VERTEX-IRF TRANSFORMATIONS, DYNAMICAL QUANTUM GROUPS AND HARMONIC ANALYSIS

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Abstract. It is shown that a dynamical quantum group arising from a vertex-IRF transformation has a second realization with untwisted dynamical multiplication but nontrivial bigrading. Applied to the SL(2; C) dynamical quantum group, the second realization is naturally described in terms of Koornwinder’s twisted primitive elements. This leads to an intrinsic explanation why harmonic analysis on the “classical” SL(2; C) quantum group with respect to twisted primitive elements, as initiated by Koornwinder, is the same as harmonic analysis on the SL(2; C) dynamical quantum group.

Dedicated to Tom Koornwinder on the occasion of his 60th birthday

1. Introduction

In the remarkable paper [17], Koornwinder obtained in 1993 a subclass of the most general, classical family of basic hypergeometric orthogonal polynomials, known as Askey-Wilson polynomials, as spherical functions on the SL(2; C) quantum group. Koornwinder’s results have been refined (see e.g. [24], [12], [13]) and have been successfully extended to higher rank (see e.g. [22], [23]), leading to the interpretation of Macdonald-Koornwinder polynomials [20], [16] as spherical functions on higher rank quantum symmetric spaces. In this paper we provide a natural reinterpretation for some of Koornwinder’s [17] techniques and results in terms of the trigonometric SL(2; C) dynamical quantum group.

In 1998 and 1999, Etingof and Varchenko [6], [7] introduced the notion of a dynamical quantum group. A dynamical quantum group is a Hopf algebroid obtained by twisting a Hopf algebra by a dynamical twist. Dynamical twists occur naturally in conformal field theory as universal fusion matrices, see e.g. [8]. In §3 we give the construction of a dynamical quantum group starting from a five-tuple \((\mathcal{U}, \mathcal{A}, \langle \cdot, \cdot \rangle, T, J(\lambda))\) with \(\langle \cdot, \cdot \rangle\) a Hopf algebra pairing between two Hopf algebras \(\mathcal{U}\) and \(\mathcal{A}\), \(T\) a finite abelian subgroup of the group-like elements of \(\mathcal{U}\) and \(J(\lambda)\) a dynamical twist for \(\mathcal{U}\) with respect to \(T\). The dynamical quantum group construction in [7] is the special case that \(\mathcal{A}\) is the space of matrix coefficients of a suitable tensor category of \(\mathcal{U}\)-representations.

We are specifically interested in dynamical quantum groups with associated dynamical twists arising from vertex-IRF transformations [5]. In statistical mechanics vertex-IRF transformations arise as gauge transformations relating vertex models to Interaction-Round-a-Face (IRF) models. The notions of vertex-IRF transformations and dynamical twists are recalled in §2. We show in §3 that the dynamical quantum group arising from a
vertex-IRF transformation has, besides its usual realization with twisted dynamical multiplication and trivial bigrading, a second realization with untwisted dynamical multiplication and twisted bigrading.

The trigonometric $\text{SL}(2; \mathbb{C})$ dynamical quantum group is one of the few known nontrivial examples of a dynamical quantum group arising from a vertex-IRF transformation [2, 3]. We show in §4 that the alternative realization of the trigonometric $\text{SL}(2; \mathbb{C})$ dynamical quantum group, with untwisted dynamical multiplication and twisted bigrading, can be naturally formulated in terms of the eigenspace decomposition of the $\text{SL}(2; \mathbb{C})$ quantum group with respect to Koornwinder’s [17] twisted primitive elements. The key fact needed here is the observation of Rosengren [25] that the vertex-IRF transformation conjugates a group with respect to Koorwinder’s [17] twisted primitive elements.

As a consequence, we obtain an intrinsic link between Koelink’s and Rosengren’s [14] harmonic analysis on the trigonometric SL(2; $\mathbb{C}$) dynamical quantum group and harmonic analysis on the SL(2; $\mathbb{C}$) quantum group as initiated by Koornwinder [17] and extended by Noumi, Mimachi [24] and Koelink [12]. Such link was predicted by Koelink and Rosengren in the introduction of [14] after a direct comparison of their harmonic analytic results to the corresponding results in [17], [24] and [12]. We explore this intrinsic link in some detail by deriving several harmonic analytic results from [17], [24] and [12] as a direct consequence of the corresponding results on the SL(2; $\mathbb{C}$) dynamical quantum group from [14].

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2. Dynamical twists and vertex-IRF transformations

This section follows closely Etingof and Nikshych [5 §2.1], see also [4 §4.1].

Let $(\mathcal{U}, m_{\mathcal{U}}, 1_{\mathcal{U}}, \Delta_{\mathcal{U}}, \epsilon_{\mathcal{U}}, S_{\mathcal{U}})$ be a Hopf-algebra over $\mathbb{C}$ with multiplication $m_{\mathcal{U}}$, unit $1_{\mathcal{U}}$, comultiplication $\Delta_{\mathcal{U}}$, counit $\epsilon_{\mathcal{U}}$ and antipode $S_{\mathcal{U}}$. For $u \in \mathcal{U}^{\otimes k}$ with $k \leq n$ and for an ordered $k$-tuple $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ we write $u_{i_1 \ldots i_k}$ for the element of $\mathcal{U}^{\otimes n}$ obtained by mapping the $j$th tensor component of $\mathcal{U}^{\otimes k}$ to the $i_j$th tensor component of $\mathcal{U}^{\otimes n}$, where $\otimes = \otimes_{\mathbb{C}}$ denotes the usual tensor product over $\mathbb{C}$. For instance, $(u \otimes v)_{31} = v \otimes 1 \otimes u$ when regarded as element in $\mathcal{U}^{\otimes 3}$.

For $j \in \{1, \ldots, n\}$ we define algebra homomorphisms $\Delta_{\mathcal{U}^j} : \mathcal{U}^{\otimes n} \to \mathcal{U}^{\otimes (n+1)}$ and $\epsilon_{\mathcal{U}^j} : \mathcal{U}^{\otimes n} \to \mathcal{U}^{\otimes (n-1)}$ by letting $\Delta_{\mathcal{U}}$ and $\epsilon_{\mathcal{U}}$ act on the $j$th tensor component. Similarly we define the map $S_{\mathcal{U}^j} : \mathcal{U}^{\otimes n} \to \mathcal{U}^{\otimes n}$ by letting the antipode $S_{\mathcal{U}}$ act on the $j$th tensor component. With these notations, the Hopf-algebra identities are

$$\begin{align*}
\Delta_{\mathcal{U}^1} \circ \Delta_{\mathcal{U}} &= \Delta_{\mathcal{U}^2} \circ \Delta_{\mathcal{U}},
\epsilon_{\mathcal{U}^1} \circ \Delta_{\mathcal{U}} &= \text{Id}_{\mathcal{U}} = \epsilon_{\mathcal{U}^2} \circ \Delta_{\mathcal{U}},
m_{\mathcal{U}} \circ S_{\mathcal{U}^1} \circ \Delta_{\mathcal{U}}(\cdot) &= \epsilon_{\mathcal{U}}(\cdot) 1_{\mathcal{U}} = m_{\mathcal{U}} \circ S_{\mathcal{U}^2} \circ \Delta_{\mathcal{U}}(\cdot).
\end{align*}$$

We define the iterated comultiplication $\Delta_{\mathcal{U}}^{(n-1)} : \mathcal{U} \to \mathcal{U}^{\otimes n}$ inductively by $\Delta_{\mathcal{U}}^{(0)} = \text{Id}_{\mathcal{U}}$ for $n = 1$ and $\Delta_{\mathcal{U}}^{(n-1)} = \Delta_{\mathcal{U}^1} \circ \Delta_{\mathcal{U}}^{(n-2)}$ for $n \in \mathbb{Z}_{>1}$.
For a unital associative \(\mathbb{C}\)-algebra \(L\) we denote \(\mathcal{U}_L = L \otimes \mathcal{U}\) for the Hopf algebra over \(L\) obtained by extending the Hopf-algebra maps \(L\)-linearly. We keep the notation \(1_\mathcal{U}\) for the unit of \(\mathcal{U}_L\) and \(m_\mathcal{U}, \Delta_\mathcal{U}, \epsilon_\mathcal{U}\) and \(S_\mathcal{U}\) for the \(L\)-linear extended Hopf algebra maps of \(\mathcal{U}_L\).

Let \(\mathcal{U}^\times\) be the multiplicative group of invertible elements in \(\mathcal{U}\). The group-like elements
\[
G(\mathcal{U}) = \{ u \in \mathcal{U} \mid \Delta_\mathcal{U}(u) = u \otimes u, \ \epsilon_\mathcal{U}(u) = 1 \}
\]
is a subgroup of \(\mathcal{U}^\times\) since \(S_\mathcal{U}(u)\) is the inverse of a group-like element \(u\). Let \(T \subseteq G(\mathcal{U})\) be a finite abelian subgroup and write \(\hat{T}\) for the character group of \(T\). The value of a character \(\alpha \in \hat{T}\) at \(t \in T\) is denoted by \(t^\alpha\).

We suppose that \(\mathcal{U}\) is \(\text{ad}(T)\)-semisimple, where \(\text{ad}(u)\)
\[
\sum u(1)vS_\mathcal{U}(u(2)), \quad u, v \in \mathcal{U}
\]
is the analogue of the adjoint action. Consequently, \(\mathcal{U}\) is \(\hat{T}\)-graded,
\[
\mathcal{U} = \bigoplus_{\alpha \in \hat{T}} \mathcal{U}[\alpha]
\]
with \(\mathcal{U}[\alpha]\) the elements \(u \in \mathcal{U}\) satisfying \(\text{ad}(t)u = t^\alpha u\) for all \(t \in T\). Since \(T\) is abelian we have \(T \subseteq \mathcal{U}[0]\). Furthermore, for \(\alpha, \beta \in \hat{T}\) with \(\alpha \neq 0\),
\[
\epsilon_\mathcal{U}(\mathcal{U}[\alpha]) = 0, \quad \Delta_\mathcal{U}(\mathcal{U}[\beta]) \subseteq \bigoplus_{\gamma \in \hat{T}} \mathcal{U}[\gamma] \otimes \mathcal{U}[\beta - \gamma], \quad S_\mathcal{U}(\mathcal{U}[\beta]) \subseteq \mathcal{U}[\beta].
\]

The primitive idempotents of \(T\) are
\[
\pi_\alpha = \frac{1}{\# T} \sum_{t \in T} t^{-\alpha} t \in \mathcal{U} \quad (\alpha \in \hat{T}).
\]
Their basic properties are
\[
\sum_{\gamma \in \hat{T}} \pi_\gamma = 1_\mathcal{U}, \quad t\pi_\alpha = t^\alpha \pi_\alpha = \pi_\alpha t, \quad u\pi_\alpha = \pi_{\alpha + \beta} u
\]
for \(t \in T, u \in \mathcal{U}[\beta]\) and \(\alpha, \beta \in \hat{T}\), and
\[
\pi_\alpha \pi_\beta = \delta_{\alpha, \beta} \pi_\alpha, \quad \Delta_\mathcal{U}(\pi_\alpha) = \sum_{\beta \in \hat{T}} \pi_\beta \otimes \pi_{\beta - \alpha},
\]
\[
\epsilon_\mathcal{U}(\pi_\alpha) = \delta_{\alpha, 0}, \quad S_\mathcal{U}(\pi_\alpha) = \pi_{-\alpha}
\]
for \(\alpha, \beta \in \hat{T}\) with \(\delta_{\alpha, \beta}\) the Kronecker delta function on \(\hat{T} \times \hat{T}\).

Let \(F\) be the \(\mathbb{C}\)-algebra of complex valued functions on \(\hat{T}\) and set \(K = F \otimes F\). An element \(f \in K\) is denoted by \(f(\lambda, \mu) = \sum j f_j(\lambda) \otimes f'_j(\mu)\), with \(f_j, f'_j \in F\). The variables \(\lambda, \mu \in \hat{T}\) thus indicate the dependence in the first and second tensor component of \(K = F \otimes F\), respectively. In particular, we have the subalgebras \(F_1\) and \(F_2\) of \(K\) consisting of elements \(f(\lambda) = f \otimes 1\) and \(g(\mu) = 1 \otimes g\), respectively. Similar notations will be used for elements in \(\mathcal{U}_K, \mathcal{U}_{F_1}\) and \(\mathcal{U}_{F_2}\).
For \( u(\lambda, \mu) \in \mathcal{U}_K^{\otimes n} \) and \( j \in \{1, \ldots, n\} \) we denote
\[
\begin{align*}
  u(\lambda \pm h^{(j)}, \mu) &= \sum_{\alpha \in \hat{T}} u(\lambda \pm \alpha, \mu) \pi_{\alpha j} \in \mathcal{U}_K^{\otimes n}, \\
  u(\lambda, \mu \pm h^{(j)}) &= \sum_{\alpha \in \hat{T}} u(\lambda, \mu \pm \alpha) \pi_{\alpha j} \in \mathcal{U}_K^{\otimes n}.
\end{align*}
\]

We are now in a position to give the definition of a dynamical twist, see [5, Def. 2.3].

**Definition 2.1.** (i) We say that \( u \in \mathcal{U}_K^{\otimes n} \) is of zero \( T \)-weight if
\[
[u, \Delta^{(n-1)}_U(t)] = 0, \quad \forall t \in T.
\]

(ii) An invertible element \( J(\lambda) \in \mathcal{U}_{F_1}^{\otimes 2} \) is called a dynamical twist with respect to \( T \) if \( J(\lambda) \) is of zero \( T \)-weight satisfying
\[
\begin{align*}
  \epsilon_U(J(\lambda)) &= 1_U = \epsilon_U(J(\lambda)), \\
  \Delta_U(J(\lambda))J_{12}(\lambda + h^{(3)}) &= \Delta_U(J(\lambda))J_{23}(\lambda).
\end{align*}
\]

Observe that \( \mathcal{U}_K[0] \) is the set of zero \( T \)-weighted elements in \( \mathcal{U}_K \), and \( \bigoplus_{\alpha \in \hat{T}} K \otimes \mathcal{U}[\alpha] \otimes \mathcal{U}[-\alpha] \) is the set of zero \( T \)-weighted elements in \( \mathcal{U}_K^{\otimes 2} \).

**Example 2.2.** Suppose \( \mathcal{U} \) is the Drinfeld-Jimbo quantized universal enveloping algebra of a complex semisimple Lie algebra. A dynamical twist \( J(\lambda) \) then naturally arises as the universal fusion matrix of \( \mathcal{U} \), see \([8] \) and \([19] \).

**Definition 2.3.** Let \( x(\lambda) \in \mathcal{U}_{F_1} \) be invertible and suppose that \( \epsilon_U(x(\lambda)) = 1 \). If \( x(\lambda) \) is of zero \( T \)-weight then \( x(\lambda) \) is called a gauge transformation with respect to \( T \).

Gauge transformations can be used to “gauge” dynamical twists, see \([5] \) §2.1.

**Lemma 2.4.** Let \( J(\lambda) \in \mathcal{U}_{F_1}^{\otimes 2} \) be a dynamical twist and \( x(\lambda) \in \mathcal{U}_{F_1} \) be a gauge transformation with respect to \( T \). Then
\[
J_x(\lambda) = \Delta_U(x(\lambda))J(\lambda)x_2^{-1}(\lambda)x_1^{-1}(\lambda + h^{(2)})
\]

is a dynamical twist with respect to \( T \).

On the other hand, a dynamical twist “almost” gives rise to a gauge transformation, cf. \([7] \) §4.2, \([4] \) §4.3 and \([8] \) Lem. 2.12.

**Lemma 2.5.** If \( J(\lambda) \in \mathcal{U}_{F_1}^{\otimes 2} \) is a dynamical twist with respect to \( T \), then
\[
K^J(\lambda) = m_U(S_{U1}(J(\lambda))), \quad Q^J(\lambda) = m_U(S_{U2}(J^{-1}(\lambda))
\]

are of zero \( T \)-weight and satisfy
\[
Q^J(\lambda + h)K^J(\lambda) = 1_U,
\]

where \( Q^J(\lambda + h) = \sum_{\alpha \in \hat{T}} Q^J(\lambda \pm \alpha) \pi_\alpha \).
Proof. The only nontrivial part of Lemma 2.5 is (2.4), which follows from applying $m_{\mathcal{U}} \circ (m_{\mathcal{U}} \otimes \text{Id}_{\mathcal{U}}) \circ S_{\mathcal{U}^2}$ to the reformulation

$$\Delta_{\mathcal{U}^1}(J(\lambda)) = \Delta_{\mathcal{U}^2}(J(\lambda))J_{12}^{-1}(\lambda + \hbar^{(3)})$$

of the second equality of (2.2), and by using (2.1) and the first equality of (2.2). \square

If $\mathcal{U}_{F_1}$ does not have zero divisors, then it follows from Lemma 2.5 that $K^J(\lambda)$ and $Q^J(\lambda)$ are gauge transformations with respect to $T$, and that $(K^J)^{-1}(\lambda) = Q^J(\lambda + \hbar)$. A vertex-IRF transformation is the following generalization of a gauge transformation, see [5, Def. 2.6].

Definition 2.6. Let $x(\lambda) \in \mathcal{U}_{F_1}$ be invertible and suppose that $\epsilon_{\mathcal{U}}(x(\lambda)) = 1$. We say that $x(\lambda)$ is a vertex-IRF transformation with respect to $T$ if

$$j_x(\lambda) = \Delta_{\mathcal{U}}(x(\lambda))x_2^{-1}(\lambda)x_1^{-1}(\lambda + \hbar^{(2)}) \in \mathcal{U}^{\otimes 2}_{F_1}$$

is of zero $T$-weight.

A vertex-IRF transformation gives rise to a dynamical twist, see [5, Prop. 2.5].

Proposition 2.7. If $x(\lambda) \in \mathcal{U}_{F_1}$ is a vertex-IRF transformation with respect to $T$ then $j_x(\lambda) \in \mathcal{U}^{\otimes 2}_{F_1}$ is a dynamical twist with respect to $T$.

Key examples of vertex-IRF transformations arise in the theory of exactly solvable lattice models as gauge transformations relating vertex models to Interaction-Round-a-Face (IRF) models, see e.g. [9] and references therein.

We end this section by considering the special case of quasi-triangular Hopf-algebra’s $\mathcal{U}$. In that case the Hopf algebra $\mathcal{U}$ has a universal $R$-matrix $R \in \mathcal{U} \otimes \mathcal{U}$, which is invertible and satisfies

$$\Delta_{\mathcal{U}}^{op}(\cdot) = R \Delta_{\mathcal{U}}(\cdot) R^{-1}, \quad \Delta_{\mathcal{U}^1}(R) = R_{13}R_{23}, \quad \Delta_{\mathcal{U}^2}(R) = R_{13}R_{12},$$

with $\Delta_{\mathcal{U}}^{op}(u) = (\Delta_{\mathcal{U}}(u))_{21}$ the opposite comultiplication. In particular, $R$ satisfies the quantum Yang-Baxter equation (QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

in $\mathcal{U}^{\otimes 3}$. Gauging the universal $R$-matrix by a dynamical twist leads to a solution of a dynamical version of the quantum Yang-Baxter equation. The precise result is as follows, see [3, §3].

Proposition 2.8. Let $J(\lambda) \in \mathcal{U}^{\otimes 2}_{F_1}$ be a dynamical twist with respect to $T$. Then

$$R^J(\lambda) = J^{-1}(\lambda)R_{21}J_{21}(\lambda) \in \mathcal{U}^{\otimes 2}_{F_1}$$

satisfies

$$R^J_{12}(\lambda + \hbar^{(3)})R^J_{13}(\lambda)R^J_{23}(\lambda + \hbar^{(1)}) = R^J_{23}(\lambda)R^J_{13}(\lambda + \hbar^{(2)})R^J_{12}(\lambda).$$
As Koornwinder emphasizes in [18], the proof of Proposition 2.8 is a direct consequence of the identities

\begin{align*}
J_{12}^{-1}(\lambda + h^{(3)})\Delta_{u1}(R^J(\lambda))J_{12}(\lambda) &= R^J_{23}(\lambda)R^J_{13}(\lambda + h^{(2)}), \\
J_{23}^{-1}(\lambda)\Delta_{u2}(R^J(\lambda))J_{23}(\lambda + h^{(1)}) &= R^J_{12}(\lambda + h^{(3)})R^J_{13}(\lambda)
\end{align*}

(2.8)

which are dynamical analogues of \(\Delta_{u1}(R_{21}) = R_{32}R_{31}\) and \(\Delta_{u2}(R_{21}) = R_{21}R_{31}\) respectively.

**Definition 2.9.** Equation (2.8) is called the dynamical quantum Yang-Baxter equation (DQYBE). A solution \(R(\lambda) \in U_{F_1}^{\otimes 2}\) is called a dynamical universal R-matrix.

The DQYBE is also known as the Gervais-Neveu-Felder equation, and is closely related to Baxter’s star-triangle relation, see e.g. [11] and [9].

**Example 2.10.** In the setup of example 2.2, \(U\) is quasi-triangular, and the universal R-matrix \(R^J(\lambda)\) associated to the universal fusion matrix \(J(\lambda)\) is the universal exchange matrix of \(U\), see [8] and [10].

The complete integrability of vertex models and IRF models are governed by the quantum Yang-Baxter equation and Baxter’s star-triangle identity, respectively. For quasi-triangular Hopf-algebras \(U\) with vertex-IRF transformation \(x(\lambda)\), Proposition 2.8 shows that these basic integrability conditions are interrelated by twisting the corresponding universal R-matrix \(R\) of \(U\) with the dynamical twist \(j_x(\lambda)\). Note that the corresponding universal dynamical R-matrix \(R^{j_x}(\lambda) = j_x^{-1}(\lambda)R_{21}(j_x)21(\lambda)\) can alternatively be written as

\begin{equation}
R^{j_x}(\lambda) = x_1(\lambda + h^{(2)})x_2(\lambda)R_{21}x_1^{-1}(\lambda)x_2^{-1}(\lambda + h^{(1)}),
\end{equation}

(2.9)

see [8] Cor. 2.11].

3. DYNAMICAL QUANTUM GROUPS

Etingof and Varchenko [6, 7] gave a general construction of a Hopf algebroid starting from a given nondegenerate, polarized Hopf algebra \(U\) and a suitable tensor category of \(U\)-representations. The notion of a Hopf algebroid is closely related to weak Hopf algebras and quantum groupoids, see [5]. The constructed Hopf algebroids are modeled on the space of matrix coefficients of the \(U\)-representations from the given tensor category. The Hopf algebroid structures are governed by the fusion matrices of \(U\), or, when \(U\) is quasi-triangular, by the exchange matrices for \(U\). These Hopf algebroids are called exchange dynamical quantum groups, or simply dynamical quantum groups. A different, but closely related construction was given in [6] §4.3.

In this section we give the construction of dynamical quantum groups in a slightly different setup. The input data is a five-tuple \((U, A, \langle \cdot, \cdot \rangle, T, J(\lambda))\) with \(U\) and \(A\) Hopf algebras, with \(\langle \cdot, \cdot \rangle\) a Hopf pairing between \(U\) and \(A\), with \(T \subset G(U)\) a finite abelian subgroup such that \(U\) is ad\((T)\)-semisimple and such that \(A\) is \((T - T)\)-semisimple with respect to the left and right regular \(U\)-action on \(A\), and with \(J(\lambda) \in U_{F_1}^{\otimes 2}\) a dynamical twist for \(U\) with respect to \(T\).
The resulting Hopf algebroid $\mathcal{A}^J$ is now modeled on the space $\mathcal{A}_K = K \otimes \mathcal{A}$. We study $\mathcal{A}^J$ in more detail when the dynamical twist $J(\lambda)$ arises from a vertex-IRF transformation.

3.1. Hopf algebroids. This subsection follows closely [7, §3.1], see also [14, §2.1], [4, §2.2] and references therein.

Let $\hat{T}$ be the character group of some finite abelian group $T$, with group operation written additively. Let $F$ be the unital $\mathbb{C}$-algebra of complex valued functions on $\hat{T}$. We denote $1_F$ for the unit of $F$. Translation over $\alpha \in \hat{T},$

$$(T_\alpha f)(\lambda) = f(\lambda + \alpha), \quad f \in F,$$

defines an automorphism of $F$.

**Definition 3.1.** A $T$-algebra is a complex associative algebra $A$ with unit $1_A$ which is $\hat{T}$-bigraded,

$$A = \bigoplus_{\alpha, \beta \in \hat{T}} A_{\alpha \beta},$$

and which is endowed with two unital algebra embeddings $\mu_l = \mu_l^A, \mu_r = \mu_r^A : F \to A_{00}$ satisfying

$$\mu_l(f) \circ \mu_r(g) = \mu_r(g) \circ \mu_l(f),$$

$$\mu_l(f) \circ a = a \circ \mu_l(T_\alpha f),$$

$$\mu_r(f) \circ a = a \circ \mu_r(T_\beta f)$$

for $f, g \in F$ and $a \in A_{\alpha \beta}$, where $\circ$ denotes the multiplication of the algebra $A$. The algebra embeddings $\mu_l$ and $\mu_r$ are called the left and right moment maps of $A$, respectively.

A morphism $\phi : A \to B$ between $T$-algebras $A$ and $B$ is an algebra homomorphism satisfying

$$\phi(A_{\alpha \beta}) \subseteq B_{\alpha \beta}, \quad \phi(\mu_l^A(f)) = \mu_l^B(f), \quad \phi(\mu_r^A(f)) = \mu_r^B(f)$$

for $\alpha, \beta \in \hat{T}$ and $f \in F$.

**Example 3.2.** The formal $|\hat{T}|$-dimensional $F$-vector space $I = \bigoplus_{\alpha \in \hat{T}} FT_\alpha$ is a unital, associative $\mathbb{C}$-algebra with multiplication $(fT_\alpha) \circ (gT_\beta) = (f(T_\alpha g))T_{\alpha + \beta}$ and unit $T_0$. It naturally acts on $F$ as difference operators on $\hat{T}$ with coefficients from $F$. The algebra $I$ is a $T$-algebra with $(\alpha, \beta)$-bigraded piece

$$I_{\alpha \beta} = \begin{cases} 0 & \text{when } \alpha \neq \beta, \\ FT_{-\alpha} & \text{when } \alpha = \beta \end{cases}$$

and moment maps $\mu_l(f) = \mu_r(f) = fT_0$.

Any $T$-algebra $A$ has the structure of a $(F - F)$-bimodule,

$$f \star a \star g = \mu_l(f) \circ \mu_r(g) \circ a, \quad f, g \in F, \ a \in A.$$
The bigraded pieces $A_{\alpha\beta} \subseteq A$ are $(F - F)$-submodules. We define the tensor product of two $T$-algebras $A$ and $B$ by

$$A \hat{\otimes} B = \bigoplus_{\alpha, \beta \in T} (A \hat{\otimes} B)_{\alpha\beta}, \quad (A \hat{\otimes} B)_{\alpha\beta} = \bigoplus_{\gamma \in T} A_{\alpha\gamma} \otimes_F B_{\gamma\beta}.$$

The balancing condition for the tensor product thus is $(\mu_r(f) \circ a) \otimes_F b = f \otimes_F (\mu_l(f) \circ b)$ for $f \in F$, $a \in A_{\alpha\gamma}$ and $b \in A_{\gamma\beta}$. Then $A \hat{\otimes} B$ becomes a $T$-algebra with multiplication

$$(a \otimes_F b) \circ (a' \otimes_F b') = a \circ a' \otimes_F b \circ b',$$

with unit $1_A \otimes_F 1_B$, with $(\alpha, \beta)$-bigraded piece $(A \hat{\otimes} B)_{\alpha\beta}$ and with moment maps

$$\mu(f) = \mu^A_l(f) \otimes_F 1_B, \quad \mu_r(f) = 1_A \otimes_F \mu^B_r(f).$$

We define for two morphisms $\phi : A \to A'$ and $\psi : B \to B'$ of $T$-algebras a morphism $\phi \hat{\otimes} \psi : A \hat{\otimes} A' \to B \hat{\otimes} B'$ by the usual formula

$$(\phi \hat{\otimes} \psi)(a \otimes_F b) = \phi(a) \otimes_F \psi(b).$$

It is now straightforward to check that the category of $T$-algebras is a tensor category with tensor product $\hat{\otimes}$, unit object $I$, the obvious associativity constraint and unit constraints $l_A : I \hat{\otimes} A \to A$, $r_A : A \hat{\otimes} I \to A$ given by

$$l_A(fT_{-\alpha} \otimes_F a) = \mu^A_l(f) \circ a, \quad r_A(a \otimes_F fT_{-\beta}) = \mu^A_r(f) \circ a, \quad f \in F, \ a \in A_{\alpha\beta}.$$

In the remainder of the paper we use the unit constraints to identify the $T$-algebras $I \hat{\otimes} A$ and $A \hat{\otimes} I$ with $A$.

**Definition 3.3.** A $T$-bialgebroid is a $T$-algebra $A$ equipped with two morphisms $\Delta : A \to A \hat{\otimes} A$ and $\epsilon : A \to I$ satisfying the familiar coalgebra axioms

$$(\Delta \otimes I_A)\Delta = (I_A \otimes \Delta)\Delta, \quad (\epsilon \otimes I_A)\Delta = I_A = (I_A \otimes \epsilon)\Delta.$$

The definition of a $T$-Hopf algebroid is a bit more subtle. Suppose $A$ is a $T$-algebra and suppose that $\phi : A \to A$ is a $\mathbb{C}$-linear map satisfying

$$\phi(\mu_r(f) \circ a) = \phi(a) \circ \mu_l(f), \quad \phi(a \circ \mu_l(f)) = \mu_r(f) \circ \phi(a)$$

for $a \in A$ and $f \in F$. Let $\psi : A \to A$ be a morphism of $T$-algebras. Then there exist unique $\mathbb{C}$-linear maps, denoted suggestively by $m(\phi \otimes \psi), m(\psi \otimes \phi) : A \hat{\otimes} A \to A$ with $m$ the multiplication map of $A$, such that

$$m(\phi \otimes \psi)(a \otimes_F b) = \phi(a) \circ \psi(b), \quad m(\psi \otimes \phi)(a \otimes_F b) = \psi(a) \circ \phi(b)$$

for $a \in A_{\alpha\gamma}$ and $b \in A_{\gamma\beta}$.

For a difference operator $a \in I$ we denote $a1_F \in F$ for the function obtained by applying $a$ to the constant function $1_F \in F$. In other words, $a1_F = \sum_{\alpha} a_{\alpha}(\lambda)$ when $a = \sum_{\alpha} a_{\alpha}(\lambda)T_{-\alpha}$. The following definition of an antipode is from \cite{14} Def. 2.1.
Definition 3.4. An antipode $S$ for a $T$-bialgebroid $A$ is a $\mathbb{C}$-linear map $S : A \to A$ satisfying

$$S(\mu_r(f) \circ a) = S(a) \circ \mu_l(f), \quad S(a \circ \mu_l(f)) = \mu_r(f) \circ S(a)$$

for $f \in F$ and $a \in A$ and satisfying the antipode axioms

$$m(Id_A \otimes S)(\Delta(a)) = \mu_l(\epsilon(a)1_F), \quad m(S \otimes Id_A)(\Delta(a)) = \mu_r(T_\alpha(\epsilon(a)1_F))$$

for $a \in A_{\alpha\beta}$. The pair $(A, S)$ is called a $T$-Hopf algebroid.

Definition 3.5. A morphism $\phi : A \to B$ of $T$-Hopf algebroids $A$ and $B$ is a $T$-algebra morphism satisfying

$$\epsilon_B(\phi(a)) = \epsilon_A(a), \quad \Delta_B(\phi(a)) = (\phi \bowtie \phi)(\Delta_A(a)), \quad S_B(\phi(a)) = \phi(S_A(a))$$

for $a \in A$.

For $T = \{1\}$ the trivial group, the definition of a $T$-Hopf algebroid reduces to the familiar definition of a Hopf algebra over $\mathbb{C}$.

3.2. The bigraded Hopf algebra. Let $U$ be a Hopf algebra over $\mathbb{C}$ as considered in §2. Let $(A, m_A, 1_A, \Delta_A, \epsilon_A, S_A)$ be a Hopf algebra over $\mathbb{C}$ and suppose that there exists an Hopf-algebra pairing $\langle \cdot, \cdot \rangle : U \times A \to \mathbb{C}$ between $U$ and $A$, which we fix once and for all. Then

$$\langle u \cdot a, a_1 \rangle a_2 = \sum \langle u, a(a_1) a_2 \rangle, \quad a \cdot \langle u, a_2 \rangle = \sum \langle u, a(a_1) a_2 \rangle$$

for $u \in U$ and $a \in A$ defines a $(U - U)$-bimodule structure on $A$. We call (3.1) the left and right regular action of $U$ on $A$, respectively.

Example 3.6. Take $A = U^*$ the Hopf-algebra dual of $U$ with Hopf algebra pairing

$$\langle u, a \rangle = a(u), \quad u \in U, \quad a \in U^*.$$  

The associated $(U - U)$-bimodule structure on $U^*$ is the regular action

$$(u \cdot a \cdot u')(v) = a(u'vu), \quad a \in U^*, \quad u, u', v \in U.$$

Let $T \subseteq G(U)$ be a finite abelian subgroup such that $U$ is ad$(T)$-semisimple, cf. §2. Recall that $K = F \otimes F$ with $F$ the algebra of complex valued functions on $\widehat{T}$. We extend the Hopf-algebra maps of $U$ and $A$ $K$-linearly to arrive at Hopf algebras $U_K$ and $A_K$ over the $\mathbb{C}$-algebra $K$ respectively, cf. §2. We denote $ab = m_A(a \otimes b)$ for the multiplication of the two elements $a, b \in A_K$ in the $K$-algebra $A_K$. The extended comultiplication $\Delta_A$ can be viewed as map $\Delta_A : A_K \to A_K \otimes K A_K$ as well as map $\Delta_A : A_K \to K \otimes A \otimes A$ via the canonical identification $A_K \otimes K A_K \simeq A_K^{\otimes 2} = K \otimes A \otimes A$. We extend the $(U - U)$-bimodule structure $K$-linearly to a $(U_K - U_K)$-bimodule structure on $A_K$. The actions are given by the formula (3.1), with the Hopf pairing $\langle \cdot, \cdot \rangle$ extended $K$-bilinearly to a Hopf pairing between $U_K$ and $A_K$. Similarly, we have a componentwise extension of the bimodule structure on $A_K$ to a $(U_K^{\otimes n} - U_K^{\otimes n})$-bimodule structure on $A_K^{\otimes n}$.

In the following lemma we list some basic properties of the regular $U_K$-action on $A_K$. The straightforward proof is left to the reader.
Lemma 3.7. The \((U_K - U_K)\)-bimodule structure on \(A_K\) has the following properties:

\[
\begin{align*}
    u \cdot 1_A \cdot v &= \epsilon_U(u)\epsilon_U(v)1_A, \\
    u \cdot m_A(a \otimes_K b) \cdot v &= m_A\left(\Delta_U(u) \cdot (a \otimes_K b) \cdot \Delta_U(v)\right), \\
    \Delta_A(u \cdot a \cdot v) &= u_2 \cdot \Delta_A(a) \cdot v_1, \\
    \epsilon_A(u \cdot a) &= \epsilon_A(a \cdot u), \\
    u \cdot S_A(a) \cdot v &= S_A(S_U(v) \cdot a \cdot S_U(u))
\end{align*}
\]  

(3.2)

for \(u, v \in U_K\) and \(a \in A_K\).

We assume from now on that \(A\) is \((T - T)\)-semisimple,

\[
A = \bigoplus_{\alpha, \beta \in \hat{T}} A[\alpha, \beta],
\]  

(3.3)

with \(A[\alpha, \beta]\) consisting of elements \(a \in A\) satisfying \(t \cdot a = t^\beta a\) and \(a \cdot t = t^\alpha a\) for all \(t \in T\). Note that \(1_A \in A[0, 0]\) by the first equality of (3.2). The direct sum decomposition (3.3) defines a \(\hat{T}\)-bigrading of \(A\) due to the second equality of (3.2).

Note that the primitive idempotents \(\pi_\alpha \in U\) (\(\alpha \in \hat{T}\)) of \(T\) act on \(A\) by

\[
\pi_\alpha \cdot a \cdot \pi_\beta = \delta_{\alpha, \delta} \delta_{\beta, \gamma} a, \quad a \in A_K[\gamma, \delta]
\]

and that the \(\hat{T}\)-bigrading of \(A\) is compatible with the \(\hat{T}\)-grading of \(U\),

\[
U[\alpha] \cdot A[\beta, \gamma] \subseteq A[\beta, \alpha + \gamma], \quad A[\beta, \gamma] \cdot U[\alpha] \subseteq A[\beta - \alpha, \gamma]
\]

Lemma 3.7 implies that the \(\hat{T}\)-bigrading of \(A_K\) is compatible with the Hopf-algebra maps of \(A_K\).

Corollary 3.8. Let \(\alpha, \beta \in \hat{T}\). Then

\[
\begin{align*}
    \Delta_A(A_K[\alpha, \beta]) &\subseteq \bigoplus_{\gamma \in \hat{T}} A_K[\alpha, \gamma] \otimes_K A_K[\gamma, \beta], \\
    \epsilon_A(A_K[\alpha, \beta]) &= \{0\} \quad \text{unless} \quad \alpha = \beta, \\
    S_A(A_K[\alpha, \beta]) &\subseteq A_K[-\beta, -\alpha].
\end{align*}
\]

The straightforward proof of Corollary 3.8 is left to the reader.

3.3. The dynamical quantum group \(A^J\). The constructions in this subsection are motivated by the dynamical quantum group constructions of Etingof and Varchenko [7, §4] and Etingof and Nikshych [4, §4.3]. Since the proofs in this subsection are quite analogous to the ones in [7, §4], we only indicate their main steps.

We keep the conventions and notations of the previous subsection. We fix a dynamical twist \(J(\lambda)\) for \(U\) with respect to \(T\).
Lemma 3.9. The \( \mathbb{C} \)-vectorspace \( A_K \) is a \( T \)-algebra with multiplication
\[
m^J(a(\lambda, \mu) \otimes b(\lambda, \mu)) = m_A(J(\mu) \cdot (a(\lambda + \alpha, \mu + \beta) \otimes b(\lambda, \mu)) \cdot J^{-1}(\lambda))
\]
for \( a(\lambda, \mu) \in A_K \) and \( b(\lambda, \mu) \in A_K[\alpha, \beta] \), with unit \( 1_A \), with \( (\alpha, \beta) \)-bigraded pieces \( A_K[\alpha, \beta] \) \((\alpha, \beta \in \hat{T})\) and with moment maps
\[
\mu_l(f) = f(\lambda)1_A, \quad \mu_r(f) = f(\mu)1_A
\]
for \( f \in F \), where (recall) \( f(\lambda) = f \otimes 1_F \in K \) and \( f(\mu) = 1_F \otimes f \in K \).

Proof. The fact that \( J(\lambda) \) is of zero \( T \)-weight implies that \( A_K = \bigoplus_{\alpha, \beta} A_K[\alpha, \beta] \) defines a \( \hat{T} \)-bigrading with respect to the new multiplication \( m^J \). The second line of (2.2) implies the associativity of \( m^J \). The first line of (2.2) implies that \( 1_A \) is the unit element with respect to \( m^J \). The axioms for the moment maps are straightforward.

We call \( m^J \) the \( J \)-twisted dynamical multiplication on \( A_K \). We write \( A^J \) for the \( \mathbb{C} \)-vectorspace \( A_K \), viewed as \( T \)-algebra by Lemma 3.9. The following proposition provides the link between the \( T \)-algebra \( A^J \) and the Faddeev-Reshetikhin-Takhtajan (FRT) type construction of dynamical quantum groups.

Proposition 3.10. Suppose \( U \) is quasi-triangular with universal \( R \)-matrix \( R \). Let \( R^J(\lambda) \) (see (2.6)) be the corresponding dynamical universal \( R \)-matrix. Then
\[
m^J(R^J(\mu) \cdot (a \otimes b)) = m^J((b \otimes a) \cdot R^J_{21}(\lambda)), \quad \forall a, b \in A \subset A^J,
\]
where we use the convention that the \( \mu \) and \( \lambda \) dependence of the action of the dynamical universal \( R \)-matrices end up in the second tensor component. In other words,
\[
m_A(J(\mu)R^J(\mu) \cdot (a \otimes b) \cdot J^{-1}(\lambda)) = m_A(J(\mu) \cdot (b \otimes a) \cdot R^J_{21}(\lambda)J^{-1}(\lambda)), \quad \forall a, b \in A.
\]

Proof. This follows directly from the well known FRT type commutation relations
\[
m_A(R_{21} \cdot (b \otimes a)) = m_A((a \otimes b) \cdot R)
\]
for \( a, b \in A \).

Let \( A_K \bigotimes_{F} A_K \) be the \( \mathbb{C} \)-linear vector space defined by taking the tensor product over \( F \) with respect to the balancing condition \( (f(\mu)a) \bigotimes_F b = a \bigotimes_F (f(\lambda)b) \) for \( f \in F \) and \( a, b \in A_K \). The restricted tensor product \( A^J \bigotimes_{F} A^J \) naturally identifies as \( \mathbb{C} \)-vectorspace with the subspace
\[
\bigoplus_{\alpha, \beta, \gamma \in \hat{T}} A_K[\alpha, \gamma] \bigotimes_F A_K[\gamma, \beta]
\]
of \( A_K \bigotimes_{F} A_K \). Let
\[
\pi : K \otimes A \otimes A \to A_K \bigotimes_{F} A_K
\]
be the unique \( \mathbb{C} \)-linear map satisfying \( \pi(f(\lambda) \otimes g(\mu) \otimes a \otimes b) = (f(\lambda)a) \bigotimes_F (g(\mu)b) \) for \( f, g \in F \) and \( a, b \in A \).
Proposition 3.11. The $T$-algebra $\mathcal{A}^J$ is a $T$-bialgebroid with coalgebroid maps $\Delta : \mathcal{A}^J \rightarrow \mathcal{A}^J \otimes \mathcal{A}^J$ and $\varepsilon : \mathcal{A}^J \rightarrow I$ defined by
\[
\Delta(a) = \pi(\Delta_A(a)),
\]
\[
\varepsilon(a) = T_{-a}(m_F(\varepsilon_A(a)))T_{-a},
\]
for $a \in \mathcal{A}_K[\alpha, \beta]$, where $m_F : K = F \otimes F \rightarrow F$, $m_F(f(\lambda) \otimes g(\mu)) = f(\lambda)g(\lambda)$ is the multiplication map of the $\mathbb{C}$-algebra $F$.

Proof. Corollary 3.8 implies that the image of $\Delta$ is contained in $\mathcal{A}^J \otimes \mathcal{A}^J$ and that $\Delta$ and $\varepsilon$ preserve the $\tilde{T}$-bigrading. The maps $\varepsilon$ and $\Delta$ are clearly compatible with the moment maps. To prove that $\varepsilon : \mathcal{A}^J \rightarrow I$ and $\Delta : \mathcal{A}^J \rightarrow \mathcal{A}^J \otimes \mathcal{A}^J$ are morphisms of $T$-algebras it thus remains to show that they are algebra homomorphisms. This follows from the compatibility of the counit $\varepsilon_A$ and comultiplication $\Delta_A$ with the $(U_K - U_K)$-action on $\mathcal{A}_K$, see Lemma 3.7. The remainder of the proof is straightforward. \hfill \Box

In the following proposition we define an antipode $S^J$ for the $T$-bialgebroid $\mathcal{A}^J$ using the two zero $T$-weighted elements $K^J$ and $Q^J$ of $U_K$ associated to $J$, see Lemma 2.5.

Proposition 3.12. The $T$-bialgebroid $\mathcal{A}^J$ is a $T$-Hopf algebroid with antipode $S^J : \mathcal{A}^J \rightarrow \mathcal{A}^J$ defined by
\[
S^J(a(\lambda, \mu)) = S_A(K^J(\lambda - \beta) \cdot a(\mu - \alpha, \lambda - \beta) \cdot Q^J(\mu)), \quad a(\lambda, \mu) \in \mathcal{A}_K[\alpha, \beta].
\]
In particular,
\[
S^J(a) = S_A(K^J(\lambda - h) \cdot a \cdot Q^J(\mu)) \quad \forall a \in \mathcal{A}.
\]

Proof. Lemma 3.7 and the fact that $K^J$ and $Q^J$ are of zero $T$-weight imply that $S^J(\mathcal{A}_K[\alpha, \beta]) \subseteq \mathcal{A}_K[-\beta, -\alpha]$.

A straightforward computation then shows that the linear map $S^J : \mathcal{A}^J \rightarrow \mathcal{A}^J$ defined by (3.5) satisfies the required compatibility conditions with respect to the moment maps of $\mathcal{A}^J$. Hence it suffices to prove the antipode identities for $a \in \mathcal{A} \subset \mathcal{A}^J$. The required antipode identities then reduce to
\[
m^J(\text{Id}_{\mathcal{A}^J} \otimes S^J)(\Delta(a)) = \varepsilon_A(a)1_A = m^J(S^J \otimes \text{Id}_{\mathcal{A}^J})(\Delta(a)),
\]
which can be proven by direct computations using Lemma 3.7, the antipode axioms for $\mathcal{U}$ and (2.24). \hfill \Box

Definition 3.13. We call the $T$-Hopf algebroid $(\mathcal{A}^J = \oplus_{\alpha, \beta} \mathcal{A}_K[\alpha, \beta], m^J, 1_A, \Delta, \varepsilon, S^J)$ the dynamical quantum group associated to the five-tuple $(\mathcal{U}, \mathcal{A}, \langle \cdot, \cdot \rangle, T, J(\lambda))$.

Remark 3.14. Associated to the four-tuple $(\mathcal{U}, \mathcal{A}, \langle \cdot, \cdot \rangle, T)$ we always have the trivial dynamical quantum group $\mathcal{A}^1$, whose associated dynamical twist is the unit element $1 = 1_{U_K^{\otimes 2}} \in U_K^{\otimes 2}$. The multiplication and antipode of $\mathcal{A}^1$ are
\[
m^1(a(\lambda, \mu) \otimes b(\lambda, \mu)) = m_A(a(\lambda + \alpha, \mu + \beta) \otimes b(\lambda, \mu)),
\]
\[
S^1(b(\lambda, \mu)) = S_A(b(\mu - \alpha, \lambda - \beta))
\]
for \( a(\lambda, \mu) \in \mathcal{A}^1 \) and \( b(\lambda, \mu) \in \mathcal{A}_K[\alpha, \beta] \), which are the trivial dynamical extensions of the multiplication and antipode of \( \mathcal{A} \).

### 3.4. Gauge equivalent dynamical quantum groups

Let \( x(\lambda) \in \mathcal{U}_{\hat{T}_1} \) be a gauge transformation and \( J(\lambda) \in \mathcal{U}^{2}_{\hat{T}_1} \) a dynamical twist with respect to \( T \). Recall that the corresponding gauged dynamical twist is given by

\[
J_x(\lambda) = \Delta_U(x(\lambda))J(\lambda)x_2^{-1}(\lambda)x_1^{-1}(\lambda + h^{(2)}) \in \mathcal{U}^{2}_{\hat{T}_1}.
\]

**Proposition 3.15.** The dynamical quantum groups \( \mathcal{A}^J \) and \( \mathcal{A}^{Js} \) are isomorphic. The corresponding \( T \)-Hopf algebroid isomorphism \( \phi_x : \mathcal{A}^J \rightarrow \mathcal{A}^{Js} \) is

\[
\phi_x(a) = x(\mu) \cdot a \cdot x^{-1}(\lambda), \quad a \in \mathcal{A}^J.
\]

**Proof.** Since \( x(\lambda) \) is of zero weight, we have \( \phi_x(\mathcal{A}_K[\alpha, \beta]) = \mathcal{A}_K[\alpha, \beta] \) for \( \alpha, \beta \in \hat{T} \). The fact that \( \epsilon_U(x(\lambda)) = 1 \) implies \( \phi_x(1_{\mathcal{A}}) = 1_{\mathcal{A}} \). It follows now directly that \( \phi_x \) respects the moment maps. The map \( \phi_x \) is an algebra homomorphism since for \( a \in \mathcal{A} \) and \( b \in \mathcal{A}[\alpha, \beta] \),

\[
\phi_x(m^J(a \otimes b)) = x(\mu) \cdot (m_A(J(\mu) \cdot (a \otimes b) \cdot J^{-1}(\lambda))) \cdot x^{-1}(\lambda)
\]

\[
= m_A(\Delta_U(x(\mu)J(\mu) \cdot (a \otimes b) \cdot J^{-1}(\lambda)\Delta_U(x^{-1}(\lambda)))
\]

\[
= m_A(J_x(\mu) \cdot (x(\mu + \beta) \cdot a \cdot x^{-1}(\lambda + \alpha) \otimes x(\mu) \cdot b \cdot x^{-1}(\lambda)) \cdot J_x^{-1}(\lambda))
\]

\[
= m^{Js}(\phi_x(a) \otimes \phi_x(b)),
\]

hence \( \phi_x \) is an isomorphism of \( T \)-algebras. The proof that \( \phi_x \) is an isomorphism of \( T \)-Hopf algebroids follows from Lemma 3.17 by direct computations. We give the computations for the comultiplication and the antipode. For the compatibility of \( \phi_x \) with respect to the comultiplication we compute for \( a \in \mathcal{A} \),

\[
(\phi_x \otimes \phi_x)(\Delta(a)) = \sum (x(\mu) \cdot a^{(1)} \cdot x^{-1}(\lambda)) \otimes_F (x(\mu) \cdot a^{(2)} \cdot x^{-1}(\lambda))
\]

\[
= \sum (a^{(1)} \cdot x^{-1}(\lambda)) \otimes_F (x(\mu) \cdot a^{(2)} \cdot x(\lambda)x^{-1}(\lambda))
\]

\[
= \sum (a^{(1)} \cdot x^{-1}(\lambda)) \otimes_F (x(\mu) \cdot a^{(2)})
\]

\[
= (\pi \circ \Delta_A)(x(\mu) \cdot a \cdot x^{-1}(\lambda))
\]

\[
= \Delta(\phi_x(a)).
\]

For the compatibility of \( \phi_x \) with respect to the antipode we first note that

\[
K^{Js}(\lambda) = S_U(x^{-1}(\lambda - h))K^J(\lambda)x^{-1}(\lambda),
\]

\[
Q^{Js}(\lambda) = x(\lambda - h)Q^J(\lambda)S_U(x(\lambda)).
\]

For \( a \in \mathcal{A}[\alpha, \beta] \) we then have

\[
\phi_x(S^J(a)) = x(\mu) \cdot S_A(K^J(\lambda - \beta) \cdot a \cdot Q^J(\mu)) \cdot x^{-1}(\lambda)
\]

\[
= S_A(S_U(x^{-1}(\lambda))K^J(\lambda - \beta) \cdot a \cdot Q^J(\mu)S_U(x(\mu)))
\]

\[
= S_A(K^{Js}(\lambda - \beta)x(\lambda - \beta) \cdot a \cdot x^{-1}(\mu - \alpha)Q^{Js}(\mu))
\]

\[
= S^{Js}(\phi_x(a)),
\]
as desired. \hfill \Box

3.5. The dynamical quantum group associated to a vertex-IRF transformation. For a vertex-IRF transformation \( x(\lambda) \in \mathcal{U}_F \), with respect to \( T \) we define

\[
\mathcal{A}_{\alpha \beta}^{x} = x^{-1}(\mu) \cdot \mathcal{A}_K[\alpha, \beta] \cdot x(\lambda), \quad \alpha, \beta \in \hat{T}.
\]  

Note that in general \( \mathcal{A}_{\alpha \beta}^{x} \neq \mathcal{A}_K[\alpha, \beta] \) since \( x(\lambda) \) is not necessarily of zero \( T \)-weight. We define a \( T \)-Hopf algebroid structure on \( \mathcal{A}_K \) such that \( \mathcal{A}_K = \oplus_{\alpha, \beta} \mathcal{A}_{\alpha \beta}^{x} \) is the associated \( \hat{T} \)-bigrading. Recall the dynamical twist \( j_x(\lambda) = \Delta_U(x(\lambda))x_{2}^{-1}(\lambda)x_{1}^{-1}(\lambda + h^{(2)}) \)

associated to the vertex-IRF transformation \( x(\lambda) \).

Theorem 3.16. Let \( x(\lambda) \in \mathcal{U}_F \) be a vertex-IRF transformation with respect to \( T \).

a. The \( \mathbb{C} \)-vectorspace \( \mathcal{A}_K \) is a \( T \)-Hopf algebroid with multiplication

\[
m_x(a(\lambda, \mu) \otimes b(\lambda, \mu)) = m_A(a(\lambda + \alpha, \mu + \beta) \otimes b(\lambda, \mu)), \quad a \in A, \ b(\lambda, \mu) \in \mathcal{A}_{\alpha \beta}^{x},
\]

with unit \( 1_A \), with \( (\alpha, \beta) \)-bigraded pieces \( \mathcal{A}_{\alpha \beta}^{x} \) for \( \alpha, \beta \in \hat{T} \), with moment maps

\[
\mu_1(f) = f(\lambda)1_A, \quad \mu_r(f) = f(\mu)1_A, \quad f \in F,
\]

with comultiplication \( \Delta = \pi \circ \Delta_A \), with counit

\[
\epsilon_x(a) = T_{-\alpha}(m_F(\epsilon_A(a)))T_{-\alpha}, \quad a \in \mathcal{A}_{\alpha \beta}^{x}
\]

and with antipode

\[
S_x(a(\lambda, \mu)) = S_A(a(\mu - \alpha, \lambda - \beta)), \quad a(\lambda, \mu) \in \mathcal{A}_{\alpha \beta}^{x}.
\]

We write \( \mathcal{A}^x \) for \( \mathcal{A}_K \) viewed as \( T \)-Hopf algebroid in this way.

b. The map \( \phi_x : \mathcal{A}^x \rightarrow \mathcal{A}^{J^x} \) defined by

\[
\phi_x(a) = x(\mu) \cdot a \cdot x^{-1}(\lambda), \quad a \in \mathcal{A}^x
\]

is an isomorphism of \( T \)-Hopf algebroids.

Proof. There is a unique \( T \)-Hopf algebroid structure on \( \mathcal{A}_K \) turning the \( K \)-linear isomorphism \( \phi_x : \mathcal{A}_K \rightarrow \mathcal{A}^{J^x} \), defined by \( \phi_x(a) = x(\mu) \cdot a \cdot x^{-1}(\lambda) \), into an isomorphism of \( T \)-Hopf algebroids. A direct computation, which is similar to the proof of Proposition 3.15 for the special case that \( J(\lambda) = 1 \) is the trivial dynamical twist, proves that the resulting \( T \)-Hopf algebroid structure on \( \mathcal{A}_K \) is defined by the explicit formulas as given in part a of the theorem. \hfill \Box

For \( x(\lambda) \) a gauge transformation with respect to \( T \) we have \( \mathcal{A}^1 = \mathcal{A}^x \simeq \mathcal{A}^{J^x} \) as \( T \)-Hopf algebroids, with \( \mathcal{A}^1 \) the trivial dynamical quantum group associated to \((\mathcal{U}, \mathcal{A}, \langle \cdot, \cdot \rangle, T)\), cf. Remark 3.14. In case of a vertex-IRF transformation \( x(\lambda) \) the bigrading of \( \mathcal{A}^x \) is a nontrivial twisting of the trivial bigrading of \( \mathcal{A}^1 \), but the remaining \( T \)-Hopf algebroid structures of \( \mathcal{A}^x \) have the same untwisted form as the \( T \)-Hopf algebroid structures of \( \mathcal{A}^1 \).
4. Askey-Wilson polynomials and the $\text{SL}(2)$ (dynamical) quantum group

The work of Babelon [2] (see also [3]) implies that the trigonometric $\text{SL}(2; \mathbb{C})$ dynamical quantum group arises from a vertex-IRF transformation. In this section we describe the two different realizations of the $\text{SL}(2; \mathbb{C})$ dynamical quantum group (due to Theorem 3.16), leading to an intrinsic link between the work of Koelink, Rosengren [14] on the trigonometric $\text{SL}(2; \mathbb{C})$ dynamical quantum group and the work of Koornwinder [17], Noumi, Mimachi [24] and Koelink [12], [13] on the $\text{SL}(2; \mathbb{C})$ quantum group. The important additional property of the vertex-IRF transformation which is needed here is Rosengren’s [25] observation that the vertex-IRF transformation conjugates the $T$-action to an action by Koornwinder’s [17] twisted primitive elements (the explicit identification of Rosengren’s [25] generalized group element with Babelon’s [2] vertex-IRF transformation is an unpublished observation of Rosengren, see also the introduction of [14]).

We fix a deformation parameter $q \in \mathbb{C}^*$ which is not a root of unity. Let $q^{1/2}$ be a fixed choice of square root of $q$.

4.1. The $\text{SL}(2)$ quantum group. Let $\mathcal{U} = \mathcal{U}_q(\mathfrak{sl}(2))$ be the unital associative $\mathbb{C}$-algebra with generators $k^{\pm 1}$, $e$, $f$ and relations
\[
kk^{-1} = k^{-1}k = 1, \\
ke = qek, \quad kf = q^{-1}fk, \\
ef - fe = \frac{k^2 - k^{-2}}{q - q^{-1}}.
\]

The algebra $\mathcal{U}$ is a Hopf-algebra with comultiplication
\[
\Delta_{\mathcal{U}}(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1}, \\
\Delta_{\mathcal{U}}(e) = k \otimes e + e \otimes k^{-1} \\
\Delta_{\mathcal{U}}(f) = k \otimes f + f \otimes k^{-1},
\]
with counit
\[
\epsilon_{\mathcal{U}}(k^{\pm 1}) = 1, \quad \epsilon_{\mathcal{U}}(e) = \epsilon_{\mathcal{U}}(f) = 0
\]
and with antipode
\[
S_{\mathcal{U}}(k^{\pm 1}) = k^{\mp 1}, \quad S_{\mathcal{U}}(e) = -q^{-1}e, \quad S_{\mathcal{U}}(f) = -qf.
\]
We take $T = \{k^m \mid m \in \mathbb{Z}\} \subset G(\mathcal{U})$ as abelian subgroup of the group-like elements in $\mathcal{U}$.

The role of the character group $\hat{T}$ is taken over by the integers $\mathbb{Z}$, viewed as characters of $T$ by
\[
(k^m)^\alpha = q^{m\alpha/2}, \quad m, \alpha \in \mathbb{Z}.
\]

Note that $\mathcal{U}$ is $\text{ad}(T)$-semisimple with spectrum contained in $2\mathbb{Z}$. 

The type 1 irreducible, finite dimensional \( \mathcal{U} \)-representations are parametrized by \( \mathbb{Z}_{\geq 0} \). For \( m \in \mathbb{Z}_{\geq 0} \) the corresponding spin \( \frac{m}{2} \) representation \( V_m \) is a \( m + 1 \)-dimensional representation with basis \( v_r^m \) (\( r = -m, 2 - m, \ldots, m - 2, m \)) and action

\[
\begin{align*}
k^{\pm 1} v_r^m &= q^{\pm \frac{1}{2} m} v_r^m, \\
e v_r^m &= \sqrt{q^m - q} \frac{q^{-\frac{1}{2}(m+r+2)} - q^{\frac{1}{2}(m+r+2)}}{q^{-1} - q} v_{r+2}^m, \\
f v_r^m &= \sqrt{q^m - q} \frac{q^{-\frac{1}{2}(m+r)} - q^{\frac{1}{2}(m+r)}}{q^{-1} - q} v_{r-2}^m
\end{align*}
\]

where \( v_r^{m+2} = v_r^{m-2} = 0 \) by convention. Type 1 refers to the fact that the modules \( V_m \) are \( T \)-semisimple with spectrum contained in \( \mathbb{Z} \). We denote \( (\cdot, \cdot) : V_m \otimes V_m \to \mathbb{C} \) for the \emph{bilinear} pairing such that \( (v_r^m, v_s^n) = \delta_{r,s} \). Then

\[
(4.1) \quad (X v, w) = (v, X^\dagger w), \quad X \in \mathcal{U}, \ v, w \in V_m
\]

with \( \dagger : \mathcal{U} \to \mathcal{U} \) the unital \( \mathbb{C} \)-linear antiinvolution determined by

\[
(k^{\pm 1})^\dagger = k^{\mp 1}, \quad e^\dagger = f, \quad f^\dagger = e.
\]

Allowing suitable completions, \( \mathcal{U} \) is a quasi-triangular Hopf algebra. We denote \( \mathcal{R} \) by the corresponding Drinfeld universal \( R \)-matrix. Its action on \( V_1 \otimes V_1 \) is given by the matrix

\[
(4.2) \quad \mathcal{R}|_{V_1 \otimes V_1} = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}
\]

with respect to the ordered basis \( \{ v_1^1 \otimes v_1^1, v_1^1 \otimes v_{-1}^1, v_{-1}^1 \otimes v_1^1, v_{-1}^1 \otimes v_{-1}^1 \} \).

The quantized function algebra \( \mathcal{A} = \mathcal{A}_q[\text{SL}(2)] \) is the Hopf-subalgebra of the Hopf-dual \( \mathcal{U}^* \) spanned by the matrix coefficients of the finite dimensional type 1 \( \mathcal{U} \)-representations. The Peter-Weyl decomposition of \( \mathcal{A} \) is

\[
\mathcal{A} = \bigoplus_{m=0}^{\infty} W(m),
\]

\[
W(m) = \text{span}_\mathbb{C} \{ t_{rs}^m(\cdot) \mid r, s = -m, 2 - m, \ldots, m - 2, m \},
\]

where \( t_{rs}^m \) is the matrix coefficient \( t_{rs}^m(\cdot) = (\cdot v_s^m, v_r^m) \in \mathcal{A}[r,s] \). The Peter-Weyl decomposition is the irreducible decomposition of \( \mathcal{A} \), viewed as \( (\mathcal{U} - \mathcal{U}) \)-bimodule. Clearly \( \mathcal{A} \) is \( (T - T) \)-semisimple. Its \( (\alpha, \beta) \)-bigraded piece \( \mathcal{A}[\alpha, \beta] \) is nonzero when \( \alpha \) and \( \beta \) are integers having the same parity.

The Hopf-algebra \( \mathcal{A} \) is generated as unital \( \mathbb{C} \)-algebra by the matrix coefficients of the two-dimensional representation \( V_1 \), which we denote by

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} (\cdot v_1^1, v_1^1) & (\cdot v_{-1}^1, v_1^1) \\ (\cdot v_1^1, v_{-1}^1) & (\cdot v_{-1}^1, v_{-1}^1) \end{pmatrix}.
\]
Note that $\alpha \in \mathcal{A}[1, 1]$, $\beta \in \mathcal{A}[1, -1]$, $\gamma \in \mathcal{A}[-1, 1]$ and $\delta \in \mathcal{A}[-1, -1]$. The characterizing commutation relations are governed by the FRT relations \([3, 4]\) for $a, b \in \{\alpha, \beta, \gamma, \delta\}$ and by the determinant relation
\[
\delta \alpha - q^{-1} \beta \gamma = 1_{\mathcal{A}}.
\]
Explicitly, the FRT commutation relations give the relations
\[
\alpha \beta = q \beta \alpha, \quad \alpha \gamma = q \gamma \alpha, \quad \beta \delta = q \delta \beta, \quad \gamma \delta = q \delta \gamma, \quad \beta \gamma = \gamma \beta,
\]
\[
\alpha \delta = - \delta \alpha = (q - q^{-1}) \beta \gamma.
\]
Since for $m \in \mathbb{Z}_{\geq 0}$ and $r, s \in \{-m, 2 - m, \ldots, m - 2, m\}$,
\[
\text{span}_{\mathbb{C}}\{t_{rs}^m\} = W(m) \cap \mathcal{A}[r, s],
\]
the study of the matrix coefficients $t_{rs}^m$ relates to harmonic analysis on the $\text{SL}(2; \mathbb{C})$ quantum group $\mathcal{A}$ with respect to $T$. Considering the $t_{rs}^m$ ($r, s = -m, 2 - m, \ldots, m - 2, m$) as the matrix coefficients of a finite dimensional $\mathcal{A}$-corepresentation, the study of the $t_{rs}^m$ relates to harmonic analysis on $\mathcal{A}$ with respect to the standard quantum analogue of the Cartan subalgebra of $\mathfrak{sl}(2; \mathbb{C})$.

A first example relating basic hypergeometric series to harmonic analysis on quantum groups is the expression of the coefficients $t_{rs}^m$ ($r, s \in \{-m, 2 - m, \ldots, m - 2, m\}$) in terms of little $q$-Jacobi polynomials, see \([26, 15]\) and \([21]\). As a special case we recall the formula for the matrix coefficients $t_{rs}^m$ with integers $r, s, m$ having the same parity and satisfying $-m \leq r \leq s \leq -r \leq m$, given by
\[
t_{rs}^m = C_{rs}^m \frac{1 - q^{2m}}{q^{-r} - q^{r}} \gamma \frac{\varphi_1}{2} \left( q^{-r-m}, q^{-r+m-2}; q^2, -q \beta \gamma \right)
\]
for some explicit nonzero constant $C_{rs}^m$. Here
\[
r+1 \varphi_r \left( a_1, \ldots, a_{r+1}; q, z \right) = \sum_{m=0}^{\infty} \frac{(a_1; q)_m \cdots (a_{r+1}; q)_m}{(q; q)_m (b_1; q)_m \cdots (b_r; q)_m} z^m,
\]
with $(a; q)_m = \prod_{j=0}^{m-1} (1 - aqu^j)$ ($m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$) the $q$-shifted factorial, is the $r+1 \varphi_r$ basic hypergeometric series, see \([10]\). The element $\beta \gamma$ is an algebraic generator of the unital $\mathbb{C}$-subalgebra $\mathcal{A}[0, 0]$ of $\mathcal{A}$, and it is “quasi-central” in $\mathcal{A}$ (it quasi-commutes with the four generators $\alpha, \beta, \gamma$ and $\delta$). The quasi-centrality can be best reformulated in dynamical terms: $q^{\lambda+\mu} \beta \gamma$ is a central element of the trivial dynamical quantum group $\mathcal{A}^1$, cf. Remark \([3, 4]\).

\section{The $\text{SL}(2)$ dynamical quantum group.}

The four-tuple $(\mathcal{U}, \mathcal{A}, \langle \cdot, \cdot \rangle, T)$ as constructed in §4.1 does not quite fit into the formal algebraic setup of §3 since the abelian group $T$ is not finite. The results and constructions of §3 though still hold true in the present setup by interpreting the action of the idempotents $\pi_\alpha$ ($\alpha \in \mathbb{Z}$) on $\mathcal{A}_K$ as the projection operators
\[
\pi_\alpha \cdot a \cdot \pi_\beta = \delta_{\alpha, \beta} \delta_{\beta, \gamma} a, \quad a \in \mathcal{A}[\gamma, \delta].
\]
Furthermore it is convenient to replace the role of the function algebra $F$ in §3 by the field $F$ of meromorphic functions on $\mathbb{C}$ and accordingly we take $K = F \otimes F$.

Besides this formal extension of the setup of §3, we also need to work with a suitable completion of the algebra $\mathcal{U}_K$, which we do not specify here in detail. All explicit formulas given below will have an obvious, functional calculus type meaning when acting on $\mathcal{A}_K$ via the left or right regular action on $\mathcal{A}$. In particular, all infinite sums below become finite when acting on $\mathcal{A}$ since both $e \in \mathcal{U}$ and $f \in \mathcal{U}$ act locally nilpotently on $\mathcal{A}_K$.

The upshot is that all universal expressions and universal identities in (a suitable completions of) $\mathcal{U}_K$ given below should be interpreted within $\text{End}_K(\mathcal{A}_K)$ through the representation maps of the left and right regular $\mathcal{U}_K$-action on $\mathcal{A}_K$, and as such the results of §3 hold true.

Babelon [2], see also [3 §2] and [19 §7], considered the element

$$x(\lambda) = \sum_{l,m=0}^{\infty} \frac{(-q^{-\lambda})^{l+m}}{(q^{-2\lambda}; q^2)_l} \frac{q^{2l+2m-2m}(1 - q^2)^{l+m}}{(q^2; q^2)_m} \frac{(f k^{-1})^l (ek^{-1})^m}{(e^{-1}; q^2)_l}.$$  

The element $x(\lambda)$ is directly related to Rosengren’s [25 Prop. 3.3] group element $U_{\lambda \mu}$ by

$$x(\lambda) = (U_{q^{-\lambda-1}, q^{-\lambda-1}})\dagger,$$

where we identify $q^\frac{1}{2}, K^\pm, X_+, X_- \in \mathcal{U}_K$ with $q, k^\pm, e, f$ respectively, and where $\dagger$ is the $K$-linear antiinvolution of $\mathcal{U}_K$ defined by

$$(k^\pm)^\dagger = k^\pm, \quad e^\dagger = -q^\frac{1}{2} f, \quad f^\dagger = -q^{-\frac{1}{2}} e.$$

Thus [25 Prop. 3.5] implies that $x(\lambda)$ is invertible (as element of $\text{End}_K(\mathcal{A}_K)$) with inverse given by

$$x^{-1}(\lambda) = \sum_{m=0}^{\infty} \frac{q^{-m\lambda} q^{-m}}{(q^2; q^2)_m} (1 - q^2)^m (k^{-1} f)^m \times \sum_{l=0}^{\infty} \frac{q^{-l\lambda} q^{2l - 2l}}{(q^2; q^2)_l} (1 - q^2)^l (k^{-1} e)^l \frac{(q^{-2l\lambda}; q^2)_\infty}{(q^{-2l(\lambda+1)} k^{-4}; q^2)_l (q^{-2\lambda k^{-4}}; q^2)_\infty}.$$  

Babelon [2], see also [3], observed that $x(\lambda)$ is a vertex-IRF transformation with respect to $T$. The corresponding dynamical twist $j_x(\lambda)$, cf. [3 §1], is directly related to the universal fusion matrix for $\mathcal{U}$, see [8] and [19]. The explicit expressions for $j_x(\lambda)$ and $j_x^{-1}(\lambda)$, regarded as elements of $\text{End}_K(\mathcal{A}_K)^{\otimes 2}$ through the left and right regular representation of $\mathcal{U}_K$, are

$$j_x(\lambda) = \sum_{l=0}^{\infty} (1 - q^2)^{2l} \frac{q^{2l\lambda + 2l^2 - 4l}}{(q^2; q^2)_l} \left( k^{-1} f^l \otimes \frac{1}{(q^{2l(\lambda-1)} k^{-4}; q^2)_l} k^{-3l} e^l \right),$$

$$j_x^{-1}(\lambda) = \sum_{l=0}^{\infty} (1 - q^2)^{2l} \frac{q^{2l\lambda + 2l^2 - 3l}}{(q^2; q^2)_l} \left( k^{-1} f^l \otimes \frac{1}{(q^{-2\lambda k^{-4}}; q^2)_l} k^{-3l} e^l \right),$$

see [3 §2] and [19 §7].
The corresponding universal dynamical \( R \)-matrix \( \mathcal{R}^{j_x}(\lambda) \), acting on the representation space \( K \otimes V_1 \otimes V_1 \), can be computed explicitly using (2.23) and (4.2). With respect to the ordered basis \( \{ v_1^1 \otimes v_1^1, v_1^1 \otimes v_{-1}^1, v_{-1}^1 \otimes v_1^1, v_{-1}^1 \otimes v_{-1}^1 \} \) it is given by

\[
(4.7) \quad \mathcal{R}^{j_x}(\lambda)|_{V_1 \otimes V_1} = q^{-\frac{1}{2}} \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \frac{q^{-1}-q}{q^{2(\lambda+1)}-1} & \frac{(q^{2(\lambda+1)}-q^2)(q^{2(\lambda+1)}-q^{-2})}{(q^{2(\lambda+1)}-1)^2} & 0 \\
0 & 0 & 0 & q
\end{pmatrix}.
\]

The dynamical quantum group \( \mathcal{A}^{j_x} \) can now directly be related to Koelink’s and Rosengren’s [14] Def. 2.4 trigonometric \( SL(2; \mathbb{C}) \) dynamical quantum group as follows. Observe that the dynamical quantum group \( \mathcal{A}^{j_x} \) is generated as unital algebra by \( \mu_i(F), \mu_r(F) \) and the matrix coefficients \( \alpha, \beta, \gamma, \delta \) of the two-dimensional \( \mathcal{U} \)-representation \( V_1 \). By abuse of notation we denote \( f(\lambda) \) and \( g(\mu) \) for the elements \( \mu_i(f) \) and \( \mu_r(g) \) in \( \mathcal{A}^{j_x} \). The defining commutation relations (with multiplication denoted by \( \circ \) to distinguish it from the multiplication in \( \mathcal{A}_K \)), are

\[
\begin{align*}
\quad f(\lambda) \circ \alpha & = \alpha \circ f(\lambda + 1), & \quad f(\lambda) \circ \beta & = \beta \circ f(\lambda + 1), \\
\quad f(\lambda) \circ \gamma & = \gamma \circ f(\lambda - 1), & \quad f(\lambda) \circ \delta & = \delta \circ f(\lambda - 1), \\
\quad f(\mu) \circ \alpha & = \alpha \circ f(\mu + 1), & \quad f(\mu) \circ \beta & = \beta \circ f(\mu - 1), \\
\quad f(\mu) \circ \gamma & = \gamma \circ f(\mu + 1), & \quad f(\mu) \circ \delta & = \delta \circ f(\mu - 1),
\end{align*}
\]

the dynamical FRT commutation relations for \( \alpha, \beta, \gamma \) and \( \delta \) (see Proposition 3.10), and finally the dynamical determinant identity

\[
\delta \circ \alpha - q^{-1} \left( \frac{q^2 - q^{2(\lambda+1)}}{1 - q^{2(\lambda+1)}} \right) \circ \beta \circ \gamma = 1_{\mathcal{A}},
\]

which is simply the determinant identity (4.3) rewritten in terms of the \( j_x \)-twisted dynamical multiplication \( m^{j_x} \). The dynamical FRT commutation relations can be expressed in the familiar form

\[
\sum_{y,y'} L_{y' \xi'} \circ L_{y \xi} \circ R^{y'y}_{\eta \eta'}(\lambda) = \sum_{y,y'} L_{\eta \eta'} \circ L_{\eta' y'} \circ R^{\xi \xi'}_{y'y}(\mu)
\]

with the indices from \( \{ \pm 1 \} \), where the \( L_{\xi \eta} \) are given by

\[
L_{1,1} = \alpha, \quad L_{1,-1} = \beta, \quad L_{-1,1} = \gamma, \quad L_{-1,-1} = \delta,
\]

and with the coefficients \( R^{\xi \xi'}_{\eta \eta'}(\lambda) \) defined by

\[
\mathcal{R}^{j_x}(\lambda)(v_{\xi}^1 \otimes v_{\xi}^1) = \sum_{\eta, \eta'} R^{\xi \xi'}_{\eta \eta'}(\lambda)v_{\eta}^1 \otimes v_{\eta'}^1.
\]

The equivalence with the \( SL(2; \mathbb{C}) \) dynamical quantum group of Koelink and Rosengren [14] now follows by identifying the generators \( (f(\lambda), g(\mu), \alpha, \beta, \gamma, \delta) \) in [14] Def. 2.4 with \( (f(-\lambda - 2), g(-\mu - 2), \delta, \gamma, \beta, \alpha) \).
Harmonic analysis on the dynamical quantum group $A^d$ with respect to the standard quantum analogue of the Cartan subalgebra of $\mathfrak{sl}(2; \mathbb{C})$ still amounts to the study of the matrix coefficients $t^m_{rs}$, now viewed as matrix coefficients of a tempered corepresentation of the dynamical quantum group $A^d$ (see [14, §3]). In analogy with the harmonic analysis of the ordinary quantum group $A$, we now express the matrix coefficient $t^m_{rs} \in W(m) \cap A_K[r, s]$ in terms of a central element $\Xi \in A^d$ which, together with $\mu_l(F)$ and $\mu_r(F)$, generate $A_K[0, 0]$ as unital $\mathbb{C}$-algebra. The element $\Xi$ is given explicitly by

$$\Xi = q^{-\lambda+\mu+1} + q^{\lambda-\mu-1} - q^{\lambda+\mu+2} (1 - q^{-2\lambda})(1 - q^{-2(\mu+2)}) \circ \beta \circ \gamma,$$

see [14, Lem. 3.3]. The matrix coefficient $t^m_{rs}$ for integers $r, s, m$ having the same parity and satisfying $-m \leq r \leq s \leq -r \leq m$ is then given by

$$t^m_{rs} = C^m_{rs}(\lambda, \mu) \circ \delta^m \circ \cdot \cdots \circ \delta (\Xi; q^{\lambda-\mu+s}, q^{-\lambda+\mu-s}, q^{-\lambda-\mu+r}, q^{\lambda+\mu-r}, q^{\lambda+\mu+3}, q^{-1})$$

for some explicit nonzero meromorphic function $C^m_{rs}(\lambda, \mu)$, with $p_n$ the Askey-Wilson polynomial of degree $n$,

$$p_n(z + z^{-1}; a, b, c, d; q) = \phi_3 \left( q^{-n}, abcdq^{n-1}, az, az^{-1} \mid ab, ac, ad \right),$$

see [14, Thm. 3.5]. Here we used the notation $\delta^m = \delta \circ \cdots \circ \delta (m$ times) and similarly for $\gamma$. The polynomial expression in $\Xi$ in formula (4.9) has the obvious interpretation as element in the commutative subalgebra $A_K[0, 0]$ of the dynamical quantum group $A^d$.

4.3. The alternative realization $A^x$. We define a Cartan type element $X(\lambda) \in U_{F_1}$ and a twisted primitive element $Y(\lambda) \in U_{F_1}$ by

$$X(\lambda) = \frac{q^{k-1}(k^2 - 1) + q^\lambda k^2 - 1}{q - q^{-1}},$$

$$Y(\lambda) = f k - e k + \left( \frac{q^{\lambda-1} + q^\lambda}{q - q^{-1}} \right) (k^2 - 1).$$

The twisted primitive element $Y(\lambda)$ was introduced by Koornwinder [17] as an one-parameter family of quantum analogues of Lie-algebra generators for nonstandard Cartan subalgebras of $\mathfrak{sl}(2; \mathbb{C})$.

A key property of the vertex-IRF transformation $x(\lambda)$ (see (4.3)), proven by Rosengren [25], is the fact that

$$x(\lambda)Y(\lambda)x^{-1}(\lambda) = X(\lambda)$$

viewed as identity in $\text{End}_K(A_K)$ via the left or right regular representation of $U_K$, see [25, (4.8)].

We now study the dynamical quantum group $A^x$ using the $T$-Hopf algebroid isomorphism $\phi_x : A^x \to A^d$,

$$\phi_x(a) = x(\mu) \cdot a \cdot x^{-1}(\lambda), \quad a \in A^x,$$

see Theorem 3.16. As observed in §3.5, the bigrading of $A^x$ is nontrivial, but the other $T$-Hopf algebroid structures have the same untwisted form as the $T$-Hopf algebroid structures.
of the trivial dynamical quantum group $A^1$. Thus an explicit description of the bigraded pieces $A^x_{\alpha \beta}$ completely clarifies the $T$-Hopf algebroid structure of $A^x$.

**Proposition 4.1.** Denote
$$\nu_\alpha(\lambda) = \frac{q^{-\lambda-1}(q^{-\alpha} - 1) + q^{\lambda+1}(q^\alpha - 1)}{q - q^{-1}} \in F$$
for $\alpha \in \mathbb{Z}$. Then
$$A^x_{\alpha \beta} = \{ a \in A_K \mid Y(\mu) \cdot a = \nu_\beta(\mu)a, \ a \cdot Y(\lambda) = \nu_\alpha(\lambda)a \}$$
for $\alpha, \beta \in \mathbb{Z}$.

**Proof.** This follows from (4.10) and the observe that
$$A_K[\alpha, \beta] = \{ a \in A_K \mid X(\mu) \cdot a = \nu_\beta(\mu)a, \ a \cdot X(\lambda) = \nu_\alpha(\lambda)a \}.$$

Denote for $m \in \mathbb{Z}_{\geq 0}$ and $r, s \in \{-m, 2 - m, \ldots, m - 2, m\}$,
$$a^m_{rs}(\lambda, \mu) = \phi_x^{-1}(t^m_{rs}) = x^{-1}(\mu) \cdot t^m_{rs} \cdot x(\lambda),$$
then $W_K(m) \cap A^x_{rs}$ is an one-dimensional $K$-subspace of $A^x$, spanned by $a^m_{rs}(\lambda, \mu)$. Harmonic analysis on $A$ with respect to Koornwinder’s twisted primitive elements precisely amounts to the study of the matrix coefficients $a^m_{rs}(\lambda, \mu)$. Thus the isomorphism $\phi_x$ yields the equivalence between harmonic analysis on $A^x$ with respect to the standard quantum Cartan subalgebra in [14], and the harmonic analysis on $A$ with respect to Koornwinder’s twisted primitive elements as studied in [17], [24], [12].

To be more concrete, we end this article by translating the results of the previous subsection to the twisted primitive picture and linking it to the results in [17], [24], [12] and [13]. The generators
$$\alpha(\lambda, \mu) = a^1_{1,1}(\lambda, \mu) = x^{-1}(\mu) \cdot \alpha \cdot x(\lambda),$$
$$\beta(\lambda, \mu) = a^1_{-1,1}(\lambda, \mu) = x^{-1}(\mu) \cdot \beta \cdot x(\lambda),$$
$$\gamma(\lambda, \mu) = a^1_{1,-1}(\lambda, \mu) = x^{-1}(\mu) \cdot \gamma \cdot x(\lambda),$$
$$\delta(\lambda, \mu) = a^1_{-1,-1}(\lambda, \mu) = x^{-1}(\mu) \cdot \delta \cdot x(\lambda)$$
of $A^x$ can be rewritten in terms of the standard generators $\alpha, \beta, \gamma$ and $\delta$ of $A_K$ by
\begin{align*}
\alpha(\lambda, \mu) &= \frac{1}{(1 - q^{-2(\mu+1)})}(\alpha + q^{-\mu-\frac{1}{2}} \beta - q^{-\lambda-\frac{3}{2}} \gamma - q^{-\lambda-\mu-2} \delta), \\
\beta(\lambda, \mu) &= \frac{1 - q^{-2\mu}}{(1 - q^{-2(\mu+1)})}(q^{-\mu-\frac{3}{2}} \alpha + \beta - q^{-\lambda-\mu-3} \gamma - q^{-\lambda-\frac{3}{2} \delta}), \\
\gamma(\lambda, \mu) &= \frac{1}{(1 - q^{-2(\mu+1)})(1 - q^{-2\lambda})}(-q^{-\lambda-\frac{1}{2}} \alpha - q^{-\lambda-\mu-1} \beta + \gamma + q^{-\mu-\frac{3}{2}} \delta), \\
\delta(\lambda, \mu) &= \frac{(1 - q^{-2\mu})}{(1 - q^{-2\lambda})(1 - q^{-2(\mu+1)})}(-q^{-\lambda-\mu-2} \alpha - q^{-\lambda-\frac{1}{2} \beta} + q^{-\mu-\frac{3}{2}} \gamma + \delta). \tag{4.11}
\end{align*}
These expressions are easily derived using the explicit expression (4.11) for the vertex-IRF transformation \( x(\lambda) \). Identifying the elements \( \alpha, \beta, \gamma, \delta, A, B, C, D, q^\alpha, q^\beta \) in \([12] \) with
\[
\alpha, q^\frac{1}{2} \beta, q^{-\frac{1}{2}} \gamma, \delta, k, q^{-\frac{1}{2}} e, q^\frac{1}{2} f, k^{-1}, \sqrt{-1} q^{-\mu -1}, \sqrt{-1} q^{-\lambda -1},
\]
the elements (4.11) correspond, up to \( K \)-normalization, to the elements \( k^{-1} \cdot \alpha_{r,s}, k^{-1} \cdot \beta_{r,s}, k^{-1} \cdot \gamma_{r,s} \) and \( k^{-1} \cdot \delta_{r,s} \) of \([12] \) Prop. 6.5.

We denote \( \rho(\lambda, \mu) = \phi_x^{-1}(\Xi) \in \mathcal{A}^x \), so
\[
(4.12) \quad \rho(\lambda, \mu) = q^{-\lambda+\mu+1} + q^{\lambda-\mu-1} - q^{\lambda+\mu+2}(1-q^{-2\lambda})(1-q^{-2(\mu+2)})\beta(\lambda-1, \mu+1)\gamma(\lambda, \mu).
\]

By a direct computation using (4.11), the element \( \rho(\lambda, \mu) \) can be explicitly represented as a quadratic expression in \( \alpha, \beta, \gamma \) and \( \delta \). Identifying the generators of \([12] \) with ours as indicated above, \( \rho(\lambda, \mu) \) equals \( 2k^{-1} \cdot \rho_{r,s}\), with \( \rho_{r,s} \) the element as defined in \([12] \) Thm. 5.1 (it was initially written down explicitly in \([17] \)). The pre-image under \( \phi_x \) of the expression (4.9) for integers \( r, s, m \) having the same parity and satisfying \( -m \leq r \leq s \leq -r \leq m \) yields
\[
a_{rs}^m(\lambda, \mu) = D_{rs}^m(\lambda, \mu) \prod_{i=0}^{\frac{s+r-1}{2} - \frac{r-s-1}{2}} \delta(\lambda + r + 1 + i, \mu + s + 1 + i)
\[
\times \prod_{j=0}^{\frac{s+r-1}{2} - \frac{r-s-1}{2}} \gamma(\lambda + \frac{r}{2} - \frac{s}{2} + 1 + j, \mu - \frac{r}{2} + \frac{s}{2} - 1 - j)
\times p_{\frac{m+r}{2}}(\rho(\lambda, \mu), q^{\lambda-\mu+1}, q^{-\lambda+\mu+1-r+s}, q^{-\lambda-\mu-1-r-s}, q^{\lambda+\mu+3}, q^2)
\]
for some non-zero meromorphic function \( D_{rs}^m(\lambda, \mu) \), where \( \prod_{i=0}^j b_i \) with \( b_i \in \mathcal{A}_K \) equals \( 1_A \) when \( j < 0 \) and equals \( b_0 b_1 \cdots b_j \) when \( j \geq 0 \). This formula is in accordance with the expressions derived by Koornwinder \([17] \) (in case \( r = s = 0 \) and \( m \) even), and by Noumi, Mimachi \([24] \) and Koelink \([12] \) for arbitrary \( r, s, m \), compare for instance with the expressions in \([12] \) Cor. 7.8(i) and \([12] \) Cor. 6.8.

The above dynamical quantum group interpretation of the harmonic analysis with respect to twisted primitive elements leads to natural interpretations and new proofs of several other known facts. For instance, the statement \([12] \) Prop. 6.5 amounts to a reformulation of the fact that \( \mathcal{A}^x = \oplus_{\alpha, \beta} \mathcal{A}_{\alpha, \beta}^x \) defines a bigrading with respect to the untwisted dynamical multiplication \( m^x \). The factorized form (4.12) of \( \rho(\lambda, \mu) \) is precisely \([13] \) Prop. 4.1.7. The centrality of \( \rho(\lambda, \mu) \) in the dynamical quantum group \( \mathcal{A}^x \) is \([13] \) Cor. 4.1.8. The interpretation of the \( t_{rs}^m \) as a matrix corepresentation of the dynamical quantum group \( \mathcal{A}^x \),
\[
\Delta(t_{rs}^m) = \sum_t t_{ri}^m \otimes_F t_{is}^m;
\]
\[
\epsilon(t_{rs}^m) = \delta_{r,s} T_{-r},
\]
implies via the isomorphism \( \phi_x : A^x \to A^j_x \) that the \( a_{rs}^m(\lambda, \mu) \) define a matrix corepresentation of the dynamical quantum group \( A^x \),
\[
\Delta(a_{rs}^m(\lambda, \mu)) = \sum_l a_{rl}^m(\lambda, \mu) \otimes F a_{ls}^m(\lambda, \mu),
\]
\[
\epsilon^x(a_{rs}^m(\lambda, \mu)) = \delta_{r,s} T_{-r}.
\]
The latter formulas directly relate to [13, Prop. 6.1.1].

References

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