Localized Modes in Nonlinear Fractional Systems with Deep Lattices

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1. Introduction

Fractional calculus, different forms of which have been elaborated in mathematics, is, essentially, a theory of fractional differentiation and integration.[1–5] More recently, it has found applications to modeling various phenomena in quantum mechanics,[6–8] optics,[9,10] ultracold atomic gases,[11,12] and condensed matter.[13,14] In particular, the formulation of quantum mechanics based on Feynman path integrals, if applied to particles moving by stochastic Levy flights, instead of the usual Brownian motion, leads to the fractional Schrödinger equation[6–8] (see also a recent book[15] summarizing this theory).

In the framework of these studies, the fractional Schrödinger equation and its extensions in the form of nonlinear fractional Schrödinger equations (NLFSes) have drawn a great deal of interest, revealing a variety of linear and nonlinear wave patterns in diverse physical media.[16–40,42,43] Recently, novel periodic potentials, such as Moiré lattices,[66,67] which feature flat-band spectra, similar to those induced by deep lattices, were introduced in this context.

The nonlinear Schrödinger equation with normal (non-fractional) diffraction and a deep lattice potential is often replaced, in the tight-binding approximation, by its discrete version with the nearest-neighbor coupling. The accuracy of this limit is confined to the first finite bandgap,[68] and GSs correspond to discrete solitons with the staggered structure.[69] However, to address GSs in fractional models including deep OL potentials, which is the subject of the present work, it is necessary to deal with NLFSes in the continuous form, as the fractional diffraction is represented by a nonlocal integral operator. Recently, a discrete approximation for the 1D and 2D NLFS with the self-attractive nonlinearity was introduced in refs.[71–73]. It takes the form of a nonlocal discrete equation with a complex form of the long-range interaction, which produces discrete solitons. However, in the case of self-repulsion, the staggered ansatz does not apply to discrete equations in which the interaction cannot be limited to the nearest neighbors.

In this work, we address, by means of numerical and analytical methods, the existence and stability of various localized modes, including 1D and 2D GSs and gap-vortex modes (cf. ref.[74]) in 2D, in the first and second finite bandgaps. In particular, the GSs
are always found as highly confined modes nearly squeezed into a single lattice cell, even if they are excited in the second bandgap, which the discrete limit would fail to grasp. On the contrary, GSs supported by shallow or moderately deep lattice potentials normally cover several (or many) cells. The same is true for discrete solitons, including ones in the fractional model with the self-attraction.\footnote{Stability of the solitons in the system under the consideration is explored by means of the linear-stability analysis and direct simulations.} Definition of the solitons in the system under the consideration is explored by means of the linear-stability analysis and direct simulations.

2. The Model

Our setting is based on the continuous NLFSE of the Gross-Pitaevskii type,\footnote{We search for soliton solutions with chemical potential \( \mu \) in the usual form, \( V_0 \sin^2 x \) in the 1D space, or \( V_0 (\sin^2 x + \sin^2 y) \) in 2D (in the latter case, the full depth is \( 2V_0 \)). The self-repulsive nonlinearity in Equation (5) is expected to create GS modes populating finite bandgaps of the linear Bloch-wave spectrum induced by the OL. In terms of optics, Equation (5), with \( t \) replaced by the scaled propagation distance, applies to the spatial-domain propagation of light in media with the effective fractional diffraction induced by means of the scheme which converts the spatial optical beam into the Fourier space and applies an appropriate phase shift in it, then getting back into the spatial domain.\footnote{In that case, norm (7) is the scaled power of the optical beam, and the deep spatially periodic potential can be realized in a photonic crystal composed of alternating circular or planar layers (in the 2D and 1D cases, respectively) with high and low values of the refractive index.} Thus, the solitons predicted here should be within the reach of the available experimental techniques.} for a mean-field wave function \( U(X, Y, T) \) of particles moving by Lévy flights. With time \( T \) and coordinates \((X, Y)\) measured in physical units, the equation is written as

\[
i \hbar \frac{\partial U}{\partial T} = D_a (-\hbar^2 \nabla^2)^{\alpha/2} U + W_{\text{OL}}(X, Y, U) + \frac{2\sqrt{\pi} \hbar^2 a}{ma} |U|^2 U \tag{1}
\]

where \( D_a \) is the Laskin’s coefficient,\footnote{Here \( f(R) \) is a arbitrary function of coordinates \((X, Y)\), \( f(K) \) is its Fourier transform, and \( \alpha \) is the Lévy index (LI), taking values \( 0 < \alpha \leq 2 \). In the 1D case, the corresponding operator \(-\partial^2 / \partial x^2)^{\alpha/2} \) is defined by the straightforward counterpart of Equation (3). In 1D systems with the attractive cubic nonlinearity, which corresponds to \( a < 0 \), interval \( 0 < \alpha < 1 \) is not considered, as the collapse occurs in it, and in the same 2D setting all values \( \alpha \leq 2 \) lead to the collapse for \( a < 0 \).\footnote{We search for soliton solutions with chemical potential \( \mu \) in the usual form, \( \psi = \phi(x, y) e^{-i \omega t} \). Substituting this in Equation (5) leads to the equation for the stationary wave function:}}

\[
\left[ -\nabla^2 \right]^{\alpha/2} f(R) \equiv F^{-1}\left[ |K|^{\alpha} F(f) \right] = \frac{1}{(2\pi)^\alpha} \int \left| K \right|^{\alpha} f(K) \exp(i K \cdot R) dK \tag{3}
\]

Equation (1) can be made dimensionless by setting

\[
(X, Y) \equiv \frac{d}{\pi} (x, y), T \equiv \frac{\pi \hbar}{2 D_a}, W_{\text{OL}} \equiv D_a \left( \frac{\pi \hbar}{d} \right)^{\alpha} V_{\text{OL}}.
\]

\[
U(X, Y, T) \equiv U_0 \psi(x, y, t), U_0 = \frac{\pi^{(1/2)(\alpha-1/2)}}{2(D_0 a)^{\alpha/2}} \sqrt{\frac{D_0 m a}{a}}
\tag{4}
\]

The accordingly rescaled Equation (1) is

\[
i \frac{\partial \psi}{\partial t} = \frac{1}{2} (-\nabla^2)^{\alpha/2} \psi + V_{\text{OL}}(x, y) \psi + |\psi|^2 \psi \tag{5}
\]

The number of atoms in the condensate, \( N \), whose evolution is governed by Equation (5), is expressed in terms of norm \( N \) of the scaled wave function as per Equation (4):

\[
N = \left( U_0 d / \pi \right)^2 N \tag{6}
\]

\[
N \equiv \iint |\psi(x, y)|^2 dx dy \tag{7}
\]

Taking into regard Equation (2), it is easy to check that expression (6) is dimensionless, as it should be. For the condensate of \( ^{87}\text{Rb} \) atoms, loaded in the OL potential with period \( d = 500 \) nm, Equation (4) shows that \( t = 1 \) corresponds, in physical units, to \( \approx 0.04 \) ms, and, with the transverse confinement size \( a_x \approx 5 \mu m \), \( N = 1 \) represents \( \approx 300 \) atoms. Similar rescalings are available in the 1D case, with the single coordinate, \( x \).

Previously reported experimental observations of effectively 1D GSs in the condensate of \(^{87}\text{Rb} \) demonstrated that the soliton was built of ca. 250 atoms.\footnote{More extended “gap waves” were observed as chains of several closely overlapped GSs, containing \( \approx 5000 \) atoms.\footnote{Thus, the solitons predicted here should be within the reach of the available experimental techniques.} Thus, the solitons predicted here should be within the reach of the available experimental techniques.} More extended “gap waves” were observed as chains of several closely overlapped GSs, containing \( \approx 5000 \) atoms.\footnote{Thus, the solitons predicted here should be within the reach of the available experimental techniques.} Thus, the solitons predicted here should be within the reach of the available experimental techniques.} Thus, the solitons predicted here should be within the reach of the available experimental techniques. We search for soliton solutions with chemical potential \( \mu \) in the usual form, \( \psi = \phi(x, y) e^{-i \omega t} \). Substituting this in Equation (5) leads to the equation for the stationary wave function:
Figure 1. Linear bandgaps spectra, shown by values of chemical potential $\mu$ versus the lattice depth $V_0$ (a), Lévy index $\alpha$ (b), and quasi-momentum $K$ (c) in the 1D model. Other parameters are fixed as $\alpha = 1.2$ in (a, c), and $V_0 = 40$ in (b, c) (these values are designated by vertical dashed lines in (b) and (a), respectively). Here and below in Figures 2–5, regions I and II represent, respectively, the first and second finite bandgaps. Purple dotted lines in the first three Bloch bands in (a) represent the approximate relation given by Equation (17), that is, $\mu = CV_0^{3/8}$ for $\alpha = 1.2$, where the fitting constants are $C = 0.64$, 1.60, and 2.31 for the three lines. The middle plot: norm $N$ of the GSs (gap solitons) in the first finite bandgap supported by the deep OL is shown versus $V_0$ by the continuous line, for $\mu = 6$ and $\alpha = 1.2$. The blue dashed line shows the same dependence as predicted by the TF (Thomas–Fermi) approximation. Typical profiles of 1D GSs in the first finite bandgap are displayed by the bottom plots at $V_0 = 12$ (left, $N = 5.42$) and $V_0 = 40$ (right, $N = 2.34$). Here and in similar figures below, alternating gray and white stripes in the bottom panels designate periods of the lattice potential.

The underlying solution is stable if all eigenvalues produced by numerical solution of Equations (12) and (13)] have zero real parts, $\lambda_R = 0$.

The numerical analysis presented below first produces solutions $\phi(x, y)$ of Equation (10), using the modified squared-operator method.[79] Then, their linear stability is explored for eigenmodes of small perturbations obtained as solutions of Equations (12) and (13). Finally, the dynamical stability is tested in direct simulations of Equation (5) employing the fourth-order Runge–Kutta method based on the fast Fourier transform.

3. The 1D Bandgap Spectrum and Gap Solitons (GSs)

To explore the nonlinear dynamics of GSs trapped in the lattice potential, the bandgap spectrum of the underlying linearized equation for the Floquet–Bloch modes is required. They are defined as usual:

$$\phi_k(x) = \Phi_k(x) \exp(iKx)$$ (14)

where $K$ is the quasi-momentum, and $\Phi(x)$ is a periodic function.[45–48] The spectrum was produced by a numerical
Figure 2. Norm $N$ as a function of chemical potential $\mu$ (a) and LI $\alpha$ (b) for 1D GSs in the fractional nonlinear system with the deep OL potential (the depth is fixed as $V_0 = 40$). To quantify stability of the solitons, the largest real part of the perturbation growth rate, $\lambda$, is shown as a function of $\mu$ by the red dashed line in (a). Other parameters are $\alpha = 1.2$ in (a) and $\mu = 4$ in (b). Panel (a) shows the results for the GSs in the two lowest bandgaps, separated by the flat Bloch band at $\mu \approx 6.37$. The result produced by the TF (Thomas-Fermi) approximation is shown by the blue dashed line in (a). The TF-predicted value of $N$, which does not depend on $\alpha$, is shown as a reference one by the horizontal blue dashed line in (b). Profiles of GSs with different values of $\mu$ (c) and $\alpha$ (d), which correspond, respectively, to the marked points in (a) and marked circles in (b), are displayed in the bottom panels.

Solution of the linearized version of Equation (10). In Figure 1a, the 1D bandgap spectrum is plotted for the OL potential (8), varying lattice depth $V_0$ at a fixed LI, $\alpha = 1.2$. It is seen that widths of the first two finite bandgaps stay nearly constant as $V_0$ varies in the interval of $[10, 30]$. At a fixed value of $V_0$ from this interval, the bandgap widens with the increase of $\alpha$, as shown in Figure 1b. The latter feature is similar to its counterpart found in fractional systems with shallow OLs.\cite{23,26,32} Figure 1c shows the bandgap spectrum versus quasimomentum $K$ for $\alpha = 1.2$ and $V_0 = 40$. Note that the Bloch bands are very flat for the deep lattice, because the tunnel coupling between adjacent potential wells is weak. The flat-band phenomenology, including the wave propagation and formation of solitons, has been recently developed in models based on the usual nonlinear Schrödinger equation ($\alpha = 2$) with other periodic potentials,\cite{80,84} including recent works with Moiré lattices.\cite{66,67}

The nearly constant values of $\mu$ corresponding to the flat bands cannot be found analytically. However, it is possible to predict their dependence on the potential’s depth, $V_0$. Indeed, individual deep wells in lattice potentials (8) and (9) may be approximated by harmonic-oscillator (HO) potentials, viz.,

$$V_{\text{HO}}^{(1D)}(x) \approx V_0 x^2$$ \hspace{1cm} (15)

in the 1D case, and

$$V_{\text{HO}}^{(2D)}(x, y) \approx V_0 (x^2 + y^2)$$ \hspace{1cm} (16)

in 2D. Then, values of $\mu$ which determine the flat bands may be approximately found as eigenvalues of the linearized version of Equation (10) with the HO potential (15) or (16). The latter problem does not admit an exact solution in the fractional space.\cite{85} Nevertheless, it is easy to predict an exact scaling in the dependence of these eigenvalues on the potential’s strength, $V_0$ (it is the same for the 1D and 2D settings):

$$\mu_{\text{flat-band}} \sim V_0^{\alpha/(2+\alpha)}$$ \hspace{1cm} (17)

Simultaneously, the size of the wave function trapped in potential (15) or (16) scales as $x_0 \sim V_0^{1/(2+\alpha)}$. In the case of $\alpha = 2$, these scaling relations correspond to the ones commonly known for the HO in quantum mechanics, $\mu_{\text{flat-band}} \sim \sqrt{V_0}$ and $x_0 \sim V_0^{-1/4}$ (the latter one represents the standard HO length). In Figure 1a, we have plotted the dependence predicted by Equation (17) for the first three Bloch bands, showing a very close match.

Usually, in deep OLs localized nonlinear modes occupy, essentially, a single cell of the lattice (which suggests a possibility of approximating them by the discrete lattice models, as mentioned above).\cite{68,86} This property is corroborated by the modal profiles of the solutions...
most completely stable in the two lowest finite bandgaps, narrow instability intervals occurring at top edges of both bandgaps. Two typical examples of stable GSs residing in these bandgaps are presented in Figure 2c. These examples clearly confirm that the GSs keep an almost tailless shape, in accordance with Equation (18), being effectively confined in a single OL cell. On the contrary, GSs in nonlinear fractional systems with a shallow lattice potential are usually broad modes extending over several lattice cells.\textsuperscript{[32,33]} For the GSs with fixed $\mu$, the increase of LI results in a decrease of the norm necessary for the existence of the solitons, as seen in Figure 2b.

The fact that the solitons are tightly confined in the single potential cell makes it natural to compare their shapes to the prediction of the Thomas-Fermi (TF) approximation, which neglects the diffraction term in Equation (10)\textsuperscript{[25]}

$$\phi^2_T(x) = \begin{cases} 
\mu - V_0 \sin^2 x, & \text{at } |x| < \arcsin\left(\sqrt{\mu/V_0}\right) \\
0, & \text{at } |x| > \arcsin\left(\sqrt{\mu/V_0}\right)
\end{cases}$$

(19)

The respective norm is given by

$$N_{TF} = 2 \int_0^{\arcsin(\sqrt{\mu/V_0})} \phi_T^2(x)dx = \sqrt{\mu(V_0 - \mu)} - (V_0 - 2\mu) \arcsin\left(\sqrt{\frac{\mu}{V_0}}\right)$$

(20)

For the case of the deep OL, a practically important particular case is the one with $\mu \ll V_0$. In this case, expression (20) simplifies to

$$N_{TF} \approx 4\mu^{3/2} / \left(3\sqrt{V_0}\right)$$

(21)

The TF predictions produced by Equations (20) and (21) are displayed by the blue dashed lines in Figure 2a,b. In fact, the numerical findings are most interesting in the case when they are conspicuously different from the TF approximation, that is, at $\mu$ not too large, and at $\alpha$ not too close to 0, as the proximity to the TF approximation, which ignores the fractional diffraction, implies that it is not an essential factor.

4. 2D Gap Solitons and Solitary Vortices

To address 2D localized gap modes in the fractional setting under the consideration, the corresponding bandgap structure for potential (9) should be produced at first. The “egg-carton” shape of the potential is displayed in Figure 3a, its first Brillouin zone for the wave vector $(k_x, k_y)$ is shown, in the reciprocal lattice space, in Figure 3c, and the resulting bandgap structure for the deep OL with $V_0 = 40$ is exhibited in the plane of $(\alpha, \mu)$ in Figure 3b, where the vertical red dashed line represents the fixed value of LI, $\alpha = 1.6$. In Figure 3b, higher-order quasi-flat bands and gaps between them are not plotted, as we here focus on the localized modes in the first two bandgaps. The study of modes in higher-order gaps may be a subject for a separate work. We see in Figure 3d that the 2D deep lattice, as well as its 1D counterpart, gives rise to the flat-band spectrum, which also obeys scaling relation (17).

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Figure 3. a) The stereographic representation of the 2D optical lattice potential, at the bottom of which bosonic atoms are trapped, in real space. c) The corresponding first reduced Brillouin zone in the reciprocal lattice space; $X$, $M$, and $\Gamma$ denote the high-symmetry points in the irreducible zone. b) Dependencies of chemical potential $\mu$ on Lévy index $\alpha$, and d) on Bloch quasi-momentum $K$, at $\alpha = 1.6$ [this value is designated by the vertical dashed red line in (b)]. In panels (b) and (d), the dependencies are plotted for lowest Bloch bands. Higher-order bands and gaps between them, which occupy the top left corner in (a) (as marked in that panel) are not shown here in detail. Here and below, the lattice depth is $V_0 = 40$. The blue dashed lines in panel d) correspond to the linear-Bloch spectrum for a shallow lattice with $V_0 = 4$. 

displayed in the third line of Figure 1. The GSs shown in it for $\mu = 6$, $\alpha = 1.2$ belong to the first finite bandgap, as per Figure 1a. At a moderate lattice depth, $V_0 = 12$, the GS features undulating tails, while at a very deep lattice, with $V_0 = 40$, produces a GS with a compact shape strongly confined to a single cell, which is a typical feature of localized modes sustained in deep lattices. A simple asymptotic consideration of Equation (10) demonstrates that, at large $|x|$, the tails decay as

$$\phi^2(x) \sim \exp\left(-\text{const} \cdot V_0^{1/4} |x|^{-1+2/\alpha}\right)$$

(18)

i.e., as a super-Gaussian at $\alpha < 2$. In the 2D case, the same asymptotic expression (18) is valid, with $|x|$ replaced by radial coordinate $r$.

Hereafter, we set $V_0 = 40$ [as marked by the red dashed line in Figure 1a] and $\alpha = 1.2$ [marked by the red dashed line in Figure 1b] for the study of the localized modes in the NLFSE with the 1D deep lattice.

Dependencies of norm $N$ on chemical potential $\mu$ and LI $\alpha$ for the family of 1D GSs sustained by the self-defocusing nonlinearity and deep OL in the two lowest bandgaps of the fractional system are shown in Figures 2a and 2b, respectively. One can see from Figure 2a that dependence $N(\mu)$ agrees with the necessary stability condition for solitons in settings with repulsive nonlinearities, viz., the “anti-Vakhitov–Kolokolov” (anti-VK) criterion, $dN/d\mu > 0$,\textsuperscript{[86]} which was tested in many other models\textsuperscript{[32,52,54,87,88]} (the VK criterion per se, $dN/d\mu < 0$, applies to systems with self-attraction\textsuperscript{[89–91]}) Also displayed in Figure 2a are results of the linear-stability analysis, which demonstrates that the GSs are all
4.1. 2D Fundamental GSs

The simplest species of 2D gap modes represents fundamental GSs with a single peak, similar to their 1D counterparts considered above. The family of such GSs in the two lowest bandgaps is characterized by $N(\mu)$ and $\mu(\alpha)$ curves plotted in Figures 4a and 4b, respectively. Note that the former curve agrees with the above-mentioned anti-VK criterion, $dN/d\mu > 0$.

Two representative examples of the fundamental GSs, belonging to the first and second finite bandgaps, are displayed in Figures 4c,e and 4d, respectively. In addition, their 1D cross-section along $x = 0$ are plotted in the bottom plots of Figure 4, indicating that the 2D fundamental GSs, as well as their 1D counterparts reported above, are essentially confined to a single cell of the lattice, with virtually invisible tails. In agreement with Equation (18), the absence of conspicuous tails is a specific feature of the fractional nonlinear systems with the lattice potential and $\alpha < 2$, the situation being different for traditional GSs in the case of the normal diffraction ($\alpha = 2$).

The TF approximation can be applied to the 2D fundamental GSs as well. It yields

$$\phi_{\text{TF}}^2(x, y) = \begin{cases} \mu - V_0 \left( \sin^2 x + \sin^2 y \right), & \text{at } \sin^2 x + \sin^2 y < \mu/V_0 \\ 0, & \text{at } \sin^2 x + \sin^2 y > \mu/V_0 \end{cases}$$

(22)

The respective norm was found numerically as

$$N_{\text{TF}}^{(2D)} = \int \left( \int \phi_{\text{TF}}^2(x, y) dx \right)$$

(23)

where the integral should be calculated numerically over the area corresponding to the top line in Equation (22). Figure 4a shows that the norm predicted by this approximation is essentially different from the numerically exact counterpart, hence we conclude that the results for the 2D solitons are nontrivial, in the sense that they are essentially affected by the fractional diffraction.
4.2. Vortex Solitons

The usual 2D nonlinear Schrödinger equation with the square-shaped OL potential (9) and self-defocusing cubic nonlinearity gives rise, in addition to fundamental solitons, to ones with embedded integer vorticity $S$. Here we aim to construct localized gap vortices composed of four peaks, onto which the overall phase circulation $2\pi$, corresponding to $S = 1$, is imprinted. It is known that such patterns represent robust gap-vortex modes in many physical settings with the usual (non-fractional) diffraction. Very recently, similar vortex solitons were found in a discrete fractional model with the self-focusing nonlinearity (unlike the defocusing nonlinear term considered here).

The four-peak configuration of the gap vortices can be arranged in the shape of a densely packed square (alias an intersite-centered vortex), or a loosely packed rhombus (an onsite-centered one), with an empty site at the center. Examples of these patterns are displayed in Figures 5c and 5d, respectively. Families of the vortex GSs, with $S = 1$, are characterized by the respective dependences $N(\mu)$ and $\mu(\alpha)$, which are shown in Figure 5a for the “squares”, and Figure 5b for the “rhombuses”. It is seen that the former species of the gap vortices has a larger norm, which may be attributed to stronger interactions between the four peaks, separated by an essentially smaller distance than in the rhombuses, as seen in Figures 5c and 5d.

4.3. Dynamics of 2D Fundamental GSs and Vortex Solitons

In Figures 6a and 6b, we show profiles and perturbed evolution of a stable 2D fundamental GS existing in the first finite bandgap, and of an unstable one found near the upper edge of the same bandgap. It is seen that the GS, which was predicted to be stable by the analysis of small perturbations, is indeed stable in direct simulations, while the unstable one develops regular oscillations. In the bottom line of both panels, the dependence of the solitons’ norm $N$ on time corroborates these conclusions. It is seen that the instability transforms the 2D soliton into a breather, but does not destroy it.

Typical profiles of two kinds of vortex GSs, that is, the above-mentioned squared and vortices, are plotted in Figures 6c–f. They display stable and unstable gap vortices, respectively, in the first finite bandgap and near the upper edge of the same bandgap. The corresponding perturbed evolution and dependence of the norm on time are depicted in the second and third lines of the panels. It is seen, in particular, that unstable vortices develop random oscillations, but survive as topologically organized patterns. We stress...
that, being consistent with the 1D case, the 2D GSs and vortices are exceptionally stable, except for near edges of the bandgaps.

5. Conclusion and Discussion

We have presented the analysis of the existence, structure, and dynamics of localized gap modes, including 1D and 2D fundamental GSs (gap solitons), as well as 2D gap vortices, in the fractional system including the deep OL (optical-lattice) potential and self-repulsive nonlinearity. The system’s spectrum features finite bandgaps separated by nearly flat bands. Fractional solitons in such deep periodic potentials were not studies previously, and, unlike the model with the normal (non-fractional) diffraction, they cannot be obtained in the discrete approximation. We have found that the 1D GSs, belonging to the first and second finite bandgaps of the underlying spectral structure, are strongly confined in a single lattice cell, on the contrary to multiblock GSs supported by shallow OLs. The linear-stability analysis and direct simulations of the perturbed evolution have identified stability and instability regions for the 1D GSs. They are stable inside the first and second finite bandgaps and unstable in narrow regions near the gap edges. The 2D GSs and vortices found here are, generally, stable too, being unstable only very close to edges of the bandgaps. The predicted localized modes may be observed in experiments with BEC loaded into deep OLs, as well as in optical waveguides composed of alternating layers with large and small values of the refractive index.

It may be interesting to develop the analysis for vortex GSs with higher values of the winding number, $S \geq 2$. This work may be extended for physical systems modeled by the NLFSEs (nonlinear fractional Schrödinger equation) for temporal optical GSs. It will also be natural to consider possible GSs in bimodal fractional models, such as those based on dual-core optical couplers.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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