Simple parameters spaces analysis of roto-orbital integrable Hamiltonians for an axisymmetric rigid body

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Abstract. The present work presents a test of two Hamiltonians that produce integrable models recently proposed to study the roto-orbital motion of an axisymmetric rigid body in motion under a central gravitational field. The dynamics assumed here is approached by the motion of an axisymmetric rigid body orbiting another massive spherical one. Based on the concept of intermediary, both models are treated in Hamiltonian formalism, as perturbation of the Keplerian-Eulerian motion, using canonical variables associated to the total angular momentum. An analysis of parameters introduced to visualize possible different applications are made, in this case with special focus in binary asteroid type dynamics. The parameters space analysis present comparisons of two recently proposed intermediaries with respect to the original non-analytically integrable model and with respect to each other. In conclusion, both models behave well in regions of the parameters space where they were proposed to be valid.

1. Introduction

The roto-orbital dynamics is a particular challenge of the full two body problem (see [10,11] and references therein). It is usually one of the most difficult problems in celestial mechanics, once it is not possible to find an exact general analytical solution for it. This fact results from the large number of orbital and rotational variables involved in the problem, resulting in equations of motion with many degrees of freedom (DOF).

In the attempt of find approximate solutions for the general motions, different authors have dealt with this problem using different techniques, as the cases of [3, 6–8] and references therein. In the present work, we develop a simple analysis of the space of parameters where recent particular integrable solutions for the problem are defined. These approximate analytical solutions are built using the concept of “Intermediary”, which a Hamiltonian built without fast angles of the original problem, allowing its analytical integration (see more details about the concept an Intermediaries in [1,5,9] and references therein). The two intermediaries analyzed in the present work are the ones developed by [3] and [7]. One of the main differences of the two Hamiltonian models presented in these two works is that the canonical transformation called “elimination of the parallax” is applied in [7], and it is made before the process of building the roto-orbital intermediary. In the present work of we focus on performing an analysis
of the impact of important parameters of the problem over the parameters space where the Hamiltonians are defined. In section 2 we present the roto-orbital Hamiltonians and their equations of motion. Following, the section 3 presents the plots with comparisons of these Hamiltonians and discussions of the results obtained. They are compared under changes of the values of attitude angle, shape, orbital eccentricity and distance. In general, both models present good precision in wide regions of the dynamical parameters spaces defined for the analysis. The section 4 presents a few conclusions and ideas for future work. Finally, the Appendix section presents results of the parameters spaces analysis.

2. Roto-orbital Hamiltonians

The Hamiltonian in terms of roto-orbital canonical variables adopted here is $\mathcal{H} \sim (r, \phi, \psi, \theta, \delta, \nu, R, \Phi, \Psi, \Theta, \Delta, N)$, where $(r, \theta, R, \Theta)$ are Hill-Whittaker variables [5], $(\nu, N)$ are Andoyer-Deprit variables [4], and the remaining ones are built from these two sets of variables. The roto-orbital variables $(r, \phi, \psi, \theta, \delta, \nu)$ are connected with their associated canonical momenta $(R, \Phi, \Psi, \Theta, \Delta, N)$ by means of Hamilton’s equations: $\dot{x} = \frac{\partial H(x, X)}{\partial X}$ and $\dot{X} = -\frac{\partial H(x, X)}{\partial x}$, where $x$ is the canonical variable and $X$ is its associated canonical momentum. The momenta are defined as follows: $R$ is the radial velocity of the center of mass; $\Psi$ is the magnitude of the total angular momentum in space frame and rotational angular momentum of the rigid body; $\Phi$ and $N$ are the third components of the total angular momentum vector in space frame and rotational angular momentum of the rigid body in the body frame (principal axes of inertia), respectively. These variables are properly defined and discussed in previous works [1, 2, 6, 7]. Thus, it is possible deal directly with the “original roto-orbital Hamiltonian” of the problem and two alternative Hamiltonians models that provide approximate analytical solutions for the motion defined by the $\mathcal{H}$ (1) is called radial intermediary (see more details about this type of intermediary in [5] and references therein). This equations of motion for $\mathcal{I}_o$ are [1–3]:

$$\mathcal{I}_o = \mathcal{I}_o + \frac{(A - C)k}{8r^3} \left[ (1 - 3c_s^2) \left( 1 - 3c_{s\sigma}^2 \right) \right] = \mathcal{I}_o(r, -,-,-, R, -,-, \Phi, \Theta, \Delta, N)$$  \hspace{1cm} (2)

The second Hamiltonian analyzed in this work is the radial intermediary proposed by [7], which is expressed in primed variables due to a preparatory canonical transformation called elimination of the parallax and that was used to obtain this model.

$$\mathcal{I}_p = \mathcal{I}_p + \frac{(A - C)k^2}{8r^3} \left[ (1 - 3c_s^2) \left( 1 - 3c_{s\sigma}^2 \right) \right] = \mathcal{I}_p(r', -,-,-, R', -,-, \Psi', \Theta', \Delta', N')$$  \hspace{1cm} (3)
where it is worth of mention that the differences between the original and primed variables are very small because the canonical transformation used is a contact transformation [1,5], and the process is able to preserves the fast angles of the problem in the generating function of the transformation [2,7]. This generating function can be used for short-period correction after the propagation of the equations of motion [5,7].

For the intermediary \( I_\mu \), we have the following 1-DOF system of equations of motion (see more details [2]): 
\[
(\dot{\psi}, \dot{\delta}, \dot{\nu}, \dot{\theta}, \dot{r}, -\dot{R}) = \frac{\partial (\psi, \theta, \nu, r, R)}{\partial (\psi, \theta, \nu, \psi, r, R)},
\]

which joint with \( \dot{\phi} = \dot{\Phi} = \dot{\Psi} = \dot{\Theta} = \Delta' = \dot{N}' = 0 \) can provide an analytical solution in terms of elliptic functions [3]. For the intermediary \( I_p \), the 1-DOF differential system is [2]: 
\[
(\dot{\psi}', \dot{\delta}', \dot{\nu}', \dot{\theta}', \dot{r}', -\dot{R}') = \frac{\partial I_p}{\partial (\psi', \theta', \nu', \psi', r', R')},
\]

joint with \( \dot{\phi}' = \dot{\Phi}' = \dot{\Psi}' = \dot{\Theta}' = \Delta' = \dot{N}' = 0 \), a system that presents an analytical solution in terms of trigonometric functions [7].

3. Results and discussion

In this section, the plots of the comparisons between the parameters spaces due to the original Hamiltonian and the two intermediaries are shown. The initial conditions for the simulations are set following the data for the physical system presented in Table 1 of [2] (see also pages 131 and 132 of [1]). For instance, the unity defined for the distance \( r \) is 13,000 km. The system parameters are fixed, with exception of the ones along the axes, denoting a static state analysis.

The essential for the present study is that the momentum of inertia \( C \) is set as 0.0007, meaning that \( A = C \) defines a spherical rigid body. The cases of \( A < C \) and \( A > C \) define oblate and prolate rigid bodies, respectively. In fact, it is easy to verify loss of precision with the decrease of the oblateness, meaning an increase in the perturbation due to more prolate shapes (Figures 1, 3 - 5). As \( C \) is along the \( z \)-axis, the results show, physically speaking, the expected stability when the rotation is around the maximum moment of inertia. Additionally, the definition of the dynamics and its analysis considering prolate bodies seem to be a topic to be evaluated very carefully in a future study. In this regime the differences are more evident in the figures cited before. The need of deeper investigation of this regime for this problem was pointed out by [2]. The results for different attitude angles \( \sigma \) (Figures 1 - 9, but specially Figures 1, 2 and 6) highlight this conclusion. They show that the parameters space defined by the original Hamiltonian is usually efficiently approached by the ones defined by the intermediaries, with exception of prolate regimes appearing with errors of the order of 20% in Figure 1.

Figures 2 - 5 and 7 - 9 show that the parameters spaces defined by the intermediaries are robust to changes in eccentricity, despite not being recommendable the use of these models to study very eccentric systems. This last restriction is related to the fact that significant orbital eccentricities present problems in convergence of the numerical solution [2]. The same explanation can be applied to results regarding close distances as \( r \approx 0.6 \) or less (Figures 6 - 9).

4. Conclusions and future work

In this study, it is presented a list of plots that provide comparisons of parameters spaces of three different Hamiltonians for the roto-orbital problem of an axisymmetric rigid body under a central gravitational field: the original Hamiltonian in expression (1); the intermediary \( I_\mu \) in expression (2); and the intermediary \( I_p \) in expression (3). The last two Hamiltonians, called radial intermediaries, were built with the purpose to approach the first one, meanwhile providing an analytical approximate solution for the dynamical system described by the first one, and which is not analytically integrable. Thus, different plots of errors between these Hamiltonians are presented for static state analysis of the systems, with variables not varying with time. The results shown here are in good accordance with the study developed by [2], where the

\(^{1}\) Hypothetical binary system composed by a pair of asteroids sized as a few hundreds of km. The orbiting body is a slow-rotating axisymmetric rigid body in an eccentric relative orbit, meaning a motion significantly perturbed.
dynamics of these Hamiltonians is numerically explored by several numerical simulations using their associated equations of motion. As continuation of this research, it is possible to perform a deeper analysis of the dynamics in regions with picks of deviations for each intermediary and the original Hamiltonian. The analysis can be extended to other kinds of intermediaries, choosing different dynamics, sets of canonical variables and also parameters, for instance, the case of mass parameters (see [1,2]). Thus, the analysis developed in the present study seems to be an useful fast tool to predict regions where a proposed integrable Hamiltonian model might be more or less precise when approaching the original Hamiltonian model describing the full dynamics.

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6. Appendix: Results of the parameters spaces analysis

Figure 1. Parameters space errors varying A and σ.
Figure 2. Parameters space errors varying $e$ and $\sigma$

Figure 3. Parameters space errors varying $e$ and $A$ for $\sigma = 1.0^\circ$
Figure 4. Parameters space errors varying $e$ and $A$ for $\sigma = 45.1^\circ$

Figure 5. Parameters space errors varying $e$ and $A$ for $\sigma = 90.1^\circ$
Figure 6. Parameters space errors varying $r$ and $\sigma$

Figure 7. Parameters space errors varying $r$ and $e$ for $\sigma = 1.0^\circ$
Figure 8. Parameters space errors varying $r$ and $e$ for $\sigma = 45.1^\circ$

Figure 9. Parameters space errors varying $r$ and $e$ for $\sigma = 90.1^\circ$