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Stability in the Energy Space of the Sum of $N$ Peakons for the Degasperis-Procesi Equation

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Abstract
The Degasperis-Procesi equation possesses well-known peaked solitary waves that are called peakons. Their stability has been established by Lin and Liu in [5]. In this paper, we localize the proof (in some suitable sense detailed in Section 3) of the stability of a single peakon. Thanks to this, we extend the result of stability to the sum of $N$ peakons traveling to the right with respective speeds $c_1, \ldots, c_N$, such that the difference between consecutive locations of peakons is large enough.

1 Introduction
The Degasperis-Procesi (DP) equation
\begin{equation}
    u_t - u_{txx} + 4uu_x = 3u_xu_{xx} + uu_{xxx}, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}
\end{equation}
is completely integrable (see [1]) and possesses, among others, the following invariants
\begin{equation}
    E(u) = \int_{\mathbb{R}} yv \quad \text{and} \quad F(u) = \int_{\mathbb{R}} u^3,
\end{equation}
where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$. Substituting $u$ by $4v - v_{xx}$ in (1.2) and using integration by parts (we suppose that $u(\pm\infty) = v(\pm\infty) = v_x(\pm\infty) = 0$), the conservation laws can be rewritten as
\begin{equation}
    E(u) = \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \quad \text{and} \quad F(u) = \int_{\mathbb{R}} (-v_{xx}^3 + 12vv_{xx}^2 - 48v^2v_{xx} + 64v^3).
\end{equation}
One can see that the conservation law $E(\cdot)$ is equivalent to $\|\cdot\|_{L^2(\mathbb{R})}^2$. Indeed, using integration by parts
\begin{equation}
    \|u\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} u^2 = \int_{\mathbb{R}} (4v - v_{xx})^2 = \int_{\mathbb{R}} (16v^2 + 8v_x^2 + v_{xx}^2) \leq 4E(u),
\end{equation}
and applying Plancherel-Parseval identity
\begin{equation}
    E(u) = \int_{\mathbb{R}} yv = \int_{\mathbb{R}} \frac{1 + \omega^2}{4 + \omega^2} |\hat{u}(\omega)|^2 \leq \int_{\mathbb{R}} |\hat{u}(\omega)|^2 = \|u\|_{L^2(\mathbb{R})}^2,
\end{equation}
where $\hat{u}$ denotes the Fourier transform of $u$. In the sequel we will denote
\begin{equation}
    \|u\|_H = \sqrt{E(u)}.
\end{equation}

1
Note that, by reversing the operator $(1 - \partial_x^2)(\cdot)$ in (1.1), the DP equation can be rewritten in conservation form as
\[
  u_t + \frac{1}{2} \partial_x^2 u^2 + \frac{3}{2}(1 - \partial_x^2)^{-1} \partial_x u^2 = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{1.7}
\]

The DP equation possesses solitary waves called peakons (see Fig. 1a) and defined by
\[
  u(t, x) = \varphi_c(x - ct) = c \varphi(x - ct) = c e^{-|x - ct|}, \quad c \in \mathbb{R}^+, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.8}
\]
but they are not smooth since $\varphi_c \notin C^1(\mathbb{R})$ (see Fig. 1b). The peakons are only global weak solutions of (1.7). It means, for any smooth test function $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$, it holds
\[
  \int_0^{+\infty} \int_{\mathbb{R}} \varphi_c(x - ct) \phi_t(t, x) dtdx + \frac{1}{2} \int_0^{+\infty} \int_{\mathbb{R}} \varphi_c^2(x - ct) \phi_x(t, x) dtdx \tag{1.11}
\]
\[
  + \frac{3}{2} \int_0^{+\infty} \int_{\mathbb{R}} (1 - \partial_x^2)^{-1} \varphi_c^2(x - ct) \phi_x(t, x) dtdx + \int_{\mathbb{R}} \varphi_c(x) \phi(0, x) dx = 0.
\]

The goal of our work is to prove that ordered trains of peakons are stable under small perturbations in the energy space $\mathcal{H}$ (equivalent to $L^2$).

**Definition 1.1 (Stability).** Let $c > 0$ be given. The peakon $\varphi_c$ is said stable in $\mathcal{H}$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if
\[
  \|u_0 - \varphi_c\|_{\mathcal{H}} \leq \delta,
\]
then for all $t \geq 0$, there exists $\xi(t)$ such that
\[
  \|u(t, \cdot) - \varphi_c(\cdot - \xi(t))\|_{\mathcal{H}} \leq \varepsilon,
\]
where $u(t)$ is the solution to (1.1) emanating from $u_0$.

Lin and Liu proved in [5] the stability of a single peakon under the additional condition that $(1 - \partial_x^2) u_0 \in \mathcal{M}^+(\mathbb{R})$. Using this result and the general strategy introduced by Martel, Merle and Tsai in [7] for the generalized Korteweg-de Vries (gKdV) equation and adapted by El Dika and Molinet in [3] and [2] for the Camassa-Holm (CH) equation, we prove here the stability of the sum of $N$ peakons for the DP equation.

Before stating the main result we introduce the function space where will live our class of solutions to the equation. For $I$ a finite or infinite time interval of $\mathbb{R}$, we denote by $\mathcal{X}(I)$ the function space
\[
  \mathcal{X}(I) = \{ u \in C(I; H^1(\mathbb{R})) \cap L^\infty(I; W^{1,1}(\mathbb{R})) \text{, } u_x \in L^\infty(I; BV(\mathbb{R})) \}.
\]

The main result of the present paper is the following theorem.

**Theorem 1.1 (Stability of the Sum of $N$ Peakons).** Let be given $N$ velocities $c_1, \ldots, c_N$ such that $0 < c_1 < \ldots < c_N$. Let $u \in \mathcal{X}([0, T])$, with $0 < T < +\infty$, be a solution of the DP equation. There exist $C > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ only depending on the speeds $(c_i)_{i=1}^N$, such that if
\[
  y_0 = (1 - \partial_x^2) u_0 \in \mathcal{M}^+(\mathbb{R}) \tag{1.12}
\]
and
\[
  \left\| u_0 - \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i^0) \right\|_{\mathcal{H}} \leq \varepsilon^2, \quad \text{with} \quad 0 < \varepsilon < \varepsilon_0, \tag{1.13}
\]
for some $z_1^0, \ldots, z_N^0$ satisfying
\[
  z_1^0 < \ldots < z_N^0 \quad \text{and} \quad z_i^0 - z_{i-1}^0 \geq L, \quad \text{with} \quad L > L_0 > 0, \quad i = 2, \ldots, N, \tag{1.14}
\]

$W^{1,1}(\mathbb{R})$ is the space of $L^1(\mathbb{R})$ functions with derivatives in $L^1(\mathbb{R})$ and $BV(\mathbb{R})$ is the space of function with bounded variation.
Remark 2.1 Moreover, \( E \) and \( M \) are defined in Subsection 4.1.

2 Preliminaries

Theorem 2.1 (Global Weak Solution; See [4] and [6]). Assume that \( u_0 \in L^2(\mathbb{R}) \) with \( y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \). Then the DP equation has a unique global weak solution \( u \in \mathcal{X}(\mathbb{R}_+) \) such that

\[
y(t, \cdot) = (1 - \partial_x^2)u(t, \cdot) \in \mathcal{M}^+(\mathbb{R}), \quad \forall t \in \mathbb{R}_+. \tag{2.1}
\]

Moreover \( E(\cdot) \) and \( F(\cdot) \) are conserved by the flow.

Remark 2.1 (Control of \( L^\infty \) Norm by \( L^2 \) Norm). From (2.1), it holds

\[
u(x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^{x'} y(x') dx' + \frac{e^x}{2} \int_{x}^{+\infty} e^{-x'} y(x') dx',
\]

and

\[
u_x(x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^{x'} y(x') dx' + \frac{e^x}{2} \int_{x}^{+\infty} e^{-x'} y(x') dx',
\]

which lead to

\[
|\nu_x(x)| \leq \nu(x), \quad \forall x \in \mathbb{R}. \tag{2.2}
\]

Then, using the Sobolev embedding of \( H^1(\mathbb{R}) \) into \( L^\infty(\mathbb{R}) \) and (2.2), we infer that there exists a constant \( C_S > 0 \) such that

\[
\|\nu\|_{L^\infty(\mathbb{R})} \leq C_S \|\nu\|_{H^1(\mathbb{R})} \leq 2C_S \|\nu\|_{L^2(\mathbb{R})}. \tag{2.3}
\]

Lemma 2.1 (Positivity; See [6]). Let \( u \in H^1(\mathbb{R}) \) with \( y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R}) \). If \( k_1 \geq 1 \), then we have

\[
(k_1 \pm \partial_x)\nu(x) \geq 0, \quad \forall x \in \mathbb{R}. \tag{2.4}
\]

Lemma 2.2 (Positivity; See [6]). Let \( w(x) = (k_1 \pm \partial_x)\nu(x) \). Assume that \( u \in H^1(\mathbb{R}) \) with \( y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R}) \). If \( k_1 \geq 1 \) and \( k_2 \geq 2 \), then we have

\[
(k_2 \pm \partial_x)(4 - \partial_x^2)^{-1}w(x) \geq 0, \quad \forall x \in \mathbb{R}. \tag{2.5}
\]

3 Stability of a single peakon

The proof of Lin and Liu in [5] is not entirely suitable for our work, because it involves all local extrema of the function \( v = (4 - \partial_x^2)^{-1}u \) on \( \mathbb{R} \), and thus is not local. For our work, we have to localize the estimates. Therefore, we need to modify a little the proof of Lin and Liu. We do this first for a single peakon.
Theorem 3.1 (Stability of Peakons). Let \( u \in \mathcal{X}([0, T]) \), with \( 0 < T \leq +\infty \), be a solution of the DP equation and \( \varphi_c \) be the peakon defined in (1.8), traveling to the right at the speed \( c > 0 \). There exist \( C > 0 \) and \( \varepsilon_0 > 0 \) only depending on the speed \( c > 0 \), such that if

\[
y_0 = (1 - \partial_x^2) u_0 \in \mathcal{M}^+(\mathbb{R})
\]

and

\[
\|u_0 - \varphi_c\|_H \leq \varepsilon, \quad \text{with} \quad 0 < \varepsilon < \varepsilon_0,
\]

then

\[
\|u(t, \cdot) - \varphi_c(\cdot - \xi(t))\|_H \leq C\sqrt{\varepsilon}, \quad \forall t \in [0, T],
\]

where \( \xi(t) \in \mathbb{R} \) is any point where the function \( v(t, \cdot) = (4 - \partial_x^2)^{-1} u(t, \cdot) \) attains its maximum.

To prove this theorem we first need the following lemma that enables to control the distance of \( E(u) \) and \( F(u) \) to respectively \( E(\varphi_c) \) and \( F(\varphi_c) \).

Lemma 3.1 (Control of Distances Between Energies). Let \( u \in H^1(\mathbb{R}) \) with \( y = (1 - \partial_x^2) u \in \mathcal{M}^+(\mathbb{R}) \). If \( \|u - \varphi_c\|_H \leq \varepsilon \), then

\[
|E(u) - E(\varphi_c)| \leq O(\varepsilon^2)
\]

and

\[
|F(u) - F(\varphi_c)| \leq O(\varepsilon^2),
\]

where \( O(\cdot) \) only depends on the speed \( c \).

Proof. For the first estimate, applying triangular inequality, and using that \( \|u - \varphi_c\|_H \leq \varepsilon \) and \( \|\varphi_c\|_H = c/\sqrt{3} \), we have

\[
|E(u) - E(\varphi_c)| = \|u\|_H - \|\varphi_c\|_H (\|u\|_H + \|\varphi_c\|_H) \\
\leq \|u - \varphi_c\|_H (\|u\|_H + \|\varphi_c\|_H) \\
\leq \varepsilon^2 \left( \varepsilon^2 + \frac{2c}{\sqrt{3}} \right) \\
\leq O(\varepsilon^2).
\]

For the second estimate, applying the Hölder inequality, and using that \( \|u - \varphi_c\|_H \leq \varepsilon \) and (2.3), we have

\[
|F(u) - F(\varphi_c)| \leq \int_{\mathbb{R}} |u^3 - \varphi_c^3| \\
\leq \int_{\mathbb{R}} |u - \varphi_c|(u^2 + u\varphi_c + \varphi_c^2) \\
\leq \|u - \varphi_c\|_{L^2(\mathbb{R})} \left( \int_{\mathbb{R}} (u^2 + u\varphi_c + \varphi_c^2)^2 \right)^{1/2} \\
= \|u - \varphi_c\|_{L^2(\mathbb{R})} \left( \int_{\mathbb{R}} (u^4 + 2u^2\varphi_c + 3u^2\varphi_c^2 + 2u\varphi_c^3 + \varphi_c^4) \right)^{1/2} \\
\leq \|u - \varphi_c\|_{L^2(\mathbb{R})} \left( 4C_S^2\|u\|_{L^2(\mathbb{R})}^4 + 4cC_S\|u\|_{L^2(\mathbb{R})}^2 + 3c^2\|u\|_{L^2(\mathbb{R})}^2 + \frac{8}{3}c^3C_S\|u\|_{L^2(\mathbb{R})} + \frac{1}{2}c^4 \right)^{1/2} \\
\leq O(\varepsilon^2),
\]

where we also use that the \( L^2 \) norm of \( u \) is bounded and the following measures of peakon:

\[
\|\varphi_c\|_{L^\infty(\mathbb{R})} = c, \quad \|\varphi_c\|_{L^2(\mathbb{R})} = \sqrt{\frac{2}{3}} c \quad \text{and} \quad \|\varphi_c\|_{L^4(\mathbb{R})} = \frac{1}{\sqrt{2}} c.
\]
This proves the lemma.

Now, to prove Theorem 3.1, by the conservation of $E(\cdot)$, $F(\cdot)$ and the continuity of the map $t \mapsto u(t)$ from $[0, T]$ to $H^1(\mathbb{R}) \hookrightarrow H$ (since $H \simeq L^2$ and $\|u\|_{L^2(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})}$), it suffices to prove that for any function $u \in H^1(\mathbb{R})$ satisfying $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, (3.4) and (3.5), if

$$\inf_{\xi \in \mathbb{R}} \|u - \varphi_c(\cdot - \xi)\|_H \leq \varepsilon^{1/4},$$

then

$$\|u - \varphi_c(\cdot - \xi_1)\|_H \leq C\sqrt{\varepsilon},$$

where $\xi_1 \in \mathbb{R}$ is any point where the function $v = (4 - \partial_x^2)^{-1}u$ attains its maximum.

We divide the proof of Theorem 3.1 into a sequence of lemmas. In the sequel, we will need to introduce the following smooth-peaks defined for all $x \in \mathbb{R}$ by:

$$\rho_c(x) = c\rho(x) = (4 - \partial_x^2)^{-1}\varphi_c(x) = \frac{c}{3}e^{-|x|} - \frac{c}{6}e^{-2|x|},$$

One can check that $\rho_c \in H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$ (by the Sobolev embedding) since $\varphi_c \in H^1(\mathbb{R})$. Indeed, we have

$$\|\rho_c\|_{H^3(\mathbb{R})} = \int_{\mathbb{R}} (1 + \omega^2)^3 |\hat{\varphi}_c(\omega)|^2 \leq \int_{\mathbb{R}} (1 + \omega^2)|\hat{\varphi}_c(\omega)|^2 \leq \|\varphi_c\|_{L^1(\mathbb{R})}^2 = 2c^2.$$

Moreover, $\rho_c$ is a positive even function which decays to 0 at infinity, and admits a single maximum at point 0 (see Fig. 1a-1c).

**Lemma 3.2 (Uniform Estimates).** Let $u \in H^1(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, and $\xi \in \mathbb{R}$. If $\|u - \varphi_c(\cdot - \xi)\|_H \leq \varepsilon^{1/4}$, then

$$\|u - \varphi_c(\cdot - \xi)\|_{L^\infty(\mathbb{R})} \leq O(\varepsilon^{1/8})$$

and

$$\|v - \rho_c(\cdot - \xi)\|_{L^\infty(\mathbb{R})} \leq O(\varepsilon^{1/4}),$$

where $v = (4 - \partial_x^2)^{-1}u$ and $\rho_c$ is defined in (3.8).

**Proof.** For the second estimate, applying the Hölder inequality and using assumption, we get for all $x \in \mathbb{R}$,

$$|v(x) - \rho_c(x - \xi)| \leq \frac{1}{4} \int_{\mathbb{R}} e^{-2|x'|} |u(x - x') - \varphi_c [(x - x') - \xi]| dx'$$

$$\leq \frac{1}{4} \left( \int_{\mathbb{R}} e^{-4|x'|} dx' \right)^{1/2} \left( \int_{\mathbb{R}} |u(x') - \varphi_c(x' - \xi)|^2 dx' \right)^{1/2}$$

$$\leq \frac{1}{2\sqrt{2}} \|u - \varphi_c(\cdot - \xi)\|_H$$

$$\leq O(\varepsilon^{1/4}).$$

For the first estimate, note that the assumption $y = (1 - \partial_x^2)u \geq 0$ implies that $u = (1 - \partial_x^2)^{-1} y \geq 0$ and satisfies (2.2). Then, applying triangular inequality, and using that $|\varphi'_c| = \varphi_c$ on $\mathbb{R}$ and (2.3), we have

$$\|u - \varphi_c(\cdot - \xi)\|_{H^1(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})} + \|\varphi_c\|_{H^1(\mathbb{R})}$$

$$\leq 2\|u\|_{L^2(\mathbb{R})} + 2\|\varphi_c\|_{L^2(\mathbb{R})}$$

$$\leq 2\|u - \varphi_c(\cdot - \xi)\|_{L^2(\mathbb{R})} + 4\|\varphi_c\|_{L^2(\mathbb{R})}$$

$$\leq O(\varepsilon^{1/4}) + O(1).$$
Now, applying the Gagliardo-Nirenberg inequality and using assumption, we obtain
\[
\|u - \varphi_c(\cdot - \xi)\|_{L^\infty(\mathbb{R})} \leq C_G \|u - \varphi_c(\cdot - \xi)\|_{L^2(\mathbb{R})}^{1/2} \|u - \varphi_c(\cdot - \xi)\|_{H^2(\mathbb{R})}^{1/2}
\]
\[
\leq O(\varepsilon^{1/8}) \left( O(\varepsilon^{1/8}) + O(1) \right)
\]
\[
\leq O(\varepsilon^{1/8}).
\]
This proves the lemma.

\[\square\]

Figure 1: Variation of peakon and smooth-peakon at initial time with the speed \(c = 1\).

\textbf{Lemma 3.3} (Quadratic Identity; See [5]). For any \(u \in L^2(\mathbb{R})\) and \(\xi \in \mathbb{R}\), it holds
\[
E(u) - E(\varphi_c) = \|u - \varphi_c(\cdot - \xi)\|_{H^2}^2 + 4c \left( v(\xi) - \frac{c}{6} \right),
\]
where \(v = (4 - \partial_x^2)^{-1}u\) and \(c/6 = \rho_c(0) = \max_{x \in \mathbb{R}} \rho_c(x - \xi)\).
Thus, combining (3.10), (3.21) and proceeding as for the estimate (3.18), we infer (3.19).

We claim that

\[ \alpha, \beta \]

and

By construction

\[ \alpha, \beta \]

must have at least one local maximum on \([\alpha, \beta]\). Assume that on \([\alpha, \beta]\) the function \(v\) admits \(k + 1\) points \((\xi_j)_{j=1}^{k+1}\) with local maximal values for some integer \(k \geq 0\), where \(\xi_1\) is the first local maximum point and \(\xi_{k+1}\) the last local maximum point. Then between \(\xi_1\) and \(\xi_{k+1}\), the function \(v\) admits \(k\) points \((\eta_j)_{j=1}^k\) with local minimal values. We rename \(\alpha = \eta_0\) and \(\beta = \eta_{k+1}\) so that it holds

\[ \eta_0 < \xi_1 < \eta_1 < \ldots < \xi_j < \eta_j < \xi_{j+1} < \eta_{j+1} < \ldots < \eta_k < \xi_{k+1} < \eta_{k+1}. \]  

Let

\[ M_j = v(\xi_j), \quad j = 1, \ldots, k + 1, \quad \text{and} \quad m_j = v(\eta_j), \quad j = 1, \ldots, k. \]

By construction

\[ v_x(x) \geq 0, \quad \forall x \in [\eta_{j-1}, \xi_j], \quad j = 1, \ldots, k \]

and

\[ v_x(x) \leq 0, \quad \forall x \in [\xi_j, \eta_j], \quad j = 1, \ldots, k. \]

We claim that

\[ v(x) \leq \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus [\eta_0, \eta_{k+1}], \]

\[ u(x) \leq \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus [\eta_0, \eta_{k+1}], \]

and there exists \(C_0 > 0\) such that

\[ [\eta_0, \eta_{k+1}] \subset [\xi - C_0, \xi + C_0]. \]

Indeed, for some \(0 < \varepsilon \ll 1\) fixed, using (3.11) we have

\[ \rho_c(\eta_0 - \xi) = v(\eta_0) + O(\varepsilon^{1/4}) = \frac{c}{2400} + O(\varepsilon^{1/4}) \leq \frac{c}{350}. \]

Please note that, we abuse notation by writing that the difference between \(v\) and \(\rho_c(\cdot - \xi)\) is equal to \(O(\varepsilon^{1/4})\). Therefore, using that \(\rho_c(\cdot - \xi)\) is increasing on \([- \infty, \xi]\), it holds for all \(x \in [- \infty, \eta_0]\),

\[ v(x) = \rho_c(x - \xi) + O(\varepsilon^{1/4}) \leq \frac{c}{300} + O(\varepsilon^{1/4}) \leq \frac{c}{300}. \]

Proceeding in the same way for \(x \in [\eta_{k+1}, + \infty]\), we obtain (3.18).

One can remark that for all \(x \in \mathbb{R},\)

\[ \varphi_c(x) - 6\rho_c(x) = ce^{-|x|} - 6 \left( \frac{c}{3} e^{-|x|} - \frac{c}{6} e^{-2|x|} \right) = -ce^{-|x|} + ce^{-2|x|} \leq 0. \]  

(3.21)

Thus, combining (3.10), (3.21) and proceeding as for the estimate (3.18), we infer (3.19).

Finally, from (3.11) we have

\[ \rho_c(\eta_0 - \xi) = v(\eta_0) + O(\varepsilon^{1/4}) = \frac{c}{2400} + O(\varepsilon^{1/4}) \geq \frac{c}{300}. \]

\[ 2 \text{In the case of an infinite countable number of local maximal values, the proof is exactly the same.} \]
Therefore, since \( \rho_c = (c/3)e^{-|x|} - (c/6)e^{-|x|} \) and that \( x \mapsto (1/3)e^{-|x|} - (1/6)e^{-2|x|} \) is a positive even function decreasing to 0 on \( \mathbb{R}_+ \) (see Fig. 1a), there exists a universal constant \( C_0 > 0 \) such that (3.20) holds.

We now are ready to establish the connection between the conservation laws. Please note that, we will change the order of the extrema of \( v = (4 - \partial^2_x)^{-1}u \) while keeping the same notations as in (3.15).

**Lemma 3.4** (Connection Between \( E(\cdot) \) and the Local Extrema of \( v \)). Let \( u \in H^1(\mathbb{R}) \) and \( v = (4 - \partial^2_x)^{-1}u \in H^3(\mathbb{R}) \). Define the function \( g \) by

\[
g(x) = \begin{cases} 
2v + v_{xx} - 3v_x, & x < \xi_1, \\
v + v_{xx} + 3v_x, & \xi_j < x < \eta_j, \\
2v + v_{xx} - 3v_x, & \eta_j < x < \xi_{j+1}, \\
2v + v_{xx} + 3v_x, & x > \xi_{k+1},
\end{cases} \tag{3.22}
\]

Then it holds

\[
\int_{-\infty}^{\infty} g^2(x)dx = E(u) - 12 \left( \sum_{j=0}^{k} M^2_{j+1} - \sum_{j=1}^{k} M^2_j \right). \tag{3.23}
\]

**Proof.** We have

\[
\int_{-\infty}^{\infty} g^2(x)dx = \int_{-\infty}^{\xi_1} g^2(x)dx + \sum_{j=1}^{k} \int_{\xi_j}^{\xi_{j+1}} g^2(x)dx + \int_{\xi_{k+1}}^{\infty} g^2(x)dx. \tag{3.24}
\]

For \( j = 1, \ldots, k \),

\[
\int_{\xi_j}^{\xi_{j+1}} g^2(x)dx = \int_{\eta_j}^{\eta_{j+1}} (2v + v_{xx} + 3v_x)^2 + \int_{\xi_j}^{\xi_{j+1}} (2v + v_{xx} - 3v_x)^2
\]

\[
= J + I.
\]

Let us compute \( I \),

\[
I = \int_{\eta_j}^{\xi_{j+1}} (4v^2 + v_{xx}^2 + 9v_x^2 + 4vv_{xx} - 12vv_x - 6v_xv_{xx})
\]

\[
= \int_{\eta_j}^{\xi_{j+1}} (4v^2 + v_{xx}^2 + 9v_x^2) + 4 \int_{\eta_j}^{\xi_{j+1}} vv_{xx} - 12 \int_{\eta_j}^{\xi_{j+1}} v_xv_x - 6 \int_{\eta_j}^{\xi_{j+1}} v_xv_x
\]

\[
= \int_{\eta_j}^{\xi_{j+1}} (4v^2 + v_{xx}^2 + 9v_x^2) + I_1 + I_2 + I_3.
\]

Applying integration by parts and using that \( v_x(\xi_j) = v_x(\eta_j) = 0 \), we get

\[
I_1 = -4 \int_{\eta_j}^{\xi_{j+1}} v_{xx}^2, \quad I_2 = -6 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v^2) = -6v^2(\xi_{j+1}) + 6v^2(\eta_j) \quad \text{and} \quad I_3 = -3 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v_x^2) = 0.
\]

Therefore

\[
I = \int_{\eta_j}^{\xi_{j+1}} (4v^2 + 5v_x^2 + v_{xx}^2) - 6v^2(\xi_{j+1}) + 6v^2(\eta_j). \tag{3.25}
\]

Similar computations lead to

\[
J = \int_{\xi_j}^{\xi_1} (4v^2 + 5v_x^2 + v_{xx}^2) - 6v^2(\xi_j) + 6v^2(\eta_j), \quad \int_{-\infty}^{\xi_j} g^2(x)dx = \int_{-\infty}^{\xi_1} (4v^2 + 5v_x^2 + v_{xx}^2) - 6v^2(\xi_1) \tag{3.26}
\]

8
and
\[
\int_{\xi_{k+1}}^{+\infty} g^2(x) \, dx = \int_{\xi_{k+1}}^{+\infty} (4v^2 + 5v_x^2 + v_{xx}^2) - 6v^2(\xi_{k+1}). \tag{3.27}
\]

Adding \( I \) and \( J \), and summing over \( j \in \{1, \ldots, k\} \), we obtain
\[
\int_{\xi_1}^{\xi_{k+1}} g^2(x) \, dx = \int_{\xi_1}^{\xi_{k+1}} (4v^2 + 5v_x^2 + v_{xx}^2) - 6 \sum_{j=1}^{k} v^2(\xi_{j+1}) - 6 \sum_{j=1}^{k} v^2(\xi_j) + 12 \sum_{j=1}^{k} u^2(\eta_j). \tag{3.28}
\]

The lemma follows by combining (3.24) and (3.26)-(3.28). \( \square \)

**Lemma 3.5** (Connection Between \( F(\cdot) \) and the Local Extrema of \( v \)). Let \( u \in H^1(\mathbb{R}) \) and \( v = (4 - \partial_x^2)^{-1} u \in H^3(\mathbb{R}) \). Define the function \( h \) by
\[
h(x) = \begin{cases} 
- v_{xx} - 6v_x + 16v, & x < \xi_1, \\
v_{xx} + 6v_x + 16v, & \xi_1 < x < \eta_j, \\
v_{xx} - 6v_x + 16v, & \eta_j < x < \xi_{j+1}, \\
v_{xx} + 6v_x + 16v, & x > \xi_{k+1}, 
\end{cases} \tag{3.29}
\]

Then it holds
\[
\int_{\mathbb{R}} h(x)g^2(x) \, dx = F(u) - 144 \left( \sum_{j=0}^{k} M_{j+1}^3 - \sum_{j=1}^{k} m_{j}^3 \right). \tag{3.30}
\]

**Proof.** We have
\[
\int_{\mathbb{R}} h(x)g^2(x) \, dx = \int_{-\infty}^{\xi_1} h(x)g^2(x) \, dx + \sum_{j=1}^{k} \int_{\xi_j}^{\xi_{j+1}} h(x)g^2(x) \, dx + \int_{\xi_{k+1}}^{+\infty} h(x)g^2(x) \, dx. \tag{3.31}
\]

For \( j = 1, \ldots, k \),
\[
\int_{\xi_j}^{\xi_{j+1}} h(x)g^2(x) \, dx = \int_{\xi_j}^{\eta_j} (-v_{xx} - 6v_x + 16v)(2v + v_{xx} - 3v_x)^2
\]
\[
+ \int_{\eta_j}^{\xi_{j+1}} (-v_{xx} + 6v_x + 16v)(2v + v_{xx} + 3v_x)^2
\]
\[
= J + I.
\]

Let us compute \( I \),
\[
I = \int_{\eta_j}^{\xi_{j+1}} (-v_x^3 + 12v_x^2v_{xx} + 64v^3 + 60v^2v_{xx}) \, dx - 54 \int_{\eta_j}^{\xi_{j+1}} v_x^3 + 27 \int_{\eta_j}^{\xi_{j+1}} v_{xx}^2v_x^2
\]
\[
- 108 \int_{\eta_j}^{\xi_{j+1}} vv_xv_{xx} - 216 \int_{\eta_j}^{\xi_{j+1}} v^2v_x^2 + 216 \int_{\eta_j}^{\xi_{j+1}} vv_x^2
\]
\[
= \int_{\eta_j}^{\xi_{j+1}} (-v_x^3 + 12v_x^2v_{xx} + 64v^3 + 60v^2v_{xx})
\]
\[
- 54 \int_{\eta_j}^{\xi_{j+1}} v_x^3 + I_1 + I_2 + I_3 + I_4.
\]

Applying integration by parts and using that \( v_x(\xi_j) = v_x(\eta_j) = 0 \), we get
\[
I_1 = 9 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v_x^3) = 0, \quad I_2 = 54 \int_{\eta_j}^{\xi_{j+1}} v_x^3, \quad I_3 = -72 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v^3) = -72v^3(\xi_{j+1}) + 72v^3(\eta_j)
\]

9
and
\[ I_1 = 108 \int_{\eta}^{\xi} \partial_x (v^2) v_x = -108 \int_{\eta}^{\xi} v^2 v_{xx}. \]

Therefore
\[ I = \int_{\eta}^{\xi} \left( -v_x^3 + 12v v_x^2 + 64v^3 + 60v^2 v_x \right) - 72v^3 (\xi + 1) + 72v^3 (\eta). \tag{3.32} \]

Similar computations lead to
\[ J = \int_{\eta}^{\xi} \left( -v_x^3 + 12v v_x^2 + 64v^3 + 60v^2 v_x \right) - 72v^3 (\xi) \tag{3.33} \]
\[ \int_{-\infty}^{\xi} h(x) g^3(x) dx = \int_{-\infty}^{\xi} \left( -v_x^3 + 12v v_x^2 + 64v^3 + 60v^2 v_x \right) - 72v^3 (\xi) \tag{3.34} \]
and
\[ \int_{\xi}^{+\infty} h(x) g^3(x) dx = \int_{\xi}^{+\infty} \left( -v_x^3 + 12v v_x^2 + 64v^3 + 60v^2 v_x \right) - 72v^3 (\xi) \tag{3.35} \]

Adding (3.35) and (3.33), and summing over \( j \in \{1, \ldots, k\} \), we obtain
\[ \int_{\xi}^{\xi+1} h(x) g^3(x) dx = \int_{\xi}^{\xi+1} \left( -v_x^3 + 12v v_x^2 + 64v^3 + 60v^2 v_x \right) - 72 \sum_{j=1}^{k} v^3 (\xi_j) \]
\[ - 72 \sum_{j=1}^{k} v^3 (\eta_j). \tag{3.36} \]

The lemma follows by combining (3.31) and (3.34)-(3.36).

**Lemma 3.6 (Connection Between \( E(\cdot) \) and \( F(\cdot) \)).** Let \( u \in H^1(\mathbb{R}) \), with \( y = (1 - \partial_x^2) u \in M^+(\mathbb{R}) \), that satisfies (3.6) for some \( \xi \in \mathbb{R} \). Assume that \( v = (4 - \partial_x^2)^{-1} u \) satisfies (3.13)-(3.20), with local extrema on \([\eta_0, \eta_{k+1}]\) arranged in decreasing order in the following way:
\[ M_1 \geq M_2 \geq \ldots \geq M_{k+1} \geq 0, \quad m_1 \geq m_2 \geq \ldots \geq m_k \geq 0, \quad M_{j+1} \geq m_j, \quad j = 1, \ldots, k. \tag{3.37} \]

There exists \( \varepsilon_0 > 0 \) only depending on the speed \( c \), such that if \( 0 < \varepsilon < \varepsilon_0 \), then it holds
\[ M_1^3 - \frac{1}{4} E(u) M_1 + \frac{1}{72} F(u) \leq 0. \tag{3.38} \]

**Proof.** The key is to show that \( h \leq 18M_1 \) on \( \mathbb{R} \). Note that by (3.6) we know that \( 18M_1 \geq c/4 \). We rewrite the function \( h \) as
\[ h(x) = \begin{cases} 
- v_{xx} - 6v_x + 16v, & x < \eta_0, \\
- (\partial_x^2 + 3\partial_x + 2) v - 3v_x + 18v, & \eta_0 < x < \xi_1, \\
- (\partial_x^2 - 3\partial_x + 2) v + 3v_x + 18v, & \xi_j < x < \eta_j, \\
- (\partial_x^2 + 3\partial_x + 2) v - 3v_x + 18v, & \eta_j < x < \xi_{j+1}, \\
- (\partial_x^2 - 3\partial_x + 2) v + 3v_x + 18v, & \xi_{k+1} < x < \eta_{k+1}, \\
- v_{xx} + 6v_x + 16v, & x > \eta_{k+1}.
\end{cases} \]

First, one can remark that for all \( x \in \mathbb{R} \),
\[ v(x) = \frac{e^{-2x}}{4} \int_{-\infty}^{x} e^{2x'} u(x') dx' + \frac{e^{2x}}{4} \int_{x}^{+\infty} e^{-2x'} u(x') dx'. \]
and

\[ v_x(x) = -\frac{e^{-2x}}{2} \int_{-\infty}^{x} e^{2x'} u(x') dx' + \frac{e^{2x}}{2} \int_{x}^{+\infty} e^{-2x'} u(x') dx'. \]

Then using that \( u = (1 - \partial_x^2)^{-1} y \geq 0 \) on \( \mathbb{R} \), we get

\[ |v_x(x)| \leq 2v(x), \quad \forall x \in \mathbb{R}. \tag{3.39} \]

Next, if \( x \in \mathbb{R} \setminus [\eta_0, \eta_{k+1}] \), using that \( v_{xx} = 4v - u \), (3.18), (3.19) and (3.39), it holds

\[ h \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c}{g}. \]

If \( \eta_0 < x < \xi_1 \), then \( v_x \geq 0 \), and using that \( y = (1 - \partial_x^2)u \geq 0 \), it follows from Lemma 2.2 that

\[ h = -(\partial_x^2 + 3\partial_x + 2)v - 3v_x + 18v \]
\[ = -(2 + \partial_x)(4 - \partial_x^2)^{-1}(1 + \partial_x)u - 3v_x + 18v \leq 18v. \]

If \( \xi_j < x < \eta_j \), then \( v_x \leq 0 \), and similarly using that \( y = (1 - \partial_x^2)u \geq 0 \), it follows from Lemma 2.2 that

\[ h = -(\partial_x^2 - 3\partial_x + 2)v + 3v_x + 18v \]
\[ = -(2 - \partial_x)(4 - \partial_x^2)^{-1}(1 - \partial_x)u + 3v_x + 18v \leq 18v. \]

Therefore, it holds

\[ h(x) \leq 18 \max_{x \in \mathbb{R}} v(x) = 18M_1, \quad \forall x \in \mathbb{R}. \tag{3.40} \]

Now, combining (3.23), (3.30) and (3.40), we get

\[ F(u) - 144 \left( \sum_{j=0}^{k} M_{j+1}^3 - \sum_{j=1}^{k} m_j^3 \right) \]
\[ = \int_{\mathbb{R}} h(x) g^2(x) dx \]
\[ \leq \|h\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} g^2(x) dx \]
\[ \leq 18M_1 \left[ E(u) - 12 \left( \sum_{j=0}^{k} M_{j+1}^2 - \sum_{j=1}^{k} m_j^2 \right) \right]. \]

For \( j = 1, \ldots, k \), we set

\[ A_j = M_{j+1}^3 - m_j^3 \quad \text{and} \quad B_j = M_{j+1}^2 - m_j^2, \]

and our inequality becomes

\[ M_1^3 - \frac{1}{4} E(u) M_1 + \frac{1}{72} F(u) \leq 2 \sum_{j=1}^{k} \left( A_j - \frac{3}{2} M_1 B_j \right). \tag{3.41} \]

On the other hand, using that \( M_{j+1} \geq m_j \), we have

\[ A_j - \frac{3}{2} M_1 B_j = -\frac{1}{2} (M_{j+1} - m_j) \left( 3M_1 m_j + 3M_1 M_{j+1} - 2M_{j+1} m_j - 2M_{j+1}^2 - 2m_j^2 \right) \leq 0. \tag{3.42} \]
Finally, combining (3.41) and (3.42), we obtain the lemma. □

**Proof of Theorem 3.1.** We argue as El Dika and Molinet in [3]. As noticed after the statement of the theorem, it suffices to prove (3.7) assuming that $u \in H^1(\mathbb{R})$ satisfies (3.1), (3.2) and (3.4)-(3.6). We set $M_1 = v(\xi_1) = \max_{x \in \mathbb{R}} v(x)$ and $\delta = c/6 - M_1$. We first remark that if $\delta \leq 0$, combining (3.4) and (3.12), it holds

$$
\|u - \varphi_c(\cdot - \xi_1)\|_{\mathcal{H}} \leq |E(u_0) - E(\varphi_c)|^{1/2} \leq O(\varepsilon),
$$

that yields the desired result. Now suppose that $\delta > 0$, that is the maximum of the function $v$ is less than the maximum of $\rho_c$. Combining (3.4), (3.5) and (3.38), we get

$$
M_1^2 - \frac{1}{4} E(\varphi_c) M_1 + \frac{1}{12} F(\varphi_c) \leq O(\varepsilon^2).
$$

Using that $E(\varphi_c) = c^2/3$ and $F(\varphi_c) = 2c^3/3$, our inequality becomes

$$
\left( M_1 - \frac{c}{6} \right)^2 \left( M_1 + \frac{c}{3} \right) \leq O(\varepsilon^2).
$$

Substituting $M_1$ by $c/6 - \delta$ and using that $(M_1 + c/3)^{-1} < 3/c$, it holds

$$
\delta^2 \leq O(\varepsilon^2) \Rightarrow \delta \leq O(\varepsilon).
$$

Finally, combining (3.4), (3.12) and (3.43), we obtain

$$
\|u - \varphi_c(\cdot - \xi_1)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon},
$$

where $C > 0$ only depends on the speed $c$. This completes the proof of the stability of a single peakon.

### 4 Stability of the trains of peakons

For $\gamma > 0$ and $L > 0$, we define the following neighborhood of all the sums of $N$ peakons of speed $c_1, ..., c_N$ with spatial shifts $z_i$ that satisfied $z_i - z_{i-1} \geq L$,

$$
U(\gamma, L) = \left\{ u \in H^1(\mathbb{R}); \inf_{z_i - z_{i-1} > L} \left\| u - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i) \right\|_{\mathcal{H}} \leq \gamma \right\}.
$$

By the continuity of the map $t \mapsto u(t)$ from $[0, T]$ into $H^1(\mathbb{R}) \hookrightarrow \mathcal{H}$, to prove Theorem 1.1 it suffices to prove that there exist $A > 0$, $\varepsilon_0 > 0$ and $L_0 > 0$ such that for all $L > L_0$ and $0 < \varepsilon < \varepsilon_0$, if $u_0$ satisfies (1.12)-(1.14), and if for some $0 < t_0 < T$,

$$
u(t) \in U \left( A(\sqrt{\varepsilon} + L^{-1/8}), \frac{L}{2} \right), \quad \forall t \in [0, t_0], \tag{4.2}
$$

then

$$
u(t_0) \in U \left( A \left( \frac{1}{2} (\sqrt{\varepsilon} + L^{-1/8})^2, \frac{2L}{3} \right) \right). \tag{4.3}
$$

Therefore, in the sequel of this section we will assume (4.2) for some $0 < \varepsilon < \varepsilon_0$ and $L > L_0$, with $A$, $\varepsilon_0$ and $L_0$ to be specified later, and we will prove (4.3).

**Remark 4.1** (Distance Between $v$ and the Sum of $N$ Smooth-peakons). From the definition of $E(\cdot)$ and $\mathcal{H}$ (see respectively (1.3) and (1.6)), one can clearly see that $\|u\|_{\mathcal{H}}$ is equivalent to $\|u\|_{H^2(\mathbb{R})}$, where $v = (4 - \partial_x^2)^{-1} u$. Let $t_1 \in [0, t_0]$ fixed, if $u(t_1) \in U(\gamma, L/2)$, then there exists $\tilde{Z} = (\tilde{z}_i)_{i=1}^{N}$ with $\tilde{z}_i - \tilde{z}_{i-1} \geq L/2$, such that $v(t_1) \in H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$ stays close to $\sum_{i=1}^{N} \rho_{c_i}(\cdot - \tilde{z}_i)$ in the $H^2$ norm, where $\rho_{c_i}$ is defined in (3.8).
4.1 Control of the distance between the peakons

In this subsection, we want to prove that the different bumps of \( u \) (respectively of \( v \)) that are individually close to a peakon (respectively a smooth-peakon) get away from each others as time is increasing. This is crucial in our analysis since we do not know how to manage strong interactions.

**Lemma 4.1** (Decomposition of the Solution Around \( \varphi_{c_i} \)). Let \( u_0 \) satisfying (1.12)-(1.14). There exist \( \gamma_0 > 0 \), \( L_0 > 0 \) and \( C_0 > 0 \) such that for all \( 0 < \gamma < \gamma_0 \) and \( 0 < L_0 < L \), if \( u(t) \in U(\gamma, L/2) \) on \([0, t_0]\) for some \( 0 < t_0 < T \), then there exist \( N \) \( C^1 \) functions \( \tilde{x}_1, \ldots, \tilde{x}_N \) defined on \([0, t_0]\) such that

\[
\left\| u(t) - \sum_{i=1}^{N} \varphi_{c_i} (\cdot - \tilde{x}_i(t)) \right\|_{\mathcal{H}} \leq O(\gamma),
\]

\[
\left\| v(t) - \sum_{i=1}^{N} \rho_{c_i} (\cdot - \tilde{x}_i(t)) \right\|_{C^1(\mathbb{R})} \leq O(\gamma),
\]

\[
|\tilde{x}_i(t) - c_i| \leq c_i^{-2} \left( O(\gamma) + O(e^{-L/4}) \right), \quad i = 1, \ldots, N,
\]

and

\[
\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geq \frac{3L}{4} + \frac{(c_i - c_{i-1})t}{2}, \quad i = 2, \ldots, N.
\]

Moreover, for \( i = 1, \ldots, N \), setting \( J_i = [y_i(t), y_{i+1}(t)] \), with

\[
\begin{align*}
y_1(t) &= -\infty, \\
y_i(t) &= \frac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2}, \quad i = 2, \ldots, N, \\
y_{N+1}(t) &= +\infty,
\end{align*}
\]

it holds

\[
|\xi_1^i(t) - \tilde{x}_i(t)| \leq \frac{L}{12}, \quad i = 1, \ldots, N,
\]

where \( \xi_1^i(t), \ldots, \xi_N^i(t) \) are any point such that

\[
v \left( t, \xi_1^i(t) \right) = \max_{x \in J_i} v(t, x), \quad i = 1, \ldots, N,
\]

and where \( v = (4 - \partial_x^2)^{-1} u \) and \( O(\cdot) \) only depends on the speeds \( c_i \).

**Proof.** We will slightly modify the construction done by El Dika and Molinet in [3]. One can remark that the peakons \( \varphi_{c_i} (\cdot - c_i t) \) and the smooth-peakons \( \rho_{c_i} (\cdot - c_i t) \) travel at the same speed \( c_i \), thanks to this, we will do our construction with \( v = (4 - \partial_x^2)^{-1} u \) instead of \( u \). We do that because the \( \mathcal{H} \) (equivalent to \( L^2 \)) approximation (4.2) does not permit us to construct a \( C^1 \) function, which is crucial for application of the Implicit Function Theorem. We note that the same approach can also be used for the CH equation.

For \( Z = (z_1, \ldots, z_N) \in \mathbb{R}^N \) fixed such that \( |z_i - z_{i-1}| > L/2 \), we set

\[
R_Z(\cdot) = \sum_{i=1}^{N} \rho_{c_i} (\cdot - z_i) \quad \text{and} \quad S_Z(\cdot) = \sum_{i=1}^{N} \varphi_{c_i} (\cdot - z_i).
\]

For \( 0 < \gamma < \gamma_0 \), we define the function

\[
\mathcal{Y} : (-\gamma, \gamma)^N \times B_{\mathcal{H}^2}(R_Z, \gamma) \to \mathbb{R}^N,
\]

\[
(y_1, \ldots, y_N, v) \mapsto (\mathcal{Y}^1(y_1, \ldots, y_N, v), \ldots, \mathcal{Y}^N(y_1, \ldots, y_N, v))
\]

13
with
\[ Y^i(y_1, \ldots, y_N, v) = \int_{\mathbb{R}} \left( v - \sum_{j=1}^{N} \rho_{c_j}(\cdot - z_j - y_j) \right) \partial_x \rho_{c_i}(\cdot - z_i - y_i). \]

\( Y \) is clearly of class \( C^1 \). For \( i = 1, \ldots, N \),
\[ \frac{\partial Y^i}{\partial y_i}(y_1, \ldots, y_N, v) = -\int_{\mathbb{R}} \left( v - \sum_{1 \leq j \leq N, j \neq i} \rho_{c_j}(\cdot - z_j - y_j) \right) \partial^2_x \rho_{c_i}(\cdot - z_i - y_i) \]
and for \( j \neq i \),
\[ \frac{\partial Y^i}{\partial y_j}(y_1, \ldots, y_N, v) = \int_{\mathbb{R}} \partial_x \rho_{c_j}(\cdot - z_j - y_j) \partial_x \rho_{c_i}(\cdot - z_i - y_i). \]

Hence
\[ \frac{\partial Y^i}{\partial y_i}(0, \ldots, 0, R_Z) = \| \partial_x \rho_{c_i} \|_{L^2(\mathbb{R})} = \frac{c_1^2}{54} \leq \frac{c_1^2}{54} \]
and for \( j \neq i \), using the exponential decay of \( \varphi_{c_i} \) and that \( |z_i - z_{i-1}| > L/2 \), for \( L > L_0 > 0 \) with \( L_0 \gg 1 \), it holds
\[ \left| \frac{\partial Y^i}{\partial y_j}(0, \ldots, 0, R_Z) \right| \]
\[ = \left| \int_{\mathbb{R}} \partial_x \rho_{c_j, \alpha}(\cdot - z_j) \partial_x \rho_{c_i, \alpha}(\cdot - z_i) dx \right| \]
\[ = \left| \int_{\mathbb{R}} \rho_{c_j, \alpha}(\cdot - z_j) \partial^2_x \rho_{c_i, \alpha}(\cdot - z_i) dx \right| \]
\[ \leq \frac{1}{9} (c_i - \alpha)(c_j - \alpha) \left\{ \int_{\mathbb{R}} e^{-|x-z_j|-|x-z_i|} dx + 2 \int_{\mathbb{R}} e^{-|x-z_j|-2|x-z_i|} dx \right\} \]
\[ + \frac{1}{2} \int_{\mathbb{R}} e^{-2|x-z_j|-|x-z_i|} dx + \int_{\mathbb{R}} e^{-2|x-z_j|-2|x-z_i|} dx \]
\[ \leq O(e^{-L/4}). \]

We deduce that, for \( L > 0 \) large enough, \( D_{(y_1, \ldots, y_N)}Y(0, \ldots, 0, R_Z) = D + P \) where \( D \) is an invertible diagonal matrix with \( \| D^{-1} \| \leq (c_1/3\sqrt{6})^{-2} \) and \( \| P \| \leq O(e^{-L/4}) \). Hence there exists \( L_0 > 0 \) such that for \( L > L_0 \), \( D_{(y_1, \ldots, y_N)}Y(0, \ldots, 0, R_Z) \) is invertible with an inverse matrix of norm smaller than \( 2(c_1/3\sqrt{6})^{-2} \).

From the Implicit Function Theorem we deduce that there exists \( \beta_0 > 0 \) and \( C^1 \) functions \( (y_1, \ldots, y_N) \) from \( B_{H^2}(R_Z, \beta_0) \) to a neighborhood of \( (0, \ldots, 0) \) which are uniquely determined such that
\[ Y(y_1, \ldots, y_N, v) = 0, \quad \forall v \in B_{H^2}(R_Z, \beta_0). \]

In particular, there exists \( C_0 > 0 \) such that if \( v \in B_{H^2}(R_Z, \beta) \), with \( 0 < \beta \leq \beta_0 \), then
\[ \sum_{i=1}^{N} |y_i(v)| \leq C_0 \beta. \quad (4.12) \]

Note that \( \beta_0 \) and \( C_0 \) only depend on \( c_1 \) and \( L_0 \) and not on the point \((z_1, \ldots, z_N)\). For \( v \in B_{H^2}(R_Z, \beta_0) \) we set \( \tilde{x}_i(v) = z_i + y_i(v) \). Assuming that \( \beta_0 \leq L_0/8C_0 \), \((\tilde{x}_1(v), \ldots, \tilde{x}_N(v))\) are thus \( C^1 \) functions on \( B_{H^2}(R_Z, \beta) \) satisfying
\[ \tilde{x}_i(v) - \tilde{x}_{i-1}(v) = z_i - z_{i-1} + y_i(v) - y_{i-1}(v) > \frac{L}{2} - 2C_0 \beta > \frac{L}{4}. \quad (4.13) \]
For $L > L_0$ and $0 < \gamma < \gamma_0 < \beta_0/2$ to be chosen later, we define the modulation of $v$ in the following way: we cover the trajectory of $v$ by a finite number of open balls in the following way:

$$\{ v(t), t \in [0, t_0] \} \subset \bigcup_{k=1, \ldots, M} B_{H^2}(R_{Z_k}, 2\gamma).$$

This is possible thanks to Remark 4.1. It is worth noticing that, since $0 < \gamma < \gamma_0 < \beta_0/2$, the functions $\tilde{x}_i(v)$ are uniquely determined for $v \in B_{H^2}(R_{Z_k}, 2\gamma) \cap B_{H^2}(R_{Z_k'}, 2\gamma)$. We can thus define the functions $t \mapsto \tilde{x}_i(t)$ on $[0, t_0]$ by setting $\tilde{x}_i(t) = \tilde{x}_i(v(t))$. By construction

$$\int_\mathbb{R} \left( v(t, \cdot) - \sum_{j=1}^N \rho_{c_j}(\cdot - \tilde{x}_j(t)) \right) \partial_x \rho_{c_i}(\cdot - \tilde{x}_i(t)) = 0. \quad (4.14)$$

For $0 < \gamma < \gamma_0$, with $\gamma_0 \ll 1$, using that $u \in U(\gamma, L/2)$ and (4.12), we have

$$\| u(t) - S_{\tilde{x}_i(t)} \|_H$$

$$\leq \| u(t) - S_{\tilde{x}_i(t)} \|_H + \sum_{i=1}^N \| \varphi_{c_i}(\cdot - z_i) - \varphi_{c_i}(\cdot - z_i - y_i(v(t))) \|_{L^2(\mathbb{R})}$$

$$\leq \gamma + \sqrt{2} \sum_{i=1}^N \left( \int_{\mathbb{R}} \varphi_{c_i}^2(x) dx - \int_{\mathbb{R}} \varphi_{c_i}(x - z_i) \varphi_{c_i}(x - z_i - y_i(v(t))) dx \right)^{1/2}$$

$$= \gamma + \sqrt{2} \sum_{i=1}^N c_i \left( 1 - e^{-|y_i(v(t))|^2} - |y_i(v(t))| e^{-|y_i(v(t))|^2} \right)^{1/2}$$

$$\leq \gamma + \sum_{i=1}^N O(|y_i(v(t))|)$$

$$\leq O(\gamma),$$

where we apply two time the mean value theorem with the function $\varphi$ on $[0, |y_i(v(t))|]$ for substituting $(1 - e^{-|y_i(v(t))|^2})$ by $|y_i(v(t))| e^{-\theta}$, with $\theta \in [0, |y_i(v(t))|]$, and this proves (4.4) (see Fig. 2a-2b).

The estimate (4.5) follows directly by using (4.4), Remark 4.1 and the Sobolev embedding of $H^2(\mathbb{R})$ into $C^1(\mathbb{R})$.

To prove that the speed of $\tilde{x}_i(\cdot)$ stays close to $c_i$, we set

$$S_j(t) = \varphi_{c_j}(\cdot - \tilde{x}_j(t)), \; \varepsilon_1(t) = u(t) - \sum_{j=1}^N S_j(t)$$

and

$$R_j(t) = \rho_{c_j}(\cdot - \tilde{x}_j(t)), \; \varepsilon_2(t) = v(t) - \sum_{j=1}^N R_j(t).$$

One can notice that

$$\partial_x^2 R_i = 4 R_i - S_i, \quad (4.15)$$

and using the Fourier transformation

$$(1 - \partial_x^2)^{-1} (4 - \partial_x^2)^{-1} (\cdot) = \mathcal{F}^{-1} \left[ \frac{1}{(1 + \omega^2) (4 + \omega^2)} \right] (\cdot)$$

$$= \mathcal{F}^{-1} \left[ \frac{1}{3(1 + \omega^2)} - \frac{1}{3(4 + \omega^2)} \right] (\cdot)$$

$$= \frac{1}{3} (1 - \partial_x^2)^{-1} (\cdot) - \frac{1}{3} (4 - \partial_x^2)^{-1} (\cdot). \quad (4.16)$$

15
Differentiating (4.14) with respect to time and using (4.15), we get
\[
\int_\mathbb{R} \partial_t \varepsilon_2 \partial_x R_i = \dot{x}_i(t) \left( 4 \int_\mathbb{R} \varepsilon_2 R_i - \int_\mathbb{R} \varepsilon_2 S_i \right)
\]
and thus
\[
\left| \int_\mathbb{R} \partial_t \varepsilon_2 \partial_x R_i \right| \leq |\dot{x}_i(t)| \left( 4 \| \varepsilon_2 \|_{L^\infty(\mathbb{R})} \| R_i \|_{L^1(\mathbb{R})} + \| \varepsilon_2 \|_{L^\infty(\mathbb{R})} \| S_i \|_{L^1(\mathbb{R})} \right)
\]
\[
\leq |\dot{x}_i(t) - c_i| O(\gamma) + O(\gamma), \quad (4.17)
\]
we point out that \( \| S_i \|_{L^1(\mathbb{R})} = c \) and \( \| R_i \|_{L^1(\mathbb{R})} = c/2 \). Substituting \( u \) by \( \varepsilon_1 + \sum_{j=1}^N S_j \) in (1.7) and using that \( S_j \) satisfies
\[
\partial_t S_j = - (\dot{x}_j(t) - c_j) \partial_x S_j - \frac{1}{2} \partial_x S_j^2 - \frac{3}{2} (1 - \partial_x^2)^{-1} \partial_x S_j^2,
\]
we infer that \( \varepsilon_1 \) satisfies on \([0, t_0]\),
\[
\partial_t \varepsilon_1 - \sum_{j=1}^N (\dot{x}_j(t) - c_j) \partial_x S_j = - \frac{1}{2} \partial_x \left[ \left( \varepsilon_1 + \sum_{j=1}^N S_j \right)^2 - \sum_{j=1}^N S_j^2 \right]
\]
\[
- \frac{3}{2} \partial_x (1 - \partial_x^2)^{-1} \partial_x \left[ \left( \varepsilon_1 + \sum_{j=1}^N S_j \right)^2 - \sum_{j=1}^N S_j^2 \right].
\]
Multiplying by \( (4 - \partial_x^2)^{-1}(\cdot) \) and using (4.16), we get
\[
\partial_t \varepsilon_2 - \sum_{j=1}^N (\dot{x}_j(t) - c_j) \partial_x R_j = - \frac{1}{2} \partial_x (1 - \partial_x^2)^{-1} \left[ \left( \varepsilon_1 + \sum_{j=1}^N S_j \right)^2 - \sum_{j=1}^N S_j^2 \right].
\]
Taking the $L^2$ scalar product with $\partial_x R_i$, integrating by parts, we find

$$-\langle \dot{x}_i(t) - c_i \rangle \int_R (\partial_x R_i)^2 = -\int_R \partial_t \varepsilon_1 \partial_x R_i + \sum_{1 \leq j \leq N} \langle \dot{x}_j(t) - c_j \rangle \int_R (\partial_x R_i)(\partial_x R_j) + \frac{1}{2} \int_R (1 - \partial_x^2)^{-1} \left[ (\varepsilon_1 + \sum_{j=1}^N S_j) - \sum_{j=1}^N S_j^2 \right] \partial_x^2 R_i,$$

We set

$$Q = \left( \varepsilon_1 + \sum_{j=1}^N S_j \right)^2 - \sum_{j=1}^N S_j^2 = \varepsilon_1^2 + 2\varepsilon_1 \left( \sum_{j=1}^N S_j \right) + \sum_{1 \leq i, j \leq N} S_i S_j,$$

then

$$(1 - \partial_x^2)^{-1} Q = (1 - \partial_x^2)^{-1} \varepsilon_1^2 + 2 \sum_{j=1}^N (1 - \partial_x^2)^{-1} (\varepsilon_1 S_j) + \sum_{1 \leq i, j \leq N} (1 - \partial_x^2)^{-1} (S_i S_j) = I + J + K.$$

We have the following estimates

$$I = \frac{1}{2} \int_R e^{-|x'|} \varepsilon_1^2(x') \leq \frac{1}{2} \left\| \varepsilon_1^2 \right\|_{L^\infty(R)} \left\| \varepsilon_1 \right\|_{L^2(R)}^2 = \frac{1}{2} \left\| \varepsilon_1 \right\|_{L^2(R)}^2,$$

$$J = \sum_{j=1}^N \int_R e^{-|x'|} \varepsilon_1(x') S_j(x') \leq \left\| \varepsilon_1 \right\|_{L^\infty(R)} \left\| \varepsilon_1 \right\|_{L^2(R)} \sum_{j=1}^N \left\| S_j \right\|_{L^2(R)} = \left( \sum_{j=1}^N c_j \right) \left\| \varepsilon_1 \right\|_{L^2(R)}$$

and

$$K = \frac{1}{2} \sum_{1 \leq i, j \leq N} \int_R e^{-|x'|} S_j(x') S_i(x') \leq \frac{1}{2} \sum_{1 \leq i, j \leq N} \int_R S_j(x') S_i(x').$$

Thus, using (4.4) and the exponential decay of $S_j$, it holds

$$\| (1 - \partial_x^2)^{-1} Q \|_{L^\infty(R)} \leq O(\gamma) + O(e^{-L/4})$$

and then

$$\left| \frac{1}{2} \int_R [(1 - \partial_x^2)^{-1} Q] \partial_x^2 R_i \right| \leq \frac{1}{2} \left\| (1 - \partial_x^2)^{-1} Q \right\|_{L^\infty(R)} \left\| \partial_x^2 R_i \right\|_{L^1(R)} \leq O(\gamma) + O(e^{-L/4}),$$

where $\left\| \partial_x^2 R_i \right\|_{L^1(R)} = c/3$. Now, combining (4.17), (4.19), and using the exponential decay of $\partial_x R_i$, it holds

$$|\dot{x}_i(t) - c_i| \left\| \partial_x R_i \right\|_{L^2(R)}^2 \leq |\dot{x}_i(t) - c_i| O(\gamma) + O(\gamma) + O(e^{-L/4}),$$

then

$$|\dot{x}_i(t) - c_i| \left( \frac{c_i^2}{54} - O(\gamma) \right) \leq O(\gamma) + O(e^{-L/4}),$$
which yields (4.6).

Taking $0 < \gamma < \gamma_0$ and $L > L_0 > 0$ with $\gamma_0 \ll 1$ and $L_0 \gg 1$, combining (1.12)-(1.14), (4.6) and (4.13), we deduce that

$$\bar{x}_i(t) - \bar{x}_{i-1}(t) = \bar{x}_i(0) - \bar{x}_{i-1}(0) + (c_i - c_{i-1})t$$

$$\geq L - 2C_0\gamma_0 + \frac{(c_i - c_{i-1})t}{2}$$

$$\geq \frac{3L}{4} + \frac{(c_i - c_{i-1})t}{2},$$

this proves (4.7).

From (4.5), we infer that

$$v(x) = \sum_{j=1}^{N} \rho_{c_j} (x - \bar{x}_j) + O(\gamma), \quad \forall x \in \mathbb{R},$$

please note that we abuse notation by writing $\varepsilon_2(x) = O(\gamma)$. Applying this formula with $x = \xi_1$ and $v(\xi_1) = \max_{x \in J_i} v(x)$, and using (4.7), it holds

$$v(\xi_1) = \max_{x \in J_i} \left\{ \sum_{j=1}^{N} \rho_{c_j} (x - \bar{x}_j) \right\} + O(\gamma)$$

$$= \frac{c_i}{6} + O(e^{-L/4}) + O(\gamma)$$

$$\geq \frac{c_i}{7}. $$

On the other hand, for $x \in J_i \setminus [\bar{x}_i(t) - L/12, \bar{x}_i(t) + L/12]$, we get

$$v(x) \leq \frac{c_i}{3}e^{-L/12} + O(e^{-L/4}) + O(\gamma) \leq \frac{c_i}{8}.$$

This ensures that $\xi_1 \in [\bar{x}_i(t) - L/12, \bar{x}_i(t) + L/12]$, and this concluded the proof of the lemma.

**4.2 Monotonicity property**

Thanks to the preceding lemma, for $\varepsilon_0 > 0$ small enough and $L_0 > 0$ large enough, one can construct $N$ $C^1$ functions $\bar{x}_1, \ldots, \bar{x}_N$ defined on $[0, t_0]$ such that (4.4)-(4.8) are satisfied. In this subsection, we state the almost monotonicity of functionals that are very close to the energy at the right of $i$th bump, $i = 1, \ldots, N - 1$ of $u$ (respectively of $v$). Let $\psi$ be a $C^\infty$ test-function (see Fig. 3) such that

$$\begin{cases} 0 < \psi(x) < 1, \quad \psi'(x) > 0, & x \in \mathbb{R}, \\ |\psi^{(q)}(x)| \leq 10\psi'(x), & q = 2, 3, 4, 5, \quad x \in [-10, 10], \end{cases}$$

(4.20)

and

$$\psi(x) = \begin{cases} e^{-|x|}, & x < -10, \\ 1 - e^{-|x|}, & x > 10. \end{cases}$$

(4.21)

Setting $\psi_K = \psi(/K)$, we introduce for $i = 2, \ldots, N$,

$$\mathcal{J}_{i,K}(t) = \int_{\mathbb{R}} \left( 4v^2 + 5v_x^2 + v_{xx}^2 \right) \psi_{i,K}(t),$$

(4.22)
where \( \psi_{i,K}(t,x) = \psi_K(x - y_i(t)) \) with \( y_i \)’s as in (4.8). Note that \( J_{i,K}(t) \) is close to \( \|u(t)\|_{H(x>y_i(t))}^2 \) (respectively to \( \|v(t)\|_{H^2(x>y_i(t))}^2 \)) and thus measures the energy at the right of the \( (i-1) \)th bump of \( u \) (respectively of \( v \)). Finally, we set

\[
\sigma_0 = \frac{1}{4} \min \{ c_1, c_2 - c_1, \ldots, c_N - c_{N-1} \}. \tag{4.23}
\]

We have the following monotonicity result.

**Proposition 4.1** (Exponential Decay of the Functional \( J_{i,K}(t) \)). Let \( u \in X([0,T[) \), with \( 0 < T \leq +\infty \), be a solution of equation (1.1) that satisfies (1.12)-(1.14) and (4.4)-(4.5). There exist \( \gamma_0 > 0 \) and \( L_0 > 0 \) only depending on \( c_1 \) such that if \( 0 < \gamma < \gamma_0 \) and \( L > L_0 > 0 \), then for any \( 4 \leq K < \sqrt{L} \),

\[
\tau i,K(t) - J_{i,K}(0) \leq O(e^{-\frac{\tau}{L}}, \forall t \in [0,t_0], \ i = 2, \ldots, N. \tag{4.24}
\]

We prove Proposition 4.1 relies on the following Virial type identity.

**Lemma 4.2** (Virial Type Identity). Let \( u \in X([0,T[) \), with \( 0 < T \leq +\infty \), be a solution of equation (1.1) that satisfies (1.12)-(1.14). For any smooth space function \( g : \mathbb{R} \rightarrow \mathbb{R} \), it holds

\[
\frac{d}{dt} \int_{\mathbb{R}} (4v^2 + 5v'^2 + v''^2) g = 2 \int_{\mathbb{R}} u^3g' - 4 \int_{\mathbb{R}} u^2v g' + \frac{1}{2} \int_{\mathbb{R}} u^2vg''' + \frac{1}{2} \int_{\mathbb{R}} u^2v^2g'' + \int_{\mathbb{R}} uhg' + \frac{1}{2} \int_{\mathbb{R}} uh_2g'' - \frac{5}{2} \int_{\mathbb{R}} vh_2g'' = 2 \int_{\mathbb{R}} v_2h_2g'' + \frac{1}{2} \int_{\mathbb{R}} vh_2g(4) \tag{4.25}
\]

where \( y = (1 - \partial_x^2)u \), \( v = (4 - \partial_x^2)^{-1}u \) and \( h = (1 - \partial_x^2)^{-1}u^2 \).

The full proof of Lemma 4.2 is given in the Appendix 4.4.

**Proof of Proposition 4.1.** We first note that, combining (4.6) and (4.8), it holds for \( i = 2, \ldots, N \),

\[
\dot{y}_i(t) = \frac{\dot{x}_{i-1}(t) + \dot{x}_i(t)}{2} = \frac{c_{i-1} + c_i}{2} + O(\gamma) \geq c_{i-1} + O(\gamma) \geq \frac{c_1}{2}. \tag{4.26}
\]
Recall that the assumption (1.12) ensures that $u \geq 0$ and $v \geq 0$ on $\mathbb{R}$. Now, applying the Virial type identity (4.25) with $g = \psi_{i,K}$ and using (4.26), we get

\[
\frac{d}{dt} J_{i,K}(t) = - \hat{y}_i \int_{\mathbb{R}} \left( 4v^2 + 5v_x^2 + v_{xx}^2 \right) \psi'_{i,K} + \frac{2}{3} \int_{\mathbb{R}} u^3 \psi_{i,K} - 4 \int_{\mathbb{R}} u^2 v \psi'_{i,K} \\
\quad - \frac{1}{2} \int_{\mathbb{R}} u^2 v \psi''_{i,K} + \frac{1}{3} \int_{\mathbb{R}} u^2 v \psi''_{i,K} + \int_{\mathbb{R}} u h \psi'_{i,K} + \frac{1}{2} \int_{\mathbb{R}} u h_{x} \psi''_{i,K} \\
\quad - \frac{5}{2} \int_{\mathbb{R}} v h_{x} \psi''_{i,K} - 2 \int_{\mathbb{R}} v \psi''_{i,K} + \frac{1}{2} \int_{\mathbb{R}} v h_{x} \psi''_{i,K} \\
\leq - \hat{y}_i \int_{\mathbb{R}} \left( 4v^2 + 5v_x^2 + v_{xx}^2 \right) \psi'_{i,K} + \frac{2}{3} \int_{\mathbb{R}} u^3 \psi_{i,K} - \frac{1}{2} \int_{\mathbb{R}} u^2 v \psi''_{i,K} \\
\quad + \frac{1}{2} \int_{\mathbb{R}} u^2 v \psi''_{i,K} + \int_{\mathbb{R}} u h \psi'_{i,K} + \frac{1}{2} \int_{\mathbb{R}} u h_{x} \psi''_{i,K} - \frac{5}{2} \int_{\mathbb{R}} v h_{x} \psi''_{i,K} \\
\quad - 2 \int_{\mathbb{R}} v \psi''_{i,K} + \frac{1}{2} \int_{\mathbb{R}} v h_{x} \psi''_{i,K} \\
\leq - \frac{c_1}{2} \int_{\mathbb{R}} \left( 4v^2 + 5v_x^2 + v_{xx}^2 \right) \psi'_{i,K} + \sum_{k=1}^{8} J_k. 
\]

(4.27)

We claim that for $k = 1, \ldots, 8$, it holds

\[
J_k \leq \frac{c_1}{20} \int_{\mathbb{R}} \left( 4v^2 + 5v_x^2 + v_{xx}^2 \right) \psi'_{i,K} + \frac{C}{K} \|u_0\|_\mathcal{H}^3 e^{-\frac{1}{8} (\sigma_0 t + L/8)}. 
\]

(4.28)

We divide $\mathbb{R}$ into two regions $D_i$ and $D_i^c$ with

\[
D_i = \left[ \bar{x}_{i-1}(t) + \frac{L}{4}, \bar{x}_i(t) - \frac{L}{4} \right], \quad i = 2, \ldots, N.
\]

Combining (4.7) and (4.8), one can check that for $x \in D_i^c$,

\[
| x - y_i(t) | \geq \frac{\bar{x}_i(t) - \bar{x}_{i-1}(t)}{2} - \frac{L}{4} \\
\quad \geq \frac{c_i - c_{i-1}}{4} t + \frac{L}{8} \\
\quad \geq \sigma_0 t + \frac{L}{8}. 
\]

(4.29)

Let us begin by an estimate of $J_1$. Using (2.3), (4.29) and the exponential decay of $\psi'_{i,K}$ on $D_i^c$, we get

\[
\frac{2}{3} \int_{\mathbb{R}} u^3 \psi'_{i,K} = \frac{2}{3} \int_{D_i} u^3 \psi'_{i,K} + \frac{2}{3} \int_{D_i^c} u^3 \psi'_{i,K} \\
\quad \leq \frac{2}{3} \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} u^2 \psi'_{i,K} + \frac{2}{3} \|\psi'_{i,K}\|_{L^\infty(D_i^c)} \|u\|_{L^\infty(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}^2 \\
\quad \leq \frac{2}{3} \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} u^2 \psi'_{i,K} + \frac{C}{K} \|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{8} (\sigma_0 t + L/8)}. 
\]

(4.30)

Note that, using the exponential decay of $|\psi''_{j,K}|$ on $D_i^c$, $|\psi''_{i,K}| \leq (10/K^2) \psi_{i,K}$ on $D_i$, with $K \geq 4$, and
that \( \|u\|_H = \|u_0\|_H \), we have

\[
\int \psi_i' v_{i,K}^2 = \int (4v - v_{xx})^2 \psi_i' K
\]

\[
= 16 \int v^2 \psi_i' K + \int v_x^2 \psi_i' K - 8 \int v v_x \psi_i' K
\]

\[
= 16 \int v^2 \psi_i' K + \int v_x^2 \psi_i' K + 4 \int v_x \psi_i'' K
\]

\[
= 16 \int v^2 \psi_i' K + \int v_x^2 \psi_i' K + 4 \int v_x \psi_i'' K
\]

\[
\leq \int (16v^2 + 8v_x^2 + v_x) \psi_i' K + 4 \int v^2 |\psi_i'' K| + 4 \int v_x |\psi_i'' K|
\]

\[
\leq 5 \int (4v^2 + 5v_x^2 + v_x) \psi_i' K + \frac{40 K^2}{K} \int \psi_i' K + \frac{C}{K} \|u_0\|_2^2 \|u\|^2 L^{-1} e^{-\frac{\gamma_0 t + 1}{8}}.
\]

(4.31)

Now, using the exponential decay of \( \varphi_c \) on \( D_i \), (4.4), and proceeding as for the estimate (3.10) (see Lemma 3.2), it holds

\[
\|u\|_{L^\infty(D_i)} \leq \left\| u - \sum_{j=1}^N \varphi_j (\cdot - \bar{x}_j(t)) \right\|_{L^\infty(D_i)} + \sum_{j=1}^N \|\varphi_j (\cdot - \bar{x}_j(t))\|_{L^\infty(D_i)}
\]

\[
\leq O(\gamma^{1/2}) + O(e^{-L t/8}).
\]

(4.32)

Therefore, for \( 0 < \gamma < \gamma_0 \) and \( L > L_0 > 0 \), with \( \gamma_0 \ll 1 \) and \( L_0 \gg 1 \), combining (4.30)-(4.31), we obtain

\[
J_1 \leq \frac{c_1}{20} \int \left( 4v^2 + 5v_x^2 + v_x^2 \right) \psi_i' K + \frac{C}{K} \|u\|_2 L^{-1} e^{-\frac{\gamma_0 t + 1}{8}}.
\]

Next, the estimate of \( J_2 \) on \( D_i \) gives us

\[
\left| \frac{1}{2} \int D_i \left( 4v^2 + 5v_x^2 + v_x^2 \right) \psi_i' K \right| \leq \frac{1}{2} \|\psi_i'' K\|_{L^\infty(D_i)} \|v\|_{L^\infty(\mathbb{R})} \|u\|^2_{L^2(\mathbb{R})}.
\]

Note that, applying the Hölder inequality, we have for all \( x \in \mathbb{R} \),

\[
v(x) = \frac{1}{4} \int e^{-2|x-x'|} u(x') dx'
\]

\[
\leq \frac{1}{4} \left( \int e^{-4|x-x'|} dx' \right)^{1/2} \left( \int |u(x')| dx' \right)^{1/2}
\]

\[
= \frac{1}{4\sqrt{2}} \|u\|_{L^2(\mathbb{R})}.
\]

(4.33)

and thus, using (4.33) and the exponential decay of \( |\psi_i'' K| \) on \( D_i \), it holds

\[
\left| \frac{1}{2} \int D_i \left( 4v^2 + 5v_x^2 + v_x^2 \right) \psi_i' K \right| \leq \frac{1}{8\sqrt{2}K^3} \|u\|^3_{L^2(\mathbb{R})} e^{-\frac{\gamma_0 t + 1}{8}}.
\]

(4.34)

Using that \( |\psi_i'' K| \leq (10/K^2) \psi_i' K \) on \( D_i \), the estimate of \( J_2 \) on \( D_i \) leads to

\[
\left| \frac{1}{2} \int D_i \left( 4v^2 + 5v_x^2 + v_x^2 \right) \psi_i' K \right| \leq \frac{5}{K^2} \|u\|_{L^\infty(D_i)} \int \psi_i' K.
\]

(4.35)
Also, one can notice that, using the exponential decay of $|\psi''_{i,K}|$ on $D_c^\varepsilon$, $|\psi'''_{i,K}| \leq (10/K^2)\psi''_{i,K}$ on $\mathbb{R}$, with $K \geq 4$, and that $\|u\|_H = \|u_0\|_H$, we have

$$
\int_R uv\psi'_{i,K} = \int_R (4v - v_{xx})v\psi'_{i,K}
= 4 \int_R v^2\psi'_{i,K} + \int_R \partial_x(vv'_{i,K})v_x
= 4 \int_R v^2\psi'_{i,K} + \int_R v^2\psi''_{i,K} + \int_R vuv\psi''_{i,K}
= 4 \int_R (v_0^2 + v_x^2)\psi'_{i,K} - \frac{1}{2} \int_R v^2\psi'''_{i,K}
\leq \int_R (v_0^2 + v_x^2)\psi'_{i,K} + \frac{1}{2} \int_{D_c^\varepsilon^*} v^2|\psi'''_{i,K}| + \frac{1}{2} \int_{D_c^\varepsilon^*} v^2|\psi'''_{i,K}|
\leq \int_R (v_0^2 + v_x^2)\psi'_{i,K} + \frac{5}{K^2} \int_\mathbb{R} v^2\psi'_{i,K} + \frac{C}{K^2} \|u_0\|^2 e^{-\frac{1}{4}(\sigma_0 t + L/8)}
\leq 2 \int_R (v_0^2 + v_x^2)\psi'_{i,K} + \frac{C}{K^2} \|u_0\|^2 e^{-\frac{1}{4}(\sigma_0 t + L/8)}. \tag{4.36}
$$

Therefore, for $0 < \gamma < \gamma_0$ and $L > L_0 > 0$, with $\gamma_0 \ll 1$ and $L_0 \gg 1$, combining (4.32), (4.34)-(4.36), it holds

$$J_2 \leq \frac{c_1}{20} \int_R (4v^2 + v_x^2 + v_{xx}^2)\psi'_{i,K} + \frac{C}{K^2} \|u_0\|^3 e^{-\frac{1}{4}(\sigma_0 t + L/8)}. \tag{4.38}
$$

In the same way, using that $|v_x| \leq 2v$ on $\mathbb{R}$ (see (3.39)), and the definition of $\psi_{i,K}$ (see (4.20) and (4.21)), we deduce the estimate of $J_3$.

Let us tackle now the estimate of $J_4$. On $D_c^\varepsilon$ we have

$$
\int_{D_c^\varepsilon} uh\psi'_{i,K} \leq \|\psi'_{i,K}\|_{L^\infty(D_c^\varepsilon)} \int_R uh
= \|\psi'_{i,K}\|_{L^\infty(D_c^\varepsilon)} \int_R u[(1 - \partial_x^2)^{-1}u^2]
= \|\psi'_{i,K}\|_{L^\infty(D_c^\varepsilon)} \int_R u^2[(1 - \partial_x^2)^{-1}u]
\leq \|\psi'_{i,K}\|_{L^\infty(D_c^\varepsilon)} \|1 - \partial_x^2\|_{L^\infty(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}^2.
$$

Remark that, applying the Hölder inequality, we have for all $x \in \mathbb{R}$,

$$
(1 - \partial_x^2)^{-1}u(x) \leq \frac{1}{2} \int_\mathbb{R} e^{-|x-x'|} |u(x')|dx'
\leq \frac{1}{2} \left( \int_\mathbb{R} e^{-2|x-x'|} dx' \right)^{1/2} \left( \int_\mathbb{R} |u(x')|^2 dx' \right)^{1/2}
= \frac{1}{2} \|u\|_{L^2(\mathbb{R})} \tag{4.37}
$$
and thus, using (4.37) and the exponential decay of $\psi'_{i,K}$ on $D_c^\varepsilon$, it holds

$$
\int_{D_c^\varepsilon} uh\psi'_{i,K} \leq \frac{1}{2K} \|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{4}(\sigma_0 t + L/8)}. \tag{4.38}
$$

22
The estimate of $J_4$ on $D_t$ leads to
\[
\int_{D_t} uh_\psi'_{i,K} \leq \|u\|_{L^\infty(D_t)} \int_{\mathbb{R}} \psi'_{i,K} [(1 - \partial_x^2)^{-1} u^2] = \|u\|_{L^\infty(D_t)} \int_{\mathbb{R}} u^2 [(1 - \partial_x^2)^{-1} \psi'_{i,K}].
\] (4.39)

On the other hand, using that $|\psi''_{i,K}| \leq (10/K^2)\psi'_{i,K}$ on $\mathbb{R}$, we have
\[
(1 - \partial_x^2)\psi'_{i,K}(x) = \psi'_{i,K}(x) - \psi''_{i,K}(x) \geq \left(1 - \frac{10}{K^2}\right)\psi'_{i,K}(x), \forall x \in \mathbb{R},
\]
and since $K \geq 4$, it holds
\[
(1 - \partial_x^2)^{-1}\psi'_{i,K}(x) \leq \left(1 - \frac{10}{K^2}\right)^{-1}\psi'_{i,K}(x), \forall x \in \mathbb{R}.
\] (4.40)

Therefore, for $0 < \gamma < \gamma_0$ and $L > L_0 > 0$, with $\gamma_0 \ll 1$ and $L_0 \gg 1$, combining (4.32), (4.38)-(4.40), it holds
\[
J_4 \leq C_4 \sum_{i=1}^{N} \int_{\mathbb{R}} (4u_x^2 + 5v_x^2 + v^2_{xx}) \psi_{i,K} + \frac{C}{K} \|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{3}(\sigma_0 t + L/8)}.
\]

Noticing that for all $x \in \mathbb{R}$,
\[
h(x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^{x'} u^2(x') dx' + \frac{e^{x}}{2} \int_{x}^{+\infty} e^{-x'} u^2(x') dx',
\]
and
\[
h_x(x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^{x'} u^2(x') dx' + \frac{e^{x}}{2} \int_{x}^{+\infty} e^{-x'} u^2(x') dx',
\]
we infer that
\[
|h_x(x)| \leq h(x), \forall x \in \mathbb{R}.
\] (4.41)

Then, combining (4.20), (4.41), and proceeding as for the estimate of $J_4$, we deduce the estimate of $J_5$.

Now, combining (4.33) and (4.37), we have for all $x \in \mathbb{R}$,
\[
(1 - \partial_x^2)^{-1}v(x) = \frac{1}{3} (1 - \partial_x^2)^{-1}u(x) - \frac{1}{3} v(x)
\leq \frac{1}{3} \|1 - \partial_x^2\|^{-1}_L u \|L^\infty(\mathbb{R})\| + \frac{1}{3} \|v\|_{L^\infty(\mathbb{R})}
\leq \frac{4 + \sqrt{2}}{24} \|u\|_{L^2(\mathbb{R})},
\] (4.42)

and using the exponential decay of $\rho_{c_i}$ on $D_t$ and (4.5), it holds
\[
\|v\|_{L^\infty(D_t)} \leq \left\|v - \sum_{j=1}^{N} \rho_{c_j}(\cdot - \tilde{x}_j(t))\right\|_{L^\infty(D_t)} + \sum_{j=1}^{N} \|\rho_{c_j}(\cdot - \tilde{x}_j(t))\|_{L^\infty(D_t)}
\leq O(\gamma) + O(e^{-L/8}).
\] (4.43)

Therefore, combining (3.39), (4.20), (4.41)-(4.43), and proceeding as for the estimate of $J_4$, we deduce the estimates of the remaining terms.

Finally, combining (4.27), (4.28) and using that $\|u\|_{L^2(\mathbb{R})} \sim \|u_0\|_{\mathcal{H}}$, it holds actually
\[
\frac{d}{dt} \mathcal{J}_{i,K}(t) \leq \frac{C}{K} \|u_0\|_{\mathcal{H}}^3 e^{-\frac{1}{3}(\sigma_0 t + L/8)}.
\]
Integrating between 0 and \( t \), we obtain
\[
\mathcal{J}_{i,K}(t) - \mathcal{J}_{i,K}(0) \leq \frac{C}{K} \| u_0 \|^2 \left( -\frac{K}{\sigma_0} e^{-\frac{t}{\sigma_0} (\sigma_0 t + L/8)} + \frac{K}{\sigma_0} e^{-\frac{t}{\sigma_0}} \right)
\]
\[
\leq \frac{C}{\sigma_0} \| u_0 \|^2 e^{-\frac{t}{\sigma_0}},
\]
and this proves the proposition for smooth initial solutions.

For \( u \in \mathcal{X}([0,T]) \), we will use that for any \( T_0 > 0 \) and any sequence \((u_{0,n})_{n \geq 1} \subset L^2(\mathbb{R})\) such that \((u_{0,n} - \partial_x^2 u_{0,n})_{n \geq 1} \subset M^+(\mathbb{R})\) and \( u_{0,n} \to u_0 \) in \( L^2(\mathbb{R}) \), the sequence of emanating global weak solutions \((u_n)_{n \geq 1}\) to the DP equation satisfies
\[
\lim_{n \to +\infty} u_n \to u \quad \text{in} \quad C\left([0,T_0]; L^2(\mathbb{R})\right),
\]  
(4.44)
where \( u \) is the global weak solution emanating from \( u_0 \). This fact can be easily deduced from the proof of the existence of the global weak solutions in [4]. Indeed, by the same arguments developed in this proof, we obtain that, up to a subsequence, \((u_n)_{n \geq 1}\) converges in \( C\left([0,T_0]; L^2(\mathbb{R})\right)\) towards a solution of the DP equation emanating from \( u_0 \). (4.44) then follows by the uniqueness result. Combining (4.44) and Remark 4.1, it follows that
\[
\lim_{n \to +\infty} \sup_{0 \leq t < T} \left| \mathcal{J}_{i,K}(t) - \mathcal{J}_{i,K}(t) \right| = 0.
\]  
(4.47)
Let \( t \in [0,T] \) be fixed, we compute
\[
\mathcal{J}_{i,K}^n(t) - \mathcal{J}_{i,K}(t) = 4 \int_{\mathbb{R}} (v_n^2 - v^2) \psi_{i,K}(t) + 5 \int_{\mathbb{R}} (v_n^2 - v^2) \psi_{i,K}(t) + 5 \int_{\mathbb{R}} (v_{n,xx}^2 - v_{xx}^2) \psi_{i,K}(t)
\]
\[
= K_1^n(t) + K_2^n(t) + K_3^n(t).
\]  
(4.48)
Then it is easy to check that
\[
|K_1^n(t)| \leq 4 \int_{\mathbb{R}} |v_n - v| (v_n + v) \psi_{i,K}(t)
\]
\[
= 4\|v_n - v\|_{L^2(\mathbb{R})} \|v_n + v\|_{L^2(\mathbb{R})} \|\psi_{i,K}\|_{L^\infty(\mathbb{R})}
\]
\[
\leq O(\|v_n - v\|_{L^2(\mathbb{R})})
\]
\[
\to 0 \quad \text{as} \quad n \to +\infty,
\]  
(4.49)
and
\[
|K_2^n(t)| \leq 5 \int_{\mathbb{R}} |v_{n,x} - v_x| \psi_{i,K}(t)
\]
\[
= 4\|v_{n,x} - v_x\|_{L^2(\mathbb{R})} \|v_{n,x} + v_x\|_{L^2(\mathbb{R})} \|\psi_{i,K}\|_{L^\infty(\mathbb{R})}
\]
\[
\leq O(\|v_{n,x} - v_x\|_{L^2(\mathbb{R})})
\]
\[
\to 0 \quad \text{as} \quad n \to +\infty.
\]  
(4.50)
Recalling that $v_{xx} = 4v - u$ and thus $v_{xx}^2 = 16v^2 + u^2 - 8uv$, we also get

$$|K_2^n(t)| \leq 16 \int_{\mathbb{R}} |v_n - v| (v_n + v) \psi_{1,K}(t) + \int_{\mathbb{R}} |u_n - u| (u_n + u) \psi_{1,K}(t)$$

$$+ 8 \int_{\mathbb{R}} u_n |v_n - v| \psi_{1,K}(t) + 8 \int_{\mathbb{R}} |v_n - v| \psi_{1,K}(t)$$

$$\leq 16 \|v_n - v\|_{L^2(\mathbb{R})} \|v_n + v\|_{L^2(\mathbb{R})} \|\psi_{1,K}\|_{L^\infty(\mathbb{R})}$$

$$+ \|u_n - u\|_{L^2(\mathbb{R})} \|u_n + u\|_{L^2(\mathbb{R})} \|\psi_{1,K}\|_{L^\infty(\mathbb{R})}$$

$$+ 8 \|v\|_{L^2(\mathbb{R})} \|u_n - u\|_{L^2(\mathbb{R})} \|\psi_{1,K}\|_{L^\infty(\mathbb{R})}$$

$$\leq O(\|u_n - u\|_{L^2(\mathbb{R})}) + O(\|v_n - v\|_{L^2(\mathbb{R})})$$

$$\to 0 \text{ as } n \to +\infty.$$  \hfill (4.51)

Combining (4.48)-(4.51), we obtain (4.47).

Thanks to (4.47), the monotonicity formula (4.24) holds for any $u \in X([0,T])$, with $0 < T \leq +\infty$. \hfill \Box

### 4.3 A localized and a global estimate

Let $K = \sqrt{L}/8$ and define the function $\phi_i = \phi_i(t,x)$ (see Fig. 4) by

$$\begin{align*}
\phi_1 &= 1 - \psi_{2,K} = 1 - \psi_K(\cdot - y_2(t)), \\
\phi_i &= \psi_{i,K} - \psi_{i+1,K} = \psi_K(\cdot - y_i(t)) - \psi_K(\cdot - y_{i+1}(t)), \quad i = 2, \ldots, N - 1, \\
\phi_N &= \psi_{N,K} = \psi_K(\cdot - y_N(t)),
\end{align*}$$  \hfill (4.52)

where $\psi_{i,K}$’s and $y_i$’s are defined in Subsection 4.2. One can see that the $\phi_i$’s are positive functions and that $\sum_{i=1}^{N} \phi_i = 1$. We take $L/K > 4$ so that $\phi_i$ satisfies for $i = 1, \ldots, N$,

$$|1 - \phi_i| \leq 2e^{-\frac{y_i}{8K}} \text{ on } \left[y_i + \frac{L}{8}, y_{i+1} - \frac{L}{8}\right],$$  \hfill (4.53)

and

$$|\phi_i| \leq 2e^{-\frac{y_i}{8K}} \text{ on } \mathbb{R} \setminus \left[y_i - \frac{L}{8}, y_{i+1} + \frac{L}{8}\right].$$  \hfill (4.54)

We will use the following localized version of the conservation laws defined for $i = 1, \ldots, N$ by

$$E_i(t) = \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i(t) \quad \text{and} \quad F_i(t) = \int_{\mathbb{R}} (-v_{xx}^3 + 12v v_{xx}^2 - 48v^2 v_x^2 + 64v^3) \phi_i(t).$$  \hfill (4.55)

One can remark that the functional $E_i(\cdot)$ and $F_i(\cdot)$ do not depend on time in the statement below since we fix $-\infty = y_1 < y_2 < \ldots < y_N < y_{N+1} = +\infty$.

For $i = 1, \ldots, N$, we set $\Omega_i = [y_i - L/8, y_{i+1} + L/8]$. First, one can notice that

$$\sum_{j=1}^{N} \rho_{\iota_i}(x - \tilde{x}_j) = \rho_{\iota_i}(x - \tilde{x}_i) + O(e^{-L/4}), \quad \forall x \in \Omega_i,$$  \hfill (4.56)

we abuse notation by writing $\rho_{\iota_i}(x - \tilde{x}_i) = O(e^{-L/4})$ for all $x \in \mathbb{R} \setminus \Omega_i$. We will now decompose this interval according to the variation of $v = (4 - \partial_x^2)^{-1} u$ in the same way as in Section 3. We set

$$\alpha_i = \sup \left\{ x < \tilde{x}_i, \ v(x) = \frac{c_i}{2400} \right\} \quad \text{and} \quad \beta_i = \inf \left\{ x > \tilde{x}_i, \ v(x) = \frac{c_i}{2400} \right\}.$$  \hfill (4.57)
According to Lemma 4.1, we know that $v$ is close to $\sum_{i=1}^{N} \rho_{c}(\cdot - \xi_{i})$ in $L^{\infty}$ norm with $\rho_{c}(0) = c_{i}/6$. Therefore $v$ must have at least one local maximum on $[\alpha_{i}, \beta_{i}]$. Assume that on $[\alpha_{i}, \beta_{i}]$ the function $v$ admits $k_{i} + 1$ points $(\xi_{j})_{j=1}^{k_{i}+1}$ with local maximal values for some integer $k_{i} \geq 0$, where $\xi_{1}$ is the first local maximum point and $\xi_{k_{i}+1}$ the last local maximum point. Then between $\xi_{1}$ and $\xi_{k_{i}+1}$, the function $v$ admits $k_{i}$ points $(\eta_{j})_{j=1}^{k_{i}}$ with local minimal values. We rename $\alpha_{i} = \eta_{0}$ and $\beta_{i} = \eta_{k_{i}+1}$ so that it holds

$$\eta_{0} < \xi_{1} < \eta_{1} < \ldots < \xi_{j} < \eta_{j} < \xi_{j+1} < \ldots < \eta_{k_{i}} < \xi_{k_{i}+1} < \eta_{k_{i}+1}. \quad (4.58)$$

Let

$$M_{j} = v(\xi_{j}), \quad j = 1, \ldots, k_{i} + 1, \quad \text{and} \quad m_{j} = v(\eta_{j}), \quad j = 1, \ldots, k_{i}. \quad (4.59)$$

By construction

$$v_{x}(x) \geq 0, \quad \forall x \in [\eta_{j-1}, \xi_{j}], \quad j = 1, \ldots, k_{i} \quad (4.60)$$

and

$$v_{x}(x) \leq 0, \quad \forall x \in [\xi_{j}, \eta_{j}], \quad j = 1, \ldots, k_{i} + 1. \quad (4.61)$$

Proceeding as for (3.18)-(3.20), we also have

$$v(x) \leq \frac{c_{i}}{300} \quad \forall x \in \Omega_{i} \setminus [\eta_{0}, \eta_{k_{i}+1}], \quad (4.62)$$

and

$$u(x) \leq \frac{c_{i}}{300} \quad \forall x \in \Omega_{i} \setminus [\eta_{0}, \eta_{k_{i}+1}], \quad (4.63)$$

and taking $L > L_{0} > 8C_{0}$, it holds

$$[\eta_{0}, \eta_{k_{i}+1}] \subset [\bar{x}_{i} - C_{0}, \bar{x}_{i} + C_{0}] \subset \left[ y_{i} + \frac{L}{8}, y_{i+1} - \frac{L}{8} \right], \quad (4.64)$$

where $C_{0} > 0$ is the universal constant appearing in (3.20).

We now derive versions of Lemma 3.4, Lemma 3.5 and Lemma 3.6 where the global functional $E(\cdot)$ and $F(\cdot)$ are replaced by their localized versions $E_{i}(\cdot)$ and $F_{i}(\cdot)$. Please note that, we will change the order of the extrema of $v = (4 - \partial_{x}^{2})^{-1}u$ while keeping the same notations as in (4.59).

---

Figure 4: Localization-function $\phi_{\text{green}}(x) = \psi(x - 15) - \psi(x - 65)$ (at time $t = 10$) profile. Also, the peakon $4\varphi(x - 40)$ and the smooth-peakon $4\rho(x - 40)$ (at time $t = 10$ with speed $c = 4$) profiles. In this example, one can see that $\phi_{\text{green}}$ is close to 1 on $[25, 55]$, and decays exponentially to 0 on $\mathbb{R} \setminus [10, 70]$.

---

3In the case of an infinite countable number of local maximal values, the proof is exactly the same.
Lemma 4.3 (Connection Between $E_i(\cdot)$ and the Local Extrema of $v$). Let $u \in H^1(\mathbb{R})$ and $v = (4 - \partial^2_x)^{-1}u \in H^3(\mathbb{R})$. For $i = 1, \ldots, N$, define the function $g_i$ by

$$
g_i(x) = \begin{cases} 
2v + v_{xx} - 3v_x, & x < \xi_i^1, \\
2v + v_{xx} + 3v_x, & \xi_i^1 < x < \eta_i^j, \\
2v + v_{xx} - 3v_x, & \eta_i^j < x < \xi_{i+1}^j, \\
2v + v_{xx} + 3v_x, & x > \xi_{k_i+1}^1,
\end{cases}
$$

(4.65)

Then it holds

$$
\int_{\mathbb{R}} g_i^2(x)\phi_i(x)dx = E_i(u) - 12 \left( \sum_{j=0}^{k_i} (M_j^i)^2\phi_i(\xi_{j+1}^i) - \sum_{j=1}^{k_i} (m_j^i)^2\phi_i(\eta_j^i) \right) + \|u\|^2_{L^2}O(L^{-1/2}).
$$

(4.66)

Proof. We have

$$
\int_{\mathbb{R}} g_i^2(x)\phi_i(x)dx = \int_{-\infty}^{\xi_i^1} g_i^2(x)\phi_i(x)dx + \int_{\xi_i^1}^{\eta_i^j} g_i^2(x)\phi_i(x)dx + \int_{\eta_i^j}^{\xi_{i+1}^j} g_i^2(x)\phi_i(x)dx + \int_{\xi_{i+1}^j}^{+\infty} g_i^2(x)\phi_i(x)dx.
$$

(4.67)

For $j = 1, \ldots, k_i$,

$$
\int_{\xi_i^j}^{\xi_{i+1}^j} g_i^2(x)\phi_i(x)dx = \int_{\xi_i^j}^{\eta_i^j} (2v + v_{xx} + 3v_x)^2\phi_i(x)dx + \int_{\eta_i^j}^{\xi_{i+1}^j} (2v + v_{xx} - 3v_x)^2\phi_i(x)dx
$$

$$
= J + I.
$$

Computing $I$, we obtain

$$
I = \int_{\eta_i^j}^{\xi_{i+1}^j} \left( 4v^2 + v_{xx}^2 + 9v_x^2 + 4vv_{xx} - 12vv_x - 6v_xv_{xx} \right)\phi_i
$$

$$
= \int_{\eta_i^j}^{\xi_{i+1}^j} (4v^2 + v_{xx}^2 + 9v_x^2)\phi_i + 4 \int_{\eta_i^j}^{\xi_{i+1}^j} vv_{xx}\phi_i - 12 \int_{\eta_i^j}^{\xi_{i+1}^j} vv_x\phi_i - 6 \int_{\eta_i^j}^{\xi_{i+1}^j} v_xv_{xx}\phi_i
$$

$$
= \int_{\eta_i^j}^{\xi_{i+1}^j} (4v^2 + v_{xx}^2 + 9v_x^2)\phi_i + I_1 + I_2 + I_3
$$

(4.68)

with

$$
I_1 = -4 \int_{\eta_i^j}^{\xi_{i+1}^j} \partial_x(v\phi_i)v_x = -4 \int_{\eta_i^j}^{\xi_{i+1}^j} v_x^2\phi_i - 4 \int_{\eta_i^j}^{\xi_{i+1}^j} vv_x\phi_i' = -4 \int_{\eta_i^j}^{\xi_{i+1}^j} v_x^2\phi_i - 2 \int_{\eta_i^j}^{\xi_{i+1}^j} (v^2)_x\phi_i'
$$

$$
= -2v^2(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) + 2v^2(\eta_j^i)\phi_i'(\eta_j^i) - 4 \int_{\eta_i^j}^{\xi_{i+1}^j} v_x^2\phi_i + 2 \int_{\eta_i^j}^{\xi_{i+1}^j} v_x^2\phi_i''
$$

(4.69)

$$
I_2 = -6 \int_{\eta_i^j}^{\xi_{i+1}^j} \partial_x(v^2)\phi_i = -6v^2(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) + 6v^2(\eta_j^i)\phi_i(\eta_j^i) + 6 \int_{\eta_i^j}^{\xi_{i+1}^j} v^2\phi_i'
$$

(4.70)

and

$$
I_3 = -3 \int_{\eta_i^j}^{\xi_{i+1}^j} \partial_x(v_x^2)\phi_i = 3 \int_{\eta_i^j}^{\xi_{i+1}^j} v_x^2\phi_i'.
$$

(4.71)
Adding (4.68)-(4.71), we get

\[ I = \int_{\eta_j}^{\xi_{j+1}} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i - 6v^2(\xi_{j+1}) \phi_i(\xi_{j+1}) + 6v^2(\eta_j) \phi_i(\eta_j) \]
\[ - 2v^2(\xi_{j+1}) \phi_i'(\xi_{j+1}) + 2v^2(\eta_j) \phi_i'(\eta_j) + R_1, \]  

(4.72)

where using that \( K = \sqrt{L}/8 \), we have

\[ |R_1| \leq 6(\|\phi_i\|_{L^\infty(\mathbb{R})} + \|\phi_i''\|_{L^\infty(\mathbb{R})}) \int_{\eta_j}^{\xi_{j+1}} (v^2 + v_x^2) \leq O(L^{-1/2}) \int_{\eta_j}^{\xi_{j+1}} (v^2 + v_x^2). \]

Similar computations lead to

\[ J = \int_{\eta_j}^{\xi_{j+1}} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i - 6v^2(\xi_{j+1}) \phi_i(\xi_{j+1}) + 6v^2(\eta_j) \phi_i(\eta_j) \]
\[ + 2v^2(\xi_{j+1}) \phi_i'(\xi_{j+1}) - 2v^2(\eta_j) \phi_i'(\eta_j) + R_2, \]  

(4.73)

\[ \int_{-\infty}^{\xi_{j+1}} g^2(x) \phi_i(x) dx = \int_{-\infty}^{\xi_{j+1}} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i - 6v^2(\xi_{j+1}) \phi_i(\xi_{j+1}) - 2v^2(\xi_j) \phi_i'(\xi_j) + R_3 \]  

(4.74)

and

\[ \int_{\xi_{j+1}}^{+\infty} g^2(x) \phi_i(x) dx = \int_{\xi_{j+1}}^{+\infty} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i - 6v^2(\xi_{j+1}) \phi_i(\xi_{j+1}) + 2v^2(\xi_{j+1}) \phi_i'(\xi_{j+1}) + R_4. \]  

(4.75)

with

\[ |R_2| \leq O(L^{-1/2}) \int_{\eta_j}^{\xi_{j+1}} (v^2 + v_x^2), \quad |R_3| \leq O(L^{-1/2}) \int_{-\infty}^{\xi_{j+1}} (v^2 + v_x^2) \quad \text{and} \quad |R_4| \leq O(L^{-1/2}) \int_{\xi_{j+1}}^{+\infty} (v^2 + v_x^2). \]

Then, adding (4.72) and (4.73), and summing over \( j \in \{1, \ldots, k_i\} \), we infer that

\[ \int_{\xi_{j+1}}^{\xi_{j+1}} g^2(x) \phi_i(x) dx = \int_{\xi_{j+1}}^{\xi_{j+1}} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i - 6 \sum_{j=1}^{k_i} v^2(\xi_{j+1}) \phi_i(\xi_{j+1}) - 6 \sum_{j=1}^{k_i} v^2(\xi_j) \phi_i(\xi_j) \]
\[ + 12 \sum_{j=1}^{k_i} v^2(\eta_j) \phi_i(\eta_j) - 2 \sum_{j=1}^{k_i} v^2(\xi_{j+1}) \phi_i'(\xi_{j+1}) + 2 \sum_{j=1}^{k_i} v^2(\xi_j) \phi_i'(\xi_j) + R, \]  

(4.76)

with

\[ |R| \leq O(L^{-1/2}) \int_{\xi_{j+1}}^{\xi_{j+1}} (v^2 + v_x^2). \]

Finally, adding (4.74)-(4.76), and recalling that \( \|v\|_{H^1} \leq \|u\|_{H^1} \), we obtain the lemma.

\[ \square \]

**Lemma 4.4** (Connection Between \( F_i(\cdot) \) and the Local Extrema of \( v \).) Let \( u \in H^1(\mathbb{R}) \) and \( v = (4 - \partial_x^2)^{-1}u \in H^3(\mathbb{R}) \). For \( i = 1, \ldots, N \), define the function \( h_i \) by

\[ h_i(x) = \begin{cases} 
- v_{xx} - 6v_x + 16v, & x < \xi_i, \\
- v_{xx} + 6v_x + 16v, & \xi_i < x < \eta_j, \\
- v_{xx} - 6v_x + 16v, & \eta_j < x < \xi_{j+1}, \\
- v_{xx} + 6v_x + 16v, & x > \xi_{j+1}, 
\end{cases} \]  

(4.77)
Then it holds
\[ \int_R h_i(x)g_i^2(x)\phi_i(x) = F_i(u) - 144 \left( \sum_{j=0}^{k_i} (M_i^{j+1})^3 \phi_i(\xi_j^{i+1}) - \sum_{j=1}^{k_i} (m_i^j)^3 \phi_i(\eta_j^i) \right) + \|u\|_H^3 O(L^{-1/2}). \] (4.78)

**Proof.** We have
\[
\int_R h_i(x)g_i^2(x)\phi_i(x)dx = \int_{-\infty}^{\xi_i^j} h_i(x)g_i^2(x)\phi_i(x)dx + \sum_{j=1}^{k_i} \int_{\xi_i^j}^{\xi_i^{j+1}} h_i(x)g_i^2(x)\phi_i(x)dx \\
+ \int_{\xi_i^{j+1}}^{+\infty} h_i(x)g_i^2(x)\phi_i(x)dx.
\] (4.79)

For \( j = 1, \ldots, k_i \),
\[
\int_{\xi_i^j}^{\xi_i^{j+1}} h_i(x)g_i^2(x)\phi_i(x)dx = \int_{\eta_i^j}^{\eta_i^{j+1}} (-v_{xx} \cdot 6v_x + 16v_x^2) (2v + v_{xx} - 3v_x^2) \phi_i \\
+ \int_{\eta_i^{j+1}}^{\xi_i^{j+1}} (-v_{xx} + 6v_x + 16v_x^2) (2v + v_{xx} + 3v_x^2) \phi_i
\]
\[ = J + I. \]

Computing \( I \), we obtain
\[
I = \int_{\eta_i^j}^{\xi_i^{j+1}} (-v_{xx}^3 + 12v_{xx}^2 + 64v_x^3 + 60v_x^2v_{xx}) \phi_i - 54 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x^3\phi_i + 27 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x^2v_{xx}\phi_i \\
- 108 \int_{\eta_i^j}^{\xi_i^{j+1}} v_xv_{xx}v_{xx}\phi_i - 216 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x^2v_{xx}\phi_i + 216 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x^2v_x\phi_i
\]
\[ = \int_{\eta_i^j}^{\xi_i^{j+1}} (-v_{xx}^3 + 12v_{xx}^2 + 64v_x^3 + 60v_x^2v_{xx}) \phi_i - 54 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x^3\phi_i + I_1 + I_2 + I_3 + I_4 \] (4.80)

with
\[
I_1 = 9 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x^2\phi_i - 9 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x^3\phi_i, \]
(4.81)
\[
I_2 = -54 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x\partial_x(v_x^3)\phi_i = 54 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x\partial_x(v_x^2)\phi_i = 54 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x^2\phi_i + 54 \int_{\eta_i^j}^{\xi_i^{j+1}} v_x^2\phi_i', \]
(4.82)
\[
I_3 = -72 \int_{\eta_i^j}^{\xi_i^{j+1}} \partial_x(v^3)\phi_i = -72v^3(\xi_i^{j+1})\phi_i(\xi_i^{j+1}) + 72v^3(\eta_i^j)\phi_i(\eta_i^j) + 72 \int_{\eta_i^j}^{\xi_i^{j+1}} v^3\phi_i', \]
(4.83)
and
\[
I_4 = 108 \int_{\eta_i^j}^{\xi_i^{j+1}} \partial_x(v^2)\phi_i - 108 \int_{\eta_i^j}^{\xi_i^{j+1}} v^2\partial_x(v_x\phi_i) = -108 \int_{\eta_i^j}^{\xi_i^{j+1}} v^2v_{xx}\phi_i - 108 \int_{\eta_i^j}^{\xi_i^{j+1}} v^2v_x\phi_i \\
= -108 \int_{\eta_i^j}^{\xi_i^{j+1}} v^2v_{xx}\phi_i - 36 \int_{\eta_i^j}^{\xi_i^{j+1}} \partial_x(v^3)\phi_i
\]
\[ = -36v^3(\xi_i^{j+1})\phi_i(\xi_i^{j+1}) + 36v^3(\eta_i^j)\phi_i(\eta_i^j) - 108 \int_{\eta_i^j}^{\xi_i^{j+1}} v^2v_{xx}\phi_i + 36 \int_{\eta_i^j}^{\xi_i^{j+1}} v^3\phi_i''. \] (4.84)
Adding (4.80)-(4.84), we get

$$I = \int_{\eta_j^1}^{\xi_{j+1}} (-v_{xx}^3 + 12v_{xx}^2 + 64v^3 - 48v^2v_{xx}) \phi_i - 72v^3(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) + 72v^3(\eta_j^i)\phi_i(\eta_j^i)$$

$$- 36v^3(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) + 36v^3(\eta_j^i)\phi_i'(\eta_j^i) + R,$$

where using that $\|v\|_{C^1(\mathbb{R})} \leq C'_S\|v\|_{H^2(\mathbb{R})}$ (with $C'_S$ the constant of Sobolev), and $\|v\|_{H^2(\mathbb{R})} \sim \|u\|_H$, the estimate of $R$ leads to

$$|R| \leq (\|\phi_i\|_{L^\infty(\mathbb{R})} + \|\phi_i'\|_{L^\infty(\mathbb{R})})(\|v\|_{L^\infty(\mathbb{R})} + \|v_x\|_{L^\infty(\mathbb{R})}) \int_{\eta_j^1}^{\xi_{j+1}} (v^2 + v_x^2)$$

$$\leq O(L^{-1/2})\|u\|_H \int_{\eta_j^1}^{\xi_{j+1}} (v^2 + v_x^2).$$

Similar computations lead to

$$J = \int_{\xi_j^1}^{\eta_j^1} (-v_{xx}^3 + 12v_{xx}^2 + 64v^3 - 48v^2v_{xx}) \phi_i - 72v^3(\xi_j^i)\phi_i(\xi_j^i) + 72v^3(\eta_j^i)\phi_i(\eta_j^i)$$

$$+ 36v^3(\xi_j^i)\phi_i'(\xi_j^i) - 36v^3(\eta_j^i)\phi_i'(\eta_j^i) + O(L^{-1/2})\|u\|_H \int_{\xi_j^1}^{\eta_j^1} (v^2 + v_x^2),$$

$$\int_{-\infty}^{\xi_j^1} h_i(x)g_i^2(x)\phi_i(x) = \int_{-\infty}^{\xi_j^1} (-v_{xx}^3 + 12v_{xx}^2 + 64v^3 - 48v^2v_{xx}) \phi_i - 72v^3(\xi_j^i)\phi_i(\xi_j^i)$$

$$- 36v^3(\xi_j^i)\phi_i'(\xi_j^i) + O(L^{-1/2})\|u\|_H \int_{-\infty}^{\xi_j^1} (v^2 + v_x^2)$$

and

$$\int_{\xi_{k+1}}^{\infty} h_i(x)g_i^2(x)\phi_i(x) = \int_{\xi_{k+1}}^{\infty} (-v_{xx}^3 + 12v_{xx}^2 + 64v^3 - 48v^2v_{xx}) \phi_i - 72v^3(\xi_{k+1}^i)\phi_i(\xi_{k+1}^i)$$

$$+ 36v^3(\xi_{k+1}^i)\phi_i'(\xi_{k+1}^i) + O(L^{-1/2})\|u\|_H \int_{\xi_{k+1}}^{\infty} (v^2 + v_x^2).$$

Adding (4.85) and (4.86), and summing over $j \in \{1, \ldots, k_i\}$, we get

$$\int_{\xi_j^1}^{\xi_{k+1}} h_i(x)g_i^2(x)\phi_i(x)dx$$

$$= \int_{\xi_j^1}^{\xi_{k+1}} (-v_{xx}^3 + 12v_{xx}^2 + 64v^3 - 48v^2v_{xx}) \phi_i - 72 \sum_{j=1}^{k_i} v^3(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i)$$

$$- 72 \sum_{j=1}^{k_i} \phi_i(\xi_j^i) + 144 \sum_{j=1}^{k_i} v^3(\eta_j^i)\phi_i(\eta_j^i) - 36 \sum_{j=1}^{k_i} v^3(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i)$$

$$+ 36 \sum_{j=1}^{k_i} v^3(\xi_j^i)\phi_i'(\xi_j^i) + O(L^{-1/2})\|u\|_H \int_{\xi_j^1}^{\xi_{k+1}} (v^2 + v_x^2).$$

Finally, adding (4.87)-(4.89), we obtain the lemma. □
Lemma 4.5 (Connection Between $E_i(\cdot)$ and $F_i(\cdot)$). Let $u \in H^1(\mathbb{R})$, with $y = (1 - \partial_x^2)u \in M^+(\mathbb{R})$, that satisfies (4.2). Let be given $N - 1$ real numbers $-\infty = y_1 < y_2 < \ldots < y_N < y_{N+1} = +\infty$ with $y_i - y_{i-1} \geq 2L/3$. For $i = 1, \ldots, N$, assume that $v = (4 - \partial_x^2)^{-1}u$ satisfies (4.57)-(4.64), with local extrema on $[\eta_i, \eta_{k_i+1}]$ arranged in decreasing order in the following way:

$$M_1 \geq M_2 \geq \ldots \geq M_{k_i+1} \geq 0, \quad m_1 \geq m_2 \geq \ldots \geq m_{k_i} \geq 0, \quad M_{j+1} \geq m_j, \quad j = 1, \ldots, k_i. \quad (4.90)$$

There exist $\gamma_0 > 0$ and $L_0 > 0$ only depending on the speeds $(c_i)_{i=1}^N$, such that if $0 < \gamma < \gamma_0$ and $L > L_0 > 0$, then defining the functional $E_i(\cdot)$'s and $F_i(\cdot)$'s as in (4.52)-(4.55), it holds

$$F_i(u) = 18M_i^3 \lVert E_i(u) \rVert_{H_1}^3 + \lVert u \rVert_{H_1}^3O(L^{-1/2}), \quad i = 1, \ldots, N. \quad (4.91)$$

Proof. Combining (4.53), (4.64) and (4.66) with $K = \sqrt{L}/8$, we get

$$\int_\mathbb{R} g_i^2(x)\phi_i(x) = E_i(u) - 12 \left( \sum_{j=0}^{k_i} (M_{j+1}^i)^2 - \sum_{j=1}^{k_i} (m_j^i)^2 \right) + \lVert u \rVert_{H_1}^2O(L^{-1/2}). \quad (4.92)$$

Similarly, combining (4.53), (4.64) and (4.78), we get

$$\int_\mathbb{R} h_i(x)g_i^2(x)\phi_i(x)dx = F_i(u) - 144 \left( \sum_{j=0}^{k_i} (M_{j+1}^i)^3 - \sum_{j=1}^{k_i} (m_j^i)^3 \right) + \lVert u \rVert_{H_1}^3O(L^{-1/2}). \quad (4.93)$$

Now, let us show that $h_i \leq 18M_i^3$ on $\Omega_i$. Note that by (4.5) and (4.56), one can check that $18M_i^3 \geq c_i/4$.

We rewrite the function $h_i$ as

$$h_i(x) = \begin{cases} 
-v_{xx} - 6v_x + 16v, & x < \eta_0^i, \\
(\partial_x^2 + 3\partial_x + 2) v - 3v_x + 18v, & \eta_0^i < x < \xi_1^i, \\
(\partial_x^2 - 3\partial_x + 2) v + 3v_x + 18v, & \xi_1^i < x < \eta_1^j, \\
(\partial_x^2 + 3\partial_x + 2) v - 3v_x + 18v, & \eta_1^j < x < \xi_{j+1}^i, \\
(\partial_x^2 - 3\partial_x + 2) v + 3v_x + 18v, & \xi_{j+1}^i < x < \eta_{k_i+1}^i, \\
-v_{xx} + 6v_x + 16v, & x > \eta_{k_i+1}^i.
\end{cases} \quad (4.94)$$

Then, if $x \in \Omega_i \setminus [\eta_0^i, \eta_{k_i+1}^i]$, using that $v_{xx} = 4v - u$, (3.39), (4.64) and (4.65), it holds

$$h_i \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c_i}{9}. \quad (4.95)$$

If $\eta_0^i < x < \xi_1^i$, then $v_x \geq 0$, and using that $y = (1 - \partial_x^2)u \geq 0$, it follows from Lemma 2.2 that

$$h_i = -(\partial_x^2 + 3\partial_x + 2)v - 3v_x + 18v \\
= -(2 + \partial_x)(4 - \partial_x^2)^{-1}(1 + \partial_x)u - 3v_x + 18v \\
\leq 18v. \quad (4.96)$$

If $\xi_1^i < x < \eta_1^j$, then $v_x \leq 0$, and similarly using that $y = (1 - \partial_x^2)u \geq 0$, it follows from Lemma 2.2 that

$$h_i = -(\partial_x^2 - 3\partial_x + 2)v + 3v_x + 18v \\
= -(2 - \partial_x)(4 - \partial_x^2)^{-1}(1 - \partial_x)u + 3v_x + 18v \\
\leq 18v. \quad (4.97)$$
Therefore, it holds

\[ h_i(x) \leq 18 \max_{x \in \Omega_i} v(x) = 18M_i^1, \quad \forall x \in \Omega_i. \quad (4.94) \]

Now, taking \( \phi_i \equiv 1 \) on \( \mathbb{R} \) in (4.66), we have \( \|g_i\|_{L^2(\mathbb{R})} \leq \|u\|_{H_i} \). Also, from the definition of \( h_i \), and using (2.3) and Remark 4.1, we have \( \|h_i\|_{L^{\infty}(\mathbb{R})} \leq \|u\|_{L^{\infty}(\mathbb{R})} + 32\|v\|_{L^{\infty}(\mathbb{R})} \leq O(\|u\|_{H_i}) \). Then, combining (4.92)-(4.94), we obtain

\[
F_i(u) - 144 \left( \sum_{j=0}^{k_i} (M_{j+1}^i)^3 - \sum_{j=1}^{k_i} (m_j^i)^3 \right) \\
= \int_{\Omega_i} h_i(x)g_i^2(x)\phi_i(x)dx + \|u\|_{H_i}^3O(L^{-1/2}) \\
= \int_{\Omega_i} h_i(x)g_i^2(x)\phi_i(x)dx + \int_{\Omega_i} h_i(x)g_i^2(x)\phi_i(x)dx + \|u\|_{H_i}^3O(L^{-1/2}) \\
\leq 18M_i^1 \int_{\Omega_i} g_i^2(x)\phi_i(x)dx + \|h_i\|_{L^{\infty}(\mathbb{R})} \|g_i\|_{L^{2}(\mathbb{R})} \|\phi_i\|_{L^{\infty}(\mathbb{R})} + \|u\|_{H_i}^3O(L^{-1/2}) \\
\leq 18M_i^1 \left[ E_i(u) - 12 \left( \sum_{j=0}^{k_i} (M_{j+1}^i)^2 - \sum_{j=1}^{k_i} (m_j^i)^2 \right) \right] + \|u\|_{H_i}^3O(L^{-1/2}).
\]

Therefore, using that \( M_{j+1}^i \geq m_j^i \) and proceeding as in Lemma 3.6 (see (3.42)), we infer that

\[
F_i(u) \leq 18M_i^1E_i(u) - 72(M_i^1)^3 + 144 \sum_{j=0}^{k_i} \left[ (M_{j+1}^i)^3 - (m_j^i)^3 \right] - \frac{3}{2} M_i^1 \left( [M_{j+1}^i]^2 - (m_j^i)^2 \right) \\
+ \|u\|_{H_i}^3O(L^{-1/2}) \\
\leq 18M_i^1E_i(u) - 72(M_i^1)^3 + \|u\|_{H_i}^3O(L^{-1/2}).
\]

This proves the lemma. \( \square \)

The lemma below is the generalization of Lemma 3.3.

**Lemma 4.6 (General Quadratic Identity).** Let \( Z = \{z_i\}_{i=1}^N \in \mathbb{R}^N \) with \( |z_i - z_{i-1}| \geq L/2, \) and \( u \in L^2(\mathbb{R}) \).

It holds

\[ E(u) - \sum_{i=1}^{N} E(\varphi_{z_i}) = \|u - S_Z\|_{H_i}^2 + 4 \sum_{i=1}^{N} c_i \left( v(z_i) - \frac{c_i}{6} \right) + O(e^{-L^4}), \quad (4.95) \]

where \( S_Z \) is defined in (4.11) and \( O(\cdot) \) only depends on \( (c_i)_{i=1}^N \).

**Proof.** Let us compute

\[
\|u - S_Z\|_{H_i}^2 = \int_{\mathbb{R}} [(1 - \partial_x^2)(u - S_Z)] [(4 - \partial_x^2)^{-1} (u - S_Z)] \\
= \|u\|_{H_i}^2 + \|S_Z\|_{H_i}^2 - 2 \int_{\mathbb{R}} [(1 - \partial_x^2)S_Z] [(4 - \partial_x^2)^{-1} u] \\
= \|u\|_{H_i}^2 + \|S_Z\|_{H_i}^2 - 2 \sum_{i=1}^{N} c_i \int_{\mathbb{R}} [(1 - \partial_x^2)\varphi_{z_i}(x - z_i)]u \\
= \|u\|_{H_i}^2 + \|S_Z\|_{H_i}^2 - 4 \sum_{i=1}^{N} c_i v(z_i), \quad (4.96)
\]
Lemma 4.7 (Control of the Distances Between Local and Global Energies at \(t = 0\)). Let \(u_0 \in H^1(\mathbb{R})\) satisfying (1.12)-(1.14). Then it holds

\[ |E(u_0) - \sum_{i=1}^{N} E(\varphi_{c_i})| \leq O(\varepsilon^2) + O(e^{-L/4}), \]

(4.100)

\[ |E_i(u_0) - E(\varphi_{c_i})| \leq O(\varepsilon^2) + O(e^{-\sqrt{\varepsilon}}), \quad i = 1, \ldots, N, \]

(4.101)

and

\[ |F_i(u_0) - F(\varphi_{c_i})| \leq O(\varepsilon^2) + O(e^{-\sqrt{\varepsilon}}), \quad i = 1, \ldots, N, \]

(4.102)

where \(O(\cdot)\) only depend on \((c_i)_{i=1}^{N}\).

Proof. For the first estimate, applying triangular inequality and (1.14), we have

\[ |E(u_0) - E(S_{Z^0})| = \|u_0\|_{\mathcal{H}} - \|S_{Z^0}\|_{\mathcal{H}} + \|u_0 - S_{Z^0}\|_{\mathcal{H}} \]

\[ \leq \|u_0 - S_{Z^0}\|_{\mathcal{H}} (\|u_0\|_{\mathcal{H}} + \|S_{Z^0}\|_{\mathcal{H}}) \]

\[ \leq \varepsilon^2 + \frac{2}{\sqrt{3}} \sum_{i=1}^{N} c_i. \]

(4.103)
Thus, combining (4.99) and (4.103), it holds
\[
\left| E(u_0) - \sum_{i=1}^{N} E(\varphi_{c_i}) \right| \leq |E(u_0) - E(S_{Z^0})| + \left| E(S_{Z^0}) - \sum_{i=1}^{N} E(\varphi_{c_i}) \right| \\
\leq \varepsilon^2(\varepsilon^2 + O(1)) + O(e^{-L/4}) \\
\leq O(\varepsilon^2) + O(e^{-L/4}).
\]

For the second estimate, using the exponential decay of \(\varphi_{c_i}\)'s and the \(\phi_i\)'s, and the definition of \(E_i(\cdot)\), we have
\[
|E_i(u_0) - E(\varphi_{c_i})| \\
\leq \left| \|u_0\|^2_{H(\Omega_i)} - \|\varphi_{c_i}\|^2_{H(\Omega_i)} \right| + O(e^{-\sqrt{T}}) \\
= \left| \|u_0\|_{H(\Omega_i)} - \|\varphi_{c_i}\|_{H(\Omega_i)} \right| \left( \|u_0\|^2_{H(\Omega_i)} + \|\varphi_{c_i}\|^2_{H(\Omega_i)} \right) + O(e^{-\sqrt{T}}) \\
\leq \left( \|u_0 - S_{Z^0}\|_{H(\Omega_i)} + \sum_{1 \leq j \leq N} \|\varphi_{c_j}\|_{H(\Omega_i)} \right) \left( \|u_0 - S_{Z^0}\|_{H} + \frac{2}{\sqrt{3}} \sum_{j=1}^{N} c_j \right) + O(e^{-\sqrt{T}}) \\
\leq \left( \varepsilon^2 + O(e^{-L/8}) \right) (\varepsilon^2 + O(1)) + O(e^{-\sqrt{T}}) \\
\leq O(\varepsilon^2) + O(e^{-\sqrt{T}}).
\]

Similarly, for the third estimate, using the exponential decay of \(\varphi_{c_i}\)'s and the \(\phi_i\)'s, and the definition of \(F_i(\cdot)\), we have
\[
|F_i(u_0) - F(\varphi_{c_i})| \\
\leq \int_{\Omega_i} \left( u_0^3 - \varphi_{c_i}^3 \right) + O(e^{-\sqrt{T}}) \\
\leq \int_{\Omega_i} |u_0 - \varphi_{c_i}| (u_0^2 + u_0 \varphi_{c_i} + \varphi_{c_i}^2) + O(e^{-\sqrt{T}}) \\
\leq \|u_0 - \varphi_{c_i}\|_{L^2(\Omega_i)} \left( \int_{\Omega_i} (u_0^2 + u_0 \varphi_{c_i} + \varphi_{c_i}^2)^2 \right)^{1/2} + O(e^{-\sqrt{T}}) \\
\leq \left( \|u_0 - S_{Z^0}\|_{L^2(\Omega_i)} + \sum_{1 \leq j \leq N} \|\varphi_{c_j}\|_{L^2(\Omega_i)} \right) \cdot O(1) + O(e^{-\sqrt{T}}) \\
\leq \left( \varepsilon^2 + O(e^{-L/8}) \right) \cdot O(1) + O(e^{-\sqrt{T}}) \\
\leq O(\varepsilon^2) + O(e^{-\sqrt{T}}).
\]

This proves the lemma. \(\Box\)

### 4.4 End of the proof of Theorem 1.1

Let \(u \in X([0, T])\), with \(0 < T \leq +\infty\), be a solution of the DP equation satisfying (1.12)-(1.14) and (4.2) for some \(t_0 \in [0, T]\). Let \(M_i = u(t_0, \xi_i(t_0)) = \max_{x \in J_i} v(t_0, x)\), with \(J_i\)'s as in (4.8), and \(\delta_i = c_i/6 - M_i\). First, from (4.7) and (4.9), we know that for \(i = 2, \ldots, N\),
\[
\xi_i(t_0) - \xi_{i-1}(t_0) \geq \frac{2L}{3} > \frac{L}{2}.
\]
Applying (4.95) and (4.100) with $u(t_0)$, we get
\[
\left\| u(t_0) - \sum_{i=1}^{N} \varphi c_i (\cdot - \xi_i(t_0)) \right\|_H \leq 4 \sum_{i=1}^{N} c_i \delta_i + O(\varepsilon^2) + O(e^{-L/4}). \tag{4.104}
\]
In the same way, from (4.91) we get
\[
F_i(u(t_0)) \leq 18 M_i E_i(u(t_0)) - 72 (M_i^3) + O(L^{-1/2}),
\]
which leads to
\[
F(u(t_0)) = \sum_{i=1}^{N} F_i(u(t_0)) \leq 18 \sum_{i=1}^{N} M_i E_i(u(t_0)) - 72 \sum_{i=1}^{N} (M_i^3) + O(L^{-1/2}), \tag{4.105}
\]
by summing over $i \in \{1, \ldots, N\}$.

Now, we will use the following notation: for a function $f : \mathbb{R}_+ \mapsto \mathbb{R}$, we set
\[
\Delta_0^i f = f(t_0) - f(0). \tag{4.106}
\]
From (4.105) and the fact that $E(\cdot)$ and $F(\cdot)$ are conservation laws for $u$, we obtain
\[
0 = \Delta_0^i F(u) = \sum_{i=1}^{N} \Delta_0^i F_i(u) \leq 18 \sum_{i=1}^{N} M_i \Delta_0^i E_i(u) + \sum_{i=1}^{N} \left[ -72 (M_i^3) + 18 M_i E_i(u_0) - F_i(u_0) \right] + O(L^{-1/2}). \tag{4.107}
\]
Note that, from (4.101) and (4.102), for $0 < \varepsilon < \varepsilon_0$ and $L > L_0 > 0$ with $\varepsilon_0 \ll 1$ and $L_0 \gg 1$, it holds
\[
\sum_{i=1}^{N} \left[ -72 (M_i^3) + 18 M_i E_i(u_0) - F_i(u_0) \right] = -72 \sum_{i=1}^{N} \delta_i^2 \left( M_i^3 + \frac{c_i}{3} \right) + O(\varepsilon^2) + O(e^{-\sqrt{T}}). \tag{4.108}
\]
Combining (4.107) and (4.108), we get
\[
\sum_{i=1}^{N} \delta_i^2 \left( M_i^3 + \frac{c_i}{3} \right) \leq \frac{1}{4} \sum_{i=1}^{N} M_i \Delta_0^i E_i(u) + O(\varepsilon^2) + O(L^{-1/2}),
\]
and using the Abel transformation with $M_i^0 = 0$, we obtain
\[
\sum_{i=1}^{N} \delta_i^2 \left( M_i^3 + \frac{c_i}{3} \right) \leq \frac{1}{4} \sum_{i=2}^{N} (M_i - M_i^{i-1}) \Delta_0^i J_{i,K} + O(\varepsilon^2) + O(L^{-1/2}), \tag{4.109}
\]
where $J_{i,K}(t)$ is defined in (4.22). From (4.2) we know that $u(t_0) \in U(\gamma, L/2)$, on account of Lemma 4.1 there exists $\tilde{\mathbf{X}} = (\tilde{x}_1, \ldots, \tilde{x}_N)$ with $\tilde{x}_i \in J_i$ such that $E(u(t_0) - S_{\tilde{X}}) \leq O(\gamma^2)$, where $S_{\tilde{X}}$ is defined in (4.11). Recalling that $\max_{x \in J_i} v(t_0, x) = \max_{x \in J_i} v(t_0, \xi_i)$ and using (4.95), we obtain $E(u(t_0) - S_{\xi}) \leq O(\gamma^2) + O(e^{-L/4})$, with $\xi_i = (\xi_1^i, \ldots, \xi_N^i)$. From (4.5), we deduce that
\[
\left\| v(t_0) - \sum_{j=1}^{N} \rho c_j (\cdot - \xi_j(t_0)) \right\|_{L^\infty(\mathbb{R})} \leq O(\gamma) + O(e^{-L/8}).
\]
Thus, we infer that
\[ v(x) = \sum_{j=1}^{N} \rho_{c_j} (x - \xi_j^1(t_0)) + O(\gamma) + O(e^{-L/8}), \quad \forall x \in \mathbb{R}, \]
and applying this formula with \( x = \xi_1^1(t_0) \) and using that \( \xi_1^1(t_0) - \xi_1^{j-1}(t_0) > L/2 \), we get
\[ v(\xi_1^1(t_0)) = \sum_{j=1}^{N} \rho_{c_j} (\xi_1^1(t_0) - \xi_j^1(t_0)) + O(\gamma) + O(e^{-L/8}) \]
\[ = \frac{c_1}{6} + \sum_{1 \leq j \leq N \atop j \neq i} \rho_{c_j} (\xi_1^1(t_0) - \xi_j^1(t_0)) + O(\gamma) + O(e^{-L/8}) \]
\[ = \frac{c_1}{6} + O(\gamma) + O(e^{-L/8}). \]

We take \( \gamma = A(\sqrt{\varepsilon} + L^{-1/8}) \), then \( M^i_1 = c_i/6 + O(\sqrt{\varepsilon}) + O(L^{-1/8}) \). Therefore, for \( 0 < \varepsilon < \varepsilon_0 \) and \( L > L_0 > 0 \), with \( \varepsilon_0 \ll 1 \) and \( L_0 \gg 1 \), it holds
\[ 0 < M^1_1 < M^2_1 < \ldots < M^N_1. \]

Combining (4.109), (4.110) and using the monotonicity estimate (4.24), it holds
\[ \sum_{i=1}^{N} \delta_i^2 \left( M^i_1 + \frac{c_i}{6} \right) \leq O(\varepsilon^2) + O(L^{-1/8}). \]

Therefore, using that \( (M^i_1 + c_i/3)^{-1} < 3/c_i \), there exists \( C > 0 \) only depending on \( (c_i)_{i=1}^{N} \) such that
\[ \delta_i \leq C(\varepsilon + L^{-1/4}), \quad i = 1, \ldots, N. \]

Now, combining (4.104) and (4.111), we obtain
\[ \left\| u(t_0) - \sum_{i=1}^{N} \varphi_{c_i} (\cdot - \xi_1^i(t_0)) \right\|_H \leq C(\sqrt{\varepsilon} + L^{-1/8}), \]
and the theorem follows by choosing \( A = 2C \).

**Remark 4.2** (The Role of the Number of Extrema). In the case where \( v = (4 - \partial_x^2)^{-1} u \) admits a countable infinite number of local maximal values on some \([\alpha_i, \beta_i]\) (see (4.57)), with \( i \in \{1, \ldots, N\} \), it suffices to change the finite sums over \( j \) by infinite sums in Lemmas 3.4-3.5 and Lemmas 4.3-4.4.

**Appendix. Proof of Lemma 4.2**

The aim of this subsection is to prove Lemma 4.2. Let us first assume that \( u \) is smooth solution. The case \( u \in X([0, T]) \) will follow by a density argument.

We compute the time variation of the following energy:
\[
\frac{d}{dt} \int_{\mathbb{R}} y v g = \int_{\mathbb{R}} y_t v g + \int_{\mathbb{R}} y v_t g = I + J.
\]
Applying the operator \((1 - \partial_x^2)(\cdot)\) on both sides of equation (1.7), we get
\[
y_t = -\frac{1}{2}(1 - \partial_x^2)\partial_x u^2 - \frac{3}{2}\partial_x u^2
\]
and substituting \( y_l \) by this value, \( I \) becomes

\[
I = -\frac{1}{2} \int \left[ (1 - \partial^2_x) \partial_x u^2 \right] v g - \frac{3}{2} \int \left[ \partial_x (u^2) \right] v g
= I_1 + I_2.
\]

By computing

\[
I_2 = \frac{3}{2} \int u^2 \partial_x (v g) = \frac{3}{2} \int u^2 v_x g + \frac{3}{2} \int u^2 v g'
\]

and

\[
I_1 = \frac{1}{2} \int \left[ (1 - \partial^2_x) u^2 \right] \partial_x (v g)
= \frac{1}{2} \int \left[ (1 - \partial^2_x) u^2 \right] v_x g + \frac{1}{2} \int \left[ (1 - \partial^2_x) u^2 \right] v g'
= I_3 + I_4
\]

with

\[
I_3 = \frac{1}{2} \int u^2 v_x g - \frac{1}{2} \int \partial_x^2 (u^2) v_x g
= \frac{1}{2} \int u^2 v_x g + \frac{1}{2} \int \partial_x (u^2) \partial_x (v_x g)
= \frac{1}{2} \int u^2 v_x g + \frac{1}{2} \int \partial_x (u^2) v_{xx} g + \frac{1}{2} \int \partial_x (u^2) v_x g'
= \frac{1}{2} \int u^2 v_x g - \frac{1}{2} \int u^2 \partial_x (v_{xx} g) - \frac{1}{2} \int u^2 \partial_x (v_x g')
= \frac{1}{2} \int u^2 v_x g - \frac{1}{2} \int u^2 v_{xxx} g - \int u^2 v_{xx} g' - \frac{1}{2} \int u^2 v_x g''
\]

and

\[
I_4 = \frac{1}{2} \int u^2 v g' - \frac{1}{2} \int \partial_x^2 (u^2) v g'
= \frac{1}{2} \int u^2 v g' + \frac{1}{2} \int \partial_x (u^2) \partial_x (v g')
= \frac{1}{2} \int u^2 v g' + \frac{1}{2} \int \partial_x (u^2) v_x g' + \frac{1}{2} \int \partial_x (u^2) v g''
= \frac{1}{2} \int u^2 v g' - \frac{1}{2} \int u^2 \partial_x (v_x g') - \frac{1}{2} \int u^2 \partial_x (v g'')
= \frac{1}{2} \int u^2 v g' - \frac{1}{2} \int u^2 v_{xx} g' - \int u^2 v_x g'' - \frac{1}{2} \int u^2 v g'''.
\]

Adding (4.113) and (4.114), we get

\[
I_1 = \frac{1}{2} \int u^2 v_x g - \frac{1}{2} \int u^2 v_{xxx} g - \int u^2 v_{xx} g' - \frac{1}{2} \int u^2 v_x g''
+ \frac{1}{2} \int u^2 v g' - \frac{1}{2} \int u^2 v_{xx} g' - \int u^2 v_x g'' - \frac{1}{2} \int u^2 v g'''
= \frac{1}{2} \int u^2 v_x g - \frac{1}{2} \int u^2 v_{xxx} g - \frac{3}{2} \int u^2 v_{xx} g' + \frac{1}{2} \int u^2 v g'
+ \frac{1}{2} \int u^2 v g' - \frac{3}{2} \int u^2 v_g'' - \frac{1}{2} \int u^2 v g'''
\]
and adding (4.112) and (4.115), we get

\[ I = 2 \int_R u^2 v_x g - \frac{1}{2} \int_R u^2 v_{xxx} g - \frac{3}{2} \int_R u^2 v_{xx} g' + 2 \int_R u^2 v g' - \frac{3}{2} \int_R u^2 v_x g'' - \frac{1}{2} \int_R u^2 v g''' \\
= \frac{1}{2} \int_R u^2 [(4 - \partial_x^2) v_x] g + \frac{3}{2} \int_R u^2 \left[ \left( \frac{4}{3} - \partial_x^2 \right) v \right] g' - \frac{3}{2} \int_R u^2 v_x g'' - \frac{3}{2} \int_R u^2 v g'''. \]

The first two integrals give us

\[ \frac{1}{2} \int_R u^2 [(4 - \partial_x^2) v_x] g = \frac{1}{6} \int_R \partial_x (u^3) g = -\frac{1}{6} \int_R u^3 g' \]

and

\[ \frac{3}{2} \int_R u^2 \left[ \left( \frac{4}{3} - \partial_x^2 \right) v \right] g' = \frac{3}{2} \int_R u^2 \left[ (4 - \partial_x^2) v - \frac{8}{3} v \right] g' = \frac{3}{2} \int_R u^3 g' - 4 \int_R u^2 v g'. \]

Finally, we obtain

\[ I = \frac{4}{3} \int_R u^3 g' - 4 \int_R u^2 v g' - \frac{3}{2} \int_R u^2 v_x g'' - \frac{1}{2} \int_R u^2 v g'''. \] (4.116)

We set \( h = (1 - \partial_x^2)^{-1} u^2 \). Applying the operator \((4 - \partial_x^2)^{-1}(\cdot)\) on both sides of equation (1.7) and using (4.16), we get

\[ v_t = -\frac{1}{2} (1 - \partial_x^2)^{-1} \partial_x u^2 = -\frac{1}{2} h_x. \] (4.117)

Substituting \( v_t \) by this value, \( J \) becomes

\[ J = -\frac{1}{2} \int_R y h_x g + \frac{1}{2} \int_R y h g' = J_1 + J_2. \]

By computing

\[ J_2 = \frac{1}{2} \int_R (u - u_{xx}) h g' \]

\[ = \frac{1}{2} \int_R u h g' - \frac{1}{2} \int_R u_{xx} h g' \]

\[ = \frac{1}{2} \int_R u h g' + \frac{1}{2} \int_R u_{xx} h_x g' + \frac{1}{2} \int_R u_x h g'' \]

\[ = \frac{1}{2} \int_R u h g' - \frac{1}{2} \int_R u_{xx} h_x g' + \frac{1}{2} \int_R u_{xx} h_x g' - \frac{1}{2} \int_R u_{xx} h_x g'' - \frac{1}{2} \int_R u_{xx} h_x g'' - \frac{1}{2} \int_R u_{xx} h_x g''' \] (4.118)
and

\[ J_1 = \frac{1}{2} \int_R (u_x - u_{xxx}) hg \]
\[ = \frac{1}{2} \int_R u_x h g - \frac{1}{2} \int_R u_{xxx} h g \]
\[ = \frac{1}{2} \int_R u_x h g + \frac{1}{2} \int_R u_x \partial_x (hg) \]
\[ = \frac{1}{2} \int_R u_x h g + \frac{1}{2} \int_R u_{xx} h_x g + \frac{1}{2} \int_R u_x h g' \]
\[ = J_3 + J_4 + J_5 \]

with

\[ J_3 = \frac{1}{2} \int_R u \partial_x (hg) = - \frac{1}{2} \int_R u h_x g - \frac{1}{2} \int_R u h g' \]

(4.119)

\[ J_5 = - \frac{1}{2} \int_R u \partial_x (hg') \]
\[ = \frac{1}{2} \int_R u_x g' - \frac{1}{2} \int_R u h g'' \]
\[ = \frac{1}{2} \int_R u \partial_x (h_x g') + \frac{1}{2} \int_R u \partial_x (hg') \]
\[ = \frac{1}{2} \int_R u h_x g' + \int_R u h_x g'' + \frac{1}{2} \int_R u h g''' \]

(4.120)

and

\[ J_4 = - \frac{1}{2} \int_R u \partial_x (h_x g) \]
\[ = - \frac{1}{2} \int_R u_x h_x g - \frac{1}{2} \int_R u_h g' \]
\[ = \frac{1}{2} \int_R u \partial_x (h_x g) + \frac{1}{2} \int_R u \partial_x (h_x g') \]
\[ = \frac{1}{2} \int_R u h_x g + \int_R u h_x g' + \frac{1}{2} \int_R u h g''. \]

(4.121)

Adding (4.118)-(4.121), we get

\[ J = - \frac{1}{2} \int_R u h_x g + \frac{1}{2} \int_R u h_x x g + \int_R u h_x x g' + \frac{1}{2} \int_R u h_x g''. \]

Using that \( h_{xx} = -u^2 + h \) and \( h_{xxx} = -2u u_x + h_x \), we have

\[ \int_R u h_x x g' = \int_R u(-u^2 + h) g' = - \int_R u^3 g' + \int_R u h g' \]

and

\[ \frac{1}{2} \int_R u h_x x g = \frac{1}{2} \int_R u(-2u u_x + h_x) g \]
\[ = - \int_R u^2 u_x g + \frac{1}{2} \int_R u h_x g \]
\[ = - \frac{1}{3} \int_R \partial_x (u^3) g + \frac{1}{2} \int_R u h_x g \]
\[ = \frac{1}{3} \int_R u^3 g' + \frac{1}{2} \int_R u h_x g. \]
At this stage it is worth noticing that the term \( \int R^* u_h x g \) cancels with the one in \( J \). Finally, we obtain

\[
J = -\frac{2}{3} \int R^* u^3 g' + \int R^* u h g' + \frac{1}{2} \int R^* u h_x g''.
\] (4.122)

Combining (4.116) and (4.122), we get

\[
\frac{d}{dt} \int R v v_g = \frac{2}{3} \int R^* u^3 g' - 4 \int R^* u^2 v g' - \frac{3}{2} \int R^* u^2 v_x g'' - \frac{1}{2} \int R^* u^2 v g'' + \int R v h g' + \frac{1}{2} \int R^* u h_x g''.
\] (4.123)

Now, substituting \( u \) by \( 4v - v_{xx} \) and using integration by parts, we rewrite the energy as

\[
\int R v v_g = \int R v \left[ (1 - \partial^2_x) (4v - v_{xx}) \right] g = 4 \int R^* u^2 g - 5 \int R^* v v_{xx} g + \int R v (\partial^3_x v) g = 4 \int R^* u^2 g + K_1 + K_2.
\]

By computing

\[
K_1 = 5 \int R^* \partial_x(v g) v_x
= 5 \int R^* v_x^2 g + 5 \int R^* v v_{xx} g' = 5 \int R^* v_x^2 g + \frac{5}{2} \int R^* \partial_x(v^2) g' = 5 \int R^* v_x^2 g - \frac{5}{2} \int R^* v^2 g''
\] (4.124)

and

\[
K_2 = -\int R^* \partial_x(v g) v_{xxx}
= -\int R v_x v_{xxx} g - \int R v v_{xxx} g' = K_3 + K_4
\] (4.125)

with

\[
K_3 = \int R^* \partial_x(v x g) v_{xx}
= \int R^* v_{xx}^2 g + \int R v_x v_{xx} g' = \int R^* v_{xx}^2 g + \frac{1}{2} \int R \partial_x(v^2) g' = \int R^* v_{xx}^2 g - \frac{1}{2} \int R^* v^2 g''
\] (4.126)
and

\[ K_4 = \int_{\mathbb{R}} \partial_x (vg')v_{xx} \]
\[ = \int_{\mathbb{R}} v_x v_{xxx} g' + \int_{\mathbb{R}} vv_{xxx} g'' \]
\[ = \frac{1}{2} \int_{\mathbb{R}} \partial_x (v_x^2) g' - \int_{\mathbb{R}} \partial_x (v_x^2 v_x) \]
\[ = -\frac{1}{2} \int_{\mathbb{R}} v_x^2 g'' - \int_{\mathbb{R}} v_x^2 g'' - \int_{\mathbb{R}} vv_x g''' \]
\[ = -\frac{3}{2} \int_{\mathbb{R}} v_x^2 g'' - \frac{1}{2} \int_{\mathbb{R}} \partial_x (v_x^2) g''' \]
\[ = -\frac{3}{2} \int_{\mathbb{R}} v_x^2 g'' + \frac{1}{2} \int_{\mathbb{R}} v_x^2 g^{(4)}. \]  

(4.127)

Combining (4.124)-(4.127), we get

\[ \int_{\mathbb{R}} yvg = \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) g + \frac{1}{2} \int_{\mathbb{R}} v^2 (g^{(4)} - 5g'') - 2 \int_{\mathbb{R}} v_x^2 g'' \]

and differentiating with respect to time

\[ \frac{d}{dt} \int_{\mathbb{R}} yvg = \frac{d}{dt} \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) g + L_1 + L_2. \]

(4.128)

Using (4.117), we have

\[ L_1 = \int_{\mathbb{R}} v v_t (g^{(4)} - 5g'') = \frac{5}{2} \int_{\mathbb{R}} v h_x g'' - \frac{1}{2} \int_{\mathbb{R}} v h_x g^{(4)} \]

(4.129)

and

\[ L_2 = -4 \int_{\mathbb{R}} v_x v_{xxx} g'' = 2 \int_{\mathbb{R}} v_x h_{xx} g'' = -2 \int_{\mathbb{R}} u^2 v_x g'' + 2 \int_{\mathbb{R}} v_x h g''. \]

(4.130)

Lemma 4.2 follows by combining (4.123) and (4.128)-(4.130). □

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41
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