Asymptotic Enumeration of Integer Matrices with Constant Row and Column Sums

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Abstract

Let $s, t, m, n$ be positive integers such that $sm = tn$. Let $M(m, s; n, t)$ be the number of $m \times n$ matrices over $\{0, 1, 2, \ldots\}$ with each row summing to $s$ and each column summing to $t$. Equivalently, $M(m, s; n, t)$ counts 2-way contingency tables of order $m \times n$ such that the row marginal sums are all $s$ and the column marginal sums are all $t$. A third equivalent description is that $M(m, s; n, t)$ is the number of semiregular labelled bipartite multigraphs with $m$ vertices of degree $s$ and $n$ vertices of degree $t$. When $m = n$ and $s = t$ such matrices are also referred to as $n \times n$ magic or semimagic squares with line sums equal to $t$. We prove a precise asymptotic formula for $M(m, s; n, t)$ which is valid over a range of $(m, s; n, t)$ in which $m, n \to \infty$ while remaining approximately equal and the average entry is not too small. This range includes the case where $m/n, n/m, s/n$ and $t/m$ are bounded from below.

1 Introduction

Let $m, s, n, t$ be positive integers such that $ms = nt$. Let $M(m, s; n, t)$ be the number of $m \times n$ matrices over $\{0, 1, 2, \ldots\}$ with each row summing to $s$ and each column summing to $t$. Figure 1 shows an example.

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The matrices counted by $M(m, s; n, t)$ arise frequently in many areas of mathematics, for example enumeration of permutations with respect to descents, symmetric function theory, and statistics. The last field in particular has an extensive literature in which such matrices are studied as contingency tables or frequency tables; see [17] as an example of enumerative work, [22] for a lengthy bibliography, and [14] for a survey of applications. The matrices counted by $M(m, s; n, t)$ have also regularly been the topic of papers whose primary focus is algorithmic, both with respect to the problem of generating contingency tables with prescribed margins at random, and with respect to approximate counting. For random generation see, for example [15, 31]; for approximate counting see the recent [5]. Other important recent studies of both random sampling and approximate counting include [11, 24].

A historically early study of integer matrices with specified row and column sums appeared in MacMahon [25], where the numbers $M(m, s; n, t)$ are identified as coefficients when certain functions are expanded in terms of standard bases of symmetric functions. In a later paper [26], MacMahon studied “general magic squares”—$n \times n$ integer matrices with all row and column sums equal to a prescribed value $t$. Stanley [34, Chap. 4] refers to these as magic squares with line sums equal to $t$, and Stanley’s [35, Chap. 1] provides further history of this topic. For fixed $n$, $M(n, t; n, t)$ is a polynomial in $t$ known as the Ehrhart polynomial of the Birkhoff polytope. This has been computed in closed form for $n \leq 9$ [2]. (The Birkhoff polytope is the set of all doubly stochastic real matrices.) The results in the present paper enable us to give the first asymptotic formula for the volume of this famous polytope [9].

Our focus in this paper is the asymptotic value of $M(m, s; n, t)$. The first significant result on asymptotics was that of Read [33], who obtained the asymptotic behavior for $s = t = 3$. In the four year period from 1971 to 1974 three published papers [3, 4, 16] gave the asymptotic value for the case $s, t$ bounded. To our knowledge, this was the extent of published, rigorously proven formulas until the present work. In a paper currently in preparation [21], Greenhill and McKay obtain a formula valid in the sparse range $st = o((mn)^{1/2})$.

Define the density $\lambda = s/n = t/m$ to be the average entry in the matrix. As early as [18, p. 100], Good implicitly gave the estimate $M(m, s; n, t) \approx G(m, s; n, t)$, where

$$
\begin{array}{cccc|c}
5 & 4 & 0 & 11 & 20 \\
2 & 8 & 9 & 1 & 20 \\
8 & 3 & 6 & 3 & 20 \\
\hline
15 & 15 & 15 & 15 & 20
\end{array}
$$

Figure 1: Example of a table counted by $M(3, 20; 4, 15)$. 

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Good spelt this out explicitly in [19] and again with Crook in [20]. In the first paper, Good
gave a heuristic argument for this approximation based on steepest-descent, “leaving aside
finer points of rigor.” His calculation also yielded another approximation [19, Eqn. B2.22]
that is very similar when \( m \approx n \) but larger otherwise. We will show in this paper
that \( G(m, s; n, t) \) is a remarkably accurate approximation of \( M(m, s; n, t) \), being out by
a constant factor over a wide range and perhaps always. Another important estimate of
\( M(m, s; n, t) \) was developed by Diaconis and Efron [13]. It aims for accuracy when the
sum of the matrix entries \( \lambda mn \) is very large, and is quite poor when that condition is not
met.

Our main result is as follows.

**Theorem 1.** Let \( s = s(m, n), t = t(m, n) \) be positive integers satisfying \( ms = nt \). Define
\( \lambda = s/n = t/m \). Let \( a, b > 0 \) be constants such that \( a + b < \frac{1}{2} \). Suppose that \( m, n \to \infty \) in
such a way that

\[
\frac{(1 + 2\lambda)^2}{4\lambda(1 + \lambda)} \left( 1 + \frac{5m}{6n} + \frac{5n}{6m} \right) \leq a \log n. \tag{1.1}
\]

Then

\[
M(m, s; n, t) = G(m, s; n, t) \exp\left( \frac{1}{2} + O(n^{-b}) \right) \tag{1.2}
\]

\[
= \frac{(\lambda^{-\lambda}(1 + \lambda)^{1+\lambda})^{mn}}{(4\pi A)^{(m+n-1)/2}m^{(n-1)/2}n^{(m-1)/2}}
\times \exp\left( \frac{1}{2} - \frac{1 + 2A}{24A} \left( \frac{m}{n} + \frac{n}{m} \right) + O(n^{-b}) \right). \tag{1.3}
\]

**Corollary 1.** Under the conditions of Theorem 1, if in addition \( mn/\lambda^2 \to 0 \),

\[
M(m, s; n, t) = \left( \lambda + \frac{1}{2} \right)^{(m-1)(n-1)} \frac{(mn)!}{m!m!} \exp\left( \frac{1}{2} + O(mn/\lambda^2 + n^{-b}) \right).
\]

**Remark 1.** The following is inspired by a similar observation made by Good [19]. The
number of \( k \)-tuples of nonnegative integers \( n_i \) satisfying \( n_1 + \cdots + n_k = K \) is well
known to be \( \binom{K + k - 1}{k - 1} = \binom{K + k - 1}{k} \). This allows an instructive interpretation of (1.2) as
\( M(m, s; n, t) = NP_1P_2E \), where

\[
N = \binom{mn + \lambda mn - 1}{\lambda mn}, \quad P_1 = N^{-1} \binom{n + s - 1}{s}^m, \quad P_2 = N^{-1} \binom{m + t - 1}{t}^n,
\]

\[
E = E(m, s; n, t) = \exp\left( \frac{1}{2} + O(n^{-b}) \right).
\]
Clearly $N$ is the number of tables whose entries sum to $\lambda mn = sm = tn$. In the uniform probability space on these $N$ tables, $P_1$ is the probability of the event that all the row sums are equal to $s$, and $P_2$ is the probability of the event that all the column sums are equal to $t$. The final quantity $E$ is thus a correction to account for the non-independence of these two events.

In an earlier paper [8] we showed how $B(m, s; n, t)$, the number of such matrices whose entries are taken from $\{0, 1\}$, can be computed exactly for small $m, n$. That algorithm is easily adaptable for efficient computation of exact values of $M(m, s; n, t)$, and the values so computed suggest the following conjecture.

**Conjecture 1.** Consider a 4-tuple of positive integers $m, s, n, t$ such that $ms = nt$. Define
\[
\Delta(m, s; n, t) = G(m, s; n, t) \left( \frac{m + 1}{m} \right)^{(m-1)/2} \left( \frac{n + 1}{n} \right)^{(n-1)/2}
\times \exp\left( -\frac{1}{2} + \frac{\Delta(m, s; n, t)}{m + n} \right).
\]
Then $0 < \Delta(m, s; n, t) < 2$.

The factor $(1 + 1/m)^{(m-1)/2}$, which approaches $e^{1/2}$ as $m \to \infty$, and the similar factor $(1 + 1/n)^{(n-1)/2}$, appear naturally in the analysis of $M(m, s; n, t)$ when one of $m, n$ goes to $\infty$ much faster than the other. This can be seen in [17, eqn. (3.5)] and will be extended rigorously in a forthcoming paper [10]. Conjecture 1 has been proved in several cases: (a) for $m = n \leq 9$, using the exact values from [2]; (b) for sufficiently large $m, n$ when $st = o((mn)^{1/5})$, using the asymptotics derived in [21]; (c) for several thousand values of $(m, s; n, t)$ for $m, n \leq 30$. It has also been established to a high degree of confidence for many larger sizes using a simulation similar to the one described in [8], which was a variation on a method of Chen, Diaconis, Holmes and Liu [11].

An interesting point of contrast between the integer and 0-1 cases is suggested by the above. Let $E_0(m, s; n, t)$ be the quantity corresponding to $E(m, s; n, t)$ in Remark 4 when the number of binary tables is decomposed in the same manner (see [7]). It was proved by Ordentlich and Roth [32] that $E_0(m, s; n, t) \leq 1$. A corollary of Corollary 4 would be that the opposite bound $E(m, s; n, t) \geq 1$ holds in the integer case. (In fact the smallest value we have found for $s, t > 0$ is $E(2, 3; 3, 2) = 539/450$.)

Some numerical examples comparing our estimates to other estimates and the correct values appear in Table 4.

Throughout the paper, the asymptotic notation $O(f(m, n))$ refers to the passage of $m$ and $n$ to $\infty$. Generally, the constant implied by the $O$ may depend on $a, b$, and the sufficiently small and fixed constant $\varepsilon$ introduced later in the definition (3.1) of region
Under the hypothesis (1.1) of Theorem 1, we have \( \log m = \) provided that for some constant \( c \)
\( M \), its value by the saddle-point method.

Applying Cauchy’s Theorem we have \( \tilde{c} \), \( \tilde{1} \) \( M \) It is clear that \( \tilde{1} \left\{ \begin{array}{l} 10,20,10,20 \\ 7.434 \times 10^{58} \\ 1.059 \times 10^{59} \\ 5.157 \times 10^{127} \\ 7.850 \times 10^{127} \\ 1.404 \times 10^{92} \\ 2.223 \times 10^{92} \\ 1.192 \times 10^{7} \\ 1.680 \times 10^{7} \\ 2.788 \times 10^{21} \\ 4.88 \times 10^{21} \\ 9.021 \times 10^{58} \\ (1.119 \pm 0.056) \times 10^{59} \\ 5.109 \times 10^{130} \\ (8.065 \pm 0.224) \times 10^{127} \\ 2.315 \times 10^{92} \end{array} \right\} \) \( \tilde{c} \), \( \tilde{1} \) \( m, s, n, t \)

Under the hypothesis (1.1) of Theorem 1, we have \( \log m \sim \log n, n = O(m \log m), m = O(n \log n) \), and \( \lambda^{-1} = O(\log n) \). Consequently, in the newly introduced notation, \( n = \tilde{O}(m), m = \tilde{O}(n) \), and \( \lambda^{-1} = \tilde{O}(1) \). In general, if \( c_1, c_2, c_3, c_4 \) are constants, then \( m^{c_1+c_2} n^{c_3+c_4} = \tilde{O}(m^{c_1} n^{c_3}) = \tilde{O}(n^{c_1+c_3}) = \tilde{O}(m^{c_1+c_3}) \).

### 2 An integral for \( M(m, s; n, t) \)

We express \( M(m, s; n, t) \) as an integral in \( (m+n) \)-dimensional complex space then estimate its value by the saddle-point method.

It is clear that \( M = M(m, s; n, t) \) is the coefficient of \( x_1^t \cdots x_m^t y_1^t \cdots y_n^t \) in

\[
\prod_{j=1}^{m} \prod_{k=1}^{n} (1 - x_j y_k)^{-1}.
\]

Applying Cauchy’s Theorem we have

\[
M = \frac{1}{(2\pi i)^{m+n}} \int \cdots \int \frac{\prod_{j,k} (1 - x_j y_k)^{-1}}{x_1^{s+1} \cdots x_m^{s+1} y_1^{t+1} \cdots y_n^{t+1}} \, dx_1 \cdots dx_m \, dy_1 \cdots dy_n, \tag{2.1}
\]
where each contour circles the origin once in the anticlockwise direction.

It will suffice to take the contours to be circles; specifically, we will put \( x_j = re^{i\theta_j} \) and \( y_k = re^{i\phi_k} \) for each \( j, k \), where

\[
 r = \sqrt{\frac{\lambda}{1 + \lambda}}.
\]

This gives

\[
 M = \frac{1}{(2\pi)^{m+n}} \left( \lambda^{-\lambda}(1 + \lambda)^{1+\lambda} \right)^{mn} I(m, n),
\]

where

\[
 I(m, n) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{j,k} (1 - \lambda(e^{i(\theta_j + \phi_k)} - 1))^{-1} \frac{d\theta}{e^{is\sum_j \theta_j + it\sum_k \phi_k}},
\]

where \( \theta = (\theta_1, \ldots, \theta_m) \) and \( \phi = (\phi_1, \ldots, \phi_n) \). Let \( F(\theta, \phi) \) denote the integrand in equation \((2.3)\).

### 3 Evaluating the integral

This section follows closely the corresponding section in [7]. However, all proofs are intended to be complete, and not require the reader to refer to the latter work, except for the following lemma which we restate here without proof.

**Lemma 1.** Let \( \varepsilon', \varepsilon'', \varepsilon'', \varepsilon, \Delta \) be constants such that \( 0 < \varepsilon' < \varepsilon'' < \varepsilon''' \), \( \varepsilon \geq 0 \), and \( 0 < \Delta < 1 \). The following is true if \( \varepsilon''' \) and \( \varepsilon \) are sufficiently small.

Let \( \hat{A} = \hat{A}(N) \) be a real-valued function such that \( \hat{A}(N) = \Omega(N^{-\varepsilon'}) \). Let \( \hat{a}_j = \hat{a}_j(N), \hat{B}_j = \hat{B}_j(N), \hat{C}_{jk} = \hat{C}_{jk}(N), \hat{E}_j = \hat{E}_j(N), \hat{F}_{jk} = \hat{F}_{jk}(N) \) and \( \hat{J}_j = \hat{J}_j(N) \) be complex-valued functions \( 1 \leq j, k \leq N \) such that \( \hat{B}_j, \hat{C}_{jk}, \hat{E}_j, \hat{F}_{jk} = O(N^{\varepsilon}), \hat{a}_j = O(N^{1/2+\varepsilon}), \) and \( \hat{J}_j = O(N^{-1/2+\varepsilon}) \), uniformly over \( 1 \leq j, k \leq N \). Suppose that

\[
 f(z) = \exp( -\hat{A}N \sum_{j=1}^{N} z_j^2 + \sum_{j=1}^{N} \hat{a}_j z_j^2 + N \sum_{j=1}^{N} \hat{B}_j z_j^3 + \sum_{j,k=1}^{N} \hat{C}_{jk} z_j z_k^2 + N \sum_{j=1}^{N} \hat{E}_j z_j^4 + \sum_{j,k=1}^{N} \hat{F}_{jk} z_j^2 z_k^2 + \sum_{j=1}^{N} \hat{J}_j z_j + \delta(z) )
\]

is integrable for \( z = (z_1, z_2, \ldots, z_N) \in U_N \) and \( \delta(N) = \max_{z \in U_N} |\delta(z)| = o(1) \), where

\[
 U_N = \{ z \mid |z_j| \leq N^{-1/2+\varepsilon} \text{ for } 1 \leq j \leq N \},
\]

where \( \hat{\varepsilon} = \hat{\varepsilon}(N) \) satisfies \( \varepsilon'' \leq 2\hat{\varepsilon} \leq \varepsilon''' \). Then, provided the \( O() \) term in the following converges to zero,

\[
 \int_{U_N} f(z) \, dz = \left( \frac{\pi}{AN} \right)^{N/2} \exp(\Theta_1 + \Theta_2 + O((N^{-\Delta} + \delta(N))\hat{Z})),
\]
where

\[
\Theta_1 = \frac{1}{2AN} \sum_{j=1}^{N} \hat{a}_j + \frac{1}{4A^2N^2} \sum_{j=1}^{N} \hat{a}_j^2 + \frac{15}{16A^3N} \sum_{j=1}^{N} \hat{B}_j^2 + \frac{3}{8A^3N^2} \sum_{j,k=1}^{N} \hat{B}_j \hat{C}_{jk} \\
+ \frac{1}{16A^3N^3} \sum_{j,k,\ell=1}^{N} \hat{C}_{jk} \hat{C}_{j\ell} + \frac{3}{4A^2N} \sum_{j=1}^{N} \hat{E}_j + \frac{1}{4A^2N^2} \sum_{j,k=1}^{N} \hat{F}_{jk}
\]

\[
\Theta_2 = \frac{1}{6A^3N^3} \sum_{j=1}^{N} \hat{a}_j^3 + \frac{3}{2A^3N^2} \sum_{j=1}^{N} \hat{a}_j \hat{E}_j + \frac{45}{16A^4N} \sum_{j=1}^{N} \hat{B}_j^2 \\
+ \frac{1}{4A^3N^3} \sum_{j,k=1}^{N} (\hat{a}_j + \hat{a}_k) \hat{F}_{jk} + \frac{3}{4A^2N} \sum_{j=1}^{N} \hat{B}_j \hat{J}_j + \frac{1}{4A^2N^2} \sum_{j,k=1}^{N} \hat{C}_{jk} \hat{J}_j \\
+ \frac{1}{16A^4N^4} \sum_{j,k,\ell=1}^{N} (\hat{a}_j + 2\hat{a}_k) \hat{C}_{jk} \hat{C}_{j\ell} + \frac{3}{8A^4N^3} \sum_{j,k=1}^{N} (2\hat{a}_j + \hat{a}_k) \hat{B}_j \hat{C}_{jk}
\]

\[
\hat{Z} = \exp\left(\frac{1}{4A^2N^2} \sum_{j=1}^{N} \text{Im}(\hat{a}_j)^2 + \frac{15}{16A^3N} \sum_{j=1}^{N} \text{Im}(\hat{B}_j)^2 \\
+ \frac{3}{8A^3N^2} \sum_{j,k=1}^{N} \text{Im}(\hat{B}_j) \text{Im}(\hat{C}_{jk}) + \frac{1}{16A^3N^3} \sum_{j,k,\ell=1}^{N} \text{Im}(\hat{C}_{jk}) \text{Im}(\hat{C}_{j\ell})\right).
\]

We use the notation \( R^c \) for the complement of a region \( R \). To evaluate the integral \( I(m, n) \) defined in (2.3), we proceed as follows:

\[
I(m, n) = \int_R F + \int_{R^c} F = \int_{R'} F + O(1) \int_{R^c} |F|, \quad R' \supseteq R.
\]

The region \( R \subseteq [-\pi, +\pi]^{m+n} \) is defined below in (3.1). The larger containing region \( R' \) is defined in (3.7). The asymptotic value of the integral \( \int_{R'} F \) is obtained by substituting \( R' \) for the variable \( R'' \) in equation (3.2), and applying the main result of this section, Theorem [2] In the next section we show that for a certain constant \( c_5 > 0 \)

\[
\int_{R^c} |F| = O(e^{-c_5 \min(m^{2\varepsilon}, n^{2\varepsilon})/\log n}) \int_{R'} F.
\]

(Again, the quantity \( \varepsilon \) is a small positive constant arising in the definition (3.1) of the region \( R \).) This is the complete summary of how we shall evaluate \( I(m, n) \), and now we may proceed to the technical details.

For any region \( R'' \) we set

\[
I_{R''}(m, n) = \int_{R''} F.
\]

In the integral (2.3), it is convenient sometimes to think of \( \theta_j, \phi_k \) as points on the unit circle. We wish to define “averages” of the angles \( \theta_j, \phi_k \). To do this cleanly we make
the following definitions, as in [8]. Let \( C \) be the ring of real numbers modulo \( 2\pi \), which we can interpret as points on a circle in the usual way. Let \( z \) be the canonical mapping from \( C \) to the real interval \((-\pi, \pi]\). An open half circle is \( C_t = (t - \pi/2, t + \pi/2) \subseteq C \) for some \( t \). Now define

\[
\hat{C}^N = \{ x = (x_1, \ldots, x_N) \in C^N \mid x_1, \ldots, x_N \in C_t \text{ for some } t \in \mathbb{R} \}.
\]

If \( x = (x_1, \ldots, x_N) \in C_0^N \) then define \( \bar{x} = z^{-1} \left( \frac{1}{N} \sum_{j=1}^{N} z(x_j) \right) \).

More generally, if \( x \in C_t^N \) then define \( \bar{x} = t + (x_1 - t, \ldots, x_N - t) \). The function \( x \mapsto \bar{x} \) is well-defined and continuous for \( x \in \hat{C}^N \).

For some sufficiently small \( \varepsilon > 0 \), let \( R \) denote the set of vector pairs \( \theta, \phi \in \hat{C}^m \times \hat{C}^n \) such that

\[
|\hat{\theta} + \hat{\phi}| \leq (1 + \lambda)^{-1} (mn)^{-1/2+2\varepsilon}
\]

region \( R : \)

\[
|\hat{\theta}_j| \leq (1 + \lambda)^{-1} n^{-1/2+\varepsilon}, 1 \leq j \leq m
\]

\[
|\hat{\phi}_k| \leq (1 + \lambda)^{-1} m^{-1/2+\varepsilon}, 1 \leq k \leq n,
\]

where \( \hat{\theta}_j = \theta_j - \bar{\theta} \) and \( \hat{\phi}_k = \phi_k - \bar{\phi} \). In this definition, values are considered in \( C \).

Let \( \hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_{m-1}) \), \( \hat{\phi} = (\hat{\phi}_1, \ldots, \hat{\phi}_{n-1}) \), and define \( T_1 \) to be the transformation \( T_1(\hat{\theta}, \hat{\phi}, \mu, \delta) = (\theta, \phi) \) given by

\[
\mu = \bar{\theta} + \bar{\phi}, \quad \delta = \bar{\theta} - \bar{\phi},
\]

together with \( \hat{\theta}_j = \theta_j - \bar{\theta} \) (1 \leq j \leq m - 1) and \( \hat{\phi}_k = \phi_k - \bar{\phi} \) (1 \leq k \leq n - 1). We also define the 1-many transformation \( T_1^* \) by

\[
T_1^*(\hat{\theta}, \hat{\phi}, \mu) = \bigcup_{\delta} T_1(\hat{\theta}, \hat{\phi}, \mu, \delta).
\]

After applying the transformation \( T_1 \) to \( I_R(m, n) \), the new integrand is easily seen to be independent of \( \delta \), so we can multiply by the range of \( \delta \) and remove it as an independent variable. Therefore, we can continue with an \((m+n-1)\)-dimensional integral over the region \( S \) defined by \( R = T_1^*(S) \). More generally, if \( S'' \subseteq [-\frac{1}{2}\pi, \frac{1}{2}\pi]^{m+n-2} \times [-2\pi, 2\pi] \) and \( R'' = T_1^*(S'') \), we have

\[
I_{R''}(m, n) = 2\pi mn \int_{S''} G(\hat{\theta}, \hat{\phi}, \mu) d\theta d\hat{\phi} d\mu,
\]

(3.2)
where \( G(\hat{\theta}, \hat{\phi}, \mu) = F(T_1(\hat{\theta}, \hat{\phi}, \mu, 0)) \). The factor \( 2\pi mn \) combines the range of \( \delta \), which is \( 4\pi \), and the Jacobian of \( T_1 \), which is \( mn/2 \).

Note that \( \mathcal{S} \) is defined by virtually the same inequalities as define \( \mathcal{R} \). The first inequality is now \( |\mu| \leq (1 + \lambda)^{-1} (mn)^{-1/2} \) and the bounds on 
\[
\hat{\theta}_m = -\sum_{j=1}^{m-1} \hat{\theta}_j \text{ and } \hat{\phi}_n = -\sum_{k=1}^{n-1} \hat{\phi}_k
\]
still apply even though these are no longer variables of integration.

Our main result in this section is the following.

**Theorem 2.** Define \( A = \frac{1}{2}\lambda(1 + \lambda) \). Under the conditions of Theorem 1, there is a region \( \mathcal{S}^\prime \supseteq \mathcal{S} \) such that
\[
\int_{\mathcal{S}^\prime} G(\hat{\theta}, \hat{\phi}, \mu) \, d\hat{\theta}d\hat{\phi}d\mu = (mn)^{-1/2}\left(\frac{\pi}{A_{mn}}\right)^{1/2}\left(\frac{\pi}{A_n}\right)^{(m-1)/2}\left(\frac{\pi}{A_m}\right)^{(n-1)/2}
\times \exp\left(\frac{1}{2} - \frac{1 + 2A}{24A}\left(\frac{m}{n} + \frac{n}{m}\right) + O(n^{-b})\right).
\]

**Proof.** For \( x \) real and \(|\lambda x| \) small,
\[
(1 - \lambda(e^{ix} - 1))^{-1} = \exp\left(\lambda ix - Ax^2 - iA_3x^3 + A_4x^4 + O((1 + \lambda)^5|x|^5)\right)
\]
with
\[
A = \frac{1}{2}\lambda(1 + \lambda), \quad A_3 = \frac{1}{6}\lambda(1 + \lambda)(1 + 2\lambda), \quad A_4 = \frac{1}{24}\lambda(1 + \lambda)(1 + 6\lambda + 6\lambda^2).
\]

We are about to state an estimate for \( G \), and some further estimates \((3.3)\) in a moment, all of which hold uniformly in the region \( \mathcal{S} \). Let us alert the reader that we shall be defining a larger region, see \((3.8)\) below, which contains \( \mathcal{S} \) and in which all of these estimates continue to hold true. Uniformly in \( \mathcal{S} \), where all \((1 + \lambda)|\mu + \hat{\theta}_j + \hat{\phi}_k| \) are small,
\[
G = \exp\left\{-A\sum_{j,k} (\mu + \hat{\theta}_j + \hat{\phi}_k)^2 - iA_3\sum_{j,k} (\mu + \hat{\theta}_j + \hat{\phi}_k)^3
+ A_4\sum_{j,k} (\mu + \hat{\theta}_j + \hat{\phi}_k)^4 + O((1 + \lambda)^5\sum_{j,k} |\mu + \hat{\theta}_j + \hat{\phi}_k|^5)\right\}.
\]

Here and below, the undelimited summation over \( j, k \) runs over \( 1 \leq j \leq m, 1 \leq k \leq n \), and we continue to use the abbreviations \( \hat{\theta}_m = -\sum_{j=1}^{m-1} \hat{\theta}_j \) and \( \hat{\phi}_n = -\sum_{k=1}^{n-1} \hat{\phi}_k \).

We now proceed to a second change of variables, \((\hat{\theta}, \hat{\phi}, \mu) = T_2(\sigma, \tau, \mu)\) given by
\[
\hat{\theta}_j = \sigma_j + c\mu_1, \quad \hat{\phi}_k = \tau_k + d\nu_1,
\]
where, for \( 1 \leq h \leq 4 \), \( \mu_h \) and \( \nu_h \) denote the power sums \( \sum_{j=1}^{m-1} \sigma_j^h \) and \( \sum_{k=1}^{n-1} \tau_k^h \), respectively. The scalars \( c \) and \( d \) are chosen to eliminate the second-degree cross-terms \( \sigma_j^h \sigma_k^i \) and
\( \tau_k, \tau_k, \) and thus diagonalize the quadratic in \( \sigma = (\sigma_1, \ldots, \sigma_{m-1}) \) and \( \tau = (\tau_1, \ldots, \tau_{n-1}) \). Suitable choices for \( c, d \) are

\[
c = -\frac{1}{m + m^{1/2}}, \quad d = -\frac{1}{n + n^{1/2}},
\]

and we find the following:

\[
\sum_{j,k} (\mu + \hat{\theta}_j + \hat{\phi}_k)^2 = mn\mu^2 + n\mu_2 + mv_2
\]
\[
\sum_{j,k} (\mu + \hat{\theta}_j + \hat{\phi}_k)^3 = 3\mu(n\mu_2 + mv_2) + n(\mu_3 + 3c\mu_1) + m(\nu_3 + 3d\nu_2\nu_1)
\]
\[
\quad + \tilde{O}((1 + \lambda)^{-1/2})
\]
\[
\sum_{j,k} (\mu + \hat{\theta}_j + \hat{\phi}_k)^4 = 6\mu_2\nu_2 + n\mu_4 + mv_4 + \tilde{O}((1 + \lambda)^{-4}n^{-1/2})
\]
\[
\sum_{j,k} |\mu + \hat{\theta}_j + \hat{\phi}_k|^5 = \tilde{O}((1 + \lambda)^{-1/2}).
\] (3.3)

The Jacobian of the matrix \( T_2 \) is \( (mn)^{-1/2} \), and so

\[
\int_S G = (mn)^{-1/2} \int_{T_2^{-1}(S)} E_1,
\] (3.4)

where \( E_1 = \exp(L_2 + \tilde{O}(n^{-1/2})) \), and

\[
L_2 = -An\mu_2 - Amv_2 - Amn\mu^2 - iA_3\mu_3 - iA_3mv_3 - 3iA_3\mu_2\mu_1 - 3iA_3\mu_2\nu_2
\]
\[
- 3iA_3\nu_2\mu_1 - 3iA_3\nu_2\nu_1 + A_4\mu_4 + A_4mv_4 + 6A_4\mu_2\nu_2.
\] (3.5)

Define the regions \( Q, M, S', \) and \( R' \) by

\[
Q = \{ |\sigma_j| \leq 2(1 + \lambda)^{-1}n^{-1/2+\epsilon}, \ j = 1, \ldots, m-1 \}
\]
\[
\cap \{ |\tau_k| \leq 2(1 + \lambda)^{-1}m^{-1/2+\epsilon}, \ k = 1, \ldots, n-1 \},
\]
\[
\cap \{ |\mu| \leq 2(1 + \lambda)^{-1}(mn)^{-1/2+2\epsilon} \}
\]
\[
M = \{ |\mu_1| \leq (1 + \lambda)^{-1}m^{1/2}n^{-1/2+2\epsilon} \} \cap \{ |\nu_1| \leq (1 + \lambda)^{-1}n^{1/2}m^{-1/2+2\epsilon} \},
\]
\[
S' = T_2(Q \cap M),
\] (3.6)

and

\[
R' = T_1^n(S').
\] (3.7)

Summing for \( 1 \leq j \leq m-1 \) the equation \( \hat{\theta}_j = \sigma_j + c\mu_1 \), and inserting the value of \( c \), we find

\[
m^{-1/2}\mu_1 = \sum_{j=1}^{m-1} \hat{\theta}_j.
\]
In the region $S$ we have $|\sum_{j=1}^{m-1} \hat{\theta}_j| \leq (1 + \lambda)^{-1} n^{-1/2+\varepsilon}$, and so in $T_2^{-1}(S)$ we have
$$|\mu_1| \leq (1 + \lambda)^{-1} m^{1/2} n^{-1/2+\varepsilon}.$$ 
Using this, and the dual inequality for $|\nu_1|$, the reader can check that
$$T_2^{-1}(S) \subseteq Q \cap M.$$

It will be convenient if we can apply the bounds (3.3) throughout the expanded region $S' = T_2(Q \cap M)$, rather than only in $S$. To see that this is valid, note that the calculations leading to these bounds hold equally well if the coefficient of $\varepsilon$ in an exponent of $m$ or $n$ is increased; or, if an assumption such as $|\mu| \leq (1 + \lambda)^{-1} (mn)^{1/2+2\varepsilon}$ is made more permissive by a multiplicative constant: $|\mu| \leq 2(1 + \lambda)^{-1} (mn)^{1/2+2\varepsilon}$. Therefore, it suffices to note that $S'$ lies inside the region

$$|\mu| \leq 2(1 + \lambda)^{-1} (mn)^{-1/2+2\varepsilon},$$

$$|\hat{\theta}_j| \leq 3(1 + \lambda)^{-1} n^{-1/2+\varepsilon}, \quad 1 \leq j \leq m - 1$$

$$|\hat{\theta}_m| \leq (1 + \lambda)^{-1} m^{-1/2+2\varepsilon},$$

$$|\hat{\phi}_k| \leq 3(1 + \lambda)^{-1} m^{-1/2+\varepsilon}, \quad 1 \leq k \leq n - 1$$

$$|\hat{\phi}_n| \leq (1 + \lambda)^{-1} m^{-1/2+2\varepsilon}.$$

(3.8)

Define $E_2 = \exp(L_2)$. We have shown that in the region $Q \cap M$ the integrand $E_1$ satisfies $E_1 = E_2(1 + O(n^{-1/2}))$. We can approximate our integral as follows:

$$\int_{Q \cap M} E_1 = \int_{Q \cap M} E_2 + \tilde{O}(n^{-1/2}) \int_{Q \cap M} |E_2|$$

$$= \int_{Q \cap M} E_2 + \tilde{O}(n^{-1/2}) \int_{Q} |E_2|$$

$$= \int_{Q} E_2 + O(1) \int_{Q \cap M^c} |E_2| + \tilde{O}(n^{-1/2}) \int_{Q} |E_2|.$$ (3.9)

It suffices to estimate each of the three integrals in the last line of (3.9).

We first compute the integral of $E_2$ over $Q$. We proceed in three stages, starting with integration with respect to $\mu$. For the latter, we can use the formula

$$\int_{(mn)^{-1/2+2\varepsilon}}^{(mn)^{-1/2+2\varepsilon}} \exp(-A m \mu^2 - i \beta \mu) d\mu = \left(\frac{\pi}{Amn}\right)^{1/2} \exp\left(-\frac{\beta^2}{4Amn} + O(n^{-1})\right),$$

provided $\beta = o(A mn^{1/2+2\varepsilon})$. In our case, $\beta = 3A_3(n \mu_2 + m \nu_2)$, which is small enough because $m = O(n \log n)$ and $n = O(m \log m)$. Integration over $\mu$ contributes

$$\left(\frac{\pi}{Amn}\right)^{1/2} \exp\left(-\frac{9A_3^2(n \mu_2 + m \nu_2)^2}{4Amn} + O(n^{-1})\right).$$ (3.10)
The second step is to integrate with respect to $\sigma$ the integrand

\[
\exp\left( -A n \mu_2 - \frac{9 A^2 n}{4 A m} \mu_2^2 - i A_3 n \mu_3 - 3 i A_3 c n \mu_1 \mu_2 \right) + \left( 6 A_4 - \frac{9 A_3^2}{2 A} \right) \mu_2 \nu_2 + A_4 n \mu_4 + O(n^{-1}) \right).
\]

(3.11)

This is accomplished by an appeal to Lemma [1]. We must take the $N$ of that lemma equal to $m - 1$, the number of variables. This dictates that the limits of integration be $\pm (m - 1)^{1/2 + \varepsilon}$, but our limits, based on the definition of $Q$, are $\pm 2(1 + \lambda)^{-1} n^{-1/2 + \varepsilon}$. Thus, we make a change of scale,

\[
\sigma_j \leftrightarrow \frac{\sigma_j}{(1 + \lambda)(n/m)^{1/2}}
\]

before integrating. This change of scale will introduce a multiplicative factor $(1 + \lambda)^{-(m-1)} (m/n)^{(m-1)/2}$ into our evaluation of the integral. In the terminology of that lemma, we have $N = m - 1$, $\delta(N) = O(n^{-1})$, $\varepsilon' = \frac{3}{5} \varepsilon$, $\varepsilon'' = \frac{5}{3} \varepsilon$, $\varepsilon''' = 3 \varepsilon$, and $\varepsilon(N) = \varepsilon + o(1)$ is defined by $2 m^{-1/2 + \varepsilon} = (m - 1)^{-1/2 + \varepsilon}$. Furthermore, taking account of the scale, we find

\[
\hat{A} = \frac{A}{(1 + \lambda)^2 m^{-1}}, \quad \hat{A}_j = \frac{12 A A_4 - 9 A_3^2}{2 A (1 + \lambda)^2} m \nu_2,
\]

\[
\hat{B}_j = -\frac{i A_3}{(1 + \lambda)^{1/2} n^{1/2} m^{-1}}, \quad \hat{C}_{jj'} = -\frac{3 i A_3}{(1 + \lambda)^3} \frac{m^{3/2}}{n^{3/2}},
\]

\[
\hat{E}_j = \frac{A_4}{(1 + \lambda)^4 n m^{-1}}, \quad \hat{E}_{jj'} = -\frac{9 A_3^2}{4 A (1 + \lambda)^4} \frac{m}{n}, \quad \hat{j}_j = 0.
\]

We can take $\Delta = \frac{4}{5}$ and calculate that

\[
\frac{3}{4 A^2 N} \sum_{j=1}^{N} \hat{E}_j + \frac{1}{4 A^2 N^2} \sum_{j,j'=1}^{N} \hat{E}_{jj'} = \frac{m}{n} \left( \frac{3 A_4}{4 A^2} - \frac{9 A_3^2}{16 A^3} \right) + O(n^{-1})
\]

\[
\frac{15}{16 A^3 N} \sum_{j=1}^{N} \hat{B}_j^2 + \frac{3}{8 A^3 N^2} \sum_{j,j'=1}^{N} \hat{B}_j \hat{C}_{jj'} + \frac{1}{16 A^3 N^3} \sum_{j,j',j''=1}^{N} \hat{C}_{jj'} \hat{C}_{j''} = -\frac{3 A_3^2 m}{8 A^3 n} + O(n^{-1})
\]

\[
\frac{1}{2 A N} \sum_{j=1}^{N} \hat{a}_j + \frac{1}{4 A^2 N^2} \sum_{j=1}^{N} \hat{a}_j^2 = \frac{m}{n} \left( \frac{3 A_4}{A} - \frac{9 A_3^2}{4 A^3} \right) \nu_2 + \tilde{O}(n^{-1}) \quad (3.12)
\]

\[
\hat{Z} = Z_1 = \exp\left( \frac{3 A_3^2 m}{8 A^3 n} + O(n^{-1}) \right) = O(1) \exp\left( \frac{(1 + 2 \lambda)^2 m}{24 A n} \right).
\]

After checking that $\Theta_2 = \tilde{O}(n^{-1}) = o(m^{-4/5} Z_1)$, we conclude that integration with respect to $\sigma$ contributes a $\tau$-free factor

\[
\left( \frac{\pi}{A n} \right)^{(m-1)/2} \exp\left( \left( \frac{3 A_4}{4 A^2} - \frac{15 A_3^2}{16 A^3} \right) \frac{m}{n} + O(m^{-4/5} Z_1) \right).
\]

(3.13)
By the conditions of Theorem 1, \( Z_1 \leq n^{1/5} \), so \( m^{-4/5}Z_1 = o(1) \) as required by Lemma 1.

Finally, we need to integrate over \( \tau \). Collecting the remaining terms from (3.13), and the terms involving \( \tau \) from (3.10) and (3.12), we have an integrand equal to

\[
\exp \left( -Am\nu_2 + \left( 3A_4 \frac{m}{An} - \frac{9A_3^2}{4A^2n} \right) \nu_2 - \frac{9A_3^2}{4An} \nu_2 \right) + A_4m\nu_4 - iA_3m\nu_3 - 3iA_3dm\nu_2 \nu_1 + O(m^{-4/5}Z_1). \]

Again we use Lemma 1. This time, the factor due to scaling is \( (1+\lambda) \), and \( N = n - 1 \), \( \delta(N) = O(m^{-4/5}Z_1) \), \( \varepsilon' = \frac{3}{2}\varepsilon \), \( \varepsilon'' = \frac{5}{3}\varepsilon \), \( \varepsilon''' = 3\varepsilon \), \( \tilde{\varepsilon} = 4\varepsilon \), and \( \tilde{\varepsilon}(N) = \varepsilon + o(1) \), as defined by \( 2n^{-1/2+\varepsilon} = (n - 1)^{-1/2+\tilde{\varepsilon}} \). The substitution table this time reads

\[
\hat{A} = \frac{A}{(1+\lambda)^2} n, \quad \hat{a}_k = \frac{12AA_4 - 9A_3}{4A^2(1+\lambda)^2},
\]

\[
\hat{B}_k = \frac{iA_3}{(1+\lambda)^3} \frac{n^{1/2}}{m^{1/2}n - 1}, \quad \hat{C}_{kk'} = -\frac{3iA_3}{(1+\lambda)^3} \frac{dm}{m^{3/2}},
\]

\[
\hat{E}_k = \frac{A_1}{(1+\lambda)^3} \frac{n}{m^{1/2}m - 1}, \quad \hat{F}_{kk'} = -\frac{9A^2}{4A(1+\lambda)^3} \frac{n}{m}, \quad \hat{j}_k = 0.
\]

This time we take \( \Delta_2 > 4/5 \) so that \( n^{-\Delta_2} = o(\delta(N)) \); calculations similar to the previous case lead to

\[
\frac{3}{4A^2} \sum_{k=1}^N \hat{E}_k + \frac{1}{4A^2N^2} \sum_{k,k'=1}^N \hat{F}_{kk'} = \frac{1}{m} \left( \frac{3A_4}{4A^2} - \frac{9A_3^2}{16A^3} \right) + \tilde{O}(n^{-1})
\]

\[
\frac{15}{16A^3N} \sum_{k=1}^N \hat{B}_k^2 + \frac{3}{8A^3N^2} \sum_{k,k'=1}^N \hat{B}_j \hat{C}_{kk'} + \frac{1}{16A^3N^3} \sum_{k,k',k''=1}^N \hat{C}_{kk'} \hat{C}_{kk''} = -\frac{3A^2_3 n}{8A^3 m} + \tilde{O}(n^{-1})
\]

\[
\frac{1}{2A} \sum_{k=1}^N \hat{a}_k + \frac{1}{4A^2N^2} \sum_{k=1}^N \hat{a}_k^2 = -\frac{9A_3^2}{8A^4} + \frac{3A_4}{2A^2} + \tilde{O}(n^{-1})
\]

\[
\hat{Z} = Z_2 = \exp \left( \frac{3A_2^2 n}{8A^3 m} + \tilde{O}(n^{-1}) \right) = O(1) \exp \left( \frac{(1+2\lambda)^2 n}{24Am} \right).
\]

Again \( \Theta_2 = \tilde{O}(n^{-1}) \), implying that \( \Theta_2 = o(m^{-4/5}Z_1Z_2) \). Including the contributions from (3.10) and (3.13), we obtain

\[
\int_Q E_2 = \left( \frac{\pi}{Anm} \right)^{1/2} \left( \frac{\pi}{An} \right)^{(m-1)/2} \left( \frac{\pi}{An} \right)^{(n-1)/2} \times \exp \left( -\frac{9A_3^2}{8A^4} + \frac{3A_4}{2A^2} + \left( \frac{m}{n} + \frac{n}{m} \right) \left( \frac{3A_4}{4A^2} - \frac{15A_3^2}{16A^3} \right) \right) + O(m^{-4/5}Z_1Z_2) \quad \text{(3.14)}
\]
Since \( Z_1 Z_2 = O(1) \exp \left( \frac{3A_2^2}{8A^3} \left( \frac{m}{n} + \frac{n}{m} \right) \right) \),

and since 
\[
\frac{3A_2^2}{8A^3} = \frac{(1 + 2\lambda)^2}{24A},
\]

it follows from the main hypothesis (1.1) of Theorem 1 that 
\( Z_1 Z_2 = O(n^{6a/5}) \). The condition \( a + b < 1/2 \) implies 
\[-4/5 + 6a/5 < -b - 1/5; \]

hence, substituting the values of \( A, A_3, A_4 \), we conclude that
\[
\int Q \mathbb{E}^2 = \left( \pi \left( \frac{m}{n} + \frac{n}{m} \right) \right)^{1/2} \left( \frac{A_2}{A_1} \right)^{(m-1)/2} \left( \frac{A_2}{A_1} \right)^{(n-1)/2} 
\times \exp \left( \frac{1}{2} - \frac{1 + 2A}{24A} \left( \frac{m}{n} + \frac{n}{m} \right) + O(n^{-b-1/5}) \right).
\]

We next infer an estimate of \( \int Q |E_2| \). The calculation that lead to (3.14) remains valid if we set \( A_3 \) to zero, which is the same as replacing \( L_2 \) by its real part. Since 
\( |E_2| = \exp(\text{Re}(L_2)) \), this gives
\[
\int Q |E_2| = \exp \left( \frac{(1 + 2\lambda)^2}{8A} \left( 1 + \frac{5n}{6m} + \frac{5m}{6n} \right) + o(1) \right) \int Q E_2
\]
\[
= O(n^a) \int Q E_2
\]

under the assumptions of Theorem 1. The third term of (3.9) can now be identified:
\[
\tilde{O}(n^{-1/2}) \int Q |E_2| = \tilde{O}(n^{-1/2+a}) \int Q E_2 = O(n^{-b}) \int Q E_2.
\]

Finally, we consider the second term of (3.9), namely
\[
\int_{Q \cap M^\varepsilon} |E_2|,
\]

which we will bound as a fraction of \( \int_Q |E_2| \) using a statistical technique. The following is a well-known result of Hoeffding [23].

**Lemma 2.** Let \( X_1, X_2, \ldots, X_N \) be independent random variables such that \( EX_i = 0 \) and \( |X_i| \leq M \) for all \( i \). Then, for any \( t \geq 0 \),
\[
\text{Prob} \left( \sum_{i=1}^N X_i \geq t \right) \leq \exp \left( -\frac{t^2}{2NM^2} \right).
\]
Now consider $|E_2| = \exp(\text{Re}(L_2))$. Write $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$, where $\mathcal{M}_1 = \{|\mu_1| \leq m^{1/2}n^{-1/2+2\varepsilon}\}$ and $\mathcal{M}_2 = \{|\nu_1| \leq n^{1/2}m^{-1/2+2\varepsilon}\}$. For fixed values of $\mu$ and $\sigma$, $\text{Re}(L_2)$ separates over $\tau_1, \tau_2, \ldots, \tau_{n-1}$ and therefore, apart from normalization, it is the joint density of independent random variables $X_1, X_2, \ldots, X_{n-1}$ which satisfy $EX_k = 0$ (by symmetry) and $|X_k| \leq 2(1 + \lambda)^{-1}m^{-1/2+\varepsilon}$ (by the definition of $Q$). By Lemma 2, the fraction of the integral over $\tau$ (for fixed $\mu, \sigma$) that has $\nu_1 \geq (1 + \lambda)^{-1}n^{1/2}m^{-1/2+2\varepsilon}$ is at most $\exp(-m^2\varepsilon/8)$. By symmetry, the same bound holds for $\nu_1 \leq -(1 + \lambda)^{-1}n^{1/2}m^{-1/2+2\varepsilon}$. Since these bounds are independent of $\mu$ and $\sigma$, we have

$$\int_{Q \cap M_2} |E_2| \leq 2 \exp(-m^2\varepsilon/8) \int_Q |E_2|. \tag{3.18}$$

Therefore we have in total that

$$\int_{Q \cap M_1} |E_2| \leq 2 \left( \exp(-m^2\varepsilon/8) + \exp(-n^2\varepsilon/8) \right) \int_Q |E_2| \leq O(n^{-b}) \int_Q E_2, \tag{3.18}$$

using again (3.15). Applying (3.10) with (3.15), (3.17) and (3.18), we find that $\int_{Q \cap M} E_1$ is given by (3.15) with the error term replaced by $O(n^{-b})$. Multiplying by the Jacobian of the transformation $T_2$, we find that Theorem 2 is proved for $S$ given by (3.6).

### 4 Concentration of the integral

Recall that $F(\theta, \phi)$ is the integrand in equation (2.3) defining $I(m, n)$, and that $R$ is the region given by (3.1). In the previous section we estimated the integral of $F(\theta, \phi)$ over a particular superset $R' \supseteq R$. In this section we show that the integral of $F(\theta, \phi)$ outside $R$ is negligible in comparison if $\lambda$ is polynomially bounded. Larger values of $\lambda$ will be handled in the following section.

**Theorem 3.** Suppose that $m, n \to \infty$ in such a way that (1.1) holds and $\lambda = n^{O(1)}$. Define $I_0$ by

$$I_0 = (mn)^{1/2} \left( \frac{\pi}{Amn} \right)^{1/2} \left( \frac{\pi}{An} \right)^{(m-1)/2} \left( \frac{\pi}{Am} \right)^{(n-1)/2} \exp \left( -\frac{1 + 2A}{24A} \frac{m}{n} + \frac{n}{m} \right). \tag{4.1}$$

Then, for sufficiently small $\varepsilon > 0$,

$$\int_{R^c} |F| = O(e^{-n^\varepsilon})I_0.$$
We begin with two technical lemmas whose proofs are omitted.

**Lemma 3.** The absolute value of the integrand of $I(m, n)$ is

$$|F(\theta, \phi)| = \prod_{j,k} f(\theta_j + \phi_k),$$

where

$$f(z) = (1 + 4A(1 - \cos z))^{-1/2}.$$

Moreover, for all real $z$ with $|z| \leq \frac{1}{10}(1 + \lambda)^{-1}$,

$$0 \leq f(z) \leq \exp(-Az^2 + (\frac{1}{12}A + A^2)z^4). \quad \square$$

**Lemma 4.** Define $N = \lceil 6000(1 + \lambda) \rceil$, $\delta = 2\pi/N$ and $g(x) = -Ax^2 + (\frac{9}{4}A + 27A^2)x^4$. Then, uniformly for $\lambda > 0$, $K \geq 1$,

$$\int_{-3\delta}^{3\delta} \exp(Kg(x)) \, dx \leq \sqrt{\pi/(AK)} \exp(O(K^{-1} + (AK)^{-1})). \quad \square$$

**Proof of Theorem 3.** Let $N$ and $\delta$ be as given in Lemma 4. Define the region $A$ to be the set of those $(\theta, \phi)$ such that

$$\cos(\theta_j + \phi_k) \leq \cos \delta$$

for at least $\frac{1}{3}\min(mn^\varepsilon, m^\varepsilon n)$ pairs $(j, k)$. Define $x_0, x_1, \ldots, x_{N-1}$ by $x_\ell = \ell \delta$.

If $X \subseteq (\pi, \pi]$, we denote by $N_{\theta}(X)$ the number of values of $\theta$ such that $\theta_j \in X$, and similarly define $N_{\phi}(X)$. Define region $R_1(\ell)$ to be the set of those $(\theta, \phi)$ such that $N_{\theta}([x_\ell - 4\delta, x_\ell + 4\delta]) \geq m - m^\varepsilon$ and $N_{\phi}([-x_\ell - 4\delta, -x_\ell + 4\delta]) \geq n - n^\varepsilon$.

Let $U = \bigcup_{\ell=0}^{N-1} R_1(\ell)$. The proof of the theorem consists in proving these three relations:

$$A \cup U = [-\pi, \pi]^{m+n} \quad \text{(4.2)}$$

$$\int_A |F| = O(e^{-n})I_0 \quad \text{(4.3)}$$

$$\int_{U \cap \mathcal{R}^{\varepsilon}} |F| = O(e^{-n^\varepsilon})I_0. \quad \text{(4.4)}$$

To begin the proof of (4.2), we show that any point $(\theta, \phi)$ for which $N_{\theta}([x_\ell - \delta, x_\ell + \delta]) \geq m^\varepsilon$ belongs to $A \cup U$. Indeed, if such a point does not belong to $A$, then it must have $N_{\phi}([-x_\ell - 2\delta, -x_\ell + 2\delta]) \geq \frac{2}{3}n$. This in turn forces $N_{\theta}([x_\ell - 3\delta, x_\ell + 3\delta]) \geq m - m^\varepsilon$, which forces $N_{\phi}([-x_\ell - 4\delta, -x_\ell + 4\delta]) \geq n - n^\varepsilon$. In particular, $(\theta, \phi) \in R_1(\ell)$.

To complete the proof of (4.2), we show that $(\theta, \phi)$ belongs to $A$ if $N_{\theta}([x_\ell - \delta, x_\ell + \delta]) \leq m^\varepsilon$ for all $\ell$. Let $a$ be a minimum-length interval $[x_\ell, x_\ell]$ such that $N_{\theta}(a) \geq m^\varepsilon$. Then
\(N_\theta(a) \leq 2m^\varepsilon\) and the complementary interval \(\overline{p}\) has \(N_\theta(\overline{p}) \geq m - 2m^\varepsilon\). If \(b = [x_{\ell+2}, x_{\ell-2}]\) (a subinterval of \(\overline{p}\)), then \(N_\theta(b) \geq m - 4m^\varepsilon\). Thus, there are at least \(m^\varepsilon\) disjoint pairs \((j, j')\) with \(\theta_j \in a\) and \(\theta_j \in b\). (By disjoint we mean there are \(2m^\varepsilon\) distinct indices \(j\) involved.) Because of the \(2\delta\) spaces between arcs \(a\) and \(b\), for each \(k\) and each pair \((j, j')\) at least one of \(\cos(\theta_j + \phi_k)\) or \(\cos(\theta_j + \phi_k)\) is bounded above by \(\cos\delta\). This implies that \((\theta, \phi) \in A\), as claimed, and completes the proof of (4.2).

We turn next to (4.3). Since \(A\delta^2 = \Theta(\lambda(1 + \lambda)^{-1})\), Lemma 3 implies

\[
|F(\theta, \phi)| \leq \exp(-c_1\lambda(1 + \lambda)^{-1}\min(m^\varepsilon n, mn^\varepsilon))
\]

for all \((\theta, \phi) \in A\) and some \(c_1 > 0\). The volume of \(A\) is no more than \((2\pi)^{m+n}\), and \(\lambda(1 + \lambda)^{-1} > (\log n)^{-1}\) by (1.1), so

\[
\int_A |F| \leq (2\pi)^{m+n}\exp(-c_1(\log n)^{-1}\min(m^\varepsilon n, mn^\varepsilon))
\]

From (1.1), which implies that \(A = O(\log n)\), and the assumption that \(\lambda = n^{O(1)}\), we have that \(I_0 = \exp(O(m \log n + n \log m))\). Relation (4.3) follows.

Finally we come to (4.4). For \((\theta, \phi) \in R_1(\ell)\) we define \(S_0 = S_0(\theta), S_1 = S_1(\theta)\) and \(S_2 = S_2(\theta)\) to be the set of indices \(j\) such that \(|\theta_j - x_\ell|\) : less than or equal to \(4\delta\), in the interval \((\delta, 5\delta]\), and larger than \(5\delta\), respectively. The index sets \(T_0 = T_0(\phi), T_1 = T_1(\phi)\) and \(T_2 = T_2(\phi)\) are defined similarly. Define \(R_1(\ell; m_2, n_2)\) to be that subregion of \(R_1(\ell)\) for which \(|S_2| = m_2\) and \(|T_2| = n_2\). Define \(U(m_2, n_2)\) by

\[
U(m_2, n_2) = \bigcup_{\ell=0}^{N-1} R_1(\ell; m_2, n_2).
\]

We note that \(m_2\) and \(n_2\) vary over the ranges \([0, m^\varepsilon]\) and \([0, n^\varepsilon]\), respectively. Define \(U_0 = U(0, 0)\) and \(U_* = U \setminus U_0\).

Suppose \((\theta, \phi) \in R_1(0; m_2, n_2)\). If \(|\theta_j|\) and \(|\phi_k|\) are both less than or equal to \(5\delta\), then Lemma 3 is applicable to \(f(\theta_j + \phi_k)\). If one of the two is less than or equal to \(4\delta\) and the other exceeds \(5\delta\), then \(\cos(\theta_j + \phi_k) \leq \cos\delta\). Thus, there exists \(c_2 > 0\) such that for \((\theta, \phi) \in R_1(0; m_2, n_2)\)

\[
f(\theta_j + \phi_k) \leq \begin{cases} 
\exp\left(-A(\theta_j + \phi_k)^2 + \left(\frac{1}{12}A + A^2\right)(\theta_j + \phi_k)^4\right), & \text{if } (j, k) \in (S_0 \cup S_1) \times (T_0 \cup T_1), \\
\exp\left(-2c_2 \lambda(1 + \lambda)^{-1}\right), & \text{if } (j, k) \in (S_0 \times T_2) \cup (S_2 \times T_0), \\
1, & \text{otherwise.}
\end{cases}
\]

(4.5)

Since \((\theta, \phi) \in U\), we have \(|S_0| \geq m - m^\varepsilon\) and \(|T_0| \geq n - n^\varepsilon\); thus, the size of \((S_0 \times T_2) \cup (S_2 \times T_0)\) exceeds \(\frac{1}{2}(mn_2 + mn_2)\). Integrating the upper bound on \(|F|\) implied by (4.5),
we find
\[ \int_{R_1(0; m_2, n_2)} |F| \leq (2\pi)^{m_2+n_2} \binom{m}{m_2} \binom{n}{n_2} \exp(-c_2\lambda(1+\lambda)^{-1}(mn_2 + m_2n)) I'_2(m_2, n_2) \quad (4.6) \]
with
\[ I'_2(m_2, n_2) = \int_{-5\delta}^{5\delta} \cdots \int_{-5\delta}^{5\delta} \exp\left(-A \sum'\sum''(\theta_j + \phi_k)^2 + \left(\frac{1}{12} + A^2\right) \sum''(\theta_j + \phi_k)^4\right) d\theta'' d\phi'', \]
where the double-primes denote restriction to \( j \in S_0 \cup S_1 \) and \( k \in T_0 \cup T_1 \). The factor \((2\pi)^{m_2+n_2}\) comes from integrating over \( \theta_j \) for \( j \in S_2 \) and \( \phi_k \) for \( k \in T_2 \), while the binomial coefficients account for the choices of \( S_2 \) and \( T_2 \).

Set \( m' = m - m_2 \), \( n' = n - n_2 \). As implied by the notation, the value of \( I'_2(m_2, n_2) \) is independent of which specific variables constitute the sets \( S_0 \cup S_1 \) and \( T_0 \cup T_1 \).

Summing on \( m_2 + n_2 \geq 1 \) yields an upper bound for the integral of \( |F| \) over the region \( \bigcup_{m_2+n_2 \geq 1} R_1(0; m_2, n_2) \). Notice, however, that the transformation \( \theta_j \mapsto \theta_j - x_t \), \( \phi_k \mapsto \phi_k + x_t \) is a volume-preserving bijection of \( R_1(\ell; m_2, n_2) \) with \( R_1(0; m_2, n_2) \) which leaves \( F(\theta, \phi) \) invariant. Thus, introducing an additional factor of \( N \) to account for all values of \( \ell \), we have
\[ \int_{\mathcal{U}_s} |F| \leq N \sum_{m_2+n_2 \geq 1} \binom{m}{m_2} \binom{n}{n_2} (2\pi)^{m_2+n_2} \times \exp(-c_2\lambda(1+\lambda)^{-1}(mn_2 + m_2n)) I'_2(m_2, n_2). \quad (4.7) \]

We now analyze \( I'_2(m_2, n_2) \) more closely. Define \( \bar{\theta}' = (m')^{-1} \sum'' \theta_j \), \( \bar{\theta}_j = \theta_j - \bar{\theta}' \) for \( j \in S_0 \cup S_1 \), \( \bar{\phi}' = (n')^{-1} \sum'' \phi_k \), \( \bar{\phi}_k = \phi_k - \bar{\phi}' \) for \( k \in T_0 \cup T_1 \), \( \bar{\mu}' = \bar{\phi}' + \bar{\theta}' \) and \( \nu' = \bar{\phi}' - \bar{\theta}' \). In terms of \( \mu', \nu', \bar{\theta}_j, \bar{\phi}_k \) we have
\[ \sum''(\theta_j + \phi_k)^2 = m'n'\mu'^2 + n'\sum_j \bar{\theta}_j^2 + m'\sum_k \bar{\phi}_k^2 \]
and
\[ \sum''(\theta_j + \phi_k)^4 \leq 27m'n'\mu'^4 + 27n'\sum_j \bar{\theta}_j^4 + 27m'\sum_k \bar{\phi}_k^4. \]
The latter follows from the inequality \((x + y + z)^4 \leq 27(x^4 + y^4 + z^4)\) valid for all \( x, y, z \). It follows that
\[ I'_2(m_2, n_2) \leq \int_{-5\delta}^{5\delta} \cdots \int_{-5\delta}^{5\delta} \exp(m'n'g(\mu') + n'\sum_j g(\bar{\theta}_j) + m'\sum_k g(\bar{\phi}_k)) d\theta'' d\phi''. \quad (4.8) \]
For the moment we are thinking of \( \mu', \nu', \bar{\theta}_j \), and \( \bar{\phi}_k \) (\( 1 \leq j \leq m', 1 \leq k \leq n' \)) as functions of the variables of integration. Arbitrarily choose one element \( j_* \in S_0 \cup S_1 \) and one element \( k_* \in T_0 \cup T_1 \). Define \( S_3 = (S_0 \cup S_1) - \{j_*\} \) and \( T_3 = (T_0 \cup T_1) - \{k_*\} \). We claim that the...
inequality (4.8) remains valid when $\sum_j \equiv \sum_{j \in S_0 \cup S_1}$ is replaced by $\sum_{j \neq j_*}$, and similarly for $k_*$. To verify this assertion, note that since $|\theta_j| \leq 5\delta$ for all $j \in S_0 \cup S_1$ it must be the case that $|\hat{\theta}_j| \leq 5\delta$, too, and so $|\hat{\theta}_j| \leq 10\delta$. The claim now follows because, as is readily checked, $g(x) \leq 0$ for $|x| \leq 10\delta$.

As in the previous section, the next step is to make a change of variables to $\mu', \nu', \hat{\theta}_j (j \in S_3)$, and $\phi_k$ ($k \in T_3$). The region corresponding under this change of variables to $[-5\delta, 5\delta]^{m' + n'}$ is not exactly a product, but it is contained in the product $[-10\delta, 10\delta]^{m' + n'}$. (The argument given a few lines earlier to show $|\hat{\theta}_j| \leq 10\delta$ applies equally well to $|\hat{\theta}_j|, j \in S_3$.) Hence,

$$I_2'(m_2, n_2) \leq O(\delta) m' n' \int_{-10\delta}^{10\delta} \cdots \int_{-10\delta}^{10\delta} \exp\left( m'n' \mu' + n' \sum_{j \in S_3} g(\hat{\theta}_j) + m' \sum_{k \in T_3} \phi(\phi_k) \right) d\hat{\theta}_j d\phi_k d\mu' .$$

Here, the factor $O(\delta) m' n'$ comes from the integration of $\nu'$ over $[-10\delta, 10\delta]$ and the Jacobian $m'n'/2$ of the transformation. Now the integral splits into a product of $m' + n' - 1$ one-dimensional integrals. Each of these factors can be bounded by Lemma 4, and we find that

$$I_2'(m_2, n_2) \leq O(\delta) m' n' \left( \frac{\pi}{Am'n} \right)^{1/2} \left( \frac{\pi}{Am} \right)^{(n'-1)/2} \left( \frac{\pi}{An'} \right)^{(m'-1)/2} \times \exp \left( o(1) + O(1 + A^{-1})(m'/n' + n'/m') \right).$$

By $m' \sim m, n' \sim n$ and (1.1), the $\exp(\cdots)$ term above equals $n^{O(1)}$. Also,

$$\left( \frac{\pi}{Am} \right)^{(n'-1)/2} = \left( \frac{\pi}{Am} \right)^{(n-1)/2} n^{O(m_2 + n_2)} ,$$

and similarly for the other terms. Noting that $\delta N = O(1)$, we find from (4.7) that

$$\int_{U_*} |F| \leq \sum_{m_2 + n_2 \geq 1} n^{O(m_2 + n_2)} \exp(-c_3(1 + \lambda)^{-1}(mn_2 + nm_2)) I_0 .$$

Since $\lambda = \Omega((\log n)^{-1})$, the sum is dominated by the terms with $m_2 + n_2 = 1$. We conclude that

$$\int_{U_*} |F| = O(e^{-n^{1-\epsilon}}) I_0 . \quad (4.9)$$

It remains to consider $U_0 \cap R^c$. As was argued before in obtaining (4.6) we have in the $m_2 = n_2 = 0$ case

$$\int_{R^c(0,0) \cap R^c} |F| \leq \int_{[-5\delta, 5\delta]^{m+n} \cap R^c} \exp \left( -A \sum_{j} (\theta_j + \phi_k)^2 + \left( \frac{1}{12} A + A^2 \right) \sum_{j} (\theta_j + \phi_k)^4 \right) d\theta d\phi ,$$

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where now the sums are over all \( j, k \). Observe that the transformation \( \theta_j \mapsto \theta_j - x_t, \phi_k \mapsto \phi_k + x_t \) takes \( R_1(\ell; 0, 0) \cap R_c \) bijectively to \( R_1(0; 0, 0) \cap R_c \), since \( R, R_c \) are invariant under this mapping. Hence, the integral over \( U_0 \cap R_c \) of \(|F|\) is no larger than \( N \) times the integral on the right side of the last displayed relation.

The region of integration, \([-5\delta, 5\delta]^{m+n} \cap R_c\), is contained in the union \( \bigcup_{h=1}^{m+n+1} P_h \), where \( P_h \) equals the product \([-5\delta, 5\delta]^{m+n} \) intersected with that part of \( R_c \) where the \( h \)th inequality of the negation of definition (3.1) fails. As was done in the \( U \) case, we may bound \( \int_{P_h} \cdots \) by a product of \( m+n-1 \) one-dimensional integrals. However, in the present situation, when we make the change of variables to \((\mu', \nu', \theta_j, \phi_k)\), the transformed region of integration, albeit still contained in the product \([-10\delta, 10\delta]^{m+n-1}\) (we are omitting \( \nu' \) from the discussion), has the additional property that one of the latter intervals, the one corresponding to the index \( h \), is missing a neighborhood of size \( (1 + \lambda)^{-1}(mn)^{-1/2+2\varepsilon} \), \( (1 + \lambda)^{-1}n^{-1/2+\varepsilon} \), or \( (1 + \lambda)^{-1}m^{-1/2+\varepsilon} \) about 0.

Throughout \([-10\delta, 10\delta]\) we have \( g(x) \leq Ax^2/2 \). Because of the inequality

\[
\int_{(1+\lambda)^{-1}K^{-1/2+\alpha}} e^{-KAx^2/2} dx \leq \frac{\pi}{(KA)^{1/2}} \exp\left(-\frac{1}{5} \lambda(1 + \lambda)^{-1} K^{2\alpha}\right),
\]

\((K\) being one of \( mn, m, \) or \( n; \) and \( \alpha \) being, respectively, \( 2\varepsilon, \varepsilon, \) or \( \varepsilon \)), one of the \( m+n-1 \) factors is smaller than \( (\pi/Amn)^{1/2}, (\pi/Am)^{1/2}, \) or \( (\pi/An)^{1/2} \) (respectively, depending on \( K \)) times \( \exp\left(-\frac{1}{5} \lambda(1 + \lambda)^{-1} \min(m^{2\varepsilon}, n^{2\varepsilon})\right)\). (One may assure that neither of the indices \( j, k \) chosen for omission in the change of variables coincides with the index connected to the \( h \)th inequality of the negation of definition (3.1).) The factor \( \exp\left(-\frac{1}{5} \cdots \right) \) is, of course, independent of \( h \). Allowing \( N \) values of \( \ell \) and \( m+n+1 \) values of \( h \), we have

\[
\int_{U_0 \cap R_c} |F| \leq (m + n + 1)NO(\delta)n^{O(1)} \left(\frac{\pi}{Amn}\right)^{1/2} \left(\frac{\pi}{Am}\right)^{(n-1)/2} \left(\frac{\pi}{An}\right)^{(m-1)/2} \times \exp\left(-\frac{1}{5} \lambda(1 + \lambda)^{-1} \min(m^{2\varepsilon}, n^{2\varepsilon})\right) \leq e^{-n^\varepsilon} I_0.
\]

This inequality and (4.9) together imply (14), thus completing the proof of Theorem 2. \( \square \)

## 5 Proof of Theorem 1

Once again we remind the reader that \( F(\theta, \phi) \) denotes the integrand in equation (2.3) defining \( I(m, n) \). We continue to use \( I_0 \) for the quantity defined in equation (4.1).
Proof of Theorem 1. Let $a, b$ be given positive numbers satisfying $a + b < \frac{1}{2}$, and let $(m, n, s, t)$ be a sequence of 4-tuples such that $m, n \to \infty$ in such a way that hypothesis \( (1.1) \) is satisfied.

We first note that both \( (1.3) \) and Corollary 1 are equivalent to \( (1.2) \) for the respective ranges of $\lambda$ indicated. Consequently, to prove Theorem 1 and Corollary 1 it suffices to prove \( (1.3) \) for $\lambda = O(n^5)$ and Corollary 1 for $\lambda \geq n^5$.

We will divide the proof into two ranges of the parameter $\lambda$. First we assume that $\lambda = O(n^5)$, where Theorem 3 can be applied, then we use a different method for $\lambda \geq n^5$.

First suppose that $\lambda = O(n^5)$. As explained in Section 3, $M(m, s; n, t) = \frac{1}{2\pi}(\lambda^{-\lambda}(1 + \lambda)^{1+\lambda})^{mn}(\int_{R} F + O(1) \int_{R} |F|)$, \( (5.1) \)

where $R$ is defined by \( (3.1) \) and $R' \supseteq R$ is defined by \( (3.7) \). By \( (3.2) \) and Theorem 2, the first integral on the right side of \( (5.1) \) equals $2\pi \exp\left(\frac{1}{2} + O(n^{-b})\right)I_0$, and by Theorem 3, the second equals $O(n^{-1})I_0$. These values yield Theorem 1 for this case in the form given by \( (1.3) \).

For the remainder of the proof, we assume that $\lambda \geq n^5$. By the first part of this proof, Corollary 1 holds for $\lambda = O(n^5)$; hence,

$$
\frac{M(m, 2n^6; n, 2mn^5)}{M(m, n^6; n, mn^5)} = 2^{(m-1)(n-1)}(1 + O(n^{-b})).
$$

Let $s_0 = \text{lcm}(m, n)/m$, $t_0 = \text{lcm}(m, n)/n$, and let $P$ be the convex polytope of $m \times n$ real, nonnegative matrices whose rows sum to $s_0$ and whose columns sum to $t_0$. For every pair $(s, t)$ such that $ms = nt$ there is an integer $q$ such that $(s, t) = (qs_0, qt_0)$, and $M(m, s; n, t)$ equals the number of integer lattice points in the dilated polytope $qP$. The latter count, as a function of the positive integer $q$, is called the polytope enumerator for $P$, and is denoted $L_P(q)$, \( [1] \). Thus, we have

$$
M(m, s; n, t) = L_P(q).
$$

The polytope $P$, an example of a transportation polytope, has integral vertices \( [1] \), and so $L_P(q)$ is a polynomial in $q$, the Ehrhart polynomial. The degree of $L_P$ is $d$, the dimension of the polytope $P$; in our case, $d = (m-1)(n-1)$. By a theorem of Stanley (an algebraic proof first appeared as Proposition 4.5 in \( [37] \); a more geometric proof is given as Theorem 2.1 in \( [36] \)) the representation of $L_P(q)$ in a particular basis,

$$
L_P(q) = \sum_{i=0}^{d} h_{d-i}\left(\frac{q + i}{d}\right),
$$

where $h_{d-i}$ are the coefficients.
has all its coefficients \( h_0, \ldots, h_d \) nonnegative. For \( q \geq d \), we have the expansion

\[
\binom{q+i}{d} = \binom{q}{d} \prod_{j=0}^{d-1} \left( 1 + \frac{i}{q-j} \right) = \binom{q}{d} \sum_{X \subseteq [0,d-1]} \frac{i^{|X|}}{\prod_{j \in X} (q-j)},
\]

and so

\[
L_P(q) = \binom{q}{d} \sum_{X \subseteq [0,d-1]} \frac{g(|X|)}{\prod_{j \in X} (q-j)}
\]

where \( g(k) = \sum_{i=0}^{d} h_{d-i} k^i \geq 0 \). Note that \( P \), and therefore \( g(k) \) for each \( k \), depend only on \( m \) and \( n \). Also note that \( q/\prod_{j \in X} (q-j) \) is decreasing as a function of \( q \) for \( q \geq d \) and any non-empty \( X \subseteq [0,d-1] \).

Since \( q = \lambda \gcd(m,n) \), we conclude that there is a function \( \alpha(m,n,\lambda) \) such that

\[
M(m, \lambda n; n, \lambda m) = \left( \frac{\lambda \gcd(m,n)}{(m-1)(n-1)} \right) g(0) \left( 1 + \alpha(m,n,\lambda)/\lambda \right), \tag{5.3}
\]

\[
\alpha(m,n,\lambda) \geq 0 \quad \text{for } q \geq d \tag{5.4}
\]

\[
\alpha(m,n,\lambda) \text{ is decreasing in } \lambda \text{ for fixed } m, n \text{ and } q \geq d. \tag{5.5}
\]

For \( \lambda \geq n^5 \), since \( d = \tilde{O}(n^2) \),

\[
\binom{q}{d} = \frac{d}{q^n} (1 + \tilde{O}(n^{-1})).
\]

Hence, by (5.3)

\[
\frac{M(m, 2n^6; n, 2mn^5)}{M(m, n^6; n, mn^5)} = 2^{(m-1)(n-1)} \left( 1 + \tilde{O}(n^{-1}) \right) \frac{1 + \frac{1}{2} \alpha(m,n,2n^5)/n^5}{1 + \alpha(m,n,n^5)/n^5}. \tag{5.6}
\]

Comparing this to (5.2) and noting from (5.4, 5.5) that \( 0 \leq \alpha(m,n,2n^5) \leq \alpha(m,n,n^5) \), we conclude that \( \alpha(m,n,n^5) = O(n^{5-b}) \). This implies by (5.3) that \( \alpha(m,n,\lambda) = O(n^{5-b}) \) for \( \lambda \geq n^5 \). Using this information about \( \alpha(m,n,\lambda) \) with (5.3) gives

\[
\frac{M(m, \lambda n; n, \lambda m)}{M(m, n^6; n, mn^5)} = \left( \frac{\lambda}{n^5} \right)^{(m-1)(n-1)} (1 + O(n^{-b})).
\]

Combining this with Corollary \( \square \) for \( \lambda = n^5 \) shows that Corollary \( \square \) holds for all \( \lambda \geq n^5 \). \( \square \)
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