Serber symmetry, Large $N_c$, and Yukawa-like One Boson Exchange Potentials

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The Serber force has relative orbital parity symmetry and requires vanishing NN interactions in partial waves with odd angular momentum. We illustrate how this property is well fulfilled for spin triplet states with odd angular momentum and violated for odd singlet states for realistic potentials but fails for chiral potentials. We analyze how Serber symmetry can be accommodated within a large $N_c$ perspective when interpreted as a long distance symmetry. A prerequisite for this is the numerical similarity of the scalar and vector meson resonance masses. The conditions under which the resonance exchange potential can be approximated by a Yukawa form are also discussed. While these masses arise as poles on the second Riemann in $\pi\pi$ scattering, we find that within the large $N_c$ expansion the corresponding Yukawa masses correspond instead to a well defined large $N_c$ approximation to the pole which cannot be distinguished from their location as Breit-Wigner resonances.

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I. INTRODUCTION

The modern theory of nuclear forces [1] aims at a systematic and model-independent derivation of the forces between nucleons in harmony with the symmetries of Quantum Chromodynamics. Actually, an outstanding feature of nuclear forces is their exchange character. Many years ago, Serber postulated [2] an interesting symmetry for the nucleon-nucleon system based on the observation that at low energies the proton-proton and neutron-proton differential cross section are symmetric functions in the Center of Mass (CM) scattering angle around 90°. This orbital parity symmetry corresponds to the transformation $\theta \rightarrow \pi - \theta$ in the scattering amplitude and was naturally explained by assuming that the potential was vanishing for partial waves with odd angular momentum. Specific attempts were directed towards the verification of such a property [3] (See Refs. [4] and [5] for early and comprehensive reviews.). This symmetry was shown to hold for the NN system, up to relatively high energies [6]. However, such a force was also found to be incompatible with the requirement of nuclear matter saturation [7] as well as with the underlying meson forces mediated by one and two pion exchange [8]. These puzzling inconsistencies were cleared up when it was understood that only singular Serber forces could provide saturation [9]. Old phase shift analyses [10] confirm the rough Serber exchange character of nuclear forces. Many nuclear structure [11], nuclear matter [12], nuclear reactions [13, 14, 15], use Serber forces both for their simplicity as well as their phenomenological success in the low and medium energy region. The possibility of implementing Serber forces in the nuclear potential was envisaged in Skyrme’s seminal paper [16]. Modern versions (SLy4) of the Skyrme effective interactions [17] implement the symmetry explicitly. In a recent paper [18] a novel fitting strategy has been proposed for the coupling constants of the nuclear energy density functional, which focus on single-particle energies rather than ground-state bulk properties, yielding naturally an almost perfect fulfillment of Serber symmetry.

A vivid demonstration of the Serber symmetry is shown in Fig. 1 where the np differential cross section is plotted for several LAB energies using the Partial Wave Analysis and the high quality potentials [19, 20] carried out by the Nijmegen group. While discrepancies regarding the comparison between forward and backward directions show that this symmetry breaks down at short distances, the intermediate region does exhibit Serber symmetry. In any case it is interesting to see that even in the intermediate energy region departures from the symmetry can be seen, the most important one is the fact that the symmetry point is shifted a few degrees towards lower values than 90° for increasing energies. While these are well established features of the NN interaction, it is amazing that such a time honoured force and gross feature of the NN interaction, even if it does not hold in the entire range, has no obvious explanation from the more fundamental and QCD motivated side. To our knowledge this topic has not been explicitly treated in any detail in the literature and no attempts have been made to justify this evident but, so far, accidental symmetry. The present paper tries to fill this gap unveiling Serber symmetry at the relevant scales from current theoretical approaches to the NN problem, looking for its consequences in nuclear physics and analyzing its possible origin. Of course, a definite explanation might finally be given by lattice QCD calculations for which incipient results exist already in the case of S-wave interactions [21, 22].
point of view, the implications for Skyrme forces and the resonance saturation of chiral forces.

In any case, the evidence for both even-L Wigner and odd-L Serber symmetries is so overwhelming that we feel a pressing need for an explanation more closely based on our current knowledge of strong interactions and QCD. Actually, central to our analysis will be the use of the large $N_c$ expansion [26, 27] (for comprehensive reviews see e.g. [28, 29, 30]). Here $N_c$ is the number of colours and in this limit the strong coupling constant scales as $\alpha_s \sim 1/N_c$. For color singlet states the picture is that of infinitely many stable mesons and glueballs, which masses behave as $m \sim N_c^0$ and widths as $\Gamma \sim 1/N_c$, and heavy baryons, with their masses scaling as $M \sim N_c$. This limit also fixes the interactions among hadrons. Meson-meson interactions are weak since they scale as $1/N_c$, meson-baryon interactions $\sim N_c^0$ and baryon-baryon interactions are strong as they scale as $\sim N_c$. While the pattern of $SU(4)$-symmetry breaking complies to the large $N_c$ expectations [31, 32], a somewhat unexpected conclusion, we also pointed out that Serber symmetry, while not excluded for odd-L waves, was not a necessary consequence of large $N_c$. The search for an explanation of the Serber force requires more detailed information than in the case of Wigner symmetry.

In Section IV we approach the Serber symmetry from a large $N_c$ perspective, and make explicit use of the fact that the meson exchange picture seems justified [33]. Actually, within such a realization a necessary prerequisite for the validity of the symmetry would be a numerical similarity of the scalar and vector meson masses. This poses a puzzle since, as is well known, these mesons arise as resonances in $\pi\pi$ scattering as poles in the second Riemann sheet yielding the values \[ m_{\sigma} = m_{\sigma} - i\Gamma_{\sigma}/2 = 441^{+16}_{-8} - i272^{+9}_{-12}\text{MeV} \] and \[ m_{\rho} = m_{\rho} - i\Gamma_{\rho}/2 = 775.49 \pm 0.34 - i149.4 \pm 1.0\text{MeV} \] [35]. The scalar and vector masses and widths are sufficiently different as to make one question if one is close to the Serber limit. In Section V we review the role of two pion exchange and analyze the resonances as well as the generation of Yukawa potentials from the exchange of $\pi\pi$ resonances from a large $N_c$ viewpoint. An important result of the present paper is to show that the Yukawa masses are determined as large $N_c$ approximations to the pole position which cannot be distinguished from the Breit-Wigner resonance. On the light of this result it is possible indeed from the large $N_c$ side to envisage a rationale for the Serber symmetry. Finally, in Section VI we come to the conclusions and summarize our main points.

II. LONG DISTANCE SYMMETRY AND WEIGHTED AVERAGE POTENTIALS

When discussing and analyzing symmetries in Nuclear Physics quantitatively we find it convenient to delineate the scale where they supposedly operate. As we see from Fig. II Serber symmetry does not work all over the range

FIG. 1: Differential cross section for np scattering at several LAB energies. The error band reflects the PWA and high quality potentials of the Nijmegen group [19, 20]. Serber symmetry implies that the functions should be symmetric in the CM scattering angle around 90°.
and equally well for all energies. Thus, we expect to see the symmetry in the medium and long distance region, precisely where a reliable theoretical description in terms of potentials becomes possible. While this is easily understood, it is less trivial to implement these features in a model independent formulation. Furthermore, we see from Fig. 1 there are some small deviations and it may be advisable to find out not only the origin of the symmetry but also the sources of this symmetry breaking.

For the purpose of the present discussion we will separate the NN potential as the sum of central components and non-central ones which will be assumed to be small,

$$V_{NN} = V_0 + V_1,$$

where $\langle \hat{L}, V_0 \rangle = 0$ whereas $\langle \hat{J}, V_1 \rangle = 0$ and $\langle \hat{L}, V_1 \rangle \neq 0$. Specifically, for the central part we take

$$V_0 = V_C + \tau W_C + \sigma V_S + \tau \sigma W_S,$$

while the non-central part is

$$V_1 = (V_T + \tau W_T)S_{12} + (V_L + \tau W_L)\hat{L} \cdot \hat{S}.$$ (3)

Here $\tau = \tau_1 \cdot \tau_2$ and $\sigma = \sigma_1 \cdot \sigma_2$ with $\sigma_1$ and $\tau_1$ are the Pauli matrices representing the spin and isospin respectively of the nucleon $i$. The tensor operator is $S_{12} = 3\sigma_1 \cdot \hat{x} \sigma_2 \cdot \hat{x} - \sigma_1 \cdot \sigma_2$ while $\hat{L} \cdot \hat{S}$ corresponds to the spin-orbit term. The total potential commutes with the total angular momentum $J = L + S$. However, we will start assuming that the potential is central, and that the breaking in orbital angular momentum is small.

We proceed in first order perturbation theory, by using the central symmetric distorted waves as the unperturbed states. Note that $\tau = \tau_1 \cdot \tau_2 = 2T(T+1) - 3$ and $\sigma = \sigma_1 \cdot \sigma_2 = 2S(S+1) - 3$ while Pauli principle requires $(-)^{S+T+L} = -1$. The corresponding zeroth order wave function is of the form

$$\Psi(\bar{x}) = \frac{u^{ST}_L(r)}{r} \chi_{LM_L}(\bar{x}) \chi^{SM_S} \phi^{TM_T},$$

with $\chi^{SM_S}$ and $\phi^{TM_T}$ spinors and isospinors with good total spin $S = 0, 1$ and isospin $T = 0, 1$ respectively. The radial wave functions satisfy the asymptotic boundary conditions

$$u^{ST}_L(r) \rightarrow \sin \left( kr - \frac{L\pi}{2} + \delta^{ST}_L \right),$$

where $k$ is the CM momentum. From the central potential assumption it is clear that partial waves do not depend on the total angular momentum and so we would have e.g. $\delta^{p_0}_L = \delta^{p_1}_L = \delta^{p_2}_L$ and so on, in complete contradiction to the data. As it is well-known the spin-orbit interaction lifts the independence on the total angular momentum, via the operator $\hat{L} \cdot \hat{S}$. Moreover, the tensor coupling operator, $S_{12}$, mixes states with different orbital angular momentum, so to account for the $J$-dependence we proceed in first order perturbation theory in the spin-orbit and tensor potentials using the orbital symmetric distorted waves as the unperturbed states. Note that this is not the standard Born approximation where all components of the potential are treated perturbatively. According to a previous calculation (see Appendix D of Ref. [23]) the correction to the phase shift to first order reads

$$\Delta \delta^{ST}_{JL} = -\frac{M}{p} \int_0^\infty dr u^{ST}_L(r) \Delta V u^{ST}_L(r),$$

so that the perturbed eigenphases become

$$\delta^{ST}_{JL} = \delta^{ST}_L + \Delta \delta^{ST}_{JL}.$$ (7)

Note that to this order the mixing phases vanish, $\Delta \epsilon_J = 0$, and there is no difference between the eigen phase shifts or the nuclear bar phase shifts. The spin-orbit interaction lifts the independence on the total angular momentum, via the operator $\hat{L} \cdot \hat{S}$. Further, the tensor coupling operator, $S_{12}$, mixes states with different orbital angular momentum. Nonetheless, these two perturbations leave the center of the orbital multiplets un-
changed. Actually, since
\[ \sum_{J=L-1}^{L+1} (2J+1)(\Delta V^1_{J,L}) = 0, \quad (8) \]
one has
\[ \sum_{J=L-1}^{L+1} (2J+1)\delta^{ST}_{L,J} = 0. \quad (9) \]
As a consequence
\[ \delta^{ST}_L = \frac{\sum_{J=L-1}^{L+1} (2J+1)\delta^{ST}_{L,J}}{(2L+1)^3} = \delta^{ST}_L, \quad (10) \]
Thus, to first order we may define a common mean phase obtained as the one obtained from a mean potential
\[ \bar{V}_{L}(r) = \frac{\sum_{J=L-1}^{L+1} (2J+1)V_{L,J}(r)}{3(2L+1)}. \quad (11) \]
It is in terms of these potentials where we expect to formulate the verification of a given symmetry. This is nothing but the standard procedure of verifying a symmetry between multiplets by defining first the center of the multiplet \(^1\).
Now, Serber symmetry requires
\[ V_{L}(r) = V_{L}(r) = 0 \quad \text{odd} - L, \quad (12) \]
while Wigner symmetry requires
\[ V_{L}(r) = V_{L}(r) \quad \text{all} - L. \quad (13) \]
Clearly these two requirements are incompatible except when all potentials vanish. In Fig. 2 we plot the Argonne V-18 potentials \(^\bullet\) for the center of the orbital multiplets. Thus the potentials suggest instead that for \( r > 1.5 \text{fm} \)
\[ V_{L}(r) \ll V_{L}(r) \quad \text{odd} - L \quad (14) \]
\[ V_{L}(r) \sim V_{L}(r) \quad \text{even} - L, \quad (15) \]
i.e. Wigner symmetry is fulfilled for even-L states while Serber symmetry holds for odd-L triplet states at distances above 1.5fm in agreement with the expectations spelled out at the beginning of this section. The parallel statements for phase shifts have been developed in detail in Ref. \(^22\) (see Fig. 3) where the relation of long distance symmetry and renormalization has been stressed. The remarkable aspect, already discussed there, is that the symmetry pattern while incompatible with Wigner symmetry for odd-L states is fully compatible with large \( N_c \) expectations \(^32\). It does not explain, however, why Serber symmetry is a good one.

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\(^1\) A familiar example is provided by the verification of SU(3) in the baryon spectrum. While the symmetry is rough the Gell-Mann-Okubo formula works rather well after it has been broken by a symmetry breaking term.

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III. SEARCHING THE SYMMETRY

Most modern potential models of the NN interaction include OPE as the dominant longest range contribution. However, they differ at short distances where many effects compete and are written in quite different forms (energy dependent, momentum dependent, angular momentum dependent, etc.). These ambiguities are of course compatible with the inverse scattering problem and manifest mainly in the off-shell behaviour of the NN forces. The relevant issue within the present context and which we analyze below regards the range and form of current NN interactions from the view point of long distance symmetries. Any potential fitting the elastic scattering data must possess the symmetries displayed by the phase-shifts sum rules as we see in Fig. 3. However, it is not obvious that potentials display the symmetry explicitly.

A. One Pion Exchange

The OPE potential reads
\[ V^\pi(r) = \tau(\sigma W_{5}^\pi(r) + S_{12} W_{T}^\pi(r)). \quad (16) \]
While OPE complies to the Wigner symmetry it does not embody exactly the Serber symmetry. Actually we get
for even-L waves
\[ V_{2L}^\pi(r) = V_{2L}^\pi(r) = V_{2L}^\pi(r) = -3W_S^\pi(r), \]
\[ V_{2S}^\pi(r) = V_{2S}^\pi(r) = V_{2S}^\pi(r) = -3W_S^\pi(r) , \]
while for odd-L waves we have
\[ V_{1L}^\pi(r) = V_{1L}^\pi(r) = V_{1L}^\pi(r) = 9W_S^\pi(r) , \]
\[ V_{1S}^\pi(r) = V_{1S}^\pi(r) = V_{1S}^\pi(r) = W_S^\pi(r). \]

The factor 9 for the singlet to triplet ratio is nonetheless a close approximation to the Serber limit in a region where the potential is anyhow small. These OPE relations are verified in practice for distances above 3–4fm. As we see from Fig. 2 the vanishing of \( \delta_3/p \) potential in the region down to the region around 1.5fm. For lower distances, potential models start deviating from each other (see e.g. [20]) but this vanishing of \( \delta_3/p \) potential is a common feature which occurs beyond the validity of OPE.

**B. Boundary conditions (alias \( V_{\text{high } \pi} \))**

We now analyze the symmetry issue for the highly successful PWA\[14\] of the Nijmegen group. There, a OPE potential is used down to \( r_c = 1.4 \text{fm} \) and the interaction below that distance is represented by a boundary condition determined by a square well potential with an energy dependent height,

\[ 2\mu V_{S,\beta}(k^2) = \sum_{n=0}^{N} a_{n,\beta}k^{2n} , \]

where \( \beta \) stands for the corresponding channel, so that the total potential reads

\[ V_\beta(r) = [V_{\beta}(r) + V_{\beta}^\text{int}(r)] \theta(r - r_c) + V_{S,\beta}(k^2) \theta(r_c - r) , \]

where \( V_{\beta}^\text{int}(r) \) is a phenomenological intermediate range potential which acts in the region \( 1.4 \text{fm} \leq r \leq 2.0 \text{fm} \). Then, for the center of the L-multiplets ( \( V \) in MeV and \( k \) in fm) we have

\[ V_{S,1}\pi(k^2) = 139.448 - 23.417k^2 + 2.479k^4, \]
\[ V_{S,2}\pi(k^2) = 14.666 + 0.92k^2 + 0.029k^4, \]
\[ V_{S,1}\pi(k^2) = 248.73, \]
\[ V_{S,2}\pi(k^2) = -33.08 + 5.90k^2, \]

where, again, we see that Serber symmetry takes place since \( V_{S,2}\pi(k^2) \ll V_{S,1}\pi(k^2) \) and \( V_{S,3}\pi(k^2) \ll V_{S,1}\pi(k^2) \). Actually, the factor is strikingly similar to the 1/9 of the OPE interaction which in the analysis of holds up to \( r_c = 1.4 \text{fm} \). Thus, in the Nijmegen PWA decomposition of the interaction we find the remarkable relation

\[ V_{3L}(r) \ll V_{1L}(r) \quad \text{odd} - L, \quad \text{all} \ r \]

showing that there is Serber symmetry in the short range piece of the potential. On the other hand, the even partial waves yield

\[ V_{S,1}\pi(k^2) = -17.813 - 1.016k^2 + 2.564k^4, \]
\[ V_{S,2}\pi(k^2) = -40.955 + 4.714k^2 + 1.779k^4, \]
\[ V_{S,3}\pi(k^2) = 61.42 - 15.678k^2, \]
\[ V_{S,3}\pi(k^2) = 28.869 - 3.579k^2, \]
\[ V_{S,1}\pi(k^2) = 466.566, \]
\[ V_{S,3}\pi(k^2) = 0, \]

where we clearly see the violations of Wigner symmetry at short distances, i.e. we only have

\[ V_{3L}(r) \sim V_{1L}(r) \quad \text{even} - L, \quad \text{all} \ r \geq r_c. \]

This simple analysis suggests that Serber symmetry, when it works, holds to shorter distances than the Wigner symmetry. Our previous analysis in terms of mean phases\[23\] fully supports this fact. Indeed, higher partial waves with angular momentum \( l \) are necessarily small at small momenta due to the well known \( \delta_3/p \sim -\alpha_p^{|l+1/2|} \) threshold behaviour. In fact, this is the case for \( \delta_1/p \) and \( \delta_1/f \). However, Serber symmetry implies that \( \delta_1/p \) and \( \delta_1/f \) are rather small not only in the threshold region but also in the entire elastic region as can be clearly seen from Fig. 3.

**C. Potentials and \( V_{\text{low } \pi} \)**

A somewhat different perspective arises from a Wilsonian analysis of the NN interaction which corresponds to a coarse graining of the potential. This viewpoint was implemented in Ref.\[37\] where the so-called \( V_{\text{low } \pi} \) approach has been pursued, and corresponds to integrating out high momentum modes below a given cut-off \( k \leq \Lambda \) from the Lippmann-Schwinger equation. It was found that high quality potential models, i.e. fitting the NN data to high accuracy and also incorporating OPE, collapse into a unique self-adjoint nonlocal potential for \( \Lambda \sim 400 \text{MeV} \). This is a not unreasonable result since all the potentials provide a rather satisfactory description of elastic NN scattering data up to \( p \sim 400 \text{MeV} \). Moreover, the potential which comes out from eliminating high energy modes can be accurately represented as the sum of the truncated original potential and a polynomial in the momentum\[38,\]

\[ V_{\text{low } \pi}(k', k) = V_{\text{NN}}(k', k) + V_{\text{CT}}^A(k', k), \quad (k, k') \leq \Lambda, \]

where \( V_{\text{NN}}(k', k) \) is the original potential in momentum space for a given partial wave channel and \( V_{\text{CT}}^A(k', k) \) is the effect of the high energy states,

\[ V_{\text{CT}}^A(k', k) = k'k'' \left[ C_0''(\Lambda) + C_2''(\Lambda)(k^2 + k'^2) + \ldots \right] \]
where the coefficients $C_n^\mu (\Lambda)$ play the role of counterterms. It should be noted that here $V_NN(k',k)$ is cut-off independent whereas $V_{CT}^\Lambda(k',k)$ does depend on $\Lambda$. When the potential given by Eq. (35) is plugged into the truncated Lippmann-Schwinger equation, i.e. intermediate states $q \leq \Lambda$, the phase shifts corresponding to the full original potential $V_{NN}(k',k)$ are reproduced. In Fig. 11 the corresponding diagonal $V_{lowk}(p,p)$ mean potentials are plotted for the Argonne-V18 force. As we see both Wigner and Serber symmetries are, again, vividly seen. The important observation here is that the separation assumed by Eq. (35) does not manifestly display the symmetry. Actually, a more convenient representation would be to separate off all polynomial dependence explicitly from the original potential 

$$V_{lowk}(k',k) = \bar{V}_{NN}(k',k) + \bar{V}_{CT}^\Lambda(k',k), \quad (k,k') \leq \Lambda,$$  

so that if $\bar{V}_{CT}^\Lambda(k',k)$ contains up to $O(p^n)$ then $\bar{V}_{NN}(k',k)$ starts off at $O(p^{n+1})$, i.e. the next higher order. This way the departures from a pure polynomial may be viewed as true and explicit effects due to the potential. In terms of these polynomials, Wigner and Serber symmetries are formulated from the coefficients

$$\bar{C}_0 = C_0 + C_0^{\text{high}}(\Lambda)$$  

constructed from the sum of the potential and the integrated out contribution below a cut-off $\Lambda$, namely

$$\bar{C}_{0;4L} = \bar{C}_{0;4L}, \quad \text{even} - L,$$

$$\bar{C}_{0;4L} = 0, \quad \text{odd} - L. \quad (39)$$

It should be noted that the $V_{lowk}$ approach is in spirit nothing but the momentum space version of the PWA of the Nijmegen group in coordinate space where short distances, $r \leq r_c$, are integrated out and parameterized by means of an energy dependent boundary condition. From this viewpoint the similarities as regards the Wigner and Serber symmetries are not surprising. This is why the standard boundary condition approach might be also denominated $V_{high R}$ (see also Ref. [39] for further discussions).

**D. Chiral two pion exchange**

The chiral Two Pion Exchange (TPE) potentials computed in Ref. [40] are understood as direct consequences of the spontaneous chiral symmetry breaking in QCD. Actually, the TPE contribution takes over the OPE one at about $r = 2fm$. At very long distances one has

$$V_{2\pi}^{\text{ChPT}}(r) = (1 + \frac{2\tau_1 \cdot \tau_2}{r}) e^{-2m_\pi r} \frac{3g_A^4m_\pi^2}{1024f_\pi^4M_N\pi^2} + \ldots \quad (40)$$

$^2$ We thank Scott K. Bogner for kindly providing the numbers of Ref. [38].
momentum transfer as
where we are working in the CM system and we take the
C terms above is as follows: $\frac{C_0}{M^2}$ for the $S$-wave states when the chiral cut-offs $\Lambda = 500\text{MeV}$ and $\Lambda = 600\text{MeV}$ are used. As we see there is a 5% violation of Wigner symmetry in the second case.

IV. ARE COUNTERTERMS FINGERPRINTS OF LONG DISTANCE SYMMETRIES?

Given the fact that both Wigner and Serber symmetries can be interpreted as long distance symmetries which roughly materialize at low energies in the potentials (see Fig. 2), the phase shifts (see Fig. 3) and the $V_{\text{lowk}}$ potentials (see Fig. 4) we find it appropriate to discuss how these results fit into renormalization ideas and the role played by the corresponding counterterms.

A. The perturbative point of view

As we have mentioned in the previous section, chiral potentials are generally used to describe NN scattering with the additional implementation of counterterms which cannot directly be determined from chiral symmetry alone. On the other hand, one expects these counterterms to encode short distance physics and hence to be related to the exchange of heavier mesonic degrees of freedom alike those employed in the One Boson Exchange (OBE) potentials [45]. The idea is quite naturally based on the resonance saturation hypothesis of the exchange forces (see e.g. [10] for a discussion in the $\pi\pi$ scattering case). This is achieved by integrating out the heavy fields using their classical equations of motion, and expanding the exchanged momentum between the nucleons as compared to the resonance mass case [47] [48]. Schematically it corresponds to power expand the Yukawa-like NN potentials (we ignore spin and isospin for simplicity),

$$\frac{g^2}{q^2 + M^2} = \frac{g^2}{M^2} - \frac{g^2}{M^2} \frac{q^2}{M^2} + \cdots$$

$$= C_0 + C_2 (p^2 + p'^2) + C_1 p \cdot p' + \cdots (41)$$

where we are working in the CM system and we take the momentum transfer as $q = p - p'$. The meaning of the terms above is as follows: $C_0$ is an s-wave zero range, $C_2$ is an s-wave finite range, $C_1$ is a p-wave etc. More generally, Eq. (41) corresponds to a power series expansion of the potential in momentum space. Obviously, we expect such a procedure to be meaningful whenever the scattering process can be treated perturbatively, like e.g. the case of peripheral waves. As is well known, central s-waves cannot be treated perturbatively as the corresponding scattering amplitudes have poles very close to threshold corresponding a virtual state in the $^1S_0$ channel and the deuteron in the $^3S_1 - ^3D_1$ channel. This does not mean, however, that the potential cannot be represented in the polynomial form of Eq. (41), but rather that the coefficients cannot be computed directly as the Fourier components of the potential.

B. The Wilsonian point of view

The momentum space $V_{\text{lowk}}$ approach [35] makes clear that the long distance behaviour of the theory is not determined by the low momentum components of the original potential only, one has to add virtual high energy states which also contribute to the interaction at low energies in the form of counterterms, as outlined by Eqs. (35) and (37). Alternatively the more conventional coordinate space boundary condition (alias $V_{\text{highk}}$) method shows that the low energy behaviour of the theory is not determined only by the long distance behaviour of the potential, one has to include the contribution from integrated out short distances in the form of boundary conditions. A true statement is that the low momentum features of the interaction in the $V_{\text{lowk}}(p, p)$ potential can be mapped into long distance characteristics of the potential $V(r)$. The symmetries are formulated in terms of the conditions in Eq. (39).

C. Long distance symmetries in Nuclear Potentials

In order to substantiate our points further, let us note that in Ref. [38] it was suggested that the $V_{\text{lowk}}$ was a viable way of determining the effective interactions which could be further used in shell model calculations for finite nuclei. Actually, this interpretation when combined with our observation of Fig. 4 that Serber symmetry shows up quite universally has interesting consequences. Schematically, this can be implemented as a Skyrme type effective (pseudo)potential [10]

$$V(\vec{r}) = t_0 (1 + x_0 P_\sigma) \delta^{(3)}(\vec{r}) + t_1 (1 + x_1 P_\sigma) \left\{ -\nabla^2, \delta^{(3)}(\vec{r}) \right\}$$

$$- t_2 (1 + x_2 P_\sigma) \nabla \delta^{(3)}(\vec{r}) \nabla + \cdots (42)$$

where $P_\sigma = (1 + \sigma_1 \cdot \sigma_2)/2$ is the spin exchange operator with $P_\sigma = 1$ for spin single $S = 0$ and $P_\sigma = 1$ for spin triplet $S = 1$ states. The dots stand for spin-orbit, tensor interaction, etc. It should be noted the close resemblance
of the momentum space version of this potential

\[
V(p', p) = t_0(1 + x_0 P_\sigma) + t_1(1 + x_1 P_\sigma)(p'^2 + p^2) + t_2(1 + x_2 P_\sigma)p' \cdot p + \ldots
\]

(43)

with Eq. (37) after projection onto partial waves, where only S- and P-waves have been retained. Traditionally, binding energies have been used to determine the parameters \(t_1\) and \(x_1\) within a mean field approach and many possible fits arise depending on the chosen observables (see e.g. Ref. [17]) possibly displaying some spurious short distance sensitivity beyond the range of applicability of Eq. (42). The low momentum character of the Skyrme force, suggests using the longest possible wave-length properties. Actually, inclusion of tensor force and a new fitting strategy to single particle energies \[18\] yields \(x_2 = -0.99\) which is an almost perfect Serber force for spin-triplets \((P_\sigma = 1)\) and reproduces the so-called SLy4 form of the Skyrme functional \[17\]. On the light of our discussion this result seems quite natural as single particle energies are determined from direct fits to the NN data when the chiral potential is added. The spread of values found in Ref. \[47\] might possibly reflect an inadequacy of the method used to characterize the long distance coarse grained NN dynamics where, as we have shown, Serber symmetry becomes quite accurate. Actually, the matching of counterterms between, say, the OBE potential and the chiral potentials is done in terms of objects which have a radically different large \(N_c\) behaviour (see Section V for further details). For instance, while \(C^{\text{OBE}}_0 \sim g^2_M/m^2_M \sim N_c\) because \(g_M \sim N^\frac{2}{3}_c\) and \(m_M \sim N^0_c\) one has \(C^{\text{chiral}}_0 \sim g^4_M m^2_M/f^4_N \sim N^2_c\) since \(m_N \sim N^0_c, f_N \sim N^\frac{2}{3}_c\) and \(g_A \sim N_c\). In fact, the value of the counterterms determined from resonance exchange is generally not simply determined by the coefficients of the power series expansion of potential in momentum space, as schematically given by Eq. (44), since they undergo renormalization and hence run with the scale.

D. Matching OBE potentials to chiral potentials

In Ref. \[47\] a systematic determination of counterterms has been carried out for a variety of realistic potentials which successfully fit the NN data by reading off the Fourier components of the potential (see e.g. Eq. (11)). The counterterms so obtained are then compared to those determined from direct fits to the NN data when the chiral potential is added. The spread of values for these counterterms found in Ref. \[47\] for realistic potentials, however, does not comply to the fact that all those potentials provide a quite satisfactory description of the phase shifts. Moreover, in Ref. \[47\] it is found that for the OBE Bonn potential \[15\]

\[
C_{1P} = +0.454 \times 10^{-4}\text{GeV}^{-4},
\]

\[
C_{3P} = \frac{1}{9}(C_{1P} + 3C_{3P} + 5C_{3P}) = -0.140 \times 10^{-4}\text{GeV}^{-4}.
\]

(45)

Thus, the triplet to singlet ratio is \(C_{3P}/C_{1P} \sim -0.33\) in this case. For the CD Bonn potential \[19\] one has \(C_{3P}/C_{1P} \sim -0.7\) whereas Argonne AV-18 \[36\] yields \(C_{3P}/C_{1P} \sim -0.54\). These large factors contrast with the much smaller factor \(V_{\Sigma 2P}(k^2)/V_{\Sigma 1P}(k^2) \sim 1/10\) of the PWA sketched above in Sect. IIIB. They also disagree with the almost vanishing ratio \(V_{\Sigma P}(p, p)/V_{\Sigma P}(p, p)\) found for the \(V_{\text{lowk}}\) potentials described in Section IIIC, which yield a universal result (see also Fig. 4 for the particular AV-18 potential). The reason is that the correct formulation of the symmetry conditions is given by Eq. (39) which are made up from the potential plus the contribution of the high energy tail. Thus, it appears that in the approach of Ref. \[17\] Serber symmetry is more strongly violated at short distances than expected from other means. In our view the spread of values found in Ref. \[47\] might possibly reflect an inadequacy of the method used to characterize the long distance coarse grained NN dynamics where, as we have shown, Serber symmetry becomes quite accurate. Actually, the matching of counterterms between, say, the OBE potential and the chiral potentials is done in terms of objects which have a radically different large \(N_c\) behaviour (see Section V for further details). For instance, while \(C^{\text{OBE}}_0 \sim g^2_M/m^2_M \sim N_c\) because \(g_M \sim N^\frac{2}{3}_c\) and \(m_M \sim N^0_c\) one has \(C^{\text{chiral}}_0 \sim g^4_M m^2_M/f^4_N \sim N^2_c\) since \(m_N \sim N^0_c, f_N \sim N^\frac{2}{3}_c\) and \(g_A \sim N_c\). In fact, the value of the counterterms determined from resonance exchange is generally not simply determined by the coefficients of the power series expansion of potential in momentum space, as schematically given by Eq. (44), since they undergo renormalization and hence run with the scale.

E. Long distance symmetry and off-shellness

The previous analysis shows that nothing forbids to have a potential which breaks the symmetry strongly on the one hand and being able to simultaneously fit the scattering data which manifestly display the symmetry on the other hand. Actually, this can only be achieved by some degree of compensating symmetry violation between long and short distances.\(^3\) However, it is somewhat unnatural as it does not reflect the true character of the theory and relegates the role of the symmetry to be an accidental one. As it is widely accepted, unveiling symmetries is not mandatory but makes life much easier.\(^4\)

Of course, these observations are true as long as we restrict to on-shell properties, such as NN scattering. However, would these symmetries have any consequence for off-shell nucleons? One may clearly have arbitrary short distance parameterizations of the force without a sizeable change of the phase shifts. However, the universality of long distance potentials above \(\sim 1.5\text{fm}\) or, equivalently, a coarse graining of the interaction with the given length scale \(\sim \pi/\Lambda\) such as \(V_{\text{lowk}}\) is by definition based on in-

\(^3\) This is the case for instance of chiral TPE potentials, see Section IIIC where the potential \[10\] breaks the symmetry above \(1.6\text{fm}\) but the data can be described \[14\] with this truncated potential plus suitable energy dependent boundary conditions.

\(^4\) The above discussion is somewhat similar to the use of regularization schemes in EFT; while it is possible to break the symmetry by all allowed counterterms, final physical results will depend on redundant combinations of parameters expressing the symmetry. In practice it is far more convenient to use a regularization scheme where the symmetry is preserved manifestly.
sensitivity to shorter wavelengths. Our discussion above on effective forces illustrates the fact that these redefinitions of the potential in the NN scattering problem cannot affect the effective force and so a violation of the Serber symmetry has a physical significance for wavelengths larger than the coarse graining scale.

V. SERBER FORCE FROM A LARGE $N_c$ PERSPECTIVE

Up till now, in this paper we have provided evidence that long distance symmetries such as Wigner’s and Serber’s do really take place in the two nucleon system. From now on we are concerned with trying to determine whether those symmetries are purely accidental ones or obey some pattern following more closely from QCD. Actually, we found [23] that large $N_c$ limit [26, 27] (for comprehensive reviews see e.g. [28, 29, 30]) provides a rationale for Wigner symmetry. The fact that Serber symmetry holds when Wigner symmetry fails suggests analyzing the large $N_c$ consequences more thoroughly. While we do not find a completely unique answer regarding the origin of Serber symmetry, the analysis does show interesting features, as will be discussed.

A. The large $N_c$ expansion

In this section we want to analyze these long distance Serber and Wigner symmetries within the two nucleon system from the large $N_c$ expansion [26, 27] (for comprehensive reviews see e.g. [28, 29, 30]). One feature of large $N_c$ which becomes relevant for the NN problem is that is does not specially hold for long or short distances. This allows in particular to switch from perturbative quarks and gluons at short distances to the non-perturbative hadrons, the degrees of freedom of interest to nuclear physics. This quark-hadron duality makes possible the applicability of large $N_c$ counting rules directly to baryon-meson interactions, at distances where explicit quark-gluon effects are not expected to be crucial. The procedure provides utterly a set of consistency conditions from which the contracted SU(4) symmetry is deduced [28, 29, 30]. Thus, while the large $N_c$ scaling behaviour and spin-flavour structure of the NN potential, Eq. (46), is directly established in terms of quarks and gluons [32], quark-hadron duality at distances larger than the confinement scale requires an identification of the corresponding exchanged mesons, and hence a link to the OBE potentials is provided. However, for internal consistency of the hadronic version of the large $N_c$ expansion, these counting rules should hold regardless of the number of exchanged mesons between the nucleons. Actually, naive power counting suggests huge violations of the NN counting rules. The issue has been clarified after the work of Banerjee, Cohen and Gelman [33] for all meson exchange cases with spin 0 and spin 1 [33] where the necessary cancellations between meson retardation in direct box diagrams and crossed box diagrams was indeed shown to take place. In the TPE case the $\Delta$-isobar embodying the contracted SU(4) symmetry was explicitly needed. Although the exchange of three and higher mesons appeared initially to present puzzling inconsistencies [54] a possible outcome was outlined in Ref. [51] by noting that large $N_c$ counting rules apply to energy independent and hence self-adjoint potentials.

B. Large $N_c$ potentials

Based on the contracted SU(4) large $N_c$ symmetry the spin-flavour structure of the NN interaction was analyzed by Kaplan, Savage and Manohar [31, 32] who found that the leading $N_c$ nucleon-nucleon potential indeed scales as $N_c$ and has the structure

$$ V(r) = V_C(r) + \sigma_1 \cdot \tau_1 \cdot \tau_2 W_S(r) + S_{12} \tau_1 \cdot \tau_2 W_T(r). $$

(46)

It is noteworthy that the tensor force appears at the leading order in the large $N_c$ expansion. From the large $N_c$ potential, Eq. (46), we have for the center of multiplet potentials the sum rules

$$ V_{1L} = V_{3L} = V_C(r) - 3 W_S(r), \quad (-1)^L = +1 $$

(47)

$$ V_{1L} = V_C(r) + 9 W_S(r), \quad (-1)^L = -1 $$

(48)

$$ V_{3L} = V_C(r) + W_S(r), \quad (-1)^L = -1 $$

(49)

where as we see $V_{1L} \neq V_{3L}$ for odd-$L$. Thus, large $N_c$ implies Wigner symmetry in even-$L$ channels and allows a violation of Serber symmetry in odd-$L$ partial waves while it allows a violation of Serber symmetry in spin singlet channels. The question is whether or not large $N_c$ implies Serber symmetry in spin triplet channels as we observe both for the potentials in Fig. 2 as well as for the phase shifts in Fig. 2. On the other hand, from the odd-waves we see from Fig. 2 that the mean triplet phase is close to null, thus one might attribute this feature to an accidental symmetry where the odd-waves potentials are likewise negligible. In the large $N_c$ limit this means $V_C + 9W_S \gg V_C + W_S$, a fact which should be verified.

C. OBE large $N_c$ potentials

According to Refs. [31, 32] in the leading large $N_c$ one has $V_C \sim W_S \sim N_c$ while $V_S \sim W_C \sim 1/N_c$. To proceed further and gain some insight we write the potentials in terms of leading single meson exchanges (see also Ref. [33]) one has Yukawa like potentials (we use the no-
tation of Ref. [43])

\[
V_C(r) = -\frac{g_{\pi NN}^2}{4\pi} \frac{e^{-m_\pi r}}{r} + \frac{g_{\sigma NN}^2}{4\pi} \frac{e^{-m_\sigma r}}{r},
\]

\[
W_S(r) = \frac{1}{12} \frac{g_{\pi NN}^2}{4\pi} \frac{m_\pi^2 e^{-m_\pi r}}{\Lambda_N^2} - \frac{1}{6} \frac{f_{\rho NN}^2}{4\pi} \frac{m_\rho^2 e^{-m_\rho r}}{\Lambda_N^2},
\]

(50)

where \( \Lambda_N = 3M_N/N_c \) is a scale which is numerically equal to the nucleon mass and is \( O(N_c^0) \). All meson couplings scale as \( g_{\pi NN}, g_{\sigma NN}, g_{\omega NN}; f_{\rho NN} \sim \sqrt{N_c} \) whereas all meson masses scale as \( m_\pi, m_\sigma, m_\rho, m_\omega \sim N_c^0 \). In principle there would be infinitely many contributions but we stop at the vector mesons. A relevant question which will be postponed to the next Section regards what values of Yukawa masses should one take. This is particularly relevant for the \( m_\sigma \) case. Note that the tensorial structure of the potential Eq. (49) is complete to \( O(N_c^{-1}) \). This leaves room for \( O(N_c^0) \) corrections to the NN potential without generating new dependences triggered by sub-leading mass shifts \( \Delta m_\sigma = O(N_c^{-1}) \) and sub-leading vertex corrections \( \Delta g_{\pi NN} = O(N_c^{-1/2}) \).

As we have mentioned above, to obtain Serber symmetry we must get a large cancellation. At short distances the Yukawa OBE potentials have Coulomb like behavior \( V \rightarrow C/(4\pi r) \) with the dimensionless combinations

\[
C_{V_c + W_S} = -g_{\pi NN}^2 + g_{\sigma NN}^2 + \frac{f_{\rho NN}^2 m_\rho^2}{6M_N^2},
\]

\[
C_{V_c + 9W_S} = -g_{\pi NN}^2 + g_{\sigma NN}^2 + \frac{3f_{\rho NN}^2 m_\rho^2}{2M_N^2},
\]

(51)

where the small OPE contribution has been dropped. To resemble Serber symmetry we should have \( C_{V_c + W_S} \ll C_{V_c + 9W_S} \). There are several scenarios where this can be achieved. For instance, if we impose the OPE 1/9-rule for the full potential we have \( g_{\pi NN}^2 = g_{\sigma NN}^2 \). Using SU(3), \( \Delta g_{\rho NN} = g_{\rho NN}, \) Sakurai’s universality \( g_{\rho NN} = g_{\pi NN}/2 \), the current-algebra KSFR relation, \( \sqrt{2}g_{\rho \pi \pi f_\pi} = m_\rho \), and the scalar Goldberger-Treiman relation, \( g_{\rho NN} f_\pi = M_N \), one would get \( M_N = N_c m_\rho/(2\sqrt{2}) \) a not unreasonable result. This only addresses the cancellation at short distances. The cancellation would be more effective at intermediate distances if \( m_\sigma \) and \( m_\pi \) would be numerically closer. In this regard, let us note that there are various schemes where an identity between scalar and vector meson masses are explicitly verified \cite{52, 53, 54}. In reality, however, the scalar and vector masses are sufficiently different \( m_\sigma = 444 \text{ MeV} \) vs \( m_\rho = 770 \text{ MeV} \). In the next Section we want to analyze this apparent contradiction.

VI. FROM \( \pi\pi \) RESONANCES TO \( NN \) YUKAWA POTENTIALS

A. Correlated two pion exchange

As we have already mentioned TPE is a genuine test of chiral symmetry. On the other hand, it is well known that the iterated exchange of two pions may become in the \( t \)-channel either a \( \sigma \) or a \( \rho \) resonance for isoscalar and isovector states respectively. While the interactions to this collectiveness are controlled to a great extent by chiral symmetry \cite{55, 56, 57}, the resulting contributions to the NN potential in terms of boson exchanges bear a very indirect relation to it. The relation of the ubiquitous scalar meson in nuclear physics and NN forces in terms of correlated two pion exchange has been pointed out many times \cite{47, 58, 59, 60} (see e.g. Refs. \cite{48, 51, 62, 63} for a discussion in a chiral context). Attempts to introduce chiral Lagrangians in nuclear physics have been numerous \cite{64, 65, 66} but the implications for the OBE potential are meager despite the fact that useful relations among couplings can be deduced \cite{5}. As we will see, they are complementary information to the large \( N_c \) requirements.

Note that the leading term generating the scalar meson is \( g_\lambda f_\pi^2 \sim N_c \) but occurs first at \( N_c^3 \)LO in the chiral counting. The central potential reads \cite{49, 61, 62, 63},

\[
V_{NN}(r) = -\frac{32\pi}{3m_\pi^2} \int \frac{d^3q}{(2\pi)^3} e^{iq \cdot x} [\sigma_{\pi N}(-q^2)]^2 t_0 (q^2)
\]

(52)

where \( \sigma_{\pi N}(s) \) is the \( \pi N \) sigma term and \( t_0(s) \) the \( \pi\pi \) scattering amplitude in the \( I = J = 0 \) channel as a function of the \( \pi\pi \) CM energy \( \sqrt{s} \) (see also Eq. (53)). Under the inclusion of \( \Delta \) resonance contributions Eq. (52) is modified by an aditive redefinition of \( g_{\pi NN} \) to include those \( \Delta \)-states \cite{61}. In the large \( N_c \) limit, \( t_{\pi \pi}(s) \sim 1/N_c \) while \( g_{\pi NN}(s) \sim N_c^{1/2} \) yielding \( V_{NN} \sim N_c \) as expected \cite{32}. Actually, at the sigma pole

\[
\sigma_{\pi N}(s) \sim \frac{32\pi}{3m_\pi^2} [G_{\pi N}(s)]^2 t_{\pi \pi}(s) \rightarrow \frac{g_{\pi NN}^2}{s - (m_\sigma - i\Gamma_\sigma/2)^2}
\]

\[
\rightarrow \frac{g_{\pi NN}^2}{s - m_\sigma^2},
\]

(53)

where in the second step we have taken the large \( N_c \) limit. This yields \( g_{\sigma NN} \sim 1/\sqrt{N_c} \) provided \( m_\sigma \sim N_c^0 \) and \( \Gamma_\sigma \sim 1/N_c \). The “fictitious” narrow \( \sigma \) exchange has been attributed to \( N_c \Delta + \Delta \Delta \) intermediate states \cite{40}, to \( 2\pi \) iterated pions \cite{62} or both \cite{61}. This identification is based on fitting the resulting r-space potentials to a Yukawa function in a reasonable distance range.

\footnote{We should mention the Goldberger-Treiman relation for pions \( g_{\lambda M_N} = g_{\pi NN} f_\pi \) and scalars \( M_N = g_{\rho NN} f_\pi \) which yields \( g_{\pi NN} = 12.8 \) and \( g_{\rho NN} = 10.1 \) with the KSFR-universality relation which yields \( g_{\rho NN} = g_{\pi NN}/2 = m_\rho / f_\pi / \sqrt{8} = 2.9 \).}
B. Exchange of Pole Resonances

In this section we separate the resonance contribution to the $NN$ potential from the background, neglecting for simplicity the vertex correction in Eq. (52). The most obvious definition of the $\sigma$ or $\rho$ propagator is via the $\pi\pi$ scattering amplitude in the scalar-isoscalar and vector-isovector channels, $(J,I) = (0,0)$ and $(J,I) = (1,1)$ respectively. Using the definition

$$t_{IJ}(s) = \frac{1}{2i\rho_{\pi\pi}}\left(e^{2i\delta_{IJ}(s)} - 1\right),$$  \hspace{1cm} (54)

with $\rho_{\pi\pi}(s) = \sqrt{1 - 4m^2_\pi^2/s}$ the phase space in our notation. Taking into account the fact that on the second Riemann sheet (taking $\sigma$ as an example) the amplitude has a pole

$$t^I_{00}(s) \rightarrow \frac{R_\sigma}{s - s_\sigma},$$  \hspace{1cm} (55)

with $\sqrt{s_\sigma} = M_\sigma - i\Gamma_\sigma/2$ the pole position and $R_\sigma$ the corresponding residue. Here we define, as usual, the analytical continuation as

$$t^I_{00}(z)^{-1} - t^I_{00}(z)^{-1} = 2i\rho_{\pi\pi}(z).$$  \hspace{1cm} (56)

By continuity $t_{00}(s) \equiv t_{00}(s \pm i0^+)$, $t_{00}^{II}(s \mp i0^+)$ and thus unitarity requires $\rho_{\pi\pi}(s + i0^+) = -\rho_{\pi\pi}(s - i0^+)$. One has for the (suitably normalized) scalar propagator

$$D_\sigma(s) = \frac{t_{00}(s)}{R_\sigma},$$  \hspace{1cm} (57)

in the whole complex plane. In particular

$$D_\sigma^{II}(s) = \frac{t_{00}^{II}(s)}{|R_\sigma|} \rightarrow \frac{e^{i\varphi_\sigma}}{s - (M_\sigma - i\Gamma_\sigma/2)^2},$$  \hspace{1cm} (58)

where the phase $\varphi_\sigma$ is defined as $e^{i\varphi_\sigma} = R_\sigma/|R_\sigma|$ and is related to the background, i.e. the non-pole contribution. In Appendix A we discuss a toy model for $\pi\pi$ scattering which proves quite useful to fix ideas. The function $D_\sigma(s)$ is analytic in the complex $s$-plane with a $2\pi$ right cut along the $(4m^2_\pi,\infty)$ line stemming from unitarity in $\pi\pi$ scattering and a left cut running from ($-\infty,0$) due to particle exchange in the $u$ and $t$ channels. Assuming the scattering amplitude to be proportional to this propagator the corresponding $\pi\pi$ phase shift is then given by

$$e^{2i\delta_{00}(s)} = \frac{t_{00}(s + i0^+)}{t_{00}(s - i0^+)} = \frac{D_\sigma(s + i0^+)}{D_\sigma(s - i0^+)}.\hspace{1cm} (59)$$

The propagator satisfies the unsubtracted dispersion relation \[65\],

$$D_\sigma(q^2) = \int_{4m^2_\pi}^{\infty} d\mu^2 \frac{\rho_\sigma(\mu^2)}{\mu^2 - q^2},$$  \hspace{1cm} (60)

where the spectral function is related to the discontinuity across the unitarity branch cut \[66\]

$$\rho_\sigma(s) = \frac{1}{2i\pi} \text{Disc}D_\sigma(s + i0^+) \hspace{1cm} (61)$$

$$= \frac{1}{\pi |R_\sigma|} \rho_{\pi\pi}(s)|t_{00}(s)|^2,$$  \hspace{1cm} (62)

which satisfies the normalization condition

$$\int_{4m^2_\pi}^{\infty} d\mu^2 \rho_\sigma(\mu^2) = Z_\sigma.$$  \hspace{1cm} (63)

where $Z_\sigma$ is the integrated strength. Thus, the Fourier transformation of the propagator is

$$D_\sigma(r) = \int \frac{d^3q}{(2\pi)^3} e^{iqr} D_\sigma(-q^2)$$

$$= -\frac{1}{4\pi} \int_{4m^2_\pi}^{\infty} d\mu^2 \rho_\sigma(\mu^2) e^{-\mu r}.$$  \hspace{1cm} (64)

According to Eq. (63), $D_\sigma(r) \sim -Z_\sigma/(4\pi r)$ for small distances. We define the analytic function $\rho_\sigma(z) e^{-\sqrt{z}}$ for $r > 0$ in the cut plane without ($-\infty,0$) and $(4m^2_\pi,\infty)$ where

$$\rho_\sigma(z) = \frac{1}{\pi |R_\sigma|} \rho_{\pi\pi}(z)t_{00}(z)t_{00}^{II}(z),$$  \hspace{1cm} (65)

and fulfilling the boundary value condition $\rho_\sigma(s) \equiv \rho_\sigma(s + i0^+)$. This function has a pole at the complex point $z = s_\sigma = (M_\sigma - i\Gamma_\sigma/2)^2$ and a square root branch cut at $z = 4m^2_\pi$ triggered by the phase space factor only since $t_{00}(z)t_{00}^{II}(z)$ is continuous, so that $\rho_\sigma(s + i0^+) = -\rho_\sigma(s - i0^+)$. Thus, we can write the spectral integral, Eq. (64) as running below the unitarity cut and by suitably deforming the contour in the fourth quadrant in the

\[66\] Defined as Disc $t(s + i0^+) = t(s + i0^+) - t(s - i0^+) = 2\text{Im} t(s)$ for a real function below $\pi\pi$ threshold, $0 < s < 4m^2_\pi$.\[67\]
second Riemann Sheet, as shown in Fig. 6, we can separate explicitly the contribution from the pole and the $2\pi$ background yielding

$$D_S(r) = D_\sigma(r) + D_{2\pi}(r).$$

(66)

While, in principle, both contributions are complex, the total result must be real and their imaginary parts cancel identically (see Appendix A for a specific example). Using that Eq. (60) implies $2\pi i_{0}(s_0)\rho_{\pi}(s_0) = 1$ the $\sigma$-pole contribution is effectively given by

$$\Re D_{\sigma}(r) = -\frac{Z_{\sigma}e^{-M_{\sigma}r}}{4\pi r} \times \left[ \cos \left( r \frac{\G_{\sigma}}{2} \right) - \tan \varphi_{\sigma} \sin \left( r \frac{\G_{\sigma}}{2} \right) \right],$$

(67)

which is an oscillating function damped by an exponential. In the narrow resonance limit, $\G_{\sigma} \to 0$, one has $\varphi_{\sigma} = \mathcal{O}(\G_{\sigma})$ yielding

$$\Re D_{\sigma}(r) \sim -\frac{Z_{\sigma}e^{-M_{\sigma}r}}{4\pi r} \left[ 1 + \mathcal{O}(\G_{\sigma}^2) \right],$$

(68)

which is a Yukawa potential. The $2\pi$ background reads

$$\Re D_{2\pi}(r) = -\frac{1}{4\pi r^2} i \int_0^\infty dy_{\sigma} (4m_{\pi}^2 - i y) e^{-r\sqrt{4m_{\pi}^2 + iy}} - \int_0^\infty dy_{\sigma} (4m_{\pi}^2 + i y) e^{-r\sqrt{4m_{\pi}^2 + iy}}.$$  

(69)

At large distances the integral is dominated by the small $y$ region, and we get the distinct TPE behaviour $\sim e^{-2m_{\pi}r}$. The pre-factor is obtained by expanding at small $y$ and using that unitarity imposes the spectral density to be proportional to the phase space factor, Eq. (62). Close to threshold, $s \to 4m_{\pi}^2$, involves the $\pi\pi$ scattering length $a_{00}$ defined as $\delta_{00}(s) \sim a_{00}\sqrt{s - 4m_{\pi}^2}$ yielding

$$\rho_{\pi}(s) = \frac{2m_{\pi}a_{00}}{\pi |R_{\sigma}|} \sqrt{s - 4m_{\pi}^2} + \ldots.$$  

(70)

We therefore get

$$D_{2\pi}(r) = -\frac{K_2(2m_{\pi}r)}{r^2} \frac{4m_{\pi}a_{00}^2}{\pi^2 |R_{\sigma}|} + \ldots$$

$$\sim -\frac{e^{-2m_{\pi}r}}{r^2} \frac{2a_{00}^2 m_{\pi}^2}{\pi |R_{\sigma}|}.$$  

(71)

In Appendix A the pole-background decomposition in Eq. (60) is checked explicitly in a toy model. The resonance contribution saturates the normalization completely, the $2\pi$ continuum background yielding a vanishing contribution to the integrated strength. On the other hand, the resonance produces a Yukawa tail with an oscillatory modulation which alternates between attraction and repulsion, although the region where the oscillation is relevant depends largely on $\varphi_\sigma$.

C. $\pi\pi$ resonances at large $N_c$

The large $N_c$ analysis also opens up the possibility to a better understanding of the role played by the ubiquitous scalar meson. This is an essential ingredient accounting phenomenologically for the mid range nuclear attraction and which, with a mass of $\sim 500$ MeV, was originally proposed in the fifties [54] to provide saturation and binding in nuclei. Along the years, there has always been some arbitrariness on the “effective” or “fictitious” scalar meson mass and coupling constant to the nucleon, partly stimulated by lack of other sources of information 7. During the last decade, the situation has steadily changed, and insistence and efforts of theoreticians [70], have finally culminated with the inclusion of the $0^{++}$ resonance (commonly denoted by $\sigma$) in the PDG [71] as $m_\sigma = 500$ MeV seen as a $\pi\pi$ resonance, with a wide spread of values ranging from 400 – 1200 MeV for the mass and a 600 – 1200 MeV for the width are displayed [72]. These uncertainties have recently been sharpened by a benchmark determination based on Roy equations and chiral symmetry [22] yielding the value $m_\sigma - i\G_\sigma/2 = 441^{+16}_{-8} - i727^{+9}_{-12}$ MeV. Once the formerly fictitious sigma has become a real and well determined lowest resonance of the QCD spectrum it is mandatory to analyze its consequences all over. Clearly, these numbers represent the value for $N_c = 3$, but large $N_c$ counting requires that for mesons $m_\sigma \sim N_c^0$ and $\G_\sigma \sim 1/N_c$.

In this regard the large $N_c$ analysis may provide a clue of what value should be taken for the $\sigma$ mass [73]. Of course, similar remarks apply to the width of other mesons, such as $\rho$, as well. If we make use of the large $N_c$ expansion according to the standard assumption ($M(5) \sim N_c^{-k}$)

$$M_\sigma = M_\sigma^{(0)} + M_\sigma^{(1)} + \mathcal{O}(N_c^{-2}),$$

(72)

$$\G_\sigma = \G_\sigma^{(1)} + \mathcal{O}(N_c^{-2}),$$

(73)

the pole contribution becomes

$$D_\sigma(r) = -\frac{e^{-m_\sigma r}}{4\pi r} + \mathcal{O}(N_c^{-2}),$$

(74)

7 For instance, in the very successful Charge Dependent (CD) Bonn potential [52] any partial wave $2S + 1L_J$-channel is fitted with a different scalar meson mass and coupling.

8 Large $N_c$ studies in $\pi\pi$ scattering based on scaling and unitarization with the Inverse Amplitude Method (IAM) of ChPT amplitudes provide results which regarding the troublesome scalar meson depend on details of the scheme used. While the one loop coupled channel approach [74] yields any possible $m_\sigma$ and a large width (in apparent contradiction with standard large $N_c$, counting [53], the (presumably more reliable) two loop approach [75], yields a large mass shift (a factor of 2) for the scalar meson when going from $N_c = 3$ to $N_c = \infty$ yielding $m_\sigma \to 900$ MeV, but a small shift in the case of the $\rho$ meson. One should note the large uncertainties of the two loop IAM method documented in Ref. [53]. Based on the Bethe-Salpeter approach to lowest order [55] we have estimated $m_\rho \to 500$ MeV [73].
where $m_\sigma = M_\sigma^{(0)} + M_\sigma^{(1)}$, representing the resonance mass to NLO in the $1/N_c$ expansion, should be used. Note that the width does not contribute to this order. Thus, for all purposes we may use a Yukawa potential to represent the exchange of a resonance. However, what is the numerical value of this $m_\sigma$ one should use for the NN problem? Model calculations based on $N_c$ scaling of $\pi\pi$ chiral unitary phase shifts for $N_c = 3$ suggest sizeable modifications as compared with the accurately determined pole position when $N_c$ is varied but the numerical results are not very robust \(76\) \(^9\). An alternative viewpoint where, to the same accuracy, the large $N_c$-NLO pole contribution could be replaced by the equivalent Breit-Wigner resonance mass to the same approximation, since according to \(77\) we may take

$$\delta_{00}(m_\sigma^2) = \frac{\pi}{2} + \mathcal{O}(1/N_c^3) \quad (75)$$

Thus, at LO and NLO in the large $N_c$ limit the exchange of a resonance between nucleons can be represented at long distances as a Yukawa potential with the Breit-Wigner mass to $\mathcal{O}(1/N_c^2)$. The vertex correction $\sigma_{\pi N}$, see e.g. Eq. (62), just adds a coupling constant yielding

$$V_\sigma(r) = -\frac{g_{\pi NN}^2}{4\pi} \frac{e^{-m_\sigma r}}{r} + \mathcal{O}(1/N_c). \quad (76)$$

Of course, the same type of arguments apply to the $\rho$-meson exchange, with the only modification

$$\delta_{11}(m_\rho^2) = \frac{\pi}{2} + \mathcal{O}(1/N_c^3), \quad (77)$$

where now $m_\rho = M_\rho^{(0)} + M_\rho^{(1)}$. In Fig. 7 we show the data for $\pi\pi$ phase shifts, where we see that the true Breit-Wigner masses or not very different. Of course, these arguments do not imply that the Yukawa masses should exactly coincide, but at least suggest that one should expect a large shift for the $\sigma$ mass from the pole position and a very small one for the $\rho$ meson mass when the next-to-leading $1/N_c$ correction to the pole masses are considered. The identity of scalar and vector masses has been deduced from several scenarios based on algebraic chiral symmetry \(52, 80\). Actually, it has been shown that $m_\rho = m_\sigma$ without appealing to the strict large $N_c$ limit but assuming the narrow resonance approximation (See also Ref. \(81\)).

### VII. CONCLUSIONS

Serber symmetry seems to be an evident but puzzling symmetry of the NN system. Since it was proposed more than 70 years ago no clear explanation based on the more fundamental QCD Lagrangean has been put forward.

In the present paper we have analyzed the problem from the viewpoint of long distance symmetries, a concept which has proven useful in the study of Wigner $SU(4)$ spin-flavour symmetry. Actually, Serber symmetry is clearly seen in the np differential cross section implying a set of sum rules for the partial wave phase shifts which are well verified to a few percent level in the entire elastic region. While this situation corresponds to scattering of on-shell nucleons, it would be rather interesting to establish the symmetry beyond this case. Therefore, we have formulated these sum rules at the level of high quality potentials, i.e. potentials which describe elastic NN scattering with $\chi^2/\text{DOF} \sim 1$ which are also well verified at distances above 1fm. This suggests that a coarse graining of the NN interaction might also display...
the symmetry. The equivalent momentum space Wilsonian viewpoint is implemented explicitly by the $V_{\text{lowk}}$ approach by including all modes below a certain cut-off $\Lambda \sim 400$. By analyzing existing $V_{\text{lowk}}$ calculations for high quality potentials we have shown that Serber symmetry is indeed fulfilled to a high degree. We remind that within the $V_{\text{lowk}}$ approach this symmetry has direct implications in shell model calculations for finite nuclei since the $V_{\text{lowk}}$ potential corresponds to the effective nuclear interaction.

A surprising finding of the present paper is that chiral potentials, while implementing extremely important QCD features, do not fulfill the symmetry to the same degree as current high quality potentials. This effect must necessarily be compensated by similar symmetry violations in the counterterms encoding the non-chiral and unknown short distance interaction and needed to describe NN phase shifts where the symmetry does indeed happen. While this is not necessarily a deficiency of the chiral approach it is disturbing that the symmetry does not manifest at long distances, unlike high quality potentials. This may be a general feature of chiral potentials which requires further investigation.

In an attempt to provide a more fundamental understanding of the striking but so far accidental Serber symmetry, we have also speculated how it might arise from QCD within the large $N_c$ perspective on the second part of the paper. The justification for advocating such a possible playground is threefold. Firstly, the NN potential tensorial structure is determined with a relative $1/N_c^2$ accuracy, which naively suggests a bold 10%. In the second place, the meson exchange picture is justified. Finally, we have found previously that such an expansion provides a rationale for the equally accidental and pre-QCD Wigner $SU(4)$ symmetry. Actually, we found that large $N_c$ predicts the NN channels where Wigner symmetry indeed works and fails phenomenologically. The intriguing point is that when Wigner symmetry fails, as allowed by large $N_c$ considerations, Serber symmetry holds instead. In the present paper we have verified the previous statements at the level of potentials at large distances or using $V_{\text{lowk}}$ potentials, reinforcing our previous conclusions based on just a pure phase shift analysis. Under those circumstances, it is therefore natural to analyze to what extent and even if Serber symmetry could be justified at all from a large $N_c$ viewpoint. In practical terms we have shown that within a One Boson Exchange framework, the fulfillment of the symmetry at the potential level is closely related to having not too dissimilar values of $\sigma$ and $\rho$ meson masses as they appear in Yukawa potentials. Actually, these $\sigma$ and $\rho$ states are associated to resonances which are seen in $\pi\pi$ scattering and can be uniquely defined as poles in the second Riemann sheet of the scattering amplitude at the invariant mass $\sqrt{s} = M_R - i\Gamma_R/2$. We have therefore analyzed the meaning of those resonances within the large $N_c$ picture, by assuming the standard mass $m_R \sim N_c^{3/2}$ and width $\Gamma_R \sim 1/N_c$ scaling. We have found that, provided we keep terms in the potential to NLO, meson widths do not contribute to the NN potential, as they are $O(N_c^{-1})$, i.e. a relative $1/N_c^2$ correction to the LO contribution. This justifies using a Yukawa potential where the mass corresponds to an approximation to the pole mass $m_R = M_R^{(0)} + M_R^{(1)}$ which cannot be distinguished from the Breit-Wigner mass up to $O(N_c^{-2})$. This suggests that the masses $m_\sigma$ and $m_\rho$ which appear in the OBE potential could be interpreted as an approximation to the pole mass rather than its exact value. This supports the customary two-Yukawa representation of complex-pole resonances pursued in phenomenological approaches since it was first proposed, since in practice only the lowest Yukawa mass contributes significantly. The question on what numerical value should be used for the Yukawa mass is a difficult one, and at present we know of no other direct way than NN scattering fits for which $m_\sigma = 520(40)$MeV might be acceptable when the uncorrelated $2\pi$ contribution is disregarded.

On a more fundamental level, however, lattice QCD calculations at variable $N_c$ values (see e.g. Ref. [40] for a review) might reliably determine the Yukawa mass parameters appearing in the large $N_c$ potential. A recent quenched QCD lattice calculation yields [41] $m_\rho/\sqrt{\sigma} = 1.670(24) - 0.22(23)/N_c^2$ with $\sqrt{\sigma}$ the string tension, which for $\sqrt{\sigma} = 444$MeV yields $m_\rho = 740$MeV for $m_\sigma = 0$ (see also Ref. [42]). The extension of those calculations to compute the needed $1/N_c$ mass shift would be most welcome and would require full dynamical quarks. Of course, one should not forget that Serber symmetry holds to great accuracy in the real $N_c = 3$ world, and in this sense it represents a stringent test to lattice QCD calculations in $P$-waves. Amazingly, the only existing of S-wave potential calculation [21] displays Wigner symmetry quite accurately.

In any case the large $N_c$ form of the potential Eq. [40] can be retained with relative $1/N_c^2$ accuracy since meson widths enter beyond that accuracy as sub-leading corrections, on equal footing with many other effects (spin-orbit, relativistic and other mesons), independently on how large the $\sigma$ width is in the real $N_c = 3$ world. In our view this paves the way for further investigations on the relevance of large $N_c$ based ideas for the two nucleon system.

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APPENDIX A: TOY MODEL FOR $\pi\pi$ SCATTERING

In this appendix we illustrate with a specific example our discussion of Section [7] and in particular the pole-background decomposition of Eq. (60). According to Ref. [67], the finite width of the scalar meson can be modelled by the propagator

$$D_s(s) = \frac{1}{s - m_\sigma^2 - i m_\sigma \gamma_\sigma \sqrt{s - 4m_\sigma^2 - 4m_\pi^2}}, \quad (A1)$$

for $t \geq 4m_\pi^2$. Below the elastic scattering threshold we use the standard definition $\sqrt{t - 4m_\pi^2} = -i \sqrt{t - 4m_\pi^2} e^{i\theta}$ where $0 \leq \theta = \text{Arg}(t - 4m_\pi^2) < 2\pi$. This defines the propagator in the first Riemann sheet, the second Riemann sheet is determined from the usual continuity equation $D_I(s + i0^+) = D_S(s - i0^+)$. The pole position is given by

$$s_\sigma = (M_\sigma - i\Gamma_\sigma/2)^2 = m_\sigma^2 - \gamma_\sigma^2 m_\sigma^2 \frac{4\Gamma_\sigma m_\sigma}{2m_\sigma^2 - 8m_\pi^2} - i\gamma_\sigma m_\sigma \sqrt{4(m_\sigma^2 - 2m_\pi^2)^2 - \gamma_\sigma^2 m_\sigma^2} \frac{2m_\sigma^2 - 8m_\pi^2}{2m_\sigma^2 - 8m_\pi^2}. \quad (A2)$$

In the small width limit the position of the pole and width are

$$M_\sigma = m_\sigma - \frac{\gamma_\sigma^2 m_\sigma^2}{8m_\sigma^2 m_\pi^2 - 4m_\pi^2} + O(\gamma_\sigma^4), \quad (A3)$$

$$\Gamma_\sigma = \gamma_\sigma + O(\gamma_\sigma^3). \quad (A4)$$

Despite the large $\sigma$-width $m_\sigma \sim \gamma_\sigma$ this expansion works because the $1/\sigma^4$-factor yields (the next correction has a numerical $1/128$, see below). Assuming the scattering amplitude to be proportional to this propagator the corresponding $\pi\pi$ phase shift is then given by

$$e^{2i\delta_0(s)} = \frac{s - m_\sigma^2 - im_\sigma \gamma_\sigma \sqrt{s - 4m_\sigma^2 - 4m_\pi^2}}{s - m_\sigma^2 + im_\sigma \gamma_\sigma \sqrt{s - 4m_\sigma^2 - 4m_\pi^2}}. \quad (A5)$$

The parameterization is such that the standard Breit-Wigner definition of the resonance is fulfilled for the bare parameters,

$$\delta_0(m_\sigma^2) = \frac{\pi}{2}, \quad \gamma_\sigma = \frac{1}{m_\sigma \delta_0'(m_\sigma^2)} \quad (A6)$$

Of course, in the limit of narrow resonances both definitions are indistinguishable and we have $M_\sigma \to m_\sigma$ and $\Gamma_\sigma \to \gamma_\sigma$. If we use the pole position in the second Riemann sheet of the S-matrix or equivalently the zero in the first Riemann sheet from [14] yielding the value $M_\sigma - i\Gamma_\sigma/2 = 441_{-8}^{+16} - i272^{+9}_{-12} \text{MeV}$ we get

$$m_\sigma = 567(10) \text{MeV} \quad \gamma_\sigma/2 = 276(10) \text{MeV}. \quad (A7)$$

From the small width expansion, Eq. (A1), one gets $m_\sigma = 554(10) \text{MeV}$, despite the apparent large width. From Ref. [93] one has the magnitude of the residue $|R_\sigma| = 0.218^{+0.022}_{-0.012} \text{GeV}^2$ whereas we get $|R_\sigma| = 0.430 \text{GeV}^2$. Note the $120(20) \text{MeV}$ shift between the Breit-Wigner and the pole position. With the above parameters the scattering length is $a_{00}m_\sigma = 0.36$ which is clearly off the value $a_{00}m_\pi = 0.220(2)$ deduced from ChPT. The propagator satisfies the unsubtracted dispersion in Eq. (60) where the spectral function is given by

$$\rho(\mu^2) = \frac{1}{\pi} \frac{\gamma_\sigma m_\sigma \sqrt{m_\sigma^2 - 4m_\pi^2} \sqrt{\mu^2 - 4m_\sigma^2}}{(m_\sigma^2 - 4m_\pi^2)(\mu^2 - m_\sigma^2)^2 + \gamma_\sigma^2 m_\sigma^2 (\mu^2 - 4m_\sigma^2)} \quad (A8)$$

and satisfies the normalization condition given by Eq. (65) with $Z_\sigma = 1$. Thus, using the Fourier transformation of the propagator and separating explicitly the contribution from the poles $D_\sigma(r)$ and the $2\pi$ background $D_{2\pi}(r)$ in Eq. (66). This yields the result depicted in Fig. (8) which illustrates and checks the pole-background decomposition and shows that the total contribution, although describable by a Yukawa shape does not correspond to the pole piece. In addition the cancellation of imaginary parts, $\text{Im} D_\sigma(r) = -\text{Im} D_{2\pi}(r)$, is explicitly verified. Using the inverse relations of Eq. (A4), in the narrow width approximation the pole contribution becomes

$$\text{Re} D_\sigma(r) = -\frac{e^{-M_\sigma r}}{4\pi r} \left(1 + \frac{r \Gamma_\sigma^2}{8} \left(\frac{2M_\sigma}{M_\sigma^2 - 4m_\pi^2} - r\right) + \ldots\right), \quad (A9)$$

in qualitative agreement with Eq. (68). On the other hand the $2\pi$ contribution at long distances becomes

$$D_{2\pi}(r) = -\frac{K_2(2m_\pi r)}{r^2} \frac{\gamma_\sigma m_\sigma^2 m_\pi}{\pi^2 (m_\sigma^2 - 4m_\pi^2)^2} + \ldots$$

$$= -\frac{e^{-2m_\sigma r}}{r^2} \frac{\gamma_\sigma m_\sigma^2 m_\pi}{\pi^2 (m_\sigma^2 - 4m_\pi^2)^2} + \ldots \quad (A10)$$

The asymptotic form in the first line reproduces 95% accuracy the full result Eq. (66) for $r > 5 \text{fm}$. Finally, the $\rho$ meson propagator and the associated $(I, J) = (1, 1)$ phase shift can be dealt with mutatis mutandis by using

$$[D_V(s)]^{-1} = s - m_\rho^2 - im_\rho \gamma_\rho \left[\frac{s - 4m_\rho^2}{m_\rho^2 - 4m_\pi^2}\right]^2 \quad (A11)$$

where the $p$-wave character of the $\rho \to 2\pi$ decay can be recognized in the phase space factor.

APPENDIX B: REALISTIC SCALAR-ISOSCALAR $\pi\pi$ SCATTERING

Realistic parameterizations of the $\pi\pi$ scattering data have been proposed based on the conformal map-
For the three sets of $B_{0,1,2}$ parameters discussed in Ref. [78] the resulting complex pole position is slightly higher than the Roy equation value $\sqrt{s_0} = 441^{+16}_{-13} - i272^{+9}_{-7}$ MeV [34]. If we use the dispersive representation for $D(r)$ given in Eq. (64) and cut the integral at the $KK$ threshold $\mu = 2m_K$ we get a function which can be fitted in the range $1 \text{fm} \leq r \leq 5 \text{fm}$ by a Yukawa potential with $m_c = 600(50)$ MeV. The uncertainty corresponds to changing the $B_{0,1,2}$ parameters within errors [78] as well as the varying the fitting interval. This is the modern version of the result found long ago [67] using a relativistic Breit-Wigner form (see Appendix A).

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