Solving Linear Systems of Equations by Gaussian Elimination Method Using Grover’s Search Algorithm: An IBM Quantum Experience

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Quantum algorithm plays a notable role in solving linear systems of equations with an exponential speedup over the classical algorithm. Here we demonstrate Gaussian elimination method for solving system of equations by using the well known Grover’s quantum search algorithm. The elimination method mainly involves elementary row operations which can be performed by applying particular matrices that can be obtained from Grover’s algorithm. We explicitly illustrate the whole process by taking a simple example consisting of a set of equations.

I. INTRODUCTION

It is well known that a quantum computer has superior power than a classical one to perform various computational tasks [1] due to the harnessing of quantum mechanics. Grover search algorithm [2], Shor’s algorithm [3] and Deutsch-Jozsa algorithm [4] have significant contributions in demonstrating the advantage of quantumness over their classical counterparts. Quantum computers are also used to simulate quantum mechanical problems [5–10] which show exponential speedups.

Linear equations can be used to describe a number of phenomena that occur in nature. Hence, solving linear systems of equations becomes an integral part to understand the way the mysteries of nature unfold. Linear equations are used vastly in all the fields of Science and Engineering. Researchers have come up with many ways to solve any set of linear equations [11]. Here we propose a way to perform Gaussian elimination to solve the set of linear equations by the usage of Grover’s search algorithm. Calculating the solutions for N linear equations in N unknowns requires a time scale of $O(N^2)$ due to the harnessing of quantum mechanics. Grover search algorithm [2], Shor’s algorithm [3] and Deutsch-Jozsa algorithm [4] have significant contributions in demonstrating the advantage of quantumness over their classical counterparts. Quantum computers are also used to simulate quantum mechanical problems [5–10] which show exponential speedups.

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The rest of the paper is organized as follows. Sec. II discusses about Gaussian elimination method, following which Grover’s search algorithm is geometrically visualized. Sec. IV applies both of the above methods to solve a given linear system of equations. Finally, we conclude in Sec. V by discussing future applications of our work.

II. GAUSSIAN ELIMINATION

Gaussian elimination or row reduction algorithm is usually used to solve systems of linear equations. The algorithm consists of a sequence of operations which are applied to the matrix representing the set of equations.

We first write the set of equations in a matrix form $(A|x⟩ = |y⟩)$. Then we make the elements of the lower triangle of the matrix $A$ as zero using appropriate row transformations that are also applied to $y$. These row operations are performed in $N$ steps, then we can find each value in $x$ by continuous subtraction and division. Here, $N$ represents the number of rows and columns in $A$. 

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III. GROVER’S SEARCH ALGORITHM

Grover’s search algorithm is the fastest way to transform a vector from its initial state to a final desired state. This is achieved by a repeated application of an operator known as Grover operator. Before understanding it mathematically, let us first visualize this algorithm geometrically.

A. Visualizing Grover’s search algorithm

Suppose $|\psi\rangle$ is the initial vector, then we can represent $|\psi\rangle$ as a superposition of the solution vector (the final desired state) and all other corresponding basis vectors orthogonal to the solution vector.

![FIG. 1. Geometric representation of Grover’s search algorithm](image)

In Fig. $\theta/2$ is the angle between the initial state and the states perpendicular to the solution state ($|\beta\rangle$). Here, $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal to each other and both together form the complete set of computational basis.

The state $|\psi\rangle$ is initialized to an equal superposition of all the computational basis states. Hence for $n$ qubits, $|\psi\rangle$ will be an equal superposition of $2^n$ computational basis states. The solution state, $|\beta\rangle$ is usually an equal superposition of $M$ computational basis states where $0 < M < N$. After certain approximations, it is evident that the number of times the Grover iteration has to be repeated in order to get arbitrarily close to $|\beta\rangle$ is $O(\sqrt{N/M})$.

IV. THE ORACLE

Any row operation on a matrix can be thought of as a matrix being multiplied with the given matrix. Hence, we can construct a matrix $U$ which when multiplied on $A$ and $|y\rangle$, performs the Gaussian elimination process. To construct the matrix $U$, we can decompose it into a product of $N$ matrices $U_1, U_2, U_3, ..., U_N$. i.e.,

$$U = U_NU_{N-1}...U_2U_1$$ (1)

Each of the matrix $U_i$ converts the column $i$ to its Gaussian form. To achieve this purpose, we first need to represent the matrix such that in each column the sum of square of each element is 1.

$$\sum_{i=1}^{N} (A_{ij})^2 = 1$$ (2)

where, $A_{ij}$ denotes an element of $i^{th}$ row and $j^{th}$ column of the matrix $A$.

It is to be noted that, $|y\rangle$ must also be normalized. i.e.,

$$\sum_{i=1}^{N} y_i^2 = 1$$ (3)

Now, we can represent each column in matrix $A$ as $|\psi_i\rangle$. Hence,

$$A = |\psi_1\rangle |\psi_2\rangle |\psi_3\rangle ... |\psi_N\rangle$$

After the application of $U$,

$$UA = [U|\psi_1\rangle U|\psi_2\rangle U|\psi_3\rangle ... U|\psi_N\rangle]$$

By knowing $|\psi_i\rangle$, we can calculate $U_i$ and hence perform Gaussian elimination.

A. Example

This method of solving linear system of equations works well especially for matrices, which have $|\psi_i\rangle$ orthogonal to each other. i.e., $\langle \psi_i | \psi_j \rangle = \delta_{ij}$. Let’s consider a matrix $A$ of the following form.

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Then,

$$|\psi_1\rangle = \frac{1}{2}[1111]^t, \ |s\rangle = [1000]^t$$

where, $[a \ b \ c \ d]^t$ represents the transpose of the row vector $[a \ b \ c \ d]$ and $a$, $b$, $c$ and $d$ are arbitrary constants.

$$(2|\psi_1\rangle \langle \psi_1| - I)(I - 2|s\rangle \langle s|) = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$
Let’s define $U_1 = (2|ψ_1⟩⟨ψ_1| - I)(I - 2|s⟩⟨s|)$. The operation of $U_1$ on $A$ leads to the following row echelon form.

\[
U_1 A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

Now we can perform $U_1$ operation on $|y⟩$ to complete the procedure for Gaussian elimination.

Fortunately, here we get a perfectly diagonalized matrix because, the unitary operation acts like a basis transformation. It is not expected that we will always get a matrix of a diagonalized form. Sometimes, depending upon the order of $|ψ_i⟩$, the final matrix’s columns may be interchanged. However, after changing the order of the variables appropriately, we can get the diagonalized matrix.

Experimentally, $U_1$ can be realized by the following circuit depicted in Fig. 2. The matrix implemented by the circuit is not exactly $U$, however, it gives the same result for the given input with an external phase factor for $|ψ_2⟩$. This matrix is represented by $U'$.

\[
U' = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1
\end{bmatrix}
\]

\[
U'A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

In this case, $θ/2$ was $30^o$. Hence, when Grover iteration was applied once, the vector got rotated by $60^o$ and lined up perfectly with the solution vector. However, in any other general case, we can calculate the angle $θ/2$ and correspondingly decide the number of iterations that gives the best result. We can decide the number of times we have to repeat the Grover iterations based on the following formula.

\[
R = \frac{(2k + 1) \pi}{2 \theta} - \frac{1}{2} \tag{4}
\]

where $k$ is a positive integer such that $R$ is an integer (or as close to an integer as possible). Based on this relation, we can apply the Grover iteration $R + \frac{1}{2}$ number of times to get as close to the solution as possible. This gives us the integer closest to $R$.

V. CONCLUSION

Here we have illustrated an approach for solving any system of linear equations by using Gaussian elimination method and the so-called Grover’s search algorithm. We have explicitly taken an example and worked out the step. We have designed the equivalent quantum circuit for the Grover operator. We have verified the experimental result and hence our approach has found to be successful. In future, this work can be extended for solving more general system of equations including non-linearity etc.
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