The generalized Crewther relation: the peculiar aspects
of the analytical perturbative QCD calculations.

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Abstract
We summarize the current status of our understanding of the structure of the perturbative
QCD expressions for the QCD generalizations of the Crewther relation.

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1 Introduction

From time to time the detailed study of the results of the analytical multiloop calculations allow one to reveal the existence of the internal symmetries and of the definite properties of the gauge theories under investigation. In fact these typical features of the concrete models can be hidden in the explicit expressions of the coefficients of the perturbative series for the renormalization group quantities. The classical example of the immediate influence of the outcomes of the analytical calculations to the further development of the understanding of the structure of the perturbative series as the whole is provided by the evaluation of the 3-loop contribution to the $\beta$-function of the $N = 4$ supersymmetric Yang-Mills theory [1]. Indeed, in Ref.[1] it was found by the direct diagram-by-diagram calculation that the corresponding $\beta$-function is zero at the 3-loop order. In its turn, this foundation pushed ahead theoretical works, which resulted in the formulation of the proof of the validity of this interesting property of the $N = 4$ supersymmetric gauge model in all orders of the perturbation theory (see e.g. Ref.[2]). Therefore, in spite of the fact that the calculation of the related $\beta$-function coefficients presumes the introduction of the regularization (definitely speaking, supersymmetric one) and thus the renormalization procedure, the property of the conformal symmetry is preserved in this model.

However, it is known, that in such realistic gauge theories, as massless QCD or QED, the procedure of the renormalization is disrupting the initial conformal invariance and gives rise to an anomaly in the trace of the energy-momentum tensor [3, 4]. Its explicit expression [4] shows that the factor $\beta(a)/a$ is the measure of the breaking of the conformal invariance within the framework of the perturbation theory expansion in the coupling constant $a$, where in QCD and QED we will normalize $a$ as $a_s = \alpha_s/\pi$ and $a = \alpha/\pi$ respectively. This means, that in QCD and QED the conformal symmetry can be effectively restored only in the vicinity of the hypothetical perturbative fixed points, which satisfy the condition $\beta(a^*) = 0$. In view of this it is rather interesting to understand whether there are any manifestations of the properties of the initial conformal symmetry and its violation by the procedure of the renormalization in the structures of any perturbative series.

By what discussed beyond we will try to convince the readers that the answers to these questions are positive. We will show that the made in Ref.[5] careful analysis of the analytical structure of the perturbative QCD predictions to the certain characteristics of the $e^+e^- \rightarrow$ hadrons and deep-inelastic scattering processes allowed to reveal the existence of the definite already proved [4] and still non-proved relations between the coefficients of the perturbative series for the Adler $D$-function of the non-singlet axial currents (or vector currents) and the polarized Bjorken sum rule (or the Gross-Llewellyn Smith sum rule). The first quantity was evaluated analytically at the next-to-leading order (NLO) of perturbative QCD in Ref.[7] (for the identical result of the independent numerical and analytical calculations see Ref.[8] and Ref.[9] respectively) and at the next-to-next-to-leading order (NNLO) in Ref.[10] (for the identical result of the semi-independent calculations see Ref.[11]). The NLO coefficient of the perturbative series for the deep-inelastic scattering sum rules we will be interested in is known from the results of the analytical calculation of Ref.[12] (later on independently confirmed in Ref.[13]), while the NNLO corrections to the polarized Bjorken sum rule and to the Gross-Llewellyn Smith sum rule were obtained in Ref.[14].

Starting from the beginning of the INR multiloop analytical calculating project, launched in 1978-1979 [13, 14], nobody was expecting that there are any relations between the characteristics of the annihilation and deep-inelastic processes. However, in 1990, when it was already understood by the second and the third authors of Ref.[10], that the published results of the analytical calculations of the 4-loop contributions to the QED $\beta$-function in the \( \overline{\text{MS}} \)-scheme [10]
and to the Adler $D$-function in QCD \cite{17} are wrong (for the discussion see e.g. Ref.\cite{18}), but the 4-loop QED results of Ref.\cite{19} and the NNLO QCD results of Refs.\cite{10,11,14} were not yet obtained, we were informed in the quite non-formal form \cite{20} about the existence of the fundamental Crewther relation \cite{3}. This relation is connecting in the conformal-invariant limit the anomalous 3-point function of the axial-vector-vector non-singlet currents with the product of the quark-parton expressions for the polarized Bjorken sum rule and the $e^+e^-$-annihilation Adler $D$-function. However, in order to answer the constructive question: “What is the status of the Crewther relation in QCD?” \cite{21} it is necessary to go beyond the framework of the conformal-invariant limit and to analyze the structure of the perturbative QCD corrections to both sides of the Crewther relation.

Here we will summarize our present understanding of the current status of the answers to this question using the results of the works of Refs.\cite{5,6,21} and of the old work of Ref.\cite{22}, which became known to us only recently.

## 2 What is the Crewther relation?

Before starting the presentation of the QCD foundations of Ref.\cite{5} let us briefly discuss the definite steps of the derivation of the Crewther relation \cite{3}, repeated also in Ref.\cite{22}. In order to complete the analysis of Refs.\cite{3,22}, performed in the $x$-space, we will follow the studies of Ref.\cite{6} and use the language of the momentum space. To our mind, this will allow to demonstrate more obviously some basic points, which were not previously clarified in Refs.\cite{3,22}.

Consider the 3-point function

$$T_{\mu\alpha\beta}^{abc}(p, q) = i \int <0 | T A^a_\mu(y) V^b_\alpha(x) V^c_\beta(0) | 0 > e^{ipx + iqy} dxdy = d^{abc} T_{\mu\alpha\beta}(p, q)$$

where $A^a_\mu(x) = \bar{\psi}_\mu \gamma_5 (\lambda^a / 2) \psi$, $V^a_\mu(x) = \bar{\psi}_\mu (\lambda^a / 2) \psi$ are the axial and vector non-singlet quark currents. The r.h.s. of Eq.(1) can be expanded in a basis of 3 independent tensor structures under the condition $(pq) = 0$ as

$$T_{\mu\alpha\beta}(p, q) = \xi_1(q^2, p^2) \epsilon_{\mu\alpha\beta\tau} P^\tau + \xi_2(p^2, q^2) (q_\alpha \epsilon_{\mu\beta\rho\tau} P^\rho q^\tau - q_\beta \epsilon_{\mu\alpha\rho\tau} P^\rho q^\tau) + \xi_3(p^2, q^2) (p_\alpha \epsilon_{\mu\beta\rho\tau} P^\rho q^\tau + p_\beta \epsilon_{\mu\alpha\rho\tau} P^\rho q^\tau).$$

Taking now the divergency of axial current one can get the following relation for the invariant amplitude $\xi_1(q^2, p^2)$:

$$q_\beta T_{\mu\alpha\beta}(p, q) = \epsilon_{\mu\alpha\beta\rho} q^\rho \xi_1(q^2, p^2)$$

while the property of the conservation of the vector currents implies that

$$\lim_{p^2 \to \infty} p^2 \xi_3(q^2, p^2) = -\xi_1(q^2, p^2)$$

(see Ref.\cite{23} for the discussions of the details of the derivation of Eqs.(2)-(4)).

In order to clarify the meaning of the second invariant amplitude, namely $\xi_2(q^2, p^2)$, let us first define the characteristics of the deep-inelastic processes, namely the polarized Bjorken sum rule

$$Bjp(Q^2) = \int_0^1 [g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2)] dx = \frac{1}{6} \frac{g_A}{g_V} |C_{Bjp}(a)|$$

and the Gross-Llewellyn Smith sum rule

$$GLS(Q^2) = \frac{1}{2} \int_0^1 F_3^{ep+\tau p}(x, Q^2) dx = 3C_{GLS}(a).$$
The coefficient function $C_{Bjp}(a_s)$ can be found from the operator-product expansion of two non-singlet vector currents \[12\]

$$i \int TV^a(x)V^b_\alpha(0)e^{ipx}dx |_{p^2 \to \infty} \approx C^{P,abc}_{\alpha\beta\rho} A^c_\rho(0) + \text{other structures}$$  \(7\)

where

$$C^{P,abc}_{\alpha\beta\rho} \sim i\delta^{abc} \epsilon_{\alpha\beta\rho\sigma} \frac{P^\sigma}{P^2} C_{Bjp}(a_s)$$  \(8\)

and $P^2 = -p^2$. In the case of the definition of the coefficient function of the Gross-Llewellyn Smith sum rule one should consider the operator-product expansion of the axial and vector non-singlet currents

$$i \int TA^a_\mu(x)V^b_\nu(0)e^{iqx}dx |_{q^2 \to \infty} \approx C^{V,ab}_{\mu\nu\alpha} V^\alpha_\alpha(0) + \text{other structures}$$  \(9\)

where

$$C^{V,ab}_{\mu\nu\alpha} \sim i\delta^{ab} \epsilon_{\mu\nu\alpha\beta} \frac{q^\beta}{Q^2} C_{GLS}(a_s)$$  \(10\)

and $Q^2 = -q^2$. The third important quantity, which will enter into our analysis, is the QCD coefficient function $C_D^{NS}(a_s)$ of the Adler $D$-function of the non-singlet axial currents

$$D^{NS}(a_s) = -12\pi^2 q^2 \frac{d}{dq^2} \Pi_{NS}(q^2) \sim C_D^{NS}(a_s)$$  \(11\)

where $\Pi_{NS}(q^2)$ is defined as

$$i \int <0|TA^a_\mu(x)A^b_\nu(0)|0> e^{iqx}dx = \delta^{ab}(g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi_{NS}(q^2).$$  \(12\)

At this point we will stop with definitions of the basic quantities and return to the consideration of the 3-point function of Eq.(1). Following Ref.[3] one can apply to this correlation function an operator-product expansion in the limit $|p^2| >> |q^2|$, $p^2 \to \infty$, namely expand first the $T$-product of two non-singlet vector currents via Eq.(7) and then take the vacuum expectation value of the $T$-product of two remaining non-singlet axial currents defined through Eq.(12). It was shown in Ref.[6] that these studies imply that

$$\xi_2(q^2,p^2)|_{|p^2| \to \infty} \sim \frac{1}{p^2} C_{Bjp}(a_s) \Pi_{NS}(a_s)$$  \(13\)

and thus

$$q^2 \frac{d}{dq^2} \xi_2(q^2,p^2)|_{|p^2| \to \infty} \sim \frac{1}{p^2} C_{Bjp}(a_s) C_D^{NS}(a_s).$$  \(14\)

Equations (13),(14) reflect the physical meaning of the invariant amplitude $\xi_2(q^2,p^2)$ and should be considered together with the relations for the invariant amplitudes $\xi_1(q^2,p^2)$ (see Eq.(3)) and $\xi_3(q^2,p^2)$ (see Eq.(4)).

On the other hand, it was shown in Ref.[24] that in a conformal invariant (c-i) limit the three-index tensor of Eq.(1) is proportional to the fermion triangle one-loop graph, constructed from the massless fermions, namely that

$$T^{abc}_{\mu\alpha\beta}(p,q)|_{c-i} = d^{abc}K(a_s) \Delta^{1-loop}_{\mu\alpha\beta}(p,q).$$  \(15\)
In other words, in a conformal invariant limit one has

\[ \xi_{c-i}(q^2, p^2) = K(a_s) \xi_{1-loop}^{c-i}(q^2, p^2), \]

\[ \xi_{c-i}(q^2, p^2) = K(a_s) \xi_{2-loop}^{c-i}(q^2, p^2), \]

\[ \xi_{c-i}(q^2, p^2) = K(a_s) \xi_{3-loop}^{c-i}(q^2, p^2). \]

(16)

Moreover, in view of the Adler-Bardeen theorem [25], which is insuring the invariant amplitude \( \xi_1(q^2, p^2) \), related to the divergency of axial current (see Eq.(3)), from the renormalizability, one has \( K(a_s) = 1 \). The 3-loop light-by-light-type scattering graphs, which were calculated in Ref.[26] and analyzed in Ref.[27], do not affect this conclusion. Indeed, in the case of the 3-point function of the non-singlet axial-vector-vector currents they are contributing to the higher order QED corrections, while the QCD corrections of the similar origin are appearing only in the 3-point function with the singlet axial current in one of the vertexes, which will be not discussed here.

In the case of the consideration of the 3-point function of the non-singlet axial-vector-vector currents the property \( K(a_s) = 1 \) is allowing to derive the fundamental Crewther relation

\[ C_{Bjp}(a_s(Q^2))C_{D}^{NS}(a_s(Q^2))|_{c-i} = 1 \]

(17)

which should be valid in the conformal invariant limit in all orders of perturbation theory. The similar relation is also true for the coefficient function \( C_{GLS}(a_s) \), defined by Eqs.(6),(9),(10) [22]. Indeed, considering first the operator-product expansion of the axial and vector non-singlet currents (see Eq.(9),(10)) in the 3-point function of Eq.(1), taking the \( T \)-product of the remaining vector currents and repeating the discussed above analysis, one can find that in the conformal invariant limit the following identity takes place:

\[ C_{GLS}(a_s(Q^2))C_{V}^{NS}(a_s(Q^2))|_{c-i} = 1 \]

(18)

where \( C_{D}^{V}(a_s) \) is the coefficient function of the Adler \( D \)-function of two vector currents.

### 3 The QCD generalization of the Crewther relation

It is well known, that the calculations of the perturbative theory corrections of the Green functions in the renormalizable quantum field models face the necessity of the introduction of the concrete regularization of the ultraviolet divergencies and of the subsequent application of the renormalization procedures, which as already mentioned, are breaking the conformal symmetry of the massless free theories. The immediate consequence of the application of this modern perturbative machinery is the appearance of the renormalization group \( \beta \)-functions, which are governing the energy behavior of the running coupling constants and are responsible for getting out from the scale-invariant limit. In QCD the \( \beta \)-function

\[ Q^2 \frac{da_s}{dQ^2} = \beta(a_s) = -\sum_{i \geq 0} \beta_i a_s^{i+2} \]

(19)

was analytically calculated at the 3-loop order in Ref.[27] and confirmed in Ref.[28] in the framework of the dimensional regularization [30] and the class of minimal subtractions schemes [31]. In our further discussions we will be interested in the expressions of the first two renormalization-scheme invariant coefficients \( \beta_0 \) and \( \beta_1 \), which are expressed through the Casimir operators \( C_A, T_T N_f \) and \( C_F \) as

\[ \beta_0 = \left( \frac{11}{3} C_A - \frac{4}{3} T_T N_f \right) \frac{1}{16} \]

\[ \beta_1 = \left( \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_T N_f - 4 C_F T_T N_f \right) \frac{1}{16}. \]

(20)
In general, it is possible to understand that

- the perturbative expression for the QCD $\beta$-function does not contain the terms proportional to $C^K_F a^K_s (K \geq 1)$ and
- the deviation from the conformal invariant limit $\beta(a_s) = 0$ is related to the fact that $T_f N_f \neq 0$ and $C_A \neq 0$.

Moreover, in perturbative series for physical quantities the latter Casimir operators can appear only starting from the NLO. Therefore, to study the theoretical consequences of the property of the conformal symmetry breaking in the massless gauge models it is necessary to consider the higher-order perturbative theory approximations for the physical quantities.

In order to get the non-conformal variants of the Crewther relations of Eqs.(16),(17) it is necessary to consider the NNLO approximations of the corresponding basic quantities. Due to the existence of the dimensional regularization [30], class of the minimal subtraction schemes [31] and the developments in the field of the creation of the multiloop calculating methods [15, 32], it became possible to obtain the concrete NNLO results for the Adler function of the electromagnetic quark currents [10, 11] by means of the classical symbolic manipulations program SCHOONSCHIP [33] and for the QCD coefficient functions of the polarized Bjorken sum rule and the Gross-Llewellyn Smith sum rule [14] with the help of its more computer-educated younger follower FORM [34]. Another problem, solved in the process of the NLO calculations of $C_{Bjp}(a_s)$ and $C_{GLS}(a_s)$ of Ref.[12] and of the NNLO ones of Ref.[14] is the proper definition of the axial non-singlet current.

Indeed, it is known, that within dimensional regularization the analog of the $\gamma_5$-matrix and therefore the axial Ward identities are not well defined. The most straightforward way, which is allowing to restore the axial Ward identities presumes the application of the additional finite renormalization of the axial currents [35], which in the non-singlet case has the following form

$$A^a_\mu \to Z_5^{NS} A^a_\mu = \tilde{A}^a_\mu \ .$$

In the $\overline{MS}$-scheme the expression for $Z_5^{NS}$ was calculated at the NLO level in Ref.[12] and NNLO level in Ref.[14]. It is possible to show, that this additional finite renormalization allows one to restore the validity of the Adler-Bardeen theorem in QCD for the anomalous triangle diagram of the axail-vector-vector non-singlet currents (for the related discussions see Refs.[36-38]). Moreover, using Eq.(21) one can find, that the NNLO approximation of the QCD coefficient function of the Adler $D$-function of the electromagnetic quark currents $C^{EM}_D(a_s)$ differs from the NNLO approximation of $C^{NS}_D(a_s)$ only by the 4-loop light-by-light type diagrams, which have the singlet nature:

$$C^{EM}_D(a_s) = C^{NS}_D(a_s) + C^{SI}_4(a_s) \ .$$

The last scheme-independent contribution has the following form [10, 11]

$$C^{SI}_4(a_s) = \frac{(\sum Q_f)^2}{n_c \sum Q_f^2} \left( \frac{11}{192} - \frac{1}{8} \zeta(3) \right) d^{abc} d^{abc} a_s^3 \ .$$

where $n_c = 3$ is the number of colours. In the $\overline{MS}$-scheme the NNLO analytical expression of $C^{NS}_D(a_s)$ is known from the results of calculations of Refs.[10,11]:

$$C^{NS}_D(a_s) = 1 + \frac{3}{4} C_F a_s + \left[ - \frac{3}{32} C_F^2 + \left( \frac{123}{32} - \frac{11}{4} \zeta(3) \right) C_F C_A + \left( - \frac{11}{8} + \zeta(3) \right) C_F T_f N_f \right] a_s^2 $$
\[
+ \left[ -\frac{69}{128} C_F^3 + \left( -\frac{127}{64} - \frac{143}{16} \zeta(3) + \frac{55}{4} \zeta(5) \right) C_F^2 C_A \right] \tag{24}
\]

\[
+ \left( \frac{90445}{3456} - \frac{2737}{144} \zeta(3) - \frac{55}{24} \zeta(5) \right) C_F C_A^2 + \left( -\frac{29}{64} + \frac{19}{4} \zeta(3) - 5 \zeta(5) \right) C_F^2 T_f N_f \right]
\]

\[
+ \left( -\frac{485}{27} + \frac{112}{9} \zeta(3) + \frac{5}{6} \zeta(5) \right) C_F C_A T_f N_f + \left( \frac{151}{54} - \frac{19}{9} \zeta(3) \right) C_F T_f^2 N_f^2 \right] a_s^3
\]

The similar $\overline{MS}$-scheme result for the coefficient function $C_{Bjp}(a_s)$ was obtained in Ref.[14]

and reads

\[
C_{Bjp}(a_s) = 1 - \frac{3}{4} C_F a_s + \left[ \frac{21}{32} C_F^2 - \frac{23}{16} C_F C_A + \frac{1}{2} C_F T_f N_f \right] a_s^2
\]

\[
+ \left[ -\frac{3}{128} C_F^3 + \left( \frac{1241}{576} - \frac{11}{12} \zeta(3) \right) C_F^2 C_A + \left( -\frac{5437}{864} + \frac{55}{24} \zeta(5) \right) C_F C_A^2 \right]
\]

\[
+ \left( -\frac{133}{576} - \frac{5}{12} \zeta(3) \right) C_F^2 T_f N_f + \left( \frac{3535}{864} + \frac{3}{4} \zeta(3) - \frac{5}{6} \zeta(5) \right) C_F C_A T_f N_f - \frac{115}{216} C_F T_f^2 N_f^2 \right] a_s^3
\]

while the NNLO coefficient function $C_{GLS}(a_s)$ recieves the additional NNLO singlet-type contribution [14]

\[
C_{GLS}(a_s) = C_{Bjp}(a_s) + C_{GLS}^{SI}(a_s)
\]

where

\[
C_{GLS}^{SI}(a_s) = \frac{N_f}{n_c} \left( -\frac{11}{192} + \frac{1}{8} \zeta(3) \right) d^{abc} d^{abc} a_s^3 . \tag{27}
\]

From the first superface glance to the pairs of the results of Eqs.(24),(25) (or to the related ones of Eqs.(22),(26)) one can think that there is nothing in common between the beautiful, but rather complicated expressions for the NNLO approximations of the characteristics of the annihilation processes (see Eqs.(22)-(24)) and the ones of the deep-inelastic processes (see Eqs.(25)-(27)). However, after constructing the NNLO QCD variants of the Crewther formulae of Eqs.(17),(18) and looking to these products more carefully the following relation was found [1]:
\[ C_{Bjp}(a_s(Q^2))C^{NS}_{D}(a_s(Q^2)) = 1 + \frac{\beta(2)(a_s)}{a_s} \left[ S_1 C_F a_s + \left( S_2 T_f N_f + S_3 C_A + S_4 C_F \right) C_F a_s^2 \right] + O(a_s^4) \]  
(28)

where \( \beta(2)(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 \) is the 2-loop expression for the QCD \( \beta \)-function with \( \beta_0 \) and \( \beta_1 \) defined in Eq.(20) and \( S_1, S_2, S_3 \) and \( S_4 \) are the analytical numbers, which contain the transcendental Riemann \( \zeta \)-functions:

\[
\begin{align*}
S_1 &= -\frac{21}{2} + 3\zeta(3), \\
S_2 &= \frac{163}{24} - \frac{19}{3} \zeta(3), \\
S_3 &= -\frac{629}{32} + \frac{221}{12} \zeta(3), \\
S_4 &= \frac{397}{96} + \frac{17}{2} \zeta(3) - 15\zeta(5).
\end{align*}
\]  
(29)

The transformation to the non-abelian case of QED can be made by taking \( C_F = 1, C_A = 0 \) and \( T_f N_f = N \) where \( N \) is the number of massless leptons with identical charges.

The authors of Ref.[5] noticed in Eqs.(28),(29) the following “seven wonders” of the generalized Crewther relation, or to be more precise, of the NNLO non-conformal discrepancy \( \Delta_{n-c} = (C^{NS}_{D}(a_s(Q^2)))C_{Bjp}(a_s(Q^2)) - 1 \) from the conformal-invariant relations of Eqs.(17),(18):

1. The leading order terms cancels in \( \Delta_{n-c} \), which thus do not contain any \( C_F a_s \)-corrections.

2. The NLO corrections give no \( C_F^2 a_s^2 \) terms in \( \Delta_{n-c} \).

3. The NNLO corrections give no \( C_F a_s^3 \) terms in \( \Delta_{n-c} \). These three foundations have lead to the observation that in the so-called quenched limit the zero-fermion-loop abelian terms in the NLO and NNLO approximations of the polarized Bjorken sum rule and the Gross-Llewellyn Smith sum rule (explicitly calculated in Ref.[12] and Ref.[14] respectively) can be obtained by inverting the expression of the one-fermion-loop QED contribution to \( C^{NS}_{D}(a) \), which is related to the scheme-independent Baker-Johnson QED \( F_1 \)-function, calculated at the 3-loop level in Ref.[39] (this result was confirmed in Ref.[40]) and at the 4-loop level in Ref.[19]:

\[ C_{Bjp}(a)|_{\text{quenched QED}} = 1 - \frac{3}{4} a + \frac{21}{32} a^2 - \frac{31}{128} a^3 \]

where the factor in the numerator is defined by the 1-loop contribution to the \( F_1 \)-function.

4. The NNLO light-by-light terms of Eqs.(23),(27) are cancelling in \( \Delta_{n-c} \) (taking equal quark charges in Eq.(23) to obtain the contribution to \( C_{Bjp}(a_s) \)).

5. The NLO corrections give \( C_F C_A a_s^2 \) and \( C_F T_f N_f \) terms in \( \Delta_{n-c} \) that are the same ratio as the \( C_A \) and \( T_f N_f \) terms in \( \beta_0 \). This reveals the exact factorization of the \( \beta_0 a_s^2 \)-contribution in the NLO correction to \( \Delta_{n-c} \).

6. The NNLO corrections to \( \Delta_{n-c} \) can be expressed as the sum of the terms, proportional to \( \beta_0 a_s^3 \) and \( \beta_1 a_s^3 \) without introduction of the \( \beta_0^2 a_s^3 \)-terms, which can be expected from the simple power-counting arguments.

7. At the NNLO level the \( \beta_1 a_s^3 \) contribution to \( \Delta_{n-c} \) occurs with the same coefficient that multiplies \( \beta_0 a_s^2 \) term at the NLO.

Before proceeding our discussions let us make several comments, related to the first three “wonders” mentioned above. It can be shown [4] that these three properties are the consequence of the Adler-Bardeen theorem and of the discussed in Sec.2 initial Crewther relation [3], which is valid in the conformal invariant limit \( T_f N_f = 0, C_A = 0 \). In fact they were already
effectively discovered in Ref.[22], where the validity of the 3-loop analog of Eq.(30) was conjectured. The result of the NLO calculations of $C_{Bjp}(a_s)$ (see Ref.[12]), taken in the limit $C_F = 1$, $T_J N_f = 0$, $C_A = 0$, is confirming the prediction of Ref.[22] for the abelian zero-fermion-loop NLO contribution to this sum rule, while the results of the NNLO calculations of the $F_1$-function [13] and $C_{Bjp}(a_s)$ [14] are extending the obtained in Ref.[22] 3-loop relation to the 4-loop level. Moreover, it is also possible to understand [3], that the properties of the cancellation of the $C_F a_s^K$-terms ($1 \leq K \leq 3$) in the discovered in Ref.[5] NNLO analog of the Crewther relation (see Eqs.(28),(29)) can be generalized to the arbitrary order $K$ of the perturbation theory.

It is worth reminding here the physical meaning of the Baker-Johnson QED $F_1$-function. In the process of the study of the finite QED program it was proved that if the condition $F_1(a_*) = 0$ takes place, than the property of the finiteness of QED can be realized in the vicinity of the point $a_*$ [11, 24]. Thus, the explicite calculations of the coefficients of the perturbative series for the $F_1$-function can be considered as the important “experimental” ingredients of the analysis of the question of the existence (or, to be more precise, non-existence) of the perturbative ultraviolet fixed point of the massless QED. It is interesting to mention, that the 3-loop analytical calculations of Ref.[39] revealed the cancellation of the $\zeta(3)$-functions, which do appear at the intermediate stages of these calculations. As was mentioned later on in Ref.[40] without any reference, the fact that the 3-loop coefficient of the $F_1$-function turned out to be rational can be related to the property of the conformal symmetry of the massless QED in the quenched approximation.

It was also pointed out in the process of the discussions of the 4-loop results of the 1987-year calculation of the $F_1$-function [16], that it is rather doubtful, that the result of Ref.[16] (which was found later on to be in error) contained the $\zeta(3)$-term [13]. Now we know, that in spite of the appearance of the $\zeta(3)$, $\zeta(4)$ and $\zeta(5)$-terms at the intermediate stages of calculations of the 4-loop corrections to the $F_1$-function, they are indeed cancelling out in the ultimate correct result of Ref.[19]. Moreover, due to the appearance of the personal understanding of the place of the conformal symmetry in the derivation of the Crewther relation [3, 22, 6], we can agree with the comment of Ref.[40], that the perturbative theory expression for the $F_1$-function is constrained by the conformal invariant limit of the massless QED. We can also generalize the proposed in Ref.[40] hypothesis to arbitrary orders of perturbation theory and make the conjecture, that it is rather natural to expect that all coefficients of the $F_1$-function can be rational numbers and will not contain Riemann $\zeta$-functions. The arguments in favour of the yet non-proved in detail relations between the appearance of the Riemann $\zeta$-functions in other results of multiloop QED calculations, say the 2-loop and 3-loop analytical results for the anomalous magnetic moment of electron $a_e$ (see Ref.[44] and Ref.[45] correspondingly) and the theory of knots are given in the works of Ref.[46].

Let us now return to the discussions of the other four “wonders” of Eq.(28), which were discovered in Ref.[5]. In fact, since it is possible to understand (see Ref.[22] and the discussions in Sec.2) that

$$C_{Bjp}(a_s(Q^2))C_{NS}^N(a_s(Q^2)) = C_{GLS}(a_s(Q^2)) C_{V}^N(a_s(Q^2))$$  (31)

there is no place for the singlet-type contributions to the QCD generalization of the Crewther relation for the 3-point function of the axial-vector-vector non-singlet currents. The mentioned in the item 4 cancellation of the light-by-light-type order $\alpha_s^3$-diagrams, which produce the colour structure $(d^{abc})^2$, is nothing more than the consequence of Eq.(31). The same equation is insuring this product from the contribution of the 3-loop triangle-type singlet diagrams, calculated in Ref.[47] and re-calculated with the same result in the works of Ref.[48] (for the details see Ref.[49]).
Three last “wonders” are leading to the striking indications on the factorization of the term $(\beta^2(a_s)/a_s)$ in the NNLO generalization of the Crewther relation of Eq.(28). It should be stressed, that we prefer to think about the theoretical origine of the factorization of this special term, but not simply of the 2-loop approximation of the QCD $\beta$-function $\beta^2(a_s)$, since in case of the cancellation of of $a_s$ in the denominator of the conformal symmetry breaking term $(\beta^2(a_s)/a_s)$ by the extra power of $a_s$ in the square bracket of the r.h.s. of Eq.(28), we will get in it free Casimir operator $C_F$, which is not multiplied by the coupling constant $a_s$.

In its turn, this will contradict the general property of the gauge structure of the concrete expressions for the perturbative contributions, which presumes the appearence of the Casimir operators only in the combinations, proportional to $C_F a_s, C_F^2 a_s^2, C_F C_A a_s^2, C_F T_f N_f a_s^2$, etc.

Another yet non-proved property of the QCD generalization of the Crewther relation is following from the “wonder” N6. Indeed, the absence of the NNLO contribution to $\Delta_n$ in $\Delta_n$ can not be sure, that the factor $(\beta(a_s)/a_s)$ can be factored out of $\Delta_n$ beyond the NNLO, it seems to the authors of Ref.[5] most likely, that at any given order $a_s^K (K \geq 1)$, one will encounter in $\Delta_n$ only the coefficients $a_s$, multiplied by linear combinations of colour factors.

In fact we think, that the hypothesis $\Delta_n \sim (\beta(a_s)/a_s)$ merits close attention. In Ref.[6] the attempt to study the theoretical consequencies of this hypothesis in more detail was made. Using the assumption, that the factorized factor, which is appearing in the r.h.s. of Eq.(28) has the origin, similar to the one of the measure of the conformal invariance breaking within perturbation theory framework $(\beta(a_s)/a_s)$ in the explicite expression for the anomaly of the energy momentum tensor $\xi_1, \xi_2, \xi_3$ (32), it was proposed in Ref.[6] to rewrite Eqs.(16) for the QCD expressions of the tensor structures of the 3-point function of Eq.(2) as

$$
\begin{align*}
\xi_1(q^2,p^2) & = K(a_s)\xi_1^{1-loop}(q^2,p^2), \\
\xi_2(q^2,p^2) & = \left(K(a_s) + \frac{\beta(a_s)}{a_s}v_2(q^2,p^2,a_s)\right)\xi_2^{1-loop}(q^2,p^2), \\
\xi_3(q^2,p^2) & = \left(K(a_s) + \frac{\beta(a_s)}{a_s}v_3(q^2,p^2,a_s)\right)\xi_3^{1-loop}(q^2,p^2).
\end{align*}
$$

(32)

where $v_2$ and $v_3$ are dimensionless functions, which can be constrained from the identity

$$
q^2 \frac{d}{dq^2}\xi_2(q^2,p^2) = -p^2 \frac{d}{dp^2}\xi_3(q^2,p^2) - \xi_2(q^2,p^2).
$$

(33)

This equation can be obtained from the derived in Ref.[23] Ward identity

$$
-\xi_1(q^2,p^2) = q^2\xi_2(q^2,p^2) + p^2\xi_3(q^2,p^2)
$$

(34)

after noting that according to the Adler-Bardeen theorem the function $\xi_1(q^2,p^2)$ is simply the unrenormalizable number. Keeping in mind Eqs.(32), which were written down after using the still non-proved assumptions that the factor $(\beta(a_s)/a_s)$ is (a) indeed factorized in the QCD generalization of the Crewther relation and (b) it can really manifest itself in the perturbative expressions of $\xi_2(q^2,p^2)$ and $\xi_3(q^2,p^2)$ in the form, suggested in Eqs.(32) it is possible to show, that in QCD the Crewther product of Eq.(17) takes the following form $C^N_{Bjp}(a_s(Q^2))C^NS_{D}(a_s(Q^2)) = 1 + \frac{\beta(a_s)}{a_s}r(a_s)$

(35)

where $r(a_s)$ is a polynomial in powers of $a_s$, which is not fixed in the approach of Ref.[6]. Therefore, the non-proved assumptions (a) and (b) are closely related.
Of course, a lot of work should be still done in order to find the really proved theoretical support in favour of these assumptions. Indeed, it is still necessary to understand on the diagrammatic language the origin of the appearance of the term, proportional to $(\beta(2)(a_s)) / a_s$ in the discovered in Ref.[5] QCD generalization of the Crewther relation (see Eq.(28)). In view of the presented in Ref.[6] considerations we think, that the analysis of this problem should be started from the explicite calculations of the NLO perturbative QCD corrections to the invariant amplitude $\xi_2(q^2, p^2)$ of the 3-point function of Eq.(1). The second non-solved question is related to the necessity of the understanding, whether the factor $(\beta(a_s)) / a_s$, presumably related to the property of the conformal symmetry breaking in massless QCD, is indeed factorized in the generalized Crewther relation in all orders of perturbation theory. We believe, that the attraction of the formalizm of the non-conformal Ward identities, developed in Ref.[50], can be rather useful for the study of this interesting problem.

4 The generalzied Crewther relation and the “commensurate scale relations”

It should be stressed that the QCD generalization of the Crewther relation of Eq.(28) was discovered in Ref.[5] using the NNLO $\overline{\text{MS}}$-scheme results for $C_D^{\text{NS}}(a_s)$ [10, 11] and the similar ones for $C_{Bjp}(a_s)$ [12]. However, it is known that on the contrary to the QED on-shell scheme, which is distinguished by the kinematics of the experimental measurements of the magnetic moments of electron and muon, the $\overline{\text{MS}}$-scheme is, regorously speaking, not physical one and is chosen as the reference scheme in the QCD phenomenology only by the convention between theoreticians and experimentalists. In view of this the number of approaches of dealing with the existing problem of fixing the scheme-dependence uncertainties in QCD was proposed. Among these methods are the principle of minimal sensitivity [51], the effective charges approach [52, 53] (which is known to be a posteriori equivalent to the scheme-invariant perturbation theory, developed in Refs.[54,55]) and the BLM approach [54], generalized to the NNLO level in Ref.[57] (see also Ref.[58]).

Following the work of Ref.[21] we will show that it is possible to apply the methods of Refs.[51-56] in order to rewrite the generalized Crewther relation of Eq.(28) in the different form and to derive the variants of the obtained in Ref.[59] so-called “commensurate scale relations”. Let us first define the effective charges of the coefficient functions $C_D^{\text{NS}}(a_s)$ and $C_{Bjp}(a_s)$ as

$$
\begin{align*}
C_D^{\text{NS}}(a_s) &= 1 + \hat{a}_D(Q^2/\Lambda_D^2), \\
C_{Bjp}(a_s) &= 1 - \hat{a}_{Bjp}(Q^2/\Lambda_{Bjp}^2),
\end{align*}
$$

(36)

where $\hat{a}_D = \frac{3C_F}{4} a_s^D$ and $\hat{a}_{Bjp} = \frac{3C_F}{4} a_s^{Bjp}$ can be related to the $\overline{\text{MS}}$-scheme results using the following equation

$$
a_{Bjp}^{D/(D)}(Q^2/\Lambda_{Bjp(D)}^2) = a_s(Q^2/\Lambda_{\overline{\text{MS}}}^2) + \left( A_{1(2)} + B_{1(2)} \beta_0 \right) a_s^2(Q^2/\Lambda_{\overline{\text{MS}}}^2)$$

$$
+ \left( C_{1(2)} + D_{1(2)} \beta_0 + E_{1(2)} \beta_1 \right) a_s^3(Q^2/\Lambda_{\overline{\text{MS}}}^2).
$$

(37)

Here $\Lambda_{Bjp(D)}^2 = \Lambda_{\overline{\text{MS}}}^2 e^{\exp\left(\frac{A_{1(2)} + B_{1(2)} \beta_0}{\beta_0}\right)}$ are the effective scales of $C_{Bjp}(a_s)$ and $C_D^{\text{NS}}(a_s)$-coefficient functions, $\beta_0$ and $\beta_1$ are the scheme-invariant coefficients of the QCD $\beta$-function (see Eq.(20)) and the exact expressions of the terms $A_i, B_i, C_i, D_i, E_i$ ($i = 1, 2$) can be extracted from the NNLO results of Eqs.(25),(24). Two effective charges $a_s^{Bjp}$ and $a_s^D$ can be related as
The scale of the single-scale generalization of the BLM method, which leads to the following redefinition of the QED. At the NNLO level this aim can be achieved with the help of the developed in Ref.[57] single-scale generalization of the BLM method, which leads to the following redefinition of the scale of the $a_s^D$-effective coupling constant from $Q^2$ to $\overline{Q}^2$ [21]:

$$
\ln \left( \frac{Q^2}{\overline{Q}^2} \right) = -B_{12} + [\beta_0(B_{12}^2 - E_{12}) + 2A_{12}B_{12} - D_{12}]a_s^D (Q^2/\Lambda_D^2).$$

It should be stressed, that this procedure is effectively eliminating the dependence of the r.h.s. of Eq.(38) from the $\beta_0$-coefficient. Moreover, since the coefficient before the $\beta_1$-term in the NNLO contribution to Eq.(38) can be chosen to be equal to the coefficient $B_{12}$ of the NLO correction, the absorption of the proportional to $N_f$ (and thus to $\beta_0$) NLO term into the scale $\overline{Q}$ automatically leads to the nullification of the proportional to $\beta_1$ NNLO contribution in this relations between effective charges.

Taking now into account the concrete expressions for the effective charges $a_s^D$ and $a_s^{Bjp}$ (see Eqs.(24),(25)) one can find [21] $\ln \left( \frac{Q^2}{\overline{Q}^2} \right) = \frac{5}{2} - 4\zeta(3) + a_s^D (\overline{Q}^2) \left[ \left( \frac{11}{12} + \frac{56}{3} \zeta(3) - 16 \zeta^2(3) \right) \beta_0 + \frac{13}{15} C_A - \frac{2}{3} C_A \zeta(3) - \frac{145}{12} C_F - \frac{46}{3} \zeta(3) C_F + 20 \zeta(5) C_F \right]$ and

$$\hat{a}_{Bjp}(Q) = \hat{a}_D(\overline{Q}^2) - \hat{a}_D^2(\overline{Q}^2) + \hat{a}_D^3(\overline{Q}^2) + \ldots.$$  (39)

Equation (39) is representing the example of the single-scale variant of the “commensurate scale relation” of Ref.[59]. It is the consequence of the following variant of the generalized Crewther relation of Eq.(28):

$$C_{Bjp}(a_s^{Bjp}(Q^2))C_D^{NS}(a_s^D(\overline{Q}^2)) = 1$$  (40)

where the non-conformal term $\Delta_{n-c}$ in Eq.(28) is absorbed into the scale of the coefficient function $C_D^{NS}$. It should be stressed, that the property of the factorization of the term $(\beta_2(a_s)/a_s)$ in the expression of Eq.(28) for $\Delta_{n-c}$, discovered in Ref.[5], turns out to be the necessary and the sufficient condition, which allowed the authors of Ref.[21] to rewrite the generalized Crewther relation of Eq.(28) in the form of the geometric progression of Eq.(39). Note also, that the less convenient for practical applications multi-scale variant of the “commensurate scale relation” of Eq.(39), which was previously derived in Ref.[59], is also the consequence of the definite variant of the generalized Crewther relation of Eq.(28).

5 The generalized Crewther relation and the experiment

After reading the previous Sections one can be interested in getting the understanding whether it is possible to build the bridge between the pure theoretical studies presented above and the existing experimental data for the characteristics of the $e^+e^-$-annihilation and deep-inelastic
processes, which enter in the Crewther relation and its QCD generalizations. The first attempts to analyze this problem was made in Ref.[21] and in the closely related work of Ref.[58]. In this Section we will follow the original considerations of Ref.[21], leaving the comments to the outcomes of Ref.[58] for the possible future presentation.

Let us first mention, that while the deep-inelastic scattering sum rules are measured in the Euclidean region, the real experimental information for the basic characteristic of the $e^+e^-$-annihilation channel $R_{e^+e^-}(s) = \sigma_{tot}(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ is coming from the measurements in the Minkowskian region. If one is sufficiently far from the resonance production thresholds, it is possible to relate the perturbative expression for $R_{e^+e^-}(s)$ with the coefficient function $C^{EM}_D(Q^2)$ by the following relation

$$R_{e^+e^-}(s) = \frac{3\sum_f Q_f^2}{2\pi i} \int_{s-i\varepsilon}^{s+i\varepsilon} \frac{d\tau}{\tau} C^{EM}_D(a_s(\tau)) \ .$$

(41)

In general this procedure results in the appearance of the $\pi^2$-like terms in the coefficient function of the $R$-ratio starting from the NNLO-level (see e.g. Ref[60]). In fact they can be taken into account in the formulae of Eqs.(37),(38) using the following shifts

$$E_2 \to E_2 - \pi^2/3, \quad E_{12} \to E_{12} + \pi^2/3.$$

The higher-order $\pi^2$-contributions to $R_{e^+e^-}(s)$ were calculated explicitely in Refs.[61,62]. However, since at the present stage we are interested in the consequences of the NNLO generalizations of the Crewther relation, we will not sum-up these higher-order $\pi^2$-contributions (like it was proposed to do e.g. in Ref.[63]), but truncate the corresponding perturbative series in the time-like region at the NNLO level. In order to obtain the relation between the time-like scale $\sqrt{s^*}$ of the $R_{e^+e^-}$-ratio and the space-like scales of the deep-inelastic scattering sum rules it is not only necessary to take into account the NNLO $\pi^2$-contributions to the $R$-ratio, but to perform the replacements of the effective charges $a^D_s(Q^2) \to a^R_s(\sqrt{s^*})$ as well. Then the resulting relation reads [21]:

$$\ln\left(\frac{Q^2}{s^*}\right) = -\frac{7}{2} + 4\zeta(3) - a^R_s(s^*) \left\{ \frac{11}{12} + \frac{56}{3}\zeta(3) - 16\zeta^2(3) - \frac{\pi^2}{3}\right\} \beta_0$$

$$+ \frac{13}{18} C_A - \frac{2}{3} C_A\zeta(3) - \frac{145}{72} C_F - \frac{46}{3}\zeta(3)C_F + 20\zeta(5)C_F \right\} \ .$$

(42)

The variants of the generalized Crewther relation can be then written down in the following form [21]

$$\frac{1}{3\sum_f Q_f^2} C_{Bj}(a_s^{Bj}(Q^2)) R_{e^+e^-}(a_s^R(s^*)) = 1 + C^_{SI}(a_s^R(s^*)) \ ,$$

(43)

$$\frac{1}{3\sum_f Q_f^2} C_{GLS}(a_s^{GLS}(Q^2)) R_{e^+e^-}(a_s^R(s^*)) = 1 + C^_{SI}(a_s^R(s^*)) + C^ {SI}_{GLS}(a_s^{GLS}(Q^2)) \ ,$$

where $C^ {SI}$ abd $C^ {SI}_{GLS}$ are the singlet-type contributions to $C^{EM}_D(a_s)$ and $C^{GLS}_{D}(a_s)$ which are defined in Eq.(23) and Eq.(27) respectively. In fact, since the numerical values of these contributions are very small, it is reasonable to neglect them in the phenomenologically oriented discussions and thus assume, that $C^{EM}_D(a_s) \approx C^{NS}_D(a_s)$ and $C_{Bj}(a_s) \approx C^{GLS}_{D}(a_s)$.

It should be stressed, that the theoretical expressions of Eq.(43) are derived in the framework of the perturbation theory and do not involve the non-perturbative contributions to $C^{EM}_D(a_s)$ [64] (and thus $R$-ratio), and to the deep-inelastic scattering sum rules, theoretically calculated in Ref.[65] and numerically estimated using the QCD sum-rules formalizm in Ref.[66]. In fact it is known, that these contributions are very important for the analysis of the low-energy experimental data for the $R$-ratio [67] and the Gross-Llewellyn Smith
and the polarized Bjorken sum rules (see Refs. [68, 69] and Refs. [70, 71] respectively). Therefore, in the estimates, which have the aim to clarify the experimental status of the perturbative QCD generalizations of the Crewther relation, it is necessary to chose the scales in the regions of energies, where the non-perturbative effects can be safely neglected. For the $e^+e^-$-annihilation $R$-ratio the lowest energy region, which is satisfying this criterion, is $4 \text{ GeV} \leq \sqrt{s} \leq 8 \text{ GeV}$. It is typical to the case of taking $N_f = 4$ numbers of active flavours and is lying above the thresholds of the production of the $\sigma^-$-bound states $\sqrt{s} \approx 3 \text{ GeV}$ and beyond the thresholds of the production of $b\bar{b}$-bound states $\sqrt{s} \approx 10 \text{ GeV}$. The detailed QCD analysis of the $e^+e^-$-data for the $R$-ratio in this region was made previously in the number of the works on the subject (see e.g. Refs. [72, 73]). In Ref. [21] the result of the most recent similar fit of Ref. [74] was used, which provide the authors of Ref. [21] with the constraint 

\[
\frac{1}{3} \sum_f Q_f^2 R_{e^+e^-} (\sqrt{s} = 5.0 \text{ GeV}) \approx 1.08 \pm 0.03 \text{ and thus } a_s^R (\sqrt{s} = 5.0 \text{ GeV}) \approx 0.08 \pm 0.03.
\]

Equations (42), (43) then imply that the corresponding estimates for the coefficient functions of the sum rules lie on the segment line which connects the following three values [21]:

\[
\begin{align*}
C_{Bjp} (Q = 11.13 \text{ GeV}) & \approx C_{GLS} (Q = 11.13 \text{ GeV}) \approx 0.952, \\
C_{Bjp} (Q = 12.33 \text{ GeV}) & \approx C_{GLS} (Q = 12.33 \text{ GeV}) \approx 0.926, \\
C_{Bjp} (Q = 13.53 \text{ GeV}) & \approx C_{GLS} (Q = 13.53 \text{ GeV}) \approx 0.900.
\end{align*}
\]

The corresponding values of the effective coupling constants lie in the line segment, which connects the three related points

\[
\begin{align*}
a_s^{Bjp} (Q = 11.13 \text{ GeV}) & \approx a_s^{GLS} (Q = 11.13 \text{ GeV}) \approx 0.048, \\
a_s^{Bjp} (Q = 12.33 \text{ GeV}) & \approx a_s^{GLS} (Q = 12.33 \text{ GeV}) \approx 0.074, \\
a_s^{Bjp} (Q = 13.53 \text{ GeV}) & \approx a_s^{GLS} (Q = 13.53 \text{ GeV}) \approx 0.1.
\end{align*}
\]

The appearence of three corresponding values of Eqs. (44) and Eqs. (45) are related to the uncertainties in the definition of $a_s^R (\sqrt{s} = 5 \text{ GeV})$, which are translated by Eq. (42) into three related numbers. The predictions of Eq. (45) could be tested experimentally.

It should be stressed, that at present the measurements of the polarized Bjorken sum rule are allowing to obtain its precise experimental values in the regions of energies of over $3 \text{ GeV}^2$ (see Ref. [75]) and $10 \text{ GeV}^2$ (see Ref. [76]). The recent measurements of the Gross-Llewellyn Smith sum rule are also preformed at relatively small values of $Q^2$ (see Ref. [69]). However, for the estimates, aimed to the comparison with the results of Eqs. (45), it was proposed in Ref. [21] to use the results of the theoretical extrapolation [7] of the available experimental data of the CCFR collaboration [78] to the wide region of energies. The indirect determination of the values of the Gross-Llewellyn Smith sum rule for $3 \text{ GeV}^2 \leq Q^2 \leq 500 \text{ GeV}^2$ of Ref. [77] gives the value

\[
a_s^{GLS} (Q = 12.25 \text{ GeV}) \approx 0.093 \pm 0.042.
\]

This interval crosses the line of the estimates of Eq. (25). To the point of view of the authors of Ref. [21] this fact gives empirical support for the generalized Crewther relation, written down in the form of Eq. (40).

In order to clarify the meaning of these numbers it seems to us rather instructive to return to the commonly used language of the $\overline{MS}$-scheme. Let us first analyse the question of the extraction of the value of $\Lambda_{\overline{MS}}$ from the fits of the $e^+e^-$-data of Ref. [74]. The results of the NLO calculations of Refs. [10, 11] relate the effective charge of the $e^+e^-$-annihilation to the coupling constant $a_s$ in the $\overline{MS}$-scheme as $a_s^R = a_s [1 + 1.524 a_s - 11.52 a_s^2]$. Here $N_f = 4$ is taken and the small contribution of $C_4^{SI} (a_s)$ is neglected. Using this relation, we find that the used
by us experimentally motivated number for $a_s^{\text{MS}}(\sqrt{s} = 5 \text{ GeV})$ corresponds to the following value of the coupling constant $\alpha_s$ in the $\overline{\text{MS}}$-scheme: $\alpha_s(\sqrt{s} = 5 \text{ GeV}) = 0.238^{+0.062}_{-0.087}$. It is known that the coupling constant $\alpha_s$ can be expressed through the QCD scale parameter $\Lambda_{\overline{\text{MS}}}$ as

$$\frac{\alpha_s}{4\pi} = \frac{1}{\beta_0 \ln(\alpha_s^2/\Lambda_{\overline{\text{MS}}}^2)} - \frac{\beta_1 \ln\ln(Q^2/\Lambda_{\overline{\text{MS}}}^2)}{\beta_0^2 \ln^2(Q^2/\Lambda_{\overline{\text{MS}}}^2)} + \frac{\beta_1^3 \ln\ln(Q^2/\Lambda_{\overline{\text{MS}}}^2) - \beta_1^2 \ln\ln(Q^2/\Lambda_{\overline{\text{MS}}}^2) + \beta_2 \beta_0 - \beta_1^3}{\beta_0^3 \ln^3(Q^2/\Lambda_{\overline{\text{MS}}}^2)} .$$ (47)

Combining Eq.(47) with the obtained above value of $\alpha_s(\sqrt{s})$ (normalized to $N_f = 4$ numbers of active flavours) we find the following interval of the values of the parameter $\Lambda_{\overline{\text{MS}}}^{(4)} = 410^{+320}_{-330} \text{ MeV}$. Evolving this result through the threshold of the production of the $b\bar{b}$-bound $M = 2m_b \approx 9 \text{ GeV}$ using the approximate formula of Ref.[79] we get $\Lambda_{\overline{\text{MS}}}^{(5)} = 275^{+246}_{-230} \text{ MeV}$ and

$$\alpha_s(M_Z) = 0.123^{+0.014}_{-0.027} .$$ (48)

This value is compatible with the result $\alpha_s(M_Z) = 0.124 \pm 0.021$, which comes from the recent detailed fit of the available data in $e^+e^-$-annihilation from $\sqrt{s} = 20 \text{ GeV}$ to $65 \text{ GeV}$ (for the detailed discussions of this result see the recent review of Ref.[81]). Thus the current measurements of $R_{e^+e^-}$ suffer from sizable experimental uncertainties.

The similar situation also holds in the case of the existing experimentally motivated data for the Gross-Llewellyn Smith sum rule at the energies, higher than $Q^2 > 5 \text{ GeV}^2$ (see, for example, the results of the recent analysis of Ref.[69]). In order to demonstrate explicitly that the used outcomes of the extrapolation-extraction of the Gross-Llewellyn Smith sum rule value at $Q^2 = 150 \text{ GeV}^2$ (see Ref.[77]) has definite theoretical and experimental uncertainties, we transform the results of Eq.(46) into the $\overline{\text{MS}}$-scheme and obtain $\alpha_s(Q = 12.25 \text{ GeV}) \approx 0.220^{+0.078}_{-0.088}$. This estimate might be weekly senistive to the change of $N_f$ from $N_f = 4$ (which is typical to the fits of the deep-inelastic data) to $N_f = 5$, which is more appropriate to the scale $Q^2 = 150 \text{ GeV}^2$. Therefore, we will extract from the high-energy results of Ref.[77] the value of $\Lambda_{\overline{\text{MS}}}^{(5)}$ and will get

$$\alpha_s(M_Z) = 0.119^{+0.010}_{-0.018} .$$ (49)

One can see that this estimate has larger uncertainties, than the value $\alpha_s(M_Z) = 0.109 \pm 0.003(stat) \pm 0.005(syst) \pm 0.003(thor)$, recently extracted in Ref.[82] from the analysis of the CCFR data for $xF_3$ structure function using the information about the NNLO corrections to the coefficient functions [83] and the results of the recent calculations of the NNLO corrections to the anomalous dimensions of the non-singlet operators [84]. Therefore, the current experimentally-motivated estimate of the value of the Gross-Llewellyn Smith sum rule at $Q^2 = 150 \text{ GeV}^2$ (see Ref.[77]) also suffers from the definite theoretical and experimental uncertainties.

These discussions are giving the additional arguments in favour of the physical conclusions of Ref.[21]: in order to check the consequences of the perturbative generalizations of the Crewther relation at more high confidence level it can be rather helpful, first, to reduce the experimental error of the measurements of $R_{e^+e^-}$ at $\sqrt{s} \approx 5 \text{ GeV}$ and, secondly, to have more precise information on the value of the Gross-Llewellyn Smith sum rule or of the Bjorken polarized sum rule at $Q^2 \approx 150 \text{ GeV}^2$. The first problem can be attacked after starting the operation of the $c - \tau$-factory, while the possible future study of the deep-inelastic scattering with both polarized electron and proton beams can open the window for direct measurements of the polarized Bjorken sum rule at high momentum transfer [85].
Another interesting, to our point of view, proposal is to try to measure the value of the polarized Bjorken sum rule (or the Gross-Llewellyn Smith sum rules) at the scale \( Q^2 = 25 \text{ GeV}^2 \). It can be useful for the study of the status of the perturbative generalization of the Crewther relation, written down in the form of Eq.(28), derived in Ref.[5]. We hope to return to the more detailed analysis of this problem in our possible future work.

6 Is there any generalization of the Crewther relation for the 3-point function of singlet axial-non-singlet vector-vector currents?

As was already explained in Sec.2, the classical Crewther relation was originally derived in Ref.[3] for the 3-point function of the axial-vector-vector non-singlet currents. For the sake of completeness let us now discuss the case of the analogous 3-point function with the axial singlet current:

\[
T_{\mu\alpha\beta}^{ab}(p, q) = i \int <0|TA_\mu(y)V_\alpha^a(x)V_\beta^b(0)|0>e^{ipx+iqy}dxdy \quad \text{(50)}
\]

where \( A_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi \). In this case it is also possible to try to repeat the considerations of Sec.2 and to think about the possibility of the derivation of the Crewther-type relations. Indeed, keeping the singlet structure in the operator-product expansion of the two non-singlet vector currents, one can get

\[
i \int TV_\alpha^aV_\beta^b e^{ipx}dx|p^2\to\infty \approx C_{\alpha\beta\rho}^{SI,ab}A_\rho(o) + \text{other structures} \quad \text{(51)}
\]

In analogy with Eq.(10) one finds that

\[
C_{\mu\alpha\rho}^{SI,ab} \sim i\delta^{ab}\epsilon_{\mu\alpha\rho} \frac{Q^3}{Q^2}C_{EJ}^{SI}(a_s) \quad \text{(52)}
\]

The coefficient function \( C_{EJ}^{SI}(a_s) \) is entering in the definition of the so-called Ellis-Jaffe sum rule as \[50\]

\[
EJ(Q^2) = \int_0^1 g_1^{p(n)}(x, Q^2) = \left( \pm \frac{1}{12}|g_A| + \frac{1}{36}a_s \right)C_{EJ}^{NS}(a_s) + \frac{1}{9}\Delta\Sigma(\mu^2)\times AD \times C_{EJ}^{SI}(a_s) \quad \text{(53)}
\]

where \( C_{EJ}^{NS}(a_s) = C_{Bjp}(a_s), \text{ AD } = \exp \left( \int_{a_s(\mu^2)}^{a_s(Q^2)} \frac{g_\Sigma(x)}{\beta(x)}dx \right) \) and \( |g_A| = \Delta_u - \Delta d, a_s = \Delta u + \Delta d - 2\Delta s, \Delta\Sigma = \Delta u + \Delta d + \Delta s, \Delta u, \Delta d, \Delta s \) can be interpreted as the measure of the polarization of quarks in a nucleon, \( C_{EJ}^{NS}(a_s) = C_{Bjp}(a_s) \) and \( \gamma^{SI}(a_s) \) is the anomalous dimension of the singlet axial current, which can be calculated within the \( \overline{MS} \)-scheme provided the additional finite renormalization of the singlet axial current is made:

\[
A_\mu \to Z_5^{SI}A_\mu = \tilde{A}_\mu \quad \text{(54)}
\]

It was shown in Ref.[38] that the finite renormalization constant \( Z_5^{SI} \) is different from the defined in Eq.(21) “non-singlet axial charge” due to the additional contribution to \( Z_5^{SI} \) of the 3-loop light-by-light-type triangle diagrams. At the 2-loop order this difference is \( Z_5^{SI} = Z_5^{NS} + \frac{3}{32}C_F N_f a_s^2 \). The similar difference is also appearing in the coefficient function.
$C_D^{SI}(a_s)$ for the Adler $D$-function of two singlet currents, which can be constructed from the following correlation function

$$i \int <0|TA_\mu(x)A_\nu(0)|0>e^{iqx}dx = \Pi^{SI}_{\mu\nu}(q^2) \ .$$

(55)

Staring from the 3-loop order the coefficient function $C_D^{SI}(a_s)$ differs from the coefficient function $C_D^{NS}(a_s)$ by the light-by-light-type diagrams, first calculated in Ref.[47] in the limit of the heavy top-quark mass $m_t$:

$$C_D^{SI}(a_s) = C_D^{NS}(a_s) + \Delta C_D^{SI}(a_s) \ .$$

(56)

In the case of the lighter quarks $m_q << m_t$ the result for the 3-loop light-by-light-type contribution can be extracted from the first paper of Ref.[49].

Let us now return to the discussion of the singlet contribution to the Ellis-Jaffe sum rule. The NLO corrections to the coefficient function $C_{EJ}^{SI}(a_s)$ was calculated in Ref.[86]. It is different from the result of the calculation of Ref.[12] due to the appearance of the light-by-light-type singlet diagrams

$$C_{EJ}^{SI}(a_s) = C_{EJ}^{NS}(a_s) + \Delta C_{EJ}^{SI}(a_s) = C_{Bjp}(a_s) + \Delta C_{EJ}^{SI}(a_s)$$

(57)

where at the NLO level $\Delta C_{EJ}^{SI}(a_s) = (\zeta(3) + \frac{1}{24}C_F T_f N_f)a_s^2$. This result is in agreement with the calculation of Ref.[83]. The anomalous dimension $\gamma^{SI}(a_s)$ was calculated in the $\overline{MS}$-scheme at the 3-loop order in Ref.[38]. In Ref.[27] the order $a_s^2$-term was related to the calculated in Ref.[26] light-by-light-type diagrams, contributing to the 3-point function of Eq.(50).

Let us now consider the conformal-invariant limit of all discussed in this Section results. In this case we have the Crewther-type relation

$$C_{EJ}^{SI}(a_s(Q^2))C_D^{SI}(a_s(Q^2))|_{c-i} = 1 \ .$$

(58)

We do not still know, whether there exist its QCD generalization, analogous to the one of Eq.(28), discovered in Ref.[5]. From the general grounds we expect, that both sides of Eq.(58) should be multiplied by the anomalous dimension term $AD$, defined in Eq.(53). It is interesting to study the problem of the possibility of the existence of any relations between the extra light-by-light-type terms $\Delta C_{EJ}^{SI}(a_s)$, $\Delta C_{EJ}^{SI}(a_s)$ and the light-by-light-type QCD contributions to the r.h.s. of Eq.(58). The most surprising fact is that the explicite analytical expression for $\Delta C_{EJ}^{SI}(a_s)$ contains the $\zeta(3)$-term, while the 3-loop correction $\Delta C_{EJ}^{SI}(a_s)$ does not contain it (see the first work of Ref.[49]).

We think that the calculations of the NNLO corrections to the Ellis-Jaffe sum rule might be useful not only for the comparison of their results with the scheme-invariant estimates of Ref.[88], but for the analysis of the possibility of the existence of new “wonders” in the non-conformal deviations from the Crewther-type relation of Eq.(58).
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