Finite Temperature behavior of the CPT-even and parity-even electrodynamics of the Standard Model Extension

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In this work, we examine the finite temperature properties of the CPT-even and Lorentz-invariance-violating (LIV) electrodynamics of the standard model extension, represented by the term \( W^{\alpha\nu\rho\phi} F^{\alpha\nu} F^{\rho\phi} \). We begin analyzing the hamiltonian structure following the Dirac’s procedure for constrained systems and construct a well-defined and gauge invariant partition function in the functional integral formalism. Next, we specialize for the non-birefringent coefficients of the tensor \( W^{\alpha\nu\rho\phi} \). In the sequel, the partition function is explicitly carried out for the parity-even sector of the tensor \( W^{\alpha\nu\rho\phi} \). The modified partition function is a power of the Maxwell’s partition function. It is observed that the LIV coefficients induce an anisotropy in the black body angular energy density distribution. The Planck’s radiation law, however, retains its frequency dependence and the Stefan-Boltzmann law keeps the usual form, except for a change in the Stefan-Boltzmann constant by a factor containing the LIV contributions.

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I. INTRODUCTION

The researches about Lorentz and CPT violation are commonly performed under the framework of the standard model extension (SME) developed by Colladay and Kostelecky [1]. The SME is an enlarged version of the usual standard model that embraces all Lorentz-invariance-violating coefficients whose tensor contractions yield Lorentz scalars in the observer frame, and in the particle frame are seen as sets of independent numbers. A strong motivation to study the SME is the necessity to get some information about underlying physics at Planck scale where both the Lorentz and CPT symmetries can be broken due to quantum gravity effects such as it is suggested by string theory [2]. The photon sector of the SME has been intensively studied in the latest years with a double purpose: the determination of new electromagnetic effects induced by the LIV interactions and the imposition of stringent upper bounds for the magnitudes of the LIV coefficients. Such investigations have connections with the Carroll-Field-Jackiw electrodynamics [3],[4], consistency aspects [5], polarization deviations for light traveling over large cosmological distances [3],[6,7], Cerenkov radiation [8], radiative corrections [9], electromagnetostatics and classical solutions [10,11,12,13], radiation spectrum of the electromagnetic field and CMB [14,15], photon interactions and quantum electrodynamics processes [16,17,18,19,20,21], and synchrotron radiation [22]. For a large and interesting review on the photon sector and related issues, see Ref. [23]. Lorentz violation and its implications have been studied in several diverse respects [24,25] and also in other theoretical environments [26].

The most general renormalizable form of the Lorentz-covariance-violating electrodynamics of the SME photon sector can be expressed by the following Lagrangian density

\[
\mathcal{L} = -\frac{1}{4} F_{\alpha\nu} F^{\alpha\nu} - \frac{1}{4} \epsilon^{\beta\alpha\rho\phi} (k_{AF})_{\beta} A_{\alpha} F^{\rho\phi} - \frac{1}{4} W^{\alpha\nu\rho\phi} F_{\alpha\nu} F^{\rho\phi},
\]

where \( \epsilon^{\beta\alpha\rho\phi} \) is the totally antisymmetric Levi-Civita tensor (\( \epsilon^{0123} = 1 \)), \( F_{\alpha\nu} \) is the electromagnetic field tensor, \( A_{\alpha} \) is the vector potential, \( (k_{AF})_{\beta} = (0,0,0,0) \) has the dimension of mass and describes a super-renormalizable (dimension 3) coupling, \( W^{\alpha\nu\rho\phi} \) is a renormalizable, dimensionless coupling giving raise to a dimension 4 operator. The tensor \( W^{\alpha\nu\rho\phi} \) has the same symmetries of the Riemann tensor \( [W^{\alpha\nu\rho\phi} = -W^{\nu\alpha\rho\phi}, W^{\alpha\nu\rho\phi} = -W^{\alpha\rho\phi\nu}, W^{\alpha\nu\rho\phi} = W^{\rho\phi\alpha\nu}] \) and a double null trace which yields only 19 independent components.

The term \( \epsilon^{\mu\nu\lambda\kappa} (k_{AF})_{\mu} A_{\nu} F_{\lambda\kappa} \) is CPT-odd and it was first introduced by Carroll-Field-Jackiw (CFJ) [3], who studied the modifications produced by this term (including vacuum birefringence) in the classical Maxwell electrodynamics. It yields a causal, stable, and unitary electrodynamics only for a purely spacelike background (see work of Adam &
Klinkhamer in Ref. \[5\]). The term \(W^{\mu \nu \lambda} F_{\mu \nu} F_{\kappa \lambda}\), composed of 19 elements, is CPT-even and was much investigated in Refs. \([6, 7, 12, 17, 20, 21, 23, 27]\). It yields an electrodynamics not plagued with stability illness.

As the LIV terms alter the light propagation, it is natural to infer that the thermodynamical properties of the theory is modified as well. In a recent work \([14]\), it was investigated the influence of the CFJ term on the thermodynamics of the Maxwell field, using the usual formalism of finite temperature field theory \([28]\). It was first analyzed the Hamiltonian structure of the model using the Dirac formalism in order to define the partition function of this theory without ambiguities. In the sequel, the LIV corrections induced on the black body Planck distribution were carried out, properly examined, and related to the cosmic background radiation (CMB). A similar investigation remains to be done for the CPT-even photonic sector of the SME, being it the main purpose of the present work.

This paper is organized as follows. In Sec. II, we discuss the Hamiltonian structure of the CPT-even electrodynamics, investigating the constraints structure of the theory (by means of the Dirac formalism). In Sec. III, we write the general partition function of this electrodynamics into the functional integral formalism by using the constraint structure and the gauge fixing conditions established in Sec. II. This partition function is then particularized and explicitly evaluated for the parity-even sector of the theory. The modified partition function is written as a function of the usual Maxwell partition function. In Sec. IV, we present our final remarks discussing the modifications induced on the Maxwell theory and comparing it with the Carroll-Field-Jackiw model at finite temperature results. In the Appendix, we evaluate the dispersion relations for the parity-even sector of the CPT-even electrodynamics, which corroborate the results obtained for the partition function of this work.

II. THE CPT-EVEN AND LIV ELECTRODYNAMICS OF THE STANDARD MODEL EXTENSION

In the present work, we just study the CPT-even and LIV electrodynamics of the SME, so that we will consider \(k_{AF} \beta = 0\). Therefore, the Lagrangian density given by Eq. (1) is reduced to

\[
\mathcal{L} = -\frac{1}{4} F_{\alpha \nu} F^{\alpha \nu} - \frac{1}{4} W^{\alpha \nu \rho \varphi} F_{\alpha \nu} F_{\rho \varphi},
\]

which yields the following Euler-Lagrange equation for the gauge field

\[
\partial_{\nu} F^{\nu \mu} - W^{\nu \mu \rho \varphi} \partial_{\nu} F_{\rho \varphi} = 0.
\]

A. The Hamiltonian structure

In order to accomplish the Hamiltonian analysis of this model, we begin defining the canonical conjugate momentum of the gauge field as

\[
\pi^{\mu} = -F^{0 \mu} - W^{0 \mu \rho \varphi} F_{\rho \varphi},
\]

with which we can write the fundamental Poisson brackets (PB): \(\{A_{\mu}(x), \pi^{\nu}(y)\} = \delta_{\mu}^{\nu} \delta(x - y)\).

From the Eq. (4), it is easy to note that \(\pi^{0} = 0\). Such a null momentum yields a primary constraint \(\phi_{1} = \pi^{0} \approx 0\) (into the Dirac formalism, the symbol \(\approx\) denotes a weak equality). Also, the momenta \(\pi^{k}\) are defined via the following dynamic relation

\[
\pi^{k} = D_{kj} F_{0j} - W^{0kj l} F_{jl},
\]

where the nonsingular and symmetric matrix \(D_{kj}\) is defined by

\[
D_{kj} = \delta_{kj} - 2 W_{0k0j}.
\]

Then, the velocities \(\dot{A}_{k}\) are given as

\[
\dot{A}_{k} = \partial_{k} A_{0} + (D^{-1})_{kj} \left[\pi^{j} + W^{0j mn} F_{mn}\right],
\]
Following the usual Dirac procedure, we introduce the primary Hamiltonian \( H = \frac{1}{2} [\pi^k + W^{0kmn} F_{mn}] (D^{-1})_{kj} [\pi^j + W^{0jmn} F_{mn}] + \pi^k \partial_k A_0 + \frac{1}{4} (F_{jk})^2 + \frac{1}{4} W^{kjm} F_{kj} F_{lm}. \) (8)

Following the usual Dirac procedure, we introduce the primary Hamiltonian \( H_p \) by adding to the canonical Hamiltonian all the primary constraints, \( H_p = H_C + \int d^3 y \, C \pi^0, \) where \( C \) is a bosonic Lagrange multiplier. The consistency condition of the primary constraint, \( \dot{\pi}^0 = \{ \pi^0, H_p \} \approx 0, \) gives a secondary constraint

\[ \phi_2 = \partial_k \pi^k \approx 0. \] (9)

It means that the Gauss’s law structure is not modified by the CPT-even and LIV background. Nevertheless, expressing it in terms of the electric and magnetic fields, we can note the explicit coupling between the electric and magnetic sectors even in the electrostatic regime [11, 12, 13].

The consistency condition of the Gauss’s law gives \( \dot{\phi}_2 = 0. \) Thus, the secondary constraint is automatically conserved and there are no more constraints in this model. The bosonic multiplier of the primary constraint remains undetermined, being an evidence for the existence of first-class constraints. This is verified by computing the PB between the primary and the secondary constraints: \( \{ \pi^0, \partial_k \pi^k \} = 0. \) The constraints \( \phi_1 = \pi^0 \approx 0 \) and \( \phi_2 = \partial_k \pi^k \approx 0 \) reveal that the CPT-even and LIV electrodynamics has a similar constraint structure as the Maxwell electrodynamics.

### B. Equations of motion and gauge fixing conditions

Following the Dirac conjecture, we define the extended Hamiltonian \( (H_E) \) by adding all the first-class constraint to the primary Hamiltonian,

\[ H_E = H_C + \int d^3 y \, [C \phi_1 + \Lambda \phi_2]. \] (10)

Under this Hamiltonian, we compute the time evolution of the field variables of the system

\[ \dot{A}_0 = \{ A_0, H_E \} = C, \] (11)

\[ \dot{A}_k = \{ A_k, H_E \} = (D^{-1})_{kj} [\pi^j + W^{0jmn} F_{mn}] + \partial_k A_0 - \partial_k \Lambda, \] (12)

Both equations show that the dynamic of gauge field \( A_\mu \) remains arbitrary. However, the second equation is similar to the Lagrangian equation [7] if and only if \( \Lambda = 0. \) Thus, we should impose a gauge condition in such a way to fix \( \Lambda = 0. \) As it is well-known, the Dirac algorithm requires a number of gauge conditions equal to the number of first-class constraints in the theory. However, those gauge conditions must be compatible with the Euler-Lagrange equations, such that they should fix \( \Lambda = 0 \) and determine the Lagrangian multiplier \( C. \) The gauge conditions together with the first-class constraints should form a second-class set.

From the equation of motion for \( A_0, \)

\[ D_{jk} \partial_j \partial_k A_0 - W_{0ijk} \partial_i F_{jk} - \partial_0 (D_{jk} \partial_j A_k) = 0, \] (13)

we set as our two gauge fixing conditions

\[ \psi_1 = D_{jk} \partial_j A_k \approx 0, \quad \psi_2 = D_{jk} \partial_j \partial_k A_0 - W_{0ijk} \partial_i F_{jk} \approx 0. \] (14)

The consistency condition for \( \psi_1 \) gives \( D_{jk} \partial_j \partial_k \Lambda = 0, \) which fixes \( \Lambda = 0. \) The consistency condition for \( \psi_2 \) gives an equation for the multiplier \( C \)

\[ D_{jk} \partial_j \partial_k C - W_{0ijk} \partial_i F_{jk} \approx 0. \] (15)

Consequently, we have determined all the Lagrange multipliers. Therefore, the set \( \Sigma_a = \{ \phi_1, \phi_2, \psi_1, \psi_2 \} \) is a second-class one.
The next step is to compute the Dirac brackets to know the field algebra. Thus, after a long computation, we find that the non-null Dirac brackets are
\[
\{ A_k (x) , \pi^j (y) \}_D = \delta_{kj} \delta (x - y) + D_{mj} \partial_m \partial^j \bar{G} (x - y),
\]
\[
\{ A_0 (x) , \pi^j (y) \}_D = -2 W_{0ijk} \partial^i \partial^j \bar{G} (x - y),
\]
where the matrix \( D_{jk} \) is given by Eq. (16) and \( \bar{G} (x - y) \) is the Green function for the modified Poisson equation:
\[
D_{jk} \partial_j \partial_k \bar{G} (x - y) = -\delta (x - y).
\]

The Dirac brackets for the physical degree of freedom (10) do not reflect the transverse character of the gauge field, however if we choose \( \partial A_k \approx 0 \) as a gauge condition, the DB is reduced to the usual transverse commutation relation
\[
\{ A_k (x) , \pi^j (y) \}_D = \left( \delta_{kj} - \frac{\partial^i \partial^j}{\nabla^2} \right) \delta (x - y).
\]

Here, we need to observe that, at quantum level, the transverse character of the gauge field can be explicitly proven by computing the Ward identity for its 1PI 2-point function, \( \Gamma_{\mu\nu} (x - y) = (\square g^{\mu\nu} - \partial^\mu \partial^\nu + \xi^{-1} \partial^\mu \partial^\nu - S^{\mu\nu}) \delta (x - y) \), thus \( \partial_{[\mu} \Gamma_{\nu]} (x - y) = \xi^{-1} \partial^\rho \delta (x - y) \).

Under the Dirac brackets, the canonical Hamiltonian \( (8) \) reads as
\[
H = \int \, dy \left\{ \frac{1}{2} E^k D_{kj} E^j + \frac{1}{2} B^2 + \frac{1}{4} W^{klij} F_{kj} F_{li} \right\}.
\]

In general, for a sufficiently small \( W^{\mu\nu\rho\sigma} \), a positive-definite Hamiltonian is guaranteed, thus providing a stable quantum theory and a well-defined partition function associated with the CPT-even and LIV electrodynamics. Now, we proceed to the computation of the partition function, performing the analysis of its implications to the black body radiation problem.

### III. THE PARTITION FUNCTION

The next step is to study the thermodynamical properties of the CPT-even photon sector of the SME. The fundamental object for this analysis is the partition function. The Hamiltonian analysis performed in the previous section allows to define the partition function (in a correct way) into the functional integral representation
\[
Z (\beta) = \int \mathcal{D} A_\mu \mathcal{D} \pi^\mu \delta (\phi_1) \delta (\phi_2) \delta (\psi_1) \delta (\psi_2) \left| \det \{ \Sigma_a (x) , \Sigma_b (y) \} \right|^{1/2} \exp \left\{ \int_\beta dx \left( i \pi^\mu \partial_\tau A_\mu - \mathcal{H}_C \right) \right\},
\]
where \( \Sigma_a = \{ \phi_1 , \phi_2 , \psi_1 , \psi_2 \} \) is a second-class set formed by the first-class constraints and the gauge fixing conditions, \( M_{ab} (x,y) = \{ \Sigma_a (x) , \Sigma_b (y) \} \) is the constraint matrix whose determinant is \( \det ( - D_{jk} \partial_j \partial_k ) \). Given the bosonic character of the gauge field, its functional integration can be performed over all the fields satisfying periodic boundary conditions in the \( \tau \)-variable: \( A_\mu (\tau, \mathbf{x}) = A_\mu (\tau + \beta, \mathbf{x}) \). The short notation \( \int_\beta dx \) denotes \( \int_0^\beta d \tau \int d^3 \mathbf{x} \), and \( \mathcal{H}_C \) is the canonical Hamiltonian given by Eq. (8), and \( \beta = 1/k_B T \), where \( k_B \) is the Boltzmann constant.

By performing the integrations over the canonical conjugate momenta and doing the following redefinitions: \( F_{rk} = \partial_\tau A_k - \partial_k A_\tau = -F_{kr} \) and
\[
W_{0kj} = -W_{kjr} , \quad W_{0kmm} = i W_{\tau kmn} ,
\]
we find the partition function for the CPT-even photonic sector of the SME as
\[
Z (\beta) = N \det ( - D_{jk} \partial_j \partial_k ) \int \mathcal{D} A_n \, \delta (D_{jk} \partial_j A_k) \exp \left\{ \int_\beta dx \left( -\frac{1}{4} F_{ab} F_{ab} - \frac{1}{4} W_{abcd} F_{ab} F_{cd} \right) \right\}.
\]

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\]
where \( a, b, c, d = \tau, 1, 2, 3 \). This partition function is not explicitly covariant. However, it is well-known that if the covariance is explicit, the calculation process becomes more manageable. The procedure to pass from a non-covariant gauge to a covariant one (like the Lorentz gauge \( \partial_\alpha A_\alpha = 0 \)) can be performed using the Faddeev-Popov ansatz. Thus, choosing the following Lorentz gauge \( G[A_\alpha] = -\xi^{-1/2}\partial_\alpha A_\alpha + f \), where \( f \) is an arbitrary scalar function and \( \xi \) is a gauge parameter. Therefore, after some algebra, we find the partition function to be

\[
Z(\beta) = \int DA_a \det \left( \frac{-\Box}{\sqrt{\xi}} \right) \exp \left\{ \int \beta dx - \frac{1}{2}A_a \left[ -\Box \delta_{ab} - \left( \frac{1}{\xi} - 1 \right) \partial_a \partial_b + S_{ab} \right] A_b \right\},
\]

where \( \Box = \partial_\alpha \partial_\alpha = (\partial_r)^2 + \nabla^2 \). We have also defined the symmetric LIV operator

\[
S_{ab} = 2W_{acdb} \partial_c \partial_d = S_{ba}.
\]

For convenience, we choose the Feynman gauge \( \xi = 1 \). Performing the gauge field integration, we find

\[
Z(\beta) = \det (-\Box) \left[ \det (-\Box \delta_{ab} + S_{ab}) \right]^{-1/2}.
\]

It is illustrative to mention that this result is similar to that obtained for the Carroll-Field-Jackiw electrodynamics, where \( S_{ab} = \epsilon_{acdb}(\kappa_{AF})_\beta \partial_\beta \).

Given the high complexity of the CPT-even term, in order to turn feasible the explicit evaluation of the partition function, the tensor \( W_{acdb} \) should be specialized for simpler configurations. It is done at zero temperature in Refs. 23, from which one knows some useful parametrization for the tensor \( W_{\mu\nu\alpha\beta} \) in terms of four \( 3 \times 3 \) matrices, \( \kappa_{DE}, \kappa_{HB}, \kappa_{DB}, \kappa_{HE} \):

\[
(\kappa_{DE})_{jk} = -2W^{0j0k}, (\kappa_{HB})_{jk} = \frac{1}{2} \epsilon^{lpq} \epsilon^{klm} W_{pqlm}, (\kappa_{DB})_{jk} = -(\kappa_{HE})_{jk} = \epsilon^{lpq} W^{0jpq}.
\]

The matrices \( \kappa_{DE} \) and \( \kappa_{HB} \) contain together 11 independent components, while \( \kappa_{DB} \) and \( \kappa_{HE} \) possess together 8 components, which sums the 19 independent elements of the tensor \( W_{acdb} \). Such coefficients can be parameterized in terms of four traceless matrices and one trace element. The parity-odd sector is written as

\[
(\tilde{\kappa}_{o+})_{jk} = \frac{1}{2}(\kappa_{DB} + \kappa_{HE})_{jk}, \quad (\tilde{\kappa}_{o-})_{jk} = \frac{1}{2}(\kappa_{DB} - \kappa_{HE})_{jk},
\]

while the parity-even sector is read in terms of two matrices and one trace element,

\[
(\tilde{\kappa}_{e+})_{jk} = \frac{1}{2}(\kappa_{DE} + \kappa_{HB})_{jk}, \quad (\tilde{\kappa}_{e-})_{jk} = \frac{1}{2}(\kappa_{DE} - \kappa_{HB})_{jk} - n\delta_{jk}, \quad n = \frac{1}{3}\tr(\kappa_{DE}).
\]

The matrix \( \kappa_{o+} \) is antisymmetric while the other three are symmetric. Ten of the 19 elements of the tensor \( W_{\alpha\nu\rho\phi} \) (5 belonging to \( \tilde{\kappa}_{o-} \) and 5 to \( \tilde{\kappa}_{e+} \)) are strongly constrained by birefringence data (at the level of 1 part in \( 10^{32} \)) 24, 27. From the nine remaining nonbirefringent coefficients, three are contained in the parity-odd matrix \( \tilde{\kappa}_{o+} \). The parity-even sector encloses six elements (five in the matrix \( \tilde{\kappa}_{e-} \) and the trace element, \( n \)).

The prescriptions (27), taking into account the finite temperature redefinitions (22), are read as

\[
(\kappa_{DE})_{jk} = 2W_{\tau k\tau j}, \quad (\kappa_{HB})_{jk} = \frac{1}{2} \epsilon_{kpq} \epsilon_{jmn} W_{pqmn}, \quad (\kappa_{DB})_{jk} = -(\kappa_{HE})_{jk} = W_{\tau kpq} \epsilon_{jpq}.
\]

We should now carry out the determinant of the operator \((-\Box \delta_{ab} + S_{ab})\) for the six non-birefringent components of the parity-even part of the \( W_{acdb} \) tensor.

**A. The parity-even sector**

The parity-even sector is composed of an isotropic component and five anisotropic components - the elements of matrix \( \tilde{\kappa}_{e-} \). We now evaluate the partition function for this sector.
1. The isotropic contribution

We first isolate the isotropic part of the parity-even sector by imposing \((\tilde{\kappa}_{e-})_{jk} = 0\), retaining only the component \(n\). The functional determinant for the operator \((-\Box \delta_{ab} + S_{ab})\) is now given as

\[
\det (-\Box \delta_{ab} + S_{ab}) = \det (n + 1)^2 [-\Box]^2 \det \left[ -\Box + \frac{2n}{n + 1} \nabla^2 \right]^2,
\]

while the partition function becomes

\[
\ln Z(\beta) = -\text{Tr} \ln \left[ -\Box + \frac{2n}{n + 1} \nabla^2 \right].
\]

We can evaluate the involved trace by writing the gauge field in terms of a Fourier expansion,

\[
A_a(\tau, x) = \left( \frac{\beta}{V} \right)^{\frac{1}{2}} \sum_{n, p} e^{i(\omega_n \tau + x \cdot p)} \tilde{A}_a(n, p),
\]

where \(V\) designates the system volume and \(\omega_n\) are the bosonic Matsubara’s frequencies, \(\omega_n = \frac{2n\pi}{\beta}\), for \(n = 0, 1, 2, \ldots\).

The contributions of the two modes of the gauge field are expressed as

\[
\ln Z(\beta) = -V \int \frac{d^3p}{(2\pi)^3} \sum_{m=-\infty}^{+\infty} \ln \beta^2 \left[ (\omega_m)^2 + \frac{1 - n}{1 + n} p^2 \right],
\]

Here, it should hold \(|n| < 1\) for yielding a well-defined partition function. By performing the rescaling \(p_i \rightarrow p_i \sqrt{\frac{1 + n}{1 - n}}\), we obtain

\[
\ln Z(\beta) = \left( \frac{1 + n}{1 - n} \right)^{3/2} \ln Z_A,
\]

where \(Z_A\) is the partition function of the Maxwell’s electrodynamics, given by

\[
\ln Z_A = -\frac{V}{\pi^2} \int_0^\infty d\omega \ \omega^2 \ln (1 - e^{-\beta \omega}) = V \frac{\pi^2}{45 \beta^3}.
\]

From (35), we see that the LIV partition function is obviously a power of \(Z_A\),

\[
Z(\beta) = (Z_A)^{\alpha(n)},
\]

for \(\alpha(n) = ((1 + n)/(1 - n))^{3/2}\). With this result, it is easy to show that both the modified Planck’s radiation and the Stefan-Boltzmann’s law of the isotropic sector are those of the Maxwell electrodynamics multiplied by the factor \(\alpha(n)\). Here, the energy density per solid-angle element remains isotropic.

2. The anisotropic contribution

The anisotropic coefficients of the parity-even sector are represented by the terms of the matrix \((\tilde{\kappa}_{e-})\). They can be isolated by setting \(n = 0\). For evaluating the functional determinant, we should express the matrix \((\tilde{\kappa}_{e-})\) in a suitable way. As the matrix \((\tilde{\kappa}_{e-})\) is symmetric and traceless, it can be parameterized in terms of two orthogonal 3D vectors, \(a\) and \(b\), as

\[
(\tilde{\kappa}_{e-})_{jk} = \frac{1}{2} (a_j b_k + b_j a_k),
\]

with \(a \cdot b = 0\) and \(\det (\tilde{\kappa}_{e-}) = 0\). Then, the functional determinant of the operator \((-\Box \delta_{ab} + S_{ab})\) is

\[
\det (-\Box \delta_{ab} + S_{ab}) = \det \left(1 - \frac{1}{4} a^2 b^2\right) \det (-\Box)^2 \det (-\Box - \nabla^2) \det (-\Box - \nabla^2),
\]

(39)
where the operators $\nabla^2_+$ and $\nabla^2_-$ are given as

\[
\begin{align*}
\nabla^2_+ &= \frac{4 (\mathbf{a} \cdot \nabla) (\mathbf{b} \cdot \nabla) + \mathbf{b}^2 (\mathbf{a} \cdot \nabla)^2 + a^2 (\mathbf{b} \cdot \nabla)^2}{4 - a^2 b^2}, \\
\nabla^2_- &= (\mathbf{a} \cdot \nabla) (\mathbf{b} \cdot \nabla).
\end{align*}
\]

(40)

(41)

With all these definitions, the partition function becomes

\[
\ln Z(\beta) = -\frac{1}{2} \ln \det \left[ -\Box - \nabla^2_+ \right] - \frac{1}{2} \ln \det \left[ -\Box - \nabla^2_- \right],
\]

(42)

representing the contributions of the two polarization modes of the gauge field. Let us observe that if we consider only the first order contribution of the LIV background, we have

\[
\nabla^2_+ \approx (\mathbf{a} \cdot \nabla) (\mathbf{b} \cdot \nabla), \quad \nabla^2_- = (\mathbf{a} \cdot \nabla) (\mathbf{b} \cdot \nabla).
\]

(43)

It means that the dispersion relation at first order are the same for both modes of the gauge field, once both modes give the same contribution to the partition function at first order. For an alternative evaluation of the dispersion relations, see Appendix. This result is compatible with the statements of Ref. [23].

Again, the functional trace is carried out by means of the Fourier expansion (33) of the gauge field. The contributions of the two modes of the gauge field are expressed as

\[
\begin{align*}
\ln Z_+(\beta) &= -\frac{1}{2} \int d\Omega \int_0^\infty d\omega \int_{-\infty}^{+\infty} \ln \beta^2 \left[ (\omega_m)^2 + \mathbf{p}^2 + \frac{b^2 (\mathbf{a} \cdot \mathbf{p})^2 + a^2 (\mathbf{b} \cdot \mathbf{p})^2 + 4 (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p})}{4 - a^2 b^2} \right], \\
\ln Z_-(\beta) &= -\frac{1}{2} \int d\Omega \int_0^\infty d\omega \int_{-\infty}^{+\infty} \ln \beta^2 \left[ (\omega_m)^2 + \mathbf{p}^2 + (\mathbf{a} \cdot \mathbf{p}) (\mathbf{b} \cdot \mathbf{p}) \right].
\end{align*}
\]

(44)

(45)

In order to perform the momentum integrations, we consider the following coordinate system: the vector $\mathbf{a}$ is aligned with the $x-$ axis, the vector $\mathbf{b}$ with the $y-$ axis, so that $\mathbf{a} \times \mathbf{b}$ points along the $z-$ axis. Expressing the momentum in spherical coordinates [$\mathbf{p} = \omega (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$], we achieve the following mode contributions:

\[
\begin{align*}
\ln Z_+(\beta) &= -\frac{1}{2} \frac{V}{(2\pi)^3} \int d\Omega \int_0^\infty d\omega \int_{-\infty}^{+\infty} \ln \beta^2 \left[ (\omega_m)^2 + \omega^2 \left( 1 + \frac{a^2 b^2 \sin^2 \theta + 2ab \sin^2 \theta \sin 2\phi}{4 - a^2 b^2} \right) \right], \\
\ln Z_-(\beta) &= -\frac{1}{2} \frac{V}{(2\pi)^3} \int d\Omega \int_0^\infty d\omega \int_{-\infty}^{+\infty} \ln \beta^2 \left[ (\omega_m)^2 + \omega^2 \left( 1 + \frac{1}{2}ab \sin^2 \theta \sin 2\phi \right) \right],
\end{align*}
\]

(46)

(47)

where $d\Omega = \sin \theta d\theta d\phi$ is the solid-angle element, $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$. By doing a rescaling of the variable $\omega$ and performing the summation, we obtain

\[
\begin{align*}
\ln Z_+(\beta) &= \frac{1}{8\pi} \ln Z_A \int d\Omega \left( 1 + \frac{\lambda^2 \sin^2 \theta + 2\lambda \sin^2 \theta \sin 2\phi}{4 - \lambda^2} \right)^{-3/2}, \\
\ln Z_-(\beta) &= \frac{1}{8\pi} \ln Z_A \int d\Omega \left( 1 + \frac{1}{2} \lambda \sin^2 \theta \sin 2\phi \right)^{-3/2},
\end{align*}
\]

(48)

(49)

where we have defined $\lambda = ab$. Taking into account the outcome of Eq. (30), and noting that the angular integrations can be exactly solved, the partition functions become

\[
\begin{align*}
\ln Z_+(\beta) &= \frac{1}{4} (4 - \lambda^2)^{1/2} \ln Z_A(\beta), \\
\ln Z_-(\beta) &= (4 - \lambda^2)^{-1/2} \ln Z_A(\beta).
\end{align*}
\]

(50)

(51)

For having a well-defined partition function, the product $ab = \lambda$ must be bounded as $0 < ab < 2$. Remembering that

\[
\ln Z(\beta) = \ln Z_+(\beta) + \ln Z_-(\beta),
\]

we can also show that the LIV partition function can be written as a power of the Maxwell's one,

\[
Z(\beta) = (Z_A)^4(\lambda),
\]
with
\[ \delta(\lambda) = \frac{1}{4} (4 - \lambda^2)^{1/2} + (4 - \lambda^2)^{-1/2}. \] (52)

Similarly to the parity-odd case, we observe that the modified Planck’s radiation law and the Stefan-Boltzmann’s law are those of the Maxwell electrodynamics multiplied by the factor \( \delta(\lambda) \). However, the energy density distribution per solid angle,
\[ u(\beta, \Omega) = \frac{\pi}{120\beta^4} \left[ \left( 1 + \frac{1}{2} \lambda \sin^2 \theta \sin 2\phi \right)^{-3/2} + \left( 1 + \frac{\lambda^2 \sin^2 \theta + 2\lambda \sin^2 \theta \sin 2\phi}{4 - \lambda^2} \right)^{-3/2} \right], \] (53)
possesses an explicitly dependence on \( \phi \) and \( \theta \) which reveals a higher degree of the anisotropy induced by LIV, as it can be shown at leading order
\[ u(\beta, \Omega) = \frac{\pi}{120\beta^4} \left[ 2 - \frac{3}{2} \lambda \sin 2\phi \sin^2 \theta \right]. \] (54)
The \( \lambda \) linear dependence of the energy density may lead to an attainment of upper-bounds on the \( \kappa_{e-} \) parameters using polarization data of the cosmic microwave background.

IV. CONCLUSIONS AND REMARKS

We have initiated this work establishing the Hamiltonian structure of the CPT-even sector of the electrodynamics of the SME. The constraint analysis allows the construction of a well-defined partition function which is given in [20] for an arbitrary and sufficiently small tensor \( W_{\text{bdec}} \). At once, we specialize our analysis for the non-birefringent components of the parity-even parts, for which we compute exactly the partition function. The expression (52) shows that it is a power of the partition function of the Maxwell electrodynamics, where the power is a pure function \( \lambda \) of the LIV parameters. This way, the Planck radiation law retains its usual functional dependence in the frequency and the Stefan-Boltzmann law remains the same one, apart from a multiplicative global factor containing the LIV coefficients. It is observed that the LIV induces an anisotropic angular distribution for the black body energy density for the anisotropic parity-even \((\kappa_{e-})\) coefficients. The anisotropic character of the angular radiation distribution reflects local energy density variations in relation to the Maxwell pattern induced by Lorentz violation. Despite such differences, the Stefan-Boltzmann law keeps the usual temperature behavior. This means that, notwithstanding small local fluctuations, the global radiation law maintains the \( T^4 \) – behavior.

Since the LIV coefficients are constrained by very stringent upper bounds, the lower order non-null LIV contribution for the Maxwell thermodynamics would give a very good information about the thermodynamical properties of the non-birefringent sector of the model. It is observed that the isotropic contribution gives a linear correction in \( n \), whereas the anisotropic contribution coming from the matrix \( \kappa_{e-} \) only is manifest at fourth order, as it is shown by Eq. (50). Hence, the pure anisotropic contribution is irrelevant when compared with the isotropic one.

Moreover, we must highlight the differences between the thermodynamical properties of the CPT-even and the CPT-odd electrodynamics, first investigated in Ref. [14]. Such difference stems from the Dirac’s algebra of the physical variables. For the CPT-odd electrodynamics, in the Coulomb gauge, we have attained
\[ \{A_k(x), \pi_j(y)\}_D = -\left[ \delta_{kj} - \frac{\partial_k \partial_j}{\nabla^2} \right] \delta(x - y), \] (55)
\[ \{\pi^h(x), \pi^j(y)\}_D = \frac{1}{2} \left[ \epsilon^{0kli} (k_{AF})_l \frac{\partial^i \partial^j}{\nabla^2} - \epsilon^{0jli} (k_{AF})_l \frac{\partial^i \partial^j}{\nabla^2} \right] \delta(x - y). \] (56)

Nevertheless, for the CPT-even case and for the Maxwell electrodynamics, the Dirac algebra is given only by Eq. (55). The noncommutativity of the physical momenta, expressed in Eq. (50), is the fundamental reason for this sector to have different thermodynamical properties when it is compared with its CPT-even counterpart. Also, since the background \( k_{AF} \) is a dimensional parameter, the temperature dependence of the logarithm of the partition function at order \((k_{AF})^{2n}\) changes as \( T^{3-2n} \). It has as a consequence that the CPT-odd partition function can not be expressed as a power of the Maxwell one such as it happens in the CPT-even case.
A. APPENDIX: Dispersion relations

In this Appendix, we write the dispersion relations for this CPT-even electrodynamics as a procedure to confirm the evaluation of the partition functions. A general evaluation for the dispersion relations may be developed from Eq. (2) and the matrix prescriptions (27). In terms of the matrices $\kappa_{DE}, \kappa_{DB}, \kappa_{HB}$ the non-homogenous Maxwell equations (in the absence of sources) are

$$\partial_j E_j + (\kappa_{DE})_{ja} \partial_j E_a - (\kappa_{DB})_{ja} \partial_j B_a = 0,$$  

$$\partial_0 E_k - \epsilon_{kja} \partial_j B_a + (\kappa_{DE})_{kj} \partial_0 E_j - (\kappa_{DB})_{kj} \partial_0 B_j - (\kappa_{DB})_{ab} \epsilon_{bklj} \partial_j E_a - (\kappa_{HB})_{ab} \epsilon_{bklj} \partial_j B_a = 0,$$  

while the homogeneous ones remain the same, $\partial_0 B_k + \epsilon_{kab} \partial_a E_b = 0$, $\partial_a B_a = 0$. The wave equation for the electric field is

$$(\partial_t)^2 E_k - \nabla^2 E_k + \text{tr}(\kappa_{DE}) \nabla^2 E_k + (\kappa_{HB})_{ab} \partial_a \partial_0 E_k + \partial_k \partial_0 E_k + (\kappa_{DE})_{ka} (\partial_t)^2 E_a$$

$$- \text{tr}(\kappa_{DE}) \partial_k \partial_0 E_a - (\kappa_{HB})_{ka} \nabla^2 E_a - (\kappa_{HB})_{ka} \partial_b \partial_a E_a + (\kappa_{DE})_{ka} \partial_0 \partial_a E_a$$

$$+ (\kappa_{DB})_{kc} \epsilon_{cba} \partial_t \partial_a E_a - (\kappa_{DB})_{ab} \epsilon_{bkc} \partial_0 \partial_a E_a = 0.$$  

We now specialize the wave equation (59) for the parity-even case, setting $\kappa_{DB} = 0$. We then express the matrices $\kappa_{DE}, \kappa_{HB}$ in terms of the $\kappa_{e+}$ and $\kappa_{e-}$

$$(\kappa_{DE})_{ab} = (\kappa_{e+})_{ab} + (\kappa_{e-})_{ab} + n\delta_{ab},$$  

$$(\kappa_{HB})_{ab} = (\kappa_{e+})_{ab} - (\kappa_{e-})_{ab} - n\delta_{ab}.$$  

Birefringence data impose $(\kappa_{e+})_{ab} = 0$, so that

$$(\partial_t)^2 E_k + n (\partial_t)^2 E_k + (\kappa_{e-})_{ka} (\partial_t) E_a - \nabla^2 E_k + n \nabla^2 E_k - (\kappa_{e-})_{ka} \nabla^2 E_a$$

$$- (\kappa_{e-})_{ab} \partial_a \partial_0 E_k + \partial_k \partial_0 E_a - n \partial_0 \partial_a E_a + (\kappa_{e-})_{kb} \partial_b \partial_a E_a + (\kappa_{e-})_{ka} \partial_0 \partial_a E_a = 0.$$  

Retaining only the isotropic component ($n \neq 0, \kappa_{e-} = 0$), we have

$$[(1 + n) (\partial_t)^2 - (1 - n) \nabla^2] E_k + (1 - n) \partial_0 \partial_a E_a = 0.$$  

Using now the first Maxwell equation, $(1 + n) \partial_0 E_a = 0$, we obtain $[(1 + n) (\partial_t)^2 - (1 - n) \nabla^2] E_k = 0$, which in Fourier space, is reads as

$$[(1 + n) p_0^2 - (1 - n) k^2] \tilde{E}_k = 0.$$  

This equation yields the following dispersion relation:

$$(1 + n) p_0^2 - (1 - n) k^2 = 0.$$  

This is the same expression contained in Eqs. (31, 32, 33), confirming our previous result:

$$\omega_{\pm} = \pm |p| \sqrt{(1 - n) / (1 + n)}.$$  

Here, we see that the phase velocity associated with the modes of the photon field is the same, showing explicitly the the nonbirefringent character of the isotropic coefficient of the parity-even sector, which is in full accordance with the statements of Ref. [23]. Moreover, we note the existence of positive and negative frequencies, $\omega_{+}$ and $\omega_{-}$.

We should finally consider the anisotropic components of the parity-even sector ($n = 0, \kappa_{e-} \neq 0$). In this case, the wave equation (59) reads as

$$[\square \delta_{ka} + (\kappa_{e-})_{ka} \square] E_a - [(\kappa_{e-})_{cb} \partial_c \partial_0 \delta_{ka} - (\kappa_{e-})_{kb} \partial_b \partial_a] E_a = 0.$$  

In momentum space, we have

$$\{ [p^2 - (\kappa_{e-})_{cb} p_c p_b] \delta_{ka} + (\kappa_{e-})_{kb} [p^2 \delta_{ab} + p_b p_a] \} E_a = 0.$$  

(68)
We now use the same parametrization of Eq. (68), where \( \mathbf{a} \) and \( \mathbf{b} \) are two orthogonal 3D vectors. Then, we have

\[
(\mathbf{\kappa}_c -)_{eb} p_e p_b = (\mathbf{a} \cdot \mathbf{p})(\mathbf{b} \cdot \mathbf{p}),
\]

(69)

\[
(\mathbf{\kappa}_c -)_{kc} [p^2 \delta_{jc} + p_e p_j] = \frac{1}{2} (a_k b_j + a_j b_k) p^2 + \frac{1}{2} a_k p_j (\mathbf{b} \cdot \mathbf{p}) + \frac{1}{2} b_k p_j (\mathbf{a} \cdot \mathbf{p}).
\]

(70)

With it, Eq. (68) is read as

\[
M_{kj} E_j = 0,
\]

(71)

with

\[
M_{kj} = [p^2 - (\mathbf{a} \cdot \mathbf{p})(\mathbf{b} \cdot \mathbf{p})] \delta_{kj} + \frac{1}{2} (a_k b_j + a_j b_k) p^2 + \frac{1}{2} a_k p_j (\mathbf{b} \cdot \mathbf{p}) + \frac{1}{2} b_k p_j (\mathbf{a} \cdot \mathbf{p}).
\]

(72)

The dispersion relations are obtained from \( \det \mathbb{M} = 0 \). Computing the determinant, we get

\[
\det \mathbb{M} = \left(1 - \frac{1}{4} a^2 b^2\right) p^2 [p^2 - (\mathbf{a} \cdot \mathbf{p})(\mathbf{b} \cdot \mathbf{p})]\left[p^2 - \frac{4(\mathbf{a} \cdot \mathbf{p})(\mathbf{b} \cdot \mathbf{p}) + (\mathbf{a} \cdot \mathbf{p})^2 b^2 + a^2(\mathbf{b} \cdot \mathbf{p})^2}{4 - a^2 b^2}\right].
\]

(73)

By this way, we attain the exact dispersion relations

\[
p^2 = (\mathbf{a} \cdot \mathbf{p})(\mathbf{b} \cdot \mathbf{p}),
\]

(74)

\[
p^2 = [4(\mathbf{a} \cdot \mathbf{p})(\mathbf{b} \cdot \mathbf{p}) + (\mathbf{a} \cdot \mathbf{p})^2 b^2 + a^2(\mathbf{b} \cdot \mathbf{p})^2][4 - a^2 b^2]^{-1}.
\]

(75)

These expressions confirm that the partition function for this case is the one stated in Eqs. (42, 44, 45). At leading order, these dispersion relations are the same one,

\[
\omega = \pm |\mathbf{p}||1 + \frac{1}{2} \frac{(\mathbf{a} \cdot \mathbf{p})(\mathbf{b} \cdot \mathbf{p})}{|\mathbf{p}|^2}|,
\]

(76)

implying absence of birefringence at leading order such as stablished in Ref. [23].

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