On quantum shuffle and quantum affine algebras

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Abstract

A construction of the quantum affine algebra $U_q(\hat{g})$ is given in two steps. We explain how to obtain the algebra from its positive Borel subalgebra $U_q(b^+)$, using a construction similar to Drinfeld’s quantum double. Then we show how the positive Borel subalgebra can be constructed with quantum shuffles.

Introduction

Let $\hat{g}$ be an affine Lie algebra over a simple finite dimensional Lie algebra $g$. The quantum affine algebra $U_q(\hat{g})$ is defined by generators and relations in the standard Drinfeld-Jimbo presentation. However, it is well known that the Kac-Moody affine algebra $\hat{g}$ has a natural realization as a central extension of the loop algebra $g \otimes \mathbb{C}[t, t^{-1}]$. Among few attempts to generalize this to the quantum case, another realization of the quantum affine algebra is given by Drinfeld in [3]. The algebra structure of $U_q(\hat{g})$ is given in term of generating series, where $\hat{g}$ is an untwisted affine Kac-Moody algebra. However the Drinfeld-Jimbo coalgebra structure leads to very complicated formulas,
which cannot be expressed in closed form using generating functions. A new coalgebra structure was given by Drinfeld (in an unpublished note), with a quite simple formulation. Ding, Iohara and Frenkel used this new Drinfeld comultiplication in [1] and [2]. In the first part of this work, we show that the algebra structure of $U_q(\hat{g})$ can be derived from a construction adapted from Drinfeld’s quantum double, using the comultiplication expressed in closed form in the new realization. On the other hand, in his paper [7], Nichols examined the structure of Hopf bimodules. A particular subalgebra of a cotensor construction was used by Rosso in [9], where the algebra structure is completely described with the help of a braiding defining the quantum shuffle product. By choosing suitable Hopf bimodules, Rosso showed that the positive Borel subalgebra of the quantized Hopf algebras $U_q(g)$ can be obtained using this shuffle construction. In the second part of this paper, we show that the positive affine subalgebra $U_q(b^+)$ can be constructed using a similar method.

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### 1 A Hopf algebra structure on $U_q(\hat{g})$

#### 1.1 Drinfeld’ new realization of $U_q(\hat{g})$

Let $A = (a_{ij})$ be a symmetrizable Cartan matrix corresponding to a simple Lie algebra $g$. Let $\hat{g}$ be the corresponding non twisted affine Kac-Moody algebra. The normalized symmetric invariant form on $\hat{h}^*$ will be noted $(\cdot|\cdot)$. The set of simple roots in $\hat{h}^*$ is $\alpha_1, \ldots, \alpha_n$. If $q$ is a non zero generic complex number, i.e. $q$ is not a root of 1, we put $q_i = q^{(\alpha_i|\alpha_i)/2}$ and $q_{ij} = q^{(\alpha_i|\alpha_j)}$. Remark that $q_{ij} = q_{ji}$.

Now, let $f_{ij}(t) = (q_{ij}t-1)/(t-q_{ij})$ a complex valued function. Let $g \in \mathbb{C}[[t]]$ be the formal series $\sum_{n \geq 0} g^{(ij)}_n t^n$ where the coefficients $g^{(ij)}_n$ are defined by the Taylor series of $f$ at zero, i.e. $f(t) = \sum_{n \geq 0} g^{(ij)}_n t^n$ for $|t| \ll 1$.

Remark that $f_{ij}(t)f_{ji}(t^{-1}) = 1$. But even by embedding $\mathbb{C}[[t]]$ in $\mathbb{C}[[t, t^{-1}]]$ in the canonical way, the same relation does not hold for the formal series $g$, because we cannot have both $|t| \ll 1$ and $|t| \gg 1$. By doing this work, we had in mind a “functional” point of view, as in [3], where the authors
speak of functional algebras, or in \([3]\), where the authors have relations such as \(x_i(z)x_j(w) = R_{ij}(w, z)x_j(w)x_i(z)\). For such relations to hold under the permutation of \(z\) and \(w\), the exchange factors \(R_{ij}\) must satisfy the relation \(R_{ij}(z, w)R_{ji}(w, z) = 1\). But as we shall soon see, commutation relations cannot be stated in such a clean way in our case. However, we have the two following useful relations in \(\mathbb{C}[[t, t^{-1}]]:\)

\[
\begin{align*}
(q_{ij} - t)g(t) &= (q_{ij}t - 1), \\
(q_{ij}t - 1)g(t^{-1}) &= (q_{ij} - t).
\end{align*}
\]

**Definition 1 (\([2]\))** \(U_q(\mathfrak{g})\) is an associative algebra with unit 1 and generators \(\{x_{i,n}^+, x_{i,n}^-, \varphi_{i,-k}, \psi_{i,k}, q^{\pm\frac{1}{2}} | i = 1 \ldots n - 1, n \in \mathbb{Z}, k \in \mathbb{N}\}\), satisfying the following relations expressed in terms of generating functions in formal variables \(z\) or \(w\):

- \(q^{\pm\frac{1}{2}}\) are central and mutually inverse,
  \[q_{i,0}^0 = \psi_{i,0}^0 \in \mathbb{C}, \quad q_{i,j}^0 q_{i,j}^{-1} = 1, \quad j \in \mathbb{Z}\]
- for any \(i \neq j\),
  \[\varphi_{i}(z)\varphi_{j}(w) = \varphi_{j}(w)\varphi_{i}(z), \quad \psi_{i}(z)\psi_{j}(w) = \psi_{j}(w)\psi_{i}(z),
  \]
- \(g_{ij}(zw^{-1}q^{-1})\varphi_{i}(z)\psi_{j}(w) = g_{ij}(zw^{-1}q^{-1})\psi_{j}(w)\varphi_{i}(z),
  \]
- \(\psi_{i}(x_{j}^\pm(w)) = g_{ij}(zw^{-1}q^{\pm\frac{1}{2}})\psi_{j}(w)\varphi_{i}(z),
  \]
- \([x_{i}^+(z), x_{j}^-(w)] = \frac{\delta_{ij}}{q - q^{-1}} \left(\delta(zw^{-1}q^{-1})\psi_{i}(wq^{\frac{1}{2}}) - \delta(zw^{-1}q^{-1})\varphi_{i}(zq^{\frac{1}{2}})\right),
  \]
- \((z - q_{ij}w)x_{i}^+(z)x_{j}^+(w) = (q_{ij}z - w)x_{j}^+(w)x_{i}^+(z),
  \]
- \(\sum_{r=0}^{1-a_{ij}} (-1)^r [\frac{r}{a_{ij}}]_{q} \text{ Sym}_z \left(x_{i}^+(z_1) \cdots x_{i}^+(z_r)x_{j}^+(w)x_{i}^+(z_{r+1}) \cdots x_{i}^+(z_{1-a_{ij}})\right),
  \]

where \(\varphi_{i}(z) = \sum_{k \leq 0} \varphi_{i,k}z^{-k}, \psi_{i}(z) = \sum_{l \geq 0} \psi_{i,l}z^{-l}\) and \(x_{i}^+(z) = \sum_{k \in \mathbb{Z}} x_{i,k}^+z^{-k}\), where \(\delta(z) = \sum_{k \in \mathbb{Z}} z_{i}^{-k}\). The operator \(\text{Sym}_z\) denotes symmetrization with respect to \(z_1, \ldots, z_{1-a_{ij}}\), and \(\delta\) is the formal distribution with support at 1, that is:

- \(\delta(z) = \sum_{k \in \mathbb{Z}} z^{-k}\).

As usual, \([n]_q\) is the quantum binomial coefficient, with \(\frac{[n]}{[q]} = \frac{[n]_q!}{[p]_q! [n-p]_q!}\), \([n]_q! = [1]_q[2]_q \cdots [n]_q\) and \([n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\).

Relations (3) are the so-called Serre relations.

Remark that products such as \(\delta(zw^{-1}q^{-c})\psi_{i}(wq^{\frac{1}{2}})\) are well defined and can be easily computed.
1.2 A Hopf algebra structure for $U_q(\hat{\mathfrak{g}})$

As the formulas for the coproduct, counit and antipode for this algebra will involve infinite expressions, we shall make some topological completion on the underlying vector space and on the tensor product. However, exhibiting an inverse system on $U_q(\hat{\mathfrak{g}})$ is not straightforward, due to some non-homogeneous relations (e.g. the commutation relation (I)). Therefore we shall follow an indirect path.

Let $\overline{U}_q(\hat{\mathfrak{g}})$ be the free algebra with same generators as $U_q(\hat{\mathfrak{g}})$. Now give $\varphi_{i,-k}$ and $\psi_{i,k}$ degree $k$ for $k \geq 0$. All other elements get degree 0. We then extend the degree on all the elements of the algebra by summation on the monomials. For $i \geq 0$, let $\overline{U}_i$ be the ideal of $\overline{U}_q(\hat{\mathfrak{g}})$ generated by elements of degree greater than $i$. We then get an inverse system $(\overline{U}_q(\hat{\mathfrak{g}})/\overline{U}_i, p_i)$, where $p_i$ is the natural projection $\overline{U}_q(\hat{\mathfrak{g}})/\overline{U}_i \rightarrow \overline{U}_q(\hat{\mathfrak{g}})/\overline{U}_{i-1}$ obtained using the following diagram, where the rows are exact sequences:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \overline{U}_{i+1} & \longrightarrow & \overline{U}_q(\hat{\mathfrak{g}}) & \longrightarrow & \overline{U}_q(\hat{\mathfrak{g}})/\overline{U}_{i+1} & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow p_{i+1} & & & & \\
0 & \longrightarrow & \overline{U}_i & \longrightarrow & \overline{U}_q(\hat{\mathfrak{g}}) & \longrightarrow & \overline{U}_q(\hat{\mathfrak{g}})/\overline{U}_i & \longrightarrow & 0
\end{array}
\]

The completion of $\overline{U}_q(\hat{\mathfrak{g}})$ is then $\overline{U}_q(\hat{\mathfrak{g}})^\tau = \varprojlim \overline{U}_q(\hat{\mathfrak{g}})/\overline{U}_i$ (τ stands for topological), which leads us to the completion of $U_q(\hat{\mathfrak{g}})$: let $U_q(\hat{\mathfrak{g}})^\tau = \overline{U}_q(\hat{\mathfrak{g}})^\tau/I$, where $I$ is the two-sided ideal generated by the relations in definition [1]. There is a canonical injection from $U_q(\hat{\mathfrak{g}})$ into $U_q(\hat{\mathfrak{g}})^\tau$.

In order to complete the tensor product we need a weaker topology (see remarks below) on $U_q(\hat{\mathfrak{g}}) \otimes U_q(\hat{\mathfrak{g}})$. As before, we first consider $\overline{U}_q(\hat{\mathfrak{g}})$. Then we give $q^{\pm \frac{1}{2}}$ degree 0. The generators $x_{i,\pm k}^+$, $x_{i,\pm k}^-$, $\varphi_{i,-k}$ and $\psi_{i,k}$ get degree $k$ for $k \geq 0$. After having extended the degree the usual way on monomials, we denote by $\overline{\mathcal{V}}_i$ the two sided ideal of $\overline{U}_q(\hat{\mathfrak{g}})$ of elements of degree at least $i$. Remark that $\overline{\mathcal{V}}_i$ is a strict subset of $\overline{\mathcal{V}}_i$. The tensor product $\overline{U}_q(\hat{\mathfrak{g}}) \otimes \overline{U}_q(\hat{\mathfrak{g}})$ can now be completed the following way: we get an inverse system by setting $\overline{S}_i = \overline{\mathcal{V}}_k \otimes \overline{\mathcal{V}}_i$. The inverse limit $\overline{U}_q(\hat{\mathfrak{g}}) \hat{\otimes} \overline{U}_q(\hat{\mathfrak{g}}) = \varprojlim \overline{U}_q(\hat{\mathfrak{g}}) \hat{\otimes} \overline{U}_q(\hat{\mathfrak{g}})/\overline{S}_i$ is then a completion of the usual tensor product (the symbol $\hat{\otimes}$ stands for completed tensor product). Now $U_q(\hat{\mathfrak{g}}) \otimes U_q(\hat{\mathfrak{g}})$ can be completed by

\[
U_q(\hat{\mathfrak{g}}) \hat{\otimes} U_q(\hat{\mathfrak{g}}) = \overline{U}_q(\hat{\mathfrak{g}}) \hat{\otimes} \overline{U}_q(\hat{\mathfrak{g}})/((\overline{U}_q(\hat{\mathfrak{g}}) \otimes I + I \otimes \overline{U}_q(\hat{\mathfrak{g}}))).
\]

Using those completions we can now put a Hopf algebra structure on $U_q(\hat{\mathfrak{g}})^\tau$, though in a weak sense:
Proposition 1 \( U_q(\hat{\mathfrak{g}})^T \) is a Hopf algebra for the coproduct \( \Delta : U_q(\hat{\mathfrak{g}})^T \to U_q(\hat{\mathfrak{g}})^T \otimes U_q(\hat{\mathfrak{g}})^T \), the antipode \( S : U_q(\hat{\mathfrak{g}})^T \to U_q(\hat{\mathfrak{g}})^T \) and the counit \( \varepsilon : U_q(\hat{\mathfrak{g}})^T \to \mathbb{C} \) satisfying the following relations:

\[
\begin{align*}
\Delta(q^{\pm \frac{c_1}{2}}) &= q^{\pm \frac{c_1}{2}} \otimes q^{\pm \frac{c_1}{2}}, \\
\Delta(x_i^+(z)) &= x_i^+(z) \otimes 1 + \varphi_i(z) q^{\frac{c_1}{2}} x_i^+(z q^{c_1}), \\
\Delta(x_i^-(z)) &= 1 \otimes x_i^-(z) + x_i^-(z q^{c_1}) \otimes \psi_i(z q^{\frac{c_1}{2}}), \\
\Delta(\varphi_i(z)) &= \varphi_i(z) q^{\frac{c_1}{2}} \otimes \varphi_i(z) q^{\frac{c_1}{2}}, \\
\Delta(\psi_i(z)) &= \psi_i(z) q^{\frac{c_1}{2}} \otimes \psi_i(z) q^{\frac{c_1}{2}}, \\
S(q^{\pm \frac{c_2}{2}}) &= q^{\mp \frac{c_2}{2}}, \\
S(x_i^+(z)) &= -\varphi_i(z) q^{-\frac{c_2}{2}} x_i^+(z q^{-c_2}), \\
S(x_i^-(z)) &= -x_i^-(z q^{-c_2}) \psi_i(z q^{-\frac{c_2}{2}})^{-1}, \\
S(\varphi_i(z)) &= \varphi_i(z)^{-1}, \\
S(\psi_i(z)) &= \psi_i(z)^{-1}, \\
\varepsilon(q^{\pm \frac{c_2}{2}}) &= \varepsilon(\varphi_i(z)) = \varepsilon(\psi_i(z)) = 1, \\
\varepsilon(x_i^+(z)) &= 0,
\end{align*}
\]

where \( c_1 = c \otimes 1 \) and \( c_2 = 1 \otimes c \).

A proof of this proposition can be found in [4]. Although the authors of this paper do not give many details about topological completion, all the formulas appearing in their proof fit well within our framework.

Remark that the coefficient of \( z^n \) in all those formulas involve finite sums, except in (7), (8), (9) and (10). But all infinite sums converge thanks to the inverse limit topology on \( U_q(\hat{\mathfrak{g}})^T \) and on the topological tensor product.

We could ask why the topological completions for \( U_q(\hat{\mathfrak{g}}) \) and for the tensor product are not the same? First, remark that we have to complete the algebra so that the antipode has a valid definition. The completion is weak enough to allow us writing expressions such as \( \varphi_i(z) x_i^+(z) \) in \( U_q(\hat{\mathfrak{g}})^T \), but also strong enough to forbid expressions such as \( g_{ij}(z w^*) x_i^+(z) x_j^+(w) \). Without this last obstruction, the Serre relations would have been a trivial consequence of the commutation relations (3) (we shall come back on that later). The topology on the tensor product is a weaker one, in the sense that expressions such as \( g_{ij}(z w^*) x_i^+(z) \otimes x_j^+(w) \) are allowed, but not \( g_{ij}(z w^*) x_i^+(z) x_j^+(w) \otimes 1 \). This discrimination will be one of the key tool in the sequel. We shall give more details later.
We would like to obtain the whole algebra $U_q(\hat{\mathfrak{g}})^T$ using Drinfeld’s double construction, by giving a Hopf pairing between suitable “positive” and “negative” subalgebras of $U_q(\hat{\mathfrak{g}})^T$. But we shall see in the next part that such a pairing cannot exist in our case. We will nonetheless exhibit a construction similar to Drinfeld’s one and sharing many properties (but not all) with it. This can be thought of as a weak quantum double construction, where we get the algebra structure of $U_q(\hat{\mathfrak{g}})$ using a weak pairing, but of course nothing for the coalgebra structure because the topology is the main obstruction to the mere existence of the pairing. Nevertheless, the restriction of the coproduct to the non-completed subalgebra $U_q(\hat{\mathfrak{g}})$ of $U_q(\hat{\mathfrak{g}})^T$ coincide with the formula for the coproduct on the tensor product of the positive and negative borel subalgebras. Moreover, the pairing will be a non-degenerate one, which will be of essential importance for the last part of this work.

### 1.3 A weak Hopf pairing between $U_q(b^+)$ and $U_q(b^-)$

Let $U_q(n^+)$ be the free algebra generated by $x_{i,n}^+$ and $\varphi_{i,k}$ for $i = 1, \ldots, n-1$, $n \in \mathbb{Z}$ and $k < 0$. The upper triangular part of $U_q(\hat{\mathfrak{g}})$ is $U_q(b^+)$ with relations (1), (2) and (4) given in definition 1. The free algebra with the same basis is $U_q(b^+) = U_q(n^+) \otimes \mathbb{C}[\varphi_{i,0}^{\pm 1}, q^{\pm c'}]$. On the negative side, we get the algebra $U_q(n^-)$ generated by $x_{i,m}^-$ and $\psi_{i,l}$ for $i = 1, \ldots, n - 1$, $m \in \mathbb{Z}$ and $l > 0$. The lower triangular part of $U_q(\hat{\mathfrak{g}})$ is $U_q(b^-)$, with the appropriate relations. The corresponding free algebra is $U_q(b^-) = U_q(n^-) \otimes \mathbb{C}[\varphi_{i,0}^{\pm 1}, q^{\pm c''}]$, where $c'$ behave as $c$. In the sequel, we shall also consider the sub algebra $U_q(b^+)$ of $U_q(b^-)$ generated by $\varphi_{i,k}$, $\varphi_{i,0}^{-1}$ and $q^{\pm \frac{1}{2}}$ for $i = 1, \ldots, n-1$ and $k \leq 0$. Its negative counterpart, $U_q(b^-)$ is constructed similarly. Remark that $U_q(b^+)$ and $U_q(b^-)$ are Hopf algebras, because no topological completion is needed for the coproduct and the antipode on them. On the contrary, it is necessary to consider the topological algebras $U_q(b^+)^T$, $U_q(b^-)^T$, $U_q(b^+)^T$ and $U_q(b^-)^T$ if we want to have an Hopf algebra structure.

Let $Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$ be the root lattice, and $Q_+ = \bigoplus_{i=1}^{n-1} \mathbb{N}\alpha_i$ be the positive root lattice. We want to put a $Q$-gradation on the algebras above. If $\alpha = \sum m_i \alpha_i$ is a root, let us write $\varphi_\alpha = \varphi_1^{m_1} \cdots \varphi_{n-1}^{m_{n-1}}$ for an expression of type $\varphi_1^{m_1}(z_1) \cdots \varphi_{n-1}^{m_{n-1}}(z_N)$ where the formal variables $z_i$ are all distinct but we do not want to emphasize on their actual name.

The gradation is simply obtained by giving $x_i^\pm$ degree $\pm \alpha_i$, and degree 0 to $\varphi_i$ and $\psi_j$, the degree on a monomial being computed as usual. As a result, we get a $Q$-gradation on $U_q(\hat{\mathfrak{g}})$, $U_q(b^\pm)$ and $U_q(n^\pm)$ (and their correspond-
ing topological algebras. Moreover, these algebras are direct sums of their subspaces of fixed degree.

Before going further, let us define the main tool of this part:

**Definition 2 (Weak Hopf pairing)** Let $A$ and $B$ be algebras over a field $k$, both embedded in topological Hopf algebras $A^T$ and $B^T$ with invertible antipodes. A weak Hopf pairing between $A$ and $B$ is a bilinear form $\langle \cdot, \cdot \rangle : A \times B \to k$ such that

1. for all $a \in A$, for all $b, b' \in B$,
   $$\langle a, bb' \rangle = \sum \langle a(1), b \rangle \langle a(2), b' \rangle,$$

2. for all $a, a' \in A$, for all $b \in B$,
   $$\langle aa', b \rangle = \sum \langle a, b(2) \rangle \langle a', b(1) \rangle,$$

3. for all $a \in A$, for all $b \in B$,
   $$\langle a, 1_B \rangle = \varepsilon(a) \quad \text{and} \quad \langle 1_A, b \rangle = \varepsilon(b),$$

4. for all $a \in A$, for all $b \in B$,
   $$\langle S(a), b \rangle = \langle a, S^{-1}(b) \rangle,$$

with the further conditions that in the sums in 1. and 2. (taken over $\Delta_{A^T}(a)$ and $\Delta_{B^T}(b)$ respectively) there is a finite number of non zero summands. Similarly, in the expression in 4. there is a finite number of non zero terms on both sides of the equation.

Remark that the condition of finiteness imposed on the sums and on the antipode are important, and must be verified for all combinations of elements of $A$ and $B$. Thus, when we check for a bilinear form to be a weak Hopf pairing, we must also check that only finite expressions are involved.

Given such a weak Hopf pairing between two algebras $A$ and $B$, we can consider their left and right kernel. As in the usual case, we have the following result:
Proposition 2. Let $I_A$ and $I_B$ be the left and right kernel of the weak pairing. Then $I_A$ and $I_B$ are called weak Hopf ideals and satisfy

1. $I_A$ and $I_B$ are two sided ideals,
2. $I_A$ and $I_B$ are coideals:
   \[
   \Delta_{A^r}(I_A) \subset I_A \hat{\otimes} A + A \hat{\otimes} I_A, \\
   \Delta_{B^r}(I_B) \subset I_B \hat{\otimes} B + B \hat{\otimes} I_B,
   \]
3. $I_A$ and $I_B$ are weakly invariant under the antipode:
   \[
   S(I_A) \subset I_A^T, \\
   S(I_B) \subset I_B^T,
   \]
where $I_A^T$ (resp. $I_B^T$) is just the usual topological closure of $I_A$ in $A^r$ (resp. $I_B$ in $B^r$).

Remark that $A^r/I_A$ is a completion of $A/I_A$, so that we get naturally an induced weak Hopf pairing between $A/I_A$ and $B/I_B$.

Let us consider the topological Hopf algebras $(\overline{U}_q(b^+)^r, \Delta_+)$ and $(\overline{U}_q(b^-)^r, \Delta_-)$. To define a weak Hopf pairing between $\overline{U}_q(b^+)$ and $\overline{U}_q(b^-)$, we will have to use a slightly generalized version of the classical result of [10]. The pairing is defined on the generators, and by using the fact that the coproduct is a (possibly infinite) sum of tensor product of generators, we can check that the pairing can be computed on any two monomials in a finite number of steps. Incidentally, we shall check that the infinite sums in the coproduct lead only to finite sums after having applied the pairing on each of its summands (remember the finiteness conditions in definition 2).

Proposition 3. There is an unique $Q$-graded weak Hopf pairing between $\overline{U}_q(b^+) \hookrightarrow (\overline{U}_q(b^+)^r, \Delta_+)$ and $\overline{U}_q(b^-) \hookrightarrow (\overline{U}_q(b^-)^r, \Delta_-)$ satisfying the fol-
Following relations:

\[
\langle q^{\pm} \gamma, q^{\pm} \gamma' \rangle = 1,
\]

\[
\langle q^{\pm} \gamma, \psi_{i,k} \rangle = \langle \varphi_{i,-k}, q^{\pm} \gamma \rangle = \delta_{k,0},
\]

\[
\langle q^{\pm} \gamma, \psi^{-1}_{i,0} \rangle = \langle \varphi^{-1}_{i,0}, q^{\pm} \gamma' \rangle = 1,
\]

\[
\langle q^{\pm} \gamma, x_{i,n}^- \rangle = \langle x_{i,n}^+, q^{\pm} \gamma \rangle = 0,
\]

\[
\langle \varphi_{i,l}, x_{j,n}^+ \rangle = \langle x_{j,n}^-, \psi_{i,k} \rangle = 0,
\]

\[
\langle \varphi_{i,l}, \psi_{j,k} \rangle = g^{(ij)}_k \delta_{k,-l},
\]

\[
\langle \varphi^{-1}_{i,0}, \psi_{j,0} \rangle = \langle \varphi_{i,0}, \psi_{j,0}^{-1} \rangle = (g^{(ij)}_0)^{-1},
\]

\[
\langle x_{i,n}^+, x_{j,m}^+ \rangle = -\delta_{i,j} \delta_{n,-m} \frac{q + q^{-1}}{q - q^{-1}},
\]

for \(n, m \in \mathbb{Z}, k \geq 0, l \leq 0\) and \(i = 1, \ldots, n - 1\).

Moreover, relations (3), (4) and (5) (the action of \(\varphi_i\) on \(x_i^+\)) are in the left annihilator ideal \(I_+\) of this weak pairing, with a similar result for the right annihilator ideal \(I_-\).

Looking at all pairing involving \(q^{\pm} \gamma\), we see that for any element \(Y \in U_q(b^-)\) we have \(\langle q^{\pm} \gamma, Y \rangle = \varepsilon(Y)\). Thus, for \(X\) and \(Y\) being any elements of \(U_q(b^+)\) and \(U_q(b^-)\) we have:

\[
\langle X q^{\pm} \gamma, Y \rangle = \sum \langle X, Y_{(2)} \rangle \langle q^{\pm} \gamma, Y_{(1)} \rangle
\]

\[
= \sum \langle X, Y_{(2)} \rangle \varepsilon(Y_{(1)})
\]

\[
= \langle X, Y \rangle.
\]

Similarly, we can show that \(\langle X, q^{\pm} \gamma' Y \rangle = \langle X, Y \rangle\) for any \(X\) and \(Y\) in \(U_q(b^+)\) and \(U_q(b^-)\). Therefore, when involved in pairing computations, \(q^{\pm} \gamma\) is completely irrelevant (i.e. can be considered as the unit 1). The same behavior holds for \(q^{\pm} \gamma'\), so that in all the following computations we shall omit any reference to those elements in order to simplify the notations.

Before we go to the general existence and uniqueness proof, let us show that all elements defined by relations (2), (3), (4) leading to \(U_q(b^+)\) are in the kernel of this weak Hopf pairing (i.e. their pairing with any other element is well defined and null). It is sufficient to show that each relation \(r\) verifies \(\langle r, y \rangle = 0\) where \(y\) is a generator of \(U_q(b^-)\). We get the general result by
\[ \langle r, y_1 \ldots y_k \rangle = \prod_{i=1}^{k} \langle r^{(i)}, y_i \rangle \]. The sum can of course be infinite, but each summand is zero because at least one \( r^{(i)} \) is in the Hopf ideal generated by all the relations.

1. We have already shown that \( q^{\pm \frac{c}{2}} \) and \( q^{\pm \frac{c'}{2}} \) are central elements.

2. We have \( r_1 = \varphi_1 \varphi_1^{-1} - 1 = 0 \):
   
   (a) \( \langle r_1, 1 \rangle = 0 \),
   
   (b) \( \langle r_1, \psi_{j,k} \rangle = \sum_{r+s=k} \langle \varphi_1, \psi_{j,r} \rangle \langle \varphi_1^{-1}, \psi_{j,s} \rangle = g_{0}^{(ij)} (g_{0}^{(ij)})^{-1} - 1 = 0 \),
   
   (c) \( \langle r_1, x_{-m} \rangle = 0 \).

3. We have \( r_2 = \varphi_{i,n} \varphi_{j,m} - \varphi_{j,m} \varphi_{i,n} = 0 \):
   
   (a) \( \langle r_2, 1 \rangle = \delta_{n,0} \delta_{m,0} - \delta_{m,0} \delta_{n,0} = 0 \),
   
   (b) \( \langle r_2, \psi_{k,o} \rangle = \sum_{r+s=o} \left( \langle \varphi_{i,n}, \psi_{k,r} \rangle \langle \varphi_{j,m}, \psi_{k,s} \rangle - \langle \varphi_{j,m}, \psi_{k,r} \rangle \langle \varphi_{i,n}, \psi_{k,s} \rangle \right) 
   
   \quad = \sum_{r+s=o} \left( g_{r}^{(ik)} \delta_{n,r} g_{s}^{(jk)} \delta_{m,s} - g_{r}^{(jk)} \delta_{m,r} g_{s}^{(ik)} \delta_{n,s} \right) 
   
   \quad = \delta_{n+m,-o} \left( g_{-n}^{(jk)} g_{-m}^{(ik)} - g_{-m}^{(jk)} g_{-n}^{(ik)} \right) 
   
   \quad = 0 \),
   
   (c) \( \langle r_2, x_{-k,o} \rangle = \langle \varphi_{i,n}, x_{-k,o} \rangle \langle \varphi_{j,m}, 1 \rangle - \langle \varphi_{j,m}, x_{-k,o} \rangle \langle \varphi_{i,n}, 1 \rangle 
   
   \quad + \sum_{l \leq 0} \left( \langle \varphi_{i,n}, \psi_{k,l} \rangle \langle \varphi_{j,m}, x_{-k,o-l} \rangle - \langle \varphi_{j,m}, \psi_{k,l} \rangle \langle \varphi_{i,n}, x_{-k,o-l} \rangle \right) 
   
   \quad = 0 \).

4. We have \( \varphi_{i}(z) x_{j}^{+}(w) = g_{ij} (zw^{-1} q^{-\frac{c}{2}}) x_{j}^{+}(w) \varphi_{i}(z) \), i.e. in term of underlying generators: \( r_3 = \varphi_{i,n} x_{j}^{+} - \sum_{0 \leq o \leq -n} g_{0}^{(ij)} q^{-o \frac{c}{2}} x_{j,m-o} \varphi_{i,n+o} = 0 \):
   
   (a) \( \langle r_3, 1 \rangle = 0 \),
   
   (b) \( \langle r_3, \psi_{k,p} \rangle = \sum_{r+s=p} \left( \langle \varphi_{i,n}, \psi_{k,r} \rangle \langle x_{j,m}^{+}, \psi_{k,s} \rangle 
   
   \quad - \sum_{0 \leq o \leq -n} g_{0}^{(ij)} \langle x_{j,m-o}^{+}, \psi_{k,r} \rangle \langle \varphi_{i,n+o}, \psi_{k,s} \rangle \right) = 0 \).
\[(c) \langle r_3, x_{k,o}^- \rangle = \langle \varphi_{i,n}, x_{k,o}^- \rangle \langle x_{j,m}^+, 1 \rangle \]
\[\quad - \sum_{0 \leq p \leq -n} g_p^{(ij)} \langle x_{j,m-p}^+, x_{k,o}^- \rangle \langle \varphi_{i,n+p}, 1 \rangle \]
\[\quad + \sum_{i \geq 0} \left( \langle \varphi_{i,n}, \psi_{k,l} \rangle \langle x_{j,m}^+, x_{k,o-l}^- \rangle \right) \]
\[\quad - \sum_{0 \leq p \leq -n} g_p^{(ij)} \langle x_{j,m-p}^+, \psi_{k,l} \rangle \langle \varphi_{i,n+p}, x_{k,o-l}^- \rangle \right) \]
\[= \frac{1}{q + q^{-1}} g_p^{(ij)} \delta_{k,j} \delta_{m-p,0} \delta_{n+p,0} - \frac{1}{q + q^{-1}} g_l^{(ia)} \delta_{k,j} \delta_{l,-n} \delta_{m,l-k} \]
\[= \frac{1}{q + q^{-1}} g_p^{(ij)} \delta_{k,j} \delta_{m+n,0} - \frac{1}{q + q^{-1}} g_l^{(ij)} \delta_{k,j} \delta_{m+n,0} \]
\[= 0. \]

Of course the same computations can be made for the corresponding relations in \( \mathcal{U}_q(b^-) \).

In order to show the existence of the weak pairing, we have to prove that it is well defined on any two monomials, as was done in [10]. But we have to be careful with comultiplications involving infinite sums.

A first step towards that goal is given by the following result: given two monomials \( X = x_{i_1,n_1}^+ \cdots x_{i_k,n_k}^+ \) and \( Y = x_{j_1,m_1}^- \cdots x_{j_l,m_l}^- \), \( \Phi \) a monomial in \( \mathcal{U}_q(h^+) \) and \( \Psi \) a monomial in \( \mathcal{U}_q(h^-) \), we have
\[ \langle X \Phi, Y \Psi \rangle = \langle X, Y \rangle \langle \Phi, \Psi \rangle \]

Indeed, we have \( \Delta_+(X) = X \otimes 1 + R \), where \( R \) is an (infinite) sum of elements having at least one \( x^+ \) generator in the right side of the tensor product. For this reason we have \( \langle R \Delta_+(\Phi), Y \otimes \Psi \rangle = 0 \). So we get
\[ \langle X \Phi, Y \Psi \rangle = \sum \langle X \Phi_{(1)}, Y \rangle \langle \Phi_{(2)}, \Psi \rangle, \]
where this sum is taken over \( \Delta_+(\Phi) \) and so is finite. Using the same argument with \( \Delta_- \), we get:
\[ \langle X \Phi, Y \Psi \rangle = \sum \langle X \Phi_{(1)}, Y \rangle \langle \Phi_{(2)}, \Psi \rangle \]
\[= \sum \langle X, Y \rangle \langle \Phi_{(1)}, 1 \rangle \langle \Phi_{(2)}, \Psi \rangle \]
\[= \sum \langle X, Y \rangle \varepsilon(\Phi_{(1)}) \langle \Phi_{(2)}, \Psi \rangle \]
\[= \langle X, Y \rangle \langle \Phi, \Psi \rangle. \]
Now, the second pair involves only finite sums, so it can be computed using weak Hopf pairing properties in a finite number of operations. For the first one, we have to be more careful:

Let $\alpha$ be the degree of $X$ and $\beta$ be the degree of $Y$. We have

$$\Delta^{(p)}_+(x_{i_1,n_1}^+) = x_{i_1,n_1}^+ \otimes 1 \cdots \otimes 1 + \sum_{k_1 \leq 0} \varphi_{i_1,k_1} q^{2c} x_{i_1,n_1-k_1}^+ \otimes 1 \cdots \otimes 1 + \ldots +$$

$$\sum_{\substack{k_1 \leq 0 \\ r_{n_1-k_1} \leq 0 \\ r_{n_2-k_1} \leq 0 \cdots \ r_{n_{k-1}-k_1} \leq 0}} \varphi_{i_1,r_{n_1-k_1}} q^{2c} \cdots \varphi_{i_1,r_{n_{k-1}-k_1}} q^{2c} x_{i_1,n_1-k_1}^+ \otimes 1 \cdots \otimes 1.$$

We do not care for the coefficients in $q^{2c}$, because those are irrelevant as regards weak Hopf pairings. Then, it appears that in

$$\langle X, Y \rangle = \langle \Delta^{(k-1)}(X), x_{j_1,m_1}^- \otimes \cdots \otimes x_{j_k,m_k}^- \rangle,$$

the only non-zero term can be those where each $x_{i,n}^+$ is paired with a corresponding $x_{i,m}^-$. Therefore, let $\sigma \in \mathfrak{S}_k$ be a permutation satisfying $i_{\sigma(l)} = j_l$ for all $1 \leq l \leq k$. The non-zero terms in the weak Hopf pairing appear when $x_{j_1,m_1}^- \otimes \cdots \otimes x_{j_k,m_k}^-$ is paired with

$$\prod_{l=1}^{k} \left( \sum_{\substack{k_l \leq 0 \\ r_{l,1} + \cdots + r_{l,\sigma(l)-1} = k_l}} \varphi_{i_1,r_{l,1}} q^{2c} \cdots \varphi_{i_1,r_{l,\sigma(l)-1}} q^{2c} x_{i_1,n_1-k_l}^+ \otimes 1 \cdots \otimes 1.\right)$$

As we want to rearrange this expression to compute all the pairings by applying relations given in definition 2, we set $\mu = \sigma^{-1}$ in $\mathfrak{S}_k$. The last expression then becomes:

$$\sum_{\substack{1 \leq l \leq k \\ r_{l,1} + \cdots + r_{l,\sigma(l)-1} = k_l \\ p \leq m \leq k \\ \mu(m) < k \\ \mu(m) > k}} \left( \prod_{p=1}^{k} \varphi_{\mu(m),r_{\mu(m)},p} \cdot x_{\mu(m),n_{\mu(m)}-k_{\mu(p)}}^+ \otimes \cdots \otimes 1.\right)$$

This sum still involves an infinite number of terms. But remark that in expressions of the form $\varphi_{i_1,\ldots,\varphi_{i_{n+1},\ldots,\varphi_{i_{n+1},\ldots,\varphi_{i_{n+1}}}}$, we have $\sigma(n_{i+1}) > \sigma(l)$ for all $1 \leq i \leq p$. Now, it is easy to see that a pairing of the form

$$\langle \varphi_{i_1,n_1} \cdots \varphi_{i_{n+1},n_{n+1}} \varphi_{i_{n+1},n_{n+1}} \cdots \varphi_{i_{n+1},n_{n+1}} \varphi_{i_{n+1},n_{n+1}} \rangle$$

is always zero but for $i = j$, $n_{r+1} + \cdots + n_p = 0$ and $n = m - (n_1 + \cdots + n_r)$. Starting with $l = k$, we get $n_{\mu(k)} - k_{\mu(k)} = m_k$ for the only non-zero pairing. Thus, $k_{\mu(k)}$ is fixed and the $r_{k,i}$ for $1 \leq i \leq k$ range in a finite number of values. Thereafter for $l = k - 1$, we see that $k_{\mu(k-1)}$ can take only a finite number of values (if we
want the pairing to be non-zero), so that \( r_{k-1,i} \) for \( 1 \leq i \leq k - 1 \) range over a finite domain also. It is now easy to continue backward until \( l = 1 \) and conclude that the pairing is non-zero only for a finite number of terms in the above sum. Thus, the pairing of any two expression is computable in a finite number of steps, and the result is well defined, as stated in the proposition.

We have proved that \( \langle X\Phi, Y\Psi \rangle \) is well defined and unique. Because of relation (4), the pairing between any monomials \( \langle X', Y' \rangle \) can be put under the above form in a finite number of steps, thus finishing the proof.

### 1.4 The weak quantum double \( \mathcal{D}(U_q(b^+), U_q(b^-)) \)

Using our weak Hopf pairing, we get:

**Proposition 4 (Weak quantum double)** There is an algebra structure on \( \mathcal{D}(U_q(b^+), U_q(b^-)) = U_q(b^+) \otimes U_q(b^-) \) where

\[
\begin{align*}
(a \otimes 1)(1 \otimes b) &= a \otimes b, \\
(1 \otimes b)(a \otimes 1) &= \sum \langle a_{(1)}, S(b_{(1)}) \rangle \langle a_{(3)}, b_{(3)} \rangle a_{(2)} \otimes b_{(2)}, \quad (11) \\
(a \otimes 1)(a' \otimes 1) &= aa' \otimes 1, \\
(1 \otimes b)(1 \otimes b') &= 1 \otimes bb',
\end{align*}
\]

with unit \( 1 \otimes 1 \). Moreover, we have natural embeddings

\[
\begin{align*}
U_q(b^+) &\to \mathcal{D}(U_q(b^+), U_q(b^-)) & U_q(b^-) &\to \mathcal{D}(U_q(b^+), U_q(b^-)) \\
a &\mapsto a \otimes 1 & b &\mapsto 1 \otimes b
\end{align*}
\]

which are algebra morphisms.

The proof (mainly the associativity of the multiplication) is similar to the non-weak case, so it will not be developed here. Though, because we are using a weak Hopf pairing, we need to be careful about the sum appearing in the multiplication (11). We will postpone this verification until the proof of next proposition.

Remark that the coefficient \(-1/(q + q^{-1})\) for the Hopf pairing between \( x^+_{i,n} \) and \( x^-_{j,n} \) is here so that we get the right commutation relations in the weak quantum double. In fact, we have :

**Proposition 5** In \( \mathcal{D}(U_q(b^+), U_q(b^-)) \) the following commutation relations hold:

\( q^\pm \frac{c}{2} \) and \( q^\pm \frac{c'}{2} \) are central,
Proof. The verifications are quite long, but straightforward. We have

1. $q^\frac{1}{2}$ is central (same demonstration for $q^\frac{1}{4}$):
   
   \[
   \begin{align*}
   (1 \otimes q^\frac{1}{2})(q^\frac{1}{2} \otimes 1) &= q^\frac{1}{2} \otimes q^\frac{1}{2}, \\
   (1 \otimes \psi_{j,k})(q^\frac{1}{2} \otimes 1) &= \sum_{u+v+w=k} \langle 1, S(\psi_{j,u}) \rangle (1, \psi_{j,w}) q^\frac{1}{2} \otimes \psi_{j,v} q^{(w-v)\frac{1}{2}} \\
   &= q^\frac{1}{2} \otimes \psi_{j,k},
   \end{align*}
   \]

   (c) We have
   
   \[
   \Delta_-(x^-_{j,m}) = 1 \otimes 1 \otimes x^-_{j,m} + \sum_{k \leq 0} 1 \otimes x^-_{j,m-k} \otimes \psi_{j,k} q^{-(m-\frac{1}{2})c'} + \sum_{k \leq 0} x^-_{j,m-k} \otimes \psi_{j,r} q^{-(m-s-\frac{1}{2})c'} \otimes \psi_{j,s} q^{-(m-\frac{1}{2})c'},
   \]

   so we get
   
   \[
   \begin{align*}
   (1 \otimes x^-_{j,m})(q^\frac{1}{2} \otimes 1) &= \langle 1, 1 \rangle (1, x^-_{j,m}) q^\frac{1}{2} \otimes 1 + \sum_{k \leq 0} \langle 1, 1 \rangle (1, \psi_{j,k}) q^\frac{1}{2} \otimes x^-_{j,m-k} \\
   &\quad + \sum_{k \leq 0} \langle 1, S(x^-_{j,m-k}) \rangle (1, \psi_{j,s}) q^\frac{1}{2} \otimes \psi_{j,r} q^{-(m-s-\frac{1}{2})c'} \\
   &= 0 + \delta_{k,0} q^\frac{1}{2} \otimes x^-_{j,m-k} + 0 \\
   &= q^\frac{1}{2} \otimes x^-_{j,m},
   \end{align*}
   \]

2. Relation $g_{ij}(zw^{-1}q^\frac{1}{2} \otimes \frac{1}{2})\varphi_i(z)\psi_j(w) = g_{ij}(zw^{-1}q^\frac{1}{2} \otimes \frac{1}{2})\psi_j(w)\varphi_i(z)$. We have

   \[
   \begin{align*}
   \Delta_+(\varphi_{i,n}) &= \sum_{r+s+t=n} \varphi_{i,r} q^{-(s+t)\frac{1}{2}} \otimes \varphi_{i,s} q^{(r-t)\frac{1}{2}} \otimes \varphi_{i,t} q^{(r+s)\frac{1}{2}}, \\
   \Delta_-(\psi_{j,m}) &= \sum_{u+v+w=m} \psi_{j,u} q^{(v+u)\frac{1}{2}} \otimes \psi_{j,v} q^{(w-u)\frac{1}{2}} \otimes \psi_{j,w} q^{-(u+v)\frac{1}{2}},
   \end{align*}
   \]
so that we get
\[
(1 \otimes \psi_{j,m})(\varphi_{i,n} \otimes 1) = \sum_{r+s+t=n \atop u+v+w=m} \langle \varphi_{i,r}, S(\psi_{j,u}) \rangle \langle \varphi_{i,t}, \psi_{j,v} \rangle 
\]
where we set \( S(\psi_j(z)) = \psi_j(z)^{-1} = \sum_{k \geq 0} S(\psi_{j,k})z^{-k} \). Now, it is easy to see (using suitable properties of weak Hopf pairings) that
\[
\langle \varphi_{i,t} q^{(r+s)\frac{t}{2}}, \psi_{j,w} q^{-(u+v)\frac{t}{2}} \rangle = \langle \varphi_{i,t}, \psi_{j,w} \rangle \delta_{w,-t} g_w^{(ij)}. 
\]
For the other pair, we proceed as follows: \( S(\psi_{j,u}) = P(\psi_{j,0}^{-1}, \psi_{j,0}, \ldots, \psi_{j,u}) \), where \( P \) is a polynomial such that each of its monomial \( \psi_{j,n_1} \cdots \psi_{j,n_k} \) satisfies \( \sum_p m_p n_p = u \). So we get exactly
\[
\langle \varphi_{i,r} q^{-(s+t)\frac{r}{2}}, S(\psi_{j,u}) q^{-(v+w)\frac{r}{2}} \rangle = \langle \varphi_{i,r}, P(\psi_{j,0}^{-1}, \psi_{j,0}, \ldots, \psi_{j,u}) \rangle 
\]
\[
= P(\langle \varphi_{i,r}, \psi_{j,0}^{-1} \rangle, \langle \varphi_{i,r}, \psi_{j,0} \rangle, \ldots, \langle \varphi_{i,r}, \psi_{j,u} \rangle) 
\]
\[
= P(g_0^{(ij)} \delta_{r,-u}, \ldots, g_u^{(ij)}) \delta_{r,-u} 
\]
Combining those two results, we get
\[
(1 \otimes \psi_{j,m})(\varphi_{i,n} \otimes 1) = \sum_{k \geq 0 \atop r+s=k} \left( g_r^{(ij)} g_s^{(ij)} q^{-(a+c)\frac{t}{2}} \otimes q^{-(a+c)\frac{t}{2}} \right) \varphi_{i,n+k} \otimes \psi_{j,m-k} 
\]
\[
= \sum_{k \geq 0} h_{k}^{(ij)} \varphi_{i,n+k} \otimes \psi_{j,m-k}, 
\]
where \( h_{ij}(t) = \sum_{k \geq 0} h_{k}^{(ij)} t^k = g_{ij}(t q^{\frac{t}{2}} \otimes q^{\frac{t}{2}})g_{ij}(t q^{-\frac{t}{2}} \otimes q^{-\frac{t}{2}})^{-1} \). Using currents, the last statement gives
\[
\psi_j(w) \varphi_i(z) = h(z w^{-1}) \varphi_i(z) \psi_j(w),
\]
which is exactly what we wanted to show.

3. Relation \( x_j^{-}(w) \varphi_i(z) = g_{ij}(z w^{-1} q^{\frac{t}{2}}) \varphi_i(z) x_j^{-}(w) \), i.e.
\[
(1 \otimes x_{j,m}^{-})(\varphi_{i,n} \otimes 1) = \sum_{0 \leq k \leq -n} g_{k}^{(ij)} q^{k \frac{t}{2}} \varphi_{i,n+k} \otimes x_{j,m-k}^{-}.
\]
We have

\[(1 \otimes x_{j,m}^-)(\varphi_{i,n} \otimes 1)\]

\[= \sum_{r+s+t=n} \langle \varphi_{i,r}, S(1) \rangle \langle \varphi_{i,t}, x_{j,m}^- \rangle \varphi_{i,s} q^{(r-t)\frac{q}{2}} \otimes 1 \]

\[+ \sum_{r+s+t=n} \langle \varphi_{i,r}, S(1) \rangle \langle \varphi_{i,t}, \psi_{j,k} \rangle \varphi_{i,s} q^{(r-t)\frac{q}{2}} \otimes x_{j,m-k}^- \]

\[+ \sum_{k \leq 0, n+v=k} \langle \varphi_{i,r}, S(x_{j,m-k}^-) \rangle \langle \varphi_{i,t}, \psi_{j,v} \rangle \varphi_{i,s} q^{(r-t)\frac{q}{2}} \otimes \psi_{j,u} q^{-(m-v-\frac{q}{2})c'} \]

\[= 0 + \sum_{r+s+t=n} \delta_{r,0} g_k^{(ij)} \delta_{t,-k} \varphi_{i,s} q^{(r-t)\frac{q}{2}} \otimes x_{j,m-k}^- + 0 \]

\[= \sum_{0 \leq k \leq n} g_k^{(ij)} \varphi_{i,n+k} q^{k\frac{q}{2}} \otimes x_{j,m-k}^- .\]

4. Relation \([x_i^+(z), x_j^-(w)] = \delta_{i,j} \frac{q-1}{q} \left( \delta(zw^{-1}q^{-c'}) \psi_i(wq \frac{1}{2} c') - \delta(zw^{-1}q^{-c'}) \varphi_i(zq \frac{1}{2} c') \right)\]

i.e.

\[ (1 \otimes x_{j,m}^-)(x_{i,n}^+ \otimes 1) = x_{i,n}^+ \otimes x_{j,m}^- + \frac{\delta_{i,j}}{q+q^{-1}} 1 \otimes \psi_{j,n+m} q^{\frac{n-m}{2}} c' \]

\[ - \frac{\delta_{i,j}}{q+q^{-1}} \varphi_{i,n+m} q^{\frac{m-n}{2}} c' \otimes 1 .\]
We have

\[
(1 \otimes x_{i,n}^{-})(x_{i,n}^{+} \otimes 1) = \langle x_{i,n}^{+}, 1 \rangle \langle 1, x_{j,m}^{-} \rangle 1 \otimes 1 \\
+ \sum_{k \geq 0} \langle x_{i,n}^{+}, 1 \rangle \langle 1, \psi_{j,k} \rangle 1 \otimes x_{j,m-k}^{-} \\
+ \sum_{k \geq 0, r+s=k} \langle x_{i,n}^{+}, S(x_{j,m-k}^{-}) \rangle \langle 1, \psi_{j,s} \rangle 1 \otimes \psi_{j,r} q^{-(m-s-\frac{q}{2})c'} \\
+ \sum_{k \geq 0} \langle \varphi_{i,t}, 1 \rangle \langle 1, x_{j,m}^{-} \rangle x_{i,n-l}^{+} \otimes x_{j,m-k}^{-} \\
+ \sum_{l \leq 0, k \geq 0} \langle \varphi_{i,t}, 1 \rangle \langle x_{i,n-l}^{+}, x_{j,m}^{-} \rangle \varphi_{i,u} q^{-(n-t-\frac{u}{2})c} \otimes x_{j,m-k}^{-} \\
+ \sum_{l \leq 0, t+u=l, k \geq 0} \langle \varphi_{i,t}, 1 \rangle \langle x_{i,n-l}^{+}, \psi_{j,k} \rangle \varphi_{i,u} q^{-(n-t-\frac{u}{2})c} \otimes x_{j,m-k}^{-} \\
+ \sum_{l \leq 0, t+u=l, k \geq 0} \langle \varphi_{i,t}, S(x_{j,m-k}^{-}) \rangle \langle x_{i,n-l}^{+}, \psi_{j,s} \rangle \varphi_{i,u} q^{-(n-t-\frac{u}{2})c} \otimes \psi_{j,r} q^{-(m-s-\frac{q}{2})c'}. 
\]

Among the 9 summands of the right hand side of this equation, it is trivial to see that the first, second, fourth and eighth are 0. For the remaining summands, remark that

\[
S(x_{j,m-k}^{-}) = - \sum_{p \geq 0} x_{j,m-k-p}^{-} S(\psi_{j,p}) q^{(m-k-\frac{q}{2})c'},
\]

so that \( \langle \varphi_{i,t}, S(x_{j,m-k}^{-}) \rangle = 0 \) and \( \langle x_{i,n}^{+}, S(x_{j,m-k}^{-}) \rangle = \frac{\delta_{i,t}}{q+s} \delta_{n,m-k} \). This shows that the sixth and ninth terms are also zero. Now, the fifth term is exactly \( x_{i,n}^{+} \otimes x_{j,m}^{-} \), the third term is

\[
\sum_{k \geq 0, r+s=k} \frac{\delta_{i,j}}{q} \delta_{n,m-k} \delta_{s,0} 1 \otimes \psi_{j,k} q^{-(m-s-\frac{q}{2})c'},
\]

which is equal to \( \frac{\delta_{i,j}}{q+s} 1 \otimes \psi_{j,n+m} q^{\frac{m-n}{2}c'} \). Finally, the seventh and last
Proof. For the sake of this proof, we will denote by $\text{annihilator ideal } I$ are in the annihilator ideal $\Delta^+ + \varphi$. Recall that a quasi-primitive element of $Q$ is not the case, and we have the following result:

**Proposition 6** The elements

\[(z - q_{ij}w)x_i^+(z)x_j^+(w) - (q_{ij}z - w)x_j^+(w)x_i^+(z)\]  

are in the annihilator ideal $\overline{I}_+$ of the weak Hopf pairing between $\overline{U}_q(b^+)$ and $\overline{U}_q(b^-)$. Moreover, the two-sided ideal they generate is a weak Hopf ideal. A similar statement holds for the $x_i^-(z)$ ($i = 1, \ldots, n-1$) and the corresponding annihilator ideal $I_-$. 

Proof. For the sake of this proof, we will denote by $I_0^+$ and $I_0^-$ the weak Hopf ideals generated by the relations in proposition (11). We can now work in the algebra $D(U_q(b^+)/I_0^+, U_q(b^-)/I_0^-)$ with the induced weak Hopf pairing and the induced $Q$-gradation (the relation defining $I_0^+$ and $I_0^-$ are $Q$-homogeneous). 

Recall that a quasi-primitive element of $U_q(b^+)/I_0^+$ is an element $x$ such that $\Delta_+(x) = x \otimes h + h' \otimes x$, where $h$ and $h'$ are in the subalgebra generated by $\varphi_i(z)$ for $i = 1, \ldots, n-1$ and $q^{\pm \frac{c}{2}}$, i.e. $U_q(\mathfrak{h}^+)$. It now easy to check that (13) is quasi-primitive:

\[
\Delta_+ \left( (z - q_{ij}w)x_i^+(z)x_j^+(w) - (q_{ij}z - w)x_j^+(w)x_i^+(z) \right) \\
= \left[ (z - q_{ij}w)x_i^+(z)x_j^+(w) - (q_{ij}z - w)x_j^+(w)x_i^+(z) \right] \otimes 1 \\
+ \varphi_i(zq^{\frac{c}{2}})\varphi_j(wq^{\frac{c}{2}}) \otimes \left[ (z - q_{ij}w)x_i^+(zq^{c_1})x_j^+(wq^{c_1}) - (q_{ij}z - w)x_j^+(wq^{c_1})x_i^+(zq^{c_1}) \right] \\
+ \left[ (z - q_{ij}w)\varphi_i(zq^{\frac{c}{2}})x_j^+(w) - (q_{ij}z - w)x_j^+(w)\varphi_i(zq^{\frac{c}{2}}) \right] \otimes x_i^+(zq^{c_1}) \\
+ \left[ (z - q_{ij}w)x_i^+(z)\varphi_j(wq^{\frac{c}{2}}) - (q_{ij}z - w)\varphi_j(wq^{\frac{c}{2}})x_i^+(z) \right] \otimes x_j^+(wq^{c_1})
\]
The last two elements of this sum are zero thanks to the commutation relations in \( \overline{U}_q(b^+)/I_0^+ \). Now, quasi-primitive elements of \( \overline{U}_q(b^+)/I_0^+ \) are orthogonal to decomposable elements of \( \overline{U}_q(b^-)/I_0^- \). Remark that all elements of degree \(-\alpha_i - \alpha_j\) are decomposable. Using the fact that the weak Hopf pairing is \( Q \)-graded, we get the first result.

We can then consider the two sided ideal \( I_1^+ \) of \( U_q(b^+) \) generated by \( I_0^+ \) and the above relation. Likewise, we have a two sided ideal \( I_1^- \) on the negative side. Those two ideals are again weak Hopf ideals.

Unlike the classical case, the Serre relations are not quasi-primitive in \( U_q(b^+) \) and \( U_q(b^-) \). But they are quasi-primitive modulo commutation relations \([5]\) between the \( x_i \)'s. Thus we have:

**Proposition 7** The Serre relations

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1}{r} \right]_q \text{Sym}_z \left( x_i^+(z_1) \cdots x_i^+(z_r)x_j^+(w)x_i^+(z_{r+1}) \cdots x_i^+(z_{1-a_{ij}}) \right)
\]

are in the annihilator ideal \( T_+ \) of the weak Hopf pairing. Moreover, the two-sided ideal generated by \( I_1^+ \) and \( [14] \) is a weak Hopf ideal. A similar statement holds for the \( x_i^- (z) \) \((i = 1, \ldots, n - 1) \) and \( T_- \).

We will show that the Serre elements are quasi-primitive in \( \overline{U}_q(b^+)/I_1^+ \) and \( \overline{U}_q(b^-)/I_1^- \). We have an induced weak Hopf pairing between these two algebras, and the commutation relation \([3]\) holds on top of \([4]\).

Remark that a proof for \( a_{ij} = -1, -2 \) or \(-3\) is sufficient. Here we just give a straightforward computation in the case \( a_{ij} = -1 \), covering in particular the case of \( U_q(\hat{sl}(n)) \). The two remaining cases were handled using a Computer Algebra System (Maple V). Let us give some new notations. For \( 1 \leq k \leq n \) and \( \varepsilon \in \{0, 1\} \) we put:

\[
X_{k,\varepsilon} = \begin{cases} 
\hat{x}_i^+(z) \hat{1} & \text{if } k < n \text{ and } \varepsilon = 1, \\
\varphi_i(zq^{-1}) \hat{x}_i^+(zq^{-1}) & \text{if } k < n \text{ and } \varepsilon = 0, \\
\hat{x}_j^+(w) \hat{1} & \text{if } k = n \text{ and } \varepsilon = 1, \\
\varphi_j(wq^{-1}) \hat{x}_j^+(wq^{-1}) & \text{if } k = n \text{ and } \varepsilon = 0.
\end{cases}
\]

A similar definition can be given for \( Y_{k,\varepsilon} \) in \( \overline{U}_q(b^-)/I_1^- \otimes \overline{U}_q(b^-)/I_1^- \). Now for \( \sigma \in S_n \) and \( \varepsilon = (\varepsilon_1 \ldots \varepsilon_n) \in \{0, 1\}^n \) we put \( P_{\sigma,\varepsilon}^+ = \prod_{k=1}^n X_{\sigma(k),\varepsilon_{\sigma(k)}} \).
$C_{\sigma}^n = (-1)^{\sigma^{-1}(n)+1} \left[ \frac{n-1}{\sigma^{-1}(n)-1} \right] q_i$ and $S_{\varepsilon}^+ = \sum_{\sigma \in \mathfrak{S}_n} C_{\sigma}^n P_{\sigma, \varepsilon}$. The coproduct of (14) in $U_q(b^+)/I^1_+ \otimes U_q(b^+)/I^1_-$ is just

$$\sum_{\varepsilon \in \{0,1\}^{a_{ij}}} S_{\varepsilon}^+.$$ 

Let us give an example: take $\varepsilon = (1,0,0)$. We get

$$P_{id, \varepsilon} = X_{1,1}X_{2,0}X_{3,0} = x_i^+(z_1) \otimes 1 \cdot \varphi_i(z_2) \otimes x_i^+(z_2) \cdot \varphi_j(w) \otimes x_j^+(w).$$

Thus,

$$P_{id, \varepsilon} = x_i^+(z_1)\varphi_i(z_2)\varphi_j(w) \otimes x_i^+(z_2)x_j^+(w).$$

We want to show that all the elements $S_{\varepsilon}^+$ are zero but the two extremal one (i.e. when all $\varepsilon_i$ are 0 or all $\varepsilon_i$ are 1). Recall the remark we made on page 3. We observed that the topological completion on the tensor product is just weak enough in order to allow expressions of the form $g_{ij}(zw^{-1})x_i^+(z) \otimes x_j^+(w)$ but not $g_{ij}(zw^{-1})x_i^+(z)x_j^+(w) \otimes 1$. To sketch the forthcoming proof, let us just say that the completion allows us to use commutation relation between $x_i^+$ and $x_j^+$ only when they are not at the same side of the tensor product. The only cases where such elements cannot be found are the two “extremal” one. This is why the completion had to be carefully chosen. We need the following lemmas:

**Lemma 1** Let $E = \sum_{n,m \in \mathbb{Z}} L_n \hat{\otimes} R_m z^{-n}w^{-m}$ be some generating series with $L_n$ and $R_m$ in $U_q(b^+)/I^1_+$. Assume that for $n$ and $m$ fixed in $\mathbb{Z}$, the degree of $L_{n+k} \hat{\otimes} R_{m-k}$ goes to $+\infty$ (according to the tensor product filtration) when $k \to +\infty$. Then $(az - bw)E = 0$ implies $E = 0$ (where $a,b \in \mathbb{C}^*$).

Fix $n$ and $m$ in $\mathbb{Z}$. If $(az - bw)E = 0$ then we have $aL_{n+1} \hat{\otimes} R_m = bL_n \hat{\otimes} R_{m+1}$ for all $n,m \in \mathbb{Z}$. Going inductively, we get $(\frac{b}{a})^k L_{n+k} \hat{\otimes} R_{m-k} = L_n \hat{\otimes} R_m$. As the left hand side degree goes to $+\infty$ when $k \to +\infty$ and the right hand side is of constant finite degree, we must have $L_n \hat{\otimes} R_m = 0$. As this is true for all $n, m$ in $\mathbb{Z}$, we get $E = 0$. 

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Lemma 2 In $\overline{U}_q(b^+)/I_+^1 \otimes \overline{U}_q(b^+)/I_+^1$ we have:

$$(q^{-1}z_1 - w)(q^{-1}z_2 - w)S^+_{\{1,1,0\}} = 0$$
$$(q^2z_1 - z_2)(q^{-1}w - z_2)S^+_{\{1,0,1\}} = 0$$
$$(q^2z_2 - z_1)(q^{-1}w - z_1)S^+_{\{0,1,1\}} = 0$$
$$(q^2z_1 - z_2)(q^{-1}z_1 - w)S^+_{\{0,1,0\}} = 0$$
$$(q^2z_2 - z_1)(q^{-1}z_2 - w)S^+_{\{0,0,1\}} = 0$$
$$(q^{-1}w - z_1)(q^{-1}w - z_2)S^+_{\{0,0,1\}} = 0$$

This lemma is the actual computation (which is difficult to handle by hand for $a_{ij} = -2$ or $-3$). We will only show how to handle the first relation. The remaining cases can be computed likewise. Using only the action of $\varphi_j(wq^\frac{1}{2})$ on $x^+_i(z_k)$ (i.e. the commutation relations $(w - q^{-1}z_k)\varphi_j(wq^\frac{1}{2})x^+_i(z_k) = (q^{-1}w - z_k)x^+_i(z_k)\varphi_j(wq^\frac{1}{2})$) we get:

$$(q^{-1}z_1 - w)(q^{-1}z_2 - w)S^+_{\{1,1,0\}} =$$

$$- wq^{-2}(q - q^{-1}) \left[(z_1 - q^2z_2)x^+_i(z_1)x^+_i(z_2)\varphi_j(wq^\frac{1}{2})
- (z_2 - q^2z_1)x^+_i(z_2)x^+_i(z_1)\varphi_j(wq^\frac{1}{2}) \right] \hat{\otimes} x^+_j(cq^{11}).$$

Now using (3), (which is a valid relation because we do this computation in $\overline{U}_q(b^+)/I_+^1 \otimes \overline{U}_q(b^+)/I_+^1$) we see the left operand of the tensor product is zero.

Now remark that all the factor of the form $(az - bw)$ in lemma 2 are precisely those satisfying the condition in lemma 1. Thus we get that $S^+_{\varepsilon} = 0$ for $\varepsilon \neq \{0,0,0\}$ and $\varepsilon \neq \{1,1,1\}$. So the Serre relations are quasi-primitive in $\overline{U}_q(b^+)/I_+^1$. Continuing as in proposition 3 we see the Serre relations are in the annihilator ideal $\overline{T}_+$. A similar result holds for the negative part.

Let $I_+^3$ and $I_-^3$ be the two sided ideals generated by all the commutation relations we have seen so far. These are weak Hopf ideals, and we denote by $U_q(b^\pm)$ the quotient of $\overline{U}_q(b^\pm)$ by $I_+^3$. The induced Hopf pairing leads to the weak quantum double $\mathcal{D}(U_q(b^+), U_q(b^-))$. The new Hopf pairing will be noted as the former, and let $I_+^3$ and $I_-^3$ be the annihilator ideals of this new pairing.
1.5 An algebra morphism between $U_q(\hat{g})$ and $\mathcal{D}(U_q(b^+), U_q(b^-))$

Remark that in $\mathcal{D}(U_q(b^+), U_q(b^-))$, the elements $q^x \otimes q^{-y}$ and $\varphi_i \otimes \psi_i$ are central and group like. Moreover, let $I$ be the two-sided ideal generated by $q^x \otimes q^{-y} - 1 \otimes 1$ and $\varphi_i \otimes \psi_i - 1 \otimes 1$. We know there is an algebra morphism from $U_q(\hat{g})$ onto $\mathcal{D}(U_q(b^+), U_q(b^-))/I$. Actually, we have a little bit more:

**Proposition 8** There is an Hopf algebra isomorphism $\Phi$ between $U_q(\hat{g})$ and $\mathcal{D}(U_q(b^+), U_q(b^-))/I$.

*Proof.* Because of the remark above we know that $\Phi$ is onto. It remains to show it is an isomorphism as a vector space. We have $\mathcal{D}(U_q(b^+), U_q(b^-))/I = U_+ \otimes \mathbb{C}[q^{\pm \frac{1}{2}}] \otimes U_-$, where $U_+$ is generated by the $x_i^+(z)$ and $\varphi_i(z)$ for $i = 1, \ldots, n-1$, and $U_-$ is constructed accordingly. Now remark that any element of $U_q(\hat{g})$ can be assumed to be in $U_+ \otimes \mathbb{C}[q^{\pm \frac{1}{2}}] \cdot U_-$. Indeed, all the commutation relations necessary for this operation involve finite sums (actually we get infinite sums only when we commute $x_i^+(z)$ and $x_j^-(w)$, but we do not have to do that here). Moreover, the above decomposition is unique, which means $U_q(\hat{g}) \simeq U_+ \otimes \mathbb{C}[q^{\pm \frac{1}{2}}] \otimes U_-$. Rosso proved this assertion for $U_q(\mathfrak{g})$ in [8]. His proof can be easily extended to our case in a straightforward matter, so it will not be done here. Finally we have $U_q(\hat{g}) \simeq U_+ \otimes \mathbb{C}[q^{\pm \frac{1}{2}}] \otimes U_- = \mathcal{D}(U_q(b^+), U_q(b^-))/I$, and the result follows.

**Proposition 9** The Hopf pairing between $U_q(b^+)$ and $U_q(b^-)$ is non degenerate.

*Proof.* Let $X^+$ be an element of $I_+$ of minimal degree $\alpha \in \mathbb{Q}_+$. In $\Delta(X^+)$, an element of degree $(\beta, \beta')$ with $\beta$ and $\beta'$ non zero is in $I^+ \otimes U(b^+) + U(b^+) \otimes I^+$, so because the $\beta + \beta'$ is minimal this element must be 0. Then $\Delta(X^+)$ is the sum of two elements of degree $(\alpha, 0)$ and $(0, \alpha)$. More specifically, we have $\Delta(X^+) = X^+ \otimes k + k' \otimes X^+$ with $k$ and $k'$ in $U_q(b^+)$, i.e. $X^+$ is quasi primitive. Now, $X^+$ is quasi-commutative: if $Y \in U(b^-)$ we have $X^+ Y = \lambda Y X^+$, with $\lambda \in \mathbb{C}$. To see that, we compute $Y X^+$ in the weak quantum double, to get

$$Y X^+ = \sum \langle X^+_{(1)}, S(Y_{(1)}) \rangle \langle X^+_{(3)}, Y_{(3)} \rangle X^+_{(2)} Y_{(2)}.$$  

But $X^+$ being in the kernel of the weak Hopf pairing, only the terms involving the element of degree $(0, \alpha, 0)$ in $\Delta^2(X^+)$ can be non zero. If $Y$ is of degree
\[ \beta \text{ then because the weak Hopf pairing is } Q\text{-graded, we can eliminate all the terms of the sum except those involving the element of degree } (0, \beta, 0) \text{ in } \Delta^2(Y). \] Combining this, we have

\[ YX^+ = \langle X^+_{(1)}, S(Y_{(1)}) \rangle \langle X^+_{(3)}, Y_{(3)} \rangle X^+ Y = \lambda X^+ Y. \]

Let \( \Lambda \) in \( Q_+ \) and \( L(\Lambda) \) be a highest weight module with highest weight vector \( v_\Lambda \). Actually we have \( L(\Lambda) = U(b^-)v_\Lambda \). Now \( X^+v_\Lambda = 0 \), and so \( X^+L(\Lambda) = 0 \) because \( X^+ \) is quasi-commutative. So \( X^+ \) is in the annihilator of every highest weight module. That is:

\[ X^+ \in \bigcap_{\Lambda \in Q^+} \text{Ann}_{U(\hat{g})} L(\Lambda). \]

But this intersection is zero according to [5], which ends the proof.

We have some kind of “functorial” construction giving the whole algebra from its positive part. Still, this Borel subalgebra \( U_q(b^+) \) is given in term of generators and relations. Rosso exhibited an interesting construction which we apply here, allowing us to construct \( U_q(b^+) \) from a suitable Hopf bimodule.

## 2 Quantum shuffle construction of \( U_q(b^+) \)

We begin by recalling some facts about quantum shuffle algebras, following Marc Rosso’s point of view. For more details, see [9].

### 2.1 Tensor algebra and cotensor coalgebra

The following facts are due to Nichols. More details can be found in [9]. Let \( H \) be a \( k \)-Hopf algebra over a commutative field \( k \) with invertible antipode \( S \), and \( M \) a Hopf bimodule over \( H \) (i.e. \( M \) is a \( H \)-bimodule and a \( H \)-bicomodule). Let \( \delta_L \) and \( \delta_R \) be the left and right coaction.

We have two dual constructions over \( H \) and \( M \): the tensor algebra is

\[ T_H(M) = H \oplus \bigoplus_{n \geq 1} M^\otimes n, \]

where the multiplication is given by concatenation over \( H \) for elements of non zero degree, and left or right module action when one element is in \( H \). The
tensor algebra as an universal property from which $T_H(M)$ can be endowed with a Hopf algebra structure, where the coproduct is the unique algebra map extending the coproduct on $H$ and $\delta_L + \delta_R$ on $M$.

Dually, the cotensor coalgebra is defined as

$$T_H^c(M) = H \oplus \bigoplus_{n \geq 1} M \square_H^n,$$

where $M \square_H M$ is the kernel of $\delta_R \otimes \text{Id} - \text{Id} \otimes \delta_L : M \otimes M \to M \otimes H \otimes M$. The coproduct is induced by the coproduct on $H$; on $M \square_H^n$ the component of bidegree $(i,j)$ of the coproduct is given by the $\delta_L \otimes \text{Id}$ when $i = 0$, $\text{Id} \otimes \delta_R$ when $j = 0$, and is induced by the map $(m_1 \otimes \cdots \otimes m_n) \mapsto (m_1 \otimes \cdots \otimes m_i) \otimes (m_{i+1} \otimes \cdots \otimes m_n)$ otherwise. The counit is $\varepsilon_H \circ \pi$ where $\pi$ is the projection onto degree zero. Here again the cotensor coalgebra has an universal property making it an Hopf algebra, where the multiplication is the unique coalgebra map extending the usual multiplication on $H$ and defined by the module structure maps on degree $H \otimes M + M \otimes H$.

Let $S_H(M)$ be the sub-Hopf algebra of $T_H^c(M)$ generated by $H$ and $M$. It is a Hopf bimodule, and it can be also obtained by the following dual construction: the universal property on $T_H^c(M)$ allows us to define an unique Hopf algebra map $\Theta$ from $T_H(M)$ to $T_H^c(M)$ induced by the natural isomorphisms on elements of degree zero and one. Then $S_H(M)$ is the image of $\Theta$. If $I$ is the kernel of $\Theta$, then we have also $S_H(M) \simeq T_H(M)/I$.

### 2.2 The quantum shuffle algebra

In [9], Rosso brought Nichols work a step further by considering a braiding introduced by Woronowicz in [11]. This braiding allows us to give a precise description of the coproduct in $T_H(M)$ and, dually, the product in $T_H^c(M)$.

Let us consider the subspaces of left and right coinvariants of $M$:

$$M^L = \{ m \in M \mid \delta_L(m) = 1 \otimes m \}$$
$$M^R = \{ m \in M \mid \delta_R(m) = m \otimes 1. \}$$

We know that $M$ is isomorphic to $M^R \otimes H$ with trivial right module and comodule structure. Moreover, $M^R$ is a sub left comodule of $M$, and a left module for the left adjoint action $h \cdot m = \sum h_{(1)} m S(h_{(2)})$. Similar properties hold for $M^L$ and the right adjoint coaction.
The braiding $\sigma$ introduced by Woronowicz sends $M^R \otimes M^R$ to himself. It is defined by

$$\sigma(m \otimes m') = \sum m_{(-2)} m' S(m_{(-1)}) \otimes m_{(0)},$$

and it satisfies the usual braid equation

$$(\text{Id} \otimes \sigma)(\sigma \otimes \text{Id})(\text{Id} \otimes \sigma) = (\sigma \otimes \text{Id})(\text{Id} \otimes \sigma)(\sigma \otimes \text{Id}).$$

Let us note $V = M^R$. We denote by $S_n$ the symmetric group of $\{1, \ldots, n\}$, and by $s_i$ the transposition $(i, i+1)$ for $i = 1, \ldots, n-1$. If $p_1 + \cdots + p_k = n$, we denote by $S_{p_1, \ldots, p_k}$ the set of $w \in S_n$ such that $w(1) < w(2) < \cdots < w(p_1)$, $w(p_1 + 1) < \cdots < w(p_1 + p_2)$, $\ldots$, and $w(p_1 + p_2 + \cdots + p_k) < \cdots < w(p_1 + \cdots + p_k)$. Such a $w$ is called a $(p_1, \ldots, p_k)$-shuffle.

The braid group $B_n$ acts on $V^\otimes n$ in the usual way: for $i = 1, \ldots, n-1$, we associate the $i$-th generator $\sigma_i$ of $B_n$ with $\text{Id}_V \otimes \sigma \otimes \text{Id}_V \otimes \cdots \otimes \text{Id}_V$ on $V^\otimes n$.

Let $w$ be a permutation of the set $\{1, \ldots, n\}$. Then the lift of $w$ in $B_n$ is $T_w = \sigma_{i_1} \cdots \sigma_{i_k}$, where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression of $w$. The corresponding Hopf bimodule isomorphism in $V^\otimes n$ will also be denoted by $T_w$.

**Proposition 10** Let $\star$ be the product on $T(V)$ defined by:

$$(x_1 \otimes \cdots \otimes x_p) \star (x_{p+1} \otimes \cdots \otimes x_n) = \sum_{w \in S_{p,n}} T_w(x_1 \otimes \cdots \otimes x_n),$$

where $x_1, \ldots, x_n \in V$ and $S_{p,n}$ is the set of $(p, n-p)$-shuffles.

Then $(T(V), \star)$ is an associative algebra.

The product $\star$ is called a quantum shuffle product. This construction is similar to the classical shuffle product, the usual twist in $V \otimes V$ being replaced by $\sigma$. The algebra $T(V)$ is then called a quantum shuffle algebra.

Remark that $T(V)$ is build on $V = M^R$. In order to have a more general construction on $M = V \otimes H$, we consider $T(V) \otimes H$, on which we put the following structure: it is an $H$-comodule, with $\delta_L$ given by the diagonal coaction of $H$ on each $V^\otimes n$. We put the crossed product algebra structure on $T(V) \otimes H$, with $H$ acting diagonally on $T(V)$. Finally, the coalgebra structure is given by

$$\Delta(v_1 \otimes \cdots \otimes v_n \otimes h) = \sum_{k=0}^n (v_1 \otimes \cdots \otimes v_k \otimes v_{k+1}(-1) \cdots v_{n(-1)}h_{(1)}) \otimes (v_{k+1}(0) \otimes \cdots \otimes v_n(0) \otimes h_{(2)}),$$

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where $v_1, \ldots, v_n$ are in $V$, and $h \in H$. Those structures are compatible, and $T(V) \otimes H$ becomes a Hopf algebra.

### 2.3 The quantum symmetric algebra

Rosso showed that the cotensor coalgebra $T^c_H(M)$ is isomorphic to $T(V) \otimes H$ as a right module and comodule. Furthermore, the image of $T(V)$ in $T^c_H(M)$ is the subalgebra of right coinvariants of $T^c_H(M)$.

Furthermore the sub Hopf algebra $S_H(M)$ of $T^c_H(M)$ generated by $H$ and $M$, is a Hopf bimodule, and it is isomorphic to the crossed product of $H$ by $S_{\sigma}(V)$, where $S_{\sigma}(V)$ is the subalgebra of $T(V)$ generated by $V$. Actually, $S_{\sigma}(V)$ is isomorphic to the right coinvariants of $S_H(M)$, via the previous isomorphism.

The Hopf algebra $S_H(M)$ is called a quantum symmetric algebra, and is the main object of our study in the sequel of this paper.

### 2.4 Construction of $S_H(M)$

In order to apply Rosso’s construction to $U_q(b^+)$, we will have to exhibit a sub Hopf algebra of it and put some Hopf bimodule structure on $\hat{U}_q(b^+)$ over this subalgebra. What we gain by using this functorial construction is a smaller number of generators and relations (those necessary to describe the subalgebra), which leads to simpler computations.

Let $H = U_q(h^+)$ be the sub Hopf algebra of $U_q(b^+)^T$ generated (algebraically, there is no need for a completion to make $H$ an Hopf algebra) by $\varphi_{i,l}$, $\psi_{i,0}$ and $q^{\pm \frac{c}{2}}$, for $i = 1, \ldots, n - 1, l \leq 0$. Let $V$ be the subspace of $U_q(\hat{\mathfrak{g}})$ generated by $x_i^{+,k}$, for $i = 1, \ldots, n - 1$ and $k \in \mathbb{Z}$. We would like to have a $H$-Hopf bimodule $V \otimes H$, with a left action of $H$ reflecting relation (4) in definition [1]. But as this Hopf bimodule will have to reflect the Hopf algebra structure of $U_q(b^+)^T$ (see remarks below), we shall consider $V \hat{\otimes} H$, the completion being similar to the one in [1].

**Proposition 11** Let $M$ be the completed tensor product $V \hat{\otimes} H$. $M$ becomes an $H$-Hopf bimodule the following way: $V \hat{\otimes} H$ is a trivial right module and
comodule. The left action of $H$ on $M$ is given by
\[
\varphi_{i,n}(x_{j,p}^+ \hat{\otimes} \varphi_{l,q}) = \sum_{k \geq 0} g_k^{(j)} x_{j,p-k}^+ \hat{\otimes} q^{-k} \varphi_{i,n+k} \varphi_{l,q},
\] (15)
and the left coaction of $H$ on $V$ by
\[
\delta_L(x_i^+(z)) = \varphi_i(z q^c) \hat{\otimes} x_i^+(z q^{c1}).
\]
The left coaction of $H$ on $M$ is then the diagonal coaction on $V \hat{\otimes} H$.

Remark 1. If we identify $x_i^+(z) \hat{\otimes} 1$ with $x_i^+(z)$ and $1 \hat{\otimes} \varphi_i(z)$ with $\varphi_i(z)$ in $M$, then the left action of $H$ on $M^R = V \hat{\otimes} 1$ can be written in the more satisfying form of relation (14).

Remark 2. If we compute $(\delta_L + \delta_R)(x_i^+(z))$, we recognize the expression of $\Delta(x_i^+(z))$ in $U_q(b^+)$. This is due to the fact that in the quantum symmetric algebra $S_H(M)$, the coproduct is just $\delta_L + \delta_R$ on elements of degree 1.

Proof. We have to check two things. It is trivial to see that $\delta_L$ and $\delta_R$ commutes, i.e. $(\delta_L \otimes \text{Id})\delta_R = (\text{Id} \otimes \delta_R)\delta_L$. It remains to show that $\delta_L$ and $\delta_R$ are morphisms of $H$-bimodules. This can be done by a direct calculation, but generating series are not well suited for that. Therefore we have to use the following relations, which are translations of all the former relations involving those series:
\[
\varphi_{i,n} \varphi_{j,m} = \varphi_{j,m} \varphi_{i,n},
\]
\[
(x_{i,n}^+ \hat{\otimes} \varphi_{j,m}) \varphi_{k,p} = x_{i,n}^+ \hat{\otimes} \varphi_{j,m} \varphi_{k,p},
\]
\[
\delta_R(x_{i,n}^+ \hat{\otimes} 1) = (x_{i,n}^+ \hat{\otimes} 1) \hat{\otimes} 1,
\]
\[
\delta_L(x_{i,n}^+) = \sum_{k \leq 0} \varphi_{i,k} q^{k \frac{c}{2} - nc} \hat{\otimes} x_{i,n-k}^+,
\] (16)
\[
\Delta(\varphi_{i,n}) = \sum_{r+s=n} \varphi_{i,r} q^{-s \frac{c}{2}} \otimes \varphi_{i,s} q^{r \frac{c}{2}}.
\] (17)
All verifications are now straightforward.

In the next part, we will compute the braiding associated to $M$.

2.5 The braiding on $M^R \otimes M^R$

Recall that the braiding $\sigma$ on $M^R \otimes M^R$ is defined by
\[
\sigma(m \otimes m') = \sum m_{(-2)} m' S(m_{(-1)}) \otimes m_{(0)}.
\]
Proposition 12 The braiding $\sigma$ is given by:

$$\sigma(x^+_i(z) \hat{\otimes} x^+_j(w)) = g_{ij}(zw^{-1})x^+_j(w) \hat{\otimes} x^+_i(z).$$

Proof. We want to compute $(\text{Id} \hat{\otimes} S \hat{\otimes} \text{Id})(\Delta \hat{\otimes} \text{Id})\delta_L(x^+_i(z))$. That is

$$(\text{Id} \hat{\otimes} S \hat{\otimes} \text{Id})(\Delta \hat{\otimes} \text{Id})(\phi)(\text{Id} \hat{\otimes} S \hat{\otimes} \text{Id})(\Delta \hat{\otimes} \text{Id}) = (\text{Id} \hat{\otimes} S \hat{\otimes} \text{Id})(\Delta \hat{\otimes} \text{Id})(\phi)$$

Therefore the braiding becomes:

$$\sigma(x^+_i(z) \hat{\otimes} x^+_j(w)) = \sum_{n \leq 0, m \in \mathbb{Z}} \varphi_{i,r} q^{-n \frac{s}{2} - mc} x^+_i(z) S(\varphi_{i,s}) q^{s \frac{s}{2} + mc - n} x^+_j(w) S(\varphi_{i,s}) q^{s \frac{s}{2} + mc - n} x^+_j(w)$$

Using the $H$-Hopf bimodule $M$ with the braiding described above, we take a more precise look at the quantum symmetric algebra we get.

2.6 An isomorphism between $S_H(M)$ and $U_q(b^+)$

Now we state the main result of this paper.
Theorem 1 There is a Hopf algebra isomorphism between $U_q(b^+)$ and $S_H(M)$.

Remark 3. Considering the definition of $H$ and $M$, the existence of such an isomorphism is not a surprise. Almost all the work is already done, and the interesting fact is the actual existence of a quantum symmetric algebra which is isomorphic to $U_q(b^+)$.

Proof. There is an obvious map going from the associative algebra with unit 1 and generators $\{x_{i,k}, \varphi_{i,l}, \varphi_{i,0}^{-1}, q^\pm \hat{z} | i = 1, \ldots, n-1, k \in \mathbb{Z}, l \leq 0\}$ to $S_H(M)$ with the shuffle product. We now have to check that the relations defining $U_q(b^+)$ are verified in $S_H(M)$ for the quantum shuffle product. Relation (2) of definition 1 is true by construction of $H$ and we already examined relation (4) in remark 1. We get the commutation relation (13) between $x^+_i(z)$ and $x^+_j(w)$ as follows:

$$x^+_i(z) \ast x^+_j(w) = x^+_i(z) \otimes x^+_j(w) + g_{ij}(zw^{-1})x^+_j(w) \otimes x^+_i(z),$$

while on the other side:

$$x^+_j(w) \ast x^+_i(z) = x^+_j(w) \otimes x^+_i(z) + g_{ji}(wz^{-1})x^+_i(z) \otimes x^+_j(w).$$

Then we get the result by using the relations $(z-q_{ij}w)g_{ij}(zw^{-1}) = (q_{ij}z-w)$ and $(q^{11}z-w)g_{ji}(wz^{-1}) = (z-q_{ij}w)$. Now it remains to show the quantum Serre relations (14). This is done using the same computation than in proposition 7 when we wanted to show that the Serre elements are quasi primitives. That is, we show that the Serre relation multiplicated by suitable factors is zero. This is easily done by applying the commutation relation (13) between $x^+_i(z)$ and $x^+_j(w)$. Now, because of the nature of the shuffle product any multiplicative factor we used satisfy the condition in lemma 1. Thus the Serre relations are zero.

Now we have an algebra morphism $\Phi$ from $U_q(b^+)$ to $S_H(M)$. It is easy to show it is actually a Hopf algebra morphism (see remark 2). This morphism is onto by construction. To achieve the proof of the theorem it remains to show that the morphism is one to one. The quantum symmetric algebra $S_H(M)$ is a graded algebra. The elements of $H$ are given degree 0, and those of $V = M^R$ are given degree 1. We then use the following lemma:

Lemma 3 Let $x$ be an element of degree at least 2 in the kernel of the morphism $\Phi$ from $U_q(b^+)$ to $S_H(M)$. Then $x$ is 0.

This is done by induction. We know that elements of degree 0 and 1 are not in the kernel of $\Phi$. Let us suppose that there are no element of degree at
most $n$ in the kernel, and let $x$ be an element of degree $n+1$. Then we have

$$\Delta(x) = \delta_L(x) + \sum x_{(0)} \hat{\otimes} x_{(1)} + \delta_R(x).$$

But the kernel is a Hopf ideal and the elements $x_{(0)}$ and $x_{(1)}$ are of degree at most $n$. So we get finally

$$\Delta(x) = \delta_L(x) + \delta_R(x),$$

which means $\Delta(x)$ is in $H \hat{\otimes} M + M \hat{\otimes} H$. But we know that elements of degree at least 2 with such a coproduct are in the kernel of the weak Hopf pairing between $U_q(b^+)$ and $U_q(b^-)$. Using the non degeneracy of this weak Hopf pairing, we get that those elements are null in $U_q(b^+)$. 

### 2.7 Work of B. Enriquez

After the preparation of the preliminary version of this manuscript, the work of Enriquez came to our attention.

Though the author discusses a shuffle algebra description of the positive part of $U_q(\mathfrak{g})$, his approach is completely different from the one being considered here. Enriquez describes some vanishing conditions on the correlation functions of Drinfeld currents of the positive nilpotent part of $U_q(\mathfrak{g})$. Those conditions are then used to give an isomorphism between the positive part of $U_q(\mathfrak{g})$ and some shuffle algebra construction, though his shuffle algebra seems to be different from ours. In his framework the problem of topological completion can be completely avoided because only highest weight modules are considered. Moreover, Enriquez does not consider the case with non zero central part (i.e. he assumes that $c = 0$).

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