POROSITY AND REGULARITY IN METRIC MEASURE SPACES

ANTTI KÄENMÄKI

ABSTRACT. This is a report of a joint work with E. Järvenpää, M. Järvenpää, T. Rajala, S. Rogovin, and V. Suomala. In \cite{3}, we characterized uniformly porous sets in \( s \)-regular metric spaces in terms of regular sets by verifying that a set \( A \) is uniformly porous if and only if there is \( t < s \) and a \( t \)-regular set \( F \supset A \). Here we outline the main idea of the proof and also present an alternative proof for the crucial lemma needed in the proof of the result.

1. Setting and result

We assume that \( X = (X, d) \) is a complete separable metric space. If \( s > 0 \) then a measure \( \mu \) on \( X \) is called \( s \)-regular on a set \( A \subset X \) provided that there are constants \( 0 < a_\mu \leq b_\mu < \infty \) and \( r_\mu > 0 \) such that

\[
a_\mu r^s \leq \mu(B(x, r)) \leq b_\mu r^s
\]

for every \( x \in A \) and \( 0 < r < r_\mu \). A set \( A \subset X \) is \( s \)-regular if there is a measure \( \mu \) on \( X \) which is \( s \)-regular on \( A \) and \( \mu(X \setminus A) = 0 \). We call a measure \( \mu \) on \( X \) doubling if there are constants \( c_\mu \geq 1 \) and \( r_\mu > 0 \) such that

\[
0 < \mu(B(x, 2r)) \leq c_\mu \mu(B(x, r)) < \infty
\]

for every \( x \in X \) and \( 0 < r < r_\mu \). Clearly, if \( \mu \) is \( s \)-regular on \( X \) then it is also doubling. Finally, a set \( A \subset X \) is called uniformly (lower) \( \varrho \)-porous provided that there is a constant \( r_p > 0 \) such that

\[
\text{por}(A, x, r) > \varrho
\]

for all \( x \in A \) and \( 0 < r < r_p \), where

\[
\text{por}(A, x, r) = \sup \{ \varrho \geq 0 : B(y, \varrho r) \subset B(x, r) \setminus A \text{ for some } y \in X \}.
\]

It is easy to see that if \( \mu \) is a doubling measure on \( X \) and \( 0 < \alpha < 1 \) then

\[
\mu(B(x, \alpha r)) \geq c_\mu^{\lfloor \log_2 \alpha \rfloor} \mu(B(x, r))
\]

whenever \( x \in X \) and \( 0 < r < r_\mu \). Here \( \lfloor a \rfloor = \max \{ n \in \mathbb{Z} : n \leq a \} \) for \( a \in \mathbb{R} \). It is also straightforward to show that if \( A \subset X \) is uniformly \( \varrho \)-porous for some
\( \varrho > 0 \) then \( \mu(A) = 0 \) for any doubling measure \( \mu \) on \( X \). This is true essentially because the points of \( A \) are nondensity points.

Observe that, by [5, Theorem 5.7] and [1, Theorem 3.16], both the Hausdorff dimension and the packing dimension of \( A \) equal to \( s \) provided that \( A \subset X \) is \( s \)-regular. An unpublished result of Saaranen shows that if \( X \) is \( s \)-regular then for each \( 0 < t < s \) there exists a \( t \)-regular set \( A \subset X \). This is the only place where we need the completeness of \( X \). The separability assumption is natural since the Hausdorff dimension of any nonseparable metric space is infinity. Also, it can be shown that no \( \sigma \)-finite doubling measures exist in nonseparable spaces.

Our result under consideration is the following. A complete proof with a slightly better result can be found from [3].

**Theorem 1.1.** Suppose \( X \) is \( s \)-regular. Then \( A \) is uniformly \( \varrho \)-porous for some \( \varrho > 0 \) if and only if there is \( t < s \) and a \( t \)-regular set \( F \supset A \).

2. **Alternative approach**

We begin with the following lemma that gives an alternative proof for [3, Corollaries 4.5 and 4.6]. The proof was found while preparing the article [3] and is also credited by the authors of that article. Observe, however, that in [3, Corollary 4.6] we were able to find a better exponent \( \delta \). The lemma generalizes the argument of [4, Lemma 2.8].

Given a set \( A \subset X \), we define \( A(t_1, t_2) = \{ x \in X : t_1 < \text{dist}(x, A) \leq t_2 \} \) as \( r > 0 \).

**Lemma 2.1.** Suppose that \( \mu \) is a doubling measure on \( X \). If \( A \subset X \) is uniformly \( \varrho \)-porous then

\[
\mu\left( (A \cap B(x_0, r_0))(r) \right) \leq C(c_{\mu}, \varrho) \mu(B(x_0, r_0))(\frac{r}{r_0})^\delta
\]

for every \( x_0 \in X \) and \( 0 < r < r_0 \leq r_{\mu} \), where \( \delta = C(c_{\mu}) \varrho \log_2 c_{\mu}/\log(1/\varrho) \).

Moreover, if \( \mu \) is \( s \)-regular then \( \delta = C(a_{\mu}, b_{\mu}, s) \varrho^s/\log(1/\varrho) \).

**Proof.** We may assume that \( \varrho \leq \frac{1}{2} \). Let us denote for \( 0 < t_1, t_2 < \infty \)

\[
A(t_1, t_2) = \{ x \in X : t_1 < \text{dist}(x, A) \leq t_2 \} = A(t_2) \setminus A(t_1).
\]

Furthermore, we denote

\[
\alpha_k = \mu\left( A(r_0 \varrho^{3k}, r_0 \varrho^{3(k-1)}) \right)
\]

as \( k \in \mathbb{N} \).

We shall first find a number \( 0 < \gamma < 1 \) such that \( \alpha_k \geq 2 \sum_{i=k+1}^{\infty} \alpha_i \) whenever \( k \in \mathbb{N} \). To do so, fix \( k \) and notice that for every \( x \in A \) there is \( y \in B(x, r_0 \varrho^{3k-2}) \).
such that
\[ B(y, r_0 \theta^{3k-1}) \subset B(x, r_0 \theta^{3k-2}) \setminus A. \] (2.1)

We claim that this implies
\[ A(r_0 \theta^{3k}) \subset (A(r_0 \theta^{3k-1}, r_0 \theta^{3k-2}))(2r_0 \theta^{3k-2}). \] (2.2)

To see this, pick \( z \in A(r_0 \theta^{3k}) \). Since \( \text{dist}(z, A) \leq r_0 \theta^{3k} \), there is \( x \in A \) such that \( d(z, x) < r_0 \theta^{3k-1} \). According to (2.1), there is \( y \in A(r_0 \theta^{3k-1}, r_0 \theta^{3k-2}) \) for which \( d(x, y) < r_0 \theta^{3k-2} \). Therefore
\[ d(z, y) \leq d(z, x) + d(x, y) < r_0 \theta^{3k-1} + r_0 \theta^{3k-2} < 2r_0 \theta^{3k-2} \]
and hence (2.2) follows.

Attaching for each \( z \in A(r_0 \theta^{3k-1}, r_0 \theta^{3k-2}) \) a ball \( B(z, \frac{1}{3} r_0 \theta^{3k-2}) \), we find, using
the \( 5r \)-covering theorem (see [2, Theorem 1.2] and [5, Theorem 2.1]), an index set \( I \) and points \( z_i \in A(r_0 \theta^{3k-1}, r_0 \theta^{3k-2}), \ i \in I \), such that
\[ A(r_0 \theta^{3k-1}, r_0 \theta^{3k-2}) \subset \bigcup_{i \in I} B(z_i, r_0 \theta^{3k-2}) \] (2.3)
and \( B(z_i, \frac{1}{3} r_0 \theta^{3k-2}) \cap B(z_j, \frac{1}{3} r_0 \theta^{3k-2}) = \emptyset \) for \( i \neq j \). Since for each \( i \in I \) and for
every \( x \in B(z_i, \frac{1}{3} r_0 \theta^{3k-1}) \) we have \( \text{dist}(x, A) \leq d(x, z_i) + \text{dist}(z_i, A) \leq r_0 (\frac{1}{3} \theta^{3k-1} + \theta^{3k-2}) \leq r_0 \theta^{3(k-1)} \) and \( \text{dist}(x, A) \geq \text{dist}(z_i, A) - d(z_i, x) > r_0 \theta^{3k-1} - \frac{1}{3} r_0 \theta^{3k-1} \geq r_0 \theta^{3k} \), it follows that
\[ B(z_i, \frac{1}{2} r_0 \theta^{3k-1}) \subset A(r_0 \theta^{3k}, r_0 \theta^{3(k-1)}) \]
whenever \( i \in I \). On the other hand, we have
\[ A(r_0 \theta^{3k}) \subset \bigcup_{i \in I} B(z_i, r_0 \theta^{3k-2}) \]
by (2.2) and (2.3), and hence, using (1.1),
\[ \alpha_k \geq \sum_{i \in I} \mu(B(z_i, \frac{1}{2} r_0 \theta^{3k-1})) \geq \gamma \sum_{i \in I} \mu(B(z_i, r_0 \theta^{3k-2})) \]
\[ \geq \gamma \mu(A(r_0 \theta^{3k})) = \gamma \sum_{i=k+1}^\infty \alpha_i \] (2.4)
where \( \gamma = c_\mu^{[\log_2(\theta/6)]} \). Moreover, if \( \mu \) is \( s \)-regular then we have \( \gamma = \frac{a_2}{a_0} (\theta/6)^s \).

Next we shall derive a growth inequality for the numbers \( \alpha_k \). Notice that \( \alpha_1 \geq \gamma \sum_{i=2}^\infty \alpha_i \geq \gamma a_2 \) by (2.3). Assuming inductively
\[ \alpha_1 \geq \gamma (\gamma + 1)^{k-1} \sum_{i=k+1}^\infty \alpha_i \geq \gamma (\gamma + 1)^{k-1} \alpha_{k+1} \]
for \( k \in \mathbb{N} \), we get
\[
\alpha_1 \geq \gamma (\gamma + 1)^{k-1} \sum_{i = k+1}^{\infty} \alpha_i = \gamma (\gamma + 1)^{k-1} \left( \alpha_{k+1} + \sum_{i = k+2}^{\infty} \alpha_i \right)
\]
\[
\geq \gamma (\gamma + 1)^{k-1} \left( \gamma \sum_{i = k+2}^{\infty} \alpha_i + \sum_{i = k+2}^{\infty} \alpha_i \right)
\]
\[
= \gamma (\gamma + 1)^k \sum_{i = k+2}^{\infty} \alpha_i \geq \gamma (\gamma + 1)^k \alpha_{k+2}
\]
using (2.4) again. Thus we have shown that
\[
\alpha_k \leq \gamma^{-1} (\gamma + 1)^{2-k} \alpha_1 \quad (2.5)
\]
whenever \( k \in \mathbb{N} \).

Now let \( 0 < r < r_0 \) and choose \( k_0 \in \mathbb{N} \) such that \( r_0 \delta^{3(k_0+1)} \leq r < r_0 \delta^{3k_0} \). Then we have
\[
\mu(A(r)) \leq \mu(A(r_0 \delta^{3k_0})) \leq \mu(A) + \sum_{i = k_0+1}^{\infty} \alpha_i
\]
\[
= \sum_{i = k_0+1}^{\infty} \alpha_i \leq \gamma^{-1} \alpha_{k_0} \leq \gamma^{-2} (\gamma + 1)^{2-k} \alpha_1
\]
\[
\leq \left( \frac{(\gamma + 1)}{\gamma} \right)^2 (\gamma + 1)^{-k_0} \mu(A(r_0))
\]
\[
\leq \left( \frac{(\gamma + 1)}{\gamma} \right)^2 \delta^{-k_0} \mu(B(x_0, 2r_0))
\]
\[
\leq \left( \frac{(\gamma + 1)}{\gamma} \right)^2 \frac{c_\delta^{-1} \mu(B(x_0, r_0)) (r/r_0)^{\log(1/\delta)}}{\log(1/\delta)}
\]
by using (2.4), (2.5), and the fact that \( \mu(A) = 0 \). Recalling the definition of \( \gamma \), the claim follows. □

3. Main ideas

To show that a \( t \)-regular set \( A \) is uniformly \( \rho \)-porous for some \( \rho > 0 \) whenever \( 0 < t < s \), fix \( x \in A \) and small \( r > 0 \) and consider balls of radius \( 2^{-k} r \) with given \( k \in \mathbb{N} \). It follows from \( s \)-regularity that the number of such balls needed to cover \( B(x, r) \) is at least a constant times \( 2^{ks} \). Taking any sufficiently separated (by means of the \( 5r \)-covering theorem) subcollection from those balls so that each ball intersects \( A \), it follows from \( t \)-regularity that the cardinality of such a subcollection is at most a constant times \( 2^{kt} \). Choosing \( k \) large enough gives the claim.

To sketch the proof of the other direction, let \( \delta = c \rho^{s} / \log(1/\rho) \) be as in Lemma 2.1 and take \( s - \delta < t < s \). We will next construct the set \( F \supset A \). Assume for simplicity that \( r_\mu \) and \( r_\rho \) are large. For all \( j \in \mathbb{N} \) we use the \( 5r \)-covering theorem.
to find a collection of disjoint balls \( \{ B(x_{ji}, (\frac{1}{2}r)^j) \} \) so that 5 times bigger balls cover \( A \). Using the uniform \( \rho \)-porosity, choose \( B(z_{ji}, \rho(\frac{1}{2}r)^j) \subset B(x_{ji}, (\frac{1}{2}r)^j) \setminus A \) for each \( i \). Recalling the result of Saaranen, we construct a \( t \)-regular set \( F_{ji} \) on each \( B(z_{ji}, (\frac{1}{2}r)^{j+1}) = B_{ji} \) and define \( F = A \cup \bigcup_{i,j} F_{ji} \). It suffices to show that \( \nu \), defined to be the sum of all \( t \)-regular measures of \( F_{ji} \), is \( t \)-regular on \( F \). We will show that there is a constant \( c \geq 1 \) such that \( \nu(B(x, r)) \leq c r^t \) for every \( x \in A \) and \( r > 0 \) small enough. To conclude that \( \nu \) is \( t \)-regular is then straightforward.

Take \( x \in A \), fix a scale \( k \in \mathbb{N} \), and denote \( N_j = \{ i : B_{ji} \cap B(x, (\frac{1}{2}r)^k) \neq \emptyset \} \). It is clear that \( B_{ji} \subset A((\frac{1}{2}r)^j) \) for every \( j \) and \( i \). If \( j \leq k - 1 \) then \( B_{ji} \cap B(x, (\frac{1}{2}r)^k) = \emptyset \) for all \( i \). If \( j \geq k \) then \( B_{ji} \subset B(x, 4(\frac{1}{2}r)^{k+1}) \) as \( i \in N_j \).

Using these observations and Lemma 2.1 we have

\[
\begin{align*}
    c \# N_j \rho^{(j+1)s} & \leq \sum_{i \in N_j} \mu(B_{ji}) \leq \mu \left( \left( A \cap B(x, 4(\frac{1}{2}r)^{k+1}) \right) \left( \frac{1}{2}r \right)^j \right) \\
 & \leq c \mu(B(x, 4(\frac{1}{2}r)^{k+1}) \left( \frac{1}{2}r \right)^j)\left( \frac{1}{4(\frac{1}{2}r)^{k+1}} \right)^\delta \\
 & \leq c \rho^{(s-\delta)(k+1)+j\delta}
\end{align*}
\]

where \( \mu \) is the \( s \)-regular measure, yielding \( \# N_j \leq c \rho^{(s-\delta)-j(s-\delta)} \) for every \( j \geq k \).

Here \( c \) denotes a constant whose value may vary on each instance even within a line. This implies, for any \( x \in A \) and \( (\frac{1}{2}r)^{k+1} \leq r < (\frac{1}{2}r)^k \),

\[
\begin{align*}
\nu(B(x, r)) & \leq \nu(B_{ji} \cap B(x, (\frac{1}{2}r)^k)) \\
 & \leq c \sum_{j=k}^{\infty} \# N_j \rho^{(j+1)t} \\
 & \leq c \rho^{t+k(s-\delta)} \sum_{j=k}^{\infty} \rho^{j(s-\delta)} = c \rho^{t+k(s-\delta)} \frac{\rho^{kt-k(s-\delta)}}{1 - \rho^{k(s-\delta)}} \leq cr^t
\end{align*}
\]

which is exactly what we wanted.

References

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Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland

E-mail address: antakae@maths.jyu.fi