On nonnegatively graded Weierstrass points

André Contiero, Aislan Leal Fontes, Jan Stevens and Jhon Quispe Vargas

Abstract
We provide a new lower bound for the dimension of the moduli space of smooth pointed curves with prescribed Weierstrass semigroup at the marked point, derived from the Deligne–Greuel formula and Pinkham’s equivariant deformation theory. Using Buchweitz’s description of the first cohomology module of the cotangent complex for monomial curves, we show that our lower bound improves a recently one given by Pflueger. By allowing semigroups running over suitable families of symmetric semigroups of multiplicity six, we show that this new lower bound is attained, and that the corresponding moduli spaces are non-empty and of pure dimension.

1 Introduction
Given a smooth projective pointed curve \((C, P) \in M_{g,1}\) of genus \(g > 1\) defined over an algebraically closed field \(k\), its associated Weierstrass semigroup \(S\) consists of the set of non-negative integers \(n\), called nongaps, such that there is a rational function on \(C\) whose pole divisor is \(nP\). Equivalently, \(n\) is a nongap if and only if \(H^0(C, \mathcal{O}_C(n−1)P) \subsetneq H^0(C, \mathcal{O}_C(nP))\).
The Riemann–Roch Theorem assures that the set of positive integers that are not in \(S\) (the set of gaps) has size exactly \(g\). In addition, we say that \(P \in C\) is a Weierstrass point if its associated Weierstrass semigroup is different from the ordinary one \(\{0, g + 1, g + 2, \ldots\}\).

For each numerical semigroup \(S\) of genus \(g > 1\), let \(M_{g,1}^S\) be the space parameterizing pointed smooth curves whose associated Weierstrass semigroup at the marked point is \(S\). It is well known that \(M_{g,1}^S\) can be empty, but if it is nonempty, since the \(i\)-th gap of a Weierstrass semigroup is an upper semicontinuous function, then it is a locally closed subspace of \(M_{g,1}\), so we get get a stratification \(M_{g,1} = \bigsqcup_S M_{g,1}^S\), where \(S\) runs over all the semigroups of genus \(g\).

In this paper we focus on the problem of computing the dimension of \(M_{g,1}^S\). In characteristic zero there are two known general bounds, namely the Rim–Vitulli upper bound \([RV77, \S6]\), that dates back to the 70’s, and more recently Pflueger’s lower bound \([Pf18]\),

\[
\begin{align*}
3g - 2 - \text{ewt}(S) & \leq \dim M_{g,1}^S \\
& \leq 2g - 2 + \lambda(S),
\end{align*}
\]

Pflueger’s bound \hspace{10cm} Rim–Vitulli bound

\[\text{(1)}\]

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where $\text{ewt}(S)$ is the effective weight of $S$ [Pf18, Def. 1.1] and $\lambda(S)$ is the number of gaps $\ell$ of $S$ such that $\ell + n \in S$ whenever $n$ is a nongap. The methods used to get the above two bounds come from deformations of suitable curves: while the Rim–Vitulli bound is derived from equivariant deformations of monomial curves, Pflueger’s bound is derived by using deformations of stable curves and limit linear series. It is important to highlight that the two bounds in (1) coincide if and only if $S$ is negatively graded, i.e. the first cohomology module $T^1(k[S])$ of the cotangent complex associated to the semigroup algebra $k[S]$ is negatively graded, cf. [RV77, Thm. 4.7] and [Pf18, Prop. 2.11].

The main result of this paper (Theorem 2.4) is that, if $M_{g,1}^s$ is nonempty,

$$2g - 2 + \lambda(S) - \dim T^{1,+}(k[S]) \leq \dim X,$$

for any irreducible component $X$ of $M_{g,1}^s$, where $\dim T^{1,+}(k[S])$ stands for the dimension of the positive graded part of $T^1(k[S])$. This bound is derived from Pinkam’s construction [Pi74] of $M_{g,1}^s$ and the Deligne–Greuel formula [Del73, Gr82] for smoothing components of curve singularities. In addition, using Buchweitz’ description [Bu80] of $T^1$ for monomial curves, we show that the lower bound (2) is not smaller than Pflueger’s one (Proposition 2.8). Hence, assuming that $M_{g,1}^s$ is nonempty we can write

$$3g - 2 - \text{ewt}(S) \leq 2g - 2 + \lambda(S) - \dim T^{1,+}(k[S]) \leq \dim M_{g,1}^s \leq 2g - 2 + \lambda(S).$$ (3)

We show examples where our new lower bound does not attain the dimension of $M_{g,1}^s$. Examples are also provided where the first inequality in (3) is strict. In particular, for the semigroups $\langle 6, 7, 8 \rangle$ and $\langle 6, 7, 15 \rangle$, Pflueger’s bound does not provide the exact dimension of the $M_{g,1}^s$, but our lower bound in Theorem 2.4 does. These two particular semigroups are symmetric, i.e. their largest gap are the biggest possible, namely $\ell_g = 2g - 1$, suggesting that symmetric semigroups can be an interesting class of semigroups to study the dimension of their associated moduli spaces $M_{g,1}^s$.

In Section 3 we briefly recall a rather explicit construction of a compactification of $M_{g,1}^s$ when $S$ is non-hyperelliptic and symmetric. This construction was given by Stoehr [St93], then improved by Contiero–Stoehr [CSt13] and generalized by Contiero–Fontes [CF18]. All these constructions can be viewed as a variant of Hauser’s algorithm [Hau83, Hau85] to compute versal deformation spaces, see also [Stev13]. Next, in Section 4, we use this construction to give explicit equations for $M_{g,1}^s$ when $S$ runs over the following families of symmetric semigroups,

$$\langle 6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau, 8 + 6\tau \rangle \text{ and } \langle 6, 1 + 6\tau, 2 + 6\tau, 3 + 6\tau, 14 + 6\tau \rangle$$

and we show that $M_{g,1}^s$ is not empty in these cases by showing that the associated affine monomial curves $\mathbb{C}_S$ can be negatively smoothable, c.f. Theorems 4.5 and 4.10. Using a result of Contiero and Stoehr [CSt13] we find an upper bound for the dimension of $M_{g,1}^s$, which coincides with the lower bound, hence we are able to conclude that $M_{g,1}^s$ is of pure dimension in these cases, Corollaries 4.7 and 4.11.
2 A new lower bound

The connection between the moduli space \( M_{g}^{\mathcal{S}} \) and the negative part of the deformation space of the monomial curve \( \mathcal{C}_{g} \) was first observed by Pinkham [Pi74, Ch. 13]. Here \( \mathcal{S} = (n_{1}, \ldots, n_{r}) \) is a numerical semigroup of genus \( g > 1 \) with semigroup ring \( k[\mathcal{S}] := \oplus_{n \in \mathcal{S}} k t^{n} \) and \( \mathcal{C}_{g} := \text{Spec} k[\mathcal{S}] \) is its associated affine monomial curve. In the sequel we assume that \( k \) is an algebraically closed field of characteristic zero.

Since \( \mathcal{C}_{g} \) has a unique singular point, there exists a versal deformation, c.f. [Ar76], say

\[
\mathcal{X}_{t_{0}} \cong \mathcal{C}_{g} \quad \longrightarrow \quad \mathcal{X}
\]

\[
\downarrow \quad \quad \quad \quad \downarrow
\]

\[
(t_{0}) = \text{Spec} k \quad \longrightarrow \quad \mathcal{T}
\]

with \( \mathcal{T} = \text{Spec} \mathcal{A} \), where \( \mathcal{A} \) is a local, complete noetherian \( k \)-algebra. Pinkham [Pi74] showed that this deformation can be taken to be equivariant: the natural \( \mathbb{G}_{m} \)-action on \( \mathcal{C}_{g} \), given by \( (\alpha, X_{i}) \mapsto \alpha^{n_{i}}X_{i} \), can be extended to the total and parameter spaces, \( \mathcal{X} \) and \( \mathcal{T} \). This induces a natural grading on the tangent space \( T_{i}^{1} = T^{1}(k[\mathcal{S}]) \) to \( \mathcal{T} \). A deformation has negative weight \( -e \) if it decreases the weights of the equations of the curve and the corresponding deformation variable has then (positive) weight \( e \). A numerical semigroup \( \mathcal{S} \) is called \emph{negatively graded} if \( T^{1}(k[\mathcal{S}]) \) has no positive graded part.

Let \( \mathcal{I} \) be the ideal of \( \mathcal{A} \) generated by the images of the deformation variables of negative weight, i.e. those elements of \( \mathcal{A} \) that correspond to the elements in the positive graded part \( T^{1} \) of \( T^{1}(k[\mathcal{S}]) \). Then \( \mathcal{T}^{-} := \text{Spec} \mathcal{A}/\mathcal{I} \) is the subspace of \( \mathcal{T} \) in negative degrees and the restriction \( \mathcal{X}^{-} \rightarrow \mathcal{T}^{-} \) is the versal deformation in negative degrees. Both \( \mathcal{X}^{-} \) and \( \mathcal{T}^{-} \) are defined by polynomials and we use the same symbols for the corresponding affine varieties.

The deformation \( \mathcal{X}^{-} \rightarrow \mathcal{T}^{-} \) can be fiberwise compactified to \( \overline{\mathcal{X}^{-}} \rightarrow \overline{\mathcal{T}^{-}} \); each fibre is an integral curve in a weighted projective space with one point \( P \) at infinity and this is a point with semi-group \( \mathcal{S} \). All the fibres over a given \( \mathbb{G}_{m} \) orbit of \( \mathcal{T}^{-} \) are isomorphic, and two fibres are isomorphic if and only if they lie in the same orbit. This is proved in [Pi74] for smooth fibres and in general in the Appendix of [Lo84].

Let \( \mathcal{C} \) be a possible singular integral complete curve of arithmetic genus \( g > 1 \) defined over \( k \). Given a smooth point \( P \) of \( \mathcal{C} \), let \( \mathcal{S} \) be the Weierstrass semigroup of \( \mathcal{C} \) at \( P \), that is the set of nonnegative integers \( n \in \mathcal{S} \) such that there is a rational function \( x_{n} \) on \( \mathcal{C} \) whose pole divisor \( nP \). Consider the line bundle \( \mathcal{L} = \mathcal{O}(P) \) and form the section ring \( \mathcal{R} = \oplus_{i=0}^{\infty} \mathcal{H}^{0}(\mathcal{C}, \mathcal{L}^{i}) \). This leads to an embedding of \( \mathcal{C} = \text{Proj} \mathcal{R} \) in a weighted projective space, with coordinates \( X_{0}, \ldots, X_{r} \) with \( \deg X_{0} = 1 \). The space \( \text{Spec} \mathcal{R} \) is the corresponding quasi-cone in affine space. Setting \( X_{0} = 0 \) defines the monomial curve \( \mathcal{C}_{g} \), all other fibres are isomorphic to \( \mathcal{C} \setminus P \). In particular, if \( \mathcal{C} \) is smooth, this construction defines a smoothing of \( \mathcal{C}_{g} \).

2.1 Theorem ([Pi74, Thm. 13.9]) Let \( \mathcal{X}^{-} \rightarrow \mathcal{T}^{-} \) be the equivariant miniversal deformation in negative degrees of the monomial curve \( \mathcal{C}_{g} \) for a given semigroup \( \mathcal{S} \) and denote
by \((\mathcal{T}^-)_s\) the open subset of \(\mathcal{T}^-\) given by the points with smooth fibers. Then the moduli space \(M_{g,1}^S\) is isomorphic to the quotient \(M_{g,1}^S = (\mathcal{T}^-)_s / G_m\) of \((\mathcal{T}^-)_s\) by the \(G_m\)-action.

The assumption on the characteristic is essential for this result. For examples in finite characteristic see [Na16]. The moduli space \(M_{g,1}^S\) is non-empty if and only if the monomial curve \(\mathcal{C}_S\) can be smoothed negatively. Deligne [Del73, Thm 2.27] established a formula for the dimension of smoothing components of curve singularities in general (and in arbitrary characteristic). A smoothing component is an irreducible component of the versal deformation space whose fiber over its generic point is smooth. Deligne’s formula simplifies for quasihomogeneous curves (in characteristic zero) [Gr82].

2.2 Theorem (Deligne–Greuel formula) For any smoothing component \(E\) of a quasihomogeneous curve \(\text{Spec} \mathcal{O}
\]
\[\dim E = \mu + t - 1 .\]

Here \(\mu = 2\delta - r + 1\) is the Milnor number, where \(\delta := \dim_k \mathcal{O}/\mathcal{O}\) the singularity degree of the curve at \(P\), \(r\) is the number of branches and \(t\) is the type \(\dim_k \text{Ext}_1^k(\mathcal{O}, \mathcal{O})\). In particular, for Gorenstein curves \(t = 1\). In the special case of monomial curves, where \(\delta = g\) and \(r = 1\), the formula for the dimension of smoothing components already occurs in [Bu80]. In [Bu80, 4.1.2] a combinatorial formula is stated for \(t\), which shows that \(t = \lambda(S)\), the number of gaps \(\ell\) of \(S\) such that \(\ell + n \in S\) whenever \(n\) is a nongap. More formally, let \(\text{End}(S) = \{n \in \mathbb{N} \mid n + S \cap 0 \subset S\}\). Then \(\lambda(S) = \#(\text{End}(S) - S)\).

As each smoothing component of \(\mathcal{T}^-\) is contained in a smoothing component of the total versal deformation the Deligne–Greuel formula gives an upper bound for the dimension of \(M_{g,1}^S = (\mathcal{T}^-)_s / G_m\), first stated by Rim and Vitulli [RV77, §6].

2.3 Theorem (Rim–Vitulli upper bound) For any numerical semigroup \(S\)
\[\dim M_{g,1}^S \leq 2g - 2 + \lambda(S) .\]

This upper bound is attained: Rim and Vitulli showed, that if \(S\) is negatively graded, then \(\mathcal{C}_S \) is negatively smoothable [RV77, Cor. 5.1], so \(\dim M_{g,1}^S = 2g - 2 + \lambda(S)\). A complete list of all negatively graded semigroups can also be found in [RV77, Thm. 4.7]. In general the Rim–Vitulli bound is far from being tight, for examples in low genus see Table 1. It also happens for the families of semigroups in Section 4.

Let us assume that \(\mathcal{C}_S\) can be smoothed negatively. Let \(E^-\) be a smoothing component of \(\mathcal{T}^-\), then there is smoothing component \(E\) of the versal deformation space of \(\mathcal{C}_S\) such that \(E^-\) is a component of \(E \cap \mathcal{T}^-\). As \(E \cap \mathcal{T}^-\) is obtained by adding \(\dim T^{1,+}(k[S])\) linear equations to its defining equations, we have that the dimension of \(Y\) is not smaller than \(\dim E - \dim T^{1,+}(k[S])\). With the Deligne–Greuel formula we obtain

2.4 Theorem Let \(S\) be a numerical semigroup \(S\) of genus bigger than 1. If \(M_{g,1}^S\) is nonempty, then for any irreducible component \(X\) of \(M_{g,1}^S\)
\[2g - 2 + \lambda(S) - \dim T^{1,+} \leq \dim X .\]
To describe a dimension formula for the graded parts of $T^1(k[S])$ we start from a result due to Herzog that assures that the ideal of $C_S := \{(t^{a_1}, \ldots, t^{a_r}) ; t \in k\} \subset A^r$ can be generated by isobaric polynomials $F_i$ that are differences of two monomials, namely

$$F_i := X_1^{\alpha_{i1}} \cdots X_r^{\alpha_{ir}} - X_1^{\beta_{i1}} \cdots X_r^{\beta_{ir}},$$

with $\alpha_i \cdot \beta_i = 0$. As usual, the weight of $F_i$ is $d_i := \sum_j n_{ij} \alpha_{ij} = \sum_j n_{ij} \beta_{ij}$. For each $i$, let $v_i := (\alpha_{i1} - \beta_{i1}, \ldots, \alpha_{ir} - \beta_{ir})$ be the vector in $k^r$ induced by $F_i$.

2.5 Theorem (cf. Thm. 2.2.1 of \cite{Bu80}) Let $k[S]$ be the semigroup ring of a numerical semigroup $S$. For $\ell \in \mathbb{Z}$ let $A_\ell := \{i \in \{1, \ldots, r\} \mid n_i + \ell \not\in S\}$ and let $V_\ell$ be the vector subspace of $k^r$ generated by the vectors $v_i$ such that $d_i + \ell \not\in S$. Then for $\ell \notin \text{End}(S)$

$$\dim T^1(k[S])_\ell = \#A_\ell - \dim V_\ell - 1$$

while $\dim T^1(k[S])_s = 0$ for $s \in \text{End}(S)$.

Pflueger produced in \cite[Thm. 1.2]{Pf18} an upper bound for the codimension of $M^S_{g,1}$ as locally closed subset of $M_{g,1}$, improving a bound by Eisenbud and Harris in \cite{EH87}. He introduced the effective weight \cite[Def. 1.1]{Pf18} of a numerical semigroup $S$ with a minimal system of generators, $S = \langle n_1, \ldots, n_r \rangle$

$$\text{ewt}(S) := \sum_{\text{gaps } \ell_i} (\# \text{ generators } n_j < \ell_i).$$

With this substitute for the classical weight $\text{wt}(S) := \sum \ell_i - i$, which is equal to the sum over the gaps of $\# \text{ nongaps } n < \ell_i$, appearing in the Eisenbud–Harris bound Pflueger established the following result.

2.6 Theorem (Pflueger’s bound) If the moduli space $M^S_{g,1}$ is nonempty, and $X$ is any irreducible component of it, then

$$\dim X \geq 3g - 2 - \text{ewt}(S).$$

We compare this bound with our bound from Theorem 2.4. Using the notation of Theorem 2.5 we first give a different formula for $\text{ewt}(S)$.

2.7 Lemma Let $n_1 < \cdots < n_r$ be a minimal system of generators for $S$. We have

$$\text{ewt}(S) = \sum_{\ell \in \text{End}(S)} \#A_\ell.$$
Proof. By definition
\[
\text{ewt}(S) = \sum_{\ell \notin S} \# \{ n_i < \ell \} = \# \{ (n_i, \ell) \mid n_i < \ell \}.
\]

On the other hand,
\[
\sum_{\ell \notin \text{End}(S)} \# A_\ell = \sum_{\ell \in S} \# A_\ell,
\]
where \( \# A_\ell \) is \( \# \{ n_i + \ell / \in S \} \), so this is the number of pairs \( (n_i, \ell) \) such that \( n_i + \ell \) is a gap. Since the map \( (n, \ell) \mapsto (n, \ell - n) \) is a bijection from the first set of pairs to the second, their cardinality is the same. \( \square \)

2.8 Proposition. For any numerical semigroup \( S \) of genus \( g \geq 1 \) the bound of Theorem 2.4 is not less then Pflueger’s lower bound:
\[
3g - 2 - \text{ewt}(S) \leq 2g - 2 + \lambda(S) - \dim T^{1+}(S).
\]

Proof. Using Theorem 2.5 and the above lemma we obtain
\[
\dim T^{1+}(k[S]) = \sum_{\ell \in \text{End}(S)} (\# A_\ell - \dim V_\ell - 1) = \text{ewt}(S) - \sum_{\ell \in \text{End}(S)} \dim V_\ell - \# (N \setminus \text{End}(S))
\]
and \( \# (N \setminus \text{End}(S)) = \# (N \setminus S) - \# (\text{End}(S) \setminus S) = g - \lambda(S) \). So
\[
3g - 2 - \text{ewt}(S) + \sum_{\ell \in \text{End}(S)} \dim V_\ell = 2g - 2 + \lambda(S) - \dim T^{1+}(S).
\]

\( \square \)

An example where Pflueger’s bound does not provide the exact dimension of \( \mathcal{M}^S_{g,1} \) is given by Pflueger himself, cf. [Pf18, 2F]. This example, the symmetric semigroup \( S := \langle 6, 7, 8 \rangle \) of genus 9, fits in a more general context. Let \( S \) be a symmetric semigroup generated by less than 5 elements. Then the affine monomial curve \( \text{Spec } k[S] \) is a complete intersection, or if \( S = \langle a_1, a_2, a_3, a_4 \rangle \) a quasi-homogeneous version of Buchsbaum-Eisenbud’s structure theorem for Gorenstein ideals of codimension 3 (see [BE77, p. 466]) applies. In both cases one can deduce that \( \text{Spec } k[S] \) can be negatively smoothed without any obstructions ( [Wal79] and [Wal80, Satz 7.1]), hence
\[
\overline{M}^S_{g,1} = \mathbb{P}(T^{1-}(k[S]), \tag{5}
\]
and therefore, \( \dim \mathcal{M}^S_{g,1} = \dim \mathbb{P}(T^{1-}(k[S])) \). For \( S = \langle 6, 7, 8 \rangle \) one computes \( \dim V_3 = 1 \) and \( \text{ewt}(S) = 12 \), so \( \dim \mathcal{M}^S_{g,1} = 14 \), Pflueger’s bound gives 13, while the Rim–Vitulli bound provides \( 2g - 1 = 17 \). In the same way, for the symmetric semigroup \( \langle 6, 7, 15 \rangle \) of genus 12, Pflueger’s bound gives 17, while \( \dim V_2 = 1 \) and \( \dim \mathcal{M}^S_{g,1} = 18 \).
For each numerical semigroup of genus not bigger than 6 the dimension of \( M_{g,1}^S \) is equal to Pflueger’s bound (see [Pf18, 2C]), hence it is also equal to that given by Theorem 2.4. In Table 1 we collect for all all numerical semigroups of genus \( g \leq 6 \), which are not negatively graded, the name of the semigroup in the list of Nakano [Na08], the gaps, the dimension of \( M_{g,1}^S \), which is also equal to both lower bounds, the value of the Rim–Vitulli bound and \( \dim T^{1,+}(k[S]) \).

### Table 1: non-negatively graded semigroups of genus \( \leq 6 \)

| \[Na08\] | gaps | \( \dim M_{g,1}^S \) | R–V | \( \dim T^{1,+} \) |
|---|---|---|---|---|
| N(5)\(_3\) | 1, 2, 4, 5, 8 | 9 | 10 | 1 |
| N(5)\(_5\) | 1, 2, 3, 5, 7 | 10 | 11 | 1 |
| N(5)\(_7\) | 1, 2, 3, 6, 7 | 9 | 10 | 1 |
| N(6)\(_3\) | 1, 2, 4, 5, 7, 10 | 11 | 12 | 1 |
| N(6)\(_4\) | 1, 2, 4, 5, 8, 11 | 10 | 11 | 1 |
| N(6)\(_6\) | 1, 2, 3, 5, 6, 9 | 12 | 13 | 1 |
| N(6)\(_7\) | 1, 2, 3, 5, 6, 10 | 11 | 12 | 1 |
| N(6)\(_8\) | 1, 2, 3, 5, 7, 9 | 11 | 13 | 2 |
| N(6)\(_9\) | 1, 2, 3, 5, 7, 11 | 10 | 11 | 1 |
| N(6)\(_10\) | 1, 2, 3, 6, 7, 11 | 10 | 11 | 1 |
| N(6)\(_12\) | 1, 2, 3, 4, 6, 8 | 13 | 14 | 1 |
| N(6)\(_13\) | 1, 2, 3, 4, 6, 9 | 12 | 13 | 1 |
| N(6)\(_15\) | 1, 2, 3, 4, 7, 8 | 12 | 13 | 1 |
| N(6)\(_16\) | 1, 2, 3, 4, 7, 9 | 11 | 12 | 1 |
| N(6)\(_17\) | 1, 2, 3, 4, 8, 9 | 10 | 12 | 2 |

### 2.9 Remark
Nakano [Na08] computed \( M_{g,1}^S \) using Pinkham’s theorem by determining the base space of the miniversal deformation in negative degrees of the monomial curve \( C_S \). In all cases he succeeded except one \( M_{g,1}^S \) has the structure of a projective quasi-cone over \( \mathbb{P}^1 \times \mathbb{P}^3 \). In case \( N(6)_8 \) with semigroup \( \langle 4, 6, 11, 13 \rangle \) the base space of the total versal deformation is, after a coordinate transformation, given by

\[
\text{Rk} \begin{pmatrix} a_9 & a_{11} & a_{16} & a_{18} \\ a_{-1} & a_1 & a_6 & a_8 \end{pmatrix} \leq 1 ,
\]

where the entries are deformation variables indexed by their weight. The remaining variables have weights 14, 12, 10, 8, 6, 4, 4, 2 and −3. So this base space is irreducible, but the intersection with \( \mathcal{T}^- \) is given by \( a_{-1} = 0 \) and consists two components, one smoothing component, and the other not: the general fiber over the component \( a_1 = a_6 = a_8 = 0 \) is a curve with a double point.
2.10 Remark There are semigroups where our new lower bound given by Theorem 2.4 is not attained. For example, if $M^S_{g,1}$ is not of pure dimension, then the dimension of the biggest component does not attain our new bound. The first examples are given by Pflueger [Pf21, Thm 1.1] in his detailed study of the moduli variety $M^S_{g,1}$ when $S$ is a Castelnuovo semigroup, a semigroup generated by consecutive suitable positive integers: $S_{r,d} := \langle d - r + 1, \ldots, d \rangle$, with $d \geq 2r - 1$. Castelnuovo semigroups also provide many examples where Pflueger’s bound (Theorem 2.6) are not attained, see [Pf21, Prop 6.3].

In addition, there are symmetric semigroups where the lower bound $2g - 2 + \lambda(S) - \dim T^{1,+}(k[S]) = 2g - 1 - \dim T^{1,+}$ is negative. For example, for the symmetric semigroups $(29, 30, \ldots, 42, 57)$, $(31, 32, \ldots, 45, 61)$ and $(33, 34, \ldots, 48, 64)$, of genus 43, 46 and 49, the number $2g - 1 - \dim T^{1,+}$ is $-6$, $-14$ and $-23$, respectively.

3 Weierstrass points on Gorenstein curves

Let $C$ be a possible singular integral complete Gorenstein curve of arithmetic genus $g > 1$ defined on an algebraically closed field $k$. Given a smooth point $P$ of $C$, let $S$ be the Weierstrass semigroup of $C$ at $P$, that is the set of nonnegative integers $n \in S$ such that there is a rational function $x_n$ on $C$ whose pole divisor $nP$. Let us assume that the semigroup $S$ is symmetric, i.e. $n \in S$ if and only if $\ell_g - n \notin S$, where $\ell_g$ is the last gap. Equivalently, $\ell_g$ is the largest possible, $\ell_g = 2g - 1$. A basis for the vector space $H^0(C, \mathcal{O}_C((2g - 2)P))$ is $\{x_{n_0}, x_{n_1}, \ldots, x_{n_g - 1}\}$, and thus $\mathcal{O}_C((2g - 2)P) \cong \mathcal{O}_C$, where $\mathcal{O}_C$ is the dualizing sheaf of $C$. By assuming that $C$ is nonhyperelliptic, the canonical morphism

$$\{(x_{n_0} : x_{n_1} : \ldots : x_{n_g - 1}) : C \hookrightarrow \mathbb{P}^{g-1}$$

is an embedding. Thus $C$ becomes a curve of genus $g$ of degree $2g - 2$ in $\mathbb{P}^{g-1}$ and the integers $\ell_i - 1$ are the contact orders of the curve with the hyperplanes at $P = (0 : \ldots : 0 : 1)$. Conversely, any nonhyperelliptic symmetric semigroup $S$ can be realized as the Weierstrass semigroup of the canonical Gorenstein monomial curve

$$\mathcal{C}_S := \{(s^{n_0}t^{g-1} : s^{n_1}t^{g-1} : \ldots : s^{n_g - 2}t^{g-1} : s^{n_g - 1}t^{g-1}) | (s : t) \in \mathbb{P}^{1}\} \subset \mathbb{P}^{g-1}$$

at its unique point $P = (0 : \ldots : 0 : 1)$ at the infinity.

Now we recall the construction of a compactification of $M^S_{g,1}$, when $S$ is symmetric, that was first introduced by Stoehr [St93], then improved by Contiero–Stoehr [CST13], and generalized by Contiero–Fontes [CF18]. Let us start with a pointed canonical Gorenstein curve $(C, P)$ whose Weierstrass semigroup $S$ at $P$ is symmetric. We know from [Ol91, Theorem 1.3] that each nongap $s \leq 4g - 4$ can be written as a sum of two others nongaps, namely

$$s = a_s + b_s, \quad a_s \leq b_s \leq 2g - 2.$$

By choosing $a_s$ the smallest possible, the $3g - 3$ rational functions $x_{a_s}x_{b_s}$ form a P-hermitian basis of the global sections $H^0(C, \mathcal{O}_C(2(2g - 2)P))$ of the bicanonical divisor. The homomorphism

$$k[X_{n_0}, \ldots, X_{n_g - 1}]_2 \rightarrow H^0(C, \mathcal{O}_C(2(2g - 2)P))$$

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induced by the substitutions $X_{n_i} \mapsto x_{n_i}$ is surjective and the kernel is the vector space of quadratic forms in the ideal of $C \subset \mathbb{P}^{g-1}$.

Now, given a nongap $s \leq 4g - 4$, let us consider all the partitions of $s$ as sum of two nongaps not greater than $2g - 2$,

$$s = a_{s_i} + b_{s_i}, \quad \text{with } a_{s_i} \leq b_{s_i} \quad (i = 0, \ldots, \nu_s), \quad \text{where } a_{s_0} := a_s.$$ 

Hence, given a nongap $s \leq 4g - 4$ and $i = 1, \ldots, \nu_s$ we can write

$$x_{a_{s_i}}x_{b_{s_i}} = \sum_{n=0}^{s} c_{s_in}x_{a_n}x_{b_n},$$

where $a_n$ and $b_n$ are nongaps of $S$ whose sum is equal to $n$, and $c_{s_in}$ are suitable constants in $k$. By normalizing the coefficients $c_{s_is} = 1$, it follows that the $(g+1) \cdot (3g - 3) = (g - 2)(g - 3)/2$ quadratic forms

$$F_{s_i} = X_{a_{s_i}}X_{b_{s_i}} - X_{a_i}X_{b_i} - \sum_{n=0}^{s-1} c_{s_iyn}X_{a_n}X_{b_n},$$

vanish identically on the canonical curve $C$, where the coefficients $c_{s_iyn}$ are uniquely determined constants. They are linearly independent, hence they form a basis for the space of quadratic relations in $I(C)$.

We need to make some assumptions on the symmetric semigroup $S$ to assure that the ideal $I(C)$ is generated by its quadratic relations $F_{s_i}$. Precisely, we have to assume that $S$ satisfies $3 < n_1 < g$ and $S \neq \langle 4, 5 \rangle$. According to [CF18, Lemma 3.1], both the conditions $n_1 \neq 3$ and $n_1 \neq g$ avoid possible trigonal Gorenstein curves whose Weierstrass semigroup at $P$ equal to $S = \langle 3, g+1 \rangle$ and $S = \langle g, g+1, \ldots, 2g-2 \rangle$, respectively. This two avoided cases are also treated by similar techniques in [CF18], but suitable cubic forms are required to compute the ideal of the canonical Gorenstein curve $C$. So, making the above assumptions on the semigroup $S$, it follows by the Enriques–Babbage Theorem that $C$ is nontrigonal and not isomorphic to a plane quintic. Hence the ideal of $C$ is generated by the $(g - 2)(g - 3)/2$ quadratic forms $F_{s_i}$, c.f. [CSt13, Theorem 2.5].

On the other hand, starting with a symmetric semigroup $S$ with $3 < n_1 < g$ and $S \neq \langle 4, 5 \rangle$, let us introduce the following $(g - 2)(g - 3)/2$ quadratic forms

$$F_{s_i} = X_{a_{s_i}}X_{b_{s_i}} - X_{a_i}X_{b_i} - \sum_{n=0}^{s-1} c_{s_iyn}X_{a_n}X_{b_n}, \quad (6)$$

where $c_{s_iyn}$ are constants to be determined in order that the intersection of the $\mathbb{V}(F_{s_i})$ in $\mathbb{P}^{g-1}$ is a canonical Gorenstein curve of genus $g$ whose Weierstrass semigroup at $P$ is $S$. We attach to the variable $X_n$ the weight $n$ and to each coefficient $c_{s_iyn}$ the weight $s - n$. If we consider $F_{s_i}$ as a polynomial expression, not only in the variables $X_n$, but also in the coefficients $c_{s_iyn}$, then it becomes quasi-homogeneous of weight $s$. 

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Since the coordinates functions $x_n$, $n \in S$ and $n \leq 2g - 2$, are not uniquely determined by their pole divisor $nP$, we may transform

$$X_{n_i} \mapsto X_{n_i} + \sum_{j=0}^{i-1} c_{n_i,n_{i-j}}X_{n_{i-j}},$$

for each $i = 1, \ldots, g - 1$, and so we can normalize $\frac{1}{2}g(g - 1)$ of the coefficients $c_{s|n}$ to be zero, see [St93, Proposition 3.1]. Due to these normalizations and the normalizations of the coefficients $c_{s|n} = 1$ with $n = s$, the only freedom left to us is to transform $x_{n_i} \mapsto c^n x_{n_i}$ for $i = 1, \ldots, g - 1$.

Let

$$F_{si}^{(0)} := X_{a_{si}}X_{b_{si}} - X_{a_s}X_{b_s}$$

be the quadratic forms that generate the ideal of the canonical monomial curve $C_S$, cf. [CSt13, Lemma 2.2]. One of the keys to construct a compactification of $M_{g,1}$ is the following lemma.

### 3.1 Syzygy Lemma (cf. [CSt13])

For each one of the $\frac{1}{2}(g - 2)(g - 5)$ quadratic binomials $F_{s'|i'}^{(0)}$ different from $F_{n_i+2g-2,1}^{(0)}$ ($i = 0, \ldots, g - 3$), there is a linear syzygy of the form

$$X_{2g-2}F_{s|i'}^{(0)} + \sum_{n|s} \varepsilon_{n|s|s_i}^{(s'|i')} X_n F_{s|s}^{(0)} = 0 \quad (7)$$

where the coefficients $\varepsilon_{n|s|s_i}^{(s'|i')}$ are integers equal to 1, -1 or 0, and the sum is taken over the nongaps $n < 2g - 2$ and the double indexes $s_i$ such that $n + s = 2g - 2 + s'$.

The explicit construction of a compactification of $M_{g,1}$ starts by replacing the initials binomials $F_{s'|i'}^{(0)}$, and $F_{s|s}^{(0)}$ in equation (7) by the corresponding forms $F_{s'|i'}^{(0)}$ and $F_{s|s}^{(0)}$ displayed in equation (6), obtaining a linear combination of cubic monomials of weight $< s' + 2g - 2$. By virtue of [CSt13, Lemma 2.4] and its proof this linear combination of cubic monomials admits the following decomposition

$$X_{2g-2}F_{s'|i'} + \sum_{n|s} \varepsilon_{n|s|s_i}^{(s'|i')} X_n F_{s|s} = \sum_{n|s} \eta_{n|s|s_i}^{(s'|i')} X_n F_{s|s} + R_{s'|i'}$$

where the sum on the right hand side is taken over the nongaps $n \leq 2g - 2$ and the double indexes $s_i$ with $n + s < s' + 2g - 2$, the coefficients $\eta_{n|s|s_i}^{(s'|i')}$ are constants, and where $R_{s'|i'}$ is a linear combination of cubic monomials of pairwise different weights $< s' + 2g - 2$.

For each nongap $m < s' + 2g - 2$, let $\rho_{s'|i'm}$ be the unique coefficient of $R_{s'|i'}$ of weight $m$. It is a quasi-homogeneous polynomial expression of weight $s' + 2g - 2 - m$ in the coefficients $c_{s|n}$.

### 3.2 Theorem [CSt13, Theorem 2.6]

Let $S$ be a symmetric semigroup of genus $g$ satisfying $3 < n_1 < g$ and $S \neq \{1, 5\}$. The isomorphism classes of the pointed complete integral
Gorenstein curves with Weierstrass semigroup $S$ correspond bijectively to the orbits of the $\mathbb{G}_m(k)$-action
\[(c; \ldots, c_{\sin}, \ldots) \mapsto (\ldots, c^{s-n}c_{\sin}, \ldots)\]
on the affine quasi-cone of the vectors whose coordinates are the coefficients $c_{\sin}$ of the normalized quadratic $F_{s\mathbb{1}}$ satisfying the quasi-homogeneous equations $\rho_{s/1} = 0$.

3.3 Remark This construction can be viewed as a variant of Hauser’s algorithm to compute versal deformation spaces [Hau83, Hau85], see also [Stev13]. The standard approach in deformation theory is to successively lift infinitesimal deformations to higher order, collecting the obstructions at each stage. In Hauser’s algorithm the defining equations of a singularity are first unfolded, an unobstructed problem, and only then flatness is imposed by lifting relations using a division procedure. The unique remainder leads to equations on the unfolding parameters. In general there are infinitely many such parameters, but if the singularity is quasi-homogeneous, we can restrict to the non-positive part of the unfolding and we obtain equations in finitely many variables for the versal deformation in non-positive weight.

By setting $X_0 = 1$ we see that the equations $\rho_{s/1} = 0$ above give the miniversal deformation in negative weight of the affine monomial curve $C_S$ in a non-minimal embedding in $\mathbb{A}^{g-1}$. The compactification of the moduli space $M^S_{g,1}$ in Theorem 3.2 corresponds to Pinkham’s theorem 2.1.

The advantage of working with a non-minimal embedding is that all equations become quadratic, but at the price of an increase in the number of variables.

4 Families of symmetric 6-semigroups

In this section we apply the techniques briefly described in the above section to deal with families of symmetric semigroups. If the symmetric semigroup is generated by less than five elements, the dimension of the moduli variety $M^S_{g,1}$ is well known, as noted in Section 2. So, we must consider symmetric semigroups of multiplicity greater than 5, just because a symmetric semigroup of multiplicity $m$ can be generated by $m - 1$ elements.

4.1 A first family

For each positive integer $\tau$ consider the semigroup
\[S = (6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau, 8 + 6\tau)\]
\[= 6\mathbb{N} \sqcup \bigcup_{j \in \{3, 4, 7, 8\}} (j + 6\tau + 6\mathbb{N}) \sqcup (11 + 12\tau + 6\mathbb{N}).\]

of multiplicity 6 minimally generated by five elements. Counting the number of gaps of $S$ and picking up the largest one, we find
\[g = 3 + 6\tau \quad \text{and} \quad \ell_g = 12\tau + 5 = 2g - 1,\]
showing that $S$ is a symmetric semigroup.

Let $C$ be a complete integral Gorenstein curve and $P$ be a smooth point of $C$ whose Weierstrass semigroup at $P$ is $S$. For each $n \in S$, let $x_n$ be a rational function on $C$ with pole divisor $nP$. We abbreviate

$$x := x_6 \quad \text{and} \quad y_j := x_{j+6\tau} \ (j = 3, 4, 7, 8)$$

and normalize

$$x_{6i} = x^i \quad \text{and} \quad x_{j+6\tau+6i} = x^i y_j, \ \forall i \geq 1.$$ 

By considering the above the normalizations we see that the functions $x, y_3, y_4, y_7, y_8$ generate the ring $\bigoplus_k H^0(C, kP)$ and therefore induce an embedding

$$(1 : x : y_3 : y_4 : y_7 : y_8) : C \hookrightarrow \mathbb{P} := \mathbb{P}(1, 6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau, 8 + 6\tau)$$

into a weighted projective space $\mathbb{P}$, whose image we call $D$. Instead of studying the ideal of the canonical curve $C \subset \mathbb{P}^{g-1}$, which has $(g-2)(g-3)/2$ quadratic generators, we study the ideal (and the relations between its generators) of the curve $D \subset \mathbb{P}$. The advantage is that the number of generators of the ideal of $D$ does not depend on the genus $g$, see Lemma 4.1 below.

We work in the affine chart on $\mathbb{P}$ obtained by setting the first coordinate equal to 1. Let $X, Y_3, Y_4, Y_7, Y_8$ be indeterminates whose weights are $6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau, 8 + 6\tau$, respectively. For each $n \in S$, we introduce a monomial $Z_n$ of weight $n$ as follows

$$Z_{6i} = X^i, \quad Z_{j+6\tau+6i} = Y_j X^i \quad \text{and} \quad Z_{11+12\tau+6i} = Y_8 Y_7 X^i.$$ 

By writing the nine products $y_i y_j$, $(i, j) \neq (3, 8)$ as linear combination of the basis elements, we obtain nine polynomials in the indeterminates $X, Y_3, Y_4, Y_7, Y_8$ that vanish identically on the affine curve $D \cap \mathbb{A}^5$, say

$$F_i = F_i^{(0)} + \sum_{j=0}^{12\tau+i} f_{ij} Z_{12\tau+1-i-j} \quad (i = 6, 7, 8, 10, 11, 12, 14, 15, 16), \quad (9)$$

where

$$F_6^{(0)} = Y_3^2 - X^{2\tau+1}, \quad F_7^{(0)} = Y_4 Y_7 - X^{\tau} Y_7, \quad F_8^{(0)} = Y_8^2 - X^{\tau} Y_8, \quad F_9^{(0)} = Y_4 Y_8 - X^{2\tau+2}, \quad F_{10}^{(0)} = Y_3 Y_7 - X^{\tau+1} Y_4, \quad F_{11}^{(0)} = Y_4 Y_7 - Y_3 Y_8, \quad F_{12}^{(0)} = Y_4 Y_8 - X^{2\tau+2}, \quad (10)$$

$$F_{14}^{(0)} = Y_7^2 - X^{\tau+1} Y_8, \quad F_{15}^{(0)} = Y_7 Y_8 - X^{\tau+2} Y_3, \quad F_{16}^{(0)} = Y_8^2 - X^{\tau+2} Y_4,$$

and the index $j$ only varies through integers with $12\tau + i - j \in S$. The proof of the next lemma is very similar to [CSt13, Lemma 4.1].

4.1 Lemma The ideal of the affine curve $D \cap \mathbb{A}^5$ is equal to the ideal $I$ generated by the forms $F_i$ $(i = 6, 7, 8, 10, 11, 12, 14, 15, 16)$. In particular the ideal of the affine monomial curve

$$C_S = \{(t^6, t^{3+6\tau}, t^{4+6\tau}, t^{7+6\tau}, t^{8+6\tau}) \mid t \in k\}$$

is generated by the initial forms $F_i^{(0)}$ $(i = 6, 7, 8, 10, 11, 12, 14, 15, 16)$.
Proof It is clear that \( J \subseteq I(D \cap A^5) \). Let \( f \) be a polynomial in the variables \( X, Y_3, Y_4, Y_7, Y_8 \). By applying induction on the degree of \( f \) in the indeterminate \( Y_3, Y_4, Y_7, Y_8 \) we note that, modulo the ideal generated by the nine forms \( F_i \), the monomials of this polynomial \( f \) are not divisible by the nine products \( Y_i Y_j, (i, j) \neq (3, 8) \), hence the class of \( f \) is a sum \( \sum c_n Z_n \) of monomials \( Z_n \) of pairwise different weights with \( n \in S \) and \( c_n \in k \). Thus the polynomial \( f \) belongs to \( I(D \cap A^5) \) if and only if the linear combination \( \sum c_n Z_n \) vanishes identically on the curve \( D \cap A^5 \) and by taking the corresponding linear combination \( \sum c_n x_n \) of rational functions on \( k(C) \) we have \( c_n = 0 \) for each \( n \in S \), hence \( f \) belongs to \( J \).

Now let us invert the above situation. Given the above fixed symmetric semigroup \( S \), we introduce nine isobaric polynomials like in (9):

\[
F_i = F_{i}^{[0]} + \sum_{j=0}^{12r+1} f_{ij} Z_{12r+1-j} \quad (i = 6, 7, 8, 10, 11, 12, 14, 15, 16) \tag{11}
\]

in the polynomial ring \( k[X, Y_3, Y_4, Y_7, Y_8] \), where the coefficients \( f_{ij} \) have weight \( j \). We want relations on the \( f_{ij} \) in order that they give rise to a Gorenstein curve in \( \mathbb{P} \) whose Weierstrass semigroup is \( S \) at the marked point \( P = (0 : 0 : 0 : 0 : 0 : 1) \). Equivalently, we want equations on the \( f_{ij} \) such that the polynomials (11) define the miniversal deformation of the affine monomial curve \( C_S \).

To express these conditions in a concise manner we first note that the coordinate ring of the monomial curve \( C_S \) is a free \( k[X] \)-module generated by \( 1, Y_3, Y_4, Y_7, Y_8 \) and \( Y_3 Y_4 \); this corresponds to the decomposition of the semigroup in (8). We write the polynomials (11) as polynomials in the \( Y_i \) with coefficients in \( k[X] \):

\[
\begin{align*}
F_6 &= Y_3^2 - X^{2r+1} + f_{6}^{(2)} Y_4 + f_{6}^{(3)} Y_3 + f_{6}^{(4)} Y_8 + f_{6}^{(5)} Y_7 + f_{6}^{(6)} \\
F_7 &= Y_3 Y_4 - X^{r} Y_7 + f_{7}^{(1)} + f_{7}^{(3)} Y_4 + f_{7}^{(4)} Y_3 + f_{7}^{(5)} Y_8 + f_{7}^{(6)} Y_7 \\
F_8 &= Y_4^2 - X^{r} Y_8 + f_{8}^{(1)} Y_7 + f_{8}^{(2)} + f_{8}^{(4)} Y_4 + f_{8}^{(5)} Y_3 + f_{8}^{(6)} Y_8 \\
F_{10} &= Y_3 Y_7 - X^{r+1} Y_4 + f_{10}^{(1)} Y_3 + f_{10}^{(2)} Y_8 + f_{10}^{(3)} Y_7 + f_{10}^{(4)} + f_{10}^{(6)} Y_4 \\
F_{11} &= Y_4 Y_7 - Y_3 Y_8 + f_{11}^{(1)} Y_4 + f_{11}^{(2)} Y_3 + f_{11}^{(3)} Y_8 + f_{11}^{(4)} Y_7 + f_{11}^{(5)} Y_3 \tag{12} \\
F_{12} &= Y_3 Y_8 - X^{2r+2} + f_{12}^{(1)} Y_3 Y_8 + f_{12}^{(2)} Y_4 + f_{12}^{(3)} Y_3 + f_{12}^{(4)} Y_8 + f_{12}^{(5)} Y_7 + f_{12}^{(6)} Y_7 \\
F_{14} &= Y_7^2 - X^{r+1} Y_8 + f_{14}^{(1)} Y_7 + f_{14}^{(2)} + f_{14}^{(3)} Y_3 Y_8 + f_{14}^{(4)} Y_4 + f_{14}^{(5)} Y_3 + f_{14}^{(6)} Y_8 \\
F_{15} &= Y_7 Y_8 - X^{r+2} Y_3 + f_{15}^{(1)} Y_8 + f_{15}^{(2)} Y_7 + f_{15}^{(3)} + f_{15}^{(4)} Y_3 Y_8 + f_{15}^{(5)} Y_4 + f_{15}^{(6)} Y_3 \\
F_{16} &= Y_8^2 - X^{r+2} Y_4 + f_{16}^{(1)} Y_3 + f_{16}^{(2)} Y_8 + f_{16}^{(3)} Y_7 + f_{16}^{(4)} + f_{16}^{(5)} Y_3 Y_8 + f_{16}^{(6)} Y_4 
\end{align*}
\]

Here \( f_{ij}^{(j)} = \sum_{k=0}^{\rho} f_{i,j+6k} X^{\rho-k} \) with \( \rho \) determined by the condition that the polynomials are isobaric. If \( i - j = 6 \epsilon \), then \( \rho = 2 \tau + \epsilon \); if \( i - j = 6 \epsilon + 1 \) or \( i - j = 6 \epsilon + 2 \) then \( \rho = \tau - 1 + \epsilon \); if \( i - j = 6 \epsilon + 3 \) or \( i - j = 6 \epsilon + 4 \) then \( \rho = \tau + \epsilon \) and finally \( \rho = 0 \) for \( i - j = 11 \).
Some of the coefficients can be made to vanish by homogeneous coordinate transformations of the form

\[\begin{align*}
X &\mapsto X + c_6 \\
Y_3 &\mapsto Y_3 + \sum_{i=0}^{\tau} c_{3+i} X^{\tau-i} \\
Y_4 &\mapsto Y_4 + c_1 Y_3 + \sum_{i=0}^{\tau} c_{4+i} X^{\tau-i} \\
Y_7 &\mapsto Y_7 + c_2 Y_4 + c_4 Y_3 + \sum_{i=0}^{\tau+1} c_{1+i} X^{\tau+1-i} \\
Y_8 &\mapsto Y_8 + c_1 Y_7 + c_4 Y_4 + c_5 Y_3 + \sum_{i=0}^{\tau+1} c_{2+i} X^{\tau+1-i},
\end{align*}\]

where \(c_i\) and \(c'_i\) are constants of weight \(i\). We normalize

\[\begin{align*}
f_7^{(3)} &= f_1^{(4)} = f_3^{(1)} = f_1^{(2)} = 0, \\
f_1^{(3)} &= f_4^{(5)} = f_1^{(6)} = 0. \\
\end{align*}\]  

(13)

There are still three normalizations left, which can be use to make the first coefficient in an \(f_1^{(1)}\) to zero: we can take \(f_{8,1} = f_{12,4} = f_{8,6} = 0\). Then the only freedom left is given by the \(\mathbb{G}_m(\mathbb{k})\)-action.

The Syzygy Lemma applied to the nine initial isobaric forms in (10), give rises to only eight syzygies of the affine monomial curve \(\mathcal{C}_8\), namely

\[\begin{align*}
Y_4 F_6^{(0)} - Y_3 F_7^{(0)} - X^4 F_{10}^{(0)} &= 0 \\
Y_4 F_7^{(0)} - Y_3 F_8^{(0)} + X^4 F_{10}^{(0)} &= 0 \\
Y_4 F_{10}^{(0)} - Y_7 F_8^{(0)} + X^{\tau+1} F_{14}^{(0)} - X^4 F_{14}^{(0)} &= 0 \\
Y_4 F_{11}^{(0)} - Y_7 F_8^{(0)} + Y_8 F_7^{(0)} &= 0 \\
Y_4 F_{12}^{(0)} - Y_8 F_8^{(0)} - Y_8 F_{16}^{(0)} &= 0 \\
Y_4 F_{14}^{(0)} - Y_8 F_{10}^{(0)} - Y_7 F_{14}^{(0)} &= 0 \\
Y_4 F_{15}^{(0)} - Y_7 F_{12}^{(0)} + X^{\tau+2} F_{7}^{(0)} &= 0 \\
Y_4 F_{16}^{(0)} - Y_8 F_{12}^{(0)} + X^{\tau+2} F_{8}^{(0)} &= 0.
\end{align*}\]  

(14)

The total number of syzygies is 16. The other syzygies can easily be found by lifting the syzygies of the zero-dimensional ring obtained by setting \(X = 0\).

Now we replace in the syzygies the initial isobaric forms \(F_i^{(0)}\) by the associated unfolded \(F_i\) in (12), with the normalizations (13). Next we apply the division algorithm to quadratic monomials in the \(Y_i\). For the first syzygy we get:

\[\begin{align*}
Y_4 F_6 - Y_3 F_7 - X^4 F_{10} - f_{10}^{(2)} F_8 + f_{11}^{(3)} F_7 - f_{1}^{(4)} F_{12} + f_{12}^{(4)} F_6 - f_{7}^{(5)} F_{11} + f_{8}^{(6)} F_{10} &= \\
&= R_5 Y_3 Y_8 + R_2 Y_8 + R_3 Y_7 + R_6 Y_4 + R_1 Y_3 + R_4
\end{align*}\]  

(15)
where

\[ R_5 = f_6^{(5)} - f_7^{(5)} \]
\[ R_2 = X f_6^{(2)} - X^2 f_1^{(2)} + f_6^{(4)} f_7^{(4)} - f_6^{(3)} f_7^{(3)} - f_6^{(2)} f_7^{(6)} + f_7^{(6)} f_1^{(2)} - f_6^{(5)} f_7^{(4)} - f_6^{(4)} f_7^{(4)} \]
\[ R_3 = X f_6^{(3)} - X^2 f_1^{(3)} + f_6^{(5)} f_7^{(4)} - f_6^{(3)} f_7^{(6)} - f_6^{(2)} f_7^{(4)} + f_1^{(6)} f_7^{(3)} - f_6^{(4)} f_7^{(5)} \]
\[ R_6 = f_6^{(6)} - X^2 f_1^{(6)} - X^2 f_1^{(10)} + f_6^{(6)} f_7^{(4)} - f_6^{(2)} f_7^{(4)} + f_7^{(6)} f_1^{(10)} - f_6^{(5)} f_7^{(4)} - f_6^{(4)} f_7^{(4)} \]
\[ R_1 = -f_7^{(1)} - f_7^{(1)} f_8^{(5)} - f_7^{(1)} f_8^{(4)} f_7^{(3)} \]
\[ R_4 = X^{2+2} f_6^{(4)} - X^{2+1} f_1^{(4)} - X^2 f_1^{(10)} - f_6^{(3)} f_7^{(1)} + f_6^{(6)} f_7^{(4)} - f_6^{(2)} f_8^{(2)} + f_7^{(6)} f_1^{(10)} \]

The condition that the syzygy between the \( F_0 \) lifts to a syzygy between the \( F_1 \) is that the coefficients \( R_k \) vanish. This in turn leads to quasi-homogeneous equations between the coefficients \( f_{ij} \). We observe that the vanishing of the coefficient \( R_5 \) of \( Y_3 Y_8 \) gives the linear equation \( f_6^{(5)} = f_7^{(5)} \). The same happens for the other 15 syzygies. We get the 16 linear equations

\[ f_{15}^{(1)} = 0, \quad f_{11}^{(1)} = f_{14}^{(1)}, \]
\[ f_{16}^{(2)} = f_{15}^{(2)} = f_{16}^{(2)}, \quad f_{14}^{(2)} = f_{12}^{(2)}, \]
\[ f_{10}^{(3)} = 0, \quad f_{11}^{(3)} = -f_{14}^{(3)}, \quad f_{12}^{(3)} = f_{14}^{(3)}, \]
\[ f_{12}^{(4)} = f_{16}^{(4)} = f_{12}^{(4)}, \]
\[ f_{16}^{(5)} = f_{14}^{(5)}, \quad f_{14}^{(5)} = f_{7}^{(5)}, \quad f_{12}^{(5)} = f_{14}^{(5)}, \]
\[ f_{10}^{(6)} = f_{14}^{(6)} = f_{16}^{(6)} = f_{15}^{(6)}. \]

We use these equations to reduce the number of variables. Together with the normalizations \( (13) \) our polynomials \( (12) \) reduce then to

\[ F_6 = Y_3^2 - X^{2+1} + f_{10}^{(2)} Y_4 - f_{11}^{(3)} Y_3 + f_{6}^{(4)} Y_8 + f_{7}^{(5)} Y_7 + f_{6}^{(6)} \]
\[ F_7 = Y_3 Y_4 - X^2 Y_7 + f_{7}^{(1)} + f_{12}^{(4)} Y_3 + f_{7}^{(5)} Y_8 + f_{8}^{(6)} Y_7 \]
\[ F_8 = Y_2^2 - X^2 Y_8 + f_{8}^{(1)} Y_7 + f_{8}^{(4)} Y_4 + f_{12}^{(4)} Y_4 + f_{12}^{(5)} Y_3 + f_{8}^{(6)} Y_8 \]
\[ F_{10} = Y_3 Y_7 - X^{2+1} Y_4 + f_{10}^{(2)} Y_8 + f_{14}^{(4)} Y_4 + f_{14}^{(6)} Y_4 \]
\[ F_{11} = Y_4 Y_7 - Y_3 Y_8 + f_{14}^{(4)} Y_4 + f_{11}^{(3)} Y_8 + f_{15}^{(5)} \]
\[ F_{12} = Y_4 Y_8 - X^{2+2} + f_{16}^{(2)} Y_4 + f_{12}^{(3)} Y_3 + f_{12}^{(4)} Y_8 + f_{12}^{(5)} Y_7 + f_{16}^{(6)} \]
\[ F_{14} = Y_2^2 - X^2 Y_8 + f_{14}^{(1)} Y_7 + f_{14}^{(4)} Y_4 + f_{14}^{(4)} Y_4 + f_{14}^{(5)} Y_3 + f_{16}^{(6)} Y_8 \]
\[ F_{15} = Y_7 Y_8 - X^{2+2} Y_3 + f_{15}^{(2)} Y_7 + f_{15}^{(3)} + f_{15}^{(4)} Y_4 + f_{15}^{(6)} \]
\[ F_{16} = Y_2^2 - X^{2+2} Y_4 + f_{16}^{(1)} Y_3 + f_{16}^{(2)} Y_8 + f_{12}^{(3)} Y_7 + f_{16}^{(4)} + f_{15}^{(6)} Y_4 \]

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involving 25 coefficients \( f_i^{(j)} \).

To proceed further we note that the equations (16) still include \( X \), which is a function on the monomial curve, in an explicit manner. It turns out to be more convenient to use a local coordinate at infinity on this curve. We put

\[
X = t^{-6}, \quad Y_3 = t^{-6-3\tau}, \quad Y_4 = t^{-6-4\tau}, \quad Y_7 = t^{-6-7\tau}, \quad Y_8 = t^{-8-3\tau}
\]

in the functions \( F_i \) and the syzygies and clear denominators. We give the variable \( t \) the weight \(-1\). We consider the polynomials

\[
f_i := \sum_{r=1}^{12\tau+i} F_i(t^{-6}, t^{-6-3\tau}, t^{-6-4\tau}, t^{-6-7\tau}, t^{-8-3\tau}) t^{-1+12\tau}
\]

and we write each one as the sum of its partial polynomials

\[
f_i^{(j)} = \sum_{r+j \equiv 0 \mod 6} f_i t^r, \quad (j = 1, \ldots, 6),
\]

which are defined by collecting all terms whose exponents are in the same residue class modulo 6. In particular, the weight of each \( f_i^{(j)} \) is now zero. Our previous \( f_i^{(j)} = \sum_{k=0}^{\rho} f_{i+j+6k} X^{p-k} \) becomes \( f_i^{(j)} = \sum_{k=0}^{\rho} f_{i+j+6k} t^{j+6k} \), justifying the use of the same notation. By using the same substitution on the syzygies, like (15), the left hand side becomes identically equal to the right hand side, as they give the unique expression of the restriction of both sides to the curve. The advantage is that the variable \( t \) does no longer occur explicitly.

We give the resulting equation from the first of the eight syzygies (14):

\[
f_6 - f_7 - f_{10} - f_{10}^{(2)} f_8 + f_1^{(3)} f_7 - f_6^{(4)} f_{12} + f_5^{(4)} f_6 - f_7^{(5)} f_{11} + f_6^{(6)} f_{10} = 0
\]

Writing out these equations for all syzygies in terms of the partial polynomials leads to 60 equations, some of which are zero while others coincide. We further eliminate variables. Equation \( R_1 \) in (15) becomes (using the linear equations) \( f_7^{(1)} = -f_{10}^{(2)} f_{12}^{(5)} - f_{12}^{(4)} f_5^{(3)} \). Similarly we can eliminate all other \( f_i^{(j)} \) with \( i - j \equiv 0 \mod 6 \), which are the ones in (17) that are not coefficients of an \( Y_1 \).

\[
\begin{align*}
    f_6^{(6)} &= f_6^{(1)} + f_6^{(6)} + f_7^{(5)} f_6^{(1)} + f_6^{(4)} f_6^{(2)} - f_6^{(14)} f_8^{(6)} \\
    f_7^{(1)} &= -f_6^{(4)} f_6^{(3)} - f_6^{(2)} f_6^{(5)} \\
    f_8^{(2)} &= -f_7^{(5)} f_6^{(3)} - f_6^{(3)} f_6^{(5)} \\
    f_9^{(4)} &= f_6^{(4)} f_6^{(2)} - f_7^{(5)} f_6^{(1)} - f_6^{(4)} f_6^{(6)} \\
    f_10^{(5)} &= f_6^{(5)} + f_6^{(1)} f_6^{(4)} + f_6^{(2)} f_6^{(3)} + f_6^{(3)} f_6^{(4)} - f_6^{(1)} f_6^{(6)} \\
    f_11^{(6)} &= f_6^{(6)} + f_6^{(6)} - f_6^{(1)} f_6^{(5)} - f_6^{(3)} f_6^{(3)} - f_6^{(2)} f_6^{(4)} - f_6^{(1)} f_6^{(6)} \\
    f_12^{(2)} &= f_6^{(2)} + f_6^{(4)} f_6^{(4)} - f_6^{(3)} f_6^{(4)} - f_6^{(2)} f_6^{(6)} \\
    f_13^{(3)} &= f_6^{(3)} + f_6^{(4)} f_6^{(5)} - f_6^{(3)} f_6^{(6)} \\
    f_14^{(4)} &= f_6^{(4)} f_6^{(3)} - f_6^{(2)} f_6^{(4)} - f_6^{(3)} f_6^{(5)} \\
    f_15^{(5)} &= f_6^{(5)} f_6^{(5)}
\end{align*}
\]

(18)
Inserting these values reduces the number of equations to five.

4.2 Lemma The remaining 16 partial polynomials $f_{ij}$ satisfy the following five equations identically in $t$:

\[ f_{16}^{(1)}(1 - f_8^{(6)}) + f_8^{(1)}(1 - f_{15}^{(6)}) + f_{12}^{(3)}f_{14}^{(4)} + f_{16}^{(2)}f_{12}^{(5)} = 0 \]
\[ f_{8}^{(1)}(1 - f_{14}^{(6)}) - f_{14}^{(4)}(1 - f_8^{(6)}) + f_{7}^{(5)}f_{16}^{(4)} + f_{6}^{(4)}f_{12}^{(3)} = 0 \]
\[ f_{11}^{(3)}(1 - f_8^{(6)}) + f_{16}^{(1)}f_{10}^{(2)} - f_{7}^{(5)}f_{12}^{(4)} + f_{6}^{(4)}f_{12}^{(3)} = 0 \]
\[ f_{6}^{(4)}(1 - f_{15}^{(6)}) - f_{14}^{(4)}(1 - f_8^{(6)}) - f_{12}^{(4)}(1 - f_{14}^{(6)}) - f_{10}^{(2)}f_{12}^{(6)} = 0 \]
\[ f_{7}^{(5)}(1 - f_{15}^{(6)}) + f_{14}^{(5)}(1 - f_8^{(6)}) - f_{12}^{(5)}(1 - f_{14}^{(6)}) + f_{10}^{(2)}f_{12}^{(3)} = 0. \]

The partial polynomials $f_{6}^{(4)}$, $f_{7}^{(5)}$ and $f_{8}^{(6)}$ contain $\tau$ variables, $f_8^{(1)}$, $f_{12}^{(3)}$, $f_{11}^{(2)}$, $f_{10}^{(2)}$, $f_{14}^{(4)}$, and $f_{16}^{(2)}$ contain $\tau + 1$ variables, $f_{14}^{(1)}$, $f_{15}^{(6)}$, $f_{14}^{(1)}$, $f_{14}^{(1)}$, $f_{16}^{(2)}$ contain $\tau + 2$ variables and $f_{16}^{(1)}$ has $\tau + 3$ variables. Using the remaining normalizations $f_{8,1} = f_{12,4} = f_{8,6} = 0$ the total number of variables is reduced by three to $16\tau + 21$.

To understand the equations (19) we return to the $f_{ij}^{(j)}$ in terms of the variable $X$ and write the equations somewhat differently.

\[ (f_{16}^{(1)} + X^2f_8^{(1)})(X^\tau - f_8^{(6)}) = f_8^{(1)}(f_{15}^{(1)} - X^2f_8^{(6)}) - f_{12}^{(3)}f_{14}^{(4)} - f_{16}^{(2)}f_{12}^{(5)} \]
\[ (f_{14}^{(4)} - Xf_8^{(1)})(X^\tau - f_8^{(6)}) = f_{14}^{(1)}f_{16}^{(5)} - f_{16}^{(1)}f_{14}^{(6)} - Xf_8^{(6)} + f_{6}^{(4)}f_{12}^{(3)} \]
\[ f_{11}^{(3)}(X^\tau - f_8^{(6)}) = f_{7}^{(5)}f_{12}^{(4)} - f_{16}^{(1)}f_{10}^{(2)} - f_{6}^{(4)}f_{12}^{(3)} \]
\[ (f_{14}^{(4)} + Xf_{12}^{(4)} - X^2f_6^{(4)})(X^\tau - f_8^{(6)}) = f_{14}^{(4)}(f_{15}^{(6)} - Xf_8^{(6)}) - f_{14}^{(4)}(f_{15}^{(6)} - X^2f_8^{(6)}) - f_{10}^{(2)}f_{16}^{(2)} \]
\[ (f_{14}^{(5)} - Xf_{12}^{(5)} + X^2f_7^{(5)})(X^\tau - f_8^{(6)}) = f_{14}^{(5)}(f_{15}^{(6)} - X^2f_8^{(6)}) - f_{14}^{(5)}(f_{15}^{(6)} - X^2f_8^{(6)}) - f_{10}^{(2)}f_{12}^{(3)}. \]

All these equations are of the form

\[ L \cdot (X^\tau - f_8^{(6)}) = R \]

with $L$ and $R$ polynomials in $X$ satisfying $\deg_X(R) \leq \deg_X(L) + \tau$. Division with remainder gives $R = Q(X^\tau - f_8^{(6)}) + \overline{R}$, and therefore we can solve $L = Q$ and find as equations for the base space that the coefficients of $\overline{R}$ have to be zero. In other words, the condition leading to the equations of the base space is that the right hand side of the equations (20) is divisible by $X^\tau - f_8^{(6)}$. A similar structure first appeared for the base spaces of rational surface singularities of multiplicity four [dJvS].

4.3 Theorem Let $S$ be the semigroup generated by $6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau$ and $8 + 6\tau$ where $\tau$ is a positive integer. The $5\tau$ equations of the base space $T^-$ of the versal deformation in negative degrees of the monomial curve $C_S$ are given by the condition on the $11\tau + 8$ coefficients $f_{1k}$ occurring in the polynomials $f_{ij}^{(j)}$, that the Pfaffians of the skew-symmetric
are divisible by $X^r - f_8^{(6)}$. In particular, the dimension of $T^{1,-}(k[S])$ is $11\tau + 8$.

**Proof** The Pfaffians of the matrix give the same set of equations as the right hand side of the equations (20). They involve only the polynomials $f_6^{(4)}, f_7^{(5)}, f_8^{(6)}, f_8^{(1)}, f_8^{(5)}, f_9^{(2)}, f_{12}^{(4)}, f_{12}^{(6)}, f_{12}^{(3)}, f_{15}^{(6)}$ and $f_{16}^{(2)}$, all others are eliminated. The divisibility condition leads to 5τ equations without linear parts. By counting the number of coefficients and taking the remaining three normalizations into account we find that the dimension of $T^{1,-}(k[S])$ is $11\tau + 8$. □

### 4.4 Corollary
The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $S$ correspond bijectively to the orbits of the $G_m$-action on $\mathcal{J}$ described in the theorem above.

### 4.5 Theorem
The monomial curve $C_S$ is negatively smoothable. Therefore the moduli space $M_{g,1}^5$ is non-empty.

**Proof** The equations (21) can be solved by setting all variables equal to zero except for those involved in $f_8^{(6)}, f_{14}^{(6)}$ and $f_{15}^{(6)}$. We take only the variables of highest weight non-zero. Using (18) we find the following generators of the ideal:

\[
\begin{align*}
Y_3^2 - (X^{r+1} - b)(X^r - a), & \quad Y_3 Y_4 - (X^r - a)Y_7, & \quad Y_4^2 - (X^r - a)Y_8, \\
Y_3^2 Y_7 - (X^{r+1} - b)Y_4, & \quad Y_4 Y_7 - Y_3 Y_8, & \quad Y_4 Y_8 - (X^{r+2} - c)(X^r - a), \\
Y_2^2 - (X^{r+1} - b)Y_8, & \quad Y_7 Y_8 - (X^{r+2} - c)Y_3, & \quad Y_2^2 - (X^{r+2} - c)Y_4,
\end{align*}
\]

(22)

where a, b and c are non-zero constants such that the polynomials $X^r - a, X^{r+1} - b$ and $X^{r+2} - c$ have pairwise no common roots. To show that this curve is smooth we compute the Jacobi-matrix of the above system of equations. First consider the submatrix

\[
\begin{pmatrix}
0 & f_{16}^{(1)} & f_{12}^{(4)} & f_{14}^{(6)} & f_8^{(6)} & f_6^{(6)} \\
-f_{16}^{(2)} & 0 & f_{12}^{(3)} & f_{15}^{(6)} & f_{14}^{(6)} & f_8^{(6)} \\
-f_8^{(1)} & -f_{15}^{(6)} & 0 & f_{12}^{(5)} & f_8^{(5)} & f_8^{(5)} \\
f_6^{(4)} & -f_8^{(6)} + X^2 f_8^{(6)} & -f_8^{(5)} & 0 & f_{10}^{(2)} & f_8^{(5)} \\
-f_6^{(4)} & -f_8^{(6)} + X f_8^{(6)} & -f_8^{(5)} & -f_8^{(2)} & 0 & 0
\end{pmatrix}
\]

(21)

The lower right $3 \times 3$ subdeterminant is equal to $Y_8^2(2Y_3^2 + (X^{r+2} - c)Y_4)$, which is congruent to $3Y_8^3$ modulo the last equation. A necessary condition for a singular point is therefore
\[ Y_3 Y_8 = 0. \] With the other equations it then follows that \( Y_7 Y_8 = 0 \) and therefore \( Y_7 = (X^{\tau+1} - b) Y_8 = 0 \). From the submatrix again we also get that \( X^\tau (X^\tau - a) Y_8 = 0 \). Because \( \gcd(X^\tau(X^\tau - a), X^{\tau+1} - b) = 1 \) it follows that \( Y_8 = 0 \) and then \( Y_4 = 0 \). The Jacobi-matrix simplifies, and we can conclude that \( X^\tau Y_8 = 0 \). From the submatrix again we also get that \( X^\tau (X^\tau - a) Y_8 = 0 \). Because \( \gcd(X^\tau(X^\tau - a), X^\tau + 1 - b) = 1 \) it follows that \( Y_8 = 0 \) and then \( Y_4 = 0 \). The Jacobi-matrix simplifies, and we can conclude that \( X^\tau Y_8 = 0 \). But then also \( (X^{\tau+1} - b)(X^\tau - a) = 0 \), contradicting the choice of \( a, b \) and \( c \). Therefore this curve provides a negative smoothing. By Pinkham’s theorem \( M_{g,1}^8 \) is then non-empty.

Although we get explicit equations for the base space, it is difficult to understand the structure of this space, or even to find its dimension. We note that the dimension is equal to the dimension of the tangent cone. Among the equations of the tangent cone we have the quadratic part of our \( 5 \tau \) equations. They define an affine quadratic cone \( Q \subset A^{11 \tau+8} \), that contains the tangent cone. In this way we get an upper bound for the dimension; this is the method presented in [CSt13, Section 3].

Note that division by \( X^\tau - f^6_8 \) introduces higher order monomials involving the coefficients of \( f^6_8 \). So to obtain the quadratic part of the equations we have to divide by \( X^\tau \).

4.6 Theorem The quadratic quasi-cone \( Q \) is isomorphic to the direct product

\[ Q = M \times N, \]

where \( M \) is the \((\tau + 8)\)-dimensional weighted affine space of weights \( 2, 2, 3, 5, 6, 6, 8, 9, 12 \) and \( 6i, i = 1, \ldots, \tau - 1 \), and \( N \) is the quadratic quasi-cone consisting of vectors

\[ (\omega_1, \ldots, \omega_{10}) = \left( \sum_{j=0}^{\tau-1} \omega_{1j} X^j, \ldots, \sum_{j=0}^{\tau-1} \omega_{10j} X^j \right), \]

such that the Pfaffians of the matrix

\[
\begin{pmatrix}
0 & \omega_1 & \omega_2 & \omega_3 & \omega_4 \\
-\omega_1 & 0 & \omega_5 & \omega_6 & \omega_7 \\
-\omega_2 & -\omega_5 & 0 & \omega_8 & \omega_9 \\
-\omega_3 & -\omega_6 & -\omega_8 & 0 & \omega_{10} \\
-\omega_4 & -\omega_7 & -\omega_9 & -\omega_{10} & 0 \\
\end{pmatrix}
\]

are zero in the artinian algebra \( k[X]/(X^\tau) \).

Proof We first put \( X^\tau = 0 \) in the matrix (21). Then we have a skew-symmetric matrix whose ten entries above the diagonal are polynomials of degree \( \tau - 1 \) with in total \( 10 \tau \) linearly independent coefficients, so a generic matrix of this type. By an obvious substitution we obtain the matrix in the statement.

By the computation in the proof of [CSt13, Cor. 4.5] the dimension of \( N \) is \( 7 \tau \). Using the lower bound from Theorem 2.4 we obtain

4.7 Corollary We have \( \dim Q = 8 \tau + 8 \). The moduli space \( M_{g,1}^8 \) is of pure dimension \( 8 \tau + 7 \).
4.2 A second family

We apply the same method as above for the following particular family of symmetric semigroups. For each $\tau \geq 1$, let

$$S = \langle 6, 1 + 6\tau, 2 + 6\tau, 3 + 6\tau, 4 + 6\tau \rangle$$

$$= \mathbb{N} \sqcup \bigcup_{j \in \{1, 2, 3, 4\}} (j + 6\tau + 6\mathbb{N}) \sqcup (5 + 12\tau + 6\mathbb{N}),$$

be a symmetric semigroup of genus $g = 6\tau$ generated minimally by five elements.

4.8 Remark If $\tau = 1$, the ideal of the canonical monomial curve $C_S \subset \mathbb{P}^5$ cannot be generated by quadratic forms only, see the recent preprint by Contiero and Fontes [CF18]. In this case there are two natural compactifications of the monomial curve in $\mathbb{A}^5$: in $\mathbb{P}^5$ the equations of the next lemma define the ideal of the canonical monomial curve; it is a trigonal curve whose ideal can be generated by 6 quadratic and 3 cubic equations. For Pinkham’s construction we compactify the affine curve in a weighted projective space with weights $(1, 6, 7, 8, 9, 10)$, and the results below are also valid for $\tau = 1$.

Since the methods are the same as in the preceding subsection, we do not give proofs.

We introduce variables $X, Y_1, Y_2, Y_3, Y_4$ of weight $6, 1 + 6\tau, 2 + 6\tau, 3 + 6\tau, 4 + 6\tau$. Similar to Lemma 4.1 we have

4.9 Lemma The ideal of the affine monomial curve

$$C_S = \{(t^6, t^{1+6\tau}, t^{2+6\tau}, t^{3+6\tau}, t^{4+6\tau}) : t \in \mathbb{k}\}$$

is generated by the forms $F_i^{(0)}$ and $G_i^{(0)}$

$$F_2^{(0)} = Y_1^2 - X^T Y_2,$$
$$F_3^{(0)} = Y_1 Y_2 - X^T Y_3,$$
$$F_4^{(0)} = Y_1 Y_3 - X^T Y_4,$$
$$F_5^{(0)} = Y_1 Y_4 - Y_2 Y_3,$$
$$F_6^{(0)} = Y_2 Y_4 - X^{2\tau+1},$$
$$F_7^{(0)} = Y_3 Y_4 - X^{\tau+1} Y_1,$$
$$F_8^{(0)} = Y_4^2 - X^{\tau+1} Y_2.$$
We have three normalizations left, which we leave for later on. These are the following:

\[ F_2 = Y_1^2 - X^2 Y_2 + f_2^{(1)} Y_1 + f_2^{(2)} + f_2^{(4)} Y_4 + f_2^{(5)} Y_3 + f_2^{(6)} Y_2 \]

\[ F_3 = Y_1 Y_2 - X^2 Y_3 + f_3^{(1)} Y_2 + f_3^{(2)} Y_1 + f_3^{(4)} Y_4 + f_3^{(5)} Y_3 + f_3^{(6)} Y_2 \]

\[ F_4 = Y_1 Y_3 - X^2 Y_4 + f_4^{(1)} Y_3 + f_4^{(2)} Y_2 + f_4^{(3)} Y_1 + f_4^{(4)} Y_4 + f_4^{(5)} Y_3 + f_4^{(6)} Y_2 \]

\[ G_4 = Y_2^2 - X^2 Y_4 + g_4^{(1)} Y_3 + g_4^{(2)} Y_2 + g_4^{(3)} Y_1 + g_4^{(4)} + g_4^{(6)} Y_4 \]

\[ F_5 = Y_1 Y_4 - Y_2 Y_3 + f_5^{(1)} Y_4 + f_5^{(2)} Y_3 + f_5^{(3)} Y_2 + f_5^{(4)} Y_1 + f_5^{(5)} \]

\[ F_6 = Y_2 Y_4 - X^{2r+1} + f_6^{(1)} Y_3 + f_6^{(2)} Y_4 + f_6^{(3)} Y_3 + f_6^{(4)} Y_2 + f_6^{(5)} Y_1 + f_6^{(6)} \]

\[ G_6 = Y_3^2 - X^{2r+1} + g_6^{(1)} Y_3 + g_6^{(2)} Y_4 + g_6^{(3)} Y_3 + g_6^{(4)} Y_2 + g_6^{(5)} Y_1 + g_6^{(6)} \]

\[ F_7 = Y_3 Y_4 - X^{r+1} Y_1 + f_7^{(1)} + f_7^{(2)} Y_2 Y_3 + f_7^{(3)} Y_4 + f_7^{(4)} Y_3 + f_7^{(5)} Y_2 + f_7^{(6)} Y_1 \]

\[ F_8 = Y_4^2 - X^{r+1} Y_2 + f_8^{(1)} Y_1 + f_8^{(2)} + f_8^{(3)} Y_2 Y_3 + f_8^{(4)} Y_4 + f_8^{(5)} Y_3 + f_8^{(6)} Y_2 \]

Here \( f_1^{(j)} \) or \( g_1^{(j)} \) is a polynomial in \( X \) with deformation variables as coefficients, of degree \( 2r \) in \( X \) if \( i = j \), \( 2r+1 \) if \( i = j + 6 \), degree 0 if \( i = j + 5 \) and otherwise of degree \( r + \lfloor \frac{j - 1}{6} \rfloor \) in \( X \).

We normalize the coefficients

\[ f_6^{(1)} = g_6^{(1)} = f_7^{(2)} = f_8^{(3)} = 0, \quad f_4^{(1)} = f_5^{(2)} = f_7^{(3)} = f_5^{(4)} = 0. \]

We have three normalizations left, which we leave for later on. These are the following: we can make \( f_{2,6}, f_{3,6} \) or \( f_{8,6} \) equal to zero, we can make \( f_{2,1}, g_{4,1} \) or \( f_{8,1} \) equal to zero and we can make \( f_{4,2} \) or \( f_{5,2} \) equal to zero (in fact also other coefficients, but they will turn out to be equal to one of these).

With these normalizations the equations become similar to the normalized equations (12). In fact, if we replace \( Y_1 \) by \( Y_7 \) and \( Y_2 \) by \( Y_8 \), and rename the equations accordingly:

equations (23): \( F_2, F_3, F_4, G_4, F_5, F_6, G_6, F_7, F_8 \)

equations (12): \( F_{14}, F_{15}, F_{16}, F_{11}, F_{12}, F_6, F_7, F_8 \)

then the equations become identical as written but the symbols \( f_1^{(j)} \) have a slightly different meaning, and the powers of \( X \) are somewhat different. The equations for the partial polynomials in \( t \) become the same. Therefore we find the equations for the base space by renaming the partial polynomials in (19), and we find

\[ g_4^{(1)} (1 - f_8^{(6)}) + f_8^{(1)} (1 - f_8^{(6)}) + f_6^{(3)} f_4^{(1)} + g_4^{(2)} f_6^{(5)} = 0 \]

\[ f_8^{(1)} (1 - f_8^{(6)}) - f_2^{(1)} (1 - f_8^{(6)}) + f_7^{(4)} g_4^{(2)} + g_4^{(4)} f_6^{(3)} = 0 \]

\[ f_6^{(3)} (1 - f_8^{(6)}) + f_8^{(1)} f_4^{(2)} - f_7^{(5)} f_6^{(4)} + g_6^{(4)} f_6^{(5)} = 0 \]

\[ g_6^{(4)} (1 - f_3^{(6)}) - f_2^{(4)} (1 - f_8^{(6)}) - f_6^{(4)} (1 - f_2^{(6)}) - f_4^{(2)} g_4^{(2)} = 0 \]

\[ f_7^{(5)} (1 - f_3^{(6)}) + f_2^{(5)} (1 - f_8^{(6)}) - f_6^{(5)} (1 - f_2^{(6)}) + f_4^{(2)} f_6^{(3)} = 0. \]
These equations, written with the variable $X$, are less suited as the term we have to divide with is $X^{\tau+1} - f_8^{(6)}$. We replace the system of equations with an equivalent one, where the second and third equation are the following:

$$g_4^{(1)}(1 - f_2^{(6)}) + f_2^{(1)}(1 - f_3^{(6)}) - f_6^{(4)} f_2^{(4)} + g_4^{(2)} f_2^{(5)} = 0$$

$$f_5^{(3)}(1 - f_3^{(6)}) - g_4^{(1)} f_4^{(2)} + f_6^{(4)} f_2^{(5)} + f_2^{(4)} f_6^{(5)} = 0$$

We now can use division with $X^\tau - f_3^{(6)}$. We write

$$(f_8^{(1)} + g_4^{(1)})(X^\tau - f_3^{(6)}) = g_4^{(1)}(f_8^{(6)} - X f_3^{(6)}) - f_6^{(4)} f_6^{(4)} - f_4^{(2)} f_6^{(5)}$$

$$(g_4^{(1)} + f_2^{(1)})(X^\tau - f_3^{(6)}) = g_4^{(1)}(f_2^{(6)} - f_3^{(6)}) + f_6^{(4)} f_4^{(2)} - f_2^{(4)} f_6^{(5)}$$

$$(g_6^{(4)} - X f_2^{(4)} - f_6^{(4)})(X^\tau - f_3^{(6)}) = -f_2^{(4)}(f_8^{(6)} - X f_3^{(6)}) - f_6^{(4)}(f_2^{(6)} - f_3^{(6)}) + f_4^{(2)} g_4^{(2)}$$

$$(f_6^{(5)} - f_7^{(5)} - f_2^{(5)})(X^\tau - f_3^{(6)}) = -f_2^{(5)}(f_8^{(6)} - X f_3^{(6)}) + f_6^{(5)}(f_2^{(6)} - f_3^{(6)}) + f_4^{(2)} f_3^{(6)}$$

We write the right hand side as Pfaffians of the following skew-symmetric matrix

$$\begin{pmatrix}
0 & g_4^{(2)} & f_6^{(4)} & -f_2^{(4)} \\
-g_4^{(1)} & 0 & f_6^{(3)} & f_8^{(6)} - X f_3^{(6)} \\
-f_6^{(4)} & -f_8^{(6)} + X f_3^{(6)} & 0 & f_2^{(5)} \\
-f_2^{(4)} & -f_6^{(4)} + f_6^{(6)} & -f_4^{(2)} & 0
\end{pmatrix}$$

We conclude that the base space has the same structure as for the first family.

**4.10 Theorem** Let $S$ be the semigroup generated by $6, 1+6\tau, 2+6\tau, 3+6\tau$ and $4+6\tau$ where $\tau$ is a positive integer. The $5\tau$ equations of the base space $\mathcal{T}^-$ of the versal deformation of the monomial curve $C_S$ in negative degrees are given by the condition on the $11\tau + 4$ coefficients occurring in the matrix (26) that the Pfaffians of this matrix are divisible by $X^\tau - f_3^{(6)}$. In particular, the dimension of $\mathcal{T}^{1-, (k[S])}$ is $11\tau + 4$.

As for the curves in the first family we can show that the monomial curve is negatively smoothable.

**4.11 Corollary** The dimension of the affine cone $Q_S$ given by the quadratic part of the equations is $8\tau + 4$ and $M^S_{g,1}$ has pure dimension $8\tau + 3$.

**4.12 Remark** Using the upper bound obtained by Contiero and Stoehr in [CSt13, Cor. 4.5] for the symmetric semigroup generated minimally by $6, 2+6\tau, 3+6\tau, 4+6\tau$ and $5+6\tau$ for $\tau \geq 1$ we find that also in this case $M^S_{g,1}$ has pure dimension $2g - 1 - \dim T^{1+, (k[S])} = 8\tau + 7$, if non-empty.
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André Contiero  
contiero@ufmg.br  
Università Federal de Minas Gerais  
Belo Horizonte, MG, Brazil  

Jan Stevens  
stevens@chalmers.se  
Department of Mathematical Sciences,  
Chalmers University of Technology and  
University of Gothenburg.  
SE 412 96 Gothenburg, Sweden

Aislan L. Fontes  
aislan@ufs.br  
Università Federal de Sergipe  
Itabaian, SE, Brazil

Jhon Quipse Vargas  
jhon.quispe@gmail.com  
Università Federal de Goias  
Catalão, GO, Brazil