Macdonald operators and homological invariants of the colored Hopf link

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Abstract

Using a power sum (boson) realization for the Macdonald operators, we investigate the Gukov, Iqbal, Kozycz and Vafa (GIKV) proposal for the homological invariants of the colored Hopf link, which include Khovanov–Rozansky homology as a special case. We prove the polynomiality of the invariants obtained by GIKV’s proposal for arbitrary representations. We derive a closed formula of the invariants of the colored Hopf link for antisymmetric representations. We argue that a little amendment of GIKV’s proposal is required to make all the coefficients of the polynomial non-negative integers.

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1. Introduction and notation

In the setup of string theory, the invariants of the colored Hopf link are identified as topological open string amplitudes on the deformed conifold $T^*S^3$ as follows: the $U(N)$ Chern–Simons theory is realized by topological string on $T^*S^3$ with $N$ topological $D$-branes wrapping on the base Lagrangian submanifold $S^3$. The Hopf link in $S^3$ consisting of two knots $K_1$ and $K_2$ can be introduced by a pair of new $D$-branes wrapping on Lagrangian three-cycles $L_1$ and $L_2$ such that $S^3 \cap L_i = K_i$ [1]. The topological open string amplitude of this brane system is supposed to give the invariants of the Hopf link. The coloring or the representation attached to each knot $K_i$ is related to the boundary states of the open string ending on $L_i$ by the Frobenius relation. After geometric transition or by the large $N$ duality [2–4], this brane configuration is mapped to the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$. The $D$-branes wrapping on $S^3$ disappear, but a pair of Lagrangian $D$-branes remains as a remnant of the Hopf link. We can describe the resulting $D$-brane system in terms of the toric diagram and compute the corresponding amplitude by the method of a topological vertex [5, 6].

The aim of this paper is to investigate the conjecture of Gukov, Iqbal, Kozycz and Vafa (GIKV) [7] on the superpolynomial $\mathcal{P}_{\lambda,\mu}(a, q, t)$ of the homological invariants of the Hopf link.
colored by two representations $\lambda$ and $\mu$.\(^1\) The superpolynomial of our interest is a polynomial in $(a, q, t) \in \mathbb{C}^3$, such that a specialization $a = q^N$ leads to the Poincaré polynomial of the $\mathfrak{sl}(N)$ link homology $P_{\mathfrak{sl}(N); \lambda, \mu}(q, t)$, which is a two parameter $(q, t)$ version of the $\mathfrak{sl}(N)$ link invariants. When the coloring is the $N$-dimensional defining representation, it is known as the Khovanov–Rozansky homology \cite{8}. In \cite{7}, they argued the relation of the homological invariants with the refined topological vertex \cite{9, 10} and the superpolynomial $P_{\lambda, \mu}(a, q, t)$ was expressed as a summation over all the partitions (see section 4). However, it is totally unclear whether the proposed expression actually gives a polynomial in $a$ and a specialization $a = q^N$ leads to the polynomial in $q$ and $t$. Moreover, the coefficients of the polynomial after the specialization are expected to be (nonnegative) integers due to the following reasons. In \cite{11}, it was argued that homological link invariants are related to a refinement of the BPS state counting in topological open string theory. The GIKV conjecture on homological link invariants of the Hopf link was based on this proposal. This means for the Hopf link state counting in topological open string theory. The GIKV conjecture on homological link invariants is expected to be (nonnegative) integer coefficients.

\[^1\] We also use another set of parameters $(Q, q, t)$ defined by $(a; q, t) = (Q^{-\frac{1}{2}}; \gamma^{\frac{1}{4}}, \frac{1}{\gamma^{\frac{1}{4}}})$.

\[^2\] We use a slightly different definition from that in \cite{16}.

The physical interpretation of $P_{\mathfrak{sl}(N); \lambda, \mu}$ as the Hilbert space of BPS states leads to a prediction on the dependence of the link homologies on the rank $N - 1$. It has been conjectured \cite{12, 13} that there exists a superpolynomial $P_{\lambda, \mu}(a, q, t)$ which is a rational function in three variables such that

$$P_{\mathfrak{sl}(N); \lambda, \mu}(q, t) = P_{\lambda, \mu}(a = q^N, q, t).$$  \hspace{1cm}(1.2)$$

We see that (1.1) and (1.2) imply that the specialization $a = q^N$ leads to a polynomial with (non-negative) integer coefficients.

In this paper, we prove the following.

(i) The GIKV’s proposal $P_{\lambda, \mu}^{\text{GIKV}}(a, q, t)$ for the superpolynomial $P_{\lambda, \mu}(a, q, t)$ of the homological invariants of the colored Hopf link gives really a polynomial in $a$ for arbitrary representations.

(ii) $P_{\mathfrak{sl}(N); \lambda, \mu}(q^N, q, t)$ vanishes for sufficiently small $N \in \mathbb{N}$, if one representation is antisymmetric and the other is arbitrary.

(iii) $P_{\mathfrak{sl}(N); \lambda, \mu}(q^N, q, t)$ with arbitrary $N \in \mathbb{N}$ is a polynomial in $q$ and $t$ with non-negative integer coefficients, if both representations are antisymmetric.

Furthermore, we perform the summation over the partitions and show a closed formula without assuming the condition $|t| < 1$ in \cite{14} for the superpolynomial of the homological invariants for antisymmetric representations.

The key ingredients of our proof are the Macdonald operators and a non-standard scalar product for the Hall–Littlewood polynomials. The Macdonald operators are a set of difference operators which commute with each other on the space of symmetric functions. Their simultaneous eigenfunctions are the Macdonald polynomials which are a two-parameter $(q, t)$ deformation of the Schur polynomials. By using a power sum realization for the Macdonald operators \cite{15, 16}, we investigate the polynomiality and integrality of the superpolynomial of the homological invariants obtained by GIKV’s proposal.

On the other hand, the Hall–Littlewood polynomial is the symmetric polynomial obtained from the Macdonald polynomial by letting $q = 0$. The higher Macdonald operator is realized by...
using a (non-standard) scalar product for which the Hall–Littlewood polynomials are the basis dual to dominant monomials [17]. We show the pairwise orthogonality of the Hall–Littlewood polynomials for this scalar product and use it for the calculation of the homological invariants.

For non-antisymmetric representations, the GIKV’s proposal for the superpolynomial of the homological invariants calls for some improvement because it has negative integer coefficients in general. We find that this positivity problem may be overcome by replacing the Schur function by the Macdonald function with an appropriate specialization.

The paper is organized as follows. In section 2, we discuss two kinds of scalar products for the Hall–Littlewood polynomials, which are used in the following sections. In section 3, we give a power sum realization for the Macdonald operators which are defined in appendix C. Our main results are stated and proved in section 4. We review the Gukov et al’s conjectures on the homological invariants of the colored Hopf link and prove some of them. Section 5 is devoted to the discussion on the positivity problem. Appendix A contains a brief summary of the symmetric functions. Proofs of the key theorems are shown in appendices B and D. In appendix E, we comment on a relation between the torus knot and the Macdonald polynomial.

Notations. The following notations are used through this paper. Let \( \lambda \) be a Young diagram, i.e. a partition \( \lambda \equiv (\lambda_1, \lambda_2, \ldots) \), which is a sequence of non-negative integers such that \( \lambda_i \geq \lambda_{i+1} \) and \( |\lambda| := \sum \lambda_i < \infty \). \( \lambda^\prime \) is its conjugate (dual) diagram. \( \ell(\lambda) := \lambda_1^\prime \) is the length.

Let \( p = (p_1, p_2, \ldots) \) be the power sum symmetric functions in \( x = (x_1, x_2, \ldots) \) defined as \( p_n(x) := \sum_{i \geq 1} x_i^n \). We treat any symmetric function \( f(x) \) in \( x \) as a function in \( p \) unless otherwise stated, and sometimes denote it as \( f(x(p)) \). \( P_\lambda(x; q, t), P_\lambda(x; t), s_\lambda(x) \) and \( e_\lambda(x) \) are the Macdonald, the Hall–Littlewood, the Schur and the elementary symmetric function in \( x \), respectively. We define the following specialization:

\[
p_\lambda(q^i t^p) := \sum_{i=1}^{\ell(\lambda)} (q^{i\lambda_i} - 1)t^{n_{1,i} - i} + \frac{1}{t^2 - t^{-2}} = \sum_{i=1}^{\ell(\lambda)} q^{i\lambda_i}t^{n_{1,i} - i} + \frac{t^{-n} - t^n}{t^2 - t^{-2}}.
\]

which is independent of \( N \) for any \( N \geq \ell(\lambda) \). We do not have to assume that \(|t| > 1\). Let \( p_n(x, y) := p_n(x) + p_n(y) \); then,

\[
p_n(cq^x t^p, Lt^{-p}) = c^n \sum_{i=1}^{\ell(\lambda)} (q^{i\lambda_i} - 1)t^{n_{1,i} - i} + c^n - L^n \frac{t^{-n} - t^n}{t^2 - t^{-2}}, \quad c, L \in \mathbb{C}.
\]

For \( N \in \mathbb{Z} \) and \( N \geq \ell(\lambda) \), \( p_n(q^{i\lambda+\beta}, t^{-\beta}) = \sum_{i=1}^{\ell(\lambda)} q^{i\lambda_i}t^{n_{1,i} - i} \) is the power sum in \( N \) variables \( \{q^{i\lambda_i+\beta}, t^{-\beta}\}_{i=1}^{\ell(\lambda)} \).

We do not have to use any analytic continuation or approximation because we treat any functions as formal power series. For example, \( \Pi(x; y, q, t) := \exp \left\{ \sum_{n=0}^{\infty} \frac{t^{n-1}}{n} p_n(x)p_n(y) \right\} \) and \( \Delta(x; t) := \exp \left\{ -\sum_{n=0}^{\infty} \frac{t^{n-1}}{n} \sum_{j<i} \frac{q^n}{j!} \right\} \) are power series in formal variables \( p_n(x) \) and \( p_n(y) \) respectively. We also let \( g_k := \prod_{(i,j) \in \lambda} (-1)^{q^{j-i} t^{j-i} - t^{-j+i}} \) and \( v := (q/t)^{1/2} \). The \( q \)-integer \( [N]_q := \frac{1-v^N}{1-v} \) and the \( q \)-binomial coefficient \( [N]_q \) \( \ell(\lambda) \) are polynomials in \( t \) with non-negative integer coefficients for \( N, r \in \mathbb{N} \).

2. Scalar product for the Hall–Littlewood polynomial

First we discuss two kinds of scalar products for symmetric polynomials, which we will use later. In this section, we treat symmetric functions as symmetric polynomials in finite number of variables \( x = (x_1, \ldots, x_N) \) by setting \( x_i = 0 \) for \( i \geq N + 1 \). For a partition \( \lambda = (\lambda_1, \ldots, \lambda_N) \), we denote \( x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N} \) and \( m_j := \# \{ \lambda_i | \lambda_i = j \} \), i.e. \( \lambda = (0^{m_0} 1^{m_1} 2^{m_2} \cdots) \) with
Here \( \sum_{i \geq 0} m_i = N \). The Hall–Littlewood polynomials \( P_\lambda(x; t) \) are obtained from the Macdonald polynomials \( P_\lambda(x; q, t) \) defined in appendix A by letting \( q = 0 \):

\[
P_\lambda(x; t) := P_\lambda(x; 0, t)
\]

\[
v_\lambda(t) := \prod_{j \geq 0} [m_j] !
\]

\[
\Delta(x; t) := \exp \left\{- \sum_{n>0} \frac{1-t^n}{n} \sum_{i<j} \frac{x_i^m}{x_j} \right\} = \prod_{i<j} \left( 1 - \frac{x_i}{x_j} \right) \sum_{n \geq 0} \left( \frac{t x_i}{x_j} \right)^n
\]

(2.1)

with the symmetric group \( S_N \) on \( N \) variables\(^3\). Here \( \tilde{\lambda} := (x_N, x_{N-1}, \ldots, x_1) \), \([N]_t := [1]_t [2]_t \cdots [N]_t \) and \([N]_t := (1-t^n)/(1-t) \). The canonical scalar product \( \langle \ast, \ast \rangle \) is defined from \( \langle \ast, \ast \rangle_{q, t} \) in appendix A by

\[
\langle P_\lambda(x; t), P_\mu(x; t) \rangle_t := \langle P_\lambda(x; 0, t), P_\mu(x; 0, t) \rangle_{0, t} = \delta_{\lambda, \mu} \prod_{j \geq 1} \prod_{i=1}^{m_j} \frac{1}{1-t^i}.
\]

(2.2)

We abbreviate it to \( \langle P_\lambda, P_\mu \rangle_t \). The Cauchy formula (A.5) is now

\[
\sum_{\lambda} \frac{1}{(P_\lambda(x; t) P_\lambda(y; t))} = \exp \left\{ \sum_{n \geq 0} \frac{1-t^n}{n} p_n(x)p_n(y) \right\}
\]

which is summed over all partitions \( \lambda \).

For functions \( f \) and \( g \) in \( x \), let us define a second scalar product as \(^1\)

\[
\langle f(x), g(x) \rangle_{N, t} := \int \prod_{j=1}^{N} \frac{dx_j}{2\pi i x_j} f(\tilde{x})^{-1} \Delta(x; t) g(x).
\]

(2.3)

Here \( x_j \)'s are formal parameters, and \( f \) and \( g \) are defined as the Taylor expansion in \( f \). For a Laurent series \( f(x) \) in \( x \), \( \int \frac{dx}{2\pi i x} f(x) \) denotes the constant term in \( f \), i.e., \( \int \frac{dx}{2\pi i x} \sum_{n \in \mathbb{Z}} f_n x^n = f_0 \). Note that the kernel function \( \Delta(x; t) \) is not symmetric in \( x \). Then, \( \langle f(x), g(x) \rangle_{N, t} = \langle g(\tilde{x}), f(\tilde{x}) \rangle_{N, t} \). Hence, the second scalar product is symmetric only for the symmetric functions. The Hall–Littlewood polynomials \( P_\lambda(x; t) \) with \( \ell(\lambda) \leq N \) are pairwise orthogonal for the second scalar product and we have\(^4\)

**Theorem 2.1.**

\[
\langle P_\lambda(x; t), P_\mu(x; t) \rangle_{N, t}^{\mu} = \langle P_\mu(x; t), P_\lambda(x; t) \rangle_{N, t}^{\mu} = \delta_{\lambda, \mu} v_\lambda(t)^{-1}[N]_t!
\]

(2.5)

A proof is given in appendix B. Since \( P_\lambda(x; t) \) coincides with the elementary symmetric polynomial \( e_\lambda(x) \), we obtain \( \langle P_\lambda(x; t), e_\ell(x) \rangle_{N, t}^{\ell} = \delta_{\lambda, \ell} \frac{[N]_t!}{[\ell]_t!} \), with \([n]_t := \prod_{i=1}^{t} \frac{1-t^n}{1-t} \). Note that from (A.12) it follows that

\[
\frac{\langle e_\ell, e_\ell \rangle_{N, t}^{\ell}}{\langle e_\ell, e_\ell \rangle_{N, t}^{\ell}} = (-1)^t t^{\frac{1}{2}} \frac{e_{N-t}(t^\ell)}{e_N(t^\ell)}, \quad e_\ell(t^\ell) = (-1)^t t^\frac{1}{2} \langle e_\ell, e_\ell \rangle_{N, t}^{\ell}.
\]

(2.6)

which we will use later.

3. Power sum realization for Macdonald operators

In this section, we give a power sum (boson) realization for the Macdonald operators which are defined in appendix C. Let \( p := (p_1, p_2, \ldots) \) be the power sum symmetric functions in infinite number of variables \( x = (x_1, x_2, \ldots) \) defined as \( p_n(x) := \sum_{i \geq 1} x_i^n \). We treat any symmetric function \( f(x) \) in \( x \) as a function in \( p \) unless otherwise stated.

\(^3\) Note that \( \Delta(x; t) = \prod_{\ell \in \mathbb{Z}, t} \frac{1-t^{\ell+1}}{1-t^{\ell}} \) by the analytic continuation.

\(^4\) It was shown in \([17]\) that the Hall–Littlewood polynomials are the basis dual to dominant monomials.

4
3.1. Macdonald operators by the power sum

First, we define a commutative family of the difference operators in power sums, whose eigenfunctions are the Macdonald functions. Let \( z = (z_1, \ldots, z_r) \). For \( r = 0, 1, 2, \ldots \), let \( H \) and \( H' \) be \( H := \sum_{r \geq 0} w^r H^r, H^0 := 1 \) and

\[
H^r := e_r(t^p) \oint \frac{dz_0}{2\pi i z_0} \Delta(z; t^{-1}) \varphi^r(z), \quad \varphi^r(z) := \varphi_+^r(z) \varphi_-^r(z)
\]

(3.1)

\[\begin{align*}
\varphi_+^r(z) & := \exp \left\{ \sum_{n=0}^{r} \frac{1-t^{-n}}{n} \sum_{a=1}^{r} z_a^n p_a \right\}, \\
\varphi_-^r(z) & := \exp \left\{ \sum_{n=0}^{r} \frac{1-t^{-n}}{n} \sum_{a=1}^{r} z_a^n p_a' \right\}
\end{align*}\]

with \( e_r(t^p) = \prod_{i=1}^{r} \frac{t^p_i}{t^p_i - 1} \) from (A.12) and \( p_a := \frac{1-t^{-a}}{a} n \frac{\partial}{\partial p_a} \). Here \( z_0 \) and \( p_a \) are formal variables and \( \oint \frac{dz}{2\pi i} f(z) \) denotes the constant term in \( f \). Note that \( H' \) is written by the second scalar product (2.4) as \( H' = e_r(t^p)(\varphi_+^r(z^{-1}), \varphi_-^r(z))' \). We also denote them as \( H(x), H'(x) \) and \( \varphi(z; x) \) if they act on the power sums \( p_n(x) \) in \( x \), but they are independent of the number of variables \( x_i \).

The Cauchy formula (2.3) for the Hall–Littlewood function leads to

\[
\varphi^r(z) = \sum_{i=1 \leq r} \frac{P_x(x(p); t^{-1}) P_x(z; t^{-1})}{P_x(z; t^{-1})} \sum_{\lambda \in \mathcal{I}_r} \ell(\lambda) \frac{P_x(z^{-1}; t^{-1}) P_x(x(p); t^{-1})}{P_x(z; t^{-1})}.
\]

(3.2)

Here \( P_x(x(p); t) \) is the Hall–Littlewood function in terms of the power sums \( p \) and \( \ell(\lambda) := \lambda'_i \) is the length of \( \lambda_i \). Thus, \( H' \) is also realized by the Hall–Littlewood function as

\[
H' = e_r(t^p) \sum_{i=1 \leq r} P_x(x(p); t^{-1}) \frac{t^{[\lambda]}(P_x, P_x)_{1 \leq r}^n}{(P_x, P_x)^2_{1 \leq r}} P_x(x(p); t^{-1}).
\]

(3.3)

Then, we have \([16] \) (\([15] \)) for \( r = 1 \)^5

**Theorem 3.1.** The Macdonald function \( P_x(x; q, t) \) is an eigenfunction for \( H' \):

\[
HP_x(x; q, t) = P_x(x; q, t) E_x,
\]

\[
H' P_x(x; q, t) = P_x(x; q, t) e_x(q^t t^p),
\]

\[
E_x := \exp \left\{ - \sum_{n=0}^{\infty} \frac{(-w)^n}{n} p_n(q^t t^p) \right\} = \sum_{r \geq 0} w^r e_x(q^t t^p),
\]

(3.4)

and therefore, \( H' \) commute with each other \([H', H'] = 0 \) on the space of symmetric functions.

The proof is given by comparing \( H' \) with the Macdonald operators.

3.2. Action on the Cauchy kernel

Second, we discuss the properties of \( H' \), which we will use in subsection 4.3. For the Cauchy kernel \( P(x, y; q, t) \) in (A.5), we obtain

\[
\frac{\varphi^r(z) \Pi(x, y; q, t)}{\Pi(x, y; q, t)} = \exp \left\{ \sum_{n=0}^{r} \frac{1-t^{-n}}{n} \left( p_n(x) p_n(z) + p_n(ty) p_{-n}(z) \right) \right\},
\]

(3.5)

^5 Our definition is slightly different from \([16] \). Ours is not symmetric in \( z_a \)'s, and also the integration contour may be different. Thus, we give our proof in appendix D.
with $p_n(z) = \sum_{\alpha=1}^{r} \alpha^\alpha (\alpha \in \mathbb{Z})$. Since
\[
\Delta(z; t^{-\rho}) \Phi(z; x) \Pi(x, y; q, t) = \Delta(t \bar{z}^{-1}; t^{-1}) \Phi(t \bar{z}^{-1}; y) \Pi(x, y; q, t),
\]
(3.6)
with $\bar{z} := (z_2, z_{r-1}, \ldots, z_1)$, the following important duality holds for any variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, even though their number of components are different:
\[
H'(x) \Pi(x, y; q, t) = H'(y) \Pi(x, y; q, t).
\]
(3.7)
In a way similar to (3.2) and (3.3), we obtain
\[
\frac{\Phi'(z; x) \Pi(x, y; q, t)}{\Pi(x, y; q, t)} = \sum_{\alpha=1}^{r} P_{\alpha}(x(p); t^{-1}) P_{\alpha}(z; t^{-1}) \sum_{\beta=1}^{r} P_{\beta}(z^{-1}; t^{-1}) P_{\beta}(y(p); t^{-1}),
\]
(3.8)
\[
\frac{H'(x) \Pi(x, y; q, t)}{\Pi(x, y; q, t)} = e_r(t^\rho) \sum_{\alpha=1}^{r} P_{\alpha}(x(p); t^{-1}) P_{\alpha}(y(p); t^{-1}).
\]
(3.9)
When $x = t^{-\rho}$, we have\footnote{This proposition was proved in [14] by assuming $|t| < 1$. However, here we do not have to assume it.}

**Proposition 3.2.**
\[
H'(x) \Pi(x, y; q, t) \bigg|_{x = t^{-\rho}} = g_V e_r(y, t^{-\rho}),
\]
(3.10)
with $\sum_{n \geq 0} (-w)^n e_r(x, y) := \exp \left\{ - \sum_{n \geq 0} p_n(x, y) w^n/n \right\}$ and $g_V := (-1)^r t^{-\sum_{i=1}^{r} \phi_i}$. 

**Proof.** From (3.9) and $P_{\alpha}(t^{\rho}; t) = \delta_{\alpha, 1} e_r(t^\rho)$, it follows that
\[
lhs = e_r(t^\rho) \sum_{\alpha=1}^{r} P_{\alpha}(t^{-\rho}; t^{-1}) (P_{\alpha}, P_{\alpha})_{1-t^{-1}} (P_{\alpha}, P_{\alpha})_{1-t^{-1}} (P_{\alpha}, P_{\alpha})_{1-t^{-1}}.
\]
(3.11)
But from (2.6) and (A.12), we obtain
\[
lhs = (-1)^r \prod_{i=1}^{r} t^{1-i} \sum_{j=0}^{r} e_{r-j}(t^{-\rho}) e_r(y).
\]
(3.12)
Then, $\sum_{i=0}^{r} e_{r-i}(x) e_r(y) = e_r(x, y)$ proves the proposition. \hfill $\square$

### 3.3. Product of Macdonald operators

Next we consider the product of the $H''$’s which we will use in subsection 4.4. For a partition $\lambda$, let $\ell := \ell(\lambda)$, $H'' := H'' H'' \cdots H''$, $\bar{z} := (z_1', z_2', \ldots, z_{\ell}'$) and $z := (z_1, z_2, \ldots, z_{\ell}) := (z_1', z_2', \ldots, z_1, z_2, \ldots, z_{\ell}');$ then,
\[
\Delta(z; t) = \exp \left\{ - \sum_{n \geq 0} \frac{1 - t^n}{n} \sum_{i < j} p_n(z^i) p_{-n}(z^j) \right\} \prod_{i=1}^{\ell} \Delta(z^i; t).
\]
(3.13)
By the OPE relations (as difference operators)
\[
\Phi'(w) \Phi'(z) = \exp \left\{ \sum_{n \geq 0} \frac{(1 - t^{-n}) (q^n - 1)}{n} p_{-n}(w) p_n(z) \right\} \Phi'_L(w) \Phi'_r(z) \Phi'_L(z),
\]
(3.14)
with \( z = (z_1, \ldots, z_r) \) and \( w = (w_1, \ldots, w_s) \), we obtain
\[
\frac{\phi_{\lambda}(z'; x) \cdot \phi_{\lambda}(z^1; x) \Pi(x, y; q, t)}{\Pi(x, y; q, t)}
= \exp \left\{ \sum_{n=0}^{N} \frac{1 - t^{-n}}{n} \left\{ p_n(x)p_n(z) + p_n(y)p_{-a}(z) + (q^n - 1) \sum_{i<j} p_n(qz_i) p_{-a}(z') \right\} \right\}
= \exp \left\{ \sum_{n=0}^{N} \frac{1 - t^{-n}}{n} \left\{ p_n(x)p_n(z) + p_n(y)p_{-a}(z) + \sum_{i<j} p_n(qz_i) p_{-a}(z') \right\} \right\}
\times \frac{\Delta(z; t^{-1})}{\prod_{a=1}^{\ell} \Delta(z^a; t^{-1})}.
\] (3.15)

\( \text{When} \ (x, y) = (t^\rho, ct^{-\rho}) \) with \( c \in \mathbb{C}, \) since \( p_n(t^\rho) = (t_i^2 - t_i^{-2})^{-1} \) it follows that
\[
\prod_{a=1}^{\ell} \Delta(z^a; t^{-1}) \times \frac{\phi_{\lambda}(z'; x) \cdot \phi_{\lambda}(z^1; x) \Pi(x, ct^{-\rho}; q, t)}{\Pi(x, ct^{-\rho}; q, t)} \bigg|_{x=t^\rho}
= \Delta(z; t^{-1}) \exp \left\{ \sum_{n=0}^{N} \frac{1 - t^{-n}}{n} \left\{ p_n(qz_i) p_{-a}(z') \right\} \prod_{a=1}^{\ell} \frac{1 - t_{-a}^2 z_{a}^\rho}{1 - t_{-a}^{-2} z_{a}} \right\},
\] (3.16)
which is a polynomial of degree \( |\lambda| \) in \( c. \) For abbreviation, we write \( 1/(1-z) \) instead of \( \sum_{n \geq 0} z^n. \) Therefore, by taking the constant term in \( z, \) we have

**Proposition 3.3.**
\[
H^\rho(x) \Pi(x, ct^{-\rho}; q, t) \bigg|_{x=t^\rho}
= \frac{\prod_{a=1}^{\ell} dz_a}{2\pi i z_a} \frac{1 - c/z_a}{1 - z_a} \prod_{a < \beta} \frac{1 - z_a/z_\beta}{1 - z_a/t z_\beta} \prod_{i<j} \prod_{a=1}^{\ell} \prod_{b=1}^{\ell} \frac{1 - q_{a\beta}^i / t z_{\beta}^j}{1 - q_{a\beta}^i / z_{\beta}^j},
\] (3.17)
is a polynomial of degree \( |\lambda| \) in \( c \) and a polynomial in \( q \) and \( 1/t \) with integer coefficients and vanishes when \( c = 1. \)

**Proof.** The rhs of (3.17) reduces to
\[
\prod_{a=1}^{\ell} \frac{dz_a}{2\pi i z_a} \left( 1 - c/z_a \right) \sum_{n=0}^{a} (z_a)^n \times \prod_{a < \beta} \left( 1 - z_a/z_\beta \right) \sum_{n=0}^{a} \left( z_a / t z_\beta \right)^n \times \prod_{i<j} \prod_{a=1}^{\ell} \prod_{b=1}^{\ell} \left( 1 - q_{a\beta}^i / z_{\beta}^j \right) \sum_{n=0}^{[\lambda]} \left( q_{a\beta}^i / z_{\beta}^j \right)^n,
\] (3.18)
with \( |z_{\beta}^j| := \alpha + \sum_{k=1}^{j-1} \lambda_k, \) so this is a polynomial in \( q \) and \( 1/t \) with integer coefficients. When \( c = 1, \) since \( (1 - 1/z_a) \sum_{n \geq 0} (z_a)^n = 1/z_a, \) the integrand in (3.17) has no constant term in the last variable \( z_{\beta}^j; \) thus, (3.17) vanishes.

Let
\[
\prod_{j=1}^{\lambda_1} (1 - c/t_{j-1}) := \frac{H^\rho(x) \Pi(x, ct^{-\rho}; q, t) \bigg|_{x=t^\rho}}{\prod_{a=1}^{\ell} \Pi(x, ct^{-\rho}; q, t) \bigg|_{x=t^\rho}}.
\] (3.19)
Then, for example,
\[ \tilde{Z}^{(3)} = 1, \]
\[ \tilde{Z}^{(2,1)} = (1 - cq) - c(1 - q)/t^2; \]
\[ \tilde{Z}^{(1,1,1)} = (1 - cq)(1 - cq^2) - c(1 - q)(1 + 2q)/t - c(1 - q)^2/t^2 + c^2(1 - q)^2/t^3. \]

Direct calculation of the functions \( \tilde{Z}^\lambda \) for \( |\lambda| \leq 7 \) suggests the following conjecture.

**Conjecture 3.4.** If \( c = t^N, N \in \mathbb{Z} \) with \( 0 \leq N < \lambda_1 \), then (3.17) vanishes.

4. Homological link invariants from GIKV conjecture

In this section, we introduce the conjecture on homological link invariants of the colored Hopf link by Gukov, Iqbal, Köhler, and Vafa [7] and prove some of them.

4.1. GIKV conjecture

First, we recall the GIKV conjecture and present our main theorem. Following [7] let us consider

\[ Z_{\lambda, \mu}(Q; q, t) := \sum_{\eta} s_\lambda(q^\eta t^\rho)s_\mu(q^\eta t^\rho) \prod_{i,j \in \eta} Q \left( 1 - q^{n-i+1}t^{\rho_i-i} \right) \left( 1 - q^{n+j-\rho_j+i+1} \right). \]  

(4.1)

Here, \( s_\lambda(q^\eta t^\rho) \) is the Schur function in the power sum \( p_\lambda(q^\eta t^\rho) \). From (A.10) and (A.8), it is written by the Macdonald functions \( P_\lambda(x; q, t) \) defined in appendix A as follows:

\[ Z_{\lambda, \mu}(Q; q, t) = \sum_{\eta} (-v^{-1} Q)^{|\eta|} P_\eta(q^\eta; q) P_\mu(q^\eta; q) s_\lambda(q^\eta t^\rho) s_\mu(q^\eta t^\rho) \]
\[ = \sum_{\eta} \frac{Q^{|\eta|}}{[P_\eta, P_\mu]_{q,t}} P_\eta(t^{-\rho}; q, t) P_\mu(t^\rho; q, t) s_\lambda(q^\eta t^\rho) s_\mu(q^\eta t^\rho). \]  

(4.2)

Note that \( Z^{\text{inst}}(Q; q, t) = \Pi(Q^\rho, t^{-\rho}; q, t) \). Then, \( Z^{\text{inst}}_{\lambda, \mu}(Q; q, t) := Z_{\lambda, \mu}(Q; q, t)/Z^{\text{inst}}_\ast(Q; q, t) \) is a power series in \( Q \) and a meromorphic function in \( q \) and \( t \). But \( Z^{\text{inst}}_{\lambda, \mu}(Q; q, t) \) is expected to give the superpolynomial of the homological invariants and Gukov, Iqbal, Köhler, and Vafa claimed the following results.

**Conjecture 4.1** [7].

1. \( Z^{\text{inst}}_{\lambda, \mu}(Q; q, t) \) is a finite polynomial in \( Q \).
2. \( Z^{\text{inst}}_{\lambda, \mu}(t^N; q, t) \) vanishes for sufficiently small \( N \in \mathbb{Z}_{\geq 0} \).
3. \( Z^{\text{inst}}_{\lambda, \mu}(t^N; q, t)(-1)^{|\lambda|+|\mu|} (N \in \mathbb{Z}_{\geq 0}) \) is a finite polynomial in \( q \) and \( t \) with integer coefficients.
4. \( Z^{\text{inst}}_{\lambda, \mu}(t^N; q, t)(-1)^{|\lambda|+|\mu|} \) for sufficiently large \( N \) coincides with the \( \text{sl}(N) \) homological invariants of the Hopf link colored by the representations \( \lambda \) and \( \mu \) up to an overall factor.

Note that if conjecture 4.1.1 is established, one can easily calculate \( Z^{\text{inst}}_{\lambda, \mu}(t^N; q, t) \) for any given \( (\lambda, \mu; N) \) and check (2) and (3). In the following subsections, we will show part of the GIKV conjecture by proving a series of propositions.\(^7\)

\(^7\) Our \( (Q, G^{\text{GKV}}(Q; q, t; a); Q; q, t) \) equals \( (Q, \sqrt{Q/q}, Q; q^{-1}; q, t) \) of [7]. In [7], \( \tilde{P}_{\lambda, \mu}(a; q, t) = (-a)^{|\lambda|+|\mu|} t^{|\lambda|+|\mu|} G_{\lambda, \mu}(-Q^{\text{GKV}}; q, t) \) and \( G_{\lambda, \mu}(-Q^{\text{GKV}}; q, t) = Z^{\text{inst}}_{\lambda, \mu}(Q; q, t) \), where \( (a; q, t) := (Q^{-1}, t^{-1}, -(t/q)) \). Note that \( a = q^{\rho} \) and \( t = -1 \) are equivalent to \( Q = t^N \) and \( q = t \), respectively.

\(^8\) In [14], we have shown (ii)–(iv) by assuming the condition \( |l| < 1 \).
Theorem 4.2.

(i) Conjecture (1) holds for arbitrary $(\lambda, \mu)$ (proposition 4.10).
(ii) Conjecture (2) holds for arbitrary $(\lambda, \mu)$ with $|\lambda| + |\mu| \leq 7$ or $\mu = 1'$ (propositions 4.10 and 4.8).
(iii) Conjecture (3) holds for $(\lambda, \mu) = (\lambda, 1')$ with $|\lambda| \leq 7$ or $\lambda = 1'$ (propositions 4.8 and 4.6).
(iv) Conjecture (4) holds for $(\lambda, \mu) = (1', 1)$ (proposition 4.6).

4.2. $q = t$ case

When $q = t$, we have

Proposition 4.3.

$$Z_{\lambda, \mu}(Q; q, q) \over Z_{\bullet, \bullet}(Q; q, q) = f_{\lambda} f_{\mu} s_\lambda(Q q^\rho, q^{-\rho}) s_\mu(Q q^\lambda q^-\rho, q^{-\rho}),$$

which is a polynomial of degree $|\lambda| + |\mu|$ in $Q$. Here, $f_\lambda := \prod_{(i,j)\in \lambda} (-1)^{q^{\lambda_i} q^{-\lambda_j} + i - j}$.

Proof. First we have the following nontrivial identity:

$$s_\lambda(q^\rho) s_\mu(q^{\lambda+\rho}) = \sum_{\nu} q^{|\nu|} s_{\lambda/\nu}(q^{-\rho}) s_{\mu/\nu}(q^{-\rho}) = s_\mu(q^\rho) s_\lambda(q^{\lambda+\rho}),$$

which is proved by $s_{\lambda/\nu}(q^{-\rho}) = s_\lambda(q^{-\rho}) = -s_\lambda(q^{-\rho})$ and the cyclic symmetry of the topological vertex [6]. Since the above identity and $s_\lambda(q^\rho) = f_\lambda (-q^\rho)$, it follows that

$$Z_{\lambda, \mu}(Q; q, q) = f_{\lambda} f_{\mu} \sum_{\nu} \Pi(Q q^{-\rho}, q^{-\rho}; q, q) s_\nu(q^{\rho}).$$

The Cauchy formula (A.5) and the adding formula (A.4) yield

$$Z_{\lambda, \mu}(Q; q, q) = f_{\lambda} f_{\mu} s_\mu(Q q^\lambda q^{-\rho}, q^{-\rho}) \Pi(Q q^\lambda q^{-\rho}, q^{-\rho}; q, q) s_\lambda(q^\rho).$$

But ((2.12) and (5.20) of [21])

$$\frac{\Pi(Q q^\rho, q^{-\rho}; q, q)}{\Pi(Q q^{\lambda+\rho}, q^{-\rho}; q, q)} = f_{\lambda} \frac{s_\lambda(q^\rho)}{s_\lambda(q^\rho)},$$

which completes the proof.

Note that for $N \in \mathbb{Z}$ and $N \geq \ell(\lambda)$, $p_\mu(q^{\lambda+\rho}, q^{-\rho}) = \sum_{i < N} q^{\rho(\lambda_i + i - \frac{1}{2})}$. Thus $s_\mu(q^{\lambda+N+\rho}, q^{-\rho})$ is the Schur polynomial in $N$ variables $[q^{\rho(i+\frac{1}{2})}]_{i < N}$, which is a polynomial in $q$ with non-negative integer coefficients and vanishes for $\ell(\lambda) \leq N < \ell(\mu)$. Therefore, when $Q = q^N$, we have

Proposition 4.4. If $N \in \mathbb{Z}_{\geq 0}$,

$$(-1)^{|\lambda| + |\mu|} \frac{Z_{\lambda, \mu}(Q^N; q, q)}{Z_{\bullet, \bullet}(q^N; q, q)} = (-1)^{|\lambda| + |\mu|} f_{\lambda} f_{\mu} s_\mu([q^{\lambda_i + i - \frac{1}{2}}]_{1 < i < N}) s_\lambda([q^{\lambda_i + i - \frac{1}{2}}]_{1 < i < N}),$$

which is a polynomial in $q$ with non-negative integer coefficients and vanishes for $0 \leq N < \max(\ell(\lambda), \ell(\mu))$. This coincides with the colored Hopf link invariant by [18] up to the overall factor$^9$.

$^9$ This was shown in [7] for $N \rightarrow \infty$. 

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Proof. If \( 0 \leq N < \ell(\lambda) \), \( s_r(q^{N+\rho}, q^{-\rho}) = 0 \). If \( \ell(\lambda) \leq N < \ell(\mu) \), \( s_r(q^{N+\rho}, q^{-\rho}) = 0 \). By the symmetry \( Z_{j,\lambda}(Q; q, q) = Z_{j,\mu}(Q; q, q) \), we have \( Z_{j,\lambda}(Q; q, q) = 0 \) for \( 0 \leq N < \max\{\ell(\lambda), \ell(\mu)\} \). On the other hand, if \( N \geq \max\{\ell(\lambda), \ell(\mu)\} \), \( s_r(q^{N+\rho}, q^{-\rho}) \) is the Schur polynomial in \( N \) variables \( \{q^{N+\rho_i}; 1 \leq i \leq N\} \), which is a polynomial in \( q \) with non-negative coefficients. \( \square \)

When \( q \neq t \), it is difficult to calculate \( Z_{j,\lambda}(Q; q, t) \) explicitly for lack of the cyclic symmetry of the refined topological vertex [7].

4.3. \((\lambda, \mu) = (1^r, 1^s)\) case

For \( \lambda = 1^r \) and \( \mu = 1^s \), we have

**Proposition 4.5.**

\[
\frac{Z_{1^r,1^s}(Q; q, t)}{Z_{1^s,1^r}(Q; q, t)} = g_{1^r}g_{1^s}e_r(Q^\rho; t^{-\rho})e_s(Q^{\rho}; t^{-\rho}),
\]

(4.9)

which is a polynomial in degree \( r + s \) in \( Q \). Here, \( g_I := (-1)^t t^{-\binom{r-1}{2}} \).

**Proof.** Since \( s_I(x) = P_I(x; q, t) = e_I(x) \), it follows that

\[
Z_{1^r,1^s}(Q; q, t) = \sum_{\eta} \frac{Q^{\eta}}{(P_{\eta}, P_{\eta})_{q,t}} P_{\eta}(t^{-\rho}; q, t) P_I(t^{\rho}; q, t) P_I(q^\rho; q, t) e_I(q^{\rho} t^\rho).
\]

The symmetry \( P_I(t^{\rho}; q, t) P_I(q^\rho; q, t) = P_I(t^{\rho}; q, t) P_I(q^\rho; q, t) \) in (A.11) leads to

\[
Z_{1^r,1^s}(Q; q, t) = P_I(t^{\rho}; q, t) \sum_{\eta} \frac{Q^{\eta}}{(P_{\eta}, P_{\eta})_{q,t}} P_{\eta}(t^{-\rho}; q, t) P_I(q^{-\rho}; q, t) e_I(q^{\rho} t^\rho).
\]

(4.10)

By (3.4), we can replace \( e_I(q^{\rho} t^\rho) \) by \( H^\rho \) as follows:

\[
Z_{1^r,1^s}(Q; q, t) = P_I(t^{\rho}; q, t) \sum_{\eta} \frac{1}{(P_{\eta}, P_{\eta})_{q,t}} P_{\eta}(Q q^{\rho^\prime} t^{\rho}; q, t) H^\rho P_{\eta}(x; q, t) |_{x = e^{-\rho}}.
\]

(4.11)

The Cauchy formulas (A.5) and (3.10) yield

\[
Z_{1^r,1^s}(Q; q, t) = P_I(t^{\rho}; q, t) H^\rho \Pi(x, Q q^{\rho^\prime} t^{\rho}; q, t) |_{x = e^{-\rho}}
\]

\[
= g_I P_I(t^{\rho}; q, t) \Pi(t^{-\rho}; Q q^{\rho^\prime} t^{\rho}; q, t) e_I(Q q^{\rho^\prime} t^{\rho}; t^{-\rho}).
\]

(4.12)

But ((2.12) and (5.20) of [21])

\[
\frac{\Pi(Q q^{\rho^\prime} t^{\rho}; t^{-\rho}; q, t)}{\Pi(t^{\rho}; q, t)} = g_I \frac{P_I(Q q^{\rho^\prime} t^{\rho}; q, t)}{P_I(t^{\rho}; q, t)},
\]

(4.13)

which completes the proof. \( \square \)

Note that for \( N \in \mathbb{Z} \) and \( N \geq \ell(\lambda) \), \( p_n(q^{N+\rho}, t^{-\rho}) = \sum_{i=1}^N q^{n \rho_i + \binom{N+1}{2}} \). Thus, \( e_r(q^{N+\rho}, t^{-\rho}) \) is the elementary symmetric polynomial in \( N \) variables \( \{q^{N+\rho_i}; 1 \leq i \leq N\} \), which is a polynomial in \( q \) and \( t \) with non-negative integer coefficients and vanishes for \( \ell(\lambda) \leq N < r \). Therefore, when \( Q = t^N \), we have

**Proposition 4.6.** If \( N \in \mathbb{Z}_{\geq 0} \),

\[
(-1)^r \frac{Z_{1^r,1^s}(Q^N; q, t)}{Z_{1^s,1^r}(Q^N; q, t)} = (-1)^r g_{1^r}g_{1^s}e_r([t^{\rho}_i]_{1 \leq i \leq N}) e_s([q^{\rho_i}]_1^{N+\rho_i} [t^{\rho_i}]_{1 \leq i \leq N}).
\]

(4.14)

which is a polynomial in \( q \) and \( t \) with non-negative integer coefficients and vanishes for \( 0 \leq N < \max\{r, s\} \) (theorem 4.2 (iii), \( \lambda = 1^r \) case).
proposed that the homological invariants for the colored Hopf link are written as \(^{10}\)

\[ \text{Proposition 4.7.} \]

For \((\lambda, \mu) = (1', 1')\),

\[ \bar{T}^{\text{col}}(Q; q^{-1}, t^{-1}) = \bar{c}_{\lambda, \mu}(t^\rho) s_\mu(Qq^\rho t^\rho, t^{-\rho}) N_{\lambda, *}(Q; q, t), \]

which coincides with (4.9) up to the overall factor \((-1)^{s+2}Q^{(r+s)/2}v^{-2rs}\).

4.4. \((\lambda, \mu) = (\lambda, 1')\) case

Since the set of the Macdonald functions \(\{P_\lambda(x; q, t)\}_{\lambda \in \mathbb{P}}\) is a basis of the homogeneous symmetric functions of degree \(d\), we can write the Schur function \(s_\lambda(x)\) by the Macdonald functions as

\[ s_\lambda(x) = \sum_{\mu \subseteq \lambda} U_{\lambda, \mu} P_\mu(x; q, t), \quad U_{\lambda, \mu} := \sum_{\nu(\lambda, \beta) \subseteq \rho(\mu, \delta)} u_{\lambda, \mu}(q, q) (u^{-1})_{\lambda, \mu}(q, t), \]

where \(u_{\lambda, \mu}(q, t)\) is defined by (A.3). Note that \(U_{\lambda, \mu}\) is a rational function in \(q\) and \(t\). Then, we have

**Proposition 4.7.**

\[ \frac{Z_{\lambda, 1'}(Q; q, t)}{Z_{\lambda, *}(Q; q, t)} = g_1^{1'} \sum_{\mu \subseteq \lambda} U_{\lambda, \mu} g_\mu P_\mu(Qq^\rho t^\rho, t^{-\rho}) e_\lambda(Qq^\rho t^\rho, t^{-\rho}), \]

which is a polynomial of degree \(|\lambda| + s\) in \(Q\). Here, \(g_\lambda := \prod_{(i,j) \in \lambda}(1 - q_i^{\lambda_i - j} t_j^{-\lambda_j + i})\).

**Proof.** First we have

\[ Z_{\lambda, 1'}(Q; q, t) = \sum_{\mu \subseteq \lambda} U_{\lambda, \mu} \tilde{Z}_{\mu, 1'}(Q; q, t), \]

where

\[ \tilde{Z}_{\mu, 1'}(Q; q, t) := \sum_{\eta} \frac{Q^{|\eta|}}{(P_\eta, P_\eta)_q} P_\eta(t^{-\rho}; q, t) P_\eta(t^\rho; q, t) P_\eta(q^\rho t^\rho; q, t) s_{1'}(q^\rho t^\rho). \]

Then, (A.11) leads to

\[ \tilde{Z}_{\mu, 1'}(Q; q, t) = \sum_{\eta} \frac{Q^{|\eta|}}{(P_{\eta+1}, P_{\eta+1})_q} P_\eta(t^{-\rho}; q, t) P_\eta(t^\rho; q, t) P_\eta(q^\rho t^\rho; q, t) s_{1'}(q^\rho t^\rho). \]

\(^{10}\) Our \((q, t) = (q^{-1}, t^{-1})\) of [20].
We can proceed in the same way as the last subsection. Namely, \( s_t(x) = e_s(x) \), and (3.4) and (3.10) yield
\[
\tilde{Z}_{\mu, \nu}(Q; q, t) = P_{\mu}(t^\rho; q, t) \sum_{tN} \frac{Q^n}{(P_{\mu}, P_{\nu})_{q,t}} P_{\nu}(q^\mu t^\rho; q, t) H^\nu P_{\eta}(x; q, t)|_{x=\rho\gamma}.
\]

But from ((2.12) and (5.20) of [21])
\[
\frac{\Pi(\mu, t^{-\rho}; q, t)}{\Pi(t^{\rho}, t^{-\rho}; q, t)} = g_{\mu} P_{\mu}(t^\rho, t^{-\rho}; q, t),
\]
we conclude that
\[
\frac{\tilde{Z}_{\mu, \nu}(Q; q, t)}{Z_{\nu}(Q; q, t)} = g_{\nu} e_s(t^{i^2}) \sum_{\mu \leq \lambda} U_{\lambda, \mu} g_{\mu}(t^{i^2}) e_s(q^\mu t^{i^2}).
\]

Note that for \( N \in \mathbb{Z} \) and \( N \geq \ell(\lambda) \), \( P_{\mu}(q^\mu t^{N+\rho}, t^{-\rho}) \) is the Macdonald polynomial in \( N \) variables \( \{q^\mu t^{N+\rho}\} \) and vanishes for \( \ell(\lambda) < N < \ell(\mu) \). Therefore, when \( Q = t^\rho \), we have

**Proposition 4.8.** If \( N \in \mathbb{Z}_{\geq 0} \),
\[
\frac{Z_{\mu, \nu}(t^N; q, t)}{Z_{\nu}(t^N; q, t)} = g_{\nu} e_s(t^{i^2}) \sum_{\mu \leq \lambda} U_{\lambda, \mu} g_{\mu}(t^{i^2}) e_s(q^\mu t^{i^2}),
\]
which vanishes for \( 0 < N < \max(\ell(\lambda), s) \) (theorem 4.2 (ii), \( \mu = 1^s \) case).

Finally, we should make a remark on the fact that the transition function \( U_{\lambda, \mu}(q, t) \) in (4.19) is a rational function in \( q \) and \( t \). It is not obvious that the formulas in the above propositions are in fact polynomials in \( q \) and \( t \). However, we have checked the following conjecture up to \( d = 7 \) by direct calculation.

**Conjecture 4.9.** For \( |\lambda| = d, \sum_{\mu, \nu} U_{\lambda, \mu} g_{\mu}(U^{-1})_{\mu, \nu} \) is a polynomial with integer coefficients of degree \( d(d-1)/2 \) in \( q \) and of degree \( d(d-1)/2 \) in \( t^{-1} \).

On the assumption that the above conjecture is true, if \( N \in \mathbb{Z}_{\geq 0} \) then
\[
\frac{Z_{\mu, \nu}(t^N; q, t)}{Z_{\nu}(t^N; q, t)} = g_{\nu} e_s(q^{\mu + \rho}, t^{-\rho}) \sum_{\mu \leq \lambda} U_{\lambda, \mu} g_{\mu}(U^{-1})_{\mu, \nu} e_s(q^\mu t^{\mu + \rho}, t^{-\rho})
\]
is a polynomial in \( q \) and \( t \) with integer coefficients because \( s_t(x) \) is a function in \( x \) with non-negative integer coefficients (theorem 4.2 (iii), \( |\lambda| \leq 7 \) case).

4.5. General \( (\lambda, \mu) \) case

From (A.1) and (A.3), we have
\[
s_t(x) = \sum_{\mu \leq \lambda} V_{\lambda, \mu} e_{\mu^1}(x), \quad V_{\lambda, \mu} := \sum_{\nu(\lambda \triangleright \nu \triangleright \mu)} u_{\lambda, \nu}(a^{-1})_{\nu, \mu},
\]
where \( u_{\lambda, \mu} := u_{\lambda, \mu}(q, a) \) and \( a_{\lambda, \mu} \) is defined in (A.1). Note that \( |\mu^\circ| = |\lambda| \) in the above equation. More precisely, we have the Jacobi–Trudy formula \( s_t(x) = \det(e_{\mu^1}(x))_{1 \leq i, j \leq k} \) with \( e_{-r} = 0 \) for \( r > 0 \). Then, we have
Proposition 4.10.

\[ \frac{Z_{\lambda,\mu}(Q; q, t)}{Z_{\star,\star}(Q; q, t)} = \sum_{\sigma \in \mathcal{S}} v_{\lambda,\mu,\sigma} \frac{H^{\nu}(x)H^{\rho}(x)\Pi(x, Q^{-\rho}; q, t)}{\Pi(x, Q^{-\rho}; q, t)} |_{x=t^0}, \]  

where is a polynomial of degree \(|\lambda| + |\mu|\) in \(Q\) and vanishes when \(Q = 1\) (theorem 4.2 (i)).

Proof. (3.4) and (A.5) yield

\[ Z_{\lambda,\mu}(Q; q, t) = \sum_{\sigma \in \mathcal{S}} \left| \frac{Q^{[\sigma]}}{(P_{Q^1}, P_{Q^2})_{Q,t}} \right| P_{\eta}(t^{-\rho}; q, t) \sum_{\tau \in \mathcal{S}} V_{\lambda,\mu,\sigma} H^{\nu}(x)H^{\rho}(x)P_{\eta}(x; q, t) |_{x=t^0}. \]  

Then, the proposition in subsection 3.3 completes the proof. □

Although \(Z_{\lambda,\mu}(Q; q, t)\) is a power series in \(Q\), we need its partial sum with degree \(|\lambda| + |\mu|\) to calculate \(Z_{\lambda,\mu}(Q; q, t) / Z_{\star,\star}(Q; q, t)\).

On the assumption that the conjecture 3.4 is true, if \(Q = t^N, N \in \mathbb{Z}\) with \(0 \leq N < \max\{\ell(\lambda), \ell(\mu)\}\), then (4.30) would vanish (theorem 4.2 (ii), \(|\lambda| + |\mu| \leq 7\) case).

5. Toward a resolution of positivity problem

We now make a comment on the positivity problem of the GIKV’s proposal. Let \(a = q^N\) with \(N \in \mathbb{N}\), the superpolynomial of the homological invariants of the colored Hopf link reduces to \(\sum_{j=0}^{N} q^{j} U_{ij} H_{ij}^{(N)}\lambda,\mu\) with certain doubly graded homology \(H_{ij}^{(N)}\lambda,\mu\) [7]. Therefore, it should be by definition a polynomial in \(q\) and \(t\) with non-negative integer coefficients. However, in general, \(Z_{\lambda,\mu}^{\text{new}}(t^N; q, t)(-1)^{|\lambda|^+|\mu|}\) is not so. For example,

\[ -Z_{\lambda,\mu}^{\text{new}}(t^2; q, t) = q^3(t^6 + t^5) + q^2(t^5 - t^4) + q(t^5 + t^4). \]  

A solution to this positivity problem may be given by replacing the Schur function in (4.1) by the Macdonald function \(P_{\lambda}(z; \tilde{q}, \tilde{t})\) with \(\tilde{t} = 0\) and appropriately chosen \(\tilde{q}\). Let

\[ \tilde{Z}_{\lambda,\mu}(Q; q, t) := \sum_{\eta} (-v^{-1} Q)^{[\eta]} P_{\eta}(q^N; t, q)P_{\eta}(t^{-\rho}; q, t)P_{\eta}(q^N t^{-\rho}; \tilde{q}, 0) P_{\eta}(q^N t^{-\sigma}; \tilde{q}, 0), \]  

and \(\tilde{Z}_{\lambda,\mu}^{\text{new}}(Q; q, t) := \tilde{Z}_{\lambda,\mu}(Q; q, t) / Z_{\star,\star}(Q; q, t)\). Note that \(\tilde{Z}_{\lambda,\mu}^{\text{new}}(Q; q, t) |_{\tilde{q}=0} = Z_{\lambda,\mu}^{\text{new}}(Q; q, t)\) and \(Z_{\lambda,\mu}^{\text{new}}(Q; q, t) = Z_{\lambda,\mu}^{\text{new}}(Q; q, t)\), because of \(P_{\lambda}(x; 0, 0) = s_{\lambda}(x)\) and \(P_{\lambda}(x; \tilde{q}, \tilde{t}) = s_{\lambda}(x)\), respectively. Since by (A.1) and (A.3),

\[ P_{\lambda}(x; \tilde{q}, \tilde{t}) = \sum_{\mu \leq \lambda} \tilde{U}_{\lambda,\mu} P_{\mu}(x; q, t), \quad \tilde{U}_{\lambda,\mu} := \sum_{v \leq \lambda \geq u \geq \mu} u_{v,\mu}(\tilde{q}, \tilde{t})(a^{-1})_{v,\mu}(q, t), \]

all propositions in subsections 4.3 and 4.4 hold with these \(\tilde{U}_{\lambda,\mu}\) and \(\tilde{V}_{\lambda,\mu}\).

If we choose the parameter \(\tilde{q}\) appropriately, it may overcome the positivity problem. For example, when \(\tilde{q} = q\),

\[ -t^2 \tilde{Z}_{\lambda,\mu}^{\text{new}}(t^2; q, t) |_{\tilde{q}=q} = q^3(t^6 + t^5 + t^4) + q(q + 1)(t^5 + t^4). \]  

\[ 44 \]
More generally, for $Q = t^N$,
\[
-t^2 \sum^\text{int}_J(\tau, q, t) = \tilde{q}^3 \frac{[N]}{3} + \tilde{q}(\tilde{q} + 1) \left( qQ \frac{[N]}{2} + t^2 \frac{[N]}{3} \right) [2]_t \\
+ q^3 \tilde{Q} t^2 [N]_t + q^3 Q t^2 (t^2 - 1) \left( \frac{[N]}{3} + qQ \frac{[N]}{2} + t^3 \frac{[N]}{3} \right).
\] (5.5)

But if we choose $\tilde{q} = q$, then the negative coefficient in the $q^2$-term vanishes as
\[
-t^2 \sum^\text{int}_J(\tau, q, t)_{|q = q} = q^3 \left( Q^2 [N]_t + qQ \frac{[N]}{2} + t^2 \frac{[N]}{3} \right) t \\
+ q(q + 1) \left( Q \frac{[N]}{2} + t^2 \frac{[N]}{3} \right) [2]_t + t^3 \frac{[N]}{3}.
\] (5.6)

Note that the $q$-integer $[N] := \frac{1 - q^N}{1 - q}$ and the $q$-binomial coefficient $\binom{N}{r} := \prod_{i=0}^{r-1} \frac{1 - q^{N-i}}{1 - q^{-i}}$ are polynomials in $t$ with non-negative integer coefficients for $N, r \in \mathbb{N}$. We checked that $(-1)^{|\lambda| + |\mu|} \sum^\text{int}_J(\tau, q, t)_{|q = q} (N \in \mathbb{Z}_{\geq 0})$ is a polynomial in $q$ and $t$ with non-negative integer coefficients for $|\lambda| + |\mu| \leq 5$ (see appendix F of arXiv:0910.0083).

For non-antisymmetric representations, the specialization $\tilde{q} = q$ fails to solve the positivity problem. For example,
\[
\tilde{Z}^\text{int}_{\mu}(|q^2; q, t)_{|q = q} = q^3 t^2 [2]_t + q^3 t^3 (t^2 - 1) + q^2 t^3 (t^2 + 3t + 1) \\
+ q^2 t^3 (t^2 + 3t + 1) + q^2 t^3 (t^2 + 3t + 1) + q^2 t^3 (t^2 + 3t + 1) + q^2 t^3 (t^2 + 3t + 1)
\] (5.7)

However, if we choose $\tilde{q} = (1 + qt)q/t + p$, then
\[
\tilde{Z}^\text{int}_{\mu}(|q^2; q, t)_{|q = q} = q^3 t^2 [2]_t + q^3 t^3 (t^2 - 1) + q^2 t^3 (t^2 + 3t + 1) + q^2 t^3 (t^2 + 3t + 1) + q^2 t^3 (t^2 + 3t + 1)
\] (5.8)

We checked that $(-1)^{|\lambda| + |\mu|} \sum^\text{int}_J(\tau, q, t)_{|q = q} (N \in \mathbb{Z}_{\geq 0})$ is a polynomial in $q$, $t$ and $p$ with non-negative integer coefficients for $|\lambda| + |\mu| \leq 5$ (see appendix F of arXiv:0910.0083).

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Appendix A. Symmetric functions

Here we recapitulate basic properties of the symmetric functions in $x = (x_1, x_2, \ldots)$ [22]. The monomial symmetric function is defined by $m_s := \sum x_1^{s_1} x_2^{s_2} \cdots$, where the
summation is over all distinct permutations of \((\lambda_1, \lambda_2, \ldots)\). The power sum symmetric function is \(p_n := \sum_{i=1}^{\infty} t^i\). The elementary symmetric function defined by \(e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots\) and \(\sum_{\ell \geq 0} w^\ell e_\ell := \prod_{\mu \geq 1} (1 + x_\mu w) = \exp \left\{ - \sum_{n \geq 0} \frac{(-w)^n}{n} p_n \right\}\) enjoys
\[
e_\lambda = \sum_{\mu \leq \lambda} a_{\lambda\mu} m_\mu, \quad a_{\lambda\lambda} = 1, \quad a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}. \tag{A.1}\]

For any symmetric functions \(f\) and \(g\), in power sums \(p_n\)'s, we define a scalar product as
\[
\langle f(p), g(p) \rangle_{q,t} := f(p^n) g(p^{\ast})_{\text{constant part}} \text{ with } p_n^{\ast} := n! q^\ell t^{\rho}.
\]

The Macdonald symmetric function \(P_\lambda(x; q, t)\) is uniquely specified by the following orthogonality and normalization:
\[
\langle P_\lambda(x; q, t), P_\mu(x; q, t) \rangle_{q,t} = 0 \quad \text{if } \lambda \neq \mu, \tag{A.2}
\]
\[
P_\lambda(x; q, t) = \sum_{\mu \leq \lambda} u_{\lambda\mu}(q, t)m_\mu(x), \quad u_{\lambda\lambda}(q, t) = 1, \quad u_{\lambda\mu}(q, t) \in \mathbb{Q}(q, t). \tag{A.3}
\]

Here we used the dominance partial ordering on the Young diagrams defined as \(\lambda \triangleright \mu \Leftrightarrow |\lambda| = |\mu| \) and \(\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i\) for all \(i\). Note that \(P(x; q^{-1}, t^{-1}) = P(x, q, t)\). The scalar product is given by \(\langle P_\lambda(x; q, t), P_\mu(x; q, t) \rangle_{q,t} = \prod_{i,j \in \lambda} (1 - q^i t^j)^{-1}\). We abbreviate it to \(\langle P_\lambda, P_\mu \rangle_{q,t}\). The skew-Macdonald symmetric function \(P_{\lambda/\mu}(x; q, t)\) is defined by \(P_{\lambda/\mu}(x; q, t) := P_\lambda(x; q, t) P_\mu(x; q, t) / \langle P_\mu, P_\mu \rangle_{q,t}\). Let \(x = (x_1, x_2, \ldots)\) and \(y = (y_1, y_2, \ldots)\) be two sets of variables. Then, we have
\[
\sum_{\mu} P_{\lambda/\mu}(x; q, t) P_{\mu/\nu}(y; q, t) = P_{\lambda/\nu}(x, y; q, t), \tag{A.4}
\]
where \(P_{\lambda/\nu}(x; y; q, t)\) denotes the skew-Macdonald function in the set of variables \((x_1, x_2, \ldots, y_1, y_2, \ldots)\). The following Cauchy formula is especially important:
\[
\sum_{\lambda} \frac{\langle P_\mu, P_\lambda \rangle_{q,t}}{\langle P_\lambda, P_\lambda \rangle_{q,t}} P_{\lambda/\mu}(x; q, t) P_\lambda(y; q, t) = \Pi(x; y; q, t) P_\mu(y; q, t) \tag{A.5}
\]
with \(\Pi(x; y; q, t) := \exp \left\{ \sum_{n \geq 0} \frac{1 - e^n}{1 - q e^n} p_n(x) p_n(y) \right\}\).

We denote
\[
p_n(q^\ell t^\rho) := \sum_{i=1}^{\ell(\lambda)} (q^{\rho_i} - 1) n^{(i-1)} \frac{1}{t^i - t^{-i}} \sum_{i=1}^{N} q^{\rho_i} n^{(i-1)} \frac{t^{-nN}}{t^i - t^{-i}}. \tag{A.6}
\]
for any \(N \geq \ell(\lambda)\). Let \(p_n(x, y) := p_n(x) + p_n(y)\); then,
\[
p_n(e^\ell q^\rho, L t^{-\rho}) = e^n \sum_{i=1}^{\ell(\lambda)} (q^{\rho_i} - 1) n^{(i-1)} \frac{c^\mu - L^n}{t^i - t^{-i}}, \quad c, L \in \mathbb{C}. \tag{A.7}
\]

Note that \([10]\)
\[
P_{\mu/\nu}(-t^{-\rho}/q^\rho; t, q) = \frac{\langle P_{\mu/\nu}, P_\mu \rangle_{q,t}}{\langle P_\mu, P_\mu \rangle_{q,t}} P_{\mu}(q^{-\ell} t^{-\rho}; q, t). \tag{A.8}
\]

The Macdonald function in the power sums \(p_n = (1 - L^n) / (t^2 - t^{-2})\) is \([22]\) (chapter VI.6)
\[
P_\ell(t^\rho, L t^{-\rho}; q, t) = \prod_{(i,j) \in \lambda} (-1)^i q^{i-1} \frac{1 - Lq^{i-1} t^{i-1}}{1 - q^{i-1} t^{i-1}}. \tag{A.9}
\]
for a generic \( L \in \mathbb{C} \). Note that
\[
P_{\lambda}(t^\rho; q, t)P_{\nu}(q^\rho; t, q) = P_{\lambda}(t^{-\rho}; q, t)P_{\nu}(q^{-\rho}; t, q)
\]
where
\[
(q/t)^i \sum_{(i,j) \in \lambda} \frac{1}{(1 - q^{-1})} (1 - q^{i+j})^{\gamma^1 + j + 1}(1 - q^{i+j+1})\gamma^2 .
\] (A.10)

If \( L = t^{-N} \) with \( N \in \mathbb{N} \) and \( N \geq \ell(\lambda) \), then \( p_{\lambda}(q^s t^\rho, t^{-N-\rho}) = \sum_{n=1}^{\infty} q^{n^2 t^m} n^{\ell(\lambda)} \) is the power sum symmetric polynomial in \( N \) variables \( \{ q^{i+j} t^{-\ell} \} \in \mathbb{N} \); hence, \( p_{\lambda}(t^\rho, t^{-N-\rho}; q, t) \) reduces to the Macdonald symmetric polynomial in \( N \) variables. Therefore,
\[
P_{\lambda}(t^\rho, t^{-N-\rho}; q, t) = 0 \text{ for } \ell(\lambda) > N \in \mathbb{N}.
\]
For \( N \in \mathbb{N} \), there is a symmetry \([22]\) (chapter VI.6)
\[
P_{\lambda}(t^\rho, t^{-N-\rho}; q, t)P_{\mu}(q^s t^\rho, t^{-N-\rho}; q, t) = P_{\mu}(t^\rho, t^{-N-\rho}; q, t)P_{\lambda}(q^s t^\rho, t^{-N-\rho}, q, t).
\] (A.11)

The Hall–Littlewood and Schur functions are defined by \( P_{\lambda}(x; t) := P_{\lambda}(x; 0, t) \) and \( s_{\lambda}(x) := P_{\lambda}(x, q, t) \), respectively. Note that \( p_{\lambda}(x), s_{\lambda}(x) \) and \( e_{\lambda}(x) = P_{\lambda}(x, q, t) \) are symmetric functions in \( x \) with non-negative integer coefficients. Note that \( \sum_{x \in \mathbb{N}} e_{\sigma}(x) = \sigma(x, y) \).

For \( \lambda = 1^r \), (A.9) reduces to
\[
e_{\lambda}(t^\rho, L_l t^{-\rho}) = \prod_{i=1}^{\ell}(1 - t^{l-1})^{1 - L_l^{i-1}} = L_l^{i-1}
\] (A.12)

Note also that \( p_{\lambda}(t^\rho; t) = \delta_{\lambda,1} t^{(\ell(\lambda))} \). The \( q \)-integer \( [N]_q := \frac{1 - t^N}{1 - t} \) and the \( q \)-binomial coefficient \( \left[ \begin{array}{c} N \end{array} \right]_q := \prod_{i=1}^{N} \frac{1 - t^{N-q}}{1 - t} \) are polynomials in \( t \) with non-negative integer coefficients for \( N, r \in \mathbb{N} \).

Note that \( t^{-N} = [N]_q \) and \( t^{-N^2} e_s(t^{N^2+\rho}, t^{-\rho}) = t^{-N^2} \). \( t^{-N^2} e_s(t^{N^2+\rho}, t^{-\rho}) = t^{-N^2} \left[ \begin{array}{c} N \end{array} \right]_t \).

**Appendix B. Proof of (2.5)**

Here we prove (2.5). For \( \sigma \in S_N \), let \( d(\sigma) := \#\{(i, j)| i < j, \sigma(i) > \sigma(j)\} \) be the inversion number and let
\[
\Delta_{\sigma} := t^{d(\sigma)} \exp \left\{ \sum_{n=0}^{\infty} \frac{t^n}{n} \sum_{\sigma(i) > \sigma(j)} x_{\sigma(i)}^n \right\}.
\] (B.1)

Then,
\[
\Delta(\bar{x}; t)^{-1} \Delta_{\sigma} = t^{d(\sigma)} \exp \left\{ \sum_{n=0}^{\infty} \frac{1-t^n}{n} \left( \sum_{\sigma(i) > \sigma(j)} x_{\sigma(j)}^n \right) - t^{-n} \sum_{\sigma(i) > \sigma(j)} x_{\sigma(j)}^n \right\}.
\] (B.2)

which is a formal power series in \( \{x_{\sigma}/x_{\lambda}\}_{\lambda} \) and is equivalent to \( \sigma \Delta(\bar{x}; t)^{-1} = \prod_{\lambda(i) \leq \lambda(j)} 1 = \frac{1 - x_{\sigma(i)}/x_{\lambda(\sigma(i))}}{1 - x_{\sigma(j)}/x_{\lambda(\sigma(j))}} \) by the analytic continuation. Since \( P_{\lambda}(x; t) \) is a polynomial in \( x \), it follows that
\[
P_{\lambda}(x; t) = v_{\lambda}^{-1}(t) \Delta(\bar{x}; t)^{-1} \sum_{\sigma \in S_N} \Delta_{\sigma} \prod_{i=1}^{N} x_{\sigma(i)}^{v_{\lambda}(i)} \quad \text{and}
\]
\[
v_{\lambda}(t) P_{\lambda}(x; t) \Delta(\bar{x}; t) = \sum_{\lambda \neq \lambda} \sum_{\sigma \in S_N} u_{\lambda,\sigma}^{\lambda} \prod_{i=1}^{N} x_{\sigma(i)}^{v_{\lambda}(i)}, \quad u_{\lambda,\sigma}^{\lambda} = t^{d(\sigma)}.
\] (B.3)
Here \( v \) is a sequence of \( N \) integers \( v = (v_1, v_2, \ldots, v_N) \), \( v_i \in \mathbb{Z} \) and \( \geq \) is the dominance partial ordering defined as \( v \geq \lambda \iff \sum_i v_i = \sum_i \lambda_i \) and \( v_1 + \cdots + v_k \geq \lambda_1 + \cdots + \lambda_k \) for all \( k \). Thus, we have

\[
v_2(t)(P_n(x; t), x^\lambda)_{N,t}^{\mu} = 1,
\]

\[
v_2(t)\left( P_n(x; t), \prod_{i} x^t_{\lambda(i)} \right)_{N,t}^{\mu} = t^d(\sigma), \quad (B.5)
\]

\[
v_2(t)\left( P_n(x; t), \prod_{i} x^t_{\lambda(i)} \right)_{N,t}^{\mu} = 0, \quad \mu < \lambda.
\]

Therefore, by (A.3) we conclude that

\[
v_2(t)(P_n(x; t), P_\mu(x; t))_{N,t}^{\mu} = \delta_{\lambda,\mu}[N]. \quad (B.6)
\]

Here we use the identity \( \sum_{\sigma \in S_n} \delta(\sigma) = [N]! \), which is proved by the induction in \( N \). This completes the proof of (2.5).

**Appendix C. Macdonald operators**

Here we define (higher order) Macdonald operators which are compatible with tending the number of variables to infinity \([16, 23, 14]\). For each integer \( r \) such that \( 0 \leq r \leq N \), let \( D_N^r \) be the Macdonald operators in \( N \) variables \( x = (x_1, \ldots, x_N) \), \( D_N^0 := 1 \) and

\[
D_N^r := t^{r(r-1)/2} \prod_{k \neq l} \frac{1 - x_k x_l}{1 - x_k x_l} \prod_{i \in I} T_{q, x_i},
\]

summed over all \( r \)-element subsets \( I \) of \( \{1, 2, \ldots, N\} \). We set \( D_N^r := 0 \), \( r > N \). Here \( T_{q, x} \) is the \( q \)-shift operator such that \( T_{q, x} f(x) = f(qx) \). Let \( D_N(\tilde{w}) := \sum_{r=0}^{N} D_N^r \tilde{w}^r \); then, the Macdonald polynomial is the eigenfunction for \( D_N^r \):

\[
D_N(\tilde{w}) P_n(x; q, t) = P_n(x; q, t) \varepsilon_{n, \lambda},
\]

\[
\varepsilon_{n, \lambda} := \prod_{i=1}^{N} (1 + \tilde{w} q^{N-i} t^{N-i}) = \sum_{\rho} \tilde{w}^{\rho} e_\rho (q^r t^{N-\frac{1}{2}} t^{\frac{1}{2}} t^{-\rho}). \quad (C.2)
\]

Therefore, \( D_N^r \) are simultaneously diagonalized by the Macdonald polynomials

\[
D_N^r P_n(x; q, t) = P_n(x; q, t) e_\rho (q^r t^{N-\frac{1}{2}} t^{\frac{1}{2}} t^{-\rho});
\]

thus, \( D_N^r \) commute with each other \([D_N^r, D_N^s] = 0 \) on the space of the symmetric function in \( N \) variables. Note that \( D_N^r \) is not compatible with the restriction of the variables defined by setting \( x_N = 0 \):

\[
D_N^r |_{x_N=0} = t^r D_N^{r-1} + t^{r-1} D_N^{r-1},
\]

\[
D_N^r(\tilde{w}) |_{x_N=0} = (1 + \tilde{w}) D_N^{r-1}(t \tilde{w}). \quad (C.4)
\]

So we need to modify it to take \( N \to \infty \). By using \( p_n(t^{\frac{1}{n}}) = 1/(t^n - 1) \) and \( \exp \left\{ \sum_{n \geq 0} \frac{1}{n} \left( -\tilde{w} t \right)^n \right\} = \sum_{\rho \geq 0} \tilde{w}^{\rho} e_\rho (\tilde{w} t) \), let

\[
H_N := D_N \exp \sum_{n \geq 0} \frac{1}{n} \left( -\tilde{w} t \right)^n =: \sum_{\rho \geq 0} \tilde{w}^{\rho} H_N^\rho, \quad w := \tilde{w} t^{-\frac{1}{n}},
\]

\[
H_N^r = \sum_{s=0}^{\min(r, N)} t^s \times \rho \times e_{r-s}(t^{\rho}) D_N^s, \quad r = 0, 1, 2, \ldots. \quad (C.5)
\]
Then, by (A.6), we obtain
\[
E_{N,\lambda} := \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{(-\bar{w})^n}{1-t^n} \right\} E_{N,\lambda} \equiv \sum_{r \geq 0} w^r e_r(q^r t^\alpha),
\]
which is independent of $N$ for any $N \geq \ell(\lambda)$. Thus,
\[
H_N P_n(x; q, t) = P_n(x; q, t) E_{N,\lambda},
\]
\[
H_N' P_n(x; q, t) = P_n(x; q, t)e_r(q^r t^\alpha).
\]
(C.7)

**Appendix D. Proof of (3.4)**

Here we prove (3.4) by comparing $H'$ with $H_N'$. In this subsection, we suppose that the number of variables $x = (x_1, \ldots, x_N)$ is finite by setting $x_i = 0$, $i \geq N + 1$ and $p_n(x) = \sum_{i=1}^N x_i^n$, $n \in \mathbb{N}$. For each integer $r$, we denote $\tilde{H}_N' := t^r H'/e_r(t^\alpha)$,
\[
\tilde{H}_N' = \oint \prod_{a=1}^r \frac{dz_a}{2\pi i z_a} \prod_{a=1}^r \prod_{j=1}^N \frac{1}{1-x_j z_a} \frac{1}{1-x_j z_a} \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{\partial^{\alpha_n}}{\partial p_n} \right\}.
\]
(D.1)

Here, $z_a$ and $p_n$ are formal parameters. For abbreviation, we write $1/(1-z)$ instead of $\sum_{n \geq 0} z^n$ for the formal parameter $z$. Note that the operators $\tilde{H}_N'$ are compatible with the restriction of the variables defined by setting $x_N = 0$, $\tilde{H}_{N-1}' = t^{-r} \tilde{H}_N'|_{x_N=0}$. We also denote $\tilde{H}_{N-1,0}' := t^{-r} \tilde{H}_N'|_{x_N=0}$.

Instead of the rational function $(t x_i - x_j)/ (x_i - x_j)$, we use
\[
[i, j] := \frac{1-t x_i / x_j}{1-x_i / x_j} := (1-t x_i / x_j) \sum_{n \geq 0} (x_i / x_j)^n,
\]
\[
[j, i] := \frac{t-x_i / x_j}{1-x_i / x_j} := (t-x_i / x_j) \sum_{n \geq 0} (x_i / x_j)^n, \quad i > j.
\]
(D.2)

Then, we have the following recurrence relation.

**Lemma D.1.** Suppose $x_i \neq x_j$ if $i \neq j$. For any $N \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{N}$,
\[
\tilde{H}_N' = \tilde{H}_{N-1}' + t^{-1} (r-1) \sum_{i=1}^N \tilde{H}_{N-1,0}'\prod_{j \neq i} [i, j] T_{q,\lambda}.
\]
(D.3)

**Proof.** For $p_n = \sum_i x_i^n$, $n \in \mathbb{N}$, since $T_{q,\lambda} p_n = (q^n - 1) x_i^n + p_n$, we have for any function $f$ in $p$
\[
T_{q,\lambda} f(p(x)) = \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{\partial^{\alpha_n}}{\partial p_n} \right\} f(p(x))|_{z=x_i^{-1}}.
\]

The constant term in $z_r$ is represented as the contour integral surrounding $\infty$ and is written by the summation of the residues at $z_r = \infty$ and $1/x_i$ with $x_i \neq 0$ as follows (figure D.1):
\[
\tilde{H}_N' = \oint \prod_{a=1}^r \frac{dz_a}{2\pi i z_a} \prod_{a=1}^r \prod_{j=1}^N \frac{1}{1-x_j z_a} \frac{1}{1-x_j z_a} \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{\partial^{\alpha_n}}{\partial p_n} \right\}
\]
\[
+ (r-1) \sum_{i=1}^N \oint \prod_{a=1}^r \frac{dz_a}{2\pi i z_a} \prod_{j \neq i} [i, j] \prod_{a=1}^r \frac{1}{1-x_a x_i / t} T_{q,\lambda}.
\]
Lemma D.2. Then, we have the following recurrence relation.

\[ \sum_{i=1}^{N} \prod_{j \neq i} [i, j] \cdot D_{N-1, I} T_{q, x_i} = t^{r-1} \frac{t^{r+1} - 1}{t - 1} D_{N}^{r+1}. \]  

Proof.

\[ t^{-r(r-1)/2} \times \text{lhs} = \sum_{i=1}^{N} \prod_{j \neq i} [i, j] \cdot T_{q, x_i} \sum_{l, \ell} \prod_{k \notin l} [k, \ell] \prod_{k \in I} T_{q, x_k} \]

\[ = \sum_{i=1}^{N} \sum_{l, \ell} \prod_{k \in I} [i, j] \cdot T_{q, x_i} \prod_{k \notin l} [k, \ell] \prod_{k \in l} T_{q, x_k}, \quad I := I \oplus \{i\}, \]

\[ = \sum_{l, \ell} \prod_{k \in l} [l, j] \prod_{k \notin l} [k, \ell] \prod_{k \in l} T_{q, x_k}. \]  

Thus, it is sufficient to show that \( \sum_{i=1}^{N} \prod_{j \neq i} [i, j] = \sum_{i=0}^{r} i^r \), which is proved as follows.

(i) Since the residues at \( x_j = x_j \) vanish, so the lhs is a constant.

(ii) By putting \( x_i = \epsilon^i \) and taking \( \epsilon \to \infty \), we obtain the rhs. \( \square \)
Hence, we have

**Lemma D.3.**

\[ \tilde{H}_N^r = \sum_{k=0}^{r} D_N^k \prod_{i=0}^{k-1} (q^{n-i} - 1). \]  
(D.9)

**Proof.** We proceed by induction on \( r \). When \( r = 0 \), since \( \tilde{H}_N^0 = 1 \), (D.9) holds. So assume that the result is true for \( r - 1 \geq 0 \). From (D.3) we obtain

\[ \tilde{H}_N^r = \sum_{k=0}^{r-1} \prod_{s=0}^{k-1} (q^{n-s} - 1) \left( D_N^k + t^{r-k-1} \sum_{i=1}^{N} D_N^{N-1,i} \prod_{j \neq i} [i, j] q_{t, s} \right) \]
\[ = \sum_{k=0}^{r-1} \prod_{s=0}^{k-1} (q^{n-s} - 1) \left( D_N^k + (t' - t^{r-k-1}) D_N^{k+1} \right) \]
\[ = \left( \sum_{k=0}^{r-1} (q^{n-k} - 1) + \sum_{k=1}^{r} (q' - t^{r-k}) \right) D_N^k \prod_{s=1}^{r-1} (q^{n-s} - 1). \]  
(D.10)

\[ \square \]

Therefore,

**Proposition D.4.**

\[ H' = \sum_{j=0}^{\min(r, N)} t^{n-j} e_{r-s}(t^q) D_N^j, \quad r = 0, 1, 2, \ldots . \]  
(D.11)

This completes the proof of (3.4) by taking the limit \( N \to \infty \).

Since the Macdonald functions for all partitions form a basis of the ring of symmetric functions, \( H' \) commute with each other on the space of symmetric functions.

**Appendix E. Torus knot**

Not only for the Hopf link but also the homological invariants for other link may be related with the refined topological vertex or the Macdonald polynomial. For example, in [12], a reduced polynomial for the torus knot \( T_{m,n} \) is conjectured as

\[ P(T_{m,n}) := \lim_{m \to \infty} P(T_{m,n}) = \frac{1 - a^2u}{1 - t^{-2}u^2} \frac{1 - a^2u^2}{1 - t^{-2}u^4} \cdots \frac{1 - a^2u^{\rho-1}}{1 - t^{-2}u^{2\rho}} \]  
(E.1)

with \( u := (qt)^2 \). But this is nothing but the following specialization of the Macdonald polynomial:

\[ P_{(n-1)}(Q t^{\frac{1}{2}+\rho}, t^{-\frac{1}{2}-\rho}; q, t) = \prod_{i=0}^{n-2} \frac{1 - qtQ}{1 - qt} \]  
(E.2)

with \( (Q; q, t) = (a^2t/q^2, q^2t^2, q^4t^2) \). Note that \( p_n(Q t^{\frac{1}{2}+\rho}, t^{-\frac{1}{2}-\rho}) = (1 - (Qt)^n)/(1 - t^n) \).
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