Stability of compressible Taylor-Couette flow. Asymptotical analysis

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Abstract. A temporal stability analysis of compressible Taylor–Couette flow is analyzed. The studied flow is contained between two concentric cylinders of infinite length, which are rotating with different angular velocities and are kept at different surface temperatures. It is shown that for large Reynolds numbers regimes of flow instability exist for which the influence of viscosity is negligible for the first approximation. Corresponding linear equations describing temporal development of vortices are deduced. For linear problems numerical results are presented.

1. Introduction
Compressible Taylor-Couette flow for large Reynolds numbers is analyzed. There are many papers devoted to the incompressible Taylor-Couette flow analysis. At the same time both in industrial applications and in fundamental aerodynamic compressible flow analysis is important as well. Traditionally main tool of analysis is associated with CFD. Therefore some papers were published where Navier-Stokes equations for compressible flows have been solved [1].

In this paper asymptotical analysis leading to the more simple mathematical models is used. General classification of the Taylor-Goertler instability in compressible laminar boundary layers has been presented in [2-3]. Some conclusions derived in this paper can be used in the Couette- Taylor compressible flow analysis.

2. Problem formulation
The flow studied is contained between two concentric cylinders of infinite length, which are rotating with different angular velocities and are kept at different surface temperatures. The cylindrical polar coordinate system is used.

Next we define \( tR_1/W_1, zR_1, rR_1, \omega, uW_1, vW_1, wW_1, \rho_0W_1, \rho, HW_1, \mu, aW_1 \) for the time, for axial, radial and angular coordinates and for corresponding velocity components, for static pressure, for density, for total enthalpy, for the viscosity coefficient and for the speed of sound. The subscript “1” corresponds to the inner cylinder parameters (if an inner cylinder is at rest then reference velocity the outer cylinder is used as indicated by the subscript “2”). The subscript “1” corresponds to the thermodynamic functions of gas filling the volume between the cylinders in the absence of rotation.

The mean flow is described by the next system of equations
\[
\frac{d}{dr} \left[ \mu_0 \frac{d}{dr} \left( \frac{w_0^2}{r} \right) \right] + 2\mu_0 \frac{d}{dr} \left( \frac{w_0}{r} \right) = 0
\]  
(1)

\[
\frac{d}{dr} \{ \mu_0 \frac{d}{dr} \left[ \frac{H_0}{\sigma} + (1 - \frac{1}{2\sigma})w_0^2 \right] \} - \frac{d}{dr} \left[ \mu_0 w_0 \frac{d}{dr} (rw_0) \right] = 0
\]

(2)

\[- \rho_0 w_0^2 + \frac{dp_0}{dr} = 0 \]

(3)

\[p_0 = \frac{(\gamma - 1)}{\gamma} \rho_0 C_p T_0, \quad C_p T_0 = H_0 - \frac{w_0^2}{2} \]

(4)

\[\mu_0 = (H_0 - \frac{w_0^2}{2})^\beta \]

(5)

\[a_0^2 = (\gamma - 1)(H_0 - \frac{w_0^2}{2}) \]

(6)

where \( \sigma \) - is the Prandtl number, \( \gamma \) - is the specific heat ratio.

The boundary conditions may be written as follows

\[r = 1 \quad w_0 = 1, \quad H_0 = H_{w1} \]

(7)

\[r = \frac{R_2}{R_1} \quad w_0 = b_2, \quad H_0 = H_{w2} \]

(8)

Equation of energy (equation for full enthalphy) may be integrated to get next result

\[H = - (\sigma - 1) \frac{w^2}{2} + \sigma \int r w^2 \frac{w}{r} dr + c_1 r + (H_{w1} - c_1) \]

where \[c_1 = \frac{1}{(R_2 / R_1 - 1)} \left( H_{w2} + \frac{\sigma - 1}{2} b_2^2 - \sigma \int \frac{w^2}{r} dr - H_{w1} \right) \]

It may be shown that for large Reynolds numbers different regimes of disturbed flow may exist. These regimes correspond to different vortex wavelength scales.

3. Linearized problem

In this paper one regime is analyzed for which wavelength scale is comparable with the distance between cylinders. Analysis of the Navier-Stokes equations shows that, for large Reynolds numbers in the core flow, the influence of viscosity is negligible in the first approximation. Viscous effects are important in thin layers located on the walls. But for the linearly disturbed flows influence of these boundary layers on the core flow is insignificant. The development of nonlinear vortices can lead to the wall boundary layer separation and to the subsequent strong influence on the core flow.

For linearly disturbed axisymmetrical flow all flow functions can be presented in the forms

\[u(x, r, \omega) = Au_1(x, r) + \ldots \]

\[v(x, r, \omega) = Av_1(x, r) + \ldots \]

\[w(x, r, \omega) = w_0(r) + Aw_1(x, r) + \ldots \]

(9)

\[p(x, r, \omega) = p_0(r) + Ap_1(x, r) + \ldots \]

\[\rho(x, r, \omega) = \rho_0(r) + A\rho_1(x, r) + \ldots \]

\[H(x, r, \omega) = H_0(r) + AH_1(x, r) + \ldots \]

The linear system of equations may be presented as follows
\begin{align*}
\rho_0 \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial x} &= 0 \quad \text{(10)} \\
\rho_0 \frac{\partial v_1}{\partial t} - \frac{2 \rho_0 w_0 v_1}{r} + \frac{\partial p_1}{\partial r} &= 0 \quad \text{(11)} \\
\rho_0 \frac{\partial w_1}{\partial t} + \frac{\rho_0 w_0 v_1}{r} &= 0 \quad \text{(12)} \\
\frac{\partial p_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial t} + \rho_0 \frac{\partial v_1}{\partial r} + \frac{\rho_0 v_1}{r} &= 0 \quad \text{(13)} \\
\rho_0 \frac{\partial H_1}{\partial t} + \rho_0 v_1 \frac{dH_0}{dr} &= \frac{\partial p_1}{\partial t} \quad \text{(14)}
\end{align*}

Coefficients in this linear system of equations are functions only of the radial coordinate. Therefore, the solution may be posited in the next form
\begin{align*}
u_1 &= U(r) \exp(\alpha t) \sin(\lambda x) \quad \text{(15)} \\
v_1 &= V(r) \exp(\alpha t) \sin(\lambda x) \\
w_1 &= W(r) \exp(\alpha t) \cos(\lambda x) \\
p_1 &= P(r) \exp(\alpha t) \sin(\lambda x) \\
\rho_1 &= R(r) \exp(\alpha t) \sin(\lambda x) \\
H_1 &= H(r) \exp(\alpha t) \sin(\lambda x)
\end{align*}

Substituting this normal mode approximation into the linear system of equations, one arrives at the system of ordinary differential equations
\begin{align*}
\rho_0 \alpha U + \lambda P &= 0 \quad \text{(16)} \\
\rho_0 \alpha V - \frac{2 \rho_0 w_0 W}{r} + P' &= 0 \quad \text{(17)} \\
\rho_0 \alpha W + \rho_0 V' + \frac{\rho_0 w_0 V}{r} &= 0 \quad \text{(18)} \\
r \alpha \rho - \rho_0 \alpha \lambda U + \rho_0 V + \rho_0 r V' &= 0 \quad \text{(19)} \\
\rho_0 \alpha H + \rho_0 V H' &= \alpha P \quad \text{(20)}
\end{align*}

This system of equations may be reduced to a single differential equation of second order for the radial velocity. The solution should satisfy two uniform (zero) boundary conditions on the surface of cylinders.

For numerical analysis, however, it is convenient to transform this system of equations to two first order differential equations as follows
\begin{align*}
V' &= \frac{V}{ra_0^2}[-(\gamma - 1)H'_0 - \frac{a_0^2}{\alpha} + w_0(\gamma - 1)(w'_0 + \frac{w_0}{r})] - \frac{P \alpha}{\rho_0 ra_0^2} \left[1 + \frac{\lambda^2 a_0^2}{\alpha^2}\right] \quad \text{(21)} \\
P' &= -\left[V \left(\rho_0 \alpha + \frac{2 \rho_0 w_0}{ar} \left(w'_0 + \frac{w_0}{r}\right)\right)\right] \quad \text{(22)}
\end{align*}

4. Numerical solution
The problem is to find eigensolutions of this system of equations having uniform boundary conditions. The dispersion relation may be expressed as
\[ \alpha = F(R_2 / R_1, b_2, H_{w1}, H_{w2}, \gamma, \sigma, \beta) \]

and may be derived as a result of numerical solution of system of equations for the mean flow and for disturbed flow.

Below some results of numerical solution are presented. Solutions have been obtained for the next parameters: \( b_2 = 0.5, \ H_{w1} = 1., \ H_{w2} = 1., \ \sigma = 0.74, \ \beta = 0.7 \) for which part of the flow is supersonic and part is subsonic. On the figures solutions for the first mode (figure 1.) and for the second mode (figure 2.) are presented for \( \lambda = 1 \). This variant corresponds to the supersonic velocity near the inner cylinder (\( M=5. \)) and to the subsonic velocity near the outer cylinder (\( M=0.71 \)).

![Figure 1. First mode \( \alpha = 0.0505 \)](image1)

![Figure 2. Second mode \( \alpha = 0.0252 \)](image2)

The next series of computations corresponds parameters for which the entire flow is supersonic, namely \( b_2 = 0.5, \ H_{w1} = 1., \ H_{w2} = 0.5, \ \sigma = 0.74, \ \beta = 0.7 \). On the figures solutions for the first mode (figure 3.) and for the second mode (figure 4.) are presented for \( \lambda = 1 \). This variant corresponds to the supersonic velocity near the inner cylinder (\( M=5. \)) and to the supersonic velocity near the outer cylinder (\( M=1.67 \)). It may be concluded that the Mach number (outer cylinder) enlargement leads to a reduction of the temporal growth rate both for the first and for the second mode.

Computations showed as well that the temporal growth of instability strongly depends on the temperature of cylinders.

![Figure 3. First mode \( \alpha = 0.0438 \)](image3)

![Figure 4. Second mode \( \alpha = 0.0225 \)](image4)
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References
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