Masses and orbital inclinations of planets in the PSR B1257+12 system

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ABSTRACT

We present measurements of the true masses and orbital inclinations of the two Earth-mass planets in the PSR B1257+12 system, based on the analysis of their mutual gravitational perturbations detectable as microsecond variations of the arrival times of radio pulses from the pulsar. The 6.2-millisecond pulsar, PSR B1257+12, has been regularly timed with the Arecibo telescope since late 1990. Assuming the standard pulsar mass of 1.4 \(M_\odot\), the derived masses of planets B and C are 4.3±0.2 \(M_\oplus\) and 3.9±0.2 \(M_\oplus\), respectively. The corresponding orbital inclinations of 53°±4° and 47°±3° (or 127° and 133°) imply that the two orbits are almost coplanar. This result, together with the known near 3:2 resonance between the orbits of the two planets, strongly supports the hypothesis of a disk origin of the PSR B1257+12 planetary system. The system’s long-term stability is guaranteed by the low, Earth-like masses of planets B and C.

Subject headings: planetary systems — pulsars: individual (PSR B1257+12)

1. Introduction

The first extrasolar planetary system consisting of three planets orbiting a neutron star, the 6.2-ms radio pulsar PSR B1257+12, has been systematically observed since the time

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of its discovery (Wolszczan & Frail 1992; Wolszczan 1994; Wolszczan et al. 2000). Pulsar planets comprise the only known system of terrestrial-mass planets beyond the Sun. They exhibit a striking dynamical similarity to the inner solar system (Mazeh & Goldman 1995). The pulsar planets have also provided the first demonstration of orbital near resonance in an extrasolar planetary system with the resulting, measurable gravitational perturbations (Wolszczan 1994; Konacki, Maciejewski, & Wolszczan 1999). In the recent years, a variety of resonances and commensurabilities have been detected in planetary systems around normal stars (Nelson & Papaloizou 2002; Fischer et al. 2003). A microsecond precision of the pulsar timing offers a possibility to detect additional, even lower mass planets around PSR B1257+12 and other neutron stars observable as radio pulsars (Wolszczan 1997).

An accurate knowledge of masses and orbital inclinations of the PSR B1257+12 planets is critically important for our understanding of the origin and evolution of this unique planetary system. In this case, the near 3:2 mean motion resonance (MMR) between planets B and C in the pulsar system and the existence of detectable gravitational perturbations between the two planets (Rasio, Nicholson, Shapiro, & Teukolsky 1992; Malhotra, Black, Eck, & Jackson 1992; Peale 1993; Wolszczan 1994; Konacki, Maciejewski, & Wolszczan 1999) provide the mechanism to derive their masses without an a priori knowledge of orbital inclinations. An approximate analytical model which includes the effect of gravitational interactions between planets B and C has been published by Malhotra (1993). Konacki, Maciejewski, & Wolszczan (2000) have developed a new semi-analytical model in which perturbations between the two planets are parametrized in terms of the two planetary masses and the mutual orientation of the orbits with a sufficient precision to make a practical application of this approach feasible. Using the simulated data, they have demonstrated that the planet masses and hence their orbital inclinations can be derived from a least-squares fit of this model to the pulse times-of-arrival (TOA) measurements spanning a sufficiently long period of time.

In this paper, we describe the results of putting this model to a practical test by applying it to the TOA measurements of PSR B1257+12 made with the 305-m Arecibo telescope over a 12-year period between 1990 and 2003. In Section 2, we present a brief summary of the timing model published by Konacki, Maciejewski, & Wolszczan (2000). Section 3 contains the description of Arecibo timing measurements of PSR B1257+12 and the details and results of data analysis. Consequences of our successful determination of masses and orbital inclinations of planets B and C in the PSR B1257+12 system and future prospects for new pulsar planet detections are discussed in Section 4.
2. The timing model

As the first approximation, the orbits of the PSR B1257+12 system can be described in terms of the sum of the Keplerian motions of its planets, in which case the direct and indirect gravitational interactions between planets are negligible. However, as predicted by Rasio, Nicholson, Shapiro, & Teukolsky (1992), Malhotra, Black, Eck, & Jackson (1992), and Peale (1993), the 3:2 commensurability of the orbital periods of planets B and C does lead to observable deviations from the simple Keplerian model (Wolszczan 1994; Konacki, Maciejewski, & Wolszczan 1999).

In the classical approach, such departures from the Keplerian dynamics can be described by means of the so-called osculating orbital elements by invoking the Keplerian orbital elements that are no longer constant but they change in time due to mutual interactions between planets. For PSR B1257+12 timing, the difficulty with this approach is that only a radial component of the spatial motion of the pulsar can be measured and a complete information on the orientation of orbits of the planets is not available. Below, we summarize an approach developed by Konacki, Maciejewski, & Wolszczan (2000) that addresses this problem and allows to accurately calculate the osculating elements of planetary orbits in the presence of gravitational perturbations.

In the new timing model, we define the TOA variations $\Delta \tau(t)$ as a sum of the two components

$$\Delta \tau(t) = \Delta \tau_{\text{kep}}(x_j^0, e_j^0, \omega_j^0, P_j^0, T_{pj}^0, t) + \delta \tau_{\text{int}}(\Delta x_j, \Delta e_j, \Delta \omega_j, \Delta P_j, \Delta T_{pj}, t) \quad (1)$$

Here, $\Delta \tau_{\text{kep}}$ describes the TOA variations due to the Keplerian part of the motion and is a function of the instantaneous values of the Keplerian elements of planets at the moment $t_0$. For a $j$-th planet, these are the projected semi-major axis of the pulsar orbit $x_j^0$, eccentricity $e_j^0$, longitude of the periastron $\omega_j^0$, orbital period $P_j^0$, and time of the periastron passage $T_{pj}^0$, respectively. The second term in Eqn. (1), $\delta \tau_{\text{int}}$, describes the TOA variations caused by changes in the osculating orbital elements, $\Delta x_j, \Delta e_j, \Delta \omega_j, \Delta P_j, \Delta T_{pj}$. These variables are functions of masses of the pulsar and the planets and of the relative geometry of the orbits.

In the model, the masses are expressed in terms of two parameters, $\gamma_B = m_B/M_{\text{psr}}$, and $\gamma_C = m_C/M_{\text{psr}}$, where $m_B, m_C, M_{\text{psr}}$ are the masses of the two planets and the pulsar, respectively. As shown in Konacki, Maciejewski, & Wolszczan (2000), geometry of the orbits can be described in terms of a relative position of the node of the orbits, $\tau$, if their relative inclination is small ($I \leq 10^\circ$). The parameter $\tau$ is related to the longitudes of the ascending nodes, $\Omega_B, \Omega_C$, and the orbital inclinations $i_B, i_C$ through the following set of equations (see
also Fig. 1 in Konacki, Maciejewski, & Wolszczan 2000)

\[
\cos(I/2)\sin(\tau/2) = \sin((\Omega_C - \Omega_B)/2)\cos((i_C + i_B)/2),
\]

\[
\cos(I/2)\cos(\tau/2) = \cos((\Omega_C - \Omega_B)/2)\cos((i_C - i_B)/2)
\]

(2)

As long as the above small-angle approximation is valid, the parameters \(\gamma_B, \gamma_C\), and \(\tau\) represent an accurate description of the time evolution of the osculating orbital elements. Consequently, Eqn. (1) can be rewritten as

\[
\Delta \tau(t) = \Delta \tau_{\text{kep}}(x_j^0, e_j^0, \omega_j^0, P_j^0, T_{pj}^0, t) + \delta \tau_{\text{int}}(\gamma_B, \gamma_C, \tau, t)
\]

(3)

to define, in general terms, a modified timing model which includes familiar quantities, \(x_j^0, e_j^0, \omega_j^0, P_j^0, T_{pj}^0\), to parametrize Keplerian orbits and introduces three additional parameters, \(\gamma_B, \gamma_C\), and \(\tau\), to account for the perturbations between planets B and C. The correctness of this approach has been verified by extensive simulations described by Konacki, Maciejewski, & Wolszczan (2000).

3. Observations and data analysis

PSR B1257+12 has been observed with the Arecibo telescope since its discovery in 1990 (Wolszczan 1990). In the years 1990-1994, before the Arecibo upgrade, the pulsar had been timed at 430 MHz and 1400 MHz with the Princeton Mark-III backend that utilizes two 32-channel filterbanks (for a description see Stinebring et al. 1992). Starting in 1994, just before the beginning of the Arecibo upgrade, the pulsar had also been timed with a 128-channel filterbank-based Penn State Pulsar Machine (PSPM; Cadwell 1997) at 430 MHz. These observations were very useful in determining the timing offsets between the Mark-III and the PSPM data sets and in eliminating another offset that was found between the data acquired with the PSPM before and after the Arecibo upgrade. Observations with the PSPM were resumed after the upgrade in November 1997. Since then, the pulsar has been systematically timed at 430 MHZ and 1400 MHz at 3-4 week intervals. A more detailed description of the data acquisition and the TOA measurement process can be found in Wolszczan et al. (2000).

The timing model included the pulsar spin and astrometric parameters, Keplerian elements of the orbits of planets A, B, and C, and the three variables \(\gamma_B, \gamma_C\), and \(\tau\) introduced to parametrize perturbations between planets B and C, as described above. A propagation delay and its long-term decline due to the varying line-of-sight electron density were parametrized in terms of the dispersion measure (DM) and its first three time derivatives. Low-amplitude DM variations on the timescales of hundreds of days have been
removed by means of direct measurements of local DM values averaged over consecutive 3 month intervals. The most recent version of the timing analysis package TEMPO (see http://pulsar.princeton.edu/tempo), was modified to incorporate this model and to least-squares fit it to the observed topocentric TOAs. The final best-fit residuals for daily-averaged TOAs are characterized by a ∼3.0 μs rms noise which is consistent with a predicted value of ∼2.0 μs based on the observing parameters and the system performance (e.g. Foster & Backer 1990). The residuals for three fits to data involving different sets of parameters are shown in Fig. 1 and the model parameters for the final fit of the full timing model are listed in Tables 1 and 2.

The new timing model for PSR B1257+12 offers further improvement of the accuracy of the determination of the standard pulsar and planetary parameters and, most importantly, it includes highly significant values for the three perturbation-related parameters, γ_B, γ_C, and τ (Fig. 2). From \( m_B = γ_B M_{psr} \), \( m_C = γ_C M_{psr} \), one obtains the masses of planets B and C to be 4.3±0.2\( M_\oplus \) and 3.9±0.2\( M_\oplus \), respectively, using the canonical pulsar mass, \( M_{psr} = 1.4 M_\odot \). Since the scatter in the measured neutron star masses is small (Thorsett & Chakrabarty 1999), it is unlikely that a possible error in the assumed pulsar mass would significantly affect these results. Because of the \( \sin(i) \) ambiguity, there are four possible sets of the orbital inclinations for the planets B and C: (53°, 47°), (127°, 133°) corresponding to the difference in the ascending nodes \( \Omega_C - \Omega_B \approx 0° \) (relative inclination \( I \approx 6° \)), and (53°, 133°), (127°, 47°), corresponding to the difference in the ascending nodes \( \Omega_C - \Omega_B \approx 180° \) (relative inclination \( I \approx 174° \)). Obviously, in both cases the planets have nearly coplanar orbits, but in the latter one, their orbital motions have opposite senses. Because our numerical simulations of the system’s dynamics show that this situation leads to distinctly different perturbative TOA variations that are not observed, only the first two sets of the orbital inclinations, 53°±4° and 47°±3° or 127° and 133° are plausible. This implies that the two planets move in nearly coplanar orbits in the same sense. In addition, with the known value of τ (Table 2), one obtains \( \Omega_C - \Omega_B \approx 3° \) or \( \Omega_C - \Omega_B \approx -3° \) from equation (2). Since it is reasonable to assume that the inner planet A is in the same plane, its mass given in Table 2 has been calculated for orbital inclination of 50°. Although the formal errors of the orbital inclinations allow their relative inclination, \( I \), to be as high as ∼13°, such a departure from the model assumption of \( I \leq 10° \) would have little effect on the best-fit masses of the planets (Konacki, Maciejewski, & Wolszczan 2000).
4. Discussion

The results described in this paper demonstrate that, under special circumstances created by the existence of measurable gravitational perturbations between planets B and C in the PSR B1257+12 system, it is possible to determine their true masses and orbital inclinations. A near 3:2 MMR between the orbits of the two planets and the fact that they are nearly coplanar imply that the pulsar system has been created as the result of a disk evolution similar to that invoked to describe planet formation around normal stars (Boss 2003). This represents a firm observational constraint which requires that any viable theory of the origin of the pulsar planets provides means to create a circumstellar disk of matter that survives long enough and has a sufficient angular momentum to enable planet formation. Continuing timing observations of PSR B1257+12 will eventually settle the problem of a fourth, more distant planet (or planets) around it (Wolszczan et al. 2000) and provide further constraints on the origin and evolution of this planetary system.

Another important consequence of the determination of true masses of planets B and C is the implied long-term stability of the pulsar system. This problem has been investigated by Rasio, Nicholson, Shapiro, & Teukolsky (1992) and Malhotra, Black, Eck, & Jackson (1992), who have established that the two planets would have to be as massive as 2-3 Jupiter masses to render the system dynamically unstable on a $10^4 - 10^5$ yr timescale. Obviously, the measured, terrestrial masses of the planets (Table 2) are much too low to create such a condition. In fact, this conclusion is not surprising, given another result of Malhotra, Black, Eck, & Jackson (1992), who have calculated that, if the masses of the two planets were about 20-40$M_\oplus$, the system would be locked in the exact 3:2 MMR and the character of perturbations would be very different from the observed near-resonance configuration.

The early theories of pulsar planet formation have been summarized by Podsiadlowski (1993) and further discussed by Phinney & Hansen (1993). More recently, Miller & Hamilton (2001) and Hansen (2002) have examined the conditions of survival and evolution of pulsar protoplanetary disks. They have concluded that an initially sufficiently massive (> $10^{28}$g) disk would be able to resist evaporation by the pulsar accretion flux and create planets on a typical, $\sim 10^7$-year timescale. A quick formation of a massive disk around the pulsar could, for instance, be accomplished by tidal disruption of a stellar companion (Stevens, Rees, & Podsiadlowski 1992; Phinney & Hansen 1993) or, possibly, in the process of a white dwarf merger (Podsiadlowski, Pringle, & Rees 1991; Livio, Pringle, & Saffer 1992). Both these processes, although entirely feasible, cannot be very common. In fact, with the exception of PSR B1257+12, no planetary companions have emerged from the precision timing of 48 galactic millisecond pulsars (Lorimer 2001), implying their rarity, independently of the specific formation mechanism.
Since the current evidence points to isolated millisecond pulsars as best candidates for a presence of planetary companions around them (Wolszczan 1997; Miller & Hamilton 2001), new detections of such objects by the ongoing and future pulsar surveys will be very important. So far, only 10 solitary millisecond pulsars, including PSR B1257+12, have been discovered. This remains in stark contrast with the sample of about 2000 solar-type stars that are included in the Doppler surveys for extrasolar planets (Marcy, Butler, Fischer, & Vogt 2002). Factors to be taken into account while designing pulsar planet searches must include the fact that such pulsars are less common and appear to be intrinsically even fainter than more typical, binary millisecond pulsars (Bailes et al. 1997). In addition, if the high space velocity of PSR B1257+12 (∼300 km s⁻¹, Table 1) and the fact that it has planets were causally connected, such objects would spend most of the time near turnover points of their galactic orbits, which would make them difficult to detect. Altogether, it appears that further improvement of the statistics of neutron star planetary systems may be a lengthy process, even if they are similar to those established for the occurrence of giant planets around normal stars (Marcy, Butler, Fischer, & Vogt 2002).

M. K. is a Michelson Postdoctoral Fellow and is partially supported by the Polish Committee for Scientific Research, Grant No. 2P03D 001 22. A. W. is supported by the NASA grant NAG5-4301 and by the NSF under grant AST-9988217. The authors thank the referee, Renu Malhotra, for insightful comments on the manuscript.

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Fig. 1.— The best-fit, daily-averaged TOA residuals for three timing models of PSR B1257+12 observed at 430 MHz. The filled circles and triangles represent TOAs measured with the Mark-III and the PSPM backends, respectively. The solid line marks the predicted TOA variations for each timing model. (a) TOA residuals after the fit of the standard timing model without planets. TOA variations are dominated by the Keplerian orbital effects from planets B and C. (b) TOA residuals for the model including the Keplerian orbits of planets A, B and C. Residual variations are determined by perturbations between planets B and C. (c) Residuals for the model including all the standard pulsar parameters and the Keplerian and non-Keplerian orbital effects.

Fig. 2.— Determination of the non-Keplerian parameters describing perturbations between planets B and C in the timing model for PSR B1257+12. (a) $\Delta \chi^2 = \chi^2 - \chi^2_{\text{min}}$ as a function of $\tau$. (b) $\Delta \chi^2$ as a function of planet masses, $m_B$ and $m_C$, obtained from $m_B = \gamma_B M_{\text{psr}}$, $m_C = \gamma_C M_{\text{psr}}$ for the pulsar mass $M_{\text{psr}} = 1.4M_\odot$. Both functions exhibit well-defined minima.
Fig. 1.—

PSR B1257+12, Arecibo, 430 MHz

(a)

(b)

(c)
Fig. 2.—
Table 1. Timing Parameters and Derived Quantities for PSR B1257+12

| Parameter                          | PSR B1257+12               |
|------------------------------------|-----------------------------|
| Right ascension, $\alpha$ (J2000) | 13$^h$00$^m$03.5767(1)     |
| Declination, $\delta$ (J2000)     | 12$^\circ$40'56.4721(3)    |
| Proper motion in $\alpha$, $\mu_\alpha$ (mas/yr) | 45.50(4)                  |
| Proper motion in $\delta$, $\mu_\delta$ (mas/yr) | -84.70(7)               |
| Period, $P$ (ms)                   | 6.21853194840048(3)        |
| Period derivative, $\dot{P}$ ($10^{-20}$) | 11.43341(4)                |
| Epoch (MJD)                        | 49750.0                    |
| Dispersion measure, $DM$ (cm$^{-3}$pc) | 10.16550(3)              |
| $\dot{DM}$ (cm$^{-3}$pc/yr)       | -0.001141(7)              |
| $\ddot{DM}$ (cm$^{-3}$pc/yr$^2$)  | 0.000121(3)               |
| $DM^{(3)}$ (cm$^{-3}$pc/yr$^3$)   | 0.000011(1)               |
| $DM$ distance, $d$ (kpc)           | 0.6(1)                     |
| Transverse velocity, $V_t$ (km/s)  | 273(45)                    |
| Kinematic correction, $\dot{P}_k$ ($10^{-20}$) | 8(3)                      |
| Characteristic age, $t_c$ (Gyr)    | 3(3)                       |
| Surface magnetic field, $B_s$ ($10^8$ G) | 5(2)                      |

*Figures in parentheses are the formal 1\(\sigma\) uncertainties in the last digits quoted. The distance, $d$, is based on the Taylor & Cordes (1993) galactic electron distribution model. $\dot{P}_k$ corrects $\dot{P}$ for accelerations due to proper motion and to vertical and differential accelerations in the Galaxy (Shklovskii 1970; Camilo, Thorsett, & Kulkarni 1994); $t_c = P/2\dot{P}$, $B_s = 3.2 \times 10^{19}(P\dot{P})^{1/2}$. \(\text{1}^\text{a}\)
Table 2. Orbital and Physical Parameters of Planets

| Parameter                                      | Planet A       | Planet B       | Planet C       |
|------------------------------------------------|----------------|----------------|----------------|
| Projected semi-major axis, $x^0$ (ms)          | 0.0030(1)      | 1.3106(1)      | 1.4134(2)      |
| Eccentricity, $e^0$                            | 0.0            | 0.0186(2)      | 0.0252(2)      |
| Epoch of pericenter, $T^0_p$ (MJD)             | 49765.1(2)     | 49768.1(1)     | 49766.5(1)     |
| Orbital period, $P^0_p$ (d)                    | 25.262(3)      | 66.5419(1)     | 98.2114(2)     |
| Longitude of pericenter, $\omega^0$ (deg)     | 0.0            | 250.4(6)       | 108.3(5)       |
| Mass ($M_\oplus$)                              | 0.020(2)       | 4.3(2)         | 3.9(2)         |
| Inclination, solution 1, $i^0$ (deg)           | ...            | 53(4)          | 47(3)          |
| Inclination, solution 2, $i^0$ (deg)           | ...            | 127(4)         | 133(3)         |
| Planet semi-major axis, $a^0_p$ (AU)           | 0.19           | 0.36           | 0.46           |

Non-Keplerian Dynamical Parameters

| Parameter                                      |                |                |                |
|------------------------------------------------|----------------|----------------|----------------|
| $\gamma_B$ ($10^{-6}$)                        |                | 9.2(4)         |                |
| $\gamma_C$ ($10^{-6}$)                        |                | 8.3(4)         |                |
| $\tau$ (deg)                                  |                | 2.1(9)         |                |

\(^a\)Figures in parentheses are the formal 1σ uncertainties in the last digits quoted.