Spherical Morita contexts and relative Serre functors

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Abstract
The Morita context provided by an exact module category over a finite tensor category gives a two-object bicategory with duals. Right and left duals of objects in the module category are given by internal Homs and coHoms, respectively. We express the double duals in terms of relative Serre functors, which leads to a Radford isomorphism for module categories. There is a bicategorical version of the Radford $S^4$ theorem: on the bicategory of a Morita context, the relative Serre functors assemble into a pseudo-functor, and the Radford isomorphisms furnish a trivialization of the square of this pseudo-functor, i.e. of the fourth power of the duals.

We also show that the Morita bicategories coming from pivotal exact module categories are pivotal as bicategories, leading to the notion of pivotal Morita equivalence. This equivalence of tensor categories amounts to the equivalence of their bicategories of pivotal module categories. Furthermore, we introduce the notion of a spherical module category; it ensures that all categories in the Morita context of a spherical module category are spherical. Our results are motivated by and have applications to topological field theory.
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1 Introduction

Two familiar mathematical structures with countless applications are (finite-dimensional) algebras and symmetric Frobenius algebras. It is important to notice that a Frobenius algebra is an algebra with additional structure (which can be expressed in different ways, e.g. as a Frobenius form or as a homotopy fixed point). Topological field theory in two dimensions relates these two types of structures to two different flavors of topological field theories: algebras to framed theories, and symmetric Frobenius algebras to oriented theories. Algebras and symmetric Frobenius algebras are thus associated with very different geometry. Algebraically, they should be carefully told apart as well.

A similar pattern is present in higher dimensions: finite tensor categories are related to framed modular functors, while pivotal finite tensor categories are related to oriented modular functors. Here a pivotal structure on a finite tensor category is the additional datum of a monoidal trivialization of the double dual. Finite tensor categories and pivotal finite tensor categories should be carefully told apart, too.

It is natural to think about finite tensor categories as a tricategory, with bimodule categories as 1-morphisms, bimodule functors as 2-morphisms, and bimodule natural transformations as 3-morphisms. Invertible bimodule categories provide a particularly important subcategory. We could call this subcategory the Morita groupoid of finite tensor categories. An invertible bimodule category is in particular exact. Now an indecomposable exact left module category over a finite tensor category \( A \) is automatically also a right module category over the dual finite tensor category \( A^* \), and is thus a bimodule category. In fact – as we will show in Theorem 3.9 – it is even an invertible bimodule category and thus provides an invertible 1-morphism. This theory is well-understood (see [EGNO], and [Mü] for the case of \( * \)-categories). Furthermore, it has applications to invertible topological defects between framed modular functors of Turaev-Viro type [DSS2].

Pivotal finite tensor categories are not only important for topological field theory, but e.g. also [Sh2, FuS] as a prolific source of Frobenius algebras in tensor categories. It is therefore highly desirable to have a similar Morita groupoid of pivotal finite tensor categories at hand. Several concepts pertinent to such a structure have appeared recently, such as the notion of a relative Serre functor [FSS] which allows one to introduce the notion of a pivotal module category. However, so far they have not been assembled to a consistent pivotal Morita theory.1

The primary contribution of this paper is to develop such a theory.

A summary of our approach and of the organization of the paper is as follows. The category of finite-dimensional modules over a finite-dimensional algebra is a finite abelian category. The basic stimulus for Morita theory is to see this category as more fundamental than the algebra itself. Moreover, the Eilenberg-Watts theorem implies that Morita equivalence of algebras can be captured in terms of bimodules. This story repeats itself in higher categorical dimensions, replacing algebras by finite tensor categories \( A \) and bimodules by bimodule categories over them. This leads to the notion of categorical Morita equivalence. It is formulated in terms of exact module categories, i.e. \( A \)-module categories \( \mathcal{M} \) for which the internal Hom functor \( \text{Hom}(m,-): \mathcal{M} \to A \) is exact for every object \( m \in \mathcal{M} \).

After recalling some pertinent background in Section 2 so as to fix notation, we review and

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1 This has not prevented physicists from computing successfully with structures that correspond to a pivotal Morita context as if it were just a pivotal monoidal category, see e.g. [BL].
extend this categorical Morita theory in Section 3. We show in Theorem 3.3 that a Morita context gives rise to a bicategory with two objects. Known coherence results for bicategories then lead to strictification results for Morita contexts. Next we show that any exact module category induces a Morita context. This Morita context is actually strong, i.e. the bimodule functors are equivalences. In Theorem 3.11 we provide a converse: any strong Morita context comes from an exact module category.

Section 4 starts our study of pivotal Morita theory. We first show that the bicategory induced by an exact module category is a bicategory with duals. To this end it is crucial to realize that the duals of objects in the module category are module functors given by internal Homs and coHoms, respectively, see Definition 4.1. When thinking about the bimodule functors in the Morita context as “mixed tensor products”, one can work with the duals in the bicategory associated with a strong Morita context in much the same way as in the familiar case of a single rigid monoidal category. Among the resulting insights are the statements that left and right duals are inverse and define a pseudo-functor from the bicategory to the bicategory with reversed 1- and 2-morphisms (Proposition 4.3), and that all internal Homs and coHoms can be expressed in terms of duals and mixed tensor products (Proposition 4.6 and Remark 4.7). We proceed to compute the double dual endofunctors on the module categories. In Proposition 4.11 we show that they are given by the relative Serre functors. A consequence is a Radford isomorphism for module categories:

**Theorem 4.15.** Let \( \mathcal{A} \) be a finite tensor category and \( \mathcal{M} \) an exact \( \mathcal{A} \)-module. There is a natural isomorphism

\[
\mathbb{D}_\mathcal{A} \Delta - \mathbb{D}_\mathcal{A}^{-1} \Delta \cong S^\mathcal{A}_\mathcal{M} \circ S^\mathcal{A}_\mathcal{M}
\]

of (twisted) bimodule functors, where \( S^\mathcal{A}_\mathcal{M} \) is the relative Serre functor of \( \mathcal{M} \) and \( \mathbb{D}_\mathcal{A} \) and \( \mathbb{D}_\mathcal{A}^{-1} \) are the distinguished invertible objects of \( \mathcal{A} \) and \( \mathcal{A}^* \), respectively.

The relative Serre functors of the categories in a Morita context form a pseudo-functor on its associated bicategory (see Definition 4.13). In analogy to Theorem 4.15 there are Radford isomorphisms for all the categories in the Morita context bicategory. In Theorem 4.16 we show that these assemble into a trivialization of the square of the relative Serre pseudo-functor. Altogether these results establish natural rigid duality structures for Morita bicategories.

Whenever one deals with dualities, the question arises whether the double dual admits a trivialization in an appropriate sense. Such a trivialization, known as a pivotal structure, has important consequences. For instance, it is needed to define oriented topological field theories. Consequently, in order to study dualities for oriented topological field theories, it is central to understand Morita equivalence in a pivotal setting. The interaction between Morita theory and pivotal structures is the subject of Section 5. The notion of a pivotal structure for monoidal and module categories is well understood, see Definition 5.1. The general definition of a pivotal bicategory then directly leads to Definition 5.7 of a pivotal Morita context. These different aspects of pivotality fit together well:

**Theorem 5.9.** The Morita context associated with a pivotal module category \( \mathcal{M} \) over a pivotal tensor category \( \mathcal{A} \) is a pivotal Morita context.

This result is by no means trivial: the pivotality of a module category \( \mathcal{M} \) just imposes a restriction on \( \mathcal{M} \) as a one-sided module; in contrast, the pivotality of the Morita context
implies that $\mathcal{M}$ is pivotal as a bimodule category.

Our study of pivotality culminates in the following statements:

**Theorem 5.12.** Two pivotal tensor categories are pivotal Morita equivalent if and only if their associated 2-categories of pivotal module categories, module functors and module natural transformations are equivalent as pivotal bicategories.

**Theorem 5.16.** If two pivotal categories are pivotal Morita equivalent, then their Drinfeld centers are equivalent as pivotal braided tensor categories.

Section 5.6 is devoted to the property of a pivotal structure being spherical. Lately the perspective on sphericality has undergone a change: while trace-sphericality emphasized the equality of left and right traces for endomorphisms in a pivotal category, a more recent notion of sphericality [DSS2] requires the pivotal structure to square to Radford’s trivialization of the quadruple dual. The two notions agree in the semisimple case. In Definition 5.18 we propose a notion of sphericality for pivotal module categories which is in the same spirit as the one for tensor categories in [DSS2]. Our definition receives its full justification in Proposition 5.20 according to which in the Morita context given by a spherical module category, all monoidal and module categories are spherical.

In the final Section 6 we extend some of the main results of the previous sections to the equivariant case.

## 2 Background

In this section we fix notation and conventions and revisit some pertinent concepts and structures. Throughout the text, all categories we consider are supposed to be linear abelian over an algebraically closed field $\mathbb{k}$ of characteristic zero. A linear abelian category where every object is of finite length and all morphism spaces are finite-dimensional is said to be locally finite. A finite $\mathbb{k}$-linear abelian category is a linear abelian category that is equivalent to the category of finite-dimensional modules over a finite-dimensional $\mathbb{k}$-algebra.

### 2.1 Tensor categories and module categories

We first recall a few standard definitions in the theory of tensor categories (see [EGNO]). A multi-tensor category is a locally finite rigid monoidal category $\mathcal{A}$ whose tensor product functor $\otimes$ is bilinear. $\mathcal{A}$ is said to be a tensor category iff in addition its monoidal unit $1_{\mathcal{A}}$ is a simple object. We take without loss of generality a monoidal category to be strict to simplify the exposition. The monoidal opposite $\overline{\mathcal{A}}$ of a monoidal category $\mathcal{A}$ is the monoidal category with the same underlying category as $\mathcal{A}$, but with reversed tensor product, i.e. $a \otimes \overline{b} = \overline{b} \otimes a$ and with the respectively adjusted associators. When convenient we denote the object in $\overline{\mathcal{A}}$ that corresponds to an object $a \in \mathcal{A}$ by $\overline{a}$.

Concerning dualities on a monoidal category $\mathcal{A}$ our conventions are as follows. A right dual $a^\vee$ of an object $a \in \mathcal{A}$ comes equipped with evaluation and coevaluation morphisms

$$\text{ev}_a : \ a^\vee \otimes a \to 1_{\mathcal{A}} \quad \text{and} \quad \text{coev}_a : \ 1_{\mathcal{A}} \to a \otimes a^\vee. \quad (2.1)$$
Similarly, a left dual $^\vee a$ and of $a \in \mathcal{A}$ comes with evaluation and coevaluation morphism
\[
\tilde{ev}_a : \ a \otimes ^\vee a \rightarrow 1_{\mathcal{A}} \quad \text{and} \quad \tilde{coev}_a : \ 1_{\mathcal{A}} \rightarrow ^\vee a \otimes a.
\] (2.2)

In every finite tensor category $\mathcal{A}$ there is a distinguished invertible object $\mathbb{D}_A$, which comes with a monoidal natural isomorphism
\[
\tilde{r}_A : \ \mathbb{D}_A \otimes - \otimes \mathbb{D}_A^{-1} \xrightarrow{\cong} (-)^{\vee \vee \vee}
\] (2.3)
known as the Radford isomorphism. A finite tensor category $\mathcal{A}$ is said to be unimodular iff $\mathbb{D}_A$ is isomorphic to the monoidal unit $1_{\mathcal{A}}$.

A (left) module category over a tensor category $\mathcal{A}$, or (left) $\mathcal{A}$-module, for short, is a category $\mathcal{M}$ together with an exact module action functor
\[
\triangleright : \ \mathcal{A} \times \mathcal{M} \rightarrow \mathcal{M}
\] (2.4)
and a mixed associator obeying a pentagon axiom. In order to indicate the tensor category over which $\mathcal{M}$ is a module, we also denote it by $\mathcal{A}\mathcal{M}$. Invoking an analogue for module categories of Mac Lane’s strictification theorem (see [EGNO, Rem. 7.2.4]), we assume strictness also for module categories. In the case that $\mathcal{A}$ is finite, we require that $\mathcal{M}$ is finite as a linear category as well.

A right $\mathcal{B}$-module $\mathcal{N}$ is defined as a left $\mathcal{B}$-module; its module action functor is denoted by
\[
\triangleright : \ \mathcal{N} \times \mathcal{B} \rightarrow \mathcal{N},
\] (2.5)
and we also denote it by $\mathcal{N}_B$. Similarly, for finite tensor categories $\mathcal{A}$ and $\mathcal{B}$, an $(\mathcal{A}, \mathcal{B})$-bimodule category is defined as a (left) module category over the Deligne product $\mathcal{A} \boxtimes \mathcal{B}$.

An $\mathcal{A}$-module category $\mathcal{M}$ is called exact iff $p \triangleright m$ is projective in $\mathcal{M}$ for any projective object $p \in \mathcal{A}$ and any object $m \in \mathcal{M}$. A module category which is not equivalent to a direct sum of two non-trivial module categories is said to be indecomposable.

We denote by $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{N})$ the category of module functors between two $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$, and by $\text{Rex}_\mathcal{A}(\mathcal{M}, \mathcal{N})$ and $\text{Lex}_\mathcal{A}(\mathcal{M}, \mathcal{N})$ the categories of right exact and left exact module functors, respectively. In the case that $\mathcal{M}$ is exact every $H \in \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{N})$ is an exact module functor.

**Lemma 2.1.** [DSS1, Cor. 2.13] Let $H : \mathcal{M} \rightarrow \mathcal{N}$ be an $\mathcal{A}$-module functor. If its underlying functor admits a right (left) adjoint functor, then $H$ admits a right (left) adjoint $\mathcal{A}$-module functor such that the unit and counit of the adjunction are module natural transformations.

The dual tensor category $\mathcal{A}_{\mathcal{M}}^*$ of a tensor category $\mathcal{A}$ with respect to an $\mathcal{A}$-module $\mathcal{M}$ is the category of right exact module endofunctors, $\mathcal{A}_{\mathcal{M}}^* = \text{Rex}_\mathcal{A}(\mathcal{M}, \mathcal{M})$, with tensor product given by composition of functors. If $\mathcal{M}$ is exact $\mathcal{A}_{\mathcal{M}}^*$ is rigid and in case $\mathcal{M}$ indecomposable, then the identity functor $id_{\mathcal{M}}$ is simple, making $\mathcal{A}_{\mathcal{M}}^*$ a tensor category. The evaluation of a functor on an object turns $\mathcal{M}$ into a $\mathcal{A}_{\mathcal{M}}^*$-module category.

The category of module functors has the structure of a bimodule category. More specifically, for given bimodule categories $\mathcal{A}\mathcal{M}_B$, $\mathcal{A}\mathcal{N}_C$ and $\mathcal{D}\mathcal{L}_B$, $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{N})$ becomes a $(\mathcal{B}, \mathcal{C})$-bimodule category via the actions
\[
b \triangleright H := H \circ (- \triangleright b) \quad \text{and} \quad H \triangleright c := (- \triangleright c) \circ H
\] (2.6)
for $H \in \text{Fun}_A(\mathcal{M}, \mathcal{N})$, $b \in \mathcal{B}$ and $c \in \mathcal{C}$, while $\text{Fun}_B(\mathcal{M}, \mathcal{L})$ inherits a $(\mathcal{D}, \mathcal{A})$-bimodule category structure given by
\[
d \r H := (d \r -) \circ H \quad \text{and} \quad H \l a := H \circ (a \l -).
\] (2.7)

Analogously, the categories of right exact and left exact module functors are endowed with the structure of a bimodule category as well.

The opposite category $\mathcal{M}^{\text{opp}}$ of the linear category $\mathcal{M}$ that underlies a bimodule $\mathcal{A}\mathcal{M}_\mathcal{B}$ can be endowed in many different ways with the structure of a $(\mathcal{B}, \mathcal{A})$-bimodule category, by twisting the actions with odd powers of duals.

**Definition 2.2.** Let $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule category over tensor categories $\mathcal{A}$ and $\mathcal{B}$. We define $\#\mathcal{M}$ as the $(\mathcal{B}, \mathcal{A})$-bimodule category with underlying category $\mathcal{M}^{\text{opp}}$ and actions given by
\[
b \r m \l a := \overline{a \r m \l b}
\] (2.8)
for $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $m \in \mathcal{M}^{\text{opp}}$. Similarly, $\mathcal{M}^\#$ is defined to be the $(\mathcal{B}, \mathcal{A})$-bimodule with actions twisted by right duals, i.e.
\[
b \r m \l a := \overline{a^\vee \r m \l b^\vee}
\] (2.9)
for $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $m \in \mathcal{M}^{\text{opp}}$.

### 2.2 Relative Deligne product

Let $\mathcal{M}$ be a right module and $\mathcal{N}$ a left module over a finite tensor category $\mathcal{B}$.

(i) A **$\mathcal{B}$-balancing** on a bilinear functor $F: \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ into a linear category $\mathcal{L}$ is a natural family of isomorphisms
\[
F(m \l b, n) \xrightarrow{\cong} F(m, b \r n)
\] (2.10)
for $b \in \mathcal{B}$, $m \in \mathcal{M}$ and $n \in \mathcal{N}$, obeying an obvious pentagon coherence condition (for details, see for instance [Sc, Def. 2.12]). A bilinear functor endowed with a balancing is called balanced functor.

(ii) A **balanced natural transformation** between balanced functors is a natural transformation between the underlying functors that commutes with the respective balancings.

(iii) Balanced functors $F: \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ together with balanced natural transformations form a category, denoted by $\text{Bal}(\mathcal{M} \times \mathcal{N}, \mathcal{L})$. Its full subcategory of right exact balanced functors is denoted by $\text{Bal}^{\text{re}}(\mathcal{M} \times \mathcal{N}, \mathcal{L})$.

(iv) The **relative Deligne product** of $\mathcal{M}_\mathcal{B}$ and $\mathcal{B}\mathcal{N}$ is a linear category $\mathcal{M} \boxtimes_\mathcal{B} \mathcal{N}$ equipped with a right exact $\mathcal{B}$-balanced functor $\boxtimes_\mathcal{B}: \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_\mathcal{B} \mathcal{N}$ such that for every linear category $\mathcal{L}$ the functor
\[
\text{Rex}(\mathcal{M} \boxtimes_\mathcal{B} \mathcal{N}, \mathcal{L}) \xrightarrow{\cong} \text{Bal}^{\text{re}}(\mathcal{M} \times \mathcal{N}, \mathcal{L}),
\]
\[
F \mapsto F \circ \boxtimes_\mathcal{B}
\] (2.11)
is an equivalence of categories. The relative Deligne product exists, different realizations are known, e.g. [DSS1, FSS] and Corollary 2.4 below.
(v) The relative Deligne product of bimodule categories is naturally endowed with the structure of a bimodule category. Accordingly, the equivalence (2.11) descends to an equivalence between the categories of bimodule right exact functors and balanced right exact bimodule functors (see [Sc, Prop. 3.7]).

We call an object of the form \( m \boxtimes n \in \mathcal{M} \boxtimes_B \mathcal{N} \) a \( \boxtimes \)-factorized object of \( \mathcal{M} \boxtimes_B \mathcal{N} \).

**Lemma 2.3.** Every object in the relative Deligne product \( \mathcal{M} \boxtimes_B \mathcal{N} \) is isomorphic to a finite colimit of \( \boxtimes \)-factorized objects.

**Proof.** As shown in [DSS1, Thm. 3.3], the relative Deligne product can be realized as a category of bimodules internal to \( \mathcal{B} \). Moreover, as pointed out there in the proof, any such bimodule can be written as a coequalizer of \( \boxtimes \)-factorized objects. \( \square \)

**Proposition 2.4.** [DSS2, Prop. 2.4.10] Let \( A \mathcal{M}_B, A \mathcal{N}_C \) and \( D \mathcal{L}_B \) be bimodule categories over finite tensor categories. The balanced functor

\[
\text{Rex}_A(\mathcal{M}, A) \times \mathcal{N} \longrightarrow \text{Rex}_A(\mathcal{M}, \mathcal{N}), \quad (H, n) \mapsto H(-) \triangleright n
\]

induces an adjoint equivalence

\[
\text{Rex}_A(\mathcal{M}, A) \boxtimes_A \mathcal{N} \simeq \text{Rex}_A(\mathcal{M}, \mathcal{N})
\]

of \( (B, C) \)-bimodule categories. Similarly, there is an adjoint equivalence

\[
\mathcal{L} \boxtimes_B \text{Rex}_B(\mathcal{M}, B) \simeq \text{Rex}_B(\mathcal{M}, \mathcal{L})
\]

of \( (D, A) \)-bimodule categories.

**Corollary 2.5.** [DSS2, Cor. 2.4.11] Let \( A \mathcal{M}_B \) and \( B \mathcal{N}_C \) be bimodule categories over finite tensor categories. There are equivalences

\[
\mathcal{M} \boxtimes_B \mathcal{N} \simeq \text{Rex}_B(\mathcal{M}^\#, \mathcal{N}) \quad \text{and} \quad \mathcal{M} \boxtimes_B \mathcal{N} \simeq \text{Rex}_B(\mathcal{N}^\#, \mathcal{M})
\]

of \( (A, C) \)-bimodule categories between relative Deligne products and categories of right exact module functors.

### 2.3 Internal Hom and coHom

Given a module category \( A \mathcal{M} \) over a finite tensor category, for every object \( m \in \mathcal{M} \) the action functor \( - \triangleright m : A \rightarrow \mathcal{M} \) is exact and therefore comes with a right adjoint \( \text{Hom}_A(m, -) : \mathcal{M} \rightarrow A \), i.e. there are natural isomorphisms

\[
\text{Hom}_\mathcal{M}(a \triangleright m, n) \xrightarrow{\cong} \text{Hom}_A(a, \text{Hom}_\mathcal{M}(m, n))
\]

for \( a \in A \) and \( m, n \in \mathcal{M} \). This extends to a left exact functor

\[
\text{Hom}_\mathcal{M}^A(-, -) : \mathcal{M}^{opp} \times \mathcal{M} \rightarrow A,
\]
which is called the internal Hom functor of $\mathcal{M}$. We denote the internal Hom also by $\text{Hom}_\mathcal{M}$ or just by $\text{Hom}$ when it is clear from the context which module category is meant. Additionally, there are canonical natural isomorphisms

$$\text{Hom}_\mathcal{M}(m, a \triangleright n) \xrightarrow{\cong} a \otimes \text{Hom}_\mathcal{M}(m, n)$$  \hspace{1cm} (2.18)

which turn $- \triangleright m \rightharpoondown \text{Hom}_\mathcal{M}(m, -)$ into an adjunction of $\mathcal{A}$-module functors, as well as

$$\text{Hom}_\mathcal{M}(a \triangleright m, n) \xrightarrow{\cong} \text{Hom}_\mathcal{M}(m, n) \otimes a^\vee.$$  \hspace{1cm} (2.19)

Together these naturally endow the internal Hom with a bimodule functor structure

$$\text{Hom}_\mathcal{M}(-, -) : \# \times \mathcal{M} \rightarrow \mathcal{A}.$$  \hspace{1cm} (2.20)

The internal Hom is well-behaved with respect to adjunctions of module functors:

**Lemma 2.6.** [FuS, Lemma 3] Let $\mathcal{A}$ be a finite tensor category, $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{A}$-module categories, and $L : \mathcal{M} \rightarrow \mathcal{N}$ and $R : \mathcal{N} \rightarrow \mathcal{M}$ be $\mathcal{A}$-module functors. Then $L$ is left adjoint to $R$ if and only if there are natural isomorphisms

$$\text{Hom}_\mathcal{N}(L(m), n) \xrightarrow{\cong} \text{Hom}_\mathcal{M}(m, R(n))$$  \hspace{1cm} (2.21)

for $m \in \mathcal{M}$ and $n \in \mathcal{N}$.

The monoidal category $\mathcal{A}_\mathcal{M}$ acts on a module category $\mathcal{M}$ by evaluation of a functor on an object. If $\mathcal{A}\mathcal{M}$ is exact, then the dual tensor category is rigid, and a left dual for $F \in \mathcal{A}_\mathcal{M}$ is given by its left adjoint $F^{\text{la}}$. In this situation the isomorphisms from Lemma 2.6 turn the internal Hom (2.20) into an $\mathcal{A}_\mathcal{M}$-balanced bimodule functor.

Dually, for $m \in \mathcal{M}$ the action functor $- \triangleright m : \mathcal{A} \rightarrow \mathcal{M}$ has a left adjoint

$$\text{coHom}_\mathcal{M}(-, a \triangleright m) : \mathcal{M}^{\text{opp}} \times \mathcal{M} \rightarrow \mathcal{A}.$$  \hspace{1cm} (2.22)

called the internal coHom, where $a \in \mathcal{A}$ and $m, n \in \mathcal{M}$. This extends to a right exact functor $\text{coHom}_\mathcal{M}(-, -) : \mathcal{M}^{\text{opp}} \times \mathcal{M} \rightarrow \mathcal{A}$. The internal Hom and coHom are related by

$$\text{coHom}_\mathcal{M}(m, n) \cong ^\vee \text{Hom}_\mathcal{M}(n, m)$$  \hspace{1cm} (2.23)

for $m, n \in \mathcal{M}$.

Hence analogously to (2.18) and (2.19), there are coherent natural isomorphisms

$$\text{coHom}_\mathcal{M}(m, a \triangleright n) \xrightarrow{\cong} a \otimes \text{coHom}_\mathcal{M}(m, n)$$  \hspace{1cm} (2.24)

and

$$\text{coHom}_\mathcal{M}(a \triangleright m, n) \xrightarrow{\cong} \text{coHom}_\mathcal{M}(m, n) \otimes ^\vee a$$  \hspace{1cm} (2.25)

turning the internal coHom into an $\mathcal{A}$-bimodule functor

$$\text{coHom}_\mathcal{M}(-, -) : \mathcal{M}^{\#} \times \mathcal{M} \rightarrow \mathcal{A}.$$  \hspace{1cm} (2.26)

In view of the relation (2.23), Lemma 2.6 takes the following form:

**Lemma 2.7.** Let $\mathcal{M}$ and $\mathcal{N}$ be module categories over a finite tensor category $\mathcal{A}$, and $L : \mathcal{M} \rightarrow \mathcal{N}$ and $R : \mathcal{N} \rightarrow \mathcal{M}$ be $\mathcal{A}$-module functors. Then $L$ is left adjoint to $R$ if and only if there are natural isomorphisms

$$\text{coHom}_\mathcal{N}(R(n), m) \xrightarrow{\cong} \text{coHom}_\mathcal{N}(n, L(m))$$  \hspace{1cm} (2.27)

for $n \in \mathcal{N}$ and $m \in \mathcal{M}$.

Again, if $\mathcal{M}$ is exact, the isomorphisms (2.27) provide an $\mathcal{A}_\mathcal{M}$-balancing to the internal coHom bimodule functor (2.26).
2.4 Relative Serre Functors

A Serre functor $S$ of a linear additive category $\mathcal{X}$ with finite dimensional morphism spaces furnishes natural isomorphisms between the vector spaces $\text{Hom}_\mathcal{X}(x, y)$ and $\text{Hom}_\mathcal{X}(y, S(x))^\ast$. In case $\mathcal{X} = \mathcal{M}$ is finite abelian, a Serre functor only exists if $\mathcal{X}$ is semisimple. But in the case that $\mathcal{X} = \mathcal{M}$ is a finite module category over a finite tensor category $\mathcal{A}$, then there is internalized version of the Serre functor which exists beyond the semisimple case:

**Definition 2.8.** [FSS, Def. 4.22] Let $\mathcal{M}$ be a left $\mathcal{A}$-module category. A (right) relative Serre functor on $\mathcal{M}$ is an endofunctor $S^A_\mathcal{M} : \mathcal{M} \to \mathcal{M}$ together with a family

$$\text{Hom}^A_\mathcal{M}(m, n) \xrightarrow{\cong} \text{Hom}^A_\mathcal{M}(n, S^A_\mathcal{M}(m))$$

of natural isomorphisms for $m, n \in \mathcal{M}$. Similarly, a (left) relative Serre functor $\overline{S}^A_\mathcal{M}$ comes with a family

$$\text{Hom}^A_\mathcal{M}(m, n) \xrightarrow{\cong} \text{Hom}^A_\mathcal{M}(\overline{S}^A_\mathcal{M}(n), m)$$

of natural isomorphisms.

According to [FSS, Prop. 4.24] a module category admits relative Serre functors if and only if it is exact. In that case the left and right relative Serre functors are quasi-inverses of each other and can be uniquely identified (see also [Sh2, Lemmas 3.3-3.5]) by the formulas

$$S^A_\mathcal{M}(m) \cong \text{Hom}^A_\mathcal{M}(m, -)^{\text{la}}(1_A) \quad \text{and} \quad \overline{S}^A_\mathcal{M}(m) \cong \text{coHom}^A_\mathcal{M}(m, -)^{\text{ra}}(1_A).$$

**Proposition 2.9.** For $\mathcal{M}$ an exact $(\mathcal{A}, \mathcal{B})$-bimodule category, there are coherent natural isomorphisms

$$S^A_\mathcal{M}(a \triangleright m) \cong a \triangleright S^A_\mathcal{M}(m) \quad \text{and} \quad S^A_\mathcal{M}(m \triangleleft b) \cong S^A_\mathcal{M}(m) \triangleright b^{\triangleright}$$

which turn the relative Serre functor $S^A_\mathcal{M}$ into a twisted bimodule equivalence. In a similar manner, its quasi-inverse comes with coherent natural isomorphisms

$$\overline{S}^A_\mathcal{M}(a \triangleright m) \cong \triangleright a \triangleright \overline{S}^A_\mathcal{M}(m) \quad \text{and} \quad \overline{S}^A_\mathcal{M}(m \triangleleft b) \cong \overline{S}^A_\mathcal{M}(m) \triangleleft b^{\triangleright}$$

making of $\overline{S}^A_\mathcal{M}$ a twisted bimodule equivalence.

**Proof.** For left module categories this was already shown in [FSS, Lemma 4.23]. Now for $b \in \mathcal{B}$ we have an adjunction $(- \triangleleft b) \dashv (- \triangleright b^{\triangleright})$ of $\mathcal{A}$-module functors. It follows that there is a chain

$$\text{Hom}^A_\mathcal{M}(n, S^A_\mathcal{M}(m \triangleleft b)) \cong \text{Hom}^A_\mathcal{M}(m \triangleleft b, n)^{\triangleright} \cong \text{Hom}^A_\mathcal{M}(m, n \triangleright b^{\triangleright})$$

$$\cong \text{Hom}^A_\mathcal{M}(n \triangleright b^{\triangleright}, S^A_\mathcal{M}(m)) \cong \text{Hom}^A_\mathcal{M}(n, S^A_\mathcal{M}(m) \triangleright b^{\triangleright})$$

of natural isomorphisms, where $m, n \in \mathcal{M}$ and $b \in \mathcal{B}$. Hence the desired family of isomorphisms is granted by the Yoneda Lemma for internal Hom’s, and by construction these are coherent with respect to the tensor product. \qed
Remark 2.10. A right module category $\mathcal{N}_B$ can be seen as a left module category $\mathcal{N}_B$ over the monoidal opposite of $\mathcal{B}$. Since right duals in $\mathcal{B}$ correspond to left duals in its monoidal opposite and vice versa, the module structure on the relative Serre functors is twisted according to

$$S^A_{\mathcal{N}_B}(n \triangleleft b) \cong S^A_{\mathcal{N}_B}(n) \triangleleft b^\vee$$

for $n \in \mathcal{N}$ and $b \in \mathcal{B}$. In the case of a bimodule category $\mathcal{A}_B$, the actions of $\mathcal{A}$ are twisted by

$$S^A_{\mathcal{N}_B}(a \triangleright n) \cong a \triangleright S^A_{\mathcal{N}_B}(n)$$

and

$$S^A_{\mathcal{N}_B}(a \triangleright n) \cong a^\vee \triangleright S^A_{\mathcal{N}_B}(n).$$

(2.34)

Proposition 2.11. Given exact $\mathcal{A}$-module categories $\mathcal{M}$ and $\mathcal{N}$, for every module functor $F: \mathcal{M} \to \mathcal{N}$ there is a natural isomorphism

$$\Lambda_F : \quad S^A_{\mathcal{N}_B} \circ F \cong F^{\text{tra}} \circ S^A_{\mathcal{M}}$$

(2.36)

of twisted module functors, where $F^{\text{tra}}$ is the double right adjoint of $F$. $\Lambda_F$ is compatible with composition of module functors, i.e. the diagram

$$S^A_{\mathcal{N}_B} \circ H \circ F \xrightarrow{\Lambda_{H \circ F}} (H \circ F)^{\text{tra}} \circ S^A_{\mathcal{M}}$$

for $H \in \text{Fun}_{\mathcal{A}}(\mathcal{N}, \mathcal{L})$ and $F \in \text{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$. Analogously, in the case of bimodule categories and a bimodule functor $F$, the natural isomorphism (2.36) is an isomorphism of twisted bimodule functors.

Proof. The first part of the statement referring to left modules is shown in [Sh2, Thm. 3.10]. For bimodules $\mathcal{A}_M_B$ and $\mathcal{A}_N_B$ and $F: \mathcal{M} \to \mathcal{N}$ a bimodule functor, we now check that the isomorphism $\Lambda_F$ in (2.36) is compatible with the $\mathcal{B}$-module functor structures: Consider the diagram

$$S^A_{\mathcal{N}_B} \circ F(m \triangleleft b) \xrightarrow{\psi} S^A_{\mathcal{N}_B}(F(m) \triangleleft b) \xrightarrow{\Lambda_{F(-) \triangleleft b}} S^A_{\mathcal{N}_B} \circ F(m) \triangleleft b^\vee$$

$$F^{\text{tra}} \circ S^A_{\mathcal{M}}(m \triangleleft b) \xrightarrow{\text{id} \circ \Lambda_{-\triangleleft b}} F^{\text{tra}} \left(S^A_{\mathcal{M}}(m) \triangleleft b^\vee\right) \xrightarrow{\Lambda_{F(-) \triangleleft b}} F^{\text{tra}} \circ S^A_{\mathcal{M}}(m) \triangleleft b^\vee$$

where $\psi_{m,b} : F(m \triangleleft b) \cong F(m) \triangleleft b$ denotes the $\mathcal{B}$-module functor structure on $F$ and $\Lambda_{-\triangleleft b}$ is the twisted $\mathcal{B}$-module structure of the relative Serre functor. Both of the triangles in (2.38) are realizations of diagram (2.37) and are thus commutative. The middle square in (2.38) commutes owing to naturality of (2.36) with respect to $F$.

Remark 2.12. An analogous statement as in Proposition 2.11 holds for left relative Serre functors: Given $F \in \text{Fun}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$, there is a natural isomorphism

$$\overline{\Lambda}_F : \quad \overline{S}^A_{\mathcal{N}_B} \circ F \cong F^{\text{lila}} \circ \overline{S}^A_{\mathcal{M}}$$

(2.39)
of twisted module functors, compatible with composition of module functors in a similar manner as in (2.37). These isomorphisms are related with (2.36) by the commutative diagram

\[
\begin{array}{ccc}
\otimes & \otimes & \otimes \\
S_N^A \circ F & \cong & F^{\text{lla}} \circ S_M^A \\
\cong & \cong & \cong \\
S_N^A \circ F \circ S_M^A \circ S_M^A & \cong & \text{id}_A \circ F^{\text{lla}} \circ S_M^A
\end{array}
\] (2.40)

considering that the left and right relative Serre functors are mutual quasi-inverses. Likewise also the statements for bimodule functors hold.

**Remark 2.13.** Given a finite linear category \( \mathcal{X} \), its right exact *Nakayama functor* is the image of the identity functor under the Eilenberg-Watts correspondence \([FSS, \text{Def. 3.14}]\), i.e. explicitly

\[
N^r_X := \int_{x \in \mathcal{X}} \text{Hom}_\mathcal{X}(-, x)^* \otimes x. \tag{2.41}
\]

It comes equipped with a family of natural isomorphisms

\[
N^r_X \circ F \cong F^{\text{lla}} \circ N^r_X \tag{2.42}
\]

for every \( F \in \text{Rex}(\mathcal{X}, \mathcal{X}) \) having a right adjoint which is right exact as well. The left exact analogue of the Nakayama functor is a left adjoint to \( N^r_X \) which is explicitly given by

\[
N^l_X := \int_{x \in \mathcal{X}} \text{Hom}_\mathcal{X}(x, -) \otimes x. \tag{2.43}
\]

By (2.42) the Nakayama functor of an exact module category \( \mathcal{M} \) over a finite tensor category \( \mathcal{A} \) is endowed with a twisted module functor structure. The relative Serre functors of \( \mathcal{M} \) are related to the Nakayama functors by

\[
\mathcal{D}_\mathcal{A} \otimes N^r_M \cong S^A_M \quad \text{and} \quad \mathcal{D}_\mathcal{A}^{-1} \otimes N^l_M \cong S^A_M \tag{2.44}
\]
as twisted module functors \([FSS, \text{Thm. 4.26}]\). A direct computation shows that for a bimodule category these are indeed isomorphisms of twisted bimodule functors.

### 3 Categorical Morita context

In the theory of rings or \( k \)-algebras the notion of Morita equivalence finds a generalization in the structure of a Morita context or pre-equivalence data \([Ba, \text{Ch.2 \S3}]\). These data involving two rings provide an adjunction (though not necessarily an equivalence) between their categories of modules. We now study the analogue of this notion for categorical Morita equivalence of finite tensor categories.

**Definition 3.1.**

(i) A (categorical) *Morita context* consists of the following data:

1. Two finite (multi-)tensor categories \( \mathcal{A} \) and \( \mathcal{B} \).  


2. Two bimodule categories $\mathcal{A} \mathcal{M}_B$ and $\mathcal{B} \mathcal{N}_A$.

3. Two bimodule functors

$$\Box : \mathcal{M} \boxtimes \mathcal{B} \mathcal{N} \longrightarrow \mathcal{A} \quad \text{and} \quad \Box : \mathcal{N} \boxtimes \mathcal{A} \mathcal{M} \longrightarrow \mathcal{B}. \quad (3.1)$$

4. Two bimodule natural isomorphisms $\alpha$ and $\beta$ of the form

$$\begin{align*}
(M \boxtimes_B N) \boxtimes_A \mathcal{M} & \xrightarrow{\cong} M \boxtimes_B (N \boxtimes_A \mathcal{M}) \\
A \boxtimes_A \mathcal{M} & \xrightarrow{\alpha} M \boxtimes_B \mathcal{B} \\
\mathcal{M} & \xrightarrow{\beta} \mathcal{M} \boxtimes_B \mathcal{B}
\end{align*} \quad (3.2)$$

and

$$\begin{align*}
(N \boxtimes_A \mathcal{M}) \boxtimes_B \mathcal{N} & \xrightarrow{\cong} N \boxtimes_A (M \boxtimes_B \mathcal{N}) \\
B \boxtimes_B \mathcal{N} & \xrightarrow{\beta} N \boxtimes_A \mathcal{A} \\
\mathcal{N} & \xrightarrow{\alpha} \mathcal{N} \boxtimes_A \mathcal{A}
\end{align*} \quad (3.3)$$

These data are required to fulfill the condition that the pentagon diagrams

$$\begin{align*}
\phi_{m_1 \triangleright n_1, m_2 \triangleright n_2} & \quad (m_1 \triangleright n_1) \otimes_A (m_2 \triangleright n_2) \\
(m_1 \triangleright (n_1 \boxtimes m_2)) \boxtimes n_2 & \xrightarrow{\cong} m_1 \boxtimes ((n_1 \boxtimes m_2) \triangleright n_2)
\end{align*} \quad (3.4)$$

and

$$\begin{align*}
\psi_{n_1 \boxtimes m_1, n_2 \boxtimes m_2} & \quad (n_1 \boxtimes m_1) \otimes_B (n_2 \boxtimes m_2) \\
(n_1 \boxtimes (m_1 \boxtimes n_2)) \boxtimes m_2 & \xrightarrow{\cong} n_1 \boxtimes ((m_1 \boxtimes n_2) \triangleright m_2)
\end{align*} \quad (3.5)$$

commute for all $m_1, m_2 \in \mathcal{M}$ and $n_1, n_2 \in \mathcal{N}$, where $\phi$ and $\phi'$ are the bimodule structures of the functor $\Box$ and $\psi$ and $\psi'$ are the bimodule structures of $\Box$. 

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(ii) We say that a Morita context is strict iff $A$ and $B$ are strict tensor categories, $M$ and $N$ are strict bimodules, $\ominus$ and $\oplus$ have strict bimodule functor structures and $\alpha$ and $\beta$ are identities.

(iii) A Morita context is said to be strong iff $\ominus$ and $\oplus$ are equivalences.

Remark 3.2.

(i) Via the equivalence (2.11) coming from the universal property of the relative Deligne product, the bimodule functors (3.1) in a Morita context correspond to bimodule balanced functors

$$\ominus : \mathcal{M} \times \mathcal{N} \to A \quad \text{and} \quad \oplus : \mathcal{N} \times \mathcal{M} \to B.$$  

(3.6)

As a consequence, a Morita context can be equivalently defined by replacing (3.2) and (3.3) with bimodule balanced natural isomorphisms

$$\mathcal{M} \times \mathcal{N} \times \mathcal{M} \xrightarrow{\text{id} \times \ominus} \mathcal{M} \times B \quad \text{and} \quad \mathcal{N} \times \mathcal{M} \times \mathcal{N} \xrightarrow{\text{id} \times \ominus} \mathcal{N} \times A$$

(3.7)

obeying relations analogous to (3.4) and (3.5).

(ii) The four categories in a Morita context interact with each other via the tensor products of $A$ and $B$ and their actions on the bimodule categories $M$ and $N$. Accordingly the functors $\ominus$ and $\oplus$ play the role of mixed products between the categories $M$ and $N$.

3.1 The bicategory of a Morita context

In the spirit of [Mü, Rem. 3.18] the data of a Morita context form a bicategory. The construction is as follows. Given a Morita context $(A, B, M, N, \ominus, \oplus, \alpha, \beta)$, define a bicategory $\mathcal{M}$ consisting of two objects $\{0, 1\}$ and Hom-categories

$$\mathcal{M}(0, 0) := A,$$

$$\mathcal{M}(0, 1) := M,$$

$$\mathcal{M}(1, 1) := B,$$

$$\mathcal{M}(1, 0) := N.$$  

(3.8)

The horizontal composition

$$\circ_{i,j,k} : \mathcal{M}(j, k) \times \mathcal{M}(i, j) \to \mathcal{M}(i, k) \quad \text{for} \quad i, j, k \in \{0, 1\}$$

(3.9)

in the bicategory $\mathcal{M}$ is given by the eight functors

$$\otimes_A : A \times A \to A,$$

$$\otimes_B : B \times B \to B,$$

$$\triangleright_M : A \times M \to M,$$

$$\triangleright_N : B \times N \to N,$$

$$\triangleleft_N : N \times A \to N,$$

$$\triangleleft_M : M \times B \to M,$$

$$\ominus : \mathcal{M} \times \mathcal{N} \to A,$$

$$\oplus : \mathcal{N} \times \mathcal{M} \to B,$$

(3.10)

i.e. by the tensor products, module actions and mixed products in the Morita context.
The associativity constraints in $\mathcal{M}$ are natural isomorphisms

$$\begin{align*}
\mathcal{M}(j, k) \times \mathcal{M}(i, j) \times \mathcal{M}(h, i) & \xrightarrow{\text{id} \times \circ_{h, i, j}} \mathcal{M}(j, k) \times \mathcal{M}(h, j) \\
\mathcal{M}(i, k) \times \mathcal{M}(h, i) & \xrightarrow{\circ_{h, i, k}} \mathcal{M}(h, k)
\end{align*}$$

for each quadruple of objects $h, i, j, k \in \{0, 1\}$. We require these sixteen constraints to be the following:

- The two associativity constraints from the tensor products of $\mathcal{A}$ and $\mathcal{B}$.
- The six module associativity constraints from the actions on the bimodules $\mathcal{M}$ and $\mathcal{N}$.
- Four constraints coming from the bimodule functor structures of $\oplus$ and $\ominus$.
- The two balancings of $\oplus$ and $\ominus$.
- The coherence data of the Morita context, i.e. the two isomorphisms $\alpha$ and $\beta$.

These data provide indeed a bicategory:

**Theorem 3.3.** *The data of a Morita context form a bicategory $\mathcal{M}$ with two objects.*

**Proof.** Clearly, the data given above are those needed for a bicategory. Thus it remains to check that the associators defined for $\mathcal{M}$ obey the relevant axiom in a bicategory, i.e. that the diagram

$$\begin{align*}
(f_{k,l} \circ f_{j,k}) \circ (f_{i,j} \circ f_{h,i}) & \xrightarrow{\gamma_{h,k,j,l}} f_{k,l} \circ (f_{j,k} \circ (f_{i,j} \circ f_{h,i})) \\
(f_{k,l} \circ (f_{j,k} \circ f_{i,j})) \circ f_{h,i} & \xrightarrow{\gamma_{h,i,j,k}} f_{k,l} \circ ((f_{j,k} \circ f_{i,j}) \circ f_{h,i})
\end{align*}$$

commutes for all $h, i, j, k, l \in \{0, 1\}$ and every quadruple of 1-morphisms $f_{k,l} \in \mathcal{M}(k, l)$, $f_{j,k} \in \mathcal{M}(j, k)$, $f_{i,j} \in \mathcal{M}(i, j)$ and $f_{h,i} \in \mathcal{M}(h, i)$. It can be verified that these conditions correspond to the following thirty-two diagrams:

1. Two pentagons obeyed by the associativity constraints of $\mathcal{A}$ and $\mathcal{B}$.
2. Four pentagons fulfilled by the left and right module constraints of $\mathcal{M}$ and $\mathcal{N}$.
3. Four pentagons fulfilled by the middle associativity constraints of the bimodules $\mathcal{M}$ and $\mathcal{N}$.
4. Six compatibility conditions on the bimodule functor structures of $\oplus$ and $\ominus$.
5. The two diagrams that describe the conditions on the balancings of $\oplus$ and $\ominus$.
6. Four conditions corresponding to the compatibility between the bimodule structures and the balancings of $\oplus$ and $\ominus$. 

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7. Four compatibility conditions characterizing $\alpha$ and $\beta$ as bimodule natural transformations.
8. Four conditions defining $\alpha$ and $\beta$ as balanced natural transformations.
9. The diagrams (3.4) and (3.5), which exhibit the compatibility between the coherence data $\alpha$ and $\beta$ of the Morita context.

Thus all the conditions are satisfied by construction.

Remark 3.4.

(i) The notion of a Morita context as a bicategory studied in [Mu] is an instance of Theorem 3.3 where a (Frobenius) algebra is chosen to realize the bimodules in the Morita context.

(ii) Theorem 3.3 justifies Definition 3.1. Coherence results for bicategories, such as Corollary 2.7 of [Gu], imply that all paths between any two possible bracketings of a product of multiple objects in the Morita context through associativity constraints are the same isomorphism.

As Theorem 3.3 indicates, the bicategorical setting emerges as the natural framework for studying Morita contexts. Let us recall a few pertinent notions from this setting. A pseudo-functor $U : \mathcal{F} \to \mathcal{G}$ between bicategories consists of assignments at the level of objects and at the level of Hom-categories together with invertible 2-morphisms that witness their compatibility with horizontal composition and which obey a pentagon axiom. A pseudo-natural equivalence $\eta : U \Rightarrow V$ between pseudo-functors amounts to an invertible 1-morphism $\eta_x : U(x) \Rightarrow V(x)$ for every object $x \in \mathcal{F}$ and an invertible 2-morphism $\eta_f : \eta_y \circ U(f) \Rightarrow V(f) \circ \eta_x$ for every 1-morphism $f : x \to y$, respecting horizontal composition. Complete definitions can for instance be found in [Sc, App. A.2].

Definition 3.5. An equivalence of Morita contexts is a pseudo-equivalence between their bicategories.

Corollary 3.6 (Coherence for Morita contexts). Every Morita context is equivalent to a strict Morita context.

Proof. Using that every bicategory is equivalent to a strict 2-category [Gu Cor. 2.7], this follows directly from Theorem 3.3.

3.2 The Morita context derived from an exact module category

Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ an exact $\mathcal{A}$-module. There is a strong Morita context associated to $\mathcal{A}\mathcal{M}$. To see this, first recall that the category $\mathcal{A}\mathcal{M}^\ast$ of module endofunctors has the structure of a tensor category and that the evaluation of a functor on an object turns $\mathcal{M}$ into an exact $\mathcal{A}\mathcal{M}^\ast$-module category. More specifically, hereby $\mathcal{M}$ becomes an $(\mathcal{A}, \mathcal{A}\mathcal{M}^\ast)$-bimodule category.

As described in (2.6), the category $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$ of module functors is naturally endowed with the structure of an $(\mathcal{A}\mathcal{M}^\ast, \mathcal{A})$-bimodule category, via the actions

$$F \triangleright H := H \circ F \quad \text{and} \quad H \triangleleft a := H(-) \otimes a = (- \otimes a) \circ H$$

for $H \in \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$, $F \in \mathcal{A}\mathcal{M}^\ast$ and $a \in \mathcal{A}$.

Having obtained two tensor categories and two bimodules categories from the exact $\mathcal{A}$-module $\mathcal{M}$, we can proceed to define the mixed products in the Morita context:
Definition 3.7. The $\mathcal{A}_M$-valued mixed product of $\mathcal{M}$ is the functor
\[
\boxdot : \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A}) \times \mathcal{M} \longrightarrow \mathcal{A}_M,
\]
\[(H, m) \longmapsto H(-) \triangleright m = (- \triangleright m) \circ H , \tag{3.14}\]
which is a special case of (2.13). The $\mathcal{A}$-valued mixed product of $\mathcal{M}$ is the functor given by evaluation
\[
\odot : \mathcal{M} \times \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A}) \longrightarrow \mathcal{A},
\]
\[(m, H) \longmapsto H(m). \tag{3.15}\]

Lemma 3.8. The mixed product $\boxdot$ is an $\mathcal{A}$-balanced $\mathcal{A}_M$-bimodule functor, and the mixed product $\odot$ is an $\mathcal{A}_M$-balanced $\mathcal{A}$-bimodule functor.

Proof. First notice that the functor $\boxdot$ comes with an $\mathcal{A}$-balancing given by the module associativity constraints of $\mathcal{A}$:
\[(H \triangleleft a) \boxdot m = (H(-) \otimes a) \triangleright m \cong H(-) \triangleright (a \triangleright m) = H \boxdot (a \triangleright m). \tag{3.16}\]
Furthermore, the identities
\[F \triangleright (H \boxdot m) = (- \triangleright m) \circ H \circ F = (F \triangleright H) \boxdot m \tag{3.17}\]
and the isomorphisms
\[(H \boxdot m) \triangleleft F = F(H(-) \triangleright m) \cong H(-) \triangleright F(m) = H \boxdot F(m) \tag{3.18}\]
that come from the module functor structure of $F$ endow $\boxdot$ with the structure of an $\mathcal{A}_M$-bimodule functor. These are compatible with the balancing because the module functor structure of $F$ is compatible with the associativity constraints of $\mathcal{A}$.
Similarly, for the functor $\odot$ the identity morphisms $(m \triangleleft F) \odot H = H \circ F(m) = m \odot (F \triangleright H)$ provide an $\mathcal{A}_M$-balancing. Moreover, there is a natural $\mathcal{A}$-bimodule functor structure on $\odot$, namely
\[a \otimes (m \odot H) = a \otimes H(m) \cong H(a \triangleright m) = (a \triangleright m) \odot H \tag{3.19}\]
and
\[(m \odot H) \otimes a = H(m) \otimes a = m \odot (H \triangleleft a) \tag{3.20}\]
given by the module functor structure on $H$ and the identity morphisms, respectively. □

Theorem 3.9. Let $\mathcal{M}$ be an exact module category over a finite tensor category $\mathcal{A}$. Then the data
\[
(\mathcal{A}, \mathcal{A}_M, \mathcal{M}, \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A}), \odot, \boxdot) \tag{3.21}
\]
form a Morita context.

Proof. Notice that the diagram
\[
\begin{array}{ccc}
\mathcal{M} \times \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A}) \times \mathcal{M} & \xrightarrow{\text{id} \times \boxdot} & \mathcal{M} \times \mathcal{A}_M \\
\downarrow \circ \times \text{id} & & \downarrow \circ \\
\mathcal{A} \times \mathcal{M} & \xrightarrow{\triangleright} & \mathcal{M}
\end{array}
\tag{3.22}
\]
commutes strictly owing to \((m \odot H) \triangleright n = H(m) \triangleright n = m \triangleleft (H \boxdot n)\), and thus the identities serve as the bimodule natural isomorphism (3.2). On the other hand, the diagram

\[
\begin{array}{ccc}
\text{Fun}_A(\mathcal{M}, \mathcal{A}) \times \mathcal{M} \times \text{Fun}_A(\mathcal{M}, \mathcal{A}) & \xrightarrow{\text{id} \times \odot} & \text{Fun}_A(\mathcal{M}, \mathcal{A}) \times \mathcal{A} \\
\otimes \times \text{id} & \downarrow & \times \\
\mathcal{A} \mathcal{M} \times \text{Fun}_A(\mathcal{M}, \mathcal{A}) & \xrightarrow{\triangleright} & \text{Fun}_A(\mathcal{M}, \mathcal{A})
\end{array}
\]

(3.23)

commutes up to the isomorphism

\[
(H_1 \boxdot m) \triangleright H_2 = H_2(H_1(-) \triangleright m) \cong H_1(-) \otimes H_2(m) = H_1 \triangleleft (m \odot H_2)
\]

(3.24)

that comes from the module functor structure of \(H_2\). One can directly verify that the associated conditions (3.4) and (3.5) are satisfied. \(\square\)

**Remark 3.10.** As we will see in Proposition 4.9, the Morita context derived from an exact module category \(\mathcal{M}\) is in fact a strong Morita context.

### 3.3 Characterization of strong Morita contexts

Exact module categories provide examples of strong Morita contexts, as stated in Remark 3.10. It turns out that, conversely, every strong Morita context in the sense of Definition 3.1(iii) is equivalent to the Morita context of an exact module category.

**Theorem 3.11.** Let \(\mathcal{A}\) and \(\mathcal{B}\) be finite tensor categories and \(\mathcal{A}\mathcal{M}_\mathcal{B}\) and \(\mathcal{B}\mathcal{N}_\mathcal{A}\) be finite bimodule categories. Consider a strong Morita context \((\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \odot, \boxdot, \alpha, \beta)\). Then the following statements hold:

(i) \(\mathcal{M}\) and \(\mathcal{N}\) are exact indecomposable bimodule categories.

(ii) The assignments

\[
\mathcal{N} \xrightarrow{\cong} \text{Fun}_A(\mathcal{M}, \mathcal{A}) \quad \text{and} \quad \mathcal{N} \xrightarrow{\cong} \text{Fun}_B(\mathcal{M}, \mathcal{B})
\]

\[
n \mapsto - \odot n \quad \text{and} \quad n \mapsto n \boxdot -
\]

(3.25)

are equivalences of \((\mathcal{B}, \mathcal{A})\)-bimodule categories.

(iii) The assignments

\[
\mathcal{M} \xrightarrow{\cong} \text{Fun}_B(\mathcal{N}, \mathcal{B}) \quad \text{and} \quad \mathcal{M} \xrightarrow{\cong} \text{Fun}_A(\mathcal{N}, \mathcal{A})
\]

\[
m \mapsto - \boxdot m \quad \text{and} \quad m \mapsto m \odot -
\]

(3.26)

are equivalences of \((\mathcal{A}, \mathcal{B})\)-bimodule categories.

(iv) The assignments

\[
R_M : \quad \mathcal{B} \xrightarrow{\cong} \text{Fun}_A(\mathcal{M}, \mathcal{M}) \quad \text{and} \quad L_N : \quad \mathcal{B} \xrightarrow{\cong} \text{Fun}_A(\mathcal{N}, \mathcal{N})
\]

\[
b \mapsto - \triangleleft b \quad \text{and} \quad b \mapsto b \triangleright -
\]

(3.27)

are equivalences of tensor categories and of \(\mathcal{B}\)-bimodule categories.
(v) The assignments

\[ R_N : \mathcal{A} \xrightarrow{\simeq} \text{Fun}_B(\mathcal{N}, \mathcal{N}) \quad \text{and} \quad L_M : \mathcal{A} \xrightarrow{\simeq} \text{Fun}_B(\mathcal{M}, \mathcal{M}) \]

\[ a \mapsto - \triangleleft a \quad \text{and} \quad a \mapsto a \triangleright - \]

are equivalences of tensor categories and of \( \mathcal{A} \)-bimodule categories.

(vi) The commutativity of the diagrams

\[
\begin{array}{ccc}
\mathcal{M} \times \mathcal{N} & \xrightarrow{\otimes} & \mathcal{A} \\
\downarrow (m,n) & & \downarrow \text{id}_A \\
\mathcal{M} \times \text{Fun}_A(\mathcal{M}, \mathcal{A}) & \xrightarrow{\otimes} & \mathcal{A}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{N} \times \mathcal{M} & \xrightarrow{\boxdot} & \mathcal{B} \\
\downarrow (n,m) & & \downarrow R_M \\
\text{Fun}_A(\mathcal{M}, \mathcal{A}) \times \mathcal{M} & \xrightarrow{\boxdot} & \mathcal{A}_\mathcal{M}
\end{array}
\]

is witnessed by the identity and by \( \alpha_{-,-,m} : - \triangleleft (n \boxdot m) \xrightarrow{\simeq} (-\boxdot n) \triangleright m \), respectively.

**Proof.** First notice that (i) follows from (iv) and (v). For instance, assuming that \( R_M \) is an equivalence, rigidity of \( \mathcal{B} \) implies that every \( \mathcal{A} \)-module endofunctor of \( \mathcal{M} \) is exact and thus \( \mathcal{A} \mathcal{M} \) is an exact module [EGNO, Prop. 7.9.7(2)]. Also, since the monoidal unit of \( \mathcal{B} \) is simple, then \( \mathcal{M} \) is an indecomposable \( \mathcal{A} \)-module. To prove the remaining statements we consider categories of right exact module functors. Once the statements are checked, exactness of the bimodules \( \mathcal{M} \) and \( \mathcal{N} \) will follow, and thus that every module functor is exact.

We now prove (ii). That \( \boxdot : \mathcal{M} \times \mathcal{N} \to \mathcal{A} \) is a balanced bimodule functor implies that \( - \boxdot n \) has an \( \mathcal{A} \)-module structure and that the functor

\[
\text{Rex}_\mathcal{A}(\mathcal{M}, \mathcal{A}) \to \mathcal{N}
\]

\[ n \mapsto - \boxdot n \]

is endowed with a \((\mathcal{B}, \mathcal{A})\)-bimodule functor structure. On the other hand, we have an equivalence \( \boxdot : \mathcal{N} \boxdot \mathcal{A} \mathcal{M} \simeq \mathcal{B} \), and in view of Lemma [23] there is an object \( \text{colim}_{i \in I} n_i \boxdot m_i \in \mathcal{N} \boxdot \mathcal{A} \mathcal{M} \) such that \( 1_\mathcal{B} \cong \boxdot (\text{colim}_{i \in I} n_i \boxdot m_i) = \text{colim}_{i \in I} n_i \boxdot m_i \). Then the functor

\[
\text{Rex}_\mathcal{A}(\mathcal{M}, \mathcal{A}) \to \mathcal{N}
\]

\[ H \mapsto \text{colim}_{i \in I} n_i \triangleleft H(m_i) \]

defines a quasi-inverse to the bimodule functor [30]. Indeed, given \( n \in \mathcal{N} \) there is a natural isomorphism

\[
\text{colim}_{i \in I} n_i \triangleleft (m_i \boxdot n) \xrightarrow{\cong} \text{colim}_{i \in I} (n_i \boxdot m_i) \triangleright n \cong 1_\mathcal{B} \triangleright n = n.
\]

Similarly, for a module functor \( H \in \text{Rex}_\mathcal{A}(\mathcal{M}, \mathcal{A}) \) and an object \( n \in \mathcal{N} \) we have a natural isomorphism

\[
\text{colim}_{i \in I} n \boxdot (n_i \triangleleft H(m_i)) \cong \text{colim}_{i \in I} (n \boxdot n_i) \otimes H(m_i) \cong \text{colim}_{i \in I} H((n \boxdot n_i) \triangleright m_i) \]

\[ \xrightarrow{\alpha} \text{colim}_{i \in I} H(n \triangleleft (n_i \boxdot m_i)) \cong H(n \triangleleft 1_\mathcal{B}) = H(n),
\]

where the first isomorphism is the right \( \mathcal{A} \)-module structure on \( \boxdot \), the second isomorphism is the module structure of \( H \), and the last isomorphism after \( \alpha \) expresses the fact that the colimit
is preserved since $H$ is right exact, the action functor is exact and $\boxdot$ is an equivalence. The proof for the functor

$$\mathcal{N} \rightarrow \text{Rex}_B(\mathcal{M}, \mathcal{B})$$

$$n \mapsto n \boxdot -$$

is obtained by merely exchanging the roles of $\ominus$ and $\boxdot$. Similarly, the assertion (iii) follows by symmetry.

Next we prove (iv). We have a $\mathcal{B}$-bimodule equivalence

$$\phi : \mathcal{B} \xrightarrow{\boxdot^{-1}} \mathcal{N} \boxtimes_A \mathcal{M} \xrightarrow{[3.30]} \text{Rex}_A(\mathcal{M}, \mathcal{A}) \boxtimes_A \mathcal{M} \xrightarrow{[2.13]} \text{Rex}_A(\mathcal{M}, \mathcal{M}).$$

(3.35)

We write $\phi(1_B) =: I$ and follow the argument given in [ENOM, Prop. 4.2]. The bimodule structure on $\phi$ provides natural isomorphisms

$$\phi(b) \cong b \triangleright I = I \circ (- \langle b)$$

and

$$\phi(b) \cong I \triangleleft b = (- \langle b) \circ I$$

(3.36)

for $b \in \mathcal{B}$. Since $\phi$ is an equivalence, there exists an object $B \in \mathcal{B}$ such that $\text{id}_\mathcal{M} \cong I \circ (- \langle B) \cong (- \langle B) \circ I$, which means that $I$ is invertible and therefore the endofunctor $(- \circ I) : \text{Rex}_A(\mathcal{M}, \mathcal{M}) \rightarrow \text{Rex}_A(\mathcal{M}, \mathcal{M})$ is an equivalence. From (3.36) we have in particular $\phi \cong (- \circ I) \circ R_M$, and thus $R_M$ must be an equivalence as well. The same type of argument can be applied to the bimodule equivalence

$$\mathcal{B} \xrightarrow{\boxdot^{-1}} \mathcal{N} \boxtimes_A \mathcal{M} \xrightarrow{[3.26]} \mathcal{N} \boxtimes_A \text{Rex}_A(\mathcal{N}, \mathcal{A}) \xrightarrow{[2.11]} \text{Rex}_A(\mathcal{N}, \mathcal{N}),$$

(3.37)

leading to the second part of the statement. And similarly assertion (v) follows by symmetry. Finally, (vi) holds by direct calculation.

\[\square\]

**Remark 3.12.** Theorem 3.11 implies in particular that every strong Morita context is equivalent to the Morita context of an exact module category. More explicitly, together with the isomorphisms (3.29) the equivalences $\mathcal{N} \xrightarrow{\cong} \text{Fun}_A(\mathcal{M}, \mathcal{A})$ and $R_M : \mathcal{B} \xrightarrow{\cong} \overline{\mathcal{A}}_M$ furnish an equivalence

$$(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \ominus, \boxdot) \xrightarrow{\cong} (\mathcal{A}, \mathcal{A}^\vee, \mathcal{M}, \text{Fun}_A(\mathcal{M}, \mathcal{A}), \ominus, \boxdot)$$

(3.38)

between Morita contexts, i.e. a pseudo-equivalence between the associated bicategories.

## 4 Dualities in a Morita context

There is a notion of dualities for a 1-morphism in a bicategory $\mathcal{F}$, see for instance [Sc, App. A.3]: A right dual (or right adjoint) to a 1-morphism $a \in \mathcal{F}(x, y)$ consists of a 1-morphism $a^\vee \in \mathcal{F}(y, x)$ and 2-morphisms $1_y \Rightarrow a \circ a^\vee$ and $a^\vee \circ a \Rightarrow 1_x$ that fulfill the appropriate snake relations; left duals are defined similarly. A bicategory for which every 1-morphism has both duals is said to be a bicategory with dualities. It is worth noting that the existence of duals is a property, rather than extra structure, of a bicategory. In the present section we explore this notion of dualities for the bicategory $\mathcal{M}$ described in Theorem 3.3 that is associated to the Morita context given by an exact module category, cf. Theorem 3.9.
4.1 Existence of dualities in the bicategory $\mathcal{M}$

For $\mathcal{M}$ an exact module category over a finite tensor category $\mathcal{A}$ we consider its (strong) Morita context $\left(\mathcal{A}, \mathcal{A}_M, \mathcal{M}, \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A}), \odot, \square\right)$. Since the tensor categories $\mathcal{A}$ and $\mathcal{A}_M$ are rigid, their objects come with dualities. Also, since $\mathcal{M}$ is exact, an object $H \in \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$ has left and right adjoints. We now show that in the whole Morita context every object is equipped with dualities, in such a way that the associated bicategory $\mathcal{M}$ is a bicategory with dualities.

**Definition 4.1.** Let $\mathcal{M}$ be an exact module category over a finite tensor category $\mathcal{A}$ and $\left(\mathcal{A}, \mathcal{A}_M, \mathcal{M}, \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A}), \odot, \square\right)$ the associated Morita context. 

(i) The right dual of an object $m \in \mathcal{M}$ is the triple $(m^\vee, \text{ev}_m, \text{coev}_m)$ consisting of the following data: First, the $\mathcal{A}$-module functor $m^\vee := \text{Hom}_\mathcal{M}(m, -) \in \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$. 

$$(4.1)$$

Second, the evaluation morphisms with which the internal Hom comes naturally endowed, which form a module natural transformation

$$\text{ev}_m : \ m^\vee \bullet m = \text{Hom}_\mathcal{M}(m, -) \circ m \Longrightarrow \text{id}_\mathcal{M},$$

$$(4.2)$$

defined as the counit of the adjunction $(- \circ m) \dashv \text{Hom}_\mathcal{M}(m, -)$. And third, the coevaluation morphism $\text{coev}_m : 1_\mathcal{A} \longrightarrow \text{Hom}_\mathcal{M}(m, m) = m \odot m^\vee$, 

$$(4.3)$$

which is defined as the component of the unit of the adjunction $(- \circ m) \dashv \text{Hom}_\mathcal{M}(m, -)$ corresponding to $1_\mathcal{A}$.

(ii) The left dual of an object $m \in \mathcal{M}$ is the triple $(^\vee m, \text{ev}_m, \text{coev}_m)$ consisting of the following data: First, the $\mathcal{A}$-module functor $^\vee m := \text{coHom}_\mathcal{M}(m, -) \in \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$.

$$(4.4)$$

Second, the evaluation morphism $\text{ev}_m : \ m \odot ^\vee m = \text{coHom}_\mathcal{M}(m, m) \longrightarrow 1_\mathcal{A}$,

$$(4.5)$$

defined as the component of the counit of the adjunction $\text{coHom}_\mathcal{M}(m, -) \dashv (- \circ m)$ that corresponds to $1_\mathcal{A}$. And third, the coevaluation morphism $\text{coev}_m : \text{id}_{\mathcal{M}} \Longrightarrow \text{coHom}_\mathcal{M}(m, -) \circ m = ^\vee m \bullet m$,

$$(4.6)$$

defined as the unit of the adjunction $\text{coHom}_\mathcal{M}(m, -) \dashv (- \circ m)$.

(iii) The right dual of an object $H \in \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$ is the triple $(H^\vee, \text{ev}_m, \text{coev}_m)$ consisting of: The object $H^\vee := H^{\text{ra}}(1_\mathcal{A}) \in \mathcal{M}$,

$$(4.7)$$

together with an evaluation morphism $\text{ev}_H : H^\vee \circ H = H \circ H^{\text{ra}}(1_\mathcal{A}) \longrightarrow 1_\mathcal{A}$

$$(4.8)$$

given by the component of the counit of the adjunction $H \dashv H^{ra}$ corresponding to $1_A$, and with a coevaluation morphism

$$\text{coev}_H : \ id_M \Longrightarrow H^{ra} \circ H \cong H(-) \triangleright H^{ra}(1_A) = H \Box H^\vee$$

(4.9)
given by the unit of the adjunction $H \dashv H^{ra}$.

(iv) The left dual of an object $H \in \text{Fun}_A(\mathcal{M}, \mathcal{A})$ is the triple $(\vee H, \tilde{\text{ev}}_m, \tilde{\text{coev}}_m)$ consisting of: The object

$$\vee H := H^{la}(1_A) \in \mathcal{M}$$

(4.10)
together with an evaluation morphism

$$\tilde{\text{ev}}_H : H \Box \vee H = H(-) \triangleright H^{la}(1_A) \cong H^{la} \circ H \Longrightarrow id_M$$

(4.11)
provided by the adjunction $H^{la} \dashv H$ and with a coevaluation morphism

$$\tilde{\text{coev}}_H : 1_A \longrightarrow H \circ H^{la}(1_A) = \vee H \Box H$$

(4.12)
being the component of the unit of the adjunction $H^{la} \dashv H$ corresponding to $1_A$.

Since they are the unit and counit of an adjunction, the morphisms (4.2) and (4.3), respectively (4.5) and (4.6), obey the snake relations for a right and left duality. For the same reason, the morphisms (4.8) and (4.9), respectively (4.11) and (4.12), obey the relevant snake relations as well. In summary, the left and right duals introduced in Definition 4.1 indeed provide proper dualities, so that we have determined dualities for all 1-morphisms in the bicategory $\mathcal{M}$.

We have thus established:

**Theorem 4.2.** Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ be an exact $\mathcal{A}$-module category. The associated bicategory $\mathcal{M}$ is a bicategory with dualities.

### 4.2 Properties of duals in a Morita context

Some further properties familiar from the duals in tensor categories are again fulfilled by duals in a Morita context. For instance, an object serves as left (right) dual of its right (left) dual, and the dual of a product is the product of the duals in the reversed order up to isomorphism.

**Proposition 4.3.** There are natural isomorphisms

\[
\begin{align*}
(\text{i}) \quad \vee(m^\vee) &\cong m \cong (\vee m)^\vee, \\
(\text{ii}) \quad \vee(H^\vee) &\cong H \cong (\vee H)^\vee, \\
(\text{iii}) \quad (a \triangleright m)^\vee &\cong m^\vee \triangleleft a^\vee, \\
(\text{iv}) \quad (m \triangleright F)^\vee &\cong F^\vee \triangleright m^\vee, \\
(\text{v}) \quad (F \triangleright H)^\vee &\cong H^\vee \triangleleft F^\vee, \\
(\text{vi}) \quad (H \rhd a)^\vee &\cong a^\vee \triangleright H^\vee, \\
(\text{vii}) \quad (m \rhd H)^\vee &\cong H^\vee \triangleright m^\vee, \\
(\text{viii}) \quad (H \triangleright a)^\vee &\cong m^\vee \triangleleft H^\vee 
\end{align*}
\]

(4.13)
for $a \in \mathcal{A}$, $m \in \mathcal{M}$, $H \in \text{Fun}_A(\mathcal{M}, \mathcal{A})$ and $F \in \text{Fun}_M^-$. Analogous relations are valid for left duals.
Proof. All these isomorphisms follow immediately from the definitions and Theorem 4.2. We provide nonetheless the proof for some of the statements. For instance, the definition of duals directly implies (i):

\[ \forall(m^\vee) = \text{Hom}_M^A(m, -)^{la}(1_A) = 1_A \triangleright m \cong m. \]  

(4.14)

Similarly, (ii) follows with the help of Lemma 2.6:

\[ (\forall H)^\vee = \text{Hom}_M^A(H^{la}(1_A), -) \cong \text{Hom}_M^A(1_A, H(-)) = H(-) \otimes 1_A \cong H. \]  

(4.15)

The isomorphism in (iii) corresponds to the module functor structure on the internal Hom given by (2.19). Finally, (viii) comes from the composition

\[ (H \boxdot m)^\vee = [(- \triangleright m) \circ H]^{ra} = H^{ra} \circ \text{Hom}_M^A(m, -) \]

\[ \cong H^{ra} \left( \text{Hom}_M^A(m, -) \otimes 1_A \right) \]

\[ \cong \text{Hom}_M^A(m, -) \triangleright H^{ra}(1_A) = m^\vee \boxdot H^\vee, \]  

where we first identify the adjoint of a composite with the composition of the adjoints in reversed order, and where the last isomorphism corresponds to the module functor structure of \( H^{ra}. \) \qed

Dualities in a tensor category provide an equivalence between its opposite category and its monoidal opposite. Analogously, the duals we just defined for a module category exhibit how the bimodule category \( \text{Fun}_A(M, A) \) plays the role of an opposite to \( M. \)

**Proposition 4.4.** Let \( M \) be an exact \( A \)-module category. The dualities on the bicategory \( \mathcal{M} \) induce equivalences

\[ (-)^\vee: \mathcal{M} \rightleftarrows \text{Fun}_A(M, A)^\# \quad \text{and} \quad ^\vee(-): \mathcal{M} \rightleftarrows \# \text{Fun}_A(M, A) \]  

(4.17)

of \( (A, \mathcal{A}_M^\bullet) \)-bimodule categories, and equivalences

\[ (-)^\vee: \text{Fun}_A(M, A) \rightleftarrows \mathcal{M}^\# \quad \text{and} \quad ^\vee(-): \text{Fun}_A(M, A) \rightleftarrows \# \mathcal{M} \]  

(4.18)

of \( (\mathcal{A}_M^\bullet, A^\bullet) \)-bimodule categories.

**Proof.** Proposition 4.3 shows that right and left duals are mutual quasi-inverses, and the isomorphisms from Proposition 4.3(iii) and 4.3(iv) endow the duality functor with a bimodule structure. The statements for the remaining equivalences follow by analogous reasoning. \qed

There are further relations involving the dualities which constitute a duality calculus in the Morita context, a powerful tool for performing calculations such as the computation of relative Serre functors.

**Remark 4.5.**

(i) Recall that for the regular module \( \mathcal{A}_A \) we have

\[ \text{Hom}_A^A(a, b) = b \otimes a^\vee \quad \text{and} \quad \text{coHom}_M^A(a, b) = b \otimes ^\vee a. \]  

(4.19)

for \( a, b \in \mathcal{A}. \) Similarly it follows from the definition of the mixed products and of the dualities that

\[ \text{Hom}_M^A(m, n) = n \otimes m^\vee \quad \text{and} \quad \text{coHom}_M^A(m, n) = n \otimes ^\vee m. \]  

(4.20)

for \( m, n \in M. \)
(ii) Acting with dualities on the bimodules in the Morita context leads to the adjunctions
\[(a^\lor 	riangleright -) \dashv (\rhd \triangleright -) \quad \text{and} \quad (- \bowtie F^{\lambda a}) \dashv (- \bowtie F) \dashv (- \bowtie F^a)\] (4.21)
as well as
\[(F^{\lambda a} \triangleright -) \dashv (F \triangleright -) \dashv (F^{\lambda a} \triangleright -) \quad \text{and} \quad (- \bowtie \forall a) \dashv (- \bowtie a) \dashv (- \bowtie a^\lor)\] (4.22)
for \(a \in \mathcal{A}\) and \(F \in \mathcal{A}^*_M\).

It turns out that the relations given in Remark 4.5 can be extended to the entire Morita context:

**Proposition 4.6.** There are natural isomorphisms
\[
\begin{align*}
(i) & \quad \text{Hom}_\mathcal{M}(a \rhd m, n) \cong \text{Hom}_\mathcal{A}(a, n \otimes m^\lor), \\
(ii) & \quad \text{Hom}_\mathcal{M}(n, a \rhd m) \cong \text{Hom}_\mathcal{A}(n \otimes ^\lor m, a), \\
(iii) & \quad \text{Hom}_\mathcal{M}(m \bowtie F, n) \cong \text{Hom}_{\mathcal{A}^*_M}(F, ^\lor m \bowtie n), \\
(iv) & \quad \text{Hom}_\mathcal{M}(n, m \bowtie F) \cong \text{Hom}_{\mathcal{A}^*_M}(m^\lor \bowtie n, F), \\
(v) & \quad \text{Hom}_{\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})}(F \bowtie H_1, H_2) \cong \text{Hom}_{\mathcal{A}^*_M}(F, H_2 \bowtie H_1^\lor), \\
(vi) & \quad \text{Hom}_{\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})}(H_2, F \bowtie H_1) \cong \text{Hom}_{\mathcal{A}^*_M}(H_2 \bowtie ^\lor H_1, F), \\
(vii) & \quad \text{Hom}_{\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})}(H_1 \bowtie a, H_2) \cong \text{Hom}_\mathcal{A}(a, ^\lor H_1 \bowtie H_2), \\
(viii) & \quad \text{Hom}_{\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})}(H_2, H_1 \bowtie a) \cong \text{Hom}_\mathcal{A}(H_1^\lor \bowtie H_2, a)
\end{align*}
\] (4.23)
for \(a \in \mathcal{A}\), \(m, n \in \mathcal{M}\), \(H_1, H_2 \in \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})\) and \(F \in \overline{\mathcal{A}^*_M}\).

**Proof.** (i) and (ii) follow directly from the definition of the duals and the defining properties of the internal Hom and coHom. The bijection
\[
\text{Hom}_\mathcal{M}(F(m), n) \longrightarrow \text{Nat}_{\text{mod}}(F, \text{coHom}(m, -) \rhd n)\] (4.24)
in (iii) is an arrow assigning to \(f: F(m) \to n\) a module natural transformation whose component at \(l \in \mathcal{M}\) is given by the composition
\[
F(l) \xrightarrow{F(\text{coev})} F(\text{coHom}(m, l) \rhd m) \cong \text{coHom}(m, l) \rhd F(m) \xrightarrow{\text{id} \bowtie f} \text{coHom}(m, l) \rhd n.\] (4.25)
The inverse of (4.24) is given by the assignment
\[
\eta \longmapsto F(m) \xrightarrow{\eta m} \text{coHom}(m, m) \rhd n \xrightarrow{\overline{m \bowtie \text{id}_n}} n\] (4.26)
for \(\eta \in \text{Nat}_{\text{mod}}(F, \text{coHom}(m, -) \rhd n) = \text{Hom}_{\mathcal{A}^*_M}(F, ^\lor m \bowtie n)\). The bijection in (iv) is defined in a similar fashion. To prove (v) consider the bijection
\[
\text{Nat}_{\text{mod}}(H_1 \bowtie F, H_2) \longrightarrow \text{Nat}_{\text{mod}}(F, H_1^{\lambda a} \bowtie H_2)\] (4.27)
which assigns to \( \eta: H_1 \circ F \Rightarrow H_2 \) the natural transformation

\[
F \mapsto H_1^{ra} \circ H_1 \circ F \xrightarrow{id \circ \eta} H_1^{ra} \circ H_2.
\]  

(4.28)

Given \( \gamma \in \text{Nat}_{\text{mod}}(F, H_1^{ra} \circ H_2) = \text{Hom}_{\mathcal{A}_M}(F, N_{H_1} \Box H_1) \) the assignment

\[
\gamma \mapsto H_1 \circ F \xrightarrow{id \circ \gamma} H_1 \circ H_1^{ra} \circ H_2 \Rightarrow H_2
\]

(4.29)
serves as inverse of \( (1.27) \). The isomorphism (vi) is defined analogously. To obtain (vii), consider the function

\[
\text{Nat}_{\text{mod}}((- \otimes a) \circ H_1, H_2) \rightarrow \text{Hom}_{\mathcal{A}}(a, H_2(H_1^{la}(1_{A})))
\]

(4.30)

\[
\eta \mapsto a \mapsto (- \otimes a) \circ H_1 \circ H_1^{la}(1_{A}) \xrightarrow{\eta_{H_1^{la}(1_{A})}} H_2(H_1^{la}(1_{A})),
\]

which has as inverse the arrow that assigns to \( f : a \rightarrow H_2(H_1^{la}(1_{A})) \) the composition

\[
(- \otimes a) \circ H_1 \xrightarrow{(- \otimes f) \circ \text{id}} (- \otimes H_2(H_1^{la}(1_{A}))) \circ H_1 \simeq H_2 \circ H_1^{la} \circ H_1 \Rightarrow H_2,
\]

(4.31)

where the isomorphism is the module structure of \( H_2 \circ H_1^{la} \) and the last arrow is the counit of the adjunction \( H_1^{la} \dashv H_1 \).

Remark 4.7. Notice that the adjunctions in Proposition \( (4.6) \) describe the internal Homs and coHoms of the bimodule categories in the Morita context in terms of the products and dualities:

\[
\begin{align*}
\text{(i)} & \quad \text{Hom}_{\mathcal{A}}^{A}(m, n) = n \odot m^\vee, \\
\text{(ii)} & \quad \text{coHom}_{\mathcal{A}}^{A}(m, n) = n \odot m, \\
\text{(iii)} & \quad \text{Hom}_{\mathcal{A}^*}^{A}(m, n) = m^\vee \boxdot n = \text{coHom}_{\mathcal{A}}^{A}(m, -) \triangleright n, \\
\text{(iv)} & \quad \text{coHom}_{\mathcal{A}^*}^{A}(m, n) = m^\vee \boxdot n = \text{Hom}_{\mathcal{A}}^{A}(m, -) \triangleright n, \\
\text{(v)} & \quad \text{Hom}_{\text{Fun}_{A}(\mathcal{M}, A)}(H_1, H_2) = H_2 \boxdot H_1^\vee = H_1^{ra} \circ H_2, \\
\text{(vi)} & \quad \text{coHom}_{\text{Fun}_{A}(\mathcal{M}, A)}^{A}(H_1, H_2) = H_2 \boxdot H_1 = H_1^{la} \circ H_2, \\
\text{(vii)} & \quad \text{Hom}_{\text{Fun}_{A}(\mathcal{M}, A)}^{A}(H_1, H_2) = H_1^\vee \odot H_2 = H_2 \circ H_1^{la}(1_{A}), \\
\text{(viii)} & \quad \text{coHom}_{\text{Fun}_{A}(\mathcal{M}, A)}^{A}(H_1, H_2) = H_1^\vee \odot H_2 = H_2 \circ H_1^{ra}(1_{A}).
\end{align*}
\]

(4.32)

In particular the formulas (iii) and (iv) relate the internal Homs and coHoms of the module categories \( \mathcal{A}\mathcal{M} \) and \( \mathcal{A}_M^* \mathcal{M} \).

Lemma 4.8. Let \( \mathcal{M} \) be an exact module category over a finite tensor category \( \mathcal{A} \). Then the assignment

\[
\text{Fun}_{A}(\mathcal{M}, A) \rightarrow \text{Fun}_{\mathcal{A}^*\mathcal{M}}(\mathcal{M}, \mathcal{A}^{\vee})
\]

(4.33)

\[
H \mapsto H \boxdot -
\]

is an equivalence of \( (\mathcal{A}^*\mathcal{M}, A) \)-bimodule categories.
Proof. According to Lemma 3.8 the functor $\boxdot$ is a balanced bimodule functor. It follows that $H\boxdot -$ is an $\overline{\mathcal{A}_M}$-module functor and that the functor (4.33) has an $(\overline{\mathcal{A}_M}, \mathcal{A})$-bimodule structure. The following functor is a quasi-inverse to (4.33):

$$\text{Fun}_{\overline{\mathcal{A}_M}}(\mathcal{M}, \overline{\mathcal{A}_M}) \longrightarrow \text{Fun}_A(\mathcal{M}, \mathcal{A}),$$

$$K \longmapsto {^\vee}[K^{la}(\text{id}_\mathcal{M})] = \text{coHom}^A_M(K^{la}(\text{id}_\mathcal{M}), \cdot)$$

Indeed, given $H \in \text{Fun}_A(\mathcal{M}, \mathcal{A})$ we have

$$\forall [K^{la}(\text{id}_\mathcal{M})] \cong {^\vee}(H^{\vee} \circ \text{id}_\mathcal{M}) \cong H,$$

where the first isomorphism comes from Proposition 4.6(iii) and the second isomorphism from Proposition 4.3(ii). Conversely, for $K \in \text{Fun}_{\overline{\mathcal{A}_M}}(\mathcal{M}, \overline{\mathcal{A}_M})$ and $n \in \mathcal{M}$ we have the chain

$$\forall [K^{la}(\text{id}_\mathcal{M})] \uplus n = \text{coHom}^A_M(K^{la}(\text{id}_\mathcal{M}), \cdot) \uplus n \cong \text{Hom}^A_M(K^{la}(\text{id}_\mathcal{M}), n) \cong \text{Hom}^A_M(\text{id}_\mathcal{M}, K(n)) \cong K(n)$$

of natural isomorphisms, where the first isomorphism is from Remark 4.7(iii), while the second is Lemma 2.6.

**Proposition 4.9.** Let $\mathcal{M}$ be an exact module category over a finite tensor category $\mathcal{A}$. Then the Morita context $(\mathcal{A}, \overline{\mathcal{A}_M}, \mathcal{M}, \text{Fun}_A(\mathcal{M}, \mathcal{A}), \odot, \boxdot)$ from Theorem 3.9 is strong.

Proof. By invoking Proposition 2.4 it follows immediately that the mixed product $\boxdot$ descends to an equivalence $\boxdot : \text{Fun}_A(\mathcal{M}, \mathcal{A}) \boxtimes \mathcal{A} \mathcal{M} \rightarrow \overline{\mathcal{A}_M}$. Thus it remains to verify that $\odot$ descends to an equivalence $\odot : \mathcal{M} \boxtimes \mathcal{B} \text{Fun}_A(\mathcal{M}, \mathcal{A}) \rightarrow \mathcal{A}$, with $\mathcal{B} := \overline{\mathcal{A}_M}^*$, as well. To see this, consider the canonical equivalence [EGNO, Thm. 7.12.11]

$$\text{can} : \mathcal{A} \xrightarrow{\cong} (\mathcal{A}_M)^* \mathcal{M}, \quad a \mapsto a \uplus -$$

and the bimodule equivalence from Lemma 4.8. The diagram

$$\begin{array}{ccc}
\mathcal{M} \boxtimes \mathcal{B} \text{Fun}_A(\mathcal{M}, \mathcal{A}) & \xrightarrow{\odot} & \mathcal{A} \\
\downarrow^{(4.33)} & & \downarrow^{\text{can}} \\
\mathcal{M} \boxtimes \mathcal{B} \text{Fun}_\mathcal{B}(\mathcal{M}, \mathcal{B}) & \xrightarrow{(2.11)} & \text{Fun}_\mathcal{B}(\mathcal{M}, \mathcal{M})
\end{array}$$

commutes strictly: we have $m \diamond (H \boxdot n) = H(m) \uplus n = (m \odot H) \uplus n$ for $n \in \mathcal{M}$. Since all other functors in the diagram are equivalences, it follows that $\odot$ is an equivalence, too.

**Remark 4.10.** Above we have focused our attention on Morita contexts associated to exact module categories. The reason for this is the following: According to Proposition 4.9 the Morita context derived from an exact module category, as described in Theorem 3.9, is strong. In view of Theorem 3.11 every strong Morita context is of this type. As a consequence, Theorem 4.2 and related statements, valid for the Morita context of an exact module category, also hold for any arbitrary strong Morita context.
4.3 Double duals and relative Serre functors

Given a tensor category \( \mathcal{A} \), the relative Serre functor of the regular module \( \mathcal{A} \) corresponds to the double right dual, \( S^A_{\mathcal{A}}(a) \cong a^{\vee \vee} \). Similarly, the double duals of objects in the Morita context of an exact module \( \mathcal{A} \) admit the following description involving the relative Serre functors:

**Proposition 4.11.** For \( \mathcal{M} \) an exact module category over a finite tensor category \( \mathcal{A} \), let \( m \in \mathcal{M} \) and \( H \in \text{Fun}_A(\mathcal{M}, \mathcal{A}) \). We have isomorphisms

\[
\begin{align*}
(i) & \quad m^{\vee \vee} \cong S^A_{\mathcal{M}}(m), & (iii) & \quad H^{\vee \vee} \cong H^{\text{rra}} \cong S^A_{\text{Fun}_A(\mathcal{M}, \mathcal{A})}(H), \\
(ii) & \quad \vee m \cong S^A_{\mathcal{M}}(m), & (iv) & \quad \vee H \cong H^{\text{lla}} \cong S^A_{\text{Fun}_A(\mathcal{M}, \mathcal{A})}(H).
\end{align*}
\]

(4.39)

**Proof.** Combining the realization (2.30) of the relative Serre functor with the description of right duals in Definition 4.1(i) and (iii) we directly get

\[
S^A_{\mathcal{M}}(m) \cong \text{Hom}^A_{\mathcal{M}}(m, -)^{\text{rra}}(1_{\mathcal{A}}) = m^{\vee \vee}.
\]

(4.40)

Similarly for a module functor we have

\[
H^{\vee \vee} = \text{Hom}^A_{\mathcal{M}}(H^{\text{rra}}(1_{\mathcal{A}}), -) \cong (\vee (H^{\text{rra}}(1_{\mathcal{A}})))^{\text{rra}} \cong H^{\text{rra}}.
\]

(4.41)

The second isomorphism in (iii) follows as

\[
S^A_{\text{Fun}_A(\mathcal{M}, \mathcal{A})}(H) \cong \text{Hom}^A_{\text{Fun}_A(\mathcal{M}, \mathcal{A})}(H, -)^{\text{rra}}(\text{id}_M) \cong (\vee H^{\vee})^{\text{rra}}(\text{id}_M) \cong \text{id}_M \times H^{\vee \vee}
\]

(4.42)

with the help of Remark 4.7(v) and Proposition 4.6(vi). The statements for double left duals follow in a similar manner.

**Corollary 4.12.** The relative Serre functors of \( \mathcal{M} \) are related by

\[
S^A_{\mathcal{M}}(m) \cong S^A_{\mathcal{M}}(m) \quad \text{and} \quad S^A_{\mathcal{M}}(m) \cong S^A_{\mathcal{M}}(m).
\]

(4.43)

**Proof.** The statements follow by considering again the standard realization of the relative Serre functors together with the duality calculus in the Morita context. For instance, combining Remark 4.7(iii), Proposition 4.6(iv) and Proposition 4.11(ii) yields the first isomorphism:

\[
S^A_{\mathcal{M}}(m) \cong \text{Hom}^A_{\mathcal{M}}(m, -)^{\text{rra}}(\text{id}_M) \cong (\vee m \-square \ -)^{\text{rra}}(\text{id}_M) \cong \vee m \circ \text{id}_M \cong S^A_{\mathcal{M}}(m).
\]

(4.44)

The second isomorphism is obtained in a similar manner.

Double duals in a tensor category are compatible with tensor products: the double dual of a product is isomorphic to the product of the double duals of the factors. This property extends to any bicategory \( \mathcal{F} \) with dualities. Moreover, the double duals of 1-morphisms form a pseudo-equivalence \((-)^{\vee \vee} : \mathcal{F} \to \mathcal{F} \).

In the case of the bicategory \( \mathcal{M} \) associated with the Morita context of a module category, Proposition 4.11 implies that the double-dual functors are isomorphic to relative Serre functors.
The compatibility between double duals and products ensures that there are coherent natural isomorphisms

\[
\begin{align*}
(i) & \quad S^A_M(a \triangleright m) \cong a \circ \circ S^A_M(m), \\
(ii) & \quad (F \triangleright H)^{\text{tra}} \cong F^{\text{tra}} \triangleright H^{\text{tra}}, \\
(iii) & \quad (m \odot H)^{\circ \circ} \cong S^A_M(m) \circ H^{\text{tra}}, \\
(iv) & \quad S^A_M(m \triangleleft F) \cong S^A_M(m) \triangleleft F^{\text{tra}}, \\
(v) & \quad (H \ll a)^{\text{tra}} \cong H^{\text{tra}} \ll a^{\circ \circ}, \\
(vi) & \quad H^{\text{tra}} \boxdot S^A_M(m) \cong (H \boxdot m)^{\text{tra}}
\end{align*}
\]

for \( a \in A, \ m \in M, \ H \in \text{Fun}_A(M, A) \) and \( F \in \overline{A_M^*} \). These isomorphisms can be obtained by iterating the isomorphisms from Proposition 4.3. In particular, (i) and (iv) recover the twisted bimodule functor structure of \( S^A_M \). Put differently, the isomorphisms (4.45) relate the value of the relative Serre functor of a product with the product of the relative Serre functors evaluated in the corresponding factors. For instance, (iii) exhibits the coherence data

\[
S^A_M(m \odot H) \cong (m \odot H)^{\circ \circ} \cong S^A_M(m) \circ H^{\text{tra}} \cong S^A_M(m) \circ S^A_{M(M, M)}(H)
\]

for the composition of \( m \in M \) and \( H \in \text{Fun}_A(M, A) \) in \( M \). In this spirit the relative Serre functors of the categories in \( M \) assemble into a pseudo-equivalence:

**Definition 4.13** (Relative Serre pseudo-functor).

Let \( M \) be an exact module category over a finite tensor category \( A \). The relative Serre pseudo-functor on the bicategory \( M \) consists of the assignment

\[
S : \quad M \xrightarrow{\cong} M,  \\
x \mapsto x, \\
M(x, y) \ni a \mapsto S_{M(x, y)}(a) \in M(x, y)
\]

(4.47)

with the natural isomorphisms (4.45), which witness the compatibility with the horizontal composition in \( M \).

### 4.4 The Radford pseudo-equivalence

For a finite tensor category \( A \), conjugation by its distinguished invertible object \( D_A \) is naturally isomorphic to the functor of taking the fourth power of the right dual. We explore how this extends to the entirety of the Morita context of an exact module category. We first point out that the computation of the relative Serre functors of \( M \) leads to a description of the distinguished invertible object of the dual tensor category.

**Proposition 4.14.** Let \( M \) be an exact module category over a finite tensor category \( A \). There is an isomorphism

\[
D_{A^*_M} \cong D_A \triangleright (S^A_M)^2 \cong N^f_M \circ S^A_M
\]

(4.48)

of module functors, where \( D_{A^*_M} \) is the distinguished invertible object of the dual tensor category \( A^*_M \).

**Proof.** We can regard \( M \) both as a left \( A \)-module and as a left \( A^*_M \)-module. Therefore from (2.44) we obtain an isomorphism

\[
D_{A^*_M}^{-1} \triangleright S^A_M \cong N^f_M \cong D_{A^*_M}^{-1} \triangleright S^A_M
\]

(4.49)
of bimodule functors. The result now follows by taking into account that $S_M^A M \cong S_M^A$. The second isomorphism in (4.48) comes from the isomorphism $N_{M}^i \cong D_A \circ S_M^A$ in (2.44).

We find that Radford’s theorem can be extended to exact module categories, with the relative Serre functor playing the role of the double right dual functor.

**Theorem 4.15** (Radford isomorphism of a module category).

Let $A$ be a finite tensor category and $M$ an exact $A$-module. There is a natural isomorphism

$$r_M : D_A \circ S_M^A \cong S_M^A \circ S_M^A$$

of twisted bimodule functors.

**Proof.** The statement follows from Proposition 4.14 by reformulating the description of $D_{A_M}^A$. From the isomorphism (4.48) we obtain

$$D_{A_M}^A \circ S_M^A \circ S_M^A \cong N_{M}^i \circ S_M^A \circ D_{A_M}^A \circ S_M^A \cong N_{M}^i \circ S_M^A \cong D_A \circ -,$$

where we use the fact that $S_M^A$ and $\overline{S}_M^A$ are quasi-inverses and the isomorphism coming from (2.44).

Note that Equation (4.50) takes a very symmetric form because we see the $A$-module category $M$ as an $(A, \overline{A_M})$-bimodule: the distinguished invertible objects of both $A$ and $\overline{A_M}$ enter in (4.50) on the same footing.

There are similar Radford isomorphisms for the categories $\text{Fun}_A(M, A)$ and $\overline{A_M}$ in the Morita context $M$, where again the corresponding relative Serre functors play the role of the double right dual. These Radford isomorphisms assemble into a trivialization of the square of the relative Serre pseudo-functor (4.47), i.e. a trivialization of the fourth power of the pseudo-functor of dualities of $M$.

**Theorem 4.16** (Radford pseudo-equivalence of a Morita context).

Let $M$ be an exact module category over a finite tensor category $A$ and $M$ the bicategory associated to its Morita context. There is a pseudo-natural equivalence

$$R : \text{id}_M \cong S^2,$$

where $S$ is the relative Serre pseudo-functor (4.47).

**Proof.** To construct the pseudo-natural equivalence, consider the following data:

(i) For the objects 0 and 1 of $M$ the distinguished invertible 1-morphisms

$$R_0 := D_A \quad \text{and} \quad R_1 := D_{A_M}.$$ (4.53)

(ii) For 1-morphisms in $M$, the following invertible 2-morphisms:
\[ \mathcal{R}_a : \mathbb{D}_A \otimes a \xrightarrow{\simeq} a^\vee \otimes \mathbb{D}_A \] (4.54)

\[ \mathcal{R}_m : \mathbb{D}_A \triangleright m \xrightarrow{\simeq} \mathbb{S}_A \circ \mathbb{S}_M(m) \triangleleft \mathbb{D}_A^* \] (4.55)

coming from (2.3) and from (4.50), respectively.

\[ \mathcal{R}_F : \mathbb{D}_A^* \circ \text{opp} F \xrightarrow{\simeq} F^\text{irra} \circ \text{opp} \mathbb{D}_A^* \] (4.56)

given by the composite
\[ F \circ \mathbb{D}_A^* \cong F \circ \mathbb{D}_A \triangleright (\mathbb{S}_M^A)^2 \cong \mathbb{D}_A \triangleright (\mathbb{S}_M^A)^2 \circ F^\text{irra} \cong \mathbb{D}_A^* \circ F^\text{irra}, \] (4.57)

where the first and last isomorphisms come from (4.48) and the middle isomorphism uses the module structure of \( F \) and the twisted structure \((2.39)\) of the relative Serre functor twice.

Analogously, for any \( H \in \text{Fun}_A(\mathcal{M}, \mathcal{A}) \) a natural isomorphism
\[ \mathcal{R}_H : \mathbb{D}_{A_M}^* \triangleright H \xrightarrow{\simeq} H^\text{irra} \triangleleft \mathbb{D}_A \] (4.58)

given by the composite
\[ H \circ \mathbb{D}_{A_M}^* \cong H \circ \mathbb{D}_A \triangleright (\mathbb{S}_M^A)^2 \cong \mathbb{D}_A \otimes^\vee (\mathcal{M}) \circ H^\text{irra} \cong (- \otimes \mathbb{D}_A) \circ H^\text{irra}, \] (4.59)

where again we first use (4.48) and then the twisted structure \((2.39)\) of the relative Serre functor (also taking into account that \( S_A^A \cong \vee (-) \)), and where the last isomorphism is the Radford isomorphism of \( A \).

The claim now reduces to making the routine check of the commutativity of the diagram
\[
\begin{array}{ccc}
\mathbb{D} \circ s \circ t & \xrightarrow{\mathcal{R}_{sot}} & (s \circ t)^\vee \circ \mathbb{D} \\
\mathcal{R}_{soid} \downarrow & & \downarrow (4.40) \\
s^\vee \otimes \mathbb{D} \circ t & \xrightarrow{\text{id} \mathbb{R}_t} & s^\vee \otimes t^\vee \circ \mathbb{D}
\end{array}
\] (4.60)

where the symbol \( \circ \) denotes the horizontal composition in the bicategory \( \mathcal{M} \) and \( s \) and \( t \) are composable 1-morphisms.

The bicategorical formulation in Theorem 4.16 unifies Radford’s theorems for tensor and module categories. It also offers a natural home to the invertible objects in Radford-type theorems, since they become part of the data of a pseudo-natural equivalence.

Remark 4.17. Theorem 4.16 can be extended to the bicategory \( \text{Mod}^\text{ex}(\mathcal{A}) \) of exact module categories over a finite tensor category \( \mathcal{A} \). \( \text{Mod}^\text{ex}(\mathcal{A}) \) is a bicategory with dualities for 1-morphisms, given by \( H^* := H^\text{la} \) and \( ^*H := H^\text{ra} \) for every module functor \( H \in \text{Fun}_A(\mathcal{M}, \mathcal{N}) \). The isomorphisms coming from (4.48) and (2.39) induce a natural isomorphism
\[ \mathcal{R}_H : \mathbb{D}_{A_N}^* \triangleright H \xrightarrow{\simeq} H^{\text{Hill}} \triangleleft \mathbb{D}_A^* \] (4.61)

for each such module functor, and these together assemble into a pseudo-natural equivalence \( \mathcal{R} : \text{id}_{\text{Mod}^\text{ex}(\mathcal{A})} \xrightarrow{\simeq} (-)^{****} \).

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5 On pivotality and Morita theory

5.1 Pivotal module categories

An additional structure that a tensor category $\mathcal{A}$ can carry is a \textit{pivotal structure}, that is, a monoidal natural isomorphism $\mathcal{p}: \text{id}_\mathcal{A} \xrightarrow{\sim} (-) \vee \vee$ between the identity functor and the double-dual functor. The monoidal opposite $\overline{\mathcal{A}}$ of a pivotal tensor category is endowed with a canonical pivotal structure, given by

$$\overline{\mathcal{p}}_a := \mathcal{p}^{-1}_a : \overline{a} \xrightarrow{\cong} \overline{a} \vee \vee.$$

A pivotal structure on a module category over a pivotal tensor category can be defined as follows:

\textbf{Definition 5.1.} Let $\mathcal{A}$ and $\mathcal{B}$ be pivotal finite tensor categories.

(i) ([Sc Def. 5.2] and [Sh2 Def. 3.11]) A \textit{pivotal structure} on an exact left $\mathcal{A}$-module category $\mathcal{M}$ is a natural isomorphism $\tilde{\mathcal{p}}: \text{id}_\mathcal{M} \xrightarrow{\cong} \mathcal{S}^{\mathcal{A}}_\mathcal{M}$ such that the diagram

$$\begin{array}{c}
   a \triangleright m \\
   \downarrow \mathcal{p}_a \triangleright \mathcal{m} \quad \overline{\mathcal{p}}_a \triangleright \mathcal{m} \\
   \mathcal{S}^{\mathcal{A}}_\mathcal{M}(a \triangleright m)
\end{array}$$

commutes for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$. A module category together with a module structure is said to be a \textit{pivotal module category}.

(ii) An exact right $\mathcal{B}$-module category $\mathcal{N}$ is said to be \textit{pivotal} if the left module category $\overline{\mathcal{B}}\mathcal{N}$ has a pivotal structure.

(iii) A \textit{pivotal bimodule category} is an exact bimodule $\mathcal{A}\mathcal{M}\mathcal{B}$ together with the structure $\tilde{\mathcal{p}}: \text{id}_\mathcal{M} \xrightarrow{\cong} \mathcal{S}^{\mathcal{A}}_\mathcal{M}$ of a pivotal $\mathcal{A}$-module and the structure $\tilde{\mathcal{q}}: \text{id}_\mathcal{M} \xrightarrow{\cong} \mathcal{S}^{\mathcal{B}}_\mathcal{M}$ of a pivotal $\mathcal{B}$-module, such that the diagrams

$$\begin{array}{c}
   m \triangleright b \\
   \downarrow \mathcal{p}_m \triangleright \mathcal{q}_b \quad \tilde{\mathcal{p}}_m \triangleright \mathcal{q}_b \\
   \mathcal{S}^{\mathcal{A}}_\mathcal{M}(m \triangleright b)
\end{array}$$

and

$$\begin{array}{c}
   a \triangleright m \\
   \downarrow \mathcal{q}_a \triangleright \mathcal{m} \quad \tilde{\mathcal{q}}_a \triangleright \mathcal{m} \\
   \mathcal{S}^{\overline{\mathcal{B}}}_\mathcal{M}(a \triangleright m)
\end{array}$$

commute for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $m \in \mathcal{M}$.

The dual tensor category of a pivotal module inherits the structure of a pivotal finite multi-tensor category:
Proposition 5.2. Let $A$ be a pivotal tensor category and $M$ a pivotal $A$-module category. Then

(i) [Sh2, Thm. 3.13] The dual tensor category $A^*_M$ has a pivotal structure given by the composite

$$q_F : F \xrightarrow{\text{id} \circ p} F \circ S_A^M \xrightarrow{2.36} S_A^M \circ F \xrightarrow{\bar{p}^{-1} \circ \text{id}} F^\text{lla}$$

for a module endofunctor $F \in A^*_M$, with $\bar{p}$ the pivotal structure of $M$.

(ii) If $A$ is in addition unimodular, then the dual tensor category $A^*_M$ is unimodular.

Proof. The statement in [Sh2, Thm. 3.13] concerns $A^*_M$, the monoidal opposite of the dual tensor category, with its pivotal structure described by

$$\mathcal{T}_F : F \xrightarrow{\bar{p} \circ \text{id}} S_A^M \circ F \xrightarrow{\text{2.36}} F^\text{lla} \circ S_A^M \xrightarrow{\text{id} \circ \bar{p}^{-1}} F^\text{lla}. \quad (5.6)$$

Considering the opposite pivotal structure $(5.1)$ on $A^*_M$ we obtain $(5.5)$.

Now suppose that $A$ is unimodular. Then it follows from Proposition 4.14 that the pivotal structure of $M$ and a trivialization of the distinguished invertible object of $A$ furnish an isomorphism $\mathbb{D}_{A^*_M} \cong \text{id}_M$. \hfill $\square$

5.2 Pivotality of the category of module functors

Let $A$ be a finite tensor category and $M$ and $N$ exact $A$-module categories. Since $M$ is a right $A^*_M$-module, the category $\text{Fun}_A(M, N)$ of module functors is a left $A^*_M$-module, with action given by

$$F \triangleright H := H \circ F \quad \text{for} \quad F \in A^*_M \text{ and } H \in \text{Fun}_A(M, N). \quad (5.7)$$

More generally, if $M$ and $N$ are bimodules, then $\text{Fun}_A(M, N)$ becomes a bimodule category with actions given by $(2.6)$.

Lemma 5.3. Let $M$ and $N$ be exact module categories over a finite tensor category $A$.

(i) The relative Serre functors of the module category $\overline{A^*_M} \text{Fun}_A(M, N)$ are given by

$$\overline{S}_{\text{Fun}_A(M, N)}(H) \cong H^\text{lla} \quad \text{and} \quad \overline{S}_{\text{Fun}_A(M, N)}(H) \cong H^\text{lla}$$

for $H \in \text{Fun}_A(M, N)$.

(ii) The Nakayama functors of $\text{Fun}_A(M, N)$ are given by

$$\overline{N}_{\text{Fun}_A(M, N)}(H) \cong N_N^A \circ H \circ S_A^M \cong S_A^M \circ H \circ N_M^A$$

and

$$N_{\text{Fun}_A(M, N)}(H) \cong N_N^A \circ H \circ S_A^M \cong S_A^M \circ H \circ N_M^A$$

for $H \in \text{Fun}_A(M, N)$.

(iii) For exact bimodules $A_M B$ and $A_N C$, the relative Serre functors of the bimodule category $\text{Fun}_A(M, N)_C$ are given by

$$\overline{S}_{\text{Fun}_A(M, N)}(H) \cong S_A^M \circ H \circ S_A^M, \quad \overline{S}_{\text{Fun}_A(M, N)}(H) \cong S_A^M \circ H \circ S_A^M,$$

for $H \in \text{Fun}_A(M, N)$.
Proof.

(i) There is a bijection analogous to \((4.27)\) for module functors in \(\text{Fun}_A(\mathcal{M}, \mathcal{N})\). The statement thus follows in complete analogy to the computation for \(\text{Fun}_A(\mathcal{M}, \mathcal{A})\) in Proposition 4.11.

(ii) We show \((5.10)\); the isomorphisms \((5.9)\) follow in the same manner. According to Proposition 4.14 the distinguished object of the dual tensor category is \(D_A^* \cong N_M^t \circ S_A^M\). Thus

\[
N^t_{\text{Fun}_A(\mathcal{M}, \mathcal{N})}(H) \cong D^*_A \circ S^M_{\text{Fun}_A(\mathcal{M}, \mathcal{N})}(H) \cong H^{\text{fa}} \circ N_M^t \circ S_A^M \cong N_N^t \circ H \circ S_M^A,
\]

where the second isomorphism uses the second isomorphism in \((5.8)\) and the last isomorphism is the twisted module structure of the Nakayama functor. Moreover, a factor of \(D_A^*\) can be juggled between the Nakayama functor of \(N\) and the relative Serre functor of \(M\) using the module structure of \(H\), leading to the second isomorphism in \((5.10)\).

(iii) All four isomorphisms follow from the relation \((2.44)\) between relative Serre and Nakayama functors. For instance, the last isomorphism in \((5.11)\) is given by the composite

\[
S_B^C \circ \text{Fun}_A(\mathcal{M}, \mathcal{N})(H) \cong D_B^* \circ N^t_{\text{Fun}_A(\mathcal{M}, \mathcal{N})}(H) \cong S_A^M \circ H \circ N^t_M(-) \circ D_B^* \cong S_A^M \circ H \circ S_B^M,
\]

where the second isomorphism comes from \((5.9)\) and the last one is an instance of \((2.44)\).

\(\Box\)

Proposition 5.4. Let \(A, B\) and \(C\) be pivotal finite tensor categories, and consider exact bimodules \(A\mathcal{M}_B\) and \(A\mathcal{N}_C\).

(i) Suppose that \(A\mathcal{M}\) and \(A\mathcal{N}_C\) are pivotal modules. Then \(\text{Fun}_A(\mathcal{M}, \mathcal{N})\) has the structure of a pivotal \(C\)-module category.

(ii) Suppose that \(A\mathcal{N}^c\) and \(A\mathcal{M}_B\) are pivotal modules. Then \(\text{Fun}_A(\mathcal{M}, \mathcal{N})\) has the structure of a pivotal \(B\)-module category.

(iii) If \(A\mathcal{M}_B\) and \(A\mathcal{N}_C\) are pivotal bimodules, then \(\text{Fun}_A(\mathcal{M}, \mathcal{N})\) has the structure of a pivotal \((B, C)\)-bimodule category.

Proof. To show (i) denote by \(\tilde{q}: \text{id}_{\mathcal{N}} \sim \Rightarrow S_N^C\) and by \(\tilde{p}: \text{id}_{\mathcal{M}} \sim \Rightarrow S_M^A\) the pivotal structures of the pivotal modules \(A\mathcal{M}\) and \(A\mathcal{N}_C\). Define for \(H \in \text{Fun}_A(\mathcal{M}, \mathcal{N})\) a natural isomorphism

\[
\tilde{Q}_H : H \xrightarrow{\tilde{q} \circ \text{id}_H \circ \tilde{p}} S_N^C \circ H \circ S_M^A \cong S^C_{\text{Fun}_A(\mathcal{M}, \mathcal{N})}(H);
\]

this serves as a \(C\)-pivotal structure for \(\text{Fun}_A(\mathcal{M}, \mathcal{N})\). It remains to check that the diagram

\[
\begin{array}{ccc}
H \langle c \rangle & \xrightarrow{\tilde{Q}_H \langle c \rangle} & S^C_{\text{Fun}_A(\mathcal{M}, \mathcal{N})}(H \langle c \rangle) \\
\downarrow\tilde{Q}_H \downarrow & \Rightarrow & \downarrow\text{RH} \\
S^C_{\text{Fun}_A(\mathcal{M}, \mathcal{N})}(H) & \xrightarrow{\vee^C} & \ldots
\end{array}
\]

is commutative.
commutes for every $c \in C$ and $H \in \text{Fun}_A(\mathcal{M}, \mathcal{N})$, where $\overline{q}$ is the pivotal structure of $\overline{\mathcal{C}}$. Now indeed, for every $m \in \mathcal{M}$ the diagram

$$
\begin{array}{ccc}
H(m) \triangleleft c & \xrightarrow{\tilde{q}_{H(m)} \circ c} & S^\mathcal{N}_\mathcal{N}(H(m) \triangleleft c) \\
& & \downarrow \text{(5.16)} \\
& \xrightarrow{\tilde{q}_{H(m)} \circ \text{id}} & S^\mathcal{N}_\mathcal{N}(H(m) \triangleleft \text{id})
\end{array}
$$

commutes: the square on the left corresponds to the condition fulfilled by $\tilde{q}$ of being a $\overline{\mathcal{C}}$-pivotal structure for $\mathcal{N}$, while the square on the right commutes owing to naturality of $\overline{\tilde{p}}$.

The claim (ii) follows analogously by considering as $\mathcal{B}$-pivotal structure for $\text{Fun}_A(\mathcal{M}, \mathcal{N})$ the natural isomorphism

$$
\overline{P}_H : \ H \xrightarrow{\tilde{p} \circ \text{id}_H \circ \overline{\tilde{q}}} S^\mathcal{A}_\mathcal{N} \circ H \circ S^\mathcal{B}_\mathcal{M} \cong S^{\text{Fun}_A(\mathcal{M}, \mathcal{N})}_\mathcal{N}(H),
$$

where $\tilde{p} : \text{id}_\mathcal{N} \cong S^\mathcal{A}_\mathcal{N}$ and $\overline{\tilde{q}} : \text{id}_\mathcal{M} \cong S^\mathcal{B}_\mathcal{M}$ are the corresponding pivotal structures of $\mathcal{A}\mathcal{N}$ and $\mathcal{M}\mathcal{B}$.

Assertion (iii) is verified by making the routine check of the diagrams (5.3) and (5.4).

**Remark 5.5.** According to Corollary 2.5 the category of module functors is a model for the relative Deligne product. Therefore Proposition 5.4 implies that the product of two pivotal bimodules inherits a pivotal structure.

### 5.3 Pivotal Morita theory

Given a bicategory $\mathcal{F}$, the existence of dualities for 1-morphisms extends to a pseudo-functor

$$
(-)^\vee : \mathcal{F} \to \mathcal{F}^{\text{op}, \text{op}},
$$

$$
x \mapsto x, \\
(a : x \to y) \mapsto (a^\vee : y \to x).
$$

A **pivotal structure** on a bicategory $\mathcal{F}$ with dualities is a pseudo-natural equivalence

$$
P : \text{id}_\mathcal{F} \xrightarrow{\cong} (-)^{\vee \vee}
$$

obeying $P_x = \text{id}_x$ for every object $x \in \mathcal{F}$. A bicategory together with a pivotal structure is called a **pivotal bicategory**.

**Remark 5.6.** A tensor category $\mathcal{A}$ can be seen as a bicategory with a single object $\mathcal{A}$. The requirement that $P_x = \text{id}_x$ imposed on the pseudo-natural equivalence (5.19) ensures that a pivotal structure on the bicategory $\mathcal{A}$ recovers a pivotal structure on the tensor category $\mathcal{A}$. A pseudo-natural equivalence (5.19) without this requirement corresponds to the notion of a **quasi-pivotal** structure on $\mathcal{A}$ [Sh1, Sec. 4], that is, a pair $(d, \gamma)$ where $d \in \mathcal{A}$ is an invertible object and $\gamma = \{\gamma_d : d \otimes a \xrightarrow{\cong} a^{\vee \vee} \otimes d\}$ is a twisted half-braiding.
Definition 5.7. A Morita context \((A, B, M, N, \odot, \boxdot)\) is said to be pivotal iff its associated bicategory \(\mathcal{M}\) is pivotal.

Next we establish that the Morita context of a pivotal module is indeed pivotal, thus justifying the terminology.

Lemma 5.8. Let \(M\) be a pivotal module category over a pivotal tensor category \(A\). Then we have:

(i) \(M\) has the structure of a pivotal \((A, A^\ast M)\)-bimodule category.

(ii) For every \(A\)-module category \(N\), the \(A^\ast M\)-module category \(\text{Fun}_A(M, N)\) inherits a pivotal structure.

(iii) The functor category \(\text{Fun}_A(M, A)\) has the structure of a pivotal \((A^\ast M, A)\)-bimodule category.

Proof. Denote by \(\tilde{p}: \text{id}_M \sim \Rightarrow S_A^M\) the pivotal structure of \(A^\ast M\). According to Corollary 4.12 we have \(S_A^A \sim S_M^A\), so that for any \(m \in M\) we can define a natural isomorphism

\[
\tilde{q}_m := \tilde{p}^{-1}_{S_M^A(m)} : m \sim \Rightarrow S_A^A(m) \cong S_M^A(m).
\]  

(5.20)

Recall from Proposition 5.2 that \(A^\ast M\) is endowed with the pivotal structure (5.5). In view of Remark 2.12 this pivotal structure coincides with the composition

\[
q_F : F \xrightarrow{\tilde{q} \circ \text{id}} S_M^A \circ F \xrightarrow{2.39} F^{\text{lla}} \circ S_M^A \xrightarrow{\text{id} \circ \tilde{q}^{-1}} F^{\text{lla}}.
\]  

(5.21)

Now we verify that \(\tilde{q}\) is compatible with this pivotal structure, i.e. that the diagram

\[
\begin{array}{ccc}
F(m) & \xrightarrow{\tilde{q}_F(m)} & S_M^A \circ F(m) \\
\downarrow{q_F \circ q_m} & & \downarrow{2.39} \\
F^{\text{lla}} \circ S_M^A(m) & & \\
\end{array}
\]  

(5.22)

commutes for every \(F \in A^\ast_M\) and \(m \in M\). In fact, by invoking the relevant definitions, the diagram translates to

\[
\begin{array}{ccc}
F(m) & \xrightarrow{\tilde{q}_F(m)} & S_M^A \circ F(m) \\
\downarrow{\tilde{q}_F(m)} & & \downarrow{2.39} \\
S_M^A \circ F(m) & \xrightarrow{2.39} & F^{\text{lla}} \circ S_M^A(m) \\
\downarrow{\text{id} \circ \tilde{q}_m} & & \downarrow{\text{id} \circ \tilde{q}_m} \\
F^{\text{lla}} \circ S_M^A(m) & & \\
\end{array}
\]  

(5.23)

which commutes trivially. Thus it is established that \(A^\ast_M\mathcal{M}\) is a pivotal bimodule category.

Statement (ii) follows from (i) and Proposition 5.4(ii). Explicitly, the pivotal structure is the composite

\[
\hat{p}_H : H \xrightarrow{\tilde{N} \circ \text{id}} S_N^A \circ H \xrightarrow{2.39} H^{\text{tra}} \circ S_M^A \xrightarrow{\text{id} \circ (\tilde{p}_M)^{-1}} H^{\text{tra}} \cong S_{\mathcal{F}^M_{\text{Fun}_A(M, N)}}^A(H)
\]  

(5.24)
for a module functor $H \in \text{Fun}_A(M, \mathcal{N})$.

Similarly, claim (iii) follows from (i) and Proposition 5.4(iii) by considering $\mathcal{A} \mathcal{N}_A = A$ as the regular bimodule category. In this situation the pivotal structures are explicitly given by

$$
\hat{p}_H : H \xrightarrow{p \circ \text{id}} (-)^{\vee \vee} \circ H \xrightarrow{(2.36)} H^{\text{rra}} \circ S^A_M \xrightarrow{\text{id} \circ \hat{p}^{-1}} H^{\text{rra}} \cong S^{A^*_M}_{\text{Fun}_A(M, A)}(H) \quad (5.25)
$$

and

$$
\hat{q}_H : H \xrightarrow{\text{id} \circ \tilde{p}} H \circ S^A_M \xrightarrow{(2.36)} (-)^{\vee \vee} \circ H^{\text{lla}} \xrightarrow{p^{-1} \circ \text{id}} H^{\text{lla}} \cong S^{A^*_M}_{\text{Fun}_A(M, A)}(H) \quad (5.26)
$$

for a module functor $H \in \text{Fun}_A(M, A)$.

**Theorem 5.9.** For $\mathcal{M}$ a pivotal module category over a pivotal tensor category $\mathcal{A}$, its Morita context $(\mathcal{A}, \overline{\mathcal{A}}^*_M, \mathcal{M}, \text{Fun}_A(M, A), \circ, \sqot)$ is a pivotal Morita context.

**Proof.** Let $\mathcal{M}$ be the bicategory associated to the Morita context of $\mathcal{M}$. We need to construct a pseudo-natural equivalence $P: \text{id}_{\mathcal{M}} \overset{\cong}{\rightarrow} (-)^{\vee \vee}$, subject to the condition that the components of $P$ on any object is the identity. For any 1-morphism $a \in \mathcal{A}$ and any $F \in \overline{\mathcal{A}}^*_M$, $m \in \mathcal{M}$ and $H \in \text{Fun}_A(M, A)$ we define

(i) $P_a : a \overset{\cong}{\rightarrow} a^{\vee \vee}$ as the pivotal structure $p_a$ of $\mathcal{A}$;

(ii) $P_F : F \overset{\cong}{\rightarrow} F^{\text{rra}}$ as the isomorphism $\overline{p}_F$ in (5.6) which serves as a pivotal structure of $\mathcal{A}^*_M$;

(iii) $P_m : m \overset{\cong}{\rightarrow} S^A_M(m)$ as the pivotal structure $\tilde{p}_m$ of the module $\mathcal{A}\mathcal{M}$;

(iv) $P_H : H \overset{\cong}{\rightarrow} H^{\text{rra}}$ as the isomorphism $\hat{p}_H$ in (5.25) which serves as a pivotal structure of the module $\overline{\mathcal{A}}^*_M\text{Fun}_A(M, A)$.

These assignments are natural for 2-morphisms in $\mathcal{M}$. The compatibility with composition of 1-morphisms reduces to the commutativity of the diagram

$$
\begin{array}{ccc}
  s \circ t & \overset{P_{s \circ t}}{\cong} & (s \circ t)^{\vee \vee} \\
  s^{\vee \vee} \circ t^{\vee \vee} & \xrightarrow{P_s \circ P_t} & (s \circ t)^{\vee \vee} \\
  s^{\vee \vee} \circ t^{\vee \vee} & \xrightarrow{\sim} & (s \circ t)^{\vee \vee}
\end{array}
$$

(5.27)

for $s$ and $t$ composable 1-morphisms in $\mathcal{M}$. These translate into the following eight conditions, which are indeed all satisfied:

The tensor categories $\mathcal{A}$ and $\overline{\mathcal{A}}^*_M$ are pivotal, i.e.:

(i) the monoidality condition of $p: \text{id}_A \overset{\cong}{\rightarrow} (-)^{\vee \vee}$ in the case $s = a$ and $t = b$ in $\mathcal{A}$;

(ii) the monoidality condition of $\overline{p}: \text{id}_{\mathcal{A}^*_M} \overset{\cong}{\rightarrow} (-)^{\text{rra}}$ in the case $s = F_1$ and $t = F_2$ in $\overline{\mathcal{A}}^*_M$.

Further, the bimodules $\mathcal{M}$ and $\text{Fun}_A(M, A)$ are pivotal ($\tilde{p}$ and $\hat{p}$ are bimodule natural transformations), i.e.
(iii) in case $s = a \in A$ and $t = m \in M$, the condition (5.2) fulfilled by $\tilde{p}$;
(iv) for $s = m \in M$ and $t = F \in \overline{A_M}$, the condition (5.3) fulfilled by $\tilde{p}$;
(v) in case $s = F \in \overline{A_M}$ and $t = \text{Fun}_A(M, A)$, the condition (5.2) fulfilled by $\hat{p}$;
(vi) for $s = H \in \text{Fun}_A(M, A)$ and $t = a \in A$, the condition (5.3) fulfilled by $\hat{p}$.

Finally, two additional conditions involving the mixed products:
(vii) for $m \in M$ and $H \in \text{Fun}_A(M, A)$ the commutativity of the diagram
\[
\begin{array}{ccc}
m \odot H & \xrightarrow{p_{m \odot H}} & (m \odot H)^{\vee \vee} \\
\tau_m \odot \tilde{p}_H & \swarrow & \searrow \\
S_A^A(m) \odot H^{\text{tra}} & \end{array}
\]  
which can be rewritten in the more explicit form
\[
\begin{array}{cccc}
H(m) & \xrightarrow{p_{H(m)}} & H(m)^{\vee \vee} \\
\downarrow \scriptstyle{p \circ \text{id}} & & \downarrow \scriptstyle{\tilde{p}_H} \\
(-)^{\vee \vee} \circ H(m) & \xrightarrow{\hat{p}_H m} & (H(m)^{\vee \vee})^{\text{tra}} \\
\downarrow\scriptstyle{H^{\text{tra}} \circ S_A^A(m)} & \downarrow\scriptstyle{id \circ \tilde{p}_m} & \downarrow\scriptstyle{id \circ p_m} \\
H^{\text{tra}} \circ S_A^A(m) & \xrightarrow{id} & H^{\text{tra}} \circ S_A^A(m)
\end{array}
\]  
which trivially commutes;
(viii) similarly, for $m \in M$ and $H \in \text{Fun}_A(M, A)$ the commutativity of the diagram
\[
\begin{array}{ccc}
H \boxdot m & \xrightarrow{\tau_{H \boxdot m}} & (H \boxdot m)^{\text{tra}} \\
\tilde{p}_H \boxdot \tilde{p}_m & \searrow \scriptstyle{\approx} & \\
H^{\text{tra}} \boxdot S_A^A(m) & \end{array}
\]  
which is the same as the diagram
\[
\begin{array}{ccc}
(- \triangleright m) \circ H & \xrightarrow{\tilde{p} \circ \text{id}} & S_A^A \circ (- \triangleright m) \circ H \\
\approx & & \approx \\
(- \triangleright S_A^A(m)) \circ (-)^{\vee \vee} \circ H & \xrightarrow{\approx} & (- \triangleright S_A^A(m)) \circ H^{\text{tra}} \circ S_A^A \\
\xrightarrow{\approx} & & \xrightarrow{\approx}
\end{array}
\]  
which is commutative: the left triangle corresponds to the condition of $\tilde{p}$ being a pivotal structure for $\mathcal{M}$, the rightmost square commutes due to naturality, and the square in the middle is the compatibility (2.37).
This shows that $P : \text{id}_M \xrightarrow{\sim} (-)^{\vee\vee}$ is a pivotal structure on the bicategory $\mathcal{M}$, and thus the claim is proven. \hfill $\square$

**Definition 5.10.** Two pivotal tensor categories $\mathcal{A}$ and $\mathcal{B}$ are said to be **pivotal Morita equivalent** iff there exists a pivotal $\mathcal{A}$-module category $\mathcal{M}$ together with a pivotal equivalence $\mathcal{B} \cong \mathcal{A}^\mathcal{M}$.

**Proposition 5.11.** Let $\mathcal{A}$ be a pivotal category and $\mathcal{M}$ a pivotal $\mathcal{A}$-module.

(i) The tensor equivalence

$$\mathcal{A} \cong \mathcal{A}^\mathcal{A} \text{,} \quad a \mapsto - \otimes a$$

from [EGNO, Ex. 7.12.3] is pivotal.

(ii) The canonical tensor equivalence

$$\text{can : } \mathcal{A} \cong (\mathcal{A}^\mathcal{M})^\mathcal{M} \text{,} \quad a \mapsto a hd -$$

from [EGNO, Thm. 7.12.11] is pivotal.

**Proof.**

(i) The pivotal structure for the functor $- \otimes a$ in $\mathcal{A}^\mathcal{A}$ is given by

$$q_{- \otimes a} : (- \otimes a) \xrightarrow{p_{- \otimes a}} (- \otimes a)^{\vee\vee} \xrightarrow{p^{-1} \otimes \text{id}} (- \otimes a)^{\vee\vee} \xrightarrow{\text{id} \otimes p} a^{\vee\vee} \xrightarrow{\text{id} \otimes p^{-1}} (a^{\vee\vee} \triangleright -) \xrightarrow{\text{can}} (- a^{\vee\vee}) \text{.}$$

But since $p : \text{id}_\mathcal{A} \xrightarrow{\sim} (-)^{\vee\vee}$ is monoidal, we have $q_{- \otimes a} = \text{id}_- \otimes p_a$.

(ii) According to Corollary [4.12] there is an isomorphism $\mathcal{S}_{\mathcal{M}}^{\mathcal{A}^\mathcal{M}} \cong \mathcal{S}_{\mathcal{M}}^\mathcal{A}$. Taking this into consideration the pivotal structure on $(\mathcal{A}^\mathcal{M})^\mathcal{M}$ is given by

$$\overline{\eta}_{a \triangleright -} : (a \triangleright -) \xrightarrow{\overline{\eta} \circ \text{id}} \mathcal{S}_\mathcal{M}^\mathcal{A} (a \triangleright -) \xrightarrow{\mathcal{S}_\mathcal{M}^\mathcal{A}} a^{\vee\vee} \triangleright \mathcal{S}_\mathcal{M}^\mathcal{A} \xrightarrow{\text{id} \circ \overline{\eta}^{-1}} (a^{\vee\vee} \triangleright -) = (a \triangleright -)^{\text{id}} \text{.}$$

But in a similar manner the defining condition (5.2) of $\overline{\eta}$ being a pivotal structure for $\mathcal{M}$ implies that $\text{can}(p_a) = \overline{\eta}_{a \triangleright -}$.

\hfill $\square$

### 5.4 The bicategory of pivotal modules

Pivotal modules over a pivotal tensor category form a pivotal bicategory, as presented in [Sc, Def. 5.2] in terms of inner-product module categories. Here we express this fact in the language of relative Serre functors. Let $\mathcal{A}$ be a pivotal tensor category, and denote by $\text{Mod}^{\text{piv}}(\mathcal{A})$ the 2-category that has pivotal $\mathcal{A}$-module categories as objects, $\mathcal{A}$-module functors as 1-morphisms and module natural transformations as 2-morphisms. Since pivotal modules are exact [FSS, Prop. 4.24], every module functor $H : \mathcal{A}\mathcal{N}_1 \rightarrow \mathcal{A}\mathcal{N}_2$ comes with adjoints

$$H^* := H^{\text{id}} : \mathcal{A}\mathcal{N}_2 \rightarrow \mathcal{A}\mathcal{N}_1 \quad \text{and} \quad H^* := H^\text{ra} : \mathcal{A}\mathcal{N}_2 \rightarrow \mathcal{A}\mathcal{N}_1 \text{.}$$
These turn $\text{Mod}^{\text{piv}}(A)$ into a bicategory with dualities for 1-morphisms. Moreover, $\text{Mod}^{\text{piv}}(A)$ is endowed with a pivotal structure (5.19). Indeed, given any 1-morphism $H: \mathcal{A} N_1 \to \mathcal{A} N_2$ in $\text{Mod}^{\text{piv}}(A)$, define

$$P_H: \quad H \xrightarrow{id \circ \tilde{p}_i} H \circ S_{N_1}^{A} \xrightarrow{2.36} S_{N_2}^{A} \circ H^{\text{lla}} \xrightarrow{(\tilde{p}_2)^{-1} \circ id} H^{\text{lla}}, \quad (5.37)$$

where $\tilde{p}_i$ are the pivotal structures of the module categories $\mathcal{N}_i$. The 2-morphisms $P_H$ are invertible and natural in $H$. Moreover, (2.37) implies that they are compatible with the composition of module functors. Therefore $P$ constitutes a pivotal structure on the 2-category $\text{Mod}^{\text{piv}}(A)$.

**Theorem 5.12.** Two pivotal tensor categories $A$ and $B$ are pivotal Morita equivalent if and only if $\text{Mod}^{\text{piv}}(A)$ and $\text{Mod}^{\text{piv}}(B)$ are equivalent as pivotal bicategories.

**Proof.** Given a pivotal $A$-module $\mathcal{M}$, according to Lemma 5.8(ii) for every $\mathcal{N} \in \text{Mod}^{\text{piv}}(A)$ the $\overline{A}_\mathcal{M}$-module $\text{Fun}_A(\mathcal{M}, \mathcal{N})$ is endowed with a pivotal structure. By Theorem 7.12.16 of [EGNO], this assignment extends to a 2-equivalence

$$\Psi: \quad \text{Mod}^{\text{piv}}(A) \longrightarrow \text{Mod}^{\text{piv}}(\overline{A}_\mathcal{M}), \quad \mathcal{N} \longmapsto \text{Fun}_A(\mathcal{M}, \mathcal{N}). \quad (5.38)$$

Moreover, $\Psi$ preserves the pivotal structure: To a 1-morphism $H: \mathcal{A} N_1 \to \mathcal{A} N_2$ in $\text{Mod}^{\text{piv}}(A)$ it assigns

$$\Psi(H): \quad \text{Fun}_A(\mathcal{M}, N_1) \xrightarrow{H \circ -} \text{Fun}_A(\mathcal{M}, N_2). \quad (5.39)$$

The component at $\Psi(H)$ of the pivotal structure of $\text{Mod}^{\text{piv}}(\overline{A}_\mathcal{M})$ is the composite

$$P_{\Psi(H)}: \quad (H \circ -) \xrightarrow{id \circ \tilde{p}_i} H \circ (-)^{\text{ra}} \xrightarrow{\cong} (H^{\text{lla}} \circ -)^{\text{ra}} \xrightarrow{(\tilde{p}_2)^{-1} \circ id} (H^{\text{lla}} \circ -), \quad (5.40)$$

where, for $i = 1, 2$, $\tilde{p}_i$ is the pivotal structure of the module category $\text{Fun}_A(\mathcal{M}, \mathcal{N}_i)$ given by (5.21). The task at hand is to verify that the diagram

$$\Psi(H) \xrightarrow{P_{\Psi(H)}} \Psi(H)^{\text{lla}} \xrightarrow{\cong} \Psi(H)$$

commutes for every 1-morphism $H$ in $\text{Mod}^{\text{piv}}(A)$. By inserting the definitions, this diagram translates to

$$\begin{align*}
(H \circ -) \xrightarrow{id \circ \tilde{p}_N \circ id} (H \circ S_{N_1}^{A} \circ -) & \xrightarrow{\cong} H \circ (-)^{\text{ra}} \circ S_{\mathcal{M}}^{A} \xrightarrow{id \circ (\tilde{p}_M)^{-1}} H \circ (-)^{\text{ra}} \\
\text{id} \circ \tilde{p}_N \circ \text{id} & \xrightarrow{id} \text{id} \\
(H \circ S_{N_1}^{A} \circ -) & \xrightarrow{\cong} (H^{\text{lla}} \circ -)^{\text{ra}} \\
(S_{N_2}^{A} \circ H^{\text{lla}} \circ -) & \xrightarrow{\tilde{p}_N \circ \text{id}} (H^{\text{lla}} \circ -)^{\text{ra}} \xrightarrow{\cong} (H^{\text{lla}} \circ -)^{\text{ra}} \circ S_{\mathcal{M}}^{A}
\end{align*} \quad (5.42)$$
This is indeed a commutative diagram: The pentagon in the middle commutes owing to the compatibility \([2.37]\), and the triangle and squares in the periphery commute trivially.

To see the converse implication, consider a pivotal 2-equivalence

\[
\Phi : \text{Mod}^{\text{piv}}(B) \xrightarrow{\sim} \text{Mod}^{\text{piv}}(A).
\]

Define \(\mathcal{M}\) as the image of the regular pivotal module \(\mathcal{B}\) under \(\Phi\). Since \(\Phi\) is a 2-equivalence, we obtain an equivalence

\[
\Phi : \mathcal{B}^* = \text{Fun}_B(\mathcal{B}, \mathcal{B}) \xrightarrow{\sim} \text{Fun}_A(\mathcal{M}, \mathcal{M}) = \mathcal{A}_\mathcal{M}^*
\]

of categories. In addition, amongst the data of the 2-functor \(\Phi\) there is a natural isomorphism

\[
\text{Fun}_B(\mathcal{B}, \mathcal{B}) \times \text{Fun}_B(\mathcal{B}, \mathcal{B}) \xrightarrow{\circ} \text{Fun}_A(\mathcal{M}, \mathcal{M})
\]

whereby \(\Phi_{B,B}\) is endowed with a tensor structure. Furthermore, \(\Phi_{B,B}\) is pivotal. To see this, notice that an object \(F \in \mathcal{B}_\mathcal{M}^*\) is a 1-morphism in \(\text{Mod}^{\text{piv}}(A)\). Now on the one hand the pivotal structure \([5.5]\) of the tensor category \(\mathcal{B}_\mathcal{M}^*\) provides an isomorphism \(q_F : F \xrightarrow{\sim} F_{\text{lla}}\), while on the other hand the pivotal structure \([5.37]\) in the bicategory \(\text{Mod}^{\text{piv}}(A)\) for \(F\) is an isomorphism \(P_F : F \xrightarrow{\sim} F_{\text{lla}}\), and in fact it coincides with \(q_F\). The same argument holds for \(\mathcal{B}^*\); since \(\Phi\) preserves \(P\), it follows that \(\Phi_{B,B}\) is a pivotal equivalence. We have thus obtained an equivalence

\[
\mathcal{B} \xrightarrow{\Phi_{B,B}} \mathcal{B}_\mathcal{M}^* \xrightarrow{\Phi_{B,B}} \mathcal{A}_\mathcal{M}^*
\]

of pivotal tensor categories; hence \(A\) and \(B\) are pivotal Morita equivalent.

\[\square\]

Remark 5.13. Theorem 5.12 implies that pivotal Morita equivalence is indeed an equivalence relation on pivotal tensor categories. Proposition 5.11 already shows reflexivity and symmetry.

5.5 Pivotality of the center and pivotal Morita equivalence

Recall that the Drinfeld center of a tensor category \(A\) is the braided tensor category \(\mathcal{Z}(A)\) whose objects are pairs \((a, \sigma)\) consisting of an object \(a \in A\) and a half-braiding \(\sigma\), i.e. a natural isomorphism \(\sigma_{b,a} : b \otimes a \xrightarrow{\sim} a \otimes b\) for \(b \in A\) obeying the appropriate hexagon axiom. According to [EGNO, Prop. 8.10.10] the Drinfeld center \(\mathcal{Z}(A)\) is unimodular. A pivotal structure \(p_a : a \xrightarrow{\sim} a^{\text{viv}}\) on \(A\) induces a pivotal structure on \(\mathcal{Z}(A)\) via \(p_{(a,\sigma)} := p_a\) [EGNO, Ex. 7.13.6].

The following result relates the Drinfeld center and module categories: Given a module category \(\mathcal{M}\) over a tensor category \(A\) there is a braided equivalence \(\mathcal{Z}(A) \simeq \mathcal{Z}(\overline{A}_\mathcal{M})\). Explicitly it can be given, without making a choice of an algebra in \(A\), as [Sh3, Thm. 3.13]

\[
\Sigma : \mathcal{Z}(A) \xrightarrow{\sim} \mathcal{Z}(\overline{A}_\mathcal{M}), \quad (a, \sigma) \longmapsto (a \triangleright -, \gamma),
\]

where \(\sigma\) endows \(a \triangleright -\) with an \(A\)-module functor structure, and for \(F \in \overline{A}_\mathcal{M}\) the half-braiding \(\gamma_{F,a\triangleright -} : a \triangleright F \xrightarrow{\sim} F(a \triangleright -)\) is given by the module functor structure of \(F\).
Lemma 5.14. Let $A$ be a finite tensor category and $M$ an $A$-module, and let $(a, \sigma) \in \mathcal{Z}(A)$.

(i) There is a natural isomorphism $\iota_a : a \triangleright \mapsto a \triangleright$ in $\mathcal{Z}(A)$.

(ii) The diagram

\[
\begin{array}{ccc}
a \triangleright S^A_M & \xrightarrow{\iota_a \triangleright \text{id}} & \triangleright a \triangleright S^A_M \\
\downarrow & & \downarrow \\
S^A_M(a \triangleright -) & & S^A_M(a \triangleright -)
\end{array}
\]  

commutes, where the natural isomorphism $(2.36)$ is applied to the module functor $a \triangleright -$.

Proof. The desired isomorphism in (i) comes from the composition

\[
a \triangleright \otimes D_A \xrightarrow{2.3} D_A \otimes \triangleright a \xrightarrow{\triangleright \sigma} \triangleright a \otimes D_A.
\]

Now denote by $F_a := a \triangleright -$ the module functor induced by $(a, \sigma)$. Its double adjoints are $F^\text{dla}_a = a \triangleright -$ and $F^\text{tra}_a = \triangleright a \triangleright -$. Assertion (ii) is thus implied by the commutativity of the diagram

\[
\begin{array}{ccc}
a \triangleright S^A_M & \xrightarrow{\iota_a \triangleright \text{id}} & \triangleright a \triangleright S^A_M \\
\downarrow & & \downarrow \\
S^A_M(a \triangleright -) & & S^A_M(a \triangleright -)
\end{array}
\]

where $(2.42)$ and $(2.36)$ are applied to the functor $F_a$. The pentagon at the top of this diagram commutes owing to naturality and the definition of $\iota_a$. Commutativity of the pentagon on the left is the condition that $(2.44)$ is an isomorphism of twisted module functors. Similarly, the pentagon on the right is secretly the diagram

\[
\begin{array}{ccc}
F^\text{tra}_a \otimes D_A \triangleright N^r_M & \xrightarrow{2.44} & F^\text{tra}_a \otimes S^A_M \\
\downarrow & & \downarrow \\
D_A \triangleright N^r_M \otimes F_a & & S^A_M \circ F_a
\end{array}
\]

which again commutes because $(2.44)$ is an isomorphism of twisted bimodule functors.

Proposition 5.15. For any pivotal $A$-module category $M$ the braided equivalence $\Sigma$ between $\mathcal{Z}(A)$ and $\mathcal{Z}(\overline{A_M})$ defined in $(5.47)$ is pivotal.
Proof. From Lemma 5.14 it follows that the pivotal structure in \( Z(\mathcal{A}_M) \) for the object \( \Sigma(a, \sigma) \) is given by the composite
\[
\begin{array}{ccc}
a \triangleright - & \overset{id \circ \tilde{p}}{\longrightarrow} & S_M^A(a \triangleright -) \overset{2.31}{\longrightarrow} a^\vee \triangleright S_M^A \overset{id \circ (\tilde{p})^{-1}}{\longrightarrow} (a^\vee \triangleright -) .
\end{array}
\]

The condition (5.2) on \( \tilde{p} \) means that \( \Sigma(p_a) \) coincides with the morphism (5.52), and thus the assertion holds.

Proposition 5.15 immediately implies

**Theorem 5.16.** If two pivotal categories \( \mathcal{A} \) and \( \mathcal{B} \) are pivotal Morita equivalent, then their Drinfeld centers \( Z(\mathcal{A}) \) and \( Z(\mathcal{B}) \) are equivalent as pivotal braided tensor categories.

### 5.6 Sphericality of module categories

A notion of sphericality for pivotal tensor categories that is defined through the Radford isomorphism (2.3) has been studied, under the assumption of unimodularity, in [DSS2]. In the semisimple case this notion is equivalent to trace-sphericity [DSS2, Prop. 3.5.4], i.e. to the property that right and left traces coincide. If \( \mathcal{A} \) is a unimodular finite tensor category, then (2.3) becomes a monoidal isomorphism
\[
\begin{array}{ccc}
\tau_A : & id_A & \overset{\cong}{\longrightarrow} & (-)^{\vee \vee \vee \vee} .
\end{array}
\]

**Definition 5.17.** [DSS2, Def. 3.5.2] A unimodular pivotal tensor category \( \mathcal{A} \) is called spherical iff the diagram
\[
\begin{array}{ccc}
id_A & \tau_A & \times \ \ (-)^{\vee \vee \vee \vee} \\
\downarrow \tilde{p} & \downarrow \rho & \downarrow \rho^\vee \vee \vee \\
(-)^{\vee \vee \vee \vee} & & \end{array}
\]

commutes, where \( \tilde{p} \) is the pivotal structure of \( \mathcal{A} \).

We explore sphericality for module categories in a similar vein. Let \( \mathcal{M} \) be a pivotal module category over a unimodular pivotal category \( \mathcal{A} \). According to Proposition 5.2(ii), \( \mathcal{A}_M^* \) is also unimodular. Then (4.50) becomes an \( \mathcal{A} \)-module isomorphism
\[
\begin{array}{ccc}
\tau_M : & id_M & \overset{\cong}{\longrightarrow} & S_M^A \circ S_M^A ,
\end{array}
\]
where \( S_M^A \circ S_M^A \) is considered to be a module functor via \( \tau_A \).

**Definition 5.18.** A pivotal module \( \mathcal{M} \) over a spherical category \( \mathcal{A} \) is called spherical iff the diagram
\[
\begin{array}{ccc}
id_M & \tau_M & \times \ \ S_M^A \circ S_M^A \\
\downarrow \tilde{p} & \downarrow id \circ \tilde{p} & \downarrow id \circ \tilde{p} \\
S_M^A \circ S_M^A & & \end{array}
\]

commutes, where \( \tilde{p} \) is the pivotal structure of \( \mathcal{M} \).
Remark 5.19. The reason for requiring $\mathcal{A}$ in Definition 5.18 to be spherical is as follows. The triangle (5.56) is a diagram of module natural isomorphisms, hence the module structures on the functors involved are significant. Now $S_{\mathcal{M}}^A \circ S_{\mathcal{M}}^A$ can be regarded as a module functor in two ways, by untwisting its module structure either via $r_A$ or via $p^\vee \circ p$. Sphericality of $\mathcal{A}$ precisely means that the resulting two module functors coincide.

Proposition 5.20. Let $\mathcal{M}$ be a spherical $\mathcal{A}$-module category. Then the following hold:

(i) The pivotal structure (5.5) on the dual tensor category $\mathcal{A}_\mathcal{M}^\ast$ is spherical.

(ii) $\mathcal{M}$ is a spherical $\overline{\mathcal{A}}_{\mathcal{M}}^\ast$-module category.

(iii) $\text{Fun}_A(\mathcal{M}, \mathcal{A})$ is a spherical $(\overline{\mathcal{A}}_{\mathcal{M}}, \mathcal{A})$-bimodule category.

Proof. To prove (i) we need to check the commutativity of the triangle

$$
\begin{array}{ccc}
\text{id}_{\mathcal{A}_\mathcal{M}^\ast} & \xrightarrow{\text{id} \circ r_{\mathcal{M}}^\ast} & (-)^{\text{lla}} \\
\downarrow q & & \downarrow \text{id} \circ \tilde{p} \circ \text{id} \\
(-)^{\text{lla}} & & (-)^{\text{lla}} \\
\end{array}
$$

where $q$ is the pivotal structure of the dual tensor category given by the isomorphism (5.5). Inserting the definitions, this diagram reads, for $F \in \mathcal{A}_\mathcal{M}^\ast$,

$$
\begin{array}{c}
F \xrightarrow{\text{id} \circ \tilde{p}^\ast \circ \text{id}} F \circ S_{\mathcal{M}}^A \circ S_{\mathcal{M}}^A \xrightarrow{\text{id} \circ \tilde{p}^\ast \circ \text{id}} F \circ S_{\mathcal{M}}^A \circ S_{\mathcal{M}}^A \xrightarrow{\tilde{p}^{-1} \circ \text{id}} F \circ S_{\mathcal{M}}^A \circ S_{\mathcal{M}}^A \xrightarrow{\text{id} \circ \tilde{p}^\ast \circ \text{id}} F \circ S_{\mathcal{M}}^A \circ S_{\mathcal{M}}^A \xrightarrow{\text{id} \circ \tilde{p}^\ast \circ \text{id}} F \circ S_{\mathcal{M}}^A \circ S_{\mathcal{M}}^A \xrightarrow{\text{id} \circ \tilde{p}^\ast \circ \text{id}} \\
\end{array}
$$

This diagram indeed commutes: the upper-left and upper-right triangles commute because $\mathcal{M}$ is spherical (diagram (5.56) commutes); the remaining squares commute due to functoriality of functor composition.

Assertion (ii) follows immediately from the commutativity of (5.56), once one recalls from Corollary 4.12 that $S_{\mathcal{M}}^{\ast \mathcal{M}} \cong S_{\mathcal{M}}^A$.

Statement (iii) follows analogously by checking that, for $H \in \text{Fun}_A(\mathcal{M}, \mathcal{A})$, the diagram

$$
\begin{array}{c}
H \xrightarrow{\text{id} \circ \tilde{p}^\ast} H \circ S_{\mathcal{M}}^A \circ S_{\mathcal{M}}^A \xrightarrow{\text{id} \circ \tilde{p}^\ast} \xrightarrow{\text{id} \circ \tilde{p}^\ast} H \circ S_{\mathcal{M}}^A \circ S_{\mathcal{M}}^A \xrightarrow{\text{id} \circ \tilde{p}^\ast} \xrightarrow{\text{id} \circ \tilde{p}^\ast} \xrightarrow{\text{id} \circ \tilde{p}^\ast} \xrightarrow{\text{id} \circ \tilde{p}^\ast} \xrightarrow{\text{id} \circ \tilde{p}^\ast} \xrightarrow{\text{id} \circ \tilde{p}^\ast} \\
\end{array}
$$

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commutes. Once more this is the case: the upper-left triangle is the sphericality of $\mathcal{M}$; the upper-right triangle commutes because $\mathcal{A}$ is spherical; and the remaining squares commute due to functoriality of functor composition. This establishes that $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$ is spherical as an $\mathcal{A}$-module category. Sphericality of $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$ as an $\mathcal{A}^\ast$-module category follows in a similar manner. 

**Remark 5.21.** Given a unimodular spherical tensor category $\mathcal{A}$, the sphericality condition on a pivotal module category $\mathcal{A}\mathcal{M}$ can be interpreted in the bicategorical setting as the statement that the pivotal structure of $\mathcal{M}$ squares to the Radford pseudo-natural equivalence (4.52). Beyond unimodularity, the non-triviality of the distinguished invertible objects in (4.52) suggests that such a bicategorical interpretation could be studied in the framework of quasi-pivotal structures, as considered in Remark 5.6.

### 6 Dualities and pivotality for equivariant Morita Theory

In [GJS] the notion of categorical Morita equivalence has been extended to finite tensor categories graded by a finite group $G$. The intention of the present section is to study the interaction of pivotal structures and Morita theory in the equivariant setting.

#### 6.1 Graded module categories and graded Morita theory

Let $G$ be a finite group. A $G$-grading on a tensor category $\mathcal{A}$ consists of a decomposition

$$\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

into a direct sum of full abelian subcategories such that for $g, h \in G$ the tensor product restricts to a bifunctor $\otimes: \mathcal{A}_g \times \mathcal{A}_h \rightarrow \mathcal{A}_{gh}$. If $\mathcal{A}_g \neq 0$ for every $g \in G$ the $G$-grading is called faithful. We will only consider faithful gradings. A *G-graded module category over $\mathcal{A}$* is an $\mathcal{A}$-module category with a decomposition

$$\mathcal{M} = \bigoplus_{g \in G} \mathcal{M}_g$$

into a direct sum of full abelian subcategories, with $\mathcal{M}_g \neq 0$ for every $g \in G$, and such that for $g, h \in G$ the $\mathcal{A}$-action restricts to $\triangleright: \mathcal{A}_g \times \mathcal{M}_h \rightarrow \mathcal{M}_{gh}$. Given $G$-graded module categories $\mathcal{M}$ and $\mathcal{N}$, the category of module functors decomposes as [GJS, Prop. 4.8]

$$\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{N}) = \bigoplus_{g \in G} \text{Fun}_\mathcal{A}(\mathcal{M}_g, \mathcal{N})_g,$$

where $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{N})_g$ is the category of *homogeneous module functors* of degree $g$, i.e. module functors $H: \mathcal{M} \rightarrow \mathcal{N}$ satisfying $H(\mathcal{M}_x) \subseteq \mathcal{N}_{xg}$ for every $x \in G$. A grading preserving module functor is a homogeneous module functor of trivial degree. The decomposition (6.2) turns $\mathcal{A}_\mathcal{M}$ into a $G$-graded tensor category and $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{N})$ into a $G$-graded $\mathcal{A}_\mathcal{M}$-module category.

**Definition 6.1.** [GJS, Def. 4.10] Two $G$-graded tensor categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *graded Morita equivalent* iff there exists a $G$-graded $\mathcal{A}$-module category $\mathcal{M}$ together with a $G$-graded tensor equivalence $\mathcal{B} \simeq \mathcal{A}_\mathcal{M}^\ast$. 

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One expects that the equivalence data for the notion of graded Morita equivalence are endowed with a graded structure in a compatible manner. That is, the categories in the Morita context of a $G$-graded module category should be graded and be related via grading preserving actions. Indeed we have

**Proposition 6.2.** Let $\mathcal{A}$ be a $G$-graded tensor category and $\mathcal{M}$ a $G$-graded $\mathcal{A}$-module category and consider $(\mathcal{A}, \overline{\mathcal{A}}, \mathcal{M}, \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A}), \odot, \boxdot)$ the Morita context associated to it.

(i) $\overline{\mathcal{A}}$ is a $G$-graded tensor category.

(ii) $\mathcal{M}$ and $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$ are $G$-graded bimodule categories.

(iii) The mixed products $\odot$ and $\boxdot$ are compatible with the group law of $G$.

**Proof.** That $\overline{\mathcal{A}}$ is a $G$-graded tensor category and $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$ is a $G$-graded $\overline{\mathcal{A}}$-module is part of the statement in Proposition 4.8 of [GJS]. The remaining assertions follow from the fact that, by the definition of the gradings, we have

\[
m \odot H = H(m) \in \mathcal{M}_g,
\]

\[
H \Box m = H(-) \rhd m \in \left(\overline{\mathcal{A}}\right)_g
\]

for all $m \in \mathcal{M}_g$, $a \in \mathcal{A}_h$, $H \in \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})_g$, and $F \in \left(\overline{\mathcal{A}}\right)_g$. □

**Remark 6.3.** Proposition 6.2 can be seen as a statement about the bicategory $\mathcal{M}$ associated to the Morita context of an exact $G$-graded module category: $\mathcal{M}$ is a bicategory enriched in the (non-symmetric) monoidal 2-category of $G$-graded linear abelian categories and grading preserving functors.

A feature of a $G$-graded tensor category is that the duals of a homogeneous object are again homogeneous, with inverse degree. This holds for the Morita context of an exact $G$-graded module category as well:

**Proposition 6.4.** Let $\mathcal{A}$ be a $G$-graded finite tensor category and $\mathcal{M}$ an exact $G$-graded $\mathcal{A}$-module category.

(i) For a homogeneous object $m \in \mathcal{M}_g$, the duals $m^\vee$ and $^\vee m$ are homogeneous of degree $g^{-1}$ in $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$.

(ii) For a homogeneous object $H \in \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})_g$, the duals $H^\vee$ and $^\vee H$ are homogeneous of degree $g^{-1}$ in $\mathcal{M}$.

(iii) The relative Serre functors of $\mathcal{M}$ and of $\text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})$ are grading preserving.

**Proof.** According to [GJS] Prop. 4.2 we have $\text{Hom}_\mathcal{A}(m, n) \in \mathcal{A}_{hg^{-1}}$ for $m \in \mathcal{M}_g$ and $n \in \mathcal{M}_h$. This also implies that $\text{coHom}_\mathcal{M}(m, n) \cong \text{Hom}_\mathcal{M}(n, m) \in \mathcal{A}_{hg^{-1}}$, which proves (i).

To show (ii), notice that the adjoints of a module functor $H \in \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{A})_g$ are homogeneous module functors in $\text{Fun}_\mathcal{A}(\mathcal{A}, \mathcal{M})_{g^{-1}}$. Since $1 \in \mathcal{A}_e$, it follows that $H^\odot(1), H^\boxdot(1) \in \mathcal{M}_{g^{-1}}$.

Assertion (iii) follows from (i) and (ii) together with Proposition 4.11. □

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6.2 De-equivariantization and the equivariant center

For a $G$-graded finite tensor category $\mathcal{A}$ there is a construction called the equivariant center \cite[Sec. 3]{GN} which assigns a braided $G$-crossed tensor category $Z_G(\mathcal{A})$ to $\mathcal{A}$. This is an instance of a procedure known as de-equivariantization, which is well studied \cite[Sect. 4.4]{DGNO}:

Let $D$ be a braided tensor category together with a braided fully faithful functor $\text{Rep}(G) \rightarrow D$. The group $G$ acts by left translations on the set $\text{Fun}(G, k)$ of functions, thereby turning it into an object in $\text{Rep}(G)$. Moreover, this object has a canonical structure of a commutative special Frobenius algebra in $\text{Rep}(G)$. Denote by $L$ the image in $D$ of $\text{Fun}(G, k)$ under the functor $\text{Rep}(G) \rightarrow D$. The de-equivariantization of $D$ is a braided $G$-crossed tensor category $D_G$ whose underlying tensor category is the category of modules $L \text{Mod}(D)$ with tensor product $\otimes_L$.

Now the Drinfeld center of a $G$-graded finite tensor category $\mathcal{A}$ is a braided tensor category endowed with a fully faithful braided functor $\text{Rep}(G) \rightarrow Z(\mathcal{A})$, $(k^n, \rho) \mapsto (1^n, \gamma_{-1^n})$, \eqref{eq:DrinfeldCenter}.

The equivariant center $Z_G(\mathcal{A})$ can be characterized as the de-equivariantization of $Z(\mathcal{A})$, i.e. $Z_G(\mathcal{A}) = L \text{Mod}(Z(\mathcal{A}))$.

**Proposition 6.5.** \cite[Prop. 4.20]{GJS} Let $\mathcal{M}$ be an exact $G$-graded $\mathcal{A}$-module category. The braided equivalence $Z(\mathcal{A}) \simeq Z(\overline{\mathcal{A}_\mathcal{M}})$ given by \eqref{eq:Equivalence} induces, via de-equivariantization, an equivalence $Z_G(\mathcal{A}) \simeq Z_G(\overline{\mathcal{A}_\mathcal{M}})$ \eqref{eq:Deequivariantization} of braided $G$-crossed tensor categories.

6.3 Pivotality in the equivariant picture

We consider now a $G$-graded tensor category $\mathcal{A}$ endowed with a pivotal structure.

**Definition 6.6.** An exact $G$-graded module $\mathcal{M}$ over a $G$-graded pivotal category $\mathcal{A}$ is said to be pivotal iff the underlying module category has the structure of a pivotal module.

**Proposition 6.7.** Let $\mathcal{M}$ be an exact $G$-graded module category over a $G$-graded pivotal tensor category $\mathcal{A}$. A pivotal structure on $\mathcal{A}_\mathcal{M}$ is the same as a pivotal structure on $\mathcal{A}_e \mathcal{M}_e$.

**Proof.** First notice that for $m, n \in \mathcal{M}_e$ we have $\underline{\text{Hom}}^A_M(m, n) = \underline{\text{Hom}}^A_M(m, n)$, and thus the restriction of the relative Serre functor obeys $\underline{S}^A_M|_{\mathcal{M}_e} = \underline{S}^A_{\mathcal{M}_e}$. The pivotal structure of $\mathcal{A}$ turns both $\underline{S}^A_M$ and $\underline{S}^A_{\mathcal{M}_e}$ into module functors. According to Proposition 6.4, relative Serre functors are grading preserving, and thus a pivotal structure on $\mathcal{A}_\mathcal{M}$ is an isomorphism $\underline{id}_M \overset{\simeq}{\to} \underline{S}^A_M$ in $(\mathcal{A}_\mathcal{M})_e$. Now Theorem 3.3 of \cite{Ga} implies that restriction induces an equivalence $(\mathcal{A}_\mathcal{M})_e = \text{Fun}_\mathcal{A}(\mathcal{M}, \mathcal{M})_e \simeq \text{Fun}_{\mathcal{A}_e}(\mathcal{M}_e, \mathcal{M}_e) = (\mathcal{A}_e)_\mathcal{M}_e$ \eqref{eq:PivotalEquivalence} under which a module natural isomorphism $\underline{id}_M \overset{\simeq}{\to} \underline{S}^A_M$ corresponds to a module natural isomorphism $\underline{id}_{\mathcal{M}_e} \overset{\simeq}{\to} \underline{S}^A_{\mathcal{M}_e}$. \hfill $\square$
Definition 6.8. Two $G$-graded pivotal categories $\mathcal{A}$ and $\mathcal{B}$ are said to be graded pivotal Morita equivalent iff there exists a $G$-graded pivotal $\mathcal{A}$-module category $\mathcal{M}$ together with a $G$-graded pivotal equivalence $\mathcal{B} \cong \mathcal{A}^\mathcal{M}$.

The Drinfeld center of a pivotal tensor category inherits a pivotal structure \cite[Ex. 7.13.6]{EGNO}. In particular for a $G$-graded pivotal tensor category $\mathcal{A}$, the Drinfeld center $Z(\mathcal{A})$ has a canonical pivotal structure $p : \text{id}_{Z(\mathcal{A})} \cong (\cdot)^\vee$. According to \cite[Thm. 1.17]{KO}, $p$ serves as pivotal structure for $\_\text{Mod}(Z(A)) = Z_G(A)$.

Proposition 6.9. Let $\mathcal{M}$ be a $G$-graded pivotal $\mathcal{A}$-module category. The braided $G$-crossed equivalence (6.6) is pivotal.

Proof. By Proposition 5.15 the equivalence (5.47) is pivotal. Further, (6.6) is induced by (5.47) and thus preserves $p$ as well.

Corollary 6.10. If two $G$-graded pivotal categories $\mathcal{A}$ and $\mathcal{B}$ are graded pivotal Morita equivalent, then their equivariant centers $Z_G(\mathcal{A})$ and $Z_G(\mathcal{B})$ are equivalent as pivotal braided $G$-crossed tensor categories.

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