Abstract

It is a known fact that a stationary subdivision scheme generates the full space of polynomials of degree up to \( N \) if and only if its symbol satisfies sum rules of order \( N + 1 \). This property is, in general, only necessary for the associated limit function to have approximation order \( N + 1 \) and for being \( C^N \)-continuous. But, the polynomial reproduction property of degree \( N \) (i.e. the capability of a subdivision scheme to reproduce in the limit exactly the same polynomials from which the data is sampled) is sufficient for having approximation order \( N + 1 \). The aim of this short paper is to show that, when dealing with non-stationary subdivision schemes, the crucial role played by polynomials and sum rules is taken by exponential polynomials and approximate sum rules. More in detail, we here show that for a non-stationary subdivision scheme the reproduction of \( N \) exponential polynomials implies fulfillment of approximate sum rules of order \( N \). Furthermore, generation of \( N \) exponential polynomials implies fulfillment of approximate sum rules of order \( N \) if asymptotical similarity to a convergent stationary scheme is also assumed together with reproduction of a single exponential polynomial. We additionally show that reproduction of an \( N \)-dimensional space of exponential polynomials, jointly with asymptotical similarity, implies approximation order \( N \). To show this we also prove the convergence of the sequence of basic limit functions of the non-stationary scheme to the basic limit function of the asymptotically similar stationary one.

Keywords: Subdivision schemes; exponential polynomial generation and reproduction; asymptotical similarity; approximate sum rules; approximation order.

1. Introduction

Subdivision schemes are tools for the design of smooth curves and surfaces in many applicative areas such as computer–aided geometric design, curve and surface reconstruction, signal/image processing. They are obtained as the limit of a simple iterative procedure based on the repeated application of refinement rules starting with an initial set of discrete data \([1, 2, 3] \). Given \( f^0 \), the sequence of refined data \( \{f^k, k > 0\} \) is constructed recursively via the subdivision operators \( S_{a^{[0]}}, \ldots, S_{a^{[k-1]}} \) associated with the finitely supported masks \( a^{[k-1]} := \{a_i^{[k-1]}, i = -M, \ldots, M\} \), as

\[
  f_i^k := (S_{a^{[k-1]}} f_{i}^{[k-1]}) := \sum_{j \in \mathbb{Z}} a_i^{[k-1]} j^{[k-1]}, \quad i \in \mathbb{Z}, \quad k > 0.
\]

In case the refinement rules are level dependent, the subdivision scheme is named non-stationary and denoted by \( \{S_{a^k}, k \geq 0\} \), otherwise stationary and simply denoted as \( S_a \). For subdivision analysis and applications it is of importance to know features of the limit curve or surface such as regularity, approximation order, capability of representing required shapes e.g. circles, spirals or polynomial/trigonometric...
curves \([1, 8, 15]\). All these features can be deduced via an *a priori* analysis of the so-called subdivision symbols, \(a^{[k]}(z) := \sum_{i \in \mathbb{Z}} a_i^{[k]} z^i, \ z \in \mathbb{C} \setminus \{0\}\), also providing information about the *generation properties* of the subdivision scheme, which is the subdivision capability to provide specific type of limit functions.

In stationary subdivision, generation of polynomials of order \(N\) is equivalent to the fact that the subdivision symbol satisfies sum rules of order \(N + 1\), a necessary condition for the associated limit function to have approximation order \(N + 1\) and \(C^N\)-continuity. In addition, generation of polynomials guarantees the existence of the so called difference schemes whose behaviour is strongly connected with convergence and regularity of the subdivision scheme. Therefore, generation of polynomials is a necessary condition for convergence/regularity of stationary subdivision schemes and for their approximation order properties (see, for example, \([2, 8, 10, 12, 13]\) and references therein). In contrast, the polynomial *reproduction property* of degree \(N\) -i.e. the capability of a subdivision scheme to reproduce in the limit exactly the same polynomial from which the data is sampled- is sufficient for having approximation order \(N + 1\).

The aim of this short paper is to show that, when dealing with non-stationary subdivision schemes, the crucial role played by sum rules and polynomial generation is taken by approximate sum rules and exponential polynomial generation, which also guarantee the existence of difference schemes. But, since the contractivity of the first-level difference scheme is not enough even to conclude convergence, in the literature additional assumptions on the sequence of subdivision masks are required, the most restrictive of them being the asymptotical equivalence to the mask of a convergent stationary scheme \([4]\). Recently, weaker conditions have been considered which are reproduction of constants together with asymptotical similarity to the mask of a convergent stationary scheme \([4]\) or, even weaker, fulfillment of approximate sum rules together with asymptotical similarity to the mask of a convergent stationary scheme \([8]\). Inspired by the latter result, in this paper we show that the reproduction of \(N\) exponential polynomials implies fulfillment of approximate sum rules of order \(N\). Even more, generation of \(N\) exponential polynomials implies fulfillment of approximate sum rules of order \(N\) if asymptotical similarity to a convergent stationary scheme is also assumed together with reproduction of a single exponential polynomial. In view of the results in \([8]\), approximate sum rules of order \(N\) and asymptotical similarity to a stationary \(C^{N-1}\) subdivision scheme thus provide sufficient conditions for \(C^{N-1}\) regularity of non-stationary subdivision schemes.

In this paper we also show that reproduction of an \(N\)-dimensional space of exponential polynomials, jointly with asymptotical similarity, implies approximation order \(N\), and thus we extend the recent approximation order results based on asymptotical equivalence given in \([11]\). To show this we also prove a result which we believe to be interesting by itself: the convergence of the sequence of basic limit functions of the non-stationary scheme to the basic limit function of the asymptotically similar stationary one.

### 2. Exponential polynomials and approximate sum rules

For a convergent subdivision scheme it is obviously important to establish the features of the limit function \(g_{0,0}\) also in terms of the initial sequence \(f^{[0]}\). This is particularly true in case \(f^{[0]}\) is made of samples of a special type of functions, e.g., polynomial functions or exponential polynomial functions. Indeed, in this paper we show that the response of the subdivision scheme to these types of starting sequences is not only important for the design of shapes of practical interest in applications \([15]\), but is also strongly connected to the approximation order of any limit function generated via the subdivision scheme (see also \([11]\)).

**Definition 1.** Given a set \(\Lambda = \{\lambda_n \in \mathbb{C}, \ n = 1, \ldots, N\}\), we select all distinct elements from \(\Lambda\), and define the set \(\mathcal{L}_\Lambda := \{ (\lambda_n, \beta), \ n = 1, \ldots, \eta, \ \beta = 0, \ldots, \mu_\eta - 1\}\), with \(\mu_\eta \in \mathbb{N}\) and \(\mu_1 + \cdots + \mu_\eta = N\). Here, \(\eta\) represents the number of all distinct elements of \(\Lambda\), and \(\mu_n\) indicates the duplication of each distinct element in \(\Lambda\). The space of the exponential polynomials associated with \(\Lambda\) (as well as with \(\mathcal{L}_\Lambda\)) is defined by

\[
\mathcal{E}_\Lambda := \text{span}\{x^\beta e^{\lambda_n x}, \ (\lambda_n, \beta) \in \mathcal{L}_\Lambda\}.
\]  

For a fixed set \(\Lambda\), and for the corresponding space \(\mathcal{E}_\Lambda\), we recall the following definitions of exponential polynomial generation and exponential polynomial reproduction.
**Definition 2.** A convergent subdivision scheme \( \{S_n, n \geq 0\} \) is said to be \( \mathcal{E}_\Lambda \)-generating if for \( f \in \mathcal{E}_\Lambda \) there exists an initial sequence \( f^0 := \{f(i), i \in \mathbb{Z}\} \), \( f \in \mathcal{E}_\Lambda \), such that

\[
\lim_{k \to \infty} S_n f S_{n-1} \cdots S_{n-\ell} f^0 = f.
\]

Moreover, it is \( \mathcal{E}_\Lambda \)-reproducing if the initial sequence is sampled from \( f \), i.e. \( f^0 := \{f(i), i \in \mathbb{Z}\} \).

The authors of [3] and [11] provide algebraic conditions to be satisfied from the subdivision symbols in order to guarantee the construction of limit functions with the property of exponential polynomial generation and reproduction. In the following theorem we recall the set of conditions as given in the second reference.

**Theorem 3.** [11, Theorem 2.3] Suppose that \( \{S_n, n \geq 0\} \) is a non-stationary subdivision scheme with Laurent polynomials \( \{a[k](z), k \geq 0\} \). Then, \( \{S_n, n \geq 0\} \) reproduces the exponential polynomials in \( \mathcal{E}_\Lambda \) with \( \#\Lambda = N \), if and only if for some Laurent polynomial \( c[k](z) \)

\[
2 - a[k](z^{1+\nu}) z^\nu = c[k](z) \prod_{n=1}^\eta (1 - e^{\lambda_n z^{-k-1-n}} z) \mu_n, \quad \text{with} \quad \nu = 0, \text{if } N \text{ even; } \\
\nu = 1, \quad \text{if } N \text{ odd.} \tag{2.2}
\]

**Remark 4.** It is important to point out that in case of reproduction of a single exponential polynomial in \( \mathcal{E}_\Lambda \), say \( e^{\lambda z} \), it is \( \nu = 0 \) if the value of \( p \) such that \( a[k](e^{-\lambda z^{2-k-1}}) = 2(e^{-\lambda z^{2-k-1}})^p \) is in \( \mathbb{Z} \), whereas it is \( \nu = 1 \) if such \( p \) is in \( \mathbb{Z}/2 \). More precisely, it is easy to see that indeed, for a \( k \)-level subdivision symbol \( a[k](z) \) satisfying (2.2) with \( \eta = 1, \lambda_1 = \lambda \) and \( \mu_1 = 1 \), it is always \( p \in \{0, \frac{1}{2}\} \).

Before proceeding by showing that, under the hypothesis of asymptotical similarity to a convergent stationary scheme, the sequence of masks \( \{c[k], k \geq 0\} \) is bounded uniformly independent of \( k \), we recall the definitions of asymptotical similarity and equivalence between subdivision schemes (see [3]).

**Definition 5.** The sequence of subdivision masks \( \{a[k], k \geq 0\} \) and the subdivision mask \( a \) are respectively asymptotically similar and asymptotically equivalent if

\[
\lim_{k \to \infty} \|a[k] - a\| = 0, \quad \text{respectively } \sum_{k=0}^{\infty} \|a[k] - a\| < \infty.
\]

Note that here and in the sequel, \( \| \cdot \| \) stands for the infinity norm for subdivision operators, sequences or functions, i.e., \( \|S_{a[k]}\| := \max \left\{ \sum_{\ell \in \mathbb{Z}} |a[k]|, \sum_{\ell \in \mathbb{Z}} |a[k]| \right\}, \|f\| := \sup_{x \in \mathbb{R}} |f(x)| \) and \( \|F\| := \sup_{x \in \mathbb{R}} |F(x)| \).

**Lemma 6.** Let \( \{S_n, n \geq 0\} \) be a non-stationary subdivision scheme and let \( \lim_{k \to \infty} a[k] = a \) with \( S_n \) a convergent stationary scheme. Suppose that \( a[k](z) \)

\[
2 - a[k](z^{1+\nu}) z^\nu = c[k](z)(1 - e^{\lambda z^{-k-1-n}} z) \tag{2.3}
\]

for some Laurent polynomial \( c[k](z) \), where \( \nu = 0 \) or \( \nu = 1 \). Then, \( \{c[k], k \geq 0\} \) is bounded uniformly independent of \( k \).

**Proof:** First consider the case \( \nu = 0 \). Since the mask \( a[k] \) is finitely supported in \([-M, M]\), we can write

\[
2 - a[k](z) = \sum_{\ell=-M}^M \tilde{a}_\ell z^\ell. \quad \text{Clearly } \{\tilde{a}_\ell]\}, \quad \ell \in \mathbb{Z} \}
\]

is uniformly bounded. Then, we can induce from (2.2) that \( c[k](z) \) is of the form \( c[k](z) = \sum_{\ell=-M}^{-1} c_\ell z^\ell \). Comparing the coefficients of the same power of \( z \) on both sides of equation (2.3), we get \( c_{-M} = \tilde{a}_{-M} \). Moreover, all coefficients \( c[k] \) are defined recursively as \( c[k+1] = c[k] + e^{\lambda z^{-k-1}} c[k], \quad \ell = -M, \ldots, M - 2 \). The latter clearly proves that \( \{c[k], k \geq 0\} \) is uniformly bounded independent of \( k \). The case \( \nu = 1 \) can be handled similarly.

Now, let \( \Phi := \{\varphi_0, \ldots, \varphi_{N-1}\} \) be a set of \( N \) linearly independent exponential polynomials in \( \mathcal{E}_\Lambda \). We define the \( N \times N \) Wronskian matrix of \( \Phi \) as \( W_\Phi(x) := \begin{pmatrix} \frac{d^n \varphi(x)}{dx^n} |_{x=0} \end{pmatrix}_{n=0}^{N-1} \). Throughout this paper,
we assume the Wronskian matrix $W_\Phi(x)$ to be invertible for any $x$ in a neighborhood of zero. Under this condition, for each $\beta = 0, \cdots, N - 1$, we introduce two functions $\mathcal{P}_{j,\beta}$ for $j = 0, 1$ defined by

$$
\mathcal{P}_{j,\beta}(x) := \sum_{n=0}^{N-1} \mu_{j,n} \varphi_n(x),
$$

so that the coefficient vector $m_\beta = (\mu_{j,n}, n = 0, \cdots, N - 1)$ is obtained by solving the Hermite interpolation problem

$$
\mathcal{P}_{j,\beta}(j2^{-k-1}) = \delta_{j,\ell}(-1)^{\ell!}, \quad \ell = 0, \ldots, N - 1,
$$

translating into the linear system $W_\Phi(j2^{-k-1}) \cdot m_\beta = c_\beta^T$, with $c_\beta := (\delta_{j,\ell}(-1)^{\ell!}, \ell = 0, \cdots, N - 1)$. Note that, if $j = 1$, the coefficient vector $m_\beta$ indeed depends on $k$ and $j$, but we abbreviate it to simplify the notation. It is clear that for a given $\beta$ the vector $m_\gamma$ can be bounded independently of $k$. It implies that the $\alpha$-th derivative of $\mathcal{P}_{j,\beta}$ for $\alpha = 0, \ldots, N - 1$ is uniformly bounded on any compact set in $\mathbb{R}$. With this setting, we now prove that $\{S_{\alpha}(k), k \geq 0\}$ satisfies the approximate sum rules of order $N$, whose definition is here recalled for completeness (see [3]).

**Definition 7.** Let $N > 0$. The sequence of symbols $\{a^{[k]}(z), k \geq 0\}$ satisfies approximate sum rules of order $N$ if

$$
\sum_{k=0}^{\infty} |a^{[k]}(1) - 2| < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} 2^{k(N-1)} \sigma_k < \infty, \quad \text{for} \quad \sigma_k := \max_{\beta=0, \ldots, N-1} 2^{-k} \left| \frac{d^\beta}{dz^\beta} a^{[k]}(-1) \right|.
$$

**Theorem 8.** Let $\{a^{[k]}(z), k \geq 0\}$ be the Laurent polynomials associated with a non-stationary scheme $\{S_{\alpha}(k), k \geq 0\}$ which reproduces $N$ linearly independent exponential polynomials in $\mathcal{E}_\Lambda$, where $\dim \mathcal{E}_\Lambda = N$. Then, for any $\beta = 0, \cdots, N - 1$, we have

$$
|a^{[k]}(1) - 2| = O(2^{-kN}), \quad \left| \frac{d^\beta}{dz^\beta} a^{[k]}(-1) \right| = O(2^{-k(N-\beta)}), \quad k \to \infty.
$$

**Proof:** We first note that the $\beta$-th derivative of $a^{[k]}(z)$ evaluated at $z = -1$ can be expressed as

$$
\frac{d^\beta}{dz^\beta} a^{[k]}(-1) = \sum_{\ell=1}^{\beta} \theta_{\beta,\ell} \sum_{i \in \mathbb{Z}} a_i^{[k]} z^\ell(-1)^i
$$

for some constants $\theta_{\beta,\ell}$ with $\ell = 1, \cdots, \beta$. Thus, to verify (2.7), it is sufficient to show that

$$
\sum_{i \in \mathbb{Z}} a_i^{[k]} - 2 = O(2^{-kN}), \quad \sum_{i \in \mathbb{Z}} (-1)^i \varphi_i a_i^{[k]} = O(2^{-k(N-\beta)}), \quad k \to \infty.
$$

Our approach for this task is to separate the summation in the above equations into two parts and then estimate them separately. To do this, for simplicity, we introduce the notation

$$
Q_{j,\beta} := \sum_{i \in \mathbb{Z}} a_i^{[k]} \left( (j2^{-1} - i)2^{-k} \right)^\beta - \mathcal{P}_{j,\beta}(j2^{-k-1}), \quad j = 0, 1,
$$

with $\mathcal{P}_{j,\beta}(j2^{-k-1})$ in (2.5). Then, noticing that $\mathcal{P}_{0,\beta}(0) = \mathcal{P}_{1,\beta}(2^{-k-1}) = \delta_{\beta,0}$, we get the following relation

$$
\sum_{i \in \mathbb{Z}} a_i^{[k]} - 2 = Q_{0,0} + Q_{1,0} \quad \text{and} \quad 2^{-\beta(k+1)} \sum_{i \in \mathbb{Z}} (-1)^i \varphi_i a_i^{[k]} = Q_{0,\beta} - Q_{1,\beta}.
$$
Now, we first estimate $Q_{0, \beta}$. Then, the other term $Q_{1, \beta}$ can be handled similarly. Since $P_{0, \beta}$ is a linear combination of exponential polynomials in $E_{\lambda}$ and the non-stationary scheme $\{S_{n[k]}, \ k \geq 0\}$ reproduces $E_{\lambda}$, we get the identity
\[
P_{0, \beta}(0) = \sum_{i \in \mathbb{Z}} a_{i}[k] P_{0, \beta}(i2^{-k}).
\]

Plugging this into (2.9) for $j = 0$ leads to $Q_{0, \beta} = \sum_{i \in \mathbb{Z}} a_{i}[k]((i2^{-k})^\beta - P_{0, \beta}(i2^{-k}))$. Here, we will use the arguments of Taylor expansion for $P_{0, \beta}$. Let $T_{\beta}$ be the Taylor polynomial of the function $P_{0, \beta}$ of degree $N-1$ around 0, that is, $T_{\beta}(x) := T_{P_{0, \beta}, N, 0}(x) := \sum_{\ell=0}^{N-1} \frac{x^\ell}{\ell!} P_{0, \beta}^{(\ell)}(0)$. Then, we replace $P_{0, \beta}$ in $Q_{0, \beta}$ by its Taylor polynomial $T_{\beta}$ plus the remainder term (say, $R_{\beta}$), such that we have the form $P_{0, \beta}(i2^{-k}) = T_{\beta}(i2^{-k}) + R_{\beta}(i2^{-k})$. In fact, from the Hermite interpolation conditions, we find that $T_{\beta}(i2^{-k}) = (i2^{-k})^\beta$. Hence, it leads to the equations $Q_{0, \beta} = -\sum_{i \in \mathbb{Z}} a_{i}[k] R_{\beta}(i2^{-k})$. On the other hand, as discussed before, the coefficient vector $m_{\beta}$ for $P_{0, \beta}$ in (2.4) is bounded independently of $k$. It follows that each derivative of $P_{0, \beta}$ is uniformly bounded around the origin. Consequently, we get $R_{\beta}(i2^{-k}) = O(2^{-kN})$ and hence, $Q_{0, \beta} = O(2^{-kN}), \ k \to \infty$. Similarly, we can prove the same convergence rate for $Q_{1, \beta}$, namely $Q_{1, \beta} = O(2^{-kN})$ as $k \to \infty$. Combining these two convergence properties and applying the two equations in (2.10), we finally get
\[
\left| \sum_{i \in \mathbb{Z}} a_{i}[k] - 2 \right| = O(2^{-kN}) \quad \text{and} \quad \left| \sum_{i \in \mathbb{Z}} (-1)^i \beta a_{i}[k] \right| = O(2^{-k(N-\beta)}), \ k \to \infty.
\]

Referring back to the identity in (2.8), the proof is completed. \(\blacksquare\)

**Remark 9.** It is easy to see that (2.7) are not sufficient for the reproduction of exponential polynomials. As a counter example consider the following level-dependent perturbation of quadratic B-splines which is not reproducing any exponential polynomial while satisfying (2.7) with $N = 1$.
\[
a[k](z) = \frac{1}{4} + 2^{-k} + \left(\frac{3}{4} - 2^{-k}\right) z + \left(\frac{3}{4} - 2^{-k}\right) z^2 + \left(\frac{1}{4} - 2^{-k}\right) z^3, \quad a[k](1) = 2(1-2^{-k}), \quad a[k](-1) = 2^{-k+1}.
\]

We continue by replacing the assumption of reproduction of $N$ exponential polynomials plus reproduction of one of them, combined with asymptotical similarity to a convergent stationary scheme. Therefore, we here consider subdivision schemes $\{S_{n[k]}, \ k \geq 0\}$ satisfying the first condition in Definition 5.

**Theorem 10.** Let $\{a[k](z), k \geq 0\}$ be the Laurent polynomials associated with a non-stationary subdivision scheme $\{S_{n[k]}, \ k \geq 0\}$ which generates $N$ exponential polynomials in $E_{\lambda}$, where dim $E_{\lambda} = N$, and reproduces one of them. Moreover let $\lim_{k \to \infty} a[k] = a$ with $S_{a}$ a convergent stationary subdivision scheme. Then, for any $\beta = 0, \ldots, N-1$, we have
\[
|a[k](1) - 2| = O(2^{-k}), \quad \left| \frac{d^\beta}{dz^\beta} a[k](-1) \right| = O(2^{-k(N-\beta)}), \quad k \to \infty.
\]

**Proof:** Since all exponential polynomials in $E_{\lambda}$ are generated by the non-stationary subdivision scheme $\{S_{n[k]}, \ k \geq 0\}$, by (1.4), Theorem 1, we can write
\[
a[k](z) = b[k](z) \prod_{n=1}^{N} (1 + e^{\lambda_{n}2^{-k}z})^{\mu_{n}}, \quad k \geq 0.
\]

Letting $k \to \infty$ the left hand side of the equation above tends, by assumption, to $a(z)$ while the product $\prod_{n=1}^{N} (1 + e^{\lambda_{n}2^{-k}z})^{\mu_{n}}$ in the right hand side of (2.14) tends to $(1 + z)^{N}$. Therefore, we can conclude that
lim_{k \to \infty} b^{[k]}(z) = b(z) for a suitable b(z). The latter means that, for k large enough, \(|b^{[k]}| < c\) where c > 0 is a constant independent of k. Next, we observe that each \(\frac{d^k}{dz^k} a^{[k]}(z)\) contains at least \(N - \beta\) factors of the form \((1 + e^{k_0 z^{-2-k-1}} z)\) and since \(1 - e^{k_0 z^{-2-k-1}} = O(2^{-k})\) we conclude that \(\frac{d^k}{dz^k} a^{[k]}(-1) = O(2^{-k(N-\beta)})\) as \(k \to \infty\), thus proving the second part of the claim. To prove the first part of the claim, we use Theorem 5 with \(N = 1\) so completing the proof. \(\blacksquare\)

In view of \(\ref{3}\) we can thus state the following

**Corollary 11.** Let \(\{S_{n[k]}, k \geq 0\}\) be a non-stationary subdivision scheme which generates \(N\) exponential polynomials in \(E_\Lambda\) and reproduces one of them. Moreover, let the non-stationary scheme be asymptotically similar to a \(C^\ell\)-convergent subdivision scheme. Then, setting \(\rho := \min \{\ell, N - 1\}\) the non-stationary subdivision scheme \(\{S_{n[k]}, k \geq 0\}\) is at least \(C^\rho\)-convergent with Hölder exponent \(\rho + \alpha, \alpha \in (0, 1)\).

**Remark 12.** The following example aims at emphasizing that the assumptions in Theorem 10 do not guarantee asymptotical equivalence between \(\{S_{n[k]}, k \geq 0\}\) and \(S_n\). Let \(r_k = e^{k_0 z^{-2-k-1}}\). The subdivision mask

\[
a^{[k]} = \left\{ \frac{1}{(r^{-1}_k + r_k)} + \frac{1}{r^{-1}_k + r_k} + \frac{1}{r^{-1}_k + r_k}, \frac{(r^{-1}_k + r_k) + (1 - r^{-1}_k)}{r^{-1}_k + r_k} + \frac{1}{r^{-1}_k + r_k} \right\}
\]

is such that \(\lim_{k \to \infty} a^{[k]} = \left\{ \frac{1}{1 + \frac{1}{2}}, \frac{1}{1 + \frac{1}{2}} \right\}\). The associated symbols satisfy \(a^{[k]}(r_k^{-1}) = a^{[k]}(r_k^{-1}) = 0\) as well as \(a^{[k]}(r_k^{-1}) = 2r_k^{-1}\), so that \(e^{\lambda x_k}\) are generated whereas only \(e^{\lambda x_k}\) is reproduced. Thus, in view of Theorem \(\ref{14}\) we have \(|a^{[k]}(1 - 2) = O(2^{-k})\), \(|a^{[k]}(1)| = O(2^{-2k})\), \(|\frac{d}{dz} a^{[k]}(-1)| = O(2^{-k})\), \(k \to \infty\). However, since

\[
\sum_{k=0}^\infty \left| \frac{1}{(r^{-1}_k + r_k)} + \frac{1}{r^{-1}_k + r_k} + \frac{1}{r^{-1}_k + r_k} \right| \text{ is not convergent, then } \sum_{k=0}^\infty \|S_{n[k]} - S_n\| = \infty. \text{ Hence } \{S_{n[k]}, k \geq 0\}\text{ and } S_n\text{ are not asymptotically equivalent.}
\]

As previously emphasized in Remark \(\ref{4}\) for a non-stationary subdivision scheme \(\{S_{n[k]}, k \geq 0\}\) reproducing \(e^{\lambda x_k}\), we denote by \(p \in \{0, \frac{1}{2}\}\) the number such that \(a^{[k]}(r_k^{-1}) = 2r_k^{-p}\) with \(r_k = e^{k_0 z^{-2-k-1}}\). For these choices of \(p\), let \(\{S_{n[p]}, k \geq 0\}\) be the non-stationary subdivision scheme with \(k\)-level symbol

\[
h_p^{[k]}(z) = \frac{r_k^{-p}}{r_k^{-1} + r_k} z^{-1}(1 + r_k^{-1} z)(1 + r_k z), \quad k \geq 0.
\]

Since \(1 - r_k \leq c 2^{-k}\) for some \(c > 0\), it is immediate that the non-stationary scheme with symbol \(h_p^{[k]}(z)\) is asymptotically equivalent to the stationary scheme of the degree 1 B-spline, say \(B_1\), which is \(C^0\) and stable. Thus, the corresponding non-stationary scheme is also \(C^0\) and stable, and further, \(H_m \to B_1\) uniformly as \(m \to \infty\) \(\ref{3}\) Lemma 15]. For this scheme, we denote by \(\{H_m, m \geq 0\}\) the corresponding basic limit functions defined by

\[
H_m := H_{m, p} := \lim_{k \to \infty} S_{n[p]}(r_k^{m+k}) \ldots S_{n[p]}(r_k), \quad \delta = \{d_{0,i}, i \in \mathbb{Z}\}.
\]

**Remark 13.** It is clear from \(\ref{15}\) that \(h_p^{[k]}(-r_k^{-1}) = 0\) and \(h_p^{[k]}(r_k^{-1}) = 2r_k^{-p}\) with \(p \in \{0, \frac{1}{2}\}\). Also, the basic limit function \(H_m\) is supported in \([-1, 1]\) and \(H_m(0) = (e^{\lambda x_m} - p)\) such that if \(p = 0\), \(H_m\) is an interpolatory function and otherwise, it approximates to an interpolatory function as \(m \to \infty\).

Based on this observation, we will get a theorem extending \(\ref{4}, \text{Theorem 11}\) where the unnecessary assumption of reproduction of constants was assumed. The proof of this result follows standard arguments and it is similar to the one in the quoted reference after replacing the classical notion of difference operator with

\[
(\Delta_1^{[k]} f[k]) := f[k] - f[k]_{r_k-1} - f[k]_{r_k}, \quad k \geq 0 \text{ where } r_k := e^{k_0 z^{-2-k-1}}.
\]

Thus, in the proof of the following theorem we only provide an explicit description of those parts of the proof that differ from \(\ref{4}, \text{Theorem 11}\).
Theorem 14. Let \( \{S_{\lambda[k]}, k \geq 0\} \) be a non-stationary subdivision scheme reproducing \( e^{\lambda x} \) and let also \( \lim_{k \to \infty} \lambda[k] = \lambda \) with \( S_{\lambda[k]}, k \geq 0 \) a convergent stationary scheme. Further, let \( a[k](r_k^{-1}) = 2r_k^{-p} \) with \( r_k = e^{\lambda x^{-k}} - 1 \). Then, the scheme \( \{S_{\lambda[k]}, k \geq 0\} \) is convergent and there exist \( \mu \in (0,1) \) and \( K \) large enough such that for \( m \geq 0 \),
\[
\left\| \lim_{k \to \infty} S_{\lambda[m+k]} \cdots S_{\lambda[m]} f^{(0)} - F_m^{k+1} \right\| \leq C \mu^k, \quad k \geq K,
\]
where \( F_m^k \) is the function defined by \( F_m^k := \sum_{i \in \mathbb{Z}} (S_{\lambda[m+k-1]} \cdots S_{\lambda[m]} f^{(0)}), H_{m+k}(2^k \cdot -i) \) with \( H_{m+k} \) in (2.16) and \( C > 0 \) independent of \( m \).

Proof: We start by observing that the generation of \( e^{\lambda x} \) implies the existence of the difference scheme \( \{S_{\lambda[k]}, k \geq 0\} \) such that for \( f_n^{(k+1)} := S_{\lambda[m+k]} \cdots S_{\lambda[m]} f^{(0)} \) and for the difference operator in (2.17) we have \( \Delta_{\lambda[k]} f_n^{(k+1)} = S_{\lambda[k]} \Delta_{\lambda[k]} f_n^{(1)} \). Due to the fact that both \( a[k](z) \) and \( h_p[k](z) \) assume the same values at \( z = -r_k^{-1} \) and \( r_k^{-1} \) (i.e. 0 and \( 2r_k^{-p} \), respectively), the difference mask \( d[k] := a[k] - h_p[k] \) satisfies \( d[k](z) = (1 - r_k^2z^2)e[k](z) \) for some Laurent polynomial \( e[k](z) \). Using the reproduction properties of \( \{S_{\lambda[k]}, k \geq 0\} \) and the asymptotical similarity to the convergent scheme \( S_{\lambda[k]} \), it is not difficult to see that \( \|a[k]\| \leq C \) with a constant \( C > 0 \) independent of \( k \). Moreover, from \( \|a[k]\| \leq C \), we can prove that for the sequence \( \{f_n^{(k+1)}, k \geq 0\} \) we have
\[
\|\Delta_{\lambda[k]} f_n^{(k+1)}\| \leq C \mu^k, \quad \mu \in (0,1),
\]
with a constant \( C > 0 \) independent of \( m \) and \( k \) (see [4, Lemma 5]). The inequality (2.18) allows us to show that for the sequence of functions \( \{F_{m+1}, k \geq 0\} \) there exists \( K > 0 \) such that for \( k \geq K \), \( \|F_{m+1} - F_m\| \leq C \mu^k \) with \( \mu \in (0,1) \). Thus we can write
\[
\|F_{m+1} - F_m\| = \sum_{j=0}^{m-1} \|F_{k+j+1} - F_{k+j}\| \leq \sum_{j=0}^{m-1} C \mu^{k+j} \leq C \mu^k \frac{1}{1 - \mu},
\]
from which it follows that \( \{F_{m+1}, k \geq 0\} \) is a Cauchy sequence in the \( L_\infty \)-norm and therefore converges uniformly to a continuous limit. To show that such limit is exactly \( \lim_{k \to \infty} S_{\lambda[m+k]} \cdots S_{\lambda[m]} f^{(0)} \) we use standard arguments (see [6, Lemma 14]): since \( \{H_{m+k}, k \geq 0\} \) is a sequence of continuous, stable, compactly supported functions which approximate partition of unity uniformly, the uniform convergence of the sequence \( \{F_{m+1}, k \geq 0\} \) implies the uniform convergence of the subdivision scheme \( \{S_{\lambda[k]}, k \geq 0\} \) and also
\[
\lim_{k \to \infty} F_{m}^k = \lim_{k \to \infty} \sum_{i \in \mathbb{Z}} (S_{\lambda[m+k-1]} \cdots S_{\lambda[m]} f^{(0)}) H_{m+k}(2^k \cdot -i) = \lim_{k \to \infty} S_{\lambda[m+k-1]} \cdots S_{\lambda[m]} f^{(0)}
\]
so concluding the proof. \( \square \)

Theorem 15. Let \( a[k](z) \) be the Laurent polynomial at level \( k \) associated with a non-stationary scheme \( \{S_{\lambda[k]}, k \geq 0\} \) reproducing exponential polynomials in \( E_\lambda \) with \( \#\lambda = N \). Suppose that \( \lim_{k \to \infty} a[k] = a \), where \( a \) is the mask of a stationary scheme satisfying \( \sum_{i \in \mathbb{Z}} a[i+2j] = 1 \) for \( j = 0, 1 \). Then
\[
\sum_{i \in \mathbb{Z}} a[i+2j] = O(2^{-kN}), \quad k \to \infty \quad \text{for} \quad j = 0, 1.
\]

Proof: To simplify the notation let \( A_k,j := \sum_{i \in \mathbb{Z}} a[i+2j] \) and \( A_j := \sum_{i \in \mathbb{Z}} a[j+2i] \) for \( j = 0, 1 \). Using the fact that \( A_0 + A_1 = 2 \) and setting \( z = 1 \) in (2.2), in view of Theorem 3 and Lemma 6 we first get
\[
(A_0 - A_{k,0}) + (A_1 - A_{k,1}) = O(2^{-kN}), \quad k \to \infty
\]
Moreover, from Theorem 3 (with \( \beta = 0 \)) we obtain \( a[k]^{-1} = A_{k,0} - A_{k,1} = O(2^{-kN}) \) which is, of course, equivalent to \( (A_0 - A_{k,0}) - (A_1 - A_{k,1}) = O(2^{-kN}) \). Combining it with (2.21), the proof is concluded. \( \square \)

We are finally in a position to prove the theorem treating the convergence of the sequence of basic limit functions \( \{\phi_n, m \geq 0\} \), \( \phi_n := \lim_{k \to \infty} S_{\lambda[m+k]} \cdots S_{\lambda[m]} \delta \) as \( m \to \infty \) where \( \delta = \{\delta_{0,i}, i \in \mathbb{Z}\} \).
Theorem 16. Let \( \{S_{m|k}, k \geq 0\} \) be a convergent non-stationary subdivision scheme reproducing \( e^{\lambda z} \) with associated sequence of basic limit functions \( \{\phi_m, m \geq 0\} \). Let \( \lim_{m \to \infty} a^{|k|} = a \) with \( S_a \) a convergent stationary scheme with basic limit function \( \phi \). Then \( \lim_{m \to \infty} \|\phi_m - \phi\| = 0 \).

Proof: Let \( \ell \) be a non-negative integer, \( \delta_m^{[\ell]} := S_{m+|\ell|-1} \cdots S_{m|} \delta \) and \( \phi_m^{[\ell]} := \{\phi_m(i2^{-\ell}), i \in \mathbb{Z}\} \). Then we can write

\[
S_{m|k} \cdots S_{m|} \delta - S_{m|k+1} \delta = \sum_{j=0}^{k-1} S_{j}^{\ell} (S_{m|k-j} - S_{m|}) \delta_{m-j}^{[\ell]} + S_{m|k}^{\ell} (S_{m|} - S_{m|}) \delta.
\]

The last term on the right hand side of the above equation can be estimated as

\[
\|S_{m|}^{\ell} (S_{m|} - S_{m|}) \delta\| \leq \|S_{m|}^{\ell}\| \|a^{[m]} - a\|.
\]

Due to the convergence assumption of \( S_{m|} \), there exists \( C > 0 \) such that for all \( k \), \( \|S_{m|}^{\ell}\| < C \) (see [6, Section 2]). Hence, from the asymptotical similarity of \( \{a^{[k]}, k \geq 0\} \) and \( a \), it is immediate that

\[
\lim_{m \to \infty} \|S_{m|}^{\ell} (S_{m|} - S_{m|}) \delta\| = 0.
\]

To estimate the summation on the right hand side of (2.22) let us write \((S_{m|k-j} - S_{m|}) \delta_{m-j}^{[\ell]}\) as

\[
(S_{m|k-j} - S_{m|}) \delta_{m-j}^{[\ell]} = \sum_{\ell \in \mathbb{Z}} (a_{m-j+2\ell} - a_{m-j-2\ell}) \delta_{m-j}^{[\ell]}.
\]

Here, our approach for estimating the above expression is to approximate \( \delta_{m-j}^{[\ell]} \) by using the values in the basic limit functions \( \phi_m \) on the grids \( 2^{-(k-j)} \mathbb{Z} \), that is, \( \phi_{m-j}^{[\ell]} \). Specifically, we express \( \delta_{m-j}^{[\ell]} \) for each \( \ell \in \mathbb{Z} \) as

\[
(\delta_{m-j}^{[\ell]} = (\delta_{m-j}^{[\ell]} - (\phi_{m-j}^{[\ell]})) + (\phi_{m-j}^{[\ell]} - (\phi_{m-j}^{[\ell]})) + (\phi_{m-j}^{[\ell]}).
\]

In view of (2.23), one should aware that since the masks \( a^{[k]} \) and \( a \) have the same finite support, we need to consider \( \ell \) only around \( i \), say \( |i - \ell| \leq 2M \) for some \( M > 0 \). Now, to estimate the first term in the right hand side of (2.24), let \( p \) be the number such that \( a^{[k]}(r_k^{-1}) = 2r_k^{-p} \) with \( r_k = e^{\lambda 2^{-k-1}} \), and consider the sequence of functions \( \{F_{m|k}, k \geq 0\} \) defined by

\[
F_{m|k} := F_{m|p} := \sum_{i \in \mathbb{Z}} (\delta_{m-j}^{[\ell]}), \quad H_{m+k}(2^k, -i).
\]

where \( H_{m+k} \) is the basic limit function of the non-stationary scheme with symbol \( h_{p}^{z}(z), s \geq m + k \) in (2.15). Using the facts that \( H_{m+k} \) is supported in \([-1,1]\) and \( H_{m+k}(0) = (e^{\lambda 2^{-m-k}} - p) \) with \( p \in \{0, 1/2\} \) (see Remark [13]), we can get the bound

\[
|((\delta_{m-j}^{[\ell]} - F_{m|(2^k)}}| \leq 2|1 - e^{-p 2^{-m-k})| |(\delta_{m-j}^{[\ell]} - F_{m|(2^k)}|) \leq |C 2^{-m-k} \|\delta_{m-j}^{[\ell]}\|.
\]

Hence, applying this inequality and Theorem [14] to the term \( (\delta_{m-j}^{[\ell]} - (\phi_{m-j}^{[\ell]})) \) yields the bound

\[
|((\delta_{m-j}^{[\ell]} - (\phi_{m-j}^{[\ell]}))| \leq C 2^{(k-j) + \mu^{k-j-1}}, \quad \mu \in (0, 1),
\]

for some constance \( C > 0 \) independent of \( m \). Moreover, to estimate \( (\phi_{m-j}^{[\ell]} - (\phi_{m-j}^{[\ell]})) \) with \( |i - \ell| \leq 2M \) for some \( M > 0 \), we apply the Hölder continuity of \( \phi_m \) (see Corollary [14]) to get the bound

\[
|\phi_m(2^{-(k-j)}) - \phi_m(i2^{-(k-j)})| \leq 2M 2^{-\alpha(k-j)}, \quad \alpha \in (0, 1).
\]
Clearly, the function $\psi$ singularity of this linear system is guaranteed by the assumed condition on the Wronskian matrix.

For the given smoothness, $\psi$ is of the form $c$ with $m = \infty$ in the space $S_{\alpha_d}$ associated norm is defined by $\|f\|_{S_{\alpha_d}} := \sum_{t=0}^{\infty} ||f^{(t)}||_{L_{\infty}({\mathbb R})}$. Therefore, under the assumption that the initial data $f^{[0]}$ is of the form $f^{[0]}(i) = f(2^{-m}i), i \in \mathbb{Z}$ for some $m \geq 0$ with $f$ denoting a smooth function in $W_\infty^m(\mathbb{R})$, in the sequel we estimate the convergence order of the error $f - g^{[0]}$ as $m \to \infty$ where $g^{[0]}$ is the limit of the subdivision scheme. We recall that, in view of the linearity of the subdivision operators, $g^{[0]} = \sum_{i \in \mathbb{Z}} \phi_m(2^m \cdot -i)$, with $\phi_m$ the $m$-th basic limit function.

We emphasize that the approximation order result discussed in the following theorem extends the result in \cite{11} where asymptotical equivalence rather than similarity is assumed.

**Theorem 17.** Assume that a non-stationary scheme $\{S_{\alpha_d}, k \geq 0\}$ reproduces the exponential polynomials in the $N$-dimensional space $\mathcal{E}_\lambda$. Assume further that $\{\phi_m, m \geq 0\}$ is uniformly bounded. If the initial data is of the form $f^{[0]} := \{f^{[0]}_i = f(2^{-m}i), i \in \mathbb{Z}\}$ for some $m \geq 0$ and for a function $f \in W_\infty^m(\mathbb{R})$, $\gamma \in \mathbb{N}$, then

$$||g^{[0]} - f||_{L_\infty(\mathbb{R})} \leq c_f 2^{-\min(\gamma,N)m},$$

with $c_f > 0$ denoting a constant depending on $f$ but independent of $m$.

**Proof:** It is enough to consider the case $\gamma \leq N$. Let $x$ be a fixed point in $\mathbb{R}$ and let $\mathcal{E}_\gamma := \text{span}\{\varphi_1, \ldots, \varphi_N\}$. For the given smoothness of the function $f$, choose $\gamma$ linearly independent exponential polynomials from the space $\mathcal{E}_\gamma$, say $\Phi_\gamma := \{\varphi_1, \ldots, \varphi_\gamma\}$, so that the corresponding Wronskian matrix $W_\Phi(x)$ is invertible at $x = 0$, i.e., $\det(W_\Phi(0)) \neq 0$. Then our idea to prove this theorem is to employ another auxiliary function $\psi$ defined as a linear combination of $\varphi_\gamma(\cdot - x), \ldots, \varphi_\gamma(\cdot - x)$, that is, $\psi := \sum_{n=1}^{\gamma} \psi_1 \varphi_n(\cdot - x)$, so that the coefficient vector $d := (d_i, i = 1, \ldots, \gamma)$ is obtained by solving the linear system

$$\psi^{(r)}(x) = f^{(r)}(x), \quad r = 0, \ldots, \gamma - 1,$$

which can be written in the matrix form $W_\Phi(0) \cdot d^T = f^T$ with $f := (f^{(r)}(x), r = 0, \ldots, \gamma - 1)^T$. The non-singularity of this linear system is guaranteed by the assumed condition on the Wronskian matrix $W_\Phi(0)$. Clearly, the function $\psi$ also belongs to the space $\mathcal{E}_\gamma$. Then, since the non-stationary scheme $\{S_{\alpha_d}, k \geq 0\}$
requires an estimate from the bound \(3.2\). Thus, using the expression of \(g_{f^{[0]}}\) in terms of basic limit functions we now estimate the difference \(f - g_{f^{[0]}}\). By the construction of \(\psi\) in (3.1), \(f(x) = \psi(x)\). Then

\[
|f(x) - g_{f^{[0]}}(x)| = \sum_{i \in \mathbb{Z}} \phi_m(2^m x - i) (f(2^{-m} i) - \psi(2^{-m} i) - f(2^{-m} i)).
\]

Now, let \(T_{\psi}\) be the degree-\((\gamma - 1)\) Taylor polynomial of a function \(g \in C^{\gamma-1}(\mathbb{R})\) around \(x\), that is, \(T_{\psi} := \sum_{\ell=0}^{\gamma-1} (\gamma - \ell) g^{(\ell)}(x)/\ell!\). Then, consider Taylor expansions \(T_{\psi}\) and \(T_{f}\) of both functions \(\psi\) and \(f\). Due to the condition \(\psi^{(r)}(x) = f^{(r)}(x)\) for \(r = 0, \ldots, \gamma - 1\) in (3.1), it is obvious that \(T_{\psi}(2^{-m} i) = T_{f}(2^{-m} i)\). Hence, applying the remainder form of Taylor expansion, we get

\[
|f(x) - g_{f^{[0]}}(x)| \leq 2^{-m \gamma} \sum_{i \in \mathbb{Z}} |\phi_m(2^m x - i) (f(2^{-m} i) - \psi(2^{-m} i) - f(2^{-m} i))|
\]

for some \(\xi_i\) between \(x\) and \(\pm 2^{-m}\). By (3.1) we are able to write \(|\psi^{(r)}(\xi_i)| \leq c \|f\|_{\gamma, R}\) for some constant \(c > 0\) independent of \(x\) and \(\pm 2^{-m}\). Thus, it is immediate that

\[
|f(x) - g_{f^{[0]}}(x)| \leq c \gamma 2^{-m \gamma} \|f\|_{\gamma, R} \sum_{i \in \mathbb{Z}} |\phi_m(2^m x - i)(2^m x - i)\gamma|
\]

for some \(c_\gamma > 0\). By Theorem 10, \(\phi_m\) is uniformly bounded independent of \(m\). Moreover, since \(\phi_m\) is compactly supported, \(\# \{i \in \mathbb{Z}: \phi_m(2^m x - i) \neq 0\} \leq c\) for any \(x\) and \(m\). Therefore we can induce the required estimate from the bound (3.2). \(\blacksquare\)

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