by homotopy classes of non trivial non peripheral simple closed curves in $S_{g, n}$. A maximal simplex has $3g - 3 + n$ element. A free homotopy class $\alpha$ of a curve in $S_{g, n}$ determines the length function $\ell_\alpha$ on a Teichmüller space $\mathcal{T}_{g, n}$ namely on an element $(X, f)$ we set a value of $\ell_\alpha$ to be a length of a geodesic homotopic to $f(\alpha)$. For a simplex $\sigma \in C(S_{g, n})$ we define a space $\mathcal{T}(\sigma) := \{ X \mid \ell_\alpha(X) = 0 \iff \alpha \in \sigma \}$. The completion of the Teichmüller space with respect to

\footnote{Sometimes also called a Hadamard space.}
\footnote{Note that in op. cit it is assumed that $CAT(0)$-spaces are not necessarily complete.}
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the Weil-Petersson metric $d_{WP}$ will called the augmented Teichmüller space and denoted by $\mathcal{T}_{g,n}$ [Abi77]. We have the following [Mas76]:

**Theorem 6.1.2.** The boundary $\partial \overline{\mathcal{T}}_{g,n} := \mathcal{T}_{g,n} \setminus \mathcal{T}_{g,n}$ of the Weil-Petersson completion of the Teichmüller space $\mathcal{T}_{g,n}$ is given by:

$$\partial \overline{\mathcal{T}}_{g,n} \cong \bigcup_{\sigma \in C(S_{g,n})} \mathcal{T}(\sigma)$$

The important result that the augmented Teichmüller space $\overline{\mathcal{T}}_{g,n}$ is CAT(0)-space [Yam]. Note that a closure $\overline{T(\sigma)} := \bigcup_{\gamma \subset \sigma} \mathcal{T}(\gamma)$ of a stratum $T(\sigma)$ is a convex closed subspace $i_\sigma: \overline{T(\sigma)} \to \overline{\mathcal{T}}_{g,n}$ and hence the nearest point projection (Lemma 6.1.1) is defined:

$$\Pi_{\overline{\mathcal{T}}}: \overline{\mathcal{T}}_{g,n} \to \overline{T(\sigma)}$$

Note that this morphism is left dual to the canonical closed inclusion:

$$i_\sigma: \overline{T(\sigma)} \to \overline{\mathcal{T}}_{g,n}$$

Let $Diff^+(S_{g,n})$ be a group of diffeomorphisms of the topological surface. By $Diff^0(S_{g,n})$ we denote the connected component of identity i.e. diffeomorphisms isotopic to identity. We define the mapping class group $MCG_{g,n}$ of a surface of genus $g$ with $n$ marked points by the rule:

$$MCG_{g,n} := Diff^+(S_{g,n})/Diff^0(S_{g,n})$$

The mapping class group acts on the Teichmüller space by the rule: $g(X, f) := (X, f g^{-1})$. This action is properly discontinuous. We have an equivalence of orbifolds:

$$\mathcal{M}_{g,n} \cong [\mathcal{T}_{g,n}/MCG_{g,n}]$$

Note that the action of the mapping class group extends to the augmented Teichmüller space through it is not longer properly discontinues. Following [Har74] we have :

**Theorem 6.1.3.** For any $g$, and $n$ such that $2g + n - 2 > 0$ we have an equivalence of topological stacks:

$$\overline{\mathcal{M}}_{g,n} \cong [\overline{\mathcal{T}}_{g,n}/MCG_{g,n}]$$

Let $\sigma \in C(S_{g,n})$ we denote by $MCG^\sigma_{g,n}$ the subgroup of $MCG_{g,n}$ consisting of those elements which have representative $g: S_{g,n} \to S_{g,n}$, such that $g(\gamma)$ is homotopic to $\gamma$ for any $\gamma \in \sigma$ and which fixes each component of $S_{g,n} \setminus \sigma$. It is easy to see that a morphism [18] is $(MCG^\sigma_{g,n}, MCG_{g,n})$-equivariant and hence we have the morphism between the corresponding quotients:

$$i_\sigma: [\overline{T(\sigma)}/MCG^\sigma_{g,n}] \cong D_{\Gamma(\sigma)} \to \overline{\mathcal{M}}_{g,n}$$

Here $D_{\Gamma}$ is a component of a boundary of the Deligne-Mumford moduli stack $\overline{\mathcal{M}}_{g,n}$ i.e. closure of locus of stable curves with a dual graph being $\Gamma(\sigma)$ (this graph depends on $\sigma$ in a usual way).
6.2. Teichmüller space $B\mathcal{T}_{g,n}$. Let $\Sigma$ be a topological surface of genus $g$ with $n$-boundary components. By $D\Sigma_{g,n} := \Sigma_{g,n} \cup \Sigma_{g,n}$ we denote the double of this surface i.e. the closed topological surface of genus $2g + n - 1$ defined by gluing the surface with the oppositely oriented one along the boundary. We have an involutive diffeomorphism $\tau: D\Sigma_{g,n} \rightarrow D\Sigma_{g,n}$ which act by interchanging $\Sigma_{g,n}$ and $\Sigma_{g,n}$. An element $\tau$ generates a subgroup in a mapping class group of $D\Sigma_{g,n}$ which by abuse of notation we will denote by the symbol $\tau \in MCG_{2g+n-1}$. As an element of the mapping class group $\tau$ acts on the Teichmüller space $\mathcal{T}_{2g+n-1} :$

\begin{align}
\tau: & \mathcal{T}_{2g+n-1} \rightarrow \mathcal{T}_{2g+n-1} \\
\tau: & (X, f) \mapsto (X, f\tau^{-1}).
\end{align}

We consider the $\tau$-invariant subspace $\mathcal{T}^\tau_{2g+n-1}$ By the definition elements in this space are pairs $(X, f)$ where $X$ is a Riemann surface of genus $2g+n-1$ and $f: D\Sigma_{g,n} \rightarrow X$ is a diffeomorphism such that $f\tau f^{-1}: X \rightarrow X$ is isotopic to a biholomorphic morphism, modulo the usual equivalence. We denote the locus of fixed points under the diffeomorphism $f\tau f^{-1}$ by $X^\mathbb{R}$.

**Definition 6.2.1.** For a bordered topological surface $\Sigma_{g,n}$ we define the Teichmüller space $B\mathcal{T}_{g,n}$ of $\Sigma_{g,n}$ as elements $(X, f) \in \mathcal{T}_{2g+n-1}^\tau$ with a choice of a component in $X \setminus X^\mathbb{R}$.

By the construction we have a finite morphism $B\mathcal{T}_{g,n} \rightarrow \mathcal{T}_{2g+n-1}^\tau$ via this morphism we equip the Teichmüller space $B\mathcal{T}_{g,n}$ with a topology. Let $MCG_{2g+n-1}$ be a mapping class group of a topological surface $D\Sigma_{g,n}$. We give:

**Definition 6.2.2.** For a bordered surface $\Sigma_{g,n}$ we define the mapping class group $MCG^B_{g,n}$ of $\Sigma_{g,n}$ as a centralizer of $\tau$ in the mapping class group of $D\Sigma_{g,n}$:

$$MCG^B_{g,n} := C_{MCG_{2g+n-1}}(\tau)$$

Note that $\mathcal{T}_{2g+n-1}^\tau$ is stable under the action of the mapping class group. Let $(X, f) \in \mathcal{T}_{2g+n-1}^\tau$ and $g \in MCG^B_{g,n}$ then we have $(X, fg^{-1})$ and $fg^{-1}\tau gf^{-1}$ since $\tau g$ is isotopic to $g\tau$ we get that $fg^{-1}\tau gf^{-1}$ is isotopic to a biholomorphic morphism. We lift the action of the mapping class group to $B\mathcal{T}_{g,n}$ by the obvious rule. We have the following:

**Proposition 6.2.3.** For any $g, n$ such that $2g + n - 2 > 0$ we have an equivalence of orbifolds:

$$\mathcal{N}_{g,n} \cong \left[B\mathcal{T}_{g,n}/MCG^B_{g,n}\right].$$

**Proof.** By the construction elements in $\left[B\mathcal{T}_{g,n}/MCG^B_{g,n}\right]$ are triples $(X, f, C)$ such that $f\tau f^{-1}: X \rightarrow X$ is an involutive biholomorphic morphism i.e. a real structure on $X$ and $C$ is a component of $X \setminus X^\mathbb{R}$. This datum is one to one corresponds to a bordered surface of genus $g$ with $n$ boundary components $\mathcal{C}_{g,n}$. The rest follows from the construction of topology on $\mathcal{N}_{g,n}$ [Lin20] (cf. [49]).

Note that the action (51) extends to the augmented Teichmüller space and we consider the corresponding fixed point locus $\mathcal{T}^\tau_{2g+n-1}$. For an element $(X, f) \in \mathcal{T}^\tau_{2g+n-1}$ we consider the normalization of every singularity in $X^\mathbb{R}$, which will be a union of connected components $X_i$. Hence we give:
Definition 6.2.4. For a bordered topological surface $\Sigma_{g,n}$ we define the **augmented Teichmüller space** $B\mathcal{T}_{g,n}$ of $\Sigma_{g,n}$ as an element $(X,f) \in \mathcal{T}_{2g+n-1}$ with a choice of a component in $X_i \setminus X_i^R$ for each component $X_i$. 

Via a forgetful morphism $B\mathcal{T}_{g,n} \to \mathcal{T}_{2g+n-1}$ we equip the augmented Teichmüller space of a bordered surface with a topology. We have the following:

Proposition 6.2.5. We have an equivalence of topological stacks:

$$[B\mathcal{T}_{g,n}/\text{MCG}] \cong \mathcal{N}_{g,n}^L$$

Proof. The result follows from [Har74] and the construction of Liu’s compactification [Liu20].

Note that the the space $\mathcal{T}_{2g+n-1}$ is $\text{CAT}(0)$-space since it is given as a fixed point set of the group acting by isometries. The augmented Teichmüller space $B\mathcal{T}_{g,n}$ of a bordered surface inherits properties of this metric. In particular $\sigma \in \mathcal{C}(D\Sigma_{g,n})$ be a simplex which is invariant under $\tau$ i.e. $\tau(\gamma)$ is homotopic to $\gamma$ for any $\gamma \in \sigma$. Analogically to the definition of the stratum in the Teichmüller space we define the stratum $B\mathcal{T}(\sigma)$. Hence we define the nearest point projection:

$$\Pi_{\mathcal{T}(\sigma)}: B\mathcal{T}_{g,n} \to B\mathcal{T}(\sigma)$$

by the rule $\Pi_{\mathcal{T}(\sigma)}(X,f,(X_i)) := (\Pi_{\mathcal{T}(\sigma)}(X,f),(\Pi_{\mathcal{T}(\sigma)}(X_i))$.

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MAX PLANCK INSTITUT FÜR MATHEMATIK IN DEN NATURWissenschaftEN, INSELSTRASSE 22, 04103 LEIPZIG, GERMANY

Email address: alexey.kalugin@msi.mpg.de