Confluence Results for a Quantum Lambda Calculus with Measurements

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Abstract
A strong confluence result for $Q^*$, a quantum $\lambda$-calculus with measurements, is proved. More precisely, confluence is shown to hold both for finite and infinite computations. The technique used in the confluence proof is syntactical but innovative. This makes $Q^*$ different from similar quantum lambda calculi, which are either measurement-free or provided with a reduction strategy.

1 Introduction
It is well known that the measurement-free evolution of a quantum system is deterministic. As a consequence it is to be expected that a good measurement-free quantum lambda calculus enjoys confluence. This is the case of $Q$, by the authors [5] and of the lambda calculus recently introduced by Arrighi and Dowek [1]. The situation becomes more complicated if we introduce a measurement operator. In fact measurements break the deterministic evolution of a quantum system\footnote{at the present theoretical status of quantum mechanics, measurement is not reconcilable with the quantum theory, this is the famous measurement problem [7]}, in presence of measurements the behaviour becomes irretrievably probabilistic.

An explicit measurement operator in the syntax allows an observation at an intermediate step of the computation: this feature is needed if we want, for example, to write algorithms such as Shor’s factorization. In quantum calculi the intended meaning of a measurement is the observation of a (possibly superimposed) quantum bit, giving as output a classical bit; the two possible outcomes (i.e., the two possible values of the obtained classical bit) can be observed with two probabilities summing to 1. Since measurement forces a probabilistic evolution in the computation, it is not surprising that we need probabilistic instruments in order to investigate the main features of the language.

In this paper, we study an extension of $Q$ obtained by endowing the language of terms with a suitable measurement operator and coherently extending the reduction relation, which becomes probabilistic for the reasons we have just explained. We investigate the resulting calculus, called $Q^*$, focusing, in particular, on confluence.

In $Q^*$ and $Q$, states are formalized by configurations, i.e., triples in the form $[Q, QV, M]$, where $M$ is a lambda term, $Q$ is a quantum state, and $QV$ is a set of names of quantum variables. So, control is classical ($M$ is simply a term) while data is quantum ($Q$ is an element of a finite-dimensional Hilbert space).

We are interested in the following question: what happens to properties such as confluence in presence of measurements? And moreover: is it possible to preserve confluence in the probabilistic...
setting induced by measurements? Apparently, the questions above cannot receive a positive answer: as we will see in section 4, it is possible to exhibit a configuration $C$ such that there are two different reductions starting at $C$ and ending in two essentially different configurations in normal form $[1,0,0]$ and $[1,0,1]$. In other words, confluence fails in its usual form. But the question now becomes: are the usual notions of computations and confluence adequate in this setting?

In $Q^*$, there are two distinct sources of divergence:

- On the one hand, a redex involving the measurement operator can be reduced in two different ways, i.e., divergence can come from a single redex.
- On the other hand, a term can contain more than one redex and $Q^*$ is not endowed with a reduction strategy. As a consequence, some configurations can be reduced in different ways due to the presence of distinct redexes in a term.

We cannot hope to be confluent with respect to the first source of divergence, but we can anyway ask ourselves whether all reduction strategies are somehow equivalent. More precisely, we say that $Q^*$ is confluent if for every configuration $C$ and for every configuration in normal form $D$, there is a fixed real number $p$ such that the probability of observing $D$ when reducing $C$ is always $p$, independently of the reduction strategy.

This notion of confluence can be easily captured by analyzing rewriting on mixed states rather than rewriting on configurations. A mixed state is a probabilistic distribution on configurations whose support is finite. Rewriting on configurations naturally extend to rewriting on mixed states. Rewriting on mixed states is not a probabilistic relation, and confluence is the usual confluence coming from rewriting theory [16].

In this paper, we prove that $Q^*$ is indeed confluent in this sense. Technically, confluence is proved in an innovative way. The key point is that we need a new definition of computation. The usual notion of computation as a sequence of configurations is not adequate here. A notion of probabilistic computation replaces it, as something more general than a linear sequence of configurations but less general than the reduction tree: a probabilistic computation is a (possibly) infinite tree, in which binary choice (a node can have at most two children) corresponds to the two possible outcomes of a measurement. This new notion of computation is needed, because proving confluence directly on mixed states is non-trivial. As by-products, we prove other results in the style of confluence.

Another important property of any quantum lambda calculus with measurements is the importance of infinite computations. In the case of standard lambda calculus, the study of infinite computations is strongly related to the study of infinite lambda terms. This is not the case of $Q^*$ (and in general of quantum calculi with measurements). This phenomenon forced us to extend the study of confluence to the case of infinite probabilistic computations. The proposed analysis is not standard and is based on new techniques.

The rest of this paper is structured as follow:

- in Section 2 the quantum $\lambda$-calculus $Q^*$ is introduced;
- in Section 4 we introduce the confluence problem in an informal way;
- in Section 5 we give the definition of a probabilistic computation;
- in Section 6 a strong confluence result on probabilistic computations is given;
- in Section 7 mixed states and mixed computations are introduced, and we give a confluence theorem for mixed computations.

2 The Calculus $Q^*$

In [5] we have introduced a measurement-free, untyped quantum $\lambda$-calculus, called $Q$, based on the quantum data and classical control paradigm (see e.g. [13] [14]). In this paper we generalize $Q$ by extending the class of terms with a measurement operator, obtaining $Q^*$.

As for $Q$, $Q^*$ is based on the notion of a configuration (see Section 3.1), namely a triple $[Q,QV,M]$ where $Q$ is a quantum register, $QV$ is a finite set of names, called quantum vari-

\footnote{the "empty" quantum register will be denoted with the scalar number 1.}
ables, and \( M \) is an untyped \textit{term} based on the linear lambda-calculus defined by Wadler \cite{17} and Simpson \cite{15}.

Quantum registers are systems of \( n \) qubits, that, mathematically speaking, are normalized vectors of finite dimensional Hilbert spaces. In particular, a quantum register \( Q \) of a configuration \( [Q, QV, M] \), is a normalized vector of the Hilbert space \( \ell^2([0, 1]) \), denoted here with \( \mathcal{H}(QV) \).

Roughly speaking, the reader could think that quantum variables are references to qubits in a quantum register.

There are three kinds of operations on quantum registers: (i) the creation of a new qubit; (ii) \textit{unitary operators}: each unitary operator \( U_{\langle q_1, \ldots, q_n \rangle} \) corresponds to a pure quantum operation acting on qubits with names \( q_1, \ldots, q_n \) (mathematically, a unitary transform on the Hilbert space \( \mathcal{H}([q_1, \ldots, q_n]) \), see \cite{3}); (iii) one \textit{qubit measurement} operations \( M_{r,0}, M_{r,1} \) responsible of the probabilistic reduction of the quantum state plus the destruction of the qubit referenced by \( r \): given a quantum register \( Q \in \mathcal{H}(QV) \), and a quantum variable name \( r \in QV \), we allow the (destructive) measurement of the qubit with name \( r \) (see Section 3 for more details).

We conclude this short overview with few words on the set of \textit{elementary unitary operators}.

We say that a class \( \{U_i\}_{i \in I} \) is \textit{elementary} iff for every \( i \in I \), the unitary operator \( U_i \) is realizable, either physically (i.e. by a laser or by other apparatus) or by means of a computable devices, such as a Turing machine. Different classes of elementary operators could be defined, among these, a remarkable class is those of \textit{computable operators}, see e.g. \cite{3, 4, 9, 10, 11}.

2.1 Terms, Judgements and Well-Formed-Terms

Let \( U \) an elementary set of unitary operators. Let us associate to each elementary operator \( U \in U \) a symbol \( U \). The set of \textit{term expressions}, or \textit{terms} for short, is defined by the following grammar:

\[
\begin{align*}
x & ::= x_0, x_1, \ldots & \text{classical variables} \\
r & ::= r_0, r_1, \ldots & \text{quantum variables} \\
\pi & ::= x \mid \langle x_1, \ldots, x_n \rangle & \text{linear patterns} \\
\psi & ::= \pi \mid \pi & \text{patterns} \\
B & ::= 0 \mid 1 & \text{boolean constants} \\
U & ::= U_0, U_1, \ldots & \text{unitary operators} \\
C & ::= B \mid U & \text{constants} \\
M & ::= x \mid r \mid ![M] \mid C \mid \text{new}(M) \mid M_1 M_2 \mid \text{meas}(M) \mid \text{if } N \text{ then } M_1 \text{ else } M_2 \mid \langle M_1, \ldots, M_n \rangle \mid \lambda \psi.M & \text{terms (where } n \geq 2) \\
\end{align*}
\]

We assume to work modulo variable renaming, i.e. \textit{terms are equivalence classes modulo }\( \alpha \)-\textit{conversion}. Substitution up to \( \alpha \)-equivalence is defined in the usual way. Since we are working modulo \( \alpha \)-conversion, we are authorized to use the so called Barendregt Convention on Variables (shortly, BCV) \cite{2}: in each mathematical context (a term, a definition, a proof...) the names chosen for bound variables will always differ from those of the free ones.

Let us denote with \( Q(M_1, \ldots, M_k) \) the set of quantum variables occurring in \( M_1, \ldots, M_k \).

Notice that:

- Variables are either \textit{classical} or \textit{quantum}: the first ones are the usual variables of lambda calculus (and can be bound by abstractions), while each quantum variable refers to a qubit in the underlying quantum register (to be defined shortly).

- There are two sorts of constants as well, namely \textit{boolean constants} (0 and 1) and \textit{unitary operators}: the first ones are useful for generating qubits and play no significant rôle in classical computation, while unitary operators are applied to (tuples of) quantum variables when performing quantum computation.

- The term constructor \textit{new}(\cdot) creates a new qubit when applied to a boolean constant.

- The term constructor \textit{meas}(\cdot) perform a single qubit measurement when applied to a quantum variable.

\[\text{Footnote:}^3\text{see [5] for a full discussion of } \mathcal{H}(QV) \text{ and [12] for a general treatment of } \ell^2(S) \text{ spaces.}\]

3
• The syntax allows the so called pattern abstraction. A pattern is either a classical variable, a tuple of classical variables, or a “banged” variable (namely an expression $lx$, where $x$ is a classical variable). In order to allow an abstraction of the kind $\lambda x.M$, environments (see below) can contain $!$-patterns, denoting duplicable or erasable variables.

For each $\text{qvs} \mathcal{QV}$ and for each quantum variable $r \in \mathcal{QV}$, we assume to have two, measurement based, linear transformation of quantum registers: $\mathcal{M}_{r,0}, \mathcal{M}_{r,1} : \mathcal{H}(\mathcal{QV}) \to \mathcal{H}(\mathcal{QV} - \{r\})$ (see Section 3 for more details).

Judgements are defined from various notions of environments, that take into account the way the variables are used. Following common notations in type theory and proof theory, a set of variables $\{x_1, \ldots, x_n\}$ is often written simply as $x_1, \ldots, x_n$. Analogously, the union of two sets of variables $X$ and $Y$ is denoted simply as $X, Y$.

• A classical environment is a (possibly empty) set of classical variables. Classical environments are denoted by $\Delta$ (possibly with indexes). Examples of classical environments are $x_1, x_2$ or $x, y, z$ or the empty set $\emptyset$. Given a classic environment $\Delta = x_1, \ldots, x_n$, $!\Delta$ denotes the set of patterns $!x_1, \ldots, !x_n$.

• A quantum environment is a (possibly empty) set (denoted by $\Theta$, possibly indexed) of quantum variables. Examples of quantum environments are $r_1, r_2, r_3$ and the empty set $\emptyset$.

• A linear environment is (possibly empty) set (denoted by $\Lambda$, possibly indexed) in the form $\Lambda, \Theta$ where $\Delta$ is a classical environment and $\Theta$ is a quantum environment. The set $x_1, x_2, r_1$ is an example of a linear environment.

• An environment (denoted by $\Gamma$, possibly indexed) is a (possibly empty) set in the form $\Delta, \Theta$ where each classical variable $x$ occurs at most once (either as $lx$ or as $x$) in $\Gamma$. For example, $x_1, r_1, !x_2$ is an environment, while $x_1, !x_1$ is not an environment.

• A judgement is an expression $\Gamma \vdash M$, where $\Gamma$ is an environment and $M$ is a term.

#### Figure 1: Well–Forming Rules

![Figure 1: Well–Forming Rules](image)

We say that a judgement $\Gamma \vdash M$ is well–formed (notation: $\triangleright \Gamma \vdash M$) if it is derivable from the well–forming rules in Figure 1. The rules app and tens are subject to the following constraint: for each $i \neq j$ $\Delta_i \cap \Delta_j = \emptyset$ (notice that $\Delta_i$ and $\Delta_j$ are sets of linear and quantum variables, being linear environments). With $d \triangleright \Gamma \vdash M$ we mean that $d$ is a derivation of the well–formed judgement $\Gamma \vdash M$. If $\Gamma \vdash M$ is well–formed, the term $M$ is said to be is well–formed with respect to the environment $\Gamma$. A term $M$ is well–formed if the judgement $\mathcal{Q}(M) \vdash M$ is well–formed.

**Remark 1** $\mathcal{Q}^*$ comes equipped with two constants $0$ and $1$, and an $if$ ($\cdot$) then ($\cdot$) else ($\cdot$) constructor. However, these constructors can be thought of as syntactic sugar. Indeed, $0$ and $1$ can be encoded as pure terms: $0 = \lambda x.\lambda y.y$ and $1 = \lambda x.\lambda y.x$. In doing so, $if \ M \ then \ N \ else \ L$ becomes $M!N!L$. The well–forming rule if (see Figure 1) of $\mathcal{Q}^*$ fully agrees with the above encodings.
3 Quantum Registers and Measurements

Before giving the definition of destructive measurement used in this paper we must clarify something about quantum spaces.

The smallest quantum space is \( \mathcal{H}(\emptyset) \), which is (isomorphic to) the field \( \mathbb{C} \). The so called empty quantum register is nothing more than a unitary element of \( \mathbb{C} \) (i.e., a complex number \( c \) such that \(|c| = 1\)). We have chosen the scalar number 1 as the canonical empty quantum register. In particular the number 1 represents also the computational basis of \( \mathcal{H}(\emptyset) \).

It is easy to show that if \( \mathcal{Q} \mathcal{V} \cap \mathcal{R} \mathcal{V} = \emptyset \) then there is a standard isomorphism

\[
\mathcal{H}(\mathcal{Q} \mathcal{V}) \otimes \mathcal{H}(\mathcal{R} \mathcal{V}) \cong \mathcal{H}(\mathcal{Q} \mathcal{V} \cup \mathcal{R} \mathcal{V}).
\]

In the rest of this paper we will assume to work up-to such an isomorphism\(^4\). Note that the previous isomorphism holds even if either \( \mathcal{Q} \mathcal{V} \) or \( \mathcal{R} \mathcal{V} \) is empty.

Since a quantum space \( \mathcal{H}(\mathcal{Q} \mathcal{V}) \) is an Hilbert space, \( \mathcal{H}(\mathcal{Q} \mathcal{V}) \) has a zero element \( 0_{\mathcal{Q} \mathcal{V}} \) (we will omit the subscript, when this does not cause ambiguity). In particular, if \( \mathcal{Q} \mathcal{V} \cap \mathcal{R} \mathcal{V} = \emptyset \), \( \mathcal{Q} \in \mathcal{H}(\mathcal{Q} \mathcal{V}) \) and \( \mathcal{R} \in \mathcal{H}(\mathcal{R} \mathcal{V}) \), then \( \mathcal{Q} \otimes 0_{\mathcal{R} \mathcal{V}} = 0_{\mathcal{Q} \mathcal{V}} \otimes \mathcal{R} = 0_{\mathcal{Q} \mathcal{V} \cup \mathcal{R} \mathcal{V}} \in \mathcal{H}(\mathcal{Q} \mathcal{V} \cup \mathcal{R} \mathcal{V}) \).

Definition 1 (Quantum registers) Given a quantum space \( \mathcal{H}(\mathcal{Q} \mathcal{V}) \), a quantum register is any \( \mathcal{Q} \in \mathcal{H}(\mathcal{Q} \mathcal{V}) \) such that either \( \mathcal{Q} = 0_{\mathcal{Q} \mathcal{V}} \) or \( \mathcal{Q} \) is a normalised vector.

Let \( \mathcal{Q} \mathcal{V} \) be a qvs with cardinality \( n \geq 1 \). Moreover, let \( \mathcal{Q} \in \mathcal{H}(\mathcal{Q} \mathcal{V}) \) and let \( r \in \mathcal{Q} \mathcal{V} \). Each state \( \mathcal{Q} \) may be represented as follows:

\[
\mathcal{Q} = \sum_{i=1}^{2^{n-1}} \alpha_i |r \mapsto 0 \rangle \otimes b_i + \sum_{i=1}^{2^{n-1}} \beta_i |r \mapsto 1 \rangle \otimes b_i
\]

where \( \{b_i\}_{i=1}^{2^{n-1}} \) is the computational basis\(^5\) of \( \mathcal{H}(\mathcal{Q} \mathcal{V} - \{r\}) \). Please note that if \( \mathcal{Q} \mathcal{V} = \{r\} \), then \( \mathcal{Q} = \alpha |r \mapsto 0 \rangle \otimes 1 + \beta |r \mapsto 1 \rangle \otimes 1 \), that is, via the previously stated isomorphism, \( \alpha |r \mapsto 0 \rangle + \beta |r \mapsto 1 \rangle \).

Definition 2 (Destructive measurements) Let \( \mathcal{Q} \mathcal{V} \) be a qvs with cardinality \( n = \|\mathcal{Q} \mathcal{V}\| \geq 1 \), \( r \in \mathcal{Q} \mathcal{V} \), \( \{b_i\}_{i=1}^{2^{n-1}} \) be the computational basis of \( \mathcal{H}(\mathcal{Q} \mathcal{V} - \{r\}) \) and \( \mathcal{Q} \) be \( \sum_{i=1}^{2^{n-1}} \alpha_i |r \mapsto 0 \rangle \otimes b_i + \sum_{i=1}^{2^{n-1}} \beta_i |r \mapsto 1 \rangle \otimes b_i \in \mathcal{H}(\mathcal{Q} \mathcal{V}) \). The two linear functions

\[
m_{r,0}, m_{r,1} : \mathcal{H}(\mathcal{Q} \mathcal{V}) \rightarrow \mathcal{H}(\mathcal{Q} \mathcal{V} - \{r\})
\]

such that

\[
m_{r,0}(\mathcal{Q}) = \sum_{i=1}^{2^{n-1}} \alpha_i b_i \quad m_{r,1}(\mathcal{Q}) = \sum_{i=1}^{2^{n-1}} \beta_i b_i
\]

are called destructive measurements. If \( \mathcal{Q} \) is a quantum register, the probability \( p_c \) of observing \( c \in \{0,1\} \) when observing \( r \) in \( \mathcal{Q} \) is defined as \( \langle \mathcal{Q} | m_{r,c} | m_{r,c} | \mathcal{Q} \rangle \).

The just defined measurement operators are general measurements \([8, 9]\):

Proposition 1 (Completeness Condition) Let \( r \in \mathcal{Q} \mathcal{V} \) and \( \mathcal{Q} \in \mathcal{H}(\mathcal{Q} \mathcal{V}) \). Then \( m_{r,0}^\dagger m_{r,0} + m_{r,1}^\dagger m_{r,1} = \text{Id}_{\mathcal{H}(\mathcal{Q} \mathcal{V})} \).
**Proof.** In order to prove the proposition we will use the following general property of inner product spaces: let \( \mathcal{H} \) be an inner product space and let \( A : \mathcal{H} \to \mathcal{H} \) be a linear map. If for each \( x, y \in \mathcal{H} \), \( \langle Ax, y \rangle = \langle x, y \rangle \) then \( A \) is the identity map.\(^6\) Let \( Q, R \in \mathcal{H}(QV) \). If \( \{b_i\}_{i \in [1,2^n]} \) is the computational basis of \( \mathcal{H}(QV - \{r\}) \), then:

\[
\begin{align*}
Q &= \sum_{i=1}^{2^n} \alpha_i |r \mapsto 0\rangle \otimes b_i + \sum_{i=1}^{2^n} \beta_i |r \mapsto 1\rangle \otimes b_i \\
R &= \sum_{i=1}^{2^n} \gamma_i |r \mapsto 0\rangle \otimes b_i + \sum_{i=1}^{2^n} \delta_i |r \mapsto 1\rangle \otimes b_i.
\end{align*}
\]

We have:

\[
\langle (m_{r,0}^\dagger m_{r,0} + m_{r,1}^\dagger m_{r,1})(Q), R \rangle = \langle m_{r,0}^\dagger m_{r,0}(Q), R \rangle + \langle m_{r,1}^\dagger m_{r,1}(Q), R \rangle
\]

\[
= \langle m_{r,0}(Q), m_{r,0}(R) \rangle + \langle m_{r,1}(Q), m_{r,1}(R) \rangle
\]

\[
= \left( \sum_{i=1}^{2^n} \alpha_i b_i, \sum_{i=1}^{2^n} \gamma_i b_i \right) + \left( \sum_{i=1}^{2^n} \beta_i b_i, \sum_{i=1}^{2^n} \delta_i b_i \right)
\]

\[
= \sum_{i=0}^{2^n} \alpha_i \gamma_i + \sum_{i=0}^{2^n} \beta_i \delta_i
\]

\[
= \langle Q, R \rangle.
\]

This concludes the proof. \( \square \)

For \( c \in \{0,1\} \), the measurement operators \( m_{r,c} \) enjoys the following properties:

**Proposition 2** Let \( Q \in \mathcal{H}(QV) \). Then:

1. \( m_{r,c}(Q \otimes |q \mapsto d\rangle) = \langle m_{r,c}(Q) \rangle \otimes |q \mapsto d\rangle \) if \( r \in QV \) and \( q \notin QV \);
2. \( \langle Q \otimes |s \mapsto d\rangle |m_{r,c}^\dagger m_{r,c}Q \otimes |s \mapsto d\rangle \rangle = \langle Q, m_{r,c}^\dagger m_{r,c}Q \rangle \); if \( r \in QV \) and \( r \neq s \);
3. \( m_{r,c}(m_{r,d}(Q)) = m_{r,d}(m_{r,c}(Q)) \); if \( r, q \in QV \).

**Proof.** 1. Given the computational basis \( \{b_i\}_{i \in [1,2^n]} \) of \( \mathcal{H}(QV - \{r\}) \), we have that:

\[
Q \otimes |q \mapsto d\rangle = \sum_{i=1}^{2^n} \alpha_i |r \mapsto 0\rangle \otimes b_i \otimes |q \mapsto d\rangle + \sum_{i=1}^{2^n} \beta_i |r \mapsto 1\rangle \otimes b_i |r \mapsto d\rangle
\]

and therefore

\[
m_{r,0}(Q \otimes |q \mapsto d\rangle) = \sum_{i=1}^{2^n} \alpha_i (b_i \otimes |r \mapsto d\rangle)
\]

\[
= \left( \sum_{i=1}^{2^n} \alpha_i b_i \right) \otimes |q \mapsto d\rangle
\]

\[
= \langle m_{r,0}(Q) \rangle \otimes |q \mapsto d\rangle.
\]

In the same way we prove the equality for \( m_{r,1} \).

2. Just observe that:

\[
\langle Q \otimes |s \mapsto d\rangle |m_{r,c}^\dagger m_{r,c}Q \otimes |s \mapsto d\rangle \rangle = \langle Q \otimes |s \mapsto d\rangle, m_{r,c}^\dagger m_{r,c}(Q \otimes |s \mapsto d\rangle) \rangle
\]

\[
= \langle m_{r,c}(Q), m_{r,c}(Q) \rangle
\]

\[
= \langle Q, m_{r,c}^\dagger m_{r,c}Q \rangle = \langle Q, m_{r,c}^\dagger m_{r,c}Q \rangle.
\]

\(^6\)such a property is an immediate consequence of the Riesz representation theorem, see e.g. [12].
3. Given the computational basis \( \{ b_i \}_{i \in [1,2^n]} \) of \( \mathcal{H}(\mathcal{QV} - \{ r,q \}) \), we have that:

\[
\mathcal{Q} = \sum_{i=1}^{2^n} \alpha_i | r \mapsto 0 \rangle \otimes | q \mapsto 0 \rangle \otimes b_i + \sum_{i=1}^{2^n} \beta_i | r \mapsto 0 \rangle \otimes | q \mapsto 1 \rangle \otimes b_i + \\
\sum_{i=1}^{2^n} \gamma_i | r \mapsto 1 \rangle \otimes | q \mapsto 0 \rangle \otimes b_i + \sum_{i=1}^{2^n} \delta_i | r \mapsto 1 \rangle \otimes | q \mapsto 1 \rangle \otimes b_i.
\]

Let us show that \( m_{q,0}(m_{r,0}(\mathcal{Q})) = m_{r,0}(m_{q,0}(\mathcal{Q})) \), the proof of other cases follow the same pattern.

\[
m_{r,0}(m_{q,0}(\mathcal{Q})) = \sum_{i=1}^{2^n} \alpha_i b_i = m_{q,0} \left( \sum_{i=1}^{2^n} \alpha_i | q \mapsto 0 \rangle \otimes b_i + \sum_{i=1}^{2^n} \beta_i | q \mapsto 1 \rangle \otimes b_i \right)
\]

\[
= \sum_{i=1}^{2^n} \alpha_i b_i = m_{q,0}(m_{r,0}(\mathcal{Q})).
\]

This concludes the proof. \( \square \)

Given a qvs \( \mathcal{QV} \) and a variable \( r \in \mathcal{QV} \), we can define two linear maps:

\[ \mathcal{M}_{r,0}, \mathcal{M}_{r,1} : \mathcal{H}(\mathcal{QV}) \rightarrow \mathcal{H}(\mathcal{QV} - \{ r \}) \]

which are “normalized” versions of \( m_{r,0} \) and \( m_{r,1} \) as follows:

1. if \( (\mathcal{Q}| m_{r,c} m_{r,c} | \mathcal{Q}) = 0 \) then \( \mathcal{M}_{r,c}(\mathcal{Q}) = m_{r,c}(\mathcal{Q}) \);
2. if \( (\mathcal{Q}| m_{r,c} m_{r,c} | \mathcal{Q}) \neq 0 \) then \( \mathcal{M}_{r,c}(\mathcal{Q}) = \frac{m_{r,c}(\mathcal{Q})}{\sqrt{\langle \mathcal{Q}| m_{r,c} m_{r,c} | \mathcal{Q} \rangle}} \).

**Proposition 3** Let \( \mathcal{Q} \in \mathcal{H}(\mathcal{QV}) \) be a quantum register. Then:

1. \( \mathcal{M}_{r,c}(\mathcal{Q}) \) is a quantum register;
2. \( \mathcal{M}_{r,c}(\mathcal{Q} \otimes | r \mapsto d \rangle) = (\mathcal{M}_{r,c}(\mathcal{Q})) \otimes | r \mapsto d \rangle \), with \( q \in \mathcal{QV} \) and \( q \neq r \);
3. \( \mathcal{M}_{q,c}(\mathcal{M}_{r,d}(\mathcal{Q})) = \mathcal{M}_{r,d}(\mathcal{M}_{q,c}(\mathcal{Q})) \), with \( q,r \in \mathcal{QV} \);
4. if \( q,r \in \mathcal{QV} \), \( \mathcal{M}_{q,c}(\mathcal{Q} \otimes | r \mapsto d \rangle) = (\mathcal{M}_{q,c}(\mathcal{Q})) \otimes | r \mapsto d \rangle \), with \( q,r \in \mathcal{QV} \);
5. \( \mathcal{M}_{r,c}(\mathcal{Q} \otimes | r \mapsto d \rangle) = (\mathcal{M}_{q,c}(\mathcal{Q})) \otimes | r \mapsto d \rangle \), with \( q,r \in \mathcal{QV} \);

**Proof.** The proofs of 1, 2, and 3 are immediate consequences of Proposition 2 and of general basic properties of Hilbert spaces. About 4 and 5 if \( \mathcal{Q} = 0_{\mathcal{QV}} \) then the proof is trivial; if either \( p_{r,c} = 0 \) or \( p_{q,d} = 0 \) (possibly both), observe that \( s_{r,c} = s_{q,d} = 0 \) and \( M_{q,r,d}(\mathcal{Q}) = \mathcal{M}_{r,d}(\mathcal{Q}) \). Given the computational basis \( \{ b_i \}_{i \in [1,2^n]} \) of \( \mathcal{H}(\mathcal{QV} - \{ r,q \}) \), we have that:

\[
\mathcal{Q} = \sum_{i=1}^{2^n} \alpha_i | r \mapsto 0 \rangle \otimes | q \mapsto 0 \rangle \otimes b_i + \sum_{i=1}^{2^n} \beta_i | r \mapsto 0 \rangle \otimes | q \mapsto 1 \rangle \otimes b_i + \\
\sum_{i=1}^{2^n} \gamma_i | r \mapsto 1 \rangle \otimes | q \mapsto 0 \rangle \otimes b_i + \sum_{i=1}^{2^n} \delta_i | r \mapsto 1 \rangle \otimes | q \mapsto 1 \rangle \otimes b_i
\]

Let us examine the case \( c = 0 \) and \( d = 0 \) (the other cases can be handled in the same way).

\[
p_{r,0} = \sum_{i=1}^{2^n} | \alpha_i |^2 + \sum_{i=1}^{2^n} | \beta_i |^2; \quad p_{q,0} = \sum_{i=1}^{2^n} | \alpha_i |^2 + \sum_{i=1}^{2^n} | \gamma_i |^2.
\]
\[
\begin{align*}
Q_{r,0} &= \mathcal{M}_{r,0}(Q) = \frac{\sum_{i=1}^{2^n} \alpha_i |q \mapsto 0\rangle \otimes b_i + \sum_{i=1}^{2^n} \beta_i |q \mapsto 1\rangle \otimes b_i}{\sqrt{p_{r,0}}} \\
Q_{q,0} &= \mathcal{M}_{q,0}(Q) = \frac{\sum_{i=1}^{2^n} \alpha_i |r \mapsto 0\rangle \otimes b_i + \sum_{i=1}^{2^n} \gamma_i |r \mapsto 1\rangle \otimes b_i}{\sqrt{p_{q,0}}}
\end{align*}
\]

Now let us consider the two states:

\[
\begin{align*}
Q^r_{q,0} &= m_{q,0}(Q_{r,0}) = \frac{\sum_{i=1}^{2^n} \alpha_i b_i}{\sqrt{p_{r,0}}} \\
Q^r_{q,0} &= m_{r,0}(Q_{q,0}) = \frac{\sum_{i=1}^{2^n} \alpha_i b_i}{\sqrt{p_{q,0}}}
\end{align*}
\]

By definition:

\[
\begin{align*}
s_{q,0} &= \frac{\sum_{i=1}^{2^n} |\alpha_i|^2}{p_{q,0}} \\
s_{r,0} &= \frac{\sum_{i=1}^{2^n} |\alpha_i|^2}{p_{r,0}}
\end{align*}
\]

and therefore \( p_{r,0} \cdot s_{q,0} = p_{q,0} \cdot s_{r,0} \). Moreover, if \( QV = \emptyset \) then \( M_{q,0}(Q_{r,0}) = M_{r,0}(Q_{q,0}) = 1 \), otherwise:

\[
\begin{align*}
M_{q,0}(Q_{r,0}) &= \frac{Q^r_{q,0}}{\sqrt{p_{q,0}}} = \frac{\sum_{i=1}^{2^n} \alpha_i b_i}{\sqrt{p_{r,0} \cdot p_{q,0}}} = \frac{\sum_{i=1}^{2^n} \alpha_i b_i}{\sqrt{p_{r,0} \cdot \sqrt{p_{q,0}}}} = \frac{\sum_{i=1}^{2^n} |\alpha_i|^2}{\sqrt{\sum_{i=1}^{2^n} |\alpha_i|^2}} \\
M_{r,0}(Q_{q,0}) &= \frac{Q^r_{q,0}}{\sqrt{p_{r,0}}} = \frac{\sum_{i=1}^{2^n} \alpha_i b_i}{\sqrt{p_{r,0} \cdot p_{q,0}}} = \frac{\sum_{i=1}^{2^n} \alpha_i b_i}{\sqrt{p_{q,0} \cdot \sqrt{p_{r,0}}}} = \frac{\sum_{i=1}^{2^n} |\alpha_i|^2}{\sqrt{\sum_{i=1}^{2^n} |\alpha_i|^2}}
\end{align*}
\]

and therefore \( M_{q,0}(Q_{r,0}) = M_{r,0}(Q_{q,0}) \).

### 3.1 Computations

In \( Q^* \) a computation is performed by reducing configurations. A preconfiguration is a triple \( [Q, QV, M] \) where:

- \( M \) is a term;
- \( QV \) is a finite quantum variable set such that \( Q(M) \subseteq QV \);
- \( Q \in \mathcal{H}(QV) \).

Let \( \theta : QV \to RV \) be a bijective function from a (nonempty) finite set of quantum variables \( QV \) to another set of quantum variables \( RV \). Then we can extend \( \theta \) to any term whose quantum variables are included in \( QV \); \( \theta(M) \) will be identical to \( M \), except on quantum variables, which are changed according to \( \theta \) itself. Observe that \( Q(\theta(M)) \subseteq RV \). Similarly, \( \theta \) can be extended to a function from \( \mathcal{H}(QV) \) to \( \mathcal{H}(RV) \) in the obvious way.

**Definition 3 (Configurations)** Two preconfigurations \( [Q, QV, M] \) and \( [R, RV, N] \) are equivalent iff there is a bijection \( \theta : QV \to RV \) such that \( R = \theta(Q) \) and \( N = \theta(M) \). If a preconfiguration \( C \) is equivalent to \( D \), then we will write \( C \equiv D \). The relation \( \equiv \) is an equivalence relation. A configuration is an equivalence class of preconfigurations modulo the relation \( \equiv \). Let \( \text{Conf} \) be the set of configurations.

**Remark 2** The way configurations have been defined, namely quotienting preconfigurations over \( \equiv \), is very reminiscent of usual \( \alpha \)-conversion in lambda-terms.

Let \( \mathcal{L} = \{ \text{uq, new, l, \beta, q, \beta, c, \beta, l, cm, r, cm, if_{l}, if_{r}, meas_{r}} \} \). For every \( \alpha \in \mathcal{L} \) and for every \( p \in \mathbb{R}_{[0,1]} \), we define a relation \( \rightarrow_{\alpha}^p \subseteq \text{Conf} \times \text{Conf} \) by the set of contractions in Figure 2. The notation \( C \rightarrow_{\alpha}^p D \) stands for \( C \rightarrow_{\alpha}^1 D \).

In order to be consistent with the so-called non-cloning and non-erasing properties, we adopt surface reduction \( [13][5] \): reduction is not allowed in the scope of any ! operator. Furthermore, as
The confluence problem is central for any quantum λ-calculus with measurements, as stressed in the introduction. Let us consider the following configuration:

\[ C = [1, \emptyset, (\lambda x. (\text{if } x \text{ then } 0 \text{ else } 1))(\text{meas}(H(\text{new}(0))))]. \]

If we focus on reduction sequences, it is easy to check that there are two different reduction
sequences starting with $C$, the first ending in the normal form $[1, \emptyset, 0]$ (with probability 1/2) and the second in the normal form $[1, \emptyset, 1]$ (with probability 1/2). But if we reason with mixed states, the situation changes: the mixed state $\{1 : C\}$ (i.e., the mixed state assigning probability 1 to $C$ and 0 to any other configuration) rewrites \textit{deterministically} to $\{1/2 : [1, \emptyset, 0], 1/2 : [1, \emptyset, 1]\}$ (where both $[1, \emptyset, 0]$ and $[1, \emptyset, 1]$ have probability 1/2). So, confluence seems to hold.

**Confluence in Other Quantum Calculi.** Contrarily to the measurement-free case, the above notion of confluence is \textit{not} an expected result for a quantum lambda calculus. Indeed, it does not hold in the quantum lambda calculus $\lambda_{sv}$ proposed by Selinger and Valiron\cite{SV}. In $\lambda_{sv}$, it is possible to exhibit a configuration $C$ that gives as outcome the distribution $\{1 : [1, \emptyset, 0]\}$ when reduced call-by-value and the distribution $\{1/2 : [1, \emptyset, 0], 1/2 : [1, \emptyset, 1]\}$ if reduced call-by-name. This is a \textit{real} failure of confluence, which is there even if one uses probability distributions in place of configurations. The same phenomenon cannot happen in $Q^*$ (as we will show in Section\ref{sec:finite}): this fundamental difference can be traced back to another one: the linear lambda calculus with surface reduction (on which $Q^*$ is based) enjoys (a slight variation on) the so-called diamond property\cite{SV}, while in usual, pure, lambda calculus (on which $\lambda_{sv}$ is based) confluence only holds in a weaker sense.

**Finite or infinite rewriting?** In $Q^*$, an infinite computation can tend to a configuration which is essentially different from the configurations in the computation itself. For example, a configuration $C = [1, \emptyset, M]$ can be built\footnote{$M \equiv (Y!(M f.\lambda x. \text{if } x \text{ then } 0 \text{ else } f(\text{meas}(H(\text{neu}(0)))))))(\text{meas}(H(\text{neu}(0))))$, where $Y$ is a fix point operator.} such that:

- after a finite number of reduction steps $C$ rewrites to a distribution in the form $\{\sum_{1 \leq i \leq n} \frac{1}{2} : [1, \emptyset, 0], 1 - \sum_{1 \leq i \leq n} \frac{1}{2} : D\}$
- only after infinitely many reduction steps the distribution $\{1 : [1, \emptyset, 0]\}$ is reached.

Therefore finite probability distributions of finite configurations could be obtained by means of infinite rewriting. We believe that the study of confluence for infinite computations is important.

**Related Work.** In the literature, probabilistic rewriting systems have been already analyzed. For example, Bournez and Kirchner\cite{Bournez-Kirchner} have introduced the notion of a probabilistic abstract rewriting system as a structure $A = ([A], \cdot \rightsquigarrow \cdot)$, where $[A]$ is a set and $\cdot \rightsquigarrow \cdot$ is a function from $[A]$ to $\mathbb{R}$ such that for every $a \in [A]$, $\sum_{b \in [A]} [a \rightsquigarrow b]$ is either 0 or 1. Then, they define a notion of \textit{probabilistic confluence} for a PARS: such a structure is probabilistically locally confluent iff the probability to be locally confluent, in a classical sense, is different from 0. Unfortunately, Bournez and Kirchner’s analysis does not apply to $Q^*$, since $Q^*$ is \textit{not} a PARS. Indeed, the quantity $\sum_{b \in [A]} [a \rightsquigarrow b]$ can in general be any natural number. Similar considerations hold for the probabilistic lambda calculus introduced by Di Pierro, Hankin and Wiklicky in\cite{DPHW}.

\section{A Probabilistic Notion of Computation}

We represent computations as (possibly) infinite trees. In the following, a (possibly) infinite tree $T$ will be an $(n + 1)$-tuple $[R, T_1, \ldots, T_n]$, where $n \geq 0$, $R$ is the root of $T$ and $T_1, \ldots, T_n$ are its immediate subtrees.

**Definition 4** A set of (possibly) infinite trees $\mathcal{I}$ is said to be a set of probabilistic computations if $P \in \mathcal{I}$ iff (exactly) one of the following three conditions holds:

1. $P = [C]$ and $C \in \text{Conf}$.
2. $P = [C, R]$, where $C \in \text{Conf}$, $R \in \mathcal{I}$ has root $D$ and $C \rightarrow_{n.\#} D$.
3. $P = [(p, q, C), R, Q]$, where $C \in \text{Conf}$, $R, Q \in \mathcal{I}$ have roots $D$ and $E$, $C \rightarrow_{\text{meas}_p} D$, $C \rightarrow_{\text{meas}_q} E$ and $p, q \in [0, 1]$.

The set of all (respectively, the set of finite) probabilistic computations is the largest set $\mathcal{I}$ (respectively, the smallest set $\mathcal{F}$) of probabilistic computations with respect to set inclusion. $\mathcal{I}$ and $\mathcal{F}$ exist because of the Knappe-Tarski Theorem.
We will often say that the root of $P = [(p, q, C), R, Q]$ is simply $C$, slightly diverging from the above definition without any danger of ambiguity.

**Definition 5** A probabilistic computation $P$ is maximal if for every leaf $C$ in $P$, $C \in \mathcal{NF}$. More formally, (sets of) maximal probabilistic computations can be defined as in Definition 4, where clause [2] must be restricted to $C \in \mathcal{NF}$.

We can give definitions and proofs over finite probabilistic computations (i.e., over $\mathcal{F}$) by ordinary induction. An example is the following definition. Notice that the same is not true for arbitrary probabilistic definitions, since $\mathcal{P}$ is not a well-founded set.

**Definition 6** Let $P \in \mathcal{P}$ be a probabilistic computation. A finite probabilistic computation $R \in \mathcal{F}$ is a sub-computation of $P$, written $R \sqsubseteq P$ iff one of the following conditions is satisfied:

- $R = [C]$ and the root of $P$ is $C$.
- $R = [C, Q]$, $P = [C, S]$, and $Q \sqsubseteq S$.
- $R = [(p, q, C), Q, S]$, $P = [(p, q, C), U, V]$, $Q \sqsubseteq U$ and $S \sqsubseteq V$.

Let $\delta : \text{Conf} \rightarrow \{0, 1\}$ be a function defined as follows: $\delta(C) = 0$ if the quantum register of $C$ is 0, otherwise, $\delta(C) = 1$.

**Quantitative Properties of Computations.** The outcomes of a probabilistic computation $P$ are given by the configurations which appear as leaves of $P$. Starting from this observation, the following definitions formalize some quantitative properties of probabilistic computations. For every finite probabilistic computation $P$ and every $C \in \mathcal{NF}$ we define $\mathcal{P}(P, C) \in \mathbb{R}_{[0, 1]}$ by induction on the structure of $P$:

- $\mathcal{P}([C], C) = \delta(C)$;
- $\mathcal{P}([C], D) = 0$ whenever $C \neq D$;
- $\mathcal{P}([C, P], D) = \mathcal{P}(P, D)$;
- $\mathcal{P}([(p, q, C), P, R], D) = p\mathcal{P}(P, D) + q\mathcal{P}(R, D)$;

Similarly for $\mathcal{N}(P, C) \leq \aleph_0$:

- $\mathcal{N}([C], C) = 1$;
- $\mathcal{N}([C], D) = 0$ whenever $C \neq D$;
- $\mathcal{N}([C, P], D) = \mathcal{N}(P, D)$;
- $\mathcal{N}([(p, q, C), P, R], D) = \mathcal{N}(P, D) + \mathcal{N}(R, D)$.

Informally, $\mathcal{P}(P, C)$ is the probability of observing $C$ as a leaf in $P$, and $\mathcal{N}(P, C)$ is the number of times $C$ appears as a leaf in $P$.

The definitions above can be easily modified to get the probability of observing any configuration (in normal form) as a leaf in $P$, $\mathcal{P}(P)$, or the number of times any configuration appears as a leaf in $P$, $\mathcal{N}(P)$. Since $\mathbb{R}_{[0, 1]}$ and $\mathbb{N} \cup \{\aleph_0\}$ are complete lattices (with respect to standard orderings), we extend the above notions to the case of arbitrary probabilistic computations, by taking the least upper bound over all finite sub-computations. If $P \in \mathcal{P}$ and $C \in \mathcal{NF}$, then:

- $\mathcal{P}(P, C) = \sup_{R \sqsubseteq P} \mathcal{P}(R, C)$;
- $\mathcal{N}(P, C) = \sup_{R \sqsubseteq P} \mathcal{N}(R, C)$;
- $\mathcal{P}(P) = \sup_{R \sqsubseteq P} \mathcal{P}(R)$;
- $\mathcal{N}(P) = \sup_{R \sqsubseteq P} \mathcal{N}(R)$.  

The following lemmas involve finite computations and can be proved by induction.

**Lemma 1** If $P \sqsubseteq R$, then $\mathcal{P}(P) \leq \mathcal{P}(R)$ and $\mathcal{N}(P) \leq \mathcal{N}(R)$. Moreover, $\mathcal{P}(P, C) \leq \mathcal{P}(R, C)$ and $\mathcal{N}(P, C) \leq \mathcal{N}(R, C)$ for every $C \in \mathcal{NF}$.

**Proof.** A trivial induction on $P$. □

**Lemma 2** If $P \sqsubseteq R$ and $P$ is maximal, then $R$ is maximal.

**Proof.** A trivial induction on $P$. □
6 A Strong Confluence Result

In this Section, we will prove a strong confluence result in the following form: any two maximal probabilistic computations \( P \) and \( R \) with the same root have exactly the same quantitative and qualitative behaviour, that is to say, the following equations hold for every \( C \in \text{NF} \):

\[
\begin{align*}
P(P, C) &= P(R, C); \\
N(P, C) &= N(R, C); \\
P(P) &= P(R); \\
N(P) &= N(R).
\end{align*}
\]

**Remark 3** Please notice that equalities like the ones above do not even hold for the ordinary lambda calculus. For example, the lambda term \((\lambda x. \lambda y . y)\Omega\) is the root of two (linear) computations, the first having one leaf \( \lambda y . y \) and the second having no leaves. This is the reason why the confluence result we prove here is dubbed as strong.

Before embarking in the proof of the equalities above, let us spend a few words to explain their consequences. The fact \( P(P, C) = P(R, C) \) whenever \( P \) and \( R \) have the same root can be read as a confluence result: the probability of observing \( C \) is independent from the adopted strategy. On the other hand, \( P(P) = P(R) \) means that the probability of converging is not affected by the underlying strategy. The corresponding results on \( N(\cdot , \cdot) \) and \( N(\cdot) \) can be read as saying that the number of (not necessarily distinct) leaves in any probabilistic computation with root \( C \) does not depend on the strategy.

**Lemma 3 (Uniformity)** For every \( M, N \) such that \( M \rightarrow_\alpha N \), exactly one of the following conditions holds:

1. \( \alpha \neq \text{new} \) and \( \alpha \neq \text{meas} \), and there is a unitary transformation \( U_{M, N} : \mathcal{H}(\mathbb{Q}(M)) \rightarrow \mathcal{H}(\mathbb{Q}(M)) \) such that \( [Q, QV, M] \rightarrow_\alpha [R, RV, N] \) iff \( [Q, QV, M] \in \text{Conf} \), \( RV = QV \) and \( R = (U_{M, N} \otimes I_{QV - Q(M)})Q \).
2. \( \alpha = \text{new} \) and there is a constant \( c \) and a quantum variable \( r \) such that \( [Q, QV, M] \rightarrow_\text{new} [R, RV, N] \) iff \( [Q, QV, M] \in \text{Conf} \), \( RV = QV \cup \{r\} \) and \( R = Q \otimes r \rightarrow c \).
3. \( \alpha = \text{meas} \), and there is a constant \( c \) and a probability \( p_c \in \mathbb{R}_{[0,1]} \) such that \( [Q, QV, M] \rightarrow_\text{meas} [R, RV, N] \) iff \( [Q, QV, M] \in \text{Conf} \), \( R = M r, c(Q) \) and \( RV = QV - \{r\} \).

**Proof.** We go by induction on \( M \). \( M \) cannot be a variable nor a constant nor a unitary operator nor a term \( \lambda L \). If \( M \) is an abstraction \( \lambda \psi . L \), then \( N \equiv \lambda \psi . P \) and the thesis follows from the inductive hypothesis. If \( M \) is \( \text{meas}(L) \) and \( N \equiv \text{meas}(P) \) then \( L \rightarrow_\alpha P \) and the thesis follows from the inductive hypothesis. Similarly if \( M \equiv \text{new}(L) \) and \( N \equiv \text{new}(P) \). And again if \( M \) is \( \langle M_1, \ldots, L, \ldots, M_n \rangle \) and \( N \equiv \langle M_1, \ldots, P, \ldots, M_n \rangle \). If \( M \equiv LQ \), then we distinguish a number of cases:

- \( N \equiv PQ \) and \( L \rightarrow_\alpha P \). The thesis follows from the inductive hypothesis.
- \( N \equiv LS \) and \( Q \rightarrow_\alpha S \). The thesis follows from the inductive hypothesis.
- \( L \equiv U, \ Q \equiv \langle r_1, \ldots, r_n \rangle \) and \( N \equiv \langle r_1, \ldots, r_n \rangle \). Then case 1 holds. In particular, \( Q(M) = \{r_1, \ldots, r_n \} \) and \( U_{M, N} = U_{\langle r_1, \ldots, r_n \rangle} \).
- \( L \equiv \lambda x . R \) and \( N = R(Q/x) \). Then case 1 holds. In particular \( U_{M, N} = I_{Q(M)} \).
- \( L \equiv \lambda(x_1, \ldots, x_n) . R, Q = \langle r_1, \ldots, r_n \rangle \) and \( N \equiv R(r_1/x_1, \ldots, r_n/x_n) \). Then case 1 holds and \( U_{M, N} = I_{Q(M)} \).
- \( L \equiv \lambda x . R \), \( Q = \tau \) and \( N \equiv \tau(T/x) \). Then case 1 holds and \( U_{M, N} = I_{Q(M)} \).
- \( Q \equiv \langle \lambda \pi . R \rangle \), \( N \equiv \langle \lambda \pi . L \rangle \). Then case 1 holds and \( U_{M, N} = I_{Q(M)} \).
- \( L \equiv \langle \lambda \pi . R \rangle \), \( N \equiv \langle \lambda \pi . RQ \rangle \). Then case 1 holds and \( U_{M, N} = I_{Q(M)} \).

If \( M \equiv \text{new}(c) \) then \( N \) is a quantum variable \( r \) and case 2 holds. If \( M \equiv \text{meas}(r) \) then there are a constant \( c \) and a probability \( p_c \) such that \( N \) is a term \( lc \) and case 3 holds. This concludes the proof. \( \square \)
Notice that $U_{M,N}$ is always the identity function when performing classical reduction. The following technical lemma will be useful when proving confluence:

Lemma 4 Suppose $[Q, QV, M] \rightarrow_{\alpha} [R, RV, N]$.
1. If $[Q, QV, M\{L/x\}] \in \text{Conf}$, then
   $$[Q, QV, M\{L/x\}] \rightarrow_{\alpha} [R, RV, N\{L/x\}].$$
2. If $[Q, QV, M\{r_1/x_1, \ldots, r_n/x_n\}] \in \text{Conf}$, then
   $$[Q, QV, M\{r_1/x_1, \ldots, r_n/x_n\}] \rightarrow_{\alpha} [R, RV, N\{r_1/x_1, \ldots, r_n/x_n\}].$$
3. If $x, \Gamma \vdash L$ and $[Q, QV, L\{M/x\}] \in \text{Conf}$, then
   $$[Q, QV, L\{M/x\}] \rightarrow_{\alpha} [R, RV, L\{N/x\}].$$

Proof. Claims 1 and 2 can be proved by induction on the proof of $[Q, QV, M] \rightarrow_{\alpha} [R, RV, N]$. Claim 3 can be proved by induction on $N$. □

We prove now that $Q^*$ enjoys a slightly variation of the so-called diamond property, whose proof is fully standard (it is a slight extension of the analogous proof given in [5] for $Q$). As for $Q$, $Q^*$ does not enjoy the diamond property in a strict sense, due to the presence of commutative reduction rules (see, e.g., case 2 of the following Proposition). But thanks to Lemma 5 below, this does not have harmful consequences.

Proposition 4 (Quasi-One-step Confluence) Let $C, D, E$ be configurations with $C \rightarrow_{\alpha}^p D$, $C \rightarrow_{\beta}^q E$. Then:
1. If $\alpha \in \mathcal{X}$ and $\beta \in \mathcal{N}$, then either $D = E$ or there is $F$ with $D \rightarrow_{\mathcal{X}} F$ and $E \rightarrow_{\mathcal{X}} F$.
2. If $\alpha \in \mathcal{X}$ and $\beta \in \mathcal{N}$, then either $D \rightarrow_{\mathcal{X}} E$ or there is $F$ with $D \rightarrow_{\mathcal{X}} F$ and $E \rightarrow_{\mathcal{X}} F$.
3. If $\alpha \in \mathcal{N}$ and $\beta = \text{meas}_p$, then there is $F$ with $D \rightarrow_{\mathcal{N}}^p F$ and $E \rightarrow_{\mathcal{X}} F$.
4. If $\alpha \in \mathcal{N}$ and $\beta = \text{meas}_p$, then either $D = E$ or there is $F$ with $D \rightarrow_{\mathcal{N}} F$ and $E \rightarrow_{\mathcal{X}} F$.
5. If $\alpha \in \mathcal{N}$ and $\beta = \text{meas}_q$ (r $\neq$ q), then there are $t, u \in \mathbb{R}_{[0,1]}$ and a $F$ such that $pt = su$,
   $$D \rightarrow_{\mathcal{N}}^t F$$
   and $E \rightarrow_{\mathcal{N}}^u F$.
6. If $\alpha = \text{meas}_q$ and $\beta = \text{meas}_q$ (r $\neq$ q), then there are $t, u \in \mathbb{R}_{[0,1]}$ and a $F$ such that $pt = su$,
   $$D \rightarrow_{\mathcal{N}}^t F$$
   and $E \rightarrow_{\mathcal{N}}^u F$.

Proof. Let $C \equiv [Q, QV, M]$. We go by induction on $M$. $M$ cannot be a variable nor a constant nor a unitary operator. If $M$ is an abstraction $\lambda x.N$, then $D \equiv [R, RV, \lambda x.S]$, $E \equiv [S, SV, \lambda x.T]$ and

$$[Q, QV, N] \rightarrow_{\alpha} [R, RV, S]$$
$$[Q, QV, N] \rightarrow_{\beta} [S, SV, T]$$

The IH easily leads to the thesis. Similarly when $M \equiv \lambda x.N$, and when $M \equiv \text{meas}(N)$ or $M \equiv \text{if } N \text{ then } P \text{ else } Q$ with $N \neq 0, 1$. If $M \equiv NL$, we can distinguish a number of cases depending on the last rule used to prove $C \rightarrow_{\alpha}^p D$, $C \rightarrow_{\beta}^p E$:

- $D \equiv [R, RV, SL]$ and $E \equiv [S, SV, NR]$ where $[Q, QV, N] \rightarrow_{\alpha} [R, RV, S]$ and $[Q, QV, L] \rightarrow_{\beta} [S, SV, R]$. We need to distinguish several sub-cases:

- If $\alpha, \beta = \text{new}$, then, by Lemma 3 there exist two quantum variables $s, t \notin QV$ and two constants $d, e$ such that $RV = QV \cup \{s\}$, $SV = QV \cup \{t\}$, $R = Q \otimes \{s \leftrightarrow d\}$ and $S = Q \otimes \{t \leftrightarrow e\}$. Applying 3 again, we obtain

$$D \rightarrow_{\text{new}} [Q \otimes \{s \leftrightarrow d\} \otimes \{u \leftrightarrow e\}, QV \cup \{s, u\}, SR\{u/t\}] \equiv F;$$
$$E \rightarrow_{\text{new}} [Q \otimes \{t \leftrightarrow e\} \otimes \{v \leftrightarrow d\}, QV \cup \{t, v\}, S\{u/s\}R] \equiv G.$$
• If \( \alpha = \text{new} \) and \( \beta \neq \text{new}, \text{meas}_r \), then, by Lemma 3 there exist a quantum variable \( r \) and a constant \( c \) such that \( RV = QV \cup \{r\}, R = Q \otimes |r \mapsto c\rangle, SV = QV \) and \( S = (U_{L,R} \otimes I_{QV-\{Q(L)\}})Q \). As a consequence, applying Lemma 3 again, we obtain
\[
D \rightarrow_\beta [(U_{L,R} \otimes I_{QV-\{Q(L)\}})(Q \otimes |r \mapsto c\rangle), QV \cup \{r\}, SR] \equiv F;
E \rightarrow_{\text{new}} [(U_{L,R} \otimes I_{QV-\{Q(L)\}})(Q \otimes |r \mapsto c\rangle), QV \cup \{r\}, SR] \equiv G.
\]

As can be easily checked, \( F \equiv G \).

• If \( \alpha \neq \text{new}, \text{meas}_r \) and \( \beta = \text{new} \), then we can proceed as in the previous case.

• If \( \alpha, \beta \neq \text{new}, \alpha \neq \text{meas}_r, \beta \neq \text{meas}_q \) \( (r,q \text{ not necessarily distinct}) \), then by Lemma 3 there exist \( SV = RV = QV, R = (U_{N,S} \otimes I_{QV-\{Q(N)\}})Q \) and \( S = (U_{L,R} \otimes I_{QV-\{Q(L)\}})Q \). Applying Lemma 3 again, we obtain
\[
D \rightarrow_\beta [(U_{L,R} \otimes I_{QV-\{Q(L)\}})((U_{N,S} \otimes I_{QV-\{Q(N)\}})Q), QV, SR] \equiv F;
E \rightarrow_\alpha [(U_{L,R} \otimes I_{QV-\{Q(L)\}})((U_{N,S} \otimes I_{QV-\{Q(N)\}})Q), QV, SR] \equiv G.
\]

As can be easily checked, \( F \equiv G \).

• If \( \alpha = \text{meas}_r, \beta = \text{meas}_q \) \( (r \neq q) \) then, by Lemma 3 there exist two constants \( d,e \) and two probabilities \( t,u \) such that \( RV = QV \cup \{r\}, SV = QV \cup \{q\}, R = M_{r,d}(Q) \) and \( S = M_{q,e}(Q) \). Remember that the quantum variable \( q \) occurs in the subterm \( N \) and the quantum variable \( r \) occurs in the subterm \( L \). Starting from \( D \equiv [M_{r,d}(Q), QV \cup \{q\}, SL] \) and \( E \equiv [M_{q,e}(Q), QV \cup \{r\}, NR] \), applying Lemma 3 again, we obtain
\[
D \rightarrow_{\text{meas}_q} [M_{q,e}(M_{r,d}(Q)), QV \cup \{q\} \rightarrow \{q\}, SR] \equiv [M_{q,e}(R), RV \cup \{q\} \rightarrow \{q\}, SR] \equiv F;
E \rightarrow_{\text{meas}_r} [M_{r,d}(M_{q,e}(Q)), QV \cup \{q\} \rightarrow \{q\}, SR] \equiv [M_{r,d}(S), SV \cup \{q\} \rightarrow \{q\}, SR] \equiv G.
\]

Clearly, \( QV \cup \{q\} \rightarrow \{q\} \equiv QV \rightarrow \{q\} \rightarrow \{q\} \) and by Proposition 3 case 2 \( M_{q,e}(M_{r,d}(Q)) \equiv M_{r,d}(M_{q,e}(Q)) \). Then \( F \equiv G \). Moreover by Proposition 3 case 3 \( pt = su \).

• If \( \alpha = \text{new}, \beta = \text{meas}_r \), then, by Lemma 3 there exists a quantum variable \( q \) \( (q \neq r) \) two constants \( d \) and \( e \) and a probability \( p_e \) such that \( RV = QV \cup \{q\}, R = Q \otimes |q \mapsto d\rangle, SV = QV \cup \{r\} \) and \( S = M_{r,e}(Q) \). As a consequence, starting from \( D \equiv [QV \cup \{q\}, Q \otimes |q \mapsto d\rangle, SL] \) and \( E \equiv [M_{r,e}(Q), QV \cup \{r\}, NR] \) applying Lemma 3 again, we obtain
\[
D \rightarrow_{\text{meas}_r} [M_{r,e}(Q \otimes |q \mapsto d\rangle), QV \cup \{q\} \rightarrow \{r\}, SR] \equiv [M_{r,e}(R), QV \cup \{q\} \rightarrow \{r\}, SR] \equiv F;
E \rightarrow_{\text{new}} [(M_{r,e}(Q)) \otimes |q \mapsto d\rangle, QV \cup \{r\} \cup \{q\}, SR] \equiv [(S) \otimes |q \mapsto d\rangle, SV \cup \{q\}, SR] \equiv G.
\]

Clearly, \( QV \cup \{q\} \rightarrow \{r\} \equiv QV \rightarrow \{r\} \cup \{q\} \). By Proposition 3 case 2 it is possible to commute the measurement of the quantum variable \( r \) with the creation of the quantum variable \( q \), in fact they are distinct quantum variables. Then \( M_{r,e}(Q \otimes |q \mapsto d\rangle) \) and \( (M_{r,e}(Q)) \otimes |q \mapsto d\rangle \) give the same quantum register. We can conclude that \( F \equiv G \).

• If \( \alpha = \text{meas}_r, \beta = \text{new} \), the case is symmetric to the previous one.

• If \( \alpha = \text{meas}_r, \beta \neq \text{new}, \text{meas}_q \), then by Lemma 3 there exists a constant \( c \) and a probability \( p_c \) such that \( R = M_{r,c}(Q), RV = QV \cup \{r\}, SV = QV \) and \( S = (U_{L,R} \otimes I_{QV-\{Q(L)\}})Q \). As a consequence, starting from \( D \equiv [M_{r,c}(Q), QV \cup \{r\}, SL] \) and \( E \equiv [(U_{L,R} \otimes I_{QV-\{Q(L)\}})Q, QV, NR] \), applying Lemma 3 again, we obtain
\[
D \rightarrow_\beta [(U_{L,R} \otimes I_{QV-\{Q(L)\}})(M_{r,c}(Q)), QV \cup \{r\}, SR] \equiv [(U_{L,R} \otimes I_{QV-\{Q(L)\}})(R), RV, SR] \equiv F;
E \rightarrow_{\text{meas}_r} [M_{r,c}((U_{L,R} \otimes I_{QV-\{Q(L)\}})Q), QV \cup \{r\}, SR] \equiv [M_{r,c}(S), QV \cup \{r\}, SR] \equiv G.
\]
Note that the operators \((U_{L,R} \otimes I_{QV}) \circ \mathcal{M}_{r,c}\) and \(\mathcal{M}_{r,c} \circ (U_{L,R} \otimes I_{QV})\) act on \(Q\) in the same way, by means of \(\text{Proposition 3}\) case 5. We can conclude that \(F \equiv G\).

- If \(\alpha = \text{new}, \beta = \text{meas}_r\), the case is symmetric to the previous one.
- \(D \equiv [\mathcal{R}, \mathcal{RV}, SL]\) and \(E \equiv [\mathcal{S}, \mathcal{SV}, TL]\), where \([Q, QV, N] \rightarrow [\mathcal{R}, \mathcal{RV}, S]\) and \([Q, QV, N] \rightarrow [\mathcal{S}, \mathcal{SV}, T]\). Here we can apply the inductive hypothesis.
- \(D \equiv [\mathcal{R}, \mathcal{RV}, NR]\) and \(E \equiv [\mathcal{S}, \mathcal{SV}, NU]\), where \([Q, QV, L] \rightarrow [\mathcal{R}, \mathcal{RV}, R]\) and \([Q, QV, L] \rightarrow [\mathcal{S}, \mathcal{SV}, U]\). Here we can apply the inductive hypothesis as well.
- \(N \equiv (\lambda x, P), D \equiv [Q, QV, P(L/x)]\), \(E \equiv [\mathcal{R}, \mathcal{RV}, NR]\), where \([Q, QV, L] \rightarrow_{\beta} [\mathcal{R}, \mathcal{RV}, R]\). Clearly \([Q, QV, P(L/x)] \in \text{Conf}\) and, by \(\text{Lemma 4}\) \([Q, QV, P(L/x)] \rightarrow [\mathcal{R}, \mathcal{RV}, P(R/x)]\). Moreover, \([\mathcal{R}, \mathcal{RV}, NR] \equiv [\mathcal{R}, \mathcal{RV}, (\lambda V)P] \rightarrow [\mathcal{R}, \mathcal{RV}, P(R/x)]\).

If \(M\) is in the form \(\text{new}(c)\), then \(D \equiv E\).

\textbf{Remark 4} 
Unfortunately, Proposition 4 does not translate into an equivalent result on mixed states, because of commutative reduction rules. As a consequence, it is more convenient to first study confluence at the level of probabilistic computations.

Note that, even if the calculus is untyped, we cannot build an infinite sequence of commuting reductions:

\textbf{Lemma 5} 
The relation \(\rightarrow_\mathcal{K}\) is strongly normalizing. In other words, there cannot be any infinite sequence \(C_1 \rightarrow_\mathcal{K} C_2 \rightarrow_\mathcal{K} C_3 \rightarrow_\mathcal{K} \ldots\).

\textbf{Proof.}\ Define the size \(|M|\) of a term \(M\) as the number of symbols in it. Moreover, define the abstraction size \(|M|_\lambda\) of \(M\) as the sum over all subterms of \(M\) in the form \(\lambda x. N\), of \(|N|\). Clearly \(|M|_\lambda \leq |M|^2\). Moreover, if \([Q, QV, M] \rightarrow_\mathcal{K} [Q, QV, N]\), then \(|N| = |M|\) but \(|N|_\lambda > |M|_\lambda\). This concludes the proof. \(\square\)

We define the weight \(W(P)\) and the branch degree \(B(P)\) of every finite probabilistic computation \(P\) by induction on the structure of \(P\):

- \(W([C]) = 0\) and \(B([C]) = 1\).
- \(B([C, P]) = B(P)\). Moreover, let \(D\) be the root of \(P\). If \(C \rightarrow_\mathcal{K} D\), then \(W([C, P]) = W(P)\), otherwise \(W([C, P]) = B(P) + W(P)\).
- \( B(((p, C), P, R]) = B(P) + B(R) \), while \( W(((p, C), P, R]) = B(P) + B(R) + W(P) + W(R) \).

Please observe that \( B(P) \geq 1 \) for every \( P \).

Now we propose a probabilistic variation on the classical strip lemma of the \( \lambda \)-calculus. It will have a crucial role in the proof of strong confluence (Theorem 1).

**Lemma 6 (Probabilistic Strip Lemma)** Let \( P \) be a finite probabilistic computation with root \( C \) and positive weight \( W(P) \).

- If \( C \rightarrow^* D \), then there is \( R \) with root \( D \) such that \( W(R) < W(P) \), \( B(R) \leq B(P) \) and for every \( E \in NF \), it holds that \( \mathcal{P}(R, E) \geq \mathcal{P}(P, E) \), \( \mathcal{N}(R, E) \geq \mathcal{N}(P, E) \), \( \mathcal{P}(R) \geq \mathcal{P}(P) \) and \( \mathcal{N}(R) \geq \mathcal{N}(P) \).
- If \( C \rightarrow^* D \), then there is \( R \) with root \( D \) such that \( W(R) \leq W(P) \), \( B(R) \leq B(P) \) and for every \( E \in NF \), it holds that \( \mathcal{P}(R, E) \geq \mathcal{P}(P, E) \), \( \mathcal{N}(R, E) \geq \mathcal{N}(P, E) \), \( \mathcal{P}(R) \geq \mathcal{P}(P) \) and \( \mathcal{N}(R) \geq \mathcal{N}(P) \).

**Proof.** By induction on the structure of \( P \):

- \( P \) cannot simply be \( [C] \), because \( W(P) \geq 1 \).
- If \( P = [C, S] \), where \( S \) has root \( F \) and \( C \rightarrow^* F \), then:
  - Suppose \( C \rightarrow^* D \). If \( D = F \), then the required \( R \) is simply \( S \). Otherwise, by Proposition 4, there is \( G \) such that \( D \rightarrow^* G \) and \( F \rightarrow^* G \). Now, if \( S \) is simply \( [F] \), then the required probabilistic computation is simply \([D]\), because neither \( F \) nor \( D \) are in normal form and, moreover, \( W([D]) = 0 < 1 = W(P) \). If, on the other hand, \( S \) has positive weight we can apply the IH to it, obtaining a probabilistic computation \( T \) with root \( G \) such that \( W(T) < W(S) \), \( B(T) \leq B(S) \), \( \mathcal{P}(T, H) \geq \mathcal{P}(S, H) \) and \( \mathcal{N}(T, H) \geq \mathcal{N}(S, H) \) for every \( H \in NF \). Then, the required probabilistic computation is \([D, T]\), since
    
    \[
    W([D, T]) = B(T) + W(T) < B(T) + W(S) \\
    \leq B(S) + W(S) = W(P); \\
    \mathcal{P}([D, T], H) = \mathcal{P}(T, H) \geq \mathcal{P}(S, H) \\
    = \mathcal{P}(P, H); \\
    \mathcal{N}([D, T], H) = \mathcal{N}(T, H) \geq \mathcal{N}(S, H) \\
    = \mathcal{N}(P, H).
    \]
  - Suppose \( C \rightarrow^* D \). By Proposition 4, one of the following two cases applies:
    - There is \( G \) such that \( D \rightarrow^* G \) and \( F \rightarrow^* G \) Now, if \( S \) is simply \( [F] \), then the required probabilistic computation is simply \([D, [G]]\), because \( W([D, [G]]) = 1 = W(P) \). If, on the other hand, \( S \) has positive weight we can apply the IH to it, obtaining a probabilistic computation \( T \) with root \( G \) such that \( W(T) \leq W(S) \), \( B(T) \leq B(S) \) and \( \mathcal{P}(T, H) \geq \mathcal{P}(T, H) \) for every \( H \in NF \). Then, the required probabilistic computation is \([D, T]\), since
      
      \[
      W([D, T]) = B(T) + W(T) \leq B(T) + W(S) \\
      \leq B(S) + W(S) = W(P) \\
      \mathcal{P}([D, T], H) = \mathcal{P}(T, H) \geq \mathcal{P}(S, H) \\
      = \mathcal{P}(P, H); \\
      \mathcal{N}([D, T], H) = \mathcal{N}(T, H) \geq \mathcal{N}(S, H) \\
      = \mathcal{N}(P, H).
      \]
    - \( D \rightarrow^* F \). The required probabilistic computation is simply \([D, S]\). Indeed:
      
      \[
      W([D, S]) = B(S) + W(S) = W([C, S]) = W([P]).
      \]
Suppose \( C \vdash^q_{\text{meas}} D \) and \( C \vdash^p_{\text{meas}} E \). By Proposition 4, there are \( G \) and \( H \) such that \( D \rightarrow^H G, E \rightarrow^H H, F \vdash^q_{\text{meas}} G, F \vdash^p_{\text{meas}} H \). Now, if \( S \) is simply \( F \), then the required probabilistic computations are simply \([D]\) and \([E]\), because neither \( F \) nor \( D \) nor \( E \) are in normal form and, moreover, \( W([D]) = W([E]) = 0 < 1 = W(P) \). If, on the other hand, \( S \) has positive weight we can apply the IH to it, obtaining probabilistic computations \( T \) and \( U \) with roots \( G \) and \( H \) such that \( W(T) < W(S), W(U) < W(S) \), \( B(T) \leq B(S) \), \( B(U) \leq B(S) \), \( qP(T,H) + (p)P(U,H) \geq P(S,H) \) and \( N(T,H) + N(U,H) \geq N(S,H) \) for every \( H \in \text{NF} \).

Then, the required probabilistic computations are \([D,T]\) and \([E,U]\), since

\[
\begin{align*}
W([D,T]) &= B(T) + W(T) < B(T) + W(S) \\
&\leq B(S) + W(S) = W(P); \\
W([E,U]) &= B(U) + W(U) < B(U) + W(S) \\
&\leq B(S) + W(S) = W(P).
\end{align*}
\]

Moreover, for every \( H \in \text{NF} \)

\[
qP([D,T],H) + pP([E,U],H) = qP(T,H) + pP(U,H) \geq P(S,H) = P(P,H)
\]

\[
N([D,T],H) + N([E,U],H) = N(T,H) + N(U,H) \geq N(S,H) = N(P,H)
\]

The other cases are similar. \( \square \)

The following Proposition follows from the probabilistic strip lemma. It can be read as a simulation result: if \( P \) and \( R \) are maximal and have the same root, then \( P \) can simulate \( R \) (and vice versa).

**Proposition 5** For every maximal probabilistic computations \( P \) and for every finite probabilistic computation \( R \) such that \( P \) and \( R \) have the same root, there is a finite sub-computation \( Q \) of \( P \) such that for every \( C \in \text{NF} \), \( P(Q,C) \geq P(R,C) \) and \( N(Q,C) \geq N(R,C) \). Moreover, \( P(Q) \geq P(R) \) and \( N(Q) \geq N(R) \).

**Proof.** Given any probabilistic computation \( S \), its \( \mathcal{X} \)-degree \( n_S \) is the number of consecutive commutative rules you find descending \( S \), starting at the root. By Lemma 5, this is a good definition. The proof goes by induction on \( \langle W(R), n_R \rangle \), ordered lexicographically:

- If \( W(R) = 0 \), then \( R \) is just \([D]\) for some configuration \( D \). Then, \( Q = R \) and all the required conditions hold.

- If \( W(R) > 0 \), then we distinguish three cases, depending on the shape of \( P \):
  - If \( P = [D,S] \), \( E \) is the root of \( S \) and \( D \rightarrow^H E \), then, by Proposition 6 there is a probabilistic computation \( T \) with root \( E \) such that \( W(T) < W(R) \) and \( P(T,C) \geq P(R,C) \) for every \( C \in \text{NF} \). By the inductive hypothesis applied to \( S \) and \( T \), there is a sub-probabilistic computation \( U \) of \( S \) such that \( P(U,C) \geq P(T,C) \) and \( N(U,C) \geq N(T,C) \) for every \( C \in \text{NF} \). Now, consider the probabilistic computation \([D,U]\). This is clearly a sub-probabilistic computation of \( P \). Moreover, for every \( C \in \text{NF} \):
    \[
    \begin{align*}
    P([D,U],C) &= P(U,C) \\
    &\geq P(T,C) \geq P(R,C) \\
    N([D,U],C) &= N(U,C) \\
    &\geq N(T,C) \geq N(R,C).
    \end{align*}
    \]

- If \( P = [D,S] \), \( E \) is the root of \( S \) and \( D \rightarrow^H E \), then, by Proposition 6 there is a probabilistic computation \( T \) with root \( E \) such that \( W(T) \leq W(R) \) and \( P(T,C) \geq P(R,C) \) for every \( C \in \text{NF} \). Now, observe we can apply the inductive hypothesis to \( S \) and \( T \), because \( W(T) \leq W(R) \) and \( n_S < n_P \). So, there is a sub-probabilistic computation \( U \) of \( S \) such
that \( \mathcal{P}(U,C) \geq \mathcal{P}(T,C) \) and \( \mathcal{N}(U,C) \geq \mathcal{N}(T,C) \) for every \( C \in \text{NF} \). Now, consider the probabilistic computation \([D,U]\). This is clearly a sub-probabilistic computation of \( P \). Moreover, for every \( C \in \text{NF} \):

\[
\begin{align*}
\mathcal{P}([D,U],C) & = \mathcal{P}(U,C) \\
& \geq \mathcal{P}(T,C) \geq \mathcal{P}(R,C) \\
\mathcal{N}([D,U],C) & = \mathcal{N}(U,C) \\
& \geq \mathcal{N}(T,C) \geq \mathcal{N}(R,C).
\end{align*}
\]

- \( P = [(p,q,D),S_1,S_2] \), \( E_1 \) is the root of \( S_1 \) and \( E_2 \) is the root of \( S_2 \), then, by Proposition 6 there are probabilistic computations \( T_1 \) and \( T_2 \) with root \( E_1 \) and \( E_2 \) such that \( W(T_1),W(T_2) < W(R) \) and \( p\mathcal{P}(T_1,C) + q\mathcal{P}(T_2,C) \geq \mathcal{P}(R,C) \) for every \( C \in \text{NF} \). By the inductive hypothesis applied to \( S_1 \) and \( T_1 \) (to \( S_2 \) and \( T_2 \), respectively), there is a sub-probabilistic computation \( U_1 \) of \( S_1 \) (a sub-probabilistic computation \( U_2 \) of \( S_2 \), respectively) such that \( \mathcal{P}(U_1,C) \geq \mathcal{P}(T_1,C) \) for every \( C \in \text{NF} \) (\( \mathcal{P}(U_2,C) \geq \mathcal{P}(T_2,C) \) for every \( C \in \text{NF} \), respectively). Now, consider the probabilistic computation \([p,q,D,U_1,U_2]\). This is clearly a sub-probabilistic computation of \( P \). Moreover, for every \( C \in \text{NF} \):

\[
\begin{align*}
\mathcal{P}([p,q,D,U_1,U_2],C) & = p\mathcal{P}(U_1,C) + q\mathcal{P}(U_2,C) \\
& \geq p\mathcal{P}(T_1,C) + q\mathcal{P}(T_2,C) \geq \mathcal{P}(R,C).
\end{align*}
\]

This concludes the proof.

The main theorem is the following:

**Theorem 1 (Strong Confluence)** For every maximal probabilistic computation \( P \), for every maximal probabilistic computation \( R \) such that \( P \) and \( R \) have the same root, and for every \( C \in \text{NF} \), \( \mathcal{P}(P,C) = \mathcal{P}(R,C) \) and \( \mathcal{N}(P,C) = \mathcal{N}(R,C) \). Moreover, \( \mathcal{P}(P) = \mathcal{P}(R) \) and \( \mathcal{N}(P) = \mathcal{N}(R) \).

**Proof.** Let \( C \in \text{NF} \) be any configuration in normal form. Clearly:

\[
\begin{align*}
\mathcal{P}(P,C) &= \sup_{Q \subseteq P} \{\mathcal{P}(Q,C)\} \\
\mathcal{P}(R,C) &= \sup_{S \subseteq R} \{\mathcal{P}(S,C)\}
\end{align*}
\]

Now, consider the two sets \( A = \{\mathcal{P}(Q,C)\}_{Q \subseteq P} \) and \( B = \{\mathcal{P}(S,C)\}_{S \subseteq R} \). We claim the two sets have the same upper bounds. Indeed, if \( x \in \mathbb{R} \) is an upper bound on \( A \) and \( S \subseteq R \), by Proposition 5 there is \( Q \subseteq P \) such that \( \mathcal{P}(Q,C) \geq \mathcal{P}(S,C) \), and so \( x \geq \mathcal{P}(S,C) \). As a consequence, \( x \) is an upper bound on \( B \). Symmetrically, if \( x \) is an upper bound on \( B \), it is an upper bound on \( A \). Since \( A \) and \( B \) have the same upper bounds, they have the same least upper bound, and \( \mathcal{P}(P,C) = \mathcal{P}(R,C) \). The other claims can be proved exactly in the same way. This concludes the proof.

\[\square\]

## 7 Computing with Mixed States

**Definition 7 (Mixed State)** A mixed state is a function \( \mathcal{M} : \text{Conf} \rightarrow \mathbb{R}_{[0,1]} \) such that there is a finite set \( S \subseteq \text{Conf} \) with \( \mathcal{M}(C) = 0 \) except when \( C \in S \) and, moreover, \( \sum_{C \in S} \mathcal{M}(C) = 1 \). Mix is the set of mixed states.

In this paper, a mixed state \( \mathcal{M} \) will be denoted with the linear notation \( \{p_1 : C_1, \ldots, p_k : C_k\} \) or as \( \{p_i : C_i\}_{1 \leq i \leq k} \), where \( p_i \) is the probability \( \mathcal{M}(C_i) \) associated to the configuration \( C_i \).

**Definition 8 (Reduction)** The reduction relation \( \equiv \) between mixed states is defined in the following way: \( \{p_1 : C_1, \ldots, p_m : C_m\} \equiv \mathcal{M} \) iff there exist \( m \) mixed states \( \mathcal{M}_1 = \{q_1^1 : D_1^1, \ldots, q_m^1 : D_m^1\} \) such that:

1. For every \( i \in [1,m] \), it holds that \( 1 \leq n_i \leq 2 \);
2. If \( n_i = 1 \), then either \( C_i \) is in normal form and \( C_i = D_1^i \) or \( C_i \equiv_{\mathcal{M}} D_1^i \);
3. If \( n_i = 2 \), then \( C_i \equiv_{\text{meas}}\, D_1^i, C_i \equiv_{\text{meas}}\, D_1^i, p \in \mathbb{R}_{[0,1]} \), and \( q_i^1 = p, q_i^2 = q \);
4. \( \forall D \in \text{Conf}, \mathcal{M}(D) = \sum_{i=1}^{n} p_i \cdot \mathcal{M}_i(D) \).

Given the reduction relation \( \implies \), the corresponding notion of computation (that we call mixed computation, in order to emphasize that mixed states play the role of configurations) is completely standard.

Given a mixed state \( \mathcal{M} \) and a configuration \( C \in \text{NF} \), the probability of observing \( C \) in \( \mathcal{M} \) is defined as \( \mathcal{M}(C) \) and is denoted as \( \mathcal{P}(\mathcal{M}, C) \). Observe that if \( \mathcal{M} \implies \mathcal{M}' \) and \( C \in \text{NF} \), then \( \mathcal{P}(\mathcal{M}, C) \leq \mathcal{P}(\mathcal{M}', C) \). If \( \{\mathcal{M}_i\}_{i<\omega} \) is a mixed computation, then

\[
\sup_{i<\omega} \mathcal{P}(\mathcal{M}_i, C)
\]

always exists, and is denoted as \( \mathcal{P}(\{\mathcal{M}_i\}_{i<\omega}, C) \).

Please notice that a maximal mixed computation is always infinite. Indeed, if \( \mathcal{M} = \{p_i : C_i\}_{1 \leq i \leq n} \) and for every \( i \in [1, n], C_i \in \text{NF} \), then \( \mathcal{M} \implies \mathcal{M} \).

**Proposition 6** Let \( \{\mathcal{M}_i\}_{i<\omega} \) be a maximal mixed computation and let \( C_1, \ldots, C_n \) be the configurations on which \( \mathcal{M}_0 \) evaluates to a positive real. Then there are maximal probabilistic computations \( P_1, \ldots, P_n \) with roots \( C_1, \ldots, C_n \) such that \( \sup_{i<\omega} \mathcal{M}_i(D) = \sum_{i=1}^{n} (\mathcal{M}_0(C_i) \mathcal{P}(P_i, D)) \) for every \( D \).

**Proof.** Let \( \{\mathcal{M}_i\}_{i<\omega} \) be a maximal mixed computation. Observe that \( \mathcal{M}_0 \implies^m \mathcal{M}_m \) for every \( m \in \mathbb{N} \). For every \( m \in \mathbb{N} \) let \( \mathcal{M}_m \) be

\[
\{ p_i^m : C_1^m, \ldots, p_n^m : C_n^m \}
\]

For every \( m \), we can build maximal probabilistic computations \( P_1^m, \ldots, P_n^m \), generatively: assuming \( P_1^{m+1}, \ldots, P_n^{m+1} \) are the probabilistic computations corresponding to \( \{\mathcal{M}_i\}_{m+1 \leq i < \omega} \), they can be extended (and possibly merged) into some maximal probabilistic computations \( P_1^m, \ldots, P_n^m \) corresponding to \( \{\mathcal{M}_i\}_{m \leq i < \omega} \). But we can even define for every \( m, k \in \mathbb{N} \) with \( m \leq k \), some finite probabilistic computations \( Q_1^{m,k}, \ldots, Q_n^{m,k} \) with root \( C_1, \ldots, C_n \) and such that, for every \( m, k, \)

\[
Q_i^{m,k} \subseteq P_i^m
\]

\[
\mathcal{M}_k(D) = \sum_{i=1}^{n} (\mathcal{M}_0(C_i) \mathcal{P}(Q_i^{m,k}, D)).
\]

This proceeds by induction on \( k - m \). We can easily prove that for every \( S \subseteq P_m^m \) there is \( k \) such that \( S \subseteq Q_i^{m,k} \): this is an induction on \( S \) (which is a finite probabilistic computation). But now, for every \( D \in \text{NF}, \)

\[
\sup_{j<\omega} \mathcal{M}_j(D) = \sup_{j<\omega} \sum_{i=1}^{n} (\mathcal{M}_0(C_i) \mathcal{P}(Q_i^{0,j}, D))
\]

\[
= \sum_{i=1}^{n} (\mathcal{M}_0(C_i) \sup_{j<\omega} \mathcal{P}(Q_i^{0,j}, D))
\]

\[
= \sum_{i=1}^{n} (\mathcal{M}_0(C_i) \mathcal{P}(P_i^0, D))
\]

This concludes the proof. \( \square \)

**Theorem 2** For any two maximal mixed computations \( \{\mathcal{M}_i\}_{i<\omega} \) and \( \{\mathcal{M}'_i\}_{i<\omega} \) such that \( \mathcal{M}_0 = \mathcal{M}'_0 \), the following condition holds: for every \( C \in \text{NF}, \mathcal{P}(\{\mathcal{M}_i\}_{i<\omega}, C) = \mathcal{P}(\{\mathcal{M}'_i\}_{i<\omega}, C) \).

**Proof.** A trivial consequence of Proposition 6. \( \square \)
8 Conclusions

The quantum lambda calculus $Q^*$ is proved to enjoy confluence in a very strong form, both for finite and for infinite computations. The proof seems to be quite independent on the particular rewriting system under consideration. Actually, the authors believe that any rewriting system enjoying properties like Proposition 4 enjoys confluence in the same sense as the one used here. Indeed, this constitutes an interesting topic for further work, which anyway lies outside the scope of this paper.

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