The global smooth symmetric solution to 2-D full compressible Euler system of Chaplygin gases

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Abstract

For one dimensional or multidimensional compressible Euler system of polytropic gases, it is well known that the smooth solution will generally develop singularities in finite time. However, for three dimensional Chaplygin gases, due to the crucial role of “null condition” in the potential equation which is derived by the irrotational and isentropic flow, P.Godin in [9] has proved the global existence of a smooth 3-D spherically symmetric flow with variable entropy when the initial data are of small smooth perturbations with compact supports to a constant state. It is noted that there are some essential differences on the global solution or blowup problems between 2-D and 3-D hyperbolic systems. In this paper, we will focus on the global symmetric solution problem of 2-D full compressible Euler system of Chaplygin gases. Through carrying out involved analysis and finding an appropriate weight we can derive some uniform weighted energy estimates on the small symmetric solution to 2-D compressible Euler system of Chaplygin gases and further establish the global existence of smooth solution by continuous induction method.

Keywords: Full compressible Euler system, Chaplygin gases, global existence, null condition, ghost weight, weighted energy estimate

Mathematical Subject Classification 2000: 35L05, 35L72

\(\S 1.\) Introduction and main results

In this paper, we are concerned with the global existence of a smooth symmetric solution to
2-D full compressible Euler system of Chaplygin gases. The 2-D full Euler system is

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, & \text{(Conservation of mass)}, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P &= 0, & \text{(Conservation of momentum)}, \\
\partial_t \left( \rho(e + \frac{|u|^2}{2}) \right) + \text{div} \left( (\rho(e + \frac{|u|^2}{2}) + P)u \right) &= 0, & \text{(Conservation of energy)}, \\
P &= P(\rho, S), & \text{(Equations of state)},
\end{align*}
\]

where \( t \geq 0, x = (x_1, x_2) \in \mathbb{R}^2 \), \( \nabla = (\partial_1, \partial_2) \), and \( u = (u_1, u_2) \), \( \rho, P, e, S \) stand for the velocity, density, pressure, internal energy, specific entropy respectively. Moreover, the pressure function \( P = P(\rho, S) \) and the internal energy function \( e = e(\rho, S) \) are smooth in their arguments. In particular, \( \partial_\rho P(\rho, S) > 0 \) and \( \partial_S e(\rho, S) > 0 \) for \( \rho > 0 \). When \( P(\rho, S) = A \rho^{\gamma} e^{\gamma e} \) and \( e(\rho, S) = \frac{A}{\gamma - 1} \rho^{\gamma - 1} e^{\gamma e} \) hold for some positive constants \( A, c_\gamma \) and \( \gamma (1 < \gamma < 3) \), such flows are then called the polytropic gases.

For the Chaplygin gases, the equation of pressure state (one can see [7] and so on) is given by

\[ P = P_0 - \frac{A(S)}{\rho}, \]

where \( P_0 > 0 \) is a positive constant, \( A(S) \) is a positive smooth function of \( S \), and \( P > 0 \) for \( \rho > 0 \). If \( (\rho, u, S) \in C^1 \) is a solution of (1.1) with \( \rho > 0 \), then (1.1) admits the following equivalent form

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t u + u \cdot \nabla u + \frac{\nabla P}{\rho} &= 0, \\
\partial_t S + u \cdot \nabla S &= 0.
\end{align*}
\]

We pose the symmetric initial data of (1.3) as follows:

\[
\begin{align*}
\rho(0, x) &= \bar{\rho} + \varepsilon \rho_0(r), \\
u(0, x) &= \varepsilon U_0(r) \frac{x}{r}, \\
S(0, x) &= \bar{S} + \varepsilon S_0(r),
\end{align*}
\]

where \( \bar{\rho} > 0 \) and \( \bar{S} \in \mathbb{R} \) are constants, \( \varepsilon > 0 \) is a small parameter, \( r = \sqrt{x_1^2 + x_2^2} \) and \( (\rho_0(r), U_0(r), S_0(r)) \in C^\infty_0(B(0, M)) \) (here and below \( B(0, M) \) stands for a ball centered at the origin with a radius \( M > 0 \)). Moreover, \( \rho(0, x) > 0, P_0 - \frac{A(\bar{S})}{\bar{\rho}} > 0 \) and \( A'(\bar{S}) \neq 0 \) hold.

Our main result in this paper is

**Theorem 1.1.** Under the assumptions above, if \( \varepsilon > 0 \) is small enough, then (1.2)-(1.4) has a global \( C^\infty([0, \infty) \times \mathbb{R}^2) \) solution \( (\rho(t, x), u(t, x), S(t, x)) \) which admits such a symmetric structure: \( (\rho(t, x), u(t, x), S(t, x)) = (\rho(t, r), U(t, r) \frac{x}{r}, S(t, r)) \).
Remark 1.1. For the polytropic gases, it is well-known that the bounded smooth solution of 1-D or multidimensional full compressible Euler system will generally develop singularities in finite time whether the vacuum states appear or not, one can see [1-2], [6-8], [14], [17] [19-23] and the references therein. However, for the Chaplygin gases, by the results of 1-D case in [17], 3-D symmetric case in [9] and our Theorem 1.1 for 2-D symmetric case, the small perturbed symmetric flows will exist globally. Here we point out that the main difference for 2-D (or 3-D) compressible Euler systems of polytropic gases and of Chaplygin gases is: if one neglects the influences of rotations and entropy, the resulting potential equation of polytropic gases, which is a 2-D (or 3-D) quasilinear wave equation, does not fulfill the “null conditions” put forward in [4], [5] and [12] but the related potential equation of Chaplygin gases does. In fact, when the 2-D or 3-D quasilinear wave equations satisfy the null conditions, it is well-known that the small data smooth solutions will exist globally (see [4-5], [11], [12] and [16]), otherwise the smooth solution will blow up in finite time (one can see [3], [10-11] and the references therein).

Remark 1.2. As in Remark 2.2 of [9], Theorem 1.1 still holds when the initial data (1.4) is replaced by \((ρ(0, x), u(0, x), S(0, x)) = (\hat{ρ} + \varepsilon ρ_0(r), \hat{u} + \varepsilon U_0(r) \frac{x}{r}, \tilde{S} + \varepsilon S_0(r))\) where \(\hat{u} \in \mathbb{R}^2\) is a constant vector since the compressible Euler system is invariant under the translation transformation.

Let us give some comments on the proof of Theorem 1.1. As the usual first step to prove the global existence or blowup of a smooth small data solution for the quasilinear hyperbolic equation, one should construct a suitable approximate solution to deal with the related compressible Euler system (1.3) so that both null conditions and the approximate solution can be obtained. Motivated by the methods in [4], we will look for a suitable “ghost weight” to deal with the related compressible Euler system (1.3) so that both null conditions and the
variable entropy can be simultaneously considered. Here we point out that the ghost weight introduced in [4] will not be applied for our case directly since (1.3) can not be changed into a scalar quasilinear wave equation due to the influence of variable entropy, and thus the ghost weight in [4] should be suitably adjusted for our uses (one can see more detailed explanations in Remark 4.1 of §4 below). On the other hand, although some procedures in this paper are somewhat analogous to those in [8-9] for considering the 3-D symmetric Euler system, our analysis is more involved since the decay rate of solution to 2-D free wave equation is lower than that in 3-D case as well as the treatments on both null conditions in 2-D quasilinear wave equation are more complicated than the treatments on one null condition in 3-D quasilinear wave equation (one can compare the reference [4] with [5] and [12]).

The rest of the paper is organized as follows. In §2, we will construct an approximate solution of (1.2)-(1.4) and give some useful preliminary knowledge. In §3 and §4, we will establish the uniform higher order energy estimates on the solution (ρ, u, S) near the light cone and in the whole time-space respectively. Based on these uniform estimates, Theorem 1.1 will be proved by continuous induction method.

In what follows, we will use the following convention as in [8-9] and [21]:

\[ x_0 = t (\geq 0) \] denotes the time variable, \( \partial = (\partial_t, \partial_1, \partial_2) = (\partial_t, \nabla) \);

\[ \Omega = x^+ \cdot \nabla \] with \( x^+ = (-x_2, x_1) \), \( X = t \partial_t + r \partial_r, L_j = t \partial_j + x_j \partial_t \) for \( j = 1, 2 \);

\[ \Lambda = (\Lambda_1, \ldots, \Lambda_7) = (\partial, \Omega, X, L_1, L_2), \Gamma = (\Gamma_1, \ldots, \Gamma_4) = (\partial, X) \);

\[ Z = (Z_1, Z_2) = (\partial_1 + \omega_1 \partial_t, \partial_2 + \omega_2 \partial_t) = \frac{t - r}{t}(\partial_1, \partial_2) + \frac{x_1, x_2}{rt}X + (-x_2, x_1)\Omega \] with \( \omega_i = \frac{x_i}{r} (i = 1, 2) \) and \( \omega = (\omega_1, \omega_2) \).

For \( \mathbb{R}^1 \)-valued smooth functions \( f(t, x) \) and \( \tilde{f}(t, x) \), we put

\[ |f(t)| = \sup_{x \in \mathbb{R}^2} |f(t, x)| \quad |f(t)|_\pm = \sup_{D_\pm} |f(t, x)|, \]

\[ \langle f, \tilde{f} \rangle(t) = \int_{\mathbb{R}^2} f(t, x) \cdot \tilde{f}(t, x) \, dx, \quad \langle f, \tilde{f} \rangle_\pm(t) = \int_{D_\pm(t)} f(t, x) \cdot \tilde{f}(t, x) \, dx, \]

and

\[ \|f(t)\| = \langle f, f \rangle^{1/2}(t), \quad \|f(t)\|_\pm = \langle f, f \rangle_\pm^{1/2}(t), \]

where \( D_-(t) = \{x \in \mathbb{R}^2 : r \leq \frac{t}{2} + M + 1\} \) and \( D_+(t) = \{x \in \mathbb{R}^2 : r \geq \frac{t}{2} + M + 1\} \);

If \( g \) is only a function of \( x \), then \( \|g\| \) represents the usual \( L^2 \) norm;

Define \( \langle \xi \rangle = 1 + |\xi| \) for \( \xi \in \mathbb{R} \) and \( \sigma(x) = \langle x \rangle, \sigma_\pm(t, x) = \langle t \pm |x| \rangle \).

§2. The construction of approximate solution to (1.2)-(1.4) and some preliminaries

Without loss of generality, we will assume the sound speed \( \bar{c} = (\partial_x P(\bar{\rho}, \bar{S}))^{1/2} = 1 \) in the whole paper. As in [8-9], denote \( (\rho_a, u_a, \bar{S}) \) by a solution of (1.3)-(1.4) satisfying the initial data condition \( u_a(0, x) = u_0(0, x) \) and \( P_a(0, x) = P(0, x) \), which means the initial density \( \rho_a(0, x) = \rho_a(0, r) = \frac{A(\bar{S})\rho(0, x)}{A(S(0, x))} \). At this time, since the initial data of \( (\rho_a, u_a, \bar{S}) \) are symmetric and isentropic, one can introduce a potential \( \phi_a(t, r) \) such that \( u_a = \nabla \phi_a \) and \( \phi_a \) satisfies
the following potential equation

\[
\begin{aligned}
\phi_a(0, r) &= -\varepsilon \int_r^M U_0(s) ds, \\
\phi_a(0, r) &= -\frac{1}{2} \varepsilon^2 U_0^2(r) - h(\rho_a(0, r)), \\
\end{aligned}
\]

where $h(\rho) = \frac{1}{2} - \frac{A(\bar{S})}{2\rho^2}$ is the enthalpy. Meanwhile, the density $\rho_a$ is determined by the Bernoulli’s law $h(\rho_a) = -\partial_t \phi_a - \frac{1}{2} |\nabla \phi_a|^2$.

It is easy to verify that (2.1) satisfies both null conditions in two space dimensions (i.e., the first null condition and second null condition, which have been illustrated in §1) posed in [4]. Then (2.1) has a global smooth solution $\phi_a$ in terms of [4] and we have

**Lemma 2.1.** If $\varepsilon > 0$ is small enough, then

1. $|\Lambda^\alpha (\rho_a - \bar{\rho})| + |\Lambda^\alpha u_a(t, x)| \leq C_\alpha \varepsilon \sigma_-(t, x)^{-1} \sigma_+(t, x)^{-1/2}$;
2. $|\partial \Lambda^\alpha (\rho_a - \bar{\rho})| + |\partial \Lambda^\alpha u_a(t, x)| \leq C_\alpha \varepsilon \bar{t}^{5/2}$ if $|x| \leq \bar{t} + M$ with $0 \leq \bar{t} < 1$;
3. $|X^k u_a(t, x)| \leq C_k \varepsilon \bar{t}^{-5/2 - \delta}$ if $0 \leq \delta < 1$ and $|x| \leq C(t)^\delta$,

where $C_\alpha$, $C_k$ and $C$ are some generic positive constants independent of $\varepsilon$ and $(t, x)$.

**Proof.** Since (2.1) satisfies both null conditions in 2-D spaces, then one has by the result of [4] or [13]

\[
|\Lambda^\alpha \phi_a| \leq C_\alpha \varepsilon \sigma_-(t, x)^{-1} \sigma_+(t, x)^{-1/2}.
\]

From this, (1) and (2) can be obtained directly.

In addition, by $X^k u_a(t, x) = (X^k \partial_t \phi_a)(t, r) \frac{x}{r} = \left( \int_0^r \partial_\lambda (X^k \partial_\lambda \phi_a)(t, \lambda) d\lambda \right) \frac{x}{r}$ and (2.2), then (3) holds obviously.

As in [8] and [21], we set $\theta(t, x) = 1 - \frac{A(S(t, x)) \bar{\rho}}{A(S) \rho(t, x)}$, $w(t, x) = u(t, x)$, $z(t, x) = \frac{A(\bar{S})}{A(S(t, x))} - 1$, then it follows from (1.3)-(1.4) that

\[
\begin{aligned}
\partial_t \theta + w \cdot \nabla \theta + (1 - \theta) \nabla \cdot w &= 0, \\
\partial_t w + w \cdot \nabla w + (1 - \theta)(1 + z) \nabla \theta &= 0, \\
\partial_t z + w \cdot \nabla z &= 0, \\
\theta(0, x) &= 1 - \frac{A(\bar{S} + \varepsilon S_0(r)) \bar{\rho}}{A(S) (\bar{\rho} + \varepsilon \rho_0(r))}, \\
w(0, x) &= \varepsilon U_0(r) \frac{x}{r}, \\
z(0, x) &= \frac{A(\bar{S})}{A(S + \varepsilon S_0(r))} - 1.
\end{aligned}
\]
Corresponding to the approximate solution \((\rho_a, u_a, S)\), define \(\theta_a(t, x) = 1 - \frac{\tilde{\rho}}{\rho_a(t, x)}\) and \(w_a(t, x) = u_a(t, x)\). Then a direct computation yields by Lemma 2.1 that

**Lemma 2.2.** For small \(\varepsilon > 0\), we have

1. \(|\Lambda^\alpha \theta_a + \Lambda^\alpha w_a|(t, x) \leq C_\alpha \varepsilon \sigma_-(t, x)^{-1} \sigma_+(t, x)^{-1/2} \); 
2. \(|\partial \Lambda^\alpha \theta_a(t, x)| + |\partial \Lambda^\alpha w_a(t, x)| \leq C_\alpha \varepsilon(t)^{-5/2} \) if \(|x| \leq lt + M\) with \(0 \leq l < 1\); 
3. \(|X^k w_a(t, x)| \leq C_k \varepsilon(t)^{-5/2 + \delta} \) if \(0 \leq \delta < 1 \) and \(|x| \leq C(t)^{\delta}\).

Let \(\hat{\theta} = \theta - \theta_a, \hat{w} = w - w_a\) and \(\hat{z} = z - z(0, x)\), then we have by (2.3)

\[
\begin{align*}
\partial_t \hat{\theta} + \nabla \cdot \hat{w} &= -(w \cdot \nabla \theta - w_a \cdot \nabla \theta_a) + (\theta \nabla \cdot w - \theta_a \nabla \cdot w_a), \\
\partial_t \hat{w} + \nabla \hat{\theta} &= -(w \cdot \nabla w - w_a \cdot \nabla w_a) + (\theta \nabla \theta - \theta_a \nabla \theta_a) - (1 - \theta) z \nabla \theta, \\
\partial_t \hat{z} &= -w \cdot \nabla (z(0, x) + \hat{z}), \\
\hat{\theta}(0, x) &= 0, \quad \hat{w}(0, x) = 0, \quad \hat{z}(0, x) = 0.
\end{align*}
\]

(2.4)

To establish the global existence of solution to (2.4) in subsequent sections, we require to give some preliminary analysis on the related energies. As usual, we define the energy for \(n \in \mathbb{N} \cup \{0\}\)

\[E_n(t) = \sum_{|\alpha| \leq n} (||\Gamma^\alpha \hat{\theta}(t)||^2 + ||\Gamma^\alpha \hat{w}(t)||^2 + ||\Gamma^\alpha \hat{z}(t)||^2).\]

We also set \(Q_n(t) = \sum_{|\alpha| \leq n-1} \left(||\sigma_-(t) \nabla \Gamma^\alpha \hat{\theta}(t)|| + ||\sigma_-(t) \partial_t \Gamma^\alpha \hat{\theta}(t)|| + ||\sigma_-(t) \nabla \cdot \Gamma^\alpha \hat{w}(t)||\right)\)

for \(n \geq 1\), and define \(\hat{Q}_n(t) = Q_n(t) + E_{n-1}^{1/2}(t)\) for \(n \geq 1\) and \(\hat{Q}_n(t) = Q_n(t) + E_{n-2}^{1/2}(t)\) for \(n \geq 2\) as in [8-9] and [21].

In addition, for our requirements to treat the 2-D full Euler system (1.3)-(1.4), it is necessary to introduce some kinds of “interior energies” as follows:

Choose a smooth function

\[
\hat{\chi}(s) = \begin{cases} 
1, & \text{if } s \leq 1/2, \\
0, & \text{if } s \geq 3/4,
\end{cases}
\]

and set \(\chi(t, x) = \hat{\chi} \left(\frac{|x|}{t + 2M + 2}\right)\), then we define for \(n \geq 1\)

\[
Q_n^{-}(t) = \sum_{|\alpha| \leq n-1} \left(||\sigma_-(t) \nabla \Gamma^\alpha (\chi \hat{\theta})(t)|| + ||\sigma_-(t) \partial_t \Gamma^\alpha (\chi \hat{\theta})(t)|| + ||\sigma_-(t) \nabla \cdot \Gamma^\alpha (\chi \hat{w})(t)||\right),
\]

\[
\hat{Q}_n^{-}(t) = Q_n^{-}(t) + E_{n-1}^{1/2}(t), \quad \text{for } n \geq 1,
\]

\[
\hat{Q}_n^{-}(t) = Q_n^{-}(t) + E_{n-2}^{1/2}(t), \quad \text{for } n \geq 2.
\]

The following relations and properties on the above defined energies will be repeatedly utilized later on.
Lemma 2.3.

(1) \(|\sigma_-(t)\nabla v| \leq C(\|\sigma_-(t)\nabla \cdot v\| + \|v\|) \) if \(v \in C^\infty_0(\mathbb{R}^2, \mathbb{R}^2)\) and \(\nabla \cdot v = 0\),

(2) \(|\sigma^{1/2}\Gamma^\alpha w(t)| + |\sigma^{1/2}\Gamma^\alpha \dot{\theta}(t)| + |\sigma^{1/2}\Gamma^\alpha \dot{z}(t)| \leq C_\alpha E^{1/2}_{|\alpha|+2}(t),

(3) \(|\sigma^{1/2}\sigma_-(t)\nabla\Gamma^\alpha \dot{\theta}(t)| \leq C_\alpha Q_{|\alpha|+3}(t),

(4) \(|\sigma^{1/2}\sigma_-(t)\nabla\Gamma^\alpha (\chi \dot{\theta})(t)| \leq C_\alpha Q_{|\alpha|+3}(t),

(5) \(|\sigma^{1/2}\sigma_-(t)\nabla\Gamma^\alpha \dot{w}(t)| \leq C_\alpha \tilde{Q}_{|\alpha|+3}(t),

(6) \(|\sigma^{1/2}\sigma_-(t)\nabla\Gamma^\alpha (\chi \dot{w})(t)| \leq C_\alpha \tilde{Q}_{|\alpha|+3}(t),

(7) \(|\sigma^{1/2}\sigma_-(t)\Gamma^\alpha \dot{\theta}(t)| \leq C_\alpha \tilde{Q}_{|\alpha|+2}(t),

(8) \(|\sigma^{1/2}\sigma_-(t)\Gamma^\alpha (\chi \dot{\theta})(t)| \leq C_\alpha \tilde{Q}_{|\alpha|+2}(t),

(9) \(|\sigma^{1/2}\sigma_-(t)\Gamma^\alpha \dot{w}(t)| \leq C_\alpha \tilde{Q}_{|\alpha|+2}(t),

(10) \(|\sigma^{1/2}\sigma_-(t)\Gamma^\alpha (\chi \dot{w})(t)| \leq C_\alpha \tilde{Q}_{|\alpha|+2}(t).

Proof. (1) comes from the formula (6.7) in [21] directly.

In addition, according to Lemma 1 of [21], one has for any smooth function \(f(x)\)

\[
\sigma(x)^{1/2}|f(x)| \leq C \sum_{j=0}^{1} \sum_{|\alpha|=0}^{2-j} \|\nabla^\alpha \Omega^j f\|,
\]

\[
\sigma(x)^{1/2}|\sigma_-(t,x)|f(x)| \leq C \sum_{j=0}^{1} \sum_{|\alpha|=0}^{2-j} \|\sigma_-(t)\nabla^\alpha \Omega^j f\|,
\]

\[
\sigma(x)^{1/2}|\sigma_-(t,x)^{1/2}|f(x)| \leq C \sum_{j=0}^{1} \left( \|\Omega^j f\| + \sum_{|\alpha|=1}^{2-j} \|\sigma_-(t)\nabla^\alpha \Omega^j f\| \right).
\]

This, together with the facts of \(\nabla \cdot (\Gamma^\alpha (\chi \dot{w})) = 0\) and \([\Omega, \Gamma^\alpha] = \sum_{|\beta| \leq |\alpha|} C_{\alpha \beta} \Gamma^\beta\), yields (2) – (10) directly. \(\square\)

Applying \(\partial^\alpha (X + 1)^k\) to (2.4) and setting \(\Gamma^\mu = \partial^\alpha X^k\), then we have

\[
\begin{align*}
\partial_t \Gamma^\mu \dot{\theta} + \nabla \cdot \Gamma^\mu \dot{w} &= h^\mu_0 \equiv \sum_{1 \leq j \leq 6} f^\mu_j, \\
\partial_t \Gamma^\mu \dot{w} + \nabla \Gamma^\mu \dot{\theta} &= h^\mu \equiv \sum_{7 \leq j \leq 13} f^\mu_j,
\end{align*}
\]

\[
f^\mu_1 = -\sum_{\nu \leq \mu} \binom{\mu}{\nu} \Gamma^\nu w_\alpha \cdot \nabla \Gamma^{\mu-\nu} \dot{\theta}, \quad f^\mu_2 = -\sum_{\nu \leq \mu} \binom{\mu}{\nu} \Gamma^\nu \dot{w} \cdot \nabla \Gamma^{\mu-\nu} \theta_\alpha,
\]
\[ f_3^\mu = - \sum \left( \frac{\mu}{\nu} \right) \Gamma^\nu \dot{w} \cdot \nabla \Gamma^{\mu-\nu} \dot{\theta}, \quad f_4^\mu = \sum \left( \frac{\mu}{\nu} \right) \Gamma^\nu \partial_a \cdot \Gamma^{\mu-\nu} \dot{w}, \]
\[ f_5^\mu = \sum \left( \frac{\mu}{\nu} \right) \Gamma^\nu \partial \nabla \cdot \Gamma^{\mu-\nu} w, \quad f_6^\mu = \sum \left( \frac{\mu}{\nu} \right) \Gamma^\nu \dot{w} \cdot \nabla \Gamma^{\mu-\nu} w, \]
\[ f_7^\mu = - \sum \left( \frac{\mu}{\nu} \right) \Gamma^\nu w \cdot \nabla \Gamma^{\mu-\nu} \dot{w}, \quad f_8^\mu = - \sum \left( \frac{\mu}{\nu} \right) \Gamma^\nu \dot{w} \cdot \nabla \Gamma^{\mu-\nu} w, \]
\[ f_9^\mu = - \sum \left( \frac{\mu}{\nu} \right) \Gamma^\nu \dot{w} \cdot \nabla \Gamma^{\mu-\nu} \dot{w}, \quad f_{10}^\mu = \sum \left( \frac{\mu}{\nu} \right) \Gamma^\nu \partial_a \nabla \Gamma^{\mu-\nu} \dot{\theta}, \]
\[ f_{11}^\mu = \sum \left( \frac{\mu}{\nu} \right) \Gamma^\nu \dot{\theta} \nabla \Gamma^{\mu-\nu} \dot{\theta}, \quad f_{12}^\mu = \sum \left( \frac{\mu}{\nu} \right) \Gamma^\nu \dot{\theta} \nabla \Gamma^{\mu-\nu} \dot{\theta}, \]
\[ f_{13}^\mu = - \partial^\alpha (X + 1)^k (1 - \theta) \partial \nabla \theta, \]

Remark 2.1. For the free wave equation with compactly supported initial data, it is well-known that the decay rate of smooth solution on the time 2 in 2-D case is slower than that in 3-D case, therefore the author in [9] can obtain an energy estimate of \( Q_n(t) \) similar to (2.10) in the whole space \( \mathbb{R}^3 \) (one can see Proposition 4.1 of [9]) but at present we only get (2.10) for \( Q_n(t) \) which is a kind of interior energy.
Proof. By (2.6), we have for $|\mu| \leq n-1$

$$
\begin{align*}
\partial_t \Gamma^\mu(\chi \dot{\theta}) + \nabla \cdot \Gamma^\mu(\chi \dot{w}) &= \tilde{h}_0^\mu = \sum_{0 \leq j \leq 6} \tilde{f}_j^\mu, \\
\partial_t \Gamma^\mu(\chi \dot{w}) + \nabla \Gamma^\mu(\chi \dot{\theta}) &= \tilde{h}^\mu = \sum_{7 \leq j \leq 14} \tilde{f}_j^\mu,
\end{align*}
$$

(2.11)

where

$$
\tilde{f}_0^\mu = \partial^\mu (X + 1)^k \left\{ \frac{\chi'}{t + 2M + 2} \left( -\frac{r}{t + 2M + 2} \dot{\theta} + \dot{w} \cdot \omega + w \cdot \omega \dot{\theta} - \theta \omega \cdot \dot{w} \right) \right\},
$$

$$
\tilde{f}_3^\mu = -\sum_{\nu \leq \mu} \left( \mu \nu \right) \Gamma^{\nu} \dot{w} \cdot \nabla \Gamma^{\mu-\nu}(\chi \dot{\theta}), \quad \tilde{f}_6^\mu = \sum_{\nu \leq \mu} \left( \mu \nu \right) \Gamma^{\nu} \dot{\theta} \nabla \cdot \Gamma^{\mu-\nu}(\chi \dot{w}),
$$

and for $i = 1, 2, 4, 5, 7, 8, 10, 11$, the expressions of $\tilde{f}_i^\mu$ are the same as $f_i^\mu$ in (2.6) when $\dot{\theta}$ or $\dot{w}$ in $f_i^\mu$ is replaced by $\chi \dot{\theta}$ or $\chi \dot{w}$ respectively.

By (2.11) together with the similar expressions of (2.7)-(2.9), an easy computation yields

$$
Q_n^\nu(t) \leq C \left( E_n^{1/2}(t) + \sum_{|\mu| \leq n-1} t \| \tilde{h}_0^\mu(t) \| + \sum_{|\mu| \leq n-1} \langle t \rangle \| \tilde{h}^\mu(t) \| \right).
$$

(2.12)

We now focus on the estimates of $\| \tilde{h}_0^\mu(t) \|$ and $\| \tilde{h}^\mu(t) \|$ in the right hand side of (2.12). According to Lemma 2.2 and by direct observations, we can obtain

$$
\| \tilde{f}_j^\mu \| \leq C_n \varepsilon(t)^{-3/2} E_n^{1/2}(t), \quad j \in \{ 1, 2, 4, 5, 7, 8, 10, 11 \},
$$

(2.13)

and

$$
\| \tilde{f}_0^\mu(t) \| + \| \tilde{f}_1^\mu(t) \| \leq C_n(t)^{-1} E_n^{1/2}(t).
$$

(2.14)

In addition, applying Lemma 2.3 (2) and (4) respectively yield that

if $|\nu| \leq |\mu - \nu|$, then

$$
\| \Gamma^{\nu} \dot{w} \cdot \nabla \Gamma^{\mu-\nu}(\chi \dot{\theta})(t) \| \leq C(t)^{-1} |\Gamma^{\nu} \dot{w}(t) \cdot |\sigma_{-} (t) \nabla \Gamma^{\mu-\nu}(\chi \dot{\theta})(t) \| \leq C_n(t)^{-1} E_n^{1/2}(t) Q_{n-1}^{-}(t);
$$

if $|\nu| > |\mu - \nu|$, then

$$
\| \Gamma^{\nu} \dot{w} \cdot \nabla \Gamma^{\mu-\nu}(\chi \dot{\theta})(t) \| \leq |\nabla \Gamma^{\mu-\nu}(\chi \dot{\theta})(t) \cdot |\Gamma^{\nu} \dot{w}(t) \| \leq C_n(t)^{-1} Q_{[\frac{n}{2}]+2}(t) E_n^{1/2}(t).
$$

Therefore, one has

$$
\| \tilde{f}_3^\mu \| \leq C_n(t)^{-1} \left( E_n^{1/2}(t) Q_{n-1}^{-}(t) + Q_{[\frac{n}{2}]+2}(t) E_n^{1/2}(t) \right).
$$

(2.15)
Similarly,
\[
\|\tilde{f}_j^\mu\| \leq C_n \langle t \rangle^{-1} (E_n^{1/2} + 2) Q_n^{-\mu} (t) + \tilde{Q}_n^{-\mu} (t) E_n^{1/2} (t), \quad j \in \{6, 9, 12\}. \tag{2.16}
\]

Finally, we deal with \(\tilde{f}_{13}^\mu\).

Set
\[
\begin{align*}
L_{1\nu}(t) &= \|\Gamma^\nu z(0, x) \nabla \Gamma^{\mu - \nu}(\chi_{\theta_a}) (t)\|, \\
L_{2\nu}(t) &= \|\Gamma^\nu \dot{z}(0, x) \nabla \Gamma^{\mu - \nu}(\chi \dot{\theta}) (t)\|, \\
L_{3\nu}(t) &= \|\Gamma^\nu \dot{z} \nabla \Gamma^{\mu - \nu}(\chi_{\theta_a}) (t)\| + \|\partial^\nu (X + 1)^k \left( \frac{\chi'}{t + 2M + 2} \dot{z} \theta_{a\omega} \right) (t)\|, \\
L_{4\nu}(t) &= \|\Gamma^\nu \dot{z} \nabla \Gamma^{\mu - \nu}(\chi \dot{\theta}) (t)\| + \|\partial^\nu (X + 1)^k \left( \frac{\chi'}{t + 2M + 2} \dot{z} \dot{\theta} \omega \right) (t)\|.
\end{align*}
\]

It is easy to obtain
\[
\begin{align*}
L_{1\nu}(t) &\leq C_n \varepsilon^2 \langle t \rangle^{-5/2}, \\
L_{2\nu}(t) &\leq C_n \varepsilon \langle t \rangle^{-1} Q_n^{-\mu} (t), \\
L_{3\nu}(t) &\leq C_n \varepsilon \langle t \rangle^{-5/2} E_n^{1/2} (t), \\
L_{4\nu}(t) &\leq C_n \langle t \rangle^{-1} \left( E_n^{1/2} (t) Q_n^{-\mu} (t) + Q_n^{-\mu} (t) E_n^{1/2} (t) \right) + C_n \langle t \rangle^{-3/2} E_n^{1/2} (t) E_n^{1/2} (t).
\end{align*}
\]

This, together with the expression of \(\tilde{f}_{13}^\mu\), yields
\[
\|\tilde{f}_{13}^\mu\| \leq C_n \left( 1 + E_n^{1/2} \left[ \frac{1}{t} \right] + 2 \right) F_{n-1} (t) + C_n E_n^{1/2} (t) F_{[\frac{1}{t}] + 1} (t), \tag{2.17}
\]
where \(F_{j-1}(t)\) is defined as
\[
\begin{align*}
F_{j-1}(t) &= \varepsilon^2 \langle t \rangle^{-5/2} + E_n^{1/2} (t) \varepsilon \langle t \rangle^{-5/2} + \langle t \rangle^{-1} Q_n^{-\mu} (t) + \langle t \rangle^{-1} Q_n^{-\mu} \varepsilon (t) + E_n^{1/2} (t) E_n^{1/2} (t) F_{j-1} (t) \quad \text{for } j \geq 1.
\end{align*}
\]

Due to \(E_n^{1/2} (t) \leq \eta\), (2.12) together with (2.13)-(2.17) derives
\[
Q_n^{-\mu} (t) \leq C_n \varepsilon (1 + Q_n^{-\mu} (t)) + C_n \varepsilon^2 \langle t \rangle^{-3/2},
\]
which means for small \(\eta\)
\[
Q_n^{-\mu} (t) \leq C_n \left( \eta + \frac{\varepsilon^2}{\langle t \rangle^{3/2}} \right). \tag{2.18}
\]

And hence, \(F_{n-1}(t) \leq C_n \langle t \rangle^{-1} \left( Q_n^{-\mu} (t) (\varepsilon + \eta) + \varepsilon E_n^{1/2} (t) + \frac{\varepsilon^2}{\langle t \rangle^{3/2}} \right) \) and \(F_{[\frac{1}{t}] + 1} (t) \leq C_n \langle t \rangle^{-1} \left( \eta (\varepsilon + \eta) + \frac{\varepsilon^2}{\langle t \rangle^{3/2}} \right) \). Substituting this into (2.17) and further combining with (2.12)-(2.16) yield
\[
Q_n^{-\mu} (t) \leq C_n \left( E_n^{1/2} (t) + \frac{\varepsilon^2}{\langle t \rangle^{3/2}} \right).
\]
\[\square\]
Next, we derive the decay estimate of solution $(\hat{\theta}, \hat{w})$ inside the light cone, which is analogous to Proposition 4.2 in [9]. This is relatively easier to be obtained than the one near the cone in subsequent §3.

**Lemma 2.5.** For fixed $\lambda \in \mathbb{N}$ with $\lambda \geq 4$, if $(\hat{\theta}, \hat{w}, \hat{z})$ is a smooth solution of (2.4) for $(t, x) \in [0, T] \times \mathbb{R}^2$ with $E_\lambda(t) \leq \varepsilon^2$, and $\tilde{Q}_{|\mu|+2}(t) \leq \eta(t)^{1/2}$ holds for $|\mu| \geq \lambda - 1$ and small constant $\eta > 0$, then

1. $|\nabla \Gamma(\hat{\theta})(t)| + |\nabla \cdot \Gamma(\hat{w})(t)| \leq C|\chi(t)|$ for $|\mu| \leq 2\lambda - 4$, where and below $\chi(t) \equiv \tilde{Q}_{k+3}(t)/(t)^{3/2} + \varepsilon^2/(t)^{5/2}$.

2. $|\partial_t \Gamma(\hat{\theta})(t)| + |\partial_t \Gamma(\hat{w})(t)| \leq C|\chi(t)|$ for $|\mu| \leq 2\lambda - 4$.

3. $|\Gamma(\hat{w})(t)| \leq C|\chi(t)|$ for $|\mu| \leq 2\lambda - 3$ and $\mu_2 + \mu_3 \neq 0$.

4. $|\Gamma(\hat{w})(t, x)| \leq C|\chi(t)|$ for $|\mu| \leq 2\lambda - 4$, $\mu_2 + \mu_3 = 0$, and $r \leq t/M + 1$.

**Proof.** (1) If $t \leq 8M + 7$, (1) can be got directly by Lemma 2.3 (4) and (6). Otherwise, we know that $t^2 - r^2$ is equivalent to $t^2$ when $r \leq \frac{3}{4}t + \frac{3}{2}(M + 1)$. Set $m_\mu(t) = |\nabla \Gamma(\hat{\theta})(t)| + |\nabla \cdot \Gamma(\hat{w})(t)|$ and $M_\mu(t) = \sum_{|\mu| \leq |\mu|} m_\nu(t)$. According to the expressions (2.7)-(2.9) and Lemma 2.3, we have

$$m_\mu(t) \leq C \left( \frac{\tilde{Q}_{|\mu|+3}(t)}{(t)^{3/2}} + \sum_{j=1}^{13} |\tilde{f}_j^{\mu}(t)| \right), \tag{2.19}$$

due to the applied $|\tilde{f}_j^{\mu}(t)| + |\tilde{f}_j^{\mu}(t)| \leq C(t)^{-3/2}E_{|\mu|+2}(t) \leq C(t)^{-3/2}Q_{|\mu|+3}(t)$.

We now treat $\sum_{j=1}^{13} |\tilde{f}_j^{\mu}|$.

If $\Gamma^\nu = \partial^\nu X^k$, then $\Gamma^\nu(\hat{w})(t, x) = \frac{1}{r} \left( \int_0^r s(\nabla \cdot \Gamma^\nu(\hat{w}))(s) \, ds \right) \frac{x}{r}$. At this time, it follows the inequality in Lemma 2.1 (c) of [1] that

$$|\partial^\nu \Gamma^\nu(\hat{w})(t)| \leq C_{\alpha} \sum_{j \leq |\alpha|-1} |\nabla \cdot \partial^\alpha_j \Gamma^\nu(\hat{w})(t)|, \tag{2.20}$$

This, together with Lemma 2.3 and the expressions of $\tilde{f}_j^{\mu}$ ($1 \leq j \leq 12$), yields

$$\sum_{j=1}^{12} |\tilde{f}_j^{\mu}(t)| \leq C \left( \frac{\varepsilon \tilde{Q}_{|\mu|+2}(t)}{(t)^{3/2}} + E_{|\mu|+2}(t)M_\mu(t) + \frac{\varepsilon}{(t)^{3/2}}M_\mu(t) \right), \tag{2.21}$$

or

$$\sum_{j=1}^{12} |\tilde{f}_j^{\mu}(t)| \leq C \left( \frac{\varepsilon \tilde{Q}_{|\mu|+2}(t)}{(t)^{3/2}} + \tilde{Q}_{|\mu|+2}(t)M_\mu(t) + \frac{\varepsilon}{(t)^{3/2}}M_\mu(t) \right). \tag{2.22}$$

Noticing $\tilde{f}_j^{\mu} = -\partial^\mu(Xt + 1)^k(\chi z \nabla \theta) + \partial^\mu(Xt + 1)^k(\chi \theta z \nabla \theta)$ and $\tilde{Q}_{|\mu|+2}(t) \leq C\varepsilon$ for $|\mu| \leq \lambda - 2$ in terms of Lemma 2.4, then by the assumption $\tilde{Q}_{|\mu|+2}(t) \leq \eta(t)^{1/2}$ for $|\mu| \geq \lambda - 1$, the lemma holds.
Lemma 2.3 and Lemma 2.2, we can obtain that $|\partial^{\alpha}X^{k}\theta(t)|$ is bounded for $|\alpha| \leq |\alpha|$, $k \leq k$ and further arrive at

$$
|\tilde{f}_{13}^{\mu}(t)| \leq C \left( \varepsilon \langle t \rangle^{-5/2} + \varepsilon M_{\mu}(t) + \varepsilon \langle t \rangle^{-5/2} E_{|\mu|+2}^{1/2}(t) + \sum_{\nu \leq \mu} E_{|\mu|+2}^{1/2}(t) m_{\mu-\nu}(t) \right)
$$

$$
+ \langle t \rangle^{-2} E_{|\mu|+2}^{1/2} Q^{-}_{|\mu|+3}(t).
$$

(2.23)

For $|\mu| \leq \lambda - 2$, collecting (2.21), (2.23) yields

$$
\sum_{j=1}^{13} |\tilde{f}_{j}^{\mu}(t)| \leq C \left( \frac{\varepsilon}{\langle t \rangle^{2}} Q^{-}_{|\mu|+3}(t) + \frac{\varepsilon^{2}}{\langle t \rangle^{5/2}} + \varepsilon M_{\mu}(t) \right),
$$

(2.24)

which together with (2.19) means (1) holds.

For $\lambda - 1 \leq |\mu| \leq 2\lambda - 4$, due to $E_{|\mu|+2}^{1/2}(t) m_{\mu-\nu}(t) \leq \varepsilon M_{\mu}(t)$ if $|\nu| \leq \lambda - 2$ and $E_{|\mu|+2}^{1/2}(t) m_{\mu-\nu}(t) \leq C \varepsilon t^{-3/2} E_{|\mu|+2}^{1/2}(t)$ otherwise, then by (2.22), (2.23)

$$
\sum_{j=1}^{13} |\tilde{f}_{j}^{\mu}(t)| \leq C \left( \frac{\varepsilon}{\langle t \rangle^{2}} Q^{-}_{|\mu|+3}(t) + (\varepsilon + \eta) M_{\mu}(t) + \frac{\varepsilon^{2}}{\langle t \rangle^{5/2}} + \frac{\varepsilon}{\langle t \rangle^{3/2}} E_{|\mu|+2}^{1/2}(t) \right).
$$

This, together with (2.19), also yields (1).

(2) If $t$ is small, the estimate can be obtained by Lemma 2.3 easily. Otherwise, (2) follows from (2.11) and (1).

(3) If $\mu_{2} + \mu_{3} \neq 0$, as shown in (2.20), then we have for $|\mu| \leq 2\lambda - 3$

$$
|\Gamma^{\mu}(\chi\tilde{w})(t)| \leq C \sum_{|\alpha| \leq \mu_{2} + \mu_{3} - 1} |\nabla \cdot \partial_{x}^{\alpha} \partial_{t}^{\mu_{1}} X^{\mu_{4}}(\chi\tilde{w})(t)| \leq C \chi_{|\mu|-1}(t).
$$

(iv) If $\mu_{2} + \mu_{3} = 0$, then $\Gamma^{\mu}(\chi\tilde{w})(t, x)$ is bounded for $|x| \geq r(t)$ by the compact support property of $\tilde{z}(0, x)$.

When $\varepsilon > 0$ is small enough, as in [8], we next show that (1.3) is isentropic outside the fixed ball $B(0, M + 1)$ for any time $t$.

**Lemma 2.6.** For small $\varepsilon > 0$, if $(\tilde{\theta}, \tilde{w}, \tilde{z})$ is a smooth solution to (2.4) for $(t, x) \in [0, T] \times \mathbb{R}^{2}$ and $E_{\lambda}(t) \leq \varepsilon^{2}$ with $\lambda \geq 4$, then $\tilde{z}(t, x) = 0$ for $|x| \geq M + 1$.

**Proof.** From the third equation in (2.3), we know that $\partial_{t} z + W(t, r) \partial_{r} z = 0$, where $W(t, r) = w(t, r) \cdot \frac{x}{r}$. Define the characteristics $r = r(t)$ by $r(t) = W(t, r(t))$ with $r(0) = M$, then it is easy to see that $z(t, x) = 0$ for $|x| \geq r(t)$ by the compact support property of $\tilde{z}(0, x)$.

By Lemma 2.2 (1) and Lemma 2.3 (9) together with Lemma 2.4, we have $|w(t)| \leq |w_{a}(t)| + |\tilde{w}(t)| \leq C \varepsilon^{3/2} + C(t)^{-1/2} Q^{-}_{2}(t) \leq C \varepsilon^{3/2}$. In addition, it follows from Lemma 2.3 (2) that $|w(t, x)| \leq |w_{a}(t, x)| + |\tilde{w}(t, x)| \leq C(t)^{-1/2}(\varepsilon + E_{2}^{1/2}(t)) \leq C \varepsilon(t)^{-1/2}$ holds for $|x| \geq \frac{t}{2} + M + 1$. Therefore, $|r(t)| \leq (M + 1)(t)^{1/2}$. From this, we can take
use of Lemma 2.2 (3) and Lemma 2.5 (4) to get \(|W(t, r(t))| \leq C \varepsilon(t)^{-1}\), and hence \(|r(t)| \leq (M + 1) \varepsilon(t)^{\delta}\) for any \(0 < \delta < \frac{1}{2}\). Applying Lemma 2.2 (3) and Lemma 2.5 (4) again, we have \(|r(t)| \leq C \varepsilon(t)^{-3/2+\delta}\) and further \(|r(t) - M| \leq C \varepsilon\). Consequently, \(z(t, r) \equiv 0\) as \(|x| \geq M + 1\), so does \(\dot{z}\).

\[\] 

§3. Analysis on \((\dot{\theta}, \dot{w}, \dot{z})\) near light cone and establishment of a crucial energy estimate

From Lemma 2.6, we know that \(z(t, x) \equiv 0\) and then \(S \equiv \tilde{S}\) holds when \(|x| \geq M + 1\). Assume that \((\dot{\theta}, \dot{w}, \dot{z})\) is a smooth solution to (2.4) for \(0 \leq t \leq T\), and let \(\phi \in C^\infty_{\text{loc}}([0, T] \times \mathbb{R}^2 : |x| \geq M + 1)\) be the corresponding potential vanishing in \(|x| \geq M + 1\) and satisfying \(\nabla \phi = u, \partial_t \phi = -\frac{1}{2}|u|^2 - h(\rho)\). For \((\lambda, y), (\tilde{\lambda}, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^2\), we define

\[F_1(\lambda, y) = \frac{1}{2}(\lambda^2 - |y|^2),\]
\[F_2((\lambda, y), (\tilde{\lambda}, \tilde{y})) = \frac{1}{2}(\lambda \tilde{\lambda} - y \cdot \tilde{y}).\]

Let \(\dot{\phi} = \phi - \phi_a, \xi = (\theta, w), \xi_a = (\theta_a, w_a)\) and \(\dot{\xi} = \xi - \xi_a = (\dot{\theta}, \dot{w})\), where \(\phi_a\) and \((\theta_a, w_a)\) are given in (2.1) and Lemma 2.2 respectively. Then we have

\[\begin{align*}
\theta_a(t, x) &= -\partial_t \phi_a(t, x) + F_1(\xi_a)(t, x), \\
\theta(t, x) &= -\partial_t \phi(t, x) + F_1(\xi)(t, x) \quad \text{for} \quad |x| \geq M + 1, \\
\dot{\theta}(t, x) &= -\partial_t \dot{\phi}(t, x) + F_2(\dot{\xi}, 2\xi_a + \dot{\xi})(t, x) \quad \text{for} \quad |x| \geq M + 1.
\end{align*}\]

In order to make use of both null conditions introduced in [4] for the second order quasilinear wave equations, we will pay more attention to the forms of \(F_1(\xi_a)\) and \(F_2(\dot{\xi}, 2\xi_a + \dot{\xi})\), that is, if \(|x| \geq M + 1\), then

\[\begin{align*}
F_1(\xi_a) &= \frac{1}{2}(\partial_t \phi_a)^2 - \frac{1}{2}|\nabla \phi_a|^2 + F_1(\xi_a)(\theta_a - \frac{1}{2}F_1(\xi_a)), \\
F_2(\dot{\xi}, 2\xi_a + \dot{\xi}) &= \frac{1}{2}\partial_t \dot{\phi}(2\partial_t \phi_a + \partial_t \dot{\phi}) - \frac{1}{2} \nabla \dot{\phi} \cdot (2\nabla \phi_a + \nabla \dot{\phi}) + \frac{1}{2} F_2(\dot{\xi}, 2\xi_a + \dot{\xi})(2\theta_a + \dot{\theta}) \\
&\quad - \frac{1}{2} (F_2(\dot{\xi}, 2\xi_a + \dot{\xi}) - \theta) (2F_1(\xi_a) + F_2(\xi, 2\xi_a + \dot{\xi})).
\end{align*}\]

Next we cite two fundamental estimates on the first and second null conditions which are shown in [4], Lemma 6.64-Lemma 6.65 of [11] or [13] (one can see Lemma 3.1-Lemma 3.5 of [13] for some details).

**Lemma 3.1.** Assume that \(g_{ij}^{jk} \in \mathbb{R}\) with \(g_{ij}^{kj} = g_{ij}^{kj}\) \((0 \leq i, j, k \leq 2)\) and \(\sum_{i,j,k=0}^{2} g_{ij}^{ik} p_i p_j p_k = 0\) for any \(p = (p_0, p_1, p_2) \in \mathbb{R}^3\) satisfying \(p_0^2 = p_1^2 + p_2^2\). Then, there exists a positive constant \(C\) such that for any \(f, g \in C^\infty([0, T] \times \mathbb{R}^2)\),

\[|\sum_{i,j,k=0}^{2} g_{ij}^{jk}(\partial_t f \partial_{jk}^2 g)(t, x)| \leq C(|Z f(t, x)||\partial^2 g(t, x)| + |\partial f(t, x)||Z \partial g(t, x)|),\]
and
\[
\sum_{|\alpha| \leq n} |\Gamma^\alpha (\sum_{i,j,k=0}^2 g^{ij}_k \partial_i f \partial_{jk}^2 f)(t, x) - \sum_{i,j,k=0}^2 g^{ij}_k (\partial_i f \partial_{jk}^2 \Gamma^\alpha f)(t, x)| \\
\leq C_n \sum_{|\beta+\nu| \leq n+1, |\beta|, |\nu| \leq n} |Z \Gamma^\beta f(t, x)||\Gamma^\nu \partial f(t, x)|. \tag{3.4}
\]

**Lemma 3.2.** Assume that \(g_{ij}^{kl} \in \mathbb{R}\) with \(g_{ij}^{kl} = g_{ij}^{lk}\) and \(g_{ij}^{kl} = g_{ij}^{lk}\) (0 ≤ \(i, j, k, l ≤ 2\)), and \(\sum_{i,j,k,l=0}^2 g_{ij}^{kl} p_i p_j p_k p_l = 0\) for any \(p = (p_0, p_1, p_2) \in \mathbb{R}^3\) satisfying \(p_0^2 = p_1^2 + p_2^2\). Then, there exists a positive constant \(C\) such that for any \(f, g, h \in C^\infty([0, T] \times \mathbb{R}^2)\),
\[
|\sum_{i,j,k,l=0}^2 g_{ij}^{kl} (\partial_i f \partial_j g \partial_{kl}^2 h)(t, x)| \leq C(|\partial f(t, x)||\partial g(t, x)||Z \partial h(t, x)| \\
+ |\partial f(t, x)||Z g(t, x)||\partial h(t, x)| + |Z f(t, x)||\partial g(t, x)||\partial h(t, x)|). \tag{3.5}
\]

Based on Lemma 3.1 and Lemma 3.2, and noting \(|Z \varphi(t, x)| \leq C\left(\frac{|x| - t}{t} |\partial \varphi(t, x)| + \frac{1}{t} (|X \varphi(t, x)| + |\Omega \varphi(t, x)|)\right)\), then one can obtain the following estimates under the assumptions of Lemma 3.1 and Lemma 3.2:
\[
|\sum_{i,j,k=0}^2 g^{ij}_k (\partial_i \Gamma^\alpha \phi_a \partial_j \Gamma^\beta \phi)(t, x)| \leq \frac{C\varepsilon}{t(t)^{1/2}} \sum_{|\nu| \leq |\beta|+1} |\Gamma^\nu \partial \phi(t, x)|, \tag{3.6}
\]
\[
|\sum_{i,j,k=0}^2 g^{ij}_k (\partial_i \Gamma^\alpha \phi_a \partial_j \Gamma^\beta \phi_a \partial_{kl}^2 \Gamma^\nu \phi)(t, x)| \leq \frac{C\varepsilon}{t(t)^{1/2}} \left( \sum_{|\nu| \leq |\alpha|+1} |(\sigma^{-1} \Gamma^\nu \phi)(t, x)| + \sum_{|\nu| \leq |\alpha|} |\Gamma^\nu \partial \phi(t, x)| \right), \tag{3.7}
\]
\[
|\sum_{i,j,k,l=0}^2 g_{ij}^{kl} (\partial_i \Gamma^\alpha \phi_a \partial_j \Gamma^\beta \phi_a \partial_{kl}^2 \Gamma^\nu \phi_a)(t, x)| \leq \frac{C\varepsilon^2}{t(t)} \sum_{|\nu| \leq |\mu|+1} |\Gamma^\nu \partial \phi(t, x)|, \tag{3.8}
\]
\[
|\sum_{i,j,k,l=0}^2 g_{ij}^{kl} (\partial_i \Gamma^\alpha \phi_a \partial_j \Gamma^\beta \phi_a \partial_{kl}^2 \Gamma^\nu \phi_a)(t, x)| \leq \frac{C\varepsilon^2}{t(t)} \left( \sum_{|\nu| \leq |\beta|+1} |(\sigma^{-1} \Gamma^\nu \phi)(t, x)| + \sum_{|\nu| \leq |\beta|} |\Gamma^\nu \partial \phi(t, x)| \right). \tag{3.9}
\]
and

\[
\left| \sum_{i,j,k=0}^{2} g_{ik}^{jk}(\partial_{i} \Gamma^{\alpha} \dot{\phi} \partial_{jk}^{2} \Gamma^{\beta} \dot{\phi})(t, x) \right| \leq \frac{C}{t} \left( \sum_{|p| \leq |\alpha|, \ |q| \leq |\beta|} \left| (\sigma_\Gamma \partial \phi \partial \Gamma^{q} \partial \phi)(t, x) \right| \right.
\]

\[
+ \sum_{|p| \leq |\alpha| + 1, \ |q| \leq |\beta|} \left| (\Gamma^{p} \partial \phi \partial \Gamma^{q} \partial \phi)(t, x) \right| + \sum_{|p| \leq |\alpha|, \ |q| \leq |\beta| + 1} \left| (\Gamma^{p} \partial \phi \partial \Gamma^{q} \partial \phi)(t, x) \right| \right),
\]

(3.10)

\[
\left| \sum_{i,j,k,l=0}^{2} g_{ij}^{kl}(\partial_{i} \Gamma^{\alpha} \dot{\phi} \partial_{j} \Gamma^{\beta} \dot{\phi} \partial_{kl} \Gamma^{\mu} \dot{\phi})(t, x) \right|
\]

\[
\leq \frac{C \varepsilon}{t(t)^{1/2}} \left( \sum_{|p| \leq |\alpha|, \ |q| \leq |\mu| + 1} \left| (\Gamma^{p} \partial \phi \partial \Gamma^{q} \partial \phi)(t, x) \right| + \sum_{|p| \leq |\beta|, \ |q| \leq |\mu| + 1} \left| (\sigma_\Gamma^{-1} \Gamma^{p} \partial \phi \partial \Gamma^{q} \partial \phi)(t, x) \right| \right),
\]

(3.11)

\[
\left| \sum_{i,j,k,l=0}^{2} g_{ij}^{kl}(\partial_{i} \Gamma^{\alpha} \dot{\phi} \partial_{j} \Gamma^{\beta} \dot{\phi} \partial_{kl} \Gamma^{\mu} \phi)(t, x) \right|
\]

\[
\leq \frac{C \varepsilon}{t(t)^{1/2}} \left( \sum_{|p| \leq |\beta| + 1, \ |q| \leq |\alpha|} \left| (\sigma_\Gamma^{-1} \Gamma^{p} \partial \phi \partial \Gamma^{q} \partial \phi)(t, x) \right| + \sum_{|p| \leq |\beta|, \ |q| \leq |\mu|} \left| (\sigma_\Gamma^{-1} \Gamma^{p} \partial \phi \partial \Gamma^{q} \partial \phi)(t, x) \right| \right),
\]

(3.12)

\[
\left| \sum_{i,j,k,l=0}^{2} g_{ij}^{kl}(\partial_{i} \Gamma^{\alpha} \dot{\phi} \partial_{j} \Gamma^{\beta} \dot{\phi} \partial_{kl} \Gamma^{\mu} \phi)(t, x) \right|
\]

\[
\leq \frac{C}{t} \left( \sum_{|p| \leq |\alpha|, \ |q| \leq |\beta|, s \leq |\mu|} \left| (\sigma_\Gamma \partial \phi \partial \Gamma^{q} \partial \phi \partial \Gamma^{s} \partial \phi)(t, x) \right| \right.
\]

\[
+ \sum_{|s| \leq |\mu| + 1, \ |q| \leq |\beta|, |p| \leq |\alpha|} \left| (\Gamma^{p} \partial \phi \partial \Gamma^{q} \partial \Gamma^{s} \partial \phi)(t, x) \right| \right.
\]

\[
+ \sum_{|q| \leq |\beta| + 1, \ |s| \leq |\mu|, |p| \leq |\alpha|} \left| (\Gamma^{p} \partial \phi \partial \Gamma^{q} \partial \Gamma^{s} \partial \phi)(t, x) \right| \right.
\]

\[
+ \sum_{|q| \leq |\beta|, \ |s| \leq |\mu|, \ |p| \leq |\alpha| + 1} \left| (\Gamma^{p} \partial \phi \partial \Gamma^{q} \partial \Gamma^{s} \partial \phi)(t, x) \right| \right) \left(3.13\right)
\]

To treat each term better in the right hand side of (3.6)-(3.13), we require the following two results (Lemma 3.3 and Lemma 3.4):
Lemma 3.3. Suppose that \( f(t, x) \in C^1(\mathbb{R}^+ \times \mathbb{R}^2) \) and \( \text{supp} \ f \subset \{(t, x) : |x| \leq M + t\} \). Then there exists a positive constant \( C \) independent of \( t \) such that

\[
\| (\sigma^{-1} f)(t, \cdot) \|_{L^2(|x| \geq M+1)} \leq C \| \nabla f(t, \cdot) \|_{L^2(|x| \geq M+1)}. \tag{3.14}
\]

Remark 3.1. Here we point out that the inequality \( \| (\sigma^{-1} f)(t, \cdot) \|_{L^2} \leq C \| \nabla f(t, \cdot) \|_{L^2} \) in the whole space has been established in [15].

Proof. Due to \( \text{supp} \ f \subset \{|x| \leq M + t\} \), one then has

\[
f(t, x) = - \int_{|x|}^{M+t} \nabla f(t, s\omega) \cdot \omega ds.
\]

This derives

\[
|f(t, x)|^2 \leq \left( \int_{|x|}^{M+t} |\nabla f(t, s\omega)|^2 (1 + |t - s|)^{1/2} ds \right) \int_{|x|}^{M+t} (1 + |t - s|)^{-1/2} ds.
\]

With easy computations, we have \( \int_{|x|}^{M+t} (1 + |t - s|)^{-1/2} ds \leq C (1 + t - |x|)^{1/2} \) for \( t > |x| \), and \( \int_{|x|}^{M+t} (1 + |t - s|)^{-1/2} ds \leq C \) for \( t \leq |x| \). Hence, if \( t \geq M + 1 \), then

\[
\int_{M+1}^{M+t} |f(t, r\omega)|^2 (1 + |t - r|)^{-2} r dr 
\]

\[
\leq C \int_{M+1}^{t} \left( \int_{r}^{M+t} |\nabla f(t, s\omega)|^2 (1 + t - s)^{1/2} ds \right) (1 + t - r)^{-3/2} r dr 
+ C \int_{M+1}^{t} \left( \int_{t}^{M+t} |\nabla f(t, s\omega)|^2 ds \right) (1 + t - r)^{-3/2} r dr 
+ C \int_{t}^{M+t} \left( \int_{r}^{M+t} |\nabla f(t, s\omega)|^2 ds \right) r (1 + r - t)^{-2} dr 
\leq C \int_{M+1}^{M+t} s |\nabla f(t, s\omega)|^2 ds. \tag{3.15}
\]

If \( 1 \leq t \leq M + 1 \), then

\[
\int_{M+1}^{M+t} |f(t, r\omega)|^2 (1 + |t - r|)^{-2} r dr \leq C \int_{M+1}^{M+t} \int_{r}^{M+t} |\nabla f(t, s\omega)|^2 r ds dr 
\leq C \int_{M+1}^{M+t} s |\nabla f(t, s\omega)|^2 ds. \tag{3.16}
\]

Combining (3.15) with (3.16) yields (3.14). \( \square \)
Lemma 3.4. Suppose that \( f(t, x) \in C^2(\mathbb{R}_+ \times \mathbb{R}^2) \) and \( \text{supp } f \subset \{(t, x) : |x| \leq M + t\} \). Then there exists a positive constant \( C \) independent of \( t \) such that for \( |x| \geq M + 1 \)

\[
\sigma(x)^{1/2} \sigma_-(t, x)^{-1} |f(t, x)| \leq C \sum_{j=0}^{1} \sum_{|\alpha|=0}^{2-j} \| \partial \nabla^\alpha \Omega^j f(t, \cdot) \|_{L^2(|x| \geq M+1)}. \tag{3.17}
\]

Proof. Similar to the proof of Lemma 1 (3.1) in [21], we have for \( |x| \geq M + 1 \) and \( g(t, x) \in C^2(\mathbb{R}_+ \times \mathbb{R}^2) \) with \( \text{supp } g \subset \{(t, x) : |x| \leq M + t\} \)

\[
\sigma(x)^{1/2} |g(t, x)| \leq C \sum_{j=0}^{1} \sum_{|\alpha|=0}^{2-j} \| \nabla^\alpha \Omega^j g(t, \cdot) \|_{L^2(|x| \geq M+1)}. \tag{3.18}
\]

Choosing \( g(t, x) = \sigma_-(t, x)^{-1} f(t, x) \) in (3.18) and applying Lemma 3.3 yield (3.17) directly. \( \square \)

Next, we extend Lemma 2.4 so that an energy estimate in the whole space is established as in Proposition 4.1 of [9].

Lemma 3.5. For fixed \( n \in \mathbb{N} \) with \( n \geq 5 \), if \( (\dot{\theta}, \dot{w}, \dot{z}) \) is a smooth solution of (2.4) for \( (t, x) \in [0, T] \times \mathbb{R}^2 \), and \( E_{n+1}^{1/2}(t) \leq \eta \) holds for small positive constant \( \eta \), then \( Q_n(t) \leq C(E_n^{1/2}(t) + \frac{\varepsilon^2}{(t)^{n/2}}) \) for \( 0 \leq t \leq T \).

Proof. If \( t \leq 1 \), it is easy to see \( Q_n(t) \leq C E_n^{1/2}(t) \). We now focus on the case of \( t \geq 1 \).

It is noted that for \( |\nu| \leq n \)

\[
\| \Gamma^\nu \dot{\phi}(t) \|_+ \leq C(\| \Gamma^\nu \dot{\theta}(t) \|_+ + \| \Gamma^\nu \dot{w}(t) \|_+) \leq C E_n^{1/2}(t), \tag{3.19}
\]

here Lemma 2.3 (2) is applied in the first inequality.

Set \( Q_j^+(t) = \sum_{|\mu| \leq n-1} \| \sigma_-(t) \nabla \Gamma^\mu \dot{\theta}(t) \|_+ + \| \sigma_-(t) \partial_\mu \Gamma^\mu \dot{w}(t) \|_+ + \| \sigma_-(t) \nabla \cdot \Gamma^\mu \dot{w}(t) \|_+ \). Similar to (2.12), the relationship between \( E_j^+(t) \) and \( Q_j^+(t) \) \((1 \leq j \leq n)\) is

\[
Q_j^+(t) \leq C(E_{j-1}^{1/2}(t) + \sum_{|\mu| \leq j-1} t \| h^\mu_0(t) \|_+ + \sum_{|\mu| \leq j-1} \langle t \rangle \| h^\mu(t) \|_+ ). \tag{3.20}
\]

When \( |\mu| \leq j - 1 \), we have

\[
\| h^\mu_0(t) \|_+ \leq \sum_{\nu \leq \mu} C_{\mu \nu} (\| I_1^{\mu\nu}(t) \|_+ + \| I_2^{\mu\nu}(t) \|_+ + \| I_3^{\mu\nu}(t) \|_+ ) + \sum_{\nu \leq \mu} C_{\mu \nu} \| J_3^{\mu\nu}(t) \|_+, \tag{3.21}
\]

where

\[
I_1^{\mu\nu}(t, x) = \sum_{i=1}^{2}(\Gamma^\nu \partial_i \phi_\alpha)(\partial_i \Gamma^{\mu-\nu} \partial_i \dot{\phi}) - (\Gamma^\nu \partial_i \phi_\alpha)(\partial_i \Gamma^{\mu-\nu} \partial_i \dot{\phi}) \).
\[ I_2^{\mu}(t, x) = \sum_{i=1}^{2} \left( (\Gamma^{\nu} \partial_i \dot{\phi})(\partial_i \Gamma^{\mu-\nu} \partial_i \phi_a) - (\Gamma^{\nu} \partial_i \dot{\phi})(\partial_i \Gamma^{\mu-\nu} \partial_i \phi_a) \right)(t, x), \]

\[ I_3^{\mu}(t, x) = \sum_{i=1}^{2} \left( (\Gamma^{\nu} \partial_i \dot{\phi})(\partial_i \Gamma^{\mu-\nu} \partial_i \phi_a) - (\Gamma^{\nu} \partial_i \dot{\phi})(\partial_i \Gamma^{\mu-\nu} \partial_i \phi_a) \right)(t, x), \]

and using (3.1)-(3.2) to get \( J_{\mu\nu}(t, x) = \sum_{j=1}^{5} \sum_{|\alpha|+|\beta|+|\gamma| \leq \mu} C_{\alpha\beta\gamma}^{\nu} J_j^{\alpha\beta\gamma}(t, x) + \tilde{J}_{\mu\nu}(t, x) \) with

\[ J_1^{\alpha\beta\gamma} = \frac{1}{2} \sum_{k=1}^{2} (\partial_t \Gamma^\alpha \phi_a)(\partial_t \Gamma^\beta \phi_a)(\partial_k^2 \Gamma^\gamma \dot{\phi}) - \frac{1}{2} \sum_{k,l=1}^{2} (\partial_t \Gamma^\alpha \phi_a)(\partial_t \Gamma^\beta \phi_a)(\partial_k^2 \Gamma^\gamma \dot{\phi}) \]

\[ - \sum_{k=1}^{2} (\partial_t \Gamma^\alpha \phi_a)(\partial_k \Gamma^\beta \phi_a)(\partial_k \partial_t \Gamma^\gamma \dot{\phi}) + \sum_{k,l=1}^{2} (\partial_t \Gamma^\alpha \phi_a)(\partial_k \Gamma^\beta \phi_a)(\partial_k^2 \Gamma^\gamma \dot{\phi}), \]

\[ J_2^{\alpha\beta\gamma} = -\sum_{k=1}^{2} (\partial_t \Gamma^\beta \phi_a)(\partial_k \Gamma^\gamma \dot{\phi})(\partial_k \partial_t \Gamma^\alpha \phi_a) + \sum_{k,l=1}^{2} (\partial_t \Gamma^\beta \phi_a)(\partial_k \Gamma^\gamma \dot{\phi})(\partial_k^2 \Gamma^\alpha \phi_a) \]

\[ - \sum_{k=1}^{2} (\partial_k \Gamma^\beta \phi_a)(\partial_k \Gamma^\gamma \dot{\phi})(\partial_k \partial_t \Gamma^\alpha \phi_a) + \sum_{k,l=1}^{2} (\partial_k \Gamma^\beta \phi_a)(\partial_k \Gamma^\gamma \dot{\phi})(\partial_k^2 \Gamma^\alpha \phi_a) \]

\[ + \sum_{k=1}^{2} (\partial_k \Gamma^\beta \phi_a)(\partial_k \Gamma^\gamma \dot{\phi})(\partial_k^2 \Gamma^\alpha \phi_a) - \sum_{k,l=1}^{2} (\partial_k \Gamma^\beta \phi_a)(\partial_k \Gamma^\gamma \dot{\phi})(\partial_k^2 \Gamma^\alpha \phi_a), \]

\[ J_3^{\alpha\beta\gamma} = -\sum_{k=1}^{2} (\partial_k \Gamma^\gamma \dot{\phi})(\partial_t \Gamma^\beta \phi_a)(\partial_k \partial_k \Gamma^\alpha \dot{\phi}) + \sum_{k,l=1}^{2} (\partial_k \Gamma^\gamma \dot{\phi})(\partial_t \Gamma^\beta \phi_a)(\partial_k^2 \Gamma^\alpha \dot{\phi}) \]

\[ - \sum_{k=1}^{2} (\partial_t \Gamma^\gamma \dot{\phi})(\partial_k \Gamma^\beta \phi_a)(\partial_k \partial_k \Gamma^\alpha \dot{\phi}) + \sum_{k,l=1}^{2} (\partial_t \Gamma^\gamma \dot{\phi})(\partial_k \Gamma^\beta \phi_a)(\partial_k^2 \Gamma^\alpha \dot{\phi}) \]

\[ + \sum_{k=1}^{2} (\partial_t \Gamma^\gamma \dot{\phi})(\partial_k \Gamma^\beta \phi_a)(\partial_k^2 \Gamma^\alpha \dot{\phi}) - \sum_{k,l=1}^{2} (\partial_t \Gamma^\gamma \dot{\phi})(\partial_k \Gamma^\beta \phi_a)(\partial_k^2 \Gamma^\alpha \dot{\phi}), \]

\[ J_4^{\alpha\beta\gamma} = -\sum_{k=1}^{2} (\partial_k \Gamma^\gamma \dot{\phi})(\partial_t \Gamma^\alpha \phi_a)(\partial_k \partial_k \Gamma^\beta \phi_a) + \sum_{k,l=1}^{2} (\partial_k \Gamma^\gamma \dot{\phi})(\partial_t \Gamma^\alpha \phi_a)(\partial_k^2 \Gamma^\beta \phi_a) \]

\[ + \frac{1}{2} \sum_{k=1}^{2} (\partial_k \Gamma^\gamma \dot{\phi})(\partial_t \Gamma^\alpha \phi_a)(\partial_k^2 \Gamma^\beta \phi_a) - \frac{1}{2} \sum_{k,l=1}^{2} (\partial_k \Gamma^\gamma \dot{\phi})(\partial_t \Gamma^\alpha \phi_a)(\partial_k^2 \Gamma^\beta \phi_a), \]
\[ J_3^{\alpha \beta \gamma} = -2 \sum_{k=1}^2 (\partial_k \Gamma^\gamma \dot{\phi})(\partial_\Gamma^\beta \dot{\phi}) (\partial_k \partial_\Gamma^\alpha \dot{\phi}) + \sum_{k,l=1}^2 (\partial_k \Gamma^\gamma \dot{\phi})(\partial_l \Gamma^\beta \dot{\phi}) (\partial_{kl} \Gamma^\alpha \dot{\phi}) \]
\[ + \frac{1}{2} \sum_{k=1}^2 (\partial_k \Gamma^\gamma \dot{\phi})(\partial_\Gamma^\beta \dot{\phi}) (\partial_{kl}^2 \Gamma^\alpha \dot{\phi}) - \frac{1}{2} \sum_{k,l=1}^2 (\partial_k \Gamma^\gamma \dot{\phi})(\partial_k \Gamma^\beta \dot{\phi}) (\partial_{kl}^2 \Gamma^\alpha \dot{\phi}), \]

and \( \tilde{J}_{\mu \nu} \) is a higher order error term whose explicit expression is not needed.

In terms of Lemma 3.3 and (3.19), it is easy to find that
\[ \| I_1^{\mu \nu}(t) \|_+ + \| I_2^{\mu \nu}(t) \|_+ \leq \frac{C \varepsilon}{\langle t \rangle^{3/2}} E_j^{1/2}(t), \]  \hspace{1cm} (3.22)
\[ \| J_1^{\alpha \beta \gamma}(t) \|_+ + \| J_2^{\alpha \beta \gamma}(t) \|_+ \leq \frac{C \varepsilon^2}{\langle t \rangle^{2}} E_j^{1/2}(t). \]  \hspace{1cm} (3.23)

To deal with \( I_3^{\mu \nu} \), we should pay attention to (3.10). If \( |\alpha| \leq |\beta| \) and \( |\alpha + \beta| \leq j - 1 \), then we can obtain with the help of Lemma 3.4
\[ \sum_{|p| \leq |\alpha|+1, \ |q| \leq |\beta|} \| (\Gamma^p \dot{\phi})(\partial \Gamma^q \dot{\phi})(t) \|_+ \leq \sum_{|p| \leq |\alpha|+1, \ |q| \leq |\beta|} |\sigma^{-1} \Gamma^p \dot{\phi}(t)|_+ \| \sigma^{-1} \partial \Gamma^q \partial \phi(t) \|_+ \]
\[ \leq \frac{C}{\langle t \rangle^{1/2}} E_j^{1/2}(t) \tilde{Q}_j(t). \]

If \( |\alpha| > |\beta| \), by using Lemma 2.3 and Lemma 3.3 we have
\[ \sum_{|p| \leq |\alpha|+1, \ |q| \leq |\beta|} \| (\Gamma^p \dot{\phi})(\partial \Gamma^q \dot{\phi})(t) \|_+ \leq \sum_{|p| \leq |\alpha|+1, \ |q| \leq |\beta|} |\sigma^{-1} \Gamma^p \dot{\phi}(t)|_+ \| \sigma^{-1} \Gamma^p \dot{\phi}(t) \|_+ \]
\[ \leq \frac{C}{\langle t \rangle^{1/2}} \tilde{Q}_{j+2}(t) E_j^{1/2}(t). \]

Similarly, we can arrive at for \( |\alpha + \beta| \leq j - 1 \)
\[ \sum_{|p| \leq |\alpha|, \ |q| \leq |\beta|} \| \sigma^{-1} (\Gamma^p \dot{\phi})(\partial \Gamma^q \dot{\phi})(t) \|_+ + \sum_{|p| \leq |\alpha|, \ |q| \leq |\beta|+1} \| (\Gamma^p \dot{\phi})(\Gamma^q \dot{\phi})(t) \|_+ \]
\[ \leq \frac{C}{\langle t \rangle^{1/2}} \left( E_j^{1/2}(t) \tilde{Q}_j(t) + \tilde{Q}_{j+2}(t) E_j^{1/2}(t) \right), \]
\[ \sum_{|p| \leq |\beta|+1, \ |q| \leq |\alpha|} \| \sigma^{-1} \Gamma^p \dot{\phi} \Gamma^q \dot{\phi}(t) \|_+ \leq \frac{C}{\langle t \rangle^{1/2}} \left( E_j^{1/2}(t) \tilde{Q}_j(t) + \tilde{Q}_{j+2}(t) E_j^{1/2}(t) \right). \]

Therefore,
\[ \| I_3^{\mu \nu}(t) \|_+ \leq \frac{C \varepsilon}{\langle t \rangle^{3/2}} \left( E_j^{1/2}(t) \tilde{Q}_j(t) + \tilde{Q}_{j+2}(t) E_j^{1/2}(t) \right), \]  \hspace{1cm} (3.24)
\[ \| J_3^{\alpha \beta \gamma}(t) \|_+ \leq \frac{C \varepsilon^2}{\langle t \rangle^{2}} \left( E_j^{1/2}(t) \tilde{Q}_j(t) + \tilde{Q}_{j+2}(t) E_j^{1/2}(t) \right), \]  \hspace{1cm} (3.25)
$$\| J_4^{\alpha\beta\gamma}(t) \|_+ \leq \frac{C\varepsilon}{(t)^2} \left( E_{\left[\frac{t}{2}\right]+2}^{1/2}(t) E_{\left[\frac{t}{2}\right]+1}^{1/2}(t) + E_{\left[\frac{t}{2}\right]+1}^{1/2}(t) E_{\left[\frac{t}{2}\right]+1}^{1/2}(t) \right).$$  \hspace{1cm} (3.26)

Analogously, we can also get the estimate of $J_5^{\alpha\beta\gamma}$ by means of comparing the size of $|p|$, $|q|$ and $|s|$ in (3.13), that is,

$$\| J_5^{\alpha\beta\gamma}(t) \|_+ \leq \frac{C}{(t)^2} \left( \tilde{Q}_{\left[\frac{t}{2}\right]+2}^{\mu}(t) E_{\left[\frac{t}{2}\right]+1}^{1/2}(t) + E_{\left[\frac{t}{2}\right]+1}^{1/2}(t) \tilde{Q}_{\left[\frac{t}{2}\right]+2}^{\mu}(t) \right) + \frac{C\eta}{(t)^{3/2}} Q_{\left[\frac{t}{2}\right]+5}^{\mu}(t).$$  \hspace{1cm} (3.27)

In addition, it is noted that $\tilde{J}_{\mu\nu}$ only contains the high-order error terms, then by a direct verification one can derive that the $L^2$ norm of $\tilde{J}_{\mu\nu}$ near the light cone can be controlled by those terms in the right hand sides of (3.22)-(3.27).

Due to $Q_{\mu}^{-}(t) \leq C \left( E_{\mu}^{1/2}(t) + \frac{\varepsilon^2}{(t)^{3/2}} \right)$ in terms of Lemma 2.4, we apply the estimates (3.21)-(3.27) to obtain $\| h_0^{\mu}(t) \|_+ \leq \frac{C}{(t)^{3/2}} \eta + \frac{\varepsilon^2}{(t)^{3/2}} Q_{\left[\frac{t}{2}\right]+5}^{\mu}(t)$ when $|\mu| \leq \left[ \frac{n+5}{2} \right] - 1$ and $\eta$ is small enough. In addition, because of $z(t, x) \equiv 0$ for $|x| \geq M + 1$, then we can get the same result for $h^{\mu}$ by the analogous analysis to $h_0^{\mu}$, that is, $\| h^{\mu}(t) \|_+ \leq \frac{C}{(t)^{3/2}} \eta + \frac{\varepsilon^2}{(t)^{3/2}} + \frac{C\eta}{(t)^{3/2}} Q_{\left[\frac{t}{2}\right]+5}^{\mu}(t)$ for $|\mu| \leq \left[ \frac{n+5}{2} \right] - 1$. Hence, $Q_{\mu}^{\mu}(t) \leq C\eta$ can be derived by utilizing (3.20). Taking $j = n$ in the estimates in (3.22)-(3.27) and applying (3.20) again, we obtain

$$Q_{n}^{\mu}(t) \leq C \left( E_{n}^{1/2}(t) + \frac{\varepsilon^2}{(t)^{5/2}} \right).$$

This, together with Lemma 2.4, yields Lemma 3.5. \hfill \Box

\section{Global energy estimates and proof of Theorem 1.1}

From (2.4), as in [9], we have

$$ (\partial_t + w \cdot \nabla) \Gamma^\mu \hat{\theta} + (1 - \theta) \nabla \cdot \Gamma^\mu \hat{w} = \hat{h}_0^{\mu} \equiv \sum_{j \in \{2, 5\}} f_j^{\mu} + \sum_{j \in \{1, 3, 4, 6\}} \hat{f}_j^{\mu}, \hspace{1cm} (4.1) $$

$$ (\frac{1}{1 + z} \partial_t + \frac{w}{1 + z} \cdot \nabla) \Gamma^\mu \hat{w} + (1 - \theta) \nabla \Gamma^\mu \hat{\theta} = \frac{\hat{h}_0^{\mu}}{1 + z} \equiv \frac{1}{1 + z} \left( \sum_{j \in \{8, 11\}} f_j^{\mu} + \sum_{j \in \{7, 9, 10, 12, 13\}} \hat{f}_j^{\mu} \right), \hspace{1cm} (4.2) $$

$$ (\partial_t + w \cdot \nabla) \Gamma^\mu \hat{z} = \hat{g}_\mu, \hspace{1cm} (4.3) $$
where $f^\mu_j (j = 2, 5, 8, 11)$ have been defined in (2.6); if $j \neq 13$, then $\hat{f}^0_j = 0$, and $\hat{f}^\mu_j$ with $\mu \neq 0$ are defined as $f^\mu_j$ but with the supplementary restriction condition $\nu \neq 0$ in the sum; if $j = 13$, then $\hat{f}^\mu_j = f^\mu_j + (1 - \theta)z \nabla \Gamma^\mu \hat{\theta}$. In addition, $\hat{g}^\mu = \sum_{j \in \{1, 2\}} \hat{g}^\mu_{aj} + \sum_{j \in \{1, 2\}} \hat{g}^\mu_j$ with $\hat{g}^\mu_{a1} = -\sum_{\nu \leq \mu} \left( \frac{\mu}{\nu} \right) \Gamma^\nu w_a \cdot \nabla \Gamma^\nu \hat{z}(0, x), \hat{g}^\mu_{a2} = -\sum_{0 < \nu \leq \mu} \left( \frac{\mu}{\nu} \right) \Gamma^\nu w_a \cdot \nabla \Gamma^\nu \hat{z}$ and $\hat{g}^\mu_1 = -\sum_{\nu \leq \mu} \left( \frac{\mu}{\nu} \right) \Gamma^\nu \hat{w} \cdot \nabla \Gamma^\nu \hat{z}$ if $\mu \neq 0$, otherwise, $\hat{g}^\mu_{a2} = \hat{g}^\mu_2 = 0$ for $\mu = 0$.

Set

$$\zeta^\mu = \left( \begin{array}{c} \Gamma^\mu \hat{\theta} \\ \Gamma^\mu \hat{w} \\ \Gamma^\mu \hat{z} \end{array} \right), \quad F^\mu = \left( \begin{array}{c} \hat{h}^\mu_0 \\ \hat{h}^\mu \\ 1 + z \hat{g}^\mu \end{array} \right), \quad A_0 = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right),$$

$$A_1 = \left( \begin{array}{cccc} w_1 & 1 - \theta & 0 & 0 \\ 1 - \theta & w_1 & 0 & 0 \\ 0 & 1 + z & w_1 & 0 \\ 0 & 0 & 1 + z & w_1 \end{array} \right), \quad A_2 = \left( \begin{array}{cccc} w_2 & 0 & 1 - \theta & 0 \\ 0 & w_2 & 0 & 0 \\ 1 - \theta & 0 & 1 + z & w_2 \\ 0 & 0 & 0 & w_2 \end{array} \right),$$

where $w = (w_1, w_2)$.

In this case, (4.1)-(4.3) can be written as $A_0 \partial_t \zeta^\mu + A_1 \partial_1 \zeta^\mu + A_2 \partial_2 \zeta^\mu = F^\mu$. Taking inner product with $\zeta^\mu$ in the space $\mathbb{R}^2$ and applying integration by parts, we arrive at

$$\frac{d}{dt} \langle A_0 \zeta^\mu, \zeta^\mu \rangle(t) = 2 \langle F^\mu, \zeta^\mu \rangle(t) + \sum_{0 \leq j \leq 2} \langle (\partial_j A_j) \zeta^\mu, \zeta^\mu \rangle(t). \quad (4.4)$$

As in [9], we set $H^\mu_j = \langle f^\mu_j, \Gamma^\mu \hat{\theta} \rangle$ if $j = 2, 5$, $H^\mu_j = \langle \hat{f}^\mu_j, \Gamma^\mu \hat{\theta} \rangle$ if $j = 1, 3, 4, 6$, $H^\mu_j = \langle \hat{f}^\mu_j, \Gamma^\mu \hat{w} \rangle$ if $j = 8, 11$, $H^\mu_j = \langle \hat{f}^\mu_j, \Gamma^\mu \hat{w} \rangle$ if $j = 7, 9, 10, 12, 13$, $\hat{H}^\mu_{aj} = \langle \hat{g}^\mu_{aj}, \Gamma^\mu \hat{z} \rangle$ and $\hat{H}^\mu_j = \langle \hat{g}^\mu_j, \Gamma^\mu \hat{z} \rangle$ for $j = 1, 2$. Then

$$\langle F^\mu, \zeta^\mu \rangle = \sum_{1 \leq j \leq 13} H^\mu_j + \sum_{j=1, 2} \hat{H}^\mu_{aj} + \sum_{j=1, 2} \hat{H}^\mu_j. \quad (4.5)$$

Although the local existence of solution to (2.4) has been early established (for example, one can see [18]), we still give some detailed illustrations for later uses.

**Lemma 4.1.** If $\varepsilon > 0$ is small, then (2.4) has a unique smooth solution $(\hat{\theta}, \hat{w}, \hat{z})$ for $t \leq 2/\varepsilon$ and $E^\mu_m(t) \leq C\varepsilon^2$. 
Lemma 2.3 and Lemma 3.5, yields
\[\| \sum j=0 \partial_j A_j(t, x) \| \leq C(\| \nabla \theta(t, x) \| + | \nabla \cdot w(t, x) | + | w \cdot \nabla z(t, x) |) \leq \frac{C}{(t)^{1/2}} (\varepsilon + \tilde{Q}_3(t)) \leq \frac{C \varepsilon}{(t)^{1/2}}.\]

In addition, a direct computation yields
\[|H_j^\mu(t)| \leq \frac{C \varepsilon}{(t)^{1/2}} E_m(t) \quad \text{for} \quad |\mu| \leq m \quad \text{and} \quad j \in \{1, 2, 4, 5, 7, 8, 10, 11\}. \tag{4.6}\]

With respect to $H_3^\mu(t)$ for $|\mu| \leq m$, we will treat it under two kinds of cases:

If $|\nu| \leq |\mu - \nu| \leq m - 1$, then by Lemma 2.3 and Lemma 3.5

\[|\langle \Gamma^\nu \dot{w} \cdot \nabla \Gamma^\mu \hat{\theta}, \Gamma^\mu \hat{\theta} \rangle(t) | \leq \frac{C}{(t)^{1/2}} |\langle \sigma^{1/2} \Gamma^\nu \sigma_\cdot \nabla \Gamma^\mu \hat{\theta}, \Gamma^\mu \hat{\theta} \rangle(t) | \leq \frac{C \varepsilon}{(t)^{1/2}} E_m(t) + \frac{C \varepsilon^3}{(t)^2} E_m^{1/2}(t).\]

Similarly, for $|\nu| > |\mu - \nu|$, \[|\langle \Gamma^\nu \dot{w} \cdot \nabla \Gamma^\mu \hat{\theta}, \Gamma^\mu \hat{\theta} \rangle(t) | \leq \frac{C}{(t)^{1/2}} |\langle \sigma^{1/2} \sigma_\cdot \nabla \Gamma^\mu \hat{\theta} \cdot \Gamma^\nu \dot{w}, \Gamma^\mu \hat{\theta} \rangle(t) | \leq \frac{C \varepsilon}{(t)^{1/2}} E_m(t).\]

Therefore,
\[|H_3^\mu(t)| \leq \frac{C \varepsilon}{(t)^{1/2}} E_m(t) + \frac{C \varepsilon^3}{(t)^2} E_m^{1/2}(t). \tag{4.7}\]

By the same way, we have
\[|H_j^\mu(t)| \leq \frac{C \varepsilon}{(t)^{1/2}} E_m(t) + \frac{C \varepsilon^3}{(t)^2} E_m^{1/2}(t) \quad \text{for} \quad |\mu| \leq m \quad \text{and} \quad j \in \{6, 9, 12\}. \tag{4.8}\]

On the other hand,
\[\| \partial^\alpha (X + 1)^k ((1 - \theta)z(0, x)\nabla \theta_a) \| + \| \partial^\alpha (X + 1)^k ((1 - \theta)z\nabla \theta_a) \| \leq \frac{C \varepsilon^2}{(t)^{5/2}} + \frac{C \varepsilon}{(t)^{5/2}} E_m^{1/2}(t) \tag{4.9}\]

and
\[\| \partial^\alpha (X + 1)^k ((1 - \theta)z \nabla \hat{\theta}) - (1 - \theta)z \nabla \Gamma^\mu \hat{\theta} \| \leq \frac{C}{(t)^{5/2}} \sum_{0 < \nu \leq \mu} \| \Gamma^\nu ((1 - \theta)z) \cdot \sigma_\cdot \nabla \Gamma^{\mu - \nu} \hat{\theta} \| \leq \frac{C \varepsilon}{(t)^{5/2}} E_m^{1/2}(t) + \frac{C \varepsilon^3}{(t)^{5/2}}. \tag{4.10}\]
which mean
\[ |H_{13}(t)| \leq \frac{C\varepsilon^2}{(t)^{5/2}} E_m^{1/2}(t) + \frac{C\varepsilon}{(t)} E_m(t). \] (4.11)

Moreover, one can easily obtain
\[ \|\hat{g}_1(t)\| + \|\hat{g}_2(t)\| \leq \frac{C\varepsilon^2}{(t)^{5/2}} + C\varepsilon E_m^{1/2}(t), \]
\[ \|\hat{g}_2(t)\| + \|\hat{g}_2(t)\| \leq \frac{C\varepsilon}{(t)^{5/2}} E_m^{1/2}(t) + C\varepsilon E_m^{1/2}(t) \]
and then
\[ \sum_{j=1,2} |\hat{H}_{aj}| + \sum_{j=1,2} |\hat{H}_{aj}| \leq \frac{C\varepsilon^2}{(t)^{5/2}} E_m^{1/2}(t) + C\varepsilon E_m(t). \] (4.12)

By (4.6)-(4.9) and (4.11)-(4.12), we have from (4.4)
\[ \frac{d}{dt} E_m(t) \leq \frac{C\varepsilon^2}{(t)^{2}} E_m^{1/2}(t) + C\varepsilon E_m(t), \]
and then for sufficiently small \( \varepsilon \),
\[ E_m^{1/2}(t) \leq C\varepsilon^2 e^{C\varepsilon t} \leq C\varepsilon^2. \]

Therefore, Lemma 4.1 is proved by the local existence of solution and continuous induction method. \( \square \)

As in Lemma 4.1, in order to prove Theorem 1.1 by the continuous induction method, we require to establish a uniform estimate on the solution \((\rho, w, z)\). To this end, we will derive the a priori estimate on the related potential \( \phi \) in the domain \( \{x : |x| \geq M + 1\} \).

**Lemma 4.2.** Assume that \( k \) and \( \lambda \) are integers with \( \left[ \frac{k + 1}{2} \right] + 3 \leq \lambda \leq k \) and \( k \geq 7 \), \((\dot{\theta}, \dot{w}, \dot{z})\) is a smooth solution of (2.4) for \( (t, x) \in \left[ \frac{1}{\varepsilon}, T \right] \times \mathbb{R}^2 \). If \( \lambda \leq \varepsilon^2 \), then one can find a positive number \( C \) independent of \( \varepsilon \) and \( T \) such that the potential \( \phi(t, x) \) of velocity \( w(t, x) \) in the domain \( \{x : |x| \geq M + 1\} \) satisfies

\[
\sum_{|\mu| \leq k} \int_{1/\varepsilon}^{t} \int_{D_+(\tau)} \sigma_-(t, x)^{-1} |Z\Gamma^\mu_\phi(\tau, x)|^2 d\tau dx 
\leq C \left( \varepsilon^4 + \sum_{|\mu| \leq k} \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} \int_{\{x : |x| \geq M + 1\}} |\Gamma^\mu_\phi(\tau, x)|^2 d\tau dx \right) \] (4.13)

**Remark 4.1.** In order to apply energy integral method to derive (4.13), we require to choose a different “ghost weight” from the one in [4] due to the following reason: notice that the both null conditions of 2-D quasilinear wave equation are fulfilled in the whole space \( \mathbb{R}^2 \) in [4].
but for our 2-D compressible Euler system (1.3), the null conditions hold only in the exterior domain \( \{|x| \geq M + 1\} \), and thus it is natural for us to multiply a smooth cut-off function on the potential \( \phi \) so that a suitable weighted energy estimate can be obtained. Due to this way of doing, the resulted “ghost weight” should be reconsidered by comparison with that in [4]. More concretely speaking, the author in [4] can obtain the a priori estimate \( |\partial \Gamma^\mu v(t, x)| \leq C\varepsilon^{-\frac{1}{2}}(t, x) \sigma_-(t, x) \) for the solution \( v \) of quasilinear wave equation 
\[
\partial^2_{tt} v - \Delta v + \sum_{0 \leq i, j \leq 2} g_{ij}(\partial v)\partial^2_{ij} v = 0 \text{ when the both null conditions hold, however, here we only get } |\partial \Gamma^\mu \phi(t, x)| \leq C\varepsilon^{-\frac{1}{2}}(t, x) \sigma_-(t, x) \text{ for the potential } \phi \text{ (see (4.17) below) which will lead to a different choice of the ghost weight from that in [4].}
\]

**Proof.** As in (2.1), the function \( \phi(t, x) \) satisfies for \( |x| \geq M + 1 \)
\[
\partial^2_{tt} \phi - \Delta \phi + 2 \sum_{j=1}^2 \partial_j \phi \partial_k \partial_j \phi + 2 \sum_{j,k=1}^2 \partial_j \phi \partial_k \phi \partial^2_{jk} \phi - (2 \partial_t \phi + |\nabla \phi|^2) \Delta \phi = 0. \tag{4.14}
\]

Define
\[
\varphi(t, x) = \varphi(|x| - t) = \int_{|x| - t}^{\infty} \frac{1}{(1 + |\rho|)^{3/2}} d\rho,
\]
then \( 0 \leq \varphi \leq 4 \) and \( \varphi'(|x| - t) = \sigma_-(t, x)^{-3/2} \) hold. In addition, we set \( \tilde{\sigma}(t, x) = (1 + |x| - t - M)^2 \) and \( \chi(t, x) = \tilde{\chi}(\frac{2r}{t + 2M + 2}) \) with the smooth function
\[
\tilde{\chi}(s) = \begin{cases} 
0, & \text{if } s \leq \frac{1}{2}, \\
1, & \text{if } s \geq 1.
\end{cases}
\]
Let the corresponding energy be denoted by
\[
\tilde{E}_n(t) = \sum_{|\mu| \leq n} \int_{\mathbb{R}^2} \tilde{\sigma}(t, x)^{-1/2} e^{\varphi(t, x)} \chi(t, x) (|\partial_t \Gamma^\mu \phi(t, x)|^2 + |\nabla \Gamma^\mu \phi(t, x)|^2) dx
\]
for \( n \in \mathbb{N} \cup \{0\} \). Motivated by the terminology in [4], the weight function \( \tilde{\sigma}(t, x)^{-1/2} e^{\varphi(t, x)} \chi(t, x) \) is also called the ghost weight by us, which will display the null conditions and decay rate simultaneously.

Notice that
\[
\partial^2_{tt} \Gamma^\mu \phi - \Delta \Gamma^\mu \phi + 2 \sum_{j=1}^2 \partial_j \phi \partial_t \partial_j \Gamma^\mu \phi + 2 \sum_{j,k=1}^2 \partial_j \phi \partial_k \phi \partial^2_{jk} \Gamma^\mu \phi - (2 \partial_t \phi + |\nabla \phi|^2) \Delta \Gamma^\mu \phi
\]
\[
= \sum_{\nu \leq \mu} C_{\nu} \Gamma^\nu \left( -2 \sum_{j=1}^2 \partial_j \phi \partial_t \partial_j \phi - 2 \sum_{j,k=1}^2 \partial_j \phi \partial_k \phi \partial^2_{jk} \phi + (2 \partial_t \phi + |\nabla \phi|^2) \Delta \phi \right)
\]
+ 2 \sum_{j=1}^2 \partial_j \phi \partial_t \partial_j \Gamma^\mu \phi + \sum_{j,k=1}^2 \partial_{j} \phi \partial_{k} \phi \partial_{j,k}^2 \Gamma^\mu \phi - (2 \partial_t \phi + |\nabla \phi|^2) \Delta \Gamma^\mu \phi. \quad (4.15)

Multiplying (4.15) by $\tilde{\sigma}^{-1/2}(t,x)e^{\phi(t,x)}\chi(t,x)\partial_t \Gamma^\mu \phi$, integrating in the space $\mathbb{R}^2$ and using integration by parts, we can get

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\phi} \chi(\Gamma^\mu \phi^2 + |\nabla \Gamma^\mu \phi|^2) dx + \frac{1}{2} \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\phi} \chi|Z \Gamma^\mu \phi|^2 dx
$$
$$
+ \frac{d}{dt} \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\phi} \chi \left( \partial_t \phi |\nabla \Gamma^\mu \phi|^2 - \frac{1}{2} \sum_{i,j=1}^2 \partial_i \phi \partial_j \phi (\partial_j \Gamma^\mu \phi)(\partial_i \Gamma^\mu \phi) + \frac{1}{2} |\nabla \phi|^2 |\nabla \Gamma^\mu \phi|^2 \right) dx
$$
$$
- \frac{1}{4} \int_{\mathbb{R}^2} (|x| - t - M) \tilde{\sigma}^{-5/2} e^{\phi} \chi |Z \Gamma^\mu \phi|^2 dx + \sum_{i=2}^5 L_i^\mu(t) = L_1^\mu(t), \quad (4.16)
$$

where

$$L_1^\mu(t) = \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\phi} \chi \partial_t \Gamma^\mu \phi \left\{ \sum_{\nu \leq \mu} C_{\nu} \Gamma^\nu \left( - \sum_{j=1}^2 \partial_j \phi \partial_t \partial_j \phi - \sum_{j,k=1}^2 \partial_j \phi \partial_k \phi \partial_{j,k}^2 \phi + (2 \partial_t \phi + |\nabla \phi|^2) \Delta \phi \right) + 2 \sum_{j=1}^2 \partial_j \phi \partial_t \partial_j \Gamma^\mu \phi + \sum_{j,k=1}^2 \partial_j \phi \partial_k \phi \partial_{j,k}^2 \Gamma^\mu \phi - (2 \partial_t \phi + |\nabla \phi|^2) \Delta \Gamma^\mu \phi \right\} dx,$$

$$L_2^\mu(t) = - \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\phi} \chi \left\{ \sum_{i=1}^2 \omega_i \partial_i \phi (\partial_i \Gamma^\mu \phi)^2 - 2 \sum_{i=1}^2 \omega_i \partial_i \phi (\partial_i \Gamma^\mu \phi) (\partial_i \Gamma^\mu \phi) \right\} dx,$$

$$L_3^\mu(t) = - \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\phi} \chi \left\{ \sum_{i=1}^2 \omega_i \partial_i \phi (\partial_i \Gamma^\mu \phi)^2 - 2 \sum_{i=1}^2 \omega_i \partial_i \phi (\partial_i \Gamma^\mu \phi) (\partial_i \Gamma^\mu \phi) \right\} dx,$$

$$L_4^\mu(t) = (t + 2M + 2)^{-1} \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\phi} \chi \left\{ \frac{r}{t + 2M + 2} ((\partial_t \Gamma^\mu \phi)^2 + |\nabla \Gamma^\mu \phi|^2) - 2 \sum_{i=1}^2 \omega_i \partial_i \phi (\partial_i \Gamma^\mu \phi)^2 \right\} dx.$$
\[
+ 2 \sum_{i=1}^{2} \omega_i (\partial_i \Gamma^\mu \phi) (\partial_i \Gamma^\mu \phi) + 4 \sum_{i=1}^{2} \omega_i \partial_i \phi (\partial_i \Gamma^\mu \phi) (\partial_i \Gamma^\mu \phi) + 2r(t + 2M + 2)^{-1} \partial_i \phi |\nabla \Gamma^\mu \phi|^2
\]
\[- 2 \sum_{i,j=1}^{2} \omega_i \partial_i \phi \partial_j (\partial_j \Gamma^\mu \phi) (\partial_i \Gamma^\mu \phi) - r(t + 2M + 2)^{-1} \partial_i \phi \partial_j (\partial_j \Gamma^\mu \phi) (\partial_i \Gamma^\mu \phi)
\]
\[+ 2 \sum_{i=1}^{2} \omega_i |\nabla \phi|^2 (\partial_i \Gamma^\mu \phi) (\partial_i \Gamma^\mu \phi) + r(t + 2M + 2)^{-1} |\nabla \phi|^2 |\nabla \Gamma^\mu \phi|^2 \}
\]
\[
L^\mu_5(t) = \frac{1}{2} \int_{\mathbb{R}^2} \left( (|x| - t - M)\tilde{\sigma}^{-5/2} e^{\varphi} \chi \sum_{i=1}^{2} \omega_i \partial_i \phi (\partial_i \Gamma^\mu \phi)^2 - 2 \sum_{i=1}^{2} \omega_i \partial_i \phi (\partial_i \Gamma^\mu \phi) (\partial_i \Gamma^\mu \phi)
\right)
\]
\[+ \sum_{i=1}^{2} \partial_i \phi (\partial_i \Gamma^\mu \phi)^2 + \sum_{i,j=1}^{2} \omega_i \partial_i \phi \partial_j (\partial_j \Gamma^\mu \phi) (\partial_i \Gamma^\mu \phi) + \frac{1}{2} \sum_{i,j=1}^{2} \partial_i \phi \partial_j (\partial_i \Gamma^\mu \phi) (\partial_j \Gamma^\mu \phi)
\]
\[- \sum_{i=1}^{2} \omega_i |\nabla \phi|^2 (\partial_i \Gamma^\mu \phi) (\partial_i \Gamma^\mu \phi) - \frac{1}{2} |\nabla \phi|^2 |\nabla \Gamma^\mu \phi|^2 \}
\]
\]

By \( \partial_i \phi = -\dot{\theta} + F_2(\xi, 2\xi_a + \dot{\xi}) \) and \( \nabla \phi = \dot{\omega} \) for \(|x| \geq M + 1\), then by Lemma 2.3 and Lemma 3.5 we have

\[
|\partial \Gamma^\mu \phi(t, x)| \leq C \varepsilon \sigma_- (t, x)^{-1/2} \sigma(x)^{-1/2} \quad \text{for} \quad |\nu| \leq \lambda - 2 \quad \text{and} \quad |x| \geq M + 1. \quad (4.17)
\]

Similarly,

\[
|\partial^2 \Gamma^\nu \phi(t, x)| \leq C \varepsilon \sigma_- (t, x)^{-1} \sigma(x)^{-1/2} \quad \text{for} \quad |\nu| \leq \lambda - 3 \quad \text{and} \quad |x| \geq M + 1. \quad (4.18)
\]

In addition, it follows Lemma 3.4 that

\[
|\Gamma^\nu \phi(t, x)| \leq C \varepsilon \sigma(x)^{-1/2} \sigma_- (t, x) \quad \text{for} \quad |\nu| \leq \lambda - 2 \quad \text{and} \quad |x| \geq M + 1. \quad (4.19)
\]

Therefore, we derive from (4.16)-(4.19) that

\[
\tilde{E}_k(t) + \sum_{|\mu| \leq k} \int_{1/\varepsilon}^{t} \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\varphi} \chi \varphi' |Z \Gamma^\mu \phi|^2 dxd\tau
\]
\[ - \sum_{|\mu| \leq k} \int_{1/\varepsilon}^{t} \int_{\mathbb{R}^2} (|x| - t - M)\tilde{\sigma}^{-5/2} e^{\varphi} \chi |Z \Gamma^\mu \phi|^2 dxd\tau
\]
\[\leq C \left( \tilde{E}_k \left( \frac{1}{\varepsilon} \right) + \sum_{|\mu| \leq k} \int_{1/\varepsilon}^{t} |L^\mu_i(\tau)| d\tau \right). \quad (4.20)
\]

Next we deal with each term \( L^\mu_i(1 \leq i \leq 5) \) in the right hand side of (4.20). In this process, we will often use the fact that \( \tilde{\sigma}(t, x) \) is equivalent to \( \sigma_- (t, x) \).
First, we take $g_{1i} = g_{i1} = -1$, $g_{ji} = 2$, $g_{ij} = g_{ij} = -\frac{1}{2}$, and $g_{ij} = 1$ for $i, j = 1, 2$, and the others are 0, then we have by Lemma 3.1 and Lemma 3.2 together with (4.17)-(4.19)

$$
|L_1^\mu(t)| \leq \int_{D_+(t)} \bar{\sigma}^{-1/2} e^\varphi \chi |\Gamma^\mu( \sum_{i,j,k=0}^2 g_{ik}^j \partial_i \phi \partial_j \phi) - \sum_{i,j,k=0}^2 g_{ik}^j \partial_i \phi \partial_j \phi \partial_k \Gamma^\mu \phi \partial_i \Gamma^\mu \phi| dx
$$

$$
+ \sum_{\nu \leq \mu} \int_{\mathbb{R}^2} \bar{\sigma}^{-1/2} e^\varphi \chi |C_{\nu} \Gamma^\nu( \sum_{i,j,k=0}^2 g_{ik}^j \partial_i \phi \partial_j \phi) - \sum_{i,j,k=0}^2 g_{ik}^j \partial_i \phi \partial_j \phi \partial_k \Gamma^\nu \phi \partial_i \Gamma^\nu \phi| dx
$$

$$
+ \sum_{\nu < \mu} \int_{D_+(t)} \bar{\sigma}^{-1/2} e^\varphi \chi |C_{\nu} \Gamma^\nu( \sum_{i,j,k=0}^2 g_{ik}^j \partial_i \phi \partial_j \phi)\partial_i \Gamma^\nu \phi| dx
$$

$$
\leq C\{ t^{-1} \int_{D_+(t)} \bar{\sigma}^{-1/2} e^\varphi \chi(\sigma_- \sum_{|\nu| \leq \frac{k+1}{2}} |\partial \Gamma^\nu \phi| + \sum_{|\nu| \leq \frac{k+1}{2}+1} |\Gamma^\nu \phi| \sum_{|\nu| \leq k} |\partial \Gamma^\nu \phi|^2 dx
$$

$$
+ \int_{D_-(t)} \bar{\sigma}^{-1/2} e^\varphi \chi \sum_{|\nu_1| \leq \frac{k+1}{2}} |\partial \Gamma^\nu_1 \phi| \sum_{|\nu_2| \leq k} |\partial \Gamma^\nu_2 \phi|^2 dx
$$

$$
+ \int_{\mathbb{R}^2} \bar{\sigma}^{-1/2} e^\varphi \chi \sum_{|\nu_1| \leq \frac{k+1}{2}} |\partial \Gamma^\nu_1 \phi|^2 \sum_{|\nu_2| \leq k} |\partial \Gamma^\nu_2 \phi|^2 dx
$$

$$
+ \int_{D_+(t)} \bar{\sigma}^{-1/2} e^\varphi \chi \sum_{|\nu_1| \leq \frac{k+1}{2}} |\partial \Gamma^\nu_1 \phi| \sum_{|\nu_2| \leq k} |Z \Gamma^\nu_2 \phi| \sum_{|\nu_3| \leq k} |\partial \Gamma^\nu_3 \phi| dx\}
$$

$$
\leq C\varepsilon \sum_{|\nu| \leq k} \int_{\mathbb{R}^2} \bar{\sigma}^{-1/2} e^\varphi \chi |\Gamma^\nu \phi|^2 dx + C\varepsilon(t)^{-1} \sum_{|\nu| \leq k} \int_{\mathbb{R}^2} e^\varphi \chi |\partial \Gamma^\nu \phi|^2 dx,
$$

(4.22)

here we give some explanations for the derivation process from (4.21) to (4.22), for example, in order to treat the last term in (4.21), one can make use of (4.17) to get

$$
\int_{D_+(t)} \bar{\sigma}^{-1/2} e^\varphi \chi \sum_{|\nu_1| \leq \frac{k+1}{2}} |\partial \Gamma^\nu_1 \phi| \sum_{|\nu_2| \leq k} |Z \Gamma^\nu_2 \phi| \sum_{|\nu_3| \leq k} |\partial \Gamma^\nu_3 \phi| dx
$$

$$
\leq C\varepsilon \int_{D_+(t)} \bar{\sigma}^{-\frac{1}{2}} e^\varphi \chi(\sigma_- \sum_{|\nu| \leq k} |Z \Gamma^\nu \phi| \sum_{|\nu| \leq k} |\partial \Gamma^\nu \phi| dx
$$

$$
\leq C\varepsilon \sum_{|\nu| \leq k} \int_{\mathbb{R}^2} \bar{\sigma}^{-1/2} e^\varphi \chi(\varphi') \sum_{|\nu| \leq k} \int_{\mathbb{R}^2} e^\varphi \chi |\partial \Gamma^\nu \phi|^2 dx + C\varepsilon(t)^{-1} \sum_{|\nu| \leq k} \int_{\mathbb{R}^2} e^\varphi \chi |\partial \Gamma^\nu \phi|^2 dx
$$

(\sigma_- is equivalent to \bar{\sigma} in $D_+(t)$).
Analogously,

\[
|L_2^\mu(t)| \leq C \left\{ \int_{D_+^1(t)} \tilde{\sigma}^{-1/2} e^{\varphi} \chi \left| \sum_{i=1}^{2} \omega_i Z_i \phi (\partial_t \Gamma^\mu \phi)^2 - \partial_t \phi \sum_{i=1}^{2} (Z_i \Gamma^\mu \phi)^2 \right| dx \\
+ \int_{D_-^1(t)} \tilde{\sigma}^{-1/2} e^{\varphi} \chi \left| \partial_\phi \right| \left| \partial \Gamma^\mu \phi \right|^2 dx + \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\varphi} \chi \left| \partial_\phi \right| \left| \partial \Gamma^\mu \phi \right|^2 dx \right\}
\]

\[
\leq C \varepsilon (t)^{-1/2} \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\varphi} \chi \left| \partial \Gamma^\mu \phi \right|^2 dx + C \varepsilon (t)^{-1} \int_{\mathbb{R}^2} e^{\varphi} \chi \left| \partial \Gamma^\mu \phi \right|^2 dx,
\]

(4.23)

\[
|L_3^\mu(t)| \leq C \left\{ \int_{D_+^1(t)} \tilde{\sigma}^{-1/2} e^{\varphi} \chi \left| \sum_{i=1}^{2} \omega_i Z_i \phi (\partial_t \Gamma^\mu \phi)^2 + (\partial \Gamma^\mu \phi) (Z \Gamma^\mu \phi) (\partial_\phi) \right| dx \\
+ \int_{D_-^1(t)} \tilde{\sigma}^{-1/2} e^{\varphi} \chi \left| \partial_\phi \right| \left| \partial \Gamma^\mu \phi \right|^2 dx + \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\varphi} \chi \left| \partial_\phi \right| \left| \partial \Gamma^\mu \phi \right|^2 dx \right\}
\]

\[
\leq C \varepsilon \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\varphi} \chi \left| \partial \Gamma^\mu \phi \right|^2 dx + C \varepsilon (t)^{-1} \int_{\mathbb{R}^2} e^{\varphi} \chi \left| \partial \Gamma^\mu \phi \right|^2 dx,
\]

(4.24)

\[
|L_4^\mu(t)| \leq C (t)^{-1} \int_{\{|x| \geq M+1\}} e^{\varphi} \left| \partial \Gamma^\mu \phi \right|^2 dx,
\]

(4.25)

\[
|L_5^\mu(t)| \leq C \left\{ - \int_{D_+^1(t)} (|x| - t - M) \tilde{\sigma}^{-5/2} e^{\varphi} \chi \left| \sum_{i=1}^{2} \omega_i Z_i \phi (\partial_t \Gamma^\mu \phi)^2 - \partial_t \phi \sum_{i=1}^{2} (Z_i \Gamma^\mu \phi)^2 \right| dx \\
+ \int_{D_-^1(t)} \tilde{\sigma}^{-3/2} e^{\varphi} \chi \left| \partial_\phi \right| \left| \partial \Gamma^\mu \phi \right|^2 dx + \int_{\mathbb{R}^2} \tilde{\sigma}^{-3/2} e^{\varphi} \chi \left| \partial_\phi \right| \left| \partial \Gamma^\mu \phi \right|^2 dx \right\}
\]

\[
\leq -C \varepsilon (t)^{-1/2} \int_{\mathbb{R}^2} (|x| - t - M) \tilde{\sigma}^{-5/2} e^{\varphi} \chi \left| \partial \Gamma^\mu \phi \right|^2 dx + C \varepsilon (t)^{-1} \int_{\mathbb{R}^2} e^{\varphi} \chi \left| \partial \Gamma^\mu \phi \right|^2 dx.
\]

(4.26)

Substituting (4.22)-(4.26) into (4.20) yields

\[
\tilde{E}_k(t) + \sum_{|\mu| \leq k} \int_{1/\varepsilon}^{t} \int_{\mathbb{R}^2} \tilde{\sigma}^{-1/2} e^{\varphi} \chi \varphi' \left| Z \Gamma^\mu \phi \right|^2 dx d\tau
\]

\[- \sum_{|\mu| \leq k} \int_{1/\varepsilon}^{t} \int_{\mathbb{R}^2} (|x| - t - M) \tilde{\sigma}^{-5/2} e^{\varphi} \chi \left| Z \Gamma^\mu \phi \right|^2 dx d\tau
\]

\[
\leq C \tilde{E}_k(t) \left( \frac{1}{\varepsilon} \right) + C \int_{1/\varepsilon}^{t} (\tau)^{-1} \sum_{|\mu| \leq k} \int_{\{|x| \geq M+1\}} \left| \partial \Gamma^\mu \phi \right|^2 dx.
\]

(4.27)

This, together with Lemma 4.1, yields (4.13). \qed

From (4.13) and (2.2), we have for \( \dot{\phi} = \phi - \phi_u \)

\[
\sum_{|\mu| \leq k} \int_{1/\varepsilon}^{t} \int_{D_+^1(\tau)} \sigma_-(t, x)^{-2} \left| Z \Gamma^\mu \dot{\phi}(\tau, x) \right|^2 dx d\tau
\]
\[
\leq C \left( \varepsilon^2 \ln t + \sum_{|\mu| \leq k} \int_1^{t/\varepsilon} \langle \tau \rangle^{-1} \int_{\{ |x| \geq M+1 \}} |\Gamma^\mu \partial \dot{\phi}(\tau, x)|^2 dx d\tau \right). \tag{4.28}
\]

**Lemma 4.3.** Assume that \( k \) and \( \lambda \) are integers with \( \frac{k}{2} \leq \lambda \leq k \) and \( k \geq 7 \), \((\hat{\theta}, \hat{w}, \hat{z})\) is a smooth solution of (2.4) for \((t, x) \in [1/\varepsilon, T] \times \mathbb{R}^2\). If \( E_\lambda(t) \leq \varepsilon^2 \), then one can find a positive number \( C \) independent of \( \varepsilon \) and \( T \) such that for \( t \in [1/\varepsilon, T] \) and sufficient small \( \varepsilon > 0 \), \( E_k(t) \leq C \varepsilon^2 \langle t \rangle^{C_\varepsilon} \) holds.

**Proof.** First, we come to estimate \( H_j^\mu \) in (4.5) when \(|\mu| \leq k\) and \( j = 1, \cdots, 13 \). It is easy to conclude that
\[
\sum_{|\mu| \leq k} \sum_{i=1}^{13} |H_i^\mu(t)| \leq C \varepsilon(t)^{-1} E_k(t) + C \varepsilon^2(t)^{-5/2} E_k(t)^{1/2}, \tag{4.29}
\]
here the estimate on \( H_i^\mu \) can be obtained as in (4.11).

We now continue to use the analogous notations as in (3.21), namely,
\[
\hat{h}_0^\mu = \sum_{0 < \nu \leq \mu} C_{\mu \nu}(I_1^\mu + I_3^\mu) + \sum_{0 \leq \nu \leq \mu} C_{\mu \nu} I_2^\mu + K^\mu,
\]
where
\[
K^\mu = \sum_{0 < \nu \leq \mu} C_{\mu \nu} \left( - \Gamma^\nu \hat{w} \cdot \nabla \Gamma^{\mu - \nu} F_2(\hat{\xi}, 2\xi_a + \hat{\xi}) + \Gamma^\nu F_1(\xi) \nabla \cdot \Gamma^{\mu - \nu} \hat{w} \right) + \sum_{0 \leq \nu \leq \mu} C_{\mu \nu} \left( - \Gamma^\nu \hat{w} \cdot \nabla \Gamma^{\mu - \nu} F_1(\xi_a) + \Gamma^\nu F_2(\hat{\xi}, 2\xi_a + \hat{\xi}) \nabla \cdot \Gamma^{\mu - \nu} w_a \right).
\]

It follows from the similar analysis of (3.22) and (3.24) together with (4.28) that
\[
\sum_{0 < \nu \leq \mu} C_{\mu \nu} \| I_1^\mu(t) \|_+ \leq C \varepsilon(t)^{-3/2} E_k^{1/2}(t), \tag{4.30}
\]
\[
\varepsilon^{-1} \sum_{0 < \nu \leq \mu} C_{\mu \nu} \int_{1/\varepsilon}^{t} \langle \tau \rangle \| I_2^\mu(\tau) \|^2_+ d\tau
\]
\[
\leq C \varepsilon^{-1} \int_{1/\varepsilon}^{t} \langle \tau \rangle \int_{D_+(\tau)} \sum_{|\nu| + |\nu_2| \leq k} \left( |Z\Gamma^{\nu_1} \dot{\phi}|^2 |\partial^2 \Gamma^{\nu_2} \phi_a|^2 + |\partial \Gamma^{\nu_1} \dot{\phi}|^2 |Z \partial \Gamma^{\nu_2} \phi_a|^2 \right) dx d\tau
\]
\[
\leq C \varepsilon \int_{|\nu| \leq k} \int_{1/\varepsilon}^{t} \int_{D_+(\tau)} \sigma_-(t, x)^2 |Z \Gamma^{\nu} \dot{\phi}(t, x)|^2 dx d\tau + C \varepsilon \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} E_k(\tau) d\tau
\]
\[
\leq C \varepsilon^3 \ln t + C \varepsilon \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} E_k(\tau) d\tau, \tag{4.31}
\]
\[
\varepsilon^{-1} \sum_{0 < \nu \leq \mu} C_{\mu \nu} \int_{1/\varepsilon}^{t} \langle \tau \rangle \| I_3^\mu(\tau) \|^2_+ d\tau
\]
\[ \leq C\varepsilon^{-1} \int_{1/\varepsilon}^{t} \langle \tau \rangle \int_{D_{4}(\tau)} \left( |Z\Gamma^{\nu_1} \hat{\phi}|^{2} |\partial^{2} \Gamma^{\nu_2} \hat{\phi}|^{2} + |\partial \Gamma^{\nu_1} \hat{\phi}|^{2} |Z \partial \Gamma^{\nu_2} \hat{\phi}|^{2} \right) dx d\tau \]

\[ \leq C\varepsilon \sum_{|\nu| \leq k-1} \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-2} \int_{D_{4}(\tau)} |\sigma_{-} \partial \Gamma^{\nu} \partial \phi|^{2} dx d\tau \]

\[ + \sum_{|\nu| \leq k} \int_{1/\varepsilon}^{t} \int_{D_{4}(\tau)} \sigma_{-}(x, t)^{-2} |Z \Gamma^{\nu} \hat{\phi}(t, x)|^{2} dx d\tau + C\varepsilon \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} E_{k}(\tau) d\tau \]

\[ \leq C\varepsilon^{3} \ln t + C\varepsilon \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} E_{k}(\tau) d\tau. \] (4.32)

By using Lemma 2.2 and Lemma 2.3, we can get

\[ |K^{\mu}(t)|_{+} \leq C\varepsilon^{2} \langle t \rangle^{-1} \sum_{|\nu| \leq k} \left( |\Gamma^{\nu} \hat{\theta}(t)|_{+} + |\Gamma^{\nu} \hat{w}(t)|_{+} \right). \] (4.33)

Hence, we have

\[ \int_{1/\varepsilon}^{t} \sum_{|\nu| \leq k} \sum_{i=1}^{6} \left| H_{i}^{\mu}(\tau) \right|_{+} d\tau \leq C\varepsilon^{3} \ln t + C\varepsilon \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} E_{k}(\tau) d\tau, \]

and similarly,

\[ \int_{1/\varepsilon}^{t} \sum_{|\nu| \leq k} \sum_{i=7}^{12} \left| H_{i}^{\mu}(t) \right|_{+} d\tau \leq C\varepsilon^{3} \ln t + C\varepsilon \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} E_{k}(\tau) d\tau. \]

Second, we deal with the terms \( \hat{H}_{i}^{\mu} \) and \( \tilde{H}_{i}^{\mu} \) in (4.5) for \( j = 1, 2 \) and \( |\mu| \leq k \).

Due to \( \text{supp} \hat{z} \subset \{|x| \leq M + 1\} \), it is obvious that for \( j = 1, 2 \)

\[ |\hat{H}_{i}^{\mu}(t)| \leq C\varepsilon \langle t \rangle^{-5/2} E_{k}^{1/2}(t)(E_{k}^{1/2}(t) + \varepsilon). \] (4.34)

To treat the terms \( \hat{H}_{j}^{\mu} (j = 1, 2) \), we will use the analogous method in \S 5 of [9] (see pages 101 of [9]). For this end, we set \( \lambda^{\mu}_{\nu} = \Gamma^{\nu} \hat{w} \cdot \nabla \Gamma^{\mu-\nu} z(0, x) \) and \( \lambda^{\mu\nu} = \Gamma^{\nu} \hat{w} \cdot \nabla \Gamma^{\mu-\nu} z \).

If \( \Gamma^{\nu} = \partial \Gamma^{d} \) with \( |d| = |\nu| - 1 \), then

\[ \| \lambda^{\mu}_{\nu}(t) \| \leq C\varepsilon \langle t \rangle^{-1} \| (\sigma_{-} \partial \Gamma^{d} \hat{w})(t) \| \leq C\varepsilon \langle t \rangle^{-1} (E_{k}^{1/2}(t) + \varepsilon^{2} \langle t \rangle^{-3/2}) \] (4.35)

and

\[ \| \lambda^{\mu\nu}(t) \| \leq \left( \sup_{|x| \leq M + 1} \| \partial \Gamma^{d} \hat{w}(t, x) \| \right) \| \nabla \Gamma^{\mu-\nu} \hat{z}(t) \| \leq C\varepsilon \langle t \rangle^{-1} E_{k}^{1/2}(t) \quad \text{for} \quad |\nu| \leq |\mu - \nu|, \] (4.36)

\[ \| \lambda^{\mu\nu}(t) \| \leq C \left( \sup \| \nabla \Gamma^{\mu-\nu} \hat{z}(t, x) \| \right) \langle t \rangle^{-1} \| \sigma_{-} \partial \Gamma^{d} \hat{w}(t) \| \leq C\varepsilon \langle t \rangle^{-1} (E_{k}^{1/2}(t) + \varepsilon^{2} \langle t \rangle^{-3/2}) \]
for \(|\nu| > |\mu - \nu|\). \hfill (4.37)

If \(\Gamma^\nu = X|\nu|\) with \(|\nu| \leq k - 1\), due to \(\Gamma^\nu \hat{w}(t, x) = \left(\Gamma^\nu W(t, r)\right)\frac{x}{r}\) holds for some smooth function \(W(t, r)\), then we have

\[
\sup_{|x| \leq M + 1} |\Gamma^\nu \hat{w}(t, x)| \leq C \langle t \rangle^{-1} \|\sigma_- \nabla \cdot \Gamma^\nu \hat{w}(t, x)\| \leq C \langle t \rangle^{-1} Q_{|\rho|+1}(t). \hfill (4.38)
\]

Hence, we derive from (4.38) that for \(|\nu| \leq k - 1\)

\[
\|\lambda^\nu_{\alpha^j}(t)\| \leq C \langle t \rangle^{-1} Q_{|\rho|+1}(t) \leq C \langle t \rangle^{-1} \left( E_k^{1/2}(t) + \varepsilon^2(t)^{-3/2} \right), \hfill (4.39)
\]

\[
\|\lambda^\nu(t)\| \leq C \langle t \rangle^{-1} Q_{|\rho|+1}(t) E_k^{1/2}(t) \leq C \langle t \rangle^{-1} E_k^{1/2}(t) \quad \text{for} \quad |\nu| \leq |\mu - \nu|, \hfill (4.40)
\]

and

\[
\|\lambda^\nu(t)\| \leq C \langle t \rangle^{-1} \left( E_k^{1/2}(t) + \varepsilon^2(t)^{-3/2} \right) \quad \text{for} \quad |\nu| > |\mu - \nu|. \hfill (4.41)
\]

If \(\Gamma^\nu = X^k\), then exactly as in the proof of (5.10) in [9] (here we will apply Lemma 2.5) together with the related estimates (4.35)-(4.41), we can obtain

\[
|\hat{H}_j^\mu(t) - \frac{d}{dt} G_j(t)| \leq C \langle t \rangle^{-1} \left( E_k(t) + \varepsilon^2(t)^{-3/2} E_k^{1/2}(t) + \varepsilon^5(t)^{-3} \right) \quad \text{for} \quad j = 1, 2, \hfill (4.42)
\]

where \(|G_j(t)| \leq C \langle t \rangle E_k(t) + \varepsilon^3(t)^{-3/2} E_k^{1/2}(t) + \varepsilon^6(t)^{-3}\).

Combining (4.34) with (4.42) yields

\[
\sum_{|\mu| \leq k} \sum_{j=1, 2} (\hat{H}_{n_2 j}^\mu(t) + \hat{H}_{n_2 j}^\mu(t)) = \frac{d}{dt} \tilde{G}_1(t) + \tilde{G}_2(t), \hfill (4.43)
\]

where \(|\tilde{G}_1(t)| \leq C \langle t \rangle E_k(t) + \varepsilon^3(t)^{-3/2} E_k^{1/2}(t) + \varepsilon^6(t)^{-3}\), and \(|\tilde{G}_2(t)| \leq C \langle t \rangle^{-1} \left( E_k(t) + \varepsilon^5(t)^{-3} + \varepsilon^{-3/2} E_k^{1/2}(t) \right)\).

Third, according to the expressions of \(A_j\) for \(j = 0, 1, 2\), it is easily known that

\[
\sum_{j=0}^2 |\partial_j A_j(t)|_\alpha \leq C \left( |\nabla \theta(t)|_\alpha + |\nabla \cdot w(t)|_\alpha + |w \cdot \nabla z(t)|_\alpha \right) \leq C \langle t \rangle^{-3/2},
\]

and then

\[
\left| \sum_{j=0}^2 (\nabla_j A_j) (\zeta^\mu, \zeta^\mu)_\alpha(t) \right| \leq C \langle t \rangle^{-3/2} E_k(t). \hfill (4.44)
\]

For the case of \(|x| \geq \frac{t}{2} + M + 1\), we know \(z(t, x) = 0\) by Lemma 2.6 and

\[
\sum_{j=0}^2 (\nabla_j A_j) (\zeta^\mu, \zeta^\mu)_\alpha(t)
\]
and in the same way, one has

\[ \int_{1/\varepsilon}^{t} \langle (\Delta \phi) \Gamma^\mu \nabla \phi - (\Gamma^\mu \partial_t \phi) \nabla \partial_t \phi, \Gamma^\mu \dot{w} \rangle_+ (\tau) d\tau \leq C_{\varepsilon}^3 \ln t + C_{\varepsilon} \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} E_k(\tau) d\tau \]

and

\[ \langle (\nabla \cdot w) \Gamma^\mu F_2(\xi, 2\xi_a + \xi \dot{\xi}) - \nabla F_1(\xi) \cdot \Gamma^\mu \dot{w}, \Gamma^\mu \dot{\theta} \rangle_+ (t) + \langle (\Gamma^\mu F_2(\xi, 2\xi_a + \xi \dot{\xi}) - \Gamma^\mu \dot{\theta} \nabla F_1(\xi) - \Gamma^\mu F_2(\xi, 2\xi_a + \xi \dot{\xi}) \nabla \theta, \Gamma^\mu \dot{w} \rangle_+ (t). \]

Substituting (4.46)-(4.48) into (4.45) and further combining with (4.44) yield

\[ \int_{1/\varepsilon}^{t} \sum_{j=0}^{2} \langle (\partial_j A_j) \zeta^\mu, \zeta^\mu \rangle(\tau) d\tau \leq C_{\varepsilon}^3 \ln t + C_{\varepsilon} \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} E_k(\tau) d\tau \]

Based on the estimates of the above three steps, if we set \( N(t) = \sum_{|\mu| \leq k} \langle A_0 \zeta^\mu, \zeta^\mu \rangle(t) - 2\tilde{G}_1(t) \), then

\[ N(t) \leq C_{\varepsilon}^3 \ln t + C_{\varepsilon} \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} E_k(\tau) d\tau + C_{\varepsilon}^2 \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-5/2} E_{k}^{1/2}(\tau) d\tau. \]

Note that \( \tilde{N}(t) = N(t) + \varepsilon^4 \langle t \rangle^{-3} \) is equivalent to \( E_k(t) + \varepsilon^4 \langle t \rangle^{-3} \), then one can derive from (4.50)

\[ \tilde{N}(t) \leq C_{\varepsilon}^3 \ln t + C_{\varepsilon} \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-1} \tilde{N}(\tau) d\tau + C_{\varepsilon}^2 \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-5/2} \tilde{N}^{1/2}(\tau) d\tau. \]
Set \( g(t) = C\varepsilon^3 \ln t + C\varepsilon \int_{1/\varepsilon}^{t} \frac{1}{\langle \tau \rangle}^{-1} N(\tau) d\tau + C\varepsilon^2 \int_{1/\varepsilon}^{t} \langle \tau \rangle^{-5/2} \tilde{N}(\tau) d\tau \), it follows from (4.51) that

\[ g'(t) \leq C\varepsilon(t)^{-1} (g(t) + \varepsilon^2) + C\varepsilon^2 (t)^{-5/2} g(t)^{1/2}, \]

furthermore, if we set \( \tilde{g}(t) = g(t) + \varepsilon^2 \), then

\[ \tilde{g}'(t) \leq C\varepsilon(t)^{-1} \tilde{g}(t) + C\varepsilon^2(t)^{-5/2} \tilde{g}(t)^{1/2}, \]

thus, \( g(t) \leq \tilde{g}(t) \leq C\varepsilon^2(t) C\varepsilon \) and further \( E_k(t) \leq C\varepsilon^2(t) C\varepsilon \). \( \square \)

Based on Lemma 4.3, we next derive the uniform energy estimate on the solution \( (\hat{\theta}, \hat{w}, \hat{z}) \) of (2.4).

**Lemma 4.4.** For a fixed integer \( \lambda \) with \( \lambda \geq 9 \), it is assumed that \( (\hat{\theta}, \hat{w}, \hat{z}) \) is a smooth solution of (2.4) for \( (t, x) \in [1/\varepsilon, T] \times \mathbb{R}^2 \). If \( E_\lambda(t) \leq \varepsilon^2 \) for \( t \in [1/\varepsilon, T] \), then \( E_\lambda(t) \leq \frac{1}{3}\varepsilon^2 \) holds for small \( \varepsilon > 0 \).

**Proof.** Similar to the proof of Lemma 4.3, we will divide the whole proof procedure into three steps. In the following, we always assume \( |\mu| \leq \lambda \) and apply the same notations in (4.5).

First, by Lemma 2.2 and the assumption of \( E_\lambda(t) \leq \varepsilon^2 \) for \( t \in [1/\varepsilon, T] \), we can get

\[ |H_j^\mu(t)|_\varepsilon \leq C\varepsilon^3 (t)^{-5/2}, \quad j \in \{1, 2, 4, 5, 7, 8, 10, 11\}. \] \hspace{5cm} (4.52)

Since \( |\Gamma^\nu \hat{w}(t)|_\varepsilon + |\Gamma^\nu \hat{\theta}(t)|_\varepsilon \leq C(t)^{-1/2} Q_{\mu+2}(t) \leq C\varepsilon^2(t)^{-1/2+\delta} \) holds by Lemma 4.3 for sufficient small positive number \( \delta \), we have

\[ |H_j^\mu(t)|_\varepsilon \leq C\varepsilon^3 (t)^{-3/2+\delta}, \quad j \in \{3, 6, 9, 12\}. \] \hspace{5cm} (4.53)

In addition, from (4.9) and (4.10) we can obtain \( \| \dot{J}_{15}^\mu \| \leq C\varepsilon^2(t)^{-1} \) and further

\[ |H_{15}^\mu(t)| \leq C\varepsilon^2(t)^{-1} \sup_{|x| \leq M+1} |\Gamma^b \hat{w}| \leq C\varepsilon^3(t)^{-3/2+\delta}. \] \hspace{5cm} (4.54)

On the other hand, it follows (3.21) and (3.23)-(3.27) that

\[ \sum_{j=1}^{12} |H_j^\mu(t)|_+ \leq C\varepsilon^3(t)^{-3/2+\delta}. \] \hspace{5cm} (4.55)

Second, it is easy to get

\[ |\dot{H}_j^\mu(t)| \leq C\varepsilon^3(t)^{-5/2} \quad \text{for} \quad j = 1, 2. \] \hspace{5cm} (4.56)

Lemma 2.5 gives \( |\Gamma^\nu \hat{w}(t, x)| \leq C\varepsilon(t)^{-3/2+\delta} \) for \( |x| \leq M+1 \) and \( |\nu| \leq \mu \), and hence

\[ |\ddot{H}_j^\mu(t)| \leq C\varepsilon^3(t)^{-3/2+\delta} \quad \text{for} \quad j = 1, 2. \] \hspace{5cm} (4.57)

Third, as in the proof of Lemma 4.3, one has

\[ \sum_{j=0}^{2} |(\partial_j A_j) \zeta^\mu, \zeta^\mu| \leq C\varepsilon^3(t)^{-3/2}. \] \hspace{5cm} (4.58)
Similar to the estimate (3.24) on \( I_3^{\mu\nu} \), we have

\[
\| (\partial_t \nabla \phi \cdot \Gamma^\mu \nabla \dot{\phi} - \Delta \phi \Gamma^\mu \partial_t \dot{\phi} ) (t) \|_+ \leq C \varepsilon (t)^{-3/2} E_{\lambda+1}^{1/2} \leq C \varepsilon^2 (t)^{-3/2+\delta} \tag{4.59}
\]

and

\[
\| (\Delta \phi \Gamma^\mu \nabla \dot{\phi} - \Gamma^\mu \partial_t \dot{\phi} \nabla \partial_t \phi ) (t) \|_+ \leq C \varepsilon (t)^{-3/2} E_{\lambda+1}^{1/2} \leq C \varepsilon^2 (t)^{-3/2+\delta}. \tag{4.60}
\]

In addition, we can decompose

\[
\langle (\nabla \cdot w) \Gamma^\mu F_2 (\xi, 2\xi_a + \xi) - \nabla F_1 (\xi) \cdot \Gamma^\mu \dot{w}, \Gamma^\mu \dot{\theta} \rangle_+ (t) + \langle (\Gamma^\mu F_2 (\xi, 2\xi_a + \xi) - \Gamma^\mu \dot{\theta} ) \nabla F_1 (\xi) - \Gamma^\mu F_2 (\xi, 2\xi_a + \xi) \nabla \theta, \Gamma^\mu \dot{w} \rangle_+ (t)
\]

\[
\equiv A_1^\mu (t) + A_2^\mu (t) + A_3^\mu (t), \tag{4.61}
\]

where

\[
|A_1^\mu (t)| = | \langle \Delta \phi \Gamma^\mu F_2 (\xi, 2\xi_a + \xi) - 2 \nabla F_1 (\xi) \cdot \Gamma^\mu \nabla \dot{\phi}, \Gamma^\mu F_2 (\xi, 2\xi_a + \xi) \rangle_+ | \leq C \varepsilon^3 (t)^{-3/2} E_\lambda (t) \leq C \varepsilon^5 (t)^{-3/2}, \tag{4.62}
\]

\[
|A_2^\mu (t)| = | \langle \nabla \partial_t \phi \cdot \Gamma^\mu \nabla \dot{\phi} - \Delta \phi \Gamma^\mu \partial_t \dot{\phi}, \Gamma^\mu F_2 (\xi, 2\xi_a + \xi) \rangle_+ | \leq C \varepsilon^4 (t)^{-2+\delta}, \tag{4.63}
\]

\[
|A_3^\mu (t)| = | 2 \langle \nabla F_1 (\xi) \cdot \Gamma^\mu \nabla \dot{\phi}, \Gamma^\mu \partial_t \dot{\phi} \rangle_+ | \leq C \varepsilon^2 (t)^{-3/2} E_\mu (t) \leq C \varepsilon^4 (t)^{-3/2}, \tag{4.64}
\]

here we point out that the estimate of \( \nabla F_1 (\xi) \in A_3^\mu (t) \) follows from

\[
|\partial_t F_1 (\xi) (t) \rangle_+ = | (\partial_t \phi \partial_t \partial_t \phi - \nabla \phi \cdot \nabla \partial_t \phi ) (t) + \partial_t (F_1 (\xi) (\theta - \frac{1}{2} F_1 (\xi))) (t) \rangle_+ | \leq C \varepsilon^2 (t)^{-3/2}.
\]

Therefore, by substituting (4.62)-(4.64) into (4.61) and further combining with (4.59)-(4.60), it follows from (4.45) that

\[
\left| \sum_{j=0}^{2} \langle (\partial_j A_j) \zeta^\mu, \zeta^\mu \rangle_+ (t) \right| \leq C \varepsilon^3 (t)^{-3/2+\delta}. \tag{4.65}
\]

Finally, inserting (4.52)-(4.58) and (4.65) into (4.4) and combining with basic energy inequalities (similar to (2.12) and (3.22)) yield

\[
\frac{d}{dt} E_\mu (t) \leq C \varepsilon^3 (t)^{-3/2+\delta}. \tag{4.66}
\]

In addition, we have \( E_\mu (\frac{1}{\varepsilon}) \leq C \varepsilon^4 \) by Lemma 4.1. This, together with (4.66), derives

\[
E_\mu (t) \leq C \varepsilon^3 \quad \text{when we choose} \quad \delta < \frac{1}{2}. \quad \text{Therefore,} \quad E_\mu (t) \leq \frac{1}{2} \varepsilon^2 \quad \text{holds for small} \quad \varepsilon > 0. \quad \square
\]

Finally, we start to show Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 4.1 and Lemma 4.4, we know that (2.4) admits a global smooth solution \((\theta, w, z)\) in terms of the continuity induction method. Thus, (2.3) has a global smooth solution \((\theta, w, z)\) since the smooth solution of (2.1) exists globally. Therefore, Theorem 1.1 is proved. \( \square \)
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