DEFINABLE MAXIMAL INDEPENDENT FAMILIES

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Abstract. We study maximal independent families (m.i.f.) in the projective hierarchy. We show that (a) the existence of a \( \Sigma^1_2 \) m.i.f. is equivalent to the existence of a \( \Pi^1_1 \) m.i.f., (b) in the Cohen model, there are no projective maximal independent families, and (c) in the Sacks model, there is a \( \Pi^1_1 \) m.i.f. We also consider a new cardinal invariant related to the question of destroying or preserving maximal independent families.

1. Introduction

In descriptive set theory, a recurrent line of inquiry is whether objects defined in a non-constructive way can exist on given levels of the projective hierarchy. For example, an ultrafilter cannot be \( \Sigma^1_1 \). Mathias [9] proved that there are no \( \Sigma^1_1 \) maximal almost disjoint (mad) families. If \( V = L \) then there are \( \Delta^1_2 \) ultrafilters and \( \Pi^1_1 \) mad families.

In this paper, we look at maximal independent families from the definable point of view. Our main results are Theorem 2.1, Theorem 3.1 and Theorem 4.1, stating that the existence of a \( \Sigma^1_2 \) m.i.f. is equivalent to the existence of a \( \Pi^1_1 \) m.i.f., that in the Cohen model there are no projective m.i.f.'s, and that in the Sacks model there is a \( \Pi^1_1 \) m.i.f., respectively.

Definition 1.1. A family \( I \subseteq [\omega]^\omega \) is independent if for all \( a_1, \ldots, a_n \in I \) and different \( b_1, \ldots, b_\ell \in I \)

\[ a_1 \cap \ldots a_n \cap (\omega \setminus b_1) \cap \cdots \cap (\omega \setminus b_\ell) \text{ is infinite.} \]

A family \( I \subseteq [\omega]^\omega \) is called a maximal independent family (m.i.f.) if it is independent and maximal with regard to this property.

We will typically use the following abbreviation:

\[ \sigma(\bar{a}; \bar{b}) := a_1 \cap \ldots a_n \cap (\omega \setminus b_1) \cap \cdots \cap (\omega \setminus b_\ell) \]

where it will be assumed that all of the \( a_i \) are different from all of the \( b_j \). Note that maximality of \( I \) is equivalent to:

\[ \forall X \in [\omega]^\omega \exists a_1, \ldots, a_n, b_1, \ldots, b_\ell \in I \text{ s.t. } \sigma(\bar{a}; \bar{b}) \subseteq^* X \text{ or } \sigma(\bar{a}; \bar{b}) \cap X =^* \emptyset. \]
By identifying the space $[\omega]^{\omega}$ with $2^{\omega}$ via characteristic functions, one can consider independent families as subsets of the reals and study their complexity in the projective hierarchy.

**Lemma 1.2.** If $\mathcal{I}$ is a $\Sigma^1_n$ m.i.f. then it is $\Delta^1_n$.

**Proof.** Suppose $\mathcal{I}$ is a $\Sigma^1_n$ m.i.f. Then $X \notin \mathcal{I}$ iff:

$$\exists a_1, \ldots, a_n, b_1, \ldots, b_l \in \mathcal{I} \text{ s.t. } X \notin \{a_1, \ldots, a_n, b_1, \ldots, b_l\} \text{ and } \sigma(\bar{a}, \bar{b}) \subseteq^* X \text{ or } \sigma(\bar{a}, \bar{b}) \cap X =^* \emptyset.$$ 

This statement is easily seen to be $\Sigma^1_n$. □

**Theorem 1.3** (Miller; [10]). There is no analytic m.i.f.

An analysis of Miller’s proof shows that it really only uses the Baire property of analytic sets. In particular, if we use $\Sigma^1_n(C)$ to denote the statement “all $\Sigma^1_n$ sets have the Baire property”, then Miller’s proof shows that for any $n$, $\Sigma^1_n(C)$ implies that there are no $\Sigma^1_n$ m.i.f. (see Corollary [X]). It follows that

1. In the Hechler model, as well as the Amoeba or Amoeba-for-category model, there is no $\Sigma^1_2$ m.i.f.,
2. In the Solovay model (the Lévy-collapse of an inaccessible), as well as Shelah’s model for the Baire Property without inaccessibles [11], there is no projective m.i.f.,
3. In $L(\mathbb{R})$ of the above two models, there is no m.i.f. at all, and
4. $AD \Rightarrow$ there is no m.i.f. (this follows because under $AD$ all sets of reals have the property of Baire, see again [X]).

In this paper, we prove a stronger result, namely, that in the Cohen model there is no projective m.i.f., and in the $L(\mathbb{R})$ of the Cohen model there is no m.i.f. at all. Notice that since $\Sigma^1_1(C)$ is false in the Cohen model, this shows that the converse implication “$\Sigma^1_1(C) \iff \#\Sigma^1_n$-m.i.f.” consistently fails.

On the other hand, it is easy to construct a m.i.f. by induction using a wellorder of the reals, and thus, it is not hard to see that in $L$ there exists a $\Sigma^1_2$ m.i.f. In [10] Miller used sophisticated coding techniques to show that, in fact, this m.i.f. can be constructed in a $\Pi^1_1$ fashion. Building on an idea of Asger Törnquist [13], we show that in fact this proof is unnecessary since one can derive, directly in $\text{ZFC}$, that if there exists a $\Sigma^1_2$ m.i.f. then there exists a $\Pi^1_1$ m.i.f.

A construction originally attributed to Eisworth and Shelah (see [3]), implicitly appearing in Shelah’s proof of $i < \mu$ [12] and elaborated in [6], yields a forcing for generically adding a Sacks-indestructible m.i.f. In this paper we show that this family can be defined in a $\Sigma^1_2$ way in $L$. Therefore, in the countable support iteration of Sacks forcing, as well as the product of Sacks forcing, starting from $L$, there exists a $\Sigma^1_2$ m.i.f., and hence a $\Pi^1_1$ m.i.f. In fact, a slight modification produces a family which is indestructible by the poset used in [12], which shows that the consistency of $i < \mu$ can be witnessed by a $\Pi^1_1$ m.i.f.
2. $\Sigma^1_2$ and $\Pi^1_1$ M.I.F.'s

**Theorem 2.1.** If there exists a $\Sigma^1_2$ m.i.f. then there exists a $\Pi^1_1$ m.i.f.

**Proof.** Suppose $I_0$ is a $\Sigma^1_2$ maximal independent family. Let $F_0 \subseteq (\omega^\omega)^2$ be a $\Pi^1_1$ set such that $I_0$ is the projection of $F_0$. Consider the space $\omega \cup 2^{<\omega}$ as a disjoint union, and consider the mapping

$$g : (\omega^\omega)^2 \rightarrow \mathcal{P}(\omega \cup 2^{<\omega})$$

$$g : (x, y) \mapsto x \cup \{\chi_y | n \in \omega \}$$

where $\chi_y$ is the characteristic function of $y$. It is not hard to see that $g$ is a continuous function (in the sense of the space $\mathcal{P}(\omega \cup 2^{<\omega})$).

By $\Pi^1_1$-uniformization, there exists a $\Pi^1_1$ set $F \subseteq F_0$ which is the graph of a function, i.e., $\forall \omega \in I_0 \exists y ((x, y) \in F)$. We set $I := g[F]$ and claim that $I$ is a $\Pi^1_1$ m.i.f.

To see that $I$ is $\Pi^1_1$, note that for $z \in [\omega \cup 2^{<\omega}]^\omega$, there is an explicit way to recover $x$ and $y$ such that $g(x, y) = z$, if such $x$ and $y$ exist. More precisely: $z \in I$ if and only if

1. $z \cap 2^{<\omega}$ is a single branch,
   i.e., $\forall n \exists s \in z \cap 2^n$ and $\forall s, t \in z \cap 2^{<\omega} (|s| < |t| \rightarrow s \subset t)$,
2. $\forall \omega \in z \cap 2^{<\omega}$ there is an explicit way to recover $x$ and $y$ such that $g(x, y) = z$, if such $x$ and $y$ exist.

This gives a $\Pi^1_1$ definition of $I$.

To see that $I$ is independent, suppose we have $z_1, \ldots, z_n$ and $w_1, \ldots, w_\ell \in I$, the $z$'s being different from the $w$'s. Write $a_i := z_i \cap \omega$ and $b_j := w_j \cap \omega$. Then all $a_i$ and $b_j$ are in dom$(F) = I_0$, and moreover, since $F$ is a function, the $a_i$'s are different from the $b_j$'s. But then we have that $\sigma(z_1, \ldots, z_n; w_1, \ldots, w_\ell) \supseteq \sigma(a_1, \ldots, a_n; b_1, \ldots, b_\ell)$ is finite, since the latter set is infinite by the independence of $I_0$.

To show maximality of $I$, suppose $W \in [\omega \cup 2^{<\omega}]^\omega$ and $W \notin I$. Let $A := W \cap \omega$. By maximality of $I_0$, there are $a_1, \ldots, a_n \in I_0$ and different $b_1, \ldots, b_\ell \in I_0$ such that $\sigma(a_1, \ldots, a_n; A; b_1, \ldots, b_\ell)$ is finite or $\sigma(a_1, \ldots, a_n; b_1, \ldots, b_\ell, A)$ is finite, w.l.o.g. the former. Then there are $z_1, \ldots, z_n$ and different $w_1, \ldots, w_\ell$ such that $a_i = z_i \cap \omega$ and $b_j = w_j \cap \omega$. To make sure that the $\omega^{2^{<\omega}}$-part of the $z$'s and the $w$'s does not make the intersection infinite, we pick two additional $t_0 \neq t_1 \in I$, different from the $z$'s and the $w$'s. Let $t_0 = g(x_0, y_0)$ and $t_1 = g(x_1, y_1)$. If $y_0 = y_1$, then $(t_0 \setminus t_1) \cap 2^{<\omega} = \emptyset$, hence $\sigma(x_1, \ldots, x_n, W, t_0; w_1, \ldots, w_\ell, t_1)$ is finite. If, on the other hand, $y_0 \neq y_1$, then the sets $\{\chi_{y_0} | n < \omega\}$ and $\{\chi_{y_1} | n < \omega\}$ are almost disjoint, so $(t_0 \cap t_1) \cap 2^{<\omega}$ is finite. In that case, $\sigma(x_1, \ldots, x_n, W, t_0, t_1; w_1, \ldots, w_\ell)$ is finite. So in any case, $I \setminus \{W\}$ is not independent, completing the proof. \qed

Clearly the above proof also holds pointwise for every parameter, i.e., if there is a $\Sigma^1_2(a)$ m.i.f. then there is a $\Pi^1_1(a)$ m.i.f. However, since $\Pi^1_1$-uniformisation is essential, the following natural question remains open:

**Question 2.2.** Does the existence of a $\Sigma^1_n$ m.i.f. imply the existence of a $\Pi^1_{n-1}$ m.i.f., for $n > 2$?

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1Following the suggestion of the anonymous referee, an alternative argument is as follows: Extend $g$ to $g'$ defined on the compact set $\mathcal{P}(\omega)^2$ in such a way that $g'$ is injective. Then $g'$ is a homeomorphism between $\mathcal{P}(\omega)^2$ and the compact set $g'(\mathcal{P}(\omega)^2)$, thus $I = g'[F] = g[F]$ is a $\Pi^1_1$ subset of the compact set $g'([\mathcal{P}(\omega)^2])$. 

3. Projective M.I.F.’s

The main result of this section is:

**Theorem 3.1.** In the Cohen model $W$ (that is, the model obtained by adding at least $\omega_1$ Cohen reals over any model of set theory) there are no projective m.i.f.’s. Furthermore, in $L(\mathbb{R})^W$, there are no m.i.f.’s.

The theorem is proved in three steps. First, we isolate a new regularity property which was implicit in Miller’s original proof ([10] proof of 10.28]).

**Definition 3.2.** A tree $T \subseteq 2^{<\omega}$ is called perfect almost disjoint (perfect a.d.) if it is a perfect tree and $\forall x \neq y \in [T]$ the set $\{n \mid x(n) = y(n) = 1\}$ is finite. A tree $S \subseteq 2^{<\omega}$ is called perfect almost covering (perfect a.c.) if it is a perfect tree and $\forall x \neq y \in [T]$, the set $\{n \mid x(n) = y(n) = 0\}$ is finite.

**Definition 3.3.** A set $X \subseteq 2^{\omega}$ satisfies the perfect-a.d.-a.c. property, abbreviated by $\mathcal{S}_{ad-ac}$, if there exists a perfect a.d. tree $T$ with $[T] \subseteq X$, or there exists a perfect a.c. tree $S$ with $[S] \subseteq X$. 

**Remark 3.4.** Note that one could also define the symmetric property: $X \subseteq 2^{\omega}$ satisfies the perfect-a.c.-a.d. property if there exists a perfect a.c. tree $S$ with $[S] \subseteq X$, or there exists a perfect a.d. tree $T$ with $[T] \cap X = \emptyset$. A curious aspect of our proof is that either of these two properties yields the proof in an analogous fashion, but since one of them is sufficient we pick the former.

**Lemma 3.5.** Let $\Gamma$ be a pointclass closed under existential quantification over reals. Then $\Gamma(\mathcal{S}_{ad-ac})$ implies that there is no m.i.f. in $\Gamma$. In particular, $\Sigma^1_n(\mathcal{S}_{ad-ac}) \Rightarrow \#\Sigma^1_n$-m.i.f.

**Proof.** Assume $\mathcal{I}$ is a $\Gamma$ m.i.f. Define

$$H := \{X \mid \exists \bar{a}, \bar{b} \subset \mathcal{I} \text{ disjoint s.t. } \sigma(\bar{a}; \bar{b}) \subseteq^* X\}$$

$$K := \{X \mid \exists \bar{a}, \bar{b} \subset \mathcal{I} \text{ disjoint s.t. } \sigma(\bar{a}; \bar{b}) \cap X =^* \emptyset\}.$$ 

The maximality of $\mathcal{I}$ implies that $[\omega]^{\omega} = H \cup K$. Moreover, $H$ is a $\Gamma$ set ($K$ also is, but this turns out to be irrelevant).

From $\Gamma(\mathcal{S}_{ad-ac})$ we then obtain a perfect almost disjoint tree $T$ with $[T] \subseteq H$, or a perfect almost covering tree $S$ with $[S] \cap H = \emptyset$, hence $[S] \subseteq K$. Assume the former.

For each $X \in [T]$ let $a_1^X, \ldots, a_n^X$ and $b_1^X, \ldots, b_l^X$ witness the fact that $X \in H$. Applying the $\Delta$-systems lemma to the family $\{\{a_1^X, \ldots, a_n^X, b_1^X, \ldots, b_l^X\} \mid X \in [T]\}$, find an uncountable subset of $[T]$ with a fixed root $R$. Moreover, an uncountable sub-family of this family has the property that the elements of the root $R$ have the same function in the sense of “being an $a_i^X$” or “being a $b_j^X$”. It follows that, for distinct $X, Y$ from this family, we have:

$$\{a_1^X, \ldots, a_n^X, a_1^Y, \ldots, a_{n_Y^*}\} \cap \{b_1^X, \ldots, b_l^X, b_1^Y, \ldots, b_{l_Y}^Y\} = \emptyset.$$ 

But then the boolean combination $\sigma(\bar{a}^X \cup \bar{a}^Y; \bar{b}^X \cup \bar{b}^Y) \subseteq^* X \cap Y =^* \emptyset$, contradicting the independence of $\mathcal{I}$. 

It is well-known that Cohen forcing adds perfect almost disjoint and perfect almost covering trees of Cohen reals (see [10] Lemma 10.29] or [2] proof of Lemma 3.6.23)). We include a proof for the sake of completeness.
Lemma 3.6. Let \([s]\) be a basic open set and \(c \in [s]\) a Cohen real over the ground model \(V\). Then in \(V[c]\) there exists a perfect almost disjoint set and a perfect almost covering set of Cohen reals over \(V\), contained inside \([s]\).

Proof. For simplicity assume that \([s] = 2^\omega\), and we only prove the case with almost disjoint trees since the other case is similar. Let \(P\) denote the partial order consisting of finite trees \(T \subseteq 2^{\omega \times \omega}\) with the property that there are \(0 = k_0 < k_1 < \cdots < k_\ell\) such that \(T \subseteq 2^{\leq k_\ell}\) and for every \(i < \ell\), there is at most one \(t \in T\) where \(t|(k_i, k_{i+1})\) is not constantly 0. The trees are ordered by end-extension.

If \(G\) is \(P\)-generic, let \(T_G\) denote the naturally defined limit of the trees in \(G\). By a standard genericity argument, \([T_G]\) must be perfect. Given any two branches \(x, y \in [T_G]\), the construction ensures that for all \(n\) after the point where \(x\) and \(y\) split, either \(x(n) = 0\) or \(y(n) = 0\), therefore \(x\) and \(y\) are almost disjoint. To show that every \(x \in [T_G]\) is Cohen over \(V\), let \(D\) be Cohen-dense and \(T \subseteq \mathcal{P}\) fixed. Enumerate all terminal nodes of \(T \subseteq 2^{\leq k_\ell}\) by \(\{t_1, \ldots, t_j\}\). Extend \(T\) to \(T' \subseteq 2^{\leq k_{\ell+1}}\) such that each terminal node \(t_i \in 2^{k_\ell}\) gets extended to \(t'_i \in 2^{k_{\ell+1}}\) such that \(t'_i\) is constantly zero on all intervals except \([k_{\ell+i}, k_{\ell+i+1})\) and \(t'_i|k_{\ell+i+1}\) \(\in D\). Thus every branch of \(T_G\) must meet \(D\).

Since \(P\) is countable, it is isomorphic to Cohen forcing. Therefore, if \(V[c]\) is a Cohen extension of \(V\), it is also a \(P\)-generic extension of \(V\), so there exists a perfect almost disjoint set \([T_G]\) of Cohen reals.

Corollary 3.7. Let \([s]\) be basic open and assume \(A\) is comeager in \([s]\). Then \(A \cap [s]\) contains a perfect a.d. set and a perfect a.c. set.

Proof. Let \(M\) be a countable model containing \(A\). Apply the previous lemma in \(M[c]\) and note that the perfect a.d. and perfect a.c. sets of Cohen reals must be contained in \(A \cap [s]\) by comeagerness.

Corollary 3.8. Let \(\Gamma\) be a pointclass closed under existential quantification over reals. Then \(\Gamma(\mathbb{C})\) implies that there is no m.i.f. in \(\Gamma\). In particular, \(\Sigma^1_4(\mathbb{C}) \Rightarrow \#\Sigma^1_4\text{-m.i.f.}\)

Proof. Immediate using Lemma 3.6 and Corollary 3.7.

Using Lemmata 3.5 and 3.6, we can complete the proof of the theorem.

Proof of Theorem 3.7. Let \(W := V^\mathbb{C}_\kappa\) (for any \(\kappa > \omega\)), and let \(A\) be a set in \(W\) defined by a formula \(\Phi(x)\) with real or ordinal parameters, w.l.o.g. all of which are in \(V\). In \(W\), let \(c\) be Cohen over \(V\), and assume w.l.o.g. that \(\Phi(c)\). Then \(V[c] \models \text{“}p \models_Q \Phi(\dot{c})\text{”}\), where \(Q\) is the remainder forcing leading from \(V[c]\) to \(W\) and \(p\) is some \(Q\)-condition. However, since \(\mathbb{C}_\kappa\) is the product forcing, \(Q\) is isomorphic to \(\mathbb{C}_\kappa\). Moreover, since \(\mathbb{C}_\kappa\) is homogeneous we can assume that \(p\) is the trivial condition, hence we really have:

\[ V[c] \models \text{“} p \models_{\mathbb{C}_\kappa} \Phi(\dot{c}) \text{”} \]

Let \([s]\) be a Cohen condition with \(c \in [s]\) forcing this statement in \(V\). By Lemma 3.6 first we find a perfect a.d. tree \(T\) with \(T \subseteq V[c]\), \([T] \subseteq [s]\) and such that all \(x \in [T]\) are Cohen over \(V\). Note that this fact remains true in \(W\), since “being a perfect set of Cohen reals” is upwards absolute. Now, for any such \(x \in [T]\) (in \(W\)), we have that \(x \in [s]\), and therefore \(V[x]\) satisfies whatever \([s]\) forces, in particular

\[ V[x] \models \text{“} p \models_{\mathbb{C}_\kappa} \Phi(\dot{x}) \text{”} \]
But, again, the remainder forcing leading from $V[x]$ to $W$ is isomorphic to $\mathbb{C}_X$, and it follows that $W \models \Phi(x)$.

Similarly, we also find a perfect a.c. tree $S$ with exactly the same properties. Thus $A$ satisfies both $S_{ad-ac}$ and $S_{ac-ad}$, and the rest follows by Lemma 3.6. 

\section{\Pi^1_1 \text{-m.i.f. in the Sacks model}}

In contrast to the above, this section is devoted to the following result:

\bf{Theorem 4.1.} \textit{In the countable-support iteration of Sacks forcing, as well as the countable-support product of Sacks forcing, starting from $L$, there exists a $\Pi^1_1$ m.i.f.}

As a consequence, we obtain $\text{Con}(\exists \Pi^1_1$-m.i.f. of size $< 2^{\aleph_0})$, and in fact even $\text{Con}(\exists \Pi^1_1$-m.i.f. + $i < 2^{\aleph_0})$. Another consequence is the consistency of $\exists \Pi^1_1$-m.i.f. together with “all $\Sigma^1_3$ sets have the Marczewski-property” (where $X \subseteq 2^{\omega}$ has the Marczewski-property if every perfect set $P$ contains a perfect subset $P'$ with $P' \subseteq X$ or $P' \cap X = \emptyset$), see [5, Theorem 7.1].

The construction we use appeared implicitly in [12] where, among other things, a forcing notion $\mathbb{P}$ for generically adding a Sacks-indestructible m.i.f. was isolated. These ideas were elaborated and studied further in [6]. Here we show that the combinatorics of this forcing can also be used to explicitly define a Sacks-indestructible m.i.f. in a model of CH, and that in $L$, such a Sacks-indestructible m.i.f. can be defined in a $\Sigma^1_3$-fashion. We start by recalling some technical definitions from [6].

To reduce cumbersome notation, in this section the following will be useful:

\bf{Notation 4.2.} If $\mathcal{I} \subseteq [\omega]^{\omega}$ then
- $\text{FF}(\mathcal{I}) := \{ h : \mathcal{I} \to 2 \mid |\text{dom}(h)| < \omega \}$, and
- For $h \in \text{FF}(\mathcal{I})$ we write

\[ \sigma(h) := \cap \{ A \mid A \in \text{dom}(h) \land h(A) = 1 \} \cap \cap \{ \omega \mid A \in \text{dom}(h) \land h(A) = 0 \}. \]

\bf{Definition 4.3.} An independent family $\mathcal{I}$ is called a densely maximal independent family if for all $X \subseteq \omega$, for all $h \in \text{FF}(\mathcal{I})$ there exists $h' \in \text{FF}(\mathcal{I})$ with $h' \supseteq h$ such that $\sigma(h') \subseteq \sigma(X) \lor \sigma(h') \cap X = \emptyset$.

\bf{Definition 4.4.} Let $\mathcal{I}$ be an independent family. The density ideal of $\mathcal{I}$ is

\[ \text{id}(\mathcal{I}) := \{ X \subseteq \omega \mid \forall h \in \text{FF}(\mathcal{I}) \exists h' \in \text{FF}(\mathcal{I}) (h' \supseteq h \land \sigma(h') \cap X = \emptyset) \}. \]

The dual filter is denoted by $\text{id}^*(\mathcal{I})$.

\bf{Lemma 4.5.} If $\mathcal{I} \subseteq \mathcal{I}'$ then $\text{id}(\mathcal{I}) \subseteq \text{id}(\mathcal{I}')$, and if $\mathcal{I} = \bigcup_{\alpha < \kappa} \mathcal{I}_\alpha$, for a regular uncountable $\kappa$, where the $\mathcal{I}_\alpha$ form a continuous increasing chain with $|\mathcal{I}_\alpha| < \kappa$, then $\text{id}(\mathcal{I}) = \bigcup_{\alpha < \kappa} \text{id}(\mathcal{I}_\alpha)$.

\bf{Proof.} The first statement is straightforward, and for the second statement, if $X \in \text{id}(\mathcal{I})$ then we can let $\alpha < \kappa$ be the least ordinal closed under the $h \mapsto h'$ operation given by the definition of $\text{id}(\mathcal{I})$. 

Recall that a filter $\mathcal{F}$ on $\omega$ is a $p$-filter iff for every $\{X_n : n < \omega\} \subseteq \mathcal{F}$ there exists $X \in \mathcal{F}$ with $X \subseteq^* X_n$ for all $n$ (“$X$ is a pseudointersection of the $X_n$’s”). A filter $\mathcal{F}$ on $\omega$ is a $q$-filter if for every partition of $\omega$ into finite sets $\mathcal{E} = \{E_n : n < \omega\}$, there is $X \in \mathcal{F}$ such that $|X \cap E_n| \leq 1$ for all $n$ (“$X$ is a semiselector for $\mathcal{E}$”). A filter $\mathcal{F}$ is a Ramsey filter if it is both a $p$-filter and a $q$-filter (cf. [2, Section 4.5.A]).
Theorem 4.6 ([12]. [6, Corollary 37]). Let $I$ be a densely maximal independent family, such that the dual filter $id^*(I)$ is generated by a Ramsey filter and the filter of cofinite sets (Fréchet filter). Then $I$ remains maximal after a countable-support iteration of Sacks forcing, as well as a countable-support product of Sacks forcing.

Definition 4.7. Let $P$ be the forcing poset of all pairs $(a, A)$ where $a$ is a countable independent family, $A \in [\omega]^{\omega}$, and for all $h \in FF(a)$, $\sigma(h) \cap A$ is infinite. The ordering is given by $(a, A) \leq (a', A')$ iff $a' \supseteq a$ and $A' \subseteq A$.

In [6] [12] this forcing was used to generically add a Sacks-indestructible m.i.f. Here, rather than forcing with $P$ we will be using it in a purely combinatorial fashion to construct a Sacks-indestructible m.i.f. in a model of CH, and, in particular, a $\Sigma_2^1$ m.i.f. in $L$.

The following properties of $P$ were proved in [12] [6]:

Lemma 4.8.

(a) $P$ is $\sigma$-closed.
(b) If $(a, A) \in P$ then there exists $B \subseteq A$ such that $B \notin a$ and $(a \cup \{B\}, A) \leq (a, A)$.
(c) If $Y \subseteq \omega$ is an arbitrary set, then for every $(a, A) \in P$ there exists $(B, B) \leq (a, A)$ such that
$$\forall h \in FF(a) \exists h' \in FF(a) \text{ s.t. } h' \supseteq h \text{ and } \sigma(h') \subseteq^* Y \text{ or } \sigma(h') \cap Y =^* \emptyset.$$ (d) Let $E := \{E_n \mid n < \omega\}$ be a partition of $\omega$ into finite sets. Then for every $(a, A) \in P$ there is $B \subseteq A$ such that $(a, B) \leq (a, A)$ and $|B \cap E_n| \leq 1$ for all $n$ ("$B$ is a semiselectora for $E$.")
(e) For all $(a, A) \in P$, if $X \in id(a)$ then there is $B$ such that $(a, B) \leq (a, A)$ and $B \cap X = \emptyset$.

Proof. See Proposition 15, Lemma 17, Corollary 19 and Lemma 14 from [6], respectively.

Definition 4.9. We call $\{(a, \alpha) \mid \alpha < \omega\}$ an indestructibility tower, if it is a strictly decreasing sequence of $P$-conditions and, letting $a := \bigcup_{\alpha \in \omega} a_\alpha$, the following four requirements are satisfied:

(1) For every $Y \subseteq \omega$, for every $h \in FF(a)$ there is $h' \in FF(a)$ with $h' \supseteq h$ such that $\sigma(h') \subseteq^* Y$ or $\sigma(h') \cap Y =^* \emptyset$.
(2) For every partition $E := \{E_n \mid n < \omega\}$ of $\omega$ into finite sets, there is $\alpha < \omega_1$ such that $|A_\alpha \cap E_n| \leq 1$ for all $n$ ($A_\alpha$ is a semiselectora for $E$).
(3) For each $\alpha < \omega_1$ there is an infinite $A \subseteq^* A_\alpha$ such that $A \in a_{\alpha+1} \setminus a_\alpha$.
(4) For every $X \in id(a)$ there is an $\alpha < \omega_1$ such that $X \cap A_\alpha =^* \emptyset$.

Lemma 4.10. If $\{(a, \alpha) \mid \alpha < \omega\}$ is an indestructibility tower, then $a := \bigcup_{\alpha \in \omega} (a_\alpha)$ is a m.i.f. which remains maximal after a countable-support iteration and a countable-support product of Sacks forcing.

Proof. In light of Theorem 4.6 it suffices to show that $a$ is a densely maximal family and that $id^*(I)$ is generated by a Ramsey filter and the filter of cofinite sets. Denseness follows immediately from condition (1) of Definition 4.9. For the second property, we show the following:

Claim. $id(a)$ is generated by $\{\omega \setminus A_\alpha \mid \alpha < \omega_1\}$ and $[\omega]^{<\omega}$.
Proof of claim. Since by condition (4) of Definition 4.9 for every \(X \subseteq^* \omega \setminus A_\alpha\), there exists \(\alpha\) such that \(X \subseteq^* \omega \setminus A_\alpha\), it suffices to show that \(\omega \setminus A_\alpha \in \text{id}(\mathcal{A})\) for every \(\alpha\). Let \(h \in \text{FF}(\mathcal{A})\) be arbitrary. Let \(\beta \geq \alpha\) be such that \(h \in \text{FF}(\mathcal{A}_\beta)\). By (3) there is an infinite \(B \subseteq^* A_\beta\) such that \(B \in \mathcal{A}_\beta \setminus \mathcal{A}_\beta\). In particular, \(B \notin \text{dom}(h)\), so we can extend \(h\) to form \(h' := h \cup \{(B, 1)\}\). Then \(h' \in \text{FF}(\mathcal{A}_{\beta+1})\), and moreover \(\sigma(h') \subseteq B \subseteq^* A_\beta \subseteq^* A_\alpha\). This shows that \(\omega \setminus A_\alpha \in \text{id}(\mathcal{A})\) and completes the proof. \(\square\)

Notice that since \(\{A_\alpha \mid \alpha < \omega_1\}\) is a tower, the filter it generates is a p-filter. Moreover, by condition (2), it is a q-filter, and thus a Ramsey filter, as we had to show. \(\square\)

**Theorem 4.11.**

1. If CH holds then there exists an indestructibility tower.
2. If \(V = L\) then there exists a \(\Sigma^1_1\)-definable indestructibility tower.

**Proof.** We give a detailed proof of the first assertion and then show how to adapt it to get a \(\Sigma^1_2\) construction in \(L\).

1. Let \(\{X_\alpha \mid \alpha < \omega_1\}\) enumerate all subsets of \(\omega\) and let \(\{E_\alpha \mid \alpha < \omega_1\}\) enumerate all partitions of \(\omega\) into finite sets.

Let \((\mathcal{A}_0, A_0) \in \mathcal{P}\) be any condition. At stage \(\alpha\), suppose \((\mathcal{A}_\beta, A_\beta)\) for all \(\beta \leq \alpha\) has been constructed. The new condition is designed in four steps:

- Consider the sets \(\{X_\beta \mid \beta \leq \alpha\}\). By repeatedly applying Lemma 4.8 (3) in countably many steps, followed by \(\sigma\)-closure which holds due to Lemma 4.8 (1), we find an extension \((\mathcal{A}'_\alpha, A'_\alpha) \leq (\mathcal{A}_\alpha, A_\alpha)\) such that, for all \(\beta \leq \alpha\), for all \(h \in \text{FF}(\mathcal{A}_\alpha)\) (not necessarily for all \(h \in \text{FF}(\mathcal{A}'_\alpha)\)) there exists \(h' \in \text{FF}(\mathcal{A}'_\alpha)\), such that \(h' \supseteq h\) and \(\sigma(h') \subseteq X_\beta\) or \(\sigma(h') \cap X_\beta = \emptyset\).

- Consider the partition \(\mathcal{E}_\alpha = \{E_\alpha^n \mid n < \omega\}\). By Lemma 4.8 (4) we find an extension \((\mathcal{A}''_\alpha, A''_\alpha) \leq (\mathcal{A}'_\alpha, A'_\alpha)\) such that \(|A''_\alpha \cap E_\alpha^n| \leq 1\) for all \(n\) \((A''_\alpha\) is a semi-selector for \(\mathcal{E}_\alpha\)).

- Consider (again) the sets \(\{X_\beta \mid \beta \leq \alpha\}\). By repeatedly applying Lemma 4.8 (5) in countably many steps, followed by \(\sigma\)-closure, we find a further extension \((\mathcal{A}'''_\alpha, A'''_\alpha) \leq (\mathcal{A}''_\alpha, A''_\alpha)\) such that, for every \(\beta\), if \(X_\beta \in \text{id}(\mathcal{A}_\alpha)\) (which implies that \(X_\beta \in \text{id}(\tilde{A})\) for any \(\tilde{A}\) extending \(\mathcal{A}_\alpha\), then \(A'''_\alpha \cap X_\beta = \emptyset\).

- Finally, use Lemma 4.8 (2) to find a \(B \subseteq^* A'''_\alpha\), such that \(B \notin \mathcal{A}'''_\alpha\), and \((\mathcal{A}'_\alpha \cup \{B\}, A'_\alpha)\) is a condition. We let \((\mathcal{A}_{\alpha+1}, A_{\alpha+1})\) be that condition.

This completes the construction of the induction step (in steps 2 and 3 we could in fact have taken \(\mathcal{A}'''_\alpha = \mathcal{A}''_\alpha = \mathcal{A}'_\alpha\) but that is not relevant). At limit stages \(\lambda\), use \(\sigma\)-closure to again find a condition \((\mathcal{A}_\lambda, A_\lambda)\) which extends all \((\mathcal{A}_\alpha, A_\alpha)\) for \(\alpha < \lambda\).

It is now easy to verify that \(\{(\mathcal{A}_\alpha, A_\alpha) \mid \alpha < \omega_1\}\) satisfies conditions (1)–(4), where for (4) we use the fact that if \(X \in \text{id}(\mathcal{A})\) then \(X \in \text{id}(\mathcal{A}_\alpha)\) for some \(\alpha < \omega_1\), see Lemma 4.8.

2. If \(V = L\) then repeat the same proof, but additionally, pick the canonical well-order \(<_L\) of the reals of \(L\) to well-order the sequences \(\{X_\alpha \mid \alpha < \omega_1\}\) and \(\{E_\alpha \mid \alpha < \omega_1\}\). At each step \(\alpha\) of the construction, the preceding proof shows how
to find an \((\mathcal{A}_{\alpha+1}, A_{\alpha+1})\) satisfying certain requirements. Now, we make sure to always pick the \(\leq_L\)-least condition \((\mathcal{A}_{\alpha+1}, A_{\alpha+1})\) satisfying the same requirements.

This way, it follows that the construction at each step \(\alpha\) only depends on the preceding \(\beta \leq \alpha\) and is thus absolute between \(L\) and an \(L_{\delta}\) for some appropriate \(\delta < \omega_1\). More precisely, if \(\mathcal{A} = \{(\mathcal{A}_\alpha, A_\alpha) \mid \alpha < \omega_1\}\), then there is a formula \(\Phi\) defining \(\mathcal{A}\) in an absolute way, i.e., \((\mathcal{A}, A) \in \mathcal{A}\) iff \(\Phi(\mathcal{A}, A)\) iff there exists \(\delta < \omega_1\) such that \(L_{\delta} \models \Phi(\mathcal{A}, A)\).

Let \(\text{ZFC}\) be a sufficiently large fragment of \(\text{ZFC}\) such that if a transitive model \(M\) satisfies \(\text{ZFC} + V = L\) then \(M = L_\xi\) for some \(\xi\). Now we can write \(\Phi(\mathcal{A}, A)\) iff \(\exists E \subseteq \omega \times \omega\) such that

- \(E\) is well-founded,
- \((\omega, E) \models \text{ZFC} + V = L,
- \((\omega, E) \models \Phi(\pi^{-1}(\mathcal{A}, A))\), where \(\pi: (\omega, E) \cong (M, \epsilon)\) is the transitive collapse of \((\omega, E)\).

By standard methods (cf. [3] Proposition 13.8 ff.) the two latter statements are arithmetic and well-foundedness is \(\Pi^1_1\). Thus \(\Phi(\mathcal{A}, A)\) is equivalent to a \(\Sigma^1_3\) statement. \(\square\)

Proof of Theorem 4.1. Let \(\mathcal{A} = \{(\mathcal{A}_\alpha, A_\alpha) \mid \alpha < \omega_1\}\) be a \(\Sigma^1_3\)-definable indestructibility tower in \(L\). If \(V\) is the extension in the iteration/product of Sacks forcing, then \(\mathcal{A} = \bigcup_{\alpha < \omega_1} \mathcal{A}_\alpha\) is still a maximal independent family with a \(\Sigma^1_2\) definition. By Theorem 2.1 there exists a \(\Pi^1_1\) m.i.f. as well. \(\square\)

Remark 4.12. In Shelah’s proof of the consistency of \(i < u\) [12] a forcing closely related to Sacks was used, which increases \(u\) as well as the continuum (note that in the Sacks model \(u < 2^{\aleph_0}\)). By a slight modification of the method in this section, it is easy to construct a \(\Sigma^1_2\) m.i.f. which is not only Sacks-indestructible, but indestructible by the poset from [12]. This shows that the witness for the m.i.f. in the proof of the consistency of \(i < u\) can in fact be \(\Pi^1_1\)-definable.

5. \(\aleph_1\)-Borel and \(\aleph_1\)-closed m.i.f.’s

The question of definable m.i.f.’s is closely related to questions concerning certain cardinal invariants (compare with [4]).

Definition 5.1.

(1) \(i\) is the least size of a m.i.f.

(2) \(\text{cl}_1\) is the least \(\kappa\) such that there exists a collection \(\{C_\alpha \mid \alpha < \kappa\}\), where each \(C_\alpha\) is a \textit{closed} independent family, and \(\bigcup_{\alpha < \kappa} C_\alpha\) is a m.i.f.

(3) \(\text{cl}_B\) is the least \(\kappa\) such that there exists a collection \(\{B_\alpha \mid \alpha < \kappa\}\), where each \(B_\alpha\) is a \textit{Borel} independent family, and \(\bigcup_{\alpha < \kappa} B_\alpha\) is a m.i.f.

It is clear that \(\text{cl}_B \leq \text{cl}_1 \leq i\). It is also known that \(r \leq i, d \leq i\) [3] Proposition 8.12 and Theorem 8.13, and \(\text{non}(M) \leq i\) [11] Theorem 3.6, where \(d, r,\) and \(\text{non}(M)\) denote the dominating and reaping numbers, and the smallest size of a nonmeager set, respectively. Notice that if \(\text{cl}_B > \aleph_1\), then there are no \(\Sigma^1_2\) m.i.f.’s (since \(\Sigma^1_2\)-sets are \(\aleph_1\)-unions of Borel sets). \(\text{cov}(M)\) is the least cardinality of a family of meager sets covering the real line.

Theorem 5.1. \(\text{cov}(M) \leq i\).
Proof. Let $\kappa < \text{cov}(\mathcal{M})$ and let $\{B_\alpha \mid \alpha < \kappa\}$ be a collection of Borel independent families. We need to show that $I := \bigcup_{\alpha < \kappa} B_\alpha$ is not maximal.

Suppose otherwise, and for every finite $E \subseteq \kappa$ define

$$H_E := \{ X \mid \exists \bar{a}, \bar{b} \in \bigcup_{\alpha \in E} B_\alpha \text{ s.t. } \sigma(\bar{a}; \bar{b}) \subseteq^* X \}$$

$$K_E := \{ X \mid \exists \bar{a}, \bar{b} \in \bigcup_{\alpha \in E} B_\alpha \text{ s.t. } \sigma(\bar{a}; \bar{b}) \cap X =^* \emptyset \}.$$ 

Notice that by maximality of $I = \bigcup_{\alpha < \kappa} B_\alpha$, we have

$$\bigcup \{ H_E \cup K_E \mid E \in [\kappa]^<\omega \} = [\omega]^\omega.$$ 

Since $\kappa < \text{cov}(\mathcal{M})$, there must exist a finite $E \subseteq \kappa$ such that $H_E \cup K_E \notin \mathcal{M}$. Suppose $H_E \notin \mathcal{M}$: since $H_E$ is analytic, there exists a basic open $[s]$ with $[s] \subseteq^* H_E$ (where $\subseteq^*$ means “modulo meager”). By Corollary 3.7, we can construct a perfect a.d. tree $T$ with $[T] \subseteq H_E$. But then, by the argument from Lemma 3.5, it follows that $\bigcup_{\alpha \in E} B_\alpha$ is not independent, contrary to the assumption. Likewise, if $K_E \notin \mathcal{M}$ then using Corollary 3.7, there exists a perfect a.c. tree $S$ with $[S] \subseteq K_E$, and the rest is the same. \hfill \square

We end this paper with the following open questions:

**Question 5.2.**

1. Is the existence of a $\Pi^1_1$ m.i.f. consistent with $i > \aleph_1$? Is it consistent with $\mathfrak{d} > \aleph_1$, $r > \aleph_1$, or $\text{non}(\mathcal{M}) > \aleph_1$?
2. What about a $\Pi^1_2$ m.i.f.?
3. Is it consistent that $i_{cl} < \mathfrak{d}$ or $i_B < \mathfrak{d}$?
4. Is it consistent that $i_{cl} < r$ or $i_B < r$?
5. Is it consistent that $i_{cl} < \text{non}(\mathcal{M})$ or $i_B < \text{non}(\mathcal{M})$?
6. Is it consistent that $i_{cl} < i$ or $i_B < i$?

We note that a positive answer to either of (3), (4), or (5) implies a positive answer to (6). Also, (1) is closely related to (3) – (6): if, e.g., $i_B \geq \mathfrak{d}$ in ZFC, then the existence of a $\Pi^1_1$ m.i.f. implies $\mathfrak{d} = \aleph_1$ because $\Pi^1_1$ sets are $\aleph_1$ Borel. If, on the other hand, $i_B < \mathfrak{d}$ is consistent, then, by Zapletal’s work [14], this consistency should hold in the Miller model; that is, a countable support iteration of Miller forcing should preserve a witness for $i_B = \aleph_1$, and doing such an iteration over $L$ then would preserve such a witness with a $\Sigma^1_2$ definition. By Theorem 2.1, the consistency of the existence of a $\Pi^1_1$ m.i.f. with $\mathfrak{d} > \aleph_1$ would hold in the Miller model.

Similar results have been proved for mad families in [4]: while $b \leq a$ in ZFC, $a_B = a_{cl} < b$ is consistent and so is the existence of a $\Pi^1_1$ mad family with $b > \aleph_1$. However, definable m.i.f.’s are more easily destroyed than definable mad families: for example, the fact that “ZF + there are no mad families” is equiconsistent with ZFC has only been proved recently by Horowitz and Shelah [7], and it is a well-known open question whether AD implies that there are no mad families (whereas for maximal independent families, both statements are an easy consequence of Corollary 3.8).
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