Coefficient problem for certain subclasses of bi-univalent functions defined by convolution

Şahsene Altınkaya and Sibel Yalçın

Abstract. In this paper, we consider a general subclass $H_\lambda^\Sigma(h,\beta)$ of bi-univalent functions. Bounds on the first two coefficients $|a_2|$ and $|a_3|$ for functions in $H_\lambda^\Sigma(h,\beta)$ are given. The main results generalize and improve a recent one obtained by Srivastava [18].

1. Introduction

Let $A$ denote the class of functions $f$ which are analytic in the open unit disk $U = \{z : |z| < 1\}$ with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Let $S$ be the subclass of $A$ consisting of the form (1) which are also univalent in $U$.

For $f(z)$ defined by (1) and $\Phi(z)$ defined by

$$\Phi(z) = z + \sum_{n=2}^{\infty} \Phi_n z^n, \quad (\Phi_n \geq 0), \quad (2)$$

the Hadamard product $(f*\Phi)(z)$ of the functions $f(z)$ and $\Phi(z)$ defined by

$$\quad (f*\Phi)(z) = z + \sum_{n=2}^{\infty} a_n \Phi_n z^n \quad (3)$$

For $0 \leq \beta < 1$ and $\lambda \in \mathbb{C}$, we let $Q_\lambda(h,\beta)$ be the subclass of $A$ consisting of functions $f(z)$ of the form (1) and functions $h(z)$ given by

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n, \quad (h_n > 0) \quad (4)$$

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and satisfying the analytic criterion:

\[
Q_{\lambda}(h, \beta) = \left\{ f \in A : \Re \left( (1 - \lambda) \frac{(f * h)(z)}{z} + \lambda(f * h)'(z) \right) > \beta \right\},
\]

(5)

\[0 \leq \beta < 1, \ z \in U\}.

The Koebe one-quarter theorem [8] states that the image of \(U\) under every function \(f\) from \(S\) contains a disk of radius \(\frac{1}{4}\). Thus every such univalent function has an inverse \(f^{-1}\) which satisfies

\[f^{-1}(f(z)) = z, \ (z \in U)\]

and

\[f(f^{-1}(w)) = w, \ \left( |w| < r_0(f), \ r_0(f) \geq \frac{1}{4} \right),\]

where

\[f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.\]

A function \(f(z) \in A\) is said to be bi-univalent in \(U\) if both \(f(z)\) and \(f^{-1}(z)\) are univalent in \(U\). Let \(\Sigma\) denote the class of bi-univalent functions defined in the unit disk \(U\). For a brief history and interesting examples in the class \(\Sigma\), see [18]. Examples of functions in the class \(\Sigma\) are

\[z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1 - z^2}\]

and so on. However, the familiar Koebe function is not a member of \(\Sigma\). Other common examples of functions in \(S\) such as

\[z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1 - z^2}\]

are also not members of \(\Sigma\) (see [18]).

In [16] the authors defined the classes of functions \(P_m(\beta)\) : let \(P_m(\beta)\), with \(m \geq 2\) and \(0 \leq \beta < 1\), denote the class of univalent analytic functions \(P\), normalized \(P(0) = 1\), and satisfying

\[\int_0^{2\pi} \left| \frac{\Re P(z) - \beta}{1 - \beta} \right| \, d\theta \leq m\pi,\]

where \(z = re^{i\theta} \in U\).

For \(\beta = 0\), we denote \(P_m = P_m(0)\), hence the class \(P_m\) represents the class of functions \(p\) analytic in \(U\), normalized with \(p(0) = 1\), and having the representation

\[p(z) = \int_0^{2\pi} \frac{1 - ze^{it}}{1 + ze^{it}} \, d\mu(t),\]

where \(\mu\) is a real valued function with bounded variation, which satisfies

\[\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m, \ m \geq 2.\]
Clearly, $P = P_2$ is the well known class of Caratheodory functions, i.e. the normalized functions with positive real part in $U$.

Lewin [13] studied the class of bi-univalent functions, obtaining the bound $1.51$ for modulus of the second coefficient $|a_2|$. Netanyahu [15] showed that $\max |a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Brannan and Taha [5] introduced certain subclasses of the bi-univalent function class $\Sigma$ similar to the familiar subclasses. $S^*(\beta)$ and $K(\beta)$ of starlike and convex function of order $\beta$ ($0 \leq \beta < 1$) respectively (see [15]). The classes $S^*_a(\beta)$ and $K_\Sigma(\beta)$ of bi-starlike functions of order $\alpha$ and bi-convex functions of order $\beta$, corresponding to the function classes $S^*(\beta)$ and $K(\beta)$, were also introduced analogously. For each of the function classes $S^*_a(\beta)$ and $K_\Sigma(\beta)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1, 3, 9, 10, 14, 17, 18, 19, 20]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2, 7, 11, 12]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, \ldots\}$) is still an open problem.

**Definition 1.1.** A function $f \in \Sigma$ is said to be $H^\lambda_\Sigma(h, \beta)$, if the following conditions are satisfied:

$$(1 - \lambda) \frac{(f * h)(z)}{z} + \lambda (f * h)'(z) \in P_m(\beta); \quad 0 \leq \beta < 1, \ m \geq 2, \ z \in U$$

and

$$(1 - \lambda) \frac{(f * h)^{-1}(w)}{w} + \lambda ((f * h)^{-1})'(w) \in P_m(\beta);$$

$0 \leq \beta < 1, \ m \geq 2, \ w \in U,$

where the function $h(z)$ is given by (4), a number $\lambda \in \mathbb{C}$ and $(f * h)^{-1}(w)$ are defined by:

$$(f * h)^{-1}(w) = w - a_2h_2w^2 + (2a_2^2h_2^2 - a_3h_3)w^3$$

$$- (5a_2^3h_2^3 - 5a_2h_2a_3h_3 + a_4h_4)w^4 + \cdots.$$  

We note that for $\lambda = 1, \ m = 2$ and $h(z) = \frac{z}{1-z}$, the class $H^\lambda_\Sigma(h, \beta)$ reduce to the class $H_\Sigma(\beta)$ studied by Srivastava et al. [18].

The object of the present paper is to find for the first two coefficients $|a_2|$ and $|a_3|$ for functions in $H^\lambda_\Sigma(h, \beta)$. The main results generalize and improve a recent one obtained by Srivastava [18].

In order to derive our main results, we require the following lemma.

**Lemma 1.1.** [6] Let the function $\varphi(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$, $z \in U$, such that $\varphi \in P_m(\beta)$. Then

$$|h_n| \leq m(1 - \beta), \quad n \geq 1.$$
2. Main results

**Theorem 2.1.** Let \( f \) given by (1) be in the class \( H_{\Sigma}^\lambda(h, \beta) \), where the function \( h(z) \) is given by (4). If \( h_2, h_3 \neq 0 \) and \( \lambda \in \mathbb{C} \setminus \{-1; -\frac{1}{2}\} \), then

\[
|a_2| \leq \min \left\{ \sqrt{\frac{m(1-\beta)}{|1+2\lambda||h_2|^2}}, \frac{m(1-\beta)}{|1+\lambda||h_2|} \right\},
\]

\[
|a_3| \leq \min \left\{ \frac{m(1-\beta) + m^2(1-\beta)^2}{|1+2\lambda||h_3|} \right\},
\]

where \( p, q \in P_m(\beta) \). Using the fact that the functions \( p \) and \( q \) have the following Taylor expansions

\[
p(z) = 1 + p_1 z + p_2 z^2 + \cdots,
\]

\[
q(w) = 1 + q_1 w + q_2 w^2 + \cdots,
\]

and it follows from (6) and (7) that

\[
(1+\lambda) a_2 h_2 = p_1,
\]

\[
(1+2\lambda) a_3 h_3 = p_2,
\]

\[
(1+2\lambda) (2a_2^2 h_2^2 - a_3 h_3) = q_2.
\]

Since \( p, q \in P_m(\beta) \), according to Lemma 1.1, the next inequalities hold:

\[
|p_k| \leq m(1-\beta), \quad k \geq 1,
\]

\[
|q_k| \leq m(1-\beta), \quad k \geq 1,
\]

and thus, from (9) and (10), by using the inequalities (11) and (12)

\[
|a_2|^2 \leq \frac{|p_2| + |q_2|}{2 |1+2\lambda||h_2|^2} \leq \frac{m(1-\beta)}{|1+\lambda||h_2|^2}, \quad \text{for } \lambda \in \mathbb{C} \setminus \left\{-\frac{1}{2}\right\}.
\]

From (8), by using (11) we have

\[
|a_2| \leq \frac{m(1-\beta)}{|1+\lambda||h_2|}, \quad \text{for } \lambda \in \mathbb{C} \setminus \{-1\}.
\]

From (9), by using (11) we have

\[
|a_3| \leq \frac{m(1-\beta)}{|1+2\lambda||h_3|}, \quad \text{for } \lambda \in \mathbb{C} \setminus \left\{-\frac{1}{2}\right\}.
\]
Also, subtracting (10) from (9), we have
\[ 2(1 + 2\lambda)(a_3h_3 - a_2^2h_2^2) = p_2 - q_2, \]
and using (8), (11) and (12), we finally obtain
\[ |a_3| \leq \frac{m(1 - \beta)}{|1 + 2\lambda||h_3|} + \frac{m^2(1 - \beta)^2}{|1 + \lambda|^2|h_3|}, \quad \text{for } \lambda \in \mathbb{C}\backslash \left\{-1, -\frac{1}{2}\right\} \]
which completes our proof. \(\square\)

Taking \(\lambda = 0\) and \(\lambda = 1\) in Theorem 2.1 we get following special cases, respectively.

**Corollary 2.1.** Let \(f\) given by (1) be in the class \(H_\Sigma(h, \beta)\), where the function \(h(z)\) is given by (4). If \(h_2, h_3 \neq 0\), then
\[ |a_2| \leq \frac{\sqrt{m(1 - \beta)}}{|h_2|}, \]
\[ |a_3| \leq \frac{m(1 - \beta)}{|h_3|}. \]

**Corollary 2.2.** Let \(f\) given by (1) be in the class \(H_1^\Sigma(h, \beta)\), where the function \(h(z)\) is given by (4). If \(h_2, h_3 \neq 0\), then
\[ |a_2| \leq \min \left\{ \frac{\sqrt{m(1 - \beta)}}{3|h_2|^2}, \frac{m(1 - \beta)}{2|h_2|} \right\}, \]
\[ |a_3| \leq \frac{m(1 - \beta)}{3|h_3|}. \]

If we put \(\lambda = 1\), \(m = 2\) and \(h(z) = \frac{z}{1-z}\) in Theorem 2.1, we deduce next corollary.

**Corollary 2.3.** Let \(f\) given by (1) be in the class \(H_\Sigma(\beta)\), then
\[ |a_2| \leq \begin{cases} \sqrt{\frac{2(1 - \beta)}{3}}, & \text{if } 0 \leq \beta \leq \frac{1}{3}, \\ (1 - \beta), & \frac{1}{3} < \beta < 1, \end{cases} \]
\[ |a_3| \leq \frac{2(1 - \beta)}{3}, \]
\[ |2a_2^2 - a_3| \leq \frac{2(1 - \beta)}{3}. \]

**Remark 2.1.** For the special case \(\frac{1}{3} < \beta < 1\), the above first inequality, and the second one for all \(0 \leq \beta < 1\), improve the estimates given by Srivastava et al. in ([18], Theorem 2).
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**Şahsene Altınkaya**  
Department of Mathematics  
Uludag University  
16059 Bursa  
Turkey  
E-mail address: sahsene@uludag.edu.tr

**Sibel Yalçın**  
Department of Mathematics  
Uludag University  
16059 Bursa  
Turkey  
E-mail address: syalcin@uludag.edu.tr