SOME CRITERIA TO CHECK IF A PROJECTIVE HYPERSURFACES IS SINGULAR OR SMOOTH

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Abstract. In this paper we present some properties for projective hypersurfaces, smooth and singular, to be criteria for identification. To make the decision with these criteria, we have included procedures written in Singular language.

1. Introduction

Let \( S = \mathbb{C}[x_0, ..., x_n] \) be the graded ring of polynomials in \( x_0, ..., x_n \) with complex coefficients and denote by \( S_d \) the vector space of homogeneous polynomials in \( S \) of degree \( d \). For any polynomial \( f \in S_d \) we define the Jacobian ideal \( J_f \subset S \) as the ideal spanned by the partial derivatives \( f_0, ..., f_n \) of \( f \) with respect to \( x_0, ..., x_n \).

The Hilbert-Poincaré series of a graded \( S \)-module \( M \) of finite type is defined by

\[
HP(M)(t) = \sum_{k \geq 0} \dim M_k \cdot t^k
\]

and it is known, to be a rational function of the form

\[
HP(M)(t) = \frac{P(M)(t)}{(1-t)^{n+1}}.
\]

For any polynomial \( f \in S_d \) we define the hypersurface \( V(f) \) given by \( f = 0 \) in \( \mathbb{P}^n \) and the corresponding graded Milnor (or Jacobian) algebra by

\[
M(f) = S/J_f.
\]

Smooth hypersurfaces \( V(f_s) \) of degree \( d \) have all the same Hilbert-Poincaré series.

Proposition 1.1. The following statements are equivalent:
(i) The hypersurface \( V(f) \) is smooth.
(ii) The Hilbert-Poincaré series is

\[
HP(M(f))(t) = \frac{(1 - t^{d-1})^{n+1}}{(1-t)^{n+1}} = (1 + t + t^2 + \ldots + t^{d-2})^{n+1}
\]

For a proof see, for instance, \cite{3}, p. 109.

As soon as the hypersurface \( V(f) \) acquires some singularities, the series \( HP(M(f)) \) is an infinite sum.

Beyond the smooth case, there is just one general situation in which we know explicit formulas for \( HP(M(f)) \). For polynomials \( f \) such that the saturated Jacobian

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ideal \( \tilde{J}_f \) of the Jacobian ideal \( J_f \) is a complete intersection ideal of multidegree \((d_1, \ldots, d_n)\), then it is known that
\[
(1.4) \quad HP(M(f))(t) = \frac{(1 - t^{d-1})^{n+1} + t^{(n+1)(d-1)} - \sum d_i (1 - t^{d_i}) \cdots (1 - t^{d_n})}{(1 - t)^{n+1}},
\]
see Proposition 4, in [5].

For some curves or surfaces, we can find effective singularities and classify.

If there is at least one singular point and possibly how many they are. It is a very hard work to obtain manually these informations, even if we consider curves and surfaces with low degree. In the last part, we present procedures in Singular languages.

2. Smooth Hypersurfaces

Smooth hypersurfaces have the same Hilbert-Poincaré series associated to Fermat type
\[
F(t) = (1 + t + t^2 + \ldots + t^{d-2})^{n+1} = \sum_{k=0}^{T} a_k t^k, \text{ where } T = (n + 1)(d - 2).
\]
To the best of our knowledge, the only results are for \( n = 2 \):
\[
a_k = \begin{cases} 
\binom{k+2}{2}, & 0 \leq k \leq d - 2 \\
\binom{k+2}{2} - 3 \binom{k+3-d}{2}, & d - 1 \leq k \leq T/2 \\
a_{T-k}, & \frac{T}{2} < k \leq T.
\end{cases}
\]

In this section we show explicit formulas for the coefficients \( a_k \). To compute these coefficients, we need additional informations.

Lemma 2.1. The number of solutions in positive integers \((b_1 \geq 1, k \geq n + 1)\) of the equation \(b_0 + b_1 + \ldots + b_n = k\) is \(\binom{k-1}{n}\).

Proof. Let be the sequences 111 \ldots 111 with \( k \) bits and \( k - 1 \) spaces. If we keep only \( n \) spaces (and all other will be deleted), we have \( n + 1 \) groups of successive bits. There are obviously \(\binom{k-1}{n}\) ways of doing this. \(\square\)

Corollary 2.2. The number of solutions in nonnegative integers \((b_i \geq 0, k \geq 0)\) of the equation \(b_0 + b_1 + \ldots + b_n = k\) is \(\binom{n+k}{n}\).

Proof. The transformation \(b_i = c_i - 1, i = 0, 1, \ldots, n\) yields an equation \( c_0 + c_1 + \ldots + c_n = n + k + 1 \) with all \( c_i \geq 1 \). The number of solutions of this equivalent equation is \(\binom{n+k}{n}\). \(\square\)

Remark 2.3. If \( k \in \mathbb{N}, k \leq d - 2 \), the number of solutions in nonnegative integers, \(0 \leq b_i \leq d - 2\) of the equation \(b_0 + b_1 + \ldots + b_n = k\) is \(\binom{n+k}{n}\). We note that condition \( b_i \leq d - 2 \) is not necessary.

Remark 2.4. If \( k \in \mathbb{N}, k > (d - 2)(n + 1) \), then the equation \(b_0 + b_1 + \ldots + b_n = k\) with \( 0 \leq b_i \leq d - 2 \), has no solutions.

Lemma 2.5. The number of solutions in nonnegative integers \((b_i \geq 0)\) of the equation \(b_0 + b_1 + \ldots + b_n = k\) with \( m \geq d - 1 \) variables and the other \( n + 1 - m \leq d - 2 \) variables is \(\binom{n+k-m(d-1)}{n}\).
Proposition 2.6. Let $b_i = c_i + (d-1)$, $i = 0, 1, \ldots, m-1$ and $b_i = c_i$ for $i = m, \ldots, n$, yields an equation $c_0 + c_1 + \ldots + b_n = k - m(d-1)$. Because all $c_i \geq 0$ we have $\binom{n+k-m(d-1)}{n}$ solutions. □

Now we can prove the main result.

**Proof.** We can permute the variables, and first take $a = m$ elements. Because $b = k$ solutions for the equation: $b = k(n+1)$. Therefore we have $\binom{n}{k}$ solutions for the equation: $b = k(n+1)$. For the second part, $k \geq d-1$, is necessary to involve principle of inclusion-exclusion. If $B$ is the set of solutions for the equation, without the restrictions $b_i \leq d-2$, then $B$ has $\binom{n+k}{n}$ elements. Because $k \geq d-1$, we have some $i$ with $b_i \geq d-1$. Let $B_i$ be the set of this solutions with $b_i \geq d-1$ then $B_1, \ldots, B_n$ have the same cardinal, $\binom{n+k-(d-1)}{n}$ and the intersection of $m$ sets from $B_0, B_1, \ldots, B_n$ has $\binom{n+k-m(d-1)}{n}$ elements. If we have $m$ variables with $b_i \geq d-1$, then $m \leq \frac{k}{d-1}$ (otherwise, $b_0 + b_1 + \ldots + b_n > k$), so $1 \leq m \leq \left[ \frac{k}{d-1} \right]$. Hence $M_k$ is the difference of $B$ and the union of $B_0, B_1, \ldots, B_n$, based on inclusion-exclusion principle, we obtain:

$$a_k = Card(M_k) = \binom{n+k}{n} + \sum_{m=1}^{q} (-1)^m \binom{n+1}{m} \cdot \binom{n+k-m(d-1)}{n}.$$

□

**Formulas for some cases**

We need to compute the $a_k$ coefficients only for $0 \leq k \leq \frac{T}{2}$, because $a_k = a_{T-k}$ for $k \geq \frac{T}{2}$.

- **Case $n = 3$**
  $$a_k = \binom{k+3}{3}$$ for $0 \leq k \leq d-2$.
  $$a_k = \binom{k+3}{3} - 4 \cdot \binom{k+4-d}{3}$$ for $d-1 \leq k \leq \frac{T}{2} = 2d - 4 < 2d - 3$.

- **Case $n = 4$**
  $$a_k = \binom{k+4}{4}$$ for $0 \leq k \leq d-2$.
  $$a_k = \binom{k+4}{4} - 5 \cdot \binom{k+5-d}{4}$$ for $d-1 \leq k \leq 2d - 3$.
  $$a_k = \binom{k+4}{4} - 5 \cdot \binom{k+5-d}{4} + 10 \cdot \binom{k+6-2d}{4}$$ for $2d - 2 \leq k \leq \frac{T}{2} < 3d - 4$.

3. **Some criteria and examples**

In this section we present some criteria to check if the projective hypersurfaces is singular or smooth.
3.1. Critical points. For some curves or surfaces, we can find effective singularities and classify. For low degree and simple polynomials, we can find the critical points by solving the system of equations:

\[
\frac{\partial f}{\partial x_0} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0.
\]

Because \( f \) is homogenous of degree \( r \), by Euler’s formula we have: \( x_0 \frac{\partial f}{\partial x_0} + \cdots + x_n \frac{\partial f}{\partial x_n} = rf \). Obviously, if all partial derivatives of polynomial \( f \) vanish at \( p \), then the polynomial \( f \) vanishes at \( p \) too. Therefore, for finding singular points it is enough to get critical points and classify.

For example, in \( \mathbb{P}^2 \), these points \((0:0:1), (1:i:0), (1:-i:0)\) are nodes (type \( A_1 \)) for Lemniscate of Bernoulli: \((x^2 + y^2)^2 - 2(x^2 - y^2)z^2 = 0\) and cusps (type \( A_2 \)) for Cardioid: \((x^2 + y^2 + xz)^2 - (x^2 + y^2)z^2 = 0\).

Remark

It is sufficient to find just one coefficient which is different in the two series to infer

\[
\text{Remark}
\]

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3.2. Hilbert-Poincaré series. The Hilbert-Poincaré series is computed with two methods: combinatorial and based on a free resolution. For other examples see [6, 7].

To compute the Hilbert-Poincaré series, first recall that the quotient rings \( S/I \) and \( S/LT(I) \) have the same series, for any monomial ordering, where \( LT(I) \) is the ideal of leading terms of the ideal \( I \).

The projective equation of the Cissoid of Diocles is given by \( x^3 + xy^2 - y^2 z = 0 \).

The leading ideal of the jacobian \( J_f \) is: \( LI = (y^2, xy, x^2) \).

For the graded Milnor algebra, \( M = \oplus_{k \geq 0} M_k \) we show the bases for the homogeneous components: \( M_0 = \{1\}, M_1 = \{z, y, x\}, M_2 = \{z^2, yz, xz\}, M_3 = \{z^3, xz^2\} \), \( M_4 = \{z^4, xz^3\} \) and \( M_k = \{z^k, xz^{k-1}\} \) for all \( k \geq 3 \).

Finally, if we count the number of monomials in each homogeneous bases, we find the Hilbert-Poincaré series \( S(t) = 1 + 3t + 3t^2 + 2t^3 + \ldots \)

The projective equation of the curve called Kappa is \( f = (x^2 + y^2)y^2 - x^2 z^2 = 0 \).

Here is the minimal graded free resolution of Milnor algebra \( M \):

\[
0 \to R_3 \to R_2 \to R_1 \to R_0 \to M \to 0
\]

where \( R_0 = S, R_1 = S^3(-3), R_2 = S(-5) \oplus S^3(-6) \) and \( R_3 = S^2(-7) \).

To get the formulas for the Hilbert-Poincaré series, we start with the resolution \( \mathbb{B}_2 \) and get \( HP(M)(t) = HP(R_0)(t) - HP(R_1)(t) + HP(R_2)(t) - HP(R_3)(t) \).

Then we use the well-known formulas \( HP(N_1 \oplus N_2)(t) = HP(N_1)(t) + HP(N_2)(t) \), \( HP(N^p(-q))(t) = pt^q HP(N)(t) \), \( HP(S)(t) = \frac{1}{(1-t)^3} \) and we obtain:

\[
HP(M)(t) = \frac{1-3t^3 + t^6 - 2t^7}{(1-t)^3} = \frac{1 + 3t + 3t^2 - t^3 - 2t^4}{1-t} = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 4t^5 + \ldots
\]

Remark

It is sufficient to find just one coefficient which is different in the two series to infer that \( V(f) \) is singular.
For smooth curve of degree three, the Hilbert-Poincaré series is polynomial, $F_3(t) = (1 + t)^3 = 1 + 3t + 3t^2 + t^3$. For the Cissoid of Dioctes curve, $M_3 = (z^3, xz^2)$ with $\dim(M_3) = 2$ so the Cissoid is singular.

Another criteria is based on degree $T = (n + 1)(d - 2)$ of the polynomial for the smooth case. If $\dim(M_{T+1}) > 0$ the hypersurfaces is singular.

3.3. **Complete intersection.** We present projective hypersurfaces $V(f)$ for which the saturated Jacobian ideal $\hat{J}_f$ is a complete intersection of multidegree $(d_1, ..., d_n)$, so these $V(f)$ are singular and their Hilbert-Poincaré series $HP(M(f))(t)$ are infinite. We display the type of complete intersection $(d_1, ..., d_n)$, the generators for the saturated Jacobian ideal $\hat{J}_f$ and the Hilbert-Poincaré series.

- **singular curves in $\mathbb{P}^2$**
  
  $V(f) : f = x^4 + y^4 + xyz^2 = 0$ with $\hat{J}_f$ of type $(1,1)$, and two generators: $(x, y)$.
  
  $$HP(M(f))(t) = \frac{(1-t)^3 + t^4(1-t)(1-t)}{(1-t^3)^4} = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + (t^6 + t^7 + \ldots)$$
  
  (this curve has a single node).

- **singular surfaces in $\mathbb{P}^3$**
  
  $V(f) : f = x^2y^2 + xz^2 + yz^3 = 0$ with $\hat{J}_f$ of type $(2,2)$, and two generators: $(xy, z^2)$.
  
  $$HP(M(f))(t) = \frac{(1-t)^2 + t^4(1-t)^2}{(1-t^2)^4} = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 4(t^5 + t^6 + \ldots)$$
  
  (this curve has 2 cusps).

3.4. **Number of singularities.** In this subsection we assume that $V(f)$ has at most isolated singularities.

We consider the following ideals: $J = J_f$ the jacobian ideal and $I$ the radical of $J$ with $G_1 = \text{std}(J)$ and $G_2 = \text{std}(I)$ their respective standard Groebner basis, $\text{mult}_1$ the degree of $G_1$ and $\text{mult}_2$ the degree of $G_2$.

**Proposition 3.5.** We have the following statements:

- If $\text{mult}_1 < (d - 1)^{n+1}$ the hypersurfaces is singular and $\text{mult}_2$ is the number of singularities.
- If $\text{mult}_1 = (d - 1)^{n+1}$ the hypersurfaces is smooth.

**Proof.** Indeed, when $V(f)$ has isolated singularities, then one has $\text{mult}_1 = \tau(V(f))$, the total Tjurina number of $V(f)$, which is the sum of the Tjurina numbers $\tau(V(f), p)$ over all singular points $p$ of $V(f)$. This follows from the definition of the multiplicity.
of a homogeneous ideal and the equality \( \dim M(f)_k = \tau(V(f)) \), for \( k \) large, see [1]. Now, at each singular point we have the inequality \( \tau(V(f), p) \leq \mu(V(f), p) \), and by taking the sum we get the inequality \( \tau(V(f)) \leq \mu(V(f)) \) between the total Tjurina and the total Milnor numbers. Moreover, it is known that \( \mu(V(f)) \leq (d-1)^n \), see for instance Proposition (3.25), page 90 as well as the discussion on page 161 in [4]. It follows that in fact, for singular hypersurfaces we have the stronger inequality

\[
mult_1 \leq (d-1)^n.
\]

For the smooth case, the Hilbert-Poincaré series is polynomial, \( F(t) = (1 + t + t^2 + \ldots + t^{d-2})^{n+1} \) and we have \( \mult_1 = \sum_{k \geq 0} \dim M_k \cdot t^k = F(1) = (d-1)^{n+1} \).

**Corollary 3.6.** If the hypersurfaces is singular and \( \mult_1 = \mult_2 \) then all the singularities are nodes.

If there are two type of singularities on \( V(f) \), say \( A_p \) and \( A_q \) it is possible to find the number \( x, y \) of each type of singularities, by solving the system:

\[
\begin{align*}
px + qy &= \mult_1 \\
x + y &= \mult_2
\end{align*}
\]

For some case, there are many configuration, for example, if \( \mult_1 = 5 \) and \( \mult_2 = 3 \) we have two configurations \( A_1 + A_2 \) and \( 2A_2 + A_3 \).

In the following table we present some examples for projective surfaces \( f = 0 \) of degree three.

| polynomial \( f \) | \( \mult_1 \) | \( \mult_2 \) | Type     |
|-------------------|-------------|-------------|---------|
| \( wz + y^3 \)    | 6           | 3           | \( 3A_2 \) |
| \( w(xy + xz + yz) + xyz \) | 4           | 4           | \( 4A_1 \) |
| \( wz + (x + y)y^2 \) | 5           | 3           | \( A_1 + 2A_2 \) |
| \( wz + (x + z)y^2 \) | 5           | 3           | \( 2A_1 + A_3 \) |
| \( wz + y^2(x + y + z) \) | 4           | 3           | \( 2A_1 + A_2 \) |
| \( wz + (x + z)(y^2 - x^2) \) | 4           | 2           | \( A_1 + A_3 \) |
| \( wz + y^2z + yx^2 \) | 5           | 2           | \( A_1 + A_4 \) |
| \( x^3 + y^3 + z^3 + w^3 \) | 16          | -           | smooth  |

**3.7. Computation of genus.** The genus of a smooth irreducible curve defined by a polynomial of degree \( d \) is given by the formula: \( g_s = \frac{(d-1)(d-2)}{2} \).

The genus of a singular irreducible projective curve \( C : f = 0 \) (i.e. the genus of its smooth model) is given by \( g(C) = g_s - \delta \), where \( \delta \) is the sum of all local delta-invariants \( \delta_p \) of the singularities \( p \in C \). If \( C \) is smooth, then \( \delta = 0 \).

The Hermitian curve \( x^d + y^d = z^d \) and Fermat curve \( x^d + y^d + z^d = 0 \) are smooth in \( \mathbb{P}^2 \), of genus \( \frac{(d-1)(d-2)}{2} \).

**Remark** Every irreducible projective curves with genus \( g = \frac{(d-1)(d-2)}{2} - 1 \) is either nodal of type \( A_1 \) or has exactly one cusp \( A_2 \).

Several computer algebra packages are able to compute the genus of a plane curve.

In the following table we present some computations with procedure genus() from Singular library normal.lib.
### Table: Criteria to Check If a Projective Hypersurfaces Is Smooth or Singular

| Polynomial $f$                                                                 | Genus | Comments               |
|--------------------------------------------------------------------------------|-------|------------------------|
| $x^3 + xy^2 - y^2z$                                                             | 0     | singular ($A_2$)       |
| $x^3y + y^3z + xz^3$                                                            | 3     | smooth (Klein quartic) |
| $x^3yz + y^5 + z^5$                                                             | 5     | singular ($A_1$)       |
| $(x^2 + y^2)^2 - 2(x^2 - y^2)z^2$                                               | 0     | singular (Lemniscate of Bernoulli) |
| $(x^2 + y^2 + xz)^2 - (x^2 + y^2)z^2$                                           | 0     | singular (Cardioid)    |
| $(x^2 + y^2)y^2 - x^2z^2 = 0$                                                   | 0     | singular (Kappa)       |

The Lemniscate of Bernoulli, the Cardioid and the Kappa curve are irreducible with genus = 0 hence these are singular rational (i.e. parameterizable) curves.

### 4. Programs in Singular Language

For mathematical computations, we can use any CAS (Computer Algebra System) software like Mathematica, Matlab or Maple, but for Algebraic Geometry, the best are perhaps Singular, Macaulay2 or CoCoA.

Singular is a package developed at the University of Kaiserslautern, see [2] and [8].

```
proc criteria_mult(){
  // compute the number of singularities for projective hypersurfaces
  int n=2; // n+1 variables, you can change
  ring R=0,(x(0..n)),dp;
  poly f;
  // you can change polynomial f
  f=(x(0)^2+x(1)^2)^2-2*(x(0)^2-x(1)^2)*x(2)^2; // Lemniscata
  int d=deg(f);
  ideal J=jacob(f);
  ideal G1=std(J);
  ideal G2=std(radical(J));
  int mult1=mult(G1);
  int mult2=mult(G2);
  if(mult1==(d-1)^(n+1)) {print("this curve is smooth!");}
  else {print("this curve is singular! ");}
  if(mult1==mult2) {print("and nodal! ");}
}
```

```
proc criteria_ci(){
  // complete intersection criteria
  int n=2; // n+1 variables, you can change
  LIB "elim.lib";
  ring R=0,(x(0..n)),dp;
  poly f;
  // you can change polynomial f
  f=x(0)^3*x(1)^5+x(1)^8+x(2)^8 ; //(4,7)
  int d=deg(f);
  ```
ideal J=jacob(f);
ideal I=std(sat(J,maxideal(1))[[1]]);
if(size(I)!=n){
    print("Not Complete intersection: ");
    return();
}
print("Complete intersection with "+string(n)+" generators: ");
print(I);
int i, a;
intvec v;
a=(n+1)*(d-1);
for(i=1;i<=n;i++) {v[i]=deg(I[i]); a=a-v[i];}
ring R=0,t,ds;
poly P, Q;
Q=1;
for(i=1;i<=n;i++) {Q=Q*(1-t^v[i]);}
P=(1-t^(d-1))^(n+1)+t^a*Q;
print("Hilbert Poincare series P(t)/(1-t)^(n+1) where P(t) is: ");
print(P);
}

criteria_hp();
// compute Hilbert-Poincare series
int n=2; // n+1 variables, you can change
ring R=0,(x(0..n)),dp;
poly f;
// you can change polynomial f
f=(x(0)^2+x(1)^2+x(0)*x(2))^2 - (x(0)^2+x(1)^2)*x(2)^2; // cardioid
int d=deg(f);
ideal J=jacob(f);
ideal G=std(J);
int i, T;
intvec v;
T=(n+1)*(d-2);
for (i=1; i<=T+2;i++) {
    v[i]=size(kbase(G,i-1));
}
if (v[T+2]==0) {print("Smooth curve! "); return();}
if (v[T+2]>0) {print("The hypersurfaces is singular! ");}
if (v[T+2]<v[T+3]) {print("Not finite isolate singularities! ");return();}

ring R=0,t,ds;
poly F, S;
F=((1-t^(d-1))/(1-t))^(n+1);
S=0;
for(i=1;i<=T+2;i++) {S=S+v[i]*t^(i-1);}
print("Hilbert Poincare series is S(t)= "); S;
print("Hilbert Poincare series smooth (Fermat) case F(t)= "); F;
}

proc criteria_genus (){  
// genus, only for projective curve
LIB "normal.lib";
ring r=0,(x,y,z),dp;
poly f;
// you can change polynomial here
f=(x^2+y^2)^2-2*(x^2-y^2)*z^2; // lemniscata -> genus=0
int d=deg(f);
ideal I=f;
int g=genus(I);
if(g==(d-1)*(d-2)/2) {print("this curve is smooth !");}
else {print("this curve is singular !");}
}

REFERENCES

[1] A. D. R. Choudary, A. Dimca, Koszul complexes and hypersurface singularities, Proc. Amer. Math. Soc. 121 (1994), 1009-1016.
[2] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, Singular - A computer algebra system for polynomial computations. Available at http://www.singular.uni-kl.de.
[3] A. Dimca, Topics on Real and Complex Singularities, Vieweg Advanced Lecture in Mathematics, Friedr. Vieweg und Sohn, Braunschweig, 1987.
[4] A. Dimca, Singularities and Topology of Hypersurfaces, Universitext, Springer-Verlag, 1992.
[5] A. Dimca, Syzygies of Jacobian ideals and defects of linear systems, Bull. Math. Soc. Sci. Math. Roumanie Tome 56(104) No. 2, 2013, 191-203.
[6] A. Dimca, G. Sticlaru, On the syzygies and Alexander polynomials of nodal hypersurfaces, Math. Nachr. 285(2012), 2120-2128.
[7] A. Dimca, G. Sticlaru, Koszul complexes and pole order filtrations, P Edinburgh Math Soc, accepted 2013.
[8] G.-M. Greuel, G. Pfister, A Singular Introduction to Commutative Algebra (with contributions by O. Bachmann, C. Lossen, and H. Schnemann). Springer-Verlag 2002 (second edition 2007).

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