On the Weyl projective curvature tensor of the projective semi-symmetric connection in an $SP$-Sasakian manifold

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Abstract

The objective of the present paper is to study the $W_2$-curvature tensor of the projective semi-symmetric connection in an $SP$-Sasakian manifold. It is shown that an $SP$-Sasakian manifold satisfying the conditions $\vec{P} \cdot \vec{W}_2 = 0$ is an Einstein manifold and $\vec{W}_2 \cdot \vec{P} = 0$ is a quasi Einstein manifold.

Key words and phrases: Projective semi-symmetric connection, $SP$-Sasakian manifold, quasi-Einstein manifold, $W_2$-curvature tensor, Einstein manifold.

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1 Introduction

The study of semi-symmetric connections is a very attractive field for investigations in the past many decades. Semi-symmetric connection was introduced by A. Friedmann and J. A. Schouten in 1924. In 1930, E. Bartolotti extended a geometrical meaning to such a connection. Further, H. A. Hayden studied a metric connection with torsion on Riemannian manifold. After a long gap, in 1970, the study of semi-symmetric...
connections was resumed by K. Yano\textsuperscript{13}. In particular, he studied semi-symmetric metric connections. Afterwards several researchers have been carried out the study of semi-symmetric connections in a variety of directions such as \textsuperscript{10, 11, 16, 17}. The studies on projective semi-symmetric connections have been further extended by P. Zhao\textsuperscript{15}, S.K. Pal\textsuperscript{7} and others.

In this article, we consider the projective semi-symmetric connection on an SP-Sasakian manifold and study some properties of $W_2$-tensor fields. The paper is organised as follows: We present a brief account of SP-Sasakian manifold in section 2. The subsequent section 3 is devoted to the brief description of the projective semi-symmetric connection and its properties. In section 4, we study the $W_2$-curvature tensor of projective semi-symmetric connection in an SP-Sasakian manifold. In section 5, we consider the two condition $\tilde{p} \cdot \tilde{W}_2 = 0$ and $\tilde{W}_2 \cdot \tilde{p} = 0$ respectively Einstein manifold and quasi Einstein manifold.

2 Preliminaries:

The notion of an (almost) para-contact manifold was introduced by I. Sato\textsuperscript{9}. An $n$-dimensional differentiable manifold $M$ is said to have almost para-contact structure $(\phi, \xi, \eta)$, where $\phi$ is a tensor field of type $(1,1)$, $\xi$ is a vector field known as characteristic vector field and $\eta$ is a 1-form satisfying the following relations

$$\phi^2(X) = X - \eta(X)\xi,$$  \hspace{0.5cm} \text{(2.1)}

$$\eta(\tilde{X}) = 0,$$  \hspace{0.5cm} \text{(2.2)}

$$\phi(\xi) = 0,$$  \hspace{0.5cm} \text{(2.3)}

and

$$\eta(\xi) = 1.$$  \hspace{0.5cm} \text{(2.4)}

A differentiable manifold with almost para-contact structure $(\phi, \xi, \eta)$ is called an almost para-contact manifold. Further, if the manifold $M$ has a Riemannian metric $g$ satisfying

$$\eta(X) = g(X, \xi),$$  \hspace{0.5cm} \text{(2.5)}

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$  \hspace{0.5cm} \text{(2.6)}

Then the set $(\phi, \xi, \eta, g)$ satisfying the conditions to is called an almost para-contact Riemannian structure and the manifold $M$ with such a structure is called an almost para-contact Riemannian manifold \textsuperscript{2, 9}.

Now, let $(M, g)$ be an $n$-dimensional Riemannian manifold with a positive definite metric $g$ admitting a 1-form $\eta$ which satisfies the conditions

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0$$  \hspace{0.5cm} \text{(2.7)}

and

$$(\nabla_X \nabla_Y \eta)(Z) = -g(X, Z)\eta(Y) - g(X, Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z),$$  \hspace{0.5cm} \text{(2.8)}

where $\nabla$ denotes the operator of covariant differentiation with respect to the metric tensor $g$. Moreover, If $(M, g)$ admits a vector field $\xi$ and a $(1,1)$ tensor field $\phi$ such that

$$g(X, \xi) = \eta(X), \quad \eta(\xi) = 1 \quad \text{and} \quad \nabla_X \xi = \phi(X),$$  \hspace{0.5cm} \text{(2.9)}

then it can be easily verified that the manifold under consideration becomes an almost para-contact Riemannian
manifold. Such a manifold is called a para-Sasakian manifold or briefly a $P$-Sasakian manifold\(^1\). It is a special case of almost para-contact Riemannian manifold introduced by I. Sato. It is known\(^1\) that on a $P$-Sasakian manifold the following relations hold:

\[
\eta(R(X,Y,Z)) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X),
\]
\[
R(\xi,X,Y) = -R(X,\xi,Y) = \eta(Y)X - g(X,Y)\xi,
\]
\[
R(\xi,X,\xi) = X - \eta(X)\xi,
\]
\[
R(X,Y,\xi) = \eta(X)Y - \eta(Y)X,
\]
\[
S(X,\xi) = -(n-1)\eta(X),
\]
\[
Q\xi = -(n-1)\xi,
\]
\[
r = -n(n-1),
\]
where $r$ is the scalar curvature.

A $P$-Sasakian manifold satisfying
\[
(\nabla_X \eta)(Y) = -g(X,Y) + \eta(X)\eta(Y)
\]
for all vector fields $X$ and $Y$, where $a$ and $b$ are scalars with $b \neq 0$. $A$ is a non-zero 1-form such that $g(X,\xi) = A(X)$, for all vector fields $X$ and $\xi$ being a unit vector.

Quasi Einstein manifolds, introduced by M.C. Chaki and R.K. Maity\(^4\), are natural generalisations of Einstein manifolds. According to them, a non-flat Riemannian manifold $(M, g)$ $(n > 2)$ is a quasi-Einstein manifold\(^4\) if its Ricci tensor $S$ of type $(0,2)$ is not identically zero and satisfies the following condition
\[
S(X, Y) = ag(X,Y) + bA(X)A(Y),
\]
for all vector fields $X$ and $Y$, where $a$ and $b$ are scalars with $b \neq 0$. $A$ is a non-zero 1-form such that $g(X,\xi) = A(X)$,

The $W_2$ curvature tensor is defined by\(^8\)
\[
W_2(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1} [g(X,Z)QY - g(Y,Z)QX],
\]
where $Q$ is a Ricci tensor of type $(1,1)$, i.e., $S(X,Y) = g(QX,Y)$; $S$ being the type $(0,2)$ Ricci tensor. The Weyl projective curvature tensor $P$ is defined by\(^12\)
\[
P(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1} [S(Y,Z)X - S(X,Z)Y],
\]
where $S$ being the type $(0,2)$ Ricci tensor.
3 Projective semi-symmetric connection:

In this section, we give a brief account of projective semi-symmetric connection and study it on a SP-Sasakian manifold.

A linear connection $\nabla$ on an $n$-dimensional Riemannian manifold $(M, g)$ is called a semi-symmetric connection, if its torsion tensor $T$ given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

has the form

$$T(X, Y) = \pi(Y)X - \pi(X)Y.$$  \hspace{1cm} (3.1)

where $\pi$ is a 1-form associated with a vector field $\rho$, i.e.,

$$\pi(X) = g(X, \rho).$$  \hspace{1cm} (3.2)

Further, a connection $\nabla$ is a metric connection if it satisfies

$$(\nabla_X g)(Y, Z) = 0.$$  \hspace{1cm} (3.3)

If the geodesic with respect to $\nabla$ are always consistent with those of the Levi-Civita connection $\nabla$ on a Riemannian manifold, then $\nabla$ is called a connection projectively equivalent to $\nabla$. If $\nabla$ is linear connection projective equivalent to $\nabla$ as well as a semi-symmetric one, we call $\nabla$ is called projective semi-symmetric connection.

Now, we consider a projective semi-symmetric connection $\nabla$ introduced by P. Zhao and H. Song given by

$$\nabla_X Y = \nabla_X Y + \Psi(Y)X + \Psi(X)Y + \Phi(Y)X - \Phi(X)Y,$$ \hspace{1cm} (3.4)

for arbitrary vector fields $X$ and $Y$, where the 1-forms $\Psi$ and $\Phi$ are given through the following relations:

$$\Psi(X) = \frac{n - 1}{2(n + 1)}\pi(X) \quad \text{and} \quad \Phi(X) = \frac{1}{2}\pi(X).$$ \hspace{1cm} (3.5)

It is easy to see that the equations (3.4) and (3.5) give us

$$(\nabla_X g)(Y, Z) = \frac{1}{n + 1}[2\pi(X)g(Y, Z) - n\pi(Y)g(X, Z) - n\pi(Z)g(X, Y)],$$ \hspace{1cm} (3.6)

which shows that the connection $\nabla$ given by (3.4) is a metric one.

We denote by $\tilde{R}$ and $R$ the curvature tensors of the manifold relative to the projective semi-symmetric connection connections $\nabla$ and the Levi-Civita connection $\nabla$. It is known that $\tilde{R}$ and $R$ are the tensors of type $(0, 2)$ given by the following relations

$$\tilde{R}(X, Y, Z) = R(X, Y, Z) + \alpha(X, Z)Y - \alpha(Y, Z)X + \beta(X, Y)Z,$$ \hspace{1cm} (3.7)

$$\alpha(X, Y) = \Psi'(X, Y) + \Phi'(Y, X) - \Psi(X\Phi(Y) - \Psi(Y)\Phi(X))$$ \hspace{1cm} (3.8)

$$\beta(X, Y) = \Psi'(X, Y) - \Psi'(Y, X) + \Phi'(Y, X) - \Phi'(X, Y).$$ \hspace{1cm} (3.9)

The tensors $\Psi'$ and $\Phi'$ of type $(0, 2)$ are defined by the following two relations.

$$\Psi'(X, Y) = (\nabla_X \Psi)(Y) - \Psi(X)\Psi(Y).$$ \hspace{1cm} (3.10)

and

$$\Phi'(X, Y) = (\nabla_X \Phi)(Y) - \Phi(X)\Phi(Y).$$ \hspace{1cm} (3.11)
Contraction of the vector field $X$ in the equation (3.7) yields a relation between Ricci tensors of the manifold relative to the two connections $\nabla$ and $\tilde{\nabla}$ which is given by
\[
\tilde{S}(Y,Z) = S(Y,Z) + \beta(Y,Z) - (n - 1)\alpha(Y,Z) \tag{3.12}
\]
Also, from the above equation, we get the following equation relating scalar curvatures $\tilde{r}$ and $r$ of the manifold with respect to the two connections $\tilde{\nabla}$ and $\nabla$
\[
\tilde{r} = r + b - (n - 1)a. \tag{3.13}
\]
where $b = \sum_{i=1}^{n} \beta(e_i,e_i)$ and $a = \sum_{i=1}^{n} \alpha(e_i,e_i)$.

In order to extend the studies of the projective semi-symmetric connection $\tilde{\nabla}$ on SP-Sasakian manifold, we identify the 1-form $\pi$ of the connection $\tilde{\nabla}$ with the 1-form $\eta$ of the P-Sasakian manifold. In view of this equality between $\pi$ and $\eta$ and the equations (3.5), we find that the expression (3.4) for the projective semi-symmetric connection $\tilde{\nabla}$ reduces to
\[
\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(c + 1)\eta(Y)X + \frac{1}{2}(c - 1)\eta(X)Y, \tag{3.14}
\]
where the constant $c$ is given by $c = \frac{n-1}{n+1}$. Now, it can be seen that
\[
(\tilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y) - c\eta(X)\eta(Y). \tag{3.15}
\]
On an SP-Sasakian manifold, we have $(\nabla_X \eta)(Y) = \nabla_Y \eta(X)$. Therefore, the above equation yields
\[
(\tilde{\nabla}_X \eta)(Y) = (\tilde{\nabla}_Y \eta)X.
\]
Thus, the connection $\tilde{\nabla}$ given by the equation (3.4) becomes special projective semi-symmetric connection studied by S.K. Pal et al.\(^7\). It can also be verified very easily that for such a projective semi-symmetric connection the tensor $\beta$ vanishes and the tensor $\alpha$ is symmetric, i.e.,
\[
\beta(X,Y) = 0, \text{ and } \alpha(X,Y) = \alpha(Y,X). \tag{3.16}
\]
As a consequence of these, the expressions for curvature tensor, the tensor $\alpha$, Ricci tensors and scalar curvatures given by (3.7), (3.8), (3.12) and (3.13) takes the following simpler forms
\[
\tilde{R}(X,Y,Z) = R(X,Y,Z) + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{3.17}
\]
\[
\alpha(X,Y) = \mu(\nabla_X \eta)(Y) - \mu^2\eta(X)\eta(Y), \tag{3.18}
\]
and
\[
\tilde{S}(Y,Z) = S(Y,Z) - (n - 1)\alpha(Y,Z). \tag{3.19}
\]
and
\[
\tilde{r} = r - (n - 1)a, \tag{3.20}
\]
where $\mu = \frac{1}{2}(c + 1)$. In view of the equations (3.16) and (3.19), it follows that the Ricci tensor $\tilde{S}(Y,Z)$ of the special projective semi-symmetric connection is symmetric.

Now, we derive some of the results concerning the tensor $\alpha$ which we shall need in subsequent sections. It is easy to see that in view of the equation (2.17), the tensor $\alpha$ takes the following form
\[
\alpha(X,Y) = -\mu g(X,Y) + \nu\eta(X)\eta(Y), \tag{3.21}
\]
where we have put $\nu = \mu - \mu^2$. This, in view of the equations (2.4) and (2.5) and symmetry of tensor $\alpha$,
produces easily that
\[ \alpha(\xi, Y) = \alpha(Y, \xi) = \lambda \eta(Y), \tag{3.22} \]
where by \( \lambda \) we mean \(-\mu^2\).

Now, putting \( \xi \) for each of the vector fields \( X, Y \) and \( Z \) in the equation (3.17) and using the equations (2.11), (2.13) and (3.22), we obtain the followings:
\[ \bar{R}(\xi, Y, Z) = \lambda' \eta(Z)Y - \theta(Y, Z)\xi, \tag{3.23} \]
and
\[ \bar{R}(X, \xi, Z) = \theta(X, Z)\xi - \lambda' \eta(Z)X, \tag{3.24} \]
where \( \lambda' = (1 + \lambda) \) and the \( \theta \) is a type (0,2) symmetric tensor given by
\[ \theta(Y, Z) = g(Y, Z) + \alpha(Y, Z). \tag{3.26} \]
Also, using the equation (3.22) in the above, we at once get
\[ \theta(Y, \xi) = \lambda' \eta(Y). \tag{3.27} \]
On account of the equation (3.21), the expressions (3.17) and (3.19) for curvature tensor and Ricci tensor assumes the following forms:
\[ \bar{R}(X,Y,Z) = R(X,Y,Z) - \mu[g(X,Z)Y - g(Y,Z)X] \]
\[ + \nu[\eta(X)Y - \eta(Y)X] \eta(Z), \tag{3.28} \]
and
\[ \bar{S}(Y,Z) = S(Y,Z) + (n - 1)\left[\mu g(Y,Z) - \nu \eta(Y)\eta(Z)\right], \tag{3.29} \]
which produces the following expression for (1,1) type Ricci tensor \( \bar{Q} \)
\[ \bar{Q}Y = QY + \mu(n - 1)Y - \nu(n - 1) \eta(Y)\xi. \tag{3.30} \]
Also, contraction of the equation (3.29) yields
\[ \bar{r} = r + (n - 1)(\mu \eta - \nu) \tag{3.31} \]
Taking inner product of the equation (3.17) with \( \eta \) we get the following
\[ \eta(\bar{R}(X,Y,Z)) = \eta(R(X,Y,Z)) + \alpha(X,Z)\eta(Y) - \alpha(Y,Z)\eta(X), \tag{3.32} \]
which, due to the equations (2.10) and (3.21), gives
\[ \eta(\bar{R}(X,Y,Z)) = -(\mu - 1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \tag{3.33} \]
In (3.29) equation put \( Z = \xi \) and using (2.14), (2.9) we get
\[ \bar{S}(Y,\xi) = (\mu - \nu - 1)(n - 1)\eta(Y). \tag{3.34} \]

4 The \( W_2 \) Curvature tensor:

In this section, we consider the \( W_2 \) curvature tensor of projective semi-symmetric connection in an \( SP \)-Sasakian manifold.

The \( W_2 \) curvature tensor field in an \( SP \)-Sasakian manifold is given by the following relation:
\[ W_2(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX]. \tag{4.1} \]
If we put \( \xi \) for \( X, Y \) and \( Z \) respectively in the above equation, then in view of the equations (2.9), (2.11), (2.13)
and (2.15), we get
\[ W_2(\xi, Y, Z) = \eta(Z)Y + \frac{1}{n-1}\eta(Z)QY, \]  
(4.2)
\[ W_2(X, \xi, Z) = -\eta(Z)X - \frac{1}{n-1}\eta(Z)QX \]  
(4.3)
and
\[ W_2(X, Y, \xi) = \eta(X)Y - \eta(Y)X + \frac{1}{n-1} [\eta(X)QY - \eta(Y)QX]. \]  
(4.4)

Similar to the definition (4.1), we define the \( \widehat{W}_2 \) curvature tensor of projective semi-symmetric connection \( \nabla \) in SP-Sasakian manifold by
\[ \widehat{W}_2(X, Y, Z) = \bar{R}(X, Y, Z) + \frac{1}{n-1} [g(X, Z)\bar{Q}Y - g(Y, Z)\bar{Q}X], \]  
(4.5)

Also, the type (0, 4) tensor field \( \prime \widehat{W}_2 \) is given by
\[ \prime \widehat{W}_2(X, Y, Z, U) = \prime \bar{R}(X, Y, Z, U) + \frac{1}{n-1} [g(X, Z)\bar{S}(Y, U) - g(Y, Z)\bar{S}(X, U)], \]  
(4.6)
where we have put
\[ \prime \bar{W}_2(X, Y, Z, U) = g(\bar{W}_2(X, Y, Z), U) \]
and
\[ \prime \bar{R}(X, Y, Z, U) = g(\bar{R}(X, Y, Z), U) \]
for the arbitrary vector fields \( X, Y, Z, U \).

With the help of (3.17), (3.30) in (4.5), we get
\[ \bar{W}_2(Y, Z, W) = R(Y, Z, W) + \alpha(Y, W)Z - \alpha(Z, W)Y + \frac{1}{(n-1)} \]  
\[ \{g(Y, W) [QZ - \nu(n-1)\eta(Z)\xi + \mu(n-1)Z] \} \]  
(4.7)
\[ \{QY - \nu(n-1)\eta(Y)\xi + \mu(n-1)Y] \}, \]
which using the equations (3.21) and (4.1), yields
\[ \bar{W}_2(Y, Z, W) = W_2(Y, Z, W) + \nu[\eta(Y)Z - \eta(Z)Y]\eta(W) \]  
\[ + \nu[\eta(Y)g(Z, W) - \eta(Z)g(Y, W)]\xi. \]  
(4.8)

Now, taking \( \xi \) for each of the vector field \( Y, Z \) and \( W \) in the above equation and using (4.2), (4.3) and (4.4), we get
\[ \bar{W}_2(\xi, Z, W) = (1 + \nu)\eta(W)Z + \frac{1}{n-1}\eta(W)QZ \]  
\[ - 2\nu\eta(Z)\eta(W)\xi + \nu g(Z, W)\xi, \]  
(4.9)
\[ \bar{W}_2(Y, \xi, W) = -(1 + \nu)\eta(W)Y - \frac{1}{n-1}\eta(W)QY \]  
\[ + 2\nu\eta(Y)\eta(W)\xi - \nu g(Y, W)\xi \]  
(4.10)
and
\[
\mathcal{W}_2(Y, Z, \xi) = (1 + \nu)\eta(Y)Z - (1 + \nu)\eta(Z)Y \\
+ \frac{1}{n-1}\{\eta(Y)QZ - \eta(Z)QY\}.
\] (4.11)

Taking inner product with \(\xi\) in the equation (4.1) and using (2.14) and (2.10), we get

\[
\eta(\mathcal{W}_2(Y, Z, W)) = 0.
\] (4.12)

Similarly, from the equations (2.14), (3.33), (4.5) and (4.8), we obtain

\[
\eta(\tilde{\mathcal{W}}_2(Y, Z, W)) = \nu\eta(Y)g(Z, W) - \nu\eta(Z)g(Y, W).
\] (4.13)

5 The Weyl Projective Curvature tensor:

In this section, we consider the Weyl projective curvature tensor of projective semi-symmetric connection in an \(SP\)-Sasakian manifold and derive two theorems related to curvature conditions of semi-symmetry type. The Weyl projective curvature tensor in an \(SP\)-Sasakian manifold is given by the following relation:

\[
P(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y].
\] (5.1)

Similar to the definition (5.1), we define the Weyl projective curvature tensor of projective semi-symmetric connection \(\tilde{\nabla}\) in \(SP\)-Sasakian manifold by

\[
\tilde{P}(X, Y, Z) = \tilde{R}(X, Y, Z) - \frac{1}{n-1}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y].
\] (5.2)

Also the type (0,4) tensor field \(\tilde{\mathcal{P}}\) is defined by

\[
\tilde{\mathcal{P}}(X, Y, Z, U) = \tilde{\mathcal{R}}(X, Y, Z, U) - \frac{1}{n-1}[g(X, U)\tilde{S}(Y, Z) - g(Y, U)\tilde{S}(X, Z)],
\] (5.3)

where we have put

\[
\tilde{\mathcal{P}}(X, Y, Z, U) = g(\tilde{P}(X, Y, Z), U)
\]

and

\[
\tilde{\mathcal{R}}(X, Y, Z, U) = g(\tilde{R}(X, Y, Z), U)
\]

for the arbitrary vector fields \(X, Y, Z, U\).

With the help of (3.21), (3.17), (3.29) and (5.1), we get

\[
\tilde{P}(Y, Z, W) = P(Y, Z, W) = R(Y, Z, W) - \frac{1}{n-1}[S(Z, W)Y - S(Y, W)Z].
\] (5.4)

Now taking \(\xi\) for each of the vector fields \(Y, Z\) and \(W\) in the above equation and using (2.11), (2.14), (2.13), we get

\[
\tilde{P}(\xi, Z, W) = -g(Z, W)\xi + \frac{1}{n-1}S(Z, W)\xi,
\] (5.5)

\[
\tilde{P}(Y, \xi, W) = g(Y, W)\xi + \frac{1}{n-1}S(Y, W)\xi
\] (5.6)

and

\[
\tilde{P}(Y, Z, \xi) = 0.
\] (5.7)

Taking inner product with \(\xi\) in the equation (5.1) and using (2.5), (2.10), we get
\[ \eta(P(Y, Z, W)) = \eta(Z)[g(Y, W) + \frac{1}{n-1}S(Y, W)] - \eta(Y)[g(Z, W) + \frac{1}{n-1}S(Z, W)]. \quad (5.8) \]

Similarly, from the equations (2.5), (3.33), (3.29) and (5.2), we obtain
\[ \eta(\bar{P}(Y, Z, W)) = g(Y, W)\eta(Z) - g(Z, W)\eta(Y) + \frac{1}{n-1}[S(Y, W)\eta(Z) - S(Z, W)\eta(Y)]. \quad (5.9) \]

**Theorem 5.1**: If an \(SP\)-Sasakian manifold admitting a projective semi-symmetric connection \(\tilde{\nabla}\) satisfies \(\overline{W}_2(\xi, X) \cdot \bar{P} = 0\), then it is a quasi Einstein manifold.

**Proof**: Let
\[ (\overline{W}_2(\xi, X) \cdot P)(Y, Z, W) = 0. \]

Therefore, we get from the above equation
\[ 0 = \overline{W}_2(\xi, X, \bar{P}(Y, Z, W)) - \bar{P}(\overline{W}_2(\xi, X, Y), Z, W) - \bar{P}(Y, \overline{W}_2(\xi, X, Z), W) - \bar{P}(Y, Z, \overline{W}_2(\xi, X, W)). \]

Now using (4.9), (5.5), (5.6), (5.7), (5.9) in the above equation, then taking inner product with \(\xi\) in the equation and using the equations (2.14), (5.4), (5.9), we get
\[ '\bar{R}(Y, Z, W, X)\nu = -\frac{1}{n-1}\eta(W)\eta(Z)S(X, Y) + \frac{1}{(n-1)}\eta(W)\eta(Y)S(X, Z) + \frac{1}{(n-1)}(2 + \nu)S(X, Y)\eta(W)\eta(Z) - \frac{1}{(n-1)}(2 + \nu)S(X, Z)\eta(W)\eta(Y) - (1 + \nu)\eta(W)\eta(Y)g(Z, X) + \nu g(X, Z)g(Y, W) - \nu g(Z, W)g(X, Y) + (1 + \nu)\eta(W)\eta(Z)g(X, Y). \]

Now, contracting the above equation with respect to \(X\) and \(Y\), we obtain
\[ S(Z, W) = (1 - n)g(Z, W) + \frac{(1 - n)}{\nu}\eta(W)\eta(Z). \]

which proves that the manifold is a quasi Einstein manifold.

**Theorem 5.2**: If an \(SP\)-Sasakian manifold admitting a projective semi-symmetric connection \(\tilde{\nabla}\) satisfies \(\bar{P}(\xi, X) \cdot \overline{W}_2 = 0\), then it is an Einstein manifold.

**Proof**: Let
\[ (\bar{P}(\xi, X) \cdot \overline{W}_2)(Y, Z, W) = 0. \]

Then we have
0 = \tilde{P}(\xi, X, \tilde{W}_2(Y, Z, W)) - \tilde{W}_2(\tilde{P}(\xi, X, Y), Z, W) - \tilde{W}_2(Y, \tilde{P}(\xi, X, Z), W) - \tilde{W}_2(Y, Z, \tilde{P}(\xi, X, W)).

Now using (4.9), (4.10), (4.11) and (5.5) in the above equation, then taking inner product with \( \xi \) in the above equation and using the equations (3.17), (3.29), (4.6) and (4.8), we get

\[ g(X, R(Y, Z, W)) + \alpha(Y, W)g(X, Z) - \alpha(Z, W)g(X, Y) + \frac{1}{(n-1)}g(Y, W)S(Z, X) + \nu\eta(Z)\eta(Y)g(Y, W) + g(W, Y)[\mu g(X, Z) - \nu\eta(Z)\eta(X)] - \frac{1}{(n-1)}g(Z, W)S(Y, X) \]
\[ -\nu\eta(Y)\eta(Z)g(Z, W) - g(W, Z)[\mu g(Y, X) - \nu\eta(Y)\eta(X)] + \frac{1}{(n-1)}[S(X, W_2(Y, Z, W)] \]
\[ = -\nu\eta(Z)\eta(W)g(X, Y) + \nu g(X, Y)g(Z, W) + \frac{1}{(n-1)}\nu S(X, Y)g(Z, W) + \nu\eta(Y)\eta(W)g(X, Z) - \nu g(X, Z)g(Y, W) - \frac{1}{(n-1)}\nu g(Y, W)S(Z, X). \]

Putting \( Y = \xi \) in the above equation and using (2.11), (3.22), (4.2), we obtain

\[ S(X, Z) = \frac{-(1 + \mu + \lambda)}{(1 + \nu)}g(X, Z) \]

which proves that the manifold is an Einstein manifold.

6 Scope of Future Research and Applications:

The study on the \( W_2 \)-curvature tensor of the projective semi-symmetric connection in an \( SP \)-Sasakian manifold is our future research in this topic, which we are going to be published very soon.

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