On J-Holomorphic Variational Vector Fields and Extremal Discs

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Abstract
We prove that every $J$-holomorphic variational vector field can be realized as derivation $\frac{d}{dt}|_{t=0} f_t$ where $(f_t)$ is a one parametric family of $J$-holomorphic discs. Furthermore, we discuss properness of an extremal $J$-holomorphic disc in a bounded pseudoconvex domain.

Keywords J-holomorphic discs · Almost complex manifolds · Extremal discs · Stationary discs · Holomorphic variations

Mathematics Subject Classification 32Q65 · 32F45

1 Introduction
Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc and $J$ a smooth almost complex structure defined on $\mathbb{R}^{2n}$, that is, a matrix function satisfying $J^2 = -id$. We denote by $J_{st}$ the standard structure on $\mathbb{R}^{2n}$ corresponding to the multiplication by the imaginary unit and call a $C^1$ map $f : \mathbb{D} \to \mathbb{R}^{2n}$ a $J$-holomorphic disc if for all $\zeta = x + iy \in \mathbb{D}$ we have

$$df(\zeta) \circ J_{st} = J(f(\zeta)) \circ df(\zeta) \iff f_x + J(f)f_y = 0.$$  \hspace{1cm} (1)

In this paper, we discuss variations of such discs.

Let $(f_t)_{t \in \mathbb{R}}$ be a one parameter family of $J$-holomorphic discs in $\mathbb{R}^{2n}$. We denote by $f = f_0$. It follows from (1) that the vector field

$$V(\zeta) := \frac{d}{dt}|_{t=0} f_t(\zeta)$$
satisfies the following equation

\[ V_x + J(f)V_y + d_f J(V) f_y = 0, \]  

(2)

where \( d_f J(V) \) denotes the linearization of \( g \to J(g) \) at \( f \). We call a \( C^1 \)-smooth solution of (2) a \( J \)-holomorphic variational \( \textit{vector field along} \) \( f \). We prove that every such vector field can be realized as a derivation of a one parametric family of \( J \)-holomorphic discs.

**Theorem 1** Assume that \( \det(J + J_{st}) \neq 0 \). Let \( f \in C^{1,\alpha}(\overline{D}) \) be a \( J \)-holomorphic disc and let \( V \in C^{1,\alpha}(\overline{D}) \) be a \( J \)-holomorphic variational vector field along \( f \). There exists a family of \( J \)-holomorphic discs \( (f_t) \subset C^{1,\alpha}(\overline{D}) \) such that \( V = \frac{d}{dt} |_{t=0} f_t \) and \( f_0 = f \).

We prove this statement in Sect. 2 using non-linear techniques developed in [1] and linear theory from [10]. In 3 a refined version of this theorem is used in the following application concerning extremal discs.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^{2n} \). A \( J \)-holomorphic disc from \( D \) to \( \Omega \) is called **extremal** if for every \( J \)-holomorphic disc \( g : D \to \Omega \) such that \( g(0) = f(0) \) and \( g'(0) = \lambda f'(0) \) for some \( \lambda \geq 0 \), it follows that \( \lambda \leq 1 \). We denote here by \( f' \) the velocity vector field \( f_x \) which, in the usual holomorphic case, agrees with the standard complex derivative. When \( J \equiv J_{st} \) and \( \Omega \) is strongly convex, these discs were discussed in the celebrated paper of Lempert [7]. He proved that they are uniquely determined by the pair \((f(0), f'(0))\) and can be extended to a proper smooth embedding attached to the boundary \( \partial \Omega \). However, the theory becomes less clear when dealing with pseudoconvex domains. For instance, Sibony’s examples in [8] show that extremal discs are not unique and that the relation between \( f(\partial D) \) and \( \partial \Omega \) is less trivial. Nevertheless, using a variational approach, Poletsky showed that \( C^1 \)-closed extremal discs are proper when \( \Omega \) is bounded by a \( C^2 \)-plurisubharmonic function [9] (see also [2, Theorem 1]).

We provide a generalization of this last statement for the case of non-integrable structures. That is, we improve [4, Proposition 4.1].

**Theorem 2** Assume that \( \det(J + J_{st}) \neq 0 \). Let \( \Omega = \{\rho < 0\} \subset \mathbb{R}^{2n} \) be a bounded domain defined by a \( C^2 \)-smooth \( J \)-plurisubharmonic function. Let \( f \in C^{2,\alpha}(\overline{D}) \) be an extremal \( J \)-holomorphic disc. Then \( f(\partial D) \subset \partial \Omega \).

When an extremal disc is proper, one can discuss its relation to the so-called stationary discs that were also introduced by Lempert in [7]. In the standard case, this refers to biholomorphic invariants that can be defined in two equivalent ways: as discs that admit a certain proper meromorphic lift to the cotangent bundle [11]; as stationarity solutions of the Euler–Lagrange equation [2,8,9]. As explained in [4], in the non-integrable case we fancy the second approach, since the first one does not provide us with the necessary conditions for a disc to be extremal. However, such theory of \( J \)-stationary discs was developed only for small perturbations of \( J_{st} \). Using our Theorem 1 this can now be improved. We discuss this briefly at the end of the paper.
2 The Implicit Function Theorem

Let \( f: \mathbb{D} \to \mathbb{R}^{2n} \) be a \( J \)-holomorphic disc. If \( \det (J + J_{st}) \neq 0 \) along the image \( f(\mathbb{D}) \), the \( J \)-holomorphicity condition (1) can be turned into a non-linear Cauchy–Riemann system

\[
f_{\bar{\zeta}} + A(f)f_{\zeta} = 0,
\]

where \( A(z)(v) = (J(z) + J_{st})^{-1}(J(z) - J_{st})(\bar{v}) \) is complex linear in \( v \in \mathbb{C}^n \) for every \( z \in \mathbb{C}^n \) and can be treated as a complex matrix function; see [12]. In this notation, a variational \( J \)-holomorphic vector field corresponds to a solution of the following complex equation

\[
V_{\bar{\zeta}} + A(f)V_{\zeta} + d_f A(V)f_{\zeta} = 0.
\]

Moreover, this equation can be rewritten into a linear condition

\[
V_{\bar{\zeta}} + A(f)V_{\zeta} + B_1^f V + B_2^f \bar{V} = 0
\]

determined by smooth matrix functions \( B_1^f \) and \( B_2^f \), depending on \( A \) and \( f \).

Note that in Theorem 1 we can assume that \( f \) is an embedding. Otherwise we consider its graph \( \tilde{f} \) that is a \( J_{st} \otimes J \)-holomorphic disc in the almost complex space \( (\mathbb{R}^{2n+2}, J_{st} \otimes J) \) and interpolate the variational vector field \( \tilde{V}(\zeta) := (\zeta, V(\zeta)) \). Therefore, after a change of coordinates, we can assume that \( A(f) = 0 \). Indeed, see e.g. the Appendix to [5]. This means that the linear condition (5) can be associated with the theory of so-called generalized vectors (see e.g. [3,10]).

Let us turn now towards the proof of Theorem 1. Suppose that \( J \equiv J_{st} \). Then \( A(f) \equiv 0 \) and, instead of (3) and (5), we have the usual holomorphicity condition for \( f \) and \( V \). Thus the required family of discs is

\[
f_t = f + tV.
\]

We will mimic this idea using the non-linear techniques for Banach spaces. That is, we rely on the following standard theorem (see e.g. [6, p. 298]).

**Theorem 3** Let \( X \) and \( Y \) be Banach spaces and let \( \mathcal{F}: \mathbb{B}_X(x_0, \epsilon) \to Y \) be a \( C^1 \) map defined on some ball in \( X \). Suppose there is \( C > 0 \) such that:

(i) Given \( x \in \mathbb{B}_X(x_0, \epsilon) \) and \( v \in Y \) the operator \( d_x \mathcal{F}: X \to Y \) is surjective and the equation \( d_x \mathcal{F}(u) = v \) admits a solution with \( \|u\|_X \leq C \|v\|_Y \).

(ii) For any \( x_1, x_2 \in \mathbb{B}_X(x_0, \epsilon) \) we have \( \|d_{x_1} \mathcal{F} - d_{x_2} \mathcal{F}\| \leq \frac{1}{2C} \).

Then the ball \( \mathbb{B}_Y (\mathcal{F}(x_0), \frac{\epsilon}{2C}) \) lies in \( \mathcal{F}(\mathbb{B}_X(x_0, \epsilon)) \).

Let us fix \( 0 < \alpha < 1 \). The classical Cauchy–Green operator given by

\[
T(f)(z) = -\frac{1}{\pi} \int \int_{\mathbb{D}} \frac{f(\xi)}{\xi - z} \, dx \, dy(\xi)
\]
is continuous when mapping between Hölder spaces of vector functions $T : C^{0, \alpha}(\overline{D}) \to C^{1, \alpha}(\overline{D})$. Thus, we can define a continuous operator

$$\mathcal{F}(f) = f + T(A(f)T_\xi)$$  \hspace{1cm} (6)$$
mapping the space $C^{1, \alpha}(\overline{D})$ to itself. Moreover, $T$ solves the usual $\bar{\partial}$-equation since $(Tu)_\overline{z} = u$. Hence, given a $J$-holomorphic disc $f$ the map $\mathcal{F}(f)$ is a $J$-holomorphic vector function.

As explained above, without loss of generality, the derivative of $\mathcal{F}$ at $f$ can be assumed to be equal to

$$d_f \mathcal{F}(V) = V + T(B_1V + B_2\overline{V}) .$$

This singular integral operator is known to be Fredholm with a possibly non-trivial cokernel [10]. Therefore, we have to linearly perturb $\mathcal{F}$ in order to meet requirements of Theorem 3.

**Lemma 4** There exists a locally invertible perturbation $\tilde{\mathcal{F}}$ of the operator given in (6) that maps $J$-holomorphic disc to the usual holomorphic ones. In particular, there is $C > 0$ such that for every $W \in C^{1, \alpha}(\overline{D})$ there exists $V \in C^{1, \alpha}(\overline{D})$ satisfying $d_f \tilde{\mathcal{F}}(V) = W$ and $\|V\|_{1, \alpha} \leq C\|W\|_{1, \alpha}$.

**Proof** This statement is proved in [10]. For completeness, we include some details. We introduce the real inner product of vector functions:

$$\langle f, g \rangle = \sum_{j=1}^{n} \int \int_{\overline{D}} f_j \overline{g_j} dxdy.$$ 

Let $H \subset C^{k, \alpha}(\overline{D})$ be the set of (usual) holomorphic vector functions. Then

$$H + \text{Range}(d_f \mathcal{F}) = C^{1, \alpha}(\overline{D}).$$

Indeed, let $V \in \text{Range}(d_f \mathcal{F})^\perp = \ker d_f \mathcal{F}^*$. The adjoint map of $d_f \mathcal{F}$ equals

$$d_f \mathcal{F}^*(V) = V - B_1^T \overline{T(V)} - B_2^T \overline{V} .$$

Hence the vector $W = \overline{T(V)}$ satisfies the equation

$$W_\overline{z} - B_1^T W - B_2^T \overline{W} = 0$$

and is a so-called generalized analytic vector. Moreover, if $V$ is also orthogonal to the space $H$, then $W$ vanishes on $\mathbb{C} \setminus \overline{D}$. This implies that $W \equiv 0$ and thus $V \equiv 0$ (see [10, Corollary 3.4]).
Let \( N = \dim \ker d_f \mathcal{F} \). There exist \( h_1, h_2, \ldots, h_N \in H \) such that
\[
\text{Span}_R (h_1, h_2, \ldots, h_N) \oplus \text{Range}(d_f \mathcal{F}) = C^{1,\alpha}(\overline{\mathbb{D}}).
\]
Thus, if \( V_1, \ldots V_n \) form a basis of \( \ker d_f \mathcal{F} \), the operator given by
\[
\tilde{\mathcal{F}}(f) = \mathcal{F}(f) + \sum_{j=1}^{N} \text{Re}(f, V_j) h_j
\]
still maps \( J \)-holomorphic discs into \( J_{st} \)-holomorphic ones and admits an invertible linearization. Hence for \( C \) one can set the norm of \( d_f \tilde{\mathcal{F}}^{-1} \). \( \square \)

Let us prove that, close to a fixed disc \( f \), the operator \( \tilde{\mathcal{F}} \) satisfies (ii) from Theorem 3. The size of the neighborhood depends on the \( (1, \alpha) \)-norm of \( f \).

**Lemma 5** Given a \( J \)-holomorphic disc \( f \) with \( A(f) \equiv 0 \) and \( \| f \|_{1,\alpha} < C \) there is \( \epsilon_0 > 0 \) such that \( \| g - f \|_{1,\alpha} < \epsilon_0 \) implies \( \| d_g \tilde{\mathcal{F}} - d_f \tilde{\mathcal{F}} \| < \frac{1}{4C} \).

**Proof** Since, in general \( A(g) \neq 0 \), we have
\[
d_g \tilde{\mathcal{F}}(V) - d_f \tilde{\mathcal{F}}(V) = I_1 + I_2 + I_3,
\]
where
\[
\begin{aligned}
I_1 &= T \left( A(g) \overline{V}_{\xi} \right), \\
I_2 &= T \left( B_1^g - B_1^{\tilde{f}} \right) V + (B_2^g - B_2^{\tilde{f}}) \overline{V}. \\
I_3 &= \sum_{j=1}^{N} \text{Re}(f - g, V_j) h_j
\end{aligned}
\]
Let us assume that \( \| g - f \|_{1,\alpha} < 1 \). Then there exist \( C_k > 0, k \in \{1, 2, 3\} \), such that
\[
\| I_k \|_{0,\alpha} \leq C_k \| g - f \|_{1,\alpha} \| V \|_{1,\alpha}.
\]
Indeed, the bound for \( I_3 \) is obvious. Furthermore, \( T : C^{0,\alpha}(\overline{\mathbb{D}}) \to C^{1,\alpha}(\overline{\mathbb{D}}) \) is bounded, and hence we only have to bound the \( (0, \alpha) \)-norm of its arguments in \( I_1 \) and \( I_2 \). In \( I_1 \) the required \( \alpha \)-bound depends on the coefficients of \( A \) and \( \| g \|_{1,\alpha} < 1 + C \). However, in \( I_2 \) we have
\[
B_1^{\tilde{f}} V + B_2^{\tilde{f}} \overline{V} = \left( \sum_{j=1}^{n} \frac{\partial A}{\partial z_j} (f) V_j + \frac{\partial A}{\partial \overline{z}_j} (f) \overline{V}_j \right) \overline{f}_{\xi}.
\]
Thus the required \( \alpha \)-bound depends on values and derivatives of \( A, f \) and \( g \). Finally, we set \( \epsilon_0 = \frac{1}{4C} \min \{1, C_1 + C_2 + C_3\} \). \( \square \)
Proof of Theorem 1} We apply Theorem 3 for $\tilde{F}$. Let us set

$$C = \max \left\{ \|f\|_{1,\alpha}, 2 \cdot \|d_f \tilde{F}^{-1}\| \right\}$$

and let $\epsilon_0 > 0$ be such as in Lemma 5. The set of invertible operators is open. Thus there is $\epsilon_1 > 0$ such that for $\|f - g\|_{1,\alpha} < \epsilon_1$ the inverse $d_g \tilde{F}^{-1}$ exists and its norm is bounded by $C$. For $\epsilon = \min\{\epsilon_0, \epsilon_1\}$ the operator $\tilde{F}$ satisfies (i) and (ii) in Theorem 3, and hence there is $t_0 > 0$ depending on $C$, $\epsilon$ and $V$ such that given a $J$-holomorphic variational vector field $V$ field along $f$ the family

$$f_t = \tilde{F}^{-1} \left( \tilde{F}(f) + t d_f \tilde{F}(V) \right)$$

is well defined and $J$-holomorphic for $|t| < t_0$. Moreover, $f_0 = f$. \hfill $\square$

It is worth noting that the size of the neighborhood on which the Implicit function theorem can be applied actually depends on the $(1,\alpha)$-norm of $f$ and the operator norm of $d_f \tilde{F}^{-1}$. Indeed, this two norms provide the appropriate constants $C$ and $\epsilon_0$ in Lemmas 4 and 5. Therefore, we have to uniformly bound them if we want to apply Theorem 1 for more than one $J$-holomorphic disc. We will do that in the next section when dealing with discs $f_r(\zeta) = f(r \zeta)$, $r \in [\frac{1}{2}, 1]$. However, this case is trivial since one can rescale the integral operator $T$ and rely on the fact that $[\frac{1}{2}, 1]$ is compact.

3 Proof of Theorem 2

We start by proving the following lemma and a corollary of Theorem 1.

**Lemma 6** Let $\phi = a + ib : \mathbb{D} \to \mathbb{C}$ be a standard holomorphic map. If $f \in C^{2,\alpha}(\mathbb{D})$ is a $J$-holomorphic disc, then

$$V = \phi \cdot f' := af' + b J(f) f' \in C^{1,\alpha}(\mathbb{D})$$

is a $J$-holomorphic variational vector field along $f$.

**Proof** Recall from (2) that $f_y = J(f) f'$ and that the equation defining a $J$-holomorphic variational vector field $V$ along $f$ can be written in the form

$$D(V) = V_x + J(f) V_y + d_f J(V) f_y = 0.$$

Note that both, $f_x$ and $f_y$, are $J$-holomorphic variational vector fields along $f$ and therefore we have $D(f_x) = D(f_y) = 0$. Together with $a_x = b_y$ and $a_y = -b_x$ this yields that

$$D(af_x + bf_y) = a_x f_x + a_y J(f) f_x + b_x f_y + b_y J(f) f_y = 0.$$

Since $J(f)^2 = -id$ this completes the proof. \hfill $\square$
Corollary 7  Let \( f \in C^{1,\alpha}(\overline{\mathbb{D}}) \) be a \( J \)-holomorphic disc and let \( V \in C^{1,\alpha}(\overline{\mathbb{D}}) \) be a \( J \)-holomorphic variational vector field along \( f \) and satisfying \( V(0) = 0 \). There exists a family \( (f_t) \subset C^{1,\alpha}(\overline{\mathbb{D}}) \) of \( J \)-holomorphic discs such that \( f_0 = f \), \( f_t(0) = f(0) \), \( \frac{d}{dt}|_{t=0} f_t = V \) and \( f_t'(0) = f'(0) + tV'(0) \).

Proof  As said, this is just a refined version of Theorem 1. Indeed, as mentioned, for \( J(f) = J_{st} \) we have \( f'(0) = f_{\xi}(0) = f_{\bar{\xi}}(0) \). Hence we can repeat the same proof by using the following normalization of the Cauchy Green operator

\[
T_0(f)(\zeta) = T(f)(\zeta) - T(f)(0) - \zeta \left[ T(f) \right]_{\zeta}(0).
\]

This yields that \( \mathcal{F} \) defined as in (6) but with \( T_0 \) satisfies \( \mathcal{F}(f)(0) = f(0) \) and \( [\mathcal{F}(f)]'(0) = f'(0) \). Moreover, it can be made locally invertible again by calculating a new adjoint operator of \( df \mathcal{F} \) and finding \( h_1, \ldots, h_N \) that span the complement of \( \text{Range}(df \mathcal{F}) \) and satisfy \( h_j(0) = 0 \) and \( h'_j(0) = 0 \).

Proof of Theorem 2  We follow the proof in [4]. Assume that \( f \) is not proper. Then there exists an open interval \( P \subset \partial \mathbb{D} \) such that \( f(P) \cap \partial \Omega = \emptyset \). The idea is to shrink the domain of \( f \) in order to assure that its perturbations will stay in \( \Omega \) but, on the other hand, enlarge its values on \( P \) so that they will contradict the extremality of \( f \).

Given \( r \in [\frac{1}{2}, 1] \) we define \( f_r(\zeta) := f(r \zeta) \). Let \( K \subset \Omega \) be a compact set such that \( f_r(\zeta) \in K \) for every \( \zeta \in P \) and every \( r \in [\frac{1}{2}, 1] \). For \( R > 0 \) that will be fixed later we consider a smooth function \( \chi_R : \partial \mathbb{D} \to \mathbb{R} \) compactly supported in \( P \) such that \( \chi_R \equiv R \) on a slightly smaller closed subinterval \( P_1 \subset P \). Using the Poisson kernel we construct a standard holomorphic function \( \phi_R \) defined on \( \mathbb{D} \) and continuous up to the boundary with properties \( \text{Im}(\phi_R)(0) = 0 \) and \( \text{Re}(\phi_R) = \chi_R \) on \( \partial \mathbb{D} \). By Lemma 6 the vector function

\[
V^R_r(\zeta) := (\zeta \exp(\phi_R(\zeta))) \cdot f'_r(\zeta) \in C^{1,\alpha}(\overline{\mathbb{D}})
\]

is a \( J \)-holomorphic variational vector field along \( f_r \). Thus, by Corollary 7, there exists a family \( (h^R_{r,t})_t \) of \( J \)-holomorphic discs such that

\[
\begin{align*}
  h^R_{r,0}(0) & = f_r(0) = f(0) \\
  h^R_{r,t}'(0) & = f'_r(0) + t \ V^R_r'(0) = r(1 + t \exp(\phi_R(0))) f'(0).
\end{align*}
\]

Note that one can assume that these families are defined for \( |t| < t_0(R) \). That is, this bound does not depend on \( r \in [\frac{1}{2}, 1] \). Indeed, using a rescaling argument for the family \( f_r \), the constants \( C \) and \( \epsilon_0 \) in Lemmas 4 and 5 can be chosen uniformly for all \( f_r, r \in [\frac{1}{2}, 1] \).

Let \( l \) denote the length of \( P_1 \). Then \( \exp(\phi_R(0)) \geq \exp(lR/2\pi) \). Hence if

\[
t > \frac{1 - r}{r} \exp(-lR/2\pi) := t_1(r, R)
\]

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we have
\[ (h_{r,t}^R)'(0) = \lambda f'(0) \quad \text{for some } \lambda > 1. \]

That is, if this holds for some $0 < t < t_0(R)$ such that $h_{r,t}^R(\mathbb{D}) \subset \Omega$ this yields a contradiction to the extremality of $f$. Hence, let us seek the condition under which the image of $h_{r,t}^R$ is in $\Omega$.

First note that $\rho \circ f_r$ is negative on $\mathbb{D}$. Hence, there is $d(K, \rho) > 0$ such that for $r \in [\frac{1}{2}, 1]$ and $\zeta \in P$ we have
\[ \rho \circ f_r(\zeta) \leq -d(K, \rho). \]

Moreover, there exist positive constants $C_1(R) > 0$ and $t_1(R) > 0$ such that for every $|t| < t_1(R)$ and every $r \in [\frac{1}{2}, 1]$ we have
\[ |\rho \circ h_{r,t}^R(\zeta) - \rho \circ f_r(\zeta)| \leq C_1(R)|t|. \]

Hence, if $|t| < \min \{t_1(R), d(K, \rho)/C_1(R)\}$, we have $h_{r,t}^R(\zeta) \in \Omega$ for $\zeta \in P$.

Next, due to the subharmonicity of $\rho \circ f_r$ there is $C_2 > 0$ depending only on $\rho$ and $f(0)$ such that for every $\zeta \in \mathbb{D}$ and every $r \in [0, 1)$ we have
\[ \rho \circ f_r(\zeta) \leq -C_2(1 - |\zeta|) < -C_2(1 - r). \]

Since $\text{Re}(\phi_R) \equiv 0$ on $\partial \mathbb{D} \setminus P$ there is a constant $C_3 > 0$ independent of $R > 0$ and a constant $t_2(R) > 0$ such that for every $r \in [\frac{1}{2}, 1]$, $\zeta \in \partial \mathbb{D} \setminus P$ and $|t| < t_2(R)$ we have
\[ |\rho \circ h_{r,t}^R(\zeta) - \rho \circ f_r(\zeta)| \leq C_3|t|. \]

Therefore, if $|t| \leq C_2(1 - r)/C_3 := t_3(r)$ and $\zeta \in \partial \mathbb{D} \setminus P$, then $h_{r,t}^R(\zeta) \in \Omega$.

Finally, we fix $R > 0$ such that
\[ \exp\left(-\frac{lR}{2\pi}\right) < \frac{C_2}{2C_3}. \]

Then $t_1(r, R) < t_3(r)$. Moreover, we can take $r$ sufficiently close to 1 so that $t_3(r) < \min \{t_0(R), t_1(R), d(K, \rho)/C_1(R), t_2(R)\}$. Thus for such $R$ and a parameter $t$ satisfying $t_1(r, R) < t < t_3(r)$ the disc $h_{r,t}^R$ is well defined, maps $\mathbb{D}$ into $\Omega$ and yields the desired contradiction. 

**Final Remark** It was proved in [8, Theorem 2.31] that every $C^2$-closed extremal disc mapping into a strongly pseudoconvex domain vanishes certain first-order variations determined by the boundary function. Based on this property, a family of $J$-stationary discs was introduced in [4]. As explained earlier, when dealing with non-integrable structures, such a necessary condition for extremal discs is more appropriate than the geometric one provided by Tumanov [11]. In particular, the authors provide an

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explicit example of a stationary disc which admits such a required meromorphic lift but is not extremal. In contrast, using the variational approach, every extremal disc is \( J \)-stationary. However, as mentioned, their theory is restricted to small perturbations of \( J_{st} \) for which Theorem 1 can be obtained from the holomorphic case by using the Implicit function theorem. That is, this paper provides the main technical tool needed in order to remove this restriction.

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