RATIONAL CURVES ON PRIMITIVE SYMPLECTIC VARIETIES OF
OG\textsuperscript{6}-TYPE

VALERIA BERTINI AND ANNALISA GROSSI

Abstract. We prove that any ample class on a primitive symplectic variety that is
locally trivial deformation of O’Grady’s singular 6-dimensional example is proportional
to the first Chern class of a uniruled divisor. This result answers a question of Lehn–
Mongardi–Pacienza [LMP21, Remark 4.7], extending their result [LMP21, Theorem
1.3] for primitive symplectic varieties of this deformation type.

Contents
1. Introduction 1
2. Preliminaries 3
3. Positive uniruled divisors on O’Grady’s moduli spaces 7
4. Monodromy orbits 11
References 15

1. Introduction

1.1. Background. An irreducible holomorphic symplectic manifold, in short ihs manifold,
is a compact complex Kähler manifold $X$ that is simply connected and such that $H^0(X, \Omega^2_X)$
is generated by an everywhere non-degenerate holomorphic 2-form, called the symplectic
form of $X$. Ihs manifolds of dimension 2 are K3 surfaces, and Beauville [Bea83] exhibited
two examples in each even complex dimension: Hilbert schemes of zero dimensional
subschemas of length $n$ on a K3 surface and generalized Kummer manifolds, whose
deformation types are called $K3^{[n]}$-type and $K_n(A)$-type respectively. O’Grady [O’G03,
O’G99] gave two further examples in dimension six and ten. Ihs manifolds deformation
equivalent to the O’Grady’s examples are called of OG\textsuperscript{6}-type and OG\textsuperscript{10}-type respectively.

In recent years, singular symplectic varieties have been object of great interest and their
theory has been very much explored and developed; we refer to §2.1 for the definitions
in the singular setting. As for the smooth case, singular symplectic varieties appear as
factors of normal Kähler spaces with mild singularities and trivial first Chern class, giving
a singular version of the Beauville-Bogomolov decomposition theorem [GKP16] [HP19]
[BGL22]. Ways to produce singular symplectic varieties are to consider quotients of
smooth ihs manifolds by a finite group of symplectic automorphisms [Bea00, Proposition 2.4], or to consider singular moduli spaces of sheaves on surfaces with trivial canonical bundle [PR18, Theorem 1.19].

The class of singular symplectic varieties that we consider throughout the paper are **primitive symplectic varieties**, see Definition 2.2. The notion of primitive symplectic variety generalizes the definition of ihs manifold to the singular setting, and in fact smooth primitive symplectic variety do coincide with ihs manifold, [BL18, Lemma 3.3] and [Sch20b, Theorem 1]. The theory of primitive symplectic varieties behaves similarly to the one of ihs manifolds in many relevant aspects, as the lattice structure on second integral cohomology group (see Theorem 2.8), their deformation theory and Torelli type results [BL18].

1.2. **Rational curves on primitive symplectic varieties.** In this paper we focus on another interesting property shared by ihs manifolds and primitive symplectic varieties: the existence of rational curves on them, and more precisely, the existence of ample uniruled divisors (see Definition 3.7). This problem has been firstly investigated in the smooth setting, where the existence of ample uniruled divisors on an ihs manifold allows to build a canonical subgroup of the 0-Chow group of the ihs manifold [CMP19, Theorem 1.5], giving a first evidence of Voisin’s geometrical realization of the conjectural Bloch-Beilinson filtration of the 0-Chow group [Voi16]. Uniruled divisors are known to exist in almost any ample linear system on ihs manifolds of $K3^{[n]}$-type [CMP19] and of $K_n(A)$-type [MP17] [MP19], and in any ample linear system with some fixed numerical invariants for ihs manifolds of $OG_{10}$-type [Ber21]. The key ingredient in all these cases is a result by Charles-Mongardi-Pacienza [CMP19] about deformations of rational curves ruling a divisor on an ihs manifold. This result allows to pass from the construction of some explicit examples of ample uniruled divisors to the existence of uniruled divisors in ample classes which are deformation of the ones explicitly constructed.

1.3. **Main results.** Lehn-Mongardi-Pacienza recently extended the Charles-Mongardi-Pacienza deformation result to the singular setting, i.e. for primitive symplectic varieties [LMP21, Theorem 1.1], and they started the search of ample uniruled divisors on them. In this paper we focus on primitive symplectic varieties that are locally trivial deformation of the singular 6-dimensional example of O’Grady, that we call of $OG_s^6$-type, see Definition 2.7. We prove that any ample class on a primitive symplectic variety of $OG_s^6$-type is proportional to the first Chern class of a uniruled divisor.

**Theorem 1.1.** Let $(X,h)$ be a polarized primitive symplectic variety of $OG_s^6$-type. Then there exists a positive integer $m$ such that $mh$ is the first Chern class of a uniruled divisor.

Let $\mathcal{M}_{OG_s^6}$ be the moduli space of polarized primitive symplectic varieties of $OG_s^6$-type, see Definition 3.5. Lehn-Mongardi-Pacienza prove in [LMP21, Theorem 1.3] that $\mathcal{M}_{OG_s^6}$ has infinitely many connected components whose points have polarization that is proportional to the first Chern class of a uniruled divisor, but they can not give any characterization of these components. With Theorem 1.1 we conclude that every connected component of $\mathcal{M}_{OG_s^6}$ has this property, answering their question [LMP21, Remark 4.7].

Theorem 1.1 is based on the existence of positive effective irreducible divisors on some specific $OG_s^6$-type primitive symplectic varieties, called $OG_s^6$ moduli spaces, as
stated in Theorem 3.8. The conclusion follows by deforming the rational curves ruling these positive effective irreducible divisors via [LMP21, Theorem 1.3], and any polarized primitive symplectic variety of OG$_6$-type is reached in this way: two of such varieties are one locally trivial polarized deformation of the other exactly when their polarizations have the same square and the same divisibility (see §4.1), and all possible squares and divisibilities arise on OG$_6$ moduli spaces by Proposition 4.5.

1.4. Structure of the paper. This paper is organized as follows. In §2 we introduce the notion of primitive symplectic varieties and the OG$_6$ deformation class, together with few fundamental results about them. In §3.1 we recall the definition of a certain regular morphism from a K3$^3$-type ihs manifold to a OG$_6$-type primitive symplectic variety introduced by Mongardi–Rapagnetta–Saccà, that we use in §3.2 to conclude the existence of uniruled divisors in any positive effective class on a OG$_6$ moduli space, see Theorem 3.8. In §3.2 we introduce the notion of polarized moduli space of OG$_6$-type primitive symplectic varieties $M_{\text{OG}_6}$. In §4 we pass to any polarized OG$_6$-type primitive symplectic variety in $M_{\text{OG}_6}$: as first in §4.1 we recall that the connected components of $M_{\text{OG}_6}$ are characterized by degree and divisibility of the polarization of an element, then in §4.2 we exhibit a positive effective class on a OG$_6$ moduli space for all possible squares and divisibilities, see Lemma 4.4 and Proposition 4.5, and we prove Theorem 1.1. We devote §4.3 to some considerations on the smooth case, i.e. ihs manifolds of OG$_6$-type.

Acknowledgements. We wish to thank Christian Lehn, Giovanni Mongardi, Claudio Onorati and Gianluca Pacienza for their hints and suggestions.

2. Preliminaries

2.1. Primitive symplectic varieties and O’Grady’s example. We start recalling the basics definitions in the smooth setting.

Definition 2.1 (Holomorphic symplectic manifolds - smooth setting). Let $X$ be a complex manifold.

1. A form $\sigma \in H^0(X, \Omega^2_X)$ is called holomorphic symplectic if it is everywhere non-degenerate on $X$.

2. $X$ is a holomorphic symplectic manifold if there exists a holomorphic symplectic form on $X$.

3. $X$ is an irreducible holomorphic symplectic (in short, ihs) manifold if it is a simply connected compact Kähler holomorphic symplectic manifold such that $h^0(X, \Omega^2_X) = 1$.

In the singular setting there are several notions of symplectic varieties, whose nomenclature is not uniform in literature; for our definitions we refer to [BL18, §3] and [PR18, §1.1, §1.2]. Let $X$ be a normal variety and $j : X^{\text{sm}} \hookrightarrow X$ the embedding of its smooth locus. For any integer $0 \leq p \leq \dim(X)$ we define

$$\Omega^p_X := j_* \Omega^p_{X^{\text{sm}}} = (\Omega^p_X)^{\vee\vee}$$

to be the sheaf of reflexive $p$-forms on $X$; here $\Omega^p_X = \text{Hom}_{\mathcal{O}_X}(\Omega^p_X, \mathcal{O}_X)$ is the dual sheaf as sheaf of $\mathcal{O}_X$-modules.

Definition 2.2. [Symplectic varieties - singular setting] Let $X$ be a normal variety.
(1) A reflexive form \( \sigma \in H^0(X, \Omega_X^{(2)}) \) is called symplectic if its restriction \( \sigma|_{X^{sm}} \) is a holomorphic symplectic form on \( X^{sm} \).

(2) A pair \((X, \sigma)\) is a symplectic variety if \( \sigma \) is a symplectic form on \( X \) and for some (hence any) resolution of singularities \( f : \tilde{X} \to X \) the pullback \( f^*\sigma|_{X^{sm}} \) extends holomorphically to \( \tilde{X} \).

(3) Given a symplectic variety \((X, \sigma)\) a resolution of singularities \( f : \tilde{X} \to X \) is called a symplectic resolution if \( f^*\sigma|_{X^{sm}} \) extends to a holomorphic symplectic form on \( \tilde{X} \).

(4) A compact Kähler symplectic variety \((X, \sigma)\) is called a primitive symplectic variety if \( h^1(X, \mathcal{O}_X) = 0 \) and \( h^0(X, \Omega_X^{(2)}) = 1 \). If furthermore \( X \) is projective, then \((X, \sigma)\) is called a Namikawa symplectic variety.

When we don’t need to keep track of the symplectic form \( \sigma \) we drop it from the notation.

A particular class of primitive symplectic varieties are the irreducible-holomorphic symplectic (often called simply irreducible symplectic) varieties, introduced by [GKP16, Definition 8.16]; they are particularly relevant as they arise as one of the three building blocks in the Beauville-Bogomolov decomposition theorem in the singular setting we have recalled in §1.1. We will not be interested in this class of primitive varieties, so we omit their definition.

By [Bea00, Proposition 1.3] a symplectic variety \( X \) has rational singularities, i.e. it is normal and for any resolution of singularities \( f : \tilde{X} \to X \) it holds \( R^if_*\mathcal{O}_{\tilde{X}} = 0 \) for any \( i \geq 1 \). As consequence one deduces the following.

**Proposition 2.3.** [PR18, Proposition 1.9] Let \( X \) be a connected projective symplectic variety admitting a symplectic resolution \( f : \tilde{X} \to X \) such that \( \tilde{X} \) is an ihs manifold. Then \( X \) is a Namikawa symplectic variety.

We denote by \( H^2(X, \mathbb{Z})_{tf} \) the torsion-free part of the abelian group \( H^2(X, \mathbb{Z}) \). Another consequence of the fact that a symplectic variety has rational singularities is the following.

**Proposition 2.4.** [BL18, Lemma 2.1, Corollary 3.5] Let \( X \) be a compact symplectic variety. Then \( H^2(X, \mathbb{Z})_{tf} \) carries a pure weight 2 Hodge structure. If furthermore \( f : \tilde{X} \to X \) is a symplectic resolution of \( X \), then the pullback \( f^* : H^2(X, \mathbb{Z}) \to H^2(\tilde{X}, \mathbb{Z}) \) is an inclusion of pure Hodge structures; in particular, \( H^2(X, \mathbb{Z}) \) is torsion-free.

The examples of (projective) ihs manifolds and primitive symplectic varieties we are going to work with are defined as moduli spaces of sheaves on canonical trivial surfaces. We are going to briefly recall here their definition and some results about them; for further details we refer to [PR18, §1.3].

Let \( S \) be a projective K3 surface or an abelian surface. Given a coherent sheaf \( \mathcal{F} \in \text{Coh}(S) \) the Mukai vector of \( \mathcal{F} \) is defined as

\[
\nu(\mathcal{F}) := \text{ch}(\mathcal{F}) \sqrt{\text{td}(S)} \in H^2(S, \mathbb{Z}) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}).
\]

For any \( v = (r, l, s) \in H^2(S, \mathbb{Z}) \) we say that \( v \) is a Mukai vector if \( r \geq 0 \) and \( l \in \text{NS}(S) \), and if \( r = 0 \) then \( l \) is the first Chern class of an effective divisor or \( l = 0 \) and \( r > 0 \). Given a polarization \( h \) on \( S \), i.e. \( h = c_1(H) \) is the first Chern class of an ample divisor \( H \) on \( S \), there is a notion of \( v \)-genericity for \( h \), that we are not going to recall here; when
rk Pic(S) = 1 then any polarization is v-generic, and this is the only case we are going to consider in the following sections.

**Definition 2.5.** Let S be a projective K3 or abelian surface, v a Mukai vector and h a v-generic polarization.

1. We call $M_v(S,h)$ the moduli space of S-equivalence classes of Gieseker h-semistable sheaves $F \in \text{Coh}(S)$ with Mukai vector $v(F) = v$.

2. When $S$ is an abelian surface we call $K_v := a_v^{-1}(0_S, O_S) \subset M_v(S,h)$, where $a_v$ is the isotrivial fibration $a_v : M_v(S,h) \to S \times S \to F \mapsto (\text{Alb}(c_2(F)), \det(F) \otimes \det(F_0)^{-1})$

Here $\text{Alb} : \text{CH}_0(S) \to S$ is the Albanese morphism and $[F_0] \in M_v(S,h)$ is the S-equivalence class of a fixed coherent sheaf.

When no confusion on $S$ and $h$ is possible, we drop them from the notation.

The moduli spaces $M_v$ and $K_v$ are projective, connected, normal varieties, and results on them are known by the work of several authors. We collect in the following theorem results about the only case we will be interested in in what follows.

**Theorem 2.6.** Let $A$ be an abelian surface, $v$ a Mukai vector and $h$ a $v$-generic polarization.

1. [O’G03] If $v = (2, 0, -2) \in H^{2*}(A, \mathbb{Z})$ then $K_v := K_6$ is singular and it admits a symplectic resolution $\tilde{\pi} : \tilde{K}_6 \to K_6$, where $\tilde{K}_6$ is a projective ihs manifold of dimension 6.

2. [LS06][PR13] Let $v$ be a Mukai vector such that $v = 2w$ with $w = (r, l, s)$ primitive and $l^2 - 2rs = 2$. Then $K_v$ is a singular symplectic variety and it admits a symplectic resolution $\tilde{\pi} : \tilde{K}_v \to K_v$, where $\tilde{K}_v$ is a projective ihs manifold deformation equivalent to $\tilde{K}_6$.

3. [PR18, Theorem 1.17] Let $v_1$ and $v_2$ be Mukai vectors as in (2). Then $K_{v_1}$ and $K_{v_2}$ are locally trivial deformation one of the other.

**Definition 2.7.** A moduli space $K_v$ as in the hypotheses of Theorem 2.6.(2) is called a OG$_{6}$ moduli space. A primitive symplectic variety that is a locally trivial deformation of $K_6$ is called of OG$_{6}$-type.

As a consequence of Proposition 2.3 the moduli spaces $K_v$ in the theorem above are Namikawa symplectic varieties.

A desingularized moduli space $\tilde{K}_v$ as in Theorem 2.6.(2) is called a OG$_{6}$ moduli space, and it is by definition an OG$_{6}$-type ihs manifold.

### 2.2. Lattice theory on primitive symplectic varieties

A **lattice** is a free $\mathbb{Z}$-module $L$ of finite rank endowed with a non-degenerate symmetric bilinear form $q_L : L \times L \to \mathbb{Z}$.

A lattice is called **even** if $q_L(x) := q_L(x,x)$ is even for any $x \in L$. Associated to any lattice $L$ there is a finite abelian group called the **discriminant group** and defined as $A_L = L^\vee / L$. 

where $L \rightarrow L' = \text{Hom}_\mathbb{Z}(L, \mathbb{Z})$ is the canonical embedding. Observe also that
\begin{equation}
L' \cong \{ v \in L \otimes \mathbb{Q} \mid (v, w) \in \mathbb{Z}, \forall w \in L \}.
\end{equation}

Given $v \in L$ we denote by $[v]$ its class in $A_L$ and by $\text{div}(v)$ the divisibility of $v$, i.e. the positive integer generator of the ideal $(v, L) \subseteq \mathbb{Z}$. Given generators $\{v_i\}_{i=1}^{rk(L)}$ of $L$, the classes $\{[\frac{v_i}{\text{div}(v_i)}]\}_{i=1}^{rk(L)}$ generate $A_L$.

The order of the discriminant group $A_L$ equals $|\det(q_L)|$ and this number is called discriminant of the lattice $L$. A lattice $L$ is called unimodular if $A_L = \{\text{id}\}$.

Because of the isomorphism (1), the pairing on $L$ induces by $\mathbb{Q}$-linear extension a pairing with $\mathbb{Q}$-values on $L'$ and hence a pairing on $A_L$ with values in $\mathbb{Q}/\mathbb{Z}$. If the lattice $L$ is even, then the $\mathbb{Q}$-valued quadratic form on $L'$ gives rise to a quadratic form
\[ q_A_L : A_L \rightarrow \mathbb{Q}/2\mathbb{Z}, \]
called the discriminant quadratic form of $L$.

The torsion-free part of second integral cohomology of a primitive symplectic variety has a lattice structure, as we recall in the following theorem.

**Theorem 2.8.** Let $(X, \sigma)$ be a primitive symplectic variety of dimension $2n$.

(1) [Bea83][Kir15][Mat15][Sch20a][BL18] There is a quadratic form $q_X$ on $H^2(X, \mathbb{Z})$, called Beauville-Bogomolov-Fujiki (in short, BBF) form, which is defined up to scaling by $^2$
\[ q_X(\alpha) := \frac{n}{2} \int_X (\sigma^2)^{n-1} \alpha^2 + (1 - n) \int_X \sigma^n \sigma^{n-1} \frac{\alpha}{2} \int_X \sigma^{n-1} \sigma^n \alpha \]
and that is non-degenerate of signature $(3, b_2(X) - 3)$. Such form gives a lattice structure to $H^2(X, \mathbb{Z})$. Such form gives a lattice structure to $H^2(X, \mathbb{Z})$.

(2) [Fuj87, Theorem 4.7][Sch20a, Theorem 2][BL18, Theorem 5.20]. There is a positive real number $c_X$, called Fujiki constant, such that for any $\alpha \in H^2(X, \mathbb{C})$ the following relation, called Fujiki relation, holds true:
\[ \int_X \alpha^{2n} = c_X \cdot q_X(\alpha)^n. \]

As consequence of item (2) in Theorem 2.8 the BBF form $q_X$ and the Fujiki constant $c_X$ are deformation invariant. The definition of the BBF quadratic form and its compatibility with the Hodge structure on $H^2(X, \mathbb{Z})$ are the starting point for Torelli theorems on primitive symplectic varieties proved in [BL18].

**Remark 2.9.** Let $X$ be a primitive symplectic variety and $f : \tilde{X} \rightarrow X$ a symplectic resolution such that $\tilde{X}$ is an ihs manifold. A natural question is whether the inclusion $f^* : H^2(X, \mathbb{Z}) \rightarrow H^2(\tilde{X}, \mathbb{Z})$ in Proposition 2.4 is compatible with the BBF-lattice structures,

\footnote{This is the reference for the same result in the smooth setting, i.e. the case of ihs manifolds.}
\footnote{For a complex compact variety $X$ the integration over its singular top cohomology group $H^{2n}(X, \mathbb{Z})$ is defined as $\int_X \alpha := \{X\} \cap \alpha$, where $\{X\} \in H_{2n}(X, \mathbb{Z})$ is the fundamental class. Integration commutes with pullbacks by bimeromorphic maps.}
\footnote{This is the reference for the same result in the smooth setting, i.e. the case of ihs manifolds.}
\footnote{Note that the terminology used in [Sch20a] is different from the one we have chosen, as he refers to primitive symplectic varieties as irreducible symplectic varieties, cfr. Definition 1.(3) therein.}
i.e. if \( q_X(\alpha) = q_\tilde{X}(f^*\alpha) \) for any \( \alpha \in H^2(X, \mathbb{Z}) \). The answer to this question is positive [Sch20a, Corollary 24]. In fact this approach was the first one to be used to define a lattice structure on \( H^2(X, \mathbb{Z}) \) in the particular case of primitive symplectic varieties admitting a symplectic resolution given by an ihs manifold, see [Nam01, Theorem 8.2] and [PR13] in the case of moduli spaces of sheaves.

As a consequence, under the same hypotheses one concludes \( c_X = c_\tilde{X} \), i.e. the Fujiki constant does not change under symplectic resolutions.

**Example 2.10.** When \( X \) is a primitive symplectic variety of \( OG_6^s \)-type then \( H^2(X, \mathbb{Z}) \) is torsion-free by Proposition 2.4 and it is an even lattice of signature \((3, 4)\). More precisely, [PR13, Theorem 1.7] gives

\[
H^2(X, \mathbb{Z}) \cong U^{\mathbb{Z}^3} \oplus [-2] =: \mathbb{L}_{OG_6^s}
\]

where \( U \) is the hyperbolic lattice and \([-2]\) is a rank-one lattice generated by an element of square \(-2\); we call \( \sigma \) this element of square \(-2\). Let \( \mathbb{A}_{OG_6^s} \) be the discriminant group of \( \mathbb{L}_{OG_6^s} \). Observe that \( \sigma \) has divisibility 2 in \( \mathbb{L}_{OG_6^s} \) and \( U^{\mathbb{Z}^3} \) is unimodular, hence \([\sigma/\text{div}(\sigma)]\) generates the discriminant group \( \mathbb{A}_{OG_6^s} \cong \mathbb{Z}/2\mathbb{Z} \). With this notation it holds that

\[
q_{\mathbb{A}_{OG_6^s}} \left( \left[ \frac{\sigma}{\text{div}(\sigma)} \right] \right) = \left[ \frac{3}{2} \right] \in \mathbb{Q}/2\mathbb{Z}.
\]

### 3. Positive uniruled divisors on O’Grady’s moduli spaces

Let \( K_v \) be an \( OG_6^s \) moduli space, which is a Namikawa symplectic variety of \( OG_6^s \)-type. The goal of this section is to prove Theorem 3.8, i.e. the existence of uniruled divisors in any linear system defined by a positive effective divisor on \( K_v \), up to a positive multiple.

**3.1. Mongardi-Rapagnetta-Saccà model.** We consider here a construction firstly introduced by Rapagnetta [Rap07] in the case of Mukai vector \((0, 2\theta, -2)\), and then generalized by Mongardi-Rapagnetta-Saccà [MRS18] for any Mukai vector as in the assumptions of Theorem 2.6.(2), i.e. the ones giving an \( OG_6^s \) moduli space. We briefly recall here the main steps of this construction; the interested reader can find any further detail in the original work [MRS18].

Let \( K_v \) be the symplectic resolution of \( K_v \) as in Theorem 2.6, that is an ihs manifold of \( OG_6^s \)-type. Lehn and Sorger [LS06] prove that \( \tilde{K}_v \cong \text{Bl}_{\Sigma_v} K_v \), where \( \Sigma_v \) is the singular locus of \( K_v \); we call \( \Sigma_v \) the exceptional divisor of this blow-up. Rapagnetta [Rap07, Theorem 3.3.1] proves that \( \tilde{\Sigma}_v \) is divisible by two in Pic(\( \tilde{K}_v \)), hence (being Pic(\( \tilde{K}_v \)) torsion free) there exists a unique normal projective variety \( \tilde{Y}_v \) with a double cover \( \tilde{\varepsilon}_v : \tilde{Y}_v \to \tilde{K}_v \) branched over \( \tilde{\Sigma}_v \).

By [MRS18, Theorem 4.2] this construction transfers to the variety \( K_v \), i.e. there exists a normal projective variety \( Y_v \) with a finite 2:1 morphism \( \varepsilon_v : Y_v \to K_v \) with branch locus \( \Sigma_v \); the singular locus of \( Y_v \) is denoted by \( \Gamma_v \) and it consists of 256 points. Finally, we consider the blow-ups \( \tilde{Y}_v := \text{Bl}_{\Gamma_v} Y_v \) and \( \tilde{K}_v := \text{Bl}_{\Sigma_v} K_v \), where \( \Omega_v \subset K_v \) is the singular locus of \( \Sigma_v \); we call \( \Gamma_v \subset \tilde{Y}_v \) the exceptional locus of the first blow-up. By [MRS18, Corollary 4.3] there exists a finite 2:1 morphism \( \tilde{\varepsilon}_v : \tilde{Y}_v \to \tilde{K}_v \) branched over
\[ \Sigma_v := \text{Bl}_{\Omega_v} \Sigma_v \text{ lifting } \varepsilon_v : Y_v \to K_v, \text{ i.e. } \text{ forming a commutative diagram together with the blow-up morphisms defining } Y_v \text{ and } K_v. \]

The variety \( Y_v \) is smooth and \( Y_v \) is normal. It follows that the morphisms \( \varepsilon_v \) and \( \tau_v \) induce regular involutions \( \tau_v : Y_v \to Y_v \) and \( \tau_v : Y_v \to Y_v \) respectively; as \( K_v \) and \( K_v \) are normal, the morphisms \( \varepsilon_v \) and \( \varepsilon_v \) are indeed the quotient maps with respect to the involutions \( \tau_v \) and \( \tau_v \), see [MRS18, Remark 4.7]. We summarize the picture we obtained so far in the following commutative diagram:

\[
\begin{array}{ccc}
\tau_v & \rightarrow & Y_v = \text{Bl}_{\Gamma_v} Y_v \\
\downarrow & & \downarrow \\
\tau_v & \rightarrow & Y_v = \text{Bl}_{\Omega_v} K_v \\
\end{array}
\]

The exceptional divisor \( \Gamma_v \subset Y_v \) consists of the union of 256 varieties \( I_{v,i} \) which are isomorphic to the incidence variety \( P(V) \times P(V) \), where \( V \) is a complex vector space of dimension 4. Let \( p_{v,i} : I_{v,i} \to P(V) \) be one of the two natural projections, which is a \( P^2 \)-fibration; the normal bundle of \( I_{v,i} \) in \( Y_v \) has degree -1 on the fibers of \( p_{v,i} \), hence by Nakano’s contraction theorem there exists a complex manifold \( Y_v \) with a morphism of complex manifolds \( h_v : Y_v \to Y_v \) having exceptional locus \( \Gamma_v \) and such that \( J_v := h_v(\Gamma_v) \) is the union of 256 copies of \( P^3 \). Furthermore, \( h_v \) realizes \( Y_v \) as the blow-up of \( Y_v \) along \( J_v \).

The fundamental point of this last construction is that \( Y_v \) is an ihs manifold of \( K3^{[3]} \)-type, i.e. an ihs manifold deformation equivalent to the Hilbert of \( n \) points on a \( K3 \) surface [MRS18, Proposition 5.3]. As the involution \( \tau_v \) on \( Y_v \) sends \( \Gamma_v \) to itself, it descends to a rational involution \( \tau_v : Y_v \to Y_v \), regular on \( Y_v \setminus J_v \). By construction of \( Y_v \), there is a regular birational morphism \( f_v : Y_v \to Y_v \) contracting \( J_v \) to \( \Gamma_v \) and such that the following diagram is commutative:

\[
\begin{array}{ccc}
\tau_v & \rightarrow & Y_v \\
\downarrow & & \downarrow \\
\tau_v & \rightarrow & Y_v \\
\end{array}
\]

As observed in [MRS18, Remark 5.4] the regular involution \( \tau_v \) exchanges the two \( P^2 \)-fibrations on each \( I_{v,i} \), hence \( \tau_v \) can not be extended out of \( Y_v \setminus J_v \), as \( Y_v \) is obtained from \( Y_v \) by contracting only one of them; therefore the indeterminacy locus of \( \tau_v \) is exactly \( J_v \).

We call

\[ \Phi_v := \varepsilon_v \circ f_v : Y_v \to K_v \]

which is a generically 2:1 regular morphism contracting \( J_v \subset Y_v \) to points. The \( \text{OG}_6^4 \)-type Namikawa symplectic variety \( K_v \) is in this sense realized as a ‘quotient’ of the \( K3^{[3]} \)-type ihs manifold \( Y_v \) via the rational involution \( \tau_v \).

We conclude relating the BBF-lattice structure of \( K_v \) and \( Y_v \) via the morphism \( \Phi_v \).
Proposition 3.1. For any $\alpha \in H^2(K_v, \mathbb{Z})$ we have
\[ q_{Y_v}(\Phi_v^*(\alpha)) = 2 \cdot q_{K_v}(\alpha) \]
where $q_{Y_v}$ and $q_{K_v}$ are the BBF-forms on $Y_v$ and $K_v$ respectively.

Proof. We have
\[ c_{Y_v} \cdot q_{Y_v}(\Phi_v^*(\alpha)) = \Phi_v^*(\alpha) \]
by Theorem 2.8.(2) on $Y_v = 2 \cdot \alpha$ as $\Phi_v$ is a generically 2:1 finite morphism
\[ = 2 \cdot c_{K_v} \cdot q_{K_v}(\alpha) \]
by Theorem 2.8.(2) on $K_v$.

As observed in Remark 2.9, we have $c_{K_v} = c_{\tilde{K}_v}^5$. As $c_{\tilde{K}_v} = 60$ [Rap07, Theorem 3.5.1] and $c_{Y_v} = 15$ [Bea83] we obtain the statement. □

3.2. Positive uniruled divisors on $K_v$. We start recalling the definition of (polarized) locally trivial deformations of primitive symplectic varieties, that we will use in the main result of this section. We also introduce the related notions of moduli space of marked or polarized primitive symplectic varieties, as we are going to use them in the following sections.

Definition 3.2. A locally trivial family is a flat proper morphism $\pi : X \to S$ of complex spaces where $S$ is connected and such that for any $p \in X$ there exist open neighborhoods $U \subseteq X$ of $p$ and $S_p \subseteq S$ of $\pi(p)$ such that
\[ U \cong U \times S_p \]
over $S_p$, where $U := U \cap \pi^{-1}(p)$. A complex variety $X_1$ is a locally trivial deformation of another complex variety $X_2$ if there exists a locally trivial family $\pi : X \to S$ and points $s_1, s_2 \in S$ such that $X_{s_1} \cong X_1$ and $X_{s_2} \cong X_2$.

Every small locally trivial deformation of a primitive symplectic variety is a primitive symplectic variety [BL18, Corollary 4.11]. Let $\delta$ be a fixed locally trivial deformation type of primitive symplectic varieties; a primitive symplectic variety of this class is called of $\delta$-type.

Let $L$ be a rank $n$ lattice of signature $(3, n - 3)$. A $L$-marked primitive symplectic variety is a pair $(X, \mu)$ with $X$ a primitive symplectic variety and $\mu$ an isometry $\mu : H^2(X, \mathbb{Z})_{ff} \to L$, called $L$-marking; here $H^2(X, \mathbb{Z})_{ff}$ has the BBF-lattice structure defined in Theorem 2.8. An isometry of $L$-marked primitive symplectic varieties $(X, \mu)$ and $(X', \mu')$ is an isomorphism $f : X \to X'$ such that $\mu' = \mu \circ f^*$, where $f^*$ is the pullback on the second cohomology group.

Let $\delta$ be a fixed locally trivial deformation type of primitive symplectic varieties. By Theorem 2.8 all primitive symplectic varieties of $\delta$-type have (the torsion-free part of) their integral second cohomology group isometric to the same abstract lattice $L_\delta$.


\[5\] PR20, Theorem 1.7] computes the Fujiki constant for any singular moduli space on a surface with trivial canonical bundle, but in this particular case, i.e. when the singular moduli space admits a symplectic resolution given by an ihs manifold, the computation is easier as the Fujiki constant equals the one of its resolution.
Definition 3.3. Let $\mathcal{M}_{L_δ}$ be the moduli space of $L_δ$-marked primitive symplectic varieties of $δ$-type, i.e. its elements are $L_δ$-marked primitive symplectic varieties $(X, \mu)$ where $X$ is of $δ$-type.

The moduli space $\mathcal{M}_{L_δ}$ has a (non-necessarily Hausdorff) topology and a complex structure obtained by gluing together the Kuranishi spaces $\text{Def}(X)$ of locally trivial deformations of a primitive symplectic variety of $δ$-type $X$, which are smooth of dimension $h^{1,1}(X)$ [BL18, Theorem 4.7]; the crucial point in the gluing is that miniversal locally trivial deformations are in fact universal [BL18, Lemma 4.9]. It follows that $\mathcal{M}_{L_δ}$ is a smooth manifold.

A polarized variety is a pair $(X, h)$ where $X$ is a projective variety and $h$ is a primitive polarization, i.e. the first Chern class of a primitive ample line bundle on $X$.

Definition 3.4. A polarized locally trivial family is a locally trivial family $\pi : \mathcal{X} \to S$ together with a line bundle $\mathcal{L} \in \text{Pic}(\mathcal{X})$. A polarized variety $(X_1, h_1)$ is a polarized locally trivial deformation of another polarized variety $(X_2, h_2)$ if there exists a polarized locally trivial family $\pi : \mathcal{X} \to S$, $\mathcal{L} \in \text{Pic}(\mathcal{X})$ and points $s_1, s_2 \in S$ such that $(\mathcal{X}_{s_1}, c_1(\mathcal{L}_{s_1})) \cong (X_1, h_2)$ and $(\mathcal{X}_{s_2}, c_1(\mathcal{L}_{s_2})) \cong (X_2, h_2)$ as polarized pairs.

A polarized primitive symplectic variety is a polarized variety $(X, h)$ such that $X$ is primitive symplectic.

Definition 3.5. For any $d \in \mathbb{N}$ let $\mathcal{M}_{δ,d}$ be the coarse moduli space of polarized primitive symplectic varieties of $δ$-type and degree $d$, i.e. its elements are polarized primitive symplectic varieties $(X, h)$ where $X$ is of $δ$-type and the BBF-square of $h$ equals $d$.

By [BL18, Proposition 8.7 and Lemma 8.8] the space $\mathcal{M}_{δ,d}$ is a quasi-projective scheme. Finally, we call

$$\mathcal{M}_{δ} := \coprod_{d \in \mathbb{N}} \mathcal{M}_{δ,d}.$$ 

Observe that two polarized primitive symplectic varieties of $δ$-type that are one deformation of the other (as polarized varieties) belong to the same component $\mathcal{M}_{δ,d}$ of $\mathcal{M}_{δ}$, but the vice versa is not true in general (see §4.1). In what follows we will be interested in the case $δ = OG^{6}$, i.e. to the moduli space $\mathcal{M}_{OG}^{6}$.

We observe here a fact we are going to use frequently in what follows.

Remark 3.6. Let $X$ be a primitive symplectic variety and $h$ an effective class that is positive with respect to the BBF-form of $X$. Consider a small locally trivial deformation $(X', h')$ of the pair $(X, h)$ such that $\text{Pic}(X') \cong \mathbb{Z}$ and $h'$ is still effective; as the BBF-form is deformation invariant, the class $h'$ is positive on $X'$. By the projectivity criterion [Huy99, Theorem 3.11] [BL18, Theorem 1.2] we conclude that $X'$ is projective, hence $h'$ is ample. We conclude that a pair $(X, h)$ as above always deforms to a pair $(X', h')$ with $h'$ ample class on the primitive symplectic variety $X'$. Observe that if the starting variety $X$ is an ihs manifold then the variety $X'$ is also ihs manifold, as it is a deformation of the first one [Bea83].

Definition 3.7. Let $D \subseteq X$ be an irreducible divisor. We say that $D$ is uniruled if there exists a $\text{dim}(D) - 1$ variety $Y$ and a finite dominant morphism $\mathbb{P}^{1} \times Y \dashrightarrow D$.

---

This is the reference in the smooth setting, i.e. for ihs manifolds.
We are finally ready to prove the main result of this section.

**Theorem 3.8.** Let $K_v$ be an $\text{OG}_6^s$ moduli space, with BBF-form $q_{K_v}$. For any effective and $q_{K_v}$-positive\(^7\) divisor $D$ on $K_v$ there exists a positive integer $m$ such that the linear system $|mD|$ contains an irreducible uniruled divisor.

**Proof.** Consider the generically $2:1$ regular morphism defined in (3):
\[
\Phi_v: Y_v \to K_v
\]
where $Y_v$ is an ihs manifold of $K3^{[3]}$-type. As $c_1(D) \in H^2(K_v, \mathbb{Z})$ is by assumption a $q_{K_v}$-positive class, the pull-back $\Phi_v^*c_1(D) \in H^2(Y_v, \mathbb{Z})$ is $q_{Y_v}$-positive by Proposition 3.1. By Remark 3.6 there exists a deformation $(Y'_v, c_1(D'))$ of $(Y_v, \Phi_v^*c_1(D))$ such that $D'$ is an ample divisor. Using [CMP19, Theorem 1.1 and Remark 1.3(iii)] on the polarized $K3^{[3]}$-type ihs manifold $(Y'_v, c_1(D'))$ we deduce that there exists a positive integer $k$ such that the linear system $|kD'|$ contains an irreducible uniruled divisor $D''$. Then the divisor $\Phi_v(D'')$ on $K_v$, taken with reduced schematic structure, is irreducible and uniruled, and its class in $H^2(K_v, \mathbb{Z})$ is a multiple of $c_1(D)$.

\[\square\]

4. **Monodromy orbits**

In this section we prove our main result Theorem 1.1, which states that any point of $\mathfrak{M}_{\text{OG}_6^s}$ corresponds to a polarized $\text{OG}_6^s$-type primitive symplectic variety whose polarization is proportional to the first Chern class of a uniruled divisor. The first step in this direction is to characterize the connected components of $\mathfrak{M}_{\text{OG}_6^s}$, which are given by the locally-trivial monodromy group.

4.1. **Monodromy of $\text{OG}_6^s$ and Eichler’s criterion.** Let $X$ be a primitive symplectic variety of $\text{OG}_6^s$-type. We denote by $\text{Mon}^2(X)$ the locally trivial monodromy group, i.e. the group of automorphisms of $H^2(X, \mathbb{Z})$ given by parallel transport operators along locally triavial families (see [MR19, Definition 2.11]). It holds that $\text{Mon}^2(X) \subseteq O^+(H^2(X, \mathbb{Z}))$ [MR19, Lemma 2.12], where $O^+(H^2(X, \mathbb{Z}))$ is the group of orientation preserving isometries, which are the isometries preserving the orientation of the positive cone in $H^2(X, \mathbb{Z})$. The monodromy group is invariant up to locally trivial deformations, and we denote by $\text{Mon}^2(\text{L}_{\text{OG}_6^s})$ the monodromy group acting on $\text{L}_{\text{OG}_6^s}$ (see Example 2.10 for the definition of $\text{L}_{\text{OG}_6^s}$), defined via a $\text{L}_{\text{OG}_6^s}$-marking $\mu: H^2(X, \mathbb{Z}) \to \text{L}_{\text{OG}_6^s}$.

Connected components of the moduli space $\mathfrak{M}_{\text{OG}_6^s}$ are given by orbits under the action of the locally trivial monodromy group. In other words, two elements in $\mathfrak{M}_{\text{OG}_6^s}$ are in the same connected component if their polarizations, seen in $\text{L}_{\text{OG}_6^s}$, are in the same $\text{Mon}^2(\text{L}_{\text{OG}_6^s})$-orbit.

The locally trivial monodromy group of primitive symplectic varieties of $\text{OG}_6^s$-type is known, and as in the case of $\text{OG}_6$-type ihs manifolds it turns out to be maximal.

**Proposition 4.1** ([MR19, Proposition 4.12]). Let $X$ be a primitive symplectic variety of $\text{OG}_6^s$-type, then
\[\text{Mon}^2(X) = O^+(H^2(X, \mathbb{Z})).\]

\(^7\)A divisor is positive if so is its first Chern class.
Thanks to the previous result, the following criterion gather a direct way to compute the Mon^2(\text{LOG}_6)-orbits of elements in the lattice \text{LOG}_6.

**Proposition 4.2** (Eichler’s criterion, [GHS09, Proposition 3.3(i)]). Let \( L \) be an even lattice that contains two orthogonal copies of the hyperbolic plane, and assume that the discriminant \( A_L \) is a cyclic group. If two vectors \( u \) and \( v \) in \( L \) have the same square and divisibility in \( L \), then there exist a transvection\(^8\) \( \tau \) such that \( \tau(u) = v \).

**Corollary 4.3.** Connected components of the moduli space \( \mathfrak{M}_{\text{OG}^6} \) are determined by the degree and the divisibility of the polarization of their elements.

**Proof.** Every transvection of \( \text{LOG}_6 \) is an orientation preserving isometry [GHS09, §3.1], hence a monodromy operator by **Proposition 4.1.** The lattice \( \text{LOG}_6 \) contains two orthogonal copies of the hyperbolic plane and it has cyclic discriminant group (see **Example 2.10**), hence we can apply **Proposition 4.2** obtaining that the orbit of a vector in \( \text{LOG}_6 \) with respect to the monodromy group is determined by its degree and divisibility. \( \square \)

### 4.2. Main results

We start analyzing the possible combinations of degree and divisibility appearing in the lattice \( \text{LOG}_6 \) introduced in **Example 2.10**. The goal of the first part of this section is to exhibit representatives for any possible combination of divisibility and positive square in \( \text{LOG}_6 \), constructed as the class of an (up to multiple) effective divisor on some \( \text{OG}^6 \) moduli space \( K_v \), see **Proposition 4.5.**

**Lemma 4.4.** Let \((X, h)\) be a polarized primitive symplectic variety of \( \text{OG}^6 \)-type, with BBF-form \( q_X \). We call \( q_X(h) = 2d \); if \( \text{div}(h) = 2 \) in \( H^2(X, \mathbb{Z}) \) then \( d \equiv 3 \mod 4 \).

**Proof.** Fix a marking \( \eta: H^2(X, \mathbb{Z}) \to \text{LOG}_6 \); by abuse of notations we call \( h \) the class \( \eta(h) \in \text{LOG}_6 \). Throughout the proof we assume \( \text{div}(h) = 2 \). Consider the class \([h/\text{div}(h)] \in \text{A}_{\text{LOG}_6}\). By (2), the possible values of the induced quadratic form \( q_{\text{A}_{\text{LOG}_6}} \) on \( \text{A}_{\text{LOG}_6} \) are \( \{0, \frac{1}{2}\} \subset \frac{\mathbb{Q}}{2\mathbb{Z}} \) and the only class of square 0 is the zero class of the discriminant group. Assume that \( d \equiv 0 \mod 4 \); it follows that

\[
q_{\text{A}_{\text{LOG}_6}} \left( \left[ \frac{h}{\text{div}(h)} \right] \right) = \frac{d}{2} = 0 \in \frac{\mathbb{Q}}{2\mathbb{Z}}
\]

hence \([h/\text{div}(h)]\) is the zero class of \( \text{A}_{\text{LOG}_6} \); we conclude that \([h/\text{div}(h)]\) is an element of the lattice \( \text{LOG}_6 \) hence \( \text{div}(h) = 1 \), which contradicts our assumption. If \( d \equiv 1 \) or \( 2 \mod 4 \) then the same computation shows that the class \([h/\text{div}(h)]\) has \( q_{\text{A}_{\text{LOG}_6}} \)-square \( \frac{1}{2} \) or \( 1 \in \frac{\mathbb{Q}}{2\mathbb{Z}} \) respectively, and we have already noticed that this case does not occur. \( \square \)

**Proposition 4.5.** Let \((X, h)\) be a polarized primitive symplectic variety of \( \text{OG}^6 \)-type, with BBF-form \( q_X \); we call \( q_X(h) = 2d \) and \( \text{div}(h) = e \) in \( H^2(X, \mathbb{Z}) \). Then there exists a \( \text{OG}^6 \) moduli space \( K_v \) and a class \( \alpha \in H^2(K_v, \mathbb{Z}) \) such that \( q_{K_v}(\alpha) = 2d \) and \( \text{div}(\alpha) = e \) in \( H^2(K_v, \mathbb{Z}) \), where \( q_{K_v} \) is the BBF-form on \( K_v \). Furthermore, \( \alpha \) or \( 2\alpha \) is an effective class.

**Proof.** Assume that \( \text{div}(h) = e = 1 \). Consider an abelian surface \( A \) of degree \( 2d \), i.e. having a polarization \( h_A \) of degree \( 2d \). The image of \( h_A \) via the primitive inclusion

\(^8\)For the definition of transvection, or Eichler transvection, we refer to [GHS09, Lemma 3.1]
$H^2(A, \mathbb{Z}) \hookrightarrow H^2(K_{(2,0,-2)}(A), \mathbb{Z})$ [Rap07, Theorem 3.5.1] gives an effective divisor of square $2d$ and divisibility 1 on the $\text{OG}_6^s$ moduli space $K_{(2,0,-2)}(A)$.

For the case $\text{div}(h) = e = 2$, we fix an abelian surface $A$ of degree 2, i.e. having a polarization $h_A$ of degree 2. By abuse of notation we call $h_A$ the image of it via the inclusion $\mu : H^2(A, \mathbb{Z}) \hookrightarrow H^2(K_{(2,0,-2)}(A), \mathbb{Z})$. Let $B \subseteq K_{(2,0,-2)}(A)$ be the Weil effective divisor parametrizing non-locally free sheaves and call $b := \frac{1}{2}c_1(2B)$. As consequence of [PR18, Theorem 2.4 and Theorem 2.5] we have:

$$H^2(K_{(2,0,-2)}(A), \mathbb{Z}) = \mu(H^2(A, \mathbb{Z})) \oplus \mathbb{Z} \cdot b$$

and the class $b$ has square -2. Taking $A$ general we can assume that $\text{NS}(K_{(2,0,-2)}(A)) = \mu(\text{NS}(A)) \oplus \mathbb{Z} \cdot b$ having BBF-matrix

$$\begin{pmatrix} h_A^2 & h_A \cdot b \\ h_A \cdot b & b^2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$  

For any odd integer $n \geq 1$ we consider the class

$$\alpha_n := (n + 1)h_A + nb \in H^2(K_{(2,0,-2)}, \mathbb{Z})$$

and observe that by construction $2\alpha_n$ is an effective class. Using the above written decomposition of the lattice $H^2(K_{(2,0,-2)}(A), \mathbb{Z})$ a straightforward computation gives that $\alpha_n$ has divisibility 2 in it. Writing $n = 2l + 1$ and $q_{K_{(2,0,-2)}}(\alpha_n) = 2d$ we have

$$d = 2n + 1 = 4l + 3 \equiv 3 \mod 4$$

and varying $l \in \mathbb{N}$ (hence $n \geq 1$ odd integer) we obtain all possible positive integers of this form. By Lemma 4.4 such $2d$ are the only possible square for a class of divisibility 2 on an $\text{OG}_6^s$-type primitive symplectic variety, hence the statement is proved.  

We are finally ready to prove our main theorem.

**Proof of Theorem 1.1.** For any $(X, h)$ as in the statement consider a pair $(K_v, \alpha)$ given by Proposition 4.5, i.e. where $K_v$ is a $\text{OG}_6^s$ moduli space and $\alpha$ is a effective class on $K_v$, having the same square and divisibility of $h$. Furthermore, $\varepsilon \alpha$ is an effective class, where $\varepsilon = 1$ if $\text{div}(\alpha) = 1$ and $\varepsilon = 2$ if $\text{div}(\alpha) = 2$. By Remark 3.6, there exists a polarized locally trivial deformation $(X', h')$ of $(K_v, \varepsilon \alpha)$ such that $h'$ is an ample class, with the same square and divisibility of $\varepsilon \alpha$. In conclusion, we have obtained a polarized primitive symplectic variety $(X', h')$ having the same numerical invariants of $(X, \varepsilon h)$ and that is a polarized locally trivial deformation of $(K_v, \varepsilon \alpha)$.

By Corollary 4.3 we have that $(X, \varepsilon h)$ and $(X', h')$ are in the same connected component of the moduli space $\mathcal{M}_{\text{OG}_6^s}$, hence we obtain one from the other with a polarized locally trivial deformation; we conclude that $(X, \varepsilon h)$ is also a polarized locally trivial deformation of $(K_v, \varepsilon \alpha)$. By Theorem 3.8 there exists a positive integer $m'$ such that the class $m' \varepsilon \alpha$ is the first Chern class of an irreducible uniruled divisor. We can finally apply [LMP21, Theorem 1.1], concluding that there exists a positive integer $k$ such that also the class $km' \varepsilon h$ is the first Chern class of a uniruled divisors, hence we have the statement for $m = km' \varepsilon$. 

\[\Box\]
4.3. The smooth case. It is natural to ask if our methods to prove Theorem 1.1 can be applied to obtain results about the existence of ample uniruled divisors on smooth $\text{OG}_6$-type ihs manifold. Also the smooth setting the monodromy group is maximal [MR19, Theorem 5.4], hence the same result as in Corollary 4.3 holds true for the moduli space of polarized ihs manifolds of $\text{OG}_6$-type. Following the proof of Theorem 1.1, what one needs to show is the existence of uniruled representatives in each positive effective class on a $\text{OG}_6$ moduli space $\tilde{K}_v$, in analogy with Theorem 3.8, combined with the fact that positive and (up to multiple) effective classes on $\tilde{K}_v$ cover all possible squares and divisibilities of polarizations on a $\text{OG}_6$-type ihs manifold, as in Proposition 4.5 for the singular case. This last point is indeed true, as we show in the following.

**Proposition 4.6.** Let $(X, h)$ be a polarized ihs manifold of $\text{OG}_6$-type, with BBF-form $q_X$; we call $q_X(h) = 2d$ and $\text{div}(h) = e$ in $H^2(X, \mathbb{Z})$. Then there exists a $\text{OG}_6$ moduli space $\tilde{K}_v$ and a class $\alpha \in H^2(\tilde{K}_v, \mathbb{Z})$ such that $q_{\tilde{K}_v}(\alpha) = 2d$ and $\text{div}(\alpha) = e$ in $H^2(\tilde{K}_v, \mathbb{Z})$, where $q_{\tilde{K}_v}$ is the BBF-form of $\tilde{K}_v$. Furthermore, $\alpha$ or $2\alpha$ is an effective class.

**Proof.** The proof follows the same steps as the proof of Proposition 4.5. As first, we prove an analogue of Lemma 4.4. By [Rap07, Theorem 3.5.1] the lattice of an $\text{OG}_6$-type ihs manifold is isomorphic to the lattice

$$L_{\text{OG}_6} := U^{\oplus 3} \oplus [-2] \oplus [-2].$$

We call $\sigma_1$ and $\sigma_2$ the generators of square $-2$ of the rank-one lattices in the decomposition above, and $A_{\text{OG}_6}$ the discriminant group of $L_{\text{OG}_6}$. Hence $A_{\text{OG}_6} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is generated by $[\sigma_1/\text{div}(\sigma_1)]$ and $[\sigma_2/\text{div}(\sigma_2)]$, and its BBF-form is given in this basis by the matrix

$$[A_{\text{OG}_6}] = \begin{pmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \in M_{2,2}(\mathbb{Q}/2\mathbb{Z}).$$

Let $h \in L_{\text{OG}_6}$ be an element of divisibility equal to $2$, and call $q_{L_{\text{OG}_6}}(h) = 2d$. Then $d \equiv 2, 3 \pmod{4}$: possible squares in $A_{\text{OG}_6}$ are $\{0, 1, \frac{3}{2}\} \subseteq \mathbb{Q}/2\mathbb{Z}$, hence arguing as in the proof of Lemma 4.4 we can exclude this time only that $d \equiv 0, 1 \pmod{4}$, getting our assertion.

The class $\alpha$ as in the statement is obtained in the case of $\text{div}(\alpha) = 1$ or $\text{div}(\alpha) = 2$ and $d \equiv 3 \pmod{4}$ using the inclusion of lattices $\tilde{\pi}^* : H^2(K_v, \mathbb{Z}) \hookrightarrow H^2(\tilde{K}_v, \mathbb{Z})$ (see Remark 2.9) and then Proposition 4.5. The only remaining case is the one of divisibility $2$ and $d \equiv 2 \pmod{4}$, and we proceed as follows. Again by [Rap07, Theorem 3.5.1] we have

$$H^2(\tilde{K}_{(2,0,-2)}, \mathbb{Z}) = \mu(H^2(A, \mathbb{Z})) \oplus \mathbb{Z} \cdot (\tilde{b} + a) \oplus \mathbb{Z} \cdot a$$

where $\mu$ is an inclusion of lattices, $a$ is the first Chern class of half of the exceptional divisor of the resolution $\tilde{\pi} : \tilde{K}_{(2,0,-2)} \to K_{(2,0,-2)}$ and $\tilde{b}$ is the first Chern class of the strict transform via $\tilde{\pi}$ of the divisor of non-locally free sheaves in $K_{(2,0,-2)}$. Furthermore, this decomposition gives the isomorphism with the lattice $L_{\text{OG}_6}$, respecting the direct sum decompositions of the lattices as written above. Let $A$ be an abelian surface of degree $2$ and let $h_A \in NS(\tilde{K}_{(2,0,-2)}(A))$ be the image of this polarization via $\mu$. For any odd integer $n \geq 1$ we finally consider the class

$$\beta_n := (n + 1)h_A + n(\tilde{b} + a) + a$$
and observe that $2\beta_n$ is effective; a straightforward computation gives $\text{div}(\beta_n) = 2$. Writing $n = 2l + 1$ and $q_{K_{(2,0,-2)}}(\beta_n) = 2d$ we have
\[ d = 2n = 4l + 2 \equiv 2 \mod 3 \]
and varying $l \in \mathbb{N}$ we obtain all possible positive integer of this form, hence the statement is proved. \hfill \Box

To conclude, we show why our method to prove Theorem 3.8 breaks down in the smooth case, going over the steps of its proof in the singular setting. Our starting point has been the regular morphism $\Phi_v : Y_v \to K_v$ defined in (3), that we used to pullback an effective divisor $D$ on $K_v$ to obtain that it has a positive multiple linearly equivalent to an irreducible uniruled divisor. The irreducibility of the divisor is crucial, as the ruling curve needs to be reduced and irreducible to apply the deformation result [LMP21, Theorem 1.1] in the proof of Theorem 1.1. The uniruled divisor in $K_v$ is obtained as image of a uniruled divisor on $Y_v$, whose existence is ensured by [CMP19, Theorem 1.1] but whose explicit definition is not given. A natural way to transfer this argument to the smooth setting is to consider the rational map
\[
\tilde{\Phi}_v : Y_v \dashrightarrow \tilde{K}_v
\]
given by the composition of $\Phi_v$ with the inverse of the birational morphism $\tilde{\pi} : \tilde{K}_v \to K_v$, hence in other words, to pullback to $\tilde{K}_v$ the uniruled divisor on $K_v$ via the resolution $\tilde{\pi}$. The problem of this strategy is that we have no control on the irreducibility of the resulting divisors: a pulled-back divisor could contain $m$ copies of the (uniruled) exceptional divisor $\tilde{\Sigma}_v \subseteq \tilde{K}_v$, as a uniruled divisor on $K_v$ could contain the singular locus $\Sigma_v \subseteq K_v$. Since we have no explicit description of the uniruled divisors on $Y_v$, and hence on $K_v$, we can not control whether the uniruled divisors on $K_v$ contain $\Sigma_v$ or not. If we knew the number $m$ of copies of $\tilde{\Sigma}_v$ in the pullback of a uniruled divisor, we could produce a new irreducible uniruled divisor (the one obtained subtracting $m$ copies of $\tilde{\Sigma}_v$ from the pullback). Anyway in order to compute the lattice invariants of such divisor we need to know its cohomology class, i.e. the integer $m$, as we know the lattice invariants of the pulled-back divisors.

We also remark that in our case we can not use [LMP21, Proposition 4.5], as it only holds true for singular moduli spaces with terminal singularities, which is not the case for $K_v$, see [PR18, §1.1].

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