Perturbative corrections to the semileptonic $b$-decay moments: $E_{\text{cut}}^\ell$ dependence and running-$\alpha_s$ effects in the OPE approach

Nikolai Uraltsev*

INFN, Sezione di Milano, Milano, Italy
and
Department of Physics, University of Notre Dame du Lac, Notre Dame, IN 46556, USA

Abstract

We have calculated the perturbative corrections to all the structure functions in the semileptonic decays of a heavy quark. Assuming an arbitrary gluon mass as a technical tool allowed to obtain in parallel all the BLM corrections. We report the basic applications, viz. perturbative corrections to the hadronic mass and energy moments with full dependence on the charge lepton energy cut. In the adopted scheme with the OPE momentum scale separation around 1 GeV the perturbative corrections to $\langle M_X^2 \rangle$ are small and practically independent of $E_{\text{cut}}$; the BLM corrections are small, too. The corrections to the second mass squared moment show some decrease with $E_{\text{cut}}$ consistent with the effect of the Darwin operator, within the previously estimated theoretical uncertainty. Perturbative corrections in the pole-type schemes appear significant and vary with $E_\ell$, decreasing the moments at higher cuts. The hardness of hadronic moments is quantitatively illustrated for different cuts on $E_\ell$.

*On leave of absence from St. Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia
1 Introduction

Inclusive decays of the heavy flavor hadrons are one of the most developed dynamic uses of the fundamental QCD in the short-distance regime. Application of the OPE allowed to address them at the nonperturbative level through the expansion in the inverse mass of the decaying quark: absence of the potentially largest $O(\Lambda_{QCD}/m_Q)$ corrections to the decay rates was established [1] and the leading power corrections to inclusive decay distributions were obtained through the expectation values of the local heavy quark operators in the decaying hadron [2–4]. As it has been known from the early days of QCD [5], the nonperturbative treatment is best applied where usual perturbative corrections are comparable, or subdominant to the leading nonperturbative effects. This environment is often realized in the beauty decays once the OPE in the Wilsonian implementation is applied, isolating effects of large distances even from the traditional ‘perturbative’ corrections.

High precision checks of theory and practical use of the inclusive decay distributions are now performed with new data of better reliability and of qualitatively different statistics. Matching this progress requires refinement of the theoretical predictions. Perturbative corrections to a number of decay distributions, first of all charge lepton spectrum in semileptonic decays of $b$ quarks, have been calculated long ago [6], along with many other characteristics [7–9]. Most of the perturbative calculations, although often rather sophisticated, however aimed at a particular observable (total width, semileptonic spectrum or its moments, angular asymmetries, etc.). As a result, in spite of the significant intellectual effort invested, they often remained highly specialized, without the possibility to be applied to only slightly modified observables.

The hadronic mass and energy distributions of the final state in the semileptonic decays appeared very promising for scrutinizing nonperturbative QCD in the heavy quark system. In practice, experiments typically have to apply lower cuts on the energy of the charged lepton, to suppress backgrounds; this complicates the kinematics. Since, following the first papers on the dynamic heavy quark expansion, the power corrections were obtained directly for the semileptonic decay structure functions [4,10], calculating nonperturbative effects for any inclusive moment became straightforward, more or less regardless of the lepton energy cut.

The perturbative effects, on the other hand, appeared in the role of the weak link – the perturbative corrections fully incorporating the cut in the lepton energy have generally been unknown for hadronic moments.

To eliminate such an obstacle once and forever, we have calculated the perturbative corrections directly to the semileptonic decay structure functions. This allows straightforward evaluation of the perturbative correction to all possible inclusive semileptonic distributions. Moreover, we allowed for an arbitrary fictitious gluon mass in the calculations; this opens the possibility to calculate BLM corrections to an arbitrary order, or even to perform complete BLM resummation similar to the one accomplished in Refs. [11, 12], on the parallel footing. To be most universal for possible applications, following Ref. [4] we compute separately the structure functions induced by the vector and the axial-vector weak current, as well as their interference ($w_3$). Likewise all five structure functions are
calculated, so the results can be readily applied to the semileptonic decays into $\tau$-leptons, etc. The explicit structure functions can also be used for computing possible averages with the weight varying depending on the lepton energy cut. This is one of the possibilities to optimize the trade-off between the sensitivity to the heavy quark parameters and suppressing experimental backgrounds.

When this study was in the completing phase, the preprint by M. Trotty [13] appeared, where the first results on the perturbative structure functions proper were presented. This is clearly an important step in completing the theoretical toolbox required for precision analysis of $B$ decays. While addressing in general closely related subjects, we differ with Ref. [13] in a number of points calculation-wise, and also in our applications. Aiming for ultimate flexibility in future applications, we have calculated all five structure functions separately for the vector and the axial-vector currents. Moreover, our results are equally applicable to the first-order perturbative corrections and to all BLM corrections.

Although having independent analytic calculations is important for cross-checks, at this point we have not attempted to compare our results with those reported in Ref. [13], even for the pure one-loop limit. Programming alternative expressions anew for numerical evaluation would represent a time-consuming process; it is not easily safeguarded against possible typos at this technical, yet unavoidable step. To check our calculations we had used other opportunities mentioned in Sect. 5, made available through our previous studies of perturbative and power corrections in the semileptonic decays of heavy flavors, or provided by the OPE applied to perturbation theory.

This paper reports a certain progress in the project presently carried out together with Paolo Gambino, dedicated to the precision analysis of the inclusive $B$ decays. It uses the Wilsonian implementation of the OPE to combine perturbative and power-suppressed effects. In this paper we mostly address the qualitative features and report only the basic applications important for the ongoing experimental analyses, which complement our recent publication [14].\footnote{Our notations follow that paper; for the nonperturbative operators we consistently use notations of Ref. [15].} The comprehensive presentation is planned for the forthcoming paper [16].

## 2 Inclusive $B$ decays in the perturbative expansion

The semileptonic structure functions are defined as the absorptive part of the covariant structures appearing in the decomposition of the forward scattering amplitude of the two weak currents off the $B$ meson:

$$h_{\mu\nu}(q^2, q_0) = \frac{1}{2M_B} \langle B | \int \mathrm{d}^4x \, e^{-iqx} \, iT \left\{ J_\mu(x), J_\nu^\dagger(0) \right\} | B \rangle,$$

where $J_\mu$ is generally the $\bar{c}\gamma_\mu b$ or/and $\bar{c}\gamma_\mu\gamma_5 b$ current ($b \to u$ decay simply corresponds to $m_c \to 0$). Following the standard notations of Ref. [4] we put

$$h_{\mu\nu} = -h_1 g_{\mu\nu} + h_2 v_\mu v_\nu - i h_3 \epsilon_{\mu\nu\alpha\beta} v^\alpha q^\beta + h_4 q_\mu q_\nu + h_5 (q_\mu v_\nu + v_\mu q_\nu),$$

\footnote{Our notations follow that paper; for the nonperturbative operators we consistently use notations of Ref. [15].}
\[ w_i(q^2, q_0) = 2 \text{Im } h_i(q^2, q_0), \]  

with \( v_\mu \) denoting the 4-velocity of the decaying meson (or quark, for perturbative calculations); \( q^2 \) and \( q_0 \) have the meaning of the invariant mass squared and combined energy of the lepton pair, respectively. All the decay distributions with light leptons are expressed, for example in terms of the first three structure functions [4]:

\[
\frac{d^3\Gamma}{dE_\ell dq^2 dq_0} = \frac{G_F^2 |V_{cb}|^2}{32\pi^4} \vartheta\left(q_0 - E_\ell - \frac{q^2}{4E_\ell}\right) \vartheta(E_\ell) \vartheta(q^2) \times \left\{2q^2w_1 + [4E_\ell(q_0 - E_\ell) - q^2]w_2 + 2q^2(2E_\ell - q_0)w_3\right\};
\]

the total integrated width without cuts on lepton energy depends only on \( w_1 \) and \( w_2 \):

\[
\Gamma_{sl} = \frac{G_F^2 |V_{cb}|^2}{16\pi^4} \int_0^{m_b^2} dq^2 \int_{q_0 > \sqrt{q^2}}^{m_b} dq_0 \sqrt{q_0^2 - q^2} \left(q^2w_1(q^2, q_0) + \frac{1}{3}(q_0^2 - q^2)w_2(q^2, q_0)\right). \tag{5}
\]

Perturbative structure functions to order \( \alpha_s^4 \) consist of virtual corrections \( \propto \delta(q_0 - \frac{m_t^2 + q^2 - m_b^2}{2m_b}) \), and bremsstrahlung contributions with \( q_0 < \frac{m_t^2 + q^2 - (m_c + \lambda)^2}{2m_b} \), where \( \lambda \) is the gluon mass. In the limit \( \lambda^2 \to 0 \) virtual corrections logarithmically diverge. The divergent part is given precisely by the tree-level (free quark) structure functions, see Ref. [4], Eqs. (A1–A11), with the universal coefficient depending on \( q^2 \). The real emission part at small \( \lambda^2 \) has a singularity near the free quark kinematics,

\[
w_i(q^2, q_0) \propto \frac{1}{q_0 - \frac{m_t^2 + q^2 - m_b^2}{2m_b}}, \tag{6}
\]

with the commensurate coefficient, so that the divergence cancels once integration over \( q_0 \) is performed. At \( q^2 \) approaching \( (m_b - m_c)^2 \) the coefficient vanishes, therefore shrinking domain of integration over \( q_0 \) does not affect the cancellation of divergences.

Perturbative structure functions themselves are strongly infrared-sensitive in the dominant domain close to the free quark decay kinematics. However, the integrals entering the experimentally measured inclusive moments are not. Moreover, their infrared sensitivity is governed by the OPE, and it is greatly reduced when using the Wilsonian separation between perturbative and power corrections based on the momentum scale a particular contribution originates from. This separation is in practice done directly for the moments rather than for the structure functions themselves.

The analytic expressions for the \( \mathcal{O}(\alpha_s) \) bremsstrahlung structure functions \( w_k(q^2, q_0) \) are relatively simple containing at worst logs of the kinematics-related square roots, even at arbitrary gluon mass. Yet they are lengthy for separate structure functions, consisting of many terms at \( \lambda^2 \neq 0 \). They simplify significantly at \( \lambda^2 = 0 \). Virtual corrections at \( \lambda^2 \to 0 \) are well known one loop renormalization of quark currents containing dilogs at worst. At arbitrary \( \lambda^2 \), however the one-loop vertices with three different internal masses and a general momentum transfer are too special functions. In practice, we represent them as one-dimensional integrals over a single Feynman parameter \( u \), with \( 0 \leq u \leq 1 \).
Obtaining a general moment with the lower cut on lepton energy we, therefore integrate the explicit structure functions over $q_0$ and $q^2$ (for bremsstrahlung), or over $u$ and $q^2$ (for virtual corrections), with the weight which generally is a polynomial in $(m_b - q_0)$ and $(m_b^2 + q^2 - 2m_bq_0 - m_c^2)$. The lepton cut enters through the concrete weight following from Eq. (4): it is obtained by integrating the coefficients from $E_{\ell 1}$ to $E_{\ell +}$, where

$$E_{\ell \pm} = \frac{q_0 \pm \sqrt{q_0^2 - q^2}}{2}, \quad E_{\ell 1} = \max\{E_{\text{cut}}, E_{-}\};$$

(7)

the weights are simple polynomials in $E_{\ell 1}$ or $E_{\ell +}$.

3 Applications

Recent experiments at $B$ factories provide data of impressive precision and quality, which allows for stringent tests of the heavy quark expansion for inclusive decays, direct experimental extraction of many heavy quark parameters and for robust defendable extraction of $V_{cb}$ and $V_{ub}$ from the integrated decays rates. The current precision requires to fully implement the applied cut on the charge lepton energy in the theoretical calculations. In the recent publication [14] the OPE-based predictions were given for various moments with cuts, in the framework which does not rely on assuming charm to be a heavy quark, but only expanding in $1/m_b$. Perturbation theory-wise we used the Wilsonian approach which assumes excluding soft gluon contributions from the coefficient functions. This rendered the perturbative corrections small in size and presumably stable against higher-order corrections.

For hadronic mass moments Ref. [14] evaluated the perturbative corrections without the cut on lepton energy, since the full perturbative corrections with cuts had not been known. Although some simpler parts of the corrections had been calculated, including them would not be consistent: in hadronic moments the terms proportional to different powers of $M_B - m_b$ essentially mix under renormalization. In particular, for the average hadronic mass square,

$$\langle M_X^2 \rangle = m_c^2 + (M_B - m_b)^2 + 2(M_B - m_b)\langle E_x \rangle + \langle m_x^2 - m_c^2 \rangle$$

(8)

the last term having the meaning of the excess of the combined parton invariant mass at the quark level over $m_c^2$, is fed under renormalization by the preceding term driven by $\langle E_x \rangle$, the average hadron energy at the parton level [14].

The justification for approximation neglecting $E_{\ell}$-cut in the perturbative corrections was, again, using the Wilsonian version of the OPE. The absolute size of the perturbative corrections is suppressed here, so even their noticeable variation with the cut was not expected to produce a significant bias. Moreover, it is the soft parton processes that are most sensitive to the particular kinematics. The truly hard gluon effects contributing the perturbative corrections in our approach, are expected to be less dependent on the details of the kinematics as long as the process in question remains sufficiently ‘hard’.
With full perturbative expressions available, we can check validity of those temporary assumptions. We indeed found that in our scheme the perturbative corrections to $\langle M^2_X \rangle$ are practically independent of $E_{\text{cut}}$ in the whole domain $E_{\text{cut}} \lesssim 1.4 \text{ GeV}$. A more noticeable – yet still not very significant – variation with $E_{\text{cut}}$ is observed for the second hadronic moment with respect to average, $\langle (M^2_X - \langle M^2_X \rangle)^2 \rangle$. This is expected, since this higher moment is more sensitive to the Darwin operator, for which we did not actually remove the corresponding soft contributions from the leading-order coefficient function. The observed $E_{\text{cut}}$-dependence, in particular the sign and the size are compatible with such a contribution from the Darwin expectation value: its coefficient function is negative and increases in magnitude for larger $E_{\text{cut}}$ [14], see, e.g. Table 6 of Ref. [14].

The full corrections are illustrated in Figs. 1 and 2 as functions of $E_{\text{cut}}$. For comparison we also show the corresponding effect in the ‘usual’ (pole-type) perturbative corrections without Wilsonian separation. It is evident that for the first moment the effect is dramatically different.

The third moment of the hadronic invariant mass squared places, for the perturbative corrections, most weight on harder gluons. In its soft part it is mostly sensitive to the $D = 6$ Darwin operator. Therefore, using the Wilsonian prescription only for the effects scaling like $1/m_b$ and $1/m_b^2$ does not make a noticeable difference, Fig. 3. The formal perturbative contribution is then significantly dependent on the cut;\footnote{Ref. [14] evaluating the third hadronic mass moment did not include the perturbative corrections to the terms $\propto (M_B - m_b)^k$ with $k \geq 1$, even at zero cut. We have computed them, and numerically they turned out to be about 1 GeV$^2$ (the dashed lines excluding the highest one), decreasing fast from $E_{\text{cut}} \gtrsim 500 \text{ MeV}$.} one should keep in mind, though that a significant fraction of it still comes from gluon momenta below 1 GeV. (For illustration we show the result of subtracting the contribution of the Darwin expectation value of 0.062 GeV$^3$ corresponding to $\mu = 0.9 \text{ GeV}$ in fixed-order perturbation theory). In practice, the third moment so far is important mainly as an estimate of the
Figure 2: Perturbative corrections to $\langle [M_X^2 - \langle M_X^2 \rangle]^2 \rangle$, for $\mu = 1$ GeV (red curve), and their breakdown showing the separate contributions $\propto (M_B - m_b)^k$, $k = 0, 1, 2$ (dashed curves). Blue line (top) refers to the pole scheme. Orange dashed-dotted line shows subtracting the soft piece of 0.062 GeV$^3$ of the Darwin expectation value; it would correspond to Wilsonian $\rho_D^3$ normalized at 0.9 GeV.

scale of $\tilde{\rho}_D^3$; as had been pointed out [17,18] other unaccounted effects introduce theoretical uncertainties of the same magnitude.

4 Running $\alpha_s$ and BLM corrections

We are also in the position to calculate the so-called BLM corrections – the effects accounting for running of the strong coupling $\alpha_s$ in one-loop perturbative diagrams. Such effects are typically quite significant where the perturbative calculations do not eliminate explicitly the low-momentum domain; however, they are moderate, or even small in size in the appropriate Wilsonian scheme. A dedicated discussion for the case of the total semileptonic width was given in Refs. [11,12]. We leave a detailed implementation of this method for the future publication [16], and here place the emphasis on the qualitative features. They allow to assess the possible accuracy of the theoretical predictions without much numerology.

The technique for the BLM summation in the Wilsonian approach was first discussed in Ref. [19]; it is concisely presented in recent Refs. [11,12], and we do not describe it here. Neither we detail numerical predictions or address all experimentally interesting observables. We also largely leave aside more subtle theoretical aspects related to full BLM resummation for higher moments with only a few lowest nonperturbative operators included. Our main interest lies in the most sensitive case for theory, viz. the width and the first hadronic moment $\langle M_X^2 \rangle$ with the lepton energy cut.

We start with the decay width itself when the lepton energy cut is imposed. A convenient way to visualize the contribution of various momentum scales $Q^2$ in one-loop
perturbation theory for an observable $A$ is provide by the distribution

$$
\frac{dA_{\text{pert}}}{d\ln Q^2} = -\frac{dA^{(1)}(\lambda^2)}{d\ln \lambda^2}
\bigg|_{\lambda^2=Q^2},
$$

where $A^{(1)}(\lambda^2)$ is the first-order perturbative coefficient calculated with the fictitious non-zero gluon mass [20, 21]. This representation has some limitations, yet is quite suitable for qualitative purposes and in adopted in the plots presented below.

Fig. 4 shows the distribution over the gluon virtualities for the two extreme cases, $E_{\text{cut}} = 0$ (no cut) and $E_{\text{cut}} = 1.5$ GeV. Without the scale separation, dashed lines, the perturbative corrections are clearly not too well behaved, since the major contribution comes from the gluon momenta noticeably below 1 GeV. This is an artefact brought in by the pole masses [22,23] used by conventional perturbative diagrams. Fig. 4 shows that applying Wilsonian procedure quite effectively eliminates this domain (solid line).

At first glance, there is no much difference between the perturbative corrections for $E_{\text{cut}} = 1.5$ GeV compared to the total width without kinematic restrictions. However, the low-momentum tail is much higher for $\Gamma_{\text{sl}}(E^\ell > 1.5 \text{ GeV})$. This leads to a far more significant impact of the soft physics, the fact deduced independently from the growth of the Wilson coefficient for the higher-dimension nonperturbative operators (e.g., Darwin). As anticipated, the effect is more significant for moments of the distributions. We will now look closer at the average hadron invariant mass squared $\langle M_X^2 \rangle$.

In the OPE, $M_X^2$ consists of two distinct dynamic pieces (they mix under the renormalization of the effective low-scale QCD for heavy quark):

$$
\langle M_X^2 \rangle = m_c^2 + (M_B - m_b)^2 + (M_B - m_b)\langle 2E_x \rangle + (m_x^2 - m_c^2)
E_x \equiv m_b - q_0 ,
\Delta_c = m_x^2 - m_c^2 \equiv m_b^2 + q^2 - 2m_b q_0 - m_c^2.
$$

Figure 3: Similar plot for the third invariant mass moment $\langle [M_X^2 - \langle M_X^2 \rangle]^3 \rangle$. Total perturbative corrections are reduced (orange dashed-dotted line) using $\rho_3(0.9 \text{ GeV})$ compared to the case of $\tilde{\rho}_3$ (red solid line). Blue solid line corresponds to the pole scheme.
Figure 4: Distribution over the gluon momentum for the total width without a cut (blue) and for $E_{\text{cut}} = 1.5$ GeV (red). Solid lines correspond to the cutoff scale $\mu = 1$ GeV, dashed lines show the unsuppressed contributions in the pole-type schemes. The area bounded by a curve yields the overall first-order perturbative coefficient.

The first one, $(M_B - m_b) \langle 2E_x \rangle$ dominates corrections to $\langle M_X^2 \rangle$ [24]. However, among the perturbative corrections those to $\langle \Delta_c \rangle$ appear most significant. Therefore, we start with the second piece, $\langle \Delta_c \rangle$.

At zero lepton cut, $\Delta_c$ which in the conventional perturbative diagrams is described by real gluon emission, is very soft, see Fig. 5: the four-body phase space for $b \to cg + \ell\nu$ is very sensitive to the gluon mass even when the latter is only a few hundred MeV. Presence of such corrections would be a disaster for precision calculations. However, in QCD the phase space-unsuppressed soft gluon emissions feeding $m_x^2$ are related by gauge symmetry to the Coulomb self-energy of the heavy quark [15]. The Wilsonian treatment then effectively eliminates the infrared domain, as illustrated by the solid curve corresponding to the separation scale $\mu = 0.9$ GeV.

The similar distribution for the cut at 1.5 GeV is shown in Fig. 6. Applying the same Wilsonian cutoff at 0.9 GeV we eliminate the deep infrared domain as well, yet paying the price of quite unphysical spike in the contribution of gluons at the scale around 1 GeV. It disappears only when $\mu$ is lowered down to 0.55 GeV. This behavior does not mean that the masses and higher-dimension operators normalized at, say 1 GeV cannot be used for perturbative calculations. The perturbative result can be expressed in terms of any short-distance masses, for instance $\bar{m}_b(m_b)$ and $\bar{m}_c(m_b)$ corresponding to $\mu$ in a few GeV range. It rather demonstrates that the effect of the domain of momenta between 600 MeV and 1 GeV is not fairly described by only the leading $1/m_b$ and $1/m_b^2$ contributions – the terms scaling like higher powers of $1/m_b$ are comparable, or even dominate them. This is a direct consequence of the deteriorating hardness of the inclusive width with the high cut, which effectively introduces another, much lower than $m_b$ mass scale parameter in the OPE [17, 18, 25].

For milder cuts with lower cutoff energy the convergence of the power expansion improves; for instance, Fig. 7 shows the similar distribution at $E_{\text{cut}} = 0.9$ GeV where it
still looks perfect at $\mu=0.8\,\text{GeV}$.

The similar scrutiny can be applied to another short-distance component of $\langle M_X^2 \rangle$, the perturbative corrections to $\langle 2E_x \rangle$. Its naive $O(\alpha_s)$ evaluation turns out quite small and might be thought to be of no practical interest. However, we see from Figs. 8 and 9 that this is a result of cancellations between softer and harder gluons; as such it is vitiated already when running of $\alpha_s$ is accounted for. The effect of the higher-order corrections, taken at face value appears dramatic here. However, in the Wilsonian approach all these corrections can be readily accounted for, and lead only to minute changes in $\langle M_X^2 \rangle$ for reasonably placed cuts on $E^\ell$.

Similar in spirit and technically anatomy can, in principle, be applied to higher hadronic moments with cuts as well, where the terms accompanying different powers of $(M_B-m_b(\mu))$ should be analyzed separately (say, for the second mass squared moment there are three terms with the power from 0 to 2). We have to bear in mind, however, that the higher moments more seriously depend on the Darwin operator. Therefore, ac-
counting for running of $\alpha_s$ would not be too meaningful without extending the Wilsonian treatment to higher-dimension operators, first of all to the Darwin operator. We will present the corresponding results in the forthcoming publication [16].

Here we illustrate the numerical results for $\langle M^2_X \rangle$. Fig. 10 shows the breakdown of the predictions for the perturbative corrections to $\langle M^2_X \rangle$ as a function of the cut on lepton energy, using the fixed-order perturbative estimates with $\alpha_s = 0.3$ [14], a generally appropriate choice for beauty decays. For comparison we show also the predictions obtained by combining the first-order $\mathcal{O}(\alpha_s)$ term evaluated with admittedly too low a value $\alpha_s^{\overline{\text{MS}}} = 0.22$, with the second-order BLM correction, $\mathcal{O}(\beta_0 \alpha_s^2)$:

\[
\langle M^2_X \rangle = m_c^2(\mu) + (M_B - m_b(\mu))^2 \\
+ (M_B - m_b(\mu))m_b \left\{ E_1^{(0)} + C_F \left[ \frac{\alpha_s}{\pi} E_1^{(1)} + \frac{\beta_0}{2} \left( \frac{\alpha_s}{\pi} \right)^2 E_1^{(2)} + \left( \frac{\beta_0}{2} \right)^2 \left( \frac{\alpha_s}{\pi} \right)^3 E_1^{(3)} + ... \right] \right\} \\
+ m_b^2 C_F \left[ \frac{\alpha_s}{\pi} \Delta_1^{(1)} + \frac{\beta_0}{2} \left( \frac{\alpha_s}{\pi} \right)^2 \Delta_1^{(2)} + \left( \frac{\beta_0}{2} \right)^2 \left( \frac{\alpha_s}{\pi} \right)^3 \Delta_1^{(3)} + ... \right],
\]  

(11)
\[ \alpha_s \equiv \alpha_s^\infty(m_b), \quad C_F = \frac{4}{3}, \quad \beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f = 9. \]

Such an approximation is routinely applied in lieu of a more thoughtful choice of \( \alpha_s \) where the corresponding BLM correction is known. We can even add the higher-order BLM terms (dotted lines in Fig. 10).

It is clear that the higher-order corrections are under good control here. Taking the results literally, the effective value of \( \alpha_s \) to be used in the fixed-order calculations is only slightly larger than 0.3 slowly increasing at higher cuts, and does not exceed 0.38, well within the interval allowed for in Ref. [14].

To have an unbiased comparison, however one should recall that the part of the shift when using the BLM-improved form is actually offset by the change in the commensurate value of the effective Darwin expectation value:

\[
\bar{\rho}_D^3 \simeq \begin{cases} 
\rho_D^3(1 \text{ GeV}) - 0.085 \text{ GeV}^3 & \text{fixed order, } \alpha_s = 0.3 \\
\rho_D^3(1 \text{ GeV}) - 0.12 \text{ GeV}^3 & \mathcal{O}(\beta_0 \alpha_s^2), \quad \alpha_s = 0.22 \\
\rho_D^3(1 \text{ GeV}) - 0.16 \text{ GeV}^3 & \mathcal{O}(\beta_0^2 \alpha_s^3), \quad \alpha_s = 0.22
\end{cases}
\]

For the first-order BLM improvement this shift lies below theoretical accuracy and we usually neglect it, but it is relevant when assessing the net effect of the BLM improvement. For higher orders it becomes significant.

Table 1. Perturbative coefficients for \( \langle M_X^2 \rangle \) at \( m_\pi = 1.16 \text{ GeV}, \ m_b = 4.6 \text{ GeV}, \ \mu = 1 \text{ GeV} \)

| \( E_{\text{cut}}, \text{GeV} \) | \( \Delta_1^{(1)} \) | \( \Delta_1^{(2)} \) | \( \Delta_1^{(3)} \) | \( \Delta_1^{(4)} \) | \( E_0^{(0)} \) | \( E_1^{(0)} \) | \( E_1^{(1)} \) | \( E_1^{(2)} \) | \( E_1^{(3)} \) | \( E_1^{(4)} \) |
|----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0              | 0.021       | 0.036       | 0.065       | 0.3         | 0.846       | 0.0098      | 0.061       | 0.097       | 0.49        |
| 0.6            | 0.02        | 0.035       | 0.064       | 0.3         | 0.839       | 0.0085      | 0.058       | 0.095       | 0.49        |
| 0.9            | 0.019       | 0.034       | 0.063       | 0.3         | 0.827       | 0.0069      | 0.056       | 0.095       | 0.51        |
| 1.2            | 0.018       | 0.036       | 0.067       | 0.33        | 0.811       | 0.0069      | 0.06         | 0.11        | 0.58        |
| 1.5            | 0.023       | 0.046       | 0.087       | 0.44        | 0.796       | 0.015       | 0.088       | 0.15        | 0.8         |

Relegating the detailed analysis of other inclusive semileptonic averages including higher hadronic moments, to the dedicated publication [16], here we give a few perturbative coefficients, including BLM corrections, for the first, Table 1, and second, Tables 2a and 2b, hadronic mass square moments, at different lepton energy cut. The coefficients for the second moment are defined in analogy to the case of the first hadronic moment, Eq. (12), however they proliferate:

\[
\langle [M_X^2 - \langle M_X^2 \rangle]^2 \rangle = (M_B - m_b)^2 m_b^2 \left\{ E_2^{(0)} + C_F \left[ \frac{\alpha_s}{\pi} E_2^{(1)} + \frac{\beta_0}{2} \left( \frac{\alpha_s}{\pi} \right)^2 E_2^{(2)} + \ldots \right] \right\} \\
+ (M_B - m_b) m_b C_F \left[ \frac{\alpha_s}{\pi} \Delta_1^{(1)} + \frac{\beta_0}{2} \left( \frac{\alpha_s}{\pi} \right)^2 \Delta_1^{(2)} + \left( \frac{\beta_0}{2} \right)^2 \left( \frac{\alpha_s}{\pi} \right)^3 \Delta_1^{(3)} + \ldots \right] \\
+ m_b^4 C_F \left[ \frac{\alpha_s}{\pi} \Delta_2^{(1)} + \frac{\beta_0}{2} \left( \frac{\alpha_s}{\pi} \right)^2 \Delta_2^{(2)} + \left( \frac{\beta_0}{2} \right)^2 \left( \frac{\alpha_s}{\pi} \right)^3 \Delta_2^{(3)} + \ldots \right].
\]
It should be stressed that higher-order coefficients are quoted only for illustration, and should not really be added to the corrections without carefully adjusting for the effect of the residual infrared piece.

The estimated perturbative corrections to the second $\langle (M_X^2 - \langle M_X^2 \rangle)^2 \rangle$ and to the third $\langle (M_X^2 - \langle M_X^2 \rangle)^3 \rangle$ hadronic mass moments depending on the lepton cut, and their breakdown into separate terms are shown in Figs. 2 and 3, both in fixed-order perturbation theory. We also show the results in the pole scheme ($\mu = 0$) assuming, however, the same values of the quark masses. For numerical evaluations to order $\alpha_s^3$ we commonly use $\alpha_s = 0.3$, and assume the values of the short-distance heavy quark masses

$$m_c(1 \text{ GeV}) = 1.16 \text{ GeV}, \quad m_b(1 \text{ GeV}) = 4.60 \text{ GeV}$$

preferred by experiment.

Table 2a. Perturbative coefficients for $\langle (M_X^2 - \langle M_X^2 \rangle)^2 \rangle$, in the same setting

| $E_{\text{cut}}$ | $\Delta_2^{(1)}$ | $\Delta_2^{(2)}$ | $\Delta_2^{(3)}$ | $\Delta_2^{(4)}$ | $\Delta_{11}^{(1)}$ | $\Delta_{11}^{(2)}$ | $\Delta_{11}^{(3)}$ | $\Delta_{11}^{(4)}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0               | -0.00082        | -0.0035         | -0.0077         | -0.046          | -0.0023         | -0.01            | -0.023          | -0.14           |
| 0.6             | -0.0016         | -0.0044         | -0.0091         | -0.052          | -0.0043         | -0.013           | -0.027          | -0.16           |
| 0.9             | -0.0026         | -0.0057         | -0.011          | -0.06           | -0.007          | -0.016           | -0.033          | -0.18           |
| 1.2             | -0.0036         | -0.0073         | -0.014          | -0.073          | -0.0099         | -0.02            | -0.039          | -0.21           |
| 1.5             | -0.0046         | -0.0091         | -0.017          | -0.089          | -0.012          | -0.024           | -0.046          | -0.24           |

Table 2b. The coefficients for the term $\propto (M_B - m_b)^2$

| $E_{\text{cut}}$ | $E_2^{(0)}$ | $E_2^{(1)}$ | $E_2^{(2)}$ | $E_2^{(3)}$ | $E_2^{(4)}$ |
|-----------------|-------------|-------------|-------------|-------------|-------------|
| 0               | 0.194       | -0.0038     | -0.017      | -0.029      | -0.17       |
| 0.6             | 0.2         | -0.0054     | -0.02       | -0.032      | -0.17       |
| 0.9             | 0.208       | -0.0076     | -0.024      | -0.035      | -0.18       |
| 1.2             | 0.218       | -0.01       | -0.03       | -0.041      | -0.2        |
| 1.5             | 0.226       | -0.013      | -0.035      | -0.047      | -0.23       |

The first-order perturbative corrections to the second mass squared moment happen to be rather suppressed near zero cut on $E^\ell$. To some extent this is accidental and takes place just for our choice of $\mu$ around 1 GeV: it would be nearly complete for a somewhat smaller $\mu$, and softer for a larger separation scale. The BLM corrections at small $E_{\text{cut}}$ then seem to have a large relative impact, nearly doubling the perturbative correction. However, as seen from Fig. 2, the cancellation fades out at a relatively low cut, and the significance of the BLM corrections becomes moderate already at $E_{\text{cut}} > 600$ MeV. Above 0.9 GeV the face value of the effective $\alpha_s$ for the one-loop contribution is about 0.35 – even discarding the possible effect of redefining the appropriate Darwin expectation value $\rho_D^3$. Therefore, it seems likely that the actual perturbative corrections to the second moment is even flatter than follows from Fig. 2.
It has been pointed out in Ref. [14] that, in view of possible cancellations in the first-order perturbative coefficients, it is unsafe to estimate the uncertainty associated with uncalculated perturbative corrections simply as a fixed fraction of the one-loop result. This may particularly apply to the schemes other than the pole one. Ref. [14] suggested that an additional uncertainty should be added obtained examining certain ‘perturbative’ variations of the nonperturbative expectation values, to evade such special cases. We see that this recipe works well here: while even a 50% variation in the perturbative corrections to the second mass moment near zero cut might easily underestimate their actual uncertainty, considering the effect of the Darwin operator gives the right estimate of the size, in fact on the safer side.

5 Discussions

We have calculated the one-loop perturbative corrections to all the semileptonic decay structure functions of the heavy quarks, in the form allowing to obtain all BLM corrections on the same footing. This parallels the work done in Ref. [26] almost a decade ago which, however was applicable only to the total semileptonic width. With the complete expressions for the power corrections to the structure functions through order $1/m_Q^3$ available [4,10], there remain no practical obstacles in calculating all sufficiently inclusive semileptonic distributions with significant precision. As has been emphasized in Ref. [12] in respect to precision determination of $|V_{cb}|$, now the real limitation in many cases becomes perturbative corrections to the Wilson coefficients of the power-suppressed nonperturbative operators; sometimes the effects of the nonperturbative four-fermion averages with charm fields (called ‘intrinsic charm’ in Ref. [12]) may be noticeable.

We think that the program of precision extraction of higher nonperturbative heavy quark parameters of $B$ mesons will be on agenda regardless of calculation the perturbative corrections with all kinematic constraints – in many instances already power corrections proper lead to significant uncertainties. In this respect studying at $B$-factories the modified higher hadronic mass-energy moments $\mathcal{N}^k_X$ detailed in Ref. [14] looks promising. From the perturbative point of view they are similar, or even simpler than the standard moments of $M^2_X$, and our analysis applies to them in full.

We plan to present both perturbative and nonperturbative corrections in an (possibly cumbersome) analytic form in the forthcoming publication [16], and to simultaneously provide ready-to-use computer programs making numerical evaluations an automated procedure.

Calculation of the perturbative $b$-decay structure functions to order $\alpha_s^1$ is rather straightforward if only lengthy. The main problem is to control possible typos, in particular in the computer program which would evaluate the corrections. A single omission can both radically change the emerging numerical result, or to be nearly invisible numerically in a particular kinematic setting. To safeguard our results against such complications, we have applied a number of checks.

Firstly, a regular limit for a massless gluon was verified for all the moments, and the
logarithmic dependence of separately the virtual and the bremsstrahlung contributions at given $q^2$ was checked to match the known classical bremsstrahlung radiation divergence. Furthermore, the limit $\lambda^2 \to 0$ in the sum of the contributions for a particular moment, is slow containing the terms $\propto \sqrt{\lambda^2}$, rather than only those scaling like $\lambda^2$. These terms, however are controlled by the OPE – they reside solely in the corresponding dependence of the pole quark masses on $\lambda^2$. Moreover, the OPE ensures that in the first-loop corrections terms $\propto \lambda^2 \ln \lambda^2$ do not appear. This provided the serious test for the moments – once passing to the Wilsonian scheme at a given gluon mass (and such a translation involves coefficients with a nontrivial kinematic dependence), we observed a straight linear in $\lambda^2$ behavior at the small gluon mass.

At arbitrary gluon mass of order $m_c$ or below we observed that the total width we numerically calculate precisely coincides with the same width we used in Refs. [11, 12]; those expressions were derived in Ref. [26] using, generally, the different set of integration variables. The total width, however depends only on $w_1$ and $w_2$, see Eq. (5). We also compared the numerical results of the perturbative corrections to the width with an arbitrary lepton energy cut at $\lambda^2 \to 0$ with the computations used previously [14] based on the well-established analytic expressions of Ref. [6]; this verified $w_3$ as well. Together with a number of more technical cross checks, a confidence has been gained that possible typos in the program for evaluating moments were all eliminated.

Although perturbative effects are conceptually simple, getting accurate predictions requires thoughtful combining perturbative and nonperturbative corrections in the optimal way. Our experience commencing with an early paper [27] suggests that using the literal Wilsonian prescription which separates various contributions based on their intrinsic momentum scale, is the way to get reliable predictions. In return, applying it often yields accurate results even with minimal computational efforts, or using simplifying approximations.

In the present paper we have reported some results for the dependence of the perturbative corrections to the hadronic mass moments, on the charge lepton energy cut; an extensive analysis is in preparation [16]. In accord with the expectations which considered peculiarity of the adopted OPE implementation, we found weak variations, for the safe intervals in $E_{\text{cut}}$, well within our estimates of the overall accuracy theory in its present form can realistically provide [14]. The perturbative corrections to $\langle M_X^2 \rangle$ actually turned out nearly constant, varying even much less than could be anticipated. The $E_{\text{cut}}$-variation in the second mass moment, $\langle (M_X^2 - \langle M_X^2 \rangle)^2 \rangle$ is more noticeable, consistent in sign and magnitude with the cut-dependent contribution from the Darwin operator. We actually found indications that they may be further flatten when the BLM corrections are incorporated: the latter eliminate a somewhat accidental cancellation at very low $E_{\text{cut}}$.

It should be emphasized that such a moderate sensitivity resulted from eliminating soft gluon effects from the perturbative diagrams; in the schemes without such a separation (we generically refer to them as pole-type) the variation, as a rule, is far more pronounced. We also find that sensible BLM improvement of the one-loop estimates is possible within the Wilsonian approach, although it does not seem to have dramatic impact on the a
priori safe moments. Nevertheless, we point out that there are strong cancellations in the perturbative corrections to the hadron energy $\langle E_x \rangle$ between different domains of integration. They are vitiated once running of $\alpha_s$ is accounted for, which strongly enhances their effect. Their significance have often been underestimated.

Our predictions [14,25] both for the absolute values of the hadronic moments, and for their cut-dependence were in a qualitatively good agreement with the preliminary data reported by BaBar and CLEO. Having calculated the full perturbative corrections, we did not find effects which would be unexpectedly large or would show a surprising behavior; therefore, we expect the agreement to persist, or possibly even strengthen. More precise verifications of the theory including fits to many measured observables, is to follow from the dedicated experimental analyses.

Acknowledgments: This study would be impossible without close collaboration with P. Gambino, including joint work directly on the subjects addressed here. I am especially grateful to him for his share in obtaining a number of numerical theory predictions long awaited by experiment, presented above. I am indebted to many experimental colleagues from BaBar, in particular to Oliver Buchmuller, Vera Luth and Urs Langenegger for important discussions and communications which initiated this study, and for encouraging exchanges. It is my pleasure to thank Ikaros Bigi for collaboration on closely related issues and for important suggestions. I would like to thank R. Zwicky and G. Uraltsev for their help with Mathematica. This work was supported in part by the NSF under grant number PHY-0087419.

References

[1] I. Bigi, N. Uraltsev and A. Vainshtein, Phys. Lett. B293 (1992) 430; B. Blok and M. Shifman, Nucl. Phys. B399 (1993) 441 and 459.

[2] I.I. Bigi, B. Blok, M. Shifman, N.G. Uraltsev and A. Vainshtein, DPF Conf. 1992:610-613; hep-ph/9212227.

[3] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, Phys. Rev. Lett. 71 (1993) 496.

[4] B. Blok, L. Koyrakh, M. Shifman and A. Vainshtein, Phys. Rev. D49 (1994) 3356.

[5] M. Shifman, A. Vainshtein and V. Zakharov, Phys. Lett. 78B (1978) 443 [Reprinted in The Standard Model Higgs Boson, Ed. M. Einhorn (North-Holland, Amsterdam, 1991), p. 84].

[6] M. Jezabek and J. H. Kuhn, Nucl. Phys. B 320 (1989) 20; A. Czarnecki, M. Jezabek and J. H. Kuhn, Acta Phys. Polon. B 20 (1989) 961; A. Czarnecki and M. Jezabek, Nucl. Phys. B 427 (1994) 3.

[7] M. Gremm and I. Stewart, Phys. Rev. D55 (1997) 1226.
[8] A. F. Falk, M. E. Luke and M. J. Savage, *Phys. Rev.* D53 (1996) 2491.

[9] A. F. Falk and M. E. Luke, *Phys. Rev.* D57 (1998) 424.

[10] M. Gremm and A. Kapustin, *Phys. Rev.* D55 (1997) 6924.

[11] N. Uraltsev, *Int. Journ. Mod. Phys. Lett.* A17 (2002) 2317.

[12] D. Benson, I. Bigi, Th. Mannel and N. Uraltsev, *Nucl. Phys.* B665 (2003) 367.

[13] M. Trott, hep-ph/0402120.

[14] P. Gambino and M. Uraltsev, *Europ. Phys. Journal* C, to appear; hep-ph/0401063.

[15] I.I. Bigi, M. Shifman, N.G. Uraltsev and A. Vainshtein, *Phys. Rev.* D52 (1995) 196.

[16] P. Gambino and N. Uraltsev, paper in progress.

[17] N. Uraltsev, Proc. of the 31st International Conference on High Energy Physics, Amsterdam, The Netherlands, 25-31 July 2002 (North-Holland – Elsevier, The Netherlands, 2003), S. Bentvelsen, P. de Jong, J. Koch and E. Laenen Eds., p. 554; hep-ph/0210044.

[18] N. Uraltsev, hep-ph/0308165; talk at Int. Conference “Flavor Physics & CP Violation 2003”, June 3-6 2003, Paris. To appear in the Proceedings.

[19] N.G. Uraltsev, *Nucl. Phys.* B491 (1997) 303.

[20] P. Ball, M. Beneke and V.M. Braun, *Nucl. Phys.* B452 (1995) 563.

[21] Yu.L. Dokshitzer, G. Marchesini and B.R. Webber, *Nucl. Phys.* B469 (1996) 93.

[22] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, *Phys. Rev.* D50 (1994) 2234.

[23] M. Beneke, V.M. Braun and V.I. Zakharov, *Phys. Rev. Lett.* 73 (1994) 3058.

[24] I. Bigi and N. Uraltsev, *Nucl. Phys.* B423 (1994) 33.

[25] I.I. Bigi and N. Uraltsev, *Phys. Lett.* B579 (2004) 340.

[26] P. Ball, M. Beneke and V. Braun, *Phys. Rev.* D52 (1995) 3929.

[27] N.G. Uraltsev, *Int. J. Mod. Phys.* A11 (1996) 515.