L-KURAMOTO-SIVASHINSKY SPDEs ON AND IN \( \{\mathbb{R}_+ \times \mathbb{R}^d\}_{d=1}^3 \):
THE L-KS KERNEL, SHARP HÖLDER REGULARITY, AND SWIFT-HOHENBERG LAW EQUIVALENCE

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Abstract. Generalizing our L-Kuramoto-Sivashinsky (L-KS) kernel—introduced earlier in [7]—we give a novel explicit-kernel formulation useful for a large class of fourth order deterministic, stochastic, linear, and nonlinear PDEs in multispatial dimensions. These include pattern formation equations like the Swift-Hohenberg (SH) and many other prominent and new PDEs. We establish existence, uniqueness, and sharp dimension-dependent spatio-temporal Hölder regularity for the canonical L-KS SPDE, driven by white noise on \( \{\mathbb{R}_+ \times \mathbb{R}^d\}_{d=1}^3 \). The spatio-temporal Hölder exponents are exactly the same as the striking ones we proved for our recently introduced Brownian-time Brownian motion (BTBM) stochastic integral equation—linked to totally different time-fractional, memoryful, and positive biLaplacian fourth order PDEs. From our recent results in [1], the spatial Hölder regularity interval length and the third dimension random field limit here are maximal among equations driven by space-time white noise that are first order in time and high order in space. The challenge in establishing our Hölder exponents here is that, unlike the positive BTBM density, the L-KS kernel is the Gaussian average of a highly oscillatory complex modified Schrödinger propagator. Thus, we modify our BTBM methods by using a combination of harmonic analysis and delicate analysis, including adaptations of our techniques in [2], to get the necessary estimates. Attaching order parameters to the L-KS spatial operator and the noise term, we give a dimension-dependent order parameters ratio that controls the limiting interaction between the two opposing forces. In particular, this critical ratio decides whether the \( L^p \) distance between an L-KS SPDE and its corresponding L-KS PDE uniformly vanishes or whether we get a finite-time \( L^2 \) blowup of L-KS SPDEs as the ratio goes to zero or infinity, respectively. Finally, we give a change-of-measure equivalence between the canonical L-KS SPDE and nonlinear L-KS SPDEs on \( \{\mathbb{R}_+ \times \mathbb{R}^d\}_{d=1}^3 \) and subsets thereof. In particular, we prove the law equivalence of the SH SPDE and the canonical L-KS SPDE on compact subsets. Hence, all these equations have the same regularity.

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1. Introduction and statements of results

We give a novel, unifying, and very useful explicit-kernel (mild) formulation for a large class of linear, nonlinear, deterministic, and stochastic fourth order PDEs that includes many new, as well as prominent, equations. We focus in this article on the L-Kuramoto-Sivashinsky (L-KS) stochastic PDEs (1.1):

\[
\begin{align*}
\frac{\partial U}{\partial t} &= -\frac{\varepsilon}{8} (\Delta + 2\vartheta)^2 U + b(U) + a(U) \frac{\partial^{d+1} W}{\partial t \partial x}, \\
U(0, x) &= u_0(x),
\end{align*}
\]

where \((\varepsilon, \vartheta) \in (0, \infty) \times \mathbb{R}\) is a pair of parameters, \(a, b : \mathbb{R} \rightarrow \mathbb{R}\) and \(u_0 : \mathbb{R}^d \rightarrow \mathbb{R}\) are Borel measurable, and \(\partial^{d+1} W/\partial t \partial x\) is the space-time white noise corresponding to the real-valued Brownian sheet \(W\) on \(\mathbb{R}_+ \times \mathbb{R}^d, d = 1, 2, 3\). In particular, \(b(u)\) may be a polynomial of (1) Allen-Cahn type \(b(u) = \sum_{k=0}^{2p-1} c_k u^k\) for \(p \in \mathbb{N}\) and for \(c_{2p-1} < 0\), to get many interesting fourth order SPDEs with an Allen-Cahn type nonlinearity (including a generalized Swift-Hohenberg SPDE when \(\varepsilon, \vartheta > 0\)), or of (2) KPP type \(b(u) = \sum_{k=0}^{2p} c_k u^k\) for \(p \in \mathbb{N}\) and for \(c_{2p} < 0\), to get new fourth order SPDEs. The name comes from the fundamental role of the linearized KS operator \(-\frac{\varepsilon}{8} (\Delta + 2\vartheta)^2\) in the nonlinear SPDE (1.1).  

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SPDEs with a KPP type nonlinearity. We then use our explicit-kernel formulation to obtain, among other things, existence, uniqueness, and dimension-dependent Hölder regularity results with sharp spatio-temporal Hölder exponents for versions of the fourth order SPDE (1.1). More specifically, our first result Theorem 1.1 establishes existence, uniqueness, and sharp dimension-dependent Hölder regularity for the zero drift ($b \equiv 0$ or canonical L-KS SPDE) fixed ($\varepsilon, \vartheta$) version of (1.1); Theorem 1.2 gives dimension-dependent order parameters limiting results on the competing interaction between the linearized Kuramoto-Sivashinsky (L-KS) operator $-\frac{\varepsilon}{2} (\Delta + 2\vartheta)^2$ and the white noise $\mathcal{W}$ in (1.1) with $b \equiv 0$ and $\vartheta$ fixed, and it gives the precise order parameters rate controlling whether the $L^p$ distance between an L-KS SPDE and its corresponding L-KS PDE uniformly vanishes (as the rate goes to zero) or whether there is a finite-time $L^2$ blowup of L-KS SPDEs (as the rate goes to infinity); and Theorem 1.3 adapts our earlier space-time change of measure results, with widely applicable conditions—from the second order SPDEs case [13, 12, 11] to our fourth order SPDEs setting here—to transfer uniqueness in law and establish the law equivalence between the zero drift ($b \equiv 0$) and the nonlinear nonzero drift versions of (1.1) on $\mathbb{R}^+ \times \mathbb{R}^d_{d=1}^3$ and compact rectangles thereof. This allows us to transfer almost sure properties of solutions—including regularity—between linear and nonlinear L-KS SPDEs in spatial dimensions $d = 1, 2, 3$. An important special case covered by Theorem 1.3 is the aforementioned Swift-Hohenberg SPDE.

We note here that the deterministic Swift-Hohenberg PDE (both real and complex) models numerous pattern formation phenomena in physics, chemistry, and optics (see e.g. [19, 24, 34, 38, 43, 44, 45, 50]). These include the Taylor-Couette flow, the Rayleigh-Bénard convection problem in a horizontal fluid layer in the gravitational field, large-scale flows and spiral core instabilities, and some chemical reactions. Also, in optics, this equation is connected to spatial structures in large aspect lasers and synchronously pumped optical parametric oscillators. The noisy Swift-Hohenberg PDE (or SPDE) treated here in Theorem 1.3 is at least as interesting and applicable. We also remark that we use our kernel representational approach in separate papers to investigate time asymptotics and other qualitative behavior of a class of fourth order equations with different nonlinearities.

**Notation 1.1.** We sometimes denote by $e^{(\varepsilon, \vartheta)}_{LKS}(a, b, u_0)$ the SPDE (1.1) on subsets of $\mathbb{R}^+ \times \mathbb{R}^d_{d=1}^3$. Similarly, the zero drift case is denoted by $e^{(\varepsilon, \vartheta)}_{LKS}(a, 0, u_0)$.

Before precisely stating our results, it is instructive to motivate and put together the building blocks—and give the different interesting links—in our approach; and then give our solution formulation and definition.

### 1.1. The L-KS PDE and L-KS kernel.

We start with what we call the $(\varepsilon, \vartheta)$ linearized Kuramoto-Sivashinsky (L-KS) PDE

\[
(1.2) \quad \frac{\partial u}{\partial t} = -\frac{\varepsilon}{2} (\Delta + 2\vartheta)^2 u; \quad t > 0, \ x \in \mathbb{R}^d, \ d \in \mathbb{N} = \{1, 2, 3 \ldots \},
\]

and we observe that it is a fundamental part of and/or intimately connected to a large family of interesting linear, nonlinear, deterministic, and stochastic PDEs. This family includes, but is not limited to, both prominent and new compelling fourth order equations—including pattern formation equations—like

\[3\text{The corresponding deterministic PDEs are, of course, obtained by simply setting } a \equiv 0.\]
(1) The PDE\footnote{Throughout the article we alternate freely between the notations \( \partial_x \) and \( \partial / \partial x \) (or \( d / dx \)) for partial (or full) derivatives, with respect to any variable \( x \), for aesthetic and typesetting reasons.} \( \partial_t u = -\frac{\epsilon}{8} (\Delta + 2\partial)^2 u + b(u) \), which includes the Swift-Hohenberg (SH) PDE (when \( b(u) \) is an Allen-Cahn type nonlinearity and \( \epsilon, \partial > 0 \)) as well as many other interesting equations;

(2) variants/versions of the Cahn-Hilliard PDE \( \partial_t u = -\frac{\epsilon}{8} \Delta^2 u + \Delta b(u) \), where \( b \) may be an Allen-Cahn type nonlinearity \( b(u) = \sum_{k=0}^{2p-1} c_k u^k, p \in \mathbb{N}, \) and \( c_{2p-1} < 0, \epsilon > 0; \)

(3) variants/versions of the Kuramoto-Sivashinsky (KS) PDE like

\begin{align}
\partial_i u &= -\frac{\epsilon}{8} \Delta^2 u - \alpha_1 \Delta u - \alpha_2 \nabla (u, u), \\
\partial_i u &= -\frac{\epsilon}{8} (\Delta + 2\partial)^2 u - \frac{1}{2} \nabla (u, u),
\end{align}

where \( \alpha_1, \alpha_2, \epsilon, \partial > 0; \)

and the stochastic versions of all the above PDEs, as well as many more new and intriguing fourth order (S)PDEs. Some of these nonlinear equations mentioned have been studied, and continue to be studied, extensively in the deterministic literature (e.g., \[22, 23, 30, 31, 33, 36, 47\] and the SH references above) and is catching up on the still growing stochastic side (e.g., \[28, 29, 49, 51, 50\]), where the effect of the noise on the qualitative behavior of the underlying PDEs is of great interest.

When they are studied in the presence of a driving space-time white noise—\textit{with} only few exceptions like \[25\] and, recently, our work on higher order stochastic equations \[11, 2, 6\]—these fourth order equations are invariably restricted to one spatial dimension \( d = 1 \). On the other hand, in our earlier work \[10, 9, 7\], we introduced and connected a large class of processes—in which the time parameter is replaced in different ways by a Brownian motion—to new memory-preserving (\textit{memoryful}) fourth order PDEs and to the linearized KS PDE \[12\] with \( \epsilon = \partial = 1; \)

\begin{equation}
\begin{cases}
\partial u / \partial t = -\frac{1}{8} \Delta^2 u - \frac{1}{2} \Delta u - \frac{1}{2} u, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^d, \\
u(0, x) = u_0(x), \quad x \in \mathbb{R}^d,
\end{cases}
\end{equation}

in all spatial dimension\footnote{This is important to note since one of the major challenges in the study of the nonlinear KS equation is that the existence of solutions in spatial dimensions \( d \geq 2 \) is unsettled, even in the noiseless deterministic case (see \[26\]).} \( d \geq 1 \) for suitably regular initial data \( u_0 \). At the heart of our approach in \[7\] is the kernel \( K_{t,x,y} \)—associated with what in \[7\] we call the imaginary-Brownian-time-Brownian-angle process (IBTBAP)—defined by

\begin{align}
K^{\text{ASP}}_{t,x,y} &= \exp \left( is \right) \frac{e^{-|x-y|^2/2is}}{(2\pi is)^{d/2}}, \\
K^{\text{LKS}}_{t,x,y} &= \int_{-\infty}^{0} K^{\text{ASP}}_{t,s,x,y} K^{\text{RM}}_{t,s} ds + \int_{0}^{\infty} K^{\text{ASP}}_{t,s,x,y} K^{\text{RM}}_{t,s} ds
\end{align}

where \( i = \sqrt{-1} \) and \( K^{\text{RM}}_{t,s} = e^{-s^2/2} \sqrt{2\pi} \). Since \( K^{\text{LKS}}_{t,x,y} \), obtained by setting \( y = 0 \in \mathbb{R}^d \) in \( K^{\text{LKS}}_{t,x,y} \), is the fundamental solution of the L-KS PDE in \[14\], we also call it the L-KS kernel. Quantum mechanics experts will quickly recognize that, except for
the exp (is) angle term, \( K_{t,x,y}^{\text{AS}} \) in the definition of the L-KS kernel in (1.5) is a \( d \)-dimensional version of the free propagator associated with Schrödinger’s equation. It is then proved in Theorem 1.1 of [7] that, for \( u_0 \in C^2(R^d, \mathbb{R}) \),

\[
(1.6) \quad u(t, x) = \int_{\mathbb{R}^d} K_{t,x,y}^{\text{LKS}} u_0(y) dy
\]
solves the linearized Kuramoto-Sivashinsky PDE (1.4) and hence that the kernel \( K_{t,x}^{\text{LKS}} \) solves the PDE L-KS in (1.4) with initial condition \( \delta(x) \).

1.2. Imaginary-Brownian-time-Brownian-angle and Schrödinger links. An important intuitive ingredient in the formulation and proof of Theorem 1.1 of [7], and in arriving at the kernel \( K_{t,x}^{\text{LKS}} \), was the use of the intimate connection between the Brownian-time processes and their densities in [10, 9] and the imaginary-Brownian-time-Brownian-angle process and its kernel in [7]. Our IBTBAP process, starting at \( u_0 : \mathbb{R}^d \to \mathbb{R} \), was given in [7] by

\[
(1.7) \quad A_{u_0}^{X,B}(t, x) := \begin{cases} u_0(x^t(iB(t))) \exp(iB(t)), & B(t) \geq 0; \\ u_0(1x^{-i\pi}(-iB(t))) \exp(iB(t)), & B(t) < 0; \end{cases}
\]

where the process \( X^x \) is an \( \mathbb{R}^d \)-valued Brownian motion (BM) starting from \( x \in \mathbb{R}^d \), \( X^{-ix} \) is an independent \( i\mathbb{R}^d \)-valued BM starting at \( -ix \) (so that \( iX^{-ix} \) starts at \( x \)), and both are independent of the inner standard \( \mathbb{R} \)-valued Brownian motion \( B \) starting from 0. The clock of the outer Brownian motions \( X^x \) and \( X^{-ix} \) is replaced by an imaginary positive Brownian time; and the angle of \( A_{u_0}^{X,B}(t, x) \) is the Brownian motion \( B \). We think of the imaginary-time processes \( \{X^x(is), s \geq 0\} \) and \( \{iX^{-ix}(-is), s \leq 0\} \) as having the same complex Gaussian distribution on \( \mathbb{R}^d \) with the corresponding complex distributional density (or Schrödinger propagator)

\[
K_{1s,x,y}^{\text{SIP}} = \frac{1}{(2\pi is)^{d/2}} e^{-|x-y|^2/2is}.
\]

We may then think of \( u \) in (1.6) in terms of complex expectation by first conditioning on \( B(t) = s \) and then removing the conditioning (by integrating over \( s \)) and defining \( u(t, x) := \mathbb{E}_s[A_{u_0}^{X,B}(t, x)] \). Viewed this way, \( K_{t,x}^{\text{LKS}} \) is the expectation kernel of the IBTBAP. Since \( K_{t,x,y}^{\text{AS}} = e^{3s}K_{t,x,y}^{\text{SIP}} \) an extra angle \( s \in \mathbb{R} \setminus \{0\} \), where \( s \) is also the real-valued time on the imaginary axis (is), we call \( K_{t,x,y}^{\text{SIP}} \) the \( d \)-dimensional \( \mathbb{R} \)-time-angled propagator. The L-KS kernel \( K_{t,x}^{\text{LKS}} \) in (1.5) is thus the Gaussian average of an \( \mathbb{R} \)-time-angled Schrödinger propagator.

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7The compact support assumption on \( u_0 \) here and in Theorem 1.1 below is for convenience only and may be replaced with more relaxed integrability conditions à la those given for the Brownian-time Brownian sheet in [3].

8In particular, Theorem 1.2 in [9] was crucial in arriving at our IBTBAP and its kernel \( K_{t,x}^{\text{LKS}} \). Of course, the L-KS kernel \( K_{t,x}^{\text{LKS}} \) is not a proper probability density in the standard sense. But, it has a nice Fourier transform, as we shall see shortly.

9In our fourth order setting we have two notions of time: the standard time \( t \) and the Brownian-time and Brownian-angle \( B(t) \) (each \( s \) in \( K_{t,x,y}^{\text{AS}} \) represents a possible value for the BM \( B \) in our IBTBAP at some time \( t \), \( B(t) = s \in \mathbb{R} \)).
1.3. The $(\varepsilon, \vartheta)$ L-KS kernel formulation. In this article, we start by using our L-KS kernel to formulate the notion of a mild kernel solution to the $(\varepsilon, \vartheta)$ L-KS (S)PDEs in \((1.1)\). We first generalize slightly $K_{t,x,y}^{LKS}$ in \((1.3)\) by scaling the time $t$ with a parameter $\varepsilon > 0$ and scaling the angle $s$ in the $\mathbb{R}$-time-angled propagator $K_{i,x,y}^{(\varepsilon,\vartheta)}$ by another parameter $\vartheta \in \mathbb{R}$ to obtain the $(\varepsilon, \vartheta)$ L-KS kernel:

\[
K_{t,x,y}^{(\varepsilon, \vartheta)} = \int_{-\infty}^{\infty} \frac{e^{i\vartheta s e^{-|x-y|^2/2is}}}{(2\pi is)^{d/2}} K_{t,s}^{BM} ds + \int_{0}^{\infty} \frac{e^{i\vartheta s e^{-|x-y|^2/2is}}}{(2\pi is)^{d/2}} K_{s,t}^{BM} ds,
\]

which, when setting $y = 0 \in \mathbb{R}^d$, is the fundamental solution $K_{t,x}^{(\varepsilon, \vartheta)}$ to the $(\varepsilon, \vartheta)$ L-KS PDE in equation \((1.2)\). As we will see in Section 2, despite the involved expression in \((1.8)\), the kernel $K_{t,x,y}^{(\varepsilon, \vartheta)}$ has a rather nice (and revealing) Fourier transform:

\[
\hat{K}_{t,x,y}^{(\varepsilon, \vartheta)} = (2\pi)^{-\frac{d}{2}} e^{-\frac{i}{\varepsilon} (-2\vartheta + |\xi|^2)^2}; \quad \varepsilon > 0, \quad \vartheta \in \mathbb{R},
\]

which, upon inverting yields the simpler form of $K_{t,x,y}^{(\varepsilon, \vartheta)}$:

\[
K_{t,x,y}^{(\varepsilon, \vartheta)} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\frac{i}{\varepsilon} (-2\vartheta + |\xi|^2)^2} e^{i\xi \cdot x} d\xi;
\]

\[
K_{t,x,y}^{(\varepsilon, \vartheta)} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{i}{\varepsilon} (-2\vartheta + |\xi|^2)^2} \cos (\xi \cdot x) d\xi;
\]

The last equality in \((1.10)\) follows since $\int_{\mathbb{R}^d} e^{-\frac{i}{\varepsilon} (-2\vartheta + |\xi|^2)^2} \sin (\xi \cdot x) d\xi = 0$. Thus, the effect of the Gaussian average of the propagator in \((1.8)\) is to “average out” the imaginary part of the kernel, and $K_{t,x,y}^{(\varepsilon, \vartheta)}$ is real-valued. We now give the new kernel formulation for the class of $(\varepsilon, \vartheta)$ L-KS (S)PDEs \((1.1)\) which includes, among other (S)PDEs, the Swift-Hohenberg (S)PDE.

**Definition 1.1** $(\varepsilon, \vartheta)$ L-KS kernel (mild) formulation of $(\varepsilon, \vartheta)$ L-KS (S)PDEs \((1.1)\). Fix $\varepsilon > 0$ and $\vartheta \in \mathbb{R}$. We call the pair $(U, \mathcal{W})$ on a usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ a $(\varepsilon, \vartheta)$ L-KS kernel solution to \((1.1)\) on $\mathbb{R}_+ \times \mathbb{R}^{d}$ whenever $\mathcal{W}$ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^{d}$; the random field $U$ is progressively measurable, and with $U(0, x) = u_0(x)$; and the pair $(U, \mathcal{W})$ satisfies the $(\varepsilon, \vartheta)$ L-KS kernel (mild) formulation:

\[
U(t, x) = \int_{\mathbb{R}^{d}} K_{t,x,y}^{(\varepsilon, \vartheta)} u_0(y) dy + \int_{\mathbb{R}^{d}} \int_{0}^{t} K_{s,x,y}^{(\varepsilon, \vartheta)} [b(U(s, y)) dy + a(U(s, y)) \mathcal{W}(ds \times dy)]
\]

$^{10}$Clearly, using the notation of the-just-introduced $(\varepsilon, \vartheta)$ L-KS kernel, we note that $K_{t,x,y}^{(1,1) \text{LKS}} = K_{t,x,y}^{(1,1) \text{LKS}}$.

$^{11}$See Section 2 for a Fourier argument. We also briefly note that with $\varepsilon > 0$, we always have the dissipative negative biLaplacian $-\Delta^2$ in \((1.2)\). On the other hand, the case $\vartheta > 0$ leads to a dissipative second order $\Delta$; whereas $\vartheta > 0$ leads to the non-dissipative second order $-\Delta$, which is the case in L-KS PDEs like the Swift-Hohenberg and Kuramoto-Sivashinsky for example.

$^{12}$We assume throughout the article that filtrations $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions, and we often simply say that $U$ is a kernel solution to \((1.2)\) to mean the same as the definition of a mild solution above.
for \( t > 0 \) and for every (or almost every) \( x \in \mathbb{R}^d \), almost surely \( \mathbb{P} \). Weak and strong—in the probability sense—solutions are defined in the usual way: we call a \((\varepsilon, \vartheta)\) L-KS kernel solution weak if the white noise \( W \) and \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) on which it’s defined are freely chosen—along with \( U \)—so as to satisfy (1.1); and the solution is strong if \( W \) and \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) are fixed and \( \{\mathcal{F}_t\} \) is the augmentation of the natural filtration for \( W \) under \( \mathbb{P} \). The solution is continuous if \( U \) has continuous paths on \( \mathbb{R}_+ \times \mathbb{R}^d \) almost surely \( \mathbb{P} \).

Uniqueness in law holds for (1.1) if the laws \( W^{(i)} \) of \( U^{(i)} \) under \( \mathbb{P}^{(i)}; i = 1, 2 \), are the same whenever \((U^{(i)}, W^{(i)}), (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}); i = 1, 2\), are \((\varepsilon, \vartheta)\) L-KS kernel solutions to (1.1). Pathwise uniqueness holds if \( U^{(1)} \) and \( U^{(2)} \) are \( \mathbb{P} \)-indistinguishable \((\mathbb{P} [U^{(1)} = U^{(2)}] = 1) \) whenever \((U^{(i)}, W^{(i)}) \) are \((\varepsilon, \vartheta)\) L-KS kernel solutions to (1.1) with respect to the same white noise \( W \) and on the same probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\).

A \((\varepsilon, \vartheta)\) L-KS kernel solution \( U \) to the deterministic version of (1.1) is obtained from (1.1) by setting \( a \equiv 0 \).

**Remark 1.1.** Although we focus in this article on the SPDE in (1.1) on \( \{\mathbb{R}_+ \times \mathbb{R}^d\}_0^\alpha \) and subsets thereof, the utility of our new L-KS kernel formulation goes well beyond just (1.1). We show separately how to adapt it to formulate the class of PDEs discussed in Section 1.1 (Cahn-Hilliard, Kuramoto-Sivashinsky, and many other fourth order equations) and their stochastic versions. We also use this explicit kernel approach in separate articles to analyze the time-asymptotic [14] and other qualitative behaviors of several fourth order L-KS type equations.

1.4. **Three main theorems.** In this article, we establish three main theorems on versions of the \((\varepsilon, \vartheta)\) L-KS SPDE (1.1). We now detail and state our main results.

1.4.1. **Theorem (1.1): existence, uniqueness, and sharp dimension-dependent Hölder regularity.** In our first result; we obtain sharp, dimension-dependent, spatio-temporal Hölder continuity regularity results for the L-KS SPDE (1.1) with \( b \equiv 0 \) (zero drift or canonical L-KS SPDE):

\[
\begin{align*}
\frac{\partial U}{\partial t} &= -\frac{\varepsilon}{8} (\Delta + 2\vartheta)^2 U + a(U) \frac{\partial^{d+1} W}{\partial t \partial x}, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^d; \\
U(0, x) &= u_0(x), \quad x \in \mathbb{R}^d.
\end{align*}
\]

In particular, for any fixed \( \varepsilon, T > 0 \) and \( \vartheta \in \mathbb{R} \), we obtain the existence of a unique real-valued solution \( U \) that is \( L^p(\Omega) \)-bounded on \([0, T] \times \mathbb{R}^d \) for all \( p \geq 2 \) and that has Hölder continuous paths in time and space. In time, the Hölder exponent is \( \gamma_t \in (0, (4-d)/8) \) and in space it is \( \gamma_x \in (0, [(4-d)/2]) \) for spatial dimensions \( d = 1, 2, 3 \). We first obtained the same striking spatio-temporal Hölder regularity profile in [2] for a different class of memoryful fourth order stochastic integral equations (SIEs) associated with the Brownian-time Brownian motion (BTBM)—see [10, 9, 5] and the discussion in [2]—which we introduced as BTBM SIEs. What is remarkable about this Hölder regularity profile is that, not only random field solutions exist in spatial dimensions \( d = 1, 2, 3 \) (not just for \( d = 1 \)) in the presence

\[14\] All solutions \( U \) in this article have continuous paths \((U \in C(\mathbb{D}; \mathbb{R}), \text{ where } \mathbb{D} \subset \mathbb{R}_+ \times \mathbb{R}^d) \). The law \( \mathcal{L}^U_{\omega} \) of the random field \( U \) under \( \mathbb{P} \) is the probability measure induced on the Borel \( \sigma \)-field of continuous function by the recipe: \( \mathcal{L}^U_{\omega}(A) = \mathbb{P}(U \in A), A \in \mathcal{B}(C(\mathbb{D}; \mathbb{R})) \)

\[15\] See [5] for a similar approach in studying random attractors for the second order Allen-Cahn case.
of the rough driving space-time white noise but these random field solutions are spatially twice as smooth as the underlying Brownian sheet in \( d = 1, 2 \). In the followup article [1], we showed that this third dimensionality limit on random field existence and the above spatial Hölder smoothness are maximal in equations driven by a space-time white noise that are first order in time and high order in space—no matter how high the order is in these equations.

Although our L-KS SPDEs here have the same spatio-temporal Hölder profile as the BTBM SIE of [2], proving it by directly adapting our methods in [2] to the L-KS kernel is demanding. The difficulty lies in the fact that the L-KS kernel in (1.8) is the Gaussian average of the highly oscillatory angled complex propagator; whereas the BTBM probability density

\[
\mathbb{K}_{t,x,y}^{\text{BTBM}} = 2 \int_0^\infty \frac{e^{-|x-y|^2/2s}e^{-s^2/2t}}{(2\pi s)^{d/2}\sqrt{2\pi t}} ds
\]

is a Gaussian average of another non-oscillatory Gaussian density. Also, the L-KS kernel is not a proper probability density function as the BTBM density. Thus, we proceed differently by applying a harmonic analytic step to the L-KS kernel at the outset. This turns out to be a useful first step towards obtaining the required regularity estimates. We then use delicate analysis, including comparing the nonzero \( \vartheta \) angle case to that of the simpler \( \vartheta = 0 \) case and adapting the probabilistic-analytic arguments from [2] to our setting here, to prove the estimates needed for the proof of Theorem 1.1.

In the process, we give a harmonic analytic explanation of why the BTBM density—which is associated with the quite different memory-preserving positive biLaplacian fourth order PDE

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{\Delta u_0}{\sqrt{8\pi t}} + \frac{1}{8} \Delta^2 u; & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
u(0, x) = u_0(x); & x \in \mathbb{R}^d,
\end{cases}
\]

and its equivalent time-fractional PDE

\[
\begin{cases}
\frac{1}{\Gamma(\frac{\vartheta}{2})} \frac{\partial^{\vartheta} u}{\partial t^{\vartheta}} = \frac{1}{\sqrt{8t}} \Delta u; & t \in (0, \infty), x \in \mathbb{R}^d, \\
u(0, x) = u_0(x); & x \in \mathbb{R}^d,
\end{cases}
\]

where \( \partial^{\vartheta} \) is the Caputo fractional derivative—has the same regularizing effect as that of the L-KS kernel and its associated PDE (1.4). This harmonic explanation is

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\[\text{15}\] This is in contrast to second order PDEs driven by space-time white noise whose random field solution exists only in \( d = 1 \). Also, it is noteworthy that, with very few exceptions (e.g., [1, 2, 5, 28]), space-time white noise driven SPDEs, even higher order ones, are not treated in more than one spatial dimension.

\[\text{16}\] Our article [2] gave the first example of space-time white noise driven equations whose solutions are smoother in either time or space than the underlying Brownian sheet corresponding to the driving white noise.

\[\text{17}\] Maximal if the spatial dimension is integer.

\[\text{18}\] See [3, 36, 37]. For more time half-derivative connections, including the half derivative generator, and for a discussion of interesting aspects of these PDEs and their history see also [10] and the introduction in [1]. In the recent multiparameter-time case the reader is referred to [4, 3]. The BTBM scaling and its nonstandard PDEs connection have now attracted a lot of attention, even outside probability and PDEs, as evidenced by the recent physics and financial articles [20, 21].
The L-KS kernel, first proved differently in [7], is used in Section 2 to sketch a different proof of the L-KS PDE (1.4) connection to SPDEs here. The Fourier transform of the L-KS kernel, and its inverse, are also given in Section 2 below. A different probabilistic heuristic argument was given in [2] as to why the BTBM-SIEs in [5] are cousins of the L-Kuramoto-Sivashinsky SPDEs here. The Fourier transform of the L-KS kernel, and its inverse, are also given in Section 2 below. A different probabilistic heuristic argument was given in [2] as to why the BTBM-SIEs in [5] are cousins of the L-Kuramoto-Sivashinsky SPDEs; including the Swift-Hohenberg and its generalization (1.1), variants of the KS, and many more.

Throughout this paper, fix an arbitrary $T > 0$, and let $T = [0, T]$. We denote by $H^{\gamma_1, \gamma_2} (\mathbb{T} \times \mathbb{R}^d; \mathbb{T})$ the space of real-valued locally Hölder functions on $\mathbb{T} \times \mathbb{R}^d$ whose time and space Hölder exponents are in $(0, \gamma_1)$ and $(0, \gamma_2)$, respectively. We now state our first existence, uniqueness, and regularity result.

**Theorem 1.1** (Existence/uniqueness and sharp Hölder regularity for the canonical $(\epsilon, \vartheta)$ L-KS SPDE (1.12) in $d = 1, 2, 3$). Fix $\varepsilon > 0$ and $\vartheta \in \mathbb{R}$. Assume that

$$
\begin{aligned}
(a) \quad & |a(u) - a(v)| \leq C |u - v|, \quad u, v \in \mathbb{R}; \\
(b) \quad & a^2(u) \leq C(1 + |u|^2), \quad u \in \mathbb{R}; \\
(c) \quad & u_0 \in C^2_{\alpha, \gamma} (\mathbb{R}^d; \mathbb{R}) \text{ and nonrandom } \forall \ d \in \{1, 2, 3\}.
\end{aligned}
$$

Then, there exists a strong $(\epsilon, \vartheta)$ L-KS kernel solution solution $(U, \mathbb{W})$ to the L-KS SPDE (1.12) on $\mathbb{R}^+ \times \mathbb{R}^d$, for $d = 1, 2, 3$, which is $L^p(\Omega)$-bounded on $\mathbb{T} \times \mathbb{R}^d$ for all $p \geq 2$ ($M_p(t) := \sup_{x \in \mathbb{R}^d} \mathbb{E}|U(t, x)|^p \leq C_T$ for $t \in \mathbb{T}$) and

$$
U \in H^{\frac{4 - d}{4}} (\mathbb{T} \times \mathbb{R}^d; \mathbb{T}) \quad \text{for } d = 1, 2, 3, \text{ almost surely.}
$$

If $(V, \mathbb{W}')$ is another such solution, with respect to the same white noise $\mathbb{W}$, then, for any $d \in \{1, 2, 3\}$, $U$ and $V$ are indistinguishable: $\mathbb{P} [U(t, x) = V(t, x); (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d] = 1$.

**Remark 1.2.** We note here that we can adapt our lattice arguments and K-martingale approach in [2] to prove existence of lattice-limits solutions to our L-KS SPDE with the same $L^p$ and Hölder regularity as those of Theorem 1.1 under the weaker non-Lipschitz conditions:

$$
\begin{aligned}
(a) \quad & a(u) \text{ is continuous in } u, \quad u \in \mathbb{R}; \\
(b) \quad & \text{and (c) same as in (Lip)}.
\end{aligned}
$$

The details are left to the interested reader.

---

19We remind the reader again that the compact support condition on $u_0$ may be replaced with more relaxed integrability conditions like those given for the Brownian-time Brownian sheet in [3].

20These lattice arguments have their roots in our second order SPDE works [14] [8].
show that this competition is controlled in the limit by the critical ratio $\varepsilon$ against the roughening force of the space-time white noise. In Theorem 1.2 we states that the $L^1$ addition, we use the same almost sure (1.12) (linear PDE part) to the nonzero-drift case (1.1) (nonlinear PDE part). In of this article, to transfer results and properties from the zero drift L-KS SPDE results, from the second order equations in [13, 12, 11] to the four order equations of the SPDE under consideration. This makes them conveniently adaptable to change of measure.

1.4.2. Theorem 1.3: L-KS vs white noise, vanishing distance or $L^2$ blowup? Consider the $(\varepsilon, \vartheta)$ L-KS SPDE (1.12). Fix $\vartheta \in \mathbb{R}$, let $\varepsilon = \varepsilon_1$ be an order parameter, and attach another order parameter $\varepsilon_2$ to the white noise term to obtain the L-KS SPDE

\[
\begin{align*}
\frac{\partial U}{\partial t} &= -\frac{\varepsilon_1}{8} (\Delta + 2\vartheta)^2 U + \varepsilon_2 a(U) \frac{\partial^{d+1} W}{\partial t^d}, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^d; \\
U(0, x) &= u_0(x), \quad x \in \mathbb{R}^d.
\end{align*}
\]

We next use the order parameters $\varepsilon_1, \varepsilon_2$ to study the limiting competition between the regularizing force of the spatial fourth order operator $-\frac{\varepsilon_1}{8} (\Delta + 2)^2$ as it pushes against the roughening force of the space-time white noise. In Theorem 1.2 we show that this competition is controlled in the limit by the critical ratio $\varepsilon_2/\varepsilon_1^{d/8}$ as $\varepsilon_1, \varepsilon_2 \rightarrow 0$ (or $\varepsilon_1, \varepsilon_2 \not\rightarrow \infty$), for dimensions $d = 1, 2, 3$. In particular, Theorem 1.2 states that the $L^{2q}$ distance between a solution to the SPDE (1.16) and a solution to its deterministic version ($a \equiv 0$) goes to zero, uniformly on $[0, T] \times \mathbb{R}^d$, as $\varepsilon_1, \varepsilon_2$, and $\varepsilon_2/\varepsilon_1^{d/8} \rightarrow 0$ for $d = 1, 2, 3$ and $q > 1$. It also gives a finite-time blowup result (in the $L^2(\Omega)$ sense) as $\varepsilon_1, \varepsilon_2 \not\rightarrow 0$ (or $\varepsilon_1, \varepsilon_2 \not\rightarrow \infty$) such that the ratio $\varepsilon_2/\varepsilon_1^{d/8} \not\rightarrow \infty$ for $d = 1, 2, 3$. We now state our result.

**Theorem 1.2** (L-KS vs white noise in (1.16): the critical order parameter ratio in $d = 1, 2, 3$). Fix $\vartheta \in \mathbb{R}$. Assume that the conditions in (Lap) are in force and that $(U_{\varepsilon_1, \varepsilon_2}^d)$ is the unique strong solution to the L-KS SPDE (1.16).

(i) (Uniformly vanishing $L^{2q}$ distance between SPDE and PDE) Suppose that $u_{\varepsilon_1}$ is the solution to the deterministic L-KS PDE obtained from (1.16) by setting $a \equiv 0$, then

\[
\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E} \left| U_{\varepsilon_1, \varepsilon_2}(t, x) - u_{\varepsilon_1}(t, x) \right|^{2q} \rightarrow 0; \quad \forall q \geq 1, \quad T > 0
\]

as $\varepsilon_1, \varepsilon_2$, and $\varepsilon_2/\varepsilon_1^{d/8} \rightarrow 0$ for $d = 1, 2, 3$.

(ii) (Finite-time $L^2$ blowup) Suppose there are constants $K_l, K_u > 0$ such that $K_l \leq a(v) \leq K_u$ for all $v \in \mathbb{R}$; then,

\[
\sup_{0 \leq s \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E} \left| U_{\varepsilon_1, \varepsilon_2}(s, x) \right|^2 \not\rightarrow \infty, \quad \forall T > 0
\]

as $\varepsilon_1, \varepsilon_2 \not\rightarrow 0$ (or $\varepsilon_1, \varepsilon_2 \not\rightarrow \infty$) such that the ratio $\varepsilon_2/\varepsilon_1^{d/8} \not\rightarrow \infty$ for $d = 1, 2, 3$.

1.4.3. Theorem 1.4: from canonical L-KS SPDEs to nonlinear L-KS SPDEs via change of measure. At their core, our space-time change of measure theorems in [13, 12, 11] are “noise” results that are independent of both the type and order of the SPDE under consideration. This makes them conveniently adaptable to different SPDEs settings. We use this fact to adapt our earlier change of measure results, from the second order equations in [13, 12, 11] to the fourth order equations of this article, to transfer results and properties from the zero drift L-KS SPDE (1.12) (linear PDE part) to the nonzero-drift case (1.1) (nonlinear PDE part). In addition, we use the same almost sure $L^2$ condition on the drift/diffusion ratio as
in our work [12] [11] to transfer uniqueness in law and establish law equivalence between solutions to (1.12) and (1.1). As observed in [12], this is a much weaker condition than the traditional Novikov condition for change of measure; and this allows us to transfer results and properties from the canonical L-KS SPDEs (1.12) to many nonlinear L-KS SPDEs (1.1), including the Swift-Hohenberg SPDE, driven by space-time white noise on subsets of \( \mathbb{R}_+ \times \mathbb{R}^d \), \( d = 1, 2, 3 \).

Now, we turn to the setting of our final main result of this paper. Recall that we denote the zero-drift L-KS SPDE (1.12) by \( e_{\text{LKS}}^{(a, 0, u_0)}(a, b, u_0) \) while the SPDE (1.1) is denoted by \( e_{\text{LKS}}^{(0, 0, u_0)}(a, b, u_0) \). We fix \( T, L_1, L_2, L_3 > 0 \), let \( \mathbb{T} = [0, T] \), and we consider both equations \( e_{\text{LKS}}^{(a, 0, u_0)}(a, b, u_0) \) and \( e_{\text{LKS}}^{(0, 0, u_0)}(a, b, u_0) \) on the time-space domain \( \mathbb{T} \times \mathbb{S} \), where either \( \mathbb{S} = \mathbb{R}^d \) or \( \mathbb{S} = \prod_{i=1}^d [0, L_i] \), \( d = 1, 2, 3 \). In the case \( \mathbb{S} = \prod_{i=1}^d [0, L_i] \) the equations are supplemented with suitable boundary conditions\(^{21} \) the nature of which is irrelevant to our change of measure results. Also irrelevant to our results below is whether solutions are defined as mild kernel solutions like in Definition 1.1—appropriate modifications to \( K_{t,x}^{(\varepsilon, \vartheta)\text{LKS}} \) to account for the boundary conditions\(^{22} \) in the case \( \mathbb{S} = \prod_{i=1}^d [0, L_i] \)—or whether solutions are defined as weak; i.e., given in the test functions formulation (TFF). For concreteness, and to also give the independently useful TFF for \( e_{\text{LKS}}^{(a, 0, u_0)}(a, b, u_0) \), we take the TFF as our definition of solutions to \( e_{\text{LKS}}^{(a, 0, u_0)}(a, b, u_0) \) and we now proceed to define it (see Remark 1.3 below about the equivalence of the two formulations). Let the Dirichlet test functions space be given by\(^{23} \)

\[
\Phi_{c, \text{Dir}}^\infty(\mathbb{S}; \mathbb{R}) := \left\{ \varphi \in C^\infty(\mathbb{R}^d; \mathbb{R}); \varphi = \Delta \varphi = 0 \text{ on } \partial \mathbb{S} \right\}; \quad \mathbb{S} = \prod_{i=1}^d [0, L_i],
\]

where \( d = 1, 2, 3 \).

**Definition 1.2** (Test function solutions to \( e_{\text{LKS}}^{(a, b, u_0)}(a, b, u_0) \)). We say that the pair \( (U, \mathcal{W}) \) defined on the usual probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}) \) is a test function solution to \( e_{\text{LKS}}^{(a, b, u_0)}(a, b, u_0) \) on \( \mathbb{R}_+ \times \mathbb{S} \) if \( \mathcal{W} \) is a space-time white noise on \( \mathbb{R}_+ \times \mathbb{S} \); the random field \( U \) is predictable (as in [18]), and with \( U(0, x) = u_0(x) \); and the pair \( (U, \mathcal{W}) \) satisfies the test function formulation:

\[
(U(t) - u_0, \varphi) = \int_0^t \left[ - \left( U(s), \frac{\varepsilon}{2} (\Delta + 2 \vartheta) \right)^2 + (b(U(s)), \varphi) \right] ds + \int_0^t \int_\mathbb{S} a(U(s, y)) \varphi(y) \mathcal{W}(ds \times dy); \quad \forall \varphi \in \Phi_{c, \text{Dir}}^2, \quad t > 0, \text{ a.s. } \mathbb{P},
\]

where \((\cdot, \cdot)\) denotes the usual inner product on \( L^2(\mathbb{S}; \mathbb{R}) \). The test function solution is continuous if \( U \) has continuous paths on \( \mathbb{R}_+ \times \mathbb{S} \). Weak and strong—in the

\(^{21}\)E.g., boundary conditions of Neumann type \( \partial U/\partial n = \partial U/\partial n = 0 = 0 \) or Dirichlet type conditions \( U = \Delta U = 0 \) on \( \partial \mathbb{S} \) and \( d = 1, 2, 3 \).

\(^{22}\)E.g., in the Neumann (Dirichlet) case, the propagator \( e^{-|x-y|^2/2\tau s/(2\pi\varsigma)} \) in the definition of the \( (\varepsilon, \vartheta) \) L-KS kernel \( K_{t,x}^{(\varepsilon, \vartheta)\text{LKS}} \) is replaced with the propagator with reflection (absorption) at \( \partial \mathbb{S} \), respectively.

\(^{23}\)Of course, the Dirichlet choice, which is assumed throughout the article whenever \( \mathbb{S} = \prod_{i=1}^d [0, L_i] \), is without loss of generality and for concreteness only. The Neumann (and other) boundary conditions are just as easily handled.
We also use $\lambda$ (including the Swift-Hohenberg and many more. In particular, the generalized Swift-Hohenberg SPDE—of measure equivalence above between the canonical LKS SPDE in (1.11) (with spatial set $\mathbb{R}$ or $\mathbb{R} = \prod_{i=1}^{d} [0, L_i]$) and test function formulation in $\mathbb{T}$ under local boundedness assumptions on $a$ and $b$.

For any function $u : \mathbb{T} \times \mathbb{S}$, we use the following notation for the drift/diffusion ratio function:

$$R_u(t, x) := \frac{b(u(t, x))}{a(u(t, x))}; (t, x) \in \mathbb{T} \times \mathbb{S}.$$  

(1.19)

We also use $\lambda$ to denote the Lebesgue measure on $\mathcal{B}(\mathbb{T} \times \mathbb{R}^d)$.

---

**Theorem 1.3** (From canonical L-KS to nonlinear L-KS SPDEs on subsets of $\{\mathbb{R} \times \mathbb{R}^d\}_{d=1}^3$ via change of measure). Assume that either $\mathbb{S} = \mathbb{R}^d$ or $\mathbb{S} = \prod_{i=1}^{d} [0, L_i]$, $d = 1, 2, 3$. Suppose that the ratios $R_U$ and $R_V$ are in $L^2(\mathcal{T} \times \mathbb{S}, \lambda)$, almost surely, whenever the continuous random fields $U$ and $V$ solve (weakly or strongly) $e^{(c, d)}_{\text{LKS}}(a, 0, u_0)$ and $e^{(c, d)}_{\text{LKS}}(a, b, u_0)$, respectively, on $\mathbb{T} \times \mathbb{S}$. Then,

(i) uniqueness in law holds for $e^{(c, d)}_{\text{LKS}}(a, b, u_0)$ if and only if uniqueness in law holds for $e^{(c, d)}_{\text{LKS}}(a, 0, u_0)$; and

(ii) if uniqueness in law holds for $e^{(c, d)}_{\text{LKS}}(a, 0, u_0)$, $U$ is a solution to $e^{(c, d)}_{\text{LKS}}(a, 0, u_0)$, and $V$ is a solution to $e^{(c, d)}_{\text{LKS}}(a, b, u_0)$ on $\mathbb{T} \times \mathbb{S}$; then the laws of $U$ and $V$ on $\mathcal{B}(C(\mathcal{T} \times \mathbb{S}; \mathbb{R}))$ are equivalent (mutually absolutely continuous).

In particular, let $\mathbb{S} = \prod_{i=1}^{d} [0, L_i]$, $d = 1, 2, 3$; and assume that $a(u) = \kappa \in \mathbb{R} \setminus \{0\}$, $b(u) = \sum_{k=0}^{N} c_k u^k$ for $c_k \in \mathbb{R}$, $k = 0, \ldots, N$, and $N \geq 0$, and $u_0 \in C^0_{\kappa, \gamma}(\mathbb{S}; \mathbb{R})$ and nonrandom $\forall \, d \in \{1, 2, 3\}$. If $U$ and $V$ are continuous solutions to $e^{(c, d)}_{\text{LKS}}(a, 0, u_0)$ and $e^{(c, d)}_{\text{LKS}}(a, b, u_0)$, respectively, on $\mathbb{T} \times \mathbb{S}$; then, the conclusions in (i) and (ii) above hold. Consequently,

$$V \in H^{-\frac{4-d}{2}} \left(\mathbb{T} \times \mathbb{S}; \mathbb{R}\right) \text{ a.s.} \iff \left(\frac{4-d}{2} \wedge 1\right)^{-} \in H^{\frac{4-d}{8}} \left(\mathbb{T} \times \mathbb{S}; \mathbb{R}\right) \text{ a.s.}$$

for $d = 1, 2, 3$.

---

**Remark 1.4.** When $\mathbb{S} = \prod_{i=1}^{d} [0, L_i]$, $d = 1, 2, 3$, Theorem 1.3 gives us the change of measure equivalence above between the canonical LKS SPDE $e^{(c, d)}_{\text{LKS}}(a, 0, u_0)$ in (1.12) and L-KS SPDEs $e^{(c, d)}_{\text{LKS}}(a, b, u_0)$ in (1.11) with polynomial nonlinearities, including the Swift-Hohenberg and many more. In particular, the generalized Swift-Hohenberg SPDE—$e^{(c, d)}_{\text{LKS}}(a, b, u_0)$ with $b(u) = \sum_{k=0}^{p-1} c_k u^k$ and with $p \in \mathbb{N}$ and $c_{2p-1} < 0$—admits uniqueness in law, is law equivalent to $e^{(c, d)}_{\text{LKS}}(a, 0, u_0)$, and
has the same Hölder regularity as the canonical L-KS SPDE $e_{\text{L-KS}}^{(\varepsilon,\vartheta)}(a,0,u_0)$ on $\mathbb{T} \times \prod_{i=1}^{d} [0,L_i]$, $d = 1,2,3$, whenever $a(u) = \kappa \in \mathbb{R} \setminus \{0\}$ and $u_0 \in C^{2,\gamma}(\mathbb{S};\mathbb{R})$ and nonrandom $\forall d \in \{1,2,3\}$.

We note that the conclusions of the last part of Theorem 1.3 hold also in the multiplicative noise case $a(u) = \kappa u$ and $b(u) = \sum_{k=1}^{N} c_k u^k$, where $\kappa, c_1 \in \mathbb{R} \setminus \{0\}$, (which covers the standard Allen-Cahn nonlinearity $u(1-u^2)$ encountered in the SH equation). We note here that all is needed is (1) uniqueness in law for $e_{\text{L-KS}}^{(\varepsilon,\vartheta)}(a,0,u_0)$, which holds since the stronger pathwise uniqueness holds because $a(u) = \kappa u$ satisfies (Lip) in Theorem 1.1 and (2) the ratios $R_U$ are $R_V$ are clearly in $L^2(\mathbb{T} \times \mathbb{S}, \lambda)$ by the continuity assumption on $U$ and $V$ and the nonzero assumption on the constants $\kappa$ and the $c_i$.

2. A HARMONIC CONNECTION BETWEEN THE L-KS AND THE BTBM KERNELES

2.1. Fourier transforms and $(\varepsilon, \vartheta)$ L-KS PDEs links. We start by obtaining the spatial Fourier transform for the Brownian-time Brownian motion (BTBM) and the $(\varepsilon, \vartheta)$ L-KS kernels. This reveals and captures both similarities and differences between both kernels and the PDEs corresponding to them.

Lemma 2.1 (Spatial Fourier transforms of the BTBM and the $(\varepsilon, \vartheta)$ L-KS kernels). Let $K_{t,x}^{\text{BTBM}}$ and $K_{t,x}^{(\varepsilon, \vartheta)\text{L-KS}}$ be the BTBM and $(\varepsilon, \vartheta)$ L-KS kernels, respectively.

(i) The spatial Fourier transform of the BTBM density in (1.3) is given by

$$\mathcal{F}_{t,x}^{\text{BTBM}}(\xi) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{t} \xi^2} \left[ \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} dr \right].$$

(ii) The spatial Fourier transform of the $(\varepsilon, \vartheta)$ LKS kernel in (1.8) is given by

$$\mathcal{F}_{t,x}^{(\varepsilon, \vartheta)\text{L-KS}}(\xi) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{t} (-2\vartheta + |\xi|^2)^2} ; \frac{\epsilon}{\vartheta} > 0, \vartheta \in \mathbb{R}.$$

Proof. Starting with the BTBM kernel Fourier transform, we have

$$\mathcal{F}_{t,x}^{\text{BTBM}} = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} 2 \int_{0}^{\infty} K_{s,x}^{\text{BM}} R_{t,s}^{\text{BM}} ds e^{-\frac{1}{t} K_{s,x}^2} dx$$

$$= (2\pi)^{-\frac{d}{2}} 2 \int_{0}^{\infty} e^{-\frac{s^2}{2t}} \frac{1}{\sqrt{2\pi s}} e^{-\frac{1}{2} |\xi|^2} ds$$

$$= (2\pi)^{-\frac{d}{2}} \left[ \frac{2 e^{\frac{1}{2} |\xi|^2}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-r^2} dr \right].$$

$^{24}$Of course we take $R_{U}\mid_{U=0} := \lim_{U \to 0} R_U = \lim_{V \to 0} R_V = c_1/\kappa := R_V\mid_{V=0}$ under our assumptions in this particular multiplicative case.

$^{25}$We use the symmetric definition of the Fourier transform. From a Physics point of view, the Fourier transform is taken over position to get energy.
proving part (i). The Fourier transform of the \((\varepsilon, \vartheta)\) L-KS kernel is now given by
\[
\hat{K}_x^{(\varepsilon, \vartheta)\text{LKS}}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}\setminus\{0\}} \frac{e^{i\xi \cdot s} e^{-|x-y|^2/2s}}{(2\pi is)^{d/2}} K_{\varepsilon t,s}^{\text{BM}} ds \, dx
\]
(2.4)

The Fourier transform of the Dirac initial condition \(\delta(x)\) is given by
\[
\hat{K}_x^{(\varepsilon, \vartheta)\text{LKS}}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{e^{-|x|^2/2\xi}}{\sqrt{2\pi \xi}} K_{\varepsilon t,s}^{\text{BM}} ds \, dx
\]
completing the proof of Lemma 2.1 □

Remark 2.1. The extra factor \(\frac{2}{\pi} \int_{\mathbb{R}^d} e^{-s^2} d\tau\) in the BTBM transform (2.4) capture the memoryful property of the PDE (1.14) (the inclusion of \(u_0\) and the plus sign of the term \(t|\xi|^4/8\) is because of the positive biLaplacian in (1.14).

Inverting the Fourier transform in Lemma 2.1, we immediately get the more-convenient form for \(K_{t,x}^{(\varepsilon, \vartheta)\text{LKS}}\) in (1.10), which can easily be verified to be a solution to the \((\varepsilon, \vartheta)\) L-KS PDE in (1.2) with Dirac initial condition \(\delta(x)\). In particular, the special case \((\varepsilon, \vartheta) = (1, 1)\) confirms that our L-KS kernel \(K_{t,x}^{\text{LKS}}\) is the fundamental solution of the L-KS PDE in (1.3). Let \(u\) be given by
\[
(2.5)
\]
and assume \(u_0\) satisfies the regularity conditions in \((\text{Lip})\) (c). The dominated convergence theorem plus a bit of analysis\(^{26}\) then give us that
\[
\partial_t u(t, x) = \int_{\mathbb{R}^d} \frac{\partial}{\partial\xi} K_{t,x}^{(\varepsilon, \vartheta)\text{LKS}} u_0(y)dy = \int_{\mathbb{R}^d} \frac{\partial}{\partial\xi} K_{t,x}^{(\varepsilon, \vartheta)\text{LKS}} u_0(y)dy = -\frac{\xi}{8} (\Delta + 2\vartheta)^2 u_0(y)dy
\]
(2.6)
and \(u(0, x) = u_0(x)\). Thus, we obtain the following theorem summarizing the PDEs connections.

Theorem 2.1. The \((\varepsilon, \vartheta)\) L-KS kernel solves the initial value \((\varepsilon, \vartheta)\) L-KS PDE
\[
(2.7)
\]
Moreover, if \(u\) is given by (2.4) and \(u_0\) satisfies the condition in \((\text{Lip})\) (c), then \(u\) solves the \((\varepsilon, \vartheta)\) L-KS PDE in (2.7) with \(u(0, x) = u_0(x)\).

Setting \(\varepsilon = \vartheta = 1\) in (2.6) in the argument leading to Theorem 2.1 gives us an alternative proof of our Theorem 1.1 of [7] connecting the linearized KS PDE (1.4)

---

\(^{26}\)See for example Lemma 2.1 in [7]. We leave the very similar details to the interested reader.
to the L-KS kernel $K^{\text{LKS}^d}_{t,x}$. On the other hand, setting $\epsilon = 1$ and $\vartheta = 0$ in (2.3), we get that the simpler kernel

$$K^{\text{SFO}^d}_{t,x} := \mathbb{F}_{t,x}^{(1,0)\text{LKS}^d} = \frac{e^{-\frac{i}{2}(\xi^2)}}{(2\pi)^\frac{d}{2}} e^{-\frac{i}{8}(2+|\xi|^2)^2} e^{ik \cdot x} d\xi$$

obtained by removing the angle $e^{ikx}$ from the L-KS kernel $K^{\text{LKS}^d}_{t,x}$ in (1.5), is the fundamental solution of the simpler fourth order PDE

$$\frac{\partial u}{\partial t} = -\frac{1}{8} \Delta^2 u, \quad (t,x) \in (0, +\infty) \times \mathbb{R}^d;$$

$$u(0,x) = \delta(x), \quad x \in \mathbb{R}^d$$

as was shown for the case $d = 1$ in Hochberg and Orsinger (33) (see also the different approach in Funaki (32), also for $d = 1$). Clearly, the Fourier transforms $\hat{K}^{\text{LKS}^d}_{t,x}$ and $\hat{K}^{\text{SFO}^d}_{t,x}$ of $K^{\text{LKS}^d}_{t,x}$ and $K^{\text{SFO}^d}_{t,x}$, and their inverses are now given as an immediate corollary to Lemma 2.1. Taking $(\epsilon, \vartheta) = (1,1)$ and $(\epsilon, \vartheta) = (1,0)$, respectively in Lemma 2.1 (ii) and using a dominated convergence argument, we get

\textbf{Corollary 2.1.}

$$\hat{K}^{\text{LKS}^d}_{t,x} = \frac{e^{-\frac{i}{8}(\xi^2)}}{(2\pi)^\frac{d}{2}}, \quad \mathbb{F}^{\text{LKS}^d}_{t,x} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{i}{8}(\xi^2)} e^{ik \cdot x} d\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{i}{8}(\xi^2)} \cos (\xi \cdot x) d\xi;$$

$$\hat{K}^{\text{SFO}^d}_{t,x} = \frac{e^{-\frac{i}{8}|\xi|^4}}{(2\pi)^\frac{d}{2}}, \quad \mathbb{F}^{\text{SFO}^d}_{t,x} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{i}{8}|\xi|^4} e^{ik \cdot x} d\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{i}{8}|\xi|^4} \cos (\xi \cdot x) d\xi.$$

\textbf{2.2. A revealing kernels $L^2$ energy.} To understand why the L-KS and the BTBM kernels $K^{\text{LKS}^d}_{t,x}$ and $K^{\text{BTBM}^d}_{t,x}$ have very similar regularizing effects on the L-KS SPDE (1.12) above (with $(\epsilon, \vartheta) = (1,1)$) and the BTBM SIE introduced in (2) (and obtained from (1.11) by replacing $K^{(\epsilon, \vartheta)\text{LKS}^d}$ with $F^{\text{BTBM}^d}_{t,x}$ and setting $b \equiv 0$), we first observe that the regularity of the L-KS PDE (1.4) is dictated by the bi-Laplacian term and that the family

$$\left\{F^{(\epsilon, \vartheta)\text{LKS}^d}_{t,x}\right\}_{\epsilon > 0, \vartheta \in \mathbb{R}}$$

of all $(\epsilon, \vartheta)$ L-KS kernels in (1.8) and (1.10)—including $K^{\text{LKS}^d}_{t,x}$ and $K^{\text{SFO}^d}_{t,x}$—share the same regularizing effect on the L-KS SPDE (1.12).

As we will see shortly, the $L^2$ quantity

$$\int_{\mathbb{R}^d} |\mathbb{F}^{\text{SFO}^d}_{t,x}|^2 dx = \int_{\mathbb{R}^d} |\mathbb{F}^{\text{SFO}^d}_{t,x}|^2 d\xi = \int_{\mathbb{R}^d} e^{-\frac{i}{8}|\xi|^4} d\xi = C_d t^{-d/4}; d = 1, 2, 3,$$

for the $(\epsilon, \vartheta) = (1,0)$ L-KS kernel $\mathbb{F}^{\text{SFO}^d}_{t,x}$—where we used the Parseval-Plancherel theorem and where $C_d$ is a dimension dependent constant—is key to understanding the regularity of our L-KS SPDE (1.12). By the above discussion (see

$$2^7 C_1 = 1/2 \Gamma \left(\frac{d}{4}\right), C_2 = 1/4 \sqrt{\pi}, \text{ and } C_3 = \Gamma \left(\frac{d}{4}\right) / \pi^2 \sqrt{\pi}.$$
also Lemma 3.1 below), it is clear that \( \int_{\mathbb{R}^d} |K_{LKS}^{t;x}|^2 \, dx \) is of the same order \(^{28}\). On the other hand as was shown in Lemma 2.2 in [2], there is a dimension dependent constant \( c_d \) such that

\[
\int_{\mathbb{R}^d} |K_{BTBM}^{t;x}|^2 \, dx = c_d t^{-d/4}; \quad t > 0, \quad d = 1, 2, 3.
\]

Equations (2.11) and (2.12) are the fundamental analytic reason why the regularity for our L-KS SPDE in our first result Theorem 1.1 above is the same as that of the BTBM SIE in Theorem 1.1 of [2], albeit here we have real solutions to a negative bi-Laplacian equation and the BTBM SIE in [2] has real solutions to a positive bi-Laplacian equation with memory (see [2]).

3. Proof of Theorem 1.1

Since both \( \varepsilon > 0 \) and \( \vartheta \in \mathbb{R} \) are fixed in Theorem 1.1, and since all the main conclusions are unaffected by the specific values of \( \varepsilon > 0 \) and \( \vartheta \in \mathbb{R} \); we will simplify our notation and exposition by assuming throughout this section (and its subsections)—without loss of generality—that either \((\varepsilon, \vartheta) = (1, 1)\) (capturing the general biLaplacian, Laplacian, and zero order term case) or \((\varepsilon, \vartheta) = (1, 0)\) (the biLaplacian term, without the lower order terms, case \(^{29}\)).

3.1. Key regularity estimates for the L-KS kernel.

Here, we prove several \( L^2 \) estimates\(^{30}\) on the L-KS kernel and its temporal and spatial differences that are key in proving our regularity results in Theorem 1.1. Again, these fundamental estimates for the L-KS kernel are very similar to those for the BTBM density in the corresponding estimates in [2], but the proofs proceed differently due to the oscillatory nature of the modified propagator part of the L-KS kernel.

**Lemma 3.1 (Kernel’s \( L^2 \)).** Fix any arbitrary \( T > 0 \). There are constants \( C_{LKS}^{(d)} \) and \( C_{BTBM}^{(d)} \) depending only on the spatial dimension \( d \) and \( T \) and a constant \( C_d \) depending only on \( d \) such that

\[
C_{LKS}^{(d)} t^{-d/4} \leq \int_{\mathbb{R}^d} |K_{LKS}^{t;x}|^2 \, dx \leq C_{BTBM}^{(d)} t^{-d/4}.
\]

\(^{28}\)In fact, in \( d = 2 \)

\[
\int_{\mathbb{R}^d} |\tilde{K}_{LKS}^{t;\xi}|^2 \, d\xi = \left[ 1 + \psi(\sqrt{t}) \right] \int_{\mathbb{R}^d} |\tilde{K}_{SFO}^{t;\xi}|^2 \, d\xi, \quad t > 0.
\]

where \( \psi(u) := (2/\sqrt{\pi}) \int_0^u e^{-r^2} \, dr \). See also Lemma 3.1 below.

\(^{29}\)It should be clear that our methods extend with only minor notational changes to any fixed values for \( \varepsilon > 0 \) and \( \vartheta \in \mathbb{R} \). The case \((\varepsilon, \vartheta) = (1, 1)\) is the simplest representative case, and we include it explicitly in this subsection since it is useful in Lemma 3.1 to obtain the fundamental \( L^2 \) estimates for the more interesting \((\varepsilon, \vartheta) = (1, 1)\) case.

\(^{30}\)Lemma 3.1 is stated only for \( 0 < t \leq T \), since we only need it for intervals \([0, T]\). In fact, for \( d = 2 \), we show that the estimates hold, with the same constants \( C_{LKS}^{(d)} \) and \( C_{BTBM}^{(d)} \), for all \( t > 0 \).
for $0 < t \leq T$, $d \in \{1, 2, 3\}$ and hence

$$
\int_0^t \int_{\mathbb{R}^d} \left| K_{LKS}^{(d)} \right|^2 \, dx \, ds = C_d \frac{4-d}{4}; \text{ and}
$$

(3.2)

$$
C_t^{(d)} \frac{4-d}{4} \leq \int_0^t \int_{\mathbb{R}^d} \left| K_{LKS}^{(d)} \right|^2 \, dx \, ds \leq C_u^{(d)} \frac{4-d}{4};
$$

for $0 < t \leq T$, $d \in \{1, 2, 3\}$.

**Proof.** The equalities in (3.1) and in (3.2) follow immediately from (2.11). Using the Parseval-Plancherel theorem, we have

(3.3)

$$
\int_{\mathbb{R}^d} \left| K_{LKS}^{(d)} \right|^2 \, dx = \int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{t}{4}(-2+|\xi|^2)^2} \, d\xi,
$$

for every $s > 0$. Let $d = 2$ and $\psi(u) := \left(\frac{2}{\sqrt{\pi}}\right) \int_0^u e^{-r^2} \, dr$. We then have

(3.4)

$$
\frac{1}{4\sqrt{\pi} s} \leq (2\pi)^{-2} \int_{\mathbb{R}^2} e^{-\frac{t}{4}(-2+|\xi|^2)^2} \, d\xi = \frac{1 + \psi(\sqrt{s})}{4\sqrt{\pi}} \frac{1}{\sqrt{s}}
$$

and the assertions in (3.1) and its immediate consequence (3.2) are established for $d = 2$.

For dimensions $d = 1, 3$, we get the desired estimates by comparing $\int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi$ with $\int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi$ (see (2.10) and (2.11) above). To start, we use (2.10) and observe that

(3.5)

$$
\lim_{t \to 0^+} \frac{\int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi}{\int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi} = \lim_{t \to 0^+} \frac{\int_{\mathbb{R}^d} e^{-\frac{t}{4}(-2+|\xi|^2)^2} \, d\xi}{\int_{\mathbb{R}^d} e^{-\frac{t}{4}(-2+|\xi|^2)^2} \, d\xi} = 1; \quad d = 1, 2, 3.
$$

From (2.10), (2.11), and (3.5), we then easily have

$$
C_{\min}^{(d)} := \inf_{0 < t \leq T} \frac{\int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi}{\int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi} = \inf_{0 < t \leq T} \frac{\int_{\mathbb{R}^d} e^{-\frac{t}{4}(-2+|\xi|^2)^2} \, d\xi}{\int_{\mathbb{R}^d} e^{-\frac{t}{4}(-2+|\xi|^2)^2} \, d\xi} > 0 \quad \text{and}
$$

(3.6)

$$
C_{\max}^{(d)} := \sup_{0 < t \leq T} \frac{\int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi}{\int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi} = \sup_{0 < t \leq T} \frac{\int_{\mathbb{R}^d} e^{-\frac{t}{4}(-2+|\xi|^2)^2} \, d\xi}{\int_{\mathbb{R}^d} e^{-\frac{t}{4}(-2+|\xi|^2)^2} \, d\xi} < \infty,
$$

for $d = 1, 2, 3$. So, for $d = 1, 3$, and $0 < s \leq T$, we use the Parseval-Plancherel theorem together with (3.6) and (2.11) to get the desired lower and upper bounds as follows:

$$
\int_{\mathbb{R}^d} \left| K_{LKS}^{(d)} \right|^2 \, dx = \int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi \geq C_{\min}^{(d)} \int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi = C_{\min}^{(d)} C_d s^{-\frac{d}{4}},
$$

(3.7)

$$
\int_{\mathbb{R}^d} \left| K_{LKS}^{(d)} \right|^2 \, dx = \int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi \leq C_{\max}^{(d)} \int_{\mathbb{R}^d} \left| \hat{K}_{LKS}^{(d)} \right|^2 \, d\xi = C_{\max}^{(d)} C_d s^{-\frac{d}{4}}.
$$
The assertions in (3.1) and its immediate consequence (3.2) are thus established for \( d = 1, 3 \) and the proof is complete.

**Remark 3.1.** In \( d = 1 \), there is a critical \( t_c > 1 \) such that

\[
(\text{3.8}) \quad \left\{ \begin{array}{ll}
\int_{\mathbb{R}} \left| \hat{K}_{t\xi}^{d_{\text{SFO}}} \right|^2 d\xi < \int_{\mathbb{R}} \left| \hat{K}_{t\xi}^{d_{\text{LKS}}} \right|^2 d\xi, & t < t_c \\
\int_{\mathbb{R}} \left| \hat{K}_{t\xi}^{d_{\text{SFO}}} \right|^2 d\xi \geq \int_{\mathbb{R}} \left| \hat{K}_{t\xi}^{d_{\text{LKS}}} \right|^2 d\xi, & t \geq t_c,
\end{array} \right.
\]

with equality at \( t = t_c \). If \( T \leq t_c \), then using (2.11) and (3.8) the lower bound of (3.1) immediately holds with \( C_1^{(1)} = C_1 = 1/2\Gamma\left(\frac{d}{2}\right) \), where \( C_1 \) is the constant in (2.11) for \( d = 1 \). On the other hand, as in the case \( d = 2 \) (see (3.4) above), when \( d = 3 \) we have

\[
(\text{3.9}) \quad \int_{\mathbb{R}^3} \left| \hat{K}_{t\xi}^{d_{\text{SFO}}} \right|^2 d\xi < \int_{\mathbb{R}^3} \left| \hat{K}_{t\xi}^{d_{\text{LKS}}} \right|^2 d\xi; \; t > 0,
\]

which, when combined with (2.11), gives us the lower bound with the constant \( C_3^{(3)} = C_3 = \Gamma\left(\frac{d}{2}\right)/\pi^2 \sqrt{3} \).

**Lemma 3.2** (Kernel’s \( L^2 \) temporal difference). Fix any arbitrary \( T > 0 \). There are constants \( \tilde{C}_{u}^{(d)} \), depending only on \( d \) and \( T \) such that

\[
(\text{3.10}) \quad \int_0^T \int_{\mathbb{R}^d} \left| \hat{K}_{s}^{d_{\text{LKS}}} - \hat{K}_{s-x}^{d_{\text{LKS}}} \right|^2 dxds \leq \tilde{C}_{u}^{(d)}(t-r)^{d-4/4}; 0 < r < t \leq T, \quad d = 1, 2, 3,
\]

with the convention that \( \hat{K}_{t\xi}^{d_{\text{LKS}}} = 0 \) if \( t < 0 \). The same estimate holds, with possibly different constants, when replacing \( \hat{K}_{t\xi}^{d_{\text{LKS}}} \) with \( \hat{K}_{t\xi}^{d_{\text{SFO}}} \).

**Proof.** Throughout the proof, unless otherwise specified, the spatial dimension \( d \in \{1, 2, 3\} \). For \( u, v > 0 \) let

\[
(\text{3.11}) \quad \hat{K}_{u+v}^{(d)} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{u}{2}(|\xi|^2)} e^{-\frac{v}{2}(|\xi'|^2)} d\xi.
\]

By the Parseval-Plancherel theorem, we have

\[
(\text{3.12}) \quad \int_0^t \int_{\mathbb{R}^d} \left| \hat{K}_{s+t-r}^{d_{\text{LKS}}} - \hat{K}_{s-x}^{d_{\text{LKS}}} \right|^2 dxds = \int_0^t \int_{\mathbb{R}^d} \left| \hat{K}_{s+t-r}^{d_{\text{LKS}}} - \hat{K}_{s-x}^{d_{\text{LKS}}} \right|^2 d\xi ds
\]

\[= \int_0^t \left[ \hat{K}_{s}^{(d)}(t-r) - 2\hat{K}_{s+t-r}^{(d)} + \hat{K}_{s}^{(d)} \right] ds
\]

\[= \left[ \int_0^{t-r} \hat{K}_{s}^{(d)} ds - \int_{t-r}^{t} \hat{K}_{s}^{(d)} ds + \int_{t}^{t+\frac{t-r}{2}} \hat{K}_{s}^{(d)} ds + \int_{t+\frac{t-r}{2}}^{2t-r} \hat{K}_{s}^{(d)} ds \right].
\]

\[\text{3.11} \quad t_c \approx 1.506188. \text{ It is interesting to note that this is only a one-dimensional phenomenon (see (3.4) and (3.8)).} \]
It is clear from (3.11) that \( \tilde{\Phi}_{2\alpha}^{(d)} \) is decreasing in \( s \). Thus, the sum of the last three terms of (3.12) is \( \leq 0 \) and we have

\[
\int_0^t \int_{\mathbb{R}^d} \left| \mathcal{L}_{KS,x}(t-r;x) - \mathcal{L}_{KS,x} \right|^2 dx ds \leq \int_0^{\frac{t-r}{t}} \tilde{\Phi}_{2\alpha}^{(d)} ds
\]

(3.13)

\[
= \int_0^{\frac{t-r}{t}} \int_{\mathbb{R}^d} \left| \mathcal{L}_{KS,x} \right|^2 dx ds \leq \tilde{C}_u^{(d)} (t-r)^{\frac{4-d}{4}}; 0 < r < t \leq T, d \in \{1, 2, 3\},
\]

where we used the definition of \( \tilde{\Phi}_{2\alpha}^{(d)} \) in (3.11), Parseval-Plancherel theorem, and Lemma 3.1. The proof of the simpler \( K_{LKS}^{FO} \) case follows the same steps, with obvious trivial changes, and will be omitted. The lemma is established. \( \square \)

**Lemma 3.3 (Kernel’s \( L^2 \) spatial difference).** For \( d \in \{1, 2, 3\} \), there are intervals \( I_1 = (0, 1), I_2 = (0, 1), \) and \( I_3 = (0, 1/2) \); positive numbers \( \{\alpha_d \in I_d\}_{d=1}^3 \); constants \( \{C_u^{(d)}\}_{d=1}^3 \) depending only on \( d \) and \( \alpha_d \in I_d \) such that

\[
\int_0^t \int_{\mathbb{R}^d} \left| \mathcal{L}_{KS,x} - \mathcal{L}_{KS,x+t}^{(d)} \right|^2 dx ds \leq C_u^{(d)} |z|^{2\alpha_d - 1 \frac{4}{2}}; \forall \alpha_d \in I_d, t > 0,
\]

(3.14)

where \( 0 < C_u^{(d)} < \infty \) for every \( \alpha_d \in I_d \) for \( d = 1, 2, 3 \). The same estimate holds, with possibly different constants, when replacing \( K_{LKS}^{(d)} \) with \( K_{SFO}^{(d)} \).

**Proof.** We first observe from (2.4) that

\[
\tilde{\Phi}_{x+\xi}^{(d)} = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}|\xi|^2} e^{i\xi \cdot x}.
\]

Suppose \( d \in \{1, 2, 3\} \), and let \( \mathbb{B}^d_{\sqrt{2}} \) := \( \{\xi \in \mathbb{R}^d, |\xi| < \sqrt{2}\} \). Again, the Parseval-Plancherel theorem tells us that the quantity we want to estimate is

\[
\int_0^t \int_{\mathbb{R}^d} \left| \mathcal{L}_{KS,x} - \mathcal{L}_{KS,x+t}^{(d)} \right|^2 dx ds
\]

(3.15)

\[
= (2\pi)^{-d} \int_0^t \int_{\mathbb{R}^d} \left| e^{-\frac{1}{2}(-2+|\xi|^2)} [1 - e^{i\xi \cdot z} ] \right|^2 d\xi ds
\]

\[
= 2(2\pi)^{-d} \int_0^t \int_{\mathbb{R}^d} e^{-\frac{1}{2}(-2+|\xi|^2)} [1 - \cos (z \cdot \xi) ] d\xi ds
\]

(3.16)

\[
= 8(2\pi)^{-d} \int_{\mathbb{B}^d_{\sqrt{2}}} \left[ \frac{1-e^{-\frac{1}{2}(-2+|\xi|^2)}}{(-2+|\xi|^2)} \right] [1 - \cos (z \cdot \xi) ] d\xi
\]

\[
+ 8(2\pi)^{-d} \int_{\mathbb{R}^d \setminus \mathbb{B}^d_{\sqrt{2}}} \left[ \frac{1-e^{-\frac{1}{2}(-2+|\xi|^2)}}{(-2+|\xi|^2)} \right] [1 - \cos (z \cdot \xi) ] d\xi.
\]

\[\text{The constants } \tilde{C}_u^{(d)} = \left[ \frac{d + 1}{2} \right] C_u^{(d)}, \text{ where the constants } C_u^{(d)} \text{ are those in Lemma 3.1.} \]
We make use of the following two sets of elementary inequalities for all \( d \geq 1 \), the first of which uses the Cauchy-Schwarz inequality to obtain the last bound

\[
1 - \cos (z \cdot \xi) \leq 2 \left( 1 \wedge |z \cdot \xi|^{2\alpha} \right) \leq 2 \left( 1 \wedge |z|^{2\alpha} |\xi|^{2\alpha} \right) ; \quad 0 < \alpha \leq 1,
\]

\[
1 - e^{-4(-2+|\xi|^2)^2} \leq \left( 1 \wedge \frac{1}{4} \right) \frac{1 - e^{-(-2+|\xi|^2)}}{(-2 + |\xi|^2)^2} ; \quad t \geq 0.
\]

We now treat the cases \( d = 1, 2, 3 \) separately. Using (3.18), (3.19), and changing to polar coordinates in \( d = 2 \) and to spherical coordinates in \( d = 3 \) we can bound our desired quantity

\[
\int_0^t \int_{\mathbb{R}^d} \left| \hat{K}_{s \cdot \xi}^{LKS} - \hat{K}_{s \cdot \xi}^{LKS} \right|^2 dx ds
\]

from above by

\[
\frac{16(1 \vee \frac{1}{2})}{2\pi} |z|^{2\alpha} \left[ \int_{-\sqrt{2}}^{\sqrt{2}} |\xi|^{2\alpha} d\xi + 2 \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1 - e^{-(-2+|\xi|^2)^2}}{(-2 + |\xi|^2)^2} |\xi|^{2\alpha} d\xi \right]
\]

\[
\leq C(1)|z|^{2\alpha} ; \quad 0 < \alpha \leq 1 \text{ for } d = 1,
\]

\[
\frac{16(1 \vee \frac{1}{2})}{(2\pi)^2} |z|^{2\alpha} \left[ \int_0^{2\pi} \int_0^{\sqrt{2}} r^{2\alpha} r dr d\theta + \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{1 - e^{-(-2+r^2)^2}}{(-2+r^2)^2} r^{2\alpha} r dr d\theta \right]
\]

\[
\leq C(2)|z|^{2\alpha} ; \quad 0 < \alpha < 1 \text{ for } d = 2,
\]

and

\[
\frac{16(1 \vee \frac{1}{2})}{(2\pi)^3} |z|^{2\alpha} \left[ \int_0^{\pi} \int_0^{2\pi} \int_0^{\sqrt{2}} r^{2\alpha} r^2 \sin(\theta) dr d\theta d\theta + \int_0^{\pi} \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{1 - e^{-(-2+r^2)^2}}{(-2+r^2)^2} r^{2\alpha} r^2 \sin(\theta) dr d\theta d\theta \right]
\]

\[
\leq C(3)|z|^{2\alpha} ; \quad 0 < \alpha < \frac{1}{2} \text{ for } d = 3.
\]

In particular, when \( d = 1, \alpha \) may be taken to be 1; in \( d = 2, \alpha \in (0, 1) \); and in \( d = 3, \alpha \in (0, 1/2) \). Our dimension-dependent upper bound constant \( C(d) \) is independent of \( t \) if \( t \leq 4 \) and increases with \( t \) if \( t > 4 \). The proof of the simpler \( K_{1:x}^{LKS} \) case follows the same steps, with obvious trivial changes, and will be omitted.

In the next two subsections, we complete the proof of Theorem 1.1. We do so by first establishing the Hölder regularity results without imposing any Lipschitz conditions on \( a \), assuming the \( L^p \) boundedness of solutions on \( T \times \mathbb{R}^d \). We then add a Lipschitz condition on \( a \) and obtain the strong (stochastically) existence and uniqueness result for the L-KS SPDE (1.12) together with the \( L^p \) boundedness assumed before; thus, we obtain the Hölder regularity with no \( L^p \) boundedness assumptions\[33\]. With Lemma 5.4, Lemma 6.3 in hand, the rest of the proof of Theorem 1.1 is a straightforward adaptation of our corresponding arguments in 2 to our setting here. For the convenience of the reader and to make the article as self

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\[33\] As we mentioned in Remark 1.2 the existence of lattice limit solutions along with the regularity results in Theorem 1.1 (including both \( L^p \) boundedness on \( T \times \mathbb{R}^d \) and Hölder regularity) can be proven under the non-Lipschitz conditions \( \text{NLip} \), as we did in 1.2.
3.2. Sharp dimension-dependent Hölder regularity of solutions. Recalling that the initial data \( u_0 \) is assumed deterministic and writing \( U \) in the kernel formulation (1.11) in terms of its deterministic and random parts, we note that the deterministic part \( U_D(t,x) = \int_{\mathbb{R}^d} \mathcal{K}_{t}^{KS} u_0(y) dy \) is smooth in time and space, under the assumptions on \( u_0 \), since it is a classical solution to the LKS PDE (1.12) for \( (\varepsilon, \vartheta) = (1,1) \). We start with the spatial difference. Using Burkholder inequality and the proof.

Lemma 3.4 (Spatial and temporal differences). Assume that (Lip) and (3.21) are in force. There exists a constant \( \tilde{C}_d \) depending only on \( q, \max_x |u_0(x)| \), the spatial dimension \( d \in \{1,2,3\} \), \( \alpha_d \), and \( T \) such that

\[
\mathbb{E} \left[ |U(t,x) - U(t,y)|^{2q} \right] \leq \tilde{C}_d |x - y|^{2q \alpha_d}; \quad \alpha_d \in I_d,
\]

for all \( x,y \in \mathbb{R}^d \), \( t \in \mathbb{T} \), and \( d \in \{1,2,3\} \); where \( \alpha_d \) and \( I_d \) are as in Lemma 3.5.

Also, there exists a constant \( \tilde{C}_d \) depending only on \( q, \max_x |u_0(x)| \), the spatial dimension \( d \in \{1,2,3\} \), and \( T \) such that

\[
\mathbb{E} \left[ |U(t,x) - U(t,y)|^{2q} \right] \leq \tilde{C}_d |t - r|^{\frac{(4-d)q}{4}},
\]

for all \( x \in \mathbb{R}^d \), for all \( t, r \in \mathbb{T} \), and for \( 1 \leq d \leq 3 \).

Proof. We start with the spatial difference. Using Burkholder inequality and the linear growth condition on \( a \) ((b) in (Lip)), we have

\[
\mathbb{E} \left[ |U(t,x) - U(t,y)|^{2q} \right] \leq C \mathbb{E} \left[ \int_{\mathbb{R}^d} \int_{0}^{t} \left[ \mathcal{K}_{t-s}^{KS} - \mathcal{K}_{t-s}^{\infty} \right] ^2 \left( 1 + |U(s,z)|^2 \right) dsdz \right] ^q
\]

for any \((t,x,y) \in \mathbb{T} \times \mathbb{R}^{2d}\). Now, for any arbitrary fixed point \((t,x,y) \in \mathbb{T} \times \mathbb{R}^{2d}\) define the measure \( \rho_t^{x,y} \) on \([0,t] \times \mathbb{R}^d\) by

\[
d\rho_t^{x,y}(s,z) = \left[ \mathcal{K}_{t-s}^{KS} - \mathcal{K}_{t-s}^{\infty} \right] ^2 dsdz
\]

with \( |\rho_t^{x,y}| = \rho_t^{x,y}([0,t] \times \mathbb{R}^d) < \infty \), for \( 1 \leq d \leq 3 \), by (3.13). Now, apply Jensen's inequality to the probability measure \( \rho_t^{x,y} / |\rho_t^{x,y}| \) and use the definition of \( M_q(\varepsilon) \) together with (3.22) to obtain

\[
\mathbb{E} \left[ |U(t,x) - U(t,y)|^{2q} \right] \leq C \int_{[0,t] \times \mathbb{R}^d} \left( 1 + M_{2q}(\varepsilon) \frac{d\rho_t^{x,y}(s,z)}{|\rho_t^{x,y}|} \right) |\rho_t^{x,y}|^q
\]
Using the boundedness assumption (3.21) on \(M_q\) on \(T\) for \(1 \leq d \leq 3\), we get
\[
\mathbb{E} |U_R(t,x) - U_R(t,y)|^{2q} \leq C |\rho^{(y)}|^{q} \leq \left[ C^{(d)}_u \left(1 \vee \frac{T}{4}\right) \right]^{q} |x-y|^{2q_m d}
\]
\[
\leq \tilde{C}_d |x-y|^{2q_m d}; \quad t \in T, \alpha_d \in I_d, d = 1, 2, 3.
\]

where the last inequality follows from Lemma 3.3, and where the last constant \(\tilde{C}_d = \left[ C^{(d)}_u \left(1 \vee \frac{T}{4}\right) \right]^{q} < \infty\).

We now turn to the temporal difference. Assume without loss of generality that \(r < t\). Using Burkholder inequality, the linear growth condition on \(a\), and using the change of variable \(\rho\), where the last inequality follows from Lemma 3.3, and where the last constant \(\tilde{C}_d\) is an L-KS kernel solution to \((3.21)\) holds. Then
\[
E |U_R(t,x) - U_R(r,x)|^{2q} \leq C (|\mu_{t,r}^{x}|^{q} + |\eta^{x}|^{q}) \leq C(t-r)^\left(\frac{4-d}{q}\right),
\]
for \(d \in \{1, 2, 3\}\), where
\[
d\mu_{t,r}^{x}(s,z) = \left|\mathbb{E}_{\Delta_{t-s,x,z}}^{LKS} - \mathbb{E}_{\Delta_{r-s,x,z}}^{LKS}\right|^2 dsdz
\]
\[
d\eta^{x}(\rho, z) = \left|\mathbb{E}_{\Delta_{(r,s),z}}^{LKS}\right|^2 d\rho dz
\]
and \(|\mu_{t,r}^{x}| \in L^{\infty}([0, r] \times \mathbb{R}^d)\) and \(|\eta^{x}| \in L^q([0, t-r] \times \mathbb{R}^d)\). The last inequality in (3.25) follows from Lemma 3.1 and Lemma 3.2.

We now have the desired Hölder regularity result as the following corollary.

**Corollary 3.1.** Assume that \((U, \mathcal{W})\) is an L-KS kernel solution to (1.12) on \([\mathbb{R}_+ \times \mathbb{R}^d])_{d-1}\). Suppose further that the \(L^p\) boundedness in (3.21) holds. Then
\[
U \in H^{\frac{4-d}{8} - \left(\frac{4-d}{2} \wedge 1\right)}(T \times \mathbb{R}^d; \mathbb{R}); \quad \text{for } d = 1, 2, 3,
\]
almost surely.

**Proof.** First, we recall that since \(U(t,x) = U_D(t,x) + U_R(t,x)\), where the deterministic part \(U_D \in C^{1,4}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R})\). By Lemma 3.4 we easily have
\[
\begin{align*}
E |U_R(t,x) - U_R(t,y)|^{2n+2d} &\leq C_d |x-y|^{(2n+2d)\alpha_d}, \\
E |U_R(t,x) - U_R(r,x)|^{2m+4d} &\leq \tilde{C}_d |t-r|^{(4-d)(m+2d)/4}.
\end{align*}
\]
for \(1 \leq d \leq 3\). Thus, by standard results, the spatial Hölder exponent is \(\gamma_s \in \left(0, \frac{2(\alpha_d+1)}{2n+2d}\right)\) and the temporal exponent is \(\gamma_t \in \left(0, \frac{m(1-d/4)\alpha_d+1-d/2}{2m+4d}\right)\) \forall m, n.

Taking the limits as \(m, n \to \infty\), we get \(\gamma_t \in \left(0, \frac{4-d}{8}\right)\) and \(\gamma_s \in \left(0, \alpha_d\right)\), for \(1 \leq d \leq 3\) and \(\alpha \in I_d\) as in Lemma 3.3. The proof is now complete. \(\square\)
3.3. Existence, uniqueness, and $L^p$ boundedness. The final piece needed for the proof of Theorem 1.1 is now given by the following Lemma, which also removes the $L^p$ boundedness assumption (3.21) by asserting that it automatically holds under the conditions (Lip)\(^3\).

**Lemma 3.5.** Under the conditions in (Lip), there exists a strong and pathwise unique solution to the L-KS SPDE (1.12) on $\mathbb{R}_+ \times \mathbb{R}^d$ that is $L^p$-bounded on $T \times \mathbb{R}^d$, for every $p \geq 2$ and every $d = 1, 2, 3$.

**Proof.** The proof follows exactly the same steps as the proof on pp. 27–29 in [2] for the BTBM SIE, with now obvious and minor changes from the BTBM setting of [2] to our L-KS setting here, we omit the details and point the interested reader to [2] for the specifics. The proof of Theorem 1.1 is now complete. □

4. Proof of Theorem 1.2

We now turn to the proof of Theorem 1.2. Again, without loss of generality, it is enough to fix $\vartheta = 1$. We first need the $\varepsilon_1$-time-scaled L-KS Kernel $K_{\varepsilon_1}^{LKS}$, which—upon taking $\vartheta = 1$ in (1.10) reduces to

\[
K_{\varepsilon_1}^{LKS,t;x} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon_1}{4} (\xi^2 + 2\varepsilon_1 t \xi \cdot x)} d\xi,
\]

(4.1)

and which, by Theorem 2.1, solves the L-KS PDE

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\frac{\varepsilon_1}{4} \Delta^2 u - \frac{\vartheta}{4} \Delta u - \frac{\varepsilon_1}{2} u, \quad (t,x) \in (0, +\infty) \times \mathbb{R}^d; \\
u(0,x) &= \delta(x); \\
x &\in \mathbb{R}^d.
\end{align*}
\]

(4.2)

Also, exactly as in Lemma 3.1 above, $K_{\varepsilon_1}^{LKS,t;x}$ satisfies the bound\(^3\)

\[
\frac{C_u^d(t \frac{4-d}{4})}{\varepsilon_1^{d/4}} \geq \int_0^t \int_{\mathbb{R}^d} |K_{\varepsilon_1}^{LKS,t;x}|^2 dx ds \geq \frac{C_1^d(t \frac{4-d}{4})}{\varepsilon_1^{d/4}}
\]

(4.3)

**Proof of Theorem 1.2.** Let $T > 0$ and $q \geq 1$ be fixed and arbitrary, and let $(t,x) \in [0,T] \times \mathbb{R}^d$ and $d = 1, 2, 3$.

\(^{34}\)Actually, (3.21) automatically holds under the weaker assumptions (NLip), just as was shown for the BTBM SIE in [2].

\(^{35}\)Again, Theorem 1.2 holds for any fixed value of $\vartheta \in \mathbb{R}$. Fixing $\vartheta$ value to 1 is for convenience and for simplifying notation and exposition in the proof only. The case $\vartheta = 1$ captures the general case of the 4th order biLaplacian term together with the second and zero order terms.

\(^{36}\)Of course, the case $\vartheta = 0$ satisfies (4.3), with only the biLaplacian term, without the Laplacian and without the zero order terms; and satisfies (4.3) with equality.
(i) Under the conditions [Lip], we have by Theorem 1.1 a unique solution $U_{\varepsilon_1, \varepsilon_2}$ to the L-KS SPDE (1.16) that is $L^{2q}(\Omega)$-bounded on $\mathbb{T} \times \mathbb{R}^d$, for every $q \geq 1$. Let $\mu_{\varepsilon_1}^{t,x}$ be the measure on $[0,t] \times \mathbb{R}^d$ defined by

$$d\mu_{\varepsilon_1}^{t,x}(s,y) = \left| K_{\varepsilon_1}^{\text{LKS}}(t-s;x,y) \right|^2 ds \, dy$$

and let $|\mu_{\varepsilon_1}^{t,x}| = \mu_{\varepsilon_1}^{t,x}([0,t] \times \mathbb{R})$. Taking the $2q$-th moment of the difference between our scaled L-KS SPDE and its deterministic counterpart—whose solution we denote by $u_{\varepsilon_1}$; using Burkholder’s inequality followed by Jensen’s inequality applied to the probability measure $d\mu_{\varepsilon_1}^{t,x}(s,y)/|\mu_{\varepsilon_1}^{t,x}|$, then using the linear growth condition ((b) in [Lip]) on $a$, the $L^{2q}(\Omega)$-boundedness of $U_{\varepsilon_1, \varepsilon_2}$ on $\mathbb{T} \times \mathbb{R}^d$, and the upper bound in (4.3), we get

$$\mathbb{E} \left[ |U_{\varepsilon_1, \varepsilon_2}(t,x) - u_{\varepsilon_1}(t,x)|^{2q} \right] \leq C \mathbb{E} \left[ \int_{\mathbb{R}^d} \int_0^t \left| K_{\varepsilon_1}^{\text{LKS}}(t-s;x,y) \varepsilon_2 a(U_{\varepsilon_1, \varepsilon_2}(s,y)) \mathcal{W}(ds,dy) \right|^{2q} \right]$$

(4.4)

$$\leq C \varepsilon_2^{2q} \int_{\mathbb{R}^d} \int_0^t \mathbb{E} \left[ \left. \left| K_{\varepsilon_1}^{\text{LKS}}(t-s;x,y) \varepsilon_2 a(U_{\varepsilon_1, \varepsilon_2}(s,y)) \right|^{2q} \right| \mu_{\varepsilon_1}^{t,x} \right] ds \, dy$$

$$\leq \frac{CT}{\varepsilon_1^{(4d/4)q} \varepsilon_2^{2q}} \rightarrow 0$$

as $\varepsilon_1, \varepsilon_2$, and $\varepsilon_2/\varepsilon_1^{d/4}$ approach 0.

(ii) We prove it by contradiction. So, assume there is a $T > 0$ such that

$$\lim_{\varepsilon_1, \varepsilon_2 \downarrow 0} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E} U_{\varepsilon_1, \varepsilon_2}^2(s,x) < \infty; \quad d = 1, 2, 3$$

and assume without loss of generality that $u_0 \equiv 0$. Observe that

$$\mathbb{E} \left[ |U_{\varepsilon_1, \varepsilon_2}(t,x)|^2 \right] = \mathbb{E} \left[ \int_{\mathbb{R}^d} \int_0^t \left| K_{\varepsilon_1}^{\text{LKS}}(t-s;x,y) \varepsilon_2 a(U_{\varepsilon_1, \varepsilon_2}(s,y)) \mathcal{W}(ds,dy) \right|^2 \right]$$

(4.5)

$$\leq \varepsilon_2^2 \int_{\mathbb{R}^d} \int_0^t \left| K_{\varepsilon_1}^{\text{LKS}}(t-s;x,y) \right|^2 \mathbb{E} a^2(U_{\varepsilon_1, \varepsilon_2}(s,y)) ds \, dy$$

$$\geq K_t^2 \varepsilon_2^2 \int_{\mathbb{R}^d} \int_0^t \left| K_{\varepsilon_1}^{\text{LKS}}(t-s;x,y) \right|^2 ds \, dy \geq \frac{\tilde{c}(d) \varepsilon_2^2 t^{4-d}}{\varepsilon_1^{d/4}}; \quad d = 1, 2, 3,$$

where we used the lower bound assumption $0 < K_t \leq a(u)$ and the lower bound in (4.3) to get the last two inequalities in (4.6). Using the assumption in (4.3), we arrive at the desired contradiction by taking the limit as $\varepsilon_1, \varepsilon_2 \searrow 0$ in (4.6) such that $\varepsilon_2/\varepsilon_1^{d/4} \not\rightarrow \infty$ for $d = 1, 2, 3$. The proof for the case or $\varepsilon_1, \varepsilon_2 \not\rightarrow \infty$ follows exactly the same steps.

The proof is complete. \qed
5. Proof of Theorem 1.3

Here, we prove the change of measure transfer of properties from the canonical L-KS to nonlinear L-KS fourth order SPDEs, including the Swift-Hohenberg SPDE on subsets of \( \mathbb{R}^d \times \mathbb{R}^d \).

We say that a progressively measurable random field \( X \) on the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) satisfies Novikov’s condition on \( T \times S \) if

\[
E_P \left[ \exp \left( \frac{1}{2} \int_{T \times S} X^2(t,x) dtdx \right) \right] < \infty.
\]

Proof of Theorem 1.3.

(i) (Transfer of law uniqueness) We prove the more interesting direction (from zero to nonzero drift). The proof of the reverse direction is similar and is omitted. Suppose that uniqueness in law holds for the zero-drift L-KS SPDE \( e^{(\varepsilon,0)}_{\text{LKS}}(a,0,u_0) \), and assume that \( (V^{(i)}, \tilde{W}^{(i)}), (\Omega^{(i)}, \mathcal{F}^{(i)}, \{\mathcal{F}^{(i)}_t\}, \tilde{P}^{(i)}); i = 1, 2 \)

are solutions to the nonzero-drift L-KS SPDE \( e^{(\varepsilon,b)}_{\text{LKS}}(a,b,u_0) \). By assumption, we have

\[
\tilde{P}^{(i)} \left( \int_{T \times S} R^2_{V^{(i)}}(t,x) dtdx < \infty \right) = 1, \ i = 1, 2.
\]

Define the sequence of stopping times \( \{\tau_n^{(i)}\} \) by

\[
\tau_n^{(i)} := T \wedge \inf \left\{ 0 \leq t \leq T; \int_{[0,t] \times S} R^2_{V^{(i)}}(s,x) dsdx = n \right\}; \ n \in \mathbb{N}, \ i = 1, 2,
\]

and let \( \mathcal{W}^{(i)} = \{\mathcal{W}^{(i)}_t(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(S)\} \) be given by

\[
\mathcal{W}^{(i)}_t(B) := \tilde{W}^{(i)}_t(B) + \int_{[0,t] \times B} R_{V^{(i)}}(s,x) dsdx; \ i = 1, 2.
\]

Then, Novikov’s condition (5.1) and Girsanov’s theorem for white noise (see Theorem 2.2, Corollary 2.3, and Lemma 2.4 in [13]) immediately gives us that

\[
\mathcal{W}^{(i)}_n = \{\mathcal{W}^{(i)}_{T \wedge \tau_n^{(i)}}(B), \mathcal{F}_{T \wedge \tau_n^{(i)}}; 0 \leq t \leq T, B \in \mathcal{B}(S)\}
\]

is a white noise stopped at time \( \tau_n^{(i)} \), under the probability measure \( \tilde{P}^{(i)}_n \) defined on \( \mathcal{F}^{(i)}_T \) by

\[
\frac{d\tilde{P}^{(i)}_n}{d\tilde{P}^{(i)}} = \frac{R_{V^{(i)}}(\tau_n^{(i)})}{\tilde{W}^{(i)}_{T \wedge \tau_n^{(i)}}(S)}; \ n \in \mathbb{N}, \ i = 1, 2,
\]
where the Radon-Nikodym derivative is given by

\[
\begin{align*}
\gamma_{T \land \tau_n^{(i)}(B)} &= \exp \left\{ \int_{[0,T \land \tau_n^{(i)}(B)] \times B} \left[ R_{V^{(i)}}(s,x) \mathcal{W}^{(i)}(ds, dx) - \frac{1}{2} R_{V^{(i)}}^2(s,x)ds dx \right] \right\};
\end{align*}
\]

0 \leq t \leq T, B \in \mathcal{B}(\mathbb{S}). Consequently, \( (V^{(i)}, \mathcal{W}^{(i)}) \), \( (\Omega^{(i)}, \mathcal{F}^{(i)}_T, \{\mathcal{F}^{(i)}_t\}, \mathbb{P}^{(i)}_n) \) is a solution to the zero-drift L-KS SPDE \( e_{LKS}^{(\varepsilon, \vartheta)}(a,0,u_0) \) on \( [0,T \land \tau_n^{(i)}] \times \mathbb{S} \) for each \( i = 1, 2 \) and \( n \in \mathbb{N} \). Clearly, for \( i = 1, 2 \),

\[
\frac{d\mathbb{P}^{(i)}_n}{d\mathbb{P}^{(1)}_n} = \frac{\mathbb{E}[\mathbb{P}^{(1)}_n]}{\mathbb{E}[\mathbb{P}^{(1)}_n]}(\mathcal{S})
\]

\[
\mathbb{P}^{(2)}_n \left[ V^{(1)} \in \Lambda, \tau_n^{(1)} = T \right] = \mathbb{E}^{(1)}_n \left[ 1_{\{V^{(1)} \in \Lambda, \tau_n^{(1)} = T\}} R_{V^{(1)}}^{(i)}(\mathcal{S}) \right]
\]

\[
\mathbb{P}^{(2)}_n \left[ V^{(2)} \in \Lambda, \tau_n^{(2)} = T \right] = \mathbb{E}^{(1)}_n \left[ 1_{\{V^{(2)} \in \Lambda, \tau_n^{(2)} = T\}} R_{V^{(2)}}^{(i)}(\mathcal{S}) \right]
\]

where we have used the uniqueness in law assumption on \( e_{LKS}^{(\varepsilon, \vartheta)}(a,0,u_0) \) (comparing the \( V^{(i)}'s \) only on \( \Omega^{(i)}_n := \{\tau_n^{(i)} = T\} \) for each \( n \), \( \mathbb{E}^{(1)}_n \) to get the second equality in \( \mathbb{E}^{(2)}_n \). By \( \mathbb{E}^{(1)}_n \) and \( \mathbb{E}^{(2)}_n \), we obtain that

\[
\lim_{n \to \infty} \mathbb{P}^{(i)}_n[\tau_n^{(i)} = T] = 1 \quad \text{for} \quad i = 1, 2.
\]

Thus, taking the limit as \( n \to \infty \) in \( \mathbb{P}^{(1)}_n \) yields that the law of \( V^{(1)} \) under \( \mathbb{P}^{(1)}_n \) is the same as that of \( V^{(2)} \) under \( \mathbb{P}^{(2)}_n \). I.e., we have uniqueness in law for the non-zero drift L-KS SPDE \( e_{LKS}^{(\varepsilon, \vartheta)}(a,b,u_0) \).

\( \text{(ii) (Law equivalence)} \) Let \( (V, \mathcal{W}) \) be a solution (weak or strong) to the nonzero-drift L-KS SPDE \( e_{LKS}^{(\varepsilon, \vartheta)}(a,b,u_0) \) on \( \Omega^{(1)}, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q} \); and let \( (U, \mathcal{W}) \) be a solution (weak or strong) to the zero-drift L-KS SPDE \( e_{LKS}^{(\varepsilon, \vartheta)}(a,0,u_0) \) on \( \Omega^{(2)}, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P} \). Then, uniqueness in law for the L-KS SPDE \( e_{LKS}^{(\varepsilon, \vartheta)}(a,b,u_0) \) follows from the uniqueness in law assumption for \( e_{LKS}^{(\varepsilon, \vartheta)}(a,0,u_0) \), the almost sure \( L^2(\mathbb{T} \times \mathbb{S}, \lambda) \) condition on \( R_V \), and part (i) of Theorem \( \mathbb{E}^{(3)}_n \).

Replacing \( V^{(i)} \) in \( \mathbb{E}^{(3)}_n \) by \( U \) and then \( V \), we get the definitions of the stopping times sequences \( \{\tau_n^{V}\} \) and \( \{\tau_n^{U}\} \), respectively. Let \( \mathcal{W} = \{\mathcal{W}_t(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(\mathbb{S})\} \) be given by

\[
\mathcal{W}_t(B) := \mathcal{W}_t^{(2)}(B) - \int_{[0,t] \times B} R_U(s,x)ds dx.
\]
Then, Novikov’s condition and Girsanov’s theorem for white noise (see Theorem 2.2, Corollary 2.3, and Lemma 2.4 in [13]) immediately give us that, for $n \in \mathbb{N}$, $\tilde{W}_n = \{ W_{t\wedge \tau_n^U}(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(S) \}$ is a white noise stopped at time $\tau_n^U$, under the probability measure $\tilde{P}_n$ defined on $\mathcal{F}_T$ by the Radon-Nikodym derivative

$$
\frac{d\tilde{P}_n}{dP} = \Xi_{T \wedge \tau_n^U}(S) \exp \left\{ \int_{[0,T \wedge \tau_n^U] \times S} R_U(s,x) \mathcal{W}^{(2)}(ds, dx) - \frac{1}{2} \int_{[0,T \wedge \tau_n^U] \times S} R_U^2(s,dx)ds \right\}.
$$

(5.6)

Thus, $(U, \tilde{W}_n)$, $(\Omega^{(2)}, \mathcal{F}_T, \{\mathcal{F}_t\}, \tilde{P}_n)$ is a solution to the nonzero-drift L-KS SPDE $e_{LKS}^{(c,\theta)}(a,b,u_0)$ on $[0,T \wedge \tau_n^U] \times S$, for each $n \in \mathbb{N}$. As a result, for any set $\Lambda \in \mathcal{B}(C(T \times S; \mathbb{R}))$ we get

$$
\mathbb{Q}[V \in \Lambda, \tau_n^V = T] = \tilde{P}_n[U \in \Lambda, \tau_n^U = T] = \mathbb{E}_P \left[ 1_{\{U \in \Lambda, \tau_n^U = T\}} \Xi_{T \wedge \tau_n^U}(S) \right] ; \quad n \in \mathbb{N}.
$$

(5.7)

To see (5.7) note that, on the event $\Omega_U^n := \{ \omega \in \Omega^{(2)}; \tau_n^U(\omega) = T \}$, $(U, \tilde{W}_n)$ is a solution to $e_{LKS}^{(c,\theta)}(a,b,u_0)$ on $T \times S$, under $\tilde{P}_n$. Thus, the first equality in (5.7) follows from the uniqueness in law for $e_{LKS}^{(c,\theta)}(a,b,u_0)$ and the definitions of $\tau_U^n$ and $\tau_V^n$. By the $L^2$ assumption on $R_U$ and the definition of $\tau_V^n$, we have $\lim_{n \to \infty} \mathbb{Q}[\tau_V^n = T] = 1$; so, taking limits in (5.7) we get

$$
\mathbb{Q}[V \in \Lambda] = \lim_{n \to \infty} \tilde{P}_n[U \in \Lambda, \tau_n^U = T] = \lim_{n \to \infty} \mathbb{E}_P \left[ 1_{\{U \in \Lambda, \tau_n^U = T\}} \Xi_{T \wedge \tau_n^U}(S) \right].
$$

(5.8)

Obviously, if $\mathbb{P}[U(\cdot, \cdot) \in \Lambda] = 0$ then $\mathbb{E}_P \left[ 1_{\{U(\cdot, \cdot) \in \Lambda, \tau_U^1 = T\}} \Xi_{T \wedge \tau_U^1}(S) \right] = 0$ for each $n$; thus,

$$
\mathbb{Q}[V(\cdot, \cdot) \in \Lambda] = \lim_{n \to \infty} \mathbb{E}_P \left[ 1_{\{U(\cdot, \cdot) \in \Lambda, \tau_U^1 = T\}} \Xi_{T \wedge \tau_U^1}(S) \right] = 0.
$$

(5.9)

I.e., $\mathcal{L}_Q^U$ is absolutely continuous with respect to $\mathcal{L}_P^U$ on $\mathcal{B}(C(T \times S; \mathbb{R}))$. The absolute continuity of $\mathcal{L}_Q^U$ with respect to $\mathcal{L}_P^U$ is proved by a similar argument, and we omit it.

We turn now to the proof of the last part of Theorem 1.3. For the remainder of the proof $S = \prod_{i=1}^d [0,L_i]$, $d = 1, 2, 3$; $\mathbb{T} = [0,T]$ for some fixed but arbitrary $T > 0$; $a(u) = \kappa$ for $\kappa \in \mathbb{R} \setminus \{0\}$; $b(u) = \sum_{k=0}^N c_k u^k$, for some $c_i \in \mathbb{R}$ and $N \geq 0$, $i = 0, \ldots, N$; and $u_0 \in C^2_c(S; \mathbb{R})$ and nonrandom $\forall \ d \in \{1, 2, 3\}$. Let $U$ and $V$ be continuous solutions to the L-KS SPDEs $e_{LKS}^{(c,\theta)}(a,0,u_0)$ and $e_{LKS}^{(c,\theta)}(a,b,u_0)$ on $\mathbb{T} \times S$, respectively; then, the square of the drift/diffusion ratios given by the random fields

$$
R_U^2(t,x) = \sum_{k=0}^{2N} \tilde{c}_k U^k(t,x) \quad \text{and} \quad R_V^2(t,x) = \sum_{k=0}^{2N} \tilde{c}_k V^k(t,x)
$$

for $\tilde{c}_i \in \mathbb{R}$

(5.9)

are continuous and thus almost surely bounded on the compact set $[0,T] \times S$. Therefore, $R_U$ and $R_V$ are in $L^2(\mathbb{T} \times S, \lambda)$, almost surely, and part (i) of Theorem 1.3 implies that uniqueness in law for zero-drift L-KS SPDE $e_{LKS}^{(c,\theta)}(a,0,u_0)$ is equivalent...
to uniqueness in law for the L-KS SPDE $e_{LKS}^{(\varepsilon, \theta)}(a, b, u_0)$ with the above polynomial drift nonlinearity $b$.

In addition, Lemma A.1 gives us pathwise uniqueness, and hence uniqueness in law, for $e_{LKS}^{(\varepsilon, \theta)}(a, 0, u_0)$; and, thus, part (ii) gives us equivalence in law between $e_{LKS}^{(\varepsilon, \theta)}(a, 0, u_0)$ and $e_{LKS}^{(\varepsilon, \theta)}(a, b, u_0)$ on $\mathcal{B}(C(\mathbb{T} \times \mathbb{S}; \mathbb{R}))$. This law equivalence, in turns, implies that

\[
V \in H^{\frac{4-d}{8}} \left( \frac{4-d}{2} \wedge 1 \right)^{-} (\mathbb{T} \times \mathbb{S}; \mathbb{R}) \text{ a.s.}
\]

\[
\Longleftrightarrow U \in H^{\frac{4-d}{8}} \left( \frac{4-d}{2} \wedge 1 \right)^{-} (\mathbb{T} \times \mathbb{S}; \mathbb{R}) \text{ a.s.}
\]

for $d = 1, 2, 3$. \(\Box\)

### APPENDIX A. Uniqueness lemma

Throughout this Appendix, we reserve the notation $\mathbb{S}$ solely for the $d$-dimensional rectangle $\prod_{i=1}^{d} [0, L_i]$, $d = 1, 2, 3$; and, as before, $\mathbb{T} = [0, T]$ for some fixed but arbitrary $T > 0$. We now prove pathwise uniqueness for $e_{LKS}^{(\varepsilon, \theta)}(a, 0, u_0)$ with Dirichlet boundary conditions. The following lemma is useful for the last part of Theorem 1.3.

**Lemma A.1.** Pathwise uniqueness, and hence uniqueness in law, holds for the zero-drift L-KS SPDE $e_{LKS}^{(\varepsilon, \theta)}(a, 0, u_0)$ on $\mathbb{R}^{+} \times \mathbb{S}$ whenever $a \equiv \kappa \neq 0$.

**Proof.** Assume without loss of generality that $d = 1$, $\mathbb{S} = [0, 1]$, and fix $\varepsilon > 0, \vartheta \in \mathbb{R}$. Assume further that $(U^{(1)}, \mathcal{W})$ and $(U^{(2)}, \mathcal{W})$ are two continuous solutions to $e_{LKS}^{(\varepsilon, \theta)}(a, 0, u_0)$ on $\mathbb{R}^{+} \times \mathbb{S}$ on the same usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and with respect to the same space-time white noise $\mathcal{W}$. Let $D(t, x) = U^{(1)}(t, x) - U^{(2)}(t, x)$ for $(t, x) \in \mathbb{R}^{+} \times \mathbb{S}$, and let

\[
\Phi_{Dir}^{\infty}(\mathbb{S}; \mathbb{R}) := \{ \varphi \in C^{\infty}(\mathbb{R}^{d}; \mathbb{R}); \varphi = \vartheta \varphi = 0 \text{ on } \partial \mathbb{S} \}.
\]

Now, the continuous difference random field $D$ satisfies

\[
\int_{0}^{1} D(t, x) \varphi(x) dx \quad (A.1)
\]

\[
= -\frac{\varepsilon}{8} \int_{0}^{t} \int_{0}^{1} D(s, x) \left( \varphi^{(4)}(x) + 4 \vartheta \varphi^{(2)}(x) + 4 \vartheta^{2} \varphi(x) \right) dx ds
\]

for every $\varphi \in \Phi_{Dir}^{\infty}(\mathbb{S}; \mathbb{R})$, $t \in \mathbb{T}$, a.s. $\mathbb{P}$. Manifestly, this implies that $D(t, x) = 0$ on $[0, T] \times [0, 1]$ a.s. $\mathbb{P}$. To see this, choose $\varphi_{m}(x) = \sin(m \pi x)$ for $m \in \mathbb{N}$ and let

\[
C_{m}(t) := \int_{0}^{1} D(t, x) \varphi_{m}(x) dx; \forall m \in \mathbb{N}, t \in \mathbb{T}.
\]

So, by (A.1), we have

\[
C_{m}(t) = \left( -\frac{\varepsilon m^{4} \pi^{4}}{8} + \frac{\varepsilon \vartheta m^{2} \pi^{2}}{2} - \frac{\varepsilon \vartheta^{2}}{2} \right) \int_{0}^{t} C_{m}(s) ds; \text{ a.s. } \mathbb{P},
\]

for all $t \in [0, T]$ and $m \in \mathbb{N}$, which obviously implies that, for each $m$, $C_{m}(t) = 0$ for all $t \in \mathbb{T}$ a.s. $\mathbb{P}$. Now, since all the Fourier Sine coefficients, $C_{m}(t)$, for $D(t, x)$
are zero and the continuous solutions of the SPDE \( \varepsilon \) vanish at \( x = 0 \) and \( x = 1 \) for all \( t \) (and hence \( D(t, 0) = D(t, 1) = 0 \) \( \forall t \)), we get that \( D(t, x) = 0 \) on \( [0, T] \times [0, 1] \) a.s. \( \mathbb{P} \). The arbitrariness of \( T \) now completes the proof. □

APPENDIX B. Frequent acronyms and notations key

I. Acronyms

1. BTBM: Brownian-time Brownian motion.
2. BTBM SIE: BTBM stochastic integral equation.
3. IBTBAP: imaginary Brownian-time Brownian angle process.
4. KS: Kuramoto Sivashinsky.
5. SH: Swift-Hohenberg.

II. Notations

1. \( K_{\varepsilon,0}^{LKS} \) the \( (\varepsilon, \vartheta) \) L-KS kernel (see (1.8)).
2. \( K_{1,1}^{LKS} \) the canonical (or (1, 1)) L-KS kernel (see (1.5) and (1.8)).
3. \( K_{1,0}^{SFO} \) the zero-angle canonical (or simple fourth order) kernel (see (2.8) and (1.8)).
4. \( C_{k,\gamma} \) the set of functions with \( \gamma \)-Hölder continuous \( k \)-th derivative, \( k = 0, 1, 2, \ldots \)
5. \( C_{b,\gamma} \) the set of functions with \( \gamma \)-Hölder continuous and bounded \( k \)-th derivative, \( k = 0, 1, 2, \ldots \)
6. \( T = [0, T] \) for some fixed and arbitrary \( T > 0 \)
7. \( M_p(t) := \sup_{x \in \mathbb{R}^d} \mathbb{E} |U(t, x)|^p \)

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