Supersolid in Bose–Bose–Fermi mixtures subjected to a square lattice

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Abstract

Two-component Bose condensates with repulsive interaction are stable when $g_1^2 > g_{12}^2$ is satisfied. By tuning the interactions, we show that the instability corresponding to Bose–Bose phase separation always happens at a higher temperature than that corresponding to Bose–Fermi phase separation. Moreover, we find the transition temperatures $T_{DW}$ of both the supersolid and coherence peak at $k_{DW}$ are enhanced in the mixtures studied. These will make the observation of a supersolid in experiments more reachable.

(Some figures may appear in colour only in the online journal)

1. Introduction

Supersolids, a concept simultaneously exhibiting superfluidity and crystalline order, have been intensely studied over five decades [1–4]. Theoretically, researchers mainly focus on lattice models of interacting bosons and fermions such as the Hubbard model and its various generalizations and have obtained many important results by numerical analysis [5, 6]. Experimentally, Kim and Chan recently reported the discovery of nonclassical rotational inertia, which should be direct evidence of a supersolid based on Leggett’s suggestion in solid $^4$He [7, 8], however, it has also been pointed out that this observation may not be due to a supersolid but could be due to other reasons, such as an increase in shear modulus of bulk solid helium [9], and has triggered an intense debate [10, 11].

Besides the study of supersolids in condensed matter systems, ultracold atoms in optical lattices [12] have emerged as a parallel platform to study supersolids because they are highly controllable. Trapped Bose–Einstein condensates with dipole–dipole interaction can produce a ‘roton’ minimum in the excitation spectrum [13–15], and this led to the prediction of a supersolid upon softening the roton excitation energy [16, 17]. Recently, on the basis of the off-resonant dressing of atomic Bose–Einstein condensates to high-lying Rydberg states, researchers have found that the effective atomic interactions resulting from such a Rydberg dressing can also produce a roton minimum and, therefore, provide a clean realization of the available model for supersolidity [18, 19].

In this work, we consider that the two kinds of bosons are two hyperfine states of $^{87}$Rb, and the fermions are a hyperfine state of $^{40}$K, and investigate Bose–Bose–Fermi mixtures in a square lattice. For the Bose–Fermi mixtures subjected to a square lattice, it has been pointed out that the density wave instability introduced by fermions will establish a crystalline order, while the condensate bosons exhibit superfluidity, so a supersolid phase emerges at finite temperature [17]. For the Bose–Bose–Fermi mixtures studied here, besides the density wave instability introduced by fermions, there is another instability between the two-component Bose condensates when $g_1 g_2 > g_{12}^2$, where $g_1$, $g_2$ are the repulsive intraspecies interaction and $g_{12}$ is the interspecies interaction [20]. When $g_1 g_2 > g_{12}^2$, the Bose condensates are mixed and stable. When $g_1 g_2 < g_{12}^2$, the Bose condensates are unstable and tend to either phase separation or collapse depending on $g_{12} > 0$ or $< 0$. In this paper, we assume the Bose condensates are initially mixed and stable, and we find that Bose–Bose phase separation always happens before Bose–Fermi phase separation when we decrease the temperature. Moreover, we find the transition temperatures $T_{DW}$ of both the supersolid and coherence peak at $k_{DW}$ are enhanced in comparison to the Bose–Fermi mixtures case [17].
The paper is organized as follows. In section 2, we consider that the two kinds of bosons are two hyperfine states of $^{87}\text{Rb}$, and the fermions are a hyperfine state of $^{40}\text{K}$, and investigate Bose–Bose–Fermi mixtures in a square lattice, and give the fermionic response in the static limit. In section 3, we give details of the instabilities and different phases induced by the instabilities, and give a mean field description of the supersolid phase. Some conclusions are presented in section 4.

2. Bose–Bose–Fermi mixtures

The Hamiltonian for the Bose–Bose–Fermi mixtures takes the form $H = H_0 + H_{\text{int}}$ with ($\alpha = \uparrow, \downarrow$)

$$H_0 = \sum_\alpha \int dx \psi_\alpha^\dagger \left( -\frac{\hbar^2}{2m_\alpha} \nabla^2 + V_\alpha (x) \right) \psi_\alpha,$$

$$H_{\text{int}} = \int dx \left \{ g_\uparrow \psi_\uparrow^\dagger \psi_\uparrow \psi_\uparrow + g_\downarrow \psi_\downarrow^\dagger \psi_\downarrow \psi_\downarrow + g_m (\psi_\uparrow^\dagger \psi_\uparrow + \psi_\downarrow^\dagger \psi_\downarrow) \right \},$$

where $\psi_\alpha^\dagger$ are the bosonic field operators and $\psi_\alpha$ is the fermionic field operator. In order to assure that the mixtures are stable, we assume that all interactions between bosons are repulsive, with $g_{\alpha,\beta} = 4\pi a_{\alpha,\beta} \hbar^2/m (\alpha, \beta \in \{\uparrow, \downarrow\})$. In this work, we use $\alpha, \beta$ to label the interactions, densities and phases of the bosons, and use $\uparrow, \downarrow$ only to label the bosonic operators, moreover, when $\alpha = \beta$ we only keep $\alpha$ for convenience.

where $g_{\alpha,\beta}$ is the mass of bosons and $\mu$ is the relative mass. $V_\alpha (x) = V_\alpha [\sin^2 (\pi x / a) + \sin^2 (\pi y / a)]$ is the periodic potential produced by the optical lattice with wavelength $\lambda = 2a$ and $n_a$ is the mass of bosons and fermions. As the bosons are two hyperfine states of $^{87}\text{Rb}$, it is justified to assume $m_\uparrow = m_\downarrow$ and $V_\uparrow = V_\downarrow$ for simplicity in the following. Since the fermions are single components, the interaction between them can be neglected due to Pauli’s exclusion principle.

In order to obtain the Hamiltonian in momentum space, we follow the procedures used in [17] and expand the bosonic and fermionic field operators $\psi_\alpha$ in the forms

$$\psi_{\alpha \uparrow} (x) = \sum_{k \in K} b_{k \alpha \uparrow} \left( \psi_{k \alpha \uparrow} (x) \right),$$

$$\psi_0 (x) = \sum_{k \in K} c_k \psi_k (x),$$

where $K$ denotes the first Brillouin zone, $b_{k \alpha \uparrow}$ and $c_k$ are the bosonic and fermionic annihilation operators, while $\psi_{k \alpha \uparrow} (x)$ and $\psi_k (x)$ are the Bloch wave functions corresponding to a single boson ($\uparrow$ or $\downarrow$) or fermion in the periodic potential $V_\alpha$, respectively. Since $m_\uparrow = m_\downarrow$ and $V_\uparrow = V_\downarrow$, $\psi_{k \alpha \uparrow} (x)$ should be equal to $\psi_{k \alpha \downarrow} (x)$. Therefore, we use $\psi_k (x)$ to denote both of them for convenience. Substituting equation (2) into equation (1) and restricting in the lowest Bloch band, we obtain the Hamiltonian in momentum space as

$$H = \sum_{k \in K, \sigma} \epsilon_{\alpha \sigma} b_{k \alpha \sigma}^\dagger b_{k \sigma} + \sum_{[k, k', q, \sigma]} \frac{U_{\alpha \beta}}{2N} b_{k \alpha \sigma}^\dagger b_{k' \beta \sigma}^\dagger b_{q \beta \sigma} b_{q' \sigma} + \sum_{Q \in K} \epsilon_Q c_Q^\dagger c_Q$$

$$+ \frac{U_{\alpha \beta}}{N} \sum_{[k, k', q, \sigma]} b_{k \alpha \sigma}^\dagger b_{k' \beta \sigma}^\dagger b_{q \beta \sigma} b_{q' \sigma},$$

where $N$ is the number of unit cells, $\epsilon_{\alpha \sigma}(k)$ denote the energy dispersion of the fermions and bosons, respectively, while $U_{\alpha \beta} = g_{\alpha \beta} \int dx \left[ \tilde{w}^2 \right] \tilde{n}^2$ and $U_{\alpha \sigma} = g_\sigma \int dx \tilde{W}$, with $\tilde{w}(x)$ and $\tilde{n}(x)$, the Wannier functions associated with the Bloch band $\psi_k (x)$ and $\psi_k (x)$. In a deep optical lattice, the Wannier functions $\tilde{w}(x)$ and $\tilde{n}(x)$ are well localized around the minimum of $V_\alpha$. As a result, the Hamiltonian reduces to a familiar Bose–Fermi–Hubbard model, and for $\epsilon_{\alpha \sigma}(k)$, only nearest-neighbour hopping survives,

$$\epsilon_{\alpha \sigma}(q) = 2J_{\alpha \sigma} (2 \cos (q_a) - \cos (q_b)),$$

$$\epsilon_{\sigma}(q) = -2J_{\sigma} (\cos (q_a) + \cos (q_b)),$$

where $J_{\alpha \sigma}$ is the hopping energy for fermions and bosons, respectively. (In the above, we have assumed that the wave functions are well-localized and the parameters $J$ and $U$ are density-independent. However, we have to remember that this assumption works well not only when the lattice is deep, but also when the density is not too high. This can be seen directly since the interaction energy is proportional to the square of the density, and high density will make the localization of the wavefunction unfavourable in energy. From the relation between the wavefunction and parameters, it can be directly seen that the delocalization of the wavefunction will directly change the parameters $J$ and $U$.) The bosonic dispersion relation implies $\mu_B = -4\hbar^2$ and the bosons will form a zero-momentum Bose–Einstein condensation for sufficiently low temperature. The fermionic dispersion relation implies the Fermi surface at half-filling $n_F = 1/2$ (where $\mu_F = 0$ in this work. $n_F$ and $n_q$ denote the number of particles per unit cell) and exhibits perfect nesting for $k_{m_a} = (\pi/a, \pi/a)$ and van Hove singularities at $k = (0, \pm \pi/a, 0).$

Integrating out the fermions produces two effects (in this work, we focus on weak interaction, i.e. $U_{\alpha \beta} \ll J_{\alpha \sigma}$, as $\epsilon_{\alpha \sigma}(q)$ is proportional to $J_{\alpha \sigma}$, $\chi (T, q)$ shown in equation (6) below is proportional to $1/J_{\alpha \sigma}$ (in fact, $\chi (T, q)$ is of order $O(0.1/J_{\alpha \sigma})$ when $q$ is away from the singularities), therefore, an expansion in $U_{\alpha \beta} / J_{\alpha \sigma}$ and a cut at the first order of $U_{\alpha \beta} / J_{\alpha \sigma}$ are justified). To first order in $U_{\alpha \beta}$ (zero order of $U_{\alpha \beta} / J_{\alpha \sigma}$), the fermions simply produce a (trivial) shift of the bosonic chemical potential $\mu_{\alpha \beta} \rightarrow \mu_{\alpha \beta} - U_{\alpha \beta} n_\sigma$. To second order in $U_{\alpha \beta}$ (the first order of $U_{\alpha \beta} / J_{\alpha \sigma}$), the fermions provide an effective interaction for the bosons which depends on the temperature $T$ of the fermionic atom gas,
with
\[ U_B(T, q - q') = U_{B0} + U_{B0}^2 x(T, q - q'), \]
\[ U_{B1}(T, q - q') = U_{B1} + U_{B1}^2 x(T, q - q'). \]

The fermionic response in the static limit is given by the Lindhard function
\[ \chi(T, q) = \int d\vec{k} \frac{f[\epsilon_\vec{k}(q)] - f[\epsilon_\vec{k}(q + q)]}{\epsilon_\vec{k}(q + q) + i\eta}, \]
where $\nu_0 = (2\pi/\alpha)^2$ is the volume of the first Brillouin zone, $f(\epsilon) = 1/[1 + \exp(\epsilon/T)]$ (with $\nu_0 = 0$ at half filling) is the Dirac–Fermi distribution function. The static limit is justified if the fermions are much faster than the bosons ($J_c \gg J_0$), so that the fermionic response occurs on much faster timescales than the movement of the bosons, and one can safely neglect retardation effects [21]. Using the fermionic dispersion relation, equation (4), the Lindhard function exhibits two logarithmic singularities at $q = 0$ and $k_{BW}$. The singularity at $q = 0$ is purely due to the logarithmic van Hove singularity in the density of states, and the singularity at $k_{BW}$ is due to the combination of van Hove singularities and perfect nesting. The singularity at $q = 0$ induces an instability towards a series of phase separation, while the singularity at $k_{BW}$ induces an instability towards density wave formation and provides a supersolid phase. The two instabilities are competing with each other.

3. Instabilities and phases

For the weak interaction, when the temperatures is well below the superfluid transition temperature $T_{cr}$ of the bosons, the Lindhard function at $q = 0$ reduces to $\chi(T \to 0, 0)$ and takes the form [17]
\[ \chi(T \to 0, 0) = \int d\epsilon N(\epsilon) \delta f(\epsilon) \sim -N_0 \ln \frac{16c_1 J}{T}, \]
with $N(\epsilon) \sim N_0 \ln \left[16\epsilon_c/\epsilon\right], N_0 = 1/(2\pi^2 J)$ and $c_1 = 2 \exp(C)/\pi \approx 1.13$. As $\chi(T \to 0, 0)$ is always negative, the coupling between the bosons and the fermions induces an attractive interaction, which is proportional to $U_{B0}^2 x(T, 0)$, between the bosons (see equation (5)). This attractive interaction has the effect of reducing the repulsive interactions $U_{B1,2}$ between bosons to $U_{B1,2} - U_{B0}^2 x(T, 0)$. As a result, even $U_{B1,2} > U_{B0}^2 x(T, 0)$ (equivalent to $g_{B1,2} > g_{B0}^2$) initially, $U_{B1,2}$ can be tuned to equal $U_{B0}^2 x(T, 0)$ by lowering the temperature to some value. Moreover, for a superfluid condensate at low temperatures to be stable requires a positive effective interaction $U_{B1,2} > 0$. If we take $U_{B1,2}$ as the energy unit, and define the ratios $U_{B1,2}/U_{B1,1}, U_{B1,2}/U_{B0,1}$ and $U_{B0,1}/U_{B1,1}$ as $\gamma, \lambda$ and $\kappa$, respectively. The condition $U_{B1,2}^2 > U_{B1,1}^2$ defines the critical temperature $T_{BW,PS}$ for Bose–Bose phase separation,
\[ T_{BW,PS} = 16c_1 J \exp \left[ \frac{\lambda^2 - \gamma}{N_0 \kappa^2 (1 + \gamma - 2\lambda)} \right]. \]

The condition $U_{B0,1} = 0$ defines two critical temperatures $T_{BW,PS}$ and $T_{BW,PS}^2$ for Bose–Fermi phase separation;
\[ T_{BW,PS} = 16c_1 J \exp \left[ \frac{-1}{N_0 \kappa^2} \right], \]
\[ T_{BW,PS}^2 = 16c_1 J \exp \left[ \frac{-\gamma}{N_0 \kappa^2} \right]. \]

When $\lambda \neq 1$ and $\gamma$, it can be directly shown that $(\gamma - \lambda^2)/(1 + \gamma - 2\lambda)$ is always smaller than $\min[1, \gamma]$ under the constraint $\lambda^2 < \gamma$, which is the condition that the Bose condensates are initially mixed (see figure 1). $(\gamma - \lambda^2)/(1 + \gamma - 2\lambda) < \min[1, \gamma]$ indicates that when we lower the temperature, it is always easier for the Bose–Bose mixtures to be unstable and phase separated (or collapse, see figure 1) than the Bose–Fermi mixtures. Moreover, when $\gamma$ gets close to the boundary $\sqrt{\gamma}$, $(\gamma - \lambda^2)/(1 + \gamma - 2\lambda)$ decreases very fast, as a result, $T_{BW,PS}$ increases exponentially to values much larger than $\max[T_{BW,PS}, T_{BW,PS}^2]$, which is easy to reach in experiments. Therefore, such a Bose–Bose phase separation induced by fermions should be easy to observe in experiments. If we continue to lower the temperature after the Bose–Bose mixtures are phase separated, we can expect that Bose–Fermi phase separation will happen and all the components will distribute separately in space at last.

Now, we discuss the second instability induced by the singularity in the Lindhard function at $k_{BW}$. Using equation (6) and the perfect nesting $\epsilon_\nu(q + k_{BW}) = -\epsilon_\nu(q)$, the Lindhard function becomes [17]
\[ \chi(T, k_{BW}) = \int d\epsilon N(\epsilon) \frac{\tanh(\epsilon/2T)}{-2\epsilon} \sim -\frac{N_0}{2} \left[ \ln \frac{16c_1 J}{T} \right]^2. \]

The combination of van Hove singularities and perfect nesting produces a $[\ln T]^2$ singular behaviour. Such a singular behaviour can produce a roton minimum at $k_{BW}$. Within Bogoliubov theory, the bosonic quasi-particle becomes
\[ E_{B1,2}(q) = \epsilon_\nu^2(q) + \epsilon_\nu(q) n_B U_{B1}(T, q) + U_{B2}(T, q) \]
\[ \pm \left[ \epsilon_\nu^2(q) n_B^2 U_{B1}(T, q) - U_{B2}(T, q)^2 \right]^{1/2} + 4\epsilon_\nu^2(q) U_{B2}(T, q)^2. \]
here we have assumed \( n_{\text{a}} = n_{\text{b}} = n_0 \). The induced attraction proportional to \( U_{\text{a}}(T, k_{\text{m}}) \) reduces the energy of quasi-particles at \( k_{\text{m}} \) from a pure bosonic maximum (when \( U_{\text{a}} \) is the maximum) of \( E_{\text{n}}(q) \) locates at \( k_{\text{m}} \) to an induced zero roton minimum \( E_{\text{n}}(k_{\text{m}}) = 0 \) at the critical temperature

\[
T_{\text{cr}} = 16c_{\text{I}}J_{\text{F}} \exp \left[-\sqrt{\frac{t_g + 2t_\gamma + 4\gamma - 4\gamma^2}{2\hbar^2 k_{\text{F}}^2(1 + \gamma + t_\gamma - 2\gamma^2)}}\right]
\]  

(12)

with \( t_g = 8J_0/n_0 \). As \( E_{\text{n}}(k_{\text{m}}, T_{\text{cr}}) = E_{\text{n}}(k = 0) = 0 \), we can expect the boson modes \( b_{\text{a,b}} \) to become macroscopically occupied just like the boson mode \( b_{\text{a,b}} \) below this critical temperature. Comparing this result to the one obtained in [17],

\[
T_{\text{cr}} = 16c_{\text{I}}J_{\text{F}} \exp \left[-\sqrt{\frac{2 + t_\gamma}{\hbar^2 n_0}}\right],
\]

we find \( T_{\text{cr}} \) is always higher than \( T_{\text{cr}} \) when parameters appearing in both methods take the same values (see figure 1). Moreover, since \( T_{\text{cr}} \) depends on \( \sqrt{\frac{2 + t_\gamma}{\hbar^2 n_0}} \) exponentially (12), a small change of this term may induce a great change of \( T_{\text{cr}} \). Therefore, such an enhancement of critical temperature can be large. However, \( T_{\text{cr}} \) cannot increase as greatly as \( T_{\text{cr}} \), since \( t_g \) has to be larger than a critical value \( t_g^* \approx 1.3 \), below which a Mott insulating phase emerges and the above picture fails [12]. As a comparison, we calculate \( T_{\text{cr}} \) based on the parameters used in [17] and find \( T_{\text{cr},\text{Mott}} > T_{\text{cr}} \) (if Bose–Bose phase separation happens first, \( T_{\text{cr}} \) reduces to \( T_{\text{cr}} \), and the enhancement effect of \( T_{\text{cr}} \) misses). Such an enhancement of \( T_{\text{cr}} \) is of realistic meaning, since the lower the temperature is, the harder it is to reach in cold atomic experiments.

For temperatures well below \( T_{\text{cr}} \) and \( T_{\text{cr}} \), both the boson mode \( b_{\text{a,b}} \) and \( b_{\text{a,b}} \) are macroscopically occupied, therefore, it is justified to use mean fields \( \langle b_{\text{a,b}} \rangle \) and \( \langle b_{\text{a,b}} \rangle \) to substitute them. Introducing the mean fields \( \langle b_{\text{a,b}} \rangle = \sqrt{n_{\text{a,b}}} \langle \psi_{\text{a,b}} \rangle \) and \( \langle b_{\text{a,b}} \rangle = \langle \psi_{\text{a,b}} \rangle \), we obtain the bosonic densities as

\[
n_{\text{a}}(x,y) = n_{\text{a}} + \frac{\Delta_{\text{a}} \cos \theta_{\text{a}}}{U_{\text{a}}} \left[ \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \right]
\]

(13)

with \( \theta_{\text{a}} = \varphi_{\text{a,b}} - \varphi_{\text{a}} \). The phase difference \( \Delta \theta = \theta_1 - \theta_2 \) between the two bosonic density waves determines whether they are constructive or destructive. Introducing \( \langle b_{\text{a,b}} \rangle \) and \( \langle b_{\text{a,b}} \rangle \) to Hamiltonian (3) and neglecting terms independent of \( \Delta_{\text{a,b}} \), the Hamiltonian per unit cell is given as

\[
H = \frac{2\hbar}{N} \left[ \frac{\Delta_{\text{a}}^2 + \Delta_{\text{b}}^2}{n_{\text{a,b}}} + \frac{\Delta_{\text{a}}^2 \cos^2 \theta_1}{2U_{\text{a}}} + \frac{\Delta_{\text{b}}^2 \cos^2 \theta_2}{2U_{\text{b}}} + \frac{U_{\text{a}} \Delta_{\text{a}} \Delta_{\text{b}} (\cos^2 \theta_1 + \cos^2 \theta_2 + 2 \cos \theta_1 \cos \theta_2)}{4U_{\text{a,b}}} \right] + \frac{H_{\text{F}}}{N} + \alpha(\Delta^4).
\]

(14)

The terms in the first and second lines describe the increase in the kinetic and interaction energies of the bosons due to the modulation of densities triggered by the boson modes \( b_{\text{a,b}} \), while \( H_{\text{F}} \) takes the form

\[
H_{\text{F}} = \frac{1}{2} \sum_{q \neq 0} \left[ c_{\text{q}, \text{q}}^* \right] \left( \begin{array}{ccc} \epsilon_{\text{r}}(q) & \Delta(\theta_1, \theta_2) & \epsilon_{\text{r}}(q) \\ \Delta(\theta_1, \theta_2) & \Delta(\theta_1, \theta_2) & \epsilon_{\text{r}}(q) \\ \epsilon_{\text{r}}(q) & \Delta(\theta_1, \theta_2) & \epsilon_{\text{r}}(q) \end{array} \right) \left[ c_{\text{q}, \text{q}} \right] \]

(15)

with a constraint \( q' = q - k_{\text{m}} + k_\text{a} \). The reciprocal lattice vector \( k_\text{a} \) ensures the constraint \( q' \in K \) and \( \Delta(\theta_1, \theta_2) = \Delta_1 \cos \theta_1 + \Delta_2 \cos \theta_2 \). Diagonalizing the fermionic Hamiltonian, we obtain the fermionic quasi-particle excitation spectrum \( E_{\text{F}}(k, \Delta) = \pm |\epsilon_{\text{F}}(k)| + (\Delta_1 \cos \theta_1 + \Delta_2 \cos \theta_2)^2|^{1/2} \). To determine the phase difference \( \Delta \theta \), we minimize the thermodynamic potential \( \Omega(T, \Delta_1, \Delta_2, \theta_1, \theta_2) \) and find a constraint between \( \theta_1 \) and \( \theta_2 \): \( \theta_1 = \theta_2 = s \pi \), s an integer. Therefore, the phase difference \( \Delta \theta = 0 \), the two bosonic density waves are completely constructive and produce a stronger density wave. A stronger density wave makes the crystalline order favourable, therefore, such a phase-locking effect is favourable for forming a supersolid phase. As \( \theta_1 = \theta_2 = s \pi \), \( \Delta_1 = \Delta_2 \) is in fact the gap. Introducing \( \Delta_1 = \Delta_1 \pm \Delta_2 \) and rewriting equation (14), the self-consistency relations \( \alpha, \Omega = 0 \) take the form

\[
(1 + \gamma + t_g - 2\lambda) \Delta_1 = (\gamma - 1) \Delta_1 \pm \frac{1}{2N_{\text{a,b}}} \left[ (1 + \Delta_1) + 2\lambda \right].
\]

(16)

Setting \( \Delta_1(T_{\text{cr}}) = 0 \) and combining the two equations above, we reproduce the critical temperature in equation (12). This confirms the picture that upon softening \( E_{\text{n}}(k_{\text{m}}) \) to zero, the bosonic density waves characterized by \( \langle b_{\text{a,b}} \rangle \) do not emerge with a breaking of the discrete symmetry of the optical lattice is correct. Furthermore, using the density of states \( N_{\text{a,b}}(\epsilon) = N(\sqrt{\epsilon^2 - \Delta_1^2})/\sqrt{\epsilon^2 + \Delta_1^2} \), the gap at \( T = 0 \) becomes

\[
\Delta_{\text{a,b}}(0) = 32c_{\text{I}}J_{\text{F}} \exp \left[-\sqrt{\frac{t_g + 2t_\gamma + 4\gamma - 4\gamma^2}{2\hbar^2 k_{\text{F}}^2(1 + \gamma + t_\gamma - 2\gamma^2)}}\right].
\]

(17)

and the standard BCS relation \( 2\Delta_{\text{a,b}}(0)/T_{\text{cr}} = 2\pi/e^2 \approx 3.58 \) holds. This relation implies that the density wave has the characteristic of the superfluid, which is evidence of a supersolid. Therefore, \( T_{\text{cr}} \) is just the critical temperature of the supersolid to emerge.

In experiments, the supersolid can be detected via the usual coherence peak of a bosonic condensate in an optical lattice. The appearance of a coherence peak at \( k_{\text{m}} \) is a symbol that the supersolid appears. Since the weight of this coherence peak is proportional to the number of bosons condensed at \( k_{\text{m}} \), the larger \( \Delta_\text{e} \) (here equivalent to \( \langle b_{\text{a,b}} \rangle \)) is, the sharper the peak. Therefore, based on the similarity of the forms between \( \Delta_{\text{a,b}}(0) \) and \( T_{\text{cr}} \), we find a sharper coherence peak at \( k_{\text{m}} \) appears in Bose–Bose–Fermi mixtures compared to the one appearing in Bose–Fermi mixtures [17] when parameters
appearing in both systems take the same values. Based on the results above, we can reach the conclusion that it is more favourable to observe the supersolid in Bose–Bose–Fermi mixtures than in Bose–Fermi mixtures.

4. Conclusions

In this paper, we have investigated a Bose–Bose–Fermi mixture subjected to a square lattice and found that the instability corresponding to Bose–Bose phase separation always happens at a higher temperature than the one corresponding to Bose–Fermi phase separation. Moreover, we find the transition temperatures $T_{DW}$ of both the supersolid and coherence peak at $k_{DW}$ are enhanced in the mixtures studied. These will make the observation of supersolids in experiments more reachable.

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