No Quantum Super-minisuperspace with $\Lambda \neq 0$

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Abstract

We show that the quantum super-minisuperspace of $N=1$ supergravity with $\Lambda \neq 0$ has no non-trivial physical states for class $A$ Bianchi models. Hence, in super quantum cosmology, the vanishing of $\Lambda$ is a condition for the existence of the universe. We argue that this result implies that in full supergravity with $\Lambda$ there are no non-trivial physical states with a finite number of fermionic fields. We use the Jacobson canonical formulation.

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I. Introduction

In a recent paper, the super-minisuperspace of canonical N=1 supergravity in the formulation given by Jacobson [1] was introduced [2]. The quantization of all Bianchi type A models was carried out, in the triad representation. Exploiting the fact that the system has a finite number of degrees of freedom, the wavefunction can be expanded in (even) powers of the gravitino field, up to sixth order. It was found that, in general, the physical states of the quantum theory have a very restricted form. Only the parts of the wavefunction of zero and sixth order in the gravitino field are non-vanishing.

In this paper, we point out that if one includes a cosmological constant $\Lambda$ in super-minisuperspace (first considered in this context in [3, 4]), the only physical state allowed is trivial, i.e. $\Psi = 0$, for any factor ordering. Since we find $\Psi = 0$, this will also be true in any representation. The technical reason for this is that, in the presence of $\Lambda$, the quantum supersymmetry constraints mix parts of the wavefunction of different order in the gravitino field. This mixing does not allow to fulfill the coupled system of quantum constraint equations, except with the trivial solution.

From this result one arrives to the conclusion that, if one is confident in the super-minisuperspace approximation, the price of introducing a cosmological constant in super-minisuperspace is the universe itself! To recall Einstein’s words, to introduce a cosmological constant would be a big blunder, indeed. Moreover, we expect the same conclusion to hold if one adds (supersymmetric) matter to this models, because the same mixing of parts of the wavefunction of different order in the gravitino fields will continue to take place.

The implications for the full theory are that in the presence of a non-vanishing cosmological constant, the only state with a finite number of fermionic fields is trivial. We can offer two arguments for this. The first is that the above mentioned mixing will still be present. The second argument is that if non-trivial states with a finite number of fermionic fields were allowed in the full theory, they would certainly show up in the mini-superspace models. From these arguments, it follows that physical states for supergravity with $\Lambda$ contain an infinite number of fermionic fields. (Carroll et al. reach the same conclusion for the case $\Lambda = 0$ [5]. See, however, Ref. [6] for a different point of view.)

It is interesting to note that these quantum states are not accessible in the super-minisuperspace approximation. Therefore, our result can be considered as an explicit example of the intrinsic limitations of the method of quantizing the homogenous sector of the classical theory.
The paper is organized as follows. In Sect. II, we recall briefly the necessary elements of canonical supergravity with \( \Lambda \), and we specialize to homogenous solutions. In Sect. III, we quantize class A Bianchi models in the triad representation. In Sect. IV, we give the explicit proof that only \( \Psi = 0 \) solves the quantum supersymmetry constraints. We conclude in Sect. V with some final remarks about previous work in this topic.

II. Super-minisuperspace with \( \Lambda \neq 0 \)

As canonical coordinates for \( N=1 \) supergravity we take a complex traceless \( SL(2, \mathbb{C}) \) spatial connection, \( A_{iA}^B \), and a traceless vector density of weight 1, \( \tilde{\sigma}_{ij}^A \), together with the spatial anti-commuting gravitino field \( \psi_i^A \), and its conjugate momentum the anti-commuting \( \tilde{\pi}_i^A \). (Small latin letters from the middle of the alphabet denote spatial indices, \( i, j, \ldots = 1, 2, 3 \). Capital latin letters denote \( SL(2, \mathbb{C}) \) indices \( A, B, \ldots = 0, 1 \). These indices are raised and lowered with the anti-symmetric symbol \( \epsilon_{AB} \), and its inverse \( \epsilon_{BA} \), according to the rules \( \lambda^A = \epsilon_{AB} \lambda^B \), \( \lambda_A = \lambda^B \epsilon_{BA} \).

The connection \( A_{iA}^B \) is the spatial pull-back of the left-handed spin connection. The vector density \( \tilde{\sigma}_{ij}^A \) may be interpreted as the (densitized) spatial triad, in the sense that the covariant (doubly densitized) spatial metric is given by \( \det q^{ij} = \tilde{\sigma}_{iA} \tilde{\sigma}_{jB} \). The momentum \( \tilde{\pi}_i^A \) is related to the complex conjugate of the spatial gravitino field, \( \bar{\psi}_i^A \).

In the presence of a cosmological constant, \( \Lambda = -4m^2 \), the supersymmetry constraints are given by [1, 2]

\[
R^A := S^A + S^A_m = 0, \quad (1)
\]

\[
R^{*A} := S^{*A} + S^{*A}_m = 0. \quad (2)
\]

We write separately the parts that correspond to a vanishing cosmological constant, given by,

\[
S^A := D_{k} \tilde{\pi}^A_k,
\]

\[
S^{*A} := (\tilde{\sigma}^i \tilde{\sigma}^j D_{[ij]} \psi_k)_k^A. \quad (4)
\]

Here \( D_i \) is the covariant derivative of \( A_{iA}^B \). We are using the convention that suppressed spinor indices are contracted from upper left to lower right, e.g. \( (\tilde{\sigma}^i \tilde{\sigma}^j)^{AB} = \tilde{\sigma}^{iAC} \tilde{\sigma}^j_C^B \). Hermitian conjugation is defined with respect to some Hermitian metric \( n^{AA'} \).

The parts proportional to the cosmological constant are

\[
S^A_m := i2\sqrt{2m}(\tilde{\sigma}^i \psi_i)^A, \quad (5)
\]

\[
S^{*A}_m := -i2\sqrt{2m}(\tilde{\pi}_i \bar{\psi}_i)^A. \quad (6)
\]
There are also constraints corresponding to the additional invariance of the theory, i.e. $SL(2, C)$ and diffeomorphism invariance, but we will not need their explicit form here (see [1]).

In the case of spatial homogeneity, we consider a kinematical triad of vectors, $X^i_a$, which commute with the three Killing vectors on the spatial hypersurface \cite{8}. (Latin letters from the beginning of the alphabet, $a,b,c,...$ label the triad vectors.) The triad satisfies $[X_a^i, X_b^j] = C_{abc}^j X_c^i$, where $C_{abc}^j$ denote the structure constants of the Bianchi type under consideration. The basis dual to $X^i_a$ is defined with $X_i^a \chi^i_a = \delta^i_a$. We restrict to type A Bianchi models by assuming that $C_{abc}^j = \epsilon_{abc} M_{dc}$, with $M_{ab}$ symmetric. (For our reasons to exclude class B Bianchi models, see \cite{2}).

The Jacobson phase space variables may be expanded with respect to the kinematical triad $X^i_a$ (or $\chi^i_a$), as,

$$A_{iA}^B = A_{aA}^B \chi^a_i,$$

$$\tilde{\sigma}^{iAB} = (det \chi) \sigma^{aAB} X^i_a,$$

$$\psi^A_i = \psi^A_a \chi^a_i,$$

$$\tilde{\pi}^i_A = (det \chi) \pi^{aA} X^i_a,$$

where we introduce $det \chi$ to de-densitize the momentum variables.

Inserting the expansion in the supersymmetry constraints (1) and (2), for Bianchi class A models they reduce to

$$R'^A = S'^A + S'^m,$$  

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where

$$S'^A = - \frac{1}{2} \epsilon_{abcd} M_{dc} (\sigma^a \sigma^b \psi_c)^A + (\sigma^a \sigma^b A_a \psi_b)^A,$$  

$$S^A = A_{aAB} \pi^A_B,$$  

$$S'^m = - i2 \sqrt{2} m h^{-1/2} (\sigma_a \pi^A)^A,$$  

$$S^m = i2 \sqrt{2} m (\sigma_a \psi_a)^A.$$  

We denote with $h$ the determinant of $h_{ab} := Tr(\sigma_a \sigma_b)$, and $\sigma_a$ is the inverse of $\sigma^a$. Note that we have rescaled the constraints by the appropriate factor of $det \chi$ to de-densitize them.

III. Quantization in the Triad Representation
Now we turn to the quantization of class A Bianchi models in the triad representation. Quantum states may be represented by wavefunctions that depend on the triad, and on the gravitino field, \( \Psi = \Psi(\sigma, \psi) \). The variables \( \sigma^{aAB} \) and \( \psi^a_A \) turn into ‘position’ operators. Their momenta are represented with \( \hat{A}_a^{AB} \Psi = (1/\sqrt{2})(\delta \Psi/\delta \sigma^{aAB}) \), \( \hat{\pi}_a^A \Psi = (1/\sqrt{2})(\delta \Psi/\delta \psi^a_A) \).

The translation of the supersymmetry constraints to their quantum version gives

\[
\hat{R}^A \Psi = (\hat{S}^A + \hat{S}^A_m) \Psi = 0, \tag{13}
\]
\[
\hat{R}^{tA} \Psi = (\hat{S}^{tA} + \hat{S}^{tA}_m) \Psi = 0, \tag{14}
\]

where

\[
\hat{S}^{tA} \Psi = \left[ -\frac{1}{2} \varepsilon_{abd} M^{dc} (\sigma^a \sigma^b \psi^c)_A + \frac{1}{\sqrt{2}} (\sigma^a \delta^{b} \psi^b)_A + \frac{1}{\sqrt{2}} \alpha \sigma^a AC \psi^a_C \right] \Psi
\]
\[
\hat{S}^A \Psi = \frac{1}{2} \frac{\delta^2 \Psi}{\delta \sigma^{aAB} \delta \psi^B_a}
\]
\[
\hat{S}^A_m \Psi = -i2m h^{-1/2} (\sigma^a \delta \Psi_{\psi^a})^A
\]
\[
\hat{S}^{tA}_m = i2\sqrt{2} m (\sigma^a \psi^a)_A \Psi.
\]

The constant \( \alpha \) parametrizes the factor ordering ambiguity in (9).

In addition, a physical state must also satisfy \( \hat{J}^{AB} \Psi = 0 \), where \( J^{AB} \) is the generator of \( SL(2, C) \) transformations. It follows that the wavefunction must be an \( SL(2, C) \) scalar. Invariance under diffeomorphisms of a physical state follows automatically from (13,14), because of the well known fact that \( \{ R^{tA}, R^A \} \propto H^{AB} \), where \( H^{AB} \) are the diffeomorphism constraints.

Since the hamiltonian system is finite dimensional, the wavefunction may be expanded in powers of the anti-comuting gravitino fields. The requirement of \( SL(2, C) \) invariance dictates that only terms of even power of the gravitino field will appear. There are six components in \( \psi^a_A \), so the expansion stops at order six. Therefore we can write, symbolically

\[
\Psi(\sigma, \psi) = \Psi(0)(\sigma) + \Psi(2)(\sigma, \psi) + \Psi(4)(\sigma, \psi) + \Psi(6)(\sigma, \psi), \tag{15}
\]

where the subscript indicates the number of gravitino fields.

When this decomposition is inserted in Eqs. (13, 14), one finds the following equations, that show explicitly the mixing,

\[
\hat{S}^{tA} \Psi(n) + \hat{S}^{tA}_m \Psi(n+2) = 0, \tag{16}
\]
\[
\hat{S}^A \Psi(n+2) + \hat{S}^A_m \Psi(n) = 0, \tag{17}
\]
where \( n = 0, 2, 4 \).

At this point, it is convenient to identify the irreducible spin 1/2 and spin 3/2 parts of the gravitino field. Let,

\[
\psi^{ABC} := \psi_A^a \sigma^{aBC} = \psi^{A(BC)}.
\]

Then,

\[
\psi^{ABC} = \rho^{ABC} + \epsilon^{A(BC)}
\]

where \( \rho^{ABC} = \rho^{(ABC)} \), represents the spin 3/2 part, and \( \epsilon^A = (2/3) \psi_B^{A} \), the spin 1/2 part. It is important to note that they can be specified independently.

The wavefunction can depend on the gravitino field only in Lorentz invariant combinations. Thus, the most general expression for the terms \( \Psi^{(n)} \) in (15) is

\[
\Psi^{(0)} = E(h),
\Psi^{(2)} = F_1(h)[\rho^2] + F_2(h)[\beta^2],
\Psi^{(4)} = G_1(h)[\rho^2]^2 + G_2(h)[\rho^2][\beta^2],
\Psi^{(6)} = H(h)[\rho^2][\beta^2],
\]

where the functions \( E, F, G, H \) depend on \( \sigma^{aAB} \) only in the combination \( h^{ab} = Tr(\sigma^a \sigma^b) \), and \( [\rho^2] := \rho^{ABC} \rho^{ABC} \), \( [\beta^2] := \beta^A \beta_A \).

IV. Physical States

In this section we give the explicit proof that all \( \Psi^{(n)} \) terms are zero, if they have to satisfy the quantum supersymmetric constraints equations.

Our first step is to express the constraints \( \hat{S}^{\dagger A} \) and \( \hat{S}^{\dagger A}_m \) in terms of the independent quantities \( \beta \) and \( \rho \), acting on each term of the expansion (15). We find

\[
\sqrt{2} \hat{S}^{\dagger A} \Psi^{(0)} = \hat{T}^A(\rho) E + \beta^A [h^{ab} \frac{\delta E}{\delta h^{ab}} - \frac{1}{2} h^{-1/2} M^{ab} h_{ab} E + \frac{3}{2} \alpha E],
\]

\[
\sqrt{2} \hat{S}^{\dagger A} \Psi^{(2)} = [\rho^2] \hat{T}^A(\rho) F_1 + [\beta^2] \hat{T}^A(\rho) F_2 + \beta^A [\rho^2] [h^{ab} \frac{\delta F_1}{\delta h^{ab}} + \frac{1}{6} F_2 + (-\frac{1}{2} h^{-1/2} M^{ab} h_{ab} + \frac{3}{2} \alpha + 2) F_1],
\]

\[
\sqrt{2} \hat{S}^{\dagger A} \Psi^{(4)} = [\rho^2][\beta^2] \hat{T}^A(\rho) G_2 + \beta^A [\rho^2]^2 [h^{ab} \frac{\delta G_1}{\delta h^{ab}} + \frac{1}{6} G_2 + (-\frac{1}{2} h^{-1/2} M^{ab} h_{ab} + \frac{3}{2} \alpha + 2) G_1],
\]

\[
\hat{S}^{\dagger A}_m \Psi^{(2)} = 4imh^{-1/2} \beta^A F_2,
\]
\[ \hat{S}_m^{A\Psi}(4) = 4imh^{-1/2}\beta^A[\rho^2]G_2, \]
\[ \hat{S}_m^{A\Psi}(6) = 4imh^{-1/2}\beta^A[\rho^2]^2H, \]

where we have defined the operator
\[ \hat{T}^A := -\sigma^{aAB}\sigma^{bCD} \rho_{BCD} \frac{\delta}{\delta h^{ab}} - h^{-1/2}M^{ab}\sigma_a^{AB}\sigma_b^{CD} \rho_{BCD}. \]  

For \( \hat{S}_m^{A\Psi}(2) \) and \( \hat{S}_m^{A\Psi}(4) \), one has,
\[ \hat{S}_m^{A\Psi}(2) = 2\sigma^{aAB}\sigma^{bCD} \rho_{BCD} \frac{\delta F_1}{\delta h^{ab}} - \frac{2}{3}\beta^A[\rho^2]h^{ab}\frac{\delta F_2}{\delta h^{ab}} + 4F_2 - 3F_1, \]
\[ \hat{S}_m^{A\Psi}(4) = 4[\rho^2]\sigma^{aAB}\sigma^{bCD} \rho_{BCD} \frac{\delta G_1}{\delta h^{ab}} + 2[\beta^2]\sigma^{aAB}\sigma^{bCD} \rho_{BCD} \frac{\delta G_2}{\delta h^{ab}} \]
\[ + \beta^A[\rho^2][-\frac{2}{3}h^{ab}\frac{\delta G_2}{\delta h^{ab}} - \frac{13}{3}G_2 + 2G_1], \]
\[ \hat{S}_m^{A\Psi}(0) = 3\sqrt{2}im\beta^A E, \]
\[ \hat{S}_m^{A\Psi}(2) = 3\sqrt{2}im\beta^A[\rho^2]F_1, \]
\[ \hat{S}_m^{A\Psi}(4) = 3\sqrt{2}im\beta^A[\rho^2]^2G_1. \]

Our second step is to insert these expressions in (16), (17). Using the fact that \( \beta \) and \( \rho \) can be specified independently, we arrive at the equations,
\[ h^{ab}\frac{\delta F_1}{\delta h^{ab}} + \frac{1}{6}F_2 + (- \frac{1}{2}h^{-1/2}M^{ab}h^{ab} + \frac{3}{2}\alpha + 2)F_1 + 4\sqrt{2}imh^{-1/2}G_2 = 0, \]
\[ h^{ab}\frac{\delta G_1}{\delta h^{ab}} + \frac{1}{6}G_2 + (- \frac{1}{2}h^{-1/2}M^{ab}h^{ab} + \frac{3}{2}\alpha + 2)G_1 + 4\sqrt{2}imh^{-1/2}H = 0, \]
\[ h^{ab}\frac{\delta E}{\delta h^{ab}} - \frac{1}{2}h^{-1/2}M^{ab}h^{ab}E + \frac{3}{2}\alpha E + 4\sqrt{2}imh^{-1/2}F_2 = 0, \]
\[ h^{ab}\frac{\delta H}{\delta h^{ab}} - \frac{2}{3}h^{-1/2}M^{ab}h^{ab}H + \frac{3}{2}\alpha H + 4\sqrt{2}imH^{-1/2}F_2 = 0, \]
and
\[ \hat{T}^Af = 0, \]
\[ \frac{\delta g}{\delta h^{ab}} \propto h_{ab}. \]
with \( f = E, F_1, F_2, G_2, g = F_1, G_1, G_2, H \).

For \( M^{ab} \neq 0 \), using Eq. (28) in (27), it follows immediately that
\[
F_1 = G_2 = 0. \tag{29}
\]
Using this in the remaining equations (21), (22), (23), (24), respectively, one finds that
\[
F_2 = G_1 = H = E = 0. \tag{30}
\]

The case \( M^{ab} = 0 \) (Bianchi I) must be treated separately. However, it is easy to see that the specialization of Eqs. (21) through (28) gives that \( \Psi \) must vanish also for this special case.

V. Final remarks

To conclude, we would like to compare our results with previous work.

Using the triad ADM canonical formulation, D’Eath and collaborators have reached our same conclusion for Bianchi I and Bianchi IX models with \( \Lambda \) \cite{9, 10}. The FRW solution is also considered in \cite{10}, where a non-vanishing physical state is found. This is a consequence of the insistence on imposing by hand supersymmetry. This leads to throw away enough degrees of freedom of the gravitino field that a solution survives. In light of our result, this seems to be a pathological case. A small amount of anisotropy will excite the neglected gravitino degrees of freedom, and force the wavefunction to vanish.

In the full theory with \( \Lambda \), in the connection representation, and for a special factor ordering, a single physical state has been proposed, that contains an infinite number of fermionic fields \cite{11, 12}. This state generalizes to supergravity the Chern-Simons type physical state of Refs. \cite{13, 14}. It is our hope that this paper will prompt a more systematic investigation of such physical states.

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