Window Expressions for Stream Data Processing

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Traditional ways of storing and querying data do not work well in scenarios where data is being generated continuously and quick decisions need to be taken. For example, in hospital intensive care units, signals from multiple devices need to be monitored and the occurrence of any anomaly should raise alarms immediately. A typical design would take the average from a window of say 10 seconds (time-based) or 10 successive (count-based) readings and look for sudden deviations. Existing stream processing systems either restrict the windows to time or count-based windows or let users define customized windows in imperative programming languages. These are subject to the implementers’ interpretation of what is desired and hard to understand for others.

We introduce a formalism for specifying windows based on Monadic Second Order logic. It offers several advantages over adhoc definitions written in imperative languages. We demonstrate four such advantages. First, we illustrate how practical streaming data queries can be easily written with precise semantics. Second, we can get different but expressively equivalent formalisms for defining windows. We use one of them (regular expressions) to design an end-user-friendly language for defining windows. Third, we use another expressively equivalent formalism (automata) to design a processor that automatically generates windows according to specifications. The fourth advantage we demonstrate is more sophisticated. Some window definitions have the problem of too many windows overlapping with each other, overwhelming the processing engine. This is handled in different ways by different engines, but all the options are about what to do when this happens at runtime. We study this as a static analysis question and prove that it is undecidable to check whether such a scenario can ever arise for a given window definition. We identify a fragment for which the problem is decidable.

Additional Key Words and Phrases: Streaming Data, Window Definitions, Regular Expressions, Automata based approach

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1 INTRODUCTION

Stream Processors. Stream Processors are programs that consume and produce streams of data. They are applied in many areas, ranging from detecting who is controlling the ball in soccer matches [19] to detecting irregularities in heartbeat rhythms in implantable cardioverter-defibrillators (ICDs) [2] and continuous analysis of RFID readings to track valid paths of shipments in inventory management systems [23]. One common aspect is that they produce output within a bounded amount of time, during which they can only read a bounded portion of the input.

Windows. Windows define a span of positions along a stream that the program can use as a unit to perform computations on and are fundamental to stream processors. Stream processors allow the end-users to specify windows using their specification language, however, not all of them allow end-users to define customized windows. The survey [6,
Table IV] mentions thirty-four processors, of which six allow user-defined windows. Four of them allow only time-based or count-based windows. The other two (Esper [11] and IBM System S [14]) allow customizable windows based on other criteria, but they must be written in imperative programming languages.

The syntax of many languages used in stream processors extend database query languages. Relational database query processing arguably derives a significant portion of its robustness from the fact that it is based on relational algebra that is expressively equivalent to first-order logic [5]. But for processing streaming data, the fundamental linear order of arrival is not part of the syntax. This may not be important for all applications, but we demonstrate many real-world streaming data queries that are simplified when written using the linear order.

Window Expressions. We introduce a novel method of defining windows called Window Expressions based on Monadic Second Order (MSO) logic of one successor, which includes the linear order of arrival as a basic building block. Apart from easily expressing practical queries, we get other advantages from concepts and constructs in formal language theory. The equivalence with regular expressions allows us to design a language for defining windows that is easier to understand for end users compared to logic-based syntax [21]. In particular, we give two equivalent definitions one based on MSO and another based on regular expressions. The equivalence with automata allows us to design a procedure that automatically produces windows from a data stream according to specifications.

Overlapping Windows. It is possible that a window specification results in a large number of windows overlapping at the same position of a data stream, overwhelming the stream processor. This is usually handled by engines using load shedding — dropping off information items from the stream at runtime when load becomes too high. This may be acceptable if it is caused by high input rate, but not if there is a design fault in the window definition. We study the crucial problem of checking whether the number of windows overlapping at a single position is potentially unbounded for a given window definition. This problem is undecidable in general for infinite alphabets. We show that this is decidable for finite alphabets and also prove the decidability for window expressions where the alphabet theory has the so-called completion property.

Data Streams and Symbolic MSO. Data elements in a stream are usually numerical or similar values from an infinite domain. Symbolic MSO [7] is meant to deal with infinite alphabets. The atomic formulas of this logic can check the properties of input symbols using predicates over the infinite alphabet. This also has automata counterparts, called symbolic automata [8] and symbolic regular expressions [22]. We will define these terms in the next section. Our window expressions can be expressed using a guarded variant Symbolic MSO that we later define formally. The high expressiveness that comes with using Symbolic MSO has advantages that outweigh the disadvantage of unboundedness being undecidable for Symbolic MSO. Moreover, we identify a fragment for which it is decidable, even for symbolic automata.

Structure of the paper. In Section 2 we give the preliminary definitions. We define window expressions using different formalisms in Section 3. We provide a skeleton algorithm of our stream processor in Section 4. In Section 5, we provide a few applications of window expressions to specify complicated windowing. Finally, we prove the decidability of the unboundedness of overlapping windows for restricted classes of window expressions in Section 6.

Related works. Regular expressions are extended in [16] with operators to handle quantitative data for processing data streams. Defining windows is a basic construct in our formalism, but in [16], windows have to be defined as derived operators or written in external code. A model called data transducers is used in [3] for implementing data stream processors, but again windows are not part of the core specification language. In [12], the windows are restricted to simple windows such as tumbling windows and sliding windows. In [13], a formal framework based on models of
We assume that data streams are infinite sequences of letters from an infinite alphabet \( \Sigma \). Given a word \( w \in \Sigma^* \) and \( i, j \in \mathbb{N} \), \( w[i : j] \) represents the contiguous substring of \( w \) starting from \( i \)th index to \( j \)th index, both inclusive. We start indexing from 0. Let \( w[i] = w[i : i] \) and \( w[i : k] \) be the suffix of \( w \) of length \( k \).

We use symbolic automata and MSO. The results given in this section are already known, or easy adaptations of similar results for finite alphabets. Proofs and some details are moved to the appendix due to space constraints. We refer to [7, 8, 22] for details. In standard automata and MSO, transitions, and atomic formulas can check that the symbol at a position is equal to some particular letter in a finite alphabet. In Symbolic automata and MSO, we can instead check that the symbol at a position satisfies some property specified in first-order logic. For example, an atomic formula of symbolic MSO can check that an input symbol is an even number, which can be specified in first-order logic over \( \mathbb{N} \) with addition.

**Definition 2.1.** An alphabet theory is a tuple \( \mathcal{A} = (\Sigma, V, \Psi_V) \) such that \( \Sigma \) is an infinite alphabet and \( \Psi_V \) is a set of first-order formulas with free variables \( V \) closed under boolean connectives \( \{\lor, \land, \neg\} \) with \( \bot, \top \in \Psi_V \). Given \( \psi \in \Psi_V \) and a valuation \( \nu : V \rightarrow \Sigma \), it should be decidable to check whether \( \nu \models_{\mathcal{A}} \psi \).

We use \( \models_{\mathcal{A}} \) to denote the models relation in the alphabet theory, to distinguish it from models relation in other logics. We use a variation of symbolic automata that can read symbols in the previous \( k \) positions, for a fixed \( k \), in addition to the current symbol. They are called \( k \)-symbolic lookback automata (\( k \)-SLA), similar to \( k \)-symbolic lookback transducers introduced in [10].

**Definition 2.2.** A \( k \)-SLA is a tuple \( S = (\mathcal{A}, Q, q_0, F, \delta) \) where \( \mathcal{A} = (\Sigma, V, \Psi_V) \) is an alphabet theory with \( V = (x_{-k}, \ldots, x_0) \) as a set of \( k + 1 \) lookback variables, \( Q \) is a finite set of states, \( q_0 \in Q \) is the initial state, \( F \subseteq Q \) is the set of final states, and \( \delta : Q \times Q \rightarrow \Psi_V \) is the transition function.

Given a word \( w \) of length \( |w| = k + 1 \), \( \nu V(w) \) denotes the valuation such that \( \nu V(w)(x_{-j}) = w[k - j] \) for all \( j \in [0, k] \).

We define the run of \( S \) on a word \( w \in \Sigma^* \) of length \( n \geq k + 1 \) to be a sequence \( (q_0, q_1, \ldots, q_{n-k-1}) \) of states such that it starts from the initial state \( q_0 \), and for all \( i \in [k, n-1] \) we have \( \nu V(w[i - k : i]) \models_{\mathcal{A}} \delta(q_{i-k}, q_{i-k+1}) \).

If \( \delta(q_i, q_j) = \varphi \), then \( \varphi \) is called the guard of the transition \( q_i \rightarrow q_j \). A \( k \)-SLA is deterministic if for every \( q, q^\prime, q^\prime\prime \in Q \) with \( q^\prime \neq q^\prime\prime \), \( \delta(q, q^\prime) \land \delta(q, q^\prime\prime) \) is unsatisfiable. We say a \( k \)-SLA is clean if every \( \delta(q, q^\prime) \) is either \( \bot \) or is satisfiable. We can easily construct an equivalent clean \( k \)-SLA for any \( k \)-SLA by replacing all the unsatisfiable guards by \( \bot \). The language accepted by \( S \) is \( L(S) = \{ w \in \Sigma^* \mid w \text{ is accepted by } S \} \). The usual closure properties are satisfied by \( k \)-SLA.

**Lemma 2.3 (Determinization).** Given a \( k \)-SLA \( S = (\mathcal{A}, Q, q_0, F, \delta) \), we can construct a deterministic \( k \)-SLA \( S' = (\mathcal{A}, Q', q'_0, F', S') \) such that \( L(S) = L(S') \).

We complement languages of \( k \)-SLAs with respect to strings of length at least \( k + 1 \): \( \overline{L(S)} = (\Sigma^{k+1} \cdot \Sigma^*) \setminus L(S) \). The following result follows from the previous one.
Lemma 2.4 (Complementation). Given a k-SLA $S$, we can construct a k-SLA $S'$ such that $L(S') = \overline{L(S)}$.

Product construction works on k-SLA as usual. Given two k-SLA $S_1 = (\mathcal{A}, Q_1, q_0^1, F_1, \delta_1)$ and $S_2 = (\mathcal{A}, Q_2, q_0^2, F_2, \delta_2)$, the product of $S_1$ and $S_2$ is $S = (\mathcal{A}, Q \times Q', (q_0, q'_0), F, \delta)$ where $F = \{(q, q') | q \in F_1 \text{ and } q' \in F_2\}$ and $\delta((q_1, q_2), (q'_1, q'_2)) = \delta_1(q_1, q'_1) \wedge \delta_2(q_2, q'_2)$ for all $q_1, q'_1 \in Q$ and $q_2, q'_2 \in Q_2$.

Lemma 2.5 (Intersection). Given two k-SLA $S_1 = (\mathcal{A}, Q_1, q_0^1, F_1, \delta_1)$ and $S_2 = (\mathcal{A}, Q_2, q_0^2, F_2, \delta_2)$, let the product of $S_1$ and $S_2$ be $S$. Then, $L(S) = L(S_1) \cap L(S_2)$.

The languages of k-SLA are also closed under union, which can be proved as usual by taking disjoint union of two automata. We use a variant of concatenation, k-concatenation, denoted by $\cdot_k$. Given two strings $w_1 = wo$ and $w_2 = w\omega'$ with $w, \omega' \neq \varepsilon$ and $|w| = k$, we define $w_1 \cdot_k w_2 = w\omega'$. If the last $k$ letters of $w_1$ do not exactly match with first $k$ letters of $w_2$, or length of either word is less than $k + 1$, then the concatenation is undefined. The k-concatenation of two languages $L_1$ and $L_2$ is defined as $L_1 \cdot_k L_2 = \{w\omega' \in \Sigma^* | wo \in L_1, \omega' \in L_2 \text{ and } |w| = k\}$.

Lemma 2.6. The languages of k-SLA are closed under k-concatenation.

Symbolic regular expressions are regular expressions over infinite alphabets.

Definition 2.7. Given an alphabet theory $\mathcal{A} = (\Sigma \cup \{e\}, V, \Psi_V)$, the set of Symbolic Regular Expressions (SRE) is defined by the following grammar: $R := \varphi | R + R | R \cdot_k R | R^*$, where $\varphi \in \Psi_V$.

The semantics of SRE are defined as follows: $L(\varphi) := \{w | |w| = k + 1, v_V(w) \models_{\mathcal{A}} \varphi\}, L(R_1 + R_2) := L(R_1) \cup L(R_2)$, $L(R_1 \cdot_k R_2) := L(R_1) \cdot_k L(R_2)$ and $L(R^*) := \bigcup_{n \in \mathbb{N}} L(R^n)$.

Using the same constructions as those used for finite alphabets, we can prove that symbolic regular expressions and k-SLA are equally expressive. We recall Symbolic Monadic Second Order (S-MSO) logic from [7].

Definition 2.8. Given an alphabet theory $\mathcal{A} = (\Sigma, V, \Psi_V)$. The syntax of S-MSO over $\mathcal{A}$ is defined by the following grammar: $\phi ::= [\varphi](x) | x < y | X(x) | \neg \phi | \phi \wedge \phi | \exists x \phi | \exists X \phi$, where $\varphi \in \Phi_V$, lower case letters $x, y, z$ are first-order variables and upper case letters $X, Y, Z$ are second order variables.

Let $\phi$ be an S-MSO formula with free variables $FV(\phi)$. Consider a word $w \in \Sigma^*$ with $|w| \geq k$ and a map $\theta : FV(\phi) \rightarrow [k - 1, |w| - 1] \cup 2^{[k - 1, |w| - 1]}$, where the first order variables are mapped to $[k - 1, |w| - 1]$ and second order variables are mapped to $2^{[k - 1, |w| - 1]}$. Given a $w_{\text{block}} = a_0 \ldots a_k$ of size $k + 1$, we define $v(\theta_{\text{block}})(x_i) = a_i$ for all $i$. If $w_{\text{block}} = \varepsilon$, then we define $v(\varepsilon)(x) = \varepsilon$. The semantics of S-MSO with k-lookback is as follows. For atomic formulas $[\varphi](x), w, \theta \models [\varphi](x) \iff v_V(w(\theta(x) - k : \theta(x))) \models_{\mathcal{A}} \varphi$. The semantics is extended to the rest of the syntax as usual, the details of which can be found in the appendix. To prove that k-SLA and S-MSO are equally expressive, the following extension of alphabet theories are helpful.

Definition 2.9. The extension of an alphabet theory $\mathcal{A} = (\Sigma, V, \Psi_V)$ with a boolean variable $x$ is a new alphabet theory $\mathcal{A}_x = (\Sigma \times \{0, 1\}, V, \Psi_V \times (x = 0, x = 1))$ such that for any $a \in \Sigma$ and $b, b' \in \{0, 1\}$, $(a, b) \models (x, x = b')$ if and only if $a \models_{\mathcal{A}} \varphi$ and $b = b'$. We denote the extension of $n$ boolean variables $x_1, x_2, \ldots, x_n$ by $\mathcal{A}(x_1, x_2, \ldots, x_n)$.

Given a word $w$ and a map $\theta$, let us define $w_\theta \in (\Sigma \times \{0, 1\})^*$ as

$$w_\theta[i] = (w[i], e_1(\theta(x_1), i), \ldots, e_1(\theta(x_n), i), e_2(\theta(X_1), i), \ldots, e_2(\theta(X_m), i)).$$
where \( e_1(n,i) = 1 \) if \( n = i \) else 0, and \( e_2(I,i) = 1 \) if \( i \in I \) else 0. Let \( \phi(x_1, \ldots, x_m, X_1, \ldots, X_n) \) be a S-MSO formula with first order free variables \( \{x_1, \ldots, x_m\} \) and second order free variables \( \{X_1, \ldots, X_n\} \). A construction similar to that of MSO over finite alphabets will give a \( k \)-SLA \( S \) over the extended alphabet theory \( A(x_1, \ldots, x_m, X_1, \ldots, X_n) \). Such a construction will give a \( k \)-SLA \( S \) over the extended alphabet theory \( \Sigma \), such that \( w \models \phi \) if and only if the word \( w_\emptyset \in (\Sigma \times \{0,1\}^{n+m})^* \) is accepted by \( S \).

Conversely, given a \( k \)-SLA \( S \) over the alphabet theory \( A = (\Sigma, V, \Phi_V) \), we can construct a S-MSO formula \( \phi \) over \( A \) with no free variables such that \( w \in \Sigma^* \) is accepted by \( S \) if and only if \( w \models \phi \) using the standard automata to MSO construction and replacing the letters with predicates \( \phi \in \Phi_V \).

The expressive power of symbolic lookback automata and symbolic MSO are useful for designing parsing algorithms and specification languages as we will see subsequently. But the expressive power is enough to simulate Turing machines and static analysis problems are undecidable.

**Theorem 2.10.** The problems of checking non-emptiness of languages of \( k \)-SLAs and satisfiability of S-MSO formulas are undecidable.

**Sketch.** The automata work over infinite domains and transitions can relate values at a position with previous values. This can be used to simulate counter machines. The domain is the set \( \mathbb{N} \) of natural numbers and the counters are simulated by numerical fields in the input stream. An incrementing transition of the counter machine can be simulated by a transition of a \( k \)-SLA, by requiring that the next value of the corresponding field is one more than the previous one. Decrementing and zero testing transitions can be similarly simulated. This is a standard trick used for models dealing with infinite domains, e.g., [9, 10].

### 3 Defining Windows with S-MSO

A window in a data stream is a pair \((i_b, i_e)\) of indices that indicate where the window begins and ends. In this section, we explain how windows can be defined with S-MSO over an alphabet theory \( A \), and also show an expressively equivalent representation using symbolic regular expressions. Some proofs have been moved to the appendix due to space constraints.

We designate first-order variables \( x_b, x_e \) for denoting the beginning and ending indices of windows. We use S-MSO formulas to specify which indices can begin and end windows. The end of a window should be detected as soon as it arrives in the stream, so the decision about whether a position is the end of a window should be made based only on the stream data that has been read so far. We enforce this in S-MSO formulas by guarding the quantifiers.

\[
\phi := \exists x \left[ x < x' \right] \left[ X(x) \right] \land \neg \phi \lor \phi \land \exists x_e \phi \land \exists X \subseteq [0, x_e] \phi
\]  

(1)

The above syntax is a guarded fragment of S-MSO and \( \exists x \leq x_e \phi \) is syntactic sugar for \( \exists x(x \leq x_e \land \phi) \) and \( \exists X \subseteq [0, x_e] \phi \) is syntactic sugar for \( \exists X(\forall y(X(y) \Rightarrow y \leq x_e) \land \phi) \).

**Definition 3.1.** Let \( x_b \leq x_e \land \phi(x_b, x_e) \) be a S-MSO formula in the guarded fragment given in (1), with \( x_b \) and \( x_e \) being free variables. A pair \((i_b, i_e)\) of indices of a word \( w \) is said to be a window recognized by \( x_b \leq x_e \land \phi(x_b, x_e) \) if \( i_b \leq i_e \) and \( w \models \phi(i_b, i_e) \).

To reduce clutter, we don’t explicitly write the condition \( x_b \leq x_e \) but assume that it is present in all window specifications. Whether a pair \((i_b, i_e)\) is recognized as a window by \( \phi(x, y) \) in a word \( w \) depends only on the word \( w[0 : i_e] \).

\[1\]  
The authors thank an anonymous referee for suggesting this way of defining windows, and its equivalence with union of pairs of regular expressions.

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In this section, we show a skeleton stream processor that takes window expressions and incoming streaming data and produces windows as specified in the expressions. For modeling purposes, we treat a stream as an infinite string in $\Sigma^\omega$. The data present in a window is usually processed to produce an aggregate value, such as the average of some field, sum of all entries in a window etc. If some positions of the stream belong to multiple windows, performing the same computation multiple times is inefficient. One way to avoid this is to sub-divide windows into panes [15] and then aggregate the values of those panes that make up a window. For example, suppose the average of a numerical field is to be computed for every window. If within the span of a window, other windows start, then the window is sub-divided into panes as shown below.

For each pane, the average and number of entries in the tuple is computed and stored. When the window ends, these can be used to compute the average of the whole window. The exact computation to be performed in panes is application dependent. We assume that a class is provided to do that computation. In our skeleton processor, we ensure that the class methods are called at the correct positions in the stream.

End users of streaming data processors may not be familiar with logic based languages. It has been observed [21] that specifications based on regular expressions are easier to understand compared to those based on logic. Next we give a way of defining windows based on symbolic regular expressions, that is expressively equivalent to the one above based on S-MSO.

Definition 3.2. A window expression is a set $R = \{(r_1, r'_1), \ldots (r_k, r'_k)\}$. For every $i$, $r_i, r'_i$ are symbolic regular expressions over an alphabet theory $A$.

Given a word $w \in \Sigma^*$, a pair $(i_b, i_e) \in [k, |w| - 1]^2$ is said to be recognized as a window by $R$ if there exists $(r, r') \in R$ such that $w[0 : i_b - 1] \in L(r)$ and $w[i_b : i_e] \in L(r')$.

Lemma 3.3. Given a S-MSO formula $\phi(x_b, x_e)$ in the guarded fragment given in (1), we can effectively construct a window expression $R = \{(r_1, r'_1), \ldots (r_k, r'_k)\}$ such that for every word $w$, the set of windows recognized by $\phi(x_b, x_e)$ is same as that recognized by $R$.

Sketch. We take the automaton corresponding to the S-MSO formula $\phi$ and split it at a transition that reads the symbol at position $x_b$. Each such split results in a pair of expressions. \hfill \Box

Lemma 3.4. Given a window expression $R = \{(r_1, r'_1), \ldots (r_k, r'_k)\}$ over $A$, we can effectively construct a S-MSO formula $\phi(x_b, x_e)$ in the guarded fragment given in (1) such that, for every word $w$, the set of windows recognized by $R$ is same as that recognized by $\phi(x_b, x_e)$.

4 STREAM PROCESSOR FOR WINDOW EXPRESSIONS

In this section, we show a skeleton stream processor that takes window expressions and incoming streaming data and produces windows as specified in the expressions. For modeling purposes, we treat a stream as an infinite string in $\Sigma^\omega$. The data present in a window is usually processed to produce an aggregate value, such as the average of some field, sum of all entries in a window etc. If some positions of the stream belong to multiple windows, performing the same computation multiple times is inefficient. One way to avoid this is to sub-divide windows into panes [15] and then aggregate the values of those panes that make up a window. For example, suppose the average of a numerical field is to be computed for every window. If within the span of a window, other windows start, then the window is sub-divided into panes as shown below.

For each pane, the average and number of entries in the tuple is computed and stored. When the window ends, these can be used to compute the average of the whole window. The exact computation to be performed in panes is application dependent. We assume that a class is provided to do that computation. In our skeleton processor, we ensure that the class methods are called at the correct positions in the stream.
The skeleton processor is shown in Algorithm 1. For simplicity of presentation, we show the processor for one pair of expressions \((r, r')\). We further assume that the pair has been converted to a pair \((PA, WA)\) of prefix and window \(k\)-SLA, both deterministic. It is routine to extend the processor to handle multiple pairs. The variable \(\text{PrefixState}\) stores the current state of the prefix automaton \(PA\). The variable \(\text{WindowStartIndices}\) stores a mapping \(\text{WindowStartIndices} : Q \rightarrow 2^\mathbb{N}\), which tracks multiple copies of the window automaton \(WA\), as explained next. A copy of \(WA\) is started at any position of the input stream that is potentially a start position of a window. If such a copy started at position \(x\) is currently in state \(q\), then \(x \in \text{WindowStartIndices}(q)\). In other words, all the copies of \(WA\) that are currently in state \(q\) are tracked by storing their starting positions in \(\text{WindowStartIndices}(q)\). The formal proof of the correctness of the processor is established in the following two results.

**Lemma 4.1 (Main Loop Invariant).** The following hold at the start of every iteration of the main loop of the processor (line 9), where \(n\) is the position of the last symbol read from the input channel \(I\):

1. \(\text{PrefixState}\) stores the state of \(PA\) after reading \(w[0 : n]\).
2. For every position \(i < n\) such that \(PA\) reaches a final state on reading \(w[0 : i]\), if \(\text{init}_{WA} \xrightarrow{w[i+1:n]} q\) and \(q\) is not a dead-state, then \(i + 1 \in \text{WindowStartIndices}(q)\).

**Corollary 4.2.** The processor shown in Algorithm 1, given the input pair \((PA, WA)\) and the stream \(w\), will add \((x+1, y)\) to the output stream iff \(w[0 : x] \in L(PA)\) and \(w[x + 1 : y] \in L(WA)\).

### 5 Examples from Practical Applications

In this section, we give some examples from practical applications of stream data processing, to illustrate how S-MSO or symbolic regular expressions can express queries with clearly specified semantics.

1. **Genome Sequencing:** As the size of data involved in analyzing DNA sequences is increasing, recently there has been interest in employing streaming algorithms [17] to analyze them. A fundamental object in genome sequence analysis is \(k\)-mer. We next explain what they are and how they can be naturally modeled as windows. We first present the definition in words, as they are usually presented:

   1. We have a set of DNA fragments, where each fragment is called a read. Each read is a string of ‘A’, ‘T’, ‘C’, and ‘G’.
   2. A \(k\)-mer is a contiguous substring of length \(k\) of a read.
   3. The \(p\)-minimizer of a \(k\)-mer is the lexicographically smallest substring in the \(k\)-mer with length \(p\).
   4. A supermer is the merge of all the consecutive \(k\)-mers that have same \(p\)-minimizer.

   Given a DNA as a string \(m = m_1m_2 \ldots m_p\), we can convert it into an integer using 2-bit compression. Let \(\Sigma' = \{00, 01, 10, 11\}\). We define the 2-bit compression via a monoid homomorphism \(f : \Sigma^* \rightarrow \Sigma'^*\) defined as \(f(A) \rightarrow 00\), \(C \rightarrow 01\), \(G \rightarrow 10\), and \(T \rightarrow 11\). The string \(m\) will be mapped to a 2\(m\) bit-sized binary string \(f(m)\), which we interpret as an integer. We reserve the variable \(k\) as we use it for \(k\)-SLA, and use the term \(t\)-mer instead. We will now describe the window expression that can be used to capture \(t\)-mers and supermers in our formalism:

   1. A read is a string \(w \in \Sigma^*\), where \(\Sigma = \{A, T, C, G\}\) from the input stream \(I\).
   2. We apply the window expression \((\text{True})^*, (\text{True})^p\) on \(I\) and pass on the output to \(O_{\text{MIN}}\). We also output an aggregate function for the windows, which will be the application of 2-bit compression \(f\). We put this in a new integer field \(x\) in the output stream, which will carry the lexicographic index of the substring in the last \(p\) positions.
Input: A pair of k-PLAs $\mathcal{R} = (PA, WA)$, channels $I, O$ for input, output
Result: Stream from $I$ decomposed into windows according to $\mathcal{R}$ and output into $O$
Nomenclature: $Q, Q'$: set of states of $PA, WA$

Function `main`

```plaintext
PrefixState := q0;
   /* $q_0$ is the initial state of $PA$ */
WindowStartIndices($q'$) := \emptyset for all $q' \in Q'$;
w_block := read first $k$ elements of $I$;
x := k − 1;
p := new Pane(StartIndex = 0, EndIndex = k − 1);
```

Update value of p to the aggregate of $w_block$;

```plaintext
P := \{p\};
   /* $P$ is the set of Panes */
```

while input stream $I$ is live do
    $\sigma$ := next item in $I$;
w_block := w_block[$−k + 1 : 0]$ $\cdot \sigma$;
startNewPane := False;
if PrefixState is a final state of $PA$ then
    Add the index $(x + 1)$ to WindowStartIndices($WA$) (init$WA$ is the initial state of $WA$);
    startNewPane := True;
end
foreach $\{q'_f \in Q' \mid q'_f$ is a final state of $WA, WindowStartIndices(q'_f) \neq \emptyset\}$ do
    Add $(x', x)$ to the output stream $O$ for each index $x' \in WindowStartIndices(q'_f)$, along with the aggregate of those panes in $P$ that are between $x'$ and $x$;
    startNewPane := True;
end
foreach $\{q_d \in Q' \mid q_d$ is a dead state of $WA\}$ do
    WindowStartIndices(q_d) := \emptyset;
end
if startNewPane == True then
    $p$.EndIndex := $x$; $p$ := newPane(); $P$.add($p$);
end
$p$.add_element($\sigma$);
   /* update the current pane with the newly read symbol */
```

Let $x_{\min}$ be the minimum of the indices in the range of WindowStartIndices;
From $P$, delete those panes that end before $x_{\min}$;
Update PrefixState to new state of $PA$ by reading $w_block$;

```plaintext
foreach $\{q \in Q'\}$ do
    $Q_{\text{pred}}$ := $\{q' \in Q' \mid q$ is the $w_block$ successor of $q'$ in $WA\}$;
    WindowStartIndices($q$) := $\bigcup_{q' \in Q_{\text{pred}}} WindowStartIndices(q')$;
end
```

WindowStartIndices := WindowStartIndices';
   /* simultaneous update */
x := 1;
end
```

Algorithm 1: Skeleton stream processor to extract windows from a stream
Window Expressions for Stream Data Processing

(3) Let $k = t$ be the number of lookback variables, and define the notation $\min(x[a, b]) := \min(x_a, x_{a+1}, \ldots, x_b)$ where $-t \leq a < b \leq 0$. We apply the window expression $(PA_2, WA_2)$ on $O_{MIN}$ and pass on the output to $O_{KMER_MIN}$, where $PA_2 = True + ((True)^* \cdot k \cdot \min(x[-k + p, -1]) \neq \min(x[-k + p + 1, 0]))$ and $WA_2 = (\min(x[-k + p, -1]) = \min(x[-k + p + 1, 0]))^* \cdot k \cdot (\min(x[-k + p, -1]) \neq \min(x[-k + p + 1, 0]))$.

(4) As we can only decide if the minimizer of two consecutive $t$-mers are different by reading to the point where the minimizers change, we will have to post-process the output stream $O_{KMER_MIN}$ as follows: for each window $(x, y) \in O_{KMER_MIN}$, the window $(x - 1, y - 1)$ is passed to the output stream $O$.

The final output stream $O$ gives us all the supermers w.r.t to the $t$-mers and the $p$-minimizers.

2. Arrhythmia Detection: Implantable Cardioverter Defibrillator (ICD) is a battery-powered device that delivers an electric shock to restore a normal heartbeat if it detects abnormal heart rhythm, called arrhythmia. One of the algorithms used in ICDs is Wavelet Peak Maxima (WPM), which we write using window expressions here. The medical background needed to understand this algorithm can be found in [1], which also explains the structure of the data stream fed to the WPM algorithm. For our purposes, it is enough to assume that the data stream is a sequence of pairs $(t, r)$ where $t$ represents time and $r$ is a real number representing the value of some spectrogram at time $t$. In a stream $(t_1, r_1), (t_2, r_2), \ldots$, a pair $(t_i, r_i)$ is said to be peaking if $r_i \geq \max(r_{i-1}, r_{i+1})$ (i.e., $r_i$ is a local maximum) and $r_i \geq p$ for some pre-defined threshold $p$.

1. The first step in WPM is to check whether a pair is peaking or not. This can be done by using the theory of reals as alphabet theory with three lookback variables $V = \{x_{-2}, x_{-1}, x_0\}$. Given an element $a = (t, r)$, let us define $T(a) = t$ and $R(a) = r$. Let $O_{PEAK}$ be the output stream on processing the input stream with the window expression $R = (PA, WA)$, where $PA = (True)^* \cdot k \cdot (R(x_{-1}) > R(x_{-2}) \land R(x_{-1}) > R(x_0) \land R(x_{-1}) > p)$.

2. The second step is to find the time difference between two consecutive peaks. Note that each element of the stream $O_{PEAK}$ is a window $w_i = (t, r)(t', r')$, such that the first pair $(t, r)$ is peaking. Let us denote the first pair of this window by $w_i[0]$. We process this stream with the window expression $(PA', WA')$ where $PA' = (True)^* \cdot k \cdot (True)^2$ and $WA' = (True)^2$, and the aggregate function of the pane handler, given the new window $w_i \cdot w_{i+1}$, which is the concatenation of two consecutive windows obtained in the previous step, outputs $T(w_{i+1}[0]) - T(w_i[0])$.

The full implementation of WPM involves some more details but these two are the main steps.

3. Trends in stock markets: Sequence of stock prices as they are traded in the market is a natural data stream. Detecting trends in such streams is widely used in both stock markets and algorithmic trading [19]. Suppose $P_1, P_2, \ldots$ is a sequence of prices of a stock being traded and one wishes to check whether the current price has gone up compared to the previous one. This can be done by using the theory of reals with one lookback variable $V = \{x_{-1}\}$ as the alphabet theory. We use the window expression $(PA, WA)$ where $PA = True^*$ and $WA = (x_0 > x_{-1})$. If one wishes to check whether the price have gone up consecutively $k$ times, $k$ lookback variables can be used with a similar expression. One can write quite complicated patterns by using appropriate window expressions.

6. MEMORY REQUIREMENT

In the skeleton processor shown in Algorithm 1, $WindowStartIndices$ and $P$ are variables that store starting positions of windows and panes respectively. If the number of starting positions or the number of panes is too big, the memory required to store them will be large. Here we study whether the memory requirement can be unbounded. The answer to this depends on whether the input data stream follows any pattern in relation to the window expressions. For example,
Let $\Sigma = \{a, b\}$ and consider the window expression $(a + b)^*a^*b$. A window can start at any position and ends at the letter $b$. If this is applied to the input stream $a^c$, an instance of the window automaton $WA$ will be started at every position, but none of them will ever be removed, since $b$ never occurs. This will cause unboundedness. However, the same expression will be bounded if we apply it to a stream not having $a^c$ as an infix, for some constant $c$.

The automata used to specify what input streams are expected will have a particular structure. Indeed, suppose a processor has read an input stream up to some position. Should this be deemed acceptable according to the specification of what input streams are expected? The answer would be “yes” if it is possible to extend the stream so that the resulting string is accepted by the specifying automaton. Hence, any prefix of an acceptable string is also acceptable. So the specifying automaton is prefix closed. Any state from which an accepting state can be reached is also an accepting state. If the automaton reaches a non-accepting state, it will never reach an accepting state again. We can think of a specifying automaton $S$ as being in “accepting zone” or in “rejecting zone”, it starts in accepting zone and switches to rejecting zone at most once. We call such automata input specifiers. A stream is said to conform to an input specifier if it never enters the rejecting zone while reading the stream.

Given a window expression and an input specifier, we address the question of whether an input stream that conforms to the input specifier can potentially cause the stream processor to require unbounded memory.

**Definition 6.1 (Bounded memory).** A window expression given in the form of a pair $(PA, WA)$ of $k$-SLAs is said to process any input stream conforming to a given input specifier $S$ within bounded memory, if the size of $WindowStartIndices$ and the set of panes $P$ in Algorithm 1 are bounded by some numbers $N_1, N_2 \in \mathbb{N}$ respectively, while processing any stream that conforms to $S$.

Checking the bounded memory property is undecidable, just like the problem of checking non-emptiness of languages of $k$-SLAs. We identify a fragment for which it is decidable. We consider alphabet theories $(\Sigma, V, \Psi_V)$ in which $\Psi_V$ is restricted to having only Boolean combinations of atomic formulas of the form $R(x_1, \ldots, x_t)$, where $R$ is a relation symbol. Even under this restriction, checking non-emptiness of the languages of $k$-SLAs is undecidable, if the equality relation and one more relation symbol are present (see [9, Theorem 10.1]).

We further restrict alphabet theories by adapting the concept of completion property [9, Section 4]. Suppose $\Phi$ is a set of atomic formulas over the set of variables $V$, $V' \subseteq V$ and $\Phi \upharpoonright V' \subseteq \Phi$ is the set of those formulas in $\Phi$ that only use variables in $V'$. An alphabet theory is said to have the completion property if for every satisfiable set of formulas $\Phi$, for every subset $V' \subseteq V$ and every partial valuation $\sigma' : V' \rightarrow \Sigma$ that satisfies all the constraints in $\Phi \upharpoonright V'$, there exists an extension $\sigma : V \rightarrow \Sigma$ of $\sigma'$ that satisfies all the constraints in $\Phi$. The theory of integers with the binary relation $<$ does not satisfy the completion property. Consider the set of formulas $\{x < y, x < z, z < y\}$ and a partial valuation $\sigma' : \{x \mapsto 1, y \mapsto 2\}$. It satisfies $x < y$, but it cannot be extended to include a mapping for $z$ such that $x < z$ and $z < y$ (i.e., $z$ is strictly between $x$ and $y$), since there is no integer strictly between 1 and 2. The theory of rational or real numbers with $<$ satisfy the completion property. For theories with linear orders, completion property is closely related to denseness of the domain [9, Lemma 5.3].

Now we give a characterization in terms of automata for checking the bounded memory property.

**Lemma 6.2.** Fix an alphabet theory having the completion property. Given a window expression in the form of a pair $(PA, WA)$ of deterministic $k$-SLAs and an input specifier $S$, the size of $WindowStartIndices$ and the set of panes $P$ in Algorithm 1 will be unbounded on processing streams conforming to $S$ if there exist words $w_1, w_2, w_3$ and a state $q$ of $WA$ such that
(1) There is a path from $q$ to a final state via a path in which no transition has the guard $\bot$ (we say that $q$ is not a dead state in this case),

(2) in the input specifier $S$, $initS \xrightarrow{w_1} s \xrightarrow{w_2} s \xrightarrow{w_0} s$, such that $s$ is an accepting state,

(3) $initPA \xrightarrow{p} p \xrightarrow{p} p$, where $p$ is a final state of $PA$,

(4) $initWA \xrightarrow{w_0} q, q \xrightarrow{w_0} w_0$, and

(5) the runs $s \xrightarrow{w_0} s \xrightarrow{p} p \xrightarrow{w_0} w_0 \xrightarrow{w_0} q$ satisfy the following: the sequence of transitions used while reading the first $k$ letters of $w_2$ is the same sequence used for reading the first $k$ letters of $w_3$.

**Proof.** ($\Rightarrow$) Let $w_n$ be a stream conforming to $S$ for which the processor adds at least $n$ indices in the $WindowStartIndices$ map. There must exist an increasing sequence of positions (or time instants) $(t_i)_i$ such that on processing $w_n[0 : t_i]$, the processor adds a position to $WindowStartIndices(initWA)$ for the $i^{th}$ time that will not be removed from the map on further processing, at least till the $n^{th}$ position is added.

Let $Q_{PA}$, $Q_{WA}$, $Q_S$ be the set of states in $PA$, $WA$, $S$ respectively. Let $T_{PA}$, $T_{WA}$ and $T_S$ be the set of $k$-tuples of transitions in $PA$, $WA$ and $S$ respectively. Let $T$ be the set of all functions of the form $Q_{WA} \rightarrow T_{WA}$. We will use the set of colors $C = T \times Q_{WA} \times Q_{WA}$ to color the edges of a graph we define later. Let $r$ be the Ramsey number $R(n_1, n_2, \ldots, n_{|C|})$, where $n_1 = n_2 = \ldots = n_{|C|} = 2|Q_{PA}|/|Q_{S}|/|T_{PA}|/|T_S| + 1$. We set $n = r$ and construct a complete graph $G$ with the set $(t_i)$ as vertices.

For every two instants $t_i, t_j$ with $i < j$, we add an edge with color $(T_{t_i}, \{(q_1, q_2) \mid q_1 < Q_{WA} \cdot q_1 < Q_{WA} \cdot q_2 \})$, where $T_{t_i}$ is the function such that $T_{t_i}(q) = T_{t_i}(q)$ is the sequence of $k$ transitions executed in $WA$ if it starts at the instant $t_i$ in state $q$. We infer from Ramsey’s theorem that the graph $G$ will have a monochromatic clique of size $2|Q_{PA}|/|Q_{S}|/|T_{PA}|/|T_S| + 1$. We infer from Pigeon Hole Principle that this monochromatic clique contains at least three time instants, say $t_1 < t_2 < t_3$, in which $PA, S$ are in the same pair of states, say $p, s$, which are accepting states of $PA, S$. Also, the sequence of $k$ transitions executed in $PA$ (resp. $S$) from $t_1, t_2, t_3$ are same. Now, consider the instances of window automata $WA$ initiated at the instants $t_i, t_j$ and $t_i$. Let $initWA \xrightarrow{w_{0,1}} w_{0,2} \xrightarrow{w_{0,3}} \cdots \xrightarrow{w_{0,n}} q$ for some state $q$. Since the edges between $t_i, t_j, t_1$ all have the same color, $initWA \xrightarrow{w_{0,1}} w_{0,2} \xrightarrow{w_{0,3}} \cdots \xrightarrow{w_{0,n}} q$. Since $WA$ is deterministic and $t_i < t_j < t_1$, we can split $initWA \xrightarrow{w_{0,1}} w_{0,2} \xrightarrow{w_{0,3}} \cdots \xrightarrow{w_{0,n}} q$ into $initWA \xrightarrow{w_{0,1}} \cdots \xrightarrow{w_{0,i-1}} q$ and $q \xrightarrow{w_{0,i}} \cdots \xrightarrow{w_{0,n}} q$. Since $(t_i, t_j)$ and $(t_j, t_1)$ have the same colour, we infer that $q \xrightarrow{w_{0,i}} \cdots \xrightarrow{w_{0,n}} q$. Let $w_1 = w_{0,1} [0 : t_1]$, $w_2 = w_{0,1} [t_1 + 1 : t_2]$ and $w_3 = w_{0,1} [t_2 + 1 : t_3]$. We have $initWA \xrightarrow{w_{0,1}} p \xrightarrow{p} \cdots \xrightarrow{p} p, initS \xrightarrow{s} \cdots \xrightarrow{s} s, p, (resp. s)$ is accepting in $PA (resp. S)$, $initWA \xrightarrow{w_{0,1}} q, q \xrightarrow{w_{0,2}} \cdots \xrightarrow{w_{0,3}} q$ and the runs $s \xrightarrow{w_{0,1}} s \xrightarrow{p} \cdots \xrightarrow{p} p, q \xrightarrow{w_{0,2}} \cdots \xrightarrow{w_{0,3}} q$ satisfy the following: the sequence of transitions used while reading the first $k$ letters of $w_2$ is the same sequence used for reading the first $k$ letters of $w_3$.

$(\Leftarrow)$ We will prove that for every $i \geq 0$, there is a word $w_i$ satisfying the following properties. Let $w_i = w_{i,1} \cdot w_{i,2} \cdots w_{i,2}$. We will prove that for all $i \geq 0$, $initWA \xrightarrow{w_{i,1}} p$ and $initWA \xrightarrow{w_{i,2}} q \xrightarrow{w_{i,3}} \cdots \xrightarrow{w_{i,n}} q$. For every $i$, the processor will add the index $[w_{i,1} + w_{i,2} + \cdots + w_{i,2}]$ to $WindowStartIndices(initWA)$. The state $initWA$ will be updated to $q$ after reading $w_{i,2} + 1$ and keeps coming back to $q$ after reading $w_{i,2}$ for $j > i + 1$. None of these added indices will be removed, since $q$ is not a dead state. Therefore, there will be $n$ indices in the map $WindowStartIndices$ after reading $w_{1} \cdot w_{0,2} \cdots w_{n+1}$. All of $w_{i,2}$ will be built from $w_2$ using the completion property, ensuring that $w_{i,2}$ makes the automaton behave exactly like $w_2$ did. We will do this with an inductive construction, for which we need to introduce some terminology.

Suppose $w$ is a stream. Consider the substring of $w$ between positions $i - k$ to $i$. We would like to capture the constraints put on this substring by some transition $r$ executed by an automaton at the $(i+j)^{th}$ position, where $j \in [0, k]$. Manuscript submitted to ACM
For that transition, the values for lookback variables $x_{-k}, \ldots, x_{-j}$ are given by $w[i - k + j], \ldots, w[i]$ respectively. For a transition $\tau$ and $j \in [0, k]$, let $\Phi(\tau) \vdash j$ be the set of all atomic formulas $\phi$ occurring in the guard of $\tau$ such that only the lookback variables $x_{-k}, \ldots, x_{-j}$ are used in $\phi$. For each atomic formula $\phi$, let $\phi \vdash j$ be the formula obtained from $\phi$ by replacing every lookback variable $x_{-\ell}$ by $x_{-\ell+j}$ (this results in the values for lookback variables $x_{-k+j}, \ldots, x_0$ of $\phi \vdash j$ being given by $w[i - k + j], \ldots, w[i]$ respectively). In the run $init_S \xrightarrow{w_0} s \xrightarrow{w_0} s \xrightarrow{w_0} s$, let $\tau_1$ be the transition executed while reading the $i$th letter of $w_2w_3$. For $i \in [0, |w_2| - 1]$, the constraints satisfied by the substring $w_1w_2[[w_1[i + i: |w_1| + i]]$ for this run is $\cup j \in [0, k] \{ \phi \vdash j | \phi \in \Phi(\tau_{i+j}) \uparrow j, v_1 (w_1w_2[[w_1[i + i: |w_1| + i]]) \vdash \phi \vdash j \} \cup \cup j \in [0, k] \{ \neg \phi \vdash j | \phi \in \Phi(\tau_{i+j}) \uparrow j, v_1 (w_1w_2[[w_1[i + i: |w_1| + i]]) \notin \phi \vdash j \}$. Let us call this set $\Gamma^S_i$. We similarly define the set $\Gamma^P_i$ for the run $init_P \xrightarrow{w_1} p \xrightarrow{w_1} p \xrightarrow{w_0} p$. In the run $q \xrightarrow{w_1[k]w_2} q \xrightarrow{w_0} q'$, let $\tau_i$ be the transition executed while reading the $i$th letter of $w_2w_3$. For $i \in [0, |w_2| - 1]$, the constraints satisfied by the substring $(w_1[i: k]w_2)[i : i + k]$ for this run is $\cup j \in [0, k] \{ \phi \vdash j | \phi \in \Phi(\tau_{i+j}) \uparrow j, v_2 ((w_1[i: k]w_2)[i : i + k]) \vdash \phi \vdash j \} \cup \cup j \in [0, k] \{ \neg \phi \vdash j | \phi \in \Phi(\tau_{i+j}) \uparrow j, v_2 ((w_1[i: k]w_2)[i : i + k]) \notin \phi \vdash j \}$. Let us call this set $\Gamma^W_i$.

We claim that for every $i \geq 0$, there exists string $w_2$ such that $|w_2| = |w_2|$ and the following property is satisfied. Recall that we defined $w' = w_1w_2 \ldots w_j$. We let $w'^{-1} = w_1$ for convenience. For every $i \geq 0$ and every $j \in [0, |w_2| - 1]$, we claim that $\forall (w')[[w'^{-1}[j: k]]$ satisfies all the constraints in $\Gamma^S_i \cup \Gamma^P_i \cup \Gamma^W_i$. This is sufficient to establish the result, since all the automata $S, PA, WA$ can repeat the same sequence of transitions for $w'^{+1}_2$ as the sequence for $w_2$. We will prove the claim by induction on $i$. For the base case $i = 0$, we set $w'_2 = w_2$ and the claim is satisfied by definition. For the induction step, suppose we have defined up to $w'^{+1}_2$ as claimed. We define $w'^{+1+1}_2[i]$ for every $i \in [0, |w_2| - 1]$ by secondary induction on $j$. For the base case, $j = 0$. By the primary induction hypothesis, $\forall (w')[[w'^{-1}[k + 1]]$ satisfies all the constraints in $\Gamma^S_{|w_0| - 1} \cup \Gamma^P_{|w_0| - 1} \cup \Gamma^W_{|w_0| - 1}$. Let $V' = V \setminus \{x_0\}$. Since the first $k$ transitions for $w_3$ are same as the first $k$ transitions for $w_2$, we infer that $\forall (w')[[k]]$ satisfies those formulas in $\Gamma^S_0 \cup \Gamma^P_0 \cup \Gamma^W_0$ that don’t use $x_0$. By the completion property, $\forall (w')[[k]]$ can be extended to include a valuation for $x_0$ so that the resulting valuation satisfies all the formulas in $\Gamma^S_0 \cup \Gamma^P_0 \cup \Gamma^W_0$. This new valuation for $x_0$ is the value we set for $w'^{+1}_2[0]$. The induction step for $j + 1$ is similar. This completes the induction step and hence establishes the result.

The above characterization can be used to obtain a decision procedure for checking the bounded memory property, provided the alphabet theory itself is decidable. This involves checking that there exist words $w_1, w_2, w_3$ as claimed above and can be done using symbolic models, which decompose the problem into an automata-theoretic problem over finite alphabets and satisfiability of formulas in the alphabet theory. A full description of this method is beyond the scope of this paper. Decidability for bounded memory property for decidable alphabet theories that have the completion property can be proven by using techniques similar to those of [9, Theorem 4.4] (attributed originally to [4]), which when converted to terminology used in this paper, states that checking non-emptiness of the language of a $k$-SLA has the same complexity as checking satisfiability of a finite set of formulas in the alphabet theory, provided it has the completion property.

7 DISCUSSION AND FUTURE WORK

A common feature is to consider data from multiple streams simultaneously, to perform computations. It would be interesting to explore how to integrate this feature into the formal framework. Another open problem is characterizing fragments with decidable bounded memory problems, besides the completion property. Moreover, we would like to study the complexity of this problem, the heuristics and algorithms that can perform well in practice.
It is trivial to observe the number of distinct windows on processing \( n \)-data elements of a stream is upper bounded by \( n^2 \). Another crucial problem is giving an exact asymptotic bound on the number of windows w.r.t. a given window expression on reading \( n \)-data elements of any stream conforming to a given input specifier.

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A PROOFS FROM SECTION 2

Proof of Lemma 2.3. We can assume, WLOG, that \( S \) is clean. Let \( q \in Q \) and \( q \subseteq Q \). We will first define the following useful notation.

\[
\delta_S(q) = \{(q, \delta(q, q'), q') \neq \perp \mid q' \in Q\}
\]

\[
\delta_S(q) = \bigcup_{q \in Q} \delta_S(q)
\]

\( Target(t) = \{q' \mid (q, q', q') \in t\} \)

\[
Cond((q, q', q')) = \perp
\]

Let \( Q' = 2^Q \). To define the transition function \( \delta' \), we will define the outgoing transition from each \( q \subseteq Q \). For each subset \( t \subseteq \delta_S(q) \) let

\[
\varphi_t = \left( \bigwedge_{t \in t} Cond(t) \right) \land \left( \bigwedge_{t \in \delta_S(q) \setminus t} \neg Cond(t) \right)
\]

If \( \varphi_t \) is satisfiable, then define \( \delta'(q, Target(t)) = \varphi_t \), otherwise \( \delta'(q, Target(t)) = \perp \). For all sets \( q' \subseteq Q \) which are not equal to \( Target(t) \) for any \( t \subseteq \delta_S(q) \), we define \( \delta'(q, q') = \perp \).

Define \( q_0' = (q_0) \) and \( F' = \{q \subseteq Q \mid \exists q \in \delta(q, F)\} \).

To see that \( S' \) is deterministic, note that for every two outgoing transition from a state \( q \in Q' \) with guard \( \varphi_t \) and \( \varphi_{t'} \) with \( t \neq t' \), \( \varphi_t \land \varphi_{t'} \) is unsatisfiable. It follows because, WLOG, if \( t \in t \) and \( t \notin t' \), then \( \varphi_t \) will have the conjunct \( Cond(t) \) while \( \varphi_{t'} \) will have \( \neg Cond(t) \).

It’s trivial to verify that \( S' \) accepts the same language as \( S \). \( \square \)

Proof of Lemma 2.4. Let \( L \) be any \( k \)-SLA-language, and let \( S = (A, Q, q_0, F, \delta) \) be a \( k \)-SLA accepting \( L \). Assume, WLOG, that \( S \) is deterministic. Define \( S' = (A, Q, q_0, Q \setminus F, \delta) \). We claim that \( S' \) accepts the language \( L \). Consider any word \( w \in L \). It will have a unique run in \( S \) to a final state, and hence it will be rejected by \( S' \). If \( w \notin L \) and \( \left| w \right| \geq k + 1 \), then it will have a unique run in \( S \) to a non-final state, which will be a final state in \( S' \), and hence \( w \) will be accepted by \( S' \). \( \square \)

Proof of Lemma 2.6. Let \( k \)-SLA \( S_1 = (A, Q_1, q_0^1, F_1, \delta_1) \) and \( S_2 = (A, Q_2, q_0^2, F_2, \delta_2) \). The construction is similar to that in case of finite automata. Formally, construct a \( k \)-SLA \( S = (A, Q_1 \cup Q_2, q_0^1 \cup F_2, \delta) \) with

\[
\delta(q, q') = \begin{cases} 
\delta_1(q, q'), & \text{if } q, q' \in Q_1 \\
\delta_2(q, q'), & \text{if } q, q' \in Q_2 \\
\delta_2(q_0^2, q'), & \text{if } q \in F_1, q' \in Q_2 \\
\perp, & \text{otherwise}
\end{cases}
\]

In the third case above, \( S \) non-deterministically switches from \( S_1 \) to \( S_2 \). The proof of correctness is routine. \( \square \)
Semantics of S-MSO.

\[ w, \theta \models [\varphi](x) \iff \nu_\theta(w[\theta(x) - k]) \models \varphi \]

\[ w, \theta \models x < y \iff \theta(x) < \theta(y) \]

\[ w, \theta \models X(x) \iff \theta(x) \in \theta(X) \]

\[ w, \theta \models \neg \phi \iff w, \theta \not\models \phi \]

\[ w, \theta \models \phi_1 \land \phi_2 \iff w, \theta \models \phi_1 \text{ and } w, \theta \models \phi_2 \]

\[ w, \theta \models \exists x \phi(x) \iff \exists i \in |k, |w| - 1| \text{ such that } w, \theta[X \mapsto i] \models \phi(x) \]

\[ w, \theta \models \exists X \phi(X) \iff \exists i \in 2^{|k, |w| - 1|} \text{ such that } w, \theta[X \mapsto i] \models \phi(X) \]

B DETAILS FROM SECTION 3

Proof of Lemma 3.3. From section 2, given an S-MSO formula \( \phi(x_b, x_e) \), we can construct a \( k \)-SLA \( S \) over the extended alphabet theory \( \mathcal{A}(x_b, x_e) \), such that for any word \( w \in \Sigma^k \Sigma^* \) and any map \( \theta : \{x_b, x_e\} \rightarrow [k, |w| - 1], \)

\( w, \theta \models \phi \) if and only if the word \( w_0 \in (\Sigma \times \{0, 1\})^* \) is accepted by \( S \).

Guards of transitions in \( S \) are of the form \( \varphi, x_b = a_b, x_e = a_e \), where \( \varphi \) is a formula from the alphabet theory and \( a_b, a_e \in \{0, 1\} \). Let us denote by \( S \downarrow x_b = 0 \) the \( k \)-SLA obtained from \( S \) by removing transitions whose guards have \( x_b = 1 \). Similarly, we define \( S \downarrow x_e = 0 \). For every pair of transitions \( (t_b, t_e) \) in \( S \) such that the guard of \( t_b \) (resp. \( t_e \)) has \( x_b = 1 \) (resp. \( x_e = 1 \)), we construct the following two \( k \)-SLAs:

1. \( PA_{(b, e)} \): We start with the \( k \)-SLA \( S \downarrow x_b = 0 \). Set the initial state to be the same as that of \( S \). Add a new state \( p_f \) and set it as the only final state. Suppose the transition \( t_b \) is from state \( q \) to \( p \). In \( PA_{(b, e)} \), add a transition from \( q \) to \( p_f \) with the same guard as \( t_b \). This is intended to accept prefixes of windows.

2. \( WA_{(b, e)} \): We start with the \( k \)-SLA \( S \downarrow x_e = 0 \). Set \( p \) as the initial state, where \( p \) is the target state of \( t_b \). Add a new state \( p_e \) and set it as the only final state. Suppose the transition \( t_e \) is from \( q' \) to \( p' \). In \( WA_{(b, e)} \), add a transition from \( q' \) to \( p_e \) with the same guard as \( t_e \). This is intended to accept windows.

We denote by \( SRE(S) \) the symbolic regular expression equivalent to the \( k \)-SLA \( S \). Define \( R = \{(SRE(PA_{(b, e)}), SRE(WA_{(b, e)})) \mid \) guard of \( t_b \) has \( x_b = 1 \), guard of \( t_e \) has \( x_e = 1 \}\). We shall now prove that for all words \( w \in \Sigma^k \Sigma^* \), the set of windows recognized in \( w \) by \( \phi(x, y) \) and \( R \) are same.

\( \Rightarrow \): Let \( (i_b, i_e) \) be any window recognized by \( \phi(x_b, x_e) \) in \( w \), so \( w[0 : i_e], \{x_b \mapsto i_b, x_e \mapsto i_e\} \models \phi(x_b, x_e) \). Let \( w' = w[0 : i_e] \). By construction \( w'_0 \) will be accepted by the \( k \)-SLA \( S \). Consider an accepting run \( \rho : q_0 \xrightarrow{(0, 0)} q_1 \ldots q_i \xrightarrow{(1, 1)} q_{i+1} \ldots q_{i-1} \xrightarrow{(1, 1)} q_i \) of \( S \) on \( w'_0 \). As \( x_b, x_e \) are first-order variables, there would be exactly two positions, \( i_b \) and \( i_e \) where they take the value 1 respectively in the word \( w_0 \), and elsewhere they would be 0.

Consider the pair \( (SRE(PA_{(b, e)}), SRE(WA_{(b, e)})) \in R \) where \( t_b = q_i \xrightarrow{(1, 0)} q_{i+1} \ldots q_{i-1} \xrightarrow{(1, 1)} q_i \). It follows by construction that \( PA_{(b, e)} \) accepts \( w[0 : i_b - 1] \) and \( WA_{(b, e)} \) accepts \( w[i_b - k : i_e] \).

\( \Leftarrow \): Let \( (i_b, i_e) \) be a window accepted by the pair

\[ (SRE(PA_{(b, e)}), SRE(WA_{(b, e)})) \in R \]

with \( t_b := q \quad \rho, t_e := q' \). Therefore, we have \( w[0 : i_b - 1] \in L(PA_{(b, e)}) \) and \( w[i_b - k : i_e] \in L(WA_{(b, e)}) \). Let \( \rho_1 = q_0 \xrightarrow{(0, 0)} q_1 \ldots q_{i_b - 1} \xrightarrow{(1, 1)} q_{i_b} = p_f \) and \( \rho_2 = q_i \xrightarrow{(1, 1)} q_{i+1} \ldots q_{i_e} = p' \) be the
accepting runs of the $k$-SLAs $PA_{(b,e)}$, $WA_{(b,e)}$ on $w[0 : i_b - 1], w[i_b - k : i_e]$ respectively. We can combine the runs $\rho_1$ and $\rho_2$ by replacing the last transition in the runs with $t_b, t_e$ respectively and merging the runs to get a run $\rho$ in $S$ for the word $w[0 : i_e]_{\emptyset}$ with $\emptyset := \{x \rightarrow i_b, y \rightarrow i_e\}$.

Proof of Lemma 3.4. Consider a pair $(r, r') \in R$. Let $\psi, \psi'$ be the $S$-MSO sentences corresponding to $r, r'$ respectively. The sentence $\psi$ partitions the set of positions of a word into multiple parts, each part corresponding to a state of the automaton for the expression $r$. The sentence $\psi'$ further verifies that the partition forms a valid run according to the transition rules of the automaton. We modify $\psi$ to partition the set of positions $[0, x_b - 1]$ instead and verify the validity of the partition. Let us call this modified formula $\psi \upharpoonright x_b$. It is routine to verify that $\psi \upharpoonright x_b$ can be written in the guarded fragment described above. Similarly, $\psi' \upharpoonright [x_b : x_e]$ will partition and verify the positions in $[x_b, x_e]$. The required formula $\phi(x_b, x_e)$ for $(r, r')$ is $(x_b \leq x_e) \land (\psi \upharpoonright x_b) \land (\psi' \upharpoonright [x_b : x_e])$. The required formula is the disjunction of all such formulas for all the pairs in $R$. □

C DETAILS FROM SECTION 4

Proof of Lemma 4.1. We prove this by induction on $n$. Before starting to read symbols from $I$, 1 and 2 hold due to the initializations done before entering the main loop at line 9. Assuming that they hold at the start of the loop after reading $n$ symbols, we will prove that after execution of the loop, the invariants would still hold.

In the loop, if PrefixState is final in $PA$, then we add the next position to WindowStartIndices$(init_{WA})$ in line 14. Also, all the states already in WindowStartIndices are updated according to the transition relation in lines 33 and 35. Hence, the invariants continue to hold after updating the position variable $x$ in the line following 35. □

Proof of Corollary 4.2. Suppose $(x, y)$ is added to the output, then using Lemma 4.1 (main loop invariant), we get $w[0 : x - 1] \in L(PA)$ and $w[x : y] \in L(WA)$. Conversely, suppose $w[0 : x - 1] \in L(PA)$ and $w[x : y] \in L(WA)$. Then Lemma 4.1 implies that PrefixState stores the state of $PA$ after reading $w[0 : x - 1]$, which must be a final state, and hence we would have added $x$ to WindowStartIndices$(init_{WA})$ in line 14. Now, on reading up to the position $y$, we will have $x \in WindowStateIndices(q)$ such that $q$ is the state of $WA$ after reading the word $w[x : y]$, as implied by Lemma 4.1.

Since $w[x : y] \in L(WA)$, $q$ would be a final state in $WA$, and hence the processor must add $(x, y)$ in the output stream $O$ in line 18. □

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