Operator Ordering Ambiguity and Third Quantization

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Abstract

In this paper, we will constrain the operator ordering ambiguity of Wheeler-DeWitt equation by analyzing the quantum fluctuations in the universe. This will be done using a third quantized formalism. It is expected that the early stages of the universe are dominated by quantum fluctuations. Furthermore, it is also expected that these quantum fluctuations get suppressed with the expansion of the universe. We will show that this desired behavior of quantum fluctuations could be realized by a wide ranges of the factor ordering parameters. We will examined two different cosmological models, and observe that a similar range of factor ordering parameters produces this desired behavior in both those cosmological models.

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1 Introduction

The information about the quantum state of the universe can be obtained from the wave function of the universe [1]-[4]. The wave function of the universe can be viewed as a solution to the Wheeler-DeWitt equation [5]-[6]. However, there are serious problems with the interpretation of quantum cosmology [7]-[10]. The Wheeler-DeWitt equation is a hyperbolic second order differential equation, so that the square of the absolute value of the wave function of the universe cannot be interpreted as the probability density. This problem is analogous to the problem which occurs in the Klein-Gordon equation. However, the problem with Klein-Gordon equation can be resolved by second quantizing the Klein-Gordon equation. There are several other problems with first quantization, which are resolved by using second quantization. So, just as several problems with the first quantization are resolved by going to second quantization, it has been proposed that third quantization will resolve several problems associated with the second quantized Wheeler-DeWitt equation [10]-[12]. Third quantization is basically a quantum field theory of geometries in the superspace. Thus, in third quantized gravity, the creation and annihilation operators create and annihilate geometries. So, it is possible to study the creation of universe using third quantization [13]-[14]. As the third quantization of gravity can create and annihilate geometries, it is possible to use third quantization to study multiverse [15]-[16]. It has been demonstrated that in such a theory, the quantum state of the multiverse is consistent with standard cosmological boundary conditions. The quantum state of such a multiverse is found to be squeezed, and can be related to accelerating universes. Recently, it has been argued that the third quantization can be used to study the evolution of the physical constants in classically disconnected universes, which are quantum-mechanically entangled [17]. Thus, third quantized gravity is an important approach to quantum gravity.

It may be noted that third quantization has been been applied in various approaches to quantum gravity. The studies in loop quantum gravity, have led to the development of group field theory [18]-[19], and group field cosmology [20]-[23], both of which are third quantized theories. Even the third quantization of string theory has been used to to properly analyze different aspects of the string theory, and this third quantized string theory is called as the string field theory [24]-[25]. The third quantization has been used to analyses the transitions of a string vacuum state to a cosmological solution [26]. This was done by analyzing the creation of a pair of two universes from a string vacuum state. As third quantization has been used in various different approaches to quantum gravity, the study of third quantization is a very important in quantum gravity.
It may be noted that third quantization of modified theories of gravity has also been analyzed. The third quantization of Brans-Dicke theories [27], \( f(R) \) gravity theories [28]-[29] and Kaluza-Klein theories [30] has been studied. It is important to study the suppression of quantum fluctuation in such cosmological models. The quantum uncertainty in third quantization has been studied, and it has been observed that such the quantum fluctuations are suppressed during expansion of the universe [31]-[32]. Thus, at the beginning of the universe, quantum fluctuations dominate, but they are suppressed as the universe expands. It has been demonstrated that this behavior occurs only for certain values of the factor ordering parameter [33]. In this paper, we will generalize these results to obtain a range of values for the factor ordering parameter, which satisfy this desired behavior. We will analyze two cosmological models, and observe that they have similar ranges for the factor ordering parameter.

In section 2, we review the formulation of the third quantized gravity and apply it to the universe which is filled by a cosmological constant. In section 3, quantum fluctuations of the universe will be investigated using the uncertainty principle. In section 4, the ranges of the factor ordering parameter will be calculated, which satisfy the desired behavior. In section 5, another cosmological model will be studied, to investigate the possibility of model dependence of the above range of factor order parameters. In section 6, we will summarize our results.

2 Third Quantized Theory

In order to analyze the third quantization of cosmological models, we need to identify the scale factor of the universe with a 'time' parameter for this third quantized system. Then we would expect that the quantum fluctuations would be suppressed at late times, and the universe would be described by a classical geometry. However, at the beginning of the universe quantum fluctuations would dominate. This requirement can be used to constrain the operator ordering ambiguity of the Wheeler-DeWitt equation [33]. In fact, such quantum fluctuations for a geometry can be analyzed in the third quantized formalism using the uncertainty principle [31]-[32].

Now let the Wheeler-DeWitt equation be given by \( H\psi(h,\phi) = 0 \), where \( h \) is the induced three metric, \( \phi \) is the value of the matter field on the boundary, and \( H \) is the Hamiltonian constraint obtained from general relativity [5]-[6]. Then we can write the third quantized Lagrangian for this system as \( \bar{L}_{3Q} = \varphi(h,\phi)H\psi(h,\phi) \). When this system is quantized we will obtain creation \( b^\dagger \) and annihilation operators \( b \), such that for vacuum state \( |0 \rangle \), we
would have $b|0 \geq 0$. These creation and annihilation operators will create and annihilate geometries. We have used $b$, for the annihilation operator to distinguish it from the scalar factor of the universe, which is denote by $a$.

Now for specific minisuperspace models, we can identify the scale factor of the universe $a$, with the time of this quantum system [7]-[8]. So, when this scale factor is small quantum fluctuations should dominate this system, and when this scale factor is large the quantum fluctuations should be suppressed.

Now as an example, in the cosmological model, where the universe is filled by a cosmological constant [34]-[35], a flat Friedmann-Lemaitre-Robertson-Walker metric can be written as

$$ds^2 = -dt^2 + a^2(t) \sum_{k=1}^{3}(dx^k)^2.$$  \hspace{1cm} (2.1)

Here $a(t)$ is the scale factor of the universe, and $a(t)$ denotes the cosmological evolution of this system and also the size of the universe. It may be noted that the Wheeler-DeWitt equation for this system can be written as (here we set $8\pi G = 1$)

$$\left[ \frac{1}{a^{p_0}} \frac{d}{da} a^{p_0} \frac{d}{da} + 12\Lambda a^4 \right] \psi(a) = 0.$$  \hspace{1cm} (2.2)

We observe that there is a factor ordering ambiguity due to the parameter $p_0$ in such minisuperspace models [36]-[41]. However, it has been demonstrated that such factor ordering can be constrained by the physics of this system. This is because the quantum fluctuations dominate at the early times and are suppressed at the later times, only for certain values of operator ordering parameter [33]. However, it is important to know the exact ranges of the factor ordering parameter for which the universe evolves as desired. Furthermore, it is important to know if this result hold for different cosmological models, or if it is a model dependent result. So, in this paper, we will analyze two different cosmological models, and observe that, since these two models have very wide common ranges of $p_0$ which produce the correct desirable behavior, there is the possibility that there exists some desirable model independent operator ordering parameter $p_0$.

Now we can use the formalism of third quantization and write the third quantized Lagrangian for this quantum system [11]-[12],

$$\mathcal{L}_{3Q} = \frac{1}{2} \left[ a^{p_0} \left( \frac{d\psi(a)}{da} \right)^2 - 12\Lambda a^{p_0+4} \psi(a)^2 \right].$$  \hspace{1cm} (2.3)

Using the standard formalism of third quantization, we can write the third
quantized Schrödinger equation for this system as [33]

\[
\begin{align*}
    i \frac{\partial \Psi(a, \psi)}{\partial a} &= \hat{H}_{3Q} \Psi(a, \psi), \\
    \hat{H}_{3Q} &= \frac{1}{2} \left[ -\frac{1}{a p^o} \frac{\partial^2}{\partial \psi^2} + 12 \Lambda a^{p_o+4} \psi^2 \right].
\end{align*}
\] (2.4)

Here we ignored the operator ordering problem in the first term of \( \hat{H}_{3Q} \) for simplicity. Now the \( \Psi(a, \psi) \) is the third quantized wave function of the universes. The wave function of the universes \( \Psi(a, \psi) \) can be obtained as a solution to the third quantized Schrödinger equation, instead of the Wheeler-DeWitt equation.

3 Quantum Fluctuations

As we have assumed that the quantum fluctuations are suppressed at later times, and dominate at earlier times, it is important to analyze these quantum fluctuations. These quantum fluctuations can be analyzed using the uncertainty principle for these minisuperspace models. In this section, we will analyze such quantum fluctuations for a universe filled with the cosmological constant. The scale factor for such a universe can be identified with the time variable which describes the evolution of the quantum system and the size of this geometry. So, we can denote the initial state of this quantum system by the limit \( a \to 0 \), and this quantum system is expected to evolve to \( a \to \infty \). Now it is expected that quantum fluctuations should dominate the limit \( a \to 0 \). Furthermore, as the universe at later times is represented by a classical geometry, we expect that these quantum fluctuations are suppressed in the limit \( a \to \infty \).

To analyze the uncertainty for this third quantized quantum system, we first assume a Gaussian form of the solution

\[
\Psi(a, \psi) = C \exp \left\{ -\frac{1}{2} A(a) [\psi - \eta(a)]^2 + i B(a) [\psi - \eta(a)] \right\}, \quad (3.1)
\]

where \( C \) is a real constant, \( A(a) \equiv D(a) + i I(a) \), and \( A(a), B(a), \eta(a) \) should be determined from Eq. (2.4). It is possible to define an inner product for two third quantized wave functions, \( \Psi_1 \) and \( \Psi_2 \) as follows,

\[
\langle \Psi_1, \Psi_2 \rangle = \int_{-\infty}^{\infty} d\psi \, \Psi_1^*(a, \psi) \Psi_2(a, \psi). \quad (3.2)
\]

Now we can use this equation to obtain the uncertainty for this third quantized quantum system. This can be done by first writing the dispersion of \( \psi \).
as
\[(\Delta \psi)^2 \equiv \langle \psi^2 \rangle - \langle \psi \rangle^2, \quad \langle \psi \rangle = \frac{\langle \Psi, \psi \rangle}{\langle \Psi, \Psi \rangle}. \quad (3.3)\]

Furthermore, we can also write the dispersion of \(\pi\) as
\[(\Delta \pi)^2 \equiv \langle \pi^2 \rangle - \langle \pi \rangle^2, \quad \langle \pi \rangle = \frac{\langle \Psi, \pi \rangle}{\langle \Psi, \Psi \rangle}. \quad (3.4)\]

We can write the the uncertainty for these geometries as [33]
\[(\Delta \psi)^2 (\Delta \pi)^2 = \frac{1}{4} \left( 1 + \frac{I^2(a)}{D^2(a)} \right). \quad (3.5)\]

It may be noted that the equation for \(A(a)\) can be written as
\[-\frac{i}{2} \frac{dA(a)}{da} = -\frac{1}{2a\rho_o} A(a)^2 + 6\Lambda a^{\rho_o+4}. \quad (3.6)\]

This equation for \(A(a)\) is sufficient to obtain the uncertainty in geometry as \(A(a) = D(a) + i I(a)\).

This cosmological model with \(\rho_o \neq 1\) has been studied [33]. Now if we define
\[z \equiv 2\sqrt{\frac{\Lambda}{3}} a^3, \quad (3.7)\]
we obtain the following solution for \(A(z)\),
\[A(z) = -i 6 \sqrt{\frac{\Lambda}{3}} \left( z \frac{\rho_o + \nu}{2 \sqrt{\frac{\Lambda}{3}}} \right)^{\frac{\nu + 2}{4}} \frac{c_J J_{\frac{\nu}{6} - \rho_o}(z) + c_Y Y_{\frac{\nu}{6} - \rho_o}(z)}{c_J J_{\frac{1}{6} - \rho_o}(z) + c_Y Y_{\frac{1}{6} - \rho_o}(z)}, \quad (3.8)\]
where \(J\nu\) and \(Y\nu\) are Bessel functions of order \(\nu\) and \(c_J\) and \(c_Y\) are arbitrary complex constants. Now using this equation, it is possible to obtain both \(D\) and \(I\).

Now assuming \(c_J c_Y^* - c_Y^* c_Y \neq 0\), we get [33]
\[
\frac{I(z)^2}{D(z)^2} = -\frac{\pi^2 z^2}{4(c_J c_Y^* - c_Y^* c_Y)^2}
\times \left[ 2|c_J|^2 J_{\frac{\nu}{6} - \rho_o}(z) J_{\frac{1}{6} - \rho_o}(z) + 2|c_Y|^2 Y_{\frac{\nu}{6} - \rho_o}(z) Y_{\frac{1}{6} - \rho_o}(z)
+ (c_J c_Y^* + c_Y^* c_Y) \left( J_{\frac{\nu}{6} - \rho_o}(z) Y_{\frac{1}{6} - \rho_o}(z) + J_{\frac{1}{6} - \rho_o}(z) Y_{\frac{\nu}{6} - \rho_o}(z) \right) \right]^2.
\quad (3.9)\]

Thus, the uncertainty of the quantum system can be obtained. So, we can now use the requirements for quantum fluctuations to constrain the ranges of the factor ordering operator for this quantum system.
4 Operator Ordering

Now we can analyze specific ranges of the operator ordering parameter for this cosmological model. It may be noted that as this quantum system evolves to \( a \to \infty \), it also evolves to \( z \to \infty \), and in this limit, we have [42]

\[
J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos \left(z - \frac{\nu \pi}{2} - \frac{\pi}{4}\right), \quad Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin \left(z - \frac{\nu \pi}{2} - \frac{\pi}{4}\right),
\]

(4.1)

where \( \nu = \frac{-5-p_0}{6} \) and \( \frac{1-p_0}{6} \). Now we can also write

\[
\frac{I(z)^2}{D(z)^2} \sim -\frac{1}{(c_j c_Y^* - c_Y^* c_Y)^2}
\]

\[
\times \left[ 2|c_j|^2 \cos \left(z + \frac{p_0+2}{12} \pi \right) \cos \left(z + \frac{p_0-4}{12} \pi \right) \right.
\]

\[
+ 2|c_Y|^2 \sin \left(z + \frac{p_0+2}{12} \pi \right) \sin \left(z + \frac{p_0-4}{12} \pi \right) \right.
\]

\[
+ (c_j c_Y^* + c_Y^* c_Y) \sin \left(2z + \frac{p_0-1}{6} \pi \right) \right]^2
\]

(4.2)

\[
\sim O(1).
\]

Thus, as \( a \to \infty \), we obtain a classical geometry, and this occurs as the quantum fluctuations are suppressed in this limit.

Now the initial state of this quantum system will be denoted by \( a \to 0 \), and this also corresponds to \( z \to 0 \). It is important to analyze the ranges of \( p_0 \) for which the uncertainty becomes of order one, and the ranges for which it tends to infinity. The uncertainty of order one corresponds to a classical geometry, and the uncertainty of order infinity corresponds to a state for which the geometry is dominated by quantum fluctuations.

Now we simplify the notation and define,

\[
\nu_1 = \frac{1-p_0}{6}, \quad \nu_2 = \frac{-5-p_0}{6}; \quad \nu_1 = \nu_2 + 1.
\]

(4.3)

So, we consider the limit \( z \to 0 \), and use the relations [42]

\[
\begin{aligned}
J_\nu(z) &\sim \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu \quad (\nu \neq -1, -2, -3, \ldots), \\
J_{-n}(z) &= (-1)^n J_n(z), \quad Y_{-n}(z) = (-1)^n Y_n(z) \quad (n = 1, 2, 3, \ldots), \\
Y_0(z) &\sim 2 \ln z, \quad Y_\nu(z) \sim -\frac{1}{\pi} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu} \quad (\text{Re } \nu > 0),
\end{aligned}
\]

(4.4)
along with
\[ Y_\nu(z) = \frac{J_\nu(z) \cos(\nu \pi) - J_{-\nu}(z)}{\sin(\nu \pi)} \quad (\nu \neq \text{integer}). \quad (4.5) \]

Now, we divide the ranges of \( \nu_1, \nu_2 \) as
1) \( \nu_1 = 0 \) or \( \nu_2 = 0, \)
2) \( \nu_1 > 0, \nu_2 > 0, \)
3) \( \nu_1 > 0, \nu_2 < 0, \)
4) \( \nu_1 < 0, \nu_2 < 0. \)

It may be noted that as \( \nu_1 = \nu_2 + 1, \) we do not need to consider \( \nu_1 < 0, \nu_2 > 0. \)

Let us first consider the case, when \( \nu_1 = 0 \) or \( \nu_2 = 0. \) We first note that
\( \nu_1 = 0, \) implies \( p_0 = 1, \) and as we have assumed \( p_0 \neq 1, \) we can omit this case. So, now \( \nu_2 = 0, \) implies \( p_0 = -5 \) and \( \nu_1 = 1. \) Now when \( z \to 0, \) we can write
\[
\begin{align*}
J_0(z) & \sim 1, \quad J_1(z) \sim \left(\frac{z}{2}\right) \frac{1}{\Gamma(2)} \to 0, \\
Y_0(z) & \sim \frac{2}{\pi} \ln z \to -\infty, \quad Y_1(z) \sim -\frac{\Gamma(1)}{\pi} \left(\frac{z}{2}\right)^{-1} \to -\infty.
\end{align*}
\]
(4.6)

The largest term in Eq. (3.9) for this case is proportional to \( Z, \) where
\[
Z = z^2[Y_0(z)Y_1(z)]^2 \sim \frac{16}{\pi^4} (\ln z)^2 \to \infty. \quad (4.7)
\]
So, when \( p_o = -5, \) we obtain
\[
\Delta \psi \cdot \Delta \pi \to \infty \quad (z \to 0). \quad (4.8)
\]

Now let us consider the case, when \( \nu_1 > 0, \) and \( \nu_2 > 0. \) In this case, we can again write the largest term in Eq. (3.9) proportional to \( Z, \) where
\[
Z = z^2[Y_{\nu_1}(z)Y_{\nu_2}(z)]^2 \sim \left[ \frac{1}{\pi^2} \Gamma(\nu_1) \Gamma(\nu_2) \left( \frac{1}{2} \right)^{-(\nu_1+\nu_2)} \right]^2 z^{2-2(\nu_1+\nu_2)}.
\]
(4.9)

Now for \( \nu_1 > 0, \) and \( \nu_2 > 0 \) implies \( p_o < -5, \) and so we obtain \( 2-2(\nu_1+\nu_2) < 0. \) This term becomes infinity when \( z \to 0. \) For \( p_o < -5, \) we also obtain
\[
\Delta \psi \cdot \Delta \pi \to \infty \quad (z \to 0). \quad (4.10)
\]

Let us also consider \( \nu_1 > 0, \) and \( \nu_2 < 0. \) This case implies \( 0 < \nu_1 < 1, \) \( -1 < \nu_2 < 0, \) and we know that \( \nu_1 \) and \( \nu_2 \) are not integer. Now when
\[ z \to 0, \text{ we obtain} \]
\[
\begin{align*}
  J_{\nu_1}(z) & \sim \frac{1}{\Gamma(\nu_1 + 1)} \left( \frac{z}{2} \right)^{\nu_1} \to 0, & J_{\nu_2}(z) & \sim \frac{1}{\Gamma(\nu_2 + 1)} \left( \frac{z}{2} \right)^{\nu_2} \to \infty, \\
  Y_{\nu_1}(z) & \sim -\frac{1}{\pi} \frac{\Gamma(\nu_1)}{\Gamma(\nu_1 + 1)} \left( \frac{z}{2} \right)^{-\nu_1} \to -\infty, & J_{-\nu_2}(z) & \sim \frac{1}{\Gamma(-\nu_2 + 1)} \left( \frac{z}{2} \right)^{-\nu_2} \\
  & & & \to 0,
\end{align*}
\]
and we also obtain
\[
Y_{\nu_2} \sim \left\{ \begin{array}{ll}
\cos(\nu_2 \pi) \frac{1}{\sin(\nu_2 \pi)} \frac{1}{\Gamma(\nu_2 + 1)} \left( \frac{z}{2} \right)^{\nu_2} \to +\infty & (-1 < \nu_2 < -\frac{1}{2}), \\
\cos(\nu_2 \pi) \frac{1}{\sin(\nu_2 \pi)} \frac{1}{\Gamma(\nu_2 + 1)} \left( \frac{z}{2} \right)^{\nu_2} \to -\infty & \left(-\frac{1}{2} < \nu_2 < 0\right), \\
-\frac{J_{\nu_2}(z)}{\sin(-\frac{\pi}{2})} \sim \frac{1}{\Gamma(\frac{3}{2})} \left( \frac{z}{2} \right)^{\frac{1}{2}} \to 0 & \left(\nu_2 = -\frac{1}{2}\right).
\end{array} \right.
\]
Thus, the term which is the largest in Eq. (3.9) in this case is proportional to \( Z_1, Z_2, Z_3 \), such that
\[
Z_1 = z^2 [J_{\nu_2}(z)Y_{\nu_1}(z)]^2 \sim \left[ \frac{1}{\Gamma(\nu_2 + 1)} \left( -\frac{\Gamma(\nu_1)}{\pi} \right) \left( \frac{1}{2} \right)^{\nu_1+\nu_2} \right]^2 z^{2(\nu_1+\nu_2)} \\
\sim O(1).
\]
\[
Z_2 = z^2 [Y_{\nu_1}(z)Y_{\nu_2}(z)]^2 \sim \left[ -\frac{\Gamma(\nu_1)}{\pi} \frac{\cos(\nu_2 \pi)}{\sin(\nu_2 \pi)} \frac{1}{\Gamma(\nu_2 + 1)} \left( \frac{1}{2} \right)^{\nu_1+\nu_2} \right]^2 \times z^{2(\nu_1+\nu_2)} \\
\sim O(1).
\]
\[
Z_3 = z^2 J_{\nu_2}(z)Y_{\nu_1}(z)Y_{\nu_1}(z)Y_{\nu_2}(z) \\
\sim \left( \frac{1}{\Gamma(\nu_2 + 1)} \right)^2 \left( -\frac{\Gamma(\nu_1)}{\pi} \right)^2 \frac{\cos(\nu_2 \pi)}{\sin(\nu_2 \pi)} \left( \frac{1}{2} \right)^{-2\nu_1+2\nu_2} z^{2(\nu_1+\nu_2)} \\
\sim O(1).
\]
Here we have used, $2 + 2(-\nu_1 + \nu_2) = 0$. It may be noted that the case $\nu_2 = -1/2$ has not been considered in Eqs. (4.14) and (4.15). Now $0 < \nu_1 < 1$ implies $-5 < p_o < 1$, and so for $-5 < p_o < 1$, we obtain

$$\Delta \psi \cdot \Delta \pi \to O(1) \quad (z \to 0). \quad (4.16)$$

Now let us consider the values $\nu_1 < 0$, and $\nu_2 < 0$. To analyze the initial state of the quantum system for this case, we need to analyze the behavior of Bessel functions in the limit $z \to 0$. Now we can write $\nu = \nu_1$ or $\nu_2$, so we can also write $\nu < 0$. When $z \to 0$, we can write

$$\begin{align*}
J_\nu(z) &\sim \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu \to \infty \quad (\nu \neq -1, -2, -3, \ldots), \\
J_{-\nu}(z) &\sim \frac{1}{\Gamma(-\nu + 1)} \left(\frac{z}{2}\right)^{-\nu} \to 0,
\end{align*} \quad (4.17)$$

We can also write

$$J_{-n}(z) = (-1)^n J_n(z) \sim (-1)^n \frac{1}{\Gamma(n + 1)} \left(\frac{z}{2}\right)^n \to 0 \quad (n = 1, 2, 3, \ldots). \quad (4.18)$$

Using the relation [43]

$$\Gamma(\nu)\Gamma(1 - \nu) = \frac{\pi}{\sin(\pi \nu)}, \quad (4.19)$$

we observe that as $z \to 0$,

$$Y_\nu(z) \sim \frac{\cos(\nu \pi)}{\sin(\nu \pi)} \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu = \cos(\nu \pi) \frac{\Gamma(1 - \nu)}{\nu \pi} \left(\frac{z}{2}\right)^\nu \quad (-\infty < \nu < 0)$$

$$\begin{align*}
\to \begin{cases} 
-\infty & (-\frac{1}{2} < \nu < 0) \\
+\infty & (-2n + \frac{1}{2} < \nu < -2n + 1, 
-2n + 1 < \nu < -2n + \frac{3}{2}) \\
-\infty & (-2n - \frac{1}{2} < \nu < -2n, 
-2n < \nu < -2n + \frac{1}{2})
\end{cases} \quad (4.20)
\end{align*}$$

$$Y_{-n}(z) = (-1)^n Y_n(z) \sim (-1)^n + 1 \frac{\Gamma(n)}{\pi} \left(\frac{z}{2}\right)^{-n} \to (-1)^{n+1} \infty \quad (4.21)$$

$$Y_{-n + \frac{1}{2}}(z) = -\frac{J_{n-\frac{1}{2}}(z)}{\sin\left((-n + \frac{1}{2})\pi\right)} = (-1)^{n+1} J_{n-\frac{1}{2}}(z) \to 0, \quad (4.22)$$

where $n = 1, 2, 3, \cdots$. 

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Using the above relations and Eq. (4.3), it is seen that the terms in Eq. (3.9) which could be large for this case, include the terms that are proportional to $Z_1, Z_2$. For $Z_1$, we note that

$$Z_1 = z^2[J_{\nu_1}(z)J_{\nu_2}(z)]^2 \sim \left[ \frac{(1/2)^{\nu_1+\nu_2}}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)} \right]^2 \left[ \frac{\Gamma(n_1)\Gamma(n_2)}{\pi^2} \left( \frac{1}{2} \right)^{n_1-n_2} \right] \sim \infty \quad (1 < -\nu_1 - \nu_2).$$

Here we have omitted the case when $\nu_1$ or $\nu_2$ is a negative integer, namely $p_o = 6n + 1$ ($n = 1, 2, 3, \cdots$). Since $\nu_1 < 0$, $\nu_2 < 0$, this implies that $p_o > 1$, and $1 < -\nu_1 - \nu_2$ holds. Thus, this $Z_1$ can become infinity, when $p_o > 1$ and $p_o \neq 6n + 1$ ($n = 1, 2, 3, \cdots$). For $Z_2$, we note that

$$Z_2 = z^2[Y_{-\nu_1}(z)Y_{-\nu_2}(z)]^2 \sim \left[ \frac{\Gamma(n_1)\Gamma(n_2)}{\pi^2} \left( \frac{1}{2} \right)^{n_1-n_2} \right]^2 z^{2-2(n_1+n_2)} \rightarrow \infty \quad (n_1 = 1, 2, 3, \cdots; n_2 = n_1 + 1),$$

where $\nu_1 = -n_1$, $\nu_2 = -n_2$. As $n_1 = 1, 2, 3, \cdots$ implies that $p_o = 6n_1 + 1$, this term becomes infinity, when $p_o = 6n + 1$ ($n = 1, 2, 3, \cdots$). So, we obtain that Eq. (3.9) becomes infinity for both these cases, when $p_o > 1$. Therefore, we observe that when $p_o > 1$,

$$\Delta \psi \cdot \Delta \pi \rightarrow \infty \quad (z \rightarrow 0).$$

Let us summarize above consideration. We obtain that

when $p_o > 1$ or $p_o \leq -5$, \quad $\Delta \psi \cdot \Delta \pi \rightarrow \infty \quad (z \rightarrow 0)$,

which means that when $p_o > 1$ or $p_o \leq -5$ the quantum fluctuations dominate the universe at the early times. On the other hand we obtain that

when $-5 < p_o < 1$, \quad $\Delta \psi \cdot \Delta \pi \rightarrow O(1) \quad (z \rightarrow 0)$,

which means that when $-5 < p_o < 1$ universe can become classical at the early times. Since we expect that the quantum fluctuations dominate the universe at the early times, $p_o > 1$ or $p_o \leq -5$ is desirable. Note that we have assumed $p_o \neq 1$, and above result is consistent with Ref. [33].

5 Model Dependence

It is important to analyze if these desirable values of factor ordering depend on a specific cosmological model, or if they are model independent. So, in
In this section, we will perform a similar analysis for a different cosmological model. In this cosmological model, a closed universe is filled with a constant vacuum energy density \( \rho_v \) and radiation \( \epsilon \), and the Wheeler-DeWitt equation for this model can be written as [44]-[45]

\[
\left[ \frac{d^2}{da^2} + \frac{p_o}{a} \frac{da}{da} - k_2 a^2 + k_4 \rho_v a^4 + k_6 \epsilon \right] \psi(a) = 0, \tag{5.1}
\]

where \( a \) is the scale factor for this closed universe, \( p_o \) is the operator ordering parameter for this cosmological model, and

\[
k_2 = \frac{9\pi^2}{4G^2\hbar^2}, \quad k_4 = \frac{6\pi^3}{G\hbar^2}, \quad k_6 = \frac{6\pi^3}{G\hbar^2}. \tag{5.2}
\]

It may be noted that the wave function of the universe for this cosmological model has been discussed, so we can perform the above analysis for this cosmological model [45].

Now in this cosmological model, we again assume a Gaussian form of solution for the third quantized Schrödinger equation. So, uncertainty in its geometry can also be obtained using the same formalism [33]. In this reference we found that at the late times for any \( p_o \) the universe becomes classical, since the quantum fluctuation becomes minimum. Now at the early times for \( p_o \neq 1 \), we can write

\[
z \equiv \sqrt{k_6 \epsilon} a. \tag{5.3}
\]

So, initial state for this quantum system can be written as \( a \rightarrow 0 \), and this also corresponds to \( z \rightarrow 0 \). For this initial state, we obtain [33],

\[
\begin{align*}
\frac{I(z)^2}{D(z)^2} &= -\frac{\pi^2 z^2}{4(c_J c_Y - c_J^* c_Y)^2} \\
&\times \left[ 2|c_J|^2 J_{\frac{1-p_o}{2}}(z) J_{\frac{1-p_o}{2}}(z) + 2|c_Y|^2 Y_{\frac{1-p_o}{2}}(z) Y_{\frac{1-p_o}{2}}(z) \\
&+ (c_J c_Y^* + c_J^* c_Y) \left( J_{\frac{1-p_o}{2}}(z) Y_{\frac{1-p_o}{2}}(z) + J_{\frac{1-p_o}{2}}(z) Y_{\frac{1-p_o}{2}}(z) \right) \right]^2.
\end{align*} \tag{5.4}
\]

Now depending on the range of \( p_o \), this quantum system is either dominated by quantum fluctuations, or the quantum fluctuations are suppressed and it is represented by a classical geometry. To analyze this range, we first define,

\[
\nu_1 = \frac{1 - p_o}{2}, \quad \nu_2 = \frac{-1 - p_o}{2}; \quad \nu_1 = \nu_2 + 1. \tag{5.5}
\]
Now we can perform a similar analysis to the one done in the previous section. Thus, we can analyze various case for this system.

Let us start by considering $\nu_1 = 0$ or $\nu_2 = 0$. We observe that for $p_o = -1$, we can write

$$\Delta \psi \cdot \Delta \pi \to \infty \ (z \to 0). \quad (5.6)$$

Now let us also consider the case $\nu_1 > 0$, and $\nu_2 > 0$. For $p_o < -1$, we obtain

$$\Delta \psi \cdot \Delta \pi \to \infty \ (z \to 0). \quad (5.7)$$

For the $\nu_1 > 0$, and $\nu_2 < 0$, we observe that when when $-1 < p_o < 1$,

$$\Delta \psi \cdot \Delta \pi \to O(1) \ (z \to 0). \quad (5.8)$$

Now for the case $\nu_1 < 0$, $\nu_2 < 0$, when $p_o > 1$, we obtain

$$\Delta \psi \cdot \Delta \pi \to \infty \ (z \to 0). \quad (5.9)$$

Summarizing above discussions, we obtain that

when $p_o > 1$ or $p_o \leq -1$, $\Delta \psi \cdot \Delta \pi \to \infty \ (z \to 0)$, \quad (5.10)

which means that when $p_o > 1$ or $p_o \leq -1$ the quantum fluctuations dominate the universe at the early times. On the other hand we obtain that

when $-1 < p_o < 1$, $\Delta \psi \cdot \Delta \pi \to O(1) \ (z \to 0)$, \quad (5.11)

which means that when $-1 < p_o < 1$ universe can become classical at the early times. Since we expect that the quantum fluctuations dominate the universe at the early times, $p_o > 1$ or $p_o \leq -1$ is desirable. Note that we have also assumed $p_o \neq 1$, and above result is consistent with Ref. [33].

Comparing this section and previous section, we find that in both models there exist the common ranges for physically desirable $p_o$, that is from Eqs. (4.26) and (5.10)

when $p_o > 1$ or $p_o \leq -5$ , $\Delta \psi \cdot \Delta \pi \to \infty \ (a \to 0)$, \quad (5.12)

which means that when $p_o > 1$ or $p_o \leq -5$ the quantum fluctuations dominate the universe at the early times, $a \to 0$. Since these ranges of $p_o$ are very wide, we could expect that there might exist some model independent desirable operator ordering parameter $p_o$ in the Wheeler-DeWitt equation. Note that, since our analysis is based on the assumption $p_o \neq 1$, there remains the possibility that $p_o = 1$ might be also the model independent desirable operator ordering parameter.
6 Conclusion

In this paper, we have analyzed the creation of universe using third quantization. At the beginning of the universe, the geometry of the universe is dominated by quantum fluctuations. These fluctuations are suppressed as this universe evolves, resulting in a classical geometry of our universe. We have used these two physical requirements to constraint the range of factor ordering for two different cosmological models. It was observed that both these cosmological models satisfy the desired evolution only for the common ranges of $p_o$, when $p_o > 1$ or $p_o \leq -5$, $\Delta \psi \cdot \Delta \pi \rightarrow \infty$ $(a \rightarrow 0)$. Thus, it seems that for the values $p_o > 1$ or $p_o \leq -5$ the quantum fluctuations dominate initial state of the universe $a \rightarrow 0$, and a classical geometry will form at later stages of the evolution of the universe. It may be noted that as we have obtained the very wide common ranges for the desirable operator ordering parameter $p_o$ for two different cosmological models, it indicates that there is a possibility that there exists some desirable $p_o$ which is independent of the specifics details of a cosmological model. However, it would be important to analyze many other different cosmological models to verify the model independence of this value. Our analysis is based on the assumption $p_o \neq 1$, so it is possible that $p_o = 1$ might also be a valid value for the operator ordering parameter.

It may be noted that the third quantization has been generalized to loop quantum gravity, and this has led to the development of group field theory [18]-[19], and group field cosmology [20]-[23]. It would be interesting to generalize the results of this paper to these third quantized models of loop quantum gravity. Furthermore, the third quantization of string theory has also been used to study the creation of a pair of universes from string vacuum state [26]. It would be interesting to use the formalism developed in this paper to analyze the creation of universe using string theoretical solutions. We would also like to point out that the third quantization Horava-Lifshitz gravity has also been discussed [46]-[48]. It would be interesting to analyze the operator ordering ambiguity for such a cosmological model. It may be noted as this is an non-trivial modification of gravity, if we obtain similar ranges for the values of the operator ordering parameter, then this would be a strong indication of the existence of the model independent operator ordering parameter.

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