Intrinsic decoherence and classical-quantum correspondence in two coupled delta-kicked rotors

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We show that classical-quantum correspondence of center of mass motion in two coupled delta-kicked rotors can be obtained from intrinsic decoherence of the system itself which occurs due to the entanglement of the center of mass motion to the internal degree of freedom without coupling to external environment.

Classical-quantum correspondence in a classically chaotic system has been one of the most interesting problem in physics for a long time [1]. In quantum mechanics, the time evolution of a wave function follows a linear Schrödinger equation, and so there is no possibility of a sensitivity on the initial condition, a trademark of classical chaos. Also, chaotic diffusion is suppressed by quantum localization [2]. It has been revealed that some crossover time \( t_r = \log(I/h)/\lambda \) (\( I \) is a characteristic action and \( \lambda \) is a Lyapunov exponent) exists so that classical-quantum correspondence breaks down for \( t > t_r \) [3]. As \( h \to 0 \) the crossover time increases indefinitely and classicality is recovered. However, the problem is that \( \hbar \) is a nonzero constant and \( t_r \) is not sufficiently large considering its logarithmic dependence of \( \hbar \) [4].

Recently, the relation between decoherence and the classical-quantum correspondence has been investigated extensively [5,6]. Decoherence breaks the purity of initial superposition, which should be conserved in the absence of coupling to the environment, and thus only the partial fraction of whole Hilbert space, namely pointer states, are selected by the environment [4]. The dynamics of the system coupled to the environment shows the unique characteristics of the system independent of the coupling strength as long as it is not too large or small [10]. In other words, with appropriate coupling to environment, the Lyapunov exponent or entropy production rates, which are important physical quantities characterizing a chaotic system, can be reproduced quantum mechanically. However, there has been some debate on whether decoherence from environment is indispensable to obtain the classical-quantum correspondence for classically chaotic systems [5].

In this letter, we show that decoherence can occur naturally in composed system even when we ignore the coupling to the outer environment. The initially pure center of mass states become dynamically entangled to the internal degrees of freedom, which effectively acts like the environment. Let us consider a classical object governed by a Hamiltonian \( H_0 = \frac{P^2}{2M} + V(X) \), where \( M \) is the mass of the classical object. Since classical objects are composed of many particles, a complete Hamiltonian will be given by \( H = \sum \frac{p_i^2}{2m_i} + V_1(x_i) + V_2(x_1, x_2, \cdots) \), where \( p_i \) and \( x_i \) are momenta and coordinates of the constituents, respectively. We assume that the mass of each constituent \( m_i \) is equal to \( m \). If the force \( f_i = -dV_1(x_i)/dx_i \) is linear, we can ignore the motion of the individual constituent particles to describe the center of mass dynamics of the macroscopic classical object since \( M\dot{X} = \sum f_i = k \sum x_i = NkX \), where \( X, N, \) and \( k \) are a position of center of mass, the number of constituents, and a constant characterizing the linear force, respectively. If the force \( f_i \) is nonlinear, however, the description of the motion of the classical object by using only \( H_0 \) is not immediately clear. We already know from our experience, nevertheless, that the classical dynamics of the macroscopic object is well described by the Hamiltonian \( H_0 \). It means that \( \sum f_i \) should almost coincide with \( F(X) = -dV(X)/dX \) in values. Thus, we can ignore the difference between \( H \) and \( H_0 \) to consider the motion of classical objects. However, the minor differences between \( H \) and \( H_0 \) play an important role in quantum mechanics since the internal degree of freedom can induce decoherence on the center of mass motion. In this case, the correct classical-quantum correspondence of the center of mass motion will be obtained not from \( H_0 \) but from \( H \).

Only due to computational difficulty, in this letter we consider a two particle system. Even with this simple model, we have observed some evidence of the intrinsic decoherence originating from the entanglement to internal dynamics. We will show the delocalization of wavefunctions in momentum space and the coincidence of classical and quantum entropy production rates. In the previous study of two coupled quantum kicked tops, the entanglement of two initially decoupled kicked tops increases linearly in time with a rate which is a linear function of the sum of the positive Lyapunov exponents [13]. However, the classical limit has not been taken, and thus the classical-quantum correspondence has not been considered although signature of classical chaos has been observed for low quantum numbers. A model of two interacting spins was also studied to investigate the correspondence of classical and quantum Liouville dynamics.
But, this correspondence was considered to have nothing to do with the decoherence effect.

The main purpose of this letter is to elucidate that the intrinsic decoherence due to the entanglement of subsystems plays an important role to understand the classical-quantum correspondence of composed systems. For this purpose, we consider the center of mass motion of the delta-kicked rotors composed of two particles of which a governing Hamiltonian is given by

\[ H = \frac{1}{2m}(p_1^2 + p_2^2) + U(r_1, r_2) + k[\cos(r_1) + \cos(r_2)] \sum_i \delta(t - iT), \]  

(1)

where \( r_1 \) and \( r_2 \) are angle variables with range 2\( \pi \) radians and \( U(r_1, r_2) \) is the interaction potential which confines two particles within a distance \( w \). If the confinement width \( w \) goes to zero, then the system reduces to a usual delta-kicked rotor.

To investigate the center of mass motion, we introduce canonical transforms, \( R = (r_1 + r_2)/2, r = r_1 - r_2, P = p_1 + p_2, \) and \( p = (p_1 - p_2)/2 \). Then, we obtain the following Hamiltonian,

\[ H = \frac{P^2}{2M} + \frac{\mu^2}{2\mu} + U(r) + K \cos(R) \cos\left(\frac{T}{2}\right) \sum_i \delta(t - iT), \]  

(2)

where \( M = 2m, \mu = m/2, \) and \( K = 2k \). Here, \( U(r) \) corresponds to the confining potential with impenetrable walls at \( r = \pm w \). Delta-kicks described by the last term of Eq. (3) yield the interaction between the center of mass motion and the internal degree of freedom, i.e., the motion of the reduced mass \( \mu \).

The time evolution of a wave function \( \Psi(R, r, t) \) is given by simple maps. Between each kick occurring at \( t = iT \) (\( i \) is an integer), the center of mass motion and the internal motion evolve independently, so that we obtain

\[ \psi(R, r, NT + 0^-) = \exp\left(-i\frac{P^2}{2M} T\right) \frac{1}{\sqrt{M}} \psi(R, r, (N - 1)T + 0^+). \]  

(3)

The wave functions just before and after the delta-kick at \( t = NT \) are related by the following mapping:

\[ \psi(R, r, NT + 0^+) = \exp\left[-iK \cos(R) \cos\left(\frac{T}{2}\right)\right] \psi(R, r, NT + 0^-). \]  

(4)

Combining two maps given in Eq. (3) and (4), we numerically calculate the evolution of the wave function \( \Psi(R, r, t) \) for various \( \hbar \) and \( w \). For most cases, the number of basis states used for the motion of coordinate \( R \) and \( r \) are 16384 and 256 respectively, while 32768 and 512 basis are used respectively for small \( \hbar \). The initial condition is chosen to be the ground state \( \Psi(R, r, 0) = 1/\sqrt{w} \cos(\pi r/2w) \).

Classical evolution is obtained from a four dimensional map for \( R, P, r \) and \( p \) derived from Hamiltonian (3). We consider an ensemble of particles which are distributed from \( R = 0 \) to \( R = 2\pi \) uniformly with \( P = 0 \), and from \( r = -w \) to \( r = w \) with a probability \( \cos^2(\pi r/2w)/w \) with \( p = \pm p_0 = \pm(\pi\hbar)/(2w) \). From this initial condition, the classical evolution is simulated, and the ensemble average of the normalized variance of center of mass momentum, \( \Delta^2 = 2\left(\langle P^2 \rangle - \langle P \rangle^2\right)/MK^2 \) is computed.

As a reference, let us mention the case with a single delta-kicked rotor governed by the Hamiltonian \( H = P^2/2M + K \cos(R) \sum_i \delta(t - iT) \), which corresponds to the case with \( w = 0 \). For \( K = 5 \), the system is fully chaotic, and classical kinetic energy increases diffusively. But the diffusion is suppressed quantum mechanically, and the classical-quantum correspondence breaks down, which is the well-known dynamical localization (3).

Now, we consider two delta-kicked rotors. The inset in Fig. 1 shows the differences of classical momentum variances between single and two delta-kicked rotors for various \( w \). As \( w \) is decreased, the difference between them vanishes. Meanwhile, shown in Fig. 1 are the differences between the classical and the quantum variances for various \( w \) with \( \hbar = 0.07 \). One clearly sees that the quantum localization breaks down when \( w \) is increased. The difference of \( \Delta_q \) and \( \Delta_{qm} \) nearly disappears for \( w = 0 \). The break-down of quantum localization, i.e., delocalization, is directly visible in the momentum distribution function in Fig. 2, where the probability distribution of the center of mass momentum \( P \) are shown at \( n = 500 \) for several \( \hbar \) and \( w \). The exponential localization is changed into a rather broad Gaussian-like profile as we decrease \( \hbar \) or increase \( w \). Let us note that the delocalization of wavefunctions alone is not enough to prove the occurrence of decoherence. In fact, the delocalization was also observed in the study of two interacting particles (TIP) in a random potential [4,13]. However, it was not attributed to the decoherence.

Next, we consider the reduced density matrix for the center of mass motion in order to show that the observed classical-quantum correspondence is ascribed to the intrinsic decoherence caused from the interaction with the internal degree of freedom. The reduced density matrix \( \rho_R(R_1, R_2) \) is given by \( Tr_r(\rho) = \sum_r \psi^*(R_1, r) \psi(R_2, r) \). As a quantitative measure of decoherence, we calculate the linear entropy \( s_l = Tr(\rho_R - \rho_R^2) / Tr(\rho_R) \). Note that for a pure state \( s_l = 0 \), while for a maximum decoherence \( s_l = 1 \). Figure 3 shows that for large \( w \) the entropy \( s_l \) rapidly approaches 1, i.e., a maximum decoherence. As we decrease \( w \), the entropy \( s_l \) shows rather reduced values and slowly increases in time. In fact, an energy level spac-
ing of the internal dynamics described by $p^2/2\mu + U(r)$ is proportional to $1/w^2$, which means the smaller $w$, the larger the level spacing. For a given perturbation determined by $K$ it is more difficult to excite the internal dynamics in the case of large spacing, which leads to effective decoupling between the center of mass motion and the internal degree of freedom, and eventually to the decrease of decoherence. This is consistent with the previous result that the break-down of the localization and the classical-quantum correspondence is easily obtained for large $w$, i.e., strong decoherence.

The $\cos(r/2)$ in the last term of Hamiltonian (2) can be regarded as an amplitude noise of the kick onto the center of mass motion. As $w$ decreases, the noise is reduced and so is the effect of decoherence. Ott at al. studied the effect of noise on the single delta-kicked rotor, which showed that, for moderate noise and $\bar{\nu}$, the momentum results (dots) show $0.5 \log(n)$ dependence, consistent with the classical prediction. These results also confirm the above proposition that the term $\cos(r/2)$ in the Hamiltonian (2) can be treated as noise and thus induces decoherence.

Finally, we consider the classical and the quantum entropy production rates of which coincidence has been the criteria for the classical-quantum correspondence. Quantum mechanically, the von Neumann entropy is given by $S_{vn} = Tr[\rho_R \log(\rho_R)]$. Due to the difficulty of computing the eigenvalues of large matrix, we use a smaller number of basis states than that used in the previous simulation, 4096 for the motion of $R$, and 256 for $r$. We choose $\bar{h} = 0.07$ and can obtain the quantum evolution up to $n = 100$ with this smaller number of basis. The time evolution of the von Neumann entropy consists of two different regimes [17] as shown in Fig. 5. After the initial transient, the von Neumann entropy increases logarithmically on time, which exactly corresponds to its classical counterpart [11]. Let us consider the phase space $(R,P)$ for the center of mass motion. Classical distributions are uniform in $R$ and Gaussian in $P$. The variance of momentum increases diffusively. As a result, the classical entropy is given by $S_{cl} \sim 0.5 \log(n)$. In Fig. 5, the quantum results (dots) show $0.5 \log(n)$ dependence, consistent with the classical prediction.

In summary, we have studied the dynamics of the center of mass motion of two coupled delta-kicked rotors and examined its classical-quantum correspondence using the momentum distribution and the entropy production rate.
FIG. 1. The differences between classical and quantum momentum variances, $\Delta_{\text{cl}} - \Delta_{\text{qm}}$, are plotted with $M = 1$, $\mu = 0.25$, $K = 5$, $T = 1$, and $\bar{\hbar} = 0.07$. From top to bottom, $w = 0.0$ (i.e. single kicked rotor), 0.1, 0.2, 0.3, 0.4, 0.5, 0.6 and 0.7. (Inset) The difference between variances of single and two coupled delta-kicked rotors, $\Delta_{\text{single}} - \Delta_{\text{two}}$ for various $w$ are shown. From top to bottom, $w = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ and 0.8.

FIG. 2. Momentum distribution functions. (a) $\bar{\hbar} = 0.25$, $w = 0.2$ and 1. (b) $w = 0.2$, $\bar{\hbar} = 0.1$ and $\bar{\hbar} = 0.25$. Other parameters are the same as in Fig. 1.

FIG. 3. Linear entropies for various $w$. From bottom to top, $w = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$ and 0.7. Other parameters are the same as in Fig. 1.

FIG. 4. Quantum diffusion coefficient $D_{\text{qm}}$ as a function of $(wK/\bar{\hbar})$ with $\bar{\hbar} = 0.25$. The solid line corresponds to $D_{\text{qm}} \propto (wK/\bar{\hbar})^4$. Other parameters are the same as in Fig. 1.

FIG. 5. The von Neumann entropies obtained from the reduced density matrix $\rho_R$ with $\bar{\hbar} = 0.07$. For a reference, the $0.5 \log(n)$ dependence of $S_{\text{cl}}$ is represented by the solid line. From top to bottom, $w = 0.2, 0.4, 0.6, 0.8$ and 1.0.