MODELING FUZZY GEOGRAPHIC OBJECTS WITHIN FUZZY FIELDS

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ABSTRACT  To improve the current GIS functions in describing geographic objects with fuzziness, this paper begins with a discussion on the distance measure of spatial objects based on the theory of sets and an introduction of dilation and erosion operators. Under the assumption that changes of attributes in a geographic region are gradual, the analytic expressions for the fuzzy objects of points, lines and areas, and the description of their formal structures are presented. The analytic model of geographic objects by means of fuzzy fields is developed. We have shown that the 9-intersection model proposed by Egenhofer and Franzosa (1991) is a special case of the model presented in the paper.

1 Introduction

The representation of spatial objects is one of the key issues in current research on the spatial database theory of GIS (Burrough and McDonnell, 1998). Spatial objects in nature are classified as two kinds: objects with distinctive boundaries and objects with transitional or fuzzy boundaries (Burrough and McDonnell, 1998). The first type of objects such as artificial buildings are commonly represented by object models, while the latter group such as arable land, land cover, forest and desert zones is generally represented by field models. Correspondingly, there are two different data formats: vector and raster data (Goodchild, 1989). The fundamental graphic elements of vector data are points, lines and polygons, while the basic unit for raster data is a single grid cell (Burrough and McDonnell, 1998).

Uncertainties of spatial data are partly derived from the conversion of spatial objects in the real world into GIS, including conceptualization of spatial objects, capture of spatial data and so on (Burrough and McDonnell, 1998). The focus of current research is on errors of positional data (Goodchild, 1991). For geographical objects with clear-cut boundaries, the uncertainties of spatial data are of randomness. This kind of objects will be regarded as points, lines or polygons in random fields. Issues related to these objects have been systematically discussed by Liu (1995, 1998) and Liu, et al. (1998), and a random uncertainty theory of spatial data was developed. On the other hand, there has been a lack of attention to the representation of geographical objects that defy easy and clear definitions (Liu, et al., 2000). Nevertheless, such geographical objects are fairly common in the fields of land surveying, environmental and resource sciences, agronomy, and earth sciences (Burrough and McDonnell, 1998).

Many techniques for capturing spatial data of geographical objects produce boundaries with uncer-
Taking remote sensing images as an example, the procedure of data capture includes both recognition of spatial objects and segmentation of images in discrete thematic categories. The transitional nature of surface features, mixed signals within pixels of fairly large sizes, and interference of atmospheric and other extraneous factors often introduce various levels of uncertainties to the positional data of thematic maps derived from remotely sensed data. Spatial objects of this type will be regarded as points, lines or polygons in fuzzy fields. In order to develop a mathematical model for describing geographical objects in fuzzy fields, this paper focuses on issues of fuzzy uncertainty of positional data, including new mathematical expressions and methods of formal description of fuzzy points, lines and polygons in GISs.

2 Definitions

2.1 Metric spaces

Let S be a set, whose elements are points denoted as $P_1, P_2, P_3, \ldots$, and a mapping $(P_1, P_2) \rightarrow d(P_1, P_2)$ of $S \times S$ into $R$ having the following properties (Liu, 1995):

1. $d(P_1, P_2) = 0$ if and only if $P_1 = P_2$
2. $d(P_1, P_2) = d(P_2, P_1)$
3. $d(P_1, P_2) \leq d(P_1, P_3) + d(P_3, P_2)$

Then the pair consisting of $S$ and $d$ is defined as a metric space, denoted as $(S, d)$. $d$ is called a metric and $d(P_1, P_2)$ is called the distance between the points $P_1$ and $P_2$. Distance between two points $P_i(x_{i1}, x_{i2}, \ldots, x_{in})$ in $R^n$ is described in terms of the Minkowski $d_1$-metric:

$$d_1(P_1, P_2) = \left( \sum_{j=1}^{n} |x_{ij} - x_{2j}|^t \right)^{1/t}$$ (1)

Conventional Euclidean distance is defined by the $d_2$-metric.

2.2 Distance measure between sets

In vector GIS, the geometric relations between graphic elements include line segments connecting two points and polygons made of multiple line segments. Moreover, any point as a set has only one element, while a line segment or a polygon as a set includes two or more point elements respectively. In addition, all points, lines and simple polygons can be denoted by closed sets. Thus, we may define the distance between any two spatial objects as:

$$d_2(S_1, S_2) = \bigcup_{s_1 \in S_1, s_2 \in S_2} \min d_2(s_1, s_2)$$ (2)

If $S_1$ is a point set like $\{P_{1i}\}$, the above Eq. (2) will be simplified into:

$$d_2(P_{1i}, S_2) = \min_{s_2 \in S_2} d_2(P_{1i}, s_2)$$ (3)

Furthermore, when $S_1, S_2$ are both point sets, Eq. (2) represents the distance between two points, denoted as $d_2(P_{1i}, P_{2i})$. Geometrically, Eq. (3) defines the vertical distance between a point and a point, a line or a polygon, while Eq. (2) represents its extension.

2.3 Morphological operation of sets

2.3.1 Dilation

$$A \oplus B(e) = \{x : x = a + b : \forall a \in A \land \exists b \in B(e)\}$$ (4)

where $\oplus$ denotes morphological dilation operator, $A$ is an object set, and $B(e)$ is a set of structure elements. $A_b$ is defined as the transition of the set $A$ by vector $b$ according to a given scale and direction, i.e., $A_b = \{x + b : x \in A\}$. As dilation is an expansion of the object set, $A \oplus B(e)$ may be regarded as its extended set (Henk, et al., 1998).

2.3.2 Erosion

$$A \ominus B(e) = \{x : a + b \in A, \forall b \in B(e)\} = \bigcap_{b \in B(e)} A_{-b}$$ (5)

where $\Theta$ is a morphological erosion operator, and $A_{-b}$ is defined as the erosion of an object set $A$, i.e., $A_{-b} = \{x - b : x \in A\}$, which is regarded as a contraction of the set. According to the topological properties of morphological operations, dilation and erosion sets are both closed sets (Henk, et al., 1998).

3 Modeling fuzzy geographic objects

3.1 Models of fuzzy points

3.1.1 The membership function of a fuzzy point

In set theory, a set is expressed by its characteristic function (Klir and Folger, 1988). When a point is certain or its errors can be neglected, it can be denoted with $P(x^0, y^3)$. Its characteristic function is expressed as:
Mathematically, denoting a point with fuzzy uncertainty as \( \tilde{P} \) is equivalent to treating the point as a fuzzy set, and its uncertainties depend on the size of its boundary region. Obviously, if its boundary region is large, the precision of the set will be relatively low (Klir and Folger, 1988). The boundary region can be generated through \( \varepsilon \)-buffer operation, designated as \( P \oplus B(\varepsilon) \) in Eq. (4) and shown in Fig. 1.

Thus, the precision measure of the set for a fuzzy point can be defined as follows:

\[
P_a = \{(x,y)/d^2((x,y),P) = \varepsilon\}
\]

where \( d^2((x,y),P) = \min \{d^2((x,y),(x_i,y_i)), V i \} \) and its geometric meaning is shown in Fig. 2. Hence, the fuzzy line \( \tilde{L} \) will be defined by its membership function as:

\[
\mu_{\tilde{L}}(x,y) = \begin{cases} 
1 - d^2((x,y),\tilde{L})/\varepsilon, d^2((x,y),\tilde{L}) \leq \varepsilon \\
0, & d^2((x,y),\tilde{L}) > \varepsilon
\end{cases}
\]

According to Eq. (14), we may generate fuzzy bands with different membership functions as shown in Fig. 2(b).

3.2 Models of fuzzy lines

3.2.1 The membership function of a fuzzy line

Denoting a certain linear feature with \( L \), when its position includes fuzzy uncertainties, we call it as a fuzzy line \( \tilde{L} \), which is generated by \( \varepsilon \)-buffer operation, i.e., \( L \oplus B(\varepsilon) \). According to the topological properties of morphological dilation, if the set \( L \) and the structure element \( B(\varepsilon) \) are both closed sets, its dilated set is also a closed set. Therefore, we can define the precision measure of the set for fuzzy line \( \tilde{L} \) as:

\[
\tilde{L} = \{(x,y)/d^2((x,y),\tilde{L}) = \varepsilon\}
\]

where \( d^2((x,y),\tilde{L}) = \min \{d^2((x,y),(x_i,y_i)), V i \} \), and its geometric meaning is shown in Fig. 2. Hence, the fuzzy line \( \tilde{L} \) will be defined by its membership function as:

\[
\mu_{\tilde{L}}(x,y) = \begin{cases} 
1 - d^2((x,y),\tilde{L})/\varepsilon, d^2((x,y),\tilde{L}) \leq \varepsilon \\
0, & d^2((x,y),\tilde{L}) > \varepsilon
\end{cases}
\]
3.2.2 Formal description of fuzzy lines

In order to analyze the morphological structure of the fuzzy set \( \tilde{L} \), we introduce a fuzzy cut set parameter \( \alpha \). The morphological structure of the fuzzy set \( \tilde{L} \) can then be defined as follows:

\[
\begin{align*}
\partial L_\alpha &= \{ (x,y) / \mu L(x,y) = \alpha \} \\
L_\alpha^0 &= \{ (x,y) / \mu L(x,y) > \alpha \} \\
P_\alpha^- &= \{ (x,y) / \mu L(x,y) < \alpha \}
\end{align*}
\]

which are the boundary, interior and exterior of the cut set \( L_\alpha \). Obviously, when \( \alpha \) takes the value of 0, \( d_2((x,y),L) = \varepsilon \). Then, \( \partial L_0, L_0^0 \) and \( L_0^- \) denote the boundary, interior and exterior of a fuzzy region generated by \( \varepsilon \)-buffer operation of a fuzzy line feature.

3.3 Models of fuzzy polygons

3.3.1 Membership functions of fuzzy polygons

For a certain polygon \( O \), when its position has fuzzy uncertainties, designated as \( \tilde{O} \), it can be generated by an \( \varepsilon \)-buffer operation, \( O \oplus \varepsilon \), as shown in Fig. 3. Considering the possibility of a point belonging to a polygon is 0.5 when it is a part of the boundary of the polygon, we may express the membership function of the fuzzy polygon \( \tilde{O} \) as:

\[
\mu_\tilde{O}(x,y) = \begin{cases} 
1, & (x,y) \in O \text{ and } d_2((x,y),B) > \varepsilon \\
\frac{1}{2} + \frac{d_2((x,y),B)}{\varepsilon}, & (x,y) \notin O \text{ and } d_2((x,y),B) \leq \varepsilon \\
\frac{1}{2} - \frac{d_2((x,y),B)}{\varepsilon}, & (x,y) \notin O \text{ and } d_2((x,y),B) \leq \varepsilon \\
0, & (x,y) \notin O \text{ and } d_2((x,y),B) > \varepsilon 
\end{cases}
\]

(18)

In order to analyze the morphological structure of a fuzzy polygon, i.e. the three elements of its formal description: boundary, interior and exterior, we define the concepts of inner and outer boundaries as follows:

\[
B_1 = \{ (x,y) / d_2((x,y),B) = \varepsilon, (x,y) \in O \}
\]

(19)

\[
B_0 = \{ (x,y) / d_2((x,y),B) = \varepsilon, (x,y) \notin O \}
\]

(20)

We use \( A_1 \) and \( A_0 \) below to represent the two regions formed by an inner boundary \( B_1 \) and an outer boundary \( B_0 \). At the same time, from Eq. (18), we further define the inner buffer region and the outer buffer region of a fuzzy polygon \( \tilde{O} \), denoted as \( IO \) and \( EO \), as follows:

\[
IO = \{ (x,y) / d_2((x,y),B) \leq \varepsilon, (x,y) \in O \}
\]

(21)

\[
EO = \{ (x,y) / d_2((x,y),B) \leq \varepsilon, (x,y) \notin O \}
\]

(22)

Then, we have \( A_0 = A_1 \cup IO \cup EO \). Obviously, when a point lies in the region \( A_1 \), this point must be inside the polygon \( O \). When lying in the region \( IO \), the possibility of the point in the polygon is \([0.5,1] \). When it is in the region \( EO \), its possibility of falling in the polygon is \([0,0.5] \). In terms of polygon attributes, it can be interpreted as follows: when one point lies in \( A_1 \), this point has the same attributes as the polygon. While a point lies in the inner buffer region \( IO \), the degree that point attributes are similar to the polygon is \([0.5,1] \) and the possibility of this point belonging to the polygon \( O \) is \([0,0.5] \). The condition that a point lies in the outer buffer region \( EO \) can be interpreted in a similar fashion.

3.3.2 Formal description of fuzzy polygons

Since uncertainties of fuzzy spatial objects are mainly from the extension of spatial ranges, i.e.
from uncertainties of boundary elements belonging to a set, we may express the boundary region of a fuzzy polygon as:

\[ \partial O = A_0 - A_1 = \partial I \cup \partial E \quad (23) \]

or written as

\[ \partial O = \{(x,y)/d_2((x,y),B) < \varepsilon\} \quad (24) \]

Obviously, \( \partial O \) is a transitional region with a width of \( 2\varepsilon \), where regional attributes change gradually. However, for analysis of land use categories and output of thematic maps, taken a threshold \( a \), fuzzy polygons are represented by the degree of its attributes belonging to a certain polygon. Therefore, we may express the three elements of formal description for the polygon \( O \) as follows:

\[ \partial \mathcal{O}_a = \{(x,y)/\mu_a(x,y) = a\} \quad (25) \]

\[ O^0_a = \{(x,y)/\mu_a(x,y) > a\} \quad (26) \]

\[ O^-_a = \{(x,y)/\mu_a(x,y) < a\} \quad (27) \]

Apparently, geometric location or spatial ranges vary with parameters \( \varepsilon \) and \( a \). However, the range of uncertainty of the polygon is always the boundary region \( \partial \mathcal{O} \). This leads to the conclusion that the position of any point in a fuzzy region can be potentially identified as the boundary, interior or exterior to a certain degree. In other words, the spatial ranges of the three elements overlap each other partially. Therefore, we have:

\[ \partial \mathcal{O} = \partial A_1 \cup \partial O \quad (28) \]

\[ \mathcal{O}^0 = A_1^0 \cup (\partial \mathcal{O})^0 \quad (29) \]

\[ \mathcal{O}^- = A_0^- \cup (\partial \mathcal{O})^- \quad (30) \]

From Eqs. (28), (29) and (30), we know that the intersections of the three formal elements of a fuzzy polygon are not an empty set (\( \emptyset \)), that is, they do not satisfy the exclusion law of a Boolean set operation, which is an important property of a fuzzy set. If the boundaries of a polygon are certain, i.e. \( A_1 = A_0 \), and its boundary region is an empty set, the formal description of a fuzzy polygon is the same as that of a certain polygon based on the point-set topology as proposed by Egenhofer and Franzosa (1991). In short, this formal description is an extension of certain objects.

4 Conclusion

Fuzzy points, lines and polygons can be expressed by their membership functions respectively and the properties of fuzzy sets can be described by the three elements of their formal description. Spatial ranges of fuzzy points and fuzzy lines are generated by means of a dilation operator, while spatial range of a fuzzy polygon is generated by means of both dilation and erosion operators. Moreover, properties of dilation and erosion sets are both closed sets.

When there is no fuzziness in the representation of points, lines and polygons, their formal description will be degenerated to the model based on the point-set topology as proposed by Egenhofer and Franzosa (1991).

The theory on randomness is extensively used in research on spatial objects with random uncertainties. Similarly, the theory on fuzziness forms the basis of studying spatial objects with fuzzy uncertainties. Since fuzzy objects are quite common in GIS applications, accurate formal description of this kind of spatial object will be a necessary prerequisite for productive research on fuzzy spatial relationships and realization of fuzzy spatial query.

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