A non-perturbative renormalization group study of the stochastic Navier–Stokes equation

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We study the renormalization group flow of the average action of the stochastic Navier–Stokes equation with power-law forcing. Using Galilean invariance we introduce a non-perturbative approximation adapted to the zero frequency sector of the theory in the parametric range of the Hölder exponent $4 - 2\varepsilon$ of the forcing where real-space local interactions are relevant. In any spatial dimension $d$, we observe the convergence of the resulting renormalization group flow to a unique fixed point which yields a kinetic energy spectrum scaling in agreement with canonical dimension analysis. Kolmogorov’s $-5/3$ law is thus, recovered for $\varepsilon = 2$ as also predicted by perturbative renormalization. At variance with the perturbative prediction, the $-5/3$ law emerges in the presence of a saturation in the $\varepsilon$-dependence of the scaling dimension of the eddy diffusivity.

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Kolmogorov’s K41 theory is the cornerstone of current understanding of fully developed turbulence in Newtonian fluids. A modern formulation of the theory is based on the asymptotic solution of the Kármán-Howarth-Monin equation, expressing energy balance, for stochastic incompressible Navier–Stokes equation

$$(\partial_t + v \cdot \nabla - \kappa \partial_x^2) v = f - \partial_x P,$$

with $f$ Gaussian, incompressible, zero average time-decorrelated with correlation

$$\langle f(x_1, t_1) \otimes f(x_2, t_2) \rangle = \delta(t_{12}) F(x_{12}).$$

Here $\langle \rangle$ denotes the ensemble average, $\otimes$ the tensor product, $x_{ij} := x_i - x_j$, $t_{ij} := t_i - t_j$ and $P$ is a pressure term enforcing incompressibility: $\partial_x \cdot v = 0$. The solution of Kármán-Howarth-Monin equation predicts in any spatial dimension strictly larger than two that the energy injected by the external stirring $(f)$ around a typical spatial scale $L$ is conserved across an inertial range of scales through a constant-flux transfer mechanism, the “energy cascade”, before being dissipated by molecular viscosity. In two dimensions, energy and enstrophy conservation across the inertial range calls for a distinct analysis of the Kármán-Howarth-Monin equation formalizing the ideas introduced by Kraichnan in \[7\]. The solution predicts a constant flux inverse energy cascade from the injection scale towards the fluid integral scale. Below the injection scale a constant flux enstrophy cascade towards the dissipative scale may take place (see e.g. \[8\]). The very existence and properties of the enstrophy cascade are, however, sensitive to the boundary conditions imposed on (1) and the eventual presence and shape of large scale friction mechanisms. Dimensional considerations based on the solution of the Kármán-Howarth-Monin equation lead then to scaling predictions for statistical indicators of the flow, including the $-5/3$ exponent for the 3d kinetic energy spectrum. These predictions convincingly account for a wide range of experimental and numerical observations (see e.g. \[3, 11\] and references therein). Their first-principle derivation is therefore a well-grounded research question. A useful tool to pursue this goal is offered by the renormalization group, although its application to the inquiry of Navier–Stokes turbulence is ridden by challenges. Renormalization group analysis \[12–14\] can be applied only far from the turbulent regime and for a very special choice of the random Gaussian field $f$. This latter needs to have in any spatial dimension $d$ a power-law spectrum with Hölder exponent $4 - 2\varepsilon$:

$$F(\mathbf{x}_{12}; m, M) = \int_{\mathbb{R}^d} \frac{d^d p}{(2\pi)^d} e^{i p \cdot \mathbf{x}_{12}} \frac{T(p)}{d - 1} F(p; m, M),$$

$$F(\lambda p; \lambda m, \lambda M) = \lambda^{4 - d - 2\varepsilon} F(p; m, M),$$

with $T(p) = 1 - p \otimes p/p^2$ the transverse projector, $p := \| p \|$ and $m \ll M$ respectively the inverse integral and ultra-violet scales of the forcing. The rationale for the choice is that for vanishing $\varepsilon$ the canonical scaling dimensions of the convective (i.e. $\partial_x v$, and $v \cdot \partial_x v$) and dissipative (i.e. $\partial_x^2 v$) terms in the Navier–Stokes equation tend to the same value. This fact suggests that for $\varepsilon$ equal zero canonical scaling dimensions may coincide with the exact scaling dimensions. In this sense, the vanishing $\varepsilon$ case defines a marginal scaling limit around which it may be possible to determine scaling dimensions by means of a perturbative expansion in $\varepsilon$ in analogy to
what is done for critical phenomena described by a Boltzmann equilibrium (see e.g. [15, 16]). For the stochastic Navier–Stokes equation the situation is, however, not conclusive. Renormalization group yields in any spatial dimension a kinetic energy spectrum

\[ \mathcal{E}(p) \propto p^{p_0 - 2 \varepsilon}, \quad p_0 = 1 - 4 \varepsilon / 3 \]  

(4) (see also [17] for an exhaustive review). In (4) the exponent labeling emphasizes the possibility of sub-leading corrections. Fully developed turbulence in 3d should correspond to an infra-red dominated spectrum of the stirring force as it occurs for \( \varepsilon \geq 2 \). Interestingly, (4) recovers Kolmogorov’s result for \( \varepsilon \) equal two. Consistence with Kolmogorov theory then requires the exponent in (4) to freeze for \( \varepsilon \) larger than two to the value \(-5/3\). Within the perturbative renormalization group framework, the occurrence of such non-analytic behavior can only be argued [18]. Direct numerical simulations [19, 20] exhibited, within a \( 512^3 \)-lattice accuracy, a transition in the \( \varepsilon \)-dependence of \( \eta_2 \) which is consistent with the freezing scenario. The situation is, however, completely different in two dimensions [21, 22]. On the one hand, perturbative renormalization group analysis [23] upholds the validity of (4) for any \( \varepsilon \). On the other hand, the asymptotic solution of the Kármán-Howarth-Monin equation [22] shows that (4) is always sub-dominant with respect to the inverse energy cascade spectrum \( \mathcal{E}(p) \propto p^{-5/3} \), for \( \varepsilon \leq 2 \) i.e. even in the regime where renormalization group analysis should apply. Direct numerical simulations up to \( 2048^3 \) resolution give clear evidence of the inverse cascade [21, 22]. A scenario reconciling these findings may be that the Kraichnan-Kolmogorov inverse cascade corresponds to a renormalization group non-perturbative fixed point which does not bifurcate from the Gaussian fixed point at marginality. Evidences of the occurrence of such an “exotic” phenomenon, have been given in models of wetting transitions by “non-perturbative approximations” of the Wilsonian renormalization group [24, 25]. More recently, similar methods gave evidence of the existence of a strong coupling fixed point in the Kardar-Parisi-Zhang model of interfacial growth [26], yielding scaling predictions favorably comparing with direct numerical simulations. Motivated by these results, in the present contribution we derive the exact renormalization group equations for the stochastic Navier-Stokes equation. We then investigate them using a “non-perturbative approximations” similar to the one used in [20]. By this we mean, as often done in non-perturbative renormalization [27, 28], truncations of the flow equations based on some assumption on the physical properties of the inquired system. Specifically, we investigate the consequences of the simplest closure compatible with Galilean invariance and with the number of relevant interactions identified by perturbative renormalization at small \( \varepsilon \). The second requirement guarantees the existence of a limit where the closure becomes exact in the sense that it recovers the perturbative renormalization group fixed point. As in [26, 29], we focus on the exact renormalization group equations for the average action or thermodynamic potential defined by the stochastic Navier-Stokes equations. In striking contrast with the compressible stochastic dynamics studied in [26], we do not find any evidence of a non-perturbative fixed point which may be associated to constant flux solutions in general and to the two dimensional inverse cascade in particular. The truncation we consider reproduces instead the expected correct scaling behavior in the regime dominated by real-space local interactions i.e. \( d = 3 \) and \( \varepsilon \leq 3 / 2 \). Interestingly, we observe in any dimension a transition at \( \varepsilon = 3 / 2 \) in the scaling behavior of the eddy-diffusivity. This latter deviates from the renormalization group scaling prediction by freezing from there on in \( \varepsilon \) to its \( \varepsilon = 3 / 2 \) value. This result was previously derived by different methods in [30]. In spite of the eddy diffusivity saturation, we obtain a kinetic energy spectrum scaling in agreement to (4) with no saturation for \( \varepsilon > 2 \). This latter fact is not entirely surprising since the particle irreducible vertices contributing to the approximated renormalization group flow, are only a subset of those needed to fully reconstruct the flux i.e. the chief statistical indicator in Kolmogorov’s theory. The structure of the paper is as follows. In section I we briefly recall the Kármán-Howarth-Monin equation and its predictions for power law forcing. In section II we derive the exact renormalization group average action for the model. The scope of these sections is to provide basic background on turbulence and functional renormalization to facilitate the reading by researchers familiar with one of these subjects but not the other. Using the Ward identities imposed by Galilean and translational invariance in section III we introduce our approximations of the exact flow. We write the resulting equations in section IV where we also outline their qualitative analysis. To simplify the discussion we detail auxiliary formulas in appendix A. An advantage of our formalism is that by preserving the structure of the exact renormalization group flow it guarantees the “realizability” of the “closure” that we impose [31]. In V we describe the analytic solution of our equations in a simplified limit. Section VI reports the result of the numerical integration of our equations respectively in the three and two-dimensional cases. Finally we turn in VII to discussion and conclusions.

I. SCALING PREDICTIONS BASED ON THE KÁRMÁN-HOWARTH-MONIN EQUATION

The Kármán-Howarth-Monin equation describes the energy balance in the putative unique steady-state to which Galilean invariant statistical indicators are expected to converge. Specifically, if we consider the two-point equal time correlation tensor

\[ C_2(x_1, t) = \propto v(x_1, t) \otimes v(x_2, t) \geq , \]  

(5)
and the three point equal time structure tensor
\[ S_3(x_{12},t) := \langle \delta v(x_{12},t) \otimes \delta v(x_{12}) \otimes \delta v(x_{12}) \rangle, \]
\[ \delta v(x_{12}) := v(x_1,t) - v(x_2,t), \]
a straightforward calculation using incompressibility and the inertial range translational and parity invariance yields
\[ \partial_t C + \frac{1}{2} \partial_x \cdot S - 2 \kappa \partial_x^2 C = F, \]
for \( C := \text{tr} C_2, F := \text{tr} \mathbf{F} \) and \( S^a := \mathbf{S}^{a_1 a_2 a_3} \) and Einstein convention on repeated indices. In any spatial dimension strictly larger than two, (8) admits an asymptotic solution under the hypotheses (see e.g. [3] for a detailed discussion) that \((i-1)\) statistical indicators attain a unique steady state and hence \( \partial_t C = 0, (i-1) \) they are smooth for any finite molecular viscosity but \((iii-1)\) the inviscid limit of the energy dissipation exhibits a dissipative anomaly
\[ 0 < -2 \lim_{||x||, \kappa \rightarrow 0} \lim_{\kappa \rightarrow 0} \kappa \partial_x^2 C = -2 \lim_{||x||, \kappa \rightarrow 0} \lim_{\kappa \rightarrow 0} \kappa \partial_x^2 C = 0. \]
Under these hypotheses, if the dominant contribution to the forcing correlation comes from wave numbers of the order \( m \), Kolmogorov’s classical result [1]
\[ \lim_{||x||, \kappa \rightarrow 0} \lim_{\kappa \rightarrow 0} \partial_x S_3^{a_1 a_2 a_3}(x) = -\frac{2 \bar{\varepsilon}}{d(d+2)} P_\alpha \{ \delta^{a_1} \delta^{a_2} \delta^{a_3} \} , \]
holds true for \( m ||x|| \ll 1, P_\alpha \) being the index cyclical permutation operation over \( \alpha = (a_1, a_2, a_3) \). In other words, the leading scaling exponent of (6) is
\[ \zeta_{3:0} = 1. \]
Dimensional considerations then yield for the kinetic energy spectrum scaling exponent the Kolmogorov’s scaling law
\[ \eta_{2:0} = -5/3. \]
If instead the forcing correlation is a power-law within the range of scales \( M^{-1} \ll ||x|| < m^{-1} \) with Hölder exponent \( 4 - 2 \varepsilon \), we should distinguish two situations. If \( \varepsilon < 2 \), the forcing correlation (6) remains well-defined in the limit of infinite integral scale \( m^{-1} \). In such a case (10) holds for \( M^{-1} \ll ||x|| \ll \ell \) where we introduced \( \ell = \kappa / \sqrt{F(0)} \), the typical scale below which molecular dissipation dominates. Under the present hypotheses \( \ell \propto \kappa M^{-2} \), the omitted proportionality factor being a dimensional constant independent of \( \kappa \) and \( M \). This range of scales is not accessible by perturbative ultra-violet renormalization group methods. Latter may describe instead the range \( ||x|| \gg M^{-1} \) where the asymptotic solution of (6) states that the leading scaling exponent of (6) is
\[ \zeta_{3:0} = -3 + 2 \varepsilon. \]
Dimensional analysis based on (13) then recovers the renormalization group prediction (1) for the kinetic energy spectrum. A different scenario occurs for \( \varepsilon > 2 \): the forcing correlation has a finite limit if the ultra-violet scale \( M \) tends to infinity for any finite value of the inverse integral scale \( m \). In the range \( m^{-1} \ll ||x|| \ll \ell, \ell \propto \kappa m^{-2} \), (10) holds with possible sub-dominant terms with scaling dimension (13). To summarize, the hint coming from the Kármán-Howarth-Monin equation for spatial dimensions \( d > 2 \) is that the \( -5/3 \) exponent stems from the dominance for \( \varepsilon > 2 \) of the constant flux over the dimensional scaling asymptotic solution of (6). Within renormalization group theory two mechanisms may provide for this phenomenon, either the fixed point of the renormalization group undergoes a bifurcation (see e.g. [32] for an analytically tractable example of non-perturbative bifurcation) or irrelevant composite operators dominating the limit of vanishing \( m \) of the two point correlation which become relevant above a threshold value of \( \varepsilon \). Examples of such operators are known [17] but the scaling dimensions become marginal exactly at \( \varepsilon \) equal zero two so nothing conclusive can be said about freezing.

The asymptotic analysis of (6) in \( 2d \) must be treated apart in order to take into account enstrophy conservation. In particular [4, 5], Kraichnan’s theory [7] is epitomized by a more restrictive version of \((i-1)\), which we will refer to as \((i-2)\), requiring only Galilean invariant quantities to reach a steady state. In other words, \( \partial_t C \) does not vanish. Furthermore, \((iii-1)\) is replaced by a new hypothesis \((iii-2)\) ruling out the occurrence of dissipative anomaly for the kinetic energy dissipation:
\[ \lim_{||x||, \kappa \rightarrow 0} \lim_{\kappa \rightarrow 0} \kappa \partial_x^2 C = \lim_{||x||, \kappa \rightarrow 0} \lim_{\kappa \rightarrow 0} \kappa \partial_x^2 C = 0. \]
It is worth noting that \((iii-2)\) can be rigorously proved to hold true in some setup for the deterministic Navier–Stokes (10) (see also discussion in [33]). We refer the reader to [22] for a detailed analysis of the two-dimensional Kármán-Howarth-Monin equation in the power-law case also corroborated by direct numerical simulations of (1). Here we only summarize the results. In the range of scales which can be investigated by perturbative ultra-violet renormalization group methods, three distinct regimes may set in depending upon the value of \( \varepsilon \). For \( \varepsilon < 2 \), the ultra-violet cut-off gives the dominant contribution to the total energy \( F(0) \propto M^{4-2\varepsilon} \) and enstrophy \( -\langle \partial_x^2 F \rangle(0) \propto M^{6-2\varepsilon} \). Correspondingly, the inviscid limit in the range \( M ||x|| \gg 1 \) predicts for the leading and sub-leading scaling exponents of (6)
\[ \zeta_{3:0} = 1 \quad \& \quad \zeta_{3:1} = -3 + 2 \varepsilon. \]
This is in agreement with Kraichnan’s theory which predicts the onset of an inverse energy cascade for wave-numbers smaller than the one characteristic of the (total) input. The ensuing dimensional prediction for the kinetic energy spectrum scaling exponent is (12) while \( \eta_{2:1} = 1 - 4/3\varepsilon \) only describes a sub-leading correction.
For $2 < \varepsilon < 3$, $F(0) \propto m^{4-2\varepsilon}$ and $-(\partial_x^2 F)(0) \propto M^{6-2\varepsilon}$ indicate that in the region $m^{-1} \gg \|x\| \gg M^{-1}$ the third order structure tensor is sustained by an input of enstrophy from larger wave-numbers and an input of energy from smaller wave-numbers. As a result, the flux balances locally in real space with the forcing so that (13) holds true. Finally for $\varepsilon > 3$ and in the presence of a large-scale hypo-friction (22) both energy $\bar{F}(0) \propto m^{4-2\varepsilon}$ and enstrophy $-(\partial_x^2 F)(0) \propto M^{6-2\varepsilon}$ inputs are dominated by the infra-red mass scale $m$. As a consequence, a direct enstrophy cascade sets in for $m \|x\| \ll 1$ and

$$\zeta_{3:0} = 3 \quad \text{and} \quad \zeta_{3:1} = -3 + 2 \varepsilon \ .$$

Again, dimensional analysis based on (16) predicts

$$\eta_{2:0} = -3 \quad \text{and} \quad \eta_{2:1} = 1 - 4/3 \varepsilon \ , \quad (17)$$

with the leading scaling exponent “freezing” at the threshold value attained at $\varepsilon = 3$. With these results in mind, we turn now to the formulation of a non-perturbative renormalization group theory with the aim of collating scaling predictions for the energy spectrum.

II. RENORMALIZATION GROUP FLOW FOR THE AVERAGE ACTION

A. Thermodynamic formalism

For finite infra-red $m$ and ultra-violet $M$ cut-offs of the Gaussian forcing (2) it is reasonable to assume that the generating function

$$Z_{(f,\bar{f})} := \exp \{ F(f + \bar{f}) \} \ , \quad (18)$$

is well defined. The average in (18) is over the Gaussian statistics of the forcing, $v(f + \bar{f})$ is the solution of (11) for any fixed realization of $f$ shifted by an arbitrary source field $\bar{f}$, and * denotes the $L^2(\mathbb{R}^d \times \mathbb{R})$ scalar product

$$j \ast v(f) := \int_{\mathbb{R}^d \times \mathbb{R}} dx \partial_t j(x, t) \cdot v(x, t) \ . \quad (19)$$

Functional derivatives at zero external sources $(f, \bar{f})$ of (18) yield the expressions of the correlation and response (to variations of $v$ with respect to $f$) tensors of any order. The generating function of connected correlations

$$W_{(f,\bar{f})} := \log Z_{(f,\bar{f})} \ , \quad (20)$$

is equal to minus the free energy of the field theory. In particular, with these conventions we have

$$C_{\alpha_1 \alpha_2}(x_{12}, t_{12}) \equiv [W^{(2)}(\bar{f})]_{\alpha_1 \alpha_2}(x_{12}, t_{12})$$

$$:= \frac{\delta^2 W_{(f,\bar{f})}}{\delta_{f^{\alpha_1}}(x, t_1) \delta_{f^{\alpha_2}}(x, t_2)} \bigg|_{f=0} \ . \quad (21)$$

Analogously, the second order response function is

$$\frac{\delta W^{(2)}(f, \bar{f})}{\delta_{f^{\alpha_1}}(x_1, t_1) \delta_{f^{\alpha_2}}(x_2, t_2)} := \frac{\delta^2 W_{(f,\bar{f})}}{\delta_{f^{\alpha_1}}(x, t_1) \delta_{f^{\alpha_2}}(x, t_2)} \bigg|_{f=0} \ . \quad (22)$$

The Legendre transform of the free energy (20) specifies the average action or the thermodynamic potential of the statistical field theory:

$$U(u, \bar{u}) := \sup_{(f, \bar{f})} \{ \int \{ f \ast u + \bar{f} \ast \bar{u} - W_{(f, \bar{f})} \} \} \ . \quad (23)$$

The Legendre anti-transform of (20) reconstruct the convex envelope of the free energy (20). In this sense the average action may be interpreted as an ultra-violet regularization of the theory. The average action is a functional of the fields $(u, \bar{u})$, which are Legendre conjugate to the external sources $(f, \bar{f})$ and which as customary will be referred to as “classical fields”. As extensively discussed in [27, 34] the average action provides a convenient starting point for non-perturbative renormalization. Dealing with it is conceptually equivalent to working with the Wilsonian effective action as done by Polchinski in [35]. Namely, the corresponding equations can in principle be converted in one another by a Legendre transform if one identifies the running cut-off. The average action offers, as we will see below, some technical advantages [28] which significantly simplify the formalism.

B. Flow equations

A stationary phase approximation to (15) in the weak stirring limit $F \downarrow 0$ (see appendix [13]) yields with logarithmic accuracy

$$U_M \sim \bar{u} \ast (\partial_t + u \cdot \partial_x - \kappa \partial_x^2)u - \frac{\bar{u} \ast F \ast \bar{u}}{2} \ , \quad (24a)$$

$$\partial_x \cdot u = \partial_x \cdot \bar{u} = 0 \ . \quad (24b)$$

The limit $F = 0$ describes the trivial steady state of the decaying Navier-Stokes equation. We posit that (24) provides the initial condition for the renormalization group flow of the running average action $U_{m_r}$. This flow describes the building up of the exact average action $U$ of (18) as a function of an infra-red cut-off suppressing any interaction above an infra-red scale $m_r$ and recovering $U$ in the limit of vanishing $m_r$. These conditions can be matched [27, 30] if we replace in (11) the molecular viscosity with an “hyper-viscous” term, local in wave number space,

$$\kappa \mapsto \tilde{\kappa} := \kappa + \kappa_m \tilde{R} \left( \frac{p}{m_r} \right) \ , \quad (25)$$

with $\tilde{R}$ a function rapidly decaying for large values of its argument and diverging at the origin. A convenient choice [26] is

$$\tilde{R}(p) = \frac{1}{e^{p^2} - 1} \ . \quad (26)$$

In [25] we also introduced the “running” viscosity $\kappa_{m_r}$. We will use this extra degree of freedom to constrain the
flow to satisfy a renormalization condition on the eddy diffusivity. As for the viscosity, we then apply a high-pass filter to the Gaussian forcing

$$f \mapsto \tilde{f},$$  

such that

$$\langle \tilde{f}_1 \otimes \tilde{f}_2 \rangle = \delta(t_{12}) \sum_{i=0}^{1} F_i(x_{12}; m_r),$$

where we defined

$$F_{(0)}(x_{12}; m_r) = F(x_{12}; m_r, \infty),$$

$$\operatorname{tr} F_{(0)}(p; m_r) = F_o m_{r}^{4-d-2\varepsilon} (d-1) \chi(0) \left( \frac{p}{m_r} \right)$$

and

$$\tilde{F}_{(1)}(p; m_r) = F_m, \chi(1)(p; m_r) T(p),$$

$$\chi(1)(p; m_r) := p^2 e^{-\frac{p^2}{m_r}}.$$  

This latter term describes a local (in the infra-red or for \( m_r = O(1) \)) perturbation of the measure progressively suppressed as \( m_r \) decreases. Locality entitles us to interpret this term as a renormalization counter-term in the sense of \( 23, 37, 38 \). Again, we will use the extra freedom introduced by \( F_m \) to impose a renormalization condition on the flow. The replacements \( 25, 27 \) turn into a family of generating functions differentiable with respect to the parameter \( m_r \). A straightforward calculation (see appendix \( \text{A1} \)) yields

$$m_r \partial_{m_r} Z_{(j,\beta)} = \int d^d x_1 d^d x_2 dt \left\{ \left( m_r \partial_{m_r} \tilde{F} \right)^{\alpha_1 \alpha_2} (x_{12}) \frac{\delta^2 Z_{(j,\beta)}}{2 \delta^2 \alpha_1 (x_1, t) \delta^2 \alpha_2 (x_2, t)} + (m_r \partial_{m_r} \kappa_m, R)(x_{12}) \partial_{x_2}^2 \frac{\delta^2 Z_{(j,\beta)}}{2 \delta^2 \alpha_1 (x_1, t) \delta^2 \alpha_2 (x_2, t)} \right\}. \tag{31}$$

Upon defining

$$R(x_{12}, t_{12}) :=$$

$$\delta(t_{12}) \left[ \begin{array}{cc} 0 & \kappa_m, R(x_{12}) \partial_{x_1}^2 \\ \kappa_m, R(x_{12}) \partial_{x_1}^2 & \tilde{F}(x_{12}) \end{array} \right], \tag{32}$$

and

$$W_{(j,\beta)}^{(2)}(x_1, x_2, t_1, t_2) :=$$

$$\left[ \begin{array}{cc} W_{(j,\beta)}^{(2,0)} & W_{(j,\beta)}^{(2,1)} \\ W_{(j,\beta)}^{(1,0)} & W_{(j,\beta)}^{(1,1)} \end{array} \right] \circ (x_1, x_2, t_1, t_2), \tag{33}$$

we can recast \( 31 \) into the form of an equation for the free energy which, in compact form, reads

$$m_r \partial_{m_r} W_{(j,\beta)} =$$

$$-\frac{1}{2} \operatorname{tr} \left\{ (m_r \partial_{m_r} \mathcal{R}) \ast \left( W_{(j,\beta)}^{(2)} - W_{(j,\beta)}^{(1)} \right) \right\}. \tag{34}$$

Functional derivatives at zero sources of \( 34 \) spawn a hierarchy of equations satisfied by the full set of connected correlation of the theory. From \( 34 \) we derive the average action flow using the following two observations. First, the very definition of Legendre transform \( 23, 38 \) implies

$$m_r \partial_{m_r} W_{(j,\beta)} = -m_r \partial_{m_r} U_{(u, u)} \left( \frac{F_{(0)}}{m_r} \right). \tag{35}$$

Second, the evolution of \( 33 \) at zero sources restores translational invariance:

$$W^{(2)}(x_{12}, t_{12}) := \left[ \begin{array}{cc} W^{(2,0)} & W^{(2,1)} \\ W^{(1,1)} & 0 \end{array} \right] \circ (x_{12}, t_{12}). \tag{36}$$

The matrix elements of \( 36 \) are specified by the second order correlation and response functions \( 21, 22 \). We may refer to them as indicators of the "Gaussian" part of the statistics of \( 1 \). We can use \( 36 \) and the general relation

$$I = W_{(j,\beta)}^{(2)} + U_{(u, u)}^{(2)} \tag{37}$$

following from the Legendre transform \( 23 \), to decouple the average action into a Gaussian and an interaction part \( 39 \):

$$I := W_{(j,\beta)}^{(2)} \ast \left[ W^{(2)} - U^{(2)} \right] \ast U_{(u, u)}^{(2)} \ast W^{(2)}, \tag{38}$$

Solving this latter relation for \( W_{(j,\beta)}^{(2)} \)

$$W_{(j,\beta)}^{(2)} = \left[ I + W^{(2)} \ast U^{(2)} \ast W^{(2)} \right]^{-1} \ast W^{(2)}, \tag{39}$$

allows us to finally derive the equation for the average action:

$$m_r \partial_{m_r} \left\{ U_{(u, u)} - \frac{1}{2} [u, u] \ast (m_r \partial_{m_r} \mathcal{R}) \ast \left[ \begin{array}{c} u \end{array} \right] \right\} =$$

$$-\frac{1}{2} \operatorname{tr} \left( \frac{(m_r \partial_{m_r} \mathcal{R})}{2} \ast \left[ \sum_{n=0}^{\infty} (-W^{(2)} \ast U^{(2)} \ast W^{(2)}) \right] \ast W^{(2)} \right). \tag{40}$$

Some observations are in order. First, the flow equation \( 40 \) is effectively an equation for the reduced average action obtained by subtracting the quadratic counter-terms associated to the running infra-red cut-off. This is desirable because all physical information is indeed contained in the reduced average action. Second, the flow in \( 40 \) does not depend upon the theory under consideration which instead specify the initial conditions for the evolution. This is a formalization of Wilson’s idea of renormalization group as a flow in the space of the probability measures. The fixed point of the flow does not depend on the details of the microscopic theory used as initial condition for \( m_r = M \). It depends instead on the basin of attraction to which the initial condition belongs. Finally, solving \( 40 \) exactly is equivalent to solve an infinite non-close hierarchy of equations. Perturbative renormalization tells us, however, that there are only a finite number of relevant coupling, at most two for \( \varepsilon \ll 1 \) and \( d \geq 2 \), \( 17, 33, 35 \) determining the scaling properties of the stochastic Navier–Stokes \( 1 \). Based on this observation, we now turn to the derivation of a truncation of the right hand side of \( 40 \) in order to derive explicit scaling predictions.
III. GALILEAN INVARIANCE AND APPROXIMATION

Perturbative renormalization identifies the number of relevant couplings by diagram power counting in unit of the ultra-violet cut-off [13]. Relevant couplings correspond to proper vertices $U^{(i,j)}$ proportional to powers of $M$ larger or equal than zero. For the stochastic Navier–Stokes equations only $U^{(1,1)}$ for any $d$ and $U^{(0,2)}$, for $d \geq 2$ have non-negative ultra-violet degree. We can use this information to hypothesize that $[40]$ converges towards an average action of the form

$$U(u,\dot{u}) = u \ast U^{(1,1)} \ast \ddot{u} + \frac{1}{2} U^{(0,2)}(\ast \ddot{u})^2 + \frac{1}{2} (u \ast)^2 U^{(2,1)} \ast \ddot{u} . \quad (41)$$

Clearly, the Ansatz closes the hierarchy of equations spawned by [40] since it is straightforward to verify that

$$U^{(2,1)} \ast U^{(1,1)} = \begin{bmatrix} U^{(2,1)} \ast \ddot{u} & u \ast U^{(2,1)} \end{bmatrix} + \begin{bmatrix} 0 \\ (u \ast U^{(2,1)})^\dagger \end{bmatrix} , \quad (42)$$

and by [23]

$$W^{(1,1)} = U^{(1,1)} \dagger - 1 , \quad (43a)$$

$$W^{(2,0)} = -W^{(1,1)} \ast U^{(0,2)} \ast W^{(1,1)} \dagger . \quad (43b)$$

Note that

$$E^{(1,1)}_{\omega} \langle x, t \rangle = - e^{ip \cdot \omega} \ast f \ast \frac{\delta \langle f \ast \ddot{u} \rangle}{\delta \dot{f}(x, t)} > , \quad (44)$$

implies that $W^{(i,0)} = U^{(i,0)} = 0$ for any integer $i$. To further evince the rationale behind [41] we observe that

$$\bar{U}^{(1,1)}(p_1, \omega_1 \mid p_2, \omega_2) = (2\pi)^{d+1} \tilde{g}^{(1,1)}(\omega) \sum_{i=1}^{2} \bar{p}_i \gamma(\omega) \left( \delta \omega_{1-2} \right) , \quad (45)$$

corresponds to a “dressing” of the quadratic coupling in [23]. Differentiating with respect to $p_1^2$ at zero wave-number and frequency the translational invariant part of [45] provides a convenient non-perturbative definition of the eddy diffusivity. We will therefore refer to [45] as the “eddy diffusivity” vertex. Also the “interaction” vertex

$$\bar{U}(p_1, \omega_1, p_2, \omega_2 \mid p_3, \omega_3) = (2\pi)^{d+1} \delta(d) \sum_{i=1}^{3} \tilde{g}^{(1,1)}(p_1, \omega_1, p_2, \omega_2) \times \hat{P}(p_1) \hat{P}(p_2) \hat{P}(p_3) \hat{P}(p_1) , \quad (46)$$

admits a similar direct interpretation from [21]. Finally, comparison with [21] evinces that the “force” vertex

$$\bar{U}^{(0,2)}(p_1, \omega_1, p_2, \omega_2) = - (2\pi)^{d+1} \delta(d) \sum_{i=1}^{2} \tilde{g}^{(0,2)}(p_1, \omega_1) \hat{T}(p_1) \quad (47a)$$

gives

$$g^{(0,2)}(p_1, \omega_1) := \frac{1}{d-1} \sum_{i=0}^{1} \text{tr} \hat{F}_{(i)}(p_1, m_r) + \bar{g}^{(0,2)}(p_1, \omega_1) , \quad (47b)$$

describes (minus) the effective forcing correlation. The three vertices are, however, not completely independent. Galilean invariance constrains the average action to satisfy the Ward identity (see e.g. [42] and appendix A.2)

$$0 = \tilde{r} \ast \ddot{u} + \frac{\delta \dot{U}}{\delta \dot{u}} \ast (r \cdot \partial u - \dot{r}) + \delta \dot{u} \ast r \cdot \partial u , \quad (48)$$

whence it follows after standard manipulations [15]

$$\tilde{U}^{(1,1)}(p_1, \omega_1, 0, 0 \mid p_3, \omega_3) = p_1 \frac{\partial_{\omega_1} \tilde{U}^{(1,1)}(p_1, \omega_1 \mid p_3, \omega_3)}{\omega_1} . \quad (49)$$

In the context of perturbative renormalization [49] is used to show that if a parameter fine-tuning ensures that $U^{(1,1)}$ is finite in the limit $M$ tending to infinity so must be $\tilde{U}^{(1,1)}$. In general [49] is not sufficient to fully specify the form of the interaction vertex in terms of $U^{(1,1)}$. If we, furthermore, hypothesize

$$g^{(2,1)} = 1 , \quad (50)$$

then [49] implies

$$g^{(1,1)}(p_1, \omega) = g^{(1,1)}(p) . \quad (51)$$

Such an approximation is too rough to give a self-consistent model for the full second order statistics. Our goal here is more restrictive as it is only to derive self-consistent scaling predictions at scales much larger than the dissipative. We therefore posit that [41] and [50] may serve for a self-closure able to capture the scaling behavior of the zero frequency sector of the theory. We also notice that a consequence of imposing [50], is that a generalized Taylor hypothesis [2] is verified by the two point correlation function for which the dispersion relation

$$\omega = \kappa \mp p^2 g^{(1,1)}(p) , \quad (52)$$

holds true. As a final step in the derivation of our approximation we rewrite the vertices [45], [47a] to decouple explicitly the functional dependence on the cut-off. Thus, we cough the eddy-diffusivity vertex into the form

$$g^{(1,1)}(p; m_r) := \frac{\kappa m_r}{\kappa} \left[ \gamma(1,1)(p; m_r) + \hat{R} \left( \frac{p}{m_r} \right) \right] . \quad (53)$$

where now $\gamma(1,1)$ is an unknown non-dimensional function which our renormalization group equation will determine. Similarly we write

$$\bar{g}^{(0,2)}(p; m_r) := \left[ \lambda_0 m_r^{2-d-2}\bar{r} + \lambda_1 \right] p^2 \gamma^{(0,2)}(p; m_r) , \quad (54)$$

where we defined the Grashof numbers

$$\lambda_0 := \frac{\Omega_d F_0}{(2\pi)^d \kappa^3 m_r^2 \bar{r}^{2\epsilon} \bar{r}} , \quad (55a)$$
\[ \lambda_1 = \frac{\Omega_d}{(2\pi)^d} \frac{F_{m_r}}{\kappa_{m_r}^3 m_r^{d-2}} , \quad (55b) \]

measuring the intensity of the non-local and local components of the stochastic forcing. In the context of perturbative renormalization the pair \((55)\) specifies the running coupling constant of the model \([17, 37, 38]\). In \((55)\) we denoted

\[ \Omega_d = \frac{2\pi^{d/2}}{\Gamma \left( \frac{d}{2} \right)} , \quad (56) \]

IV. APPROXIMATED RENORMALIZATION GROUP FLOW

The Ansatz

\[ U(u, \bar{u}) = u \star U^{(1,1)} \star \bar{u} + \frac{1}{2} U^{(0,2)}(u \star \bar{u})^2 + (\bar{T} \bar{u}) \cdot [\partial (T u) \cdot \partial (T \bar{u})] , \quad (57) \]

with \(T\) the transverse projector and \((45), (47a)\) specifying the Fourier representation of the order two vertices summarizes the approximations described in the previous section. The insertion of \((57)\) into the exact renormalization group equation \((40)\) yields the equations

\[ m_r \partial_{m_r} \left\{ (d-1) \kappa_{m_r} p^2 \gamma(1,1)(p, m_r) \right\} = \]

\[ -\frac{1}{2} \text{tr} \left\{ \widetilde{W}^{(2)} \star \frac{\delta U^{(2) \text{int}}(u \star \bar{u})}{\delta \alpha_1} \star W^{(2)} \star \frac{\delta U^{(2) \text{int}}(u \star \bar{u})}{\delta \alpha_2} \right\} \bigg|_{\omega=0} \]

\[ -\frac{1}{2} \text{tr} \left\{ W^{(2)} \star \frac{\delta U^{(2) \text{int}}(u \star \bar{u})}{\delta \alpha_1} \star W^{(2)} \star \frac{\delta U^{(2) \text{int}}(u \star \bar{u})}{\delta \alpha_2} \right\} \bigg|_{\omega=0} \]

\[ \frac{1}{2} \text{tr} \left\{ \frac{\delta U^{(2) \text{int}}(u \star \bar{u})}{\delta \alpha_1} \star \frac{\delta U^{(2) \text{int}}(u \star \bar{u})}{\delta \alpha_2} \right\} \bigg|_{\omega=0} \]

\[ m_r \partial_{m_r} \left\{ (d-1) \tilde{g}^{(0,2)}(p, m_r) \right\} = \]

\[ -\frac{1}{2} \text{tr} \left\{ \widetilde{W}^{(2)} \star \frac{\delta U^{(2) \text{int}}(u \star \bar{u})}{\delta \alpha_1} \star \frac{\delta U^{(2) \text{int}}(u \star \bar{u})}{\delta \alpha_2} \right\} \bigg|_{\omega=0} \]

\[ -\frac{1}{2} \text{tr} \left\{ W^{(2)} \star \frac{\delta U^{(2) \text{int}}(u \star \bar{u})}{\delta \alpha_1} \star \frac{\delta U^{(2) \text{int}}(u \star \bar{u})}{\delta \alpha_2} \right\} \bigg|_{\omega=0} \]

\[ m_r \partial_{m_r} \left\{ \tilde{g}^{(0,2)}(p, m_r) \right\} = \]

\[ m_r \partial_{m_r} \left\{ \tilde{g}^{(0,2)}(p, m_r) \right\} = \frac{1}{2} \frac{d}{d \ln m_r} \ln \kappa_{m_r} \quad \text{&} \quad \eta_F := m_r \frac{d}{d \ln m_r} \ln F_{m_r} \quad (63) \]

determined by the fixed point of the renormalization group flow and the canonical dimensions

\[ d_{F_0} = 4 - d - 2 \varepsilon \quad \& \quad d_{F_1} = 2 \quad (64) \]

In other words, \((58)\) imply that the functional vector field driving the renormalization group flow with our approximation is obtained by taking the variation of the mode coupling equations in a way adapted to \((52)\). We summarize this calculation in appendix \(\text{C}\). Here, we notice instead that after turning to non-dimensional variables \((p \mapsto \bar{p}/m_r)\) we can rewrite \((58)\) as

\[ \left\{ m_r \partial_{m_r} - \frac{\partial}{\partial \bar{p}} + \eta_\kappa \gamma^{(1,1)}(p) \right\} = \eta_F G_{\kappa}^{(1,1)}(p) - \eta_\kappa G_{\kappa}^{(1,1)}(p) - G_{\kappa}^{(1,1)}(p) , \quad (65a) \]

\[ \left\{ m_r \partial_{m_r} - \frac{\partial}{\partial \bar{p}} + \eta_F \gamma^{(0,2)}(p) \right\} = \eta_F G_{\kappa}^{(0,2)}(p) - \eta_\kappa G_{\kappa}^{(0,2)}(p) - G_{\kappa}^{(0,2)}(p) , \quad (65b) \]

where

\[ \eta_\kappa := m_r \frac{d}{d \ln m_r} \ln \kappa_{m_r} \quad \& \quad \eta_F := m_r \frac{\partial}{\partial \ln m_r} \ln F_{m_r} \quad (63) \]

\[ \eta_\kappa := m_r \frac{d}{d \ln m_r} \ln \kappa_{m_r} \quad \& \quad \eta_F := m_r \frac{d}{d \ln m_r} \ln F_{m_r} \quad (61a) \]

\[ \frac{1}{2} (d-1) \frac{d}{d \ln m_r} \left\{ \frac{m_r}{\kappa_{m_r}} \gamma^{(1,1)}(p, m_r) \right\} = \]

\[ \eta_F \left\{ \lambda_0 m_r^{d-2} + \left( \lambda_1 + \lambda_0 \right) \gamma^{(0,2)}(p, m_r) \right\} = \]

\[ \frac{1}{2} \frac{d}{d \ln m_r} \left\{ \lambda_0 + \lambda_1 \right\} \quad (66) \]
The set of the $G_{k}^{(i,j)}$'s are non-linear convolutions of the unknown functions $\gamma^{(1,1)}, \gamma^{(0,2)}$ with certain integral kernels specified by the dynamics. We detail the form of these convolutions in appendices C1 and C2. In order to fully specify the dynamics we need to associate to two renormalization conditions specifying the coefficients $\hat{G}$, $\hat{G}_{*}^{(i,j)}$. We require

$$\gamma^{(1,1)}(p_o) = \gamma^{(0,2)}(p_o) = 1 ,$$

where $p_o$ is the renormalization scale, i.e., the reference intra-red scale where we suppose to measure the eddy-diffusivity and the force amplitude. Solving the renormalization condition (67) for $\eta_F, \eta_c$ we obtain

$$\eta_c = G_{*F}^{(1,1)} \frac{\hat{G}_{*}^{(i,j)}(p_o)}{G_{*F}^{(1,1)}},$$

$$\eta_F = \frac{1 + G_{*F}^{(1,1)}}{G_{*F}^{(1,1)}} \eta_c - \frac{\hat{G}_{*}^{(i,j)}}{G_{*F}^{(1,1)}},$$

where $G_{*F}^{(i,j)} = G_{*F}^{(i,j)}(p_o)$ for all $i, j, k$ and

$$\hat{G}_{*}^{(1,1)} := (p \cdot \partial p \hat{\gamma}^{(1,1)})(p_o) - G_{o}^{(1,1)}(p_o) ,$$

$$\hat{G}_{*}^{(0,2)} := (p \cdot \partial p \hat{\gamma}^{(0,2)})(p_o) - \frac{2 - d - 2 \varepsilon}{\lambda_{(1)}} G_{o}^{(0,2)}(p_o).$$

The physical motivation behind the renormalization conditions (67) is the following. When the running cut-off $m_r$ is of the order of the ultra-violet cut-off $M$ the average action tends to the limit as the integral scale $m$ tends to zero, we must observe in the scaling range

$$\gamma^{(1,1)}(p) \sim p^{-\frac{2 \varepsilon}{d}} ,$$

and

$$\gamma^{(0,2)}(p) \sim p^{2 - d - 2 \varepsilon} .$$

We expect this behavior to be the physically correct for $0 < \varepsilon \ll 1$ and $d > 2$. Perturbative renormalization in two dimensions also predicts the attainment of this fixed point.

C. Fixed point for $\lambda_{(0)} > 0, \lambda_{(1)} = 0$

The approximated renormalization group flow equations remain well defined in the limit $\lambda_{(1)} \to 0$. In such a case $G_{*F}^{(i,j)}(p) = 0$ and decouples from (65b). Furthermore the renormalization conditions yield, self-consistently,

$$\eta_F = 0 .$$

In other words, the renormalization group equation has only one relevant coupling, the eddy diffusivity. This is the situation usually faced in perturbative renormalization under the assumption that the spatial dimension is bounded away from two. In such a case only $U^{(1,1)}$ has non-negative ultra-violet degree. This implies that there is no need to introduce a local counter-term in $U^{(0,2)}$ so that $F_{m_r}$ is set to zero a priori. The approximated, non-perturbative flow here devised reproduces these features. It is readily seen that the scaling predictions are then the same as in case IV E.

D. Fixed point for $\lambda_{(0)} = 0, \lambda_{(1)} > 0$

A similar fixed point, if attained, describes an energy input dominated by its ultra-violet component independently of $\varepsilon$. It is tempting to associate a similar scenario
with the 2d inverse cascade. The attainment of such fixed point implies
\[ \eta_F = 2 - d + 3 \eta_\kappa . \] (76)
The value of \( \eta_\kappa \) here needs to be determined dynamically.

In order to check the realizability of the aforementioned scenarios we resorted to the numerical solution of the coupled set of equations (65), (68) and (71).

V. A SIMPLIFIED MODEL

Before turning to the numerical solution of (65), it is expedient to analyze a simplified version of the flow. We therefore set
\[ F_{m_r} = \lambda_{(1)} = R = 0 , \] (77)
and hypothesize a sharp infra-red cut-off for the power-law forcing
\[ F(p; m_r) = H(p - m_r) F_0 p^{4-d-2 \varepsilon} . \] (78)
Since perturbative ultra-violet renormalization forbids non-local counter-terms \[ 37, 38 \], these approximations are adapted only to the case \( d > 2 \). As a consequence, we expect (65) to converge to the fixed point of section IV C

\[(p \cdot \partial p + \frac{2 \varepsilon}{3}) \gamma^{(1,1)}(p) = \frac{G^{(1,1)}_o(p)}{\lambda_{(0)}} , \] (79a)
\[ [p \cdot \partial p - (2 - d - \varepsilon)] \gamma^{(0,2)}(p) = \frac{G^{(0,2)}_o(p)}{\lambda_{(0)}} , \] (79b)

with \( G^{(1,1)}_o \), \( G^{(0,2)}_o \) respectively specified by
\[ \frac{G^{(1,1)}_o(p)}{\lambda_{(0)}} = \frac{C_d}{2 P^2} \int_{-1}^{1} d\phi \left( \frac{1 - \phi^2}{P^2} \right)^{\frac{d-1}{2}} \times \frac{[(d-1)p^3 - 2p(2 - d) + (d-3)p^2 + 2\phi p]}{g^{(1,1)}(1)g^{(1,1)}(1 + P^2 g^{(1,1)}(P))} , \] (80)
and
\[ \frac{G^{(0,2)}_o(p)}{\lambda_{(0)}} = \frac{C_d}{2} \int_{-1}^{1} d\phi \left( \frac{1 - \phi^2}{P^2} \right)^{\frac{d+1}{2}} g^{(0,2)}(P) \times \frac{[(d-1)p^2 - 2d \phi k + 2k^2 (d+2 \phi^2 - 2)]}{g^{(1,1)}(1)g^{(1,1)}(P)g^{(1,1)}(1 + P^2 g^{(1,1)}(P))} . \] (81)

In (80), (81) we used the notation
\[ P := \sqrt{1 + p^2 + 2 \phi p} . \] (82)
In the limit \( p \gg 1 \) we can approximate (79a) as
\[ \left( p \partial_p + \frac{2 \varepsilon}{3} \right) \gamma^{(1,1)}(p) \approx \frac{(d-1)}{2 d p^2 \gamma^{(1,1)}(p) \gamma^{(0,1)}(1)} . \] (83)

The value of \( \gamma^{(0,1)} \) needs to be determined dynamically.

In (80), (81) we used the notation
\[ p \approx \frac{d - 1}{2 d p^2 \gamma^{(1,1)}(p) \gamma^{(0,1)}(1)} . \] (83)

whence we infer the leading scaling behavior
\[ \gamma^{(1,1)}(p) p^\gamma \begin{cases} p^{-\frac{2d}{3}} & 0 < \varepsilon < \frac{3}{2} \\ p^{-1} & \frac{3}{2} < \varepsilon \end{cases} , \] (84)
under the self-consistency condition
\[ p \gg \frac{(d-1)}{2 d c + \gamma^{(1,1)}(1) \left( 1 - \frac{2\varepsilon}{3} \right)} \approx 1 . \] (85)

Logarithmic corrections may be possible at \( \varepsilon = 3/2 \). Similarly, we can approximate (79b) as
\[ (p \partial_p + d + 2 \varepsilon - 2) \gamma^{(0,2)}(p) \approx \frac{p^{2-d-2 \varepsilon} + \gamma^{(0,2)}(p)}{\gamma^{(1,1)}(p)} \left( p \partial_p + \frac{2 \varepsilon}{3} \right) \gamma^{(1,1)}(p) , \] (86)
in the non-dimensional wave-number range defined by (85). The corresponding scaling prediction is
\[ \gamma^{(0,2)}(p) p \gg \begin{cases} p^{2-d-2 \varepsilon} & 0 < \varepsilon < \frac{3}{2} \\ p^{2-d-2 \varepsilon + \frac{2d}{3} - 1} & \frac{3}{2} < \varepsilon \end{cases} . \] (87)

The conclusion is that the model problem kinetic energy spectrum should scale in agreement with the prediction of the perturbative renormalization group:
\[ \mathcal{E}(p) \approx p^{d-1} p^{2-d-2 \varepsilon} + \frac{\gamma^{(0,2)}(p)}{\gamma^{(1,1)}(p)} \sim p^{1-\frac{4 \varepsilon}{3}} . \] (88)

The eddy diffusivity and the force vertices, however, individually deviate from the perturbative renormalization group prediction. In particular the eddy diffusivity as observed first in [40] saturates to an \( \varepsilon \) independent value for \( \varepsilon > 3/2 \). In Fig 5 we show that the above predictions compare favorably with the numerical integration of (79). For \( 0 < \varepsilon < 2 \), these results are also consistent with the direct numerical simulations of [14, 20].
We integrated numerically the set of equations \( \frac{2}{55} \) for the eddy diffusivity and the renormalized forcing amplitude, and equations \( \frac{71}{} \) for the coupling constants, together with the renormalization conditions \( \frac{65}{} \).}
All the eddy diffusivity scales as the perturbative renormalization prediction. In particular, in this regime both fields deviate individually from the perturbative solution, as obtained in (73) and (74). Instead, for $\varepsilon < 3/2$, the eddy diffusivity and the forcing amplitude $E$ are shown in the middle panel) as a function of $\varepsilon$ (blue solid circles) for the $d = 3$ simplified model of section §3. The red open squares in the upper panel correspond to $\eta_\gamma$. The solid and dashed lines correspond to the predicted scaling of equations (84), (87) and (88), for $\varepsilon > 3/2$ respectively.

To determine the ultra-violet scaling law as a function of $\varepsilon$, we computed

$$\Lambda^{(x,y)} \equiv \lim_{p \to \infty} \frac{\log \gamma^{(x,y)}}{\log p}, \tag{89}$$

for $(x, y) = (1, 1)$ or $(0, 2)$, which defines the scaling exponent of the respective function. We denote with $\Lambda^{(E)}$ the analogous measure for the energy spectrum.

In Fig. 3 we show the scaling exponents $\eta_\gamma$ (red open squares in the upper panel) and $\eta_E$ (red open squares in the middle panel) as a function of $\varepsilon$. Our numerical results are in excellent agreement with the theoretical predictions (72) (solid lines), meaning that our closure yields the perturbative renormalization scaling. In the same figure we also show the scaling exponent of the dimensionless renormalized functions $\gamma^{(1,1)}$ (upper panel), $\gamma^{(0,2)}$ (middle panel) and of the energy spectrum $E$ as a function of $\varepsilon$ (blue solid circles), for $\varepsilon < 3/2$ and $\varepsilon > 3/2$ respectively.

We observe two different regimes. In the first regime, for $\varepsilon < 3/2$, the eddy diffusivity and the forcing amplitude scale in agreement with perturbative renormalization, as obtained in (73) and (74). Instead, for $\varepsilon > 3/2$, both fields deviate individually from the perturbative renormalization prediction. In particular, in this regime the eddy diffusivity scales as $\gamma^{(1,1)} \sim p^{-1}$ independently of $\varepsilon$. This saturation has been predicted first in (30). More interestingly, the deviation of the forcing amplitude is such that the energy spectrum scaling is in agreement with perturbative renormalization, i.e., $E \sim p^{-2/3}$, for all $\varepsilon$. Moreover, the deviations of the eddy diffusivity and the forcing amplitude from the perturbative renormalization coincide with those predicted by our simplified model, equations (84) and (87).

Finally, we would like to remark some properties of the convergence of the numerical scheme that we have used. As we mentioned above, the initial seed for the integration scheme comprises the initial value of the Grashof numbers. We have chosen this initial numbers by drawing $\lambda_{(0)}$ and $\lambda_{(1)}$ as random values in the domain $[0.01, 10]$. By doing this, we found that the solution of our numerical scheme always converged to the fixed point when $\varepsilon < 3$. However, for larger $\varepsilon$, we noticed that this was no longer the case. For $\varepsilon > 3$ some of the initial conditions failed to converge. This can be seen in Fig. 4 in which we show as yellow (light grey) dots, those initial conditions that converged to the fixed point. We notice that the basin of attraction, limited to the $[0.01, 10] \times [0.01, 10]$ domain, shrinks as $\varepsilon$ grows. While we have no ultimate explanation for this behavior, it may be due either to the very small values that $\lambda_{(0)}$ attain for $\varepsilon > 3$ or, more trivially,
FIG. 6. (Color online) Dependence of the fixed point \((\lambda_0, \lambda_1)\) (blue dots) on \(\varepsilon\) and \(d = 2\). The fixed point tends toward \((0, 0)\) as \(\varepsilon \to 0\).

to the fact that our numerical scheme fails to converge to the fixed point (shown as the blue (dark grey) circle), when the initial condition is too far from it.

B. Single renormalization condition

We have solved the simplified model of section V simply by setting \(\eta_F = 0\) and using the numerical scheme described above, by integrating equations (65), (71a) and (68a). In Fig. 6 we show the results that corroborate the predicted behavior of equations (84), (87) and (88).

In summary, we have obtained that the stationary solution to equations (65) is described by equations (84), (87) and (88), irrespectively if we impose the system to either one or two renormalization conditions.

C. 2d

In two dimensions the results are in perfect agreement with the predictions of equations (84), (87) and (88), meaning that the fixed point in two dimensions is qualitatively the same as in three dimensions, namely the fixed point \((\lambda_0, \lambda_1)\) tends to \((0, 0)\) as \(\varepsilon\) tends to zero; for \(\varepsilon \lesssim 1\), \(\lambda_{(1)} < 0\) and becomes positive for a value of \(\varepsilon\) between 1 and 1.25; for \(\varepsilon > 2\), \(\lambda_{(0)}\) decreases exponentially.

In Fig. 7 we show the scaling exponent \(\eta_{\kappa}\) (red open squares in the upper panel) as a function of \(\varepsilon\), in agreement with the prediction (72). Moreover, we also show the scaling exponent of the dimensionless renormalized functions \(\gamma^{(1,1)}(p)\) and \(\gamma^{(0,2)}(p)\) and of the energy spectrum \(E\) as a function of \(\varepsilon\) (blue solid circles) and \(d = 2\). We also show in red open squares the dependence of the scaling exponents \(\eta_{\kappa}\) (upper panel) and \(\eta_F\) (middle panel) on \(\varepsilon\). The solid and dashed lines correspond to the predicted scaling of equations (84), (87) and (88), for \(\varepsilon < 3/2\) and \(\varepsilon > 3/2\) respectively.

Finally, as it was the case in three dimensions, in two dimensions we also observed that the basin of attraction shrinks for \(\varepsilon \gtrsim 3\), as is seen in Fig. 8.

VII. CONCLUSIONS

The accuracy of the predictions of non-perturbative truncations of the renormalization group flow obviously depends upon the available physical information about the fixed point to which the flow is expected to converge. In the case of the stochastic Navier–Stokes equations, we used the simplest possible Ansatz for the critical average action satisfying Galilean-invariance in order to investigate the infra-red behavior of the theory at zero frequency. This is of course a drastic approximation, certainly a priori inadequate to study the multi-time behavior of correlation functions. We were, however, motivated by the results of [20] where it was shown that a similar approximation appears to be able to capture
the existence of non-perturbative fixed point for the KPZ stochastic partial differential equation. This latter model shares with the stochastic Navier–Stokes equation invariance under Galilean transformations and convergence towards a non-Boltzmann steady state. An important difference between these two models resides, however, in the non-locality of the interactions that the incompressibility condition brings forth. In retrospective, this difference explains why our approximation can reproduce and extend the predictions of the perturbative renormalization group to a finite range of values of the Hölder exponent. On the one hand, direct numerical simulations corroborate these predictions in the physical range (0 < ε < 2 for d > 2 and 2 < ε < 3 in d = 2) where phenomenological insights support the idea that the critical theory should be described by an average action with relevant local (in real space) interactions. On the other hand, in the regime of fully developed turbulence, all evidences point at the fact that the local-interaction energy spectrum becomes a sub-leading correction to the constant-flux −5/3 kinetic energy spectrum. Our approximations fall short of explaining this behavior. There are two types of mechanisms which may account with renormalization theory for the constant flux scaling predictions. Either the dominance of statistical indicators by composite operators perturbing the scaling relations dictated by the local average action fixed point (see discussion in [17]) or the onset of a non-perturbative fixed point, the scenario that we tried to explore in the present contribution. Fully developed turbulence is believed (see discussion in [3]) to be characterized by "localness" of interactions. This means interactions which are local not in real space but in wave-number space. Localness of interactions was the motivation behind the attempts to derive scaling in turbulence models by implementing renormalization group transformations through progressive resummation of the infra-red degrees of freedom (see e.g. [42, 43]). Our results indicate that further explorations of non-perturbative renormalization of the stochastic Navier–Stokes equation in the localness of interaction regime should necessarily encompass the full set of proper vertices contributing to the flux. As a conclusive remark we observe that non-perturbative renormalization methods may have spin-offs for engineering applications: in three dimension for 0 < ε < 2 the results of the present paper already provide a simple model for the computation of eddy-diffusivities which can be used in large eddy simulations [45].
preserve ordinary calculus. Using (A3) we can write

\[
< e^{r_v \delta (j \star v)} \star (m_r \partial_m, \kappa_m, R) \star \partial^2 v > \\
= e^{r_v \delta f} \star (m_r \partial_m, \kappa_m, R) \star \partial^2 v > \\
= \text{tr}(m_r \partial_m, \kappa_m, R) \star \partial^2 Z^{(1,1)}_{(j, \bar{j})}. 
\]

Furthermore, a functional integration by parts yields

\[
< e^{r_v \delta (j \star v)} \star (m_r \partial_m, f') > \\
= \frac{1}{2} < (m_r \partial_m, f') \star \delta^2 e^{r_v} \delta \frac{\partial}{\partial j} >, 
\]

the factor 1/2 being a consequence of Stratonovich convention.

2. Ward identity

Let \( r_t : \mathbb{R} \rightarrow \mathbb{R}^d \) a smooth path. The generalized Galilean transformation

\[
\bar{x} = x + \varepsilon r_t, \\
\bar{v} = v + \varepsilon \dot{r}_t,
\]

leaves (1) invariant in form when if accompanied by the redefinition of the forcing \( \bar{f} = f + \varepsilon \dot{r}_t \). We must have therefore

\[
Z^{(e)}_{(j, \bar{j})} = Z_{(j, \bar{j})}. 
\]

When we differentiate this equality at \( \varepsilon \) equal zero and use (A2) we obtain after standard manipulations (see e.g.

\[
0 = \dot{r} \star \left( \frac{\delta W_{(j, \bar{j})}}{\delta \bar{j}} \right) + \\
\dot{\bar{j}} \left( r \cdot \partial \frac{\delta W_{(j, \bar{j})}}{\delta j} - \dot{r} \right) + \dot{\bar{j}} \star \left( r \cdot \partial \frac{\delta W_{(j, \bar{j})}}{\delta j} \right). 
\]

An alternative way to derive the results of this appendix is based on the Janssen–De Dominicis \[16,47\] path integral representation of (1). We refer to \[42\] for a detailed presentation.

Appendix B: Janssen–De Dominicis path integral and optimal fluctuation

The Janssen–De Dominicis \[16,47\] representation is the formal measure on path space obtained by requiring through an infinite dimensional product of Dirac \( \delta \)-functions that at any space-time point \( \mathbb{H} \) be satisfied. The resulting expression is then averaged over the realizations of the stochastic forcing. We obtain

\[
A = \bar{v} \star F \star \bar{v} - j \star v \\
- iv \star [(\partial_t - \kappa \partial_x^2)v + T(v \cdot \partial_x v) - \bar{j}]. 
\]

A precise meaning to (B1) can be given on a space-time lattice using a pre-point discretization \( dt (\bar{v} \cdot \partial_t v) \sim \bar{v}(t_i) \cdot [v(t_{i+1}) - v(t_i)] \), \( dt f(\bar{v}(t), v(t)) \sim dt f(\bar{v}(t_i), v(t_i)) \) for all other terms in (B1b). Notice that in the limit of vanishing stirring \( F \downarrow 0 \), (B1b) recovers the Fourier representation of a product of Dirac \( \delta \)-functions localizing the measure over the deterministic decaying dynamics. In this sense (B1b) remains meaningful also as a formal measure inclusive of compressible fluctuations. From (B1b) a stationary phase approximation yields the weak noise limit of the free energy \( W_{(j, \bar{j})} \) around an optimal fluctuation \( v^* \). As usual (B8), the stationary phase condition is derived by closing a contour in the complex variables

\[
\bar{v} = \bar{v}_R + iv_\Im, 
\]

which decomposes (B1b) into the real and imaginary parts

\[
\Re A_{(j, \bar{j})} = \bar{v}_R \star F \star \bar{v}_R - j \star v + \\
\Im A_{(j, \bar{j})} = \bar{v}_\Im \star \left\{ (\partial_t - \kappa \partial_x^2) v + T(v \cdot \partial_x v) - \frac{1}{2} F \star \bar{v}_\Im - \bar{j} \right\}. 
\]

\[
\Re A_{(j, \bar{j})} = -\bar{v}_R \star \left\{ (\partial_t - \kappa \partial_x^2) v + T(v \cdot \partial_x v) - F \star \bar{v}_\Im - \bar{j} \right\}. 
\]

The stationary phase condition \( \Re A_{(j, \bar{j})} = 0 \) can then be solved for \( \bar{v}_\Im \) and leaves with a convex functional of the principal field \( v \). Assuming that we can minimize such functional for some assigned boundary condition, we find within logarithmic accuracy

\[
W_{(j, \bar{j})} \sim j \star v^* \\
- \| (\partial_t - \kappa \partial_x^2) v^* + T(v \cdot \partial_x v)^* - \bar{j} \|^2_x / 2, 
\]

where \( \| v \|^2_x \) stands for \( \| v \|^2_x = v^* F^{-1} v \). The Legendre transform gives the conditions

\[
u = v^*, 
\]

\[
u = F^{-1} \star \left\{ (\partial_t + u \cdot \partial_x - \kappa \partial_x^2) u - \frac{1}{2} \bar{u} \star F \star \bar{u} \right\}. 
\]

whence we finally obtain

\[
U \sim \bar{u} \star (\partial_t + u \cdot \partial_x - \kappa \partial_x^2) u - \frac{1}{2} \bar{u} \star F \star \bar{u}. 
\]

Appendix C: Explicit expression of the convolutions

An alternative derivation of the renormalization group equations is obtained if we observe that we may interpret the free energy defined by the Ansatz for the average
action \cite{57} as solution of a formal Janssen-De Dominicis path integral

\[ \mathcal{W}(\varphi, \bar{\varphi}) = \lim_{\varepsilon \to 0} e^{\frac{i}{\hbar} \int_{\mathbb{R}^d} \bar{\varphi}(x) \mathcal{J}(x) \varphi(x) - \mathcal{H}(x) dx} \]  

(C1)

Computing the right hand side in a perturbative expansion in powers of the interaction vertex \cite{46, 50} we obtain by standard diagrammatic techniques

\[ \kappa_m, p^2 \gamma^{(1,1)}(p/m_r) = \int \frac{d^d k}{(2 \pi)^d} \frac{1 - \phi^2}{2 g^{(1,1)}(k) D_1(p, k, \phi)} \]  

(C2)

and

\[ [\lambda(0)]^2 p_r^2 = \lambda(1) g^{(0,2)}(p/m_r) = \int \frac{d^d k}{(2 \pi)^d} \frac{1 - \phi^2}{4 g^{(1,1)}(k) g^{(1,1)}(Q) D_1(p, k, \phi)} \]  

(C3)

We recover equations \cite{65a} by taking the logarithmic derivative \( \partial_m \partial_m \) of both sides of (C2), (C3). Note that in (C2), (C3) we denoted

\[ Q := p - k \]  

(C4)

and \( \phi \) the cosine between the external \( p \) and the integration \( k \) wave-numbers:

\[ \phi := \frac{p \cdot k}{|p|} \]  

(C5)

We also defined the auxiliary integrand factors

\[ D_1(p, k, \phi) = k^2 g^{(1,1)}(k) + Q^2 g^{(1,1)}(Q) \]  

(C6)

\[ D_2(p, k, \phi) = 2 k^2 g^{(1,1)}(k) + Q^2 g^{(1,1)}(Q) \]  

(C7)

and the constants

\[ C_{d-1} = (d - 1) \int_{-1}^1 d\phi (1 - \phi^2)^\frac{d-3}{2} \]  

(C8)

Finally, the convolutions depends upon certain integral kernels which stem from the expansion up to one loop accuracy of the Ansatz average action \cite{57}. These are

\[ N^{(1,1)}(p, k, \phi) := \frac{(d - 1) p^2 (p - 2 \phi k) + k^2 p [(d - 3) p + 2 \phi k]}{k^2 (p^2 + k^2 - 2 p k \phi)} \]  

(C9a)

\[ \tilde{N}^{(1,1)}(p, k, \phi) := \frac{p k [(d - 1) p k - 2 (p^2 + k^2 - 2 p k \phi) \phi]}{k^2 (p^2 + k^2 - 2 p k \phi)} \]  

(C9b)

for the eddy diffusivity vertex, \cite{65b} will be needed below, and

\[ N^{(0,2)}(p, k, \phi) := \frac{p^2 [(d - 1) p^2 - 2 d p k \phi + 2 k^2 (d + 2 \phi^2 - 2)]}{k^2 (p^2 + k^2 - 2 p k \phi)^2} \]  

(C10)

for the force vertex. Finally in \cite{65a} there appear terms of the form

\[ \frac{C_{d-1}}{2 p^2} \int_0^\infty \frac{dk}{k} \int_1^\infty \frac{d\phi}{1 - \phi^2} V^{(i,j)}(p, k, \phi) \]  

(C11)

with \( l \) taking values \{F, \kappa, \omega\} and \( V^{(i,j)}(p, k, \phi) \) the non-linear convolutions specified below.

1. Equation for the eddy diffusivity vertex

The following three non-linear convolutions enter \cite{65b}:

\[ V^{(1,1)}_F(p, k, \phi) := \frac{N^{(1,1)}(p, k, \phi) \lambda(1) \chi(1)}{g^{(1,1)}(k) D_1(p, k, \phi)} \]  

(C12)

with coefficient \( \eta_F \),

\[ V^{(1,1)}_\kappa(p, k, \phi) := \frac{\overline{R}(k)}{|D_1(p, k, \phi)|^2} \times \left\{ \frac{D_2(p, k, \phi) N^{(1,1)}(p, k, \phi) g^{(0,2)}(k)}{|g^{(1,1)}(k)|^2} + \frac{k^4 \tilde{N}^{(1,1)}(p, k, \phi) g^{(0,2)}(Q)}{Q^2 g^{(1,1)}(Q)} \right\} \]  

(C13)

with coefficient \( \eta_{\kappa} \), and

\[ V^{(1,1)}_\omega(p, k, \phi) := \frac{N^{(1,1)}(p, k, \phi) \lambda(1) \chi(1)}{g^{(1,1)}(k) D_1(p, k, \phi)} \frac{(k \cdot \partial_k \overline{R})(k)}{|D_1(p, k, \phi)|^2} \left[ \frac{D_2(p, k, \phi) N^{(1,1)}(p, k, \phi) g^{(0,2)}(k)}{|g^{(1,1)}(k)|^2} + \frac{k^4 \tilde{N}^{(1,1)}(p, k, \phi) g^{(0,2)}(Q)}{Q^2 g^{(1,1)}(Q)} \right] \]  

(C14)

with coefficient equal to the unity.

2. Equation for the force vertex

The following three non-linear convolutions enter \cite{65b}:

\[ V^{(0,2)}_F(p, k, \phi) := \frac{N^{(0,2)}(p, k, \phi) g^{(0,2)}(Q) \chi(1)}{g^{(1,1)}(Q) g^{(1,1)}(k) D_1(p, k, \phi)} \]  

(C15)

with coefficient \( \eta_F \),

\[ V^{(0,2)}_\kappa(p, k, \phi) := \frac{N^{(0,2)}(p, k, \phi) \times}{g^{(0,2)}(Q) g^{(0,2)}(k) \overline{R}(k) D_2(p, k, \phi)} \frac{g^{(0,2)}(Q) g^{(1,1)}(k)}{|g^{(1,1)}(k)|^2 |D_1(p, k, \phi)|^2} \]  

(C16)
with coefficient $\eta_n$, and

$$V^{(0,2)}_\omega(p,k,\phi) := \frac{N^{(0,2)}(p,k,\phi) g^{(0,2)}(Q)}{g^{(1,1)}(Q) g^{(1,1)}(k) D_1(p,k,\phi)} \times \left\{ \sum_{i=0}^{\infty} \lambda_{(i)} \left( k \cdot \partial_k - d_{L(i)} \right) \chi_{(i)}(k,\mu) - \frac{(k \cdot \partial_k \tilde{R})(k) g^{(0,2)}(k)}{g^{(1,1)}(k)} \frac{D_2(p,k,\phi)}{D_1(p,k,\phi)} \right\}, \quad (C17)$$

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