Classification of Gauge Orbit Types for $SU_n$-Gauge Theories

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Abstract

A method for determining the orbit types of the action of the group of gauge transformations on the space of connections for gauge theories with gauge group $SU_n$ in space-time dimension $d \leq 4$ is presented. The method is based on the 1:1-correspondence between orbit types and holonomy-induced Howe subbundles of the underlying principal $SU_n$-bundle. It is shown that the orbit types are labelled by certain cohomology elements of space-time satisfying two relations. Thus, for every principal $SU_n$-bundle the corresponding stratification of the gauge orbit space can be determined explicitly. As an application, a criterion characterizing kinematical nodes for physical states in $2 + 1$-dimensional Chern-Simons theory proposed by Asorey et al. is discussed.
1 Introduction

One of the basic principles of modern theoretical physics is the principle of local gauge invariance. Its application to the theory of particle interactions gave rise to the standard model, which proved to be a success from both theoretical and phenomenological points of view. The most impressive results of the model were obtained within the perturbation theory, which works well for high energy processes. On the other hand, the low energy hadron physics, in particular, the quark confinement, turns out to be dominated by nonperturbative effects, for which there is no rigorous theoretical explanation yet.

The application of geometrical methods to non-abelian gauge theories revealed their rich geometrical and topological properties. In particular, it showed that the configuration space of such theories, which is the space of gauge group orbits in the space of connections, may have a highly nontrivial structure. In general, the orbit space possesses not only orbits of the so called principal type, but also orbits of other types, which may give rise to singularities of the configuration space. This stratified structure of the gauge orbit space is believed to be of importance for both classical and quantum properties of non-abelian gauge theories in nonperturbative approach, and it has been intensively studied in recent years. Let us discuss some aspects indicating its physical relevance.

First, studying the geometry and topology of the generic (principal) stratum, one gets a deeper understanding of the Gribov-ambiguity and of anomalies in terms of index theorems. In particular, one gets anomalies of purely topological type, which can not be seen by perturbative quantum field theory. These are well-known results from the eighties. Moreover, there are partial results and conjectures concerning the relevance of nongeneric strata. First of all, nongeneric gauge orbits affect the classical motion on the orbit space due to boundary conditions and, in this way, may produce nontrivial contributions to the path integral. They may also lead to localization of certain quantum states, as it was suggested by finite-dimensional examples [10]. Further, the gauge field configurations belonging to nongeneric orbits can possess a magnetic charge, i.e. they can be considered as a kind of magnetic monopole configurations, which are responsible for quark confinement. This picture was found in 3-dimensional gauge systems [3], and it is conjectured that it can hold for 4-dimensional theories as well [4]. Finally, it was suggested in [16] that non-generic strata may lead to additional anomalies.

Most of the problems mentioned here are still awaiting a systematic investigation. In a series of papers we are going to make a new step in this direction. We give a complete solution to the problem of determining the strata that are present in the gauge orbit space for SU$n$ gauge theories in compact Euclidean space-time of dimension $d = 2, 3, 4$. Our analysis is based on the results of a paper by Kondracki and Rogulski [23], where it was shown that the gauge orbit space is a stratified topological space in the ordinary sense, cf. [22] and references therein. Moreover, these authors found an interesting relation between orbit types and certain bundle reductions, which we are going to use. We also refer to [14] for the discussion of a very simple, but instructive special example (orbit types of SU2-gauge theory on $S^4$).

The paper is organized as follows. In Section 2 we introduce the basic notions related to
the action of the group of gauge transformations on the space of connections, state the
definitions of stabilizer and orbit type and recall basic results concerning the stratification
structure of the gauge orbit space. In Section 3 we introduce the notions of Howe subgroup
and holonomy-induced Howe subbundle and establish the relation between orbit types
and holonomy-induced Howe subbundles. Section 4 is devoted to the study of the Howe
subgroups of $\text{SU}_n$. In Section 5 we give a classification of the Howe subbundles of $\text{SU}_n$-
bundles for space-time dimension $d \leq 4$. In Section 6 we prove that any Howe subbundle
of $\text{SU}_n$-bundles is holonomy-induced. In Section 7 we implement the equivalence relation
of Howe subbundles due to the action of $\text{SU}_n$. As an example, in Section 8 we determine
the orbit types for gauge group $\text{SU}_2$. Finally, in Section 9 we discuss an application to
Chern-Simons theory in $2 + 1$ dimensions.

In two subsequent papers we shall investigate the natural partial ordering on the set of
orbit types and the structure of another, coarser stratification, see [24], obtained by first
factorizing with respect to the so called pointed gauge group and then by the structure
group.

2 Gauge Orbit Types and Stratification

We consider a fixed topological sector of a gauge theory with gauge group $G$ on a Riemann-
nian manifold $M$. In the geometrical setting it means that we are given a smooth right
principal fibre bundle $P$ with structure group $G$ over $M$. $G$ is assumed to be a compact
connected Lie group and $M$ is assumed to be compact, connected, and orientable.

Denote the sets of connection forms and gauge transformations of $P$ of Sobolev class $W^k$
(locally square integrable up to the $k$-th derivative) by $\mathcal{A}^k$ and $\mathcal{G}^k$, respectively. Here
$k$ is a nonnegative integer. For generalities on Sobolev spaces of cross sections in fibre
bundles, see [28]. Provided $2k > \dim M$, $\mathcal{A}^k$ is an affine Hilbert space and $\mathcal{G}^{k+1}$ is a
Hilbert Lie group acting smoothly from the right on $\mathcal{A}^k$ [23, 26, 32]. We shall even
assume $2k > \dim M + 2$. Then, by the Sobolev Embedding Theorem, connection forms
are of class $C^1$ and, therefore, have continuous curvature. If we view elements of $\mathcal{G}^{k+1}$ as
$G$-space morphisms $P \to G$, the action of $g \in \mathcal{G}^{k+1}$ on $A \in \mathcal{A}^k$ is given by

$$A^g = g^{-1}Ag + g^{-1}dg.$$  \hspace{1cm} (1)

Let $\mathcal{M}^k$ denote the quotient topological space $\mathcal{A}^k/\mathcal{G}^{k+1}$. This space represents the con-
figuration space of our gauge theory.

For this to make sense, $\mathcal{M}^k$ should not depend essentially on the purely technical pa-
rameter $k$. Indeed, let $k' > k$. Then one has natural embeddings $\mathcal{G}^{k+1} \hookrightarrow \mathcal{G}^{k+1}$ and
$\mathcal{A}^{k'} \hookrightarrow \mathcal{A}^k$. As a consequence of the first, the latter projects to a map $\varphi : \mathcal{M}^{k'} \to \mathcal{M}^k$.

Since the image of $\mathcal{A}^{k'}$ in $\mathcal{A}^k$ is dense, so is $\varphi(\mathcal{M}^{k'})$ in $\mathcal{M}^k$. To see that $\varphi$ is injective,
let $A_1, A_2 \in \mathcal{A}^{k'}$ and $g \in \mathcal{G}^{k+1}$ such that $\varphi(A_2) = \varphi(A_1)^g$. Then (1) implies

$$dg = g\varphi(A_2) - \varphi(A_1)g.$$  \hspace{1cm} (2)

Due to $2k' > 2k > \dim M$, by the multiplication rule for Sobolev functions, the rhs. of (2)
is of class $W^{k+1}$. Then $g$ is of class $W^{k+2}$. This can be iterated until the rhs. is of class
$W^{k'}$. Hence, $g \in G^{k'+1}$, so that $A_1$ and $A_2$ are representatives of the same element of $\mathcal{M}^{k'}$. This shows that $\mathcal{M}^{k'}$ can be identified with a dense subset of $\mathcal{M}^k$. Another question is whether the stratification structure of $\mathcal{M}^k$, which will be discussed in a moment, depends on $k$. Fortunately, the answer to this question is negative, see Theorem 3.3.

In general, the orbit space of a smooth Lie group action does not admit a smooth manifold structure w.r.t. which the projection is smooth. The best one can expect is that it admits a stratification. For the notion of stratification of a topological space, see [22] or [23, §4.4]. For the gauge orbit space $\mathcal{M}^k$, a stratification was constructed in [23], using a method which is known from compact Lie group actions on completely regular spaces [7]. In order to explain this, let us recall the notions of stabilizer and orbit type. The stabilizer, or isotropy subgroup, of $A \in A^k$ is the subgroup

$$G_A^{k+1} = \{ g \in G^{k+1} \mid A^g = A \}$$

of $G^{k+1}$. It has the following transformation property: For any $A \in A^k$ and $g \in G^{k+1}$,

$$G_A^{(g)} = g^{-1}G_A^{k+1} g.$$  

Thus, there exists a natural map, called type map, assigning to each element of $\mathcal{M}^k$ the conjugacy class in $G^{k+1}$ made up by the stabilizers of its representatives in $A^k$. Let $\text{OT}(A^k, G^{k+1})$ denote the image of this map. The elements of $\text{OT}(A^k, G^{k+1})$ are called orbit types. The set $\text{OT}(A^k, G^{k+1})$ carries a natural partial ordering: $\tau \leq \tau'$ iff there are representatives $G_A^{k+1}$ of $\tau$ and $G_A'^{k+1}$ of $\tau'$ such that $G_A^{k+1} \supseteq G_A'^{k+1}$. Note that this definition is consistent with [7] but not with [23] and several other authors who define it just inversely. As was shown in [23], the subsets $\mathcal{M}_\tau^k \subseteq \mathcal{M}^k$, consisting of gauge orbits of type $\tau$, can be equipped with a smooth Hilbert manifold structure and the family

$$\{ \mathcal{M}_\tau^k \mid \tau \in \text{OT}(A^k, G^{k+1}) \}$$

(3)

is a stratification of $\mathcal{M}^k$. Accordingly, the manifolds $\mathcal{M}_\tau^k$ are called strata. In particular,

$$\mathcal{M}^k = \bigcup_{\tau \in \text{OT}(A^k, G^{k+1})} \mathcal{M}_\tau^k,$$

where for any $\tau \in \text{OT}(A^k, G^{k+1})$ there holds

$$\mathcal{M}_{\tau_0}^k \text{ is open and dense in } \bigcup_{\tau' \leq \tau} \mathcal{M}_{\tau'}^k.$$  

(4)

Similarly to the case of compact Lie groups, there exists a principal orbit type $\tau_0$ obeying $\tau_0 \geq \tau$ for all $\tau \in \text{OT}(A^k, G^{k+1})$. Due to (3), the corresponding stratum $\mathcal{M}_{\tau_0}^k$ is open and dense in $\mathcal{M}^k$. For this reason it is called the generic stratum.

The above considerations show that the set $\text{OT}(A^k, G^{k+1})$ together with its natural partial ordering carries the information about which strata occur and how they are patched together.

To conclude, let us remark that instead of using Sobolev techniques one can also stick to smooth connection forms and gauge transformations. Then one obtains essentially analogous results about the stratification of the corresponding gauge orbit space where, roughly speaking, one has to replace 'Hilbert manifold' and 'Hilbert Lie group' by 'tame Fréchet manifold' and 'tame Fréchet Lie group', see [1, 2].
3 Correspondence between Orbit Types and Bundle Reductions

In this section, let \( p_0 \in P \) be fixed. For \( A \in \mathcal{A}^k \), let \( H_A \) and \( P_A \) denote the holonomy group and holonomy subbundle, respectively, of \( A \) based at \( p_0 \). We assume \( 2k > \dim M + 2 \).

Then, by the Sobolev Embedding Theorem, \( A \) is of class \( C^1 \) so that \( P_A \) is a subbundle of \( P \) of class \( C^2 \). In particular, it is a subbundle of class \( C^0 \).

For any \( g \in G_{k+1} \), let \( \vartheta_g \) denote the associated vertical automorphism of \( P \).

Then \( \vartheta_g (p) = p \cdot g(p) \quad \forall \ p \in P \). \( \ \ (5) \)

Let \( C_G(H) \) denote the centralizer of \( H \subseteq G \) in \( G \). We abbreviate \( C^2_G(H) = C_G(C_G(H)) \).

Let \( A \in \mathcal{A}^k \) and \( g \in G_{k+1} \). Since the elements of \( G_{k+1} \) map \( A \)-horizontal paths in \( P \) to \( A \)-horizontal paths they are constant on \( P_A \). Conversely, any gauge transformation which is constant on \( P_A \) leaves \( A \) invariant. Thus, for any \( g \in G_{k+1} \) one has

\[ g \in G_{k+1} \quad \iff \quad g|_{P_A} \text{ is constant.} \quad \ (6) \]

This suggests the idea to characterize orbit types by certain classes of subbundles of \( P \).

For any subgroup \( S \subseteq G_{k+1} \) define a subset \( \Theta(S) \subseteq P \) by

\[ \Theta(S) = \{ p \in P \mid g(p) = g(p_0) \ \forall g \in S \}. \quad \ (7) \]

Lemma 3.1

(a) For any \( A \in \mathcal{A}_k \), \( \Theta(G_{k+1}^A) = P_A \cdot C^2_G(H_A) \).

(b) Let \( A, A' \in \mathcal{A}^k \). Then \( \Theta(G_{k+1}^A) = \Theta(G_{k+1}^{A'}) \) implies \( G_{k+1}^A = G_{k+1}^{A'} \).

(c) Let \( g \in G_{k+1}^k \). For any subgroup \( S \subseteq G_{k+1} \), \( \Theta(gSg^{-1}) = \vartheta_g(\Theta(S)) \cdot g(p_0)^{-1} \).

Remark: In \( [23] \), \( P_A \cdot C^2_G(H_A) \) is called the evolution bundle generated by \( A \). Thus, in this terminology, (a) states that \( \Theta \) maps the stabilizer of a connection to its evolution bundle.

Proof:

(a) Let \( A \in \mathcal{A}_k \). Recall that \( P_A \) has structure group \( H_A \). Hence, in view of (3), the equivariance property of gauge transformations implies

\[ \{ g(p_0) \mid g \in G_{k+1}^A \} = C_G(H_A). \quad \ (8) \]

Thus, by equivariance again,

\[ g \in G_{k+1}^A \quad \Longrightarrow \quad g|_{P_A \cdot C^2_G(H_A)} \text{ is constant.} \quad \ (9) \]

This shows \( P_A \cdot C^2_G(H_A) \subseteq \Theta(G_{k+1}^A) \). Conversely, let \( p \in P \) such that \( g(p) = g(p_0) \) for all \( g \in G_{k+1}^A \). There exists \( a \in G \) such that \( p \cdot a^{-1} \in P_A \). Then (3) yields

\[ g(p_0) = g(p \cdot a^{-1}) = ag(p)a^{-1} = ag(p_0)a^{-1}, \]
for all \( g \in \mathcal{G}_A^{k+1} \). Due to (3), then \( a \in C^2_G(H_A) \). Hence, \( p = (p \cdot a^{-1}) \cdot a \in P_A \cdot C^2_G(H_A) \).

(b) Let \( A, A' \in \mathcal{A}^k \) be given. For any \( g \in \mathcal{G}_A^{k+1} \), we have

\[
g \in \mathcal{G}_A^{k+1} \iff g|_{\mathcal{G}_A^{k+1}} \text{ is constant.}
\]

Here implication from left to right is due to (3) and assertion (a), the inverse implication follows from \( P_A \subseteq \Theta(\mathcal{G}_A^{k+1}) \) and (3). Since a similar characterization holds for \( \mathcal{G}_A^{k+1} \), the assertion follows.

(c) Let \( p \in P, h \in S \). Using (3) we compute

\[
g(p_0)^{-1}g(p)h(p)g(p)^{-1}g(p_0) = h(\vartheta_{g^{-1}}(p) \cdot g(p_0)) .
\]

This allows us to write down the following chain of equivalences:

\[
p \in \Theta(gSg^{-1}) \iff g(p)h(p)g(p)^{-1} = g(p_0)h(p_0)g(p_0)^{-1}
\]

\[
\iff g(p_0)^{-1}g(p)h(p)g(p)^{-1}g(p_0) = h(p_0)
\]

\[
\iff h(\vartheta_{g^{-1}}(p) \cdot g(p_0)) = h(p_0)
\]

\[
\iff \vartheta_{g^{-1}}(p) \cdot g(p_0) \in \Theta(S).
\]

This proves assertion (c).

\[\Box\]

**Definition 3.2** Let \( G \) be a Lie group and let \( P \) be a principal \( G \)-bundle. A subgroup \( H \subseteq G \) is called Howe subgroup iff there exists a subset \( K \subseteq G \) such that \( H = C_G(K) \). A reduction of \( P \) to a Howe subgroup of \( G \) will be called Howe subbundle. A subbundle \( Q \subseteq P \) will be called holonomy-induced of class \( C^r \) iff it contains a connected subbundle \( \bar{Q} \subseteq Q \) of class \( C^r \) with structure group \( \bar{H} \) such that

\[
Q = \bar{Q} \cdot C^2_G\left(\bar{H}\right) .
\]

(10)

We remark that the notion of Howe subgroup is common in the literature, cf. [27]. The notions of Howe subbundle and holonomy-induced subbundle are, to our knowledge, new. Moreover, we note that \( H \subseteq G \) is a Howe subgroup iff \( C^2_G(H) = H \). In particular, the subset \( K \subseteq G \) in the definition can be chosen to be \( C_G(H) \).

The set of holonomy-induced Howe subbundles of \( P \) of class \( C^0 \) is acted upon in a natural way by the group of continuous gauge transformations \( \mathcal{G}^{C^0} \) and by the structure group \( G \).

Let \( \text{Howe}_s(P) \) denote the corresponding set of conjugacy classes. We note that the actions of \( \mathcal{G}^{C^0} \) and \( G \) commute. Moreover, two subbundles of class \( C^0 \) of \( P \) are conjugate under the action of \( \mathcal{G}^{C^0} \) iff they are isomorphic. \( \text{Howe}_s(P) \) carries a natural partial ordering, which is defined similarly to that of orbit types: \( \eta \geq \eta' \) iff there are representatives \( Q \) of \( \eta \) and \( Q' \) of \( \eta' \) such that \( Q \subseteq Q' \).

**Theorem 3.3** Let \( M \) be compact, \( \dim M \geq 2 \). Then the assignment \( \Theta \) induces, by passing to quotients, an order-preserving bijection from \( \text{OT}(\mathcal{A}^k, \mathcal{G}^{k+1}) \) onto \( \text{Howe}_s(P) \).
Proof: Let $\tau \in \text{OT} (A^k, \mathcal{G}^{k+1})$ and choose a representative $S \subseteq \mathcal{G}^{k+1}$. There exists $A \in A^k$ such that $S = \mathcal{G}^{k+1}_A$. According to Lemma $3.1(a)$, $\Theta(S)$ can be obtained by extending the subbundle $P_A \subseteq P$ to the structure group $C^2_G(H_A)$. Since $P_A$ is of class $C^0$, so is $\Theta(S)$. Since $P_A$ is connected, $\Theta(S)$ is holonomy-induced of class $C^0$. Moreover, it is obviously Howe. According to Lemma $3.1(c)$, when passing to quotients the class of $\Theta(S)$ does not depend on the chosen representative $S$ of $\tau$. Thus, indeed, $\Theta$ projects to a map from $\text{OT} (A^k, \mathcal{G}^{k+1})$ to $\text{Howe}_x(P)$.

To check that this map is surjective, let $Q \subseteq P$ be a holonomy-induced Howe subbundle of class $C^0$. Let $Q \subseteq Q$ be a connected subbundle of class $C^0$, with structure group $\bar{H}$, such that (III) holds. Due to well-known smoothing theorems [17, Ch. I, §4], $\bar{Q}$ and $Q$ are $C^0$-isomorphic to $C^\infty$-subbundles of $P$. Hence, up to the action of $\mathcal{G}^{C^0}$, we may assume that $\bar{Q}$ and $Q$ are of class $C^\infty$ themselves. Moreover, up to the action of $G$, $p_0 \in \bar{Q}$. Since $M$ is compact and $\dim M \geq 2$, $\bar{Q}$ carries a $C^\infty$-connection with holonomy group $\bar{H}$ [21, Ch. II, Thm. 8.2]. This connection prolongs to a unique (smooth) $A \in A^k$ obeying $P_A = \bar{Q}$ and $H_A = \bar{H}$. Then Lemma $3.1(a)$ and (III) imply

$$\Theta (\mathcal{G}^{k+1}_A) = \bar{Q} \cdot C^2_G \left( \bar{H} \right) = Q.$$  

This proves surjectivity.

To show that the projected map is injective, let $\tau, \tau' \in \text{OT} (A^k, \mathcal{G}^{k+1})$. Choose representatives $S, S'$ and assume that there exist $g \in G$, $a \in G$ such that

$$\Theta (S') = \vartheta_g (\Theta (S)) \cdot a.$$  

(11)

Consider the following lemma.

Lemma 3.4 Let $A \in A^k$ and let $Q \subseteq P$ be a subbundle of class $C^\infty$. If there exists $h \in \mathcal{G}^{C^0}$ such that

$$\Theta (\mathcal{G}^{k+1}_A) = \vartheta_h (Q)$$  

(12)

then $h$ may be chosen from $\mathcal{G}^{k+1}$.

Before proving the lemma, let us assume that it holds and finish the arguments. Again, due to smoothing theorems, $\Theta(S)$ is $C^0$-isomorphic to some $C^\infty$-subbundle $Q \subseteq P$. Then there exists $h \in \mathcal{G}^{C^0}$ such that $\Theta(S) = \vartheta_h (Q)$. Due to Lemma 3.4, we can choose $h \in \mathcal{G}^{k+1}$. Moreover, due to (III), $\Theta(S') = \vartheta_{gh} (Q \cdot a)$. By application of Lemma 3.4 again, we can achieve $gh \in \mathcal{G}^{k+1}$. This shows that we may assume, from the beginning, $g \in \mathcal{G}^{k+1}$.

Now consider (III). Since $p_0 \in \Theta(S)$, $\vartheta_g (p_0) \cdot a = p_0 \cdot (g(p_0)a) \in \Theta(S')$. Since also $p_0 \in \Theta(S')$, $g(p_0)a$ is an element of the structure group of $\Theta(S')$. Then $\Theta(S') \cdot (a^{-1}g(p_0)^{-1}) = \Theta(S')$, so that (III) and Lemma 3.4(c) yield

$$\Theta (S') = \vartheta_g (\Theta (S)) \cdot g(p_0)^{-1} = \Theta (gSg^{-1}).$$

Due to Lemma 3.4(b), then $S' = gSg^{-1}$. This proves injectivity.
Proof of Lemma 3.4: Let $A$ and $Q$ be given. Under the assumption that (12) holds, $\Theta (G^{k+1}_A)$ and $Q$ have the same structure group $H$. There exist an open covering $\{U_i\}$ and local trivializations

$$\xi_i : U_i \times H \rightarrow \Theta (G^{k+1}_A)|_{U_i}, \quad \eta_i : U_i \times H \rightarrow Q|_{U_i}$$

of $\Theta (G^{k+1}_A)$ and $Q$, respectively. These define local trivializations

$$\tilde{\xi}_i, \tilde{\eta}_i : U_i \times G \rightarrow P|_{U_i}$$

of $P$ over $\{U_i\}$. Here $\eta_i, \tilde{\eta}_i$ can be chosen from the class $C^\infty$. As for $\xi_i$ and $\tilde{\xi}_i$, we note that $\Theta (G^{k+1}_A)$ contains the holonomy bundle $P_A$. A standard way of constructing local cross sections in $P_A$ goes as follows (cf. the proof of Lemma 1 in [21, Ch. II, §7.1]): Choose a local chart about some $x \in M$ and a point $p$ in the fibre of $P_A$ over $x$. Take the pre-images, under the local chart map, of straight lines running through $x$ and lift them to $A$-horizontal paths running through $p$. Since $A$ is of class $W^k$, the lifts are of class $W^{k+1}$. Hence, so is the local cross section and, therefore, the local trivialization defined by that cross section. This shows that $\xi_i$ and $\tilde{\xi}_i$ may be chosen from the class $W^{k+1}$.

Due to (12), the family $\{\vartheta \circ \eta_i\}$ defines a local trivialization of class $C^0$ of $\Theta (G^{k+1}_A)$ over $\{U_i\}$. Hence, there exists a vertical automorphism $\vartheta'$ of class $C^0$ of $\Theta (G^{k+1}_A)$ such that $\xi_i = \vartheta' \circ \vartheta \circ \eta_i \forall i$. By equivariant prolongation, $\vartheta'$ defines a unique element $h' \in G^{C^0}$. Since $\vartheta_{h'}$ leaves $\Theta (G^{k+1}_A)$ invariant, $\Theta (G^{k+1}_A) = \vartheta_{h'h}(Q)$. Thus, by possibly redefining $h$ we may assume that $h' = 1$, i.e., that $\vartheta'$ is trivial. Then

$$\tilde{\xi}_i = \vartheta \circ \tilde{\eta}_i \quad \forall i.$$  \hspace{1cm} (13)

As we shall argue now, (13) implies $h \in G^{k+1}$. To see this, note that $h \in G^{k+1}$ iff the local representatives $h_i = h \circ \tilde{\eta}_i \circ \iota$ are of class $W^{k+1}$, see [20]. Here $\iota$ denotes the embedding $U_i \hookrightarrow U_i \times G, x \mapsto (x, 1)$. Using $\tilde{\eta}_i(x, h_i(x)) = \vartheta \circ \tilde{\eta}_i(x, 1), \forall x \in U_i$, we find

$$h_i = \text{pr}_2 \circ \tilde{\eta}_i \circ \vartheta \circ \tilde{\eta}_i \circ \iota,$$  \hspace{1cm} (14)

where $\text{pr}_2$ is the canonical projection $U_i \times G \rightarrow G$. Inserting (13) into (14) yields

$$h_i = \text{pr}_2 \circ \tilde{\eta}_i^{-1} \circ \tilde{\xi}_i \circ \iota \quad \forall i.$$  

Here $\tilde{\xi}_i$ is of class $W^{k+1}$ and the other maps are of class $C^\infty$. Thus, according to the composition rules of Sobolev mappings, $h_i$ is of class $W^{k+1}$. It follows $h \in G^{k+1}$. This proves Lemma 3.4 and, therefore, Theorem 3.3. \hfill $\blacksquare$

Remarks:

1. As one important consequence of Theorem 3.3, $\text{OT} (A^k, G^{k+1})$ does not depend on $k$.
2. General arguments show that $\text{Hom}_*(P)$ is countable, see [23, §4.2]. Hence, so is $\text{OT} (A^k, G^{k+1})$. Countability of $\text{OT} (A^k, G^{k+1})$ is a necessary condition for this set to define a stratification in the sense of [22]. It was first stated in Theorem 4.2.1 in [23]. In
fact, the proof of this theorem already contains most of the arguments needed to prove Theorem 3.3. Unfortunately, although in the proof of Theorem 4.2.1 the authors used that isomorphy of evolution subbundles implies conjugacy under the action of $G^{k+1}$, they did not give an argument for that. Such an argument is provided by our Lemma 3.4.

3. Theorem 3.3 also shows that the notion of holonomy-induced Howe subbundle may be viewed as an abstract version of the notion of evolution subbundle generated by a connection.

4. The geometric ideas behind the proof of Theorem 3.3 are also contained in [13, §2]. However, a rigorous proof was not given there.

In view of Theorem 3.3, we are left with the problem of determining the set $\text{Howe}_n(P)$ together with its partial ordering. This leads us to the following

Programme

Step 1  Determination of the Howe subgroups of $G$. Since $G$-action on subbundles conjugates the structure group, classification up to conjugacy is sufficient.

Step 2  Determination of Howe subbundles of $P$. Since subbundles are conjugate by $G^{c0}$ if and only if they are isomorphic, classification up to isomorphy is sufficient.

Step 3  Specification of the Howe subbundles which are holonomy-induced.

Step 4  Factorization by $G$-action

Step 5  Determination of the natural partial ordering.

In the present paper, we perform steps 1–4 for the group $G = \text{SU}_n$. As already noted, the determination of the natural partial ordering, which includes the study of the natural partial ordering of Howe subgroups, will be published in a subsequent paper.

4  The Howe Subgroups of $\text{SU}_n$

Let $\text{Howe}(\text{SU}_n)$ denote the set of conjugacy classes of Howe subgroups of $\text{SU}_n$. In order to derive $\text{Howe}(\text{SU}_n)$, we consider $\text{SU}_n$ as a subset of the general linear algebra $\text{gl}(n, \mathbb{C})$, viewed as an associative algebra.

In the literature it is customary to consider, instead of Howe subgroups, Howe dual pairs. A Howe dual pair is an ordered pair of subgroups $(H_1, H_2)$ of $G$ such that

$$H_1 = C_G(H_2), \quad H_2 = C_G(H_1).$$

The assignment $H \mapsto (H, C_G(H))$ defines a 1 : 1-relation between Howe subgroups and Howe dual pairs. One also defines Howe subalgebras and Howe dual pairs in an associative algebra. We remark that, usually, one restricts attention to reductive Howe dual pairs. This means, one requires that any finite dimensional representation of the members be completely reducible. In our case this condition is automatically satisfied, because $\text{SU}_n$ is compact and Howe subgroups are always closed. Reductive Howe dual pairs play an important role in the representation theory of Lie groups. This was first observed by
R. Howe [18]. Although for SU\(n\) it is not necessary to go into the details of the classification theory of reductive Howe dual pairs, we note that there exist, essentially, two methods. One applies to the isometry groups of Hermitian spaces and uses the theory of Hermitian forms [27, 29, 31]. The other method applies to complex semisimple Lie algebras and uses root space techniques [30].

Let \(K(n)\) denote the collection of pairs of sequences (of equal length) of positive integers

\[ J = (k, m) = ((k_1, \ldots, k_r), (m_1, \ldots, m_r)) , \quad r = 1, 2, 3, \ldots, n , \]

which obey

\[ k \cdot m = \sum_{i=1}^{r} k_im_i = n . \quad (15) \]

For a given element \(J = (k, m)\) of \(K(n)\), let \(g\) denote the greatest common divisor of the integers \(m_1, \ldots, m_r\). Define a sequence \(\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_r)\) by \(g\tilde{m}_i = m_i \forall i\). Moreover, for any permutation \(\sigma\) of \(r\) elements, define \(\sigma J = (\sigma k, \sigma m)\).

Any \(J \in K(n)\) generates a canonical decomposition

\[ \mathbb{C}^n = (\mathbb{C}^{k_1} \otimes \mathbb{C}^{m_1}) \oplus \cdots \oplus (\mathbb{C}^{k_r} \otimes \mathbb{C}^{m_r}) . \quad (16) \]

This decomposition, in turn, induces an injective homomorphism

\[ \text{gl}(k_1, \mathbb{C}) \times \cdots \times \text{gl}(k_r, \mathbb{C}) \rightarrow \text{gl}(n, \mathbb{C}) \]

\[ (D_1, \ldots, D_r) \mapsto (D_1 \otimes 1_{m_1}) \oplus \cdots \oplus (D_r \otimes 1_{m_r}) . \quad (17) \]

Written as a matrix w.r.t. the canonical basis, the elements of the image look like

\[
\begin{pmatrix}
D_1 & 0 & \cdots & 0 \\
0 & D_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_1 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & D_r \\
\end{pmatrix}
\]

\((m_1 \text{ times } D_1, m_2 \text{ times } D_2, \text{ etc.})\). We denote the image of this homomorphism by \(\text{gl}(J, \mathbb{C})\), its intersection with \(U_n\) by \(UJ\) and its intersection with \(SU_n\) by \(SUJ\). Note that \(UJ\) is the image of the restriction of \((17)\) to \(U_{k_1} \times \cdots \times U_{k_r}\).

**Lemma 4.1**

(a) A \(*\)-subalgebra of \(\text{gl}(n, \mathbb{C})\) is Howe if and only if it is conjugate, under the action of \(SU_n\) by inner automorphisms, to \(\text{gl}(J, \mathbb{C})\) for some \(J \in K(n)\).

(b) A subgroup of \(U_n\) (resp. \(SU_n\)) is Howe if and only if it is conjugate, under the action of \(SU_n\) by inner automorphisms, to \(UJ\) (resp. \(SUJ\)) for some \(J \in K(n)\).
Remark: Assertion (a) is a version of the structure theorem for finite-dimensional von Neumann algebras.

Proof: (a) For any $J \in K(n)$, $gl(J, \mathbb{C})$ is a $*$-subalgebra of $gl(n, \mathbb{C})$ containing $1_n$. Thus, $gl(J, \mathbb{C})$ is a von Neumann algebra and the Double Commutant Theorem says that $gl(J, \mathbb{C})$ is Howe. Conversely, let $L$ be a Howe $*$-subalgebra of $gl(n, \mathbb{C})$. Let $L'$ denote the centralizer of $L$ in $gl(n, \mathbb{C})$ and let $\tilde{L}$ denote the subalgebra generated by $L$ and $L'$. Decompose

$$\mathbb{C}^n = V_1 \oplus \cdots \oplus V_r$$

(18)

into mutually orthogonal, $\tilde{L}$-irreducible subspaces. Due to Schur's Lemma, for each $i$,

$$V_i = W_i \otimes \mathbb{C}^{m_i},$$

(19)

where $W_i$ is $L$-irreducible and $m_i$ is a positive integer. Moreover,

$$L'|_{V_i} \subseteq \{id_{W_i} \otimes D'_i \mid D'_i \in gl(m_i, \mathbb{C})\},$$

(20)

$$L|_{V_i} \subseteq \{D_i \otimes 1_{m_i} \mid D_i \in \text{End}(W_i)\}.$$  

(21)

Since $L'$ is the full centralizer of $L$, one has equality in (20). As a consequence, since $L$ is the full centralizer of $L'$, there holds equality in (21). Now let $k_i = \dim W_i$ and $J = (k, m)$. By identifying $W_i \cong \mathbb{C}^{k_i}$, we arrive at a decomposition of $\mathbb{C}^n$ of the form (14). There exists $T \in SU_n$ transforming this decomposition into the canonical one induced by $J$. Then the inner automorphism of $gl(n, \mathbb{C})$ defined by $T$ transforms $L$ into $gl(J, \mathbb{C})$.

(b) We give only the proof for $SU_n$. Let $H$ be a Howe subgroup of $SU_n$. Then

$$H = C_{SU_n}(K) = C_{gl(n, \mathbb{C})}(K) \cap SU_n$$

for some subset $K \subseteq SU_n$. By construction, $C_{gl(n, \mathbb{C})}(K)$ is a Howe $*$-subalgebra of $gl(n, \mathbb{C})$. Hence, by virtue of (a), it is conjugate, under $SU_n$-action, to $gl(J, \mathbb{C})$ for some $J$. Then $H$ is conjugate, in $SU_n$, to $gl(J, \mathbb{C}) \cap SU_n = SUJ$. Conversely, let $J \in K(n)$. Due to (a), there exists $J' \in K(n)$ and $T \in SU_n$ such that $gl(J, \mathbb{C}) = C_{gl(n, \mathbb{C})}(Tgl(J', \mathbb{C})T^{-1})$. Taking into account that $(Tgl(J', \mathbb{C})T^{-1}) \cap SU_n$ spans $(Tgl(J', \mathbb{C})T^{-1})$, we obtain

$$SUJ = gl(J, \mathbb{C}) \cap SU_n$$

$$= C_{gl(n, \mathbb{C})}(Tgl(J', \mathbb{C})T^{-1} \cap SU_n) \cap SU_n$$

$$= C_{SU_n}(Tgl(J', \mathbb{C})T^{-1} \cap SU_n).$$

This shows that $SUJ$ is Howe.

Lemma 4.2 Let $J, J' \in K(n)$. Then $gl(J, \mathbb{C})$ and $gl(J', \mathbb{C})$ are conjugate under the action of $SU_n$ by inner automorphisms if and only if there exists a permutation $\sigma$ of $1, \ldots, r$ such that $J' = \sigma J$. 

\[\square\]
Proof: Let $T \in SU_n$ such that $\text{gl}(J', \mathbb{C}) = T^{-1}\text{gl}(J, \mathbb{C})T$. Then $\text{gl}(J, \mathbb{C})$ and $\text{gl}(J', \mathbb{C})$ are isomorphic. Hence, there exists a permutation $\sigma$ of $1, \ldots, r$ such that $k' = \sigma k$. Moreover, $T$ is an isomorphism of the representations

$$\text{gl}(k_1, \mathbb{C}) \times \cdots \times \text{gl}(k_r, \mathbb{C}) \overset{J}{\longrightarrow} \text{gl}(n, \mathbb{C}),$$

$$\text{gl}(k_1, \mathbb{C}) \times \cdots \times \text{gl}(k_r, \mathbb{C}) \overset{\sigma}{\longrightarrow} \text{gl}(k_{\sigma(1)}, \mathbb{C}) \times \cdots \times \text{gl}(k_{\sigma(r)}, \mathbb{C}) \overset{J'}{\longrightarrow} \text{gl}(n, \mathbb{C}),$$

where $J, J'$ indicate the embeddings (17) defined by $J$ and $J'$, respectively. In particular, it does not change the multiplicities of the irreducible factors. This implies $m' = \sigma m$. It follows $J' = \sigma J$. Conversely, assume that $J' = \sigma J$ for some permutation $\sigma$ of $1, \ldots, r$. Consider the canonical decompositions (16) defined by $J$ and $J'$, respectively. There exists $T \in SU_n$ which maps the factors $C^k_i \otimes C^{l_i'}$ of the second decomposition identically onto the factors $C^k_{\sigma(i)} \otimes C^{m_{\sigma(i)}}$ of the first one, for $i = 1, \ldots, r$. It is not difficult to see that $\text{gl}(J', \mathbb{C}) = T^{-1}\text{gl}(J, \mathbb{C})T$. 

As a consequence of Lemma 4.2, we introduce an equivalence relation on the set $K(n)$ by writing $J \sim J'$ iff there exists a permutation of $1, \ldots, r$ such that $J' = \sigma J$. Let $\hat{K}(n)$ denote the set of equivalence classes.

**Theorem 4.3** The assignment $J \mapsto SUJ$ induces a bijection from $\hat{K}(n)$ onto Howe($SU_n$).

**Proof:** According to Lemma 4.1, the assignment $J \mapsto SUJ$ induces a surjective map $K(n) \to Howe(SU_n)$. Due to Lemma 4.2 this map projects to $\hat{K}(n)$ and the projected map is injective.

This concludes the classification of Howe subgroups of $SU_n$, i.e., Step 1 of our programme.

**5 The Howe Subbundles of $SU_n$-Bundles**

In this chapter, let $P$ be a principal $SU_n$-bundle over $M$, $\dim M \leq 4$. We are going to derive the Howe subbundles of $P$ up to isomorphy. Since the action of $SU_n$ on subbundles conjugates the structure group, it suffices to consider reductions of $P$ to the subgroups $SUJ, J \in K(n)$. Thus, let $J \in K(n)$ be fixed. Let $\text{Bun}(M, SUJ)$ denote the set of isomorphism classes of principal $SUJ$-bundles over $M$ (where principal bundle isomorphisms are assumed to commute with the structure group action and to project to the identical map on the base space). Moreover, let $\text{Red}(P, SUJ)$ denote the set of isomorphism classes of reductions of $P$ to the subgroup $SUJ \subseteq SU_n$.

We shall first derive a description of $\text{Bun}(M, SUJ)$ in terms of suitable characteristic classes and then give a characterization of the subset $\text{Red}(P, SUJ)$. The classification of $\text{Bun}(M, SUJ)$ will be performed by constructing the Postnikov tower of the classifying space $BSUJ$ up to level 5. For the convenience of the reader, the basics of this method will be briefly explained below.

Note that in the sequel maps of topological spaces are always assumed to be continuous, without explicitly stating this.
5.1 Preliminaries

Universal Bundles and Classifying Spaces Let $G$ be a Lie group. As a basic fact in bundle theory, there exists a so-called universal $G$-bundle

$$G \hookrightarrow EG \to BG$$

with the following property: For any $CW$-complex (hence, in particular, any manifold) $X$ the assignment

$$[X, BG] \to \text{Bun}(X, G), \ f \mapsto f^*EG,$$

is a bijection [19]. Here $[\cdot, \cdot]$ means the set of homotopy classes of maps and $f^*$ denotes the pull-back of bundles. Both $EG$ and $BG$ can be realized as $CW$-complexes. They are unique up to homotopy equivalence. $BG$ is called the classifying space of $G$. The homotopy class of maps $X \to BG$ associated to $P \in \text{Bun}(X, G)$ by virtue of (23) is called the classifying map of $P$. We denote it by $f_P$. Note that a principal $G$-bundle is universal iff its total space is contractible. In particular,

$$\pi_i(G) \cong \pi_{i+1}(BG), \ i = 0, 1, 2, \ldots.$$  \hspace{1cm} (24)

This is an immediate consequence of the exact homotopy sequence of fibre spaces [8].

Associated Principal Bundles Defined by Homomorphisms Let $\varphi : G \to G'$ be a Lie group homomorphism and let $P \in \text{Bun}(X, G)$. By virtue of the action $G \times G' \to G'$, $(a, a') \mapsto \varphi(a)a'$,

$G'$ becomes a left $G$-space and we have an associated bundle $P[\varphi] = P \times_G G'$. Observe that $P[\varphi]$ can be viewed as a principal $G'$-bundle with right $G'$-action

$$P[\varphi] \times G' \to P[\varphi], \ \left(\left[(p, a'), b'\right]\right) \mapsto \left[(p, a'b')\right].$$

One has the natural bundle morphism

$$\psi : P \to P[\varphi], \ p \mapsto \left[(p, 1_{G'})\right].$$ \hspace{1cm} (25)

It obeys $\psi(p \cdot a) = \psi(p) \cdot \varphi(a)$ and projects to the identical map on $X$.

In the special case where $\varphi$ is a Lie subgroup embedding, the natural bundle morphism (25) is an embedding of $P$ onto a subbundle of $P[\varphi]$. In this case, if no confusion about $\varphi$ can arise, we shall often write $P[G']$ instead of $P[\varphi]$. Note that $P[\varphi]$ is the extension of $P$ by $G'$ and $P$ is a reduction of $P[\varphi]$ to the subgroup $(G, \varphi)$.

Classifying Maps Associated to Homomorphisms Again, let $\varphi : G \to G'$ be a homomorphism. There exists a map $B \varphi : BG \to BG'$, associated to $\varphi$, which is defined as the classifying map of the principal $G'$-bundle $(EG)[\varphi]$ associated to the universal $G$-bundle $EG$. It has the following functorial property: For $\varphi : G \to G'$ and $\varphi' : G' \to G''$ there holds

$$B(\psi \circ \varphi) = B\psi \circ B\varphi.$$ \hspace{1cm} (26)
Using $B\varphi$, the classifying map of $P[\varphi]$ can be expressed through that of $P$:

$$f_{P[\varphi]} = B\varphi \circ f_P. \quad (27)$$

In the special case where $\varphi$ is a normal Lie subgroup embedding, the short exact sequence of Lie group homomorphisms

$$1 \rightarrow G \rightarrow G' \overset{p}{\rightarrow} G/G' \rightarrow 1 \quad (28)$$

induces a principal bundle

$$G'/G \hookrightarrow B(\varphi) \rightarrow B G'. \quad (29)$$

The classifying map of this bundle is $Bp$, where $p$ denotes the natural projection.

**Characteristic Classes** Let $G$ be a Lie group. Consider the cohomology ring $H^*(BG, \pi)$ of the classifying space with values in some Abelian group $\pi$. For any $P \in \text{Bun}(X, G)$, the homomorphism $(f_P)^*$, induced on cohomology, maps $H^*(BG, \pi)$ to $H^*(X, \pi)$. Therefore, given $\gamma \in H^*(BG, \pi)$, one can define a map

$$\chi_\gamma : \text{Bun}(X, G) \rightarrow H^*(X, \pi), \quad P \mapsto (f_P)^* \gamma. \quad (30)$$

This is called the characteristic class for $G$-bundles over $X$ defined by $\gamma$. By construction, one has the following universal property of characteristic classes: Let $f : X \rightarrow X'$ be a map and let $P' \in \text{Bun}(X', G')$. Then

$$\chi_\gamma (f^* P') = f^* \chi_\gamma (P'). \quad (31)$$

Observe that if two bundles are isomorphic then their images under arbitrary characteristic classes coincide, whereas the converse, in general, does not hold. This is due to the fact that characteristic classes can control maps $X \rightarrow BG$ only on the level of the homomorphisms induced on cohomology. In general, the latter do not give sufficient information on the homotopy properties of the maps. In certain cases, however, they do. For example, such cases are obtained by specifying $G$ to be $U(1)$ or discrete, or by restricting $X$ in dimension. In these cases there exist sets of characteristic classes which classify $\text{Bun}(X, G)$. In the sequel we will utilize this to determine $\text{Bun}(M, SU(J))$.

**Eilenberg-MacLane Spaces** Let $\pi$ be a group and $n$ a positive integer. An arcwise connected $CW$-complex $X$ is called an Eilenberg-MacLane space of type $K(\pi, n)$ iff $\pi_n(X) = \pi$ and $\pi_i(X) = 0$ for $i \neq n$. Eilenberg-MacLane spaces exist for any choice of $\pi$ and $n$, provided $\pi$ is commutative for $n \geq 2$. They are unique up to homotopy equivalence.

The simplest example of an Eilenberg-MacLane space is the 1-sphere $S^1$, which is of type $K(\mathbb{Z}, 1)$. Two further examples, $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}_p, 1)$, are briefly discussed in Appendix A. Note that Eilenberg-MacLane spaces are, apart from very special examples, infinite dimensional. Also note that, up to homotopy equivalence one has

$$K(\pi_1 \times \pi_2, n) = K(\pi_1, n) \times K(\pi_2, n). \quad (32)$$
Now assume $\pi$ to be commutative also in the case $n = 1$. Due to the Universal Coefficient Theorem, $\text{Hom}(H_n(K(\pi,n)),\pi)$ is isomorphic to a subgroup of $H^n(K(\pi,n),\pi)$. Due to the Hurewicz Theorem, $H_n(K(\pi,n)) \cong \pi_n(K(\pi,n)) = \pi$. It follows that $H^n(K(\pi,n),\pi)$ contains elements which correspond to isomorphisms $H_n(K(\pi,n)) \to \pi$. Such elements are called characteristic. If $\gamma \in H^n(K(\pi,n),\pi)$ is characteristic then for any CW-complex $X$, the map
\[
[X, K(\pi,n)] \to H^n(X,\pi), \quad f \mapsto f^*\gamma,
\] (33)
is a bijection [3, §VII.12]. In this sense, Eilenberg-MacLane spaces provide a link between homotopy properties and cohomology. We remark that the bijection (33) induces an Abelian group structure on the set $[X,K(\pi,n)]$.

**Path-Loop-Fibration** Let $X$ be an arcwise connected topological space. Consider the path-loop fibration over $X$
\[
\Omega(X) \hookrightarrow P(X) \twoheadrightarrow X,
\] (34)
where $\Omega(X)$ and $P(X)$ denote the loop space and the path space of $X$, respectively (both based at some point $x_0 \in X$). Since $P(X)$ is contractible, the exact homotopy sequence induced by the fibration (34) implies $\pi_i(\Omega(X)) \cong \pi_{i+1}(X)$, $i = 0, 1, 2, \ldots$. Thus, for $X = K(\pi,n)$,
\[
\pi_i(\Omega(K(\pi,n+1))) \cong \pi_{i+1}(K(\pi,n+1)) = \begin{cases} 
\pi & | i = n \\
0 & | \text{otherwise}
\end{cases}
\]
Hence, $\Omega(K(\pi,n+1)) = K(\pi,n) \forall n$ and the path-loop fibration over $K(\pi,n+1)$ reads
\[
K(\pi,n) \hookrightarrow P(K(\pi,n+1)) \twoheadrightarrow K(\pi,n+1).
\] (35)

**Postnikov Tower** A map $f : X \to X'$ of topological spaces is called an $n$-equivalence iff the homomorphism induced on homotopy groups $f_* : \pi_i(X) \to \pi_i(X')$ is an isomorphism for $i < n$ and surjective for $i = n$. One also defines the notion of an $\infty$-equivalence, which is often called weak homotopy equivalence.

Let $f : X \to X'$ be an $n$-equivalence and let $Y$ be a CW-complex. Then the map $[Y,X] \to [Y,X']$, $g \mapsto f \circ g$, is bijective for $\dim Y < n$ and surjective for $\dim Y = n$ [3, Ch. VII, Cor. 11.13].

A CW-complex $Y$ is called $n$-simple iff it is arcwise connected and the action of $\pi_1(Y)$ on $\pi_i(Y)$ is trivial for $1 \leq i \leq n$. It is called simple iff it is $n$-simple for all $n$.

The following theorem describes how a simple CW-complex can be approximated by $n$-equivalent spaces constructed from Eilenberg-MacLane spaces.

**Theorem 5.1** Let $Y$ be a simple CW-complex. There exist
(a) a sequence of CW-complexes $Y_n$ and principal fibrations
\[
K(\pi_n(Y),n) \hookrightarrow Y_{n+1} \xrightarrow{q_n} Y_n, \quad n = 1, 2, 3, \ldots,
\] (36)
induced by maps $\theta_n : Y_n \to K(\pi_n(Y),n+1)$,
(b) a sequence of $n$-equivalences $y_n : Y \to Y_n$, $n = 1, 2, 3, \ldots$, such that $Y_1 = *$ (one point space) and $q_n \circ y_{n+1} = y_n$ for all $n$. 

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Proof: The assumption that \( Y \) be a simple CW-complex implies that the constant map \( Y \to * \) is a simple map (see [3, Ch. VII, Def. 13.4] for a definition of the latter). Thus, the assertion is a consequence of a more general theorem about simple maps given in [3, Ch. VII, Thm. 13.7].

Remarks:
1. The sequence of spaces and maps \( (Y_n, y_n, q_n), \ n = 1, 2, 3, \ldots, \) is called a Postnikov tower, or Postnikov system, or Postnikov decomposition of \( Y \).
2. For the principal fibrations (36) to be induced by a map \( \theta_n : Y_n \to K(\pi_n(Y), n + 1) \) means that they are given as pull-back of the path-loop fibration (35) over \( K(\pi_n(Y), n+1) \).

The theorem allows one to successively construct \( Y_n \) for given \( n \), starting from \( Y_1 = * \). For example, for \( n = 5 \) such constructions have been carried out for \( Y = \text{BU}_k \) in [3, §4.2], or for \( Y = \text{BPU}_k \), where \( \text{PU}_k \) denotes the projective unitary group, in [34]. In the sequel, we shall construct \( (\text{BSU}_J)_{5^*} \). For this purpose, we need information about the low-dimensional homotopy groups of \( \text{SU}_J \).

5.2 The Homotopy Groups of \( \text{SU}_J \)

For \( a \) being a positive integer, denote the canonical embedding \( \mathbb{Z}_a \hookrightarrow \text{U}_1 \) by \( j_a \) and the endomorphism of \( \text{U}_1 \) mapping \( z \mapsto z^a \) by \( p_a \). Moreover, let \( j_J \) and \( i_J \) denote the natural embeddings \( \text{SU}_J \hookrightarrow \text{U}_J \) and \( \text{U}_J \hookrightarrow \text{U}_n \), respectively.

Using the natural projections \( \text{pr}^{U_j}_i : U_J \to \text{U}_{k_i} \), we define a homomorphism

\[
\lambda_J : U_J \to \text{U}_1, \quad D \mapsto \prod_{i=1}^r p_{\tilde{m}_i} \circ \text{det}_{U_{k_i}} \circ \text{pr}^{U_j}_i(D) .
\]  
(37)

The following diagram commutes:

\[
\begin{array}{ccc}
\text{U}_J & \xrightarrow{j_J} & \text{U}_1 \\
\downarrow{\lambda_J} & & \downarrow{p_g} \\
\text{U}_1 & \xrightarrow{\text{det}_{U_n}} & \text{U}_1
\end{array}
\]  
(38)

Accordingly, the restriction of \( \lambda_J \) to the subgroup \( \text{SU}_J \) takes values in \( j_g(\mathbb{Z}_g) \subseteq \text{U}_1 \). Thus, we can define a homomorphism \( \lambda_J^S : \text{SU}_J \to \mathbb{Z}_g \) by the following commutative diagram:

\[
\begin{array}{ccc}
\text{SU}_J & \xrightarrow{j_J} & \text{U}_J \\
\downarrow{\lambda_J^S} & & \downarrow{\lambda_J} \\
\mathbb{Z}_g & \xrightarrow{j_g} & \text{U}_1
\end{array}
\]  
(39)

Let \( (\text{SU}_J)_0 \) denote the arcwise connected component of the identity. Note that it is also a connected component, because \( \text{SU}_J \) is a closed subgroup of \( \text{GL}(n, \mathbb{C}) \).
Lemma 5.2 The homomorphism $\lambda^S_j$ projects to an isomorphism $SUJ/(SUJ)_0 \to \mathbb{Z}_g$.

Proof: Consider the homomorphism $\lambda^S_j : SUJ \to \mathbb{Z}_g$. The target space being discrete, $\lambda^S_j$ must be constant on connected components. Hence $(SUJ)_0 \subseteq \ker \lambda^S_j$, so that $\lambda^S_j$ projects to a homomorphism $SUJ/(SUJ)_0 \to \mathbb{Z}_g$. The latter is surjective, because $\lambda^S_j$ is surjective. To prove injectivity, we show $\ker \lambda^S_j \subseteq (SUJ)_0$. Let $D \in \ker \lambda^S_j$ and denote $D_i = \text{pr}_{i}^{SUJ} \circ j_{j}(D)$. Define a homomorphism

$$\varphi : U1^r \to U1, (z_1, \ldots, z_r) \mapsto z_1^{\tilde{m}_1} \cdots z_r^{\tilde{m}_r}.$$  

Then

$$\lambda^S_j(D) = \varphi (\det_{U1_k} D_1, \ldots, \det_{Uk_r} D_r).$$  

By assumption, $(\det_{U1_k} D_1, \ldots, \det_{Uk_r} D_r) \in \ker \varphi$. Since the exponents defining $\varphi$ have greatest common divisor 1, $\ker \varphi$ is connected. Thus, there exists a path $(\gamma_1(t), \ldots, \gamma_r(t))$ in $\ker \varphi$ running from $(\det_{U1_k} D_1, \ldots, \det_{Uk_r} D_r)$ to $(1, \ldots, 1)$. For each $i = 1, \ldots, r$, define a path $G_i(t)$ in $Uk_i$ as follows: First, go from $D_i$ to $(\det_{U1_k} D_i) \oplus 1_{k_i-1}$, keeping the determinant constant, thus using connectedness of $SUk_i$. Next, use the path $\gamma_i(t) \oplus 1_{k_i-1}$ to get to $1_{k_i}$. By construction, the image of $(G_1(t), \ldots, G_r(t))$ under the embedding (17) is a path in $SUJ$ connecting $D$ with $1_n$. This proves $\ker \lambda^S_j \subseteq (SUJ)_0$.

Theorem 5.3 The homotopy groups of $SUJ$ are

$$\pi_i(SUJ) \cong \begin{cases} \mathbb{Z}_g & | \ i = 0 \\ \mathbb{Z}^{\oplus(r-1)} & | \ i = 1 \\ \pi_i(Uk_1) \oplus \cdots \oplus \pi_i(Uk_r) & | \ i > 1. \end{cases}$$

In particular, $\pi_1(SUJ)$ and $\pi_3(SUJ)$ are torsion-free.

Proof: For $i = 0$, $\pi_0(SUJ) = SUJ/(SUJ)_0$. This group is given by Lemma 5.2. For $i > 1$, the assertion follows immediately from the exact homotopy sequence induced by the bundle $SUJ \hookrightarrow UJ \xrightarrow{\det_{UJ}} U1$. For $i = 1$, consider the following portion of this sequence:

$$
\pi_1(U1) \longrightarrow \pi_1(SUJ) \longrightarrow \pi_1(U1) \xrightarrow{\det_{U1}*} \pi_1(U1) \longrightarrow \pi_0(SUJ) \longrightarrow \pi_0(UJ) \\
0 \longrightarrow \pi_1(SUJ) \longrightarrow \mathbb{Z}^{\oplus r} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}.
$$

One has $\mathbb{Z}^{\oplus r} / \ker((\det_{U1})*) \cong \im((\det_{U1})*)$. Exactness implies $\ker((\det_{U1})*) \cong \pi_1(SUJ)$ and $\im((\det_{U1})*) = g\mathbb{Z} \cong \mathbb{Z}$. It follows $\pi_1(SUJ) \cong \mathbb{Z}^{\oplus(r-1)}$, as asserted.

5.3 The Postnikov Tower of $BSUJ$ up to Level 5

Let $r^*$ denote the number of indices $i$ for which $k_i > 1$.

Theorem 5.4 The 5th level of the Postnikov tower of $BSUJ$ is given by

$$(BSUJ)_5 = K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2) \times \prod_{j=1}^{r^*} K(\mathbb{Z}, 4).$$  

(40)
Proof: First, we check that $\text{BSU}J$ is a simple space. To see this, note that any inner automorphism of $\text{SU}J$ is generated by an element of $(\text{SU}J)_0$, hence is homotopic to the identity automorphism. Consequently, the natural action of $\pi_0(\text{SU}J)$ on $\pi_i(\text{SU}J)$, $i = 1, 2, 3, \ldots$, induced by inner automorphisms, is trivial. Since the natural isomorphisms $\pi_{i-1}(\text{SU}J) \cong \pi_i(\text{BSU}J)$ transform this action into that of $\pi_1(\text{BSU}J)$ on $\pi_i(\text{BSU}J)$, the latter is trivial, too. Thus, we can apply Theorem 5.1 to construct the Postnikov tower of $\text{BSU}J$ up to level 5. According to Theorem 5.3, the relevant homotopy groups are

$$
\pi_1(\text{BSU}J) = \mathbb{Z}_g, \quad \pi_2(\text{BSU}J) = \mathbb{Z}^\oplus(r-1), \quad \pi_3(\text{BSU}J) = 0, \quad \pi_4(\text{BSU}J) = \mathbb{Z}^{\oplus r^*}. \quad (41)
$$

Moreover, we note that $H^*(K(\mathbb{Z}, 2), \mathbb{Z})$ is torsion-free, and that

$$
H^{2i+1}(K(\mathbb{Z}, 2), \mathbb{Z}) = 0, \quad H^{2i+1}(K(\mathbb{Z}_g, 1), \mathbb{Z}) = 0, \quad i = 0, 1, 2, \ldots, \quad (42)
$$

see Appendix A. We start with $(\text{BSU}J)_1 = *$.

$(\text{BSU}J)_2$: Being a fibration over $(\text{BSU}J)_1$, $(\text{BSU}J)_2$ must coincide with the fibre:

$$(\text{BSU}J)_2 = K(\mathbb{Z}_g, 1). \quad (43)$$

$(\text{BSU}J)_3$: In view of (43) and (44), $(\text{BSU}J)_3$ is the total space of a fibration

$$K(\mathbb{Z}^\oplus(r-1), 2) \hookrightarrow (\text{BSU}J)_3 \xrightarrow{\theta_2} K(\mathbb{Z}_g, 1) \quad (44)$$

induced from the path-loop fibration over $K(\mathbb{Z}^\oplus(r-1), 3)$ by some map $\theta_2 : K(\mathbb{Z}_g, 1) \to K(\mathbb{Z}^\oplus(r-1), 3)$. Note that $K(\mathbb{Z}^\oplus(r-1), 1) = \prod_{i=1}^{r-1} K(\mathbb{Z}, 1) \forall n$. Then, due to (43),

$$[K(\mathbb{Z}_g, 1), K(\mathbb{Z}^\oplus(r-1), 3)] = \prod_{i=1}^{r-1} H^3(K(\mathbb{Z}_g, 1), \mathbb{Z}).$$

Here the rhs. is trivial by (42). Hence, $\theta_2$ is homotopic to a constant map, so that the fibration (44) is trivial. It follows

$$(\text{BSU}J)_3 = K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2). \quad (45)$$

$(\text{BSU}J)_4$: In view of (44), $(\text{BSU}J)_4$ is given by a fibration over $(\text{BSU}J)_3$ with fibre $K(0, 3) = *$. Hence, it just coincides with the base space.

$(\text{BSU}J)_5$: According to (44) and (45), $(\text{BSU}J)_5$ is the total space of a fibration

$$K(\mathbb{Z}^{\oplus r^*}, 4) \hookrightarrow (\text{BSU}J)_5 \xrightarrow{\theta_4} K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2), \quad (46)$$

which is induced by a map $\theta_4$ from the base to $K(\mathbb{Z}^{\oplus r^*}, 5)$. Similarly to the case of $\theta_2$,

$$\left[ K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2), K(\mathbb{Z}^{\oplus r^*}, 5) \right] = \prod_{i=1}^{r^*} H^5 \left( K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2), \mathbb{Z} \right). \quad (47)$$
Since $H^*(K(\mathbb{Z}, 2), \mathbb{Z})$ is torsion-free, we can apply the Künneth Theorem for cohomology \[25\text{ Ch. XIII, Cor. 11.3}\] to obtain
\[
H^5 \left( K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2), \mathbb{Z} \right) \cong \bigoplus_{j+j_1+\cdots+j_{r-1}=5} H^j(\mathbb{Z}_g, 1), \mathbb{Z} \otimes H^{j_1}(\mathbb{Z}, 2), \mathbb{Z} \otimes \cdots \otimes H^{j_{r-1}}(\mathbb{Z}, 2), \mathbb{Z}. \]

By (12), each summand of the rhs. is trivial, because it contains a tensor factor of odd degree. Hence, (17) is trivial, and so is the fibration (16). This proves the assertion. 

The fact that $(\text{BSU}J)_5$ is a direct product of Eilenberg-MacLane spaces immediately yields the following corollary.

**Corollary 5.5** Let $J \in K(n)$ and $\dim M \leq 4$. Let $P, P' \in \text{Bun}(M, \text{SU}J)$. Assume that for any characteristic class $\alpha$ defined by an element of $H^1(\text{BSU}J, Z_g), H^2(\text{BSU}J, \mathbb{Z})$, or $H^4(\text{BSU}J, \mathbb{Z})$ there holds $\alpha(P) = \alpha(P')$. Then $P$ and $P'$ are isomorphic.

**Proof:** Let $\text{pr}_1, \text{pr}_{21}, \ldots, \text{pr}_{2r-1},$ and $\text{pr}_{41}, \ldots, \text{pr}_{4r}$ denote the natural projections of the direct product $K(\mathbb{Z}_g, 1) \times \prod_{i=1}^{r-1} K(\mathbb{Z}, 2) \times \prod_{i=1}^{r} K(\mathbb{Z}, 4)$ onto its factors. Let $\gamma_1, \gamma_2,$ and $\gamma_4$ be characteristic elements of $H^1(K(\mathbb{Z}_g, 1), \mathbb{Z}_g), H^2(K(\mathbb{Z}, 2), \mathbb{Z}),$ and $H^4(K(\mathbb{Z}, 4), \mathbb{Z})$, respectively. Consider the map
\[
\varphi : [M, \text{BSU}J] \to [M, (\text{BSU}J)_5] \\
\to [M, K(\mathbb{Z}_g, 1)] \times \prod_{i=1}^{r-1} [M, K(\mathbb{Z}, 2)] \times \prod_{i=1}^{r} [M, K(\mathbb{Z}, 4)] \\
\to H^1(M, \mathbb{Z}_g) \times \prod_{i=1}^{r-1} H^2(M, \mathbb{Z}) \times \prod_{i=1}^{r} H^4(M, \mathbb{Z}) ,
\]
\[
 f \mapsto \left( \{ f^*(\text{pr}_1 \circ y_5)^* \gamma_1 \}, \{ f^*(\text{pr}_{2i} \circ y_5)^* \gamma_2 \}_{i=1}^{r-1}, \{ f^*(\text{pr}_{4i} \circ y_5)^* \gamma_4 \}_{i=1}^{r} \right), \quad (48)
\]
where $y_5 : \text{BSU}J \to (\text{BSU}J)_5$ is the 5-equivalence provided by Theorem 3.4. According to Theorem 3.4, the second step of $\varphi$ and, therefore, the whole map, is a bijection.

Now let $P, P' \in \text{Bun}(M, \text{SU}J)$ as proposed in the assertion. Then, by assumption, the homomorphisms $(f_P)^*$ and $(f_{P'})^*$, induced on $H^1(\text{BSU}J, Z_g), H^2(\text{BSU}J, \mathbb{Z}),$ and $H^4(\text{BSU}J, \mathbb{Z})$, coincide. This implies $\varphi(f_P) = \varphi(f_{P'})$. Hence, $f_P$ and $f_{P'}$ are homotopic. This proves the corollary. 

We remark that, of course, the cohomology elements $(\text{pr}_1 \circ y_5)^* \gamma_1, (\text{pr}_{2i} \circ y_5)^* \gamma_2, i = 1, \ldots, r - 1,$ and $(\text{pr}_{4i} \circ y_5)^* \gamma_4, i = 1, \ldots, r$ define a set of characteristic classes which classifies $\text{Bun}(M, \text{SU}J)$. These classes are independent and surjective. However, they are hard to handle, because we do not know the homomorphism $y_5^*$ explicitly. Therefore, we prefer to work with characteristic classes defined by some natural generators of the cohomology groups in question. The price we have to pay for this is that the classes so constructed are subject to a relation and that we have to determine their image explicitly.
5.4 Generators for $H^*(BU_J, \mathbb{Z})$

Instead of generators for the groups $H^2(BU_J, \mathbb{Z})$ and $H^4(BU_J, \mathbb{Z})$ only, we can construct generators for the whole cohomology algebra $H^*(BU_J, \mathbb{Z})$ without any additional effort. Consider the homomorphisms and induced homomorphisms

$$H^*(BU_J, \mathbb{Z}) \xrightarrow{\text{pr}^U_J} U_k$$

where $i = 1, \ldots, r$. Recall that the cohomology algebra $H^*(BU_k, \mathbb{Z})$ is generated freely over $\mathbb{Z}$ by elements $\gamma_{U_k}^{(2i)} \in H^{2j}(BU_k, \mathbb{Z})$, $j = 1, \ldots, k$, see [3]. We denote

$$\gamma_{U_k} = 1 + \gamma_{U_k}^{(2)} + \cdots + \gamma_{U_k}^{(2k)}.$$  \hfill (50)

The generators $\gamma_{U_k}^{(2j)}$ define elements

$$\gamma_{J, i}^{(2j)} = (\text{Bpr}_i^{U_J})^* \gamma_{U_k}^{(2j)} , \quad (51)$$

$$\gamma_{J, i}^{(2j)} = (B_j^{U_J} (\text{Bpr}_i^{U_J}))^* \gamma_{U_k}^{(2j)} , \quad (52)$$

of $H^{2j}(BU_J, \mathbb{Z})$ and $H^{2j}(BU_J, \mathbb{Z})$, respectively. We denote

$$\tilde{\gamma}_{J, i} = 1 + \tilde{\gamma}_{J, i}^{(2j)} + \cdots + \tilde{\gamma}_{J, i}^{(2k)} , \quad i = 1, \ldots, r , \quad (53)$$

$$\gamma_{J, i} = 1 + \gamma_{J, i}^{(2j)} + \cdots + \gamma_{J, i}^{(2k)} , \quad i = 1, \ldots, r , \quad (54)$$

as well as $\tilde{\gamma}_J = (\tilde{\gamma}_{J, 1}, \ldots, \tilde{\gamma}_{J, r})$ and $\gamma_J = (\gamma_{J, 1}, \ldots, \gamma_{J, r})$.

**Lemma 5.6** The cohomology algebra $H^*(BU_J, \mathbb{Z})$ is generated freely over $\mathbb{Z}$ by the elements $\tilde{\gamma}_{J, i}^{(2j)}$, $j = 1, \ldots, k$, $i = 1, \ldots, r$.

**Proof:** Consider the isomorphism and induced isomorphism

$$H^*(BU_J, \mathbb{Z}) \xrightarrow{d_r} H^*(\Pi_i U_J) \xrightarrow{\Pi_i \text{pr}^U_J} H^*(\Pi_i BU_k, \mathbb{Z}) . \quad (55)$$

where $d_r$ denotes $r$-fold diagonal embedding and $\Pi_i$ is a shorthand notation for $\Pi_{i=1}^r$. Due to the Künneth Theorem for cohomology [23, Ch. XIII, Cor. 11.3], $H^*(\Pi_i BU_k, \mathbb{Z})$ is generated freely over $\mathbb{Z}$ by the elements

$$1_{BU_k} \times \cdots \times 1_{BU_{k-1}} \times \gamma_{U_k}^{(2j)} \times 1_{BU_{k+1}} \times \cdots \times 1_{BU_r} , \quad j = 1, \ldots, k, i = 1, \ldots, r .$$

Here $\times$ stands for the cohomology cross product, and $1_{BU_k}$ denotes the generator of $H^0(BU_k, \mathbb{Z})$. The assertion follows by applying (55) to these generators and using

$$d_r^*(\alpha_1 \times \cdots \times \alpha_r) = \alpha_1 \bowtie \cdots \bowtie \alpha_r \quad \forall \alpha_i \in H^*(BU_k, \mathbb{Z}) . \quad (56)$$
Lemma 5.7 \((B_{jJ})^*\) is surjective.

Proof: According to (29) and due to \(UJ/SUJ \cong U1\), \(B_{jJ}\) is the projection in a principal bundle
\[ U1 \hookrightarrow BSUJ \xrightarrow{B_{jJ}} BUJ. \] (57)

Denote this bundle by \(\eta\). Due to \(\pi_1(BUJ) \cong \pi_0(UJ) = 0\), \(\eta\) is orientable, see [3, Def. 7.3.3]. Therefore, it induces a Gysin sequence, see [3, §7.3.1],
\[
H^1(BUJ, \mathbb{Z}) \xrightarrow{(B_{jJ})^*} H^1(BSUJ, \mathbb{Z}) \xrightarrow{\sigma^*} H^0(BUJ, \mathbb{Z}) \xrightarrow{c_1(\eta)} H^2(BUJ, \mathbb{Z}) \rightarrow \cdots.
\] (On the level of differential forms, \(\sigma^*\) is given by integration over the fibre.) If \(\eta\) was trivial, \(\pi_1(BSUJ)\) would coincide with \(\pi_1(BUJ \times U1) = \mathbb{Z}\), which would contradict Theorem 5.3. Hence, \(\eta\) is nontrivial, so that \(c_1(\eta) \neq 0\). Since \(H^*(BUJ, \mathbb{Z})\) has no zero divisors by Lemma 5.6, it follows that multiplication by \(c_1(\eta)\) is an injective operation on \(H^*(BUJ, \mathbb{Z})\). Then exactness of the Gysin sequence (58) implies that the homomorphism \(\sigma^*\) is trivial and, therefore, \((B_{jJ})^*\) is surjective.

Lemmas 5.6 and 5.7 imply the following corollary.

Corollary 5.8 The cohomology algebra \(H^*(BSUJ, \mathbb{Z})\) is generated over \(\mathbb{Z}\) by the elements \(\gamma_{ij}^{(2)}\), \(j = 1, \ldots, k_i\), \(i = 1, \ldots, r\). ■

Remark: The generators \(\gamma_{ij}^{(2)}\) of \(H^*(BSUJ, \mathbb{Z})\) are subject to a relation which is, however, irrelevant for our purposes. For the sake of completeness, we derive this relation in Appendix B.

5.5 Generator for \(H^1(BSUJ, \mathbb{Z}_g)\)

Since \(BSUJ\) is connected, \(H^1(BSUJ, \mathbb{Z}_g)\) can be computed by means of the Hurewicz and the Universal Coefficient Theorems:
\[
H^1(BSUJ, \mathbb{Z}_g) \cong \text{Hom}(H_1(BSUJ), \mathbb{Z}_g) \oplus \text{Ext}(H_0(BSUJ), \mathbb{Z}_g)
\cong \text{Hom}(\pi_1(BSUJ), \mathbb{Z}_g) \oplus \text{Ext}(\mathbb{Z}, \mathbb{Z}_g).
\]

Due to Theorem 5.3, \(\pi_1(BSUJ) \cong \pi_0(SUJ) \cong \mathbb{Z}_g\). Moreover, \(\text{Ext}(\mathbb{Z}, \mathbb{Z}_g) = 0\). Hence, \(H^1(BSUJ, \mathbb{Z}_g)\) is isomorphic to \(\mathbb{Z}_g\). It is therefore generated by a single element. Apparently, we are free to choose any of the generators to work with. However, there exists a relation between this generator and the generators \(\gamma_{ij}^{(2)}\) of \(H^2(BSUJ, \mathbb{Z})\). This can be seen as follows. Consider the short exact sequence
\[
0 \rightarrow \mathbb{Z} \xrightarrow{\mu_g} \mathbb{Z} \xrightarrow{\nu_g} \mathbb{Z}_g \rightarrow 0,
\] (59)
where \( \mu_g \) denotes multiplication by \( g \) and \( g \) reduction modulo \( g \). This induces a long exact sequence (see \[8\] \( \S IV.5 \))

\[
\cdots \to \beta_g^* H^i(\cdot, \mathbb{Z}) \xrightarrow{\mu_g} H^i(\cdot, \mathbb{Z}) \xrightarrow{\varphi_g} H^i(\cdot, \mathbb{Z})_g \xrightarrow{\beta_g} H^{i+1}(\cdot, \mathbb{Z}) \xrightarrow{\mu_g} \cdots. \tag{60}
\]

Here we have denoted the coefficient homomorphisms induced by \( \mu_g \) and \( g \) by the same letters, i.e., \( \mu_g \) maps \( \alpha \mapsto g\alpha \) and \( g \) maps \( \alpha \mapsto \alpha \mod g \). Usually, the connecting homomorphism \( \beta_g \) is called Bockstein homomorphism. Application of \( \beta_g \) to an arbitrary generator of \( H^1(\text{BSU}_j, \mathbb{Z}_g) \) yields an element of \( H^2(\text{BSU}_j, \mathbb{Z}) \) which can be expressed in terms of the \( \gamma^{(2)}_{j_1} \). In order to keep track of this relation, we have to choose a specific generator of \( H^1(\text{BSU}_j, \mathbb{Z}_g) \). This will be constructed now.

Consider the homomorphism \( \lambda^g_{\text{SU}_1} : \text{SU}_1 \to \mathbb{Z}_g \) and the induced homomorphism

\[
(\lambda^g_{\text{SU}_1})^* : H^1(\mathbb{Z}_g, \mathbb{Z}) \to H^1(\text{BSU}_j, \mathbb{Z}_g). \tag{61}
\]

Due to Lemma \[5.2\], \( \lambda^g_{\text{SU}_1} : \pi_0(\text{SU}_1) \to \pi_0(\mathbb{Z}_g) \) is an isomorphism. Hence, so is \( (\lambda^g_{\text{SU}_1})^* : \pi_1(\text{BSU}_j) \to \pi_1(\mathbb{Z}_g) \). Then the Hurewicz and Universal Coefficient Theorems imply that \( (61) \) is an isomorphism. Thus, generators of \( H^1(\text{BSU}_j, \mathbb{Z}_g) \) can be obtained as the images of generators of \( H^1(\mathbb{Z}_g, \mathbb{Z}) \) under \( (\lambda^g_{\text{SU}_1})^* \).

**Lemma 5.9** There exists a unique element \( \delta_g \in H^1(\mathbb{Z}_g, \mathbb{Z}_g) \) such that

\[
\beta_g(\delta_g) = (Bj_g)^* \gamma^{(2)}_{U_1}. \tag{62}
\]

This element is a generator of \( H^1(\mathbb{Z}_g, \mathbb{Z}_g) \).

**Proof:** First we notice that both \( \beta_g(\delta_g) \) and \( (Bj_g)^* \gamma^{(2)}_{U_1} \) are elements of \( H^2(\mathbb{Z}_g, \mathbb{Z}_g) \) so that Eq. (62) makes sense.

Next, consider the following portion of the exact sequence (60):

\[
\cdots \to H^1(\mathbb{Z}_g, \mathbb{Z}) \xrightarrow{\varphi_g} H^1(\mathbb{Z}_g, \mathbb{Z}_g) \xrightarrow{\beta_g} H^2(\mathbb{Z}_g, \mathbb{Z}_g) \xrightarrow{\mu_g} H^2(\mathbb{Z}_g, \mathbb{Z}) \to \cdots. \tag{63}
\]

To determine the cohomology groups, we note that a model for \( \mathbb{Z}_g \) is given by the lens space \( L_g, \) see Appendix \[A\]. Thus, from Eq. (139) in this appendix we infer

\[
H^1(\mathbb{Z}_g, \mathbb{Z}) = 0 \tag{64}
\]

\[
H^2(\mathbb{Z}_g, \mathbb{Z}) = \mathbb{Z}_g. \tag{65}
\]

Due to (64), \( \beta_g \) is injective in (63). Due to (65), \( \mu_g \) is trivial in (63) so that \( \beta_g \) is also surjective there. Hence, we can define \( \delta_g = \beta_g^{-1} \circ (Bj_g)^* \gamma^{(2)}_{U_1} \). In order to check that this is a generator, we consider \( J^g = ((1), (g)) \in K(g) \). We observe that \( \mathbb{Z}_g \cong \text{SU}_j^0, U_1 \cong UJ^0, \) and that \( j_g \) corresponds, by virtue of these isomorphisms, to \( j_{J^g} : \text{SU}_j^0 \to UJ^0 \). Then Lemma \[5.7\] implies that \( (Bj_g)^* \) is surjective. In particular, \( H^2(\mathbb{Z}_g, \mathbb{Z}) \) is generated by \( (Bj_g)^* \gamma^{(2)}_{U_1} \). Hence, \( H^1(\mathbb{Z}_g, \mathbb{Z}_g) \) is generated by \( \delta_g \).

We define

\[
\delta_J = (B\lambda^g_{\text{SU}_1})^* \delta_g. \tag{66}
\]

Then Lemma \[5.9\] yields the following corollary.
Corollary 5.10  \( H^1(\text{BSU}_J, \mathbb{Z}_y) \) is generated by \( \delta_J \), where \( \beta_y(\delta_J) = (B\lambda^S)^* (Bj_g)^* \gamma^{(2)}_{U_1} \). ■

5.6  The Relation between Generators

In this subsection, we are going to derive the relation between \( \delta_J \) and the \( \gamma^{(2)}_{J,i} \), i.e., to compute \( \beta_g(\delta_J) \) in terms of the latter.

For any topological space \( X \), let \( H^\text{even}_0(X, \mathbb{Z}) \) denote the subset of \( H^\text{even}(X, \mathbb{Z}) \) consisting of elements of the form \( 1 + \alpha^{(2)} + \alpha^{(4)} + \ldots \). For any finite sequence of nonnegative integers \( \alpha = (\alpha_1, \ldots, \alpha_s) \), define a polynomial function

\[
E_\alpha : \prod_{i=1}^s H^\text{even}_0(X, \mathbb{Z}) \to H^\text{even}_0(X, \mathbb{Z}), \quad (\alpha_1, \ldots, \alpha_s) \mapsto \alpha_1^{\alpha_1} \cdots \alpha_s^{\alpha_s},
\]

where powers are taken w.r.t. the cup product. By construction, for any map \( f : X \to Y \),

\[
f^*E_\alpha(\alpha_1, \ldots, \alpha_s) = E_\alpha(f^*\alpha_1, \ldots, f^*\alpha_s).
\]

Let us derive explicit expressions for the components of \( E_\alpha \) of 2nd and 4th degree.

Lemma 5.11  Let \( \alpha = (\alpha_1, \ldots, \alpha_s) \in \prod_{i=1}^s H^\text{even}_0(X, \mathbb{Z}) \). Then

\[
E^{(2)}_\alpha(\alpha) = \sum_{i=1}^s a_i \alpha_i^{(2)},
\]

\[
E^{(4)}_\alpha(\alpha) = \sum_{i=1}^s a_i \alpha_i^{(4)} + \sum_{i=1}^s \frac{a_i(a_i-1)}{2} \alpha_i^{(2)} \alpha_i^{(2)} + \sum_{i<j}^s a_ia_j \alpha_i^{(2)} \alpha_j^{(2)}.
\]

Proof: This is a straightforward computation which we only indicate here. Let \( \alpha \) be given. Without loss of generality we may assume that the components of \( \alpha_i \) of degree higher than 4 vanish. For the cup product of elements of this form one has the following formula:

\[
(1 + \beta^{(2)} + \beta^{(4)}) \ast (1 + \gamma^{(2)} + \gamma^{(4)}) = 1 + (\beta^{(2)} + \gamma^{(2)}) + (\beta^{(4)} + \gamma^{(4)} + \beta^{(2)} \ast \gamma^{(2)}).
\]

We can iterate this to obtain

\[
\alpha_i^{a_i} = 1 + (a_i \alpha_i) + \left( a_i \alpha_i^{(4)} + \frac{a_i(a_i-1)}{2} \alpha_i^{(2)} \alpha_i^{(2)} \right)
\]

and then to compute the product of all the factors \( \alpha_i^{a_i} \). This yields the assertion. ■

As an immediate consequence of (69), for any \( l \in \mathbb{Z} \),

\[
E^{(2)}_{\alpha} = l E^{(2)}_{\alpha}.
\]
Lemma 5.12 The following two formulae hold:
\[(Bi_J)^* \gamma_{U_n} = E_m (\tilde{\gamma}_J), \quad (B\lambda_J)^* \gamma_{U_1}^{(2)} = E_m^{(2)} (\tilde{\gamma}_J). \] (72) (73)

Proof: First, consider (72). We decompose \(i_J\) as follows:
\[i_J : UJ \xrightarrow{d_r} \Pi_i UJ \xrightarrow{\Pi_i p_{BU}^U} \Pi_i UJ_i \xrightarrow{\Pi_i d_{mi}} \Pi_i (UJ_i \times \cdots \times UJ_i) \xrightarrow{j} UJ. \] (74)

Here \(d_r, d_{mi}\) denote \(r\)-fold and \(m_i\)-fold diagonal embedding, respectively, and \(j\) stands for the natural (blockwise) embedding. According to (74), \((Bi_J)^*\) decomposes as
\[(Bi_J)^* : H^* (BU_n, \mathbb{Z}) \xrightarrow{g} H^* (\Pi_i (BUJ_i \times \cdots \times BUJ_i), \mathbb{Z}) \xrightarrow{d_r} H^* (\Pi_i (BUJ_i), \mathbb{Z}) \xrightarrow{d_r^*} H^* (BUJ, \mathbb{Z}). \] (75)

Under the assumption that we have chosen the generators \(\gamma_{U_{U_k}}^{(2)}\) for different \(k\) in a consistent way (namely, such that the universal properties of Chern classes hold for the characteristic classes defined by these elements), there holds the relation
\[(Bi_J)^* \gamma_{U_n} = (\gamma_{U_{U_{U_k}} \cdots \gamma_{U_{U_k}}}) \times \cdots \times (\gamma_{U_{U_{U_k}} \cdots \gamma_{U_{U_k}}}). \] (76)

Using this, as well as (50), we obtain
\[(Bi_J)^* \gamma_{U_n} = d_r^* \circ (\Pi_i p_{BU}^U)^* \circ (\Pi_i d_{mi})^* \circ (Bi_J)^* \gamma_{U_n} \]
\[= d_r^* \circ (\Pi_i p_{BU}^U)^* \circ (\Pi_i d_{mi})^* \left( (\gamma_{U_{U_{U_k}} \cdots \gamma_{U_{U_k}}}) \times \cdots \times (\gamma_{U_{U_{U_k}} \cdots \gamma_{U_{U_k}}}) \right) \]
\[= d_r^* \circ (\Pi_i p_{BU}^U)^* \left( (\gamma_{U_{U_{U_k}} \cdots \gamma_{U_{U_k}}}) \times \cdots \times (\gamma_{U_{U_{U_k}} \cdots \gamma_{U_{U_k}}}) \right) \]
\[= d_r^* \left( \tilde{\gamma}_{i,1}^{m_{i}} \cdots \tilde{\gamma}_{i,r}^{m_{i}} \right) \]
\[= \tilde{\gamma}_{i,1}^{m_{i}} \cdots \tilde{\gamma}_{i,r}^{m_{i}}. \]

This yields (72). Now consider (73). The commutative diagram (38) implies
\[(B\lambda_J)^* (Bp_g)^* \gamma_{U_1}^{(2)} = (Bi_J)^* (B\det u_n)^* \gamma_{U_1}^{(2)}. \] (77)

We have
\[(B\det u_n)^* \gamma_{U_1}^{(2)} = \gamma^{(2)}_{U_1}, \quad (Bp_g)^* \gamma_{U_1}^{(2)} = g \gamma^{(2)}_{U_1}. \] (78) (79)

Formula (79) follows, by virtue of the Hurewicz and the Universal Coefficient Theorems, from the fact that the homomorphism \((Bp_g)_* : \pi_2 (BU1) \rightarrow \pi_2 (BU1)\) is given by multiplication by \(g\). Inserting Eqs. (78) and (79) into (77) we obtain
\[g (B\lambda_J)^* \gamma_{U_1}^{(2)} = E_m^{(2)} (\tilde{\gamma}_J) \quad \text{by (74)} \]
\[= g E_m^{(2)} (\tilde{\gamma}_J) \quad \text{by (73).} \]

Since this relation holds in \(H^2 (BUJ, \mathbb{Z})\) which is free Abelian, it implies (78).
Theorem 5.13 There holds the relation $\beta_g(\delta_J) = E^{(2)}_m(\gamma_J)$.

Proof: We compute

$$
\beta_g(\delta_J) = (B\lambda_j)^* (B_j g)^* \gamma_{\alpha_1}^{(2)} \big| \text{ by Corollary 5.10}
= (B_j g)^* (B\lambda_j)^* \gamma_{\alpha_1}^{(2)} \big| \text{ by (39)}
= (B_j g)^* E_{m,1}^{(2)} (\gamma_J) \big| \text{ by (73)}
= E_{m,1}^{(2)} (\gamma_J).
$$

5.7 Characteristic Classes for SUJ-Bundles

Using the cohomology elements $\gamma_{J,i}^{(2)}$ and $\delta_J$ constructed above, we define the following characteristic classes for SUJ-bundles over a manifold $M$:

$$
\alpha_{J,i} : \text{Bun}(M, SUJ) \to H^0_{\text{even}}(M, Z), \quad Q \mapsto (fQ)^* \gamma_{J,i}, \quad i = 1, \ldots, r \quad (80)
$$

$$
\xi_J : \text{Bun}(M, SUJ) \to H^1(M, Z), \quad Q \mapsto (fQ)^* \delta_J. \quad (81)
$$

Sorted by degree, $\alpha_{J,i}(Q) = 1 + \alpha_{J,i}^{(2)}(Q) + \cdots + \alpha_{J,i}^{(2k_i)}(Q)$, where $\alpha_{J,i}^{(2)}(Q) = (fQ)^* \gamma_{J,i}^{(2)}$. Moreover, we introduce the notation $\alpha_J(Q) = (\alpha_{J,1}(Q), \ldots, \alpha_{J,r}(Q))$. Then

$$
\alpha_J(Q) = (fQ)^* \gamma_J \quad (82)
$$

and $\alpha_J$ can be viewed as a map from Bun($M, SUJ$) to the set

$$
H^{(J)}(M, Z) = \prod_{i=1}^r \prod_{j=1}^{k_i} H^{2j}(M, Z). \quad (83)
$$

By construction, the relation which holds for $\gamma_J$ and $\delta_J$ carries over to the characteristic classes $\alpha_J$ and $\xi_J$. By virtue of (88), from Theorem 5.13 we infer

$$
E_{m,1}^{(2)} (\alpha_J(Q)) = \beta_g (\xi_J(Q)) \quad \forall Q \in \text{Bun}(M, SUJ). \quad (84)
$$

In order to derive expressions for $\alpha_J$ and $\xi_J$ in terms of the ordinary characteristic classes for $Uk_i$-bundles and $Z_g$-bundles, let $Q \in \text{Bun}(M, SUJ)$. There are two kinds of principal bundles associated in a natural way to $Q$: The $Uk_i$-bundles $Q_{[\text{pr}_i J, oj]}$, $i = 1, \ldots, r$, and the $Z_g$-bundle $Q^{[\lambda_j]}$. For the first ones, using (27) and (52) we compute

$$
c \left( Q_{[\text{pr}_i J, oj]} \right) = \left( f_{Q_{[\text{pr}_i J, oj]}} \right)^* \gamma_{Uk_i} = (fQ)^* \circ (B_j g)^* \circ (\text{pr}_i J)^* \gamma_{Uk_i} = (fQ)^* \gamma_{J,i},
$$

so that

$$
\alpha_{J,i}(Q) = c \left( Q_{[\text{pr}_i J, oj]} \right), \quad i = 1, \ldots, r. \quad (85)
$$

As for the second one, let $\chi_g$ denote the characteristic class for $Z_g$-bundles over $M$ defined by the generator $\delta_g \in H^1(BZ_g, Z_g)$, i.e.,

$$
\chi_g(R) = (fR)^* \delta_g, \quad R \in \text{Bun}(M, Z_g). \quad (86)
$$

Then (27) and (56) yield $\chi_g \left( Q^{[\lambda_j]} \right) = \left( f_{Q^{[\lambda_j]}} \right)^* \delta_g = (fQ)^* \circ (B\lambda_j)^* \delta_g = (fQ)^* \delta_J$. Consequently,

$$
\xi_J(Q) = \chi_g \left( Q^{[\lambda_j]} \right). \quad (87)
$$
5.8 Classification of SUJ-Bundles

We denote

\[ K(M)_J = \left\{ (\alpha, \xi) \in H^J(M, \mathbb{Z}) \times H^1(M, \mathbb{Z}_g) \mid E^{(2)}_m(\alpha) = \beta_g(\xi) \right\}. \]  

(88)

**Theorem 5.14** Let \( M \) be a manifold, \( \dim M \leq 4 \), and let \( J \in K(n) \). Then the characteristic classes \( \alpha_J \) and \( \xi_J \) define a bijection from \( \text{Bun}(M, SU) \) onto \( K(M)_J \).

*Proof:* The map is injective by Corollary 5.13. In the following lemma we prove that it is also surjective.

**Lemma 5.15** Let \( M \) be a manifold, \( \dim M \leq 4 \), and let \( J \in K(n) \). Let \( (\alpha, \xi) \in K(M)_J \). Then there exists \( Q \in \text{Bun}(M, SU) \) such that \( \alpha_J(Q) = \alpha \) and \( \xi_J(Q) = \xi \).

*Proof:* We give a construction of \( Q \) in terms of \( U_k \) and \( Z_g \)-bundles. There exists \( R \in \text{Bun}(M, Z_g) \) such that \( \chi_g(R) = \xi \). Due to \( \dim M \leq 4 \), there exist also \( Q_i \in \text{Bun}(M, U_k) \) such that \( c(Q_i) = \alpha_i, \ i = 1, \ldots, r \). Define \( \tilde{Q} = Q_1 \times_M \cdots \times_M Q_r \) (Whitney, or fibre, product). By identifying \( U_1 \times \cdots \times U_k \) with \( U \), \( \tilde{Q} \) becomes a UJ-bundle. Then

\[ \tilde{Q}^{[pr_U]_i} \cong Q_i, \ i = 1, \ldots, r. \]  

(89)

Consider the U1-bundles \( \tilde{Q}^{[\lambda_J]} \) and \( R^{[j_g]} \) associated to \( \tilde{Q} \) and \( R \), respectively. Assume, for a moment, that they are isomorphic. Then \( R \) is a subbundle of \( \tilde{Q}^{[\lambda_J]} \) with structure group \( Z_g \). Let \( Q \) denote the pre-image of \( R \) under the natural bundle morphism \( \tilde{Q} \to \tilde{Q}^{[\lambda_J]} \), see (25). This is a subbundle of \( \tilde{Q} \) with structure group being the pre-image of \( Z_g \) under \( \lambda_J \), i.e., with structure group SUJ. Using (87), \( Q^{[j_j]} = \tilde{Q} \), and (89), for \( i = 1, \ldots, r \),

\[ \alpha_{j_i}(Q) = c \left( Q^{[pr_U]_i} \right) = c \left( (\tilde{Q})^{[pr_U]_i} \right) = c(\tilde{Q}) = c(Q_i) = \alpha_i. \]

Moreover, by construction of \( Q \), \( Q^{[\lambda_J]} \cong R^{[j_g]} \). Thus, (87) yields \( \xi_J(Q) = \chi_J(R) = \xi \).

It remains to prove \( \tilde{Q}^{[\lambda_J]} \cong R^{[j_g]} \). We compute

\[ c_1 \left( \tilde{Q}^{[\lambda_J]} \right) = (f_{\tilde{Q}}^{[\lambda_J]})^* \gamma^{(2)}_{U_1} \]

\[ = (f_{\tilde{Q}})^* \circ (B_{\lambda_J})^* \gamma^{(2)}_{U_1} \]

\[ = (f_{\tilde{Q}})^* E^{(2)}_m(\tilde{\gamma}_J) \]

\[ = E^{(2)}_m \left( (f_{\tilde{Q}})^* \tilde{\gamma}_J \right) \]

\[ = E^{(2)}_m \left( (f_{\tilde{Q}})^* \circ (B_{pr_U}^{[U_J]})^{\gamma_{U_1}}, \cdots, (f_{\tilde{Q}})^* \circ (B_{pr_U}^{[U_J]})^{\gamma_{U_1}} \right) \]

\[ = E^{(2)}_m \left( (f_{\tilde{Q}_1})^* \gamma_{U_1}, \cdots, (f_{\tilde{Q}_r})^* \gamma_{U_1} \right) \]

\[ = E^{(2)}_m \left( c(Q_1), \cdots, c(Q_r) \right) \]

\[ = E^{(2)}_m(\alpha), \]
as well as

\[ c_1(R^{[j_\alpha]}) = (f_R^{[j_\alpha]})^* \gamma^{(2)}_{U_1} \]
\[ = (f_R)^* \circ (B_j)^* \gamma^{(2)}_{U_1} \quad \text{by (27)} \]
\[ = (f_R)^* \circ \beta_g(\delta_g) \quad \text{by (28)} \]
\[ = \beta_g \circ (f_R)^* \delta_g \quad \text{by (68)} \]
\[ = \beta_g(\gamma_g(R)) \quad \text{by (80)} \]
\[ = \beta_g(\xi). \]

Thus, due to \((\alpha, \xi) \in K(M)_J\), \(c_1(\tilde{Q}^{[\lambda_j]}) = c_1(R^{[j_\alpha]})\). It follows that, indeed, \(\tilde{Q}^{[\lambda_j]} \cong R^{[j_\alpha]}\).

This proves the lemma and, therefore, the theorem. \(\blacksquare\)

### 5.9 Classification of SU\(J\)-Subbundles of SU\(_n\)-Bundles

Let \(P\) be a principal SU\(_n\)-bundle over a manifold \(M\) and let \(J \in K(n)\). We are going to characterize the subset \(\text{Red}(P, SUJ) \subseteq \text{Bun}(M, SUJ)\) in terms of the characteristic classes \(\alpha_J\) and \(\xi_J\). Recall that for \(Q \in \text{Bun}(M, SUJ)\), \(Q^{[SUJ]}\) denotes the extension of \(Q\) by SU\(_n\).

**Lemma 5.16** For any \(Q \in \text{Bun}(M, SUJ)\), \(c(Q^{[SUJ]}) = E_m(\alpha_J(Q))\).

**Proof:** Note that \(c(Q^{[SUJ]}) = c(Q^{[Un]}) = c(Q^{[1]}).\) Hence,

\[ c(Q^{[SUJ]}) = (f_Q)^* \circ (B_j)^* \circ (B_j)^* \gamma_{Un} \quad \text{by (68)} \]
\[ = (f_Q)^* \circ (B_j)^* \gamma_{Jj} \quad \text{by (68)} \]
\[ = E_m((f_Q)^* \gamma_{Jj}) \quad \text{by (68)} \]
\[ = E_m(\alpha_J(Q)) \quad \text{by (82)}. \]

We define

\[ K(P)_J = \{ (\alpha, \xi) \in K(M)_J \mid E_m(\alpha) = c(P) \}. \]

**Theorem 5.17** Let \(P\) be a principal SU\(_n\)-bundle over a manifold \(M\), \(\dim M \leq 4\), and let \(J \in K(n)\). Then the characteristic classes \(\alpha_J, \xi_J\) define a bijection from \(\text{Red}(P, SUJ)\) onto \(K(P)_J\).

**Proof:** Let \(Q \in \text{Bun}(M, SUJ)\). Then \((\alpha_J(Q), \xi_J(Q)) \in K(M)_J\). Lemma 5.16 implies that \((\alpha_J(Q), \xi_J(Q)) \in K(P)_J\) if and only if \(c(Q^{[SUJ]}) = c(P)\). Due to \(\dim M \leq 4\), the latter is equivalent to \(Q^{[SUJ]} \cong P\), i.e., to \(Q \in \text{Red}(P, SUJ)\). \(\blacksquare\)

The equation \(E_m(\alpha) = c(P)\) actually contains the two equations \(E_m(2)(\alpha) = 0\) and \(E_m(4)(\alpha) = c_2(P)\). However, under the assumption that \((\alpha, \xi) \in K(M)_J\), the first one
is redundant, because then, due to (71), \( E_m^{(2)}(\alpha) = g E_{m'}^{(2)}(\alpha) = g \beta_g(\xi) = 0 \). Thus, the relevant equations are

\[
\begin{align*}
E_m^{(2)}(\alpha) &= \beta_g(\xi), \\
E_m^{(4)}(\alpha) &= c_2(P),
\end{align*}
\]

where \( \alpha \in H^{(j)}(M, \mathbb{Z}), \xi \in H^1(M, \mathbb{Z}_g) \). The set of solutions of Eq. (71) yields \( K(M)_J \), hence \( \text{Bun}(M, \text{SU}_J) \). The set of solutions of both Eqs. (71) and (72) yields \( K(P)_J \) and, therefore, \( \text{Red}(P, \text{SU}_J) \).

This concludes the classification of Howe subbundles of \( P \), i.e., Step 2 of our programme.

### 5.10 Examples

We are going to determine \( K(P)_J \), i.e., to solve the system of equations (71) and (72), for several choices of \( J \) and for base manifolds \( M = S^4, S^2 \times S^2, T^4, \) and \( L_p^3 \times S^1 \). Here \( L_p^3 \) denotes the 3-dimensional lens space which is defined to be the quotient of the restriction of the natural action of \( U_1 \) on the sphere \( S^3 \subset \mathbb{C}^2 \) to the subgroup \( \mathbb{Z}_p \). Note that \( L_p^3 \) is orientable. We remark that there are more general lens spaces, even in 3 dimensions.

#### Preliminary Remarks

Due to compactness and orientability, \( H^4(M, \mathbb{Z}) \cong \mathbb{Z} \). Let us derive the Bockstein homomorphism \( \beta_g : H^1(M, \mathbb{Z}_g) \rightarrow H^2(M, \mathbb{Z}) \). Since for products of spheres the integer-valued cohomology is torsion-free, \( \beta_g \) is trivial here. On the other hand, consider \( M = L_p^3 \times S^1 \). From the exact homotopy sequence induced by the fibration \( \mathbb{Z}_p \hookrightarrow S^3 \rightarrow L_p^3 \) we infer

\[
\pi_1(L_p^3) \cong \mathbb{Z}_p.
\]

According to the Hurewicz and Universal Coefficient Theorems, then

\[
H^1(L_p^3, \mathbb{Z}_g) \cong \text{Hom}(\pi_1(L_p^3), \mathbb{Z}_g) \cong \text{Hom}(\mathbb{Z}_p, \mathbb{Z}_g) \cong \mathbb{Z}_{(p,g)},
\]

where \((p,g)\) denotes the greatest common divisor of \( p \) and \( g \). Let \( \gamma_{L_p^3,\mathbb{Z}_g}^{(1)} \) and \( \gamma_{S^1}^{(1)} \) be generators of \( H^1(L_p^3, \mathbb{Z}_g) \) and \( H^1(S^1, \mathbb{Z}) \), respectively. Due to the Künneth Theorem for cohomology [25, Ch. XIII, Thm. 11.2] and (73),

\[
H^1(L_p^3 \times S^1, \mathbb{Z}_g) \cong H^1(L_p^3, \mathbb{Z}_g) \otimes H^0(S^1, \mathbb{Z}) \oplus H^0(L_p^3, \mathbb{Z}_g) \otimes H^1(S^1, \mathbb{Z})
\]

\[
\cong \mathbb{Z}_{(p,g)} \oplus \mathbb{Z}_g,
\]

where the first factor is generated by \( \gamma_{L_p^3,\mathbb{Z}_g}^{(1)} \times 1_{S^1} \) and the second one by \( 1_{L_p^3,\mathbb{Z}_g} \times \gamma_{S^1}^{(1)} \).

Consider the following portion of the exact sequence (60):

\[
H^1(L_p^3, \mathbb{Z}) \xrightarrow{\partial_g} H^1(L_p^3, \mathbb{Z}_g) \xrightarrow{\beta_g} H^2(L_p^3, \mathbb{Z}).
\]

One has, see [8, §10, p. 363],

\[
H^1(L_p^3, \mathbb{Z}) = 0, \quad H^2(L_p^3, \mathbb{Z}) \cong \mathbb{Z}_p
\]

(97)
(the first equality follows also from (93)). In view of (97) and (94), (96) implies
\[
\beta_g \left( \gamma_{1_p, Zg}^{(1)} \right) = \frac{p}{\langle p, g \rangle} \gamma_{1_p, Z}^{(2)},
\]
where \( \gamma_{1_p, Z}^{(2)} \) is an appropriately chosen generator of \( H^2(L^3_p, \mathbb{Z}) \). Moreover, due to (97), \( \beta_g(1_{L^3_p, Zg}) = 0 \). Thus, we obtain
\[
\beta_g \left( \gamma_{1_p, Zg}^{(1)} \times 1_{S^1} \right) = \frac{p}{\langle p, g \rangle} \gamma_{1_p, Z}^{(2)} \times 1_{S^1}, \quad \beta_g \left( 1_{L^3_p, Zg} \times \gamma_{S^1}^{(1)} \right) = 0.
\]
Finally, note that via the Künneth Theorem, (97) implies \( H^2(L^3_p \times S^1, \mathbb{Z}) \cong \mathbb{Z}_p \). Since \( H^4(L^3_p \times S^1, \mathbb{Z}) \) is torsion-free, then the cup product \( \alpha^{(2)}_1 \smile \alpha^{(2)}_2 \) is trivial.

Now let us discuss some special choices for \( J \). For brevity, we write \( J = (1|n) \in K(n) \) where \( SUJ = \mathbb{Z}_n \), the center of \( SUn \). Moreover, \( g = n \). Variables are \( \xi \in H^1(M, \mathbb{Z}_n) \) and \( \alpha = 1 + \alpha^{(2)}_i, \alpha^{(2)}_i \in H^2(M, \mathbb{Z}) \). The system of equations (91) and (92) reads
\[
\alpha^{(2)} = \beta_n(\xi), \quad \frac{n(n-1)}{2} \alpha^{(2)}_1 \smile \alpha^{(2)}_2 = c_2(P).
\]
Note that we have used Lemma 5.11. Eq. (100) merely expresses \( \alpha^{(2)} \) in terms of \( \xi \). In particular, it yields \( n \alpha^{(2)} = 0 \), so that Eq. (101) requires \( c_2(P) = 0 \). As a result, \( K(P)_J \) is nonempty iff \( P \) is trivial and is then parametrized by \( \xi \). This coincides with what is known about \( \mathbb{Z}_n \)-subbundles of \( SUn \)-bundles.

\( J = (n|1) \in K(n) \) Here \( SUJ = SUn \), the whole group. Due to \( g = 1 \), the only variable is \( \alpha = 1 + \alpha^{(2)} + \alpha^{(4)} \), where \( \alpha^{(2j)} \in H^{2j}(M, \mathbb{Z}) \). The system of equations (91) and (92) is
\[
\alpha^{(2)}_1 + \alpha^{(2)}_2 = \beta_2(\xi), \quad \alpha^{(4)} = c_2(P).
\]
Thus, \( K(P)_J \) consists of \( P \) itself.

\( J = (1, 1|2, 2) \in K(4) \) One can check that \( SUJ \) has connected components
\[
\{ \text{diag}(z, z, z^{-1}, z^{-1}) | z \in U1 \}, \quad \{ \text{diag}(z, z, -z^{-1}, -z^{-1}) | z \in U1 \}.
\]
It is, therefore, isomorphic to \( U1 \times \mathbb{Z}_2 \). Variables are \( \xi \in H^1(M, \mathbb{Z}_2) \) and \( \alpha_i = 1 + \alpha^{(2)}_i, \alpha^{(2)}_i \in H^2(M, \mathbb{Z}) \). The system of equations under consideration is
\[
\alpha^{(2)}_1 + \alpha^{(2)}_2 = \beta_2(\xi), \quad \alpha^{(2)}_1 \smile \alpha^{(2)}_2 + 4 \alpha^{(2)}_1 \smile \alpha^{(2)}_2 = c_2(P).
\]
We solve Eq. (102) w.r.t. \( \alpha^2 \) and insert it into Eq. (103). Since \( H^4(M, \mathbb{Z}) \) is torsion-free, products including \( \beta_2(\xi) \) vanish. Thus, we obtain

\[
-2\alpha_1^{(2)} \sim \alpha_1^{(2)} = c_2(P).
\] (104)

For base manifold \( M = S^1 \), \( H^2(M, \mathbb{Z}) = 0 \). Hence, \( K(P)_J \) is nonempty iff \( c_2(P) = 0 \), in which case it contains the (necessarily trivial) \( U1 \times \mathbb{Z}_2 \)-bundle over \( S^1 \).

For \( M = L^\mathbb{Z}_p \times S^1 \), in case \( c_2(P) = 0 \), \( K(P)_J \) is parametrized by \( \xi \in H^1(M, \mathbb{Z}_p) \cong \mathbb{Z}(2p) \oplus \mathbb{Z}_2 \) and \( \alpha_1^{(2)} \in H^2(M, \mathbb{Z}) \cong \mathbb{Z}_p \). Otherwise it is empty.

For \( M = S^2 \times S^2 \), \( H^1(M, \mathbb{Z}_p) = 0 \). Let \( \gamma_s^{(2)} \) be a generator of \( H^2(S^2, \mathbb{Z}) \). Due to the Künneth Theorem, \( H^2(M, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \), where it is generated by \( \gamma_s^{(2)} \times 1_{S^2} \) and \( 1_{S^2} \times \gamma_s^{(2)} \).

Moreover, \( H^4(M, \mathbb{Z}) \) is generated by \( \gamma_s^{(2)} \times \gamma_s^{(2)} \). Writing

\[
\alpha_1^{(2)} = a \gamma_s^{(2)} \times 1_{S^2} + b \ 1_{S^2} \times \gamma_s^{(2)}
\] (105)

with \( a, b \in \mathbb{Z} \), Eq. (104) becomes

\[
-4ab \gamma_s^{(2)} \times \gamma_s^{(2)} = c_2(P).
\]

If \( c_2(P) = 0 \), there are two series of solutions: \( a = 0 \) and \( b \in \mathbb{Z} \) as well as \( a \in \mathbb{Z} \) and \( b = 0 \). Here \( K(P)_J \) is infinite. If \( c_2(P) = 4l \gamma_s^{(2)} \times \gamma_s^{(2)} \), \( l \neq 0 \), then \( a = q \) and \( b = -l/q \), where \( q \) runs through the (positive and negative) divisors of \( l \). Hence, in this case, the cardinality of \( K(P)_J \) is twice the number of divisors of \( l \). If \( c_2(P) \) is not divisible by 4 then \( K(P)_J \) is empty.

Finally, for \( M = T^4 \) one has \( H^1(M, \mathbb{Z}) \cong \mathbb{Z}^{\mathbb{Z}_4} \) and \( H^2(M, \mathbb{Z}) \cong \mathbb{Z}^{\mathbb{Z}_6} \). Moreover, \( H^2(M, \mathbb{Z}) \) is generated by elements \( \gamma_{12}^{(2)} \), \( 1 \leq i < j \leq 4 \), where

\[
\gamma_{12}^{(2)} = \gamma_{11}^{(1)} \times \gamma_{11}^{(1)} \times 1_{S^1} \times 1_{S^1}, \quad \gamma_{13}^{(2)} = \gamma_{22}^{(1)} \times 1_{S^1} \times \gamma_{22}^{(1)} \times 1_{S^1}, \quad \text{etc.,}
\]

whereas \( H^4(M, \mathbb{Z}) \) is generated by \( \gamma_{14}^{(4)} = \gamma_{11}^{(1)} \times \gamma_{11}^{(1)} \times \gamma_{11}^{(1)} \times \gamma_{11}^{(1)} \). One can check

\[
\gamma_{12}^{(2)} \sim \gamma_{13}^{(2)} = \epsilon_{ijkl} \gamma_{14}^{(4)},
\] (106)

where \( \epsilon_{ijkl} \) denotes the totally antisymmetric tensor in 4 dimensions. Writing

\[
\alpha_1^{(2)} = \sum_{1 \leq i < j \leq 4} a_{ij} \gamma_{12}^{(2)}
\] (107)

and using (106), Eq. (104) yields

\[
-4(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) \gamma_{14}^{(4)} = c_2(P).
\]

Hence, we find that \( K(P)_J \) is again nonempty iff \( c_2(P) \) is divisible by 4, in which case it now has always infinitely many elements.

To conclude, let us point out that, when passing to \( \tilde{K}(P) \), one has to identify the symbol containing \( (\alpha_1, \alpha_2) \) with that containing \( (\alpha_2, \alpha_1) \).
\( J = (1,1|2,3) \in K(5) \) The subgroup SUJ of SU5 consists of matrices of the form diag\((z_1, z_1, z_2, z_2, z_2)\), where \( z_1, z_2 \in U1 \) such that \( z_1^2 z_2^3 = 1 \). We can parametrize \( z_1 = z^3, z_2 = z^{-2}, z \in U1 \). Hence, SUJ is isomorphic to U1. Variables are \( \alpha_i = 1 + \alpha_{i}^{(2)}, i = 1, 2 \). The equations to be solved read
\[
2\alpha_{1}^{(2)} + 3\alpha_{2}^{(2)} = 0 \quad (108)
\]
\[
\alpha_{1}^{(2)} \sim \alpha_{1}^{(2)} + 3\alpha_{2}^{(2)} \sim \alpha_{2}^{(2)} + 6\alpha_{1}^{(2)} \sim \alpha_{2}^{(2)} = c_2(P) \quad (109)
\]
Eq. (108) can be parametrized by \( \alpha_{1}^{(2)} = 3\eta, \alpha_{2}^{(2)} = -2\eta \), where \( \eta \in H^2(M, \mathbb{Z}) \). Then Eq. (109) becomes
\[
-15\eta \sim \eta = c_2(P).
\]
The discussion of this equation is analogous to that of Eq. (104) above.

\( J = (2,3|1,1) \in K(5) \) Here SUJ \( \cong S(U2 \times U3) \), the symmetry group of the standard model. In the grand unified SU5-model this is the subgroup to which SU5 is broken by the heavy Higgs field. Moreover, the subgroup SUJ is the centralizer of the subgroup SU\((1,1|2,3)\) discussed above. Variables are \( \alpha_i = 1 + \alpha_{i}^{(2)} + \alpha_{i}^{(4)}, \) where \( \alpha_{i}^{(2)} \in H^2(M, \mathbb{Z}), i, j = 1, 2 \). Eqs. (110) and (111) read
\[
\alpha_{1}^{(2)} + \alpha_{2}^{(2)} = 0, \quad (110)
\]
\[
\alpha_{1}^{(4)} + \alpha_{2}^{(4)} + \alpha_{1}^{(2)} \sim \alpha_{2}^{(2)} = c_2(P). \quad (111)
\]
Using (110) to replace \( \alpha_{2}^{(2)} \) in (111) we obtain for the latter
\[
\alpha_{2}^{(4)} = c_2(P) - \alpha_{1}^{(4)} \sim \alpha_{1}^{(2)}.
\]
Thus, K\((P)_J \) can be parametrized by \( \alpha_1 \) (or \( \alpha_2 \)), i.e., by the Chern class of one of the factors U2 or U3. This has been known for a long time [20].

\( J = (2|2) \) The subgroup SUJ of SU4 consists of matrices \( D \oplus D \), where \( D \in U2 \) such that \( (\det D)^2 = 1 \). Hence, it has connected components \( \{ D \oplus D | D \in SU2 \} \) and \( \{(iD) \oplus (iD) | D \in SU2 \} \). One can check that the map SU\(2 \times \mathbb{Z}_4 \rightarrow SUJ, (D, a) \mapsto e^{2\pi i a/4}D \), induces an isomorphism from \( SU2 \times \mathbb{Z}_4 / \mathbb{Z}_2 \) onto SUJ.

Now consider K\((P)_J \). Variables are \( \xi \in H^1(M, \mathbb{Z}_2) \) and \( \alpha = 1 + \alpha_{(2)} + \alpha^{(4)} \). We have
\[
\alpha_{(2)} = \beta_2(\xi) \quad (112)
\]
\[
\alpha_{(2)} \sim \alpha_{(2)} + 2\alpha^{(4)} = c_2(P) \quad (113)
\]
Eq. (112) fixes \( \alpha_{(2)} \) in terms of \( \xi \). For example, in case \( M = L_4^3 \times S^1 \), by expanding \( \xi = \xi_L \gamma_{L,4;3}^{(1)} \times 1_{S^1} + \xi_S 1_{L,4;3} \times \gamma_{S;1}^{(1)} \), Eqs. (99) and (112) imply
\[
\alpha_{(2)} = \begin{cases} 
q\xi_L \gamma_{L,4;3}^{(2)} \times 1_{S^1} & | \ p = 2q \\
0 & | \ p = 2q + 1.
\end{cases}
\]
For general \( M \), due to (112), Eq. (113) becomes
\[
2\alpha^{(4)} = c_2(P).
\]
Thus, \( K(P) \) is nonempty iff \( c_2(P) \) is even and is then parametrized by \( \xi \in H^1(M, \mathbb{Z}_2) \).

Let us point out the following. Consider Eq. (112) which defines the set \( K(M) \) classifying SUJ-bundles over \( M \). In case \( M \) is a product of spheres (or, more generally, has trivial \( \beta_2 \) in degree 1), this classification coincides with that of \((\text{SU}2 \times \mathbb{Z}_2)\)-bundles, although this group is only locally isomorphic to SUJ. In case \( M = L^3_p \times S^1 \), there is a difference though. Here the SUJ-bundles which are nontrivial over the factor \( L^3_p \) have a magnetic charge, whereas an \((\text{SU}2 \times \mathbb{Z}_2)\)-bundle can never have one.

### 6 Holonomy-Induced Howe Subbundles

In the next step of our programme we have to specify the Howe subbundles which are holonomy-induced. However, in fact, this turns out not to be necessary here because for Howe subbundles of principal \( \text{SU}n \)-bundles the condition of being holonomy-induced is redundant. This will be proved now.

**Lemma 6.1** Let \( H \subseteq H' \subseteq \text{SU}n \) be Howe subgroups. If \( \dim H = \dim H' \) then \( H = H' \).

**Proof:** There exist \( J, J' \in K(n) \) such that \( H \) and \( H' \) are conjugate to \( \text{SU}J \) and \( \text{SU}J' \), respectively. Consider \( UJ \) and \( UJ' \). Due to \( H \subseteq H' \), there exists \( D \in \text{SU}n \) such that \( D^{-1}UJD \subseteq UJ' \). Moreover,
\[
\dim UJ' = \dim SUJ' + 1 = \dim H' + 1 = \dim H + 1 = \dim SUJ + 1 = \dim UJ.
\]
(114)

\( UJ' \) being connected and \( D^{-1}UJD \) being closed in \( UJ' \), (114) implies \( D^{-1}UJD = UJ' \). Then \( D^{-1}\text{SU}JD = D^{-1}(UJ \cap \text{SU}n)D = (D^{-1}UJD) \cap \text{SU}n = UJ' \cap \text{SU}n = SUJ' \). It follows \( H = H' \). \( \blacksquare \)

**Theorem 6.2** Any Howe subbundle of a principal \( \text{SU}n \)-bundle is holonomy-induced.

**Proof:** Let \( P \) be a principal \( \text{SU}n \)-bundle and let \( Q \) be a Howe subbundle of \( P \) with structure group \( H \). Denote the structure group of a connected component of \( Q \) by \( \tilde{H} \). It is easily seen that \( Q \) is holonomy-induced provided
\[
C_{\text{SU}n}(\tilde{H}) = H.
\]
(115)

Since \( H \) is a Howe subgroup, we have \( \tilde{H} \subseteq C_{\text{SU}n}(\tilde{H}) \subseteq H \). Since \( \tilde{H} \) and \( H \) are of the same dimension, so are \( C_{\text{SU}n}(\tilde{H}) \) and \( H \). By virtue of Lemma 6.1 this implies (115). \( \blacksquare \)

For the reader who wonders whether there exist Howe subbundles which are not holonomy-induced we give an example here. Consider the Lie group SO3. One checks that the subgroup \( H = \{ \text{I}_3, \text{diag}(-1, -1, 1) \} \) is Howe. Thus, the subbundle \( Q = M \times H \) of the bundle \( M \times \text{SO3} \) is Howe. Any connected subbundle \( \tilde{Q} \) of \( Q \) has the center \( \{ \text{I}_3 \} \) as its structure group. Since the center is Howe itself, \( \tilde{Q} \cdot C_{\tilde{Q}}(\{ \text{I}_3 \}) = \tilde{Q} \neq Q \).
7 Factorization by SU\textsuperscript{n}-Action

In Step 4 of our programme to determine Howe\textsubscript{\ast}(P), we actually have to take the disjoint union of Red(P,H) over all Howe subgroups H of SU\textsubscript{n} and to factorize by the action of SU\textsubscript{n}. Since SU\textsubscript{n}-action on subbundles conjugates their structure groups, however, it suffices to take the union only over SU\textsubscript{J}, J \in K(n):

\[ \bigsqcup_{J \in K(n)} \text{Red}(P, SU_J). \] (116)

According to this, define

\[ K(P) = \bigsqcup_{J \in K(n)} K(P)_J. \] (117)

We shall denote the elements of K(P) by L and write them in the form 

\[ L = (J; \alpha, \xi), \]

where \( J \in K(n) \) and \( (\alpha, \xi) \in K(P)_J \). Due to Theorem 5.17, the collection of characteristic classes \( \{ (\alpha_J, \xi_J) | J \in K(n) \} \) defines a bijection from (116) onto K(P). Now we reverse this bijection: Let \( L \in K(P), L = (J; \alpha, \xi) \). Then define \( Q_L \) to be the isomorphism class of SU\textsubscript{J}-subbundles of P which obey

\[ \alpha_J(Q_L) = \alpha, \quad \xi_J(Q_L) = \xi. \] (118)

**Lemma 7.1** Let \( L, L' \in K(P) \), where \( L = (J; \alpha, \xi) \) and \( L' = (J'; \alpha', \xi') \). There exists \( D \in SU_n \) such that \( Q_{L'} \cong Q_L \cdot D \) (up to isomorphy) if and only if \( \xi' = \xi \), as well as \( J' = \sigma J \) and \( \alpha' = \sigma \alpha \) for some permutation \( \sigma \) of 1, \ldots, r.

**Proof:** For \( J_1, J_2 \in K(n) \) and \( D \in SU_n \) such that \( D^{-1} SU_1 D \subseteq SU_2 \), define embeddings

\[ h_{J_1 J_2}^D : UJ_1 \to UJ_2, \quad C \mapsto D^{-1} CD \]

\[ h_{J_1 J_2}^{DS} : SUJ_1 \to SUJ_2, \quad C \mapsto D^{-1} CD. \]

First, assume that we are given \( D \in SU_n \) such that \( Q_{L'} \cong Q_L \cdot D \). Then \( D^{-1} SUJ D = SUJ' \), so that we can consider the isomorphisms \( h_{J_1 J_2}^{DS} : SUJ \to SUJ' \) and \( h_{J_1 J_2}^D : UJ \to UJ' \).

One can check that

\[ Q_L \cdot D \cong Q_L [h_{J_1 J_2}^{DS}]^{-1}. \] (119)

Moreover, there exists a permutation \( \sigma \) of 1, \ldots, r such that \( h_{J_1 J_2}^D \) maps the \( \sigma(i) \)-th factor of UJ isomorphically onto the \( i \)-th factor of UJ'. Then, in particular, \( J' = \sigma J \). Moreover, there exists \( C \in SUn \) such that \( pr_{i}^{UJ'} \circ h_{J_1 J_2}^{DS} = pr_{i}^{UJ} \) \( \forall i \) (in fact, \( C \) has been constructed in the proof of Lemma 4.2). Then

\[ pr_{i}^{UJ'} \circ j_J \circ h_{J_1 J_2}^{DS} = pr_{i}^{UJ} \circ j_J. \] (120)

Next, define \( B = DC^{-1} \in SU_n \). The corresponding homomorphism \( h_{J_1 J_2}^D \) is an automorphism of UJ which leaves each factor invariant separately. One can check that it is an
Analogously, using (123), we find $\xi$.

To prove the converse implication, assume $\xi' = \xi$ and let $\sigma$ be a permutation of $1, \ldots, r$ such that $J' = \sigma J$ and $\alpha' = \sigma \alpha$. Due to Lemma 7.1, there exists $D \in \text{SU}n$ such that $\text{SU}J' = D^{-1} \text{SU}J \cdot D$. As shown in the proof of this lemma, $D$ can be chosen in such a way that (120) holds. Then (122) holds, too. We compute the characteristic classes of $Q_L \cdot D$:

$$
\alpha_{J', i} (Q_L \cdot D) = \alpha_{J', i} (Q_L[H_{J', J'}]) \quad | \quad \text{by (119)}
$$

$$
= \alpha_{J, \sigma(i)} (Q_L) \quad | \quad \text{by (122)}
$$

$$
= \alpha_{\sigma(i)} \quad | \quad \text{by (118)}
$$

$$
= \alpha' \quad | \quad \text{by assumption}
$$

$$
= \alpha_{J', i} (Q_L') \quad | \quad \text{by (118)}.
$$

Analogously, using (123), we find $\xi' (Q_L \cdot D) = \xi (Q_L')$. It follows $Q_L \cdot D \cong Q_{L'}$.

As suggested by Lemma 7.1, we introduce an equivalence relation on the set $K(P)$: Let $L, L' \in K(P)$, where $L = (J; \alpha, \xi)$ and $L' = (J'; \alpha', \xi')$. Write $L \sim L'$ iff $\xi' = \xi$ and there exists a permutation $\sigma$ of $1, \ldots, r$ such that $J' = \sigma J$ and $\alpha' = \sigma \alpha$. The set of equivalence classes will be denoted by $K(P)$.
Theorem 7.2  The assignment \( L \mapsto Q_L \) induces a bijection from \( \hat{K}(P) \) onto Howe\(_*\)(\( P \)).

Proof: The assignment \( L \mapsto Q_L \) induces a map \( K(P) \to Howe\(_*\)(P) \). This map is surjective by construction. Due to Lemma 7.1, it projects to \( \hat{K}(P) \) and the projected map is injective.

With Theorem 7.2 we have accomplished the determination of Howe\(_*\)(\( P \)) and, therefore, of the set of orbit types \( OT(A^k, G^{k+1}) \). Calculations for the latter can now be performed entirely on the level of the classifying set \( \hat{K}(P) \).

8 Example: Gauge Orbit Types for SU2

In this section, we are going to determine \( OT(A^k, G^{k+1}) \) for an SU2-gauge theory over the base manifolds discussed in Subsection 5.10. The set \( K(2) \) contains the elements

\[
J_a = (1|2), \quad J_b = (1, 1|1, 1), \quad J_c = (2|1).
\]

Here \( SUJ_a = \{ \pm 1 \} \cong \mathbb{Z}_2 \) is the center, \( SUJ_b = \{ \text{diag}(z, z^{-1}) \mid z \in U1 \} \cong U1 \) is the toral subgroup and \( SUJ_c = SU2 \). The strata corresponding to the elements of \( K(P)_a \), \( K(P)_b \), \( K(P)_c \) are, in the respective order, those with stabilizers isomorphic to \( SU2, U1 \), and the generic stratum. Accordingly, we shall refer to the first class as SU2-strata and to the second class as U1-strata. We have

\[
K(P) = K(P)_a \cup K(P)_b \cup K(P)_c
\]

(disjoint union). As we already know, \( K(P)_a \) is parametrized by \( \xi \in H^1(M, \mathbb{Z}_2) \) in case \( P \) is trivial and is empty otherwise. Moreover, \( K(P)_c \) consists of \( P \) itself. For \( K(P)_b \), variables are \( \alpha_i = 1 + \alpha_i^{(2)} \), \( \alpha_i^{(2)} \in H^2(M, \mathbb{Z}) \), \( i = 1, 2 \). The system of equations (91), (92) reads

\[
\begin{align}
\alpha_1^{(2)} + \alpha_2^{(2)} &= 0 \quad (124) \\
\alpha_1^{(2)} \sim \alpha_2^{(2)} &= c_2(P) \quad (125)
\end{align}
\]

Using (124) to replace \( \alpha_2^{(2)} \) in (125) we obtain

\[
- \alpha_1^{(2)} \sim \alpha_1^{(2)} = c_2(P). \quad (126)
\]

Note that here \( \alpha_1^{(2)} \) is just the first Chern class of a reduction of \( P \) to the subgroup U1. According to this, Eq. (126) has been derived in the discussion of spontaneous symmetry breaking of SU2 to U1, see [20]. Note also that when passing from \( K(P) \) to \( \hat{K}(P) \), the pairs \( (\alpha_1, \alpha_2) \) and \( (\alpha_2, \alpha_1) \) label the same class of subbundles. Hence, solutions \( \alpha_1^{(2)} \) of (126) have to be identified with their negative.

Let us now discuss Eq. (126) as well as the set \( \hat{K}(P) \) in dependence of \( P \) for some specific base manifolds \( M \).
\( M = S^4 \) Since \( H^2(M,\mathbb{Z}) = 0 \), Eq. (126) requires \( c_2(P) = 0 \). Thus, in case \( P \) is trivial, \( \hat{K}(P) \) contains the \( \mathbb{Z}_2 \)-bundle, the U1-bundle (both necessarily trivial) and \( P \) itself. Accordingly, in the gauge orbit space there exist, besides the generic stratum, an SU2-stratum and a U1-stratum.

We remark that, due to the base manifold being a sphere, this result can be obtained much easier by homotopy arguments. It is, therefore, well known. The structure of the gauge orbit space in the present situation has been studied in great detail in [14]. It has been shown that the two nongeneric strata of the gauge orbit space can be parametrized by means of an affine subspace of \( \mathcal{A}^k \) which is acted upon by the Weyl group of SU2 (the latter being just \( \mathbb{Z}_2 \)).

In case \( P \) is nontrivial, both \( K(P)_a \) and \( K(P)_b \) are empty, so that \( \hat{K}(P) \) contains only the bundle \( P \) itself. Accordingly, there do not exist nongeneric strata in the gauge orbit space.

\( M = S^2 \times S^2 \) We write \( \alpha_1^{(2)} \) in the form (103). Then Eq. (126) becomes

\[-2ab \gamma_2^{(2)} \times \gamma_2^{(2)} = c_2(P).\]

Thus, if \( P \) is trivial then \( \hat{K}(P) \) contains, in addition to \( P \) itself, the \( \mathbb{Z}_2 \)-bundle, which is trivial, and the U1-bundles labelled by \( a \geq 0, b = 0 \) and \( a = 0, b \geq 0 \), i.e., which are trivial over one of the 2-spheres. The corresponding nongeneric strata in the gauge orbit space are one SU2-stratum and infinitely many U1-strata.

In case \( c_2(P) = 2l \gamma_2^{(2)} \times \gamma_2^{(2)}, l \neq 0 \), \( \hat{K}(P) \) contains the U1-bundles with \( a = q \) and \( b = -l/q \), where \( q \) is a (positive) divisor of \( m \). Hence, here the nongeneric part of the gauge orbit space consists of finitely many U1-strata.

Finally, in case \( c_2(P) = (2m + 1) \gamma_2^{(2)} \times \gamma_2^{(2)}, \hat{K}(P) \) contains only \( P \) itself and the gauge orbit space consists only of the generic stratum.

\( M = T^4 \) We use the notation introduced in Subsection 5.10. Writing \( \alpha_1^{(2)} \) in the form (107) and using (106), Eq. (126) reads

\[-2 \left( a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \right) \gamma_4^{(4)} = c_2(P).\]

Hence, the result is similar to that for the case \( M = S^2 \times S^2 \). The only difference is that, due to \( H^1(M,\mathbb{Z}_2) \cong \mathbb{Z}_2^4 \), there exist 16 different \( \mathbb{Z}_2 \)-bundles which are all contained in \( \hat{K}(P) \) for \( P \) being trivial. Accordingly, in this case the gauge orbit space contains 16 SU2-strata. Moreover, in case \( c_2(P) = 2m \gamma_4^{(4)}, m \neq 0 \), the number of solutions of (127) is infinite. Therefore, in this case there exist infinitely many U1-strata in the gauge orbit space.

We remark that pure Yang-Mills theory on \( T^4 \) has been discussed in [12, 13], where the authors have studied the maximal Abelian gauge for gauge group SU\( n \). They have found that in case \( P \) is nontrivial this gauge fixing is necessarily singular on Dirac strings joining magnetically charged defects in the base manifold. It is an interesting question whether there is a relation between such defects and the nongeneric strata of the gauge orbit space.
$M = L^3_p \times S^1$ Here (126) requires $c_2(P) = 0$. Thus, in case $P$ is trivial $\hat{K}(P)$ contains, in addition to $P$ itself, the $\mathbb{Z}_2$-bundles over $L^3_p \times S^1$, which are labelled by the elements of

$$H^1(L^3_p \times S^1, \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & p \neq 0, \text{ even} \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

as well as the $U_1$-bundles over $L^3_p \times S^1$, which are labelled by the elements of $H^1(L^3_p \times S^1, \mathbb{Z}_2) \cong \{ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \mid p \neq 0, \text{ even} \}$.

In case $P$ is trivial there do not exist nongeneric strata in the gauge orbit space.

9 Application: Kinematical Nodes in 2+1 Dimensional Chern-Simons Theory

Following [3], we consider Chern-Simons theory with gauge group $SU_n$ in the Hamiltonian approach. The Hamiltonian in Schrödinger representation is given by

$$H = -\frac{\Lambda}{2} \int_{\Sigma_s} \frac{d^2x}{\sqrt{h}} \text{Tr} \left[ \frac{\delta}{\delta A_\mu} + \frac{i\ell}{4\pi} \epsilon^{\mu\nu} A_\nu \right]^2 + \frac{1}{4\Lambda} \int_{\Sigma_s} d^2x \sqrt{h} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) .$$

Geometrically, $A_\mu$ and $F_{\mu\nu}$ are the local representatives of a connection $A$ in a given $SU_n$-bundle $P$ over the 2 dimensional space $\Sigma_s$ and its curvature $F$, respectively. We assume $\Sigma_s$ to be a Riemann surface of genus $s$. Note that, due to the dimension of $\Sigma_s$, $c_2(P) = 0$, i.e., $P$ is trivial. The configuration space is the gauge orbit space $M_k$ associated to $P$. Physical states are given by cross sections of a complex line bundle $\eta$ over (the topological space) $M^k$ with first Chern class $c_1(\eta) = \ell$. Correspondingly, they can be thought of as functionals $\psi$ on the space $A^k$ of connections in $P$ which are subject to the Gauss law condition

$$\nabla^\mu_A \frac{\delta}{\delta A^\mu(x)} \psi(A) = \frac{i\ell}{4\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu(x) \psi(A) . \tag{128}$$

Here $\nabla_A$ denotes the covariant derivative w.r.t. $A$. In [3] it has been shown that if $A$ carries a nontrivial magnetic charge, i.e., if it can be reduced to some subbundle of $P$ with nontrivial first Chern class, all physical states obey $\psi(A) = 0$. Such a connection is called a kinematical node. (Note that, due to $\eta$ being nontrivial, there exist also dynamical nodes, which differ from state to state.) The authors of [3] argue that nodal gauge field configurations are relevant for the confinement mechanism. In the following we shall show that being a node is a property of strata. For that purpose, we reformulate the result of [3] in our language.

**Theorem 9.1** Let $A \in A^k$ have orbit type $[L] \in \hat{K}(P)$, where $L = (J; \alpha, \xi)$. If $\alpha_i^{(2)} \neq 0$ for some $i$ then $A$ is a kinematical node, i.e., $\psi(A) = 0$ for all physical states $\psi$. 

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Proof: The proof follows the lines of [3]. By assumption, $A$ can be reduced to a connection on an $SU_J$-subbundle $Q \subseteq P$ of class $(\alpha_J(Q), \xi_J(Q)) = (\alpha, \xi)$. $\Sigma_s$ being a compact orientable 2-manifold, $H^2(\Sigma_s, \mathbb{Z}) \cong \mathbb{Z}$. Hence, $\alpha_s^{(2)} = c_i \gamma^{(2)}$, where $c_i \in \mathbb{Z}$ and $\gamma^{(2)}$ is a generator of $H^2(\Sigma_s, \mathbb{Z})$. We define a 1-parameter subgroup $\{\tilde{\Phi}_t | t \in \mathbb{R}\}$ of $SU_n$ by

$$\tilde{\Phi}_t = \left( \exp \left\{ \frac{c_1 k_1}{i} t \right\} 1_{k_1} \otimes 1_{m_1} \right) \oplus \cdots \oplus \left( \exp \left\{ \frac{c_r k_r}{i} t \right\} 1_{k_r} \otimes 1_{m_r} \right).$$

Using

$$(m_1 c_1 + \cdots + m_r c_r) \gamma^{(2)} = g \left( m_1 \alpha_1^{(2)} + \cdots + m_r \alpha_r^{(2)} \right)$$

$$= g E^{(2)}_{m} (\alpha) \quad \text{by (39)}$$

$$= g \beta_g (\xi) \quad \text{by (51)}$$

$$= 0 \quad \text{by } g \beta_g = 0,$$

one checks $\det \tilde{\Phi}_t = \exp \{it(m_1 c_1 + \cdots + m_r c_r)\} = 1$. Commuting with $SU_J$ for all $t$, $\tilde{\Phi}_t$ defines a 1-parameter subgroup $\{\Phi_t | t \in \mathbb{R}\}$ of $G^{k+1}$ by

$$\Phi_t(q) = \tilde{\Phi}_t \quad \forall \quad q \in Q, t \in \mathbb{R}.$$

Each element of this subgroup is constant on $Q$. Hence, so is the generator $\phi = \dot{\Phi}(0)$:

$$\phi(q) = i \left[ \left( \frac{c_1}{k_1} 1_{k_1} \otimes 1_{m_1} \right) \oplus \cdots \oplus \left( \frac{c_r}{k_r} 1_{k_r} \otimes 1_{m_r} \right) \right] \quad \forall \quad q \in Q. \quad (129)$$

In particular,

$$\nabla_A \phi = 0. \quad (130)$$

According to this, for any state $\psi$,

$$\int_{\Sigma_s} \text{Tr} \left( \phi \nabla_A^{\mu} \frac{\delta}{\delta A^{\mu}} \right) \psi(A) = - \int_{\Sigma_s} \text{Tr} \left( \nabla_A^{\mu} \phi \frac{\delta}{\delta A^{\mu}} \right) \psi(A) = 0.$$

For physical states, the Gauss law implies

$$\int_{\Sigma_s} \text{Tr} (\phi dA) \psi(A) = 0. \quad (131)$$

Using (130), as well as the structure equation $F = dA + \frac{1}{2} [A, A]$, we obtain

$$\int_{\Sigma_s} \text{Tr} (\phi dA) = \int_{\Sigma_s} \text{Tr} (\phi dA - \nabla_A \phi \wedge A)$$

$$= \int_{\Sigma_s} \text{Tr} (\phi dA - d \phi \wedge A - [A, \phi] \wedge A)$$

$$= \int_{\Sigma_s} \text{Tr} (2 \phi dA + [A, \phi])$$

$$= 2 \int_{\Sigma_s} \text{Tr} (\phi F). \quad (132)$$
Since \( A \) is reducible to \( Q \), \( F \) has block structure \((F_1 \otimes \mathbb{1}_{m_1}) \oplus \cdots \oplus (F_r \otimes \mathbb{1}_{m_r})\) with \( F_j \) being \((k_j \times k_j)\)-matrices. Thus, using \((129)\),

\[
\int_{\Sigma_s} \text{Tr} (\phi F) = \sum_{j=1}^r m_j \int_{\Sigma_s} \text{Tr} F_j . \tag{133}
\]

Now \( c_j \) being just the first Chern classes of the elementary factors of the \( U k_1 \times \cdots \times U k_r \)-bundle \( Q^{[UJ]} \), we have

\[
\int_{\Sigma_s} \text{Tr} F_j = -2\pi i c_j , \quad j = 1, \ldots, r .
\]

Inserting this into \((133)\) and the latter into \((132)\) we obtain

\[
\int_{\Sigma_s} \text{Tr} (\phi dA) = 4\pi \sum_{j=1}^r m_j \frac{k_j}{k_j} (c_j)^2 . \tag{134}
\]

Thus, in view of \((131)\),

\[
4\pi \sum_{j=1}^r m_j \frac{k_j}{k_j} (c_j)^2 \psi(A) = 0 . \tag{135}
\]

It follows that if one of the \( c_j \) is nonzero then \( \psi(A) = 0 \) for all physical states \( \psi \), i.e., \( A \) is a kinematical node. This proves the theorem.

Remark: Let us compare \((134)\) with Formula (6) in \([3]\). Define \( k_i' = k_i m_i \) and \( m_i' = 1 \). Then \( J' = (k', m') \in K(n) \) and \( U J \subseteq U J' \). Let \( \varphi : U J \rightarrow U J' \) denote the canonical embedding. Consider the subbundle \( Q^{[\varphi]} \subseteq P \). One can check that the elementary factors of this subbundle have first Chern classes \( c_i' = m_i c_i \). Inserting \( k_i' \), \( l_i' \), and \( c_i' \) into \((134)\) one obtains Formula (6) in \([3]\). In fact, the authors of \([3]\) use that \( A \) is reducible to \( Q' \), rather than that it is even reducible to \( Q \). This argument being "coarser" than ours, it still suffices to prove that any connection which is reducible to a subbundle with nontrivial magnetic charge is a kinematical node.

As a consequence of Theorem \(9.1\), one can speak of nodal and nonnodal strata. This information can be read off directly from the labels of the strata. Let us discuss this in some more detail. Let \( J \in K(n) \) be given and consider Eqs. \((91)\) and \((92)\). Variables are \( \xi \in H^1(\Sigma_s, \mathbb{Z}_g) \) and \( \alpha_i^{(2)} \in H^2(\Sigma_s, \mathbb{Z}), \ i = 1, \ldots, r \). Since \( H^2(\Sigma_s, \mathbb{Z}) \) is torsion-free, \( \beta_g \) is trivial. Moreover, due to \( H^4(\Sigma_s, \mathbb{Z}) = 0, a_i \sim a_j \) vanishes. Thus, the system of equations \((91), (92)\) reduces to

\[
E_{m}^{(2)}(\alpha) = 0 . \tag{135}
\]

Writing \( \alpha_i^{(2)} = c_i \gamma^{(2)} \) again, \((135)\) becomes

\[
\sum_{i=1}^r \tilde{m}_i c_i = 0 .
\]
The set of solutions of this equation is a subgroup \( G_{\tilde{m}} \subseteq \mathbb{Z}^{\oplus r} \). According to Theorem 9.1, the nonnodal strata are parametrized by \( \xi \) and the neutral element of \( G_{\tilde{m}} \), whereas the nodal strata are labelled by \( \xi \) and all the other elements of \( G_{\tilde{m}} \). For example, in the case of SU2 we obtain the following.

\( J = (1|2) \): Here \( G_{\tilde{m}} = \{0\} \subseteq \mathbb{Z} \), hence all strata are nonnodal.

\( J = (1,1|1,1) \): We have \( G_{\tilde{m}} = \{(c, -c)|c \in \mathbb{Z}\} \subseteq \mathbb{Z}^{\oplus 2} \). Since also \( \xi = 0 \), each value of \( c \) labels one stratum. That corresponding to \( c = 0 \) is nonnodal, the others are nodal.

\( J = (2|1) \): Here we have the generic stratum, which is nonnodal.

10 Summary

Starting from a principal SU\( n \)-bundle \( P \) over a compact connected orientable Riemannian 4-manifold \( M \), we have derived a classification of the orbit types of the action of the group of gauge transformations of \( P \) on the space of connections in \( P \). Orbit types are known to label the elements of the natural stratification, given by Kondracki and Rogulski [23], of the gauge orbit space associated to \( P \). The interest in this stratification is due to the fact that the role of nongeneric strata in gauge physics is not clarified yet.

In order to accomplish the classification, we have utilized that orbit types are 1:1 with a certain class of subbundles of \( P \) (called holonomy-induced Howe subbundles), factorized by the natural actions of vertical automorphisms of \( P \) and of the structure group. We have shown that such classes of subbundles are labelled by symbols \([\langle J; \alpha, \xi \rangle]\), where

\[ J = ((k_1, \ldots, k_r), (m_1, \ldots, m_r)) \]

is a pair of sequences of positive integers obeying

\[ \sum_{i=1}^{r} k_i m_i = n, \]

\( \alpha = (\alpha_1, \ldots, \alpha_r) \) where \( \alpha_i \in H^*(M, \mathbb{Z}) \) are admissible values of the Chern class of \( U_{k_i} \)-bundles over \( M \), and \( \xi \in H^1(M, \mathbb{Z}_g) \) with \( g \) being the greatest common divisor of \( (m_1, \ldots, m_r) \). The cohomology elements \( \alpha_i \) and \( \xi \) are subject to the relations

\[ \sum_{i=1}^{r} m_i \frac{\alpha_i(2)}{g} = \beta_g(\xi), \]

\[ \alpha_1^{m_1} \cdots \alpha_r^{m_r} = c(P), \]

where \( \beta_g : H^1(M, \mathbb{Z}_g) \rightarrow H^2(M, \mathbb{Z}) \) is the connecting homomorphism associated to the short exact sequence of coefficient groups in cohomology

\[ 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_g \rightarrow 0. \]

Finally, for any permutation \( \sigma \) of \( \{1, \ldots, r\} \), the symbols

\[ [\langle (k_1, \ldots, k_r), (m_1, \ldots, m_r); (\alpha_1, \ldots, \alpha_r), \xi \rangle], \]

\[ [\langle (k_{\sigma(1)}, \ldots, k_{\sigma(r)}), (m_{\sigma(1)}, \ldots, m_{\sigma(r)}); (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)}), \xi \rangle] \]
have to be identified.
The result obtained enables one to determine which strata are present in the gauge orbit space, depending on the topology of the base manifold and the topological sector, i.e., the isomorphism class of $P$. For some examples we have discussed this dependence in detail.
We have also shown that our result can be used to reformulate a sufficient condition on a connection to be a node for all physical states (when the latter are viewed as functionals of connections subject to the Gauss law). This condition has been derived in [3] and applies to Chern-Simons Theory in $2 + 1$ dimensions.
In this way, our result may be viewed as one more step towards a systematic investigation of the physical effects related to nongeneric strata of the gauge orbit space.
We remark that orbit types still carry more information about the stratification structure. Namely, their partial ordering encodes how the strata are patched together in order to build up the gauge orbit space (cf. [23, Thm. (4.3.5)]). A derivation of this partial ordering will be published separately.

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A The Eilenberg-MacLane Spaces $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}_g, 1)$

In this appendix, we construct a model for each of the Eilenberg-MacLane spaces $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}_g, 1)$ and derive the integer-valued cohomology of these spaces. Consider the natural free action of $U_1$ on the sphere $S^\infty$ which is induced from the natural action of $U_1$ on $S^{2n-1} \subset \mathbb{C}^n$. The orbit space of this action is the complex projective space $\mathbb{CP}^\infty$.
Moreover, by viewing $\mathbb{Z}_g$ as a subgroup of $U_1$, this action gives rise to a natural free action of $\mathbb{Z}_g$ on $S^\infty$. The orbit space of the latter is the lens space $L^\infty_g$. By construction, one has principal bundles

$$U_1 \hookrightarrow S^\infty \twoheadrightarrow \mathbb{CP}^\infty,$$

$$\mathbb{Z}_g \hookrightarrow S^\infty \twoheadrightarrow L^\infty_g.$$  \hspace{1cm} (136) \hspace{1cm} (137)

Due to $\pi_i(S^\infty) = 0 \ \forall i$, the exact homotopy sequences induced by (136), (137) yield

$$\pi_i(\mathbb{CP}^\infty) = \pi_{i-1}(U_1) = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & i \neq 2, \end{cases}$$

$$\pi_i(L^\infty_g) = \pi_{i-1}(\mathbb{Z}_g) = \begin{cases} \mathbb{Z}_g & i = 1 \\ 0 & i = 2, 3, \ldots, \end{cases}$$
respectively. As a consequence, $\mathbb{C}P^{\infty}$ is a model of $K(\mathbb{Z}, 2)$ and $L^{\infty}_{g}$ is a model of $K(\mathbb{Z}_{g}, 1)$. In particular,

$$H^{i}(K(\mathbb{Z}, 2), \mathbb{Z}) = H^{i}(\mathbb{C}P^{\infty}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}, \quad (138)$$

see [3] Ch. VI, Prop. 10.2, and

$$H^{i}(K(\mathbb{Z}_{g}, 1), \mathbb{Z}) = H^{i}(L^{\infty}_{g}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_{g} & i \neq 0, \text{ even} \\ 0 & i \neq 0, \text{ odd} \end{cases}, \quad (139)$$

see [11], §24, p. 176.

We notice that the vanishing of all homotopy groups of $S^{\infty}$ also implies that the bundles $(136)$ and $(137)$ are universal for $U_{1}$ and $\mathbb{Z}_{g}$, respectively. Hence, $\mathbb{C}P^{\infty}$ and $L^{\infty}_{g}$ are models of $BU_{1}$ and $B\mathbb{Z}_{g}$, respectively. For $B\mathbb{Z}_{g}$, this has been used in the proof of Lemma 5.9.

**B The Cohomology Algebra $H^{*}(BSU_{J}, \mathbb{Z})$**

Let $J \in K(n)$, $J = (k, m)$. Recall from Corollary 5.8 that the cohomology algebra $H^{*}(BSU_{J}, \mathbb{Z})$ is generated by the elements $\gamma^{(2j)}_{j,i}$, $j = 1, \ldots, k$, $i = 1, \ldots, r$, defined in (52). Here we are going to derive the relations these generators are subject to.

**Proposition B.1** The generators $\gamma^{(2j)}_{j,i}$ of $H^{*}(BSU_{J}, \mathbb{Z})$ are subject to the relation

$$E_{m}^{(2)}(\gamma_{J}) = 0. \quad (140)$$

**Proof:** Consider the homomorphism $(Bj_{j})^{*} : H^{*}(BU_{J}, \mathbb{Z}) \to H^{*}(BSU_{J}, \mathbb{Z})$ induced by the embedding $j_{J} : SU_{J} \to U_{J}$. By construction, $\gamma^{(2j)}_{j,i} = (Bj_{j})^{*} \tilde{\gamma}^{(2j)}_{j,i}$. Since there are no relations between the $\tilde{\gamma}^{(2j)}_{j,i}$, see Lemma 5.4, all the relations between the $\gamma^{(2j)}_{j,i}$ are derived from elements of the kernel of $(Bj_{j})^{*}$. We claim that

$$\ker (Bj_{j})^{*} = H^{*}(BU_{J}, \mathbb{Z}) \sim E_{m}^{(2)}(\tilde{\gamma}_{J}), \quad (141)$$

i.e., that $\ker (Bj_{j})^{*}$ is generated as an ideal by the single element $E_{m}^{(2)}(\tilde{\gamma}_{J})$. This implies the assertion.

In order to prove (141), consider the U1-bundle $\eta$ given in (57). The Gysin sequence (58) induced by $\eta$ being exact, it yields

$$\ker (Bj_{j})^{*} = H^{*}(BU_{J}, \mathbb{Z}) \sim c_{1}(\eta).$$

Let us compute $c_{1}(\eta)$. We note that $\eta$ is induced as a principal bundle of the form (28) by the short exact sequence of Lie group homomorphisms

$$1 \longrightarrow SU_{J} \xrightarrow{j_{J}} U_{J} \xrightarrow{\det_{U_{J}}} U_{1} \longrightarrow 1,$$
cf. \((28)\). Hence, it has classifying map \(B \text{det}_U \circ B i_J\). Accordingly,
\[
c_1(\eta) = (B i_J)^* \circ (B \text{det}_U)^* \gamma^{(2)}_{U1} \\
= (B i_J)^* \gamma^{(2)}_{U n} \quad | \text{by (78)} \\
= E^{(2)}_m (\tilde{\gamma}_{J1}, \ldots, \tilde{\gamma}_{Jr}) \quad | \text{by (72)}
\]
This proves the proposition.

As a consequence of Proposition \([B,1]\), \(H^*(BSU J, \mathbb{Z})\) is isomorphic to the polynomial ring
\[
\mathbb{Z}[x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}]/(m_1 x_{11} + \cdots + m_r x_{r1}),
\]
where \(\deg(x_{ij}) = 2j\).

We remark that the relation \((140)\) is a consequence of the relation between \(\delta_J\) and \(\gamma_J\) given in Theorem 5.13. Namely, using \((71)\), Theorem 5.13, and \(g \beta_g = 0\) we find
\[
E^{(2)}_m (\gamma_J) = g E^{(2)}_m (\tilde{\gamma}_J) = g \beta_g (\delta_J) = 0.
\]
This ensures, in particular, that \((140)\) does not generate a relation independent of \((84)\) on the level of the characteristic classes \(\alpha_J\). (Nonetheless, we have already proved directly in Lemma 5.15 that there are no relations independent of \((84)\).)

C The Cohomology Algebra \(H^*(BSU J, \mathbb{Z}_g)\)

Let \(J \in K(n), J = (k, m)\). In this appendix, we derive \(H^*(BSU J, \mathbb{Z}_g)\). Recall that \(g\) denotes the greatest common divisor of \(m\). Moreover, recall that \(\varrho_g : H^*(BSU J, \mathbb{Z}) \to H^*(BSU J, \mathbb{Z}_g)\) denotes reduction modulo \(g\). We start with some technical lemmas.

Lemma C.1
(a) Let \(\alpha \in H^*(BSU J, \mathbb{Z})\) such that \(\varrho_g (\alpha \sim \beta_g (\delta_J)) = 0\). Then \(\varrho_g (\alpha) = 0\) as well as \(\alpha \sim \beta_g (\delta_J) = 0\).
(b) One has \(\text{im} \beta_g \subseteq H^*(BSU J, \mathbb{Z}) \sim \beta_g (\delta_J)\).
(c) The homomorphism \(\varrho_g \circ \beta_g : H^{2j+1}(BSU J, \mathbb{Z}_g) \to H^{2j+2}(BSU J, \mathbb{Z}_g)\) is injective for \(j = 0, 1, 2, \ldots\).

Proof: (a) Let \(\alpha\) be given as proposed. Then there exists \(\alpha' \in H^*(BSU J, \mathbb{Z})\) such that
\[
\alpha \sim \beta_g (\delta_J) = g \alpha'. \tag{142}
\]
Moreover, there exist \(\tilde{\alpha}, \tilde{\alpha}' \in H^*(BU J, \mathbb{Z})\) such that \(\alpha = (B j_J)^* \tilde{\alpha}\) and \(\alpha' = (B j_J)^* \tilde{\alpha}'\). Due to Theorem 5.13 and \((82)\),
\[
\beta_g (\delta_J) = (B j_J)^* E^{(2)}_m (\tilde{\gamma}_J) \tag{143}
\]
Using this, as well as \((141)\), \((142)\) implies that there exists \(\tilde{\alpha}'' \in H^*(BU J, \mathbb{Z})\) such that
\[
\tilde{\alpha} \sim E^{(2)}_m (\tilde{\gamma}_J) = g \tilde{\alpha}' + \tilde{\alpha}'' \sim E^{(2)}_m (\tilde{\gamma}_J) \\
= g \left( \tilde{\alpha}' + \tilde{\alpha}'' \sim E^{(2)}_m (\tilde{\gamma}_J) \right). \tag{144}
\]
Taking into account that \( H^* (BUJ, \mathbb{Z}) \) is torsion-free, that the elements \( \tilde{m}^{(2j)}_i \) are generators, and that the greatest common divisor of the integers \( \tilde{m}_i \) is 1, from (144) we infer that there exists \( \tilde{\alpha}'' \in H^* (BUJ, \mathbb{Z}) \) such that \( \tilde{\alpha} = \tilde{g} \tilde{\alpha}'' \). Then \( \alpha = (B_{jj})^* \tilde{\alpha} = \tilde{g} (B_{jj})^* \tilde{\alpha}'' \). It follows \( \varrho_g (\alpha) = 0 \). Due to \( \varrho_g (\beta_{j}) = 0 \), also \( \alpha \sim \beta_g (\delta_{j}) = 0 \), as asserted.

(b) Let \( \alpha \in \text{im} \beta_g \). By exactness of (153), \( \varrho_g \alpha = 0 \). Let \( \tilde{\alpha} \in H^* (BUJ, \mathbb{Z}) \) such that \( \alpha = (B_{jj})^* \tilde{\alpha} \). Then \( g \tilde{\alpha} \in \ker (B_{jj})^* \). According to (141), there exists \( \tilde{\alpha}' \in H^* (BUJ, \mathbb{Z}) \) such that

\[
g \tilde{\alpha} = \tilde{\alpha}' \sim E^{(2)}_m (\tilde{\gamma}_j) = g \left( \tilde{\alpha}' \sim E^{(2)}_m (\tilde{\gamma}_j) \right).
\]

Since \( H^* (BUJ, \mathbb{Z}) \) is free Abelian, this implies \( \tilde{\alpha} = \tilde{\alpha}' \sim E^{(2)}_m (\tilde{\gamma}_j) \). Using (143) we obtain

\[
\alpha = (B_{jj})^* \tilde{\alpha}' \sim \beta_g (\delta_{j}).
\]

(c) We note that (a) and (b) immediately imply that \( \text{im} \beta_g \cap \ker \varrho_g = \{0\} \). Moreover, by exactness of (153), \( \beta_g \) is injective in odd degree. This proves (c).

Lemma C.2 The homomorphism \( H^{2j} (BSUJ, \mathbb{Z}_g) \to H^{2j+1} (BSUJ, \mathbb{Z}_g) \), \( \alpha \mapsto \alpha \sim \delta_{j} \), is an isomorphism for \( j = 0, 1, 2, \ldots \).

Proof: First, we check injectivity. Let \( \alpha \in H^{2j} (BSUJ, \mathbb{Z}_g) \) such that \( \alpha \sim \delta_{j} = 0 \). Consider the homomorphism \( \varrho_g \circ \beta_g \). As an immediate consequence of the definition of \( \beta_g \) (namely, as the connecting homomorphism in the long exact sequence (50)), this is a skew-derivation, i.e., for any \( \alpha_i \in H^{j,i} (BSUJ, \mathbb{Z}_g) \), \( i = 1, 2 \), one has

\[
\varrho_g \circ \beta_g (\alpha_1 \sim \alpha_2) = (\varrho_g \circ \beta_g (\alpha_1)) \sim \alpha_2 + (-1)^{j_1} \alpha_1 \sim (\varrho_g \circ \beta_g (\alpha_2)). \tag{145}
\]

Hence, due to \( \beta_g \) being trivial in even degree,

\[
\varrho_g \circ \beta_g (\alpha \sim \delta_{j}) = \alpha \sim \varrho_g \circ \beta_g (\delta_{j}). \tag{146}
\]

Since \( \varrho_g \) is surjective in even degree, there exists \( \alpha' \in H^{2j} (BSUJ, \mathbb{Z}) \) such that \( \alpha = \varrho_g (\alpha') \). Inserting this on the rhs. of (146) we obtain

\[
\varrho_g \circ \beta_g (\alpha \sim \delta_{j}) = \varrho_g (\alpha' \sim \beta_g (\delta_{j})). \tag{147}
\]

By assumption, the lrhs. of (147) vanishes. Then Lemma C.1(a) implies \( \alpha = \varrho_g (\alpha') = 0 \). This proves injectivity. To show surjectivity, let \( \alpha \in H^{2j+1} (BSUJ, \mathbb{Z}_g) \). Due to Lemma C.1(b), there exists \( \alpha' \in H^* (BSUJ, \mathbb{Z}) \) such that \( \beta_g (\alpha) = \alpha' \sim \beta_g (\delta_{j}) \). Then

\[
\varrho_g \circ \beta_g (\alpha) = \varrho_g (\alpha' \sim \beta_g (\delta_{j}))
= \varrho_g (\alpha') \sim \varrho_g \circ \beta_g (\delta_{j})
= \varrho_g \circ \beta_g (\alpha' \sim \delta_{j}),
\]

where the last equality is due to (145) and the fact that \( \beta_g \) is trivial in even degree. As a consequence, Lemma C.1(c) implies \( \alpha = \varrho_g (\alpha') \sim \delta_{j} \). This shows surjectivity and, therefore, concludes the proof of the lemma.

Lemma C.3 There holds \( \delta_{j} \sim \delta_{j} = \begin{cases} 0 & g = 2l + 1 \\ l \varrho_g \circ \beta_g (\delta_{j}) & g = 2l \end{cases} \).
Proof: We notice that both the collections of maps
\[ \theta_1 : H^1(\cdot, \mathbb{Z}_g) \rightarrow H^2(\cdot, \mathbb{Z}_g), \quad \alpha \mapsto \alpha \circ \alpha, \]  
\[ \theta_2 : H^1(\cdot, \mathbb{Z}_g) \rightarrow H^2(\cdot, \mathbb{Z}_g), \quad \alpha \mapsto \varrho_g \circ \beta_g(\alpha), \]  
(148)  
(149)
define a natural transformation of cohomology functors for $CW$-complexes. Such transformations are called cohomology operations of type $(1, \mathbb{Z}_g; 2, \mathbb{Z}_g)$ or, more generally, of type $(i_1, \pi_1; i_2, \pi_2)$ if they map $H^{i_1}(\cdot, \pi_1) \rightarrow H^{i_2}(\cdot, \pi_2)$ [8, Ch. VII, Def. 12.2]. Here $\pi_1, \pi_2$ are Abelian groups. One should note that cohomology operations need not consist of group homomorphisms. Nevertheless, the Abelian group structure of $H^{i_2}(\cdot, \pi_2)$ induces an according structure on the set of cohomology operations of type $(i_1, \pi_1; i_2, \pi_2)$. Due to a theorem of Serre [8, Ch. VII, Thm. 12.3], there exists a group isomorphism from the group so defined onto $H^{i_2}(K(\pi_1, i_1), \pi_2)$. This is given by evaluating the cohomology operations at some fixed characteristic element of $H^{i_1}(K(\pi_1, i_1), \pi_1)$.

To apply this theorem, we choose $L_g^\infty$ as a model of $K(\mathbb{Z}_g, 1)$ and $\delta_g$ as a characteristic element of $H^1(K(\mathbb{Z}_g, 1), \mathbb{Z}_g)$. Assume, for a moment, that there holds
\[ \theta_1(\delta_g) = \begin{cases} 0 & g = 2l + 1 \\ l \theta_2(\delta_g) & g = 2l. \end{cases} \]  
(150)

Then the cohomology operations are related by
\[ \theta_1 = \begin{cases} 0 & g = 2l + 1 \\ l \theta_2 & g = 2l. \end{cases} \]

Thus, in order to prove the lemma, we have to show (150). Although the cohomology of the spaces $L_g^\infty$ is well known, this particular relation can rarely be found in textbooks. An exception is the case $g = 2$, where $L_2^\infty = \mathbb{RP}^\infty$. For the sake of completeness, in the following we derive (150) for general $g$ from the case $g = 2$.

One can show that $H^2(L_g^\infty, \mathbb{Z}_g) \cong \mathbb{Z}_g$ and that it is generated by $\varrho_g \circ \beta_g(\delta_g)$, where $\delta_g$ is a generator of $H^1(L_g^\infty, \mathbb{Z}_g)$ [14]. Thus, there exists $a \in \mathbb{Z}_g$ such that
\[ \theta_1(\delta_g) = \delta_g \circ \delta_g = a \varrho_g \circ \beta_g(\delta_g). \]

Since $2\delta_g \circ \delta_g = 0$, $2a = 0$. Thus, if $g$ is odd then $a = 0$. If $g$ is even then either $a = l$, where $g = 2l$, or $a = 0$. To rule out the second case, it suffices to find $\delta \in H^1(L_g^\infty, \mathbb{Z}_g)$ such that $\delta \circ \delta \neq 0$. Consider the composite homomorphism
\[ H^1(L_g^\infty, \mathbb{Z}_g) \xrightarrow{\varrho_2} H^1(L_g^\infty, \mathbb{Z}_2) \xrightarrow{p} H^1(\mathbb{RP}^\infty, \mathbb{Z}_2). \]  
(151)

Here $\varrho_2$ is reduction modulo 2 and $p$ is the projection in the principal bundle
\[ \mathbb{Z}_l \hookrightarrow \mathbb{RP}^\infty \xrightarrow{p} L_g^\infty \]
which arises by factorizing $\mathbb{RP}^\infty = S^\infty/\mathbb{Z}_2$ by the residual action of $\mathbb{Z}_g/\mathbb{Z}_2 \cong \mathbb{Z}_l$. We check that (151) is surjective. For the part of $\varrho_2$ this is obvious, due to $g$ being even.

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For the part of $p^*$, we note that, by virtue of the Hurewicz and the Universal Coefficient Theorem, $p^*$ is the $\mathbb{Z}_2$-dual of the homomorphism $p_* : \pi_1 \mathbb{RP}^\infty \to L_\infty$. The latter is easily seen to be injective. Therefore, $p^*$ is surjective, as asserted. As a consequence, there exists $\delta \in H^1(L_\infty, \mathbb{Z}_2)$ such that $p^* \circ \varrho_2(\delta) = \delta_2$, where $\delta_2$ denotes the generator of $H^1(\mathbb{RP}^\infty, \mathbb{Z}_2)$. We compute

$$p^* \circ \varrho_2(\delta) = (p^* \circ \varrho_2(\delta)) \sim (p^* \circ \varrho_2(\delta)) = \delta_2 \sim \delta_2.$$ 

Here the rhs. is known to be nontrivial [8, Ch. VI, Prop. 10.2]. Hence, $\delta \sim \delta_2$ must have been nontrivial. This proves the Lemma. 

**Proposition C.4** $H^*(\text{BSU}, \mathbb{Z}_2)$ is generated over $\mathbb{Z}_2$ by the elements $\delta_j$ and $\varrho_g \left( \gamma_{i,j,i}^{(2i)} \right)$, $j = 1, \ldots, k_i$, $i = 1, \ldots, r$. The generators are subject to the relation

$$\delta_j \sim \delta_j = \begin{cases} 0 & g = 2l + 1 \\ l E_m^{(2)}(\gamma_{j,i}) & g = 2l \end{cases} \quad (152)$$

**Proof:** Consider the long exact sequence (60) for the space $\text{BSU}$. Since the integer-valued cohomology of $\text{BSU}$ is trivial in odd degree, this sequence splits into exact portions

$$0 \xrightarrow{\varrho_g} H^{2j+1}(\text{BSU}, \mathbb{Z}_2) \xrightarrow{\beta_g} H^{2j+2}(\text{BSU}, \mathbb{Z}_2) \xrightarrow{\mu_g} H^{2j+2}(\text{BSU}, \mathbb{Z}_2) \xrightarrow{\beta_g} 0, \quad (153)$$

where $j = 0, 1, 2, \ldots$. As an immediate consequence, $\varrho_g$ is surjective in even degree. Thus, $H^{\text{even}}(\text{BSU}, \mathbb{Z}_2)$ is generated by $\varrho_g \left( \gamma_{i,j,i}^{(2i)} \right)$, $j = 1, \ldots, k_i$, $i = 1, \ldots, r$. Then Lemma C.2 implies that the whole of $H^*(\text{BSU}, \mathbb{Z}_2)$ is generated by these elements together with $\delta_j$. It remains to determine the relations the generators are subject to. First, consider relations among the generators $\varrho_g \left( \gamma_{i,j,i}^{(2i)} \right)$. These arise from the relation (140) and from the elements of ker $\varrho_g$. Due to $E_m^{(2)} = g E_m^{(2)}$, the mod $g$-reduction of (140) is trivially satisfied. Moreover, relations generated by elements of ker $\varrho_g$ are already taken into account by taking $\mathbb{Z}_g$ as the base ring. Thus, there is no relation among the generators $\varrho_g \left( \gamma_{i,j,i}^{(2i)} \right)$. Next, consider relations involving $\delta_j$. One such relation is provided by Lemma C.3. It covers all relations in even degree, because the latter must contain an even power of $\delta_j$, hence can be written without $\delta_j$. Relations in odd degree, on the other hand, are of the form $\alpha \sim \delta_j = 0$, where $\alpha$ is of even degree. Due to Lemma C.2 then $\alpha = 0$. Consequently, there are no further relations.

Due to Proposition C.4 if $g$ is odd then $H^*(\text{BSU}, \mathbb{Z}_2)$ is isomorphic to the polynomial ring

$$\mathbb{Z}_g[x, x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}],$$

whereas if $g$ is even then it is isomorphic to

$$\mathbb{Z}_g[x, x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}] / (x^2 - (\tilde{m}_1 x_{11} + \cdots + \tilde{m}_r x_{r1})),$$

where $g = 2l$. Here $\deg(x) = 1$ and $\deg(x_{ij}) = 2j$. 

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