Collective excitations of atoms and field modes in coupled cavities

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Abstract
The exact solution for the system formed from two or three q-bits doped in coupled cavities is discussed. The problem of indistinguishability between the excited radiators and the photons is analyzed using the intrinsic symmetry of the system. It is demonstrated that the solution is drastically simplified when the radiators and photons are considered as new polariton excitations. The exact solution of the Schrödinger equation is obtained for single and two excitations in each cavity considering the indistinguishability principle. This approach opens new possibilities for the interpretation of quantum entangled states in comparison with the traditional distinct situation (see e.g. Napoli and Messina 2001 Fortschr. Phys. 49 1059; Enaki and Bazgan 2013 Phys. Scr. T153 014022) due to the decrease in the number of degrees of freedom in the system. Considering that the energy of coupling between the radiators and the photons is larger than that of the coupling with an external vacuum field, we have found the master equation for the dumping of collective excitations. The time dependence of the population for new dressed quasi-levels of energy is obtained by solving the master equation analytically and numerically.

Keywords: indistinguishability between optical excitations, cooperative excitations in coupled optical cavities, symmetry of excitations in coupled cavities

(Some figures may appear in colour only in the online journal)

1. Introduction

In recent years, the cooperative interaction of \( N \) two-level radiators with the cavity electromagnetic field (EMF) was the focus of many experimental and theoretical research studies [1–3]. This is connected with the major application of the two-level system as a q-bit in quantum processing of information. In many cases, distinct ensembles of q-bits are used in the realization of quantum registers [4, 5]. According to the indistinguishability principle for atoms [6], \( 2^N \) states of \( N \) two-level radiators can be reduced to \( N+1 \) states in the process of coherent excitation. As a particular case, the cooperative effect between two undistinguished radiators occurs, since two states of the single excitation (the atom \( A \) in the excited state and \( B \) in the ground state and, respectively, atom \( A \) in the ground state and \( B \) in the excited state) are considered as the same collective state. Recently, the symmetry properties of the Dicke model have been the center of attention in the literature [7]. As has been demonstrated in the literature the new effects are possible to observe, when the atoms are placed in the single-mode cavity. In this case, the Dicke model shows new peculiarities such as giant quantum oscillators [8], atom–field entanglement [9] and phase transition [10]. It is not difficult to observe that the number of collective excitations is drastically reduced with the increasing number of atoms.

It is attractive from the physical viewpoint to apply this principle to atoms placed in coupled cavities. The combination of atom–cavity physics and photon systems offers new opportunities in this field for better device functionality and for probing emulators of condensed-matter systems. In [11], a single-polariton approximation was proposed for the study of periodical photonic systems. In this approximation, the authors of this paper have applied Bloch states to the uniformly tuned Jaynes–Cummings–Hubbard model to analytically determine the energy-band structure. In [12] the photon-blockade-induced Mott’s transitions and XY spin models in coupled cavity arrays were studied. It was found that a range of many-body effects such as a Mott transition for polaritons obeying mixed statistics could be observed in the optical systems of individually addressable coupled cavity arrays interacting with two-level systems.
We propose using the symmetry principle for coupled cavities. In order to apply the proposed symmetry, we introduce with atomic states and the second with EMF states in each subsystems in each cavity. The first of them is connected reducing the number of degrees of freedom of two and three atoms situated in vertices of the equidistant triangle, we can obtain four collective states \( |\mathbf{g}\rangle \), \( |\mathbf{e}\mathbf{g}\rangle \), \( |\mathbf{g}\mathbf{e}\rangle \), and \( |\mathbf{e}\rangle \). As is observed, the number of degrees of freedom from the system formed from three coupled cavities is reduced as in the Dicke problem for three radiators situated in the volume with dimensions less than the radiation wavelength. In this section, we apply the properties of the symmetry for two and three coupled cavities. Thanks to this, we have obtained the exact solution of the Schrödinger equation in the interaction picture. Introducing the losses from the cavities, the master equation for indistinguishable excitations from two and more cavities is studied in section 3.

2. Collective excitation of coupled cavities

In this section, we apply the properties of the symmetry of indistinguishable atoms and photons packed in coupled cavities. For comparison, we consider the system of atoms situated within a distance less than the emission wavelength as in the Dicke [6] model. For simplicity, we consider the system of two atoms. These distinct atoms are characterized by four states: \( |\mathbf{e}_1\mathbf{e}_2\rangle \), \( |\mathbf{e}_1\mathbf{g}_2\rangle \), \( |\mathbf{g}_1\mathbf{e}_2\rangle \), and \( |\mathbf{g}_1\mathbf{g}_2\rangle \). Here \( \mathbf{e}_1 \) and \( \mathbf{g}_1 \) are the excited and ground states of the \( l \) atom, \( l = 1, 2 \). Considering that one of the atoms from the ensemble is prepared in the excited state, we can construct the cooperative state \( |\mathbf{e}\mathbf{g}\rangle = \frac{1}{\sqrt{3}} (|\mathbf{e}_1\mathbf{g}_2\rangle + |\mathbf{e}_2\mathbf{g}_1\rangle) \) which together with the other two states \( |\mathbf{e}\mathbf{e}\rangle \) and \( |\mathbf{g}\mathbf{g}\rangle \) form three atomic states. It is important to emphasize that in the rotation of this ensemble at the angle \( \pi \), these collective states remain the same. For three atoms situated in vertices of the equidistant triangle, we can obtain four collective states \( |\mathbf{g}\mathbf{g}\mathbf{g}\rangle \), \( |\mathbf{g}\mathbf{g}\mathbf{e}\rangle \), \( |\mathbf{g}\mathbf{e}\mathbf{g}\rangle \) and \( |\mathbf{e}\mathbf{e}\mathbf{e}\rangle \) from eight distinct \( |\mathbf{g}_1\mathbf{g}_2\mathbf{e}_3\rangle \), \( |\mathbf{g}_1\mathbf{e}_2\mathbf{g}_3\rangle \), \( |\mathbf{g}_2\mathbf{g}_3\mathbf{e}_1\rangle \), \( |\mathbf{g}_2\mathbf{e}_1\mathbf{g}_3\rangle \), \( |\mathbf{g}_3\mathbf{g}_1\mathbf{e}_2\rangle \), \( |\mathbf{g}_3\mathbf{e}_2\mathbf{g}_1\rangle \) and \( |\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\rangle \). As follows from this description of two collective states \( |\mathbf{g}\mathbf{g}\rangle \) and \( |\mathbf{e}\mathbf{e}\rangle \), we observe that they can be regarded as a superposition of three distinguished states \( (|\mathbf{g}_1\mathbf{g}_2\mathbf{e}_3\rangle + |\mathbf{g}_1\mathbf{e}_2\mathbf{g}_3\rangle + |\mathbf{g}_2\mathbf{g}_3\mathbf{e}_1\rangle) / \sqrt{3} \) and \( (|\mathbf{g}_1\mathbf{g}_2\mathbf{e}_3\rangle + |\mathbf{g}_1\mathbf{e}_2\mathbf{g}_3\rangle + |\mathbf{e}_1\mathbf{g}_2\mathbf{e}_3\rangle) / \sqrt{3} \), and are invariant under the angle rotations \( 2\pi / 3 \) and \( 4\pi / 3 \).

Let us return to the coupled ensemble of \( n \) doped cavities. In comparison with the above situation, we have two different subsystems in each cavity. The first of them is connected with atomic states and the second with EMF states in each cavity. In order to apply the proposed symmetry, we introduce the Hamiltonian of such coupled cavities into the interaction with the external EMF. Indeed, considering the \( n \) single-mode coupled cavities doped with radiators as is represented in figure 1, we can introduce the following Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{H}_1 \), in which the free part describes the energies of the atomic inversion, single-mode EMF of each cavity

\[
\hat{H}_0 = \hbar \omega \sum_{i=1}^{n} (\hat{R}_i^+ + \hat{a}_i^\dagger \hat{a}_i).
\]

The interaction part of the Hamiltonian contains the following couplings: between the atom and the mode of each cavity and between the cavities

\[
\hat{H}_1 = h g \sum_{i=1}^{n} (\hat{R}_i^+ \hat{a}_i + \hat{a}_i^\dagger \hat{R}_i^-) + h \chi \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} (\hat{a}_i \hat{a}_j^\dagger + \text{h.c.})
\]

where \( \omega \) is the resonant frequency between two-level radiators and modes of the EMF, \( \hat{a}_i^\dagger \) and \( \hat{a}_i \) are the creation and annihilation boson operators of the EMF and \( \hat{R}_i^+ \), \( \hat{R}_i^- \) and \( \hat{R}_{ij} \) are the excitation, de-excitation and inversion atomic operators for the \( i \)-cavity. These operators satisfy the commutation relationship \( [\hat{R}_i^+, \hat{R}_j^-] = 2\delta_{ij} \hat{R}_j^- \), \( [\hat{R}_{ij}, \hat{R}_{ij}^+] = \pm \hat{R}_{ij}^\pm \). The coupling between the cavities is achieved by the second term of the interaction Hamiltonian described by the exchange parameter \( \chi \).

Following the idea that the system formed from \( n \) cavities must remain in the same state after the application of rotation symmetry to a regular polygon in the vertices of which are situated in the centers of coupled cavities, we introduce the wave functions of the Hamiltonian (1). Let us neglect in the first approximation the coupling of the cavities with external EMF. Solving the problem in a higher \( Q \) cavity limit, we will introduce the losses from the cavities in the next section. Let us below apply these properties of symmetry for two and three cavities.

2.1. Two cavity case

In the good cavity limit, for single excitation in both cavities it is easy to observe four wave functions for distinct atoms and photons packed in coupled cavities. As follows from the distinctive description of the quantum systems becomes complicated due to the increasing number of degrees of freedom with the increasing number of coupled cavities in the system. In this case, the number of states is connected to the number of coupled quantum oscillators: atoms and cavity modes. Considering that the excitations of atoms and cavity modes become indistinguishable, we can use the symmetry transformation of such a system considering the invariance of the states after the actions of rotations corresponding to the local symmetry group. For example, the wave function of a single excitation of the radiator or mode inside one of the three cavities and collective excitations of cavity modes and atoms.

![Figure 1. N radiators in interaction with cavity modes of coupled cavities and collective excitations of cavity modes and atoms.](image-url)

\[\text{Figure 1.} \]
functions
\[
\begin{align*}
|\psi_{11}\rangle &= \frac{1}{\sqrt{2}} \left[ |e, 0\rangle_1 |g, 0\rangle_2 + e^{i\phi} |g, 0\rangle_1 |e, 0\rangle_2 \right], \\
|\psi_{12}\rangle &= \frac{1}{\sqrt{2}} \left[ |g, 1\rangle_1 |g, 0\rangle_2 + e^{i\phi} |g, 0\rangle_1 |g, 1\rangle_2 \right].
\end{align*}
\]
\tag{3}

For two excitations in two cavities instead of the eight wave functions \(|g, 1\rangle_1 |g, 1\rangle_2, |g, 2\rangle_1 |g, 0\rangle_2, |g, 0\rangle_1 |g, 2\rangle_2, |e, 0\rangle_1 |e, 0\rangle_2, |e, 1\rangle_1 |g, 0\rangle_2, |g, 0\rangle_1 |e, 1\rangle_2, |e, 0\rangle_1 |g, 1\rangle_2, |g, 1\rangle_1 |e, 0\rangle_2\) we obtain five collective states
\[
\begin{align*}
|\psi_{22}\rangle &= \frac{1}{\sqrt{2}} \left[ |e, 1\rangle_1 |g, 0\rangle_2 + e^{i\phi} |g, 0\rangle_1 |e, 1\rangle_2 \right], \\
|\psi_{23}\rangle &= \frac{1}{\sqrt{2}} \left[ |e, 0\rangle_1 |g, 1\rangle_2 + e^{i\phi} |g, 1\rangle_1 |e, 0\rangle_2 \right], \\
|\psi_{24}\rangle &= \frac{1}{\sqrt{2}} \left[ |g, 2\rangle_1 |g, 0\rangle_2 + e^{i\phi} |g, 0\rangle_1 |g, 2\rangle_2 \right], \\
|\psi_{25}\rangle &= |e, 0\rangle_1 |e, 0\rangle_2, \quad |\psi_{25}\rangle = |g, 1\rangle_1 |g, 1\rangle_2.
\end{align*}
\tag{4}
\]

Here \(e^{i\phi} = 1(\phi = 0)\) for symmetrical functions and \(-1(\phi = \pi, -\pi)\) for the anti-symmetrical case. \(|\psi_{ia}\rangle = |a_1, n_1\rangle |a_2, n_2\rangle\) is the state of the \(g\) degenerate level of energy, where \(\alpha = n_1 + n_2 + \delta_{a_1 + a_2} + \delta_{a_2}\), \(\delta_{a}\) is the number of excitations in the system and \(a_1\) and \(n_1\) indicate the atomic and field excitations in the \(a\) cavity, respectively. These wave functions (3) and (4) are invariant under the rotation of the system with angle \(\pi\). For this rotation through the angle \(\pi\), we must replace the indices of the cavities between them \(1 \leftrightarrow 2\).

It is easy to observe this for the one quanta of energy we have and the other superposition functions for two coupled cavities (one photon in the first and the atom excitation in the second and vice versa). However, these wave functions represent a superposition of collective excitations of atomic and field states of both cavities.

Let us consider that the degenerate states for single (3) and two excitations (4) are weakly split by the interaction Hamiltonian (2), so that this splitting is less than the energy distance between the first and second excited states of the cavities. Representing the non-stationary wave function through the superpositions of the single (3) or two excitations (4) states \(|\psi(t)\rangle = \sum_{ia} \exp[-i\lambda_{i\alpha} t] |\psi_{ia}\rangle\), we can find the new dressed states of the system solving the stationary Schrödinger equation in the interaction picture \(H_i |\psi_i\rangle = \lambda_i |\psi_i\rangle\). The wave functions for single collective excitations \(|\psi_{1i}(t)\rangle = \sum_{j=1,2} c^{(1)}_{i,j} \exp[-i\lambda_{i,j} t] |\psi_{ij}\rangle\). Here \(j\) indicates the new energy level; the eigenvalues of interaction Hamiltonian are
\[
\lambda_{1,1(-1)} = \frac{1}{2} \left( e^{i\phi} \sqrt{\lambda^2 + \sqrt{g^2 + \lambda^2}} \right)
\tag{5}
\]
and the superposition coefficients
\[
c^{(1)}_{i,j} = \sqrt{g^2 / (g^2 + \lambda^2)}, \quad c^{(1)}_{i,j} = \sqrt{\lambda^2 / (g^2 + \lambda^2)}
\tag{6}
\]
are labeled so that the first index indicates the new level of energy, and the index in parentheses indicates the number of excitations from the cavities.

For two excitations in the system we have the wave function \(|\psi_{2i}(t)\rangle = \sum_{j=1,2} c^{(2)}_{i,j} \exp[-i\lambda_{i,j} t] |\psi_{ij}\rangle\). Here the solutions of the characteristic equation are
\[
\lambda_{2,1(-1)} = \pm \frac{1}{\sqrt{2}} \sqrt{5g^2 + 3\lambda^2 - \sqrt{(3g^2 + \lambda^2)^2 + 12(1 + e^{i\phi})g^2 \lambda^2}},
\]
\[
\lambda_{2,2(-2)} = \pm \frac{1}{\sqrt{2}} \sqrt{5g^2 + 3\lambda^2 + \sqrt{(3g^2 + \lambda^2)^2 + 24g^2 \lambda^2}},
\]
\[
\lambda_{2,0} = 0
\tag{7}
\]
and the new coefficients for two collective excitations can be found from the normalization condition
\[
c^{(2)}_{i,2} = \frac{\chi}{\sqrt{2g}} \left( 1 - \frac{3g^2}{2g^2 + g^2 - \lambda_{2,2}^2} \right) c^{(2)}_{i,1},
\]
\[
c^{(2)}_{i,3} = \frac{\lambda_{2,1} - \lambda_{2,3}}{\sqrt{2g}} c^{(2)}_{i,1}, \quad c^{(2)}_{i,4} = \frac{-3\lambda \chi c^{(2)}_{i,1}}{\sqrt{2(2g^2 + g^2 - \lambda_{2,2}^2)}},
\tag{8}
\]
\[
c^{(2)}_{i,5} = \left( 1 - \frac{3g^2}{2g^2 + g^2 - \lambda_{2,2}^2} \right) c^{(2)}_{i,1}
\tag{8}
\]
for \(i = -2 \ldots 2\).

Here the indices of \(c^{(m)}_{i,j}: m, n\) and \(i\) indicate the number of quanta of energy in the system, the new number of the quasi-energy level corresponding to the energy \(h(nm\omega + \lambda_n)\) and the state \(|\psi_{mi}\rangle\) used in superpositions. The expressions (5), (7) and (6), (8) represent the frequencies of new quasi-levels of energy and wave function coefficients for single and two excitations.

### 2.2 Three cavity case

For three coupled doped cavities we can obtain the wave function of the single excitation in a similar way. This methodology can be extended for larger numbers of excitations representing the eigenstates and wave functions numerically. Taking this into account only the master equation is studied for one excitation in the system, for which it is possible to obtain analytical solutions. The degenerate states for one excitation are
\[
|\psi_{11}\rangle = \frac{1}{\sqrt{3}} \left( |e, 0\rangle_0 |g, 0\rangle_1 + |g, 0\rangle_0 |e, 0\rangle_1 \right) + |g, 0\rangle_0 |g, 0\rangle_0 |e, 0\rangle_0,
\]
\[
|\psi_{12}\rangle = \frac{1}{\sqrt{3}} \left( |g, 1\rangle_0 |g, 0\rangle_0 |e, 0\rangle_0 + |g, 0\rangle_0 |g, 0\rangle_0 |g, 0\rangle_0 + |g, 0\rangle_0 |g, 0\rangle_0 |e, 0\rangle_0 \right)
\tag{9}
\]
We observe other superposition functions of the single excitation in the three cavities. These are the superposition with one photon in one of them and atomic excitation
in another. Similarly, the eigenstates and eigenvalues of the interaction Hamiltonian are found:

\[ \lambda_{1,1(-1)} = \chi \pm \sqrt{\chi^2 + g^2}, \]
\[ c_{1,1} = \sqrt{g^2/(g^2 + \lambda_{1,1}^2)}, \]
\[ c_{1,2} = \sqrt{\chi^2/(g^2 + \lambda_{1,1}^2)} \quad \text{for} \quad i = -1, 1. \]

So, the analyzed interaction allows us to get two new quasi-energy levels. For two excitations we observe five collective states:

\[ |\psi_{21}⟩ = \frac{1}{\sqrt{3}} (|e, 0⟩_1 |e, 0⟩_2 |g, 0⟩_3 + |e, 0⟩_1 |g, 0⟩_2 |e, 0⟩_3 + |g, 0⟩_1 |e, 0⟩_2 |e, 0⟩_3), \]
\[ |\psi_{22}⟩ = \frac{1}{\sqrt{6}} (|e, 0⟩_1 (|g, 1⟩_2 |g, 0⟩_3 + |g, 0⟩_2 |g, 1⟩_3) + |g, 1⟩_1 |e, 0⟩_2 |g, 0⟩_3 + |g, 0⟩_1 |e, 0⟩_2 |g, 1⟩_3 + (|g, 1⟩_1 |g, 0⟩_2 |g, 0⟩_3 + |g, 0⟩_1 |e, 0⟩_2 |g, 1⟩_3), \]
\[ |\psi_{23}⟩ = \frac{1}{\sqrt{3}} (|g, 1⟩_1 |g, 1⟩_2 |g, 0⟩_3 + |g, 1⟩_1 |e, 0⟩_2 |g, 1⟩_3 + |g, 0⟩_1 |g, 1⟩_2 |g, 1⟩_3), \]
\[ |\psi_{24}⟩ = \frac{1}{\sqrt{3}} (|e, 1⟩_1 |g, 0⟩_2 |g, 0⟩_3 + |g, 0⟩_1 |e, 1⟩_2 |g, 0⟩_3 + |g, 0⟩_1 |e, 0⟩_2 |g, 1⟩_3), \]
\[ |\psi_{25}⟩ = \frac{1}{\sqrt{3}} (|g, 2⟩_1 |g, 0⟩_2 |g, 0⟩_3 + |g, 0⟩_1 |g, 2⟩_2 |g, 0⟩_3 + |g, 0⟩_1 |g, 0⟩_2 |g, 2⟩_3), \]

and the same number of solutions for the characteristic equation for \( g = \chi \): \( \lambda_{2,1} = -2.43065, \lambda_{2,2} = -0.771049, \lambda_{2,3} = 0.294764, \lambda_{2,4} = 1.73598 \) and \( \lambda_{2,5} = 4.17096 \). Here \( \lambda_{i,j} = \lambda_{i,j}/g \). For three excitations, from 38 states we obtain 10 collective states. It is clear that for two or more excitations in three cavities we can solve the Schrödinger equation only numerically.

2.3. \( N \) cavity case

Let us assume that we have \( N \) cavities placed in the vertices of a regular polygon with \( N \) equal sides. If the number of cavities is large, as is represented in figure 2, according to the interaction Hamiltonian (2) for one indistinguishable excitation of atoms or photons we observe the following degenerate wave functions:

\[ |ψ_1⟩ = \frac{1}{\sqrt{N}} (|e_1 g_2 .. g_N⟩ + ... + |g_1 g_2 .. e_N⟩) |0⟩_{02 .. 0N}, \]
\[ |ψ_2⟩ = \frac{1}{\sqrt{N}} |g_1 g_2 .. g_N⟩ (|1⟩_{02 .. 0N} + ... + |0⟩_{02 .. 1N}). \]

It is not difficult to observe that the actions of the interaction Hamiltonian on these functions are \( H_t |ψ_1⟩ = \hbar g |ψ_1⟩ \) and \( H_t |ψ_2⟩ = \hbar g |ψ_1⟩ + 2hχ |ψ_2⟩ \). In this case, the eigenvalues and eigenfunctions of the interaction Hamiltonian are equivalent to the three-cavity case \( \lambda_{1,1(-1)} = \chi \pm \sqrt{\chi^2 + g^2} \) and \( c_{1,1} = \sqrt{g^2/(g^2 + \lambda_{1,1}^2)}, \quad c_{1,2} = \sqrt{\chi^2/(g^2 + \lambda_{1,1}^2)} \quad \text{for} \quad i = -1, 1. \)

In conclusion, we observe the following behavior of the excited atoms in two or more cavities. Two excitations in two cavities are described by five collective states. In three cavities, three excitations are described by ten collective states. So, we can observe that for a small number of excitations the following expression can be applied for the number of the collective states \( N = n_x^2 + 1 \), where \( n_x \) is the number of excitations in the system. This behavior is applicable to the number of excitations \( n_x \) which are less than the number of cavities. This law is violated beginning with four excitations. Indeed for four excitations it is not difficult to observe that we can construct 19 collective states. According to our representation, this corresponds to the number \( n_x^2 + 3 \). In our opinion, this modification is connected with the packing of the cavities in space, and the atomic and photon excitations between cavities. For example, four cavities can be placed within the circle and, of course, these cavities can be packed in the vertices of a regular tetrahedron. So, we have two different types of symmetry, and two cases with different numbers of collective states for four excitations.

3. The master equation and behavior of the population of new quasi-energy levels

Let us now obtain the master equation for collective excitations of two radiators placed in two coupled cavities.
with the losses. In order to find the master equation for such a system let us introduce in the Hamiltonian (2) the coupling of cavity modes with the vacuum of EMF $\hat{H}_0^e = \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} \kappa \hat{b}^\dagger_{jk} \hat{b}_{jk} + \text{h.c.}$. Here the photon loss $\kappa$ indicates the transformation of cavity photons into the free EMF photons described by the new annihilation and creation boson operators $\hat{b}_j$ and $\hat{b}_j^\dagger$. Using the collective excitation states (6) and (8) obtained in the first section, we can represent the cavity field operators through the new collective states. Considering that the energy is measured from the relative level $E = 0$ (see figure 1), the Hamiltonian for two coupled cavities can be represented as $\hat{H} = \hat{H}_0 + \hat{H}_1$, where

$$\hat{H}_0 = \sum_{j=1}^{2} \hbar \omega \vert i i \rangle \langle i i \vert + \sum_{k} \hbar \omega_{k} \hat{b}_{jk}^\dagger \hat{b}_{jk} ,$$

$$\hat{H}_1 = \sum_{k} \kappa \hat{b}_{jk}^\dagger \sum_{i=1}^{2} \sqrt{2} c_{i,j}(0) \langle i i \vert + \text{h.c.}$$

(11)

Here $\vert i i \rangle = \sum_{j=1,2} c_{i,j}^{(1)} \vert \psi_{ej} \rangle$ are the Hilbert vectors of collective excitation, and the re-normalized frequencies $\omega \pm \lambda_1 = \omega + \frac{1}{2} \kappa$ and $\lambda_1 = \frac{1}{2} \sqrt{4 \omega^2 + \kappa^2}$, $c_{(2)} = c_{(1)}^{(1)}$. In the Hamiltonian, we have considered that the action of the creation operators in the state with the maximum number of excitations $m = 1$ is neglected. $\hat{a}_j^\dagger \vert \psi_{n+m} \rangle = 0$ since such an excitation is impossible for the system in interaction with the external vacuum field.

Considering the large number of degrees of freedom of the free EMF vacuum, we can eliminate from the density matrix equation the boson operators of external EMF. In the interaction picture the density matrix equation is $i\hbar \frac{\delta \hat{\rho}(t)}{\delta t} = [\hat{H}_I(t), \hat{\rho}(t)]$ where $\hat{H}_I(t) = \text{exp}[i\hat{H}_0/\hbar] \hat{H}_I \times \text{exp}[-i\hat{H}_0/\hbar]$ is the Hamiltonian in the interaction picture. Using the method of the projection operator on the vacuum field $\mathcal{P} = \{0\} \{0\} \text{Tr}_{\text{vac} - 1}$, we can represent the density matrix through slower $\hat{\rho}_I(t) = \mathcal{P} \hat{\rho}(t) \mathcal{P}$ and rapidly oscillating $\hat{\rho}_0(t) = \mathcal{P} \hat{\rho}(t) \mathcal{P}$ parts, respectively. $\mathcal{P} = 1 - \mathcal{P}$. It can be shown that $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{P} \mathcal{P} = 0$. Following the well-known procedure of the elimination of the rapid oscillatory part of the density matrix [14], one can obtain the equation of the density matrix $\hat{W}(t) = \text{Tr}_{\text{vac}}[\hat{\rho}(t)]$ for the cavity excitations

$$\frac{d\hat{W}(t)}{dt} - i\lambda_1 \hat{W}(t) (\hat{U}_{1,1}^\dagger(t) - \hat{U}_{1,1}^\dagger(t)) = -y [\hat{W}(t)c_{1,0}^\dagger(t) + c_{2,0}^\dagger(t)] + c_{1,0}^\dagger(t) + \text{h.c.}$$

(12)

Here $y = \sum_{k} 2 \pi \delta (\omega - \omega_k) \kappa^2/\hbar$ and $\hat{U}_{1,0}^\dagger(t) = \vert \alpha \rangle \langle y \vert$ is the Ladder operator which indicates the transition between the new quasi-levels for a single cavity–atom excitation. These operators satisfy the commutation relationship $[\hat{U}_{1,0}^\dagger, \hat{U}_{1,0}^\dagger] = \delta_{\alpha,0} \hat{U}_{1,0}^\dagger - \delta_{\alpha,0} \hat{U}_{1,0}^\dagger$. The generalized equation (12) takes into consideration the cooperative decay processes of two cavities in interaction with the external vacuum field.

Let us study the decay rate of a single collective excitation placed in two or three cavities. Considering that the new collective states $|1, 1\rangle$ and $|1, -1\rangle$ can be easily prepared, we find the following closed system of equations for the population and correlation functions of photons and radiators:

$$\frac{dx(t)}{dt} = -x(t) - pu(t),$$

$$\frac{dy(t)}{dt} = -p^2 y(t) - pu(t),$$

$$\frac{du(t)}{dt} = -2qu(t) - \left[ p^2 + 1 \right] u(t) - \frac{p}{2} x(t) + y(t),$$

$$\frac{dw(t)}{dt} = 2qu(t) - \left[ p^2 + 1 \right] w(t).$$

Here $x = t / \tau_2$, $p = c_{2,1}$, $q = \lambda_1 / \tau_2$, $x = (U_{1,1}^\dagger(t), y = (U_{1,1}^\dagger) - (U_{1,1}^\dagger) / 2$ and $w = ((U_{1,1}^\dagger) - (U_{1,1}^\dagger))/2i$. For simplicity, we have represented the relaxation times of two transitions from the two upper to the ground states through the constants $1 / \tau_{1,2} = 2y c_{2,1} / 2$. The evolution of the population of new quasi-levels of energy is studied considering the situation when only the state $|1, 1\rangle$, $((U_{1,1}^\dagger) = 1)$ is populated while the second excited state $|1, -1\rangle$ is unpopulated. From the physical viewpoint the partial transfer of excitation from the state $|1, 1\rangle$ to the state $|1, -1\rangle$ is remarkable. This transfer is represented in figure 2, in which (b) represents the nutation of the excitation between the upper states $|1, 1\rangle$ and $|1, -1\rangle$. This nutation is accompanied by the transfer of the excitation to the state $|1, -1\rangle$ which in the processes of interaction with the vacuum field becomes entangled with the state $|1, 1\rangle$. During the decay time, both states $|1, 1\rangle$ and $|1, -1\rangle$ become unpopulated. In order to obtain the entropy of excitations in the cavities let us represent the density matrix through the excitation correlations $\hat{W}(t) = (1 - x(t) + y(t)) U_{1,0}^\dagger(t) + (y(t) U_{1,1}^\dagger(t) + (u(t) + iw(t)) U_{1,1}^\dagger(t)$.

The entropy of the cavity excitations is described by the expression

$$S = - (\hat{U}_{1,0}^\dagger(t) \log(\hat{U}_{1,0}^\dagger(t))) - U_{1,0} \log U_{1,0} - U_{1,2} \log U_{1,2},$$

where $U_{1,2} = (x + y) / 2 + \sqrt{(x - y)^2 / 4 + w^2 + u^2}$, $U_{1,2} = (x + y) / 2 - \sqrt{(x - y)^2 / 4 + w^2 + u^2}$ and $\langle \hat{U}_{1,0}^\dagger(t) \rangle = 1 - x - y$. 

Figure 3. The relative time dependence of entropy for initial conditions $\langle \hat{U}_{1,1}^\dagger(t) \rangle = 0$, $\langle \hat{U}_{1,1}^\dagger(t) \rangle = 1$ (continue line), $\langle \hat{U}_{1,0}^\dagger(t) \rangle = 1/2$, $\langle \hat{U}_{1,1}^\dagger(t) \rangle = 1/2$ (dashed line), $q = 3$, $p = 1$. 

\[ S = \sum_{j} (\hat{U}_{1,j}^\dagger(t) \log(\hat{U}_{1,j}^\dagger(t))) - U_{1,j} \log U_{1,j} - U_{1,2} \log U_{1,2}, \]

where \[ U_{1,2} = (x + y) / 2 + \sqrt{(x - y)^2 / 4 + w^2 + u^2}, \]

\[ U_{1,2} = (x + y) / 2 - \sqrt{(x - y)^2 / 4 + w^2 + u^2} \] and \[ \langle \hat{U}_{1,0}^\dagger(t) \rangle = 1 - x - y. \]
From the time dependence of entropy and inversion, we observe that when the population of excited and ground states becomes equal, the entropy takes the maximum value. This maximum value corresponds to the creation of an entangled state between excited and ground states (see figure 3). Another interesting effect is the decay process of excited states. Here two situations are possible, the first of which corresponds to the preparation of excited states in the superposition of the states $|1, 1\rangle$ and $|1, -1\rangle$. In this case, the entropy increases from a non-zero value $S(0) = 0.5$. Preparing the system in only one excited state, we can observe during the process of spontaneous emission that the excitation partially ‘jumps’ in the second excited state $|1, 1\rangle$ when the population of another state $(\hat{U}^{1, -1}_{1, 1}(0)) = 0$. In this case, after the decay process of spontaneous emission, the electron jumps to the unpopulated state. A similar effect is known from other investigations and is experimentally observed in three level systems [15].

4. Conclusions

We have introduced the cooperative excitations of the EMF of coupled cavities in interaction with two-level radiators (9) and (10). With the increasing number of excitations, the solution of the large number of indistinguishable atoms and photons becomes difficult in the analytical representation (see two cavities with two excitations (7) and (8)). Unlike the traditional approach [2, 16] it examines the behavior of two collective excitations of coupled cavities considering the local symmetry of this system (see (7) and (8)). Considering the indistinguishability principle between the atomic and photonic excitations, in this paper it is proposed that the collective excitations for the description of the interaction $q$-bits be introduced. For this, we have proposed the exact solutions for two excitations in two cavities (9) and (10). Considering that the energy of the dressed states of the atom + cavity field is larger than the decay rates from the cavities, we have introduced the new interaction Hamiltonian of cooperative excitations with the external vacuum field (11).

From exact and numerical solutions it follows that the time evolution of a damped system of equations depends on the initial preparation of the collective states $(\hat{U}^{1, 1}_{1, 1}(0)) = \alpha^2_1$ and $(\hat{U}^{1, -1}_{1, -1}(0)) = \alpha^2_2$. In comparison with other approaches, we observe the damped oscillations of the population inversion between two excited states, as is represented in figure 2. Two limits are observed for this dependence. The first limit $c_2/c_1 \rightarrow 1$ corresponds to the bad coupling between the cavities and the second limit $c_2/c_1 \rightarrow 0$ is connected to high coupling between cavities.

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