REGULARITY OF ASYMPTOTICALLY CONICAL RICCI-FLAT
KÄHLER METRICS

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Abstract. Using methods of A. Grigor’yan and L. Saloff-Coste we prove that
on a manifold with a conical end the heat kernel has a Gaussian bound. This
result is applied to asymptotically conical Kähler manifolds. It is a result of
the author and R. Goto that a crepant resolution \( \pi : Y \to X \) of a Ricci-flat
Kähler cone \( X \) admits a Ricci-flat Kähler metric asymptotic to the cone metric
in every Kähler class. We prove the sharp rate of convergence of the metric
to the cone metric. For compact Kähler classes this is the same as for the
Ricci-flat ALE metrics of P. Kronheimer and D. Joyce.

1. Introduction

This article considers Riemannian manifolds with a conical end. A metric cone
is a manifold \((C(S), g)\) with \(C(S) = \mathbb{R}_{>0} \times S\) and \(g = dr^2 + r^2 g_S\). We consider
manifolds with an end for which the metric is approximated by a cone metric. This
can be considered a generalization of ALE manifolds. But in this case the compact
manifold \((S, g_S)\) is arbitrary. We will also consider Kähler manifolds with an end
approximated by a Kähler cone. If \((C(S), g)\) is Kähler, then \((S, g_S)\) is a Sasaki
manifold. So in this case \((S, g_S)\) is far from arbitrary, and such manifolds have
been studied extensively [4]. In particular we will consider Ricci-flat asymptotically
conical Kähler manifolds.

Ricci-flat Kähler manifolds with an asymptotically conical end, and in particular
resolutions of a Ricci-flat Kähler cone \((C(S), g)\), have been of particular interest
recently due to their relevance to the AdS/CFT correspondence. See [21] for the
construction of many explicit examples which are resolutions of Ricci-flat Kähler
cones. See also [22, 23] for more on the relevance of these manifolds to AdS/CFT.

Asymptotically conical Ricci-flat Kähler manifolds have been extensively studied
by solving the Monge-Ampère equation in various cases. There are the existence results of G. Tian and S.-T. Yau [28], and independently S. Bando and R.
Kobayashi [2], on Ricci-flat Kähler metrics on quasi-projective manifolds \( X \setminus D \).
And there are the results of D. Joyce on the existence of Ricci-flat ALE metrics [16].
The author [31] and R. Goto [11] have proved existence results on resolutions of
Ricci-flat Kähler cones. This article shows what solutions to the Monge-Ampère
equation behave essentially as in the ALE case due to D. Joyce. In particular, we
have the following for resolutions of cones.

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manifold.
Theorem 1.1. Let $\pi : \hat{X} \to X = C(S) \cup \{o\}$ be a crepant resolution of a Ricci-flat Kähler cone $(X, \bar{g})$ of complex dimension $m$. Then in every Kähler class in $H^2_c(\hat{X}, \mathbb{R}) \subset H^2(\hat{X}, \mathbb{R})$ there is a unique Ricci-flat Kähler metric $g$ asymptotic to $\bar{g}$ as follows. There exists an $R > 0$ such that for each $k \geq 0$
\begin{equation}
|\nabla^k(\pi_* g - \bar{g})| = O(r^{-2m-k}) \quad \text{on } \{x \in C(S) : r(x) > R\},
\end{equation}
where $\nabla$ and the point-wise norm are with respect to $\bar{g}$.

Furthermore, in every Kähler class in $H^2(\hat{X}, \mathbb{R}) \setminus H^2_c(\hat{X}, \mathbb{R})$ there is a Ricci-flat metric $g$ asymptotic to $\bar{g}$. In this case there exists an $R > 0$ such that for each $k \geq 0$
\begin{equation}
|\nabla^k(\pi_* g - \bar{g})| = O(r^{-2-k}) \quad \text{on } \{x \in C(S) : r(x) > R\}.
\end{equation}

Remark 1.2 Both rates of convergence in (1) and (2) are sharp. The contribution of this article is in proving the sharp convergence in (1). The author proved a weaker version in [31] where the exponent in (1) is $-2m + \delta$, $\delta > 0$. This was also proved by R. Goto [11] along with a clever proof of the existence for non-compact Kähler classes.

The first part of this article considers the more general case of arbitrary real manifolds with a conical end. Using methods of A. Grigor’yan and L. Saloff-Coste we prove that every such manifold satisfies the parabolic Harnack inequality, and as a consequence we have a Gaussian bound on the heat kernel. This follows from the scale-invariant Poincaré inequality, and the volume doubling condition, which is proved using a discretization technique. We use this to prove some results on the Laplacian on weighted Hölder spaces that will be needed later.

In the second part we consider asymptotically conical Kähler manifolds. The main result is a version of the Calabi conjecture for asymptotically conical Kähler manifolds. The proof due to D. Joyce [16] for ALE Kähler manifolds goes through as is using the Sobolev inequality and the results on the Laplacian on weighted Hölder space in this context. The result is that solutions to the Monge-Ampère equation on asymptotically conical Kähler manifolds behave as on ALE Kähler manifolds.

The final section gives an overview of some examples of Ricci-flat asymptotically conical manifolds given by Theorem 1.1. One is considering cones over Sasaki-Einstein manifolds which have crepant resolutions. Examples can be found from hypersurface singularities. Many such examples of Sasaki-Einstein manifolds are known (cf. [3, 5, 4]). We also briefly discuss the toric case. The existence problem of Sasaki-Einstein metrics in this case is solved [10]. Therefore it is an easy source of examples.

Theorem 1.1 does not settle the existence problem of asymptotically conical Ricci-flat Kähler metrics. More generally, one can consider crepant resolutions $\pi : \hat{X} \to X$ of more general affine varieties $X$ such that $\hat{X}$ is a quasi-projective variety with an end which is diffeomorphic to a cone, but not holomorphically.

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Notation. We will use ‘⊂’ to denote set inclusion, proper and otherwise. The geodesic ball centered at \( x \in M \) of radius \( r \) will be denoted by \( B(x, r) \), and we will use \( V(x, r) \) to denote its volume with respect to the Riemannian measure.

2. Asymptotically conical manifolds and

2.1. Asymptotically conical manifolds. We cover some analysis on asymptotically conical manifolds. This will be used in the sequel in the proof of the Calabi conjecture. But it may be of independent interest.

Let \( (X, g) \) be a complete Riemannian manifold. Using the metric \( g \) and Levi-Civita connection \( \nabla \) we have the Sobolev spaces \( L^q_k(X) \), the \( C^k \)-spaces \( C^k(X) \), and the Hölder spaces \( C^{k,\alpha}(X) \). But we will also need weighted Sobolev spaces and weighted Hölder spaces.

Let \( o \in X \) be a fixed point and \( d(o, x) \) the Riemannian distance to \( x \in X \). Then we define a weight function \( \rho(x) = (1 + d(o, x)^2)^{1/2} \).

**Definition 2.1.** Let \( q \geq 1, \beta \in \mathbb{R}, \) and \( k \) a nonnegative integer. We define the weighted Sobolev space \( L^q_{k,\beta} \) to be the set of functions \( f \) on \( X \) which are locally integrable and whose weak derivatives up to order \( k \) are locally integrable and for which the norm

\[
\| f \|_{L^q_{k,\beta}} := \left( \sum_{j=0}^{k} \int_X |\rho^{-\beta} \nabla^j f|^q \rho^{-\alpha} d\mu_g \right)^{1/q}
\]

is finite. Then \( L^q_{k,\beta} \) is a Banach space with this norm.

**Definition 2.2.** For \( \beta \in \mathbb{R} \) and \( k \) a nonnegative integer we define \( C^k_\beta(X) \) to be the space of continuous functions \( f \) with \( k \) continuous derivatives for which the norm

\[
\| f \|_{C^k_\beta} := \sum_{j=0}^{k} \sup_X |\rho^{-\beta} \nabla^j f|
\]

is finite. Then \( C^k_\beta(X) \) is a Banach space with this norm.

Let \( \text{inj}(x) \) be the injectivity radius at \( x \in X \), and \( d(x, y) \) the distance between \( x, y \in X \). Then for \( \alpha, \gamma \in \mathbb{R} \) and \( T \) a tensor field define

\[
[T]_{\alpha,\gamma} := \sup_{\substack{x \neq y \\text{ \text{d}(x,y) < \text{inj}(x)}}} \left[ \min(\rho(x), \rho(y))^{-\gamma} \cdot \frac{|T(x) - T(y)|}{d(x, y)^\alpha} \right],
\]

where \( |T(x) - T(y)| \) is defined by parallel translation along the unique geodesic between \( x \) and \( y \).

For \( \alpha \in (0, 1) \) define the weighted Hölder space \( C^{k,\alpha}_\beta(X) \) to be the set of \( f \in C^k_\beta(X) \) for which the norm

\[
\| f \|_{C^{k,\alpha}_\beta} := \| f \|_{C^k_\beta} + \| \nabla^k f \|_{\alpha,\beta-k-\alpha}
\]

is finite. Then \( C^{k,\alpha}_\beta(X) \) is a Banach space with this norm.

Define \( C^\infty_\beta(X) \) to be the intersection of the \( C^k_\beta(X) \), for \( k \geq 0 \).

It will be convenient to use a different weight function \( \rho(x) \) on the manifolds we consider but it will define equivalent norms. See \( [7] \) for an introduction to the theory of weighted Hölder spaces.
Definition 2.3. Let \((S, g_S)\) be a compact Riemannian manifold. The cone over \((S, g_S)\) is the Riemannian manifold \((C(S), g)\) with \(C(S) = \mathbb{R}_{>0} \times S\) and \(g = dr^2 + r^2g_S\) where \(r\) usual coordinate on \(R_{>0}\).

We will sometimes consider \((C(S), g)\) with the apex \(o \in C(S)\) at \(r = 0\). This is singular at \(o \in C(S)\) unless \(S = S^{n-1}\) is the sphere with the round metric.

Definition 2.4. Let \((C(S), g_0)\) be a metric cone. Then \((X, g)\) is asymptotically conical of order \((\delta, k+\alpha)\), if there is compact subset \(K \subset X\), a compact neighborhood \(o \in K_0 \subset C(S)\), and diffeomorphism \(\phi : X \setminus K \to C(S) \setminus K_0\) so that

\[
|\phi_* g - g_0| \in C_S^{k,\alpha} \quad \text{on} \quad C(S) \setminus K_0.
\]

We will abbreviate this by \(\text{AC}(\delta, a + \alpha)\), and always \(\delta < 0\).

Suppose \(K_0 \subset D_{r_0} = \{(r, s) \in C(S) : r < r_0\}\), and assume \(r_0 \geq 2\). Then a smooth extension of \(\phi^* r : X \setminus \phi^{-1}(D_{r_0}) \to [2, \infty)\) to \(\rho : X \to [1, \infty)\) is a radius function of the AC manifold \((X, g)\).

Remark 2.5 It will be convenient sometimes to define weighted Hölder and Sobolev spaces using the radius function as the weight function \(\rho\). In other cases we will use \(\rho(x) = (1 + d(o, x)^2)^{1/2}\) for \(o \in K\). This will mostly be a matter of convenience.

If \((X, g)\) is \(\text{AC}(\delta, 0), \delta > 0\), it is not difficult to check that \(c^{-1}\rho \leq \rho \leq c\rho\) for \(c > 0\). Thus the weighted norms are equivalent. We will denote such a relation between functions by \(\rho \sim \tilde{\rho}\).

### 2.2. The Sobolev inequality.
We give a proof of the Sobolev inequality on asymptotically conical manifolds. Recall that metrics \(g\) and \(\tilde{g}\) on \(X\) are quasi-isometric if there is a \(c > 0\) so that

\[
(4) \quad c^{-1}g_x(W, W) \leq \tilde{g}_x(W, W) \leq cg_x(W, W), \quad \forall x \in X \text{ and } \forall W \in T_xX.
\]

Theorem 2.6. Let \((X, g)\) be \(\text{AC}\) of order \((\delta, 0), \delta < 0\), or merely quasi-isometric to an AC manifold. Then there is a constant \(C > 0\) so that we have the Sobolev inequality

\[
(5) \quad \|f\|_{n/(n-1)} \leq C\|\nabla f\|_1, \quad \forall f \in C_0^\infty(X).
\]

And easy argument with the Hölder inequality gives the following.

Corollary 2.7. For any real \(p, 1 \leq p < n\) we have

\[
(6) \quad \|f\|_{np/(n-p)} \leq C\|\nabla f\|_p, \quad \forall f \in C_0^\infty(X).
\]

Note that \((5)\), and most of what follows in this section, is stable under quasi-isometries. So all that is essential in Theorem 2.6 is that the end of \((X, g)\) is quasi-isometric to a cone.

We prove Theorem 2.6 with a discretization procedure used in [13] to prove Poincaré inequalities and generalized in [23] to prove more general Poincaré-Sobolev inequalities. A proof of [13] is given in [28], but our result is more general and the proof is simpler. Let \(\mu\) denote the Riemannian measure on \((X, g)\).

Definition 2.8. Let \(A \subset A^\#\) be subsets of \(X\). A family \(\mathcal{U} = (U_i, U_i^*, U_i^\#)_{i \in I}\) consisting of subsets of \(X\) having finite measure is said to be a good covering of \(A\) in \(A^\#\) if the following is true:

\begin{enumerate}[(i)]
  \item \(A \subset \bigcup_i U_i \subset \bigcup_i U_i^\# \subset A^\#\);
\end{enumerate}
(ii) \( \forall i \in I, U_i \subset U_i^* \subset U_i^\# \);
(iii) \( \exists Q_1, \forall i_0 \in I, \text{Card}\{i \in I : U_i^\# \cap U_i^\# \neq \emptyset\} \leq Q_1 \);
(iv) For every \( (i, j) \in I^2 \) satisfying \( U_i \cap U_j \neq \emptyset \), there is an element \( k(i, j) \) such that \( U_i \cup U_j \subset U_{k(i,j)}^* \);
(v) There exists a \( Q_2 \) such that for every \( (i, j) \in I^2 \) with \( U_i \cap U_j \neq \emptyset \) we have \( \mu(U_{k(i,j)}) \leq Q_2 \min(\mu(U_i), \mu(U_j)) \).

Given a Borel set \( U \) with finite \( \mu \)-measure and a \( \mu \)-integrable function \( f \), denote by \( f_U \) the mean value of \( f \) on \( U \):

\[
    f_U = \frac{1}{\mu(U)} \int_U f d\mu.
\]

Given any good covering we have an associated weighted graph \((\mathcal{G}, m)\) as follows.

**Definition 2.9.** Let \( \mathcal{U} = (U_i, U_i^*, U_i^\#)_{i \in I} \) be a good covering of \( A \) in \( A^\# \). The associated weighted graph \((\mathcal{G}, m)\) has vertices \( \mathcal{V} = I \) and edges \( \mathcal{E} = \{\{i, j\} \subset \mathcal{V} : i \neq j, U_i \cap U_j \neq \emptyset\}\).

Measures, both denoted \( m \), are defined on \( \mathcal{V} \) and \( \mathcal{E} \) as follows:

(i) \( \forall i \in \mathcal{V}, m(i) = \mu(U_i) \);
(ii) \( \forall \{i, j\} \in \mathcal{E}, m(i, j) = \max(m(i), m(j)) \).

We will patch together Sobolev inequalities on the subsets \((U_i, U_i^*, U_i^\#)\) of a good covering using discrete Sobolev inequalities on the associated weighted graph \((\mathcal{G}, m)\).

**Definition 2.10.** Let \( 1 \leq p < \infty \), and suppose \( p < \nu \leq \infty \). We say that a good covering \( \mathcal{U} \) satisfies a continuous \( L^p \) Sobolev inequality of order \( \nu \) if there exists a constant \( S_c \) such that for every \( i \in I \)

\[
    \left( \int_{U_i} |f - f_{U_i}|^{\frac{\nu}{p}} d\mu \right)^{\frac{p}{\nu}} \leq S_c \int_{U_i^*} |\nabla f|^p d\mu, \quad \forall f \in C^\infty(U_i^*),
\]

and

\[
    \left( \int_{U_i^*} |f - f_{U_i}|^{\frac{\nu}{p}} d\mu \right)^{\frac{p}{\nu}} \leq S_c \int_{U_i^\#} |\nabla f|^p d\mu, \quad \forall f \in C^\infty(U_i^\#).
\]

**Definition 2.11.** Given \( 1 \leq p < \nu \leq \infty \), we say that the weighted graph \((\mathcal{G}, m)\) satisfies a discrete \( L^p \) Sobolev-Dirichlet inequality of order \( \nu \) if there exists a constant \( S_d \) such that for every \( f \in L^p(\mathcal{V}, m) \)

\[
    \left( \sum_{i \in \mathcal{V}} |f(i)|^{\frac{\nu}{p}} m(i) \right)^{\frac{p}{\nu}} \leq S_d \sum_{i,j \in \mathcal{E}} |f(i) - f(j)|^p m(i, j).
\]

Suppose \((\mathcal{G}, m)\) is a finite weighted graph. Then for \( f \in \mathbb{R}^\nu \), denote by \( m(f) \) the mean of \( f \)

\[
    m(f) = \frac{1}{m(\mathcal{V})} \sum_{i \in \mathcal{V}} f(i).
\]
Definition 2.12. Given $1 \leq p < \nu \leq \infty$, we say that a finite weighted graph $(G, m)$ satisfies a discrete $L^p$ Sobolev-Neumann inequality of order $\nu$ if there exists a constant $S_d$ such that for every $f \in \mathbb{R}^N$, $N = |\mathcal{V}|$,

$$\left( \sum_{i \in \mathcal{V}} |f(i) - m(f)|^{\frac{p}{\nu}} m(i) \right)^{\frac{\nu}{p}} \leq S_d \sum_{i,j \in \mathcal{E}} |f(i) - f(j)|^p m(i,j).$$

Note that the $L^p$ Sobolev inequality of order $\nu = \infty$ is the $L^p$ Poincaré inequality.

The following theorem is crucial in the proof of the Sobolev inequality on $(X, g)$. We also state the “Neumann” analogue of this theorem since we will use it in a proof of a Poincaré inequality on $(X, g)$.

Theorem 2.13 (23). Suppose $1 \leq p < \nu \leq \infty$. If a good covering $\mathcal{U}$ of $A$ in $A^\#$ satisfies the continuous $L^p$ Sobolev inequality of order $\nu$ (Definition 2.10) and the discrete $L^p$ Sobolev-Dirichlet inequality of order $\infty$ (Definition 2.11), then the following Sobolev-Dirichlet inequality is true:

$$\left( \int_A |f|^{\frac{p\nu}{p\nu - \nu}} \, d\mu \right)^{\frac{p\nu - \nu}{p\nu}} \leq S \int_{A^\#} |\nabla f|^p \, d\mu, \quad \forall f \in C^\infty_c(A).$$

Furthermore, one can choose $S = S_d Q_1^{2p - 1 + \frac{\nu}{p\nu - \nu}} (1 + S_d Q_2 (2^p Q_1^{\nu - p}))^{\frac{\nu}{p\nu - \nu}}$.

Theorem 2.14 (23). Suppose $1 \leq p < \nu \leq \infty$. If a finite good covering $\mathcal{U}$ of $A$ in $A^\#$ satisfies the continuous $L^p$ Sobolev inequality of order $\nu$ (Definition 2.10) and the discrete $L^p$ Sobolev-Neumann inequality of order $\infty$ (Definition 2.13), then the following Sobolev-Neumann inequality is true:

$$\left( \int_A |f - f_A|^{\frac{p\nu}{p\nu - \nu}} \, d\mu \right)^{\frac{p\nu - \nu}{p\nu}} \leq S \int_{A^\#} |\nabla f|^p \, d\mu, \quad \forall f \in C^\infty_c(A^\#).$$

Furthermore, one can choose $S = S_d Q_1^{2p - 1 + \frac{\nu}{p\nu - \nu}} (1 + S_d Q_2 (2^p Q_1^{\nu - p}))^{\frac{\nu}{p\nu - \nu}}$.

We will need prove discrete Sobolev inequalities to apply these theorems. The discrete Sobolev-Dirichlet inequality in Definition 2.11 follows from an isoperimetric inequality on the graph $(G, m)$.

Definition 2.15. Given a graph $G$, we define the boundary $\partial \Omega$ of a subset $\Omega \subset \mathcal{V}$ as

$$\partial \Omega := \{\{i, j\} \in \mathcal{E} : \{i, j\} \cap \Omega \neq \emptyset \text{ and } \{i, j\} \cap (\mathcal{V} \setminus \Omega) \neq \emptyset\}.$$ 

The following result is completely analogous to the situation with continuous Sobolev inequalities.

Proposition 2.16. Let $(G, m)$ be an infinite weighted graph and fix $1 < \nu \leq \infty$. Then the $L^1$ Sobolev inequality of order $\nu$

$$\left( \sum_{i \in \mathcal{V}} |f(i)|^{\frac{\nu}{\nu - 1}} m(i) \right)^{\frac{\nu - 1}{\nu}} \leq C \sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)| m(i,j), \quad \forall f \in L^1(\mathcal{V}, m),$$

is equivalent to the isoperimetric inequality of order $\nu$

$$m(\Omega)^{\frac{\nu}{\nu - 1}} \leq C m(\partial \Omega), \quad \forall \Omega \subset \mathcal{V} \text{ with } m(\Omega) < \infty.$$
In the following section we will need a good bound on the constant in the discrete Sobolev-Neumann inequality. Suppose \((G, m)\) is a finite weighted graph with \(|V| = N\). The spectral gap \(\lambda(G, m)\) is defined by

\[
\lambda(G, m) = \inf \left\{ \frac{\sum_{(i,j)\in E} |f(i) - f(j)|^2 m(i,j)}{\sum_{i\in V} |f(i) - m(f)|^2 m(i)} : f \in \mathbb{R}^N \setminus \{0\} \right\}.
\]

Clearly, if \(S_d\) is the minimum constant so that the discrete Sobolev-Neumann \(L^2\) inequality of order \(\infty\) holds, then \(S_d = 1/\lambda(G, m)\).

The Cheeger constant \(h(G, m)\) is defined by

\[
h(G, m) = \inf \left\{ \frac{\sum_{(i,j)\in E} |f(i) - f(j)| m(i,j)}{\inf_{c} \sum_{i\in V} |f(i) - c|^2 m(i)} : f \in \mathbb{R}^N \right\}.
\]

It is well-known (cf. [26]) that the Cheeger constant is equivalent to the isoperimetric inequality

\[
h(G, m) = \inf \left\{ \frac{m(\partial U)}{m(U)} : U \subset V, 0 < m(U) \leq \frac{1}{2} m(V) \right\}.
\]

And we can compare the constants \(h(G, m)\) and \(\lambda(G, m)\) by

\[
\frac{h^2(G, m)}{8m_0} \leq \lambda(G, m) \leq h(G, m),
\]

where

\[
m_0 := \max_{i\in V} \left\{ \frac{1}{m(i)} \sum_{j\in V} m(i, j) \right\}.
\]

We state the Euclidean scale-invariant Sobolev-Poincaré inequalities. They will be used in the proof of Theorem [27] and also in the proof of the scale invariant Poincaré inequality on \((X, g)\). Let \(B_r = B(o, r) \subset \mathbb{R}^n\) be the ball of radius \(r > 0\). For \(1 \leq p \leq n\) and \(n \leq \nu \leq \infty\) there exists a constant \(S_{p, \nu}\), only depending on \(p, \nu\), so that

\[
\left( \int_{B_r} |f - f_r|^\frac{np}{n-p} d\mu \right)^{\frac{n-p}{np}} \leq S_{p, \nu} r^{1 - \frac{p}{n}} \left( \int_{B_r} |\nabla f|^p d\mu \right)^{\frac{1}{p}}, \quad \forall f \in C^\infty(B_r).
\]

We may assume that the end of \((X, g)\) has the conical metric. That is, if \(\rho\) is a radius function, then there is an \(r_0 \geq 2\) so that \(\phi : X \setminus K_{r_0} \to C(S) \setminus \bar{D}_{r_0}\), with \(K_{r_0} = \{ x \in X : \rho(x) \leq r_0 \}\) and \(D_{r_0} = \{ x \in C(S) : r(x) \leq r_0 \}\), is an isometry.

**Lemma 2.17.** Fix \(R \geq r_0, \kappa > 1\) and consider the annulus \(A = A(R, \kappa R) = D_{\kappa R} \setminus D_R\). Then if we let \(A_\delta\) be the \(\delta R\)-neighborhood of \(A\), with \(0 < \delta < 1\), sufficiently small, there is a constant \(C > 0\) independent of \(R\) so that

\[
\left( \int_{A_\delta} |f - f_A|^\frac{n}{n-1} d\mu \right)^{\frac{n-1}{n}} \leq C \int_{A_\delta} |\nabla f| d\mu, \quad \forall f \in C^\infty(A_\delta).
\]

**Proof.** Set \(s = \delta R\), and let \(\{x_i\}_{i \in I}\) be a maximal subset of \(A\) such that the distance between any two of its elements is at least \(s\). Set \(V_i = B(x_i, s)\) and \(V_i^* = V_i^\# = B(x_i, 3s)\). Then \((V_i, V_i^*, V_i^\#)\) is a finite good covering of \(A\) in \(A_\delta\). Conditions [III] and [IV] are satisfied with \(Q_1 = \text{Card}(I)\) and \(Q_2 = \frac{\max\{\mu(V_i^\#) : i \in I\}}{\min\{\mu(V_i) : i \in I\}}\). And in [IV] we may take \(k(i, j) = i\).
We will apply Theorem \ref{2.14} The exponential map gives coordinates \( \phi_i : B_\nu \to V_i \) and \( \phi^*_i : \nu B_\nu \to V^*_i \). Then if \( g_0 \) is the flat metric on \( B_\nu \), by finiteness there is a \( c > 0 \) so that \( c^{-1} g_0 \leq (\phi^*_i)^* g \leq c g_0 \) for all \( i \in I \).

We will need the following which is easy to prove using the H"older inequality. Let \((U, \lambda)\) be a finite measure space, then
\[
\| f - f_{U;\lambda} \|_{L^q(U;\lambda)} \leq 2 \inf_{c \in \mathbb{R}} \| f - c \|_{L^q(U;\lambda)}.
\]

Then \((\ref{2.12})\) with \( p = 1 \) and \( \nu = n \), \((\ref{14})\), and the uniform bound on \( g \) imply that for \( V = V_i \) or \( V^*_i \)
\[
\left( \int_{V} | f - f_{V;\mu} | \frac{n-1}{n} d\mu \right)^{\frac{n}{n-1}} \leq 2 S_1 n e^n \int_{V} | \nabla f | d\nu.
\]

Thus the covering \((V_i, V^*_i, V^#_i)\) satisfies the continuous \( L^1 \) Sobolev inequality of order \( \nu = n \). It remains to show the discrete \( L^1 \) Sobolev-Neumann inequality. But this follows because \( G \) is a finite connected graph, and any two norms on a finite dimensional vector space are equivalent. This proves \((\ref{13})\). Then one can see that the same \( C \) can be used in \((\ref{13})\) for any \( R \geq R_0 \) by considering the Euler action \( \psi_a : C(S) \to C(S), a > 0 \), which acts by homotheties. We have \( \psi_a : A(R, \kappa R) \to A(aR, a\kappa R) = A(R', \kappa R') \). And if \( A_i^a \) denotes the \( \delta R \)-neighborhood of \( A(R', \kappa R') \), then \( \psi_a : A_i^a \to A_i^a \) is a homothetic diffeomorphism. Since \((\ref{13})\) is invariant under homotheties, the lemma follows. \( \square \)

**Proof of Theorem \ref{2.6}** Let \( R > 0 \) and \( \kappa > 1 \) be as above and define \( R_i = \kappa^i R \). We define a good covering of \((X, g)\). Define \( A_i = A(R_{i-1}, R_i) \) for \( i \geq 1 \), and \( A_0 = D_R = \{ x \in X : \rho(x) < R \} \). We let \( A^*_i \) be the union of all the \( A_j \) whose closure intersects \( A_i \). And similarly, \( A^#_i \) is the union of all the \( A_j \) whose closure intersects \( A^*_i \).

Lemma \ref{2.17} shows that the continuous \( L^1 \) Sobolev inequality of order \( n \) holds for the pairs \((A_i, A^*_i)\) with \( i \geq 1 \), and for \((A^*_i, A^#_i)\) with \( i \geq 2 \). Since \( A_0 = D_R \) is pre-compact one can show that \((\ref{13})\) holds with \( \Lambda = A_0 \) using the same argument in Lemma \ref{2.17} Thus the continuous \( L^1 \) Sobolev inequality holds for \((A_0, A^*_0)\), and the remaining cases are identical.

Proposition \ref{2.16} shows that the discrete \( L^1 \) Sobolev-Dirichlet inequality, of order \( \infty \), holds if
\[
\frac{m(\Omega)}{m(\partial \Omega)} \leq C, \quad \forall \Omega \subset V \text{ with } m(\Omega) < \infty.
\]

Suppose \( \Omega \subset \{0, \ldots, j\} \) with \( j \in \Omega \). Then
\[
m(\Omega) \leq \sum_{i=0}^{j} m(i)
\]
\[
= \mu(A_0) + \sum_{i=1}^{j} \mu(A(\kappa^{i-1} R, \kappa^i R))
\]
\[
= \mu(A_0) + \mu(A_1) \sum_{i=1}^{j} \kappa^{n(i-1)}
\]
\[
= \mu(A_0) + \mu(A_1) \frac{\kappa^{nj} - 1}{\kappa^n - 1}
\]
And
\[ \mu(\partial \Omega) \geq m(j + 1) = \mu(A_j) \kappa^m. \]
The isoperimetric inequality (16) holds with \( C = \frac{\mu(A_0)}{\mu(A_j)} + \frac{1}{\kappa^{n-1}} \). We then apply Theorem 2.13 and the prove is complete. \( \Box \)

Let \( V(x, r) \) be the volume of the geodesic ball \( B(x, r) \) of radius \( r \) centered at \( x \in X \). It is well known (cf. [27]) that (5) implies the following volume growth condition.

Corollary 2.18. There is a constant \( c > 0 \), depending only on \( n \) and \( C \) in (6), so that \( V(x, r) \geq cr^n \).

Remark 2.19. The above arguments can easily be adapted to prove the Sobolev inequality (5) on an AC manifold with multiple, but finitely many, ends.

2.3. Gaussian bound on the heat kernel. Recall that the heat kernel \( h(t, x, y) \) is a smooth function on \( \mathbb{R}_{>0} \times X \times X \) symmetric in \( x, y \) which is the fundamental solution to the heat diffusion equation, \( (\partial_t + \Delta_x)h = 0 \), with \( \lim_{t \to 0^+} h(t, x, y) = \delta_x \).

In this section we prove that \( h(t, x, y) \) satisfies a Gaussian bound. First we need a definition.

Definition 2.20. Let \( U \subset U' \) be relatively compact open subsets of \( (X, g) \). The Poincaré constant of the pair \( (U, U') \) is the smallest positive number \( \Lambda(U, U') \) such that
\[
\int_U |f - f_U|^2 d\mu \leq \Lambda(U, U') \int_{U'} |
abla f|^2 d\mu, \quad \forall f \in C^\infty(U').
\]
We say that \( (X, g) \) satisfies a Poincaré inequality with parameter \( 0 < \delta \leq 1 \) if there is a constant \( C_P > 0 \) so that for any ball \( B(x, r) \)
\[
\Lambda(B(x, \delta r), B(x, r)) \leq C_P r^2.
\]

Theorem 2.21. For a complete manifold \( (X, g) \) the following are equivalent.

(i) \( (X, g) \) satisfies the volume doubling condition. That is, there exists a constant \( C_D > 0 \) so that for any ball \( B(x, r) \)
\[
V(x, 2r) \leq C_D V(x, r).
\]
And \( (X, g) \) satisfies a Poincaré inequality with parameter \( 0 < \delta \leq 1 \).

(ii) The heat kernel \( h(t, x, y) \) of \( (X, g) \) satisfies the two-sided Gaussian bound
\[
\frac{c_1 \exp(-C_1 d^2(x, y)/t)}{V(x, \sqrt{t})} \leq h(t, x, y) \leq \frac{C_2 \exp(-C_2 d^2(x, y)/t)}{V(x, \sqrt{t})}.
\]

We also have the following time derivative estimates on \( h(t, x, y) \) if the equivalent conditions in the theorem hold. For any integer \( k \),
\[
|\partial_t^k h(t, x, y)| \leq \frac{A_k \exp(-C_2 d^2(x, y)/t)}{t^k V(x, \sqrt{t})},
\]
and there is an \( \epsilon > 0 \) so that
\[
|\partial_t^k (t, x, y) - \partial_t^k h(t, x, z)| \leq \frac{d(y, z)^\epsilon \exp(-C_2 d(x, y)/t)}{t^{k+\frac{\epsilon}{2}} V(x, \sqrt{t})}.
\]
Lemma 2.26. Let $\rho$ may assume that an isometry.

Proof. Per Remark 2.23 we may assume (19). In $\bar{\text{ inj}}(B)$ the chart $\exp_x$ anchored.

The main result of this section is the following.

Theorem 2.24. Suppose $(X, g)$ is AC or merely quasi-isometric to an AC metric. Then the equivalent conditions of Theorem 2.21 hold on $(X, g)$.

We will prove Theorem 2.24 by showing that $(X, g)$ satisfies the volume doubling condition and the existence of a Poincaré inequality in Theorem 2.21 are invariant under quasi-isometry. This will be used to simplify the proofs below.

Definition 2.25. Fix $o \in X$ and a parameter $0 < \epsilon \leq 1$ (the remote parameter).

(i) We say that a ball $B(x, r)$ is remote if $r \leq \epsilon^{1/2}d(o, x)$.

(ii) We say that a ball $B(o, r)$ is anchored.

Lemma 2.26. Let $(X, g)$ be AC. Then $(X, g)$ satisfies the volume doubling condition.

Proof. Per Remark 2.23 we may assume $(X, g)$ is conical outside a compact set. We may assume that $\rho$ is a radius function and there is an $r_0 \geq 2$ so that $\phi : X \setminus D_{r_0} \to C(S) \setminus K_{r_0}$, with $D_{r_0} = \{x \in X : \rho(x) \leq r_0\}$ and $K_{r_0} = \{x \in C(S) : r(x) \leq r_0\}$, is an isometry.

We first prove that volume doubling is satisfied for remote balls. Choose $\delta < \text{inj}(X, g)$ and $R > r_0$ so that $R - \delta > r_0$. Let $\{x_i\}_{i \in I}$ be a maximal set of points in $D_R$ such that the distance between any two is at least $\delta$. For each $i \in I$ let $B_i \subset TB(x_i, \delta)$ be the radius $\delta$ disk bundle. And fix an isomorphism $\beta_i : B_i \times B(x_i, \delta) \cong B_i$ linear and preserving distances on the fibers. Then define maps $\psi_i : B_i \times B(x_i, \delta) \to X$ by $\psi_i(w, x) = \exp_{x_i}(\beta_i(w, x))$. Let $g_i$ be the restriction of $\psi_i^*g$ to the fibers on $B_i \times B(x_i, \delta)$.

By an easy compactness argument it is easy to see that if $g_0$ is the flat metric on $B_\delta$, then $c^{-1}g_0 \leq g_i \leq cg_0$ for $c > 1$ uniformly in $B(x_i, \delta)$ for each $i \in I$. If $x \in D_R$, then the above arguments show that $g$ in the chart $\exp_x : B_\delta \to B(x, \delta)$ satisfies $c^{-1}g_0 \leq g \leq cg_0$. Then one can show that the volume doubling condition holds for balls $B(x, r)$ with $x \in D_R$ and $r < \frac{1}{2}\delta$, i.e. there exists a $C > 0$ so that $V(x, 2r) \leq CV(x, r)$.

Now let $x \in X$ with $\rho(x) > R$. Then by applying the Euler action $\psi_n : C(S) \to C(S)$ to the above charts, we have the chart $\exp_x : B_{\rho(x)}(\delta) \to B(x, \frac{\rho(x)}{R}\delta)$ in which $c^{-1}g_0 \leq g \leq cg_0$. Thus we have volume doubling for balls $B(x, r)$ with $r < \frac{\rho(x)}{2R}\delta$. Since $\rho \sim d(o, -)$, volume doubling holds for remote balls if we chose a small enough remote parameter $\epsilon > 0$.

There is a constant $C_0 > 0$ so that $V(o, r) \leq C_0 r^n$. And by Corollary 2.7 there is a constant $C > 0$ so that $V(x, r) \geq Cr^n$ for any $x \in X$. 

for $d(y, z) \leq \sqrt{t}$. 

Remark 2.22 It is known [15] that if a Poincaré inequality (18) holds for $0 < \delta < 1$ then it holds for all $\delta \in (0, 1]$.

A. Grigor’yan [12] and L. Soloff-Coste [25] proved that (11) is equivalent to a parabolic Harnack inequality. The equivalence of (11) and the parabolic Harnack inequality goes back to [9]. See also [24] for proofs of both of these equivalences.

Remark 2.23 It is not difficult to check that both the volume doubling condition and the existence of a Poincaré inequality in Theorem 2.21 are invariant under quasi-isometry. This will be used to simplify the proofs below.
Choose $\epsilon$ to be the remote parameter from above. Set $d(o, x) = \ell$. We consider three cases.

Case 1: If $r \leq \frac{1}{2} \ell$, then the ball $B(x, r)$ is remote and volume doubling holds.

Case 2: If $r \geq \frac{3}{4} \ell$, then we have

$$V(x, 2r) \leq V(o, \frac{8}{3} r) \leq C_o \frac{8^n}{3^n} r^n \leq \frac{8^n C_o}{3^n C} V(x, r).$$

Case 3: If $\frac{1}{2} \ell \leq r \leq \frac{3}{4} \ell$, then

$$V(x, 2r) \leq V(o, 8 \ell) \leq C_o 16^n \epsilon^n \leq \frac{C_o 16^n}{C} \epsilon^n V(x, r).$$

The proof of the following is straightforward.

**Lemma 2.27 (12).** Given $o \in X$ suppose the Poincaré inequality holds for all remote and anchored balls with parameter $0 < \delta_0 \leq 1$ and constant $C_P > 0$. That is, for any remote or anchored ball $B(x, r)$

$$\Lambda(B(x, \delta_0 r), B(x, r)) \leq C_P r^2.$$ 

Then the Poincaré inequality holds for any ball with parameter $\delta = \epsilon \delta_0 / 8$ and constant $C_P > 0$. That is, for any ball $B(x, r)$

$$\Lambda(B(x, \delta r), B(x, r)) \leq C_P r^2.$$

**Proof of Theorem 2.24** We will use the same notation used in the proof of Lemma 2.26. From the proof of Lemma 2.26 we have a remote parameter $\epsilon > 0$ so that for $r \leq \epsilon \frac{1}{2} d(o, x)$ in the chart $\exp_x : B_\delta \rightarrow B(x, r)$ the metric $g$ satisfies $c^{-1} g_0 \leq g \leq c g_0$. Then by the local Poincaré inequality, (12) with $p = 2, \nu = \infty$, there is a $C > 0$ so that

$$\int_{B(x, \epsilon)} |f - f_\epsilon|^2 d\mu \leq \int_{B(x, r)} |f - f_{B_r} d\mu| \leq C r^2 \int_{B(x, r)} \|\nabla f\|^2 d\mu, \forall f \in C^\infty(B(x, r)),$$

for any remote ball $B(x, r)$, where $f_\epsilon$ is the average with respect to the $g$ volume $d\mu_g$.

Thus by Lemma 2.27, it remains to prove the Poincaré inequality for anchored balls. As in the proof of Lemma 2.26, we may assume the end $X \setminus \bar{D}_{r_0}$ of $(X, g)$ is conical. Fix $R > r_0$ and $\kappa > 1$. And choose $\delta > 0$ so that $s = \delta R < \frac{1}{3} \text{inj}(g)$. Let $A = A(R, \kappa R)$ and $A_t$ the $\delta R$-neighborhood of $A$. Let $\{x_i\}_{i \in I}$ be a maximal subset of $A$ so that the distance between any two elements is at least $s$. Let $V_i = B(x_i, s)$ and $V_i^\# = B(x_i, 3s)$. Then $(V_i, V_i^\#)_{i \in I}$ is a finite good cover of $A$ in $A_\delta$.

By uniformly bounding $g$ as above, there is a constant $S_\epsilon$ so that $\Lambda(V_i, V_i^\#) \leq S_\epsilon$ and $\Lambda(V_i^\#, V_i^\#) \leq S_\epsilon$. In other words, the covering satisfies the continuous $L^2$ Sobolev inequality of order $\nu = \infty$. The associated graph $(G, m)$ if finite and connected, thus there is a $S \delta > 0$ so that the discrete $L^2$ Sobolev-Neumann inequality of order $\infty$ holds. Theorem 2.14 gives a constant $S > 0$ so that $\Lambda(A, A_\delta) \leq S$. By considering the homothetic action $\psi_A : A(R, \kappa R)_\delta \rightarrow A(R', \kappa R')_\delta$ with $R' = aR$ we see there is a constant $C > 0$ independent of $R$ so that

$$\Lambda(A, A_\delta) \leq CR^2, \forall R > r_0.$$
Let $R > r_0$ and $\kappa > 1$ be as above. Choose $\delta > 0$ so that $R - r_0 > \delta R$, and set $R_i = \kappa^i R$. By increasing $r_0$ if necessary, we may assume there is an $r_1 > 0$ so that $D_{R_0} \subset B(o, r_1) \subset D_{R_1}$. We define a covering with the following sets

\begin{align}
(24) & \quad A_0 = D_{R_0} \text{ and } A_i = A(R_{i-1}, R_i) \text{ for } i \geq 1, \\
(25) & \quad A_i^* = A_{i-1} \cup A_i \cup A_{i+1}, \\
(26) & \quad A_i^\# = A_{i-1}^* \cup A_i^* \cup A_{i+1}^*,
\end{align}

where we assume $A_i = \emptyset$ for $i < 0$. Choose the least $\ell \geq 1$ so that $B(o, r) \subset D_{R_\ell}$. Then $\mathcal{A} = \{(A_i, A_i^*, A_i^\#)\}_{i=0}^{n-1}$ is a good covering of $B(o, r)$ in $D_{R_{\ell+2}} \subset B(o, \kappa^{\ell+2}r)$. If we set $\bar{r} = R_\ell$, then from (23) we have $\Lambda(A_i, A_i^*) \leq C\bar{r}^2$ for $i \geq 1$ and $\Lambda(A_i^*, A_i^\#) \leq C\bar{r}^2$ for $i \geq 2$.

It remains to prove the discrete Sobolev-Neumann inequality. We will show that there is a constant $c > 0$ so that the spectral gap (28) satisfies $c < \lambda(\mathcal{G}, m)$. And by (10) and (11) it suffices to show there is a $C > 0$ independent of $\ell$ such that

\begin{equation}
(28) \quad m(U) \leq Cm(\partial U), \quad \forall U \subset \mathcal{V} \text{ such that } m(U) \leq \frac{1}{2} m(\mathcal{V}).
\end{equation}

Let $j := \max\{i \in \mathcal{V} : (i, i+1) \in \partial(U)\}$. Then

\begin{align}
m([0, j]) & = \sum_{i=0}^{j} m(i) \\
& = \mu(A_0) + \sum_{i=1}^{j} \mu(A(\kappa^{i-1} R, \kappa^i R)) \\
& = \mu(A_0) + \mu(A_1) \sum_{i=1}^{j} \kappa^{n(i-1)} \\
& = \mu(A_0) + \mu(A_1) \frac{\kappa^{nj} - 1}{\kappa^n - 1}
\end{align}

Set $C = \frac{\mu(A_0)}{\mu(A_1)} + \frac{1}{\kappa^n - 1}$. Since either $U$ or $\mathcal{V} \setminus U$ is contained in $[0, j]$,

\begin{equation}
(29) \quad m(U) \leq \min\{m(U), m(\mathcal{V} \setminus U)\} \leq m([0, j]) \leq Cm(A_1)\kappa^{nj} = Cm(j + 1) \leq Cm(\partial U).
\end{equation}

By Theorem 2.14 we have $\Delta(B(o, r), D_{R_0}) \leq C\bar{r}^2$, for $r \geq r_1$ where $C$ is independent of $\bar{r}$. From (27) there is a $C_1 > 0$ so that $\Lambda(B(o, r), B(o, r)) \leq C_1 r^2$ for $r \leq r_1$. The proof is completed by observing that $d(o, -) \sim \rho$ on $X \setminus D_{R_0}$.

\begin{remark}
Unlike Theorem 2.14, Theorem 2.24 does not generalize to AC manifolds with more than one end. For example, it is known that the Poincaré inequality does not hold on a connected sum of Euclidean spaces $\mathbb{R}^n \# \mathbb{R}^n$. See [13] for more information on this.
\end{remark}
2.4. Laplacian on Asymptotically conical manifolds. We will need some properties of the Laplacian on an asymptotically conical manifold. These will follow from some good bounds on the Green’s function which follow from Theorem 2.21. First we state an elementary lemma whose proof is an easy exercise.

**Lemma 2.29.** Let $(X, g)$ be $\mathrm{AC}(\delta, k + \alpha), k \geq 1, 0 < \alpha < 1$. Suppose $u \in C^2_\beta(X)$ and $v \in C^2_\gamma(X)$ where $\beta, \gamma \in \mathbb{R}$ with $\beta + \gamma < 2 - n$. Then

\begin{equation}
\int_X u \Delta v d\mu = \int_X v \Delta u d\mu.
\end{equation}

If $\rho$ is a radius function of $(X, g)$, then $\Delta (\rho^{2-n}) \in C^{k-1,\alpha}_\delta(X)$. And if $\Omega := \text{Vol}(S, g_S)$, where $S$ is the link in the conical end, then

\begin{equation}
\int_X \Delta (\rho^{2-n}) d\mu = (n-2)\Omega.
\end{equation}

Recall, the Green’s function satisfies

\begin{equation}
\Delta_y G(x, y) = \delta_x(y), \quad \forall x \in X.
\end{equation}

Equivalently $G(x, y)$ satisfies

\begin{equation}
\int_X G(x, y) \Delta f(y) d\mu(y) = f(x),
\end{equation}

and

\begin{equation}
\Delta_x \int_X G(x, y) f(y) d\mu(y) = f(x),
\end{equation}

for any smooth compactly supported function $f$ on $X$.

If $(X, g)$ satisfies Theorem 2.21 then $(X, g)$ admits a positive symmetric Green’s function if and only if $\int_0^\infty V(x, \sqrt{t})^{-1} dt < \infty$. And in this case

\[ G(x, y) = \int_0^\infty h(t, x, y) dt. \]

If $n > 2$, then from Corollary 2.7 and integrating (20) we have for some $C > 0$ depending only on $g$

\begin{equation}
0 < G(x, y) \leq c \int_{d(x,y)^2}^\infty \frac{dt}{V(x, \sqrt{t})} \leq Cd(x, y)^{2-n},
\end{equation}

for all $x, y \in X$, where the second inequality uses Corollary 2.7.

Similarly, by integrating (22) for $x \neq y$ and $d(y, z) \leq d(x, y)/2$ we have

\begin{equation}
\frac{|G(x, y) - G(x, z)|}{d(y, z)\epsilon} \leq c \int_{d(x,y)^2}^\infty \frac{dt}{V(x, \sqrt{t})} \leq Cd(x, y)^{2-n-\epsilon},
\end{equation}

where again we have used Corollary 2.7 in the second inequality.

If $f \in C^{k,\alpha}_\beta(X)$ with $k \geq 0, \alpha \in (0, 1)$ and $\beta < -2$, then standard regularity arguments show that $u(x) = \int_X G(x, y) f(y) d\mu(y)$ is locally in $C^{k+2,\alpha}_\beta(X)$ and $\Delta u = f$. We will extend these arguments to prove the following.

**Theorem 2.30.** Suppose $(X, g)$ is $\mathrm{AC}(\delta, \ell + \alpha), \delta < -\epsilon, \ell \geq 0, \alpha \in (0, 1)$ of dimension $n > 2$. Let $k \leq \ell$, then we have the following.
(i) Suppose $-n < \beta < -2$. There exists a $C > 0$ so that for each $f \in C^k_{\beta,2}(X)$ there is a unique $u \in C^{k+2,2}_{\beta+2}(X)$ with $\Delta u = f$ which satisfies $\|u\|_{C^{k+2,2}_{\beta+2}} \leq C\|f\|_{C^k_{\beta,2}}$.

(ii) Suppose $-n - \epsilon < \beta < -n$. There exist $C_1, C_2 > 0$ such that for each $f \in C^k_{\beta,2}(X)$ there is a unique $u \in C^{k+2,2}_{\beta+2}(X)$ with $\Delta u = f$. Furthermore, if we define

$$A = \frac{1}{(n-2)\Omega} \int_X f d\mu,$$

where $\Omega = \text{Vol}(S)$, then $u = A \rho^{2-n} + v$ with $v \in C^{k+2,2}_{\beta+2}(X)$ satisfying $|A| \leq C_1\|f\|_{C^0_{\beta}}$ and $\|v\|_{C^{k+2,2}_{\beta+2}} \leq C_2\|f\|_{C^k_{\beta,2}}$.

Proof. We define

$$u(y) = \int_X G(y,x)f(x)d\mu(x),$$

and we first prove that $u \in C^0_{\beta+2}$. We have $|f(x)| \leq \|f\|_{C^0_{\beta}} \rho(x)^\beta$, and from (38) and (35) we have

$$|u(y)| \leq C\|f\|_{C^0_{\beta}} \int_X d(y,x)^{2-n} \rho(x)^\beta d\mu(x).$$

Let $o \in X$ be a fixed point. We split the integral into three regions $R_1 = \{x \in X : 4d(o,x) \leq d(o,y)\}$, $R_2 = \{x \in X : \frac{d(o,y) < d(o,x) < 2d(o,y)\}$, and $R_3 = \{x \in X : d(o,x) > 2d(o,y)\}$. Estimating the integral over the three regions gives the following:

$$\int_{R_1} d(y,x)^{2-n} \rho(x)^\beta d\mu(x) \leq \begin{cases} C\rho(\rho)^{\beta+2} & \text{if } \beta \in (-n,-2) \\
C\rho(\rho)^{2-n} & \text{if } \beta < -n \end{cases},$$

$$\int_{R_2} d(y,x)^{2-n} \rho(x)^\beta d\mu(x) \leq \rho(\rho)^{\beta+2}, \text{ and}$$

$$\int_{R_3} d(y,x)^{2-n} \rho(x)^\beta d\mu(x) \leq \rho(\rho)^{\beta+2}.$$

This proves that $u \in C^0_{\beta+2}(X)$ in part (i).

We now consider part (ii). If $\Delta u = f$ with $u \in C^{k+2,2}_{\beta+2}(X)$ and $\beta \in (-n-\epsilon,-n)$, then

$$\int_X f d\mu = \int_X \Delta u d\mu = 0$$

by the arguments in Lemma 22. Thus for $f \in C^k_{\beta,2}(X)$ there exists $u \in C^{k+2,2}_{\beta+2}(X)$ solving $\Delta u = f$ only if $\int_X f d\mu = 0$. So suppose that $\int_X f d\mu = 0$. And we replace (38) with

$$u(y) = \int_X [G(y,x) - G(y,o)] f(x) d\mu(x).$$

Since $\int_X f d\mu = 0$, this integral is equal to (38) and thus solves $\Delta u = f$. From (38) we have

$$|u(y)| \leq C\|f\|_{C^0_{\beta}} \int_X d(x,y)^{2-n-\epsilon} \rho(x)^\beta d\mu.$$
And it is not difficult to show that
\[ \int_{\mathcal{R}_1} d(x,y)^{2-n} d(o,x) \rho(x)^3 d\mu(x) \leq \begin{cases} C' \rho(y)^{\beta+2} & \text{if } \beta \in (-n-\epsilon, -2) \\ C' \rho(y)^{2-n-\epsilon} & \text{if } \beta \leq -n-\epsilon \end{cases} \]

This shows that \( u \in C^0_{\beta+2}(X) \) if \( \beta \in (-n-\epsilon, -n) \).

In both parts (i) and (ii) we have proved that \( \|u\|_{C^0_{\beta+2}} \leq C' \|f\|_{C^0_\beta} \cdot \|f\|_{C^{k,\alpha}_\beta} \). As remarked above, we have that \( u \) is locally in \( C^{k+2,\alpha}_0(X) \). By taking an appropriate covering of \((X, g)\) using the conical structure and applying the Schauder interior estimates one can show that there is a \( C > 0 \) so that for \( \Delta u = f \) one has

\[ \|u\|_{C^{\beta+2,\alpha}_0} \leq C \left( \|u\|_{C^0_{\beta+2}} + \|f\|_{C^{k,\alpha}_\beta} \right). \]

Thus there is a \( C > 0 \) so that \( \|u\|_{C^{\beta+2,\alpha}_0} \leq C\|f\|_{C^{k,\alpha}_\beta} \).

We complete the proof of part (ii). Define \( A \) by \( (47) \). Then by Lemma 2.29 we have \( \int_X [f - A\Delta(\rho^{2-n})] d\mu = 0 \). And also by Lemma 2.29 we have \( f - A\Delta(\rho^{2-n}) \in C^{k,\alpha}_\beta(X) \). Thus by what we have already proved there is a \( v \in C^{k+2,\alpha}_{\beta+2}(X) \) with \( \Delta v = f - \Delta(\rho^{2-n}) \) with

\[ \|v\|_{C^{\beta+2,\alpha}_0} \leq C \left( \|f\|_{C^{\beta,\alpha}_0} + |A|\|\Delta(\rho^{2-n})\|_{C^{k,\alpha}_\beta} \right). \]

Note that

\[ |A| \leq \frac{1}{(n-2)\Omega} \int_X |f| d\mu \leq \frac{\|f\|_{C^0_\beta}}{(n-2)\Omega} \int_X \rho^2 d\mu \leq C_1 \|f\|_{C^0_\beta}, \]

where \( C_1 = \frac{1}{(n-2)\Omega} \int_X \rho^2 d\mu \) is finite since \( \beta \leq -n \). And this combined with (43) completes the proof. \( \Box \)

3. Kähler case

Asymptotically conical Kähler manifolds will now be considered. We will begin with some definitions and preliminary results. In particular, the link \( S \) in a Kähler cone is far from arbitrary. It is a Sasaki manifold which can be thought of as an odd dimensional analogue of a Kähler manifold. We also consider some Hodge theory which will be useful late.

3.1. Background.

**Definition 3.1.** A \( 2m - 1 \)-dimensional Riemannian manifold \((S, g)\) is Sasaki if the metric cone \((C(S), \bar{g})\), \(C(S) = \mathbb{R}_{>0} \times S, \ \bar{g} = dr^2 + r^2 g\), is Kähler.

**Remark 3.2** This is the succinct definition of a Sasaki manifold. They were originally defined as a manifold carrying a special type of metric contact structure. For more on Sasaki manifolds see the monograph [4].

It follows from the definition that the Euler vector field \( r \partial_r \) acts holomorphically, i.e. \( \mathcal{L}_{r \partial_r} J = 0 \). It is also not difficult to show that \( \xi = Jr \partial_r \) is a Killing vector field which restricts to \( S = \{ r = 1 \} \subset C(S) \). Thus \( \xi + ir \partial_r \) is a holomorphic vector field on \( C(S) \). The restriction of \( \xi \) to \( S \) is the Reeb vector field of \((S, g)\). Sasaki manifolds can be distinguished by the action of the Reeb vector field \( \xi \). If \( \xi \) generates a free action of \( U(1) \) then the Sasaki structure is regular. The Sasaki
structure is *quasi-regular* if the orbits close but there are non-trivial stabilizers. If the orbits do not close, then the Sasaki structure is *irregular*.

Let \( \eta \) be the dual 1-form to \( \xi \) with respect to \( g \), that is
\[
\eta = \frac{1}{r^2} \xi \wedge \bar{g}.
\]
Then one can check
\[
\eta = -J^* \frac{dr}{r} = 2d^c \log r,
\]
where \( d^c = i \left( \bar{\partial} - \partial \right) \). The restriction of \( \eta \) to \( S \) is a contact form with Reeb vector field \( \xi \).

Since \( \mathcal{L}_{r \partial_r} J = 0 \), the Kähler form \( \omega \) satisfies
\[
2\omega = \mathcal{L}_{r \partial_r} \omega = d(r \partial_r \cdot \omega)
= d(r^2 \eta)
= dd^c(r^2), \quad \text{using (45)}.
\]
Thus the Kähler form \( \omega \) on \( C(S) \) has potential \( \frac{1}{2} r^2 \).

We are in particular interested in Ricci-flat Kähler cones. The following easily follows from the warped product structure of \( \bar{g} \).

**Proposition 3.3.** Let \( (S, g) \) be a \( 2m-1 \)-dimensional Sasaki manifold. Then the following are equivalent.

(i) \( (S, g) \) is Sasaki-Einstein with Einstein constant \( 2n-2 \).

(ii) \( (C(S), \bar{g}) \) is Ricci-flat Kähler.

Of course, a necessary condition for \( (C(S), \bar{g}) \) to be Ricci-flat Kähler is that \( K_{C(S)}^\ell \) must be trivial for some positive integer \( \ell > 0 \). But for \( C(S) \) to admit a Ricci-flat Kähler cone metric with the given Reeb vector field \( \xi \) one must require a little more.

**Proposition 3.4.** A necessary condition for \( C(S) \) to admit a Ricci-flat Kähler cone metric with the same complex structure \( J \) and Reeb vector field \( \xi \) is that \( K_{C(S)}^\ell \), for some integer \( \ell \geq 1 \), admits a nowhere vanishing section \( \Omega \) with \( \mathcal{L}_\xi \Omega = im \Omega \).

If the condition in the proposition holds, then \( \Omega \) satisfies
\[
\frac{i}{2^{m^2}} m! \omega^m \wedge \bar{\Omega} = m! e^h \omega^m,
\]
where \( h \in C^\infty(C(S)) \) is basic, meaning that \( \xi h = r \partial_r h = 0 \). Note that the Ricci form is given by \( \text{Ric}(\omega) = dd^c h \) and is zero precisely when \( h \) is constant.

**Remark 3.5** *A priori* a Kähler cone \( C(S) \) does not contain the vertex. But it can be proved that \( C(S) \cup \{ o \} \), with the vertex \( o \), is an affine variety. See [30] for a proof of the relevant embedding theorem.

We now define AC Kähler manifolds.

**Definition 3.6.** Let \( (C(S), g_0) \) be a Kähler cone. Then we say that a Kähler manifold \( (X, g) \) is asymptotically conical of order \( (\delta, k + \alpha) \), asymptotic to \( (C(S), g_0) \), if there is compact subset \( K \subset X \), a compact neighborhood \( o \in K_0 \subset C(S) \), and a
diffemorphism $\phi : X \setminus K \to C(S) \setminus K_0$ so that

$$|\phi_* g - g_0| \in C^k_{\delta,\alpha} \quad \text{on } C(S) \setminus K_0 \quad \text{and}$$

$$|\phi_* J - J_0| \in C^k_{\delta,\alpha} \quad \text{on } C(S) \setminus K_0.$$  

We will denote this by Kähler $AC(\delta, k + \alpha)$.

In many cases the end of $(X, g)$ will be holomorphically a cone. In this case $\phi : X \setminus K \to C(S) \setminus K_0$ is a biholomorphism. And one can show that in this case $\phi$ extends to $\phi : X \to C(S) \cup \{\phi\}$, which is therefore a resolution of $C(S)$.

3.1.1. Hodge theory. We review some Hodge theory that will be needed. In particular, a Hodge decomposition will be needed in a proof of a weighted version of the $\partial\bar{\partial}$-lemma.

We define the space of $L^2$ harmonic forms, where we assume $g$ is quasi-isometric to an AC metric, to a AC metric,

$$L^2\mathcal{H}(X, g) := \{ \eta \in L^2(\Lambda^p X) : \Delta \eta = 0 \}.$$  

One can show that $L^2\mathcal{H}(X, g) = \{ \eta \in L^2(\Lambda^p X) : d\eta = d^*\eta = 0 \}$. If $\eta \in L^2\mathcal{H}(X, g)$, then $\eta \in L^2(\Lambda^k X)$ for all $k \geq 0$. So $L^2\mathcal{H}(X, g)$ consists of smooth forms. Furthermore, one can show that it is finite dimensional, and if $\tilde{g}$ is quasi-isometric to $g$, as in (51), there is a natural isomorphism $L^2\mathcal{H}(X, \tilde{g}) \cong L^2\mathcal{H}(X, g)$. See [15] for proofs of these statements.

The $L^2$ harmonic spaces have been computed in our context. For the following, note that an AC manifold $X$ can be compactified with boundary $\partial X = S$. And $H^*(X, S)$ is isomorphic to the compactly supported cohomology $H^*_c(X)$.

**Theorem 3.7** ([14]). Let $(X, g)$ be a manifold of dimension $n$ which is AC up to quasi-isometry. Then we have the natural isomorphisms

$$L^2\mathcal{H}(X, g) \cong \begin{cases} H^p(X, S), & p < n/2, \\ \text{Im}(H^p(X, S) \to H^p(X)), & p = n/2, \\ H^p(X), & p > n/2. \end{cases}$$  

Recall the Kodaira decomposition theorem, which for arbitrary manifolds gives the orthogonal decomposition

$$L^2(\Lambda^p(X, g)) = L^2\mathcal{H}(X, g) \oplus \overline{dC^\infty_0(\Lambda^{p-1})} \oplus \overline{d^*C^\infty_0(\Lambda^{p+1})},$$

where the closure in the last two summands is in $L^2$.

We need a more precise decomposition than (52). Assume from now on that $(X, g)$ is $AC(\delta, \ell + \alpha)$ with $\delta < 0$ and $\ell \geq 2$. A difficulty in improving (52) is that the operator

$$\Delta : L^2_{k+2, \beta}(\Lambda^p X) \to L^2_{k, \beta-2}(\Lambda^p X)$$

is not Fredholm for arbitrary $\beta \in \mathbb{R}$. The kernel of (53) is finite dimensional and the closure of the range has finite codimension. The difficulty is that the range is not always closed. It is a result of [19] that there is a discrete set $\mathcal{D}_\Delta \subset \mathbb{R}$ so that (53) is Fredholm precisely when $\beta \in \mathbb{R} \setminus \mathcal{D}_\Delta$. But one can define a Banach space $L^2_{k+2, \beta}(\Lambda^p X)$ so that

$$\Delta : L^2_{k+2, \beta}(\Lambda^p X) \to L^2_{k, \beta-2}(\Lambda^p X)$$

is Fredholm, and the range of (54) is the closure of the range of (53).
For each $\tau \geq \beta$ with $\tau \in \mathbb{R} \setminus \mathcal{D}_\Delta$ let $\tilde{B}_\tau$ be the closure of $L^2_{k+2,\beta}(\Lambda^p X)$ in
\begin{equation}
B_\tau = \{ \eta \in L^2_{k+2,\tau}(\Lambda^p X) : \Delta \eta \in L^2_{k,\beta-2}(\Lambda^p X) \}
\end{equation}
with respect to the norm
\begin{equation}
\|\eta\|_{B_\tau} = \|\eta\|_{L^2_{k+2,\tau}} + \|\Delta \eta\|_{L^2_{k,\beta-2}}.
\end{equation}
When $\tilde{B}_\tau$ is equipped with this norm one can show \cite{18} that
\begin{equation}
\Delta : \tilde{B}_\tau \longrightarrow L^2_{k,\beta-2}(\Lambda^p X)
\end{equation}
is Fredholm with range equal to the closure of the range of \cite{53}. One can also show that all the $\tilde{B}_\tau$ are isomorphic Banach spaces. We define $L^2_{k+2,\beta}(\Lambda^p X)$ to be any one of the $\tilde{B}_\tau$. In particular, we have $\tilde{L}^2_{k+2,\beta}(\Lambda^p X) = L^2_{k+2,\beta}(\Lambda^p X)$ with equivalent norms if $\beta \in \mathbb{R} \setminus \mathcal{D}_\Delta$. And in general
\begin{equation}
\tilde{L}^2_{k+2,\beta}(\Lambda^p X) \subset \bigcap_{\tau \geq \beta} L^2_{k+2,\tau}(\Lambda^p X).
\end{equation}

By our conventions we have $L^2_{0,-m}(\Lambda^p X) = L^2(\Lambda^p X)$ with equal norms. Thus consider
\begin{equation}
\Delta : \tilde{L}^2_{2,2-m}(\Lambda^p X) \longrightarrow L^2_{0,-m}(\Lambda^p X) = L^2(\Lambda^p X).
\end{equation}
The cokernel of \cite{59} is $L^2\mathcal{H}^p(X, g)$, so we have the following decomposition refining \cite{62}.

**Theorem 3.8.** Suppose $(X, g)$ is AC($\delta, \ell + \alpha$) with $\delta < 0$ and $\ell \geq 2$. Then we have
\begin{equation}
L^2(\Lambda^p X) = L^2\mathcal{H}^p(X, g) \oplus dd^*(\tilde{L}^2_{2,2-m}(\Lambda^p X)) \oplus d^*(\tilde{L}^2_{2,2-m}(\Lambda^p X)).
\end{equation}
In particular, if $\eta \in L^2(\Lambda^p X)$, then we have the unique $L^2$ decomposition $\eta = \sigma + d\zeta + d^*\theta$, where $\zeta \in L^2_{1,\delta}(\Lambda^{p-1} X)$ for all $\delta > 1 - m$ and $\theta \in L^2_{0,\delta}(\Lambda^{p+1} X)$ for all $\delta > 1 - m$.

Of course when $(X, g)$ is Kähler the decomposition in Theorem \cite{36} respects the decomposition into types $\Lambda^p(X) \otimes \mathbb{C} = \oplus_{\tau + s = \beta}\Lambda^\tau(X)$ as usual because $\Delta = 2\Delta_\beta$.

We now prove a weighted version of the $\ddbar$-lemma.

**Proposition 3.9.** Let $(X, g)$ be Kähler AC($\delta, \ell + \alpha$) with $\delta < 0$, $\ell \geq 2$, and $H^1(S, \mathcal{R}) = 0$. Suppose $\beta \in (-2m, -m)$ and $\eta \in C^k_\beta(\Lambda^{1,1} X)$, with $0 \leq k \leq \ell$, is a closed real $(1, 1)$-form with $[\eta] = 0$ in $H^2(X, \mathbb{R})$. Then there exists a unique real function $u \in C^{k+2,\alpha}_\beta(X)$ with $d\ddbar u = \eta$.

**Proof.** Recall that if $u$ is a smooth function, then
\begin{equation}
-m d\ddbar u \wedge \omega^{m-1} = \Delta u \wedge \omega^m.
\end{equation}
Define $f$ by $-m\eta \wedge \omega^{m-1} = f\omega^m$. So $f \in C^k_\beta(X)$. By Theorem \cite{23} there is a $u \in C^{k+2,\alpha}_{\beta+2}(X)$ with $\Delta u = f$. Then $\gamma = \eta - d\ddbar u$ is an exact 2-form in $C^{k+2,\alpha}_{\beta}(\Lambda^{1,1} X)$. And since $\gamma \wedge \omega^{m-1} = -\frac{1}{m}(f - \Delta u)\omega^m = 0$, one can show that $\gamma$ satisfies
\begin{equation}
\gamma \wedge \gamma \wedge \omega^{m-2} = \frac{1}{2m(m-1)}|\gamma|^2 \omega^m.
\end{equation}

Since $\beta < -m$, we have $\gamma \in L^2(\Lambda^{1,1} X)$. Therefore $\gamma = \sigma + d\zeta$ according to Theorem \cite{36} with $\sigma \in L^2\mathcal{H}^{1,1}(X, g)$. Since $H^1(S, \mathcal{R}) = 0$, the homomorphism $H^2(X, S) \to H^2(X)$ is an inclusion. It follows from Theorem \cite{57} that $L^2\mathcal{H}^{1,1}(X, g)$
contains no exact forms. Thus \( \gamma = d\zeta \), where \( \zeta \in L^2_{1,\delta}(\Lambda^1 X) \) with \( \delta > 1 - m \). Since \( C_0^\infty(X) \) is dense in \( L^2_{1,\delta}(\Lambda^1 X) \), we can choose a sequence \( \{\zeta_j\} \) converging to \( \zeta \) in \( L^2_{1,\delta}(\Lambda^1 X) \). From (62) we have

\[
0 = \int_X d[\zeta_j \wedge \gamma \wedge \omega^{m-2}] = \int_X d\zeta_j \wedge \gamma \wedge \omega^{m-2} \rightarrow \frac{-1}{2m(m-1)} \int_X |\gamma|^2 \omega^m, \quad \text{as} \; j \rightarrow \infty.
\]

The convergence follows because \( d\zeta_j \rightarrow \gamma \) in \( L^2_{0,\delta-1}(\Lambda^2 X) \) and we may assume \( \delta > 1 - m \) is chosen small enough that \( \delta - 1 + \beta < -2m \). Therefore \( \gamma = 0 \), and \( \eta = dd^c u \).

3.2. Calabi conjecture. On an AC Kähler manifold \((X, g, J)\) we consider the Monge-Ampère equation

\[
(\omega + dd^c \phi)^m = e^f \omega^m.
\]

If \( \phi \) is a solution to (64) and \( \omega' = \omega + dd^c \phi \), then the respective Ricci forms satisfy

\[
\text{Ric}(\omega') = \text{Ric}(\omega) - dd^c f.
\]

Equation (64) was solved by S.-T. Yau [32] for a compact Kähler manifold \((M, g, J)\) under the necessary assumption that \( \int_M (1 - e^f) \, d\mu_g = 0 \). This solved a conjecture of E. Calabi.

The Calabi conjecture for AC Kähler manifolds was solved independently by S. Bando and R. Kobayashi [2] and G. Tian and S.-T. Yau [28]. D. Joyce [16] gave a more exacting proof for the ALE case which gave more precise information on the solution. The proof of D. Joyce applies *mutatis mutandis* to this situation.

**Theorem 3.10** ([16]). Suppose \((X, g)\) is asymptotically conical Kähler of order \((\delta, j + \alpha)\), where \( \delta < -\epsilon \), \( 0 < \alpha < 1 \) and \( 3 \leq j \leq \infty \). Let \( 3 \leq k \leq j \leq \infty \).

(i) If \( \beta \in (-2m, -2) \), then for each \( f \in C^{k,\alpha}_\beta(X) \) there is a unique \( \phi \in C^{k+2,\alpha}_{\beta+2}(X) \) so that \( \omega + dd^c \phi \) is a positive \((1,1)\)-form and \( (\omega + dd^c \phi)^m = e^f \omega^m \) on \( X \).

(ii) If \( \beta \in (-2m - \epsilon, -2m) \), then for each \( f \in C^{k,\alpha}_\beta(X) \) there is a unique \( \phi \in C^{k+2,\alpha}_{2-2m}(X) \) so that \( \omega + dd^c \phi \) is a positive \((1,1)\)-form and \( (\omega + dd^c \phi)^m = e^f \omega^m \) on \( X \). Furthermore, we have \( \phi = A \rho^{2-2m} + \psi \) where \( \psi \in C^{k+2,\alpha}_{\beta+2}(X) \) and

\[
A = \frac{1}{(m-1)\Omega} \int_X (1 - e^f) \, d\mu,
\]

where \( \Omega = \text{Vol}(S) \), \( S = \{r = 1\} \subset C(S) \).

**Remark 3.11** Of course, by the local theory of elliptic operators, whenever \( f \in C^\infty(X) \) we have \( \phi \in C^\infty(X) \). The theorem is written as it is to show the precise global regularity that the proof gives.

Part (i) of Theorem 3.10 is already known and was essentially proved in [28]. See [11] for a proof in the context of manifolds with a conical end. The contribution here is part (ii) which gives a sharp estimate on solutions for rapidly decaying \( f \in C^{k,\alpha}_\beta(X) \), \( \beta < -2m \).

The proof of Theorem 3.10 goes through as in [16] §§8.6-8.7. The proof is by the continuity method, and the essential ingredients are some *a priori* estimates.
on a solution \( \phi \) of (64). The Sobolev inequality [3] is used to prove an a priori estimate on \( \| \phi \|_{C^0} \). A priori estimates on \( \| dd^c \phi \|_{C^0} \) and \( \| \nabla dd^c \phi \|_{C^0} \) depending only on \( \| \phi \|_{C^0}, \| f \|_{C^0}, \) and \( |R|_{C^1} \), where \( R \) is the curvature, due to T. Aubin [11] and S.-T. Yau [32] are applied as in [28]. Then Theorem 2.30 is applied as in [10] to show \( \phi \) is in the appropriate Hölder space.

3.3. Ricci-flat metrics. Our main motivating for proving Theorem 3.10 is the following which is an immediate consequence.

**Corollary 3.12.** Suppose \((X, g, J)\) is AC\((\delta, \ell + \alpha)\) with \(3 \leq \ell \leq \infty\). Let \(3 \leq k \leq \ell\), and suppose the Ricci form of \((X, g, J)\) satisfies

\[
\text{Ric}(\omega) = dd^c f, \quad f \in C^{k,\alpha}_\beta(X), \quad \beta < -2.
\]

Then there exists a \( \phi \in C^{k+2,\alpha}_\beta(X) \) so that \( \omega' = \omega + dd^c \phi \) is Ricci-flat, and the corresponding metric \( g' \) converges to \( g \) in \( C^{0,\alpha}_\gamma(X) \) where \( \gamma = \max(\beta, -2m) \).

Merely observe that for a solution to \((\omega + dd^c \phi)^m = e^f \omega^m\) the Ricci forms of the Kähler metrics \( \omega \) and \( \omega' = \omega + dd^c \phi \) satisfy

\[
(67) \quad \text{Ric}(\omega') - \text{Ric}(\omega) = -dd^c \log \left( \frac{\omega'}{\omega^m} \right) = -dd^c f.
\]

We have the following uniqueness result.

**Theorem 3.13.** Suppose \((X, g, J)\) is AC\((\delta, \ell + \alpha)\), \(2 \leq \ell \leq \infty\), and Ricci-flat. Suppose \( g' \) is another Ricci-flat Kähler metric which converges to \( g \) in \( C^{0,\alpha}_\gamma(X) \) with \( \gamma < -m \), i.e. \( |g' - g| \in C^{0,\alpha}_\gamma(X) \), and the Ricci forms satisfy \( [\omega'] = [\omega] \). Then \( g' = g \).

**Proof.** Set \( \eta = \omega' - \omega \). Then \( \eta \) is an exact form, and by Proposition 3.9 there is a \( \phi \in C^{4,\alpha}_\gamma(X) \) with \( dd^c \phi = \eta \). Since \( \phi \) solves \((\omega + dd^c \phi)^m = e^f \omega^m\), the uniqueness part of Theorem 3.11 gives \( \phi = 0 \). \( \Box \)

3.4. Ricci-flat metrics on resolutions. The main motivation for proving Theorem 3.10 is to construct examples of asymptotically conical Ricci-flat Kähler manifolds. In this section we give a proof of the part of Theorem 3.11 concerning compactly supported Kähler classes.

In order to apply Theorem 3.10 one must start with an AC Kähler manifold \((X, g, \omega)\) with \( c_1(X) = 0\). Thus we suppose there is a nowhere vanishing holomorphic \( n \)-form \( \Omega \) on \( X \). Recall that the Ricci form of \((X, g, J)\) is

\[
(68) \quad \text{Ric}(\omega) = dd^c \log \left( \frac{\Omega \wedge \Omega}{\omega^m} \right).
\]

If \( f = \log \left( \frac{d\Omega}{\omega^m} \right) \), then a solution \((X, g', \omega')\) to Theorem 3.10 has Ricci form \( \text{Ric}(\omega') = \text{Ric}(\omega) - dd^c f = 0 \). But order to apply Theorem 3.11 one must start with an AC Kähler manifold \((X, g, \omega)\) with Ricci potential \( f \in C^{\beta}_\beta(X) \) with \( \beta < -2 \). In general, it may be difficult to find a Kähler metric on \( X \) satisfying this.

The case of a quasi-projective variety \( X = Y \setminus D \), where \( D \) is a divisor, supporting the anti-canonical divisor \( K_Y^{-1} \), which admits a Kähler-Einstein metric was dealt with in [28] and independently in [2]. A Kähler metric \( \omega \) on \( X \) was perturbed to a Kähler metric \( \omega_0 \) whose Ricci potential \( f \) satisfies \( f \in C^{\beta}_\beta(X) \). The author considered [20] and extension of this result to some cases where \( D \) does not admit
a Kähler-Einstein metric. One essentially needs to start with an AC Kähler metric $(X, g, \omega)$ which approximates a Ricci-flat metric at infinity to high enough order.

We consider the relatively easy case of a crepant resolution $\pi : \hat{X} \to X = C(S) \cup \{o\}$ of a Ricci-flat Kähler cone $C(S)$. We will obtain Theorem 1.1 of the introduction. In the following $r$ will denote the the radius function on the cone $C(S)$.

Recall that a variety $X$ has rational singularities if for some, and it follows any, resolution $\pi : Y \to X$ $R^j\pi_*\mathcal{O}_Y = 0$ for $j > 0$.

**Proposition 3.14.** Let $C(S)$ be a Kähler cone satisfying Proposition 3.4. Then $o \in X = C(S) \cup \{o\}$ is a rational singularity. In particular, if $\pi : \hat{X} \to X$ is a resolution, then $H^j(\hat{X}, \mathbb{R}) = 0$ for $j \geq 1$.

We use the criterion of H. Laufer and D. Burns for the rationality of an isolated singularity $o \in X$. If $\Omega$ is an holomorphic $n$-form on a deleted neighborhood of $o \in X$, then $o \in X$ is rational if and only if

$$\int_U \Omega \wedge \bar{\Omega} < \infty,$$

where $U$ is a small neighborhood of $o \in X$. If $\Omega$ satisfies $\mathcal{L}_\xi = im\Omega$, then (47) is satisfied. And one easily see that the inequality (69) holds.

**Proposition 3.15.** Suppose $\omega$ is a Kähler metric on $\hat{X}$ with $[\omega] \in H^2_c(\hat{X}, \mathbb{R})$. Then there exists a Kähler metric $\omega_0$ on $\hat{X}$ with $[\omega_0] = [\omega]$ and $\bar{\omega} = \pi_*\omega_0$ on \{x $\in \hat{X} : \rho(x) > R\} for some $R > 0$, where $\bar{\omega} = C\frac{1}{2}dd^c(r^2)$, $C > 0$, is the Kähler cone metric on $C(S)$, up to homothety.

**Proof.** Let $E_i$, $i = 1, \ldots, d$, be the prime divisors in the exceptional set $E = \pi^{-1}(o) \subset \hat{X}$. Since $[\omega] \in H^2_c(\hat{X}, \mathbb{R})$, and it is Poincaré dual to $\sum_{i=1}^d a_i[E_i] \in H_{2m-2}(\hat{X}, \mathbb{R})$, for $a_i \in \mathbb{R}$. Thus there exists a closed compactly supported real (1,1)-form $\theta$ with $[\theta] = [\omega]$. Let $\eta = \omega - \theta$. Then $\eta$ is an exact real (1,1)-form on $\hat{X}$. There exists an $\alpha \in \bar{\Omega}^1$ with $d\alpha = \eta$. We have $\alpha = \alpha^{1,0} + \alpha^{0,1}$ where $\alpha^{0,1} = \frac{\alpha^{1,0}}{\mu}$. Then $\bar{\partial}\alpha^{0,1} = 0$, and by Proposition 3.14 there exists a $u \in C^\infty(\hat{X}, \mathbb{C})$ with $\bar{\partial}u = \alpha^{0,1}$. Define $v = \frac{1}{\mu}(u - \bar{u})$. Then

$$dd^c v = iid\bar{\partial}v = \frac{1}{2}(\bar{\partial}\partial v - \bar{\partial}\bar{\partial}v) = \frac{1}{2}(\bar{\partial}\partial v + \bar{\partial}\bar{\partial}v) = \frac{1}{2}(\partial\bar{\partial}\alpha^{0,1} + \bar{\partial}\alpha^{1,0}) = \eta.$$

We may assume the radius function $\rho$ on $\hat{X}$ is chosen so that $\rho(x) = \pi^*r(x)$, for $\rho(x) > 2$, and $dd^c(\rho^2) \geq 0$. Let $\mu : \mathbb{R} \to [0,1]$ be a smooth function with $\mu(t) = 0$ for $t > 1$ and $\mu(t) = 0$ for $t < 0$. Define $\omega_0 = \theta + C\frac{1}{2}dd^c(\rho^2) + dd^c[\mu(\rho - R)v]$. Choose $R$ large enough that the support of $\theta$ is contained in $\{\rho < R\} \subset \hat{X}$. Then for $C > 0$ chosen sufficiently large $\omega_0$ is a Kähler form with the required properties. \hfill $\Box$

Now suppose $X = C(S) \cup \{o\}$ be a Ricci-flat Kähler cone. And let $\pi : \hat{X} \to X$ be a crepant resolution. Recall, this means that $\pi^*K_X = K_{\hat{X}}$. Thus $K_{\hat{X}}$ is trivial. Let $\Omega$ be the holomorphic $n$-form on $X$ as in Proposition 3.3 Then $\pi^*\Omega$ is a nowhere
vanishing holomorphic form on $\hat{X}$, which we again denote by $\Omega$. If $\omega_0$ is a Kähler form as in Proposition 3.10 then a Ricci-potential of $(\hat{X}, g_0, \omega_0)$ is $f = \log \left( \frac{\Omega \wedge \bar{\Omega}}{\omega_0^m} \right)$ where we choose the constant $c = im^2 \frac{m!}{\omega_0^m}$ so that $f = 0$ outside a compact set.

Part (ii) of Theorem 3.10 gives a $\phi \in C_{2-2m}^\infty (X)$ of the form $\phi = A\rho^2 - 2m + \psi$ where $\psi \in C_{2+\beta}^\infty$, where $\beta < -2m$. And $\omega = \omega_0 + dd^c\phi$ is the Kähler form of the Ricci-flat Kähler metric in Theorem 1.1.

Suppose that $\omega'$ is another Ricci-flat Kähler form with $[\omega'] = [\omega]$ and $|\omega' - \omega| \in C_{\beta+2}^\infty(X)$ with $\beta < -m$. By Proposition 3.9 there is a smooth $u \in C_{\beta+2}^\infty(X)$ with $\omega' - \omega = dd^c u$. Then $u$ solves $(\omega + dd^c u)^m = \omega^m$. The uniqueness result of Theorem 3.10 then shows that $u = 0$.

The constant $A$ in (69) turns out to be an invariant of the Kähler class in $H_c^*(\hat{X}, \mathbb{R})$. Recall that $\omega_0 = \theta + \frac{1}{2} Cdd^c(\rho^2) + dd^c [\mu(\rho - R)v]$ where the first and third terms have compact support. Expanding and using that the integral of a compactly supported exact form is zero gives

$$\int_X (1 - e^f)\omega_0^m = \int_X \omega_0^m - e^{\Omega} \wedge \bar{\Omega}$$

$$= \int_X \theta^m + \left( \frac{1}{2} Cdd^c(\rho^2) \right)^m - \left( \frac{1}{2} Cdd^c(\rho^2) \right)^m$$

$$= \int_X \theta^m.$$

Therefore, if we consider the Kähler class $[\omega] \in H^*_c(\hat{X}, \mathbb{R})$, then

$$A = \frac{1}{(m-1)!} \int_X (1 - e^f) d\mu = \frac{1}{(m-1)!m!} [\omega]^{1+m},$$

where $\Omega = \text{Vol}(S) = \{ r = 1 \} \subset C(S)$.

We also have the following result on the Kähler potential of Ricci-flat metrics of Theorem 1.1.

**Theorem 3.16.** Let $\hat{X}$ be a crepant resolution of a Ricci-flat Kähler cone $X = C(S) \cup \{ o \}$. Then in each Kähler class on $\hat{X}$, with $C > 0$ as in Proposition 3.12, there is a unique Ricci-flat Kähler metric $g$ with Kähler form $\omega$ which satisfies

$$\pi_*(\omega) = \frac{1}{2} Cdd^c(r^2) + Add^c(r^{2-2m}) + dd^c(\psi),$$

on $\{ x \in X : r(x) > R \}$. Here $A$ is given by the Kähler class $[\omega]$ in (70), and $\psi \in C^{\infty}(X)$ with $\gamma < 2 - 2m$.

**Remark 3.17** There remains the question of the optimal $\gamma < 2 - 2m$ giving the decay of $\psi$ in Theorem 3.16. This comes down to finding the largest $\epsilon > 0$ in Theorem 2.30. In general, we only know $\epsilon > 0$. But in Theorem 3.16 the AC Kähler manifold $(\hat{X}, g_0, \omega_0)$ has “boundary” $S$ which is Einstein. The condition for the Laplacian

$$\Delta : L_{k+2, s}^p(\hat{X}) \to L_{k, s-2}^p(\hat{X})$$

to be Fredholm is well known [19]. There is a family of operators on $S$

$$I(\Delta, \lambda) = \lambda^2 - (2m - 2)\sqrt{-1}\lambda + \Delta_S,$$
for \( \lambda \in \mathbb{C} \) where \( \Delta_S \) is the Laplacian on \( S \). Then \( \text{Spec}(I, \lambda) \) is the set of \( \lambda \) for which
\[
I(\Delta, \lambda) : L^p_{k+2}(S) \rightarrow L^p_k(S)
\]
does not admit a bounded inverse. In our case
\[
\text{Spec}(I, \lambda) = \{0, (2m-2)\sqrt{-1}, \mu_j^+ \sqrt{-1}, \mu_j^- \sqrt{-1}, \ldots | j = 1, 2, \ldots \}
\]
where \( \mu_j^\pm \sqrt{-1} \) are the two solutions of \( x^2 - (2m-2)\sqrt{-1}x + \lambda_j = 0 \) with \( \lambda_j \) the \( j \)-th eigenvalue of \( \Delta_S \). It was shown in [19] that if \( \text{Im} \text{Spec}(I, \lambda) \) denote the imaginary components, then (72) is Fredholm for \( -\delta \notin \text{Im} \text{Spec}(I, \lambda) \).

If \( (S, g_S) \) is Sasaki-Einstein then by Lichnerowicz’s Theorem \( \lambda_1 \geq 2m-1 \) with equality only if \( S \) is isometric to the sphere. And one can check that \( \mu_1^\pm \geq 2m-1 \) with equality only if \( S \) is isometric to a sphere. And for \( \delta \in (-\mu_1^+, 2-2m) \) the operator (72) is Fredholm with index \(-1\). So in Theorem 3.16 one will actually have \( \psi \in C_{1-2m}^\infty(X) \). \[\text{1}\]

4. Examples

We consider some examples of asymptotically conical Ricci-flat Kähler manifolds given by Theorem 1.1. These examples are \( AC(2n, \infty) \) Ricci-flat examples, and are either resolutions of toric Kähler cones or resolutions of hypersurface singularities. See [31] for many more examples on resolutions of Kähler cones. Here we just give examples to give the reader an idea of the scope of examples.

Also in [29] examples are constructed on affine varieties which are of type \( AC(2n, k) \) for large \( k > 0 \).

4.1. Resolutions of hypersurface singularities. We describe how examples can be constructed from resolutions of weighted homogeneous hypersurface singularities. Let \( w = (w_0, \ldots, w_m) \in (\mathbb{Z}^+)^m+1 \) with \( \gcd(w_0, \ldots, w_m) = 1 \). We have the weighted \( \mathbb{C}^* \)-action on \( \mathbb{C}^{m+1} \) given by \((z_0, \ldots, z_m) \rightarrow (\lambda^{w_0} z_0, \ldots, \lambda^{w_m} z_m) \) for \( \lambda \in \mathbb{C}^* \). A polynomial \( f \in \mathbb{C}[z_0, \ldots, z_m] \) is \textit{weighted homogeneous} of degree \( d \in \mathbb{Z}_+ \) if
\[
f(\lambda^{w_0} z_0, \ldots, \lambda^{w_m} z_m) = \lambda^d f(z_0, \ldots, z_m).
\]

There is a \textit{weighted Sasaki structure} on the sphere \( S^{2m+1}_w \) for which the Reeb vector field \( \xi_w \) generates the \( S^1 \)-action induced by the above weighted action. See [30] for details. The cone \( C(S^{2m+1}_w) \) is biholomorphic to \( \mathbb{C}^{m+1} \setminus \{0\} \), but with a much different metric. If \( f \) is a weighted homogeneous polynomial, then the Kähler cone structure of \( C(S^{2m+1}_w) \) restricts to \( X_f = \{ z \in \mathbb{C}^{m+1} : f(z) = 0 \} \). And similarly the Sasaki structure on \( S^{2m+1}_w \) restricts to the link \( S_f := X_f \cap S^{2m+1} \). This is given in the diagram:

\[
\begin{align*}
X_f & \hookrightarrow C(S^{2m+1}_w) \\
\cup & \\
S_f & \hookrightarrow S^{2m+1}_w \\
\downarrow & \\
Z_f & \hookrightarrow \mathbb{C}P(w)
\end{align*}
\]

Here \( Z_f \) is a hypersurface in the weighted projective space \( \mathbb{C}P(w) \).

\[\text{1}\]Ryushi Goto pointed this out to me.
Proposition 4.1. The Kähler cone $C(S_f)$ admits a nowhere vanishing holomorphic $m$-form $\Omega$ satisfying $\mathcal{L}_\xi \Omega = \text{im} \Omega$, after possibly rescaling the Reeb vector field $\xi$, if and only if $d < |w| = \sum_{j=0}^{m} w_j$.

This is precisely the condition that the orbifold canonical bundle $K_{Z_f}$ on $Z_f$ is negative.

With Proposition 4.1 satisfied, we are interested in transversally deforming the Sasaki structure of $S_f$ to a Sasaki-Einstein structure. If $\eta$ is the contact structure of $S_f$ then $\omega^T = \frac{1}{2} d\eta$ is the Kähler structure transversal to the foliation generated by the Reeb field $\xi$. A transversal deformation is a new Sasaki structure with transversal Kähler form

$$ (\omega^T)' = \omega^T + dd^c \phi, $$

for some basic $\phi \in C^\infty_{\text{reg}}(S)$. The new contact form is $\eta' = \eta + 2d^c \phi$. And one can show that the Kähler structure on the cone becomes $\omega' = \frac{1}{2} dd^c (r')^2$ where $r' = e^\phi r$. Obtaining a Sasaki-Einstein structure is equivalent to solving the transversal Kähler-Einstein condition

$$ \text{Ric}(\omega^T)' = 2m(\omega^T)'. $$

Condition (75) implies that

$$ \text{Ric}(\omega^T) - 2m\omega^T = dd^c h. $$

And solving (77) is equivalent to solving the transversal Monge-Ampère equation

$$ (\omega^T + dd^c \phi)^m = e^{-2m\phi + h}(\omega^T)^m. $$

See [5, 3, 4] for more on solving (78) to find Sasaki-Einstein metrics. In particular, if $f$ is a Brieskorn-Pham polynomial, $f = \sum_{j=0}^{m} z_j^{a_j}$. Then [5, Theorem 34] gives simple numerical conditions on the $a_j$ for (78) to be solvable.

For example, consider

$$ f = z_0^m + z_1^m + \cdots + z_{m-1}^m + z_m^k. $$

Then these conditions are satisfied if $k > m(m-1)$, and $X_k := \{ z \in \mathbb{C} : f(z) = 0 \} \subset \mathbb{C}^{m+1}$ has a Ricci-flat Kähler cone structure.

Let $\mathbb{C}^{\tilde{m}+1}$ be the blow-up of $\mathbb{C}^{m+1}$. If $X' \subset \mathbb{C}^{\tilde{m}+1}$ is the birational transform and $E = X' \cap \mathbb{C}^{\tilde{m}+1} \subset \mathbb{C}^{\tilde{m}+1}$ is the exceptional divisor, then adjunction gives

$$ K_{X'} = \pi^*K_X + (m - \deg f)E. $$

Thus $\pi : X' \to X_k$ is crepant for $k \geq m$. It is not difficult to see that $X'$ has one singularity isomorphic to $X_{k-m}$. If $k = 0$ or 1 mod $m$, then by repeatedly blowing up $\frac{m}{2}$ times we get a smooth crepant resolution $\pi : \hat{X}_k \to X_k$. Therefore, if $k > n(n-1)$ and $k = 0$ or 1 mod $m$, then Theorem [4, 1] gives a $\frac{m}{2}$ family of Ricci-flat Kähler metrics on $\hat{X}_k$ converging to the cone metric as in (1).

4.2. Toric examples. One easy way to construct examples of Ricci-flat metrics on resolutions is to consider resolutions of toric Kähler cones.

Definition 4.2. A Kähler cone $(C(S), \bar{g})$, $\dim_C C(S) = m$, is toric if it admits an effective isometric action of the torus $T = T^m$ which preserves the Euler vector field $r \partial_r$. 

The associated Sasaki manifold \((S, g)\) is said to be toric. Let \(\mathfrak{t}\) be the Lie algebra of \(T^m\). It follows that the Reeb vector field \(\xi \in \mathfrak{t}\).

Since \(T^m\) preserves the Kähler form \(\omega = \frac{1}{2} d(r^2 \eta)\) and further preserves \(r^2 \eta\), there is a moment map

\[
\mu : C(S) \rightarrow \mathfrak{t}^*, \quad \langle \mu(x), X \rangle = \frac{1}{2} r^2 \eta(X_S(x)),
\]

where \(X_S\) denotes the vector field on \(C(S)\) induced by \(X \in \mathfrak{t}\). We have the moment cone defined by

\[
\mathcal{C} := \mu(C(S)) \cup \{0\},
\]

which from \([14]\) is a strictly convex rational polyhedral cone. Recall that this means that there are vectors \(u_i, i = 1, \ldots, d\) in the integral lattice \(\mathbb{Z}_T = \ker\{\exp(2\pi i \cdot) : \mathfrak{t} \rightarrow T\}\) such that

\[
\mathcal{C} = \bigcap_{j=1}^d \{y \in \mathfrak{t}^* : \langle u_j, y \rangle \geq 0\}.
\]

The elements \(u_j \in \mathbb{Z}_T, \ j = 1, \ldots, d\), span a cone \(\mathcal{C}^*\) in \(\mathfrak{t}\) dual to \(\mathcal{C}\). Then \(\mathcal{C}^*\) and all of its faces define a fan \(\Delta\) characterizing \(C(S) \cup \{0\}\) as an algebraic toric variety. (cf. \([24]\)) There is a nowhere vanishing holomorphic \(m\)-form satisfying Proposition \(3.4\) precisely when there is a \(\gamma \in \text{Hom}_\mathbb{Z}(\mathbb{Z}_T, \mathbb{Z})\) with \(\gamma(u_j) = 1, \ j = 1, \ldots, d\). This is the condition that \(C(S) \cup \{0\}\) is Gorenstein.

The result of A. Futaki, H. Ono, and G. Wang on the existence of Sasaki-Einstein metrics on toric Sasaki manifolds makes toric geometry a propitious source of examples.

**Theorem 4.3** ([14] [8]). Let \(C(S) \cup \{0\}\) be a Gorenstein toric Kähler cone with toric Sasaki manifold \(S\). Then we can deform the Sasaki structure by varying the Reeb vector field and then performing a transverse Kähler deformation to a Sasaki-Einstein metric. The Reeb vector field and transverse Kähler deformation are unique up to isomorphism.

Define \(H_{\gamma} = \{\gamma = 1\} \subset \mathfrak{t} \cong \mathbb{R}^m\). The intersection \(P_{\Delta} = H_{\gamma} \cap \mathcal{C}^*\) is an integral polytope in \(H_{\gamma} \cong \mathbb{R}^{m-1}\). A toric crepant resolution

\[
\pi : X_\Delta \rightarrow X_{\Delta}
\]

is given by a nonsingular subdivision \(\Delta\) of \(\Delta\) with every 1-dimensional cone \(\tau_i \in \Delta(1), i = 1, \ldots, N\) generated by a primitive vector \(u_i := \tau_i \cap H_{\gamma}\). This is equivalent to a basic, lattice triangulation of \(P_{\Delta}\). **Lattice** means that the vertices of every simplex are lattice points, and **basic** means that the vertices of every top dimensional simplex generates a basis of \(\mathbb{Z}^{n-1}\). Note that a maximal triangulation of \(P_{\Delta}\), meaning that the vertices of every simplex are its only lattice points, always exists. Every basic lattice triangulation is maximal, but the converse only holds in dimension 2.

We want Kähler structures on the resolution \(X_\Delta\). This is given by a strictly convex support function \(h \in \text{SF}(\Delta, \mathbb{R})\) on \(\Delta\). This is a real valued function which is piecewise linear on the cones of \(\Delta\). Convexity means that \(h(x+y) \geq h(x) + h(y)\) for \(x, y \in |\Delta|\), the support of \(\Delta\). Let \(l_\sigma\) define \(h\) on the \(m\)-cone \(\sigma\). **Strict convexity** means that \(\langle l_\sigma, x \rangle \geq h(x)\), for all \(x \in |\Delta|\), with equality only if \(x \in \sigma\).
The following is proved by taking a torus Hamiltonian reduction of $\mathbb{C}^N$. See [6] and also [31].

**Proposition 4.4.** For each strictly convex support function $h \in \text{SF}(\Delta, \mathbb{R})$ there is a Kähler structure $\omega_h$ so that $(X_\Delta, \omega_h)$ is a Hamiltonian Kähler manifold and the image of the moment map is the polyhedral set

$$C_h := \bigcap_{j=1}^{N} \{ y \in \mathfrak{t}^* : \langle u_j, y \rangle \geq \lambda_j \}.$$ 

If $h$ satisfies $h(u_j) = 0$ for $j = 1, \ldots, d$, then $[\omega_h] \in H^2_c(X_\Delta, \mathbb{R})$. The $u_j \in \text{Int} P_\Delta, j = d+1, \ldots, N$, correspond to the prime divisors $D_j$ in $E = \pi^{-1}(o)$. For each $j = d+1, \ldots, N$, let $c_j \in H^2_c(X_\Delta, \mathbb{R})$ be the Poincaré dual of $[D_j]$ in $H_{2n-2}(X_\Delta, \mathbb{R})$. Then

$$[\omega_h] = -2\pi \sum_{j=d+1}^{N} \lambda_j c_j.$$ 

If $[\omega_h] \in H^2_c(X_\Delta, \mathbb{R})$, then we can apply Proposition 5.15 to construct a Kähler metric $\omega_0$ with Ricci potential $f = \log \left( \frac{\omega_0^d}{\omega_0^{2n}} \right)$. If $[\omega_h]$ is not compactly supported, then an initial metric is constructed by R. Goto [11, §5] with Ricci potential $f \in C^\infty(X)$. In both cases the initial Kähler metrics $\omega_0$ and Ricci potentials $f$ can be taken $T^m$-invariant. We get the toric version of Theorem 1.1.

**Theorem 4.5.** Let $\pi : X_\Delta \to C(S) \cup \{ o \}$ be a crepant resolution of a toric Kähler cone. Then for each strictly convex $h \in \text{SF}(\Delta, \mathbb{R})$, there is a $T^m$-invariant Ricci-flat Kähler metric $g$ whose Kähler form satisfies $[\omega] = [\omega_h]$. If $[\omega_h] \in H^2_c(X_\Delta, \mathbb{R})$, then $g$ is unique and is asymptotic to the cone metric as in [7], otherwise $g$ converges as $[2]$.

If $[\omega_h] \in H^2_c(X_\Delta, \mathbb{R})$, then from (70) we have

$$A = \frac{1}{(m-1)m!\Omega} [\omega]^m = \frac{-2\pi}{(m-1)m!\Omega} \sum_{j=d+1}^{N} \lambda_j c_j [\omega]^{m-1}$$

$$= \frac{-2\pi}{(m-1)m!\Omega} \sum_{j=d+1}^{N} \lambda_j \int_{E_j} [\omega_h]^{m-1} < 0.$$ 

(84)

Note that all the quantities in (84) can be computed from $h$ in terms of values of various polytopes.

When $\dim_{\mathbb{C}} X_\Delta = 3$ there always exists a toric crepant resolution $X_\Delta$. And further, if $X_\Delta$ is not the quadric cone $\{ z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0 \} \subset \mathbb{C}^4$, then it admits a toric crepant resolution $X_\Delta$ with a strictly convex support function $h$ so that $[\omega_h] \in H^2_c(X_\Delta, \mathbb{R})$. See [30] for more details.

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