The sh-Lie algebra perturbation Lemma

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Abstract

Let $R$ be a commutative ring with 1 which contains the rationals as a subring and let $(M \xrightarrow{\nabla} g, h)$ be a contraction of chain complexes (over $R$). We denote the symmetric coalgebra functor by $S^c$, the loop Lie algebra functor by $L$, the classifying coalgebra functor by $C$, and the suspension operator by $s$. We shall establish the following.

Theorem. Let $\partial$ be an sh-Lie algebra structure on $g$, that is, a coalgebra perturbation of the differential $d$ on $S^c[s g]$. Then the given contraction and the sh-Lie algebra structure $\partial$ on $g$ determine an sh-Lie algebra structure on $M$, that is, a coalgebra perturbation $D$ of the coalgebra differential $d^0$ on $S^c[s M]$, a Lie algebra twisting cochain $\tau: S^c_D[s M] \rightarrow L S^c_D[s g]$ and, furthermore, a contraction

$$(S^c_D[s M] \xleftarrow{\tau} \Pi_\partial, C[L S^c_D[s g]], H_{\partial})$$

of chain complexes which are natural in terms of the data. The injection

$\tau: S^c_D[s M] \rightarrow C[L S^c_D[s g]]$

is then a morphism of coaugmented differential graded coalgebras.

Together with the adjoint $S^c_D[s g] \rightarrow C[L S^c_D[s g]]$ of the universal Lie algebra twisting cochain of $L S^c_D[s g]$, this yields an sh-equivalence between $(M, D)$ and $(g, \partial)$. For the special case where $M$ and $g$ are connected, we also construct an explicit extension of the retraction $\Pi_\partial$ to an sh-Lie map.

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1 Introduction

Higher homotopies are nowadays playing a prominent role in mathematics as well as in certain branches of theoretical physics. Higher homotopies often arise as follows: Suppose we are given a huge object, e.g. a chain complex, whose homology includes invariants of a certain geometric or algebraic situation. When one tries to cut such a huge object to size by passing to a smaller object, chain equivalent to the initial one, typically higher homotopies, e.g. Massey products, arise. Furthermore, under homotopy, strict algebraic structures such as e.g. the Jacobi identity of a differential graded Lie bracket are not in general preserved, and higher homotopies arise measuring e.g. the failure of the Jacobi identity in a coherent way. Even for strict structures, non-trivial higher homotopies may encapsulate additional information; this is true, e.g., for the Borromean rings: A non-trivial Massey product detects the non-trivial linking of the three rings. In physics such higher homotopies arise e.g. as anomalies or higher order correlation functions; see e.g. [13] and the references there, in particular to the seminal papers of J. Stasheff.

The ordinary perturbation lemma for chain complexes has become a standard tool to handle higher homotopies in a constructive manner. In view of a celebrated result of Kontsevich’s, sh-Lie (also known as $L_\infty$) algebras have attracted much attention, and the issue of compatibility of the perturbation lemma with a general sh-Lie algebra structure arises. The question whether certain perturbation constructions preserve algebraic structure actually shows up already when one tries to construct e.g. models for differential graded algebras. In the literature, the tensor trick [6], [15], cf. [13] for more literature, was successfully exploited to explore perturbations of free differential graded algebras and cofree differential graded coalgebras, the basic reason for that success being the fact that homotopies of morphisms of such algebras or coalgebras can then be handled concisely; this tensor trick may actually be viewed as an instance of a labelled rooted trees construction [14]. However, for differential graded cocommutative coalgebras as well as for differential graded commutative algebras, the tensor trick breaks down; indeed, as noted already in [24], the notion of homotopy of morphisms of cocommutative coalgebras is a subtle concept. The Cartan-Chevalley-Eilenberg coalgebra (or classifying coalgebra) of a differential graded Lie algebra is a differential graded cocommutative coalgebra; more generally, an sh-Lie algebra is defined in terms of a coalgebra perturbation on a differential graded cocommutative coalgebra. These objects actually arise in deformation theory, see e.g. [11] and the literature there. The purpose of the present paper is to offer ways to overcome the difficulties with the notion of homotopy in the (co)commutative case by establishing the perturbation lemma for sh-Lie algebras. As a side remark we note that, in a different context, suitable homological perturbation theory (HPT) constructions that are compatible with other algebraic structure enabled us to carry out complete numerical calculations in group cohomology [7]–[10] which cannot be done by other methods.

To explain this general perturbation lemma at the present stage somewhat informally, let $R$ be a commutative ring with 1 which contains the rational numbers as a subring and let $(M \xrightarrow{\nabla} \mathfrak{g}, h)$ be a contraction of chain complexes over $R$. Differential graded Lie algebras defined over a ring more general than a field arise in homotopy theory via Samelson brackets, cf. e.g. [2], in gauge theory, e.g. as Lie algebras of gauge transformations—
here the ground ring is the algebra of smooth functions on a smooth manifold and hence manifestly contains the rationals as a subring—and in combinatorial group theory [19]. These remarks justify, perhaps, building the theory over rings more general than a field. A version of the sh-Lie algebra perturbation lemma is the following.

**Theorem.** Given an sh-Lie algebra structure on $\mathfrak{g}$, that is, a coalgebra perturbation of the differential $d$ on $S^c[\mathfrak{g}]$, the chain complex $M$ acquires an sh-Lie algebra structure that is natural in terms of the given contraction and the sh-Lie algebra structure on $\mathfrak{g}$, and the data determine an sh-equivalence between $M$ and $\mathfrak{g}$ relative to the sh-Lie algebra structures that is natural in terms of the data.

The meaning of sh-equivalence is this: Given the coalgebra perturbation $\partial$ of the differential $d$ on $S^c[\mathfrak{g}]$, the data determine in particular a coalgebra perturbation $D$ of the coalgebra differential $d^0$ on $S^c[sM]$ and a Lie algebra twisting cochain

$$\tau: S^c_D[sM] \to LS^c_\partial[\mathfrak{g}].$$

The injection $\pi: S^c_D[sM] \to C[LS^c_\partial[\mathfrak{g}]]$ is then a morphism of coaugmented differential graded coalgebras inducing an isomorphism on homology. Together with the adjoint $S^c_\partial[\mathfrak{g}] \to C[LS^c_\partial[\mathfrak{g}]]$ of the universal Lie algebra twisting cochain of $LS^c_\partial[\mathfrak{g}]$, this yields an sh-equivalence between $(M, D)$ and $(\mathfrak{g}, \partial)$.

A special case of the theorem is the Lie algebra perturbation lemma established in a predecessor of this paper [12]. Exploiting a suitable version of the loop Lie algebra relative to a coaugmented differential graded cocommutative coalgebra, see Section 2 below for details, we will reduce the present general case to the special case in [12]. We conjecture that the theory we develop in this paper has applications to foliation theory and to the integration problem of sh-Lie algebras. The main result of the present paper includes a very general solution of the master equation or, equivalently, Maurer-Cartan equation. More comments about the relevance and history of the master equation can be found in [12], [13], and [16].

I am much indebted to Jim Stasheff for having prodded me on various occasions to pin down the general perturbation lemma for sh-Lie algebras, to M. Duflo for discussion about the PBW theorem, and to the referee for a number of comments which helped improve the exposition.

## 2 The sh-Lie algebra perturbation lemma

The ground ring, written as $R$, is a commutative ring with 1 which contains the rationals as a subring. We will take chain complex to mean differential graded $R$-module. A chain complex will not necessarily be concentrated in non-negative or non-positive degrees. The differential of a chain complex will always be supposed to be of degree $-1$. Write $s$ for the suspension operator as usual and, accordingly, $s^{-1}$ for the desuspension operator. Thus, given the chain complex $X$, $(sX)_j = X_{j-1}$, etc., and the differential $d: sX \to sX$ on the suspended object $sX$ is defined in the standard manner so that $ds + sd = 0$.

For a filtered chain complex $X$, a perturbation of the differential $d$ of $X$ is a (homogeneous) morphism $\partial$ of the same degree as $d$ such that $\partial$ lowers filtration and $(d + \partial)^2 = 0$. 
or, equivalently,
\[ [d, \partial] + \partial \partial = 0. \] (2.1)

Thus, when \( \partial \) is a perturbation on \( X \), the sum \( d + \partial \), referred to as the perturbed differential, endows \( X \) with a new differential. When \( X \) has a graded coalgebra structure such that \( (X, d) \) is a differential graded coalgebra, and when the perturbed differential \( d + \partial \) is compatible with the graded coalgebra structure, we refer to \( \partial \) as a coalgebra perturbation; the notion of algebra perturbation is defined accordingly. Given a differential graded coalgebra \( C \) and a coalgebra perturbation \( \partial \) of the differential \( d \) on \( C \), we will occasionally denote the new differential graded coalgebra by \( C_{\partial} \). Thus the differential of the latter is given by the sum \( d + \partial \).

The following notion goes back to [3]: A contraction
\[ (N \xrightarrow{\nabla} \pi M, h) \] (2.2)
of chain complexes consists of chain complexes \( N \) and \( M \), chain maps \( \pi: N \to M \) and \( \nabla: M \to N \), and a morphism \( h: N \to N \) of the underlying graded modules of degree 1; these data are required to satisfy
\[
\begin{align*}
\pi \nabla &= \text{Id}, \\
Dh &= \text{Id} - \nabla \pi, \\
\pi h &= 0, \quad h \nabla &= 0, \quad hh &= 0.
\end{align*}
\] (2.3) (2.4) (2.5)
The requirements (2.5) are referred to as annihilation properties or side conditions.

**Remark 2.1.** It is well known that the side conditions (2.5) can always be achieved. This fact relies on the standard observation that a chain complex is contractible if and only if it is isomorphic to a cone, cf. [17] (IV.1.5). Under the present circumstances, given data of the kind (2.2) such that (2.3) and (2.4) hold but not necessarily the side conditions (2.5), the operator
\[
\tilde{h} = (\text{Id} - \nabla \pi)h(\text{Id} - \nabla \pi)d(\text{Id} - \nabla \pi)h(\text{Id} - \nabla \pi)
\]
satisfies the requirements (2.4) and (2.5), with \( \tilde{h} \) instead of \( h \); when \( h \) already satisfies (2.5), \( \tilde{h} \) coincides with \( h \).

Let \( C \) be a coaugmented differential graded coalgebra with coaugmentation map \( \eta: R \to C \) and coaugmentation coideal \( JC = \text{coker}(\eta) \), the diagonal map being written as \( \Delta: C \to C \otimes C \). Recall that the counit \( \varepsilon: C \to R \) and the coaugmentation map determine a direct sum decomposition \( C = R \oplus JC \) and that the coaugmentation filtration \( \{F_nC\}_{n \geq 0} \) is given by
\[
F_nC = \ker(C \longrightarrow (JC)^{\otimes(n+1)}) \quad (n \geq 0)
\]
where the unlabelled arrow is induced by some iterate of the diagonal \( \Delta \) of \( C \). This filtration is well known to turn \( C \) into a filtered coaugmented differential graded coalgebra; thus, in particular, \( F_0C = R \). We recall that \( C \) is said to be cocomplete when \( C = \bigcup F_nC \).
Let \( C \) be a coaugmented differential graded coalgebra and \( A \) an augmented differential graded algebra, the multiplication map of \( A \) being written as \( \mu: A \otimes A \to A \) and the augmentation map as \( \varepsilon: A \to R \). Recall that, given homogeneous morphisms \( a, b: C \to A \), their cup product \( a \cup b \) is the composite

\[
C \xrightarrow{\Delta} C \otimes C \xrightarrow{a \otimes b} A \otimes A \xrightarrow{\mu} A
\]

(2.6)

where \( \mu \) refers to the multiplication map of \( A \). The cup product \( \cup \) is well known to turn \( \text{Hom}(C, A) \) into an augmented differential graded algebra, the differential being the ordinary Hom-differential. Recall also that an ordinary twisting cochain \( \tau: C \to A \) is a homogeneous morphism of the underlying graded \( R \)-modules of degree \(-1\) satisfying the identity

\[
D\tau = \tau \cup \tau
\]

(2.7)

and the requirements \( \tau \eta = 0 \) and \( \varepsilon \tau = 0 \).

Given two graded objects \( U \) and \( V \), we denote the (graded) interchange map by \( T: U \otimes V \to V \otimes U \). Recall that a graded coalgebra \( C \) is graded cocommutative when its diagonal map \( \Delta \) satisfies the condition \( T \Delta = \Delta \).

Let \( g \) be (at first) a chain complex, the differential being written as \( d: g \to g \), and let

\[
(M \xrightarrow{\nabla} g, h)
\]

(2.8)

be a contraction of chain complexes. Consider the cofree coaugmented differential graded cocommutative coalgebra (differential graded symmetric coalgebra) \( S^c = S^c[sM] \) on the suspension \( sM \) of \( M \), the existence of of that coalgebra being guaranteed by the hypothesis that the ground ring \( R \) contain the rational numbers as a subring. Further, let \( d^0 \) denote the coalgebra differential on \( S^c = S^c[sM] \) induced by the differential on \( M \). For \( b \geq 0 \), we will henceforth denote the homogeneous (tensor) degree \( b \) component of \( S^c[sM] \) by \( S^c_b \); thus, as a chain complex, \( F_b S^c = R \oplus S^c_1 \oplus \cdots \oplus S^c_b \). Likewise, as a chain complex, \( S^c = \oplus_{f=0}^\infty S^c_f \). We denote by \( \tau_M: S^c \to M \) the composite of the canonical projection \( \text{proj}: S^c \to sM \) from \( S^c = S^c[sM] \) to its homogeneous degree 1 constituent \( sM \) with the desuspension map \( s^{-1} \) from \( sM \) to \( M \). In particular, \( \tau_g: S^c[sq] \to g \) refers to the composite of the canonical projection to \( S^c[g] = sq \) with the desuspension map.

Given a homogeneous element \( x \) of a graded module, we will denote its degree by \(|x|\). Given two chain complexes \( X \) and \( Y \), recall that \( \text{Hom}(X, Y) \) inherits the structure of a chain complex by the operator \( D \) defined by \( D\phi = d\phi - (-1)^{|\phi|} \phi d \) where \( \phi \) is a homogeneous morphism of \( R \)-modules from \( X \) to \( Y \).

Let now \( C \) be a coaugmented differential graded cocommutative coalgebra and \( h \) a differential graded Lie algebra, the graded bracket being written as \([\cdot, \cdot]\). Given homogeneous morphisms \( a, b: C \to h \), with a slight abuse of the bracket notation \([\cdot, \cdot]\), their cup bracket \([a, b]\) is given by the composite

\[
C \xrightarrow{\Delta} C \otimes C \xrightarrow{a \otimes b} h \otimes h \xrightarrow{[\cdot, \cdot]} h.
\]

(2.9)

This bracket turns \( \text{Hom}(C, h) \) into a differential graded Lie algebra.
In particular, take $C$ to be the differential graded symmetric coalgebra $\mathcal{S}^c[sh]$ and define the coderivation

$$\partial: \mathcal{S}^c[sh] \longrightarrow \mathcal{S}^c[sh]$$

on $\mathcal{S}^c[sh]$ by the requirement

$$\tau_h \partial = \frac{1}{2}[\tau_h, \tau_h]: \mathcal{S}^c_2[sh] \rightarrow h. \quad (2.10)$$

Plainly $D\partial (= d\partial + \partial d) = 0$ since the Lie algebra structure on $h$ is supposed to be compatible with the differential $d$ on $h$. Moreover, the requirement that the bracket $[\cdot, \cdot]$ on $h$ satisfy the graded Jacobi identity is equivalent to the requirement that $\partial \partial$ vanish, that is, to $\partial$ being a coalgebra perturbation of the differential $d^0$ on $\mathcal{S}^c[sh]$, cf. [12] and [16].

The Lie algebra perturbation lemma (Theorem 2.1 in [12] and reproduced below as Lemma 2.2) and the sh-Lie algebra perturbation lemma (Theorem 2.5 below) both generalize this observation. Under the present circumstances, $h$ being an ordinary differential graded Lie algebra, the resulting differential graded coalgebra $\mathcal{S}^c_0[sh]$ is precisely the standard CARTAN-CHEVALLEY-EILENBERG (CCE-) or classifying coalgebra for $h$ and, following [23] (p. 291), we denote this coalgebra by $C[h]$.

As before, let $C$ be a coaugmented differential graded cocommutative coalgebra. A Lie algebra twisting cochain $t: C \rightarrow h$ is a homogeneous morphism of degree $-1$ whose composite with the coaugmentation map is zero and which satisfies the equation

$$Dt = \frac{1}{2}[t, t], \quad (2.12)$$

cf. [16], [21] and [23]. The equation (2.12) is a version of the master equation, cf. [16] and the literature there. In particular, relative to the graded Lie bracket $[\cdot, \cdot]$ on $h$, the morphism $\tau_h: C[h] \rightarrow h$ is a Lie algebra twisting cochain, the CARTAN-CHEVALLEY-EILENBERG (CCE-) or universal Lie algebra twisting cochain for $h$. Likewise, when $M$ is viewed as an abelian differential graded Lie algebra, $\mathcal{S}^c = \mathcal{S}^c[sM]$ may be viewed as the CCE- or classifying coalgebra $C[M]$ for $M$, and $\tau_M: \mathcal{S}^c \rightarrow M$ is then the universal differential graded Lie algebra twisting cochain for $M$ as well.

For intelligibility, we will now recall the main result of [12], spelled out there as Theorem 2.1.

**Lemma 2.2** (Lie algebra perturbation lemma). Suppose that $\mathfrak{g}$ carries a differential graded Lie algebra structure. Then the contraction (2.8) and the graded Lie algebra structure on $\mathfrak{g}$ determine an sh-Lie algebra structure on $M$, that is, a coalgebra perturbation $D$ of the coalgebra differential $d^0$ on $\mathcal{S}^c[sM]$, a Lie algebra twisting cochain

$$\tau: \mathcal{S}^c_D[sM] \longrightarrow \mathfrak{g} \quad (2.13)$$

and, furthermore, a contraction

$$\left( \mathcal{S}^c_D[sM] \xrightarrow{\tau} C[\mathfrak{g}], H \right) \quad (2.14)$$
of chain complexes which are natural in terms of the data so that
\[
\pi \tau = \tau_M : S^c[sM] \rightarrow M, \tag{2.15}
\]
\[
h \tau = 0. \tag{2.16}
\]

The injection \( \tau : S^c_0[sM] \rightarrow C[\mathfrak{g}] \) is then a morphism of coaugmented differential graded coalgebras.

In the statement of Lemma 2.2, the adjoint \( \tau \) of (2.13) is plainly an sh-equivalence in the sense that it induces an isomorphism on homology, \( M \) being endowed with the sh-Lie algebra structure given by \( D \). In Section 4 below we shall explain how \( \tau \) yields actually an sh-equivalence between \( g \) and \( M \) in a certain stronger sense when \( M \) and \( g \) are connected.

We will now consider the more general case where \( g \) is endowed with merely an sh-Lie algebra structure. To this end, we will denote by \( S \) the graded symmetric algebra functor in the category of \( R \)-modules. As before, let \( h \) be a differential graded Lie algebra and, \( h \) being viewed as a chain complex, let \( S[h] \) be the differential graded symmetric algebra on \( h \). Since the ground \( R \) is supposed to contain the rational numbers as a subring, the diagonal map \( h \rightarrow h \oplus h \) of \( h \) induces a diagonal map \( \Delta : S[h] \rightarrow S[h] \otimes S[h] \) that turns \( S[h] \) into a differential graded cocommutative Hopf algebra; furthermore, the obvious filtration then turns \( S[h] \) into a filtered differential graded cocommutative Hopf algebra. Consider the universal differential graded algebra \( U[h] \) associated with \( h \) and let \( j : h \rightarrow U[h] \) denote the canonical morphism of differential graded Lie algebras; it is well known that, just as for the symmetric algebra on \( h \), via the appropriate universal property, the diagonal map of \( h \) induces a diagonal map \( \Delta : U[h] \rightarrow U[h] \otimes U[h] \) turning \( U[h] \) into a differential graded cocommutative Hopf algebra which, relative to the ordinary Poincaré-Birkhoff-Witt filtration \( R = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_\ell \subseteq \ldots \) is actually a filtered differential graded cocommutative Hopf algebra. We denote the associated graded object by \( E^0U[h] \); this is a differential graded commutative and cocommutative Hopf algebra endowed with a canonical morphism \( S[h] \rightarrow E^0U[h] \) of differential graded Hopf algebras.

**Proposition 2.3.** The classical Poincaré symmetrization map

\[
e : S[h] \rightarrow U[h], \quad e(x_1 \ldots x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \pm j(x_{\sigma 1}) \ldots j(x_{\sigma n})
\]

in the category of \( R \)-modules is a functorial isomorphism of filtered differential graded \((R-)\)coalgebras which induces an isomorphism

\[
e : S[h] \rightarrow E^0U[h]
\]

of differential graded Hopf algebras. Consequently, the differential graded \((R-)\)algebra \( U[h] \) being viewed as a differential graded \((R-)\)Lie algebra via the commutator bracket as usual, the canonical morphism \( j : h \rightarrow U[h] \) of differential graded \( R \)-Lie algebras is injective, and the universal algebra \( U[h] \) is enveloping. \( \square \)
This proposition makes precise the idea that $U[\mathfrak{h}]$, viewed as a differential graded Hopf algebra, is a perturbation of $S[\mathfrak{h}]$, viewed as a differential graded Hopf algebra, the coalgebra structure being unperturbed.

Let $Y$ be a chain complex, and let $T[Y]$ be the differential graded tensor algebra on $Y$. The shuffle diagonal map is well known to turn $T[Y]$ into a differential graded cocommutative Hopf algebra and, $T[Y]$ being viewed as a differential graded Lie algebra via the commutator bracket, the free (differential graded) Lie algebra $L[Y]$ on $Y$ is the differential graded Lie subalgebra of $T[Y]$ generated by $Y$. Further, the canonical morphism of augmented differential graded algebras from $U[L[Y]]$ to $T[Y]$ is an isomorphism, cf. e.g. [2] (Proposition 2.10). This explains the differential graded Hopf algebra structure on $U[L[Y]]$ in the particular case of the differential graded Lie algebra $L[Y]$.

The submodule $\text{Prim}[Y]$ of primitive elements in the Hopf algebra $T[Y]$ is well known to be a differential graded Lie subalgebra of $T[Y]$ and, since $Y$ is manifestly contained in $\text{Prim}[Y]$, the free (differential graded) Lie algebra $L[Y]$ on $Y$ is plainly a differential graded Lie subalgebra of $\text{Prim}[Y]$. In view of a classical theorem of K. O. Friedrichs’, over a field of characteristic zero, the two coincide and, more generally, the two coincide whenever the ground ring $R$ is an integral domain of characteristic zero and $Y$ a free graded $R$-module, cf. [2] (Proposition 2.8).

Let $C$ be a coaugmented differential graded coalgebra. By construction, the loop algebra $\Omega C$ is the perturbed tensor algebra $T[\Delta][s^{-1}JC]$ on $s^{-1}JC$, the algebra perturbation $\partial_\Delta$ on $T[s^{-1}JC]$ being induced by $\Delta$. Suppose, in addition, that $C$ is cocommutative. Then $\Omega C$ acquires a differential graded Hopf algebra structure. Moreover, since the diagonal map $\Delta$ is a morphism of differential graded coalgebras, the induced morphism $J\Delta: JC \to JC \otimes JC$ is compatible with the structure, whence the algebra perturbation $\partial_\Delta$ descends to a Lie algebra perturbation on $\text{Prim}[s^{-1}JC] = \ker(J\Delta)$

which we still denote by $\partial_\Delta$, and we denote the resulting differential graded Lie algebra by $\text{Prim}_\Delta[s^{-1}JC]$. Over a field of characteristic zero, this is the loop Lie algebra on $C$, a familiar object, and the loop Lie algebra then coincides with the free Lie algebra. In general, it is not clear whether the obvious injection of the free differential graded Lie algebra $L[s^{-1}JC]$ into $\text{Prim}[s^{-1}JC]$ is onto.

**Lemma 2.4.** The coaugmented differential graded coalgebra $C$ being assumed to be graded cocommutative, the values of the Lie algebra perturbation $\partial = \partial_\Delta$, restricted to $L[s^{-1}JC]$, lie in $L[s^{-1}JC]$.

**Proof.** Write $Y = s^{-1}JC$, so that $L[s^{-1}JC] = L[Y] \subseteq \Omega C$, and so that the augmented graded algebra which underlies $\Omega C$ coincides with the tensor algebra $T[Y]$. We will use the notation $[\cdot, \cdot]$ for the graded commutator in the graded tensor algebra $T[s^{-1}JC]$. The values of the morphism $\partial - T\partial: Y \to Y \otimes Y$

lie in the submodule $[Y, Y] \subseteq Y \otimes Y$ spanned by the commutators of elements from $Y$. The algebra perturbation $\partial_\Delta$ on $T[s^{-1}JC]$ is induced by the morphism $J\Delta$ coming from the diagonal map $\Delta$ of $C$. Since the latter is cocommutative, $-T\partial$ coincides with $\partial$ whence
the values of $2\partial$, restricted to $Y$, lie in $L[Y]$. Consequently the values of the perturbation $\partial$, restricted to $Y$, lie in $L[Y]$. 

Let $C$ be a coaugmented differential graded cocommutative coalgebra. We will use the notation $\mathcal{L}C$ for $L[s^{-1}JC]$, endowed with the perturbed differential $d + \partial_{\Delta}$ and we will refer to $\mathcal{L}C$ as the loop Lie algebra over $C$. The desuspension map induces a Lie algebra twisting cochain

$$t_{\mathcal{L}}: C \longrightarrow \mathcal{L}C,$$

the universal Lie algebra twisting cochain for the loop Lie algebra. See [21] and [23] for the case where the ground ring is a field of characteristic zero. Whether or not the ground ring is a field of characteristic zero, the canonical morphism

$$U[\mathcal{L}C] \longrightarrow \Omega C$$

(2.17)
of augmented differential graded algebras is an isomorphism, and the adjoint

$$\Omega C \longrightarrow U[\mathcal{L}C]$$

of the composite of $t_{\mathcal{L}}$ with the canonical morphism $\mathcal{L}C \rightarrow U[\mathcal{L}C]$ yields the inverse for (2.17) in the category of augmented differential graded algebras.

With $C = S^c[sg]$, the isomorphism (2.17) then takes the form

$$U[\mathcal{L}S^c[sg]] \longrightarrow \Omega S^c[sg].$$

(2.18)

An sh-Lie algebra structure or $L_\infty$-structure on the chain complex $g$ is a coalgebra perturbation $\partial$ of the differential $d$ on the cofree coaugmented differential graded cocommutative coalgebra $S^c[sg]$ on $sg$, cf. [16] (Def. 2.6). Given such an sh-Lie algebra structure $\partial$ on $g$, with $C = S^c[sg]$, (2.17) yields the isomorphism

$$U[\mathcal{L}S^c_0[sg]] \longrightarrow \Omega S^c_0[sg].$$

(2.19)

In particular, via the coderivation (2.10), an ordinary graded Lie algebra structure $[\cdot, \cdot]$ determines an sh-Lie algebra structure $\partial$ and, in this case, $S^c_0[sg]$ amounts to the CCE-coalgebra $\mathcal{C}[g]$ for $(g, [\cdot, \cdot])$. Given two sh-Lie algebras $(g_1, \partial_1)$ and $(g_2, \partial_2)$, an sh-morphism or sh-Lie map from $(g_1, \partial_1)$ to $(g_2, \partial_2)$ is a morphism $S^c_{\partial_1}[sg_1] \rightarrow S^c_{\partial_2}[sg_2]$ of coaugmented differential graded coalgebras [16]; we then define a generalized sh-morphism or generalized sh-Lie map from $(g_1, \partial_1)$ to $(g_2, \partial_2)$ to be a Lie algebra twisting cochain $S^c_{\partial_1}[sg_1] \rightarrow \mathcal{L}S^c_{\partial_2}[sg_2]$.

**Theorem 2.5** (Sh-Lie algebra perturbation lemma). Let $g$ be a chain complex and let $\partial$ be an sh-Lie algebra structure on $g$, that is, a coalgebra perturbation of the differential $d$ on $S^c[sg]$. Then the contraction (2.8) and the sh-Lie algebra structure $\partial$ on $g$ determine an sh-Lie algebra structure on $M$, that is, a coalgebra perturbation $D$ of the coalgebra differential $d^0$ on $S^c[sM]$, a Lie algebra twisting cochain

$$\tau: S^c_D[sM] \longrightarrow \mathcal{L}S^c_0[sg]$$

(2.20)
and, finally, a contraction
\[
\left( \mathcal{S}^c_D [sM] \xleftarrow{\tau} \mathcal{C}[\mathcal{L}S^c_D [s\mathfrak{g}]], H_0 \right)
\]  
(2.21)

of chain complexes, and (2.20) and (2.21) are natural in terms of the data. The injection
\[
\tau : \mathcal{S}^c_D [sM] \rightarrow C[\mathcal{L}S^c_D [s\mathfrak{g}]]
\]
is then a morphism of coaugmented differential graded coalgebras.

Under the circumstances of Theorem 2.5, the twisting cochain (2.20) is a generalized morphism of sh-Lie algebras from \((M, D)\) to \((\mathfrak{g}, \partial)\), and the adjoint \(\tau\) of (2.20) is plainly an sh-equivalence in the sense that it induces an isomorphism on homology, including the brackets of all order that are induced on homology. In Section 4 below, we shall sketch an extension of the contraction (2.21) to an sh-equivalence, in a stronger sense, between these two sh-Lie algebras for the special case where \(M\) and \(\mathfrak{g}\) are connected.

3 Proof of the sh-Lie algebra perturbation lemma

Until further notice we will view \(\mathfrak{g}\) merely as a chain complex or, equivalently, as an abelian differential graded Lie algebra. The desuspension map induces the ordinary twisting cochain
\[
\tau^c : \mathcal{S}^c [s\mathfrak{g}] \rightarrow \mathcal{S}[\mathfrak{g}],
\]
and the adjoint \(\pi_S : \Omega \mathcal{S}^c [s\mathfrak{g}] \rightarrow \mathcal{S}[\mathfrak{g}]\) thereof is a surjective morphism of augmented differential graded algebras.

We will denote the reduced bar construction functor by \(B\) (defined on the category of augmented differential graded algebras).

**Lemma 3.1.** The projection \(\pi_S\) extends to a contraction
\[
\left( \mathcal{S}[\mathfrak{g}] \xleftarrow{\nabla_S} \Omega \mathcal{S}^c [s\mathfrak{g}], h_S \right)
\]
(3.1)
of chain complexes that is natural in terms of the data.

In this lemma, nothing is claimed as far as compatibility of \(\nabla_S\) and \(h_S\) with the algebra structures is concerned.

**Proof.** Consider the ordinary loop algebra contraction
\[
\left( \mathcal{S}[\mathfrak{g}] \xleftarrow{\nabla^\Omega} \Omega B \mathcal{S}[\mathfrak{g}], h^\Omega \right)
\]
(3.2)
for \(\mathcal{S}[\mathfrak{g}]\), cf. [17, 22] (2.14) (p. 17). Here the projection \(\pi^\Omega\) is the adjoint of the universal bar construction twisting cochain \(B \mathcal{S}[\mathfrak{g}] \rightarrow \mathcal{S}[\mathfrak{g}]\) and is therefore a morphism of augmented differential graded algebras. The adjoint
\[
\nabla^c_S = \overline{\tau^c} : \mathcal{S}^c [s\mathfrak{g}] \rightarrow B \mathcal{S}[\mathfrak{g}]
\]
(3.3)
of the twisting cochain $\tau^S$ is the standard coalgebra injection of $S^c[sg]$ into $BS[\mathfrak{g}]$, and a familiar construction extends (3.3) to a contraction

$$ \left( S^c[sg] \xleftarrow{\nabla^S} BS[\mathfrak{g}], h_{S^c} \right) $$

which is natural in terms of the data. Similarly, the induced morphism

$$ \Omega\nabla^S = \Omega\tau^S : \Omega S^c[sg] \longrightarrow \Omega BS[\mathfrak{g}] $$

of differential graded algebras extends to a contraction

$$ \left( \Omega S^c[sg] \xleftarrow{\Omega\nabla^S} \Omega BS[\mathfrak{g}], h_{\Omega S^c} \right) $$

which is natural in terms of the data, and $\pi_S = \pi_{\Omega} \circ \Omega\nabla^S$. Let

$$ \nabla_S = \pi_{\Omega S^c} \circ \nabla^\Omega, \quad \tilde{h} = \pi_{\Omega S^c} \circ h_{\Omega S^c} \circ \Omega\nabla^S, \quad h_S = \tilde{h} \circ d \circ \tilde{h}. $$

This yields data of the kind (3.1). In view of Remark 2.1 above, these data constitute a contraction of chain complexes that is natural in terms of the data. \hfill \Box

The chain complex $\mathfrak{g}$ still being viewed as an abelian differential graded Lie algebra, consider the loop Lie algebra $\mathcal{L} = \mathcal{L}S^c[sg] = S^c[sg]$. Let $\nabla_\mathcal{L} : \mathfrak{g} \to \mathcal{L}S^c[sg]$ be the canonical injection of chain complexes and, likewise, $\mathfrak{g}$ still being viewed as an abelian differential graded Lie algebra, let $\pi_\mathcal{L} : \mathcal{L}S^c[sg] \to \mathfrak{g}$ be the familiar adjoint of the corresponding universal Lie algebra twisting cochain $S^c[sg] \to \mathfrak{g}$; this morphism $\pi_\mathcal{L}$ is plainly a surjective morphism of differential graded Lie algebras. It admits the following elementary description: The canonical projection $s^{-1}JS^c[sg] \to \mathfrak{g}$ induces a surjective morphism $L[s^{-1}JS^c[sg]] \to L[\mathfrak{g}]$ of differential graded Lie algebras, the canonical projection $L[\mathfrak{g}] \to \mathfrak{g}$ is simply the abelianization map (of differential graded Lie algebras), and the composite

$$ L[s^{-1}JS^c[sg]] \longrightarrow \mathfrak{g} $$

of the two yields the morphism $\pi_\mathcal{L}$ of differential graded Lie algebras, manifestly surjective, $\mathfrak{g}$ being viewed abelian.

For intelligibility, we explain the details: Write $L = L[s^{-1}JS^c[sg]]$ and let $\tilde{L}$ denote the kernel of (3.7). The obvious injection of $\mathfrak{g}$ into $L$ induces a direct sum decomposition

$$ L \cong \tilde{L} \oplus \mathfrak{g} $$

of chain complexes. Moreover, the Lie algebra perturbation $\partial_\Delta$ on $L$ vanishes on the direct summand $\mathfrak{g}$ and the other direct summand $\tilde{L}$ is closed under the operator $\partial_\Delta$. Let $\tilde{\mathcal{L}} = \tilde{L}_{\partial_\Delta}$; that is to say, the graded Lie algebra which underlies $\tilde{\mathcal{L}}$ coincides with that underlying the kernel $\tilde{L}$ whereas the differential is perturbed via the diagonal map $\Delta$ of $S^c[sg]$. Thus the canonical projection from $L$ to $\mathfrak{g}$ is also compatible with the perturbed differential relative to the diagonal map of $S^c[sg]$, and $\tilde{\mathcal{L}}$ is the kernel of the resulting
projection \( \pi_L \) from \( L \) to \( g \). Furthermore, as a chain complex, \( L = L_{\partial_{\Delta}} \) decomposes as the direct sum
\[
L = \tilde{L} \oplus \nabla_L(g),
\]
and \( \tilde{L} \) is a differential graded Lie ideal of \( L \). Thus the obvious injection \( \nabla_L : g \rightarrow L \) of \( g \) into \( L \) is a chain map and the obvious projection \( \pi_L : L \rightarrow g \) of \( L \) onto \( g \) is a morphism of differential graded Lie algebras, \( g \) being viewed abelian.

For \( j \geq 0 \), we denote by \( S^j \) the \( j \)-th homogeneous constituent of the symmetric algebra functor \( S \).

**Lemma 3.2.** The homotopy \( h_S \) in the contraction (3.1) induces a homotopy \( h_L \) such that the data
\[
\left( g \xrightarrow{\nabla_L} \mathcal{L}S^c[sg], h_L \right)
\]
constitute a contraction of chain complexes.

**Proof.** Consider the perturbed objects
\[
\mathcal{L}S^c[sg] = L_{\Delta}[s^{-1}JS^c[sg]], \quad \Omega S^c[sg] = T_{\Delta}[s^{-1}JS^c[sg]],
\]
the perturbations—beware, not to be confused with the perturbation \( \partial \) defining the sh-Lie algebra structure on \( g \)—being induced by the diagonal map of \( S^c[sg] \). Relative to the corresponding perturbed differentials, the projection to the associated graded object induces an isomorphism
\[
\Omega S^c[sg] \longrightarrow R \oplus \mathcal{L}S^c[sg] \oplus S^2\mathcal{L}S^c[sg] \oplus \ldots \oplus S^\ell \mathcal{L}S^c[sg] \oplus \ldots \quad (3.9)
\]
of chain complexes. Furthermore, relative to the direct sum decomposition (3.9), for \( \ell \geq 1 \), the component
\[
S^\ell \mathcal{L}S^c[sg] \longrightarrow S^\ell \mathcal{L}S^c[sg]
\]
of the homotopy \( h_S \) in (3.1) above is itself a homotopy and, for \( \ell' \neq \ell \), a component of the kind
\[
S^{\ell'} \mathcal{L}S^c[sg] \longrightarrow S^{\ell''} \mathcal{L}S^c[sg],
\]
if non-zero, is necessarily a cycle (in the corresponding Hom-complex), since the right-hand side of (3.9) is a direct sum decomposition of chain complexes. The component
\[
\mathcal{L}S^c[sg] = S^1\mathcal{L}S^c[sg] \longrightarrow S^1\mathcal{L}S^c[sg] = \mathcal{L}S^c[sg]
\]
yields the homotopy \( h_L \) we are looking for. \( \square \)

We now prove Theorem 2.5 (the sh-Lie algebra perturbation lemma): Given the contraction (2.8), suppose that \( g \) comes with a general sh-Lie algebra structure, that is, let \( \partial \) be a general coalgebra perturbation of the differential \( d \) on \( S^c[sg] \) induced by the differential on \( g \).

The coaugmentation filtration \( \{ F_n(S^c[sg]) \}_{n \geq 0} \) of \( S^c[sg] \) turns \( \mathcal{L}S^c[sg] \) into a filtered differential graded Lie algebra \( \{ F_n(\mathcal{L}S^c[sg]) \}_{n \geq 0} \) via
\[
F_0(\mathcal{L}S^c[sg]) = 0, \quad F_n(\mathcal{L}S^c[sg]) = LF_n(S^c[sg]) \quad (n \geq 0),
\]
and we make \( \mathfrak{g} \) into a trivially filtered chain complex \( \{ F_n(\mathfrak{g}) \}_{n \geq 0} \) via \( F_0(\mathfrak{g}) = 0 \) and \( F_n(\mathfrak{g}) = \mathfrak{g} \) for \( n \geq 1 \). This turns (3.8) into a filtered contraction of chain complexes. Furthermore, the sh-Lie algebra structure \( \partial \) on \( \mathfrak{g} \) (coalgebra perturbation on \( S^c[\mathfrak{s}\mathfrak{g}] \)) perturbs the differential on \( S^c[\mathfrak{s}\mathfrak{g}] \) and hence that on \( \mathcal{L}S^c[\mathfrak{s}\mathfrak{g}] \) and, indeed, yields a Lie algebra perturbation on \( \mathcal{L}S^c[\mathfrak{s}\mathfrak{g}] \); we write this perturbation as

\[
\partial_L : \mathcal{L}S^c[\mathfrak{s}\mathfrak{g}] \longrightarrow \mathcal{L}S^c[\mathfrak{s}\mathfrak{g}].
\]

Thus perturbing the loop Lie algebra \( \mathcal{L}S^c[\mathfrak{s}\mathfrak{g}] \) on \( S^c[\mathfrak{s}\mathfrak{g}] \) via \( \partial_L \) carries the loop Lie algebra \( \mathcal{L}S^c[\mathfrak{s}\mathfrak{g}] \) to the loop Lie algebra \( \mathcal{L}S^c_0[\mathfrak{s}\mathfrak{g}] \) on \( S^c_0[\mathfrak{s}\mathfrak{g}] \). Application of the ordinary perturbation lemma (reproduced in [12] as Lemma 5.1) to the Lie algebra perturbation \( \partial_L \) on \( \mathcal{L}S^c[\mathfrak{s}\mathfrak{g}] \) and the filtered contraction of chain complexes (3.8) yields the contraction

\[
\left( \mathfrak{g} \overset{\nabla}{\underset{\pi}{\leftrightarrow}} \mathcal{L}S^c_0[\mathfrak{s}\mathfrak{g}], h_0 \right) \tag{3.10}
\]

of chain complexes. In the special case where the perturbation \( \partial \) arises from an ordinary differential graded Lie algebra structure on \( \mathfrak{g} \), the morphism \( \pi_\partial \) is the adjoint of the resulting Lie algebra twisting cochain \( \mathcal{C}[\mathfrak{g}] \rightarrow \mathfrak{g} \) relative to the Lie algebra structure on \( \mathfrak{g} \) and is therefore a morphism of differential graded Lie algebras relative to the Lie algebra structure on \( \mathfrak{g} \). Whether or not the perturbation \( \partial \) arises from an ordinary differential graded Lie algebra structure on \( \mathfrak{g} \), we now combine the contraction (3.10) with the original contraction (2.8) to the contraction

\[
\left( M \overset{\nabla}{\underset{\pi}{\leftrightarrow}} \mathcal{L}S^c_0[\mathfrak{s}\mathfrak{g}], h \right) \tag{3.11}
\]

of chain complexes where the notation \( \nabla, \pi, h \) is abused somewhat. More precisely, when the two contractions (3.10) and (2.8) are written as

\[
\left( M \overset{\nabla_1}{\underset{\pi_1}{\leftrightarrow}} \mathfrak{g}, h_1 \right), \quad \left( \mathfrak{g} \overset{\nabla_2}{\underset{\pi_2}{\leftrightarrow}} \mathcal{L}S^c_0[\mathfrak{s}\mathfrak{g}], h_2 \right),
\]

the three morphisms in the contraction (3.11) are given by

\[
\pi = \pi_1 \pi_2, \quad \nabla = \nabla_2 \nabla_1, \quad h = h_2 + \nabla_2 h_1 \pi_2.
\]

Applying the ordinary Lie algebra perturbation lemma (Lemma 2.2 above) to the contraction (3.11) relative to the differential graded Lie algebra structure on \( \mathcal{L} = \mathcal{L}S^c_0[\mathfrak{s}\mathfrak{g}] \), we obtain the perturbation \( \mathcal{D} \) on \( S^c[\mathfrak{s}M] \), the Lie algebra twisting cochain

\[
\tau : S^c_0[\mathfrak{s}M] \longrightarrow \mathcal{L},
\]

and the asserted contraction (2.21) of chain complexes, where we use the notation \( \Pi_\partial \) and \( H_\partial \) rather than the notation \( \Pi \) and \( H \), respectively, in the contraction (2.14) spelled out in the ordinary Lie algebra perturbation lemma. This completes the proof of Theorem 2.5.
4 Inverting the retraction as an sh-Lie map

We return to the situation of the ordinary Lie algebra perturbation lemma (Lemma 2.2 above). Thus \( g \) is now an ordinary differential graded Lie algebra. Let \( \tau \) be the Lie algebra twisting cochain (2.13). The retraction

\[
\Pi : C[g] \longrightarrow S_c^e[sM]
\]

for the contraction (2.14) constructed in the last section of [12] is not in general compatible with the graded coalgebra structures. As already pointed out, the reason is that the notion of homotopy of morphisms of differential graded cocommutative coalgebras is a subtle concept. We will now explain how, in the special case where \( M \) and \( g \) are connected, the retraction \( \Pi \) can be extended to a morphism of sh-Lie algebras, that is, to a morphism preserving the appropriate structure.

For intelligibility, we recall the constructions of the retraction \( \Pi \) and contracting homotopy \( H \) in (2.14) carried out in [12]: Application of the ordinary perturbation lemma (reproduced in [12] as Lemma 5.1) to the perturbation \( \partial \) on \( S_c^e[sM] \) determined by the graded Lie algebra structure on \( g \) and the induced filtered contraction

\[
\left( S_c^e[sM] \xrightarrow{S_c^e[\nabla]} S_c^e[sg], S_c^e[sh] \right)
\]

(4.1)

of coaugmented differential graded coalgebras, the filtrations being the ordinary coaugmentation filtrations, yields the perturbation \( \delta \) of the differential \( d^0 \) on \( S_c^e[sM] \) and, furthermore, the contraction

\[
\left( S_c^e[sM] \xrightarrow{\nabla} C[g], \tilde{H} \right)
\]

(4.2)

of chain complexes. Moreover, the composite

\[
\Phi : S_D^c[sM] \xrightarrow{\tau} C[g] \xrightarrow{\Pi} S_c^e[sM]
\]

(4.3)

is an isomorphism of chain complexes, and the morphisms

\[
\Pi = \Phi^{-1}\Pi : C[g] \longrightarrow S_D^c[sM],
\]

(4.4)

\[
H = \tilde{H} - \tilde{H}\tau\Pi : C[g] \longrightarrow C[g]
\]

(4.5)

complete the construction of the contraction (2.14).

In general, none of the morphisms \( \delta, \nabla, \Pi, \tilde{H}, H \) is compatible with the coalgebra structures. The isomorphism of chain complexes \( \Phi \) admits an explicit description in terms of the data as a perturbation of the identity and so does its inverse; details have been given in the last section of [12].

Consider the universal loop Lie algebra twisting cochain

\[
t_L : S_D^c[sM] \longrightarrow LS_D^c[sM].
\]

(4.6)

We recall that \( M \) to be connected means that \( M \) is concentrated either in positive or in negative degrees; in particular, the degree zero constituent of \( M \) is zero.
Lemma 4.1. Suppose that $M$ is connected. The recursive construction

$$\vartheta = t_L\Pi + \frac{1}{\vartheta}, \vartheta | H: C[\mathfrak{g}] \to \mathcal{L}\mathcal{S}_D(\mathfrak{s}M)$$

(4.7)

yields a Lie algebra twisting cochain $\vartheta: C[\mathfrak{g}] \to \mathcal{L}\mathcal{S}_D(\mathfrak{s}M)$ such that

$$\vartheta \tau = t_L : \mathcal{S}_D(\mathfrak{s}M) \to \mathcal{L}\mathcal{S}_D(\mathfrak{s}M).$$

(4.8)

Proof. The construction (4.7) being recursive means that

$$\vartheta = \vartheta_1 + \vartheta_2 + \ldots$$

where $\vartheta_1 = t_L\Pi$, $\vartheta_2 = \frac{1}{2}[\vartheta_1, \vartheta_1]H$, $\vartheta_3 = [\vartheta_1, \vartheta_2]H$, etc. The connectedness hypothesis entails the convergence, which is naive. We leave the details as an exercise. \qed

Complement I to Lemma 2.2. In view of the identity (4.8), it is manifest that the composite

$$\mathcal{S}_D(\mathfrak{s}M) \xrightarrow{\tau} C[\mathfrak{g}] \xrightarrow{\vartheta} \mathcal{L}\mathcal{S}_D(\mathfrak{s}M)$$

coincides with the universal loop Lie algebra twisting cochain (4.6). In this sense, $\vartheta$ yields an sh-retraction for the sh-morphism from $(M, \mathcal{D})$ to $\mathfrak{g}$ given by $\tau$.

To explain in which sense the other composite

$$\mathfrak{g} \xrightarrow{\vartheta} (M, \mathcal{D}) \xrightarrow{\tau} \mathfrak{g}$$

(4.9)

of these morphisms is homotopic to the identity, we need some more preparation.

Let $C$ be a coaugmented differential graded coalgebra and $A$ an augmented differential graded algebra. Recall that, given two ordinary twisting cochains $\tau_1, \tau_2: C \to A$, a homotopy $h: \tau_1 \simeq \tau_2$ of twisting cochains is a homogeneous morphism

$$h: C \to A$$

(4.10)

of degree zero such that $\varepsilon h \eta = \varepsilon \eta$ and

$$Dh = \tau_1 \cup h - h \cup \tau_2 \in \text{Hom}(C, A).$$

(4.11)

Such a homotopy $h: \tau_1 \simeq \tau_2$ of twisting cochains is well known to induce a chain homotopy

$$\overline{h}: C \to \mathcal{B}A$$

(4.12)

between the adjoints $\tau_1, \tau_2: C \to \mathcal{B}A$ into the reduced bar construction $\mathcal{B}A$ on $A$, and the homotopy $\overline{h}$ is compatible with the coalgebra structures.

Recall that the augmented differential graded algebra $A$ is complete when the canonical morphism of differential graded algebras from $A$ to $\varprojlim A/(IA)^n$ is an isomorphism; here $IA$ refers to the augmentation ideal as usual.
Lemma 4.2. Suppose the following data are given:
— coaugmented differential graded coalgebras \( B \) and \( C \);
— a contraction
\[
(B \xrightarrow{\nabla} C, h)
\]
of chain complexes, \( \nabla \) being a morphism of coaugmented differential graded coalgebras;
— an augmented differential graded algebra \( A \);
— twisting cochains \( t_1, t_2 : C \to A \);
— a homotopy \( h^B : B \to A \) of twisting cochains \( h^B : t_1 \nabla \simeq t_2 \nabla \), so that
\[
D(h^B) = (t_1 \nabla) \cup h^B - h^B \cup (t_2 \nabla).
\]
(4.13)

Suppose that the augmented differential graded algebra \( A \) is complete. Then the recursive formula
\[
h^C = h^B \pi - (t_1 \cup h^C - h^C \cup t_2)h
\]
(4.14)
yields a homotopy \( h^C : C \to A \) of twisting cochains \( h^C : t_1 \simeq t_2 \) such that \( h^C \nabla = h^B \).

The formula (4.14) being recursive means that
\[
h^C = \varepsilon \eta + h_1 + h_2 + \ldots
\]
where \( h_1 = h^B \pi - (t_1 - t_2)h \), \( h_2 = -(t_1 \cup h_1 - h_1 \cup t_2)h \), etc.

Proof. The identity \( h^C \nabla = h^B \) is obvious and, since \( t_1 \) and \( t_2 \) are ordinary twisting cochains, the morphism \( t_1 \cup h^C - h^C \cup t_2 \) is (easily seen to be) a cycle. Furthermore, since \( \nabla \) is compatible with the coalgebra structures,
\[
(t_1 \cup h^C - h^C \cup t_2) \nabla \pi = ((t_1 \nabla) \cup (h^C \nabla) - (h^C \nabla) \cup (t_2 \nabla)) \pi
= ((t_1 \nabla) \cup h^B - h^B \cup (t_2 \nabla)) \pi.
\]

Consequently
\[
Dh^C = (D(h^B)) \pi + (t_1 \cup h^C - h^C \cup t_2) Dh
= (t_1 \nabla) \cup h^B - h^B \cup (t_2 \nabla) \pi + (t_1 \cup h^C - h^C \cup t_2) - (t_1 \cup h^C - h^C \cup t_2) \nabla \pi
= t_1 \cup h^C - h^C \cup t_2
\]
as asserted. \( \square \)

Let \((\mathfrak{h}_1, \partial_1)\) and \((\mathfrak{h}_2, \partial_2)\) be two sh-Lie algebras and let
\[
\vartheta_1, \vartheta_2 : S_{\partial_1}^c [\mathfrak{sh}_1] \longrightarrow LS_{\partial_2}^c [\mathfrak{sh}_2]
\]
be two Lie algebra twisting cochains, that is, generalized sh-morphisms or generalized sh-Lie maps from \((\mathfrak{h}_1, \partial_1)\) to \((\mathfrak{h}_2, \partial_2)\). We define a homotopy of generalized sh-morphisms or homotopy of generalized sh-Lie maps from \(\vartheta_1\) to \(\vartheta_2\) to be a homotopy
\[
h : S_{\partial_1}^c [\mathfrak{sh}_1] \longrightarrow ULS_{\partial_2}^c [\mathfrak{sh}_2] = \Omega S_{\partial_2}^c [\mathfrak{sh}_2]
\]
(4.15)
of ordinary twisting cochains \( h : \vartheta_1 \simeq \vartheta_2 \). Here and below we identify a Lie algebra twisting cochain with the corresponding ordinary twisting cochain having values in the corresponding universal algebra.
Remark 4.3. Write $L = \mathcal{L}S^\Delta_2[s\mathfrak{h}_2]$. In view of the definitions, the adjoint of a homotopy $h$ of the kind (4.13) takes the form $\eta: S^\Delta_1[s\mathfrak{h}_1] \rightarrow BUL = B\Omega S^\Delta_2[s\mathfrak{h}_2]$, whence the values of the adjoint $\eta$ of the homotopy (4.13) necessarily lie in the coaugmented differential graded coalgebra $BUL$ rather than in the coaugmented differential graded cocommutative coalgebra $C[L]$, viewed as a subcoalgebra of $BUL$ via the canonical injection $C[L] \rightarrow BUL$. (4.16)

The injection (4.16), in turn, is well known to be a quasi-isomorphism, though.

Historically, the injection (4.16) has played a major role for the development of Lie algebra cohomology, cf. e.g. [1] (Ch. XIII, Theorem 7.1) for the special case of an ordinary (ungraded) Lie algebra. From the point of view of sh-Lie algebras, $C[L]$ would be the correct target for the adjoint of a homotopy of the kind (4.13). To arrive at an adjoint having values in $C[L]$, one would have to require that the values of a homotopy of twisting cochains of the kind (4.15) lie in $L$ rather than in $U[L] = \Omega S^\Delta_2[s\mathfrak{h}_2]$. Such a requirement would lead to inconsistencies, though: The requirement that a homotopy of the kind $h$ be compatible with coalgebra structures forces a condition of the kind (4.11); this condition, in turn, necessarily involves the multiplication map in the universal algebra $UL = \Omega S^\Delta_2[s\mathfrak{h}_2]$ of the corresponding differential graded Lie algebra $L$ (rather than just the graded Lie algebra structure of $L$) and hence cannot be phrased merely in terms of the graded Lie algebra structure alone, whence the values of the homotopy (4.15) cannot in general lie in $L$. Thus, strictly speaking, the notion of homotopy leaves the world of sh-Lie algebras. Again this observation reflects the fact that the notion of homotopy of morphisms of differential graded cocommutative coalgebras is a subtle concept.

Nevertheless, a cure is provided by an appropriate higher homotopies construction: A differential graded coalgebra of the kind $BUL = B\Omega S^\Delta_2[s\mathfrak{h}_2]$ is a quasi-commuted coalgebra, cf. [17] (p. 175); moreover, in the category $DCSH$, cf. [4], the injection (4.16) is an isomorphism (preserving the diagonal maps), and the diagonal map of $BUL = B\Omega S^\Delta_2[s\mathfrak{h}_2]$ is a morphism in the category. Thus, suitably rephrased, the notion of homotopy will stay within the world of sh-Lie algebras. The exploration of categories of the kind $DCSH$ was prompted by [5].

We will now exploit Lemma 4.2 in the following manner: Suppose that $M$ and $\mathfrak{g}$ are connected. Let $B = S^\Delta_2[sM]$, $C = C[\mathfrak{g}]$, consider the contraction (2.14), let $A = ULC[\mathfrak{g}] = \Omega C[\mathfrak{g}]$—notice that $A$ is connected in the sense that $A_0$ is a copy of the ground ring—, and let

$$t_1 = \mathcal{L}(\tau)\vartheta: C[\mathfrak{g}] \rightarrow L\mathcal{C}[\mathfrak{g}],$$
$$t_2 = t_\mathcal{L}: C[\mathfrak{g}] \rightarrow L\mathcal{C}[\mathfrak{g}],$$
$$h^B = \varepsilon\eta.$$

By construction,

$$t_1\tau = t_2\tau: S^\Delta_2[sM] \rightarrow L\mathcal{C}[\mathfrak{g}],$$

and Lemma 4.2 applies. These observations establish the following.

Complement II to Lemma 2.2 Suppose that $\mathfrak{g}$ is connected. The homotopy

$$h^C: C[\mathfrak{g}] \rightarrow ULC[\mathfrak{g}] = \Omega C[\mathfrak{g}]$$
of twisting cochains $h^C : t_1 \simeq t_2$ given by (1.14) yields a homotopy between the composite (1.9) and the identity of $g$, all objects and morphisms in sight being viewed as sh-objects and sh-morphisms.

Constructions of the same kind yield an explicit sh-inverse for (2.20) as a twisting cochain of the kind

$$C[LS^c_0[g]] \to LS^c_0[M]$$

as well, $M$ and $g$ still being supposed to be connected. We spare the reader and ourselves these added troubles here.

## 5 The proof of the theorem in the introduction

Let $\partial$ be an sh-Lie algebra structure on $g$, and let $D$ be the coalgebra perturbation on $S^c[sM]$ and

$$\tau : S^c_{\partial}[sM] \to LS^c_0[g]$$

the Lie algebra twisting cochain (2.20) given in the sh-Lie algebra perturbation lemma (Theorem 2.5 above). The theorem in the introduction comes down to the observation that, with the notation of the previous section, both the adjoint

$$\tau : S^c_{\partial}[sM] \to C[LS^c_0[g]] \quad (5.1)$$

of $\tau$ and the adjoint

$$\tau_L : S^c_{\partial}[sM] \to C[LS^c_0[g]] \quad (5.2)$$

of the universal loop Lie algebra twisting cochain $t_L : S^c_0[g] \to LS^c_0[g]$ yield sh-equivaleces. Under appropriate connectivity hypotheses, constructions similar to those spelled out in the previous section yield explicit sh-inverses for (5.1) and (5.2).

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