Spinning gauged boson stars in anti-de Sitter spacetime

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Abstract

We study axially symmetric solutions of the Einstein-Maxwell-Klein-Gordon equations describing spinning gauged boson stars in a 3+1 dimensional asymptotically AdS spacetime. These smooth horizonless solutions possess an electric charge and a magnetic dipole moment, their angular momentum being proportional to the electric charge. A special class of solutions with a self-interacting scalar field, corresponding to static axially symmetric solitons with a nonzero magnetic dipole moment, is also investigated.

1 Introduction

Recently there has been a lot of interest in solutions of the general relativity with a negative cosmological constant $\Lambda$ coupled to a Maxwell field and a charged scalar with mass $M$ and gauge coupling constant $q$. Working in four spacetime dimensions, this system is usually described by the action

$$I = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} g^{\mu\nu} (D_\mu \psi^* D_\nu \psi + D_\nu \psi^* D_\mu \psi) - U(|\psi|) \right], \quad (1.1)$$

where $G$ is the gravitational constant, $R$ is the Ricci scalar associated with the spacetime metric $g_{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the U(1) field strength tensor and $D_\mu \psi = \partial_\mu \psi + iqA_\mu \psi$ is the covariant derivative. $U(|\psi|)$ denotes the potential of the scalar field $\psi$, whose mass is defined by $M^2 = \frac{1}{2} \frac{\partial^2 U}{\partial |\psi|^2} \big|_{\psi=0}$.

Although (1.1) does not correspond to a consistent truncation of a more fundamental theory (unless $\psi$ is vanishing), it can be viewed however, as a simple toy model for the charged scalar dynamics of systems that appear in concrete examples of the AdS/CFT correspondence. The interest in this system (sometimes called the gravitating Abelian Higgs model) has been boosted due to Gubser’s observation [1] that the Einstein-Maxwell black holes can become unstable to forming scalar hair at low temperatures. This instability results in new branches of solutions with scalar hair, which are thermodynamically favoured over the Reissner-Nordström-AdS black holes. The charged black branes which are asymptotically
AdS in a Poincaré coordinate patch are of main interest, their physics being recently exploited to obtain a dual gravitational description of important phenomena in condensed matter physics (in particular superconductivity and phase transitions) in a three dimensional flat spacetime $R_t \times R^2$. A discussion of these aspects can be found e.g. in [2], together with a large set of references.

However, another natural arena to study solutions of the model (1.1) is to consider instead a globally AdS background, with a line element $ds^2 = -(1+r^2\ell^2)dt^2 + \frac{dr^2}{1+r^2\ell^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$, (where $t$ is the time coordinate and $r, \theta, \varphi$ are the spherical coordinates with the usual range and $\Lambda = -3/\ell^2$), the conformal boundary being a static Einstein universe $R_t \times S^2$. In this case, the simplest nonvacuum solutions of the model (1.1) are the boson stars with a vanishing gauge field, $F_{\mu\nu} = 0$. These are smooth horizonless configurations representing gravitational bound states of a complex scalar field with a harmonic time dependence, which provide us with the simplest model of a relativistic star\(^1\). Such solutions have been extensively studied for $\Lambda = 0$, i.e. an asymptotically flat spacetime background, starting with the early work of Kaup [3] and Ruffini and Bonazzola [4]. They have found interesting physical applications, being proposed as candidates for dark matter halos and as dark alternatives to astrophysical black hole candidates; also, they may help explain galaxy rotation curves – see the review work [5].

The boson starts with AdS asymptotics are believed to play an important role in holographic gauge theories through the AdS/CFT correspondence\(^2\). While the early studies restricted to spherically symmetric configurations [7], [8] (see also [9, 10]) recently there has been some progress on including the effects of rotation. Spinning AdS boson stars in $d = 4$ spacetime dimensions have been discussed in [11]. These stationary localized configurations possess a finite mass and angular momentum, their angular momentum being quantized, $J = nQ$ (with $n$ an integer and $Q$ the Noether charge), the energy density exhibiting a toroidal distribution. A particular set of higher dimensional\(^3\) spinning boson stars possessing equal angular momenta has been considered in [12], [13], for $d = 2k+1 \geq 5$ and a special multiplet scalar fields ansatz [14].

An interesting question to address in this context is the issue of rotating horizonless solutions within the full model (1.1), and, in particular, how an electric charge would affect their properties. For $\Lambda = 0$, this problem has been addressed in Ref. [16], which gave numerical evidence for the existence of spinning gauged boson stars in a Minkowski spacetime background. Similar to a Kerr-Newman black hole, these solutions possess a nonzero electric charge and a magnetic dipole moment. However, their pattern is rather similar to that of the ungauged boson stars discussed in [17], [18], [19]; in particular, one finds again a limited range for the allowed scalar field frequencies.

In this work we show that the spinning gauged boson stars in [16] can be generalized for

\(^1\)One should remark that in contrast to ordinary stars or neutron stars, generically the boson stars do not display a sharp edge. In this case, the matter is not confined in a finite region of space and the boson stars possess only an effective radius [5] (see however, the compact boson stars in [6]).

\(^2\)However, note that the boson starts do not have natural counterparts in a Poincaré coordinate patch.

\(^3\)For completeness, we mention that spinning boson stars with an AdS\(_3\) background have been studied in [15]. However, these solutions have rather special properties.
an AdS background. In some sense, these globally regular configurations can be regarded as regularized Kerr-Newman-AdS solutions, the event horizon and the singularity disappearing due to the supplementary interaction with a complex scalar field. These gauged boson stars can also be viewed as axially symmetric generalizations of the spherically symmetric gauged solutions in [8], [20], [21], [22]. Similar to that case, our results show that they exist also up to a maximal value of the gauge coupling constant. Their angular momentum is quantized, being proportional to the electric charge.

This paper is organized as follows. In the next Section we formulate a numerical approach of the problem based on a specific ansatz. The numerical results are given in Section 3, where we exhibit the physical properties of the gauged spinning boson star solutions. We conclude in Section 4 with some further remarks.

2 The problem

2.1 The equations and the ansatz

The configurations investigated in this work are solutions of the coupled Einstein-Maxwell-scalar field equations

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} - 8\pi GT_{\mu\nu} = 0,$$  \hspace{0.5cm} (2.2)

$$D_\mu D^\mu \psi = \frac{\partial U}{\partial |\psi|^2} \psi, \quad \nabla_\mu F^{\mu\nu} = iq[(D^{\nu}\psi^*)\psi - \psi^*(D^{\nu}\psi)] \equiv qj^{\nu},$$  \hspace{0.5cm} (2.3)

where $T_{\mu\nu}$ is the stress-energy tensor

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$  \hspace{0.5cm} (2.4)

$$+ (D_\mu \psi^* D_\nu \psi + D_\nu \psi^* D_\mu \psi) - g_{\mu\nu} \left[ \frac{1}{2}g^{\alpha\beta}(D_\alpha \psi^* D_\beta \psi + D_\beta \psi^* D_\alpha \psi) + U(|\psi|) \right].$$

This model is invariant under the local U(1) gauge transformation

$$\psi \rightarrow \psi e^{-iqa}, \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha,$$  \hspace{0.5cm} (2.5)

with $\alpha$ a real function.

We are interested in stationary axially symmetric configurations, with a spacetime geometry admitting two Killing vectors $\partial_t$ and $\partial_\varphi$, in a system of adapted coordinates. Then the most general line element is written as $ds^2 = G_{tt}(x)dt^2 + 2G_{t\varphi}(x)dtd\varphi + G_{\varphi\varphi}(x)d\varphi^2 + h_{ij}(x)dx^i dx^j$, with $x^i = (r, \theta)$. In the numerics, it is convenient to choose a metric gauge with $(1 + \frac{r^2}{\ell^2})h_{rr} = h_{\theta\theta}/r^2$ and $h_{r\theta} = 0$. This leads to a metric ansatz with four unknown functions, a convenient form being

$$ds^2 = -F_0(r, \theta)(1 + \frac{r^2}{\ell^2})dt^2 + F_1(r, \theta) \left( \frac{dr^2}{1+\frac{r^2}{\ell^2}} + r^2 d\theta^2 \right) + F_2(r, \theta) r^2 \sin^2 \theta \left( d\varphi - \frac{W(r, \theta)}{r} dt \right)^2,$$  \hspace{0.5cm} (2.6)
with \((F_i, W)\) (where \(i = 0, 1, 2\)) being smooth in \(r, \theta\).

For the scalar field, we adopt the stationary ansatz \([17], [18], [19]\):

\[
\psi(t, r, \theta, \varphi) = \phi(r, \theta) e^{i(n \varphi - \omega t)}, \tag{2.7}
\]

where \(\phi(r, \theta)\) is a real function, and \(\omega\) and \(n\) are real constants. Single-valuedness of the scalar field requires \(\psi(\varphi) = \psi(2\pi + \varphi)\); thus the constant \(n\) must be an integer, i.e., \(n = 0, \pm 1, \pm 2, \ldots\). In what follows, we shall take \(n \geq 0\) and \(\omega \geq 0\), without any loss of generality.

A consistent ansatz for the U(1) gauge field reads

\[
A = A_\mu dx^\mu = A_t(r, \theta) dt + A_\varphi(r, \theta) \sin \theta (d\varphi - \frac{W}{r} dt). \tag{2.8}
\]

Substituting (2.6), (2.7), (2.8) in the field equations (2.2), (2.3) results\(^4\) in a set of seven coupled non-linear PDEs of the form \(\nabla^2 F_a = J_a\) where \(F_a = (F_0, F_1, F_2, W; \phi, A_\varphi, A_t)\), \(J_a\) are ‘source’ terms depending on the functions \(F_a\) and their first derivatives, while \(\nabla^2\) is the Laplace operator associated with the auxiliary space \(d\sigma^2 = dr^2/(1 + r^2) + r^2 d\theta^2\).

Solutions of this model with a vanishing gauge field \(A_t = A_\varphi = 0\) have been discussed in [11], generalizing for the AdS spacetime the asymptotically flat rotating boson stars in \([17], [18], [19]\). Note that, in contrast to that case, the \((t, \varphi)\)-dependence of the scalar field \(\psi\) can now be gauged away by applying the local U(1) symmetry (2.5) with \(\alpha = (n \varphi - \omega t)/q\). However, this would also change the gauge field as \(A_t \to A_t - \omega/q\), \(A_\varphi \to A_\varphi + n/q\), so that it would become singular in the \(q \to 0\) limit. Therefore, in order to be able to consider this limit, we prefer to keep the \((t, \varphi)\)-dependence in the scalar field ansatz and to fix the corresponding gauge freedom by setting \(A_t = A_\varphi = 0\) at infinity.

We note also that the spherically symmetric limit is found for \(n = 0\), in which case the functions \(F_0, F_1, F_2\) and \(\phi, A_t\) depend only on \(r\), with \(F_1 = F_2\) and \(W = A_\varphi = 0\).

### 2.2 The asymptotics and boundary conditions

We are interested in horizonless, particle-like solutions of the equations (2.2), (2.3) within the ansatz (2.6), (2.7), (2.8), approaching at infinity the globally AdS background. Since this problems does not seem to possess closed form solutions, the field equations are solved numerically with suitable boundary conditions. These conditions result from a study of an approximate form of the solutions on the boundaries of the domain of integration, compatible with regularity and AdS asymptotics requirements.

For small values of \(r\), the solutions possess a power series on the form \(F_k = F_{k0} + O(r^2)\) (with \(k = 0, 1, 2\) and \(F_{10} = F_{20}\)), \(W = O(r^2)\), \(\phi = O(r^{2n})\), \(A_\varphi = O(r^2)\) and \(A_t = V + O(r^2)\). This leads to the following boundary conditions at the origin:

\[
\partial_r F_i|_{r=0} = W|_{r=0} = 0, \quad \phi|_{r=0} = 0, \quad \partial_r A_t|_{r=0} = A_\varphi|_{r=0} = 0. \tag{2.9}
\]

\(^4\)Note that the Einstein equations \(E^\mu_\nu = 0\), \(E^\nu_\nu - E^0_0 = 0\) are not automatically satisfied, yielding two constraints. However, following [23], one can show that these constraints are satisfied as a consequence of the identities \(E^\mu_\nu = 0\) plus the set of chosen boundary conditions. In practice, the constraint equations \(E^\mu_\nu\) and \(E^\nu_\nu - E^0_0\) are used to monitor the numerical accuracy of the solutions.
At infinity, the AdS background is approached, while the scalar field $\phi$ and the gauge potential $A_\mu$ vanish. Restricting to the case $M^2 \geq 0$, the matter fields decay asymptotically as

$$\phi \sim \frac{c_1(\theta)}{r^\Delta} + \ldots, \quad A_t \sim \frac{Q_e}{r} + \ldots, \quad A_\varphi \sim \frac{\mu \sin \theta}{r} + \ldots,$$

where $Q_e$, $\mu$ and $c_1(\theta)$ result from numerics, while $\Delta = \frac{3}{2} \left(1 + \sqrt{1 + \frac{4}{9} M^2 \ell^2}\right)$. Then the Einstein equations imply the following form of the metric functions as $r \to \infty$

$$F_0 = 1 + \frac{f_{03}(\theta)}{r^3} + \ldots, \quad F_1 = 1 + \frac{f_{13}(\theta)}{r^3} + \ldots, \quad F_2 = 1 + \frac{f_{23}(\theta)}{r^3} + \ldots, \quad W = \frac{w_2(\theta)}{r^2} + \ldots,$$

in terms of two functions $f_{13}(\theta)$ and $w_2(\theta)$ which result from the numerics, with $f_{03}(\theta) = -3f_{13}(\theta) - \frac{4}{3} \tan \theta f'_{13}(\theta)$, and $f_{23}(\theta) = f_{13}(\theta) + \frac{4}{3} \tan \theta f'_{13}(\theta)$. Then the corresponding boundary conditions employed in the numerics are

$$F_i|_{r \to \infty} = 1, \quad W|_{r \to \infty} = 0, \quad \phi|_{r \to \infty} = A_t|_{r \to \infty} = A_\varphi|_{r \to \infty} = 0.$$  

(2.12)

For $\theta = 0$, the study of an approximate solution of the equations (2.2), (2.3) implies the behaviour $F_k = \tilde{F}_k(\theta) + O(\theta^2)$, $W = \tilde{w}_0(\theta) + O(\theta^2)$, $\phi = O(\theta^2n)$, $A_\varphi = O(\theta)$ and $A_t = A_t^0(r) + O(\theta^2)$. A similar expansion holds also for $\theta = \pi$, with $\theta \to \pi - \theta$. Thus we require the boundary conditions

$$\partial_\theta F_i|_{\theta=0,\pi} = \partial_\theta W|_{\theta=0,\pi} = 0, \quad \phi|_{\theta=0,\pi} = \partial_\theta A_t|_{\theta=0,\pi} = A_\varphi|_{\theta=0,\pi} = 0.$$  

(2.13)

The absence of conical singularities imposes on the symmetry axis the supplementary condition $F_1|_{\theta=0,\pi} = F_2|_{\theta=0,\pi}$, which is used to verify the accuracy of the solutions.

Also, all solutions in this work are invariant under the parity transformation $\theta \to \pi - \theta$. We make use of this symmetry to integrate the equations for $0 \leq \theta \leq \pi/2$ only, the following boundary conditions being imposed in the equatorial plane

$$\partial_\theta F_i|_{\theta=\pi/2} = \partial_\theta W|_{\theta=\pi/2} = 0, \quad \partial_\theta \phi|_{\theta=\pi/2} = \partial_\theta A_t|_{\theta=\pi/2} = \partial_\theta A_\varphi|_{\theta=\pi/2} = 0.$$  

(2.14)

### 2.3 The global charges

The charged boson star solutions possess two global charges associated with the asymptotic Killing vectors $\partial_t$ and $\partial_\varphi$ of the metric, which are the mass-energy $J$ and angular momentum $I$. To compute these quantities, we employ first the quasilocal formalism in [25]. In this approach, the action (1.1) is supplemented with the Gibbons-Hawking surface term [26] and a boundary counterterm $I_{ct} = -\frac{1}{8\pi G} \int_{\partial M} d^2x \sqrt{-h} \left(\frac{\ell}{2} + \frac{\ell}{2} R\right)$ (where $R$ is the Ricci scalar for the boundary metric $h$). Then the variation of the total action with respect to $h_{ab}$ results [25] in the boundary stress tensor $T_{ab} = \frac{1}{8\pi G} (K_{ab} - Kh_{ab} - \frac{2}{3} \ell h_{ab} + \ell E_{ab})$ (where $E_{ab}$ is the

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$^5$Without any loss of generality, we suppose $U(\phi) \to 0$ as $\phi \to 0$.

$^6$However, gauged boson stars with odd parity with respect to a reflection in the equatorial plane should also exist. For $\Lambda = 0$ and a vanishing gauge field, such configurations have been studied in [24].
Einstein tensor of the boundary metric, \( K_{ab} \) is the extrinsic curvature tensor of the boundary and \( K = h_{ab} K^{ab} \). In this approach, the mass-energy and angular momentum are computed from \( T_{ab} \), being regarded as conserved charges associated with the Killing vectors \( \partial/\partial t \) and \( \partial/\partial \varphi \) of the boundary metric. For the metric ansatz (2.6) (with the asymptotics (2.11)), a straightforward computation leads to the following expressions of these quantities

\[
E = \frac{1}{8G\ell^2} \int_0^\pi d\theta \sin \theta \left( 5f_{13}(\theta) + 3f_{23}(\theta) \right), \quad J = -\frac{3}{8G} \int_0^\pi d\theta \sin^3 \theta \ w_2(\theta).
\]

(2.15)

A similar result is found when using instead the Ashtekar-Magnon-Das formalism in [27]. Moreover, the same expression for the angular momentum is found from the Komar integral,

\[
J = \frac{1}{8\pi G} \int R^t_{\varphi} \sqrt{-g} drd\theta d\varphi.
\]

Apart from \( E \) and \( J \), the system possesses two global charges associated with the matter fields. These are the Noether charge (i.e. the total particle number)

\[
Q = \int j^t \sqrt{-g} drd\theta d\varphi = 2\pi \int_0^\infty dr \int_0^\pi d\theta \ 2r^2 \sin \theta F_1 \sqrt{\frac{F_2}{F_0}} \frac{\phi^2}{2} \ (\omega - qA_t - \frac{nW}{r}),
\]

(2.16)

and the electric charge \( Q_e \), which is read from the asymptotics of the electric potential \( A_t \) as given in (2.10). However, a straightforward computation shows that both the Noether charge and the electric charge of the spinning solutions are proportional\(^7\) to the total angular momentum,

\[
J = nQ = \frac{Q_e n}{q}.
\]

(2.17)

These configurations possess also a magnetic dipole moment \( \mu \) which is read from the asymptotics (2.10) of the magnetic gauge potential.

The gauged spinning boson stars have no horizon and therefore they are zero entropy objects, without an intrinsic temperature. The first law of thermodynamics reads in this case \( dE = \omega dQ = \frac{\omega}{n} dJ = \Phi dQ_e \), with \( \Phi = \omega/q \) the electrostatic potential.

### 3 The results

In the numerics, we set \( 4\pi G = 1 \) and express all variables and quantities in natural units set by the AdS length scale \( \ell \) (for example, we scale \( r \to r/\ell, \omega \to \omega \ell, M \to M/\ell \) etc.).

The set of seven coupled non-linear elliptic partial differential equations for the functions \( F_a \) has been solved numerically subject to the boundary conditions (2.9) at \( r = 0 \), (2.12) at infinity, (2.13) for \( \theta = 0 \), and respectively (2.14) in the equatorial plane. In practice, we have compactified \( r \) to the coordinate \( x \in [0, 1] \), where \( dr = \frac{dx}{(1-x)^2} \). Also, the field equations

\(^7\)This relation is valid also for spinning solutions with a magnetic dipole in a generalization of (1.1) with an SU(2) local gauge symmetry of the matter lagrangian. However, in contrast to (1.1), a Yang-Mills-Higgs theory possesses also solutions with a non-zero magnetic flux at infinity. As discussed in [28], the total angular momentum of such non-Abelian configurations is zero.
have been discretized using a fourth order finite difference scheme, taking a uniform grid with \( N_x \times N_\theta \) points (typically \( N_x = 250, N_\theta = 30 \)). All numerical calculations have been performed by using the programs FIDISOL/CADSOL [29]. This software provides also an error estimate for each unknown function. The typical relative error for the solutions reported in this work is estimated to be of the order of \( 10^{-3} \).

In our approach, the input data are the scalar potential \( U(|\psi|) \), the frequency \( \omega \) and the winding number \( n \) in the ansatz (2.7) for the scalar field \( \psi \), the gauge coupling constant \( q \) and the AdS length scale \( \ell \). All quantities of interest (e.g. the mass-energy \( E \) and angular momentum \( J \)) are extracted from the numerical solutions. Also, for simplicity we restrict our study in this work to the case of a nodeless scalar field.

3.1 The probe limit: spinning gauged Q-balls in a fixed AdS background

Before discussing the properties of the gravitating solutions, it is useful to consider first the so-called ‘probe limit’. In this case, the backreaction on the spacetime geometry is ignored and the gauged scalar field solutions are studied in a fixed AdS background, i.e. with \( F_1 = F_2 = F_0 = 1, W = 0 \) in (2.6). Then the problem reduces to solving three PDEs for the matter functions \( \phi, A_\phi \) and \( A_t \). These solutions describe a special class of non-topological solitons—the U(1)-gauged Q-balls (see Ref. [31] for a discussion of the corresponding problem for \( \Lambda = 0 \)). The scalar field possesses in this case a self-interaction potential, the usual form in the literature being (see e.g. [19])

\[
U(\phi) = M^2 \phi^2 - \lambda \phi^4 + \nu \phi^6,
\]

with \( \lambda, \nu \) positive constants. The total mass-energy and angular momentum of the gauged Q-balls are found by integrating over the entire space the corresponding components of the energy-momentum tensor,

\[
E = - \int d^3 x T_t^t, \quad J = \int d^3 x T_t^\phi.
\]

Before discussing the properties of the solutions, we note the existence of the following virial identity

\[
\mathcal{T} + 3 \mathcal{U} + \mathcal{S} = \mathcal{E}_M + 3 \mathcal{Q},
\]

with

\[
\mathcal{T} = \int_0^\infty dr \int_0^\pi d\theta \sin^2 \theta \left[ \left( 1 + \frac{r^2}{\ell^2} \right) \phi_r^2 + \phi_\theta^2 + \frac{(n + qA_\phi \sin \theta)^2 \phi^2}{r^2 \sin^2 \theta} \right],
\]

\[
\mathcal{U} = \int_0^\infty dr \int_0^\pi d\theta \sin^2 \theta \ U(\phi), \quad \mathcal{Q} = \int_0^\infty dr \int_0^\pi d\theta \sin^2 \theta \frac{(\omega - qA_t)^2 \phi^2}{1 + \frac{r^2}{\ell^2}},
\]

\[\text{We have found this method robust and allowing for an accurate extraction of the boundary quantities. The Ref. [29] provides a detailed description of the numerical method and explicit examples (see also e.g. the Appendix in [30] for a discussion within a physical problem).}\]

\[\text{However, angular excited boson stars are also likely to exist, representing excited states of the model. Their basic properties case have been discussed in [16], for an asymptotically flat spacetime.}\]
positive terms, and

\[ \mathcal{E}_M = \int_0^\infty dr \int_0^\pi d\theta \sin^2 \theta \left[ \frac{(1 + \frac{r^2}{\ell^2}) A_{\phi r}^2}{2 r^2} + \frac{1}{2} A_{t r}^2 + \frac{A_{t \theta}^2}{2 r^2 (1 + \frac{r^2}{\ell^2})} + \frac{(A_{\phi \theta} + A_\phi)^2}{2 r^4} \right] \geq 0, \]

a term proportional to the total energy stored in the electromagnetic field. Also

\[ S = -\frac{2\Lambda}{3} \int_0^\infty dr \int_0^\pi d\theta \sin^2 \theta r^2 \left[ \phi_r^2 + \frac{(\omega - q A_t)^2 \phi^2}{(1 + \frac{r^2}{\ell^2})^2} + \frac{A_{\phi r}^2}{2 r^2} + \frac{A_{t \theta}^2}{2(1 + \frac{r^2}{\ell^2})^2 r^2} \right], \]

is a (positive) term encoding the contribution of the AdS background. The total mass-energy of the solutions is \( E = T + Q + U + \mathcal{E}_M. \)

One can see that the solutions with a strictly positive potential, \( U(\phi) > 0, \) are supported by the harmonic time dependence of the scalar field together with the contribution of the U(1) field. Also, note that the identity (3.19) does not forbid the existence of static \( \omega = A_t = 0 \) gauged configurations. However, similar to the case of a flat space-time background [31], no such solutions have been found for \( U(\phi) > 0. \)

Since the basic properties of the gauged configurations are rather similar to those of the Q-balls discussed in [11], we shall start with a brief review of that case. Supposing that \( U(\phi) > 0, \) the solutions exist for a limited range of frequencies, \( 0 < \omega_{\min} < \omega < \omega_{\max}, \) with

\[ \omega_{\max} = \frac{n + \Delta}{\ell}. \]  

(3.20)

As \( \omega \to \omega_{\max}, \) the Q-balls emerge as a perturbation around the ground state \( \phi = 0, \) while both the mass-energy and Noether charge appear to diverge\(^{10}\) as \( \omega \to \omega_{\min}. \)

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\(^{10}\)Note the difference with respect to the case of a Minkowski spacetime background. There the mass-energy and angular momentum grow without bounds as \( \omega \) approaches the limits of the allowed frequency range, with \( \omega_{\max} = M. \) In between, there is a critical value of the frequency, for which both \( E \) and \( J \) attain their minimal values and a double-branch structure is observed.
We have found that any spinning AdS Q-ball in [11] possesses generalizations with a U(1) gauged field. The solutions are found by slowly increasing the value of gauge coupling constant \( q \). For \( q \neq 0 \), the frequency dependence of the solutions is similar to the Q-ball case, see Figure 1 (the solutions there have been found for a potential of the form (3.18) with \( M = 1, \lambda = -2, \nu = 0.2 \)). The maximal value of the frequency is still given by (3.20), while \( \omega_{\text{min}} \) decreases with \( q \).

The picture found when keeping fixed all other input parameters and varying the value of \( q \) is similar to that discussed below for the gravitating case, with the existence of a maximal allowed value for \( q \).

\[ \text{Figure 2. Left: The functions } f_{13}(\theta), w_2(\theta) \text{ and } c_1(\theta) \text{ which enter the large-}r \text{ asymptotics are shown for a typical solution with } \omega \ell = 3, q = 2, n = 1. \text{ Right: The mass-energy } E \text{ is shown as a function of the frequency } \omega \text{ for gauged boson star solutions with several values of the gauge coupling constant } q. \]

3.2 Gravitating spinning gauged boson stars

To construct gravitating solutions, we use again the ungauged spinning Q-balls in [11] as seed configurations and slowly increase the value of the gauge coupling constant \( q \). Once a solution is found with a nonzero gauge field potential, we keep \( q \) constant and couple to gravity.

To simplify the picture, we restrict the discussion here to the case of a scalar field with a vanishing potential\(^\text{11} \) \( U(|\psi|) = 0 \).

\(^\text{11}\) However, we have constructed also solutions with a scalar potential of the form (3.18). Our results suggest that the qualitative properties of the solutions discussed here are generic as long as \( U(|\psi|) > 0 \). Families of solutions with a tachyonic mass of the scalar field, \( U(|\psi|) = M^2|\psi|^2 < 0 \), have been also considered. However, the study of such configurations requires a deviation from the scheme proposed in this work. For example, the scalar field possesses in this case a more general asymptotics than (2.10), which implies in general a different boundary condition in the far field region than the one in (2.12).
The basic properties of these gravitating spinning gauged boson stars solutions can be summarized as follows. First, they have $n \geq 1$, i.e., being disconnected from the spherically symmetric sector, they do not possess a slowly rotating limit.

Second, the gauged boson stars exist for a limited range of frequencies $0 < \omega_{\min} < \omega < \omega_{\max}$, emerging as a perturbation of the globally AdS spacetime for a critical frequency $\omega_{\max}$ as given by (3.20). As seen in Figure 2 (right), the minimal frequency increases with $q$. In contrast to the probe limit discussed above, the global charges stay finite as $\omega \to \omega_{\min}$. Instead, a backbending towards larger values of $\omega$ is observed there, see Figure 2 (right). One may expect that, similar to the spherically symmetric case, this backbending would lead to an inspiraling of the solutions towards a limiting configuration with $\omega_c > \omega_{\min}$.

Third, for given values of $\omega\ell$ and $n$, spinning solutions exist up to a maximal value of the gauge coupling constant only, $q = q_{\max}$, see Figure 3. The mechanism explaining this behaviour is similar to the spherically symmetric case. For $q > q_{\max}$ the charge repulsion becomes bigger than the gravitational attraction and the localized solutions cease to exist (this feature has been noticed already in the initial study [32] of the spherically symmetric, asymptotically flat charged boson stars). As seen in Figure 3, the maximal value of $q$ increases with frequency. Also, all global charges stay finite as $q \to q_{\max}$. Unfortunately, the numerical accuracy does not allow to clarify the limiting behavior\(^\text{12}\) at the maximal value of $q$. We notice only that, as $q \to q_{\max}$, the metric function $F_0$ takes very small values at $r = 0$.

\(^\text{12}\)Based on the results for $n = 0$ (see the inset in Figure 3 (left)), we expect a complicated behaviour, with the occurrence of secondary branches and an inspiraling in $q$ towards a limiting configuration with $q_c < q_{\max}$. 

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**Figure 3.** *Left:* The mass-energy $E$ is shown as a function of the gauge coupling constant $q$ for gauged boson star solutions with several frequencies. *Right:* The $(J,E)$-diagram for the same solutions.
while the other functions remain finite and nonzero (although \( F_1 \) and \( F_2 \) take large values at the origin).

Fourth, the shape of the metric functions and of the scalar field \( \phi \) is rather similar to the ungauged case. Concerning the gauge field, the electric potential \( A_t \) does not possess a strong angular dependence; however the magnetic potential \( A_\phi \) exhibits a complicated angular dependence. The energy density of all solutions has a strong peak in the equatorial plane and decreases monotonically along the symmetry axis, such that the typical energy density isosurfaces have a toroidal shape. As seen in Figure 2 (left), the functions \( f_{12} \), \( w_2 \), and \( c_1 \) which enter the large-\( r \) asymptotics, have a strong \( \theta \)-dependence.

### 3.3 The static limit: gauged scalar solitons with magnetic dipole moment

An interesting property of a set of configurations considered in this work is the possible existence of a nontrivial limit, describing static gauged solutions with a magnetic dipole moment and no electric field\(^{13}\) (thus \( A_t = W = 0 \) in this case).

These solutions are supported by a field potential which is not strictly positive definite and exist already in the probe limit, see the virial relation (3.19). We have studied such solutions for several choices of the parameters in the general potential (3.18) (the solutions in Figure 4 have \( U(|\psi|) = -\lambda |\psi|^4 \), with \( \lambda = 2.2 \)).

The limit \( q = 0 \) corresponds to the static axially symmetric scalar solitons whose existence has been reported in [11]. Again, the gauged solutions emerge from these configurations by slowly increasing the value of the gauge coupling constant. As seen in Figure 4, in contrast

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\(^{13}\)To our knowledge, all known particle-like solutions with this property exist in models with non-Abelian fields, the electroweak sphaleron with a nonvanishing mixing angle [33] being perhaps the best known example.
to the spinning case, the mass of the static solutions decreases with frequency. In the probe limit, this can be understood as follows. For small values of $q$, one can write the expansion

$$\phi = \phi^{(0)} + q^2 \phi^{(2)} + O(q^4), \quad A_\varphi = qA_\varphi^{(1)} + O(q^3),$$

where $\phi^{(0)}$ is the ungauged configuration. After inserting this into the expression of the total mass-energy and using the equations of motion (2.3), one finds

$$E = E^{(0)} - q^2 E^{(2)} + O(q^4),$$

where $E^{(2)} = \frac{1}{4} \int d^3 x F_{\mu\nu}^{(1)\mu\nu}$ and $E^{(0)}$ the total mass-energy of the ungauged spinning boson star.

Although the above physical argument for the existence of a maximal value of $q$ does not apply for these static solutions (since the electric repulsion is absent), the numerical results suggest that a maximal value of $q$ does nevertheless exist, see Figure 4. As $q \to q_{\text{max}}$, the critical solutions exhibit the same behaviour as the one noticed in the spinning case. In particular, the relevant quantities remain finite close to that point, while $F_0(0,0)$ and $1/F_1(0,0)$ take small values. Also, the maximal value of $q$ decreases again as the value of the winding number $n$ increases.

4 Further remarks

The main purpose of this work was to propose a numerical scheme for the study of spinning gauged boson stars in a four-dimensional AdS spacetime, together with a discussion of the basic properties of these configurations. In our approach, the gauged spinning boson stars emerge smoothly from the corresponding solutions with a vanishing electromagnetic field by slowly increasing the value of the gauged coupling constant. Then, as expected, their basic properties are rather similar to those of the ungauged configurations. However, the gauged coupling constant cannot be arbitrarily large.

It would be desirable to study these solutions also from a holographic dual point of view. On general grounds, one expects the $d = 4$ spinning gauged boson stars to describe zero temperature states of a conformal field theory (CFT) defined in a fixed background with $ds^2 = \gamma_{ab} dx^a dx^b = -dt^2 + \ell^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$ (i.e. a 2 + 1 Einstein universe). As usual, the asymptotic behavior of the bulk solutions determines certain properties of the dual field theory. For example, the asymptotics of the temporal component of the gauge potential (in a gauge without a time dependence of the scalar field) gives the charge density and the chemical potential of the CFT. Likewise, the field $\psi$ is dual to an operator $\mathcal{O}$ in the CFT, with a scaling dimension $\Delta$. One can also compute the expectation value of the dual CFT stress-tensor $\langle \tau_{ab} \rangle$ by using the relation [34] $\sqrt{-\gamma} \gamma^{ab} < \tau_{bc} >= \lim_{r \to \infty} \sqrt{-h} h^{ab} T_{bc}$ (with $T_{ab}$ the boundary stress tensor). This results in the interesting expression (with $x^1 = \theta$, $x^2 = \varphi$, $x^3 = t$)

$$8\pi G \ell^4 < \tau^a_b >= A_1(\theta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + A_2(\theta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix} + A_3(\theta) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \ell^2 \sin^2 \theta \\ 0 & -1 & 0 \end{pmatrix},$$

\footnote{Note, however, that finding an accurate value for $q_{\text{max}}$ has proven a difficult task (at least for the input parameters we have considered), since the numerical accuracy decreases close to that point, yielding relatively large violations for the constraint equations.}
(where \(A_1(\theta) = 2f_{13} - \tan \theta f'_{13},\ A_2(\theta) = \tan \theta f'_{13},\ A_3(\theta) = \frac{3}{2}w_2\)), which is finite, covariantly conserved and manifestly traceless.

The inclusion of finite temperature effects in this model would require to construct the corresponding black hole solutions. In this context, the numerical study in this work can be viewed as a necessary step before approaching the more complex case of a spinning black hole of the gravitating Abelian-Higgs model in a globally AdS background\(^{15}\). Different from the ungauged case, the model (1.1) possesses black hole solutions with scalar hair already in the spherically symmetric limit, see e.g. [20]. The numerical results in Ref. [36] imply that these static black holes with scalar hair possess also spinning generalizations\(^{16}\). Then, on general grounds, we expect that when putting together the solitons in this work with the (yet to be found) hairy black holes, the resulting picture will be similar to that revealed in [12] for a version of the model without gauge fields. That is, we conjecture that when the horizon size shrinks to zero, the hairy black hole solutions of the gravitating Abelian-Higgs model will reduce to the rotating gauged boson stars discussed above. Conversely, one can put a small rotating black hole inside any spinning, horizonless solution in this work. The angular velocity of the horizon of a rotating black hole is \(\Omega_H = (\omega - q\Phi_H)/n\), with \(\Phi_H\) the electrostatic potential on the horizon. We hope to return elsewhere with a systematic study of these aspects.

Finally, let us remark that the rotating gauged boson stars studied in this paper can be considered as simple prototypes of more complicated spinning configurations, possibly with non-Abelian gauge fields. Moreover, they may provide as well a fertile ground for further study of charged rotating configurations in gauged supergravity models, with a more complicated action than (1.1). Thus the study in this work gives an idea of how a similar procedure would work in those cases. Also, we expect that the numerical scheme proposed here could also be applied to the study of higher dimensional solutions of the gravitating Abelian Higgs model with rotation in a single plane.

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\(^{16}\)This follows from the explicit construction in [36] of a marginal mode of the Kerr-Newman-AdS black hole. This implies the existence of a new branch of charged stationary solutions with a nonzero scalar condensate outside the horizon. However, no explicit construction of these solutions has been reported so far in the literature.
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