Process convergence of self normalized sums of i.i.d. random variables coming from domain of attraction of stable distributions

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Abstract

In this paper we show that the continuous version of the self normalised process
\[ Y_{n,p}(t) = \frac{S_n(t)}{V_{n,p} + (nt - [nt])X_{[nt]+1}} \]
where \( S_n(t) = \sum_{i=1}^{[nt]} X_i \) and \( V_{n,p} = (\sum_{i=1}^{n} |X_i|^p)^{\frac{1}{p}} \) and \( X_i \) i.i.d. random variables belong to \( DA(\alpha) \),

has a non trivial distribution iff \( p = \alpha = 2 \). The case for \( 2 > p > \alpha \) and \( p \leq \alpha < 2 \) is systematically eliminated by showing that either of tightness or finite dimensional convergence to a non-degenerate limiting distribution does not hold. This work is an extension of the work by Csörgö et al. who showed Donsker’s theorem for \( Y_{n,2}(\cdot) \), i.e., for \( p = 2 \), holds iff \( \alpha = 2 \) and identified the limiting process as standard Brownian motion in sup norm.

Keywords and Phrases: Domain of attraction, Process convergence, Self Normalised Sums, Stable distributions.

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1 Introduction

Limit theory plays a fundamental role in probability and statistics. Various forms of limit theorems, like the strong laws of large numbers, the central limit theorems, the law of iterated logarithm and the laws of large deviations are celebrated results in this field. However restrictive assumptions like the finiteness of moments upto a

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certain order or the existence of the moment generating function in a neighbourhood of zero are necessary conditions for proving these theorems. Also the choice of the normalising factor is the standard deviation, which is typically unknown in many statistical applications. What is done instead is to estimate the unknown parameters by a sequence of random variables (the sample standard deviation like the Student’s t statistic). The normalising factor is random in this case. To see whether the above mentioned limit laws hold with random normalisation is a fruitful area of research that has yielded many interesting results in the last two decades. For example, it has been shown in [12] that even under much less assumptions an analogy of the law of iterated logarithm holds under randomised normalisation. The same thing can be shown in case of laws of large and moderate deviations, see [15].

The study of the asymptotics of the self normalised sums are also interesting. Logan et al, [13] first showed the asymptotics of the self normalised sums where the variables belong to the domain of attraction of a stable distribution. In [10], it has been shown that limiting distribution of the self normalised sums converges to Normal if and only if the constituent random variables coming from the domain of attraction of a Normal stable distribution (henceforth denoted as $DAN$). Hence they conclude the same for $t$-statistics. Csörgö et al [4] show a functional (process) convergence result in sup norm for suitably scaled products of the self normalised sums (with $L_2$ normalisation as in [10]). They also show the result holds if and only if constituent random variables come from $DAN$. Basak et al [1] showed the convergence of a suitably scaled process to an Ornstein Uhlenbeck process. There also the constituent variables come from $DAN$. The aim of this paper is to show that the only case when the asymptotic distribution of the self normalised process is non trivial is when the norming index $p$ is exactly equal to the index of stability $\alpha$ (for definition see Section 2).

This paper is organised as follows. Section 2 contains definitions and a preliminary result that is used throughout. Section 3 contains the main result of this paper. Section 4 contains various application applied to functionals of the self normalised process as corollary to the main theorem. Section 5 and Section 6 shows convergence of finite dimensional distribution of the self normalised process and tightness result respectively for various choices of $p$ and $\alpha$. This two sections together show what should be the relation between $p$ and $\alpha$ for which the resulting asymptotic distribution is non trivial. Secton 7 concludes the paper with a few possible research
2 Definition and preliminaries

Let \( \{X_i\} \) be a sequence of i.i.d. random variables. We intend to study the convergence of the process determined at time \( t \) by

\[
Y_{n,p}(t) = \frac{S_n(t)}{V_{n,p}} + (nt - [nt])X_{[nt]+1}/V_{n,p} \quad 0 < t < 1 \quad p > 0 \quad (2.1)
\]

where the process \( S_n(.) \) and \( V_{n,p}(.) \) is defined as

\[
S_n(t) = \sum_{i=1}^{[nt]} X_i \quad \text{and} \quad V_{n,p} = \left(\sum_{i=1}^{n} |X_i|^p\right)^{1/p}
\]

where \( X_i \)'s belong to the domain of attraction of a \( \alpha \)-stable family denoted by \( DA(\alpha) \) and \([x]\) is the largest integer less than or equal to \( x \). We prove process convergence by showing finite dimensional convergence and tightness.

Here we prove a lemma first for the benefit of the reader, as we would use it often (Feller, Vol. 2, \[9\]) :

**Lemma 1** If \( X \in DA(\alpha) \) then \( Y = \text{sgn}(X)|X|^{\frac{\alpha}{2}} \in DAN \).

**Proof.** To prove the lemma we need the following characterisation:

\[
Y \in DAN \text{ iff } \lim_{y \to \infty} \frac{y^2 P(|Y| > y)}{E(Y^2 I(|Y| < y))} = 0,
\]

see \[4\]

We show that the random variable \( Y \) satisfies the necessary and sufficient condition. Now,

\[
\lim_{y \to \infty} y^2 P(|Y| > y) = \lim_{y \to \infty} y^2 P(|X|^{\frac{\alpha}{2}} > y) \\
= \lim_{y \to \infty} y^2 P(|X| > y^{\frac{2}{\alpha}}) \\
= \lim_{y \to \infty} y^2 (y^{\frac{2}{\alpha}})^{-\alpha} \\
since \text{the tail of } DA(\alpha) \text{ is Paretian, i.e, } P(|X| > x) = O(x^{-\alpha}) \\
= O(1).
\]
And,

\[
E(Y^2I(|Y| < y)) = E(|X|^\alpha I(|X| \leq \frac{y^\alpha}{\alpha}) \\
= E(|X|^\alpha I(|X| \leq \frac{y^\alpha}{\alpha}) \\
= \int_0^{\frac{y^\alpha}{\alpha}} z^\alpha dF_{|X|}(z) \\
= \int_0^{\frac{y^\alpha}{\alpha}} (\int_0^z \alpha t^{\alpha-1} dt) dF_{|X|}(z).
\]

Applying Fubini’s theorem and interchanging order of integration we get

\[
E(Y^2I(|Y| < y)) = \int_0^{\frac{y^\alpha}{\alpha}} \alpha \int_t^{\frac{y^\alpha}{\alpha}} dF_{|X|}(z) t^{\alpha-1} dt \\
= \alpha \int_0^{\frac{y^\alpha}{\alpha}} P(t < |X| \leq \frac{y^\alpha}{\alpha}) t^{\alpha-1} dt \\
= \alpha \int_0^M P(t < |X| \leq \frac{y^\alpha}{\alpha}) t^{\alpha-1} dt \\
+ \alpha \int_M^{\frac{y^\alpha}{\alpha}} P(t < |X| \leq \frac{y^\alpha}{\alpha}) t^{\alpha-1} dt.
\]

Now the first integral is nonnegative and less than \(M^\alpha\) and for the limit of the second integral as \(y \to \infty\), use Monotone Convergence Theorem, to get

\[
\alpha \int_M^{\frac{y^\alpha}{\alpha}} P(t < |X| < \frac{y^\alpha}{\alpha}) t^{\alpha-1} dt \\
= \lim_{y \to \infty} \alpha \int_M^\infty I_{\{M,y^\alpha\}}(t) P(t < |X| < \frac{y^\alpha}{\alpha}) t^{\alpha-1} dt \\
= \alpha \int_M^\infty P(|X| > t) t^{\alpha-1} dt \\
= \infty, \quad \text{as } P(|X| > t) = O(t^{-\alpha}).
\]

Hence, \(\lim_{y \to \infty} \frac{y^2P(|Y| > y)}{E(Y^2I(|Y| < y))} = 0 \Longleftrightarrow Y \sim \text{DAN.}\)

We also quote a theorem due to [4]

THEOREM 1 The following statements are equivalent:

1. \(EX = 0\) and \(X\) is in the domain of attraction of the normal law.
2. \( S_{[nt_0]} / V_{n,2} \to N(0, t_0) \) for \( t_0 \in (0, 1] \).

3. \( S_{[nt]} / V_n \to W(t) \) on \( D[0, 1], \rho \), where \( \rho \) is the sup-norm metric for functions in \( D[0, 1] \), and \( \{ W(t), 0 < t < 1 \} \) is a standard Wiener process.

4. On an appropriate probability space for \( X, X_1, X_2, \ldots \) we can construct a standard Wiener process \( \{ W(t), 0 < t < \infty \} \) such that

\[
\sup_{0 \leq t \leq 1} |S_{[nt]} / V_{n,2} - W(nt) / \sqrt{n}| = o_p(1). \tag{2.2}
\]

3 Main result

Let \( X_i \) be i.i.d. symmetric observations from the domain of attraction of a \( \alpha \)-Stable distribution and \( \{ Y_{n,p}(\cdot) \} \) as defined in (2.1). Then we have the following theorem:

**Theorem 2** \( Y_{n,p}(t) \) converges weakly to Brownian motion in \( C[0, 1] \), if and only if \( p = \alpha = 2 \).

**Proof:** In Section 5 we show that for \( 0 < p < \alpha \leq 2 \) and \( 0 < p = \alpha < 2 \) the finite dimensional distributions converge in probability to a degenerate distribution at zero. A non trivial limiting distribution exists if \( p > \alpha \) and \( p = \alpha = 2 \). In sections 6 we show that the sequence \( \{ S_n / V_{n,p} \} \) of self normalised sums is tight iff \( 0 < p \leq \alpha \leq 2 \). The only case where we have both tightness and finite dimensional convergence is \( p = \alpha = 2 \). The limiting distribution of the sequence for this choice of \( p \) and \( \alpha \) was identified by [10] as Normal. Applying Prohorov’s Theorem we have the distributional convergence to the Wiener process. The convergence in the sup norm metric follows directly from (2.2) of the above theorem by [4].

In Csörgő ([1]) they are interested in the process \( S_{[nt]} / V_{n,p} \) which is in \( D([0, 1]) \) and we are interested in the process \( Y_{n,p}(t) \) which is in \( C([0, 1]) \). However from the definition of \( Y_{n,p}(t) \),

\[
|Y_{n,p}(t) - S_{[nt]} / V_{n,p}| = |(nt - [nt])X_{[nt]+1} / V_{n,2} \leq |X_{[nt]+1} / V_{n,p}|.
\]

If \( p = \alpha = 2 \) then, by Darling ([7]), we have that \( \max_{1 \leq i \leq n} |X_i| / (\sum_{i=1}^n X_i^2)^{1/2} \overset{P}{\to} 0 \). So \( |Y_{n,p}(t) - S_{[nt]} / V_{n,p}| \overset{P}{\to} 0 \). Therefore \( Y_{n,p}(t) \) takes the same limiting distribution of \( S_{[nt]} / V_{n,p} \) which is Normal.
4 Application

Here we present a few applications of the main theorem. These follows from the original extension by Erdos and Kac \[8\] to the corresponding self normalised functionals. Let \( W(t) \) be the standard Brownian motion. Define the following quantities

\[ G_1(x) = P\left( \sup_{0 < t < 1} W(t) < x \right), \quad G_2(x) = P\left( \sup_{0 < t < 1} |W(t)| < x \right) \]
\[ G_3(x) = P\left( \int_0^1 W^2(t) < x \right), \quad G_4(x) = P\left( \int_0^1 |W(t)| < x \right) \]

**Corollary 1** The following weak convergence holds iff \( p = \alpha = 2, \forall x > 0 \).

1. \( P(\max_{1 \leq k \leq n} S_k/V_{n,p} < x) \to G_1(x) \)
2. \( P(\max_{1 \leq k \leq n}) |S_k|/V_{n,p} \to G_2(x) \)
3. \( P(\frac{1}{n} \sum_{1 \leq k \leq n} (S_k/V_{n,p})^2) \to G_3(x) \)
4. \( P(\frac{1}{n} \sum_{1 \leq k \leq n} |S_k/V_{n,p}|) \to G_4(x) \)

5 Convergence of Finite Dimensional Distributions

To get the process convergence we first need to examine the convergence of finite-dimensional distributions, i.e., for \( 0 < t_1 < t_2 < \ldots < t_k, k \geq 1 \) we want to examine the convergence of the random vector \((Y_{n,p}(t_1), Y_{n,p}(t_2), \ldots, Y_{n,p}(t_k))\) as \( n \to \infty \). We will do this for \( p < \alpha, p = \alpha \) and \( p > \alpha \) separately.

5.1 Case 1: \( p < \alpha \)

Since \( X_i \in DA(\alpha) \), by SLLN, \( V_{n,p}/n^{1/p} \) converges to a positive constant, say, \( k(\alpha, p) \). Now, for \( X_i \in DA(\alpha) \), \( S_n/(n^{1/\alpha}h(n)) \) converges in distribution to a \( S(\alpha) \) random variable, where \( h \) is a slowly varying function of \( n \). Since \( p < \alpha \), \( S_n/n^{1/p} = n^{(1/\alpha)-(1/p)} S_n/n^{1/\alpha} \to 0 \), in probability, as \( n \to \infty \). Thus, \( S_n/V_{n,p} = \frac{S_n/n^{1/p}}{V_{n,p}/n^{1/p}} \to 0 \), in probability, as \( n \to \infty \). Therefore, the joint distribution would converge to a degenerate one, in this case.

5.2 Case 2: \( p = \alpha \)

Here we assume that \( X_i \) is symmetric and belongs to \( DA(\alpha) \).
Lemma 2 For $V_{n,\alpha}$ defined as in Section 2, $V_{n,\alpha} \geq V_{n,1} \geq V_{n,\beta} \geq V_{n,2}$ if $\alpha \leq 1 \leq \beta \leq 2$.

Proof. We use the inequality for $a > 0, b > 0, \alpha \leq 1, \beta \geq 1$,
\[ a^{\alpha} + b^{\alpha} \geq (a + b)^{\alpha} \quad \text{and} \quad (a + b)^{\beta} \geq a^{\beta} + b^{\beta} \]
\[ \Rightarrow (a^{\alpha} + b^{\alpha})^{\frac{\beta}{\alpha}} \geq (a + b)^{\beta} \geq (a^{\beta} + b^{\beta})^{\frac{\alpha}{\beta}}. \]

Now, take $\alpha = 1$ and $\beta \geq 1$ and then $\alpha \leq 1$ and $\beta = 1$ to get,
\[ (a^{\alpha} + b^{\alpha})^{\frac{1}{\alpha}} \geq (a + b) \geq (a^{\beta} + b^{\beta})^{\frac{1}{\beta}}. \]

Also, for $1 \leq \beta \leq 2$, it follows that $1 \leq 2/\beta \leq 2$. Hence,
\[ (a^{\beta} + b^{\beta})^{2/\beta} \geq (a^{\beta(2/\beta)} + b^{\beta(2/\beta)}) = (a^2 + b^2) \Rightarrow (a^{\beta} + b^{\beta})^{\frac{1}{2}} \geq (a^2 + b^2)^{\frac{1}{2}}. \]

The case for $n$ positive numbers can be shown in the same manner. Thus, combining above, get
\[ V_{n,2} \leq V_{n,\beta} \leq V_{n,1} \leq V_{n,\alpha}. \]

Now, we show that self-normalized sum for $p = \alpha$ converges to degenerate distribution as well.

Theorem 3 If $p = \alpha \leq 1$, $\lim_{n \to \infty} \text{Var}(\frac{S_n}{V_{n,p}}) = 0$.

Proof. Note that,
\[ E\left(\frac{\sum X_i}{V_{n,\alpha}}\right)^2 = \sum E\left(\frac{X_i^2}{V_{n,\alpha}^2}\right) + \sum_{(i,j): i \neq j} E\left(\frac{X_i X_j}{V_{n,\alpha}^2}\right) \]
\[ = \sum_i E\left(\frac{X_i^2}{V_{n,\alpha}^2}\right) + \sum_i E\left(\sum_{j \neq i} X_i E\left(\frac{X_j}{V_{n,\alpha}^2} \mid X_i, i \neq j\right)\right) \]
\[ = \sum_i E\left(\frac{X_i^2}{V_{n,\alpha}^2}\right), \quad (5.1) \]

the second term vanishes since
\[ \frac{X_j}{V_{n,\alpha}^2} = - \frac{X_j}{V_{n,\alpha}^2} \text{ in distribution.} \]
We now use the fact that $V_{n,\alpha} \geq V_{n,2}$ implies that $(\sum_{i=1}^{n} X_i^2/V_{n,\alpha})^2 \leq (\sum_{i=1}^{n} X_i^2/V_{n,2})^2 = 1$. Hence if we could show that $(\sum_{i=1}^{n} X_i^2/V_{n,\alpha})^2 \to 0$ in probability, then by Dominated convergence theorem (DCT) we have the result. Observe that, for $\alpha \leq 1$, if $X_i \sim DA(\alpha)$ then $Y_i = \text{sgn}(X_i)|X_i|^\frac{2}{\alpha} \sim DAN$. From [10] we have that $E(Y_i^2) = o(1) \to 0$ as $n \to \infty$. Hence, $(\sum_{i=1}^{n} X_i^2/V_{n,\alpha})^2 \to 0$ in probability, as it goes to zero in $\alpha$-th mean. Therefore, by DCT we conclude the proof.

We now proceed to prove the result for $p = \alpha > 1$.

**Lemma 3** If $X \sim DA(\alpha)$, then $\frac{\sum X_i^2}{V_{n,\alpha}} \to 0$.

**Proof.** Observe that, if $X_i \sim DAN$ then by [13] $\max_{1 \leq i \leq n} \frac{|X_i|}{V_{n,2}} \to 0$. Because, if $X_i \sim DA(\alpha)$ then $Y_i = \text{sgn}(X_i)|X_i|^\frac{2}{\alpha} \sim DAN$ by the lemma [11] Therefore

$$
\max_{1 \leq i \leq n} \frac{|X_i|^\frac{2}{\alpha}}{(\sum |X_i|^\alpha)^{\frac{2}{\alpha}}} \to 0 \quad \Leftrightarrow \quad \max_{1 \leq i \leq n} \frac{|X_i|^\frac{2}{\alpha}}{V_{n,\alpha}^2} \to 0 \quad (5.2)
$$

Again, from [9] [7], since $X_i \sim DA(\alpha)$, one gets $|X_i|^2 \sim DA(\alpha/2)$. Define $Y^*_n = \max_{1 \leq i \leq n} X_i^2$. Hence, for $\epsilon, \eta > 0$ choose $\delta = \frac{\epsilon}{K_\eta}$ where $K_\eta$ is chosen so that $P(\sum X_i^2/Y^*_n > K_\eta) < \eta/2$. (This is possible since by [7] $Y^*_n/\sum X_i^2$ has a limiting distribution and hence tight.)

$$
P\left(\frac{\sum X_i^2}{V_{n,\alpha}^2} > \epsilon\right) \leq P\left(\frac{\sum X_i^2}{V_{n,\alpha}^2} > \epsilon, \frac{Y^*_n}{V_{n,\alpha}^2} > \delta\right) + P\left(\frac{\sum X_i^2}{V_{n,\alpha}^2} > \epsilon, \frac{Y^*_n}{V_{n,\alpha}^2} \leq \delta\right)$$

$$
\leq P\left(\frac{Y^*_n}{V_{n,\alpha}^2} > \delta\right) + P\left(\frac{\sum X_i^2}{X^*_n} \frac{Y^*_n}{V_{n,\alpha}^2} > \epsilon, \frac{Y^*_n}{V_{n,\alpha}^2} \leq \delta\right)
$$

$$
\leq P\left(\frac{Y^*_n}{V_{n,\alpha}^2} > \delta\right) + P\left(\frac{\sum X_i^2}{Y^*_n} > \frac{\epsilon}{\delta}\right).
$$

Choose $n_0$ sufficiently large so that the first probability is less than $\eta/2$. By the choice of $\delta$ we have the second probability less than $\eta/2$. Which implies that,
\[ P(\sum X_i^2/V_{n,\alpha}^2 > \epsilon) < \eta \text{ for } n \geq n_0. \]

Hence the lemma is proved.

**Theorem 4** Let \( 1 < p = \alpha < 2 \), and \( X_i \)s are symmetric and \( X_i \sim DA(\alpha) \). Then
\[ \lim_{n \to \infty} \text{Var}(S_{n,p}/V_{n,p}) = 0. \]

Note that \( \text{Var}(S_{n,p}) = E(S_{n,p}^2) \) by symmetry of \( X_i \) and also \( E(S_{n,p}^2) = E(\sum X_i^2) \) by (5.1) in the proof of Theorem 3.

\[ V_{n,\alpha} \geq V_{n,2} \text{ for } 0 < \alpha \leq 2 \]
\[ \Rightarrow \frac{\sum X_i^2}{(\sum |X_i|^\alpha)^{2/\alpha}} \leq \frac{\sum X_i^2}{\sum X_i^2} = 1. \]

Hence, by lemma 3 and applying bounded convergence theorem,
\[ \lim_{n \to \infty} E\left( \frac{\sum X_i^2}{(\sum |X_i|^\alpha)^{2/\alpha}} \right) = 0. \]

This proves the Theorem.

**Remark 1** For \( X_i \sim DA(\alpha) \) symmetric, we, in fact, showed in theorems 3 and 4 that \( (S_n/V_{n,p}) \to 0 \) in probability, for \( 0 < p = \alpha < 2 \). Using same technique, it is immediate that for any fixed \( 0 \leq t \leq 1 \), \( (S_{[nt]}/V_{n,p}) \to 0 \), in probability, for \( 0 < p = \alpha < 2 \) as well. The result for \( k \) dimension can be obtained from the above result. Note that the joint distribution of \( (S_{[nt_1]}/V_{n,p}, S_{[nt_2]}/V_{n,p}, \ldots, S_{[nt_k]}/V_{n,p}) \) can be obtained from the joint distribution of \( (S_{[nt_1]}-S_{[nt_1]}/V_{n,p}, S_{[nt_2]}-S_{[nt_1]}/V_{n,p}, \ldots, S_{[nt_k]}-S_{[nt_{k-1}]}/V_{n,p}) \) by a linear transformation. We next show that the joint distribution of the latter converges to zero. Write \( S_1 = S_{[nt_1]}/V_{n,p}, S_2 = S_{[nt_2]}-S_{[nt_1]}/V_{n,p} \text{ and } S_k = S_{[nt_k]}-S_{[nt_{k-1}]}/V_{n,p} \). Now consider the variance of any linear combination of them \( V(a_1S_1 + a_2S_2 + \ldots + a_kS_k) \) where \( a_i \)'s are any arbitrary constants. Due to independence the cross product term vanishes and by Theorem 3 the variances are zero which implies that any linear combination tends in probability to zero. Therefore \( \phi_{S_1,S_2,\ldots,S_k}(a_1,a_2,\ldots,a_k) \to 1 \), where \( \phi_{S_1,S_2,\ldots,S_k} \) is the characteristic function. Applying continuity theorem we therefore have that the limiting joint distribution of \( (S_1, S_2, \ldots, S_k) \) and hence \( (S_{[nt_1]}/V_{n,p}, S_{[nt_2]}/V_{n,p}, \ldots, S_{[nt_k]}/V_{n,p}) \) is degenerate at 0.
5.3 Case 3: $p > \alpha$

We show that a limiting distribution exits in this case by finding the joint characteristic function of \( \left( \frac{S_{\alpha_1}}{\sqrt{n}}, \frac{S_{\alpha_2}}{\sqrt{n}}, \ldots, \frac{S_{\alpha_k}}{\sqrt{n}} \right) \) where \( 1 \leq m_1 \leq m_2 \leq \ldots m_k \leq n \). This is equivalent to finding the characteristic function of \( (S_{m_1}/n^{\frac{1}{\alpha}}, (S_{m_2} - S_{m_1})/n^{\frac{1}{\alpha}}, \ldots, (S_{m_k} - S_{m_{k-1}})/n^{\frac{1}{\alpha}}, V_{n,p}^p/n^{\frac{1}{\alpha}}) \) by virtue of a transformation. The characteristic function of the latter is

\[
E(\exp(i \frac{u_1}{n^{\frac{1}{\alpha}}} S_{m_1}) + i \frac{u_2}{n^{\frac{1}{\alpha}}} (S_{m_2} - S_{m_1}) + \ldots + i \frac{u_k}{n^{\frac{1}{\alpha}}}(S_{m_k} - S_{m_{k-1}}) + i \frac{l}{n^{\frac{1}{\alpha}}} V_{n,p})
\]

\[
= E(\exp(i \frac{u_1}{n^{\frac{1}{\alpha}}} S_{m_1}) + i \frac{l}{n^{\frac{1}{\alpha}}} V_{m_1})
\]

\[
+ \left( \sum_{i=2}^{k} \left( E(\exp(i \frac{u_i}{n^{\frac{1}{\alpha}}} (S_{m_i} - S_{m_{i-1}})) + i \frac{l}{n^{\frac{1}{\alpha}}} (V_{m_i,p} - V_{m_{i-1},p}) \right) \right)
\]

Due to independence and identical distribution of \( X' \)'s we have

\[
E(\exp(i \frac{u_1}{n^{\frac{1}{\alpha}}} S_{m_1} + i \frac{l}{n^{\frac{1}{\alpha}}} V_{m_1,p})) = E(\exp(iu_1 \frac{X}{n^{\frac{1}{\alpha}}} + il \frac{|X|^p}{n^{\frac{1}{\alpha}}}))^{m_1}
\]

and

\[
E(\exp(i \frac{u_k}{n^{\frac{1}{\alpha}}} (S_{m_k} - S_{m_{k-1}}) + i \frac{l}{n^{\frac{1}{\alpha}}} (V_{m_k,p} - V_{m_{k-1},p}))) = E(\exp(iu_k \frac{X}{n^{\frac{1}{\alpha}}} + il \frac{|X|^p}{n^{\frac{1}{\alpha}}}))^{m_k - m_{k-1}}
\]

Now,

\[
\left\{ E(\exp\left(\frac{iu}{m_1^{\frac{1}{\alpha}}} \right)^\frac{1}{\alpha} + iwx^{\frac{1}{\alpha}} (|x|^{\frac{1}{\alpha}})^p(m_1^{\frac{1}{\alpha}})^{\frac{1}{\alpha}}) \right\}^{m_1}
\]

\[
= (\int \exp(iu \frac{x}{m_1^{\frac{1}{\alpha}}} \left( \frac{1}{\alpha} + iwx^{\frac{1}{\alpha}} (|x|^{\frac{1}{\alpha}})^p(m_1^{\frac{1}{\alpha}})^{\frac{1}{\alpha}} \right) g(x)dx)^{m_1}
\]

\[
= (1 + \int (\exp(iu \frac{x}{m_1^{\frac{1}{\alpha}}} \left( \frac{1}{\alpha} + iwx^{\frac{1}{\alpha}} (|x|^{\frac{1}{\alpha}})^p(m_1^{\frac{1}{\alpha}})^{\frac{1}{\alpha}} - 1) \right) g(x)dx)^{m_1}
\]

\[
= (1 + \frac{1}{m_1} \int (\exp(iuy(m_1^{\frac{1}{\alpha}})^{\frac{1}{\alpha}} + iwy^{\frac{1}{\alpha}} (m_1^{\frac{1}{\alpha}})^{\frac{1}{\alpha}} - 1)) g(m_1^{\frac{1}{\alpha}} y)
\]
Since \( (\exp(iu\frac{X}{m_1}) - 1) \) is bounded by 2 and \( m_1^\frac{1}{\alpha} g(m_1^\frac{1}{\alpha} y) \) is integrable we can apply DCT to get

\[
\lim_{m_1 \to \infty} \mathbb{E}(\exp(iu\frac{X}{m_1} + iw|\frac{X}{m_1}|^p) - 1) = \lim_{m_1 \to \infty} (1 + \frac{c_{m_1,n}(u,w)}{m_1})(u,w)dy.
\]

where

\[
c_{m_1,n}(u,w) = \int (\exp(iuy\frac{m_1}{n} + iw|y|^p) - 1)g(m_1^\frac{1}{\alpha} y)m_1^\frac{1}{\alpha} + dy.
\]

and

\[
\lim_{m_1,n} c_{m_1,n}(u,w) = \int (\exp(iyt_1\frac{1}{\alpha} + iw|t_2|^p) - 1)\frac{K(y)}{y^{\alpha+1}}dy,
\]

where \( K(y) = \lim_{m \to \infty} (m^{\frac{1}{\alpha}} g(m^{\frac{1}{\alpha}} y)) \) which is

\[
K(y) = \begin{cases} 
  r & \text{if } y > 0 \\
  s & \text{if } y < 0 
\end{cases}
\]

The same thing can be done for \( \mathbb{E}(\exp(iu\frac{X}{n^\frac{1}{\alpha}} + il\frac{|X|^p}{n^\frac{1}{\alpha}}))^{m_k-m_{k-1}} \) and let us call it \( c_{m_{k-1},m_k,n}(u_k,l) \). The limiting characteristic function of \( (S_{m_1}/n^\frac{1}{\alpha}, S_{m_2}/n^\frac{1}{\alpha}, \ldots, S_{m_k}/n^\frac{1}{\alpha}, V_{n,p}/n^\frac{1}{\alpha}) \) is therefore (the limits are meant in such a way that \( n^\frac{1}{\alpha} \) tends to a constant as \( m_i, n \to \infty \)).

\[
\lim_{m_1,m_2,\ldots,m_k,n \to \infty} \mathbb{E}(\exp(it_1\frac{1}{n^\frac{1}{\alpha}} S_{m_1} + t_2\frac{1}{n^\frac{1}{\alpha}} (S_{m_2} - S_{m_1}) + \ldots + \frac{t_k}{n^\frac{1}{\alpha}} (S_{m_k} - S_{m_{k-1}}) + \frac{l}{n^\frac{1}{\alpha}} V_{n,p}))
\]

\[
= \lim_{m_1,n \to \infty} c_{m_1,n} \lim_{m_2,m_1,n \to \infty} c_{m_2,m_1,n} \ldots \times \lim_{m_{k-1},m_k,n \to \infty} c_{m_{k-1},m_k} \times \lim_{m_k,n \to \infty} \mathbb{E}(\exp(i\frac{|X|^p}{\alpha^\frac{1}{\alpha}}))^{n-m_k},
\]

since \( X \sim DA(\alpha) \) the last limit exists. Hence the finite-dimensional distribution converges to some (non-trivial) non-degenerate distribution for \( p > \alpha \).
Remark 2 For \( p = \alpha = 2 \) the finite dimensional distribution of can be obtained by using the fact (see [10]) that \( S \) is uncorrelated and hence using Slutsky’s Theorem and the fact that \( S \rightarrow N(0, 1) \) and \( \frac{1}{n(t)} V_{n,2}^2 \rightarrow 1 \) for a slowly varying function \( l(\cdot) \). Applying the same argument as above we see that the distribution of \((Y_{n,p}(t_1), Y_{n,p}(t_2), \ldots, Y_{n,p}(t_k))\) can be obtained from the distribution of \(\left(\frac{S_{[nt_1]}}{V_{n,p}}, \frac{S_{[nt_2]}-S_{[nt_1]}}{V_{n,p}}, \ldots, \frac{S_{[nt_k]}-S_{[nt_{k-1}]}+S_{[nt_1]}}{V_{n,p}}\right)\) by a linear transformation. Now the components in the latter are uncorrelated and hence

\[
\frac{S_{[nt_1]}}{V_{n,p}} = \frac{\sqrt{|nt_1|} l(|nt_1|)}{\sqrt{n l(n)}} \sqrt{\frac{1}{n(t)} V_{n,p}} \Rightarrow \sqrt{t_1} N(0, 1),
\]

(by using Slutsky’s Theorem and the fact that \( l(\cdot) \) is a slowly varying function). Similar thing can be done for \( \frac{S_{[nt_2]}-S_{[nt_1]}}{V_{n,p}} \) and the limiting distribution in that case will be \( \sqrt{t_2 - t_1} N(0, 1) \). If \( t_i < t_j \) then \( \text{Cov}(\frac{S_{[nt_1]}-S_{[nt_1]}}{V_{n,p}}, \frac{S_{[nt_2]}-S_{[nt_1]}}{V_{n,p}}) = \text{Cov}(\frac{S_{[nt_1]}-S_{[nt_1]}}{V_{n,p}}, \frac{S_{[nt_1]}-S_{[nt_1]}+S_{[nt_1]}}{V_{n,p}}) = V(\frac{S_{[nt_1]}}{V_{n,p}}) = t_i = \min(t_i, t_j) \). Since the Jacobian of the transformation is one the finite dimensional distribution of \((Y_{n,p}(t_1), Y_{n,p}(t_2), \ldots, Y_{n,p}(t_k))\) is a multivariate normal distribution with dispersion matrix \((v_{i,j})\) given by:

\[
v_{i,j} = \begin{cases} t_i & \text{if } i = j \\ \min(t_i, t_j) & \text{otherwise} \end{cases}
\]

In fact, the above finite dimensional convergence follows form Theorem 1 since the self normalised sums is converging in probability to the Wiener motion properly scaled in the sup norm metric.

Remark 3 Note that we have shown finite dimensional convergence results for \((S_{[nt_1]}/V_{n,p}, S_{[nt_2]}/V_{n,p}, \ldots, S_{[nt_k]}/V_{n,p})\). However this is equivalent to show finite dimensional convergence for the process \( Y_{n,p}(\cdot) \). To see this, note that

\[
E(|Y_{n,p}(t_1) - S_{[nt_1]}/V_{n,p}|^2) = E((nt_1 - |nt_1|^2|X_{[nt_1]}|^2/V_{n,p}^2)) \\
\leq E(|X_{[nt_1]}^2/V_{n,p}^2|) \\
\leq E(|X_{[nt_1]}^2/V_{n,p}^2|) \quad \forall p \leq 2 \\
= \frac{1}{n} \quad \text{since } |nt_1| < n
\]

Therefore \( Y_{n,p}(t_1) - S_{[nt_1]}/V_{n,p} \rightarrow 0 \) which implies that these two are asymptotically negligible and all limiting properties of \( S_{[nt]}/V_{n,p} \) will be shared by \( Y_{n,p}(t) \). Although we have shown the result for one dimension the result can be extended in a natural way to \( k \) dimensions, ie, we can show that \( (Y_{n,p}(t_1) - S_{[nt_1]}/V_{n,p}, Y_{n,p}(t_2) - S_{[nt_2]}/V_{n,p}, \ldots, Y_{n,p}(t_k) - S_{[nt_k]}/V_{n,p}) \rightarrow 0 \).
6 Tightness

Theorem 5 The process \( \{ Y_{n,p}(\cdot) \} \) is tight iff \( p \leq \alpha \leq 2 \).

We first prove the if part and then the only if part.

If part: The process \( Y_{n,p}(\cdot) \) is tight if \( p \leq \alpha \leq 2 \).

Proof: We first take the case that \( p \leq \alpha < 2 \). From Theorem 7.3 from [3] the process \( Y_{n,p}(\cdot) \) is tight iff \( Y_{n,p}(0) \) is tight and for all \( \epsilon > 0, \eta > 0, \exists \delta, (0 < \delta < 1) \) such that \( \lim_{n \to \infty} P(\omega Y_{n,p}(\delta) \geq \epsilon) = 0 \) where \( \omega_X(\delta) = \sup_{t-s<\delta} |X(t) - X(s)| \) is the modulus of continuity for any process \( X(\cdot) \). Also from Equation 7.11 of [3] \( P(\omega X(\delta) \geq 3\epsilon) \leq \sum_{i=1}^{v} P(\sup_{t_{i-1}<s<t_i} |X(s) - X(t_{i-1})| \geq \epsilon) \) for any arbitrary probability \( P \), \( \epsilon > 0 \) \( \delta > 0 \) process \( X(\cdot) \), and for a partition \( 0 = t_0 < t_1 < t_2 < \ldots < t_v = 1 \) such that \( \min_{1<i<v} (t_i - t_{i-1}) \geq \delta \).

Take partition \( t_i = m_i/n \) where \( 0 = m_0 < m_1 < \ldots < m_v = n \). By the definition of the process in (2.1) we have that \( \max_{m_i-1<k<m_i} |S_k - S_{m_i-1}| \). Therefore,

\[
P(\omega(Y_{n,p}, \delta) \geq 3\epsilon) \leq \sum_{i=1}^{v} P\left[ \max_{m_i-1<k<m_i} |S_k - S_{m_i-1}| \geq \epsilon V_{n,p} \right].
\]

The sequence \( \{ S_n \} \) is stationary and hence the above is same as

\[
\sum_{i=1}^{v} P\left[ \max_{k<m_i-m_{i-1}} |S_k| > \epsilon V_{n,p} \right].
\]

Choose \( m_i = m_i \) where \( m \) is an integer satisfying \( m = [n\delta] \) and \( v = [n/m] \). With this choice \( v \to 1/\delta < 2/\delta \). Therefore for sufficiently large \( n \),

\[
P(\omega(Y_{n,p}, \delta) \geq 3\epsilon) \leq v P(\max_{k \leq m} |S_k| / V_{n,p} > \epsilon)
\]

\[
\leq 2/\delta P(\max_{k \leq m} |S_k| / V_{n,p} > \epsilon).
\]

Note that \( S_k / V_{k,p} \) is a martingale (increments of independent mean zero random variables) and hence \( |S_k| / V_{k,p} \) is a non negative sub martingale. The ratio \( V_{m,p} / V_{n,p} \) has a probability limit to \( (m/n)^{1/\delta} \to \delta^{1/\delta} \). Therefore

\[
\frac{1}{\delta} P(\max_{k \leq m} |S_k| / V_{n,p} > \epsilon) = \frac{1}{\delta} P(\max_{k \leq m} \frac{|S_k|}{V_{k,p}} V_{n,p} > \epsilon)
\]

\[
\leq \frac{1}{\delta} P(\max_{k \leq m} \frac{|S_k|}{V_{k,p}} V_{n,p} > \epsilon).
\]
Writing $X_m = \max_{k \leq m} \frac{|S_k|}{V_{k,p}}$ and $Y_m = \frac{V_m}{V_{n,p}}$ we have

\[
\frac{1}{\delta} P\left( \max_{k < m} \frac{|S_k|}{V_{n,p}} > \epsilon \right) \leq \frac{1}{\delta} P(X_m Y_m > \epsilon) \\
= \frac{1}{\delta} \{ P(X_m Y_m > \epsilon, Y_m > 2\delta^\frac{1}{p}) + P(X_m Y_m > \epsilon, Y_m < 2\delta^\frac{1}{p}) \}
\leq \frac{1}{\delta} \{ P(X_m Y_m > \epsilon, < 2\delta^\frac{1}{p}) + P(Y_m > 2\delta^\frac{1}{p}) \}
\leq \frac{1}{\delta} \{ P(X_m > \epsilon/2\delta^\frac{1}{p}) + P(Y_m > 2\delta^\frac{1}{p}) \}
\leq \frac{1}{\delta} \{ P(X_m > \epsilon/2\delta^\frac{1}{p}) + \eta \} \text{ choosing sufficiently large } m \text{ such that } P(Y_m > 2\delta^\frac{1}{p}) < \eta
\leq \frac{1}{\delta} \{ 4\delta^\frac{1}{p}/\epsilon^2 V(S_m/V_{m,p}) + \eta \} \text{ by Doob’s inequality for nonnegative submartingales}
= (4\delta^\frac{1}{p}/\epsilon^2) V(S_m/V_{m,p}) + \eta/\delta , \text{ for some } \gamma > 0. \tag{6.1}
\]

Now, for $p \leq \alpha < 2$, or, $p < \alpha = 2$, $\text{Var}(S_m/V_{m,p})$ tends to zero (see section 5.1, 5.2). Tending $m \to \infty$, (since $m = \lceil n\delta \rceil$) we have that the right hand side in (6.1) can be made arbitrarily small. Hence the lemma is proven.

For the case $p = \alpha = 2$, the lemma holds by [10] since it has been shown that the self normalised sums converges to the Normal distribution for $p = \alpha = 2$.

Before proving the only if part we need the following lemma.

**Lemma 4** \{Y_{n,p}(\cdot)\} is tight $\Rightarrow \max_{1 \leq i \leq n} \frac{|X_i|}{V_{n,p}} \to 0$.

**Proof.**

We use an equivalent condition of tightness given in Theorem 4.2 of [3]. A process is tight iff $\forall \epsilon > 0, \forall \eta > 0, \exists n_0$ and $0 < \delta < 1$ such that

\[
P\left( \sup_{|t-s|<\delta} |Y_{n,p}(s) - Y_{n,p}(t)| \geq \epsilon \right) \leq \eta \quad \forall t \in [0, 1]. \tag{6.2}
\]

Assume that the hypothesis is true. Which means that for every $\epsilon, \eta > 0, \exists 0 < \delta < 1$ such that (6.2) holds. Choose $n_0$ sufficiently large so that $\frac{1}{\delta} < \delta \quad \forall n > n_0$. Then we have

\[
P\left( \sup_{|t-s|<\frac{1}{n}} |Y_{n,p}(t) - Y_{n,p}(s)| > \epsilon \right) \quad < \quad P\left( \sup_{|t-s|<\delta} |Y_{n,p}(t) - Y_{n,p}(s)| > \epsilon \right).
\]
Now by definition of the process $Y_{n,p}(\cdot)$,

$$\sup_{|t-s|<\frac{1}{n}} |Y_{n,p}(t) - Y_{n,p}(s)| < \max_{1 \leq i \leq n} \frac{|X_i|}{V_{n,p}} \forall t \in [0,1]$$

$$\Rightarrow P(\max_{1 \leq i \leq n} \frac{|X_i|}{V_{n,p}} > \epsilon) < P(\sup_{|t-s|<\delta} |Y_{n,p}(t) - Y_{n,p}(s)| > \epsilon)$$

$$\Rightarrow P(\max_{1 \leq i \leq n} \frac{|X_i|}{V_{n,p}} > \epsilon) < \eta \ \forall n > n_0, \text{by hypothesis.}$$

\[\square\]

**Remark 4** The converse is not necessarily true. To see this assume that $\max_{1 \leq i \leq n} \frac{|X_i|}{V_{n,p}} \xrightarrow{P} 0$. Assume that there exists a $\delta_1$ such that (6.2) holds. Given such a $\delta_1 > 0$, for any integer $m$ we can get an $n$ such that $\frac{m}{n} < \delta_1$. Then for such a $m,n$ we have $|Y_{n,p}(t) - Y_{n,p}(s)| \leq (\max_{1 \leq i \leq n} \sum_{j=1}^{m} |X_{i+j}|)/V_{n,p}$. But the hypothesis does not guarantee that the right hand side converges to zero in probability.

We use the above lemma to prove the necessary part in the following lemma.

**Only if part** For $2 \geq p > \alpha$ the process is not tight.

**Proof:**

For $2 \geq p > \alpha$ observe that

$$\max_{1 \leq i \leq n} \frac{|X_i|}{V_{n,p}} \xrightarrow{P} 0$$

$$\Leftrightarrow \left( \max_{1 \leq i \leq n} \frac{|X_i|}{V_{n,p}} \right)^p \xrightarrow{P} 0$$

$$\Leftrightarrow \max_{1 \leq i \leq n} \frac{|X_i|^p}{\sum |X_i|^p} \xrightarrow{P} 0.$$  

But $|X_i|^p \sim DA(\gamma)$, where $\gamma = \frac{\alpha}{p} < 1$, for which (Darling, [7], Theorem 5.1) says that if $Y_i \sim DA(\gamma)$ where $\gamma < 1$ then $\max_{1 \leq i \leq n} \frac{|Y_i|}{\sum |Y_i|}$ converges in distribution to a non-degenerate random variable $G$ whose characteristic function is identified in the same paper. Thus, $\max_{1 \leq i \leq n} \frac{|X_i|^p}{\sum |X_i|^p}$ does not go to zero in probability. Hence, $\max_{1 \leq i \leq n} \frac{X_i}{V_{n,p}}$ cannot converge to zero in probability and therefore from Lemma 4 the process cannot be tight.  

\[\blacksquare\]
7 Conclusion

The study of self normalised sums has seen a recent upsurge following the works of [10], [6], [13] and [15]. Results for functional convergence was shown only by [4] where the random variable were from the domain of attraction of a Stable(\(\alpha\)) distribution.

This paper deals with the same type of random variables but with norming index \(p \in (0, 2]\). Although it is almost intuitive that the norming index \(p\) has something to do with the stability index \(\alpha\) the relation between them has not been explored in the past. Csörgő et al [4] kept the value of the norming index \(p\) fixed at 2 and compared with various choices of \(\alpha\). This paper, to our knowledge, seems to be the first one where we simultaneously change \(p\) and \(\alpha\). Here, using simple tools of tightness and finite dimensional convergence, we show that the only non trivial case is iff \(p = \alpha = 2\). The if part was shown by Gine et al [10] and Csörgő et al [4]. Our paper shows the only if part.

To proceed further a rate of convergence would be important. A non uniform Berry–Essen bound was given in [2], when the random variables are from \(DAN\), and a bound using Saddlepoint approximation was proved in [11]. Although the process convergence is for \(p = \alpha = 2\), Logan et al [13] have shown that the self normalised sequence can converge for \(p > \alpha\). Using their techniques we have shown in Section 5.3 what the possible limiting characteristic distribution would look like. From our personal communication with Qi-Man Shao we have learned about an unpublished result on limiting finite-dimensional distribution of \(((S_{[nt_1]}/V_{n,p}, \ldots, S_{[nt_k]}/V_{n,p}), p > \alpha)\) where they have shown the limiting joint distribution as mixture of Poisson-type distribution using technique of (Csörgő and Horvath [5]). The rate of convergence for this case has not been explored to our knowledge.

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