A generalization of
dual symmetry and reciprocity
for symmetric algebras

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Abstract
Slicing a module into semisimple ones is useful to study modules. Loewy structures provide a means of doing so. To establish the Loewy structures of projective modules over a finite dimensional symmetric algebra over a field $F$, the Landrock lemma is a primary tool. The lemma and its corollary relate radical layers of projective indecomposable modules to radical layers of the $F$-duals of those modules (“dual symmetry”) and to socle layers of those modules (“reciprocity”).

We generalize these results to an arbitrary finite dimensional algebra $A$. Our main theorem, which is the same as the Landrock lemma for finite dimensional symmetric algebras, relates radical layers of projective indecomposable modules $P$ to radical layers of the $A$-duals of those modules and to socle layers of injective indecomposable modules $\nu P$ where $\nu(\cdot)$ is the Nakayama functor. A key tool to prove the main theorem is a pair of adjoint functors, which we call socle functors and capital functors.

Keywords: Loewy structure, radical layer, socle layer, symmetric algebra, socle functor, capital functor
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1. Introduction

Semisimple modules are one of the most well-understood classes of modules. Hence slicing a module into semisimple ones is a natural way to study modules. Loewy structures provide a means of doing so. To establish Loewy structures several studies has been done [1] [2] [3]. A primary tool in these studies is the Landrock lemma [3] [4], which is stated below.

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Let \( A \) be a finite dimensional algebra over a field \( F \) and \((-)^* := \text{Hom}_F(-, F)\) the \( F \)-dual functor. The opposite algebra is denoted by \( A^{\text{op}} \). The term module refers to a finitely generated right module. Recall that \( A \) is a symmetric algebra if \( A \cong A^* \) as \((A, A)\)-bimodules. For other notations see Definition 2.1.

**Theorem 1.1 (Landrock [3, Theorem B]).** For a finite dimensional symmetric algebra \( A \) over a field \( F \), let \( P_i \) and \( P_j \) be the projective covers of simple \( A \)-modules \( S_i \) and \( S_j \) respectively. Then for an integer \( n \geq 1 \) we have an \( F \)-linear isomorphism

\[
\text{Hom}_A(\text{rad}_n P_i, S_j) \cong \text{Hom}_{A^{\text{op}}}(\text{rad}_n (P_j^*), S_i^*).
\]

**Corollary 1.2.** Under the same assumptions and notations of Theorem 1.1, we have an \( F \)-linear isomorphism

\[
\text{Hom}_A(\text{rad}_n P_i, S_j) \cong \text{Hom}_A(S_i, \text{soc}_n P_j).
\]

Although these results are powerful as indicated in the beginning, they are not applicable to algebras other than finite dimensional symmetric ones. We generalize the above results to arbitrary finite dimensional algebras. To state our main theorem we let \((-)^\lor := \text{Hom}_A(-, A)\) be the \( A \)-dual functor and \( \nu(-) := ((-)^\lor)^* \) the Nakayama functor.

**Theorem 1.3.** For a finite dimensional algebra \( A \) over a field \( F \), let \( P_i \) and \( P_j \) be the projective covers of simple \( A \)-modules \( S_i \) and \( S_j \) respectively. Then for an integer \( n \geq 1 \) we have \( F \)-linear isomorphisms

\[
\text{Hom}_A(\text{rad}_n P_i, S_j) \cong \text{Hom}_{A^{\text{op}}}(\text{rad}_n (P_j^{\lor}), S_i^*)
\]

(1.1)

and

\[
\text{Hom}_A(\text{rad}_n P_i, S_j) \cong \text{Hom}_A(S_i, \text{soc}_n \nu P_j).
\]

(1.2)

This paper is organized as follows. In Section 2 we introduce key tools to prove our main theorem, socle functors and capital functors, and prove Theorem 1.3 assuming lemmas. We also derive Theorem 1.1 and Corollary 1.2 from our main theorem. Section 3 deals with a simple example to see how our main theorem looks in a concrete situation. Section 4 consists of proofs of the lemmas in Section 2.

2. Proof of Main Theorem

We introduce basic terminology of this paper and state some useful lemmas first. Then a proof of the main theorem is given. The Landrock lemma is proved as a special case of the main theorem.
Definition 2.1. For a module $V$ over an algebra, $\text{soc} V$ denotes the sum of minimal submodules of $V$ and $\text{rad} V$ denotes the intersection of maximal submodules of $V$. For an integer $n \geq 0$, the $n$th socle of $V$ is defined inductively by $\text{soc}^0 V = 0$ and

$$\text{soc}^n V = \{ v \in V \mid v + \text{soc}^{n-1} V \in \text{soc}(V/\text{soc}^{n-1} V) \}$$

if $n > 0$. For an integer $n \geq 0$, the $n$th radical of $V$ is also defined inductively by $\text{rad}^0 V = V$ and

$$\text{rad}^n V = \text{rad}(\text{rad}^{n-1} V)$$

if $n > 0$. We then write

$$\text{soc}_n V = \text{soc}^n V/\text{soc}^{n-1} V$$

and call it the $n$th socle layer of $V$ for $n \geq 1$. We also write

$$\text{rad}_n V = \text{rad}^{n-1} V/\text{rad}^n V$$

and call it the $n$th radical layer of $V$ for $n \geq 1$.

Definition 2.2. For an integer $n \geq 0$ and a module $V$ over an algebra we write $\text{cap}^n V = V/\text{rad}^n V$ and call it the $n$th capital of $V$. Since any homomorphism maps the $n$th socle into the $n$th socle and the $n$th radical into the $n$th radical, $\text{soc}^n$ and $\text{cap}^n$ define endofunctors. We call these endofunctors the $n$th socle functor and the $n$th capital functor respectively.

The next simple lemma, which does not appear in the literature to the best of author’s knowledge, is vital to prove the main theorem. The category of finitely generated right $A$-modules is denoted by $\text{mod} A$.

Lemma 2.3. Let $A$ be a finite dimensional algebra over a field. Then for any integer $n \geq 0$ the $n$th capital functor and $n$th socle functor yield an adjoint pair of functors.

$$\text{mod} A \xrightarrow{\text{cap}^n} \text{mod} A \quad \text{soc}^n \dashv \text{cap}^n.$$

For a sense of unity we adopt an alias

$$\text{cap}_n = \text{rad}_n.$$

Note that $\text{cap}_n$ and $\text{soc}_n$ define endofunctors as $\text{cap}^n$ and $\text{soc}^n$. The following lemma can essentially be found in [5, Problem 2.14.ii].

Lemma 2.4. Let $A$ be a finite dimensional algebra over a field. For any integer $n \geq 1$ we have a natural isomorphism

$$\text{soc}_n \xrightarrow{\sim} \text{mod} A \xrightarrow{(-)^*} \text{mod} A^{\text{op}} \xrightarrow{\text{cap}_n} (\text{soc}_n (-))^* \cong \text{cap}_n ((-)^*).$$
Proof of Theorem 1.3. Let us prove the reciprocity part (1.2) first. From the definitions we have two short exact sequences

\[ 0 \longrightarrow \text{cap}_n P_i \longrightarrow \text{cap}^n P_i \longrightarrow \text{cap}^{n-1} P_i \longrightarrow 0 \]
\[ 0 \longrightarrow \text{soc}^{n-1} \nu P_j \longrightarrow \text{soc}^n \nu P_j \longrightarrow \text{soc}_n \nu P_j \longrightarrow 0. \]

By applying exact functors \( \text{Hom}_A(\cdot, \nu P_j) \) and \( \text{Hom}_A(P_i, \cdot) \) we have

\[ 0 \xrightarrow{\sim} \text{Hom}_A(\text{cap}^{n-1} P_i, \nu P_j) \xrightarrow{\sim} \text{Hom}_A(\text{cap}^n P_i, \nu P_j) \xrightarrow{\sim} \text{Hom}_A(\text{cap}^n P_i, \nu P_j) \xrightarrow{\sim} 0 \]
\[ 0 \xrightarrow{\sim} \text{Hom}_A(P_i, \text{soc}^{n-1} \nu P_j) \xrightarrow{\sim} \text{Hom}_A(P_i, \text{soc}^n \nu P_j) \xrightarrow{\sim} \text{Hom}_A(P_i, \text{soc}_n \nu P_j) \xrightarrow{\sim} 0, \]

where vertical isomorphisms follow from Lemma 2.3. Hence we get

\[ \text{Hom}_A(\text{cap}_n P_i, \nu P_j) \cong \text{Hom}_A(P_i, \text{soc}_n \nu P_j). \] (2.1)

Since the left hand side of (2.1) can be transformed as

\[ \text{Hom}_A(\text{cap}_n P_i, \nu P_j) \cong \text{Hom}_A(\text{cap}(\text{cap}_n P_i), \nu P_j) \]
\[ \cong \text{Hom}_A(\text{cap}(\text{cap}_n P_i), \text{soc} \nu P_j) \quad \text{(By Lemma 2.3)} \]
\[ \cong \text{Hom}_A(\text{cap}_n P_i, S_j) \quad \text{(By [8, Lemma III.5.1.i]i.)} \]

and the right hand side of (2.1) can be transformed as

\[ \text{Hom}_A(P_i, \text{soc}_n \nu P_j) \cong \text{Hom}_A(P_i, \text{soc}(\text{soc}_n \nu P_j)) \]
\[ \cong \text{Hom}_A(\text{cap} P_i, \text{soc}_n \nu P_j) \quad \text{(By Lemma 2.3)} \]
\[ \cong \text{Hom}_A(S_i, \text{soc}_n \nu P_j), \]

we have the desired isomorphism

\[ \text{Hom}_A(\text{cap}_n P_i, S_j) \cong \text{Hom}_A(S_i, \text{soc}_n \nu P_j). \]

Now let us prove the dual symmetry part (1.1). It follows immediately from the reciprocity part.

\[ \text{Hom}_A(\text{cap}_n P_i, S_j) \cong \text{Hom}_A(S_i, \text{soc}_n \nu P_j) \]
\[ \cong \text{Hom}_{A^\circ}(\text{soc}_n \nu P_j^\circ, S_i^\circ) \quad \text{(By [5, Lemma 2.8.6.ii].)} \]
\[ \cong \text{Hom}_{A^\circ}(\text{cap}_n((\nu P_j)^\circ), S_i^\circ) \quad \text{(By Lemma 2.4)} \]
\[ \cong \text{Hom}_{A^\circ}(\text{cap}_n(P_j^\circ), S_i^\circ). \]

\[ \square \]
Remark 2.5. The above proof is element-free.

We derive Theorem 1.1 and Corollary 1.2 from the main theorem in the following. The next characterization of symmetric algebras is well-known [7, Theorem 3.1].

**Theorem 2.6.** A finite dimensional algebra over a field is symmetric if and only if \((-)^* \cong (-)^\vee\).

**Corollary 2.7.** If a finite dimensional algebra over a field is symmetric then the Nakayama functor is naturally isomorphic to the identity functor.

**Proof of Theorem 1.1.** Apply Theorem 2.6 to (1.1).

**Proof of Corollary 1.2.** Apply Corollary 2.7 to (1.2).

Remark 2.8. Okuyama and Tsushima gave a short proof of the Landrock lemma for group algebras in [6, Theorem 2].

3. Example

In this section, Theorem 1.3 is illustrated by a simple example, a connected selfinjective Nakayama algebra. Let us describe this algebra and their modules first. See [8] for terminology.

3.1. Algebra and Modules

Let \(k\) and \(\ell\) be natural numbers and \(F\) an algebraically closed field. The quiver defined by

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
k-1 & \rightarrow & 3 \\
\end{array}
\]

is denoted by \(\Delta_k\). The ideal of path algebra \(F\Delta_k\) over the field \(F\) generated by the arrows of \(\Delta_k\) is denoted by \(R_k\). Set \(N_{\ell}^k = F\Delta_k / R_k^{\ell+1}\). This is a connected selfinjective Nakayama algebra [8, Theorem IV.6.15]. The simple module corresponding to the vertex \(i\) is denoted by \(S_i\) for \(i \in \mathbb{Z}/k\mathbb{Z}\). Since \(N_{\ell}^k\) is a Nakayama algebra, the projective cover \(P_i\) of \(S_i\) and the injective envelope \(I_i\) of \(S_i\) are uniserial. The composition series, radical series, and socle series...
of those modules coincide. Those structures are described by its composition factors as below.

\[
P_i = \begin{array}{c}
S_i \\
| \\
S_{i+1} \\
| \\
\vdots \\
| \\
S_{i+\ell}
\end{array} \quad I_i = \begin{array}{c}
S_i \\
| \\
S_{i-1} \\
| \\
\vdots \\
| \\
S_i
\end{array}
\]

The Nakayama functor thus acts as \( \nu P_j = I_j = P_{j-\ell} \) for \( j \in \mathbb{Z}/k\mathbb{Z} \).

3.2. Reciprocity

The reciprocity [1,2] is then illustrated by the following.

\[
P_i = \begin{array}{c}
S_i \\
| \\
S_{i+1} \\
| \\
\vdots \\
| \\
S_{i+(n-1)} \\
| \\
S_{i+\ell}
\end{array} \quad \sim rad_n P_i \quad \sim soc_n \nu P_j \sim S_{j-(n-1)} \sim = \nu P_j
\]

\[
\dim \text{Hom}_A(\text{rad}_n P_i, S_j) = \delta_{i+(n-1),j} = \delta_{i,j-(n-1)} = \dim \text{Hom}_A(S_i, soc_n \nu P_j)
\]

3.3. Dual Symmetry

The radical layers of \( P_j^\vee \) is obtained as follows. For \( 1 \leq n \leq \ell + 1 \) we have

\[
\text{rad}_n(P_j^\vee) \cong \text{cap}_n((P_j^\vee)^{**})
\]
\[
\cong (soc_n \nu P_j)^* \quad \text{(By Lemma 2.4)}
\]
\[
\cong S_{j-(n-1)}^*
\]
The dual symmetry (1.1) is hence illustrated by the following.

\[
\begin{array}{ccc}
S_i & \cdots & S_j^* \\
| & \cdots & | \\
S_{i+1} & \cdots & S_{j-1}^* \\
| & \cdots & | \\
\vdots & \cdots & \vdots \\
| & \cdots & | \\
S_{i+\ell} & \cdots & S_{j-\ell}^* \\
\end{array}
\]

\[
P_i = S_{i+(n-1)} \cong \text{rad}_n P_i \hspace{1cm} \text{rad}_n P_j^\vee \cong S_{j-(n-1)}^* = P_j^\vee
\]

\[
\dim \text{Hom}_A(\text{rad}_n P_i, S_j) = \delta_{i+(n-1),j} = \delta_{i,j-(n-1)} = \dim \text{Hom}_{A^{\text{op}}}(\text{rad}_n P_j^\vee, S_i^*)
\]

**Remark 3.1.** The examples $N_k^\ell$ contain the symmetric cases to which the Landrock lemma and its corollary are applicable. Indeed, the algebra $N_k^\ell$ is symmetric if and only if $k$ divides $\ell$ [8, Corollary IV.6.16].

4. Proofs of Lemmas

**Proof of Lemma 2.3.** Let $U$ and $V$ be $A$-modules. Define $F$-linear maps $\eta_{U,V}$ and $\xi_{U,V}$ by the following.

\[
\begin{array}{ccc}
\text{Hom}_A(\text{cap}^n U, V) & \xrightarrow{\eta_{U,V}} & \text{Hom}_A(U, \text{soc}^n V) \\
\xi_{U,V} & \xleftarrow{=} & \text{Hom}_A(\text{cap}^n U, V)
\end{array}
\]

\[
\eta_{U,V}(f): u \mapsto f(u + \text{rad}^n U) \hspace{1cm} (f \in \text{Hom}_A(\text{cap}^n U, V))
\]

\[
\xi_{U,V}(g): u + \text{rad}^n U \mapsto g(u) \hspace{1cm} (g \in \text{Hom}_A(U, \text{soc}^n V))
\]

The well-definedness of these maps follow from $\text{rad}^n U = U(\text{rad}^n A)$ and $\text{soc}^n V = \{ v \in V \mid v(\text{rad}^n A) = 0 \}$. It is routine work to check that these yield mutually inverse natural transformations. □

We prove Lemma 2.4 in the following. Let us introduce a lemma to prove the lemma.

**Lemma 4.1.** Let $A$ be a finite dimensional algebra over a field. For any integer $n \geq 0$ we have a natural isomorphism

\[
\text{cap}^n \cong \text{mod} A \xrightarrow{(-)^*} \text{mod} A^{\text{op}} \xleftarrow{\text{soc}^n} \text{soc}^n((-)^*) \cong (\text{cap}^n(-))^*.
\]
Proof. Let $U$ be an $A$-module. Then $F$-linear maps $\eta_{n,U}$ and $\xi_{n,U}$ defined by the following yield well-defined natural transformations.

$$\text{soc}^n(U^*) \xrightarrow{\eta_{n,U}} (\text{cap}^n U)^*$$

$$\eta_{n,U}(\lambda): u + \text{rad}^n U \mapsto \lambda(u) \quad (\lambda \in \text{soc}^n(U^*))$$

$$\xi_{n,U}(\mu): u \mapsto \mu(u + \text{rad}^n U) \quad (\mu \in (\text{cap}^n U)^*) \quad (4.1)$$

□

Proof of Lemma 2.4. To obtain a desired natural transformation, it is suffice to prove a similar statement

$$\text{soc}_n((-))^* \cong (\text{cap}_n(-))^* \quad (4.2)$$

since

$$(\text{soc}_n((-))^* \cong (\text{soc}_n((-))^{**})^* \cong (\text{cap}_n((-)))^{**} \cong \text{cap}_n((-))^*) \quad (4.2)$$

For an $A$-module $U$, consider the following diagram with exact rows

$$0 \xrightarrow{} \text{soc}^{n-1}(U^*) \xrightarrow{i_{n,U}} \text{soc}^n(U^*) \xrightarrow{p_{n,U}} \text{soc}_n(U^*) \xrightarrow{} 0$$

$$0 \xrightarrow{} (\text{cap}^{n-1} U)^* \xrightarrow{j_{n,U}} (\text{cap}^n U)^* \xrightarrow{q_{n,U}} (\text{cap}_n U)^* \xrightarrow{} 0,$$

where

- vertical arrows are the ones defined in (4.1),
- $i_{n,U}$ and $p_{n,U}$ are the inclusion and the canonical projection, and
- $j_{n,U}$ and $q_{n,U}$ are homomorphisms induced from the canonical projection and the inclusion by the $F$-dual functor.

Since the left rectangle commutes and each rows are exact, we have unique well-defined homomorphisms $\bar{i}_{n,U}$ and $\bar{q}_{n,U}$ that commute the right rectangle. A chase reveals that these are mutually inverse since $p_{n,U}$ and $q_{n,U}$ are epimorphism.

Let $U$ and $V$ be $A$-modules and $f \in \text{Hom}_A(U,V)$. A chase of the following diagram, which each face except the front commutes, reveals that the isomor-
phism $\eta_{n,U}$ is natural since $p_{n,V}$ is epimorphism.

\[
\begin{array}{ccc}
soc^n(U^*) & \overset{\eta_{n,U}}{\rightarrow} & (cap^n U)^* \\
\downarrow soc^n(f^*) & & \downarrow (cap^n f)^* \\
soc^n(V^*) & \overset{\eta_{n,V}}{\rightarrow} & (cap^n V)^* \\
\downarrow p_{n,V} & & \downarrow q_{n,V} \\
soc_n(U^*) & \overset{\eta_{n,U}}{\rightarrow} & (cap_n U)^* \\
\downarrow soc_n(f^*) & & \downarrow (cap_n f)^* \\
soc_n(V^*) & \overset{\eta_{n,V}}{\rightarrow} & (cap_n V)^* \\
\end{array}
\]

\[\square\]

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