THE PEAK MODEL FOR THE TRIPLET EXTENSIONS AND THEIR TRANSFORMATIONS TO THE REFERENCE HILBERT SPACE IN THE CASE OF FINITE DEFECT NUMBERS

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Abstract. We develop the so-called peak model for the triplet extensions of supersingular perturbations in the case of a not necessarily semibounded symmetric operator with finite defect numbers. The triplet extensions in scales of Hilbert spaces are described by means of abstract boundary conditions. The resolvent formulas of Krein–Naimark type are presented in terms of the \( \gamma \)-field and the abstract Weyl function. By applying certain scaling transformations to the triplet extensions in an intermediate Hilbert space we investigate the obtained operators acting in the reference Hilbert space and we show the connection with the classical extensions.

1. Introduction

In the classical extension theory a proper extension of a densely defined, closed, symmetric operator in a Hilbert space is the adjoint operator parametrized by an ordinary boundary triple and a boundary parameter. The theory of ordinary boundary triples and their variants is studied in [1, 2, 3, 4, 5, 6, 7] and in references therein. For a symmetric operator with finite defect numbers \((d, d)\) a boundary parameter is a linear relation [8, 9, 10] in a \(d\)-dimensional Hilbert space, hence \(\mathbb{C}^d\). A proper extension is in bijective correspondence with a boundary parameter, see e.g. [3, Proposition 7.12].

The extension theory can be presented, and will be presented throughout, by using the notions from the theory of singular perturbations of self-adjoint operators. For that reason a densely defined, closed, symmetric operator \(\hat{L}_0\) is defined as a restriction of a (typically distinguished) self-adjoint operator \(L\) to the domain of elements \(u \in \text{dom } L\) such that \(\langle \hat{\varphi}, u \rangle = 0\) for some vector-valued functional \(\langle \hat{\varphi}, \cdot \rangle : \text{dom } L \to \mathbb{C}^d\).

Let \(L\) be a self-adjoint operator in a Hilbert space \(\mathcal{H}_0\) and let \((\mathcal{H}_n)_{n \in \mathbb{Z}}\) be the scale of Hilbert spaces \((\mathcal{H}_n, \langle \cdot, \cdot \rangle_n)\) associated with \(L\). Let \(\langle \varphi, \cdot \rangle\) be a vector-valued functional from \(\mathcal{H}_n (n \in \mathbb{N})\) to \(\mathbb{C}^d (d \in \mathbb{N})\). Depending on whether \(n = 1\) or \(n \geq 2\), a rank-\(d\) singular perturbation of \(L\), described by means of \(\varphi\), is respectively infinitesimally form bounded or form unbounded; for \(n \geq 3\), the perturbation is also said to be supersingular. The main aspects of the theory of singular perturbations can be found in [11, 12, 13, 14, 15, 16, 17, 18, 19] and in references therein.

When \(n = m + 2\) for \(m \in \mathbb{N}\), the classical restriction-extension theory is limited for describing singular perturbations in that a symmetric restriction \(L_{\text{min}} \subseteq L\) defined on \(f \in \mathcal{H}_{m+2}\) such that \(\langle \varphi, f \rangle = 0\) is essentially self-adjoint in \(\mathcal{H}_0\), and so there are no nontrivial self-adjoint extensions in the reference Hilbert space \(\mathcal{H}_0\). To deal with supersingular perturbations one studies triplet extensions instead. The triplet adjoint \(L_{\text{max}}\) of \(L_{\text{min}}\) corresponding to the Hilbert
triple $\mathcal{H}_m \subseteq \mathcal{H}_0 \subseteq \mathcal{H}_{-m}$ is associated with $\hat{L}_0$ and $L_{\min}$ by the similarity relations as described in Corollary 4.4.

The supersingular perturbation in the reference Hilbert space is interpreted by means of the compressed resolvent of Krein–Naimark type, provided that the corresponding Weyl (or $Q$-) function is appropriately renormalized; see [20, 21, 22] for the theory of generalized resolvents of symmetric operators. The renormalization procedure suggests that it is sufficient to consider the triplet adjoint $L_{\max}$ restricted to a finite-dimensional extension of $\mathcal{H}_m$. The extended space $\mathcal{H}$ is called an intermediate Hilbert space, in the sense that $\mathcal{H}_{-m} \subseteq \mathcal{H} \subseteq \mathcal{H}_m$.

The theory of triplet extensions in $\mathcal{H}$, which is developed in [14, 15, 16, 23, 24, 25], is referred to as the cascade model. Under appropriate conditions, the restriction $A_{\max} \subseteq L_{\max}$ to $\mathcal{H}$ is the adjoint of a densely defined, closed, symmetric operator $A_{\min} := A_{\max}^*$ in $\mathcal{H}$ (see Corollary 6.8 for the case considered in this paper).

In the present paper, however, we generalize the so-called peak model, first introduced in [26] in the case $d = 1$ and $L \geq 0$. The main difference between the two models is that the closed linear span of singular elements in $\mathcal{H}$ has the same order of singularity in the peak model, while the singular elements have different orders of singularity in the cascade model. Namely, $\mathcal{H}$ is the completion of the vector direct sum $\mathcal{H}_m + \mathfrak{K}$, where the $md$-dimensional vector space $\mathfrak{K}$ is defined as a closed linear span of singular elements from $\mathcal{H}_m \setminus \mathcal{H}_{-m+1}$ in the peak model and from $\mathcal{H}_{-m-2+j} \setminus \mathcal{H}_{-m-1+j}$ ($1 \leq j \leq m$) in the cascade model. As opposed to the cascade model, which deals with operators in Pontryagin spaces, the peak model is essentially the Hilbert space model. The space $\mathcal{H}$ in the peak model is studied in detail in Section 5.

We briefly describe the structure of the paper by simultaneously discussing some key results. In Sections 2, 3, we provide some preparatory results from the classical extension theory that we use later on. We discuss several motivational examples by presenting a regularized functional $\tilde{\varphi} \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$—which is obtained from $\varphi \in \mathcal{H}_{-m-2} \setminus \mathcal{H}_{-m-1}$ by means of the scaling transformation (isometric isomorphism) $P(L)^{1/2} := (|L| + I)^{m/2}$—in the case when $\varphi = \delta$ is the Dirac distribution.

In Sections 4, 5, 6 we extend the model initiated in [26] to the case of a not necessarily semibounded self-adjoint operator $L$, whose symmetric restriction $L_{\min}$ in the space $\mathcal{H}_m$ has finite defect numbers $(d, d)$. Whenever appropriate, we present our main findings by using the notions from the theory of boundary triples (see e.g. Corollary 6.8, Theorem 6.10). The triplet adjoints are described in a form similar to von Neumann’s formulas (Theorems 6.3, 6.7).

In the last two Sections 7, 8 we study the properties of the triplet extensions, initially defined in $\mathcal{H}$, and then transformed to the reference Hilbert space $\mathcal{H}_0$ by using the transformations of the form: a) $P(L)^{1/2}$ and b) $P_{\mathcal{H}}P(L)^{1/2}$; $P_{\mathcal{H}}$ is a bounded operator (e.g. isometric isomorphism) from $\mathcal{H}_{-m}$ onto $\mathcal{H}$.

In case a) (Section 7) we derive, among other things, a representation of the adjoint operator $\hat{L}_0^*$ in $\mathcal{H}_0$ in terms of model parameters (Theorem 7.15, formula (7.38)). Because $\hat{L}_0^*$ does not depend on $m$, the derived representation remains valid for all $m > 0$. The result turns out to be useful when transferring from the triplet extensions back to the classical extensions. Thus, in Theorems 7.17 and 7.18 we demonstrate the connection between a proper extension of $\hat{L}_0$ and the transformed triplet extension. It follows in particular that the domain of the transformed triplet extension is a core for $\hat{L}_0^*$. As it is illustrated in Section 7.5, the peak model allows one to construct an isometric boundary triple for $\hat{L}_0^*$ (see [2, 3] for the theory and applications of isometric triples).

It appears from Theorem 7.11 that the scaling transformation in case a) does not preserve the main part of the resolvent, i.e. its Weyl (or $Q$-) function. Therefore, by modifying the transformation as given in case b), in Sections 8 we study the conditions imposed on $P_{\mathcal{H}}$ upon
which the triplet extension transformed to $\mathcal{F}_0$ preserves the Weyl function. In other words, we propose a family of operators in $\mathcal{F}_0$ that share the same Weyl function with the triplet extensions in $\mathcal{H}$. The main results are Theorem 8.10 and Corollary 8.13. In the special case the transformation becomes an isometric isomorphism constructed in the proof of [22, Theorem 2.2].

Let us fix some notation. As a rule, a linear operator $T$ from a Hilbert space $\mathfrak{h}$ to a Hilbert space $\mathfrak{k}$ is identified with its graph, i.e. a single-valued linear relation. Recall that a linear relation $T$ is a subspace of the Cartesian product $\mathfrak{h} \times \mathfrak{k}$ provided with the usual inner product. When $\mathfrak{k} = \mathfrak{h}$, one considers a linear relation in $\mathfrak{h}$. Thus, if in particular $\mathfrak{h}$ is finite-dimensional (i.e. $\mathfrak{h} = \mathbb{C}^d$) then $T$ in $\mathfrak{h}$ is automatically closed. Nevertheless we sometimes write additionally that the relation is closed when we want to emphasize this property. A (closed) linear relation $T$ is densely defined. The triplet adjoint will be defined separately (Definition 4.1). The scalar product in a Hilbert space $\mathfrak{h}$ is a bijective mapping that preserves the norm (recall e.g. [27, Definition II.3.17]). However, we avoid the term “unitary” bearing in mind that it is more often understood in the case $\mathfrak{h} = \mathfrak{f}$.

The domain, the range, the kernel, and the multivalued part of a linear relation $T$ are denoted by $\text{dom} \ T$, $\text{ran} \ T$, $\text{ker} \ T$, and $\text{mul} \ T$, respectively. The eigenspace $\text{ker}(T - z)$, for $z \in \mathbb{C}$, is denoted by $\mathcal{N}_z(T)$. The resolvent set of a closed linear relation $T$ is $\text{res} \ T$. We write $T \upharpoonright_{\mathcal{F}}$ for the restriction to a subset $\mathfrak{F} \subseteq \text{dom} \ T$.

In the proofs to be presented, we repeatedly use the following definition of the adjoint: The adjoint of a linear relation $T$ from a Hilbert space $\mathfrak{h}$ to a Hilbert space $\mathfrak{k}$ is the set of elements $(y, x) \in \mathfrak{k} \times \mathfrak{h}$ such that $(\forall (u, v) \in T)(u, x)_{\mathfrak{h}} = (v, y)_{\mathfrak{k}}$. The adjoint is denoted by $T^*$, and it is an operator iff $T$ is densely defined. The triplet adjoint will be defined separately (Definition 4.1). The space of bounded operators from a Hilbert space $\mathfrak{h}$ to a Hilbert space $\mathfrak{k}$ is denoted by $\mathcal{L}(\mathfrak{h}, \mathfrak{k})$.

In Section 5 we also use $\mathfrak{F} (\text{resp. } \mathfrak{F})$ for the componentwise (resp. orthogonal componentwise) sum of linear relations; see [9, Section 2.4].

2. Scales of Hilbert spaces

2.1. Canonical scale spaces. Let $(\mathcal{F}_n = \mathcal{F}_n(L))_{n \in \mathbb{Z}}$ be the scale of Hilbert spaces associated with a (not necessarily semibounded) self-adjoint operator $L : \mathcal{F}_2 \to \mathcal{F}_0$. More precisely, for $m \in \mathbb{N}_0$, $\mathcal{F}_m$ is the Hilbert space $(\text{dom} |L|^{m/2}, \langle \cdot, \cdot \rangle_m)$ equipped with the (canonical) scalar product

$$\langle f, g \rangle_m := (P(L)^{1/2}f, P(L)^{1/2}g)_0, \quad P(L) := (|L| + I)^m$$

with $f, g \in \mathcal{F}_m$; the induced graph norm $\|f\|_m := \sqrt{\langle f, f \rangle_m}$.

The dual Hilbert space $\mathcal{F}_{-m}$ ($m \in \mathbb{N}_0$) is the completion of $\mathcal{F}_0$ with respect to the norm

$$\|f\|_{-m} := \|P(L_{-m})^{-1/2}f\|_0, \quad f \in \mathcal{F}_{-m}.$$  

The continuation $L_{-m}$ of $L$ is a bounded self-adjoint operator from $\mathcal{F}_{-m+2}$ into $\mathcal{F}_{-m}$. More generally, define a self-adjoint operator in $\mathcal{F}_n$ by

$$L_n := L \upharpoonright_{\mathcal{F}_{n+2}}: \mathcal{F}_{n+2} \to \mathcal{F}_{n}, \quad n \in \mathbb{Z}.$$  

We have $P(L_n)^{1/2} = \mathcal{F}_{n+m}$ and $P(L_n)^{1/2} \mathcal{F}_{n+m} \subseteq \mathcal{F}_n$. On the other hand, when considered as an operator in $\mathcal{F}_n$, $P(L_n)^{1/2}$ is self-adjoint (and hence closed), with the trivial kernel...
ker \( P(L_n)^{1/2} = \{0\} \); hence the range \( P(L_n)^{1/2} \mathcal{H}_{n+m} = \mathcal{H}_n \) by applying e.g. [28, Corollary 2.2].

Using in addition that \( L_l \supseteq L_n \) for integers \( l \leq n \), we therefore deduce the following result.

**Proposition 2.1.** \( (\forall l \in \mathbb{Z}_{\leq n}) \) \( P(L_l)^{1/2} \in [\mathcal{H}_{n+m}, \mathcal{H}_n] \) is an isometric isomorphism. \( \square \)

**Corollary 2.2.** \( (\forall l \in \mathbb{Z}_{\leq n-m}) \) \( P(L_l) \in [\mathcal{H}_{n+m}, \mathcal{H}_{n-m}] \) is an isometric isomorphism. \( \square \)

In a slightly different form the results stated in Proposition 2.1 and Corollary 2.2 can be found e.g. in [11, Section II.A]. Let us remark that the subscript \( l \) is usually omitted, for the operator is assumed to be defined in the generalized sense. However, we write the subscript in order to keep track of the Hilbert spaces under consideration.

Using (2.2) and Proposition 2.1 one deduces another elementary but useful proposition.

**Corollary 2.3.** Let \( m \in \mathbb{N}_0 \); then

(i) \( L_m = P(L)^{-1/2}LP(L)^{-1/2} \).

(ii) \( L_{-m} = P(L_{-m})^{1/2}LP(L_{-m+2})^{-1/2} \).

Moreover, the resolvent set \( \text{res} L_m = \text{res} L \forall n \in \mathbb{Z} \). \( \square \)

**Remark 2.4.** One should make a clear distinction between a densely defined self-adjoint operator \( P(L)^{1/2} \) in the Hilbert space \( \mathcal{H}_0 \), with domain \( \text{dom} P(L)^{1/2} = \mathcal{H}_0 \subseteq \mathcal{H}_n (m \in \mathbb{N}_0) \), and a bounded operator \( P(L)^{1/2} \) from the Hilbert space \( \mathcal{H}_m \) to the Hilbert space \( \mathcal{H}_0 \); in the latter case \( P(L)^{1/2} \) has the adjoint.

**Proposition 2.5.** When considered as an operator from the Hilbert space \( \mathcal{H}_m \) to the Hilbert space \( \mathcal{H}_0 \), \( P(L)^{1/2} \) has the adjoint \( (P(L)^{1/2})^* : \mathcal{H}_0 \to \mathcal{H}_m \), which is \( (P(L)^{1/2})^* = P(L)^{-1/2} \). \( \square \)

Throughout we assume that \( m \) is a nonnegative integer, unless stated otherwise, while a generic \( n \in \mathbb{Z} \). In what follows we do not specify \( P(L) \), as long as \( P(L)^{1/2} : \mathcal{H}_m \to \mathcal{H}_0 \) is an isometric isomorphism, which is self-adjoint as an operator in \( \mathcal{H}_0 \).

For more details concerning standard scales of Hilbert spaces from various perspectives, the reader is referred to [12, 13, 17, 18, 19].

### 2.2. Connection with equivalent scale spaces.

Let \( m \in \mathbb{N} \) and let \( \mathcal{Z}_o \subseteq \text{res} L \) be the set of at least \( m \) and at most \( 2m \) numbers \( \mu_t \in \text{res} L (1 \leq t \leq 2m) \) such that \( \mu_t \in \mathcal{Z}_o \leftrightarrow \overline{\mu_t} \in \mathcal{Z}_o \) and \( \mu_t \neq \mu_{t'} \Rightarrow t \neq t' \). Note that \( \mu_t = \overline{\mu_t} \) is permitted for some \( t \), hence the dimension \( \dim \mathcal{Z}_o \) is \( \{m, 2m\} \cap \mathbb{N} \). In particular, \( \dim \mathcal{Z}_o = m \) iff all \( \mu_t \)’s are real, i.e. iff \( \mathcal{Z}_o = \mathcal{Z}_o \cap \mathbb{R} \), while \( \dim \mathcal{Z}_o = 2m \) iff all \( \mu_t \)’s have nontrivial imaginary parts.

Pick (any) \( m \) numbers from \( \mathcal{Z}_o \) and label them by \( z_j (j \in J) \), with \( J := \{1, 2, \ldots, m\} \). The set of \( z_j \)’s is denoted by \( \mathcal{Z} \). This set is not uniquely defined for \( \dim \mathcal{Z}_o > m \). Notice that \( z_j \in \mathcal{Z} \) does not necessarily imply that also \( \overline{z_j} \in \mathcal{Z} \). Indeed, if one defines \( \mathcal{Z}^* \) as the set of \( \overline{z_j} \) such that \( z_j \neq \mathcal{Z} \forall j \in J \), then \( \mathcal{Z}^* \cap \mathcal{Z} = \emptyset \) precisely when \( \dim \mathcal{Z}_o = 2m \). Clearly \( \mathcal{Z}^* \cup \mathcal{Z} = \mathcal{Z}_o \).

For example, let \( \mathcal{Z}_o \) be a singleton (\( m = 1 \)) consisting of a real number \( \zeta \in \text{res} L \). Then all three sets, \( \mathcal{Z}_o \) and \( \mathcal{Z} \) and \( \mathcal{Z}^* \), coincide and are given by \( \{z_1 = \mu_1 = \zeta\} \). However, if \( 3\zeta \neq 0 \), then \( \mathcal{Z}_o = \{\mu_1, \mu_2\} \), where either \( \mu_1 = \zeta \) and \( \mu_2 = \overline{\zeta} \) or \( \mu_1 = \overline{\zeta} \) and \( \mu_2 = \zeta \). There are two possible sets \( \mathcal{Z} \): \( \mathcal{Z} = \{z_1 = \zeta\} \) (with \( \mathcal{Z}^* = \{\overline{\zeta}\} \)) or \( \mathcal{Z} = \{z_1 = \overline{\zeta}\} \) (with \( \mathcal{Z}^* = \{\zeta\} \)); in both cases \( \mathcal{Z}^* \cap \mathcal{Z} = \emptyset \).

Next, let us define an operator

\[
\tilde{P}(L) := \prod_{j \in J} (L - z_j)
\]

with the convention that \( \tilde{P}(L) := I \) (identity) for \( m = 0 \). Since \( z_j \in \mathcal{Z} \) can have a nontrivial imaginary part, the formal adjoint \( \tilde{P}(L)^+ \) of \( \tilde{P}(L) \) is defined by (2.3), but with \( z_j \) replaced by its complex conjugate \( \overline{z_j} \). In particular, if \( \mathcal{Z} = \mathcal{Z}_o \) (i.e. if \( \mathcal{Z}_o = \mathcal{Z}_o \cap \mathbb{R} \)) then \( \tilde{P}(L)^+ = \tilde{P}(L) \).
The inverse of (2.3) is a bounded operator in $\mathcal{F}_n$ given by (see also [26, Eq. (6.4)])

\begin{equation}
\tilde{P}(L)^{-1} = \sum_{j \in J} \frac{1}{b_j(z_j)} (L - z_j)^{-1}, \quad b_j(\cdot) := \prod_{j' \in J \setminus \{j\}} (\cdot - z_{j'}). \tag{2.4}
\end{equation}

Note that $b_1(z) := 1$ for $m = 1$ for $z \in \mathbb{C}$.

Similar to Proposition 2.1 one has Proposition 2.6 (for $m > 0$).

**Proposition 2.6.** $\tilde{P}(L_n)^{1/2} \in [\mathcal{F}_{n+m}, \mathcal{F}_n]$ is an isomorphism whose adjoint in $\mathcal{F}_n$ is an isomorphism $[\tilde{P}(L_n)^+]^{1/2} \in [\mathcal{F}_{n+m}, \mathcal{F}_n]$. \hfill $\square$

It follows that $\tilde{P}(L)^{1/2}$ is an isometry if one redefines the scalar product in (2.1) by putting there $\tilde{P}(L)$ instead of $P(L)$. The above described construction of a set $Z$ allows us to consider a not necessarily semibounded self-adjoint operator $L$. Let us recall that, in the context of triplet extensions, the case $L \geq 0$ and $Z = \mathbb{R} \cap \mathbb{R}_{<0}$ is studied in [14, 26], and the case $L \geq 0$ and $\tilde{P}(L) = P(L)$ can be found in [15].

Here, we find it more convenient to keep the scalar product in (2.1) fixed (for a given $P(L)$). We characterize equivalent scalar products by using an operator (à la “scaling parameter”)

\begin{equation}
p(L) := P(L)\tilde{P}(L)^{-1}. \tag{2.5}
\end{equation}

One has $p(L_n)^{1/2}\mathcal{F}_n = \mathcal{F}_n$ because $P(L_n)^{1/2}\mathcal{F}_{n+m} = \mathcal{F}_n$ and $\tilde{P}(L_n)^{-1/2}\mathcal{F}_n = \mathcal{F}_{n+m}$ by Propositions 2.1 and 2.6, respectively. Likewise, $p(L_n)^{-1/2}\mathcal{F}_n = \mathcal{F}_n$. Using in addition that $L_d \supseteq L_n$ for integers $d \leq n$, one deduces the following result.

**Corollary 2.7.** $(\forall l \in \mathbb{Z}_{\leq n}) p(L_l) \in [\mathcal{F}_n]$ is an isomorphism whose adjoint in $\mathcal{F}_n$ is an isomorphism $p(L_l)^+ := P(L_l)[\tilde{P}(L_l)^+]^{-1} \in [\mathcal{F}_n]$. \hfill $\square$

In particular one has $p(L)\mathcal{F}_m = \mathcal{F}_m$. Therefore, if $\langle \cdot, \cdot \rangle_m$ denotes the scalar product in (2.1) with $P(L)$ replaced by $\tilde{P}(L)$, then $\langle f, g \rangle_m = \langle f, |p(L)|g \rangle_m$ for $f, g \in \mathcal{F}_m$. Here and elsewhere, the absolute value $|p(L)| = \sqrt{p(L)^+p(L)}$ by Corollary 2.7.

### 3. Boundary Triples, $\gamma$-Field, Weyl Function

#### 3.1. Singular functional and its regularization.

Consider a family

\[ \{\varphi_\sigma \in \mathcal{F}_{d-2} \setminus \mathcal{F}_{d-1} \mid \sigma \in \mathcal{S} \} \]

of linearly independent functionals, where $m \in \mathbb{N}_0$ and $\mathcal{S}$ is an index set of dimension $d < \infty$; when $d = 1$, the index $\sigma$ is omitted. The functional $\varphi_\sigma$ acts on $\mathcal{F}_{m+2}$ via the duality pairing $\langle \cdot, \cdot \rangle : \mathcal{F}_{d-2} \times \mathcal{F}_{d+2} \to \mathbb{C}$ by

\begin{align}
\langle \varphi_\sigma, f \rangle &= \langle (|L_{d-2}| + I)^{-\frac{m}{2}} \varphi_\sigma, (|L| + I)^{\frac{m}{2}} f \rangle_0 \\
(3.1a) &= \langle \hat{\varphi}_\sigma, P(L)^{1/2} f \rangle \\
(3.1b) &= \langle \hat{\varphi}_\sigma, P(L) \tilde{P}(L)^{-1/2} \varphi_\sigma \rangle
\end{align}

for $f \in \mathcal{F}_{m+2}$. The regularized functional $\hat{\varphi}_\sigma \in \mathcal{F}_{d-1} \setminus \mathcal{F}_{d-2}$ is defined by

\begin{equation}
\hat{\varphi}_\sigma := P(L_{d-2})^{-1/2} \varphi_\sigma \tag{3.2}
\end{equation}

and the corresponding duality pairing $\langle \cdot, \cdot \rangle$ maps $\mathcal{F}_{d-2} \times \mathcal{F}_{d-2}$ into $\mathbb{C}$. We also use the vector notation

\[ \langle \varphi, \cdot \rangle := (\langle \varphi, \cdot \rangle) \in [\mathcal{F}_{m+2}, \mathbb{C}^d], \quad \langle \hat{\varphi}, \cdot \rangle := (\langle \hat{\varphi}, \cdot \rangle) \in [\mathcal{F}_{d+2}, \mathbb{C}^d] \]

Finite rank perturbations involving $\mathcal{F}_{d-2}$-independent functionals $\hat{\varphi}_\sigma \in \mathcal{F}_{d-2}$ are extensively studied in [17, Chapters 3 and 4] (see also references therein). In applications, especially in quantum mechanics, the prototypical example of a singular perturbation is the Dirac distribution $\delta$ (together with its derivatives); see e.g. [29, 30, 31, 32, 33]. For example, $\delta$ is of class $\mathcal{F}_{d-2}$.
for a three-dimensional Laplace operator. In the examples below, however, we discuss the cases when \( \delta \) (or its derivatives) is of class \( \mathfrak{S}_{-m-2} \), and then we expose its \( \mathfrak{S}_{-2} \)-regularization.

**Example 3.1 (Laplace operator with point-interaction in higher dimensions).** Let \( \Delta \) denote a \( \nu \)-dimensional Laplace operator \((\nu > 0)\). The positive definite operator \( L = -\Delta \) defined on the Sobolev space \( H^2(\mathbb{R}^\nu) \) \( (= \mathfrak{S}_{\delta}) \), also known as the \((L^2)\)-Bessel potential \([34, \text{Definition 2.39}]\), is self-adjoint in the space \( L^2(\mathbb{R}^\nu) \) \( (= \mathfrak{S}_0) \).

Let \( \delta \) be the Dirac distribution concentrated at \( 0 \in \mathbb{R}^\nu \). Then \( \delta \in \mathfrak{S}_{-n} \) for \( n \in \mathbb{N} \) such that \( 2n > \nu \).

Let us recall that this follows from the observation that the \( \nu \)-dimensional volume element in \( \mathbb{R}^\nu \) is proportional to \( r^{\nu-1} \), where \( r := |x| \) and \( x \in \mathbb{R}^\nu \). Hence the norm is finite, \( \|(L + I)^{-n/2}\|_0 < \infty \), if the integral of \( r^{\nu-1}(r^2 + 1)^{-n} \) exists for large \( r \), i.e. \( \nu - 1 - 2n < -1 \); see also \([14, \text{Section 6}]\).

For instance, assume that \( \nu = 4 \). Then \( n > 2 \) and hence \( \delta \in \mathfrak{S}_{-3} \setminus \mathfrak{S}_{-2} \). In addition, one can put \( \varphi := N\delta \), with \( N := 4\sqrt{2\pi} \), so that \( \varphi \) is normalized to unity, \( \|\varphi\|_3 = 1 \). The regularized functional \((3.2)\) is defined by

\[
\langle \hat{\varphi}, \cdot \rangle = \frac{N}{(2\pi)^2} \int_{\mathbb{R}^4} (\xi)^{-1} F[\cdot](\xi) d\xi \quad \text{on } H^2(\mathbb{R}^4)
\]

i.e. \( \hat{\varphi} \) acts as the integral operator. Here and in the next example \( F = F_{x \to \xi} \) denotes the Fourier transform and the symbol \( \langle \xi \rangle := \sqrt{1 + \xi^2} \). Let us recall that \( \langle D_x \rangle^s \) \((s \in \mathbb{R})\), defined by \( \langle D_x \rangle^s f = F^{-1}[\langle \xi \rangle^s F[f]] \) for \( f \in S'(\mathbb{R}^\nu) \) (tempered distributions), is an isomorphism \( H^s(\mathbb{R}^\nu) \rightarrow L^2(\mathbb{R}^\nu) \). Thus \( \langle D_x \rangle^s \geq P(L)^{1/2} \) for \( s = m = 1 \).

**Example 3.2 (Laplace operator with point-interaction in three dimensions).** Consider the three-dimensional Laplace operator as in Example 3.1 with \( \nu = 3 \), and put \( \varphi_j := N_j \partial_j \delta \) for some \( N_j > 0 \), for \( j \in \{1, 2, 3\} \). Here \( \partial_j \delta \) is the distributional derivative of the Dirac \( \delta \) with respect to the \( j \)th component. The functional \( \varphi_j \) is of class \( \mathfrak{S}_{-3} \setminus \mathfrak{S}_{-2} \), and the present model, with \( N_1 = N_2 = N_3 \), is studied in \([26, \text{Sec. 10}]\). The regularized functional \( \hat{\varphi}_j := (L + I)^{-1/2} \varphi_j \) is given by

\[
\langle \hat{\varphi}_j, \cdot \rangle = \frac{iN_j}{(2\pi)^3} \int_{\mathbb{R}^3} (\xi)^{-1} \xi_j F[\cdot](\xi) d\xi \quad \text{on } H^2(\mathbb{R}^3)
\]

where \( \xi_j \) is the \( j \)th component of \( \xi \in \mathbb{R}^3 \).

**Example 3.3 (Spin-orbit coupled cold molecules).** Let \( \Delta_X \) (resp. \( \Delta_x \)) be a three-dimensional Laplace operator in \( X \in \mathbb{R}^3 \) (resp. \( x \in \mathbb{R}^3 \)), and put \( L := -2\Delta_x - 2^{-1}\Delta_X \). The operator \( L \) defined on \( H^2(\mathbb{R}^6) \) \( (= \mathfrak{S}_2) \) is self-adjoint in \( L^2(\mathbb{R}^6) \) \( (= \mathfrak{S}_0) \), and it represents a familiar free two-particle Hamiltonian in the center-of-mass coordinate system \( Q = (x, X) \), where \( x \) is the relative coordinate (the distance between the two particles) and \( X \) is the center-of-mass coordinate.

For the particles interacting through the zero-range potential \((i.e. \text{Dirac distribution } \delta)\), one associates the perturbation \( \delta \) to the Laplacian \(-2\Delta_x \), and in this case \( \delta \in \mathfrak{S}_{-2}(\Delta) \setminus \mathfrak{S}_{-1}(\Delta) \); see \( e.g. [17, \text{Theorem 5.2.1}] \).

On the other hand, as it is shown in \([35]\), the Hamiltonian of the form \( L(\alpha) = L + O(\alpha) \) describes the system of two Rashba spin-orbit coupled cold atoms with point-interaction, where \( \alpha \geq 0 \) denotes the spin-orbit-coupling strength and the remaining term \( O(\alpha) \) is “small” in a certain sense.

When \( \alpha > 0 \), the system no longer reduces to the single-particle case, which means one needs to associate the singular perturbation \( \delta \) to the total two-particle operator \( L(\alpha) \); in this case \( \delta \in \mathfrak{S}_{-4}(L) \setminus \mathfrak{S}_{-3}(L) \).
Assume now that $\alpha$ is negligibly small. Then the singular perturbation of rank $d = 4$ approximates to the perturbation described by $\varphi := N\delta_{Q_0}$, where $N := 16\sqrt{2}\pi$ and $\delta_{Q_0}$ is the Dirac distribution concentrated at $Q_0 = (0, X)$ (see [35, Section 5] for the details). Then the functional $\varphi$ is normalized to unity by means of $\| (L^2 + 1)^{-1}\varphi \|_0 = 1$. The regularized functional $\hat{\varphi}$ acts by the convolution
\[
\langle \hat{\varphi}, u \rangle = (k_{\varphi} * u)(Q_0), \quad k_{\varphi}(\cdot) := \frac{N}{(2\pi)^3} \frac{K_2(|\cdot|)}{|\cdot|^2}
\]
for $u \in H^2(\mathbb{R}^6)$. Here $K_2$ denotes the modified Bessel function of the second kind (or else the Macdonald function). Note that the kernel satisfies $|k_{\varphi}(Q)| < C|Q|^{-4}$ for some $C > 0$, and hence is locally summable on $\mathbb{R}^6$.

3.2. Boundary space. For notational convenience let us recall some basic definitions that we use throughout. In the following definitions one is free to replace $\text{dom} \, T$ by $T$ whenever $T$ is identified with its graph (or is a linear relation). More details are found in [2, 3, 5, 7, 8].

**Definition 3.4.** Let $T$ be an operator in a Hilbert space $\mathfrak{h}$. The boundary form of $T$ is the sesquilinear form $\langle \cdot, \cdot \rangle_T$ on $\text{dom} \, T$ defined by
\[
[f, g]_T := \langle f, Tg \rangle_{\mathfrak{h}} - \langle Tf, g \rangle_{\mathfrak{h}}
\]
for $f, g \in \text{dom} \, T$.

**Definition 3.5.** Let $T$ be an operator in a Hilbert space $\mathfrak{h}$. One says that the boundary form of $T$ satisfies an (abstract) Green identity if
\[
[f, g]_T = \langle \Gamma_0 f, \Gamma_1 g \rangle_{\mathfrak{t}} - \langle \Gamma_1 f, \Gamma_0 g \rangle_{\mathfrak{t}} =: [\Gamma f, \Gamma g]_{\mathfrak{t} \times \mathfrak{t}}
\]
for $f, g \in \text{dom} \, T$; here $\Gamma := (\Gamma_0, \Gamma_1)$ is a mapping from $\text{dom} \, T$ to $\mathfrak{t} \times \mathfrak{t}$; $\mathfrak{t}$ is a Hilbert space.

**Definition 3.6.** Let $T$ be a closed symmetric operator in a Hilbert space $\mathfrak{h}$ with equal deficiency indices and let $T^*$ be the adjoint whose boundary form satisfies an abstract Green identity:
\[
[f, g]_{T^*} = [\Gamma f, \Gamma g]_{\mathfrak{t} \times \mathfrak{t}}
\]
for $f, g \in \text{dom} \,(T^*)$ and some Hilbert space $\mathfrak{t}$. Then the triple $(\mathfrak{t}, \Gamma_0, \Gamma_1)$, where $\Gamma := (\Gamma_0, \Gamma_1)$ is a single-valued surjective mapping $\text{dom} \,(T^*) \to \mathfrak{t} \times \mathfrak{t}$, is said to be an (ordinary) boundary triple for $T^*$.

Fix $m \in \mathbb{N}_0$ and define the restriction $L_{\min}$ of $L_m$ by
\[
L_{\min} := L_m \mid_{\text{dom} \, L_{\min}}, \quad \text{dom} \, L_{\min} = \{ f \in \mathfrak{H}_{m+2} \mid \langle \varphi, f \rangle = 0 \}.
\]
By noting that (see also Corollary 4.4)
\[
P(L)^{1/2} L_{\min} P(L)^{-1/2} \supseteq \hat{L}_0 := L \mid_{\{ u \in \mathfrak{H}_2 \mid \langle \hat{\varphi}, u \rangle = 0 \}}
\]
and $\hat{L}_0$ is a densely defined, closed, symmetric operator in $\mathfrak{H}_0$, and has defect numbers $(d, d)$ (see e.g. [17] and references therein), one concludes the following.

**Theorem 3.7.** The operator $L_{\min}$ is densely defined, closed, and symmetric in $\mathfrak{H}_m$, and has defect numbers $(d, d)$. The deficiency subspaces are given by
\[
\mathfrak{N}_z(L_{\min}^*) = \text{lin} \{ G_\sigma(z) \mid \sigma \in S \}
\]
for $z \in \text{res} \, L$, where the deficiency elements are given by
\[
G_\sigma(z) := P(L_{-m})^{-1} g_\sigma(z) \in \mathfrak{H}_m \setminus \mathfrak{H}_{m+1} \quad \text{with}
\]
\[
g_\sigma(z) := (L_{-m-2} - z)^{-1} \varphi_\sigma \in \mathfrak{H}_{-m} \setminus \mathfrak{H}_{-m+1}.
\]
The adjoint in \( \mathcal{H}_m \) is the operator \( L_{\min}^* \supseteq L_m \) defined on the domain
\[
\text{dom} \ L_{\min}^* = \mathcal{H}_{m+2} \oplus \mathcal{N}_z(L_{\min}^*). 
\]

**Corollary 3.8.** Define the surjective mapping \( \Gamma := (\Gamma_0, \Gamma_1) : \text{dom} \ L_{\min}^* \to \mathbb{C}^d \times \mathbb{C}^d \) by
\[
\begin{align*}
\Gamma_0 f & := c = (c_\sigma) \in \mathbb{C}^d, \\
\Gamma_1 f & := \langle \varphi^{ex}, f \rangle = (\langle \varphi^{ex}_\sigma, f \rangle) \in \mathbb{C}^d
\end{align*}
\]
for \( f = f^\# + G_z(c) \in \text{dom} \ L_{\min}^* \), where \( f^\# \in \mathcal{H}_{m+2} \) and
\[
G_z(c) := \sum_{\sigma \in \mathcal{S}} c_\sigma G_\sigma(z) \in \mathcal{N}_z(L_{\min}^*)
\]
and \( z \in \text{res} \ L \). The extension \( \varphi^{ex}_\sigma \) to \( \text{dom} \ L_{\min}^* \supseteq \mathcal{H}_{m+2} \) is defined by
\[
\langle \varphi^{ex}_\sigma, f \rangle := \langle \varphi, f^\# \rangle + R(z)c \quad \text{with}
\]
\[
R(z) = (R_{\sigma\sigma'}(z)) \in [\mathbb{C}^d], \quad R_{\sigma\sigma'}(z) := \langle \varphi^{ex}_\sigma, G_{\sigma'}(z) \rangle
\]
\forall (\sigma, \sigma') \in \mathcal{S} \times \mathcal{S}. Then \( (\mathbb{C}^d, \Gamma_0, \Gamma_1) \) is a boundary triple for \( L_{\min} \). \( \Box 
\]
Notice that the adjoint \( R(z)^* = R(\overline{z}) \), so the matrix \( R(z) \) is Hermitian if \( z \in \text{res} \ L \cap \mathbb{R} \). The matrix \( R(z) \) (or \( R(\overline{z}) \)) is known as the \textit{admissible matrix} for the functionals of class \( \mathcal{H}_{-2} \setminus \mathcal{H}_{-1} \); see \textit{e.g.} [12] and [17, Section 3.1]. Some properties of the matrix-valued Nevanlinna function \( R(z_j) \), for \( z_j \in \mathbb{Z} \) and \( j \in J \), are discussed in Section 6.1.

### 3.3. Proper extensions

Let \( \Gamma \) be as in Corollary 3.8. The mapping
\[
\Theta \to L_{\Theta} := \{ \hat{f} \in \mathcal{L}_{\min}^* \mid \Gamma \hat{f} \in \Theta \}
\]
where \( \Gamma \) is regarded as a mapping from (the graph of) \( \mathcal{L}_{\min}^* \) onto \( \mathbb{C}^d \times \mathbb{C}^d \), establishes a one-to-one correspondence between a linear relation \( \Theta \) in \( \mathbb{C}^d \) and a proper extension \( L_{\Theta} \) of \( \mathcal{L}_{\min} \) in \( \mathcal{H}_m \), \textit{i.e.} such that
\[
\mathcal{L}_{\min} \subseteq L_{\Theta} \subseteq \mathcal{L}_{\min}^*.
\]
Moreover, the adjoint
\[
L_{\Theta}^* \equiv (L_{\Theta})^* = L_{\Theta^*}
\]
and so \( L_{\Theta} \) is self-adjoint in \( \mathcal{H}_m \) if \( \Theta \) is self-adjoint in \( \mathbb{C}^d \) (recall \textit{e.g.} [2] and [3, Proposition 7.12]).

In view of Corollary 2.3 and \( L_m = \ker \Gamma_0 \) (that is, \( L_m = \mathcal{L}_{\min}^* \bigr|_{\ker \Gamma_0} \) in a standard operator notation), let us recall that the \textit{\gamma-field} \( \gamma \) res \( L \ni z \mapsto \gamma(z) \in [\mathbb{C}^d, \mathcal{H}_m] \) and the \textit{(abstract) Weyl function} res \( L \ni z \mapsto M(z) \in [\mathbb{C}^d] \) associated with the boundary triple \((\mathbb{C}^d, \Gamma_0, \Gamma_1)\) are analytic operator functions defined by [3, Definition 7.13], [28, Section 14.5] \( \gamma(z) := (\Gamma_0 |_{\mathcal{N}_z(L_{\min})})^{-1} \) and \( M(z) := \Gamma_1 \gamma(z) = (\langle \varphi^{ex}, \gamma(z) \rangle) \). Note that ran \( \gamma(z) = \mathcal{N}_z(L_{\min}^*) \subseteq \text{dom} \ L_{\min}^* \).

**Proposition 3.9.** Let \( \Gamma \) be as in Corollary 3.8. The \textit{\gamma-field} \( \gamma \) and the Weyl function \( M \) associated with the boundary triple \((\mathbb{C}^d, \Gamma_0, \Gamma_1)\) for \( L_{\min}^* \) are given by
\[
\begin{align*}
\gamma(z) & = G_z(\cdot), \quad M(z) = R(z)
\end{align*}
\]
for \( z \in \text{res} \ L \). For a (closed) linear relation \( \Theta \) in \( \mathbb{C}^d \), the Krein–Naimark resolvent formula holds:
\[
(L_{\Theta} - z)^{-1} = (L_m - z)^{-1} + \gamma(z)(\Theta - M(z))^{-1} \gamma(\overline{z})^*
\]
for \( z \in \text{res} \ L \cap \text{res} \ L_{\Theta} \). \( \Box \)
4. Triplet adjoint in scale spaces

We define the triplet adjoint according to [14, Section 2.1], [26, Section 3].

**Definition 4.1.** Let \( \mathfrak{h}, \mathfrak{h}', \) and \( \mathfrak{f} \) be Hilbert spaces such that \( \mathfrak{h} \subseteq \mathfrak{f} \subseteq \mathfrak{h}' \), with both inclusions being dense. Here \( \mathfrak{h}' \) is the dual of \( \mathfrak{h} \), i.e. an element of \( \mathfrak{h}' \) is a continuous linear functional on \( \mathfrak{h} \) whose action is defined via the duality pairing \( \langle \cdot, \cdot \rangle : \mathfrak{h}' \times \mathfrak{h} \to \mathbb{C} \). The duality pairing is defined by extending the scalar product \( \langle \cdot, \cdot \rangle \) in \( \mathfrak{f} \) so that \( \langle g, f \rangle \) is well-defined \( \forall g \in \mathfrak{h}' \forall f \in \mathfrak{h} \). Let \( T \) be a densely defined operator in \( \mathfrak{h} \). Then there exists the unique operator \( T^\dagger \) in \( \mathfrak{h}' \), called the **triplet adjoint** of \( T \), defined by

\[
\text{dom } T^\dagger = \{ g \in \mathfrak{h}' | (\forall g \in \text{dom } T)(\exists g' \in \mathfrak{h}') (g, T f) = \langle g', f \rangle \}.
\]

When such a \( g' \in \mathfrak{h}' \) exists, it is unique and denoted by \( T^\dagger g \).

Clearly, when \( \mathfrak{h}' \) coincides with \( \mathfrak{h} \) in the above definition, \( T^\dagger = T^* \) is just the Hilbert space adjoint of a densely defined operator \( T \) in \( \mathfrak{h} \).

The duality pairing \( \langle \cdot, \cdot \rangle : \mathfrak{H}_m \times \mathfrak{H}_m \to \mathbb{C} \) defined by extending the scalar product \( \langle \cdot, \cdot \rangle \) in \( \mathfrak{H} \) so that \( \langle g, f \rangle \) is well-defined \( \forall g \in \mathfrak{H}' \forall f \in \mathfrak{H} \). Let \( T \) be a densely defined operator in \( \mathfrak{H} \). Then there exists the unique operator \( T^\dagger \) in \( \mathfrak{H}' \), called the **triplet adjoint** of \( T \), defined by

\[
\text{dom } T^\dagger = \{ g \in \mathfrak{H}' | (\forall g \in \text{dom } T)(\exists g' \in \mathfrak{H}') (g, T f) = \langle g', f \rangle \}.
\]

When such a \( g' \in \mathfrak{H}' \) exists, it is unique and denoted by \( T^\dagger g \).

We prove the equality in (4.3). For elements \( f \in \text{dom } L_{\min} \subseteq \mathfrak{H}_m \) and \( g \in \text{dom } L_{\min} \subseteq \mathfrak{H}_m \) it holds by Definition 4.1

\[
\langle L_{\min}^\dagger g, f \rangle = \langle g, L_{\min} f \rangle = \langle g, L_m f \rangle = \langle g, L_{m-2} f \rangle = \langle L_{m-2} g, f \rangle
\]

where in the last step we implicitly use (4.1). Since \( \text{dom } L_{\min} \) is dense in \( \mathfrak{H}_m \), it follows that

\[
L_{m-2} g = L_{\min}^\dagger g + \sum_{\sigma \in S} c_\sigma \varphi_\sigma \in \mathfrak{H}_{m-2}
\]

for \( c = (c_\sigma) \in \mathbb{C}^d \). Using that \( L_{\min}^\dagger g \in \mathfrak{H}_{m-2} \) and \( g_\sigma (z) \in \mathfrak{H}_m \) for \( z \in \text{res } L \) (recall (3.6)), we have that

\[
L_{\min}^\dagger g = L_{m-2} g - \sum_{\sigma} c_\sigma \varphi_\sigma = L_{m-2} (g - \sum_{\sigma} c_\sigma g_\sigma (z)) + z \sum_{\sigma} c_\sigma g_\sigma (z)
\]

and

\[
g - \sum_{\sigma} c_\sigma g_\sigma (z) = g^* \in \mathfrak{H}_{m+2}
\]
Using that $L_{-m-2}g^2 = L_{-m}g^2 \in \mathcal{H}_{-m}$ and by Corollary 2.2
\[ g^2 \in \mathcal{H}_{-m+2} \iff g^\# := P(L_{-m})^{-1}g^2 \in \mathcal{H}_{m+2} \]
we deduce the equality in (4.2) by applying Theorem 3.7.

**Step 2.** Next we show that $L_{\min}^* \subseteq L_{\max}$. Let $f \in \text{dom } L_{\min}^*$ be as in Corollary 3.8 and let $g \in \text{dom } L_{\min}$, i.e. $g \in \mathcal{H}_{m+2}$ and $\langle \varphi, g \rangle = 0$. Then
\[
\langle L_{\max}f, g \rangle_m = \langle f, L_{\min}g \rangle_m = \langle f, L_mg \rangle_m = \langle L_{-m}^{-1}f, g \rangle_m
\]
\[
= \langle L_m f^\# + \sum_{\sigma} \sigma L_{-m-2}G_{\sigma}(z), g \rangle_m
\]
\[
= \langle L_m f^\# + \sum_{\sigma} c_{\sigma} P(L_{-m-2})^{-1}(zg_{\sigma}(z) + \varphi_{\sigma}), g \rangle_m
\]
\[
= \langle L_m f^\# + zG_{\sigma}(c), g \rangle_m + \langle c, \langle \varphi, g \rangle \rangle_{\mathbb{C}^d} = \langle L_{\min}^* f, g \rangle_m.
\]
Since $\text{dom } L_{\min} \subseteq \mathcal{H}_m$ densely, it therefore holds $L_{\max} f = L_{\min}^* f$.

Using that $L_{\max} \supseteq L_{\min}^*$ and $L_{\max}$ is densely defined in $\mathcal{H}_m$, it follows that $L_{\max}$ is densely defined in $\mathcal{H}_m$. Taking the adjoints in $\mathcal{H}_m$ the reverse inclusions $L_{\max}^* \subseteq L_{\min} \subseteq L_{\min}^*$ show that $L_{\max}$ is non-symmetric, while $L_{\max}^* \subseteq L_{\max}$ is symmetric provided that $L_{\max}$ is closed.

**Step 3.** We compute $L_{\max}^*$. The adjoint operator is defined by
\[
\text{dom } L_{\max}^* = \{ f \in \mathcal{H}_m \mid (\forall g \in \text{dom } L_{\max})(\exists f' \in \mathcal{H}_m)\langle L_{\max}g, f \rangle_{-m} = \langle g, f' \rangle_{-m}; f' := L_{\max}^* f \}.
\]
Thus, for $g \in \text{dom } L_{\max}$ as in (4.5) we have by (4.4)
\[
0 = \langle L_{\max}g, f \rangle_{-m} - \langle g, f' \rangle_{-m}
\]
\[
= \langle g^2, L_{-m-2}f - f' \rangle_{-m} + \sum_{\sigma} c_{\sigma} \langle g_{\sigma}(z), \overline{\varphi} f - f' \rangle_{-m}.
\]
The relation must be valid for all $g^2 \in \mathcal{H}_{m+2}$ and all $c \in \mathbb{C}^d$; hence $f \in \mathcal{H}_{-m+2}$ and $f' = L_{-m} f \in \mathcal{H}_{-m}$ and $(\forall \sigma \in \mathcal{S})$
\[
0 = \langle g_{\sigma}(z), \overline{\varphi} f - f' \rangle_{-m} = -\langle g_{\sigma}(z), (L_{-m} - \overline{\varphi}) f \rangle_{-m} = -\langle \varphi_{\sigma}, P(L_{-m+2})^{-1} f \rangle
\]
i.e. $P(L_{-m+2})^{-1} f \in \text{dom } L_{\min}$. Since $P(L_{-m+2})^{-1}$ is bijective by Corollary 2.2, one deduces the equality in (4.3).

By applying the above procedure one shows that $L_{\max} = L_{\max}^*$, and this completes the proof. \hfill \Box

**Corollary 4.3.** The operator $L_{\max} \supseteq L_{-m}$ extends $L_{-m}$ to the domain
\[
\text{dom } L_{\max} = \mathcal{H}_{-m+2} + \mathcal{H}_z(L_{\max})
\]
where the eigenspace is given by
\[
\mathcal{H}_z(L_{\max}) = \text{lin}\{g_{\sigma}(z) \mid \sigma \in \mathcal{S}\}
\]
and $z \in \text{res } L$.

**Proof.** This follows from Corollary 2.3, Theorem 3.7, and Proposition 4.2. \hfill \Box

For $d = 1$, the description of $L_{\max}$ given in Corollary 4.3 is found in [14, Theorem 2.1], [15, Lemma 3.1], [26, Eqs. (4.5)–(4.7)].

Below we list some useful relations in terms of the operator $\hat{L}_0$, defined in (3.4), and its adjoint $\hat{L}_0^*$ in $\mathcal{H}_0$. We examine the connection between the triplet adjoint and the operator $\hat{L}_0^*$ in more detail in Section 7.

**Corollary 4.4.** The following identities hold:
(i) $L_{\text{min}} = P(L)^{-1/2} \hat{L}_0 P(L_2)^{1/2}$.
(ii) $L_{\text{min}}^* = P(L)^{-1/2} \hat{L}_0^* P(L)^{1/2}$.
(iii) $L_{\text{max}} = P(L_m)^{-1/2} \hat{L}_0^* P(L_m)^{-1/2}$.
(iv) $P_{\text{max}} = P(L_m)^{-1/2} \hat{L}_0 P(L_{m+2})^{-1/2}$.

Proof. In view of Corollary 2.3, items (i) and (ii) follow from Theorem 3.7, while (iii) and (iv) follow from (i), (ii), and Proposition 4.2.

5. Intermediate Hilbert space

Fix $m \in \mathbb{N}$ and define the vector space

\begin{align}
(5.1a) \quad \mathcal{H} := & \mathcal{H}_m \cap \mathcal{R} \\
(5.1b) \quad \mathcal{R} := & \text{lin}\{g_\alpha := g_\sigma(z_j) | \alpha = (\sigma, j) \in S \times J\}.
\end{align}

The sum in (5.1a) is direct because of Corollary 5.2 below.

Lemma 5.1. $\mathcal{R}_{\text{min}} \subseteq \mathcal{R} \subseteq \mathcal{H}_m$, where

\begin{align}
\mathcal{R}_{\text{min}} := & (\mathcal{R} \cap \mathcal{H}_{m-2}) \setminus \mathcal{H}_{m-1} \\
& = \text{lin}\{\hat{P}(L_{m-2})^{-1}\varphi_\sigma | \sigma \in S\}.
\end{align}

Proof. An element $k \in \mathcal{R}$ is a linear combination of elements $g_{\sigma j} \in \mathcal{H}_{m-1} \setminus \mathcal{H}_{m-1+1}$, with some coefficients of expansion $d_{\sigma j}(k) \in \mathbb{C}$ for $(\sigma, j) \in S \times J$. Thus clearly $\mathcal{R} \subseteq \mathcal{H}_m$. Now, using (2.4) and that $\hat{P}(L_{m-2})^{-1}\varphi_\sigma \in \mathcal{H}_{m-2} \subseteq \mathcal{H}_{m-2-2m}$ for $n \in \mathbb{N}_0$, it follows that the minimal subset of $\mathcal{R}$, other than $\{0\}$, consists of the elements $k$ whose coefficients $d_{\sigma j}(k) = \chi_\sigma/b_j(z_j)$ for some $\chi_\sigma \in \mathbb{C}$.

Corollary 5.2. $\mathcal{R} \cap \mathcal{H}_{m-1} = \{0\}$.

The vector space $\mathcal{H}$ is made into the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ by completing $\mathcal{H}$ with respect to the norm $\|\cdot\|_{\mathcal{H}} := \sqrt{\langle \cdot, \cdot \rangle_{\mathcal{H}}}$, where the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ in $\mathcal{H}$ is defined by

\begin{align}
(5.2) \quad \langle f + k, f' + k' \rangle_{\mathcal{H}} := & \langle f, f' \rangle_m + \langle k, k' \rangle_{-m}
\end{align}

for $f, f' \in \mathcal{H}_m$ and $k, k' \in \mathcal{R}$. The obtained Hilbert space $\mathcal{H}$ is referred to as an intermediate Hilbert space, in the sense that $\mathcal{H}_m \subseteq \mathcal{H} \subseteq \mathcal{H}_m$.

The original definition of an intermediate Hilbert space $\mathcal{H}$ in the case $d = 1$ and $L \geq 0$ and $\mathcal{Z} = \mathcal{Z} \cap \mathbb{R}_{>0}$ is given in [26, Lemma 4.1]. Since $\mathcal{R}$ is spanned by the elements $\{g_\alpha\}$ of the same order of singularity, i.e., $g_\alpha \in \mathcal{H}_m \setminus \mathcal{H}_{m+1}$ $\forall \alpha$, the present approach for describing triplet extensions in the space $\mathcal{H}$ is called the peak model. This is in contrast to the cascade model developed in [14, Section 2], [15], [16, Section 4.3], where $\mathcal{R}$ (for $d = 1$) is spanned by the elements from $\mathcal{H}_{m-2+2j} \setminus \mathcal{H}_{m-1+2j}$ $\forall j \in J$.

An element $k \in \mathcal{R}$ is in bijective correspondence with an element

\begin{align}
(5.3) \quad d(k) = (d_\alpha(k)) \in \mathbb{C}^{md}, \quad d_\alpha(k) := [G^{-1} g_{\alpha} k]_{-m} \quad (\forall \alpha \in S \times J)
\end{align}

where one uses the vector notation $(g_{\alpha})_{-m} := (g_{\alpha, \tau})_{-m} : \mathcal{H}_m \rightarrow \mathbb{C}^{md}$. The Hermitian matrix $G$ is the Gram matrix of vectors generating $\mathcal{R}$.

\begin{align}
(5.4) \quad G = (G_{\alpha \alpha'}) \in [\mathbb{C}^{md}]^\prime; \quad G_{\alpha \alpha'} := (g_{\alpha}, g_{\alpha'})_{-m}
\end{align}

$\forall \alpha, \alpha' \in S \times J$. The correspondence $\mathcal{R} \ni k \leftrightarrow d(k) \in \mathbb{C}^{md}$ follows from the fact that the system $\{g_\alpha\}$ is linearly independent, i.e., the matrix $G$ is positive definite. Indeed, $\langle \forall \xi = (\xi_\alpha) \in \mathbb{C}^{md} \rangle (\exists k \in \mathcal{R}) k = \sum_\alpha \xi_\alpha g_\alpha$. And conversely, projecting the latter $k$ on $g_{\alpha'}$ yields $\langle g_{\alpha'}, k \rangle_{-m} = [G\xi]_{\alpha'} \forall \alpha' \in S \times J$, i.e., $\langle \forall k \in \mathcal{R} \rangle (\exists \xi \in \mathbb{C}^{md}) \xi = d(k)$, since $G$ is invertible.
It follows that an intermediate space $\mathcal{H}$ is isometrically isomorphic (equivalent) to an external orthogonal sum $\mathcal{H}_m \oplus \mathbb{C}^{md}$ via the bijective correspondence

\[(5.5) \quad \mathcal{H} \ni f+k \leftrightarrow (f,d(k)) \in \mathcal{H}_m \oplus \mathbb{C}^{md} \]

provided that the scalar product in $\mathcal{H}' := \mathcal{H}_m \oplus \mathbb{C}^{md}$ is given by

\[(5.6) \quad \langle (f,\xi), (f',\xi') \rangle_{\mathcal{H}'} = \langle f, f' \rangle_m + \langle \xi, \mathcal{G} \xi' \rangle_{\mathbb{C}^{md}} \]

for $(f,\xi), (f',\xi') \in \mathcal{H}_m \oplus \mathbb{C}^{md}$. To see this, it suffices to notice that the scalar product (5.6) coincides with (5.2) for $\xi = d(k)$ and $\xi' = d(k')$ given by (5.3), where $k, k' \in \mathcal{R}$.

Remark 5.3. Let us point out that an intermediate Hilbert space $\mathcal{H}$ is continuously embedded into the Hilbert space $\mathcal{H}_m$. However, we do not use this property here, and we omit the proof.

Let $\mathcal{R}_{\min} \subseteq \mathcal{R}$ be as in Lemma 5.1. An element $k \in \mathcal{R}_{\min}$ can be represented by $k = k_{\min}(c)$, where

\[(5.7a) \quad k_{\min}(c) := \sum_{\sigma \in \mathcal{S}} c_{\sigma} \hat{P}(L_{m-2})^{-1} \varphi_{\sigma}, \]

\[(5.7b) \quad = \sum_{\alpha \in \mathcal{S} \times J} [\mathcal{G}^{-1} \mathcal{G}_b c]_{\alpha} g_{\alpha} \]

for $c = (c_{\sigma}) \in \mathbb{C}^d$. The matrix $\mathcal{G}_b$ is defined by

\[(5.8) \quad \mathcal{G}_b = ([\mathcal{G}_b]_{\alpha \alpha'}) \in [\mathbb{C}^d, \mathbb{C}^{md}], \quad [\mathcal{G}_b]_{\alpha \alpha'} := \sum_{j \in J} \mathcal{G}_{\alpha,\alpha'}^{(j)} b_j(z_j)^{-1} \]

$\forall \alpha \in \mathcal{S} \times J \forall \alpha' \in \mathcal{S}$; and hence $[\mathcal{G}^{-1} \mathcal{G}_b c]_{\sigma j} = c_{\sigma} / b_j(z_j) \forall (\sigma, j) \in \mathcal{S} \times J$. This shows that $(\forall k \in \mathcal{R}_{\min}) (\exists c = (c_{\sigma}) \in \mathbb{C}^d) c_{\sigma} / b_j(z_j) = d_{\sigma j}(k)$. Conversely $(\forall c \in \mathbb{C}^d) (\exists k \in \mathcal{R}_{\min}) k = k_{\min}(c)$.

Remark 5.4. (a) For example, for $d = 1$, $\mathcal{G}_b$ is identified with the column-vector $\mathcal{G}b \in \mathbb{C}^m$, where the Gram matrix $\mathcal{G} = (\mathcal{G}_{jj'}) \in [\mathbb{C}^m]$ and where the column-vector $b := (b_j(z_j))^{-1} \in \mathbb{C}^m$.

(b) Let us remark that $(\forall k \in \mathcal{R}_{\min}) (\forall \sigma \in \mathcal{S}) \sum_{j \in J} d_{\sigma j}(k) = 0$ for $m > 1$, because of the property $\sum_{j} 1 / b_j(z_j) = 0$, where both sums run over $j \in J$. In Lemma 7.12 we show that the same applies to $k \in \mathcal{R} \cap \mathcal{H}_{-m+2}$ for $m > 0$.

We close the paragraph by writing down the properties of $\mathcal{G}_b$ $(\forall d \in \mathbb{N})$ that we use to prove Theorems 6.7-(ii) and 7.11.

**Proposition 5.5.** $\ker \mathcal{G}_b = \{0\}$.

**Proof.** Let $c = (c_{\sigma}) \in \mathbb{C}^d$, $\xi = (\xi_{\alpha}) \in \mathbb{C}^{md}$, $\xi_{\alpha} := c_{\sigma} / b_j(z_j) \forall \alpha = (\sigma, j) \in \mathcal{S} \times J$. By (5.8) it holds $\mathcal{G}_b c = \mathcal{G} \xi$. Thus $c \in \ker \mathcal{G}_b$ implies that $\xi = 0$, i.e. $\xi_{\alpha} = 0$ $\forall \alpha$, which further implies that $c = 0$. \qed

**Lemma 5.6.** Consider the adjoint matrix $\mathcal{G}_b^* \subseteq \mathbb{C}^{md} \times \mathbb{C}^d$ as a closed linear relation, which is identified with the graph of a bounded operator. Let $H_b$ be a generalized inverse of $\mathcal{G}_b$, i.e. such that $\mathcal{G}_b^* H_b \mathcal{G}_b = \mathcal{G}_b^*$. Then the inverse linear relation $(\mathcal{G}_b^*)^{-1}$ admits a decomposition

\[(5.9a) \quad (\mathcal{G}_b^*)^{-1} = H_b \hat{\phi}(\{0\} \times \ker \mathcal{G}_b^*) \quad \text{with} \]

\[(5.9b) \quad \ker \mathcal{G}_b^* = (I - H_b \mathcal{G}_b^*)^\mathbb{C}^{md} \]

**Proof.** This follows from a well-known result in linear algebra (see e.g. [36, Section 2], [37, Proposition 1.4]): For a given $\chi \in \operatorname{ran} \mathcal{G}_b^* \subseteq \mathbb{C}^d$, the equation $\mathcal{G}_b^* \xi = \chi$ has a general solution $\xi = H_b \chi + \xi'$, where $\xi' \in (I - H_b \mathcal{G}_b^*)^{\mathbb{C}^{md}}$. \qed
Lemma 5.7. Consider the linear relation \( G_b^* \subseteq \mathbb{C}^{md} \times \mathbb{C}^d \) as in Lemma 5.6. Then the inverse linear relation \((G_b^*)^{-1}\) admits a canonical decomposition
\[
(G_b^*)^{-1} = H_b^{\text{op}} \oplus \{0\} \times \ker G_b^*
\]
where the operator part of \((G_b^*)^{-1}\) is a bounded operator
\[
H_b^{\text{op}} := H_b G_b^* H_b \in [\mathbb{C}^d, \mathbb{C}^{md}] .
\]

Proof. By definition, the operator part is the set of \((\chi, \xi) \in (G_b^*)^{-1}\) such that \(\chi \perp \text{mul}(G_b^{-1})\). By Lemma 5.6, \(\chi \in \text{dom } H_b = \text{ran } G_b^*\) and \(\xi = H_b \chi + \xi'\) for some \(\xi' \in \ker G_b^*\). On the other hand, since \(\text{mul}(G_b^{-1})^\perp = (\ker G_b^*)^\perp = H_b G_b^* \mathbb{C}^{md}\) by (5.9b), it follows that \(\xi = H_b G_b^* \xi''\) for some \(\xi'' \in \mathbb{C}^{md}\). But then
\[
H_b \chi + \xi' = H_b G_b^* \xi'' \Rightarrow G_b^* H_b \chi = G_b^* \xi''
\]
by multiplying the first equation from the left by \(G_b^*\), and so \(\xi = H_b G_b^* H_b \chi\). This shows (5.10).

The domain
\[
\text{dom } H_b^{\text{opt}} = \text{dom } H_b = \text{ran } G_b^*
\]
by (5.9a) and (5.10b). But we have that
\[
\text{ran } G_b^* = \text{ran } G_b^* = (\ker G_b)^\perp = \mathbb{C}^d
\]
by Proposition 5.5, and hence \(H_b^{\text{opt}} \in [\mathbb{C}^d, \mathbb{C}^{md}]\) is bounded. \(\square\)

The range \(\text{ran } G_b^* = \text{ran } G_b^*\) is closed because \(\text{dim ran } G_b^* = d < \infty\) and \(\text{ran } G_b^*\) is a vector subspace of \(\mathbb{C}^d\). One can recall a canonical decomposition of a closed linear relation e.g. in [9, Theorem 3.9], [10, Proposition 4.4].

Lemma 5.8. Let \(G_b\) and \(H_b\) be as in Lemma 5.6. The adjoint \(H_b^* \supseteq G_b^{-1}\) satisfies
\[
H_b^* G_b = I_d , \quad G_b H_b^* \supseteq I_{\text{ran } G_b}
\]
where \(I_d\) is the identity matrix in \(\mathbb{C}^d\) and \(I_{\text{ran } G_b} : \text{ran } G_b \ni \xi \mapsto \xi\).

Proof. Using (5.9a) and that the adjoint of a componentwise sum of linear relations is the intersection of the adjoint relations (see e.g. [9, Lemma 2.6]), we have that
\[
G_b^{-1} = (H_b \oplus \{0\} \times \ker G_b^*)^* = H_b^* \cap \{0\} \times \ker G_b^*\]
\[
= H_b^* \cap (\text{ran } G_b \times \mathbb{C}^d) = H_b^* \cap (\text{dom } G_b^{-1} \times \mathbb{C}^d) ;
\]
hence \(H_b^* \supseteq G_b^{-1}\).

By using the obtained inclusion we prove (5.11): \((\forall \chi \in \mathbb{C}^d) \ G_b \chi \in \text{ran } G_b\) and hence, using that \(G_b^{-1} G_b = I_d\) because \(\ker G_b = \{0\}\), one gets that
\[
H_b^* G_b \chi = G_b^{-1} G_b \chi = \chi
\]
which shows \(H_b^* G_b = I_d\).

Likewise, using that \(G_b G_b^{-1} = I_{\text{ran } G_b}\) because \(\text{mul } G_b = \{0\}\), one gets that \((\forall \xi \in \text{ran } G_b)\)
\[
G_b H_b^* \xi = G_b G_b^{-1} \xi = \xi
\]
which shows \(G_b H_b^* \supseteq I_{\text{ran } G_b}\). \(\square\)

Corollary 5.9. The adjoint operator \((H_b^{\text{opt}})^* = H_b^*\).

Proof. Since both \(H_b\) and \(G_b^*\) are bounded, it follows from (5.10b) and (5.11) that the adjoint \((H_b^{\text{opt}})^* = H_b^* G_b H_b^* = H_b^*\). \(\square\)
6. Triplet adjoint restricted to an intermediate space

6.1. Closed restriction $A_0$. Let $m \in \mathbb{N}$ and let $A_0$ be the operator in $\mathcal{H}$ defined by

\begin{equation}
\text{dom } A_0 := \mathcal{H}_{m+2} + \mathbb{R}, \quad A_0(f + k) := L_m f + \sum_{\alpha \in \mathcal{S} \times J} [Z_d d(k)]_\alpha g_\alpha
\end{equation}

for $f \in \mathcal{H}_{m+2}$ and $k \in \mathbb{R}$, where the matrix

\begin{equation}
Z_d := Z \oplus \cdots \oplus Z \quad \text{(d times)}
\end{equation}

is the matrix direct sum of $d$ diagonal matrices.

Some key features of $A_0$ are described in Proposition 6.1.

**Proposition 6.1.** A densely defined, closed, and, in general, non-symmetric operator $A_0$ in $\mathcal{H}$ satisfies the following properties:

(i) The boundary form of $A_0$ satisfies an abstract Green identity

\begin{align}
[f, g]_{A_0} &= \langle d(k), G_Z d(k') \rangle_{C^m} - \langle G_Z d(k), d(k') \rangle_{C^m} \\
&= \langle c(k), M d(k') \rangle_{C^d} - \langle M d(k), c(k') \rangle_{C^d}
\end{align}

for $f = f^# + k \in \mathcal{H}_{m+2} + \mathbb{R}$ and $g = g^# + k' \in \mathcal{H}_{m+2} + \mathbb{R}$, where the matrices

\begin{equation}
G_Z := G Z_d \in [C^{md}]
\end{equation}

and

\begin{equation}
\mathcal{M} = (\mathcal{M}_{\sigma \alpha'}) \in [C^{md}, C^d], \quad \mathcal{M}_{\sigma \alpha'} := R_{\sigma \alpha'}(z_j')
\end{equation}

\forall \sigma \in \mathcal{S} \ \forall \alpha' = (\sigma', j') \in \mathcal{S} \times J, \text{ and the column-vector}

\begin{equation}
c(k) = (c_\sigma(k)) \in C^d, \quad c_\sigma(k) := \sum_{j \in J} d_{\sigma j}(k)
\end{equation}

\forall \sigma \in \mathcal{S}. The matrix $R(\cdot)$ is defined in (3.10b).

(ii) The adjoint $A_0^*$ in $\mathcal{H}$ is the operator defined by

\begin{equation}
\text{dom } A_0^* = \text{dom } A_0, \quad A_0^*(f + k) = L_m f + \sum_{\alpha \in \mathcal{S} \times J} [G^{-1} G_Z^* d(k)]_\alpha g_\alpha
\end{equation}

for $f \in \mathcal{H}_{m+2}$ and $k \in \mathbb{R}$; the adjoint matrix $G_Z^* \equiv (G_Z)^*$.

(iii) The boundary form of $A_0^*$ is given by $(-1)[\cdot, \cdot]_{A_0}$. Moreover, the operator $A_0$ is self-adjoint if the matrix $G_Z$ is Hermitian.

(iv) The resolvent of $A_0$ is given by

\begin{equation}(A_0 - z)^{-1} = U^* [(L_m - z)^{-1} \oplus (Z_d - z)^{-1}] U
\end{equation}

for $z \in \text{res } A_0 = \text{res } L \times Z$, where $U : \mathcal{H} \to \mathcal{H}'$ is the isometric isomorphism defined by (5.5).
Proof. (i) The first relation \((6.4a)\) follows from \((5.2), (5.3), \) and \((6.1)\) by direct computation. To prove the second relation \((6.4b)\) we show that \((6.4b)\) implies \((6.4a)\). First we rewrite \((6.4b)\) explicitly:

\[
\langle c(k), M d(k') \rangle_{C^d} - \langle M d(k), c(k') \rangle_{C^d} = \sum_{\alpha, \alpha'} d_{\alpha}(k) (M_{\alpha \alpha'} - \overline{M_{\alpha \alpha'}}) d_{\alpha'}(k')
\]

with \(\alpha = (\sigma, j)\) and \(\alpha' = (\sigma', j')\) ranging over \(S \times J\). Next, by \((3.10b)\)

\[
(R(z) - R(w))_{\sigma \sigma'} = (z - w) \langle g_{\sigma}(z), g_{\sigma'}(w) \rangle_{-m}
\]

\(\forall \sigma, \sigma' \in S \forall z, w \in \text{res } L\), and so in particular

\[
M_{\sigma \sigma'} = R_{\sigma \sigma'}(z) + (z - \overline{z}) \delta_{\sigma \sigma'}
\]

by using also \((5.4)\). Therefore

\[
M_{\sigma \sigma'} - \overline{M_{\sigma \sigma'}} = (z - \overline{z}) \delta_{\sigma \sigma'} = [G_Z - \delta Z]_{\sigma \sigma'}
\]

and \((6.4a)\) follows.

(ii) By definition, the adjoint linear relation \(A_0^*\) is given by

\[
A_0^* = \{(g_1, g_2) \in H \times H | (\forall f \in \delta_{m+2}) (\forall k \in \mathbb{R}) \langle f + k, g_2 \rangle_H = \langle A_0(f + k), g_1 \rangle_H \}.
\]

Since \(g_1, g_2 \in H\), one has \(g_1 = g_1^# + k_1\) and \(g_2 = g_2^# + k_2\) for some \(g_1^#, g_2^# \in \delta_{m}\) and \(k_1, k_2 \in \mathbb{R}\). Using \((5.3)\) and \((6.1)\), the condition defining \(A_0^*\) reads

\[
0 = \langle f + k, g_2 \rangle_H - \langle A_0(f + k), g_1 \rangle_H = \langle P(L_z)^{1/2} f, P(L_{-2})^{1/2} (g_2^# - L_{m-2} g_1^#) \rangle + \langle d(k), G d(k_2) - G_Z d(k_1) \rangle_{C^m}
\]

where \(\langle \cdot, \cdot \rangle\) is the duality pairing between \(\delta_{2}\) and \(\delta_{-2}\); see \((4.1)\). The obtained equation must hold for all \(f \in \delta_{m+2}\) and all \(k \in \mathbb{R}\). Since \(\mathbb{R} \cap \delta_{m+2} = \{0\}\), by Corollary \(5.2\), it follows that \(g_1^# \in \delta_{m+2}, g_2^# = L_{m} g_1^# \in \delta_{m}\), and \(d(k_2) = G^{-1} G_Z d(k_1)\); hence \((6.8)\) holds. Similarly, one shows that the closure \(A_0^*: = (A_0^*)^* = A_0\).

(iii) This is clear from \((6.1)\) and \((6.8)\).

(iv) For an arbitrary \(f \in \text{dom } A_0, (A_0 - z)f = g \in H\) for some \(z \in \text{res } A_0\). Using \((6.1)\) and that \(g \in H\) is of the form \(g = g^# + k\) for some \(g^# \in \delta_{m}\) and \(k \in \mathbb{R}\), we get that

\[
(A_0 - z)^{-1} g = (L_m - z)^{-1} g^# + \sum_{(\sigma, j) \in S \times J} d_{\sigma j}(k) \frac{1}{z_j - \overline{z}} g_{\sigma j}
\]

and \(\text{res } A_0 = \text{res } L \setminus \mathbb{Z}\). Now apply \((5.5)\) and deduce \((6.9)\). \(\square\)

The matrix \(M\) defined in \((6.6)\) arises in the boundary forms in Lemma \(7.14\) and \((7.43)\); see also \((6.14)\) for the connection with the admissible matrix \(R(\cdot)\); another useful relation is given in Lemma \(7.8-(iii)\).

We remark on certain properties of the matrix \(R(\cdot)\) in the case when the matrix \(G_Z\) in \((6.5)\) is Hermitian.

For \(d = 1\), the condition \(G_Z = G_Z\) yields \(Z G = G Z\), where the Gram matrix \(G = (G_{\alpha \alpha'}) \in [\mathbb{C}^m]\) and the diagonal matrix \(Z \in [\mathbb{C}^m]\) is as in \((6.3)\). The commutation relation implies that \(G\) must be diagonal (see also [26, Eq. \((5.12)\)]).

For \(d \geq 1\), the condition \(G_Z^* = G_Z = 0\) reads

\[
(\overline{z_j} - z_{j'}) G_{\sigma j, \sigma' j'} = 0
\]
\( \forall (\sigma, j), (\sigma', j') \in S \times J \). Using (5.4) and (6.10), equation (6.12) yields \( R(z_j) = R(z_{j'}) \) \( \forall j, j' \in J \), i.e.

\[
R := R(z_j) = R(z_{j'}) \quad \forall j, j' \in J; \quad R^* = R
\]

and hence \( Z = Z \cap \mathbb{R} \). As a result, for \( G_Z \) Hermitian, the Gram matrix \( G \) is necessarily diagonal in \( j \in J \) for all \( d \geq 1 \). The converse is also true, that is, (6.13) yields \( G_Z^* = G_Z \) by (6.10).

Example 6.2. Let \( d = 1 \). The Nevanlinna function \( R(z) \in \mathbb{C} (z \in \mathbb{C} \setminus \mathbb{R}) \) admits the integral representation [38, 39, 40, 41, 42]

\[
R(z) = a + bz + \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{1}{\lambda^2 + 1} \right) d\sigma(\lambda)
\]

where \( a \in \mathbb{R}, b \geq 0 \), and the function \( \sigma : \mathbb{R} \to \mathbb{R} \) is non-decreasing and satisfies \( \int_{\mathbb{R}} d\sigma(\lambda)/\left((\lambda^2 + 1)\right) < \infty \). Thus the condition (6.13) implies that

\[
b + \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{(\lambda - z_j)(\lambda - z_{j'})} = 0
\]

for \( j, j' \in J \) and \( j \neq j' \) and \( m > 1 \). For example, if \( m = 2 \) then

\[
R(z) = R + (z - z_1)(z - 2) \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{(\lambda - z)(\lambda - z_1)(\lambda - z_2)}
\]

and

\[
R = a + \int_{\mathbb{R}} \left( \frac{\lambda - z_1 - z_2}{(\lambda - z_1)(\lambda - z_2)} - \frac{1}{\lambda^2 + 1} \right) d\sigma(\lambda)
\]

for \( z_1, z_2 \in Z \).

Using that \( R^* = R \) for \( G_Z^* = G_Z \), one notices that in this case (\( \forall k \in \mathcal{H} \))

\[
\mathcal{M}d(k) = Rc(k)
\]

and the form (6.4b) vanishes. When \( m > 1 \), the latter form also vanishes for \( A_0 \) restricted to \( \mathcal{H}_{m+2} \) \( \cup \mathcal{R}_{\min} \) (see Remark 5.4-(b)), for a not necessarily Hermitian \( G_Z \).

6.2. Closable restriction \( A_{\max} \). Let \( A_{\max} \) be the restriction of \( L_{\max} \) to the domain of vectors from \( \mathcal{H} \) such that the range of \( L_{\max} \) is also contained in \( \mathcal{H} \); i.e.

\[
A_{\max} := L_{\max} \mid \mathrm{dom} A_{\max}, \quad \mathrm{dom} A_{\max} := \{ f \in \mathcal{H} \cap \mathrm{dom} L_{\max} \mid L_{\max} f \in \mathcal{H} \}.
\]

The following proposition holds (compare [26, Lemmas 5.1 and 5.2] with Theorem 6.3 and Theorem 6.5-(i) below, when applied to the case \( d = 1, Z = Z \cap \mathbb{R}_{<0}, p(L) = I \)).

**Theorem 6.3.** The operator \( A_{\max} \supseteq A_0 \) extends \( A_0 \) to the domain

\[
\mathrm{dom} A_{\max} = \mathrm{dom} A_0 + \mathcal{R}_z(A_{\max})
\]

where the eigenspace of \( A_{\max} \) is given by

\[
\mathcal{R}_z(A_{\max}) = \mathcal{R}_z(L_{\max})
\]

and \( z \in \mathrm{res} A_0; \mathcal{R}_z(L_{\max}) \) is as in (4.7).
Proof. Consider \( g \in \mathcal{H} \cap \text{dom } L_{\text{max}} \) such that \((L_{\text{max}} - z)g = f \in \mathcal{H} \) and \( f \neq 0 \) for some \( z \in \text{res } L \setminus \mathcal{Z} \). Then, by definition, \( g \in \text{dom } A_{\text{max}} \) is of the form (4.6) (see also (4.5)). On the other hand, relation \( f \in \mathcal{H} \) implies that \( f = f^\# + k \) for some \( f^\# \in \mathcal{H}_m \) and \( k \in \mathcal{R} \). By applying Corollary 4.3 (see also (4.4)) and using (5.3) we therefore get that
\[
(L_{-m} - z)g^\# = f^\# + \sum_{\alpha \in \mathcal{S} \times \mathcal{J}} d_\alpha(k)g_\alpha
\]
for \( g^\# \in \mathcal{H}_{m+2} \) as in (4.5). For \( z \in \text{res } L \setminus \mathcal{Z} \), the above equation gives \( g^\# \) in the form
\[
g^\# = g^\# + \sum_{(\sigma, \jmath) \in \mathcal{S} \times \mathcal{J}} f_{\sigma\jmath}(k)\frac{d_{\sigma\jmath}(k)}{z_j - z}(g_{\sigma\jmath} - g_\sigma(z)) \quad \text{for } g^\# := (L_m - z)^{-1}f^\# \in \mathcal{H}_{m+2}.
\]
Thus, by defining
\[
(6.18) \quad g_z(c) := \sum_\sigma c_\sigma g_\sigma(z) \in \mathcal{H}_z(L_{\text{max}})
\]
\( \forall c = (c_\sigma) \in \mathbb{C}^d \), we get that
\[
g = g^\# + g_z(c) = g_0 + g_z(c')
\]
where
\[
g_0 := g^\# + k' \in \text{dom } A_0, \quad d_{\sigma\jmath}(k') := \frac{d_{\sigma\jmath}(k)}{z_j - z}
\]
and
\[
c' = (c'_\sigma) \in \mathbb{C}^d, \quad c'_\sigma := c_\sigma - \sum_\jmath d_{\sigma\jmath}(k').
\]
This shows that
\[
(6.19) \quad \text{dom } A_{\text{max}} = \text{dom } A_0 + \mathcal{N}_z(L_{\text{max}})
\]
for \( z \in \text{res } A_0 \) (= \( \text{res } L \setminus \mathcal{Z} \)).

By using that \( A_{\text{max}} \subseteq L_{\text{max}} \), by definition (6.15), the action of \( A_{\text{max}} \) on \( g \) is then given by
\[
A_{\text{max}}g = L_{\text{max}}g = zg + f = z(g^\# + k' + g_z(c')) + f^\# + k = zg^\# + zk' + zg_z(c') + (L_m - z)g^\# + \sum_\sigma \sum_\jmath (z_j - z)d_{\sigma\jmath}(k')g_{\sigma\jmath} = zg_z(c') + L_mg^\# + \sum_\sigma \sum_\jmath z_jd_{\sigma\jmath}(k')g_{\sigma\jmath}
\]
and hence
\[
(6.20) \quad A_{\text{max}}(g_0 + g_z(c')) = A_0g_0 + zg_z(c')
\]
in view of (6.1). This shows that \( A_{\text{max}} \) is an extension of \( A_0 \) to the domain given in (6.19). In particular, it follows from (6.20) that \( \mathcal{N}_z(A_{\text{max}}) \) coincides with \( \mathcal{N}_z(L_{\text{max}}) \) for \( z \in \text{res } A_0 \), and hence (6.16) and (6.17) are verified, provided that we show that the sum in (6.16) is direct.

Thus, let \( 0 = f + k + g_z(c) \) for \( f \in \mathcal{H}_{m+2}, \ k \in \mathcal{R}, \) and \( c = (c_\sigma) \in \mathbb{C}^d \); we need to verify that the equation yields \( f = 0, \ k = 0, \) and \( g_z(c) = 0 \) (i.e. \( g_z(c) = 0 \) in (6.18)). To accomplish the task we apply the following lemma (cf. [26, Eqs. (4.10), (6.3)]).

**Lemma 6.4.** Let \( z \in \text{res } A_0 \) and put
\[
(6.21) \quad F_\sigma(z) := \frac{1}{b(z)}g_\sigma(z) \quad (\forall \sigma \in \mathcal{S}), \quad b(z) := \tilde{P}(z).
\]
Then
\begin{equation}
F_\sigma(z) = \frac{1}{b(z)} \sum_{j \in J} b_j(z) g_{\sigma j} + \widetilde{G}_\sigma(z), \quad \widetilde{G}_\sigma(z) := p(L_m)G_\sigma(z)
\end{equation}
where \( p(\cdot) \) is as in (2.5).

Proof. Formula (6.22) is obtained from (2.4) by computing
\[
\widetilde{G}_\sigma(z) = \tilde{P}(L_{-m})^{-1}g_\sigma(z)
\]
with the help of the relations
\[
b(z)/(z - z_j) = b_j(z) \quad \text{and} \quad 1 = \sum_j b_j(z)/b_j(z_j)
\]
for \( z \in \text{res } A_0 \).

By using (6.22) we get that
\[
0 = f + k + g_z(c) = f + k' + b(z)\tilde{P}(L)^{-1}g_z(c)
\]
where \( k' \in \mathcal{R} \) is defined by
\[
d_\alpha(k') := d_\alpha(k) + c_\sigma b_j(z)/b_j(z_j)
\]
\( \forall \alpha = (\sigma, j) \in S \times J \). Now \( \tilde{P}(L)^{-1}g_z(c) \in \mathcal{F}_m(\mathcal{F}_{m+1}) \) (apply e.g. Proposition 2.6); hence \( f = 0 \), \( k' = 0 \), and \( c = 0 \) by Corollary 5.2. But then also \( k = 0 \), and this accomplishes the proof of the theorem. \( \square \)

Let us emphasize that, as it follows from Lemma 6.4, an element from \( \mathcal{N}_z(A_{\text{max}}) \) can be written in terms of an element from \( \mathcal{R} \) and an element from \( \mathcal{F}_m \), i.e. \( \text{dom } A_{\text{max}} \subseteq \mathcal{H} \) in (6.16). In the following theorem, the latter inclusion is seen explicitly.

**Theorem 6.5.** The operator \( A_{\text{max}} \), for \( z \in \text{res } A_0 \), can be described as follows:

(i) It holds
\begin{align}
(6.23a) \quad \text{dom } A_{\text{max}} &= 2 \mathcal{F}_m + \mathcal{R} + \text{lin}\{\widetilde{G}_\sigma(z) \mid \sigma \in S\}, \\
(6.23b) \quad A_{\text{max}}(f + k + \widetilde{G}_z(c)) &= A_0(f + k) + z\widetilde{G}_z(c) + k_{\text{min}}(c)
\end{align}
for \( f \in \mathcal{F}_m \), \( k \in \mathcal{R} \), and \( c \in \mathbb{C}^d \). Here \( \widetilde{G}_\sigma(z) \) is defined in (6.22),
\[ G_z(c) := p(L_m)G_z(c), \]
\( G_z(c) \) is defined in (3.9), and \( k_{\text{min}}(c) \) is as in (5.7).

(ii) It holds
\begin{align}
(6.25a) \quad \text{dom } A_{\text{max}} &= p(L_m) \text{dom } L^{\ast}_{\text{min}} + \mathcal{R}, \\
(6.25b) \quad A_{\text{max}}(p(L_m)f + k) &= p(L_m)L^{\ast}_{\text{min}}f + \sum_{\alpha \in S \times J} [Z_{\alpha d}(k)]_\alpha g_\alpha + k_{\text{min}}
\end{align}
for \( f \in \text{dom } L^{\ast}_{\text{min}} \) as in Corollary 3.8, \( k_{\text{min}} = k_{\text{min}}(c) \) as in (i), \( k \in \mathcal{R} \).

Proof. (i) This follows from (6.1), Theorem 6.3, and Lemma 6.4. The sums in (6.23a) are direct because of Corollary 5.2.

(ii) Using (3.5) and Corollary 2.7
\[ \mathcal{F}_m + \mathcal{R} + \text{lin}\{\widetilde{G}_\sigma(z) \mid \sigma \in S\} = p(L_m)(\mathcal{F}_m + \mathcal{N}_z(L^{\ast}_{\text{min}})) + \mathcal{R}. \]
This shows (6.25a) in view of (3.7), (6.23a), and Corollary 5.2. Then one computes the right-hand side in (6.25b) by using (6.1), (6.23b), and Theorem 3.7, and by applying the commutation relation $L_m p(L_{m+2}) = p(L_m) L_m$. □

**Corollary 6.6.** Define the surjective mapping $\tilde{\Gamma} := (\tilde{\Gamma}_0, \tilde{\Gamma}_1)$: dom $A_{\max} \rightarrow \mathbb{C}^d \times \mathbb{C}^d$ by

\begin{align}
(6.26a) & \quad \tilde{\Gamma}_0 f := c = (c_\sigma) \in \mathbb{C}^d, \\
(6.26b) & \quad \tilde{\Gamma}_1 f := \langle \tilde{\varphi}, f' \rangle - G_k^* d(k) \in \mathbb{C}^d
\end{align}

for $f = p(L_m)f' + k \in$ dom $A_{\max}$, where $k \in \mathbb{K}$, $f' = f^* + G_k(c) \in$ dom $L_{\min}^*$, $f^* \in \mathbb{F}_{m+2}$, $G_k(c)$ as in (3.9), $\mathbb{G}_b$ as in (5.8), and $z \in \text{res } A_0$. The functional $\langle \tilde{\varphi}, \cdot \rangle := (\langle \varphi, \cdot \rangle)$: dom $L_{\min}^* \rightarrow \mathbb{C}^d$ extends the functional $\langle \varphi, |p(L_{m+2})|^2 \rangle : \mathbb{F}_{m+2} \rightarrow \mathbb{C}^d$ according to

\[ \langle \tilde{\varphi}, f' \rangle := \langle \varphi, |p(L_{m+2})|^2 f^* \rangle + \tilde{R}(z)c \quad \text{with} \quad \tilde{R}(z) = (\tilde{\sigma}_\sigma(z)) \in [\mathbb{C}^d], \ \tilde{\sigma}_\sigma(z) := \langle \varphi, G_\sigma(z) \rangle \]

$\forall (\sigma, \sigma') \in \mathcal{S} \times \mathcal{S}$.

Then, the boundary form of the operator $A_{\max}$ reads

\begin{equation}
(6.27) \quad [f, g]_{A_{\max}} = [k, k']_{A_0} + \langle \tilde{\Gamma} f, \tilde{\Gamma} g \rangle_{\mathbb{C}^d \times \mathbb{C}^d}
\end{equation}

for $f = p(L_m)f' + k \in$ dom $A_{\max}$ and $g = p(L_m)g' + k' \in$ dom $A_{\max}$, where $f', g' \in$ dom $L_{\min}^*$ and $k, k' \in \mathbb{K}$. In particular, the boundary form $[f, g]_{A_{\max}}$ of $A_{\max}$ satisfies an abstract Green identity $[\tilde{\Gamma} f, \tilde{\Gamma} g]_{\mathbb{C}^d \times \mathbb{C}^d}$ iff the matrix $G_Z$ defined in (6.5) is Hermitian.

**Proof.** To show that the mapping $\tilde{\Gamma}_j$, for $j \in \{0, 1\}$, is surjective, one considers $\tilde{\Gamma}_j$ as a linear relation from $\mathcal{H}$ to $\mathbb{C}^d$, and then computes the adjoint (cf. the proof of Proposition 7.21), which is $\tilde{\Gamma}_j^* = \{(0, 0)\}$; hence $\text{ran } \tilde{\Gamma}_j = (\ker \tilde{\Gamma}_j)^\perp = \mathbb{C}^d$. One computes the boundary form by using Theorem 3.7, Proposition 6.1-(i), and Theorem 6.5-(ii). The iff argument is due to (6.27) and Proposition 6.1-(i). □

One notices that the matrix $\tilde{R}(z)$ plays the role of the admissible matrix $R(z)$ in Corollary 3.8. The connection with the Weyl function associated with the triple $(\mathbb{C}^d, \tilde{\Gamma}_0, \tilde{\Gamma}_1)$ is established in Theorem 6.10; see also (6.53) and Theorem 8.10.

In the next paragraph we accomplish the analysis of the operator $A_{\max}$ by showing that it is not closed, in general, but closable; it is closed iff the matrix $G_Z$ is Hermitian. To compare with, an analogous construction in the cascade model is closed [14, Eq. (2.3)].

6.3. **Closed restriction** $A_{\min}$. Let us recall the set $\mathcal{Z}_0$ in Section 2.2, which is the union of elements from $\mathcal{Z}$ and their complex conjugate counterparts (if any) from $\mathcal{Z}^*$. Let us put $\mathcal{Z}_0^* := \text{res } L \setminus \mathcal{Z}_0$, so that $\mathcal{Z}_0 \subseteq \text{res } A_0$; recall res $A_0$ in Proposition 6.1-(iv).

**Theorem 6.7.** Let

\begin{equation}
(6.28) \quad A_{\min} := A_{\max}^*
\end{equation}

be the adjoint of $A_{\max}$ in $\mathcal{H}$. The following statements hold for $z \in \mathcal{Z}_0^*$:

(i) The operator $A_{\min} \subseteq A_0^*$ is given by

\begin{equation}
(6.29) \quad A_{\min} = A_0^* \big|_{\text{dom } A_0^* \cap \ker \tilde{\Gamma}_1}
\end{equation}

where the operator $A_0$ and its adjoint are described in Proposition 6.1. Moreover, the eigenspace $\mathcal{N}_z(A_{\min}) = \{0\}$. 
(ii) The adjoint $A_{\text{min}}^* \supseteq A_0$ in $\mathcal{H}$ extends the operator $A_0$ to the domain

$$\text{dom } A_{\text{min}}^* = \text{dom } A_0 + \mathfrak{N}_z(A_{\text{min}}^*)$$

where the eigenspace is given by

$$\mathfrak{N}_z(A_{\text{min}}^*) = \text{lin}\{F_\sigma(z) + \sum_{\sigma' \in \mathcal{S} \times J} \Lambda_{\sigma\sigma'}(z)g_{\sigma'} \mid \sigma \in \mathcal{S}\}$$

and the matrix $\Lambda(z) = (\Lambda_{\sigma\sigma'}(z)) \in [\mathbb{C}^d, \mathbb{C}^{md}]$ is defined by

$$\Lambda(z) := [(z\mathcal{G} - \mathcal{G}_z^*)^{-1} - (z\mathcal{G} - \mathcal{G}_Z)^{-1}]g_b.$$

(iii) The operator $A_{\text{max}}$ is closed, that is, $A_{\text{min}}^* = A_{\text{max}}$, iff the matrix $\mathcal{G}_Z$ is Hermitian.

For $\mathcal{G}_Z$ Hermitian, the above statements extend to $z \in \text{res } A_0$. The function $F_\sigma(z)$ is defined in (6.21).

**Proof.** (i) By definition, the adjoint linear relation $A_{\text{max}}^*$ is the set of the elements $(f, f') \in \mathcal{H} \times \mathcal{H}$ that satisfy

$$\langle g, f' \rangle_{\mathcal{H}} = \langle A_{\text{max}}g, f \rangle_{\mathcal{H}}$$

\forall g \in \text{dom } A_{\text{max}}. Since $f, f' \in \mathcal{H}$, one has $f = f_1 + k_1$ and $f' = f_2 + k_2$ for some $f_1, f_2 \in \mathfrak{S}_m$ and $k_1, k_2 \in \mathfrak{K}$. Since $g \in \text{dom } A_{\text{max}}$, one has $g = p(L_m)g' + k$ for some $g' \in \text{dom } L_{\text{min}}^*$ and $k \in \mathfrak{K}$, by Theorem 6.5-(ii). An element $g'$ is of the form $g' = g^\# + G_z(c)$ for some $g^\# \in \mathfrak{S}_{m+2}$ and $c = (c_\sigma) \in \mathbb{C}^d$, as described Theorem 3.7, and $G_z(c)$ as defined in (3.9). Thus, using (5.2), (5.3), (5.7), (6.25b), and then applying Theorem 3.7, the defining equation (6.33) reads

$$0 = \langle g, f' \rangle_{\mathcal{H}} - \langle A_{\text{max}}g, f \rangle_{\mathcal{H}}$$

$$= \langle p(L_{m+2})g^\# + G_z(c), f_2 \rangle_m + \langle d(k), Gd(k_2) \rangle_{\mathbb{C}^{md}}$$

$$- \langle p(L_m)L_mg^\# + zG_z(c), f_1 \rangle_m + \langle G_Zd(k) + G_0c, d(k_1) \rangle_{\mathbb{C}^{md}}$$

$$= \langle P(L_2)^{1/2}p(L_{m+2})g^\#, P(L_2)^{1/2}(f_2 - L_{m-2}f_1) \rangle$$

$$+ \langle c, \tilde{G}(z)g^\# - \tilde{G}_0d(k_1) \rangle_{\mathbb{C}^d} + \langle d(k), Gd(k_2) - G_0^*d(k_1) \rangle_{\mathbb{C}^{md}}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathfrak{S}_2$ and $\mathfrak{S}_{-2}$, as described in (4.1). Recall $\tilde{G}_z(c)$ in (6.24), and one uses the vector notation

$$\langle \tilde{G}(z), \cdot \rangle_m := \langle \tilde{G}_\sigma(z), \cdot \rangle_m : \mathfrak{S}_m \to \mathbb{C}^d$$

for $\tilde{G}_\sigma(z)$ as in (6.22).

The obtained equation must be valid $\forall g^\# \forall c \forall k$. Since the three elements arise from disjoint sets, one concludes that $f_1 \in \mathfrak{S}_{m+2}, f_2 = L-mf_1 \in \mathfrak{S}_m, d(k_2) = \mathcal{G}^{-1}\mathcal{G}_Zd(k_1)$, and

$$0 = \langle \tilde{G}(z), f_2 - \overline{f}_1 \rangle_m - \tilde{G}_0^*d(k_1) = \langle \tilde{G}(z), (L_m - \overline{f}_1) \rangle_m - \tilde{G}_0^*d(k_1)$$

$$= \langle \overline{f}, p(L_{m+2})^+f_1 \rangle - \tilde{G}_0^*d(k_1)$$

i.e. $\tilde{\Gamma}_1(f_1 + k_1) = 0$ by (6.26b). Now, formula (6.29) follows from the latter boundary condition and Proposition 6.1-(ii).

To compute the eigenspace $\mathfrak{N}_z(A_{\text{min}})$, let us consider $f_z \in \mathfrak{N}_z(A_{\text{min}})$. Since $\mathfrak{N}_z(A_{\text{min}}) \subseteq \text{dom } A_{\text{min}}$, we have by the above $f_z = f_\# + k$ for $f_\# \in \mathfrak{S}_{m+2}$ and $k \in \mathfrak{K}$ such that $\tilde{\Gamma}_1f_z = 0$.

By using (6.29), the eigenvalue equation $(A_{\text{min}} - z)f_z = 0$ reads

$$0 = (L_m - z)f_\# + \sum_{\alpha \in \mathcal{S} \times J} [(\mathcal{G}^{-1}\mathcal{G}_Z^* - z)d(k)]_\alpha g_\alpha.$$
Since \( z \in \text{res } L_m = \text{res } L \) (Corollary 2.3), it holds

\[
0 = f^\#_z + h_z, \quad h_z := (L_m - z)^{-1} \sum_{\alpha} [G^{-1}G^*_\alpha - z) d(k)]_\alpha g_\alpha .
\]

Since \( \mathcal{H}_{m+1} \cap (L_m - z)^{-1} \mathcal{R} = \{0\} \) by Corollary 5.2, and \( f^\#_z \in \mathcal{H}_{m+2} \subseteq \mathcal{H}_{m+1} \), it follows that \( f^\#_z = h_z = 0 \). Then \( h_z = 0 \), together with the boundary condition for \( f_z \), yields \( d(k) \in \ker G^*_\alpha \cap \ker (G^{-1}G^*_\alpha - z) \). But \( G^{-1}G^*_\alpha - z \) is invertible for \( z \in \mathbb{Z}^*_\alpha \), and so \( k = 0 \).

(ii) Taking the adjoints in \( A_{\min} \subseteq A^*_\alpha \) one gets that \( A^*_{\min} \supseteq A_0 \), since the operator \( A_0 \) is closed. Now \( (\forall f_0 + g_z \in \text{dom } A_0 + \mathcal{R}_z(A^*_{\min})) \)

\[
(A^*_{\min} - z)(f_0 + g_z) = (A_0 - z)f_0
\]

and so \( \text{dom } A_0 + \mathcal{R}_z(A^*_{\min}) \subseteq \text{dom } A^*_{\min} \).

To verify the reverse inclusion, assume that \( z \in \mathbb{Z}^*_\alpha \cap \text{res } A^{*}_{\min} \) and consider an arbitrary \( h \in \mathcal{H} \). Since we have \( \mathcal{R}_z(A_{\min}) = \{0\} \) by (i), it follows that \( \text{ran}(A^*_{\min} - z) = \mathcal{H} \) for \( z \) as assumed above. Therefore (\( \exists f \in \text{dom } A^*_{\min} \) ) \((\exists g \in \text{dom } A_{\min} \) ) \((\exists g_\alpha \in \mathcal{R}_z(A^*_{\min}) \) )

\[
h = (A^*_{\min} - z)f = g_\alpha + (A_{\min} - z)g .
\]

Put \( g'_z := f - (A_0 - z)^{-1}h \); then, using that \( A^*_{\min} \supseteq A_0 \) we get that

\[
(A^*_{\min} - z)g'_z = g_\alpha + (A_{\min} - z)g - (A^*_{\min} - z)(A_0 - z)^{-1}h
\]

\[
= h - h = 0
\]

i.e. \( g'_z \in \mathcal{R}_z(A^*_{\min}) \). It follows that \( f = (A_0 - z)^{-1}h + g'_z \in \text{dom } A_0 + \mathcal{R}_z(A^*_{\min}) \). This shows (6.30) for \( z \in \mathbb{Z}^*_\alpha \cap \text{res } A^{*}_{\min} \). Now (6.30) extends to \( z \in \mathbb{Z}^*_\alpha \) because of (6.35), which is valid for all \( z \in \mathbb{Z}^*_\alpha \). We show that the sum in (6.30) is direct after we prove (6.31).

Thus, let us compute the eigenspace \( \mathcal{R}_z(A^*_{\min}) \). By (i), an element \( f = f^\# + k \in \text{dom } A_{\min} \), with \( f^\# \in \mathcal{H}_{m+2} \) and \( k \in \mathcal{R} \), satisfies the boundary condition \( \Gamma_1 f = 0 \). By considering \( G^*_\alpha \) as a linear relation (Lemma 5.6), the boundary condition reads

\[
(d(k), \langle \varphi, p(L_{m+2})^+ f^\# \rangle) \in G^*_\alpha \iff (\langle \varphi, p(L_{m+2})^+ f^\# \rangle, d(k)) \in (G^*_\alpha)^{-1} .
\]

By Lemma 5.7, this implies that

\[
d(k) = H^\circ_b \langle \varphi, p(L_{m+2})^+ f^\# \rangle + \xi , \quad \xi \in \ker G^*_\alpha .
\]

Now let \( g \in \mathcal{R}_z(A^*_{\min}) \) for \( z \in \mathbb{Z}^*_\alpha \). Since \( g \in \mathcal{H} \), it must hold \( g = g^\# + k' \) for some \( g^\# \in \mathcal{H}_m \) and \( k' \in \mathcal{R} \). Using (i) it follows that (\( \forall f \) )

\[
0 = (\langle A_{\min} - z \rangle g, f)_{\mathcal{H}} = (\langle g, (A_{\min} - z \rangle f \rangle_{\mathcal{H}}
\]

\[
= (g^\#, (L_m - z \rangle f^\#)_{\text{dom}} + \langle \xi', d(k) \rangle_{\text{dom}}
\]

where

\[
\xi' := (G^*_\alpha - zG)d(k') .
\]

Substitute (6.36) in (6.37), and since (6.37) must hold \( \forall f^\# \forall \xi \), conclude that

\[
\xi' \in (\ker G^*_\alpha)^\perp = \text{ran } G^*_\alpha = \text{ran } G^*_b
\]

and (\( \forall f^\# \))

\[
0 = (g^\#, (L_m - z \rangle f^\#)_{\text{dom}} + \langle \xi', H^\circ_b \langle \varphi, p(L_{m+2})^+ f^\# \rangle_{\text{dom}} .
\]

By applying Corollary 5.9, and since dom \( H^\circ_b \supseteq \text{ran } G^*_b \) by Lemma 5.8, equation (6.40) further simplifies thus: (\( \forall f^\# \))

\[
0 = \langle g^\# + \tilde{G}_z(H^\circ_b \xi'), (L_m - z \rangle f^\#\rangle_{\text{dom}} .
\]
But \((L_m - z)\mathfrak{H}_{m+2} = \mathfrak{H}_m\), and so \(g^\# = -\tilde{G}_z(h_0^c)\). By (6.39) \((\exists c = (c_0) \in \mathbb{C}^d) \xi' = -G_0 c\), hence \(H_0^0 \xi' = -c\) by (5.11). Then \(g^\# = \tilde{G}_z(c)\) and, by using (6.38), \(g = g^\# + k'\) becomes of the form

\[
(6.41a) \quad g = \sum_{\sigma \in \mathfrak{S}} c_{\sigma} h_{\sigma}(z), \quad h_{\sigma}(z) := \tilde{G}_{\sigma}(z) + \sum_{\alpha' \in \mathfrak{S} \times J} [(zG - G_0^\sigma)^{-1}G_0]_{\alpha' \sigma} g_{\alpha'}.
\]

Using (6.22), \(h_{\sigma}(z)\) can be further rewritten thus:

\[
(6.41b) \quad h_{\sigma}(z) = F_{\sigma}(z) + \sum_{\alpha'} \Lambda_{\alpha' \sigma}(z) g_{\alpha'}
\]

where the matrix \(\Lambda(z) = (\Lambda_{\alpha' \sigma}(z))\) is defined by

\[
(6.42a) \quad \Lambda(z) := (zG - G_0^\sigma)^{-1}G_0 - \Sigma(z) \quad \text{with}
\]

\[
(6.42b) \quad \Sigma(z) = (\Sigma_{\alpha' \sigma}(z)), \quad \Sigma_{\alpha' \sigma}(z) := \frac{\delta_{\sigma' \sigma}}{(z - \tau_j)y_j(z_j)}.
\]

We show that \(\Lambda(z)\) is of the form (6.32). It follows from (6.42) that

\[
[(z - Z_d)((zG - G_0^\sigma)^{-1}G_0 - \Lambda(z))]_{\alpha' \sigma} = \delta_{\sigma' \sigma} b_j(z_j)^{-1}.
\]

Multiply the latter by \(G_{\alpha' \sigma'}\), with an arbitrary \(\alpha_1 \in \mathfrak{S} \times J\), and perform the summation of the obtained expression over \(\alpha' \in \mathfrak{S} \times J\), and deduce by using (5.8) that

\[
(zG - G_0)(z((G - G_0^\sigma)^{-1}G_0 - \Lambda(z))) = G_0
\]

from which (6.32) follows. This together with (6.41) proves (6.31).

Finally, we verify that the sum in (6.30) is direct. For this, we argue exactly the same way as when proving that the sum in (6.16) is direct: For \(f \in \mathfrak{H}_{m+2}, k \in \mathfrak{R}, \) and \(g \in \mathfrak{H}_m(A_{\text{min}}^* )\) as in (6.41), the equation \(0 = f + k + g\) reads \(0 = f + k' + \tilde{G}_z(c)\) where \(k' \in \mathfrak{R}\) is defined by

\[
d_{\alpha}(k') := d_{\alpha}(k) + c_{\alpha} b_j(z_j) + [\Lambda(z)c]_{\alpha}
\]

\(\forall \alpha = (\sigma, j) \in \mathfrak{S} \times J\). Thus \(f = 0, k' = 0, \) and \(c = 0\) by Corollary 5.2; then also \(k = 0\).

(iii) According to (6.16) and (6.30), \(A_{\text{min}}^* = A_{\text{max}}\) iff \(\mathfrak{N}_m(A_{\text{min}}^* ) = \mathfrak{N}_m(A_{\text{max}})\); but the latter holds true iff \(G_{\sigma}\) is Hermitian, by (6.17) and (6.31).

**Corollary 6.8.** Let \(G_{\sigma}\) be an Hermitian matrix. Then:

(i) The operator \(A_{0} = A_{\text{max}} |_{\ker \Gamma_{0}}\) is self-adjoint in \(\mathcal{H}\).

(ii) The operator \(A_{\text{min}} = A_{\text{max}} |_{\ker \Gamma}\) is densely defined, closed, and symmetric in \(\mathcal{H}\), and has defect numbers \((d, d)\).

(iii) The adjoint of \(A_{\text{min}}\) in \(\mathcal{H}\) is given by \(A_{\text{min}}^* = A_{\text{max}}\).

(iv) The triple \((\mathbb{C}^d, \Gamma_{0}, \Gamma_{1})\) is a boundary triple for \(A_{\text{min}}^*\).

**Proof.** The statements follow from Proposition 6.1, Corollary 6.6, and Theorem 6.7.

6.4. **Proper extensions.** Corollary 6.8 suggests that one can apply the classical extension theory to the operator \(A_{\text{min}}\), provided that the matrix \(G_{\sigma}\) is Hermitian. But before doing so, we find it instructive first to examine restrictions of \(A_{\text{max}}\) for a not necessarily Hermitian \(G_{\sigma}\).
Non-Hermitian case. Let \( \widetilde{\Gamma} \) be as in Corollary 6.6. Let \( \Theta \) be a linear relation in \( \mathbb{C}^d \). Define an operator \( A_\Theta \) in \( \mathcal{H} \) by

\[
A_\Theta := \{ \hat{f} \in A_{\max} | \hat{\Gamma}f \in \Theta \}.
\]

One assumes that the matrix \( \mathcal{G}_Z \) is not necessarily Hermitian, and so the statements of Corollary 6.8 do not necessarily apply in this case.

Further, define an operator \( A'_{\max} \) in \( \mathcal{H} \) similar to \( A_{\max} \) in Theorem 6.5-(i):

\[
\text{dom } A'_{\max} := \text{dom } A_{\max}, \quad A'_{\max}(f + k + \tilde{G}_z(c)) := A_0^*(f + k + z\tilde{G}_z(c) + k_{\min}(c))
\]

for \( f \in \mathcal{S}_{m+2}, k \in \mathbb{R}, \) and \( c \in \mathbb{C}^d \). Note that here one writes the adjoint \( A_0^* \) instead of \( A_0 \). Thus, by Proposition 6.1-(iii), \( A'^{\max} = A_{\max} \) (i.e. \( A_0^* = A_0 \)) if \( \mathcal{G}_Z = \mathcal{G}_Z \).

**Theorem 6.9.** Let \( A_\Theta \) be as in (6.43). The adjoint in \( \mathcal{H} \) is given by

\[
A_\Theta^* = \{ \hat{f} \in A'_{\max} | \hat{\Gamma}f \in \Theta^* \}
\]

provided that \( \Gamma \) is regarded as a mapping \( A'_{\max} \to \mathbb{C}^d \times \mathbb{C}^d \).

**Proof.** The adjoint linear relation \( A_\Theta^* \) is the set of the elements \((y, x) \in \mathcal{H} \times \mathcal{H}\) such that

\[
\langle f, x \rangle_\mathcal{H} = \langle A_\Theta f, y \rangle_\mathcal{H}.
\]

Using the decomposition \( x = x^# + k_x \) and \( y = y^# + k_y \) for some \( x^#, y^# \in \mathcal{S}_m \) and \( k_x, k_y \in \mathbb{R} \), as well as \( f = f^# + k + \tilde{G}_z(c) \) for some \( f^# \in \mathcal{S}_{m+2}, k \in \mathbb{R}, \) and \( c \in \mathbb{C}^d \), one has by Theorem 6.5-(i)

\[
0 = \langle f, x \rangle_\mathcal{H} - \langle A_\Theta f, y \rangle_\mathcal{H} = \langle f, x \rangle_\mathcal{H} - \langle A_{\max} f, y \rangle_\mathcal{H}
\]

\[
= \langle f^# + \tilde{G}_z(c), x^# \rangle_m + \langle k, k_x \rangle_m - \{\langle L_{m^2} f^# + z\tilde{G}_z(c), y^# \rangle_m + \langle \mathcal{G}_Z d(k) + \mathcal{G}_0 c, d(k_y) \rangle_{\mathbb{C}^d} \}
\]

and so

\[
0 = \langle f^#, P(L_{m-2})(x^# - L_{m-2}y^#) \rangle + \langle c, \tilde{G}(z), x^# - x^# y^# \rangle_m - \langle \mathcal{G}_Z^* d(k_y) \rangle_{\mathbb{C}^d}
\]

\[
= \langle f^#, P(L_{m-2}k_{\min}(r)) + \langle c, \tilde{G}(z) r - s \rangle_\mathcal{C} \rangle_\mathcal{C} - \langle d(k), \mathcal{G}_0 r \rangle_{\mathbb{C}^d}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( \mathcal{S}_{m+2} \) and \( \mathcal{S}_{m-2} \), and where \( \langle \tilde{G}(z), \cdot \rangle_m \) is defined in (6.34). Equation (6.45) must hold for all \( f \in \text{dom } A_{\max} \) such that \( \hat{\Gamma}f \in \Theta \). The boundary condition implies that

\[
\langle \tilde{G}_0 f, s \rangle_\mathbb{C} = \langle \tilde{G}_1 f, r \rangle_\mathbb{C}
\]

or equivalently

\[
0 = \langle \tilde{G}_1 f, r \rangle_\mathbb{C} - \langle \tilde{G}_0 f, s \rangle_\mathbb{C}
\]

\[
0 = \langle f^#, P(L_{m-2})k_{\min}(r) \rangle + \langle c, \tilde{G}(z) r - s \rangle_\mathcal{C} - \langle d(k), \mathcal{G}_0 r \rangle_{\mathbb{C}^d}
\]

By comparing (6.45) with (6.46) one concludes that \( \langle \exists(r, s) \in \Theta^* \rangle \)

\[
0 = \langle x^# - L_{m-2}y^# \rangle = k_{\min}(r),
\]

\[
\langle \tilde{G}(z), x^# - x^# y^# \rangle_m - \mathcal{G}_0^* d(k_y) = \tilde{R}(z) r - s,
\]

\[
\mathcal{G}_d(k_x) - \mathcal{G}_0^* d(k_y) = - \mathcal{G}_0 r.
\]

It follows from (6.47a) and \( x^# \in \mathcal{S}_m \) that \( y^# \in p(L_m) \text{ dom } L_{\min}^* \) (i.e. \( y \in \text{dom } A_{\max} \)); for only in this case \( y^# = y^# + \tilde{G}_z(c_y) \), with some \( y^# \in \mathcal{S}_{m+2} \) and \( c_y := \tilde{G}_0 y = -r \), gives

\[
x^# = L_{m} y^# + z\tilde{G}_z(c_y) \in \mathcal{S}_m.
\]
and hence by using (6.47c) \( x = A'_{\max}y \).

The second equation (6.47b) then yields \( s = -\tilde{\Gamma}_1y \), and so the boundary condition for \( y \) reads\(^1\) \( \tilde{\Gamma}y \in \Theta^\bullet \).

Since \( A_{\max} = A_{\mathbb{C}^d \times \mathbb{C}^d} \), one recovers from (6.44) the operator \( A_{\min} \) given in (6.29).

**Hermitian case.** Let \( G_Z \) be Hermitian. In analogy to (3.11), Corollary 6.8 shows that the mapping \( \Theta \rightarrow A_\Theta \) establishes a one-to-one correspondence between a (closed) linear relation \( \Theta \) in \( \mathbb{C}^d \) and a proper extension \( A_\Theta \) of \( A_{\min} \) in \( \mathcal{H} \), i.e. such that

\[
A_{\min} \subseteq A_\Theta \subseteq A_{\max}.
\]

Moreover, the adjoint (Theorem 6.9)

\[
A_\Theta^* = A_{\Theta^*}.
\]

Thus \( A_\Theta \) is self-adjoint in \( \mathcal{H} \) iff \( \Theta \) is self-adjoint in \( \mathbb{C}^d \).

Consider the \( \gamma \)-field \( \text{res } A_0 \ni z \mapsto \tilde{\gamma}(z) [\in [\mathbb{C}^d, \mathcal{H}^\dagger] \text{ and the abstract Weyl function } \text{res } A_0 \ni z \mapsto \tilde{M}(z) [\in [\mathbb{C}^d]] \text{ associated with the boundary triple } (\mathbb{C}^d, \tilde{\Gamma}_0, \tilde{\Gamma}_1). \) That is (\( \forall z \in \text{res } A_0 \))

\[
\tilde{\gamma}(z) := (\tilde{\Gamma}_0 | \gamma_{\mathrm{res}}(A_{\max}))^{-1}, \quad \tilde{M}(z) := \tilde{\Gamma}_1\tilde{\gamma}(z).
\]

Then the resolvent of a proper extension \( A_\Theta \) is found from the Krein–Naimark resolvent formula

\[
(A_\Theta - z)^{-1} = (A_0 - z)^{-1} + \tilde{\gamma}(z)(\Theta - \tilde{M}(z))^{-1}\tilde{\gamma}(z)^* \]

for \( z \in \text{res } A_0 \cap \text{res } A_\Theta \). The resolvent of \( A_0 \) is given in (6.9). The functions \( \tilde{\gamma} \) and \( \tilde{M} \) are explicitly described in Theorem 6.10.

**Theorem 6.10.** Let \( G_Z \) be an Hermitian matrix and let \( z \in \text{res } A_0 \). Define the matrix

\[
Q_g(z) = ([Q_g(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [Q_g(z)]_{\sigma\sigma'} := \sum_{j \in J} \frac{G_{\sigma j, \sigma' j}}{(z_j - z)b_j(z_j)^2}
\]

\( \forall (\sigma, \sigma') \in S \times S \). The \( \gamma \)-field \( \tilde{\gamma} \) and the Weyl function \( \tilde{M} \) associated with the boundary triple \((\mathbb{C}^d, \tilde{\Gamma}_0, \tilde{\Gamma}_1)\) for \( A_{\min}^* \) are given by

\[
\tilde{\gamma}(z) = \sum_{\sigma \in \mathcal{S}} \chi_\sigma F_\sigma(z), \quad \chi = (\chi_\sigma) \in \mathbb{C}^d,
\]

\[
\tilde{M}(z) = \tilde{R}(z) + Q_g(z).
\]

(The adjoint \( \tilde{\gamma}(z)^* = ((F_\sigma(z), \cdot)_{\mathcal{H}}) : \mathcal{H} \rightarrow \mathbb{C}^d.)

**Proof.** Formula (6.52a) is clear from the definition (6.50) by using that \( \mathcal{N}_z(A_{\max}) = \{F_\sigma(z)\} \) for \( G_Z = G_Z \) and \( z \in \text{res } A_0 \), by (6.17) and (6.31), and noting that \( \tilde{\Gamma}_0g_z(\cdot) = b(z)\tilde{\Gamma}_0G_z(\cdot) \) by (6.22) and (6.26a); here \( g_z(\cdot) \) and \( G_z(\cdot) \) are defined in (6.18) and (6.24), respectively. The adjoint \( \tilde{\gamma}(z)^* \) follows directly from (6.52a).

The Weyl function (6.52b) is found by using (6.52a), with \( F_\sigma(z) \) written as in (6.22): Given \( \chi = (\chi_\sigma) \in \mathbb{C}^d \), put \( f := \tilde{\gamma}(z)\chi \); then \( f \in \text{dom } A_{\max} \) is as in Corollary 6.6 with \( f^\# = 0 \), \( c = \chi \), \( d_{\sigma j}(k) = \chi_\sigma/[(z - z_j)b_j(z_j)] \). Now apply \( \tilde{M}(z)\chi = \tilde{\Gamma}_1f \) and deduce (6.52b) with \( Q_g(z) \) given by

\[
[Q_g(z)]_{\sigma\sigma'} = \sum_{j \in J} \frac{[G_{\sigma j, \sigma' j}^\#]}{(z_j - z)b_j(z_j)^2}.
\]

\(^1\)In deriving the present boundary condition, as well as in the similar situations everywhere else below, one uses the property \((r, s) \in \Theta \Leftrightarrow (-r, -s) \in \Theta \) for a linear relation \( \Theta \). To verify this, note that \((r, s) \in \Theta \Leftrightarrow (r, -s) \in -\Theta : = \Theta' \), because by definition \(-\Theta = \{(r, -s) \mid (r, s) \in \Theta\}\), and hence it holds \((-s, r) \in (\Theta')^{-1} \). But this is equivalent to \((-s, -r) \in -(\Theta')^{-1} = \Theta^{-1} \), from which the desired property follows.
For $G^*_Z = G_Z$, the matrix $G \in [\mathbb{C}^{md}]$ is diagonal in $j \in J$ and $z_j \in \mathbb{R} \cap \text{res } L \forall j$ (recall (6.13)); hence, by applying (5.8), the latter formula for $Q_G(z)$ reduces to (6.51).

The following Corollary 6.11, with real $\Theta = \Theta$, is an analogue of Theorem 6.1 in [26].

**Corollary 6.11.** Let $d = 1$ and $p(L) = I$, and let $G_Z$ be Hermitian. Then

$$(A_{\Theta} - z)^{-1} = (A_0 - z)^{-1} + \frac{F(z)U^*(\langle G(\overline{\sigma}), \cdot \rangle_{m} \oplus \langle b, G(z - Z)^{-1} \cdot \rangle_{cm})U}{\Theta - R(z) + \langle b, G(z - Z)^{-1} b \rangle_{cm}}$$

for $z \in \text{res } A_0 \cap \text{res } A_{\Theta}$ and $\Theta \in \mathbb{C} \cup \{\infty\}; b := (b_j(z_j)^{-1}) \in \mathbb{C}^m$. The mapping $U : \mathcal{H} \rightarrow \mathcal{H}'$ is the isometric isomorphism defined by (5.5).

Let us recall that we omit the indices $\sigma \in S$ for $d = 1$; thus the functions $F(z) := g(z)/b(z)$ and $G(z) := P(L_m)^{-1}g(z)$, where $g(\cdot) := (L_m - \cdot)^{-1}\varphi$ and $\varphi \in \mathcal{H}_m \setminus \mathcal{H}_m$. The Krein $Q$-function vs Weyl function. As above, assume that $G_Z$ is Hermitian. Fix $z_0 \in \text{res } L$ and define the Krein $Q$-function

$$\tilde{Q}(z) := \tilde{Q}_L(z) + Q_G(z), \quad z \in \text{res } A_0$$

with $Q_G$ as in (6.51) and

$$\tilde{Q}_L(z) = ([\tilde{Q}_L(z)]_{\sigma \sigma'} \in [\mathbb{C}^d], \quad [\tilde{Q}_L(z)]_{\sigma \sigma'} := (z - z_0) \langle \tilde{G}_{\sigma}(\overline{\sigma}), \tilde{G}_{\sigma}(z) \rangle_{cm})$$

$\forall \sigma, \sigma' \in S$. Then the Weyl function in Theorem 6.10 can be written thus:

$$(6.53) \quad \tilde{M}(z) = \tilde{R}(z_0) + \tilde{Q}(z).$$

To verify (6.53) it suffices to notice that the $Q$-function associated with the operator $L$ satisfies

$$\tilde{Q}_L(z) = \tilde{R}(z) - \tilde{R}(z_0).$$

Recall that the matrix $\tilde{R}(\cdot)$ defined in Corollary 6.6 is an analogue of the admissible matrix $R(\cdot)$ defined in Corollary 3.8.

In particular, under hypotheses of Corollary 6.11, the $Q$-function associated with the Gram matrix $G$ is given by $Q_G(z) = \langle b, G(Z - Z)^{-1} b \rangle_{cm}$, and so the Krein $Q$-function $\tilde{Q}$ is exactly the $Q$-function defined in [26, Eq. (6.10)]. There, various properties, including renormalization, of this $Q$-function are studied in great detail. In the next sections, however, we do not put ourselves into similar considerations in the case $d \geq 1$ and $p(L) \neq I$, but we rather concentrate on the analysis of the triplet extensions transformed to the original Hilbert space $\mathcal{H}_0$.

We close the paragraph by pointing out that the symmetric operator $A_{\text{min}}$ is simple, i.e. the closed linear span $\text{lin} \{\mathcal{N}_z(A_{\text{max}}) \mid z \in \text{res } A_0 \} = \mathcal{H}$. This follows from

$$(6.54) \quad \frac{\tilde{M}(z) - \tilde{M}(w)^*}{z - w} = \tilde{\gamma}(w)^* \tilde{\gamma}(z)$$

$\forall z, w \in \text{res } A_0$, and [22, Theorem 2.2]. Relation (6.54) is verified by computing the scalar product $\langle F_{\sigma}(w), F_{\sigma'}(z) \rangle_{\mathcal{H}} \forall \sigma, \sigma' \in S$ with the help of (5.2) and (6.22), and recalling that, for $G_Z$ Hermitian, $Z = Z \cap \mathbb{R}$ and the Gram matrix $G$ is diagonal in $j \in J$. It follows that $A_{\text{min}}$ has no eigenvalues (cf. Theorem 6.7-(i)).

7. **Triplet Adjoint in the Reference Hilbert Space**

In the present section we examine the connection between a densely defined, closed, symmetric operator $\hat{L}_0$ in $\mathcal{H}_0$ (see (3.4)) and a triplet extension restricted to an intermediate Hilbert space $\mathcal{H}$ (see (6.43)). We do not assume a priori that the matrix $G_Z$ (see (6.5)) is Hermitian, and so relations (6.48) and (6.49) do not necessarily hold in our considerations.
7.1. Adjoint operator $\hat{L}_0^*$. Let $\hat{L}_0^*$ be the adjoint of $\hat{L}_0$ in the Hilbert space $\mathfrak{H}_0$. Then $\hat{L}_0^* \supseteq L$ is defined by

\begin{equation}
\text{dom} \hat{L}_0^* = \mathfrak{H}_2 + \mathfrak{N}_z(\hat{L}_0^*),\quad \mathfrak{N}_z(\hat{L}_0^*) = \text{lin}\{\hat{g}_\sigma(z) | \sigma \in \mathcal{S}\}
\end{equation}

for $z \in \text{res} \ L$. One puts

\[ \hat{g}_\sigma(\cdot) := P(L_{-m})^{-1/2}g_\sigma(\cdot) \in \mathfrak{H}_0 \setminus \mathfrak{H}_1 \]

$\forall \sigma \in \mathcal{S}$; recall $g_\sigma(\cdot)$ in (3.6). The construction of $\hat{L}_0^*$ is due to von Neumann’s formula, and the eigenspace is found from Theorem 3.7 by applying Corollary 4.4-(ii).

Let us define the mapping $\hat{\Gamma} := (\hat{\Gamma}_0, \hat{\Gamma}_1): \text{dom} \hat{L}_0^* \rightarrow \mathbb{C}^d \times \mathbb{C}^d$ by $\hat{\Gamma} := \Gamma P(L)^{-1/2}$, where $\Gamma$ is defined in (3.8); i.e.

\begin{align}
(7.2a) & \quad \hat{\Gamma}_0 u := c = (c_\sigma) \in \mathbb{C}^d, \\
(7.2b) & \quad \hat{\Gamma}_1 u := (\hat{\varphi}^{\text{ex}}, u) := (\hat{\varphi}, u^\#) + R(z)c \in \mathbb{C}^d
\end{align}

for $u \in \text{dom} \hat{L}_0^*$ of the form

\begin{equation}
(7.3) \quad u = u^\# + \hat{g}_z(c), \quad u^\# \in \mathfrak{H}_2, \quad \hat{g}_z(c) := \sum_{\sigma \in \mathcal{S}} c_\sigma \hat{g}_\sigma(z) \in \mathfrak{N}_z(\hat{L}_0^*).
\end{equation}

Then the triple $(\mathbb{C}^d, \hat{\Gamma}_0, \hat{\Gamma}_1)$ is a boundary triple for $\hat{L}_0^*$. Without computing directly, this follows from Corollaries 3.8 and 4.4-(ii).

The $\gamma$-field $\hat{\gamma}(z)$ and the Weyl function $\hat{M}(z)$ associated with the boundary triple $(\mathbb{C}^d, \hat{\Gamma}_0, \hat{\Gamma}_1)$ are found from (3.12) by applying the scaling transformation in Corollary 4.4-(ii):

\begin{equation}
(7.4) \quad \hat{\gamma}(z) = \hat{g}_z(\cdot), \quad \hat{M}(z) = R(z)
\end{equation}

for $z \in \text{res} \ L$.

A proper extension $\hat{L}_\Theta$ of $\hat{L}_0$, $\hat{L}_\Theta := \{\hat{u} \in \hat{L}_0^* | \hat{\Gamma} \hat{u} \in \Theta\}$,

is parametrized by a linear relation $\Theta$ in $\mathbb{C}^d$; the adjoint $\hat{L}_\Theta^* = \hat{L}_\Theta^\circ$. The distinguished self-adjoint extension $\hat{L}_{(0)} \times \mathbb{C}^d$ coincides with $L$. The resolvent of $\hat{L}_\Theta$ is similar to (3.13).

7.2. Core for $\hat{L}_0^*$. Consider the operator $A_{\max}$ as described in Theorems 6.3 and 6.5. Let $\Theta$ be a linear relation in $\mathbb{C}^d$ and let $A_\Theta$ be the restriction of $A_{\max}$ defined by (6.43).

Let us define an operator $\hat{A}_\Theta$ in $\mathfrak{H}_0$ by

\begin{align}
(7.6a) & \quad \hat{A}_\Theta := P(L_{-m})^{-1/2}A_\Theta P(L_{-m})^{1/2} \\
(7.6b) & \quad \text{dom} \hat{A}_\Theta := \{u \in \mathfrak{H}_0 | P(L_{-m})^{1/2}u \in \text{dom} A_\Theta\}.
\end{align}

The main objective in this paragraph is to show that the closure of $\hat{A}_\Theta$ in $\mathfrak{H}_0$ is the adjoint operator $\hat{L}_0^*$ (Theorems 7.9 and 7.11). To achieve the goal we first derive some preparatory results.

**Proposition 7.1.** It holds

\begin{equation}
(7.7) \quad \text{dom} \hat{A}_\Theta = P(L_{-m})^{-1/2} \text{dom} A_\Theta.
\end{equation}

**Proof.** By (7.6b) $P(L_{-m})^{1/2} \text{dom} \hat{A}_\Theta \subseteq \text{dom} A_\Theta$. Since $P(L_{-m})^{1/2}: \mathfrak{H}_0 \rightarrow \mathfrak{H}_{-m}$ is bijective by Proposition 2.1, this shows the inclusion $\subseteq$ in (7.7). To show the reverse inclusion $\supseteq$, let $f \in \text{dom} A_\Theta$ and put $u := P(L_{-m})^{-1/2}f$. Then $\hat{A}_\Theta u = P(L_{-m})^{-1/2}A_\Theta f$; hence $u \in \text{dom} \hat{A}_\Theta$. □
Let us also put
\begin{equation}
\hat{A}_{\min} := P(L_m)^{-1/2}A_{\min}P(L_m)^{1/2}, \quad \hat{A}_{\max} := P(L_m)^{-1/2}A_{\max}P(L_m)^{1/2}.
\end{equation}
Note that the operator \(\hat{A}_{\max} = \hat{A}_{1 \times L \subseteq C^2}\); hence \(\hat{A}_{\Theta} \subseteq \hat{A}_{\max}\). The operator \(\hat{A}_{\min}\) is assumed to be defined on its natural domain, which is \(P(L_m)^{-1/2} \text{dom } A_{\min}\) by the above proof. For \(G_Z\) Hermitian, \(\hat{A}_{\min} = \hat{A}_{(0,0)}\) by Corollary 6.8-(ii), and in this case \(\hat{A}_{\min} \subseteq \hat{A}_{\Theta} \subseteq \hat{A}_{\max}\).

**Proposition 7.2.** \(\hat{A}_{\max} \subseteq \hat{L}_0^*\).

**Proof.** Consider \(A_{\max}\) as the operator with the exit space \(\mathcal{H}_{-m}\). Using \(A_{\max} \subseteq L_{\max}\), by (6.15), and applying Corollary 4.4-(iii), one deduces the inclusion as claimed. □

Although not needed for deriving our principal result, we find it instructive to remark on the difference between the adjoints of \(A_{\max}\) in the Hilbert spaces \(\mathcal{H}\) and \(\mathcal{H}_{-m}\).

Let us recall that the operator \(A_{\min}\) is defined as the adjoint \(A_{\max}^*\) in \(\mathcal{H}\); see (6.28). However, the adjoint in \(\mathcal{H}_{-m}\), which we denote by \(A_{\max}^*\), is different from \(A_{\max}^\ast\). The difference arises because of different scalar products in the Hilbert spaces \(\mathcal{H}\) and \(\mathcal{H}_{-m}\) and because the operator \(P(L_m)^{1/2}\) maps \(\mathcal{H}_{+m}\) to \(\mathcal{H}_{-m}\), and not to \(\mathcal{H}\). Thus the adjoint \(\hat{A}_{\max}^*\) in \(\mathcal{H}_0\) satisfies the relations
\begin{equation}
\hat{A}_{\max}^* = (A_{\max}P(L_m)^{1/2})^\ast P(L_m)^{-1/2} \supseteq P(L_m)^{-1/2}A_{\max}^\ast P(L_m)^{1/2}
\end{equation}
by applying Proposition 2.5 and definition (7.8); and the last operator on the right-hand side is different from \(\hat{A}_{\min}\). Moreover, by using Proposition 7.1, it is easy to show that the above inclusion is actually the equality.

**Proposition 7.3.** Let \(A_{\max}^\ast\) be the adjoint of \(A_{\max}\) in \(\mathcal{H}_{-m}\). Then
\begin{equation}
A_{\max}^\ast = P(L_m)^{1/2}\hat{A}_{\max}^* P(L_m)^{-1/2}, \quad \text{dom } A_{\max}^\ast = P(L_m)^{1/2} \text{dom } \hat{A}_{\max}^*.
\end{equation}

**Proof.** By definition, the adjoint linear relation \(A_{\max}^\ast\) is the set of \((f, f')\in \mathcal{H}_{-m} \times \mathcal{H}_{-m}\) such that \((\text{cf. (6.33)})\)
\begin{equation}
\langle g, f' \rangle_{-m} = \langle A_{\max}g, f \rangle_{-m}
\end{equation}
\(\forall g \in \text{dom } A_{\max}\). By applying Proposition 7.1 the latter yields (7.9). □

Taking the adjoints in \(\mathcal{H}_0\) in Proposition 7.2 one gets that \(\hat{A}_{\max}^\ast \supseteq \hat{L}_0^*\). Then by Proposition 7.3 and Corollary 4.4-(iv) one gets that \(A_{\max}^\ast \supseteq L_{\max}^\ast\) (but not \(A_{\max}^\ast \supseteq L_{\max}^\ast\)). By Theorem 7.9 it actually holds \(\hat{A}_{\max}^\ast = \hat{L}_0^*\), and so \(A_{\max}^\ast = L_{\max}^\ast\) (Corollary 7.10).

The sets \(\hat{\mathcal{R}}'\) and \(\hat{\mathcal{R}}\). Now we switch back to the analysis of \(\hat{A}_{\Theta}\), and in particular \(\hat{A}_{\max}\). Let \(m \in \mathbb{N}\) and let \(\hat{\mathcal{R}}'\) and \(\hat{\mathcal{R}}\) denote the sets
\begin{equation}
\hat{\mathcal{R}}' := P(L_m)^{-1/2}p(L_m) \text{dom } L_{\min}^*, \quad \hat{\mathcal{R}} := P(L_m)^{-1/2}\mathcal{R}.
\end{equation}
Then by Theorem 6.5-(ii) and (7.8), \(\text{dom } \hat{A}_{\max} = \hat{\mathcal{R}}' + \hat{\mathcal{R}}\); see also Lemma 7.6.

Let us study the above sets in more detail. By Proposition 2.1 and Theorem 3.7, the set \(\hat{\mathcal{R}}'\) satisfies (for \(m > 0\))
\begin{equation}
\mathcal{H}_{2m+2} \subseteq \hat{\mathcal{R}}' \subseteq \mathcal{H}_{2m} \subseteq \mathcal{H}_2
\end{equation}
so that \(\hat{\mathcal{R}}'\) is dense in \(\mathcal{H}_0\). An element \(u' \in \hat{\mathcal{R}}'\) is of the form
\begin{equation}
u' = P(L_m)^{-1/2}p(L_m)f, \quad f' \in \text{dom } L_{\min}^*,
\end{equation}
Using (7.12) and a representation of \(f' \in \text{dom } L_{\min}^*\) in (3.7), one writes \(u' \in \hat{\mathcal{R}}'\) in the form \(u' = u'(c)\), where
\begin{equation}
u'(c) := f^2 + P(L_m)^{-1/2}\tilde{G}_z(c) = f^2 + P(L)^{-1}\tilde{g}_z(c)
\end{equation}
for \( f^z \in \mathcal{H}_{2m+2} \) and \( c \in \mathbb{C}^d \) and \( z \in \text{res } L \).

By (5.1b) and (7.10), the set \( \hat{\mathcal{R}} \) is a closed linear span
\[
(7.14) \quad \hat{\mathcal{R}} = \text{lin}\{\hat{g}_\alpha := \hat{g}_\alpha(z_j) | \alpha = (\sigma, j) \in \mathcal{S} \times J\}.
\]
An element \( \hat{k} \in \hat{\mathcal{R}} \) is in one-to-one correspondence with an element (cf. (5.3))
\[
(7.15) \quad d(\hat{k}) = (d_\alpha(\hat{k})) \in \mathbb{C}^{md}, \quad d_\alpha(\hat{k}) := [\mathcal{G}^{-1}\langle \hat{\mathcal{G}}_{\hat{k}} \rangle_0]_\alpha \quad (\forall \alpha \in \mathcal{S} \times J)
\]
as discussed in Section 5; here one puts \( \langle \hat{g}, \cdot \rangle_0 := (\langle \hat{g}_\alpha, \cdot \rangle_0) : \mathcal{H}_0 \to \mathbb{C}^{md} \). Note that \( d(\hat{k}) \equiv d(k) \) for \( \hat{k} = P(L^{-1/2} k), k \in \mathcal{R} \).

**Lemma 7.4.** \( \hat{\mathcal{R}}_{\text{min}} \subseteq \hat{\mathcal{R}} \subseteq \mathcal{H}_0 \), where
\[
\hat{\mathcal{R}}_{\text{min}} := (\hat{\mathcal{R}} \cap \mathcal{H}_{2m-2}) \setminus \mathcal{H}_{2m-1} = \text{lin}\{\hat{P}(L^{-1} \phi_\sigma | \sigma \in \mathcal{S}\}.
\]

**Proof.** By using (3.2) and (7.14), this follows from Proposition 2.1 and Lemma 5.1. \(\square\)

**Corollary 7.5.** \( \hat{\mathcal{R}} \cap \mathcal{H}_{2m-1} = \{0\} \). \(\square\)

Note that \( \hat{\mathcal{R}}_{\text{min}} = \hat{\mathcal{R}} \) for \( m = 1 \). Since \( \mathcal{H}_{2m-2} \subseteq \mathcal{H}_2 \) for \( m \geq 2 \), it follows in particular that the intersection \( \hat{\mathcal{R}} \cap \mathcal{H}_2 \) is nontrivial unless \( m = 1 \).

By using (5.7), an element \( \hat{k}_{\text{min}} \in \hat{\mathcal{R}}_{\text{min}} \) is represented by \( \hat{k}_{\text{min}} = \hat{k}_{\text{min}}(c) \), where
\[
(7.16a) \quad \hat{k}_{\text{min}}(c) := \sum_{\sigma \in \mathcal{S}} c_\sigma \hat{P}(L^{-1} \phi_\sigma
\]
\[
(7.16b) \quad = \sum_{\alpha \in \mathcal{S} \times J} [\mathcal{G}^{-1} \mathcal{G}_c]_\alpha \hat{g}_\alpha
\]
for \( c = (c_\sigma) \in \mathbb{C}^d \). With this notation, \( u'(c) \) in (7.13) admits a representation
\[
(7.17) \quad u'(c) = f^z + (L_{2m-2} - z)^{-1} \hat{k}_{\text{min}}(c).
\]
We are now in a position to prove the following lemma.

**Lemma 7.6.** The operator \( \hat{A}_{\text{max}} \) is represented by
\[
(7.18) \quad \text{dom } \hat{A}_{\text{max}} = \hat{\mathcal{R}}' + \hat{\mathcal{R}}, \quad \hat{A}_{\text{max}}(u' + \hat{k}) = L_{2m-2} u' + \sum_{\alpha \in \mathcal{S} \times J} [Z_d(\hat{k})]_\alpha \hat{g}_\alpha
\]
\( \forall u' + \hat{k} \in \hat{\mathcal{R}}' + \hat{\mathcal{R}}. \)

**Proof.** As already pointed out above, the domain is due to Theorem 6.5-(ii), (7.8), and (7.10). The sets \( \hat{\mathcal{R}}' \) and \( \hat{\mathcal{R}} \) are disjoint because of (7.11) and Corollary 7.5.

Let \( f \in \text{dom } A_{\text{max}} \). By Theorem 6.5-(ii), \( f = p(L_m) f^z + k \), where \( k \in \mathcal{R} \) and \( f' \in \text{dom } L_{\text{min}}^* \) is of the form \( f' = f^\# + L_{2m-2} c \) for some \( f^\# \in \mathcal{H}_{m+2} \) and \( c \in \mathbb{C}^d \). Then by Theorem 6.5-(i), (6.1), and (7.16)
\[
P(L^{-1} A_{\text{max}} f = P(L^{-1} A_0(p(L_{m+2}) f^\# + k) + z P(L^{-1} G_{z}(c) + \hat{k}_{\text{min}}(c)
\]
\[
= P(L^{-1} L_m p(L_{m+2}) f^\# + \sum_{\alpha \in \mathcal{S} \times J} [Z_d(\hat{k})]_\alpha \hat{g}_\alpha
\]
\[+ L_{2m-2} (L_{2m-2} - z)^{-1} \hat{k}_{\text{min}}(c).
\]
By applying Corollary 2.3-(i)
\[
P(L_{m})^{-1/2} L_m p(L_{m+2}) f^\# = L_{2m} f^z, \quad f^z := P(L_{m+2})^{-1/2} p(L_{m+2}) f^\# \in \mathcal{H}_{m+2}
\]
and one deduces (7.18) from the latter and (7.17).

**Corollary 7.7.** \( \hat{\mathcal{R}} + \mathcal{R} \subseteq \text{dom } \hat{L}_0. \)

**Proof.** This follows from Proposition 7.2 and Lemma 7.6.

It is informative to verify Proposition 7.2 by using Lemma 7.6. An element

\[(7.19) \quad u(c) := u'(c) + \hat{k} \in \hat{\mathcal{R}} + \mathcal{R}, \quad c \in \mathbb{C}^d\]

can be written thus

\[
(7.20) \quad u(c) = u^\#(c) + \hat{g}_x(c(\hat{k}))
\]

with \( \alpha = (\sigma, j) \in \mathcal{S} \times J \) and \( z \in \text{res } L; \ i.e. \)

where

\[(7.21a) \quad u^\#(c) := u'(c) + \sum_{\alpha \in \mathcal{S} \times J} d_\alpha(\hat{k})(\hat{g}_\alpha - \hat{g}_x(z)) \in \mathcal{F}_2, \]

\[(7.21b) \quad c(\hat{k}) = (c_\sigma(\hat{k})) \in \mathbb{C}^d, \quad c_\sigma(\hat{k}) := \sum_{j \in J} d_{\sigma j}(\hat{k})\]

and \( \hat{g}_x(c) \in \mathcal{M}_2(\hat{L}_0^*) \) is defined in (7.3). Notice that \( u^\#(c) \in \mathcal{F}_2 \) because \( u'(c) \in \mathcal{F}_2 \) by (7.11) and because

\[(7.22) \quad \hat{g}_\alpha - \hat{g}_x(z) = (z_j - z)(L - z)^{-1}\hat{g}_\alpha \in \mathcal{F}_2. \]

Thus \( u(c) \in \mathcal{F}_2 + \mathcal{M}_2(\hat{L}_0^*) = \text{dom } \hat{L}_0^*, \) and this shows Corollary 7.7. Note also that \( c(\hat{k}) \equiv c(k) \) for \( \hat{k} = P(L_{-m})^{-1/2}k, \) \( k \in \mathcal{R}, \) in view of (6.7), (7.15), and (7.21b).

Now, by Lemma 7.6 and (7.19), (7.20), (7.21), and (7.22)

\[
\hat{A}_{\text{max}}(u'(c) + \hat{k}) = Lu^\#(c) + \sum_{\alpha} d_\alpha(\hat{k})[z_j\hat{g}_\alpha - L(\hat{g}_\alpha - \hat{g}_x(z))]
\]

\[
= Lu^\#(c) + z\hat{g}_x(c(\hat{k})) \equiv \hat{L}_0^*[u^\#(c) + \hat{g}_x(c(\hat{k}))]
\]

and hence the inclusion \( \hat{A}_{\text{max}} \subseteq \hat{L}_0^* \) follows.

**Orthogonal projection.** Let \( P \) be the orthogonal (hence self-adjoint) projection onto \( \hat{\mathcal{R}} \) in \( \mathcal{F}_0, \) and let \( P' := I - P. \) Then we have

\[(7.23) \quad \hat{\mathcal{R}} = PS_0 \quad \text{and} \quad \hat{\mathcal{R}}^\perp := \mathcal{F}_0 \ominus \hat{\mathcal{R}} = P'\mathcal{F}_0. \]

Thus an arbitrary \( u \in \mathcal{F}_0 \) can be uniquely expressed as a sum \( u = u_\perp + \hat{k}_u, \) with \( u_\perp := P'u \in \hat{\mathcal{R}}^\perp \) and \( \hat{k}_u := Pu \in \hat{\mathcal{R}}. \) Similar to (7.15), an element \( \hat{k}_u \in \hat{\mathcal{R}} \) is in bijective correspondence with an element \( d(\hat{k}_u) = G^{-1}(\hat{g}, u)_0 \in \mathbb{C}^{m+d}. \)

**Lemma 7.8.** Define the matrix

\[(7.24) \quad \mathcal{X} = (\mathcal{X}_{\alpha \sigma'}) \in [\mathbb{C}^d, \mathbb{C}^{m+d}], \quad \mathcal{X}_{\alpha \sigma'} := \sum_{j' \in J} [G^{-1}]_{\alpha, \sigma j'} \quad \forall \alpha \in \mathcal{S} \times J \forall \sigma' \in \mathcal{S}. \text{ Then:}
\]

(i) \( \ker \mathcal{X} = \{0\}. \)
Proof. (i) Let $\chi = (\chi_\alpha) \in \mathbb{C}^d$ and let $\hat{\chi} = (\hat{\chi}_\alpha) \in \mathbb{C}^{md}$, where $\hat{\chi}_\alpha := \chi_\alpha$ for $\alpha = (\sigma, j) \in \mathcal{S} \times J$. Then by (7.24) $\mathcal{X}^\dagger \chi = \mathcal{G}^\dagger \hat{\chi}$. Then $\chi \in \ker \mathcal{X}$ implies that $\hat{\chi} = 0 \Rightarrow \chi = 0$.

(ii) By definition $d(PLu) = \mathcal{G}^\dagger (\hat{g}, Lu)_0$. But $(\forall \alpha = (\sigma, j) \in \mathcal{S} \times J)$ $$(\langle \hat{g}_\alpha, Lu \rangle_0 = \langle \hat{\varphi}_\sigma, (L - \bar{z}_j)^{-1}Lu \rangle = \langle \hat{\varphi}_\sigma, u + \bar{z}_j(L - \bar{z}_j)^{-1}u \rangle = \langle \hat{\varphi}_\sigma, u \rangle + \bar{z}_j \langle \hat{g}_\alpha, u \rangle_0$$ and so the claimed formula follows from the latter by using (6.5), (7.24).

(iii) Using (6.11) one has $(\forall \alpha = (\sigma, j) \in \mathcal{S} \times J)$ $$[\mathcal{M}^\ast c(\hat{k}) + (\mathcal{G}_Z - \mathcal{G}_Z^\ast)\hat{d}(\hat{k})]_\alpha = \sum_{\alpha'}([\mathcal{M}^\ast]_{\alpha\alpha'} + [\mathcal{G}_Z - \mathcal{G}_Z^\ast]_{\alpha\alpha'})d_{\alpha'}(\hat{k})$$ $$= \sum_{\alpha'}(R_{\sigma\sigma}(z_j) + [R(z_{j'}) - R(\bar{z}_j)]_{\sigma\alpha'})d_{\alpha'}(\hat{k}) = \sum_{\alpha'}R_{\sigma\alpha'}(z_{j'})d_{\alpha'}(\hat{k}) = [\mathcal{M}d(\hat{k})]_\alpha$$ (with $\alpha' = (\sigma', j') \in \mathcal{S} \times J$). But $[\mathcal{M}d(\hat{k})]_\sigma = [\mathcal{G}\mathcal{X}\mathcal{M}d(\hat{k})]_\alpha$, and the desired formula follows. \hfill \Box

Theorem 7.9. $\widehat{A}_{\max}^* = \widehat{L}_0$.

Proof. The adjoint linear relation $\widehat{A}_{\max}^*$ is the set of $(y, x) \in \mathcal{H}_0 \times \mathcal{H}_0$ such that $$\langle u, x \rangle_0 = \langle \widehat{A}_{\max} u, y \rangle_0$$ $\forall u \in \text{dom} \widehat{A}_{\max}$. Thus, for $u = u' + \hat{k} \in \mathcal{X} \hat{\mathcal{R}} + \mathcal{H}$, it follows from Lemma 7.6 that $$0 = \langle u, x \rangle_0 - \langle \widehat{A}_{\max} u, y \rangle_0$$ (7.25) $$= \langle u', x - L_2y \rangle + \langle d(\hat{k}), \mathcal{G}d(\hat{k}_x) - \mathcal{G}_Z^\ast d(\hat{k}_y) \rangle_{\mathbb{C}^{md}}$$ where $\hat{k}_x := Px \in \mathcal{H}$, $\hat{k}_y := Py \in \mathcal{H}$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{H}_2$ and $\mathcal{H}_{-2}$. Since $\mathcal{X} \hat{\mathcal{R}} \cap \mathcal{H} = \{0\}$ and $x \in \mathcal{H}_0$, one concludes that $y \in \mathcal{H}_2$, $x = Ly$, and $d(\hat{k}_x) = \mathcal{G}^\dagger \mathcal{G}_Z^\ast d(\hat{k}_y)$. But $x = Ly$ also implies that $$d(\hat{k}_x) = \mathcal{G}^\dagger (\hat{g}, Ly)_0 = \mathcal{X} \langle \hat{\varphi}, y \rangle + \mathcal{G}^\dagger \mathcal{G}_Z^\ast d(\hat{k}_y)$$ by Lemma 7.8-(ii). Thus $\langle \hat{\varphi}, y \rangle = 0$ by Lemma 7.8-(i). \hfill \Box

Corollary 7.10. Let $A_{\max}^*$ be as in Proposition 7.3. Then $A_{\max}^* = L_{\max}^*$.

Proof. This follows from Corollary 4.3-(iv) and Theorem 7.9. \hfill \Box

A general case. Now we generalize Theorem 7.9 for an arbitrary operator $\widehat{A}_\Theta \subseteq \widehat{A}_{\max}$ in (7.6), (7.7). In analogy to (6.43) we define $\widehat{A}_\Theta$ by

(7.26) $$\widehat{A}_\Theta = \{ \hat{u} \in \widehat{A}_{\max} \mid \tilde{\hat{\varphi}} \hat{u} \in \Theta \} , \quad \tilde{\hat{\varphi}} := \tilde{\hat{\varphi}} P(L_m)^{1/2}.$$ Let $\tilde{\hat{\varphi}} := (\tilde{\hat{\varphi}}_0, \tilde{\hat{\varphi}}_1)$; then

(7.27a) $$\tilde{\hat{\varphi}}_0 := \tilde{\hat{\varphi}}_0 P(L_m)^{1/2} : \text{dom} \widehat{A}_{\max} \ni u'(c) + \hat{k}$$
The following equivalence relation holds:

\[ \text{Lemma 7.12.} \quad \text{For } u, \text{ hence } r, \text{ where } c \text{ follows from Theorem (7.28)} \]

By comparing (7.31) with (7.30) one finds that \((\exists (r, s) \in \Theta^*)\)

\[ x - L_{-2}y = P(L_{-2m-2})^2 \tilde{k}_{\min}(r), \]

\[ \langle \tilde{P}(L_{-2})^{-1} \tilde{\varphi}, (L_{-2} - \overline{z})^{-1}(x - L_{-2}y) \rangle_0 = \tilde{R}(z)^* r - s, \]

\[ G\tilde{k}_y - Gd(\tilde{k}_x) = G_\theta r. \]

Using \( x \in \mathcal{H}_0, x - L_{-2}y \in \mathcal{H}_{-2} \subseteq \mathcal{H}_{-2m-1}, \) and (7.30), one gets \( \tilde{k}_{\min}(r) = 0 \Rightarrow G_\theta r = 0 \) and hence \( r = 0 \) by Proposition 5.5. Then also \( s = 0 \) and so \( \hat{A}_\Theta = \hat{A}_{\max}^* \). The final conclusion now follows from Theorem 7.9.

7.3. A non-standard representation of \( \hat{L}_0^* \). In this paragraph we show that the adjoint operator \( \hat{L}_0^* \) can be written in terms of model parameters (i.e. the points \( z_j \in \text{res } L \) for \( 1 \leq j \leq m \)) independently of \( m > 0 \).

\[ \text{Lemma 7.12.} \quad \text{The following equivalence relation holds:} \]

\[ \hat{R} \cap \mathcal{H}_2 \ni \hat{k} \Leftrightarrow c(\hat{k}) = 0 \]

where \( c(\hat{k}) \) is defined in (7.21b).
Proof. Using $L_{-2}\hat{g}_\alpha = z_j\hat{g}_\alpha + \hat{\varphi}_\sigma \forall \alpha = (\sigma, j) \in S \times J$, one has $(\forall u \in \mathfrak{H}_0)$

$$L_{-2}u = L_{-2}u_\perp + \sum_{\alpha \in S \times J} [Zd(\hat{k}_u)]\hat{\alpha}\hat{g}_\alpha + \sum_{\sigma \in S} c_\sigma\hat{k}_u\hat{\varphi}_\sigma$$

where $u_\perp := P'u \in \hat{\mathfrak{H}}_\perp$ and $\hat{k}_u := Pu \in \hat{\mathfrak{H}}$, and where the projections are as in (7.23).

Now let $u \in \mathfrak{H}_2$. Then $u_\perp \in \hat{\mathfrak{H}}_\perp \cap \mathfrak{H}_2$ and $\hat{k}_u \in \hat{\mathfrak{H}} \cap \mathfrak{H}_2$; For $m = 1$, $\hat{k}_u = 0$ by Corollary 7.5; for $m > 1$, since $L_{-2}u = Lu \in \mathfrak{H}_0$ and $L_{-2}u_\perp = Lu_\perp \in \mathfrak{H}_0$ and $\hat{\varphi}_\sigma \in \mathfrak{H}_{-2} \cap \mathfrak{H}_{-1}$, it follows from (7.32) that $c(\hat{k}_u) = 0$.

Conversely, assume that $c(\hat{k}) = 0$ for some $\hat{k} \in \hat{\mathfrak{H}}$ and $m > 1$ (for $m = 1$, one has trivially $\hat{k} = 0$). Then

$$d_{\alpha m}(\hat{k}) = -\sum_{j=1}^{m-1} d_{\alpha j}(\hat{k})$$

$\forall \sigma \in S$, and whence $\hat{k}$ is given by

$$\hat{k} = \sum_{\sigma \in S} \sum_{j=1}^{m-1} d_{\sigma j}(\hat{k})(\hat{g}_{\alpha j} - \hat{g}_{\alpha m})$$

and is the sum of $\mathfrak{H}_2$-functions, since $\hat{g}_{\alpha j} - \hat{g}_{\alpha m} \in \mathfrak{H}_2$ $\forall (\sigma, j) \in S \times J$. Thus $\hat{k} \in \hat{\mathfrak{H}} \cap \mathfrak{H}_2$. \(\square\)

**Lemma 7.13.** Consider the operator $\hat{\tau}$ in $\mathfrak{H}_0$ defined by

$$\hat{\tau} := \{u \in \mathfrak{H}_0 \mid u_\perp \in \mathfrak{H}_0^\perp := \hat{\mathfrak{H}}_\perp \cap \mathfrak{H}_2\} ,$$

$\hat{\tau}u := Lu_\perp + \sum_{\alpha \in S \times J} [Zd(\hat{k}_u)]\hat{\alpha}\hat{g}_\alpha$, $u \in \text{dom} \hat{\tau}$

where $u_\perp := P'u \in \mathfrak{H}_0^\perp$ and $\hat{k}_u := Pu \in \hat{\mathfrak{H}}$. Then $\hat{\tau} \subseteq \hat{L}_0$.

Proof. An element $u \in \text{dom} \hat{\tau}$ is of the form $u = u^\# + \hat{g}_z(c(\hat{k}_u))$, where $\hat{g}_z(\cdot)$ is defined in (7.3) and

$$u^\# := u_\perp + \sum_{\alpha = (\sigma, j) \in S \times J} d_{\alpha}(\hat{k}_u)(\hat{g}_\alpha - \hat{g}_\sigma(z)) \in \mathfrak{H}_2$$

(see (7.22)) with $u_\perp := P'u \in \mathfrak{H}_2^\perp$, $z \in \text{res} L$. Thus $u \in \mathfrak{H}_2 + \mathfrak{H}_z(\hat{L}_0) = \text{dom} \hat{L}_0$.

Next, using the above representation of $u \in \text{dom} \hat{\tau}$ one has by (7.22) and (7.33)

$$\hat{\tau}u = Lu^\# + \sum_{\alpha} d_{\alpha}(\hat{k}_u)[z_j\hat{g}_\alpha - L(\hat{g}_\alpha - \hat{g}_\sigma(z))]$$

$$= Lu^\# + z\hat{g}_z(c(\hat{k}_u)) = \hat{L}_0^*[u^\# + \hat{g}_z(c(\hat{k}_u))].$$

\(\square\)

**Lemma 7.14.** The boundary form of the operator $\hat{\tau}$ satisfies an abstract Green identity

$$[u, v]_\hat{\tau} = [\hat{\tau}_\hat{u}, \hat{\tau}_v]_{C^\alpha \times C^\alpha}$$

$\forall u, v \in \text{dom} \hat{\tau}$, where the mapping $\hat{\tau} \subseteq \hat{\Gamma}$ is defined by $\hat{\tau} := (\hat{\Gamma}_0, \hat{\Gamma}_1)$ with

$$\hat{\tau}_0 := c(\hat{k}_u) ,$$

$$\hat{\tau}_1 := \langle \hat{\varphi}, u_\perp \rangle + \mathcal{M}d(\hat{k}_u)$$

for $u \in \text{dom} \hat{\tau}$; $u_\perp := P'u \in \mathfrak{H}_0^\perp$, $\hat{k}_u := Pu \in \hat{\mathfrak{H}}$. The matrix $\mathcal{M}$ is as in (6.6).

Proof. This follows from (7.2), Lemma 7.13, as well as the representation of $u \in \text{dom} \hat{\tau}$ given in the proof of Lemma 7.13. \(\square\)
**Theorem 7.15.** \( \hat{\tau} = \hat{L}_0^* \).

**Proof.** Let \( \Theta \) be a linear relation in \( \mathbb{C}^d \) and let \( \hat{\tau}_\Theta \subseteq \hat{\tau} \) be the restriction of \( \hat{\tau} \) to the domain \( \{ u \in \text{dom} \hat{\tau} \mid \hat{\tau}_u \in \Theta \} \). Clearly \( \hat{\tau} = \hat{\tau}_\Theta \times \mathbb{C}^d \). Also, \( \hat{L}_0 = \hat{\tau}_{\hat{\tau}(0,0)} \) because in this case \( u \in \text{dom} \hat{\tau}_{(0,0)} \) satisfies \( u \in \text{dom} \hat{\tau} \) and \( \hat{\tau} u = \{(0,0)\} \): The boundary condition \( \hat{\tau}^* u = 0 \) implies that \( u \in \mathcal{H}_2 \) by Lemma 7.12, and then \( \hat{\tau} u = Lu \) by Lemma 7.13. The boundary condition \( \hat{\tau}^* u = \langle \hat{\tau}, u \rangle = 0 \) then shows that \( u \in \text{dom} \hat{L}_0 \).

It remains to prove that the adjoint \( \hat{\tau}_\Theta = \hat{\tau}_{\Theta^*} \), and then to apply this to \( \Theta = \{(0,0)\} \).

The adjoint linear relation \( \hat{\tau}_\Theta^* \) consists of \( (y, x) \in \mathcal{H}_0 \times \mathcal{H}_0 \) such that

\[
\langle u, x \rangle_0 = \langle \hat{\tau}_\Theta u, y \rangle_0
\]

\( \forall u \in \text{dom} \hat{\tau}_\Theta \). An element \( u \in \text{dom} \hat{\tau}_\Theta \) belongs to \( \text{dom} \hat{\tau} \) (\( \subseteq \text{dom} \hat{L}_0^* \)) and satisfies the boundary condition \( \hat{\tau}^* u = \hat{\tau} u \in \Theta \); see Lemmas 7.13 and 7.14. The boundary condition shows that \( (\exists (r, s) \in \Theta^*) \)

\[
\langle \hat{\tau}_0 u, s \rangle_{\mathbb{C}^d} = \langle \hat{\tau}_1 u, r \rangle_{\mathbb{C}^d}.
\]

For \( u \) as in the proof of Lemma 7.13 (i.e. \( \hat{\tau} \) is of the form (7.3), where \( c = c(k_u) \)), the above condition reads

\[
0 = \langle u^\#, \hat{\omega} r \rangle + \langle c, R(z)^* r - s \rangle_{\mathbb{C}^d}
\]

for \( c = c(k_u) \), where \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( \mathcal{H}_2 \) and \( \mathcal{H}_{-2} \), and

\[
\hat{\omega} r := \sum_{\sigma \in S} r_\sigma \hat{\phi}_\sigma \in \mathcal{H}_{-2} \ominus \mathcal{H}_{-1}, \quad r = (r_\sigma) \in \mathbb{C}^d.
\]

On the other hand, since \( \hat{\tau}_\Theta \subseteq \hat{L}_0^* \), one has that

\[
0 = \langle u, x \rangle_0 - \langle \hat{\tau}_\Theta u, y \rangle_0 = \langle u, x \rangle_0 - \langle \hat{L}_0^* u, y \rangle_0
\]

\( \langle u^\#, x - L_{-2} y \rangle + \langle c, (\hat{g}(z), x - \bar{z} y) \rangle_{\mathbb{C}^d} \)

with vector notation \( \langle \hat{g}(z), \cdot \rangle_0 := \langle (\hat{g}_\sigma(z), \cdot) \rangle : \mathcal{H}_0 \to \mathbb{C}^d \). By comparing (7.35) with (7.37) one deduces that \( (\exists (r, s) \in \Theta^*) \)

\[
x = L_{-2} y + \hat{\omega} r, \quad \langle \hat{g}(z), x - \bar{z} y \rangle_0 = R(z)^* r - s.
\]

Using the orthogonal decomposition \( y = y_+ + \hat{k}_y \in \hat{\mathbb{R}}^I \oplus \hat{\mathbb{R}} \) and applying (7.32), the first equation yields \( y_\perp \in \mathcal{H}_2^\perp \) (i.e. \( y \in \text{dom} \hat{\tau} \)) and \( x = \hat{\tau} y \) and \( \hat{\tau}_0 y + r = 0 \). Then the second equation yields \( \hat{\tau}^* y = -s \), and so \( y \) satisfies the boundary condition \( \hat{\tau}^* y \in \Theta^* \).

Another reformulation of Theorem 7.15 is that (for \( m > 0 \))

\[
\hat{L}_0^* = LP' + \sum_{\alpha \in S \times J} [Z_0 G^{-1} (\hat{g}, P \cdot)_{0,\alpha} \hat{g}_\alpha], \quad \text{dom} \hat{L}_0^* = \{ u \in \mathcal{H}_0 \mid P'u \in \mathcal{H}_2^\perp \}.
\]

We use the theorem when transferring from the triplet extensions back to the classical extensions (Theorems 7.17 and 7.18).

**Corollary 7.16.** \( \hat{\tau}^* = \hat{\tau} \).

**Proof.** Clear from Lemma 7.14 and Theorem 7.15.
7.4. **Triplet extension vs classical extension.** In this paragraph we provide some generalizations of Theorems 7.9 and 7.11. Let $\Theta$ be a linear relation in $\mathbb{C}^d$ and put
\begin{equation}
\hat{\Gamma}^{\Theta} := \hat{\Gamma} \big|_{\text{dom} \, \hat{A}_\Theta}
\end{equation}
The definition is correct in view of Proposition 7.2. Fix another linear relation $\Theta_0$ in $\mathbb{C}^d$ and define the restriction $\widehat{S}_{\Theta, \Theta_0} \subseteq \hat{A}_\Theta$ as follows:
\begin{equation}
\widehat{S}_{\Theta, \Theta_0} := \{ \hat{u} \in \hat{A}_\Theta \mid \hat{\Gamma}^{\Theta} \hat{u} \in \Theta_0 \}.
\end{equation}
One has
\begin{equation}
\widehat{S}_{\Theta, \Theta_0} = \{ \hat{u} \in \widehat{S}_{\Theta_0} \mid \hat{\Gamma}^{\Theta} \hat{u} \in \Theta \},
\end{equation}
in analogy to (7.26). Since $\widehat{S}_{\Theta_0} \subseteq \hat{A}_\Theta$, it follows from (7.5), (7.28), and (7.40) that
\begin{equation}
\widehat{S}_{\Theta, \Theta_0} = \hat{L}_0 \big|_{\text{dom} \, \widehat{S}_{\Theta_0, \Theta_0}} = \hat{L}_{\Theta_0} \big|_{\text{dom} \, \hat{L}_{\Theta_0} \cap \text{dom} \, \hat{A}_\Theta}
\end{equation}
i.e.
\begin{equation}
\widehat{S}_{\Theta_0} \subseteq \hat{L}_{\Theta_0}.
\end{equation}
Thus an operator $\widehat{S}_{\Theta_0}$ is a restriction of a proper extension $\hat{L}_{\Theta_0}$ of $\hat{L}_0$. Let $\widehat{S}_{\Theta, \Theta_0}^*$ (resp. $\widehat{S}_{\Theta_0}^*$) denote the adjoint in $\mathfrak{S}_0$.

**Theorem 7.17.** $\widehat{S}_{\Theta_0}^* = \hat{L}_{\Theta_0}$ for a linear relation $\Theta_0$ in $\mathbb{C}^d$.

**Proof.** The adjoint linear relation $\widehat{S}_{\Theta_0}^*$ consists of $(y, x) \in \mathfrak{S}_0 \times \mathfrak{S}_0$ such that
\begin{equation}
(u, x)_0 = (\widehat{S}_{\Theta_0}^* u, y)_0
\end{equation}
$\forall u \in \text{dom} \, \widehat{S}_{\Theta_0}$. By (7.39) and (7.40) an element $u \in \text{dom} \, \widehat{S}_{\Theta_0}$ belongs to $\text{dom} \, \hat{A}_{\text{max}}$ (\subseteq $\text{dom} \, \hat{L}_0^\ast$) and satisfies the boundary condition $\hat{\Gamma}^{\Theta} u = \hat{\Gamma} u \in \Theta_0$. For $u = u' + \hat{k} \in \mathring{\mathbb{R}} + \mathring{\mathbb{K}} (= \text{dom} \, \hat{A}_{\text{max}})$, the latter can be written in the form (7.35), with $u^\#$ as in (7.21a), $c = c(\hat{k})$ (this $c$ is different from that in (7.21a)), and some $(r, s) \in \Theta_0^*$. On the other hand, the inclusion in (7.41b) shows that equation (7.42) is written in the form (7.37). Then, by arguing as in the proof of Theorem 7.15 one finds that $x = \hat{\tau} y$, $y \in \text{dom} \, \hat{\tau}$, and $\hat{\Gamma}^r y \in \Theta_0^*$. Since $\hat{\tau} = \hat{L}_0^* \lor \hat{\Gamma}^r = \hat{\Gamma}$ (Theorem 7.15 and Corollary 7.16), the claim follows.

It follows from Theorem 7.17 that, for a (closed) linear relation $\Theta_0$, the closure of $\widehat{S}_{\Theta_0}$ is an operator $\hat{L}_{\Theta_0}$. By putting $\Theta_0 = \mathbb{C}^d \times \mathbb{C}^d$ (i.e. $\Theta_0^* = \{ (0, 0) \}$) in the theorem, one recovers Theorem 7.9. The result is now generalized for an operator $\widehat{S}_{\Theta, \Theta_0}$.

**Theorem 7.18.** $\widehat{S}_{\Theta, \Theta_0}^* = \hat{L}_{\Theta_0}^*$ for the linear relations $\Theta$, $\Theta_0$ in $\mathbb{C}^d$.

**Proof.** It suffices to show that $\widehat{S}_{\Theta, \Theta_0}^* = \widehat{S}_{\Theta_0}^*$; then the conclusion follows from Theorem 7.17. But the equality is seen by substituting $u^\# = u^\#(c)$ from (7.21a) in (7.37), where $c = c(\hat{k})$ (this $c$ is different from that in (7.21a)), and then arguing as in the proof of Theorem 7.11.

By putting $\Theta_0 = \mathbb{C}^d \times \mathbb{C}^d$ in the theorem, one recovers Theorem 7.11.

**Corollary 7.19.** For the (closed) linear relations $\Theta$, $\Theta_0$ in $\mathbb{C}^d$
\begin{equation}
\Theta_0 \subseteq \Theta_0^* \Leftrightarrow \widehat{S}_{\Theta, \Theta_0} \subseteq \widehat{S}_{\Theta_0}^*.
\end{equation}
Moreover, $\widehat{S}_{\Theta, \Theta_0}$ is essentially self-adjoint in $\mathfrak{S}_0$ iff $\Theta_0$ is self-adjoint in $\mathbb{C}^d$.

Note that an essentially self-adjoint $\Theta_0$ is automatically self-adjoint.
7.5. Boundary space of $\hat{A}_{\text{max}}$. Proposition 7.2 implies that $(\forall u, v \in \text{dom } \hat{A}_{\text{max}})$ the boundary form $[u, v]_{\hat{A}_{\text{max}}}$ of the operator $\hat{A}_{\text{max}}$ satisfies an abstract Green identity $[\hat{F} u, \hat{F} v]_{c^d \times c^d}$ ($\hat{F}$ is as in (7.2)). On the other hand, one computes the boundary form of $\hat{A}_{\text{max}}$ directly by using Lemma 7.6, and finds that the form satisfies an abstract Green identity $[\hat{F}' u, \hat{F}' v]_{c^d \times c^d}$, where the mapping $\hat{F}' := (\hat{F}_0', \hat{F}_1')$ : dom $\hat{A}_{\text{max}}$ $\to$ $c^d \times c^d$ is defined by

\begin{align}
\hat{F}_0' : \hat{R}' + \hat{R}' &\ni u' + \hat{k} \mapsto c(\hat{k}), \\
\hat{F}_1' : \hat{R}' + \hat{R}' &\ni u' + \hat{k} \mapsto \langle \hat{\varphi}, u' \rangle + \mathcal{M}d(\hat{k}).
\end{align}

Recall $c(\hat{k})$ in (7.21b) and $\mathcal{M}$ in (6.6). One verifies the inclusion $\hat{F}' \subseteq \hat{F}$ by direct computation.

**Proposition 7.20.** $\hat{F}' \subseteq \hat{F}$.

**Proof.** First note that dom $\hat{F}' \subseteq$ dom $\hat{F}$ is due to Proposition 7.2, because by definition dom $\hat{F}' =$ dom $\hat{A}_{\text{max}}$ and dom $\hat{F} =$ dom $\hat{L}_0$. Next, using (7.2), (7.20), and (7.21) one deduces that $\hat{F}_0' u(c) = c(\hat{k}) = \hat{F}_0 u(c)$, $\hat{F}_1' u(c) = \langle \hat{\varphi}, u'(c) \rangle + R(z)c(\hat{k}) = \hat{F}_1 u(c)$ and hence the claim follows.

**Proposition 7.21.** $\hat{F}'$ is surjective.

**Proof.** Let $j \in \{0, 1\}$ and consider the mapping $\hat{F}_j'$ as a linear relation from $\mathcal{H}_0$ to $c^d$. The adjoint linear relation $(\hat{F}_j')^*$ consists of $(\chi, v) \in c^d \times \mathcal{H}_0$ such that $(\forall u' + \hat{k} \in \hat{R}' + \hat{R})$

$$
\langle u' + \hat{k}, v \rangle_0 = \langle \hat{F}_j'(u' + \hat{k}), \chi \rangle_{c^d}.
$$

Using (7.43) one has that

$$
0 = \langle u' + \hat{k}, v \rangle_0 - \langle \hat{F}_0'(u' + \hat{k}), \chi \rangle_{c^d} = \langle u', v \rangle_0 + \langle d(\hat{k}), \mathcal{G}d(\hat{k}_v) \rangle_{\text{cmd}} - \langle c(\hat{k}), \chi \rangle_{c^d}
$$

and

$$
0 = \langle u' + \hat{k}, v \rangle_0 - \langle \hat{F}_1'(u' + \hat{k}), \chi \rangle_{c^d} = \langle u', v \rangle_0 + \langle d(\hat{k}), \mathcal{G}d(\hat{k}_v) \rangle_{\text{cmd}} - \langle \langle \hat{\varphi}, u' \rangle + \mathcal{M}d(\hat{k}), \chi \rangle_{c^d}
$$

$$
= \langle u', v - \hat{\omega}\chi \rangle + \langle d(\hat{k}), \mathcal{G}d(\hat{k}_v) - \mathcal{M}^*\chi \rangle_{\text{cmd}}
$$

where $\hat{k}_v := P \hat{v} \in \hat{R}$, $\chi$ is defined in (7.24), $\hat{\omega}$ is as in (7.36), and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{H}_2$ and $\mathcal{H}_{-2}$.

Since $\hat{R}'$ is dense in $\mathcal{H}_0$ and ker $\chi = \{0\}$ (Lemma 7.8-(i)), it follows that $(\chi, v) = (0, 0)$ for $i = 0$, and $v = \hat{\omega}\chi$ and $d(\hat{k}_v) = \mathcal{G}^{-1}\mathcal{M}^*\chi$ for $i = 1$. But $v \in \mathcal{H}_0$, while $\hat{\omega}\chi \in \mathcal{H}_{-2} \times \mathcal{H}_{-1}$; hence $(\chi, v) = (0, 0)$ for $i = 1$ as well.

Now the claim follows from ker$(\hat{F}_j')^* = \{0\}$.

Let us consider the mapping $\hat{F}'$ as a linear relation from $\mathcal{H}_0^2 := \mathcal{H}_0 \times \mathcal{H}_0$ to $c^{2d} := c^d \times c^d$, i.e.

$$
\hat{F}' = \{(u' + \hat{k}, \hat{A}_{\text{max}}(u' + \hat{k})), (c(\hat{k}), \langle \hat{\varphi}, u' \rangle + \mathcal{M}d(\hat{k})) \mid u' + \hat{k} \in \hat{R}' + \hat{R}\}.
$$

Then ran $\hat{F}' = c^{2d}$ by Proposition 7.21. Likewise, if we define $\hat{F}$ as a linear relation from $\mathcal{H}_0^2$ to $c^{2d}$,

$$
\hat{F} = \{(u^\#, \hat{L}_0^2(u^\# + \hat{g}_z(c))), (c, \langle \hat{\varphi}, u^\# \rangle + R(z)c) \mid u^\# \in \mathcal{H}_2; c \in c^d\},
$$

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then clearly Proposition 7.20 remains valid. Moreover, the inclusion similar to that in the proposition holds for the inverse linear relations as well, and the meaning of this is the following. We demonstrate that \( \hat{\Gamma}' \) is a strictly isometric boundary relation for the closure of \( \text{dom} \hat{\Gamma}' = \hat{A}_{\text{max}} \) (here \( \hat{A}_{\text{max}} \) is identified with its graph). Recall that the closure is \( \hat{L}_0^* \) (Theorem 7.9). By saying “strictly” we mean that it cannot be lifted to the corresponding unitary boundary relation for \( \hat{L}_0^* \). This is because \( \hat{\Gamma}' \) is not closed, or equivalently, as we show below, the inverse \( (\hat{\Gamma}')^{-1} \) is a proper subset of the Krein space adjoint \( \hat{\Gamma}^{-1} \).

Let us briefly recall the basic notions that would help us explain the main findings in the present paragraph. As it is described in [2, 3, 5] (and in an extensive list of references therein), the Krein space \( (\mathfrak{h} \oplus \mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h} \oplus \mathfrak{h}}) \) associated with a Hilbert sum \( \mathfrak{h} \oplus \mathfrak{h} \) of a Hilbert space \( \mathfrak{h} \) with itself is obtained by endowing \( \mathfrak{h} \oplus \mathfrak{h} \) with an indefinite inner product

\[
(7.46) \quad [\hat{x}_1, \hat{x}_2]_{\mathfrak{h} \oplus \mathfrak{h}} := (\hat{x}_1, \sigma_2 \hat{x}_2)_{\mathfrak{h} \oplus \mathfrak{h}}
\]

for \( \hat{x}_1, \hat{x}_2 \in \mathfrak{h} \oplus \mathfrak{h} \). Here \( \langle \cdot, \cdot \rangle_{\mathfrak{h} \oplus \mathfrak{h}} \) is the canonical scalar product in \( \mathfrak{h} \oplus \mathfrak{h} \) and \( \sigma_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) is the second Pauli matrix.

The linear relation \( \hat{\Gamma}' \subseteq \mathcal{S}_0^2 \times \mathbb{C}^{2d} \) in (7.44) is also interpreted as a linear relation from the Krein space \( [\mathcal{S}_0^2]_\mathcal{K} \) to the Krein space \( [\mathbb{C}^{2d}]_\mathcal{K} \), where

\[
[\mathcal{S}_0^2]_\mathcal{K} := ([\mathcal{S}_0 \oplus \mathcal{S}_0, [\cdot, \cdot]_{\mathcal{S}_0 \oplus \mathcal{S}_0}], \quad [\mathbb{C}^{2d}]_\mathcal{K} := ([\mathbb{C}^d \oplus \mathbb{C}^d, [\cdot, \cdot]_{\mathbb{C}^d \oplus \mathbb{C}^d}]).
\]

The reason for this is that, by using (7.46), the Green identity can be written as

\[
(7.47) \quad [\hat{u}_1, \hat{u}_2]_{\hat{A}_{\text{max}}} = i\hat{[u}_1, \hat{u}_2]_{\mathcal{S}_0 \oplus \mathcal{S}_0} = i\hat{[u}_1, \hat{u}_2]_{\mathcal{C}^d \oplus \mathcal{C}^d}
\]

for \( \hat{u}_1, \hat{u}_2 \in \hat{A}_{\text{max}} \); and so \( \hat{\Gamma}' \) is isometric as a mapping from \( [\mathcal{S}_0^2]_\mathcal{K} \) to \( [\mathbb{C}^{2d}]_\mathcal{K} \). That \( \hat{\Gamma}' \) is isometric means that the inclusion \( \hat{\Gamma}' \subseteq (\hat{\Gamma}')^{|\cdot|} \) holds, where the Krein space adjoint is defined by (see e.g. [2, Eq. (2.6)], [3, Section 7.2])

\[
(7.48) \quad (\hat{\Gamma}')^{|\cdot|} := \left\{ (\hat{y}, \hat{x}) \in \mathbb{C}^{2d} \times \mathcal{S}_0^2 \mid (\forall (\hat{u}, \hat{v}) \in \hat{\Gamma}') [\hat{u}, \hat{x}]_{\mathcal{S}_0 \oplus \mathcal{S}_0} = [\hat{v}, \hat{y}]_{\mathcal{C}^d \oplus \mathcal{C}^d} \right\}.
\]

The relation \( \hat{\Gamma}' \) from \( [\mathcal{S}_0^2]_\mathcal{K} \) to \( [\mathbb{C}^{2d}]_\mathcal{K} \) is unitary iff the inclusion becomes the equality. However, as the following proposition states, this is not the case.

**Theorem 7.22.** \((\hat{\Gamma}')^{|\cdot|} = \hat{\Gamma}^{-1}\).

**Proof.** By definition (7.48), \((\hat{\Gamma}')^{|\cdot|}\) consists of \(( (\chi_1, \chi_2), (y, x) ) \in \mathbb{C}^{2d} \times \mathcal{S}_0^2 \) such that \(( \forall (\hat{u}' + \hat{k} \in \hat{\mathcal{K}}) \)

\[
\langle \hat{u}' + \hat{k}, x \rangle_0 - \langle \hat{A}_{\text{max}}(u' + \hat{k}), y \rangle_0 = \langle c(\hat{k}), \chi_2 \rangle_{\mathcal{C}_2^d} - \langle \langle \hat{\omega}, u' \rangle + \mathcal{M}d(\hat{k}), \chi_1 \rangle_{\mathcal{C}_1^d}. \nonumber
\]

With the help of (7.25), and using that \( \hat{\mathcal{K}}' \cap \hat{\mathcal{K}} = \{ 0 \} \), one finds that

\[
x = L_{-2}y - \hat{\omega} \chi_1, \quad d(\hat{k}_x) = G^{-1}(G_{-2}^d(\hat{k}_y) - \mathcal{M}^* \chi_1) + \mathcal{X} \chi_2
\]

and \( \hat{\omega} \) is defined in (7.36); \( \hat{k}_x := P x \in \hat{\mathcal{K}}, \hat{k}_y := P y \in \hat{\mathcal{K}} \). Using \( x \in \mathcal{S}_0 \) and (7.32), the first equation implies that \( y \in \text{dom} \hat{\tau}, \chi_1 = c(\hat{k}_y) =: \hat{\Gamma}_0 y \), and \( x = \tau y \) (recall \( \hat{\tau} \) in (7.33) and \( \hat{\Gamma}' \) in (7.34)). Using the latter and Lemma 7.8-(iii), the second equation yields

\[
d(\hat{k}_x) = Z_d(\hat{k}_y) + \mathcal{X} \langle \hat{\omega}, y \rangle.
\]

On the other hand, by applying Lemma 7.8-(ii), the relation \( x = \tau y \) shows that

\[
d(\hat{k}_x) = Z_d(\hat{k}_y) + \mathcal{X} \langle \hat{\omega}, y \rangle.
\]
where \( y_{\perp} := P'y \in \mathcal{H}_2^\ast \). Therefore, by using Lemma 7.8-(i)
\[
\chi_2 = \langle \hat{\varphi}, y_{\perp} \rangle + \mathcal{M}d(k_y) =: \hat{\Gamma}_0^y.
\]
The final conclusion now follows from Theorem 7.15 (and Corollary 7.16).

Since \( \hat{\Gamma}' \) is isometric (i.e., the Green identity in (7.47) holds), in the terminology of [3, Section 7.8], \( \hat{\Gamma}' \) is an isometric boundary relation for the closure \( \overline{\text{dom} \hat{\Gamma}'} = \hat{L}_0' \). In analogy to [2, Definition 1.8], let us put \( \hat{A}_s := \text{dom} \hat{\Gamma}' \). Let us also define \( \hat{A} := \ker \hat{\Gamma}' \) and \( \hat{A}_0 := \ker \hat{\Gamma}_0' \). Then \( \hat{A}_s = \hat{A}_{\text{max}} \) and
\[
\hat{A} = \hat{L}_0 \upharpoonright_{\text{dom} \hat{L}_0 \cap \text{dom} \hat{A}_{\text{max}}} , \quad \hat{A}_0 = \hat{S}(0) \times \mathcal{C}^d(= L \upharpoonright_{\mathcal{H}_2 \cap \text{dom} \hat{A}_{\text{max}}} ) .
\]
The above formulas are obtained from (7.44) by using Lemma 7.12 and then by applying Proposition 7.2, and recalling (7.40), (7.41). By Corollary 7.19, the operator \( \hat{A}_0 \) is essentially self-adjoint, whose closure \( \hat{A}_0^\ast = L \) (Theorems 7.17 and 7.18). We compute the adjoint \( \hat{A}^\ast \).

**Theorem 7.23.** \( \hat{A}^\ast = \hat{L}_0 \).

**Proof.** The linear relation \( \hat{A}^\ast \) is the set of \( (y, x) \in \mathcal{H}_0 \times \mathcal{H}_0 \) such that the equation (7.25) holds for all \( u' + k \in \hat{\mathcal{H}}' + \hat{\mathbb{R}} \) such that \( \hat{\Gamma}'(u' + k) = \{(0, 0)\} \). The latter boundary condition implies that \((\forall r, s \in \mathbb{C}^d \times \mathbb{C}^d) \)
\[
0 = \langle u', \hat{\varphi}r \rangle + \langle d(k), \mathcal{M}^\ast r \rangle \in \mathcal{C}^d , \quad 0 = \langle c(k), s \rangle \in \mathcal{C}^d = \langle d(k), G \chi s \rangle \in \mathcal{C}^d .
\]
where \( \hat{\varphi} \) is defined in (7.36) and \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( \mathcal{H}_2 \) and \( \mathcal{H}_1 \). By combining this with (7.25) one gets that
\[
x = L_{-2}y + \hat{\varphi}r , \quad d(\hat{k}_x) = G^{-1}(G_x^\ast d(\hat{k}_y) + \mathcal{M}^\ast r) + \chi s
\]
with \( \hat{k}_x := Px \in \hat{\mathcal{R}} \) and \( \hat{k}_y := Py \in \hat{\mathcal{R}} \). Now, by arguing exactly the same way as in the proof of Theorem 7.22 one concludes that \( y \in \text{dom} \hat{\varphi}, r = -\hat{\Gamma}_0^y, s = -\hat{\Gamma}_0^y, \) and \( x = \hat{\varphi}y \). Since \( \hat{\Gamma}_0^y = \hat{\Gamma}' \) is surjective, the claim in the theorem follows.

**Corollary 7.24.** The triple \((\mathbb{C}^d, \hat{L}_0, \hat{\Gamma}_1^\ast)\) is an isometric boundary triple for \( \hat{L}_0 \).

(Recall [2, Definitions 1.8 and 3.1]; there, \( A \) corresponds to our \( \hat{A}^\ast \), and not to \( \hat{A} \). Nevertheless, the corollary remains true because the adjoints of both operators, \( \hat{A}^\ast \) and \( \hat{A} \), coincide and because \( \hat{A}^\ast = L_0 \) is closed and symmetric, and even densely defined.) It follows from above that
\[
\hat{A} \subseteq \hat{A}_s \subseteq \hat{A}^\ast , \quad \hat{A}_s^\ast = \hat{A}^\ast , \quad \hat{A}_0^\ast = \hat{A}_s^\ast .
\]
From here it is seen that the triple cannot be unitary because \( \hat{A} \subseteq \hat{A}^\ast \).

The eigenspace \( \mathcal{H}_z(\hat{A}) = \mathcal{H}_z(\hat{L}_0) \) for \( z \in \text{res} L \). This is because \( \mathcal{H}_z(\hat{A}_{\text{max}}) = \mathcal{H}_z(\hat{L}_0) \), initially assumed for \( z \in \text{res} A_0 \) (see (6.17), (7.1), (7.8)), extends to all \( z \in \text{res} L \) by applying Lemma 7.6. Then, the \( \gamma \)-field \( \hat{\gamma}(z) := (\hat{\Gamma}_0^\ast |_{\mathcal{H}_z(\hat{A})})^{-1} \) and the abstract Weyl function \( \hat{M}(z) := \hat{\Gamma}_0^\ast \hat{\gamma}(z) \) coincide with \( \hat{\gamma}(z) \) and \( \hat{M}(z) \) in (7.4). The operator-valued functions are bounded because one has \( \hat{\Gamma}_0^\ast \mathcal{H}_z(\hat{A}_s) = \hat{\Gamma}_0^\ast \mathcal{H}_z(\hat{L}_0) = \mathbb{C}^d \).

To this end let us remark that the operator \( \hat{L}_0 \) (and its adjoint) does not depend on \( m \), so the triple \((\mathbb{C}^d, \hat{L}_0, \hat{\Gamma}_1^\ast)\) actually defines a family of isometric boundary triples for \( \hat{L}_0 \) for all \( m > 0 \). Let us also mention that, given an isometric boundary triple \((\mathbb{C}^d, \hat{L}_0, \hat{\Gamma}_1^\ast)\), one can construct other isometric boundary triples by applying e.g. [2, Lemma 3.12]: The new boundary operator \( V \circ \hat{\Gamma}' \) is constructed with the help of some isometry \( V \) in the Krein space \([\mathbb{C}^d]_K\). Since \( V \) is
also injective, one actually constructs ordinary boundary triples of the form $V^{-1} \circ \hat{\Gamma}$. We leave the details for future discussions though.

8. The transformation preserving the Weyl function

It appears from Corollary 7.10 (see also (6.15)) that the triplet adjoint $A_{\text{max}}$ is nonclosed when considered in the Hilbert space $\mathfrak{H}_m$. This implies in particular that the scaling transformation (see (7.6a)) does not preserve the structure of the resolvent (Theorem 7.11). In this last section, by modifying the transformation we study the class of operators in $\mathfrak{H}_0$ whose Weyl function is given by (6.52b). We consider conditions under which the transformation $\mathcal{H} \to \mathfrak{H}_0$ preserves the essential part of the resolvent of the triplet extension. Theorem 8.10 and Corollary 8.13 below can be viewed as the main results.

8.1. The defining operator. Consider a bounded operator $P_{\mathcal{H}} \in [\mathfrak{H}_m, \mathcal{H}]$ and let $P_{\mathcal{H}}^* \in [\mathcal{H}, \mathfrak{H}_m]$ be its adjoint. For example, one can think of $P_{\mathcal{H}}$ as an isomorphism. However, we do not assume that $P_{\mathcal{H}}$ is necessarily invertible in general. What we do assume though is that $P_{\mathcal{H}}$ is defined as a continuous mapping from the Hilbert space $\mathfrak{H}_m$ onto the Hilbert space $\mathcal{H}$, and not as an operator acting in $\mathfrak{H}_m \supset \mathcal{H}$; hence $P_{\mathcal{H}}^*$ is not the adjoint in $\mathfrak{H}_m$, but

\[ (8.1) \quad \langle g, P_{\mathcal{H}}^* f \rangle_{\mathfrak{H}_m} = \langle P_{\mathcal{H}} g, f \rangle_{\mathcal{H}} \]

\[ \forall f \in \mathcal{H} \quad \forall g \in \mathfrak{H}_m. \]

Define another bounded operator

\[ (8.2) \quad \Omega := P_{\mathcal{H}} P (L_{-m})^{1/2} : \mathfrak{H}_0 \to \mathcal{H} \]

(considered as a mapping from the Hilbert space $\mathfrak{H}_0$ to the Hilbert space $\mathcal{H}$). Since $P_{\mathcal{H}}$ is bounded and since the adjoint $(P(L_{-m})^{1/2})^* = P(L_{-m})^{-1/2}$ by Proposition 2.5, the adjoint $\Omega^*$ is a bounded operator given by

\[ (8.3) \quad \Omega^* = P(L_{-m})^{-1/2} P_{\mathcal{H}}^*: \mathcal{H} \to \mathfrak{H}_0. \]

That is, in view of (8.1), $\Omega^*$ is defined by

\[ (8.4) \quad \langle u, \Omega^* f \rangle_0 = \langle \Omega u, f \rangle_{\mathcal{H}} \]

\[ \forall u \in \mathfrak{H}_0 \quad \forall f \in \mathcal{H}; \text{ clearly } u \text{ in (8.4) and } g \text{ in (8.1) are related by } u = P(L_{-m})^{-1/2} g. \] We remark that, since $P(L_{-m})^{1/2}$ is bijective, $\Omega$ (resp. $\Omega^*$) is invertible  iff so is $P_{\mathcal{H}}$ (resp. $P_{\mathcal{H}}^*$).

Let $\Theta$ be a linear relation in $\mathcal{C}^d$; $A_{\Theta}$ the operator as in (6.43). The main object of interest is the operator in $\mathfrak{H}_0$ defined by

\[ (8.5) \quad \hat{A}_{\Theta}^\Omega := \Omega^* A_{\Theta} \Omega , \quad \text{dom} \hat{A}_{\Theta}^\Omega := \{ u \in \mathfrak{H}_0 | \Omega u \in \text{dom} A_{\Theta} \}. \]

The next proposition allows one to characterize proper extensions of a symmetric operator just like it was done in the case of triplet extensions in $\mathcal{H}$.

**Theorem 8.1.** Let $\Theta$ be a linear relation in $\mathcal{C}^d$ and let $\hat{A}_{\Theta}^\Omega$ be defined as in (8.5). Then the adjoint $(\hat{A}_{\Theta}^\Omega)^*$ in $\mathfrak{H}_0$ is the operator

\[ (\hat{A}_{\Theta}^\Omega)^* = \Omega^* A_{\Theta}^* \Omega \]

(defined on its natural domain $\{ u \in \mathfrak{H}_0 | \Omega u \in \text{dom} A_{\Theta} \}$). The adjoint operator $A_{\Theta}^*$ is described in Theorem 6.9.

Notice that we do not assume that the matrix $G_Z$ defined in (6.5) is Hermitian, so the equality in (6.49) does not necessarily hold.
Proof. The adjoint linear relation \((\hat{A}_\Theta^\Omega)^*\) is the set of \((y, x) \in \mathcal{H}_0 \times \mathcal{H}_0\) such that \((\forall u^\Omega \in \text{dom} \hat{A}_\Theta^\Omega)\)

\[\langle u^\Omega, x \rangle_0 = \langle \hat{A}_\Theta^\Omega u^\Omega, y \rangle_0.\]

By (8.4) and (8.5)

\[\langle \hat{A}_\Theta^\Omega u^\Omega, y \rangle_0 = \langle A_\Theta f, y^\Omega \rangle_{\mathcal{H}}, \quad f := \Omega u^\Omega \in \text{dom} A_\Theta, \quad y^\Omega := \Omega y \in \mathcal{H}.\]

This shows that \((\exists h \in \mathcal{H}) x = \Omega^* h \in \text{ran} \Omega^*\); for only in this case the scalar product \(\langle u^\Omega, x \rangle_0\) in \(\mathcal{H}_0\) is transformed (by using (8.4)) into the one in \(\mathcal{H}\):

\[\langle u^\Omega, x \rangle_0 = \langle f, h \rangle_{\mathcal{H}}.\]

Next, \(f \in \text{dom} A_\Theta\) implies that \(f = f^# + k + \tilde{G}_z(c)\) for \(f^# \in \mathcal{H}_{m+2}, k \in \mathbb{R}, c \in \mathbb{C}^d\) such that \(\tilde{\Gamma} f \in \Theta\) (see Theorem 6.5-(i) and (6.43)). The boundary condition shows that \((\exists (r, s) \in \Theta^*)\) such that (6.46) holds.

Using the representation \(h = h^# + k_h\) for \(h^# \in \mathcal{H}_m\) and \(k_h \in \mathbb{R}, c \in \mathbb{C}^d\) such as \(y^\Omega = y^# \Omega + k_y^\Omega\) for \(y^\Omega \in \mathcal{H}_m\) and \(k_y^\Omega \in \mathbb{R},\) and then applying Theorem 6.5-(i) for computation of \(A_\Theta f = A_{\text{max}} f,\) one further writes

\[
0 = \langle u^\Omega, x \rangle_0 - \langle \hat{A}_\Theta^\Omega u^\Omega, y \rangle_0 = \langle f, h \rangle_{\mathcal{H}} - \langle A_\Theta f, y^\Omega \rangle_{\mathcal{H}}
= \langle f^#, P(L_{m-2})(h^# - L_{m-2}y^# \Omega), + \langle c, \tilde{G}(z), h^# - \Gamma y^# \Omega, - \tilde{G}_0 d(k_y^\Omega) \rangle_{\mathbb{C}^d}
+ \langle d(k), Gd(k_h) - \tilde{G}_0^k d(k_y^\Omega) \rangle_{\mathbb{C}^d}
\]

(8.6)

where \(\langle \cdot, \cdot \rangle\) is the duality pairing between \(\mathcal{H}_{m+2}\) and \(\mathcal{H}_{m-2}.\) By comparing (8.6) with (6.46) one deduces the system (6.47), but where now \(x^# \) and \(k_x\) are replaced by \(h^# \) and \(k_h,\) as well as \(y^# \) and \(y^\Omega,\) and then applying Theorem 6.5-(i) for \(\text{dom} A_\Theta^\Omega = A_{\text{max}}\) \(\Omega^* x\) and \(h = A_{\text{max}}^\Omega y\). By applying Theorem 6.9 the claim follows.

Let us put

\[
\hat{A}_\Theta^{\Omega, \min} := \Omega^* A_{\text{min}} \Omega, \quad \hat{A}_\Theta^{\Omega, \max} := \Omega^* A_{\text{max}} \Omega
\]

where the operator \(A_{\text{max}}\) in (6.15) is described in Theorems 6.3 and 6.5 and Corollary 6.6, and the operator \(A_{\text{min}}\) is represented in Theorem 6.7; see also Corollary 6.8.

According to (6.28) and Theorem 8.1 the operator

\[
\hat{A}_\Theta^{\Omega, \min} = (\hat{A}_\Theta^{\Omega, \max})^*\]

is closed in \(\mathcal{H}_0,\) while \(\hat{A}_\Theta^{\Omega, \max} = \hat{A}_\Theta^{\Omega, \min} \times \mathbb{C}^d\) is closable; \(\hat{A}_\Theta^{\Omega, \max} = (\hat{A}_\Theta^{\Omega, \min})^*\) is closed iff the matrix \(G_Z^k\) is Hermitian, because only in this case \(\hat{A}_\Theta^{\Omega, \min} = \hat{A}_\Theta^{\Omega, \Omega(0, 0)}\) (Corollary 6.8). For \(G_Z\) Hermitian, the operator \(\hat{A}_\Theta^{\Omega, \min}\) is symmetric in \(\mathcal{H}_0,\) because by (8.4)

\[
[u, v]_{\hat{A}_\Theta^{\Omega, \min}} = [\Omega u, \Omega v]_{\text{A}_{\text{min}}} = 0
\]

for \(u, v \in \text{dom} \hat{A}_\Theta^{\Omega, \min},\) and \(A_{\text{min}}\) is symmetric in \(\mathcal{H}\) (Corollary 6.8). We summarize the observations in the following corollary.

**Corollary 8.2.** Let \(G_Z\) be Hermitian. Let \(\Theta\) be a linear relation in \(\mathbb{C}^d.\) Then the adjoint

\[
(\hat{A}_\Theta^{\Omega, \min})^* = \hat{A}_\Theta^{\Omega, \min}.
\]

In particular, \(\hat{A}_\Theta^{\Omega, \min}\) is symmetric (resp. self-adjoint) in \(\mathcal{H}_0\) iff \(\Theta\) is symmetric (resp. self-adjoint) in \(\mathbb{C}^d.\)

Moreover

\[
\hat{A}_\Theta^{\Omega, \min} \subseteq \hat{A}_\Theta^{\Omega, \infty} \subseteq \hat{A}_\Theta^{\Omega, \max}.
\]
That is, \( \hat{A}_\Theta^\Omega \) is a proper extension of a densely defined, closed, and symmetric operator \( \hat{A}_\Theta^\Omega \) in \( \mathcal{H}_0 \), whose adjoint in \( \mathcal{H}_0 \) is the operator \( \hat{A}_\Theta^{\Omega^*} \).

**Proof.** This follows from Theorems 6.9 and 8.1, by observing that \( \hat{A}_\Theta^\Omega \subseteq \hat{A}_\Theta^\Omega_\Theta \) for linear relations \( \Theta_1 \subseteq \Theta_2 \).

Before we move to the analysis of the boundary space of \( \hat{A}_\Theta^\Omega \), let us first examine the eigenspace \( \mathcal{N}_z(\hat{A}_\Theta^\Omega_0) \quad (z \in \mathbb{C}) \). The results will be used for constructing the \( \gamma \)-field and the Weyl function (see (8.19) for the standard definition).

### 8.2. Eigenspace

Let \( \Omega \) (resp. \( \Omega^* \)) be as in (8.2) (resp. (8.3)) and let

\[
\iota := \Omega \Omega^* = P_H P_H^* \in [\mathcal{H}] \cdot
\]

Then \( \iota = |P_H^*|^2 > 0 \) is a bounded, positive, self-adjoint operator in \( \mathcal{H} \). The next lemma shows that the eigenspace \( \mathcal{N}_z(\hat{A}_\Theta^\Omega_0) \) is the set of \( u \in \text{dom} \hat{A}_\Theta^\Omega \) such that \( \Omega u \in \mathcal{N}_z(\iota A_\Theta) \).

**Lemma 8.3.** Let \( \mathcal{G}_Z \) be Hermitian and let \( \Theta \) be a (closed) linear relation in \( \mathbb{C}^d \). The operator \( \iota A_\Theta \) is closed in \( \mathcal{H} \) and it holds

\[
\Omega \mathcal{N}_z(\hat{A}_\Theta^\Omega_0) \subseteq \mathcal{N}_z(\iota A_\Theta)
\]

for \( z \in \mathbb{C} \).

**Proof.** The operator \( \iota A_\Theta \) is closed because \( \iota \) is bounded and boundedly invertible, and because the operator \( A_\Theta \) is closed for \( \mathcal{G}_Z \) Hermitian (see (6.49)).

Consider \( u \in \mathcal{N}_z(\hat{A}_\Theta^\Omega) \), that is, \( u \in \text{dom} \hat{A}_\Theta^\Omega_0 \) and \( (\hat{A}_\Theta^\Omega_0 - z)u = 0 \). Then \( f := \Omega u \in \text{dom} A_\Theta \) and \( \hat{A}_\Theta^\Omega_0 u = \Omega^* A_\Theta f \) by definition (8.5). In view of the latter, multiply the eigenvalue equation from the left by \( \Omega \) and deduce that \( (\iota A_\Theta - z)f = 0 \), i.e. \( f \in \mathcal{N}_z(\iota A_\Theta) \).

By using the next lemma one verifies that the reverse inclusion in (8.8) does not necessarily hold.

**Lemma 8.4.** Let \( \Theta \) be a linear relation in \( \mathbb{C}^d \). Assume that the operator \( P_H \) leaves \( \text{dom} A_\Theta \) invariant:

\[
P_H \text{dom} A_\Theta \subseteq \text{dom} A_\Theta.
\]

Then

\[
\text{dom} \hat{A}_\Theta^\Omega \supseteq \text{dom} \hat{A}_\Theta \quad \text{and} \quad \text{dom} \hat{A}_\Theta^\Omega = \text{dom} A_\Theta.
\]

**Proof.** In view of Proposition 2.1 and (8.2) and (8.5)

\[
\text{dom} \hat{A}_\Theta^\Omega = P(L_{-m})^{-1/2} \mathcal{D}_\Theta, \quad \mathcal{D}_\Theta := \{ f \in \mathcal{H}_{-m} \mid P_H f \in \text{dom} A_\Theta \}.
\]

Let \( g \in \text{dom} A_\Theta \). By hypothesis (8.9), \( g \in \mathcal{D}_\Theta \), and so \( \text{dom} A_\Theta \subseteq \mathcal{D}_\Theta \). By applying (7.7) one therefore deduces the first relation in (8.10).

To prove the second relation in (8.10), first note that the inclusion \( \subseteq \) in \( \text{dom} \hat{A}_\Theta^\Omega = \text{dom} A_\Theta \) is clear from the definition in (8.5). We prove the reverse inclusion \( \supseteq \).

Let \( g \in \text{dom} A_\Theta \) as above; then \( (\exists f \in \mathcal{D}_\Theta) P_H f = g \), since \( \mathcal{D}_\Theta \supseteq \text{dom} A_\Theta \) as shown above.

Putting \( u := P(L_{-m})^{-1/2} f \) one finds that \( u \in \text{dom} \hat{A}_\Theta^\Omega \) by (8.11), and then \( g = P_H f = \Omega u \), i.e. \( g \in \Omega \text{dom} \hat{A}_\Theta^\Omega \).

Thus, assume (8.9) for \( \Theta = \mathbb{C}^d \times \mathbb{C}^d \) and consider \( f \in \mathcal{N}_z(\iota A_{\max}) \). Then \( f \in \text{dom} \iota A_{\max} = \text{dom} A_{\max}, \quad (\iota A_{\max} - z)f = 0 \), and by (8.10) \( \exists u \in \text{dom} \hat{A}_\Theta^\Omega_{\max} \) \( f = \Omega u \). In view of (8.7) the eigenvalue equation yields \( (\hat{A}_\Theta^\Omega_{\max} - z)u \in \ker \Omega \), and this shows that \( u \notin \mathcal{N}_z(\hat{A}_\Theta^\Omega_{\max}) \) in general.

By repeating the above procedure for an arbitrary \( \Theta \) one concludes the following.
Proposition 8.5. If $P_H$ is invertible and satisfies (8.9), then the inclusion in (8.8) becomes the equality.

Since $\mathcal{M}_z(iA_0)$ is obtained from $\mathcal{M}_z(iA_{\text{max}})$ by imposing on the eigenfunction an appropriate boundary condition, in what follows we concentrate on the properties of the eigenspace $\mathcal{M}_z(iA_{\text{max}})$.

Lemma 8.6. Let $\mathcal{M}_z$ denote the set of $(c, k) \in \mathbb{C}^d \times \mathfrak{K}$ such that
\begin{equation}
0 = (\i - I)z\tilde{G}_z(c) + \sum_{\alpha \in \mathfrak{S} \times J} [(Z_{dt} - z)d(k)]_{\alpha}g_{\alpha} + ik_{\min}(c)
\end{equation}
for $z \in \text{res} L$. Then
$$\mathcal{M}_z(iA_{\text{max}}) = \{f_z(c, k) := \tilde{G}_z(c) + k \mid (c, k) \in \mathcal{M}_z\}$$
for $z \in \Sigma_i := \text{res} iL_m \cap \text{res} A_0$.

Proof. First note that the operator $iL_m$ is closed (in $\mathcal{H}$), because $i$ is bounded and boundedly invertible, and because $L_m$ is closed. Thus the resolvent set res $iL_m$ is nonempty. An element from $\mathcal{M}_z(iA_{\text{max}})$ belongs to dom $A_{\text{max}}$, and so it can be written in the form described in Theorem 6.5-(i): The eigenvalue equation then yields $f = 0$ for $z \in \Sigma_i$, and (8.12) follows.

By putting $i = I$ in Lemma 8.6 and applying (6.22) one recovers $\mathcal{M}_z(A_{\text{max}})$ in Theorem 6.3; in this case res $iL_m = \text{res} L_m = \text{res} L$ (Corollary 2.3). Another definition of the set $\mathcal{M}_z(z \in \text{res} A_0)$ is given in the next proposition.

Proposition 8.7. Define the column-vector
$$C_z(c, k) = ([C_z(c, k)]_\alpha) \in \mathbb{C}^{md}, \quad [C_z(c, k)]_\alpha := d_{\alpha}(k) = \frac{c_{\alpha}}{(z - z_{j})b_{j}(z_j)}$$
for $c = (c_\alpha) \in \mathbb{C}^d, k \in \mathfrak{K}, \alpha = (\sigma, j) \in \mathfrak{S} \times J$, and $z \in \mathbb{C} \setminus \mathbb{Z}$. Then $\mathcal{M}_z$, for $z \in \text{res} A_0$, is the set of $(c, k) \in \mathbb{C}^d \times \mathfrak{K}$ such that
\begin{equation}
0 = (\i - I)z\tilde{\gamma}(z)c + \sum_{\alpha \in \mathfrak{S} \times J} [(Z_{dt} - z)C_z(c, k)]_{\alpha}g_{\alpha}.
\end{equation}
The $\gamma$-field $\tilde{\gamma}$ is as in Theorem 6.10.

Proof. Plug $\tilde{G}_\sigma(z)$ ($\forall \sigma \in \mathfrak{S}$) from (6.22) into (8.12) and apply (6.52a).

With the notation as in Lemma 8.6 and Proposition 8.7 one has by applying (6.22)
\begin{equation}
f_z(c, k) = \tilde{\gamma}(z)c + \sum_{\alpha \in \mathfrak{S} \times J} [C_z(c, k)]_{\alpha}g_{\alpha}
\end{equation}
for $c \in \mathbb{C}^d, k \in \mathfrak{K}, z \in \text{res} A_0$. Define the set
\begin{equation}
\mathcal{M}_z^0 := \{(c, k) \in \mathbb{C}^d \times \mathfrak{K} \mid C_z(c, k) = 0\}
\end{equation}
for $z \in \mathbb{C} \setminus \mathbb{Z}$. It is clear from Lemma 8.6, (8.14), (8.15) that, for $z \in \text{res} A_0$, the eigenspace $\mathcal{M}_z(iA_{\text{max}})$ is the set of $f_z(c, k)$ with $(c, k) \in \mathcal{M}_z^0$, hence (6.17).

We remark that $(\forall c) \exists k \in \mathcal{M}_z$. For example, if $k \neq 0$ is such that $C_z(c, k) = 0$ and $z \neq 0$, then by (8.13) $c \in \ker[(\i - I)\tilde{\gamma}(z)]$. But $\tilde{\gamma}(z)$ is invertible, which means that $c \in \mathbb{C}^d$ and $\mathcal{M}_z(iA_{\text{max}}) \subseteq \mathcal{M}_z(i)$. Conversely, assume that $z \in \text{res} A_0 \setminus \{0\}$ and $\mathcal{M}_z(iA_{\text{max}}) \subseteq \mathcal{M}_z(i)$. Then (8.13) shows that $\mathcal{M}_z \supseteq \mathcal{M}_z^0$, as well as $\mathcal{M}_z(iA_{\text{max}}) \supseteq \mathcal{M}_z(iA_{\text{max}}) \forall z \in \Sigma_i \setminus \{0\}$. The inclusion $\mathcal{M}_z(iA_{\text{max}}) \subseteq \mathcal{M}_z(i)$ is always satisfies if, for example, $P_H$ is an isometric isomorphism, because then $\Omega$ is also an isometric isomorphism, and so $\Omega^* = \Omega^{-1}$ and $i = I$. Below we give an example of a nontrivial $i$ whose eigenspace $\mathcal{M}_z(i)$ contains a closed subset.
Example 8.8. Let $P^*_H$ be a partial isometry; recall (8.1) for the definition of the adjoint. This means that there exists a closed subset $\mathcal{X} \subseteq \mathcal{H}$ such that i) $\langle \forall f \in \mathcal{X} \rangle \| P^*_H f \|_m = \| f \|_H$ and ii) $P^*_H \mathcal{X} = \{0\}$ for $\mathcal{X}^\perp := \mathcal{H} \ominus \mathcal{X}$. (It follows from (8.3) and i) and ii) that the adjoint $\Omega^*$ is also a partial isometry with the initial space $\mathcal{X}$ and the final space ran $\Omega^* = \Omega^* \mathcal{X}$. Now by using i) one finds that $(i - I)\mathcal{X} \subseteq \mathcal{X}^\perp$. In particular, the latter inclusion is satisfied if $\mathcal{X} \subseteq \mathfrak{S}_1(\nu)$. And this case occurs when $P_H$ is a partial isometry with the initial space $\mathcal{X}_0 := P^*_H \mathcal{X} = \text{ran} P^*_H$ and the final space ran $P_H = P_H \mathcal{X}_0 = \nu \mathcal{X} = \mathcal{X}$. Note that $\mathcal{X}^+_0 = \ker P_H$, and so $P_H \mathcal{X}^+_0 = \{0\}$. Note also that, since $\mathcal{X}$ is closed, the set $\mathcal{X}_0$ is closed by the closed range theorem.

8.3. Boundary space. In this paragraph the matrix $\mathcal{G}_Z$ is Hermitian. The boundary form of the operator $\hat{A}^\Omega_{\max}$ satisfies the Green identity: $(\forall u, v \in \text{dom } \hat{A}^\Omega_{\max})$

\begin{equation}
[u, v]_{\hat{A}^\Omega_{\max}} = [\Omega u, \Omega v]_{A_{\max}} = [\hat{\Omega} u, \hat{\Omega} v]_{C^d \times C^d}
\end{equation}

by (6.27) and (8.4). Thus, in analogy to (6.43) and (6.48), the operator

\begin{equation}
\hat{A}^\Omega_\Theta = \{ \hat{u} \in \hat{A}^\Omega_{\max} : \hat{\Omega} \hat{u} \in \Theta \}, \quad \hat{\Omega} := \hat{\Omega}^1
\end{equation}

is a proper extension of the symmetric operator $\hat{A}^\Omega_{\min}$ (see Corollary 8.2).

Theorem 8.9. Let $\mathcal{G}_Z$ be Hermitian. Assume that the operator $P_H$ leaves $\text{dom } A_{\max}$ invariant (i.e. (8.9) holds for $\Theta = C^d \times C^d$). Then the triple $(C^d, \hat{\Gamma}_0^1, \hat{\Gamma}_1^1)$ is a boundary triple for $\hat{A}^\Omega_{\max}$; here $\hat{\Gamma}_0^1 := \hat{\Gamma}_0 \Omega$ and $\hat{\Gamma}_1^1 := \hat{\Gamma}_1 \Omega$ are surjective operators from $\text{dom } \hat{A}^\Omega_{\max}$ onto $C^d$.

\textbf{Proof.} The operator $\hat{\Gamma}^1$ is surjective because of Lemma 8.4 and because $\hat{\Gamma} : \text{dom } A_{\max} \to C^d \times C^d$ is surjective. The remaining arguments are due to Corollary 8.2 and (8.16) and (8.17). \hfill $\Box$

One of the two distinguished self-adjoint extensions of $\hat{A}^\Omega_{\min}$ is the operator

\begin{equation}
\hat{A}^\Omega_0 := \hat{A}^\Omega_{\max} |_{\ker \hat{\Gamma}_0^1} = \Omega^* A_0 \Omega.
\end{equation}

The operator $A_0$ in $\mathcal{H}$ is defined in (6.1), and it is described in Proposition 6.1 and Corollary 6.8. The self-adjointness of $\hat{A}^\Omega_0$ is seen from Corollary 6.8-(i) and Theorem 8.1. Alternatively, the operator $\hat{A}^\Omega_0$ is self-adjoint because $\hat{A}^\Omega_0 = \hat{A}^\Omega_{\{0\} \times C^d}$ and $\{0\} \times C^d$ is self-adjoint in $C^d$ (Corollary 8.2). The second distinguished self-adjoint extension is $\hat{A}^\Omega_{C^d \times \{0\}}$. In what follows, however, we choose $\hat{A}^\Omega_0$ as a reference operator.

Under the hypotheses in Theorem 8.9, the $\gamma$-field $\hat{\gamma}^\Omega_\Theta \ni z \mapsto \hat{\gamma}^\Omega(z)$ and the abstract Weyl function $\hat{\gamma}^\Omega_\Theta \ni z \mapsto \hat{\gamma}^\Omega(z)$ associated with the boundary triple $(C^d, \hat{\Gamma}_0^1, \hat{\Gamma}_1^1)$ for $\hat{A}^\Omega_{\max}$ are the operator valued functions defined by

\begin{equation}
\hat{\gamma}^\Omega(z) := (\hat{\Gamma}_0^1 |_{\mathcal{N}_z(\hat{A}^\Omega_{\max})} )^{-1} \in [C^d, \mathfrak{S}_0], \quad \hat{\gamma}^\Omega(z) := \hat{\Gamma}_1^1 \gamma^\Omega(z) \in [C^d].
\end{equation}

Note that $\text{ran } \hat{\gamma}^\Omega(z) = \mathfrak{N}_z(\hat{A}^\Omega_{\max})$. For a (closed) linear relation $\Theta$ in $C^d$ the Krein–Naimark resolvent formula holds:

\[(\hat{A}^\Omega_0 - z)^{-1} = (\hat{A}^\Omega_0 - z)^{-1} + \hat{\gamma}^\Omega(z)(\Theta - \hat{\gamma}^\Omega(z))^{-1} \gamma^\Omega(z)^*\]

for $z \in \text{res } \hat{A}^\Omega_0 \cap \text{res } \hat{A}^\Omega_0$. In the next theorem we give an explicit representation of the above operator-valued functions.

Theorem 8.10. Assume the hypotheses in Theorem 8.9; let $z \in \Sigma_0 \cap \text{res } \hat{A}^\Omega_0$ in (i)–(ii).

(i) The (graph of the) $\gamma$-field is given by

\begin{equation}
\hat{\gamma}^\Omega(z) = \{(c, u) \in C^d \times \mathfrak{N}_z(\hat{A}^\Omega_{\max}) : \Omega u = f_z(c, k); (c, k) \in \mathfrak{M}_z\}.
\end{equation}
(ii) The (graph of the) Weyl function is given by the operatorwise sum
\[ \widehat{M}^\Omega(z) = \widehat{M}(z) + \Delta(z) \]
where
\[ \Delta(z) := \{(c, -\mathcal{G}^*_b C_z(c, k)) \mid (c, k) \in \mathcal{M}_z\} \]
is the graph of a bounded operator in \( \mathbb{C}^d \). The Weyl function \( \widehat{M} \) is as in Theorem 6.10.
(iii) For a (closed) linear relation \( \Theta \) in \( \mathbb{C}^d \)
\[ \Omega(\widehat{A}^\Omega_\Theta - z)^{-1} \subseteq (\iota A_\Theta - z)^{-1} \Omega \]
for \( z \in \text{res} \widehat{A}^\Omega_\Theta \cap \text{res} \iota A_\Theta \). For those \( \Theta \) for which additionally condition (8.9) holds, the inclusion in (8.22) becomes the equality.

Proof. (i) By the definition in (8.19), the linear relation \( \widehat{\gamma}^\Omega(z) \) is the set of elements \( (c, u) \) such that \( c \in \mathbb{C}^d, u \in \mathcal{H}_z(\widehat{A}^\Omega_{\mathcal{M}_z}) \), and \( \widehat{\Gamma}_0 u = c \). By Lemma 8.3, \( \widehat{\Gamma}_0 f = c \) with \( f := \Omega u \in \mathcal{H}_z(\iota A_{\mathcal{M}_z}) \). Then (8.20) is due to Lemma 8.6.

We show that \( \widehat{\gamma}^\Omega(z) \) is single-valued, i.e. that \( \widehat{\gamma}^\Omega(z) \) in (8.20) is the graph. The multivalued part
\[ \text{mul} \widehat{\gamma}^\Omega(z) = \ker \widehat{\Gamma}_0 \big|_{\mathcal{H}_z(\widehat{A}^\Omega_{\mathcal{M}_z})} = \{0\} \]
which follows from the definition in (8.19) and the Green identity in (8.16), and which is valid for \( z \in \text{res} \widehat{A}^\Omega_0 \). We verify this for \( \widehat{\gamma}^\Omega(z) \) in (8.20). It is convenient first to have at hand the following lemma.

Lemma 8.11. Let \( h_\alpha \in \mathcal{H} \setminus \{0\} \) \((\alpha \in \mathcal{S} \times J)\) and let \( k \in \mathbb{R} \) such that
\[ 0 = \sum_\alpha d_\alpha(k) h_\alpha ; \]
then \( k = 0 \).

Proof. Projecting (8.24) to \( h \in \mathcal{H} \) and applying the definition of \( d(k) \) given in (5.3) one finds that
\[ 0 = \langle \sum_\alpha \xi_\alpha(h) g_\alpha, k \rangle_{-m}, \quad \xi_\alpha(h) := \sum_{\alpha'} [\mathcal{G}^{-1}]_{\alpha \alpha'} \langle h_{\alpha'}, h \rangle_{\mathcal{H}} \]
\((\alpha \in \mathcal{S} \times J)\). Since \( h \in \mathcal{H} \) was arbitrary, \( k \in \mathbb{R} \cap \mathbb{R}^+ = \{0\} \). \( \square \)

For \( \widehat{\gamma}^\Omega(z) \) in (8.20), \( \text{mul} \widehat{\gamma}^\Omega(z) \) is the set of \( u \in \mathcal{H}_z(\widehat{A}^\Omega_{\mathcal{M}_z}) \) such that \( \Omega u = k \in \mathbb{R} \) and (8.24) holds with \( h_\alpha := (z, t - z) g_\alpha \forall \alpha = (\sigma, j) \in \mathcal{S} \times J \). Thus \( k = 0 \) by the lemma, and so it holds
\[ \text{mul} \widehat{\gamma}^\Omega(z) = \mathcal{H}_z(\widehat{A}^\Omega_{\mathcal{M}_z}) \cap \ker \Omega \]
But the right-hand side of (8.25) is exactly \( \ker \widehat{\Gamma}_0 \big|_{\mathcal{H}_z(\widehat{A}^\Omega_{\mathcal{M}_z})} \); now apply (8.23).

(ii) By the definition in (8.19) and applying (8.20), the linear relation \( \widehat{M}^\Omega(z) \) is the set of elements \( (c, \mathcal{R}(z) - \mathcal{G}^*_b d(k)) \) with \((c, k) \in \mathcal{M}_z \). Since \((\forall c)(\exists k)\) \((c, k) \in \mathcal{M}_z \), we have that \( \Delta(z) \) (and hence \( \widehat{M}^\Omega(z) \)) is bounded, and then by applying (6.52b) we deduce (8.21). (Note that the boundedness of \( \Delta(z) \) is also seen from \( \text{dom} \Delta(z) = \text{dom} \widehat{M}^\Omega(z) = \widehat{\Gamma}_0 \mathcal{H}_z(\widehat{A}^\Omega_{\mathcal{M}_z}) = \mathbb{C}^d \).

Since \( \text{mul} \Delta(z) \) is the set of \( -\mathcal{G}^*_b d(k) \) such that \( k \in \mathbb{R} \) solves (8.24), i.e. \( k = 0 \), it follows that \( \Delta(z) \) (and hence \( \widehat{M}^\Omega(z) \)) is the graph.
Consider \((\hat{A}^\Omega_\Theta - z)u = v\) for some \(z \in \text{res} \hat{A}^\Omega_\Theta\), \(u \in \text{dom} \hat{A}^\Omega_\Theta\), and \(v \in \text{ran}(\hat{A}^\Omega_\Theta - z)\). Put \(f := \Omega u \in \text{dom} A_\Theta\) and multiply the above equation from the left by \(\Omega\): \((iA_\Theta - z)f = \Omega v\). For \(z \in \text{res} iA_\Theta \cap \text{res} \hat{A}^\Omega_\Theta\) (\(\text{res} iA_\Theta\) is nonempty by Lemma 8.3), this gives

\[ f = \Omega(\hat{A}^\Omega_\Theta - z)^{-1}v = (iA_\Theta - z)^{-1}\Omega v \]

from which one deduces (8.22), because \(\Omega \text{ ran}(\hat{A}^\Omega_\Theta - z) \subseteq \text{ran}(iA_\Theta - z)\). The latter inclusion becomes the equality provided that condition (8.9) holds, since in this case one has (8.10). \(\square\)

When considered separately from (8.21a), the matrix-valued function \(\Delta(z)\) in (8.21b) is well-defined for \(z \in \text{res} A_0\).

**Proposition 8.12.** Define the set

\[ \mathfrak{M}_z^* := \{(c, k) \in \mathbb{C}^d \times \mathfrak{A} | C_z(c, k) \in \ker \mathcal{G}_0^*\} \]

for \(z \in \mathbb{C} \setminus \mathcal{Z}\). Then the following equivalence relation holds

\[ \Delta(z) = 0 \iff \mathfrak{M}_z \subseteq \mathfrak{M}_z^* \]

for \(z \in \text{res} A_0\).

**Proof.** Clear from (8.21b). \(\square\)

Recall from Lemma 5.6 that \(\ker \mathcal{G}_0^* \neq \{0\}\) in general; hence \(\mathfrak{M}_z \subseteq \mathfrak{M}_z^*\).

By putting \(\iota = I\) in (8.13), i.e. when deriving \(\mathcal{M}_z(A_{\max})\), one gets that \(\mathfrak{M}_z = \mathfrak{M}_z^*\), which means that \(\Delta(z)\) is automatically zero. In this case the inverse \(\Omega^{-1} = \Omega^*\), and so the operator \(\Omega\) is an isometric isomorphism, and one recovers from (8.22) the standard formula connecting the resolvents of similar operators \(A_\Theta\) and \(\Omega^{-1}A_0\Omega\) (provided that (8.9) holds). As the following corollary states, however, it is not necessary to take \(\iota = I\) in order to get \(\Delta(z) = 0\).

**Corollary 8.13.** Assume the hypotheses in Theorem 8.9 and in addition \(\mathfrak{M}_z \subseteq \mathfrak{M}_z^* \forall z \in \text{res} A_0\). Then the Weyl function \(\overline{M}^\Omega(z) = \overline{M}(z) \forall z \in \text{res} A_0\).

**Proof.** For \(z \in \Sigma \cap \text{res} \hat{A}^\Omega_0(\subseteq \text{res} A_0)\), this follows from Theorem 8.10-(iii) and Proposition 8.12. For analytic functions, the equality \(\overline{M}^\Omega(z) = \overline{M}(z)\) then extends to the domain of analyticity of \(\overline{M}(z)\), namely, \(\text{res} A_0\). \(\square\)

We end up the discussion by commenting on the Langer–Textorius result on \(Q\)-functions of symmetric operators [22]. Namely, if the \(Q\)-functions (or, in view of (6.53), the corresponding Weyl functions) of two densely defined, closed, symmetric, and simple operators \(\hat{A}^\Omega_{\min}\) and \(A_{\min}\) coincide, that is, \(\overline{M}^\Omega(z) = \overline{M}(z)\) under hypotheses in Corollary 8.13, then the operators \(\hat{A}^\Omega_{\min}\) and \(A_{\min}\), as well as their distinguished self-adjoint extensions \(\hat{A}^\Omega_0\) and \(A_0\), respectively, are similar. Specifically, this means that there exists an isometric isomorphism \(\Omega_1 \in [\mathfrak{H}_0, \mathcal{H}]\) such that \(\Omega_1 \mathcal{M}_z(\hat{A}^\Omega_{\max}) = \mathcal{M}_z(A_{\max})\). The latter definition entails the equalities \(\Omega_1 \text{ dom} \hat{A}^\Omega_0 = \text{ dom} A_0\) and \(\hat{A}^\Omega_0 = \Omega_1^{-1}A_0\Omega_1\) (cf. (8.18)).

We want to show that, under suitable conditions, we can choose \(\Omega_1 = \Omega\). Thus, assume that:

a) The matrix \(\mathcal{G}_Z\) is Hermitian; b) (8.9) holds for \(\Theta = \mathbb{C}^d \times \mathbb{C}^d\) and \(\Theta = \{0\} \times \mathbb{C}^d\); c) \(\Omega\) (and hence \(P_H\)) is invertible, with the inverse \(\Omega^{-1} = \Omega^*\). By hypotheses, Corollary 8.13 holds with \(\iota = I\) and \(\mathfrak{M}_z = \mathfrak{M}_z^* \subset \mathfrak{M}_z^*\). In addition, \(\Omega \text{ dom} \hat{A}^\Omega_0 = \text{ dom} A_0\) by b) and Lemma 8.4. Then \(\Omega(\hat{A}^\Omega_0 - z)^{-1} = (A_0 - z)^{-1}\Omega\) by Theorem 8.10-(iii), which is equivalent to \(\hat{A}^\Omega_0 = \Omega^{-1}A_0\Omega\).

Next, items b), c), and Proposition 8.5 yield \(\Omega \mathcal{M}_z(\hat{A}^\Omega_{\max}) = \mathcal{M}_z(A_{\max})\). As remarked by (6.54), the symmetric operator \(A_{\min}\) is simple; hence the symmetric operator \(\hat{A}^\Omega_{\min}\) is simple by c). But then \(\Omega\) is an isometric isomorphism defined in the proof of the second part of Theorem 2.2 in [22], and so \(\Omega\) can be identified with \(\Omega_1\).
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