EVIDENCE AGAINST MACROSCOPIC ASTROPHYSICAL DYADOSPHERES

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ABSTRACT

It is shown how pair production itself would almost certainly prevent the astrophysical formation of macroscopic dyadospheres, hypothetical regions extending many electron Compton wavelengths in all directions where the electric field exceeds the critical value for microscopically rapid Schwinger pair production. Pair production is a self-regulating process that would discharge a growing electric field, in the example of a hypothetical collapsing charged stellar core, before it reached 6% of the minimum dyadosphere value, keeping the pair production rate more than 26 orders of magnitude below the dyadosphere value and keeping the efficiency below $2 \times 10^{-14} (M/1 M_\odot)^{1/2}$.

Subject headings: black hole physics — elementary particles — gamma rays: bursts

1. INTRODUCTION

Ruffini and his collaborators (Damour & Ruffini 1975; Ruffini 1998, 1999a, 1999b, 2000a, 2000b, 2002a, 2002b, 2002c, 2003; Preparata et al. 1998, 2000; Ruffini et al. 1998, 1999a, 1999b, 2000, 2001a, 2001b, 2001c, 2001d, 2002a, 2002b, 2003a, 2003b, 2003c, 2003d, 2004a, 2004b, 2004c, 2004d, 2005a, 2005b, 2005c, 2005d, 2006a, 2006b; Ruffini & Xue 1998; Bianco et al. 2001; Ruffini & Vitagliano 2002; Chardonnet et al. 2003; Bernardini et al. 2004, 2005; Bianco & Ruffini 2004, 2005a, 2005b; Corsi et al. 2004; Fraschetti et al. 2004; Xue et al. 2005; Vagenas 2006) have proposed a model for explaining gamma-ray bursts that presumes the initial existence of what they call a dyadosphere, a macroscopic region of spacetime where the electric field exceeds the critical value for microscopically rapid Schwinger pair production. (Sauter 1931; Heisenberg & Euler 1936; Weisskopf 1936; Schwinger 1951; Nikishov 1970). The difficulty of producing these large electric fields is a problem with this model that has not been adequately addressed.

There are at least two strong reasons for doubting that such large electric fields can develop over astrophysical scales in all directions (i.e., over lengths scales much larger than the electron Compton wavelength or much larger than the collision regions of individual charged particles). First, it would be very difficult to develop sufficient charge imbalance for macroscopic electric fields to produce a significant number of pairs. Second, even if macroscopic pair production could somehow be achieved, I show in this paper that this process is sufficiently self-regulating that it prevents the electric field from achieving a value that would produce pairs at even $10^{-25}$ that of dyadosphere models, assuming that the field is extended over a three-dimensional region at least as large as the Schwarzschild radius of a solar-mass black hole and that the field develops over a time that is at least as long as the corresponding timescale. I conclude that it is highly implausible that macroscopic dyadospheres can form in outer space, and thereby, invoking them for models of gamma-ray bursts is not at all likely to be viable.

The first reason for being extremely doubtful of the existence of astrophysical dyadospheres is that it is very difficult for a large charge imbalance to develop over a macroscopic region astrophysically because of the very high charge-to-mass ratio of elementary particles. For example, the ratio of the electrostatic repulsion to the gravitational attraction of two protons, each of charge $q$ and mass $m_p$, is the square of their charge-to-mass ratio in Planck units, which is

$$\left(\frac{q}{m_p}\right)^2 = \frac{q^2}{4\pi\varepsilon_0 G m_p^2} \approx 1.24 \times 10^{16}. \quad (1)$$

This implies that if one had a spherical object, such as a stellar core, with a positive charge-to-mass ratio $Q/M$ greater than the inverse of the charge-to-mass ratio of the proton, $m_p/q \approx 9 \times 10^{-19}$, the electrostatic repulsive force on the protons at the surface would be greater than the gravitational attractive force, so such protons would most likely be ejected. (If the object had a negative charge, electrons of mass $m = m_e$, would be expelled if $-Q/M > m_e/q \approx 4.9 \times 10^{-22}$, lower by the factor of the mass ratio of the proton and the electron, $m_p/m_e \approx 1836$, so the maximum value of the charge of such an object would be even less if it were negative.)

If one takes the mass-to-charge ratio of the proton to be a rough estimate of the maximum charge-to-mass ratio of an astrophysical object (or else gravity would not be strong enough to hold in the protons that make up the excess charge), then using the formulas of the succeeding sections, one can readily calculate that at the surface of a spherical object of radius $R$ and mass $M$, the ratio of the electrostatic field value, $E_c$, to the critical field value of a dyadosphere, $E_c \equiv m_e^2 c^3/(\hbar q) \approx 1.32 \times 10^{16}$ V cm$^{-1}$, is

$$E_c/E_c \leq \frac{\hbar c m_p}{4 G M c^2} \frac{1 M_e}{1.2 \times 10^{-13} \frac{1 M_\odot}{M}} \leq 1.2 \times 10^{-13} \frac{1 M_\odot}{M}. \quad (2)$$

Therefore, if protons can be ejected from an astrophysical object whenever the electrostatic repulsion exceeds the gravitational attraction, then the electric field is constrained to be more than 13 orders of magnitude smaller than the critical value for a dyadosphere [if the mass is greater than 1.2 $M_\odot$, which would be a conservative lower limit on any mass that could contract]
to $2GM/(c^2R) \sim 1]$. For a negatively charged object, the corresponding limit would be more than 16 orders of magnitude smaller than the critical value for a dyadosphere.

Although it seems very unlikely, one might seek to evade the electrostatic expulsion of protons by postulating that they are bound by nuclear forces to an astrophysical object, such as a collapsing neutron star core. The critical electric field $E_c$ that Ruffini and his collaborators used to define the minimum value for a dyadosphere would give the electrostatic force on an electron or proton of magnitude $F_e = qE_c = (m_e e^2)^2/(hc) \approx 0.00132$ MeV fm$^{-1}$, whereas nuclear energies of the order of a few MeV over length scales of the order of a fermi would give nuclear forces of the order of a few MeV fm$^{-1}$, about 3 orders of magnitude larger than the electrostatic force of a minimal dyadosphere on a proton. Ruffini has proposed (Ruffini 2006) that since the gravitational force is mainly provided by the nucleons of a collapsing neutron star core with a tiny fraction of the nucleons in the form of protons that are bound by nuclear forces to the surface of the neutron star core, a sufficient fraction can give the electric field originating the dyadosphere.

My suspicion is that such a configuration would be highly unstable to pieces of the charged surface breaking off and being ejected by the huge electrostatic forces on them. However, I do not have a firm result of when this would definitely happen, and therefore I was led to do the calculations below with the second mechanism (self-regulation) for preventing the occurrence of a dyadosphere. This, as we see below, will almost certainly discharge any growing electric field well before it reaches dyadosphere values. This occurs essentially because astrophysical length scales are much greater than the electron Compton wavelength, which is the scale at which the pair production becomes significant at the critical electric field value for a dyadosphere. Therefore, the electric field will discharge astrophysically even when the pair production rate is very low on the scale of the electron Compton wavelength.

In particular, I assume that a macroscopic astrophysical electric field cannot develop faster than the fastest timescale for bulk motions, which I assume to be the final gravitational collapse timescale for the smallest stellar mass that can collapse into a black hole, on the order of $M\approx 5 \mu s$ or greater. If a macroscopic electric field cannot develop much faster than this by astrophysical processes, the discharge from the pair production will then self-regulate the field to keep it from getting near dyadosphere values.

The calculations below lead to the conclusion that it would be very difficult astrophysically to achieve, over a macroscopic region comparable to the size of a black hole or larger star, electric field values greater than a few percent of the minimum value for a dyadosphere, if that. The Schwinger pair production itself would then never exceed $10^{-26}$ times the minimum dyadosphere value.

Since the idealized model below does give pair production at macroscopically significant rates (although more than 26 orders of magnitude below that of a dyadosphere by the original definition given above), one might revise the definition of a dyadosphere to include any macroscopic electric field that gives macroscopically significant pair production. Then (assuming that sufficient charge separation can somehow be achieved by forces necessarily much stronger than gravitational forces in order to evade the limitations discussed above) the arguments and model below would not exclude the possibility of such a revised concept of a dyadosphere. However, we see below that the much weaker amount of pair production in the model below does not seem to have nearly high enough energy efficiency for models of gamma-ray bursts. Therefore, in this paper, unless otherwise stated, I stick with the original definition of a dyadosphere, given above and in more detail in § 2.

2. Pair Production Rate and Definition of a Dyadosphere

The Schwinger pair production (Sauter 1931; Heisenberg & Euler 1936; Weisskopf 1936; Schwinger 1951) gives a rate $N$ of electron-positron pairs per 4-volume (per 3-volume and per time) in a uniform electric field, that is (Nikishov 1970),

$$N = \frac{q^2 E^2}{4\pi^3} \exp\left(-\frac{\pi m^2}{qE}\right) = \frac{m^4}{4\pi} \frac{e^{-w}}{w^2},$$

(3)

where (using Planck units $G = h = c = 4\pi\epsilon_0$) $m \approx 4.185 \times 10^{-23}$ is the mass of the electron, $q = (\alpha)^{1/2} \approx 0.08542$ is the magnitude of the charge of the electron, and

$$w = \frac{\pi m^2}{qE} = \frac{\pi E_c}{E},$$

(4)

if one defines the critical electric field strength $E_c$ to be

$$E_c \equiv \frac{m^2}{q} \approx 1.323 \times 10^{16} \text{ V cm}^{-1}.$$

(5)

Ruffini (Ruffini 1998, 2000a; Preparata et al. 1998; Ruffini et al. 1998; Ruffini & Xue 1998) has defined a dyadosphere to be a region of spacetime in which the electric field $E$ is greater than the critical electric field $E_c$. Therefore, it is a region in which $w < w_c = \pi$ and the electron-positron pair production rate is

$$N > N_c = \frac{m^4}{4\pi} \frac{e^{-\pi}}{\pi^2} \approx 10^{-93} \approx 4.7 \times 10^{48} \text{ cm}^{-3} \text{ s}^{-1} \approx 7.4 \times 10^{58} M_\odot^{-4}.$$

(6)

The last number means that if one had a dyadosphere over a cube of edge length $1 M_\odot \approx 1.5$ km and over a time $1 M_\odot \approx 5 \mu s$, it would produce more than $7 \times 10^{58}$ pairs, and the positrons would have a total charge $\sim 10^{50} M_\odot$.

This shows that the minimum dyadosphere pair production rate, although only $4\pi e^{-\pi}$ in units of the electron Compton wavelength, is utterly enormous at macroscopic astrophysical scales. This strongly suggests that dyadospheres will never form over macroscopic astrophysical scales. Indeed, the calculations below confirm this for a simple spherically symmetric model and show that under extremely conservative assumptions, the pair production rate from astrophysical gravitational collapse is likely to be always less than $10^{-20}$ of that of a dyadosphere. (Under the plausible but not rigorous arguments of § 1 that the charge is ejected when the electrostatic repulsion exceeds the gravitational attraction, the upper limit in a region on the size of a black hole would be trillions of orders of magnitude weaker than a dyadosphere, a factor of more than $10^{10}$, and hence completely negligible.)

3. Crude Upper Limit of Pair Production Rate

One can make the following crude estimate of the upper limit of the pair production rate from any macroscopic astrophysical process that has an electric field $E = \pi E_c / w = \pi m^2/Eq$ extending along the field direction a macroscopic distance $L \gg L_{\text{pair}} = 2m/(qE) = 2w/(\pi m)$ (where $L_{\text{pair}}$ is the minimum distance along this field over which it is energetically possible to produce an
electron-positron pair with total rest mass $2m$) and lasting for a timescale $T \gtrsim L/c$. We assume here and henceforth that, since we are assuming a macroscopic electric field that extends over a distance much greater than $L_{\text{pair}}$, we can take it to be nearly homogeneous over the pair production distance $L_{\text{pair}}$, so that the pair production rate is given very accurately by equation (3) for a uniform field.

This pair production over a time $T$ and length $L$ along the electric field direction will lead to a charge density produced per area perpendicular to the field of magnitude $\sigma \sim qNTL$, which will discharge the field if it becomes comparable to $E(4\pi)$. Therefore, the average pair production rate is roughly limited by

$$N = \frac{m^4}{4\pi} \frac{e^2}{w^2} \gtrsim \frac{E}{4\pi qTL} \approx \frac{m^2}{4q^2wTL}.$$  

(7)

Now if for the given $T$ and $L$ one defines

$$X = \frac{m^2q^2TL}{\pi} \approx \frac{cTL}{(8 \times 10^{-10} \text{ cm})^2} \gg 1,$$  

(8)

inequality (7) implies that $we^2 > X$ or $w \gtrsim \ln X - \ln \ln X$. This then leads to the ratio of the actual pair production rate $N$ to the minimal dyadosphere rate $N_c$ being

$$\frac{N}{N_c} \lesssim \frac{\pi^2e^2}{X(\ln X - \ln \ln X)} \sim \frac{(10^{-8} \text{ cm})^2}{cTL} \ll 1$$  

(9)

for any macroscopic time and length scales, which would give $X \gg \pi^2e^2$. This strongly suggests that no matter what the situation, if the time and length scales for the electric field obey

$$cT \gtrsim L \gg \frac{\pi^{3/2}e^{\pi/2}}{mq} \sim 10^{-8} \text{ cm},$$  

(10)

then dyadospheres of such a macroscopic size will not form.

In the following sections we find a more precise upper bound on the pair production rate in an example with spherical symmetry. This example confirms the limit above and strengthens the evidence that pair production rates in macroscopic processes must almost certainly be many, many orders of magnitude below dyadosphere rates.

It must be admitted that the bound above is heuristic and is not a rigorous mathematical result. For example, if one just wants a hypothetical example of how a spatially infinite dyadosphere could develop from initial conditions without one, consider the following collision of two plane electromagnetic waves with the electromagnetic field tensor

$$\mathbf{F} = E[\theta(t-z)dx \wedge dt(z) + \theta(t+z)dx \wedge dt(z)],$$  

(11)

where $\theta$ denotes the Heaviside step function.

For $t < 0$ there are two incoming electromagnetic fields, with one front at $z = -|t|$ moving in the positive $z$-direction and the other front at $z = -t = |t|$ moving in the negative $z$-direction. These fields are nonzero and constant behind their fronts, for $|z| > |t|$, each with electric field $E$ in the positive $x$-direction, but with the field at $z < -|t|$ that is moving in the positive $z$-direction having its magnetic field, also of magnitude $E$, pointing in the positive $y$-direction, and with the field at $z > |t|$ that is moving in the negative $z$-direction having its magnetic field of magnitude $E$ pointing in the negative $y$-direction. Because the scalar invariants of the electromagnetic field vanish for $t < 0$, there is no pair production then.

However, for $t > 0$ the two incoming electromagnetic fields overlap for $|z| < |t|$ and have their magnetic fields cancel out there. In the absence of pair production, there would just be an electric field $2E$ in the $x$-direction in this region, which is infinitely extended in the $x$- and $y$-directions. Therefore, if $2E > E_c$, there would then be a dyadosphere that is infinitely large (in two directions) for $t > 0$, until the pair production reduced the field below the dyadosphere value.

This hypothetical example circumvents the heuristic argument for the limit above first by not having the timescale $T \gtrsim L/c$, but more importantly by having the incoming electromagnetic field (at $t < 0$) have Fourier components of arbitrarily short wavelength, in order to produce the sharp jump in the field from zero at $|z| < -t = |t|$ to its nonzero values at $z < -|t|$ and at $z > |t|$. If instead the fields of the incoming plane waves rose sufficiently slowly (on the microscopic scale of the electron Compton wavelength, although this certainly allows what would be considered a rapid variation on macroscopic astrophysical scales) to dyadosphere values of the electric field (although the equal magnetic field magnitudes in the perpendicular direction would prevent the field from actually being a pair-producing dyadosphere in each incoming plane wave when they are separate), then when the waves started overlapping to cancel the magnetic field and give pair production, the pair production itself would regulate the field from ever attaining dyadosphere values. Therefore, unless it is somehow possible for macroscopic astrophysical electromagnetic fields to develop significant components of microscopically large wavenumbers (and hence significant changes in the fields over length scales on the order of an electron Compton wavelength), it seems very unlikely that macroscopic dyadospheres will ever develop astrophysically.

4. SCHWINGER DISCHARGE OF A CHARGED COLLAPSING CORE

In this section we analyze the pair production and discharge of an electric field produced by the collapse of a hypothetical charged sphere or stellar core, ignoring the processes discussed in § 1 that would most probably cause almost all of the excess charge of the sphere to be electrostatically ejected. For a sphere collapsing in finite time, there is only a finite time for the electric field to be discharged by the Schwinger pair production process, so the discharge is never complete, but instead leaves a residual electric field at each moment of the collapse. We calculate an upper limit for this value and show that it is always more than a factor of 18 less than the minimum value for a dyadosphere. Because the pair production rate is exponentially damped by the inverse of the electric field value, the pair production rate itself never exceeds a value that is more than 26 orders of magnitude below that of a dyadosphere. (The factor of the order of $10^{26}$, which we derive below, comes mainly from the fine-structure constant multiplied by the square of the ratio of the Schwarzschild radius of a solar-mass black hole to the Compton wavelength of an electron, which is why this factor is so large.)

The maximum electric field is smaller for cores that collapse into larger black holes (because the discharge time during infall is greater), so for a very conservative upper limit on the electric field, we assume that the black hole that forms has $1 M_\odot$. Of course, we expect that the minimum mass of a black hole that forms astrophysically has significantly more than $1 M_\odot$, so the corresponding maximum electric field would be weaker (by a logarithmic factor such that the maximum pair production rate
would essentially vary inversely with the square of the mass of the black hole). That is, we are assuming that \( M = 1 \ M_{\odot} \) is a very conservative lower limit on the mass of any black hole that forms astrophysically in the present universe (as distinct from, say, primordial black holes that might have formed much smaller in the very early universe; our calculations do not apply to those, but they will have had by now a very long time to discharge and so would also not be expected to have significant charge today).

The maximum electric field is also smaller the slower that the core collapses (giving more time for discharge), so again to get a very conservative upper limit on the electric field, we assume that the core falls in as fast as is astrophysically possible, which is free fall with zero binding energy. We assume that it is not astrophysically possible to have the spherical outer boundary of a collapsing core moving inward with a velocity so high that it would have come from an unbound configuration with nonzero inward velocity at radial infinity, or that at smaller radii it could have nongravitational inward accelerations.

Third, the maximum electric field is of course smaller the smaller the initial charge on the collapsing core. To get the maximum electric field possible, we assume that the initial charge-to-mass ratio of the collapsing core is unity, the largest value possible for which the electrostatic repulsive forces do not overwhelm the total gravitational attractive forces on the entire core and prevent the core from collapsing (since we are excluding the possibility that the core is shot in from far away or is otherwise pushed inward by nongravitational forces, which is not at all astrophysically plausible).

Now of course if the charge-to-mass ratio of the core really were unity, the core would not fall until there was some discharge. But merely to get a conservative upper limit on the electric field as the core collapses, for simplicity we assume that the initial charge equals the mass but ignore the electrostatic repulsion, so that the core nevertheless falls in at the rate it would from purely gravitational free fall from infinity, with no reduction of the free-fall rate by the electrostatic repulsion of the charge. The gravitational effects of the electric field outside the core would also reduce the free-fall time, but we ignore this effect as well, and simply take the external gravitational field to be given by the vacuum spherically symmetric Schwarzschild metric.

First we make a rough Newtonian estimate of the discharge rate for a collapsing charged core. Next, we derive and give approximate solutions of the relativistic partial differential equations describing the process more precisely. Finally, we calculate the efficiency of the process for releasing the energy of the core.

4.1. Approximate Estimate of the Discharge Rate

We consider a positively charged spherical core of mass \( M \) freely collapsing into a black hole, with the surface at a radius \( R(t) \). To avoid having the electric field discharged by plasma (a likely occurrence in most astrophysical situations), for the sake of argument we assume a vacuum outside the core (except for the electromagnetic field and pairs produced by it, which we assume will not significantly modify the Schwarzschild geometry; including such modifications would reduce the electric field even further than the conservative upper limits we find below). Outside the surface of the core, pairs will be produced, with the positrons moving outward and the electrons moving inward. (Since the core is expected to be highly conducting, virtually all of its excess charge would be at or near the surface, so inside the core there would be a negligible macroscopic field and hence negligible pair production.) One can calculate (Page 2006a) that the interactions between individual positrons and electrons produced outside the core are utterly negligible; e.g., the probability for each particle to annihilate with an antiparticle turns out to be less than \( 10^{-26} \). In addition, the number of particles produced per mode is very small, so we need not worry about particle degeneracy and Pauli blocking in the pair production region outside the core.

As the electrons pass inward through the outer boundary of the core over time, they reduce the value of the charge of the core, \( Q(t) \), limiting the value of the electric field outside, \( E(t, r) \approx \frac{Q(t)}{R(t)^2} \), under the assumption (which can be verified from the results) that at any one time the charge contributed by the pairs outside the core radius \( R(t) \) is a small fraction of \( Q(t) \), as long as one does not go to such a huge radius that it includes the outgoing positrons emitted over a large fraction of the previous infall of the core.

We also make the assumptions, which are verified below (at least for \( M \ll 10^6 \ M_{\odot} \)), that \( Q(t) \ll M \) once the core gets within a few orders of magnitude of the Schwarzschild radius \( 2M \), and that the total energy that goes into the pairs is also much smaller than \( M \) (so that the core mass \( M \) stays very nearly constant).

Because the pair production rate per 4-volume, \( N = (m^4/4\pi)w^{-2}e^{-w} \), decreases exponentially rapidly with \( w(t, r) = \pi m^2/(qE) \approx \pi m^2 r^2/qQ(t) \), and because the pairs produced decrease \( Q(t) \) and hence increase \( w(t, r) \) at fixed \( r \), there will be a self-regulation of \( w(t, r) \) (described more precisely by the differential equations of § 4.2, although we do not need this for the approximate estimate of this subsection). In particular, if we define

\[
z(t) = \frac{w(t, R(t))}{E(t, R(t))} = \frac{\pi m^2 R(t)^2}{qQ(t)},
\]

which gives the pair production rate \( N(t) = (m^4/4\pi)z^{-2}e^{-z} \) at the surface of the core (where the rate is maximal at that time, since the electric field has an inverse \( r^2 \) falloff at greater radii \( r > R \)), the self-regulation will keep \( z(t) \) changing only slowly with \( t \) and \( R(t) \). Hence, \( Q(t) \) will vary roughly in proportion to \( R(t)^2 \).

This means that the logarithmic rate of change of \( Q(t) \) will be roughly twice the logarithmic rate of change of \( R(t) \). If we do a Newtonian analysis of free fall from rest at infinity, we find that \( dR/dt = -2(M/R)^{3/2} \), which makes the logarithmic rate of change of \( R(t) \) equal to \(-1/(2M)(2M/R)^{3/2}\). As the core approaches the horizon at \( R = 2M \), this Newtonian estimate of the logarithmic rate will approach \(-1/(2M) \), so the logarithmic rate of change of \( Q(t) \) will approximately approach \(-1/M \). (Here we are ignoring relativistic corrections, but it turns out that they make only a very small difference.)

Now we can calculate \( dQ/dt \) from the pair production rate (as a function of \( Q \)) and set it equal to \(-Q/M \) when \( R \sim 2M \) to solve for \( Q \) and hence the pair production rate. Because the logarithmic rate of change of \( Q \) is enormously less than that given above for a dyadosphere, the pair production rate must be suppressed by rather large values of \( w \) outside the core. Since \( w \approx \pi m^2 r^2/qQ \) increases proportionally to the square of the radius, the suppression will rather rapidly increase with the radius outside the core (which we are now taking to be at \( R \sim 2M \)). This means that the pair production rate will decrease roughly exponentially with the radius and will reach a value a factor of \( 1/e \) smaller than at the core surface itself (where \( w = z = \pi m^2 R^2/qQ \sim 4\pi m^2 M^2/qQ \) roughly when \( w \) decreases by one. Since the logarithmic rate of increase of \( w \) with \( r = R \) is \( 2/R \), the point at
which the pair production rate will have dropped by the factor of $1/e$ will be at $r - R \approx R/(2z) \ll R$, the last inequality coming because $z \gg 1$ from the fact that the pair production rate is much less than dyadosphere rates.

With a roughly exponential decrease in the pair production rate with radius, at a logarithmic rate greater than the logarithmic rate at which the radius grows by a factor of $2z \gg 1$, the total pair production rate per time is roughly the pair production rate per 4-volume at the surface of the core $[N(R) = (m^4/4\pi)z^{-2}e^{-z}]$ multiplied by the effective volume in which most of the pair production is occurring, which in this case is roughly the area $4\pi R^2$ of the core surface (or of the black hole formed by the collapsing core) multiplied by the radial distance $r - R \approx R/(2z)$ out to where the pair production rate per 4-volume has decreased by a factor of $1/e$. (For the pair production rate per coordinate time $t$, the relativistic correction that makes the proper radial distance greater than $r - R$ is compensated in the approximately Schwarzschild metric outside the core by the relativistic correction that makes the proper time smaller than $\Delta t$ by what is precisely the same factor in the Schwarzschild metric.)

When the number rate per time is multiplied by the charge $-q$ of each ingoing electron, one gets at $R \sim 2M$ that

$$\frac{dQ}{dt} \approx -qN(R)4\pi R^2 \frac{R}{2z} \approx \frac{-4qm^4M^3}{z^2e^z}. \quad (13)$$

For the logarithmic rate of change of $Q$ to be roughly $-1/M$, we set $dQ/dt$ equal to $-Q/M = -\pi m^2R^2/(qMz) \sim -4\pi m^2M/(qz)$. This leads to the equation for $z$,

$$z^2e^z \sim Z, \quad (14)$$

where

$Z \equiv q^2m^2M^2/\pi \equiv A \left(\frac{M}{M_\odot}\right)^2 \equiv A\mu^2 \approx 3.4 \times 10^{28}\mu^2, \quad (15)$

$$A \equiv q^2m^2M_\odot^2/\pi = \frac{\alpha}{\pi} \left(\frac{GM_\odot m^2}{hc}\right)^2 \approx 3.396 \times 10^{28}, \quad (16)$$

and

$$\mu \equiv \frac{M}{M_\odot} \quad (17)$$

is the ratio of the mass $M$ of the freely collapsing core to the mass $M_\odot$ of the Sun. One can solve equation (14) numerically for $z$ when $\mu = 1 (M = M_\odot)$ to get what I call $z_*$:

$$z_* \approx 57.6. \quad (18)$$

Then for any $|2\ln\mu| \ll \ln A$, one gets the approximate solution

$$z \approx z_* + \frac{2z_*}{z_* + 2} \ln\mu \approx 57.6 + 1.93 \ln M/M_\odot. \quad (19)$$

This means that the ratio of the electric field value $E(2M)$ at the surface of the collapsing core (when it enters the black hole) to the critical electric field value $E_c$ for the definition of a dyadosphere is

$$\frac{E(2M)}{E_c} = \frac{\pi}{2} \approx \frac{\pi}{z_*} - \frac{2\pi \ln\mu}{z_*(z_* + 2)} \approx 0.0546 - 0.00183 \ln M/M_\odot. \quad (20)$$

Thus, this approximate estimate would indicate that the electric field outside a charged collapsing core is always less than about 5.5% of that of a dyadosphere, differing from that of a dyadosphere by a factor of at least 18. (We see below that the numerical solution of the ordinary differential equation governing the charge during the collapse, including relativistic effects, agrees with the result above to about three decimal places, so the result above is quite accurate.)

From this solution, one can see that the pair production rate at the surface of the core when it crosses the horizon at $R = 2M$ is

$$N(R) \sim \frac{m^4}{4\pi}z_*^{-2}e^{-z_*} \sim \frac{m^4}{4\pi A\mu^2} = \frac{m^2}{4\pi^2M^2} = \frac{\pi^2e^\mu}{4}q^{-2}N_c \sim 7 \times 10^{-27} \left(\frac{M}{M_\odot}\right)^{-2}N_c, \quad (21)$$

which for $M > 1 M_\odot$ or $\mu = M/M_\odot > 1$ is more than 26 orders of magnitude smaller than the minimum pair production rate $N_c$ of a dyadosphere.

It is also of interest to compare the energy density $\rho_E$ of the electric field and the energy density $\rho_p$ of the particles produced with the mass-energy density $\rho_r$ of the stellar core when the core surface is crossing the event horizon. Of course, the density of the core will be affected by its infall and is likely to be inhomogeneous, but to get a quantity with which to compare $\rho_E$ and $\rho_p$, let us say the core density is its total mass-energy $M$ divided by the flat-space volume formula $4\pi R^3/3$ when $R = 2M$, which gives $\rho_r = 3/(32\pi M^3)$. Then since $\rho_E = E^2/(8\pi)$, one can readily calculate that, as the core surface crosses the event horizon,

$$\frac{\rho_E}{\rho_r} = \frac{4\pi^2m^4M^2}{3q^2z^2} \approx 1.4 \times 10^{-14} \left(\frac{M}{1M_\odot}\right)^{1.933}. \quad (22)$$

The energy density of the particles produced peaks at $r = 1.5R = 3M$ because of a trade-off between the growing electrostatic energy $qQ(R^{-1} - r^{-1})$ gained by the positrons being accelerated outward by the $Q/r^2$ electric field and the $1/r^2$ fall-off in the number density. At this peak, one can readily calculate that

$$\frac{\rho_p}{\rho_r} = \frac{64\pi^2m^4M^2}{81q^2z^2} \approx 8.2 \times 10^{-15} \left(\frac{M}{1M_\odot}\right)^{1.933}, \quad (23)$$

a factor of 16/27 times that of the electric field at $r = R = 2M$. Therefore, for the case $M \ll 3 \times 10^6 M_\odot$ in which significant discharge occurs, when the core collapses into a black hole, the electric field and pairs produced have a negligible energy density compared with the core itself, so they would not be expected to contribute significantly to the energetics of the core collapse or of any subsequent process like gamma-ray bursts.

Thus, we see that the approximate algebraic estimate is that the pair production rate is never more than about $10^{-26}$ times the minimum amount for a dyadosphere, and the energy density of the electric field and of the particles produced is only a very tiny fraction of the collapsing stellar core energy density. In § 4.2 we indeed confirm from the solutions of the general relativistic differential equations that this algebraic estimate is indeed a good approximation for the maximum electric field. This result implies that it is very unlikely that dyadospheres can form from
the collapse of charged cores, even if somehow all discharge mechanisms are eliminated other than the pair production itself.

4.2. Differential Equations for the Discharge

In this subsection we derive and give approximate solutions for the general relativistic differential equations for the discharge of the collapsing core. The differential equations are derived under the assumption that the tunneling distance for a pair to come into real existence, \( L_{\text{pair}} = 2m(qE) = 2\omega/(\pi m) \sim 10^{-9} \text{ cm} \), is much less than the astrophysical length scales for the collapsing core, which is a very good approximation. The differential equations will be solved under the approximation that this tunneling length is also significantly greater than the Compton wavelength of an electron, which implies that \( w \equiv \pi m^2/(qE) \gg 1 \) and hence that the pair production is mostly confined to a radial region \( r - R \sim R/w \ll R \), which is much smaller than the radius \( R \) of the surface of the charged collapsing core. This latter approximation is not as good but still leads to errors of only a few percent.

After the charged particles are produced in pairs, with rms transverse momenta \( (qE/\pi)^{1/2} = m/w^{1/2} \), they will be accelerated by the radial electric field (electrons inward and positrons outward, under the assumption here that the core is positively charged). Each time a charged particle travels a distance \( m/(qE) \) parallel to the electric field, it will gain additional kinetic energy equal to its rest mass. Thus, it will very quickly accelerate to a huge gamma factor and move very nearly at the speed of light [with an asymptotic error at radial infinity of \( 1 - v \approx w^2/(2\pi^2 m^2 R^2) \approx 2.87 \times 10^{-30}(M/1 M_\odot)^{0.933} \) for outgoing positrons created when the core surface reaches the horizon \( R = 2M \)] and in very nearly the radial direction [with an asymptotic angular error of about \( \omega^2/(\pi mR) \approx 3.16 \times 10^{-16}(M/1 M_\odot)^{-0.983} \) when the core enters the black hole]. As a result, one will effectively get a null 4-vector of positive current density (highly relativistic positrons) moving radially outward and a null 4-vector of negative current density (highly relativistic electrons) moving radially inward.

It is most convenient to describe this current in terms of radial null coordinates, e.g., \( U \) and \( V \), so that the approximately Schwarzschild metric outside the collapsing core can be written as

\[
ds^2 = -e^{2\sigma} dU dV + r^2(U, V)(d\theta^2 + \sin^2\theta d\phi^2). \tag{24}\]

Now we can write the nearly null outward number flux 4-vector of positrons as \( n_+ = n^U_+ \partial_U \) and the nearly null inward number flux 4-vector of electrons as \( n_- = n^U_- \partial_U \). Since each positron has charge \( q \) and each electron has charge \( -q \), the total current density 4-vector is

\[
j = q n_+ - q n_- = q n^U_+ \partial_U - q n^U_- \partial_U. \tag{25}\]

The radial electric field of magnitude \( E = Q/r^2 \) has the electromagnetic field tensor

\[
F = -\frac{1}{2} e^{2\sigma} dU \wedge dV = -\frac{Q}{2r^2} e^{2\sigma} dU \wedge dV, \tag{26}\]

where \( Q = Q(U, V) \) is the charge inside the sphere labeled by \((U, V)\) (and is a function only of these two null coordinates, because of the assumed spherical symmetry).

Then from Maxwell’s equations (essentially just Gauss’s law here), we can deduce that the null components of the current density vector are

\[
j^U_+ = -\frac{2e^{-2\sigma} Q U}{4\pi r^2} - \frac{Q V}{4\pi r^2}, \tag{27}\]

\[
j^U_- = \frac{2e^{-2\sigma} Q V}{4\pi r^2} - \frac{Q U}{4\pi r^2}. \tag{27}\]

Although of course the current density 4-vector field is conserved, the number flux 4-vectors \( n_+ \) and \( n_- \) of the positrons and electrons are not. Their 4-divergences are each equal to the pair production rate \( \mathcal{N} \) when we can neglect annihilations, as we can here with the density of pairs being sufficiently small. When these 4-divergences are written in terms of the charge \( Q(U, V) \), one gets the following partial differential equation for the pair production and discharge process:

\[
8\pi qr^2 \mathcal{N} = \frac{2q^2 Q^2}{\pi r^2} \exp\left(-\frac{\pi m^2 r^2}{qQ}\right) = 2\Box Q - \frac{2}{r} \nabla_r \nabla_r Q \tag{29}\]

or explicitly in terms of the electric field \( E = Q/r^2 \) as

\[
8\pi qr^2 \mathcal{N} = \frac{2q^2 r^2 E^2}{\pi^2} \exp\left(-\frac{\pi m^2}{qQ}\right)
\]

\[
n = 2\Box (r^2 E) = r \Box (r E) - r^3 E \left(\frac{1}{r^2}\right). \tag{30}\]

Since for the sake of argument in this section we assume that the positively charged particles at the surface of the collapsing core do not escape to the outside, and since there is no electric field inside to produce particles there, the boundary condition at the surface of the collapsing core is that there is no outward flux of positrons there, so that \( Q_{U,V} = 0 \) at the core surface. The boundary condition at infinite radii is that we assume that there are no incoming electrons there, but only outgoing positrons pair-produced by the electric field at finite radius, so at radial infinity, \( Q_{,V} = 0 \). The boundary condition in the infinite past is that we assume that \( Q \) is as large as it can be and still have the core collapse gravitationally, so there we set \( Q = M \) but then make the idealized assumption of ignoring the effect of the electrostatic repulsion on the collapse of the core.

For details of the derivation of an approximate solution of the relativistic partial differential equations (28)–(30), see Page (2006a). A brief summary of this solution is the following: Let

\[
v = \sqrt{\frac{2M}{R}}, \tag{31}\]

be used as a time coordinate on the surface of the collapsing core with Schwarzschild (circumferential) radius \( r = R \). For our assumption of the core surface freely falling, from an initial
condition corresponding to being at rest at radial infinity, along radial geodesics in the assumed exterior Schwarzschild geometry, one can readily calculate that \( r \) is the negative rate of change of \( R \) with respect to the proper time \( \tau \) along the core surface, \(-dR/d\tau\), and that it is also the magnitude of the three-velocity of the infalling core surface that an observer at rest at fixed \( r \) would observe when the core surface crosses the event horizon, which is the limit of the validity of the approximate solution.

The core surface has Schwarzschild time \( t \), Schwarzschild radius \( r \), tortoise radial coordinate \( r_* \), and proper time \( \tau \) given as explicit functions of \( v \) by

\[
t = 2M \left[ -\frac{2}{3} v^3 - 2v^{-1} - \ln (1 - v) + \ln (1 + v) \right],
\]

\[
r = \frac{2M}{v^2},
\]

\[
r_* = r + 2M \ln \left( \frac{r}{2M} - 1 \right)
\]

\[
= 2M \left[ v^2 - 2 \ln v + \ln (1 - v) + \ln (1 + v) \right],
\]

\[
\tau = -\frac{4M}{3v^3}.
\]

Next, choose null coordinates \((U, V)\) in the exterior Schwarzschild geometry that are both equal to \( v \) on the collapsing core surface. These are defined implicitly as solutions of the following equations in terms of \( t \) and \( r_* \) in the exterior region:

\[
t - r_* = 4M \left[ -\frac{2}{3} U^3 - \frac{1}{2} U^{-2} - U^{-1} + \ln U - \ln (1 - U) \right],
\]

\[
t + r_* = 4M \left[ -\frac{1}{3} V^{-3} + \frac{1}{2} V^{-2} - V^{-1} - \ln V + \ln (1 + V) \right].
\]

Then in the region exterior to the collapsing charged core (assumed to start with \( Q = Q_0 \leq M \) at \( R = \infty \), and yet with the charge having no effect on the geometry or infall rate, for the sake of argument, to give a lower limit on the discharge), the approximate solution (Page 2006a) gives

\[
Q(U, V) \approx \frac{4\pi m^2 M^2}{q U^4 z} \left\{ 1 - \frac{2}{z} \ln \left[ \frac{1}{2} \left( \frac{1}{\sqrt{1 + P + 1}} \right) \right] 
- \frac{1}{2} \left( \frac{1}{\sqrt{1 + P - 1}} - 1 \right) e^{-2z(1-U)/U^2(U-V)} \right\}.
\]

Here

\[
P = P(U) \equiv \frac{4U(1 - S)}{(1 - U)^2},
\]

where

\[
S = S(U) \equiv 1 - \frac{q^2 m^2 M^2 e^{-z} J}{\pi U^5 z^2},
\]

\[
J = J[z(U)] \equiv 2z - 2\sqrt{\pi z} e^z \text{erfc} \sqrt{z}
= 1 - \frac{3}{2z} + \frac{3(5)}{4z^2} - \frac{3(5)(7)}{8z^3} + O(z^{-4}),
\]

and where \( z = z(U) \) obeys the ordinary differential equation

\[
\frac{dz}{dU} = \frac{-8z U^2}{4U + \sqrt{(1 + U)^2 - 4U U^2}}.
\]

At the core surface, \( U = V = v \), one has \( Q(v, v) = \pi m^2 R^2/(qz) \), so \( z = z(U) = v = \pi m^2 R^2/(qQ) = \pi E_c / E \) at the surface. For \( M < 3 \times 10^6 \text{ M}_\odot \) and for initial core charge \( Q = Q_0 \) large enough that significant discharge occurs, but not too large to prevent the core from collapsing,

\[
\frac{M}{3 \times 10^6 \text{ M}_\odot} < \frac{Q_0}{M} < 1,
\]

a highly precise approximate solution of equation (38) for \( z(v) \) is (Page 2006a)

\[
z(v) \approx g(v) - 2 \ln g(v) + g(v)^{-1} \left[ 4 \ln g(v) - \frac{3}{2} + g(v)^{-2} \right]
\]

\[
\times \left[ 4 \ln^2 g(v) - 11 \ln g(v) + \frac{45}{8} \right] + g(v)^{-3}
\]

\[
\times \left[ \frac{16}{3} \ln^3 g(v) - 30 \ln^2 g(v) + \frac{89}{2} \ln g(v) - \frac{177}{8} \right],
\]

where

\[
g(v) \equiv f(v) + \ln \left( 1 + \frac{v^5 z_1(v)^2 e^{-z_1(v)}}{2J z_1(v)} \right)
+ \frac{5(1 + v)}{4z_2(v)} \left( 5(1 + 2v)(3 - 2v) \right).
\]

\[
z_1(v) \equiv \frac{4\pi m^2 M^2}{q Q_0 v^4},
\]

\[
z_2(v) \equiv f(v) - 2 \ln f(v) + \frac{4 \ln f(v) - 0.25 + 1.25v}{f(v)}
+ \frac{4 \ln^2 f(v) - (8.5 - 2.5v) \ln f(v)}{f(v)^2}
\]

\[
f(v) \equiv \ln \left( \frac{q^2 m^2 M^2}{\pi v^5} \right).
\]

A less precise but still reasonably good, and much simpler, approximate solution of the ordinary differential equation (38) is given implicitly as the solution of

\[
z^2 e^z \approx \left( \frac{4\pi m^2 M^2}{q Q_0 v^4} \right)^2 \exp \left( \frac{4\pi m^2 M^2}{q Q_0 v^4} \right) + \frac{q^2 m^2 M^2}{\pi v^5},
\]

or, equivalently term for term for \( Q = Q(R) \) at the core surface,

\[
\left( \frac{\pi E_c R^2}{Q} \right)^2 \exp \left( \frac{\pi E_c R^2}{Q_0} \right) \approx \left( \frac{\pi E_c R^2}{Q} \right)^2 \exp \left( \frac{\pi E_c R^2}{Q_0} \right) + \frac{q^2 m^2 M^2}{4\pi \sqrt{2M}}.
\]
When \( v = (2M/R)^{1/2} \) is sufficiently small (or \( R \) is sufficiently large) that the first term on the right-hand side greatly dominates, little discharge occurs and \( Q(R) \approx Q_0 \). When \( v \) gets sufficiently large, the second term on the right-hand side greatly dominates, and then the discharge and self-regulation of the charge become important. This will have occurred by the time the core reaches the horizon (\( v = 1 \)) if \( M \ll 3 \times 10^6 M_\odot \) and if \( Q_0/M > M/(3 \times 10^6 M_\odot) \) so that there was initially enough charge for the electric field to get high enough during the collapse to lead to significant discharge. Then taking \( \pi^2 R/c^2 \approx q^2 m^2 M^2/\pi \) at the horizon leads precisely to the Newtonian result of equation (14) and maximum pair production rate \( \sim 7 \times 10^{-27} (1 M_\odot/M)^2 \) times that of a minimal dyadosphere rate with \( E = E_c \). That is, the approximate solution of the general relativistic partial differential equations for the discharge leads very nearly to the same result as the approximate Newtonian estimate.

Using the more precise approximate solution given by equation (40), or the numerical solution of the ordinary differential equation (38) (both of which agreed to several decimal places), a more precise formula for the ratio of the maximum pair production rate with \( z \) to \( 1 \) if \( \mu \) is 1 is

\[
\frac{N(M)}{N_e} \approx 6.612 \times 10^{-27} \left( \frac{1 M_\odot}{M} \right)^2 \left[ 1 + 0.000555 \times \ln \left( \frac{M}{1 M_\odot} \right) - 0.000018 \ln^2 \left( \frac{M}{1 M_\odot} \right) \right].
\]  

(47)

Therefore, assuming that \( M = 1 M_\odot \) or that \( \mu = 1 \) is a very conservative lower limit on the mass of a core that can collapse into a black hole; we indeed see that even if one can somehow start with \( Q = M \) when the core is very large, and somehow not have the charge on the core itself directly ejected by the enormous electrostatic forces (other than the discharge by the pair production process), the pair production rate would always be more than 26 orders of magnitude smaller than that of a putative dyadosphere. In particular, the minimum dyadosphere value would be more than \( 1.5 \times 10^{26} \) times larger than the actual value.

Another quantity of interest is the maximum value of \( Q/M \) of the black hole when it initially forms. For \( M \ll 10^6 M_\odot \), this is then

\[
\frac{Q}{M} = \frac{4\pi m^2}{q^2(1)} \approx \frac{4\pi m^2}{q^2(1)} \left( 1 - \frac{2 \ln \mu}{z_* + 2} \right) = 4.09 \times 10^{-7} \frac{M}{1 M_\odot} \left( 1 - 0.0336 \ln \frac{M}{1 M_\odot} \right).
\]  

(48)

This implies that stellar-mass black holes would always have a very low charge-to-mass ratio, even if the star somehow had a large charge before the collapse and ensuing discharge by pair production.

Furthermore, if one were to envisage a collapsing core much larger than a stellar mass, sufficiently massive that it could collapse into a black hole without discharging significantly, it would be very hard to imagine how the positive charge (e.g., protons) could avoid being electrostatically ejected. Even if nuclear forces were somehow effective in accomplishing that Herculean feat for neutron star cores, it would seem even much more unlikely that one could form a neutron star-like core of very many solar masses, so that nuclear forces on the protons could conceivably be effective in overcoming the huge electrostatic repulsion if there were a significant charge imbalance.

Therefore, I would actually be surprised if any black holes of astrophysical masses ever formed within our universe (or our pocket universe with our values of the masses and charges of the electron, proton, and neutron) with values of their charges at all near their masses. To put it more concretely, I would predict that no astrophysical black hole ever has a detectable change in its geometry given by the energy in its macroscopic electric field. In other words, \( Q/M \) would always be so far below unity that the metric of an astrophysical Reissner-Nordstrom or Kerr-Newman black hole would be indistinguishable from a Schwarzschild or Kerr black hole.

5. ENERGY EFFICIENCY OF THE PAIR PRODUCTION

We have calculated that even with idealized conditions of a collapsing stellar core initially somehow having \( Q_0 = M \) and somehow keeping its excess protons from being driven off by the extremely large electrostatic forces (although admittedly smaller than the nuclear forces within a nucleus), one cannot get the electric field to become large enough to produce pairs at a rate per 4-volume within 26 orders of magnitude of the minimal dyadosphere rate. However, we did get astrophysically significant pair production during this idealized process (which might count as a dyadosphere under a revised definition of the term), and so one might ask what the energy efficiency of this process is, i.e., what fraction \( \epsilon \) of the mass-energy \( M \) of the stellar core is converted into outgoing positrons, to give them total energy \( \epsilon M \). Here we show that unless \( M > 1 M_\odot \), the efficiency is very small, \( \epsilon < 1.86 \times 10^{-4} (M/1 M_\odot)^{1/2} \approx 1 \).

In the approximately Schwarzschild metric (eq. [24]), as written in terms of radial null coordinates \( U \) and \( V \), if a positron is produced at some \((U, V)\) and then is accelerated to very high gamma factors essentially along the outward null line \( U = \text{const.} \), then the kinetic energy (as measured at radial infinity) gained during this acceleration by the electric field \( Q(U, V)/r^2(U, V) \) is

\[
\mathcal{E}(U, V) = \int_r^\infty qQdU/dr \int_r^\infty 2qQ(U, V)e^{2q(U, V)dV} dV^{1/2}/r^2(U, V),
\]  

(49)

where the integral is taken along the outward radial null line \( U = \text{const.} \) along which the positron approximately travels from its creation point at \((U, V)\) to radial infinity at \((U, \infty)\). Almost all of the positrons are created sufficiently deep in the electric field that their final kinetic energies, at radial infinity, far exceed their rest mass energies, so I ignore the latter. Then one can multiply the positron production rate by the energy gained by each positron and integrate over all of the exterior region where the production is occurring to get the total energy emitted during the core collapse as

\[
\epsilon M = -\frac{1}{q} \int \mathcal{E}(U, V)Q_{,UV}dUdV.
\]  

(50)

(The negative sign comes from the decrease of \( Q \) during the discharge, so \( Q_{,UV} < 0 \).)

The crucial quantity for estimating the efficiency is the radius \( R_t \) where the self-regulation of the charge starts becoming important, for instance, at the point where the terms on the right-hand sides of equations (45) and (46) become equal. For general \( \mu = M/1 M_\odot \) and \( \xi_0 = Q_0/M_\odot \), the critical radius where the self-regulation starts becoming significant is (Page 2006a)

\[
R_t = 2M/v^2 \approx 5300\mu^{0.5049}\xi_0^{0.5082} \text{ km}.
\]  

(51)
Thus, if one starts with a hypothetical charged core collapsing freely from a very large radius into a black hole with initial charge $Q_0 \sim M$, the self-regulation of the charge will start to become important when the core gets to a radius of the order of the radius of the earth, and that is when there will start to be significant pairs being produced. The proper time left during the free-fall collapse after the core surface crosses this radius is then (Page 2006a)

$$\Delta \tau = \frac{4 M}{3 \mu t} \approx 0.499 \mu^{0.257} \xi_0^{0.762} \text{ s.}$$

(52)

Therefore, if $Q_0 \sim M$ the burst of positrons that will be emitted as the core collapses from $R = R_t$ to $R = 2M$ lasts a time that is of the order of 1 s.

Then when one evaluates the integral in equation (50), one gets (Page 2006a) that the efficiency of the conversion of the core mass $M$ into outgoing positrons of energy $\epsilon M$ is

$$\epsilon \approx 0.0001855 \left(\frac{M}{M_\odot}\right)^{0.495} \left(\frac{Q_0}{M}\right)^{0.742}.$$ \hspace{1cm} (53)

If one multiplies this efficiency by the mass-energy $M$ of the core, one gets that the total energy emitted in positrons is

$$\epsilon M \approx 3.315 \times 10^{50} \left(\frac{M}{M_\odot}\right)^{1.495} \left(\frac{Q_0}{M}\right)^{0.742} \text{ ergs.}$$ \hspace{1cm} (54)

For stellar-mass cores, this is 2–3 orders of magnitude smaller than the energy of gamma-ray bursts (Ruffini et al. 1998, 1999a, 1999b, 2000; Torres & Anchordoqui 2004), essentially because the efficiency is 3–4 orders of magnitude less than unity.

Again, I should emphasize that all of these estimates and calculations give only very conservative upper limits on the efficiency and energy emitted, since they all assume that somehow one can hold the charge onto the surface of the collapsing core even when it is as large as the transition radius $R_t$, which for $\xi_0 = Q_0/M > 10^{-4}$ would be significantly greater than the size of a neutron star. For such a core the electrostatic forces of repulsion on the excess protons on its surface would be more than 14 orders of magnitude greater than the gravitational attraction of the core for these protons, and it is hard to imagine that any forces sufficiently powerful to hold in the protons (such as nuclear forces at nuclear distances) could be effective in any star larger than a neutron star. And if one does take $\xi_0 \sim 10^{-4}$ so that $R_t \sim 50 \text{ km}$, somewhat larger than a neutron star, then the upper limit on the efficiency of the pair production process would be only on the order of $2 \times 10^{-7}$, which is far too small to give a viable model for gamma-ray bursts.

6. CONCLUSIONS

It does not seem to be possible to have astrophysical dyadospheres (electric fields larger than the critical value for Schwinger pair production, over macroscopic regions much larger in all directions than the regions of high fields that might conceivably be produced by individual heavy nuclear collisions, or much thicker than possible very thin high-field regions at the surface of a strange star [Usov 1998, 2004; Usov et al. 2005] or neutron star [Ruffini 2006]). If, as is most plausible, charge carriers like protons are bound to an astrophysical object, such as a star or stellar core, primarily by gravitational forces, then the electric field cannot get within 13 orders of magnitude of the minimal dyadosphere values. (The excess charges will simply be ejected by the electrostatic repulsion when that exceeds the gravitational attraction. Then the pair production rate from the macroscopic electric field, as opposed to that from collisions of individual particles, will be trillions of orders of magnitude below dyadosphere values and so will be completely negligible.)

Even in what I consider to be the implausible scenario in which the excess charge carriers are bound by nuclear forces to a collapsing stellar core, I have shown here in a simple spherically symmetric model that the electric field has a very conservative maximum value that is more than a factor of 18 below the minimal dyadosphere value. Because the pair production rate is essentially exponential in the negative inverse of the electric field, the upper limit of the pair production rate in even this implausible scenario is more than 26 orders of magnitude below the minimal dyadosphere values.

The idealized implausible scenario considered here, to give this very conservative upper bound on the pair production rate, had the maximal amount of charge somehow bound to the surface of an idealized stellar core with maximal initial charge that undergoes free-fall collapse from radial infinity in the Schwarzschild metric (conservatively ignoring the fact that an actual astrophysical collapse would start at finite radius and fall in more slowly, giving more time for discharge, and the fact that if the initial charge were maximal, the strong electric field would modify the geometry and also give electrostatic repulsion of the core surface, both of which would also slow down the collapse and lead to greater discharge and smaller electric fields). This scenario led to the maximal electric field (occurring when the freely collapsing core enters the event horizon of the black hole that would form) being less than 5.5% of dyadosphere values. Although the pair production rate is always less than $10^{-26}$ that of dyadosphere values, in this implausible scenario of having no other mechanism of discharging the core, there would be enough pair production to keep the electric fields always more than 18 times smaller than dyadosphere values.

Although macroscopic astrophysical dyadospheres do not form in this example or in any other similar example that has been considered, in this idealized implausible scenario there is significant pair production (although at macroscopic astrophysical rates that are much, much lower than microscopic dyadosphere rates), and so I calculated what fraction of the total mass-energy $M$ of the collapsing stellar core would be converted into pairs. Here I found that this efficiency, even under the highly idealized conditions of having maximal initial charge at such large radii that it seems inconceivable that the charge carriers could be sufficiently bound to such objects so much larger than neutron stars, is always much less than unity for collapsing objects with much less mass than 3 million $M_\odot$: the efficiency is very conservatively bounded by $2 \times 10^{-4} (M/1 M_\odot)^{1/2}$. Therefore, even these idealized charged collapsing objects, unless they were enormously more massive than the Sun, would not produce enough energy in outgoing charged particles to be consistent with the observed gamma-ray bursts.

It would of course be of interest to calculate the upper limits on the electric field and on the maximum pair production rate for models in which one relaxed the spherical symmetry. Although I would readily admit that I do not have a rigorous proof that the pair production rate cannot be higher, the general arguments of §3 strongly suggest that it would be very surprising if it could be much larger. Therefore, the spherically symmetric model analyzed here leads me to conjecture that the maximal pair production rates achievable by astrophysical electric fields that are macroscopic in all directions are always less than roughly $10^{-26}$ that of hypothetical dyadosphere values.
In conclusion, macroscopic dyadospheres almost certainly cannot form astrophysically, and the much weaker pair production rates that might occur, under highly idealized and implausible scenarios, do not seem sufficient for giving viable models of gamma-ray bursts.

For a shorter version of this work from a conference proceedings, see Page (2006b). For more details, see Page (2006a).

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