Perturbative matching of lattice and continuum heavy-light currents with NRQCD heavy quarks

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The temporal and spatial components of the heavy-light vector current and the spatial components of the axial current are expressed in terms of lattice-regulated operators suitable for simulations of $B$ and $D$ mesons. The currents are constructed by matching the appropriate scattering amplitudes in continuum QCD and a lattice model to one-loop order in perturbation theory. In the lattice theory, the heavy quarks are treated using the nonrelativistic (NRQCD) formulation and the light quarks are described by the tadpole-improved clover action. The light quarks are treated as massless. Our currents include relativistic and discretization corrections through $O(a_s/M, a/M, \alpha_s^2)$, where $M$ is the heavy-quark mass, $a$ is the lattice spacing, and $\alpha_s$ is the QCD coupling. As in our previous construction of the temporal component of the heavy-light axial current, mixing between several lattice operators is encountered at one-loop order, and $O(a_s M_0)$ dimension-four improvement terms are identified.

I. INTRODUCTION

An important goal of lattice gauge theory is to provide estimates of the hadronic matrix elements, such as those of the electroweak currents and effective four-fermion operators, which are needed for precision tests of the standard model. Numerical simulations of quarks and gluons in lattice-regulated QCD currently provide the only means of calculating such matrix elements from first principles taking long-distance QCD dynamics fully into account. An important ingredient in these calculations is the construction of the lattice current operators which match the currents defined in the continuum to some desired accuracy. In this paper, we express the heavy-light vector and axial-vector currents in terms of operators defined in a lattice model in which the heavy quarks are treated using the nonrelativistic (NRQCD) formulation, the light quarks are described by the tadpole-improved clover action, and the gluons are governed by the standard Wilson action. These heavy-light currents are important in studies of heavy meson leptonic and semileptonic decays (for recent reviews, see Refs. [2,3]). The lattice currents are determined by matching scattering amplitudes in the lattice model to those in continuum QCD to one-loop order in perturbation theory. The matching is carried out through $O(1/M, \alpha_s/M, a \alpha_s)$ where $M$ is the heavy quark mass, $a$ is the lattice spacing, and $\alpha_s$ is the QCD coupling. The light quarks are treated as massless.

Each heavy-light current $J_\mu$, defined in some continuum renormalization scheme, can be written

$$J_\mu = \sum_j C_j^\mu \cdot J_{\mu, \text{lat}} + O(\alpha_s^2, a^2/2M_0^2, \alpha_s a/M_0)$$

where $J_{\mu, \text{lat}}$ are operators defined in the effective lattice theory, $M_0$ is the heavy quark (bare) mass parameter appearing in the lattice NRQCD action, and the $C_j^\mu$ coefficients are $c$-numbers which depend only on $\alpha_s$ and $aM_0$. The goal is to identify the operators $J_{\mu, \text{lat}}$ and to calculate the matching coefficients $C_j^\mu(\alpha_s, aM_0)$. The procedure for doing this was described in a previous article in which the temporal component of the axial-vector current was studied. For convenience, we reiterate the salient steps: (1) select a quark-gluon scattering process induced by the heavy-light current of interest and calculate the one-loop amplitude for this process in continuum QCD; (2) expand the amplitude in powers of $1/M$; (3) identify operators in the lattice theory, usually by inspection, which reproduce the terms in this expansion; (4) calculate the one-loop mixing matrix of these operators in the lattice theory; and (5) adjust the $C_j^\mu$ coefficients to produce a linear combination of lattice current operators whose one-loop scattering amplitude agrees with that determined in step 2 to a given order in $1/M$ and $a$. In this paper, we apply the above procedure to the spatial components $A_k$ of the axial-vector current and the spatial and temporal components of the vector current.
Vμ. We omit much of the calculational details since they are similar to those already described in Ref. 3.

In Sec. II, we start from a continuum QCD calculation and identify the current operators in the effective theory needed to reproduce the continuum current through O(αs/M). In Sec. III, we describe the lattice theory and the one-loop mixing calculation among the ψ(μ). The matching procedure is then completed in Sec. IV. In Sec. V, we discuss O(aαs) corrections in the static limit, and issues pertaining to terms that behave as αs log(aM) are dealt with in Sec. VI. The paper concludes with a summary of our results in Sec. VII.

II. OPERATOR IDENTIFICATION

The heavy-light vector and axial-vector currents are given as usual by Vμ(x) = ̅q(x) γμ h(x) and Aμ(x) = ̅q(x) γ5 γμ h(x), respectively, where q(x) is the light quark field, h(x) is the heavy quark field, and γμ are the standard Dirac γ-matrices in Euclidean space-time which satisfy {γμ, γν} = 2 δμν, and γ5 = γ0 γ1 γ2 γ3. Euclidean-space four-vectors are defined in terms of Minkowski-space four-vectors (indicated by an underline) using x0 = i ̅0 and xj = xj = -xj, for j = 1, 2, 3. For the derivative operator, ∂0 = -i ̅0 and ∂j = ̅j = ̅j. The currents are related by J0 = J0, and Jj = -i ̅j. For further details, see Refs. 6, 7.

The first step in identifying the necessary operators in the effective lattice theory is to calculate the matrix elements of the continuum QCD currents for an incoming heavy quark of momentum p and an outgoing light quark of momentum p′. Using the on-shell mass and wave function renormalization scheme in Feynman gauge and expanding in 1/M (except for u(p)), one finds to one-loop order in perturbation theory

\[ \langle q(p') | J_\mu | h(p) \rangle_{QCD} = \bar{u}_q(p') W_\mu^J(p', p) u_h(p), \]

where

\[
W_\mu^J(p', p) = a_1 \Gamma_\mu - a_2 \frac{i p_\mu}{M} \Gamma_0 \gamma_0 - a_3 \frac{p \cdot p'}{M^2} \Gamma_\mu
- a_4 \frac{i p_\mu'}{M} \Gamma_0 \gamma_0 + a_5 \frac{p' \cdot p}{M^2} \Gamma_\mu \gamma_0 + O(1/M^2),
\]

with

\[
a_1 = 1 + \frac{\alpha_s}{3\pi} \left[ 3 \ln M - \frac{11}{4} \right],
\]

\[
a_2 = \frac{\alpha_s}{3\pi} 2,
\]

\[
a_3 = \frac{\alpha_s}{3\pi} \left[ 6 \ln M - \frac{8\pi}{3} \frac{M}{\lambda} + \frac{1}{2} \right],
\]

\[
a_4 = \frac{\alpha_s}{3\pi} \left[ -2 \ln \frac{M}{\lambda} + \frac{1}{2} \right],
\]

\[
a_5 = \frac{\alpha_s}{3\pi} \left[ -4 \ln \frac{M}{\lambda} + 5 \right].
\]

For the vector current, Γμ = γμ, and for the axial-vector current, Γμ = γ5 γμ. uh(p) and u(p) are the standard spinors for the heavy and light quarks, respectively, which satisfy the Dirac equation. The light quark mass is set equal to zero. Ultraviolet divergences are treated using dimensional regularization, and fully anti-commuting γ5 matrices are used. We use a gluon mass λ to regulate infrared divergences. Note that M is the heavy-quark pole mass.

In lattice NRQCD, the heavy quark is described in terms of a two-component (in spin space) field ψ(x). The Dirac field h(x) is related to ψ(x) (and the antiquark field ψ(x)) by a unitary Foldy-Wouthuysen transformation:

\[ h(x) = U_{FW}^{\dagger} \left( \psi(x) \right) \psi(x) . \]

This transformation decouples the upper and lower components of the Dirac field, thereby separating the quark field from the antiquark field. To facilitate the identification of lattice NRQCD operators capable of matching Eq. 2, we similarly transform the external state spinor u(p) into a nonrelativistic Pauli spinor:

\[ u_h(p) = \left[ 1 - i \frac{\gamma \cdot p}{2M} \right] u_Q(p) + O(1/M^2), \]

where

\[ u_Q(p) = \begin{pmatrix} U_Q \\ 0 \end{pmatrix}, \]

and UQ is a two-component external state spinor depending only on the spin of the heavy quark. Note that we are working in the Dirac-Pauli representation. Using Eq. 4, the relation γ0 u(p) = u(p), and the Dirac equation for the light quark ̅u(p')p0 = -̅u(p')γ0 p' γ0, the spatial components of Eq. 5 may be written

\[ \langle q(p') | J_k | h(p) \rangle_{QCD} = \eta_0 \Omega_k^{(0)} + \eta_1 \Omega_k^{(1)} + \eta_2 \Omega_k^{(2)} + \eta_3 \Omega_k^{(3)} + \eta_4 \Omega_k^{(4)} + O(\alpha_s^2, 1/M^2), \]

with

\[
\Omega_k^{(0)} = ̅u_q(p') \Gamma_k^f u_Q(p),
\]

\[
\Omega_k^{(1)} = -i ̅u_q(p') \Gamma_k^f \gamma \cdot \frac{p}{2M} u_Q(p),
\]

\[
\Omega_k^{(2)} = i ̅u_q(p') \frac{\gamma \cdot p'}{2M} \gamma_0 \Gamma_k^f u_Q(p),
\]

\[
\Omega_k^{(3)} = -i ̅u_q(p') \frac{p}{2M} \gamma_0 \Gamma_k^f u_Q(p),
\]

\[
\Omega_k^{(4)} = -i ̅u_q(p') \frac{p}{2M} \Gamma_0^f u_Q(p).
\]
We use the bare quark mass or following five lattice operators should be considered:

\[ J_{k,\text{lat}}(x) = \bar{q}(x) \Gamma^j_k Q(x), \]

where

\[ q(x) \text{ is now the light quark field in the lattice theory, } M_0 \text{ is the bare heavy quark mass, and } Q(x) \text{ is related to the heavy quark field } \psi(x) \text{ in lattice NRQCD by} \]

\[ Q(x) = \begin{pmatrix} \psi(x) \\ 0 \end{pmatrix}. \]

We use the bare quark mass \( M_0 \) in the above definitions since it is the natural choice to use in lattice simulations and because the pole mass \( M \) is not well defined beyond perturbation theory. Definitions of the lattice derivatives are given in the next section.

For the temporal components of the currents \( J_0 = V_0 \) or \( A_0 \), the leading \( 1/M \) behavior is given by

\[ \langle q(p') | J_0 | h(p) \rangle_{\text{lat}} = \eta_0 \Omega_0^{(0)} + \eta_1 J_0 \Omega_1^{(1)} + \eta_2 J_0 \Omega_2^{(2)} + O(\alpha_s^2, 1/M^2), \]

with

\[ \Omega_0^{(0)} = \bar{u}(p') \Gamma_0^j u_Q(p), \]

\[ \Omega_0^{(1)} = -i \bar{u}(p') \Gamma_0^j \frac{\gamma^\mu P}{2M} u_Q(p), \]

\[ \Omega_2^{(2)} = i \bar{u}(p') \gamma^\mu P' \gamma_0 \Gamma_0^j u_Q(p). \]

The coefficients in Eq. (22) are given by

\[ \eta_0 = (a_1 + a_2) = 1 + \alpha_s \tilde{B}_0, \]

\[ \eta_1 = (a_1 - a_2) = 1 + \alpha_s \tilde{B}_1, \]

\[ \eta_2 = 2(a_3 + a_4 + a_5) = \alpha_s \tilde{B}_2, \]

where

\[ \tilde{B}_0 = \frac{1}{3\pi} \left[ 3 \ln \frac{M}{\lambda} - \frac{11}{4} \right], \]

\[ \tilde{B}_2 = \frac{1}{3\pi} \left[ 12 \ln \frac{M}{\lambda} - \frac{16\pi M}{3} + 1 \right], \]

\[ \tilde{B}_4 = \frac{1}{3\pi} \left[ -4 \ln \frac{M}{\lambda} + 1 \right]. \]

Having obtained the \( 1/M \) expansion of the above scattering amplitude in continuum QCD, the next step is to identify operators in the lattice effective theory that can reproduce the terms in this expansion. An inspection of Eq. (8) suggests immediately that matrix elements of the following five lattice operators should be considered:

\[ J_{0,\text{lat}}(x) = \bar{q}(x) \Gamma_0^j Q(x), \]

\[ J_{1,\text{lat}}(x) = \bar{q}(x) \Gamma_0^j \gamma_\alpha \nabla Q(x), \]

\[ J_{2,\text{lat}}(x) = \bar{q}(x) \Gamma_0^j \nabla \gamma_\alpha \gamma_\beta \nabla Q(x), \]

\[ J_{3,\text{lat}}(x) = \bar{q}(x) \Gamma_0^j \nabla \gamma_\alpha \gamma_\beta \gamma_\gamma \nabla Q(x), \]

\[ J_{4,\text{lat}}(x) = \bar{q}(x) \gamma_\gamma \nabla \gamma_\alpha \gamma_\beta \nabla \gamma_\gamma \nabla \Gamma_0^j Q(x), \]

\[ J_{5,\text{lat}}(x) = \bar{q}(x) \gamma_\gamma \nabla \gamma_\alpha \gamma_\beta \nabla \gamma_\gamma \nabla \Gamma_0^j \Gamma_0^j Q(x), \]

\[ J_{6,\text{lat}}(x) = \bar{q}(x) \gamma_\gamma \nabla \gamma_\alpha \gamma_\beta \nabla \gamma_\gamma \nabla \Gamma_0^j \Gamma_0^j \Gamma_0^j Q(x). \]

These operators were previously used in the expansion of the temporal component of the axial-vector heavy-light current in Refs. [8].

### III. MIXING MATRIX

Having identified the necessary operators in the lattice-regulated effective field theory, we next need to calculate the mixing matrix \( Z_{ij} \) of these operators defined by

\[ \langle q(p') | J_{\mu,\text{lat}}^{(i)} | h(p) \rangle_{\text{lat}} = \sum_j Z_{ij}^{(i)} \Omega_0^{(j)} + O(\alpha_s^2, 1/M^2), \]

where \( \Omega_0^{(j)} \) are the five \( \Omega_0^{(j)} \) defined in Eqs. (8)-(13) for \( \mu = k = 1, 2, 3 \) and the three \( \Omega_0^{(j)} \) defined in Eqs. (23)-(25) for \( \mu = 0 \). The determination of the \( Z_{ij}^{(i)} \) factors is the most difficult aspect of the current construction. These factors are determined numerically using lattice perturbation theory. The same methods used in Ref. [8] are applied here. We also use the same lattice action as in Ref. [8], but for the convenience of the reader, we
restate these actions below. The Feynman rules for our heavy and light quark lattice action and further details of lattice perturbation theory are given in Refs. 13.

For the heavy quark, the following NRQCD action density on the lattice [4] is used:

\[ a_{\text{NRQCD}} = \psi^1(x) \psi(x) \]

\[ \psi^1(x + a_i) \left( 1 - \frac{a \delta H}{2} \right) \left( 1 - \frac{a H_0}{2n} \right)^n \frac{U^\dagger(x)}{u_0} \]

where

\[ H_0 = -\frac{\Delta^{(2)}}{2M_0}, \]

\[ \delta H = -c_B \frac{g}{2M_0} \sigma \cdot B, \]

\[ U_{\mu}(x) \]

are standard gauge-field link variables, \( n \) is an integer, \( \sigma_j \) are the Pauli matrices, \( u_0 \) is the mean link parameter introduced by the tadpole improvement procedure [14], and the QCD coupling \( g \) is related to \( \alpha_s \) by \( \alpha_s = g^2/(4\pi) \). At tree level, \( c_B = 1 \); the one-loop contribution to \( c_B \) is an \( O(\alpha^2) \) effect in our mixing calculation and hence can be ignored here. The chromomagnetic field is given by \( B_j(x) = -\frac{1}{2} \epsilon_{jlm} F_{lm}(x) \), where the Hermitian and traceless field strength tensor \( F_{\mu\nu}(x) \) is defined at the sites of the lattice in terms of clover-leaf operators:

\[ F_{\mu\nu}(x) = F_{\mu\nu}(x) - \frac{1}{3} \text{Tr} F_{\mu\nu}(x), \]

\[ F_{\mu\nu}(x) = \frac{-i}{2a^2 g} \left( \Omega_{\mu\nu}(x) - \Omega_{\mu\nu}^\dagger(x) \right), \]

\[ \Omega_{\mu\nu}(x) = \frac{1}{4 u_0^2} \sum_{\{(\alpha, \beta)\}_{\mu\nu}} U_{\alpha}(x) U_{\beta}(x + a_\alpha) \]

\[ \times U_{-\alpha}(x + a_\alpha + a_\beta) U_{-\beta}(x + a_\beta), \]

with \( \{(\alpha, \beta)\}_{\mu\nu} = \{(\mu, \nu), (\nu, -\mu), (-\mu, -\nu), (-\nu, \mu)\} \) for \( \mu \neq \nu \). The lattice derivatives in our action and in the current operators are given by

\[ a\nabla_\mu O(x) = \frac{1}{2u_0} \left[ U_{\mu}(x) O(x + a_\mu) \right. \]

\[ -U_{\mu}^\dagger(x - a_\mu) O(x - a_\mu), \]

\[ O(x) a\nabla_\mu = \frac{1}{2u_0} \left[ U_{\mu}(x + a_\mu) O(x + a_\mu) \right. \]

\[ -O(x - a_\mu) U_{\mu}(x - a_\mu), \]

\[ a^2 \Delta^{(2)} O(x) = \sum_{k=1}^3 (u_0^{-1} U_k(x) O(x + a_k) \]

\[ + U_k^\dagger(x - a_k) O(x - a_k) - 2O(x)), \]

\[ a^2 \nabla^{(2)} O(x) = \sum_{\mu=0}^3 (u_0^{-1} U_{\mu}(x) O(x + a_\mu) \]

\[ + U_{\mu}^\dagger(x - a_\mu) O(x - a_\mu) - 2O(x)), \]

where \( O(x) \) is an operator defined at lattice site \( x \) with appropriate color structure.

For the light quarks, we use the clover action [2],

\[ a_{\text{light}} = \overline{q} \partial \overline{q} - \frac{r}{2} \overline{q} \partial^{(2)} q + m_0 \overline{q} q, \]

\[ -iga r \sum_{\mu,\nu} \overline{q} \gamma_\mu F_{\mu\nu} q, \]

where \( \overline{q} = \sum_\mu \gamma_\mu \overline{q} \), \( m_0 \) is the bare light quark mass, \( \gamma_\mu = \frac{1}{2} [\gamma_\mu, \gamma_\nu] \), and we set the Wilson parameter \( r = 1 \).

The one-loop correction to the clover coefficient is an \( O(\alpha^2) \) effect in our matching calculation and is neglected here.

At one-loop order, the mixing matrix elements may be written [4]

\[ Z_{ij} = \delta_{ij} + \alpha_s \left\{ \left[ (\tilde{C}_q + \tilde{C}_Q) + C_m(1 - \delta_{i0}) \right] \delta_{ij} + \tilde{c}_{ij} \right\}, \]

\[ \tilde{C}_q \text{ and } \tilde{C}_Q \text{ are the contributions from the light-} \]

and heavy-quark external leg corrections (that is, from wave function renormalization factors), and \( \tilde{c}_{ij} \) denote the contributions from the vertex corrections. Our use of an on-shell renormalization scheme with lattice operators defined in terms of the bare mass \( M_0 \) is responsible for the term proportional to \( C_m \), where

\[ M = [1 + \alpha_s C_m] M_0 + \alpha(\alpha_s^2). \]

Note that although the current operators \( J_{\mu}^{(1)} \) are defined using \( M_0 \), the pole mass \( M \) appears in \( \Omega_{\mu}^{(2)} \). Within the context of finite-order perturbation theory, the pole mass is an observable quantity. It is gauge invariant and regularization scheme independent. Hence, it is the natural heavy-quark mass definition to use when matching amplitudes in different schemes. However, in reality, a confined quark has no pole mass. Once the matching coefficients \( C_j \) are obtained in terms of \( aM \), it is then necessary to eliminate any dependence on \( M \) in favor of some quark mass which is defined beyond perturbation theory. Clearly, the mass parameter \( M_0 \) appearing in the lattice action is the most suitable and convenient choice. The relationship between \( M \) and \( M_0 \) given in Eq. (42) is then used to re-express the \( C_j \) coefficients in terms of \( M_0 \).

The factors in Eq. (41) may be further decomposed:

\[ \tilde{C}_q = C_q + \frac{2}{3\pi} \ln a \lambda + C_q^{\text{TI}}, \]

\[ \tilde{C}_Q = C_Q - \frac{4}{3\pi} \ln a \lambda, \]

\[ \tilde{C}_m = C_m + C_m^{\text{TI}}, \]

\[ \tilde{c}_{ij} = \delta_{ij} + \delta_{ij}^{\text{TI}} + \delta_{ij}^{\text{IR}}, \]

(43)
where $C_q$, $C_Q$, $C_m$, and $\zeta_{ij}$ are infrared finite and independent of the tadpole improvement factor $u_0$, and $\zeta_{ij}^{IR}$ and $\zeta_{ij}^{TI}$ contain the infrared divergences and tadpole improvement contributions, respectively, from the vertex corrections. Contributions to $C_q$ and $C_m$ from the tadpole improvement counterterms are denoted by $C_q^{TI}$ and $C_m^{TI}$, respectively. The tadpole improvement terms are

$$C_q^{TI} = -u_0^{(2)}$$

$$C_m^{TI} = -u_0^{(2)} \left(1 - \frac{3}{2 \pi a M_0}\right)$$

$$\zeta_{ij}^{J,TI} = u_0^{(2)} \delta_{ij} (1 - \delta_{0i})$$

where $u_0 = 1 - \alpha_s u_0^{(2)} + O(\alpha_s^2)$. For the usual plaquette definition $u_0 = \langle 4 \text{Tr}\{U\} \rangle^{1/4}$ in the Wilson gluonic action, $u_0^{(2)} = \pi/3$. For massless clover quarks, $C_q = 1.030$. Values for $C_Q$ and $C_m$ are given in Ref. 4.

IV. MATCHING

To complete the operator construction for $J_k$, we invert Eq. (31) to get $\Omega_{ji}^{(u)} = \sum_{j} (Z^{(u)})_{ji}^{-1} (j^{(1)})_{\mu,\nu}^{(u)}$, substitute this into Eq. (8), peel off the dependence on the external states, and use Eqs. (14), (15), (41), (42), and (43) to obtain

$$J_k = \left(1 + \alpha_s \left[ B_0 - C_{Qq} - \tau_0^{J_k} \right] \right) J_{k,\text{lat}}^{(0)}$$

$$+ \left(1 + \alpha_s \left[ B_1 - C_{Qqm} - \tau_1^{J_k} - \tau_1^{TI} \right] \right) J_{k,\text{lat}}^{(1)}$$

$$+ \alpha_s \left[ B_2 - \tau_2^{J_k} \right] J_{k,\text{lat}}^{(2)}$$

$$+ \alpha_s \left[ B_3 - \tau_3^{J_k} \right] J_{k,\text{lat}}^{(3)}$$

$$+ \alpha_s \left[ B_4 - \tau_4^{J_k} \right] J_{k,\text{lat}}^{(4)}$$

$$+ O(\alpha_s^2, a^2, 1/M^2, \alpha_s a/M)$$

where $\tau_k^{J_k} = \rho_0^{J_k} + \zeta_{ij}^{J_k}$, $\tau_1^{TI} = u_0^{(2)}$.

As expected, the infrared divergences from the various terms in the expansion coefficients cancel. Results for $\tau_0^{V_k}$ and $\tau_0^{A_k}$ for various values of $a M_0$ are listed in Tables II and II, respectively.

For the temporal component of the vector current, we substitute the inverted Eq. (32) into Eq. (22), remove the dependence on the external states, and use Eqs. (26), (27), (12), and (13) to obtain

$$V_0 = \left(1 + \alpha_s \left[ B_0^0 - C_{Qq} - \tau_0^{V_0} \right] \right) J_{0,\text{lat}}^{(0)}$$

$$+ \left(1 + \alpha_s \left[ B_1^0 - C_{Qqm} - \tau_1^{V_0} - \tau_1^{TI} \right] \right) J_{0,\text{lat}}^{(1)}$$

$$+ \alpha_s \left[ B_2^0 - \tau_2^{V_0} \right] J_{0,\text{lat}}^{(2)}$$

$$+ O(\alpha_s^2, a^2, 1/M^2, \alpha_s a/M)$$

where

$$B_0^0 = \frac{1}{\pi} \left[ \ln(a M_0) - \frac{11}{12} \right]$$

$$B_1^0 = \frac{1}{\pi} \left[ 4 \ln(a M_0) + \frac{1}{3} \right]$$

$$B_2^0 = \frac{4}{3\pi}$$

$$B_3^0 = \frac{4}{\pi} \left[ -\frac{4}{3} \ln(a M_0) + \frac{1}{3} \right]$$

$$B_4^0 = \frac{4}{\pi} \left[ -\frac{4}{9} \ln(a M_0) + \frac{1}{3} \right]$$

Again, all infrared divergences cancel. Results for $\tau_0^{V_0}$ for various $a M_0$ values are given in Table II.

From Eqs. (43) and (49), one easily obtains the matching coefficients $C^J_i$ of Eq. (1) which expresses the heavy-light currents in terms of operators suitable for numerical simulations using lattice NRQCD. These coefficients can be written in the form

$$C^J_i(\alpha_s, a M_0) = \begin{cases} 1 + \alpha_s \rho_i^J(a M_0) + O(\alpha_s^2), & (i = 0, 1), \\ \alpha_s \rho_i^J(a M_0) + O(\alpha_s^2), & (i \geq 2). \end{cases}$$

The values for $\rho_i^{V_k}$, $\rho_i^{A_k}$, and $\rho_i^{V_0}$ for various values of $a M_0$ are listed in Tables III, IV, and IV, respectively.

V. $O(\alpha s^2)$ CORRECTIONS

In this section, discretization corrections in the static limit $a M_0 \to \infty$ are discussed. To study such corrections, we calculated all $\tau^J_i$ for several large values of the heavy quark mass. With the exception of $\tau_2^J$, all of the $\tau_i^J$ tended tamely to finite values in the static limit. However, as $a M_0$ became large, we found that the magnitude of $\tau_2^J$ began to grow linearly with $a M_0$:

$$\tau_2^J \frac{a M_0 \to \infty}{a M_0} - 2 a M_0 c_{disc}^J + \text{constant}$$
where the ζ_j^{disc} are constants independent of aM_0. We determined the ζ_j^{disc} by computing τ_j aM_0 = 10.0, 25.0, 100.0, 400.0, 1000.0 and 5000.0. We then fit the results for \( -τ_j^2/(2aM_0) \) to a quadratic polynomial in \( 1/(aM_0) \) and obtained the following limiting values:

\[
\begin{align*}
ζ_{\text{disc}}^0 &= 1.029(2), \\
ζ_{\text{disc}}^V &= 1.031(1), \\
ζ_{\text{disc}}^A &= -0.063(1), \\
ζ_{\text{disc}}^V_0 &= -0.063(1).
\end{align*}
\]

The discretization factor ζ_j^{disc} for the temporal component of the axial-vector current was previously calculated in Ref. [6], but its determination was not done very accurately. We have recalculated ζ_j^{disc} here to a much higher precision.

Since the contributions proportional to ζ_j^{disc} are purely \( O(a\alpha_s) \) discretization corrections, one may absorb these terms into the lattice current operators. Improved current operators can be defined using

\[
\begin{align*}
J_{\mu,\text{lat}}(x) &= J^{(0)}_{\mu,\text{lat}}(x) + \alpha_s \, ζ_j^{disc} \, J^{(\text{disc})}_{\mu,\text{lat}}(x), \\
J_{\mu,\text{lat}}(x) &= J^{(j)}_{\mu,\text{lat}}(x) \quad (j > 0),
\end{align*}
\]

where

\[
J^{(\text{disc})}_{\mu,\text{lat}}(x) = 2aM_0 \, J^{(2)}_{\mu,\text{lat}}(x),
\]

\[
= -a \, q(x) \, γ \cdot \vec{D} \, γ_0 \, Γ_μ^{(j)}(x).
\]

The mixing of the improved operators is given by

\[
\langle q(p') | \hat{J}^{(j)}_{\mu,\text{lat}} | h(p) \rangle_{\text{lat}} = \sum_j Z_{ij}^j \, Ω_j^{(j)} \, \gamma \cdot \vec{D} \, γ_0 \, Γ_μ^{(j)}(x) + O(\alpha_s^2, a^2 M^2),
\]

where

\[
\begin{align*}
Z_{ii}^j &= Z_{ii}^0 + 2\alpha_s aM_0ζ_j^{disc} \, δ_{2i}, \\
Z_{jj}^j &= Z_{jj}^0 \quad (j > 0).
\end{align*}
\]

Then the coefficients \( \hat{C}_j^j \) of these operators are

\[
\begin{align*}
\hat{C}_2^j &= C^j_2 - 2aM_0ζ_{\text{disc}}^j \, α_s, \\
\hat{C}_4^j &= C^j_4, \quad (i \neq 2),
\end{align*}
\]

and the heavy-light currents are given by

\[
J_j = \sum_j \hat{C}_j^j \, J^{(j)}_{\mu,\text{lat}} + O(\alpha_s^2, a^2 M^2, α_s a/M_0).
\]

Terms which grow linearly with aM_0 are no longer present in \( \hat{C}_j^j \) and \( \hat{Z}_{ij}^j \). Note that in the lattice NRQCD approach, discretization and relativistic corrections are intertwined since \( O(a) \) and \( O(1/M) \) interactions are treated as equally important.

**VI. LARGE LOGARITHMS**

Another feature of the \( \hat{C}_j^j \) matching coefficients is the presence of \( \ln(aM_0) \) terms. The simulations of heavy-light systems carried out to date have used bare heavy quark mass values in the range \( aM_0 \sim 1.6 - 4.0 \) where these logarithms are not large. However, in the large \( aM_0 \) limit, these logarithms must be treated with care. The logarithms appearing in the coefficients \( \hat{C}_j^j \) for \( j > 0 \) are tamed by the \( 1/(aM_0) \) factors in their corresponding current operators. However, the logarithm appearing in the \( \hat{C}_0^j \) coefficient becomes problematic and must be dealt with using the renormalization group.

In matching calculations between QCD and various continuum effective theories, one usually encounters similar logarithms of the form \( \ln(M/μ) \), where \( μ \) is some scale introduced by the renormalization procedure. Such logarithms are summed using the renormalization group (RG) equations which follow from the requirement that physical quantities must not depend on \( μ \). Since \( μ \) appears only inside the logarithms, one ends up with simple anomalous-dimension matrices and RG equations which can be solved straightforwardly. The situation is more complicated in the present calculation. The role of \( μ \) is taken over by the inverse lattice spacing \( 1/a \), and \( a \) appears not only in the logarithms, but also in other places, such as the \( τ_j^j \) which are complicated functions of \( aM_0 \). Furthermore, the ultraviolet cutoff \( 1/a \) is an integral part of our effective theory and we cannot take \( a \to 0 \).

As discussed in Ref. [8], the observation that the left-hand side of Eq. (6) is independent of the lattice spacing can be exploited to derive an RG equation for the \( \hat{C}_j^j \) coefficients. This equation describes the change in the \( \hat{C}_j^j \) as the lattice spacing is varied. Collecting the \( \hat{C}_j^j \) coefficients into a vector, the RG equation may be written

\[
\left( a \frac{d}{da} + (γ^j)^{tr} \right) \vec{C}^j = 0,
\]

where the anomalous dimension matrix is given by

\[
γ_{ij}^j(α_s, aM_0) = \sum_k \left( a \frac{d}{da} Z_{ik}^j \right) (Z^j)_{kj}^{-1}.
\]

The right-hand side of Eq. (66) is a complicated function of \( aM_0 \). Once \( γ_{ij}^j \) is determined numerically for a large range of \( aM_0 \) values, Eq. (66) can be solved by numerical methods. Our primary concern was the determination of the matching coefficients \( \hat{C}_j^j \) for values of \( aM_0 \) relevant for simulations of \( B \) and \( D \) mesons, and for such values, RG improvement was not needed. Hence, we have not attempted to obtain the entire anomalous dimension matrices for our current operators.
VII. SUMMARY

In summary, the spatial components of the heavy-light axial-vector current $A_k$ and all components of the heavy-light vector current $V_{\mu}$ were expressed in terms of lattice operators suitable for use in simulations of $B$ and $D$ mesons. In the lattice theory, the heavy quarks were treated using the NRQCD formulation, the light quarks were described by the tadpole-improved clover action, and the standard Wilson action was used for the gluons. The light quarks were treated as massless. The expansions were carried out to $O(1/M)$ by matching appropriate scattering amplitudes to one-loop order in perturbation theory. We found

$$J_{\mu} = \sum_{j=0}^{N_{f_{\mu}}} C_j^{(j)} J_{\mu,\text{lat}}^{(j)} + O(\alpha_s^2, \alpha_s^2 1/M_0^2, \alpha_s a/M_0),$$  

where $N_{A_k} = N_{V_k} = 5$ and $N_{V_0} = 3$. The lattice current operators are given by

$$J_{\mu,\text{lat}}^{(0)}(x) = i\bar{q}(x) J^{\mu}_{\mu} Q(x),$$  

$$J_{\mu,\text{lat}}^{(1)}(x) = -\frac{1}{2M_0} \bar{q}(x) \Gamma_{\mu}^J \gamma_5 \nabla Q(x),$$  

$$J_{\mu,\text{lat}}^{(2)}(x) = -\frac{1}{2M_0} \bar{q}(x) \gamma_5 \nabla_0 \Gamma_{\mu}^J Q(x),$$  

$$J_{\mu,\text{lat}}^{(3)}(x) = -\frac{1}{2M_0} \bar{q}(x) \Gamma_{\mu}^J \nabla_k Q(x),$$  

$$J_{\mu,\text{lat}}^{(4)}(x) = \frac{1}{2M_0} \bar{q}(x) \overrightarrow{\nabla}_k \Gamma_{\mu}^J Q(x),$$

with $\Gamma_{\mu}^V = \gamma_{\mu}$ and $\Gamma_{\mu}^A = \gamma_5 \gamma_{\mu}$, and the coefficients are

$$C_j^{(j)} = \begin{cases} 1 + \alpha_s \rho_j^{aM_0} + O(\alpha_s^2), & (i = 0, 1), \\ \alpha_s \rho_j^{aM_0} + O(\alpha_s^2), & (i \geq 2). \end{cases}$$

Matching coefficients for other NRQCD actions which have appeared in the literature and for different values of $(aM_0, n)$ are also available. For example, Ref. [12] used a slightly different action, and an action with higher-order improvement terms was employed in Refs. [13,15]. In all cases, we find that in the useful range $1.0 \leq aM_0 \leq 10.0$, the one-loop coefficients exhibit only a mild dependence on $aM_0$ and do not become particularly large.

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TABLE I. Values of the coefficients $\tau_i^{V_k}$ corresponding to the spatial components of the vector current $V_k$ for various values of the bare heavy-quark mass $aM_0$ and NRQCD stability parameter $n$. Uncertainties in the determinations of these parameters due to the use of Monte Carlo integration are included.

| $aM_0$ | $n$ | $\tau_0^{V_k}$ | $\tau_1^{V_k}$ | $\tau_2^{V_k}$ | $\tau_3^{V_k}$ | $\tau_4^{V_k}$ |
|--------|-----|----------------|----------------|----------------|----------------|----------------|
| 10.0   | 1   | 0.8189(1)      | -0.588(3)      | -13.93(1)      | -0.566(4)      | -0.940(5)      |
| 7.0    | 1   | 0.7880(1)      | -0.640(2)      | -8.150(7)      | -0.508(3)      | -0.909(3)      |
| 4.0    | 1   | 0.7221(1)      | -0.757(5)      | -2.809(7)      | -0.385(7)      | -0.836(7)      |
| 4.0    | 2   | 0.7340(1)      | -0.742(4)      | -3.084(8)      | -0.422(6)      | -0.842(7)      |
| 3.5    | 2   | 0.7176(2)      | -0.775(4)      | -2.310(5)      | -0.392(6)      | -0.817(6)      |
| 3.0    | 2   | 0.698(1)       | -0.821(4)      | -1.594(4)      | -0.351(5)      | -0.791(5)      |
| 2.7    | 2   | 0.6840(2)      | -0.856(3)      | -1.196(4)      | -0.323(5)      | -0.767(5)      |
| 2.5    | 2   | 0.6732(2)      | -0.877(3)      | -0.955(4)      | -0.309(5)      | -0.750(4)      |
| 2.0    | 2   | 0.6410(1)      | -0.962(3)      | -0.427(3)      | -0.254(4)      | -0.693(3)      |
| 1.7    | 2   | 0.6170(2)      | -1.027(2)      | -0.188(2)      | -0.218(3)      | -0.647(3)      |
| 1.6    | 2   | 0.6075(2)      | -1.051(2)      | -0.123(2)      | -0.205(3)      | -0.627(2)      |
| 1.2    | 3   | 0.5768(2)      | -1.175(2)      | -0.081(2)      | -0.150(3)      | -0.534(2)      |
| 1.0    | 4   | 0.5559(2)      | -1.265(2)      | -0.148(1)      | -0.108(3)      | -0.464(2)      |
| 0.8    | 5   | 0.5249(2)      | -1.387(2)      | -0.288(1)      | -0.049(3)      | -0.363(1)      |

TABLE II. Values of the coefficients $\tau_i^{A_k}$ corresponding to the spatial components of the axial-vector current $A_k$, similar to Table I.

| $aM_0$ | $n$ | $\tau_0^{A_k}$ | $\tau_1^{A_k}$ | $\tau_2^{A_k}$ | $\tau_3^{A_k}$ | $\tau_4^{A_k}$ |
|--------|-----|----------------|----------------|----------------|----------------|----------------|
| 10.0   | 1   | 0.263(1)       | -1.098(4)      | 2.624(18)      | 0.611(6)       | -0.729(1)      |
| 7.0    | 1   | 0.2685(1)      | -1.145(3)      | 2.137(12)      | 0.636(4)       | -0.7017(2)     |
| 4.0    | 1   | 0.2817(1)      | -1.254(2)      | 1.494(6)       | 0.691(2)       | -0.6401(2)     |
| 4.0    | 2   | 0.2742(1)      | -1.197(2)      | 1.475(6)       | 0.590(2)       | -0.6407(2)     |
| 3.5    | 2   | 0.2766(1)      | -1.225(2)      | 1.339(5)       | 0.595(2)       | -0.6219(2)     |
| 3.0    | 2   | 0.2793(1)      | -1.259(1)      | 1.168(4)       | 0.598(2)       | -0.5977(2)     |
| 2.7    | 2   | 0.2812(1)      | -1.284(1)      | 1.065(3)       | 0.598(2)       | -0.5796(2)     |
| 2.5    | 2   | 0.2824(1)      | -1.303(1)      | 0.985(3)       | 0.600(2)       | -0.5654(2)     |
| 2.0    | 2   | 0.2853(1)      | -1.365(1)      | 0.750(2)       | 0.599(2)       | -0.5196(2)     |
| 1.7    | 2   | 0.2862(1)      | -1.416(1)      | 0.582(2)       | 0.595(2)       | -0.4816(2)     |
| 1.6    | 2   | 0.2860(1)      | -1.436(1)      | 0.513(2)       | 0.591(2)       | -0.4661(2)     |
| 1.2    | 3   | 0.2725(1)      | -1.510(2)      | 0.188(1)       | 0.537(2)       | -0.3823(2)     |
| 1.0    | 4   | 0.2598(1)      | -1.567(2)      | -0.025(1)      | 0.511(2)       | -0.3190(2)     |
| 0.8    | 5   | 0.2407(1)      | -1.658(2)      | -0.291(1)      | 0.500(2)       | -0.2290(2)     |
TABLE III. Values of the coefficients $\tau_i^{V_0}$ corresponding to the temporal component of the vector current $V_0$, similar to Table I.

| $aM_0$ | $n$ | $\tau_0^{V_0}$ | $\tau_1^{V_0}$ | $\tau_2^{V_0}$ | $\tau_3^{V_0}$ |
|--------|-----|----------------|----------------|----------------|----------------|
| 10.0   | 1   | 0.3549(2)      | -0.541(8)      | 1.706(19)      |                |
| 7.0    | 1   | 0.3961(2)      | -0.572(6)      | 1.339(12)      |                |
| 4.0    | 2   | 0.4868(1)      | -0.695(3)      | 0.960(6)       |                |
| 3.5    | 2   | 0.5161(1)      | -0.718(3)      | 0.907(5)       |                |
| 3.0    | 2   | 0.5542(1)      | -0.752(3)      | 0.844(4)       |                |
| 2.7    | 2   | 0.5832(2)      | -0.774(2)      | 0.815(4)       |                |
| 2.5    | 2   | 0.6058(1)      | -0.794(2)      | 0.792(3)       |                |
| 2.0    | 2   | 0.6795(2)      | -0.853(2)      | 0.744(3)       |                |
| 1.7    | 2   | 0.7417(2)      | -0.899(2)      | 0.718(2)       |                |
| 1.6    | 2   | 0.7666(2)      | -0.920(2)      | 0.707(2)       |                |
| 1.2    | 3   | 0.8832(2)      | -1.034(1)      | 0.684(1)       |                |
| 1.0    | 4   | 0.9674(2)      | -1.109(1)      | 0.682(1)       |                |
| 0.8    | 5   | 1.0853(2)      | -1.202(1)      | 0.696(1)       |                |

TABLE IV. Values of the coefficients $\rho^{V_k}_j$ defined in Eq. (51) corresponding to the spatial components of the vector current for various values of the bare heavy-quark mass $aM_0$ and NRQCD stability parameter $n$. Uncertainties in the determinations of these parameters due to the use of Monte Carlo integration are included.

| $aM_0$ | $n$ | $\rho_0^{V_k}$ | $\rho_1^{V_k}$ | $\rho_2^{V_k}$ | $\rho_3^{V_k}$ | $\rho_4^{V_k}$ |
|--------|-----|----------------|----------------|----------------|----------------|----------------|
| 10.0   | 1   | -0.3051(3)     | -0.128(5)      | 10.968(11)     | 0.991(4)       | 0.069(5)       |
| 7.0    | 1   | -0.3441(2)     | -0.202(4)      | 10.733(7)      | 0.932(3)       | 0.190(3)       |
| 4.0    | 2   | -0.3496(2)     | -0.346(5)      | 4.680(7)       | 0.810(7)       | 0.354(7)       |
| 3.5    | 2   | -0.3744(2)     | -0.382(5)      | 4.955(8)       | 0.846(6)       | 0.360(7)       |
| 3.0    | 2   | -0.3694(2)     | -0.421(5)      | 4.011(5)       | 0.816(6)       | 0.392(6)       |
| 2.7    | 2   | -0.3574(2)     | -0.443(5)      | 3.099(4)       | 0.775(5)       | 0.431(5)       |
| 2.5    | 2   | -0.3472(2)     | -0.470(4)      | 2.566(4)       | 0.747(5)       | 0.451(5)       |
| 2.0    | 2   | -0.3366(2)     | -0.495(4)      | 2.228(4)       | 0.734(5)       | 0.467(4)       |
| 1.7    | 2   | -0.3698(2)     | -0.521(3)      | 1.415(3)       | 0.678(4)       | 0.505(3)       |
| 1.6    | 2   | -0.3439(3)     | -0.564(3)      | 0.828(2)       | 0.630(3)       | 0.534(2)       |
| 1.2    | 3   | -0.3679(3)     | -0.609(3)      | 0.419(2)       | 0.574(3)       | 0.563(2)       |
| 1.0    | 4   | -0.3018(4)     | -0.606(3)      | 0.254(1)       | 0.532(3)       | 0.570(2)       |
| 0.8    | 5   | -0.1818(5)     | -0.597(4)      | 0.110(1)       | 0.473(3)       | 0.564(1)       |
TABLE V. Values of the coefficients $\rho_{A_k}^4$ corresponding to the spatial components of the axial-vector current, similar to Table IV.

| $aM_0$ | $n$ | $\rho_0^{A_k}$ | $\rho_1^{A_k}$ | $\rho_2^{A_k}$ | $\rho_3^{A_k}$ | $\rho_4^{A_k}$ |
|--------|-----|----------------|----------------|----------------|----------------|----------------|
| 10.0   | 1   | 0.0504(2)      | 0.383(6)       | 0.413(18)      | -0.186(6)     | -0.1420(2)    |
| 7.0    | 1   | -0.0246(2)     | 0.303(4)       | 0.447(12)      | -0.211(4)     | -0.0180(2)    |
| 4.0    | 1   | -0.1093(2)     | 0.151(3)       | 0.377(6)       | -0.267(2)     | 0.1578(2)     |
| 4.0    | 2   | -0.1146(2)     | 0.073(3)       | 0.397(6)       | -0.166(2)     | 0.1584(2)     |
| 3.5    | 2   | -0.1285(1)     | 0.028(3)       | 0.362(5)       | -0.171(2)     | 0.1963(2)     |
| 3.0    | 2   | -0.1396(1)     | -0.005(3)      | 0.337(4)       | -0.174(2)     | 0.2375(2)     |
| 2.7    | 2   | -0.1444(2)     | -0.043(3)      | 0.305(3)       | -0.174(2)     | 0.2641(2)     |
| 2.5    | 2   | -0.1458(2)     | -0.070(3)      | 0.288(3)       | -0.176(2)     | 0.2826(2)     |
| 2.0    | 2   | -0.1401(2)     | -0.117(3)      | 0.238(2)       | -0.174(2)     | 0.3316(2)     |
| 1.7    | 2   | -0.1253(2)     | -0.161(3)      | 0.200(2)       | -0.171(2)     | 0.3625(2)     |
| 1.6    | 2   | -0.1177(2)     | -0.180(3)      | 0.191(2)       | -0.166(2)     | 0.3727(2)     |
| 1.2    | 3   | -0.0636(3)     | -0.275(3)      | 0.151(1)       | -0.112(2)     | 0.4110(2)     |
| 1.0    | 4   | -0.0058(3)     | -0.304(3)      | 0.131(1)       | -0.087(2)     | 0.4251(2)     |
| 0.8    | 5   | 0.1023(5)      | -0.326(4)      | 0.113(1)       | -0.076(2)     | 0.4298(2)     |

TABLE VI. Values of the coefficients $\rho_{V_0}^4$ corresponding to the temporal component of the vector current, similar to Table IV.

| $aM_0$ | $n$ | $\rho_0^{V_0}$ | $\rho_1^{V_0}$ | $\rho_2^{V_0}$ | $\rho_3^{V_0}$ | $\rho_4^{V_0}$ |
|--------|-----|----------------|----------------|----------------|----------------|----------------|
| 10.0   | 1   | 0.1712(3)      | -0.387(9)      | -0.433(19)     | -0.433(19)     | -0.433(19)    |
| 7.0    | 1   | 0.0599(2)      | -0.482(7)      | -0.665(12)     | -0.665(12)     | -0.665(12)    |
| 4.0    | 1   | -0.1107(3)     | -0.672(4)      | 0.291(12)      | 0.291(12)      | 0.291(12)     |
| 4.0    | 2   | -0.1150(2)     | -0.642(5)      | 0.305(6)       | 0.305(6)       | 0.305(6)      |
| 3.5    | 2   | -0.1557(2)     | -0.691(4)      | 0.367(5)       | 0.367(5)       | 0.367(5)      |
| 3.0    | 2   | -0.2023(2)     | -0.725(4)      | 0.430(4)       | 0.430(4)       | 0.430(4)      |
| 2.7    | 2   | -0.2341(2)     | -0.765(3)      | 0.458(4)       | 0.458(4)       | 0.458(4)      |
| 2.5    | 2   | -0.2570(2)     | -0.791(3)      | 0.481(3)       | 0.481(3)       | 0.481(3)      |
| 2.0    | 2   | -0.3221(2)     | -0.842(3)      | 0.529(3)       | 0.529(3)       | 0.529(3)      |
| 1.7    | 2   | -0.3686(3)     | -0.891(3)      | 0.555(2)       | 0.555(2)       | 0.555(2)      |
| 1.6    | 2   | -0.3861(3)     | -0.907(3)      | 0.566(2)       | 0.566(2)       | 0.566(2)      |
| 1.2    | 3   | -0.4621(3)     | -0.963(3)      | 0.589(1)       | 0.589(1)       | 0.589(1)      |
| 1.0    | 4   | -0.5012(4)     | -0.974(3)      | 0.591(1)       | 0.591(1)       | 0.591(1)      |
| 0.8    | 5   | -0.5300(5)     | -0.994(3)      | 0.577(1)       | 0.577(1)       | 0.577(1)      |