STRONGLY $\psi$-2-ABSORBING SECOND SUBMODULES

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Abstract. Let $R$ be a commutative ring with identity and $M$ be an $R$-module. Let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, where $S(M)$ denote the set of all submodules of $M$. The main purpose of this paper is to introduce and investigate the notion of strongly $\psi$-2-absorbing second submodules of $M$ as a generalization of strongly 2-absorbing second and $\psi$-second submodules of $M$.

1. Introduction

Throughout this paper, $R$ will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. We will denote the set of ideals of $R$ by $S(R)$ and the set of all submodules of $M$ by $S(M)$, where $M$ is an $R$-module.

Let $M$ be an $R$-module. A proper submodule $P$ of $M$ is said to be prime if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [7]. A non-zero submodule $S$ of $M$ is said to be second if for each $a \in R$, the endomorphism of $M$ given by multiplication by $a$ is either surjective or zero [10].

Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. Anderson and Bataineh in [1] defined the notation of $\phi$-prime ideals as follows: a proper ideal $P$ of $R$ is $\phi$-prime if for $r, s \in R, rs \in P \setminus \phi(P)$ implies that $r \in P$ or $s \in P$ [1]. In [11], the author extended this concept to prime submodule. Let $M$ be an $R$-module. For a function $\phi : S(M) \to S(M) \cup \{\emptyset\}$, a proper submodule $N$ of $M$ is called $\phi$-prime if whenever $r \in R$ and $x \in M$ with $rx \in N \setminus \phi(N)$, then $r \in (N :_R M)$ or $x \in N$.

Let $M$ be an $R$-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Farshadifar and Ansari-Toroghy in [8], defined the notation of $\psi$-second submodules of $M$ as a dual notion of $\phi$-prime submodules of $M$. A non-zero submodule $N$ of $M$ is said to be a $\psi$-second submodule of $M$ if $r \in R, K$ a submodule of $M, rN \subseteq K$, and $r\psi(N) \nsubseteq K$, then $N \subseteq K$ or $rN = 0$ [8].

The concept of 2-absorbing ideals was introduced in [6]. A proper ideal $I$ of $R$ is said to be a 2-absorbing ideal of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

In [3], the authors introduced the notion of strongly 2-absorbing second submodules as a dual notion of 2-absorbing submodules and investigated some properties of this class of modules. A non-zero submodule $N$ of $M$ is said to be a strongly 2-absorbing second submodule of $M$ if whenever $a, b \in R, K$ is a submodule of $M$, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in \text{Ann}_R(N)$ [3].

Let $M$ be an $R$-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. The main purpose of this paper is to introduce and investigate the notion of strongly $\psi$-2-absorbing second submodule.
ψ-2-absorbing second submodules of $M$ as a generalization of strongly 2-absorbing second and ψ-second submodules of $M$.

2. Main results

**Definition 2.1.** Let $M$ be an $R$-module, $S(M)$ be the set of all submodules of $M$, $ψ : S(M) → S(M) ∪ \{∅\}$ be a function. We say that a non-zero submodule $N$ of $M$ is a strongly ψ-2-absorbing second submodule of $M$ if $a, b \in R, K$ a submodule of $M$, $abN ⊆ K$, and $abψ(N) ∉ K$, then $aN ⊆ K$ or $bN ⊆ K$ or $ab ∈ AnnR(N)$.

In Definition 2.1 since $abψ(N) ∉ K$ implies that $ab(ψ(N) + N) ∉ K$, there is no loss of generality in assuming that $N ⊆ ψ(N)$ in the rest of this paper.

A non-zero submodule $N$ of $M$ is said to be a weakly strongly 2-absorbing second submodule of $M$ if whenever $a, b \in R, K$ is a submodule of $M$, $abM ∉ K$, and $abN ⊆ K$, then $aN ⊆ K$ or $bN ⊆ K$ or $ab ∈ AnnR(N)$ [5].

Let $M$ be an $R$-module. We use the following functions $ψ : S(M) → S(M) ∪ \{∅\}$.

$$ψ_1(N) = (N : M Ann^1_R(N)), \forall N \in S(M), \forall i ∈ N,$$

$$ψ_0(N) = \sum_{i=1}^{∞} ψ_i(N), \forall N \in S(M),$$

$$ψ_M(N) = M, \forall N \in S(M),$$

Then it is clear that strongly ψ$_M$-2-absorbing second submodules are weakly strongly 2-absorbing second submodules. Clearly, for any submodule and every positive integer $n$, we have the following implications:

strongly 2-absorbing second $⇒$ strongly ψ$_{n-1}$-2-absorbing second

$⇒$ strongly ψ$_n$-2-absorbing second $⇒$ strongly ψ$_σ$-2-absorbing second.

For functions $ψ, θ : S(M) → S(M) ∪ \{∅\}$, we write $ψ ≤ θ$ if $ψ(N) ⊆ θ(N)$ for each $N \in S(M)$. So whenever $ψ ≤ θ$, any strongly ψ-2-absorbing second submodule is a strongly θ-2-absorbing second submodule.

**Remark 2.2.** Let $M$ be an $R$-module and $ψ : S(M) → S(M) ∪ \{∅\}$ be a function. Clearly every strongly 2-absorbing second submodule and every ψ-second submodule of $M$ is a strongly ψ-2-absorbing second submodule of $M$. Also, evidently $M$ is a strongly ψ$_M$-2-absorbing second submodule of itself. In particular, $M = Z_6 ⊕ Z_{10}$ is not strongly 2-absorbing second $Z$-module but $M$ is a strongly ψ$_M$-2-absorbing second $Z$-module of $M$.

In the following theorem, we characterize strongly ψ-2-absorbing second submodules of an $R$-module $M$.

**Theorem 2.3.** Let $N$ be a non-zero submodule of an $R$-module $M$ and $ψ : S(M) → S(M) ∪ \{∅\}$ be a function. Then the following are equivalent:

(a) $N$ is a strongly ψ-2-absorbing second submodule of $M$;

(b) for submodule $K$ of $M$ with $aN ∉ K$ and $a \in R$, we have $(K :_R aN) = Ann_R(aN) ∪ (K :_R N) ∪ (K :_R aψ(N))$;

(c) for submodule $K$ of $M$ with $aN ∉ K$ and $a \in R$, we have either $(K :_R aN) = Ann_R(aN)$ or $(K :_R aN) = (K :_R N)$ or $(K :_R aN) = (K :_R aψ(N))$.
Proof. (a) ⇒ (b). Let for a submodule $K$ of $M$ with $aN \not\subseteq K$ and $a \in R$, we have $b \in (K :_R aN) \setminus (K :_R \psi(N))$. Then since $N$ is a strongly $\psi$-2-absorbing second submodule of $M$, we have $b \in \text{Ann}_R(aN) or bN \subseteq K$. Thus $(K :_R aN) \subseteq \text{Ann}_R(aN)$ or $(K :_R aN) \subseteq K :_R N)$. Hence,

$$(K :_R aN) \subseteq \text{Ann}_R(aN) \cup (K :_R N) \cup (K :_R \psi(N)).$$

As we may assume that $N \subseteq \psi(N)$, the other inclusion always holds.

(b) ⇒ (c). This follows from the fact that if an ideal is the union of two ideals, it is equal to one of them.

(c) ⇒ (d). Let $a, b \in R$ such that $ab\psi(N) \not\subseteq abN$ and $aN \not\subseteq abN$. Then by part (c), we have either $(ab :_R aN) = \text{Ann}_R(aN)$ or $(ab :_R aN) = (ab :_R N)$. Hence, $abN = 0$ or $bN \subseteq abN$, as needed.

(d) ⇒ (a). Let $a, b \in R$ and $K$ be a submodule of $M$ such that $abN \subseteq K$ and $\psi(N) \not\subseteq K$. If $\psi(N) \subseteq abN$, then $abN \subseteq K$ implies that $ab\psi(N) \subseteq K$, a contradiction. Thus by part (d), either $abN = aN$ or $abN = bN$ or $abN = 0$. Therefore, $aN \subseteq K$ or $bN \subseteq K$ or $abN = 0$ and the proof is completed. \hfill \square

A proper submodule $N$ of an $R$-module $M$ is said to be completely irreducible if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of $M$, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$ [9].

Remark 2.4. (See [2].) Let $N$ and $K$ be two submodules of an $R$-module $M$. To prove $N \subseteq K$, it is enough to show that if $L$ is a completely irreducible submodule of $M$ such that $K \subseteq L$, then $N \subseteq L$.

Theorem 2.5. Let $M$ be an $R$-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let $N$ be a strongly $\psi$-2-absorbing second submodule of $M$ such that $\text{Ann}_R^2(N) \psi(N) \not\subseteq N$. Then $N$ is a strongly 2-absorbing second submodule submodule of $M$.

Proof. Let $a, b \in R$ and $K$ be a submodule of $M$ such that $abN \subseteq K$. If $ab\psi(N) \not\subseteq K$, then we are done because $N$ is a strongly $\psi$-2-absorbing second submodule of $M$. Thus suppose that $ab\psi(N) \subseteq K$. If $ab\psi(N) \not\subseteq N$, then $ab\psi(N) \not\subseteq N \cap K$. Hence $abN \subseteq N \cap K$ implies that $aN \subseteq N \cap K \subseteq K$ or $bN \subseteq N \cap K \subseteq K$ or $abN = 0$, as needed. So let $ab\psi(N) \subseteq N$. If $a\text{Ann}_R(N)\psi(N) \not\subseteq K$, then $a(b + \text{Ann}_R(N))\psi(N) \not\subseteq K$. Thus $a(b + \text{Ann}_R(N))N \subseteq K$ implies that $aN \subseteq K$ or $bN = (b + \text{Ann}_R(N))N = K$ or $abN = (a+b)N = 0$, as required. So let $a\text{Ann}_R(N)\psi(N) \subseteq K$. Similarly, we can assume that $b\text{Ann}_R(N)\psi(N) \subseteq K$. Since $\text{Ann}_R^2(N)\psi(N) \not\subseteq N$, there exist $a_1, b_1 \in \text{Ann}_R(N)$ such that $a_1b_1\psi(N) \not\subseteq N$. Thus there exists a completely irreducible submodule $L$ of $M$ such that $N \subseteq L$ and $a_1b_1\psi(N) \not\subseteq L$ by Remark 2.3. If $ab_1\psi(N) \not\subseteq L$, then $a(b + b_1)\psi(N) \not\subseteq L \cap K$. Thus $a(b + b_1)N \subseteq L \cap K$ implies that $aN \subseteq L \cap K \subseteq K$ or $bN = (b + b_1)N \subseteq L \cap K \subseteq K$ or $abN = (a+b)N = 0$, as needed. So let $ab_1\psi(N) \subseteq L$. Similarly, we can assume that $a_1b_1\psi(N) \not\subseteq L$. Therefore, $a(a + a_1)(b + b_1)\psi(N) \not\subseteq L \cap K$. Hence, $a(a + a_1)(b + b_1)N \subseteq L \cap K$ implies that $aN = (a + a_1)N \subseteq K$ or $bN = (b + b_1)N \subseteq K$ or $abN = (a + a_1)(b + b_1)N = 0$, as desired. \hfill \square
Let $M$ be an $R$-module. A submodule $N$ of $M$ is said to be coidempotent if $N = (0 :_M \text{Ann}_R^2(N))$. Also, $M$ is said to be fully coidempotent if every submodule of $M$ is coidempotent \[4].\]

**Corollary 2.6.** Let $M$ be an $R$-module and $\psi : S(M) \to S(M) \cup \{\emptyset \}$ be a function. If $M$ is a fully coidempotent $R$-module and $N$ is a proper submodule of $M$ with $\text{Ann}_R(\psi(N)) = 0$, then $N$ is a strongly $\psi$-2-absorbing second submodule if and only if $N$ is a strongly 2-absorbing second submodule.

**Proof.** The sufficiency is clear. Conversely, assume on the contrary that $N \neq M$ is a strongly $\psi$-2-absorbing second submodule of $M$ which is not a strongly 2-absorbing second submodule. Then by Theorem 2.5, $\text{Ann}_R^3(N) \subseteq \text{Ann}_R(\psi(N))$. Hence as $\text{Ann}_R(\psi(N)) = 0$, we have $\text{Ann}_R^3(N) = 0$. Thus since $N$ is coidempotent,

$$N = (0 :_M \text{Ann}_R^2(N)) = (0 :_M \text{Ann}_R^3(N)) = M,$$

which is a contradiction. $\square$

**Proposition 2.7.** Let $M$ be an $R$-module and $\psi : S(M) \to S(M) \cup \{\emptyset \}$ be a function. Let $N$ be a non-zero submodule of $M$. If $N$ is a strongly $\psi$-2-absorbing second submodule of $M$, then for any $a, b \in R \setminus \text{Ann}_R(N)$, we have $abN = aN \cap bN \cap ab\psi(N)$.

**Proof.** Let $N$ be a strongly $\psi$-2-absorbing second submodule of $M$ and $ab \in R \setminus \text{Ann}_R(N)$. Clearly, $abN \subseteq aN \cap bN \cap ab\psi(N)$. Now let $L$ be a completely irreducible submodule of $M$ such that $abN \subseteq L$. If $ab\psi(N) \subseteq L$, then we are done. If $ab\psi(N) \not\subseteq L$, then $aN \not\subseteq L$ or $bN \not\subseteq L$ because $N$ is a strongly $\psi$-2-absorbing second submodule of $M$. Hence $aN \cap bN \cap ab\psi(N) \subseteq L$. Now the result follows from Remark 2.3. $\square$

Let $R_i$ be a commutative ring with identity and $M_i$ be an $R_i$-module for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an $R$-module and each submodule of $M$ is in the form of $N = N_1 \times N_2$ for some submodules $N_1$ of $M_1$ and $N_2$ of $M_2$.

**Theorem 2.8.** Let $R = R_1 \times R_2$ be a ring and $M = M_1 \times M_2$ be an $R$-module, where $M_1$ is an $R_1$-module and $M_2$ is an $R_2$-module. Suppose that $\psi^i : S(M_i) \to S(M_i) \cup \{\emptyset \}$ be a function for $i = 1, 2$. Then $N_1 \times 0$ is a strongly $\psi^1 \times \psi^2$-2-absorbing second submodule of $M$, where $N_1$ is a strongly $\psi^1$-2-absorbing second submodule of $M_1$ and $\psi^2(0) = 0$.

**Proof.** Let $(a_1, a_2), (b_1, b_2) \in R$ and $K_1 \times K_2$ be a submodule of $M$ such that $(a_1, a_2)(b_1, b_2)(N_1 \times 0) \subseteq K_1 \times K_2$ and

$$(a_1, a_2)(b_1, b_2)((\psi^1 \times \psi^2)(N_1 \times 0)) = a_1b_1\psi^1(N_1) \cap a_2b_2\psi^2(0) = a_1b_1\psi^1(N_1) \times 0 \subseteq K_1 \times K_2$$

Then $a_1b_1N_1 \subseteq K_1$ and $a_1b_1\psi^1(N_1) \subseteq K_1$. Hence, $a_1b_1N_1 = 0$ or $a_1N_1 \subseteq K_1$ or $b_1N_1 \subseteq K_1$ since $N_1$ is a strongly $\psi^1$-2-absorbing second submodule of $M_1$. Therefore, we have $(a_1, a_2)(b_1, b_2)(N_1 \times 0) = 0 \times 0$ or $(a_1, a_2)N_1 \times 0 \subseteq K_1 \times K_2$ or $(b_1, b_2)N_1 \times 0 \subseteq K_1 \times K_2$, as requested.

**Theorem 2.9.** Let $M$ be an $R$-module and $\psi : S(M) \to S(M) \cup \{\emptyset \}$ be a function. Then we have the following.

(a) If $(0 :_M t) \subseteq \psi((0 :_M t))$, then $(0 :_M t)$ is a strongly 2-absorbing second submodule if and only if it is a strongly $\psi$-2-absorbing second submodule.
(b) If \((tM :_R \psi(tM)) \subseteq \text{Ann}_R(tM)\), then the submodule \(tM\) is strongly 2-absorbing second if and only if it is strongly \(\psi\)-2-absorbing second.

Proof. (a) Suppose that \((0 :_M t)\) is a strongly \(\psi\)-2-absorbing second submodule of \(M\), \(a, b \in R\), and \(K\) is a submodule of \(M\) such that \(ab(0 :_M t) \subseteq K\). If \(ab\psi((0 :_M t)) \not\subseteq K\), then since \((0 :_M t)\) is strongly \(\psi\)-2-absorbing second, we have \(a(0 :_M t) \subseteq K\) or \(b(0 :_M t) \subseteq K\) or \(ba \in \text{Ann}_R((0 :_M t))\) which implies \((0 :_M t)\) is strongly 2-absorbing second. Therefore we may assume that \(ab\psi((0 :_M t)) \subseteq K\). Clearly, \(a(b + t)(0 :_M t) \subseteq K\). If \(a(b + t)\psi((0 :_M t)) \not\subseteq K\), then we have \((b + t)(0 :_M t) \subseteq K\) or \(a(0 :_M t) \subseteq K\) or \(a(b + t) \in \text{Ann}_R((0 :_M t))\). Since \(at \in \text{Ann}_R((0 :_M t))\) therefore \(b(0 :_M t) \subseteq K\) or \(a(0 :_M t) \subseteq K\) or \(ab \in \text{Ann}_R((0 :_M t))\). Now suppose that \((b + t)\psi((0 :_M t)) \subseteq K\). Then since \(ab\psi((0 :_M t)) \subseteq K\), we have \(t\psi((0 :_M t)) \subseteq \text{Ann}_R((0 :_M t))\) if and so \(t\psi((0 :_M t)) \subseteq (K :_M a)\). Now \((0 :_M t) \subseteq \psi((0 :_M t))\) implies that \((0 :_M t) \subseteq (K :_M a)\). Thus \(a(0 :_M t) \subseteq K\), as needed. The converse is clear.

(b) Let \(tM\) be a strongly \(\psi\)-2-absorbing second submodule of \(M\) and assume that \(a, b \in R\) and \(K\) be a submodule of \(M\) with \(abM \subseteq K\). Since \(tM\) is strongly \(\psi\)-2-absorbing second submodule, we can suppose that \(ab\psi(tM) \subseteq K\), otherwise \(tM\) is strongly 2-absorbing second. Now \(abM \subseteq tM \cap K\). If \(ab\psi(tM) \not\subseteq tM \cap K\), then as \(tM\) is strongly \(\psi\)-2-absorbing second submodule, we are done. So let \(ab\psi(tM) \subseteq tM \cap K\). Then \(ab\psi(tM) \subseteq tM\). Thus \((tM :_R \psi(tM)) \subseteq \text{Ann}_R(tM)\) implies that \(ab \in \text{Ann}_R(tM)\), as requested. The converse is clear. \(\Box\)

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