FROM LIOUVILLE THEORY TO THE QUANTUM GEOMETRY OF RIEMANN SURFACES

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The aim of this note is to propose an interpretation for the full (non-chiral) correlation functions of the Liouville conformal field theory within the context of the quantization of spaces of Riemann surfaces.

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1. Introduction

Liouville theory is a conformal field theory in two dimensions which has a classical limit described by the (euclidean) action

$$ S[\ ] = \frac{1}{2} \int d^2 z \partial \phi \partial \phi + e^{2\phi} : $$

Understanding the corresponding quantum theory is an important problem in mathematical physics for at least two reasons:

Quantum Liouville theory provides the simplest example for a two-dimensional conformal field theory with continuous spectrum \(^{1,2}\). It can therefore be regarded as a paradigm for a whole new class of two-dimensional conformal field theories which are neither rational nor quasi-rational.

The quantized Liouville theory is related to quantized spaces of Riemann surfaces. This interpretation should provide the basis for a deeper understanding of two-dimensional quantum gravity \(^3\) as well as a future theory of three-dimensional quantum gravity (see e.g. \(^4,5\) and references therein).

In the following note we will be mainly concerned with the second of these two points. The expectation that quantum Liouville theory is related to the quantum geometry of Riemann surfaces goes back to Polyakov’s work \(^3\) and was formulated more precisely in \(^6,7\). This interpretation was recently given a solid ground \(^8,9\) by establishing a direct relation between the conformal blocks of quantum Liouville theory \(^2,10\) and the space of states obtained by quantizing the Teichmüller spaces of Riemann surfaces \(^11,12,13,14,15,16\).

Our aim in the present note will be to elaborate further on the geometrical interpretation of quantum Liouville theory by proposing a (partly conjectural) interpretation of the full (non-chiral) correlation functions of quantum Liouville theory within quantum Teichmüller theory.
2. The Liouville conformal field theory

Quantum Liouville theory is a conformal field theory. The space of states decomposes into irreducible representations of the (left/right) Virasoro algebras as:

\[
H = \sum \text{da} \mathcal{V}_{a; c} \quad \text{with} \quad \mathcal{V}_{a; c} \quad \text{and} \quad S = \frac{Q}{2} + \mathbb{R}^+ \cdot i
\]

where \( Q = b + b^\dagger \) and \( \mathcal{V}_{a; c} \) are the irreducible unitary representations of the Virasoro algebra with central charge \( c = 1 + 6Q^2 \) and highest weight \( a = a \oplus a \). Being a conformal field theory, quantum Liouville theory is fully characterized by the set of s-point functions

\[
\mathcal{V}_{a, \mathbb{C}} (z_1; z_2) : : : V_{a, \mathbb{C}} (z_s; z_1)
\]

of the primary fields \( \mathcal{V}_{a, \mathbb{C}} (z; z) \), a \( 2 \mathbb{C} \) with conformal dimensions \( a = a \oplus a \). The Möbius-invariance of the s-point functions allows us to assume \( z_1 = 1, z_1 = 1 \) and \( z_2 = 0 \).

The primary fields \( \mathcal{V}_{a, \mathbb{C}} (z; z) \) of quantum Liouville theory were constructed in \( 2 \mathbb{C} \). With the help of the constructions in \( 2 \mathbb{C} \), it is possible to show that the s-point functions can be represented in a holomorphically factorized form

\[
\mathcal{V}_{a, \mathbb{C}} (z_1; z_2) : : : V_{a, \mathbb{C}} (z_2; z_1) = \int_{\mathbb{S}} m (S) \mathcal{F}_{S, a, \mathbb{C}} (z; z_2) \mathcal{F}_{S, a, \mathbb{C}} (z_1; z)
\]

In order to write (2.4) compactly we have introduced the tuples of variables \( \mathbb{A} = (a_1, \ldots, a_s) \), \( \mathbb{S} = (1; \ldots; 1) \), \( \mathbb{Z} = (z_1; \ldots; z_1) \) and \( \mathbb{Z} = (z_1; \ldots; z_1) \), where \( s = 3 \). The tuple \( \mathbb{S} \) is integrated over \( \mathbb{S} \), where \( \mathbb{S} = \frac{Q}{2} + \mathbb{R}^+ \cdot i \), and the measure \( m (\mathbb{S}) \) is given as

\[
m (\mathbb{S}) = \prod_{i=1}^{s} \sin \left( \frac{\pi}{2} \mathbb{Q} \right) \sin \left( \frac{\pi}{2} \mathbb{Q} \right)
\]

The key objects in (2.4) are the conformal blocks \( \mathcal{F}_{S, a, \mathbb{C}} (z; z) \). In the remainder of this section, adapted from \( 8 \), we will briefly describe the definition and some relevant properties of the conformal blocks.

2.1. The conformal Ward identities

It is well-known that the conformal blocks are strongly constrained by the conformal Ward-identities which express the conservation of energy-momentum on the punctured Riemann-sphere \( P^1 \cap \mathbb{C} \). In order to exhibit the mathematical content of the conformal Ward identities let us consider functionals

\[
\mathcal{F}_{\mathbb{A}} : \mathcal{V}_{a_1} : : : \mathcal{V}_{a_s} \Rightarrow \mathbb{C}
\]

that satisfy the following invariance condition. Let \( \nu (z) \) be a meromorphic vector field that is holomorphic on \( P^1 \cap \mathbb{C} \). Write the Laurent-expansion of \( \nu (z) \) in an annular neighborhood of \( z_0 \) in the form \( \nu (z) = \sum_{n=2}^{\infty} \nu_n (z) \mathbb{Z}^{n-1} \), and define an operator \( T [\nu] \) on \( \mathcal{V}_{a_1} : : : \mathcal{V}_{a_s} \) by

\[
T [\nu] = \sum_{k=1}^{\infty} \sum_{n=2}^{\infty} \nu_n (k) \mathbb{L}_n (k)
\]

The conformal Ward identities can then be formulated as the condition that

\[
\mathcal{F}_{\mathbb{A}} (T [\nu] w) = 0
\]
holds for all $w$ 2 $V_{a_2}$ = $v_{a_1}$ and all meromorphic vector fields $v$ that are holomorphic on $\mathbb{C}$.

By choosing vector fields $v$ that are singular at a single point only one gets rules for moving Virasoro generators from one puncture to the other ones. In this way one may convince oneself that the functional $F_{a_3}$ is uniquely determined by the value $F_{a_3} (G_0, G_2) = F_{a_3} (v_0, 2 \mathbb{C})$ that it takes on the product of highest weight states $v_0$, $v_{a_3}$ = $v_{a_1}$.

It is well-known that the space of solutions to the condition (2.6) is one-dimensional for the case of the three-punctured sphere $s = 3$. Invariance under global conformal transformations allows one to assume that $s = 3 = \mathbb{P}^1 \cap 0, 1, \mathbb{C}$. We will adopt the normalization from $^2$ and denote $C_3$, the unique conformal block that satisfies $C_3 (v_{a_2}, v_{a_3}, v_{a_1}) = N (\alpha_3, \alpha_2, \alpha_1)$. The function $N (\alpha_3, \alpha_2, \alpha_1)$ is defined in $^2$ but will not be needed explicitly in the following.

Let us furthermore note that the case of $s = 2$ corresponds to the invariant bilinear form $h : \mathfrak{s}_n : V_n \rightarrow \mathbb{C}$ which is defined such that $h \prod_n w_i v_i = h w_i v_i = \prod_n w_i v_i$.

2.2. Sewing of conformal blocks

For $s > 3$ one may generate large classes of solutions of the conformal Ward identities by the following “sewing” construction. Let $V_i$, $i = 1, 2$ be Riemann surfaces with $m_i + 1$ punctures, and let $G_{a_i}$ and $H_{a_i}$ be conformal blocks associated to $V_i$, $i = 1, 2$ and representations labeled by $A_i = (a_{m_i}, \cdots ; a_i; a)$ and $A_1 = (a; a_{m_i}, \cdots ; a_i)$ respectively. Let $P_i$, $i = 1, 2$ be the distinguished punctures on $V_i$ that are associated to the representation $V_a$. Around $P_i$ choose local coordinates $z_i$ such that $z_i = 0$ parameterizes the points $P_i$ themselves. Let $A_{12}$ be the annuli $\mathbb{D}_j < \mathbb{C} < \mathbb{D}_j$. The conformal block $F_{a_{12}} (V_{a_{12}})$ is defined in $^2$.

Let us furthermore note that the case of $s = 2$ corresponds to the invariant bilinear form $h : \mathfrak{s}_n : V_n \rightarrow \mathbb{C}$ which is defined such that $h \prod_n w_i v_i = h w_i v_i = \prod_n w_i v_i$.

2.3. Feynman rules for the construction of conformal blocks

The sewing construction allows one to construct large classes of solutions to the conformal Ward identities from simple pieces. The resulting construction resembles the construction of field theoretical amplitudes by the application of a set of Feynman rules. Let us summarize the basic ingredients and their geometric counterparts.
**Propagator** — Invariant bilinear form:

\[ \mathcal{H}_{ij} e^{t L_0} v_i v_j \]

**Vertex** — Invariant trilinear form:

\[ C_\alpha (v_1; v_2; v_3) \]

**Gluing** — Completeness:

\[ \mathcal{H} \]

The variable \( q \) is related to \( q = e^{t L_0} \). We have introduced the dashed lines in order to take care of the fact that the rotation of a boundary circle by \( 2 \pi \) (Dehn twist) is not represented trivially. It acts by multiplication with \( e^{2 \pi \alpha} \). This describes a part of the action of the mapping class group on the spaces of conformal blocks, which will be further discussed below. The Riemann surfaces that are obtained by gluing cylinders and three-holed spheres as drawn will therefore carry a trivalent graph which we will call a Moore-Seiberg graph.

The gluing construction furnishes spaces of conformal blocks \( \mathcal{H} \) associated to a Riemann surface together with a Moore-Seiberg graph. A basis for this space is obtained by coloring the “internal” edges of the Moore-Seiberg graph with elements of \( S \), for example

\[ \text{Span} \]

In order to show that the spaces of conformal blocks associated to each two Moore-Seiberg graphs \( \mathcal{H} \) are isomorphic, one needs to find operators \( U_{\mathcal{H}} \) such that \( \mathcal{H} \) and \( \mathcal{H} \) are isomorphic. We will describe the construction of such operators below.

### 2.4. The Moore-Seiberg groupoid

The transitions between different Moore-Seiberg graphs on a surface of genus zero with \( s \) punctures generate a groupoid \( \mathcal{M}_s \) that will be called the Moore-Seiberg groupoid. This groupoid can be characterized by generators and relations. The set of generators for the groupoids \( \mathcal{M}_s \) associated to subsurfaces of genus zero is pictorially represented in Figure 1 below.
In order to define the coefficients \( \text{identity} \) transformation. However, all relations of the Moore-Seiberg groupoid can be shown to follow from a finite set of basic relations \(^{17,18}\). In order to derive the basic relations in \( \mathbb{M} S^0_\alpha \) it suffices to consider the cases \( s = 4; 5 \).

### 2.5. Representation of the Moore-Seiberg groupoid on \( \mathbb{H} \)

In order to characterize a representation of the Moore-Seiberg groupoids \( \mathbb{M} S^0_\alpha \) it suffices to specify the operators \( \mathcal{U} \) for the cases where \( 2 \) and \( 1 \) differ by an A- or B-move. In the case of the Liouville conformal blocks in genus zero this was done in \(^{2,10}\).

**A-MOVE**: In order to describe the representation of the A-move let \( \text{be the four-punctured sphere, with parameters } \Lambda = (a_1; \ldots; a_4) \) associated to the four punctures respectively. The conformal blocks corresponding to the two sewing patterns indicated on the left half of Figure 1 will be denoted \( F_{\Lambda_{A\alpha}} \) and \( G_{\Lambda_{B\alpha}} \) respectively, where \( F_{\Lambda_{A\alpha}} \) corresponds to the leftmost part of Figure 1. The A-move is then represented as an integral transformation of the following form:

\[
F_{\Lambda_{A\alpha}} = \int_{\mathcal{U}} \mathcal{U} (a_{\alpha}) F_{\Lambda_{A\alpha}} a_3 a_2 a_1 \cdot G_{\Lambda_{B\alpha}}
\]

The kernel \( F_{\Lambda_{A\alpha}} a_3 a_2 a_1 \) is given by the following expression:

\[
F_{\Lambda_{A\alpha}} a_3 a_2 a_1 = \frac{s_b (a_1)}{s_b (a_2)} \frac{s_b (a_2)}{s_b (a_3)} \frac{Z}{r} \left( \prod_{i=1}^{4} s_b (r) s_i \right)
\]

where the special function \( s_b (\mathcal{X}) \) can be defined by means of the following integral representation:

\[
\log s_b (\mathcal{X}) = \int_{\mathcal{Z} \alpha}^{1} \frac{d\mathcal{T}}{t} \frac{\sin 2\mathcal{X} t}{2 \sinh 2 \mathcal{X} t \sinh 2 \mathcal{B} t \cdot \mathcal{B}}
\]

In order to define the coefficients \( r_1, s_1, u_1 \) and \( w_1 \) let us introduce \( q_b = \frac{d^2}{2} \) and write \( a_1 = \frac{q_b}{2} + i q_b \).

\[
\begin{align*}
    r_1 &= p_2 + p_1 \quad s_1 = q_b + p_4 + p_2 \quad u_1 = p_3 + p_2 + p_1 \\
    r_2 &= p_2 + p_1 \quad s_2 = q_b + p_4 + p_2 + p_1 \\
    r_3 &= p_4 + p_3 \quad s_3 = q_b + p_4 + p_3 \\
    r_4 &= p_4 + p_3 \quad s_4 = q_b + p_4 + p_3 + p_1 \quad w_1 = p_3 + p_1 + p_4 \\
\end{align*}
\]

(2.11)

Setting \( a_\alpha = \frac{q_b}{2} + i q_b \) one may finally write the measure \( d (a_{\alpha}) \) in the form \( d (a_{\alpha}) = dp_b (q_b) \), where \( m (p_b) = 4 \sinh 2 \, b \sinh 2 \, b \, \sinh 2 \, b \).

![Figure 1. The A- and B-moves](image-url)
B-MOVE: The B-move is realized simply by the multiplication with the phase factor

$$\text{B}^{\text{Liou}} (a_1; a_2; a_3) = e^{i a_1 a_2 a_3}$$ (2.12)

where \(a_k, k = 1; 2; 3\) are the conformal dimensions \(a = a Q a\).

3. Relations between classical Liouville theory and Teichmüller theory

3.1. The semi-classical limit

In order to present first hints towards the geometrical interpretation of quantum Liouville theory let us consider a semi-classical limit of the Liouville correlation functions, following \(^{19,21}\). We will study the limit when \(b \to 0\) with \(i = h a, i = 1; \ldots; n\) fixed. Noting that the rescaling

$$r = 2 b$$ (3.13)

relates to the classical Liouville field \(r\) and \(b\) to one lead to the expectation that the semi-classical limit of the correlation functions should be of the form

$$V_{a_r} (z_a; z_b) \cdots V_{a_s} (z_s; z_t) \exp b^{-2} \log \prod_{s}^{n} \left( z_s \right)$$ (3.14)

where \(r = (z; z) \in \mathbb{R} \), \(z = (z_1; \ldots; z_n)\) is the unique solution to the euclidean Liouville equation

$$\theta \theta r = 2 e^{\log z} \theta_0 z;$$ (3.15)

with the boundary conditions

$$r (z; z) = 2 (1 \ldots z) \log j + O (1) \text{ at } \eta j! \text{ if } z = 1;$$

$$r (z; z) = 2 (1 \ldots z) \log z + O (1) \text{ at } \eta j! \text{ if } z = 1;$$ (3.16)

The divergence of \(r\) near the singular points \(z_1; \ldots; z_n\) requires the inclusion of suitable regularization terms in the definition of the classical action \(S^c l \{ r \} = \lim \frac{1}{c} S^c l \{ r \} \) where

$$S^c l \{ r \} = \frac{1}{4} \sum_{z} \int_{z}^{\eta j} g \frac{d z}{2 (1 \ldots z) \log z}$$ (3.17)

$$\text{dx} \theta r \theta_0 z \theta_0 z;$$

where \(D 1 = f z 2 C; \eta z < g, D 2 = f z 2 C; \eta z > \) and \(X = D 2 n S \ldots D 1\).

Remark 3.1. It is not rigorously proven yet that the Liouville correlation functions constructed in the previous section have a semi-classical asymptotics given by (3.14). So far it was directly verified only in the case of the three-point function \(^{19}\). Evidence for the validity of (3.14) for \(s > 3\) will follow from our discussion in \(x \delta\).
3.2. Energy-momentum tensor and accessory parameters

An important observable is the energy-momentum tensor $T(z)$. In the classical Liouville theory it may be defined as

$$T(z) = \frac{1}{2} (\gamma')^2 + \gamma'' z,$$  \hfill (3.18)

It is a classical result that the evaluation of $T(z)$ on the (unique) solution of equations (3.15) and (3.16) yields an expression of the following form:

$$T(z) = \sum_{i=1}^{s} \frac{i}{(z - z_i)^2} + \frac{C_i}{z - z_i},$$  \hfill (3.19)

where $i = \delta(1 - i)$. The asymptotic behavior of $T(z)$ near $z = 1$ may be represented as

$$T(z) = \frac{s}{z^2} + \frac{C_s}{z^3} + O(z^{-4}).$$  \hfill (3.20)

The so-called accessory parameters $C_i, i = 1; \ldots; s$ are nontrivial functions on the moduli space

$$M_0^s = \{(z_1; \ldots; z_s); z_i \neq 0; 1 \text{ and } z_i \neq z_j \text{ for } i \neq j\}.$$  \hfill (3.21)

of Riemann surfaces with genus 0 and $s$ punctures, which are restricted by the relations

$$C_1 = 0; \quad (z_i C_1 + h_1) = h_n i; \quad (z_i^2 C_1 + 2h_1 z_i) = C_s.$$  \hfill (3.22)

It is instructive to compare (3.19) to the conformal Ward-identities, which are often written in the following form

$$T(x) V_{a_1} (z_2; z_3) \ldots V_{a_1} (z_1; z_1) = \prod_{i=1}^{s} \frac{a_i}{(z - z_i)^2} + \prod_{i=1}^{s} \frac{\partial}{\partial z_i} V_{a_1} (z_2; z_3) \ldots V_{a_1} (z_1; z_1).$$  \hfill (3.23)

Validity of the asymptotic relation (3.14) would therefore imply that

$$C_i = \frac{\partial}{\partial z_i} S^{c_1} (\mathbf{x});$$  \hfill (3.24)

Equation (3.24) is a nontrivial prediction which was proven directly in $^{20,21}$. Similar relations can also be shown to hold in the case of compact Riemann surfaces of arbitrary genus $^{22}$.

3.3. Geometrical interpretation

In the case of a generic conformal field theory one usually interprets the insertion points $z_i$ as parameters for the “gravitational” euclidean background on which one studies the theory. For the case at hand, however, we may note that (3.15) implies that the metric

$$ds^2 = e^{2\gamma} |dz|^2$$  \hfill (3.25)

represents the unique metric of constant negative curvature that is compatible with the complex structure on the punctured Riemann sphere $\mathbb{P}^1 \setminus \{z_1; \ldots; z_s\}$. If one interprets $\gamma$ as describing via (3.25) the gravitational background itself, it becomes natural to study the action $S^{c_1}$ as a function of the “moduli” $z_1; \ldots; z_s$, thereby elevating the moduli to dynamical variables. Indeed, if one
fixes only the topological type of Riemann surface that one wants to work on (here by choosing the number \( s \) of operator insertions), then the moduli space \( M_0^s \) can be identified with the space of solutions of the Liouville equation (3.15).

In order to formulate the corresponding quantization problem one has to describe the symplectic structure of the relevant phase space, which will here be identified with the space of solutions of the Liouville equation (3.15) on a Riemann surface with fixed topological type. Knowing the action \( S \) as a function on phase space makes it possible to extract the corresponding symplectic structure in the usual manner. Working with the complex coordinates \( z_i, i = 1; \ldots; s \) it is of course natural to take advantage of the complex structure and to define the symplectic form associated to \( S \) as

\[
\Omega = 2 i \frac{\partial S}{\partial \bar{z}_i},
\]

where \( \partial, \bar{\partial} \) are the holomorphic and anti-holomorphic components of the de Rham differential on \( M_0^s \) respectively. This symplectic form turns out to be identical to the natural symplectic form on \( M_0^s \), the so-called Weil-Petersson form \( \Omega_{WP} \) (see e.g. 23):

Theorem 3.1. (Takhtajan-Zograf)

\[
\Omega = \Omega_{WP}
\]  

(3.27)

From this point of view one is naturally led to ask the following two questions:

Is it possible to quantize the spaces \( (M_0^s, \Omega_{WP}) \) in a natural way? In fact, it turns out to be better to ask for a quantization of the corresponding \textit{Teichmüller spaces} \( (T_0^s, \Omega_{WP}) \) which are the universal covering spaces of the moduli spaces \( M_0^s \). The nontrivial topology of the moduli spaces \( M_0^s \) may then be taken into account by requiring that the covering group (the \textit{mapping class group} \( \mathcal{MCG}_0^s \)) gets represented by unitary operators.

Is it possible to give a natural interpretation for the correlation functions in the quantum Liouville theory within the quantum theory obtained by quantizing \( (T_0^s, \Omega_{WP}) \)?

One might hope that there exists a “coherent-state” representation for the Hilbert space \( \mathcal{H}_0^s \) in which the wave-functions are holomorphic functions on the Teichmüller spaces \( T_0^s \) and the (holomorphic) coordinates \( z_n, n = 1; \ldots; s \) are realized as multiplication operators. The relations (3.24) furthermore identify the accessory parameters \( C_m, m = 1; \ldots; s \) as some sort of conjugate momenta to the holomorphic variables \( z_n \), in the sense that

\[
\{ z_n; z_m \} = 0 = \{ C_n; C_m \}, \quad \{ z_n; C_m \} = \frac{1}{2} \delta_{nm}.
\]

(3.28)

This suggests that the sought-for coherent-state representation should be such that the accessory parameters get represented by the holomorphic derivatives \( \frac{\partial}{\partial z_n} \). The correlation functions of the quantum Liouville theory, being related to the functions \( S^{cl'} \) on the phase space \( T_0^s \) in the semi-classical limit, should correspond to certain natural operators \( O_\lambda \) on \( \mathcal{H}_0^s \). An operator \( O \) on \( \mathcal{H}_0^s \) would in a holomorphic representation be represented by a kernel \( K_O \) that depends holomorphically on \( V = (v_1; \ldots; v_s) \) and anti-holomorphically on \( \bar{W} = (\bar{w}_1; \ldots; \bar{w}_s) \). Could it be that the relation

\[
V_{\lambda_1} (1; i) V_{\lambda_0} (0; 1) V_{\lambda_0} (0; 0) \prod_{i=1}^{3} V_{\lambda_i} (v_i, \bar{w}_i) = K_{\lambda_0} \quad V; \bar{W}
\]

(3.29)
holds for a certain operator \( O_A \). And indeed, the compatibility of (3.29) with the conformal Ward identities (3.23) requires that the operators \( C_n \) that correspond to the classical observables \( C_n \) should be given by

\[
C_n = \frac{k^2 \theta}{\theta z_n} \quad (3.30)
\]

We are going to propose that (3.29) holds for \( O_A = \pm \text{id} \). Unfortunately, so far we only know quantization schemes for \( T_0^s \) in which the wave-functions are represented as functions of real variables at present \( 11,12,13,14,15,16 \). However, a precise relationship between these quantization schemes and Liouville theory was exhibited in \( 8 \), as we will review in the next section before we further discuss the possible existence of a coherent-state representation for the quantized Teichmüller spaces \( T_0^s \).

4. Classical and quantized Teichmüller spaces

Throughout this section we will consider Riemann surfaces of arbitrary genus \( g \) and number of boundary components \( s \).

4.1. The Fenchel-Nielsen coordinates

A classical set of coordinates for the Teichmüller spaces \( T(\cdot) \) are the so-called Fenchel-Nielsen or length-twist coordinates, see e.g. \( 23 \) for a review. These coordinates describe the inequivalent ways of gluing hyperbolic trinions to form two-dimensional surfaces with negative constant curvature.

The basic observation underlying the definition of the Fenchel-Nielsen coordinates is the fact that for each triple \( (l_1, l_2, l_3) \) of positive real numbers there is a unique metric of constant curvature \( -1 \) on the three-holed sphere (trinion) such that the boundary components are geodesics with lengths \( l_i \), \( i = 1, 2, 3 \). A trinion with its metric of constant curvature \( -1 \) will be called hyperbolic trinion. There exist three distinguished geodesics on each hyperbolic trinion that connect the boundary components pairwise.

Let us call a trinion marked if it carries a graph like the one depicted in Figure 2. Marked trinions can be glued such that the markings glue to a three-valent graph on the resulting Riemann surface. Conversely one may decompose each surface of genus \( g \) with \( s \) circular boundaries into trinions by cutting along a maximal set of mutually non-intersecting cycles \( c_i, i = 1, \ldots, t \), where

\[
3g - 3 + s = \sum_{i=1}^{t} \chi(c_i) \quad (4.31)
\]
A trivalent graph on will be called Moore-Seiberg graph if it is isotopic to a graph that can be constructed by gluing marked trinions.

Let now \((\gamma ;\gamma)\) be a hyperbolic surface with geodesic boundary which is marked with a Moore-Seiberg graph. The Moore-Seiberg graph defines a decomposition of into hyperbolic trinions by cutting along mutually non-intersecting geodesics \(c_i, i = 1;\ldots;\) of the geodesics form half of the Fenchel-Nielsen coordinates. In order to define the remaining half let us start with the case that the geodesic \(c_i\) separates two trinions \(t_{ia}\) and \(t_{ib}\). Pick boundary components \(c_{ia}\) and \(c_{ib}\) of \(t_{ia}\) and \(t_{ib}\) respectively by starting at \(c_i\), following the marking graphs, and turning left at the vertices. As mentioned above, there exist distinguished geodesics on \(t_{ia}\) and \(t_{ib}\) that connect \(c_i\) with \(c_{ia}\) and \(c_{ib}\) respectively. Let \(\varepsilon_i\) be the signed geodesic distance between the end-points of these geodesics on \(c_i\), and let

\[
\varepsilon_i = 2 \frac{i}{l_i}
\]

be the corresponding twist-angle. In a similar way one may define \(\varepsilon_i\) in the case that cutting along \(c_i\) opens a handle.

It can be shown (see e.g.23) that the hyperbolic surface is characterized uniquely by the tuple \((l_i;\ldots;l;\varepsilon^1;\ldots;\varepsilon^s)\). In order to describe the Teichmüller space \(T(\gamma)\) of deformations of it suffices to allow for arbitrary real values of the twist angles \(\varepsilon_i\). Points in \(T(\gamma)\) are then parametrized by tuples \((l_i;\ldots;l;\varepsilon_1;\ldots;\varepsilon_s)\) \((\mathbb{R}^s)\) \(\mathbb{R}\).

The Weil-Petersson symplectic form becomes particularly simple in terms of the Fenchel-Nielsen coordinates:

**Theorem 4.1. (Wolpert)\(^25\)**

\[
\omega_{WP} = \sum_{i=1}^{X} d\varepsilon_i \wedge dl_i, \quad \varepsilon_i = \frac{1}{2} l_i i;
\]

4.2. The Moore-Seiberg groupoid

Changes of the Moore-Seiberg graph generate canonical transformations from one set of Fenchel-Nielsen coordinates to another. The transitions between different Moore-Seiberg graphs on a surface generate the Moore-Seiberg groupoids \(M S(\gamma)\). These groupoids can be characterized by generators and relations \(^{17;18}\). In the case that \(\gamma\) has genus \(g > 0\) one needs to supplement the generators depicted in Figure 1 by one additional generator only, which is shown in Figure 3 below.

![Figure 3. The S-move](image)

All relations of the Moore-Seiberg groupoid can be shown to follow from a finite set of basic relations \(^{17;18}\). In order to derive the basic relations it suffices to consider the cases \(g = 0, s = 4;5\) and \(g = 1, s = 1;2\).
The Moore-Seiberg groupoid \( M S( ) \) contains the mapping class group \( M C G( ) \) as an important subgroup. \( M C G( ) \) is generated by the Dehn-twists, which represent the operation of cutting out an annular region, twisting one end by an angle of \( 2\pi \) before re-gluing, as indicated in Figure 4. This operation will map any Moore-Seiberg graph on a surface into another one. The action of a Dehn-twist on an annulus.

4.3. Quantization of the Teichmüller spaces

The quantization of \( T( ) \) with the Weil-Petersson symplectic form \( !_W \) is not canonical in the Fenchel-Nielsen coordinates. One needs to implement the restriction that the operators \( l_i, i = 1; \ldots; g \), which represent the lengths of the closed geodesics \( c_i \), have positive spectrum. Fortunately there exists an alternative set of coordinates, introduced by Penner, which has the advantage to make the quantization of \( T( ) \) canonical. The result of these constructions is an assignment ("modular functor")

\[
! : H_T( ) \otimes A_T( ) \rightarrow T_{\mathfrak{c}_c}( ) ; \quad l \in \mathbb{R}_+^g
\]

where \( H_T( ) \) is a Hilbert space, \( A_T( ) \) is an algebra of operators on \( H_T( ) \) that quantizes the commutative algebra of functions on \( T( ) \), and \( T_{\mathfrak{c}_c}( ) \) is a representation of the mapping class group \( M C G( ) \) by unitary operators on \( H_T( ) \).

Moreover, there is a construction that allows one to obtain reasonably simple expressions for the length functions in terms of the Penner coordinates. Based on this observation it becomes possible to construct the quantum operators \( l_i, i = 1; \ldots; g \) that correspond to the geodesic length functions \( l_c \). The key technical result concerning the operators \( l_i \) was obtained in \( 11; 15; 16 \). They indeed represent positive self-adjoint operators with spectrum \( \mathbb{R}_+^g \).

It is then not difficult to show that the length operators \( l_c \) and \( l_c^0 \) associated to two non-intersecting closed geodesics \( c \) and \( c^0 \) always commute, \([l_c; l_c^0] = 0\). Specifying a Moore-Seiberg graph amounts to picking a maximal set \( c_1; \ldots; c_s \) of mutually non-intersecting closed geodesics. By simultaneous diagonalization of the corresponding length operators \( l_1; \ldots; l_g \) one may construct a basis for \( H_T( ) \) which consists of generalized eigenfunctions of \( l_1; \ldots; l_g \).

This also allows us to generalize the definition of the "modular functor" (4.34) to the case that is a Riemann surface with \( g \) geodesic boundaries of fixed length rather than punctures. We will use the notation \( L \) to indicate the dependence on the values \( \mu = (\mu_1; \ldots; \mu_s) \) of the boundary lengths. It will also be convenient to denote by \( \mu \), with \( L = (\mu_1; \ldots; \mu_g) \in \mathbb{R}_+^g \), the Moore-Seiberg graph "colored" by assigning the values \( \mu_i \) to the geodesics \( c_i \).

**Theorem 4.2.** For each Moore-Seiberg graph on a surface of genus \( g \) with \( g \) geodesic boundaries there exists a basis \( B = \{ l_1; \ldots; l_g \} \in \mathbb{R}_+^g \) of generalized eigenfunctions...
of $\mathbf{g}$ such that the completeness relation for $B$ can be written as
\begin{equation}
\mathcal{M}_{\mathcal{B}^{(1)}} = \prod_{j \mu, j \mu} \mathcal{M}(\mathcal{L}) \mu \mu j j
\end{equation}
where the measure $\mathcal{M}(\mathcal{L})$ is defined as
\begin{equation}
\mathcal{M}(\mathcal{L}) = \frac{1}{b} \sinh \frac{1}{2} \sinh \frac{1}{2b^2}.
\end{equation}

The outcome of these constructions may be considered as a quantization of the Fenchel-Nielsen coordinates associated to a fixed Moore-Seiberg graph. One may consider the operators $l_i, i = 1; \cdots ;$ as a natural set of Hamiltonians, and the corresponding one-parameter groups $e^{\pm it}$, $~ = b^2$ as quantum counterparts of the Fenchel-Nielsen twist flows.

\section{4.4. Representation of the Moore-Seiberg groupoid on $H^T(\mathcal{L})$}

In order to characterize a representation of the Moore-Seiberg groupoid it suffices to specify the operators $U_{\{i, \cdots, j\}}$ for the cases where $2$ and $1$ differ by an A-, B- or S-move. The corresponding operators will be denoted by $A$, $B$ and $S$, and will be defined below.

\textbf{A-MOVE:} In order to describe the representation of the A-move let $\mathcal{L}$ be the four-punctured sphere, with parameters $= \mathcal{L}_a, i = 1; \cdots ;$ associated to the four boundary components respectively. The basis corresponding to the Moore-Seiberg graph $^a\mathcal{L}$ depicted in the leftmost diagram in Figure 1 will be denoted by $B_a \mathcal{L} = f^a_{\mathcal{L}_a, \mathcal{L}_b} \mathcal{L}_c \mathcal{L}_d 2 \mathcal{R}^+g$, whereas the graph on the second diagram from the left in Figure 1 will be denoted by $B \mathcal{L}$, with corresponding basis $B \mathcal{L} = f^a_{\mathcal{L}_a, \mathcal{L}_b} \mathcal{L}_c \mathcal{L}_d 2 \mathcal{R}^+g$. The A-move is then represented as an integral transformation of the following form.
\begin{equation}
\int_{\mathcal{L}} f^a_{\mathcal{L}_a, \mathcal{L}_b} \mathcal{L}_c \mathcal{L}_d 2 \mathcal{R}^+g.
\end{equation}

The measure $\mathcal{M}(\mathcal{L})$ is defined by specializing (4.36) to $= 1$. The kernel $F_{a_1 a_2}^{a_3 a_4}$ turns out to be the coincide with $F_{a_1 a_2}^{a_3 a_4}$ provided that the parameters are related as
\begin{equation}
a_1 = \frac{Q}{2} + \frac{1}{4a_2}, \quad \forall \{2; 3; 4; 5\}.
\end{equation}

\textbf{B-MOVE:} The B-move is realized simply by the multiplication with the phase factor
\begin{equation}
B \mathcal{L} = e^{it} \mathcal{L}_a \mathcal{L}_b.
\end{equation}

where $a_i$ and $l_i, i = 1; 2; 3$ are related as in (4.38), and $a_i = a_i(Q, a_i).

\textbf{S-MOVE:} Let $\mathcal{I}$ be a torus with one hole of length $\mathcal{L}$. The elements of bases corresponding to the Moore-Seiberg graphs $^a\mathcal{I}$ and $^b\mathcal{I}$ shown on left and right halves of Figure 3 will be denoted by $B_a f^a_{\mathcal{L}_a, \mathcal{L}_b} \mathcal{L}_c \mathcal{L}_d 2 \mathcal{R}^+g$ and $B_b f^b_{\mathcal{L}_a, \mathcal{L}_b} \mathcal{L}_c \mathcal{L}_d 2 \mathcal{R}^+g$ respectively. The S-move is then represented as an integral transformation of the following form.
\begin{equation}
\int_{\mathcal{I}} f^a_{\mathcal{L}_a, \mathcal{L}_b} \mathcal{L}_c \mathcal{L}_d 2 \mathcal{R}^+g.
\end{equation}
The kernel $S_{l, l'} (q_e)$ is given by the following expression:

$$S_{l, l'} (q_e) = \frac{2^{3/2}}{s_b (q_e)} \int_0^Y \frac{dr}{s_b p_b + \frac{1}{2} (q_e + q_b) + r} e^{4 i p_e r}.$$  \hspace{1cm} (4.41)

We have set $l_i = 4 b p_i$ for $\{ 2 fa; b; e \}$.

**Theorem 4.3.**

(i) The operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$ generate projective representations $\mathcal{M}_S (\ )$ of the Moore-Seiberg groupoids, with central extension $\alpha$ being related to the Liouville central charge $c = 1 + 6 Q^2$ by $\alpha = e^{2\pi i c}$.

(ii) The corresponding representations $\mathcal{M}_{\mathcal{C}G} (\ )$ of the mapping class groups reproduce the classical action of $\mathcal{M}_{\mathcal{C}G} (\ )$ on the Teichmüller spaces $T ( \ )$ in the limit $b \to 0$.

(iii) The representations $\mathcal{M}_{\mathcal{C}G} (\ )$ coincide with the representations of the Moore-Seiberg groupoids on the spaces of genus zero conformal blocks in Liouville theory.

5. **Towards a coherent state quantization of $T^0_s$**

We now want to further discuss the possible existence of the coherent state quantization of $T^0_s$. To begin with, we should formulate more precisely what we consider to be defining properties of such a quantization scheme.

The most basic requirement is of course that the quantum states in this representation are represented by holomorphic functions on $T^0_s$, and that the quantum observables which correspond to analytic functions on $\mathcal{M}_{\mathcal{C}G} (\ )$ are represented as multiplication operators. We have furthermore argued that the quantum operators $C_i$ that correspond to the accessory parameters $C_i$ should be represented as

$$C_i = \frac{b^2}{\partial z_i}$$  \hspace{1cm} (5.42)

if we use the coordinates $z_i$, $i = 1; \ldots; s 3$, as local coordinates for $T^0_s$.

Another important ingredient must of course be the representation of the mapping class groups $\mathcal{M}_{\mathcal{C}G} (\ )$. Assume that we describe the states $j i$ by multi-valued wave-functions $(Z)$, $Z = (z_1; \ldots; z_n \ldots)$. For each element $m \in \mathcal{M}_{\mathcal{C}G} (\ )$, and each function $(Z)$ we may define the function $m (Z)$ via analytic continuation as the result of the covering transformation corresponding to $m$. In the previous section we have associated a unitary operator $U_m$ to each element $m \in \mathcal{M}_{\mathcal{C}G} (\ )$.

It is then natural to require that

$$U_m (Z) = m (Z):$$  \hspace{1cm} (5.43)

Let us now formulate the conjecture that we want to propose.

**Conjecture 5.1.**

(i) There exists a representation for the quantized Teichmüller spaces with the properties above. This requires in particular the existence of a measure $d (\xi)$ on $T^0_s$ such that

$$h_{2j' l i} = \int_{T^0_s} d (\xi) \cdot 2 (\xi) \cdot (\xi):$$

---

\*In the sense of $26$, $x5.7$
We have denoted by \( i(\mathbb{X}) \), \( i = 1;2 \) the analytic functions on \( \Gamma_0^0 \) that correspond to the multi-valued wave-functions \( i(\mathbb{Z}) \).

(ii) The Liouville conformal blocks \( F_{S, A}(\mathbb{Z}) \) represent the (generalized) eigenfunctions \( \mu(\mathbb{Z}) = \mathbb{Z}_j \mu_i \) of the length operators in the coherent state representation, where the sets of parameters are related via (4.38).

(iii) The vacuum expectation values of primary fields in quantum Liouville theory represent the kernel of the identity operator in the coherent state representation.

One may immediately observe that point (iii) of Theorem 4.3 strongly supports part (ii) of our conjecture. Indeed, our requirement (5.43) fixes the monodromies of the wave-functions that might represent the eigenfunctions of the length operators. We are therefore dealing with a Riemann-Hilbert type problem, for which point (iii) of Theorem 4.3 asserts the existence of a solution.

Let us furthermore observe that parts (i) and (ii) of our conjecture actually imply part (iii). This becomes clear if one notes that in the length-representation for \( H_0^0 \) one may represent the identity as in (4.35). Comparing (4.35) with the holomorphically factorized representation (2.4) for the vacuum expectation values of primary fields immediately yields part (iii) of our conjecture.

5.1. Quantization of the boundaries of the Teichmüller spaces

In the following we will consider surfaces for which all boundary components are punctures, i.e. holes of zero size. We want to show that the conjecture above can be verified quite explicitly if one restricts attention to the behavior of the relevant objects near the boundaries of the Teichmüller spaces which correspond to degenerating Riemann surfaces. This will allow us to show that the Liouville conformal blocks \( F_{S, A}(\mathbb{Z}) \) are in fact the only reasonable candidates for the eigenfunctions \( \mu(\mathbb{Z}) \) of the length operators as conjectured in part (ii) of our conjecture.

The relevant degenerations correspond to vanishing of the length \( l \) of a closed geodesic. Let us denote the (possibly disconnected) Riemann surface obtained by cutting by \( \gamma \). There always exists an annular region around the geodesic that may be modeled by \( A_{\mathbb{q}} = \mathbb{f}(\mathbb{z}) \mathbb{w} \mathbb{f}^{-1} \) for \( j_{\mathbb{q}} < 1 \). The complex parameter \( \mathbb{q} \) represents the “sewing” modulus that appears if one reconstructs \( \gamma \) as in 2.2, with \( j_{\mathbb{q}} \) corresponding to shrinking the length \( l \). \( \mathbb{q} \)

The behavior near degeneration is universal, allowing one to consider \( \mathbb{q} \) independently from the other moduli of \( \gamma \). Fortunately it is possible to calculate the asymptotic behavior for \( j_{\mathbb{q}} \) of all relevant objects explicitly. First, the relation between \( j_{\mathbb{q}} \) and the Fenchel-Nielsen coordinates \( (l, g) \) associated to \( \gamma \) is given by:

\[
1 = \frac{2}{\log(1/j_{\mathbb{q}})}; \quad 2 = \arctan(\mathbb{q}); \quad (5.44)
\]

The Weil-Petersson symplectic form \( \omega_{WP} \) can be written as:

\[
\omega_{WP} = \frac{3}{\log^3(1/j_{\mathbb{q}})} dq \wedge dq; \quad (5.45)
\]

The accessory parameter \( C_{\mathbb{q}} \) corresponding to \( \mathbb{q} \) is finally given by the expression:

\[
C_{\mathbb{q}}(j_{\mathbb{q}}) = \frac{1}{4\mathbb{q}} \frac{2}{\log^2(1/j_{\mathbb{q}})} 1; \quad (5.46)
\]
By using (5.44)-(5.46) it is straightforward to verify that
\[ f_q; C_q g = \frac{1}{2}; \]
(5.47)
It is therefore indeed natural to define the quantum operator \( C_q \) that corresponds to \( C_q \) by \( \frac{\partial^2}{\partial q^2} \), as required in (5.42). Let us furthermore note the relation
\[ \varphi C_q (q; \varphi q) = \frac{1}{4} \varphi^2 \frac{1}{4}; \]
(5.48)
which follows from (5.44) and (5.46). We propose to “quantize” this relation as
\[ q \frac{\partial}{\partial q} = \frac{1}{4} \frac{1}{4}; \]
(5.49)
where \( \varphi \) is the geodesic length operator. The motivation for this particular operator ordering will be discussed in Remark 1 below. It is now of course easy to find the eigenfunctions of \( \varphi \) in the \( q \)-representation. They are given as
\[ \varphi (q) = q_{-\frac{1}{2}} \frac{1}{4}; \]
(5.50)
This coincides precisely with the asymptotic behavior of the Liouville conformal blocks for \( j \neq 0 \).

**Remark 1.** Instead of (5.49) one might consider the more general ansatz
\[ q \frac{\partial}{\partial q} + (1 - \frac{1}{4}) \frac{1}{4}; \]
(5.51)
which parametrizes the ambiguity of ordering the operators \( q \) and \( \frac{\partial}{\partial q} \) if we assume that \( 0 \neq 1 \).

The choice \( \frac{1}{4} (z; \frac{1}{4}) \) adopted in (5.49) is the only one that is compatible with our requirement (5.43), which determines the monodromy around \( q = 0 \). We conclude that the Liouville conformal blocks \( F_{z; A} (z) \) are in fact the only candidates for the eigenfunctions \( \varphi \) (\( z \)) that are compatible with both our requirement (5.43) and with the ansatz (5.51), which determines the asymptotic behavior of eigenfunctions of the length operator \( \varphi \) near the boundary of the Teichmüller spaces.

Let us finally briefly comment on the existence of a suitable measure of integration for the definition of the scalar product. We will propose to consider an ansatz of the form
\[ \int_{\mathbb{H}} d^2 x \; (\kappa; x) \frac{1}{2} (\varphi^2; x); \]
(5.52)
where we have introduced the “uniformizing” variable \( x \) via \( q = e^{2 \varphi i x} \), and integrate \( x \) over the upper half plane \( \mathbb{H} \). In order to satisfy part (ii) of our conjecture we must have
\[ \int_{\mathbb{H}} d^2 x \; (\kappa; x) \frac{1}{2} (\varphi^2; x) = \frac{1}{m (q)} (1 \varphi); \]
(5.53)
Taking into account the explicit form (5.50) of \( \varphi (q) \) one may observe that it suffices to assume that \( (\kappa; x) \) does not depend on \( x \) in order to produce the delta-distribution in (5.53). \( (\kappa; x) \)
\[ \varphi (\varphi x) \] can then be determined in terms of \( m (q) \) by means of an inverse Laplace transformation.
6. “Exercises”

(1) Prove Conjecture 5.1.

(2) Construct the Liouville conformal blocks on higher genus Riemann surfaces and prove Conjecture 5.1 for these cases as well.

(3) Develop the theory of Teichmüller spaces for Riemann surfaces with boundaries of arbitrary shape. Quantize these spaces. Thereby gain insight into the relations between the conformal Ward identities, the geometric action of the Virasoro algebra on moduli spaces and the quantization of Teichmüller spaces.

(4) Is it possible to quantize the universal Teichmüller space of Bers in a natural way? How is this related to the solution of Exercise 3?

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