AN INVERSE PROBLEM FOR THE PSEUDO-PARABOLIC EQUATION WITH P-LAPLACIAN

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ABSTRACT. In this article, we study the inverse problem of determining the right side of the pseudo-parabolic equation with a p-Laplacian and nonlocal integral overdetermination condition. The existence of solutions in a local and global time to the inverse problem is proved by using the Galerkin method. Sufficient conditions for blow-up (explosion) of the local solutions in a finite time are derived. The asymptotic behavior of solutions to the inverse problem is studied for large values of time. Sufficient conditions are obtained for the solution to disappear (vanish to identical zero) in a finite time. The limits conditions that which ensure the appropriate behavior of solutions are considered.

1. Introduction. If the properties of the material and/or the external environment are unknown in advance, then along with the solution $u(x,t)$, one or more coefficients of the equation and/or the function of the external source $f(x,t)$ may be unknown. In mathematics, problems in the theory of partial differential equations in which both the solution of an equation and one or more of its coefficients, or (and) the right side, are unknown, are named as inverse problems. As a rule, in such problems, along with the boundary conditions characteristic of a particular direct problem (problems with known coefficients and a known right-hand side), additional information is given due to the presence of an additional unknown function (functions). In this paper, the unknown parameter is the coefficient of the right side, which depends on the time variable. Similar problems were studied in ([13],[14],[17],[23],[25]), but not for a nonlinear Sobolev type equation. In particular, the solvability of inverse problems with local and non-local redefinition conditions for Sobolev-type equations

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has been studied in many papers ([1], [2], [15], [16], [18]-[21], [24], [26]) and in a number of others.

In this paper, we consider the following pseudoparabolic equation

\[ u_t - \chi \Delta u_t - \text{div} \left( |\nabla u|^p \nabla u \right) = b(x,t)|u|^{\sigma - 2}u + f(t)h(x), \]

with the integral overdetermination condition which is considered as additional information. Inverse problems of determining the right side of the differential equation arise in mathematical modeling of some physical processes when, in addition to solving the equation, it is necessary to restore the action of external sources. To date, research on direct and inverse problems for pseudo-parabolic equations is rapidly developing in connection with the needs of modeling and controlling of processes in thermophysics, hydrodynamics and continuum mechanics. Pseudo-parabolic equations, such as those considered in this paper, arise when describing heat and mass transfer processes, the movement of non-Newtonian liquids, wave processes, and in many other areas. It should be noted that the monograph [3] considers a wide class of boundary value problems for nonlinear Sobolev type equations. The questions of local and global existence of solutions and their blow-up in a finite time were studied too. In the works [4]-[8], the initial-boundary value problems for nonlinear equations of the Kelvin-Voigt were studied. It describes the flows of incompressible, inhomogeneous, non-Newtonian liquids.

In [9], we investigated the inverse problem of determining the right side for a nonlinear parabolic equation with a non-standard growth condition with an integral overdetermination condition. The existence of local and global time solutions and their uniqueness were proved. Sufficient conditions for the blow-up (explosion) of a local solution in a finite time in a bounded domain are obtained and the asymptotic behavior of solutions to the inverse problem for large time values is studied. In the work [10] an inverse problem with an integral overdetermination condition for a parabolic type equation was considered. The existence of a local solution is proved and sufficient conditions for the blow-up of a local solution in a finite time with a bounded domain are obtained.

2. Problem statement. Let \( \Omega \) be a bounded area of space \( \mathbb{R}^N \), \( N \geq 1 \) with a sufficiently smooth boundary \( \Gamma \), \( Q_T = \Omega \times (0, T) \) of finite height \( T \), \( S = \Gamma \times (0, T) \). The inverse problem requires a pair of the functions \( u(x, t), f(t) \) which satisfy the following

\[ u_t - \chi \Delta u_t - \text{div} \left( |\nabla u|^p \nabla u \right) = b(x,t)|u|^{\sigma - 2}u + f(t)h(x), \]

(1)

\[ u(x, 0) = u_0(x), \]

(2)

\[ u|_S = 0, \]

(3)

\[ \int_{\Omega} u(x, t) \cdot h(x) dx = \varphi(t). \]

(4)

Here \( \Omega \subset \mathbb{R}^n \), \( n \geq 1 \) is a bounded area with a fairly smooth boundary \( \partial \Omega \), and \( b(x,t), h(x), u_0(x), \varphi(t) \) are given functions and \( \chi, p, \sigma \) are positive constants.

Remark 1. In the case of equation (1), the right-hand side of \( h(x) \) has a special form:

\[ h(x) = \omega(x) - \chi \Delta \omega(x), \]

(5)
where
\[ |b| \leq b_0, \quad |b_1| \leq b_0, \]
\[ \omega \in L_2(\Omega) \cap L_\sigma(\Omega) \cap W^1_2(\Omega) \cap W^2_2(\Omega), \quad \omega \in L_2(\Omega) \cap L_p(\Omega), \]
\[ 1 < p \leq 3, \quad \sigma > 2. \]

The functions \( u_0, \varphi \) satisfy the following conditions:
\[ \varphi(t) \in W^1_2(0, T), \quad |\varphi'(t)| \leq C, \quad C > 0, \]
\[ \int_\Omega u_0 \cdot hdx = \varphi(0), \quad u_0 \in W^1_2(\Omega) \cap L_p(\Omega). \]

2.1. Equivalent problem.

**Lemma 2.1.** Problem (1)-(4) is equivalent to the following problem for a nonlinear parabolic equation containing a nonlinear, nonlocal operator of the function \( u(x, t) \)
\[ u_t - \chi \Delta u_t - \text{div}\left( |\nabla u|^{\sigma-2} \nabla u \right) = b(x, t)|u|^{\sigma-2} u + F(t, u) h(x), \quad x \in \Omega, \quad t > 0, \quad (8) \]
\[ u(x, 0) = u_0(x), \quad x \in \Omega, \quad u|_S = 0. \quad (9) \]

Here
\[ F(t, u) = \varphi'(t) + \int_\Omega |\nabla u|^{\sigma-2} \nabla u \nabla \omega dx - \int_\Omega b(x, t)|u|^{\sigma-2} u \omega dx. \]

**Proof.** From equation (1) the following is obtained.
\[ \int_\Omega u_t \omega dx = \int_\Omega \Delta u_t \omega dx - \int_\Omega \text{div}\left( |\nabla u|^{\sigma-2} \nabla u \right) \omega dx = \int_\Omega b(x, t)|u|^{\sigma-2} u \omega dx + \int_\Omega f(t) h(x) \omega dx. \]

If conditions (4) and (5) are met, then
\[ F(t, u) = \varphi'(t) + \int_\Omega |\nabla u|^{\sigma-2} \nabla u \nabla \omega dx - \int_\Omega b(x, t)|u|^{\sigma-2} u \omega dx. \]

Therefore, the ratio (10) holds.

We will now consider the problem (8)-(9). If the ratio (10) holds, then the equality (12) follows from it. Then
\[ F(t, u) = \varphi'(t) + \int_\Omega |\nabla u|^{\sigma-2} \nabla u \nabla \omega dx - \int_\Omega b(x, t)|u|^{\sigma-2} u \omega dx = \varphi'(t) - \int_\Omega \text{div}\left( |\nabla u|^{\sigma-2} \nabla u \right) \omega dx - \int_\Omega b(x, t)|u|^{\sigma-2} u \omega dx. \]
As a result of (11) we obtain
\[ F(t, u) = \varphi'(t) + \int_0^1 |\nabla u|^2 |\nabla \omega dx| - \int_0^1 b(x, t)|u|^{\sigma-2}u dx = \]
\[ = \varphi'(t) - \int_0^1 u_0 |\nabla \omega dx| \cdot \chi \int_0^1 \Delta u |\nabla \omega dx| + \frac{1}{2} \int_0^1 b(x, t)|u|^{\sigma-2}u dx + \]
\[ + \int_0^1 f(t) h(x) |\nabla \omega dx| - \int_0^1 b(x, t)|u|^{\sigma-2}u dx, \]
\[ \varphi'(t) - \int_0^1 u_0 (\omega - \chi \Delta u) dx = 0. \]

Thus, in \( \frac{d}{dt} (\varphi(t) - \int_0^1 u_0 (\omega - \chi \Delta u) dx) = 0 \). Let us denote \( v(t) = \varphi(t) - \int_0^1 u_0 (\omega - \chi \Delta u) dx \). Then the function \( v(t) \) is a solution of the Cauchy problem: \( v'(t) = 0, v(0) = 0, (v(0) = 0 \) follows from the condition (7)). This problem has only trivial solution \( v(t) \equiv 0 \). Hence \( \int_0^1 u_0 (\omega - \chi \Delta u) dx = \varphi(t) \). The lemma is proved. \( \square \)

3. Definition and construction of the solution. Galerkin approximations.

**Definition 3.1.** A function \( u \) is a weak generalized solution of problem (8)-(9), if

1. \( u \in L_0(0, T; W^1_2(\Omega)), \nabla u \in L_2(Q_T), u_t \in L_2(Q_T; W^1_2(\Omega)), |\nabla u|^p |\nabla u| \in L_\infty(0, T; L_p(\Omega)), |u|^\sigma |\nabla u| \in L_\infty(0, T; L_{\sigma'}(\Omega)), \]
2. \( |u|^p \leq |u|^\sigma \) as \( t \rightarrow +0, \)
3. the integral identity
\[ \int_0^T \int_\Omega \left( \sum_{i=1}^{2N} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial i}{\partial x_i} + \frac{\partial u}{\partial t} + |\nabla u|^p |\nabla u| - b(x, t)|u|^\sigma |\nabla u| \right) dx dt = \]
\[ = \int_0^T \int_\Omega F(t, u) h dx dt, \quad (13) \]

are satisfied for every \( v(x, t) \in L_2(0, T; W^1_2(\Omega)). \)

**Theorem 3.2.** Let the conditions (6),(7) be satisfied and \( \sigma > p, 2 < p < 4, \)
\( 2 < \sigma < \frac{2N}{N-2}, N \geq 3 \). Then there exists \( T_0 \in (0, T) \) such that problem (8),(9) has at least one weak solution \( u \) on the interval \( (0, T), T < T_0, \) (respectively problem \( (1)-(4) \) has at least one weak solution \( u(x, t, f(t)) \)). Moreover,
\[ u \in L_\infty(0, T; W^1_2(\Omega)), \nabla u \in L_2(Q_T), Q_T = \Omega \times (0, T), \]
\[ u_t \in L_2(Q_T; W^1_2(\Omega)), |\nabla u|^p |\nabla u| \in L_\infty(0, T; L_{p'}(\Omega)), \]
\[ |u|^\sigma |\nabla u| \in L_\infty(0, T; L_{\sigma'}(\Omega)). \]

**Proof.** Select in \( W^1_2(\Omega) \) a certain system of functions \( \{\Psi_j(x)\} \), forming a basis in a given space. This system exists because \( W^1_2(\Omega) \) is a separable space. We will look for an approximate solution of problem (8)-(9) in the following form
\[ u_m(x, t) = \sum_{k=1}^{m} C_{mk}(t) \Psi_k(x), \quad (14) \]
where the coefficients \( C_{mk}(t) \) are determined from the conditions
\[ \sum_{k=1}^{m} \left\{ \begin{aligned} C_{mk}(t) \int_\Omega \Psi_k \Psi_j + \chi \sum_{i=1}^{m} \frac{\partial \Psi_k}{\partial x_i} \cdot \frac{\partial \Psi_j}{\partial x_i} \right. \right. \]
\[ + \int_\Omega |\nabla u_m|^p |\nabla u_m| |\nabla \Psi_j| dx - \int_\Omega b(x, t)|u_m|^\sigma |\nabla \Psi_j| dx = \int_\Omega F(t, u_m) h(x) \Psi_j dx. \]
\[ u_{m0} = u_m(0) = \sum_{k=1}^{m} C_{mk}(0) \Psi_k = \sum_{k=1}^{m} \alpha_k \Psi_k \quad (15) \]
\[ u = \sum_{k=1}^{m} C_{mk}(t) \Psi_k \quad (16) \]
with
\[ u_{m0} \rightarrow u_0 \text{ strongly in } W^1_2(\Omega) \text{ as } m \rightarrow \infty. \]

Let us denote
\[ \bar{C}_m \equiv \{C_{m1}(t), \ldots, C_{mm}(t)\}^T, \bar{\alpha} \equiv \{\alpha_1, \ldots, \alpha_m\}^T, a_{kj} = \int_\Omega [\Psi_k \Psi_j + \chi (\nabla \Psi_k, \nabla \Psi_j)] \, dx, \]
\[ b_{kj} = \int_\Omega |\nabla u_m|^{p-2} \nabla \Psi_k \nabla \Psi_j \, dx + \int_\Omega b(x, t) |u_m|^{\alpha-2} \Psi_k \Psi_j \, dx + \int_\Omega F(t, u_m) h(x) \Psi_j \, dx \]
\[ A_m (\bar{C}_m) \equiv \{a_{kj}(\bar{C}_m)\}, \bar{\alpha} \equiv \{b_{kj}(\bar{C}_m)\} \bar{C}_m. \]

Then the system of equations (15) and the condition (16) will have the matrix form
\[ A_m \bar{C}_m = \bar{\alpha}, \quad \bar{C}_m(0) = \bar{\alpha}. \] (18)

The matrix \( A_m \) is invertible. We will consider the following quadratic form
\[ \sum_{k,j=1}^m a_{kj} \xi_k \xi_j = \int_\Omega |\eta|^2 \, dx + \chi \int_\Omega |\nabla \eta|^2 \, dx, \quad \eta = \sum_{l=1}^m \xi_l \psi_l. \]

The quadratic form is equal to zero if and only if \( \eta = 0 \). Since \( \psi_l(x) \) is a Galerkin basis in \( W^1_2(\Omega) \), it is linearly independent in \( W^1_2(\Omega) \), when \( \eta = 0 \) and if and only if \( \xi = (0, \ldots, 0) \). Hence, because of the Sylvester criterion, we conclude that \( \det ||a_{kj}||_{k,j=1}^m > 0 \) for all \( m \in N \). Considering the positivity of the matrix \( A_m \), the problem (18) can be reduced to the following form
\[ \bar{C}_m\gamma = A_m^{-1} \bar{\alpha}, \quad \bar{C}_m(0) = \bar{\alpha}. \] (19)

According to Cauchy’s theorem, the problem (19) has at least one solution \( \bar{C}_m \) within some time interval \( t \in (0, T_m) \). \( T_m > 0 \).

3.1. A priori estimates. We have a priori estimates for \( u_m \) independent of \( m \) and in some cases valid for any finite \( t \). To get the first estimate, we multiply both parts of the equality (15) by \( C_{nj}(t) \) and sum both parts of the resulting equality by \( j = 1, 2, \ldots, m \). As a result, we obtain the following equality
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega \|u_m\|^2 + \chi |\nabla u_m|^2 \, dx + \int_\Omega |\nabla u_m|^p \, dx + \int_\Omega b(x, t) |u_m|^{\alpha-2} u_m \, dx + \int_\Omega F(t, u_m) h u_m \, dx. \] (20)

We will use the following embedding inequality here in after.

Lemma 3.3 ([11], p.296, Lemma 3.2). For any function \( u(x) \in W^2_2(\Omega) \) the inequality
\[ \|u\|^\sigma_{\sigma, \Omega} \leq \frac{(2N-1)^\sigma}{N-\sigma} \|\nabla u\|\sigma_{2, \Omega} \leq C_0 (\chi \|\nabla u\|^2_{2, \Omega} + \|u\|^2_{2, \Omega})^{\frac{\sigma}{2}} \] (21)
is satisfied with \( C_0 = \max(1, \frac{1}{\chi}) \frac{(2N-1)^\sigma}{N-\sigma} , \quad 2 < \sigma < \frac{2N}{N-2} , \quad N \geq 3. \quad W^2_2(\Omega) \hookrightarrow L_\sigma(\Omega) \) is valid under conditions \( 2 < \sigma < \frac{2N}{N-2} , \quad 2 < p < N, \quad N \geq 3. \)

We estimate the right hand side of (20) using the inequality (21) of lemma 2
\[ \int_\Omega b(x, t) |u_m|^\sigma \, dx \leq b_0 \|u_m\|^\sigma_{\sigma, \Omega} \leq b_0 C_0 (\chi \|\nabla u\|^2_{2, \Omega} + \|u\|^2_{2, \Omega})^{\frac{\sigma}{2}}, \] (22)
\[ \int_\Omega \|\phi'\|_{\sigma, \Omega} h u_m \, dx \leq \|\phi'\|_{\sigma, \Omega} \|h\|_{2, \Omega} \|u_m\|_{2, \Omega} \leq \frac{1}{4} \|\phi'\|^2_{2, \Omega} + \|u_m\|^2_{2, \Omega} , \] (23)
\[ \int_\Omega h u_m \, dx \int_\Omega b(x, t) |u_m|^{\alpha-2} u_m \, \omega \, dx \leq b_0 \|u_m\|^\sigma_{\sigma, \Omega} \|\omega\|_{\sigma, \Omega} \|h\|_{\sigma, \Omega} \leq \frac{1}{4} \|\phi'\|^2_{2, \Omega} + \|u_m\|^2_{2, \Omega} \] (24)
We will multiply equality (15) by \( t \).

Integrating the latter from 0 to \( T \), we obtain

\[
\|u_m\|_{L^p(\Omega)} + \frac{(2(p-1))^{p-1}}{p^p} \left( \|u_m\|^2_{L^2(\Omega)} + \|\nabla u_m\|^2_{L^2(\Omega)} \right) \leq \left( \|u_m\|^2_{L^2(\Omega)} + \chi \|\nabla u_m\|^2_{L^2(\Omega)} \right)^{\frac{p}{2}}.
\]

(25)

If we denote \( z(t) = e^{-C_2 t} (\|u_m\|^2_{L^2(\Omega)} + \chi \|\nabla u_m\|^2_{L^2(\Omega)}) \), then (26) will take the following form

\[
\frac{dz(t)}{dt} \leq C_4 e^{\frac{s-2}{2} C_2 t} |z(t)|^{\frac{s}{2}} + C_1 e^{-C_2 t}.
\]

Integrating the latter from 0 to \( t \), we obtain the inequality

\[
z(t) \leq z(0) + C_3 \int_0^t e^{\frac{s-2}{2} C_2 t} |z(s)|^{\frac{s}{2}} ds + \frac{C_1}{C_2}.
\]

We can apply the condition (5), as well as the Gronwall-Bellman-Bihari Lemma [12], if

\[
\frac{C_3}{C_2} \left( e^{\frac{s-2}{2} C_2 t} - 1 \right) < \frac{1}{(z(0) + \frac{C_1}{C_2})^{\frac{s}{2}}}.
\]

Then the inequality is true

\[
z(t) \leq \frac{z(0) + \frac{C_1}{C_2}}{1 \left( 1 - \left( z(0) + \frac{C_1}{C_2} \right)^{\frac{s-2}{2}} \left( e^{\frac{s-2}{2} C_2 t} - 1 \right) \right)^{\frac{s}{2}}}
\]

As a result, the following estimate is obtained.

\[
\|u_m(x, t)\|^2_{L^2(\Omega)} + \chi \|\nabla u_m(x, t)\|^2_{L^2(\Omega)} \leq \frac{\left( \|u_m(x_0, t)\|^2_{L^2(\Omega)} + \|u_m(x_0)\|^2_{L^2(\Omega)} + \frac{C_1}{2} \right)}{1 - \frac{\left( 1 - \left( \|u_m(x_0, t)\|^2_{L^2(\Omega)} + \chi \|\nabla u_m(x_0)\|^2_{L^2(\Omega)} + \frac{C_1}{2} \right)^{\frac{s-2}{2}} \left( e^{\frac{s-2}{2} C_2 t} - 1 \right) \right)^{\frac{s}{2}}}
\]

From these estimates and (21), (26), we can conclude that there is a \( T_0 > 0 \) such as

\[
\sup_{t \in [0, T]} \left( \|u_m\|_{L^2(\Omega)}^2 + \|u_m\|_{L^2(\Omega)}^2 + \chi \|\nabla u_m\|_{L^2(\Omega)}^2 \right) \leq C_4, \quad T < T_0,
\]

(27)

where the constant \( C_4 \) does not depend on \( m \in N \).

Going back to (26) and considering (27), we obtain another inequality:

\[
\int_0^T \int_\Omega |\nabla u_m|^p dx dt \leq C_5.
\]

(28)

We will multiply equality (15) by \( C_{m_j}(t) \) and sum by \( j = 1, m \). As a result we will have

\[
\|u_m\|^2_{L^2(\Omega)} + \chi \|\nabla u_m\|^2_{L^2(\Omega)} + \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u_m|^p dx = \frac{1}{2} \frac{d}{dt} \int_\Omega b(x, t) |u_m|^p dx - \frac{1}{2} \int_\Omega b(x, t) |u_m|^p dx + \int_\Omega F(t, u_m) h(x) \partial u_m dx.
\]

(29)
If we integrate the last one over \( t \) from 0 to \( T \), then we will obtain the relation

\[

t_0^{T} \left( \|u_m\|_{2,\Omega}^2 + \chi \|\nabla u_m'\|_{2,\Omega}^2 \right) dt + \frac{1}{p} \int_{\Omega} |\nabla u_m|^p dx = \frac{1}{p} \int_{\Omega} |\nabla u_m(x, 0)|^p dx - \frac{1}{p} \int_{\Omega} b(x, 0)|u_m(x, 0)|^p dx + \frac{1}{p} \int_{\Omega} b(x, t)|u_m|^p dx - \frac{1}{p} \int_{\Omega} b(x, \tau)|u_m|^p dx + \int_{\Omega} F(t, u_m)h(x) \partial_t u_m dx dt.
\]

We will estimate the right-hand side of (30) using (21), (10) as follows

\[
\left| \int_{\Omega} \phi_0^\prime (\tau) \omega d\tau \right| \leq \left( \int_{\Omega} \omega dx \right) \left( \int_{\Omega} \phi_0^\prime (\tau) d\tau \right) \leq \left| \int_{\Omega} \phi_0^\prime (\tau) d\tau \right| \left| \int_{\Omega} \omega dx \right| \leq C_4 \left| \int_{\Omega} \phi_0^\prime (\tau) d\tau \right| \left| \int_{\Omega} \omega dx \right|.
\]

We will substitute the obtained inequalities in the identity (30) and obtain the following estimate

\[
\int_{0}^{T} \left( \|u_m\|_{2,\Omega}^2 + \chi \|\nabla u_m'\|_{2,\Omega}^2 \right) dt + \frac{1}{p} \int_{\Omega} |\nabla u_m|^p dx \leq C_6.
\]

3.2. Passage to the limit. From the obtained estimations (27), (28), (31) the following statements of convergence of the sequence \( u_m \) follow accordingly

\[
u_m \text{ bounded in } L_\infty(0, T; W^1_2(\Omega)) \tag{32}
\]

\[
\nabla u_m \text{ bounded in } L_2(Q_T), \quad Q_T = \Omega \times (0, T) \tag{33}
\]

\[
\partial_t u_m \text{ bounded in } L_2(0, T; W^1_2(\Omega)) \tag{34}
\]

From (32) follows that there is a subsequence \( u_{m_k} \) of the sequence \( u_m \) *-weakly converging to some element \( u \in L_\infty(0, T; W^1_2(\Omega)) \), i.e.

\[
u_{m_k} \rightharpoonup u \text{ *-weakly in } L_\infty(0, T; W^1_2(\Omega)).
\]

Similarly, it follows from (33)-(34) that a sequence \( \{u_{m_k}\} \subset \{u_m\} \), such as

\[
u_{m_k} \rightharpoonup u \text{ weakly in } L_2(0, T; W^1_2(\Omega)) \exists.
\]

Because of the Rellich-Condrashov theorem, the embedding \( W^1_2(Q_T) \) in \( L_2(Q_T) \) is compact. It means that the sequence \( u_{m_k} \) can be chosen such that \( u_{m_k} \rightarrow u \) in \( L_2(Q_T) \).
norm, which means that convergence is almost everywhere. Because of lemma 1.3, proven in [22], it follows that \( b(x, t)|u_m|^{p-2}u_m \to b(x, t)|u|^{p-2}u \).

Applying estimates (27), (28) and (31), we will try to do the passage to the limit as \( m \to \infty \). According to (27), (28) and (31), we can conclude that there is a subsequence of \( u_{m_k} \) (for the sake of simplicity, we will use the notation: \( u_{m_k} = u_m \)) and the function \( u \) such that

\[
\begin{align*}
&u_m \to u, \text{ a.e. } Q_T, \quad u \in W^{1,2}(Q_T), \\
&\nabla u_m \rightharpoonup \nabla u \text{ weakly in } L_2(Q_T), \\
&u_m \rightharpoonup u \text{ weakly in } L_p(Q_T), \\
&\nabla u_m \rightharpoonup \nabla u \text{ weakly in } L_p(Q_T),
\end{align*}
\]

\( A_m := |\nabla u_m|^{p-2}\nabla u_m \to A \in L_{p'}(Q_T) \) weakly in \( A \in L_{p'}(Q_T) \).

Let \( d_{mk}(t) \in C^1[0, T] \), \( k = 1, \ldots, n \) be an arbitrary function. Multiplying (15) by \( d_{mk}(t) \), summing and integrating by \( t \) we obtain:

\[
\begin{align*}
\int_{Q_T} [u_{m}u_t + \chi \nabla u_m \nabla \mu + |\nabla u_m|^{p-2}\nabla u_m \nabla \mu - b|u_m|^{p-2}u_m \mu] dx dt &= \\
\int_{Q_T} [\varphi' + \int_I \nabla u_m |^{p-2}\nabla u_m | \omega dx - \int_I b|u_m|^{p-2}u_m \omega dx] h \mu dx dt
\end{align*}
\]

which is valid for any function \( \mu(x, t) = \sum_{k=1}^m d_{mk}(t)\Theta_k(x) \). In particular,

\[
\begin{align*}
\int_{Q_T} [u_{m}u_t + \chi \nabla u_m \nabla u_m + |\nabla u_m|^{p-2}\nabla u_m | \omega dx - \int_I b|u_m|^{p-2}u_m \omega dx] \mu dx dt &= \\
\int_{Q_T} [\varphi' + \int_I A \nabla \omega dx - \int_I b|u_m|^{p-2}u_m \omega dx] h u_m dx dt.
\end{align*}
\]

Going to the limit in (40) for \( m \to \infty \), we obtain

\[
\begin{align*}
\int_{Q_T} [u_t + \chi \nabla u \nabla \mu + A \nabla \mu - |u|^{p-2}u \mu] dx dt &= \\
\int_{Q_T} [\varphi' + \int_I A \nabla \omega dx - \int_I b|u|^{p-2}u \omega dx] h u dx dt.
\end{align*}
\]

Since the \( \Theta_m \) functions are dense in \( L_2(0, T; W^2_0(\Omega)) \), then we can assume that \( \mu = u \)

\[
\begin{align*}
\int_{Q_T} [u_t + \chi \nabla u \nabla u + A \nabla u - |u|^p] dx dt &= \\
\int_{Q_T} [\varphi' + \int_I A \nabla \omega dx - \int_I b|u|^{p-2}u \omega dx] h u dx dt
\end{align*}
\]

where \( A = \lim A_m = \lim |\nabla u_m|^{p-2}\nabla u_m \).

Now we need to prove that

\[
A = |\nabla u|^{p-2}\nabla u.
\]

To do this, we use Minty’s lemma (see [22]) and the monotonicity inequality:

\[
(b|b|^{p-2} - |a|^{p-2}a, b - a) \geq 0, \quad \forall a, b, \quad 1 < p < \infty.
\]

From monotony, for any smooth function \( \zeta \)

\[
\begin{align*}
|\nabla u|^p &= \left( |\nabla u_m|^{p-2}\nabla u_m - |\nabla \zeta|^{p-2}\nabla \zeta \right) \cdot \nabla (u_m - \zeta) + \\
&+ |\nabla \zeta|^{p-2}\zeta \cdot \nabla (u_m - \zeta) + |\nabla u_m|^{p-2}\nabla u_m \cdot \nabla \zeta \quad \geq \\
&\geq |\nabla \zeta|^{p-2}\zeta \cdot \nabla (u_m - \zeta) + |\nabla u_m|^{p-2}\nabla u_m \cdot \nabla \zeta.
\end{align*}
\]

Subtracting (41) from (43), we will have

\[
\int_{Q_T} [A \nabla u - |\nabla u_m|^p] = I_1 + I_2 + I_3,
\]

where

\[
I_1 = \int_{Q_T} (u_t - u_{m}u_{m}) dx dt, \quad I_2 = \int_{Q_T} (\nabla u_t \nabla u - \nabla u_{m} \nabla u_m) dx dt.
\]
By a lower semicontinuity of the norm we have

\[ u \]

Then it follows from (48) that

\[ \text{From (35) - (39) that} \]

and respectively

\[
\text{Moreover}
\]

Moreover

\[
I_2 = \int_0^t \frac{1}{2} \frac{d}{dt} \int_\Omega \left( |\nabla u|^2 - |\nabla u_m|^2 \right) dx ds = \frac{1}{2} \int_\Omega \left( |\nabla u|^2 - |\nabla u_m|^2 \right) dx \Big|_0^t.
\]

Using the monotonicity of (44), we can rewrite (46) as

\[
\int_{Q_T} \left( |\nabla \zeta|^{p-2} \nabla \zeta \cdot \nabla (u_m - \zeta) + |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla \zeta - A \nabla u \right) dx dt \leq I_1 + I_2 + I_3.
\]

Going to the limit for \( m \to \infty \) in the last inequality, we obtain:

\[
\int_{Q_T} \left( |\nabla \zeta|^{p-2} \nabla \zeta - A \right) \cdot \nabla (u_m - \zeta) dx dt \leq \lim_{m \to \infty} (I_1 + I_2 + I_3).
\]

From (35) - (39) that

\[
\lim_{m \to \infty} I_1 = \lim_{m \to \infty} \int_{Q_T} (u_m - u_m \omega dx h) dx dt = 0,
\]

\[
\lim_{m \to \infty} I_3 = \lim_{m \to \infty} \int_{Q_T} (|\nabla u|^2 - |\nabla u_m|^2) dx dt = 0,
\]

\[
\lim_{m \to \infty} I_5 = \lim_{m \to \infty} \int_{Q_T} (|\nabla u_m|^{p-2} \nabla u_m \cdot \nabla \zeta - A \nabla u) dx dt = 0,
\]

\[
\lim_{m \to \infty} I_6 = \lim_{m \to \infty} \int_{Q_T} b (|u_m|^{\sigma-2} u_m - |u|^{\sigma-2} u) dx dt = 0,
\]

\[
\lim_{m \to \infty} I_2 = \lim_{m \to \infty} \frac{1}{2} \int_{\Omega} \left( |\nabla u|^2 - |\nabla u_m|^2 \right) dx = 0,
\]

\[
= \frac{1}{2} \lim_{m \to \infty} \int_{\Omega} \left( |\nabla u(x, t)|^2 - |\nabla u_m(x, t)|^2 \right) dx + \int_{\Omega} \left( |\nabla u(x, 0)|^2 - |\nabla u_m(x, 0)|^2 \right) dx,
\]

and from selected \( u_m(x, 0) \):

\[
\lim_{m \to \infty} \int_{\Omega} \left( |\nabla u(x, 0)|^2 - |\nabla u_m(x, 0)|^2 \right) dx = 0.
\]

By a lower semicontinuity of the norm we have

\[
\int_{\Omega} |\nabla u(x, t)|^2 dx \leq \lim_{m \to \infty} \int_{\Omega} |\nabla u_m(x, t)|^2 dx, \ a.e. \ t \in (0, T)
\]

and respectively

\[
\lim_{m \to \infty} I_2 \leq \frac{1}{2} \lim_{m \to \infty} \int_{\Omega} \left( |\nabla u(x, 0)|^2 - |\nabla u_m(x, 0)|^2 \right) dx = 0.
\]

Then it follows from (48) that

\[
\int_{Q_T} \left( (|\nabla \zeta|^{p-2} \nabla \zeta - A) \cdot \nabla (u_m - \zeta) \right) dx dt \leq 0.
\]
Now suppose \( \zeta = u + \delta \eta \) with arbitrary \( \delta > 0 \) and \( \eta \in W \). Simplifying and setting \( \delta \to 0 \), we get the inequality
\[
\int_{Q_T} (|\nabla u|^{p-2}\nabla u - A) : \nabla \eta dx dt \geq 0, \quad \forall \eta \in W,
\]
that is possible if and only if \( |\nabla u|^{p-2}\nabla u = A \) a.e. \( Q_T \). Then from (42) follows
\[
\int_{Q_T} [u_t + \chi \nabla u \nabla \mu + |\nabla u|^{p-2}\nabla u \nabla u - |u|^{\sigma-2}u \mu] dx dt
= \int_{Q_T} [\varphi' + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \omega dx - \int_\Omega b|u|^{\sigma-2}u \omega dx] h \mu dx dt
\]
for any function \( \mu(x,t) = \sum_{k=1}^m d_m k(t) \Psi_k(x) \). Since the functions \( \mu(x,t) \) are dense in the space of test functions, then the last integral equality coincides with the definition of a weak solution.

Given the obtained inclusions and convergence, we proceed in (40) to the limit for \( m \to \infty \) and we have (13) for \( \nu = \mu \). Since the set of all functions \( \mu(x,t) \) is dense in \( W_2^1(0,T; W_2^1(\Omega)) \), then the limit relation holds for all \( v(x,t) \in L_2(0,T; W_2^1(\Omega)) \).

Let us assume that equation (1) describes a process with absorption, i.e., \( b(x,t) \leq 0 \). Then the above scheme allows us to prove the existence of a weak solution to problem (8)-(9) (respectively to problem (1)-(4)) for any finite time interval.

**Theorem 3.4.** Let the conditions (6), (7) be satisfied, and \( 0 < b_0 \leq -b(x,t) \leq b_1 < \infty, \quad 1 < \sigma < \infty, \quad 1 < p < 3, \quad 0 < b_0 \leq b(x,t) \leq b_1 < \infty, \quad 1 < \sigma \leq 2, \quad N \geq 3, \quad 2 < p < 4. \) Then problem (8), (9) has at least one weak solution \( u(x,t) \) for any finite interval \( (0,T) \), (respectively problem (1)-(4) has solution \( u(x,t), f(t) \)). Moreover the following inclusions hold:
\[
\begin{align*}
&u \in L_\infty(0,T; W^0_2(\Omega)), \nabla u \in L_2(Q_T), \quad Q_T = \Omega \times (0,T), \\
u_t \in L_2(0,T; W^0_2(\Omega)), \quad |\nabla u|^{p-2}\nabla u \in L_\infty(0,T; L^{\frac{2}{2-p}}(\Omega)), \\
&|u|^{\sigma-2}u \in L_\infty(0,T; L^{\frac{2}{2+\sigma}}(\Omega)).
&
\end{align*}
\]

4. **Blow-up of the solution in a finite time.** Now we prove will the blow-up of the weak generalized solution of the inverse problem (1)-(4). To avoid cumbersome calculations we will assume \( \varphi(t) = 1, \quad b(x,t) = 1 \).

We will introduce the following notation
\[
I(0) = \frac{1}{\sigma} \|u_0\|_{\sigma,\Omega}^\sigma - \frac{\lambda}{2} \|\nabla u_0\|_{2,\Omega}^2 - \frac{\lambda}{2} \|u_0\|_{2,\Omega}^2 - \frac{1}{p} \|u_0\|_{p,\Omega}^p,
\]
\[
B = C(\varepsilon_3) \|\nabla \omega\|_{p,\Omega}^p + C(\varepsilon_5) \|\omega\|_{\theta,\Omega}^\theta; \quad \varepsilon_3 = \frac{\sigma - p}{2p}; \quad \varepsilon_5 = \frac{\sigma - p}{2\sigma}; \quad \lambda > 0; \quad \chi > 0. \tag{50}
\]

**Theorem 4.1.** Let the conditions (49), (50) hold and
\[
\begin{align*}
&1) \quad \sigma > \max(p,4-p), \quad 2 \leq p < 4, \\
&2) \quad I(0) - \frac{B}{2} > 0.
&
\end{align*}
\]

Then, every weak solution of the inverse problem (1) - (4) blows up in the sense that is defined in (55) function \( \rho(t) \to \infty \) when \( t \to T \), where \( u(x,t) = e^{\lambda t}v(x,t), \rho(t) = \frac{1}{2} \left( \|v\|_{2,\Omega}^2 + \chi \|\nabla v\|_{2,\Omega}^2 \right) \) for some finite time \( T \leq t^* \), where
\[
t^* = \frac{8 \left( \|u_0\|_{2,\Omega}^2 + \chi \|\nabla u_0\|_{2,\Omega}^2 \right)}{(\sigma + p - 4)(\sigma + p) (I(0) - \frac{B}{2})}.
\]
Proof. If we replace \( u(x,t) = e^{\lambda t}v(x,t) \), then the inverse problem (1) - (4) will take the following form
\[
v_t - \chi \Delta v_t - \lambda \chi \Delta v + \lambda v - e^{\lambda(p-2)t} dv \left( \| \nabla v \|^p v \right) = e^{\lambda(p-2)t} |v|^p v_t + e^{-\lambda t} f(t) h(x),
\]
\[
 v(x,0) = u_0(x),
\]
\[
 v|_S = 0.
\]
\[
 \int_{\Omega} v(x,t) \cdot h(x) dx = e^{-\lambda t}.
\]

We denote by
\[
 \rho(t) = \frac{1}{2} \left( \int v^2 + \chi \| \nabla v \|^2 \right),
\]
\[
 I(t) = \frac{1}{2} e^{\lambda(p-2)t} \int |v|^p + \frac{\lambda \chi}{2} \| \nabla v \|^2_2 - \frac{\lambda}{2} \| v \|^2_2 - \frac{1}{p} e^{\lambda(p-2)t} \| \nabla v \|^p_{p,\Omega}.
\]
If we multiply equation (51) sequentially by the functions \( \omega(x), v(x,t), v_t(x,t) \) and integrate, then we obtain the following relations:
\[
e^{-\lambda t} f(t) = e^{\lambda(p-2)t} \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \omega dx - e^{\lambda(p-2)t} \int_{\Omega} |v|^{p-2} v \omega dx,
\]
\[
 \rho'(t) = e^{\lambda(p-2)t} \| v \|^p_{p,\Omega} - \chi \lambda \| \nabla v \|^2_2 - \lambda \| v \|^2_2 - e^{\lambda(p-2)t} \| \nabla v \|^p_{p,\Omega} + e^{-\lambda t} f(t),
\]
\[
 I'(t) = \| u_t \|^2 + \chi \| \nabla v \|^2_2 + \frac{\lambda \chi}{2} e^{\lambda(p-2)t} \| v \|^p_{p,\Omega} - \frac{\lambda(p-2)}{p} e^{\lambda(p-2)t} \| \nabla v \|^p_{p,\Omega} + e^{-\lambda t} f(t).
\]
Substituting the ratios (56) in (57) and (58), and estimating the summands, we obtain
\[
 |e^{\lambda(p-3)t} \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \omega dx| \leq e^{\lambda(p-3)t} \left( \int_{\Omega} |\nabla v|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |\nabla \omega|^p dx \right)^{\frac{1}{p}} \leq e^{\lambda(p-3)t} \left( \int_{\Omega} |\nabla v|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |\nabla \omega|^p dx \right)^{\frac{1}{p}} \leq \varepsilon_3 \lambda^{(p-2)t} \| \nabla v \|^p_{p,\Omega} + C(\varepsilon_3) e^{-\lambda t} \| \nabla \omega \|^p_{p,\Omega},
\]
\[
 |e^{\lambda(p-3)t} \int_{\Omega} |v|^{p-2} v \cdot \omega dx| \leq e^{\lambda(p-3)t} \left( \int_{\Omega} |v|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |\nabla \omega|^p dx \right)^{\frac{1}{p}} \leq e^{\lambda(p-3)t} \left( \int_{\Omega} |v|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |\nabla \omega|^p dx \right)^{\frac{1}{p}} \leq \varepsilon_3 e^{\lambda(p-2)t} \| v \|^p_{p,\Omega} + C(\varepsilon_3) e^{-\lambda t} \| \nabla \omega \|^p_{p,\Omega},
\]
where \( C(\varepsilon_3) = \frac{1}{\varepsilon_3} \exp \left( \frac{1}{\varepsilon_3} \right). \)

Similarly, we obtain
\[
 |e^{\lambda(p-3)t} \int_{\Omega} |\nabla v|^p dx| \leq e^{\lambda(p-3)t} \left( \int_{\Omega} |v|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |\nabla \omega|^p dx \right)^{\frac{1}{p}} \leq \varepsilon_5 e^{\lambda(p-2)t} \| v \|^p_{p,\Omega} + C(\varepsilon_5) e^{-\lambda t} \| \nabla \omega \|^p_{p,\Omega},
\]
where \( C(\varepsilon_5) = \frac{1}{\varepsilon_5} \exp \left( \frac{1}{\varepsilon_5} \right). \)

\[
 |\lambda e^{-\lambda t} f(t)| \leq \lambda \varepsilon_3 e^{\lambda(p-2)t} \| \nabla v \|^p_{p,\Omega} + \lambda C(\varepsilon_3) e^{-\lambda t} \| \nabla \omega \|^p_{p,\Omega} + \lambda \varepsilon_5 e^{\lambda(p-2)t} \| v \|^p_{p,\Omega} + C(\varepsilon_5) e^{-\lambda t} \| \nabla \omega \|^p_{p,\Omega}.
\]
Substituting the obtained estimates into the identities (57) and (58), we obtain the following inequalities
\[
 \rho'(t) \geq (1 - \varepsilon_5) e^{\lambda(p-3)t} \| v \|^p_{p,\Omega} - \lambda \chi \| \nabla v \|^2_2 - (1 + \varepsilon_3) e^{\lambda(p-2)t} \| \nabla v \|^p_{p,\Omega} - (C(\varepsilon_5) \| \nabla \omega \|^p_{p,\Omega} + C(\varepsilon_3) \| \omega \|^p_{p,\Omega}) e^{-2\lambda t}.
\]
Applying (61) again, we have

\[ I'(t) \geq \|v_t\|_{2,\Omega}^2 + \chi \|\nabla v_t\|_{2,\Omega}^2 + \]
\[ + \left( \frac{\lambda(\sigma-2)}{\sigma} - \lambda \varepsilon_5 \right) e^{\lambda(\sigma-2)t} \|v\|_{\sigma,\Omega}^\sigma - \]
\[ - \left( \frac{\lambda(p-2)}{p} + \lambda \varepsilon_5 \right) e^{\lambda(p-2)t} \|\nabla v\|_{p,\Omega}^p - \]
\[ - \left( \lambda C(\varepsilon_5) \|\nabla \omega\|_{p,\Omega}^p + \lambda C(\varepsilon_5) \|\omega\|_{\sigma,\Omega}^\sigma \right) e^{-2\lambda t}. \]

(60)

Assume \( \varepsilon_3 = \frac{\sigma-p}{2p} \); \( \varepsilon_5 = \frac{\sigma-p}{2\sigma} \). Then the inequalities (59) and (60) will take the following form

\[ \rho'(t) \geq \frac{\sigma+p}{2} I(t) - Be^{-2\lambda t}, \]  

where

\[ B = C(\varepsilon_3) \|\nabla \omega\|_{p,\Omega}^p + C(\varepsilon_5) \|\omega\|_{\sigma,\Omega}^\sigma, \]

\[ I'(t) \geq \frac{\lambda(\sigma+p-4)}{2} I(t) + \|v_t\|_{2,\Omega}^2 + \chi \|\nabla v_t\|_{2,\Omega}^2 - \lambda Be^{-2\lambda t}. \]  

(62)

When condition \( I(0) - \frac{2B}{\sigma+p} > 0 \), we have

\[ I(t) \geq e^{\frac{\lambda(\sigma+p-4)t}{2}} \left( I(0) - \frac{2B}{\sigma+p} \right) \geq e^{\frac{\lambda(\sigma+p-4)t}{2}} \left( I(0) - \frac{B}{2} \right) \geq 0. \]

Then the inequality (62) is written as

\[ I'(t) \geq \|v_t\|_{2,\Omega}^2 + \chi \|\nabla v_t\|_{2,\Omega}^2 - \lambda Be^{-2\lambda t}. \]  

(63)

Now we prove the following inequality

\[ \rho'(t) = \int_{\Omega} v \cdot v_t dx + \chi \int_{\Omega} \nabla v \cdot \nabla v_t dx \leq \|v\|_{2,\Omega} \|v_t\|_{2,\Omega} + \chi \|\nabla v\|_{2,\Omega} \|\nabla v_t\|_{2,\Omega}, \]

\[ [\rho'(t)]^2 \leq \left( \|v\|_{2,\Omega}^2 + \chi \|\nabla v\|_{2,\Omega}^2 \right) \left( \|v_t\|_{2,\Omega}^2 + \chi \|\nabla v_t\|_{2,\Omega}^2 \right). \]  

(64)

If we multiply the inequality (63) by \( \rho(t) \) and use (64), then we have

\[ I'(t) \rho(t) \geq \frac{1}{2} [\rho'(t)]^2 - \lambda Be^{-2\lambda t} \rho(t). \]

In the latter inequality, we apply (61)

\( (I'(t) + \lambda Be^{-2\lambda t}) \rho(t) \geq \frac{1}{2} \rho'(t) \left( \frac{\sigma+p}{2} I(t) - Be^{-2\lambda t} \right), \)

\[ I(t) \geq \frac{I(0) - \frac{B}{2}}{\rho(0)} [\rho(t)]^{\frac{\sigma+p}{4}} + \frac{B}{2} e^{-2\lambda t}. \]

Applying (61) again, we have

\[ \rho'(t) \geq \frac{\sigma+p}{2} \left( \frac{I(0) - \frac{B}{2}}{\rho(0)} \right)^{\frac{\sigma+p}{4}} + \frac{B(\sigma+p-4)}{4} e^{-2\lambda t}. \]

Integrating the latter we have

\[ [\rho(t)]^{\frac{\sigma+p-4}{4}} \geq \left( \frac{1}{\rho(0)} \right)^{\frac{\sigma+p-4}{4}} - \frac{(\sigma+p-4)(\sigma+p)}{8} \left( \frac{I(0) - \frac{B}{2}}{\rho(0)} \right)^{-1}. \]
5. **Exponential behavior of the solution by time.** Consider in the cylinder $Q = \{(x, t) : x \in \Omega, t \in (0, \infty)\}$ the inverse problem of determining the unknown right side of the pseudo-parabolic equation

$$\begin{align*}
 u_t - \chi \Delta u_t - a_0 \Delta u - \text{div} (|\nabla u|^{p-2} \nabla u) + |u|^\sigma - 2 u f(t) h(x), \\
 u(x, 0) = u_0(x), \\
 u|_\Gamma = 0, \\
 \int_{\Omega} u(x, t) \cdot h(x) dx = \varphi(t).
\end{align*} \tag{65}$$

Let us show the exponential behavior of this problem. We will multiply equation (65) by the functions $\omega(x)$, $u(x, t)$ and integrate over the domain $\Omega$

$$f(t) = \varphi'(t) + a_0 \int_{\Omega} \nabla u \nabla \omega dx + \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \omega dx + \int_{\Omega} |u|^\sigma - 2 u \omega dx, \tag{69}$$

$$\rho(t) = a_0 \|\nabla u\|_{2,\Omega}^2 + \|\nabla u\|_{p,\Omega}^p + \|u\|_{\sigma,\Omega}^\sigma = \varphi(t) f(t), \tag{70}$$

where $\rho(t) = \frac{1}{2} \left(\|u\|_{2,\Omega}^2 + \chi \|\nabla u\|_{2,\Omega}^2\right)$.

Substituting the ratio (69) in the identity (70), we obtain

$$\rho'(t) + a_0 \|\nabla u\|_{2,\Omega}^2 + \|\nabla u\|_{p,\Omega}^p + \|u\|_{\sigma,\Omega}^\sigma = \varphi(t) \varphi'(t) + a_0 \varphi(t) \int_{\Omega} \nabla u \nabla \omega dx + \varphi(t) \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \omega dx + \varphi(t) \int_{\Omega} |u|^\sigma - 2 u \omega dx. \tag{71}$$

Let the function $\varphi(t)$ satisfy the condition $\varphi(t) \in C^1(0, \infty), |\varphi(t)| \leq N_1, N_1 > 0$.

We will now estimate the right hand side of equality (71)

$$\begin{align*}
 |a_0 \int_{\Omega} \nabla u \nabla \omega dx| &\leq a_0 \|\nabla u\|_{2,\Omega} \|\nabla \omega\|_{2,\Omega} \leq \varepsilon_{22} \|\nabla u\|_{2,\Omega}^2 + \frac{a_0^2}{4\varepsilon_{22}} \|\nabla \omega\|_{2,\Omega}^2, \\
 |f(t) \nabla u|^{p-2} \nabla u \cdot \nabla \omega dx| &\leq (\int_{\Omega} |\nabla u|^{p} dx)^{\frac{p-1}{p}} \left(\int_{\Omega} |\nabla \omega|^p dx\right)^{\frac{1}{p}} \\
 &\leq \varepsilon_{23} \|\nabla u\|_{p,\Omega}^p + C(\varepsilon_{23}) \|\nabla \omega\|_{p,\Omega},
\end{align*}$$

here $C(\varepsilon_{23}) = \frac{1}{\rho(\varepsilon_{23})^{\frac{1}{p-1}}}.$

$$\begin{align*}
 |f(t) \nabla u|^{\sigma-2} u \cdot \omega dx| &\leq (\int_{\Omega} |u|^{\sigma} dx)^{\frac{\sigma-1}{\sigma}} \left(\int_{\Omega} |\nabla \omega|^\sigma dx\right)^{\frac{1}{\sigma}} \\
 &\leq \varepsilon_{25} \|u\|_{\sigma,\Omega}^\sigma + C(\varepsilon_{25}) \|\omega\|_{\sigma,\Omega},
\end{align*}$$

here $C(\varepsilon_{25}) = \frac{1}{\sigma(\varepsilon_{25})^{\frac{1}{\sigma-1}}}.$

Substituting the obtained estimates in (71), we obtain

$$\begin{align*}
 \rho'(t) &+ a_0 \|\nabla u\|_{2,\Omega}^2 + \|u\|_{\sigma,\Omega}^\sigma + \|\nabla u\|_{p,\Omega}^p \leq \varphi(t) \varphi'(t) + \varphi(t) \varepsilon_{22} \|\nabla u\|_{2,\Omega}^2 + \\
 &+ \varphi(t) \frac{a_0^2}{4\varepsilon_{22}} \|\nabla \omega\|_{2,\Omega}^2 + \varphi(t) \varepsilon_{23} \|\nabla u\|_{p,\Omega}^p + \varphi(t) C(\varepsilon_{23}) \|\nabla \omega\|_{p,\Omega} + \\
 &+ \varphi(t) \varepsilon_{25} \|u\|_{\sigma,\Omega}^\sigma + \varphi(t) C(\varepsilon_{25}) \|\omega\|_{\sigma,\Omega}.
\end{align*}$$
Substituting in the last inequality
\[ \varepsilon_{22} = \frac{a_0}{2N_1}, \quad \varepsilon_{23} = \frac{1}{2N_1}, \quad \varepsilon_{25} = \frac{1}{2N_1}, \]
\[ \theta = \frac{a_0N_1}{2} \| \nabla \omega \|_{2,\Omega}^2 + \frac{(2N_1(p-1))^{p-1}}{p^p} \| \nabla \omega \|_{p,\Omega}^p + \frac{(2N_1(\sigma-1))^{\sigma-1}}{\sigma^\sigma} \| \omega \|_{\sigma,\Omega}^\sigma, \]
we obtain
\[ \frac{d}{dt} \left( \| u \|_{2,\Omega}^2 + \chi \| \nabla u \|_{2,\Omega}^2 \right) + a_0 \| \nabla u \|_{2,\Omega}^2 + \| \omega \|_{\sigma,\Omega}^\sigma + \| \nabla u \|_{p,\Omega}^p \leq 2\varphi(t)(\varphi'(t)+\theta). \]  
(72)

The Poincaré inequality gives \( \| \nabla u \|_{2,\Omega}^2 \geq \lambda_1 \| u \|_{2,\Omega}^2 \). Then, we have
\[ \| \nabla u \|_{2,\Omega}^2 = \frac{1}{1+\chi\lambda_1} \| \nabla u \|_{2,\Omega}^2 + \frac{\lambda_1\chi}{1+\chi\lambda_1} \| \nabla u \|_{2,\Omega}^2 \geq \]
\[ \frac{\lambda_1}{1+\chi\lambda_1} \left( \| u \|_{2,\Omega}^2 + \chi \| \nabla u \|_{2,\Omega}^2 \right). \]  
(73)

Using only the second term in the left part (72), we consider the case when
\[ \varphi(t) \leq C_\varphi \exp(-\mu t), \quad \mu \geq C_\lambda, \quad C_\lambda = \frac{a_0\lambda_1}{1+\chi\lambda_1}. \]  
(74)

Using (73) in (72), we obtain the differential inequality for the function \( \rho(t) : \)
\[ \frac{d}{dt} \rho(t) + C_\lambda \rho(t) \leq C_\varphi \exp(-\mu t). \]

From this inequality we obtain the estimates
\[ \rho(t) \leq \exp \left( -C_\lambda t \right) \left( \rho(0) + \frac{C_\varphi}{\mu - C_\lambda} \right), \quad \mu > C_\lambda; \]  
(75)
\[ \rho(t) \leq \exp \left( -C_\lambda t \right) \left( \rho(0) + tC_\varphi \right), \quad \mu = C_\lambda. \]  
(76)

Thus, the following theorem is proved:

**Theorem 5.1.** Let the conditions \( \omega \in L_p(\Omega) \cap W^1_2(\Omega), \nabla \omega \in L_2(\Omega) \cap L_p(\Omega), \sigma > p, p \geq 2, \sigma \geq 2, \int_\Omega u_0 \omega dx = \varphi(0), u_0 \in W^1_2(\Omega) \) and (74) be satisfied. Then the solution of the problem (65)-(68) satisfies the estimate (75) or (76).

6. Localization of the solution in a finite time. We consider the case for the inverse problem (65)-(68), when
\[ 1 < \sigma < 2, \quad p > \frac{2N_1}{N_1+2}, \quad 2\varphi(t)\varphi'(t)+\theta \leq C_\varphi \left( 1 - \frac{t}{\tau} \right)^{\frac{p}{p-1}}, \]
\[ |\varphi'(t)| \leq C \left( 1 - \frac{t}{\tau} \right)^{\frac{p}{p-1}}, \quad \mu \in \left( \frac{1}{2}, 1 \right), \quad (\varphi)_+ = \max(\varphi, 0), \]  
(77)

where
\[ \theta = \frac{a_0N_1}{2} \| \nabla \omega \|_{2,\Omega}^2 + \frac{(2N_1(p-1))^{p-1}}{p^p} \| \nabla \omega \|_{p,\Omega}^p + \frac{(2N_1(\sigma-1))^{\sigma-1}}{\sigma^\sigma} \| \omega \|_{\sigma,\Omega}^\sigma. \]

From the embedding theorem (see . [11], Lemma 3.2, p.296), the following inequalities follow
\[ \| u \|_{2,\Omega} \leq C \| \nabla u \|_{p,\Omega}^\delta \| u \|_{\sigma,\Omega}^{(1-\delta)} \leq C \left( \| \nabla u \|_{p,\Omega}^\delta + \| u \|_{\sigma,\Omega}^\delta \right)^{\frac{1-\delta}{\delta}}. \]
\[ \delta = \frac{2-\sigma}{\sigma p - \frac{2p}{Np - \sigma(N-2)}} \in (0,1), \quad 1 < \sigma < 2, \quad p > \frac{2N}{N+2}. \] 

The latter leads to the following

\[ C_1 \left( \|u\|_{2, \Omega}^2 \right)^\mu \leq \|\nabla u\|_{p, \Omega}^p + \|u\|_{\sigma, \Omega}^\sigma, \quad (78) \]

where \( \mu = \frac{p\sigma}{2\delta\sigma + 2p(1-\sigma)} \leq 1 \iff p\sigma < 2\delta\sigma + 2p(1-\delta), \quad 0 < \sigma < 2. \) In the differential inequality (72), we apply (78), then we obtain

\[ \frac{d}{dt} \left( \|u\|_{2, \Omega}^2 + \chi \|\nabla u\|_{2, \Omega}^2 \right) + a_0 \|\nabla u\|_{2, \Omega}^2 + C_1 \left( \|u\|_{2, \Omega}^2 \right)^\mu \leq 2\varphi(t)(\varphi'(t) + \theta). \quad (79) \]

We will apply the following inequality

\[ \mu \frac{d}{dt} (a^\mu + b^\mu) \leq a + b^\mu, \quad 0 < \mu < 1, \quad N \geq 1, \]

which is true for any \( a \geq \mu b \), \( 0 > 0 \) and \( b \geq 0 \).

Then taking into account (77), the inequality (79) can be written as

\[ \frac{d}{dt} \left( \|u\|_{2, \Omega}^2 + \chi \|\nabla u\|_{2, \Omega}^2 \right) + C_2 \left( \|u\|_{2, \Omega}^2 + \chi \|\nabla u\|_{2, \Omega}^2 \right)^\mu \leq 2\varphi(t)(\varphi'(t) + \theta), \]

\[ \frac{d}{dt} \rho(t) + C_2 [\rho(t)]^\mu \leq C_\varphi \left( 1 - \frac{t}{t_\varphi} \right)^{\frac{1}{1-\mu}}. \quad (80) \]

If the condition

\[ C_2 [\rho(0)]^\mu - \frac{\rho(0)}{t_\varphi(1-\mu)} \leq C_\varphi \]

is met, then by the formula (1.28) in ([11]) any non-negative solution of inequality (80) satisfies the estimation

\[ \rho(t) \leq \rho(0) \left( 1 - \frac{t}{t_\varphi} \right)^{\frac{1}{1-\mu}}. \]

Thus, the following theorem is proved:

**Theorem 6.1.** Let the conditions (77), \( \omega \in L_\sigma(\Omega) \cap W_2^0(\Omega), \quad \nabla \omega \in L_2(\Omega) \cap L_p(\Omega), \)

\[ \int_{\Omega} u_0 \cdot \omega dx = \varphi(0), \quad u_0 \in W_2^1(\Omega) \) and (81) be satisfied. Then the solution of the inverse problem (65)-(68) vanishes in a finite time \( t_\varphi. \)

**Remark 2.** Note that the condition claimed in (81) connects three parameters which characterize the problem: the instant \( t_\varphi \) of vanishing of the source, the source intensity \( C_\varphi, \) and the initial value \( \rho(0). \) For this reason, given an arbitrary intensity \( t_f, \) the effect of vanishing of the solution can be provided by an appropriate choice of \( C_\varphi \) and \( \rho(0). \)

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