AN INFINITE PRESENTATION FOR THE MAPPING CLASS
GROUP OF A NON-ORIENTABLE SURFACE WITH BOUNDARY

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Abstract. We give an infinite presentation for the mapping class group of
a non-orientable surface with boundary components. The presentation is a
generalization of the presentation given by the second author [15].

1. Introduction

Let $\Sigma_{g,n}$ be a compact connected oriented surface of genus $g \geq 0$ with $n \geq 0$
boundary components. The mapping class group $\mathcal{M}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is the group
of isotopy classes of orientation preserving self-diffeomorphisms on $\Sigma_{g,n}$ fixing the
boundary pointwise. A finite presentation for $\mathcal{M}(\Sigma_{g,n})$ was given by Hatcher-
Thurston [7], Harer [6], Wajnryb [21], Gervais [5] and Labruère-Paris [11]. Ger-
vais [4] obtained an infinite presentation for $\mathcal{M}(\Sigma_{g,n})$ by using Wajnryb’s finite
presentation for $\mathcal{M}(\Sigma_{g,n})$, and Luo [14] rewrote Gervais’ presentation into a sim-
pler infinite presentation (see Theorem 2.5).

Let $N_{g,n}$ be a compact connected non-orientable surface of genus $g \geq 1$ with
$n \geq 0$ boundary components. The surface $N_g = N_{g,0}$ is a connected sum of $g$ real
projective planes. The mapping class group $\mathcal{M}(N_{g,n})$ of $N_{g,n}$ is the group of isotopy
classes of self-diffeomorphisms on $N_{g,n}$ fixing the boundary pointwise. For $g \geq 2$ and
$n \in \{0,1\}$, a finite presentation for $\mathcal{M}(N_{g,n})$ was given by Lickorish [12], Birman-
Chillingworth [2], Stukow [19]. and Paris-Szepietowski [16]. Note that $\mathcal{M}(N_1)$ and
$\mathcal{M}(N_1,1)$ are trivial (see [3, Theorem 3.4]) and $\mathcal{M}(N_2)$ is finite (see [12, Lemma 5]).
Stukow [19] rewrote Paris-Szepietowski’s presentation into a finite presentation with
Dehn twists and a “Y-homeomorphism” as generators (see Theorem 3.1). The
second author [15] gave a simple infinite presentation for $\mathcal{M}(N_{g,n})$ for $g \geq 1$ and
$n \in \{0,1\}$. The generating set consists of all Dehn twists and all “crosscap push-
ning maps” along simple loops. We review the crosscap pushing map in Section 2.

In this paper, we give a simple infinite presentation for $\mathcal{M}(N_{g,n})$ for arbitrary $g \geq
0$ and $n \geq 0$ (Theorem 4.1). The presentation is a generalization of the presentation
given by the second author [15]. We will prove Theorem 4.1 by applying Gervais’
argument to a finite presentation for $\mathcal{M}(N_{g,n})$ in Proposition 3.2.

Contents of this paper are as follows. In Section 2 we prepare some elements of
$\mathcal{M}(N_{g,n})$ and some relations among their elements in $\mathcal{M}(N_{g,n})$, and review the in-
finité presentation for $\mathcal{M}(\Sigma_{g,n})$ (Theorem 2.6) which is an improvement by Luo [14]
of Gervais’ presentation in [4]. In Section 3 we review Stukow’s finite presentation
for $\mathcal{M}(N_{g,n})$ when $n \in \{0,1\}$ (Theorem 3.1) and give a finite presentation for
$\mathcal{M}(N_{1,n})$ when $n \geq 2$ (Proposition 3.2). In the proof of the main theorem in Sec-
tion 4 we use their finite presentations for $\mathcal{M}(N_{g,n})$. In Section 4 we give the main

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we define $f_h$ in this paper and a proof of the main theorem. Finally, in Section 3 we give a proof of Proposition 3.2.

2. Preliminaries

2.1. Relations among Dehn twists and Gervais’ presentation. Let $S$ be either $N_{g,n}$ or $\Sigma_{g,n}$. We denote by $N_S(A)$ a regular neighborhood of a subset $A$ in $S$. We assume that every simple closed curve on $S$ is oriented throughout this paper, and for simple closed curves $c_1, c_2$ on $S$, $c_1 = c_2$ means $c_1$ is isotopic to $c_2$ in consideration of their orientations. Denote by $c^{-1}$ the inverse curve of a simple closed curve $c$ on $S$. Note that $(c^{-1})^{-1} = c$. For a two-sided simple closed curve $c$ on $S$, we can take two orientations $+c$ and $-c$ of $N_S(c)$. When $S$ is orientable, we take $+c$ as the orientation of $N_S(c)$ which is induced by the orientation of $S$. Then for a two-sided simple closed curve $c$ on $S$ and an orientation $\theta \in \{+c, -c\}$ of $N_S(c)$, denote by $t_{c,\theta}$ the right-handed Dehn twist along $c$ on $S$ with respect to $\theta$.

Note that $t_{c,+c} = t_{c^{-1},+c} = t_{c,-c}$. For some convenience, we write $t_{c} = t_{c,+c}$ for a two-sided simple closed curve $c$, where orientation of $N_S(c)$ is given explicitly (for instance, $S$ is an orientable surface). In particular, for a given explicit two-sided simple closed curve, an arrow on a side of the simple closed curve indicates the direction of the Dehn twist (see Figure 1). For elements $f = [\varphi], h = [\psi] \in \mathcal{M}(S)$, we define $fh := [\varphi \circ \psi] \in \mathcal{M}(S)$.

![Figure 1. The right-handed Dehn twist $t_c = t_{c,\theta}$ along a two-sided simple closed curve $c$ on $S$ with respect to the orientation $\theta \in \{+c, -c\}$ of $N_S(c)$ as in the figure.](image)

Recall the following relations in $\mathcal{M}(S)$ among Dehn twists along two-sided simple closed curves on $S$.

**Lemma 2.1.** Let $c$ be a two-sided simple closed curve $c$ on $S$ and $\theta \in \{+c, -c\}$ an orientation of $N_S(c)$. If $c$ bounds a disk or a Möbius band in $S$, then we have $t_{c,\theta} = 1$ in $\mathcal{M}(S)$.

For a two-sided simple closed curve $c$ on $S$ and $f \in \mathcal{M}(S)$, we have a bijection $f_* = (f|_{N_S(c)})_* : \{+c, -c\} \to \{+f(c), -f(c)\}$.

**Lemma 2.2** (The braid relation (i)). For a two-sided simple closed curve $c$ on $S$ and $f \in \mathcal{M}(S)$, we have

$$t_{f(c);\theta'} = ft_{c,\theta}f^{-1} \quad \text{when } f_*(\theta) = \theta'$$

$$t_{f(c);\theta'}^{-1} = ft_{c,\theta}f^{-1} \quad \text{when } f_*(\theta) \neq \theta'.$$

When $f$ in Lemma 2.2 is a Dehn twist $t_d$ along a two-sided simple closed curve $d$ and the geometric intersection number $|c \cap d|$ of $c$ and $d$ is $m$, we denote by $T_m$ the braid relation.
Let $c_1, c_2, \ldots, c_k$ be two-sided simple closed curves on $S$. The sequence $c_1, c_2, \ldots, c_k$ is a $k$-chain on $S$ if $c_1, c_2, \ldots, c_k$ satisfy $|c_i \cap c_{i+1}| = 1$ for each $i = 1, 2, \ldots, k-1$ and $|c_i \cap c_j| = 0$ for $|j-i| > 1$.

**Lemma 2.3** (The $k$-chain relation). Let $c_1, c_2, \ldots, c_k$ be a $k$-chain on $S$ and let $\delta, \delta'$ (resp. $\delta$) be distinct boundary components (resp. the boundary component) of $N^*_S(c_1 \cup c_2 \cup \cdots \cup c_k)$ when $k$ is odd (resp. even). We give an orientation $\theta \in \{+\delta, -\delta\}$ of $N^*_S(\delta)$ and the orientation of $N^*_S(c_1 \cup c_2 \cup \cdots \cup c_k)$ which is induced by $\theta$. Then we have

$$
(t \epsilon_{c_1} t \epsilon_{c_2} \cdots t \epsilon_{c_k})^{k+1} = t_{\delta, \theta} t_{\delta', \theta'} \quad \text{when } k \text{ is odd},
$$

$$
(t \epsilon_{c_1} t \epsilon_{c_2} \cdots t \epsilon_{c_k})^{2k+2} = t_{\delta, \theta} \quad \text{when } k \text{ is even},
$$

where for $c \in \{c_1, c_2, \ldots, c_k, \delta'\}$, $\epsilon_c = 1$ if $\theta_c$ coincides with the orientation of $N^*_S(c_1 \cup c_2 \cup \cdots \cup c_k)$, and $\epsilon_c = -1$ otherwise.

**Lemma 2.4** (The lantern relation). Let $\Sigma$ be a subsurface of $S$ which is diffeomorphic to $\Sigma_{0,4}$ and let $\delta_{12}, \delta_{23}, \delta_{13}, \delta_1, \delta_2, \delta_3$ and $\delta_4$ be simple closed curves on $\Sigma$ as in Figure 2. We give an orientation $\theta \in \{+\delta_4, -\delta_4\}$ of $N^*_S(\delta_4)$ and the orientation of $\Sigma$ which is induced by $\theta$. Then we have

$$
(t \epsilon_{\delta_{12}} t \epsilon_{\delta_{23}} t \epsilon_{\delta_{13}}) t \epsilon_{\delta_1} t \epsilon_{\delta_2} t \epsilon_{\delta_3} t \epsilon_{\delta_4} = t_{\delta_1, \theta} t_{\delta_2, \theta} t_{\delta_3, \theta} t_{\delta_4, \theta},
$$

where for $c \in \{\delta_{12}, \delta_{23}, \delta_{13}, \delta_1, \delta_2, \delta_3, \delta_4\}$, $\epsilon_c = 1$ if $\theta_c$ coincides with the orientation of $\Sigma$, and $\epsilon_c = -1$ otherwise.

**Figure 2.** The simple closed curves $\delta_{12}$, $\delta_{23}$, $\delta_{13}$, $\delta_1$, $\delta_2$, $\delta_3$ and $\delta_4$ on $\Sigma$.

Luo’s presentation for $\mathcal{M}(\Sigma_{g,n})$, which is an improvement of Gervais’ one, is as follows.

**Theorem 2.5** ([4, 14]). For $g \geq 0$ and $n \geq 0$, $\mathcal{M}(\Sigma_{g,n})$ has the following presentation:

- **generators:** $\{t_c : \text{s.c.c. on } \Sigma_{g,n}\}$, where s.c.c. means simple closed curve.

- **relations:**
  
  - (0') $t_{c} = 1$ when $c$ bounds a disk in $\Sigma_{g,n}$,
  
  - (I') All the braid relations $T_0$ and $T_1$,
  
  - (II) All the 2-chain relations,
  
  - (III) All the lantern relations.
2.2. Relations among the crosscap pushing maps and Dehn twists. Let \( \mu \) be a one-sided simple closed curve on \( N_{g,n} \) and let \( \alpha \) be a simple closed curve on \( N_{g,n} \) such that \( \mu \) and \( \alpha \) intersect transversally at one point. Recall that \( \alpha \) is oriented. For these simple closed curves \( \mu \) and \( \alpha \), we denote by \( Y_{\mu,\alpha} \) a self-diffeomorphism on \( N_{g,n} \) which is described as the result of pushing the Möbius band \( N_{N_{g,n}}(\mu) \) once along \( \alpha \). We call \( Y_{\mu,\alpha} \) a crosscap pushing map. In particular, if \( \alpha \) is two-sided, we call \( Y_{\mu,\alpha} \) a \( Y \)-homeomorphism (or a crosscap slide), where a crosscap means a Möbius band in the interior of a surface. Note that \( Y_{\mu,\alpha} \) is a \( Y \)-homeomorphism was originally defined by Lickorish [12]. We have the following fundamental relation in \( \mathcal{M}(N_{g,n}) \) and we also call the relation the braid relation.

**Lemma 2.6** (The braid relation (ii)). Let \( \mu \) be a one-sided simple closed curve on \( N_{g,n} \) and let \( \alpha \) and \( \beta \) be simple closed curves on \( N_{g,n} \) such that \( \mu \) and \( \alpha \) intersect transversally at one point. For \( f \in \mathcal{M}(N_{g,n}) \), suppose that \( f(\alpha) = \beta \) or \( f(\alpha) = \beta^{-1} \). Then we have

\[
Y_{f(\mu),\beta} = fY_{\mu,\alpha}f^{-1} \quad \text{when} \quad f(\alpha) = \beta,
\]

\[
Y_{f(\mu),\beta}^{-1} = fY_{\mu,\alpha}f^{-1} \quad \text{when} \quad f(\alpha) = \beta^{-1}.
\]

We describe crosscap pushing maps from a different point of view. Let \( e : D' \hookrightarrow \text{int} S \) be a smooth embedding of the unit disk \( D' \subset \mathbb{C} \). Put \( D := e(D') \). Let \( S' \) be the surface obtained from \( S - \text{int} D \) by the identification of antipodal points of \( \partial D \).

We call the manipulation that gives \( S' \) from \( S \) the blowup of \( S \) on \( D \). Note that the image \( M \subset S' \) of \( N_{S-\text{int} D}(\partial D) \subset S - \text{int} D \) with respect to the blowup of \( S \) on \( D \) is a crosscap. Conversely, the blowdown of \( S' \) on \( M \) is the following manipulation that gives \( S \) from \( S' \). We paste a disk on the boundary obtained by cutting \( S \) along the center line \( \mu \) of \( M \). The blowdown of \( S' \) on \( M \) is the inverse manipulation of the blowup of \( S \) on \( D \).

Let \( \mu \) be a one-sided simple closed curve on \( N_{g,n} \) and let \( S \) be the surface which is obtained from \( N_{g,n} \) by the blowdown of \( N_{g,n} \) on \( N_{N_{g,n}}(\mu) \). Note that \( S \) is diffeomorphic to \( N_{g-1,n} \) or \( S_{h,n} \) for \( g = 2h + 1 \). Denote by \( x_\mu \) the center point of a disk \( D_\mu \) that is pasted on the boundary obtained by cutting \( S \) along \( \mu \). Let \( e : D' \hookrightarrow D_\mu \subset S \) be a smooth embedding of the unit disk \( D' \subset \mathbb{C} \) to \( S \) such that \( D_\mu = e(D') \) and \( e(0) = x_\mu \). Let \( \mathcal{M}(S,x_\mu) \) be the group of isotopy classes of self-diffeomorphisms on \( S \) fixing the boundary \( \partial S \) and the point \( x_\mu \), where isotopies also fix the boundary \( \partial S \) and \( x_\mu \). Then we have the blowup homomorphism

\[
\varphi_\mu : \mathcal{M}(S,x_\mu) \to \mathcal{M}(N_{g,n})
\]

that is defined as follows. For \( h \in \mathcal{M}(S,x_\mu) \), we take a representative \( h' \) of \( h \) which satisfies either of the following conditions: (a) \( h'|_{D_\mu} \) is the identity map on \( D_\mu \); (b) \( h'(x) = e(e^{-1}(x)) \) for \( x \in D_\mu \), where \( e^{-1}(x) \) is the complex conjugation of \( e^{-1}(x) \in \mathbb{C} \). Such \( h' \) is compatible with the blowup of \( S \) on \( D_\mu \), thus \( \varphi_\mu(h) \in \mathcal{M}(S) \) is induced and well defined (c.f. [20 Subsection 2.3]).

The point pushing map

\[
j_{x_\mu} : \pi_1(S,x_\mu) \to \mathcal{M}(S,x_\mu)
\]

is a homomorphism that is defined as follows. For \( \gamma \in \pi_1(S,x_\mu) \), \( j_{x_\mu}(\gamma) \in \mathcal{M}(S,x_\mu) \) is described as the result of pushing the point \( x_\mu \) once along \( \gamma \). The point pushing map comes from the Birman exact sequence. Note that for \( \gamma_1, \gamma_2 \in \pi_1(S,x_\mu) \), \( \gamma_1 \gamma_2 \) means \( \gamma_1 \gamma_2(t) = \gamma_2(2t) \) for \( 0 \leq t \leq \frac{1}{2} \) and \( \gamma_1 \gamma_2(t) = \gamma_1(2t - 1) \) for \( \frac{1}{2} \leq t \leq 1 \).
Following Szepietowski [20] we define the composition of the homomorphisms:

\[ \psi_{x_{\mu}} := \varphi_{\mu} \circ j_{x_{\mu}} : \pi_1(S, x_{\mu}) \to \mathcal{M}(N_{g,n}). \]

For each closed curve \( \alpha \) on \( N_{g,n} \) which transversely intersects with \( \mu \) at one point, we take a loop \( \overline{\alpha} \) on \( S \) based at \( x_{\mu} \) such that \( \overline{\alpha} \) has no self-intersection points on \( D_\mu \) and \( \overline{\alpha} \) is the image of \( \overline{\alpha} \) with respect to the blowup of \( S \) on \( D_\mu \). If \( \alpha \) is simple, we take \( \overline{\alpha} \) as a simple loop. The next two lemmas follow from the description of the point pushing map (see [10, Lemma 2.2, Lemma 2.3]).

**Lemma 2.7.** For a simple closed curve \( \alpha \) on \( N_{g,n} \) which transversely intersects with a one-sided simple closed curve \( \mu \) on \( N_{g,n} \) at one point, we have

\[ \psi_{x_{\mu}}(\overline{\alpha}) = Y_{\mu,\alpha}. \]

**Lemma 2.8.** For a one-sided simple closed curve \( \alpha \) on \( N_{g,n} \) which transversely intersects with a one-sided simple closed curve \( \mu \) on \( N_{g,n} \) at one point, we take \( N_S(\overline{\alpha}) \) such that the interior of \( N_S(\overline{\alpha}) \) contains \( D_\mu \) and an orientation \( \theta_{\alpha} \in \{+\alpha, -\alpha\} \) of \( N_S(\overline{\alpha}) \). Denote by \( \delta_1 \) (resp. \( \delta_2 \)) the boundary component of \( N_S(\overline{\alpha}) \) on the right (resp. left) side of \( \overline{\alpha} \), and by \( \delta_i \) (\( i = 1, 2 \)) the two-sided simple closed curve on \( N_{g,n} \) which is the image of \( \delta_i \) with respect to the blowup of \( S \) on \( D_\mu \). Let \( \theta_i \in \{+\delta_i, -\delta_i\} \) (\( i = 1, 2 \)) be the orientation of \( N_{N_{g,n}}(\delta_i) \) which is induced by \( \theta_{\alpha} \) and \( \theta_i \in \{+\delta_i, -\delta_i\} \) (\( i = 1, 2 \)) the orientation of \( N_{N_{g,n}}(\delta_i) \) which is induced by \( \theta_i \) (see Figure 3). Then we have

\[ Y_{\mu,\alpha} = t_{\delta_1, \delta_2}^{-\varepsilon_{\delta_1}} t_{\delta_1, \delta_2}^{\varepsilon_{\delta_2}}, \]

where \( \varepsilon_{\delta_1} = 1 \) if \( \theta_{\delta_1} = \theta_1 \), and \( \varepsilon_{\delta_1} = -1 \) otherwise.

![Figure 3](image_url)

**Figure 3.** Simple closed curves \( \delta_1, \delta_2, \delta_1 \) and \( \delta_2 \), and orientations \( \overline{\delta_1}, \overline{\delta_2}, \theta_1 \) and \( \theta_2 \) of their regular neighborhoods. The x-mark means that antipodal points of \( \partial D_\mu \) are identified.

By the definition of the homomorphism \( \psi_{x_{\mu}} \) and Lemma 2.7, we have the following lemma.

**Lemma 2.9.** Let \( \alpha \) and \( \beta \) be simple closed curves on \( N_{g,n} \) which transversely intersect with a one-sided simple closed curve \( \mu \) on \( N_{g,n} \) at one point each. Suppose that the product \( \overline{\alpha\beta} \) of \( \overline{\alpha} \) and \( \overline{\beta} \) in \( \pi_1(S, x_{\mu}) \) is represented by a simple loop on \( S \), and \( \alpha\beta \) is a simple closed curve on \( N_{g,n} \) which is the image of the representative of \( \overline{\alpha\beta} \) with respect to the blowup of \( S \) on \( D_\mu \). Then we have

\[ Y_{\mu,\alpha\beta} = Y_{\mu,\alpha} Y_{\mu,\beta}. \]
Finally, we recall the following relation between a Dehn twist and a Y-homeomorphism.

**Lemma 2.10.** Let $\alpha$ be a two-sided simple closed curve on $N_{g,n}$ which transversely intersect with a one-sided simple closed curve $\mu$ on $N_{g,n}$ at one point and let $\delta$ be the boundary of $N_{g,n}^\pm(\alpha \cup \mu)$. Since $\pi_1^\pm(S, x_{\mu})$ is represented by a two-sided simple loop, we take $\delta_1$ and $\theta_i$ ($i = 1, 2$) as in Lemma 2.8 when $\alpha$ in Lemma 2.8 is replaced by $\alpha^\pm$ (see Figure 4). Then we have

$$Y_{\mu, \alpha}^2 = \begin{cases} t_{\delta_1} & \text{when } \delta = \delta_1, \\ t_{\delta_2} & \text{when } \delta = \delta_2. \end{cases}$$

where $\varepsilon_1 = 1$ for $i = 1, 2$ if $\theta_i = \theta_1$, and $\varepsilon_i = -1$ otherwise.

Lemma 2.10 follows from relations in Lemma 2.1, Lemma 2.8 and Lemma 2.9.

**Figure 4.** The orientation $\theta_1$ of $N_{g,n}^\pm(\delta)$ when $\delta = \delta_1$ or the orientation $\theta_2$ of $N_{g,n}^\pm(\delta)$ when $\delta = \delta_2$.

### 3. Finite presentation for $\mathcal{M}(N_{g,n})$

In this section, we review Stukow’s finite presentation for $\mathcal{M}(N_{g,n})$ when $n \in \{0, 1\}$ and give a finite presentation for $\mathcal{M}(N_{g,n})$ when $n \geq 2$. We use their finite presentations for $\mathcal{M}(N_{g,n})$ in the proof of the main theorem in Section 4.

Let $e_i : D' \hookrightarrow \partial D_0$ for $i = 1, 2, \ldots, g + n - 1$ be smooth embeddings of the unit disk $D' \subset \mathbb{C}$ to a disk $D_0$ such that $D_i := e_i(D')$ and $D_j$ are disjoint for distinct $1 \leq i, j \leq g + n - 1$. For $n \geq 1$, we take a model of $N_{g,n}$ as the surface obtained from $\partial D_{g+1}, \ldots, \partial D_{g+n-1}$ by the blowups on $D_1, \ldots, D_g$ and we describe the identification of $\partial D_i$ by the x-mark as in Figures 5. We denote by $\delta_1, \ldots, \delta_{g-1}$ and $\delta$ boundary components of $N_{g,n}$ as in Figure 5 which are obtained from $\partial D_{g+1}, \ldots, \partial D_{g+n-1}$ and $\partial D_{g+1}$, respectively. Let $\alpha_1, \ldots, \alpha_{g-1}, \beta$ and $\mu_1$ be simple closed curves on $N_{g,n}$ as in Figure 6 and let $\alpha_{i,j}$ for $1 \leq i \leq g - 1$ and $1 \leq j \leq n - 1$, $\rho_{i,j}$ for $1 \leq i \leq g$ and $1 \leq j \leq n - 1$ and $\sigma_{i,j}$, $\delta_{i,j}$ for $1 \leq i < j \leq n - 1$ be simple closed curves on $N_{g,n}$ as in Figure 5. We give orientations of regular neighborhoods of their simple closed curves as in Figure 5. Then we define the mapping classes

$$a_i := t_{\alpha_i} \quad \text{for } 1 \leq i \leq g - 1,$$

$$b := t_{\beta},$$

$$y := Y_{\mu_1, \alpha_1}.$$
\begin{align*}
d_i &:= t_{\delta_i} \quad \text{for } 1 \leq i \leq n - 1, \\
a_{i;j} &:= t_{\alpha_{i;j}} \quad \text{for } 1 \leq i \leq g - 1 \text{ and } 1 \leq j \leq n - 1, \\
r_{i;j} &:= t_{\rho_{i;j}} \quad \text{for } 1 \leq i \leq g \text{ and } 1 \leq j \leq n - 1, \\
s_{i,j} &:= t_{\sigma_{i,j}} \quad \text{for } 1 \leq i < j \leq n - 1, \\
\bar{s}_{i,j} &:= t_{\bar{\sigma}_{i,j}} \quad \text{for } 1 \leq i < j \leq n - 1, \\
\bar{s}_{j,k;i} &:= \{(a_1 a_1^{-1})^{-1} r_{2,k} \cdots (a_{i-1} a_1^{-1})^{-1} r_{i;k})^{-1} \bar{s}_{j,k} \\
 &\quad \{(a_1 a_1^{-1})^{-1} r_{2,k} \cdots (a_{i-1} a_1^{-1})^{-1} r_{i;k}) \}
&\quad \text{for } 2 \leq i \leq g \text{ and } 1 \leq j < k \leq n - 1.
\end{align*}

Remark that, for $2 \leq i \leq g$ and $1 \leq j < k \leq n - 1$, $\bar{s}_{j,k;i}$ is the Dehn twist along the simple closed curve $\bar{\sigma}_{j,k;i}$ on $N_{g,n}$ as in Figure 7.

**Figure 5.** A model of $N_{g,n}$ and simple closed curves $\alpha_1, \ldots, \alpha_{g-1}$, $\beta$ and $\mu_1$ on $N_{g,n}$.

**Figure 6.** The simple closed curves $\alpha_{i;j}$, $\rho_{i;j}$, $\sigma_{i;j}$ and $\bar{\sigma}_{i,j}$ on $N_{g,n}$.

Epstein [3] show that $\mathcal{M}(N_{1,1})$ is trivial. We define $[x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1}$. Stukow gave the following finite presentation for $\mathcal{M}(N_{g,1})$ when $g = 2$ in [17], and when $g \geq 3$ in [19] by rewriting the finite presentation in [16].

**Theorem 3.1** ([3], [17], [19]). $\mathcal{M}(N_{1,1})$ is the trivial group. $\mathcal{M}(N_{2,1})$ has the presentation
\[
\mathcal{M}(N_{2,1}) = \langle a_1, y \mid ya_1 y^{-1} = a_1^{-1} \rangle.
\]
If \( g \geq 3 \), then \( \mathcal{M}(N_{g,1}) \) admits a presentation with generators \( a_1, \ldots, a_{g-1}, y \), and \( b \) for \( g \geq 4 \). The defining relations are

\[
\begin{align*}
\text{(A1)} & \quad [a_i, a_j] = 1 & \text{for } g \geq 4, |i - j| > 1, \\
\text{(A2)} & \quad a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1} & \text{for } i = 1, \ldots, g-2, \\
\text{(A3)} & \quad [a_i, b] = 1 & \text{for } g \geq 4, i \neq 4, \\
\text{(A4)} & \quad a_ia_4b = b_4a_4 & \text{for } g \geq 5, \\
\text{(A5)} & \quad (a_ia_2a_3a_4b)^6 = (a_1a_2a_3a_4b)^6 & \text{for } g \geq 5, \\
\text{(A6)} & \quad (a_1a_2a_3a_4a_5a_6b)^{12} = (a_1a_2a_3a_4a_5a_6b)^{12} & \text{for } g \geq 7, \\
\text{(A9a)} & \quad [b_2, b] = 1 & \text{for } g = 6, \\
\text{(A9b)} & \quad [a_{g-5}, b_{g-2}, b_{g-1}, b_g] = 1 & \text{for } g \geq 8 \text{ even}, \text{where } b_0 = a_1, b_1 = b \text{ and } \\
& \quad b_{i+1} = (b_{i-1}a_2a_3+1a_2i+1a_2i+2b_3)5(b_{i-1}a_2a_3+1a_2i+1a_2i+3) - 6 & \text{for } 1 \leq i \leq \frac{g-4}{2}, \\
\text{(B1)} & \quad y(a_2a_3a_1a_2ya_2) = (a_2a_3a_1a_2ya_2) & \text{for } g \geq 4, \\
\text{(B2)} & \quad (a_2a_1y^{-1}a_2a_1y) = a_1a_2a_1y^{-1}a_1a_2a_1y & a_1, \\
\text{(B3)} & \quad [a_i, y] = 1 & \text{for } g \geq 4, i = 3, \ldots, g-1, \\
\text{(B4)} & \quad a_2(ya_2y^{-1}) = (ya_2y^{-1})a_2, \\
\text{(B5)} & \quad ya_1 = a_1^{-1}y, \\
\text{(B6)} & \quad byy^{-1} = (a_1a_2a_3(y^{-1}a_2y)a_3^{-1}a_2^{-1}a_3^{-1}a_2^{-1}) & \text{for } g \geq 4, \\
\text{(B7)} & \quad [(a_4a_2a_1a_2a_3a_1a_2ya_2) = (a_4a_2a_1a_2a_3a_1a_2ya_2) & \text{for } g \geq 6, \\
\text{(B8)} & \quad \{[a_1a_2a^{-1}a_3^{-1}a_2^{-1}a_1^{-1}b^{-1}(a_2a_3a_2a_1)] \} & \text{for } g \geq 5.
\end{align*}
\]

For \( n \geq 2 \), we have the following finite presentation for \( \mathcal{M}(N_{g,n}) \) and give the proof in Section 5.3.

**Proposition 3.2.** For \( g \geq 1 \) and \( n \geq 2 \), \( \mathcal{M}(N_{g,n}) \) has the presentation which obtained from the finite presentation for \( \mathcal{M}(N_{g,1}) \) in Theorem 3.1.1 by adding generators \( d_i (i = 1, \ldots, n-1) \), \( a_{i,j} (1 \leq i \leq g-1, 1 \leq j \leq n-1) \), \( r_{i,j} (1 \leq i \leq g, 1 \leq j \leq n-1) \), \( s_{i,j} (1 \leq i < j \leq n-1) \) and \( s_{i,j} (1 \leq i < j \leq n-1) \) and the following relations for \( 1 \leq i, m \leq g, 1 \leq j < k \leq n-1 \) and \( 1 \leq l < t < k \):

\[
\begin{align*}
\text{(D0)} & \quad [d_i, a_i] = [d_j, y] = [d_j, b] = [d_j, d_i] = [d_j, a_{i,t}] = [d_j, r_{i,t}] = [d_j, s_{i,t}] = [d_j, s_{i,t}] = 1, \\
\text{(D1a)} & \quad a_m(a_i^{-1}a_i^{-1})a_m^{-1} = \begin{cases} (a_i^{-1}a_i^{-1})(a_i^{-1}a_i^{-1}) & \text{for } m = i - 1, \\
(a_i^{-1}a_i^{-1})(a_i^{-1}a_i^{-1}) & \text{for } m = i + 1, \\
a_i^{-1}a_i^{-1} & \text{for } m \neq i - 1, i + 1, \\
(a_i^{-1}a_i^{-1})r_{i,k}d_{k}^{-1} & \text{for } i = 1, \\
(a_i^{-1}a_i^{-1})r_{i,k}d_{k}^{-1} & \text{for } i = 2, \\
a_i^{-1}a_i^{-1} & \text{for } i \geq 3, \\
\end{cases} \\
\text{(D1b)} & \quad y(a_i^{-1}a_i^{-1})y^{-1} = \begin{cases} (a_i^{-1}a_i^{-1})r_{i,k}d_{k}^{-1} & \text{for } i = 2, \\
(a_i^{-1}a_i^{-1})r_{i,k}d_{k}^{-1} & \text{for } i \geq 3, \\
\end{cases}
\end{align*}
\]
(D1c) $b(a_{i,k}a_i^{-1})b^{-1} = \begin{cases} 
(a_{3,k}a_i^{-1})(a_{1,k}a_i^{-1})^{-1}(a_{2,k}a_i^{-1})(a_{3,k}a_i^{-1}) & \text{for } i = 1, \\
(a_{3,k}a_i^{-1})(a_{1,k}a_i^{-1})^{-1}(a_{2,k}a_i^{-1})(a_{3,k}a_i^{-1}) & \text{for } i = 2, \\
(a_{4,k}a_i^{-1})(a_{1,k}a_i^{-1})^{-1}(a_{2,k}a_i^{-1}) & \text{for } i = 3, \\
(a_{4,k}a_i^{-1})(a_{1,k}a_i^{-1})^{-1}(a_{2,k}a_i^{-1}) & \text{for } i = 4, \\
(a_{i,k}a_i^{-1}) & \text{for } i \geq 5, 
\end{cases}$

(D1d) $a_{m,l}(a_{i,k}a_i^{-1})^{a_{m,l}^{-1}} = \begin{cases} 
(s_{l,k}d_l^{-1})^{-1}, (s_{m,k}d_m^{-1})^{-1}(a_{i,k}a_i^{-1}) & \text{for } m \leq i - 2, \\
(s_{l,k}d_l^{-1})^{-1}, (s_{m,k}d_m^{-1})^{-1}(a_{i,k}a_i^{-1}) & \text{for } m = i - 1, \\
(s_{l,k}d_l^{-1})^{-1}(a_{i,k}a_i^{-1}) & \text{for } m = i, \\
(s_{l,k}d_l^{-1})^{-1}(a_{i,k}a_i^{-1}) & \text{for } m = i + 1, \\
(a_{i,k}a_i^{-1})^{-1} & \text{for } m \geq i + 2, 
\end{cases}$

(D1e) $r_{m,l}(a_{i,k}a_i^{-1})r_{m,l}^{-1} = \begin{cases} 
(s_{l,k}d_l^{-1})^{-1}, (s_{m,k}d_m^{-1})^{-1}(a_{i,k}a_i^{-1}) & \text{for } m \leq i - 1, \\
(s_{l,k}d_l^{-1})^{-1}, (s_{m,k}d_m^{-1})^{-1}(a_{i,k}a_i^{-1}) & \text{for } m = i - 1, \\
(s_{l,k}d_l^{-1})^{-1}, (s_{m,k}d_m^{-1})^{-1}(a_{i,k}a_i^{-1}) & \text{for } m = i, \\
(s_{l,k}d_l^{-1})^{-1}, (s_{m,k}d_m^{-1})^{-1}(a_{i,k}a_i^{-1}) & \text{for } m = i + 1, \\
(a_{i,k}a_i^{-1})^{-1} & \text{for } m \geq i + 2, 
\end{cases}$

(D1f) $s_{l,t}(a_{i,k}a_i^{-1})s_{l,t}^{-1} = a_{i,k}a_i^{-1}.$

(D1g) $\tilde{s}_{l,t}(a_{i,k}a_i^{-1})\tilde{s}_{l,t}^{-1} = \begin{cases} 
(s_{l,k}d_l^{-1})^{-1}, (s_{l,k}d_l^{-1})^{-1}(a_{1,k}a_1^{-1}) & \text{for } i = 1, \\
(s_{l,k}d_l^{-1})^{-1}, (s_{l,k}d_l^{-1})^{-1}(a_{1,k}a_1^{-1}) & \text{for } i = 2, \\
(s_{l,k}d_l^{-1})^{-1}, (s_{l,k}d_l^{-1})^{-1}(a_{1,k}a_1^{-1}) & \text{for } i \geq 2, 
\end{cases}$

(D2a) $a_m r_{i,k} a_i^{-1} = \begin{cases} 
(r_{i,k}r_{1,k})^{-1}(a_{i-1,k}a_i^{-1})^{-1}r_{i,k}(a_{i-1,k}a_i^{-1}) & \text{for } m = i - 1, \\
(a_{i,k}a_i^{-1})^{-1}r_{i+1,k}^{-1}(a_{i,k}a_i^{-1}) & \text{for } m = i, \\
r_{i,k}^{-1} & \text{for } m \neq i - 1, i, 
\end{cases}$

(D2b) $y r_{i,k} y^{-1} = \begin{cases} 
(a_{1,k}a_1^{-1})^{-1}r_{2,k}r_{1,k}^{-1} & \text{for } i = 1, \\
(a_{1,k}a_1^{-1})^{-1}r_{2,k}r_{1,k}^{-1} & \text{for } i = 2, \\
r_{i,k}^{-1} & \text{for } i \geq 3, 
\end{cases}$
\begin{align*}
\text{(D2c)} \quad b_{r,ik}^{-1} &= \begin{dcases}
(a_{1,k}a_1^{-1})^{-1}(a_{3,k}a_3^{-1})^{-1}(a_{2,k}a_2^{-1})^{-1}r_{r,ik}^{-1}(a_{3,k}a_3^{-1})^{-1}r_{r,ik}^{-1}(a_{2,k}a_2^{-1})^{-1}r_{r,ik}^{-1}(a_{1,k}a_1^{-1})^{-1}r_{r,ik}^{-1} & \text{for } i = 1, \\
\{(a_{3,k}a_3^{-1})^{-1}(a_{1,k}a_1^{-1})^{-1}(a_{3,k}a_3^{-1})^{-1}r_{r,ik}^{-1}(a_{2,k}a_2^{-1})^{-1}r_{r,ik}^{-1}(a_{3,k}a_3^{-1})^{-1}r_{r,ik}^{-1}(a_{1,k}a_1^{-1})^{-1}r_{r,ik}^{-1} & \text{for } i = 2, \\
\{(a_{3,k}a_3^{-1})^{-1}(a_{1,k}a_1^{-1})^{-1}(a_{3,k}a_3^{-1})^{-1}r_{r,ik}^{-1}(a_{2,k}a_2^{-1})^{-1}r_{r,ik}^{-1}(a_{3,k}a_3^{-1})^{-1}r_{r,ik}^{-1}(a_{1,k}a_1^{-1})^{-1}r_{r,ik}^{-1} & \text{for } i = 3, \\
\{(a_{3,k}a_3^{-1})^{-1}(a_{1,k}a_1^{-1})^{-1}(a_{3,k}a_3^{-1})^{-1}r_{r,ik}^{-1}(a_{2,k}a_2^{-1})^{-1}r_{r,ik}^{-1}(a_{3,k}a_3^{-1})^{-1}r_{r,ik}^{-1}(a_{1,k}a_1^{-1})^{-1}r_{r,ik}^{-1} & \text{for } i = 4, \\
(a_{3,k}a_3^{-1})^{-1}(a_{1,k}a_1^{-1})^{-1}(a_{3,k}a_3^{-1})^{-1}r_{r,ik}^{-1} & \text{for } i \geq 5,
\end{dcases}
\end{align*}

\begin{align*}
\text{(D2d)} \quad a_{m,i}r_{r,ik}a_{m,i}^{-1} &= \begin{dcases}
[(s_{t,k}d_t^{-1})^{-1}, (a_{m,k}a_m^{-1})^{-1}]^{-1}r_{r,ik}[(s_{t,k}d_t^{-1})^{-1}, (a_{m,k}a_m^{-1})^{-1}] & \text{for } m \leq i - 2, \\
\{(s_{t,k}d_t^{-1})^{-1}, (a_{m,k}a_m^{-1})^{-1}\}^{-1}(a_{i-1,k}a_{i-1}^{-1})^{-1}(s_{t,k}d_t^{-1})^{-1}r_{r,ik} & \text{for } m = i - 1, \\
(a_{i-1,k}a_{i-1}^{-1})^{-1}[(s_{t,k}d_t^{-1})^{-1}, (a_{i-1,k}a_{i-1}^{-1})^{-1}]^{-1}r_{r,ik} & \text{for } m = i, \\
(a_{i-1,k}a_{i-1}^{-1})^{-1}[(s_{t,k}d_t^{-1})^{-1}, (a_{i-1,k}a_{i-1}^{-1})^{-1}]^{-1}r_{r,ik} & \text{for } m = i + 1, \\
(a_{i-1,k}a_{i-1}^{-1})^{-1}[(s_{t,k}d_t^{-1})^{-1}, (a_{i-1,k}a_{i-1}^{-1})^{-1}]^{-1}r_{r,ik} & \text{for } m = i + 2, \\
& \vdots
\end{dcases}
\end{align*}

\begin{align*}
\text{(D2e)} \quad r_{m,i}r_{r,ik}r_{m,i}^{-1} &= \begin{dcases}
[(s_{t,k}d_t^{-1})^{-1}, r_{m,i}^{-1}]^{-1}r_{r,ik}[(s_{t,k}d_t^{-1})^{-1}, r_{m,i}^{-1}] & \text{for } m \leq i - 1, \\
\{(s_{t,k}d_t^{-1})^{-1}, r_{m,i}^{-1}\}^{-1}r_{r,ik} & \text{for } m = i, \\
r_{r,ik} & \text{for } m \geq i + 1,
\end{dcases}
\end{align*}

\begin{align*}
\text{(D2f)} \quad s_{t,k}r_{r,ik}s_{t,k}^{-1} &= r_{r,ik},
\end{align*}

\begin{align*}
\text{(D2g)} \quad s_{t,k}r_{r,ik}s_{t,k}^{-1} &= \begin{dcases}
(s_{t,k}d_t^{-1})^{-1}, (s_{t,k}d_t^{-1})^{-1} & \text{for } i = 1, \\
(s_{t,k}d_t^{-1})^{-1}, (s_{t,k}d_t^{-1})^{-1} & \text{for } i = 2, \\
& \vdots
\end{dcases}
\end{align*}

\begin{align*}
\text{(D3a)} \quad a_{m}(s_{j,k}d_j^{-1})a_{m}^{-1} &= s_{j,k}d_j^{-1},
\end{align*}

\begin{align*}
\text{(D3b)} \quad y(s_{j,k}d_j^{-1})y^{-1} &= s_{j,k}d_j^{-1},
\end{align*}

\begin{align*}
\text{(D3c)} \quad b(s_{j,k}d_j^{-1})b^{-1} &= s_{j,k}d_j^{-1},
\end{align*}

\begin{align*}
\text{(D3d)} \quad a_{m}(s_{j,k}d_j^{-1})a_{m}^{-1} &= \begin{dcases}
[(s_{t,k}d_t^{-1})^{-1}, (a_{m,k}a_m^{-1})^{-1}]^{-1} & \text{for } l > j, \\
(a_{m,k}a_m^{-1})^{-1} & \text{for } l = j,
\end{dcases}
\end{align*}

\begin{align*}
\text{(D3e)} \quad r_{m,i}(s_{j,k}d_j^{-1})r_{m,i}^{-1} &= \begin{dcases}
[(s_{t,k}d_t^{-1})^{-1}, r_{m,i}^{-1}]^{-1} & \text{for } l > j, \\
r_{m,i}^{-1} & \text{for } l = j,
\end{dcases}
\end{align*}
\[(D3f)\] \[s_{l,t}(s_{j,k}d_j^{-1})s_{l,t}^{-1} = \begin{cases} \{(s_{l,k}d_t^{-1})(s_{j,k}d_j^{-1})\}^{-1}(s_{j,k}d_j^{-1})\}^{-1}((s_{l,k}d_t^{-1})(s_{j,k}d_j^{-1})) & \text{for } l = j, \\
(s_{l,k}d_t^{-1})^{-1}(s_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})^{-1}(s_{l,k}d_t^{-1})^{-1}(s_{l,k}d_t^{-1})^{-1} & \text{for } l < j < t, \\
(s_{l,k}d_t^{-1})^{-1}(s_{j,k}d_j^{-1})(s_{l,k}d_t^{-1})^{-1} & \text{for } t = j, \\
(s_{j,k}d_j^{-1})^{-1} & \text{for the other cases,} \end{cases} \]

\[(D3g)\] \[\hat{s}_{l,t}(s_{j,k}d_j^{-1})\hat{s}_{l,t}^{-1} = \begin{cases} \{(\hat{s}_{l,k}d_t^{-1})(\hat{s}_{j,k}d_j^{-1})\}^{-1}(\hat{s}_{l,k}d_t^{-1})\}^{-1}(s_{j,k}d_j^{-1})^{-1}((s_{l,k}d_t^{-1})(s_{j,k}d_j^{-1})) & \text{for } l = j, \\
(s_{l,k}d_t^{-1})^{-1}(s_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})^{-1}(s_{l,k}d_t^{-1})^{-1}(s_{l,k}d_t^{-1})^{-1} & \text{for } l < j < t, \\
(s_{l,k}d_t^{-1})^{-1}(s_{j,k}d_j^{-1})(s_{l,k}d_t^{-1})^{-1} & \text{for } t = j, \\
(s_{j,k}d_j^{-1})^{-1} & \text{for the other cases,} \end{cases} \]

\[(D4a)\] \[a_m(\hat{s}_{j,k}d_j^{-1})a_m = \begin{cases} \{r_{1,k}^{-1}(a_1:k^{-1})\}^{-1}(\hat{s}_{j,k}d_j^{-1})\}^{-1}(r_{1,k}^{-1}(a_1:k^{-1})\}^{-1} & \text{for } m = 1, \\
(\hat{s}_{j,k}d_j^{-1})^{-1} & \text{for } m \geq 2, \end{cases} \]

\[(D4b)\] \[y(\hat{s}_{j,k}d_j^{-1})y^{-1} = \begin{cases} \{r_{1,k}^{-1}(a_1:k^{-1})^{-2}r_{2,k}r_{1,k}^{-1}\}^{-1}(\hat{s}_{j,k}d_j^{-1})\}^{-1}(r_{1,k}^{-1}(a_1:k^{-1})^{-2}r_{2,k}r_{1,k}^{-1}) & \text{for } m = 1, \\
(\hat{s}_{j,k}d_j^{-1})^{-1} & \text{for } m \geq 2. \end{cases} \]

\[(D4c)\] \[b(\hat{s}_{j,k}d_j^{-1})b^{-1} = \begin{cases} \{r_{1,k}^{-1}(a_2:a_2)^{-1}r_{3,k}^{-1}(a_3:a_3^{-1})r_{3,k}^{-1}(a_3:a_3^{-1})r_{2,k}^{-1}(a_1:a_1^{-1})^{-1}(\hat{s}_{j,k}d_j^{-1})^{-1} & \text{for } m = 1, \\
\{r_{1,k}^{-1}(a_2:a_2)^{-1}r_{3,k}^{-1}(a_3:a_3^{-1})r_{3,k}^{-1}(a_3:a_3^{-1})r_{2,k}^{-1}(a_1:a_1^{-1})^{-1} & \text{for } m \geq 2, \end{cases} \]

\[(D4d)\] \[a_m(\hat{s}_{j,k}d_j^{-1})a_m^{-1} = \begin{cases} \{r_{1,k}^{-1}(a_1:k^{-1})^{-1}\}^{-1}(\hat{s}_{j,k}d_j^{-1})\}^{-1}(r_{1,k}^{-1}(a_1:k^{-1})^{-1}(\hat{s}_{j,k}d_j^{-1})^{-1} & \text{for } m = 1, \\
(\hat{s}_{j,k}d_j^{-1})^{-1} & \text{for } m \geq 2, \end{cases} \]

\[(D4e)\] \[\text{for the other cases,} \]
The main theorem in this paper is as follows:

\[ \text{(D4e) } r_{m,t}(\bar{s}_{j,k}d_j^{-1})r_{m,t}^{-1} = \begin{cases} 
\{r_{1,k}^{-1}(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})r_{1,k}^{-1}(\bar{s}_{j,k}d_j^{-1}) \}, & \text{for } m = 1, l < j, \\
\{r_{1,k}^{-1}(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})r_{1,k}^{-1}(\bar{s}_{j,k}d_j^{-1}) \}, & \text{for } m = 1, l = j, \\
\{(s_{j,k}d_j^{-1})(r_{1,k}^{-1}(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})r_{1,k}^{-1}(\bar{s}_{j,k}d_j^{-1}) \}^{-1} & \text{for } m = 1, l > j, \\
\{r_{m,j}^{-1}(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})r_{m,j}^{-1}(\bar{s}_{j,k}d_j^{-1}) \}^{-1}, & \text{for } m \geq 2, l = j, \\
\{r_{m,j}^{-1}(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})r_{m,j}^{-1}(\bar{s}_{j,k}d_j^{-1}) \}^{-1}, & \text{for } m \geq 2, l > j, \\
(\bar{s}_{j,k}d_j^{-1})^{-1} & \text{for } m \geq 2, l < j, \\
\end{cases} \]

\[ \text{(D4f) } s_{t,t}(\bar{s}_{j,k}d_j^{-1})s_{t,t}^{-1} = \begin{cases} 
(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1}) & \text{for } t = j, \\
(s_{j,k}d_j^{-1})^{-1}(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1}) & \text{for } l < t, \\
(s_{j,k}d_j^{-1})^{-1}(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1}) & \text{for } l = j, \\
(\bar{s}_{j,k}d_j^{-1})^{-1} & \text{for the other cases,} \\
\end{cases} \]

\[ \text{(D4g) } \bar{s}_{j,t}(\bar{s}_{j,k}d_j^{-1})\bar{s}_{j,t}^{-1} = \begin{cases} 
(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1}) & \text{for } t < j, \\
(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})^{-1}(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1}) & \text{for } t = j, \\
(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})^{-1}(\bar{s}_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1})^{-1}(s_{j,k}d_j^{-1}) & \text{for } l > j, \\
(\bar{s}_{j,k}d_j^{-1})^{-1} & \text{for } l < j, \\
\end{cases} \]

4. Infinite Presentation for $\mathcal{M}(N_g,n)$

The main theorem in this paper is as follows:

**Theorem 4.1.** For $g \geq 1$ and $n \geq 0$, $\mathcal{M}(N_g,n)$ has the following presentation:

- Generators: $\{t_{c,+c}, t_{c,-c} | c \text{ two-sided s.c.c. on } N_{g,n}\} \cup \{Y_{\mu,\alpha} | \mu \text{ one-sided s.c.c. on } N_{g,n}, \alpha \text{ s.c.c. on } N_{g,n}, |\mu \cap \alpha| = 1\}$.

Denote the generating set by $X$.

**Relations:**

- (i) $t_{c,+c} = 1$ when $\theta_c \in \{+, -, c\}$ and $c$ bounds a disk or a Möbius band in $N_{g,n}$,
- (ii) $t_{c,+c} = c^{-1} t_{c,-c} = t_{c,-c}^{-1}$,
- (iii) $Y_{\mu,\alpha} = Y_{\mu,\alpha}^{-1} = Y_{\mu,-\alpha}^{-1}$,
(I) All the braid relations

\[
\begin{align*}
(i) \quad t_{f(c)}^{(\alpha)} & = ft_{c,\theta}f^{-1} \quad \text{when } f_*(\theta) = \theta' \text{ and } f \in X, \\
(i) \quad t_{f(c)}^{-1} & = ft_{c,\theta}f^{-1} \quad \text{when } f_*(\theta) \neq \theta' \text{ and } f \in X, \\
(ii) \quad Y_{f(\mu),\beta} & = fY_{\mu,\alpha}f^{-1} \quad \text{when } f(\alpha) = \beta \text{ and } f \in X, \\
Y_{f(\mu),\beta}^{-1} & = fY_{\mu,\alpha}f^{-1} \quad \text{when } f(\alpha) = \beta^{-1} \text{ and } f \in X.
\end{align*}
\]

(II) All the 2-chain relations,

(III) All the lantern relations,

(IV) All the relations in Lemma 2.9, i.e. \(Y_{\mu,\alpha,\beta} = Y_{\mu,\alpha}Y_{\mu,\beta},\)

(V) All the relations in Lemma 2.8, i.e. \(Y_{\mu,\alpha} = \delta_{\mu,1}^{-1} \delta_{\mu,2}^{-1} \delta_{\mu,2} \delta_{\mu,1},\)

The second author [15] proved Theorem 4.1 when \(g \geq 1 \text{ and } n \in \{0,1\}.\) Since we do not distinguish \(t_{c;\pm,}, \ t_{c;\epsilon}, \ t_{c;\epsilon}^{-1},\) and also do not distinguish \(Y_{\mu,\alpha}^{-1}, \ Y_{\mu,\alpha}^{-1} \text{ and } Y_{\mu,\alpha}^{-1},\) in [15], the presentation in Theorem 3.1 of [15] is different from the presentation in Theorem 4.1. However, these presentation are equivalent by Relation (0)(ii) and (0)(iii). In fact, we can apply the proof of Theorem 3.1 in [15] to the presentation in Theorem 4.1. In (I) and (IV) one can substitute the right hand side of (V) for each generator \(Y_{\mu,\alpha}\) with one-sided \(\alpha.\) Then one can remove the generators \(Y_{\mu,\alpha}\) with one-sided \(\alpha\) and relations (V) from the presentation.

We denote by \(G\) the group which has the presentation in Theorem 4.1. Let \(\iota \ : \Sigma_{g,n} \hookrightarrow \Sigma_{g,n} \) be a smooth embedding and let \(G'\) be the group whose presentation has all Dehn twists along simple closed curves on \(\Sigma_{g,n}\) as generators and Relations (0'), (I), (II) and (III) in Theorem 2.5. By Theorem 2.3 \(M(\Sigma_{g,n})\) is isomorphic to \(G',\) and we have the homomorphism \(G' \to G\) defined by the correspondence of \(t_{c;\epsilon,}\) to \(t_{\iota(c);\epsilon,}\) and \(t_{\iota(c);\epsilon,}^{-1}.\) Then we remark the following.

**Remark 4.2.** The composition \(t_* : M(\Sigma_{g,n}) \to G\) of the isomorphism \(M(\Sigma_{g,n}) \to G'\) and the homomorphism \(G' \to G\) is a homomorphism. In particular, if a product \(t_{c;\epsilon,}^{\epsilon_1} t_{c;\epsilon,}^{\epsilon_2} \cdots t_{c;\epsilon,}^{\epsilon_k}\) of Dehn twists along simple closed curves \(c_1, c_2, \ldots, c_k\) on a connected compact orientable subsurface of \(N_{g,n}\) is equal to the identity map in the mapping class group of the subsurface, then \(t_{\iota(c);\epsilon,}^{\epsilon_1} t_{\iota(c);\epsilon,}^{\epsilon_2} \cdots t_{\iota(c);\epsilon,}^{\epsilon_k}\) is equal to 1 in \(G.\) That means such a relation \(t_{\iota(c);\epsilon,}^{\epsilon_1} t_{\iota(c);\epsilon,}^{\epsilon_2} \cdots t_{\iota(c);\epsilon,}^{\epsilon_k} = 1\) is obtained from Relations (0'), (I), (II) and (III).

Set \(X^\pm := X \cup \{x^{-1} | x \in X\}.\) By Relation (I), we have the following lemma.

**Lemma 4.3.** For \(f \in G,\) suppose that \(f = f_1f_2 \cdots f_k,\) where \(f_1, f_2, \ldots, f_k \in X^\pm.\) Then we have

\[
\begin{align*}
(i) \quad t_{f(c)}^{(\alpha)} & = ft_{c,\theta}f^{-1} \quad \text{when } f_*(\theta) = \theta', \\
(i) \quad t_{f(c)}^{-1} & = ft_{c,\theta}f^{-1} \quad \text{when } f_*(\theta) \neq \theta', \\
(ii) \quad Y_{f(\mu),\beta} & = fY_{\mu,\alpha}f^{-1} \quad \text{when } f(\alpha) = \beta, \\
Y_{\mu,\alpha}^{-1} & = fY_{\mu,\alpha}f^{-1} \quad \text{when } f(\alpha) = \beta^{-1}.
\end{align*}
\]

The next lemma follows from an argument of the combinatorial group theory (for instance, see [5, Lemma 4.2.1, p42]).

**Lemma 4.4.** For groups \(\Gamma, \Gamma'\) and \(F, a\) surjective homomorphism \(\pi : F \to \Gamma\) and a homomorphism \(\nu : F \to \Gamma'\) and \(\nu'(x) := \nu(\tilde{x})\) for \(x \in \Gamma,\) where \(\tilde{x} \in F\) is a lift of \(x\) with respect to \(\pi\) (see the diagram below).

Then if \(\ker \pi \subset \ker \nu, \nu'\) is well-defined and a homomorphism.
\[ \begin{array}{c}
\pi \\
\downarrow \\
\Gamma \\
\nu \\
\downarrow \\
\Gamma' \\
\end{array} \]

We start the proof of Theorem 4.1. When \( n \in \{0, 1\} \), we proved Theorem 4.1 in [15]. Assume \( g \geq 1 \) and \( n \geq 2 \). Then we obtain Theorem 4.1 if \( \mathcal{M}(N_{g,n}) \) is isomorphic to \( G \). Let \( \varphi : G \to \mathcal{M}(N_{g,n}) \) be the surjective homomorphism defined by \( \varphi(t_{c,+}) := t_{c,+} \), \( \varphi(t_{c,-}) := t_{c,-} \), and \( \varphi(Y_{\mu,\alpha}) := Y_{\mu,\alpha} \).

Denote by \( X_0 \subset \mathcal{M}(N_{g,n}) \) the generating set of the finite presentation for \( \mathcal{M}(N_{g,n}) \) in Proposition 3.2. Let \( F(X_0) \) be the free group which is freely generated by \( X_0 \) and let \( \pi : F(X_0) \to \mathcal{M}(N_{g,n}) \) be the natural projection. We define the homomorphism \( \nu : F(X_0) \to G \) by \( \nu(a_i) := a_i \), \( \nu(b) := b \), \( \nu(y) := y \), \( \nu(a_{i,j}) := a_{i,j} \), \( \nu(r_{i,j}) := r_{i,j} \), \( \nu(s_{i,j}) := s_{i,j} \) and a map \( \psi = \nu' : \mathcal{M}(N_{g,n}) \to G \) by \( \psi(a_{i,j}^\pm) := a_{i,j}^\pm \), \( \psi(b^\pm) := b^\pm \), \( \psi(y^\pm) := y^\pm \), \( \psi(a_{i,j}^\pm) := a_{i,j}^\pm \), \( \psi(r_{i,j}^\pm) := r_{i,j}^\pm \), \( \psi(s_{i,j}^\pm) := s_{i,j}^\pm \), and \( \psi(f) := \nu(f) \) for the other \( f \in \mathcal{M}(N_{g,n}) \), where \( \tilde{f} \in F(X_0) \) is a lift of \( f \) with respect to \( \pi \) (see the diagram below).

\[ \begin{array}{c}
F(X_0) \\
\pi \\
\downarrow \\
\mathcal{M}(N_{g,n}) \\
\downarrow \psi \\
G \\
\end{array} \]

If \( \psi \) is a homomorphism, \( \varphi \circ \psi = \text{id}_{\mathcal{M}(N_{g,n})} \) by the definition of \( \varphi \) and \( \psi \). Thus it is sufficient to show that \( \psi \) is a homomorphism and surjective for proving that \( \psi \) is isomorphism.

**4.1. Proof that \( \psi \) is a homomorphism.** By Lemma 4.2 if the relations of the presentation in Proposition 3.2 are obtained from Relations (0), (I), (II), (III), (IV) and (V), then \( \psi \) is a homomorphism.

Let \( N \) be the subsurface of \( N_{g,n} \) as in Figure 8. \( N \) is diffeomorphic to \( N_{g,1} \) and includes simple closed curves \( \alpha_1, \ldots, \alpha_{g-1}, \mu \) and \( \beta \). We regard \( \mathcal{M}(N) \) as a subgroup of \( \mathcal{M}(N_{g,n}) \). Relations (A1), \ldots, (A9b) and (B1), \ldots, (B8) of the presentation for \( \mathcal{M}(N_{g,n}) \) in Proposition 3.2 are relations of \( \mathcal{M}(N) \cong \mathcal{M}(N_{g,1}) \). By Theorem 3.1 in [15], Relations (A1), \ldots, (A9b) and (B1), \ldots, (B8) are obtained from Relations (0), (I), (II), (III), (IV) and (V).

By arguments in the last part of Section 4.1 and Section 5.6, Relations (D0)-(D4g) are obtained from Relations (I) and (III). We have proved that \( \psi \) is a homomorphism.

**4.2. Surjectivity of \( \psi \).** For some convenience, we write \( t_{c,+} = t_c \) in this subsection. We show that there exist lifts of \( t_c \)'s and \( Y_{\mu,\alpha} \)'s with respect to \( \psi \) for cases below, to prove the surjectivity of \( \psi \).

1. \( t_c \): \( c \) is non-separating and \( N_{g,n} - c \) is non-orientable,
2. \( t_c \): \( c \) is non-separating and \( N_{g,n} - c \) is orientable,
3. \( t_c \): \( c \) is separating,
4. \( Y_{\mu,\alpha} \): \( \alpha \) is two-sided and \( N_{g,n} - \alpha \) is non-orientable,
5. \( Y_{\mu,\alpha} \): \( \alpha \) is two-sided and \( N_{g,n} - \alpha \) is orientable,
(6) $\gamma_{m,n}$; $\alpha$ is one-sided.

Set $X^\pm_0 := X_0 \cup \left\{ x^{-1} \mid x \in X_0 \right\}$. For a simple closed curve $c$ on $N_{g,n}$, we denote by $(N_{g,n})_c$ the surface obtained from $N_{g,n}$ by cutting $N_{g,n}$ along $c$ and denote by $\Sigma$ the component of $(N_{g,n})_c$ which does not include $\delta$.

For generators of type (1), (2), (4), (5), (6), by similar arguments in Section 3.2 of [4], there exist their lifts with respect to $\psi$. We note that we use the existence for lifts of generators of type (3) for the proof of the existence for lifts of generators of type (6).

Case (3) where $\Sigma$ is diffeomorphic to $\Sigma_{0,m+1}$ for $m \geq 0$. We proceed by induction on $m \geq 0$. When $m = 0$, $t_c$ is trivial by Relation (0). When $m = 1$, $c = \delta_i$ for some $1 \leq i \leq b - 1$. Hence $d_k$ is the lift of $t_c$.

When $m = 2$, there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1$, $f_2$, $\ldots$, $f_k \in X_0^\pm$ which satisfy either $c = f(\sigma_{i,j})$ or $c = f(\overline{\sigma}_{i,j})$ for some $1 \leq i < j \leq b - 1$. Thus, if $c = f(\sigma_{i,j})$, we have

$$\psi(f s_{i,j} f^{-1}) = f_1 f_2 \cdots f_k s_{i,j} f^{-1} f_2^{-1} f_1^{-1}$$

Lem. [4],

$$= t_{f(\sigma_{i,j})} = t_{f_C}$$

where $\varepsilon$ is 1 or $-1$. Thus $f s_{i,j} f^{-1} \in \mathcal{M}(N_{g,n})$ is a lift of $t_c \in G$ with respect to $\psi$ for some $\varepsilon \in \{-1, 1\}$. By a similar argument, when $c = f(\overline{\sigma}_{i,j})$, $f \overline{s}_{i,j} f^{-1} \in \mathcal{M}(N_{g,n})$ is also a lift of $t_c \in G$ with respect to $\psi$ for some $\varepsilon \in \{-1, 1\}$.

For $m \geq 3$, there exists a simple closed curve $c'$ on $\Sigma$ such that $c'$ separates $\Sigma$ into $\Sigma'$ and $\Sigma''$ which are diffeomorphic to $\Sigma_{0,4}$ and $\Sigma_{0,m-1}$, respectively, and $c \subset \Sigma'$. By using a lantern relation on $\Sigma'$, there exist simple closed curves $c_1 = c', c_2, \ldots, c_6$ on $\Sigma'$ such that $t_c = t_{c_1}^\varepsilon_1 t_{c_2}^\varepsilon_2 \cdots t_{c_6}^\varepsilon_6 \in G$ for some $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6 \in \{-1, 1\}$. Since each $c_i$ ($i = 1, 2, \ldots, 6$) bounds a subsurface of $N_{g,n}$ which does not include $c$ and is diffeomorphic to $\Sigma_{0,m_i+1}$ for some $m_i < m$, by the inductive assumption, there exist lifts $h_1, \ldots, h_6 \in \mathcal{M}(N_{g,n})$ of $t_{c_1}, \ldots, t_{c_6} \in G$ with respect to $\psi$, respectively. Thus $h_1^\varepsilon_1 h_2^\varepsilon_2 \cdots h_6^\varepsilon_6 \in \mathcal{M}(N_{g,n})$ is a lift of $t_c$ with respect to $\psi$.

Case (3) where $\Sigma$ is diffeomorphic to $\Sigma_{h,m+1}$ for $h \geq 1$. In this case, there exists a simple closed curve $c'$ on $\Sigma$ such that $c'$ separates $\Sigma$ into $\Sigma'$ and $\Sigma''$ which are diffeomorphic to $\Sigma_{h,2}$ and $\Sigma_{0,m+1}$, respectively. Then there exists a $2h+1$-chain $c_1, c_2, \ldots, c_{2h+1}$ on $\Sigma'$ such that $N_{N_{g,n}}(c_1 \cup c_2 \cup \cdots \cup c_{2h+1}) = \Sigma'$. By the chain relation, $(t_{c_1}^\varepsilon_1 t_{c_2}^\varepsilon_2 \cdots t_{c_{2h+1}}^\varepsilon_{2h+1})^{2h+2} = t_c t_c'$ for some $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2h+1}, \varepsilon' \in \{-1, 1\}$.
Since \(t_{c_1}, t_{c_2}, \ldots, t_{c_{2h+1}}\) are Dehn twists of type (1) and \(c'\) bounds \(\Sigma''\), \(t_{c_1}, t_{c_2}, \ldots, t_{c_{2h+1}}, t_{c'} \in G\) have lifts \(h_1, h_2, \ldots, h_{2h+1}, h' \in \mathcal{M}(N_{g,n})\) with respect to \(\psi\), respectively. Thus we have

\[
\psi((h_1^{t_1} h_2^{t_2} \cdots h_{2h+1}^{t_{2h+1}})^{2h+2}(h')^{-c'}) = \left(t_{c_1}^{t_1} t_{c_2}^{t_2} \cdots t_{c_{2h+1}}^{t_{2h+1}}\right)^{2h+2}t_{c'}^{-c'}
\]

We remark that the last relation is obtained from Theorem 2.5 and Remark 4.2. Thus \((h_1^{t_1} h_2^{t_2} \cdots h_{2h+1}^{t_{2h+1}})^{2h+2}(h')^{-c'} \in \mathcal{M}(N_{g,n})\) is a lift of \(t_{c'} \in G\) with respect to \(\psi\).

**Case (3)** where \(\Sigma\) is diffeomorphic to \(N_{h,m+1}\) for \(h \geq 1\) \(m \geq 0\). We proceed by induction on \(m \geq 0\). When \(m = 0\), by similar arguments in Section 3.2 in [13], there exists their a lift of \(t_c \in G\) with respect to \(\psi\).

When \(m = 1\), we proceed by induction on \(h \geq 1\). When \(h = 1\), there exists a product \(f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})\) of \(f_1, f_2, \ldots, f_k \in X^h\) such that \(c = f(\rho_{1,j})\) for some \(1 \leq j \leq n - 1\). By a similar argument in the case where \(\Sigma\) is diffeomorphic to \(\Sigma_{0,m+1}\) in \(\Sigma\), we can obtain a lift of \(t_c\) with respect to \(\psi\). Suppose \(h \geq 2\). Then there exist simple closed curves \(c_1, c_2\) on \(\Sigma\) such that \(c_1 \cup c_2\) separates \(\Sigma\) into \(\Sigma', \Sigma''\) and \(\Sigma'''\) which are diffeomorphic to \(\Sigma_{0,4}, \Sigma_{1,1}\) and \(\Sigma_{h-1,1}\), respectively, and \(c \subset \Sigma'\). By using a lantern relation on \(\Sigma'\), there exist simple closed curves \(c_3, \ldots, c_6\) on \(\Sigma'\) such that \(t_c = t_{c_1}^{t_1} t_{c_2}^{t_2} \cdots t_{c_6}^{t_6} \in G\) for some \(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_6 \in \{-1, 1\}\). Since each \(c_i\) \((i = 1, \ldots, 6)\) is a boundary component of a subsurface of \(\Sigma\) which is diffeomorphic to an orientable surface, \(N_{h,1}\) for some \(h_i \leq h\) or \(N_{h,2}\) for some \(h_i < h\), by the inductive assumption, there exist lifts \(h_{1}, \ldots, h_{6} \in \mathcal{M}(N_{g,n})\) of \(t_{c_1}, \ldots, t_{c_6} \in G\) with respect to \(\psi\), respectively. Thus \(h_1^{t_1} h_2^{t_2} \cdots h_6^{t_6} \in \mathcal{M}(N_{g,n})\) is a lift of \(t_c\) with respect to \(\psi\).

Suppose \(m \geq 2\). Then there exist simple closed curves \(c_1, c_2\) on \(\Sigma\) such that \(c_1 \cup c_2\) separates \(\Sigma\) into \(\Sigma', \Sigma''\) and \(\Sigma'''\) which are diffeomorphic to \(\Sigma_{0,4}, \Sigma_{0,m}\) and \(\Sigma_{h,1}\), respectively, and \(c \subset \Sigma'\). By using a lantern relation on \(\Sigma'\), there exist simple closed curves \(c_3, \ldots, c_6\) on \(\Sigma'\) such that \(t_c = t_{c_1}^{t_1} t_{c_2}^{t_2} \cdots t_{c_6}^{t_6} \in G\) for some \(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_6 \in \{-1, 1\}\). Since each \(c_i\) \((i = 1, \ldots, 6)\) is a boundary component of a subsurface of \(\Sigma\) which is diffeomorphic to an orientable surface or \(N_{h,m+1}\) for some \(m_i < m\), by the inductive assumption, there exist lifts \(h_{1}, \ldots, h_{6} \in \mathcal{M}(N_{g,n})\) of \(t_{c_1}, \ldots, t_{c_6} \in G\) with respect to \(\psi\), respectively. Thus \(h_1^{t_1} h_2^{t_2} \cdots h_6^{t_6} \in \mathcal{M}(N_{g,n})\) is a lift of \(t_c\) with respect to \(\psi\).

We have completed the proof of Theorem 4.4.

### 5. Proof of Proposition 4.2 and preliminaries for the proof

In this section, we give a proof of Proposition 4.2 which is used in the proof of Theorem 4.4. The proof is given in Section 5.1. For giving the proof, we prepare Section 5.1, 5.2 and 5.3.

#### 5.1. Group presentations and short exact sequence

Let \(G\) be a group and let \(H = \langle X \mid R \rangle\), \(Q = \langle Y \mid S \rangle\) be presented groups which have the exact sequence

\[
1 \rightarrow H \xrightarrow{\iota} G \xrightarrow{\nu} Q \rightarrow 1.
\]

We take a lift \(\tilde{y} \in G\) of \(y \in Q\) with respect to \(\nu\) for each \(y \in Q\). Then we put \(\tilde{X} := \{\iota(x) \mid x \in X\} \subset G\) and \(\tilde{Y} := \{\tilde{y} \mid y \in Y\} \subset G\). Denote by \(\tilde{r}\) the word in \(\tilde{X}\) which obtained from \(r \in R\) by replacing each \(x \in X\) by \(\iota(x)\) and denote by \(\tilde{s}\) the
word in $\tilde{Y}$ which obtained from $s \in S$ by replacing each $y \in Y$ by $\tilde{y}$. We note that $\tilde{r} = 1$ in $G$. For each $s \in S$, since $\tilde{s} \in G$ is an element of $\operatorname{ker} \nu$, there exists a word $v_s$ in $\tilde{X}$ such that $\tilde{s} = v_s$ in $G$. Since $i(H)$ is a normal subgroup of $G$, for each $x \in X$ and $y \in Y$, $\tilde{y} \nu(x) \tilde{y}^{-1}$ is an element of $i(H)$. Hence there exists a word $w_{x,y}$ in $\tilde{X}$ such that $\tilde{y} \nu(x) \tilde{y}^{-1} = w_{x,y}$ in $G$. The next lemma follows from an argument of the combinatorial group theory (for instance, see [8, Proposition 10.2.1, p139]).

**Lemma 5.1.** In this situation above, the group $G$ has the following presentation: generators: $\{i(x), \tilde{y} \mid x \in X, y \in Y\}$.

relations:

(A) $\tilde{r} = 1$ for $r \in R$,
(B) $\tilde{s} = v_s$ for $s \in S$,
(C) $\tilde{y} \nu(x) \tilde{y}^{-1} = w_{x,y}$ for $x \in X$, $y \in Y$.

5.2. Extended lantern relations. Let $S$ be a connected compact surface and let $D$ be a disk on $\operatorname{int} S$ with the center point $x_0$. Then we have the point pushing map (defined in Section 2.2) $\iota_{x_0} : \pi_1(S, x_0) \to \mathcal{M}(S, x_0)$. For a two-sided simple loop $\gamma$ on $S$ based at $x_0$, we take an orientation $\theta_\gamma \in \{+\gamma, -\gamma\}$ of $\mathcal{N}_S(\gamma)$. Denote by $c_1$ (resp. $c_2$) the boundary component of $\mathcal{N}_S(\gamma)$ on the right (resp. left) side of $\gamma$, and by $\theta_i \in \{+c_i, -c_i\}$ ($i = 1, 2$) the orientation of $\mathcal{N}_S(c_i)$ which is induced by $\theta_\gamma$. We regard $\gamma$ as an element of $\pi_1(S, x_0)$. Then we have a well-known relation

$$J_{\iota_{x_0}}(\gamma) = t_{c_1, \theta_1} t_{c_2, \theta_2}^{-1}.$$ 

Let $\mathcal{L}^+ = \mathcal{L}^+(S, x_0)$ be the subset of $\pi_1(S, x_0)$ which consists of elements represented by two-sided simple loops. Then we define a map

$$\Delta = \Delta_{\iota_{x_0}} : \mathcal{L}^+ \to \mathcal{M}(S - \operatorname{int} D)$$

as follows. For any two-sided simple loop $\gamma$ on $S$ based at $x_0$, we take $\mathcal{N}_S(\gamma)$ whose interior contains $D$. Then we take $c_1$, $c_2$, $\theta_1$ and $\theta_2$ as above. Define the inclusion $\iota : S - \operatorname{int} D \to S$ and $\tilde{c}_i := \nu^{-1}(c_i)$ for $i = 1, 2$. Then we define

$$\Delta(\gamma) := t_{\tilde{c}_1, \theta_{\tilde{c}_1}} t_{\tilde{c}_2, \theta_{\tilde{c}_2}}^{-1} \in \mathcal{M}(S - \operatorname{int} D),$$

where $\theta_{c_i} := (\iota(S - \operatorname{int} D))^{-1}(\theta_{c_i}) \in \{+c_i, -c_i\}$ for $i = 1, 2$.

The next two lemmas are obtained from an argument in Section 3 of [8].

**Lemma 5.2.** Let $\alpha$ and $\beta$ be two-sided simple loops on $S$ based at $x_0$ such that $\alpha$ tangentially intersects with $\beta$ at $x_0$ only and the composition $\alpha \beta \in \pi_1(S, x_0)$ is represented by a simple loop. We take the orientation of $\mathcal{N}_S(\alpha \cup \beta)$ which is induced by the orientation of $\mathcal{N}_S - \mathcal{N}_D(\partial D)$. Then we have

$$\Delta(\alpha) \Delta(\beta) = \Delta(\alpha \beta(t^{-1}_{\partial D}),$$

where $\varepsilon = 1$ if $\alpha$ and $\beta$ are counterclockwise around $x_0$ as on the left-hand side of Figure 4 and $\varepsilon = -1$ if $\alpha$ and $\beta$ are clockwise around $x_0$ as on the right-hand side of Figure 4.

The relations in Lemma 5.2 are original lantern relations. We call the relations in Lemma 5.2 Relations (L+) when $\varepsilon = 1$ and Relations (L-) when $\varepsilon = -1$ (see Figure 3).

**Lemma 5.3.** Let $\alpha$ and $\beta$ be two-sided simple loops on $S$ based at $x_0$ such that $\alpha$ transversely intersects with $\beta$ at $x_0$ only. Then we have

$$\Delta(\alpha) \Delta(\beta) = \Delta(\alpha \beta).$$
We call the relations in Lemma 5.3 Relations (L0). We have the following lemma.

**Lemma 5.4.** Relations (L0) are obtained from the braid relations (i).

**Proof.** Let \( \alpha \) and \( \beta \) be two-sided simple loops on \( S \) based at \( x_0 \) such that \( \alpha \) transversely intersects with \( \beta \) at \( x_0 \) only. We take a representative of \( \alpha \beta \in \pi_1(S, x_0) \) by a simple loop \( \gamma \) and also take the orientations of \( N_S(\alpha \cup \beta) \subset S \) and \( N_S(\alpha \cup \beta) \cap \text{int} D \subset S - \text{int} D \) which is induced by the orientation of \( N_S \cap \partial D \). Define boundary components \( a_1 \cup a_2 = \partial N_S(\alpha) \), \( b_1 \cup b_2 = \partial N_S(\beta) \) and \( c_1 \cup c_2 = \partial N_S(\gamma) \) such that \( a_1, b_1 \) and \( c_1 \) are on the right-hand side of \( \alpha, \beta \) and \( \gamma \), respectively. We consider the case where the algebraic intersection number, with respect to the orientation of \( N_S(\alpha \cup \beta) \), of \( \alpha \) and \( \beta \) is 1 and orientations of \( N_S(\alpha \cup \beta) \cap \partial D \), \( N_S \cap \partial D(\tilde{a}_i) \) and \( N_S \cap \partial D(\tilde{c}_i) \) are compatible with the orientation of \( N_S(\alpha \cup \beta) \). Figure 10 expresses this situation. Then we have \( \Delta(\alpha) = t_{\tilde{a}_1} t_{\tilde{a}_2}^{-1}, \Delta(\beta) = t_{\tilde{b}_1} t_{\tilde{b}_2}^{-1} \) and \( \Delta(\gamma) = t_{\tilde{c}_1} t_{\tilde{c}_2}^{-1} \). For the other cases, we can prove this lemma by an argument similar to the following argument.

Since \( t_{\tilde{a}_i}^{-1} (\tilde{b}_i) = \tilde{c}_i \) for \( i = 1, 2 \), we have

\[
\Delta(\alpha \beta) = t_{\tilde{c}_1} t_{\tilde{c}_2}^{-1}
\]

\[
\overset{(i)}{=} t_{\tilde{a}_1}^{-1} (t_{\tilde{b}_1} t_{\tilde{b}_2}^{-1}) t_{\tilde{a}_2}
\]

\[
= t_{\tilde{a}_2}^{-1} t_{\tilde{b}_1} t_{\tilde{b}_2}^{-1} t_{\tilde{a}_1} \cdot (t_{\tilde{b}_1} t_{\tilde{b}_2}^{-1})^{-1} t_{\tilde{b}_1} t_{\tilde{b}_2}^{-1}
\]

\[
= t_{\tilde{a}_2}^{-1} (t_{\tilde{b}_1} t_{\tilde{b}_2}^{-1}) t_{\tilde{a}_2} (t_{\tilde{b}_1} t_{\tilde{b}_2}^{-1})^{-1} \cdot t_{\tilde{b}_1} t_{\tilde{b}_2}^{-1}
\]

\[
\overset{(i)}{=} t_{\tilde{a}_2}^{-1} t_{\tilde{a}_1} \cdot t_{\tilde{b}_1} t_{\tilde{b}_2}^{-1}
\]

\[
= \Delta(\alpha) \Delta(\beta)
\]

We have the lemma. \(\square\)

We remark that Relations (L+) and (L0) are obtained from Relations (I) and (III).

### 5.3 Generators for the subgroup of the fundamental group generated by two-sided loops

Recall that we take a model of \( N_{g,n} \) as in Figure 6 for \( n \geq 1 \). Assume \( n \geq 2 \). We regard \( N_{g,n-1} \) as the surface obtained by regluing \( D_{g+n-1} \) and \( N_{g,n} \). Put the center point \( x_0 \) of \( D_{g+n-1} \). Let \( \pi_1(N_{g,n-1}, x_0) \) be the subgroup of \( \pi_1(N_{g,n-1}, x_0) \) which consists of elements that is represented by loops such that the pushing maps along their loops preserve a local orientation of \( x_0 \). Let \( x_1, \ldots, x_g, y_1, \ldots, y_{n-2} \) be loops on \( N_{g,n-1} \) based at \( x_0 \) as in Figure 11 and we regard \( N_{g,n-1} \) as the surface in Figure 11 for some conveniences. Note that
Put formations (for instance see \[9, Proposition 4.4.5, p46\]) and relations (by the same as that of \(w\) are elements of \(\pi_1(N,g,n)\)). \(x_i\) and \(x_i^{-1}\) for \(1 \leq i \leq n - 2\) are elements of \(\pi_1(N,g,n)\). \(x_i\) and \(x_i^{-1}\) are represented by loops as in Figure 12. Since \(\pi_1(N,g,n)\) is the free group which is freely generated by \(x_1, \ldots, x_g, y_1, \ldots, y_{n-2}\), \(\pi_1(N,g,n)\) is also isomorphic to a free group. We have the following lemma.

**Lemma 5.5.** For \(g \geq 1 \) and \(n \geq 2\), \(\pi_1(N,g,n)\) is the free group which is freely generated by \(x_1^2, x_2, x_2x_1, \ldots, x_gx_{g-1}, y_1, \ldots, y_{n-2}, x_1^{-1}y_1^{-1}, \ldots, x_1^{-1}y_{n-2}^{-1}\).

**Proof.** We use the Reidemeister-Schreier method (for instance see \[9\]) for \(\pi_1(N,g,n)\) to obtain the generators for \(\pi_1(N,g,n)\). Since \(\pi_1(N,g,n)\) is an index 2 subgroup of \(\pi_1(N,g,n, x_0)\) and the non-trivial element of the quotient group \(\pi_1(N,g,n, x_0)/\pi_1(N,g,n)\) is represented by \(x_1\), the set \(U := \{1, x_1\} \subset \pi_1(N,g,n, x_0)/\pi_1(N,g,n)\) is a Schreier transversal for \(\pi_1(N,g,n)\) in \(\pi_1(N,g,n, x_0)\). Set \(X := \{x_1, \ldots, x_g, y_1, \ldots, y_{n-2}\}\). For any word \(w\) in \(X\), denote by \(\overline{w}\) the element of \(U\) whose equivalence class in \(\pi_1(N,g,n, x_0)/\pi_1(N,g,n)\) is the same as that of \(w\). Then \(\pi_1(N,g,n)\) is the free group which is freely generated by

\[
B = \{xu^{-1}xu \mid x \in X, u \in U, xu \notin U\}
\]

\[
= \{x_1, x_1^{-1}x_j, y_k, x_1^{-1}yx_k \mid i = 1, \ldots, g, j = 2, \ldots, g, k = 1, \ldots, n - 2\}.
\]

Put \(z_i := x_i^2, z_i := (x_i)_{g,x_i^{-1}}(x_i^{-1})_{g,x_i} \) for \(i = 2, \ldots, g\), \(w_i := (x_{i-1}w_{i+1})(x_i^{-1}w_i)\) for \(i = 2, \ldots, g - 1\) as words in \(B\). By using the Tietze transformations (for instance see \[9\] Proposition 4.4.5, p46) and relations \((x_i^{-1}x_i) = (x_{i+1}x_i^{-1}z_i) \) and \((x_i^{-1}x_i) = w_i(x_i^{-1}x_i)^{-1}\) for \(i \geq 2\), we have isomorphisms

\[
\langle B \rangle 
\]

\[
\cong \langle B \cup \{z_i, w_j \mid i = 1, \ldots, g, j = 1, \ldots, g - 1\} \rangle
\]

\[
\cong \langle B \cup \{z_i, w_j \mid i = 1, \ldots, g, j = 1, \ldots, g - 1\} \rangle
\]

\[
\cong \langle \{z_i, w_j, y_k, x_1^{-1}y_kx_1 \mid i = 1, \ldots, g, j = 1, \ldots, g - 1, k = 1, \ldots, n - 2\} \rangle.
\]
Note that $z_i = x_i^2$ and $w_i = x_{i+1}x_i$ as elements of $\pi_1(N_{g,n})^+$. We get this lemma. □

5.4. Proof of Proposition 3.2 Let $\mathcal{M}^+(N_{g,n-1},x_0)$ be the subgroup of $\mathcal{M}(N_{g,n-1},x_0)$ whose elements preserve a local orientation of $x_0$. For $n \geq 2$, the forgetful homomorphism $\mathcal{M}(N_{g,n-1},x_0) \to \mathcal{M}(N_{g,n-1})$ induces the following exact sequence

$$(5.1) 1 \to \pi_1(N_{g,n-1})^+ \xrightarrow{\ j_{x_0} \ } \pi_1(N_{g,n-1}) \to \mathcal{M}^+(N_{g,n-1},x_0) \to \mathcal{M}(N_{g,n-1}) \to 1.$$  

Since $\pi_1(N_{g,n-1},x_0)$ is isomorphic to a free group, the center of $\pi_1(N_{g,n-1},x_0)$ is trivial. Thus the homomorphism $j_{x_0}$ is injective by [1, Corollary 1.2].

The natural inclusion $\iota: N_{g,n} \to N_{g,n-1}$ induces the surjective homomorphism $\iota_*: \mathcal{M}(N_{g,n}) \to \mathcal{M}^+(N_{g,n-1},x_0)$. By Theorem 3.6 in [18], we have the exact sequence

$$(5.2) 1 \to \mathbb{Z}[d_{n-1}] \to \mathcal{M}(N_{g,n}) \xrightarrow{\iota_*} \mathcal{M}^+(N_{g,n-1},x_0) \to 1.$$  

We remark that for a relation $v_{i_1}^{e_{i_1}} \cdots v_{i_k}^{e_{i_k}} = w_1^{b_1} \cdots w_l^{b_l}$ of a group $G$, we call $v_{i_1}^{e_{i_1}} \cdots v_{i_k}^{e_{i_k}}(w_1^{b_1} \cdots w_l^{b_l})^{-1}$ the relator of $G$ (obtained from the relation).

Proof of Proposition 3.2. We proceed by induction on $n \geq 1$. The finite presentation for $\mathcal{M}(N_{g,1})$ is given by Theorem 3.1.

Assume $n \geq 2$. For some conveniences, we define $\iota_*(a_i) = a_i \in \mathcal{M}^+(N_{g,n-1},x_0)$, $\iota_*(y) = y \in \mathcal{M}^+(N_{g,n-1},x_0)$, $\iota_*(b) = b \in \mathcal{M}^+(N_{g,n-1},x_0)$, etc., and we take lifts $a_i$, $y$, $b$, ... $\in \mathcal{M}^+(N_{g,n-1},x_0)$ of $a_i$, $y$, $b$, ... $\in \mathcal{M}(N_{g,n-1})$, respectively. By Lemma 5.5 the inductive assumption and applying Lemma 5.1 to the exact
sequence we obtain the following presentation for $\mathcal{M}^+(N_{g,n-1}, x_0)$: 

1. $a_i (1 \leq i \leq g-1), y, b, d_i (1 \leq i \leq n-2), a_{i,j} (1 \leq i \leq g-1, 1 \leq j \leq n-2), \ r_{i,j} (1 \leq i \leq g, 1 \leq j \leq n-2), \ s_{i,j} (1 \leq i \leq n-2), \ \bar{s}_{i,j}$

2. $j_{x_0}(x_i^2) = r_{i;n-1} (1 \leq i \leq g), \ j_{x_0}(x_i x_{i+1}) = a_{i;n-1} a_i^{-1} (1 \leq i \leq g-1), \ j_{x_0}(y_i) = s_{i;n-1} d_i^{-1} (1 \leq i \leq n-2)$ and $j_{x_0}(x_i^{-1} y_i x_1) = \bar{s}_{i;n-1} d_i^{-1} (1 \leq i \leq n-2)$.

Denote by $X_1$ the set of generators in (1) and by $X_2$ the set of generators in (2).

relations:

1. For any relator $s$ of the presentation for $\mathcal{M}(N_{g,n-1})$ and the lift $\bar{s}$ of $s$ to $\mathcal{M}^+(N_{g,n-1}, x_0)$, 
   \[
   \bar{s} = v_s
   \]
   for some product $v_s$ of elements in $X_2$.

2. For $x \in X_2$ and $f \in X_1$, 
   \[
   fx f^{-1} = w_{x,f}
   \]
   for some product $w_{x,f}$ of elements in $X_2$.

Note that the generators in (1) come from the generators for $\mathcal{M}(N_{g,n-1})$ and the generators in (2) come from the generators for $\pi_1(N_{g,n-1})^{\ast}$ in Lemma 5.4.

We calculate each $v_s$ and $w_{x,f}$ in the relation (1) and (2) above. We take the subsurface $N'$ of $N_{g,n}$ which is diffeomorphic to $N_{g,n-1}$ as in Figure 13. Since every simple closed curve used in a generator of the presentation for $\mathcal{M}(N_{g,n-1})$ is isotopic to a simple closed curve on int$N'$, we regard generators of the presentation for $\mathcal{M}(N_{g,n-1})$ as elements of $\mathcal{M}(N')$. In particular, the inclusion $\iota' : N' \hookrightarrow N_{g,n}$ induces the injective homomorphism $\iota'_r : \mathcal{M}(N') \rightarrow \mathcal{M}(N_{g,n})$. By using the composition $\iota' \circ \iota'_s : \mathcal{M}(N') \rightarrow \mathcal{M}^+(N_{g,n-1}, x_0)$, we can take $v_s = 1$ for each relator $s$ of the presentation for $\mathcal{M}(N_{g,n-1})$.

Recall that $\bar{s}_{j,k;i}$ is the Dehn twist along the simple closed curve $\iota(\bar{\sigma}_{j,k;i})$ and $\bar{\sigma}_{j,k;i}$ is defined in Figure 12. For the relation (2) above, we can take $w_{x,f}$ as follows. Assume $1 \leq i, m \leq g, 1 \leq j \leq n-2, 1 \leq l < t < n-2$ and $[x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1}$.

(D1a)' $a_m(a_{i;n-1} a_i^{-1}) a_m^{-1} = \begin{cases} (a_{i;n-1} a_i^{-1})(a_{i;1;n-1} a_i^{-1}) & \text{for } m = i - 1, \\ (a_{i;1;n-1} a_i^{-1})(a_{i;n-1} a_i^{-1}) & \text{for } m = i + 1, \\ a_{i;n-1} a_i^{-1} & \text{for } m \neq i - 1, i + 1, \end{cases}$

(D1b)' $y(a_{i;n-1}^{-1} a_i^{-1}) y^{-1} = \begin{cases} (a_{i;1;n-1}^{-1})(a_{i;n-1}^{-1}) r_{2;n-1} r_{1;n-1} & \text{for } i = 1, \\ (a_{i;2;n-1}^{-1})(a_{i;n-1}^{-1}) r_{1;n-1} & \text{for } i = 2, \\ a_{i;n-1} a_i^{-1} & \text{for } i \geq 3, \end{cases}$

(D1c)' $b(a_{i;n-1} a_i^{-1}) b^{-1} = \begin{cases} (a_{3;n-1} a_i^{-1})(a_{1;1;n-1} a_i^{-1})^{-1} (a_{1;n-1} a_i^{-1})(a_{3;n-1} a_i^{-1})(a_{1;1;n-1} a_i^{-1}) & \text{for } i = 1, \\ (a_{3;n-1} a_i^{-1})(a_{1;1;n-1} a_i^{-1})^{-1} (a_{2;n-1} a_i^{-1})(a_{3;n-1} a_i^{-1})(a_{1;1;n-1} a_i^{-1}) & \text{for } i = 2, \\ (a_{i;n-1} a_i^{-1})^{-1} (a_{3;n-1} a_i^{-1})(a_{1;1;n-1} a_i^{-1}) & \text{for } i = 3, \\ (a_{i;4;n-1} a_i^{-1})(a_{3;n-1} a_i^{-1})(a_{1;1;n-1} a_i^{-1}) & \text{for } i = 4, \\ a_{i;n-1} a_i^{-1} & \text{for } i \geq 5, \end{cases}$
(D1d') \( a_{m+1}(a_{i:n-1}^{-1})a_{m;l}^{-1} = \)
\[
\begin{cases}
[(s_{l,n-1}d_{l}^{-1})^{-1}, (a_{m:n-1}a_{m}^{-1})^{-1}(a_{i:n-1}a_{i}^{-1})] & \text{for } m \leq i - 2, \\
[(s_{l,n-1}d_{l}^{-1})^{-1}, (a_{m:n-1}a_{m}^{-1})^{-1}] & \text{for } m = i - 1, \\
\{(s_{l,n-1}d_{l}^{-1})(a_{i:n-1}a_{i}^{-1})\}^{-1}(a_{i:n-1}a_{i}^{-1}) & \text{for } m = i, \\
(a_{i+1:n-1}a_{i+1}^{-1})^{-1}(s_{l,n-1}d_{l}^{-1})^{-1}(a_{i:n-1}a_{i}^{-1}) & \text{for } m = i + 1, \\
(a_{i:n-1}a_{i}^{-1})^{-1} & \text{for } m \geq i + 2,
\end{cases}
\]

(D1e') \( r_{m+1}(a_{i:n-1}a_{i}^{-1})r_{m;l}^{-1} = \)
\[
\begin{cases}
[(s_{l,n-1}d_{l}^{-1})^{-1}, r_{m:n-1}^{-1}(a_{i:n-1}a_{i}^{-1})] & \text{for } m \leq i - 1, \\
r_{i:n-1}^{-1}(s_{l,n-1}d_{l}^{-1})^{-1}r_{i:n-1}^{-1}(a_{i:n-1}a_{i}^{-1}) & \text{for } m = i, \\
r_{i+1:n-1}^{-1}(s_{l,n-1}d_{l}^{-1})^{-1}r_{i+1:n-1}^{-1}(s_{l,n-1}a_{i}^{-1}) & \text{for } m = i + 1, \\
(a_{i:n-1}a_{i}^{-1})^{-1} & \text{for } m \geq i + 2,
\end{cases}
\]

(D1f') \( s_{i,l}(a_{i:n-1}a_{i}^{-1})s_{i;l}^{-1} = a_{i:n-1}a_{i}^{-1}, \)

(D1g') \( s_{i,l}(a_{i:n-1}a_{i}^{-1})s_{i;l}^{-1} = \)
\[
\begin{cases}
[(s_{l,k}d_{l}^{-1})^{-1}, s_{l,k}d_{l}^{-1}]^{-1}(s_{l,k}d_{l}^{-1})^{-1}(a_{l:k}a_{l}^{-1})r_{l:k}(s_{l,k}d_{l}^{-1})^{-1} & \text{for } i = 1, \\
(s_{l,k}^{-1})^{-1}r_{l:k}^{-1}[(s_{l,k}d_{l}^{-1})^{-1}, (s_{l,k}d_{l}^{-1})^{-1}r_{l:k}]^{-1} & \text{for } i = 1, \\
[(s_{l,n-1}d_{l}^{-1})^{-1}, (s_{l,n-1}d_{l}^{-1})^{-1}]^{-1}(s_{l,n-1}d_{l}^{-1}, r_{l:n-1}(s_{l,n-1}d_{l}^{-1})^{-1}r_{l:n-1}^{-1})^{-1} & \text{for } i = 2, \\
(a_{i:n-1}a_{i}^{-1})^{-1} & \text{for } i = 2, \\
(s_{l,n-1}d_{l}^{-1})^{-1}[s_{l,n-1}d_{l}^{-1}, r_{l:n-1}(s_{l,n-1}d_{l}^{-1})^{-1}r_{l:n-1}^{-1}]^{-1} & \text{for } i = 2,
\end{cases}
\]

(D1h') \( d_{l}(a_{i:n-1}a_{i}^{-1})d_{l}^{-1} = a_{i:n-1}a_{i}^{-1}, \)

(D2a') \( a_{m}r_{i:n-1}a_{m}^{-1} = \)
\[
\begin{cases}
r_{i:n-1}^{-1}r_{i:n-1}^{-1}(a_{i:n-1}a_{i}^{-1})^{-1}r_{i:n-1}^{-1}(a_{i:n-1}a_{i}^{-1})^{-1} & \text{for } m = i - 1, \\
(a_{i:n-1}a_{i}^{-1})^{-1}r_{n+1:n-1}^{-1}(a_{i:n-1}a_{i}^{-1})^{-1} & \text{for } m = i, \\
r_{i:n-1}^{-1} & \text{for } m \neq i - 1, i,
\end{cases}
\]

(D2b') \( g_{r_{i:n-1}}y^{-1} = \)
\[
\begin{cases}
\{(a_{i:n-1}a_{i}^{-1})^{-1}r_{2:n-1}r_{1:n-1}^{-1}\}^{-1}r_{1:n-1}^{-1} & \text{for } i = 1, \\
\{(a_{i:n-1}a_{i}^{-1})^{-1}r_{2:n-1}r_{1:n-1}^{-1}\}^{-1}r_{2:n-1}r_{1:n-1}^{-1} & \text{for } i = 2, \\
r_{i:n-1}^{-1} & \text{for } i \geq 3,
\end{cases}
\]
\[ (D2c') \quad b_{r_{i-1}}^{-1} = \begin{cases} (a_{1,n-1}a_1^{-1})^{-1}(a_{5,n-1}a_3^{-1})^{-1}(a_{2,n-1}a_2^{-1})^{-1}r_{n-1}^{-1}(a_3,n-1a_3^{-1})r_{n-1}^{-1}(a_2,n-1a_2^{-1})r_{n-1}^{-1}(a_{1,n-1}a_1^{-1}) & \text{for } i = 1, \\ \{ (a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})^{-1}r_{n-1}^{-1}(a_{3,n-1}a_3^{-1})r_{n-1}^{-1}(a_2,n-1a_2^{-1})^{-1}r_{n-1}^{-1} \\ (a_{3,n-1}a_3^{-1})^{-1}r_{n-1}(a_2,n-1a_2^{-1})^{-1}(a_{3,n-1}a_3^{-1})^{-1}r_{n-1}^{-1}(a_{1,n-1}a_1^{-1}) & \text{for } i = 2, \\ \{ (a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})^{-1}r_{n-1}^{-1} \\ (a_{3,n-1}a_3^{-1})r_{n-1}^{-1}(a_2,n-1a_2^{-1})^{-1}r_{n-1}^{-1} \\ (a_{2,n-1}a_2^{-1})^{-1}r_{n-1}(a_{3,n-1}a_3^{-1})^{-1}r_{n-1}^{-1} \\ \{ (a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})^{-1}r_{n-1}^{-1}(a_{3,n-1}a_3^{-1})^{-1}r_{n-1}^{-1} & \text{for } i = 3, \\ r_{n-1}^{-1}(a_{2,n-1}a_2^{-1})^{-1}r_{n-1}^{-1}(a_{1,n-1}a_1^{-1})^{-1}r_{n-1}^{-1}(a_{2,n-1}a_2^{-1})^{-1}r_{n-1}^{-1}(a_{1,n-1}a_1^{-1}) & \text{for } i = 4, \\ r_{n-1}^{-1} & \text{for } i \geq 5, \end{cases} \]

\[ (D2d') \quad a_{m,l}r_{i-1}^{-1}a_{m,l}^{-1} = \begin{cases} ([s_{l,n-1}d_1^{-1}]^{-1}, (a_{m,n-1}a_m^{-1})^{-1})^{-1}r_{i-1}^{-1}([s_{l,n-1}d_1^{-1}]^{-1}, (a_{m,n-1}a_m^{-1})^{-1}) & \text{for } m \leq i - 2, \\ \{ ([s_{l,n-1}d_1^{-1}]^{-1}, (a_{m,n-1}a_m^{-1})^{-1})^{-1}r_{i-1}^{-1}([s_{l,n-1}d_1^{-1}]^{-1}, (a_{m,n-1}a_m^{-1})^{-1}) \} \text{for } m = i - 1, \\ (a_{1,n-1}a_1^{-1})^{-1}r_{i-1}^{-1}(a_{1,n-1}a_1^{-1})^{-1}r_{i-1}^{-1}([s_{l,n-1}d_1^{-1}]^{-1}, (a_{m,n-1}a_m^{-1})^{-1}) & \text{for } m = i, \\ r_{i-1}^{-1} & \text{for } m \geq i + 1, \end{cases} \]

\[ (D2e') \quad r_{m,l}r_{i-1}^{-1}r_{m,l}^{-1} = \begin{cases} ([s_{l,n-1}d_1^{-1}]^{-1}, r_{m,n-1}^{-1})^{-1}r_{i-1}^{-1}([s_{l,n-1}d_1^{-1}]^{-1}, r_{m,n-1}^{-1}) & \text{for } m \leq i - 1, \\ \{ ([s_{l,n-1}d_1^{-1}]^{-1}, r_{m,n-1}^{-1})^{-1}r_{i-1}^{-1}([s_{l,n-1}d_1^{-1}]^{-1}, r_{m,n-1}^{-1}) \} \text{for } m = i, \\ r_{i-1}^{-1} & \text{for } m \geq i + 1, \end{cases} \]

\[ (D2f') \quad s_{l,n}r_{i-1}^{-1}s_{l,n}^{-1} = r_{i-1,n}^{-1}, \]

\[ (D2g') \quad \tilde{s}_{l,n}r_{i-1}^{-1}\tilde{s}_{l,n}^{-1} = \begin{cases} ([s_{l,n-1}d_1^{-1}]^{-1}, (\tilde{s}_{l,n-1}d_1^{-1})^{-1})^{-1}[r_{i-1}^{-1}(s_{l,n-1}d_1^{-1})r_{i-1}^{-1}] = r_{i-1,n}^{-1} & \text{for } i = 1, \\ \{ ([s_{l,n-1}d_1^{-1}]^{-1}, (\tilde{s}_{l,n-1}d_1^{-1})^{-1})^{-1}[s_{l,n-1}d_1^{-1}, r_{i-1}^{-1}(\tilde{s}_{l,n-1}d_1^{-1})^{-1}r_{i-1,n}^{-1}]^{-1} \} \text{for } i \geq 2, \end{cases} \]

\[ (D2h) \quad d_{i-1}r_{i-1}^{-1}d_{i-1}^{-1} = r_{i-1,n}^{-1}, \]

\[ (D3a') \quad a_{m}(s_{j,n}d_j^{-1})a_{m}^{-1} = s_{j,n}d_j^{-1}, \]

\[ (D3b') \quad y(s_{j,n}d_j^{-1})y^{-1} = s_{j,n}d_j^{-1}, \]

\[ (D3c') \quad b(s_{j,n}d_j^{-1})b^{-1} = s_{j,n}d_j^{-1}, \]

\[ (D3d') \quad a_{m,l}(s_{j,n}d_j^{-1})a_{m,l}^{-1} = \begin{cases} ([s_{l,n-1}d_1^{-1}]^{-1}, (a_{m,n-1}a_m^{-1})^{-1})^{-1}(s_{l,n-1}d_1^{-1}) & \text{for } l \geq j, \\ (a_{m,n-1}a_m^{-1})^{-1}(s_{l,n-1}d_1^{-1})(a_{m,n-1}a_m^{-1}) & \text{for } l = j, \\ s_{j,n}d_j^{-1} & \text{for } l = j, \end{cases} \]
(D3e) \[ r_{m;t}(s_{j,n-1}d_{j}^{-1})r_{m;l}^{-1} = \begin{cases} \frac{1}{(s_{l,n-1}d_{l}^{-1})^{-1}}(s_{j,n-1}d_{j}^{-1})^{-1} \frac{1}{(s_{l,n-1}d_{l}^{-1})^{-1}}(s_{j,n-1}d_{j}^{-1})^{-1} & \text{for } l > j, \\ r_{m,n}^{-1}(s_{j,n-1}d_{j}^{-1})r_{m,n}^{-1} & \text{for } l = j, \\ s_{j,n-1}d_{j}^{-1} & \text{for } l < j, \end{cases} \]

(D3f) \[ s_{l,t}(s_{j,n-1}d_{j}^{-1})s_{l,t}^{-1} = \begin{cases} (s_{t,n-1}d_{t}^{-1})(s_{j,n-1}d_{j}^{-1})^{-1}(s_{j,n-1}d_{j}^{-1}) & \text{for } l = j, \\ (s_{t,n-1}d_{t}^{-1})^{-1},(s_{t,n-1}d_{t}^{-1})^{-1}(s_{j,n-1}d_{j}^{-1}) & \text{for } l < j < t, \\ (s_{t,n-1}d_{t}^{-1})^{-1}(s_{j,n-1}d_{j}^{-1})(s_{t,n-1}d_{t}^{-1})^{-1} & \text{for } t = j, \\ s_{j,n-1}d_{j}^{-1} & \text{for the other cases,} \end{cases} \]

(D3g) \[ s_{l,t}(s_{j,n-1}d_{j}^{-1})s_{l,t}^{-1} = \begin{cases} (s_{t,n-1}d_{t}^{-1})(s_{j,n-1}d_{j}^{-1})^{-1}(s_{j,n-1}d_{j}^{-1}) & \text{for } l > j, \\ (s_{t,n-1}d_{t}^{-1})(s_{j,n-1}d_{j}^{-1})^{-1}(s_{j,n-1}d_{j}^{-1}) & \text{for } l = j, \\ (s_{j,n-1}d_{j}^{-1})(s_{t,n-1}d_{t}^{-1})^{-1}(s_{j,n-1}d_{j}^{-1}) & \text{for } l < j < t, \\ (s_{j,n-1}d_{j}^{-1})(s_{t,n-1}d_{t}^{-1})^{-1}(s_{j,n-1}d_{j}^{-1}) & \text{for } t = j, \\ s_{j,n-1}d_{j}^{-1} & \text{for } t < j, \end{cases} \]

(D3h) \[ d_{l}(s_{j,n-1}d_{j}^{-1})d_{l}^{-1} = s_{j,n-1}d_{j}^{-1}, \]

(D4a) \[ a_{m}(s_{j,n-1}d_{j}^{-1})a_{m}^{-1} = \begin{cases} r_{1,n}^{-1}r_{2,n}^{-1}(a_{1,n-1}a_{1}^{-1})^{-1} & \text{for } m = 1, \\ s_{j,n-1}d_{j}^{-1} & \text{for } m \geq 2, \end{cases} \]

(D4b) \[ y(s_{j,n-1}d_{j}^{-1})y^{-1} = \begin{cases} r_{1,n}^{-1}(a_{1,n-1}a_{1}^{-1})^{-1}r_{2,n}^{-1}r_{1,n}^{-1}(s_{j,n-1}d_{j}^{-1}) & \text{for } m = 1, \\ 2r_{2,n}^{-1}r_{1,n}^{-1}(s_{j,n-1}d_{j}^{-1}) & \text{for } m \geq 2, \end{cases} \]

(D4c) \[ b(s_{j,n-1}d_{j}^{-1})b^{-1} = \begin{cases} r_{1,n}^{-1}(a_{2,n-1}a_{2}^{-1})^{-1}r_{1,n}^{-1}(a_{3,n-1}a_{3}^{-1}) & \text{for } m = 1, \\ (a_{2,n-1}a_{2}^{-1})^{-1}r_{2,n}^{-1}(a_{1,n-1}a_{1}^{-1})^{-1}s_{j,n-1}d_{j}^{-1} & \text{for } m \geq 2, \end{cases} \]
\[(D4d) \quad a_{m,l}(\bar{s}_{j,n-1}d_j^{-1})a_{m,l}^{-1} = \]
\[\begin{cases} 
\{{r}_{1;1}^{-1}l_1(\bar{s}_{1,n-1}a_1^{-1}\{r}_{1;1}^{-1}l_1^{-1}\bar{s}_{j,n-1}d_j^{-1}\}^{-1} & \text{for } m + 1, l < j, \\
\{l_1^{-1}\bar{s}_{1,n-1}a_1^{-1}\{r}_{1;1}^{-1}l_1^{-1}\bar{s}_{j,n-1}d_j^{-1}\}^{-1} & \text{for } m + 1, l > j, \\
\{\bar{s}_{1,n-1}\{r}_{1;1}^{-1}l_1^{-1}\bar{s}_{1,n-1}d_j^{-1}\}^{-1} & \text{for } m + 1, l = j. 
\end{cases}\]

\[(D4e) \quad m,l(\bar{s}_{j,n-1}d_j^{-1})m,l^{-1} = \]
\[\begin{cases} 
\{{r}_{1;1}^{-1}l_1(\bar{s}_{1,n-1}^{-1}d_j^{-1})^{-1}\{r}_{1;1}^{-1}l_1^{-1}\bar{s}_{j,n-1}d_j^{-1}\}^{-1} & \text{for } m + 1, l < j, \\
\{l_1^{-1}\bar{s}_{1,n-1}^{-1}d_j^{-1}\{r}_{1;1}^{-1}l_1^{-1}\bar{s}_{j,n-1}d_j^{-1}\}^{-1} & \text{for } m + 1, l > j, \\
\{\bar{s}_{1,n-1}\{r}_{1;1}^{-1}l_1^{-1}\bar{s}_{1,n-1}d_j^{-1}\}^{-1} & \text{for } m + 1, l = j. 
\end{cases}\]

\[(D4f) \quad s_{t,t}(\bar{s}_{j,n-1}d_j^{-1})s_{t,t}^{-1} = \]
\[\begin{cases} 
\{\bar{s}_{1,n-1}^{-1}\{r}_{1;1}^{-1}\bar{s}_{1,n-1}d_j^{-1}\}\{\bar{s}_{1,n-1}^{-1}\bar{s}_{j,n-1}d_j^{-1}\}^{-1} & \text{for } t = j, \\
\{\bar{s}_{1,n-1}^{-1}\{r}_{1;1}^{-1}\bar{s}_{1,n-1}d_j^{-1}\}\{\bar{s}_{1,n-1}^{-1}\bar{s}_{j,n-1}d_j^{-1}\}^{-1} & \text{for } l < j < t, \\
\{\bar{s}_{1,n-1}^{-1}\{r}_{1;1}^{-1}\bar{s}_{1,n-1}d_j^{-1}\}\{\bar{s}_{1,n-1}^{-1}\bar{s}_{j,n-1}d_j^{-1}\}^{-1} & \text{for } l = j, \\
\{\bar{s}_{1,n-1}^{-1}\bar{s}_{j,n-1}d_j^{-1}\}^{-1} & \text{for the other cases.}
\end{cases}\]
(D4g) \[ \hat{s}_{l,t}(\hat{s}_{j,n-1}d^{-1}_j)\hat{s}^{-1}_{l,t} = \begin{cases} \left( [\hat{s}_{l,n-1}^{-1}r_{1,n-1}^{-1}\hat{s}_{l,n-1}d^{-1}_j]^{-1}r_{1,n-1}^{-1}\hat{s}_{j,n-1}d^{-1}_j \right)^{-1} & \text{for } t < j, \\ \left( [\hat{s}_{l,n-1}^{-1}r_{1,n-1}^{-1}\hat{s}_{l,n-1}d^{-1}_j]^{-1}r_{1,n-1}^{-1}\hat{s}_{j,n-1}d^{-1}_j \right)^{-1} & \text{for } t = j. \end{cases} \]

Since \( j_x(f(x)) = \hat{f} j_x(f)^{-1} \) for \( x \in \pi_1(N_{g,n-1}, x_0) \) and \( f \in \mathcal{M}(N_{g,n-1}) \), it is useful for obtaining \( \pi_1(N_{g,n-1}) \) in Lemma 5.5 such that \( \hat{\pi}_1 \) is useful for obtaining \( \pi_1 \) for \( \hat{s}_{l,t} \) and \( \hat{s}_{j,n-1}^{-1} \) for \( s_{l,t} \). For example, in Relation (D1e) for \( m = i \), \( r_{i,l}(x_{i+1}x_i) \) is represented by the loop as on the right-hand side of Figure 14. Thus we have

\[ r_{i,l}(x_{i+1}x_i) = \left\{ x_i^{-2}y_i^{-1}x_i^{-2}y_i^{-1}\hat{x}_{i,l}^{-1}\left\{ x_i^{-2}y_i^{-1}x_i^{-2} \right\}^{-1} \right\} \in \pi_1(N_{g,n-1}) \]

and also have

\[ r_{i,l}(a_{i,n-1}^{-1})r_{i,l}^{-1} = \left\{ r_{i,n-1}^{-1}\left( s_{i,n-1}^{-1}d^{-1}_i \right) r_{i,n-1}^{-1}\left( s_{i,n-1}^{-1}d^{-1}_i \right) a_{i,n-1}^{-1} \right\} \left( s_{i,n-1}^{-1}d^{-1}_i \right) \left( s_{i,n-1}^{-1}d^{-1}_i \right) r_{i,n-1}^{-1} \left( s_{i,n-1}^{-1}d^{-1}_i \right) r_{i,n-1}^{-1}. \]

Therefore \( \mathcal{M}^+(N_{g,n-1}, x_0) \) has the presentation which is obtained from the finite presentation for \( \mathcal{M}(N_{g,n-1}) \) by adding generators \( a_{i,n-1} \) for \( 1 \leq i \leq g-1 \), \( r_{i,n-1}^{-1} \) for \( 1 \leq i \leq g \), \( s_{i,n-1} \) and \( \hat{s}_{i,n-1} \) for \( 1 \leq i \leq n-2 \), and Relations (D1a)-\( (D4h) \).

Next, we apply Lemma 5.1 to the exact sequence \( 0 \to X'_2 \to X'_1 \to \mathcal{M}^+(N_{g,n-1}, x_0) \) by adding generators \( a_{i,n-1}, r_{i,n-1}, s_{i,n-1} \) and \( \hat{s}_{i,n-1} \) to the presentation for \( \mathcal{M}^+(N_{g,n-1}, x_0) \). Note that \( d_{n-1} \) commutes with every element of \( \mathcal{M}(N_{g,n}) \). \( \mathcal{M}(N_{g,n}) \) admits the presentation which has the generating set \( X_1 \cup X'_2 \) and relations as follows:

(1) For each relator \( r \) of the finite presentation for \( \mathcal{M}^+(N_{g,n-1}, x_0) \) and the lift \( \hat{r} \) of \( r \) with respect to \( \iota_* \), there exists \( \varepsilon_r \in \mathbb{Z} \) such that \( \hat{r} \equiv \varepsilon_r d_{n-1}^{-1} \).

(2) For each generator \( x \) of the finite presentation for \( \mathcal{M}^+(N_{g,n-1}, x_0) \) and the lift \( \hat{x} \) of \( x \) with respect to \( \iota_* \),

\[ [d_{n-1}, \hat{x}] = 1. \]

Thus we obtain Relations (D0) from the relations in (2) above. Note that Relations (D0) are obtained from Relation (I).

We compute \( \varepsilon_r \) for each \( r \). We have the homomorphism \( \iota' : \mathcal{M}(N') \hookrightarrow \mathcal{M}(N_{g,n}) \) induced by the inclusion \( \iota' : N' \hookrightarrow N_{g,n} \). Thus for each relator \( r \) of the finite presentation for \( \mathcal{M}^+(N_{g,n-1}, x_0) \) which obtained from a relator of the finite presentation for \( \mathcal{M}(N_{g,n-1}) \), we have \( \varepsilon_r = 0 \). Hence we obtain Relations \( (A1)-(B8) \) and \( (D0)-(D4g) \) for \( 1 \leq k \leq n-2 \) in \( \mathcal{M}(N_{g,n}) \). By the inductive argument on \( n \) and the argument below, we can show that Relations \( (A1)-(B8) \) and \( (D0)-(D4g) \) for \( 1 \leq k \leq n-2 \) are obtained from Relations (I) and (III).
We remark that the lift of Relations (D1h)', (D2h)', (D3h)' and (D4h)' are obtained from Relations (D0). Hence Relations (D1h), (D2h), (D3h) and (D4h) are obtained from Relation (I).

Relations (D1a)' for \(m \neq i - 1, i + 1\), (D1b)' for \(i \geq 3\), (D1c)' for \(i \neq 4\), (D1d)' for \(m \leq i - 2, m = i\) and \(m \geq i + 2\), (D1e)' for \(m \leq i - 1\) and \(m \geq i - 2\), (D1f)', (D1g)' for \(i \geq 2\), (D1h)', (D2a)' for \(m \neq i - 1\), (D2b)' for \(i \neq 2\), (D2c)' for \(i \geq 5\), (D2d)' for \(m \leq i - 2\) and \(m \geq i + 1\), (D2e)', (D2f)', (D2g)' for \(i \geq 2\) and (D2h)' and (D3a)’-(D4h)' are obtained from the braid relations. Hence for their relators \(r\)'s, we have \(\varepsilon_r = 0\). Thus we obtain Relations (D1a) for \(m \neq i - 1, i + 1\), (D1b) for \(i \geq 3\), (D1c) for \(i \neq 4\), (D1d) for \(m \leq i - 2, m = i\) and \(m \geq i + 2\), (D1e) for \(m \leq i - 1\) and \(m \geq i - 2\), (D1f), (D1g) for \(i \geq 2\), (D2a) for \(m \neq i - 1\), (D2b) for \(i \neq 2\), (D2c) for \(i \geq 5\), (D2d) for \(m \leq i - 2\) and \(m \geq i + 1\), (D2e), (D2f), (D2g) for \(i \geq 2\) and (D3a)-(D4g) when \(k = n - 1\) by Relation (I).

For the other relators, i.e. Relations (D1a)' for \(m = i - 1, i + 1\), (D1b)' for \(i = 1, 2\), (D1c)' for \(i = 4\), (D1d)' for \(m = i - 1, i + 1\), (D1e)' for \(m = i, i + 1\), (D1f)' for \(i = 1\), (D2a)' for \(m = i - 1\), (D2b)' for \(i = 2\), (D2c)' for \(i = 1, 2, 3, 4\), (D2d)' for \(m = i - 1, i\) and (D2g)' for \(i = 1\) when \(k = n - 1\), we compute their \(\varepsilon_r\) in Section 5.5.

By the computations in Section 5.5, we can show that the remaining relations of presentation in Proposition 3.2 are obtained from Relations (I) and (III) and we have completed the proof of Proposition 3.2 except for computations of their \(\varepsilon_r\)'s.

\[\square\]

**Figure 13.** The subsurface \(N'\) of \(N_{g,n}\) which is diffeomorphic to \(N_{g,n-1}\).

**Figure 14.** Loop \(r_{i,l}(x_{i+1}x_i)\) on \(N_{g,n-1}\) based at \(x_0\) for \(1 \leq i \leq n - 2\).
5.5. Computing $\varepsilon_r$.

In this section, we compute $\varepsilon_r$ for Relations (D1a)' for $m = i - 1, i + 1$, (D1b)' for $i = 1, 2$, (D1c)' for $i = 4$, (D1d)' for $m = i - 1, i + 1$, (D1e)' for $m = i, i + 1$, (D1g)' for $i = 1$, (D2a)' for $m = i - 1$, (D2b)' for $i = 2$, (D2c)' for $i = 1, 2, 3, 4$, (D2d)' for $m = i - 1, i$ and (D2g)' for $i = 1$ when $k = n - 1$. We define

$$\bar{y}_i := x_{1}^{-1}y_i x_1$$

for $i = 1, \ldots, n - 2$,

$$\bar{y}_{i,j} := (x_2 x_1)^{-1}x_2^{-1} \cdots (x_i x_{i-1})^{-1}x_i^{-1}\bar{y}_i (x_2 x_1)^{-1}x_2^{-1} \cdots (x_i x_{i-1})^{-1}x_i^{-1}$$

for $1 \leq i \leq n - 2$, $2 \leq j \leq g$.

Remark that $\bar{y}_{i,j} = x_j^{-1}y_i x_j$. By using Relations (L+), (L-) and (L0) (see Lemma 5.2 and 5.3), when $k = n - 1$, we can compute $\varepsilon_r$ as follows: $\varepsilon_r = 2$ for Relation (D1e)' when $m = i + 1$, $\varepsilon_r = 1$ for Relation (D1d)' when $m = i + 1$, (D1h)' when $m = i - 1$ and (D2b)' when $i = 2$, $\varepsilon_r = -1$ for Relation (D1b)' when $i = 2$, (D1c)' when $i = 4$ and (D1d)' $\varepsilon_r = 2$ for Relation (D1b)' when $i = 1$, (D1e)' when $m = i$ and $\varepsilon_r = 0$ for the other relations. In particular, Relations (D1a) when $m = i - 1, i + 1$ and (D1b) when $i = 2$ are obtained from a single braid relation and one of Relations (L+), (L-) and (L0), and for Relations (D1b) when $i = 1$, (D1c) when $i = 4$, (D1d) when $m = i + 1$, (D2a) when $m = i - 1$, (D2b) when $i = 2$, (D2d) when $m = i$, we can compute $\varepsilon_r$ easily. As examples, we compute $\varepsilon_r$ for Relation (D1e)' when $m = i$ and Relation (D2c)' when $i = 2$ by using figures. For the other cases, we give computations of $\varepsilon_r$ by only deformations of the expressions.

For Relation (D1e)' when $m = i$ and $k = n - 1$, we have the following relation in $\mathcal{M}(N_{g,n})$ by Figure 15,

$$\Delta(y_i)\Delta((x_{i+1}x_i))\Delta(\bar{y}_{i;1})^{-1} \overset{(L+)}{=} \Delta((y_i(x_{i+1}x_i))\Delta(\bar{y}_{i;1})^{-1})d_{n-1}^{-1}$$

$$\Delta(y_i(x_{i+1}x_i))\bar{y}_{i;1}^{-1}d_{n-1}^{-1} \overset{(L+)}{=} \Delta(y_i(x_{i+1}x_i))\bar{y}_{i;1}^{-1}d_{n-1}^{-1}.$$

Note that $\Delta(r_{i;1}(x_i x_{i+1})) = r_{i;1}(a_{i,n-1}^{-1}a_i^{-1})r_{i;1}^{-1}$ by the braid relation. Hence we have

$$\{r_{i;1}^{-1}(s_{i,n-1}^{-1}d_i^{-1})^{-1}r_{i;1}^{-1}\} (s_{i,n-1}^{-1}d_i^{-1})(a_{i,n-1}^{-1}a_i^{-1})(s_{i,n-1}^{-1}d_i^{-1})^{-1}$$

$$\{r_{i;1}^{-1}(s_{i,n-1}^{-1}d_i^{-1})^{-1}r_{i;1}^{-1}\}^{-1}$$

$$= \{r_{i;1}^{-1}(s_{i,n-1}^{-1}d_i^{-1})^{-1}r_{i;1}^{-1}\} \Delta(y_i)\Delta((x_{i+1}x_i))\Delta(\bar{y}_{i;1})^{-1}$$

$$\{r_{i;1}^{-1}(s_{i,n-1}^{-1}d_i^{-1})^{-1}r_{i;1}^{-1}\}^{-1}$$

$$= \{r_{i;1}^{-1}(s_{i,n-1}^{-1}d_i^{-1})^{-1}r_{i;1}^{-1}\} \Delta(y_i(x_{i+1}x_i))\bar{y}_{i;1}^{-1}d_{n-1}^{-1}$$

$$\{r_{i;1}^{-1}(s_{i,n-1}^{-1}d_i^{-1})^{-1}r_{i;1}^{-1}\}^{-1}$$

$$= \Delta(\{r_{i;1}^{-1}(s_{i,n-1}^{-1}d_i^{-1})^{-1}r_{i;1}^{-1}\} (y_i(x_{i+1}x_i))\bar{y}_{i;1}^{-1})d_{n-1}^{-1}$$

$$= \Delta(\{r_{i;1}(x_i x_{i+1}))\bar{y}_{i;1}^{-1}d_{n-1}^{-1}$$

$$\overset{(I)}{=} r_{i;1}(a_{i,n-1}^{-1}a_i^{-1})r_{i;1}^{-1}d_{n-1}^{-1}.$$
in a braid relation. By Figure 16, we have $b$ by a loop as on the lower right side of Figure 21 and $\Delta(y_i) = \Delta((y_i(x_{i+1}x_i))d_{n-1}$ is obtained from Relation (L+). By Figure 17 and Figure 18, we have the following relation in $\Delta((y_i(x_{i+1}x_i))d_{n-1}$ is obtained from Relation (L+). “$\simeq$” means a deformation of the loop by a homotopy fixing $x_0$.

For Relation (D2c) when $i = 2$ and $k = n - 1$, we note that $b(x_2^2)$ is represented by a loop as on the lower right side of Figure 21 and $\Delta(b(x_2^2)) = b r_{2,n-1}^1 b^{-1}$ by the braid relation. By Figure 16, we have

\[
\begin{align*}
(a_{1:n-1} a_1^{-1})(a_{3:n-1} a_3^{-1}) r_{2,n-1} & (a_{1:n-1} a_1^{-1})^{-1} r_{2,n-1} = (a_{1:n-1} a_1^{-1}) \Delta(x_4 x_3) \Delta(x_2^2) \Delta(x_2^2) (a_{1:n-1} a_1^{-1})^{-1} d_{n-1} \\
\overset{(L+)}{=} (a_{1:n-1} a_1^{-1}) \Delta(x_4 x_3) \Delta(x_2^2) (a_{1:n-1} a_1^{-1})^{-1} d_{n-1} \\
\overset{(L+)}{=} (a_{1:n-1} a_1^{-1}) \Delta((x_4 x_3)x_2^2 x_1^2) (a_{1:n-1} a_1^{-1})^{-1} d_{n-1} \\
\overset{(1)}{=} \Delta((a_{1:n-1} a_1^{-1})(x_4 x_3)x_2^2 x_1^2))d_{n-1}^2
\end{align*}
\]

in $\mathcal{M}(N_{g,n})$. By Figure 17 and Figure 18, we have the following relation in $\mathcal{M}(N_{g,n})$.

\[
\begin{align*}
(a_{2,n-1} a_2^{-1})^{-1} r_{3,n-1} & (a_{3:n-1} a_3^{-1})^{-1} r_{4,n-1} (a_{2,n-1} a_2^{-1}) \\
= (a_{2,n-1} a_2^{-1})^{-1} r_{3,n-1} r_{4,n-1} & (a_{3:n-1} a_3^{-1})^{-1} r_{4,n-1} (a_{2,n-1} a_2^{-1}) \\
= (a_{2,n-1} a_2^{-1})^{-1} \Delta(x_3^2) & \Delta(x_4^2) r_{4,n-1} (a_{3:n-1} a_3^{-1})^{-1} r_{4,n-1} (a_{2,n-1} a_2^{-1}) \\
\overset{(L+), (2)}{=} (a_{2,n-1} a_2^{-1})^{-1} \Delta(x_3^2) & \Delta(r_{4,n-1} (a_{3:n-1} a_3^{-1})^{-1} r_{4,n-1} (a_{2,n-1} a_2^{-1}) d_{n-1} \\
\overset{(L+)}{=} (a_{2,n-1} a_2^{-1})^{-1} \Delta(x_3^2 x_4) & (a_{3:n-1} a_3^{-1})^{-1} r_{4,n-1} (a_{2,n-1} a_2^{-1}) d_{n-1}
\end{align*}
\]
Let $\zeta_1$ and $\zeta_2$ be simple closed curves on $N_{g,n}$ as in Figure 19. Since $\iota_*(t_{\zeta_1}) \in \mathcal{M}^+(N_{g,n-1}, x_0)$ fixes $x_2$ and $\Delta((a_{1:n-1}a_{1}^{-1})((x_4x_3)x_2^2x_4^2)) = t_{\zeta_1}t_{\zeta_2}^{-1}$, we have $\Delta((a_{1:n-1}a_{1}^{-1})((x_4x_3)x_2^2x_4^2))(a_2^2) = t_{\zeta_2}^{-1}(x_2^2)$. We remark that the loop as on the upper right side of Figure 20 is homotopic to the loop as on the lower right side of Figure 20 by a homotopy fixing $x_0$ as in Figure 20. By the relations above and Figure 21 we have

\[
\Delta((a_{2:n-1}a_{2}^{-1}r_{2:n-1})^{-1}(x_2^2x_4^2r_{4:n-1}^{-1}((x_4x_3)^{-1}))).
\]
Thus \( \varepsilon_r = 0 \) for Relation (D2c) when \( i = 2 \) and \( k = n - 1 \), and we obtain Relation (D2c) for \( i = 2 \) and \( k = n - 1 \) by Relations (I) and (III).

**Figure 16.** Relations \( \Delta(x^2_2)\Delta(x^2_1) = \Delta(x^2_2 x^2_1) d_{n-1} \) and \( \Delta(x_4 x_3)\Delta(x^2_2 x^2_1) = \Delta((x_4 x_3)x^2_2 x^2_1) d_{n-1} \) and loop \((a_{1;n-1}a^{-1}_1))((x_4 x_3)x^2_2 x^2_1)\).

**Figure 17.** Relation \( \Delta(x^3_2)\Delta(x^4_2) = \Delta(x^3_2 x^4_2) d_{n-1} \).

For Relation (D1d) when \( m = i - 1 \) and \( k = n - 1 \), we have

\[
\begin{align*}
&\left[ (a_{i-1;n-1}^{-1} a_{i-1}^{-1}) \right]^{-1} (s_{i,n-1} a_{i}^{-1}) (s_{i,n-1} a_{i}^{-1}) (s_{i,n-1} a_{i}^{-1}) (a_{i-1,n} a_{i}^{-1}) \\
&= \left\{ (s_{i,n-1} a_{i}^{-1}) (a_{i-1,n} a_{i}^{-1}) \right\}^{-1} \Delta(x_i x_{i-1}) \Delta(y_i) \Delta(x_{i+1} x_i) \left\{ (s_{i,n-1} a_{i}^{-1}) \right\} \\
&= \left\{ (s_{i,n-1} a_{i}^{-1}) (a_{i-1,n} a_{i}^{-1}) \right\}^{-1} \Delta(x_i x_{i-1}) \Delta(y_i) \Delta(x_{i+1} x_i) \left\{ (s_{i,n-1} a_{i}^{-1}) \right\} d_n \\
&= \left\{ (s_{i,n-1} a_{i}^{-1}) (a_{i-1,n} a_{i}^{-1}) \right\}^{-1} \Delta(x_i x_{i-1}) \Delta(y_i) \Delta(x_{i+1} x_i) \left\{ (s_{i,n-1} a_{i}^{-1}) \right\} d_n 
\end{align*}
\]
Figure 18. Relation \( \Delta(x_3^2x_4^2)\Delta(r_{4n-1}^{-1}(x_4x_3)^{-1})) = \Delta(x_3^2x_4^2r_{4n-1}^{-1}(x_4x_3)^{-1})d_{n-1}^{-1} \) and loops \( r_{4n-1}^{-1}(x_4x_3)^{-1} \) and (\( a_{2n-1}a_2^{-1})^{-1}(x_3^2x_4^2r_{4n-1}^{-1}(x_4x_3)^{-1}) \)).

Figure 19. Simple closed curves \( \zeta_1 \) and \( \zeta_2 \) on \( N_{g,n} \).

\[
\begin{align*}
&\equiv \{(s_{i,n-1}d_t^{-1})(a_{i-1:n-1}a_{i-1})\}^{-1}\Delta((x_i,x_{i-1})y_t(x_{i+1}x_i))\{(s_{i,n-1}d_t^{-1})
\end{align*}
\]

\[
\begin{align*}
&= \Delta((s_{i,n-1}d_t^{-1})(a_{i-1:n-1}d_{i-1}))^{-1}(x_i,x_{i-1})y_t(x_{i+1}x_i)d_{n-1}
\end{align*}
\]

\[
\begin{align*}
&= \Delta(a_{i-1:d(x_{i+1}x_i)}d_{n-1}-1)
\end{align*}
\]

\[
\begin{align*}
&= a_{i-1:d(a_{i-1:n-1}a_{i-1}^{-1})a_{i-1:d(n-1)}^{-1}}.
\end{align*}
\]
Thus \( \varepsilon_r = -1 \) for Relation (D1d) when \( m = i - 1 \) and \( k = n - 1 \), and we obtain Relation (D1d) when \( m = i - 1 \) and \( k = n - 1 \) by Relations (I) and (III).

For Relation (D1e) when \( m = i + 1 \) and \( k = n - 1 \), we have

\[
\begin{align*}
    r_{i+1; n-1}^{-1} s_{t,n-1}^{-1} &- r_{i+1; n-1}^{-1} (\bar{s}_{t,n-1}^{-1} + 1) d_{t}^{-1} (a_{i-1})^{-1} \\
    = & \quad r_{i+1; n-1}^{-1} \Delta(y_l) r_{i+1; n-1}^{-1} \Delta(\bar{y}_r^{-1}) \Delta(x_l^{-1}) \Delta(x_l^{-1}) \\
    & \quad \text{(I)} \\
    & \quad \Delta(r_{i+1; n-1}^{-1} (y_l)) \Delta(\bar{y}_r^{-1} (x_l^{-1} x_l)) d_{n-1}^{-1} \\
    & \quad \text{(II)} \\
    & \quad \Delta(r_{i+1; n-1}^{-1} (y_l)) \Delta(\bar{y}_r^{-1} (x_l^{-1} x_l)) d_{n-1}^{-1} \\
    & \quad r_{i+1; n-1}^{-1} (a_{i-1}^{-1} d_{l})^{-1} r_{i+1; n-1}^{-1} d_{n-1}^{-1}.
\end{align*}
\]

Thus \( \varepsilon_r = -2 \) for Relation (D1e) when \( m = i + 1 \) and \( k = n - 1 \), and we obtain Relation (D1e) when \( m = i + 1 \) and \( k = n - 1 \) by Relations (I) and (III).

For Relation (D1g) when \( i = 1 \) and \( k = n - 1 \), we have

\[
\begin{align*}
    [(s_{t,k}^{-1} d_{t}^{-1})^{-1}, s_{t,k} d_{t}^{-1}]^{-1} &- r_{1; k}^{-1} (s_{t,k} d_{t}^{-1})^{-1} (a_{1; k} a_{i-1}^{-1}) r_{1; k}^{-1} (s_{t,k} d_{t}^{-1})^{-1} \\
    &\quad r_{1; k}^{-1} [(s_{t,k} d_{t}^{-1})^{-1}, (s_{t,k} d_{t}^{-1})] \\
    = &\quad [(s_{t,k}^{-1} d_{t}^{-1})^{-1}, s_{t,k} d_{t}^{-1}]^{-1} \Delta(y_l) \Delta(x_l^{-2} x_l) r_{1; k}^{-1} \Delta(\bar{y}_r^{-1}) \Delta(y_l) r_{1; k}^{-1} \Delta(y_l) \\
    &\quad r_{1; k}^{-1} \Delta(\bar{y}_r^{-1}) r_{1; k}^{-1} [(s_{t,k}^{-1} d_{t}^{-1})^{-1}, (s_{t,k} d_{t}^{-1})] \\
    &\quad \text{r_{1; k}^{-1} \Delta(\bar{y}_r^{-1}) r_{1; k}^{-1} [(s_{t,k}^{-1} d_{t}^{-1})^{-1}, (s_{t,k} d_{t}^{-1})]}.
\end{align*}
\]
Figure 21. Relations $\Delta((a_{1:n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)) \Delta((a_{2:n-1}a_1^{-1})^{-1}$

\begin{align*}
&\Delta((a_{1:n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)) = \\
&(a_{2:n-1}a_1^{-1})^{-1}(x_3^2x_4^{-1}r_{4:n-1}((x_4x_3)^{-1}))d_{n-1}^{-1}
\end{align*}

and

\begin{align*}
&\Delta(t_{3}^{-1}(x_2^2))\Delta((a_{1:n-1}a_1^{-1})((x_4x_3)x_2^2x_1^2)) = \\
&(a_{2:n-1}a_1^{-1})^{-1}(x_3^2x_4^{-1}r_{4:n-1}((x_4x_3)^{-1}))d_{n-1}^{-1} \text{ and loop } b(x_2^2).
\end{align*}
Thus \( \varepsilon_r = 0 \) for Relation (D1g)\( i \) when \( i = 1 \) and \( k = n - 1 \), and we obtain Relation (D1g) when \( i = 1 \) and \( k = n - 1 \) by Relations (I) and (III).

For Relation (D2c) when \( i = 1 \) and \( k = n - 1 \), we have

\[
\begin{align*}
\text{(I), (L)} = & (a_{1,n-1}a_1^{-1})^{-1}(a_{3,n-1}a_3^{-1})^{-1}(a_{2,n-1}a_2^{-1})^{-1}r_{4,n-1}^{-1}(a_{3,n-1}a_3^{-1}) \\
& r_{3,n-1}^{-1}(a_{2,n-1}a_2^{-1})r_{2,n-1}^{-1}(a_{1,n-1}a_1^{-1}) \\
& = (a_{1,n-1}a_1^{-1})^{-1}(a_{3,n-1}a_3^{-1})^{-1}\Delta(x_3x_2)^{-1}\Delta(x_2)^{-1}(a_{3,n-1}a_3^{-1}) \\
r_{3,n-1}^{-1}\Delta(x_3x_2)r_{3,n-1}^{-1}\Delta(x_2)^{-1}(a_{1,n-1}a_1^{-1}) \\

\text{\( (L) \) = } (a_{1,n-1}a_1^{-1})^{-1}(a_{3,n-1}a_3^{-1})^{-1}\Delta((x_3x_2)^{-1}x_2^{-2})(a_{3,n-1}a_3^{-1}) \\
\Delta(r_{3,n-1}^{-1}(x_3x_2))\Delta(x_3^{-2}x_2^{-2})(a_{1,n-1}a_1^{-1})d_{n-1}^{-1} \\

\text{\( (L+) \) = } (a_{1,n-1}a_1^{-1})^{-1}(a_{3,n-1}a_3^{-1})^{-1}((x_3x_2)^{-1}x_2^{-2})(a_{3,n-1}a_3^{-1})^{-1} \\
\Delta((a_{3,n-1}a_3^{-1})^{-1}((x_3x_2)^{-1}x_2^{-2})(r_{3,n-1}^{-1}(x_3x_2)x_3^{-2}x_2^{-2}) \\
(a_{1,n-1}a_1^{-1}) \\

\text{\( (I) \) = } \Delta((a_{3,n-1}a_3^{-1})^{-1}((x_3x_2)^{-1}x_2^{-2})(r_{3,n-1}^{-1}(x_3x_2)x_3^{-2}x_2^{-2})) \\
\Delta(b(x_2^4)) \\
\text{\( (I) \) \( b(r_{1,n-1})b^{-1}. \)}
\end{align*}
\]

Thus \( \varepsilon_r = 0 \) for Relation (D2c)\( i \) when \( i = 1 \) and \( k = n - 1 \), and we obtain Relation (D2c) when \( i = 1 \) and \( k = n - 1 \) by Relations (I) and (III).

For Relation (D2c) when \( i = 3 \) and \( k = n - 1 \), we have

\[
\begin{align*}
\text{(I), (L)} = & \{(a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})\}^{-1}r_{4,n-1}^{-1}(a_{3,n-1}a_3^{-1})r_{3,n-1}^{-1}(a_{2,n-1}a_2^{-1}) \\
r_{3,n-1}^{-1}(a_{1,n-1}a_1^{-1})r_{1,n-1}^{-1}(a_{2,n-1}a_2^{-1})^{-1}r_{3,n-1}^{-1}(a_{3,n-1}a_3^{-1})^{-1} \\
(a_{1,n-1}a_1^{-1})^{-1}\{(a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})\} \\
= \{(a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})\}^{-1}r_{4,n-1}^{-1}(a_{3,n-1}a_3^{-1})r_{3,n-1}^{-1}(a_{2,n-1}a_2^{-1}) \\
r_{3,n-1}^{-1}\Delta(x_2x_1)r_{2,n-1}^{-1}\Delta(x_2)^{-1}\Delta(x_2)^{-1}(a_{2,n-1}a_2^{-1})^{-1}r_{3,n-1}^{-1}(a_{3,n-1}a_3^{-1})^{-1} \\
\Delta(x_2x_1)^{-1}\{(a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})\} \\
\text{\( (I) \) \( \Delta(x_2x_1)^{-1}\{(a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})\} \\
D(x_2x_1)^{-1}\{(a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})\}d^{-1}_{n-1} \\
\text{\( (L+) \) = } \{(a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})\}^{-1}r_{4,n-1}^{-1}(a_{3,n-1}a_3^{-1})r_{3,n-1}^{-1}(a_{2,n-1}a_2^{-1}) \\
\Delta(r_{2,n-1}^{-1}(x_2x_1))\Delta(x_2^{-2}x_2^{-2})(a_{2,n-1}a_2^{-1})^{-1}r_{3,n-1}^{-1}(a_{3,n-1}a_3^{-1})^{-1} \\
\Delta(x_2x_1)^{-1}\{(a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})\} \\
\text{\( (I) \) \( \{(a_{3,n-1}a_3^{-1})(a_{1,n-1}a_1^{-1})\}^{-1} \\
r_{4,n-1}^{-1}\Delta((a_{3,n-1}a_3^{-1})r_{3,n-1}^{-1}(a_{2,n-1}a_2^{-1})\{(r_{2,n-1}^{-1}(x_2x_1)x_2^{-2}x_2^{-2})r_{4,n-1}^{-1} \\
\}
\end{align*}
\]
\[ \Delta(x_3^2)^{-1}\Delta(x_2x_1)^{-1}\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \]

\[ = \{a_{3;n-1}a_3^{-1}(a_{1;n-1}a_1^{-1})\}^{-1} \]

\[ \Delta((r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})(r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1}))\{r_{2;n-1}^{-1}(x_2x_1)x_2^{-2}x_1^{-2}\}) \]

\[ \Delta(x_4^{-2}(x_2x_1)^{-1})\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\} \]

\[ = \{a_{3;n-1}a_3^{-1}(a_{1;n-1}a_1^{-1})\}^{-1} \]

\[ \Delta((r_{4;n-1}^{-1}(a_{3;n-1}a_3^{-1})r_{3;n-1}^{-1}(a_{2;n-1}a_2^{-1}))\{r_{2;n-1}^{-1}(x_2x_1)x_2^{-2}x_1^{-2}\}x_4^{-2} \]

\[ (x_2x_1)^{-1}\{(a_{3;n-1}a_3^{-1})(a_{1;n-1}a_1^{-1})\}^{-1} \]

Thus \( \varepsilon_r = 0 \) for Relation (D2c) when \( i = 3 \) and \( k = n-1 \), and we obtain Relation (D2c) when \( i = 3 \) and \( k = n-1 \) by Relations (I) and (III).

For Relation (D2c) when \( i = 4 \) and \( k = n-1 \), we have

\[ r_{4;n-1}^{-1}(a_{2;n-1}a_2^{-1})r_{1;n-1}^{-1}(a_{1;n-1}a_1^{-1})^{-1}r_{2;n-1} \]

\[ = \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3)\Delta(x_2x_1) \]

\[ = \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3x_2x_1) \]

\[ d_n^{-1} \]

\[ \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3x_2x_1) \]

\[ d_n^{-1} \]

\[ \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3x_2x_1) \]

\[ d_n^{-1} \]

\[ \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3x_2x_1) \]

\[ d_n^{-1} \]

\[ \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3x_2x_1) \]

\[ d_n^{-1} \]

\[ \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3x_2x_1) \]

\[ d_n^{-1} \]

\[ \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3x_2x_1) \]

\[ d_n^{-1} \]

\[ \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3x_2x_1) \]

\[ d_n^{-1} \]

\[ \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3x_2x_1) \]

\[ d_n^{-1} \]

\[ \Delta(x_4^{-2})\Delta(x_3x_2)\Delta(x_4^{-2})(a_{1;n-1}a_1^{-1})^{-1}(x_2^{-2})\Delta(x_2x_1)^{-1} \]

\[ \Delta((a_{4;n-1}a_4^{-1})^{-1}(x_4^{-2}))\Delta((x_3x_2)^{-1}(x_4x_3)^{-1})r_{4;n-1}^{-1}\Delta(x_4x_3x_2x_1) \]

\[ d_n^{-1} \]
\[(L_c) \quad \Delta (x_4^2(x_3x_2)x_2^2(a_{1,n-1}a_{1}^{-1})^{-1}(x_3^2)(x_2x_1)^{-1}(a_{2,n-1}a_{2}^{-1})^{-1}(x_3^2))(x_3x_2)^{-1}
\]
\[= \Delta (b(x_4^2))\]
\[\begin{equation}
\begin{aligned}
(1) & \quad b(r_{i;n-1})b^{-1}.
\end{aligned}
\end{equation}
\]
Thus \(\varepsilon_r = 0\) for Relation (D2c)\(i\) when \(i = 4\) and \(k = n - 1\), and we obtain Relation (D2c) \(i\) when \(i = 4\) and \(k = n - 1\) by Relations (I) and (III).

For Relation (D2d) \(i\) when \(m = i - 1\) and \(k = n - 1\), we have
\[(L_c) \quad \Delta (x_4^2(x_3x_2)x_2^2(a_{1,n-1}a_{1}^{-1})^{-1}(x_3^2)(x_2x_1)^{-1}(a_{2,n-1}a_{2}^{-1})^{-1}(x_3^2))(x_3x_2)^{-1}
\]
\[= \Delta (b(x_4^2))\]
\[\begin{equation}
\begin{aligned}
(1) & \quad b(r_{i;n-1})b^{-1}.
\end{aligned}
\end{equation}
\]
Thus \(\varepsilon_r = 0\) for Relation (D2d)\(i\) when \(m = i - 1\) and \(k = n - 1\), and we obtain Relation (D2d) \(i\) when \(m = i - 1\) and \(k = n - 1\) by Relations (I) and (III).
Thus $\varepsilon_i = 0$ for Relation $(D2g)_i$ when $i = 1$ and $k = n - 1$, and we obtain Relation $(D2g)$ when $i = 1$ and $k = n - 1$ by Relations (I) and (III).

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