Morphisms general properties of geometrical models

D F Kuchkarova, D A Achilova

Tashkent institute of irrigation and agricultural mechanization engineers
Tashkent, Uzbekistan, Kari Niyazi street, 39, 100000
kuchkarova-dilarom@yandex.ru

Abstract. The notion “modeling” is one of the most significant in modern science. At present the approach is dominating, under which the model is understood as a set of given collection of relations in it.
It is supposed that some abstraction of identification exists allowing to identify a real object with artificially created model; for all that it is more often used isomorphism and homomorphism for the relation identification. With the notion ‘modeling’ a principally new method of scientific knowledge is used that is the ‘mathematical experiment’, taking an intermediate place between ‘classical deductive and classical experimental research methods’. Under mathematical model of a concrete object it is understood combination of equations, inequalities and other limitations and conditions for their solving, that allow in created system of co-ordinates to identify the object with the only method. Any object, except mathematical one, can have other models: geometrical, graphical, digital, linguistic and so on. Sequence of transformation of geometrical model into mathematical one and then into digital one is the sequence of homomorphism. While modeling we compare it not only with mathematically similar object i.e. the model, but with theoretical conception as well as with other suitable objects for this conception. It is shown in this article the primary geometrical model in relation to mathematical one and the definition of geometrical modeling is given. The necessary definitions of isomorphism and homomorphism models are represented as well as examples of relations between concrete objects and their models. Definitions of full geometrical model, its complexity degrees and conditions of two models equivalence are also given.

1. Introduction

In hierarchy of models, describing some object (geometrical, mathematical, digital, graphical e.t.c) the geometrical model is always primary, as a scope of geometrical conditions, defining an object or process with required precision of geometrical transition, allowing to visualize the object and to formalize representation of the object or process in mathematical, digital, graphical, and other outlook.

The mathematical modeling of the object comes to mathematical definition of its geometrical model. As stated above, while modeling, the object is matched not only with mathematically similar object – the model, but with theoretical concept, and also with other objects, suitable for this conception.

It appears from this, that two objects are the models of one theory, and therefore they are only akin to structure but not to the nature.

The choice of one or another theory is determined, as a rule, by simplicity and substantivizes of the system of axioms, on the one hand, and definition fullness of objects under research, on the other hand.
The models being studied within the framework of some theory, can be compared towards isomorphism and homomorphism.

Let us give proper definitions.

Let the set $A$ and the set $A'$, two random sets (finite or infinite), be equivalent in general set-theoretical sense, i.e. it is possible to establish mutual unique correspondence between set members (bijection): for each element $a_i \in A$ to match one and the only one element $a_i' \in A'(\phi/a_i)$ and vice-versa, for each element $a_j \in A'$ to correspond one and the only one element of the first set $f(a_j) = a_j \in A, \phi = f^{-1}$ and $f = \phi^{-1}$.

Suppose further bijection can be set also between defined predicate systems in sets $A$ and $A'$, but such sets $A$ and $A'$ are isomorphic as well as models defined on the given sets.

Now suppose, in mentioned above isomorphism definition, the requirement of mutual image simplicity $\phi: A \rightarrow A'$ is weakened that is replaced by its simplicity condition only into one side, i.e. now it is only supposed that to each element $a \in A$ the only ‘image’ $\phi(a) = a' \in A'$ corresponds, the uniqueness of ‘counter images’ for arbitrary $a \in A$ is not supposed. In this case (all the rest points of isomorphism definition remain unchangeable) it is used to say, that $A'$ is homomorphism image of set $A$ (it also concerns mentioned above data of predicate families i.e. signature) or that $A'$ is in homomorphism relation to $A$.

For simple enough geometrical objects, represented by graphic or linguistic models, isomorphism or homomorphism connection can be demonstrated visually. For example, text ‘drawing of right-angled triangle’ that is linguistic model is always isomorphic to its graphical model, and a number of such graphic models break dots of the space $R^2$ for 1 class of equivalence. Refined linguistic model «drawing of right-angled triangle with hypotenuse equal to 10 millimeters» is also isomorphic to its graphical model, but a great number of such models break dots of space $R^2$ for classes of equivalence. The models of one class meet relations of reflexiveness, symmetry and transitivity.

The central projection of some real triangle i.e. graphical model of its physical model is always isomorphic to its prototype. A set of such central projections break space $R^2$ dots for the equivalence class.

But between the central projections of some volume body, for example, a parallelepiped and an object itself homomorphism relation exists. A set of such central projections i.e. graphic models of parallelepiped break the space $R^2$ dots for a class of tolerance. Models of this class meet relations of reflexiveness and symmetry. Transitivity relation is infringed.

Between the models (geometrical, mathematical, digital, graphical and so on) of the same object the relations of homomorphism exist, but the morphism degree may be various.

Geometrical modeling is understood as image of geometrical information $G = (\{S\}, \{M\}, \{x\})$ into the geometrical information $G' = (\{S\}', \{M\}', \{x\}')$ — that is combined space forms i.e. a set which elements are equivalence classes on the space of all dot kinds of sets, $\{M\}, \{M\}'$ - are various metrical characteristics of space forms $\{S\}, \{S\}'$ on measured infinite and finite Euclid spaces, $\{x\}, \{x\}'$ - are parameters giving location of dot sets in corresponding spaces.

Let’s give definition to a full geometric model.

Definition 1.
Let’s name a geometric model of some object (process or phenomenon) a full one, if its geometric information is full \( G = \left( \{S\}, \{M\}, \{x\} \right) \).

For objects with the complete set of geometric signs, it is always possible to make their geometric models with precision up to isomorphism. Objects of technogeneous nature of any complexity degree, as a rule, possess a set of geometrical signs and the task of making their geometrical models is only to choose this or that method for geometrical modeling.

For natural objects to make a complete geometrical model is a very difficult task to solve. In a better case, here is a model of homomorphism object image, although more complicated object images are possible in its model, without coming only to homomorphism.

Mathematical models of objects under research, such as homomorphic depictions of their geometrical models, are dissimilar logical functions. (predicates) defined on a number of distinctive conditions of a great number of parameters under consideration of these objects and taking different meanings of true.

The degree of result validity, getting out of modeling, is defined by selection of a set model validity that means: by the sphere of its definition. As a result, it is quite right to say not about two logical true meanings («O» or «1»), but about a great number of such meanings (just as it takes place in many sensible logics).

In this respect the question arises about correspondence of the model complexity degree with the complexity of the object itself and about explication of the notion ‘complexity’ in regard to geometrical models.

Definition 2.

Let’s name as complexity degree of geometrical model the quantity \( \sum w^2 + \sum \phi^2 \), i.e., the summary of absolute inner and outer peak curves of the metric (model).

At the same time, on the intuitive and even on the visual levels it is clear, that the criterion of total curving, as the degree of complication of an object, does not allow to establish equivalence of this object to another one.

For geometrical nature objects, beside equality of summary absolute curving, equivalence supposes coincidence of other geometrical signs.

Definition -3.

Let us name two geometrical models equivalent (equal in their complication), if:

1) their complexity degrees are equal i.e. the total sum of curves in absolute quantity:

\[ |\sum w^2| + |\sum \phi^2| = \sum |w^2| + \sum |\phi^2| \]

2) there is isomorphism correlation between structural lines of two models.

It is possible to show that complexity degree of geometrical model is also its complexity measure. In reality, if \( \{x_i\}_{i=1}^n \) is a set of peaks of some model \( X \), which is supposed to be counted and where each peak has the curving, then \( \{w'_i\}_{i=1}^4, j = 1, n \) out of the fact, that any model site is a complete metrical space, it follows:

\[
U\{x_i\}_{i=1}^N \subseteq X, \sum \{w_i^2\} + \sum \{\phi_i^2\} = m\{x_i\}_{i=1}^N \leq m(X)
\]
Hence, according to the measure theory, affirmation justice follows
To solve identification tasks the regularity model is suggested:

$$\Delta^2(B) = \frac{\sum_{j=1}^{N_B}(q_{\text{табл}} - q_M)^2}{\sum_{i=1}^{N}q_{\text{табл}}^2} \rightarrow \min$$

Where $\Delta^2(B)$ is the middle square mistake, counted on new points, that had not been used to get model coefficient estimation. $NB$ – is the number of points of separated checking data choosing, $q(\text{табл})$- is the table initial data values, $q_M$ are values counted for this model.

Criterion is based on ID dividing into two parts: educating part $NA$ and checking part $NB$. All identification data are ranged into a row according to values of their dispersion starting from the middle value, and this row is divided into two indicated parts ($N=NA+NB$)

For regularity criterion is usually accepted $A=2/3N,B=1/3N$, that corresponds to optimal dividing ID table into two parts. The various regularity criterion modification:

$$\Delta_{\text{cm}}^2 = \frac{\sum_i(q_A-q_B)^2}{\sum_i q_{\text{табл}}^2} \rightarrow \min;$$

$$\Delta^2(A,B) = \Delta^2(A/B) + \Delta^2(B/A) \rightarrow \min$$

and etc can be used under various requirements to the model exactness on different sites of the field and its determination.

To solve model identification tasks of topographical surfaces, identification data are divided into three parts: $N_A$- mobile comprising points incident to structural ones, $N_B$- educating parts, $N_C$ – checking ones. The following operation sequence is performed:

1) algorithms are realized to seek structural lines and triangulation of the model task field.
2) points, incident to structural lines, are selected into mobile choosing $N_A$
3) the remained points of ID table are divided into two parts depending on dispersion value starting from the middle value; new selections are being formed $N_B$ and $N_C$ so $N_A=N_B$, $N_A=N_C$
4) at points of educating selection $N_B$ the model is being created through the method chosen by a user.
5) exactness of created model is being cleared up by using points of checking selection $N_C$, on the base of regularity criterion $\Delta^2$

For geometrical models the regularity criterion accepts values lying in the range from “0” till $1(0<\Delta^2<1)$ under successful modeling method choosing; obviously, the case $\Delta^2=0$ under geometrical modeling is not reachable. At the same time regularity $\Delta^2(B)$ criterion and its modifications $\Delta^2_{\text{qm}}, \Delta^2(A,B)$ give the opportunity to determine homomorphism degree between relief site and its model:

1) $0.8 \leq \Delta^2 < 1$ is weak homomorphism; 2) $0.5 \leq \Delta^2 \leq 0.8$ is middle homomorphism; 3) $0 < \Delta^2 \leq 0.5$ is strong homomorphism.

References

1. Gastev Yu. A. Homomorphisms and models M.: Science, 1975. – 150 p
2. Shreyder Yu. A., Sharov A. A. Systems and models. M.: Radio and connection, 1982. – 152 p
3. Stoyan Yu. G., Yakovlev S. V. Mathematical models and
geometric design optimization methods. Kiyev: Nauk.Dumka, 1986. – 268 p
4. Ivakhnenko A. G. Reordered methods of model self-organization and clustering // Automathics, 1989, №4, 15-17 p