The Hodge-FVH Correspondence

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Abstract

The Hodge-FVH correspondence establishes a relationship between the special cubic Hodge integrals and an integrable hierarchy, which is called the fractional Volterra hierarchy. In this paper we prove this correspondence. As an application of this result, we give a new algorithm for computing the coefficients that appear in the gap phenomenon associated to the special cubic Hodge integrals.

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1 Introduction

The study of the deep relationship between topology of Deligne-Mumford moduli space of stable algebraic curves and integrable hierarchies started from Witten’s conjecture [30]. It states that the formal series $Z_{WK}(t; \epsilon) \in \mathbb{Q}((\epsilon^2))[t_0, t_1, \ldots]$ defined by

$$Z_{WK}(t; \epsilon) := \exp \left( \sum_{g \geq 0} \epsilon^{2g-2} \sum_{k \geq 0} \sum_{i_1, \ldots, i_k \geq 0} \frac{t_{i_1} \cdots t_{i_k}}{k!} \int_{\mathcal{M}_{g,k}} \psi_1^{i_1} \cdots \psi_k^{i_k} \right),$$

is a particular tau-function of the Korteweg-de Vries (KdV) hierarchy. Here, $\mathcal{M}_{g,k}$ denotes the Deligne-Mumford moduli space of stable algebraic curves of genus $g$ with $k$ distinct marked
points, $\psi_i$ denotes the first Chern class of the $i$-th cotangent line bundle over $\overline{\mathcal{M}}_{g,k}$, $i = 1, \ldots, k$, and $t := (t_i)_{i \geq 0}$ is the infinite vector of indeterminates. Witten’s conjecture was first proved by Kontsevich [16]; see for example [15, 18, 24, 26] for several different proofs.

The Hodge integrals over $\overline{\mathcal{M}}_{g,k}$ are some rational numbers defined by

$$\int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{i_1} \cdots \psi_k^{i_k} \lambda_{j_1} \cdots \lambda_{j_l},$$

where $\lambda_j$ is the $j$-th Chern class of the Hodge bundle $E_{g,k}$ on $\overline{\mathcal{M}}_{g,k}$, $j = 0, \ldots, g$. For the dimension reason these integrals vanish unless

$$i_1 + \cdots + i_k + j_1 + \cdots + j_l = 3g - 3 + k. \quad (1.1)$$

They reduce to the $\psi$-class intersection numbers when $j_1 = j_2 = \cdots = 0$. In this paper we consider the generating series of certain cubic Hodge integrals given by

$$Z_{\text{cubic}}(t; p, q, r; \epsilon) = e^{\sum_{g \geq 0} \epsilon^{2g-2} \mathcal{H}_g(t)}, \quad (1.2)$$

where $\mathcal{H}_g(t)$, $g \geq 0$ are the genus $g$ cubic Hodge potentials

$$\mathcal{H}_g(t; p, q, r) = \sum_{k \geq 0} \sum_{i_1, \ldots, i_k \geq 0} \frac{t_1^{i_1} \cdots t_k^{i_k}}{k!} \int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{i_1} \cdots \psi_k^{i_k} C_g(-p)C_g(-q)C_g(-r). \quad (1.3)$$

Here, $C_g(z) := \sum_{j=0}^{g} \lambda_j z^j$ denotes the Chern polynomial of $E_{g,k}$. It was proven in [6] that the formal series $u_{\text{cubic}}$ defined by

$$u_{\text{cubic}} := e^{2\partial_0^2} \log Z_{\text{cubic}}(t; p, q, r; \epsilon)$$

satisfies an integrable hierarchy of Hamiltonian PDEs, called the cubic Hodge hierarchy.

The above cubic Hodge integrals in (1.2) are called special if $p, q, r$ satisfy the following local Calabi-Yau condition:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0. \quad (1.4)$$

Geometric significance for these special cubic Hodge integrals is manifested in the localization technique of computing Gromov-Witten invariants for toric three-folds [13, 17, 25], and by the Gopakumar-Mariño-Vafa conjecture regarding the Chern-Simons/string duality [12, 23].

The conjectural Hodge-FVH correspondence [22], which is a generalization of Witten’s conjecture, states that the cubic Hodge hierarchy for the special cubic Hodge integrals is equivalent to the fractional Volterra hierarchy (FVH). The goal of this paper is to prove this conjecture.

To give a precise statement of the Hodge-FVH correspondence, let us introduce some notations. Denote

$$\mathcal{I}_1 := \{kp \mid k \in \mathbb{Z}_{>0}\}, \quad \mathcal{I}_2 := \{kq \mid k \in \mathbb{Z}_{>0}\}, \quad \mathcal{I} := \mathcal{I}_1 \cup \mathcal{I}_2. \quad (1.5)$$

The set $\mathcal{I}$ will serve as the set of indices. Also introduce the numbers $c_\mu$, $\mu \in \mathcal{I}$ by

$$c_\mu := \begin{cases} \left(-\frac{\mu}{p}\right), & \mu \in \mathcal{I}_1, \\ \left(-\frac{\mu}{q}\right), & \mu \in \mathcal{I}_2. \end{cases} \quad (1.6)$$
Theorem 1.1 (Hodge-FVH correspondence) Let $\Lambda := e^{\theta x}$. The formal series
\[
u(x;T;\epsilon) := \left(\Lambda - \frac{1}{2}\right) \left(\Lambda - \frac{1}{2}\right) \log Z_{\text{cubic}} \left(t(x,T); p, q, r; \sqrt{p+q} \epsilon\right)
\] (1.7)
satisfies the FVH, namely, for $\mu \in I$,
\[
\frac{\partial L}{\partial T_{\mu}} = \begin{cases} \left(\frac{L^i}{\Lambda}\right)_+, & \mu \in I_1, \\ \left(\frac{L^i}{\Lambda}\right)_-, & \mu \in I_2. \end{cases}
\] (1.8)
where
\[
L = \Lambda^\frac{1}{p} + e^{\mu} \Lambda^\frac{1}{q},
\]
(1.9)
\[
t_i(x, T) := \frac{1}{pq} \sum_{\mu \in I} \mu^{i+1} c_{\mu} T_{\mu} - 1 + x\delta_{i,0} + \delta_{i,1}.
\] (1.10)
Moreover, the formal series $Z(x;T;\epsilon) \in \mathbb{C}[[x-1, T]] [[\epsilon^2, \epsilon^{-2}]]$ defined by
\[
Z(x;T;\epsilon) := \exp \left(\frac{p^2 q^2}{(p+q)\epsilon^2} A\right) Z_{\text{cubic}} \left(t(x,T); p, q, r; \sqrt{p+q} \epsilon\right)
\] (1.11)
is a tau-function for the FVH, where $A$ is an element in $\mathbb{C}[[x-1, T]]$ defined by
\[
A := \frac{1}{2p^2q^2} \sum_{\mu_1, \mu_2 \in I} \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} c_{\mu_1} c_{\mu_2} T_{\mu_1} T_{\mu_2} + \frac{1}{pq} \sum_{\mu \in I} \left(x - \frac{\mu}{\mu + 1}\right) c_{\mu} T_{\mu} + \frac{1}{4} - x.
\] (1.12)
See in Section 2 for the definition of tau-function for the FVH.

In order to prove Theorem 1.1, we will first consider the rational case, i.e.,
\[
p = \frac{1}{m}, \quad q = \frac{1}{n}, \quad r = -\frac{1}{h},
\] (1.13)
where $m$ and $n$ are coprime positive integers, and $h := m + n$. The key idea of our proof is to use the uniqueness theorem of solutions to the loop equation given in [20]. We will show that there is a particular tau-function $\tau_{\text{top}}$ for the FVH satisfying a technical condition required by the uniqueness theorem of [20], and then show that this tau-function satisfies the Virasoro constraints, which leads to the loop equation and also to the proof of Theorem 1.1 in the rational case. These proofs are given in Sections 3–5. Theorem 1.1 in the general case will be proved by using a continuation argument in Section 6.

As an application of Theorem 1.1, we give a new algorithm for computing the coefficients of the gap phenomenon [20] of the special cubic Hodge integrals, which was proven in [20] for the rational case. Denote
\[
\mathcal{H}_g(x,T) := H_g \left(t(x,T); \frac{1}{m}, \frac{1}{n}, -\frac{1}{h}\right), \quad g \geq 0,
\]
where $t(x,T)$ is given in (1.10), then the gap phenomenon shown in Corollary 1.4 of [20] means that the dependence of $\mathcal{H}_g(x,T)$ on $x$ has a gap in its principal part:
\[
\mathcal{H}_1(x,T) = \frac{\sigma_1 - 1}{24} \log x \in \mathbb{C}[x][[T]],
\] (1.14)
\[ \tilde{H}_g(x, T) - \frac{R_g(\sigma_1, \sigma_3)}{x^{2g-2}} \in C[[x]][[T]], \quad g \geq 2, \]  
\[ \text{where} \]
\[ \sigma_1 = \frac{1}{h} - \frac{1}{m} - \frac{1}{n}, \quad \sigma_3 = \frac{2}{h^3} - \frac{2}{m^3} - \frac{2}{n^3}. \]

The coefficients \( R_g(\sigma_1, \sigma_3) \) are hard to compute, and we only gave the expressions of \( R_2 \) and \( R_3 \) in [20]. Our new algorithm for \( R_g \) is based on the introduction of the particular tau-function \( \tau_{\text{top}} \) for the FVH.

The tau-function \( \tau_{\text{top}} \) corresponds to a solution \( u_{\text{top}}(x, T; \epsilon) \) to the FVH, whose initial value \( V(x; \epsilon) = u_{\text{top}}(x, T = 0; \epsilon) \in \mathbb{Q}[[x - 1; \epsilon]] \) satisfies the following difference equation:

\[ \sum_{0 \leq \alpha_1 \leq \ldots \leq \alpha_m \leq n} \epsilon^{\sum_{j=1}^{m}} V(x + \alpha_j \epsilon - (j - \frac{1}{2}) n \epsilon) = \left( \frac{m + n}{m} \right) x. \]  
\[ \text{We will show in Section 3 that this equation has a unique solution of the form} \]
\[ V(x; \epsilon) = \frac{1}{m} \log x + \sum_{g \geq 1} \epsilon^{2g} \frac{P_g(m, n)}{x^{2g}}, \]

where \( P_g(m, n) \in \mathbb{Q}[m, n, m^{-1}] \).

Let us introduce a sequence of rational numbers \( C_k(m, n), k \geq 0 \) by the generating function

\[ \sum_{k \geq 0} C_k(m, n) z^{2k} := \frac{nhz^2}{4 \sinh (\frac{nz}{2}) \sinh (\frac{h}{2})} = 1 - \frac{n^2 + h^2}{24} z^2 + \frac{7n^2 + 10hn + 7h^2}{5760} z^4 + \cdots. \]

These rational numbers have the following explicit expression:

\[ C_k(m, n) = \sum_{k_1, k_2 \geq 0 \atop k_1 + k_2 \leq 2k} (-1)^{k_2} B_{k_2} B_{2k-k_1-k_2} m^{k_1} n^{k_2} (m + n)^{2k-k_1-k_2}, \quad k \geq 0 \]

with \( B_j \) the \( j \)th Bernoulli number, \( j \geq 0 \).

**Proposition 1.2** The polynomials \( R_g(\sigma_1, \sigma_3) \) introduced in Corollary 1.4 of [21] can be represented in terms of \( P_g(m, n) \) and \( C_k(m, n) \) as follows:

\[ R_g = \frac{(2g-3)!}{(mn)^g} \left( m \sum_{k=1}^{g} \frac{C_{g-k}(m, n) P_k(m, n)}{(2k-1)!} - C_g(m, n) \right), \quad g \geq 2. \]

**Remark 1.3** Recall from Corollary 1.4 of [21] that \( R_g \in \mathbb{C}[\sigma_1, \sigma_3] \) and that \( R_g \) have the form

\[ R_g = \sum_{k=0}^{g-1} \sum_{l=0}^{3g-3k} R_{g, k, l} \sigma_3^k \sigma_1^l, \quad g \geq 2. \]

From Proposition 1.2, we obtain special values of \( R_g \) with \( \sigma_1 \) and \( \sigma_3 \) given by (1.16). By using Lemma 4.1 of [21], we find that all the coefficients \( R_{g, k, l} \) can be determined by the values of \( R_g \) for sufficiently many pairs \((m, n)\).
Organizations of the paper In Section 2 we give two definitions of tau-functions for the FVH, and prove the equivalence of these definitions. In Sections 3–5 we prove Theorem 1.1. As an application of Theorem 1.1, we give in Section 6 an explicit computation for the coefficients of the gap phenomenon of the special cubic Hodge integrals.

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2 Tau-functions for the FVH

In this section, we first recall the definition of the tau-function of a solution \(u(x, T; \epsilon)\) to the FVH (1.8) that is given in [22]. Then we give an alternative definition, following the one for the KP hierarchy [3, 28, 29], of the tau-function in term of the wave functions of the integrable hierarchy, and show the equivalence of these two definitions. We will use the first definition of the tau-function in Section 3 to specify the topological tau-function \(\tau_{\text{top}}\) of the FVH, and to prove that it has the form of the genus expansion required for the derivation of the loop equation [20] from the Virasoro constraints. The second definition will be used in Section 4 to prove that the topological tau-function \(\tau_{\text{top}}\) satisfies the Virasoro constraints.

Let us introduce some notations. We denote by \(A_u\) the ring \(A_{u,[[\epsilon]]}\) of differential polynomials of \(u\), where \(A_{u,0}\) denotes the ring of smooth functions of \(u\). Note that the shift operator \(\Lambda = e^{\epsilon \partial_x}\) acts on \(A_{u,[[\epsilon]]}\). Introduce a gradation \(\deg\) on \(A_{u,[[\epsilon]]}\), defined via the following degree assignments:

\[
\deg \partial_x^k u = k, \quad \deg \epsilon = -1, \tag{2.1}
\]

where \(k \geq 1\). For convenience, we will denote

\[
\Lambda_1 = \Lambda^1, \quad \Lambda_2 = \Lambda^{1/2}, \quad \Lambda_3 = \Lambda^{1/4}. \tag{2.2}
\]

Then the operator \(L\) defined in (1.9) has the form

\[
L = \Lambda_2 + \epsilon \Lambda_1^{-1}. \tag{2.3}
\]

We also extend the gradation deg to the difference operators of the form \(\sum_{j \in \mathbb{Z}} g_j \Lambda^a_j \Lambda^a\) with \(g_j \in A_{u,[[\epsilon]]}\), \(a \in \mathbb{C}\) by assigning \(\deg \Lambda = 0\).

2.1 The definition of the tau-function

We introduced in [22] the following two-point correlation functions in \(A_{u,[[\epsilon]]}\):

\[
\Omega_{\lambda, \mu} = \begin{cases} 
\Lambda_2^{\lambda / p} \sum_{j=1}^{\lambda / p} \Lambda_3^{-j} \left( \text{Res}_{\Lambda_3} \left( \Lambda_3^{-j} L^{-\frac{\mu}{4}} \right) \right), & \lambda \in I_1, \\
\Lambda_2^{\lambda / q} \sum_{j=1}^{\lambda / q} \Lambda_3^{-j} \left( \text{Res}_{\Lambda_3} \left( L^{-\frac{\mu}{4}} \Lambda_3^j \right) \right), & \lambda \in I_2, 
\end{cases} \tag{2.4}
\]

where \(\mu \in \mathcal{I}\), and \(\text{Res}_{\Lambda_3}\) means to take “residue” in the following sense:

\[
\text{Res}_{\Lambda_3} \sum_{i \in \mathbb{Z}} a_i \Lambda_3^i := a_0. \tag{2.5}
\]
Lemma 2.2 The following relations hold true for $a, b, d$ where

\[ \frac{\partial \Omega_{\lambda, \mu}}{\partial T_\nu} = \frac{\partial \Omega_{\mu, \nu}}{\partial T_\lambda}, \] (2.4)

\[ (\lambda_3 - 1)\mathcal{A}_2^{\frac{1}{3}} \Omega_{\lambda, \mu} = \varepsilon \frac{\partial}{\partial T_\lambda} \text{res}_{\lambda_3} L^{-\frac{2}{3}}. \] (2.5)

Proposition 2.1 (22) Let $u = u(x, T; \varepsilon)$ be an arbitrary solution in $\mathbb{C}[[x - 1, T; \varepsilon]]$ to the FVH (1.8). Then the following formulae hold true for any $\lambda, \mu, \nu \in \mathcal{I}$:

\[ \Omega_{\lambda, \mu} = \Omega_{\mu, \lambda}, \quad \frac{\partial \Omega_{\lambda, \mu}}{\partial T_\nu} = \frac{\partial \Omega_{\mu, \nu}}{\partial T_\lambda}, \] (2.4)

\[ (\lambda_3 - 1)\mathcal{A}_2^{\frac{1}{3}} \Omega_{\lambda, \mu} = \varepsilon \frac{\partial}{\partial T_\lambda} \text{res}_{\lambda_3} L^{-\frac{2}{3}}. \] (2.5)

It follows from Proposition 2.1 that for an arbitrary solution $u = u(x, T; \varepsilon) \in \mathbb{C}[[x - 1, T; \varepsilon]]$ to the FVH (1.8), there exists $\tau = \tau(x, T; \varepsilon) \in \mathbb{C}((\varepsilon))[[x - 1, T]]$ satisfying the relations

\[ u = \left( \lambda_3^{\frac{1}{3}} - \lambda_3^{-\frac{2}{3}} \right) \left( \lambda_1^{\frac{1}{3}} - \lambda_1^{-\frac{2}{3}} \right) \log \tau, \] (2.6)

\[ \varepsilon (\lambda_3 - 1) \mathcal{A}_2^{\frac{1}{3}} \frac{\partial \log \tau}{\partial T_\lambda} = \text{res}_{\lambda_3} L^{-\frac{2}{3}}, \quad \lambda \in \mathcal{I}, \] (2.7)

\[ \varepsilon^2 \frac{\partial^2 \log \tau}{\partial T_\lambda^2} = \Omega_{\lambda, \mu}, \quad \lambda, \mu \in \mathcal{I}. \] (2.8)

Note that the tau-function $\tau$ is uniquely determined by (2.6)–(2.8) up to multiplying by a factor of the form

\[ \exp \left\{ ax + b + \sum_{\lambda \in \mathcal{I}} d_\lambda T_\lambda \right\}, \] (2.9)

where $a, b, d_\lambda \in \mathbb{C}((\varepsilon))$ are arbitrary constant Laurent series in $\varepsilon$.

Lemma 2.2 The following relations hold true for $\lambda, \mu \in \mathcal{I}$:

\[ \text{res}_{\lambda_3} L^{-\lambda/\rho} - c_\lambda e^{\lambda u/p} \in \varepsilon \mathcal{A}_u[[\varepsilon]], \] (2.10)

\[ \Omega_{\lambda, \mu} - \frac{1}{p + q} \lambda \mu \mathcal{C}_\lambda \mathcal{C}_\mu e^{(\lambda + \mu)u/p} \in \varepsilon \mathcal{A}_u[[\varepsilon]], \] (2.11)

\[ \deg \text{res}_{\lambda_3} L^{-\lambda/\rho} = 0, \quad \deg \Omega_{\lambda, \mu} = 0. \] (2.12)

Here the constants $c_\lambda, c_\mu$ are defined in (1.6).

Proof From the definition of the fractional powers of $L$ given in [22] we have, for any $k \in \mathbb{Z}$,

\[ \text{res}_{\lambda_3} L^{-\lambda/r} \Lambda_3^{-k} - \left( \frac{-\lambda/r}{\lambda/p - k} \right) e^{(\lambda/p - k)u} \in \varepsilon \mathcal{A}_u[[\varepsilon]], \quad \lambda \in \mathcal{I}_1, \] (2.13)

\[ \text{res}_{\lambda_3} L^{-\lambda/r} \Lambda_3^{-k} - \left( \frac{-\lambda/r}{\lambda/q + k} \right) e^{(\lambda/q + k)u} \in \varepsilon \mathcal{A}_u[[\varepsilon]], \quad \lambda \in \mathcal{I}_2, \] (2.14)

\[ \deg \text{res}_{\lambda_3} L^{-\lambda/r} \Lambda_3^{-k} = 0, \quad \lambda \in \mathcal{I}. \] (2.15)

Then by using the definition (2.2) of $\Omega_{\lambda, \mu}$ we find, for any $\lambda, \mu \in \mathcal{I}_1$, that $\deg \Omega_{\lambda, \mu} = 0$ and

\[ \Omega_{\lambda, \mu} - \frac{1}{p + q} \lambda \mu \mathcal{C}_\lambda \mathcal{C}_\mu e^{(\lambda + \mu)u/p} = \Omega_{\lambda, \mu} - \sum_{j=1}^{\lambda/p} \left( \frac{-\lambda/r}{\lambda/p - j} \right) \left( \frac{-\mu/r}{\mu/p + j} \right) e^{(\lambda + \mu)u/p} \in \varepsilon \mathcal{A}_u[[\varepsilon]]. \]
In the above equality, the following elementary identity (cf. [22]) is used:

\[
\sum_{j=1}^{\lambda/p} \frac{(-\lambda/r)}{\lambda/p-j} \left( \frac{-\mu/r}{\mu/p+j} \right) = \frac{1}{p+q} \frac{\lambda \mu}{\lambda + \mu} c_{\lambda c_{\mu}}, \quad \lambda, \mu \in \mathcal{I}_1.
\]

The proof of (2.10)–(2.12) for the cases with \( \lambda \in \mathcal{I}_1, \mu \in \mathcal{I}_2 \) or \( \lambda, \mu \in \mathcal{I}_2 \) is similar. The lemma is proved. \( \square \)

2.2 An alternative definition of the tau-function

The tau-function of a solution to the FVH can also be defined, following the approach of [3, 28, 29], via the introduction of wave functions for the FVH. To this end, we first introduce the dressing operators.

**Lemma 2.3** For an arbitrary solution \( u = u(x,T;\epsilon) \in \mathbb{C}[[x-1,T;\epsilon]] \) to the FVH (1.5), there exist difference operators \( \Phi_1, \Phi_2 \), called dressing operators, of the form

\[
\Phi_1(x,T;\epsilon) = \sum_{i \geq 0} a_{1,i}(x,T;\epsilon) \Lambda_3^{-i}, \quad \Phi_2(x,T;\epsilon) = \sum_{i \geq 0} a_{2,i}(x,T;\epsilon) \Lambda_3^{i} \tag{2.16}
\]

with \( a_{1,i}, a_{2,i} \in \mathbb{C}[[x-1,T]](\epsilon) \), \( i \geq 0 \) and \( a_{1,0}(x,T;\epsilon) = 1 \), such that

\[
L = \Phi_1 \Lambda_2 \Phi_1^{-1} = \Phi_2 \Lambda_1^{-1} \Phi_2^{-1}, \tag{2.17}
\]

\[
\epsilon \frac{\partial \Phi_1}{\partial T_{\mu}} = - \left( \Phi_1 \Lambda_3^{-\mu/p} \Phi_1^{-1} \right) \Phi_1, \quad \epsilon \frac{\partial \Phi_2}{\partial T_{\mu}} = \left( \Phi_1 \Lambda_3^{\mu/p} \Phi_1^{-1} \right) \Phi_2, \quad \mu \in \mathcal{I}_1, \tag{2.18}
\]

\[
\epsilon \frac{\partial \Phi_1}{\partial T_{\mu}} = - \left( \Phi_2 \Lambda_3^{-\mu/q} \Phi_2^{-1} \right) \Phi_1, \quad \epsilon \frac{\partial \Phi_2}{\partial T_{\mu}} = \left( \Phi_2 \Lambda_3^{\mu/q} \Phi_2^{-1} \right) \Phi_2, \quad \mu \in \mathcal{I}_2. \tag{2.19}
\]

These two operators \( \Phi_1 \) and \( \Phi_2 \) are uniquely determined by \( u \) up to right-multiplications by operators of the form

\[
\Phi_1 \mapsto \Phi_1 \left( 1 + c_{1,1} \Lambda_3^{-1} + c_{1,2} \Lambda_3^{-2} + \cdots \right), \quad \Phi_2 \mapsto \Phi_2 \left( c_{2,0} + c_{2,1} \Lambda_3 + c_{2,2} \Lambda_3^2 + \cdots \right),
\]

where \( c_{i,j} \) are arbitrary Laurent series of \( \epsilon \) with constant coefficients.

**Proof** The equation \( L = \Phi_1 \Lambda_2 \Phi_1^{-1} \) is equivalent to the following recursion relations:

\[
(\Lambda_2 - 1) a_{1,1} = -\epsilon a_{1,1},
\]

\[
(\Lambda_2 - 1) a_{1,i} = -\epsilon a_{1,i-1} a_{1,i-1}, \quad i \geq 2.
\]

On the other hand, by using the FVH (1.8) we can easily check that equations (2.17)–(2.19) for \( \Phi_1 \) are compatible, from which we know the existence of \( \Phi_1 \). If we have another operator \( \Phi'_1 \) which satisfies the same equations as for \( \Phi_1 \), then \( \Phi_1^{-1} \Phi'_1 \) satisfies the following two equations:

\[
[\Phi_1^{-1} \Phi'_1, \Lambda_2] = 0, \quad \epsilon \partial_{T_{\mu}} (\Phi_1^{-1} \Phi'_1) = 0, \quad \forall \mu \in \mathcal{I}.
\]

Hence \( \Phi_1^{-1} \Phi'_1 \) must have the form

\[
\Phi_1^{-1} \Phi'_1 = 1 + c_{1,1} \Lambda_3^{-1} + c_{1,2} \Lambda_3^{-2} + \cdots.
\]
The assertion of the lemma for $\Phi_2$ can be proved in a similar way.

Given a solution $u = u(x, T; \epsilon) \in \mathbb{C}[[x-1, T; \epsilon]]$ to the FVH, we associate the so-called wave functions $\psi_1, \psi_2$ by the following formulae:

$$\psi_1 := \Phi_1 e^{\vartheta_1}, \quad \psi_2 := \Phi_2 e^{\vartheta_1},$$

where $\Phi_1, \Phi_2$ are the dressing operators,

$$\vartheta_1 = \vartheta_1(x, T; \epsilon, z) := -\frac{r}{\epsilon} \log z + \sum_{\mu \in \mathcal{J}_1} \frac{T_{\mu}}{\epsilon} z^{\mu} + \frac{1}{2} \sum_{\mu \in \mathcal{J}_3} \frac{T_{\mu}}{\epsilon} z^{\mu},$$

$$\vartheta_2 = \vartheta_2(x, T; \epsilon, z) := \frac{r}{\epsilon} \log z - \sum_{\mu \in \mathcal{J}_2} \frac{T_{\mu}}{\epsilon} z^{\mu} - \frac{1}{2} \sum_{\mu \in \mathcal{J}_3} \frac{T_{\mu}}{\epsilon} z^{\mu},$$

and the index sets $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ are defined by

$$\mathcal{J}_1 := \mathcal{I}_1 \setminus \mathcal{I}_2, \quad \mathcal{J}_2 := \mathcal{I}_2 \setminus \mathcal{I}_1, \quad \mathcal{J}_3 := \mathcal{I}_1 \cap \mathcal{I}_2.$$ 

It is easy to verify that $\psi_1, \psi_2$ satisfy the following equations:

$$L \psi_i = \lambda_i \psi_i, \quad e^{\partial_{T_{\mu}}^i} = P_{\mu} \psi_i, \quad \mu \in \mathcal{I}, \quad i = 1, 2.$$ 

Here $\lambda_1 = z^{m/\ell}, \lambda_2 = z^{n/\ell}$, and

$$P_{\mu} := \begin{cases} 
\left( L^{-\frac{\mu}{\epsilon}} \right)^+, & \mu \in \mathcal{J}_1, \\
-\left( L^{-\frac{\mu}{\epsilon}} \right)^-, & \mu \in \mathcal{J}_2, \\
\frac{1}{2} \left( L^{-\frac{\mu}{\epsilon}} \right)^+_+ - \frac{1}{2} \left( L^{-\frac{\mu}{\epsilon}} \right)^-_-, & \mu \in \mathcal{J}_3.
\end{cases}$$

We also define the dual wave functions $\psi_1^* \psi_2$ by

$$\psi_1^* := (\Phi_1^{-1})^* e^{-\vartheta_1}, \quad \psi_2^* := (\Phi_2^{-1})^* e^{-\vartheta_2}.$$ 

Here we recall that $\Lambda^* := \Lambda^{-1}$, and for two difference operators $A$ and $B$,

$$(AB)^* := B^* A^*.$$ 

**Theorem 2.4** A formal series $u \in \mathbb{C}[[x-1, T; \epsilon]]$ satisfies the FVH if and only if the associated wave functions and their duals satisfy the following bilinear equations: $\forall \ell \in \mathbb{Z},$

$$\text{res}_z \psi_1(x, T; \epsilon, z) \psi_1^* \left( x - \frac{\ell}{r} \epsilon, T'; \epsilon, z \right) \frac{dz}{z} = \text{res}_z \psi_2(x, T; \epsilon, z) \psi_2^* \left( x - \frac{\ell}{r} \epsilon, T'; \epsilon, z \right) \frac{dz}{z}.$$ 

**Lemma 2.5** Two operator-valued formal series of the form

$$D_1(x, T) \in \mathbb{C}((\Lambda_3^{-1}, \epsilon)) [[x-1, T]], \quad D_2(x, T) \in \mathbb{C}((\Lambda_3, \epsilon)) [[x-1, T]]$$

are equal if and only if the following identity holds true: $\forall \ell \in \mathbb{Z},$

$$\text{res}_z D_1(x, T) (\psi_1(x, T; z)) \Lambda_3^\ell (\psi_1^* (x, T'; z)) \frac{dz}{z} = \text{res}_z D_2(x, T) (\psi_2(x, T; z)) \Lambda_3^\ell (\psi_2^* (x, T'; z)) \frac{dz}{z}.$$
For an arbitrary function \( \phi \), we have
\[
\phi(x, T; \epsilon; z) := \Phi_1(x, T; \epsilon) |_{\Lambda_3 \rightarrow z}, \quad \phi(x, T; \epsilon; z) := \Phi_2(x, T; \epsilon) |_{\Lambda_3 \rightarrow z^{-1}},
\]
(2.30)

Comparing the coefficients of \( \epsilon^i \), \( i = 1, 2 \), we get the following formal series:
\[
\phi_1(x, T; \epsilon; z) := \Phi_1^*(x, T; \epsilon) |_{\Lambda_3 \rightarrow z}, \quad \phi_2(x, T; \epsilon; z) := \Phi_2^*(x, T; \epsilon) |_{\Lambda_3 \rightarrow z}.
\]
(2.31)

Clearly, we have
\[\phi_1 = \sum_{i \geq 0} a_{1,i}(x, T; \epsilon) z^{-i}, \quad \phi_2 = \sum_{i \geq 0} a_{2,i}(x, T; \epsilon) z^{-i}.\]

For an arbitrary function \( f(T) \), denote
\[
f(T - [z^{-1}]_1) = \exp \left( - \sum_{\mu \in I_1} \frac{z^{-\mu}}{\mu/p} \frac{\partial}{\partial T^\mu} \right) f(T),
\]
\[
f(T - [z^{-1}]_2) = \exp \left( - \sum_{\mu \in I_2} \frac{z^{-\mu}}{\mu/q} \frac{\partial}{\partial T^\mu} \right) f(T).
\]

**Theorem 2.6** For an arbitrary solution \( u(x, T; \epsilon) \) in \( C[[x - 1, T; \epsilon]] \) to the FVH (1.3), there exists a choice of dressing operators \( \Phi_i, i = 1, 2 \) such that the associated formal series \( \phi_i, \phi_i^* \), \( i = 1, 2 \) can be represented in terms of a certain formal power series \( \tau \in C((\epsilon))[[x - 1, T]] \) as follows:
\[
\phi_1(x, T; \epsilon; z) = \frac{\tau(x, T - [z^{-1}]_1; \epsilon)}{\tau(x, T; \epsilon)}, \quad \phi_2(x, T; \epsilon; z) = \frac{\tau(x - \epsilon/r, T - [z^{-1}]_2; \epsilon)}{\tau(x, T; \epsilon)},
\]
(2.32)
\[
\phi_1^*(x, T; \epsilon; z) = \frac{\tau(x - \epsilon/r, T + [z^{-1}]_1; \epsilon)}{\tau(x - \epsilon/r, T; \epsilon)}, \quad \phi_2^*(x, T; \epsilon; z) = \frac{\tau(x, T + [z^{-1}]_2; \epsilon)}{\tau(x - \epsilon/r, T; \epsilon)}.
\]
(2.33)

We call \((\Phi_1, \Phi_2, \tau)\) a dressing triple for the FVH. We also call the formal power series \( \tau \) the Sato type tau-function of the solution \( u \). It is uniquely determined up to a factor of the form \((2.9)\).

**Proof** We prove the theorem by employing the method used in [29]. Firstly, by using the bilinear equations (2.28), we can prove that
\[
\phi_1(x, T; \epsilon; \zeta) \phi_1^* \left( x + \frac{\epsilon}{r}, T - [\zeta^{-1}]_1 - [\xi^{-1}]_1; \epsilon; \zeta \right)
= \phi_1(x, T; \epsilon; \xi) \phi_1^* \left( x + \frac{\epsilon}{r}, T - [\xi^{-1}]_1 - [\xi^{-1}]_1; \epsilon; \xi \right),
\]
(2.34)
\[
\phi_1(x, T; \epsilon; \zeta) \phi_1^* \left( x, T - [\zeta^{-1}]_1 - [\xi^{-1}]_2; \epsilon; \zeta \right)
= \phi_2(x, T; \epsilon; \xi) \phi_2^* \left( x, T - [\xi^{-1}]_1 - [\xi^{-1}]_2; \epsilon; \xi \right),
\]
(2.35)
\[
\phi_2(x, T; \epsilon; \zeta) \phi_2^* \left( x - \frac{\epsilon}{r}, T - [\zeta^{-1}]_2 - [\xi^{-1}]_2; \epsilon; \zeta \right)
= \phi_2(x, T; \epsilon; \xi) \phi_2^* \left( x - \frac{\epsilon}{r}, T - [\xi^{-1}]_2 - [\xi^{-1}]_2; \epsilon; \xi \right).
\]
(2.36)

Comparing the coefficients of \( \zeta^0 \) of the above equations (2.34), (2.35) respectively, we obtain
\[
\phi_1(x, T; \epsilon; \xi) \phi_1^* \left( x + \frac{\epsilon}{r}, T - [\xi^{-1}]_1; \epsilon; \xi \right) = 1, \quad \phi_2(x, T; \epsilon; \xi) \phi_2^* \left( x, T - [\xi^{-1}]_2; \epsilon; \xi \right) = 1.
\]
(2.37)
By using these two relations we can eliminate \( \phi_1^* \) and \( \phi_2^* \) from (2.34)–(2.36) and obtain

\[
\phi_1 \left( x, T - \left[ \xi^{-1} \right]_1 ; \epsilon ; \xi \right) \phi_1 \left( x, T ; \epsilon ; \xi \right) = \phi_1 \left( x, T - \left[ \xi^{-1} \right]_1 ; \epsilon ; \xi \right) \phi_1 \left( x, T ; \epsilon ; \xi \right), \tag{2.38}
\]
\[
\phi_1 \left( x - \frac{\epsilon}{r}, T - \left[ \xi^{-1} \right]_2 ; \epsilon ; \xi \right) \phi_2 \left( x, T ; \epsilon ; \xi \right) = \phi_2 \left( x - \frac{\epsilon}{r}, T - \left[ \xi^{-1} \right]_2 ; \epsilon ; \xi \right) \phi_1 \left( x, T ; \epsilon ; \xi \right), \tag{2.39}
\]
\[
\phi_2 \left( x - \frac{\epsilon}{r}, T - \left[ \xi^{-1} \right]_2 ; \epsilon ; \xi \right) \phi_2 \left( x, T ; \epsilon ; \xi \right) = \phi_2 \left( x - \frac{\epsilon}{r}, T - \left[ \xi^{-1} \right]_2 ; \epsilon ; \xi \right) \phi_2 \left( x, T ; \epsilon ; \xi \right). \tag{2.40}
\]

Equations (2.38) and (2.40) imply the existence of two formal series

\[
\tau_1, \tau_2 \in \mathbb{C}((\epsilon))[[x - 1, T]]
\]
satisfying the equations

\[
\phi_1(x, T; \epsilon; z) = \frac{\tau_1(x, T - \left[ z^{-1} \right]_1; \epsilon)}{\tau_1(x, T; \epsilon)}, \quad \phi_2(x, T; \epsilon; z) = \frac{\tau_2(x - \epsilon/r, T - \left[ z^{-1} \right]_2; \epsilon)}{\tau_2(x, T; \epsilon)}.
\]

Define \( f = \log \tau_2 - \log \tau_1 \), then we obtain from equation (2.39) that

\[
f \left( x - \frac{\epsilon}{r}, T - \left[ \xi^{-1} \right]_1 - \left[ \xi^{-1} \right]_2; \epsilon \right) + f(x, T; \epsilon) = f \left( x - \frac{\epsilon}{r}, T - \left[ \xi^{-1} \right]_2; \epsilon \right) + f \left( x, T - \left[ \xi^{-1} \right]_1; \epsilon \right),
\]
whose general solution in \( \mathbb{C}((\epsilon))[[x - 1, T]] \) has the following form

\[
f(x, T; \epsilon) = f_1 \left( x, T\prime; \epsilon \right) + f_2 \left( T\prime; \epsilon \right) + f_0(T''; \epsilon) + b(\epsilon),
\]

where

\[
T\prime = (T_\mu \mid \mu \in J_1), \quad T'' = (T_\mu \mid \mu \in J_2), \quad T''' = (T_\mu \mid \mu \in J_3),
\]
and \( f_0 \in \mathbb{C}((\epsilon))[[T'']] \) is linear in \( T''' \),

\[
f_1 \left( x, T\prime; \epsilon \right) \in \mathbb{C}((\epsilon))[[x - 1, T\prime]], \quad f_2 \left( T\prime; \epsilon \right) \in \mathbb{C}((\epsilon))[[T\prime]], \quad b(\epsilon) \in \mathbb{C}((\epsilon)).
\]

Let \( \tau = \tau_1 e^{h} \), then we have

\[
\log \phi_1(x, T; \epsilon; z) = \log \tau \left( x, T - \left[ z^{-1} \right]_1; \epsilon \right) - \log \tau(x, T; \epsilon),
\]
\[
\log \phi_2(x, T; \epsilon; z) = \log \tau \left( x + h \epsilon, T - \left[ z^{-1} \right]_2; \epsilon \right) - \log \tau(x, T; \epsilon) - \log \tau(x, T; \epsilon) - \log C(\epsilon; z)
\]
for some \( C(\epsilon; z) \in \mathbb{C}[[\epsilon; z^{-n}]] \).

Finally, by replacing \( \Phi_2 \) with \( \Phi_2 \circ C(\epsilon; \Lambda_3^{-1}) \), we find that \( \Phi_1 \), the new \( \Phi_2 \), and \( \tau(x, T; \epsilon) \) satisfy equations (2.32). Equations (2.33) then follow from (2.34). The uniqueness statement of the theorem can be verified straightforwardly. The theorem is proved.

2.3 The equivalence of the two definitions of the tau-function

For an arbitrary solution \( u \in \mathbb{C}[[x - 1, T; \epsilon]] \) to the FVII, let \( \tau_s \) be the tau-function defined in the Section 2.2. We prove in this subsection the equivalence of \( \tau_s \) with the tau-function defined in Section 2.1. In other words, we are to prove that \( \tau_s \) satisfies the relations (2.3)–(2.8). Let us start with the proof of the following lemma.
Lemma 2.7 The following formula holds true:

\[ \epsilon (\Lambda_3 - 1) \frac{\partial \log \tau_s}{\partial T_\mu} = \text{res}_{\Lambda_3} L^{-\mu/r}, \quad \mu \in \mathcal{I}. \tag{2.41} \]

Proof Let \( \Phi_1, \Phi_2 \) be the dressing operators for the FVH such that \( \Phi_1, \Phi_2, \tau_s \) form a dressing triple. Recall that \( \Phi_2 \) has the form \( \Phi_2 = \sum_{i \geq 0} a_{2,i} \Lambda_3^i \). By using the second formula of (2.32) we have

\[ (\Lambda_3 - 1) \log \tau_s(x, T; \epsilon) = \log \epsilon a_{2,0}(x, T; \epsilon). \tag{2.42} \]

Also, it follows from the formulae (2.18)–(2.19) that

\[ \epsilon \frac{\partial a_{2,0}}{\partial T_\mu} = (\text{res}_{\Lambda_3} L^{-\mu/r}) a_{2,0}. \]

The lemma is then proved by substituting (2.42) in the above formula. \( \square \)

Lemma 2.8 \( \forall \lambda, \mu \in \mathcal{I} \), we have the following relations:

\[ \epsilon^2 \frac{\partial^2 \log \tau_s}{\partial T_\lambda \partial T_\mu} \in \mathcal{A}_u[[\epsilon]], \tag{2.43} \]

\[ \deg \epsilon^2 \frac{\partial^2 \log \tau_s}{\partial T_\lambda \partial T_\mu} = 0. \tag{2.44} \]

Proof Let \( \psi_1, \psi_2 \) be the wave functions associated to \( \Phi_1, \Phi_2 \), respectively. It follows from (2.30) and (2.20) that

\[ \epsilon \frac{\partial T_\mu \psi_1}{\psi_1} - \epsilon \frac{\partial T_\mu \psi_1}{\psi_1} = \frac{P_\mu \psi_1}{\psi_1} - \epsilon \frac{1}{\psi_1} \left( L^{-\mu/r} \right)_- \psi_1, \quad \mu \in \mathcal{I}. \tag{2.45} \]

Observe that the operator \( \left( L^{-\mu/r} \right)_- \) can be represented in the form

\[ \left( L^{-\mu/r} \right)_- = - \sum_{k \geq 1} f_{\mu,k}(u, u_x, \ldots) L^{k \mu/r}, \tag{2.46} \]

where \( f_{\mu,k} = f_{\mu,k}(u, u_x, \ldots) \in \mathcal{A}_u[[\epsilon]] \) are given by the following recursion relations:

\[ f_{\mu,k} = - \text{res}_{\Lambda_3} \left( L^{-\mu/r} \Lambda_3^k \right) - \sum_{j=1}^{k-1} f_{\mu,j} \text{res}_{\Lambda_3} \left( L^{-j \mu/r} \Lambda_3^k \right), \quad k \geq 1. \tag{2.47} \]

Then from Lemma 2.2 it follows that \( \deg f_{\mu,k} = 0 \). Recalling \( L^{-\mu/r} \psi_1 = z \psi_1 \), we find that formula (2.45) can be written as

\[ \epsilon \frac{\partial T_\mu \phi_1}{\phi_1} = \sum_{k \geq 1} f_{\mu,k} z^{-k} \epsilon \left[ \mathcal{A}_u[[\epsilon]] \right][[z^{-1}]], \quad \mu \in \mathcal{I}. \tag{2.48} \]

From Theorem 2.6 we know that

\[ \epsilon \frac{\partial T_\mu \phi_1}{\phi_1} = - z^{-1} \epsilon^2 \frac{\partial T_\mu \partial T_\rho \log \tau_s}{2} - \left( \epsilon^2 \frac{\partial T_\mu \partial T_\rho \log \tau_s}{2} - \epsilon^3 \frac{\partial T_\mu \partial T_\rho \log \tau_s}{2} \right) + \cdots. \tag{2.49} \]

Then the relations (2.43)–(2.44) for \( \lambda \in \mathcal{I}_1, \mu \in \mathcal{I} \) follow from (2.48) and (2.49). The relations (2.43)–(2.44) for \( \lambda \in \mathcal{I}_2, \mu \in \mathcal{I} \) can be proved in a similar way. The lemma is proved. \( \square \)
Lemma 2.9 The Sato type tau-function $\tau_s$ satisfies the relation
\[
\epsilon \frac{\partial^2 \log \tau_s}{\partial T_{\lambda} \partial T_{\mu}} - \frac{1}{p + q} \frac{\lambda \mu}{\lambda + \mu} c_{\lambda} c_{\mu} e^{(\lambda + \mu)u/p} \in \epsilon A_u[[\epsilon]], \quad \forall \lambda, \mu \in \mathcal{I}.
\] (2.50)

Proof Consider the case $\lambda = kp, \mu = lp$ with $k, l \geq 1$. From formulae (2.13), (2.14) and (2.47), it follows that the differential polynomials $f_{kp,l}$ must take the form
\[
f_{kp,l} = g_{kp,lp}e^{(k+l)u} + f_{ kp,l}^{[1]}(u, u_x, \ldots ; z),
\]
where $f_{ kp,l}^{[1]} = f_{ kp,l}^{[1]}(u, u_x, \ldots ; z) \in \epsilon A_u[[\epsilon]][[z^{-1}]]$ satisfy the condition $\deg f_{ kp,l}^{[1]} = 0$, and the constants $g_{kp,l}$ are satisfy
\[
g_{kp,l} = -\left(-\frac{k p}{r} \frac{k}{k + l} \right) - \sum_{j=1}^{l-1} \left(-\frac{j p}{r} \frac{l - j}{l}ight) g_{kp,j}, \quad l \geq 1.
\]
Solving this recursion relation, we obtain
\[
g_{kp,l} = -\frac{p}{p + q} \frac{k}{k + l} c_{kp} c_{lp}.
\] (2.51)

Here we recall that the numbers $c_{\mu}, \mu \in \mathcal{I}$ are defined in (1.6). Therefore, from (2.48) we have
\[
\epsilon \frac{\partial \log \phi_1}{\partial T_{kp}} = -\sum_{l \geq 1} \frac{p}{p + q} \frac{k}{k + l} c_{kp} c_{lp} e^{(k+l)u} z^{-l} + G_k(u, u_x, \ldots ; z),
\] (2.52)
with $G_k = G_k(u, u_x, \ldots ; z) \in \epsilon A_u[[\epsilon]][[z^{-1}]]$ satisfying $\deg G_k = 0$.

On the other hand, from the definition of the Sato type tau-function, we know that the derivatives of $\log \tau_s$ with respect to $T'$s are Laurent series in $\epsilon$ with degree being greater or equal to $-2$. So from (2.32) it follows that
\[
\epsilon \frac{\partial \log \phi_1}{\partial T_{kp}} = \left(\epsilon - \sum_{l \geq 1} \frac{1}{l} \frac{p}{p + q} \frac{k}{k + l} c_{kp} c_{lp} \right) \frac{\partial \log \tau_s}{\partial T_{kp}} = -\sum_{l \geq 1} \frac{\epsilon^2}{l} \frac{\partial \log \tau_s}{\partial T_{kp}} \frac{\partial \log \tau_s}{\partial T_{lp}} z^{-l} + \tilde{G}_k(u, u_x, \ldots ; z)
\]
with $\tilde{G}_k = \tilde{G}_k(u, u_x, \ldots ; z) \in \epsilon A_u[[\epsilon]][[z^{-1}]]$ satisfying $\deg \tilde{G}_k = 0$. By comparing this formula with (2.52) we obtain
\[
\epsilon \frac{\partial^2 \log \tau_s}{\partial T_{kp} \partial T_{lp}} - \frac{p}{p + q} \frac{k l}{k + l} c_{kp} c_{lp} e^{(k+l)u} \in \epsilon A_u[[\epsilon]].
\]
Similarly, we can prove the cases when $\lambda$ and $\mu$ take other values. The lemma is proved. \qed

Proposition 2.10 The function $\tau_s(x + /2p, T; \epsilon)$ satisfies the defining relations (2.6) - (2.8) for the tau-function of the FVH.

Proof From equation (2.17) it follows that
\[
u(x, T; \epsilon) = (1 - \Lambda_1^{-1}) \log a_{2,0}(x, T; \epsilon).
\]

Therefore, by using the formula (2.42), we obtain that \( \tau_s(x + \epsilon/2 p, T; \epsilon) \) satisfies equation (2.6) (which is the first defining equation for \( \tau_{DZ} \)), i.e.,

\[
U(x, T; \epsilon) = \left( \Lambda_1^{1/2} - \Lambda_3^{-1/2} \right) \left( \Lambda_1^{1/2} - \Lambda_1^{-1/2} \right) \log \tau_s(x + \epsilon/2 p, T; \epsilon).
\]

(2.53)

Differentiating the both sides of (2.53) with respect to \( T_\mu \) and \( T_\lambda \), and noticing that the FVH (1.8) can be represented in the form

\[
\frac{\partial u}{\partial T_\mu} = \epsilon^{-1} \left( \Lambda_1^{1/2} - \Lambda_1^{-1/2} \right) \left( \Lambda_1^{-1/2} \text{res}_{\Lambda_3} L^\mu \right),
\]

(2.54)

and using (2.5) we obtain

\[
\left( \Lambda_3^{1/2} - \Lambda_3^{-1/2} \right) \left( \Lambda_1^{1/2} - \Lambda_1^{-1/2} \right) \Omega_{\lambda,\mu} = \left( \Lambda_3^{1/2} - \Lambda_3^{-1/2} \right) \left( \Lambda_1^{1/2} - \Lambda_1^{-1/2} \right) \epsilon^2 \frac{\partial^2}{\partial T_\lambda \partial T_\mu} \log \tau_s(x + \epsilon/2 p, T; \epsilon), \quad \lambda, \mu \in \mathcal{I}.
\]

Therefore,

\[
\Delta_{\lambda,\mu} := \epsilon^2 \frac{\partial^2 \log \tau_s(x + \epsilon/2 p, T; \epsilon)}{\partial T_\lambda \partial T_\mu} - \Omega_{\lambda,\mu} \in \text{Ker} \frac{\partial^2}{\partial x^2},
\]

where the Ker is taken in \( A_u[[\epsilon]] \). Since Lemmas 2.2, 2.8 imply deg \( \Delta_{\lambda,\mu} = 0 \), and Lemmas 2.2, 2.9 imply \( \Delta_{\lambda,\mu} \in \epsilon A_u[[\epsilon]] \), we find that \( \Delta_{\lambda,\mu} \) must be zero. The proposition is then proved by using Lemma 2.7.

\[\square\]

3 The topological tau-function of the FVH

In this section we restrict our discussions to the rational case, i.e.,

\[
p = \frac{1}{m}, \quad q = \frac{1}{n}, \quad r = -\frac{1}{h},
\]

(3.1)

where \( m, n \in \mathbb{N} \) are coprime, \( h = m + n \).

In our previous work [20], we showed that the formal power series \( Z(x, T; \epsilon) \) defined in (1.11) satisfies a collection of Virasoro constraints

\[
L_k Z(x, T; \epsilon) = 0, \quad k \geq 0,
\]

where

\[
L_0 = \sum_{\mu \in \mathcal{I}} \mu T_\mu \frac{\partial}{\partial T_\mu} + \frac{x^2}{2 mn h \epsilon^2} + \frac{\sigma_1}{24} - \frac{\Gamma(m) \Gamma(n)}{\Gamma(1 + h)} \frac{\partial}{\partial T_1},
\]

(3.2)

\[
L_k = \sum_{\mu \in \mathcal{I}} \mu T_\mu \frac{\partial}{\partial T_{\mu+k}} + \frac{x}{mn} \frac{\partial}{\partial T_k} - \frac{\Gamma(m) \Gamma(n)}{\Gamma(1 + h)} \frac{\partial}{\partial T_{1+k}}
\]

\[+ \frac{\epsilon^2}{2m} \sum_{\lambda+\mu = k} \frac{\partial^2}{\partial T_\lambda \partial T_\mu} + \frac{\epsilon^2}{2n} \sum_{\lambda+\mu = k} \frac{\partial^2}{\partial T_\lambda \partial T_\mu}, \quad k \geq 1,
\]

(3.3)
and $\sigma_1$ is defined in (1.16). We also know from [6] that $Z(x,T;\epsilon)$ has the genus expansion of the form
\begin{equation}
\log Z(x,T;\epsilon) = \frac{1}{mnhe^2}A + \sum_{g\geq 0}(mnh)^{g-1}\epsilon^{2g-2}\mathcal{H}_g(t(x,T))
\end{equation}
\begin{equation}
= \epsilon^{-2}\hat{F}_0(x,T) + \sum_{g\geq 1}\epsilon^{2g-2}\hat{F}_g\left(v_0(t(x,T)),v'_0(t(x,T)),\ldots,v^{(3g-2)}_0(t(x,T))\right),
\end{equation}
where $A$ is defined by (1.12), the prime “$'$” means taking derivative with respect to $x$,
\begin{equation}
\hat{F}_0(x,T) = \frac{1}{mnh}(A + \mathcal{H}_0(t(x,T))),
\end{equation}
\begin{equation}
v_0(t) := \frac{\partial^2}{\partial t^2}\mathcal{H}_0(t),
\end{equation}
and the Virasoro constraints are equivalent to the so-called loop equation for the functions $\{\hat{F}_g \mid g \geq 2\}$. The solution to the loop equation is unique, and we can find $\hat{F}_g$ ($g \geq 2$) by solving the loop equation recursively.

In order to prove Theorem 1.1 we are to construct in this section a particular tau-function $\tau_{\text{top}}$ for the FVH which has a genus expansion of the form (3.5), and to prove in the next section that it satisfies the Virasoro constraints $L_k\tau_{\text{top}} = 0$, $k \geq 0$. Then Theorem 1.1 in the rational case follows from the uniqueness theorem of the loop equation that was proven in [20]. The theorem for the general case can be proved from the one for the rational case by using a continuation argument.

Let us present the main results of this section in the following theorem.

**Theorem 3.1** For the rational case, there exists a nontrivial tau-function $\tau_{\text{top}} = \tau_{\text{top}}(x, T; \epsilon)$ of the FVH (1.8), living in the ring $\mathbb{C}(((\epsilon^2))[[x - 1, T]]$, which satisfies the following two equations:
\begin{equation}
L_0\tau_{\text{top}} = 0,
\end{equation}
\begin{equation}
K_0\tau_{\text{top}} + \frac{\tau_{\text{top}}}{24} = 0,
\end{equation}
where $L_0$ is defined in (3.2), and
\begin{equation}
K_0 := \sum_{\mu \in I}T_{\mu}\frac{\partial}{\partial T_{\mu}} + x\frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial \epsilon} - \frac{\Gamma(m)\Gamma(n)}{\Gamma(1+h)}\frac{\partial}{\partial T_1}.
\end{equation}
Moreover, $\tau_{\text{top}}$ possesses the genus expansion of the same form with (3.5).

We note that the non-trivial tau-function $\tau_{\text{top}}$ in the above theorem is **uniquely** determined by (3.6)–(3.7) up to multiplying by a constant factor that is irrelevant for the current study. We call this tau-function the **topological tau-function** for the FVH.

The construction of the tau-function $\tau_{\text{top}}$ consists of five steps:

**Step I** To fix the particular solution $u_{\text{top}}(x, T; \epsilon)$ to the FVH, corresponding to $\tau_{\text{top}}$, by using the initial value problem for the FVH which is implied by the Virasoro constraint $L_0\tau_{\text{top}} = 0$. 

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Step II. To show that the solution $u_{\text{top}}(x, T; \epsilon)$ has the following form of genus expansion:

$$u_{\text{top}}(x, T; \epsilon) = v_{\text{top}} + \sum_{g \geq 1} \epsilon^{2g} A_g \left( v'_{\text{top}}, v''_{\text{top}}, \ldots, v^{(3g)}_{\text{top}} \right),$$  \hspace{1cm} (3.9)

where $v_{\text{top}}(x, T) = u_{\text{top}}(x, T; 0)$, $v'_{\text{top}} = \partial_x v_{\text{top}}$, $v^{(k)}_{\text{top}} = \partial_x^k v_{\text{top}}$.

Step III. To show a certain transcendency of the function $v_{\text{top}}$.

Step IV. To show the existence of functions $\tilde{F}_g(z_0, z_1, \ldots, z_{3g-2})$ such that

$$A_g \left( v'_{\text{top}}, v''_{\text{top}}, \ldots, v^{(3g)}_{\text{top}} \right) = \partial_x^2 \tilde{F}_g \left( v_{\text{top}}, v'_{\text{top}}, v''_{\text{top}}, \ldots, v^{(3g)}_{\text{top}} \right), \hspace{1cm} g \geq 1. \hspace{1cm} (3.10)$$

Step V. To construct the tau function $\tau_{\text{top}}$ satisfying the conditions (2.6)-(2.8), and show that $\log \tau_{\text{top}}$ has the genus expansion of the following form:

$$\log \tau_{\text{top}} = \frac{1}{c^2} F_0(x, T) + \sum_{g \geq 1} \epsilon^{2g-2} F_g \left( v_{\text{top}}, v'_{\text{top}}, \ldots, v^{(3g-2)}_{\text{top}} \right), \hspace{1cm} (3.11)$$

where $F_0(x, T)$ satisfies the relation $v_{\text{top}}(x, T) = \partial_x^2 F_0(x, T)$.

Proof of Theorem 3.1. Let us prove the theorem step by step as outlined above.

Step I. To find the solution $u_{\text{top}}(x, T; \epsilon)$ to the FVH, let us first fix its initial value $u_{\text{top}}(x, 0; \epsilon)$ by using the condition (3.6). Assuming the existence of such a solution with the associated tau-function $\tau_{\text{top}}$, we apply the operator $\Lambda_{\text{top}}^{-1/2}$ to the both sides of the equation $L_{\tau_{\text{top}}}/\tau_{\text{top}} = 0$, and put $T = 0$, then we arrive at the equation

$$\epsilon (\Lambda_3 - 1) \Lambda_2^{-1/2} \left( x^2 \frac{2mnhc^2}{\Gamma(n)} \frac{\partial \log \tau_{\text{top}}}{\partial T} \right) \Bigg|_{T=0} = 0. \hspace{1cm} (3.12)$$

By using the defining relation (2.7) for the tau-functions and the fact that

$$\epsilon (\Lambda_3 - 1) \Lambda_2^{-1/2} = \epsilon \left( h \epsilon \partial_x + \frac{n\hbar \epsilon^2}{2} \partial_x^2 + O(\epsilon^3) \right),$$

we obtain from (3.12) the following equation:

$$\left( \text{res}_{\Lambda_3} L^h \right) \Bigg|_{T=0} = \left( \frac{m+n}{m} \right) \left( x + \frac{n\epsilon}{2} \right). \hspace{1cm} (3.13)$$

On the other hand, it follows from (1.9), (3.1) that

$$\text{res}_{\Lambda_3} L^h = \text{res}_{\Lambda_3} \left( \Lambda^m + e^{u_{\text{top}}} \Lambda^{-n} \right)^{m+n}$$

$$= \sum_{0 \leq \alpha_1 \leq \ldots \leq \alpha_m \leq n} \exp \left( \sum_{j=1}^{m} u_{\text{top}}(x + \epsilon \alpha_j m - \epsilon(j-1)n, T; \epsilon) \right). \hspace{1cm} (3.14)$$

Then by using (3.13)-(3.14) and by making a shift $x \to x - \frac{n}{2} \epsilon$, we arrive at equation (1.17) that is satisfied by $u_{\text{top}}(x, 0; \epsilon)$.

It is easy to show that there exists a unique solution $V(x; \epsilon)$ to equation (1.17) in $\mathbb{C}[x-1; \epsilon]$. Combining this with the statement that the FVH (1.8) is integrable [22], we arrive at the following proposition.
Proposition 3.2 There is a unique solution
\[ u_{\text{top}} = u_{\text{top}}(x, T; \epsilon) \in \mathbb{C}[[x-1, T; \epsilon]] \]
to the FVH satisfying the initial condition
\[ u_{\text{top}}(x, 0; \epsilon) = V(x; \epsilon), \quad (3.15) \]
where \( V(x; \epsilon) \) is the unique solution to equation (1.17) in \( \mathbb{C}[[x-1; \epsilon]] \). We call \( u_{\text{top}} \) the topological solution to the FVH.

We show next that \( V(x; \epsilon) \) and \( u_{\text{top}}(x, T; \epsilon) \) can be represented as power series of \( \epsilon^2 \). To see this let us introduce some notations. Let \( \mathcal{Y} \) denote the set of partitions of non-negative integers, i.e.,
\[ \mathcal{Y} := \left\{ (J_1, J_2, \ldots) \in \mathbb{Z}^N \mid J_1 \geq J_2 \geq \cdots \geq 0, \sum_{i \geq 1} J_i < \infty \right\}. \]
For a partition \( J = (J_1, J_2, \ldots) \in \mathcal{Y} \), denote by \( l(J) \) the number of non-zero components of \( J \), by \( |J| := J_1 + J_2 + \cdots \) the weight of \( J \), and by \( \mathcal{Y}_k \) the set of all partitions of weight \( k \). We have the following lemma.

Lemma 3.3 For each \( \lambda \in \mathcal{I} \), we have
\[ \Lambda_{1}^{\frac{1}{2}} \text{res}_{\Lambda_3} L^\lambda h = c_\lambda e^{\lambda_\mu u} \sum_{g \geq 0} \epsilon^{2g} M^g_{\lambda} \left( u', \ldots, u^{(2g)} \right), \quad (3.16) \]
where \( c_\lambda \) is defined in (1.6),
\[ M^0_{\lambda} := 1, \quad M^g_{\lambda} := M^g_{\lambda} \left( u', \ldots, u^{(2g)} \right) = \sum_{J \in \mathcal{Y}_{2g}} a_{g,J}^\lambda u^{(J)} \in \mathcal{A}_u, \ g \geq 1, \quad (3.17) \]
and
\[ u^{(k)} := \partial^k_x u, \quad u^{(J)} := u^{(J_1)} \cdots u^{(J_{l(J)})}, \quad k \geq 0, \ J = (J_1, J_2, \ldots). \]
Moreover, the coefficients \( a_{g,J}^\lambda \) (\( J \in \mathcal{Y}, g \geq 1 \)) are polynomials of \( \lambda \) with degrees less than or equal to \( l(J) + |J|/2 \).

Proof We first show that, for each \( \lambda \in \mathcal{I} \), \( \Lambda_{1}^{\frac{1}{2}} \text{res}_{\Lambda_3} L^\lambda h \in \mathcal{A}[[\epsilon^2]] \). Indeed,
\[ \Lambda_{1}^{\frac{1}{2}} \text{res}_{\Lambda_3} L^\lambda h = \Lambda_{1}^{\frac{1}{2}} \text{res}_{\Lambda_3} (\Lambda_2^{-1} + e^{\Lambda_1} u A_1)^\lambda h \]
\[ = \Lambda_{1}^{\frac{1}{2}} \text{res}_{\Lambda_3} (\Lambda_2^{-1} + e^{u A_1})^\lambda h, \quad (3.19) \]
which shows that \( \Lambda_{1}^{\frac{1}{2}} \text{res}_{\Lambda_3} L^\lambda h \) is invariant under the map \( \epsilon \mapsto -\epsilon \). Here, (3.18) is valid due to the fact that \( \text{res}_{\Lambda_3} \sum_{i \in \mathbb{Z}} \alpha_i \Lambda_3^i = \text{res}_{\Lambda_3} \sum_{i \in \mathbb{Z}} \Lambda_3^{-i} \circ \alpha_i \), and (3.19) is valid because of the fact that \( \text{res}_{\Lambda_3} L^\lambda h \big|_{u \mapsto \Lambda_1 u} = \Lambda_1 \text{res}_{\Lambda_3} L^\lambda h. \)
For $\lambda = 1/m$, we have

$$\Lambda_{1/\lambda}^{1/2} \res \Lambda^h/m = \frac{\Lambda_{1/\lambda}^{1/2} - \Lambda_{1/\lambda}^{-1/2}}{\Lambda_{1/\lambda}^{1/2} - \Lambda_{1/\lambda}^{-1/2}} \text{e}^u = \frac{h}{m} \text{e}^u \left( 1 + \sum_{g \geq 1} \epsilon^{2g} M_{1/m}^{[g]} \right),$$

(3.20)

from which it follows that $M_{1/m}^{[g]} \in \mathbb{C}[u', \ldots, u^{(2g)}]$ have the form (3.17). Note that Proposition 2.1 implies the following tau-symmetry property [6, 9]:

$$\frac{\partial}{\partial T_{1/m}} \left( \Lambda_{1/\lambda}^{1/2} \res \Lambda^h/m \right) = \frac{\partial}{\partial T_{\lambda}} \left( \Lambda_{1/\lambda}^{1/2} \res \Lambda^h/m \right), \forall \lambda \in \mathbb{I}. \quad (3.21)$$

From this equation we arrive at a certain recursion relation for $M_{1/m}^{[g]}$, $g \geq 1$, and we find that $M_{1/m}^{[g]}$ has the required form (3.17) with the coefficients $a_{g,J}(\lambda)$ being polynomials of $\lambda$ with degree less than or equal to $l(\mathcal{J}) + |\mathcal{J}|/2$. The lemma is proved. □

**Lemma 3.4** The unique solution to the difference equation (1.17) in the ring $\mathbb{Q}[x-1; \epsilon]$ can be represented in the form

$$V(x; \epsilon) = \frac{1}{m} \log x + \sum_{g \geq 1} \epsilon^{2g} P_g(m,n) x^{2g},$$

(3.22)

where $P_g(m,n) \in \mathbb{Q}[m,n,m^{-1}]$ for $g \geq 1$.

**Proof** Note that equation (1.17) can be written as

$$\res \Lambda^m \left( \Lambda + e^{V(x-\frac{1}{m} \epsilon) \Lambda^{-n}} \right)^h = \left( \frac{m+n}{m} \right)^x. \quad (3.23)$$

From Lemma 3.3 it follows that this equation can also be represented in the form

$$(\sum_{g \geq 0} \epsilon^{2g} M_{1/m}^{[g]} \left( V^{(1)}, \ldots, V^{(2g)} \right) ) e^{mV} = x,$$

(3.24)

where the coefficients $M_{1/m}^{[g]}$ are obtained recursively by using the tau-symmetry condition (3.21) with $\lambda = 1$. Then by substituting the expression (3.22) of $V(x; \epsilon)$ into (3.24), we can determine the coefficients $P_g(m,n)$ recursively and to see that they belong to $\mathbb{Q}[m,n,m^{-1}]$. The lemma is proved. □

**Remark 3.5** Although the formulae (3.23) and (3.24) are equivalent, it is more efficient to compute $P_g(m,n)$ by using (3.24) than by using (3.23). We will give some results of such a computation of $P_g(m,n)$ in Section 6.

From the above Lemmas 3.3, 3.4, we know that the right-hand-side of the equivalent form (2.54) of the FVH (1.18) is in $\mathcal{A}_u[\epsilon^2]$, and that the power series $V(x; \epsilon)$ is in $\mathbb{C}[[x-1; \epsilon^2]]$, respectively. Thus we arrive at the following corollary.

**Corollary 3.6** The topological solution $u_{\text{top}}$ is an element of $\mathbb{C}[[x-1, T; \epsilon^2]]$. 17
Step II. Let us proceed to prove that \( u_{\text{top}}(x, T; \epsilon) \) has the genus expansion form \([3.30]\) by using the Euler–Lagrange equation that it satisfies.

**Proposition 3.7** The formal power series \( u := u_{\text{top}} \) satisfies the following Euler–Lagrange equation:

\[
\sum_{\lambda \in \mathcal{I}} \lambda \bar{T}_\lambda \left( \lambda \text{res}_{\Lambda_3} L^h \right) \frac{\partial u}{\partial T_{\lambda}} = \frac{x + \frac{\epsilon}{2}}{mn} = 0, \tag{3.25}
\]

where \( \bar{T}_\lambda = T_{\lambda} - \frac{\Gamma(m)\Gamma(n)}{\Gamma(1+h)} \delta_{\lambda,1} \).

**Proof** Define a power series \( F(x, T; \epsilon) \in \mathbb{C}[\left[x - 1, T; \epsilon]\right] \) by

\[
F(x, T; \epsilon) := \sum_{\lambda \in \mathcal{I}} \lambda \bar{T}_\lambda \left( \lambda \text{res}_{\Lambda_3} L^h_{\text{top}} \right) + \frac{x + \frac{\epsilon}{2}}{mn}, \tag{3.26}
\]

where \( L_{\text{top}} := \Lambda_2 + e^{u_{\text{top}}} \Lambda_1^{-1} \). We are to show \( F(x, T; \epsilon) = 0 \). By using the relation \([3.10]\) and equation \([2.54]\) we obtain

\[
\frac{\partial F(x, T; \epsilon)}{\partial T_\mu} = \sum_{\lambda \in \mathcal{I}} \lambda \bar{T}_\lambda \left( \lambda \text{res}_{\Lambda_3} L^h_{\text{top}} \right) \frac{\partial u}{\partial T_{\lambda}} \left( \lambda \text{res}_{\Lambda_3} L^{\mu h}_{\text{top}} \right) + \mu \text{res}_{\Lambda_3} L^{\mu h}_{\text{top}}
\]

\[
= \sum_{\lambda \in \mathcal{I}} \lambda \bar{T}_\lambda \left( m \mu \frac{\partial u}{\partial T_{\lambda}} \text{res}_{\Lambda_3} L^h_{\text{top}} \right) + \mu \text{res}_{\Lambda_3} L^{\mu h}_{\text{top}}
\]

\[
= \sum_{\lambda \in \mathcal{I}} \lambda \bar{T}_\lambda \left( m \mu \frac{\partial u}{\partial T_{\lambda}} \text{res}_{\Lambda_3} L^h_{\text{top}} \right) + \mu \text{res}_{\Lambda_3} L^{\mu h}_{\text{top}}
\]

\[
= \sum_{\lambda \in \mathcal{I}} \lambda \bar{T}_\lambda \left( m \mu \frac{\partial u}{\partial T_{\lambda}} \text{res}_{\Lambda_3} L^h_{\text{top}} \right) + \mu \text{res}_{\Lambda_3} L^{\mu h}_{\text{top}}
\]

\[
= \sum_{\lambda \in \mathcal{I}} \lambda \bar{T}_\lambda \left( m \mu \frac{\partial u}{\partial T_{\lambda}} \text{res}_{\Lambda_3} L^h_{\text{top}} \right) + \mu \text{res}_{\Lambda_3} L^{\mu h}_{\text{top}}
\]

Here we denote \( u_{\text{top}} \) by \( u \), and \( \hat{F}(x, T; \epsilon) \in \mathbb{C}[\left[x - 1, T; \epsilon]\right] \) is the power series defined by

\[
\hat{F}(x, T; \epsilon) := e^{-1} (1 - \Lambda_1^{-1}) F(x, T; \epsilon) = \sum_{\lambda \in \mathcal{I}} \lambda \bar{T}_\lambda \frac{\partial u}{\partial T_{\lambda}} + \frac{1}{m}. \tag{3.28}
\]

Taking \( T = 0 \) in \([3.26]\) we obtain

\[
F(x, 0; \epsilon) = - \frac{\Gamma(m)\Gamma(n)}{\Gamma(1+h)} \text{res}_{\Lambda_3} L^h_{\text{top}} \bigg|_{T=0} + \frac{x + \frac{\epsilon}{2}}{mn},
\]

which vanishes since \( u_{\text{top}}(x, 0; \epsilon) = V(x; \epsilon) \) satisfies equation \([3.23]\) (or equivalently \([1.17]\)) so from \([3.28]\) it follows that

\[
\hat{F}(x, 0; \epsilon) = e^{-1} (1 - \Lambda_1^{-1}) F(x, 0; \epsilon) = 0. \tag{3.29}
\]
Then by mathematical induction we know from (3.27)–(3.28) that
\[
\frac{\partial^i F}{\partial T_{\mu_1} \cdots \partial T_{\mu_i}}(x, 0; \epsilon) = 0, \quad \forall i \geq 0, \quad \mu_1, \ldots, \mu_i \in \mathcal{I}.
\]
(3.30)
Thus the proposition is proved. □

Define a power series \( v_{\text{top}} \in \mathbb{C}[[x - 1, T]] \) by
\[
v_{\text{top}} = v_{\text{top}}(x, T) := u_{\text{top}}(x, T; \epsilon = 0).
\]
(3.31)
Then it satisfies the following dispersionless FVH:
\[
\frac{\partial v}{\partial T_\lambda} = \lambda c_{\lambda} e^{\lambda m v} \frac{\partial v}{\partial x}, \quad \lambda \in \mathcal{I}.
\]
(3.32)
It is clear from Lemma 3.3 and Proposition 3.7 that \( v_{\text{top}} \) also satisfies the following dispersionless Euler-Lagrange equation:
\[
\sum_{\lambda \in \mathcal{I}} \lambda c_{\lambda} \bar{T}_\lambda e^{\lambda m v} + \frac{x}{\lambda mn} = 0.
\]
(3.33)

**Lemma 3.8** There exist functions \( A_g = A_g(z_1, \ldots, z_{3g}) \in z_1^{-(4g-2)} \mathbb{C} [z_1, z_2, \ldots, z_{3g}], \quad g \geq 1, \)
(3.34)
satisfying the homogeneity condition
\[
\sum_{i \geq 1} i z_i \frac{\partial A_g}{\partial z_i} = 2g A_g, \quad g \geq 1,
\]
(3.35)
such that \( u_{\text{top}}(x, T; \epsilon) \) can be represented in the form
\[
u_{\text{top}}(x, T; \epsilon) = v_{\text{top}}(x, T) + \sum_{g \geq 1} \epsilon^{2g} A_g \left( v'_{\text{top}}(x, T), \ldots, v^{(3g)}_{\text{top}}(x, T) \right).
\]
(3.36)
Here \( v^{(k)}_{\text{top}}(x, T) = \partial^k_x v_{\text{top}}(x, T). \)

**Proof** According to Corollary 3.6, \( u_{\text{top}}(x, T; \epsilon) \) can be represented in the form
\[
u_{\text{top}}(x, T; \epsilon) = \sum_{g \geq 0} \epsilon^{2g} u^{[g]}_{\text{top}}(x, T), \quad u^{[0]}_{\text{top}}(x, T) = v_{\text{top}}(x, T),
\]
(3.37)
where \( u^{[g]}_{\text{top}}(x, T) =: u^{[g]}_{\text{top}} \in \mathbb{C}[[x - 1, T]]. \) The Euler–Lagrange equation (3.25) for \( u = u_{\text{top}} \) is then equivalent to (3.33) together with the following equations for \( u^{[g]}_{\text{top}}, \quad g \geq 1:
\[
\left( \sum_{\lambda \in \mathcal{I}} \lambda^2 c_{\lambda} \bar{T}_\lambda e^{\lambda m u_{\text{top}}} \right) u^{[g]}_{\text{top}} = \sum_{J, G, G' \in \mathcal{Y}} b_{J, G, G'} \left( \sum_{\lambda \in \mathcal{I}} \lambda c_{\lambda} a_{g, J}(\lambda) \bar{T}_\lambda e^{\lambda m u_{\text{top}}} \right) u^{[G]}_{\text{top}} \partial_J u^{[G']}_{\text{top}},
\]
(3.38)
where \( b_{J,G,G'} \) are some constants, \( a_{g,J}(\lambda) \) are the coefficients appearing in (3.37) (which are defined to be 0 when \( |J| = 0 \)), and

\[
\begin{align*}
&u^{[G]}_{\text{top}} := u^{[G_1]}_{\text{top}} \cdots u^{[G_{|J|}]}_{\text{top}}, \quad \partial_J u^{[G']}_{\text{top}} := \partial_{x^J} u^{[G_1]}_{\text{top}} \cdots \partial_{x^{|J|}} u^{[G_{|J|}]}_{\text{top}}.
\end{align*}
\]

Similarly to (3.38), define a sequence of power series \( Q_1, Q_2, \cdots \in \mathbb{C}[[x - 1, T]] \) by

\[
Q_k := \sum_{\lambda \in \mathcal{I}} \lambda^{k+1} c_\lambda \bar{T}_\lambda e^{\lambda m_{\text{top}}}, \quad k \geq 1.
\]

It is easy to prove that \( Q_k \) satisfy the following relations:

\[
Q_1 = -\frac{1}{m^2 n v'_{\text{top}}}, \quad Q_{k+1} = \frac{1}{m v'_{\text{top}}} \partial_x Q_k, \quad k \geq 1.
\]

So we have

\[
Q_k \in (v'_{\text{top}})^{1-2k} \mathbb{C} \left[ v'_{\text{top}}, \ldots, v^{(k)}_{\text{top}} \right], \quad \sum_{i \geq 1} v^{(i)}_{\text{top}} \partial_{v^{(i)}_{\text{top}}} Q_k = -Q_k, \quad k \geq 1. \tag{3.39}
\]

When \( g = 1 \), we solve from (3.38) that \( u^{[1]}_{\text{top}} = A_1 \left( v^{(1)}_{\text{top}}, v^{(2)}_{\text{top}}, v^{(3)}_{\text{top}} \right) \), where

\[
A_1(z_1, z_2, z_3) = \frac{nh}{24} \left( \frac{z_3}{z_1} - \frac{z_2}{z_1} + z_2 \right).
\]

It is obvious that \( A_1 \) satisfies the conditions (3.34)–(3.35). Assume that we have already found functions

\[
A_{g'} = A_g(z_1, \ldots, z_{3g'}) \in z_1^{-(4g'-2)} \mathbb{C}[z_1, \ldots, z_{3g'}], \quad g' \leq g - 1
\]

satisfying the homogeneity condition

\[
\sum_{i \geq 1} i z_i \frac{\partial A_{g'}}{\partial z_i} = 2g' A_{g'}, \tag{3.40}
\]

and that \( u^{[g']}_{\text{top}} := A_{g'} \left( v^{(1)}_{\text{top}}, \ldots, v^{(3g')}_{\text{top}} \right) \) solves equation (3.38) for \( g' \leq g - 1 \). According to Lemma 3.3, \( a_{g,J}(\lambda) \) are polynomials of \( \lambda \) with degree less than or equal to \( l(J) + |J|/2 \), so it follows from (3.39) that

\[
\sum_{\lambda \in \mathcal{I}} \lambda c_\lambda \bar{T}_\lambda a_{g,J}(\lambda)e^{\lambda m_{\text{top}}} \in (v'_{\text{top}})^{1-2l(J)-|J|} \mathbb{C} \left[ v'_{\text{top}}, \ldots, v^{(l(J)+|J|)}_{\text{top}} \right], \quad \forall J \in \mathbb{Y}. \tag{3.41}
\]

Thus using the fact that \( \sum_{\lambda \in \mathcal{I}} \lambda^{2c_\lambda \bar{T}_\lambda} e^{\lambda m_{\text{top}}} = Q_1 \), we solve from equation (3.38) that the unique solution \( u^{[g]}_{\text{top}} \) satisfies the properties required by the lemma. The lemma is proved. \( \square \)

**Step III.** Let us show that formula (3.36) gives the quasi-trivial transformation of the FVH \( \mathcal{F} \) [21]. Namely, for any solution \( v = v(x, T) \) to the dispersionless FVH (3.32) with \( v' \neq 0 \),

\[
u = v + \sum_{g \geq 1} \epsilon^{2g} A_g \left( v^{(1)}, \ldots, v^{(3g)} \right) \tag{3.42}
\]
gives a solution to the FVH. To this end, we first prove the following lemma on the transcendency of the solution \( v_{\text{top}} \) to the dispersionless FVH (3.32).
Lemma 3.9 Let \( F(z_0, z_1, \ldots, z_N) \in A_{z_0,0}[z_1, \ldots, z_N] \) be a polynomial in the indeterminates \( z_1, \ldots, z_N \) with coefficients depending smoothly on \( z_0 \), which satisfies the condition
\[
F\left(v_{\text{top}}(x,T), v_{\text{top}}'(x,T), \ldots, v_{\text{top}}^{(N)}(x,T)\right) = 0,
\]
then \( F(z_0, z_1, \ldots, z_N) \equiv 0 \).

Proof From the dispersionless FVH \([332]\), it follows that
\[
\frac{\partial}{\partial T}\lambda F\left(v_{\text{top}}, v_{\text{top}}', \ldots, v_{\text{top}}^{(N)}\right) = n c_\lambda \sum_{j=0}^{N} \frac{\partial F}{\partial z_j}(v_{\text{top}}, v_{\text{top}}', \ldots, v_{\text{top}}^{(N)}) \partial_x^{j+1} e^{\lambda MV_{\text{top}}}, \quad \lambda \in \mathcal{I}.
\]
Dividing the right-hand-side of the above equation by \( e^{\lambda MV_{\text{top}}} \) we obtain a polynomial in \( \lambda \) with degree \( N + 1 \). Since this polynomial vanishes for any \( \lambda \in \mathcal{I} \), all its coefficients must vanish. Thus we obtain
\[
\frac{\partial F}{\partial z_j}(v_{\text{top}}, v_{\text{top}}', \ldots, v_{\text{top}}^{(N)}) = 0, \quad j = 0, \ldots, N.
\]
From a similar reason we find that for all \( k \geq 1, \ 0 \leq j_1, \ldots, j_k \leq N \),
\[
\frac{\partial^k F}{\partial z_{j_1} \cdots \partial z_{j_k}}(v_{\text{top}}, v_{\text{top}}', \ldots, v_{\text{top}}^{(N)}) = 0,
\]
which lead to the vanishing of \( F \). The lemma is proved. \( \square \)

Now let us prove that the function \( u \) defined by \([342]\) satisfies the FVH. Indeed, for any \( \lambda \in \mathcal{I} \), it follows from Theorem 1.2 of \([19]\) the existence of a quasi-trivial transformation
\[
u = v + \sum_{g \geq 1} e^{2g A_{\lambda,g}}(v, v', \ldots, v^{(N_g)}) \],
which transforms \([332]\) to the \( \partial T_{\lambda} \)-flow of the FVH. Here \( \hat{A}_{\lambda,g} \left(z_0, \ldots, z_{N_g}\right) \in z_1^{-N'} A_{z_0,0} \left[z_1, \ldots, z_{N_g}\right] \), and \( N_g \) and \( N' \) are positive integers. Since \( u_{\text{top}} \) satisfies the FVH, from \([336]\) we know that
\[
A_g\left(v_{\text{top}}', \ldots, v_{\text{top}}^{(3g)}\right) = \hat{A}_{\lambda,g}\left(v_{\text{top}}, v_{\text{top}}', \ldots, v_{\text{top}}^{(N_g)}\right), \quad g \geq 1.
\]
Thus by using Lemma 3.9 we arrive at
\[
A_g\left(z_1, \ldots, z_{3g}\right) = \hat{A}_{\lambda,g}\left(z_0, z_1, \ldots, z_{N_g}\right), \quad \forall g \geq 1,
\]
and we proved that \([342]\) is the quasi-trivial transformation of the FVH.

Step IV. We are to prove the existence of the functions \( \tilde{F}_g \) satisfying equations \([3.10] \). To this end, we need to use the following lemma which can be proved by applying Theorem 1.2 of \([19]\) to the Hamiltonian structure of the FVH given in \([22]\).

Lemma 3.10 The quasi-trivial transformation \([3.10] \) transforms the Hamiltonian structure
\[
\frac{\partial v}{\partial T_{\lambda}} = P_0 \left(\frac{\delta H_0^{[0]}}{\delta v(x)}\right),
\]
(3.44)
\[ P_0 = \frac{nh}{m} \partial_x, \quad H^{[0]}_\lambda = \frac{c_\lambda}{\lambda h} \int e^{\lambda mv} \, dx \]  

(3.45)

for the dispersionless FVH to the one for the FVH given by

\[
\frac{\partial u}{\partial T_\lambda} = P \left( \frac{\delta H_\lambda}{\delta u(x)} \right), 
\]

(3.46)

\[
P = (\Lambda^3 - 1) \left( \frac{1}{1 - \Lambda} - \frac{1}{1} \right) \varepsilon (\Lambda^2 - 1), \quad H_\lambda = \frac{1}{\lambda h} \int_{\text{res}} \Lambda^3 L^h \, dx. 
\]

(3.47)

It follows from the above Lemma that

\[
P \frac{\delta}{\delta u} \left( \int u \, dx \right) = P_0 \delta \frac{\delta}{\delta v} \left( \int \left( v + \sum_{g \geq 1} \varepsilon^{2g} A_g \left( v', \ldots, v^{(3g)} \right) \right) \, dx \right),
\]

(3.48)

so we have

\[
\frac{\delta}{\delta v} \int A_g \left( v', \ldots, v^{(3g)} \right) \, dx \equiv \text{const}, \quad \forall g \geq 1.
\]

By using the homogeneity condition (3.35), we know that the constants in the above formula must vanish. Therefore, there exist functions

\[ B_g = B_g(z_0, \ldots, z_{3g-1}) \in A_{z_0,0} \left[ \log(z_1), z_1^{-1}, z_2, \ldots, z_{3g-1} \right], \quad g \geq 1 \]

satisfying the relations

\[
A_g = \sum_{i \geq 0} z_{i+1} \frac{\partial B_g}{\partial z_i}, \quad \sum_{i \geq 1} iz_i \frac{\partial B_g}{\partial z_i} = (2g - 1)B_g. 
\]

(3.49)

Taking derivatives with respect to \( z_0 \) on the both sides of the above equations, respectively, we obtain

\[
\sum_{i \geq 0} z_{i+1} \frac{\partial}{\partial z_i} \left( \frac{\partial B_g}{\partial z_0} \right) = 0, \quad \sum_{i \geq 1} iz_i \frac{\partial}{\partial z_i} \left( \frac{\partial B_g}{\partial z_0} \right) = (2g - 1) \frac{\partial B_g}{\partial z_0}, 
\]

(3.50)

from which it follows that \( B_g \) does not depend on \( z_0 \). By using (3.49), (3.34) we further conclude that

\[ B_g \in z_1^{-(4g-3)} C[z_1, \ldots, z_{3g-1}], \quad g \geq 1. \]

Now let us proceed to prove that \( B_g(v', v'', \ldots, v^{(3g-1)}) \) can be represented as the \( x \)-derivative of some function of \( v \) and its \( x \)-derivatives. We first prove the following lemma.

Lemma 3.11 If \( F \in (v')^{-N'} A_{v,0} \left[ v', \ldots, v^{(N)} \right] \) satisfies the equations

\[
\int Fe^{\lambda mv'} \, dx = 0, \quad \forall \lambda \in I, 
\]

(3.51)

\[
\sum_{i \geq 1} iv^{(i)} \frac{\partial F}{\partial v^{(i)}} = N'' F 
\]

(3.52)

for some positive integers \( N, N', N'' \), then \( F \equiv 0 \).
Proof For any \( \lambda \in \mathcal{I} \), it is easy to verify that \( F \) satisfies the equation

\[
\frac{\delta}{\delta v} \int F e^{\lambda mv} v'^{\mu} dx = \sum_{k=0}^{N} (-1)^k \left( (\delta_k F) \partial_x^k \left( e^{\lambda mv} v' \right) + \left( \partial_x^k F \right) \delta_k \left( e^{\lambda mv} v' \right) \right),
\]

(3.53)

where the operators \( \delta_k \) are defined by

\[
\delta_k := \sum_{i \geq 0} (-1)^i \binom{i+k}{k} \partial_x^i \frac{\partial}{\partial v^{i+k}}, \quad k \geq 0.
\]

The left-hand-side of (3.53) vanishes due to the condition (3.51). Dividing the right-hand-side of (3.53) by \( e^{\lambda mv} \), we obtain a polynomial in \( \lambda \) of degree \( N \). The vanishing of this polynomial for any \( \lambda \in \mathcal{I} \) yields \( \partial F/\partial v_k(= 1, \ldots, N) = 0 \). Thus by using the condition (3.52), we conclude that \( F \equiv 0 \). The lemma is proved. □

Let us take derivatives with respect to arbitrary \( T^\mu \), \( \lambda \in \mathcal{I} \) on the both sides of (3.42). Then by using the FVH (2.54) and its dispersionless limit (3.32), and by integrating with respect to \( x \), we arrive at the equation

\[
\epsilon^{-1} \partial_x^{-1} \left( 1 - \Lambda_1^{-1} \right) \text{res}_{\Lambda_3} L^\lambda h = n c_\lambda e^{\lambda mv} + n c_\lambda \sum_{i \geq 1} \partial_x^{i+1} \left( e^{\lambda mv} \right) \frac{\partial}{\partial v^{i}} \sum_{g \geq 1} \epsilon^{2g} B_g \left( v', \ldots, v^{(3g-1)} \right).
\]

(3.54)

Denote

\[
\Delta_\lambda := \epsilon^{-1} \partial_x^{-1} \left( 1 - \Lambda_1^{-1} \right) \text{res}_{\Lambda_3} L^\lambda h - n c_\lambda e^{\lambda mv}.
\]

Then by using Lemma 2.2 we can represent \( \Delta_\lambda \) in the form

\[
\Delta_\lambda = \sum_{k \geq 1} \epsilon^{2k} \Delta_\lambda^{[k]}(v, v', \ldots),
\]

where \( \Delta_\lambda^{[k]} \in (v')^{-m_{\lambda,k}} A_v \) for some nonnegative integers \( m_{\lambda,k} \), and \( \deg \Delta_\lambda^{[k]} = 2k \). It follows from the formula (2.5) that

\[
0 = \int \frac{\partial \Delta_\lambda}{\partial T^\mu} dx = \int \sum_{k \geq 0} \frac{\partial \Delta_\lambda}{\partial v^{(k)}} \partial_x^{k+1} (n c_\mu e^{\mu mv}) dx = \mu n c_\mu \int e^{\mu mv} v' \left( \frac{\delta}{\delta v} \int \Delta_\lambda dx \right) dx.
\]

So by using Lemma 3.11 we obtain

\[
\frac{\delta}{\delta v} \int \Delta_\lambda^{[k]} dx = 0, \quad k \geq 1,
\]

which, together with (3.54), yields the relations

\[
0 = \int \Delta_\lambda dx = \lambda n c_\lambda \int e^{\lambda mv} v' \left( \frac{\delta}{\delta v} \int \sum_{g \geq 1} \epsilon^{2g} B_g \left( v', \ldots, v^{(3g-1)} \right) dx \right) dx, \quad \forall \lambda \in \mathcal{I}.
\]

(3.55)

By using Lemma 3.11 again we have

\[
\frac{\delta}{\delta v} \int B_g \left( v', \ldots, v^{(3g-1)} \right) = 0, \quad \forall g \geq 1.
\]

By using (3.49) and the above equation we thus arrive at the following proposition.
Lemma 3.12 There exist functions
\[
\tilde{F}_1(z_0, z_1) = \frac{nh}{24} \log z_1 + \frac{nh}{24} z_0, \quad (3.56)
\]
\[
\tilde{F}_g(z_1, \ldots, z_{3g-2}) \in z_1^{-(4g-4)} \mathbb{C}[z_1, \ldots, z_{3g-2}], \quad g \geq 2, \quad (3.57)
\]
satisfying the quasi-homogeneous condition
\[
\sum_{i \geq 1} iz_i \frac{\partial \tilde{F}_g}{\partial z_i} = (2g - 2)\tilde{F}_g + \frac{nh}{24} \delta_{g,1}, \quad g \geq 1, \quad (3.58)
\]
such that the functions \(A_g\) given in Lemma 3.8 can be represented in the form
\[
A_g\left(v', \ldots, v^{(3g)}\right) = \frac{\partial^2}{\partial x^2} \tilde{F}_g\left(v, v', \ldots, v^{(3g-2)}\right), \quad (3.59)
\]

Step V. Finally, let us prove the existence of the tau-function \(\tau_{\text{top}}\). Following the approach of [5], we define \(F_0 = F_0(x, T) \in \mathbb{C}[\{x - 1, T\}]\) by the formula
\[
F_0 := \frac{mn}{2h} \sum_{\lambda, \mu \in I} \frac{\lambda \mu}{\lambda + \mu} c_\lambda c_\mu \tilde{T}_\lambda \tilde{T}_\mu e^{(\lambda + \mu)mv_{\text{top}}} + \frac{x}{h} \sum_{\mu \in I} c_\mu \tilde{T}_\mu e^{mv_{\text{top}}} + \frac{x^2}{2nh} v_{\text{top}}. \quad (3.60)
\]
By using the dispersionless Euler-Lagrange (3.33), we find that \(F_0\) has the following properties:
\[
\frac{\partial^2}{\partial x^2} F_0 = \frac{1}{nh} v_{\text{top}}, \quad (3.61)
\]
\[
\frac{\partial_x}{\partial T_\lambda} F_0 = \frac{c_\lambda}{h} e^{\lambda mv_{\text{top}}}, \quad \forall \lambda \in I, \quad (3.62)
\]
\[
\frac{\partial_{T_\lambda}}{\partial T_\mu} F_0 = \frac{mn}{h} \frac{\lambda \mu}{\lambda + \mu} c_\lambda c_\mu e^{(\lambda + \mu)mv_{\text{top}}}, \quad \forall \lambda, \mu \in I, \quad (3.63)
\]
\[
\sum_{\lambda \in I} \lambda \frac{\partial \tilde{F}_0}{\partial T_\lambda} + \frac{x^2}{2mn} = 0, \quad \sum_{\lambda \in I} \frac{\partial \tilde{F}_0}{\partial T_\lambda} + x \frac{\partial \tilde{F}_0}{\partial x} = 2F_0. \quad (3.64)
\]
By using Lemma 3.8 and Lemma 3.12, we know the existence of functions
\[
F_1 = F_1(z_0, z_1) := \frac{1}{24} \log z_1 - \frac{nh + n^2}{24nh} z_0, \quad (3.65)
\]
such that
\[
u_{\text{top}} = \left(\Lambda_3^{1/2} - \Lambda_3^{-1/2}\right) \left(\Lambda_1^{1/2} - \Lambda_1^{-1/2}\right) \left(\sum_{g \geq 0} \epsilon^{2g-2} F_g\right), \quad (3.66)
\]
\[
\sum_{i \geq 1} iz_i \frac{\partial F_g}{\partial z_i} = (2g - 2)F_g + \frac{1}{24} \delta_{g,1}, \quad g \geq 1, \quad (3.67)
\]
where \( F_g = F_g(x, T) := F_g(v_{\text{top}}, v'_{\text{top}}, \ldots, v_{3g-2}^{(3g-2)}) \in \mathbb{C}[[x - 1, T]]. \) Clearly, \( F_g \) and \( \bar{F}_g \) are related by

\[
F_g = \frac{1}{nh} \sum_{k=1}^{g} C_{g-k}(m, n) \partial^{g-2k} \bar{F}_k + \frac{C_g(m, n)}{nh} z_{2g-2}, \quad \partial := \sum_{i \geq 0} z_{i+1} \frac{\partial}{\partial z_i},
\]

where \( C_k(m, n), k \geq 0 \) are defined in (1.19).

Now let us define a formal power series \( \tau_{\text{top}} \in \mathbb{C}(\mathbb{C})[[x - 1, T]] \) by the formula

\[
\tau_{\text{top}} = \tau_{\text{top}}(x, T, \varepsilon) := \varepsilon^{\sum g \geq 0} \varepsilon^{2g-2} F_g.
\]

Then (3.69) can be written as

\[
u_{\text{top}} = \left( \Lambda_{3/2} - \Lambda_{1/2} \right) \left( \Lambda_{1/2} - \Lambda_{-1/2} \right) \log \tau_{\text{top}}.
\]

From the identities (3.64) it follows that

\[
\sum_{\lambda \in I} \lambda \bar{T}_\lambda \frac{\partial \nu_{\text{top}}}{\partial T_\lambda} + \frac{1}{m} = 0, \quad \sum_{\lambda \in I} \bar{T}_\lambda \frac{\partial \nu_{\text{top}}}{\partial T_\lambda} + x \frac{\partial \nu_{\text{top}}}{\partial x} = 0,
\]

which yield

\[
\sum_{\lambda \in I} \lambda \bar{T}_\lambda \frac{\partial \nu_{\text{top}}^{(i)}}{\partial T_\lambda} + \frac{1}{m} \delta_{i, 0} = 0, \quad \sum_{\lambda \in I} \bar{T}_\lambda \frac{\partial \nu_{\text{top}}^{(i)}}{\partial T_\lambda} + x \frac{\partial \nu_{\text{top}}^{(i)}}{\partial x} = -i \nu_{\text{top}}^{(i)}.
\]

So by using (3.65) and (3.67) we know that \( \tau_{\text{top}} \) satisfies the following conditions:

\[
L_0 \tau_{\text{top}} = 0, \quad K_0 \tau_{\text{top}} + \frac{1}{24} \tau_{\text{top}} = 0,
\]

where the operators \( L_0 \) and \( K_0 \) are defined by (3.2) and (3.8), respectively.

Now it remains to prove that the power series \( \tau_{\text{top}} \in \mathbb{C}(\mathbb{C})[[x - 1, T]] \) is a tau-function for the FVH (1.8). For any given \( \lambda, \mu \in I \), define two power series in \( \mathbb{C}(\mathbb{C})[[x - 1, T]; \varepsilon] \) by

\[
\Delta_\lambda := (\Lambda_3 - 1) \Lambda_{-1/2} \frac{\partial \log \tau_{\text{top}}}{\partial T_\lambda} - \res_{\lambda} L_{\lambda}^{\Lambda_{-1/2}}|_{u=u_{\text{top}}} ,
\]

\[
\Delta_{\lambda, \mu} := \varepsilon^2 \frac{\partial^2 \log \tau_{\text{top}}}{\partial T_\lambda \partial T_\mu} - \Omega_{\lambda, \mu}|_{u=u_{\text{top}}}.
\]

We are to show that

\[
\Delta_\lambda = 0, \quad \Delta_{\lambda, \mu} = 0.
\]

Firstly, by using the definition (3.69) of \( \tau_{\text{top}} \), equations (3.62)–(3.63), Lemma 2.2 and Lemma 3.9, we know the existence of

\[
\Delta_\lambda^* = \Delta_\lambda^*(z_0, z_1, \ldots), \quad \Delta_{\lambda, \mu}^* = \Delta_{\lambda, \mu}^*(z_0, z_1, \ldots) \in \varepsilon \mathcal{A}_{z_0, 0} \left[z_1^{-1}, z_1, z_2, \ldots \right] [[\varepsilon]]
\]

satisfying

\[
\Delta_\lambda = \Delta_\lambda^*(v_{\text{top}}, v'_{\text{top}}, \ldots), \quad \Delta_{\lambda, \mu} = \Delta_{\lambda, \mu}^*(v_{\text{top}}, v'_{\text{top}}, \ldots).
\]
Then using Lemma 2.2, Lemma 3.9, the second equality of (3.73) and the second identity of (3.72) we find

\[
\left( \sum_{i \geq 1} iz_i \frac{\partial}{\partial z_i} - \frac{\partial}{\partial \varepsilon} \right) \Delta^*_\lambda = 0,
\left( \sum_{i \geq 1} iz_i \frac{\partial}{\partial z_i} - \frac{\partial}{\partial \varepsilon} \right) \Delta^*_{\lambda,\mu} = 0.
\] (3.76)

Next, acting \( \varepsilon \partial \frac{T}{\varepsilon} \lambda \) and \( \varepsilon^2 \partial_T \lambda, \partial_T \mu \) on (3.70), we obtain

\[
\partial x \Delta^* \lambda = 0, \quad \partial^2 x \Delta^*_{\lambda,\mu} = 0.
\]

Again using Lemma 3.9, we obtain

\[
\partial \Delta^*_\lambda = 0, \quad \partial^2 \Delta^*_{\lambda,\mu} = 0.
\]

This, together with formulae (3.75)–(3.76), implies that

\[
\Delta^* \lambda = 0, \quad \Delta^*_{\lambda,\mu} = 0.
\]

Hence the identities (3.74) are true. It follows that the power series \( \tau_{\text{top}} \) satisfies the definition (2.6)–(2.8) for the tau-function for the FVH. Thus we have proved Theorem 3.1. \( \square \)

4 Virasoro constraints

In this section we prove, following the approach used by Orlov–Shulman [27] and Adler–Shiota–van Moerbeke [1], that the topological tau-function \( \tau_{\text{top}} \) of the FVH satisfies the Virasoro constraints \( L_k \tau_{\text{top}} = 0, \ k \geq 0 \), where \( L_k \) are defined in (3.2)–(3.3).

We first derive some useful formulae valid for arbitrary power series solutions to the FVH. Let \( u \) be a solution to the FVH in the ring \( \mathbb{C}[[x-1,T; \varepsilon]] \), \( (\Phi_1, \Phi_2, \tau) \) a dressing triple associated to \( u \), and \( \psi_1, \psi_2 \) the wave functions corresponding to \( \Phi_1, \Phi_2 \). (See their definitions given in Section 2.) Introduce the following notations:

1. \( \Gamma_1 \) and \( \Gamma_2 \) denote the following elements in \( \mathbb{C} \left[ \Lambda_3, \Lambda_3^{-1}, \varepsilon \right] [[x-1,T]] \):

\[
\Gamma_1 := \frac{x}{\hbar e} \Lambda_3 - \frac{m}{\varepsilon} \sum_{\mu \in J_1} \mu T_\mu \Lambda_3^{-m-1} + \frac{m}{2\varepsilon} \sum_{\mu \in J_3} \mu T_\mu \Lambda_3^{-m-1},
\]

\[
\Gamma_2 := \frac{x}{\hbar e} \Lambda_3 - \frac{n}{\varepsilon} \sum_{\mu \in J_2} \mu T_\mu \Lambda_3^{-n-1} + \frac{n}{2\varepsilon} \sum_{\mu \in J_3} \mu T_\mu \Lambda_3^{-n-1}.
\] (4.1)

2. \( M_1 \) and \( M_2 \) denote the following difference operators:

\[
M_1 := \Phi_1 \Gamma_1 \Phi_1^{-1}, \quad M_2 := \Phi_2 \Gamma_2 \Phi_2^{-1}.
\] (4.3)

Clearly, \( M_1 \in \mathbb{C}((\Lambda_3^{-1}, \varepsilon))[[x-1,T]], \ M_2 \in \mathbb{C}((\Lambda_3, \varepsilon))[[x-1,T]]. \) We call \( M_1, M_2 \) the Orlov-Schulman type operators.
3. \( N_1 \) and \( N_2 \) denote the following generating series of difference operators:

\[
N_1 = N_1(\xi, \zeta) := \sum_{k \geq 1} \frac{\zeta - \xi}{(k - 1)!} \sum_{d \in \mathbb{Z}} \xi^{-k-d} M_1^{k-1} L^{(k+d-1)h/m},
\]

(4.4)

\[
N_2 = N_2(\xi, \zeta) := \sum_{k \geq 1} \frac{\zeta - \xi}{(k - 1)!} \sum_{d \in \mathbb{Z}} \xi^{-k-d} M_2^{k-1} L^{(k+d-1)h/n}.
\]

(4.5)

Then we have

\[
N_1 \in \mathbb{C}((\Lambda_3^{-1}, \epsilon))[[x - 1, T]][[\xi, \xi^{-1}]][[\zeta - \xi]],
\]

\[
N_2 \in \mathbb{C}((\Lambda_3, \epsilon))[[x - 1, T]][[\xi, \xi^{-1}]][[\zeta - \xi]].
\]

Let us note that the operators of the form \( L^{k/m} \) belong to the space \( \mathcal{A}_u[[\epsilon]]((\Lambda_3^{-1})) \) for \( k \in \mathbb{Z} \), and that of the form \( L^{k/n} \) belong to the space \( \mathcal{A}_u[[\epsilon]]((\Lambda_3)) \). See [22] for the definition of the fractional powers of \( L \).

4. \( X_1 \) and \( X_2 \) denote the following two operators:

\[
X_1(\xi, \zeta) = e^{\vartheta_1(x, T; x; \xi) - \vartheta_1(x, T; x; \xi)} \exp \left( \sum_{\mu \in I_1} \frac{1}{\xi^{\mu n} - \xi^{\mu m}} \frac{1}{\mu n} \frac{\partial}{\partial T_\mu} \right),
\]

(4.6)

\[
X_2(\xi, \zeta) = e^{\vartheta_2(x, T; x; \xi) - \vartheta_2(x, T; x; \xi)} \exp \left( \sum_{\mu \in I_2} \frac{1}{\xi^{\mu n} - \xi^{\mu m}} \frac{1}{\mu n} \frac{\partial}{\partial T_\mu} \right),
\]

(4.7)

where \( \vartheta_1, \vartheta_2 \) are defined in [22], [22].

5. Denote by \( Y_k, k \geq 0 \) the following operators in \( \mathbb{C}((\epsilon))[[\Lambda_3, \Lambda_3^{-1}]][[x - 1, T]]: \)

\[
Y_k := \frac{1}{m} M_1 L^{(k+1/m)h} - \frac{1}{n} M_2 L^{(k+1/n)h} - \frac{\Gamma(m)\Gamma(n)}{\Gamma(1 + h)\epsilon} L^{(k+1)h}.
\]

(4.8)

**Lemma 4.1** The following formulae hold true:

\[
N_1(\xi, \zeta)\psi_1(x, T; \epsilon; z) = (\zeta - \xi) \sum_{d \in \mathbb{Z}} \xi^{-d} z^{d-1} \frac{X_1(\xi, \zeta)\tau_3(x, T; z^{-1}; \epsilon)}{\tau_3(x, T; z^{-1}; \epsilon)} \psi_1(x, T; \epsilon; z),
\]

(4.9)

\[
N_2(\xi, \zeta)\psi_2(x, T; \epsilon; z) = (\zeta - \xi) \sum_{d \in \mathbb{Z}} \xi^{-d} z^{d-1} \frac{X_2(\xi, \zeta)\tau_3(x + h\epsilon, T; z^{-1}; \epsilon)}{\tau_3(x + h\epsilon, T; z^{-1}; \epsilon)} \psi_2(x, T; \epsilon; z).
\]

(4.10)

**Proof** The generating series \( N_1 \) can be written as

\[
N_1(\xi, \zeta) := (\zeta - \xi)e^{(\zeta - \xi)M_1} \sum_{d \in \mathbb{Z}} \xi^{-d} L^{(d-1)h/m}.
\]

So we have

\[
\frac{1}{\zeta - \xi} N_1(\xi, \zeta)\psi_1(x, T; \epsilon; z) = \sum_{d \in \mathbb{Z}} \xi^{-d} z^{d-1} e^{(\zeta - \xi)\partial_x} \psi_1(x, T; \epsilon; z)
\]

(4.10)
From the definition (4.8) of $\tau_s(x, T; \epsilon)$ where $L$ operators are defined by 

\[ (4.1) \]

Lemma 4.2 We have the following relations between the operators $Y_k$ and the Virasoro operators $L_k$, $k \geq 0$:

\[
\frac{-(Y_k)_- \psi_1}{\psi_1} = \left( e^{\sum_{\mu \in Z_1} -z_{-\mu} \partial_{x_{-\mu}}} \frac{\partial}{\partial T} - 1 \right) \frac{\bar{L}_k \tau_s}{\tau_s}, \tag{4.12}
\]

\[
\frac{(Y_k)_+ \psi_2}{\psi_2} = \left( e^{\hbar \partial_{x_1} + \sum_{\mu \in Z_2} z_{-\mu} \partial_{x_{-\mu}}} \frac{\partial}{\partial T} - 1 \right) \frac{\bar{L}_k \tau_s}{\tau_s}, \tag{4.13}
\]

where

\[
\bar{L}_k := L_k - \frac{x}{2 \hbar \epsilon \delta_{k,0}} - \frac{\hbar \epsilon}{2 \hbar \epsilon \partial T_k}, \quad k \geq 0, \tag{4.14}
\]

and $L_k$ are defined by (3.2) - (3.3).

Proof From the definition (4.8) of $Y_k$ and the definitions (4.4), (4.5) of $N_1, N_2$ we obtain

\[
-(Y_k)_- = \left( \frac{1}{2 \hbar \epsilon} \right) \Gamma(m) \Gamma(n) \Gamma(1 + h) \epsilon \left( \frac{L^{(k+1)\hbar}}{\Gamma(1 + h) \epsilon} \right)_-.
\]

In order to prove the formula (4.12), we need to represent $-(L^{kh})_- \psi_1 / \psi_1, -(N_1)_- \psi_1 / \psi_1$, and $-(N_2)_- \psi_1 / \psi_1$ in terms of the tau-function $\tau_s$.

From (2.24), (2.30) and (2.32), it follows that

\[
\frac{-(L^{kh})_- \psi_1}{\psi_1} = \left( e^{\sum_{\mu \in Z_1} z_{-\mu} \partial_{x_{-\mu}}} \frac{\partial}{\partial T} - 1 \right) \frac{\bar{L}_k \tau_s}{\tau_s}, \quad k \geq 1.
\]
To calculate \(-(N_1)_{-\psi_1/\psi_1}\) and \(-(N_2)_{-\psi_1/\psi_1}\), we observe that \(N_1\) and \(N_2\) can be decomposed into the following forms:

\[
N_1 = f_1(L^{h/m}) + g_1(L^{h/n}), \quad N_2 = f_2(L^{h/m}) + g_2(L^{h/n}),
\]

where \(f_1(z), f_2(z), g_1(z)\) and \(g_2(z)\) are defined by

\[
f_1(z) := -\frac{X_1(\xi, \zeta)\tau_s(x, T - [z^{-1}]_1; \epsilon)}{\tau_s(x, T - [z^{-1}]_1; \epsilon) - 1} - \zeta/z,
\]

\[
g_1(z) := -\frac{\zeta X_1(\xi, \zeta)\tau_s(x, T - [z^{-1}]_1; \epsilon)}{\tau_s(x, T - [z^{-1}]_2; \epsilon) - 1} - \zeta/z,
\]

\[
f_2(z) := -\frac{\zeta X_2(\xi, \zeta)\tau_s(x, T - [z^{-1}]_1; \epsilon)}{\tau_s(x, T - [z^{-1}]_1; \epsilon) - 1} - \zeta/z,
\]

\[
g_2(z) := -\frac{X_2(\xi, \zeta)\tau_s(x, T - [z^{-1}]_1; \epsilon)}{\tau_s(x, T - [z^{-1}]_1; \epsilon) - 1} - \zeta/z.
\]

To prove the formulae given in (4.17), let us denote

\[
D_0 := e^{\frac{i}{\epsilon}\sum_{k \geq 1} T_k L^{kh}} \in \mathbb{C}[\Lambda_3, \Lambda_3^{-1}][[T; \epsilon^{-1}]].
\]

By applying \(D_0\) to both sides of the bilinear equations (2.28) we obtain

\[
\text{res}_z e^{\frac{i}{\epsilon}\sum_{k \geq 1} T_k L^{kh}} \frac{\tau_s(x, T - [z^{-1}]_1; \epsilon)}{\tau_s(x, T; \epsilon)} \psi_1^* (x + \ell h, T'; \epsilon; z) \frac{dz}{z} = \text{res}_z \frac{1}{\epsilon}\sum_{k \geq 1} T_k z^{\mu n} \frac{\tau_s(x, T - [z^{-1}]_2; \epsilon)}{\tau_s(x, T; \epsilon)} \psi_2^* (x + \ell h, T'; \epsilon; z) \frac{dz}{z}, \quad \forall \ell \in \mathbb{Z}. \tag{4.18}
\]

Acting the operator

\[
\frac{\tau_s(x, T - [\zeta^{-1}]_1 + [\zeta^{-1}]_1; \epsilon)}{\tau_s(x, T; \epsilon)} X_1(\xi, \zeta)
\]

on both sides of (4.18) we arrive at the identities

\[
\text{res}_z \frac{X_1(\xi, \zeta)\tau_s(x, T - [z^{-1}]_1; \epsilon)}{\tau_s(x, T - [z^{-1}]_1; \epsilon) - 1} - \zeta/z D_0 \psi_1 (x, T; \epsilon; z) \psi_1^* (x + \ell h, T'; \epsilon; z) \frac{dz}{z} = \text{res}_z \frac{X_1(\xi, \zeta)\tau_s(x, T - [z^{-1}]_2; \epsilon)}{\tau_s(x + h\epsilon, T - [z^{-1}]_2; \epsilon)} \psi_2^* (x + \ell h, T'; \epsilon; z) \frac{dz}{z}. \tag{4.19}
\]

On the other hand, by using the fact that

\[
(\xi - \zeta) \sum_{d \in \mathbb{Z}} \xi^{-d} z^{d-1} = -\frac{1 - \zeta/z}{1 - \xi/z} + \frac{\zeta}{\xi} 1 - \zeta/z,
\]

we can rewrite equation (4.19) in the form

\[
N_1 \psi_1 = f_1(z) \psi_1 + \frac{\zeta X_1(\xi, \zeta)\tau_s(x, T - [z^{-1}]_1; \epsilon)}{\tau_s(x, T - [z^{-1}]_1; \epsilon) - 1} - \zeta/z \psi_1.
\]
So it follows from (4.19) and the equations $L^{h/m} \psi_1 = z \psi_1$, $L^{h/n} \psi_2 = z \psi_2$ that

$$\text{res}_z \left[ \left( N_1 - f_1(L^{h/m}) \right) D_0 \psi_1(x, T; \epsilon; z) \right] \psi_1^*(x + \ell h \epsilon, T'; \epsilon; z) \frac{dz}{z} = \text{res}_z \left[ g_1(L^{h/n}) D_0 \psi_2(x, T; \epsilon; z) \right] \psi_2^*(x + \ell h \epsilon, T'; \epsilon; z) \frac{dz}{z}, \quad \forall \ell \in \mathbb{Z}.$$ 

From the above equations and Lemma 2.5 we obtain the first identity of (4.17). The proof for the second identity of (4.17) is similar.

By using the identities

$$[g_1 \left( L^{h/n} \right)]_\epsilon = 0, \quad [g_2 \left( L^{h/n} \right)]_\epsilon = 0$$

we know from (4.17) that

$$(N_1)_- = f_1 \left( L^{h/m} \right) - \text{res}_{A_3} f_1 \left( L^{h/m} \right), \quad (N_2)_- = f_2 \left( L^{h/m} \right) - \text{res}_{A_3} f_2 \left( L^{h/m} \right).$$

So we have the relations

$$\frac{-(N_1)_- \psi_1}{\psi_1} = -(f_1(L^{h/m}) - \text{res}_{A_3} f_1(L^{h/m})) \psi_1/\psi_1 = \frac{X_1(\xi, \zeta)\tau_\alpha(x, T - [z^{-1}] T; \epsilon) \left( 1 - \frac{z}{\xi} \right) 1 - \frac{z}{\zeta} - X_1(\xi, \zeta)\tau_\alpha(x, T; \epsilon)}{\tau_\alpha(x, T - [z^{-1}] T; \epsilon) 1 - \frac{z}{\xi} - \tau_\alpha(x, T; \epsilon)} \left( e^{-\sum_{\mu \in I_1} \frac{2 h}{m^2} \left( \frac{d}{\partial r} \right)^2} - 1 \right) \left( \frac{X_1(\xi, \zeta)\tau_\alpha(x, T; \epsilon)}{\tau_\alpha(x, T; \epsilon)} \right) + \tilde{N}_1,$$ 

where $\tilde{N}_1$ is given by

$$\tilde{N}_1 = \left( e^{-\frac{1}{2} \sum_{k \geq 1} \frac{2 h}{m^2} \left( \frac{d}{\partial r} \right)^2} - 1 \right) \frac{X_1(\xi, \zeta)\tau_\alpha(x, T; \epsilon)}{\tau_\alpha(x, T; \epsilon)}.$$ 

In a similar way we obtain

$$\frac{-(N_2)_- \psi_1}{\psi_1} = -(e^{-\sum_{\mu \in I_1} \frac{2 h}{m^2} \left( \frac{d}{\partial r} \right)^2} - 1) \left( \frac{\zeta X_2(\xi, \zeta)\tau_\alpha(x, T; \epsilon)}{\tau_\alpha(x, T; \epsilon)} \right) + \tilde{N}_2,$$ 

where $\tilde{N}_2$ is given by

$$\tilde{N}_2 = \left( e^{-\frac{1}{2} \sum_{k \geq 1} \frac{2 h}{m^2} \left( \frac{d}{\partial r} \right)^2} - 1 \right) \frac{\zeta X_2(\xi, \zeta)\tau_\alpha(x, T; \epsilon)}{\tau_\alpha(x, T; \epsilon)}.$$ 

Now by using (4.15) with (4.16), (4.20) and (4.21), we arrive at

$$\frac{-(Y_k)_- \psi_1}{\psi_1} = \left( e^{-\sum_{\mu \in I_1} \frac{2 h}{m^2} \left( \frac{d}{\partial r} \right)^2} - 1 \right) \left[ \frac{1}{\tau_\alpha(x, T; \epsilon)} \left( \frac{1}{2m} \text{res}_\xi \xi^{k+1} \partial_\xi^2 X_1(\xi, \zeta) \bigg|_{\xi = \xi} + \frac{1}{2n} \text{res}_\xi \xi^{k+1} \partial_\xi^2 \left( \frac{\zeta}{\zeta} X_2(\xi, \zeta) \right) \bigg|_{\xi = \xi} \right. 

\left. - \frac{\Gamma(m) \Gamma(n)}{\Gamma(1 + h) \partial T_{k+1}} \partial_{T_{k+1}} \right) \tau_\alpha(x, T; \epsilon) \right],$$ 

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which leads to formula (4.12) via a straightforward calculation. The proof of formula (4.13) is similar. The lemma is proved. □

We are now ready to prove the following theorem for the topological tau-function $\tau_{\text{top}}$ of FVH. In the proof of the theorem we will use the notations $\psi_1, \psi_2, Y_k, k \geq 0$, etc. defined in the beginning of this section for the tau-function $\tau_{\text{top}}$.

**Theorem 4.3** The power series $\tau_{\text{top}} \in \mathbb{C}(\langle \epsilon^2 \rangle[[x - 1, T]])$ satisfies the following Virasoro constraints:

$$L_k \tau_{\text{top}} = 0, \quad k \geq 0.$$  \hspace{1cm} (4.22)

**Proof** Recall from Proposition 2.10 that

$$\tau_{\text{top}}(x, T; \epsilon) = \tau_s \left( x + \frac{m \epsilon}{2}, T; \epsilon \right).$$

From the definition of $\tau_{\text{top}}$ in Theorem 3.1, we know that $L_0 \tau_{\text{top}} = 0$, which is, by a shift $x \mapsto x - \frac{m \epsilon}{2}$, equivalent to

$$\left( L_0 - \frac{x}{2n \epsilon} + \frac{m}{8n \epsilon} \right) \tau_s = 0.$$ 

It follows from Lemma 4.2 that

$$- (Y_0)_- \psi_1 = 0, \quad (Y_0)_+ \psi_2 = - \frac{1}{2m} \psi_2.$$ \hspace{1cm} (4.23)

Since $- (Y_0)_-$ and $(Y_0)_+$ can be represented in the form

$$- (Y_0)_- = \sum_{k \leq -1} \gamma_k L^{kh/m}, \quad (Y_0)_+ = \sum_{k \geq 0} \gamma_k L^{-kh/n},$$

where $\gamma_k \in \mathbb{C}(\langle \epsilon \rangle[[x - 1, T]])$, $k \in \mathbb{Z}$, the formulae (4.23) imply that $\gamma_k = - \frac{1}{2m} \delta_{k,0}$ for $k \in \mathbb{Z}$. Hence $Y_0 = - \frac{1}{2m}$. Then it follows from (4.8) that

$$Y_k = Y_0 L^{kh} = - \frac{1}{2m} L^{kh}, \quad \forall k \geq 1.$$

By using the above formulae together with Lemma 4.2 and the following identities:

$$\frac{- (L^{kh})_- \psi_1}{\psi_1} = \left( \sum_{\mu \in \mathcal{I}_1} \frac{x^{\mu} \partial}{\mu \partial_x} - 1 \right) \left( \frac{\epsilon \partial_{\bar{L}_k} \tau_s}{\tau_s} \right),$$

$$\frac{(L^{kh})_+ \psi_2}{\psi_2} = \left( \sum_{\mu \in \mathcal{I}_2} \frac{x^{\mu} \partial}{\mu \partial_x} - 1 \right) \left( \frac{\epsilon \partial_{\bar{L}_k} \tau_s}{\tau_s} \right),$$

we obtain for arbitrary $k \geq 1$ that

$$\frac{\left( \sum_{\mu \in \mathcal{I}_1} \frac{-x^{\mu} \partial}{\mu \partial_x} - 1 \right)}{\tau_s} \left( \frac{\bar{L}_k + \epsilon \partial_{\bar{L}_k}}{\tau_s} \right) \tau_s = 0,$$

$$\frac{\left( \sum_{\mu \in \mathcal{I}_2} \frac{-x^{\mu} \partial}{\mu \partial_x} - 1 \right)}{\tau_s} \left( \frac{\bar{L}_k + \epsilon \partial_{\bar{L}_k}}{\tau_s} \right) \tau_s = 0.$$
This implies
\[
\left( L_k - \frac{\epsilon}{2n} \frac{\partial}{\partial T_k} \right) \tau_s = c_k(\epsilon) \tau_s, \quad c_k(\epsilon) \in \mathbb{C}[\epsilon],
\]
which, by a shift \( x \mapsto x + \frac{mn}{2} \), is equivalent to
\[
L_k \tau_{\text{top}} = c_k(\epsilon) \tau_{\text{top}}.
\]
From the Virasoro commutation relations
\[
[L_0, L_k] = -k L_k,
\]
it follows that \( c_k(\epsilon) = 0 \) for any \( k \geq 1 \). The theorem is proved. \( \square \)

5 Proof of Theorem 1.1

Let us first prove the theorem for the case when \( p = 1/m, q = 1/n, r = -1/h \), where \( m, n \) are coprime positive integers and \( h = m + n \).

From Theorem 3.1 we know that \( \log \tau_{\text{top}} \) has the following genus expansion:
\[
\log \tau_{\text{top}}(x, T; \epsilon) = \sum_{g \geq 0} \epsilon^{2g-2} F_g(x, T)
= \epsilon^{-2} F_0(x, T) + \sum_{g \geq 1} \epsilon^{2g-2} F_g \left( v_{\text{top}}(x, T), v'_{\text{top}}(x, T), \ldots, v_{(3g-2)}(x, T) \right). \tag{5.1}
\]
Here \( v_{\text{top}} \) and \( F_0 \) are determined by the dispersionless Euler-Lagrange equation (3.33) and by (3.60) respectively, they are related by the formula
\[
v_{\text{top}}(x, T) = nh \partial_x^2 F_0(x, T). \tag{5.2}
\]
On the other hand, we have mentioned in the beginning of Section 5 that according to [6], \( \log Z(x, T; \epsilon) \) has the genus expansion
\[
\log Z(x, T; \epsilon) = \frac{1}{mnhe^2} A + \sum_{g \geq 0} (mn)g^{-1} \epsilon^{2g-2} H_g(t(x, T))
= \epsilon^{-2} \tilde{F}_0(x, T) + \sum_{g \geq 1} \epsilon^{2g-2} \tilde{F}_g \left( v_H(t(x, T)), v'_H(t(x, T)), \ldots, v_{(3g-2)}^H(t(x, T)) \right), \tag{5.3}
\]
where \( A \) is defined by (1.12), \( \tilde{F}_0(x, T) = \frac{1}{mn} (A + \mathcal{H}_0(t(x, T))) \), and \( v_H = \partial^2_{x^2} \mathcal{H}_0 \).

We already know that \( \tau_{\text{top}}(x, T; \epsilon) \) and \( Z(x, T; \epsilon) \) satisfy the same Virasoro constraints
\[
L_k \tau_{\text{top}} = 0, \quad L_k Z = 0, \quad k \geq 0.
\]
It is shown in [20] that these Virasoro constraints lead to the loop equation which determines \( F_g \) (\( g \geq 1 \)) from \( F_0 \) and \( \tilde{F}_g \) (\( g \geq 1 \)) from \( \tilde{F}_0 \) up to the addition of some constants. In order to prove the theorem, we need to show that
\[
F_0(x, T) = \tilde{F}_0(x, T). \tag{5.4}
\]
Let us first note the identity
\[ v_{\text{top}}(x, T) = \frac{1}{m} v_h(t(x, T)). \] (5.5)
This is due to the fact that \( v_h(t) \in \mathbb{C}[[t]] \) is determined by the equation
\[ v_h(t) = \sum_{i \geq 0} t_i v_h(t)^i \] from which it is easy to see that \( \frac{1}{m} v_h(t(x, T)) \) satisfies equation (5.3). So we have the identity (5.5). Then (5.4) follows from the fact that
\[ H_0(t) = \frac{1}{2} \sum_{i,j \geq 0} \tilde{t}_i \tilde{t}_j v_h(t)^{i+j+1} \tilde{t}_i^{i+j+1}, \quad \tilde{t}_i = t_i - \delta_{i,1}, \] (5.6)
and the definition (3.6) of \( F_0(x, T) \). From the dilaton equation satisfied by the Hodge integrals we know that \( Z(x, T; \epsilon) \) also satisfies the condition (3.7). Due to the identity (5.4) and the fact that \( \tau_{\text{top}}, Z \) satisfy the same loop equation, we conclude that
\[ Z(x, T; \epsilon) = c \tau_{\text{top}}(x, T; \epsilon), \]
where \( c \) is a nonzero constant that does not depend on \( \epsilon \). So we have proved the theorem for the special rational case.

Let us proceed to prove Theorem 1.1 for general \( p, q, r \in \mathbb{C} \) satisfying the local Calabi–Yau condition. We first prove some lemmas.

**Lemma 5.1** The power series \( u(x, T; \epsilon) \in \mathbb{C}[[x - 1, T; \epsilon]] \) defined by (1.7) satisfies a hierarchy of evolutionary PDEs of the form
\[ \frac{\partial u}{\partial T_\lambda} = \frac{\lambda}{pq} \epsilon^{\frac{\lambda}{2}} u \left( u' + \sum_{g \geq 1} \epsilon^{2g} \sum_{J \in \mathcal{Y}_{2g+1}} C_J(p, q; \lambda) u^{(J)} \right), \quad \lambda \in \mathcal{I}, \] (5.7)
where \( C_J(p, q; \lambda) \in \mathbb{C} \{ p, q, p^{-1}, q^{-1}, (p + q)^{-1} \}, J \in \mathcal{Y} \) are homogeneous rational functions in \( p \) and \( q \), and \( u^{(J)} := \partial_x^J u \cdots \partial_x^J u \).

**Proof** From [6] we know that
\[ w := -\frac{\epsilon^2}{pqr} \partial_t^2 \log Z_{\text{cubic}} \left( t; p, q, r; \sqrt{\frac{p + q}{pq}} \epsilon \right) \]
satisfies the Hodge hierarchy which has the form
\[ \frac{\partial w}{\partial t_i} = X_i(w, w', \ldots), \quad i \geq 0, \]
where \( X_i = X_i(w, w', \ldots) \in \mathcal{A}_{w,0}[w', w', \ldots][[\epsilon]] \). Moreover, the tau-symmetry property [6] of the Hodge hierarchy implies that
\[ \frac{\partial X_{i+1}}{\partial w} = X_i, \quad i \geq 0. \] (5.8)
Hence the Hodge hierarchy has the form

\[
\frac{\partial w}{\partial t_i} = \frac{w^i}{i!} w' + \sum_{g \geq 1} \epsilon^{2g} \sum_{k=0}^{i} \frac{w^{i-k}}{(i-k)!} \sum_{J \in \mathbb{N}^{g+1}} \tilde{C}_{k,J}(p,q) w(J), \quad i \geq 0,
\]  
(5.9)

where the coefficients \(\tilde{C}_{k,J}(p,q) \in \mathbb{C} [p,q,p^{-1},q^{-1},(p+q)^{-1}]\) do not depend on \(i\). Introduce a gradation on \(\mathbb{C} [p,q,p^{-1},q^{-1},(p+q)^{-1}][[t]]\) via the following degree assignments:

\[
\deg p = \deg q = 1, \quad \deg t_i = i - 1, \quad i \geq 0.
\]

By using the relations (1.11), we find that \(\deg H_g(t;p,q,r) = 3g - 3\). Consequently, we obtain \(\deg w = -1\) and

\[
\deg \tilde{C}_{k,J}(p,q) = l(J) - |J| - k.
\]

From the definition (1.7) of \(u\) and that of \(w\) we see that they are related by a Miura-type transformation

\[
u = pw + \frac{2p^2 + 2pq + q^2}{24pq^2} w'' \epsilon^2 + \cdots.\]  
(5.10)

By using the relation (1.11), we obtain equations (5.7) from equations (5.9) with

\[
C_{J}(p,q;\lambda) \in \mathbb{C} [p,q,p^{-1},q^{-1},(p+q)^{-1}].
\]

Then from equations (5.7) and the fact that \(\deg u = 0\), we know that \(C_{J}(p,q;\lambda)\) are homogeneous rational functions of \(p,q\) with degrees

\[
\deg C_{J}(p,q;\lambda) = 1 - |J|.
\]

The lemma is proved. \(\square\)

**Lemma 5.2** The flows of the FVH (1.8) can be represented in the form

\[
\frac{\partial u}{\partial T_\lambda} = \frac{\lambda}{pq} e^{\frac{\lambda}{pq} u} \left( u' + \sum_{g \geq 1} \epsilon^{2g} \sum_{J \in \mathbb{N}^{g+1}} \tilde{C}(p,q;\lambda) u(J) \right), \quad \lambda \in \mathcal{I},
\]

(5.11)

where \(\tilde{C}(p,q;\lambda) \in \mathbb{C} [p,q,p^{-1},q^{-1},(p+q)^{-1}]\) are homogeneous rational functions of \(p,q\) with degrees

\[
\deg \tilde{C}(p,q;\lambda) = 1 - |J|.
\]

**Proof** Similarly to the proof of Lemma 3.3 we can prove this lemma by induction based on the following identity:

\[
\frac{\partial}{\partial T_{1/m}} \left( \frac{\partial u}{\partial T_\lambda} \right) = \frac{\partial}{\partial T_\lambda} \left( \frac{\partial u}{\partial T_{1/m}} \right), \quad \forall \lambda \in \mathcal{I}.
\]

We omit the details here. \(\square\)
Example 5.3 We list here the explicit expressions of a few coefficients $\overline{C}_J(p, q; \lambda)$:

\[
\begin{align*}
\overline{C}_3(p, q; \lambda) &= \frac{(p + q)\lambda}{12p^2q^2}, \quad \overline{C}_2,1(p, q; \lambda) = \frac{(p + q)\lambda(2\lambda + p)}{12p^3q^2}, \\
\overline{C}_{1,1,1}(p, q; \lambda) &= \frac{(p + q)\lambda^2(\lambda + p)}{12p^4q^2}, \\
\overline{C}_5(p, q; \lambda) &= \frac{(p + q)\lambda(3(p + q)\lambda - (p^2 + pq + q^2))}{720p^4q^4}, \\
\overline{C}_{4,1}(p, q; \lambda) &= \frac{(p + q)\lambda(9(p + q)\lambda^2 + (2p^2 + 2pq - 3q^2)\lambda - p(p^2 + pq + 2q^2))}{720p^5q^4}, \\
\overline{C}_{3,2}(p, q; \lambda) &= \frac{(p + q)\lambda(3(p + q)\lambda^2 + (p^2 + pq - q^2)\lambda - pq^2)}{144p^6q^4}.
\end{align*}
\]

Lemma 5.4 For any homogeneous polynomial $f(p, q) \in \mathbb{C}[p, q]$, if $f(p, q) = 0$ holds true for

\[(p, q) \in U := \left\{ \left( \frac{1}{m}, \frac{1}{n} \right) \in \mathbb{C}^2 \mid m \text{ and } n \text{ are coprime positive integers} \right\},
\]

then $f(p, q) \equiv 0$ for any $p, q \in \mathbb{C}$.

Proof If $f(p, q) \not\equiv 0$, then the set of zero points of $f(p, q)$ can be written as

\[
\{(p, q) \in \mathbb{C}^2 \mid f(p, q) = 0\} = \bigcup_{i=1}^N \{(p, q) \in \mathbb{C}^2 \mid a_i p + b_i q = 0\},
\]

for some certain integer $N \geq 1$ and $a_i, b_i \in \mathbb{C}$. Obviously, $U$ is not a subset of this set, so $f(p, q)$ must vanish. The lemma is proved. $\square$

Now by using the validity of Theorem 1.1 for the special rational case and Lemmas 5.1 and 5.2 we know that the equalities

\[
C_J \left( \frac{1}{m}, \frac{1}{n}; \lambda \right) = \overline{C}_J \left( \frac{1}{m}, \frac{1}{n}; \lambda \right), \quad \forall J \in \mathbb{Y}
\]

hold true for arbitrary coprime positive integers $m$ and $n$. From the homogeneity of $C_J(p, q; \lambda)$, $\overline{C}_J(p, q; \lambda)$, and Lemma 5.4 it follows the validity of the identities

\[
C_J(p, q; \lambda) = \overline{C}_J(p, q; \lambda), \quad \forall J \in \mathbb{Y}
\]

for arbitrary non-zero $p, q \in \mathbb{C}$. Hence the power series $u(x, T; \epsilon)$ given by formula (1.7) is a solution to the FVH (1.8).

To complete the proof of the theorem, we need to show that the power series $Z(x, T; \epsilon)$ given by formula (1.11) is a tau-function for the FVH, i.e., to verify that it satisfies formulæ (2.7)–(2.8). Let us introduce, for any given $\lambda, \mu \in \mathbb{T}$, the following power series in $\mathbb{C}[[x - 1, T; \epsilon]]$:

\[
\Delta_\lambda := (\Lambda_3 - 1) \Lambda_2^{-\frac{1}{2}} \epsilon \frac{\partial \log Z(x, T; \epsilon)}{\partial T_\lambda} \bigg|_{u=u(x,T;\epsilon)} - \text{res}_{\lambda_3} L^{-\lambda/\rho} \bigg|_{u=u(x,T;\epsilon)},
\]

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$$\Delta_{\lambda,\mu} := e^2 \frac{\partial^2 \log Z(x, T; \epsilon)}{\partial T_\lambda \partial T_\mu} - \Omega_{\lambda,\mu\mid u=u(x, T; \epsilon)},$$

where $u(x, T; \epsilon)$ is defined by (1.14). We are to show

$$\Delta_\lambda = 0, \quad \Delta_{\lambda,\mu} = 0.$$  \hspace{1cm} (5.12)

By using the genus expansion of $\log Z_{\text{cubic}}$ given in [6], we can represent the two point correlation functions of $Z_{\text{cubic}}$ in the form

$$\frac{p + q}{p^2q^2} e^2 \frac{\partial^2 \log Z_{\text{cubic}}}{\partial t_i \partial t_j} \left( t(x, T); p, q, r; \frac{\sqrt{p + q}}{pq} \epsilon \right) = \frac{(v/p)^{i+j+1}}{(i+j+1)!} + X_{i,j}(v, v', \ldots; \epsilon)$$  \hspace{1cm} (5.13)

for $i, j \geq 0$. Here

$$v = v(x, T) := u(x, T; \epsilon) = 0 \in \mathbb{C}[[x - 1, T]],$$

and $X_{i,j}(z_0, z_1, \ldots; \epsilon) =: X_{i,j}$ are in $\epsilon A_{2,0,0} [z_1^{-1}, z_1, z_2, \ldots] [[\epsilon]]$ satisfying

$$\sum_{k \geq 1} k z_k \frac{\partial X_{i,j}}{\partial z_k} - \epsilon \frac{\partial X_{i,j}}{\partial \epsilon} = 0.$$  

Then by using the definition (1.11) of $Z(x, T; \epsilon)$ and formulae (5.13), we have

$$e^2 \frac{\partial^2 \log Z(x, T; \epsilon)}{\partial x \partial T_\lambda} = -r c_\lambda + \sum_{i \geq 0} \frac{\partial t_i}{\partial T_\lambda} e^2 \frac{\partial^2 \log Z_{\text{cubic}}}{\partial t_i \partial t_i} \left( t(x, T); p, q, r; \frac{\sqrt{p + q}}{pq} \epsilon \right)$$

$$= -r c_\lambda e^{\lambda v/p} + f_\lambda(v, v', \ldots; \epsilon),$$

$$e^2 \frac{\partial^2 \log Z(x, T; \epsilon)}{\partial T_\lambda \partial T_\mu} = - \frac{r}{pq} \frac{\lambda \mu}{\lambda + \mu} + \sum_{i,j \geq 0} \frac{\partial t_i}{\partial T_\lambda} \frac{\partial t_j}{\partial T_\mu} e^2 \frac{\partial^2 \log Z_{\text{cubic}}}{\partial t_i \partial t_j} \left( t(x, T); p, q, r; \frac{\sqrt{p + q}}{pq} \epsilon \right)$$

$$= \frac{1}{p + q \lambda + \mu} c_\lambda c_\mu e^{(\lambda + \mu) v/p} + f_{\lambda,\mu}(v, v', \ldots; \epsilon),$$

where $f_\lambda(z_0, z_1, \ldots; \epsilon) =: f_\lambda$ and $f_{\lambda,\mu}(z_0, z_1, \ldots; \epsilon) =: f_{\lambda,\mu}$ are in $\epsilon A_{2,0,0} [z_1^{-1}, z_1, z_2, \ldots] [[\epsilon]]$ satisfying

$$\sum_{k \geq 1} k z_k \frac{\partial f_\lambda}{\partial z_k} - \epsilon \frac{\partial f_\lambda}{\partial \epsilon} = 0, \quad \sum_{k \geq 1} k z_k \frac{\partial f_{\lambda,\mu}}{\partial z_k} - \epsilon \frac{\partial f_{\lambda,\mu}}{\partial \epsilon} = 0.$$  

Hence by using Lemma 2.2, we find that there exist elements

$$\Delta_\lambda = \Delta_\lambda^*(z_0, z_1, \ldots), \quad \Delta_{\lambda,\mu} = \Delta_{\lambda,\mu}^*(z_0, z_1, \ldots) \in \epsilon A_{2,0,0} [z_1^{-1}, z_1, z_2, \ldots] [[\epsilon]]$$  \hspace{1cm} (5.14)

satisfying the homogeneity conditions

$$\sum_{i \geq 1} iz_i \frac{\partial \Delta_\lambda^*}{\partial z_i} - \epsilon \frac{\partial \Delta_\lambda^*}{\partial \epsilon} = 0, \quad \sum_{i \geq 1} iz_i \frac{\partial \Delta_{\lambda,\mu}^*}{\partial z_i} - \epsilon \frac{\partial \Delta_{\lambda,\mu}^*}{\partial \epsilon} = 0,$$  \hspace{1cm} (5.15)

and

$$\Delta_\lambda = \Delta_\lambda^*(v, v', \ldots), \quad \Delta_{\lambda,\mu} = \Delta_{\lambda,\mu}^*(v, v', \ldots).$$
On the other hand, by acting $\epsilon \partial_{\lambda}$ and $\epsilon^2 \partial_{\lambda} \partial_{\mu}$ on the both sides of (1.7), and by using equations (2.5) and (2.54), we have

$$\partial_x \Delta_\lambda = 0, \quad \partial_x^2 \Delta_{\lambda, \mu} = 0.$$  

Then by using the transcendency of $v(x, T)$ (similarly with Lemma 3.9) we have the identities

$$\partial \Delta_\lambda^* = 0, \quad \partial^2 \Delta_{\lambda, \mu}^* = 0,$$

where $\partial = \sum_{i \geq 0} z^{i+1} \frac{\partial}{\partial z}$. Combining with (5.14)–(5.15), it gives $\Delta_\lambda^* = 0$, and $\Delta_{\lambda, \mu}^* = 0$. So equations (5.12) are true, namely, the power series $Z(x, T; \epsilon)$ satisfies the defining relations (2.6)–(2.8) of tau-functions for the FVH. Theorem 1.1 is proved.

### 6 The gap phenomenon

In this section, we give an application of Theorem 1.1 to the explicit computation of the coefficients of the gap condition that is given in [20]. Here we consider the case when $p = 1/m$, $q = 1/n$, where $m, n$ are coprime positive integers.

Recall that Lemma 3.4 states that the difference equation (1.17), i.e., the equation

$$\sum_{0 \leq \alpha_1 \leq \ldots \leq \alpha_m \leq n} \epsilon^{\sum_{j=1}^m V(x + \alpha_j m \epsilon - (j-\frac{1}{2})mc; \epsilon)} = \binom{m+n}{m} x,$$

has a unique solution of the form

$$V = V_{\text{gap}}(x; \epsilon) = \frac{1}{m} \log x + \sum_{g \geq 1} \epsilon^{2g} \frac{P_g(m, n)}{x^{2g}}.$$

**Proof of Proposition 1.2** We have shown in [20] that

$$\log Z_{\text{cubic}} \left( t(x, 0); \frac{1}{m}, \frac{1}{n}, \frac{1}{h}; \epsilon \right) = \epsilon^{-2} H_0 \left( t(x, 0) \right) + \frac{\sigma_1 - \frac{1}{24}}{2} \log x + \sum_{g \geq 2} \epsilon^{2g-2} \frac{R_g}{x^{2g-2}},$$

where $R_g = R_g(\sigma_1, \sigma_3) \in \mathbb{Q}[\sigma_1, \sigma_3]$, $g \geq 2$, and $\sigma_1, \sigma_3$ are given in (1.16). From Theorem 1.1 it follows that

$$u_{\text{top}}(x, T; \epsilon) = \left( \Lambda_3^{\frac{1}{2}} - \Lambda_3^{-\frac{1}{2}} \right) \left( \Lambda_1^{\frac{1}{2}} - \Lambda_1^{-\frac{1}{2}} \right) \log Z_{\text{cubic}} \left( t(x, T); \frac{1}{m}, \frac{1}{n}, -\frac{1}{h}; \epsilon \right).$$

So by using (3.13) we obtain

$$V(x; \epsilon) = \left( \Lambda_3^{\frac{1}{2}} - \Lambda_3^{-\frac{1}{2}} \right) \left( \Lambda_1^{\frac{1}{2}} - \Lambda_1^{-\frac{1}{2}} \right) \log Z_{\text{cubic}} \left( t(x, 0); \frac{1}{m}, \frac{1}{n}, -\frac{1}{h}; \epsilon \right).$$

The proposition then follows from (5.5) and (6.2)–(6.3).

**Example 6.1** Let us use Proposition 1.2 to compute $R_g$ for $g = 2, 3, 4$. To this end, we first derive the explicit expressions of $M_1^{[g]}$ for $g = 2, 3, 4$ by using (3.21). We have

$$M_1^{[1]} = \frac{mn(2mh - n)}{24} u'' + \frac{m^3 n(h + 1)}{24} (u')^2,$$

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\[ M_1^{[3]} = \frac{mn (m^2nh^2 - 4mh(2m^2 + 2mn + 7n^2) + 7n^3)}{5760} u^{(4)} \]
\[ + \frac{m^3n(h + 1) (12mnh - 4m^2 - 6mn - 9n^2)}{1440} u''u''' \]
\[ + \frac{m^2n (36m^2nh^2 - 4mh(3m^2 - 2mn + 8n^2) - 12m^3 - 8m^2n - 12mn^2 + 5n^3)}{5760} (u'')^2 \]
\[ + \frac{m^4n(h + 1) (22mnh - (8m^2 + 13n^2) - 4h)}{2880} (u')^3u'' \]
\[ + \frac{m^5n(h + 1) (5mnh - (2m^2 - 3mn + 2n^2) - 2h)}{5760} (u')^4. \]
\[ (6.6) \]

The expressions for \( M_1^{[3]} \) and \( M_1^{[4]} \) are already quite long, and are not listed here. Then by using (3.24) we obtain

\[ P_1 = \frac{mn - nh}{24m}, \]
\[ P_2 = -\frac{4m^2n^2h^2 + mnh(3m^2 - 7mn - 7n^2) - nh(4m^2 - 3mn - 3n^2) + mnh}{960m}, \]
\[ P_3 = \frac{1}{72576m} \left( 8m^3n^3h^3 + 4m^2n^2h^3 \left( 4m^2 - 19mn - 19n^2 \right) \right. \]
\[ + mnh \left( 95m^4 - 251m^3n - 156m^2n^2 + 190mn^3 + 95n^4 \right) \]
\[ - nh \left( 144m^4 - 88m^3n - 61m^2n^2 + 54mn^3 + 27n^4 \right) \]
\[ + mnh \left( 50m^2 - 13mn - 13n^2 - mnh \right), \]
\[ P_4 = \frac{1}{4147200m} \left( 3376m^4n^4h^4 - 32m^3n^3h^3 \left( 87m^2 + 62mn + 62n^2 \right) \right. \]
\[ - 40m^2n^2h^2 \left( 312m^4 - 68m^3n - 7m^2n^2 + 122mn^3 + 61n^4 \right) \]
\[ - mnh \left( 5257m^6 - 15689m^5n - 13203m^4n^2 + 3699m^3n^3 - 1333m^2n^4 \right. \]
\[ - 3819mn^5 - 1273n^6 \]
\[ + nh \left( 840m^6 - 3312m^5n - 344m^4n^2 + 5711m^3n^3 + 2293m^2n^4 - 675mn^5 - 225n^6 \right) \]
\[ - 2mnh \left( 1764m^4 - 232m^3n - 23m^2n^2 + 418mn^3 + 209n^4 \right) \]
\[ - 408m^2n^2h^2 + mnh \left( 147m^2 + 47mn + 47n^2 \right) - 2mnh \right). \]

Finally it follows from Proposition 1.2 that

\[ R_2 = -\frac{1}{1440} + \frac{13\sigma_1}{5760} + \frac{7\sigma_1^2}{5760} + \frac{1}{17280} \left( \sigma_1^3 - \sigma_3 \right), \]
\[ R_3 = \frac{1}{181440} - \frac{107\sigma_1}{362880} + \frac{145\sigma_1^2}{290304} - \frac{31\sigma_1^3}{161280} \]
\[ - \left( \frac{31}{1088640} - \frac{113\sigma_1}{4354560} \right) \left( \sigma_1^3 - \frac{\sigma_3}{2} \right) - \frac{\sigma_1^3 - \sigma_3/2}{13063680}, \]

which coincide with the ones given in [20]. In the same way we also compute \( R_4 \):

\[ R_4 = \frac{211}{10886400} + \frac{\sigma_1}{48600} - \frac{1193\sigma_1^2}{4354560} + \frac{18629\sigma_1^3}{58060800} - \frac{127\sigma_1^4}{1290240} \]
\[
+ \left( \frac{83}{3870720} - \frac{1657\sigma_1}{32659200} + \frac{6469\sigma_1^2}{261273600} \right) \left( \sigma_3^3 - \frac{\sigma_3}{2} \right)
+ \left( \frac{17}{65318400} + \frac{247\sigma_1}{1567641600} \right) \left( \sigma_1^3 - \frac{\sigma_3}{2} \right)^2 - \left( \frac{\sigma_3^3}{2351462400} \right).
\]

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