$p$-adic entropy and a $p$-adic Fuglede–Kadison determinant

Dedicated to Yuri Ivanovich Manin

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1 Introduction

In several instances the entropy $h(\varphi)$ of an automorphism $\varphi$ on a space $X$ can be calculated in terms of periodic points:

$$h(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log |\text{Fix}(\varphi^n)|.$$  \hfill (1.1)

Here $\text{Fix}(\varphi^n)$ is the set of fixed points of $\varphi^n$ on $X$. Let $\log_p : \mathbb{Q}_p^* \to \mathbb{Z}_p$ be the branch of the $p$-adic logarithm normalized by $\log_p(p) = 0$. The $p$-adic analogue of the limit (1.1), if it exists, may be viewed as a kind of entropy with values in the $p$-adic number field $\mathbb{Q}_p$,

$$h_p(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log_p |\text{Fix}(\varphi^n)|.$$  \hfill (1.2)

It depends only on the action of $\varphi$ on $X$ viewed as a set.

An earlier different approach to a $p$-adic entropy theory was mentioned to me by Amnon Besser. The usual definitions of measure theoretic or topological entropy have no obvious $p$-adic analogue since $\lim$ or $\sup$ do not make sense $p$-adically and since the cardinalities of partitions, coverings and of separating or spanning sets do not behave reasonably in the $p$-adic metric.

Instead of actions of a single automorphism $\varphi$ we look more generally at actions of a countable discrete residually finite but not necessarily amenable group $\Gamma$ on a set $X$. Let us write $\Gamma_n \to e$ if $(\Gamma_n)$ is a sequence of cofinite normal subgroups of $\Gamma$ such that only the neutral element $e$ of $\Gamma$ is contained in infinitely many $\Gamma_n$'s. Let $\text{Fix}_{\Gamma_n}(X)$ be
the set of points in $X$ which are fixed by $\Gamma_n$. If the limit:

$$h_p := \lim_{n \to \infty} \frac{1}{(\Gamma : \Gamma_n)} \log_p |\text{Fix}_{\Gamma_n}(X)|$$

exists with respect to a choice of $\Gamma_n \to e$ we call it the $p$-adic entropy of the $\Gamma$-action on the set $X$ (with respect to the sequence $(\Gamma_n)$).

In this note we show that for an interesting class of $\Gamma$-actions the $p$-adic entropy exists independently of the choice of $\Gamma_n \to e$. In these examples $X$ is an abelian group and $\Gamma$ acts by automorphisms of groups. Namely, let $(\mathbb{R}/\mathbb{Z})^\Gamma$ be the full shift on $\Gamma$ with values in the circle $\mathbb{R}/\mathbb{Z}$ and left $\Gamma$-action by $\gamma(x_{\gamma'}) = (x_{\gamma-1\gamma'})$. For an element $f = \sum_{\gamma} a_{\gamma\gamma}$ in the integral group ring $\mathbb{Z}\Gamma$ consider the closed subshift $X_f \subset (\mathbb{R}/\mathbb{Z})^\Gamma$ consisting of all sequences $(x_{\gamma})$ which satisfy the equation

$$\sum_{\gamma'} x_{\gamma'} a_{\gamma-1\gamma'} = 0 \quad \text{in } (\mathbb{R}/\mathbb{Z})^\Gamma \text{ for all } \gamma \in \Gamma.$$

In fact as in [ER01] we study more general systems defined by an $r \times r$-matrix over $\mathbb{Z}\Gamma$. However, in this introduction, for simplicity, we describe only the case $r = 1$. If $\Gamma$ is amenable, we denote by $h(f)$ the topological entropy of the $\Gamma$-action on $X_f$.

The case $\Gamma = \mathbb{Z}^d$ is classical. Here we may view $f$ as a Laurent polynomial and according to [LSW90] the entropy is given by the (logarithmic) Mahler measure of $f$

$$h(f) = m(f) := \int_{T^d} \log |f(z)| \, d\mu(z).$$

Here $\mu$ is the normalized Haar measure on the $d$-torus $T^d$. According to [LSW90] the $\mathbb{Z}^d$-action on $X_f$ is expansive if and only if $f$ does not vanish in any point of $T^d$. By a theorem of Wiener this is also equivalent to $f$ being a unit in $L^1(\mathbb{Z}^n)$. In this case $h(f)$ can be calculated in terms of periodic points, c.f. [LSW90] Theorem 7.1. See also [Sch95] for this theory.

What about a $p$-adic analogue? In [Den97] it was observed that in the expansive case $m(f)$ has an interpretation via the Deligne–Beilinson regulator map from algebraic $K$-theory to Deligne cohomology. Looking at the analogous regulator map from algebraic $K$-theory to syntomic cohomology one gets a suggestion what a (purely) $p$-adic Mahler measure $m_p(f)$ of $f$ should be, c.f. [BD99]. It can only be defined if $f$ does not vanish in any point of the $p$-adic $d$-torus $T_p^d = \{ z \in \mathbb{C}_p^d \mid |z_i|_p = 1 \}$, where $\mathbb{C}_p$ is the completion of a fixed algebraic closure $\mathcal{O}_p$ of $\mathbb{Q}_p$. In this case $m_p(f)$ is given by the convergent Snirelman integral

$$m_p(f) = \int_{T_p^d} \log_p f(z).$$

(1.5)
Recall that the Snirelman integral of a continuous function \( F : T^d_p \to \mathbb{C}_p \) is defined by the following limit if it exists:

\[
\int_{T^d_p} F(z) := \lim_{N \to \infty} \frac{1}{N^d} \sum_{\zeta \in \mu_N^d} F(\zeta).
\]

Here \( \mu_N \) is the group of \( N \)-th roots of unity in \( \mathbb{Q}_p^* \).

For example, let \( P(t) = a_m t^m + \ldots + a_r t^r \) be a polynomial in \( \mathbb{C}_p[t] \) with \( a_m, a_r \neq 0 \) whose zeroes \( \alpha \) satisfy \( |\alpha|_p \neq 1 \). Then, according to [BD99] Proposition 1.5 we have the following expression for the \( p \)-adic Mahler measure:

\[
m_p(f) = \log_p a_r - \sum_{0 < |\alpha|_p < 1} \log_p \alpha
\]

(1.6)

\[= \log_p a_m + \sum_{|\alpha|_p > 1} \log_p \alpha.
\]

For \( d \geq 2 \) there does not seem to be a simple formula for \( m_p(f) \).

In [BD99] we mentioned the obvious problem to give an interpretation of \( m_p(f) \) as a \( p \)-adically valued entropy. This is now provided by the following result:

**Theorem 1.1** Assume that \( f \in \mathbb{Z}[\mathbb{Z}^d] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \) does not vanish in any point of the \( p \)-adic \( d \)-torus \( T^d_p \). Then the \( p \)-adic entropy \( h_p(f) \) of the \( \Gamma = \mathbb{Z}^d \)-action on \( X_f \) in the sense of (1.3) exists for all \( \Gamma_n \to 0 \) and we have \( h_p(f) = m_p(f) \).

Now we turn to more general groups \( \Gamma \). In [DS] extending [Den06] it was shown that for countable residually finite amenable groups \( \Gamma \) and elements \( f \) in \( \mathbb{Z}^\Gamma \) which are invertible in \( L^1(\Gamma) \) we have

\[
h(f) = \log \det_{\mathcal{N}_\Gamma} f.
\]

(1.7)

Here \( \det_{\mathcal{N}_\Gamma} \) is the Fuglede–Kadison determinant [FK52] on the units of the von Neumann algebra \( \mathcal{N}_\Gamma \supset L^1(\Gamma) \supset \mathbb{Z}^\Gamma \) of \( \Gamma \). In fact, equation (1.7) holds without the condition of amenability if \( h(f) \) is replaced by the quantity:

\[
h_{\text{per}}(f) := \lim_{n \to \infty} \frac{1}{(\Gamma : \Gamma_n)} \log |\text{Fix}_{\Gamma_n}(X)|.
\]

For the \( \Gamma \)-action on \( X_f \) this limit exists and is independent of the choice of sequence \( \Gamma_n \to e \).

In the \( p \)-adic case, instead of working with a \( p \)-adic \( L^1 \)-convolution algebra it is more natural to work with the bigger convolution algebra \( c_0(\Gamma) \). It consists of all formal series \( x = \sum_{\gamma} x_{\gamma} \gamma \) with \( x_{\gamma} \in \mathbb{Q}_p \) and \( |x_{\gamma}|_p \to 0 \) as \( \gamma \to \infty \) in \( \Gamma \).
For $\Gamma = \mathbb{Z}^d$ it is known that $f \in \mathbb{Z}[\mathbb{Z}^d]$ does not vanish in any point of the $p$-adic $d$-torus $T^d_p$ if and only if $f$ is a unit in the algebra $c_0(\mathbb{Z}^d)$. Hence in general, it is natural to look for a $p$-adic analogue of formula (1.7) for all $f \in \mathbb{Z}\Gamma$ which are units in $c_0(\Gamma)$. In the $p$-adic case there is no analogue for the theory of von Neumann algebras and for the functional calculus used to define $\det \mathcal{A}_\Gamma$. However using some algebraic $K$-theory and the results of [FL03], [BLR] and [KLM88] we can define a $p$-adic analogue $\log_p \det f$ of $\log \det \mathcal{A}_\Gamma$ for suitable classes of groups $\Gamma$. For example we get the following result generalizing theorem 1.1:

**Theorem 1.2** Assume that the residually finite group $\Gamma$ is elementary amenable and torsion-free. Let $f$ be an element of $\mathbb{Z}\Gamma$ which is a unit in $c_0(\Gamma)$. Then the $p$-adic entropy $h_p(f)$ of the $\Gamma$-action on $X_f$ in the sense of (1.3) exists for all $\Gamma_n \rightarrow e$ and we have

$$h_p(f) = \log_p \det f.$$  

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## 2 Preliminaries

Fix an integer $r \geq 1$ and set $T = T^r = (\mathbb{R}/\mathbb{Z})^r$. For a discrete group $\Gamma$ let $T^\Gamma$ be the full shift with left $\Gamma$-action by $\gamma(x_{\gamma'}) = (x_{\gamma-1\gamma'})$. Write $M_r(R)$ for the ring of $r \times r$-matrices over a ring $R$. For an element $f = \sum a_{\gamma\gamma'} \gamma$ in $M_r(\mathbb{Z})[\Gamma] = M_r(\mathbb{Z}\Gamma)$ the closed subshift $X_f \subset T^\Gamma$ is defined as the closed subgroup consisting of all sequences with

$$\sum_{\gamma'} x_{\gamma\gamma'} a^*_{\gamma-1\gamma'} = 0 \quad \text{in } T^\Gamma \text{ for all } \gamma \in \Gamma.$$  

Here $a^*$ denotes the transpose of a matrix $a$ in $M_r(\mathbb{Z})$. The group ring $M_r(\mathbb{Z})[\Gamma]$ is equipped with an anti-involution $\ast$ defined by $f^* = \sum_{\gamma} a^*_{\gamma-1\gamma}$ for $f = \sum_{\gamma} a_{\gamma\gamma}$.

Let $\rho_f$ be right multiplication by $f^*$ on the group $T[[\Gamma]]$ of formal $T$-valued series on $\Gamma$. For $x = \sum_{\gamma} x_{\gamma\gamma}$ in $T[[\Gamma]]$ we have

$$\rho_f(x) = \sum_{\gamma} x_{\gamma\gamma} \sum_{\gamma'} a^*_{\gamma-1\gamma} = \sum_{\gamma} \left( \sum_{\gamma'} x_{\gamma\gamma'} a^*_{\gamma-1\gamma'} \right)_{\gamma'}.$$  

Hence we see that

$$X_f = \text{Ker} \left( \rho_f : T[[\Gamma]] \rightarrow T[[\Gamma]] \right)$$
where on the right hand side the group $\Gamma$ acts by left multiplication. Let $N$ be a normal subgroup of $\Gamma$ with quotient map $\sim: \Gamma \to \tilde{\Gamma} = \Gamma/N$. Set
\[
\tilde{f} = \sum_{\gamma} a_{\gamma} \tilde{\gamma} = \sum_{\delta \in \Gamma} \left( \sum_{\gamma \in \delta} a_{\gamma} \right) \delta \quad \text{in } M_r(Z)[\tilde{\Gamma}].
\]
This is the image of $f$ under the reduction map $M_r(Z)[\Gamma] \to M_r(Z)[\tilde{\Gamma}]$. Under the natural isomorphism
\[
T[[\Gamma]] \xrightarrow{\sim} \text{Fix}_N(T[[\Gamma]])
\]
mapping $\sum_\delta x_\delta \delta$ to $\sum_\gamma x_\gamma \gamma$, the action $\rho_{\tilde{f}}$ corresponds to the restriction of $\rho_f$. Hence we have
\[
\text{Fix}_N(X_f) = \ker (\rho_f : T[[\Gamma]] \to T[[\tilde{\Gamma}]]) = X_{\tilde{f}}.
\]
If we assume that $\tilde{\Gamma}$ is finite we get that
\[
\text{Fix}_N(X_f) = \rho_{f_{\widetilde{\mathbb{R}}}}^{-1}((\mathbb{Z}\tilde{\Gamma})^r/(\mathbb{Z}\tilde{\Gamma})^r)
\]
for the endomorphism $\rho_{f_{\widetilde{\mathbb{R}}}}$ of right multiplication by $\tilde{f}^*$ on $(\mathbb{R}\tilde{\Gamma})^r$. This implies the following fact, c.f. [DS], Corollary 4.3:

**Proposition 2.1** Let $\tilde{\Gamma}$ be finite. Then $\rho_{\tilde{f}}$ is an isomorphism of $(\mathbb{Q}\tilde{\Gamma})^r$ if and only if $\text{Fix}_N(X_f)$ is finite. In this case the order is given by
\[
|\text{Fix}_N(X_f)| = \pm \det \rho_{\tilde{f}}.
\]
This follows from the fact that for an isomorphism $\varphi$ of a finite dimensional real vector space $V$ and a lattice $\Lambda$ in $V$ with $\varphi(\Lambda) \subset \Lambda$ we have:
\[
|\varphi^{-1}\Lambda/\Lambda| = |\Lambda/\varphi(\Lambda)| = |\det(\varphi|V)|.
\]

For any countable discrete group $\Gamma$ let $c_0(\Gamma)$ be the set of formal series $\sum_\gamma x_\gamma \gamma$ with $x_\gamma \in \mathbb{Q}_p$ and $|x_\gamma|_p \to 0$ for $\gamma \to \infty$. This means that for any $\varepsilon > 0$ there is a finite subset $S \subset \Gamma$ such that $|x_\gamma|_p < \varepsilon$ for all $\gamma \in \Gamma \setminus S$. The set $c_0(\Gamma)$ is a $\mathbb{Q}_p$-vector space and it becomes a $\mathbb{Q}_p$-algebra with the product
\[
\sum_\gamma x_\gamma \cdots \gamma \cdot \sum_\gamma y_\gamma \gamma = \sum_\gamma \left( \sum_{\gamma \gamma'' = \gamma} x_{\gamma'} y_{\gamma''} \right) \gamma.
\]
Note that the sums
\[
\sum_{\gamma \gamma'' = \gamma} x_{\gamma'} y_{\gamma''} = \sum_{\gamma'} x_{\gamma'} y_{\gamma' \gamma^{-1}}.
\]
converge $p$-adically for every $\gamma$ since $\lim_{\gamma' \to \infty} |x_{\gamma'} y_{\gamma'-1, \gamma}|_p = 0$. The value is independent of the order of summation. Moreover, because of the inequality

$$\left| \sum_{\gamma'} x_{\gamma'} y_{\gamma'-1, \gamma} \right|_p \leq \sup_{\gamma'} |x_{\gamma'} y_{\gamma'-1, \gamma}|_p ,$$

we have

$$\lim_{\gamma \to \infty} \sum_{\gamma'} x_{\gamma'} y_{\gamma'-1, \gamma} = 0 ,$$

so that the product (2.1) is well defined. We may also view $c_0(\Gamma)$ as an algebra of $\mathbb{Q}_p$-valued functions on $\Gamma$ under convolution.

The $\mathbb{Q}_p$-algebra $c_0(\Gamma)$ is complete in the norm

$$\| \sum_{\gamma} x_{\gamma} \| = \sup_{\gamma} |x_{\gamma}|_p = \max_{\gamma} |x_{\gamma}|_p .$$

The norm satisfies the following properties:

$$\| x \| = 0 \text{ if and only if } x = 0 \quad (2.3)$$

$$\| x + y \| \leq \max(\|x\|, \|y\|) \quad (2.4)$$

$$\| \lambda x \| = |\lambda|_p \| x \| \quad \text{for all } \lambda \in \mathbb{Q}_p \quad (2.5)$$

$$\| xy \| \leq \|x\| \| y \| \text{ and } \|1\| = 1 \quad (2.6)$$

Hence $c_0(\Gamma)$ is a $p$-adic Banach algebra over $\mathbb{Q}_p$, i.e. a unital $\mathbb{Q}_p$-algebra $B$ which is complete with respect to a norm $\| \| : B \to \mathbb{R}^{\geq 0}$ satisfying conditions (2.3)–(2.6).

We will only consider Banach algebras where $\| \|$ takes values in $p\mathbb{Z} \cup \{0\}$. The subring $A = B^0$ of elements $x$ in $B$ of norm $\|x\| \leq 1$ is a $p$-adic Banach algebra over $\mathbb{Z}_p$, defined similarly as before. An example is given by

$$c_0(\Gamma, \mathbb{Z}_p) = c_0(\Gamma)^0 = \{ \sum_{\gamma} x_{\gamma} \gamma | x_{\gamma} \in \mathbb{Z}_p \text{ with } \lim_{\gamma \to \infty} |x_{\gamma}|_p = 0 \} .$$

In this case the residue algebra $A/pA$ over $\mathbb{F}_p$ is isomorphic to the group ring of $\Gamma$ over $\mathbb{F}_p$:

$$c_0(\Gamma, \mathbb{Z}_p)/pc_0(\Gamma, \mathbb{Z}_p) = \mathbb{F}_p[\Gamma] .$$

The 1-units $U^1 = 1 + pA$ form a subgroup of $A^*$ since

$$(1 + pa)^{-1} := \sum_{\nu=0}^{\infty} (-pa)^\nu$$

provides an inverse of $1 + pa \in U^1$ in $U^1$. It is easy to see that one has an exact sequence of groups

$$1 \to U^1 \to A^* \to (A/pA)^* \to 1 .$$

(2.8)
For \( A = c_0(\Gamma, \mathbb{Z}_p) \) this is the exact sequence

\[
1 \rightarrow 1 + pc_0(\Gamma, \mathbb{Z}_p) \rightarrow c_0(\Gamma, \mathbb{Z}_p)^* \rightarrow \mathbb{F}_p[\Gamma]^* \rightarrow 1.
\] (2.9)

Concerning the units of a \( p \)-adic Banach algebra over \( \mathbb{Q}_p \), we have the following known fact:

**Proposition 2.2** Let \( B \) be a \( p \)-adic Banach algebra over \( \mathbb{Q}_p \) whose norm takes values in \( p\mathbb{Z} \cup \{0\} \) and set \( A = B^0 \). If the residue algebra \( A/pA \) has no zero divisors, then we have

\[
B^* = p\mathbb{Z}A^* \quad \text{and} \quad p\mathbb{Z} \cap A^* = 1.
\]

**Proof** For \( f \) in \( B^* \) set \( g = 1/f \). Let \( \nu, \mu \) be such that \( f_1 = p^\nu f \) and \( g_1 = p^\mu g \) have norm one. The reductions \( \overline{f}_1, \overline{g}_1 \) of \( f_1, g_1 \) are non-zero. In the equation \( f_1g_1 = p^{\nu+\mu} \) we have \( \nu + \mu \geq 0 \). Reducing mod \( p \) we find that \( 0 \neq \overline{f}_1\overline{g}_1 = p^{\nu+\mu} \) mod \( p \) in \( A/pA \). Hence we have \( \nu + \mu = 0 \) and therefore \( f_1g_1 = 1 \). The first assertion follows. Because of (2.6) we have \( \|a\| = 1 \) for \( a \in A^* \) and \( \|p^\nu\| = p^{-\nu} \). This implies the second assertion. \( \square \)

**Example 2.3** For the group \( \Gamma = \mathbb{Z}^d \) the algebra \( c_0(\mathbb{Z}^d) \) can be identified with the affinoid commutative algebra \( \mathbb{Q}_p[\{t_1^{\pm 1}, \ldots, t_d^{\pm 1}\}] \) of power series \( \sum_{\nu \in \mathbb{Z}^d} x_\nu t^\nu \) with \( x_\nu \in \mathbb{Q}_p \) and \( \lim_{|\nu| \to \infty} |x_\nu|_p = 0 \). Note that these power series can be viewed as functions on \( T_p^d \). The residue algebra is \( \mathbb{F}_p[\mathbb{Z}^d] = \mathbb{F}_p[\{t_1^{\pm 1}, \ldots, t_d^{\pm 1}\}] \). It has no zero divisors and its groups of units is

\[
\mathbb{F}_p[\mathbb{Z}^d]^* = \mathbb{F}_p^* t_1^\mathbb{Z} \cdots t_d^\mathbb{Z}.
\]

The preceeding proposition and the exact sequence (2.9) now give a decomposition into a direct product of groups

\[
c_0(\mathbb{Z}^d)^* = p\mathbb{Z}\mu_{p-1}t_1^\mathbb{Z} \cdots t_d^\mathbb{Z}(1 + p c_0(\mathbb{Z}^d, \mathbb{Z}_p))
\]

**Proposition 2.4** For \( f \) in \( \mathbb{Q}_p[\mathbb{Z}^d] = \mathbb{Q}_p[\{t_1^{\pm 1}, \ldots, t_d^{\pm 1}\}] \) the following properties are equivalent:

\( a \) We have \( f(z) \neq 0 \) for every \( z \) in \( T_p^d \)

\( b \) \( f \) is a unit in \( c_0(\mathbb{Z}^d)^* \)

\( c \) \( f \) has the form \( f(t) = ct^\nu(1 + pg(t)) \) for some \( c \in \mathbb{Q}_p^* \), \( \nu \in \mathbb{Z}^d \) and \( g(t) \) in \( c_0(\mathbb{Z}^d, \mathbb{Z}_p) \).

**Proof** We have seen that \( b \) and \( c \) are equivalent and it is clear that both \( b \) and \( c \) imply \( a \). For proving that \( a \) implies \( b \) note that the maximal ideals of \( c_0(\mathbb{Z}^d) = \mathbb{Q}_p[\{t_1^{\pm 1}, \ldots, t_d^{\pm 1}\}] \) correspond to the orbits of the \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \)-operation on \( T_p^d \cap (\overline{\mathbb{Q}}_p)^d \). Hence \( f \) is not contained in any maximal ideal of \( c_0(\mathbb{Z}^d) \) by assumption \( a \) and therefore \( f \) is a unit. \( \square \)
3 The Frobenius group determinant and a proof of theorem 1.1

The map \( \mathbb{Z}\Gamma \to \mathbb{Z}\tilde{\Gamma} \) for \( \tilde{\Gamma} = \Gamma/N \) from the beginning of the last section can be extended to a homomorphism of \( \mathbb{Q}_p \)-algebras \( c_0(\Gamma) \to c_0(\tilde{\Gamma}) \) by sending \( f = \sum a_\gamma \gamma \) to \( \tilde{f} = \sum a_\gamma \tilde{\gamma} \). Note that this is well defined by the ultrametric inequality and that we have \( \| \tilde{f} \| \leq \| f \| \). If \( \tilde{\Gamma} \) is finite, we have \( c_0(\tilde{\Gamma}) = \mathbb{Q}_p\tilde{\Gamma} \) and hence we obtain a homomorphism of groups \( GL_r(c_0(\Gamma)) \to GL_r(\mathbb{Q}_p\tilde{\Gamma}) \). It follows that for \( f \) in \( M_r(\mathbb{Z}\Gamma) \cap GL_r(c_0(\Gamma)) \) the endomorphism \( \rho_{\tilde{f}} \) of \( (\mathbb{Q}\tilde{\Gamma})^r \) is an isomorphism. Together with proposition 2.1 we have shown the first equation in the following proposition:

**Proposition 3.1** Let \( \Gamma \) be a discrete group and \( N \) a normal subgroup with finite quotient group \( \tilde{\Gamma} \). For \( f \) in \( M_r(\mathbb{Z}\Gamma) \cap GL_r(c_0(\Gamma)) \) the set \( \text{Fix}_N(X_f) \) is finite and we have

\[
|\text{Fix}_N(X_f)| = \pm \det \rho_{\tilde{f}} = \pm \prod_\pi \det_{\mathbb{Q}_p} \left( \sum_\gamma a_\gamma^* \otimes \rho_\pi(\tilde{\gamma}) \right)^{d_\pi}.
\]

Here \( \pi \) runs over the equivalence classes of irreducible representations \( \rho_\pi \) of \( \tilde{\Gamma} \) on \( \mathbb{Q}_p \)-vector spaces \( V_\pi \) and \( d_\pi \) is the degree \( \dim V_\pi \) of \( \pi \).

**Proof** It remains to prove the second equation which is essentially due to Frobenius. Consider \( \mathbb{Q}_p\tilde{\Gamma} \) as a representation of \( \tilde{\Gamma} \) via the map \( \tilde{\gamma} \mapsto \rho(\tilde{\gamma}) := \text{right multiplication with } \tilde{\gamma}^{-1} \). This (“right regular”) representation decomposes as follows into irreducible representations c.f. [Sch95] I, 2.2.4

\[
\mathbb{Q}_p\tilde{\Gamma} \cong \bigoplus_\pi V_\pi^{d_\pi}.
\]

The endomorphism

\[
\rho_{\tilde{f}} = \sum_\gamma a_\gamma^* \otimes \rho(\tilde{\gamma})
\]

on

\[
(\mathbb{Q}_p\tilde{\Gamma})^r = \mathbb{Q}_p^r \otimes \mathbb{Q}_p\tilde{\Gamma}
\]

therefore corresponds to the endomorphism

\[
\bigoplus_\pi \left( \sum_\gamma a_\gamma^* \otimes \rho_\pi(\tilde{\gamma}) \right)^{d_\pi} \text{ on } \bigoplus_\pi \mathbb{Q}_p^r \otimes V_\pi^{d_\pi} = \bigoplus_\pi (\mathbb{Q}_p^r \otimes V_\pi)^{d_\pi}.
\]

Hence the formula follows. \( \square \)

**Remark** In the real case and for the Heisenberg group, Klaus Schmidt previously used the group determinant to calculate \( |\text{Fix}_{\Gamma_n}(X_f)| \) for \( f \) in \( L^1(\Gamma)^* \) and certain \( \Gamma_n \).
The following result generalizes theorem 1.1 from the introduction, at least for a particular sequence $\Gamma_n \to 0$:

**Theorem 3.2** Let $f = \sum_{\nu \in \mathbb{Z}^d} a_{\nu} t^{\nu}$ in $M_r(\mathbb{Z}[t^{\pm 1}, \ldots, t^{\pm 1}])$ be invertible in every point of the $p$-adic $d$-torus $T_p^d$. Then the $p$-adic entropy $h_p(f)$ of the $\Gamma = \mathbb{Z}^d$-action on $X_f$ exists in the sense of (1.3) for the sequence $\Gamma_n = (n\mathbb{Z})^d \to 0$ with $n$ prime to $p$, and we have

$$h_p(f) = m_p(\det f).$$

**Proof** By assumption, the Laurent polynomial $\det f$ does not vanish in any point of $T_p^d$. Hence $\det f$ is a unit in $c_0(\Gamma)$ by proposition 2.4. It follows from proposition 3.1 that we have:

$$|\text{Fix}_{\Gamma_n}(X_f)| = \prod_{\chi} \det_{\overline{\mathbb{Q}}_p} \left( \sum_{\nu \in \mathbb{Z}^d} a_{\nu}^* \otimes \chi(\nu) \right)$$

where $\chi$ runs over the characters of $\Gamma/\Gamma_n = (\mathbb{Z}/n\mathbb{Z})^d$. These correspond via $\chi(\nu) = \zeta^\nu$ to the elements $\zeta$ of $\mu_n^d$. Viewing $f$ as a matrix of functions on $T_p^d$ we therefore get the formulas

$$|\text{Fix}_{\Gamma_n}(X_f)| = \prod_{\zeta \in \mu_n^d} \det_{\overline{\mathbb{Q}}_p} \left( \sum_{\nu \in \mathbb{Z}^d} a_{\nu}^* \zeta^\nu \right)$$

$$= \prod_{\zeta \in \mu_n^d} \det_{\overline{\mathbb{Q}}_p}(f(\zeta))$$

$$= \prod_{\zeta \in \mu_n^d} (\det f)(\zeta).$$

Thus the $p$-adic entropy $h_p(f)$ of the $\Gamma$-action on $X_f$ with respect to the above sequence is given by:

$$h_p(f) = \lim_{n \to \infty} \frac{1}{(\Gamma : \Gamma_n)} \log_p |\text{Fix}_{\Gamma_n}(X_f)|$$

$$= \lim_{n \to \infty} \frac{1}{n^d} \sum_{\zeta \in \mu_n^d} \log_p (\det f)(\zeta)$$

$$= \int_{T_p^d} \log_p \det f = m_p(\det f).$$

Note here that for the Laurent polynomials $\det f$ under consideration the Snirelman integral exists by [BD99] Proposition 1.3. ☐

**Remark** A suitable generalization of that proposition would give theorems 3.2 and 1.1 for general sequences $\Gamma_n \to 0$ in $\Gamma = \mathbb{Z}^d$. We leave this to the interested reader since the general case of theorem 1.1 is also a corollary of theorem 1.2 which will be proved by a different method in section 5.
**Example** The polynomial in one variable \( f(T) = 2T^2 - T + 2 \) does not vanish in any point of the 2-adic circle \( T_2 \). In this sense, \( X_f \) is “2-adically expansive”. Consider the square root of \(-15\) in \( \mathbb{Z}_2 \) given by the 2-adically convergent series

\[
\sqrt{-15} = (1 + (-16))^{1/2} = \sum_{\nu=0}^{\infty} \binom{1/2}{\nu} (-1)^\nu 2^{4\nu}.
\]

The zeroes of \( f(T) \) in \( \mathbb{Q}_2 \) are given by \( \alpha_\pm = \frac{1}{4}(1 \pm \sqrt{-15}) \in \mathbb{Q}_2 \) where \( |\alpha_+|_2 = 2 \) and \( |\alpha_-|_2 = 1/2 \). Successive approximations for \( \alpha_+ \) coming from the series for \( \sqrt{-15} \) are \( 1/2, -3/2, -19/2, -83/2 \). By theorem \ref{3.2} and formula \ref{1.6} the 2-adic entropy of \( X_f \) is given by

\[
h_2(f) = \log_2 \alpha_+ \in \mathbb{Z}_2.
\]

Note that \( f \) viewed as a complex valued function has both its zeroes on \( S^1 \), so that \( X_f \) is not expansive in the usual sense. The topological entropy is \( h(f) = \log 2 \).

---

### 4 The logarithm on the 1-units of a \( p \)-adic Banach algebra

For a discrete group \( \Gamma \) we would like to define a homomorphism

\[
\log_p \det_{\Gamma} : c_0(\Gamma)^* \longrightarrow \mathbb{Q}_p
\]

which should be a \( p \)-adic replacement for the map

\[
\log \det_{\mathcal{N} \Gamma} : L^1(\Gamma)^* \subset (\mathcal{N} \Gamma)^* \longrightarrow \mathbb{R}.
\]

More generally, we would like to define such a map on \( \text{GL}_r(c_0(\Gamma)) \). In this section we give the definition on the subgroup of 1-units and relate \( \log_p \det_{\Gamma} \) to \( p \)-adic entropy. The extension to a map on all of \( c_0(\Gamma)^* \) will be done in the next section for suitable classes of groups \( \Gamma \) using rather deep facts about group rings.

Let \( B \) be a \( p \)-adic Banach algebra over \( \mathbb{Q}_p \) whose norm \( \| \| \) takes values in \( p^\mathbb{Z} \cup \{0\} \). A trace functional on \( B \) is a continuous linear map \( \text{tr}_B : B \rightarrow \mathbb{Q}_p \) which vanishes on commutators \([a,b] = ab - ba\) of elements in \( B \). For \( b \in B \) and \( c \in B^* \) we have

\[
\text{tr}_B(b c c^{-1}) = \text{tr}_B(b). \tag{4.1}
\]

Set \( A = B^0 = \{b \in B \mid \|b\| \leq 1\} \) and let \( U^1 \) be the normal subgroup of 1-units in \( A^* \). The logarithmic series

\[
\log : U^1 \longrightarrow A, \quad \log u = -\sum_{\nu=1}^{\infty} \frac{(1-u)^\nu}{\nu}
\]
converges and defines a continuous map. An argument with formal power series shows that we have
\[ \log uv = \log u + \log v \] (4.2)
if the elements \( u \) and \( v \) in \( U^1 \) commute with each other.

The next result is a consequence of the Campbell–Baker–Hausdorff formula.

**Theorem 4.1** The map
\[ \text{tr}_B \log : U^1 \rightarrow \mathbb{Z}_p \]
is a homomorphism. For \( u \) in \( U^1 \) and \( a \) in \( A^* \) we have
\[ \text{tr}_B \log(aua^{-1}) = \text{tr}_B \log(u) . \] (4.3)

**Proof** Formula (4.3) follows from (4.1). From [Bou72] Ch. II, § 8 we get the following information about \( \log \). Set \( G = \{ b \in B \mid \| b \| < \sqrt{p^{-1}} \} \). Then the exponential series defines a bijection \( \exp : G \simrightarrow 1 + G \) with inverse \( \exp^{-1} = \log \mid_{1+ G} \). For \( x, y \) in \( G \) we have
\[ \exp x \cdot \exp y = \exp h(x, y) \] (4.4)
where \( h(x, y) \in G \) is given by a convergent series in \( B \). It has the form
\[ h(x, y) = x + y + \text{series of (iterated) commutators} . \]
Elements \( u, v \) of \( 1 + G \) have the form \( u = \exp x \) and \( v = \exp y \). Taking the log of relation (4.4) and applying \( \text{tr}_B \) we get
\[ \text{tr}_B \log(uv) = \text{tr}_B h(x, y) = \text{tr}_B (x + y) = \text{tr}_B \log u + \text{tr}_B \log v . \] (4.5)
Hence \( \text{tr}_B \log \) is a homomorphism on the subgroup \( 1 + G \) of \( U^1 \). By assumption the norm of \( B \) takes values in \( p^2 \cup \{ 0 \} \). For \( p \neq 2 \) we therefore have \( 1 + G = U^1 \) and we are done.

For \( p = 2 \) the restriction of the map
\[ \varphi = \text{tr}_B \log : U^1 \rightarrow \mathbb{Q}_2 \]
to \( 1 + G = 1 + 4A \) is a homomorphism by (4.5). We have to show that it is a homomorphism on \( U^1 = 1 + 2A \) as well. For \( u \) in \( U^1 \) we have \( \varphi(u) = \frac{1}{2} \varphi(u^2) \) by (4.2) and \( u^2 \) lies in \( 1 + 4A \). Now consider elements \( u, v \) in \( U^1 \). Then we have
\[ \varphi(uv) = \frac{1}{2} \varphi((uv)^2) = \frac{1}{2} \varphi(uvuv) = \frac{1}{2} \varphi(u^2vuvu^{-1}) \]
\[ = \frac{1}{2} \varphi(u^2) + \frac{1}{2} \varphi(vuv^{-1}) \]
since $u^2$ and $vuvu^{-1}$ lie in $1 + 4A$ where $\varphi$ is a homomorphism. By similar arguments we get

$$\varphi(uv) = \varphi(u) + \frac{1}{2}\varphi(v^2uvu^{-1}v^{-1})$$

Thus we must show that $\varphi(uvu^{-1}v^{-1}) = 0$. By (4.3) we have $\varphi(uvu^{-1}v^{-1}) = \varphi(vu^{-1}v^{-1}u)$ and hence using that both $uvu^{-1}v^{-1}$ and $vu^{-1}v^{-1}u$ lie in $1 + 4A$ we find

$$2\varphi(uvu^{-1}v^{-1}) = \varphi(uvu^{-1}v^{-1}) + \varphi(vu^{-1}v^{-1}u)$$

$$= \varphi(u^2v^{-1}u^2v)$$

$$= \varphi(u^{-2}) + \varphi(v^{-1}u^2v)$$

$$= 0.$$

For a discrete group $\Gamma$, the map

$$\text{tr}_\Gamma : c_0(\Gamma) \to \mathbb{Q}_p, \quad \text{tr}_\Gamma(\sum a_\gamma \gamma) = a_e$$

defines a trace functional on $c_0(\Gamma)$. Let $B = M_r(c_0(\Gamma))$ be the $p$-adic Banach algebra over $\mathbb{Q}_p$ of $r \times r$-matrices $(a_{ij})$ with entries in $c_0(\Gamma)$ and equipped with the norm $\|(a_{ij})\| = \max_{ij} \|a_{ij}\|$. The composition:

$$\text{tr}_\Gamma : M_r(c_0(\Gamma)) \to c_0(\Gamma) \to \mathbb{Q}_p$$

defines a trace functional on $M_r(c_0(\Gamma))$.

The algebra $A = B^0$ is given by $M_r(c_0(\Gamma, \mathbb{Z}_p))$ and we have $U^1 = 1 + pM_r(c_0(\Gamma, \mathbb{Z}_p))$. The exact sequence (2.8) becomes the exact sequence of groups:

$$1 \to 1 + pM_r(c_0(\Gamma, \mathbb{Z}_p)) \to \text{GL}_r(c_0(\Gamma, \mathbb{Z}_p)) \to \text{GL}_r(\mathbb{F}_p\Gamma) \to 1.$$  \quad (4.6)

According to theorem 4.1 the map

$$\log_p, \det : \text{tr}_\Gamma \log : 1 + pM_r(c_0(\Gamma, \mathbb{Z}_p)) \to \mathbb{Z}_p$$  \quad (4.7)

is a homomorphism of groups.

**Example 4.2** For $\Gamma = \mathbb{Z}^d$, in the notation of example 2.3 we have a commutative diagram, c.f. [BD99] Lemma 1.1

$$\xymatrix{ c_0(\Gamma) \ar[r] \ar[d]_{\text{tr}_\Gamma} & \mathbb{Q}_p(t_1^{\pm 1}, \ldots, t_d^{\pm 1}) \ar[d]_{f_{\text{tr}_\Gamma}} \\
\mathbb{Q}_p & \mathbb{Q}_p.}$$
It follows that for a 1-unit \( f \) in \( M_r(c_0(\Gamma)) \) we have:

\[
\log_p \det f = \int_{T_p^d} \log \det f = m_p(\det f).
\] (4.8)

Here we have used the relation

\[
\text{tr} \log f = \log \det f \quad \text{in} \ c_0(\Gamma),
\] (4.9)

where \( \det : \text{GL}_r(c_0(\Gamma)) \rightarrow c_0(\Gamma)^* \) is the determinant and tr the trace for matrices over the commutative ring \( c_0(\Gamma) \). Note that \( \det \) maps 1-units to 1-units. Relation (4.9) can be proved by embedding the integral domain \( c_0(\Gamma) = \mathbb{Q}_p(t_1^{\pm1}, \ldots, t_d^{\pm1}) \) into its quotient field and applying [Har77] Appendix C, Lemma 4.1.

For finite groups \( \Gamma \) the map \( \log_p \det f \) can be calculated as follows. For \( f \) in \( M_r(c_0(\Gamma)) = M_r(\mathbb{Q}_p\Gamma) \) let \( \rho_f \) be the endomorphism of \( (\mathbb{Q}_p\Gamma)^r \) by right multiplication with \( f^* \) and \( \det_{\mathbb{Q}_p}(\rho_f) \) its determinant over \( \mathbb{Q}_p \).

**Proposition 4.3** Let \( \Gamma \) be finite. Then we have

\[
\log_p \det f = \frac{1}{|\Gamma|} \log_p \det_{\mathbb{Q}_p}(\rho_f)
\] (4.10)

for \( f \) in \( 1 + pM_r(\mathbb{Z}_p\Gamma) \).

**Remark** Since \( \rho_{fg} = \rho_f \rho_g \), the equation in the proposition shows that \( \log_p \det f \) is a homomorphism – something we know in general by theorem 4.1. For finite \( \Gamma \) the group \( \text{GL}_r(\mathbb{F}_p\Gamma) \) is finite. Hence, by (4.6) there is at most one way to extend \( \log_p \det f \) from \( 1 + pM_r(\mathbb{Z}_p\Gamma) \) to a homomorphism from \( \text{GL}_r(\mathbb{Z}_p\Gamma) \) to \( \mathbb{Q}_p \). Namely, we have to set

\[
\log_p \det_{\mathbb{Z}_p}(f) := \frac{1}{N} \log_p \det_{\mathbb{Z}_p}(f^N),
\]

where \( N \geq 1 \) is any integer with \( f^N = 1 \) in \( \text{GL}_r(\mathbb{F}_p\Gamma) \). Because of (4.2) this is well defined but it is not clear from the definition that we get a homomorphism. However, for the same \( f, N \) we have

\[
\log_p \det_{\mathbb{Q}_p}(\rho_f) = \frac{1}{N} \log_p \det_{\mathbb{Q}_p}(\rho_{f^N}).
\]

Hence equation (4.10) holds for all \( f \) in \( \text{GL}_r(\mathbb{Z}_p\Gamma) \) and it follows that \( \log_p \det f \) extends to a homomorphism on \( \text{GL}_r(\mathbb{Z}_p\Gamma) \). In the next section such arguments will be generalized to infinite groups with the help of \( K \)-theory.
Proof of 4.3] Under the continuous homomorphism of $p$-adic Banach algebras over $\mathbb{Z}_p$,

$$\rho : M_r(\mathbb{Z}_p\Gamma) \longrightarrow \text{End}_{\mathbb{Z}_p}(\mathbb{Z}_p\Gamma)^r$$

the groups of 1-units are mapped to each other. Hence we have

$$\log \rho f = \rho \log f \quad (4.11)$$

for $f$ in $1 + pM_r(\mathbb{Z}_p\Gamma)$.

On the other hand we have

$$\text{tr}_\Gamma(g) = \frac{1}{|\Gamma|} \text{tr}(\rho g) \quad (4.12)$$

for any element $g$ of $M_r(\mathbb{Q}_p\Gamma)$. This is proved first for $r = 1$ by checking the cases where $g = \gamma$ is an element of $\Gamma$. Then one extends to arbitrary $r$ by thinking of $\rho g$ as a block matrix with blocks of size $|\Gamma| \times |\Gamma|$.

Combining (4.11) and (4.12) we find:

$$\log_p \det \Gamma f = \lim_{n \to \infty} \log_p \det \Gamma(\gamma) f^{(n)} = \frac{1}{|\Gamma|} \text{tr}(\rho \log f) = \frac{1}{|\Gamma|} \text{tr}(\log \rho f) = \frac{1}{|\Gamma|} \log_p \det_{\mathbb{Q}_p}(\rho f).$$

The last equation is proved by writing $\rho f$ in triangular form in a suitable basis over $\mathbb{Q}_p$ and observing that the eigenvalues of $\rho f$ are 1-units in $\mathbb{Q}_p$. \qed

The next result is necessary to prove the relation of $\log_p \det \Gamma f$ with $p$-adic entropies.

**Proposition 4.4** Let $\Gamma$ be a residually finite countable discrete group and $\Gamma_n \to e$ a sequence as in the introduction. For $f$ in $1 + pM_r(c_0(\Gamma, \mathbb{Z}_p))$ consider its image $f^{(n)}$ in $1 + pM_r(\mathbb{Z}_p\Gamma^{(n)})$ where $\Gamma^{(n)}$ is the finite group $\Gamma^{(n)} = \Gamma/\Gamma_n$. Then we have

$$\log_p \det \Gamma f = \lim_{n \to \infty} \log_p \det \Gamma^{(n)} f^{(n)} \text{ in } \mathbb{Z}_p.$$ 

**Proof** The algebra map $M_r(c_0(\Gamma)) \to M_r(c_0(\Gamma^{(n)}))$ sending $f$ to $f^{(n)}$ is continuous since we have $\|f^{(n)}\| \leq \|f\|$. For $f$ in $1 + pM_r(c_0(\Gamma, \mathbb{Z}_p))$ we therefore get:

$$(\log f)^{(n)} = \log f^{(n)} \text{ in } M_r(c_0(\Gamma^{(n)})).$$

The next claim for $g = \log f$ thus implies the assertion. \qed
Claim 4.5 For \( g \in M_r(c_0(\Gamma)) \) we have

\[
\text{tr}_\Gamma(g) = \lim_{n \to \infty} \text{tr}_{\Gamma(n)}(g^{(n)}) .
\]

Proof We may assume that \( r = 1 \). Writing \( g = \sum a_\gamma \gamma \) with \( a_\gamma \in \mathbb{Q}_p, |a_\gamma|_p \to 0 \) for \( \gamma \to \infty \) we have

\[
|\text{tr}_\Gamma(g) - \text{tr}_{\Gamma(n)}(g^{(n)})|_p = \left| a_e - \sum_{\gamma \neq e} a_\gamma \right|_p
= \left| \sum_{\gamma \in \Gamma_n \setminus e} a_\gamma \right|_p
\leq \max_{\gamma \in \Gamma_n \setminus e} |a_\gamma|_p .
\]

For \( \varepsilon > 0 \) there is a finite subset \( S = S_\varepsilon \) of \( \Gamma \) such that \( |a_\gamma|_p < \varepsilon \) for \( \gamma \in \Gamma \setminus S \). Only \( e \) is contained in infinitely many \( \Gamma_n \)'s. Hence there is some \( n_0 \) such that \( (\Gamma_n \setminus e) \cap S = \emptyset \) i.e. \( \Gamma_n \setminus e \subset \Gamma \setminus S \) for all \( n \geq n_0 \). It follows that for \( n \geq n_0 \) we have \( |\text{tr}_\Gamma(g) - \text{tr}_{\Gamma(n)}(g^{(n)})|_p \leq \varepsilon. \)

Corollary 4.6 Let \( \Gamma \) be a residually finite countable discrete group and \( f \) an element of \( M_r(\mathbb{Z}\Gamma) \) which is a 1-unit in \( M_r(c_0(\Gamma)) \). Then the \( p \)-adic entropy \( h_p(f) \) of the \( \Gamma \)-action on \( X_f \) exists for all \( \Gamma_n \to e \) and we have

\[
h_p(f) = \log_p \det f \quad \text{in } \mathbb{Z}_p .
\]

Proof By propositions 3.1 and 4.3 we have

\[
\frac{1}{(\Gamma : \Gamma_n)} \log_p |\text{Fix}_{\Gamma_n}(X_f)| = \frac{1}{(\Gamma : \Gamma_n)} \log_p \det Q_p(\rho_f^{(n)})
= \log_p \det_{\Gamma(n)}(f^{(n)}) .
\]

Hence the assertion follows from proposition 4.4.

5 A \( p \)-adic logarithmic Fuglede–Kadison determinant and its relation to \( p \)-adic entropy

Having defined a homomorphism \( \log_p \det \Gamma \) on \( 1 + pc_0(\Gamma, \mathbb{Z}_p) \) in (4.7) one would like to use the exact sequence (2.9) to extend it to \( c_0(\Gamma, \mathbb{Z}_p)^* \). However, for infinite groups \( \Gamma \) the abelianization of the group \( \mathbb{F}_p[\Gamma]^* \) divided by the image of \( \Gamma \) is not known to be torsion in any generality – as far as I know. However, corresponding results are known
for $K_1$ of $\mathbb{F}_p[\Gamma]$ and this determines our approach which even for $r = 1$ requires the preceding considerations for matrix algebras.

For a unital ring $R$ recall the embedding $\text{GL}_r(R) \hookrightarrow \text{GL}_{r+1}(R)$ mapping $a$ to $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Let $GL_\infty(R)$ be the union of the $GL_r(R)$’s. We will view elements of $GL_\infty(R)$ as infinite matrices with 1’s on the diagonal and only finitely many further nonzero entries. The subgroup $E_r(R) \subset GL_r(R)$ of elementary matrices is the subgroup generated by matrices which have 1’s on the diagonal and at most one further nonzero entry. Let $E\subset F$ be their union and set $K_1(R) = GL_\infty(R)/E_\infty(R)$. It is known that we have $E_\infty(R) = (GL_\infty(R),GL_\infty(R))$ and hence that $K_1(R) = GL_\infty(R)^{\text{ab}}$ c.f. [Mil71] § 3. The Whitehead group over $Wh$ over $\mathbb{F}_p$ of a discrete group $\Gamma$ is defined to be

$$Wh^{\mathbb{F}_p}(\Gamma) := K_1(\mathbb{F}_p[\Gamma])/\langle \Gamma \rangle.$$ 

Here $\langle \Gamma \rangle$ is the image of $\Gamma$ under the canonical map $\mathbb{F}_p[\Gamma]^* \to K_1(\mathbb{F}_p[\Gamma])$.

We can treat groups for which $Wh^{\mathbb{F}_p}(\Gamma)$ is torsion. According to [FL03] Theorem 1.1 this is the case for torsion-free elementary amenable groups $\Gamma$. Recently, in [BLR] it has been shown for a larger class of groups that $Wh^{\mathbb{F}_p}(\Gamma)$ is torsion. Apart from the elementary amenable groups, this class comprises all word hyperbolic groups. It is closed under subgroups, finite products, colimits and suitable extensions.

**Theorem 5.1** Let $\Gamma$ be a countable discrete residually finite group such that $Wh^{\mathbb{F}_p}(\Gamma)$ is torsion. Then there is a unique homomorphism

$$\log_p \det_\Gamma : K_1(c_0(\Gamma,\mathbb{Z}_p)) \longrightarrow \mathbb{Q}_p$$

with the following properties:

**a** For every $r \geq 1$ the composition

$$1 + pM_r(c_0(\Gamma,\mathbb{Z}_p)) \hookrightarrow \text{GL}_r(c_0(\Gamma,\mathbb{Z}_p)) \to K_1(c_0(\Gamma,\mathbb{Z}_p)) \xrightarrow{\log_p \det_\Gamma} \mathbb{Q}_p$$

coincides with the map $\log_p \det_\Gamma$ introduced in [4.7].

**b** On the image of $\Gamma$ in $K_1(c_0(\Gamma,\mathbb{Z}_p))$ the map $\log_p \det_\Gamma$ vanishes.

**Proof** Set $A = c_0(\Gamma,\mathbb{Z}_p)$ and $\overline{A} = A/pA = \mathbb{F}_p[\Gamma]$. The reduction map $A \to \overline{A}$ induces an exact sequence

$$0 \to \Gamma E_\infty(A)(1 + pM_\infty(A))/\Gamma E_\infty(A) \to K_1(A)/\langle \Gamma \rangle \to K_1(\overline{A})/\langle \Gamma \rangle. \quad (5.1)$$

Here $M_\infty(A)$ is the (non-unital) algebra of infinite matrices $(a_{ij})_{i,j \geq 1}$ with only finitely many non-zero entries. Note that $1 + pM_\infty(A)$ is a subgroup of $GL_\infty(A)$ since $1 + pM_r(A)$ is a subgroup of $GL_r(A)$. Moreover $\Gamma E_\infty(A)$ is a normal subgroup of $GL_\infty(A)$. Hence the sequence (5.1) becomes an exact sequence:

$$0 \to (1 + pM_\infty(A))/\Gamma E_\infty(A) \cap (1 + pM_\infty(A)) \to K_1(A)/\langle \Gamma \rangle \to K_1(\overline{A})/\langle \Gamma \rangle. \quad (5.2)$$
Since $Q_p$ is uniquely divisible this implies the uniqueness assertion in the theorem for any group $\Gamma$ such that $WhF_p(\Gamma) = K_1(\overline{A})/(\overline{\Gamma})$ is torsion. For the existence, we first note that the homomorphisms defined in (4.7) induce a homomorphism

$$\log_p \det \Gamma : 1 + pM_\infty(A) \longrightarrow \mathbb{Z}_p.$$ 

We have to show that $\log_p \det \Gamma f = 0$ for every $f$ in $1 + pM_\infty(A)$ which also lies in $\Gamma E_\infty(A)$. Under our identification of $GL_r(A)$ with a subgroup of $GL_\infty(A)$ we find some $r \geq 1$ such that we have

$$f = i(\gamma)e_1 \cdots e_N \text{ in } 1 + pM_r(A).$$

Here the $e_i$ are elementary $r \times r$-matrices and $i(\gamma) = \left( \begin{smallmatrix} \gamma & 0 \\ 0 & 1_{r-1} \end{smallmatrix} \right)$ for some $\gamma$ in $\Gamma$. According to propositions 4.3 and 4.4 we have for any choice of sequence $\Gamma_n \rightarrow e$:

$$\log_p \det \Gamma f = \lim_{n \rightarrow \infty} \frac{1}{(\Gamma : \Gamma_n)} \log_p \det Q_p(\rho_{f(n)}) \text{.}$$

On the other hand:

$$\det Q_p(\rho_{f(n)}) = \det Q_p(\rho_{i(\gamma)(n)} e_1^{(n)} \cdots e_N^{(n)}) = \det Q_p(\rho_{i(\gamma)(n)}) \prod_i \det Q_p(\rho_{e_i^{(n)}}).$$

Let $b$ be a basis of $Q_p[\Gamma^{(n)}]$. In the basis $(b, \ldots, b)$ of $Q_p[\Gamma^{(n)}]^r$ the endomorphism $\rho_{e_i^{(n)}}$ is given by a matrix of $[\Gamma^{(n)}] \times [\Gamma^{(n)}]$-blocks. The diagonal blocks are identity matrices. At most one of the other blocks is non-zero. In particular, the matrix is triangular and we have $\det Q_p(\rho_{e_i^{(n)}}) = 1$. In the same basis $\rho_{i(\gamma)(n)}$ is a permutation matrix and hence $\det Q_p(\rho_{i(\gamma)(n)}) = \pm 1$. It follows that we have $\log_p \det \Gamma f = 0$ as we wanted to show. \hspace{1cm} $\Box$

I think that theorem 5.1 should also hold without the condition that $\Gamma$ is residually finite.

**Remark 5.2** For $\Gamma = \mathbb{Z}^d$ and $f$ in $GL_r(c_0(\Gamma, \mathbb{Z}_p))$, writing $[f]$ for the class of $f$ in $K_1$, we have

$$\log_p \det \Gamma [f] = m_p(\det f)$$

extending equation (4.8).

This follows from the uniqueness assertion in theorem 5.1. Namely, the map $[f] \mapsto m_p(\det f)$ defines a homomorphism on $K_1$ which according to equation (4.8) satisfies condition a. It satisfies condition b as well, since $\log_p$ vanishes on roots of unity and hence we have $m_p(t^\nu) = 0$ for all $\nu$ in $\mathbb{Z}^d$, c.f. example 2.3.
**Definition 5.3** For any group \( \Gamma \) as in the theorem we define the homomorphism \( \log_p \det_\Gamma \) on \( GL_r(c_0(\Gamma, \mathbb{Z}_p)) \) to be the composition

\[
\log_p \det_\Gamma : GL_r(c_0(\Gamma, \mathbb{Z}_p)) \longrightarrow K_1(c_0(\Gamma, \mathbb{Z}_p)) \xrightarrow{\log_p \det_\Gamma} \mathbb{Q}_p .
\]

If we unravel the definitions we get the following description of this map. Given a matrix \( f \) in \( GL_r(c_0(\Gamma, \mathbb{Z}_p)) \) there are integers \( N \geq 1 \) and \( s \geq r \) such that in \( GL_s(c_0(\Gamma, \mathbb{Z}_p)) \) we have \( f^N = i(\gamma)g \) with \( \epsilon \) in \( E_s(c_0(\Gamma, \mathbb{Z}_p)) \), \( g \) in \( 1 + pM_s(c_0(\Gamma, \mathbb{Z}_p)) \) and \( i(\gamma) \) the \( s \times s \)-matrix \( \begin{pmatrix} \gamma & 0 \\ 0 & 1_{s-1} \end{pmatrix} \). Then we have

\[
\log_p \det_\Gamma f = \frac{1}{N} \log_p \det g = \frac{1}{N} \text{tr}_\Gamma \log g .
\]

(5.3)

We can now prove the following extension of corollary 4.6.

**Theorem 5.4** Let \( \Gamma \) be a residually finite countable discrete group such that \( WH^{F_p}(\Gamma) \) is torsion. Let \( f \) be an element of \( M_r(\mathbb{Z}\Gamma) \cap GL_r(c_0(\Gamma, \mathbb{Z}_p)) \). Then \( h_p(f) \) exists for all \( \Gamma_n \to e \) and we have

\[
h_p(f) = \log_p \det_\Gamma f \quad \text{in} \quad \mathbb{Q}_p .
\]

**Proof** Let us write \( f^N = i(\gamma) \epsilon g \) as above. Then by proposition 3.1 we have

\[
\log_p |\text{Fix}_{\Gamma_n}(X_f)| = \log_p \det \mathbb{Q}_p(\rho_{f^{(n)}}) = \frac{1}{N} \log_p \det \mathbb{Q}_p(\rho_{i(\gamma)^{(n)}}) + \frac{1}{N} \log_p \det \mathbb{Q}_p(\rho_{\epsilon^{(n)}}) + \frac{1}{N} \log_p \det \mathbb{Q}_p(\rho_{g^{(n)}}) .
\]

Note here that the composition

\[
M_s(c_0(\Gamma)) \longrightarrow M_s(c_0(\Gamma^{(n)})) \xrightarrow{\rho} \text{End}_{\mathbb{Q}_p}(\mathbb{Q}_p\Gamma^{(n)})^s
\]

is a homomorphism of algebras.

As in the proof of theorem 5.1 we see that the terms \( \log_p \det \mathbb{Q}_p(\rho_{i(\gamma)^{(n)}}) \) and \( \log_p \det \mathbb{Q}_p(\rho_{\epsilon^{(n)}}) \) vanish. This gives

\[
\frac{1}{(\Gamma : \Gamma_n)} \log_p |\text{Fix}_{\Gamma_n}(X_f)| = \frac{1}{(\Gamma : \Gamma_n)} \frac{1}{N} \log_p \det \mathbb{Q}_p(\rho_{g^{(n)}}) = \frac{1}{N} \log_p \det_\Gamma (g^{(n)}) \quad \text{by proposition 4.3}
\]

Using proposition 4.4 we get in the limit \( n \to \infty \) that

\[
h_p(f) = \frac{1}{N} \log_p \det g \underset{5.3}{=} \log_p \det f .
\]
For groups $\Gamma$ as in theorem 5.4 whose group ring $\mathbb{F}_p\Gamma$ has no zero divisors it is possible to extend the definition of $\log_p \det_\Gamma$ from $c_0(\Gamma, \mathbb{Z}_p)^*$ to $c_0(\Gamma)^*$. Namely, by proposition 2.2 we know that

$$c_0(\Gamma)^* = p^\mathbb{Z} c_0(\Gamma, \mathbb{Z}_p)^* \quad \text{and} \quad p^\mathbb{Z} \cap c_0(\Gamma, \mathbb{Z}_p)^* = 1.$$  

Hence there is a unique homomorphism

$$\log_p \det_\Gamma : c_0(\Gamma)^* \to \mathbb{Q}_p$$

which agrees with $\log_p \det_\Gamma$ previously defined on $c_0(\Gamma, \mathbb{Z}_p)^*$ in definition 5.3 and satisfies

$$\log_p \det_\Gamma (p) = 0.$$  

Let $\Gamma$ be a torsion-free elementary amenable group. Then according to [KLM88] Theorem 1.4 the group ring $\mathbb{F}_p\Gamma$ has no zero divisors and according to [FL03] Theorem 1.1 the group $Wh^{\mathbb{F}_p}(\Gamma)$ is torsion. Hence $\log_p \det_\Gamma$ is defined on $c_0(\Gamma)^*$ and this is the map used in theorem 1.2.

**Proof of theorem 1.2** Writing $f$ in $\mathbb{Z}\Gamma \cap c_0(\Gamma)^*$ as a product $f = p^n g$ with $g$ in $c_0(\Gamma, \mathbb{Z}_p)^*$ it follows that $g \in \mathbb{Z}\Gamma$ and proposition 3.1 shows that we have

$$\log_p |\text{Fix}_{\Gamma_n}(X_f)| = \log_p \det_{\mathbb{Q}_p}(\rho_{f(n)}) = \log_p \det_{\mathbb{Q}_p}(\rho_{g(n)}) = \log_p |\text{Fix}_{\Gamma_n}(X_g)|.$$  

Note here that we have $\log_p(p) = 0$. It follows from theorem 5.4 applied to $g$ that for all $\Gamma_n \to e$ we get:

$$h_p(f) = h_p(g) = \log_p \det_\Gamma g = \log_p \det_\Gamma f.$$  

For $\Gamma = \mathbb{Z}^d$ it follows from remark 5.2 that for any $f$ in $c_0(\mathbb{Z}^d)^* = \mathbb{Q}_p \langle t_1^{\pm 1}, \ldots, t_d^{\pm 1} \rangle^*$ we have:

$$\log_p \det_\Gamma f = m_p(f).$$  

Hence theorem 1.1 is a special case of theorem 1.2.

Concerning approximations of $\log_p \det_\Gamma f$ we note that proposition 4.4 extends to more general cases.

**Proposition 5.5** Let $\Gamma$ be a residually finite countable discrete group and $\Gamma_n \to e$ as in the introduction. For $f$ in $M_r(c_0(\Gamma))$ let $f^{(n)}$ be its image in $M_r(\mathbb{Q}_p\Gamma^{(n)})$. Then the formula

$$\log_p \det_\Gamma f = \lim_{n \to \infty} \frac{1}{(\Gamma : \Gamma_n)} \log_p \det_{\mathbb{Q}_p}(\rho_{f^{(n)}})$$  

(5.4)
holds whenever \( \log_p \det_\Gamma f \) is defined. These are the cases

a where \( f \) is in \( 1 + p M_r(c_0(\Gamma, \mathbb{Z}_p)) \)

b where \( Wh_{\mathbb{F}_p}(\Gamma) \) is torsion and \( f \) is in \( GL_r(c_0(\Gamma, \mathbb{Z}_p)) \)

c where \( Wh_{\mathbb{F}_p}(\Gamma) \) is torsion, \( \mathbb{F}_p \Gamma \) has no zero divisors and \( f \) is in \( c_0(\Gamma)^\ast \).

**Proof** The assertions follow from propositions 4.3 and 4.4 together with calculations as in the proofs of theorems 5.1 and 5.4. 

We end the paper with some open questions: Is there a dynamical criterion for the existence of the limit defining \( p \)-adic entropy? Is there a notion of “\( p \)-adic expansiveness” for \( \Gamma \)-actions on compact spaces \( X \) which for the systems \( X_f \) with \( f \) in \( M_r(\mathbb{Z}_\Gamma) \) translates into the condition that \( f \) is invertible in \( M_r(c_0(\Gamma)) \), (or in \( M_r(c_0(\Gamma, \mathbb{Z}_p)) \))? In fact, I assume that \( p \)-adic entropy can only be defined for “\( p \)-adically expansive” systems, c.f. [BD99] Remark after proposition 1.3. Is there a direct proof that the limit in formula (5.4) exists?

Finally, in [BD99] a second version of a \( p \)-adic Mahler measure was defined which involves both the \( p \)-adic and the archimedian valuations of \( \mathbb{Q} \). Can this be obtained for the systems \( X_f \) by doing something more involved with the fixed points than taking their cardinalities and forming the limit (1.3)?

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