Entangling qubit registers via many-body states of ultracold atoms

R. G. Melko,1, 2 C. M. Herdman,1, 3, 4 D. Iouchtchenko,4 P.-N. Roy,4 and A. Del Maestro5, 7

1Department of Physics and Astronomy, University of Waterloo, Ontario, N2L 3G1, Canada
2Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada
3Institute for Quantum Computing, University of Waterloo, Ontario, N2L 3G1, Canada
4Department of Chemistry, University of Waterloo, Ontario, N2L 3G1, Canada
5Department of Physics, University of Vermont, Burlington, VT 05405, USA

Inspired by the experimental measurement of the Rényi entanglement entropy in a lattice of ultracold atoms by Islam et al. [Nature 528, 77 (2015)], we propose a method to entangle two spatially-separated qubits using the quantum many-body state as a resource. Through local operations accessible in an experiment, entanglement is transferred to a qubit register at the ends of a one-dimensional chain. We compute the operational entanglement, which bounds the entanglement physically transferable from the many-body resource to the register, and discuss a protocol for its experimental measurement. Finally, we explore measures for the amount of entanglement available in the register after transfer, suitable for use in quantum information applications.

Islam et al. [1] have performed a measurement of the Rényi entanglement entropy in a one-dimensional optical lattice of 87Rb atoms by exploiting a many-body analogue of the Hong-Ou-Mandel [2] photon interference effect. After interfering two proximate copies of an L-site lattice using the atomic control of a quantum gas microscope [3], a measurement of the parity of the site resolved particle occupation number provides access to the state overlap of the two copies. If the initial copies are identical, this gives the purity of the state [4]. Hence, if a globally pure state is partitioned into spatial subregions, the many-body interference/parity measurement protocol localized to a subregion yields the Rényi entropy, a measure of entanglement between subregions [5]. This provides an experimental probe of a remarkable feature of quantum mechanics with no classical analogue: complete knowledge of the global state of a composite system may not be enough to completely specify the state of a subsystem.

The advantage of measuring the Rényi entropy as in Ref. [1] is that it encodes the entanglement between subsystems in a scalar quantity that can be accessed through the expectation values of local operators [4]. This is in contrast to other entanglement measures calculated directly from the full density matrix, which is generally inaccessible in experiments without using full state tomography [6]. In particular, there is currently no scalable scheme for its reconstruction for N interacting itinerant particles. This fact makes the two-copy Rényi entropy, \( S_2(A) = -\log(\text{Tr} \rho_A^2) \), particularly well-suited for exploration in a quantum many-body system bipartitioned into a spatial region \( A \) and its complement \( \bar{A} \).

\( S_2 \) has proved fruitful for the general characterization of many-body phases and quantum phase transitions, e.g. through the exploration of its scaling with subsystem size [7]. Given that entanglement is a physical resource that can be used for quantum information processing [8, 9], it is natural to ask whether this many-body entanglement can be harnessed for these tasks [10–13].

To quantify this usable entanglement, one must take into account physical restrictions that limit the amount of entanglement that may be transferred to an external quantum register. For itinerant particles, a super selection rule (SSR) due to particle number conservation provides one key limitation [14]. Further restrictions are imposed if one wants to entangle spatially separated qubits with only local operations on the many-body system [15].

In this paper we propose the general scheme shown in Fig. 1 and present an experimental protocol, using the basic capabilities of Islam et al. [1], to transfer some of the entanglement in a many-body state of ultracold atoms to two spatially-separated qubits composing a quantum register. We emphasize the importance of the operational entanglement as a bound on the transferable entanglement, and discuss its measurement in the many-body state. The demonstration of this transfer would be proof of principle confirmation that a quantum register can be entangled in current experimental apparatuses for ultracold atoms.

The 87Rb atoms of the Islam experiment are confined to move in a deep one-dimensional optical lattice. In their weakly interacting regime, the low energy dynamics of the atoms are accurately governed by the lattice Bose-
Hubbard Hamiltonian with $N$ particles on $L$ sites:

$$H = -J \sum_{i=1}^{L-1} \left( b_i^\dagger b_{i+1} + \text{h.c.} \right) + \frac{U}{2} \sum_{i=1}^{L} n_i(n_i - 1),$$

where $b_i^\dagger (b_i)$ creates (annihilates) a boson, and $n_i = b_i^\dagger b_i$ counts the number of atoms on site $i$. $J$ sets the rate of tunneling between sites while $U$ parametrizes the strength of the on-site repulsion between atoms. In an experiment, the interaction strength between $^{87}$Rb atoms is fixed by their s-wave scattering length, while $J$ can be tuned by manipulating the height of the optical lattice. In the thermodynamic limit at unit filling ($N = L$), Eq. (1) exhibits two distinct phases: a Mott insulator for $U/J \gg 1$ and a superfluid for $U/J \ll 1$, both of which are observed experimentally. A quantum phase transition separates these two phases at $(U/J)_c \approx 3.3$.

The spatially delocalized nature of particles in the superfluid phase suggests that it should be significantly more entangled under a spatial bipartition than a Mott insulator with localized particles. This is manifest as an increase in $S_2$ accompanying the onset of delocalization at $U/J \sim O(1)$ observed in the experiment for $N = 4$ atoms [1]. The same experimental capabilities that allow for the measurement of the entanglement in an optical lattice can also be used to transfer entanglement to spatially separated qubits, that can be employed as a quantum register for information processing tasks via local operations and classical communication (LOCC).

In order to explore which parameter regime maximizes operational entanglement, we calculate $S_2^{\text{op}}$ in the Bose-Hubbard model. Experiments on $^{87}$Rb in the near future should be possible with $4 < N \lesssim 10$, and we study the ground states of systems with sizes of this order via exact numerical diagonalization of Eq. (1). In Fig. 2 we compare the two-copy Rényi entropy for a symmetric spatial bipartition to the operational entanglement for a range of $U/J$ and $N$ relevant for experiment. Unlike the entropy under a spatial bipartition, which is maximum deep in the superfluid phase (or the particle entanglement, which is maximum deep in the Mott phase [21]), $S_2^{\text{op}}$ displays a peak at an intermediate value of the interaction. While for these system sizes, the peak is not positioned directly at the thermodynamic-limit critical point $(U/J)_c \approx 3.3$, it appears to approach this value as the system size is increased. This suggests that the appropriate experimental parameters for maximizing the transfer of many-body entanglement to a system of quantum registers will be those that tune the system to near

$$S_2^{\text{op}}(A) = \sum_n P_n S_2(A_n),$$

where $S_2(A_n)$ is the Rényi entropy evaluated for the reduced density matrix $\rho_{A_n} = P_n \rho_A P_n / P_n$ projected by $P_n$ onto states of fixed local particle number $n$ in subsystem $A$. The summation is over all possible local particle number states in the subregion with $n = 0, \ldots, N$, each having probability $P_n = \langle \Psi | P_n | \Psi \rangle$. This projection is a local operation that can only decrease entanglement [14] so $S_2^{\text{op}} \leq S_2$.

Thus it is $S_2^{\text{op}}$, not $S_2$ which bounds the amount of entanglement that can be generated in the register using...
The optimal procedure to do so. This allows the many-body state to act as an entanglement resource for quantum information protocols. We concentrate on the minimal \( L = N = 6 \) Bose-Hubbard system where entanglement may be transferred to two spatially separated qubits. Each qubit is comprised of one atom occupying one of two neighboring lattice sites adjacent to the Bose-Hubbard chain; the two locations of the atom provide the logical states. Thus, the physical system we describe consists of \( 10 \) total lattice sites, which must be doubled to preserve particle number within the resource and qubits (and thus not remain in the logical subspace of the qubits), subsystem resolved particle occupation number measurements must be used to post-select states that have exactly one particle in each of \( A \) and \( B \).

Transfer of many-body entanglement to the register and its subsequent measurement can be accomplished via the three step procedure depicted in Fig. 3. The optical lattice within the array is manipulated such that large barriers (as indicated by solid lines) isolate the many-body resource. Each qubit must be constructed with exactly one particle between its two sites, with the barrier between them remaining high throughout the experiment. The many-body resource can be prepared identically to Ref. \([1]\) with the lattice strength tuned near the critical value \((U/J)_c\) to maximize the operational entanglement as discussed above. A SWAP operation (double arrow) is performed between \( A \leftrightarrow Q_A \) by applying the unitary hopping operator \( U_{1,1'}(\pi/2)U_{2,2'}(\pi/2) \), where sites 1, 2 are in region \( A \), while 1', 2' label adjacent sites in \( Q_A \). This is repeated for \( B \leftrightarrow Q_B \) and the identical procedure is performed in the copy. Thus entanglement is transferred from the many-body resource to the spatially separated qubits. To read out this entanglement, a beam-splitter operation (single arrow) is performed between the two copies of \( Q_A \) and \( Q_B \), followed by a subsystem resolved particle number measurement where instances with one atom in each qubit are post-selected.

The above procedure will transfer many-body entanglement to a quantum register. As only \( A \) and \( B \) are swapped with the register, its density matrix \( \rho_{Q_A,Q_B} \) will generically be in a mixed state, even if the initial many-body state \( (\rho_{ABC}) \) was pure. Consequently, the mutual information \( I_2(AB) = S_2(A) + S_2(B) - S_2(A|B) \) will have contributions from both classical correlations and quantum entanglement. \( I_2(AB) \) is measurable in current experiments combined with post-selection to conserve particle number in \( Q_{A/B} \).

To quantify only the desired generation of quantum entanglement between the qubits, we compute various measures of mixed state entanglement for the reduced density matrix \( \rho_{AB} \) of the many-body ground state. Unlike

\[
U_{ij}(\phi) \equiv \exp \left[ i\phi \left( b_i^* b_j + h.c. \right) \right].
\]
for pure states, where the von Neumann entropy is the unique and appropriate entanglement measure, for mixed states, there are a variety of entanglement measures with different physical meanings. For example, the entanglement of formation $E_F$, roughly defined as the amount of entanglement required to form the mixed state, can be directly computed for any two qubit density matrix \[22\]. The logarithmic negativity $E_N$ depends on the sum of the negative eigenvalues of the density matrix after a partial transpose, and thus is readily computable for any density matrix \[23\]. It provides an upper bound to the amount of entanglement that can be extracted from the mixed state using LOCC.

In Fig. 3 we have plotted $I_2(AB)$, $E_F$, and $E_N$ of $\rho_{AB}$ for the ground state of Eq. (1) in the 6-site geometry of Fig. 3 projected onto states with a single particle occupying $A$ and $B$. We find that all these measures peak near the quantum phase transition \[24\]. The peaks coincide with the parameter region of maximal operational entanglement desired for optimal transfer between resource and register. $E_N > 0$ is a necessary and sufficient condition for a two qubit state to be inseparable \[27\] such that it can be distilled to form a maximally entangled state \[28\]. This implies that near the critical point the many-body resource has entanglement that can be extracted and distilled. Although there is no general relationship between $I_2$ and the entanglement measures $E_F$ and $E_N$, in this case we can compute the relationship exactly for the Bose-Hubbard model. Thus, measurement of $I_2$ in an experimental regime where the Bose-Hubbard parameters are known will provide an estimate of the entanglement that can be generated between the qubits through the relationship calculated in Fig. 4.

In conclusion, we have introduced an experimental procedure for the transfer of entanglement from a many-body resource to spatially separated qubits forming a register suitable for quantum information processing. Conservation of particle number limits the amount of entanglement transferable from the resource, as quantified by the operational entanglement. The precise control of the current generation of quantum emulator experiments enables the faithful creation of lattice Bose-Hubbard models using ultracold atoms. This allows us to quantify the operational entanglement using exact calculations, and we find that the transferable entanglement is maximized near the quantum phase transition between the Mott insulator and superfluid phases. This is in contrast to the naive expectation that transfer should occur in the superfluid phase, where experiments have confirmed that the two-copy Rényi entanglement is largest \[1\].

We have introduced a measurement protocol to experimentally probe the entanglement transferred by this procedure that employs a variation of a many-body interference technique \[1\]. It is explicitly described for the transfer of entanglement from a 6-site resource to a register composed of two 2-site qubits – 20 lattice sites in total. It can be easily scaled to arbitrary size as experimental technology progresses. Our Bose-Hubbard calculations quantify the relationship between a mutual information accessible by this protocol and well-known measures for entanglement in mixed states.

The ability to engineer a wealth of variations of the Bose-Hubbard model will open up exciting prospects for extensions and optimizations of our results, through inhomogeneous parameters, topologies, and dimensionality. The experimental implementation of our protocol will demonstrate the potential of using many-body states of ultracold atoms as an entanglement resource for quantum information processing.

This work would not have been possible without discussions with R. Islam and A. Kaufmann. We thank J. Carrasquilla for his insights into the 1D Bose-Hubbard model and A. Brodutch for discussions about entanglement in mixed states. This research was supported by NSERC of Canada, the Canada Research Chair Program, the Perimeter Institute for Theoretical Physics (PI) and the National Science Foundation under Grant No. NSF PHY11-25915. Research at PI is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development & Innovation.

* Adrian.DelMaestro@uvm.edu

[1] R. Islam, R. Ma, P. M. Preiss, M. E. Tai, A. Lukin,
M. Rispoli, and M. Greiner, *Nature* **528**, 77 (2015).

[2] C. K. Hong, Z. Y. Ou, and L. Mandel, *Phys. Rev. Lett.* **59**, 2044 (1987).

[3] W. S. Bakr, A. Peng, M. E. Tai, R. Ma, J. Simon, J. I. Gillen, S. Fölling, L. Pollet, and M. Greiner, *Science* **329**, 547 (2010).

[4] A. J. Daley, H. Pichler, J. Schachenmayer, and P. Zoller, *Phys. Rev. Lett.* **109**, 020505 (2012).

[5] R. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[6] D. F. V. James, P. G. Kwiat, W. J. Munro, and A. G. White, *Phys. Rev. A* **64**, 052312 (2001).

[7] L. Amico, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).

[8] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).

[9] G. Vidal, *Phys. Rev. Lett.* **91**, 147902 (2003) (0301063).

[10] L. Banchi, A. Bayat, P. Verrucchi, and S. Bose, *Phys. Rev. Lett.* **106**, 140501 (2011).

[11] N. Y. Yao, L. Jiang, a. V. Gorshkov, Z.-X. Gong, A. Zhai, L.-M. Duan, and M. D. Lukin, *Phys. Rev. Lett.* **106**, 040505 (2011).

[12] S. M. Giampaolo and F. Illuminati, *New J. Phys.* **12**, 025019 (2010).

[13] L. Campos Venuti, S. M. Giampaolo, F. Illuminati, and P. Zanardi, *Phys. Rev. A* **76**, 052328 (2007).

[14] H. M. Wiseman and J. A. Vaccaro, *Phys. Rev. Lett.* **91**, 097902 (2003).

[15] M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. Lett.* **84**, 2014 (2000).

[16] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, Rev. Mod. Phys. **83**, 1405 (2011).

[17] J. Carrasquilla, S. R. Manmana, and M. Rigol, *Phys. Rev. A* **87**, 043606 (2013).

[18] G. Boeris, L. Gori, M. D. Hoogerland, A. Kumar, E. Lucioni, L. Tanzi, M. Inguscio, T. Giamarchi, C. D’Errico, G. Carleo, G. Modugno, and L. Sanchez-Palencia, (2015) 1509.04742.

[19] G. E. Astrakharchik, K. V. Krutitsky, M. Lewenstein, and F. Mazzanti, (2015) 1509.01424.

[20] Y. Aharonov and L. Susskind, *Phys. Rev.* **155**, 1428 (1967).

[21] C. M. Herdman, S. Inglis, P. N. Roy, R. G. Melko, and A. Del Maestro, *Phys. Rev. E* **90**, 013308 (2014).

[22] W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).

[23] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002) (0102117).

[24] A. Osterloh, L. Amico, G. Falci, and R. Fazio, *Nature* **416**, 608 (2002).

[25] T. Osborne and M. Nielsen, *Phys. Rev. A* **66**, 032110 (2002).

[26] I. Frérot and T. Roscilde, (2015) 1512.00805.

[27] M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. Lett.* **223**, 1 (1996).

[28] M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. Lett.* **78**, 574 (1997).