Phase synchronization on scale-free and random networks in the presence of noise

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Abstract. In this work we investigate the stability of synchronized states for the Kuramoto model on scale-free (SF) and Erdős–Rényi (ER) random networks in the presence of white noise forcing. We show that for a fixed coupling constant, the robustness of the globally synchronized state against the noise depends on the noise intensity on both kinds of networks. At low noise intensities ER networks are more robust against losing the coherency but upon increasing the noise, synchronization among the population vanishes suddenly at a specific noise strength. In contrast, on SF networks the global synchronization disappears continuously at a much larger critical noise intensity with respect to ER networks.

Keywords: network dynamics, random graphs, networks, nonlinear dynamics, stochastic processes
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1. Introduction

Collective behaviour in a population of individuals is of great importance in many areas of physics, biology, social sciences and many other disciplines [1]–[3]. One of the most celebrated collective behaviours is the case of phase synchronization among a population of interacting self-sustained oscillators, in which all members tend to oscillate coherently with more or less the same phase. A population of coupled phase oscillators with mutual sine interaction between the pairs is known as the Kuramoto model [4]. This model is introduced by Kuramoto [5] and has been extensively investigated by many authors (see [6]–[8] and references therein).

Considering a network of coupled rotors (phase oscillators), many factors such as coupling strength [9], time delayed interactions [10], individual frequency distribution [6,11], topology of network [12] and noises [6,13] affect the path toward the full synchronization.

The synchronization of the deterministic Kuramoto model with random distribution of rotor frequencies and initial phases has been studied on SF networks by Moreno and Pacheco [14]. There, it was shown that the onset of synchronization occurs through a continuous transition at a small value of coupling with a critical exponent near 0.5. This resembles the mean-field behaviour, except that in the case of SF networks the critical coupling at which the rotors begin to synchronize is much smaller than that of all-to-all networks. They have also managed to find that in the complete synchronized state, the dependence of the recovery time on the node degree is a power law function with exponent close to $-1$, which shows the robustness of highly connected nodes (hubs) against perturbations.

A comparison between the synchronizabilities of the Kuramoto model on ER and SF networks has been carried out recently by Goméz-Gardenes et al [15,16]. In these references, the authors found that while the onset of global synchronization occurs at a smaller value of coupling for SF networks, the tendency toward global coherence grows suddenly to a larger extent for ER networks at higher couplings. The reason for such behaviour is that the giant cluster in the heterogeneous (such as SF) networks—originating from a central core of high connectivity—grows continuously by attaching the smaller clusters to it upon increasing the coupling constant. In contrast, for the homogeneous
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networks, the evolution toward full synchronization would be boosted by merging many small clusters spreading uniformly throughout the networks when the coupling is large enough.

In this paper our focus is on the effect of white noise forcing on the synchronization of the Kuramoto model for two types of networks: SF networks introduced by Barabási and Albert (BA) [17] and ER [18]. The paper is organized as follows. In section 2, we introduce the stochastic Kuramoto model and express briefly some results for all-to-all networks. Simulation results are presented in section 3, and section 4 is devoted to a summary and concluding remarks.

2. The stochastic Kuramoto model on a complex network

Consider a system composed of $N$ rotors with intrinsic frequencies, $\omega_i$, $i = 1, \ldots, N$, on top of a complex network consisting of $N$ nodes. The stochastic Kuramoto model on such a network is described by the following set of equations:

$$\frac{d\theta_i}{dt} = \omega_i + \lambda \sum_j a_{ij} \sin(\theta_j - \theta_i) + \eta_i(t), \quad i = 1, \ldots, N,$$

(1)

where $\theta_i$ denotes the phase of the rotor on the node $i$, $\lambda$ is the coupling constant and $a_{ij}$ is an element of the connectivity matrix, whose value is 1 if $i$ and $j$ are linked together and 0 otherwise. $\eta_i$ is the random force applied to the $i$th rotor, and usually is chosen as a white noise with zero mean value. The spatial–temporal correlation of such a noise is given by

$$\langle \eta_i(t)\eta_j(t') \rangle = 2D\delta(t-t')\delta_{ij},$$

(2)

where $D$ is the variance of the noise.

The stochastic Kuramoto model on an all-to-all network was investigated analytically by Acebrón et al [6]. They showed that taking a one-peaked symmetrical frequency distribution, $f(\omega) = f(-\omega)$, for oscillators, there would be a critical coupling constant $\lambda_c = 2/[\pi f'(0)]$ above which the network begins to synchronize. Near this critical point, the order parameter obeys a power law relation, namely

$$r \sim \sqrt{-16(\lambda - \lambda_c)/\pi\lambda_c^2f''(0)}.$$  \hspace{1cm} (3)

They also showed that for a Lorentzian frequency distribution $f(\omega) = (\gamma/\pi)/(\omega^2 + \gamma^2)$, the incoherent solution is linearly stable for points $(\lambda, D)$ above the critical line $D = -\gamma + \lambda/2$. In terms of coupling strength, this is also linearly stable for $\lambda < \lambda_c = 2D + 2\gamma$.

The bimodal frequency distribution $g(\omega) = (\delta(\omega - \omega_0) + \delta(\omega + \omega_0))/2$ has been extensively studied using linear stability analysis and a rich phase diagram obtained [19]–[22]. According to these results, when $\lambda < 2D$ incoherence is linearly stable for all $\omega_0$ and it is always unstable for $\lambda > 4D$. For $2D < \lambda < 4D$ there are two cases: (i) for $\omega_0 < D$ incoherence is linearly stable if $\lambda < \lambda_c = 2D[1 + (\omega_0/D)^2]$ and for $\lambda > \lambda_c$ the synchronized phase—bifurcating from incoherence—is stationary and stable; (ii) for $\omega_0 > D$ the synchronized phase bifurcating from incoherence is oscillating. It has also
been shown that the order parameter behaves continuously near the phase boundary as

\[ r \sim \left[ \frac{(\lambda - \lambda_c) D (4 + (\omega_0^2/D^2))}{\lambda_c^2 (1 - (2\omega_0^2/D^2))} \right]^{1/2}. \]  

(4)

In section 3 we numerically integrate equation (1) on SF and ER networks and compare the results.

3. Simulation results for scale-free and Erdős–Rényi networks

To create an SF network with average connectivity \( \langle k \rangle = 2m \), we use the BA algorithm [17]. In this procedure, starting from \( m_0 \) initial nodes all connected to each other, at each step one attaches a newly entering node to \( m \leq m_0 \) older ones such that the nodes with higher connectivity have larger probability (proportional to their degree) of getting connected with this new one. Repeating these stages provides us with a network whose degree distribution obeys a power law function as \( P(k) \sim k^{-\gamma} \) with \( \gamma = 3 \). For producing an ER network composed of \( N \) nodes and with the same average degree per node \( \langle k \rangle = 2m \), it is enough to distribute \( Nm \) edges between a randomly chosen pair of nodes [18].

In this work, we set a delta function distribution for intrinsic frequencies, \( f(\omega) = \delta(\omega - \omega_0) \). Changing the reference frame to a rotating one with rotation frequency \( \omega_0 \) enables us to set \( \omega_i = 0 \) for all rotors in equation (1). We also pick \( \eta_i(t) \) out of a box distribution in the interval \(-W/2 < \eta < W/2\); hence its variance equals \( D = (W^2/24) \). Using Ito’s formalism for integration of a stochastic function [23], one obtains the following discrete equation from equation (1):

\[ \theta_i(t + dt) = \theta_i(t) + \lambda \left[ \sum_j a_{ij} \sin(\theta_j(t) - \theta_i(t)) \right] dt + \eta_i(t) \sqrt{dt} + O(dt^2), \]  

(5)

where in Ito’s picture, \( \eta_i(t) \) is evaluated at the initial point of the time interval \([t, t + dt]\), where \( dt \) is the simulation time step. The initial values of the \( \theta_i \) are randomly drawn from a uniform distribution in the interval \([-\pi, \pi]\). To characterize the global phase coherency, we define the following order parameter:

\[ r(t) = \left\langle \left| \frac{1}{N} \sum_{j=1}^{N} e^{i \theta_j(t)} \right| \right\rangle, \]  

(6)

where \( \langle \cdots \rangle \) means the averaging over different realizations of noise and initial conditions. The order parameter lies in the interval \( 0 \leq r \leq 1 \), where \( r = 0 \) corresponds to the disordered phase and \( r = 1 \) characterizes the full synchronized state. In the stationary regime the time argument of \( r(t) \) can be omitted and one can replace the averaging over realizations by time averaging.

In the absence of noise for the SF network, we found that the rotors synchronize for infinitesimally small values of the coupling constant. We start from a fully synchronized state and turn on the noise. On increasing the noise intensity the global coherency vanishes at a critical noise strength, \( W_c \).
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Figure 1. Order parameter versus time for different noise intensities on the SF network. The results are obtained for coupling constant $\lambda = 0.2$ and the number of nodes is $N = 10^4$.

Figure 2. Order parameter versus time for different noise intensities on the ER network. The results are obtained for coupling constant $\lambda = 0.2$ and the number of nodes is $N = 10^4$.

Figures 1 and 2 represent the time dependence of the order parameter for SF and ER networks composed of $N = 10^4$ rotors, respectively. To derive these data, we put $dt = 0.01$, $\lambda = 0.2$ and averagings were carried out over 100 different realizations of noise and initial phase configurations, for the noise intensities increasing from $W = 2.0$ in steps of $\Delta W = 2.0$. From these figures, one finds that after about 500 time steps the system reaches the stationary state for all noise intensities. Obviously the global coherency vanishes at larger coupling values for the SF network (around $W = 10.0$), comparing with the ER network (around $W = 8.0$).

In what follows, to find the dependence of the order parameter on the noise intensity, we fix the number of nodes to $N = 10^4$. And the averagings are carried out for $10^4$ time steps after skipping 2000 initial steps, where the system is certainly in the stationary state. In figures 3 and 4 we depict the order parameter, $r$, versus the noise intensity, $W$, for SF and ER networks, for three coupling constants $\lambda = 0.1, 0.15, 0.2$. Our calculations
for different network sizes show that finite-size effects are negligible in the synchronized regime; however they are traceable in the unsynchronized state. If there is no correlation between the phases of the nodes in the decoherent regime, according to the central limit theorem, we should have $r \sim N^{-1/2}$. Figures 5 and 6 show the size dependence of the order parameter on a logarithmic scale, for SF and ER networks, respectively. From these figures one finds that for $W < W_c$, the order parameter is independent of $N$, but when $W > W_c$ the order parameter decreases with increasing number of nodes. The linear fitting of these data shows a power law relation $r \sim N^{-\alpha}$ with $\alpha = 0.52 \pm 0.02$ for the ER network and $\alpha = 0.42 \pm 0.02$ for the SF one. Our results are in good agreement with the central limit theorem for the ER network, while the deviation from 0.5 for the exponent of the SF network indicates the presence of correlation among the nodes for such networks in the unsynchronized state.

It is possible to reduce the two parameters $\lambda$ and $W$ to a single coupling by the following procedure. Redefining the time as $\tau = \lambda t$ and rescaling the noise as...
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Figure 5. Size dependence of the order parameter for the SF network. The results are obtained for several noise intensities (larger and smaller than \( W_c \sim 10 \)) with \( \lambda = 0.2 \). For \( W > W_c \), the order parameter tends to zero on increasing the number of nodes through a power law relation \( r \sim N^{-\alpha} \) with \( \alpha = 0.42 \pm 0.02 \).

Figure 6. Size dependence of the order parameter for the ER network. The results are obtained for several noise intensities (larger and smaller than \( W_c \sim 6.5 \)) with \( \lambda = 0.2 \). For \( W > W_c \), the order parameter tends to zero on increasing the number of nodes through a power law relation \( r \sim N^{-\alpha} \) with \( \alpha = 0.52 \pm 0.02 \).

In conclusion, the only important coupling is \( (D/\lambda) \) (or equivalently \( g = (W^2/24\lambda) \)). Figures 7 and 8 represent the order parameter versus the reduced coupling, \( g \), for SF and ER networks, respectively. From these figures one finds that all curves in figures 3 and 4 collapse to a single curve, which verifies the above scaling argument.
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Figure 7. The order parameter versus reduced coupling $g$ for the SF network. The results are obtained for three coupling values $\lambda = 0.1, 0.15, 0.2$ and $N = 10^4$ phase oscillators.

Figure 8. The order parameter versus $g$ for the ER network. The results are obtained for three coupling values $\lambda = 0.1, 0.15, 0.2$ and $N = 10^4$ phase oscillators.

In addition to the order parameter $r$, for better specifying the transition from coherency to decoherency, we also introduce Binder’s fourth cumulant which is defined as

$$u = 1 - \frac{\langle r^4 \rangle}{3\langle r^2 \rangle^2}. \quad (9)$$

It is easy to see that in the coherent phase, where $\langle r \rangle$ is non-zero, $u$ takes the value $2/3$ in the large $N$ limit, while upon the global synchronization vanishing ($\langle r \rangle = 0$), this quantity falls to $1/3$ in the thermodynamic limit. The numerical error in Binder’s fourth cumulant is much smaller than that of the order parameter. This makes Binder’s fourth cumulant more advantageous for determination of the position and treatment of the coherence–decoherence phase transition.

For comparison, the noise intensity variations of $r$ and $u$ have been depicted for both SF and ER networks in figures 9(a) and (b). By inspecting these figures one can extract

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two essential results:

(i) Synchronizability for each kind of network depends on the reduced coupling strength $g$ such that for a fixed coupling $\lambda$, at small noise intensities the order parameter for the ER network falls more slowly than that for the SF network. Therefore at small noise intensities, the ER network is more robust than the SF network against the noise, while at large noise intensities the situation is vice versa (see figure 9(a)). The critical coupling ($g_c$) at which the transition from the synchronized to the unsynchronized state occurs is larger for the SF network. So the coherent state in the SF network is more persistent against noise than the ER network with the same average degree and coupling constant.

(ii) The coherency among the population of rotors is destroyed smoothly on increasing the noise intensity in SF networks, while in ER networks the synchronization disappears in a sudden fall at the transition point. These behaviours are more apparent from the treatment of Binder’s fourth cumulant shown in figure 9(b). Therefore the order–disorder transition in SF networks resembles the continuous transitions in equilibrium critical phenomena, while the transition in ER networks is discontinuous-like. For the SF network, we found that the behaviour of the order parameter near the critical coupling is best fitted with a power law $r \sim |g - g_c|^\beta$ with $\beta = 1.27 \pm 0.02$, much larger than the exponent $1/2$ of the all-to-all network (equation (4)).

We also repeated the above simulations with Gaussian white noise. The results are qualitatively similar to the noise with a box distribution; however the critical couplings are smaller in this case due to the larger extent of the Gaussian noise with the same variance. Referring to Goméz-Gardeñes et al [15, 16], a nice explanation of our results is as follows. In homogeneous systems such as ER networks, starting from the fully synchronized phase, some incoherent clusters with more or less the same size begin to form upon turning on the noise. At low noise intensities, these clusters are small and they are well separated; however on increasing the noise they get larger, and they become connected to each other
at intermediate noise intensities. At this point, the locally synchronized regions are no longer coherent; thus a big drop occurs in the order parameter. This is much like the case for first-order phase transitions in equilibrium statistical mechanics, where the ordered and disordered phases coexist at the transition point. On the other hand, the fully coherent state in SF networks is formed around a highly connected core. When noise is applied to such a state, the unsynchronized parts leave this giant cluster one by one, leading to continuous destruction of global coherency.

4. Conclusion

In summary, we numerically investigated the stability of the global phase synchronized state in the Kuramoto model on top of scale-free and random networks, under white noise forcing on each oscillator. Our results emphasize the fact that the stability of the synchronized phase is dependent on the noise strength, such that at low noise intensities ER networks are more stable against loss of coherency, while at intermediate noise intensities, the coherency falls abruptly in such networks. However, in SF networks the coherency among the rotors decreases smoothly and also persists up to larger extents of noise intensity. Therefore, our findings confirm the picture presented by Goméz-Gardeñes et al [15,16], that in heterogeneous networks the giant cluster formed around a core of hubs grows (falls) continuously on increasing (decreasing) the coupling or lowering (raising) the noise intensity. In contrast, in homogeneous systems such as ER networks, on increasing the noise intensity, the coalescence of unsynchronized clusters which are uniformly distributed over the network results in a sudden fall in the global synchronization at the transition point. So the more heterogeneous a network is, the more predictable it is.

Acknowledgments

This work sheds more light on the different aspects of non-linear dynamics on top of homogeneous and heterogeneous network topologies and we hope that it will promote more research on this very interesting problem.

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