Magnon Nernst Effect in Magnon Spin Hall Systems

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Magnon spin Hall systems could hardly show experimentally observable particle and thermal transport phenomena intrinsically due to the spin cancellation. Here we demonstrated that the magnon spin Hall systems can exhibit magnon Nernst effect and thermal Hall effect under external magnetic field by considering two typical systems, i.e. the antiferromagnetically (AFM) coupled bilayer honeycomb ferromagnets and monolayer collinear honeycomb antiferromagnet. The both systems experience magnetic phase transitions from AFM phase to a field-polarized phase via a spin-flop (SF) phase or directly. In both systems, there exist magnon Nernst effect and also thermal Hall effect under a longitudinal temperature gradient, which can be regarded as the indicator of the magnetic phase transitions, with the Hall conductivity dependence on magnetic field consistent with the order of magnetic phase transition.

I. INTRODUCTION

In electronic systems, the transport properties can reveal the electronic structure information and also the topological properties. One pronounced example is the Berry phase effect on the electrons.\(^1\) The Berry curvature manifests the electrons a transverse velocity, giving rise to various Hall effect, such as the anomalous Hall effect, topological Hall effect, spin Hall effect or valley Hall effect.\(^{1-8}\) In a semiclassical interpretation, the Berry curvature acts as a fictitious magnetic field in the momentum space, and the related effective Lorentz force bends the electron trajectory to the transverse direction. The observation of these effects usually requires time-reversal or inversion symmetry breaking.

The concepts of Berry phase and Berry curvature are not restricted on the electronic systems, but also applies to the bosons, such as phonons,\(^7\) photons\(^^{9-12}\) and also magnetic excitations, or magnons,\(^{13-29}\) inducing the bosonic analog of anomalous Hall effects. In this paper, we focus on the magnons as they are easily manipulated by external magnetic field. Meanwhile, the external magnetic field may drive the system into different magnetic phases, giving distinct Berry curvature distribution in bands or topological phases, thus showing different transport phenomena. For magnons, the Berry curvature mainly comes from the Dzyaloshinskii-Moriya interaction (DMI),\(^{14-18}\) dipolar interactions,\(^{19-22}\) the magnetic textures,\(^{23-25}\) or the antiferromagnetic (AFM) coupling.\(^{26-28}\) As magnon is charge neutral, it is hard to observe the Hall effect directly. But magnons can carry heat current and thus exhibit a thermal Hall effect, which is experimentally observable. Besides the magnon Hall effect, the antiferromagnetic order endows effective magnon spins, giving rise to magnonic analog of spin Hall effect when the two spin branches host opposite Berry curvature.\(^{27,28,30,31}\) Same to the electronic systems, the magnon spin Hall systems exhibit zero net magnon current in the transverse direction, thus hold a vanishing thermal Hall effect, making it hard to detect experimentally.

In this paper, we demonstrated that the magnon spin Hall systems can exhibit a magnon Nernst (Hall) and thermal Hall effects under external magnetic field. We here consider two typical systems showing magnon spin Hall effect in the absence of magnetic field. The first one is the bilayer honeycomb ferromagnets with AFM interlayer interactions. The monolayer one with nearest-nearest-neighbor DMI is quite widely investigated, which show magnon anomalous Hall effect. The second system is monolayer collinear honeycomb antiferromagnet with NNN DMI. Both of the systems host magnon spin Hall effect due to the AFM coupling and NNN DMI. Under magnetic field, both of the systems experiences magnetic phase transitions from the AFM ordering between layers or sublattices to a field-polarized (FP) phase via a first-order spin-flop (SF) transition or directly at critical values depending on the easy-axis magnetic anisotropy. In all the phases, the topology of the magnon bands are characterized by the Chern number or spin Chern number. In the AFM phase, the broken degeneracy of magnon bands give a magnon Hall effect, manifest it the Nernst effect under the temperature gradient, with the Nernst conductivity depending on the magnetic field. The conductivity also experiences a first-order transition when the both systems go into the SF phase or the FP phase from AFM phase. From the SF phase to the FP phase, the conductivity change continuously but not smoothly, consistent with the phase transition. The thermal Hall effect of magnons show the same behavior to the Nernst effect. The different behavior of Nernst effect and thermal Hall effect of magnons indicates that we can use them as the indicator of the magnetic phase transitions in magnon spin Hall systems. We further find the other magnon spin Hall systems have direct correspondence to the two typical systems in phase diagram and transport properties.

This paper is organized as follows. In Sec. II, we present the theoretical models, the method to determine the magnetic phase transition, calculation of the magnon bands and Nernst and thermal Hall conductivity. In Sec. III, we give the phase diagram, the magnon bands in different magnetic phases and the band topology. The Nernst effect and thermal Hall effect dependence on the magnetic field are also discussed. Finally, we summarize
Here we consider two typical magnon spin Hall systems, as shown in Fig. 1. The first one is the bilayer honeycomb ferromagnets with AFM interlayer coupling, which have been proved to host a $Z_2$ topological invariant.\textsuperscript{30,31} Without the Pauli exclusion, the boson usually cannot give a quantized response to the external field, the system would thus give a spin Hall effect. Here we take the monoclinic stacked bilayer CrI$_3$ as the example material,\textsuperscript{32–34} illustrated in Fig. 1 (a). We apply a perpendicular magnetic field $B = Be_z$. The spin interaction Hamiltonian for the bilayer honeycomb ferromagnets is given by

$$H_1 = -\sum_{i\neq j} J_{ij} S_{i,1} \cdot S_{j,1} + D \sum_{\langle ij \rangle} \nu_{ij} \hat{z} \cdot (S_{i,1} \times S_{j,1})$$

$$- \sum_i \left[ h S_i^z + K_z (S_i^z)^2 \right] + \sum_{\langle ij \rangle} J' S_{i,1} \cdot S_{j,2},$$

(1)

where $h = g \mu_B B$, $\mu_B$ the Bohr magneton, $g$ denotes the g-factor, $l = 1, 2$ is the layer index. $J_{ij} > 0$ characterizes intralayer FM coupling. $D$ represents the next-nearest-neighbor (NNN) Dzyaloshinskii-Moriya interaction (DMI) strength. $\nu_{ij} = \pm 1$ with $+$ (−) for counterclockwise (clockwise) circulation. $K_z$ characterizes the single-ion easy-axis anisotropy. The last term denote the AFM interlayer coupling with $J' > 0$.

The second system is the monolayer honeycomb antiferromagnet, demonstrated before to exhibit spin Nernst (Hall) effect,\textsuperscript{27,28} illustrated in Fig. 1 (b). For now, we did not find the experimentally demonstrated candidate materials. The spin interaction Hamiltonian

$$H_2 = J \sum_{\langle ij \rangle} S_i \cdot S_j + D \sum_{\langle ij \rangle} \nu_{ij} \hat{z} \cdot (S_i \times S_j)$$

$$- \sum_i \left[ h S_i^z + K_z (S_i^z)^2 \right].$$

(2)

Here $J > 0$ characterize the NN antiferromagnetic coupling. The same parameter denotes the same meaning to the first systems.

Under the external magnetic field, the AFM ordering between layers or sublattices will not always be the ground state and the both systems will experience magnetic phase transitions. We can restrict the transition in the $xz$ plane due to the anisotropic exchange interactions. The ground state magnetic ordering can be obtained by minimizing the interaction energies. The spins in the same layer for the first system and in the same sublattice for the second system will align to the same direction, as the NNN DMI does not compromise the ferromagnetic ground state.\textsuperscript{35} For the two systems, the interaction energy per unit cell can be written in a unified form, $E = N J_{AFM} S^2 \delta \cos(\theta_1 - \theta_2) - N h S (\sin \theta_1 + \sin \theta_2) - N K_z S^2 (\sin^2 \theta_1 + \sin^2 \theta_2)$, where $N = 2$ for the first system and $N = 1$ for the second system. $J_{AFM}$ is the AFM coupling strength, $S$ the spin value, $\delta = 2$ for the first system and $\delta = 3$ for the second systems. $\theta_1$ and $\theta_2$ the rotation angle of spins on the separate layer or sublattice with respect to positive $x$-axis.

The magnons are the spin excitations from the magnetic ground state. To get the dispersion relation, we apply the linear spin wave theory. We can write the spin wave vector as $\mathbf{S}_i = S_i^x \mathbf{e}_x + S_i^y \mathbf{e}_y + S_i^z \mathbf{e}_z$ on the sphere and apply the Holstein-Primakoff (HP) transformation,

$$S_i^x = S - a_i^\dagger a_i, S_i^+ = \sqrt{2} S a_i, S_i^- = \sqrt{2 S} a_i^\dagger,$$

Then we make the Fourier transformation

$$a_i = \frac{1}{\sqrt{N}} \sum_k e^{i \mathbf{k} \cdot \mathbf{r}_i} b_{\alpha,k}, a_i^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-i \mathbf{k} \cdot \mathbf{r}_i} b_{\alpha,k}^\dagger (\alpha$$(the sublattice index in a unit cell). The single-particle term vanished by applying the ground state solution. We here neglect the three and more particle term, and the Hamiltonian is expressed as $H = \frac{1}{2} \sum_k \Psi_k^\dagger H_k \Psi_k$ in the basis $\Psi_k = (\psi_k, \psi_{-k})^T$ up to the quadratic term. $\psi_k = (b_{\alpha,1,k}, b_{\alpha,1,k}, b_{\alpha,2,k}, b_{\alpha,2,k})^T$ for the first system and $\psi_k = (b_{\alpha,k}, b_{\alpha,k}, b_{\alpha,k}, b_{\alpha,k})^T$ for the second system. The form of the Hamiltonian $H_k$ depends on the system and the magnetic phases. We will list them in the next section.

Diagonalizing the quadratic form invokes the Bogoliubov transformation, $T_k^\dagger H_k T_k = diag[E_k, -E_k]$ with paraunitary eigenvectors satisfying $T_k^\dagger \tau_z T_k = \tau_z$ and $\tau_z$ is the Pauli matrix acting on the particle-hole space.\textsuperscript{36} For magnon, we usually adopt $E_k = \hbar \omega_k$ with $\omega_k$ the frequency of spin wave modes.

To describe the transport properties of magnons, the Berry curvature is essential. We introduce the Berry connection $A^\alpha_n(k) = i \text{Tr}[T^n \tau_z T^\dagger \tau_z T_k]$, and the Berry curvature $\Omega_n = \nabla_k \times A^\alpha_n(k)$ for $n$-th magnon band, where $\Gamma^n$ is the diagonal matrix taking $+1$ for n-th diagonal component and zero otherwise. Due to the external magnetic field, the topology of magnon bands are described by the Chern number, given by $C_n = \frac{1}{2\pi} \int_{BZ} \Omega_n^z(k) d^2k$ for $n$-th band.

Here we are mainly interested in the transport properties induced by the magnons. The non-vanishing Berry curvature of the magnon bands indicate that magnon...
wavepacket will experience an additional force during motion, giving rise to Hall-like effect and also thermal Hall effect under a thermal gradient. The magnon Hall current is given by

\[ j = -\frac{k_B}{\hbar} \hat{z} \times \nabla T \sum_n \int [d\mathbf{k}] |\Omega_n^z(\mathbf{k})| c_1(\rho_{n,k}) = \alpha_{xy} \hat{z} \times \nabla T, \]

where \([d\mathbf{k}] = d^2\mathbf{k}/(2\pi)^2\), \(k_B\) is the Boltzmann constant, \(h\) reduced Planck constant, \(c_1(x) = (1 + x) \log(1 + x) - x \log x\) and \(\rho_{n,k} = (e^{\beta E_{n,k}} - 1)^{-1}\) is the Bose distribution function, \(\beta = \frac{1}{k_B T}\). \(T\) is the temperature. The Nernst conductivity

\[ \alpha_{xy} = -\frac{k_B^2 T}{\hbar} \sum_n \int [d\mathbf{k}] |\Omega_n^z(\mathbf{k})| c_1(\rho_{n,k}), \quad (3) \]

and the Nernst coefficient \(N = \alpha_{xy}/B\). The motion of magnon would also give a thermal Hall effect, with the thermal Hall current given by

\[ J_Q = -\frac{k_B^2 T}{\hbar} \hat{z} \times \nabla T \sum_n \int [d\mathbf{k}] |\Omega_n^z(\mathbf{k})| c_2(\rho_{n,k}) = \kappa_{xy} \hat{z} \times \nabla T, \]

and the thermal Hall conductivity \(\kappa_{xy} = -\frac{k_B^2 T}{\hbar} \sum_n \int [d\mathbf{k}] |\Omega_n^z(\mathbf{k})| c_2(\rho_{n,k})\).

with \(c_2(x) = (1 + x)(\log \frac{1 + x}{x})^2 - (\log x)^2 - 2 L_{iz}(-x)\),

where \(L_{iz}(x)\) is the polylogarithm function.

**III. RESULTS AND DISCUSSIONS**

**A. Phase diagram and magnon bands**

The ground state magnetic ordering between layer and sublattice depends on the external magnetic field and the magnetic interaction parameters. We plot the phase diagrams in Fig. 2 (a) and also the order parameter, i.e. the magnetization. The two systems share a similar phase transition diagram. There are two distinct magnetic phase transition paths depending on the easy-axis anisotropy \(K_z\). Firstly, when \(K_z < \frac{4}{3} J_{AFM}\), the magnetic order stays in the AFM phase below a critical value \(B_{c1} = 2S\sqrt{K_z(\delta J_{AFM} - K_z)/g\mu_B}\). Further increasing the magnetic field, the both systems experience a spin-flop (SF) transition. In this phase, the spins are partly aligned to the magnetic field direction and \(\theta_2 = \pi - \theta_1\). The magnetic phase transition is a first-order between AFM and SF phases, with a jump in the order parameter \(m_z\), as shown in Fig. 2 (a). The angle \(\theta_1 = \sin^{-1} \left\{ \frac{h}{2S(\delta J_{AFM} - K_z)} \right\}\) increasing the magnetic field to another critical value \(B_{c2} = 2S(\delta J_{AFM} - K_z)/g\mu_B\), \(\theta_1 = \theta_2 = \pi/2\), the spins are fully polarized along the magnetic field. A larger magnetic field will not affect the ground state magnetic order. This transition is a second-order phase transition. Second, when \(K_z > \frac{4}{3} J_{AFM}\), the both systems will go into the FP phase directly from the AFM phase with the critical value \(B_{c3} = \delta J_{AFM} S/g\mu_B\) and the phase transition is in first-order.

In the three magnetic phases, the systems hold different ground state spin configurations. The spin excitations are expected to be different. We here discuss the magnon Hamiltonians and the topological phases in different magnetic phases. We first discuss the bilayer honeycomb ferromagnets. The magnon Hamiltonian in the AFM phase is given by

\[ H^A_{1,0} = h^A_{1,0} + \begin{pmatrix} h^A_{1,k} & 0 & 0 & h^A_{1,k} \\ 0 & h^A_{1,-k} & h^A_{1,k} & 0 \\ 0 & h^A_{1,-k} & h^A_{1,k} & 0 \\ h^A_{1,-k} & 0 & 0 & h^A_{1,k} \end{pmatrix}, \]

where \(h^A_{1,0} = (2K_z + 3J_1 + 2J' + 2J_2p_k)S + h\tau_0\sigma_0\), \(h^A_{1,k} = \mathbf{h} \cdot \sigma\) with \(h_x = -J_1 S \text{Re}(f_k)\), \(h_y = J_1 S \text{Im}(f_k)\), \(f_k = 1 + e^{-i k_\alpha a_1} + e^{i k_\alpha a_2}\), \(\mathbf{h} = -2DSg(k)\), \(g(k) = \sum_{\alpha=1}^3 \sin(k \cdot a_\alpha)\), \(p_k = \sum_{\alpha=1}^3 \cos(k \cdot a_\alpha)\), \(a_1 = (\sqrt{3}, 0, 0)\), \(a_2 = (0, \sqrt{3}, 0)\), \(a_3 = -(a_1 + a_2)\) intralayer NNN vectors. \(\tau, s, \sigma\) are pauli matrices acting on particle-hole, layer
We can decompose the Hamiltonian as the topological part. In the SF and FP phases, we can see they exhibit different topological phases in different magnetic orders. As a result, for each spin-polarized band, the Berry curvature effect does not vanish, bringing the spin Nernst effect, demonstrated in previous works. Here the split bands by magnetic field will induce imbalanced spin population at finite temperatures, expected to give rise to nonvanishing Nernst effect.

In the SF phase, the magnon Hamiltonian is in the following form,

\[
H_{2,k}^S = h_{2,0}^S + \begin{pmatrix}
h_{z,k}^S & h_{12,k}^S & \Delta_{11}^S & \Delta_{12}^S \\
0 & h_{z,k}^S & -\Delta_{12}^S & \Delta_{21}^S \\
J S f_k & 0 & h_{z,k}^S & 0 \\
0 & J S f_k & 0 & -h_{z,-k}^S \\
\end{pmatrix},
\]

where \( h_{2,0}^S = 3JS + 2K_zS + h\theta_0\sigma_z \). This Hamiltonian can be recast into two sectors even at finite \( h \). As demonstrated in former work, the two sectors give us two magnon branches, denoted as the magnon spin, \( H_{2,k}^A = H_{2,1}^A + H_{2,2}^A \). Without magnetic field, the two magnon branches are degenerate. The magnetic field breaks the degeneracy, as shown in Fig. 3 (a). Here we should notice that for each band, the Berry curvature holds the relation, \( \Omega_k = -\Omega_{-k} \), the Chern number of both bands are zero. The NNN DMI brings the asymmetry magnon bands, leading to nonreciprocal transport behavior of magnons. As a result, for each spin-polarized band, the Berry curvature effect does not vanish, bringing the spin Nernst effect, demonstrated in previous works.

From the two typical magnon spin Hall systems, we can see they exhibit different topological phases in different magnetic ordering. The Berry curvature distributions also vary for different system and magnetic ordering. Next we will prove that magnon transport properties, i.e., the Nernst conductivity and thermal Hall conductivity, are similar for the two systems, but different in the three magnetic phases. As a result, we can use the Nernst effect and thermal Hall effect of magnons to distinguish the magnetic phases experimentally for magnon spin Hall systems.
The sign
Further increase of the magnetic field, we find the Nernst conductivity
Here we should notice that the spin Nernst coefficient
FIG. 4: The magnon Nernst conductivity ((a) and (e)), Nernst coefficient ((b) and (f)), spin Nernst conductivity ((c) and (g)) and thermal Hall conductivity ((d) and (h)) with respect to the magnetic field for the first magnon spin Hall systems at different interlayer coupling strength $J'=0.3$ meV ((a)-(d)) and $J'=0.15$ meV ((e)-(h)), which give different magnetic phase transition path under magnetic field. The other parameters are adopted the same to Fig. 2 ((b)-(d)). The colors denote different temperatures.

B. Magnon Nernst effect and thermal Hall effect

We now turn to the thermal transport properties induced by the magnons for the magnon spin Hall systems under magnetic field. The magnetic field lifts the band degeneracy, thus induces imbalanced populations of magnons with opposite Berry curvatures at finite temperatures. The total effect of Berry curvature does not vanish, and thus manifests Nernst effect and thermal Hall effect under thermal gradient. We below discuss in details for two systems case by case and summarize the same behaviors.

We first concentrate on the bilayer honeycomb ferromagnets. The weak interlayer AFM coupling strength is comparable to the easy-axis anisotropy term. There are two possible phase transition paths. When $K_z < J'$, the ground state magnetic ordering is from the AFM ordering to the FP phase via a SF transition. In the AFM ordering, the magnon bands are spin-polarized. The bands with opposite spin hold opposite Berry curvature at a fixed $k$ for valence and conduction bands, $\Omega_{xy}^{\uparrow} = -\Omega_{xy}^{\downarrow}$. At zero magnetic field, it will give us a spin Nernst effect, vanishing Nernst and thermal Hall effect as $\alpha_{xy,\uparrow} = -\alpha_{xy,\downarrow}$ cancels each other. At finite field, the spin-polarized bands are separated energetically, inducing an imbalanced spin population, thus we have $\alpha_{xy,\uparrow} \neq -\alpha_{xy,\downarrow}$. A net Nernst conductivity $\alpha_{xy} = \alpha_{xy,\uparrow} + \alpha_{xy,\downarrow}$ survives, as shown in Fig. 4 (a). Here we should notice that the spin Nernst coefficient $\alpha_{xy,\uparrow}^s = \alpha_{xy,\uparrow} - \alpha_{xy,\downarrow}$ does not vanish, as shown in Fig. 4 (c). From Fig. 4 (a), we find the Nernst conductivity $\alpha_{xy}$ are linearly dependent on the magnetic field, which leads to a constant Nernst coefficient $\mathcal{N} = \alpha_{xy}/B$ in the AFM phase, as shown in Fig. 4 (b). The Nernst conductivity and coefficient depend on the temperature.

When $B_{c1} < B < B_{c2}$, the ground state magnetic ordering is in the SF phase. Across $B_{c1}$, the system experience a first-order phase transition and the magnetization change discontinuously. Correspondingly, the Nernst conductivity $\alpha_{xy}$ shows a discontinuous transition and reverses the sign, as shown in Fig. 4 (a). The sign reverse is because the lowest (highest) two magnon bands hold the same sign, different from the AFM phase. The increasing magnetic field increases the absolute value of $\alpha_{xy}$, which reaches the extreme at $B = B_{c2}$, shown in Fig. 4 (a). Further increase of the magnetic field, the spin configuration will go into the FP phase via a second-order phase transition. From the Nernst conductivity, we can also see a continuous but not smoothly change, also for the Nernst coefficient $\mathcal{N}$, consistent with the order of phase transition. Magnons can carry heat, a Nernst effect indicates a thermal Hall effect. We plot the thermal Hall conductivity $\kappa_{xy}$ with respect to the magnetic field in Fig. 4 (d). We can see $\kappa_{xy}$ behave the same the $\alpha_{xy}$. $\kappa_{xy}$ shows a linear behavior in the AFM phase and then a first-order discontinuous change to the SF phase, then increases to the extreme value and decreases in the FP phase. For the two magnetic phase transitions, we can see the transition of the Nernst conductivity and thermal Hall conductivity is consistent with order parameter...
transitions. Thus we can use the Nernst conductivity and the thermal Hall conductivity as the indicator of magnetic phase transitions.

When $K_z > J'$, the spin configuration will transit form the AFM to the FP phase directly via a first-order phase transition. In the AFM phase, the Nernst conductivity, Nernst coefficient, spin Nernst coefficient, and also the thermal Hall conductivity are expected to show same behavior to the first path, as shown in Fig. 4 (e-h). As there is no SF phase in this path, the behavior at $B > B_{c3}$ is also the same to the FP phase in the above path. $\kappa_{xy}$ and $\kappa_{xy}$ show different behavior in the SF and FP phase. If $K_z < J'$, we can use the two critical value of magnetic field to find the value of $K_z$ and $J'$. While $K_z > J'$, we can know the $J'$ via $B_{c3}$. Here we take the bilayer CrI$_3$ as the example. The neutron scattering experiments cannot fully determine magnetic interaction parameters from the fitting result of the bulk materials experiments.\cite{37,38} Our results provide an alternative way to determine the parameters.

Now we discuss the monolayer honeycomb antiferromagnets. The easy-axis anisotropy is much smaller than the AFM coupling, $K_z < \frac{3}{2}J$. The ground state spin configuration will go into the FP ordering via a SF transition under magnetic field. In the AFM ordering, the spin-polarized bands are shifted upwards or downwards by magnetic field. For each band, we have $\Omega^\uparrow_k = -\Omega^\downarrow_{-k}$, but $E_k \neq E_{-k}$ due to the DMI. The nonreciprocal bands hold net Nernst effect under thermal gradient. But for opposite spins, $\Omega^\uparrow_k \neq -\Omega^\downarrow_{-k}$, the Berry curvature is opposite at fixed $k$. A zero field give a spin Nernst effect. At finite field, the split bands give an imbalanced population for opposite spins. The Nernst conductivity is finite but small, shown in Fig. 5 (a). The same to the first system, the spin Nernst effect survives, with the spin Nernst conductivity shown in Fig. 5 (c). Here differently, the Nernst conductivity is not linearly dependent on the magnetic field and is negative-valued. Across $B_{c1}$, the Nernst and thermal Hall conductivity show a first-order-like phase transition. In the SF phase, the increase of the magnetic field increases the absolute value of $\alpha_{xy}$ and $\kappa_{xy}$, which reach the maximum absolute value at $B_{c2}$, same to the first system. Across $B_{c2}$, the $\alpha_{xy}$ and $\kappa_{xy}$ decrease as magnetic field increases, the transition is consistent with the phase transition of order parameter. Also, we can use the Nernst and thermal Hall conductivity to find the value of $J$ and $K_z$.

Above, we have demonstrated theoretically that the magnon spin Hall systems share the same magnetic phase transition diagram by considering two typical systems. The Nernst effect, also thermal Hall effect under thermal gradient show distinct behaviors in three magnetic phases but almost the same for the two systems. We can use the results of Nernst and thermal Hall conductivity as the indicator of magnetic phase transition and help us to determine the parameters of anisotropic term and AFM coupling. The other magnon spin Hall systems are expected to show similar results. In Ref.\cite{39} the bilayer kagome ferromagnets with AFM interlayer coupling is also a magnon spin Hall system. The behavior of this system under magnetic field are expected similar to the bilayer honeycomb ferromagnets. Ref.\cite{28,30} present another magnon spin Hall systems, the bilayer honeycomb antiferromagnets with AFM interlayer coupling. For AA stacking, the interaction energy per unit cell is given by $E = E_{\text{intra}} + E_{\text{inter}} + E_{Z} + E_{\text{ani}}$ with

$$E_{\text{intra}} = 3JS^2\cos(\theta_{1A} - \theta_{1B}) + 3JS^2\cos(\theta_{2A} - \theta_{2B}),$$

$$E_{\text{inter}} = J'S^2\cos(\theta_{1A} - \theta_{2A}) + J'S^2\cos(\theta_{1B} - \theta_{2B}),$$

$$E_{Z} = -hS(\sin\theta_{1A} + \sin\theta_{1B} + \sin\theta_{2A} + \sin\theta_{2B}),$$

$$E_{\text{ani}} = -K_zS^2(\sin^2\theta_{1A} + \sin^2\theta_{1B} + \sin^2\theta_{2A} + \sin^2\theta_{2B}).$$

Here the number denotes the layer index and $A(B)$ denote the sublattice. $J$ and $J'$ denote the intralayer and interlayer AFM coupling, respectively. We can see that the interchanges $\theta_{1A} \leftrightarrow \theta_{2B}, \theta_{1B} \leftrightarrow \theta_{2A}$ will not change the interaction energy. Thus the relation $\theta_{1A} = \theta_{2B}, \theta_{1B} = \theta_{2A}$ is valid for all magnetic phases. This system is equivalent to our second system by replacing $J$ with $J + \frac{J'}{4}$ for the interaction energy by considering two isolated layers. Thus the magnetic phase diagram is similar. There are four magnon bands, the topology can be characterized by the spin Chern number in the AFM phase\cite{28} and Chern number in other two magnetic phases. As a result, the Nernst conductivity and thermal Hall conductivity are expected to behave similarly to the first system under magnetic field.

FIG. 5: The Nernst conductivity (a), the Nernst coefficient (b), the spin Nernst conductivity (c) and thermal Hall conductivity (d) with respect to magnetic field for the second spin Hall systems. $D = 0.1J$, $K_z = 0.3J$. 

\begin{itemize}
  \item \textbf{a)}: The Nernst conductivity as a function of magnetic field for the second spin Hall systems.
  \item \textbf{b)}: The Nernst coefficient as a function of magnetic field for the second spin Hall systems.
  \item \textbf{c)}: The spin Nernst conductivity as a function of magnetic field for the second spin Hall systems.
  \item \textbf{d)}: The thermal Hall conductivity as a function of magnetic field for the second spin Hall systems.
\end{itemize}
IV. CONCLUSIONS

In conclusion, we theoretically find the magnon spin Hall system can exhibit Nernst effect and thermal Hall effect under perpendicular magnetic field. There are three magnetic phases, i.e., the AFM, SF and FP phases. In these phases, the two systems we considered both exhibit Nernst effect and thermal Hall effect, which show different dependence on magnetic field in the three phases. The transition behavior between the magnetic phases is consistent with the order of magnetic phase transition, indicating we can use the Nernst effect and thermal Hall effect as the indicator of magnetic phase transitions, also helping us to determine the magnetic interaction parameters. Although we only considered two models, the other magnon spin Hall systems can be mapped to the two systems in the ground state spin configuration and also the thermal transport properties, making our results universal for magnon spin Hall systems.

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