DIRAC GENERATING OPERATORS OF SPLIT COURANT ALGEBROIDS

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Abstract. Given a vector bundle $A$ over a smooth manifold $M$ such that the square root $L$ of the line bundle $\Lambda^{\top} A^* \otimes \Lambda^{\top} T^* M$ exists, the Clifford bundle associated to the split pseudo-Euclidean vector bundle $(E = A \oplus A^*, \langle \cdot , \cdot \rangle)$, admits a spinor bundle $\Lambda^* A \otimes L$, whose section space can be thought of as that of Berezinian half-densities of the graded manifold $A^*[1]$. We give an explicit construction of Dirac generating operators of split Courant algebroid (or proto-bialgebroid) structures on $A \oplus A^*$ introduced by Alekseev and Xu. We also prove that the square of the Dirac generating operator gives rise to an invariant of the split Courant algebroid.

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Introduction

The main object of this paper is split Courant algebroid, which can also be expressed as a proto-bialgebroid. In [7], the second author of this paper and Stiénon studied a special type of split Courant algebroids which are doubles of Lie bialgebroids. This paper presents a study of this problem in general.

The origin of studying proto-bialgebroids can be traced back to Drinfeld’s work on Lie bialgebras [10]. After that, with his landmark “Quantum groups” article [11], a series of follow-up studies developed rapidly. For example, Drinfeld further studied quasi-Hopf algebras that generalize the Hopf algebras defining quantum groups, and their semi-classical limits, the Lie quasi-bialgebras [12]. Kosmann-Schwarzbach also introduced the notion of quasi-Poisson Lie groups in [21, 22]. It turns out that the infinitesimal counterpart of a quasi-Poisson Lie group is a quasi-Lie bialgebra, which is another weak version of the Lie bialgebra structure. The more general case of proto-bialgebroids (called there “proto-Lie bialgebras”) is treated in [4].

Lie bialgebras are special cases of Lie bialgebroids introduced by Mackenzie and Xu in [27], where they appeared as linearization of Poisson groupoids. Likewise, proto-bialgebras are special cases of proto-bialgebroids. According to Kosmann-Schwarzbach [23] (see also [29, 31]), a proto-bialgebroid is a pair \((A, A^*)\) of vector bundles in duality, together with a degree 3 function \(\Theta\) (known as the Hamiltonian generating function) on the \((-2)-\text{shifted Poisson manifold} \) \(T^*2A[1]\) satisfying the classical master equation \(\{\Theta, \Theta\} = 0\). Unpacking this Hamiltonian generating function \(\Theta\), we obtain the following data:

- Two skew-symmetric brackets \([\cdot, \cdot]_A, [\cdot, \cdot]_s\) on \(\Gamma(A)\) and \(\Gamma(A^*)\), respectively;
- Two bundle maps (called anchors) \(a_A: A \to TM\) and \(a_s: A^* \to TM\);
- Two elements \(\tau \in \Gamma(\wedge^3 A)\) and \(\phi \in \Gamma(\wedge^3 A^*)\), which are “fluxes” in field theory [9].

Unpacking the classical master equation \(\{\Theta, \Theta\} = 0\), both \((A, [\cdot, \cdot]_A, a_A)\) and \((A^*, [\cdot, \cdot]_s, a_s)\) are skew-symmetric dull algebroids in the sense of [20], and the above data are subject to several compatibility conditions (see Definition 1.2). When both \(\tau\) and \(\phi\) vanish, it becomes a Lie bialgebroid \((A, A^*)\). The cases \(\phi = 0\) or \(\tau = 0\) correspond to Lie quasi-bialgebroids or quasi-Lie bialgebroids, respectively.

The notion of Courant algebroids also originates from Drinfeld’s observation [11] that the direct sum \(g \oplus g^*\) (called the Drinfeld double) of a Lie bialgebra \((g, g^*)\) is a canonical quadratic Lie algebra. Extending the construction of the Drinfeld double of a Lie bialgebra to the case of a Lie bialgebroid \((A, A^*)\) is a non-trivial problem. One of the solutions is provided by Liu, Weinstein and Xu [25] in terms of the Courant algebroid structure on \(A \oplus A^*\). Roughly speaking, a Courant algebroid consists of a pseudo-Euclidean vector bundle \((E, \langle\cdot, \cdot\rangle)\) over a smooth manifold \(M\), a Leibniz bracket \(\circ: \Gamma(E) \times \Gamma(E) \to \Gamma(E)\) (known as the Dorfman bracket), and a map \(\rho: E \to TM\) (called the anchor) satisfying several compatible conditions (see Definition 1.1).

Proto-bialgebroids can also be interpreted as split Courant algebroids. Given a proto-bialgebroid \((A, A^*)\), one can obtain a Courant algebroid structure on \(A \oplus A^*\) (see Section 1 for detail). Conversely, to get a proto-bialgebroid out of a Courant algebroid \(E\), one needs an additional assumption — the pseudo-Euclidean vector bundle \(E\) decomposes as the direct sum of two transverse Lagrangian sub-bundles. In other words, when the Courant structure is defined on the Whitney sum \(E = A \oplus A^*\) of a vector bundle \(A\) and its dual, where \(A\) and \(A^*\) are both co-isotropic subbundles in \(E\), we obtain a split Courant algebroid, and thus a proto-bialgebroid structure on \((A, A^*)\). In particular, the Courant algebroid structure on \(E\) can be the double of a Lie bialgebroid, a Lie quasi-bialgebroid, or a quasi-Lie bialgebroid.
Various attempts have been made to understand Courant algebroids. One method is due to Weinstein, Ševera, and Roytenberg [30, 32] — a Courant algebroid can be described as a degree 2 symplectic graded manifold together with a degree 3 Hamiltonian generating function $\Theta$ satisfying $\{\Theta, \Theta\} = 0$, where $\{\cdot, \cdot\}$ is the graded Poisson bracket induced from the graded symplectic structure. This graded Poisson bracket is called big bracket in [23]. The anchor map and the Dorfman bracket of the Courant algebroid $E$ are recovered as derived brackets.

Around the same time, in an unpublished manuscript [1], motivated by an earlier work of Cabras and Vinogradov [5], Alekseev and Xu approached Courant algebroids in terms of Dirac generating operators, an analogue of Kostant’s cubic Dirac operators [24]. Here is a quick sketch — Let $E \rightarrow M$ be a vector bundle endowed with a fiberwise nondegenerate pseudo-metric $\langle \cdot, \cdot \rangle$, and let $C(E)$ be the associated bundle of Clifford algebras. Assume that there exists a bundle of Clifford modules $S$ over the same base manifold $M$, that is, a vector bundle whose fibers are Clifford modules over fibers of $C(E)$. The natural $\mathbb{Z}_2$-grading of $\Gamma(C(E))$ induces a $\mathbb{Z}_2$-grading on the operators on $S$. For example, the multiplication by a function $f \in C^\infty(M)$ is an even operator, while the Clifford action of a section $e \in \Gamma(E)$ is odd. A Dirac generating operator is an odd operator $D$ on $\Gamma(S)$ satisfying the following properties (here and below, $\{\cdot, \cdot\}$ stands for the graded commutator on the space of graded operators on $\Gamma(S)$):

- For all $f \in C^\infty(M)$, the operator $[D, f]$ is the Clifford action of some section of $E$.
- For all $e_1, e_2 \in \Gamma(E)$, the operator $[[D, e_1], e_2]$ is the Clifford action of some section of $E$.
- The square of $D$ is the multiplication by some function on $M$.

From a Dirac generating operator $D$, the derived bracket $e_1 \circ e_2 = [[D, e_1], e_2]$ on $\Gamma(E)$ together with the anchor map $\rho: E \rightarrow TM$ given by $\rho(e)f = 2\langle[D, f], e \rangle$, define a Courant algebroid structure on $E$. Conversely, it is proved loc. cit that for a general Courant algebroid $E$ there exists a Dirac generating operator, acting on a certain spinor bundle of $(E, \langle \cdot, \cdot \rangle)$, which plays exactly the same role as the de Rham differential operator does in the Cabras-Vinogradov’s approach to the standard Courant algebroid.

Part of the motivation behind this work is to better understand the Dirac generating operators of Courant algebroids associated to proto-bialgebroids. As stated earlier that, the proto-bialgebroid structure on $(A, A^*)$ can be encoded in a certain Hamiltonian function $\Theta$ satisfying the classical master equation $\{\Theta, \Theta\} = 0$. According to [23, 29, 31], the function $\Theta$ is the sum of four homogeneous terms

$$\Theta = d_A + d_* + \tau + \phi \in C^\infty(T^*[2]A[1])$$

where $d_A$ and $d_*$ correspond to the skew-symmetric dull algebroid structure on $A$ and $A^*$, respectively. The main purpose of this paper is to prove a quantum analog of this condition.

Here is an outline of our results. Given a rank $n$ vector bundle $A$ over an $m$-dimensional smooth manifold $M$, we consider the vector bundle $E = A \oplus A^*$ which is equipped with the standard pseudo-metric (see (1)). The spin module we take is $S = \wedge^* A \otimes (\wedge^m A^* \otimes \wedge^m T^* M)^{1/2}$, which can be thought of as the space of Berezinian half densities on the graded manifold $A^*[1]$. Given a pair of skew-symmetric dull algebroids $A$ and $A^*$, and two elements $\tau \in \Gamma(\wedge^3 A)$ and $\phi \in \Gamma(\wedge^5 A^*)$, we introduce an operator

$$\bar{D} := d_* + \bar{\partial} + \tau - \iota_\phi: \Gamma(S) \rightarrow \Gamma(S).$$

Here $\bar{d}_*$ and $\bar{\partial}$ come from the skew-symmetric dull algebroid structure on $A^*$ and $A$, respectively, similar to how we define a Batalin-Vilkovisky operator. For detailed explanation of the symbols, see Section 2.1. The operator $\bar{D}$ actually comes from a formula invented by Kosmann-Schwarzbach in [23], called a deriving operator therein. Our first main theorem (Theorem 2.1) declares the
following equivalence of facts:

\((A, A^*)\) forms a proto-bialgebroid \(\iff \bar{D}^2 \in \mathcal{C}^\infty(M) \iff \bar{D}\) is a Dirac generating operator.

Recently, Grützmann, Michel, and Xu [16] studied Weyl quantization of degree 2 symplectic graded manifolds. Given a pseudo-Euclidean vector bundle \((E, \langle \cdot, \cdot \rangle)\) over \(M\), each metric connection \(\nabla\) on \(E\) determines a degree 2 symplectic graded manifold \((T^*M \oplus E[1], \omega\nabla)\). They proved that the Weyl quantization of this degree 2 symplectic graded manifold establishes a bijection between Hamiltonian generating functions and \textit{skew-symmetric} Dirac generating operators. By considering the square of the unique skew-symmetric Dirac generating operator, they also obtain a new Courant algebroid invariant. This new invariant, as a function on the base manifold, is a natural extension of the square norm of the Cartan 3-form of a quadratic Lie algebra.

In our second main result (Theorem 2.6), we give a specific expression of the Dirac generating operator \(\bar{D}\) of a proto-bialgebroid \((A, A^*)\), and prove directly that the square \(\bar{D}^2\) of \(\bar{D}\) is indeed the invariant of the proto-bialgebroid \((A, A^*)\) (without Weyl quantization). When reduced to the case \(\tau = 0\) and \(\phi = 0\), our results recover the conclusions in [7] regarding Dirac generating operators for Lie bialgebroids.

The derived brackets of the Courant algebroids and more generally, metric algebroids [33], play an important role in the generalized complex geometry developed by Hitchin [18] and Gualtieri [17], and double field theory [9, 28], where many remarkable results have been established. We hope our results will be of some use in this subject. Notably, it is shown in the papers [2, 14] that each split Courant algebroid \(E = A \oplus A^*\) corresponds to a multiplicative curved \(\mathcal{L}_\infty\)-algebra structure on \(\Gamma(\Lambda^* A)[2]\).

The paper is organized as follows. Section 1 gives a succinct account of standard facts about Courant algebroids, proto-bialgebroids and Dirac generating operators. The differential operator \(\bar{D}\) is defined in Section 2 and the main theorems are then stated without proofs. Section 3 is devoted to prove the statements in Section 2. Our results are then particularized to a few concrete situations in Section 4.

Conventions, terminologies and notations.

1. \textit{The manifold} \(M\), \textit{the ring} \(\mathcal{C}^\infty(M)\), \textit{and space of vector fields} \(\mathfrak{X}(M)\). We only work with real smooth manifolds, say \(M\). The symbol \(\mathcal{C}^\infty(M)\) denotes the algebra of real valued smooth functions on \(M\); and \(\mathfrak{X}(M) := \Gamma(TM)\) denotes the space of vector fields.

2. \textit{The tensor product} \(\otimes\) stands for \(\otimes_{\mathcal{C}^\infty(M)}\).

3. \textit{Vector bundle} \(A \to M\). A vector bundle \(A \to M\) means a real vector bundle of finite rank. Denote by \(A^*\) the dual vector bundle of \(A\).

4. \textit{Skew-symmetric dull algebroids} \(A\) and \(A^*\). The terminology of skew-symmetric dull algebroids is introduced in [20], which refers to a triple \((A, [\cdot, \cdot]_A, a_A)\) consisting of the following data

\begin{itemize}
  \item a vector bundle \(A \to M\);
  \item a bundle map \(a_A: \Gamma(A) \to \mathfrak{X}(M)\), called the anchor;
  \item an \(\mathbb{R}\)-bilinear and skew-symmetric bracket \([\cdot, \cdot]_A: \Gamma(A) \times \Gamma(A) \to \Gamma(A)\) satisfying the Leibniz rule
  \[ [x, f y]_A = f [x, y]_A + a_A(x)(f) y, \]
\end{itemize}

for all \(x, y \in \Gamma(A)\) and \(f \in \mathcal{C}^\infty(M)\). Moreover, if the Jacobi identity for \([\cdot, \cdot]_A\) holds (that is \([\cdot, \cdot]_A\) is a Lie bracket), then the skew-symmetric dull algebroid \((A, [\cdot, \cdot]_A, a_A)\) is a Lie algebroid.
Similarly, \((A^*, [\cdot, \cdot]_s, a_s)\) denotes a skew-symmetric dull algebroid whose underlying vector bundle is dual to \(A\). We do not presume any compatibility conditions between \((A, [\cdot, \cdot]_A, a_A)\) and \((A^*, [\cdot, \cdot]_s, a_s)\) unless otherwise specified.  

(5) The derivations \(d_A\) and \(d_s\). Given a skew-symmetric dull algebroid \((A, [\cdot, \cdot]_A, a_A)\), it induces a derivation 

\[
d_A : \Gamma((\wedge^\bullet A^*) \rightarrow \Gamma((\wedge^{\bullet+1} A^*))
\]

by 

\[
(d_A \omega)(x_0, x_1, \ldots, x_n) := \sum_{i=0}^{n} (-1)^i a_A(x_i)(\omega(x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n)) + \sum_{0 \leq i < j \leq n} (-1)^{i+j} \omega([x_i, x_j]_A, x_0, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n),
\]

for all \(\omega \in \Gamma((\wedge^n A^*)\) and \(x_0, x_1, \ldots, x_n \in \Gamma(A)\). It follows that \(d_A^2 = 0\) if and only if \((A, [\cdot, \cdot]_A, a_A)\) is a Lie algebroid. The derivation of the skew-symmetric dull algebroid \((A^*, [\cdot, \cdot]_s, a_s)\) is denoted by 

\[
d_s : \Gamma((\wedge^\bullet A) \rightarrow \Gamma((\wedge^{\bullet+1} A)).
\]

(6) Pairings \([\cdot, \cdot]\) and \([\cdot, \cdot]\) of \(A \oplus A^*\). For any \(\xi \in \Gamma(A^*)\), denote by \(\iota_{\xi} : \Gamma((\wedge^\bullet A) \rightarrow \Gamma((\wedge^{\bullet-1} A))\) the standard contraction defined by 

\[
(\iota_{\xi} r)(\eta_1, \eta_2, \ldots, \eta_{n-1}) := r(\xi, \eta_1, \eta_2, \ldots, \eta_{n-1}), \quad \forall r \in \Gamma((\wedge^n A), \eta_1, \ldots, \eta_{n-1} \in \Gamma(A^*).\)

For \(\xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_n \in \Gamma((\wedge^n A^*)\), we define 

\[
\iota_{\xi_1} \wedge \iota_{\xi_2} \wedge \ldots \wedge \iota_{\xi_n} := \iota_{\xi_1} \circ \iota_{\xi_{n-1}} \circ \cdots \circ \iota_{\xi_1} : \Gamma((\wedge^\bullet A) \rightarrow \Gamma((\wedge^{\bullet-n} A).\)

Similarly, for all \(x, x_1, \cdots, x_n \in \Gamma(A)\), we have contractions \(\iota_x : \Gamma((\wedge^\bullet A^*) \rightarrow \Gamma((\wedge^{\bullet-1} A^*))\) and 

\[
\iota_{x_1} \wedge \iota_{x_2} \wedge \ldots \wedge \iota_{x_n} := \iota_{x_1} \circ \iota_{x_{n-1}} \circ \cdots \circ \iota_{x_1} : \Gamma((\wedge^\bullet A) \rightarrow \Gamma((\wedge^{\bullet-n} A^*)\).

We make the following agreement: 

\[
\langle \xi_1 \wedge \ldots \wedge \xi_n \rangle_{x_1 \wedge \ldots \wedge x_n} := \iota_{x_n} \wedge \iota_{x_{n-1}} \cdots \iota_{x_1} (\xi_1 \wedge \ldots \wedge \xi_n) \\
= \iota_{\xi_n} \wedge \iota_{\xi_{n-1}} \cdots \iota_{\xi_1} (x_1 \wedge \ldots \wedge x_n) \\
= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \xi_1(x_{\sigma(1)}) \xi_2(x_{\sigma(2)}) \cdots \xi_n(x_{\sigma(n)}).
\]

There are two pairings on \(A \oplus A^*\) given by for all \(x + \xi, y + \eta \in \Gamma(A \oplus A^*)\), 

\[
\langle x + \xi | y + \eta \rangle := \xi(y) + \eta(x),
\]

and 

\[
\langle x + \xi, y + \eta \rangle := \frac{1}{2} \langle x + \xi | y + \eta \rangle = \frac{1}{2} \xi(y) + \frac{1}{2} \eta(x).
\]

We refer to the second one as the standard pseudo-metric on \(A \oplus A^*\).  

(7) The elements \(\tau\) and \(\phi\). Throughout this paper, the symbol \(\tau\) stands for an element in \(\Gamma(\wedge^3 A)\); while \(\phi\) stands for an element in \(\Gamma((\wedge^3 A^*)\).  

(8) Einstein convention is adopted throughout the paper: \(a^i b_i = \sum_i a^i b_i\).  

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1. Preliminaries

In this preliminary section, we make a succinct introduction to split Courant algebroids, proto-bialgebroids, and Dirac generating operators.
1.1. Split Courant algebroids and proto-bialgebroids. The notion of Courant algebroid was first introduced in [25].

**Definition 1.1.** A Courant algebroid is a vector bundle $E \to M$ equipped with three structures: (1) a pseudo-metric on $E$, i.e., a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\Gamma(E)$; (2) a bilinear operation $\circ$ on $\Gamma(E)$ called Dorfman bracket; and (3) a bundle map $\rho : E \to TM$ called anchor. These structure maps are subject to the following axioms:

1. $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$, for all $e_1, e_2, e_3 \in \Gamma(E)$;
2. $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$, for all $e_1, e_2 \in \Gamma(E)$;
3. $e_1 \circ (f e_2) = f(e_1 \circ e_2) + \rho(e_1)(f) e_2$, for all $e_1, e_2 \in \Gamma(E)$, $f \in \mathcal{C}^\infty(M)$;
4. $e \circ e = \mathcal{D}(e,e)$, for all $e \in \Gamma(E)$;
5. $\langle \mathcal{D}(f,e), e \rangle = \frac{1}{2} \rho(e)(f)$.

This paper is devoted to study a particular type of Courant algebroids commonly known as split Courant algebroids. In fact, they are equivalent to the objects of proto-bialgebroids. Let us firstly clarify this correspondence.

A pseudo-Euclidean vector bundle is called split, if it is isomorphic to the Whitney sum $A \oplus A^*$ for some vector bundle $A \to M$, equipped with the standard pseudo-metric given by

$$\langle x + \xi, y + \eta \rangle := \frac{1}{2} \langle x + \xi | y + \eta \rangle = \frac{1}{2} \xi(y) + \frac{1}{2} \eta(x), \quad \forall x + \xi, y + \eta \in \Gamma(A \oplus A^*).$$

(1)

A Courant algebroid is called split, if its underlying pseudo-Euclidean vector bundle is split.

Assume that $E = A \oplus A^*$ is a split Courant algebroid. Then the anchor $\rho : E \to TM$ is decomposed into $a_A : A \to TM$ and $a_* : A^* \to TM$ by

$$\rho(x + \xi) = a_A(x) + a_\ast(\xi),$$

for all $x \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$. The restriction of the Dorfman bracket on $\Gamma(A)$ determines a skew-symmetric dull algebroid $(A, [\cdot, \cdot]_A, a_A)$ and an element $\phi \in \Gamma(\wedge^2 A^*)$ by

$$x \circ y = [x, y]_A - t_y t_x \phi,$$

for all $x, y \in \Gamma(A)$. Similarly, the restriction of the Dorfman bracket on $\Gamma(A^*)$ determines skew-symmetric dull algebroid $(A^*, [\cdot, \cdot]_*, a_\ast)$ and an element $\tau \in \Gamma(\wedge^2 A^*)$ by

$$\xi \circ \eta = -t_\eta t_\xi \tau + [\xi, \eta]_\ast,$$

for all $\xi, \eta \in \Gamma(A^*)$. The Dorfman bracket $\circ$ on $\Gamma(A \oplus A^*)$ takes the form

$$(x + \xi) \circ (y + \eta) = ([x, y]_A + L_\xi y - t_\eta (d_\ast(x)) - t_\eta (d_A(\xi)) - t_\eta t_x \phi).$$

(2)

The pair $(A, A^*)$ of skew-symmetric dull algebroids is subject to several compatibility conditions induced from the axioms of the Courant algebroid $A \oplus A^*$. (See [29] for details and also [23] in terms of big bracket.) These conditions are exactly the axioms of proto-bialgebroids, which we summarize below.

**Definition 1.2** ([23,29]). A proto-bialgebroid consists of the following data:

- two skew-symmetric dull algebroids in duality, $(A, [\cdot, \cdot]_A, a_A)$ and $(A^*, [\cdot, \cdot]_*, a_\ast)$,
- an element $\tau \in \Gamma(\wedge^2 A)$,
- an element $\phi \in \Gamma(\wedge^2 A^*)$.

\[^{1}\text{Note that the definition of } \mathcal{D} \text{ in different literature may differ by a constant multiple.}\]
They are subject to the following axioms: for all \( x, y, z \in \Gamma(A) \), \( \xi, \eta, \chi \in \Gamma(A^*) \),

1. The Jacobi identity of \([\cdot, \cdot]_A\) is controlled by \( \phi \) and \( d_\ast \), i.e.,
\[
([x, y]_A, z)_A + ([y, z]_A, x)_A + ([z, x]_A, y)_A = d_\ast(\phi([x, y, z])) + \iota_\phi(d_\ast(x \wedge y \wedge z)),
\]
where the map \( d_\ast : \Gamma(\Lambda^k A) \to \Gamma(\Lambda^{k+1} A) \) is the derivation arising from the skew-symmetric dull algebroid \((A^*, [\cdot, \cdot]_s, a_\ast)\).

2. The Jacobi identity of \([\cdot, \cdot]_s\) is controlled by \( \tau \) and \( d_A \), i.e.,
\[
([\xi, \eta]_s, \chi)_s + ([\eta, \chi]_s, \xi)_s + ([\chi, \xi]_s, \eta)_s = d_A(\tau(\xi, \eta, \chi)) + \iota_\tau(d_A(\xi \wedge \eta \wedge \chi)),
\]
where the map \( d_A : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{k+1} A^*) \) is the derivation arising from the skew-symmetric dull algebroid \((A, [\cdot, \cdot]_A, a_A)\).

3. The skew-symmetric dull algebroids \((A, [\cdot, \cdot]_A, a_A, a_\ast, \tau, \phi)\) and \((A^*, [\cdot, \cdot]_s, a_\ast, \phi)\) are compatible in the sense that
\[
d_\ast([x, y]_A) = [d_\ast(x), y]_A + [x, d_\ast(y)]_A + \iota_\phi(d_\ast(x \wedge y \wedge z)).
\]

4. The element \( \phi \in \Gamma(\Lambda^3 A^*) \) is \( d_\ast \)-closed, i.e., \( d_\ast(\phi) = 0 \).

5. The element \( \tau \in \Gamma(\Lambda^3 A) \) is \( d_\ast \)-closed, i.e., \( d_\ast(\tau) = 0 \).

Such a proto-bialgebroid will be denoted by \((A, [\cdot, \cdot]_A, [\cdot, \cdot]_s, a_A, a_\ast, \tau, \phi)\).

**Remark 1.3.** Assume that \((A, [\cdot, \cdot]_A, [\cdot, \cdot]_s, a_A, a_\ast, \tau, \phi)\) is a proto-bialgebroid.

- If \( \tau \in \Gamma(\Lambda^3 A) \) vanishes, then the triple \((A^*, [\cdot, \cdot]_s, a_\ast)\) is a Lie algebroid and the six-tuple \((A^*, [\cdot, \cdot]_s, a_\ast, \phi)\) is known as a quasi-Lie bialgebroid.
- If \( \phi \in \Gamma(\Lambda^3 A^*) \) vanishes, then the triple \((A, [\cdot, \cdot]_A, a_A)\) is a Lie algebroid and the six-tuple \((A, [\cdot, \cdot]_A, a_A, a_\ast, \tau)\) is called a Lie quasi-bialgebroid.
- If both \( \tau \) and \( \phi \) vanish, then \( A \) and \( A^* \) form a Lie bialgebroid.

### 1.2. Dirac generating operators.
We now briefly recall the approach to Courant algebroids via Dirac generating operators [1, 16]. Given a pseudo-Euclidean vector bundle \((E, \langle \cdot, \cdot \rangle)\) over \( M \), let \( C(E) \to M \) be the associated bundle of Clifford algebras with the generating relation \( e_1 \otimes e_2 + e_2 \otimes e_1 = 2(\epsilon_1, \epsilon_2) \), for all \( p \in M \) and all \( e_1, e_2 \in C(E)_p \). Assume that there exists a smooth vector bundle \( S \to M \) whose fiber \( S_p \) over every point \( p \in M \) is the spin module of the Clifford algebra \( C(E)_p \). Assume further that \( S \) is \( \mathbb{Z}_2 \)-graded, i.e., \( S = S^0 \oplus S^1 \). An operator \( D \) on \( \Gamma(S) \) is said to be even (resp. odd) if \( D(S^i) \subset S^i \) (resp. \( D(S^i) \subset S^{i+1} \)). Here \( i \in \mathbb{Z}_2 \). If \( D_1 \) and \( D_2 \) are operators of degree \( i_1 \) and \( i_2 \), respectively, then their graded commutator is given by \( [D_1, D_2] = D_1 \circ D_2 - (-1)^{i_1 i_2} D_2 \circ D_1 \).

**Definition 1.4 ([1]).** A Dirac generating operator for a pseudo-Euclidean vector bundle \((E, \langle \cdot, \cdot \rangle)\) is an odd operator \( D \) on \( \Gamma(S) \) satisfying the following conditions.

(a) For all \( f \in C^\infty(M) \), we have \([D, f] \in \Gamma(E)\). This means that the operator \([D, f] \) is the Clifford action of some section of \( E \) on \( \Gamma(S) \).

(b) For all \( e_1, e_2 \in \Gamma(E) \), we have \([[[D, e_1], e_2], e_3] \in \Gamma(E)\).

(c) The square of \( D \) is the multiplication by some smooth function on \( M \), i.e., \( D^2 \in C^\infty(M) \).

**Theorem 1.5 ([1]).** Let \( D \) be a Dirac generating operator for a pseudo-Euclidean vector bundle \((E, \langle \cdot, \cdot \rangle)\) over \( M \). Then there is a canonical Courant algebroid structure on \( E \), whose anchor and Dorfman bracket are defined respectively by
\[
\rho(e)(f) = [[[D, f], e], e_1 \circ e_2 = [[[D, e_1], e_2], e_3],
\]
where \( e, e_1, e_2 \in \Gamma(E) \) and \( f \in C^\infty(M) \).
It is natural to ask which kind of Dirac generating operators generates split Courant algebroids. This question is our main concern of the paper.

2. Dirac generating operators of split Courant algebroids

In this section, we characterize Dirac generating operators of split Courant algebroids. We mainly follow the approach developed in [7], where Dirac generating operators of Courant algebroids arising from Lie bialgebroids are considered.

2.1. General settings and the first main theorem. Let $(A, \cdot, \cdot)_A, a_A)$ be a skew-symmetric dull algebroid, and $B$ a vector bundle over the same base manifold $M$. By saying an $A$-connection on $B$, we mean an $\mathbb{R}$-bilinear map

$$\nabla_f b = f \nabla b$$

satisfying

$$\nabla_x b = a_A(x)(f)b + f \nabla b,$$

for all $x \in \Gamma(A), b \in \Gamma(B), f \in C^\infty(M)$. Such an $A$-connection determines an operator called covariant derivative $d^B_A : \Gamma(\wedge^* A^*) \otimes \Gamma(B) \to \Gamma(\wedge^{*+1} A^*) \otimes \Gamma(B)$ satisfying

$$d^B_A(\omega \otimes b) = (d_A(\omega)) \otimes b + (-1)^k \omega \wedge (d^B_A(b)),$$

for all $\omega \in \Gamma(\wedge^{k} A^*), b \in \Gamma(B)$. Here and in the sequel $\otimes$ stands for $\otimes_{C^\infty(M)}$. However, an element in $\Gamma(\wedge^* A^*) \otimes \Gamma(B)$ is still denoted by $\omega \otimes b$ rather than $\omega \otimes b$.

Let $(A, \cdot, \cdot)_A, a_A)$ and $(A^*, \cdot, \cdot)_A, a_A)$ be a pair of skew-symmetric dull algebroids, where $A$ is a rank $n$ vector bundle over an $m$-dimensional manifold $M$. Inspired by [13], the line bundle $\wedge^n A^* \otimes \wedge^n T^* M$ admits a canonical $A^*$-connection. More precisely, a section $\xi \in \Gamma(A^*)$ “acts” on $\Gamma(\wedge^n A^* \otimes \wedge^m T^* M)$ by Lie derivatives

$$L_{\xi}(\xi_1 \wedge \ldots \wedge \xi_n) = \sum_{i=1}^n (\xi_1 \wedge \ldots \wedge [\xi_i, \xi]_A \wedge \ldots \wedge \xi_n) \otimes \mu + \xi_1 \wedge \ldots \wedge \xi_n \otimes L_{\xi(\xi)} \mu.$$

The square root $\mathcal{L} = (\wedge^n A^* \otimes \wedge^m T^* M)^{1/2}$, if exists, also admits an $A^*$-connection. Denote the associated covariant derivative by

$$\tilde{d}_s : \Gamma(\wedge^k A) \otimes \Gamma \mathcal{L} \to \Gamma(\wedge^{k+1} A) \otimes \Gamma \mathcal{L}.$$

Similarly, if the square root $(\wedge^n A \otimes \wedge^m T^* M)^{1/2}$ exists, it also admits an $A$-connection, and thus a covariant derivative

$$\Gamma(\wedge^k A^* \otimes (\wedge^n A \otimes \wedge^m T^* M)^{1/2}) \to \Gamma(\wedge^{k+1} A^* \otimes (\wedge^n A \otimes \wedge^m T^* M)^{1/2}).$$

Using the isomorphisms of vector bundles

$$\wedge^k A^* \cong \wedge^k A^* \otimes \wedge^n A \otimes \wedge^n A^* \cong \wedge^{n-k} A \otimes \wedge^n A^*$$

and

$$\wedge^n A^* \otimes (\wedge^n A \otimes \wedge^m T^* M)^{1/2} \cong (\wedge^n A^* \otimes \wedge^m T^* M)^{1/2},$$

one has a family of isomorphisms

$$\beta_k : \wedge^k A^* \otimes (\wedge^n A \otimes \wedge^m T^* M)^{1/2} \cong \wedge^{n-k} A \otimes (\wedge^n A^* \otimes \wedge^m T^* M)^{1/2}.$$

Therefore, one ends up with a derivation

$$\tilde{\mathcal{L}} : \Gamma(\wedge^k A) \otimes \Gamma \mathcal{L} \to \Gamma(\wedge^{k-1} A) \otimes \Gamma \mathcal{L}.$$
In the sequel, we fix two elements $\tau \in \Gamma(\wedge^3 A)$ and $\phi \in \Gamma(\wedge^3 A^*)$. They give rise to two operators

$$
\tau : \Gamma(\wedge^k A) \otimes \Gamma L \to \Gamma(\wedge^{k+3} A) \otimes \Gamma L, \quad \tau(r \otimes l) := (\tau \wedge r) \otimes l,
$$

and

$$
t_\phi : \Gamma(\wedge^k A) \otimes \Gamma L \to \Gamma(\wedge^{k-3} A) \otimes \Gamma L, \quad t_\phi(r \otimes l) := (\iota_\phi r) \otimes l.
$$

We wish to find a simple condition so that $\tau$ and $\phi$ together with the two skew-symmetric dull algebroids $A$ and $A^*$ form a proto-bialgebroid. For this purpose, the spinor bundle we choose is $S = \wedge^* A \otimes \mathcal{L}$. For the convenience of description, we introduce the odd operator

$$
\check{D} = \check{d}_x + \check{D} + \tau - t_\phi : \Gamma(\wedge^* A) \otimes \Gamma L \to \Gamma(\wedge^* A) \otimes \Gamma L.
$$

Here is our first main result, which can be regarded as an enhancement of [7, Theorem 3.1].

**Theorem 2.1.** Let $(A, [\cdot, \cdot]_A, a_A)$ and $(A^*, [\cdot, \cdot]_s, a_s)$ be two dual skew-symmetric dull algebroids. Given $\tau \in \Gamma(\wedge^3 A)$ and $\phi \in \Gamma(\wedge^3 A^*)$, let $D$ be the operator defined in (3). Then the following three statements are equivalent:

1. The septuple $(A, [\cdot, \cdot]_A, \tau, a_A, a_s, \phi)$ is a proto-bialgebroid;
2. The square of the operator $D$ satisfies $D^2 \in C^\infty(M)$;
3. The operator $\check{D} : \Gamma(\wedge^* A) \otimes \Gamma L \to \Gamma(\wedge^* A) \otimes \Gamma L$ is a Dirac generating operator for the split pseudo-Euclidean vector bundle $(\wedge^* A, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the standard metric defined in (1).

Suppose that $\check{D}$ is a Dirac generating operator. The function $\check{D}^2 \in C^\infty(M)$ is called the characteristic function of the proto-bialgebroid $(A, [\cdot, \cdot]_A, \tau, a_A, a_s, \phi)$ or of the associated split Courant algebroid $(\wedge^* A, \langle \cdot, \cdot \rangle, \circ, \rho, \tau, \phi)$. The proof of Theorem 2.1 is deferred to Section 3.

### 2.2. Modular elements and the second main theorem.

Throughout this section, we assume that $(A, [\cdot, \cdot]_A, a_A)$ and $(A^*, [\cdot, \cdot]_s, a_s)$ are skew-symmetric dull algebroids in duality. Denote by $d_A : \Gamma(\wedge^* A^*) \to \Gamma(\wedge^{*+1} A^*)$ and $d_s : \Gamma(\wedge^* A) \to \Gamma(\wedge^{*+1} A)$ the associated derivations. The Lie derivative along any element $x \in \Gamma(A)$ is an $\mathbb{R}$-linear derivation

$$
L_x : \Gamma(\wedge^* A \otimes \wedge^\circ A^*) \to \Gamma(\wedge^* A \otimes \wedge^\circ A^*)
$$

induced by $L_x y := [x, y]_A$ for all $y \in \Gamma(A)$ and by a Cartan type formula $L_x \xi := d_A(\iota_x \xi) + \iota_x (d_A \xi)$. Similarly, the Lie derivative along any element $\xi \in \Gamma(A^*)$ is an $\mathbb{R}$-linear derivation

$$
L_\xi : \Gamma(\wedge^* A \otimes \wedge^\circ A^*) \to \Gamma(\wedge^* A \otimes \wedge^\circ A^*)
$$

induced by $L_\xi \eta := [\xi, \eta]_s$ for all $\eta \in \Gamma(A^*)$ and by a Cartan type formula $L_\xi x := d_s(\iota_\xi x) + \iota_\xi (d_s x)$.

**Lemma 2.2** ([7, Proposition 4.6]). For all $x \in \Gamma(A), \xi \in \Gamma(A^*)$ and $f \in C^\infty(M)$, we have

$$
\langle d_s([f, x]_A) - [d_s(f), x]_A + [f, d_s(x)]_A, \xi \rangle = \langle (L_d f + L_d x) x | \xi \rangle = \langle [L_x, \xi] - L_x \xi, f \rangle,
$$

where $x \circ \xi := -\iota_\xi (d_s x) + L_x \xi$.

Let us choose a nowhere-vanishing section $\Omega \in \Gamma(\wedge^n A^*)$ and let $V \in \Gamma(\wedge^n A)$ be the dual of $\Omega \in \Gamma(\wedge^n A^*)$ in the sense that $\langle \Omega | V \rangle = 1$.

**Lemma 2.3** ([7, Lemmas 4.1 and 4.3]). For all $x \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$, we have

$$
L_x \Omega = -\partial(x) \Omega, \quad L_x V = \partial(x) V, \quad L_\xi \Omega = \partial(\xi) \Omega, \quad (L_x \Omega) \otimes V = -\Omega \otimes (L_x V), \quad (L_\xi \Omega) \otimes V = -\Omega \otimes (L_\xi V).
$$
The elements $\Omega$ and $V$ induce two isomorphisms:

$$
\begin{align*}
\Omega^k &: \Gamma(\wedge^k A) \to \Gamma(\wedge^{n-k} A^*), \quad r \mapsto \iota_r \Omega, \\
V^k &: \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{n-k} A), \quad \omega \mapsto \iota_\omega V,
\end{align*}
$$

which are essentially inverse to each other:

$$
\begin{align*}
V^k \circ \Omega^k &= (-1)^{k(n-1)} \text{id}_{\Gamma(\wedge^k A)}, \\
\Omega^k \circ V^k &= (-1)^{k(n-1)} \text{id}_{\Gamma(\wedge^k A^*)}.
\end{align*}
$$

Consider the operator $\partial$ induced from $d_A$ by the isomorphism $V^k$:

$$
\begin{align*}
\Gamma(\wedge^k A^*) &\xrightarrow{V^k} \Gamma(\wedge^{n-k} A) \\
\downarrow \text{\hspace{1cm}} &\downarrow \partial \\
\Gamma(\wedge^{k+1} A^*) &\xrightarrow{V^k} \Gamma(\wedge^{n-k-1} A).
\end{align*}
$$

In other words, for all $\omega \in \Gamma(\wedge^k A^*)$, we have

$$
- V^k d_A \omega = (-1)^k \partial V^k \omega,
$$

which implies that

$$
\partial r = -(-1)^{n-k} \left( V^k \circ d_A \circ (V^k)^{-1} \right) r = -(-1)^{n(k+1)} (V^k \circ d_A \circ \Omega^k) r, \quad \forall r \in \Gamma(\wedge^k A).
$$

The operator $\partial$ is not a derivation, but a second order differential operator, called a Batalin-Vilkovisky operator for the skew-symmetric dull algebroid $A$. For any $r_1 \in \Gamma(\wedge^k A)$ and $r_2 \in \Gamma(\wedge^l A)$, we have the BV relation

$$
[r_1, r_2]_A = (-1)^k \partial (r_1 \wedge r_2) - (-1)^k (\partial r_1) \wedge r_2 - r_1 \wedge (\partial r_2). \quad (10)
$$

Similarly, we also have the Batalin-Vilkovisky operator $\partial_s$ dual to $d_s$:

$$
\begin{align*}
\Gamma(\wedge^{n-k} A) &\xleftarrow{V^k} \Gamma(\wedge^k A^*) \\
\downarrow \text{\hspace{1cm}} &\downarrow \partial_s \\
\Gamma(\wedge^{n-k+1} A) &\xrightarrow{V^k} \Gamma(\wedge^{k-1} A^*).
\end{align*}
$$

Explicitly, we have for all $\omega \in \Gamma(\wedge^k A^*)$,

$$
d_s V^k \omega = (-1)^k V^k \partial_s \omega.
$$

The BV relation for $\partial_s$ reads

$$
[\Xi_1, \Xi_2]_s = (-1)^l \partial_s (\Xi_1 \wedge \Xi_2) - (-1)^l (\partial_s \Xi_1) \wedge \Xi_2 - \Xi_1 \wedge (\partial_s \Xi_2),
$$

for all $\Xi_1, \Xi_2 \in \Gamma(\wedge^l A^*)$.

**Remark 2.4.** One should be cautious that in general $\partial^2 \neq 0$. In fact, from Equation (10) one can find for all $x, y, z \in \Gamma(A)$,

$$
\partial^2 (x \wedge y) = \left[ \partial(x, y)_A + [x, \partial(y)]_A - \partial([x, y]_A),
$$

and

$$
\partial^2 (x \wedge y \wedge z) = \frac{1}{3} \partial([x \wedge y, z]_A) + \frac{1}{3} \partial([y \wedge z, x]_A) + \frac{1}{3} \partial([z \wedge x, y]_A) \\
+ \frac{1}{3} \partial([x, \partial(y)]_A) + \frac{1}{3} \partial([y, \partial(z)]_A) + \frac{1}{3} \partial([z, \partial(x)]_A) \\
+ [\partial(x), y]_A z + [\partial(y), z]_A x + [\partial(z), x]_A y.
$$

In fact, if $\partial^2 = 0$, then $A$ is a Lie algebroid.
Definition-Proposition 2.5. Let $s \in \Gamma(\wedge^n T^* M)$ be a volume form of $M$, $\Omega \in \Gamma(\wedge^n A^*)$ a nowhere-vanishing section, and $V \in \Gamma(\wedge^n A)$ be its dual such that $\langle \Omega|V \rangle = 1$.

(1) There exists a unique $X_0 \in \Gamma(A)$, called the modular element of the skew-symmetric dull algebroid $A^*$, such that

$$L_\xi(\Omega \otimes s) = (L_\xi \Omega) \otimes s + \Omega \otimes L_{a_A(\xi)} s = \langle \xi|X_0\rangle \Omega \otimes s, \quad \forall \xi \in \Gamma(A^*).$$

(12)

(2) There exists a unique $\xi_0 \in \Gamma(A^*)$, called the modular element of the skew-symmetric dull algebroid $A$, such that

$$L_x(s \otimes V) = (L_{a_A(x)} s) \otimes V + s \otimes L_x V = \langle \xi_0|x\rangle s \otimes V, \quad \forall x \in \Gamma(A).$$

(13)

When $(A, A^*)$ is a Lie bialgebroid, both $X_0$ and $\xi_0$ are Chevalley-Eilenberg 1-cocycles, called modular cocycles. Their cohomology classes are called modular classes [13].

Our second main result gives expression of the characteristic function $\bar{D}^2$ when the operator $\bar{D}$ in (3) is a Dirac generating operator.

Theorem 2.6. Assume that $(A,[\cdot,\cdot],A^*,[\cdot,\cdot],a_A,a_*,\tau,\phi)$ is a proto-bialgebroid such that the line bundle $\wedge^n A^* \otimes \wedge^n T^* M$ of the graded manifold $A^*[1]$ admits a square root $\mathcal{L}$. Then the Dirac generating operator $\bar{D}$ has the form

$$\bar{D} = d + \frac{1}{2}X_0 - \partial + \frac{1}{2}i\xi_0, \quad \phi, \quad \Gamma(\wedge^* A) \otimes \Gamma \mathcal{L} \rightarrow \Gamma(\wedge^* A) \otimes \Gamma \mathcal{L}.$$ (14)

Moreover, the characteristic function

$$\bar{f} := \bar{D}^2 = \frac{1}{4} \langle \xi_0|X_0\rangle - \frac{1}{2} \partial(X_0) - \langle \tau|\phi \rangle,$$ (15)

is an invariant of this proto-bialgebroid, which can be determined by any of the following two equations:

(a) $L_{X_0}(\Omega \otimes s) = 4(\bar{f} + \langle \tau|\phi \rangle) \Omega \otimes s$;

(b) $L_{\xi_0}(s \otimes V) = 4(\bar{f} + \langle \tau|\phi \rangle) s \otimes V$.

We postpone the proof to Section 3. In particular, when both $\tau$ and $\phi$ vanishes, we rediscover the Dirac generating operator and characteristic function of Lie bialgebroids.

Corollary 2.7 ([7, 16]). Assume that $(A,[\cdot,\cdot],A^*,[\cdot,\cdot],a_A,a_*)$ is a Lie bialgebroid. Then its Dirac generating operator $\bar{D}$ has the form

$$\bar{D} = \bar{d} + \bar{\partial} = d + \frac{1}{2}(X_0 + i\xi_0).$$

The characteristic function

$$\bar{f} = \frac{1}{4} \langle \xi_0|X_0\rangle - \frac{1}{2} \partial(X_0)$$

is an invariant of this Lie bialgebroid $(A, A^*)$.

3. Proofs of main theorems

Let $(A,[\cdot,\cdot],A,a_A)$ and $(A^*,[\cdot,\cdot],a_*)$ be two skew-symmetric dull algebroids in duality. We also fix two elements $\tau \in \Gamma(\wedge^3 A)$ and $\phi \in \Gamma(\wedge^3 A^*)$. All other notations are specified as before.
3.1. Some preparatory work.

Lemma 3.1. For all $s \in \Gamma(\wedge^k A)$ and $\omega \in \Gamma(\wedge^{n-k} A^*)$ satisfying $s = V^2(\omega) = \iota_\omega V$, we have
\[
(\partial^2 - d_\phi \circ \iota_\phi - \iota_\phi \circ d_\phi - \frac{1}{2} \iota_{X_0} \phi - \frac{1}{2} \iota_{dA(\xi_0)}) (s) = -V^2 \left(d_A^2 + L_\phi - \partial_\phi (\phi) + \frac{1}{2} \iota_{X_0} \phi + \frac{1}{2} d_A(\xi_0)\right) \omega,
\]
where $L_\phi \omega = [\phi, \omega]_s$.

Proof. Firstly, using (9) twice, we have
\[
\partial^2(s) = \partial^2(V^2(\omega)) = -V^4(d_A^2 \omega).
\]
Direct computations show that
\[
\iota_{X_0} \phi \circ s = \iota_{X_0} \phi V^2(\omega) = \iota_{(\iota_{X_0} \phi)} \wedge \omega V = V^2((\iota_{X_0} \phi) \wedge \omega),
\]
and
\[
\iota_{dA(\xi_0)} \circ s = \iota_{dA(\xi_0)} V^2(\omega) = \iota_{(dA(\xi_0))} \wedge \omega V = V^2(dA(\xi_0) \wedge \omega).
\]
Meanwhile,
\[
(d_\phi \circ \iota_\phi + \iota_\phi \circ d_\phi)(s) = (d_\phi \circ \iota_\phi + \iota_\phi \circ d_\phi)(V^2(\omega))
= d_\phi(V^2(\omega \wedge \phi)) + \iota_\phi(-1)^{n-k}V^2(\partial_\phi \omega)
= d_\phi V^2(\omega \wedge \phi) + (-1)^{n-k} \iota_\phi \partial_\phi \omega
= -(-1)^{n-k} V^2(\partial_\phi (\omega) \wedge \phi) + (-1)^{n-k} V^2((\partial_\phi \omega) \wedge \phi)
= -V^2(\partial_\phi (\omega) \wedge \phi) + \phi \wedge (\partial_\phi \omega))
= V^2(L_\phi - \partial_\phi (\phi) \omega),
\]
where the final equality follows from the BV relation (11). Combing these four equations, we conclude the proof. \hfill \qed

We introduce two operators $K$ and $L$.
\[
K(x, y) : \Gamma(A) \to \Gamma(A), \quad K(x, y)(z) := \iota_\phi(d_\phi(x) \wedge y \wedge z) - \iota_\phi(x \wedge d_\phi(y) \wedge z),
\]
and $L(\xi, \eta) : \Gamma(A^*) \to \Gamma(A^*)$, $L(\xi, \eta)(\chi) := \iota_{\tau(\xi, \chi)}(dA(\xi)) + \iota_{\tau(\omega, \xi)}(dA(\eta))$,
where $x, y, z \in \Gamma(A), \xi, \eta, \chi \in \Gamma(A^*)$.

Lemma 3.2. For $K$ and $L$ as defined above, we have
\[
\text{tr}(K(x, y)) = -2\langle d_\phi(x)|_\tau \phi \rangle + 2\langle d_\phi(y)|_\tau \phi \rangle,
\]
\[
\text{tr}(L(\xi, \eta)) = -2\langle dA(\xi)|_\tau \tau \rangle + 2\langle dA(\eta)|_\tau \tau \rangle,
\]
for all $x, y \in \Gamma(A), \xi, \eta \in \Gamma(A^*)$.

Proof. Let $\{\xi^1, \ldots, \xi^n\}$ be a local basis of $\Gamma(A)$ and $\{\theta_1, \ldots, \theta_n\}$ be the dual basis of $\Gamma(A^*)$. Then we have
\[
\text{tr}(K(x, y)) = \langle K(x, y)(\xi^n)|_\theta_n \rangle = -\langle \iota_{\theta_n}(d_\phi(x))|_\tau \phi \rangle + \langle \iota_{\theta_n}(d_\phi(y))|_\tau \phi \rangle = -2\langle d_\phi(x)|_\tau \phi \rangle + 2\langle d_\phi(y)|_\tau \phi \rangle.
\]
The second equation for $\text{tr}(L(\xi, \eta))$ can be similarly examined. \hfill \qed
Lemma 3.3. Assume that \((A, \cdot, \cdot, A, \cdot, \cdot, a_A, a_\star, \tau, \phi)\) is a proto-bialgebroid. Then
\[
(d_A(\xi_0)(x, y))\Omega \otimes s = (L_{\phi(x,y)}\Omega) \otimes s - \Omega \otimes L_{a_\star(\phi(x,y))}(s) + 2\left(\langle d_s(x)|a_\star y\phi \rangle - \langle d_s(y)|a_\star x\phi \rangle\right)\Omega \otimes s,
\]
\[
(d_s(X_0)(\xi_0, \eta))s \otimes V = s \otimes (L_{\tau(\xi, \eta)}V) - (L_{a_A(\tau(\xi, \eta))} s) \otimes V + 2\left(\langle d_A(\xi)|a_\star \eta \tau \rangle - \langle d_A(\eta)|a_\star \xi \tau \rangle\right)s \otimes V,
\]
for all \(x, y \in \Gamma(A), \xi, \eta \in \Gamma(A^*)\).

Proof. We first prove
\[
(d_A(\xi_0)(x, y))s \otimes V = -(L_{[x,y], A} - L_xL_y + L_yL_x)(s \otimes V).
\]
In fact, by Equations (12) and (13), we have
\[
(L_{[x,y], A} - L_xL_y + L_yL_x)(s \otimes V) = (\langle \xi_0[xy], A \rangle s \otimes V - L_x(\langle \xi_0|y \rangle s \otimes V + L_y(\langle \xi_0|x \rangle s \otimes V)
\]
\[
= \langle \xi_0[xy], A \rangle s \otimes V - a_A(x)(\langle \xi_0|y \rangle s \otimes V - \langle \xi_0|y \rangle \langle \xi_0|x \rangle s \otimes V
\]
\[
+ a_A(y)(\langle \xi_0|x \rangle s \otimes V + \langle \xi_0|x \rangle \langle \xi_0|y \rangle s \otimes V
\]
\[
= \left(\langle \xi_0[xy], A \rangle - a_A(x)(\langle \xi_0|y \rangle) + a_A(y)(\langle \xi_0|x \rangle)\right)s \otimes V
\]
\[
= -(d_A(\xi_0)(x, y))s \otimes V.
\]
Since for all \(x, y \in \Gamma(A)\)
\[
L_{[x,y], A} - L_xL_y + L_yL_x = L_{\phi(x,y)} + K(x,y): \Gamma(A^*) \rightarrow \Gamma(A^*),
\]

it follows that
\[
\frac{(d_A(\xi_0))(x,y)(\Omega \otimes s) \otimes V}{(17)}
\]
\[
= \Omega \otimes (L_{[x,y], A} - L_xL_y + L_yL_x)(s \otimes V)
\]
\[
= \Omega \otimes ((L_{a_A(\phi(x,y))} - L_{a_A(x)}L_{a_A(y)} + L_{a_A(y)}L_{a_A(x)}))s \otimes V
\]
\[
- \Omega \otimes s \otimes ((L_{[x,y], A} - L_xL_y + L_yL_x)V)
\]
\[
= \Omega \otimes (L_{a_A(\phi(x,y))} s) \otimes V - \Omega \otimes s \otimes ((L_{\phi(x,y)} + K(x,y))V)
\]
\[
= \Omega \otimes (L_{a_A(\phi(x,y))} s) \otimes V + (L_{\phi(x,y)}V) \otimes s \otimes V - \text{tr}(K(x,y))\Omega \otimes s \otimes V
\]
\[
\frac{\text{Lemma 3.2}}{(8)}
\]
\[
= \Omega \otimes (L_{a_A(\phi(x,y))} s) \otimes V + (L_{\phi(x,y)}V) \otimes s \otimes V
\]
\[
- (\langle -2\langle d_s(x)|a_\star y \phi \rangle + 2\langle d_s(y)|a_\star x \phi \rangle\rangle \Omega \otimes s \otimes V,
\]
Thus, we have
\[
(d_A(\xi_0)(x, y))\Omega \otimes s = (L_{\phi(x,y)}\Omega) \otimes s - \Omega \otimes (L_{a_A(\phi(x,y))} s) + 2\left(\langle d_s(x)|a_\star y \phi \rangle - \langle d_s(y)|a_\star x \phi \rangle\right)\Omega \otimes s.
\]
The second equation (16) can be verified similarly and thus omitted. \(\square\)

Lemma 3.4. Assume that \((A, \cdot, \cdot, A, \cdot, \cdot, a_A, a_\star, \tau, \phi)\) is a proto-bialgebroid. Then
\[
\partial(\tau) = \frac{1}{2}L_{\xi_0}\tau + \frac{1}{2}d_s(X_0),
\]
\[
\partial_s(\phi) = \frac{1}{2}L_{\xi_0}\phi + \frac{1}{2}d_A(\xi_0).
\]

Proof. First of all, by Equation (13), we have
\[
(i_{\xi_0}\tau)(\xi, \eta)s \otimes V = (L_{a_A(\tau(\xi, \eta))} s) \otimes V + s \otimes (L_{\tau(\xi, \eta)}V).
\]
Using Equation (10), we obtain
\[
\partial(\tau) = \tau_1 \wedge [\tau_2, \tau_3]_A + [\tau_1, \tau_3]_A \wedge \tau_2 - [\tau_1, \tau_2]_A \wedge \tau_3
+ \partial(\tau_1) \tau_2 \wedge \tau_3 - \partial(\tau_2) \tau_1 \wedge \tau_3 + \partial(\tau_3) \tau_1 \wedge \tau_2.
\]

Hence,
\[
\partial(\tau)(\xi, \eta)s \otimes V \overset{(5)}{=} \left( \tau_1 \wedge [\tau_2, \tau_3]_A + [\tau_1, \tau_3]_A \wedge \tau_2 - [\tau_1, \tau_2]_A \wedge \tau_3 \right)(\xi, \eta)s \otimes V
+ ([\tau_1, \tau_3] \wedge [\tau_2, \tau_3]_A)(\xi, \eta)s \otimes L_\tau V
+ ([\tau_2, \tau_3]_A \wedge \tau_1)(\xi, \eta)s \otimes L_\tau V
\]
\[
= \left( \tau_1 \wedge [\tau_2, \tau_3]_A + [\tau_1, \tau_3]_A \wedge \tau_2 - [\tau_1, \tau_2]_A \wedge \tau_3 \right)(\xi, \eta)s \otimes V
+ s \otimes L_\tau(\xi, \eta) = \left( a_A(\tau_1)(([\tau_2, \tau_3]_A)(\xi, \eta)) - a_A(\tau_2)(([\tau_1, \tau_3]_A)(\xi, \eta))
+ a_A(\tau_3)(([\tau_1, \tau_2]_A)(\xi, \eta)) \right)s \otimes V
\]
\[
= s \otimes L_\tau(\xi, \eta) = \langle d_A(\xi)|\tau_1 \tau_2 - \langle d_A(\eta)|\tau_1 \tau_2 \rangle \rangle s \otimes V.
\]

Here we have used the identity
\[
L_{fx}V = fL_xV - a_A(x)(f)V, \quad \forall x \in \Gamma(A).
\]

Then using Equation (16), we obtain \(\partial(\tau) = \frac{1}{2}t_{\xi_0} + d_x(X_0). \) The proof of the second identity is similar and thus omitted. \(\square\)

**Corollary 3.5.** Assume that \((A, [\cdot, \cdot]_A, \{\cdot, \cdot\}_s, a_A, a_s, \tau, \phi)\) is a proto-bialgebroid. Then
\[
\partial^2 = d_s \circ \tau_\phi + \tau_\phi \circ d_s + \frac{1}{2}t_{\tau_0} + \frac{1}{2}t_{d_A(\xi_0)} : \Gamma(\wedge A) \rightarrow \Gamma(\wedge A).
\]

**Proof.** By assumption, the Jacobiator of the bracket \([\cdot, \cdot]_A\) is controlled by \(\phi\) and \(d_s\), i.e.,
\[
d_s^2 + L_\phi = 0,
\]
where \(L_\phi = [\phi, \cdot]_s\). Combining with Equation (19) and the identity in Lemma 3.1, we obtain the desired result. \(\square\)

Without loss of generality, suppose that \(\tau = \sum \tau_{ij} \wedge \tau_{k3} \wedge \tau_3\) and \(\phi = \sum \phi_{j3} \wedge \phi_{j2} \wedge \phi_{j3}\), where \(\tau_i \in \Gamma(A)\) and \(\phi_j \in \Gamma(A^*)\). We introduce four maps
\[
Q_1 := \sum_j \left( (t_{\phi_j3} t_{\phi_j2} \tau) \wedge t_{\phi_j1} - (t_{\phi_j2} t_{\phi_j1} \tau) \wedge t_{\phi_j3} + (t_{\phi_j2} t_{\phi_j1} \tau) \wedge t_{\phi_j3} \right) : \Gamma(\wedge A) \rightarrow \Gamma(\wedge A),
\]
\[
Q_2 := \sum_j \left( -(t_{\phi_j3} \tau) \wedge t_{\phi_j2} t_{\phi_j1} + (t_{\phi_j2} \tau) \wedge t_{\phi_j3} t_{\phi_j1} - (t_{\phi_j1} \tau) \wedge t_{\phi_j3} t_{\phi_j2} \right) : \Gamma(\wedge A) \rightarrow \Gamma(\wedge A),
\]
\[
Q_3 := \sum_i \left( (t_{\tau_{i3}} t_{\tau_{i2}} \phi) \wedge t_{\tau_{i1}} - (t_{\tau_{i1}} t_{\tau_{i3}} \phi) \wedge t_{\tau_{i2}} + (t_{\tau_{i2}} t_{\tau_{i1}} \phi) \wedge t_{\tau_{i3}} \right) : \Gamma(\wedge A^*) \rightarrow \Gamma(\wedge A^*),
\]
\[
Q_4 := \sum_i \left( -(t_{\tau_{i3}} \phi) \wedge t_{\tau_{i2}} t_{\tau_{i1}} + (t_{\tau_{i2}} \phi) \wedge t_{\tau_{i3}} t_{\tau_{i1}} - (t_{\tau_{i1}} \phi) \wedge t_{\tau_{i3}} t_{\tau_{i2}} \right) : \Gamma(\wedge A^*) \rightarrow \Gamma(\wedge A^*).
\]

**Lemma 3.6.** The operators \(Q_1\) and \(Q_3\) satisfy
\[
\langle Q_1(x)|\xi \rangle + \langle x|Q_3(\xi) \rangle = 2\langle t_x \phi|\tau_\xi \rangle,
\]
for all \(x \in \Gamma(A)\) and \(\xi \in \Gamma(A^*)\).
The following lemma is a generalization of [7, Lemma 5.1].

**Lemma 3.7.** Assume that \((A, [\cdot, \cdot], \cdot, a_A, a_\ast, \tau, \phi)\) is a proto-bialgebroid. Then
\[
\langle y \vert (L_{x \ast} + [L_x, L_\xi]) \eta \rangle = \langle \iota_y d_x (x) \vert \iota_y d_A (\xi) \rangle + \phi(x, \tau(\xi, \eta), y),
\]
for all \(x, y \in \Gamma(A)\) and \(\xi, \eta \in \Gamma(A^\ast)\).

**Proof.** From the definition of the Dorfman bracket, it follows that
\[
\langle y \vert L_{z + \chi} \eta \rangle = 2 \langle y, (z + \chi) \circ \eta + \iota_y d_x (z) + \tau(\chi, \eta) \rangle = 2 \langle y, (z + \chi) \circ \eta \rangle
\]
for all \(z + \chi \in \Gamma(A \oplus A^\ast)\). Hence, we obtain
\[
\langle y \vert L_x L_\xi \eta \rangle = 2 \langle y, x \circ (\xi \circ \eta) \rangle - \phi(x, \tau(\xi, \eta), y)
\]
and
\[
\langle y \vert L_\xi L_x \eta \rangle = 2 \langle y, \xi \circ (L_x \eta) \rangle
\]
\[
= 2 \langle L_\xi y, L_x \eta \rangle - \langle L_x \eta, L_\xi \eta \rangle
\]
\[
= 2 \langle L_\xi y, x \circ \eta \rangle - \langle \xi \circ y, x \circ \eta + \iota_y d_x (x) \rangle \rangle
\]
\[
= 2 \langle y, \xi \circ (x \circ \eta) \rangle + \iota_y d_x (x) \rangle + \iota_y d_A (\xi),
\]
where we have used the equation
\[
\langle y, L_\xi (x \circ \eta) \rangle = \langle y, L_\xi (L_x \eta - \iota_y d_x (x)) \rangle = \langle y, \xi \circ (L_x \eta - \iota_y d_x (x)) + \tau(\xi, L_x \eta) - \iota_y d_x (x) d_A (\xi) \rangle.
\]
Thus, we have
\[
\langle y \vert (L_{x \circ \xi} - [L_x, L_\xi]) \eta \rangle \rangle = 2 \langle y, (x \circ \xi) \circ \eta \rangle - 2 \langle y, x \circ (\xi \circ \eta) \rangle + \phi(x, \tau(\xi, \eta), y)
\]
\[
+ 2 \langle y, \xi \circ (x \circ \eta) \rangle + \iota_y d_x (x) \rangle + \iota_y d_A (\xi) \rangle
\]
\[
= \iota_y d_x (x) \rangle + \iota_y d_A (\xi) \rangle + \phi(x, \tau(\xi, \eta), y).
\]

Consider the pair of operators
\[
D = d_\ast + \partial : \Gamma(\wedge^\ast A) \to \Gamma(\wedge^\ast A), \quad D_\ast = d_A + \partial_\ast : \Gamma(\wedge^\ast A^\ast) \to \Gamma(\wedge^\ast A^\ast).
\]
In the case of Lie bialgebroids, their squares yield a pair of Laplacian operators (see [7]):
\[
\Delta := d_\ast \partial + \partial d_\ast : \Gamma(\wedge^k A) \to \Gamma(\wedge^k A),
\]
\[
\Delta_\ast := d_A \partial_\ast + \partial_\ast d_A : \Gamma(\wedge^k A^\ast) \to \Gamma(\wedge^k A^\ast).
\]
In our situation, we also call them **Laplacians** although we do not require \((A, A^\ast)\) to be a Lie.

bialgebroid. They satisfy the following key relations.
Lemma 3.8 ([7, Proposition 4.5]). For all \( r_1, r_2 \in \Gamma(A) \), we have
\[
\Delta(r_1 \wedge r_2) - (\Delta r_1) \wedge r_2 - r_1 \wedge (\Delta r_2) = (-1)^{|r_1|} (d_\Delta([r_1, r_2]) - [d_\Delta(r_1), r_2]_A - (-1)^{|r_1|}[r_1, d_\Delta(r_2)]_A).
\]  
(21)

For all \( x \in \Gamma(A), \xi \in \Gamma(A^*), f, g \in C^\infty(M) \), we have
\[
\Delta_*(fg) - f\Delta_*(g) - g\Delta_*(f) = -[d_\Delta(f), g]* + [f, d_\Delta(g)]*,
\]  
(22)
\[
\Delta_*(f\xi) - f\Delta_*(\xi) - \Delta_*(f)\xi = d_\Delta([f, \xi]_*) - [d_\Delta(f), \xi]_* + [f, d_\Delta(\xi)]_*.
\]  
(23)

Proposition 3.9. Let \((A, [\cdot, \cdot]_A, a_A)\) and \((A^*, [\cdot, \cdot]_A, a_A)\) be two skew-symmetric dual algebroids in duality. Then for \( \tau \in \Gamma(A^3) \) and \( \phi \in \Gamma(A^3) \), the following four assertions are equivalent:

(I) For all \( x \in \Gamma(A) \) and \( \xi \in \Gamma(A^*), \Delta_*(\xi|x) = \Delta_*(\xi)|x| + \langle \xi|\Delta(x) \rangle - 2\langle \iota_\xi \phi|\xi \rangle \tau; \)

(II) For all \( x \in \Gamma(A) \) and \( \xi \in \Gamma(A^*), \Delta_*(\xi|x) = \Delta_*(\xi)|x| + \langle \xi|\Delta(x) \rangle - 2\langle \iota_\xi \phi|\xi \rangle \tau; \)

(III) For all \( x \in \Gamma(A) \) and \( \xi \in \Gamma(A^*), \) the maps \( L_{x \xi} : [L_x, L_\xi] : \Gamma(A^*) \to \Gamma(A^*) \) and \( L_{x \xi} - [L_x, L_\xi] : \Gamma(A) \to \Gamma(A) \) are \( C^\infty(M) \)-linear, and their traces coincide,
\[
\text{tr}(L_{x \xi} - [L_x, L_\xi]) = \text{tr}(L_{x \xi} - [L_x, L_\xi]) = 2\langle d_A(\xi)|d_\Delta(x) \rangle - 2\langle \iota_\xi \phi|\xi \rangle \tau.
\]
Here \( x \circ \xi := -\iota_\xi (d_\Delta x) + L_\xi x \) and \( \xi \circ x := L_\xi x - \iota_x (d_\Delta \xi) \).

(IV) For all \( f \in C^\infty(M), x \in \Gamma(A), \) and \( \xi \in \Gamma(A^*), \)
\[
\Delta(f) = \Delta_*(f) = \frac{1}{2}(L_{X_0} + L_{\xi_0})(f),
\]
\[
\Delta(x) = (\frac{1}{2}L_{X_0} + \frac{1}{2}L_{\xi_0} + Q_1)x,
\]
\[
\Delta_*(\xi) = (\frac{1}{2}L_{X_0} + \frac{1}{2}L_{\xi_0} + Q_2)(\xi).
\]

Proof. The proof consists of four steps: (I) \( \Leftrightarrow \) (II), (I) \( \Rightarrow \) (III), (III) \( \Rightarrow \) (IV), and (IV) \( \Rightarrow \) (I).

- (I) \( \Leftrightarrow \) (II): We claim that
\[
L_{d_\Delta(f)} + L_{d_A(f)} = 0 \quad \text{as a map} \quad \Gamma(A^*) \to \Gamma(A^*).
\]
(24)
In fact, for all \( f \in C^\infty(M), x \in \Gamma(A), \) and \( \xi \in \Gamma(A^*), \) we have, on the one hand,
\[
\Delta_*(\langle x|\xi \rangle) = \Delta_*(f\xi) = f\Delta_*(\xi) + f\Delta_*(\xi) + \Delta_*(f)\xi = \Delta_{d_\Delta(f)}(\langle x|\xi \rangle) - [d_\Delta(f), \langle x|\xi \rangle]_* + [f, d_\Delta(\langle x|\xi \rangle)]_*
\]
(22)
\[
= \Delta_*(\xi)|x| + f\Delta_*(\xi) - (L_{d_\Delta(f)} + L_{d_A(f)})(\langle x|\xi \rangle);
\]
On the other hand, we have
\[
\Delta_*(\langle x|\xi \rangle) \overset{\text{(I)}}{=} \langle \Delta_*(f)|\xi \rangle + \langle x|\Delta_*(f)\rangle - 2f\langle \iota_\xi \phi|\xi \rangle \tau
\]
(21)
\[
\overset{\text{(23)}}{=} f\langle \Delta_*(x)|\xi \rangle + \langle x|f\Delta_*(\xi) + \Delta_*(f)\xi + \Delta_*(f)\xi + d_\Delta([f, \xi]_*) - [d_\Delta(f), \xi]_* + [f, d_\Delta(\xi)]_*
\]
\[
- 2f\langle \iota_\xi \phi|\xi \rangle \tau
\]
\[
= f\langle \Delta_*(x)|\xi \rangle + \Delta_*(f)|\langle x|\xi \rangle - \langle x|(L_{d_\Delta(f)} + L_{d_A(f)})(\langle x|\xi \rangle) - 2f\langle \iota_\xi \phi|\xi \rangle \tau
\]
(21)
\[
\overset{\text{(1)}}{=} f\Delta_*(\langle x|\xi \rangle) + \Delta_*(f)|\langle x|\xi \rangle - \langle x|(L_{d_\Delta(f)} + L_{d_A(f)})(\langle x|\xi \rangle).
\]
Here we have used the identity
\[
\langle x|d_\Delta([f, \xi]_*) - [d_\Delta(f), \xi]_* + [f, d_\Delta(\xi)]_* = ([L_{x \xi} - L_{\xi_0}, L_x] - L_{x \xi})(f).
\]
Thus, we obtain \((L_{d_\Delta(f)} + L_{d_A(f)})(\langle x|\xi \rangle) = \langle x|(L_{d_\Delta(f)} + L_{d_A(f)})(\langle x|\xi \rangle), \) which implies Equation (24).

Note that
\[
(\Delta f)\Omega = (\partial d_\Delta f)\Omega \overset{(4)}{=} -L_{d_\Delta(f)}\Omega \overset{(24)}{=} L_{d_A(f)}\Omega \overset{(6)}{=} (\partial d_A f)\Omega = (\Delta_*(f))\Omega,
\]
which implies that $\Delta f = \Delta_s f$, for all $f \in C^\infty(M)$. Thus, we have the equivalence between (I) and (II).

- (I) $\Rightarrow$ (III): By Lemma 2.2 and Equation (24),
  $$(L_{x^\xi} - [L_x, L_{\xi}]) (f) = 0, \quad \forall f \in C^\infty(M).$$
  Thus the map
  $$L_{x^\xi} - [L_x, L_{\xi}]: \Gamma(A^*) \to \Gamma(A^*)$$
  is $C^\infty(M)$-linear. By [7, Proposition 4.7], we have
  $$(L_{x^\xi} - [L_x, L_{\xi}]) \Omega \otimes V = (2\langle d_s|d_A|\xi \rangle - \langle \Delta(x)|\xi \rangle - \langle x|\Delta_s(\xi) \rangle + \Delta_s(x|\xi)) \Omega \otimes V$$
  Using relation (I), we obtain
  $$\text{tr}(L_{x^\xi} - [L_x, L_{\xi}]) = 2\langle d_s|d_A(\xi) \rangle - 2\langle t_{x^\xi}\phi|\xi\tau \rangle.$$ 
  Similarly, it follows from (II) that
  $$\text{tr}(L_{x^\xi} - [L_x, L_{\xi}]) = 2\langle d_s|d_A(\xi) \rangle - 2\langle t_{x^\xi}\phi|\xi\tau \rangle.$$

- (III) $\Rightarrow$ (IV): We need the following relations proved in [7, Propositions 4.10 and 4.11].
  $$(2\Delta(f) - (L_{x^0} + L_{\xi_0})(f)) \Omega \otimes s \otimes V = \Omega \otimes s \otimes (L_{d_A(f)} + L_{d_A(f)} V - (L_{d_A(f)} + L_{d_A(f)} s \otimes V),$$
  $$(2\Delta(x) - (L_{x^0} + L_{\xi_0})|x|\xi) \Omega \otimes s \otimes V = 2\langle d_s|d_A(\xi) \rangle \Omega \otimes s \otimes V + (\xi|L_x, L_{\xi} - L_{x^\xi}) \Omega \otimes s \otimes V + \Omega \otimes ([L_x, L_{\xi}] - L_{x^\xi}) s \otimes V.$$ (26)

for all $x \in \Gamma(A), \xi \in \Gamma(A^*)$ and $f \in C^\infty(M)$, where $X_0$ and $\xi_0$ are modular elements defined in Equations (12) and (13).

We start by proving that
  $$\Delta(f) = \frac{1}{2}(L_{x^0} + L_{\xi_0})(f), \quad \forall f \in C^\infty(M).$$
In fact, the assumption that $L_{x^\xi} - [L_x, L_{\xi}]$ is $C^\infty(M)$-linear implies that
  $$(L_{x^\xi} - [L_x, L_{\xi}])f = 0, \quad \forall f \in C^\infty(M).$$
By Lemma 2.2, we also have
  $$L_{d_A(f)} + L_{d_A(f)} = 0$$
as a map $\Gamma(\Lambda^*A) \to \Gamma(\Lambda^*A)$, which further implies that
  $$(L_{d_A(f)} + L_{d_A(f)}) s = (L_{d_A(f)} + L_{d_A(f)}) g dq^1 \land \ldots \land dq^n$$
  $$+ g \sum_{i=1}^n dq^1 \land \ldots \land d(L_{d_A(f)} + L_{d_A(f)}) q^i \land \ldots \land dq^n$$
  $$= 0,$$
where $s = gdq^1 \land \ldots \land dq^n$ is a local coordinates expression of $s$. Thus, the right-hand side of Equation (25) vanishes, which implies that
  $$\Delta(f) = \frac{1}{2}(L_{x^0} + L_{\xi_0})(f).$$
To obtain the second equation in (IV), we use the assumption (III) and Equation (26) to obtain
  $$\langle \Delta(x) - \frac{1}{2} (L_{x^0} + L_{\xi_0})(x)|\xi \rangle = \langle t_{x^\xi}\phi|\xi\tau \rangle = t_{x^\xi}\phi\xi\tau = t_{\xi\phi}\tau = \langle t_{x^\xi}\phi|\xi \rangle,$$
which implies that \( \Delta(x) = \frac{1}{2}(L_{X_0} + L_{\xi_0})(x) + \iota_{x_0} \phi \tau \). If we assume that \( \phi = \phi_1 \land \phi_2 \land \phi_3 \), then the term
\[
\iota_{x_0} \phi \tau = (\tau(\phi_2, \phi_3)\iota_{\phi_1} - \tau(\phi_1, \phi_3)\iota_{\phi_2} + \tau(\phi_1, \phi_2)\iota_{\phi_3})(x).
\]
Thus, we obtain
\[
\Delta(x) = \left( \frac{1}{2}L_{X_0} + \frac{1}{2}L_{\xi_0} + \tau(\phi_2, \phi_3) \land \iota_{\phi_1} - \tau(\phi_1, \phi_3) \land \iota_{\phi_2} + \tau(\phi_1, \phi_2) \land \iota_{\phi_3} \right)(x)
\]
\[
= \left( \frac{1}{2}L_{X_0} + \frac{1}{2}L_{\xi_0} + Q_1 \right)(x),
\]
as desired. The proof for the third equation is similar and thus omitted.

- (IV) \( \Rightarrow \) (I): This implication follows directly from Lemma 3.6.

\[
\square
\]

**Proposition 3.10.** Assume that \((A, [\cdot, \cdot]_A, [\cdot, \cdot]_s, a_A, a_s, \tau, \phi)\) is a proto-bialgebroid. Then we have for all \(k \geq 0\),

\[
\begin{align*}
(V) \quad \Delta &= \frac{1}{2}(L_{X_0} + L_{\xi_0}) + Q_1 + Q_2 : \Gamma(\land \text{k} A) \to \Gamma(\land \text{k} A), \\
(VI) \quad \Delta_* &= \frac{1}{2}(L_{X_0} + L_{\xi_0}) + Q_3 + Q_4 : \Gamma(\land \text{k} A^*) \to \Gamma(\land \text{k} A^*).
\end{align*}
\]

**Proof.** First of all, we note that the following two assertions hold:

\[
\begin{align*}
(VII) \quad d_*([r_1, r_2]_A) &= [d_*(r_1), r_2]_A + (-1)^{|r_1|-1}[r_1, d_*(r_2)]_A + \iota_\phi(\tau \land r_1 \land r_2); \\
(VIII) \quad \Delta(r_1 \land r_2) &= (\Delta r_1) \land r_2 + r_1 \land (\Delta r_2) + (-1)^{|r_1|}\iota_\phi(\tau \land r_1 \land r_2),
\end{align*}
\]

for all \(r_1, r_2 \in \Gamma(\land \text{k} A), \omega_1, \omega_2 \in \Gamma(\land \text{k} A^*)\). In fact, (VII) is a generalized form of axiom (3) in Definition 1.2. By Equation (21), statements (VII) and (VIII) are equivalent.

Note that relations (V) and (VI) are symmetric. It suffices to prove (V). We firstly show that the relation (III) holds. In fact, by Lemma 2.2, one has
\[
(L_{x_0 \xi} - [L_x, L_{\xi}])(f \eta) = f \left( L_{x_0 \xi} - [L_x, L_{\xi}] \right)(\eta) - \langle d_*(f, x)_A - [d_* f, x]_A + [f, d_*(x)]_A \xi, \eta \rangle,
\]
for all \(x \in \Gamma(A), \xi, \eta \in \Gamma(A^*), f \in C^\infty(M)\). It follows from (VII) that \(L_{x_0 \xi} - [L_x, L_{\xi}]\) is a \(C^\infty(M)\)-linear endomorphism of \(\Gamma(\land \text{k} A^*)\).

Let \(\{\xi^1, \ldots, \xi^n\}\) be a local basis of \(\Gamma(A)\) and \(\{\theta_1, \ldots, \theta_a\}\) be the dual basis of \(\Gamma(A^*)\). It follows that
\[
\text{tr}(L_{x_0 \xi} - [L_x, L_{\xi}]) = \langle \xi^a \mid (L_{x_0 \xi} - [L_x, L_{\xi}]) (\theta_a) \rangle = \langle \iota_{\theta_a} d_*(x)[\iota_{\xi} d_*(\xi)] + \phi(x, \tau(\xi, \theta_a), \xi^a) \rangle \text{ by Lemma 3.7}
\]
\[
= 2\langle d_*(x)[d_*(\xi)] + 2\iota_{x} \phi(\iota_{\xi} \tau) \rangle.
\]

Here the last equality holds since
\[
\phi(x, \tau(\xi, \theta_a), \xi^a) = -\phi(x, \iota_{\xi} \tau, \tau(\xi, \theta_a)) = -\langle \iota_{\xi} \iota_{x} \phi(\iota_{\theta_a} \iota_{\xi} \tau) \rangle = -2\langle \iota_{x} \phi(\iota_{\xi} \tau) \rangle.
\]

In a similar manner, we see that \(L_{\xi_0 x} - [L_{\xi}, L_x]\) is \(C^\infty(M)\)-linear and
\[
\text{tr}(L_{\xi_0 x} - [L_{\xi}, L_x]) = 2\langle d_*(\xi)[d_*(x)] - 2\langle \iota_{x} \phi(\iota_{\xi} \tau) \rangle.
\]

Thus, the relation (III) holds. By Proposition 3.9, we see that relations in (IV) hold, which mean that
\[
\Delta = \frac{1}{2}(L_{X_0} + L_{\xi_0}) + Q_1 + Q_2 : \Gamma(\land 0 A) \to \Gamma(\land 0 A),
\]
and
\[
\Delta = \frac{1}{2}(L_{X_0} + L_{\xi_0}) + Q_1 + Q_2 : \Gamma(\land 1 A) \to \Gamma(\land 1 A).
\]
We now check that this relation holds as a map $\Gamma(\wedge^2 A) \to \Gamma(\wedge^2 A)$, i.e., for all $x, y \in \Gamma(A)$, we have

$$\Delta(x \wedge y) = \left( \frac{1}{2} L_{X_0} + \frac{1}{2} L_{\xi_0} + Q_1 + Q_2 \right)(x \wedge y).$$

(27)

In fact, we have

$$\Delta(x \wedge y) = \left( \frac{1}{2} L_{X_0} + L_{\xi_0} \right)(x) + \left( \frac{1}{2} L_{X_0} + L_{\xi_0} \right)(y) + \frac{1}{2} (L_{X_0} + L_{\xi_0})(x \wedge y) - \ell_{x, y} \phi \tau$$

This proves the claim (27). The cases for $k \geq 3$ follow from a standard induction argument and are omitted.

3.2. Proofs of theorems.

3.2.1. Proof of Theorem 2.1. First of all, we need to compute the square of $\bar{D}$. For all $x \in \Gamma(A), \xi \in \Gamma(A^*)$, $v \in \Gamma(\wedge^2 A)$ and $\phi \in \Gamma(\wedge^3 A^*)$, one has

$$(\tau \wedge \ell_{x, y} \circ \ell_{x, y}) (v) = (\tau \circ \phi) v - Q_1(v) - Q_2(v),$$

and

$$\partial \circ \ell_{x, y} + \ell_{x, y} \circ \partial = \ell_{\Delta x}(\phi).$$

Meanwhile, a simple computation that adapted from [7] yields that

$$\bar{d}_x (r \otimes l) = (d_x(r) + \frac{1}{2} X_0 \wedge r) \otimes l,
\bar{d}(r \otimes l) = (-\partial(r) + \frac{1}{2} \xi_0 r) \otimes l,$$

for all $r \in \Gamma(\wedge^2 A)$ and $l \in \Gamma \mathcal{L}$. Thus, the operator $\bar{D}$ is related to the modular elements $X_0$ and $\xi_0$ by

$$\bar{D} = d_x + \bar{\partial} + \tau - \ell_{x, y} = d_x - \partial + \frac{1}{2} (X_0 \wedge + \xi_0) + \tau - \ell_{x, y}.$$ 

Combining these equations with the BV identity (10) for $\partial$, one has for all $v \in \Gamma(\wedge^2 A)$,

$$\bar{D}^2(v) = (d_x^2 + \partial^2 + \frac{1}{2} d_x(X_0) + \frac{1}{2} \xi_0 \tau - \partial(\tau))(v)$$

$$+ ( - d_x \circ \ell_{x, y} - \ell_{x, y} \circ d_x + \partial^2 - \frac{1}{2} \ell_{d_1}(\phi) ) (v)$$

$$+ ( \frac{1}{2} L_{X_0} + L_{\xi_0} - \Delta + Q_1 + Q_2)(v)$$

$$+ \left( \frac{1}{2} L_{X_0} - \frac{1}{2} \partial(X_0) - \langle \tau | \phi \rangle \right) v$$

(28)

Note that the implication (3) $\Rightarrow$ (2) follows from the definition of Dirac generating operators. The rest proof of Theorem 2.1 is divided into the following three steps.

Step 1: (1) $\Rightarrow$ (2) Namely, $\bar{D}^2$ is smooth function on $M$, i.e., $\bar{D}^2 \in C^\infty(M)$ under the hypothesis that $A, [\cdot, \cdot]_A, [\cdot, \cdot], a_A, a_* , \tau, \phi$ is a proto-bialgebroid.
Recall that the Jacobiator of the bracket $[\cdot, \cdot]_A$ on $\Gamma(A^\ast)$ is controlled by the element $\tau$ and the derivation $d_A$, i.e.,
\[ d_A^2 + L_\tau = 0, \]
where $L_\tau = [\tau, \cdot]_A$. Combining with Equation (18), the first line on the right hand side of Equation (28) vanishes; by Equation (20), the second line vanishes; by $d_\ast(\tau) = 0$ and $d_A(\phi) = 0$, the third and fourth lines are zero; by (V) of Proposition 3.10, the fifth line is also zero. In summary, the above terms combine to yield
\[ \tilde{D}^2(\nu) = \tilde{f} \nu, \]
where
\[ \tilde{f} = \frac{1}{4} \langle \xi_0 | X_0 \rangle - \frac{1}{2} \partial(X_0) - \langle \tau | \phi \rangle, \]
which is exactly the characteristic function.

**Step 2:** $(2) \Rightarrow (1)$ Suppose that there exists a smooth function $g \in C^\infty(M)$ such that $\tilde{D}^2(\nu) = g \nu$ for all $\nu \in \Gamma(\Lambda^\ast A)$. To see that the septuple $(A, [\cdot, \cdot]_A, [\cdot, \cdot]_\ast, a_A, a_\ast, \tau, \phi)$ is a proto-bialgebroid, we need to check axioms in Definition 1.2.

Consider the constant function $\nu = 1$ in Equation (28). We have
\[ d_A(\phi) = 0, \quad d_\ast(\tau) = 0, \]
\[ \frac{1}{2} d_\ast(X_0) + \frac{1}{2} \iota_{\xi_0} \tau - \partial(\tau) = 0, \]
\[ d_A^2 + L_\tau = 0, \quad \frac{1}{2} L_{X_0} + \frac{1}{2} L_{\xi_0} - \Delta + Q_1 + Q_2 = 0, \]
\[ -d_\ast \circ \iota_\phi - \iota_\phi \circ d_\ast + \partial^2 - \frac{1}{2} \iota_{\xi_0 \phi} - \frac{1}{2} \iota_{d_A(\xi_0)} = 0. \]
(Axioms (4) and (5) are exactly the two equations in the first row. The Equations in (29) imply that axioms (2) and (3) are fulfilled, respectively. By Lemma 3.1 and Equation (30), we have
\[ d_A^2 + L_\phi - \partial_\ast(\phi) + \frac{1}{2} \iota_{X_0} \phi + \frac{1}{2} d_A(\xi_0) = 0. \]
Applying to the constant function 1, we get $-\partial_\ast(\phi) + \frac{1}{2} \iota_{X_0} \phi + \frac{1}{2} d_A(\xi_0) = 0$. It follows that $d_A^2 + L_\phi = 0$, which implies that axiom (1) is also satisfied.

**Step 3:** $(1) \Rightarrow (3)$ We need to verify the operator $\tilde{D}$ in (14) satisfies conditions (a) $\sim$ (c) in Definition 1.4. As condition (c) follows from Step 1, we only need to check conditions (a) and (b), i.e.,
\[ [\tilde{D}, f] \in \Gamma(A \oplus A^\ast), \quad [[\tilde{D}, x + \xi], y + \eta] \in \Gamma(A \oplus A^\ast), \]
for all $f \in C^\infty(M)$, $x, y \in \Gamma(A)$, and $\xi, \eta \in \Gamma(A^\ast)$. Firstly, a direct calculation shows that
\[ [\tau - \iota_\phi, f](v) = 0, \]
and
\[ [[\tau - \iota_\phi, x + \xi], y + \eta](v) = -((\iota_\phi \iota_\xi) + v - \iota_{\phi \iota_\phi} v), \]
for all $x, y \in \Gamma(A)$, $\xi, \eta \in \Gamma(A^\ast)$, $f \in C^\infty(M)$, and $v \in \Gamma(\Lambda^\ast A)$. For all $\nu \in \Gamma(\Lambda^\ast A)$, we have
\[ [\tilde{D}, f](v) \overset{(14)}{=} [d_\ast - \partial + \frac{1}{2}(X_0 + \iota_{\xi_0}) + \tau - \iota_\phi, f](v) \overset{(33)}{=} [d_\ast - \partial, f](v) = (d_\ast(f) + d_A(f)) \cdot v \in \Gamma(A \oplus A^\ast) \cdot v. \]
Thus, we have

\[ \text{Dirac generating operator in (14)} \]

This proves (31).

For Equation (32), we compute, for all \( x + \xi, y + \eta \in \Gamma(A \oplus A^*) \), \( v \in \Gamma(\wedge^k A) \),

\[
[\tilde{D}, x + \xi], y + \eta](v) \overset{(14)}{=} \left[ d_x - \partial + \frac{1}{2}(X_0 + \iota_{\xi_0}) + \tau - \iota_{\phi}, x + \xi, y + \eta \right](v)
\]

\[
\overset{(34)}{=} [d_x(x) + L_\xi + L_x - \partial(x) - \iota_{d_A(\xi)}(x, y + \eta)](v) - (\iota_{\phi} \xi \tau) \wedge v - \iota_{(\iota_{\phi} \xi \phi)} v
\]

\[
= \left( [x, y]_A + L_y(x) - \iota_{d_A(\xi)}(x) - \iota_{g(t_x \phi)}(v) \right) \in \Gamma(A) \wedge v
\]

\[
\overset{(2)}{=} (x + \xi) \circ (y + \eta)(v) \in \Gamma(A \oplus A^*) \cdot v.
\]

3.2.2. Proof of Theorem 2.6. In the proof of Theorem 2.1, we have obtained the expressions of the Dirac generating operator in (14) and the characteristic function in (15). We now verify that the characteristic function

\[
\tilde{f} = \frac{1}{4}(\xi_0|X_0) - \frac{1}{2}\partial(X_0) - \langle \tau|\phi \rangle.
\]

is indeed an invariant of the proto-bialgebroid, i.e., it does not depend on the choices of the volume \( s \in \Omega^m(M) \) of \( M \), the top form \( \Omega \in \Gamma(\wedge^n A^*) \) and its dual \( V \in \Gamma(\wedge^n A) \).

Assume that \( s' = gs \) and that \( \Omega' = h\Omega, V' = 1/hV \) for some nowhere vanishing smooth functions \( g, h \in C^\infty(M) \). Let \( X'_0 \in \Gamma(A), \xi'_0 \in \Gamma(A^*) \) be the modular elements associated with \( s', \Omega', V' \).

Using Equations (12), we have for all \( \xi \in \Gamma(A^*) \),

\[
\langle \xi|X'_0 \rangle \Omega' \otimes s' = L_\xi(\Omega' \otimes s') = L_\xi(gh\Omega \otimes s) = a_* (\xi|gh)(\Omega \otimes s) + ghL_\xi(\Omega \otimes s)
\]

\[
= \frac{a_* (\xi|(gh))}{gh} \Omega' \otimes s' + \langle \xi|X_0 \rangle \Omega' \otimes s' = \langle \xi|X_0 + d_A(\log(gh)) \rangle \Omega' \otimes s'.
\]

Thus, we have

\[
X'_0 = X_0 + d_A(\log(g h)) = X_0 + d_A(\log g + \log h).
\]

By a similar computation and using (13), we have

\[
\xi'_0 = \xi_0 + d_A(\log g - \log h).
\]

Denote by \( \partial' \) the BV operator induced from \( \Omega' \) and \( V' \). Then we have for all \( r \in \Gamma(\wedge^k A) \),

\[
\partial'(r) = -(1)^{(k+1)}(V'^* \circ d_A \circ \Omega'^*)(r) = -(1)^{(k+1)}1/hV'^*(d_A(h\Omega'^*)(r))
\]

\[
= \partial(r) - (1)^{(k+1)}1/hV^*(d_A(h \wedge \Omega^*)(r)) = \partial(r) - \iota_{d_A \log h}r
\]

A direct computation shows that

\[
\tilde{f}' - \tilde{f} = \frac{1}{4}(\langle \xi'_0|X'_0 \rangle - \langle \xi_0|X_0 \rangle) - \frac{1}{2}(\partial'(X'_0) - \partial(X_0)) = 0.
\]

Thus \( \tilde{f} \) is indeed an invariant of the proto-bialgebroid.

Finally, we check that \( \tilde{f} \) is completely determined by either statement (a) or (b). Note that

\[
L_{X_0}(\Omega \otimes s) \otimes V = (L_{X_0} \Omega) \otimes s \otimes V + \Omega \otimes (L_{X_0}s) \otimes V
\]

\[
\overset{(7)}{=} -\Omega \otimes s \otimes (L_{X_0}V) + \Omega \otimes (L_{X_0}s) \otimes V
\]

\[
\overset{(5)(13)}{=} -(1)^{(k+1)}1/hV^*(d_A(h \wedge \Omega^*)(r)) = (\partial(r) - \iota_{d_A \log h}r)
\]

\[
\overset{(15)}{=} 4(\tilde{f} + \langle \tau|\phi \rangle) \Omega \otimes s \otimes V.
\]
The proves statement (a). The proof of Statement (b) is similar and thus omitted.

**Remark 3.11.** Although the characteristic function \( \bar{f} = \bar{D}^2 \) is an invariant, the Dirac generating operator \( D \) does depend on the choices of \( s, \Omega \) and \( V \). In a sense, what curvatures are to connections, characteristic functions are to Dirac generating operators.

4. Examples

In this section, we consider some concrete proto-bialgebroids and compute the corresponding Dirac generating operators and characteristic functions.

4.1. Proto-bialgebroids over Euclidean spaces. We start with a proto-bialgebroid over an Euclidean space \( \mathbb{R}^m \), which can be seen as a local model of proto-bialgebroid.

Let \( (A, [\cdot, \cdot]_A, [\cdot, \cdot]_s, a_A, a_s, \tau, \phi) \) be a proto-bialgebroid, where \( A = \mathbb{R}^m \times \mathbb{R}^n \) is a product vector bundle over \( \mathbb{R}^m \) of rank \( n \). Choose a coordinate system \( \{q^1, \ldots, q^m, e^1, \ldots, e^n\} \) for base space \( \mathbb{R}^m \). Suppose that \( \{e_1, \ldots, e_n\} \) is a basis of \( \Gamma(A) \) and that \( \{e^1, \ldots, e^n\} \) is the dual basis of \( \Gamma(A^*) \). We assume that the two elements \( \phi \) and \( \tau \) are of the form

\[
\phi = \bar{\phi}_{ijk} e^i \wedge e^j \wedge e^k \in \Gamma(\land^3 A^*) \quad \text{and} \quad \tau = \bar{\tau}^{ijk} e_i \wedge e_j \wedge e_k \in \Gamma(\land^3 A),
\]

And the skew-symmetric dull algebroid structures on \( A \) and \( A^* \) can be written as follow:

\[
[e_i, e_j]_A = a_{ij}^k e_k, \quad a_A(e_i) = A_i^a \frac{\partial}{\partial q^a}, \quad e^i, e^*_s = b_{ij}^k e^k, \quad a_s(e^i) = B^i_{\alpha} \frac{\partial}{\partial q^\alpha}.
\]

It follows that the two associated derivations \( d_A \) and \( d_s \) are given by

\[
d_s(q^\alpha) = B^i_{\alpha} e^i, \quad d_s(e_i) = \frac{1}{2} b_{ij}^k e_j \wedge e_k, \quad d_A(q^\alpha) = A_i^a e^i, \quad d_A(e^i) = \frac{1}{2} a_{ij}^k e^j \wedge e^k.
\]

Assume that

\[
\Omega = e^1 \wedge \ldots \wedge e^n \in \Gamma(\land^n A^*), \quad V = e_1 \wedge \ldots \wedge e_n \in \Gamma(\land^n A), \quad s = dq^1 \wedge \ldots \wedge dq^m \in \Gamma(\land^m T^* \mathbb{R}^m).
\]

We now compute the associated BV operator \( \partial \). It is clear that \( \partial(q^\alpha) = 0 \). And

\[
\partial(e_i) = (-1)^{n-i} \partial(V^\sharp(e^1 \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^n)) \quad \overset{(9)}{=} (-1)^{i} V^\sharp(d_A(e^1 \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^n)) \\
= (-1)^{i} V^\sharp \left( \sum_{j=1}^{i-1} (-1)^{j-1} e^1 \wedge \ldots \wedge \hat{e}^j \wedge \ldots \wedge e^n \wedge d_A(e^j) \wedge e^{j+1} \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^n \right) \\
+ \sum_{k=i+1}^{n} (-1)^k e^1 \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^{k-1} \wedge d_A(e^k) \wedge e^{k+1} \wedge \ldots \wedge e^n \\
= (-1)^{i} V^\sharp \left( \sum_{j=1}^{i-1} (-1)^{j-1} e^1 \wedge \ldots \wedge \left( -\frac{1}{2} a_{ij}^j e^j \wedge e^i - \frac{1}{2} a_{jk}^j e^j \wedge e^i \right) \wedge \ldots \wedge \hat{e}^i \wedge \ldots \wedge e^n \right)
\]
Thus, we obtain are two skew-symmetric dull algebras in duality, Observe that

\[
\sum_{k=1+1}^n (-1)^k e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge \left(- \frac{1}{2} a_{ik} e^i \wedge e^k - \frac{1}{2} \bar{a}_{ik} e^k \wedge e^i\right) \wedge \ldots \wedge e^n
\]

\[
= (-1)^i V^2 ((-1)^{i-1}(-1)^{i-j} a_{ji}^j \Omega + (-1)^{k-1}(-1)^i a_{ik}^k \Omega)
\]

\[
= - i \sum_{j=1}^n a_{ij}^j + \sum_{k=1+1}^n a_{ik}^k = a_{ij}^j.
\]

Observe that

\[
\langle e^i | X_0 \rangle \Omega \otimes s \overset{(12)}{=} L_{e^i} (\Omega \otimes s) = (L_{e^i} \Omega) \otimes s + \Omega \otimes L_{a_{e^i}} s
\]

\[
= (L_{e^i} (e^1 \wedge \ldots \wedge e^n)) \otimes s + \Omega \otimes L_B (\frac{\partial q^1 \wedge \ldots \wedge dq^m}{\partial s})
\]

\[
= \sum_{j=1}^n e^1 \wedge \ldots \wedge [e^1, e^j] \wedge \ldots \wedge e^n \otimes s
\]

\[
\quad + \Omega \otimes \left( \sum_{\beta=1}^m dq^1 \wedge \ldots \wedge d(L_B (\frac{\partial q^1 \wedge \ldots \wedge dq^m}{\partial s})) \right)
\]

\[
= \langle b_{ij} + \frac{\partial B^{i\alpha}}{\partial q^\alpha} \rangle \Omega \otimes s.
\]

Thus, we obtain

\[
X_0 = (b_{ij} + \frac{\partial B^{i\alpha}}{\partial q^\alpha}) e_i.
\]

(36)

In a similar approach, we have

\[
\xi_0 = \frac{\partial A^{i\alpha}}{\partial q^\alpha} + a_{ij}^j e_i.
\]

Using Equations (35) and (36), we obtain

\[
\partial (X_0) = a_{ij}^j (b_{ik} + \frac{\partial B^{i\alpha}}{\partial q^\alpha}) - A_{ij}^\beta \frac{\partial}{\partial q^\beta} (a_{ij}^j + \frac{\partial B^{i\alpha}}{\partial q^\alpha}).
\]

Hence, by Theorem 2.6, the characteristic function in this case is given by

\[
D^2 \overset{(15)}{=} \frac{1}{4} \langle \xi_0 | X_0 \rangle - \frac{1}{2} \partial (X_0) - \langle \tau | \phi \rangle
\]

\[
= \frac{1}{4} \left( \frac{\partial A^{i\alpha}}{\partial q^\alpha} + a_{ij}^j (b_{ik} + \frac{\partial B^{i\alpha}}{\partial q^\alpha}) - \frac{1}{2} A_{ij}^\alpha \frac{\partial}{\partial q^\beta} (b_{ij}^j + \frac{\partial B^{i\alpha}}{\partial q^\alpha}) - \tilde{\phi}_{ijk} \tilde{\tau}_{ijk} \right)
\]

\[
\quad + \frac{1}{4} \left( \frac{\partial A^{i\alpha}}{\partial q^\alpha} - a_{ij}^j (b_{ik} + \frac{\partial B^{i\alpha}}{\partial q^\alpha}) + \frac{1}{2} A_{ij}^\beta \frac{\partial}{\partial q^\beta} (b_{ij}^j + \frac{\partial B^{i\alpha}}{\partial q^\alpha}) - \tilde{\phi}_{ijk} \tilde{\tau}_{ijk} \right).
\]

4.2. Proto-bialgebras.

4.2.1. r-dimensional proto-bialgebras. We focus on proto-bialgebras \((g, g^*)\), i.e., proto-bialgebroids over a single point \(M = \{*\}\). The corresponding Courant algebroid is denoted by \(g \oplus g^*\), where \(g\) and \(g^*\) are \(r\)-dimensional vector spaces in duality.

Definition 4.1. A **proto-bialgebra** is a quadruple \((g, g^*, \tau, \phi)\), where \((g, [\cdot, \cdot]_g)\) and \((g^*, [\cdot, \cdot]_{g^*})\) are two skew-symmetric dull algebras in duality, \(\tau \in \wedge^3 g^*\), and \(\phi \in \wedge^3 g^*\), satisfying the following properties:

\[
[[x, y]_g, z]_g + [[y, z]_g, x]_g + [[z, x]_g, y]_g = \iota_\phi (d_* (x \wedge y \wedge z)),
\]

(37)
Hence, we have
\begin{equation}
[[\xi, \eta]_g^*, \chi]_g^* + [[\eta, \chi]_g^*, \xi]_g^* + [[\chi, \xi]_g^*, \eta]_g^* = \iota_{\tau}(d_A(\xi \wedge \eta \wedge \chi)),
\end{equation}
\begin{equation}
[d_*(x, y)]_g + [x, d_*(y)]_g + \iota_{(\iota_{x} \wedge \phi)} \tau = d_*(x, y)_g,
\end{equation}
\begin{equation}
d_A(\phi) = 0;
\end{equation}
\begin{equation}
d_A(\tau) = 0,
\end{equation}
where the maps $d_A: \wedge^+ g \to \wedge^{*+1} g$ and $d_A: \wedge^+ g^* \to \wedge^{*+1} g^*$ are the derivations arising from the skew-symmetric dual algebras $(g^*, [\cdot, \cdot]_g^*)$ and $(g, [\cdot, \cdot]_g)$, respectively.

Let us choose a basis $\{e_1, e_2, \ldots, e_r\}$ of $g$ with dual basis $\{e^1, e^2, \ldots, e^r\}$ of $g^*$. Assume that
\[
\phi = \bar{\phi}_{ij} e^i \wedge e^j \wedge e^k \quad \text{and} \quad \tau = \bar{\tau}^{ijk} e_i \wedge e_j \wedge e_k.
\]
Let $a^k_{ij}$ and $b^k_{ij}$ be the structure constants of the two brackets $[\cdot, \cdot]_g$ and $[\cdot, \cdot]_g^*$, that is,
\[
[e_i, e_j]_g = a^k_{ij} e_k \quad \text{and} \quad [e^i, e^j]_g^* = b^k_{ij} e^k.
\]
Then the two operators $d_*$ and $\partial$ are given by
\[
d_*(e_i) = -\frac{1}{2} b^k_{ij} e_j \wedge e_k, \quad \partial(e_i) = a^k_{ij}, \quad \text{and} \quad \partial(e_i \wedge e_j) = -a^k_{ij} e_k + a^k_{ik} e_j - a^k_{jk} e_i.
\]
The Dorfman bracket of the corresponding Courant algebroid $g \oplus g^*$ reads
\[
(x + \xi) \circ (y + \eta) = ([x, y]_g + \text{ad}^*_x y - \text{ad}^*_y x - \iota_y \ell_x \eta) + ([\xi, \eta]_g^* + \text{ad}^*_x \xi - \text{ad}^*_y \xi - \iota_y \ell_x \phi),
\]
which is indeed skew-symmetric, and satisfies the Jacobi identity. Therefore, $(g \oplus g^*, \circ)$ is indeed a Lie algebra, and $(g \oplus g^*, \circ, \{\cdot, \cdot\})$ is a quadratic Lie algebra.

For expressions of modular elements $X_0 \in g$ and $\xi_0 \in g^*$, we choose the particular volume forms $\Omega = e^1 \wedge \ldots \wedge e^r \in \wedge^r g^*$ and $V = e_1 \wedge \ldots \wedge e_r \in \wedge^r g$. Thus, we have
\[
\langle X_0 | \xi \rangle = \text{tr}(\text{ad}_\xi), \quad \langle \xi_0 | x \rangle = \text{tr}(\text{ad}_x), \quad \forall \xi \in g^*, x \in g.
\]
Hence, we have
\[
X_0 = b^k_{ij} e_i \quad \text{and} \quad \xi_0 = a^k_{ij} e^i.
\]
It follows that
\[
\partial(X_0) = a^k_{ij} b^k_{ij}.
\]
The characteristic function $D^2$, which is just a real number, is given by
\[
D^2 \overset{(15)}{=} \frac{1}{4} \langle \xi_0 | X_0 \rangle - \frac{1}{2} \partial(X_0) - \langle \tau | \phi \rangle = \frac{1}{4} a^k_{ij} b^k_{ij} - \bar{\phi}_{ij} \bar{\tau}^{ijk}.
\]

4.2.2. 3-dimensional case. Finally, we study 3-dimensional proto-bialgebras in detail. In this case,
\[
\text{Equation (37) } \iff [[x, y]_g, z]_g + [[y, z]_g, x]_g + [[z, x]_g, y]_g = 0;
\]
\[
\text{Equation (38) } \iff [[\xi, \eta]_g^*, \chi]_g^* + [[\eta, \chi]_g^*, \xi]_g^* + [[\chi, \xi]_g^*, \eta]_g^* = 0;
\]
\[
\text{Equation (39) } \iff d_*(x, y)_g = \{d_*(x), y\}_g + [x, d_*(y)]_g + \iota_{(\iota_{x} \wedge \phi)} \tau.
\]
Equations (40) and (41) hold by degree reasons. Therefore, both $(g = \text{span}\{e_1, e_2, e_3\}, [\cdot, \cdot]_g)$ and $(g^* = \text{span}\{e^1, e^2, e^3\}, [\cdot, \cdot]_g^*)$ are indeed Lie bialgebras. However, the pair $(g, g^*)$ is not a Lie bialgebra unless the term $\iota_{(\iota_{x} \wedge \phi)} \tau$ in (42) vanishes, which holds if either the element $\tau$ or $\phi$ is zero. This special case has been well studied in [19].

In what follows we assume that both $\tau$ and $\phi$ are nontrivial. We assume that
\[
\phi = \bar{\phi} e^1 \wedge e^2 \wedge e^3 \in \wedge^3 g^* \quad \text{and} \quad \tau = \bar{\tau} e^1 \wedge e^2 \wedge e^3 \in \wedge^3 g,
\]
for some nonzero numbers $\bar{\phi}, \bar{\tau} \in \mathbb{R}$. Note that
\[
\begin{align*}
    d_*(e_1) &= -b_1^{12} e_1 \land e_2 - b_1^{23} e_2 \land e_3 - b_1^{31} e_3 \land e_1, \\
    d_*(e_2) &= -b_2^{12} e_1 \land e_2 - b_2^{23} e_2 \land e_3 - b_2^{31} e_3 \land e_1, \\
    d_*(e_3) &= -b_3^{12} e_1 \land e_2 - b_3^{23} e_2 \land e_3 - b_3^{31} e_3 \land e_1.
\end{align*}
\]
Substituting $(x, y) = (e_1, e_2), (e_2, e_3), (e_3, e_1)$ in Equation (42), we obtain the following constraints on structure constants:
\[
\begin{align*}
    &\begin{cases}
        a_{12}^{31} b_1^{12} - a_{23}^{12} b_1^{23} - a_{31}^{23} b_1^{31} - a_{31}^{12} b_1^{23} - a_{31}^{12} b_1^{31} + \bar{\phi} \bar{\tau} = 0, \\
        a_{12}^{12} b_1^{23} + a_{23}^{12} b_1^{23} + a_{31}^{23} b_1^{23} + a_{31}^{12} b_1^{23} + a_{31}^{12} b_1^{31} = 0;
    \end{cases} \\
    &\begin{cases}
        a_{12}^{12} b_1^{12} + a_{23}^{12} b_1^{23} + a_{31}^{23} b_1^{31} + a_{31}^{12} b_1^{31} + a_{31}^{12} b_1^{12} = 0, \\
        a_{12}^{12} b_1^{31} + a_{23}^{12} b_1^{31} - a_{31}^{23} b_1^{31} - a_{31}^{12} b_1^{31} + \bar{\phi} \bar{\tau} = 0; \tag{43}
    \end{cases} \\
    &\begin{cases}
        a_{12}^{12} b_1^{12} + a_{23}^{12} b_1^{23} + a_{31}^{23} b_1^{31} + a_{31}^{12} b_1^{31} + a_{31}^{12} b_1^{12} = 0, \\
        a_{12}^{12} b_1^{31} + a_{23}^{12} b_1^{31} - a_{31}^{23} b_1^{31} - a_{31}^{12} b_1^{31} + \bar{\phi} \bar{\tau} = 0. \tag{44}
    \end{cases}
\end{align*}
\]
and
\[
\begin{align*}
    &\begin{cases}
        a_{12}^{12} b_1^{12} + a_{23}^{12} b_1^{23} + a_{31}^{23} b_1^{31} + a_{31}^{12} b_1^{31} + a_{31}^{12} b_1^{12} = 0, \\
        a_{12}^{12} b_1^{31} + a_{23}^{12} b_1^{31} - a_{31}^{23} b_1^{31} - a_{31}^{12} b_1^{31} + \bar{\phi} \bar{\tau} = 0. \tag{45}
    \end{cases}
\end{align*}
\]
Moreover, the modular elements are $X_0 = \sum_{i,j=1}^{3} a_{ij} e_i$ and $\xi_0 = \sum_{i,j=1}^{3} a_{ij} e_i$. We also have
\[
\partial(X_0) = (a_{12}^{13} + a_{13}^{13})(b_1^{12} + b_2^{13}) + (a_{21}^{13} + a_{23}^{13})(b_1^{21} + b_3^{23}) + (a_{31}^{12} + a_{32}^{12})(b_1^{31} + b_2^{32}).
\]
Thus, the characteristic function is given by the real number
\[
\tilde{D}^2 = \frac{1}{4}(\xi_0 | X_0) - \frac{1}{2} \partial(X_0) - (\tau | \phi) = \frac{1}{4} (a_{12}^{13} + a_{13}^{13})(b_1^{12} + b_2^{13}) + \frac{1}{4} (a_{21}^{13} + a_{23}^{13})(b_1^{21} + b_3^{23}) + \frac{1}{4} (a_{31}^{12} + a_{32}^{12})(b_1^{31} + b_2^{32}) - \bar{\phi} \bar{\tau}.
\]
\[\text{Example 4.2.} \quad \text{Consider } \mathfrak{g} = sl(2; \mathbb{R}) = \text{span}\{e_1, e_2, e_3\} \text{ with the standard relations}
\]
\[
[e_1, e_2]_g = 2e_2, \quad [e_1, e_3]_g = -2e_3, \quad [e_2, e_3]_g = e_1.
\]
Suppose that $\mathfrak{g}^* = \text{span}\{e^1, e^2, e^3\}$ is also endowed with a Lie bracket: $[e^i, e^j]_g = b_{ij}^{kl} e^k$. $1 \leq i, j, k \leq 3$. A well-known choice is given by
\[
[e^1, e^2]_g = \frac{1}{4} e^2, \quad [e^1, e^3]_g = \frac{1}{4} e^1, \quad [e^2, e^3]_g = 0,
\]
which makes $(\mathfrak{g}, \mathfrak{g}^*)$ into a Lie bialgebra (see [26]).

By Equations (43), (44), and (45), we see that $(\mathfrak{g}, \mathfrak{g}^*)$ is a proto-bialgebra if and only if there exists two real numbers $\bar{\phi}$ and $\bar{\tau}$ such that the following equations hold:
\[
\begin{align*}
    &\begin{cases}
        b_1^{23} (b_1^{12} - b_3^{31} ) = 0, \\
        b_1^{31} b_1^{23} = 0, \\
        b_1^{21} b_1^{12} = 0, \\
        b_1^{12} + b_3^{31} = \bar{\phi} \bar{\tau}, \\
        b_1^{31} + b_2^{23} = 0, \\
        b_1^{12} + b_2^{23} = 0, \\
        b_3^{31} = b_2^{31}. \tag{46}
    \end{cases}
\end{align*}
\]
Moreover, if \( b_{23} = \bar{\phi} \tau \neq 0 \), then the above conditions reduce to
\[
\begin{align*}
\phi_{23} = 2 \phi_{12} = 2 \phi_{31} = \bar{\phi} \tau, \\
\phi_{12} = \phi_{31} = \phi_{23} = \phi_{31} = \phi_{32} = 0.
\end{align*}
\]
For example, we can take \( \phi_{23} = \phi_{23} = 1, \phi_{12} = \phi_{31} = \frac{1}{2}, \) i.e.,
\[
[e^1, e^2]_{g^*} = \frac{1}{2} e^2, \quad [e^1, e^3]_{g^*} = -\frac{1}{2} e^3, \quad [e^2, e^3]_{g^*} = e^1.
\]
Since
\[
\xi_0 = (a_{12}^2 + a_{13}^3) e^1 + (a_{21}^1 + a_{31}^3) e^2 + (a_{31}^1 + a_{32}^3) e^3 = 0,
\]
and \( X_0 = (b_{12}^2 + b_{13}^3) e_1 + (b_{21}^1 + b_{23}^3) e_2 + (b_{31}^1 + b_{32}^3) e_3 = 0, \)
it follows that the characteristic function reads
\[
D^2 = \tilde{f} = -\bar{\phi} \tau (\neq 0).
\]

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