All-Pass Filters for Mirroring Pairs of Complex-Conjugated Roots of Rational Matrix Functions

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Proposed Running Head

Mirroring Complex-Conjugated Roots

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Abstract

In this note, we construct real-valued all-pass filters for mirroring pairs of complex-conjugated determinantal roots of a matrix polynomial. This problem appears, e.g., when proving the spectral factorization theorem, or more recently in the literature on possibly non-invertible or possibly non-causal vector autoregressive moving average (VARMA) models. In general, it is not obvious whether the all-pass filter (and as a consequence the all-pass transformed matrix polynomial with real-valued coefficients) which mirrors complex-conjugated roots at the unit circle is real-valued. Naive constructions result in complex-valued all-pass filters which implies that the real-valued parameter space (usually relevant for estimation) is left.

Keywords: All-pass filters, Blaschke matrices, spectral factorization
1 Introduction

It is well known that there are multiple spectral factors which generate the same spectral density, see Baggio and Ferrante (2019) for a recent contribution. In the classical time series literature, Rozanov (1967) and Hannan (1970) use all-pass filters (also known as Blaschke filters) to mirror determinantal roots of spectral factors from inside to outside the unit circle when proving the spectral factorization theorem (for rational spectral densities). More recently, all-pass filters play an important role in the literature on possibly non-invertible or possibly non-causal vector autoregressive moving average (VARMA) models, see Lanne and Saikkonen (2013), Velasco and Lobato (2018), Funovits (2020).

It is not obvious whether the all-pass filter (and as a consequence the all-pass transformed polynomial or rational matrix with real-valued coefficients) which mirrors complex-conjugated roots at the unit circle is real-valued and, to the best of our knowledge, there is no proof of whether this is true available in the literature. Naive constructions, as in Gouriéroux et al (2019), result in complex-valued all-pass filters which implies that the real-valued parameter space (usually relevant for estimation) is left.

Here, we show how to obtain real-valued all-pass filters for mirroring pairs of complex-conjugated determinantal roots of a rational matrix function at the unit circle in two ways. Both constructions start from the QR decomposition of the real and imaginary part of a normalized vector in the (right-) kernel of a polynomial matrix $p(z)$ evaluated at a determinantal zero with non-trivial imaginary part. One approach parametrises consecutively unitary matrices and Blaschke factors in terms of the matrix $R$ of the QR decomposition and the real and imaginary part of the determinantal root, say $\alpha_+$, of $p(z)$. This approach leaves the real-valued parameter space in intermediary steps and it is only ensured at the end that the parameter matrices are indeed real-valued. The second and more elegant construction is based on state spaces methods and does not leave the real-valued parameter space.

The remainder of this note is structured as follows. In Section 2, we define all-pass filters and some special instances of all-pass filters that will appear in our derivations. In Section 3, we prepare the two approaches by discussing the QR decomposition of the real and imaginary part of a normalized vector in the (right-) kernel of $p(\alpha_+)$ and some implications. In Section 4, we parametrise unitary matrices and Blaschke matrices in terms of the elements of $R$ and the real and imaginary part of $\alpha_+$ such that their product has real coefficients and mirrors the given pair of complex-conjugated roots at the unit circle. In Section 5, we discuss the state space approach for constructing real-valued all-pass filters.
2 All-Pass Filters and Blaschke Matrices

A multivariate rational all-pass filter is an \((n \times n)\)-dimensional matrix \(V(z)\) whose entries are rational functions and which satisfies \(V(z)V^* \left(\frac{1}{z}\right) = V^* \left(\frac{1}{z}\right) V(z) = I_n\). The superscript asterisk takes an (arbitrary) matrix function \(m(z) = \sum_{j=-\infty}^{\infty} m_j z^j\) to its version with complex conjugated and transposed coefficient matrices, i.e. \(m^*(z) = \sum_{j=-\infty}^{\infty} m_j^* z^j\).

An elementary Blaschke factor at \(\alpha\) (which is obviously all-pass) is of the form \(B(z, \alpha) = \frac{1-\bar{\alpha}z}{-\alpha + z}\).

A squared Blaschke factor at the complex root \(\alpha_{\pm} = \alpha_r \pm i\alpha_i\) (in obvious notation) is defined as \(B_{sq}(z, \alpha_{\pm}) = \frac{1-\alpha z}{1-\bar{\alpha}z} = \frac{1-2\alpha z + |\alpha|^2 z^2}{|\alpha|^2 - 2\alpha z + z^2}\). Lastly, a bivariate Blaschke factor pertaining to the pair of complex conjugated roots \(\alpha_{\pm} = \alpha_r \pm i\alpha_i\), where \(\alpha_i > 0\), and the non-zero vector \(w \in \mathbb{C}^{2 \times 1}\) is given as \(B_2(z, \alpha_{\pm}, w) = a^{-1}(z)b(z)\), where \(a(z)\) is a diagonal matrix with entries \(B_{sq}(z, \alpha_{\pm})\), and \(b(z)\) is a \((2 \times 2)\) polynomial matrix with highest degree 2 and which is of reduced rank at \(z = \alpha^{-1}, z = \bar{\alpha}^{-1}\), \(z = \alpha\) and \(z = \bar{\alpha}\). We construct \(b(z)\) such that the column space of \(b(\alpha_+)\) is spanned by a given (non-trivial) vector \(w \in \mathbb{C}^{2 \times 1}\).

\[1\text{Sometimes, the Blaschke factor is defined with an additional factor } \alpha/|\alpha|. \text{ However, this factor is not well defined if } \alpha = 0.\]
3 QR Decomposition of Complex-Valued Right-Kernel

We start from a (normalized) vector \( v = v_r + iv_i \in \mathbb{C}^{n\times1} \) in the right-kernel of \( p(\alpha_+) \), where \( v_r, v_i \in \mathbb{R}^{n\times1} \) and will eventually result in the transformed polynomial matrix

\[
\tilde{p}(z) = p(z) \tilde{Q}
\]

where the orthogonal real matrix \( \tilde{Q} \) and the upper-triangular matrix \( R \) with positive diagonal elements will be constructed below. Note that \( \tilde{p}(z)\tilde{p}(\frac{1}{z}) = p(z)p\left(\frac{1}{z}\right) \) holds.

If \( v_r \) and \( v_i \) are linearly dependent\(^2\), then \( v = \beta \hat{v} \) for some \( \beta \in \mathbb{C} \) and \( \hat{v} \in \mathbb{R}^{n\times1} \), i.e. \( p(\alpha_+)\hat{v} = p(\alpha_-)\hat{v} = 0 \). In this case, we can apply the real-valued univariate all-pass function \( B_{sq}(z, \alpha_\pm) \) straightforwardly. Thus, we assume in the following that the matrix \( \begin{pmatrix} v_r & v_i \end{pmatrix} \) has rank 2.

From the QR decomposition, we obtain that

\[
\begin{pmatrix} v_r & v_i \end{pmatrix} = \tilde{Q} \tilde{R} = \begin{pmatrix} Q & \tilde{Q}_2 \end{pmatrix} \begin{pmatrix} R \\ 0_{(n-2)\times2} \end{pmatrix}
\]

with upper-triangular \( R \in \mathbb{R}^{2\times2} \) with positive diagonal elements, orthogonal \( \tilde{Q} \), and \( Q \in \mathbb{R}^{n\times2} \), \( \tilde{Q}_2 \in \mathbb{R}^{n\times(n-2)} \). It follows from

\[
0 = p(\alpha_+)v = p(\alpha_+) \begin{pmatrix} v_r & v_i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = [p(\alpha_+)Q] R \begin{pmatrix} 1 \\ i \end{pmatrix}
\]

that the right-kernel of \( p(\alpha_+)Q \) can be written as \( R\left(\frac{1}{i}\right) \). If the column space of \( b(\alpha_+, \alpha_\pm, w) \) is spanned by \( w = R\left(\frac{1}{i}\right) \), then \( p(\alpha_+)Qb(\alpha_+, \alpha_\pm, R\left(\frac{1}{i}\right)) = p(\alpha_-)Qb(\alpha_-, \alpha_\pm, R\left(\frac{1}{i}\right)) = 0 \) and hence all entries of \( p(z)Qb(\alpha_+, \alpha_\pm, R\left(\frac{1}{i}\right)) \) and \( p(z)Qb(\alpha_-, \alpha_\pm, R\left(\frac{1}{i}\right)) \) are divisible by the diagonal element of \( a(z, \alpha_\pm) \).

\(^2\)This occurs in particular for \( n = 1 \).
4 Parametrising All-Pass Filters Successively

Starting from the QR decomposition as described above, it is possible to construct unitary \((2 \times 2)\)-dimensional matrices \(V_\beta, V_\gamma, \text{ and } V_\delta\) such that

\[
\bar{Q} \cdot \text{diag}(V_\beta, I_{n-2}) \cdot \text{diag}(B(z, \alpha_+), I_{n-1}) \cdot \text{diag}(V_\gamma, I_{n-2}) \cdot \text{diag}(B(z, \alpha_-), I_{n-1}) \cdot \text{diag}(V_\delta, I_{n-2})
\]

has real-valued coefficient matrices.

Every unitary matrix can be parametrised with four parameters \((\phi_0, \phi_1, \phi_2, \phi_3)\) as

\[
e^{i\frac{\phi_0}{2}} \begin{pmatrix}
e^{i\phi_1} \cos(\phi_3) & e^{i\phi_2} \sin(\phi_3) \\
e^{-i\phi_2} \sin(\phi_3) & e^{-i\phi_1} \cos(\phi_3)
\end{pmatrix}.
\]

Thus, we obtain \(V_\beta\) with \((\beta_0, \beta_1) = (0, 0)\) by choosing \(\beta_2, \beta_3\) such that \((1, 1)\) is in the span of \(\begin{pmatrix}\cos(\beta_3) e^{i\beta_2} \\ \sin(\beta_3)\end{pmatrix}\). More specifically, let \(R = \begin{pmatrix}a & 0 \\ 0 & \frac{1}{a}\end{pmatrix}\), where \(a\) and \(c\) are positive by construction and \(a^2 + b^2 + c^2 = 1\). The column space spanned by \(R \begin{pmatrix}1 \\ 1\end{pmatrix}\) is equivalent to the one spanned by \(\begin{pmatrix}\frac{b}{\sqrt{-1 + a}} \\ \frac{-c}{\sqrt{-1 + a}}\end{pmatrix}\). Normalising this vector, we obtain

\[
V_\beta = \begin{pmatrix}\frac{r}{\sqrt{1+r^2}} e^{-i\arccos(\frac{\sqrt{r}}{1+r})} \\ \frac{-1}{\sqrt{1+r^2}} \end{pmatrix} \begin{pmatrix}\cos(\beta_3) e^{i\beta_2} \\ \sin(\beta_3) \end{pmatrix} = \begin{pmatrix}\cos(\beta_3) e^{i\beta_2} \\ \sin(\beta_3) \end{pmatrix}.
\]

Similarly, \(V_\gamma\) with \((\gamma_0, \gamma_1) = (0, 0)\) is determined by setting \(\gamma_2, \gamma_3\) such that \((1, -1)\) is in the span of \(V_\beta \cdot \begin{pmatrix}B(\alpha_+, \alpha_-) & 0 \\ 0 & 1\end{pmatrix}\). Note that the parameters \(\gamma_2, \gamma_3\) are functions of \(\beta_2, \beta_3\), and \(\alpha_+\). Thus, the first column of \(V_\gamma\) should be equal to a normalised version of \(\begin{pmatrix}\cos(\beta_3) e^{i\beta_2} \\ \sin(\beta_3)\end{pmatrix}\).

This leads eventually to the first column of \(V_\gamma\) being equal to

\[
\frac{1}{k} \begin{pmatrix}B(\alpha_+, \alpha_-)^{-1} \cos (\beta_2)^2 e^{-i2\beta_3} + \sin (\beta_2)^2 \\ 2i \sin (\beta_2) \cos (\beta_2) \sin (\beta_3)\end{pmatrix},
\]

where \(k\) is the normalising constant.

Last, \(V_\delta\) is chosen such that \(V_\delta \cdot \begin{pmatrix}B(1, \alpha_+) & 0 \\ 0 & 1\end{pmatrix} \cdot V_\gamma \cdot \lambda B(1, \alpha_-) & 0 \\ 0 & 1\end{pmatrix} \cdot V_\delta\) is equal to the identity matrix.

Straight-forward computation verifies that the coefficient matrices in

\[
V_\beta \cdot \begin{pmatrix}(1 - \alpha_-)(1 + \alpha -) \\ -\alpha_+(-\alpha_- - z)
\end{pmatrix} \cdot V_\gamma \cdot \begin{pmatrix}(1 - \alpha_+)(1 + \alpha + z) \\ -\alpha_+(-\alpha_- + z)
\end{pmatrix} \cdot V_\delta
\]

are indeed real.

\(^3\text{Remember that for } z = x + iy, \text{ the polar representation } z = r \cos(\phi) + i r \sin(\phi) \text{ can be obtained with } r = \sqrt{x^2 + y^2} \text{ and, for } r > 0, \phi = \arccos \left(\frac{2}{\sqrt{r^2 + y^2}}\right) \text{ when } y > 0 \text{ and } \phi = -\arccos \left(\frac{2}{\sqrt{r^2 + y^2}}\right) \text{ when } y < 0. \text{ Note that } \phi \text{ is always positive for us by construction.}\)
5 State Space Construction

The most elegant approach to construct a polynomial matrix $b(z)$ with real coefficients (such that $p(z)b(z)a^{-1}(z)$ is real as well) is a state space construction. In the following, we construct the matrices $(A, B, C, D)$ in the state space representation of the $(2 \times 2)$-dimensional, real-valued, rational, all-pass filter $B_2(z, \alpha_\pm, w) = a^{-1}(z)b(z) = C(z^{-1}I_2 - A)^{-1}B + D$ explicitly.

5.1 Fixing the Poles of the All-Pass Filter: Determining $A$

The eigenvalues of $A$ are equal to the inverse of the determinantal roots of $(I_n - Az)$, i.e. the eigenvalues of $A = \begin{pmatrix} \lambda_r & i\lambda_i \\ -\lambda_i & \lambda_r \end{pmatrix}$ are $\lambda_+ = \lambda_r + i\lambda_i = \alpha_+^{-1} = (\alpha_r + i\alpha_i)^{-1}$ and $\lambda_-$. Of course, the zeros of $a(z)$, i.e. the poles of $B_2(z, \alpha_\pm, w)$, are $\alpha_+ = \lambda_+^{-1}$ and $\alpha_- = \lambda_-^{-1}$. The all-pass filter may be factored as

$$B_2(z, \alpha_\pm, w) = C \begin{pmatrix} z^{-1} - \lambda_r & -\lambda_i \\ \lambda_i & z^{-1} - \lambda_r \end{pmatrix}^{-1} B + D$$

$$= (z^{-2} - 2\lambda_r z^{-1} + \lambda_r^2 + \lambda_i^2)^{-1} C \begin{pmatrix} z^{-1} - \lambda_r & \lambda_i \\ -\lambda_i & z^{-1} - \lambda_r \end{pmatrix} B + D$$

$$= \left(1 - 2\lambda_r z + |\lambda|^2 z^2 \right)^{-1} C \begin{pmatrix} z - \lambda_r z^2 & \lambda_i z^2 \\ -\lambda_i z^2 & z - \lambda_r z^2 \end{pmatrix} B + D$$

$$= a(z)^{-1} C \begin{pmatrix} z - \lambda_r z^2 & \lambda_i z^2 \\ -\lambda_i z^2 & z - \lambda_r z^2 \end{pmatrix} B + a(z)^{-1}a(z)D$$

$$= a(z)^{-1} C \begin{pmatrix} z - \lambda_r z^2 & \lambda_i z^2 \\ -\lambda_i z^2 & z - \lambda_r z^2 \end{pmatrix} B + a(z)D$$

5.2 Fixing the Column-Space at $\alpha_\pm$: Determining $C$

Next, we determine $C$ such that the column-space of $b(\alpha_+)$ is spanned by a given column vector $w \in \mathbb{C}^{2 \times 1}$, in our case $w = R(\frac{1}{\alpha})$. Note that

$$b(\alpha_+) = \alpha_+^2 C \begin{pmatrix} \alpha_+^{-1} - \lambda_r & \lambda_i \\ -\lambda_i & \alpha_+^{-1} - \lambda_r \end{pmatrix} B + a(\alpha_+)D$$

$$= \alpha_+^2 C \begin{pmatrix} (\lambda_r + i\lambda_i) - \lambda_r & \lambda_i \\ -\lambda_i & (\lambda_r + i\lambda_i) - \lambda_r \end{pmatrix} B = \alpha_+^2 C \begin{pmatrix} i\lambda_i & \lambda_i \\ -\lambda_i & i\lambda_i \end{pmatrix} B.$$ 

Therefore, we set $C \begin{pmatrix} i\lambda_i \\ -\lambda_i \end{pmatrix} = \lambda_i \|w\|^{-1}w$, i.e. $C = \|w\|^{-1}(w, -w_r)$ where $w_r, w_i$ denote the real and imaginary parts of $w$. 

5
5.3 Ensuring All-Pass Property: Determining B and D

Finally, we construct $B, D$ (for given $A, C$) such that the rational matrix $B_2(z, \alpha_\pm, w)$ is indeed all-pass. If $(A, B, C, D)$ is a state space realization of the real-valued all-pass filter $B_2(z, \alpha_\pm, w)$, then the product $B'_2 \left( \frac{1}{z}, \alpha_\pm, w \right) B_2(z, \alpha_\pm, w)$ (and thus $B_2(z, \alpha_\pm, w)B'_2 \left( \frac{1}{z}, \alpha_\pm, w \right)$) has a realization given by

$$
\begin{pmatrix}
A'^{-1} & A'^{-1}C'C & A'^{-1}C'D \\
0 & A & B \\
-B'A'^{-1} & (D' - B'A'^{-1}C')C & D'D - B'A'^{-1}C'D
\end{pmatrix}
$$

A state transformation results in

$$
\begin{pmatrix}
I_n & X & 0 \\
0 & I_n & 0 \\
0 & 0 & I_n
\end{pmatrix}
\begin{pmatrix}
A'^{-1} & A'^{-1}C'C & A'^{-1}C'D \\
0 & A & B \\
-B'A'^{-1} & (D' - B'A'^{-1}C')C & D'D - B'A'^{-1}C'D
\end{pmatrix}
\begin{pmatrix}
I_n & -X & 0 \\
0 & I_n & 0 \\
0 & 0 & I_n
\end{pmatrix}
= 
\begin{pmatrix}
A'^{-1} & A'^{-1}C'C + XA - A'^{-1}X & A'^{-1}C'D + XB \\
0 & A & B \\
-B'A'^{-1} & (D' - B'A'^{-1}C')C + B'A'^{-1}X & D'D - B'A'^{-1}C'D
\end{pmatrix}
$$

In order to make the (1,1) block non-controllable and the (2,2) block non-observable, we need to set the (1,2), the (1,3) and the (3,2) block on the right-hand-side equal to zero. That is,

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4Equivalently, matrices $C, D$ of an all-pass filter which is left-multiplied on a polynomial matrix to mirror roots could be constructed from given $A, B$.
5In general, the multiplication of two rational functions $k_1(z)$ and $k_2(z)$ of appropriate dimensions and parametrized as two state space systems $\left( \begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix} \right)$ and $\left( \begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix} \right)$ such that $k_1(z) \cdot k_2(z) = (C_1(z^{-1}I_n - A_1)^{-1}B_1 + D_1)(C_2(z^{-1}I_n - A_2)^{-1}B_2 + D_2)$ results in the state space system

$$
\begin{pmatrix}
A_1 & B_1C_2 \\
0 & A_2 \\
C_1 & D_1C_2
\end{pmatrix}
\begin{pmatrix}
B_1 \\
B_2 \\
D_1D_2
\end{pmatrix}
$$

The rational function $B'_2 \left( \frac{1}{z}, \alpha_\pm, w \right)$ may be represented as

$$
B'_2 \left( \frac{1}{z}, \alpha_\pm, w \right) = B' \left( z - A' \right)^{-1}C' + D' = B' \left( (zA'^{-1} - I_n) A' \right)^{-1}C' + D' = [-B'A'^{-1}] \left( I_n - zA'^{-1} \right)^{-1} C' + D' = \left( \sum_{j=1}^{\infty} A'^{-j} z^j \right) C' + D' = \left( \sum_{j=1}^{\infty} A'^{-j} z^j \right) C' + (D' - B'A'^{-1}C') = [-B'A'^{-1}] \left( \frac{1}{z} - A'^{-1} \right)^{-1} \left[ A'^{-1}C' \right] + (D' - B'A'^{-1}C')
$$

i.e.

$$
\begin{pmatrix}
A'^{-1} \\
-B'A'^{-1} \\
D' - B'A'^{-1}C'
\end{pmatrix}
$$
\[ A'^{-1}C' + XA - A'^{-1}X = 0 \iff C'C + A'XA = 0 \]

\[ A'^{-1}C'D + XB = 0 \iff (X,A'^{-1}C') \begin{pmatrix} B \\ D \end{pmatrix} = 0 \]

\[ (D' - B'A'^{-1}C')C + B'A'^{-1}X = 0 \iff (X'A^{-1} - C'CA^{-1}, C') \begin{pmatrix} B \\ D \end{pmatrix} = 0 \]

\[ \iff \begin{pmatrix} A'X', C' \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix} = 0 \]

First, we obtain \( X \) as solution of the Lyapunov equation obtained from block (1,2). Second, we obtain \( B \) as a function of \( D \) from the equation obtained from block (1,3), which contains the same information as the equation obtained from block (3,2). In particular, we obtain that \( B(D) = -X^{-1}A'^{-1}C'D \).

The (3,3) block needs to satisfy

\[ D'D - B'A'^{-1}C'D = I_n. \]

Together with the above, we have thus

\[ I_n = D'D + D'CA^{-1}X^{-1}A'^{-1}C'D = D'(I_n + CA^{-1}X^{-1}A'^{-1}C')D \]

and may obtain \( D \) from a Cholesky decomposition of \((I_n + CA^{-1}X^{-1}A'^{-1}C')\).

### 5.4 Summary of State Space Construction

It follows that we end up with the system

\[
\begin{pmatrix}
  A'^{-1} & 0 & 0 \\
  0 & A & B \\
  -B'A'^{-1} & 0 & I_n
\end{pmatrix}
\]

whose transfer function is equal to

\[
\left( -B'A'^{-1} \right) \left( I_n z^{-1} - A'^{-1} \begin{pmatrix} 0 \\ 0 & I_n z^{-1} - A \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ B \end{pmatrix} + I_n.
\]
6 Acknowledgments

Financial support by the Research Funds of the University of Helsinki as well as by funds of the Oesterreichische Nationalbank (Austrian Central Bank, Anniversary Fund, project number: 17646) is gratefully acknowledged.

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