The effect of cell-attachment on the group of self-equivalences of an $R$-localized space

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Abstract Let $R \subseteq \mathbb{Q}$ be a ring with least non-invertible prime $p$. Let $X = X^n \cup_\alpha (\bigcup_{j \in J} e^q)$ be a cell attachment with $J$ finite and $q$ small with respect to $p$. Let $\mathcal{E}(X_R)$ denote the group of homotopy self-equivalences of the $R$-localization $X_R$. We use DG Lie models to construct a short exact sequence

$$0 \to \bigoplus_{j \in J} \pi_q(X^n)_R \to \mathcal{E}(X_R) \to C^q \to 0$$

where $C^q$ is a subgroup of $\text{GL}_{|J|}(R) \times \mathcal{E}(X_R^n)$. We obtain a related result for the $R$-localization of the nilpotent group $\mathcal{E}_e(X)$ of classes inducing the identity on homology. We deduce some explicit calculations of both groups for spaces with few cells.

Keywords Homotopy self-equivalences · Quillen model · Anick model · $R$-local homotopy theory Moore space · Nilpotent group

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1 Introduction

Let $X$ be a finite, simply connected CW complex. Let $\mathcal{E}(X)$ denote the group of homotopy equivalence classes of homotopy self-equivalences of $X$. Let $\mathcal{E}_s(X)$ denote the subgroup represented by self-equivalences that induce the identity map on $H_*(X; \mathbb{Z})$. The study of the groups $\mathcal{E}(X)$ and $\mathcal{E}_s(X)$ by means of a cellular decomposition of $X$ is a difficult problem with a long history. See Rutter [16, Chapter 11] for a survey.

In [11], Dror-Zabrodsky proved $\mathcal{E}_s(X)$ is a nilpotent group. Maruyama [14] then proved $\mathcal{E}_s(X)_R \cong \mathcal{E}_s(X_R)$. These results together opened the door to the use of algebraic models for studying the localization of nilpotent self-equivalence groups. The group $\mathcal{E}_s(X)$ was studied in [5]. The rationalization of the subgroup $\mathcal{E}_s^*(X)$ of self-equivalences inducing the identity on homotopy groups has been studied extensively using Sullivan models (c.f. [3, 6, 12]).

The group $\mathcal{E}(X_\mathbb{Q})$ has emerged as a recent object of interest. Arkowitz-Lupton [4] gave the first examples of finite groups occurring as $\mathcal{E}(X_\mathbb{Q})$. Further examples were given by the first named author in [9]. Costoya-Viruel [10] then proved the remarkable result that every finite group $G$ occurs as $G \cong \mathcal{E}(X_\mathbb{Q})$ for some finite $X$. Again, all this work was accomplished using Sullivan models. The purpose of this paper is to explore the use of Anick’s and Quillen’s DG Lie algebra models for studying the groups $\mathcal{E}(X_R)$ and $\mathcal{E}_s(X_R)$.

We briefly recall the main result of Anick's and Quillen’s theories now in order to establish our overriding hypotheses. Let $R \subseteq \mathbb{Q}$ be a subring with least non-invertible prime $p \in R$. When $R = \mathbb{Q}$ set $p = +\infty$. Let $\text{CW}^{k+1}_m$ denote the category of $m$-connected, finite CW complexes of dimension no greater than $k + 1$ with $m$-skeleton reduced to a point. Let $\text{CW}^{k+1}_m(R)$ denote the category obtained by $R$-localizing the spaces in $\text{CW}^{k+1}_m$. By Anick [1, 2], when $k < \min(m + 2p - 3, mp - 1)$ the homotopy category of $\text{CW}^{k+1}_m(R)$ is equivalent to the homotopy category of $\text{DGL}^k_m(R)$ consisting of free differential graded (DG) Lie algebras $(L(V), \partial)$ in which $V$ is a free $R$-module satisfying $V_n = 0$ for $n < m$ and $n > k$. When $R = \mathbb{Q}$ the corresponding result for $m = 1$ and any $k$ is due to Quillen [15]. Summarizing, we have:

**Hypothesis 1.1** We assume that $R \subseteq \mathbb{Q}$ is a ring with least non-invertible prime $p$. With $R$ fixed, we take $1 \leq m < k$ satisfying $k < \min(m + 2p - 3, mp - 1)$. By a space $X$ we always mean an object in $\text{CW}^{k+1}_m$, an $m$-connected finite CW complex with top degree cells of dimension $\leq k + 1$ When $R = \mathbb{Q}$ we assume $m = 1$ and $k$ is finite.

Let $X$ be an object in $\text{CW}^{k+1}_m$. Write $X^n$ for the $n$-skeleton of $X$. We consider the situation in which

$$X = X^q = X^n \cup_{\alpha} \left( \bigcup_{i=1}^{j} e^q_i \right)$$

is the space obtained by attaching $q$-cells to a space $X^n$ for $n < q \leq k + 1$ by a map $\alpha : \bigcup_{i} S^{q-1} \to X^n$. 

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Theorem 1 Given $R \subseteq \mathbb{Q}$ and $X$ satisfying Hypothesis 1.1, there are short exact sequences

$$0 \to \bigoplus_{i=1}^{j} \pi_q(X^n_R) \to \mathcal{E}(X_R) \to \mathcal{C}^q \to 0$$  \hspace{1cm} (1)

$$0 \to \bigoplus_{i=1}^{j} \pi_q(X^n_R) \to \mathcal{E}_*(X_R) \to \mathcal{C}^q_* \to 0$$  \hspace{1cm} (2)

Here, $\mathcal{C}^q \subseteq \text{GL}_j(R) \times \mathcal{E}(X^n_R)$. The subgroup $\mathcal{C}^q_* \subseteq \mathcal{C}^q$ is contained in $\text{GL}_k(R) \times \mathcal{E}_*(X^n_R)$ with $k$ the rank of the linking homomorphism $\pi_{n+1}(X^n, X^{n-1})_R \to \pi_n(X^n_R, X^{n-1})_R$ in the long exact sequence of the triple. In particular, $\mathcal{C}^q_* \subseteq \mathcal{E}_*(X^n_R)$ when $q > n + 1$.

We prove Theorem 1 in Sect. 2. In Sect. 3, we deduce some consequences for $R$-localized spaces.

Given a finitely generated abelian group $G$, let $M(G, m)$ denote the Moore space. Barcus–Barrett [8] proved the homology representation $\mathcal{E}(M(G, m)) \to \text{aut}(G)$ is surjective and identified the kernel as an Ext-group. It is a classical open problem to complete this calculation [13, Problem 8] with many partial results and extensions (cf. Rutter [op. cit] and Baues [7]). We obtain the following result in Anick’s category of $R$-local spaces:

**Example 1.2** Let $G_1, \ldots, G_r$ be finitely generated abelian groups and $R \subseteq \mathbb{Q}$. Let

$$X = M(G_1, m + 1) \lor M(G_2, m + 3) \lor \cdots \lor M(G_r, m + 2r - 1)$$

with $r \leq m/2$ and $m, k = m + 2r$ and $R$ satisfying Hypothesis 1.1. Then

$$\mathcal{E}(X_R) \cong \prod_{i=1}^{r} \text{aut}((G_i)_R).$$

We obtain a related calculation for the group $\mathcal{E}_*(X_R)$:

**Example 1.3** Let $G$ be a finitely generated abelian group and $R \subseteq \mathbb{Q}$. Let $X$ in $\text{CW}_k^{k+1}$ and $R$ be as in Hypothesis 1.1 with $X$ having a cellular decomposition

$$M(G, m + 1) = X^{m+2} \subset X^{m+3} \subset \cdots \subset X^{2m} = X.$$ 

Assume that the linking homomorphisms $\pi_{r+1}(X^{r+1}, X^r)_R \to \pi_r(X^r, X^{r-1})_R$ vanish for $r = m + 2, \ldots, 2m - 1$. Let $j = \dim_R(H_{2m}(X, X^{2m-1}; R))$. Then

$$\mathcal{E}_*(X)_R \cong \bigoplus_{i=1}^{j} [G_R, G_R].$$

Here, $[G_R, G_R]$ is additive group corresponding to the sub $R$-module of $G_R \otimes G_R$ generated by elements of the form $x \otimes y - (-1)^m y \otimes x$. 

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We give a complete calculation of both groups in a simple case. Let $R^* = \text{aut}(R)$ denote the group of units of $R$.

Example 1.4 Let $m, m + n + 1$ and $R \subseteq \mathbb{Q}$ satisfy Hypothesis 1.1. Then

\[ \mathcal{E} \left( (S^{m+1} \times S^{n+1})_R \right) \cong \begin{cases} R^* \times R^* & \text{for } n \neq 2m \text{ or } m \text{ even} \\ R \oplus (R^* \times R^*) & \text{for } n = 2m \text{ and } m \text{ odd} \end{cases} \]

\[ \mathcal{E}_s(S^{m+1} \times S^{n+1})_R \cong \begin{cases} 0 & \text{for } n \neq 2m \text{ or } m \text{ even} \\ R & \text{for } n = 2m \text{ and } m \text{ odd} \end{cases} \]

We prove two results concerning the question of finiteness $\mathcal{E}(X_R)$. First:

Theorem 1.5 Let $R \subseteq \mathbb{Q}$ and $X$ in $\mathbf{CW}^{k+1}_m$ be as in Hypothesis 1.1. Suppose $\pi_{n+1}(X^{n+1}, X^n)_R \neq 0$ and the linking homomorphism $\pi_{n+1}(X^{n+1}, X^n)_R \to \pi_n(X^n, X^{n-1})_R$ vanishes. Then $\mathcal{E}(X^n_R)$ finite implies $\mathcal{E}(X^{n+1}_R)$ is infinite.

Finally, we give a calculation to indicate that finiteness of $\mathcal{E}(X_R)$ requires a reasonably large CW complex $X$.

Theorem 1.6 Let $X$ be a simply-connected finite CW complex of dimension $\leq 5$. Let $R \subseteq \mathbb{Q}$ have least invertible prime $p \geq 7$. If $p$ is finite, assume the linking homomorphisms $\pi_{r+1}(X^{r+1}, X^r)_R \to \pi_r(X^r, X^{r-1})_R$ vanish for $r = 2, 3, 4$ vanish. Then $\mathcal{E}(X(R))$ is infinite.

2 Homotopy self-equivalences of DG Lie algebras

Let $(L(V), \partial)$ be an object in $\mathbf{DGL}^k_m(R)$. Recall this means $V$ is a free graded $R$-module concentrated in degrees $n$ with $m \leq n \leq k$ and $L(V)$ is the free graded Lie algebra over $R$. Write $V_{<n} = \bigoplus_{i=m}^{n-1} V_i$. The differential $\partial$ is of degree $-1$. For each $n \leq k$, $\partial$ induces a differential $\partial_{<n}$ on $L(V_{<n})$ making $(L(V_{<n}), \partial_{<n})$ a sub DG Lie algebra. We write the homology as $H_*(L(V_{<n}))$, suppressing the differential. The linear part of $\partial$ gives a differential $d$ on $V$. The homology $H_*(V) = H_*(V, d)$ can be identified with the graded module of indecomposable generators of $(L(V), \partial)$.

Homotopies between maps $\alpha, \alpha'$: $(L(V), \partial) \to (L(W), \delta)$ in $\mathbf{DGL}^k_m(R)$ are defined by means of the Tanré cylinder (cf. [17, Ch.II.5] and [1, pp. 425–6]). Let

\[ (L(V), \partial)_I = (L(V, sV, V'), D) \]

be the DG Lie algebra with $V \cong V'$ and $(sV)_i = V_{i-1}$. Let $S$ denote the derivation of degree +1 on $L(V, sV, V')$ with $S(v) = sv$ and $S(sv) = S(v') = 0$. The differential $D$ is given by $D(v) = \partial(v)$, $D(sv) = v'$ and $D(v') = 0$. The degree zero derivation $\theta = D \circ S + S \circ D$ of $(L(V), \partial)_I$ gives rise to an automorphism $e^\theta$ of $(L(V, \partial)_I$. We note that for $v \in V$

\[ e^\theta(v) = v + v' + \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v). \]
Define $\alpha \simeq \alpha'$ if there is a DG Lie morphism

$$F : (L(V, sV, V'), D) \rightarrow (L(V, \partial))$$

satisfying $F(v) = \alpha(v)$ and $F \circ e^\theta(v) = \alpha'(v)$.

Quillen [15] and Anick [1] proved that, under Hypothesis 1.1, there is an assignment $X \mapsto (L(V, \partial))$ setting up an equivalence between the homotopy categories of $\text{CW}_m^{k+1}(R)$ and $\text{DGL}_m^k(R)$. The model $(L(V, \partial))$ recovers $R$-local homotopy invariants of $X$ via isomorphisms (with shifts)

$$\tilde{H}_*(X; R) \cong H_{*-1}(V) \quad \text{and} \quad \pi_*(X)_R \cong H_{*+1}(L(V)).$$

As for self-equivalence groups, their results directly imply identifications:

$$E(X_R) = \text{aut}(L(V, \partial)) / \cong \quad \text{and} \quad E_*(X_R) = \text{aut}_*(L(V, \partial)) / \cong$$

for these algebraic equivalence groups.

Now consider a cellular attachment $X = X^n \cup (\bigcup_{i=1}^j e_q^i)$ as in Theorem 1. Let $(L(V, \partial))$ denote the DG Lie algebra model for $X$. Then $V = V_{q-1} \oplus V_{<n}$ where $\dim V_{q-1} = j$, the number of $q$-cells attached. A homotopy self-equivalence $f : X \rightarrow X$ induces a DG Lie algebra isomorphism $\alpha : (L(V, \partial)) \rightarrow (L(V, \partial))$. Let $\tilde{\alpha}_{q-1} : V_{q-1} \rightarrow V_{q-1}$ denote the map induced on $V_{q-1}$ by restricting $\alpha$ and then projecting to $V_{q-1}$. Let $\alpha_{<n} : (L(V_{<n}, \partial_{<n})) \rightarrow (L(V_{<n}), \partial_{<n})$ denote the DG Lie algebra restriction map. We then obtain a commutative square of the form:

$$
\begin{array}{ccc}
V_{q-1} & \xrightarrow{\tilde{\alpha}_{q-1}} & V_{q-1} \\
| & & | \\
H_{q-2}(L(V_{<n})) & \xrightarrow{H(\alpha_{<n})} & H_{q-2}(L(V_{<n})) \\
| & & | \\
B_{q-1} & \xrightarrow{\partial_{q-1}} & B_{q-1}
\end{array}
$$

Here, $B_{q-1} : V_{q-1} \rightarrow H_{q-2}(L(V_{<n}))$ is given by

$$B_{q-1}(v_{q-1}) = \{\partial_{q-1}(v_{q-1})\} \in H_{q-2}(L(V_{<n})).$$

where $\{\}$ denotes the homology class of a cycle. We use this diagram to define the group $C^q$. Given a self-equivalence $\alpha$ of $L(V)$ we write $[\alpha]$ for the homotopy equivalence class in $E(L(V), \partial)$. 
**Definition 2.1** Let $C^q$ the subset of pairs $(\xi, [\gamma]) \in \text{aut}(V_{q-1}) \times \mathcal{E}(L(V_{<n}), \partial_{<n})$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
V_{q-1} & \xrightarrow{\xi} & V_{q-1} \\
\downarrow R_{q-1} & & \downarrow R_{q-1} \\
H_{q-2}(L(V_{<n})) & \xrightarrow{H(\gamma)} & H_{q-2}(L(V_{<n}))
\end{array}
\]

**Proposition 2.2** $C^q$ is a subgroup of $\text{aut}(V_{q-1}) \times \mathcal{E}(L(V_{<n}), \partial_{<n})$

**Proof** Straightforward. \(\square\)

Define $\Psi_q : \mathcal{E}(L(V_{q-1} \oplus V_{<n}), \partial) \rightarrow C^q$ by setting:

\[\Psi_q([\alpha]) = (\tilde{\alpha}_{q-1}, [\alpha_{<n}])\]

**Proposition 2.3** The map $\Psi_q$ is a surjective homomorphism

**Proof** It is easy to see $\Psi_q$ is a homomorphism. We prove surjectivity. Let $(\xi, [\gamma]) \in C^q$. Choose $\{v_i\}_{i \in J}$ as a basis of $V_{q-1}$. From the hypothesized commutative diagram we obtain $(\gamma \circ \partial - \partial \circ \xi)(v_i) \in \text{im} \partial_{<n}$. Choose $u_i \in L(V_{<n})$ of degree $q - 1$ with $(\gamma \circ \partial - \partial \circ \xi)(v_i) = \partial_{<n}(u_i)$. Define $\alpha : (L(V_{q-1} \oplus V_{<n}), \partial) \rightarrow (L(V_{q-1} \oplus V_{<n}), \partial)$ by setting $\alpha(v_i) = \xi(v_i) + u_i$ for $v_i \in V_{q-1}$ and $\alpha = \gamma$ on $V_{<n}$ and then extending. Then $\alpha$ is clearly an automorphism of $L(V)$. Observe

\[\partial \circ \alpha(v_i) = \partial(\xi(v_i)) + \partial_{<n}(u_i) = \gamma \circ \partial(v_i) = \alpha \circ \partial(v_i)\]

Thus $\alpha$ represents a class in $\mathcal{E}(L(V), \partial)$ satisfying $\Psi_q([\alpha]) = (\xi, [\gamma])$. \(\square\)

We next identify

\[\ker \Psi_q = \left\{[\alpha] \in \mathcal{E}(L(V), \partial) \mid \tilde{\alpha}_{q-1} = \text{id}_{V_{q-1}}, \alpha_{<n} \simeq \text{id}_{L(V_{<n})}\right\}\]

Let $[\alpha] \in \ker \Psi_q$. That $\tilde{\alpha}_{q-1} = \text{id}_{V_{q-1}}$ means for all $v \in V_q$ we have $\alpha(v) - v \in L_{q-1}(V_{<n})$. Here, $L_{q-1}(V_{<n})$ denotes the space of elements of $L(V_{<n})$ of degree $q - 1$. Define

$\varphi_{\alpha} : V_{q-1} \rightarrow L_{q-1}(V_{<n})$ by $\varphi_{\alpha}(v) = \alpha(v) - v \in L_{q-1}(V_{<n})$ for $v \in V_{q-1}$.

We next prove that the class $[\alpha]$ has a representative $\beta$ such that $\varphi_{\beta}(V)$ is contained in the cycles of $L_{q-1}(V_{<n})$:

**Lemma 2.4** Let $[\alpha] \in \ker \Psi_q$. Then there exists $[\beta] \in \ker \Psi_q$ satisfying

(i) $\partial(\varphi_{\beta}(v)) = 0$ for all $v \in V_{q-1}$
(ii) $\beta_{<n} = \text{id}_{L(V_{<n})}$.
(iii) $\alpha \simeq \beta$. 

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Proof Since, $\alpha_{<n} \simeq \text{id}_{L(V_{<n})}$ there is a homotopy $F: (L(V_{<n}), \partial_{<n}) \to (L(V_{<n}), \partial_{<n})$ satisfying $F(v) = v$ and $F \circ e^\theta(v) = \alpha_{<n}$ for $v \in V_{<n}$. Define $\beta$ by setting

$$
\beta(v) = \begin{cases} 
& v 
& \text{for } v \in V_{<n} \\
& \alpha(v) - F \left( \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v) \right) 
& \text{for } v \in V_{q-1}.
\end{cases}
$$

Given $v \in V_{q-1}$ we compute:

$$
\partial(\varphi_\beta(v)) = \partial \left( \alpha(v) - F \left( \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v) \right) \right) - \partial(v)
$$

$$
= \alpha_{<n}(\partial(v)) - \partial \circ F \left( \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v) \right) - \partial(v)
$$

$$
= F \circ e^\theta(\partial(v)) - F \circ D \left( \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v) \right) - \partial(v)
$$

$$
= F \circ D \circ e^\theta(v) - F \circ D \left( \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v) \right) - \partial(v)
$$

$$
= F \circ D(v + v') - \partial(v)
$$

$$
= 0.
$$

Thus $\beta$ satisfies (i). For (ii), we define $G: (L(V), \partial) \to (L(V), \partial)$ by setting $G = F$ on $(L(V_{<n}), \partial_{<n})$ while, for $v \in V_{q-1}$, we set $G(v) = \beta(v)$ and $G(v') = G(sv) = 0$. It is easy to check that $G$ is a DG Lie algebra map. Given $v \in V_{q-1}$, we have

$$
G \circ e^\theta(v) = G \left( v + v' + \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v) \right)
$$

$$
= G(v) + G \left( \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v) \right)
$$

$$
= \beta(v) + F \left( \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v) \right)
$$

$$
= \alpha(v).
$$

Using Lemma 2.4 (i), we define a map

$$
\Theta_q : \ker \Psi_q \to \text{Hom}(V_{q-1}, H_{q-1}(L(V_{<n}))) \text{ by } \Theta_q([\beta])(v) = \{\varphi_\beta(v)\} \text{ for } v \in V_{q-1}
$$

where $\beta$ is chosen as in Lemma 2.4.
Proposition 2.5 The map
\[ \Theta_q : \ker \Psi_q \to \text{Hom}(V_{q-1}, H_{q-1}(L(V_{<n}))) \]
is an isomorphism.

Proof First we prove that \( \Theta_q \) is well-defined. Suppose \( \beta \simeq \beta' \) satisfy the conclusion of Lemma 2.4. Since, both maps then restrict to the identity on \( L(V_{<n}) \), the homotopy \( F : (L(V), \partial)_I \to (L(V), \partial) \) between them can be chosen so that \( F(V_{<n}^') = F(sV_{<n}) = 0 \). Given \( v \in V_{q-1} \) suppose \( \varphi_{\beta}(v) = \{ y \} \) and \( \varphi_{\beta'}(v) = \{ y' \} \) for cycles \( y, y' \in L_{q-1}(V_{<n}) \). We then have
\[
y' - y = \beta'(v) - \beta(v)
= F \circ e^0(v) - F(v)
= F(v) + F(v') + F\left( \sum_{n \geq 1} \frac{1}{n!} (S \circ \partial)^n(v) \right) - F(v)
= F(D(sv)) + F\left( \sum_{n \geq 1} \frac{1}{n!} (S \circ \partial)^n(v) \right)
= \partial(F(sv))
\]
Thus \( y' - y \) is a boundary.

It is easy to check \( \Theta_q \) is a homomorphism. For injectivity, suppose \( \Theta_q([\beta])(v) = \Theta_q([\beta'])(v) \) in \( H_{n+1}(L(V_{<n-1}), \partial_{<n}) \) for all \( v \in V_{q-1} \). Then \( \text{im}\{\beta - \beta' : L(V) \to L(V)\} \) is contained in an acyclic sub DG Lie algebra of \( (L(V), \partial) \). Thus \( \beta \simeq \beta' \) by [17, Prop.II.5(4)].

Finally, given a homomorphism \( \psi \in \text{Hom}(V_{q-1}, H_{q-1}(L(V_{<n}), \partial_{<n})) \), we define \( \beta : (L(V), \partial) \to (L(V), \partial) \) by:
\[ \beta(v) = v + \psi(v) \text{ for } v \in V_{q-1} \text{ and } \beta = \text{id on } V_{<n}. \]
Then \( \beta \) is a DG Lie morphism with \( \Theta_q([\beta]) = \psi. \)

Summarizing, we have proven:

Theorem 2.6 Let \( (L(V), \partial) \) be an object in \( DGL_{m}^k(R) \) with \( V = V_{q-1} \oplus V_{<n} \) for \( q > n \). Then, there exists a short exact sequence of groups:
\[
0 \to \text{Hom}(V_{q-1}, H_{q-1}(L(V_{<n}))) \to \mathcal{E}(L(V), \partial) \to C^q \to 0.
\]

The first exact sequence in Theorem 1 is a direct consequence.

Proof of Theorem 1 Part (1). The result follows from Theorem 2.6, the isomorphisms \( \mathcal{E}(X_R) \cong \mathcal{E}(L(V), \partial), \pi_q(X^n)_R \cong H_{q-1}(L(V_{<n})) \) and the identification of \( V_{q-1} \) with the free \( R \)-module with generators corresponding to the \( q \)-cells of \( X \). See [1, Theorem 8.5].

\( \Box \)
We now focus on the group $\mathcal{E}_s(L(V), \partial)$ and the proof of Theorem Part (2). Again, we take $(L(V), \partial)$ to be an object in $\text{DGL}^k_m(R)$ with $V = V_{q-1} \oplus V_{<n}$ for $m \leq n < q \leq k$. When $q = n + 1$ we must take into account the linear differential $d_n: V_n \to V_{n-1}$. Since $V_n$ is a free $R$-module, we may choose a subspace $W_n$ of $V_n$ complementary to the $d_n$-cycles in $V_n$ giving $V_n = (\ker d_n) \oplus W_n$. When $q > n + 1$ we set $W_{q-1} = V_{q-1}$.

Let $\alpha \in \text{aut}_s(L(V), \partial)$ be given and, as usual, let $\tilde{\alpha}_{q-1}: V_{q-1} \to V_{q-1}$ denote the induced map. Since, $\alpha$ induces the identity on $H_*(V)$, we see $\tilde{\alpha}_{q-1}$ fixes $\ker d_{q-1} = H_{q-1}(V)$. It follows that $\alpha$ induces a map $\tilde{\alpha}'_{q-1}: W_{q-1} \to W_{q-1}$.

**Definition 2.7** Let $C^q_s$ denote the subset of pairs $(\chi, [\eta]) \in \text{aut}(W_{q-1}) \times \mathcal{E}_s(L(V_{<n}), \partial_{<n})$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
W_{q-1} & \xrightarrow{\chi} & W_{q-1} \\
B_{q-1} \downarrow & & \downarrow B_{q-1} \\
H_{q-2}(L(V_{<n})) & \xrightarrow{H(\eta)} & H_{q-2}(L(V_{<n}))
\end{array}
\]

**Remark 2.8** If $q > n + 1$ or if $d_n = 0$ then $W_n = \{0\}$. In this case $C^q_s = C^q \cap \mathcal{E}_s(L(V_{<n}), \partial_{<n})$.

We prove:

**Theorem 2.9** Let $(L(V), \partial)$ be an object in $\text{DGL}^k_m(R)$ with $V = V_{q-1} \oplus V_{<n}$ for $q > n$. Then, there is a short exact sequence:

\[
0 \to \text{Hom}(V_{q-1}, H_{q-1}(L(V_{<n}))) \to \mathcal{E}_s(L(V), \partial) \to C^q_s \to 0
\]

**Proof** Define $\Gamma_q: \mathcal{E}_s(L(V), \partial) \to C^q_s$ by $\Gamma_q(\alpha) = (\tilde{\alpha}'_{q-1}, [\alpha_{<n}])$. We claim $\Gamma_q$ is surjective. For given $(\chi, [\eta]) \in C^q_s$ as in Definition 2.7, we can extend $\chi$ to a map $\xi: V_{q-1} \to V_{q-1}$ by setting $\xi = \text{id}$ on $\ker d_{q-1}$. Then the pair $(\xi, [\eta]) \in C^q$ and so, by Proposition 2.3, there exists $[\alpha] \in \mathcal{E}(L(V), \partial)$ with $\Psi([\alpha]) = (\xi, [\eta])$ and, further, $\alpha$ may be chosen in $\text{aut}_s(L(V), \partial)$ since $\xi$ and $\eta$ fix $H_*(V)$. Thus $\Gamma_q([\alpha]) = (\chi, [\eta])$. Finally, observe that $\ker \Gamma_q = \ker \Psi_q$ and the result follows from Proposition 2.5. □

We deduce:

**Proof of Theorem 1 Part (2).** The result follows again from the Quillen-Anick identifications as in the proof of Theorem 1, above. In this case, we note that the linear differential $d_n: V_n \to V_{n-1}$ corresponds linking homomorphism $\pi_{n+1}(X^{n+1}, X^n)_R \to \pi_n(X^n, X^{n-1})_R$ in the long exact sequence of the triple. □

**Remark 2.10** The exact sequences in Theorems 2.6 and 2.9 do not split in general. One simple criterion for splitting occurs when $\partial_{<n} = 0$. □
3 Self-equivalences of $R$-local spaces

We begin with a result on the full group $\mathcal{E}(X_R)$. The following was stated in the introduction as Example 1.2.

**Theorem 3.1** Let $G_1, \ldots, G_r$ be finitely generated abelian groups. Given $R \subseteq \mathbb{Q}$, suppose $X = M(G_1, m+1) \vee M(G_2, m+3) \vee \cdots \vee M(G_r, m+2r-1)$ for $r \leq m/2$ and $X$ an object of $\mathbb{C}W_m^{k+1}$ satisfying Hypothesis 1.1. Then

$$\mathcal{E}(X_R) \cong \prod_{i=1}^{r} \text{aut}((G_i)_R).$$

**Proof** Recall $M(G, i)$ is $(i-1)$-connected and of dimension $\leq i + 1$. It follows that, in the Anick model $(L(V), \partial)$ for $X$, we have $V = V_m \oplus V_{m+1} \oplus \cdots \oplus V_{m+2r-1}$ with $\partial = d$ purely linear and taking the form

$$
v_{m+2r-1} \xrightarrow{d_{m+2r-1}} v_{m+2r-2} \xrightarrow{0} v_{m+2r-3} \xrightarrow{d_{m+2r-3}} v_{m+2r-4} \xrightarrow{0} \cdots
$$

$$
\cdots \xrightarrow{0} v_{m+3} \xrightarrow{d_{m+3}} v_{m+2} \xrightarrow{0} v_{m+1} \xrightarrow{d_m} v_{m} \xrightarrow{0} 0.
$$

Here,

$$(G_i)_R \cong \frac{V_m}{\text{im } d_{m+2i-1}}.$$  

Note $\mathcal{C}^{m+2}$ consists of pairs $(\xi_{m+1}, \lambda_m) \in \text{aut}(V_{m+1}) \times \text{aut}(V_m)$ with $d_{m+1} \circ \xi_{m+1} = \lambda_m \circ d_{m+1}$ Since $(G_1)_R \cong \frac{V_m}{\text{im } d_{m+1}}$ we deduce that $\mathcal{C}^{m+2} \cong \text{aut}(G_R)$. Since, $H_{m+1}(L(V_{\leq m})) = 0$, invoking Theorem 2.6 we deduce $\mathcal{E}(L(V_{\leq m+1})) \cong \text{aut}(G_R)$.

Now proceed by induction. Assume $r > 1$ and $r \leq m/2$, as hypothesized. The latter assumption ensures $H_{m+2r-1}(L(V_{\leq m+2r-2})) = H_{m+2r-2}(L(V_{\leq m+2r-3})) = 0$. Since, $d_{m+2r-2} = 0$ we have

$$\mathcal{C}^{m+2r-1} = \text{aut}(V_{m+2r-2}) \times \mathcal{E}(L(V_{\leq m+2r-3}), \partial_{\leq m+2r-3}).$$

Since, $H_{m+2r-2}(L(V_{\leq m+2r-3})) = 0$, by Theorem 2.6 and the induction hypothesis we obtain

$$\mathcal{E}(L(V_{\leq m+2r-2}), \partial_{\leq m+2r-2}) \cong \text{aut}(V_{m+2r-2}) \times \prod_{i=1}^{r-1} \text{aut}((G_i)_R),$$

Finally, note $\mathcal{C}^{m+2r}$ is the set of triples

$$(\xi_{m+2r-1}, \lambda_{m+2r-2}, \alpha) \in \text{aut}(V_{m+2r-1}) \times \text{aut}(V_{m+2r-2}) \times \prod_{i=1}^{r-1} \text{aut}((G_i)_R)$$
such that the pair \((\xi_{m+2r-1}, \lambda_{m+2r-2}) \in \text{aut}(V_{m+2r-1}) \times \text{aut}(V_{m+2r-2})\) gives a commutative diagram:

\[
\begin{array}{ccc}
V_{m+2r-1} & \xrightarrow{\xi_{m+2r-1}} & V_{m+2r-1} \\
d_{m+2r-1} & \downarrow & d_{m+2r-1} \\
V_{m+2r-2} & \xrightarrow{\lambda_{m+2r-2}} & V_{m+2r-2}
\end{array}
\]

We conclude \(C^{m+2r} \cong \text{aut}((G_{r})_{R}) \times \prod_{i=1}^{r-1} \text{aut}((G_{i})_{R})\). Theorem 2.6 and the fact that \(H_{m+2r-1}(L(V_{\leq m+2r-2})) = 0\) now completes the induction and the proof. \(\square\)

We deduce the following direct consequence of Theorem 1 (2).

**Corollary 3.2** Let \(R \subseteq \mathbb{Q}\) and \(X = X^{q} = X^{n} \cup_{a} \left( \bigcup_{i=1}^{j} e_{i}^{q} \right)\) satisfy Hypothesis 1.1. When \(q = n+1\) suppose further that the linking homomorphism \(\pi_{n+1}(X^{n+1}, X^{n})_{R} \rightarrow \pi_{n}(X^{n}, X^{n-1})_{R}\) vanishes. Then \(\mathcal{E}_{*}(X^{n})_{R} = 0\) implies \(\mathcal{E}_{*}(X^{q})_{R} \cong \bigoplus_{i=1}^{j} \pi_{q}(X^{n})_{R}\). \(\square\)

We apply this result to give a calculation of \(\mathcal{E}_{*}(X)_{R}\). Given \(R\)-modules \(H_{i}, H_{j}\), let \([H_{i}, H_{j}] \subseteq (H_{i} \otimes H_{j}) \oplus (H_{j} \otimes H_{i})\) denote the \(R\)-submodule given by:

\([H_{i}, H_{j}] = R(x_{i} \otimes x_{j} - (-1)^{(i-1)(j-1)}x_{j} \otimes x_{i} | x_{i} \in H_{i}, x_{j} \in H_{j}\)\]

Here, we write \(R\langle \rangle\) to denote a free \(R\)-module on given generators. Given subspaces \(V, W \subseteq L\) with \(L\) a graded Lie algebra we similarly write

\([V, W] = R\langle v, w | v \in V, w \in W\rangle\).

We prove

**Theorem 3.3** Let \(R \subseteq \mathbb{Q}\) and \(X\) in \(\text{CW}_{m}^{k+1}\) satisfy Hypothesis 1.1. Suppose \(X\) has a cellular decomposition of the form

\(X^{m+1} \subset X^{m+2} \subset \cdots \subset X^{2m} = X\)

such that that the linking homomorphisms \(\pi_{r+1}(X^{r+1}, X^{r})_{R} \rightarrow \pi_{r}(X^{r}, X^{r-1})_{R}\) vanish for \(r = m+2, \ldots, 2m-1\). Suppose \(\mathcal{E}_{*}(X^{m+1})_{R} = 0\). Then

\(\mathcal{E}_{*}(X^{i})_{R} = 0, \text{ for } i \leq 2m - 1\) and \(\mathcal{E}_{*}(X^{2m})_{R} \cong \bigoplus_{i=1}^{j} \left[ H_{m+1}(X; R), H_{m+1}(X; R) \right]\),

where \(j = \dim_{R}(H_{2m}(X, X_{2m-1}; R))\).

**Proof** By the freeness of the Anick model as DG Lie algebra over \(R\) we obtain

\(H_{m+1}(L(V_{\leq m})) = H_{m+2}(L(V_{\leq m+1})) = \cdots = H_{2m-1}(L(V_{\leq 2m-2})) = 0\).
Since, by hypothesis, $\mathcal{E}_*(X^{m+1})_R = 0$, applying Corollary 3.2 repeatedly gives

$$\mathcal{E}_*(X^{m+2})_R = \mathcal{E}_*(X^{m+3})_R = \cdots = \mathcal{E}_*(X^{2m-1})_R = 0.$$ 

Applying this result again then gives

$$\mathcal{E}_*(X^{2m})_R \cong \bigoplus_{i=1}^{j} \pi_{2m+1}(X)_R.$$

Using the Anick model, we compute

$$\pi_{2m+1}(X)_R \cong [V_m, V_m] \cong [H_{m+1}(X; R), H_{m+1}(X; R)].$$

\[\Box\]

The following result was stated as Example 1.3 in the introduction.

**Corollary 3.4** Let $G$ be a finitely generated abelian group and $R \subseteq \mathbb{Q}$ with least invertible prime $p$. Let $X$ be in $CW_m^{k+1}$ as in Hypothesis 1.1 with cellular decomposition of the form

$$M(G, m + 1) = X^{m+2} \subset X^{m+3} \subset \cdots \subset X^{2m} = X.$$

Assume that the linking homomorphisms $\pi_{r+1}(X^{r+1}, X^r)_R \to \pi_r(X^r, X^{r-1})_R$ vanish for $r = m + 2, \ldots, 2m - 1$ and that $\dim_R(H_{2m}(X, X^{2m-1}; R)) = j$. Then

$$\mathcal{E}_*(X)_R \cong \bigoplus_{i=1}^{j} [G_R, G_R]$$

where $G_R = H_{m+1}(M(G, m + 1); R)$ is of degree $m + 1$.

**Proof** By [5, Theorem 3.2], $\mathcal{E}_*(M(G, m + 1)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$. By Hypothesis 1.1, the prime $p = 2$ is invertible in $R$. Thus $\mathcal{E}_*(M(G, m + 1))_R = 0$. The result now follows from Theorem 3.3.\[\Box\]

We can easily compute both groups for a tame product of spheres. The following was stated as Example 1.4 in the introduction.

**Theorem 3.5** Let $R \subseteq \mathbb{Q}$ be a ring with least invertible prime $p$. Let $m \leq n$ be chosen so that $m, k = m + n + 1$ and $p$ satisfy Hypothesis 1.1. Then

$$\mathcal{E}_*(S^{m+1} \times S^{n+1})_R \cong \begin{cases} R^* \times R^* & \text{for } n \neq 2m \text{ or } m \text{ even.} \\ R \oplus (R^* \times R^*) & \text{for } n = 2m \text{ and } m \text{ odd.} \end{cases}$$

$$\mathcal{E}_*(S^{m+1} \times S^{n+1})_R \cong \begin{cases} 0 & \text{for } n \neq 2m \text{ or } m \text{ even.} \\ R & \text{for } n = 2m \text{ and } m \text{ odd.} \end{cases}$$

\[\Box\]
Proof Let \( X = S^{m+1} \times S^{n+1} \). We can write the Anick model as \( (L(u, v, w), \partial) \) where \( |u| = m, |v| = n, |w| = n+m+1 \) with \( \partial(u) = \partial(v) = 0 \) and \( \partial(w) = [u, v] \). Theorem 2.6 gives a short exact sequence:

\[
0 \rightarrow H_{m+n+1}(L(u, v)) \rightarrow \mathcal{E}(L(u, v), \partial) \rightarrow C^{m+n+2} \rightarrow 0.
\]

For degree reasons, \( H_{m+n+1}(L(u, v)) = 0 \). Thus \( \mathcal{E}(X_R) \cong C^{m+n+2} \) and we compute the latter group.

The group \( C^{m+n+2} \) consists of pairs \((\xi, [\alpha])\) \(\in\) \(\text{aut}(R\langle w\rangle) \times \mathcal{E}(L(u, v), 0)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
R\langle w \rangle & \xrightarrow{\xi} & R\langle w \rangle \\
B_{m+n+1} \downarrow & & \downarrow B_{m+n+1}
\end{array}
\]

\[
\begin{array}{ccc}
R\langle [u, v] \rangle & \xrightarrow{H_{m+n}(\alpha)} & R\langle [u, v] \rangle \\
H_{m+n}(\alpha) \downarrow & & \downarrow H_{m+n}(\alpha)
\end{array}
\]

The map \( \xi \) is determined by \( H_{m+n}(\alpha) \). Thus \( C^{m+n+2} \cong \mathcal{E}(L(u, v), 0) \). Applying Theorem 2.6 again gives a short exact sequence:

\[
0 \rightarrow H_n(L(u)) \rightarrow \mathcal{E}(L(u, v), 0) \rightarrow C^{n+1} \rightarrow 0.
\]

The sequence splits since the differential vanishes. As above, \( C^{n+1} \cong \text{aut}(L(u, v)) \cong R^* \times R^* \). If \( n \neq 2m \) or \( m \) odd then \( H_n(L(u)) = 0 \). Otherwise, for \( n = 2m \) and \( m \) even, \( H_n(L(u)) = R \). The result for \( \mathcal{E}(X_R) \) follows.

The proof for \( \mathcal{E}_*(X)_R \) is similar. Theorem 2.9 gives \( \mathcal{E}_*(L(u, v, w), \partial) \cong C^{m+n+2}_* \). As above, \( C^{m+n+2}_* \cong \mathcal{E}_*(L(u, v); 0) \). Applying Theorem 2.9 again gives \( \mathcal{E}_*(L(u, v); 0) \cong H_n(L(u)) \) and the result follows.

We next give a general result showing that finiteness of \( \mathcal{E}(X_R) \) is not preserved by cell attachments in consecutive degrees:

**Theorem 3.6** Let \( R \subset \mathbb{Q} \) and and \( X \in \text{CW}^{k+1} \) be as in Theorem 1. Suppose \( X = X^n \cup (\bigcup_{i=1}^j e^{n+1}_i) \) for \( j > 0 \). Assume that the linking homomorphism \( \pi_{n+1}(X^{n+1}, X^n)_R \rightarrow \pi_n(X^n, X^{n-1})_R \) vanishes. Then \( \mathcal{E}(X^n_R) \) finite implies \( \mathcal{E}(X^{n+1}_R) \) is infinite.

**Proof** Our hypothesis on the linking homomorphism ensures \( d_n : V_n \rightarrow V_{n-1} \) is zero. Thus given \( v \in V_n, \partial(v) \in L^{n-1}(V_{n-1}) \). Since, \( \mathcal{E}(X^n_R) \cong \mathcal{E}(L(V_{<n}), \partial_{<n}) \) is finite, applying Theorem 2.6 gives \( H_{n-1}(L(V_{<n})) = 0 \). It follows that the map \( B_n = 0 : V_n \rightarrow H_{n-1}(L(V_{<n})) \). Given \( v \in V_n, \gamma = \text{id} : (L(V_{<n}), \partial) \rightarrow (L(V_{<n}), \partial) \) and \( \xi^a \in \text{aut}(V_n), a \in R \) with \( \xi^a(v) = av \) for \( v \in V_n \). The following diagram is obviously commutative:

\[
\begin{array}{ccc}
V_n & \xrightarrow{\xi^a} & V_n \\
B_n = 0 \downarrow & & \downarrow B_n = 0 \\
H_{n-1}(L(V_{<n})) & \xrightarrow{H_{n-1}(\gamma)} & H_{n-1}(L(V_{<n})).
\end{array}
\]
Therefore, there exists an infinity of pairs \((\xi^a, [\text{id}]) \in C_{n+1}\). Since, \(C_{n+1}\) is infinite, 
\(\mathcal{E}(X_R^{n+1}) \cong \mathcal{E}(L(V_{\leq n}, \partial_{\leq n})\) is infinite by Theorem 2.6. \(\square\)

When \(R = \mathbb{Q}\) the result becomes:

**Corollary 3.7** Let \(X\) be a finite, simply connected CW complex. Suppose that \(\pi_{n+1}(X^{n+1}, X^n)_{\mathbb{Q}} \neq 0\). Then \(\mathcal{E}(X^n_{\mathbb{Q}})\) finite implies \(\mathcal{E}(X^{n+1}_{\mathbb{Q}})\) is infinite. \(\square\)

Finally, we show \(\mathcal{E}(X_R)\) is infinite for CW complexes of small dimension.

**Theorem 3.8** Let \(X\) be a simply-connected finite CW complex of dimension \(\leq 5\). Let \(R \subseteq \mathbb{Q}\) have least invertible prime \(p \geq 7\). If \(p\) is finite, assume the linking homomorphisms \(\pi_{r+1}(X^{r+1}, X^r) \to \pi_r(X^r, X^{r-1})_R\) vanish for \(r = 2, 3, 4\). Then \(\mathcal{E}(X_R)\) is infinite.

**Proof** Let \((L(V), \partial)\) denote the Anick model for \(X\). Our hypothesis on the linking homomorphism implies \(d = 0\). Since, \(X_R\) is not contractible, \(V \neq 0\).

When \(X = X^3\) then for degree reasons \(\partial = 0\). It follows that \(\mathcal{E}(X^3_R) \cong \text{aut}(V)\) which is infinite since \(V \neq 0\).

Next suppose \(X = X^4\). By Theorem 1 (1), it suffices to show that \(\mathcal{C}^4\) is infinite. Here, \(V = V_3 \oplus V_2 \oplus V_1\) with \(\partial(V_3) \subseteq [V_1, V_1]\) and at least one \(V_i \neq 0\). The group \(\mathcal{C}^4\) consists of pairs \((\xi, \alpha)\) where \(\alpha : L(V_3 \oplus V_1) \to L(V_2 \oplus V_1)\) is an automorphism and \(\xi : V_3 \to V_3\) makes the diagram commute:

\[
\begin{array}{ccc}
V_3 & \xrightarrow{\xi} & V_3 \\
\partial_3 & & \partial_3 \\
[V_1, V_1] & \xrightarrow{\alpha} & [V_1, V_1]
\end{array}
\]

Given \(a \neq 0\), define \(\alpha^a(v) = av\) for \(V_{\leq 2}\) Define \(\xi^{a^2}(v) = a^2 v\) for \(v \in V_3\). This gives an infinity of distinct pairs \((\alpha^a, \xi^{a^2})\) in \(\mathcal{C}^4\).

For the case \(X = X^5\) we have \(V = V_{\leq 4}\). Again \(\partial_3(V_3) \subseteq [V_1, V_1]\) by minimality. We identify \(H_4(L(V_{\leq 3}))\) as vector space:

\[
H_4(L(V_{\leq 3})) = [\ker \partial_3, V_1] \oplus [V_2, V_2] \oplus \frac{[V_2, [V_1, V_1]]}{[\partial_3(V_3), V_2]} \oplus \frac{[V_1, [V_1, [V_1, V_1]]]}{[\partial_3(V_3), [V_1, V_1]]}.
\]

By Theorem 2.6, if \(H_4(L(V_{\leq 3})) \neq 0\) then \(\mathcal{E}(X^5_{\mathbb{Q}})\) is infinite. Thus we may assume \(H_4(L(V_{\leq 3})) = 0\) which forces \(\text{dim} V_2 \leq 1\). If \(V_1 = 0\) then \(\partial = 0\) and \(\mathcal{E}(X_R) \cong \text{aut}(V)\) is infinite. So assume \(V_1 \neq 0\). Then we must have \(\ker \partial_3 = 0\) and \(\partial_3(V_3) = [V_1, V_1]\). Again, it suffices to show \(\mathcal{C}^5\) is infinite. We note that

\[
H_3(L(V_{\leq 3})) = \ker \partial_3 \oplus [V_2, V_1] \oplus \frac{[V_1, [V_1, V_1]]}{[\partial_3(V_3), V_1]} \cong [V_2, V_1]
\]

\(\square\) Springer
Then $C^5$ is the set of pairs $(\xi, [\alpha])$ with $\xi \in \text{aut}(\mathbb{Q}(w))$ and $\alpha \in \text{aut}(L(V_{\leq 3}), \partial_{\leq 3})$ making the diagram commute:

$$
\begin{array}{ccc}
V_4 & \xrightarrow{\xi} & V_4 \\
\downarrow{\alpha_4} & & \downarrow{\alpha_4} \\
[V_2, V_1] & \xrightarrow{H(\alpha)} & [V_2, V_1].
\end{array}
$$

Given $a \in R^*$ define a DG Lie map $\alpha^a : (L(V_{\leq 3}), \partial_{\leq 3}) \to (L(V_{\leq 3}), \partial_{\leq 3})$ by setting

$$
\alpha(v) = av \text{ for } v \in V_1 \quad \text{and} \quad \alpha(u) = a^2u \text{ for } u \in V_3 \oplus V_2,
$$

and extending. We then obtain an infinity of pairs $(\xi^a, [\alpha^a]) \in C^5$ where $\alpha^a(x) = a^3x$ for $x \in V_4$. \hfill \Box

We conclude by proposing a problem. By Costoya–Viruel [10], every finite group $G$ occurs as $\mathcal{E}(X_{\mathbb{Q}})$. The construction of $X$ for a given $G$ requires cells (cohomology classes) in a wide range of dimensions. This suggests the following:

**Problem 3.9** Given $R \subseteq \mathbb{Q}$, find the smallest $n \geq 1$ such that there exists a simply connected CW complex $X$ with $\dim X \leq n$, $X_{R}$ non-contractible and $\mathcal{E}(X_{R})$ is finite.

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