Hopf Hypersurfaces in pseudo-Riemannian complex and para-complex space forms

Henri Anciaux∗, Konstantina Panagiotidou†

Abstract

The study of real hypersurfaces in pseudo-Riemannian complex space forms and para-complex space forms, which are the pseudo-Riemannian generalizations of the complex space forms, is addressed. It is proved that there are no umbilic hypersurfaces, nor real hypersurfaces with parallel shape operator in such spaces. Denoting by \( J \) be the complex or para-complex structure of a pseudo-complex or para-complex space form respectively, a non-degenerate hypersurface of such space with unit normal vector field \( N \) is said to be Hopf if the tangent vector field \( JN \) is a principal direction. It is proved that if a hypersurface is Hopf, then the corresponding principal curvature (the Hopf curvature) is constant. It is also observed that in some cases a Hopf hypersurface must be, locally, a tube over a complex (or para-complex) submanifold, thus generalizing previous results of Cecil, Ryan and Montiel.

2010 MSC: 53C42, 53C40, 53B25

Introduction

The study of real hypersurfaces in complex space forms, i.e. the complex projective space \( \mathbb{CP}^n \) and the complex hyperbolic space \( \mathbb{CH}^n \), have attracted a lot of attention in the last decades (see [NR] for a survey of the subject and references therein). The complex structure \( J \) of a complex space form induces a rich structure on real hypersurface; in particular, on an arbitrary oriented hypersurface \( S \) of \( \mathbb{CP}^n \) or \( \mathbb{CH}^n \) with unit vector normal field \( N \), a canonical tangent field, called the structure vector field or the Reeb vector field, is defined by \( \xi := -JN \). If \( \xi \) is a principal direction on \( S \), i.e. an eigenvector of the shape operator, \( S \) is called a Hopf hypersurface. It turns out that the principal curvature associated to the structure vector \( \xi \) (the Hopf principal curvature) of a connected, Hopf hypersurface must be constant (this was proved in [Ma] in the projective case and in [KS] in the hyperbolic case). Moreover, in [CR], Hopf hypersurfaces in \( \mathbb{CP}^n \) are locally characterized as tubes over complex submanifolds, while in [Mo], the same statement is proved for Hopf hypersurfaces of \( \mathbb{CH}^n \) whose Hopf principal

∗Universidade de São Paulo; supported by CNPq (PDE 211682/2013-6)
†Faculty of Engineering, Aristotle University of Thessaloniki, Greece
curvature $a$ satisfies $|a| > 2$. Recently Hopf hypersurfaces of $\mathbb{CH}^n$ with small Hopf principal curvature, i.e. satisfying $|a| \leq 2$, have been studied through a kind of generalized Gauss map in [IR] and [IV], while in [Ki] a unified approach is proposed, relating Hopf hypersurfaces to totally complex (or para-complex) submanifolds of some natural quaternionic manifold.

The purpose of this paper is to address the study of real hypersurfaces in \textit{pseudo-complex space forms} $\mathbb{CP}^n_p$, which are the pseudo-Riemannian generalizations of the complex space forms, and in \textit{para-complex space form} $\mathbb{DP}^n$. The latter space is the para-complex analog of $\mathbb{CP}^n$ and is equipped with both a pseudo-Riemannian metric and a \textit{para-complex} structure, still denoted by $J$, which satisfies $J^2 = Id$. Furthermore, given a real hypersurface in $\mathbb{DP}^n$ with non-degenerate induced metric, the Hopf field is defined exactly as in the complex case. We refer to the next section for the precise definition of $\mathbb{DP}^n$ and a brief description of its geometry. Since both the pseudo-complex and the para-complex case will be studied simultaneously, we define $\epsilon$ in such way that $J^2 = - \epsilon Id$, i.e. $\epsilon = 1$ corresponds to the complex case and $\epsilon = -1$ to the para-complex case. Moreover, $\mathcal{M}$ will denote the pseudo-Riemannian complex space form $\mathbb{CP}^n_p$ or the para-complex space form $\mathbb{DP}^n$, with holomorphic or para-holomorphic curvature $4c$, where $c := \pm 1$.

Our results are:

\textbf{Theorem 1.} There exist no umbilic real hypersurface, nor real hypersurface with parallel shape operator, in $\mathcal{M}$.

\textbf{Theorem 2.} Let $S$ be a connected, non-degenerate hypersurface of $\mathcal{M}$ which is Hopf, i.e. its structure vector $\xi$ is a principal direction of $S$. Then the corresponding principal curvature $a$, i.e. defined by $A\xi = a\xi$, is constant.

\textbf{Theorem 3.} Let $S$ be a connected, non-degenerate hypersurface of $\mathcal{M}$ with unit normal $N$. Assume that $S$ is Hopf and denote by $a$ the corresponding principal curvature, i.e. $A\xi = a\xi$. Then if $c \epsilon \langle N, N \rangle = 1$, or if $c \epsilon \langle N, N \rangle = -1$ and $|a| > 2$, then $S$ is, locally, a tube over a complex or para-complex submanifold.

\textbf{Remark 1.} In the case $c = 1, \epsilon = 1$ and $p = 0$, $\mathcal{M}$ is the complex projective space $\mathbb{CP}^n$, and if $c = -1, \epsilon = 1$ and $p = n$, we have $\mathcal{M} = \mathbb{CH}^n$, the complex hyperbolic space. Hence Theorem 3 generalizes [CR] and [Mo]. Observe that in these two cases, the metric being positive, we have $\langle N, N \rangle = 1$.

This paper is organized as follows: in Section 1 the geometry of the pseudo-Riemannian complex and the para-complex space forms is described. Section 2 contains basic relations about the geometry of real hypersurfaces in $\mathcal{M}$ and the proof of Theorem 1. In Section 3 four Lemmas about real hypersurfaces and the proof of Theorem 2 are presented. Finally, in Section 4 the proof of Theorem 3 is given and at the end of the Section some open problems are proposed for further research on this area.
1 The ambient spaces: pseudo-Riemannian complex and para-complex space forms

1.1 The abstract structures

All along the paper the ambient space will be a $2n$-dimensional pseudo-Riemannian manifold $(M, \langle \cdot, \cdot \rangle, J)$ endowed with is a complex or para-complex structure $J$, i.e. a $(1, 1)$ tensor field satisfying $J^2 = -\epsilon 1d$ which is compatible with respect to $\langle \cdot, \cdot \rangle$, i.e.

$$\langle J\cdot, J\cdot \rangle = \epsilon \langle \cdot, \cdot \rangle.$$ 

In other words, $J$ is an isometry in the complex case and an anti-isometry in the para-complex case. This assumption implies that the signature of $\langle \cdot, \cdot \rangle$ must be even in the complex case and neutral in the para-complex case.

The bilinear map $\omega(X, Y) := \langle JX, Y \rangle$ is alternate and non-degenerate. Furthermore, the 2-form $\omega$ is closed, hence symplectic. Therefore, the triple $(\langle \cdot, \cdot \rangle, J, \omega)$ is a pseudo-Kähler or para-Kähler structure.

We assume furthermore that the curvature $R$ of $\langle \cdot, \cdot \rangle$ satisfies

$$R(X, Y) = c (\epsilon X \wedge Y + JX \wedge JY + 2\langle X, JY \rangle J),$$

where the notation $X \wedge Y$ denotes the operator $Z \rightarrow (X \wedge Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y$ and where $c$ is a real constant. Observe that if $X$ is a non-null vector, we have $\langle R(X, JX)JX, X \rangle = 4c$, i.e. any complex or para-complex 2-plane $Span(X, JX)$ has sectional curvature $4c$. The constant $4c$ is called the holomorphic or para-holomorphic curvature of $(M, \langle \cdot, \cdot \rangle, J)$.

Observe that the rescaled $\lambda \langle \cdot, \cdot \rangle$, where $\lambda$ is a positive constant has holomorphic curvature $\lambda^{-2}c$. On the other hand, replacing the metric $\langle \cdot, \cdot \rangle$ by its opposite $-\langle \cdot, \cdot \rangle$ leaves invariant the curvature operator $R$. It follows that if $(M, \langle \cdot, \cdot \rangle, J)$ has (para-)holomorphic curvature $4c$, then $(M, -\langle \cdot, \cdot \rangle, J)$ has (para-)holomorphic curvature $-4c$.

In the next two sections instances of such manifolds will be described explicitly.

1.2 Pseudo-Riemannian complex space forms

We consider the space $\mathbb{C}^{n+1}$ endowed with the pseudo-Hermitian form:

$$\langle \cdot, \cdot \rangle_p = -\sum_{j=1}^{p} dz_j d\bar{z}_j + \sum_{j=p+1}^{n+1} dz_j d\bar{z}_j$$

The corresponding metric $\langle \cdot, \cdot \rangle_{2p} := \text{Re} \langle \cdot, \cdot \rangle_p$ has signature $(2p, 2(n+1-p))$.

We define the hyperquadrics

$$\mathbb{S}^{2n+1}_{p, c} := \{ z \in \mathbb{C}^{n+1} | \langle z, z \rangle_{2p} = c \},$$

For example, $\mathbb{S}^{2n+1}_{0, 1} = \mathbb{S}^{2n+1}$ is the round unit sphere;
The pseudo-Riemannian complex space forms are the quotients of these hyperquadrics by the natural $S^1$-action:

$$\mathbb{C}P^n_{p,c} := \mathbb{S}^{2n+1}_{2p,c} / \sim,$$

where $z \sim z'$ if there exists $\theta \in \mathbb{R}$ such that $z' = (\cos \theta, \sin \theta)z$. In particular

- $\mathbb{C}P^n_{0,1} = \mathbb{C}P^n$ is the complex projective space;
- $\mathbb{C}P^n_{n,-1} = \mathbb{C}H^n$ is the complex hyperbolic space;

We denote by $\pi$ the canonical projection $\pi : \mathbb{S}^{2n+1}_{2p,c} \to \mathbb{C}P^n_{p,c}$. We endow $\mathbb{C}P^n$ with the metric $\langle \cdot, \cdot \rangle$ that makes the projection $\pi$ a pseudo-Riemannian submersion. The projection $\pi : \mathbb{S}^{2n+1}_{2p,c} \to \mathbb{C}P^n_{p,c}$ also induces a natural complex structure in $\mathbb{C}P^n_{p,c}$. It is easy to check that $(\mathbb{C}P^n_{p,c}, J, \langle \cdot, \cdot \rangle)$ is a pseudo-Kähler manifold and that its curvature tensor satisfies

$$R(X, Y) = c (X \wedge Y + JX \wedge JY + 2 \langle X, JY \rangle J).$$

In particular, $\mathbb{C}P^n_{p,c}$ has constant holomorphic curvature $4c$.

Observe that the involutive map $(z_1, \ldots, z_n) \mapsto (z_{p+1}, \ldots, z_n, z_1, \ldots, z_p)$ is an anti-isometry between $\mathbb{S}^{2n+1}_{2p,c}$ and $\mathbb{S}^{2n+1}_{2n+1-2p,1}$. It follows that the spaces $\mathbb{C}P^n_{p,c}$ and $\mathbb{C}P^n_{n+1-p,-c}$ are anti-isometric.

### 1.3 Para-complex space forms

The set of para-complex (or split-complex, or double) numbers $\mathbb{D}$ is the two-dimensional real vector space $\mathbb{R}^2$ endowed with the commutative algebra structure whose product rule is given by

$$(x, y) \cdot (x', y') = (xx' + yy', xy' + x'y).$$

The para-complex projective plane is the set of para-complex lines of $\mathbb{D}^{n+1}$. We consider the neutral metric

$$\langle \cdot, \cdot \rangle_* := \sum_{j=1}^{n} dx_j^2 - dy_j^2$$

and the hyperquadric

$$\mathbb{S}^{2n+1}_{n+1,-1} := \{z \in \mathbb{D}^{n+1} | \langle z, z \rangle_* = -1\}.$$

Then we define:

$$\mathbb{D}P^n := \mathbb{S}^{2n+1}_{n+1,-1} / \sim,$$

where $z \sim z'$ if there exists $\theta \in \mathbb{R}$ such that $z' = (\cosh \theta, \sinh \theta)z$. We endow $\mathbb{D}P^n$ with the metric $g$ that makes the projection $\pi : \mathbb{S}^{2n+1}_{n+1,-1} \to \mathbb{D}P^n$ a pseudo-Riemannian submersion. The metric $g$ has neutral signature $(n, n)$. For technical reasons it is convenient to introduce the "polar" space $\mathbb{D}P^n$ of $\mathbb{D}P^n$ by

$$\mathbb{D}P^n := \mathbb{S}^{2n+1}_{n+1,1} / \sim.$$
The anti-isometry $J$ of $\mathbb{D}^{n+1}$ induces canonically an anti-isometry between $\mathbb{D}\mathbb{P}^n$ and $\mathbb{D}\mathbb{P}^n$. According to [GM], the curvature operator of $\mathbb{D}\mathbb{P}^n$ is given by

$$R(X, Y) = -X \wedge Y + JX \wedge JY + 2\langle X, JY \rangle J.$$ 

In particular, $\mathbb{D}\mathbb{P}^n$ has constant para-holomorphic curvature 4 (but it is not characterized by this property). On the other hand, $\mathbb{D}\mathbb{P}^n$ has constant para-holomorphic curvature $-4$.

2 Auxiliary relations about real hypersurfaces and proof of Theorem 1

In this section let $\mathcal{S}$ be an immersed real hypersurface in $\mathcal{M}$, whose induced metric is non-degenerate. This implies the local existence of a unit normal vector field $N$. After a possible change of metric $\langle \cdot, \cdot \rangle \rightarrow -\langle \cdot, \cdot \rangle$, there is no loss of generality in assuming that $\langle N, N \rangle = 1$, and we will do so in the remainder of the paper. Observe that reversing the metric has the effecting of reversing its curvature $c$. Hence, without loss of generality, we could alternatively assume that $c = 1$ and let $\langle N, N \rangle$ take the two possible values $\pm 1$. However the first choice seems more natural.

2.1 The structure of a real hypersurface in $\mathcal{M}$

The structure vector field $\xi$ is given by

$$\xi := -\epsilon JN.$$ (1)

It follows that $N = J\xi$ and that $\langle \xi, \xi \rangle = \epsilon$. The orthogonal complement $\xi^\perp := \text{Hor}$, a $(2n - 2)$-dimensional subspace of $T\mathcal{S}$, will be refered as to the horizontal distribution. Given a vector $X$ tangent to $\mathcal{S}$, the vector $JX$ is not necessarily tangent to $\mathcal{S}$ but its tangential part, that we denote by $\varphi$, is horizontal. Introducing the one-form $\eta := \langle J\cdot, N \rangle$, we have

$$JX = \varphi X + \langle JX, N \rangle N = \varphi X + \eta(X)N.$$ (2)

Observe also that

$$\eta(X) = \langle JX, N \rangle = -\langle X, JN \rangle = \epsilon \langle X, \xi \rangle$$

and

$$\eta(\xi) = \langle J\xi, N \rangle = \langle N, N \rangle = 1.$$ (3)

On the other hand, doing $X = \xi$ in Equation (2), we get

$$\varphi \xi = 0.$$ (4)
Now, we have
\[ -\varepsilon X = J^2 X = J(\varphi X + \eta(X)N) = J(\varphi X) + \eta(X)JN = \varphi(X) + \eta(\varphi X)N + \eta(X)JN. \] (5)

Considering the tangent and normal parts of this equation, we get that
\[ \eta \circ \varphi = 0 \] and
\[ -\varepsilon X = \varphi^2 X - e\eta(X)\xi, \]
so that
\[ \varphi^2 = -\varepsilon Id + e\eta(\cdot)\xi. \] (6)

Finally, denoting by \( g \) the induced metric on \( S \), we have the following relation:
\[ \langle \varphi X, \varphi Y \rangle = \langle JX - \eta(X)N, JY - \eta(Y)N \rangle = \langle JX, JY \rangle - \eta(X)\eta(Y)\langle N, N \rangle + \eta(X)\eta(Y)\langle N, JX \rangle + \langle JY, JY \rangle - \eta(Y)\eta(X)\langle N, N \rangle + \eta(Y)\eta(X)\langle N, JX \rangle + \eta(X)\eta(Y)\langle JX, JY \rangle + \eta(Y)\eta(X)\langle JX, JY \rangle. \]

We conclude that, according to Relations (1), (3), (4), (6) and (7), the quadruple \((\varphi, \eta, \xi, \langle \cdot, \cdot \rangle)\) defines an almost contact metric structure on \( S \) when \( \varepsilon = 1 \) and an almost para-contact metric structure on \( S \) when \( \varepsilon = -1 \).

The Gauss and the Weingarten formulas are respectively given by the equations
\[ \nabla_X Y = \nabla_X Y + \langle AX, Y \rangle N \tag{8} \]
\[ \nabla_X N = -AX, \tag{9} \]
where \( \nabla \) and \( \nabla \) are the Levi-Civita connection on \( M \) and \( S \) respectively and \( A \) is the shape operator of \( S \) with respect to \( N \). Denoting by \( \overline{R} \) and \( R \) the curvature of \( \nabla \) and \( \nabla \) respectively, the Gauss equation takes the form:
\[ \langle R(X, Y)Z, W \rangle = \langle \overline{R}(X, Y)Z, W \rangle + \langle AX, Z \rangle \langle AY, W \rangle - \langle AX, W \rangle \langle AY, Z \rangle \]
for \( X, Y, Z \) and \( W \) tangent to \( S \). Hence
\[ \overline{R}(X, Y)Z = c(\varepsilon X \land Y + JX \land JY + 2\langle X, JY \rangle)Z + \langle AY, Z \rangle AX - \langle AX, Z \rangle AY \]
\[ = c(\varepsilon X \land Y + \varphi X \land \varphi Y + 2\langle X, \varphi Y \rangle)Z + (AX \land AY)Z, \]
so that
\[ \overline{R}(X, Y) = AX \land AY + c(\varepsilon X \land Y + \varphi X \land \varphi Y + 2\langle X, \varphi Y \rangle) \]
We now deal with Codazzi equation: for \( X, Y \) and \( Z \) tangent to \( S \), we have
\[ \langle R(X, Y)Z, N \rangle = \langle (\nabla_X A)Y - (\nabla_Y A)X, Z \rangle. \]

Using the expression of \( R \), we have
\[
\begin{align*}
\langle R(X, Y)Z, N \rangle &= c \left( \epsilon \langle (X \wedge Y)Z, N \rangle + \langle (JX \wedge JY)Z, N \rangle + 2 \langle X, JY \rangle \langle JZ, N \rangle \right) \\
&= c \left( \epsilon \langle Y, Z \rangle \langle X, N \rangle - \epsilon \langle X, Z \rangle \langle Y, N \rangle + \langle JY, Z \rangle \langle JX, N \rangle - \langle JX, Z \rangle \langle JY, N \rangle \right) \\
&= c \left( \langle \varphi Y, Z \rangle \eta(X) - \langle \varphi X, Z \rangle \eta(Y) + 2 \epsilon \langle X, \varphi Y \rangle \langle Z, \xi \rangle \right)
\end{align*}
\]
to get
\[
\langle (\nabla_X A)Y - (\nabla_Y A)X, N \rangle = c \left( \eta(X) \varphi Y - \eta(Y) \varphi X + 2 \epsilon \langle X, \varphi Y \rangle \xi \right).
\] (10)

### 2.2 Proof of Theorem 1

The proof is an easy consequence of the Codazzi equation.

Assume first that \( S \) is umbilic, i.e. there exists \( \lambda \in C^\infty(S) \) such that \( A = \lambda Id \). Then the Codazzi equation (10) becomes:
\[
(X \cdot \lambda)Y - (Y \cdot \lambda)X = c \left( \eta(X) \varphi Y - \eta(Y) \varphi X + 2 \epsilon \langle X, \varphi Y \rangle \xi \right),
\]
Taking \( X \) horizontal and non-vanishing, and \( Y = \xi \) yields
\[
(X \cdot \lambda) \xi - (\xi \cdot \lambda)X = c \varphi X.
\]
The inner product of the above relation with \( \varphi X \) implies \( c = 0 \), which is a contradiction.

Assume now that \( S \) has parallel shape operator, i.e. \( (\nabla_X A)Y = 0 \), for any tangent vectors \( X, Y \). Then the Codazzi equation becomes
\[
0 = c \left( \eta(X) \varphi Y - \eta(Y) \varphi X + 2 \epsilon \langle X, \varphi Y \rangle \xi \right).
\]
Taking \( X \) horizontal and non-vanishing, and \( Y = \xi \) yields \( c \varphi X = 0 \). Since \( \varphi X \) does not vanish, we get \( c = 0 \), a contradiction.

### 3 Proof of Theorem 2

Before providing the proof of Theorem some basic Lemmas which hold for real hypersurfaces in \( \mathcal{M} \) are given.
3.1 Basic Lemmas

Lemma 1. Let $S$ be a real hypersurface in $\mathcal{M}$. Then:

$$\nabla_X \xi = \epsilon \phi AX$$ \hspace{1cm} (11)

and

$$(\nabla_X \phi)Y = \eta(Y) AX - \epsilon \langle AX, Y \rangle \xi.$$ \hspace{1cm} (12)

Proof. Using successively Gauss equation, Weingarten equation and Equation (2), we first calculate

\[
\nabla_X \xi = \nabla_X \xi - \langle AX, \xi \rangle N = -\epsilon \nabla_X JN - \langle AX, \xi \rangle N = \epsilon JAX - \langle AX, \xi \rangle N = \epsilon \phi AX + \eta(AX) - \langle AX, \xi \rangle N.
\]

Taking the tangential part of this yields Equation (11).

As for Equation (12), using again the Gauss, Weingarten equation and Equation (2), we have

\[
(\nabla_X \phi)Y = \nabla_X (\phi Y) - \phi (\nabla_X Y) = \nabla_X (JY - \epsilon \eta(Y) N - \langle AX, \phi Y \rangle N - \phi (\nabla_X Y)) = J(\nabla_X Y - \epsilon \nabla_X (\eta(Y) N - \langle AX, \phi Y \rangle N - \phi (\nabla_X Y)) = J(\nabla_X Y + \langle AX, Y \rangle N) - \epsilon \langle (\nabla_X Y, \xi) N + \langle Y, \nabla_X \xi \rangle N + \langle Y, \xi \rangle \nabla_X N \rangle - \langle AX, \phi Y \rangle N - \phi (\nabla_X Y) = \eta(\nabla_X Y) N + \langle AX, Y \rangle JN - \epsilon \langle (\nabla_X Y, \xi) N + \langle Y, \nabla_X \xi \rangle N + \langle Y, \xi \rangle \nabla_X N \rangle - \langle AX, \phi Y \rangle N = -\epsilon \langle AX, Y \rangle \xi - \langle Y, \phi AX \rangle N + \epsilon \langle Y, AX \rangle + \langle \phi AX, Y \rangle N = (\eta(Y) AX - \epsilon \langle AX, Y \rangle \xi).
\]

Lemma 2. The following two relations hold on a hypersurface $S$ of $\mathcal{M}$:

$$\langle (\nabla_X A) Y - (\nabla_Y A) X, \xi \rangle = 2\epsilon \langle X, \phi Y \rangle,$$ \hspace{1cm} (13)

$$\langle (\nabla_X A) \xi, \xi \rangle = \langle (\nabla_{\xi} A) X, \xi \rangle = \langle (\nabla_{\xi} A) X, \xi \rangle.$$ \hspace{1cm} (14)

Proof. Taking the inner product of Codazzi equation (10) with $\xi$, recalling that $\langle \xi, \xi \rangle = \epsilon$, implies (13).
The first equality in (14) is a particular case of (13) making \( Y = \xi \). For the second equality in (14) we have

\[
\langle (\nabla_A X)\xi, X \rangle = \langle \nabla_A (A\xi), X \rangle - \langle A\nabla_\xi A, X \rangle - \langle \nabla_\xi A, AX \rangle
\]

\[
= \xi \cdot \langle A\xi, X \rangle - \langle A\xi, \nabla_\xi X \rangle + \langle \xi, \nabla_\xi (AX) \rangle
\]

\[
= -\langle \xi, A\nabla_\xi X \rangle + \langle \xi, \nabla_\xi (AX) \rangle
\]

\[
= \langle \xi, (\nabla_\xi A)X \rangle.
\]

\[\square\]

Lemma 3. Let \( S \) be a Hopf hypersurface in \( M \) and \( a \) the Hopf curvature, i.e. \( A\xi = a\xi \). Then the following relations hold on \( S \)

\[
\text{grad } a = \epsilon (\xi \cdot a)\xi, \quad (15)
\]

\[
A\varphi A - \frac{a}{2} (A\varphi + \varphi A) - \epsilon \varphi = 0, \quad (16)
\]

\[
(\xi \cdot a)(\varphi A + A\varphi) = 0. \quad (17)
\]

Proof. — Proof of (15): we first calculate, using several times Equation (11),

\[
(\nabla_X A)\xi = \nabla_X (A\xi) - A\nabla_X \xi
\]

\[
= (X \cdot a)\xi + a\nabla_X \xi - \epsilon A\varphi AX
\]

\[
= (X \cdot a)\xi + \epsilon a\varphi AX - \epsilon A\varphi AX.
\]

Hence we obtain

\[
(\nabla_X A)\xi = (X \cdot a)\xi + \epsilon (a\text{Id} - A)\varphi AX. \quad (18)
\]

Taking the inner product of (18) with \( \xi \) yields (taking into account that \( \langle \xi, \xi \rangle = \epsilon \))

\[
\langle (\nabla_X A)\xi, \xi \rangle = \epsilon (X \cdot a) = \epsilon \langle \text{grad } a, X \rangle. \quad (19)
\]

On the other hand, making \( X = \xi \) and recalling that \( \varphi \xi \) vanishes, we get

\[
(\nabla_\xi A)\xi = (\xi \cdot a)\xi.
\]

Putting together these last two equations, we conclude, using Lemma 2,

\[
\langle \text{grad } a, X \rangle = \epsilon \langle (\nabla_X A)\xi, \xi \rangle
\]

\[
= \epsilon \langle (\nabla_\xi A)\xi, X \rangle
\]

\[
= \epsilon \langle (\nabla_\xi A)\xi, X \rangle,
\]

from which Equation (15) follows.

— Proof of (16): first, by an easy calculation,

\[
\langle (\nabla_X A)Y, \xi \rangle = \langle (\nabla_X A)\xi, Y \rangle
\]
Then, using Equations (18), (19) and Lemma 2, we get
\[
\langle (\nabla X A) Y, \xi \rangle = \langle (\nabla X A) \xi, Y \rangle \\
= (X \cdot a) \langle \xi, Y \rangle + \epsilon \langle (aId - A) \varphi AX, Y \rangle \\
= \epsilon \langle (\nabla \xi A) \xi, Y \rangle + \epsilon \langle (aId - A) \varphi AX, Y \rangle \\
= \epsilon (\xi \cdot a) \langle \xi, Y \rangle + \epsilon \langle (aId - A) \varphi AX, Y \rangle.
\]
Interchanging \(X\) and \(Y\) and subtracting, we calculate
\[
\langle (\nabla X A) Y - (\nabla Y A) X, \xi \rangle = \epsilon(\langle (aId - A) \varphi AX, Y \rangle - \langle (aId - A) \varphi YX, X \rangle).
\]
Now, from (13) (Lemma 2) this implies
\[
\epsilon(\langle (aId - A) \varphi AX, Y \rangle - \langle (aId - A) \varphi YX, X \rangle) = 2c\epsilon\langle X, \varphi Y \rangle \langle \xi, \xi \rangle \\
= 2c\langle X, \varphi Y \rangle.
\]
It follows, using the facts that \(A\) is self-adjoint (and therefore \(aId - A\) as well) and that \(\varphi\) is skew-symmetric, that
\[
2c\epsilon\varphi = -A\varphi(aId - A) - (aId - A)\varphi A \\
= -a(A\varphi + \varphi A) + 2A\varphi A,
\]
from which Equation (16) follows.

— Proof of (17): setting \(\beta := \epsilon \xi \cdot a\) (so in particular \(\text{grad} \; a = \beta \xi\)), we have
\[
\langle \nabla X (\text{grad} \; a), Y \rangle - \langle \nabla Y (\text{grad} \; a), X \rangle = X \cdot \langle \text{grad} \; a, Y \rangle - \langle \text{grad} \; a, \nabla X Y \rangle \\
= -Y \cdot \langle \text{grad} \; a, X \rangle + \langle \text{grad} \; a, \nabla Y X \rangle \\
= X \cdot (Y \cdot a) - Y \cdot (X \cdot a) + \langle \text{grad} \; a, \nabla X Y - \nabla Y X \rangle \\
= (\langle X, Y \rangle - \nabla X Y - \nabla Y X) \cdot a \\
= 0.
\]
It follows that
\[
0 = \langle \nabla X \beta \xi, Y \rangle - \langle \nabla Y \beta \xi, X \rangle \\
= (X \cdot \beta) \langle \xi, Y \rangle - \beta \langle \varphi AX, Y \rangle - (Y \cdot \beta) \langle \xi, X \rangle - \beta \langle \varphi YX, X \rangle \\
= (X \cdot \beta) \langle \xi, Y \rangle - (Y \cdot \beta) \langle \xi, X \rangle - \beta ((\varphi A + A\varphi)X, Y).
\]
Making \(Y = \xi\) yields
\[
0 = X \cdot \beta \langle \xi, \xi \rangle - (\xi \cdot \beta) \langle \xi, X \rangle - \beta ((\varphi A + A\varphi)X, \xi) \\
= \epsilon X \cdot \beta - (\xi \cdot \beta) \langle \xi, X \rangle - \beta (A\varphi X, \xi) \\
= \epsilon X \cdot \beta - (\xi \cdot \beta) \langle \xi, X \rangle - \beta (\varphi X, \alpha \xi) \\
= \epsilon X \cdot \beta - (\xi \cdot \beta) \langle \xi, X \rangle.
\]

Hence we have $X \cdot \beta = \epsilon (\xi \cdot \beta) \langle \xi, X \rangle$, which implies that $(X \cdot \beta) \langle \xi, Y \rangle = (Y \cdot \beta) \langle \xi, X \rangle$. Combining with (20) yields

$$\beta \langle (\varphi A + A \varphi) X, Y \rangle,$$

which implies the desired identity. \( \square \)

**Lemma 4.** If $X$ is an principal vector of $A$ with principal curvature $\kappa$, then $\varphi X$ is a principal vector with principal curvature

$$\bar{\kappa} := \frac{\kappa a + 2 \epsilon}{2 \kappa - a}.$$

In particular the principal subspace $E_\kappa := \{ X \in TS \mid AX = \kappa X \}$ is $\varphi$-invariant if and only if $\kappa^2 - \kappa a - \epsilon c = 0$.

**Proof.** We write Equation (16) in the case when $AX = \kappa X$:

$$\kappa A \varphi X \kappa = \frac{\epsilon}{2} (A \varphi X + \kappa \varphi X) - \epsilon c \varphi X = 0,$$

so that

$$(\kappa - \frac{a}{2}) A \varphi X = (\epsilon c + \frac{\kappa a}{2}) \varphi X,$$

i.e.

$$A \varphi X = \frac{\kappa a + 2 \epsilon}{2 \kappa - a} \varphi X,$$

so we get the required expression for $\bar{\kappa}$ satisfying $A(\varphi X) = \bar{\kappa}(\varphi X)$. Finally, if $E_\kappa$ is $\varphi$-stable, we must have $\kappa = \frac{\kappa a + 2 \epsilon}{2 \kappa - a}$, which implies the last claim of the Lemma. \( \square \)

**Proof of Theorem 2.**

We proceed by contradiction. By Equation (15) if $a$ is not constant, then $\xi \cdot a \neq 0$. Consider $Z$ a horizontal vector. So $\eta(Z) = 0$ and, by Lemma 1, we have $\nabla_\xi A Z = 0$. It follows that

$$0 = \nabla_\xi (\varphi A + A \varphi) Z$$

$$= \varphi (\nabla_\xi A) Z + (\nabla_\xi A) \varphi Z. \quad (21)$$

We now write the Codazzi equation (10) with $X = \xi$ and $Y = Z$, yields, using (18):

$$(\nabla_\xi A) Z = (\nabla_Z A) \xi + \epsilon (\eta(\xi) \varphi Z - \eta(Z) \varphi \xi + 2 \epsilon \langle \xi, \varphi Z \rangle \xi)$$

$$= (\nabla_Z A) \xi + c \varphi Z$$

$$= (Z \cdot a) \xi + \epsilon (a \text{Id} - A) \varphi AZ + c \varphi Z$$

$$= \epsilon (a \text{Id} - A) \varphi AZ + c \varphi Z.$$
(Equation (15) implies that \((Z \cdot a)\) vanishes). It follows that
\[
\varphi(\nabla_\xi A)Z = \epsilon \varphi(a \text{Id} - A)\varphi AZ + \epsilon \varphi^2 Z
\]
\[
= \epsilon a \varphi^2 Z - \epsilon \varphi A \varphi AZ - \epsilon \varphi Z
\]
\[
= -aAZ - A^2 Z - \epsilon \varphi Z.
\]
Analogously, the Codazzi equation with \(X = \xi\) and \(Y = \varphi Z\)
\[
(\nabla_\xi A)\varphi Z = (\nabla_{\varphi Z} A)\xi + \epsilon \big(\eta(\xi)\varphi^2 Z - \eta(\varphi Z)\varphi \xi + 2\epsilon(\xi, \varphi^2 Z)\xi\big)
\]
\[
= (\nabla_{\varphi Z} A)\xi - \epsilon \varphi Z
\]
\[
= (\varphi Z \cdot a)\xi + \epsilon (a \text{Id} - A)\varphi AZ - \epsilon \varphi Z
\]
\[
= \epsilon (a \text{Id} - A)\varphi AZ - \epsilon \varphi Z
\]
\[
= -aAZ + A^2 Z - \epsilon \varphi Z.
\]
By (21) we deduce that
\[
aAZ = -\epsilon \varphi Z.
\]
This implies \(a\) does not vanish, and moreover that the restriction of \(A\) to the horizontal space is \(-\epsilon \varphi a^{-1} \text{Id}\). Since the horizontal space is \(\varphi\)-stable, it follows from Lemma 4, that
\[
(-\epsilon \varphi a^{-1})^2 - a(-\epsilon \varphi a^{-1}) - \epsilon = 0,
\]
i.e.
\[
\frac{c^2}{a^2} + \epsilon c - \epsilon c = 0.
\]
This implies \(c = 0\), a contradiction.

4 Proof of the Theorem 3

We recall that \(\mathcal{M} := \mathbb{D}^n\) or \(\mathbb{C}^n\) and that \(\tilde{\mathcal{M}} := S^{2n+1}_{n-c}\) or \(S^{2n+1}_{2p,c}\) is a bundle over \(\mathcal{M}\) with projection \(\pi\). We still denote by \(\langle \cdot, \cdot \rangle\) the metric on \(\tilde{\mathcal{M}}\). We shall denote by \(\nabla\) the Levi-Civita connection on \(\mathcal{M}\). This is nothing but the tangential part of the flat connection of \(\mathbb{R}^{2n+2}\). We denote by \(\tilde{J}\) the complex (resp. para-complex) structure of \(\mathbb{C}^{n+1}\) (resp. \(\mathbb{D}^{n+1}\)).

Let \(F : U \rightarrow \mathcal{M}\) be a local parametrization of \(\mathcal{S}\) and \(\tilde{F} : \mathcal{S} \rightarrow \tilde{\mathcal{M}}\) a lift of \(F\), i.e. \(\pi \circ \tilde{F} = F\). In particular \(\langle F, \tilde{F} \rangle = \epsilon \epsilon\). Observe that the choice of \(\tilde{F}\) is not unique, but none of them satisfies \(\langle d\tilde{F}(\cdot), \tilde{J}\tilde{F} \rangle = 0\), because the integral submanifolds of the hyperplane distribution \(\tilde{J}^1\) have at most dimension \(n\) (Legendrian submanifolds).

By a slight abuse of notation, we still denote by \(N\) the composition of the unit normal vector field on \(F(U)\) with \(F\). In other words, \(N : U \rightarrow T\mathcal{M}\). Let \(\tilde{N}\) be a lift of \(N\), i.e. \(\tilde{N} : U \rightarrow T\tilde{\mathcal{M}}\) such that
\[
d\pi_{\tilde{F}(x)} \circ \tilde{N}(x) = N(x), \forall x \in U.
\]
In particular \( \tilde{N} \in \tilde{\mathcal{M}} \) (resp. \( \tilde{M} \)) if \( \epsilon = 1 \) (resp. \( \epsilon = -1 \)) and \( \langle \tilde{N}, \tilde{N} \rangle = 1 \).

Moreover, we have
\[
(d\tilde{F}(\cdot), \tilde{N}) = 0.
\]

Since the immersion \( d\tilde{F} \) has co-dimension 2, the choice of \( \tilde{N} \) is not unique. Since moreover \( d\tilde{F} \) is transverse to the vector field \( \tilde{J} \tilde{F} \), we may choose \( \tilde{N} \) in such a way that
\[
\langle \tilde{N}, \tilde{J} \tilde{F} \rangle = 0.
\]

We denote by \( \xi \) the tangent vector field on \( U \) such that \( \Xi := dF(\xi) = -JN \).

We now assume that \( F(U) \) is Hopf, i.e. \( A\xi = a\xi \).

By Proposition 10, p. 97 of [Ani] (see also [ON]), we have
\[
\nabla_\Xi \tilde{N} = \nabla_\Xi \tilde{N} + \langle \tilde{J} \tilde{F}, \tilde{J} \tilde{F} \rangle \langle \Xi, \tilde{J} \tilde{N} \rangle \tilde{J} \tilde{F}
\]
\[
= \nabla_\Xi \tilde{N} - c \tilde{J} \tilde{F}
\]
\[
= -a \Xi - \epsilon' \tilde{J} \tilde{F}.
\]

It follows that
\[
d\tilde{f}(\xi) = \cos'(\theta) d\tilde{F}(\xi) + \sin'(\theta) d\tilde{N}(\xi)
\]
\[
= \cos'(\theta) \Xi + \sin'(\theta) \nabla_\Xi \tilde{N}
\]
\[
= -\epsilon \cos'(\theta) \tilde{J} \tilde{N} + \sin'(\theta) \left( a \epsilon \tilde{J} \tilde{N} - \epsilon' \tilde{J} \tilde{F} \right)
\]
\[
= -\epsilon \left( \epsilon' \sin'(\theta) \tilde{J} \tilde{F} + (\cos'(\theta) - \epsilon \sin'(\theta)) \tilde{J} \tilde{N} \right).
\]

We now choose \( \theta \) in order to have \( df(\xi) = 0 \), i.e. \( d\tilde{f}(\xi) \in \tilde{J} \tilde{f} \mathbb{R} \). This is equivalent to the existence of \( \lambda \in \mathbb{R} \) such that
\[
\begin{cases}
\epsilon' \sin'(\theta) = \lambda \cos'(\theta), \\
\cos'(\theta) - \epsilon \sin'(\theta) = \lambda \sin'(\theta).
\end{cases}
\]

It follows that \( \lambda = \epsilon' \tan'(\theta) \) and
\[
a = \frac{\cos'(\theta)}{\sin'(\theta)} - \lambda = \frac{(\cos'(\theta))^2 - \epsilon'(\sin'(\theta))^2}{\cos'(\theta)\sin'(\theta)} = 2\cot'(2\theta)
\]
Hence, taking

\[ \theta := \frac{1}{2} \cot \epsilon'^{-1}(a/2), \]

we get \( df(\xi) = 0 \). This is possible for all real number \( a \) if \( \epsilon' = 1 \), and if \( |a| > 2 \) if \( \epsilon' = -1 \). Hence \( f \) is constant along the integral lines of \( \xi \). In particular, the rank of \( f \) is strictly less than \( 2n - 1 \).

We now claim that the rank of \( f \) is even and that its image is a complex submanifold of \( M \). We first calculate, for a horizontal vector \( v \in \text{Hor} \):

\[
\begin{align*}
    df(v) &= \cos \epsilon'(\theta) dF(v) + \sin \epsilon'(\theta) dN(v) \\
    &= \cos \epsilon'(\theta) dF(v) + \sin \epsilon'(\theta) \tilde{\nabla} dF(v) \tilde{N} \\
    &= \cos \epsilon'(\theta) dF(v) + \sin \epsilon'(\theta) \left( -dF(Av) + \langle \tilde{J}F, \tilde{J}F \rangle dF(v), \tilde{J}N, \tilde{J}F \right) \\
    &= \cos \epsilon'(\theta) dF(v) - \sin \epsilon'(\theta) dF(Av) \\
    &= dF(\cos \epsilon'(\theta)v - \sin \epsilon'(\theta)Av).
\end{align*}
\]

Hence \( \text{Ker}(df) = \text{Ker}(\cos \epsilon'(\theta)Id - \sin \epsilon'(\theta)A) \) and therefore

\[ \text{rank}(f) = 2n - \dim \text{Ker}(\cos \epsilon'(\theta)Id - \sin \epsilon'(\theta)A) \]

Moreover, \( v \in \text{Ker}(df) \) if and only if \( Av = \cot \epsilon'(\theta)v \), i.e. \( \cot \epsilon'(\theta) \) is a principal curvature of \( F \). Observe that \( \cot \epsilon'(\theta) - \cot \epsilon'(\theta)a = \epsilon' = 0 \), so by Lemma 4 of Section 2, the corresponding eigenspace is \( \tilde{J} \)-invariant. In particular the rank of \( f \) is even.

If \( v \) does not belong to \( \text{Ker}(\cos \epsilon'(\theta)Id - \sin \epsilon'(\theta)A) \), we claim that there exists \( w \) such that \( JdF(v) = df(w) \). Since

\[
\tilde{J}d\tilde{f}(v) = d\tilde{F}(\cos \epsilon'(\theta)v - \sin \epsilon'(\theta)Av)
\]

this is equivalent to

\[ \varphi(\cos \epsilon'(\theta)Id - \sin \epsilon'(\theta)A)v = (\cos \epsilon'(\theta)Id - \sin \epsilon'(\theta)A)w. \]

Hence, we get the required relation setting

\[ w := (\cos \epsilon'(\theta)Id - \sin \epsilon'(\theta)A)^{-1} \varphi(\cos \epsilon'(\theta)Id - \sin \epsilon'(\theta)A)v. \]

This proves that \( df(TU) \) is stable with respect to \( J \), i.e. \( f(U) \) is a complex submanifold. The easy task to check that \( F(U) \) is the tube of radius \( \theta \) over \( f(U) \) is left to the reader.

\(^1\)In the case \( \epsilon' = -1 \), if we instead set

\[ \tilde{f}' := \sinh(\theta)\tilde{F} + \cosh(\theta)\tilde{N}, \]

which is valued in \( \tilde{M} \), we get again \( a = 2 \coth(2\theta) \). The map \( f' := \pi \circ \tilde{f}' \) is the polar of \( f := \pi \circ \tilde{f} \).
4.1 Open Problems

Summarizing, in this paper some basic results are presented and a characterization of real hypersurfaces with Hopf curvature satisfying $|\alpha| > 2$ in pseudo-Riemannian complex space forms and para-complex space forms is given. Therefore, a first question which is raised in a natural way is:

Are there real hypersurfaces in pseudo-Riemannian complex space forms or para-complex space forms whose Hopf curvature is small, i.e. $|\alpha| \leq 2$?

Following similar steps to those which have been done in the study of real hypersurfaces in the cases of complex space forms, complex two-plane Grassmannians, etc., a great amount of questions concerning real hypersurfaces in pseudo-Riemannian complex space forms and para-complex space forms come up. For instance, it would be interesting to answer the following:

Are there real hypersurfaces in pseudo-Riemannian complex space forms or para-complex space forms whose shape operator commutes with $\varphi$, i.e. $A\varphi = \varphi A$?

References

[An1] H. Anciaux, Minimal submanifolds in Pseudo-Riemannian geometry, World Scientific, 2010

[An2] H. Anciaux, Surfaces with one constant principal curvature in three-dimensional space forms, arXiv:1307.6735

[BD] A. Bejancu, K. L. Duggal, Real hypersurfaces of indefinite Kaehter manifolds, Internat. J. Math. and Math. Sci. 16 no. 3, (1993), 545–556

[CR] T. Cecil, P. Ryan, Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. 269 (1982), 481–499

[GM] P. M. Gadea, A. M. Montesinos Amilibia, Spaces of constant paraholomorphic curvature, Pacific J. of Maths. 136 no. 1, (1989), 85–101

[IR] T. Ivey, P. Ryan, Hopf Hypersurfaces of Small Hopf Principal Curvature in $\mathbb{C}H^2$, Geom. Dedicata 141 (2009), 147–161

[Iv] T. Ivey, A d’Alembert Formula for Hopf Hypersurfaces, Results in Maths. 60 (2011), 293–309

[KS] U.-H. Ki, Y.-J. Suh, On real hypersurfaces of a complex space form, Math. J. Okayama Univ. 32 (1990), 207–221

[Ki] M. Kimura, Hopf hypersurfaces in nonflat complex space forms, Proceedings of The Sixteenth International Workshop on Diff. Geom. 16 (2012) 25–34
[NR] R. Niebergall, P. Ryan, *Real Hypersurfaces in Complex Space Forms*, Tight and Taut Submanifolds MSRI Publications Volume 32, 1997

[Ma] Y. Madea, *On real hypersurfaces of a complex projective space*, J. Math. Soc. Japan 28 (1976), 529–540

[Mo] S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan 37 (1985), no. 3, 515–535

[ON] O’Neill, *The fundamental equations of a submersion*, Michigan Math. J., 13 (1966), 459–469

Henri Anciaux
Universidade de São Paulo, IME
1010 Rua do Matão,
Cidade Universitária
05508-090 São Paulo, Brazil
henri.anciaux@gmail.com

Konstantina Panagiotidou
Faculty of Engineering
Aristotle University of Thessaloniki
Thessaloniki 54124, Greece
kapanagi@gen.auth.gr