The Effective Lagrangian of Three Dimensional Quantum Chromodynamics

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Abstract

We consider the low energy limit of three dimensional Quantum Chromodynamics with an even number of flavors. We show that Parity is not spontaneously broken, but the global (flavor) symmetry is spontaneously broken. The low energy effective lagrangian is a nonlinear sigma model on the Grassmannian. Some Chern–Simons terms are necessary in the lagrangian to realize the discrete symmetries correctly. We consider also another parametrization of the low energy sector which leads to a three dimensional analogue of the Wess–Zumino–Witten–Novikov model. Since three dimensional QCD is believed to be a model for quantum anti–ferromagnetism, our effective lagrangian can describe their long wavelength excitations (spin waves).
1. Introduction

In this and an accompanying paper [1] we will study the low energy behaviour of Three Dimensional Quantum Chromodynamics (3DQCD) by adapting effective lagrangian methods well known [2] in the theory of strong interactions. The main motivation is to construct a lower dimensional (therefore simpler) analogue of QCD where phenomena such as chiral symmetry breaking can be studied. Furthermore three dimensional QCD is of some intrinsic interest as a high temperature limit* of QCD. Another motivation that three dimensional QCD is believed to describe quantum anti–ferromagnetism.

One point of view [3], [4] is that a quantum Heisenberg model on the plane, of spin $j$ with $SU(2)$ symmetry is described by 3DQCD with $2j$ colors and two flavors. More generally, 3DQCD with $SU(N_c)$ gauge symmetry and $SU(N)$ flavor symmetry is related to a quantum anti–ferromagnet with ‘spins’ in a representation of $SU(N)$ with $N_c$ rows in the Young Tableau. Such generalizations of the Heisenberg model have been studied in the condensed matter literature [5]. In another point of view [6], an ordered phase of the anti–ferromagnet in which the spins are periodic with a period equal to a multiple of the lattice spacing is considered. Then, it is possible to reformulate the theory in terms of a super–lattice; each point of the new lattice represents a group of spins of the original lattice. Thus the field variable on the new lattice will be a multi–component fermionic variable that satisfies certain bilinear constraints. The continuum limit of this can be QCD. Of particular interest are anti–ferromagnetism in two space dimensions, as the High $T_c$ superconductors are materials of this type. It is possible that the charge carriers of these superconductors are solitons and that the mechanism that binds them

* If we take the high temperature limit of $\text{tr} e^{-\beta H}$ in QCD, only the bosonic (gluon) fields will contribute. However, the high temperature limit of $\text{tr}(-1)^F e^{-\beta H}$ will have contributions from the quarks as well; the infra red behaviour of this free energy can be studied by our effective lagrangians.
into Cooper pairs is an exchange of spin waves. Effective lagrangians of the type we study provide a systematic framework in which such questions can be answered. Such a close connection between anti–ferromagnetism and gauge theories is an exciting prospect. We can apply well–tested methods of the theory of strong interactions to make predictions about anti–ferromagnetism. Conversely, new dynamical insights obtained from studying anti–ferromagnetism can help us understand QCD.

In this paper, we will (except here in the introduction) use the particle physics terminology. For clarity, it is important not to mix metaphors, and we will stick to the one we feel more comfortable with. For example, the particles we call pions must be the spin waves of the anti–ferromagnet. Electric charge of the anti–ferromagnetic system is to be identified with baryon number in our language. Thus the spin waves are electrically neutral. The defects in the anti–ferromagnet caused by doping them would then correspond to baryons (they are studied in an accompanying paper). Thus the charge carriers are the baryons. We discover that there is a mechanism (pion exchange) by which these charge carriers can form bound states of rather small binding energy. Perhaps this can provide a mechanism for the formation of Cooper pairs in the High $T_c$ superconductors, which have an anti–ferromagnetic phase [7].

The basic result of our paper is that 3DQCD has a parity preserving phase in which chiral symmetry is spontaneously broken. In this phase, the low energy behaviour of the theory is very analogous to QCD. There are low mass pseudo–scalars (‘pions’), somewhat more massive vector mesons ($\rho$ mesons) and fermionic solitons of large mass (‘baryons’). Although there could be anyons even in a parity invariant theory, we find that do not arise in our version of 3DQCD. Thus three dimensional QCD is a much better analogue for strong interactions than two dimensional QCD, for example. There is no spontaneous breaking of chiral symmetry in two dimensions; although a theory of mesons and baryons as their solitons can be constructed [8] [9], in the $\frac{1}{N_c}$ expansion, there is no systematic
low energy approximation.

We will first show that parity is not spontaneously broken in one version of 3DQCD. This is the theory with gauge group $SU(N_c)$ and an even number $(2n)$ of flavors:

$$L = -\frac{1}{\alpha_s} \text{tr} F^{\mu\nu} F_{\mu\nu} + \bar{q}^i \gamma \cdot \nabla q_i + m \left( \sum_{i>0} \bar{q}^i q_i - \sum_{i<0} \bar{q}^i q_i \right).$$

(1)

The quark masses have been assumed to come in pairs of equal magnitude but opposite signs, so that parity is a symmetry of the lagrangian. The proof is an adaptation of the Vafa–Witten argument in four dimensions [10].

Then, we will argue that the global symmetry $SU(2n)$ (analogous to chiral symmetry), is spontaneously broken to $S(U(n) \times U(n))$. We will then construct a low energy effective lagrangian for this phase of 3DQCD. This will a nonlinear sigma model on the Grassmannian $Gr_n = SU(2n)/S(U(n) \times U(n))$. However, it will be necessary to add some Chern–Simons terms to this action so that its doesn’t have a discrete symmetry that 3DQCD does not have [11]. The small oscillations of this effective lagrangian describes pseudo–scalars of small mass as well as vectors mesons of mass of order $N^0_c$ in the $\frac{1}{N_c}$ expansion. Some of the issues in this and the accompanying paper [1] are also discussed independently in [12].

2. Three Dimensional QCD and its Flavor Symmetries

We will study the theory (Three Dimensional Quantum ChromoDynamics, 3DQCD), defined by the action

$$S = \int \left[ -\frac{1}{\alpha_s} \text{tr} F^{\mu\nu} F_{\mu\nu} + \bar{q}^i \gamma \cdot \nabla q_i + \sum_i m_i \bar{q}^i q_i \right] d^3x.$$

(2)

Here $q_i$ are two component spinors transforming under the fundamental representation of the color group $SU(N_c)$; both spin and color indices are suppressed. The index $i = 1 \cdots N$ labels flavor; in the limit where all the masses are equal, there is a global $U(N)$
symmetry associated to flavor. The $U(1)$ part of this symmetry, (with current $J_\mu = \frac{1}{N_c} \sum_i \bar{q}_i \gamma_\mu q^i$ corresponding to baryon number), is a symmetry for any choice of quark masses. There is no distinction between left and right handed fermions in three dimensions, so the flavor symmetry does not have the chiral form $U(N)_L \times U(N)_R$ familiar from four dimensions. Although $U(N)$ symmetry allows equal masses for all the quarks, assuming parity symmetry in addition will require that the quark masses are zero. * Thus the flavor symmetry in the limit of massless quarks is $U(N) \times Z_2$. It is possible to have massive quarks in a parity conserving theory if the quarks masses come in pairs with equal magnitude but opposite sign. If the number of flavors is even $N = 2n$, the most symmetric possibility with massive quarks is $(i = -n, \ldots, -1, 1, \ldots n)$,

$$L = -\frac{1}{\alpha_s} \text{tr} F^{\mu\nu} F_{\mu\nu} + \bar{q}_i \gamma \cdot \nabla q_i + m \left( \sum_{i > 0} \bar{q}_i q_i - \sum_{i < 0} \bar{q}_i q_i \right).$$

(3)

In this case the global flavor symmetry is $S(U(n) \times U(n)) \times Z_2$. This theory is invariant under a parity operation $(P)$ that is the product of the usual parity operation $(P_1)$ with the interchange $P_2 : i \rightarrow -i$.

With an odd number of quarks, there is no way to make all of them massive without violating parity. This means that parity has a ‘global anomaly’, and terms that violate parity can be induced due to radiative corrections \cite{13}, \cite{14}. In particular, there is nothing to prevent a Chern–Simons term in the color gauge field \cite{13},

$$S_{C.S.} = K \int \text{tr}[A dA + \frac{2}{3} A^3]$$

(4)

This means that the theory exists in a phase where the color forces are screened rather than confining. Once such parity breaking is allowed, the quarks can acquire a mass of order $g^2$, the gauge coupling constant. There will be particles (quarks) carrying fractional

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* Parity is the symmetry $P_1 : A(x_1, x_2, t) \rightarrow A(-x_1, x_2, t), q(x) \rightarrow q(-x_1, x_2, t), \bar{q}(x) \rightarrow -\bar{q}(-x_1, x_2, t)$. We regard $A = A_\mu dx^\mu$ as a one form.
baryon number and exotic statistics (‘non–abelian’ fractional statistics) in the spectrum. The gluons and quarks will have masses of the same order. This phase of 3DQCD is related to the ‘generalized flux phase’ of anti–ferromagnetism [4].

This phase is very different from the one we will study in detail, in which parity is a symmetry. There are light particles (‘pions’) that carry no baryon number. There are heavier particles (‘baryons’) with integer baryon number, and fermi statistics. It is expected that all the particles that are observable at infinity in a scattering experiment in QCD are color singlets; there should be some phase in which this is true in three dimensions as well. It should therefore be possible to construct a theory equivalent to the one above, describing directly these color singlet bound states. At present this is possible only in two dimensional QCD. In three dimensional QCD, (as in four dimensions), we should be able to construct an effective lagrangian for the low mass particles of the theory, under reasonable assumptions. This is what we will do in this paper.

The first issue is whether the flavor symmetry $U(N) \times Z_2$ of the theory is spontaneously broken in the limit of massless quarks. If $N$ is even, an analogue of the Vafa–Witten argument will show that parity is not spontaneously broken.

Consider the expectation values of some operator $X$ involving only gluons that changes sign under parity (For example $X = \epsilon_{\mu\nu\rho} \mathrm{tr} F_{\mu\sigma} \nabla_\nu F_{\rho\sigma}$). We will assume that the number of flavors is even and the quarks have masses $m_i = -m_{-i} \neq 0$ that do not break parity explicitly. Then the generating function for the expectation values of $X$ is

$$e^{-W[J]} = \int D[A] \det [i\gamma \cdot \nabla + iM] e^{\int [-\frac{1}{2\alpha_s} \mathrm{tr} F_{\mu\nu} F_{\mu\nu} + iJX] dx}. \quad (5)$$

$M$ is the diagonal mass matrix for the quarks. The fermion determinant

$$\det [i\gamma \cdot \nabla + iM] = \prod_{i=1}^{n} \det [i\gamma \cdot \nabla + im_i] [i\gamma \cdot \nabla - im_i]$$

$$= \prod_{i=1}^{n} \det [-(\gamma \cdot \nabla)^2 + m_i^2]$$
is a parity invariant positive function of the gauge fields. It then follows that $W[J]$ is an even convex function of $J$. The effective potential $\Gamma[X]$ (the Legendre transform of $W[J]$) is then also an even convex function. Hence the minimum of $\Gamma[X]$ (the vacuum expectation value of $X$) is at $X = 0$. Thus parity will not be broken by the expectation value of any function of gauge fields. The argument can be extended to functions of Fermi fields as well, as remarked by Vafa and Witten. The reason this argument works is that there is a parity invariant Infra Red regularization which makes the fermion determinant positive. The fermion determinant is not positive for odd $N$, and parity might be broken by anomalies as well as spontaneously.

Now let us see if the continuous part of the flavor symmetry can be broken spontaneously. Imagine that we have a small mass term $m \bar{q} q$ added to the lagrangian. $(\epsilon = \text{diag}(1, \ldots, 1, -1, \ldots, -1))$ It is possible to calculate the two point function of flavor currents

$$< j^a_\mu(k) j^b_\nu(-k) > \sim \frac{N_c}{4\pi} \epsilon^{ab} \epsilon_{\mu\nu\rho} i k_\rho \quad \text{for} \quad k \rightarrow 0$$

from a one loop diagram. There is a theorem stating that this long distance behaviour is in fact exact to all orders in perturbation theory [15]. Moreover, it remains non-zero even as $m \rightarrow 0$. In this limit, the QCD lagrangian is invariant under $U(N)$, yet, here is a Green’s function that is not invariant under $U(N)$. This is characteristic of spontaneous symmetry breaking. (There is no anomaly in the flavor current in 2+1 dimensions, so the symmetry is not explicitly broken by radiative corrections. Thus, chiral symmetry breaking is a consequence of the absence of flavor anomalies in three dimensions. In four dimensional QCD, the ’t Hooft anomaly ‘matching conditions’ [16] are used usually to arrive at the same result.)

If $U(N) \times Z_2$ is spontaneously broken, what is the unbroken subgroup?. We will use an analogue of the Coleman–Witten argument [17] (based on the large $N_c$ limit) to show that, for even $N = 2n$, $U(2n)$ breaks to $U(n) \times U(n)$. Thus the Goldstone bosons are
described by a nonlinear sigma model on the Grassmannian \( Gr_n = U(2n)/U(n) \times U(n) \).

Consider the effective potential \( V(\Phi) \) of the order parameter \( \Phi_i^j \sim \bar{q}_i q^j \) of flavor symmetry breaking. It transforms under the adjoint representation of \( U(N) \) and \( \Phi \to -\Phi \) under \( P_1 \).

The effective potential can be expanded in terms of the invariants of \( U(N) \):

\[
V(\Phi) = \sum_{n=1} v_n \text{tr} \Phi^n + \sum_{n=1, m=1} v_{n,m} \text{tr} \Phi^n \text{tr} \Phi^m + \cdots
\]

In the large \( N_c \) limit, the terms involving products such as \( \text{tr} \Phi^n \text{tr} \Phi^m \) are suppressed: they involve more than one quark loop. To leading order in \( \frac{1}{N_c} \), therefore,

\[
V(\Phi) = \sum_{i=1}^N v(\phi_i)
\]

where \( \phi_i \) are the eigenvalues of the hermitian matrix \( \Phi \), and \( v(\phi) \) is some function of a real variable. Since \( \bar{q} q \) (and hence \( \Phi \)) changes sign under the discrete symmetry \( P_1 \), this function must be even. The vacuum expectation value of \( \Phi \) will be determined by the minima of \( v(\phi) \). If the minimum is at a non–zero value of \( \phi \), the \( U(N) \times Z_2 \) symmetry will be broken. If \( a \) is a minimum, so must be \(-a\). It is reasonable to assume that there are no other minima (i.e., that there is no accidental degeneracy). Then the vacuum expectation value of \( \Phi \) will be of the form

\[
\langle \Phi \rangle = \text{diag}(-a, -a, \cdots, a, \cdots a).
\]

This breaks \( U(N) \to U(n) \times U(N-n) \). The value of \( n \) (the number of negative eigenvalues) cannot be determined at this order of the \( \frac{1}{N_c} \) expansion: we need a term of the type \( (\text{tr} \Phi)^2 \) in the effective potential. In order to preserve parity, in some higher order of the \( \frac{1}{N_c} \) expansion, the value \( n = \frac{N}{2} \) must be selected as the ground state.

3. Effective Lagrangian for 3DQCD

The low energy effective lagrangian must describe Goldstone bosons on the Grassmannian, \( Gr_n = SU(2n)/G \) where \( G = S((U(n) \times U(n))) \). (The \( U(1) \) of baryon number acts
trivially on the Grassmannian, so that \( U(2n)/U(n) \times U(n) \) is the same as \( SU(2n)/S(U(n) \times U(n)) \). This theory can be written in many ways, some of which we will discuss in a later section. The description that seems most convenient for our purposes is to use the action,

\[
S_1(\chi, A_\mu) = \frac{F_\pi}{2} \int \text{tr} \nabla_\mu \chi \nabla^\mu \chi d^3x
\]

(10)

where \( F_\pi \) is the analogue of the pion decay constant. Here, \( \chi \in SU(2n) \) and \( A_\mu \in G \) are the basic field variables, and

\[
\nabla_\mu \chi = \partial_\mu \chi + \chi A_\mu.
\]

(11)

The gauge field is really a Lagrange multiplier that can be eliminated by its equations of motion,

\[
A_\mu = -i P_G \left( \chi^\dagger \partial_\mu \chi \right)
\]

(12)

\( P_G \) being the projection operator to the unbroken subalgebra \( G \). The Lagrangian has a gauge invariance under the right action of \( G \) on \( \chi \), so that the true degrees of freedom lie on the coset space \( SU(n)/G \). There is, also, a global symmetry under the left action of \( U(2n) \).

There are also a pair of discrete global symmetries, \( \sigma : \chi \rightarrow \chi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( P_2 : \chi \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \chi \), which are related to parity (see below). Note that, unlike \( P_2 \), \( \sigma \) does not commute with the gauge invariance: under conjugation by \( \sigma \) the gauge transformation \( \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \) is mapped to \( \begin{pmatrix} g_2 & 0 \\ 0 & g_1 \end{pmatrix} \). Thus \( \sigma \) is in the ‘normalizer’ of the gauge group, which is sufficient for its action on the right cosets to be well-defined.

Expanding \( \chi \) around the trivial solution \( \chi = 1 \) shows that there are a set of \( 2n^2 \) massless scalars, as needed. The quantity \( \Phi = \chi \epsilon \chi \dagger \) transforms like the order parameter \( \bar{q} \epsilon q \) of the symmetry breaking. \( \dagger \)

\[
\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

is an element of the Lie algebra \( SU(N) \). The Grassmannian is being considered as the co–adjoint orbit of \( \epsilon \). A point on the Grassmannian can also be parametrized by a Hermitian matrix \( \Phi \) satisfying \( \text{tr} \Phi = 0, \Phi^2 = 1 \).
However, this lagrangian has too much symmetry. The massless 3DQCD action is invariant under the parity $P_1$. Under this, the order parameter transforms as

$$P_1 : \Phi(x_1, x_2, t) \rightarrow -\Phi(-x_1, x_2, t). \quad (13)$$

In terms of $\chi$, this can be thought of as the product of two transformations,

$$P_0 : \chi(x_1, x_2, t) \rightarrow \chi(-x_1, x_2, t) \quad \sigma : \chi(x, t) \rightarrow \chi(x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (14)$$

But for our action, $P_0$ and $\sigma$ separately are symmetries. The microscopic theory, 3DQCD, does not have such a pair of symmetries, only their product is a symmetry. We need to break these separate symmetries to their product, by adding some higher derivative terms to the action. The required terms are Chern–Simons terms for the gauge field [13], [18]:

$$S_k = \frac{k}{4\pi} \int \text{tr} \left[ A_1 dA_1 + \frac{2}{3} A_1^3 \right] - \frac{k}{4\pi} \int \text{tr} \left[ A_2 dA_2 + \frac{2}{3} A_2^3 \right]. \quad (15)$$

Here, we have decomposed $A$ into components $A_1, A_2, A_3$ in the simple factors $SU(n) \oplus SU(n) \oplus U(1)_{\epsilon}$ of $G$ respectively. The operation $P_0 : A_{1,2}(x, t) \rightarrow A_{1,2}(-x_1, x_2, t)$ will change the sign of each Chern–Simons term, and $\sigma : A_1 \leftrightarrow A_2$ will interchange them. So the theory is invariant under $P_1 = P_0\sigma$. The field $A_3$ changes sign under parity. It does not acquire a Chern–Simons term as, it would break $P_1$.

The constant $k$ must be an integer for the action $e^{iS_k}$ to be gauge invariant [13]. Furthermore, comparison with 3DQCD will show that $k = N_c$. (We can calculate the long range behaviour of the flavor current two point function in QCD exactly. Comparing this to the prediction of the effective lagrangian, we see that $k = N_c$.) Once these terms are added, the gauge fields become propagating fields. In fact a linear analysis shows that the spectrum consists of $2n^2$ massless scalars, plus, a set of $n^2$ spin one particles of mass

$$\mu_A = \frac{2\pi F_{\epsilon}}{k}. \quad (16)$$

The positive helicity states come from $A_1$ and the negative helicity states from $A_2$. 

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If there is an explicit breaking of the \( U(N) \) symmetry (by the addition of a current quark mass) to \( G \), there should be a corresponding mass term,

\[
S_m = \frac{F_\pi}{2} m_\pi^2 \int \text{tr} \epsilon \epsilon^\dagger d^3 x
\]

in the lagrangian. This term is still invariant under the right action of the gauge group \( G \). Furthermore, the global symmetry is now a left action by \( G \). If we expand around the trivial solution, we get a set of pseudo-Goldstone bosons of mass \( m_\pi \). If the explicit symmetry breaking is small, \( m_\pi \ll F_\pi \), these bosons will be light (compared to the scale that determines their self–interaction, \( F_\pi \)). In addition, of course, are the vector mesons of mass \( 2\pi F_\pi \).

Thus we arrive at the following long distance effective action for 3DQCD:

\[
S = S_1 + S_k + S_m = \frac{F_\pi}{2} \int \text{tr} \nabla_\mu \chi^\dagger \nabla^\mu \chi d^3 x + \frac{k}{4\pi} \int \text{tr} \left[ A_1 dA_1 + \frac{2}{3} A_1^3 \right] - \frac{k}{4\pi} \int \text{tr} \left[ A_2 dA_2 + \frac{2}{3} A_2^3 \right] + \frac{F_\pi}{2} m_\pi^2 \int \text{tr} \epsilon \epsilon^\dagger d^3 x.
\]

The spectrum of small oscillations of this theory can be understood as mesons in the naive quark model. Let us call the quarks with a positive mass the ‘up’ quarks and the others the ‘down’ quarks. In three dimensions, the Dirac equation predicts that quark and anti–quarks states will have positive helicity for positive mass and negative helicity for negative mass. Thus the bound states of the type \( \bar{u}u \), which have positive helicity, can be identified with \( A_1 \), and the negative helicity bound states \( \bar{d}d \) with \( A_2 \). The bound states of type \( \bar{u}d \) and \( \bar{d}u \) describe the spin zero states described by \( \chi \). Thus our effective lagrangian describes the spin zero and spin one mesons of 3DQCD. It is possible to find effective lagrangians also for the case where the symmetry breaks in an uneven (parity violating) way \( SU(N) \to S(U(N - n) \times U(n)) \) by effective lagrangians. Some of these cases we will discuss in a later section. They also have quark model interpretations.
In an accompanying paper, we will show that this effective action has soliton solutions. They will be shown to be the baryons of 3DQCD.

4. Three dimensional analogue of the Wess–Zumino–Witten–Novikov model

We describe in this section some alternative approaches constructing the low energy effective lagrangian. They are equivalent as far as the properties of massless particles are concerned, but not for the massive particles. We find that these effective lagrangians do not contain enough of the short distance physics to produce baryons as solitons. They might be useful in describing three dimensional critical phenomena.

In a two dimensional space time $M$, the Wess–Zumino–Witten–Novikov model has a field variable valued in a compact Lie Group $G$. The action of the model [19], [20] can be written as

$$S = \frac{1}{2} F_\pi \int_M \text{tr} g^{-1}dg \ast g^{-1}dg + \frac{k}{12\pi^2} \int_{M_3} \text{tr} \left(g^{-1}dg\right)^3.$$  (17)

Here the second term is an integral on a three dimensional manifold $M_3$ whose boundary is space time, $\partial M_3 = M$. The last term represents the third co–homology of the configuration space (target space). This term breaks the two separate discrete symmetries

$$P_1 : g \rightarrow g^\dagger, \quad x \rightarrow -x$$  (18)

to their product $P = P_1 P_2$, so that the field variable describes a pseudo–scalar. The current algebra of this model is a Kac–Moody algebra. $F_\pi$ is a dimensionless variable so that theory is classically scale invariant. This scale invariance is broken in the quantum theory, except for two values of $F_\pi$. At $F_\pi = \infty$ there is an Ultra–Violet stable fixed point. Also, there is a non–trivial IR fixed point for this theory at which it describes a conformal field theory.

It is of some interest to find a higher dimensional generalization of this model. For example the current algebra of such a model could provide a generalization for the Kac–
Moody algebra. To find an analogous model in a three dimensional space–time, we must have a target space $M'$ with $H^4(M') \neq 0$. One obvious candidate is $CP^{N-1}$, for which $H^4(CP^{N-1}) = Z$ for $N > 2$. However, $\pi_4(CP^{N-1}) = 0$ so that this is not a sufficiently close generalization. A more appropriate generalization is a nonlinear model on the Grassmannian $Gr_{N,n} = SU(N)/S(U(N-n) \times U(n))$, with $n > 1$. Then, $\pi_4(Gr_{N,n}) = Z$ and $H^4(Gr_{N,n}) = Z \oplus Z$.

Each point on the Grassmannian $Gr_{N,n}$ describes an $n$ dimensional subspace of $C^N$. By attaching to each point the corresponding $n$ dimensional vector space, we get a natural complex vector bundle over $Gr_{N,n}$. The Chern classes of this vector bundle generate the cohomology ring of the Grassmannain. In the limit $N \to \infty$, the topology of the Grassmannian simplifies \[21\], so that the homotopy groups can be calculated: $\pi_{2l}(Gr_{\infty,n}) = Z$, $\pi_{2l+1}(Gr_{\infty,n}) = 0$. It is interesting that the dynamics of the theory also simplifies in this large $N$ limit, enough to make it exactly solvable. (This is a simple extension of the work of ref. \[22\] on the $CP^{N-1}$ model.)

The Grassmannian can also be thought of as the set of all hermitian matrices of square one, with the trace picking out the the particular connected component:

$$Gr_{N,n} = \{ \Phi | \Phi \dagger = \Phi, \ \Phi^2 = 1, \ \text{tr} \ \Phi = 2N - n \} \quad (19)$$

In our earlier parametrization as a coset, $\Phi = \chi \epsilon \chi \dagger$. The generator of the even order cohomology is

$$\omega_{2l} = \text{tr} \Phi (d\Phi)^{2l}. \quad (20)$$

Then

$$d\omega_{2l} = \text{tr} d\Phi (d\Phi)^{2l}$$

$$= \text{tr} d\Phi \Phi^2 (d\Phi)^{2l}$$

$$= (-1)^{2l+1} \text{tr} d\Phi \Phi (d\Phi)^{2l}$$

by moving one of the $\Phi$’s through all the $d\Phi$’s; (recall that $\Phi^2 = 1 \Rightarrow \Phi d\Phi = -d\Phi \Phi$). So,
\( \omega_{2l} \) is indeed closed. The two generators of \( H^4 \) are therefore \( \omega_2^2 \) and \( \omega_4 \). (\( \omega_2 \) is just the Kahler form).

The nonlinear sigma model on the Grassmannian, has action

\[
S_1 = \frac{1}{2} F_\pi \int \text{tr} d\Phi \star d\Phi. \tag{21}
\]

This is invariant under the discrete symmetries

\[
\Phi(x) \to -\Phi(x) \quad \text{and} \quad \Phi(x) \to \Phi(-x_1, x_2, t) \tag{22}
\]
separately. As usual we can get a model invariant only under the product \( P_1 \) of the two by adding topological terms. Of the two generators \( \omega_2^2 \) and \( \omega_4 \) of \( H^4 \), only \( \omega_4 \) is invariant under \( P_1 \). Thus the three dimensional analogue of the Wess–Zumino–Witten–Novikov model has action

\[
S = \frac{1}{2} F_\pi \int_M \text{tr} d\Phi \star d\Phi + \frac{k}{64\pi} \int_{M_4} \text{tr} \Phi(d\Phi)^4. \tag{23}
\]

\( M_4 \) is a four–manifold of which the space time manifold is the boundary. The numerical constant in front of the second term is chosen such that the level number \( k \) an integer.

The global symmetry under \( SU(N) \)

\[
\Phi \to g \Phi g^\dagger \tag{24}
\]

can be gauged by the Noether procedure to arrive at the effective action,

\[
S = \frac{1}{2} F_\pi \int \text{tr} (d\Phi + [V, \Phi])^\dagger (d\Phi + [V, \Phi]) + \Gamma[\Phi, V]. \tag{25}
\]

Here,

\[
\Gamma[\phi, V] = \frac{k}{64\pi} \int_{M_4} \text{tr} \Phi(d\Phi)^4
+ \frac{k}{8\pi} \int_M \text{tr} \left[ V \Phi(d\Phi)^2 + dV \Phi V + dVV \Phi + (V \Phi)^2 d\Phi + \Phi V^3 + \frac{1}{3} (V \Phi)^3 \right].
\]
It is possible to gauge the entire global symmetry $SU(N)$ unlike in the even dimensional WZWN models. Thus the topological term does not represent an anomaly. Indeed, there is no anomaly in odd dimensional space–time. The above effective action is gauge invariant only modulo some surface terms, which means that it leads to the existence of some ‘edge currents’ at the surface.

Unlike the two dimensional WZWN model, this theory is not classically scale invariant ($F_\pi$ has dimensions of mass). Hence the loop expansion of this theory is not renormalizable. However, there is another expansion method that can be used to define the quantum theory, the $\frac{1}{N}$ expansion. In the limit $N \to \infty$ keeping $n$ fixed, we have a sensible expansion. It is best to rewrite this theory in yet another parametrization,

$$\Phi = 2ZZ^\dagger - 1$$

where $Z$ is an $N \times n$ matrix with

$$Z^\dagger Z = 1.$$  

The action equivalent to the nonlinear sigma model on the Grassmannian is

$$S_1(Z, A) = \frac{1}{2} \int \text{tr} \nabla^\mu Z^\dagger \nabla_\mu Z d^3x + N \int \text{tr} \sigma(Z^\dagger Z - \frac{1}{g})d^3x$$

Here

$$\nabla_\mu Z = \partial_\mu Z + ZA_\mu$$

$A_\mu$ being a gauge field valued in $U(n)$. The WZ term can replaced by a Chern–Simons term for $A$. (This will introduce some massive vector mesons into the theory, but at low energies it will be equivalent to the earlier theory).

$$S(Z, A) = \frac{1}{2} \int \text{tr} \nabla^\mu Z^\dagger \nabla_\mu Z d^3x + N \int \text{tr} \sigma(Z^\dagger Z - \frac{1}{g})d^3x + \frac{k}{4\pi} \int \text{tr}[A dA + \frac{2}{3} A^3]$$

This theory admits a $\frac{1}{N}$ expansion that is renormalizable, a straightforward generalization of Ref. [22]. More precisely, it has a non–trivial Ultra–Violet stable fixed point. It is
possible that the Chern–Simons term will lead to the existence of a non–trivial fixed point as well.

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