Ground State in Gluodynamics
and Quark Confinement

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Abstract

The properties of the ground state wavefunctional in gluodynamics responsible for
confinement are considered. It is shown that confinement arises due to the generation
of a mass gap in the averaging over the gauge group which is necessary to ensure the
gauge invariance of the ground state. The string tension is calculated in assumption
of a particular infrared behavior of the vacuum wavefunctional.
1 Introduction

It is generally accepted that the confinement of quarks can be explained as a consequence of the linear attractive potential arising between quark and antiquark at large distances due to strong vacuum fluctuations of the gluon fields. The origin of these strong fluctuations is a complicated structure of the vacuum in QCD. In a literal sense, the vacuum is the ground state of the Hamiltonian. In this paper we consider the concrete properties of the vacuum wavefunction in gluodynamics which are responsible for confinement.

In principle, the ground state wavefunctional in QCD can be obtained by solving Schrödinger equation in some reasonable approximation. The crucial point in any practical approach to this problem is the maintenance of the gauge invariance. The method of imposing gauge invariance condition by integration over all gauges \[1, 2\] appears to be most convenient for our purposes. This method is familiar in the gauge theories at finite temperature \[3, 4\]. It turns out that the averaging over gauges provides a simple way to distinguish between Coulomb and confining behavior of the quark-antiquark potential. Confinement arises due to the absence of the long-range correlations in the integration over the gauge group. The qualitative explanation of the relationship between confinement and the generation of a mass gap is given in Ref. \[4\].

2 Ground state wavefunctional and gauge invariance

The canonical variables in pure gluodynamics (in the temporal gauge \(A_0 = 0\)) are gauge potentials \(A_i^A(x)\) and electric fields \(E_i^A(x)\):

\[
[A_i^A(x), E_j^B(y)] = i \delta^{AB} \delta_{ij} \delta(x - y).
\]

(2.1)

We consider gauge group \(SU(N)\) and sometimes use matrix notations for gauge potentials: \(A_i = A_i^A T^A\), where \(T^A\) are traceless anti-Hermitean generators of \(SU(N)\) normalized by \(\operatorname{tr} T^A T^B = - \delta^{AB}/2\).

The Hamiltonian of the Yang-Mills theory is

\[
H = \int d^3 x \left( \frac{1}{2} E_i^A E_i^A + \frac{1}{4} F_{ij}^A F_{ij}^A \right),
\]

(2.2)

where \(F_{ij}^A = \partial_i A_j^A - \partial_j A_i^A + g f^{ABC} A_i^B A_j^C\). Apart from the Schrödinger equation, the ground state wavefunction satisfies also the Gauss law constraint:

\[
D_i E_i^A \Psi = 0.
\]

(2.3)

The covariant derivative \(D_i\) acts in the adjoint representation: \(D_i^{AB} = \delta^{AB} \partial_i + g f^{ACB} A_i^C\).

In the “coordinate” representation, the electric field operators act as the variational derivatives, \(E_i^A(x) = -i \delta / \delta A_i^A(x)\), and the wavefunction \(\Psi\) is a functional of the gauge potentials. The operator on the left hand side of Eq. (2.3) generates gauge transformations

\[
A_i \longrightarrow A_i^\Omega = \Omega^i \left( A_i + \frac{1}{g} \partial_i \right) \Omega,
\]

(2.4)
so the Gauss law (2.3) is the infinitesimal form of the gauge invariance condition:

\[ \Psi[A] = \Psi[A]. \quad (2.5) \]

Strictly speaking, this is true only for topologically trivial gauge transformations – the wave-functional of the \( \theta \)-vacuum is invariant only up to a phase factor \( e^{ik\theta} \), where \( k \) is a winding number of the gauge transformation, but for simplicity we assume that \( \theta = 0 \), then Eq. (2.5) holds for any \( \Omega(x) \).

In principle, any functional of \( A_i \) can be made gauge invariant by projection on the subspace of states obeying the Gauss law. The projection can be realized by averaging over the gauge group [3]. So, the following ansatz for the ground state wavefunctional automatically satisfies (2.5):

\[ \Psi[A] = \int [DU] e^{-S[A^U]}, \quad (2.6) \]

The invariance of this expression follows simply from the fact that a gauge transformation \( A_i \rightarrow A_i^\Omega \) can be absorbed by the change of integration variables: \( U \rightarrow \Omega^U U \).

This method of imposing gauge invariance was used in the variational approach of Ref. [1]. The variational ansatz of Ref. [1] consists of restricting to the purely quadratic action

\[ S_0[A] = \frac{1}{2} \int d^3x d^3y A_i^A(x)K(x - y)A_i^A(y). \quad (2.7) \]

with some special choice of the coefficient function \( K(x - y) \). We consider more general situation when the action \( S[A] \) can contain arbitrary powers of gauge potentials. However, it is clear that the functional \( S[A] \) is defined ambiguously. Different choices of \( S \) may lead to one and the same \( \Psi \) and it is convenient to get rid of this arbitrariness demanding the quadratic part of \( S[A] \) to coincide with \( S_0[A] \). The color and the Lorentz structure of the quadratic form in (2.7) is dictated by rotational, translational and global gauge symmetries. The general principles allow also a longitudinal term, but it can be always removed without change of the wavefunctional, as shown in Appendix.

The introduction of the “bare” wavefunctional \( \exp(-S[A]) \), which is not gauge invariant and requires the averaging over gauges, nevertheless, has several advantages. There is a hope that the bare wavefunctional looks rather simple in some reasonable approximation [I], while after the integration over the gauge group one obtains sufficiently complicated gauge invariant functional. The possibility to impose the gauge invariance condition exactly on an approximate wavefunctional of a simple form makes this method convenient for variational calculations [I]. We shall show that the construction of states containing static color charges is straightforward in this framework, which allows to formulate a relatively simple criterion of confinement.

Due to the gauge invariance the normalization integral \( \langle \Psi|\Psi \rangle \) contains a group volume factor. This factor is easily extracted in the representation (2.6):

\[ \langle \Psi|\Psi \rangle = \int [DU][DU'][dA] e^{-S[A^U] - S[A'^U]} = \int [DU][dA] e^{-S[A^U] - S[A]}, \quad (2.8) \]
where $U'$ is removed by the change of integration variables: $A_i \rightarrow A_i^{U'i}$, $U \rightarrow U'U$. The same trick works for the expectation value of any gauge invariant operator \[1\]. In particular,

$$\langle \Psi | H | \Psi \rangle = \int [DU][dA] \int d^3x \left( \frac{1}{2} \frac{\delta^2 S[A]}{\delta A_i^A \delta A_i^A} - \frac{1}{2} \frac{\delta S[A]}{\delta A_i^A} \frac{\delta S[A]}{\delta A_i^A} + \frac{1}{4} F_{ij}^A F_{ij}^A \right) e^{-S[A'] - S[A]}.$$  

(2.9)

Thus, the vacuum energy is determined by the average of the operator

$$R[A] = \int d^3x \left( \frac{1}{2} \frac{\delta^2 S[A]}{\delta A_i^A \delta A_i^A} - \frac{1}{2} \frac{\delta S[A]}{\delta A_i^A} \frac{\delta S[A]}{\delta A_i^A} + \frac{1}{4} F_{ij}^A F_{ij}^A \right)$$  

(2.10)

in the statistical ensemble which is defined by the partition function (2.8). The same result is obtained by averaging of $R[A']$ – it arises when the variational derivatives in the Hamiltonian act not on the right but on the left. It is natural to preserve the symmetry of (2.8) under the interchange of $A_i$ with $A_i^{U'i}$ and to average the symmetric combination $\frac{1}{2}(R[A'] + R[A])$. Since the vacuum energy is proportional to the volume, it is also natural to add this operator to the action and to consider the partition function

$$Z = \int [DU][dA] e^{-S[A'] - S[A] + \frac{1}{2} R[A'] + \frac{1}{2} R[A]}.$$  

(2.11)

The physical parameters characterizing the QCD vacuum are simply related to the thermodynamic quantities in the statistical system defined by this partition function. At least, this is true for the gluon condensate and the string tension, as shown below.

The vacuum energy in QCD is equal to the derivative of the free energy $\ln Z$ with respect to $\lambda$:

$$E_{\text{vac}} = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\partial}{\partial \lambda} \ln Z \bigg|_{\lambda=0}.$$  

(2.12)

It is worth mentioning that the perturbation in (2.11) softens the short-distance behavior, since $R$ is constructed from dimension four operators. So, the parameter $\lambda$, which has the dimension of length, can be regarded also as an UV cutoff.

The regularized energy density, $\epsilon_{\text{vac}} = E_{\text{vac}}/V - \text{UV divergent terms}$, can be expressed through the gluon condensate [3]:

$$\epsilon_{\text{vac}} = \frac{\beta(\alpha_s)}{4\alpha_s} \left( 0 \left| F_{\mu \nu}^A F_{\mu \nu}^A \right| 0 \right),$$  

(2.13)

where $\alpha_s = g^2/4\pi$ and $\beta(\alpha_s) = -11N\alpha_s^2/6\pi + \ldots$. The gluon condensate is known to be positive. Hence, the regularized vacuum energy density is negative. Therefore, we expect that $R$ is negative definite operator after the subtraction of a field independent divergent constant $\frac{1}{2}VK(0) = V \int \frac{d^3p}{(2\pi)^3} \frac{K(p)}{2} - \text{zero point energy which comes from the second variation of the quadratic term in } S[A]$.

This explains why the sign before $\lambda$ in Eq. (2.11) is positive.

### 3 Charged states and confinement

In the presence of charges, the Gauss law (2.3) acquires a right hand side proportional to the charge density. We consider the case of two static charged sources placed at the points
$x_1$ and $x_2$ with quark and antiquark quantum numbers. The wavefunctional of such state transforms in $N \otimes \bar{N}$ representation of $SU(N)$. The Gauss law constraint for it is

$$D_i E^A_i(x) \Psi = ig \left( \delta(x-x_2) T^A_{\nu\rho} \delta_{\alpha\alpha'} - \delta(x-x_1) T^A_{\alpha\alpha'} \delta_{\nu\rho} \right) \Psi_{\alpha'\nu}, \quad (3.14)$$

which implies the following transformation law:

$$\Psi_{ab}[A^\Omega; x_1, x_2] = \Omega_{ab}^\dagger(x_1) \Omega_{\alpha'\beta}(x_2) \Psi_{\alpha'\beta}[A; x_1, x_2]. \quad (3.15)$$

The construction of the state $\Psi_{ab}[A; x_1, x_2]$ requires operators in the fundamental representation. These operators are necessarily nonlocal in terms of the gauge fields, since gluons transform in the adjoint representation. The integral formula (2.6) for the wavefunctional, in fact, introduces the fields in the fundamental representation, since $U$ transforms as $U \rightarrow \Omega^\dagger U$. This circumstance allows to construct the states with quark quantum numbers entirely from local operators at a price of the subsequent integration over all gauges. The state obeying (3.13) is constructed as follows:

$$\Psi_{ab}[A; x_1, x_2] = \int [DU] U_{ac}(x_1) U^\dagger_{cb}(x_2) e^{-S[A^\nu]}. \quad (3.16)$$

The crucial point here is that the action $S$ in bulk is the same as in the vacuum sector. The reasoning is based on the variational arguments.

The energy of the state (3.16) consists of the vacuum energy and quark-antiquark interaction potential:

$$\frac{\langle \Psi_{ab} | H | \Psi_{ab} \rangle}{\langle \Psi_{ab} | \Psi_{ab} \rangle} = E_{\text{vac}} + V(x_1 - x_2). \quad (3.17)$$

The vacuum energy is proportional to the volume, while the interaction potential remains finite in the infinite volume limit. Suppose we are trying to minimize the energies of the states $\Psi$ and $\Psi_{ab}$. Then $O(V)$ terms in the variational equations are the same in both cases. Since variational equations completely determine the ground state in each sector, the action in Eq. (3.16) coincides with that in Eq. (2.3) in bulk. In principle, in the charged sector, some extra terms localized near the points $x_1$ and $x_2$ can arise. It is more natural to include these terms in the definition of the operators $U_{ac}(x_1)$ and $U^\dagger_{cb}(x_2)$, which create external charges. So, strictly speaking, for the state (3.16) to be an eigenfunction of the Hamiltonian, the operators $U_{ac}(x_1)$ and $U^\dagger_{cb}(x_2)$ should be replaced by some dressed ones $U_{ac}(x_1)$ and $U^\dagger_{cb}(x_2)$ with the same quantum numbers. The particular form of these dressed operators is determined by $O(1)$ terms in the variational equations. However, in what follows we do not distinguish between $U$ and $\mathcal{U}$, since we are interested in a large distance behavior, which is determined by the bulk theory and thus is independent of the explicit form of the operators.

The energy of the state (3.16) can be expressed in terms of correlation functions in the statistical system defined by the partition function (2.11). Analogously to Eqs. (2.8), (2.9) we find:

$$\langle \Psi_{ab} | \Psi_{ab} \rangle = \int [DU][dA] \text{ tr } U(x_1) U^\dagger(x_2) e^{-S[A^\nu] - S[A]}, \quad (3.18)$$

$$\langle \Psi_{ab} | H | \Psi_{ab} \rangle = \int [DU][dA] \text{ tr } U(x_1) U^\dagger(x_2) \frac{1}{2} \left( R[A^\nu] + R[A] \right) e^{-S[A^\nu] - S[A]}. \quad (3.19)$$
These expressions together with Eq. (2.12) lead to the following representation for the $q\bar{q}$ potential:

$$V(x_1 - x_2) = \left. \frac{\partial}{\partial \lambda} \ln \langle \mathrm{tr} U(x_1) U^\dagger(x_2) \rangle \right|_{\lambda = 0},$$

(3.20)

where the average is defined as usual:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [DU][dA] \mathcal{O} e^{-\Gamma[A,U]},$$

(3.21)

$$\Gamma[A,U] = S[A^U] + S[A] - \frac{\lambda}{2} R[A^U] - \frac{\lambda}{2} R[A].$$

(3.22)

All subsequent consideration is based on this result.

The averaging over the gauge group in (3.21) may or may not produce a mass gap. Consider first the former case. Then the two-point correlator in Eq. (3.20) falls exponentially at large distances:

$$\langle \mathrm{tr} U(x_1) U^\dagger(x_2) \rangle = C r^\eta e^{-mr} + \ldots,$$

(3.23)

where $r = |x_1 - x_2|$, $m$ is a mass gap and $\eta$ is related to an anomalous dimension of the operator $U$. Note that the constant term in the large distance asymptotics (3.23) is forbidden by the symmetry $U(x) \to U(x)h$, which can not be broken spontaneously, since this symmetry follows from the global color invariance of the action $S[A]$.

Substituting (3.23) in Eq. (3.20) we find that $q\bar{q}$ potential grows linearly at large distances:

$$V(r) = \sigma r + \ldots$$

(3.24)

with

$$\sigma = - \left. \frac{\partial m}{\partial \lambda} \right|_{\lambda = 0}.$$

(3.25)

Thus, the generation of a mass gap in the averaging over gauges leads to the confinement of charges in the fundamental representation. It is worth mentioning that the mass gap, and thus the string tension $\sigma$, is determined by the bulk theory. This means that the confinement actually is the property of the ground state and not of the operators creating external charged sources.

The screening length in Eq. (3.23) depends only on the quantum numbers of the operators on the left hand side. This justifies the use of the bare operators $U$ instead of the dressed ones $U$ in the study of the large distance behavior of the $q\bar{q}$ potential. The above arguments are essentially based on the absence of local operators with quark quantum numbers in the pure gluodynamics and they fail for the states with adjoint charges. In this case, the state (3.16) (with the matrices $U$ and $U^\dagger$ taken in the adjoint representation) inevitably mixes with the ones of the form $\mathcal{O}^A(x_1) \mathcal{O}^B(x_2) \Psi$, where $\mathcal{O}^A$ are some local operators, for example, $\mathcal{O}^A = F^A_{ij}$. The energy of such states, roughly speaking, is determined by Eq. (3.20) with $\langle \mathrm{tr} U(x_1) U^\dagger(x_2) \rangle$ replaced by $\langle \mathcal{O}^2(x_1) \mathcal{O}^2(x_2) \rangle$. The large distance asymptotics of this correlator is determined by an expectation value of $\mathcal{O}^2$, which is generally nonzero. It is easy to see that appearance of the constant term leads to the screening of the interaction.
potential: $V_{\text{adj}}(r) \sim e^{-mr}$. However, the mixing with the states composed from purely gluonic operators can be small, then the potential for adjoint charges will exhibit a linear growth at moderate distances.

Another remark concerns the anomalous dimension $\eta$ entering Eq. (3.23). It gives rise to the logarithmic term in the $q\bar{q}$ potential, unless

$$\left. \frac{\partial \eta}{\partial \lambda} \right|_{\lambda=0} = 0.$$  \hspace{1cm} (3.26)

It is generally assumed that subleading terms in the $q\bar{q}$ potential decrease at infinity, so one may expect that Eq. (3.26) holds for the ground state in gluodynamics.

Now we turn to the case when the mass gap is absent $-m = 0$ in Eq. (3.23). Assuming the validity of (3.26) we find that the leading term in the large distance asymptotics of the pair correlator is insufficient to determine the behavior of the $q\bar{q}$ interaction potential. Retaining the next term:

$$\langle \text{tr} U(x_1)U^\dagger(x_2) \rangle = \frac{C}{r^\eta} \left( 1 - \frac{B}{r} + \ldots \right),$$  \hspace{1cm} (3.27)

we obtain the Coulomb law:

$$V(r) = \text{const} - \frac{Q^2}{4\pi r} + \ldots, \quad Q^2 = 4\pi \left. \frac{\partial B}{\partial \lambda} \right|_{\lambda=0}.$$  \hspace{1cm} (3.28)

Therefore, the presence of long-range correlations in the averaging over gauges leads to the Coulomb interaction of static charges. Unfortunately, there is no symmetry that distinguish between Coulomb and confinement phases, in contrast to the finite temperature gauge theories [3, 4].

4 Examples

Below the general formalism of the previous section is illustrated by two examples. One is $U(1)$ gauge theory. Another refers directly to gluodynamics. We show that minor assumptions about an infrared behavior of the wavefunctional allow to calculate the string tension.

4.1 QED

In the Abelian theory $U = e^{ig\varphi}$, $A^U_i = A_i + \partial_i \varphi$ and the action $S[A]$ is quadratic, i.e. coincides with (2.7), and the averaging over gauges reduces to Gaussian integrals. In the vacuum sector, the integration over $\varphi$ in Eq. (2.1) replaces $A_i$ by $A^\perp_j = \left( \delta_{ij} - \partial_i \partial_j \right) A_j$:

$$\Psi[A] = \exp \left( -\frac{1}{2} \int d^3x d^3y A^\perp_i(x)K(x-y)A^\perp_i(y) \right).$$  \hspace{1cm} (4.29)
This is an exact ground state wavefunctional of the free electromagnetic field, provided that
\( K = \sqrt{-\partial^2} \) (in the momentum space \( K(p) = |p| \)). The integration over the gauge group for
the state (3.16) containing charged sources also can be easily performed yielding:

\[
\Psi[A; x_1, x_2] = e^{igV(x_1)} e^{-igV(x_2)} \Psi[A],
\]

where

\[
V(x) = \int d^3 y \frac{1}{\partial^2} (x - y) \partial_i A_i(y).
\]

The operators \( e^{igV(x)} \) were introduced by Dirac [6]. The states of type (4.30) created by
Dirac charge operators behave properly under gauge transformations and are the eigenstates
of the Hamiltonian of the free electromagnetic field. The wavefunctional (3.16) provides a
natural nonabelian generalization of this construction.

Dropping an irrelevant constant term we get for the action (3.22) in the Abelian theory:

\[
\Gamma = \frac{1}{2} \int d^3 x d^3 y A_i(x) \left( K(x - y) + \frac{\lambda}{2} K^2(x - y) \right) A_i(y) + (A_i \leftrightarrow A_i + \partial_i \varphi) - \frac{\lambda}{4} \int d^3 x F_{ij} F_{ij}.
\]

Here \( K^2(x - y) \) denotes the kernel of the operator \( K^2 \). The gauge potentials can be integrated
out by the change of variables:

\[
A_i = \bar{A}_i - \frac{1}{2} \partial_i \varphi.
\]

The shifted gauge potentials \( \bar{A}_i \) decouple and we are left with the effective action for \( \varphi \):

\[
\Gamma_{\text{eff}} = \frac{1}{4} \int d^3 x d^3 y \partial_i \varphi(x) \left( K(x - y) + \frac{\lambda}{2} K^2(x - y) \right) \partial_i \varphi(y).
\]

So, for the two-point correlator we get:

\[
\langle e^{ig\varphi(x_1)} e^{-ig\varphi(x_2)} \rangle = e^{g^2(D(x_1 - x_2) - D(0))},
\]

where \( D \) is the propagator of the field \( \varphi \):

\[
D^{-1} = \frac{-\partial^2}{2} \left( K + \frac{\lambda}{2} K^2 \right).
\]

Differentiating the propagator with respect to \( \lambda \) we find:

\[
\left. \frac{\partial D}{\partial \lambda} \right|_{\lambda=0} = -\frac{1}{-\partial^2}.
\]

Hence, according to Eq. (3.20), we obtain the Coulomb law for the interaction potential:

\[
V(x_1 - x_2) = -g^2 \left( \frac{1}{-\partial^2 (x_1 - x_2)} - \frac{1}{-\partial^2 (0)} \right) = -\frac{g^2}{4\pi r} + \text{self-energy}.
\]
It is interesting that the explicit form of the coefficient function $K$ was not used anywhere in the derivation of this result. Nevertheless, it is useful to substitute $K = \sqrt{-\partial^2}$ in Eq. (4.36) in order to compare (4.35) with (3.27). After some calculations we obtain:

\[
\langle e^{ig\varphi(x_1)} e^{-ig\varphi(x_2)} \rangle = \left( \frac{\lambda e^{-\gamma}}{2r} \right)^2 e^{-\frac{\lambda g^2}{2\pi r} \left( \frac{2}{x} + \cos \frac{2}{x} \sin \frac{2}{x} - \sin \frac{2}{x} \right)}
\]

where $\gamma$ is the Euler constant. Comparing this expression with (3.27) we find that the constant $C$ diverges in the limit $\lambda \to 0$. This divergence reflects the fact that the self-energy of the Coulomb charge is infinite, but this infinity is regularized by $\lambda$.

### 4.2 QCD

Due to the asymptotic freedom, the short distance behavior of the $q\bar{q}$ potential is determined by the perturbation theory. At weak coupling, one can expand $U = e^{g\Phi A T^A} = 1 + g\Phi A T^A + \ldots$. Only the quadratic term in the action $S[A]$ contribute to the leading order in $g$ and the calculations are literally the same as in the Abelian theory. Since the explicit form of the coefficient function $K$ is inessential for these calculations, at short distances we get the Coulomb potential (4.38) with $g^2$ multiplied by a group factor $(N^2 - 1)/N$.

It is naturally to assume that at large distances the long wavelength approximation is valid and the action (3.22) can be approximated by its derivative expansion. The structure of the derivative expansion for $\Gamma$ depends on the behavior of the coefficient function $K(p)$ at small momenta. Again, only the quadratic term in the action is essential, since the remaining ones correspond to operators of higher dimension. Here we assume that $K(p = 0) \neq 0$, so

\[
K(p) = M + O(p^2).
\]

Neglecting $O(p^2)$ terms we obtain:

\[
\Gamma = \frac{\mu}{2} \int d^3x \left( A_i^A A_i^A + A_i U^A A_i U^A \right),
\]

where

\[
\mu = M + \frac{\lambda M^2}{2}.
\]

The magnetic term $(F_{ij}^A)^2$ is also irrelevant in this approximation, since it contains extra derivatives in comparison to $(A_i^A)^2$.

The case of purely quadratic action $S$ with the coefficient function $K$ approximated by a constant at small momenta was considered in Ref. [1]. Following this consideration, which is based on methods typical for nonlinear sigma-models, we express the string tension in terms of $M$ and the gauge coupling constant.

First, we get rid of the integration over $A_i$ by the change of variables:

\[
A_i = \tilde{A}_i - \frac{1}{2g} \partial_i U U^\dagger,
\]

9
which is a counterpart of (4.33). After this, \( \bar{A}_i \) decouples and we get the effective action for \( U \):

\[
\Gamma_{\text{eff}} = -\frac{\mu}{2g^2} \int d^3x \left( U^\dagger \partial_i U \right)^2 = \frac{\mu}{2g^2} \int d^3x \, \partial_i U^\dagger \partial_i U.
\]

(4.44)

It is convenient to integrate over \( U(N) \) instead of \( SU(N) \). This can be achieved by adding the following term to the action:

\[
\Gamma_1 = \frac{\mu}{2g^2N} \int d^3x \left( \text{tr} U^\dagger \partial_i U \right)^2.
\]

(4.45)

Since \( \Gamma_{\text{eff}} + \Gamma_1 \) is invariant under phase transformations \( U \to e^{i\varphi} U \), it depends only on \( SU(N) \) variables, for which it reduces to \( \Gamma_{\text{eff}} \). The last step consists in the replacement of the group integral by an integral over complex \( N \times N \) matrices. The unitarity condition is imposed by means of a Lagrange multiplier:

\[
\Gamma_2 = \frac{\mu}{2g^2} \int d^3x \, \text{tr} \sigma \left( U^\dagger U - 1 \right),
\]

(4.46)

where

\[
\sigma_{ab} = m^2 \delta_{ab} + v_{ab},
\]

(4.47)

and \( v \) is anti-Hermitean matrix.

The crucial point is that (4.46) introduces a mass term for \( U \), so the pair correlator falls exponentially at large distances as in Eq. (3.23), which leads to confinement. The mass gap \( m \) is determined by the equation of motion for \( \sigma \). This equation follows from the effective action obtained by integrating out \( U(x) \). The gap equation at the same time is equivalent to the unitarity condition \( \langle U(x)U^\dagger(x) \rangle = 1 \). In the one-loop approximation, we have

\[
\int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + m^2} = \frac{\mu}{2g^2N}.
\]

(4.48)

The left hand side is UV divergent and, in principle, it is necessary to introduce a cutoff in order to find the mass gap \[.\] However, the string tension appears to be cutoff independent, since for the derivative of \( m \) we obtain:

\[
- \frac{\partial m^2}{\partial \lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2 + m^2)^2} = \frac{M^2}{4g^2N},
\]

(4.49)

and, according to Eq. (3.23), the string tension is

\[
\sigma = - \frac{\partial m}{\partial \lambda} = \frac{\pi M^2}{g^2N} = \frac{M^2}{4\alpha_s N}.
\]

(4.50)

This result is not universal, it is valid under the condition that \( K(p = 0) \neq 0 \). This assumption is compatible with the variational estimates of Ref. \[.\] However, the arguments in favor of \( K(p = 0) = 0 \) also exist, and there are some indications \[] that the expansion of \( K(p) \) at small momenta begins with the term of order \( p^2 \). If such behavior actually holds, the present treatment of confinement properties should be modified.
5 Discussion

Our main result is that, in the Hamiltonian picture, the confinement arises as a consequence of the generation of a mass gap in the averaging over gauges. Although it is possible to derive the linear potential and even to calculate the string tension under some simple and not very restrictive assumptions about the infrared properties of the ground state in gluodynamics, the quantitative treatment requires a more detailed knowledge of the vacuum wavefunctional. Possibly, the variational techniques [1] can provide a reasonable approximation for the ground state in QCD incorporating both asymptotic freedom [8] and confinement.

A more detailed quantitative treatment of the ground state in gluodynamics may provide insights in some other nonperturbative properties of the low-energy QCD. In particular, the usual picture of the $q\bar{q}$ interaction is associated with a string stretched between quark and antiquark. Perhaps, this picture would correspond to a sum-over-path representation for the correlator entering Eq. (3.20):

$$\langle \text{tr} U(x_1)U^\dagger(x_2) \rangle = \int_{X(0)=x_1}^{X(1)=x_2} \mathcal{D}X \ e^{-\mathcal{H}[X(\xi)]}. \quad (5.51)$$

The large distance asymptotics (3.23) is naturally interpreted as a saddle point approximation for the path integral (5.51) and the functional $\mathcal{H}[X(\xi)]$ has a meaning of an effective string Hamiltonian.

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Appendix A Removing arbitrariness in the action

To demonstrate that the representation (2.6) is ambiguous let us consider a set of functionals $S[A;\xi]$, whose dependence on the parameter $\xi$ is determined by the equation

$$\frac{\partial S}{\partial \xi} = \int d^3xd^3y \ A_i^A(x)W(x-y)\partial_i D_{AB}^{AB} \frac{\delta S}{\delta A_j^B(y)}. \quad (A.1)$$

Here $W(x-y)$ is an arbitrary function. Then the wavefunctional

$$\Psi[A;\xi] = \int [\mathcal{D}U] \ e^{-S[U;\xi]} \quad (A.2)$$

is independent of $\xi$ up to a normalization factor, which is inessential. The proof is based on Schwinger-Dyson equations.

The measure of integration in (A.2) is invariant under left shifts:

$$\delta_\omega U = U\omega, \quad \delta_\omega U^\dagger = -\omega U^\dagger, \quad (A.3)$$
where $\omega$ is traceless anti-Hermitean matrix. The twisted gauge potentials $A^U_i$ transform under left shifts as

$$\delta_\omega A^U_i = [A^U_i, \omega] + \frac{1}{g} \partial_i \omega \equiv \frac{1}{g} D^U_i \omega. \quad \text{(A.4)}$$

The Schwinger-Dyson equation follows from the identity

$$0 = \int [DU] \delta_\omega \left( \partial_i A^U_i A(x) e^{-S[A^U]} \right)$$

$$= \frac{1}{g} \int [DU] \left( \partial_i D^U_i A^U C(x) + \partial_i A^U_i A(x) \int d^3 y \omega_c(y) D^{U CB}_j \delta S[A^U] \delta A^U j B(y) \right) e^{-S[A^U]}.$$ 

This equality holds for any $\omega_c(y)$. Taking $\omega_c(y) = \delta C W(x-y)$ and summing over $A$ we find:

$$(N^2-1) \partial^2 W(0) \Psi[A] + \int [DU] \int d^3 y \partial_i A^U_i A(x) W(x-y) D^{U AB}_j \delta S[A^U] \delta_A U j B(y) e^{-S[A^U]} = 0. \quad \text{(A.5)}$$

Combining this equation with (A.1) we obtain:

$$\frac{\partial \Psi}{\partial \xi} = - \int [DU] \int d^3 x d^3 y A^U_i A(x) W(x-y) \partial_i D^{U AB}_j \delta S[A^U] \delta_A U j B(y) e^{-S[A^U]} = c \Psi, \quad \text{(A.6)}$$

where $c = (N^2 - 1) V \partial^2 W(0)$. Thus,

$$\Psi[A; \xi] = e^{\xi c} \Psi[A; 0]. \quad \text{(A.7)}$$

The above arguments show that the solution of Eq. (A.1) for any $W$ and at arbitrary value of the parameter $\xi$ gives essentially the same wavefunctional. The action $S$ can be expanded in the power series:

$$S[A] = \frac{1}{2} \int d^3 x d^3 y A^A_i A(x) K_{ij}(x-y) A^A_j (y) + \ldots. \quad \text{(A.8)}$$

Then the equality (A.1) is rewritten as an infinite set of equations for coefficient functions. The first equation in this set contains only the two-point function and reads:

$$\frac{\partial K_{ij}}{\partial \xi} = W \partial_i \partial_k K_{kj}. \quad \text{(A.9)}$$

Here $W$ denotes an integral operator with the kernel $W(x-y)$. We can start with the general expression compatible with translational and rotational invariance taken as an initial condition for Eq. (A.9):

$$K_{ij}(0) = \left( \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) K + \frac{\partial_i \partial_j}{\partial^2} K', \quad \text{(A.10)}$$

where $K$ and $K'$ are the functions of $\partial^2$ only. Solving Eq. (A.8) we find:

$$K_{ij}(\xi) = \left( \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) K + \frac{\partial_i \partial_j}{\partial^2} e^{\xi W \partial^2} K'. \quad \text{(A.11)}$$
Taking
\[ \xi W = (\partial^2)^{-1}(\ln K - \ln K') \] (A.12)
we get the diagonal coefficient function \( K_{ij} = \delta_{ij}K \), thus proving the assertion used in Sec. 2.
The extreme case of \( \xi W \to +\infty \) corresponds to purely transversal coefficient function, which arises if the functional \( S[A] \) is gauge invariant.

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