ALMOST COMPLEX STRUCTURES THAT ARE HARMONIC MAPS

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Abstract. We find geometric conditions on a four-dimensional almost Hermitian manifold under which the almost complex structure is a harmonic map or a minimal isometric imbedding of the manifold into its twistor space.

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1. Introduction

Recall that an almost complex structure on a Riemannian manifold \((M, g)\) is called almost Hermitian if it is \(g\)-orthogonal. If a Riemannian manifold admits an almost Hermitian structure, it has many such structures. One way to see this is to consider the twistor bundle \(\pi : Z \to M\) whose fibre at a point \(p \in M\) consists of all \(g\)-orthogonal complex structures \(J_p : T_p M \to T_p M\) \((J^2_p = -Id)\) on the tangent space of \(M\) at \(p\). The fibre is the compact Hermitian symmetric space \(O(2n)/U(n)\) and its standard metric \(-\frac{1}{2}\text{Trace} J_1 \circ J_2\) is Kähler-Einstein. The twistor space admits a natural Riemannian metric \(h\) such that the projection map \(\pi : (Z, h) \to (M, g)\) is a Riemannian submersion with totally geodesic fibres. Now suppose that \((M, g)\) admits an almost Hermitian structure \(J\), i.e. a section of the bundle \(\pi : Z \to M\). Take a section \(V\) with compact support \(K\) of the bundle \(J^* \mathcal{V} \to M\), the pull-back under \(J\) of the vertical bundle \(\mathcal{V} \to Z\). There exists \(\varepsilon > 0\) such that, for every point \(I\) of the compact set \(J(K)\), the exponential map \(\exp_I\) is a diffeomorphism of the \(\varepsilon\)-ball in \(T_I Z\). Set \(J_t(p) = \exp_{J_I(p)}[tV(p)]\) for \(p \in M\) and \(t \in (-\varepsilon, \varepsilon)\). Then \(J_t\) is a section of \(Z\), i.e. an almost Hermitian structure on \((M, g)\) (such that \(J_t = J\) on \(M \setminus K\)).

Thus it is natural to seek for "reasonable" criteria that distinguish some of the almost Hermitian structures on a given Riemannian manifold (cf., for example, [5, 19, 20]). Motivated by the harmonic maps theory, C. Wood [19, 20] has suggested to consider as "optimal" those almost Hermitian structures \(J : (M, g) \to (Z, h)\) that are critical points of the energy functional under variations through sections of \(Z\), i.e. that are harmonic sections of the twistor bundle. In general, these critical points are not harmonic maps, but, by analogy, they are referred to as "harmonic almost complex structures" in [19, 20]. The almost Hermitian structures that are critical points of the energy functional under variations through all maps \(M \to Z\) are genuine harmonic maps and the purpose of this paper is to find geometric conditions

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on a four-dimensional almost Hermitian manifold \((M, g, J)\) under which the almost complex structure \(J\) is a harmonic map of \((M, g)\) into \((Z, h)\). We also find conditions for minimality of the submanifold \(J(M)\) of the twistor space. As is well-known, in dimension four, there are three basic classes in the Gray-Hervella classification \([12]\) of almost Hermitian structures - Hermitian, almost Kähler (symplectic) and Kähler structures. If \((g, J)\) is Kähler, the map \(J : (M, g) \to (Z, h)\) is a totally geodesic isometric imbedding. In the case of a Hermitian structure, we express the conditions for harmonicity and minimality of \(J\) in terms of the Lee form, the Ricci and star-Ricci tensors. Several examples illustrating these results are discussed in the last section of the paper.

2. Preliminaries

Let \((M, g)\) be an oriented Riemannian manifold of dimension four. The metric \(g\) induces a metric on the bundle of two-vectors \(\pi : \Lambda^2 TM \to M\) by the formula
\[
g(v_1 \wedge v_2, v_3 \wedge v_4) = \frac{1}{2} \text{det} [g(v_i, v_j)].
\]

The Levi-Civita connection of \((M, g)\) determines a connection on the bundle \(\Lambda^2 TM\), both denoted by \(\nabla\), and the corresponding curvatures are related by
\[
R(X \wedge Y)(Z \wedge T) = R(X, Y)Z \wedge T + Z \wedge R(X, Y)T
\]
for \(X, Y, Z, T \in TM\). The curvature operator \(\mathcal{R}\) is the self-adjoint endomorphism of \(\Lambda^2 TM\) defined by
\[
g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X, Y)Z, T)
\]

Let us note that we adopt the following definition for the curvature tensor \(R : R(X, Y) = \nabla_X (\nabla_Y - \nabla_Y \nabla_X)\).

The Hodge star operator defines an endomorphism \(*\) of \(\Lambda^2 TM\) with \(*^2 = Id\).

Hence we have the orthogonal decomposition
\[
\Lambda^2 TM = \Lambda_+^2 TM \oplus \Lambda_-^2 TM
\]
where \(\Lambda_\pm^2 TM\) are the subbundles of \(\Lambda^2 TM\) corresponding to the \((\pm 1)\)-eigenvalues of the operator \(*\).

Let \((E_1, E_2, E_3, E_4)\) be a local oriented orthonormal frame of \(TM\). Set
\[
s_1 = E_1 \wedge E_2 + E_3 \wedge E_4, \quad s_2 = E_1 \wedge E_3 + E_4 \wedge E_2, \quad s_3 = E_1 \wedge E_4 + E_2 \wedge E_3.
\]

Then \((s_1, s_2, s_3)\) is a local orthonormal frame of \(\Lambda_+^2 TM\) defining an orientation on \(\Lambda_+^2 TM\), which does not depend on the choice of the frame \((E_1, E_2, E_3, E_4)\).

For every \(a \in \Lambda^2 TM\), define a skew-symmetric endomorphism \(K_a\) of \(T_{\pi(a)}M\) by
\[
g(K_a X, Y) = 2g(a, X \wedge Y), \quad X, Y \in T_{\pi(a)}M.
\]

Note that, denoting by \(G\) the standard metric \(-\frac{1}{2}\text{Trace} PQ\) on the space of skew-symmetric endomorphisms, we have \(G(K_a, K_b) = 2g(a, b)\) for \(a, b \in \Lambda^2 TM\). If \(\sigma \in \Lambda_+^2 TM\) is a unit vector, then \(K_{\sigma}\) is a complex structure on the vector space \(T_{\pi(a)}M\) compatible with the metric and the orientation of \(M\). Conversely, the 2-vector \(\sigma\) dual to one half of the fundamental 2-form of such a complex structure is
a unit vector in $\Lambda^2_T M$. Thus the unit sphere subbundle $Z$ of $\Lambda^2_T M$ parametrizes the complex structures on the tangent spaces of $M$ compatible with its metric and orientation. This subbundle is called the twistor space of $M$.

The Levi-Civita connection $\nabla$ of $M$ preserves the bundles $\Lambda^2_T M$, so it induces a metric connection on each of them denoted again by $\nabla$. The horizontal distribution of $\Lambda^2_T M$ with respect to $\nabla$ is tangent to the twistor space $Z$. Thus we have the decomposition $T Z = H \oplus V$ of the tangent bundle of $Z$ into horizontal and vertical components. The vertical space $\nabla V = \{ V \in T \tau Z : \pi_* V = 0 \}$ at a point $\tau \in Z$ is the tangent space to the fibre of $Z$ through $\tau$. Considering $T \tau Z$ as a subspace of $T_\tau (\Lambda^2_T M)$ (as we shall always do), $\nabla V$ is the orthogonal complement of $\tau$ in $\Lambda^2_T (\pi_\tau) M$. The map $V \ni V \rightarrow K_V$ gives an identification of the vertical space with the space of skew-symmetric endomorphisms of $T_\tau (\pi_\tau) M$ that anti-commute with $K_\tau$. Let $s$ be a local section of $Z$ such that $s(p) = \tau$ where $p = \pi(\tau)$. Considering $s$ as a section of $\Lambda^2_T M$, we have $\nabla X s \in \nabla V$ for every $X \in T_p M$ since $s$ has a constant length. Moreover, $X^b_s = s_* X - \nabla X s$ is the horizontal lift of $X$ at $\tau$.

Denote by $\times$ the usual vector cross product on the oriented 3-dimensional vector space $\Lambda^2_T M$, $p \in M$, endowed with the metric $g$. Then it is easy to check that

$$g(R(a) b, c) = g(R(b \times c), a)$$

for $a \in \Lambda^2_T M$, $b, c \in \Lambda^2_T M$. It is also easy to show that for every $a, b \in \Lambda^2_T M$

$$K_a \circ K_b = -g(a, b)Id + K_{a \times b}.$$  

(4)

For every $t > 0$, define a Riemannian metric $h_t$ by

$$h_t (X^b_a + V, Y^b_a + W) = g(X, Y) + t g(V, W)$$

for $\sigma \in Z$, $X, Y \in T_{\pi(\sigma)} M$, $V, W \in \nabla \sigma$.

The twistor space $Z$ admits two natural almost complex structures that are compatible with the metrics $h_t$. One of them has been introduced by Atiyah, Hitchin and Singer who have proved that it is integrable if and only if the base manifold is anti-self-dual [2]. The other one, introduced by Eells and Salamon, although never integrable, plays an important role in harmonic maps theory [9].

The action of $SO(4)$ on $\Lambda^2\mathbb{R}^4$ preserves the decomposition $\Lambda^2 \mathbb{R}^4 = \Lambda^2_+ \mathbb{R}^4 \oplus \Lambda^2_- \mathbb{R}^4$. Thus, considering $S^2$ as the unit sphere in $\Lambda^2 \mathbb{R}^4$, we have an action of the group $SO(4)$ on $S^2$. Then, if $SO(M)$ denotes the principal bundle of the oriented orthonormal frames on $M$, the twistor space $Z$ is the associated bundle $SO(M) \times_{SO(4)} S^2$. It follows from the Vilms theorem (see, for example, [3, Theorem 9.59]) that the projection map $\pi : (\tilde{Z}, h_t) \rightarrow (M, g)$ is a Riemannian submersion with totally geodesic fibres (this can also be proved by a direct computation).

Let $(N, x_1, \ldots, x_4)$ be a local coordinate system of $M$ and let $(E_1, \ldots, E_4)$ be an oriented orthonormal frame of $T M$ on $N$. If $(s_1, s_2, s_3)$ is the local frame of $\Lambda^2_T M$ define by (1), then $\bar{x}_a = x_a \circ \pi$, $y_j(\tau) = g(\tau, (s_j \circ \pi)(\tau))$, $1 \leq a \leq 4$, $1 \leq j \leq 3$, are local coordinates of $\Lambda^2_T M$ on $\pi^{-1}(N)$.

The horizontal lift $X^h$ on $\pi^{-1}(N)$ of a vector field

$$X = \sum_{a=1}^4 X^a \frac{\partial}{\partial x^a}$$


is given by
\[
X^h = 4 \sum_{a=1}^{4} (X^a \circ \pi) \frac{\partial}{\partial x_a} - \sum_{j,k=1}^{3} y_{j}(g(\nabla X s_j, s_k) \circ \pi) \frac{\partial}{\partial y_k}.
\]  

(5)

Hence
\[
[X^h, Y^h] = [X, Y]^h + \sum_{j,k=1}^{3} y_{j}(g(R(X \wedge Y)s_j, s_k) \circ \pi) \frac{\partial}{\partial y_k}.
\]  

(6)

for every vector fields \(X, Y\) on \(N\). Let \(\tau \in \mathbb{Z}\). Using the standard identification \(T_{\tau}(\Lambda^2 T_{\pi(\tau)} M) \cong \Lambda^2 T_{\pi(\tau)} M\), we obtain from (6) the well-known formula
\[
[X^h, Y^h]_{\tau} = [X, Y]^h_{\tau} + R_p(X \wedge Y)_{\tau}, \quad p = \pi(\tau).
\]  

(7)

Denote by \(D\) the Levi-Chivita connection of \((\mathbb{Z}, h_t)\). Then we have the following

**Lemma 1.** ([7]) If \(X, Y\) are vector fields on \(M\) and \(V\) is a vertical vector field on \(\mathbb{Z}\), then
\[
(D_X Y^h)_{\tau} = (\nabla_X Y)^h_{\tau} + \frac{1}{2} R_p(X \wedge Y)_{\tau},
\]  

(8)

\[
(D_V X^h)_{\tau} = \mathcal{H}(D_X V)_{\tau} = -\frac{t}{2} (R_p(\tau \times V) X)^h_{\tau}
\]  

(9)

where \(\tau \in \mathbb{Z}\), \(p = \pi(\tau)\), and \(\mathcal{H}\) means "the horizontal component".

**Proof.** Identity (8) follows from the Koszul formula for the Levi-Chivita connection and (7).

Let \(W\) be a vertical vector field on \(\mathbb{Z}\). Then
\[
h_t(D_V X^h, W) = -h_t(X^h, D_V W) = 0
\]

since the fibres are totally geodesic submanifolds, so \(D_V W\) is a vertical vector field. Therefore \(D_V X^h\) is a horizontal vector field. Moreover, \([V, X^h]\) is a vertical vector field, hence \(D_V X^h = \mathcal{H}D_X V\). Thus
\[
h_t(D_V X^h, Y^h) = h_t(D_X V, Y^h) = -h_t(V, D_X Y^h).
\]

Now (9) follows from (8) and (3).

Let \((M, g, J)\) be an almost Hermitian manifold of dimension four. Define a section \(J\) of \(\Lambda^2 TM\) by
\[
g(J, X \wedge Y) = \frac{1}{2} g(JX, Y), \quad X, Y \in TM.
\]

Note that the section \(2J\) is dual to the fundamental 2-form of \((M, g, J)\). Consider \(M\) with the orientation induced by the almost complex structure \(J\). Then \(J\) takes its values in the twistor space \(\mathbb{Z}\) of the Riemannian manifold \((M, g)\).

Let \(J_1\) and \(J_2\) be the Atyah-Hitchin-Singer and Eells-Salamon almost complex structures on the twistor space \(\mathbb{Z}\). It is well-known (and easy to see) that the map \(J : (M, J) \to (\mathbb{Z}, J_1)\) is holomorphic if and only if the almost complex structure \(J\) is integrable, while \(J : (M, J) \to (\mathbb{Z}, J_2)\) is holomorphic if and only if \(J\) is symplectic (i.e. \((g, J)\) is an almost Kähler structure).

In this note we are going to find geometric conditions under which the map \(J : (M, g) \to (\mathbb{Z}, h_t)\) that represents \(J\) is harmonic.
Let $\mathfrak{J}^{-1}T\mathcal{Z} \to M$ be the pull-back of the bundle $T\mathcal{Z} \to \mathcal{Z}$ under the map $\mathfrak{J} : M \to \mathcal{Z}$. Then we can consider the differential $\mathfrak{J}_* : TM \to T\mathcal{Z}$ as a section of the bundle $\text{Hom}(TM, \mathfrak{J}^{-1}T\mathcal{Z}) \to M$. Denote by $D^{(3)}$ the connection on $\mathfrak{J}^{-1}T\mathcal{Z}$ induced by the Levi-Civita connection $\nabla$ on $TM$ and the connection $D^{(3)}$ on $\mathfrak{J}^{-1}T\mathcal{Z}$ induce a connection $\nabla$ on the bundle $\text{Hom}(TM, \mathfrak{J}^{-1}T\mathcal{Z})$. The map $\mathfrak{J} : (M, g) \to (\mathcal{Z}, h_\ell)$ is harmonic if

$$\text{Trace}_g \nabla \mathfrak{J}_* = 0.$$ (cf., for example, [8]). Recall also that the map $\mathfrak{J} : (M, g) \to (\mathcal{Z}, h_\ell)$ is totally geodesic if $\nabla \mathfrak{J}_* = 0$.

**Proposition 1.** For every $X, Y \in T_p M$, $p \in M$,

$$\nabla \mathfrak{J}_*(X, Y) = \frac{1}{2}(\nabla^2_{XY} \mathfrak{J} - g(\nabla^2_X \mathfrak{J}, \mathfrak{J})\mathfrak{J}(p) + \nabla^2_X \mathfrak{J} - g(\nabla^2_Y \mathfrak{J}, \mathfrak{J})\mathfrak{J}(p)$$

$$-t(R_p(3 \times \nabla \mathfrak{J})Y)^h - t(R_p(3 \times \nabla \mathfrak{J})X)^h)$$

where $\nabla^2_X \mathfrak{J} = \nabla_X \nabla \mathfrak{J} - \nabla_{\nabla \mathfrak{J}} \mathfrak{J}$ is the second covariant derivative of $\mathfrak{J}$.

**Proof.** Extend $X$ and $Y$ to vector fields in a neighbourhood of the point $p$. Take an oriented orthonormal frame $E_1, \ldots, E_4$ near $p$ such that $E_3 = J E_2$, $E_4 = J E_1$, so $\mathfrak{J} = s_3$. Define coordinates $(\tilde{x}_a, y_j)$ as above by means of this frame and a coordinate system of $M$ at $p$. Set

$$V_1 = (1 - y_2^2)^{-1/2}(y_3 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_3}),$$

$$V_2 = (1 - y_2^2)^{-1/2}(-y_1 y_2 \frac{\partial}{\partial y_1} + (1 - y_2^2) \frac{\partial}{\partial y_2} - y_2 y_3 \frac{\partial}{\partial y_3}).$$

Then $V_1, V_2$ is a $g$-orthonormal frame of vertical vector fields in a neighbourhood of the point $\sigma = \mathfrak{J}(p)$ such that $V_1 \circ \mathfrak{J} = s_1$, $V_2 \circ \mathfrak{J} = s_2$. Note also that $[V_1, V_2]_\sigma = 0$. This and the Koszul formula imply $(D_{V_1} V_1)_\sigma = 0$ since $D_{V_1} V_1$ are vertical vector fields, $k, l = 1, 2$. Thus $D_W V_l = 0$, $l = 1, 2$, for every vertical vector $W$ at $\sigma$. We have

$$\mathfrak{J}_* \circ Y = Y^h \circ \mathfrak{J} + \nabla_Y \mathfrak{J} = Y^h \circ \mathfrak{J} + \sum_{k=1}^2 g(\nabla_Y \mathfrak{J}, s_k)(V_k \circ \mathfrak{J}),$$

hence

$$D_X^{(3)}(\mathfrak{J}_* \circ Y) = (D_{\mathfrak{J}_* X} Y^h) \circ \mathfrak{J} + \sum_{k=1}^2 g(\nabla_Y \mathfrak{J}, s_k)(D_{\mathfrak{J}_* X} V_k) \circ \mathfrak{J}$$

$$+ \sum_{k=1}^2 g(\nabla_X \nabla_Y \mathfrak{J}, s_k) + g(\nabla_Y \mathfrak{J}, \nabla_X s_k)](V_k \circ \mathfrak{J})$$

This, in view of Lemma 1, implies

$$D^{(3)}_{X^h}(\mathfrak{J}_* \circ Y) = (\nabla_X Y)_\sigma^h + \frac{1}{2} R(X, Y)_\sigma$$

$$-\frac{t}{2}(R_p(3 \times \nabla \mathfrak{J})Y)^h - \frac{t}{2}(R_p(3 \times \nabla \mathfrak{J})X)^h$$

$$+ \sum_{k=1}^2 g(\nabla_X \nabla_Y \mathfrak{J}, s_k) s_k(p) + \sum_{k=1}^2 [g(\nabla_Y \mathfrak{J}, s_k)[X^h, V_k]_\sigma + g(\nabla_Y \mathfrak{J}, \nabla_X s_k)s_k(p)].$$

An easy computation using (5) gives

$$[X^h, V_1]_\sigma = g(\nabla_X s_1, s_2)s_2(p), \quad [X^h, V_2]_\sigma = g(\nabla_X s_2, s_1)s_1(p).$$
These identities imply
\[ \sum_{k=1}^{2} [g(\nabla Y \tilde{J}, s_k)[X^b, V_k] + g(\nabla Y \tilde{J}, \nabla X s_k)s_k(p)] = 0. \]

Since \( g(\nabla X \tilde{J}, \tilde{J}) = 0 \) for every \( X \in T_p M \), we have
\[ g(\nabla Y \nabla X \tilde{J}, \tilde{J}) = -g(\nabla X \tilde{J}, \nabla Y \tilde{J}) = g(\nabla X \nabla Y \tilde{J}, \tilde{J}) \]

Hence
\[ \sum_{k=1}^{2} g(\nabla X \nabla Y \tilde{J}, s_k)s_k(p) = \nabla X \nabla Y \tilde{J} - \frac{1}{2} [g(\nabla X \nabla Y \tilde{J} + \nabla Y \nabla X \tilde{J}, \tilde{J})(p). \]

It follows that
\[ \tilde{\nabla} J^* (X, Y) = D^{(2)}_{X Y} (J^* \circ Y) - (\nabla X Y)^b - \nabla \nabla X Y \tilde{J} \]
\[ = \frac{1}{2} [\nabla X \nabla Y \tilde{J} - \nabla \nabla X \nabla Y \tilde{J} - g(\nabla X \nabla Y \tilde{J}, \tilde{J})(p) \]
\[ + \nabla Y \nabla X \tilde{J} - \nabla \nabla Y X \tilde{J} - g(\nabla Y \nabla X \tilde{J}, \tilde{J})(p) \]
\[ - t(R_p(\tilde{J} \times \nabla X \tilde{J}, Y)^b - t(R_p(\tilde{J} \times \nabla Y \tilde{J}, X)^b) \]. \]

**Corollary 1.** If \((M, g, J)\) is a Kähler surface, the map \( \tilde{J} : (M, g) \rightarrow (\mathbb{Z}, h) \) is a totally geodesic isometric imbedding.

**Remark 1.** By a result of C. Wood [19, 20], \( J \) is a harmonic almost complex structure, i.e. \( \tilde{J} \) a harmonic section of the twistor space \((\mathbb{Z}, h) \rightarrow (M, g)\) if and only if \([J, \nabla^* \nabla J] = 0\) where \( \nabla^* \nabla \) is the rough Laplacian. Taking into account that \( \nabla^* \nabla J = -\text{Trace} \nabla^2 J \), one can see that the latter condition is equivalent to
\[ g(\text{Trace} \nabla^2 J, X \wedge Y - J X \wedge J Y) = 0, \quad X, Y \in TM, \]
which is equivalent to \( \nabla \text{Trace} \nabla^2 J = 0 \). Thus, by Proposition 1, \( \tilde{J} \) is a harmonic section if and only if
\[ \nabla \text{Trace} \nabla^2 \tilde{J} = 0. \]

**3. Harmonicity of \( \tilde{J} \)**

Let \( \Omega(X, Y) = g(JX, Y) \) be the fundamental 2-form of the almost Hermitian manifold \((M, g, J)\). Denote by \( N \) the Nijenhuis tensor of \( J \)
\[ N(Y, Z) = -[Y, Z] + [JY, JZ] - J[Y, JZ] - J[JY, Z]. \]
It is well-known (and easy to check) that
\[ 2g((\nabla X J)(Y), Z) = d\Omega(X, Y, Z) - d\Omega(X, JY, JZ) + g(N(Y, Z), JX), \] (10)
for all \( X, Y, Z \in TM \).
3.1. The case of integrable $J$. Suppose that the almost complex structure $J$ is integrable. This is equivalent to $(\nabla_X J)(Y) = (\nabla_J X)(J Y)$, $X, Y \in TM$ [11, Corollary 4.2]. Let $B$ be the vector field on $M$ dual to the Lee form $\theta = -\delta \Omega \circ J$ with respect to the metric $g$. Then (10) and the identity $d\Omega = \theta \wedge \Omega$ imply the following well-known formula

$$2(\nabla_X J)(Y) = g(JX, Y)B - g(B, Y)JX + g(X, Y)JB - g(JB, Y)X.$$  

(11)

We have

$$g(\nabla_X \tilde{\Omega}, Y \wedge Z) = \frac{1}{2}g((\nabla_X J)(Y), Z)$$

and it follows that

$$\nabla_X \tilde{\Omega} = \frac{1}{2}(JX \wedge B + X \wedge JB).$$  

(12)

The latter identity implies

$$\nabla^2_{XY} \tilde{\Omega} = \frac{1}{2}((\nabla_X J)(Y) \wedge B + Y \wedge (\nabla_X J)(B) + JY \wedge \nabla_X B + Y \wedge J\nabla_X B].$$  

(13)

Now recall that the $*$-Ricci tensor $\rho^*$ of the almost Hermitian manifold $(M, g, J)$ is defined by

$$\rho^*(X, Y) = \text{trace}(Z \to R(JZ, X)JY).$$

Note that

$$\rho^*(JX, JY) = \rho^*(X, X),$$  

(14)

in particular $\rho^*(X, JX) = 0$.

Denote by $\rho$ the Ricci tensor of the Riemannian manifold $(M, g)$.

**Theorem 1.** Suppose that the almost complex structure $J$ is integrable. Then the map $\tilde{\Omega} : (M, g) \to (\mathcal{Z}, h)$ is harmonic if and only if $d\theta$ is a $(1,1)$-form and $\rho(X, B) = \rho^*(X, B)$ for every $X \in TM$.

**Proof.** According to Proposition 1 the map $\tilde{\Omega}$ is harmonic if and only if

$$\forall \text{Trace} \nabla^2 \tilde{\Omega} = \forall \text{Trace} \nabla^2 \tilde{\Omega} = 0$$

and

$$\mathcal{H} \text{Trace} \nabla^2 \tilde{\Omega} = \text{Trace} \{TM \ni X \to R(\tilde{\Omega} \times \nabla \tilde{\Omega})X\} = 0.$$

It follows from identity (13) that for every $X, Y \in TM$

$$4g(\text{Trace} \nabla^2 \tilde{\Omega}, X \wedge Y) =$$

$$-g(\nabla_J X B, Y) + g(\nabla_J Y B, X) - g(\nabla X B, JY) + g(\nabla Y B, JX) + ||B||^2g(X, JY)$$

$$= -d\theta(JX, Y) - d\theta(X, JY) + ||B||^2g(X, JY).$$  

(15)

Take an orthonormal frame $E_1, E_2 = JE_1, E_3, E_4 = JE_3$ so that $\tilde{\Omega} = E_1 \wedge JE_1 + E_3 \wedge JE_3$. Then (15) implies

$$4g(\text{Trace} \nabla^2 \tilde{\Omega}, \tilde{\Omega} \wedge Y) = -||B||^2g(JX, JY).$$

Therefore

$$4g(\forall \text{Trace} \nabla^2 \tilde{\Omega}, X \wedge Y) = 4g(\text{Trace} \nabla^2 \tilde{\Omega}, X \wedge Y) - g(\text{Trace} \nabla^2 \tilde{\Omega}, \tilde{\Omega} \wedge Y)$$

$$= -d\theta(JX, Y) - d\theta(X, JY).$$  

(16)
Thus $\nabla^2 \mathcal{J} = 0$ if and only if $d\theta(JX, Y) + d\theta(X, JY) = 0$, which is equivalent to $d\theta$ being of type $(1, 1)$.

Identity (12) implies that

$$\mathcal{J} \times \nabla_X \mathcal{J} = g(\nabla_X \mathcal{J}, s_2)s_3 - g(\nabla_X \mathcal{J}, s_3)s_2 = g(\nabla_JX \mathcal{J}, s_3)s_2 + g(\nabla_JX \mathcal{J}, s_2)s_2$$

$$= \nabla_JX \mathcal{J} = -\frac{1}{2}(X \wedge B - JX \wedge JB).$$

It follows that for every $X \in TM$

$$2\mathcal{H}Trace \mathcal{J} = 2 \sum_{i=1}^{4} g(R(\mathcal{J} \times \nabla_E \mathcal{J})E_i, X) = -\rho(X, B) + \rho^*(X, B)$$

(17)

This proves the theorem.

Remark 1 and the proof of Theorem 1 give the following.

**Corollary 2.** The map $\mathcal{J} : (M, g) \rightarrow (\mathcal{Z}, h_1)$ representing an integrable almost Hermitian structure $J$ on $(M, g)$ is a harmonic section if and only if the 2-form $d\theta$ is of type $(1, 1)$.

**Remark 2.** Note that the 2-form $d\theta$ of a Hermitian surface $(M, g, J)$ is of type $(1, 1)$ if and only if the $*$-Ricci tensor $\rho^*$ is symmetric.

Indeed, let $s = Trace \rho$ and $s^* = Trace \rho^*$ be the scalar and $*$-scalar curvatures. Set

$$L(X, Y) = (\nabla_X \theta)(Y) + \frac{1}{2}\theta(X)\theta(Y), \quad X, Y \in TM.$$

Formula (3.4) in [18] implies the following identity ([16])

$$\rho(X, Y) - \rho^*(X, Y) = \frac{1}{2}[L(JX, JY) - L(X, Y)] + \frac{s - s^*}{4}g(X, Y).$$

It follows that, when $\rho$ is $J$-invariant, $\rho^*$ is symmetric if and only if $L(JX, JY) - L(JY, JX) = L(X, Y) - L(Y, X)$. This identity is equivalent to $d\theta(JX, JY) = d\theta(X, Y)$, which means that $d\theta$ is of type $(1, 1)$.

### 3.2. The case of symplectic $J$.

Recall that an almost Hermitian manifold is called almost Kähler (or symplectic) if its fundamental 2-form is closed.

Denote by $\Lambda^2_T\! TM$ the subbundle of $\Lambda^2 T\! M$ orthogonal to $\mathcal{J}$ (thus $\Lambda^2_T\! TM = \mathcal{V}_{\mathcal{J}(\rho)}$). Under this notation we have the following.

**Theorem 2.** Let $(M, g, J)$ be an almost Kähler 4-manifold. Then the map $\mathcal{J} : (M, g) \rightarrow (\mathcal{Z}, h_1)$ is harmonic if and only if the $*$-Ricci tensor $\rho^*$ is symmetric and

$$Trace \{\Lambda^2_T\! TM \ni \tau \rightarrow R(\tau)(\mathcal{N}(\tau))\} = 0.$$

**Proof.** The 2-form $\Omega$ is harmonic since $d\Omega = 0$ and $\ast \Omega = \Omega$, so by the Weitzenböck formula

$$(Trace \nabla^2 \Omega)(X, Y) = Trace\{Z \rightarrow (R(Z, Y)\Omega)(Z, X) - (R(Z, X)\Omega)(Z, Y)\},$$

$X, Y \in TM$ (see, for example, [8]). We have

$$(R(Z, Y)\Omega)(Z, X) = -\Omega(R(Z, Y)Z, X) - \Omega(Z, R(Z, Y)X)$$

$$= g(R(Z, Y)Z, JX) + g(R(Z, Y)X, JZ).$$

Hence

$$(Trace \nabla^2 \Omega)(X, Y) = \rho(Y, JX) - \rho(X, JY) + 2\rho^*(X, JY).$$
Thus
\[ 2g(\text{Trace } \nabla^2 J, X \wedge Y) = (\text{Trace } \nabla^2 \Omega)(X, Y) = \rho(Y, JX) - \rho(X, JY) + 2\rho^*(X, JY). \quad (18) \]

Let \( E_1, \ldots, E_4 \) be an orthonormal basis of a tangent space \( T_p M \) such that \( E_2 = JE_1 \), \( E_4 = JE_3 \). Define \( s_1, s_2, s_3 \) by (1). Then \( \mathfrak{J}(p) = s_1 \) and \( \mathfrak{V}(p) = \text{span}\{s_2, s_3\} \). By Proposition 1 and identity (18)
\[
g(\text{Trace } \nabla^2 \mathfrak{J}, s_2) = g(\text{Trace } \nabla^2 \mathfrak{J}, s_2) = \rho^*(E_1, E_4) - \rho^*(E_4, E_1),
\]
\[
g(\text{Trace } \nabla^2 \mathfrak{J}, s_3) = g(\text{Trace } \nabla^2 \mathfrak{J}, s_3) = -\rho^*(E_1, E_3) - \rho^*(E_2, E_4). \quad (19) \]

It follows, in view of (14), that \( \text{Trace } \nabla^2 \mathfrak{J} = 0 \) if and only if \( \rho^*(E_1, E_j) = \rho^*(E_j, E_1) \), \( i, j = 1, \ldots, 4 \).

In order to compute \( \text{Trace } \nabla^2 \mathfrak{J} = \text{Trace } \{TM \ni X \rightarrow (\mathfrak{J} \times \nabla\mathfrak{J})X^h\} \) we first note that, by (10),
\[
g((\nabla_X J)(Y), Z) = \frac{1}{2} g(N(Y, Z), JX).
\]

Then
\[
g(\nabla_X \mathfrak{J}, Y \wedge Z) = \frac{1}{2} g((\nabla_X J)(Y), Z) = \frac{1}{4} g(N(Y, Z), JX)
\]
The Nijenhuis tensor \( N(Y, Z) \) is skew-symmetric, so it induces a linear map \( \Lambda^2 TM \rightarrow TM \) which we denote again by \( N \). It follows that, for every \( a \in \Lambda^2 TM \) and \( X \in T_{\tau(a)} M \),
\[
g(\nabla_X \mathfrak{J}, a) = \frac{1}{4} g(N(a), JX). \quad (20) \]

Take an orthonormal basis \( E_1, \ldots, E_4 \) of \( T_p M \), \( p \in M \), with \( E_2 = JE_1, E_4 = JE_3 \). Define \( s_1, s_2, s_3 \) by (1). Then \( \mathfrak{J} = s_1 \) and
\[
\mathfrak{J} \times \nabla_X \mathfrak{J} = g(\nabla_X \mathfrak{J}, s_2)s_3 - g(\nabla_X \mathfrak{J}, s_3)s_2 = -\frac{1}{4} [g(JN(s_2), X)s_3 - g(JN(s_3), X)s_2]
\]
\[
= \frac{1}{4} [g(N(s_3), X)s_3 + g(N(s_2), X)s_2] = -\nabla_X \mathfrak{J},
\]
in view of (20) and the identities
\[
N(JX, Y) = N(X, JY) = -JN(X, Y), \quad X, Y \in T_p M. \quad (21) \]

Therefore
\[
g(\text{Trace } \{TM \ni X \rightarrow R(\mathfrak{J} \times \nabla\mathfrak{J})X\}, Y) = -\sum_{i=1}^{4} g(R(\nabla_{JE_i} \mathfrak{J})E_i, Y)
\]
\[
= \sum_{i=1}^{4} [g(\nabla_{JE_i} \mathfrak{J}, s_2)g(R(s_2)Y, E_i) + g(\nabla_{JE_i} \mathfrak{J}, s_3)g(R(s_3)Y, E_i)]
\]
\[
= g(\nabla_{JR(s_2)} Y \mathfrak{J}, s_2) + g(\nabla_{JR(s_3)} Y \mathfrak{J}, s_3)
\]
\[
= -\frac{1}{4} [g(N(s_2), R(s_2)Y) + g(N(s_3), R(s_3)Y)]
\]
\[
= \frac{1}{4} [g(R(s_2)(N(s_2)), Y) + g(R(s_3)(N(s_3)), Y)].
\]

Thus
\[
4g(\text{Trace } \{TM \ni X \rightarrow R(\mathfrak{J} \times \nabla\mathfrak{J})X\}, Y) = g(\text{Trace } \{\Lambda^2_0 TM \ni \tau \rightarrow R(\tau)(N(\tau))\}, Y).
\quad (22) \]

This proves the theorem.
4. Minimality of $\mathcal{J}$

The map $\mathcal{J} : M \to \mathcal{Z}$ is an imbedding and, in this section, we discuss the problem when $\mathcal{J}(M)$ is a minimal submanifold of $(\mathcal{Z}, h_t)$.

Let $D'$ be the Levi-Civita connection of the metric on $\mathcal{J}(M)$ induced by the metric $h_t$ on $\mathcal{Z}$. Let $\Pi$ be the second fundamental form of the submanifold $\mathcal{J}(M)$. Then, as is well-known (and easy to see), for every vector fields $X, Y$ on $M$

$$\mathcal{J}(X, Y) = D'_{\mathcal{J}(X)} Y + \Pi(\mathcal{J}(X), \mathcal{J}(Y)) - \mathcal{J}(\nabla_X Y).$$

Thus $\Pi(\mathcal{J}, \mathcal{J}, \mathcal{J}, Y)$ is the normal component of $\mathcal{J}(X, Y)$, in particular $\mathcal{J}(M)$ is a minimal submanifold if and only if the normal component of $\text{Trace } \nabla \mathcal{J}$ vanishes.

4.1. The case of integrable $J$.

**Theorem 3.** Suppose that the almost complex structure $J$ is integrable. Then the map $\mathcal{J} : M \to (\mathcal{Z}, h_t)$ is a minimal isometric imbedding if and only if $d\theta$ is a $(1, 1)$ form and $\rho(X, B) = \rho^*(X, B)$ for every $X \perp \{B, JB\}$.

**Proof.** Let $p \in M$ and suppose that $B_p = 0$. Then $\nabla \mathcal{J}|_p = 0$ by (12). Hence $\mathcal{J}(X) = \mathcal{J}^h_p$ for every $X \in T_pM$. Thus the tangent space of $\mathcal{J}(M)$ at the point $\mathcal{J}(p)$ is the horizontal space $\mathcal{H}_{\mathcal{J}(p)}$, while the normal space is the vertical space $\mathcal{V}_{\mathcal{J}(p)}$. Let $E_1, E_2, JE_1, E_3, JE_3$ be an orthonormal basis of $T_pM$ and define $s_1, s_2, s_3$ by formula (1). Then $\mathcal{J}(p) = s_1$ and $\mathcal{J}^h(p) = \text{span}\{s_2, s_3\}$. Hence the normal component of $\mathcal{J}(p)$ of $\text{Trace } \nabla \mathcal{J}$ vanishes if and only if

$$g(\text{Trace } \nabla \mathcal{J}, s_2) = g(\text{Trace } \nabla \mathcal{J}, s_3) = 0.$$

Applying (16), we see that this is equivalent to

$$d\theta(E_2, E_3) = -d\theta(E_1, E_4), \quad d\theta(E_2, E_4) = d\theta(E_1, E_3).$$

The latter identities are equivalent to $(d\theta)_p$ being of type $(1, 1)$.

Now assume that $B_p \neq 0$. Then we can find an orthonormal basis of $T_pM$ of the form $E, JE, ||B_p||^{-1}B, ||B_p||^{-1}JB_p$. It follows from (12) that

$$E^h_{\mathcal{J}(p)} = \frac{4}{l||B_p||^2} \nabla E \mathcal{J}, \quad (JE)^h_{\mathcal{J}(p)} = \frac{4}{l||B_p||^2} \nabla JE \mathcal{J}$$

is a $h_t$-orthogonal basis of the normal space of $\mathcal{J}(M)$ at $\mathcal{J}(p)$. Therefore, according to Proposition 1, the normal component of the vertical part of $\text{Trace } \nabla \mathcal{J}$ at $s(p)$ vanishes if and only if

$$g(\text{ Trace } \nabla^2 \mathcal{J}, \nabla E \mathcal{J}) = g(\text{ Trace } \nabla^2 \mathcal{J}, \nabla JE \mathcal{J}) = 0.$$

It follows from (12) and (16) that the latter identities hold if and only if $d\theta(X, B) = d\theta(JX, JB)$ for every $X \perp \{B_p, JB_p\}$, which is equivalent to $d\theta$ being of type $(1, 1)$.

Moreover, by (17), the normal component of the horizontal part of $\text{Trace } \nabla \mathcal{J}$ vanishes if and only if

$$-\rho(E, B) + \rho^*(E, B) = -\rho(JE, B) + \rho^*(JE, JB) = 0,$$

or, equivalently, $\rho(X, B) = \rho^*(X, B)$ for every $X \in T_pM$, $X \perp \{B_p, JB_p\}$.

**Corollary 3.** If $J$ is integrable and the Ricci tensor is $J$-invariant then, the map $\mathcal{J} : M \to (\mathcal{Z}, h_t)$ is a minimal isometric imbedding.
Proof. According to [16, Lemma 1], the Ricci tensor $\rho$ is $J$-invariant if and only if
\[ \rho - \rho^* = \frac{s - s^*}{4} g. \]
Thus $\rho^*$ is symmetric if $\rho$ is $J$-invariant, hence $d\theta$ is of type $(1,1)$ by Remark 2. Moreover, clearly $\rho(X,B) = \rho^*(X,B)$ for $X \perp B$. Thus the result follows from Theorem 3.

This proof and Corollary 2 give the following

**Corollary 4.** If $J$ is integrable and the Ricci tensor is $J$-invariant, then $\mathfrak{J} : (M,g) \to (Z,h_4)$ is a harmonic section.

**Remark 3.** By [1, Theorem 2], every compact Hermitian surface with $J$-invariant Ricci tensor is locally conformally Kähler, $d\theta = 0$. Moreover, if its first Betti number is even, it is globally conformally Kähler ([17]). It is still unknown whether there are compact complex surfaces with $J$-invariant Ricci tensor and odd first Betti number.

### 4.2. The case of symplectic $J$

Set $N_p = \text{span}\{N(X,Y) : X,Y \in T_p M\}$, $p \in M$, so $N_p = N(\Lambda^2 T_p M)$. Identity (21) implies that $N(\Lambda^2 T_p M) = 0$ and $N(\mathfrak{J}) = 0$. Hence $N_p = N(\Lambda_p^2 T_p M)$ is a $J$-invariant subspace of $T_p M$ of dimension 0 or 2.

**Theorem 4.** Let $(M,g,J)$ be an almost Kähler 4-manifold. Then the map $\mathfrak{J} : M \to (Z,h_4)$ is a minimal isometric imbedding if and only if the $*$-Ricci tensor $\rho^*$ is symmetric and for every $p \in M$

\[ \text{Trace} \{ \Lambda_p^2 T_p M \ni \tau \to R_p(\tau)(N(\tau)) \} \in N_p. \]

**Proof.** Suppose that $N_p = 0$ for a point $p \in M$. Then $\nabla \mathfrak{J}|_p = 0$ by (20). Hence $\mathfrak{J}_p(X) = X_{\mathfrak{J}(p)}$ for every $X \in T_p M$. Thus the normal space of $\mathfrak{J}(M)$ at the point $\mathfrak{J}(p)$ is the vertical space $V_{\mathfrak{J}(p)}$. Therefore the normal component at $\mathfrak{J}(p)$ of $\text{Trace} \nabla \mathfrak{J}_s$ vanishes if and only if

\[ g(\text{Trace} \nabla \mathfrak{J}_s, s_2) = g(\text{Trace} \nabla \mathfrak{J}_s, s_3) = 0, \]

where $s_2, s_3$ are defined via (1) by means of an orthonormal basis $E_1, \ldots, E_4$ of $T_p M$ such that $E_2 = JE_1$, $E_4 = JE_3$. According to (19) (and in view of (14)) the latter identities are equivalent to $\rho^*(X,Y) = \rho^*(Y,X)$ for every $X, Y \in T_p M$.

Now assume that $N_p \neq 0$. Then there exists $\tau \in \Lambda^2_p T_p M, ||\tau|| = 1$, such that $\tau \perp \mathfrak{J}(p)$ and $N(\tau) \neq 0$. Take a unit vector $E_1 \in T_p M$ and set $E_2 = JE_1$, $E_3 = K_\tau E_1$, $E_4 = K_{\mathfrak{J}(p) \times \tau} E_1$. Then $E_1, \ldots, E_4$ is an orthonormal basis of $T_p M$ such that $\mathfrak{J}(p) = s_1, \tau = s_2, \mathfrak{J}(p) \times \tau = s_3$. By (21), $N(\tau) = N(s_2) = 2N(E_1, E_3)$, $N(\mathfrak{J}(p) \times \tau) = N(s_3) = 2JN(E_1, E_3)$, thus $N(\mathfrak{J}(p) \times \tau) = JN(\tau)$. Now we set $A_1 = ||N(\tau)||^{-1} N(\tau)$, $A_2 = JA_1$, $A_3 = K_\tau A_1$, $A_4 = K_{\mathfrak{J}(p) \times \tau} A_1$. Note that $A_4 = JA_3$ by (4). In view of (20), we have for every $X \in T_p M$

\[ g(\nabla_X \mathfrak{J}, \nabla_X A_1) = \frac{1}{16} [g(N(\tau), JX)g(N(\tau), JA_1) + g(N(\mathfrak{J}(p) \times \tau), JX)g(N(\mathfrak{J}(p) \times \tau), JA_1)] \]

\[ = \frac{1}{16} ||N(\tau)||^2 g(A_1, X). \]
Using (20), we see that
\[
\phi_{t\text{ained in the following way [14, p.787]. Let }} \operatorname{a group structure by a discrete subgroup } \Gamma. \text{ The multiplication on } C \text{ vanishes if and only if }
\]

5.1. harmonic maps into twistor spaces. However, taking into account (14), we see that (23) is equivalent to

\[
g(\nabla_{X, j}, \nabla_{A_2 j}) = 0. \tag{23}
\]

Using (20), we see that

\[
\nabla_{A_1 j} = \frac{1}{4} ||N(\tau)||^2 \nabla_{A_1 j}, \quad (A_2 j)(p) - 16 \frac{1}{4} \nabla_{A_2 j}
\]

is a \( h_t \) - orthogonal basis of the normal space of \( J(M) \) at \( J(p) \). It follows from Proposition 1 that the normal component of the vertical part of \( \nabla J_\tau \) at \( s(p) \) vanishes if and only if

\[
g(\text{Trace } \nabla^2 J, \nabla_{A_1 j}) = g(\text{Trace } \nabla^2 J, \nabla_{A_2 j}) = 0. \tag{23}
\]

It follows from (18) that (23) is equivalent to the identities

\[
\rho^* (A_1, A_3) = \rho^* (A_2, A_4), \quad \rho^* (A_1, A_4) = \rho^* (A_1, A_4).
\]

Now, taking into account (14), we see that (23) is equivalent to \( \rho^*(N(\tau), X) = \rho^*(X, N(\tau)) \) for \( X \perp \{ N(\tau), JN(\tau) \} \), i.e. \( \rho^*(X, Y) = \rho^*(Y, X) \) for \( X \perp N_p, Y \in N_p \). The subspace \( N_p \) and \( N_p^\perp \) of \( T_p M \) are two-dimensional and \( J \)-invariant, and it follows from (14) that \( \rho^*(X, Y) = \rho^*(Y, X) \) for \( X, Y \in N_p \) or \( X, Y \in N_p^\perp \). Thus identity (23) is equivalent to \( \rho^* \) being symmetric.

In view of (22), the normal component of the horizontal part of \( \text{Trace } \nabla J_\tau \) vanishes if and only if

\[
\text{Trace } \{ A_0^2 TM \ni \tau \rightarrow R(\tau)(N(\tau)) \} \perp \{ A_1, A_2 \}.
\]

This proves the statement.

5. Examples

In this section we give examples of almost Hermitian structures that determine harmonic maps into twistor spaces.

5.1. Kodaira surfaces. Recall that every primary Kodaira surface \( M \) can be obtained in the following way [14, p.787]. Let \( \varphi_k(z, w) \) be the affine transformations of \( C^2 \) given by

\[
\varphi_k(z, w) = (z + a_k, w + \overline{a_k} z + b_k),
\]

where \( a_k, b_k, k = 1, 2, 3, 4, \) are complex numbers such that

\[
a_1 = a_2 = 0, \quad \text{Im}(a_3 \overline{a_4}) = mb_1 \neq 0, \quad b_2 \neq 0
\]

for some integer \( m > 0 \). They generate a group \( G \) of transformations acting freely and properly discontinuously on \( C^2 \), and \( M \) is the quotient space \( C^2 / G \).

It is well-known that \( M \) can also be describe as the quotient of \( C^2 \) endowed with a group structure by a discrete subgroup \( \Gamma \). The multiplication on \( C^2 \) is defined by

\[
(a, b)(z, w) = (z + a, w + \overline{a} z + b), \quad (a, b), (z, w) \in C^2,
\]
and $\Gamma$ is the subgroup generated by $(a_k, b_k)$, $k = 1, \ldots, 4$ (see, for example, [4]).

Further we consider $M$ as the quotient of the group $\mathbb{C}^2$ by the discrete subgroup $\Gamma$. Every left-invariant object on $\mathbb{C}^2$ descends to a globally defined object on $M$ and both of them will be denoted by the same symbol. We identify $\mathbb{C}^2$ with $\mathbb{R}^4$ by $(z = x + iy, w = u + iv) \mapsto (x, y, u, v)$ and set
\[
A_1 = \frac{\partial}{\partial x} - x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v}, \quad A_2 = \frac{\partial}{\partial y} - y \frac{\partial}{\partial u} - x \frac{\partial}{\partial v}, \quad A_3 = \frac{\partial}{\partial u}, \quad A_4 = \frac{\partial}{\partial v}.
\]
These form a basis for the space of left-invariant vector fields on $\mathbb{C}^2$. We note that their Lie brackets are
\[
[A_1, A_2] = -2A_4, \quad [A_1, A_3] = 0
\]
for all other $i, j$. It follows that the group $\mathbb{C}^2$ defined above is solvable.

Denote by $g$ the left-invariant Riemannian metric on $M$ for which the basis $A_1, \ldots, A_4$ is orthonormal.

We shall show that any integrable or symplectic almost complex structures $J$ on $M$ compatible with the metric $g$ and defined by a left-invariant almost complex structure on $\mathbb{C}^2$ is a harmonic map from $(M, g)$ to $(Z, h_t)$.

Note that by [13] every complex structure on $M$ is induced by a left-invariant complex structure on $\mathbb{C}^2$.

I. If $J$ is a left-invariant almost complex structure compatible with $g$, we have $JA_1 = \sum_{i=1}^{4} a_{ij} A_j$ where $a_{ij}$ are constants with $a_{ij} = -a_{ji}$. Let $N$ be the Nijenhuis tensor of $J$. Computing $N(A_i, A_j)$ in terms of $a_{ij}$, one can see ([15, 6]) that $J$ is integrable if and only if
\[
JA_1 = \varepsilon_1 A_2, \quad JA_3 = \varepsilon_2 A_4, \quad \varepsilon_1, \varepsilon_2 = \pm 1.
\]
Denote by $\theta$ the Lee form of the Hermitian structure $(M, g, J)$ where $J$ is defined by means of the latter identities.

The non-zero covariant derivatives $\nabla_{A_i} A_j$ are
\[
\nabla_{A_1} A_2 = -\nabla_{A_2} A_1 = -A_4, \quad \nabla_{A_1} A_4 = \nabla_{A_4} A_1 = A_2, \quad \nabla_{A_2} A_4 = \nabla_{A_4} A_2 = -A_1.
\]
This implies that the Lie form is
\[
\theta(X) = -2\varepsilon_1 g(X, A_3).
\]
Therefore
\[
B = -2\varepsilon_1 A_3, \quad \nabla \theta = 0. \quad (24)
\]

Set for short $R_{ijk} = R(A_i, A_j)A_k$. Then the non-zero $R_{ijk}$ are
\[
R_{121} = -3A_2, \quad R_{122} = 3A_1, \quad R_{111} = A_4, \quad R_{144} = -A_1, \quad R_{242} = A_4, \quad R_{244} = -A_2.
\]
Set also $\rho_{ij} = \rho(A_i, A_j)$, $\rho_{ij}^* = \rho^*(A_i, A_j)$. Then
\[
\rho_{ij} = 0 \text{ and } \rho_{ij}^* = 0 \text{ except } \rho_{11} = 2, \quad \rho_{22} = -4, \quad \rho_{11}^* = \rho_{22}^* = -3. \quad (25)
\]
It follows from (24), (25) and Theorem 1 that the complex structure $J$ is a harmonic map from $(M, g)$ to the twistor space $(Z, h_t)$. 

It is easy to describe explicitly the twistor space \((Z, h_t)\) \((6)\) since \(A^2_4 M\) admits a global orthonormal frame defined by
\[
s_1 = \varepsilon_1 A_1 \wedge A_2 + \varepsilon_2 A_3 \wedge A_4, \quad s_2 = A_1 \wedge A_3 + \varepsilon_1 \varepsilon_2 A_4 \wedge A_2, \quad s_3 = \varepsilon_2 A_1 \wedge A_4 + \varepsilon_1 A_2 \wedge A_3.
\]
It induces a natural diffeomorphism \(F : Z \cong M \times S^2, \sum_{k=1}^3 x_k s_k(p) \mapsto (p, x_1, x_2, x_3)\), under which \(J\) determines the section \(p \mapsto (p, 1, 0, 0)\). In order to find an explicit formula for the metrics \(h_t\) we need the covariant derivatives of \(s_1, s_2, s_3\) with respect to the Levi-Civita connection \(\nabla\) of \(g\). The non-zero of these are
\[
\nabla_{A_1} s_1 = -\varepsilon_2 \nabla_{A_4} s_2 = -\varepsilon_1 \varepsilon_2 s_3, \quad \varepsilon_1 \nabla_{A_1} s_3 = -\nabla_{A_2} s_2 = \varepsilon_2 s_1; \\
\varepsilon_2 \nabla_{A_2} s_1 = -\varepsilon_1 \nabla_{A_4} s_3 = s_2.
\]
It follows that \(F_\ast\) sends the horizontal lifts \(A^1_1, \ldots, A^4_1\) at a point \(\sigma = \sum_{k=1}^3 x_k s_k(p) \in Z\) to the following vectors of \(T M \oplus TS^2\)
\[
A_1 + \varepsilon_1 \varepsilon_2 (-x_3, 0, x_1), \quad A_2 + \varepsilon_2 (x_2, -x_1, 0), \quad A_3, \quad A_4 + \varepsilon_1 (0, x_3, -x_2).
\]
For \(x = (x_1, x_2, x_3) \in S^2\), set
\[
u_1(x) = \varepsilon_1 \varepsilon_2 (-x_3, 0, x_1), \quad \nu_2(x) = \varepsilon_2 (x_2, -x_1, 0), \quad \nu_3(x) = 0, \quad \nu_4(x) = \varepsilon_1 (0, x_3, -x_2).
\]
Denote the pushforward of the metric \(h_t\) under \(F\) again by \(h_t\). Then, if \(X, Y \in T_p M\) and \(P, Q \in T_x S^2\),
\[
h_t(X + P, Y + Q) = g(X, Y) + t - P - \sum_{i=1}^4 g(X, A_i) u_i(x), Q - \sum_{j=1}^4 g(Y, A_j) u_j(x) >
\]
where \(< , , >\) is the standard metric of \(\mathbb{R}^3\).

**II.** Suppose again that \(J\) is an almost complex structure on \(M\) obtained from a left-invariant almost complex structure on \(G\) and compatible with the metric \(g\). Set \(J A_i = \sum_{j=1}^4 a_{ij} A_j\). Denote the fundamental 2-form of the almost Hermitian structure \((g, J)\) by \(\Omega\). The basis dual to \(A_1, \ldots, A_4\) is
\[
o_1 = dx, \quad o_2 = dy, \quad o_3 = xdx + ydy + du, \quad o_4 = -ydx + xdy + dv.
\]
We have \(d o_1 = o_2, d o_2 = o_3, d o_3 = 2dx \wedge dy\). Hence
\[
d \Omega = a \sum_{i<j} a_{ij} o_i \wedge o_j = -2 a_{34} dx \wedge dy \wedge du.
\]
Thus \(d \Omega = 0\) is equivalent to \(a_{34} = 0\). If \(a_{34} = 0\), we have \(a_{1j} = a_{2j} = 0\) for \(j = 1, 2\), \(a_{3k} = a_{4k} = 0\) for \(k = 3, 4\), \(a_{13}^2 + a_{24}^2 = 1\), \(a_{13} a_{23} + a_{14} a_{24} = 0\), \(a_{23}^2 + a_{24}^2 = 1\). It follows that the structure \((g, J)\) is almost Kähler (symplectic) if and only if \(J\) is given by \((15, 6)\)
\[
J A_1 = -\varepsilon_1 \sin \varphi A_3 + \varepsilon_2 \varepsilon_2 \cos \varphi A_4, \quad J A_2 = -\cos \varphi A_3 - \varepsilon_2 \sin \varphi A_4, \\
J A_3 = \varepsilon_1 \sin \varphi A_1 + \cos \varphi A_2, \quad J A_4 = -\varepsilon_2 \cos \varphi A_1 + \varepsilon_2 \sin \varphi A_2,
\]
\[
\varepsilon_1, \varepsilon_2 = \pm 1, \quad \varphi \in [0, 2\pi).
\]
Let \(J\) the almost complex structure defined by these identities for some \(\varepsilon_1, \varepsilon_2, \varphi\). Set
\[
E_1 = A_1, \quad E_2 = -\varepsilon_1 \sin \varphi A_3 + \varepsilon_2 \varepsilon_2 \cos \varphi A_4, \quad E_3 = \cos \varphi A_3 + \varepsilon_2 \sin \varphi A_4, \quad E_4 = A_2.
Then $E_1, ..., E_4$ is an orthonormal frame of $TM$ for which $JE_1 = E_2$ and $JE_3 = E_4$. The only non-zero Lie bracket of these fields is

$$[E_1, E_4] = -2(\varepsilon_1 \varepsilon_2 \cos \varphi E_2 + \varepsilon_2 \sin \varphi E_3).$$

The non-zero covariant derivatives $\nabla_{E_i} E_j$ are

$$\nabla_{E_1} E_2 = -\varepsilon_1 \varepsilon_2 \cos \varphi E_4, \quad \nabla_{E_1} E_3 = \varepsilon_1 \varepsilon_2 \sin \varphi E_4,$$

$$\nabla_{E_1} E_4 = -\varepsilon_2 \cos \varphi E_3 - \varepsilon_1 \sin \varphi E_3,$$

$$\nabla_{E_2} E_4 = \varepsilon_1 \varepsilon_2 \cos \varphi E_1, \quad \nabla_{E_3} E_4 = \varepsilon_1 \varepsilon_2 \cos \varphi E_1.$$

Set $R_{ijk} = R(E_i, E_j)E_k$. We have the following table for the non-zero components of the curvature tensor $R$:

$$R_{121} = \cos^2 \varphi E_2 + \frac{1}{2}\varepsilon_1 \sin 2 \varphi E_3,$$

$$R_{122} = -\cos^2 \varphi E_1, \quad R_{123} = -\frac{1}{2} \varepsilon_1 \sin 2 \varphi E_1,$$

$$R_{131} = \frac{1}{2} \varepsilon_1 \sin 2 \varphi E_2 + \sin^2 \varphi E_3,$$

$$R_{132} = -\frac{1}{2} \varepsilon_1 \sin 2 \varphi E_1, \quad R_{133} = -\sin^2 \varphi E_1,$$

$$R_{141} = -3E_4, \quad R_{144} = 3E_1,$$

$$R_{242} = \cos^2 \varphi E_4,$$

$$R_{243} = \frac{1}{2} \varepsilon_1 \sin 2 \varphi E_4, \quad R_{244} = -\cos^2 \varphi E_2 - \frac{1}{2} \varepsilon_1 \sin 2 \varphi E_3,$$

$$R_{342} = \frac{1}{2} \varepsilon_1 \sin 2 \varphi E_4, \quad R_{343} = \sin^2 \varphi E_4, \quad R_{344} = -\frac{1}{2} \varepsilon_1 \sin 2 \varphi E_2 - \sin^2 \varphi E_3.$$

Define an orthonormal frame $s_l, l = 1, 2, 3,$ of $\Lambda^2_+ TM$ by means of $E_1, ..., E_4$. Then $s_2, s_3$ is a frame of $\Lambda^3_+ TM$ and by (21)

$$N(s_2) = 2N(E_1, E_3) = -4\varepsilon_1 \varepsilon_2 \cos \varphi E_1 + 4\varepsilon_2 \sin \varphi E_4,$$

$$N(s_3) = 2N(E_1, E_4) = 4\varepsilon_1 \varepsilon_2 \cos \varphi E_2 + 4\varepsilon_2 \sin \varphi E_3.$$

It follows that

$$Trace \{\Lambda^3_+ TM \ni \tau \rightarrow R(\tau)(N(\tau))\} = R(s_2)(N(s_2)) + R(s_3)(N(s_3)) = 0.$$

Setting $\rho^{*}_{ij} = \rho^*(E_i, E_j)$, we have

$$\rho^{*}_{11} = \rho^{*}_{22} = \cos^2 \varphi, \quad \rho^{*}_{33} = \rho^{*}_{44} = \sin^2 \varphi, \quad \rho^{*}_{14} = \rho^{*}_{41} = -\frac{1}{2} \varepsilon_1 \sin 2 \varphi$$

and the other $\rho^{*}_{ij}$ vanish. Thus, by Propostion 2, the almost Kähler structure $J$ is a harmonic map $(M, g) \rightarrow (Z, h_t)$.

As in the preceding case, it is easy to find an explicit description of the twistor space $Z$ of $M$ and the metric $h_t$ ([6]). The frame $s_1, s_2, s_3$ gives rise to an obvious diffeomorphism $F : Z \cong M \times S^2$ under which $\mathfrak{J}$ becomes the map $p \rightarrow (p, 1, 0, 0)$. We have the following table for the covariant derivatives of $s_1, s_2, s_3$:

$$\nabla_{E_1} s_1 = \nabla_{E_2} s_2 = \varepsilon_1 \varepsilon_2 \cos \varphi s_3, \quad \nabla_{E_4} s_1 = -\nabla_{E_1} s_2 = -\varepsilon_2 \sin \varphi s_3,$$

$$\nabla_{E_3} s_1 = \varepsilon_1 \varepsilon_2 \cos \varphi s_2, \quad \nabla_{E_2} s_2 = -\varepsilon_1 \varepsilon_2 \cos \varphi s_1, \quad \nabla_{E_2} s_3 = 0,$$

$$\nabla_{E_3} s_2 = \varepsilon_2 \sin \varphi s_2, \quad \nabla_{E_1} s_2 = -\varepsilon_2 \sin \varphi s_1, \quad \nabla_{E_3} s_3 = 0,$$

$$\nabla_{E_1} s_3 = -\varepsilon_1 \varepsilon_2 \cos \varphi s_1 - \varepsilon_2 \sin \varphi s_2, \quad \nabla_{E_4} s_3 = \varepsilon_2 \sin \varphi s_1 - \varepsilon_1 \varepsilon_2 \cos \varphi s_2.$$
Using this table we see that \( F_* \) sends the horizontal lifts \( E_i^h, i = 1, \ldots, 4 \), to \( E_i + u_i \)
where
\[
\begin{align*}
  u_1(x) & = (x_3 \varepsilon_1 \varepsilon_2 \cos \varphi, x_3 \varepsilon_2 \sin \varphi, -x_1 \varepsilon_1 \varepsilon_2 \cos \varphi - x_2 \varepsilon_2 \sin \varphi), \\
  u_2(x) & = (x_2 \varepsilon_1 \varepsilon_2 \cos \varphi, -x_1 \varepsilon_1 \varepsilon_2 \cos \varphi, 0), \\
  u_3(x) & = (x_2 \varepsilon_2 \sin \varphi, -x_1 \varepsilon_2 \sin \varphi, 0) \\
  u_4(x) & = (-x_3 \varepsilon_2 \sin \varphi, x_3 \varepsilon_1 \varepsilon_2 \cos \varphi, x_1 \varepsilon_1 \varepsilon_2 \sin \varphi - x_2 \varepsilon_1 \varepsilon_2 \cos \varphi).
\end{align*}
\]
for \( x = (x_1, x_2, x_3) \in S^2 \). Then, if \( X, Y \in T_p M \) and \( P, Q \in T_x S^2 \),
\[
h_t(X + P, Y + Q) = g(X, Y) + t < P - \sum_{i=1}^{4} g(X, E_i) u_i(x), Q - \sum_{j=1}^{4} g(Y, E_j) u_j(x) > .
\]

5.2. **Four-dimensional Lie groups.** We shall show that every left-invariant almost Kähler structure \( (g, J) \) with \( J \)-invariant Ricci tensor on a 4-dimensional Lie group \( M \) determines a harmonic map \( \lambda : (M, g) \to (Z, h_t) \).

These (non-integrable) structures have been determined in [10]. According to the main result therein, for any such a structure \( (g, J) \), there exists an orthonormal frame of left-invariant vector fields \( E_1, \ldots, E_4 \) such that
\[
JE_1 = E_2, \quad JE_3 = E_4
\]
and
\[
[E_1, E_2] = 0, \quad [E_1, E_3] = sE_1 + \frac{s^2}{t} E_2, \quad [E_1, E_4] = \frac{s^2 - t^2}{2t} E_1 - sE_2, \\
[E_2, E_3] = -tE_1 - sE_2, \quad [E_2, E_4] = -sE_1 - \frac{s^2 - t^2}{2t} E_2, \quad [E_3, E_4] = -\frac{s^2 + t^2}{t} E_3
\]
where \( s \) and \( t \neq 0 \) are real numbers. Then we have the following table for the Levi-Civita connection
\[
\begin{align*}
\nabla_{E_1} E_1 & = -sE_3 - \frac{s^2 - t^2}{2t} E_4, \quad \nabla_{E_2} E_1 = -\frac{s^2 - t^2}{2t} E_3 + sE_4, \quad \nabla_{E_3} E_1 = \frac{s^2 + t^2}{2t} E_2 \\
\nabla_{E_1} E_2 & = -\frac{s^2 - t^2}{2t} E_3 + sE_4, \quad \nabla_{E_2} E_2 = sE_3 + \frac{s^2 - t^2}{2t} E_4, \quad \nabla_{E_3} E_2 = \frac{s^2 + t^2}{2t} E_1 \\
\nabla_{E_1} E_3 & = sE_1 + \frac{s^2 - t^2}{2t} E_2, \quad \nabla_{E_2} E_3 = \frac{s^2 - t^2}{2t} E_1 - sE_2, \quad \nabla_{E_3} E_3 = \frac{s^2 + t^2}{t} E_4 \\
\nabla_{E_1} E_4 & = \frac{s^2 - t^2}{2t} E_1 - sE_2, \quad \nabla_{E_2} E_4 = -sE_1 - \frac{s^2 - t^2}{2t} E_2, \quad \nabla_{E_3} E_4 = -\frac{s^2 + t^2}{t} E_3 \\
\n\nabla_{E_4} E_1 = \nabla_{E_4} E_2 = \nabla_{E_4} E_3 = \nabla_{E_4} E_4 = 0.
\end{align*}
\]
This implies the following table for the components \( R_{ijk} = R(E_i, E_j)E_k \) of the curvature tensor; in this table \( \lambda = \frac{s^2 + t^2}{2t} \cdot \)
\[
\begin{align*}
  R_{121} & = 2\lambda E_2, \quad R_{122} = -2\lambda E_1, \quad R_{123} = 2\lambda E_4, \quad R_{124} = -2\lambda E_3, \\
  R_{131} & = -\lambda E_3, \quad R_{132} = \lambda E_4, \quad R_{133} = \lambda E_1, \quad R_{134} = -\lambda E_2, \\
  R_{141} & = -\lambda E_4, \quad R_{142} = -\lambda E_3, \quad R_{143} = \lambda E_2, \quad R_{144} = -\lambda E_1, \\
  R_{231} & = -\lambda E_1, \quad R_{232} = -\lambda E_4, \quad R_{233} = \lambda E_2, \quad R_{234} = -\lambda E_1, \\
  R_{241} & = \lambda E_3, \quad R_{242} = -\lambda E_4, \quad R_{243} = -\lambda E_1, \quad R_{244} = \lambda E_2, \\
  R_{341} & = 2\lambda E_2, \quad R_{342} = -2\lambda E_1, \quad R_{343} = -4\lambda E_4, \quad R_{344} = 4\lambda E_3.
\end{align*}
\]
Then the non-zero $\rho^*_{ij} = \rho^*(E_i, E_j)$ are

$$\rho^*_{11} = \rho^*_{22} = 4\lambda, \quad \rho^*_{33} = \rho^*_{44} = -2\lambda.$$ 

Therefore the $*$-Ricci tensor $\rho^*$ is symmetric.

Set $s_2 = E_1 \wedge E_3 + E_4 \wedge E_2$, $s_3 = E_1 \wedge E_4 + E_2 \wedge E_3$. Then

$$N(s_2) = 2N(E_1, E_3) = -8(sE_1 + \frac{s^2 - t^2}{2t}E_2),$$

$$N(s_3) = 2N(E_1, E_4) = 8(-\frac{s^2 - t^2}{2t}E_1 + sE_2).$$

It follows that

$$\text{Trace} \{ \Lambda^2_0 TM \ni \tau \to R(\tau)(N(\tau)) \} = 0.$$ 

Thus, according to Theorem 4, $J$ is a harmonic map.

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