We show the existence of automorphisms $F$ of $\mathbb{C}^2$ with a Fatou component $\Omega$ biholomorphic to $\mathbb{C} \times \mathbb{C}^*$ that is attracted to a Siegel curve (a curve on which $F$ is conjugated to an irrational rotation). We further show that the biholomorphism $\Omega \to \mathbb{C} \times \mathbb{C}^*$ can be chosen such that it conjugates $F$ to a translation $(z, w) \mapsto (z + 1, w)$, making $\Omega$ a non-recurrent Siegel cylinder as recently defined by L. Boc Thaler, F. Bracci and H. Peters. $F$ and $\Omega$ are obtained by blowing up a fixed point of an automorphism $\tilde{F}$ of $\mathbb{C}^2$ with a Fatou component of the same biholomorphic type attracted to that fixed point, established by F. Bracci, J. Raissy and B. Stensønes. A crucial point is the observation that the automorphism in their work can be chosen such that it fixes a coordinate axis. We can then remove the proper transform of that axis from the blow-up to obtain an $F$-stable subset of the blow-up that is biholomorphic to $\mathbb{C}^2$. Thus we can interpret $F$ as an automorphism of $\mathbb{C}^2$.  

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1. Introduction

When studying the behaviour of iterates of a holomorphic endomorphism $F$ of $\mathbb{C}^d$, $d \geq 1$, one of the basic objects of interest is the Fatou set of all points in $\mathbb{C}^d$ that admit a neighbourhood on which $\{F^n\}_{n \in \mathbb{N}}$ is normal. The connected components of the Fatou set are called the Fatou components of $F$. They are maximal connected subsets on which the behaviour of $F$ is roughly the same. A Fatou component $V$ of $F$ is invariant, if $F(V) = V$. It is recurrent if it contains an accumulation point of an orbit $\{F^n(p)\}_{n \in \mathbb{N}}$ for some $p \in V$ and non-recurrent otherwise. $V$ is attracting to a point $p \in V$ if all orbits starting in $V$ converge to $p$.

In one variable, any invariant Fatou component $V$ is either attracting to a point in $V$ or $V$ is a rotation domain, i.e. there exists a subsequence $\{F^{n_k}\}_k$ converging to the identity on $V$. In other words, the images of limit functions of convergent subsequences of $\{F^n\}_n$ on $V$ are unique and have dimension 0 or full dimension 1 respectively.

Recurrent Fatou components of endomorphisms of $\mathbb{C}^2$ have been classified for polynomial automorphisms in [BS91], [FS95] and [Ued08]. In the general case, [ABFP19] shows that if $F$ is an automorphism of $\mathbb{C}^2$ with constant Jacobian, then a Fatou component $V$ is either the basin of an attracting fixed point in $V$ (biholomorphic to $\mathbb{C}^2$ by [PVW08], Theorem 2 and the appendix of [RR88]) or a rotation domain, or a recurrent Siegel or Hermann cylinder, i.e. biholomorphic to $A \times \mathbb{C}$, with $A \subseteq \mathbb{C}$ a domain invariant under rotations, on which $F$ is conjugated to $(z, w) \mapsto (\lambda z, aw)$ with $|\lambda| = 1$ and $|a| < 1$.

By [LP14] every invariant non-recurrent Fatou component of a polynomial automorphism with sufficiently small Jacobian is attracting to a parabolic fixed point in the boundary. Moreover, by [Ued86], every invariant non-recurrent attracting Fatou component of a polynomial automorphism is biholomorphic to $\mathbb{C}^2$ and admits coordinates conjugating it to a translation $(z, w) \mapsto (z + 1, w)$. Outside the polynomial setting, the classification is far from complete and several new phenomena occur.

In [BRS] the authors construct automorphisms of $\mathbb{C}^2$ with an attracting non-recurrent invariant Fatou component biholomorphic to $\mathbb{C} \times \mathbb{C}^*$ (see also [Rep19] for multiple such components). We show in Proposition 4.8 that on these Fatou components the automorphisms are again conjugated to a translation $(z, w) \mapsto (z + 1, w)$. It is an open question whether these are the only possible biholomorphic types of non-recurrent attracting invariant Fatou components of automorphisms of $\mathbb{C}^2$ and if they all admit such a conjugation. It is not even clear that these are the only homotopy types that can occur.

In [JL04] the authors take first steps towards narrowing down the possible invariant non-recurrent Fatou components of automorphisms of $\mathbb{C}^2$. They split their discussion according to the rank of limit maps of $\{F^n\}_n$ on the Fatou component $V$. In case all limit maps have rank 0, they show that $V$ is either attracting or the images of the limit maps form an uncountable set without isolated points contained in a subvariety of fixed points. The eigenvalues in each of these points are $\{1, \alpha\}$ where $\alpha$ is a unique non-diophantine rotation. There are no known examples with more than one rank 0 limit map.
In the case of rank 1 limit maps [BTBP19] defines and gives examples of non-recurrent Siegel cylinders biholomorphic to \( \mathbb{C}^2 \) in the following sense:

**Definition 1.1.** Let \( F \) be a self-map of \( \mathbb{C}^2 \). The \( \omega \)-limit set \( \omega_F(p) \) of a point \( p \in \mathbb{C}^2 \) or \( \omega_F(U) \) of an open set \( U \subseteq \mathbb{C}^2 \) under \( F \) is the set of all accumulation points of orbits under \( F \) starting in \( p \) or \( U \) respectively.

**Remark 1.2.** A Fatou component \( U \) of \( F \) is non-recurrent, iff \( \omega_F(U) \cap U = \emptyset \).

**Definition 1.3.** An invariant non-recurrent Fatou component \( V \) of \( F \) is called a non-recurrent Siegel cylinder, if

1. the closure of \( \omega_F(V) \) contains an isolated fixed point,
2. there is an injective holomorphic map \( \Phi : V \to \mathbb{C}^2 \) conjugating \( F \) to the translation \( (z, w) \mapsto (z + 1, w) \),
3. all limit maps of \( \{F^n\}_n \) on \( \Omega \) have dimension 1.

[JL04] gives examples of Fatou components with a unique rank 1 limit map and with an uncountable family of rank 1 limit maps with identical images. The latter are a subclass of the Siegel cylinders examined in [BTBP19]. The authors further show that the images of two limit maps of rank 1 are either disjoint or intersect in a relatively open subset. There are no known examples of rank 1 limit maps with non-identical images or limit maps of different rank.

In this paper we show the following:

**Theorem 1.4.** There exist holomorphic automorphisms \( F \) of \( \mathbb{C}^2 \) possessing a non-recurrent Siegel cylinder \( \Omega \) biholomorphic to \( \mathbb{C} \times \mathbb{C}^* \) and such that the limit maps of \( \{F^n\}_n \) on \( \Omega \) all have image \( \mathbb{C}^* \times \{0\} \) and differ precisely by postcomposition with rotations of the image.

The non-recurrent Siegel cylinder in the above theorem is punctured in that it is biholomorphic to \( \mathbb{C} \times \mathbb{C}^* \) and has as its \( \omega \)-limit set a punctured Siegel curve \( \mathbb{C}^* \times \{0\} \), i.e. a holomorphic curve on which \( F \) is conjugated to an irrational rotation minus the fixed point.

The automorphisms in Theorem 1.4 have, near \( \mathbb{C} \times \{0\} \), the form

\[
F(x, y) = (\lambda^2 x + R(x, y), \lambda w(1 - xy^2/2) + yO((\|xy\|)^4)),
\]
where $R(x, y) = xO(||(xy, y)||^l) + O(||(xy, y)||^l)$ and $\lambda \in S^1$ is a Brjuno number. They are obtained by lifting to the blow-up at the origin automorphisms of the form
\[
(1.2) \quad \tilde{F}(z, w) = (\lambda z, \lambda w) \cdot \left(1 - \frac{zw}{2}\right) + wO(||(z, w)||^l)
\]
with $l \in \mathbb{N}_0$ sufficiently large, for which [BRS] established the existence of a non-recurrent Fatou component $\tilde{\Omega}$ attracted to the origin and biholomorphic to $\mathbb{C} \times \mathbb{C}^{*}$. The Siegel cylinder $\Omega$ is the proper transform of $\tilde{\Omega}$ and contains an $F$-invariant subset $B$ eventually containing any orbit in $\Omega$ given by
\[
B = \{(x, y) \in \mathbb{C}^2 \mid xy^2 \in S, |x| < \min\{|y|^{2\gamma}, |y|^{\gamma^{-1}}\}\},
\]
where $S$ is a small sector with vertex at 0 around the positive real axis in $\mathbb{C}$ and $\gamma \in (0, 1)$. Figure 1.1 shows $B$ as a product in polar decomposition (barring some truncation away from the limit set $\mathbb{C}^{*} \times \{0\}$ depending on $S$).

**Outline.** In Section 2, we construct our family of automorphisms. We first use results from [Var01] and [Var00] to show the existence of automorphisms $\tilde{F}$ of the form (1.2). We then blow up at the origin and observe that the lift $F$ of $\tilde{F}$ leaves invariant the proper transform of the $z$-axis. Removing this subvariety from the blow-up leaves us with a copy of $\mathbb{C}^2$ on which $F$ acts as an automorphism.

In Section 3 we use estimates on orbit behaviour in the Fatou component $\tilde{\Omega}$ to show that the proper transform $\Omega$ is still a Fatou component of $F$.

Then we establish coordinates on $\Omega$ conjugating $F$ to $(z, w) \mapsto (z + 1, w)$ in Section 4, which we use in Section 5 to identify the images of limit maps.

**Conventions.** We use the following notations for asymptotic behaviour (as $x \to x_0$):

- $f(x) = O(g(x))$, if $\limsup_{x \to x_0} \frac{|f(x)|}{|g(x)|} = C < +\infty$,
- $f(x) \approx g(x)$, if $f(x) = O(g(x))$ and $g(x) = O(f(x))$,
- $f(x) \sim g(x)$, if $\lim_{x \to x_0} \frac{|f(x)|}{|g(x)|} = 1$.

2. THE FAMILY OF AUTOMORPHISMS

In this section we show that the automorphisms with non-recurrent Fatou components biholomorphic to $\mathbb{C} \times \mathbb{C}^{*}$ and attracted to the origin constructed in [BRS] can be chosen such that their lift to the blow-up at the origin can be restricted to an automorphism of a subset biholomorphic to $\mathbb{C}^2$.

We first recall the framework of [BRS]: Let $F_N$ be a germ of biholomorphisms of $\mathbb{C}^2$ at the origin given by
\[
F_N(z, w) := (\lambda z, \lambda w) \cdot \left(1 - \frac{zw}{2}\right),
\]
where $\lambda \in S^1$ is Brjuno, i.e.
\[
- \sum_{\nu=1}^{\infty} 2^{-\nu} \log \omega(2^\nu) < \infty,
\]
where $\omega(m) := \min\{|\lambda^k - \lambda| \mid 2 \leq k \leq m\}$ for $m \geq 2$. 

Definition 2.1. For $r > 0$, $\theta \in (0, \pi/2)$, and $\beta \in (0, 1/2)$ let
\[
W(\beta) := \{(z, w) \in \mathbb{C}^2 \mid |z| < |zw|^{\beta}, |w| < |zw|^{\beta}\},
\]
\[
S(r, \theta) := \{u \in \mathbb{C} \mid |\arg(u)| < \theta, |u - r| < r\},
\]
and
\[
\tilde{B}(r, \theta, \beta) := \{(z, w) \in W(\beta) \mid zw \in S(r, \theta)\}.
\]

The main result in [BRS] (globalising a local result in [BZ13]) is:

Theorem 2.2. Let $l \in \mathbb{N}_{\geq 4}$, $\theta_0 \in (0, \pi/2)$, $\beta_0 \in (0, 1/2)$ such that $\beta_0(l + 1) \geq 4$. Then there exist automorphisms $\tilde{F}$ of $\mathbb{C}^2$ such that
\[
(2.2) \quad \tilde{F}(z, w) = F_N(z, w) + O(\|(z, w)\|^{l})
\]

near the origin and every automorphism of the form (2.2) has an invariant, non-recurrent Fatou component $\hat{\Omega}$ attracted to $(0, 0)$ and biholomorphic to $\mathbb{C} \times \mathbb{C}^*$, that contains a local (uniform) basin of attraction $\tilde{B} := \tilde{B}(r_0, \beta_0, \theta_0)$ for some $r_0 > 0$, that eventually contains any orbit in $\hat{\Omega}$, i.e. $\tilde{F}(\hat{B}) \subseteq \hat{B}$, $\lim_{n \to \infty} \tilde{F}^n \equiv (0, 0)$ uniformly in $\hat{B}$, and $\hat{\Omega} = \bigcup_{n \in \mathbb{N}} \tilde{F}^{-n}(\hat{B})$.

Next we show the above class of automorphisms contains elements fixing an axis. D. Varolin’s work on the density property shows in particular:

Theorem 2.3. For every invertible germ of automorphisms $G_0$ of $\mathbb{C}^2$ at the origin pointwise fixing $\{w = 0\}$ and every $l \in \mathbb{N}$, there exists an automorphism $G \in \text{Aut}(\mathbb{C}^2)$ such that
\[
(2.3) \quad G(z, w) = G_0(z, w) + wO(\|(z, w)\|^{l}).
\]

Proof. By [Var01], Theorem 5.1, the Lie algebra $\mathfrak{g}$ of holomorphic vector fields on $\mathbb{C}^2$ that vanish on $\mathbb{C} \times \{0\}$ has the density property, i.e. the complete vector fields are dense in $\mathfrak{g}$. [Var00], Theorem 1 states that for such a Lie algebra, if a germ can be interpolated up to some order $l \in \mathbb{N}$ (i.e. matched up to order $l$) by compositions of flows of vector fields in $\mathfrak{g}$, the same can be done using only flows of complete vector fields in $\mathfrak{g}$. Flows of complete vector fields in $\mathfrak{g}$ are automorphisms of $\mathbb{C}^2$ fixing $\{w = 0\}$. By [Var00] Example 1, the germs that can be interpolated in this way (to arbitrary order $l \in \mathbb{N}$) are precisely the ones fixing $\{w = 0\}$ pointwise.

Let $\Lambda = \text{diag}(\lambda, \overline{\lambda})$. Applying Theorem 2.3 to $G_0 = \Lambda^{-1} F_N$, we obtain an automorphism $\tilde{F} = \Lambda G \in \text{Aut}(\mathbb{C}^2)$ fixing $L := \{w = 0\}$ as a set and interpolating $F_N$ up to order $l$, i.e.
\[
(2.4) \quad \tilde{F}(z, w) = F_N(z, w) + wO(\|(z, w)\|^{l}).
\]

In particular, for $(z, 0) \in L$, we have $\tilde{F}(z, 0) = (\lambda z, 0) \in L$.

Finally consider the Blow-up $\Pi : \hat{\mathbb{C}}^2 \to \mathbb{C}^2$ of $\mathbb{C}^2$ at the origin. Then the lift $F$ of $\tilde{F}$ to $\hat{\mathbb{C}}^2$ leaves invariant the proper transform $\hat{L}$ of $L$ and hence its complement $\hat{\mathbb{C}}^2 \setminus \hat{L}$ which is isomorphic to $\mathbb{C}^2$ via the coordinates $(x, y) = (z/w, w)$ defined (after extending through the exceptional divisor $E$) on all of $\hat{\mathbb{C}}^2 \setminus \hat{L}$. So $F$ induces an automorphism of
\( \mathbb{C}^2 \) in these coordinates. The exceptional divisor \( E := \Pi^{-1}((0,0)) \) restricted to this \( \mathbb{C}^2 \) is \( E' = \mathbb{C} \times \{0\} \). For \((x, y) \in \mathbb{C}^2\), let \((x_n, y_n) := F^n(x, y)\). Then we have
\[
x_1 = \frac{\lambda^2 x(1 - xy^2/2) + O(\|xy\|^{1/\gamma})}{1 - xy^2/2 + O(\|xy\|^{1/\gamma})} = \lambda^2 x + R(x, y),
\]
where \( R(x, y) = xO(\|xy\|^{1/\gamma}) + O(\|xy\|^{1/\gamma}) \) near \( E' \) and hence
\[
F(x, y) = (\lambda^2 x + R(x, y), \bar{x}y(1 - xy^2/2) + yO(\|xy\|^{1/\gamma})).
\]
In particular \( E' \) is a Siegel curve for \( F \), i.e. \( F(x, 0) = (\lambda^2 x, 0) \) for all \((x, 0) \in E' \). The local basin \( \tilde{B} \) lifts to the \( F \)-invariant set
\[
B = \Pi^{-1}(\tilde{B}) = \{(x, y) \in \mathbb{C}^2 \mid xy^2 \in S(r_0, \theta_0), |x| < \min\{|y|^{2\gamma_0}, |y|^{\gamma_0-1}\}\},
\]
where \( \gamma_0 = \frac{\beta_0}{1-\beta_0} \in (0,1) \) (see Figure 1).

3. The Fatou component

In the following we will examine the dynamics of \( F \) on the lifted local basin \( B \) and show that the corresponding global basin \( \Omega := \Pi^{-1}(\tilde{\Omega}) = \bigcup_{n \in \mathbb{N}} F^{-n}(B) \) is still a Fatou component.

For \((x, y) \in B \) and \( n \in \mathbb{N} \), let \( U := 1/(xy^2) \) and \( U_n := 1/(x_ny_n^2) \). Then the local basin can be written as
\[
B = \{(x, y) \in \mathbb{C}^2 \mid U \in H(R_0, \theta_0), |x| < \min\{|y|^{2\gamma_0}, |y|^{\gamma_0-1}\}\},
\]
where \( R_0 = 1/(2r_0) \) and for \( R > 0 \) and \( \theta \in (0, \pi/2) \), the set
\[
H(R, \theta) := \{U \in \mathbb{C} \mid \Re U > R, |\arg(U)| < \theta\}
\]
is a sector “at infinity”. Now [BRS] Lemma 2.5 implies:

**Lemma 3.1.** For \((x, y) \in \Omega \), we have as \( n \to \infty \) locally uniformly
\[
(1) \ U_n \sim n,
(2) \ |y_n| \approx n^{-1/2},
(3) \ |x_n| \approx 1 \ (i.e. \ x_n \ is \ locally \ bounded \ away \ from \ 0 \ and \ \infty).
\]

Moreover the lower bound in (1) and upper bound in (2) are uniform in \( B \).

This is enough to show:

**Proposition 3.2.** \( \Omega \) is a Fatou component.

**Proof.** Lemma 3.1 parts (2) and (3) show that the family \( \{F^n\}_{n \in \mathbb{N}} \) is locally uniformly bounded on \( \Omega \), so by Montel’s theorem, it is a normal family on \( \Omega \).

Let \( V(B) \) be the Fatou component containing \( B \). Since \( \{F^n\}_{n \in \mathbb{N}} \) is normal on \( \Omega \), we have \( \Omega \subseteq V(B) \). For any limit map \( F_\infty = \lim_{k \to \infty} F^{n_k} \) for a subsequence \( \{n_k\}_{k \in \mathbb{N}} \), the image \( F_\infty(\Omega) \) is contained in the exceptional divisor \( E \), so by the identity principle, so is the image \( F_\infty(V(B)) \). In particular, for any \((x, y) \in V(B) \), we have \((z_n, w_n) = (x_ny_n, y_n) \to (0,0) \), so we can work with \( \bar{F} \) in the original \( \mathbb{C}^2 \) (before blowing up) and the proof of Theorem 5.7 in [BRS] shows that \( \Omega \) is a Fatou component. \( \square \)
4. Cylinder coordinates

In this section we show that $F$ is conjugated to the translation $(z, w) \mapsto (z + 1, w)$.

We use a Fatou coordinate and a second local coordinate introduced in [BRZ13] and [BRS] respectively to construct a global second coordinate.

In Sections 3 and 4 [BRS] shows, again setting $U = 1/(xy^2)$:

**Lemma 4.1.** There exists a map $\psi : \Omega \to \mathbb{C}$ such that $\psi(x, y) = U + c \log(U) + O(U^{-1})$ as $(x, y) \to E'$ and

$$\psi \circ F = \psi + 1,$$

and a map $\sigma : \Omega_0 := \psi^{-1}(\psi(B)) \to \mathbb{C}^*$ such that $\sigma(x, y) = y + O(U^{-\alpha})$ as $(x, y) \to E'$ with $\alpha \in (1 - \beta_0, 1) \subseteq (1/2, 1)$ and

$$\sigma \circ F = \lambda e^{-1/(2\psi)} \sigma.$$

Furthermore

$$(\psi, \sigma) : \Omega_0 \to \psi(B) \times \mathbb{C}^*$$

is biholomorphic and $\psi(\Omega_0) = \psi(B)$ sits between sectors at infinity $H(\tilde{R}, \tilde{\theta}) \subseteq \psi(B) \subseteq H(R_1, \theta_1)$ for some $\tilde{R} \geq R_1 > 0$ and $0 < \tilde{\theta} \leq \theta_1 < \pi/2$.

**Remark 4.2.** In particular, by Lemma 3.1, this implies $\psi(x_n, y_n) \sim U_n$, $\sigma(x_n, y_n) \sim y_n$, and $\psi(x_n, y_n) \sim n$ and $\sigma(x_n, y_n) \approx n^{-1/2}$ as $n \to +\infty$ where the lower bound on $\psi(x_n, y_n)$ is uniform in $\Omega_0$.

To construct our global second coordinate, we need the following lemma about the harmonic series:

**Lemma 4.3.** For $\zeta \in \mathbb{C}$ such that $\text{Re} \zeta > 0$ we have

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} \frac{1}{\zeta + j} - \log \left( \frac{\zeta + n}{\zeta} \right) = h(\zeta) = O \left( \frac{1}{\zeta} \right)$$

and both the limit and the bound are uniform for $\text{Re} \zeta > R$ for any fixed $R > 0$.

**Proof.** For $m < n$, we have

$$\left| \sum_{j=m}^{n-1} \frac{1}{\zeta + j} - \log \left( \frac{\zeta + n}{\zeta + m} \right) \right| \leq \sum_{j=m}^{n-1} \left| \frac{1}{\zeta + j} - \log \left( \frac{\zeta + j + 1}{\zeta + j} \right) \right|$$

$$= \sum_{j=m}^{n-1} \left| \frac{1}{\zeta + j} - \log \left( 1 + \frac{1}{\zeta + j} \right) \right|$$

$$= \sum_{j=m}^{n-1} O(|\zeta + j|^{-2})$$

$$= O(1/|\zeta + m|)$$

For $m \to \infty$ this shows uniform convergence and for $m = 0$ and $n \to \infty$ it follows that the limit is $O(1/\zeta)$. \[\square\]
Proposition 4.4. There exists a map \( \tau : \Omega_0 \to \mathbb{C}^* \) bijective on each fibre \( \psi^{-1}(p) \) for \( p \in \psi(B) \) such that
\[
(4.1) \quad \tau \circ F = \overline{\lambda}^n
\]
and \( \tau(x, y) = \sqrt{\psi(x, y)}\sigma(x, y) + \sigma(x, y)O(\psi(x, y)^{-1/2}) \) as \( (x, y) \to E' \).

Remark 4.5. By Remark 4.2, we have \( x_n = (\sqrt{\psi_n}y_n)^2 \sim (\tau(x_n, y_n))^2 \).

Proof. Let \( (x, y) \in \Omega_0 \) and \( n \in \mathbb{N} \). Note first that \( \psi(x_n, y_n) \in H(R_1, \theta_1) \), so the square root \( \sqrt{\psi(x_n, y_n)} \) is well-defined by choosing its values in the right half plane and we can define
\[
\tau_n(x, y) := \lambda^n \sqrt{\psi(x_n, y_n)} \sigma(x, y, y_n)
\]
\[
= \sqrt{\psi(x, y)} + n \exp \left( -\frac{1}{2} \sum_{j=0}^{n-1} \frac{1}{\psi(x, y) + j} \right) \sigma(x, y).
\]
We can consider each \( \tau_n \) as a map \( \psi(B) \times \mathbb{C}^* \to \mathbb{C}^* \) in variables \( \psi \) and \( \sigma \) given by
\[
\tau_n(\psi, \sigma) = \exp \left( \frac{1}{2} \left( \log \left( \frac{\psi + n}{\psi} \right) - \sum_{j=0}^{n-1} \frac{1}{\psi + j} \right) \right) \sqrt{\psi} \sigma,
\]
and by Lemma 4.3 we have
\[
\tau(\psi, \sigma) := \lim_{n \to \infty} \tau_n(\psi, \sigma) = \exp \left( \frac{1}{2} h(\psi) \right) \sqrt{\psi} \sigma = (1 + O(1/\psi)) \sqrt{\psi} \sigma = \sqrt{\psi} \sigma + \sigma O(\psi^{-1/2}).
\]
\( \tau \) is clearly bijective on each fibre and satisfies (4.1) since
\[
\tau_n \circ F = \sqrt{\psi \circ F + n} \exp \left( -\frac{1}{2} \sum_{j=0}^{n-1} \frac{1}{\psi \circ F + j} \right) \sigma \circ F
\]
\[
= \lambda \sqrt{\psi + n + 1} \exp \left( -\frac{1}{2} \sum_{j=0}^{n} \frac{1}{\psi + j} \right) \sigma = \lambda \tau_{n+1}.
\]

Now we can extend \( \tau \) using the functional equation (4.1):

Proposition 4.6. \( (\psi, \tau) : \Omega_0 \to \psi(B) \times \mathbb{C}^* \) extends to a biholomorphism \( \Phi : \Omega \to \mathbb{C} \times \mathbb{C}^* \) given by
\[
\Phi(x, y) = (\psi(x, y), \lambda^n \tau(F^n(x, y))
\]
for \( (x, y) \in F^{-n}(B) \) and conjugating \( F \) to \( (z, w) \mapsto (z + 1, \overline{\lambda} w) \).

Proof. \( (\psi, \tau) \) is injective on \( \Omega_0 \) by Proposition 4.4. Let \( p \in \Omega \). Then there exists \( n \in \mathbb{N} \) such that \( F^n(p) \in B \subseteq \Omega_0 \). For \( m < n \) such that \( F^m(p) \) and \( F^n(p) \) lie in \( B \), we have
\[
\lambda^n \tau(F^n(p)) = \lambda^n \tau(F^{n-m}(F^m(p))) = \lambda^m \tau(F^m(p)),
\]
so \( \Phi \) is well-defined. \( \Phi \) is moreover injective as for any \( p, q \in \Omega \) there exists \( n \in \mathbb{N} \) such that \( F^n(p) \) and \( F^n(q) \) lie in \( B \subseteq \Omega_0 \) where \( (\psi, \tau) \) is injective.

To show surjectivity take \( (\zeta, \xi) \in \mathbb{C} \times \mathbb{C}^* \). Then there exists \( n \in \mathbb{N} \) such that \( \zeta + n \in H(\tilde{R}, \tilde{\theta}) \subseteq \psi(B) \) and hence \( (\zeta + n, \lambda^{-n} \xi) \in \psi(B) \times \mathbb{C}^* = \text{im}(\psi, \tau) \), i.e. there exists \( p \in \Omega_0 \) such that \( (\psi, \sigma)(p) = (\zeta + n, \lambda^{-n} \xi) \) and hence \( \Phi(F^{-n}(p)) = (\zeta, \xi) \).
The multiplicative term $\lambda$ in the second component can always be eliminated, since the biholomorphic map $(z, w) \mapsto (z, \lambda^2 w)$ conjugates $(z, w) \mapsto (z + 1, \lambda w)$ to $(z, w) \mapsto (z + 1, w)$, yielding the following corollary:

**Corollary 4.7.** There exists a biholomorphic map $\Psi : \Omega \to \mathbb{C} \times \mathbb{C}^*$ conjugating $F$ to $(z, w) \mapsto (z + 1, w)$.

The arguments in this section rely only on the internal dynamics on $\Omega$ described by the coordinates in Lemma 4.1 that have been constructed in [BRZ13] and [BRS] for any automorphism of the form (2.2). Hence we have moreover shown:

**Proposition 4.8.** Let $\tilde{F}$ and $\tilde{\Omega}$ be as in Theorem 2.2. Then there exists a biholomorphic map $\tilde{\Psi} : \tilde{\Omega} \to \mathbb{C} \times \mathbb{C}^*$ conjugating $\tilde{F}$ to $(z, w) \mapsto (z + 1, w)$.

5. Limit sets

We use the coordinates from the previous section to identify the limit sets of orbits in $\Omega$ and the images of limit functions, concluding the proof of Theorem 1.4.

**Lemma 5.1.** For $(x, y) \in \Omega$, we have $\omega_F(x, y) = \tau(x, y)^2 S_1 \times \{0\}$.

**Proof.** By Lemma 3.1 we have $y_n \to 0$ and by Remark 4.5 we have $x_n \sim \tau(x_n, y_n)^2 = \lambda^{2n} \tau(x, y)^2$.

Since $\lambda$ is an irrational rotation, $x_n$ accumulates on all of $\tau(x, y)^2 S_1$. \hfill $\square$

**Corollary 5.2.** $\omega_F(B) = \mathbb{C}^* \times \{0\}$ and any limit function $F_\infty : \Omega \to \mathbb{C}^* \times \{0\}$ of a convergent subsequence of $\{F^n\}$ is surjective. Postcomposition of $F_\infty$ with a rotation of $\mathbb{C}^* \times \{0\}$ yields precisely all possible such limit functions.

**Proof.** The map $\tau : \Omega_0 \to \mathbb{C}^*$ is surjective, so $\omega(B) = \mathbb{C}^* \times \{0\}$. Every limit function $F_\infty$ is not constant by Lemma 5.1 and by Picard’s theorem satisfies $F_\infty(\Omega) = \mathbb{C}^* \times \{0\}$. \hfill $\square$

This concludes the proof of Theorem 1.4.

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 Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via Della Ricerca Scientifica 1, 00133, Roma, Italy
*E-mail address:* reppekus@mat.uniroma2.it