Solution of Duffing-Harmonic Oscillator by the Power Series Method

Mazen I. Qaisi
Mechanical Engineering Department, University of Jordan, Amman 11942, Jordan

Abstract: This paper provides a power series solution to the Duffing-harmonic oscillator and compares the frequencies with those obtained by the harmonic balance method. To capture the periodic motion of the oscillator, the power series expansion is used upon transforming the time variable into an “oscillating time” which reduces the governing equation to a form well-conditioned for a power series solution. A recurrence equation for the series coefficients is established in terms of the “oscillating time” frequency which is then determined by employing Rayleigh’s energy principle. The response of the oscillator is compared with a numerical solution and good agreement is demonstrated.

Key words: Nonlinear, power series, Duffing-harmonic, oscillator.

1. Introduction

Non-linear conservative oscillators having rational form of the potential energy, such as the Duffing-harmonic oscillator, model interesting physical phenomena [1-3]. For those oscillators, the usual perturbation expansion procedures do not apply, and resort is normally made to the harmonic balance method. Mickens [3] obtained a first approximation of the natural frequency of the Duffing-harmonic oscillator using the harmonic balance method. Hu et al. [4] used a first order harmonic balance method via first Fourier coefficient to construct an approximate frequency-amplitude relation for the Duffing-harmonic oscillator. The relation was in agreement with the result obtained by the Ritz procedure [5]. Recently, Wang et al. [6] applied a nonlinear time transformation to obtain analytical solutions of a generalized Duffing-harmonic oscillator.

In this paper, the power series approach [7] is used to provide a solution for this oscillator. An “oscillating time” variable is shown to transform the governing equation into a form well-conditioned for a solution by the power series method.

A recurrence equation is obtained for the series coefficients which are used to determine the oscillating time frequency by invoking Rayleigh’s energy principle.

2. Analytical Formulation

We consider the free vibration of the undamped Duffing-harmonic oscillator:

\[ \ddot{x} + \frac{x^3}{1 + x^2} = 0 \]  (1)

Subject to the initial conditions \( x(0) = A \) and \( \dot{x}(0) = 0 \). The overdot denotes differentiation with respect to time \( t \). Exact solution to this equation is not available. Note that for small \( x \), this equation approximates that of a Duffing oscillator \( \ddot{x} + x^3 = 0 \), whereas for large \( x \), it approximates that of a linear oscillator \( \ddot{x} + x = 0 \).

In order to facilitate the use of the power series method in the solution of Eq. (1), the periodic motion can be captured [7] by transforming the independent time variable \( t \) into a new independent variable:

\[ \tau = \sin (\omega t) \]  (2)

which starts at \( \tau = 0 \) when \( t = 0 \) and oscillates between the values of -1 and +1 at a frequency \( \omega \) as \( t \) is increased indefinitely. The infinite time domain \( 0 \leq t \)
\( \leq \infty \) is thereby reduced to a finite time scale \(-1 \leq \tau \leq +1\). When Eq. (1) is transformed from the \( x-t \) plane to the \( x-\tau \) plane in accordance with Eq. (2), the transformed equation and initial conditions become:

\[
\omega^2(1-\tau^2)x'' - \omega^2 \tau x' + \frac{x^3}{1 + x^2} = 0 \tag{3}
\]

where the prime denotes differentiation with respect to \( \tau \). The frequency \( \omega \) is yet undetermined. Note that this transformation changes the character of Eq. (1) from an autonomous equation into a non-autonomous one in which the time variable appears explicitly. Eq. (3) has an ordinary point at \( \tau = 0 \) and two regular singular points at \( \tau = \pm 1 \). For linear vibration (large), differential equation theory [8] guarantees a convergent power series expansion about \( \tau = 0 \) with a radius of convergence \( |\tau| \leq 1 \). This convergence interval covers the infinite time domain except the singular points. However, for non-linear differential equations, the question of convergence is still not settled. In the case of the Duffing-harmonic oscillator, it is assumed that a convergent power series expansion about \( \tau = 0 \) exists for \( |\tau| \leq 1 \) as:

\[
x(\tau) = a_1 + a_2 \tau + a_3 \tau^2 + \cdots = \sum_{k=1}^{\infty} a_k \tau^{k-1} \tag{4}
\]

where \( a_k \) is constant coefficients to be determined.

Using Eq. (4), the terms involving derivatives in Eq. (3) can be expanded as:

\[
\omega^2 x'' = \sum_{k=1}^{\infty} (k-1)(k-2)a_k \omega^2 \tau^{k-3}
\]

\[
= \sum_{k=1}^{\infty} k(k + 1)a_{k+2} \omega^2 \tau^{k-1}
\]

\[
-\omega^2 \tau x' = -\sum_{k=1}^{\infty} (k-1)(k-2)a_k \omega^2 \tau^{k-1}
\]

\[
-\omega^2 \tau x'' = -\sum_{k=1}^{\infty} (k-1)a_k \omega^2 \tau^{k-1} \tag{7}
\]

An appropriate change of index has been introduced in Eq. (5) so that all terms in the differential equation have the same form. The non-linear term in Eq. (3) involves powers of \( x \) which may also be expanded, together with the non-linear term, as follows:

\[
1 + x^2 = d_1 + d_2 \tau + d_3 \tau^2 + \cdots
\]

\[
= \sum_{k=1}^{\infty} d_k \tau^{k-1} \tag{8}
\]

\[
x^3 = c_1 + c_2 \tau + c_3 \tau^2 + \cdots
\]

\[
= \sum_{k=1}^{\infty} c_k \tau^{k-1} \tag{9}
\]

By inserting Eqs. (8) and (9) in Eq. (10) and equating coefficients of same powers results in the relation:

\[
c_n = \sum_{k=1}^{n} e_k d_{n-k+1}
\]

\[
= e_n d_1 + \sum_{k=1}^{n-1} e_k d_{n-k+1} = e_n d_1 + P \tag{10}
\]

It follows that the coefficients of the non-linear term can be calculated from those of the powers of \( x \) as:

\[
e_n = \frac{c_n - P}{d_1} \tag{11}
\]

where \( P = \sum_{k=1}^{n-1} e_k d_{n-k+1}, n > 1 \)

\( = 0 \) for \( n = 1 \)

Eqs. (8) and (9) involve double and triple multiplication of Eq. (4), respectively. Consequently, the coefficients \( c_n, d_n \) can be determined once the coefficients \( a_1, a_2, \ldots, a_n \) are known. Now, by substituting Eqs. (5)-(7) and (10) in Eq. (3) and equating the coefficients of each power to zero, the recurrence relation:

\[
a_{k+2} = \frac{(k-1)^2 \omega^2 a_k - e_k}{k(k+1)\omega^2}, k = 1, 2, \ldots \tag{12}
\]

is established among the series coefficients. It is noted that, all the coefficients \( a_3 \) and higher can be computed in terms of the fundamental coefficients \( a_1 \) and \( a_2 \) which are determined from the initial conditions, resulting in:

\[
a_1 = A, a_2 = 0 \tag{13}
\]
To initiate computing the series coefficients from the recurrence Eq. (12), we note that Eq. (11) gives the starting value $e_1 = c_1/d_1 = A^3/(1+A^2)$. It is observed that all the series coefficients depend on $a_4$ and the oscillating time frequency $\omega$. Rayleigh’s energy principle can now be used to determine the frequency $\omega$. This principle states that, for conservative systems, the maximum potential and kinetic energies are equal. For the system under consideration, the kinetic energy is given by:

$$T = \frac{1}{2} \dot{x}^2 = \frac{1}{2} \omega^2 (1 - \tau^2) x^2$$

and the potential energy $V = \frac{x^2}{2} \cdot \frac{1}{2} \ln(1 + x^2)$ (15).

The motion assumed to start at $\tau = 0$, with maximum displacement and hence maximum potential energy. The motion returns to the initial conditions every time $\tau = 0$ for which $\omega t = \pi, 2\pi, 3\pi, \ldots$. Consequently, the vibration frequency $\Omega$ is twice the oscillating time frequency ($\Omega = 2\omega$). The equilibrium position associated with maximum velocity is reached at angular positions $\Omega t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ for which $\omega t = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$ and $\tau = \pm \frac{1}{\sqrt{2}}$.

By using Eqs. (4), (14) and (15), this leads to the characteristic function:

$$A^2 \left( \frac{1}{2} \ln(1 + A^2) - \frac{\omega^2}{4} \left[ \frac{2a_3}{\sqrt{2}} + \frac{3a_4}{(\sqrt{2})^4} \right] + \frac{4a_5}{(\sqrt{2})^6} + \ldots \right)^2 = 0$$

Eq. (16) may have more than one root but at the correct root $\omega$, the characteristic function is stationary. The corresponding series coefficients $a_i$ uniquely determine the periodic motion as:

$$x(t) = a_1 + a_3 \sin^{\omega} t + a_5 \sin^{4 \omega} t + \cdots$$

It is noted that the odd powered terms are zero so that one half cycle of the oscillating time $\tau$ reproduces one full oscillation cycle.

### Table 1  Comparison of natural frequency $\Omega$.

| Initial displacement | First order Harmonic-balance [4] | Ritz procedure [5] | Power series |
|----------------------|----------------------------------|--------------------|--------------|
| $0.1$                | 0.08627                          | 0.08624            | 0.08538      |
| $0.5$                | 0.39736                          | 0.39423            | 0.38885      |
| $1$                  | 0.65465                          | 0.64359            | 0.64220      |
| $5$                  | 0.97435                          | 0.96731            | 0.99987      |
| $10$                 | 0.99340                          | 0.99095            | 0.99987      |
| $50$                 | 0.99973                          | 0.99960            | 0.99988      |

3. Numerical Illustration

The vibration frequency $\Omega$ of the Duffing-harmonic oscillator was computed for several initial displacements and the results were compared, in Table 1, with those obtained by the first order harmonic balance method [4] and those obtained by the Ritz procedure [5]. It is observed that the first order harmonic balance and the Ritz procedure provide overestimates of the vibration frequency for small amplitudes (Duffing-type) and underestimates of the vibration frequency for large amplitudes (harmonic-type).
Solution of Duffing-harmonic Oscillator by the Power Series Method

Fig. 1  Convergence of the oscillating time frequency.

Table 2  Non-zero series coefficients ($A = 1$).

| $K$   | 1   | 3   | 5   | 7   | 9   | 11  |
|-------|-----|-----|-----|-----|-----|-----|
| $a_k$ | 1.0000 | -2.4247 | 1.1514 | -0.2333 | -0.4850 | -0.2773 |
| $K$   | 13  | 15  | 17  | 19  | 21  | 23  |
| $a_k$ | 0.1273 | 0.3253 | 0.0950 | -0.0312 | -0.0355 | 0.1592 |

Fig. 1 shows the convergence of the oscillating time frequency $\omega$ as the number of non-zero series terms is increased for initial displacement $A = 1$.

The first twelve non-zero coefficients are listed in Table 2 for $A = 1$. The progressive decrease in the value of the series coefficients characterizes a convergent solution.

The displacement and velocity time responses are shown in Figs. 2 and 3 respectively for $A = 1$. They are compared with a numerical solution using the fourth order Runge-Kutta scheme. Good agreement is seen between the solutions.
Solution of Duffing-harmonic Oscillator by the Power Series Method

Fig. 2 Comparison of displacement response.

Fig. 3 Comparison of velocity response.
4. Conclusion

A power series solution of the Duffing-harmonic oscillator is presented. An “oscillating time” variable is used to transform the governing equation into a form well-conditioned for power series analysis. The solution is obtained by invoking Rayleigh’s energy principle. The natural frequencies obtained for a range of initial displacements covering the Duffing and harmonic regions, are compared with existing methods. Good agreement is demonstrated. The displacement and velocity responses are shown to be in good agreement with a numerical solution.

References

[1] Stoker, J. J. 1950. Nonlinear Vibration, Chapter 1, Section 2. New York: Interscience.

[2] Mickens, R. E. 1988. “Semi-classical Quantization Using the Method of Harmonic Balance.” Nuove Cimento 101: 359-66.

[3] Mickens, R. E. 2001. “Mathematical and Numerical Study of the Duffing-Harmonic Oscillator.” Journal of Sound and Vibration 244: 563-7.

[4] Hu, H., and Tang, J. H. 2006. “Solution of a Duffing-Harmonic Oscillator by the Method of Harmonic Balance.” Journal of Sound and Vibration 294: 637-9.

[5] Tiwari, S. B., Nageswara, B., Shivakumar, N., Sai, K. S., and Nataraja, H. R. 2005. “Analytical Study on a Duffing-Harmonic Oscillator.” Journal of Sound and Vibration 285: 1217-22.

[6] Wang, H., and Chung, K. 2012. “Analytical Solutions of a Generalized Duffing-Harmonic Oscillator by Nonlinear Time Transformation Method.” Physics Letters A 376: 1118-24.

[7] Qaisi, M. I. 1996. “A Power Series Approach for the Study of Periodic Motion.” Journal of Sound and Vibration 196 (4): 401-6.

[8] Kaplan, W. 1958 Ordinary Differential Equations. Massachusetts: Addison-Wesley Company.