HYPERCONTRACTIVITY IN NON-COMMUTATIVE
HOLOMORPHIC SPACES

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Abstract. We prove an analog of Janson’s strong hypercontractivity in a class of non-commutative “holomorphic” algebras. Our setting is the \(q\)-Gaussian algebras \(\Gamma_q\) associated to the \(q\)-Fock spaces of Bozejko, Kümmerer and Speicher, for \(q \in [-1, 1]\). We construct sub-algebras \(\mathcal{H}_q \subset \Gamma_q\), a \(q\)-Segal-Bargmann transform, and prove Janson’s strong hypercontractivity \(L^2(\mathcal{H}_q) \to L^r(\mathcal{H}_q)\) for \(r\) an even integer.

1. Introduction

As part of the work in the 1960s and 1970s to construct a mathematically consistent theory of interacting quantum fields, Nelson proved his famous hypercontractivity inequality in its initial form \([N1]\); by 1973 it evolved into the following statement, which may be found in \([N2]\).

Theorem 1.1 (Nelson, 1973). Let \(A_\gamma\) be the Dirichlet form operator for Gauss measure \(d\gamma(x) = (2\pi)^{-n/2}e^{-|x|^2/2}dx\) on \(\mathbb{R}^n\). For \(1 < p \leq r < \infty\) and \(f \in L^p(\mathbb{R}^n, \gamma)\),

\[
\|e^{-tA_\gamma}f\|_r \leq \|f\|_p, \quad \text{for} \quad t \geq t_N(p, r) = \frac{1}{2} \log \frac{r-1}{p-1}.
\] (1.1)

For \(t < t_N(p, r)\), \(e^{-tA_\gamma}\) is not bounded from \(L^p\) to \(L^r\).

(If \(p < 2\), one must first extend \(e^{-tA_\gamma}\) to \(L^p\); this can be done, uniquely, and Theorem 1.1 should be interpreted as such in this case. The same comment applies to all of the following.) It is worth noting that \(t_N\), the least time to contraction, does not depend on the dimension \(n\) of the underlying space \(\mathbb{R}^n\).

The initial purpose of such hypercontractive inequalities was to prove the semiboundedness of Hamiltonians in the theory of Boson quantum fields. (See, for example, \([Gli]\), \([N1]\), and \([Se2]\).) In \([G1]\), Gross used this inequality (through an appropriate cut-off approximation) to show that the Boson energy operator in a model of 2-dimensional Euclidean quantum field theory has a unique ground state. In that paper he also showed that if one represents the Fock space for Fermions as the \(L^2\)-space of a Clifford algebra (as in \([Se1]\), then inequalities similar to 1.1 also hold. He developed this further in \([G3]\).

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Over the subsequent three decades, Nelson’s hypercontractivity inequality (and its equivalent form, the logarithmic Sobolev inequality, invented by Gross in [G2]) found myriad applications in analysis, probability theory, differential geometry, statistical mechanics, and other areas of mathematics and physics. See, for example, the recent survey [G5].

The Fermion hypercontractivity inequality in [G3] remained unproven in its sharp form until the early 1990s. Lindsay [L] and Meyer [LM] proved that it holds \( L^2 \to L^r \) for \( r = 2, 4, 6, \ldots \) (and in the dual cases \( L^r \to L^2 \) as well). Soon after, Carlen and Lieb [CL] were able to complete Gross’ original argument with some clever non-commutative integration inequalities, thus proving the full result. (Precisely: they showed that the Clifford algebra analogs of the inequalities 1.1 hold with exactly the same constants.)

Then, in 1997, Biane [B1] extended Carlen and Lieb’s work beyond the Fermionic (Clifford algebra) setting to the \( q \)-Gaussian von Neumann algebras \( \Gamma_q \) of Bozejko, Kümmerer, and Speicher [BKS]. His theorem may be stated as follows.

**Theorem 1.2** (Biane, 1997). Let \(-1 < q < 1\), let \( N_q \) denote the number operator associated to \( \Gamma_q \), and let \( \| \cdot \|_p \) be the non-commutative \( L^p \)-norm associated to the vacuum expectation state \( \tau_q \) on \( \Gamma_q \). Then for \( 1 < p \leq r < \infty \),

\[
\| e^{-tN_q} f \|_r \leq \| f \|_p \text{ for all } f \in L^p(\Gamma_q, \tau_q) \text{ iff } t \geq t_N(p, r).
\]

Of particular interest is the case \( q = 0 \) which corresponds to free probability. Biane proved the full result (for \(-1 < q < 1\)) by first extending Carlen and Lieb’s work to the case of a system of mixed spins (in a von Neumann algebra generated by elements which satisfy some commutation and some anti-commutation relations), and then applying a central limit theorem due to Speicher [S]. The case \( q = -1 \) is Carlen and Lieb’s adaptation of Gross’ work, while the \( q = 1 \) case is Nelson’s original hypercontractive estimate (Theorem 1.1).

Concurrent to the work on non-commutative hypercontractivity, a different sort of extension of Nelson’s theorem was being developed. In 1983 Janson, [J], discovered that if one restricts the semigroup \( e^{-tA_{\gamma}} \) in Theorem 1.1 to holomorphic functions on \( \mathbb{R}^{2n} \cong \mathbb{C}^n \) then the contractivity of Equation 1.1 is attained in a shorter time than \( t_N \). Writing \( \mathcal{HL}^p = L^p(\mathbb{R}^{2n}, \gamma) \cap \text{Hol}(\mathbb{C}^n) \), Janson’s strong hypercontractivity may be stated thus.

**Theorem 1.3** (Janson, 1983). Let \( 0 < p \leq r < \infty \), and let \( f \in \mathcal{HL}^p \). Then

\[
\| e^{-tA_{\gamma}} f \|_r \leq \| f \|_p, \text{ for } t \geq t_J(p, r) = \frac{1}{2} \log \frac{r}{p}.
\]  

(1.2)

For \( t < t_J(p, r) \), \( e^{-tA_{\gamma}} \) is not bounded from \( \mathcal{HL}^p \) to \( \mathcal{HL}^r \).

Note that the least time \( t_J \) to contraction is shorter than the time \( t_N \) (if \( 1 < p < r < \infty \)). Moreover, Janson’s result holds as \( p \to 0 \), in a regime
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where the semigroup $e^{-tA}$ is not even well-defined in the full $L^p$-space. These results have been further generalized by Gross in [G4] to the case of complex manifolds.

In this paper, non-commutative algebras $\mathcal{H}_q$ will be introduced, which are $q$-deformations of the algebra of holomorphic functions. The special cases $q = \pm 1$ and $q = 0$ are already known; $\mathcal{H}_{-1}$ is defined in [BSZ], while $\mathcal{H}_0$ is isomorphic to the free Segal-Bargmann space of $[B2]$. We will construct a unitary isomorphism $\mathcal{S}_q$ from $L^2(\Gamma_q)$ to $L^2(\mathcal{H}_q)$, which is a $q$-analogue of the Segal-Bargmann transform. $\mathcal{H}_q$ itself will be constructed as a subalgebra of $\Gamma_q$, and so inherits its $p$-norms as well as its number operator $N_q$. In the context of these $q$-deformed Segal Bargmann spaces, the following theorem is our main result.

**Theorem 1.4.** For $-1 \leq q < 1$ and $r$ an even integer,

$$\|e^{-tN_q}f\|_r \leq \|f\|_2$$

for all $f \in L^2(\mathcal{H}_q, \tau_q)$ iff $t \geq t_J(2, r)$.

It is interesting that the least time to contraction, $t_J$, is independent of both the dimension of the underlying space and the parameter $q$. We fully expect the same results to hold $L^p(\mathcal{H}_q) \to L^r(\mathcal{H}_q)$ for $2 \leq p \leq r < \infty$, but standard interpolation techniques fail to work in the holomorphic algebras we consider. (In particular, the dual results that Lindsay and Meyer achieved in the full Clifford algebra do not follow in this holomorphic setting.)

This paper is organized as follows. We begin with a summary of the $q$-Fock spaces $F_q$ and the von Neumann algebras $\Gamma_q$ associated to them. We will also define the holomorphic subalgebras $\mathcal{H}_q$ and construct a $q$-Segal-Bargmann transform. In the subsequent section, we prove the appropriate strong hypercontractivity estimates for algebras with arbitrary mixed spins (mixed commutation and anti-commutation relations), much in the spirit of Biane’s approach [B1]. We then proceed to review Speicher’s central limit theorem, and apply it to prove Theorem 1.4.

## 2. The $q$-Fock space and associated algebras

We begin by briefly reviewing the $q$-Fock spaces of Bozejko, Kümmerer and Speicher, relevant aspects of the von Neumann algebras $\Gamma_q$ (which are related to the creation and annihilation operators on $F_q$), and the number operators on them. We then proceed to define the Banach algebra $\mathcal{H}_q$ which corresponds to the classical Segal-Bargmann space, and exhibit a $\ast$-isomorphism between $\mathcal{H}_0$ and the free Segal-Bargmann space $\mathcal{C}_{hol}$ defined in [B2]. We finally construct a generalized $q$-Segal-Bargmann transform, which is a unitary isomorphism $L^2(\Gamma_q) \to L^2(\mathcal{H}_q)$ that respects the action of the number operator.

### 2.1. The $q$-Fock space $F_q$ and the algebra $\Gamma_q$

Our development closely follows that found in [B1]; the details may be found in [BKS]. Let $\mathcal{H}$ be a real Hilbert space with complexification $\mathcal{H}_C$. Let $\Omega$ be a unit vector in a

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1-dimensional complex Hilbert space (disjoint from $\mathcal{H}_c$). We refer to $\Omega$ as the vacuum, and by convention define $\mathcal{H}_c^\otimes 0 \equiv C\Omega$. The algebraic Fock space $\mathcal{F}(\mathcal{H})$ is defined as

$$\mathcal{F}(\mathcal{H}) \equiv \bigoplus_{n=0}^{\infty} \mathcal{H}_c^\otimes n,$$

where the direct sum and tensor product are algebraic. For any $q \in [-1,1]$, we then define a Hermitian form $(\cdot,\cdot)_q$ to be the conjugate-linear extension of $(\Omega,\Omega)_q = 1$,

$$(f_1 \otimes \cdots \otimes f_j, g_1 \otimes \cdots \otimes g_k)_q = \delta_{jk} \sum_{\pi \in \mathcal{S}_k} q^{\iota(\pi)} (f_1 \cdot \cdot \cdot f_j, g_1 \cdot \cdot \cdot g_k),$$

for $f_i, g_i \in \mathcal{H}_c$, where $\mathcal{S}_k$ is the symmetric group on $k$ symbols, and $\iota(\pi)$ counts the number of inversions in $\pi$; that is

$$\iota(\pi) = \#\{(i,j) : 1 \leq i < j \leq k, \pi i > \pi j\}.$$ 

The reader may readily verify that $(-1)^{\iota(\pi)} = \text{parity}(\pi)$ for any permutation $\pi$. Hence, the form $(\cdot,\cdot)_{-1}$ reduces to the standard Hermitian form associated to the Fermion Fock space. Similarly, the form $(\cdot,\cdot)_1$ yields the standard Hermitian form on the Boson Fock space. In each of these cases the form is degenerate, thus requiring that we take a quotient of $\mathcal{F}(\mathcal{H})$ before completing to form the Fermion or Boson Fock space. It is somewhat remarkable that, for $-1 < q < 1$, the form $(\cdot,\cdot)_q$ is already non-degenerate on $\mathcal{F}(\mathcal{H})$.

**Proposition 2.1 (BKS).** The Hermitian form $(\cdot,\cdot)_q$ is positive semi-definite on $\mathcal{F}(\mathcal{H})$. Moreover, it is an inner product on $\mathcal{F}(\mathcal{H})$ for $-1 < q < 1$.

For $-1 < q < 1$, the $q$-Fock space $\mathcal{F}_q(\mathcal{H})$ is defined as the completion of $\mathcal{F}(\mathcal{H})$ with respect to the inner-product $(\cdot,\cdot)_q$. (It should be noted that, in the case $q = 0$, the definition of the form $(\cdot,\cdot)_0$ requires the convention that $0^0 = 1$. It follows that $\mathcal{F}_0(\mathcal{H})$ is just $\bigoplus_{n=0}^{\infty} \mathcal{H}_c^\otimes n$ with the Hilbert space tensor product and direct sum.) These spaces interpolate between the classical Boson and Fermion Fock spaces $\mathcal{F}_{\pm 1}(\mathcal{H})$, which are constructed by first taking the quotient of $\mathcal{F}(\mathcal{H})$ by the kernel of $(\cdot,\cdot)_{\pm 1}$ and then completing.

As in the classical theory, the spaces $\mathcal{F}_q$ come equipped with creation and annihilation operators. For any vector $f \in \mathcal{H} \subset \mathcal{H}_c$, define the creation operator $c_q(f)$ on $\mathcal{F}_q(\mathcal{H})$ to extend

$$c_q(f)\Omega = f$$

$$c_q(f)f_1 \otimes \cdots \otimes f_k = f \otimes f_1 \otimes \cdots \otimes f_k.$$
The annihilation operator $c_q^*(f)$ is its adjoint, which the reader may compute satisfies
\[ c_q^*(f)\Omega = 0 \]
\[ c_q^*(f)f_1 \otimes \cdots \otimes f_k = \sum_{j=1}^k q^{j-1}(f_j, f) f_1 \otimes \cdots \otimes f_{j-1} \otimes f_{j+1} \otimes \cdots \otimes f_k. \]

These are similar to the definitions of the creation and annihilation operators in the Fermion and Boson cases, where appropriate (anti)symmetrization must also be applied. One notable difference is that, in the Boson ($q = 1$) case, the operators are unbounded. For $q < 1$, the creation and annihilation operators are always bounded, and hence we may discuss the von Neumann algebra they generate without difficulties.

The operators $c_q, c_q^*$ satisfy the $q$-commutation relations, which interpolate between the canonical commutation relations (CCR) and canonical anticommutation relations (CAR) usually associated to the Boson and Fermion Fock spaces. Over the $q$-Fock space, we have
\[ c_q^*(g)c_q(f) - qc_q(f)c_q^*(g) = (f, g)\text{id}_{F_q(\mathcal{H})} \text{ for } f, g \in \mathcal{H}. \quad (2.1) \]

It is worth pausing at this point to note one significant difference between the $q = \pm 1$ cases and the $-1 < q < 1$ cases. For both Bosons and Fermions, the operators $c, c^*$ also satisfy additional (anti)commutation relations. In the Boson case, for example, $c(f)$ and $c(g)$ commute for any choices of $f$ and $g$. It is a fact, however, that if $q \neq \pm 1$ there are no relations between $c_q(f)$ and $c_q(g)$ if $(f, g) = 0$.

It is a well-known theorem that the creation and annihilation operators in the Boson and Fermion cases are irreducible; that is, they have no non-trivial invariant subspaces. That theorem is also true for the operators $c_q$ for $-1 < q < 1$, although a published proof does not seem to exist. We prove it here for completeness. The $q = 0$ case will be used in Proposition 2.5 below.

**Theorem 2.2.** For $-1 < q < 1$, the von Neumann algebra generated by \{c_q(h) ; h \in \mathcal{H}\} is $B(F_q(\mathcal{H}))$.

**Proof.** Denote by $\mathcal{W}_q$ the von Neumann algebra generated by the $c_q$’s. We consider the $q = 0$ case first. Let \{e_1, e_2, \ldots\} be an orthonormal basis for $\mathcal{H}$, and consider the operator
\[ P = \sum_{j=1}^\infty c(e_j)c^*(e_j) \]
(where $c(h) = c_0(h)$), which is in $\mathcal{W}_0$ since $\mathcal{W}_0$ is weakly closed. It is easy to calculate that $P(e_{i_1} \otimes \cdots \otimes e_{i_k}) = e_{i_1} \otimes \cdots \otimes e_{i_k}$, while $P\Omega = 0$. Thus, $P$ is the projection onto the orthogonal complement of the vacuum. So
The operator which is the rank-1 operator with image spanned by $e_{i_1} \otimes \cdots \otimes e_{i_n}$, and kernel orthogonal to $e_{j_1} \otimes \cdots \otimes e_{j_m}$. It follows that $\mathcal{W}_0$ contains all finite rank operators, and hence is the full algebra $\mathcal{B}(\mathcal{F}_0(\mathcal{H}))$.

For $q \neq 0$, it is proved in [DN] that there is a unitary map $U_q : \mathcal{F}_0 \to \mathcal{F}_q$, which preserves the vacuum and satisfies

$$U_q' \mathcal{C}_0 U_q^* \subseteq \mathcal{C}_q,$$

where $\mathcal{C}_q$ is the $C^*$-algebra generated by $\{c_q(h) : h \in \mathcal{H}\}$. As $\mathcal{W}_q$ is the weak closure of $\mathcal{C}_q$, it follows easily that $\mathcal{B}(\mathcal{F}_q(\mathcal{H})) = U_q \mathcal{B}(\mathcal{F}_0(\mathcal{H})) U_q^* = U_q \mathcal{W}_0 U_q^* \subseteq \mathcal{W}_q$ as well, and this completes the proof. \hfill \Box

For $q < 1$ and for each $f \in \mathcal{H}$, define the self-adjoint operator $X_q(f)$ on $\mathcal{F}_q(\mathcal{H})$ by $X_q(f) = c_q(f) + c_q^*(f)$. These operators are in $\mathcal{W}_q = \mathcal{B}(\mathcal{F}_q(\mathcal{H}))$, but they do not generate it. The von Neumann algebra they do generate is defined to be $\Gamma_q(\mathcal{H})$, the $q$-Gaussian algebra over $\mathcal{H}$. (In the $q = 1$ case, $\Gamma_1(\mathcal{H})$ is the von Neumann algebra generated by the operators $\varphi(X(f))$ for $\varphi \in L^\infty(\mathbb{R})$.) The notation $\Gamma_q$ is chosen to be consistent with the second quantization functor from constructive quantum field theory (see [BSZ]), which assigns to each real Hilbert space $\mathcal{H}$ a von Neumann algebra $\Gamma(\mathcal{H})$ and to each contraction $T : \mathcal{H} \to \mathcal{K}$ a unital positivity-preserving map $\Gamma(T) : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{K})$. Indeed, $\Gamma_q$ can be construed as such a functor as well.

The isomorphism classes of the von Neumann algebras $\Gamma_q(\mathcal{H})$ for $q \notin \{\pm 1, 0\}$ are not yet understood. (For some partial results, however, see [R] and [Sn].) The $\pm 1$ cases have been understood since antiquity: $\Gamma_1(\mathcal{H}) = L^\infty(M, \gamma)$ for a certain measure space $M$ with a Gaussian measure $\gamma$, while $\Gamma_{-1}(\mathcal{H})$ is a Clifford algebra modeled on $\mathcal{H}$. These facts rely upon the additional commutation relations between $c(f)$ and $c(g)$ that hold in those cases. (Indeed, in the Boson case $X(f)$ and $X(g)$ commute, resulting in a commutative von Neumann algebra $\Gamma(\mathcal{H})$. It is primarily for this reason that it is customary to begin with a real Hilbert space and complexify – if $c(f)$ were defined for all $f \in \mathcal{H}_c$, then $c(f)$ and $c(g)$ would no longer commute even in the Boson case. While there are no commutation relations between $c_q(f)$ and $c_q(g)$, it is still advantageous for us to have the real subspace $\mathcal{H} \subset \mathcal{H}_c$ in order to define the holomorphic subalgebra in section 22.) $\Gamma_0(\mathcal{H})$ was shown (in [V]) to be isomorphic to the group von Neumann algebra of a free group with countably many generators.

One known fact about the algebras $\Gamma_q(\mathcal{H})$ for $-1 < q < 1$ is that they are all type $II_1$ factors. This is a consequence (in the $\dim \mathcal{H} = \infty$ case) of the following theorem, which was proved in [BSZ].
Proposition 2.3 (Bozejko, Speicher). Let \(-1 < q < 1\). The vacuum expectation state \(\tau_q(A) = (A\Omega, \Omega)_q\) on \(\mathcal{B}(\mathcal{F}_q(\mathcal{H}))\) restricts to a faithful, normal, finite trace on \(\Gamma_q(\mathcal{H})\).

The reader may wish to verify that \(\tau_q(c_q^*c_q) = 1\), while \(\tau_q(c_qc_q^*) = 0\); hence, \(\tau_q\) is certainly not a trace on all of \(\mathcal{B}(\mathcal{F}_q(\mathcal{H}))\).

The algebra \(\Gamma_q\) can actually be included as a dense subspace of \(\mathcal{F}_q\). The map \(A \mapsto A\Omega\) is one-to-one from \(\Gamma_q\) into \(\mathcal{F}_q\). The precise action of this map will be important to us, and so it bears mentioning. The \(q\)-Hermite polynomials \(H_n^q\) are one-variable real polynomials defined so that \(H_0^q(x) = 1\), \(H_1^q(x) = x\), and satisfying the following recurrence relation:

\[
xH_n^q(x) = H_{n+1}^q(x) + \frac{q^n - 1}{q - 1}H_{n-1}^q(x),
\]

where \((q^n - 1)/(q - 1)\) is to be interpreted as \(n\) when \(q = 1\). In this case, the generated polynomials \(H_n^1\) are precisely the Hermite polynomials that play an important role in the Boson theory. When \(q = 0\), the polynomials \(H_n^0\) are the Tchebyshev polynomials, and play an analogous role in the theory of semi-circular systems (see [V]). We can express the action of the above map \(A \mapsto A\Omega\) succinctly in terms of the polynomials \(H_n^q\). The following proposition is proved in [BKS].

**Proposition 2.4.** The map \(A \mapsto A\Omega\) from \(\Gamma_q\) to \(\mathcal{F}_q\) is one-to-one, and extends to a unitary isomorphism \(L^2(\Gamma_q, \tau_q) \to \mathcal{F}_q\). If \(\{e_j\}\) are orthonormal vectors in \(\mathcal{H}\) and \(j_\ell \neq j_{\ell+1}\) for \(1 \leq \ell \leq k - 1\), then

\[
H_{n_1}^q(X_q(e_{j_1})) \cdots H_{n_k}^q(X_q(e_{j_k}))\Omega = e_{j_1}^{\otimes n_1} \cdots e_{j_k}^{\otimes n_k}.
\]

The algebraic Fock space \(\mathcal{F}(\mathcal{H})\) carries a number operator \(N\), whose action is given by

\[
N\Omega = 0, \quad N(f_1 \otimes \cdots \otimes f_n) = nf_1 \otimes \cdots \otimes f_n.
\]

This operator extends to a densely-defined, essentially self-adjoint operator \(N_q\) on \(\mathcal{F}_q(\mathcal{H})\). The algebra \(\Gamma_q\) then inherits the action of \(N_q\), via the map in Proposition 2.3. The reader may readily check that if \(\{e_j\}\) are orthonormal vectors in \(\mathcal{H}\) and \(j_\ell \neq j_{\ell+1}\) for \(1 \leq \ell \leq k - 1\), then the element

\[
H_{n_1}^q(X_q(e_{j_1})) \cdots H_{n_k}^q(X_q(e_{j_k}))
\]

is an eigenvector of \(N_q\) with eigenvalue \(n_1 + \cdots + n_k\). In the case \(\mathcal{H} = \mathbb{R}^d\), this is a precise analogy to the action of the number operator for Bosons. The algebra \(\Gamma_1(\mathbb{R}^d)\) is isomorphic to \(L^\infty(\mathbb{R}^d, \gamma)\), and the operators \(X_1(e_j)\) (for the standard basis vectors \(e_j\)) are multiplication by the coordinate functions \(x_j\). The Boson number operator \(A_\gamma\) has \(H_{n_1}^1(x_1) \cdots H_{n_k}^1(x_k)\) as an eigenvector, with eigenvalue \(n_1 + \cdots + n_k\).
The number operator generates a contraction semigroup $e^{-tN_q}$ on $L^2(\Gamma_q, \tau_q)$, which is known to restrict for $p > 2$, and extend for $1 \leq p < 2$, to a contraction semigroup on $L^p(\Gamma_q, \tau_q)$. Biane’s hypercontractivity theorem, Theorem 1.2, is an extension of these results.

2.2. the holomorphic algebra, and the $q$-Segal-Bargmann transform. Let $q < 1$. We wish to define a Banach algebra of “holomorphic” elements in $B(F_q)$. To that end, we follow a similar procedure to the formal construction of holomorphic polynomials. We begin by doubling the number of variables, and so we consider the algebra $\Gamma_q(H \oplus \bar{H})$. This algebra contains two independent copies of the variable $X(h) \in \Gamma_q(H)$:

$$Z(h) = \frac{1}{\sqrt{2}}(X(h, 0) + iX(0, h)). \quad (2.4)$$

In the case $H = \mathbb{R}$ and $q = 1$, this precisely corresponds to the holomorphic variable $z = (x + iy)/\sqrt{2}$, the normalization chosen so that $z$ is a unit vector in $HL^2(\gamma)$. We define the $q$-holomorphic algebra $H_q(H_C)$ as the Banach algebra generated by $\{Z(h) : h \in H\}$.

In [B2], Biane introduced a Banach algebra $C_{hol}$ in the $q = 0$ case which is also an analog of the algebra of holomorphic functions. His algebra is not contained in $\Gamma_0(H \oplus \bar{H})$, so it is less natural to consider an action of $\mathbb{N}_0$ on it. We introduce it here (with slightly changed notation to avoid inconsistencies) to show that it is isomorphic to $H_0$, and so the work presented here indeed generalizes Biane’s results. Consider the von Neumann algebra $B(F_0(H \oplus \bar{H}))$; it contains all the operators $c_0(h, g)$ and their adjoints, for $h, g \in H$. Define the operator

$$B(h) = c_0(h, 0) + c_0^*(0, h).$$

Let $C(H_C)$ be the von Neumann algebra generated by $\{B(h) : h \in H\}$, and $C_{hol}(H_C)$ the Banach algebra so generated. The vacuum expectation state $\tau_0(A) = (\Omega, A\Omega)$ restricts to a faithful, normal, finite trace on $C(H_C)$, and the map $h \mapsto B(h)$ is a circular system with respect to $\tau_0$ (see [V]).

Although Biane’s algebra $C_{hol}(H_C)$ is not contained in $\Gamma_0(H \oplus \bar{H})$, it is in fact isomorphic to our algebra $H_0(H_C)$, in the following strong sense.

**Proposition 2.5.** There is a $\ast$-automorphism of $B(F_0(H \oplus \bar{H}))$ which maps $\Gamma_0(H \oplus \bar{H})$ onto $C(H_C)$. In particular, it maps $Z(h)$ to $B(h)$, and so sends $H_0(H_C)$ to $C_{hol}(H_C)$. 
Proof. By Theorem 2.2 we can define an endomorphism of $B(\mathcal{F}_0(\mathcal{H} \oplus \mathcal{H}'))$ on the generators $c_0(h, g)$ by

$$\alpha(c_0(h, g)) = \frac{1}{\sqrt{2}}(c_0(h, h) + ic_0(-g, g))$$

$$\alpha(c_0^*(h, g)) = \frac{1}{\sqrt{2}}(c_0^*(h, h) - ic_0^*(-g, g)).$$

A straightforward computation verifies that the operators $\alpha(c_0(h, g))$ satisfy the 0-commutation relations of Equation 2.1. Hence, $\alpha$ is a $\ast$-homomorphism. It can also easily be checked that $\alpha$ has an inverse of the form

$$\alpha^{-1}(c_0(h, g)) = \frac{1}{\sqrt{2}}(c_0(h + g, 0) + ic_0(0, h - g))$$

$$\alpha^{-1}(c_0^*(h, g)) = \frac{1}{\sqrt{2}}(c_0^*(h + g, 0) - ic_0^*(0, h - g)),$$

which also extends to a $\ast$-homomorphism. Hence, $\alpha$ is a $\ast$-automorphism. Finally, one can calculate that $\alpha(Z(h)) = B(h)$. Whence, $\alpha$ maps $\mathcal{H}_0$ onto $\mathcal{C}_{\text{hol}}$, and so maps $W^*(\mathcal{H}_0(\mathcal{H}_0')) = \Gamma_0(\mathcal{H} \oplus \mathcal{H}')$ onto $\mathcal{C}(\mathcal{H}_0')$. □

It should be noted that Proposition 2.3 generalizes automatically to $q \neq 0$; however, we are only concerned with the $q = 0$ case for $\mathcal{C}_{\text{hol}}$.

**Corollary 2.6.** The map $\alpha$ from $\Gamma_0(\mathcal{H} \oplus \mathcal{H}')$ to $\mathcal{C}(\mathcal{H}_0')$ extends to an isometric isomorphism $L^p(\Gamma_0, \tau_0) \to L^p(\mathcal{C}, \tau_0)$ for $0 < p \leq \infty$.

**Proof.** Since $\alpha$ is a $\ast$-automorphism of the full von Neumann algebra $B(\mathcal{F}_0)$, by Wigner’s theorem it is induced (through conjugation) by either a unitary or an anti-unitary on $\mathcal{F}_0$. Suppose it is a unitary, $U$, so that $\alpha(A) = U^*AU$ for each $A \in B(\mathcal{F}_0)$. Recall that $\tau_0$ is known to restrict to a tracial state on both $W^*(\mathcal{H}_0)$ and $\mathcal{C}$. Hence, for $A \in \mathcal{H}_0$ and $p > 0$,

$$||\alpha(A)||_p^p = \tau_0(|\alpha(A)|^p) = \tau_0(\alpha(||A||^p)) = \tau_0(U^*||A||^pU) = \tau_0(||A||^p) = ||A||_p^p.$$

It follows that $\alpha$ extends to an isometric isomorphism from $L^p(\mathcal{H}_0, \tau_0)$ onto $L^p(\mathcal{C}_{\text{hol}}, \tau_0)$. The anti-unitary case is similar. □

Hence, the algebraic map which sends $Z(h)$ to $B(h)$ preserves all $L^p$ topology (even for $p < 1$), and so the analyses of the spaces $\mathcal{H}_0$ and $\mathcal{C}_{\text{hol}}$ are very much the same.

In the commutative context, one of the most powerful tools in this area is the *Segal-Bargmann transform* $\mathcal{S}$, which is a unitary isomorphism

$$\mathcal{S} : L^2(\mathbb{R}^d, \gamma) \to \mathcal{H}L^2(\mathbb{Z}^d, \gamma').$$

Here, $\gamma'$ denotes the measure whose density with respect to Lebesgue measure is (a constant multiple of) the complex Gaussian $\exp(-|z|^2)$, rather than $\exp(-|x|^2/2)$ as in $\gamma$. The Hermite polynomials $H_{n_1}^1(x_1) \cdots H_{n_d}^1(x_d)$,
appropriately normalized, form an orthonormal basis of $L^2(\mathbb{R}^d, \gamma)$, and the action of $\mathcal{S}$ on this basis is simple:

$$\mathcal{S}: H_{n_1}^1(x_1) \cdots H_{n_d}^1(x_d) \mapsto z_1^{n_1} \cdots z_d^{n_d}. \quad (2.5)$$

So $\mathcal{S}$ maps the Hermite polynomials to the holomorphic monomials.

(A note on normalization. Instead of changing the measure $\gamma \rightarrow \gamma'$, we could redefine $\mathcal{S}' : L^2(\mathbb{R}^d, \gamma) \rightarrow H^2L^2(\mathbb{C}^d, \gamma)$ by setting $\mathcal{S}'f(z) = \mathcal{S}f(z/\sqrt{2})$. This map is, of course, a unitary isomorphism. It is this point of view that we take while generalizing the Segal-Bargmann transform. After all, we have already built the factor $1/\sqrt{2}$ in to the variable $Z(h)$.)

In [B2], a free Segal-Bargmann transform is introduced, which is a unitary isomorphism $L^2(\Gamma_0, \tau_0) \rightarrow L^2(\mathcal{C}_{hol}, \tau_0)$. We will modify this transform and extend it to all $q \in [-1, 1]$, and further show that besides generalizing the classical transform $\mathcal{S}$ it respects the action of the number operator. First, we will need to understand the embedding of $H_q(\mathcal{H}_c)$ in $F_q(\mathcal{H} \oplus \mathcal{H})$ (it is injected via the map $A \mapsto A\Omega$, which is one-to-one on all of $\Gamma_q(\mathcal{H} \oplus \mathcal{H})$ by Proposition 2.4). Consider the diagonal mapping $\delta : \mathcal{H}_c \rightarrow \mathcal{H}_c \oplus \mathcal{H}_c$ defined $\delta(h) = 2^{-1/2}(h, ih)$. Since $\delta$ is isometric, it extends to an isometric embedding $\delta_q : F_q(\mathcal{H}) \hookrightarrow F_q(\mathcal{H} \oplus \mathcal{H})$ (that is, $\delta_q\delta(h_1 \otimes h_2 \otimes \cdots) = \delta(h_1) \otimes \delta(h_2) \otimes \cdots$).

**Proposition 2.7.** The map $A \mapsto A\Omega$ injecting $H_q(\mathcal{H}_c) \hookrightarrow F_q(\mathcal{H} \oplus \mathcal{H})$ extends to a unitary isomorphism $L^2(\mathcal{H}_q, \tau_q) \rightarrow \delta_q F_q(\mathcal{H})$. If $\{e_j\}$ are orthonormal vectors in $\mathcal{H}$, then

$$Z_q(e_{j_1})^{n_1} \cdots Z_q(e_{j_k})^{n_k} \Omega = \delta_q(e_{j_1}^{\otimes n_1} \otimes \cdots \otimes e_{j_k}^{\otimes n_k}). \quad (2.6)$$

**Proof.** Let $\phi = h_1 \otimes \cdots \otimes h_n \in F(\mathcal{H})$, and consider $Z_q(h)\delta(\phi)$. We may compute

$$X_q(h, 0)\delta(\phi) = (c_q(h, 0) + c_q^*(h, 0))\delta(\phi)$$

$$= (h, 0) \otimes \delta(\phi) + \sum_{j=1}^{\infty} q^{j-1} \left(2^{-1/2}(h_j, ih_j), (h, 0)\right)_{\mathcal{H} \oplus \mathcal{H}} \delta(\hat{\phi}_j)$$

$$= (h, 0) \otimes \delta(\phi) + \frac{1}{\sqrt{2}} \sum_{j=1}^{\infty} q^{j-1}(h_j, h)_{\mathcal{H}} \delta(\hat{\phi}_j),$$

where $\hat{\phi}_j = h_1 \otimes \cdots \otimes h_{j-1} \otimes h_{j+1} \otimes \cdots \otimes h_n$. A similar calculation shows that

$$X_q(0, h)\delta(\phi) = (0, h) \otimes \delta(\phi) + \frac{1}{\sqrt{2}} \sum_{j=1}^{\infty} q^{j-1}(ih_j, h)_{\mathcal{H}} \delta(\hat{\phi}_j),$$

and so in the sum $X_q(h, 0) + iX_q(0, h)$ the $c_q^*$ terms cancel. (Note, we have assumed as is standard that the complexified inner product $(h, g)$ is linear in $h$ and conjugate-linear in $g$.) Thus, we have $Z_q(h)\delta(\phi) = 2^{-1/2}(h, ih) \otimes \delta(\phi) = \delta(h \otimes \phi)$. Equation 2.6 now follows by induction, and the theorem follows since such vectors are dense in $\delta_q F_q(\mathcal{H})$. \(\square\)
By Equations 2.3 and 2.6, the action of $S_\delta$ solutions 2.4 and 2.7 give (up to the map $1 \leq k \leq$ the classical Segal-Bargmann transform. Comparing Equations 2.5 and 2.7, we see that $S_\delta$ can be seen from Proposition 2.4. Hence, we have $S_\delta$ the Fock space the conjugated action $S_\delta$ an eigenvector of $N_q$. That is, $H$ theorem), it is equal to the intersection of $L^p(H_q(\mathcal{H}), \tau_q)$, and give a combinatorial proof of a strong hypercontractivity theorem like Theorem 1.4 for it.

Since $H_q(\mathcal{H}_C)$ is contained in $\Gamma_q(\mathcal{H} \oplus \mathcal{H})$, it inherits the number operator $N_q$ from it, induced by the inclusion of $\Gamma_q(\mathcal{H} \oplus \mathcal{H})$ into $F_q(\mathcal{H} \oplus \mathcal{H})$ via the map $A \mapsto A\Omega$. From Equation 2.6, we see then that $Z_q(e_1)^{n_1} \cdots Z_q(e_k)^{n_k}$ is an eigenvector of $N_q$ with eigenvalue $n_1 + \cdots + n_k$. This is precisely matches the conjugated action $S_\delta N_q S_\delta^*$ of the number operator $N_q$ on $\Gamma_q(\mathcal{H})$, as can be seen from Proposition 2.4. Hence, we have $S_\delta N_q = N_q S_\delta$, just as in the commutative case.

Finally, we define $L^p(H_q(\mathcal{H}_C), \tau_q)$ to be the completion of $H_q(\mathcal{H}_C)$ in the $L^p(\Gamma_q(\mathcal{H} \oplus \mathcal{H}), \tau_q)$-norm. For $p \geq 2$ (the case of interest for our main theorem), it is equal to the intersection of $L^2(H_q(\mathcal{H}_C), \tau_q)$ with $L^p(\Gamma_q(\mathcal{H} \oplus \mathcal{H}), \tau_q)$. The class of Banach spaces $L^p(H_q, \tau_q)$ is a non-commutative generalization of the spaces $H L^p(C^d, \gamma')$ that occur in Janson’s Theorem 1.3. Since the algebra $H_q$ is not a von Neumann algebra, this family is not known to be complex interpolation scale. For example, in the $q = 1$ case, the family is not complex interpolation scale when $\mathcal{H}$ is infinite-dimensional (this is almost proven in [JPR]). Hence, once we have proved Theorem 1.4, it is not an easy matter to generalize to the case $p > 2, r \neq 2, 4, 6, \ldots$

3. Mixed Spin and Strong Hypercontractivity

We will consider the mixed-spin algebras $\mathcal{C}(I, \sigma)$ introduced in [3], which represent systems with some commutation and some anti-commutation relations. Such systems may be viewed as approximations to the $q$-commutation relations, in a manner which will be made precise in Section 4. We introduce a holomorphic subalgebra $\mathcal{H}(I, \sigma)$, and give a combinatorial proof of a strong hypercontractivity theorem like Theorem 1.4 for it.
3.1. **The mixed-spin algebra** $\mathcal{C}(I, \sigma)$. Let $I$ be a finite totally ordered set (with cardinality denoted by $|I|$), and let $\sigma$ be a function $I \times I \to \{-1, 1\}$ which is symmetric, $\sigma(i, j) = \sigma(j, i)$, and constantly $-1$ on the diagonal, $\sigma(i, i) = -1$. Let $\mathcal{C}(I, \sigma)$ denote the unital $\mathbb{C}$-algebra with generators $\{x_i ; i \in I\}$ and relations

$$x_i x_j - \sigma(i, j)x_j x_i = 2\delta_{ij} \quad \text{for} \quad i, j \in I. \quad (3.1)$$

(The requirement $\sigma(i, i) = -1$ forces $x_i^2 = 1$, and guarantees that $\mathcal{C}(I, \sigma)$ is finite-dimensional.) In the special case $\sigma \equiv -1$, this is precisely the complex Clifford algebra $\mathcal{C}_I$, hence our choice of notation. In the case $\sigma(i, j) = 1$ for $i \neq j$ (i.e. when different generators commute), the generators of $\mathcal{C}(I, \sigma)$ may be modeled by $|I|$ i.i.d. Bernoulli random variables, and so we reproduce the toy Fock space considered in [M]. In the general case, $\mathcal{C}(I, \sigma)$ has, as a vector space, a basis consisting of all $x_A$ with $A = (i_1, \ldots, i_k)$ increasing multi-indices in $I^k$, where $x_A = x_{i_1} \cdots x_{i_k}$, and $x_\emptyset$ denotes the identity $1 \in \mathcal{C}(I, \sigma)$. Thus, $\dim \mathcal{C}(I, \sigma) = 2^{|I|}$. Moreover, $\mathcal{C}(I, \sigma)$ has a natural decomposition

$$\mathcal{C}(I, \sigma) = \bigoplus_{n=0}^{|I|} \mathcal{C}_n(I, \sigma),$$

where $\mathcal{C}_n = \text{span}\{x_A ; |A| = n\}$ is the “$n$-particle space.” Of some importance to us will be the natural grading of the algebra,

$$\mathcal{C}(I, \sigma) = \mathcal{C}_+(I, \sigma) \oplus \mathcal{C}_-(I, \sigma),$$

where $\mathcal{C}_+ = \bigoplus\{\mathcal{C}_n ; n \text{ is even}\}$, and $\mathcal{C}_-$ is the corresponding odd subspace. The reader may readily verify that this decomposition is a grading – i.e. $\mathcal{C}_\alpha \cdot \mathcal{C}_\beta \subseteq \mathcal{C}_{\alpha\beta}$, where $\alpha, \beta \in \{+, -\}$ and their product is to be interpreted in the obvious fashion.

We equip $\mathcal{C}(I, \sigma)$ with an involution $*$, which is defined to be the conjugate-linear extension of the map $x_A^* = x_{A^*}$, where $(i_1, \ldots, i_k)^*$ is the reversed multi-index $(i_k, \ldots, i_1)$. In particular, the generators $x_i = x_i^*$ are self-adjoint, and in general $x_A^* = \pm x_A$. We also define a tracial state $\tau_\sigma$ by $\tau_\sigma(x_A) = \delta_{AB}$; that is, $\tau_\sigma(1) = 1$ while $\tau_\sigma(x_A) = 0$ for all other basis elements. It is easy to check that $\tau_\sigma(ab) = \tau_\sigma(ba)$. This allows us to define an inner product on $\mathcal{C}(I, \sigma)$ by

$$(a, b)_\sigma = \tau_\sigma(b^* a).$$

The basis $\{x_A\}$ is orthonormal with respect to $(\cdot, \cdot)_\sigma$. Following the GNS construction, the action of $\mathcal{C}(I, \sigma)$ on the Hilbert space $(\mathcal{C}(I, \sigma), (\cdot, \cdot)_\sigma)$ by left-multiplication is continuous, and yields an injection of $\mathcal{C}(I, \sigma)$ into the von Neumann algebra of bounded operators on the Hilbert space. In this way, $\mathcal{C}(I, \sigma)$ gains a von Neumann algebra structure. We denote by $L^p(\mathcal{C}(I, \sigma), \tau_\sigma)$ the non-commutative $L^p$ space of this von Neumann algebra with its trace $\tau_\sigma$. (So, in particular, $L^2(\mathcal{C}(I, \sigma), \tau_\sigma)$ is naturally isomorphic to the Hilbert space $(\mathcal{C}(I, \sigma), (\cdot, \cdot)_\sigma)$. The $L^p(\mathcal{C}(I, \sigma), \tau_\sigma)$-norm is, in fact,
just the (normalized) Schatten $L^p$-norm on the matrix algebra. This can be seen from the following Proposition.

**Proposition 3.1.** Let $tr$ denote the normalized trace on the finite-dimensional algebra $B(L^2(\mathcal{H}(I, \sigma), \tau_\sigma))$. Then for any $x \in \mathcal{H}(I, \sigma)$,

$$\tau_\sigma(x) = tr(x).$$

**Proof.** Using the orthonormal basis $\{x_A\}$ for $L^2(\mathcal{H}(I, \sigma), \tau_\sigma)$, we compute that

$$tr(x_A) = 2^{-|I|} \sum_B (x_A x_B, x_B)_\sigma = \begin{cases} 0, & \text{if } A \neq \emptyset \\ 2^{-|I|} \sum_B (x_B, x_B)_\sigma = 1, & \text{if } A = \emptyset \end{cases}$$

where the sums are taken over all increasing multi-indices $B$. So $tr(x_A) = \delta_{A\emptyset} = \tau_\sigma(x_A)$. □

Note that the trace $\tau_\sigma$ may be expressed in terms of the inner product as the pure state $\tau_\sigma(x) = (x_1, 1)_\sigma$. This formula extends to all of $B(L^2(\mathcal{H}(I, \sigma), \tau_\sigma))$, giving the pure state $\beta \mapsto (\beta 1, 1)_\sigma$. However, this state does not equal $tr$ for all bounded operators $\beta$. We will see examples in Section 4 showing that it is not tracial in general.

The algebra $\mathcal{H}(I, \sigma)$ comes equipped with a number operator $N_\sigma$ which has $\mathcal{H}_n(I, \sigma)$ as an eigenspace with eigenvalue $n$. That is,

$$N_\sigma x_A = |A|x_A.$$

This is a generalization of the action of the operator $N_{-1}$ on the Clifford algebra $\mathcal{C}_{|I|} = \Gamma_{-1}(\mathbb{R}|I|)$. $N_\sigma$ is a positive semi-definite operator on $L^2(\mathcal{H}(I, \sigma), \tau_\sigma)$, and so generates a contraction semigroup $e^{-tN_\sigma}$. It is to the study of this semigroup, restricted to a holomorphic subspace, that we devote the remainder of this section.

### 3.2. The mixed-spin holomorphic algebra $\mathcal{H}(I, \sigma)$.

Following our construction of $\mathcal{H}_q$, we will begin by doubling the number of variables. We extend $\sigma$ to the set $I \times \{0, 1\}$ by setting

$$\sigma((i, \zeta), (j, \zeta')) = \sigma(i, j),$$

and then consider the algebra $\mathcal{H}(I \times \{0, 1\}, \sigma)$. If we relabel $x_{(i,0)} \rightarrow x_i$ and $x_{(i,1)} \rightarrow y_i$, then this is tantamount to constructing the unital $\mathbb{C}$-algebra with relations

$$\begin{aligned}
&x_i x_j - \sigma(i, j)x_j x_i = 2\delta_{ij} \\
y_i y_j - \sigma(i, j)y_j y_i = 2\delta_{ij} \\
x_i y_j - \sigma(i, j)y_j x_i = 0
\end{aligned} \quad \text{for } i, j \in I. \quad (3.2)$$

Note, $\mathcal{H}(I, \sigma)$ is $*$-isomorphically embedded in $\mathcal{H}(I \times \{0, 1\}, \sigma)$ via the inclusion $x_i \mapsto x_{(i,0)}$. Hence, this relabeling should not be confusing.

We define elements $z_j \in \mathcal{H}(I \times \{0, 1\}, \sigma)$ by

$$z_j = 2^{-1/2}(x_j + iy_j) = 2^{-1/2}(x_{(j,0)} + ix_{(j,1)}), \quad \text{for } j \in I. \quad (3.3)$$
is, in fact, possible to think of $\hat{z}$ of view is that the pure state $\sigma$ in the case $2\hat{z}$ which sends $\hat{z}$ representation, the $C$ as the $z$ generated by $\hat{z}$ and the same relations (of course) hold for the $\hat{z}$.

The reader may readily verify that Equation 3.5 looks much like the $\parallel$-commutation relations of Equation 2.1. It is, in fact, possible to think of $\hat{z}$, $\hat{z}^*$ as creation and annihilation operators. That is, there is a faithful representation of $\hat{z}$, $\hat{z}^*$ in $B(L^2(\mathcal{C}(I, \sigma), \tau_\sigma))$, which sends $\hat{z}$ and $\hat{z}^*$ to the creation and annihilation operators $\beta_j, \beta_j^*$ on $L^2(\mathcal{C}(I, \sigma), \tau_\sigma)$ discussed in Section 4. (By our definition, the operators $\hat{z}, \hat{z}^*$ are a priori in the doubled space $B(L^2(\mathcal{C}(I \times \{0, 1\}, \sigma), \tau_\sigma))$.) This representation, the spin-chain representation, is discussed in [CL31] in detail in the case $\sigma \equiv -1$, and is generalized in [B31]. The problem with this point of view is that the pure state $\beta \mapsto (\beta 1, 1)_\sigma$ on $B(L^2(\mathcal{C}(I, \sigma), \tau_\sigma))$ does not correspond to the trace $\tau_\sigma$ under the representation. So, we prefer not to think of $\hat{z}$, $\hat{z}^*$ as creation and annihilation operators.

A simple calculation shows that if $|A| = n$ then $z_A \in \mathcal{C}_n(I \times \{0, 1\}, \sigma)$, and so $N_\sigma z_A = |A| z_A$. Thus, $\mathcal{H}(I, \sigma)$ is a reducing subspace for the (self-adjoint) operator $N_\sigma$ on $L^2(\mathcal{C}(I \times \{0, 1\}, \sigma), \tau_\sigma)$. Note also that the action of $N_\sigma$ on $z_A$ mirrors that of $N_\sigma$ on $x_A$. In fact, this can be stated in terms of a $\sigma$-Segal-Bargmann transform: the map $\mathcal{S}_\sigma: x_A \mapsto z_A$ is a unitary isomorphism of $L^2(\mathcal{C}(I, \sigma), \tau_\sigma)$ onto $L^2(\mathcal{H}(I, \sigma), \tau_\sigma)$, and $\mathcal{S}_\sigma N_\sigma = N_\sigma \mathcal{S}_\sigma$.

The main part of the proof of Theorem 1.4 is the following strong hypercontractivity result regarding the semigroup $e^{-tN_\sigma}$ acting on $\mathcal{H}(I, \sigma)$.

**Theorem 3.2.** For $p = 2$ and $r$ an even integer,

$$\|e^{-tN_\sigma}a\|_r \leq \|a\|_p \text{ for all } a \in \mathcal{H}(I, \sigma) \text{ iff } t \geq t_j(p, r) = \frac{1}{2} \log \frac{r}{p}.$$
We expect the theorem holds for $2 \leq p \leq r < \infty$. (The case $p < 2$ may be somewhat different from the commutative case; in a communication from L. Gross, a calculation showed that in 1-dimension the least time to contraction seems to be larger than the Janson time for some $p, r < 2$.) If $I \neq \emptyset$, it is easy to see that the Janson time cannot be improved for any $p, r > 0$, again by calculation in the 1-dimensional case.

**Proof of the ‘only if’ direction of theorem 3.2.** Let $a(\epsilon) = 1 + \epsilon \hat{z} \in \mathcal{H}(I, \sigma)$ where $\hat{z} = \hat{z}_j$ for some $j \in I$. Then $|a(\epsilon)|^2 = (1 + \epsilon \hat{z}^*)(1 + \epsilon \hat{z}) = 1 + \epsilon(\hat{z} + \hat{z}^*) + \epsilon^2 |\hat{z}|^2 = 1 + \epsilon x + \epsilon^2 |\hat{z}|^2$, where $x = x_j$. Hence,

$$|a(\epsilon)|^{2p} = (1 + \epsilon(x + \epsilon|\hat{z}|^2))^p$$

$$= 1 + p\epsilon(x + \epsilon|\hat{z}|^2) + \frac{p(p-1)}{2}\epsilon^2(x + \epsilon|\hat{z}|^2)^2 + o(\epsilon^2)$$

$$= 1 + \epsilon(px) + \epsilon^2 \left(p|\hat{z}|^2 + \frac{p(p-1)}{2}x^2\right) + o(\epsilon^2).$$

Now, $|\hat{z}|^2 = (1/2)(1 + ixy)$ where $y = y_j$, and so $\tau_{\sigma}|\hat{z}|^2 = 1/2$. Also $x^2 = 1$, and $\tau_{\sigma}x = 0$. Therefore,

$$\|a(\epsilon)\|_{2p} = (\tau_{\sigma}|a(\epsilon)|^{2p})^{1/2p} = \left(1 + \left(p \cdot \frac{1}{2} + \frac{p(p-1)}{2}\right)\epsilon^2 + o(\epsilon^2)\right)^{1/2p}$$

$$= 1 + \frac{p}{4}r^2 + o(\epsilon^2).$$

So $\|a(\epsilon)\|_p = 1 + (p/8)e^2 + o(\epsilon^2)$. Now, $e^{-tN_{\sigma}}a(\epsilon) = 1 + e^{-t}\hat{z} = a(e^{-t}\epsilon)$. Thus, in order for $\|e^{-tN_{\sigma}}a(\epsilon)\|_r \leq \|a(\epsilon)\|_p$, we must have

$$1 + \frac{1}{8}e^{-2t}r^2 + o(\epsilon^2) \leq 1 + \frac{p}{8}r^2 + o(\epsilon^2),$$

and so as $\epsilon \to 0$, it follows that $e^{-2t} \leq p/r$ — or $t \geq t_f(p, r)$. \[Q.E.D.\]

Hence, the necessity condition holds for all $r \geq p > 0$. For the sufficiency, however, the tools available to us are extremely limited (due to the fact that $\mathcal{H}(I, \sigma)$ is not a $*$-algebra). We are forced to give a combinatorial proof, which cannot reach beyond the cases when $p = 2$ and $r$ is even. The remainder of the 'if' direction of Theorem 3.2 is the main subject of Section 3.3.

### 3.3. Strong hypercontractivity for $\mathcal{H}(I, \sigma)$.

We will prove Theorem 3.2 by induction on $|I|$. Note, in the case $|I| = 0$, the algebra $\mathcal{H}(I, \sigma)$ is just $\mathbb{C}$. Since the action of $e^{-tN_{\sigma}}$ on $\mathbb{C}$ is trivial, and since all $\|\cdot\|_p$ norms are equal to the complex modulus $|\cdot|$, the sufficiency condition follows automatically in this case. Now, suppose the strong hypercontractivity result of Theorem 3.2 holds for the algebras $\mathcal{H}(I', \sigma')$ with $|I'| \leq d$. Let $I$ be a set of size $d + 1$, and $\sigma$ a spin-assignment on $I$. Select any fixed element $i \in I$. Any element $a \in \mathcal{H}(I, \sigma)$ can be uniquely decomposed as

$$a = b + \hat{z}_i c, \quad b, c \in \mathcal{H}(I - \{i\}, \sigma|_{I - \{i\}}).$$

(3.6)
For convenience, throughout we will refer to $I - \{i\}$ as $J$, and to $\hat{z}_i$ as $\hat{z}$, $x_i$ as $x$, and so forth. Since $|J| = d$, the inductive hypothesis is that $H(J, \sigma|_J)$ satisfies the strong hypercontractivity estimate of Theorem 3.2.

The quantity $|\hat{z}|^2$ will often come up in calculations, and so we give it a name: $\xi = |\hat{z}|^2 = \hat{z}^* \hat{z}$. We will also encounter $\hat{z} \hat{z}^*$, but by Equation 3.5 $\hat{z} \hat{z}^* = 1 - \xi$. The following lemma records some of the important properties of the operators $\xi$, $\hat{z}$, and $\hat{z}^*$. All of the statements may be verified by trivial calculation.

**Lemma 3.3.** The following properties hold for $\xi$, $\hat{z}$, and $\hat{z}^*$.

1. $\xi^p = \xi$ for $p > 0$.
2. $\xi$ is independent of $\mathcal{C}(J \times \{0, 1\}, \sigma|_J)$ — that is, for each $u \in \mathcal{C}(J \times \{0, 1\}, \sigma|_J)$, $\xi u = u \xi$ and $\tau_\sigma(\xi u) = \tau_\sigma(\xi) \tau_\sigma(u) = \frac{1}{2} \tau_\sigma(u)$.
3. Let $u \in \mathcal{C}(J \times \{0, 1\}, \sigma|_J)$, let $h \in \{\hat{z}, \hat{z}^*, \xi, 1 - \xi\}$, and let $p > 0$. Then $\|hu\|_p = 2^{-1/p}\|u\|_p$.
4. $\xi \hat{z} = \hat{z}^* \xi = 0$, $\hat{z} \xi = \hat{z}$, and $\hat{z} \hat{z}^* = \hat{z}^*$.

The commutativity in item 2 above follows in large part from the fact that $\xi = \frac{1}{2}(1 + ixy) \in \mathcal{C}_+(I \times \{0, 1\}, \sigma)$. The grading plays an important role in the combinatorics to follow. In fact, the grading of $\mathcal{C}(\{i\} \times \{0, 1\}, \sigma|_{\{i\}})$ induces a grading on the full algebra $\mathcal{C}(I \times \{0, 1\}, \sigma)$. We refer to this grading by

$$\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-,$$

$$\mathcal{C}_+ \cdot \mathcal{C}_- \subseteq \mathcal{C}_0.$$

So, for example, the element $y_j \xi$ ($i \neq j$) is in $\mathcal{C}_+^i(I \times \{0, 1\}, \sigma)$, even though it is in $\mathcal{C}_-(I \times \{0, 1\}, \sigma)$. Note that

$$\mathcal{C}_+^i(I \times \{0, 1\}, \sigma) = \{\hat{z} u + \hat{z}^* v ; u, v \in \mathcal{C}(J \times \{0, 1\}, \sigma|_J)\}.$$

For any such $u$, $\tau_\sigma(\hat{z} u) = (\hat{z}, u^*)\sigma = 0$, and similarly $\tau_\sigma(\hat{z}^* u) = 0$. It follows that $\tau_\sigma|_{\mathcal{C}_+^i} = 0$. Using the graded structure, this leads to the following important lemma, which aids in the calculation of moments.

**Lemma 3.4.** Let $v^0 \in \mathcal{C}_+^i(I \times \{0, 1\}, \sigma)$ and $v^1 \in \mathcal{C}_-^i(I \times \{0, 1\}, \sigma)$. Let $\eta$ be $\{0, 1\}$-sequence of length $n$, and denote by $|\eta|$ the sum $\eta_1 + \cdots + \eta_n$ of its entries (i.e. the number of 1s). Then the element $v^n = v^{n_1} \cdots v^{n_n}$ has $\tau_\sigma(v^n) = 0$ if $|\eta|$ is odd.

Now, we proceed to expand the moments of $|a|^2$. Using the decomposition in 3.6 we have $|a|^2 = (b + \hat{z} c)^*(b + \hat{z} c) = |b|^2 + b^* \hat{z} c + c^* \hat{z}^* b + c^* |\hat{z}|^2 c$. That is,

$$|a|^2 = (|b|^2 + \xi |c|^2) + (b^* \hat{z} c + c^* \hat{z}^* b) = v^0 + v^1. \quad (3.7)$$

Equation 3.7 decomposes $|a|^2$ into its $\mathcal{C}_+^i$ and $\mathcal{C}_-^i$ parts, $v^0 = |b|^2 + \xi |c|^2$ and $v^1 = b^* \hat{z} c + c^* \hat{z}^* b$. It follows immediately that

$$\|a\|^2 = \tau_\sigma(v^0) = \tau_\sigma(|b|^2 + \xi |c|^2) = \|b\|^2 + \frac{1}{2} \|c\|^2. \quad (3.8)$$
The factor of $1/2$ (unusual in Pythagoras’ formula) is due to our choice to normalize $\hat{\xi}$ in $L^\infty$ and not in $L^2$. More generally, for the $n$th moment of $|a|^2$,

$$\|a\|_{2n}^2 = \tau_\sigma(|a|^{2n}) = \tau_\sigma[(v^0 + v^1)^n] = \sum_{\eta \in 2^n} \tau_\sigma(\nu^\eta),$$

where $2^n$ denotes the set of all $\{0,1\}$-sequences of length $n$. Using Lemma 3.4, we have

$$\|a\|_{2n}^2 = \sum_{\eta \text{ even}} \tau_\sigma(\nu^\eta) = \sum_{k=0}^{|n/2|} \sum_{|\eta|=2k} \tau_\sigma(\nu^\eta).$$

Now, the term $\nu^\eta$ is a product of $n$ terms, each of which is either $|b|^2 + |c|^2$ or $b^* \hat{\xi} c + c^* \hat{\xi}^* b$. Define

$$v^{00} = |b|^2 \quad v^{01} = |c|^2 \quad v^{10} = b^* \hat{\xi} c \quad v^{11} = c^* \hat{\xi}^* b.$$  

Then we may write $v^\eta$ as

$$v^\eta = \sum_{\nu \in 2^n} v^{\nu^\eta} = \sum_{\nu \in 2^n} v^{\eta_{1}^{\nu_{1}} \cdots v^{\eta_{n}^{\nu_{n}}} \cdot}.$$  

It should be noted that many of the terms in this sum are in fact 0. For example, consider $(v^{10})^2 = b^* \hat{\xi} c b^* \hat{\xi} c$. In general, for any $u \in \mathcal{C}(J \times \{0,1\}, \sigma|_{J})$, there is a $\tilde{u} \in \mathcal{C}(J \times \{0,1\}, \sigma|_{J})$ such that $\hat{\xi} u = \tilde{u} \hat{\xi}$. Hence the term $(v^{10})^2$ contains $\hat{\xi}^2 = 0$, and so is 0. More generally, a term like $v^{10} v^{01} v^{10}$ is also 0: the $\hat{\xi}$ in $v^{10}$ can be commuted past all terms except $\xi$, at which point the product is either 0 or $\hat{\xi}$ (by Lemma 3.3), so the term is 0. On the other hand, the term $v^{11} v^{01} v^{10}$ is nonzero, since (once commuting past the $\mathcal{C}(J \times \{0,1\}, \sigma|_{J})$-terms) we have $\hat{\xi}^* \hat{\xi} = (\hat{\xi}^* \hat{\xi})^2 = \xi \neq 0$.

Let $\eta, \nu \in 2^n$. Denote by $1(\eta) \subseteq \{1, \ldots, n\}$ the set of $j$ such that $\eta_j = 1$. Then say that $\nu$ is $\eta$-alternating, $\nu \in A(\eta)$, if the subsequence $\{(\nu_j); j \in 1(\eta)\}$ is alternating. For example, let $\eta = (1, 1, 0, 1)$. Then the sequences $(0, 1, 0, 0)$ and $(0, 1, 1, 0)$ are both in $A(\eta)$, while the sequence $(0, 0, 0, 0)$ is not. Note that $v^{10}$ and $v^{11}$ are the terms containing $\hat{\xi}$ and $\hat{\xi}^*$. Hence, the $v^{\nu^\eta}$ with $\nu \in A(\eta)$ are precisely those terms in which $\hat{\xi}$ and $\hat{\xi}^*$ alternate when they occur. By the considerations in the preceding paragraph, these are the only nonzero terms in the expansion of $v^\eta$. Thus,

$$v^\eta = \sum_{\nu \in A(\eta)} v^{\nu^\eta}.$$  

In any term in the above sum, let $|\eta| = 2k$ and let $|\nu| = m$. Since $1(\eta)$ is a set of $2k$ indices, and since $\nu \in A(\eta)$, $\nu_j = 0$ for $k$ of these indices $j$, and $\nu_j = 1$ for the other $k$. Thus $\nu$ contains at least $k$ 1s and at least $k$ 0s, and
so \( k \leq m \leq n - k \). It follows that the full expansion for the \( n \)th moment is

\[
\|a\|_{2n}^{2n} = \sum_{k=0}^{[n/2]} \sum_{m=k}^{n-k} \sum_{|\nu|=2k} \sum_{\nu \in A(\eta) \atop |\nu|=m} \tau_\sigma(v^{\nu}).
\]

It will be useful to consider the cases \( k = 0 \) and \( m = 0 \) separately, and so we rewrite this moment as

\[
\|a\|_{2n}^{2n} = \tau_\sigma[(v^{00})^n] + \sum_{m=1}^{n} \sum_{|\nu|=m} \tau_\sigma(v^{0\nu}) + \sum_{k=1}^{[n/2]} \sum_{m=k}^{n-k} \sum_{|\nu|=2k} \sum_{\nu \in A(\eta) \atop |\nu|=m} \tau_\sigma(v^{\nu}). \tag{3.9}
\]

(Not, if \( \eta \equiv 0 \) then the condition \( \nu \in A(\eta) \) is vacuously satisfied for all \( \nu \in 2^n \).) Each of the \( v^{\nu} \) in Equation \( \text{(3.9)} \) is a product of terms, each of which contains some elements of \( \mathcal{C}(J \times \{0,1\}, \sigma|_J) \) and some factors of \( \hat{\zeta}, \hat{\zeta}^*, \) or \( \xi \). (Observe the only term which has no factors from \( \mathcal{C}(\{i\} \times \{0,1\}, \sigma|_{\{i\}}) \) is the first one \( (v^{00})^n \).) To estimate such terms, we introduce the following tool.

**Lemma 3.5.** Let \( u_1, \ldots, u_s \in \mathcal{C}(J \times \{0,1\}, \sigma|_J) \). Let \( U \) be a product including all of the elements \( u_1, \ldots, u_n \) together with some non-zero number of terms from \( \{\hat{\zeta}, \hat{\zeta}^*, \xi\} \). Then

\[
\tau_\sigma(U) \leq \frac{1}{2} \|u_1\|_s \cdots \|u_s\|_s. \tag{3.10}
\]

**Proof.** First note that \( \tau_\sigma(U) \) is invariant under cyclic permutations of \( U \). \( U \) may then be written in the form \( h_1 U_1 h_2 U_2 \cdots h_{\ell} U_{\ell} \), where each \( U_j \) is a product of some of the \( u_1, \ldots, u_s \), and each \( h_j \) is a product of the terms \( \hat{\zeta}, \hat{\zeta}^*, \) and \( \xi \). Let \( s_j \) be the number of terms in \( U_j \); then \( s_1 + \cdots + s_\ell = s \). So \( s_1/s + \cdots + s_\ell/s = 1 \), and when we apply Hölder’s inequality, we find

\[
\tau_\sigma(U) \leq \|h_1 U_1\|_{s/s_1} \cdots \|h_\ell U_\ell\|_{s/s_\ell}. \tag{3.11}
\]

By Lemma \[\text{3.3}\] part \( \text{II} \), any product of terms in \( \{\hat{\zeta}, \hat{\zeta}^*, \xi\} \) is either \( \hat{\zeta}, \hat{\zeta}^*, \xi, 1 - \xi, \) or \( 0 \). Thus, using Lemma \[\text{3.3}\] part \( \text{II} \) we have

\[
\|h_j U_j\|_{s/s_j} \leq 2^{-s_j/s} \|U_j\|_{s/s_j}. \tag{3.12}
\]

Now, since \( U_j \) is a product of \( s_j \) terms, say \( u_{k_1}, \ldots, u_{k_{s_j}} \), applying Hölder’s inequality again (using \( 1/s_j + \cdots + 1/s_j = 1/(s/s_j) \)) we have \( \|U_j\|_{s/s_j} \leq \|u_{k_1}\|_s \cdots \|u_{k_{s_j}}\|_s \). Combining this with Equations \( \text{3.11} \) and \( \text{3.12} \) we get

\[
\tau_\sigma(U) \leq 2^{-s_1/s} \cdots 2^{-s_\ell/s} \|u_1\|_s \cdots \|u_s\|_s,
\]

and since \( s_1 + \cdots + s_\ell = s \), this reduces to Equation \( \text{3.10} \).
We now apply Lemma 3.5 to estimate the three terms in Equation 3.9. The first term is merely \( \tau_\sigma(\|b\|^{2n}) = \|b\|^{2n}_m \). In the first sum
\[
\sum_{m=1}^{n} \sum_{|v|=m} \tau_\sigma(v^{0v}),
\]
the term \( v^{0v}, \) with \( |v| = m \), is a product containing \( m \) factors of \( v^{01} = \xi c^*c \) and \( n - m \) factors of \( v^{00} = b^*b \). So there are a total of \( 2n \) factors from the set \( \{b, b^*, c, c^*\} \subset C(J \times \{0, 1\}, \sigma|_J) \). Since \( \|u\|_{2n} = \|u^*\|_{2n} \) for each \( u \in C(J \times \{0, 1\}, \sigma|_J) \), Lemma 3.5 then implies that
\[
\tau_\sigma(v^{0v}) \leq \frac{1}{2}(\|b\|_{2n})^{2(n-m)}(\|c\|_{2n})^{2m}.
\]
Hence,
\[
\sum_{m=1}^{n} \sum_{|v|=m} \tau_\sigma(v^{0v}) \leq \frac{1}{2} \sum_{m=1}^{n} \sum_{|v|=m} (\|b\|_{2n})^{n-m}(\|c\|_{2n})^{m} = \frac{1}{2} \sum_{m=1}^{n} \left( \frac{n}{m} \right) (\|b\|_{2n})^{n-m}(\|c\|_{2n})^{m}. \tag{3.13}
\]

Now we consider the second sum
\[
\sum_{k=1}^{n/2} \sum_{m=k}^{n-k} \sum_{|\eta|=2k} \sum_{\nu \in A(\eta)} \sum_{|\nu|=m} \tau_\sigma(v^{\eta \nu}).
\]
In each term \( v^{\eta \nu}, \) since \( |\eta| = 2k \) and \( \nu \in A(\eta) \), we know that \( k \) of the terms are \( v^{10} \) and \( k \) of the terms are \( v^{11} \). So \( k \) of the 1s in \( \nu \) have been accounted for with the \( v^{11} \) terms, and since \( |\nu| = m \) precisely \( m - k \) terms must be \( v^{01} \). As the total number of terms must be \( n \), this means the remaining \( v^{00} \) terms are \( n - (2k + m - k) = n - m - k \) in number. So, there are
- \( k \) factors of \( v^{10} = b^* \hat{z}c \), so \( k \) factors each of \( b^* \hat{z} \) and \( c \),
- \( k \) factors of \( v^{11} = c^* \hat{z}b \), so \( k \) factors each of \( b \) and \( c^* \),
- \( m - k \) factors of \( v^{01} = \xi c^*c \), so \( m - k \) factors each of \( c \) and \( c^* \), and
- \( n - m - k \) factors of \( v^{00} = b^*b \), so \( n - m - k \) factors each of \( b \) and \( b^* \).
In total, then, \( v^{\eta \nu} \) contains \( 2k + 2(n - m - k) = 2(n - m) \) factors of \( b \) or \( b^* \), and \( 2k + 2(m - k) = 2m \) factors of \( c \) or \( c^* \). Applying Lemma 3.5 again,
\[
\sum_{k=1}^{n/2} \sum_{m=k}^{n-k} \sum_{|\eta|=2k} \sum_{\nu \in A(\eta)} \sum_{|\nu|=m} \tau_\sigma(v^{\eta \nu}) \leq \frac{1}{2} \sum_{k=1}^{n/2} \sum_{m=k}^{n-k} \sum_{|\eta|=2k} \sum_{\nu \in A(\eta)} \sum_{|\nu|=m} (\|b\|_{2n})^{2(n-m)}(\|c\|_{2n})^{2m}.
\]
We must now count the number of pairs \((\eta, \nu)\) with \(|\eta| = 2k, \nu \in A(\eta)\) and \(|\nu| = m\). There are \( \binom{n}{2k} \) such \( \eta \). We know that \( \nu \) is alternating on \( \mathbb{1}(\eta) \), and so the corresponding subsequence must be either 0101...01 or 1010...10, giving two choices, and exhausting \( k \) of the \( m \) 1s in \( \nu \). Finally, since \( |\mathbb{1}(\eta)| = 2k \), there are \( n - 2k \) 0s in \( \eta \), and \( \nu \) is unconstrained there;
hence, there are \( \binom{n-2k}{m-k} \) choices. Whence, the number of pairs \((\eta, \nu)\) in the sum is

\[
2 \binom{n}{2k} \binom{n-2k}{m-k}.
\]

This gives the estimate

\[
\sum_{k=1}^{[n/2]} \sum_{m=k}^{n-k} \sum_{|\nu|=m} \tau_\sigma(\nu^m)
\]

\[
\leq \sum_{k=1}^{[n/2]} \sum_{m=k}^{n-k} \left( \binom{n}{2k} \binom{n-2k}{m-k} \right) (\|b\|_2^2)^{n-m} (\|c\|_2^2)^m
\]

for the final sum in Equation 3.14. It will be convenient to reorder the terms in Equation 3.14 so that \(m\) occurs first. Since the sum (for each \(k\)) ranges from \(m=k\) to \(m=n-k\), the pairs \((k, m)\) in the sum are those with \(1 \leq k \leq \lfloor n/2 \rfloor\) and \(k \leq m \leq n-k\). The second condition gives two inequalities: \(k \leq m\) and \(k \leq n-m\). Note, if both of these are satisfied then, summing, \(2k \leq n\) – the first condition is automatically satisfied. The sum can therefore be rewritten as

\[
\sum_{m=1}^{n} \sum_{k=1}^{m \land (n-m)} \binom{n}{2k} \binom{n-2k}{m-k} (\|b\|_2^2)^{n-m} (\|c\|_2^2)^m.
\]

So, combining Equations 3.9, 3.13, and 3.15 we have the estimate

\[
\|a\|_2^2 \leq \|b\|_2^2 + \sum_{m=1}^{n} \chi_m (\|b\|_2^2)^{n-m} (\|c\|_2^2)^m,
\]

where the coefficient \(\chi_m\) is given by

\[
\chi_m = \frac{1}{2} \binom{n}{m} + \sum_{k=1}^{m \land (n-m)} \binom{n-2k}{m-k}.
\]

The following proposition shows that \(\chi_m\) is optimally bounded to yield the necessary strong hypercontractive estimate. We state it without proof; the reader may do the necessary calculations.

**Proposition 3.6.** The coefficients \(\chi_m\) satisfy

\[
\chi_m \leq \binom{n}{m} \left( \frac{n}{2} \right)^m.
\]

This inequality is an equality in the case \(m = 1\).

Applying Proposition 3.6 to Equation 3.16 we have

\[
\|a\|_2^2 \leq \|b\|_2^2 + \sum_{m=1}^{n} \left( \frac{n}{2} \right)^m (\|b\|_2^2)^{n-m} (\|c\|_2^2)^m.
\]

We now complete the proof of Theorem 3.2.
Proof of the ‘if’ direction of Theorem 3.2: For $a = b + \hat{z}c$, we have $a_t = b_t + e^{-t} \hat{z}c_t$, where $a_t = e^{-tN_\sigma} a$ and so forth. Using the estimate in Equation 3.17, we have

$$∥a_t∥_{2n}^2 ≤ ∥b_t∥_{2n}^2 + \sum_{m=1}^{n} \left( \binom{n}{m} \left( \frac{n}{2} \right)^m \right) e^{-2mt} \left( ∥b_t∥_{2n}^2 \right)^{n-m} \left( ∥c_t∥_{2n}^2 \right)^m.$$ 

Now, suppose $t ≥ t_J(2, 2n) = \frac{1}{2} \log n$. Then $e^{-2mt} ≤ n^{-m}$. Since $b, c ∈ \mathcal{H}(J, \sigma|_J)$ it follows from the inductive hypothesis that $∥b_t∥_{2n} ≤ ∥b∥_2$ and $∥c_t∥_{2n} ≤ ∥c∥_2$. Thus,

$$∥a_t∥_{2n}^2 ≤ ∥b∥_2^2 + \sum_{m=1}^{n} \left( \binom{n}{m} \left( \frac{n}{2} \right)^m \right) n^{-m} \left( ∥b∥_2^2 \right)^{n-m} \left( ∥c∥_2^2 \right)^m$$

$$= \left( ∥b∥_2^2 + \frac{1}{2} ∥c∥_2^2 \right)^n,$$

and from Equation 3.8 this equals $∥a∥_{2n}^2$. This proves the theorem. □

4. SPEICHER’S STOCHASTIC INTERPOLATION

In this final section, we consider creation and annihilation operators $\beta_j$, $\beta_j^*$ on $L^2(\mathcal{C}, \tau_\sigma)$ which bear the same relation to the generators $x_j$ in $\mathcal{C}$ as the creation and annihilation operators $c_q$, $c_q^*$ bear to the $q$-Gaussian variables $X_q ∈ \Gamma_q$. We use these operators, together with a non-commutative central limit theorem of Speicher, to approximate the $L^p(\mathcal{H}_q, \tau_q)$-norm by the norm on $L^p(\mathcal{H}, \tau_\sigma)$, and thus transfer Theorem 3.2 from the context of the mixed spin holomorphic algebras to the arena of the $q$-holomorphic algebras, proving Theorem 1.4. All of the techniques in this section are analogs of Biane’s ideas in [B1].

4.1. Creation and Annihilation operators on $L^2(\mathcal{C}, \tau_\sigma)$. Define operators $\beta_j$ on $L^2(\mathcal{C}(I, \sigma), \tau_\sigma)$ by

$$\beta_j(x_A) = \begin{cases} x_jx_A, & \text{if } j \notin A \\ 0, & \text{if } j \in A \end{cases}.$$

One may readily verify that the adjoint of $\beta_j$ is given by

$$\beta_j^*(x_A) = \begin{cases} 0, & \text{if } j \notin A \\ x_jx_A, & \text{if } j \in A \end{cases}.$$

In the case $\sigma(i, j) = 1$ for $i ≠ j$, these are the Bébé Fock operators on the toy Fock space of [M]. In general, $\beta_j$ and $\beta_j^*$ mimic the creation and annihilation operators. It is easy to see from dimension considerations that the $*$-algebra they generate is all of $\mathcal{B}(L^2(\mathcal{C}(I, \sigma), \tau_\sigma))$. We also have

$$\beta_j + \beta_j^* = x_j,$$  (4.1)
as a left-multiplication operator on $\mathcal{C}(I, \sigma)$. One can readily compute that these operators $\sigma$-commute – i.e. $\beta_i \beta_j = \sigma(i, j) \beta_j \beta_i$ if $i \neq j$. They also satisfy the $\sigma$-relations

$$\beta_i^* \beta_j - \sigma(i, j) \beta_j \beta_i^* = \delta_{ij},$$

just like the operators $\hat{z}_j \in \mathcal{C}(I \times \{0, 1\}, \sigma)$. In fact, the map $\hat{z}_j \mapsto \beta_j$ induces a $\ast$-isomorphism from $\mathcal{C}(I \times \{0, 1\}, \sigma)$ onto $\mathcal{B}(L^2(\mathcal{C}(I, \sigma)))$. (In the case $\sigma \equiv -1$, this reduces to the well known isomorphism from the complex Clifford algebra $\mathcal{C}_2$ onto the full matrix algebra $M_{2^n}(\mathbb{C})$.) Beware, however: this isomorphism does not send $\tau_\sigma$ to the normalized trace $tr$ on $\mathcal{B}(L^2(\mathcal{C}(I, \sigma)))$, as pointed out in Section 3.2.

The operators $\beta_j$, $\beta_j^*$ demonstrate concretely that the pure state $\beta \mapsto (\beta 1, 1)$, the extension of $\tau_\sigma$ to $L^2(\mathcal{C}(I, \sigma), \tau_\sigma)$, is not tracial. Indeed, it is easy to calculate that $(\beta_1^* \beta_1 1, 1) = 0$ while $(\beta_1^* \beta_2 1, 1) = 1$. These are, however, the same covariance relations that the operators $c_q$ and $c_q^*$ satisfy with respect to the pure state $A \mapsto (A \Omega, \Omega)q$ on $\mathcal{B}(\mathcal{F}_q)$. It is additionally true that $(\beta_1^* 1, 1) = (\beta_2^* 1, 1) = 0$, also in line with the operators $c_q$ and $c_q^*$.

The following lemma shows that the state $(1, 1)$ factors over naturally ordered products of the operators $\beta_j$ and $\beta_j^*$. It is proved in [11].

**Lemma 4.1.** For each $j \in I$, let $\alpha_j$ be in the $\ast$-algebra generated by $\beta_j$. Let $j_1, \ldots, j_s$ be $s$ distinct elements in $I$, then

$$(\alpha_{j_1} \cdots \alpha_{j_s} 1, 1) = (\alpha_{j_1} 1, 1) \cdots (\alpha_{j_s} 1, 1).$$

**4.2. Speicher’s central limit theorem.** Fix $q \in [-1, 1]$. We consider the family of random matrices $\mathcal{G}_q$, consisting of all those infinite symmetric random matrices $\sigma: \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \{-1, 1\}$ constantly $-1$ on the diagonal, for which $\{\sigma(i, j) : i < j\}$ are i.i.d. with $\mathbb{P}(\sigma = 1) = (1 + q)/2$. Note, then, $\mathbb{P}(\sigma = -1) = (1 - q)/2$, and so

$$\mathbb{E}(\sigma(i, j)) = \frac{1 + q}{2} \cdot 1 + \frac{1 - q}{2} \cdot -1 = q.$$

This family of random matrices features prominently in the main theorem of [3], which we will use to prove Theorem 1.4.

Let $I_n$ denote the set $\{1, \ldots, n\}$, and let $\sigma \in \mathcal{G}_q$. For convenience, we denote the algebra $\mathcal{C}(I_n \times \{0, 1\}, \sigma|_{I_n})$ as $\mathcal{C}(n, \sigma)$. The creation operators on $\mathcal{C}(n, \sigma)$ are labeled by pairs $(j, \zeta)$ where $j \in I_n$ and $\zeta \in \{0, 1\}$; to avoid confusion, we also index them as $\beta_{j, \zeta}$ to keep track of the dependence on $\sigma$. Let $d$ be a positive integer, and define new variables $\beta_{1, \zeta}^{\sigma, n}, \ldots, \beta_{d, \zeta}^{\sigma, n}$, which act on $\mathcal{C}(nd, \sigma)$, by

$$\beta_{k+1, \zeta}^{\sigma, n} = \frac{1}{\sqrt{n}} \sum_{\ell=nk+1}^{n(k+1)} \beta_{\ell, \zeta}^{\sigma}, \quad 0 \leq k \leq d - 1.$$
These operators are constructed to approximate the operators $c_q$. The intuition is: due to the expectation of the matrix $\sigma \in \mathcal{S}_q$, for large $n$ the $\beta_{j,\zeta}$ satisfy commutation relations close to the $q$-commutation relations of Equation 2.1. Speicher’s central limit theorem (Theorem 2 in [S]) makes this statement precise, but requires that the matrix of spins for the different variables have independent (upper triangular) entries. In our case, since for each pair $i,j$ the entries $\sigma((i,\zeta),(j,\zeta'))$ are the same for all choices of $\zeta, \zeta' \in \{0,1\}$, the matrix is only block-independent (with blocks of size $2 \times 2$). Nevertheless, as with the classical central limit theorem, a straightforward modification of Speicher’s proof generalizes the theorem to this case. We thus have the following theorem.

**Theorem 4.2.** Let $e_1, \ldots, e_d$ be an orthonormal basis for $\mathbb{R}^d$. Among the operators $c_q(e_j, e_k)$ on $\mathcal{F}_q(\mathbb{R}^d \oplus \mathbb{R}^d)$, denote $c_q(e_j, 0)$ as $c_{j,0}^d$, and denote $c_q(0, e_j)$ as $c_{j,1}^d$. Let $Q$ be a polynomial in $4d$ non-commuting variables. For almost every $\sigma \in \mathcal{S}_q$,

$$
\lim_{n \to \infty} (Q(\beta_{1,0}^{\sigma,n}, \ldots, \beta_{d,1}^{\sigma,n}, (\beta_{1,0}^{\sigma,n})^*, \ldots, (\beta_{d,1}^{\sigma,n})^*)1,1)_\sigma = (Q(c_{1,0}^d, \ldots, c_{d,1}^d, (c_{1,0}^d)^*, \ldots, (c_{d,1}^d)^*)\Omega, \Omega)_q.
$$

**Proof.** This follows from Speicher’s central limit theorem. The required covariance conditions for the operators $\beta_{j,\zeta}^\sigma$ were verified above, and the factorization of naturally-ordered products is the content of Lemma 4.1. □

An immediate corollary is that the moments of elements in $\mathcal{H}_q(\mathbb{C}^d)$ can be approximated by the corresponding elements in $\mathcal{C}(nd, \sigma)$. To be precise: let $x_j^{\sigma,n} = \beta_{j,0}^{\sigma,n} + (\beta_{j,0}^{\sigma,n})^*$ and let $y_j^{\sigma,n} = \beta_{j,1}^{\sigma,n} + (\beta_{j,1}^{\sigma,n})^*$. By Equation 4.11

$$
x_j^{\sigma,n} = \frac{1}{\sqrt{n}} \sum_{\ell=\sigma,n+1} x_\ell^\sigma, \quad y_j^{\sigma,n} = \frac{1}{\sqrt{n}} \sum_{\ell=\sigma,n+1} y_\ell^\sigma.
$$

Let $z_j^{\sigma,n} = 2^{-1/2}(x_j^{\sigma,n} + iy_j^{\sigma,n})$, which is in $\mathcal{H}(I_{nd}, \sigma|_{I_{nd}})$.

**Proposition 4.3.** Denote $Z_q(e_j)$ as $Z_j^q$. Let $r$ be an even integer, and let $P$ be a polynomial in $d$ non-commuting variables. For almost every $\sigma \in \mathcal{S}_q$,

$$
\lim_{n \to \infty} ||P(z_1^{\sigma,n}, \ldots, z_d^{\sigma,n})||_{L^r(\mathcal{H}_q, \tau_\sigma)} = ||P(Z_1^q, \ldots, Z_d^q)||_{L^r(\mathcal{H}_q, \tau_\sigma)}.
$$

**Proof.** Let $Q$ be the polynomial in $4d$ non-commuting variables defined by

$$
Q(\beta_{1,0}^{\sigma,n}, \ldots, \beta_{d,1}^{\sigma,n}, (\beta_{1,0}^{\sigma,n})^*, \ldots, (\beta_{d,1}^{\sigma,n})^*) = P(z_1^{\sigma,n}, \ldots, z_d^{\sigma,n})^* P(z_1^{\sigma,n}, \ldots, z_d^{\sigma,n}).
$$

Such a polynomial exists because the variable $z_j^{\sigma,n}$ is a (linear) polynomial in $\beta_{j,0}^{\sigma,n}, \beta_{j,1}^{\sigma,n}$, and their adjoints. By definition, the same polynomial yields

$$
Q(c_{1,0}^d, \ldots, c_{d,1}^d, (c_{1,0}^d)^*, \ldots, (c_{d,1}^d)^*) = P(Z_1^q, \ldots, Z_d^q)^* P(Z_1^q, \ldots, Z_d^q).
$$

Applying Theorem 4.2 to the polynomial $Q^n$, we have

$$
\lim_{n \to \infty} \tau_\sigma|P(z_1^{\sigma,n}, \ldots, z_d^{\sigma,n})|^{2m} = \tau_\sigma|P(Z_1^q, \ldots, Z_d^q)|^{2m} \quad a.s. \{\sigma\}.
$$
where we have used the fact that \((\cdot, 1)_\sigma\) reduces to \(\tau_\sigma\) when applied to elements of \(\mathcal{C}(nd, \sigma)\).

We will also need to know that the semigroup \(e^{-tN_\sigma}\) approximates \(e^{-tN_q}\).

**Proposition 4.4.** Let \(r\) be an even integer, and let \(P\) be a polynomial in \(d\) non-commuting variables. For \(t > 0\), and for almost every \(\sigma \in \mathcal{S}_q\),

\[
\lim_{n \to \infty} \|e^{-tN_\sigma} P(z_1^{n, \sigma}, \ldots, z_d^{n, \sigma})\|_r = \|e^{-tN_q} P(Z_1^q, \ldots, Z_d^q)\|_r.
\]

**Proof.** We can expand \(P(Z_1^q, \ldots, Z_d^q)\) as a linear combination of monomials \(Z_{i_1}^q \cdots Z_{i_t}^q\). Each such monomial is an eigenvector of \(e^{-tN_\sigma}\) with eigenvalue \(e^{-t\ell}\). So it is easy to see that there is a unique polynomial \(P_t\) such that

\[
P_t(Z_1^q, \ldots, Z_d^q) = e^{-tN_\sigma} P(Z_1^q, \ldots, Z_d^q).
\]

Now, consider the polynomials \(z_1^{\sigma, n} \cdots z_{i_t}^{\sigma, n}\). Since \(z_i^{\sigma, n}\) is a linear combination of \(z_1^{\sigma}, \ldots, z_{nd}^{\sigma}\), this polynomial may be expanded as a linear combination of monomials \(z_1^{\sigma} \cdots z_{j_t}^{\sigma}\) with \(1 \leq j_1, \ldots, j_t \leq nd\). From Equation 3.4, if any two indices are equal, then \(z_1^{\sigma} \cdots z_{j_t}^{\sigma} = 0\); otherwise, it is of degree \(\ell\). Hence

\[
e^{-tN_\sigma} (z_1^{\sigma} \cdots z_{j_t}^{\sigma}) = e^{-t\ell} (z_1^{\sigma} \cdots z_{j_t}^{\sigma}).
\]

It follows that \(e^{-tN_\sigma} (z_1^{\sigma, n} \cdots z_{i_t}^{\sigma, n}) = e^{-t\ell} (z_1^{\sigma} \cdots z_{j_t}^{\sigma})\). Thus, we see that

\[
P_t(z_1^{\sigma, n}, \ldots, z_{i_t}^{\sigma, n}) = e^{-tN_\sigma} P(z_1^{\sigma, n}, \ldots, z_{i_t}^{\sigma, n}).
\]

The theorem now follows by applying Proposition 4.3 to the polynomial \(P_t\).

It should be noted that this elementary argument fails in the full algebra \(\mathcal{C}(nd, \sigma)\); for example, \((x_1^q)^2 = 1\) is of degree 0, while \(X_q(e_1)^2\) is of degree 2 if \(q > -1\). The relevant statement is still true in that case, but a much more delicate argument (which can be found in [13]) is necessary to prove it.

We now conclude with the end of the proof of Theorem 1.4.

**Proof of Theorem 1.4.** First note that the sharpness of the Janson time \(t_J(p, r)\) for any \(p, r > 0\) can be confirmed by an argument identical to the one in the proof of Theorem 3.2. For sufficiency, by standard arguments it is enough to prove the theorem for the finite dimensional Hilbert space \(\mathcal{H} = \mathbb{R}^d\), and moreover it suffices to prove it for elements \(f \in L^2(\mathcal{H}_q, \tau_q)\) that are polynomials \(f = P(Z_1^q, \ldots, Z_d^q)\) of the generators. Let \(r\) be an even integer, and let \(t \geq t_J(2, r)\). By Proposition 4.3,

\[
\|e^{-tN_q} f\|_r = \lim_{n \to \infty} \|e^{-tN_\sigma} P(z_1^{\sigma, n}, \ldots, z_d^{\sigma, n})\|_r \quad a.s.[\sigma].
\]

By Theorem 3.2 applied to the algebra \(\mathcal{H}(I_{nd}, \sigma|I_{nd})\),

\[
\|e^{-tN_\sigma} P(z_1^{\sigma, n}, \ldots, z_d^{\sigma, n})\|_r \leq \|P(z_1^{\sigma, n}, \ldots, z_d^{\sigma, n})\|_2.
\]
Finally, applying Proposition 4.3, we have
\[ \lim_{n \to \infty} \| P(\sigma_1, \ldots, \sigma_d) \|_2 = \| f \|_2 \quad a.s.[\sigma]. \]
This completes the proof. \qed

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