Equations of motion in Double Field Theory: from classical particles to quantum cosmology

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Abstract
The equation of motion for a point particle in the background field of double field theory is considered. We find that the motion is described by a geodesic flow in the doubled geometry. Inspired by analysis on the particle motion, we propose a modified model of quantum string cosmology, which includes two scale factors. The report is based on Phys. Rev. D84 (2011) 124049 [arXiv:1108.5795].

1 Introduction
Double Field Theory (DFT)\textsuperscript{1} is the theory of the massless field with a higher symmetry of spacetime including dual coordinates. Through this theory, Hull, Zwiebach, and Hohm clarified the T-duality symmetry of the massless field, and new symmetry related to the theory. More recently Jeon, Lee, and Park studied the structure of DFT using projection-compatible differential geometrical methods in\textsuperscript{2}.

The present report consists of two parts. In the first part, the motion of the particle in the background field in DFT is investigated. We show that the geodesic in the $2D$-dimensional doubled-spacetime cannot be the geodesic in the $D$-dimensional spacetime. The geodesic equation in the $D$-dimensional spacetime is found to be the geodesic flow equation. In the second part, we consider the string cosmology with a bimetric model inspired by the constraint method discussed in the first part. Our method for the restriction on the metrics functions well, at least in the present reduced model for cosmology.

2 Review of projection-compatible approach
Coordinates are combined with dual coordinates to be $X^A = (\tilde{x}_a, x^\mu)^T$, where the suffixes $A, B, \ldots$ range over $1, 2, \ldots, 2D$, while $\mu, \nu, \ldots$ as well as $a, b, \ldots$ range over $1, 2, \ldots, D$. The constant metric is assumed to be expressed as the following $2D \times 2D$ matrix,

$$\eta_{AB} = \begin{pmatrix} 0 & \delta_{a\nu}^b \\ \delta_{\mu^b}^a & 0 \end{pmatrix}. \tag{1}$$

The suffixes are entirely raised and lowered by this constant metric. Of course, $\eta^{AC} \eta_{CB} = \delta^A_B$ is satisfied.

The generalized metric is defined as follows:

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{ab} & -g^{a\sigma}b_{\sigma\nu} \\ b_{\mu\sigma}g^{\sigma\nu} & g_{\mu\nu} - b_{\mu\rho}g^{\rho\sigma}b_{\sigma\nu} \end{pmatrix}. \tag{2}$$

Here, $g_{\mu\nu}$ and $b_{\mu\nu}$ are the metric in $D$ dimensions and the antisymmetric tensor, respectively. It should be noted that $\mathcal{H}^{AB}$ satisfies $\mathcal{H}^{AC} \mathcal{H}_{CB} = \delta^A_B$.

The following projection matrices are defined on the basis of the existence of two kinds of metrics. $P \equiv \frac{1}{2} (\eta + \mathcal{H})$, $\tilde{P} \equiv \frac{1}{2} (\eta - \mathcal{H})$ which satisfy $P^2 = P$, $\tilde{P}^2 = \tilde{P}$, $P \tilde{P} = \tilde{P} P = 0$. From these, one can derive the identities $P(\partial_A P) = P(\partial_A \tilde{P}) = 0$ or $P_B^D (\partial_A \mathcal{H}_{BC}) P^C_E = \tilde{P}_D^B (\partial_A \mathcal{H}_{BC}) P^C_E = 0$.

Now, the projection-compatible derivative is defined. In other words, both the metrics are “covariantly constant,” i.e., $\nabla_A \eta_{BC} = \nabla_A \tilde{H}_{BC} = 0$. So, the covariant derivative of the projection of an arbitrary...
where the three-form field

\[ U \]  

expression

\[ A \]  

is an undecided multiplier. The Euler-Lagrange equation leads to the constraint \( \bar{\dot{H}} = 0 \), where \( \bar{\dot{H}} \) is defined as

\[ \bar{\dot{H}} = \frac{1}{2} \mathcal{H}_{AB} \mathcal{X}^A \mathcal{X}^B. \]  

The multiplier can be determined from the Hamilton equation as \( \lambda^A = p_A + \lambda^G \mathcal{P}_{CA} p_B - \lambda^D \mathcal{P}_{DB} \).

The following Lagrangian is adopted, and the mechanics derived from it are considered:

\[ \Gamma_{ABC} = 2P_{[A}^{\ D} P_{B]}^{\ E} \partial_C P_{DE} + 2(\bar{P}_{[A}^{\ D} P_{B]}^{\ E} - P_{[A}^{\ D} P_{B]}^{\ E}) \partial_D P_{EC}. \]  

They also obtained the action for the generalized metric, which was previously found by Hohm et al. \[ 2 \] found that the action coincides with the projection of the covariant derivative of the tensor. Jeon et al.\[ 2 \] found that the projection has a problem. It is obviously different from the usual geodesic equation in general relativity (or differential geometry).

In general, it is understood that the usage of the projection is not the equation of motion for a particle. The geodesic equation is given by the following

\[ \mathcal{L} = \frac{1}{2} \sqrt{-g} \epsilon^{-2\phi} \left[ R + 4(\partial \phi)^2 - \frac{1}{12} \mathcal{H}^2 \right], \]

where the three-form field \( H = db \) is the field strength of the Kalb-Ramond 2-form \( b_{ij} \).

3 The geodesic equation is not the equation of motion for a particle

Next, we consider the equation of motion for a particle. The geodesic equation is given by the following expression \( \bar{U}^\mu \nabla_\mu U^\nu = 0 \), where \( U^\mu = \frac{dx^\mu}{ds} \), \( s \) being a parameter.

The corresponding equation in the projection compatible geometry of Jeon et al. is considered to be

\[ U^A \nabla_A U^B = U^A (\partial_A U^B + \Gamma_{ABC} U^C) = 0, \]

where \( U^A = (\bar{U}_a, U^\nu)^T = \frac{dX^A}{ds} \). From the project space ansatz \( \mathcal{P} U = 0 \), we are forced to use \( \bar{U}_a = g_{a\nu} U^\nu \).

Moreover, we set \( \bar{\dot{g}} = 0 \) as in the interpretation of DFT. Now, we find that the above equation reads

\[ U^\mu \partial_\mu U^\nu + \frac{1}{2} \bar{g}^{\mu\nu} (\partial_\mu g_{\nu\sigma} + \partial_\sigma g_{\nu\mu}) U^\mu U^\sigma = 0. \]  

It is obviously different from the usual geodesic equation in general relativity (or differential geometry). In general, it is understood that the usage of the projection has a problem.

4 Projection and geodesic flow

The following Lagrangian is adopted, and the mechanics derived from it are considered:

\[ L = \frac{1}{2} \mathcal{H}_{AB} \mathcal{X}^A \mathcal{X}^B + \lambda^A \mathcal{P}_{AB} \mathcal{X}^B. \]  

Here, \( \lambda^A \) is an undecided multiplier. The Euler-Lagrange equation leads to the constraint \( \mathcal{P} \mathcal{X} = 0 \).

We find that the Hamiltonian is defined as

\[ H = \frac{1}{2} \mathcal{H}_{AB} (p_A - \lambda^C \mathcal{P}_{CA}) (p_B - \lambda^D \mathcal{P}_{DB}). \]

The multiplier can be determined from the Hamilton equation as \( \lambda_A = p_A + \lambda^B M^B \), where \( M^B \) is an arbitrary vector. When this is substituted into the above Hamiltonian, we obtain a new Hamiltonian

\[ H_\ast = \frac{1}{2} \mathcal{P}_{AB} p_A p_B. \]  

Using the new Hamiltonian, we obtain

\[ \dot{\mathcal{X}}^A = \frac{\partial H_\ast}{\partial p_A} = \mathcal{P}_{AB} p_B, \quad \dot{p}_A = -\frac{\partial H_\ast}{\partial \mathcal{X}^A} = -\frac{1}{2} \partial_A \mathcal{P}^{BC} p_B p_C = -\frac{1}{4} \partial_A \mathcal{H}^{BC} p_B p_C. \]
These equations describe the geodesic flow in the system. The combined equation is found to be

$$\ddot{X}^A = \dot{X}^C (\partial_C D^{AB}) p_B - \frac{1}{2} P^{AB} \partial_B P^{CD} p_C p_D .$$  \hspace{1cm} (10)

Now, let us take the condition $\ddot{\tilde{p}}^a = 0$ for the correspondence with DFT. If we consider $\tilde{p}^a = 0$, we obtain $\ddot{x}^\mu + \left( \frac{\mu}{\lambda_s} \right) \tilde{p}^a \ddot{x}^a = 0$, the geodesic equation in a usual $D$-dimensional spacetime. We have obtained the geodesic equation in the $D$-dimensional spacetime from the geodesic flow in the 2D-dimensional space described by the generalized metric with natural assumptions.

5 A simple bi-metric model

We apply a similar method to a modified model for cosmology, which is related to the string cosmology [3]. In the model here, we consider two metrics, $g$ and $\tilde{g}$. Though our model describes a bi-metric theory, the degree of freedom is to be mildly restricted. For simplicity of the discussion, we consider $b_{\mu \nu} = 0$. The cosmological action we consider is

$$S = -\frac{\lambda_s}{2} \int \sqrt{-g} \left[ \frac{1}{8} \mathrm{Tr}(M'\eta M'\eta) + \Phi^2 + e^{-2\Phi} V \right] , \quad \text{with} \quad M_{AB} \equiv \left( \frac{\tilde{G}}{1} \frac{G}{1} \right) ,$$  \hspace{1cm} (11)

where $G$ and $\tilde{G}$ are the spatial parts of the metrics and $\Phi \equiv 2d$. Here, we add the constant potential $V$ to the Lagrangian and $\lambda_s$ is the constant that represents the scale of string theory [3]. The prime denotes differentiation with respect to $\tau$.

We now define the “pseudo”-projection matrices $P = \frac{n+M}{2}$, $\tilde{P} = \frac{n-M}{2}$ and we wish to enforce $P M' P = \tilde{P} M' P = 0$ using some constraints. Now, the Lagrangian $L_A$ with the constraint term is

$$L_A = \frac{\lambda_s}{2} \left[ -\frac{1}{8} M^{AB} M'_{AB} + \tilde{A}_{AB} \tilde{P}^{AC} M'_{CD} \tilde{P}^{DB} + \Lambda_{AB} P^{AC} M'_{CD} D^{DB} - \Phi^2 - e^{-2\Phi} V \right].$$  \hspace{1cm} (12)

The Hamiltonian of the system becomes

$$H_A = \frac{4}{\lambda_s} \left[ \Pi^{AB} - \frac{\lambda_s}{2} \left( \tilde{P}^{AC} \tilde{A}_{CD} \tilde{P}^{DB} + P^{AC} \Lambda_{CD} D^{DB} \right) \right]^2 - \frac{1}{2\lambda_s} \Pi^{\Phi} + \frac{\lambda_s}{2} e^{-2\Phi} V .$$  \hspace{1cm} (13)

where the conjugate momentum for $M_{AB}$ and $\Phi$ are represented by $\Pi^{AB}$ and $\Pi_{\Phi}$, respectively. We consider simplification by using the assumed relation, $P^2 \simeq P$ and $\tilde{P}^2 \simeq \tilde{P}$. The symbol $\simeq$ is used to indicate this assumed approximation adopted by us. Finally, we obtain the Hamiltonian

$$H_s \equiv -\frac{8}{\lambda_s} \Pi^{AB} M_{BC} D^{CD} P_{DA} - \frac{1}{2\lambda_s} \Pi_{\Phi} + \frac{\lambda_s}{2} e^{-2\Phi} V .$$  \hspace{1cm} (14)

6 “Minisuperspace” version of the bi-metric model

Next, we examine the previous procedure of modification in the minisuperspace model. We suppose

$$M_{AB} = \left( \begin{array}{cc} \tilde{A}(\tau) \delta^{ab} & 0 \\ 0 & A(\tau) \delta_{\mu \nu} \end{array} \right) .$$  \hspace{1cm} (15)

The Hamiltonian for the minisuperspace version of our modified model is found to be

$$H_s = -\frac{2}{\lambda_s D} \left( \pi_{\tilde{\pi}} + \pi_{\pi} - \tilde{A} \pi A \pi - \tilde{A} \tilde{\pi} \tilde{A} \pi \right) - \frac{1}{2\lambda_s} \Pi^{\Phi} + \frac{\lambda_s}{2} e^{-2\Phi} V .$$  \hspace{1cm} (16)

A special solution can be found for the Hamilton equations. The solution is

$$\tilde{A}(\tau) = \frac{1}{A_0} \exp \left[ -\frac{2}{\sqrt{D}} C(\tau - \tau_0) \right] , \quad A(\tau) = A_0 \exp \left[ \frac{2}{\sqrt{D}} C(\tau - \tau_0) \right] + \delta ,$$  \hspace{1cm} (17)

where $A_0$, $\tau_0$, and $\delta$ are constants. The similarity to the known string cosmological solution [3] is obvious, up to the possible constant deviation $\delta$ in $A$. For the solution, we find that $A \tilde{A} \rightarrow 1$ when $\tau \rightarrow +\infty$. 

Quantum cosmology of the bi-metric model

Quantum cosmological treatment of the string cosmology has been widely studied [3]. In our model, the minisuperspace Wheeler-DeWitt equation is obtained as

$$\left[ \frac{2}{\lambda_s D} \left( 2 \frac{\partial}{\partial A} \frac{\partial}{\partial \tilde{A}} - A \frac{\partial}{\partial A} A \frac{\partial}{\partial A} - \tilde{A} \frac{\partial}{\partial \tilde{A}} \tilde{A} \frac{\partial}{\partial \tilde{A}} \right) + \frac{1}{2 \lambda_s} \frac{\partial^2}{\partial \Phi^2} + \frac{\lambda_s}{2} e^{-2\Phi} V \right] \Psi = 0, \quad (18)$$

where $\Psi$ is the wave function of the universe. To simplify the description of the system, we use the following variables:

$$x = \sqrt{D} \ln \frac{A}{\tilde{A}}, \quad y = \sqrt{D} \ln A \frac{\partial}{\partial \tilde{A}}. \quad \text{Up to the ordering, we have}$$

$$\left[ -1 + e^{-4/\sqrt{D}}y^2 \frac{\partial^2}{\partial x^2} - \frac{1}{2} e^{-4/\sqrt{D}}y \frac{\partial}{\partial y} + \frac{\partial^2}{\partial \Phi^2} + \lambda_s^2 e^{-2\Phi} V \right] \Psi = 0. \quad (19)$$

If we assume a solution of the form, $\Psi(x, y, \Phi) = X_k(x)Y_kK(y)Z_K(\Phi)$, we find the non singular real solution for $Y_kK(y)$ at $y = 0$ as follows:

$$Y_{kK}(y) = e^{-\sqrt{k^2 - 2K^2}y} F\left(\frac{1 + \sqrt{1 - k^2} + \sqrt{k^2 - 2K^2}}{2}, \frac{1 - \sqrt{1 - k^2} + \sqrt{k^2 - 2K^2}}{2}, 1; 1 - e^{-2y}\right), \quad (20)$$

where $F(\alpha, \beta; \gamma; z)$ is the Gauss’ hypergeometric function.

If $K = \pm k$, $Y_{k \pm k}(y)$ has a maximum at $y = 0$ (see Figure 1). When we construct a wave packet for the cosmological wave function [3], the peak of this wave packet in terms of parameter $y$ is naturally located at $y = 0$. Thus, the approximate scale factor duality $A\tilde{A} \simeq 1$ is expected even at the “beginning” of the quantum universe. The detailed investigation on the behavior of the universe is left for future research.

![Figure 1: 3D-plot of $Y_{kk}(y)$.](image)

References

[1] C. Hull and B. Zwiebach, JHEP 0909 (2009) 099; JHEP 0909 (2009) 090.
[2] I. Jeon, K. Lee and J.-H. Park, JHEP 1104 (2011) 014.
[3] For a review, M. Gasperini and G. Veneziano, Phys. Rep. 373 (2003) 1.