A NOTE ON DEDEKIND ZETA VALUES AT $1/2$

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Abstract. For a number field $K$, let $\zeta_K(s)$ be the Dedekind zeta function associated to $K$. In this note, we study non-vanishing and transcendence of $\zeta_K$ as well as its derivative $\zeta'_K$ at $s = 1/2$. En route, we strengthen a result proved by Ram Murty and Tanabe in *J. Number Theory*, 2016.

1. Introduction

For a number field $K$, let $\zeta_K$ be the Dedekind zeta function associated to $K$. The non-vanishing of $\zeta_K(s)$ at $s = 1/2$ is a deep arithmetic question. Armitage [1] gave examples of number fields $K$ for which $\zeta_K(1/2) = 0$. On the other hand, it is believed that $\zeta_K(1/2) \neq 0$ when $K$ is an $S_n$-number field, that is, a number field of degree $n$ whose normal closure has Galois group $S_n$ over $\mathbb{Q}$. Furthermore, very little is known about the transcendental nature of the non-zero values of $\zeta_K(1/2)$. For instance, one has

$$\zeta(1/2) = \frac{1}{1 - \sqrt{2}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \approx -1.46035450880 \cdots ,$$

where $\zeta$ is the Riemann zeta function.

In this connection, one has a classical conjecture of Dedekind which asserts that if $L/K$ is an extension of number fields, then $\zeta_K(s)$ divides $\zeta_L(s)$, in other words, the function $\zeta_L(s)/\zeta_K(s)$ is entire. This conjecture is open in general, but holds when $L/K$ is Galois, thanks to the works of Aramata and Brauer. Also the celebrated Artin’s conjecture for holomorphicity of his $L$-functions will establish Dedekind’s conjecture.

If $L/\mathbb{Q}$ is a Galois number field with Galois group $S_n$, then $L$ contains a quadratic subfield $K$. Dedekind’s conjecture ensures that vanishing of $\zeta_K(1/2)$ will ensure vanishing of $\zeta_L(1/2)$.
In this note, we study various aspects of the derivative \( \zeta'_K(s) \) at \( s = 1/2 \). As discussed above, study of these circle of questions for quadratic fields merits special attention. We note that for quadratic fields \( K \), the non vanishing of \( \zeta_K(1/2) \) is equivalent to the non vanishing of \( L(1/2, \chi) \) for quadratic character \( \chi \). Non vanishing of such \( L(1/2, \chi) \) has been conjectured by Chowla.

We begin with the following theorem for quadratic fields.

**Theorem 1.** Let \( K \) and \( L \) be distinct quadratic fields such that \( \zeta_K(1/2) \zeta_L(1/2) \neq 0 \). Then

\[
\frac{\zeta'_K(1/2)}{\zeta_K(1/2)} \neq \frac{\zeta'_L(1/2)}{\zeta_L(1/2)}.
\]

The above theorem in particular shows that there is at most one quadratic field \( K \) for which \( \zeta_K(1/2) \neq 0 \) while the derivative \( \zeta'_K(1/2) = 0 \). We then prove the following quantitative theorem where the existence of the fictitious exception alluded to above is ruled out for both quadratic and cubic number fields.

**Theorem 2.** Let \( K \) be an algebraic number field with degree \( \leq 3 \). Then

\[ \zeta_K(1/2) = 0 \iff \zeta'_K(1/2) = 0. \]

As a corollary, we have the following courtesy of the seminal work by K. Soundararajan [19].

**Corollary 3.** For at least 87.5% of quadratic number fields \( K \) with discriminant \( 8d \) with odd positive square-free \( d \), one has \( \zeta'_K(1/2) \neq 0 \).

Let us very briefly describe the context as well as content of the work of Soundararajan indicated above. The Generalised Riemann Hypothesis (GRH) does not preclude the possibility that \( L(1/2, \chi) = 0 \) for some primitive Dirichlet character \( \chi \). But it is believed that there is no rational linear relation between the ordinates of the non-trivial zeros of the Dirichlet \( L \)-functions and consequently, \( L(1/2, \chi) \) is expected to be non-zero for any primitive Dirichlet character \( \chi \). In particular when \( \chi \) is a quadratic character, this seems to have been conjectured first by Chowla [4] as indicated earlier. In his outstanding work [19], Soundararajan showed that for at least 87.5% of the odd square-free integers \( d \geq 0 \), \( L(1/2, \chi_{8d}) \neq 0 \). Here for a fundamental discriminant (discriminant of some quadratic number field) \( d \), \( \chi_d(n) := \left(\frac{d}{n}\right) \) where \( \left(\frac{d}{n}\right) \) denotes the Kronecker symbol. Results along this direction were obtained earlier in [3], [9] and [10].

We now have the following theorem for higher degree number fields.

**Theorem 4.** Let \( K \) be an algebraic number field of degree \( n > 3 \) such that the absolute value of its discriminant \( |d_K| \in \mathbb{R} \setminus [(44.763)^n, (215.333)^n] \). Then

\[ \zeta_K(1/2) = 0 \iff \zeta'_K(1/2) = 0. \]

We now consider the analogous question for Galois number fields.
Theorem 5. Consider the following sets.

\[ X = \{ K \text{ Galois} : K \subset \mathbb{R}, \zeta_K(1/2) \neq 0, \zeta'_K(1/2) = 0 \} \]

\[ Y = \{ K \text{ Galois} : K \not\subset \mathbb{R}, \zeta_K(1/2) \neq 0, \zeta'_K(1/2) = 0 \}. \]

Then at least one of the sets \( X \) and \( Y \) is empty. Furthermore, there are at most finitely many abelian number fields for which \( \zeta_K(1/2) = 0 \) but \( \zeta_K(1/2) \neq 0 \). All such number fields (if exist) have degree less than 46369.

Remark 6. Suppose \( \zeta_K(1/2) \neq 0 \) and \( \zeta'_K(1/2) = 0 \), then degree of \( K/\mathbb{Q} \) is precisely \( \frac{\log |d_K|}{\pi/2 + \log 2 + \gamma} \) and \( \frac{\log |d_K|}{\log 2 + \gamma} \) in case of totally real and totally complex Galois number fields respectively, where \( |d_K| \) denotes the absolute discriminant of \( K \) and \( \gamma \) is the ubiquitous Euler’s constant.

The above theorem refines a result of Ram Murty and Tanabe [17, Cor 3.9]. Investigations similar to ours for Elliptic curves over \( \mathbb{Q} \) as well as Modular forms were initiated by Gun, Murty and Rath [6]. Furthermore in [17], it has been proved that there are only finitely many abelian totally real number fields \( K \) for which \( \zeta_K(1/2) \neq 0 \) while the derivative \( \zeta'_K(1/2) = 0 \). One of our objectives in this note was to further this line of investigation to arbitrary number fields, obtain some quantitative results and finally study transcendental nature of these deeply mysterious numbers. In particular, we use Baker’s seminal theorem (see [6], [7], [14] and [15] for some other applications of Baker’s theorem). In this context, we have the following theorem.

Theorem 7. Let \( K \) and \( L \) be distinct algebraic number fields of degree \( n \) and \( m \) respectively and \( \zeta_K(1/2) \zeta_L(1/2) \neq 0 \). If one of the two following conditions hold

1. \( |d_K|^m \neq |d_L|^n \);

2. \( m \frac{\zeta'_K(1/2)}{\zeta_K(1/2)} \neq n \frac{\zeta'_L(1/2)}{\zeta_L(1/2)} \),

then at least one of the following two numbers

\[ \frac{\zeta'_K(1/2)}{\zeta_K(1/2)} \quad \text{and} \quad \frac{\zeta'_L(1/2)}{\zeta_L(1/2)} \]

is transcendental.

Now, we obtain the following interesting corollaries from Theorem 7.

Corollary 8. Let \( n \) be a positive integer. Then the set

\[ \left\{ \frac{\zeta_K(1/2)}{\zeta'_K(1/2)} : \zeta_K(1/2) \neq 0, [K : \mathbb{Q}] = n \right\} \]
has at most one algebraic number. Furthermore,
\[
\frac{\zeta_K(1/2)}{\zeta_K(1/2)} - \frac{n}{2}(\log 8\pi + \gamma)
\]
is a transcendental number.

Two non-zero integers \(u\) and \(v\) are said to be multiplicatively independent if for integers \(n\) and \(m\), \(u^n = v^m\) implies \(n = m = 0\). In this context, we deduce the following corollary for arbitrary degree number fields.

**Corollary 9.** Let \(F\) be a family of number fields with pairwise multiplicatively independent discriminants. If \(\zeta_K(1/2) \neq 0\) for every \(K \in F\), then the set
\[
\left\{ \frac{\zeta'_K(1/2)}{\zeta_K(1/2)} : K \in F \right\}
\]
has at most one algebraic number.

We refer to [14] and [15] for investigations related to the non-vanishing of derivatives of \(\zeta_K(s)\) and \(L(s, f)\) at \(s = 1\), where \(f\) is a periodic arithmetic function.

### 2. Preliminaries

We fix some useful notations which will be used throughout this paper. For a number field \(K\),
- \(d_K\) = discriminant of \(K\),
- \(r_1\) = number of real embeddings of \(K\),
- \(r_2\) = number of non-conjugate complex embeddings of \(K\),
- \(n = r_1 + 2r_2\) = degree of \(K\) over \(\mathbb{Q}\),
- \(\Gamma\) = gamma function,
- \(\gamma\) = Euler–Mascheroni constant \(\approx 0.577 \cdots\).

We recall some relevant facts about the Dedekind zeta function \(\zeta_K(s)\) associated to a number field \(K\). \(\zeta_K(s)\) initially given by the following Dirichlet series
\[
\zeta_K(s) = \sum_{I \neq 0} \frac{1}{N(I)^s}
\]
for \(\Re(s) > 1\) has a meromorphic continuation to the complex plane with a simple pole at \(s = 1\). Furthermore, the function
\[
Z_K(s) := \Gamma_C(s)^{r_2}\Gamma_R(s)^{r_1}\zeta_K(s)
\]
extends meromorphically to the complex plane with simple poles at \(s = 0\) and \(s = 1\) and satisfies the functional equation \(Z_K(s) = [d_K]^{1/2-s} Z_K(1-s)\). Also \(\Gamma_C(s) = (2\pi)^{-s}\Gamma(s), \Gamma_R(s) = \)
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But we shall use the following version of the functional equation which is amenable for our purpose, namely

\[ \zeta_K(1 - s) = A_K(s)\zeta_K(s) \]

for $s \in \mathbb{C} \setminus \{1\}$ (see [12, p. 467], for instance) with the factor

\[ A_K(s) := |d_K|^{s-1/2} \left( \cos \frac{\pi s}{2} \right)^{r_1 + r_2} \left( \sin \frac{\pi s}{2} \right)^{r_2} (2(2\pi)^{-s}\Gamma(s))^n. \]

We now list some transcendental pre-requisites required for our work. We shall need Gelfond-Schneider Theorem which states the following.

**Theorem 10.** [5, 11] If $\alpha$ and $\beta$ are non-zero algebraic numbers with $\beta \neq 1$ and $\log \alpha/\log \beta \notin \mathbb{Q}$, then $\log \alpha/\log \beta$ is transcendental.

We now record the following application of Baker’s seminal theorem on linear forms in logarithms of algebraic numbers [2, Thm 2.1].

**Lemma 11.** [16, p. 154, Lemma 25.4] Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive algebraic numbers. If $c_0, c_1, \ldots, c_n$ are algebraic numbers with $c_0 \neq 0$, then

\[ c_0\pi + \sum_{j=1}^{n} c_j \log \alpha_j \]

is a transcendental number.

Another important lemma required to prove our results is the following application of Lindemann-Weierstrass theorem [11].

**Lemma 12.** [11, Cor 1.3] If $\alpha$ is an algebraic number different from 0 and 1, then $\log \alpha$ is a transcendental number where $\log$ denotes any branch of logarithmic function.

Now we quickly recall the discriminant of a quadratic number field $K$. Let $d$ be a square-free integer, then the discriminant $d_K$ of the field $K = \mathbb{Q}(\sqrt{d})$ is

\[ d_K = \begin{cases} d & \text{if } d \equiv 1 \pmod{4} \\ 4d & \text{if } d \equiv 2, 3 \pmod{4}. \end{cases} \]

In 1984, Ram Murty proved the following elegant result on lower bounds of discriminants of abelian number fields in terms of their degrees. More precisely,

**Theorem 13.** [13, Cor 2] Let $K$ be an abelian extension of $\mathbb{Q}$ of degree $n$. Then,

\[ \log |d_K| \geq \frac{n \log n}{2}. \]

We also record the following classical theorem which follows directly from the Minkowski’s bound.
Theorem 14. [18, §4.3, Thm 1] For any number field $K \neq \mathbb{Q}$, $|d_K| > 1$.

An essential ingredient for the proof of Theorem 5 is the following result due to Hermite (see, for instance, [12, ch. 3], [18]).

Theorem 15. There exist only finitely many number fields with bounded discriminant.

Finally, we state the following deep theorem of Soundararajan which we shall need to prove Corollary 3.

Theorem 16. [19, Thm 1] For at least 87.5% of the odd square-free integers $d \geq 0$, $L(1/2, \chi_{8d}) \neq 0$.

3. Proofs of the Main Theorems

As indicated in the earlier section, we shall work with the following functional equation of $\zeta_K(s)$ for $s \in \mathbb{C} \setminus \{1\}$ [12, p. 467]

$$\zeta_K(1 - s) = A_K(s) \zeta_K(s)$$

(1)

with the factor $A_K(s) := |d_K|^{s-1/2} (\cos \frac{\pi s}{2})^{r_1 + r_2} (\sin \frac{\pi s}{2})^{r_2} (2\pi)^{-s} \Gamma(s)^n$.

Let us begin with an easy, but important observation that $A_K(1/2) = 1$.

We now differentiate (1) w.r.t $s$ and substitute at $s = 1/2$ to obtain

$$\zeta'_K(1/2) = -(1/2) A'_K(1/2) \zeta_K(1/2).$$

(2)

On the other hand, taking the logarithmic derivative of $A_K(s)$ we obtain

$$\frac{A'_K(s)}{A_K(s)} = \log |d_K| - \frac{\pi}{2} (r_1 + r_2) \tan \frac{\pi s}{2} + r_2 \frac{\pi}{2} \cot \frac{\pi s}{2} - n \log 2\pi + n \frac{\Gamma'(s)}{\Gamma(s)},$$

where log is the natural logarithm.

Since the value of digamma function $\frac{\Gamma'(s)}{\Gamma(s)}$ at $s = 1/2$ is $-\gamma - 2 \log 2$ (see [20], p. 427), we have

$$A'_K(1/2) = \log |d_K| - r_1 \frac{\pi}{2} - n (\log 8\pi + \gamma).$$

(3)

3.1. Proof of Theorem 1

Let $K = \mathbb{Q}(\sqrt{d_1})$ and $L = \mathbb{Q}(\sqrt{d_2})$, where $d_1$ and $d_2$ are distinct square-free integers. Since $K$ and $L$ are distinct quadratic fields, we have $d_K \neq d_L$.

Using (2) and (3), we obtain

$$-2 \left( \frac{\zeta'_K(1/2)}{\zeta_K(1/2)} - \frac{\zeta'_L(1/2)}{\zeta_L(1/2)} \right) = \log \left( \frac{|d_K|}{|d_L|} \right) + \frac{\pi}{2} (r_1^{(L)} - r_1^{(K)}),$$

where $r_1^{(L)}$ and $r_1^{(K)}$ denote the number of real embeddings of $L$ and $K$ respectively.

It follows from Theorem 10 that $e^\pi$ is a transcendental number. So if $r_1^{(L)} - r_1^{(K)} \neq 0$, then
the right hand side of the above equation is non-zero by Theorem 10. On the other hand if \( r_1^{(L)} - r_1^{(K)} = 0 \), then the right hand side of the above equation is actually transcendental by Lemma 12.

Thus,

\[
\frac{\zeta'_K(1/2)}{\zeta_K(1/2)} - \frac{\zeta'_L(1/2)}{\zeta_L(1/2)}
\]

is non-zero.

We note that our proof along with lemma 11 gives a stronger assertion, namely the number

\[
\frac{\zeta'_K(1/2)}{\zeta_K(1/2)} - \frac{\zeta'_L(1/2)}{\zeta_L(1/2)}
\]

is actually transcendental.

### 3.2. Proof of Theorem 2

By (2), it is enough to show that \( A'_K(1/2) \neq 0 \). We have

\[
A'_K(1/2) = 0 \iff |d_K| = \exp(r_1 \frac{\pi}{2} + n(\log 8\pi + \gamma)). \tag{4}
\]

\( A'_K(1/2) \) is evidently non-zero for \( K = \mathbb{Q} \). In fact, we have \( \zeta'(1/2) = -3.922\ldots \).

So we have the following two cases.

**Case (i).** Assume \( K \) is a quadratic field. So \( r_1 \) could be either 0 or 2. At \( r_1 = 0 \), we have

\[
2003 < \exp(r_1 \frac{\pi}{2} + 2(\log 8\pi + \gamma)) < 2004.
\]

At \( r_1 = 2 \), we have

\[
46368 < \exp(r_1 \frac{\pi}{2} + 2(\log 8\pi + \gamma)) < 46369.
\]

Since \( d_K \) is always an integer, \( A'_K(1/2) \) can never be zero in case of quadratic fields.

**Case (ii).** Now we consider \( K \) a cubic field. So \( r_1 \) is either 1 or 3. At \( r_1 = 1 \), we have

\[
431471 < \exp(r_1 \frac{\pi}{2} + 3(\log 8\pi + \gamma)) < 431472.
\]

At \( r_1 = 3 \), we have

\[
9984558 < \exp(r_1 \frac{\pi}{2} + 3(\log 8\pi + \gamma)) < 9984559.
\]

Since \( d_K \) is always an integer, \( A'_K(1/2) \) can not be zero in case of cubic fields also.

### 3.3. Proof of Corollary 3

For a quadratic number field \( K \), we have

\[
\zeta_K(s) = \zeta(s) L(s, \chi_{d_K}), \quad \Re(s) > 1,
\]

where \( L(s, \chi_{d_K}) := \sum_{n=1}^{\infty} \frac{\chi_{d_K}(n)}{n^s} \). We refer the reader to [8, Ch. VII] and [12] for further details.

We recall that this generalization of Riemann zeta function also has the analytic continuation to whole complex plane except \( s = 1 \). By uniqueness of analytic continuation of complex functions,
one could get the same identity for \( s \in \mathbb{C} \setminus \{1\} \).

Now, we let \( K = \mathbb{Q}(\sqrt{2d}) \), where \( d \) is a square-free positive odd integer. It is easy to see that discriminant of \( K \) is \( 8d \) (for instance, see §5.3, [18]). Hence,

\[
\zeta_K(1/2) = \zeta(1/2)L(1/2, \chi_{8d}).
\]

Using Theorem 16, we have our desired result.

### 3.4. Proof of Theorem 4

We show that in the given interval of \( d_K \),

\[
A'_K(1/2) = \log |d_K| - r_1 \frac{\pi}{2} - n(\log 8\pi + \gamma) \neq 0.
\]

Using hypothesis, we have

\[
A'_K(1/2) < n(\log(44.763) - \log 8\pi - \gamma) < 0.
\]

Similarly,

\[
A'_K(1/2) > n(\log(215.333) - \frac{\pi}{2} - \log 8\pi - \gamma) > 0.
\]

So our result follows from (2).

### 3.5. Proof of Theorem 5

If possible, let us assume that there exist Galois number fields \( K \in X \) and \( L \in Y \) of degree \( n \) and \( m \) respectively. From (2), we have

\[
A'_K(1/2) = A'_L(1/2) = 0.
\]

From (3), we have

\[
nA'_K(1/2) = \log |d_K|^{1/n} - r_1^{(K)} \frac{\pi}{2n} - \log 8\pi - \gamma
\]  

and

\[
mA'_L(1/2) = \log |d_L|^{1/m} - r_1^{(L)} \frac{\pi}{2m} - \log 8\pi - \gamma,
\]

where \( r_1^{(K)} \) and \( r_1^{(L)} \) denote the number of real embeddings of \( K \) and \( L \) respectively.

From (5) and (6), we obtain

\[
\log |d_K|^{1/n} - r_1^{(K)} \frac{\pi}{2n} - \log |d_L|^{1/m} + r_1^{(L)} \frac{\pi}{2m} = 0.
\]

Since Galois fields are the normal extensions of their base fields, so there does not exist any complex embedding in real Galois fields. Similarly, there are no real embeddings in non-real Galois fields. Therefore, \( r_1^{(K)} = n \) and \( r_1^{(L)} = 0 \). Hence,

\[
\log \frac{|d_K|^{1/n}}{|d_L|^{1/m}} - \frac{\pi}{2} = 0,
\]

which is a contradiction as \( e^\pi \) is transcendental by Theorem 10. This completes the first part.
We now proceed with the second part of Theorem 5. From (3), we have

\[ A_K'(1/2) = \log |d_K| - r_1 \frac{\pi}{2} - n(\log 8\pi + \gamma). \]

Using Theorem 13, we obtain

\[ A_K'(1/2) \geq n \left( \log \left( \frac{n}{2} \right) - \frac{\pi}{2} - \log 8\pi - \gamma \right) > 0, \ \forall n \geq 46369. \]

This implies that for all \( n \geq 46369 \), \( \zeta_K'(1/2) = 0 \) if and only if \( \zeta_K(1/2) = 0 \).

Now we aim to prove that the set

\[ S := \{ K : \zeta_K'(1/2) = 0, \ \zeta_K(1/2) \neq 0, \ n < 46369 \} \]

has finite cardinality. By (2), we see that

\[ S \subseteq S' := \{ K : A_K'(1/2) = 0, \ n < 46369 \}. \]

So it is enough to show that the set \( S' \) has finite cardinality. By (4), we have

\[ A_K'(1/2) = 0 \iff |d_K| = \exp \left( r_1 \frac{\pi}{2} + n(\log 8\pi + \gamma) \right). \]

Since \( n \) and \( r_1 \) are bounded in the latter set, the discriminant \( d_K \) is also bounded. So \( S' \) is a set of number fields with bounded discriminant. Hence, we conclude our result by Theorem 15.

### 3.6. Proof of Theorem 7

From (2) and (3), we obtain

\[-2 \left( m \frac{\zeta_K'(1/2)}{\zeta_K(1/2)} - n \frac{\zeta_L'(1/2)}{\zeta_L(1/2)} \right) = \log \frac{|d_K|^m}{|d_L|^n} + \frac{\pi}{2} \left( nr_1(L) - mr_1(K) \right),\]

where \( r_1(L) \) and \( r_1(K) \) denote the number of real embeddings of \( L \) and \( K \) respectively.

If \( nr_1(L) - mr_1(K) \neq 0 \), then the right hand side of the above equation is a transcendental number by Lemma 11. On the other hand, if \( nr_1(L) - mr_1(K) = 0 \), then the right hand side of the above equation is a transcendental number by Lemma 12. So both real numbers

\[ \frac{\zeta_K'(1/2)}{\zeta_K(1/2)} \quad \text{and} \quad \frac{\zeta_L'(1/2)}{\zeta_L(1/2)} \]

can not be algebraic.

### 3.7. Proof of Corollary 8 and 9

If there exist two distinct numbers from the set given in Corollary 8, then it would be a contradiction to Theorem 7. So the first statement is a direct consequence of Theorem 7. Now we prove the second part of Corollary 8. Combining (2) and (3), we obtain

\[ \frac{\zeta_K'(1/2)}{\zeta_K(1/2)} - n \frac{\log 8\pi + \gamma}{2} = r_1 \frac{\pi}{4} - (1/2) \log |d_K|. \]
If $K = \mathbb{Q}$, then the right hand side of above equation is $\frac{\pi}{4}$. By Theorem 14, $|d_K| > 1$ for all $K$ different from $\mathbb{Q}$. So

$$
\frac{\zeta_K'(1/2)}{\zeta_K(1/2)} - \frac{n}{2}(\log 8\pi + \gamma)
$$

is a transcendental number by Lemma 11 and 12.

For the proof of Corollary 9, note that the hypothesis given on discriminants ensures that the first condition of Theorem 7 is satisfied. Consequently, it follows from Theorem 7.

4. Concluding remarks

We believe that for any number field $K$, one should have

$$
\zeta_K(1/2) \neq 0 \implies \zeta_K'(1/2) \neq 0.
$$

Such results hold for Elliptic curves over $\mathbb{Q}$ as well as Modular forms [6]. The nature of the functional equation in these set ups are amenable to deduce the above supposition. The classical bounds between degree and discriminant in our context do not seem to be strong enough to prove the above supposition, at least through our approach.

Furthermore, if there does exist a number field $K$ such that $\zeta_K(1/2) \neq 0$ while $\zeta_K'(1/2) = 0$, we shall have $\log \pi + \gamma$ being equal to a linear form in logarithm of algebraic numbers, an unlikely possibility from a transcendental perspective since neither $\log \pi$ nor $\gamma$ is expected to be a Baker period, that is, a $\mathbb{Q}$- linear combination of logarithms of algebraic numbers (see [16] for details on Baker periods).

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Dear N.Kandhil,

About your arXiv text on zeta(1/2).

The behaviour at $s = 1/2$ of zeta and $L$ functions is an old question. I worked briefly on that topic about 50 years ago. Most people seemed to believe that $L(1/2)$ is nonzero, and they were surprised when I suggested an explicit counter-example, related to Martinet’s work on extensions of $\mathbb{Q}$ with group the quaternion group; that example is the one of Armitage’s note. This gave rise to a very interesting theory, mainly developed by Fröhlich, in which the central point is the sign of the functional equation. Incidentally, it is strange that you do not mention that sign. In case it is +1, that shows that $L(1/2) = 0$ implies $L'(1/2) = 0$.

About $L(1/2)$ being zero or not, the situation seems to be the following:

Let $\rho$ be an irreducible complex linear representation of a Galois extension of $\mathbb{Q}$ and let $L(s, \rho)$ be the corresponding $L$ function. Then $L(1/2, \rho)$ is expected to be $\neq 0$, except in the following case:

(*) $\rho$ fixes a non degenerate antisymmetric bilinear form, and the constant of the functional equation of $L(s, \rho)$ is -1.

[ The condition that $\rho$ fixes a non degenerate bilinear form is equivalent to the character of rho being real valued. When that bilinear form is symmetric, then the constant of the functional equation is +1 by a theorem of Fröhlich-Queyrut.]

In case (*), one expects that $1/2$ is a simple zero, hence that $L'(1/2, \rho)$ is nonzero.

These expectations are not based on much evidence: merely that $L$ functions should not have “unexplained” common zeroes (or even $\mathbb{Q}$-related ones). Maybe you could mention them: I am not sure that they are in the literature.

Best wishes,
J-P. Serre

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