Checking the Model and the Prior for the Constrained Multinomial

Berthold-Georg Englert\textsuperscript{1}, Michael Evans\textsuperscript{2}, Gun Ho Jang\textsuperscript{3}, Hui Khoon Ng\textsuperscript{4}, David Nott\textsuperscript{5} and Yi-Lin Seah\textsuperscript{6}

Abstract: The multinomial model is one of the simplest statistical models. When constraints are placed on the possible values for the probabilities, however, it becomes much more difficult to deal with. Model checking and checking for prior-data conflict is considered here for such models. A theorem is proved that establishes the consistency of the check on the prior. Applications are presented to models that arise in quantum state estimation as well as the Bayesian analysis of models for ordered probabilities.

Key words and phrases: model checking, checking for prior-data conflict, constrained multinomial, quantum state estimation, ordered probabilities, Zipf-Mandelbrot distribution, marginalizing the elicitation.

1 Introduction

Suppose we have a sample of \( n \) from a multinomial(1, \( \theta_1, \ldots, \theta_{k+1} \)) distribution where \( \theta = (\theta_1, \ldots, \theta_k) \in \Theta_k \) is an unknown element of

\[
\Theta_k = \{ (\theta_1, \ldots, \theta_k) : \theta_1 + \cdots + \theta_k < 1, \theta_i > 0, i = 1, \ldots, k \}
\]

and \( \theta_{k+1} = 1 - \theta_1 - \cdots - \theta_k \). Note that throughout the paper the notation \( \theta \) always refers to first \( k \) probabilities. If \( T_n = (T_1, \ldots, T_k) \) denotes the counts from the first \( k \) categories, with \( T_{k+1} = n - T_1 - \cdots - T_k \), then \( T_n \) is a minimal sufficient statistic (mss) and \( T_n \sim \text{multinomial}(n, \theta_1, \ldots, \theta_{k+1}) \).

In some applications something is known about the true value of \( \theta \), beyond the fact that it is in \( \Theta_k \), and this is expressed in the form of a prior probability distribution \( \Pi \) on \( \Theta_k \). It is assumed that the prior is absolutely continuous with respect to volume measure on \( \Theta_k \) with the density of \( \Pi \) denoted by \( \pi \). The prior can be thought of as a way of imposing constraints on \( \theta \). These may be soft\textsuperscript{1}Centre for Quantum Technologies (CQT), Dept. of Physics, National University of Singapore and MajuLab, Singapore\textsuperscript{2}Dept. of Statistical Sciences, University of Toronto, Toronto, Ontario, Canada\textsuperscript{3}Ontario Institute for Cancer Research, Toronto, Ontario, Canada\textsuperscript{4}Yale-NUS College, CQT and MajuLab, Singapore\textsuperscript{5}Dept. of Statistics and Applied Probability, National University of Singapore, Singapore\textsuperscript{6}Centre for Quantum Technologies, Singapore
constraints such that \( \pi(\theta) \) is relatively low in parts of \( \Theta_k \), reflecting beliefs that these values are improbable, or hard constraints where \( \pi(\theta) = 0 \) for values that are ‘known’ to be impossible. The prior is to be thought of as the marginal distribution of \( \theta \) and the multinomial \((n, \theta_1, \ldots, \theta_{k+1})\) is the conditional for \( T_n \) given \( \theta \), together giving rise to the joint probability model for \((\theta, T_n)\).

Given that the model and prior are appropriate in a particular application, the inferential analysis proceeds by first obtaining the conditional distribution for \( \theta \) given \( T_n \), the posterior of \( \theta \). The posterior represents beliefs about the true value of \( \theta \) after observing the data. While a variety of approaches can be considered for inference about \( \theta \), our concern here is with whether or not the specified model and prior are appropriate.

The primary way to determine whether or not an ingredient to a statistical analysis is appropriate is to compare it somehow with the data. For example, if the observed data is surprising for every distribution in the model, then it is reasonable to question the appropriateness of the model. By surprising it is meant that the data falls in a region such that each distribution in the model gives a relatively low probability for that data’s occurrence. This assessment is usually approached via the computation of a p-value in a so-called goodness-of-fit test. For example, for the multinomial this could be assessed by a generalization of the well-known runs test as this is assessing whether or not the unadjusted data is i.i.d. In this paper it will always be assumed that i.i.d. sampling holds. This can be induced (approximately) when random sampling from large populations but otherwise needs to be checked.

It is to be noted that model checking procedures are typically based on aspects of the data beyond the values of the mss although, as subsequently discussed, some qualifications are necessary. So, for example, the goodness-of-fit test could be based on the conditional distribution of the original data given the value of the minimal sufficient statistic. This conditional distribution is completely independent of \( \theta \) and the data that was surprising for this distribution is indicating a problem with the model. In certain cases there are ancillaries that are functions of the mss, however, and then model checking can proceed by comparing the values of these ancillaries with their known distributions. For example, the introduction of hard-constraints with the multinomial can indeed produce such ancillaries.

In general, there are many ways in which model checking can proceed and it doesn’t seem possible to argue definitively for one approach over another. There are, however, some basic principles that seem necessary. For example, being careful about what aspects of the data are used in model checking seems paramount. Another basic principle would seem to be the separation of the checking of the prior from checking the model. If the ingredients fail the tests of appropriateness, then we would like to know specifically what component caused the failure. If we try to jointly check the model and prior, say by some aspect of the posterior, then failure cannot be assigned to the specific component. Also, given that the prior is implicitly dependent on the model, it isn’t meaningful to check the prior if the model fails its checks. So as argued in Evans and Moshonov (2006), it makes sense to separate the checks on the model and
prior and perform the check on the model first. Also, it is shown in Evans and Moshonov (2006, 2007) that the check on the prior can sometimes be decomposed so that individual components of the prior, as when the prior is specified hierarchically, can be checked so that an isolated aspect of the prior can be identified as causing the problem when one exists.

While the separation of the check on the prior and the model is a principle worth noting, this does not rule out the possibility of a Bayesian approach to model checking. For example, suppose hard constraints lead to the model being a subclass $M'$ of the full multinomial family $M$. It is then reasonable to place a uniform prior on $M$, equivalently a uniform prior on $\theta$, so that each multinomial has the same weight and then assess whether or not $M'$ is 'plausible'. Just how this assessment is to be carried out is a matter for some discussion but our preference is a comparison of the prior belief in $M'$ with its posterior belief. So if belief in $M'$ has increased after seeing the data, then there is evidence for $M'$ being true and if it has decreased, then there is evidence against $M'$ being true. Such an approach is discussed in Al-Labadi and Evans (2016) and Al-Labadi, Baskurt and Evans (2017) where a distance measure is introduced and the concentration of the posterior about $M'$ is compared with the concentration of the prior about $M'$ to assess whether or not there is evidence for or against $M'$. A notable aspect of this, is that while the check is Bayesian and based on a measure of statistical evidence, it does not involve the prior $\Pi$ which is only checked if there is evidence in favor of $M'$.

Note too that this approach involves conditioning on all the data and so avoids the issues that arise when there are multiple maximal ancillaries.

In Section 2 model checking is discussed for the constrained multinomial, namely, when $\theta \in \Theta$ and $\Theta$ is a proper subset of $\Theta_k$. Constrained multinomial models arise naturally in quantum state estimation and such an example is developed in the paper as well as goodness-of-fit for the multinomial model with ordered probabilities. In Section 3 methods are discussed for checking the prior and a consistency result is established for the specific check used that substantially generalizes a result established in Evans and Jang (2011b). Also, an elicitation algorithm is developed for the multinomial with ordered probabilities and this leads to a methodology for checking elicited information which is somewhat different than checking a specific prior.

\section{Checking the Model}

Suppose that we are satisfied that the data is i.i.d. multinomial(1, $\theta_1, \ldots, \theta_{k+1}$) for some $\theta = (\theta_1, \ldots, \theta_k) \in \Theta_k$. Suppose further, however, it is believed that $\theta^{\text{true}} \in \Theta$, where $\Theta$ is a proper subset of $\Theta_k$, and it is desirable for the analysis to reflect this. It is then necessary to check that indeed this constraint is appropriate for the data obtained. For example, it might be that the data was not collected correctly and this led to a failure of the model such that, while the multinomial assumption is correct, $\theta^{\text{true}} \not\in \Theta$. Note that for the model in question any elicited prior $\Pi$ on $\theta$ must satisfy $\Pi(\Theta) = 1$. By \textit{elicited prior} is
meant a prior that is selected based upon a process that reflects what is known about the true distribution being sampled from.

Consider the following examples.

Example 1. Quantum state estimation.

Counts of events are recorded associated with the state of a qubit. A quantum measurement involves $k + 1$ detectors and quantum theory leads to a distribution that corresponds to sampling from a multinomial$(1, \theta_1, \ldots, \theta_{k+1})$ for some $k$ where $\theta_i$ is the probability that a “click” is recorded from the $i$-th detector. Quantum theory dictates, depending on how the measurements are taken, that the probabilities satisfy certain constraints.

For example, when $k + 1 = 3$, then the symmetric trine model imposes the constraint $\theta_1^2 + \theta_2^2 + \theta_3^2 \leq 1/2$, with more involved constraints required for the asymmetric case as discussed in Example 3. When $k + 1 = 4$, then the cross hairs model imposes the constraints $\theta_1 + \theta_2 = 1/2, \theta_3 + \theta_4 = 1/2$ and $\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 \leq 3/8$, while the tetrahedron model corresponds to the constraint $\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 \leq 1/3$ only. When $k + 1 = 6$, then the Pauli model imposes the constraints $\theta_1 + \theta_2 = 1/3, \theta_3 + \theta_4 = 1/3, \theta_5 + \theta_6 = 1/3$ and $\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + \theta_5^2 + \theta_6^2 \leq 2/9$. More on how these models arise can be found in Shang, Ng, Sehrawat, Li and Englert (2013) but, sufficed to say, these applications produce a rich variety of constrained multinomial models.

Example 2. Contingency tables with ordered probabilities.

In some circumstances it is reasonable to suppose that the probabilities satisfy an ordering such as

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_{k+1}. \quad (1)$$

Such a model arises in contexts where systems exhibit aging as in, for example, Briegel, Englert, Sterpi and Walther (1994). Issues associated with checking (1), and with eliciting a prior on $\theta$ when this restriction is deemed correct, lead to the utility of the following result.

Lemma 1. Any $\theta \in \Theta_k$ satisfying (1) is given by, for some $\omega \in \Theta_k$,

$$(\theta_1, \ldots, \theta_{k+1})^t = A_k(\omega_1, \ldots, \omega_{k+1})^t \quad (2)$$

and any $\omega \in \Theta_k$ produces a $\theta \in \Theta_k$ satisfying (1) via (2), with

$$A_k = \begin{pmatrix} 1 & 1/2 & 1/3 & \ldots & 1/(k+1) \\ 0 & 1/2 & 1/3 & \ldots & 1/(k+1) \\ 0 & 0 & 1/3 & \ldots & 1/(k+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1/(k+1) \end{pmatrix}.$$ 

Proof: First assume $(\theta_1, \ldots, \theta_{k+1})^t = A_k(\omega_1, \ldots, \omega_{k+1})^t$ for $\omega \in \Theta_k$. Then $\theta_i = \sum_{j=1}^{k+1} \omega_j / j$ and it is clear that $0 \leq \theta_i \leq 1$ and $\theta_1 + \cdots + \theta_{k+1} = \omega_1 + \cdots + \omega_{k+1} = 1$ so $\theta \in \Theta_k$. It is also immediate that $\theta$ satisfies (1).
Now suppose \( \theta \in \Theta_k \) satisfies (1) and put \((\omega_1, \ldots, \omega_{k+1})^t = A_k^{-1}(\theta_1, \ldots, \theta_{k+1})^t\). An easy calculation shows that

\[
A_k^{-1} = \begin{pmatrix}
1 & -1 & 0 & \ldots & 0 \\
0 & 2 & -2 & \ldots & 0 \\
0 & 0 & 3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & k+1
\end{pmatrix}.
\]

Therefore, \( \omega_i = i(\theta_i - \theta_{i+1}) \) for \( i = 1, \ldots, k \) and \( \omega_{k+1} = (k + 1)\theta_{k+1} \). Since \( \theta_i \geq \theta_{i+1} \), then \( \omega_i \geq 0 \) and if \( \omega_i > 1 \) then \( \theta_i > \theta_{i+1} + 1/i > 1/i \) which by (1) implies \( \theta_1 + \cdots + \theta_i > 1 \) which is false. So \( \omega_i \in [0,1] \) for \( i = 1, \ldots, k \) and similarly \( \omega_{k+1} \in [0,1] \). Finally, \( \omega_1 + \cdots + \omega_{k+1} = \sum_{j=1}^k i(\theta_i - \theta_{i+1}) + (k + 1)\theta_{k+1} = \theta_1 + \cdots + \theta_{k+1} = 1 \) and this completes the proof.

A particular model satisfying (1) is given by the Zipf-Mandelbrot distribution where, for parameters \( \alpha > -1, \beta \geq 0 \),

\[
\theta_i = (\alpha + i)^{-\beta}/C_k(\alpha, \beta) \tag{3}
\]

for \( i = 1, \ldots, k + 1 \) where \( C_k(\alpha, \beta) = \sum_{i=1}^{k+1} (\alpha + i)^{-\beta} \). When \( \beta = 0 \) this is the uniform distribution and for fixed \( \beta \) this converges to the uniform as \( \alpha \to \infty \).

For fixed \( \alpha \) the distribution becomes degenerate on the first cell as \( \beta \to \infty \). For large \( k, \alpha = 0 \) and \( \beta > 1 \) the zeta(\( \beta \)) distribution, with \( \theta_i \propto i^{-\beta} \) for \( i = 1, 2, \ldots \), serves as an approximation. As discussed in Izsák (2006), there are a variety of applications of this distribution as in word frequency distributions in texts. The distribution given by (3) is denoted here as \( \text{ZM}_k(\alpha, \beta) \).

In dose-response models, as discussed in Chuang-Stein and Agresti (1997), there are \( I \) treatments, possibly corresponding to the frequency of a particular treatment, and \( J \) response classifications reflecting the severity of a condition from nonexistent to most severe. When there is a monotone increasing effect based on say frequency of treatment, it makes sense to assume \( \theta_{ij} \leq \theta_{2j} \leq \cdots \leq \theta_{k+1} \) for \( j = 1, \ldots, J \).

So there are many applications of the constrained multinomial.

Following the discussion in Section 1 the constrained multinomial model is checked first and, if the model passes, the prior \( \Pi \) placed on the restricted parameter space \( \Theta \) is then checked. The check on the constraints should not involve \( \Pi \) in any way since it is not involved in the production of the data. One of the advantages of this is that we can contemplate using another prior \( \Pi_0 \) on \( \Theta_k \) where \( \text{supp}(\Pi_0) = \Theta_k \), for the model checking step. This leads to formal inference methods for model checking. For example, taking \( \Pi_0 \) to be the uniform prior on \( \Theta_k \) is quite natural as this treats all multinomials equivalently and so this prior is used for the model checking step hereafter.

Suppose first that \( \Pi_0(\Theta) > 0 \). Based on the observed \( T_n = T_n(x) \), the posterior probability of \( \Theta \) is

\[
\Pi_0(\Theta | T_n(x)) = \int_\Theta f_\theta(T_n(x)) \pi_0(\theta) d\theta/m_{T_n}(T_n(x))
\]
where \( f_\theta \) is the multinomial\((n, \theta_1, \ldots, \theta_{k+1}) \) density, \( \pi_0 \) is the Dirichlet\((1, \ldots, 1) \) density and \( m_{T_n}(T_n(x)) = \int_{\Theta} f_\theta(T_n(x)) \pi_0(\theta) \, d\theta \). Note that the posterior distribution of \( \theta \) is Dirichlet\((T_1(x) + 1, \ldots, T_{k+1}(x) + 1) \) which can be used in evaluating \( \Pi_0(\Theta | T_n(x)) \). The relative belief ratio in favor of the hypothesis \( H_0 = \Theta \), namely, the hypothesis that the constraints hold, is then

\[
\text{RB}(\Theta | T_n(x)) = \frac{\Pi_0(\Theta | T_n(x))}{\Pi_0(\Theta)}
\]

and there is evidence in favor of \( H_0 \) when \( \text{RB}(\Theta | T_n(x)) > 1 \) and evidence against when \( \text{RB}(\Theta | T_n(x)) < 1 \). It seems reasonable in such circumstances to not be concerned about the model if \( \text{RB}(\Theta | T_n(x)) > 1 \) and consider that a problem has occurred whenever \( \text{RB}(\Theta | T_n(x)) < 1 \). It is possible, however, to also assess the strength of this evidence by reporting \( \Pi_0(\Theta | T_n(x)) \) as this represents our belief (as opposed to the evidence) that \( H_0 \) is true. So, if \( \text{RB}(\Theta | T_n(x)) > 1 \) and \( \Pi_0(\Theta | T_n(x)) \) is low, then there is only weak evidence in favor of \( H_0 \) while, if \( \text{RB}(\Theta | T_n(x)) < 1 \) and \( \Pi_0(\Theta | T_n(x)) \) is low, then there is strong evidence against \( H_0 \). Similarly, if \( \text{RB}(\Theta | T_n(x)) > 1 \) and \( \Pi_0(\Theta | T_n(x)) \) is high, then there is strong evidence in favor of \( H_0 \) while, if \( \text{RB}(\Theta | T_n(x)) < 1 \) and \( \Pi_0(\Theta | T_n(x)) \) is high, then there is only weak evidence against \( H_0 \). Note that it is important to separate the measurement of evidence from the measurement of belief as is done here.

In some circumstances it may arise that \( \Pi_0(\Theta) = 0 \) simply because \( \Theta \) is a lower dimensional subset of \( \Theta_k \). As such, we cannot proceed as just described. In fact, if \( \Pi_0(\Theta) \) is very small, then again the preceding approach to checking the model seems questionable as it cannot be expected that \( \Pi_0(\Theta | T_n(x)) \) will be large, and so obtain strong evidence in favor of the model when that is appropriate, unless the amount of data is very large. To deal with these problems consider the approach discussed in Al-Labadi and Evans (2016) and Al-Labadi, Baskurt and Evans (2017). For this \( d_{H_0} : H_0 \to [0, \infty) \) is specified where \( d_{H_0}(\theta) \) is a measure of the distance of \( \theta \) from \( H_0 \) with \( d_{H_0}(\theta) = 0 \) iff \( H_0 \) is true. For example, in some contexts squared Euclidean distance might make sense so that \( d_{H_0}(\theta) = \inf_{\theta' \in \Theta} ||\theta - \theta'||^2 \) for \( \theta \in \Theta_k \). Perhaps a more natural measure is obtained using Kullback-Leibler divergence, namely, \( d_{H_0}(\theta) = \inf_{\theta' \in \Theta} \text{KL}(\theta || \theta'). \)

The hypothesis \( H_0 \) is then assessed via the relative belief ratio

\[
\text{RB}_{d_{H_0}}(0 \mid T_n(x)) = \lim_{\delta \downarrow 0} \text{RB}_{d_{H_0}}([0, \delta] \mid T_n(x)).
\]

Typically the limit cannot be computed exactly so \( \delta > 0 \) is selected such that \( \text{RB}_{d_{H_0}}([0, \delta] \mid T_n(x)) \approx \text{RB}_{d_{H_0}}(0 \mid T_n(x)). \) In practice, there is a \( \delta > 0 \) such that, if \( d_{H_0}(\theta_{true}) \in [0, \delta) \), then \( H_0 \) can be regarded as true. The value of \( \delta \) can be determined via bounding the absolute error in the probabilities (squared Euclidean distance) or bounding the relative error in the probabilities (KL divergence), see Al-Labadi, Baskurt and Evans (2017). So, if \( \text{RB}_{d_{H_0}}([0, \delta] \mid T_n(x)) > 1 \) there is evidence in favor of \( H_0 \) and evidence against when \( \text{RB}_{d_{H_0}}([0, \delta] \mid T_n(x)) < 1 \). The strength of the evidence can be measured by discretizing the range of the prior distribution of \( d_{H_0} \) into \([0, \delta], [\delta, 2\delta], \ldots \) and then computing the posterior probability \( \Pi_0(\{i : \text{RB}_{d_{H_0}}([0, \delta] \mid T_n(x)) \leq \text{RB}_{d_{H_0}}([0, \delta] \mid T_n(x)) \} \mid T_n(x)) \).
Table 1: Results from two experiments based on the trine model in Example 3.

|          | $n_1$ | $n_2$ | $n_3$ |
|----------|-------|-------|-------|
| Symmetric| 7076  | 1912  | 1748  |
| Asymmetric| 6756  | 316   | 248   |

The consistency of this approach to model checking follows from results in Evans (2015). As $n \rightarrow \infty$ the relative belief ratio converges to its maximum possible value (greater than 1) and the strength goes to 1 when $H_0$ is true and the relative belief ratio and the strength go to 0 when $H_0$ is false.

Consider now applications to some examples.

**Example 3.** Goodness-of-fit for the trine model.

Table 1 contains data from two separate experiments discussed in Len, Dai, Englert and Krivitsky (2017) where models corresponding to two instances of the trine model are relevant and $n_i$ is the number of clicks on the $i$-th detector.

For these models $\Theta = \{\theta : (\theta - c)^t C (\theta - c) \leq 1\}$ where

$$c = \frac{1}{2} \left( \frac{2a}{1-a} \right), C = (1 - 2a)^{-1} \left( \frac{(1 - 1/a)^2}{2} \frac{2}{4} \right),$$

$a = 0.5 \sin^2(\cos^{-1}(\cot(2\varphi_0)))$ and $\varphi_0$ an angle associated with the experiment.

For the symmetric trine $\varphi_0 = \pi/6$ so $a = 1/3$ and $\Pi_0(\Theta) = a \sqrt{1 - 2a\pi} = 0.6046$. Under $\Pi_0$ the posterior distribution of $\theta$ is Dirichlet(3417, 1913, 1749) and sampling from this distribution shows that the entire posterior is concentrated within $\Theta$ so $\text{RB}(\Theta | T_n(x)) = 1/0.6046 = 1.6540$. So there is evidence in favor of the symmetric trine model and this is very strong evidence since the posterior content of $\Theta$ is effectively 1.

For the asymmetric trine case $\varphi_0 = 2\pi/9$ so $a = 0.48445$ and $\Pi_0(\Theta) = 0.2684$. Under $\Pi_0$ the posterior distribution of $\theta$ is Dirichlet(6193, 317, 249) and sampling from this distribution shows that the entire posterior is concentrated within $\Theta$ so $\text{RB}(\Theta | T_n(x)) = 1/0.2684 = 3.7258$. So there is evidence in favor of the symmetric trine model and again this is very strong evidence.

So with both models one can feel quite confident that the true values of the probabilities lie within the respective sets $\Theta$. The evidence is pretty definitive here because of the large amount of data collected.

**Example 4.** Goodness-of-fit for ordered probabilities.

A numerical example used in Izsák (2006), based on data concerned with fly diversity found in Papp (1992), is considered where the counts are given by $f = (145, 96, 35, 29, 20, 11, 4, 4, 4, 3, 2, 2, 1, 1, 1, 1, 1, 1)$. The question is whether or not these data can reasonably be thought of as coming from the model given by (1) and even from the submodel given by the collection of Zipf-Mandelbrot distributions. So here $k = 17$ and $n = 363$, the prior $\Pi_0$ is $\theta \sim \text{Dirichlet}(1, \ldots, 1)$ and the posterior is $\theta \sim \text{Dirichlet}(146, 97, 36, \ldots, 2)$.

Consider first checking (1). For this model $\Pi_0(\Theta) = 1/18! = 1.5619 \times 10^{-16}$ which is exceedingly small and this suggests that estimating $\Pi_0(\Theta | f)$ with any
accuracy will be very difficult. It is to be noted, however, that given the small prior probability of this set, if any of the values generated from the posterior for some feasible sample size fall in \( \Theta \), then this will give clear evidence in favor of the model. For example, in a sample of \( 10^7 \) the posterior content was estimated as \( 10^{-7} \) and this produces a relative belief ratio of \( 6.4 \times 10^8 \) but the strength of this evidence in favor is exceedingly weak. A better approach in such a problem is to group the cells into sequential subgroups such that the hypothesized monotonicity in the model is maintained. For example, here there are 18 cells and so 9 groups of size 2 or 6 groups of size 3 are possible. Also, 4 groups of size 4 are possible with a fifth group of size 2. Clearly it is always possible to group the cells in this way so that monotonicity in the probabilities is maintained. To select which grouping to use for the test it makes sense to start with the finest grouping such that the posterior content can be accurately estimated but coarser groupings can also be examined. Choosing 9 groups of size 2 worked here as the posterior content of the relevant set was estimated as \( 0.0396 \), \( 0.0402 \) and \( 0.0406 \) based on samples of sizes \( 10^4 \), \( 10^5 \) and \( 10^6 \), respectively. Since the prior content of the relevant set is \( 1/9! = 2.7557 \times 10^{-6} \), the relative belief ratio is 14726 although, based on the posterior content, this appears to be only weak evidence in favor. A coarser grouping leads to increased confidence in the model. For example, with groups of size 3 the relative belief ratio is 285.6312 and the posterior content is 0.40.

Consider now checking the ZM\(_k\) model. Since the set of ZM\(_k\) distributions has prior probability 0 with respect to the uniform distribution, the distance measure approach is necessary and the KL distance measure is used. A technical difficulty involved here is the need to find, for given \( \theta \), the value of

\[
d_{H_0}(\theta) = \inf_{\alpha > -1, \beta \geq 0} \sum_{i=1}^{k+1} \theta_i \ln(C_k(\alpha, \beta) (\alpha + i)^\beta)
\]

for each generated \( \theta \), to obtain samples from the prior and posterior distribution of \( d_{H_0} \). For this a large table of ZM\(_k\) distributions was created, computed for each element and the distribution found that minimizes this quantity.

As discussed in Example 2, there is redundancy in the parameterization of this family. For example, uniformity is well-approximated by many values of \((\alpha, \beta)\). A value of \( \delta > 0 \) is selected, however, so that if the KL distance between two distributions is less than \( \delta \), then this difference is irrelevant for the application. Note that \( \log(\theta_i/p_i) = \log(1 + (\theta_i - p_i)/p_i) \approx (\theta_i - p_i)/p_i \) when \( (\theta_i - p_i)/p_i \) is small and so, if \( \text{KL}(\theta, p) = \sum_{i=1}^{k+1} \theta_i \log(\theta_i/p_i) \leq \delta \), then the average relative difference between the probabilities given by \( \theta \) and \( p \) is immaterial. So for any \( \beta \geq 0 \) it is only necessary to consider values of \( \alpha \) such that the KL distance

\[
(k + 1)^{-1} \sum_{i=1}^{k+1} \ln(C_k(\alpha, \beta) (\alpha + i)^\beta/(k + 1))
\]

is greater than or equal to \( \delta \) and this places an upper bound on \( \alpha \). As such, this redundancy plays no role in assessing the goodness-of-fit.
Figure 1 is plot of the prior and posterior densities of $d_{H_0}$ based on Monte Carlo samples of size $10^5$, using $\delta = 0.02$ and some smoothing. It is clear from this that the posterior has become much more concentrated about the ZM$_k$ model than the prior. Furthermore, $\text{RB}((0, \delta) \mid f) = 1.75 \times 10^3$ and the strength equals 1. So there is ample evidence in favor of the ZM$_k$ model with this data set. This agrees somewhat with the finding in Izsák (2006) who conducted a goodness-of-fit test via computing a p-value based on the chi-squared statistic after grouping and found no evidence against the model. Note that with the methodology developed here, there is no need to appeal to asymptotics.

It may seem anomalous that stronger evidence is found for the ZM$_k$ model than for the bigger model of ordered probabilities. One might try to account for this by noting that different methodologies are used in the two cases, but even when the same methods are used it is possible to have evidence for a subset and evidence against the superset let alone just weaker evidence. This phenomenon can arise, when the prior probability of the subset is relatively small when compared to the prior probability of the superset as is the case here. So evidence in favor of a subset may be mitigated by the other possibilities in the bigger set. A full discussion and relevant example concerning this can be found in Evans (2015).

### 3 Checking the Prior

Suppose the model has been found to be acceptable so that now attention focuses on the elicited prior $\Pi$. A prior is inappropriate when the prior places relatively little mass in a region containing the true value. Of course the true value is not known but still there are a number of ways in which the prior can be checked.
Overall, however, all methods for checking the prior are assessing whether or not such a contradiction exists.

The approach taken in Evans and Moshonov (2006) is used here. The prior predictive density of $T_n$ is given by

$$m_{T_n}(t) = \frac{n!}{\prod_{j=1}^{k+1} t_j!} \int_{\Theta_k} \prod_{j=1}^{k+1} \theta_j^{t_j} \pi(\theta) \, d\theta.$$  (6)

Based on this density the check on the prior is to compute

$$M_{T_n}(m_{T_n}(t) \leq m_{T_n}(T_n(x))).$$  (7)

Clearly (7) is measuring where the observed $T_n(x)$ lies with respect to its prior distribution as $m_{T_n}$ is the prior density of $T_n$. If (7) is small, then $T_n(x)$ lies in a region of low prior probability for $T_n$ and it is apparent that the data and the prior are in conflict. It is also clear that the check on the prior should depend on the data only through a mss because the conditional distribution of the remaining aspects of the data beyond the mss does not depend on $\theta$ and so can reveal nothing about the adequacy of the prior.

In Evans and Jang (2011b) a general consistency result is established for (7) as it is shown there that, under conditions, this converges to $\Pi(\pi(\theta) \leq \pi(\theta^{\text{true}}))$. A small value of (7) is thus an indication that the prior is placing its mass in the wrong place as this suggests that the true value lies in a region of relatively low prior probability. One of the conditions for convergence, however, is that the prior predictive distribution be continuous. This is clearly not true in the case of the multinomial, as $M_{T_n}$ is always discrete. It was proved in Evans and Jang (2011b), however, that when $k = 1$ a continuized version of $M_{T_n}$ can be constructed that yields the consistency result in the binomial case. It is part of our purpose here to establish the general consistency result for the multinomial. This turns out to be much more difficult than the binomial case. As such the Appendix contains the proof of the following result where $\Theta_{k,c}, \Theta_{k,d}$ denote the sets of continuity and discontinuity points of $\pi$.

**Theorem 1** Let $\pi$ be a prior on the probabilities $\theta \in \Theta_k$ that satisfies the following conditions.

(A1) The prior density is bounded above, that is, there exists $B > 0$ such that $\pi(\theta) \leq B$ for all $\theta \in \Theta_k$.

(A2) The prior density is continuous almost surely with respect to volume measure, that is, $\text{vol}(\Theta_{k,d}) = 0$.

(A3) The prior probability of each level set of prior density is a null set with respect to the prior, that is, $\Pi(\{\theta : \pi(\theta) = l\}) = 0$ for any $l \geq 0$.

Then (7) converges to $\Pi(\pi(\theta) \leq \pi(\theta^{\text{true}}))$ whenever $\theta^{\text{true}} \in \Theta_{k,c}$ as $n \to \infty$.

It is to be noted that, because of the discreteness, the value of (7) is invariant under 1-1 transformations of $T$ and also under reparametrizations. The theorem states that (7) converges to $\Pi(\pi(\theta) \leq \pi(\theta^{\text{true}}))$ where $\pi$ is the prior on the
probabilities \((\theta_1, \ldots, \theta_k)\). So if we reparameterized and used some other prior to compute \(m_T\), this has no effect on the limit.

Note that the theorem requires that \(\Pi\) be a continuous probability measure and also the prior density cannot be constant on a subregion of \(\Theta_k\) having positive volume. For example, the theorem does not cover the uniform prior on \(\Theta_k\) although the result still holds there as (7) equals 1 as does \(\Pi(\pi(\theta) \leq \pi(\theta^{\text{true}}))\) and there is no need to check this prior. The following result also follows from the proof of the theorem.

**Corollary 2** If \(\pi\) satisfies (A1) and (A2) and is continuous at \(\theta^{\text{true}}\), then

\[
\Pi(\pi(\theta) < \pi(\theta^{\text{true}})) \leq \liminf_{n \to \infty} M_{T_n}(m_{T_n}(t) \leq m_{T_n}(T_n(x)))
\]

\[
\leq \limsup_{n \to \infty} M_{T_n}(m_{T_n}(t) \leq m_{T_n}(T_n(x))) \leq \Pi(\pi(\theta) \leq \pi(\theta^{\text{true}})).
\]

So, if there is no prior-data conflict in the sense that \(\Pi(\pi(\theta) < \pi(\theta^{\text{true}}))\) is not small, then \(M_{T_n}(m_{T_n}(t) \leq m_{T_n}(T_n(x)))\) for large \(n\) will reflect this.

In the following examples we make use of a parameterization of the Dirichlet referred to here as the concentration parameterization. For the Dirichlet \((\alpha_1, \ldots, \alpha_{k+1})\) distribution with all \(\alpha_i > 1\) the mode is at \((\xi_1, \ldots, \xi_{k+1})\) where \(\xi_i = (\alpha_i - 1)/\tau\) and the concentration parameter \(\tau = \alpha_1 + \cdots + \alpha_{k+1} - (k + 1)\). As \(\alpha_i = 1 + \tau \xi_i\), it is seen that the set of all Dirichlets with this mode is indexed by \(\tau > 0\). The mean and variance of the \(i\)-th coordinate equal \((1 + \tau \xi_i)/\tau + k + 1)\) and \((1 + \tau \xi_i)(\tau + k - \tau \xi_i))/\tau + k + 2)\) which converge respectively to \(\xi_i\) and 0 as \(\tau \to \infty\), and so the distribution concentrates at the mode, and as \(\tau \to 0\) the distribution converges to the uniform on the simplex.

Consider now the examples of Section 2.

**Example 5. Checking the prior for the trine.**

For a single qubit, an experimenter without any prior knowledge could assign a prior to the qubit state space that is uniform under the Hilbert-Schmidt measure. When a trine measurement is performed on the qubit, this results in the prior given by \(\pi(\theta) \propto (1 - (\theta - c)^i C(\theta - c)^i)^{1/2}\) when \(\theta \in \Theta\) and 0 otherwise, where \(c\) and \(C\) are as in Example 3. The change of variable \(\theta \to (r, \omega)\), where \(\theta = c + C^{1/2} r^{1/2}(\cos \omega, \sin \omega)^t\), has Jacobian proportional to \(r^{1/2}\). Therefore, \(\omega \sim U(0, 2\pi)\) independent of \(r \sim \beta(3/2, 3/2)\) and this provides an algorithm for generating from \(\pi\). In circumstances where \(n\) is modest, generating from \(\pi\) and averaging the likelihood can be used to compute the values \(m_{T_n}(t_1, t_2, t_3)\) needed for (7). In this case \(n\) is large so this is too inefficient due to the concentration of the likelihood near the MLE over \(\Theta_k\). The posterior under the uniform prior on \(\Theta_k\) also concentrates near the MLE and so importance sampling based on sampling from the Dirichlet \((t_1 + 1, t_2 + 1, t_3 + 1)\) is used to estimate \(m_{T_n}(t_1, t_2, t_3)\) for each \((t_1, t_2, t_3)\) in a sample drawn from \(m_{T_n}\). Sampling from \(m_{T_n}\) is carried out by generating \((\theta_1, \theta_2, \theta_3)\) from \(\pi\) and then generating \((t_1, t_2, t_3) \sim \text{multinomial}(n, \theta_1, \theta_2, \theta_3)\). The values of \(m_{T_n}(t_1, t_2, t_3)\) are compared to \(m_{T_n}(n_1, n_2, n_3)\) to estimate (7).

This procedure was carried out for the entries in Table 1 with \(10^3\) values of \((t_1, t_2, t_3)\) generated from \(m_{T_n}\) and each value of \(m_{T_n}(t_1, t_2, t_3)\) estimated using a
sample of $10^4$ from the relevant posterior based on $(t_1, t_2, t_3)$. Prior-data conflict in this example corresponds to the true value of $\theta$ lying near the boundary of the respective set $\Theta$. For the symmetric case (7) was estimated as 0.87 and for the asymmetric case (7) was estimated as 0.15. These results were quite stable over different choices of simulation sample sizes. So in neither case is there any indication of prior-data conflict.

As the sample size are large, the importance samplers are quite concentrated. For example, when estimating $m_{T_n}(n_1, n_2, n_3)$, the standard deviations of the posterior distributions of the $\theta_i$, based on the uniform prior on $\Theta_2$, are $(5.94 \times 10^{-3}, 5.28 \times 10^{-3}, 5.13 \times 10^{-3})$ in the symmetric case. So to investigate the sensitivity of the results, more diffuse importance samplers were considered and this was achieved by taking lower values of $\tau$ in the concentration parameterization of the Dirichlet. Here $\tau = n$ corresponds to using the posterior based on the uniform prior on $\Theta_2$ as the importance sampler and $\tau = 0$ corresponds to using the uniform distribution on $\Theta_2$ as the importance sampler which has standard deviations $(235.70 \times 10^{-3}, 235.70 \times 10^{-3}, 235.70 \times 10^{-3})$.

Of course, as $\tau$ drops it is to be expected that the efficiency of the importance sampling will decrease. Even when $\tau = n/100$, which has standard deviations $(57.76 \times 10^{-3}, 51.51 \times 10^{-3}, 50.11 \times 10^{-3})$, similar results were obtained with the second decimal place in the estimate of (7) changing.

**Example 6. Eliciting and checking the prior for ordered probabilities.**

It is necessary to first provide an elicitation algorithm for a prior on $\theta$ satisfying (1). For this Lemma 1 helps considerably since $(\theta_1, \ldots, \theta_{k+1})^T = A_k(\omega_1, \ldots, \omega_{k+1})^T$ and so any prior on $\omega$ will induce a prior on $\theta$ satisfying (1). Perhaps it is natural to choose a Dirichlet($\alpha_1, \ldots, \alpha_{k+1}$) prior on $\omega$ but how should the $\alpha_i$ be chosen? This of course depends upon what is known about the $\theta_i$ and various elicitation algorithms can be considered.

Perhaps a natural approach is for the investigator to specify ordered probabilities $\theta_1^* \geq \theta_2^* \geq \cdots \geq \theta_{k+1}^*$ and then use a prior with mode at this point. By Lemma 1 this can be accomplished by a Dirichlet distribution with mode at $(\xi_1, \ldots, \xi_{k+1})^T = A_k^{-1}(\theta_1^*, \ldots, \theta_{k+1}^*)^T$ and then, using the concentration parameterization, $\tau$ can be chosen to reflect belief in this mode. This requires, however, that the $\theta_i^*$ satisfy (1) as well as being a probability distribution. The following result characterizes the choices of $\theta_i^*$ that are equispaced as this seems like a somewhat natural choice although there are many other possibilities that can be characterized in similar ways.

**Lemma 2.** The probabilities $\theta_i^*$ satisfying (1) are equispaced with $\theta_i^* = \theta_1^* - (i-1)\delta$ iff $\delta = 2/k(k+1)$ and $0 \leq \delta \leq 2/k(k+1)$ and in this case $\xi_i = i\delta$ for $i = 1, \ldots, k$ and $\xi_{k+1} = 1 - k(k+1)\delta/2$.

**Proof:** By Lemma 1 the $\theta_i^* = \theta_1^* - (i-1)\delta$ give probabilities satisfying (1) for some $\delta \geq 0$ iff $\xi_i = i\delta$ for $i = 1, \ldots, k$ and $\xi_{k+1} = (k+1)(\theta_1^* - k\delta)$ and $\theta_i^* \geq k\delta$. Since $1 = \sum_{j=1}^{k+1} \xi_j = k(k+1)\delta/2 + (k+1)(\theta_1^* - k\delta) = (k+1)(\theta_1^* - k\delta/2)$ iff $\theta_1^* = k\delta/2 + 1/(k+1)$ and this satisfies $\theta_i^* \geq k\delta$ iff $\delta \leq 2/k(k+1)$.

Lemma 2 implies that $(\omega_1, \ldots, \omega_{k+1}) \sim \text{Dirichlet}(1 + \tau\delta, \ldots, 1 + k\tau\delta)$. When $\delta = 0$, then $\theta_i^* = 1/(k+1)$ for all $i$ implying $\xi_i = 0$.
for $i = 1, \ldots, k$ and $\xi_{k+1} = 1$ so $(\omega_1, \ldots, \omega_{k+1}) \sim \text{Dirichlet}(1, \ldots, 1, 1 + \tau)$. When $\delta = 2/k(k + 1)$, then $\theta_i^* = [2/(k + 1)][1 - (i - 1)/k]$ for $i = 1, \ldots, k$ implying $\xi_i = 2i/k(k + 1)$ for $i = 1, \ldots, k$ and $\xi_{k+1} = 0$ so $(\omega_1, \ldots, \omega_{k+1}) \sim \text{Dirichlet}(1 + 2\tau/k(k + 1), 1 + 4\tau/k(k + 1), \ldots, 1 + 2\tau/(k + 1), 1)$.

It remains to determine $\tau$. There are a number of ways to proceed but here it is supposed that there is an interval $(l, u)$ such that it is believed all the true probabilities lie in $(l, u)$ with virtual certainty, namely, the prior satisfies

$$
\gamma \leq \Pi(l < \theta_{k+1}, \theta_1 < u) = \Pi\left(l < \omega_{k+1}/(k + 1), \sum_{i=1}^{k+1} \omega_i/i < u\right) \tag{8}
$$

where $\gamma$ is a large probability like 0.99. Since $l < \theta_{k+1}^*, \theta_1^* < u$, the right-hand side of \ref{eq:8} goes to 1 as $\tau \to \infty$. Therefore, $\tau$ satisfying \ref{eq:8} is easily found by simulation and the smallest such value of $\tau$ is preferable as this implies the least concentration for the prior.

For the data of Example 4 and with $\delta = 0$ then $\theta_1^* = \theta_{k+1}^* = 1/18$. Then $l = 1/450, u = 1/2$ are possible and the value $\tau = 2.85$ is the estimated smallest value satisfying \ref{eq:8} with $\gamma = 0.99$. When $\delta = 2/k(k + 1)$, then $\theta_i^* = 1/9$ and $\theta_{k+1}^* = 0$ so $l = 0, u = 1/4$ is possible and the value $\tau = 16.5$ is the estimated smallest value satisfying \ref{eq:8} with $\gamma = 0.99$. Now consider checking the prior corresponding to $\delta = 0$ and $l = 1/450, u = 1/2$. Figure 2 is a plot of density histograms for the first four probabilities based on a sample of $10^5$ from the full prior. Our approach to computing \ref{eq:6} is based on importance sampling. For this particular data set $(f_1/n, \ldots, f_{18}/n) \in \Theta$ and an importance sampler on $\Theta$ with this mode and values of $\tau \approx 60$ produces reasonably stable estimates of $m(f_1, \ldots, f_{18})$. Note that Lemma 1 plays a key role in obtaining such an importance sampler.

A problem arises, however, when computing values of $m(t_1, \ldots, t_{18})$ necessary for estimating \ref{eq:6}. Values of $(t_1, \ldots, t_{18})$ are obtained by generating $(\theta_1, \ldots, \theta_{18}) \sim \pi$ and then $(t_1, \ldots, t_{18}) \sim \text{multinomial}(n, \theta_1, \ldots, \theta_{18})$. When $n$ is small relative to dimension, as is the case here, then typically $(t_1/n, \ldots, t_{18}/n) \notin \Theta$ and so the choice of importance sampler is unclear. For example, an importance sampler such as the Dirichlet($t_1 + 1, \ldots, t_{18} + 1$), which has its mode at $(t_1/n, \ldots, t_{18}/n)$, virtually never generates points in $\Theta$ and so is useless. A better approach is to use an importance sampler based on Lemma 1 where the Dirichlet on $(\omega_1, \ldots, \omega_{k+1})$ has its mode at $A^{-1}(\theta_1^*, \ldots, \theta_{k+1}^*)$ where $(\theta_1^*, \ldots, \theta_{k+1}^*)$ is the convex combination of the prior mode and $(t_1/n, \ldots, t_{18}/n)$ that just satisfies being in $\partial\Theta$. This always generates points inside $\Theta$ and should at least somewhat mimic the integrand over $\Theta$ provided the concentration $\tau$ is not chosen too large or too small. Here a representative $(t_1, \ldots, t_{18})$ was selected and $\tau$ chosen such that both smaller and larger values lead to smaller estimates of \ref{eq:6}. When this was carried out on this example the value of \ref{eq:6} was estimated as 0.36 which indicates there is no prior-data conflict. This makes sense as the naive estimate $(f_1/n, \ldots, f_{18}/n)$ is not only in $\Theta$ but also satisfies the bounds. Suppose instead the data $f = (35, 29, 20, 145, 96, 11, 4, 4, 4, 3, 2, 2, 1, 1, 1, 1, 1)$ was observed that clearly violates the monotonicity. In this case \ref{eq:6} equals 0.
and so prior-data conflict was detected as is correct. Also, with the original data and the prior determined by $l = 0, u = 0.2$ satisfying (8), then (7) equals 0 again indicating a definite prior-data conflict.

One way to avoid the computational problems encountered estimating (7) is to collect more data as the distribution of $(t_1/n, \ldots, t_{k+1}/n)$ becomes degenerate at a point in $\Theta$ as $n \to \infty$ and so will be in $\Theta$ for all $n$ large enough. Another approach to avoiding these problems is to reduce dimension by grouping as was done in Example 4. Intuitively, as the ratio of dimension to sample size decreases, the values of the generated relative frequencies are more likely to be in the relevant parameter space and this was confirmed by a simulation experiment. Implementing the importance sampling for estimating the prior predictive densities of the reduced problem, however, requires the computation of the marginal prior density as this is not in closed form. Since this would be required at each generated value from the importance sampler, the computational advantage is largely negated. The prior density of the full parameter $\theta$ is easily obtained via (2) so this is not an issue in that case.

As such, an alternative approach is considered based on the original elicitation algorithm and which could be called marginalizing the elicitation. Rather than trying to marginalize the full prior, consider being presented with the re-
duced problem and then applying the elicitation algorithm to that problem. This will not result in a prior that is the marginal of the full prior but surely checking this prior for conflict with the data is also assessing whether or not the information being used to choose the prior is appropriate. For example, the original elicitation led to the inequality $u \geq \theta_1 \geq \cdots \geq \theta_{k+1} \geq l$ holding with virtual certainty and recall that necessarily $\theta_i \leq 1/i$. So, if cells are grouped in pairs to maintain the monotonicity and to make the best use of the bounds, then supposing $k+1$ is even, $u + 1/(1+(k+1)/2) \geq \theta_1 + \theta_1 + (k+1)/2 \geq \theta_2 + \theta_2 + (k+1)/2 \geq \cdots \geq \theta_{(k+1)/2} + \theta_{k+1} \geq l + 1/(k+1)$. If $k+1$ is odd then the last group can consist of $\theta_{k+1}$ by itself and the lower bound doesn’t change. So for even modest $k$ the bounds will not increase by much and clearly this idea can be extended to groups of 3, 4 etc. Lemma 1 can then be used, as in the full problem, to obtain the prior for the parameters for the grouped problem. Supposing there are $m$ groups and $(t_1^{\text{red}}, \ldots, t_m^{\text{red}})$ denotes a value generated from the prior predictive for the reduced problem, our recommendation is that this reduction be continued until a reasonable proportion of the values $(t_1^{\text{red}}/n, \ldots, t_m^{\text{red}}/n)$ lie in the reduced parameter space. When this is the case even those that lie outside should be close to the relevant set which will improve the quality of the importance sampling. Note that this does not imply that the observed data when grouped has to lie in the reduced parameter space, as there may indeed be prior-data conflict, but because the model has been accepted, it seems likely that this point will be either in or close to this set.

For the model considered here, with $10^4$ values of $(t_1^{\text{red}}/n, \ldots, t_m^{\text{red}}/n)$ generated from the prior, the following values of $(m, p_m)$ were obtained where $p_m$ is the proportion that lay inside the relevant parameter space: $(18, 0.00)$, $(9, 0.04)$, $(6, 0.25)$, $(5, 0.44)$, $(4, 0.61)$, $(3, 0.78)$ and $(2, 0.93)$. The values 0.42 and 0.44 were obtained for $\theta$ when $m = 9$ and $m = 6$, respectively. So one can feel fairly confident that the elicited information is not in conflict with the data.

If the model and prior are deemed acceptable, then computations based on the posterior are required for inference and these can proceed using importance sampling as described. Also, a Gibbs sampling approach is available. Putting $\theta_{\neq i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_k)$ with $\alpha_0 = 1$, $\theta_0 = 1$ when $i = 1, \ldots, k - 1$, then $\theta_i | \theta_{\neq i}$ has density proportional to $\frac{\theta_i^k (1 - \theta_i - \theta_{\neq i} f_{i+1} + \alpha_{k+1} - 1)(\theta_{i-1} - \theta_i)_{\alpha_i - 1}^n (\theta_i - \theta_{i+1})_{\alpha_{i+1} - 1}^n (\theta_i + \theta_k - 1 + \theta_{\neq i} f_{i+1})_{\alpha_k - 1}^n}{\min(\{\theta_{i+1}, 1 - \theta_{\neq i} - \theta_k\}, \min(\{\theta_{i-1}, 1 - \theta_{\neq i}\})}$ and for $i = k$, then $\theta_k | \theta_{\neq k}$ has density proportional to $\frac{\theta_k^k (1 - \theta_k - \theta_{\neq k} f_s + \alpha_{k+1} - 1)(\theta_{k-1} - \theta_k)_{\alpha_k - 1}^n (\theta_{k+1} - 1 + \theta_{\neq k})_{\alpha_{k+1} - 1}^n}{(1 - \theta_{\neq k})/2, \min(\{\theta_{k-1}, 1 - \theta_{\neq k}\})}$. Sampling from these densities can proceed by generating $\theta_i/(1 - \theta_{\neq i}) | \theta_{\neq i}$ as a truncated beta($f_i + 1, f_{i+1} + \alpha_{i+1}$), which accounts for the first two factors, and then using the envelope methods described in Evans and Swartz (1998) to account for the remaining factors.

The ZM$_k$ model was also considered but a significant problem with this family remains unresolved. In particular, it is unclear how to elicit a prior on $(\alpha, \beta)$ and this is because the interpretation of these parameters is not obvious. Also, the ZM$_k$ family imposes some sharper constraints on the proba-
abilities than hold generally. For example, the maximum probabilities over all \((\alpha, \beta)\) for \(i = 1, 2, 3, 4\) are 1.00, 0.33, 0.26, 0.25, respectively, which contrasts with 1.00, 0.50, 0.33, 0.25 for the general model for ordered probabilities. So, given that it is much easier to use and interpret, the general model for ordered probabilities is recommended over the ZM\(_k\) model.

4 Conclusions

Constrained multinomial models arise in a number of interesting contexts and pose some unique challenges. The emphasis here has been on checking these models and checking for prior-data conflict. Issues associated with inference will be considered elsewhere. A significant consistency theorem has been established for the check on the prior. As a particular application a general model for ordered probabilities has been developed, together with an elicitation algorithm for a prior, and the results of the paper applied. Also, one of a variety of constrained multinomial models used in quantum state estimation has been used to illustrate the methodology.

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Appendix

The proof of the theorem proceeds via a number of lemmas which are individually of some interest. As with the binomial case it is necessary to construct a continuous probability measure that is essentially equivalent to \(M_{T_n}\).

Let \(D_{k,n} = \{n = (n_1, \ldots, n_k) : n_i \in \mathbb{N}_0, n_1 + \cdots + n_k \leq n\}\) denote the set of possible values of \(T_n\). Now construct a set of disjoint sets that cover \(\Theta_k\) and are indexed by \(n \in D_{k,n}\) such that \(n/n\) is in the set that \(n\) indexes. Let \(\Theta_k(n) = \prod_{i=1}^{k} [n_i/n - 1/2n, n_i/n + 1/2n]\) and the \(\Theta_k(n)\) are disjoint, \(n/n\) is the center of \(\Theta_k(n)\), \(\text{vol}(\Theta_k(n)) = n^{-k}\) and \(\Theta_k \subset \bigcup_{n \in D_{k,n}} \Theta_k(n) \downarrow \Theta_k\) as \(n \to \infty\).

For \(r \in \bigcup_{n \in D_{k,n}} \Theta_k(n)\) define

\[
m^*_n(r) = n^k m_{T_n}(n(r))
\]

where \(n(r) \in D_{k,n}\) is such that \(r \in \Theta_k(n(r))\). Note that for \(r \in \Theta_k(n(r))\), then \(n(r) = (n_1(r), \ldots, n_k(r))\) where \(n_i(r) = \lfloor nr_i + 1/2\rfloor\) and \(\lfloor x\rfloor\) is the biggest integer that is not greater than \(x\). Also, define \(c_n(r) = n(r)/n\) which is the center of the \(k\)-cell containing \(r\). The following result then holds.
Lemma 3  (i) \(|r - c_n(r)|| \leq \sqrt{k/n} \) so \(c_n(r) \to r\) as \(n \to \infty\). (ii) \(m_n^*\) is constant on \(\Theta_k(n)\) and takes the value \(u^k m_{T_n}(n)\) there. (iii) \(m_n^*\) is the density of an absolutely continuous probability measure \(M_n^*\) where \(M_n^*(\Theta_k(n)) = m_{T_n}(n)\) and \(M_n^*(\{t : m_{T_n}(t) \leq m_{T_n}(T_n(x))\}) = M_n^*(\{r : m_n^*(r) \leq m_n^*(T_n(x)/n)\})\).

Proof: Parts (i) and (ii) are obvious. For (iii) we have \(\{t : m_{T_n}(t) \leq m_{T_n}(T_n(x))\} = M_n^*(\{n \in (n(r) : m_n^*(r) \leq m_n^*(T_n(x)/n)\})\).

So indeed \(M_n^*\) is a continuized version of \(M_n\).

For \(s > 0\), define

\[ G_{n,s}(r) = \{\theta \in \Theta_k : \sum_{i=1}^{k+1} c_{n,i}(r) \log(\theta_i/c_{n,i}(r)) > -[(k+1+s)\log(n)])/n\} \]

\[ = \{\theta \in \Theta_k : \text{KL}(c_n(r) || \theta) < [(k+1+s)\log(n)])/n\}, \]

a Kullback-Leibler neighborhood of \(c_n(r)\). We have the following result where \(B_s(r)\) is the ball of radius \(\epsilon\) centered at \(r\).

Lemma 4  If \(r \in \Theta_k\), then, for \(\epsilon > 0\), there exists \(N\) such that for all \(n > N, r \in G_{n,s}(r) \subseteq B_s(r)\), so \(G_{n,s}(r) \to \{r\}\) as \(n \to \infty\).

Proof: Note that \(r_i \neq 0\), so for \(n\) large enough \(|nr_i + 1/2| \geq 1\) and this holds for all \(i\). Since \(nr_i + 1/2 \geq |nr_i + 1/2|\) and \(\log(nr_i/(nr_i + 1/2)) < 0\),

\[ \sum_{i=1}^{k+1} c_{n,i}(r) \log \left( \frac{r_i}{c_{n,i}} \right) = \sum_{j=1}^{k+1} \left( \frac{nr_i + 1/2}{n} \right) \log \left( \frac{nr_i}{nr_i + 1/2} \right) \]

\[ \geq \sum_{i=1}^{k+1} \left( \frac{nr_i + 1/2}{n} \right) \log \left( \frac{nr_i}{nr_i + 1/2} \right) \]

\[ = \sum_{i=1}^{k+1} \left( \frac{nr_i + 1/2}{n} \right) \log \left( 1 - \frac{1}{2(nr_i + 1/2)} \right) \]

\[ = -\frac{1}{2n} \sum_{i=1}^{k+1} \left\{ \sum_{j=1}^{\infty} \frac{2(nr_i + 1/2)^{-j+1}}{j} \right\} \]

\[ = -\frac{1}{2n} \sum_{i=1}^{k+1} \left\{ \sum_{j=1}^{\infty} \frac{2(nr_i + 1/2)^{-j+1}}{j} \right\} \]

\[ = -\frac{1}{2n} \sum_{i=1}^{k+1} \frac{1}{1 - 1/2(nr_i + 1/2)} = -\frac{1}{2n} \sum_{i=1}^{k+1} \left( \frac{1 + \frac{1}{2nr_i}}{2nr_i} \right) = -\frac{k+1}{2n} - \frac{1}{4n^2} \sum_{i=1}^{k+1} \frac{1}{r_i} \]

and clearly there is an \(N\) such that this is bounded below by \(-[(k+1+s)\log(n)])/n\) for all \(n > N\). Therefore, \(r \in G_{n,s}(r)\) for all \(n\) large enough. If \(|r - \theta| = \epsilon > 0\), then by Lemma 3(i), \(|c_n(r) - \theta| > \epsilon/2\) for all \(n\) large enough.

Since \(\sum_{j=1}^{k+1} c_{n,j}(r) \log(\theta_j/c_{n,j}(r))\) is continuous in \(c_n(r)\), bounded above by 0 and 0 iff \(c_n(r) = \theta\), then

\[ \sum_{j=1}^{k+1} c_{n,j}(r) \log(\theta_j/c_{n,j}(r)) \to \delta < 0. \]
From this we conclude that \( \theta \notin G_{n,s}(r) \) for all \( n \) large enough and this completes the proof.

If \( r \notin \Theta_k \), then clearly \( m_n^* (r) \rightarrow \pi (r) = 0 \). It is now proved that \( m_n^* (r) \rightarrow \pi (r) \) when \( r \in \Theta_k \). The following identity is useful, namely, when \( n \in D_{k,n} \), then

\[
\int_{\Theta_k} n! \prod_{j=1}^{k+1} \left( \frac{\theta^n_j}{n_j!} \right) d\theta = n!/(n + k)!. \quad (9)
\]

Lemma 5  If \( \pi \) satisfies assumptions A1 and A2 then, for \( r \in \Theta_k \) and any \( s > 0 \),

\[
\left| m_n^* (r) - \frac{n! n^k}{(n + k)!} \pi (r) \right| \leq \frac{1}{n^s} + \sup_{\theta \in G_{n,s}(r) \cap \Theta_k} |\pi (\theta) - \pi (r)|
\]

when \( n > (1 + Bk!)^2 \).

Proof: We abbreviate \( G_{n,s} \) to \( G_{n,s} \) and let \( F_{n,s} = \Theta_k - G_{n,s} = \{ \theta : \sum_{j=1}^{k+1} \log (\theta_j / n_{n,j}) \leq -(k + 1 + s) \log (n(n))/n \} \). Then

\[
\left| m_n^* (r) - \frac{n! n^k}{(n + k)!} \pi (r) \right| = \left| n^k \int_{\Theta_k} n! \prod_{j=1}^{k+1} \left( \frac{\theta^n_j}{n_j!} \right) \pi (d\theta) - n^k \pi (r) \int_{\Theta_k} n! \prod_{j=1}^{k+1} \left( \frac{\theta^n_j}{n_j!} \right) d\theta \right|
\]

\[
\leq n^k \int_{F_{n,s}} n! \prod_{j=1}^{k+1} \frac{\theta^n_j}{n_j!} \pi (d\theta) + \pi (r) n^k \int_{F_{n,s}} n! \prod_{j=1}^{k+1} \left( \frac{\theta^n_j}{n_j!} \right) d\theta + n^k \int_{G_{n,s}} n! \prod_{j=1}^{k+1} \left( \frac{\theta^n_j}{n_j!} \right) |\pi (\theta) - \pi (r)| d\theta = I_{n,1} + I_{n,2} + I_{n,3}.
\]

We will find upper bounds for the three terms \( I_{n,1}, I_{n,2}, I_{n,3} \).

First we show that \( I_{n,1} \leq n^{-s-1/2} \) for all \( n \). For \( \theta \in F_{n,s} \), putting \( n_j = n_{n,j} \),

\[
\sum_{j=1}^{k+1} n_j \log (\theta_j) \leq \sum_{j=1}^{k+1} n_j \log (n_j) - (n + k + 1 + s) \log (n),
\]

and the probability function satisfies

\[
n! \prod_{j=1}^{k+1} \frac{1}{n_j!} \times \prod_{j=1}^{k+1} \theta^n_j \leq n! \prod_{j=1}^{k+1} \frac{1}{n_j!} \times \frac{1}{n^n} \prod_{j=1}^{k+1} n_j^n \times \frac{1}{n^{k+1+s}} = \frac{n!}{n^n} \prod_{j=n_j=0}^{k+1} \frac{n_j^n}{n_j!} \times \frac{1}{n^{k+1+s}}.
\]

When \( \max(n_1, \ldots, n_{k+1}) = n \), that is, \( n_j = n \) for some \( j \) and \( n_i = 0 \) for all \( i \neq j \),

\[
\frac{n!}{n^n} \prod_{j=n_j=0}^{k+1} \frac{n_j^n}{n_j!} = \frac{n!}{n^n} n^n = 1 \leq n^{1/2}.
\]

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Assume \(\max(n_1, \ldots, n_{k+1}) < n\). Then by the Robbins (1955) result on Stirling’s approximation,

\[
1/(12n + 1) < \log(n!) - \frac{1}{2}\log(2\pi n) + n\log(n) - n < 1/12n
\]

for all \(n > 0\), and so

\[
\frac{n!}{n^n} \prod_{j:n_j>0} \frac{n_j^{n_j}}{n_j!} \leq \frac{(2\pi n)^{1/2} e^{-n} e^{1/12n}}{n} \prod_{j:n_j>0} \frac{e^{n_j} e^{-1/(12n_j+1)}}{(2\pi n_j)^{1/2}}
\]

\[
\leq \frac{n^{1/2}}{(2\pi)^{1/2}} \prod_{j:n_j>0} \frac{1}{n_j^{1/2}} < n^{1/2}
\]

since \(0 < \exp\{1/12n - \sum_{j:n_j>0} 1/(12n_j + 1)\}< 1\). Then for any prior \(\Pi\)

\[
n^k \int_{F_{n,s}} n! \prod_{j:n_j>0} \frac{\theta_j^{n_j}}{n_j!} \Pi(d\theta) \leq n^k \int_{F_{n,s}} n^{1/2} \frac{1}{n^{k+1} + s} \Pi(d\theta) \leq \frac{1}{n^{k+1/2} \Pi(F_{n,s})}
\]

Hence \(I_{n,1} \leq n^{-(s+1/2)}\) regardless of prior.

The Dirichlet\((1, \ldots, 1)\) has density equal to \(k!\) on \(\Theta\). Applying the above argument with \(\Pi\) equal to this prior implies \(I_{n,2} \leq \pi(r)k!n^{-s-1/2}\). By A2

\[
I_{n,3} = n^k \int_{G_{n,s} \cap \Theta_{k,c}} n! \prod_{j=1}^{k+1} \frac{\theta_j^{n_j}}{n_j!} |\pi(\theta) - \pi(r)| d\theta
\]

which, using (1), implies

\[
I_{n,3} \leq \frac{n!n^k}{(n+k)!} \times \sup_{\theta \in G_{n,s} \cap \Theta_{k,c}} |\pi(\theta) - \pi(r)| \leq \sup_{\theta \in G_{n,s} \cap \Theta_{k,c}} |\pi(\theta) - \pi(r)|
\]

since \(n!n^k/(n+k)! \leq 1\). Finally, for \(n \geq (1 + Bk)!^2\) and using A1,

\[
I_{n,1} + I_{n,2} + I_{n,3}
\]

\[
\leq \frac{1 + Bk!}{n^{1/2}} \frac{1}{n^s} + \sup_{\theta \in G_{n,s} \cap \Theta_{k,c}} |\pi(\theta) - \pi(r)| \leq \frac{1}{n^s} + \sup_{\theta \in G_{n,s} \cap \Theta_{k,c}} |\pi(\theta) - \pi(r)|.
\]

and the lemma is proved.

**Corollary 6** \(m_n^* \to \pi\) as \(n \to \infty\) where the convergence is almost sure with respect to volume measure.
Proof: If \( r \notin \Theta_k \), then \( m_n^*(r) \rightarrow \pi(r) = 0 \). Now suppose \( r \in \Theta_{k,c} \). For \( \epsilon > 0 \)
there exists \( \delta \) such that if \( \theta \in B_d(r) \) implies \( |\pi(\theta) - \pi(r)| < \epsilon/3 \). By Lemma 4 there exists \( N > (1 + Br)^2 \) such that for all \( n > N \), then \( G_{n,s}(r) \subset B_d(r), n^{-s} < \epsilon/3 \) and \( 1 - n!n^k/(n + k)! < \epsilon/3B \). Therefore, by Lemma 4
\[
|m_n^*(r) - \pi(r)| \leq |m_n^*(r) - n!n^k/(n + k)\pi(r)| + \left| n!n^k/(n + k)\pi(r) - \pi(r) \right| < \epsilon
\]
establishing convergence everywhere except on \( \Theta_{k,d} \cup \partial \Theta_k \) which has volume 0.

The following technical result is required for the proof of the theorem.

**Lemma 7** If \( \pi \) is continuous at \( \theta^{\text{true}} \), then as \( n \to \infty \)
\[
\sup_{\theta \in G_n,G_n(T_n(x)/n) \cap \Theta_{k,c}} |\pi(\theta) - \pi(\theta^{\text{true}})| \to 0.
\]

**Proof:** Since \( \pi \) is continuous at \( \theta^{\text{true}} \), there exists \( \delta > 0 \) such that, if \( \theta \in B_d(\theta^{\text{true}}) \), then \( |\pi(\theta) - \pi(\theta^{\text{true}})| < \epsilon \). If \( \theta \in G_n,G_n(T_n(x)/n) \), then the Kullback-
Csiszar-Kemperman inequality (see Devroye (1987), Theorem 1.4) says
\[
\sum_{j=1}^{k+1} |\theta_j - T_{n,j}/n| \leq [2\text{KL}(T_n(x)/n || \theta)]^{1/2} < [2n^{-1}(k + 1 + s) \log n]^{1/2}
\]
which implies that
\[
\sum_{j=1}^{k+1} |\theta_j - \theta_j^{\text{true}}| \leq (2n^{-1}(k + 1 + s) \log n)^{1/2} + \sum_{j=1}^{k+1} |\theta_j^{\text{true}} - T_{n,j}/n|.
\]
Therefore, the almost sure convergence \( T_n/n \to \theta^{\text{true}} \) implies that there exists \( N \) such that for all \( n > N \), if \( \theta \in G_n,G_n(T_n(x)/n) \), then \( ||\theta - \theta^{\text{true}}|| < \delta \). This proves the lemma.

The main result is now established.

**Proof of Theorem 1** Fix \( 0 < \eta < 1 \) small. Under A3, \( \pi(\theta) \) has a continuous
distribution when \( \theta \sim \pi \). Therefore, there exists \( \epsilon > 0 \) such that \( \Pi(\{ \theta : |\pi(\theta) - \pi(\theta^{\text{true}})| \leq \epsilon \}) < \eta \).

Define \( H_n = \{ r \in \Theta_{k,c} : \sup_{\theta \in G_{n,s}(r) \cap \Theta_{k,c}} |\pi(\theta) - \pi(r)| < \epsilon/6 \} \). By Lemma 4
the diameter of \( G_{n,s}(r) \) shrinks to zero. If \( r \in \Theta_{k,c} \), then sup_{\theta \in G_{n,s}(r) \cap \Theta_{k,c}} |\pi(\theta) - \pi(r)| \to 0 \) as \( n \to \infty \), so there exists \( N(r, \epsilon) > 0 \) such that \( r \in H_n \) for all \( n \geq N(r, \epsilon) \). This implies \( H_n \to \Theta_{k,c} \) and so \( \Pi(H_n) \to \Pi(\Theta_{k,c}) = 1 \) as \( n \to \infty \).

From the continuity of \( \pi \) at \( \theta^{\text{true}} \), there exists \( \delta > 0 \) such that \( |\pi(\theta) - \pi(\theta^{\text{true}})| < \epsilon/12 \) for any \( \theta \in B_d(\theta^{\text{true}}) \). The strong law of large numbers implies
\( T_n(x)/n \to \theta^{\text{true}} \) almost surely so there exits \( N_1 \) such that \( ||T_n(x)/n - \theta^{\text{true}}|| < \delta \) for all \( n > N_1 \). Therefore, if \( n > N_1 \), then \( |\pi(T_n(x)/n) - \pi(\theta^{\text{true}})| < \epsilon/12 \). Also, by Lemma 7
there exists \( N_2 \) such that for \( n > N_2 \), then
\[
\sup_{\theta \in G_{n,s}(T_n(x)/n) \cap \Theta_{k,c}} |\pi(\theta) - \pi(\theta^{\text{true}})| < \epsilon/6
\]

and \( n^{-s} < \epsilon/6 \). Therefore, using Lemma 5 for all \( n > N_3 = \max\{(1 + B_nk!)^2, N_1, N_2\},

\[
\left| m_n^* \left( \frac{T_n(x)}{n} \right) - \frac{n!n^k}{(n+k)!} \pi(\theta^{true}) \right| \\
\leq \left| m_n^* \left( \frac{T_n(x)}{n} \right) - \frac{n!n^k}{(n+k)!} \pi \left( \frac{T_n(x)}{n} \right) \right| + \frac{n!n^k}{(n+k)!} \left| \pi \left( \frac{T_n(x)}{n} \right) - \pi(\theta^{true}) \right| \\
\leq \frac{1}{n^s} + \sup_{\theta \in G_{n,s}(T_n(x)/n) \cap \Theta_{k,c}} \left| \pi(\theta) - \pi \left( \frac{T_n(x)}{n} \right) \right| + \frac{n!n^k}{(n+k)!} \left| \pi \left( \frac{T_n(x)}{n} \right) - \pi(\theta^{true}) \right| \\
\leq \frac{1}{n^s} + \sup_{\theta \in G_{n,s}(T_n(x)/n) \cap \Theta_{k,c}} \left| \pi(\theta) - \pi(\theta^{true}) \right| + \left( 1 + \frac{n!n^k}{(n+k)!} \right) \left| \pi(\theta^{true}) - \pi \left( \frac{T_n(x)}{n} \right) \right| \\
\leq \epsilon/2.
\]

Note that this implies that for all \( n > N_3 \),

\[
m_n^*(T_n(x)/n) \leq \frac{n!n^k}{(n+k)!} \pi(\theta^{true}) + \epsilon/2 \leq \pi(\theta^{true}) + \epsilon/2.
\] (10)

The prior-data conflict probability satisfies

\[
M_n^* (\{ \mathbf{r} : m_n^*(\mathbf{r}) \leq m_n^*(T_n(x)/n) \}) = M_n^* (\{ \mathbf{r} : m_n^*(\mathbf{r}) \leq m_n^*(T_n(x)/n) \} \cap \Theta_{k,c})
\]

since \( M_n^* \) is absolutely continuous and \( \Theta_{k,d} \) has volume measure 0. Also,

\[
M_n^* (\{ \mathbf{r} : m_n^*(\mathbf{r}) \leq m_n^*(T_n(x)/n) \} \cap \Theta_{k,c}) =
M_n^* (\{ \mathbf{r} : m_n^*(\mathbf{r}) \leq m_n^*(T_n(x)/n) \} \cap H_n) +
M_n^* (\{ \mathbf{r} : m_n^*(\mathbf{r}) \leq m_n^*(T_n(x)/n) \} \cap H_n^c \cap \Theta_{k,c})
\]

and \( M_n^* (\{ \mathbf{r} : m_n^*(\mathbf{r}) \leq m_n^*(T_n(x)/n) \} \cap H_n^c \cap \Theta_{k,c}) \leq M_n^* (H_n^c \cap \Theta_{k,c}) \). Using Lemma 5 and the bound \( |\pi(\theta) - \pi(\mathbf{r})| \leq B \), when \( n > (1 + Bn^k)^2 \), then

\[
M_n^* (H_n^c \cap \Theta_{k,c}) \leq \frac{n!n^k}{(n+k)!} \Pi(H_n^c \cap \Theta_{k,c}) + \frac{1}{n^s} + B \text{vol}(H_n^c \cap \Theta_{k,c}).
\]

So \( M_n^* (H_n^c \cap \Theta_{k,c}) \to 0 \) and \( M_n^* (\{ \mathbf{r} : m_n^*(\mathbf{r}) \leq m_n^*(T_n(x)/n) \} \cap H_n^c \cap \Theta_{k,c}) \to 0 \) as \( n \to \infty \).

Now put

\[
A_n = M_n^* (\{ \mathbf{r} \in H_n : m_n^*(\mathbf{r}) \leq m_n^*(T_n(x)/n) \}).
\]
There exist $N_4 > N_3$ such that for all $n > N_4$, then $0 < (1 - n!n^k/(n + k)!)B \leq \epsilon/6$. Then, for all $n > N_4$, by Lemma 5 and the definition of $H_n$, for all $\mathbf{r} \in H_n$,

$$m_n^*(\mathbf{r}) \geq \frac{n!n^k}{(n + k)!} \pi(\mathbf{r}) - \frac{1}{n^s} - \sup_{\theta \in G_{n,s}(\mathbf{r}) \cap \Theta_{k,c}} |\pi(\theta) - \pi(\mathbf{r})|$$

$$= \pi(\mathbf{r}) - \left(1 - \frac{n!n^k}{(n + k)!}\right) \frac{1}{n^s} - \sup_{\theta \in G_{n,s}(\mathbf{r}) \cap \Theta_{k,c}} |\pi(\theta) - \pi(\mathbf{r})|$$

$$\geq \pi(\mathbf{r}) - \epsilon/2.$$  \hspace{1cm} (11)

By (10) and (11), for all $n > N_4$,

$$A_n \leq M_n^*(\{\mathbf{r} \in H_n : \pi(\mathbf{r}) \leq \pi(\theta^{true}) + \epsilon\}) \leq M_n^*(\{\mathbf{r} : \pi(\mathbf{r}) \leq \pi(\theta^{true}) + \epsilon\}).$$

Putting $C_\epsilon(\theta^{true}) = \{\mathbf{r} : \pi(\mathbf{r}) \leq \pi(\theta^{true}) + \epsilon\}$, we have

$$|M_n^*(C_\epsilon(\theta^{true})) - \Pi(C_\epsilon(\theta^{true}))| = \int_{C_\epsilon(\theta^{true})} (m_n^*(\theta) - \pi(\theta)) \, d\theta$$

$$\leq \int_{\mathbb{R}} |m_n^*(\theta) - \pi(\theta)| \, d\theta \rightarrow 0$$

as $n \rightarrow \infty$ by Corollary 6 and Scheffé’s theorem. Therefore, $\limsup_{n \rightarrow \infty} A_n \leq \Pi(\pi(\mathbf{r}) \leq \pi(\theta^{true}) + \epsilon)$. Similarly, a lower bound is obtained as $\liminf_{n \rightarrow \infty} A_n \geq \Pi(\pi(\mathbf{r}) \leq \pi(\theta^{true}) - \epsilon)$. Therefore,

$$\limsup_{n \rightarrow \infty} A_n - \liminf_{n \rightarrow \infty} A_n \leq \Pi(|\pi(\theta) - \pi(\theta^{true})| \leq \epsilon) \leq \eta.$$

Since $\eta > 0$ is arbitrary, $\lim_{n \rightarrow \infty} A_n = \Pi(\{\mathbf{r} \in \Theta_{k,c} : \pi(\mathbf{r}) \leq \pi(\theta^{true})\})$ and the proof is complete.

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