SUB-WEYL BOUNDS FOR $GL(2)$ $L$-FUNCTIONS

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ABSTRACT. In this paper we obtain a sub-Weyl bound for $L(1/2 + it, f)$ for $f$ a Hecke modular form.

1. Introduction

In 1982, A. Good [6] extended the classic bound of Weyl, Hardy and Littlewood [9], [16] to degree two $L$-functions. More precisely, for holomorphic Hecke cusp forms $F$, he proved that the associated $L$-functions satisfy

$$L(1/2 + it, F) \ll t^{1/3 + \varepsilon}$$

on the central line. A simpler proof of this result was given by Jutila [8], and it was generalised to the case of Maass wave forms by Meurman [11]. (The case of general level has been addressed recently in [4].) Though in the last hundred years the bound on the Riemann zeta function has been improved, albeit mildly (Bourgain’s recent work [5] gives the exponent $1/6 - 1/84$), Good’s bound has so far remained unsurpassed. The purpose of this paper is to obtain a sub-Weyl bound using the $GL(2)$ delta method, as introduced in [13], [14]. In fact we will add one more layer in this method by introducing an extra averaging over the spectrum. This is a conductor lowering mechanism and it is effective to deal with the subconvexity problem in the $t$-aspect (and spectral aspect). We now state the main result of this paper. (We have not tried to obtain the best possible exponent.)

**Theorem 1.** Let $t > 1$. Suppose $F$ is a Hecke cusp form for $SL(2, \mathbb{Z})$, then

$$L \left( \frac{1}{2} + it, F \right) \ll t^{\frac{1}{3} - \frac{1}{240} + \varepsilon}.$$  

(1)

The reader will notice that our argument also works for Eisenstein series (as well as Maass forms) and in particular yields a weak, nevertheless sub-Weyl, bound for the Riemann zeta function with exponent $1/6 - 1/480$. But complementing our argument by the classical theory of exponent pairs (not even going beyond Titchmarsh [15]), we can obtain far better bounds. Further results in this direction will appear in an upcoming paper. Also it is conceivable that our new method can be used to break
the long standing Voronoi barrier $O(x^{1/3+\varepsilon})$ for the ‘divisor problem’ for cusp forms

$$\sum_{n\leq x} \lambda_F(n).$$

2. The set up

Suppose $F$ is a holomorphic Hecke cusp form of weight $k_0$ for $SL(2,\mathbb{Z})$ with normalised Fourier coefficients $\lambda_F(n)$, so that the Fourier expansion is given by

$$F(z) = \sum_{n=1}^{\infty} \lambda_F(n) n^{(k_0-1)/2} e(nz)$$

with $e(z) = e^{2\pi iz}$. The associated Hecke $L$-function is given by the Dirichlet series

$$L(s, F) = \sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s}$$

in the half plane $\text{Re}(s) = \sigma > 1$. This extends to an entire function and satisfies the Hecke functional equation. A consequence of which is the approximate functional equation that yields the bound

$$L\left(\frac{1}{2} + it, F\right) \ll t^{\varepsilon} \sup_N \frac{|S(N)|}{N^{1/2}} + t^{(1-\theta)/2}$$

where the supremum is taken over $t^{1-\theta} < N < t^{1+\varepsilon}$, and $S(N)$ are sums of the form

$$S(N) = \sum_{N<n\leq 2N} \lambda_F(n)n^{it}.$$ 

(In fact one has a smoothed version of this sum, but we will not require that extra advantage.) The trivial bound for the sum $S(N)$ gives the convexity bound for $L(1/2+it, F)$. The generalised Riemann Hypothesis predicts square-root cancellation in these sums. For sub-Weyl one would need to show strong cancellations in $S(N)$.

Our first step consists of introducing the Weyl shifts

$$S(N) = \sum_{N<n\leq 2N} \lambda_F(n+h)(n+h)^{it} + O(t^\varepsilon H),$$

where $h \sim H \ll \sqrt{N}t^{1/3-\delta}$ for some $\delta > 0$. (For the error term we are applying the Deligne bound, but all one needs is a Ramanujan bound on average.) It follows that

$$S(N) = S^*(N) + O(Ht^\varepsilon)$$

where

$$S^*(N) = \frac{1}{H} \sum_{h \in \mathbb{Z}} W\left(\frac{h}{H}\right) \sum_{N<n\leq 2N} \lambda_F(n+h)(n+h)^{it}$$

where $W$ is a smooth bump function supported on $[1, 2]$ with $\int W = 1$. 

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We will use the $GL(2)$ delta method to analyse the sum $S^*(N)$. Let $q$ be a prime number of size $Q$ and let $\psi$ be an odd character of $\mathbb{F}_q^\times$. Let $H_k(q, \psi)$ be the set of Hecke-normalized newforms which is an orthogonal Hecke basis of the space of cusp forms $S_k(q, \psi)$. We will use the Petersson trace formula. To this end we will follow the standard notations - $\lambda_f(n)$ will denote the normalized Fourier coefficient of the form $f$, $\omega_f^{-1}$ denotes the spectral weight, $S_\psi(a, b; c)$ is the generalized Kloosterman sum and $J_{k-1}(x)$ is the Bessel function of order $k - 1$. Consider the (Fourier) sum

$$F = \sum_{k=1}^{\infty} W\left(\frac{k-1}{K}\right) \sum_{q \in \mathcal{Q}} \sum_{\psi \bmod q \psi(-1) = -1} \sum_{f \in H_k(q, \psi)} \omega_f^{-1}$$

$$\times \sum_{m, \ell = 1}^{\infty} \lambda_F(m) \lambda_f(m) \psi(\ell) U\left(\frac{m\ell^2}{N}\right)$$

$$\times \sum_{N < n \leq 2N} \sum_{h \in \mathbb{Z}} \lambda_f(n + h)(n + h)^{it} W\left(\frac{h}{H}\right).$$

Here $U$ is a smooth function supported in $[1/2, 3]$, with $U(x) = 1$ for $x \in [1, 2]$, and satisfying $y^j U^{(j)}(y) \ll_j 1$. The diagonal term in the Petersson formula corresponds to $m = n + h$, in which case we are left with the weight function $U((n + h)\ell^2/N)$, with $N + H < n + h \leq 2N + 2H$. Considering the supports of the functions, we see that the weight function is vanishing if $\ell > 1$. Hence the diagonal term is given by

$$\sum_{k=1}^{\infty} W\left(\frac{k-1}{K}\right) \sum_{q \in \mathcal{Q}} \sum_{\psi \bmod q \psi(-1) = -1} \sum_{N < n \leq 2N} \sum_{h \in \mathbb{Z}} \lambda_F(n + h)(n + h)^{it} W\left(\frac{h}{H}\right)$$

which equals $FS^*(N)$, where

$$F = H \sum_{k=1}^{\infty} W\left(\frac{k-1}{K}\right) \sum_{q \in \mathcal{Q}} \sum_{\psi \bmod q \psi(-1) = -1} (-1)^{k+1} 1 \asymp HQ^2K.$$

Hence Petersson formula yields that

$$\frac{|S(N)|}{\sqrt{N}} \ll \frac{|F| + |O|}{\sqrt{NHQ^2K}} + \frac{H \ell^2}{N^{1/2}}$$

(3)
where $O$ stands for the (direct) off-diagonal which is given by

\begin{equation}
O = \sum_{k=1, k\text{ odd}}^\infty W\left(\frac{k-1}{K}\right) \sum_{\psi \text{ mod } q} \psi^{\psi(-1):=-1} \sum_{q \in \mathbb{Q}} \psi(n) \sum_{n \leq N} (n+h)^i W\left(\frac{h}{H}\right) \times \sum_{m, \ell=1}^\infty \lambda_F(m) \psi(\ell) U\left(\frac{m^2}{N}\right) \sum_{N<n<2N} (n+h)^i W\left(\frac{h}{H}\right) \times \sum_{c=1}^\infty \frac{i^{-k} S_\psi(n+h, m; cq)}{cq} J_{k-1}\left(\frac{4\pi \sqrt{m(n+h)}}{cq}\right).
\end{equation}

In the rest of the paper we will prove sufficient bounds for the off-diagonal $O$ and the Fourier sum $F$. In Sections 3 we will prove the following.

**Proposition 1.** Let $H = N/t^{1/3} \ll N^{1/2} t^{1/3-\delta}$. We have

\[ O \ll \sqrt{N} H Q^2 K t^{1/3-\delta} \]

if $K$ and $Q$ satisfy

\[ N \ll \min\{Q^{3/2} K^2 t^{-1/18-2\delta/3}, Q^{3/2} K^{5/2} t^{-1/6-\delta}, Q K^4 t^{-\varepsilon}\}. \]

In Sections 4-8 we will prove the following.

**Proposition 2.** Let $H = N/t^{1/3} \ll N^{1/2} t^{1/3-\delta}$. We have

\[ F \ll \sqrt{N} H Q^2 K t^{1/3-\delta} \]

if $K$ and $Q$ satisfy

\[ Q K^4 \ll N^2 t^{-1/6-\delta}, \text{ or } N \ll Q^2 t^{2/3-2\delta}, \]

and

\[ Q^{1/2} t^{\varepsilon} \ll K \ll \min\{t^{1/3-2\delta}, N t^{-1/3-2\delta}\}, \text{ and } Q K^4 \ll N^{1/2} t^{13/12-\delta}. \]

Moreover $K$, $Q$ should satisfy the inequalities (65), (68), (69), (70) and (71).

Assuming the propositions we now complete the proof of the main theorem. We pick $K$ by using the third inequality from (69). More precisely we set

\[ K = t^{5/18+\theta} \]

where $\theta > 2\delta/3 > 0$ may depend on $N$. Then we pick $Q$ by equating the first inequality in Proposition 1, namely

\[ Q = N^{2/3} t^{1/3+\eta}, \]
with \( \eta = 4\theta/3 - 4\delta/9 \). Then we need to check that all other inequalities are satisfied. The remaining inequalities in Proposition 1 are easily checked. Let \( N = t^{\alpha} \) with \( \alpha \geq 2/3 - 2\delta \). The first set of inequalities in Proposition 2 is satisfied if

\[
\frac{11}{18} + \theta + 2\delta < \alpha < \frac{11}{6} - 16\theta - 9\delta, \quad \text{and} \quad \theta < \frac{1}{18} - 2\delta.
\]

Without aiming to obtain the optimum values, we verify that all the inequalities in (65) are satisfied if we pick \( \theta = 1/72 + \delta \). Then we find that all the inequalities in (68) are satisfied if \( 21/72 + \delta < 1/3 - \delta \), i.e. \( \delta < 1/18 \). The inequalities in (69) are satisfied if \( \delta < 1/240 \). This actually comes from the last inequality in (69). The other two inequalities are satisfied under the much weaker condition \( \delta < 1/48 \). In fact with this choice of \( \delta \) we see that the remaining two sets of inequalities (70), (71) are also satisfied. This completes the proof of Theorem 1.

3. The off-diagonal

In this section we will analyse the off-diagonal \( O \) which is given in (4). Consider the sum over \( k \)

\[
\sum_{k=1}^{\infty} W \left( \frac{k-1}{K} \right) i^{-k} J_{k-1} \left( \frac{4\pi \sqrt{m(n+h)}}{cq} \right).
\]

We recall the formula (see [7])

\[
4 \sum_{u \equiv a \mod 4} g(u) J_a(x) = \int_{\mathbb{R}} \hat{g}(v) c_a(v; x) dv
\]

where

\[
c_a(v; x) = -2i \sin(x \sin 2\pi v) + 2i^{1-a} \sin(x \cos 2\pi v)
\]

and the Fourier transform is defined by

\[
\hat{g}(v) = \int g(u) e(uv) du.
\]

Hence we get that (5) is given by

\[
\int_{\mathbb{R}} \hat{W}(v) \sin \left( 2\pi x \cos \frac{2\pi v}{K} \right) dv,
\]

with \( x = 2\sqrt{m(n+h)/cq} \). This integral can be expressed as a linear combination of the integrals

\[
\int_{\mathbb{R}} \hat{W}(v) e \left( \pm x \cos \frac{2\pi v}{K} \right) dv.
\]

Since the Fourier transform \( \hat{W} \) decays rapidly, we can and will introduce a smooth function \( F \) with support \([-V, V]\) with \( F^{(j)} \ll_j V^{-j} \), \( F(v) = 1 \) for \( |v| \ll t^\epsilon \) and any \( V \).
in the range \( t^\varepsilon \ll V \ll Kt^{-\varepsilon} \), and replace the above integrals (up to negligible error terms) by
\[
\int_{\mathbb{R}^2} W(u)F(v)e\left( uv \pm x\cos \frac{2\pi v}{K} \right) \, du \, dv.
\]
Now let us recall the following result from [2], which will be used throughout this paper.

**Criterion for an exponential integral to be negligibly small:** Let \( Y \geq 1, X, Q, U, R > 0 \). Let \( W \) be a smooth weight function supported on \([\alpha, \beta]\), satisfying \( W^{(j)} \ll jXU^{-j} \), and let \( h \) be a real valued smooth function on the same interval such that \( |h'| \geq R \), and \( h^{(j)} \ll_j YQ^{-j} \) for \( j \geq 2 \). Then

\[
\int_{\mathbb{R}} W(y)e(h(y))dy \ll_A (\beta - \alpha)X \left[ \left( \frac{QR}{\sqrt{Y}} \right)^{-A} + (RU)^{-A} \right],
\]
for any \( A \geq 1 \). In particular the integral is ‘negligibly small’, i.e. \( O(t^{-A}) \) for any \( A > 1 \), if \( R \gg t^{\varepsilon} \max\{Y^{1/2}/Q, 1/U\} \) and \( t \gg [(\beta - \alpha)X]^{\varepsilon} \).

We apply this result to the \( v \)-integral, and it follows that the integral is negligibly small, i.e. \( O(t^{-J}) \) for any \( J \geq 1 \), if \( x \ll K^{2-\varepsilon} \). This analysis holds even if the weight function \( W \) has a little oscillation, say \( W^{(j)} \ll_j t^{j\varepsilon} \). In the complementary range for \( x \) we expand the cosine function into a Taylor series. Since \( x \ll N/Q \), if we assume that

\[
N \ll QK^4t^{-\varepsilon},
\]
then we only need to retain the first two terms in the expansion, and the above integral essentially reduces to
\[
e^{\pm x} \int_{\mathbb{R}^2} W(u)F(v)e\left( uv \mp \frac{4\pi^2 x v^2}{K^2} \right) \, du \, dv.
\]
To the integral over \( v \) we apply the stationary phase analysis. It turns out to be negligibly small (due to (6)) when we have + sign inside the exponential, otherwise the integral essentially reduces to
\[
e\left( x + \frac{u^2 K^2}{16\pi^2} \right) \frac{K}{\sqrt{x}} \sim e(x) \frac{K}{\sqrt{x}}
\]
with \( x \gg K^{2-\varepsilon} \) (upto an oscillatory factor which oscillates at most like \( t^{\varepsilon} \)). In any case, it follows that we can cut the sum over \( c \) in (4) at \( C \ll Nt^{\varepsilon}/QK^2 \), at a cost of a negligible error term. To get the Weyl bound it is sufficient to take \( QK^2 \gg Nt^{\varepsilon} \), so that the off-diagonal is trivially small (see [1]). But to achieve sub-Weyl one needs to take \( QK^2 \) smaller.
The direct off-diagonal $O(4)$ has reduced to

$$K \sum_{q \in \mathbb{Q}} \sum_{\psi \equiv q}^{\psi(-1)=-1} \sum_{m, \ell=1}^{\infty} \lambda_{F}(m) \psi(\ell) U \left( \frac{m \ell^{2}}{N} \right) \times \sum_{n<h, n \leq 2N} \sum_{h \in \mathbb{Z}} (n+h)^{it} W \left( \frac{h}{H} \right) \sum_{c \sim C} S_{\psi}(n+h, m; cq) e \left( \frac{2 \sqrt{m(n+h)}}{cq} \right).$$

Next we apply the Poisson summation formula on the sum over the shifts $h$. By Poisson the sum

$$\sum_{h \in \mathbb{Z}} S_{\psi}(n+h, m; cq) e \left( \frac{2 \sqrt{m(n+h)}}{cq} \right) (n+h)^{it} W \left( \frac{h}{H} \right)$$

gets transformed into

$$H \sum_{h \in \mathbb{Z}} \psi(-h) e \left( -\frac{hn}{cq} - \frac{\bar{h}m}{cq} \right) \mathcal{J}$$

where the integral is given by

$$\mathcal{J} = \int W(y) e(f(y)) dy,$$

with

$$f(y) = \frac{t}{2\pi} \log(n+Hy) + 2 \frac{\sqrt{m(n+Hy)}}{cq} - \frac{Hhy}{cq}.$$

We have the Taylor expansion

$$f(y) = \left[ \frac{t}{2\pi} \log n + 2 \frac{\sqrt{mn}}{cq} \right] + \frac{Hy}{n} \left[ \frac{t}{2\pi} + \frac{\sqrt{mn}}{cq} - \frac{hn}{cq} \right] - \frac{H^{2}y^{2}}{n^{2}} \left[ \frac{t}{4\pi} + \frac{\sqrt{mn}}{4cq} \right] + O \left( \frac{t}{(N/H)^{3}} \right).$$

To restrict the phase function up to the quadratic term we pick $H = N/t^{1/3}$. (The error term is a ‘flat’ function in the sense that $y^{3}E^{(j)}(y) \ll 1$ with respect to all the variables. It should be possible to improve our result by picking $H = N/t^{1/4}$ and allowing up to the cubic term. The argument given below might go through with some modifications. But the expressions would be much more complicated.) Set

$$A = \frac{t}{2\pi} + \frac{\sqrt{mn}}{cq} - \frac{hn}{cq}, \quad \text{and} \quad B = \frac{t}{\pi} + \frac{\sqrt{mn}}{cq},$$
then by stationary phase analysis we can replace (8) by

\[
\frac{KH}{\sqrt{NCT^1/6}} \sum_{q \in \mathbb{Q}} \sum_{\psi \mod q \psi(1) = -1} \sum_{m, \ell = 1}^{\infty} \lambda_F(m) \psi(\ell) \left( \frac{m\ell^2}{N} \right) \\
\times \sum_{c \sim C} \sum_{N < n \leq 2N} \sum_{h \equiv \ell \mod q} \psi(h) n^{it} e \left( \frac{2\sqrt{mn} \cdot cn}{cq} - \frac{hn}{cq} - \frac{\bar{m}h}{cq} + \frac{A^2}{B} \right) W \left( \frac{2nA}{HB} \right).
\]

The last weight function puts the restriction that \( A \approx t^{2/3} \), and consequently

(12) \[ \left| n - \frac{tcq}{2\pi h} \right| \ll N \max \left\{ \frac{1}{t^{1/3}}, \frac{N}{CtQt} \right\} =: N\Delta. \]

and \( h \sim tCQ/N \). Executing the sum over \( \psi \) we arrive at

(13) \[
\frac{KH\sqrt{Q}}{\sqrt{NCT^1/6}} \sum_{q \in \mathbb{Q}} \sum_{m, \ell = 1}^{\infty} \lambda_F(m) U \left( \frac{m\ell^2}{N} \right) \\
\times \sum_{c \sim C} \sum_{N < n \leq 2N} \sum_{h \equiv \ell \mod q} \psi(h) n^{it} e \left( \frac{2\sqrt{mn} \cdot cn}{cq} - \frac{hn}{cq} - \frac{\bar{m}h}{cq} + \frac{A^2}{B} \right) W \left( \frac{2nA}{HB} \right).
\]

Note that the weight function retains the restrictions we stated above.

Applying Cauchy we get that the above expression is bounded by

(14) \[ \sum_{\ell} \frac{KH\sqrt{Q}}{\sqrt{Ct^{1/6}}} \Omega^{1/2}_\ell \]

where \( \Omega_\ell \) is given by

(15) \[ \sum_{m \sim N/\ell^2} \left| \sum_{q \in \mathbb{Q}} \sum_{c \sim C} \sum_{N < n \leq 2N} \sum_{h \equiv \ell \mod q} \psi(h) \psi(h) n^{it} e \left( \frac{2\sqrt{mn} \cdot cn}{cq} \right) e \left( -\frac{hn}{cq} - \frac{\bar{m}h}{cq} + \frac{A^2}{B} \right) \right|^2 W \left( \frac{2nA}{HB} \right). \]

Next we open the absolute value square. In the case of small gap, i.e.

(16) \[ c_1 q_1 h_1 - c_2 q_2 h_2 \ll \Delta \frac{(CQ)^2}{N} =: M/\ell, \]

we estimate the sum trivially. The number of \((n_1, n_2)\) pairs is given by \( 1 + (\Delta N)^2 \) once the other variables are given. It is, however, non-trivial to count the number of \((q_i, c_i, h_i)\). Setting \( h_i \ell = 1 + g_i q_i \) we get that

(17) \[ c_1 g_1 q_1^2 - c_2 g_2 q_2^2 \ll M. \]

Let us now try to get a bound for the count. Consider the equation

(18) \[ a_1 q_1^2 - a_2 q_2^2 = v \]

with \( a_i \sim tC^2/N, q_i \sim Q \) and \( v \ll M \). Fix \( a_1, a_2 \), and set \( \alpha = \sqrt{a_2/a_1} \). We seek to count the number of \( q_i \) such that \( |q_i - \alpha q_2| \ll MN/\ell C^2 Q \). Given \( q_2 \) there are
at most one \(q_1\), and so the counting reduces to finding the number of \(q_2\) such that \(\| \alpha q_2 \| \ll MN/tC^2Q\). So the count for the number of solutions of (18) is given by

\[
\sum_{\alpha_1, \alpha_2} \sum_{q_2} 1_{\| \alpha q_2 \| \ll MN/tC^2Q}.
\]

The trivial count at this stage is given by \((tC^2/N)^2Q\), which gives a saving of \(Q\) over the easy bound \((tC^2Q/N)^2\). To get a better bound we observe that the condition on \(\| \alpha q_2 \|\) can be detected by

\[
\frac{MN}{tC^2Q} \sum_{u \sim tC^2Q/MN} e(\alpha q_2 u)\ |
\]

Then by Cauchy and large sieve we can save at least

\[
\min\{C\sqrt{t/N}, \sqrt{tQC/\sqrt{MN}}\} = \sqrt{tQC/\sqrt{MN}}.
\]

It follows that the contribution of these terms to (13) is bounded by

\[
K HQ \frac{N^{2.5/12}Q^{5/4}}{K^3}.
\]

(Note that \(K \ll t^{1/3}\).) This is a satisfactory bound if

\[
N \ll Q^{3/2}K^2t^{-1/18-28/3}.
\]

In the complementary range when the gap is not small, i.e. (16) does not hold, opening the absolute value square in (15) we apply the Poisson summation on the sum over \(m\). We get that the contributions of these terms to (15) is bounded by

\[
\frac{N}{\ell^2} \sum_{m \in \mathbb{Z}} \sum_{q_1,q_2 \in \mathbb{Q}} \sum_{c_1,c_2 \sim C} \sum_{N<n_1,n_2 \leq 2N} \sum_{h_i \equiv \pm \ell \mod q_i} \sum_{c_1q_1h_1-c_2q_2h_2 \equiv M/\ell} |S_2|\]

where the integral is given by

\[
S_2 = \int W(y) W\left(\frac{2n_1A_1}{HB_1}\right) W\left(\frac{2n_2A_2}{HB_2}\right) e\left[A_1^*y + A_2^*y^2 + A_3^*y^3 + \ldots\right] dy.
\]

The coefficients in the Taylor expansion can be computed explicitly, and we get

\[
A_1^* = \frac{2\sqrt{N}}{\ell} \left(\frac{\sqrt{n_1}}{c_1q_1} - \frac{\sqrt{n_2}}{c_2q_2}\right) + O(E)
\]

and

\[
A_2^* = \frac{N}{\ell^2} \left(\frac{\pi}{t} \left(\frac{n_1}{(c_1q_1)^2} - \frac{n_2}{(c_2q_2)^2}\right) - \frac{m}{c_1c_2q_1q_2}\right) + O(E),
\]

and \(A_j^* = O(E)\) for all \(j \geq 3\), where

\[
E = \frac{N\Delta}{Q\ell}.
\]
Note that the $j$-th derivative of the weight function in the integral is bounded by $(N/t^{1/3}QC\ell)^j \ll E_j$. Since we are in the case where (16) does not hold, we get that the first term of $A_j^\ast$ is larger than the error term $E_j$ (recall (12)), and hence it follows that the integral is negligibly small unless

$$0 \neq m \asymp \frac{(QC\ell)^2}{N} \sqrt{N} \left( \frac{\sqrt{n_1}}{c_1q_1} - \frac{\sqrt{n_2}}{c_2q_2} \right) = M_s.$$ 

In this case we estimate the integral using the second derivative bound

$$\mathcal{J} \ll \frac{C\ell Q}{\sqrt{N|m|}}.$$ 

It now remains to count the number of $(m, h_1, h_2, c_1, c_2, q_1, q_2)$ satisfying the congruence conditions. This is not an easy task, and we only seek to obtain a good upper bound. Given $(m, c_1, q_1)$, there are at most $O(tC/N)$ many $h_1$. Then $c_2q_2$ is determined modulo $c_1q_1$. Hence there are at most $O(tC)$ many $(c_2, q_2)$. Finally we get at most $O(tC/N)$ many $h_2$. It follows that the number of vectors is bounded by

$$t^\varepsilon M_s CQ \left( \frac{tC}{N} \right) \left( \frac{t}{N} \right) \ll t^\varepsilon M_s Q \left( \frac{tC}{N} \right)^2.$$ 

Then we count the number of $n_i$ using the restriction (12). It follows that the contribution of these terms without ‘small gap’, to (14) is dominated by

$$\sum_{\ell} KH \sqrt{Q} \frac{N^{1/4}C^{7/4}Q^{5/4}t^{1/3}}{\ell^{1/4}} \ll \sqrt{N} KHQ^2 t^{1/3} \frac{N t^{1/6}}{K^{5/2} Q^{3/2}},$$

which is a satisfactory bound if

$$(21) \quad N \ll Q^{3/2} K^{5/2} t^{-1/6 - \delta}.$$ 

This completes the proof of Proposition 1. (Note that the last bound is off from the expected bound by a factor of $Q^{1/2}$, as we lost a congruence modulo $q_2$ in our count.)

4. Applying functional equation: Dual side

Consider the Fourier sum $F(2)$, where we will apply the functional equation of the $L$-function $L(s, f)^2$, to dualise the sum over $(m, \ell)$. By the Mellin inversion formula we get

$$(22) \quad \sum_{m, \ell} \lambda_F(m)\lambda_f(m)\psi(\ell)U \left( \frac{m\ell^2}{N} \right) = \frac{1}{2\pi i} \int_{(2)} N^s \tilde{U}(s)L(s, f)^2 ds,$$

where $\tilde{U}$ stands for the Mellin transform of $U$. Using the functional equation of the $L$-function $L(s, F \times f)$ we get

$$\frac{1}{2\pi i} \int_{(2)} N^s \tilde{U}(s) \eta^2 \left( \frac{q}{4\pi^2} \right)^{1-2s} \frac{\gamma_k(1-s)}{\gamma_k(s)} L(1-s, \tilde{f})^2 ds,$$
where
\[ \gamma_k(s) = \Gamma\left(s + \frac{(k-k_0)}{2}\right) \Gamma\left(s + \frac{(k+k_0)}{2} - 1\right). \]
The sign of the functional equation is given by
\[ \eta^2 = i^{2k} \frac{g_\psi^2}{\lambda_f(q^2)q} \]
where \( g_\psi \) is the Gauss sum associated with the character \( \psi \). Next we expand the \( L \)-function into a Dirichlet series and take dyadic subdivision. By shifting contours to the right or left we can show that the contribution of the terms from the blocks with \( ml^2 \notin [Q^2K^4t^{-\varepsilon}/N, Q^2K^4t^{\varepsilon}/N] \) is negligibly small. Hence the sum in (22) essentially gets transformed into
\[ \varepsilon_\psi^2 \frac{N}{Q^2K^2} \frac{\gamma_k(1/2 + i\tau)}{\gamma_k(1/2 - i\tau)} \sum_{m,\ell=1}^{\infty} \sum_{\lambda_f(m)\lambda_f(mq^2)\overline{\psi}(\ell)} U\left(\frac{ml^2}{N}\right) \]
where \( \varepsilon_\psi \) is the sign of the Gauss sum \( g_\psi \) and
\[ \tilde{N} \asymp \frac{Q^2K^4}{N}. \]
(In this paper the notation \( A \asymp B \) means that \( B/t^\varepsilon \ll A \ll Bt^\varepsilon \), with implied constants depending on \( \varepsilon \).) We are keeping a \( \tau \), with \( |\tau| \ll t^\varepsilon \), in the gamma factor as we need to keep track of possible oscillation in the \( k \) aspect. Now let us study the gamma factor. Using Stirling series
\[ \Gamma(z) = \sqrt{2\pi} \left(\frac{z}{e}\right)^z \left(\sum_j a_j \frac{z^j}{z^j} + O(|z|^{-j})\right) \]
which holds for \( z = k/2 + i\tau \) as above, it turns out that this ratio of the gamma functions is essentially equivalent to
\[ e(\text{arg}([k/2 + i\tau]^{k/2+i\tau})/\pi) = e(\tau \log(k^2/4 + \tau^2)/2\pi + k \text{arg}(k/2 + i\tau)/2\pi). \]
Now expanding \( \log(k^2/4 + \tau^2) = 2 \log(k/2) + O(\tau^2/k^2) \), and
\[ \text{arg}(k/2 + i\tau) = \tan^{-1}(2\tau/k) = 2\tau/k + O(\tau^3/k^3), \]
we get that the ratio of the gamma functions essentially behaves like \( k^{4i\tau} \). So the gamma factor oscillates mildly, which can be neglected.

This reduces the analyses of the sum in (22) to that of the sums of the type
\[ D = \frac{N}{K^2} \sum_{k=1, k \text{ odd}}^{\infty} W\left(\frac{k-1}{K}\right) \sum_{n \sim N} \sum_{\psi \mod q} \varepsilon_\psi^2 \sum_{f \in H_k(q,\psi)} \sum_{\omega_f^{-1}} \lambda_f(m)\lambda_f(mq^2) \overline{\psi}(\ell) \sum_{h \in \mathbb{Z}} \lambda_f(n+h)(n+h)^\ell W\left(\frac{h}{H}\right). \]
Observe that we have dropped some of the smooth weight functions, as they do not play any role whatsoever in the upcoming analysis. Also we have dropped the $q$ sum, at the cost of a multiplier of size $Q$. Indeed the $q$ sum is not involved in the dual side at all. Next one writes $m = q^\nu m'$ with $q \nmid m'$, so that $\lambda_f(m') = \lambda_f(m) \bar{\psi}(m')$. We get

$$
\frac{N}{K^2} \sum_{\nu=0}^\infty \lambda_F(q^\nu) \sum_{k=1 \atop k \text{ odd}} W \left( \frac{k-1}{K} \right) \sum_{n \sim N} \sum_{\psi \bmod q \atop \psi(-1) = -1} \varepsilon_\psi^2 \sum_{f \in H_k(q,\psi)} \omega_{f^{-1}}^{-1}

\times \sum_{m \ell^2 \sim N/q^\nu} \lambda_F(m) \lambda_f(m) \bar{\psi}(m\ell) \sum_{h \in \mathbb{Z}} \frac{\lambda_f(q^2+\nu(n+h))(n+h)^i}{h} W \left( \frac{h}{H} \right).
$$

We apply the Petersson formula to this dual sum. There is no diagonal contribution as $\psi(m) = 0$ when $q|n$. Hence we are only left with the (dual) off-diagonal which is given by

$$
\mathcal{O}^* = \frac{N}{K^2} \sum_{\nu=0}^\infty \lambda_F(q^\nu) \sum_{k=1 \atop k \text{ odd}} W \left( \frac{k-1}{K} \right) \sum_{n \sim N} \sum_{\psi \bmod q \atop \psi(-1) = -1} \varepsilon_\psi^2

\times \sum_{m \ell^2 \sim N/q^\nu} \lambda_F(m) \lambda_f(m) \bar{\psi}(m\ell) \sum_{h \in \mathbb{Z}} \frac{\lambda_f(q^2+\nu(n+h))(n+h)^i}{h} V \left( \frac{h}{H} \right)

\times \sum_{c=1}^\infty \frac{i^{-k}}{c} S_{\psi}(q^2+\nu(n+h), m; cq) J_{k-1} \left( \frac{4\pi \sqrt{mq^\nu(n+h)}}{c} \right).
$$

Consider the sum over $k$ which is given by

$$
\sum_{k=1 \atop k \text{ odd}}^\infty W \left( \frac{k-1}{K} \right) i^{-k} J_{k-1} \left( \frac{4\pi \sqrt{mq^\nu(n+h)}}{c} \right).
$$

This sum is exactly same as we had for the direct off-diagonal before (only now it is independent of $q$ in the generic case $\nu = 0$). Notice that we have deliberately dropped the gamma factors, which were mildly oscillating. So we assume that $W^{(j)} \ll_j t^j$ in the present case. Temporarily we set $x = 2\sqrt{mq^\nu(n+h)}/c$. We choose to have

$$
QK^2 \ll K^4 t^{-\varepsilon},
$$

so that we can again restrict ourselves to the quadratic phase in the expansion, and the above sum essentially reduces to

$$
e(\pm x) \int \int_{\mathbb{R}^2} W(u)F(v)e \left( \frac{4\pi^2 xuv^2}{K^2} \right) \mathrm{d}u \mathrm{d}v.
$$

As before this reduces to

$$
e(x) \frac{K}{\sqrt{x}}.$$
with \( x \gg K^{2-\varepsilon} \). In the complementary range the integral is negligibly small. With this the off-diagonal essentially reduces to

\[
\frac{N\sqrt{\ell}}{\sqrt{C}K^2Q^{3/2}} \sum_{n \sim N} \sum_{\psi \pmod{q} \nu(1) = -1} \varepsilon_{\psi}^{2} \sum_{\nu = 0}^{\infty} \lambda_{F}(q^{\nu}) \sum_{m \sim N/q^{\nu}\ell^{2}} \lambda_{F}(m) \psi(m\ell)
\]

\[
\times \sum_{h \in \mathbb{Z}} (n + h)^{it} W \left( \frac{h}{H} \right) \sum_{c \sim C} S_{\psi}(q^{2+\nu}(n + h), m; cq) e \left( \frac{2\sqrt{mq^{\nu}(n + h)}}{c} \right).
\]

with \( C \ll Qt^{\varepsilon}/\ell \). If \( q \parallel c \) then the Kloosterman sum vanishes as \( q \nmid m \), so we necessarily have \((c, q) = 1\). The Kloosterman sum splits, and we get that

\[
\sum_{\psi \pmod{q} \nu(1) = -1} \varepsilon_{\psi}^{2} \bar{\psi}(m\ell) S_{\psi}(q^{2+\nu}(n + h), m; cq)
\]

is a difference of two terms

\[
qS(q^{\nu}(n + h), m; c) e \left( \pm \frac{c\ell}{q} \right).
\]

The last factor is non-oscillating in the generic situation, as we only need to consider \( c \) in the range \( c \ll Qt^{\varepsilon} \) (and \( \ell \) can be as small as 1). Note that we are dropping the sum over \( \ell \) as the object is essentially independent of \( \ell \). Of course we need to execute the sum over \( \ell \) trivially at the end. Also we will drop the factor \( e(c\ell/q) \) from the expressions, as the sums over \( \ell \) and \( q \) are executed trivially at the end, and in our analysis below we will not apply any summation formula on the sum over \( c \). In the expressions below always bear in mind that the \( c \) sums have some arithmetic weight of size 1, which does not depend on any other sums in the expressions. So we continue our analysis with

\[
\frac{N\sqrt{\ell}}{\sqrt{C}K^2Q^{3/2}} \sum_{n \sim N} \sum_{\nu = 0}^{\infty} \lambda_{F}(q^{\nu}) \sum_{m \sim N/q^{\nu}\ell^{2}} \lambda_{F}(m)
\]

\[
\times \sum_{h \in \mathbb{Z}} (n + h)^{it} W \left( \frac{h}{H} \right) \sum_{c \sim C} S(n + h, mq^{\nu}; c) e \left( \frac{2\sqrt{mq^{\nu}(n + h)}}{c} \right).
\]

Now we execute the sum over \( \nu \) by gluing \( q^{\nu} \) back to \( m \), and this yields

\[
\frac{N\sqrt{\ell}}{\sqrt{C}K^2} \sum_{n \sim N} \sum_{m \sim N/q^{\nu}\ell^{2}} \lambda_{F}(m)
\]

\[
\times \sum_{h \in \mathbb{Z}} (n + h)^{it} W \left( \frac{h}{H} \right) \sum_{c \sim C} S(n + h, m; c) e \left( \frac{2\sqrt{m(n + h)}}{c} \right).
\]
At this point we can apply the Voronoi summation to get a bound which is satisfactory for small values of $C$ and $N$. This plays a role for $N$ near $t^{2/3}$. Indeed applying Voronoi summation we get that (29) is bounded by

$$ N^{1/2}HQ^2Kt^{1/3} \frac{N^{3/2}C}{(QK)^2t^{1/3}} $$

which is satisfactory if

$$ C \ll \frac{(QK)^{21/3 - \delta}}{N^{3/2} \ell}. $$

5. Stationary phase analysis for dual off-diagonal

Consider the sum over $h$ in (29), which is given by

$$ \sum_{h \in \mathbb{Z}} (n + h)^itS(n + h, m; c) e \left( \frac{2\sqrt{m(n+h)}}{c} \right) W \left( \frac{h}{H} \right). $$

This is structurally different from the $h$ sum we had in the direct off-diagonal (9). We apply the Poisson summation formula with modulus $c$ to arrive at

$$ \frac{H}{c} \sum_{h \in \mathbb{Z}} \mathcal{C} \mathcal{J} $$

where the character sum is given by

$$ \mathcal{C} = \sum_{b \mod c} S(b + n, m; c) e \left( \frac{bh}{c} \right) = c \left( -\frac{hm}{c} - \frac{n}{c} \right), $$

and the integral is given by

$$ \mathcal{J} = \int W(y)e(f(y))dy, $$

where (we temporarily set)

$$ f(y) = \frac{t}{2\pi} \log(n + Hy) + \frac{2\sqrt{mn + Hy}}{c} - \frac{Hy}{c}. $$

We have the Taylor expansion

$$ f(y) = \left[ \frac{t}{2\pi} \log n + \frac{2\sqrt{mn}}{c} \right] + \frac{Hy}{n} \left[ \frac{t}{2\pi} + \frac{\sqrt{mn}}{c} - \frac{hn}{c} \right] $$

$$ - \frac{H^2y^2}{n^2} \left[ \frac{t}{4\pi} + \frac{\sqrt{mn}}{4c} \right] + O(1). $$

(The error term is a ‘flat’ function in the sense that $y^j E^{(j)}(y) \ll 1$ with respect to all the variables.) Notice that the phase function is exactly similar to what we had in the
direct off-diagonal, with the only difference that we have \( c \) in place of \( cq \). Applying
the stationary phase expansion it follows that the dual off-diagonal is given by

\[
\frac{NH\sqrt{\ell}}{\sqrt{CQK^2t^{1/6}}} \sum_{m \sim N/\ell^2} \lambda_F(m) \\
\times \sum_{c \sim C} \sum_{N < n \leq 2N} \sum_{h \sim tC/N} n^t e \left( \frac{2\sqrt{mn}}{c} - \frac{hn}{c} - \frac{\bar{h}m}{c} + \frac{A^2}{B} \right) W \left( \frac{2nA}{HB} \right),
\]

where

\[
A = \frac{t}{2\pi} + \frac{\sqrt{mn}}{c} - \frac{hn}{c}, \quad B = \frac{t}{\pi} + \frac{\sqrt{mn}}{c}
\]

are as in (13) with \( c \) in place of \( cq \). It follows that we have \( A \approx t^{2/3} \), This can be used
to conclude the following restrictions

\[
D := \frac{hn}{c} - \frac{t}{2\pi} \ll t \max \left\{ \frac{1}{t^{1/3}}, \frac{QK^2}{Ct} \right\} =: t\Delta
\]

and

\[
\left| m - \frac{D^2c^2}{n} \right| \ll \frac{Q^2K^4}{N\ell^2} \min \left\{ 1, \frac{Ct^{2/3}}{QK^2} \right\} = \tilde{N} \frac{1}{\ell^2} \frac{1}{t^{1/3}\Delta}.
\]

We derive that (33) is bounded by

\[
\sqrt{NHQ^2Kt^{1/3}} \frac{C^{3/2}K^{t^{1/6}}}{\sqrt{NQ\ell^{3/2}}},
\]

which is not sufficient for our purpose. However this bound is fine for smaller values
of \( C \), namely in the range

\[
C \ll \frac{(NQ)^{1/3}t^{-28/3}\ell^{1/3}}{K^{2/3}t^{1/9}}.
\]

Below we proceed with \( C \) which lies in the range complementary to both (30) and (37).

6. Cauchy for dual off-diagonal

We apply the Cauchy inequality to bound (33) by

\[
\frac{N^{3/2}H\sqrt{\ell}}{\sqrt{CQK^2t^{1/6}}} \Omega_{\ell}^{1/2}
\]

where now \( \Omega_{\ell} \) is given by

\[
\sum_{n \sim N} \sum_{m \sim N/\ell^2} \lambda_F(m) \sum_{c \sim C} \sum_{h \sim tC/N} e \left( \frac{2\sqrt{mn}}{c} - \frac{hn}{c} - \frac{\bar{h}m}{c} + \frac{A^2}{B} \right) W \left( \frac{2nA}{HB} \right)^2.
\]
We open the absolute square to arrive at
\[
\sum_{n \in \mathbb{Z}} W \left( \frac{n}{N} \right) \sum_{m_1, m_2 \sim \mathcal{X}/\varepsilon^2} \sum_{h_1, h_2 \sim C/N} \sum_{c_1, c_2 \sim C} \lambda_F(m_1) \lambda_F(m_2) 
\times e \left( -\frac{nh_1}{c_1} + \frac{nh_2}{c_2} + 2\sqrt{m_1n} \frac{c_1}{c_1} - 2\sqrt{m_2n} \frac{c_2}{c_2} + \frac{A_1^2}{B_1} - \frac{A_2^2}{B_2} \right) 
\times e \left( \frac{m_2 h_2}{c_2} - \frac{m_1 h_1}{c_1} \right) W \left( \frac{2nA_1}{HB_1} \right) W \left( \frac{2nA_2}{HB_2} \right)
\]
where the subscript in \( A_1 \) indicates that the related parameters are \((m_1, h_1, c_1)\) and so on. The weight function implies that
\[
\left| \frac{h_1}{c_1} - \frac{h_2}{c_2} \right| \ll \frac{t\Delta}{N}.
\]
Applying the Poisson summation formula the \( n \) sum transforms into
\[
N\Delta \sum_{n \in \mathbb{Z}} J
\]
where the integral is given by
\[
J = \int V(y)e \left( a_1 \frac{N\Delta y}{n_0} + a_2 \frac{N\Delta^2 y^2}{n_0^2} + \ldots \right) dy
\]
with \( n_0 = tc_1/2\pi h_1 \), \( V(y) \ll (t^{1/3}\Delta)^j \),
\[
a_1 = -\sqrt{n_0} \left( \frac{\sqrt{m_1}}{c_1} - \frac{\sqrt{m_2}}{c_2} \right) + n_0 c_1 h_2 \frac{c_1}{c_2} \left( \frac{h_1}{c_1} - \frac{h_2}{c_2} \right) - \frac{nn_0}{c_1 c_2} + O \left( \frac{QK^2}{C\ell t^{2/3}} \right)
\]
and
\[
a_2 = \sqrt{n_0} \left( \frac{\sqrt{m_1}}{c_1} - \frac{\sqrt{m_2}}{c_2} \right) + \left( \frac{\sqrt{m_1 n_0}}{2c_1} - \frac{h_1 n_0}{c_1} \right) \left( \frac{t}{\pi} + \frac{\sqrt{m_1 n_0}}{c_1} \right)^{-1}
- \left( \frac{\sqrt{m_2 n_0}}{2c_2} - \frac{h_2 n_0}{c_2} \right) \left( \frac{t}{\pi} + \frac{\sqrt{m_2 n_0}}{c_2} \right)^{-1} + \text{smaller order terms,}
\]
and so on. Using the congruence condition in \( \text{(42)} \) we write
\[
n = h_1 c_2 - h_2 c_1 + \mu c_1 c_2,
\]
and applying \( \text{(6)} \), we get that the integral is negligibly small if
\[
\left| \sqrt{n_0} \left( \frac{\sqrt{m_1}}{c_1} - \frac{\sqrt{m_2}}{c_2} \right) + n_0 \mu \right| \gg t\Delta^2.
\]
The case of \( \mu \neq 0 \) is easily ruled out as then we would need \( C \ll QK^2/N\ell \), which can not happen, due to \( \text{(30)} \) and \( \text{(37)} \), if we impose the condition that
\[
QK^4 \ll N^2 t^{-1/6-\delta} \quad \text{or} \quad N \ll Q^{2/3-2\delta}.
\]
Even for $\mu = 0$ the integral is negligibly small unless

$$m_1c_2^2 - m_2c_1^2 \ll \frac{(QK)^2C^2Clt\Delta^2}{N^{t^2}} ,$$

which is smaller than the generic size by a factor of $QK^2/C\ell t\Delta^2$. This results in a saving of $Q^{1/2}K/(Ct)^{1/2}t^{1/2}\Delta$, at the price of loosing the restriction (35) on one of the $m_i$’s. So effectively we save $Q^{1/2}K/(Ct)^{1/2}t^{3/3}\Delta^{3/2}$ which is not enough for our purpose, as the resulting bound is

$$\sqrt{NHQ^2Kt^{1/3}C2^{5/6}\Delta^{3/2}/\sqrt{NQ\ell}} .$$

in place of (36).

7. Second application of Cauchy on dual off-diagonal

Now we consider (40) with the restriction (45), i.e. terms with ‘small gap’. This can be dominated by

$$\sum_{m_1,m_2 \sim \tilde{N}/\ell^2} \left| \sum_{n \sim \tilde{N}/\ell^2} \sum_{c_1,c_2 \sim C} e \left(-\frac{nh_1}{c_1} + \frac{nh_2}{c_2}\right) e \left(-\frac{m_2}{c_2} - \frac{m_1}{c_1}\right) \right| ,$$

where $N_0 = QK^2C^3t^2(N\ell)$. The trivial bound for this sum is given by

$$\left(\frac{\tilde{N}}{\ell^2} C\ell t\Delta^2 \right) N\Delta \sqrt{\frac{t^2C^2\Delta}{N^2}} \times \left(\frac{Q^{1/2}K^3C^{5/4}t^{1/3}\Delta^{3/2}}{N^{3/2}\ell^{3/2}}\right)^2 ,$$

which when substituted for $\Omega_\ell$ in (38) yields the bound (40). We apply the Cauchy inequality yet again, and then open the absolute square and apply the Poisson summation on ($m_1$, $m_2$). Before the application of Poisson, the sum over $(m_1, m_2)$ is trivially bounded by $O((\tilde{N}/\ell^4) \times (C\ell t^{3/2}\Delta/QK^2))$, which is the product of the first two terms in braces on the left hand side of (38). After Poisson the sum gets transformed into

$$\left(\frac{\tilde{N}}{\ell^2}\right)^2 \sum_{m_1,m_2 \in \mathbb{Z}} \mathcal{J},$$

where the integral is given by

$$\mathcal{J} = \int \int e \left(F_1(y_1) - F_2(y_2)\right) W \left(\frac{2nA_1}{HB_1}\right) W \left(\frac{2n'A_1'}{HB_1'}\right) \times W \left(\frac{2nA_2}{HB_2}\right) W \left(\frac{2n'A_2'}{HB_2'}\right) V \left(\frac{y_1c_2^2 - y_2c_1^2}{N_0\ell^2/N}\right) V \left(\frac{y_1c_2'^2 - y_2c_1'^2}{N_0\ell^2/N}\right) dy_1 dy_2 ,$$
with
\[ F_i(u) = \frac{2 \sqrt{\tilde{N} u n}}{c_i \ell} - \frac{2 \sqrt{\tilde{N} u n'}}{c_i' \ell} + \frac{A_i^2}{B_i} - \frac{A_i'^2}{B_i'} = \frac{\tilde{N} u i u}{c_i c_i' \ell^2}. \]
(Here in \( A_i \) etc. \( m_i \) is replaced by \( \tilde{N} u / \ell^2 \).) The last two weights impose the restriction
\[ c_1 c_2' - c_1' c_2 \ll C^2 \frac{C \ell \Delta^2}{Q K^2}. \]
We set \( w = (y_1 c_2 - y_2 c_1') \tilde{N}/N_0 \ell^2 \), \( \alpha = c_2/c_1^2 \) and \( \Theta = C^3 \ell \Delta^2 / c_1^2 Q K^2 \). Then substituting for \( y_2 \), and using Taylor expansion, we arrive at the expression
\[ J = \frac{C \ell \Delta^2}{Q K^2} \iint e (F_1(y_1) - F_2(\alpha y_1) + \Theta w F_2'(\alpha y_1) - \ldots) U(y_1, w) dy_1 dw, \]
where the weight function satisfies \( U^{(j_1+j_2)} \ll (t^{1/3} \Delta)^{j_1+j_2} \). Now by repeated integration by parts we see that the integral is negligibly small if
\[ \max\{m_1, m_2\} \gg \frac{N C \ell}{Q K^2}, \quad \text{or} \quad m_1 c_1 c_2' - m_2 c_2 c_1' \gg \frac{C^4 \ell^2 t \Delta^2}{N}. \]
The last condition reduces to \( m_1 - m_2 \gg (C \ell)^2 t \Delta^2 / \tilde{N} \) because of (50). We can say more about the size of the integral if \( (m_1, m_2) \neq (0, 0) \). Let us assume that \( m_2 \neq 0 \). Given \( y_1 \), look at the integral over \( w \), which turns out to be negligibly small if
\[ |y_1 - \ast| \gg t^{1/3} \frac{1}{|m_2|} \frac{C \ell N}{Q K^2} \left( \frac{1}{t^{1/3}} + \frac{C t \Delta^2}{Q K^2} \right), \]
for some \( \ast \). In generic case this cuts down the length of the \( y_1 \) integral by \( t^{1/3} \). For \( y_1 \) in the above range, and if
\[ \sqrt{n} c_2 - \sqrt{n'} c_2' \gg \frac{N^{1/2} \Delta}{C} \quad \text{or equivalently} \quad h_2 c_2 - h_2' c_2' \gg \frac{t C^2 \Delta_0}{N} \gg \frac{t C^2 \Delta}{N}, \]
the second derivative bound for \( w \) integral yields
\[ J \ll \left( \frac{\tilde{N}}{\ell^2} \frac{1}{t^{1/3} \Delta} \right) \left( \frac{\tilde{N} C \ell t \Delta^2}{\ell^2 Q K^2} \right) \frac{1}{(t \Delta_0 \Delta)^{1/2}} \frac{1}{|m_2|} \frac{C \ell N}{Q K^2} \left( \frac{1}{t^{1/3}} + \frac{C t \Delta^2}{Q K^2} \right). \]
In the range complementary to (53), we will use the trivial bound for the integral over \( w \). Basically that would mean that we would not have the factor \( (t \Delta_0 \Delta)^{1/2} \) in the denominator in (54).

We will now consider the contribution of the zero frequency \( (m_1, m_2) = (0, 0) \). The congruence conditions in (13) imply that \( c_1 = c_1' \) and \( c_2 = c_2' \). Also \( h_i \equiv h_i' \mod c_i \), so that \( h_i - h_i' = c_i g_i \) for some integer \( g_i \). Also it follows by repeated integration by parts (either w.r.t. \( y_1 \) or \( y_2 \)) that the integral \( J \) is negligibly small in the complementary range of
\[ n - n' \ll N/t^{1/3}. \]
Combining with the existing restriction (34), namely $|n - tc_i/2\pi h_i| \ll N\Delta$ it follows that

$$h_i - h_i' \ll \frac{tC}{N}\Delta. $$

So the number of $g_i$ is given by $O(1 + t\Delta/N) = O(1 + t^{2/3}/N)$, where for the last equality we use the assumption that $QK^2 \ll \mathcal{C}N$ (see (44)). Finally we observe that (55) together with (35) imply the stronger restriction

$$\left| \frac{h_1}{c_1} - \frac{h_2}{c_2} \right| \ll \frac{t^{2/3}}{N} + \frac{t\Delta^2}{N} $$

in place of (44). Hence the contribution of the zero frequency to (17) is bounded by

$$\frac{N}{\ell^2} \left\{ \left( \frac{1}{\ell^2} \right) \left( \frac{N\mathcal{C}t\Delta^2}{QK^2} \right) \frac{N^2\Delta}{\ell^{1/3}} \frac{t^2C^2}{N^2} \left( \frac{1}{t^{1/3}} + \Delta^2 \right) \left( 1 + \frac{t^{2/3}}{N} \right)^2 \right\}^{1/2}.$$

Comparing this with (48) we observe that we have effectively saved

$$\min \left\{ 1, \frac{N}{t^{3/2}} \right\} \min \left\{ t^{1/6}, \frac{1}{\Delta} \right\} \frac{t^{3/2}C^{5/2}\ell^{1/2}\Delta^2}{Q^{1/2}KN}.$$

It follows that the overall contribution of the zero frequency to the dual off-diagonal is given by

$$\sqrt{N}\mathcal{H}Q^2Kt^{1/3} \frac{C^2\ell^{5/6}\Delta^{3/2}}{\sqrt{N}\mathcal{Q}\ell} \frac{Q^{1/4}K^{1/2}N^{1/2}}{C^{5/4}\ell^{1/4}\Delta^{3/4}} \max \left\{ \frac{1}{t^{1/12}}, \Delta^{1/2} \right\} \max \left\{ 1, \frac{t^{1/3}}{\sqrt{N}} \right\},$$

which is satisfactory for our purpose if

$$K \ll \min\{ t^{1/3-2\delta}, Nt^{-1/3-2\delta} \}, \quad \text{and} \quad QK^4 \ll N^{1/2}t^{13/12-\delta}.$$}

The last condition takes care of small values of $C$. This is redundant in the light of the existing assumption (44) if $N \ll t^{5/6}$.

8. The final counting problem

Now we will analyse the contribution of the non-zero frequencies $(m_1, m_2) \neq (0, 0)$ for $\Delta_0 \gg \Delta$ in the generic range (53). We have reduced the problem to counting the number of

$$h_i, h_i' \sim tC/N, \quad c_i, c_i' \sim C, \quad i = 1, 2,$$

satisfying the following constraints

$$\left| \frac{h_1}{c_1} - \frac{h_2}{c_2} \right|, \quad \left| \frac{h_1'}{c_1'} - \frac{h_2'}{c_2'} \right| \ll \frac{t\Delta}{N}, \quad \left| \frac{c_1'}{c_1} - \frac{c_2'}{c_2} \right| \ll \frac{C\ell t\Delta^2}{QK^2}$$

$$\left| \frac{h_1}{c_1} - \frac{h_1'}{c_1'} \right| \ll \frac{M_1}{C^2}, \quad \left| \frac{h_2}{c_2} - \frac{h_2'}{c_2'} \right| \ll \frac{M_2}{C^2}, \quad h_2c_2 - h_2'c_2' \sim \frac{tC^2\Delta_0}{N},$$

with $C\ell \ll Q, M_i \ll C\ell N/QK^2$. We set $\mathbf{v}_i = (h_i, h_i', c_i, c_i')$. Let

$$N_0(C, M_1, M_2) = \#\{ \mathbf{v}_1, \mathbf{v}_2, m_1, m_2 : \text{satisfying (59) and congruences in (49)} \}.$$
Then the contribution of the $m_i \sim M_i$ block to (47) is bounded by

$$\left( \frac{N}{\ell^2} \right)^2 \left( \frac{N}{\ell^2} \right)^{2} \frac{C \ell t^{2/3} }{Q K^2} \frac{1}{(t \Delta_0)^{1/2}} (N \Delta)^2 \frac{1}{M_2} \frac{C \ell N}{Q K^2} \left( \frac{1}{t^{1/3}} + \frac{C t \Delta^2}{Q K^2} \right) N_0(\ldots) \right]^{1/2}.$$  

(60)

We will use the last two conditions in (59) to estimate $N_0(C, M_2)$ the number of $(v_2, m_2)$ pairs. Given such a vector, then we count the number of $v_1$ satisfying the first three conditions of (59). The first two conditions reduce to

$$h_1 c_2 - h_2 c_1 = j, \quad h'_1 c'_2 - h'_2 c'_1 = j'$$

with $j, j' \ll C^2 t \Delta / N$. It follows that $c_1 \equiv -j \bar{h}_2$ mod $c_2$, and $c'_1 \equiv -j' \bar{h}'_2$ mod $c'_2$. Hence given $j, j'$ we have $O(t^\varepsilon)$ many $v_1$. Now the third condition of (59) can be rewritten as

$$\left| \frac{c'_1}{c'_2} - \frac{c_1}{c_2} \right| \ll \frac{C \ell t \Delta^2}{Q K^2}.$$

This implies that

$$j' \bar{h}'_2 c_2 - j \bar{h}_2 c'_2 \equiv j'' \text{ mod } c_2 c'_2$$

with $j'' \ll C^3 t \ell t \Delta^2 / Q K^2$. Let us first consider the case where $j'' \neq 0$. Now given such a $j''$ we get $(1 + C \ell t / N)^2$ many pairs $(j, j')$. Hence it follows that in total we have

$$O \left( t^\varepsilon \left( 1 + \frac{C \ell t \Delta}{N} \right)^2 \frac{C \ell t \Delta^2}{Q K^2} \right)$$

many $v_1$ once we have a $v_2$. Finally we count the number of $m_1$. In the light of (51), this is bounded by

$$O \left( 1 + \frac{\ell^2 t \Delta^2 N}{(Q K^2)^2} \right) = O(1).$$

(We will have $K \gg t^{1/6}$, see (65).) Hence the contribution of $j'' \neq 0$ to $N_0(C, M_1, M_2)$ is given by

$$N_{j'' \neq 0}(C, M_1, M_2) \ll t^\varepsilon \left( 1 + \frac{C \ell t \Delta}{N} \right)^2 \frac{C^3 \ell t \Delta^2}{Q K^2} N_0(C, M_2)$$

$$\ll t^\varepsilon \left( 1 + \frac{C \ell t \Delta}{N} \right)^2 \frac{C^3 \ell t \Delta^2}{Q K^2} \left( M_2 \Delta_0 \left( \frac{t C}{N} \right)^2 + M_2^2 \left( \frac{t}{N} \right)^2 \right),$$

(61)

where in the last inequality we use Theorem 4.1 of [3] and the fact that $\Delta_0 \gg \Delta$. Note that when $M_2 / C^2 < 1/2$, we need to apply Theorem 4.1 of [3], and $m_2$ is then uniquely determined. For $M_2 / C^2 \geq 1/2$, the number of $v_2$ is trivially $O(t^2 C^4 \Delta_0 / N^2)$, and the number of $m_2$ satisfying the congruence is given by $O(M_2 / C^2)$. Consequently
we get that the contribution of $j'' \neq 0$ to (60) is dominated by
\begin{equation}
\frac{\tilde{N}^2}{\ell^4} \frac{N^{1/2} (C\ell)^{5/2} t^{11/12} \Delta^2}{(QK^2)^{3/2} \ell^2} \left( \Delta_0^{1/4} C + \frac{(NC\ell)^{1/2}}{(QK^2)^{1/2} \Delta_0^{1/4}} \right) \\
\times \left( 1 + \frac{t^{2/3} C}{N} \right) \left( \frac{(QK^2)^{1/2}}{(C\ell t^{2/3})^{1/2}} + \frac{(C\ell t^{2/3})^{1/2}}{(QK^2)^{1/2}} \right).
\end{equation}

Let us now consider the two possible ranges separately. Suppose that $QK^2/C\ell t^{2/3} \gg 1$, i.e. when $\Delta = QK^2/C\ell t$. Then the above expression reduces to (as $\Delta \ll \Delta_0 \ll 1$)
\begin{equation}
\frac{\tilde{N}^2}{\ell^4} \frac{QK^2 N^{1/2}}{t^{11/12} \ell} \left( C + \frac{N^{1/2} t^{1/4} (C\ell)^{3/4}}{(QK^2)^{3/4}} \right) \left( 1 + \frac{t^{2/3} C}{N} \right).
\end{equation}

Substituting this bound for $\Omega_\ell$ in (38) we get
\begin{equation}
\sqrt{N}HQ^2 K t^{1/3} \frac{N^{1/4} K^2}{C^{1/2} t^{2/3} 24 \ell^2} \left( C^{1/2} + \frac{N^{1/4} t^{1/8} (C\ell)^{3/8}}{(QK^2)^{3/8}} \right) \left( 1 + \frac{t^{1/3} C^{1/2}}{N^{1/2}} \right)
\end{equation}
in place of (36). This turns out to be satisfactory, viz. $O(\sqrt{N}HQ^2 K t^{1/3-\delta}/\ell)$ under the condition (57). Next suppose that $QK^2/C\ell t^{2/3} \ll 1$, i.e. $\Delta = 1/t^{1/3}$. Then the expression in (62) reduces to
\begin{equation}
\frac{\tilde{N}^2}{\ell^4} \frac{(C\ell)^3 N^{1/2} t^{13/12}}{(QK^2)^2} \left( C + \frac{(NC\ell)^{1/2} t^{1/12}}{(QK^2)^{1/2}} \right) \left( 1 + \frac{t^{2/3} C}{N} \right).
\end{equation}

Substituting this bound for $\Omega_\ell$ in (38) we get
\begin{equation}
\sqrt{N}HQ^2 K t^{1/3} \frac{N^{1/4} t^{1/24}}{K t^{1/4}} \left( 1 + \frac{N^{1/4} t^{1/24}}{(K^4)^{1/2}} \right) \left( 1 + \frac{t^{1/3} Q^{1/2}}{N^{1/2}} \right).
\end{equation}

This is satisfactory if the following condition is satisfied
\begin{equation}
K \gg \max\{N^{1/4} t^{1/24+\delta}, Q^{-1/3} N^{1/3} t^{1/18+2\delta/3}, t^{5/18+2\delta/3}, Q^{1/2} N^{-1/4} t^{3/8+\delta}\}.
\end{equation}

Now suppose we carry out the same process to count in the case of $j'' = 0$. Then in (61) instead of $C^3 \ell t^2 \Delta^2/QK^2$ we will have 1. Hence we get
\begin{equation}
\sqrt{N}HQ^2 K t^{1/3} \frac{N^{1/4} K^{3/2}}{C^{3/4} Q^{1/4} t^{17/24} \ell^4} \left( C^{1/2} + \frac{N^{1/4} t^{1/8} (C\ell)^{3/8}}{(QK^2)^{3/8}} \right) \left( 1 + \frac{t^{1/3} C^{1/2}}{N^{1/2}} \right)
\end{equation}
in place of (63), and
\begin{equation}
\sqrt{N}HQ^2 K t^{1/3} \frac{N^{1/4}}{(QK)^{1/2} t^{1/4} \ell^{1/4}} \left( 1 + \frac{N^{1/4} t^{1/24}}{(QK)^{1/2}} \right) \left( 1 + \frac{t^{1/3} Q^{1/2}}{N^{1/2}} \right)
\end{equation}
in place of (64). Now (66) is satisfactory if we have
\begin{equation}
K \ll \min\{t^{49/120-7\delta/10} N^{-1/10} Q^{1/5}, t^{2/3-5\delta/4} N^{-3/8} Q^{3/4}, t^{1/4-2\delta/3} N^{1/6}\}.
\end{equation}
On the other hand (67) is satisfactory if we have

\begin{equation}
K \gg \max\left\{ N^{1/2}t^{1/4}Q^{-1}, t^{7/12+\delta}N^{-1/2}, t^{1/3+\delta}Q^{-1/2} \right\}.
\end{equation}

The first two inequalities in (69) are not that binding, as we already have (19) and the third inequality in (65). The last inequality is new, as it implies that $QK^2 \gg t^{2/3}$, where the right hand side may be larger than $N$.

Now let us consider the case where $\Delta_0$ is small, i.e. $\Delta_0 \ll \Delta$. In this case we will use the trivial bound for the $w$ integral in $\mathcal{I}$. This basically means that we need to scale up by the factor $(t\Delta_0\Delta^2)^{1/2}$ (see (54)). This does not change our estimates in the case $\Delta = t^{-1/3}$. But we need to consider the case of $\Delta = QK^2/C\ell t$, whence we have $(t\Delta_0\Delta^2)^{1/2} \ll (QK^2)^{3/2}/(C\ell t^{2/3})^{3/2}$. Consequently in this case we get

$$
\sqrt{NHQ^2}Kt^{1/3} \frac{N^{1/4}Q^{3/8}K^{11/4}}{C^78^42^{9/2}\ell^2} \left( \frac{(CQK^2)^{1/4}}{t^{1/4}} \right) + \frac{N^{1/4}t^{1/8}(C\ell)^{3/8}}{(QK^2)^{3/8}} \left( 1 + \frac{t^{1/3}C^{1/2}}{N^{1/2}} \right)
$$

in place of (63) for $j'' \neq 0$. This is satisfactory if

\begin{equation}
K \ll \min\{t^{25/66-17\delta/44}N^{-1/8}Q^{-5/44}, t^{37/84-4\delta/7}N^{-1/7}Q^{1/14}, t^{1/3-13\delta/40}N^{7/80}Q^{-7/40} \}.
\end{equation}

Also for $j'' = 0$ we get

$$
\sqrt{NHQ^2}Kt^{1/3} \frac{N^{1/4}K^{3/4}Q^{1/8}}{C^9/8t^{23/24}\ell^{17/8}} \left( \frac{(CQK^2)^{1/4}}{t^{1/4}} \right) + \frac{N^{1/4}t^{1/8}(C\ell)^{3/8}}{(QK^2)^{3/8}} \left( 1 + \frac{t^{1/3}C^{1/2}}{N^{1/2}} \right)
$$

in place (66). This is satisfactory if

\begin{equation}
K \ll \min\{t^{1/3-19\delta/40}N^{-1/80}Q^{-1/40}, t^{3/8-3\delta/4}N^{-1/8}Q^{1/4}, t^{5/18-5\delta/12}N^{1/8}Q^{-1/12} \}.
\end{equation}

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