Hadronic Light-by-Light Contribution to Muon $g - 2$ in Chiral Perturbation Theory

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Abstract

We compute the hadronic light-by-light scattering contributions to the muon anomalous magnetic moment, $a_{\mu}^{LL}(\text{had})$, in chiral perturbation theory that are enhanced by large logarithms and a factor of $N_C$. They depend on a low-energy constant entering pseudoscalar meson decay into a charged lepton pair. The uncertainty introduced by this constant is $\pm 60 \times 10^{-11}$, which is comparable in magnitude to the present uncertainty entering the leading-order vacuum polarization contributions to the anomalous moment. It may be reduced to some extent through an improved measurement of the $\pi^0 \rightarrow e^+e^-$ branching ratio. However, the dependence of $a_{\mu}^{LL}(\text{had})$ on non-logarithmically enhanced effects cannot be constrained except through the measurement of the anomalous moment itself. The extraction of information on new physics would require a future experimental value for the anomalous moment differing significantly from the 2001 result reported by the E821 collaboration.
The recently reported measurement of the muon anomalous magnetic moment $a_\mu$ by the E821 collaboration has generated considerable excitement about possible evidence for new physics. The interpretation of the result, however, depends in part on a reliable treatment of hadronic contributions to $a_\mu$ which arise at two- and three-loop order. While the vacuum polarization contribution can be constrained by $e^+e^-$ experiments and $\tau$ decays and appears to be under adequate theoretical control, recent analysis of the hadronic light-by-light scattering contribution $a^{LL}_\mu(\text{had})$ have uncovered a sign error in previous calculations of the dominant, pseudoscalar pole term. The resulting sign change reduces the $2.6\sigma$ deviation of $a_\mu$ from the Standard Model prediction reported in by one standard deviation, thereby modifying considerably the original interpretation of the result.

The commonly quoted values for $a^{LL}_\mu(\text{had})$ (after incorporating the corrected overall sign) rely on model treatments of the off shell $\pi^0\gamma^*\gamma^*$, $\eta\gamma^*\gamma^*$ and $\eta'\gamma^*\gamma^*$ interactions. While the amplitude for pseudoscalar decay into two real photons is dictated by the chiral anomaly, the off-shell amplitudes relevant for $a^{LL}_\mu(\text{had})$ are affected by non-perturbative strong interactions whose effects cannot yet be computed with sufficient precision from first principles in QCD. Similarly, the contributions from other hadronic intermediate states besides the $\pi^0$, $\eta$, and $\eta'$ cannot be computed reliably at present. The analysis of $a^{LL}_\mu(\text{had})$ falls naturally under the purview of chiral perturbation theory ($\chi$PT), which provides systematic, model-independent framework for parameterizing presently incalculable hadronic effects. The static quantity $a^{LL}_\mu(\text{had})$ has an expansion in powers of $p/\Lambda$, where $p$ is a small mass of order $m_\mu$ or $m_\pi$ and $\Lambda$ is a hadronic scale, typically taken to be $\sim 4\pi F_\pi \sim 1\text{GeV}$. The coefficients appearing in the expansion depend in part on a priori unknown “low-energy constants” (LEC’s), which parameterize the effects of non-perturbative short distance physics. In principle, the LEC’s may be determined from an appropriate set of experimental measurements.

In this Letter, we perform a $\chi$PT calculation of $a^{LL}_\mu(\text{had})$ including all the logarithmically enhanced contributions to $a^{LL}_\mu(\text{had})$ that arise at order $N_C\alpha^3p^2/\Lambda^2$, where $N_C$ is the number of quark colors. We identify the dependence of $a^{LL}_\mu(\text{had})$ on the large logarithms of $\Lambda/p$ as well as on the relevant LEC’s. The result for the large $\ln^2$ term – which is determined entirely by gauge-invariance and the chiral anomaly – was first given in Ref. and was used to uncover the sign error in previous model calculations. However, only part of the large ln term is fixed by symmetry considerations. It receives an additional contribution involving $\chi$, a LEC entering the rate for pseudoscalar decay into leptons. The $\chi$-dependent piece was also computed in Ref. An analytic calculation of the $\chi$-independent large ln term was performed by the authors of Ref., who used a model for the off-shell $P\gamma^*\gamma^*$ form factor to regulate the two-loop amplitude and assumed $m_\mu$ was almost equal to $m_\pi$. The results of the latter calculation also contain a model prediction for the non-logarithmic $O(N_C\alpha^3p^2/\Lambda^2)$ contribution to $a^{LL}_\mu(\text{had})$.

In what follows, we carry out a consistent $\chi$PT treatment of $a^{LL}_\mu(\text{had})$, providing the first complete, model-independent analysis of the terms enhanced by large logarithms through $O(N_C\alpha^3p^2/\Lambda^2)$. The sum of these terms is known, since existing measurements for $\eta \to \mu^+\mu^-$ and $\pi^0 \to e^+e^-$ branching ratios fix the value of $\chi$. However, the uncertainty in $a^{LL}_\mu(\text{had})$ from the error in $\chi$ is significant: $\pm 60 \times 10^{-11}$, which is roughly the same size as

\[1\text{We treat } m_\mu \text{ and } m_\pi \text{ to be of the same order and take both to be small compared with } \Lambda.\]
the present theoretical uncertainty in the leading-order vacuum polarization contributions to $a_\mu$. In principle, an improved measurement of the $\pi^0 \to e^+e^-$ branching ratio could reduce this source of uncertainty.

A more serious consideration involves the contributions to $a_\mu^{LL}(\text{had})$ that are beyond the order at which we compute. These include effects of order $\mathcal{O}(\alpha^3 p^2/\Lambda^2)$ which are not enhanced by a factor of $N_C$ and order $\mathcal{O}(N_C \alpha^3 p^2/\Lambda^2)$ terms that are not enhanced by large logarithms. We parameterize all these effects in terms of $\tilde{C}$. For simplicity we will refer to $\tilde{C}$ as a low energy constant even though it has nonanalytic dependence on the quark masses. At present, $\tilde{C}$ cannot be determined in a model-independent way without reliance on the measurement of $a_\mu$ itself. Even the sign of the $\tilde{C}$-dependent term cannot presently be fixed in an model-independent manner. The presence of this LEC renders the interpretation of $a_\mu$ in terms of new physics problematic, since the size of the $\tilde{C}$-dependent contribution could be as large as the present experimental error in $a_\mu$ [1]. Below we discuss the conditions under which one might still be able to extract information on new physics from an $a_\mu$ measurement.

To leading order in $N_C$, the lowest-order chiral contributions to $a_\mu^{LL}(\text{had})$, enhanced by large logarithms, arise from the two- and one-loop graphs of Fig 1. Taking $m_\pi$ and $m_\mu$ as being of $\mathcal{O}(p)$, the leading, large logarithmic contributions arise at order $N_C \alpha^3 p^2/\Lambda^2$. The two-loop graphs (Fig. 1a) contain an overall, superficial cubic divergence as well as a linearly-divergent one-loop subgraph involving two photons and a muon line. The latter must be regulated by adding the appropriate one-loop counterterm (ct) (Fig. 1b). The one-loop graphs also contain an insertion of $\chi(\mu)$. The sum of these graphs contains a residual divergence, which must be removed by the appropriate magnetic moment ct (Fig. 1c). Associated with this ct is a finite piece which, as discussed above, can only be fixed in a model-independent way by the measurement of $a_\mu$ itself. Additional contributions also arise from the graphs such as those appearing in Fig. 2. Although subdominant in $N_C$ counting the contribution of the charged pion loop in 2b (and related diagrams) is leading order in $p/\Lambda$. The order $\mathcal{O}(\alpha^3 N_C^0)$ term arising from the three loop graphs with a charged pion loop – which we denote by $a_\mu^{LL}(\text{had})_{1.o.}$ – has been computed in Ref. [15,5]. The result is finite and contains no large logarithms. We will include this contribution when we compare the theoretical prediction for $a_\mu^{LL}(\text{had})$ with experiment.

At order $\alpha^3 p^2/\Lambda^2$ there will be contributions to $a_\mu^{LL}(\text{had})$ from higher dimension operators inserted at the vertices in the charged pion loop graph of Fig.2 and from four-loop graphs containing an additional hadronic loop. Some of these should contain large logarithms. However, these are suppressed by a factor of $N_c$ compared to the logarithmically enhanced pieces that we compute. As remarked earlier we absorb these and many other subdominant contributions (e.g., Fig. 2a) into $\tilde{C}$ and do not discuss them further here.

As inputs for amplitudes of Fig. 1, we require the Wess-Zumino-Witten $P\gamma\gamma$ interaction Lagrangian [13]:

$$\mathcal{L}_{WZW} = \frac{\alpha N_C}{24\pi F_\pi} \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \left( \pi^0 + \frac{1}{\sqrt{3}} \eta \right) + \cdots \quad (1)$$

as well as the leading-order operator contributing to the decays $P \to \ell^+\ell^-$ [14]:

$$\mathcal{L}_{P\ell^+\ell^-} = \frac{i3N_C\alpha^2}{96\pi^2} \mu \gamma^\lambda \gamma_5 \mu \chi_1 \text{Tr} \left( Q^2 \Sigma^\dagger D_\lambda \Sigma - Q^2 D_\lambda \Sigma^\dagger \Sigma \right) \quad (2)$$
\[ + \chi_2 \text{Tr} \left( Q \Sigma^\dagger Q D_\lambda \Sigma - Q D_\lambda \Sigma^\dagger Q \Sigma \right) \]
\[ = - \frac{N_C a^2}{48 \pi^2 F_\pi} (\chi_1 + \chi_2) \bar{\mu} \gamma^\lambda \gamma_5 \mu \left( \partial_\lambda \pi^0 + \frac{1}{\sqrt{3}} \partial_\lambda \eta \right) + \cdots . \] (3)

where

\[ \Sigma = \exp \left( \sum \lambda^a \pi^a / F_\pi \right) \] (4)

Here, \( F_\pi = 92.4 \text{ MeV} \) is the pion decay constant, \( \lambda^a \) denote the Gell-Mann SU(3) matrices, and \( D_\lambda \) is the covariant derivative. Note that in contrast to the conventions of Ref. [14], we have made the \( N_C \)-dependence of the LEC’s \( \chi_i \) explicit for the sake of clarity.

In computing the loop amplitudes involving these operators, it is important to employ a regulator which maintains the consistent power-counting of the chiral expansion. To that end, we employ dimensional regularization, where we continue only momenta (and not Dirac matrices) into \( d = 4 - 2\epsilon \) dimensions. The relation between bare and renormalized couplings is,

\[ \chi_1^0 + \chi_2^0 = \chi_1 + \chi_2 - \frac{6}{\epsilon} \equiv \chi(\mu) - \frac{6}{\epsilon} . \] (5)

Using Eq. (5) and adding the amplitudes for Fig. 1a,b, we obtain the divergent part of the two-loop amplitude

\[ \mathcal{M} = -e \left( \frac{\alpha}{\pi} \right)^3 \left( \frac{N_C}{3} \right)^2 \left( \frac{m_\mu}{F_\pi} \right)^2 \left( \frac{1}{32\pi^2} \right) \left[ \frac{3}{2\epsilon^2} - \left( \frac{\chi(\mu)}{2} + \frac{3}{4} \right) \frac{1}{\epsilon} \right] \bar{u} i \sigma_{\alpha \beta} q^\alpha \varepsilon^\beta \frac{1}{2m_\mu} u . \] (6)

where \( q^\alpha \) and \( \varepsilon^\beta \) are the photon momentum and polarization, respectively. We remove this divergence using a magnetic moment counterterm. The bare coupling \( C_0 \) and renormalized coupling \( C(\mu) \) are related by

\[ \mathcal{M}_0 = e \left( \frac{\alpha}{\pi} \right)^3 \left( \frac{N_C}{3} \right)^2 \left( \frac{m_\mu}{F_\pi} \right)^2 \left( \frac{1}{32\pi^2} \right) C_0 \bar{u} i \sigma_{\alpha \beta} q^\alpha \varepsilon^\beta \frac{1}{2m_\mu} u \] (7)

\[ C_0 = C(\mu) + \left[ \frac{3}{2\epsilon^2} - \left( \frac{\chi(\mu)}{2} + \frac{3}{4} \right) \frac{1}{\epsilon} \right] \] (8)

The light-by-light contribution to the anomalous magnetic moment \( a_\mu^{\text{LL}}(\text{had}) \) is a physical quantity and has no dependence on the subtraction point \( \mu \). The \( \mu \)-dependence of the diagrams cancels that of the couplings \( C(\mu) \) and \( \chi(\mu) \). To obtain the \( \mu \)-dependence of the couplings we require that the bare Green’s functions corresponding to the sum of Figs. 1a-c and the to the \( P \ell^+ \ell^- \) one-loop subgraphs, respectively, be independent of the subtraction scale. Doing so leads to a coupled set of renormalization group equations for \( \chi(\mu) \) and \( C(\mu) \):

\[ \mu \frac{d \chi}{d \mu} = -12 \] (9)

\[ \mu \frac{d C}{d \mu} = -3 - \chi \] (10)

The solution is

3
\[
\chi(\mu) = 12 \ln(\mu_0/\mu) + \chi(\mu_0) \tag{11}
\]
\[
C(\mu) = 6 \ln^2(\mu_0/\mu) + [\chi(\mu_0) + 3] \ln(\mu_0/\mu) + C(\mu_0) \tag{12}
\]

At a scale \(\mu_0 = \Lambda \sim 1\ \text{GeV}\), the constants \(C(\mu_0)\) and \(\chi(\mu_0)\) contain no large logarithms of the form \(\ln^k(\Lambda/p)\) \((k = 1, 2)\) where \(p\) is around \(m_\mu\) or \(m_\pi\). For \(\mu\) of \(O(p)\), however, the Feynman diagrams contain no such large logarithms, and they live entirely in \(C(\mu)\) and \(\chi(\mu)\). Hence, the resulting expression for \(a^{LL}_\mu(\text{had})\) is, in the \(\overline{MS}\) scheme,

\[
a^{LL}_\mu(\text{had}) = a^{LL}_\mu(\text{had})_{1.0.}
+ \frac{3}{16} \left( \frac{\alpha}{\pi} \right)^3 \left( \frac{m_\mu}{F_\pi} \right)^2 \left( \frac{N_C}{3\pi} \right)^2 \left[ \ln^2 \left( \frac{\Lambda}{\mu} \right) + \left[ -f(r) + \frac{1}{2} + \frac{1}{6} \chi(\Lambda) \right] \ln \left( \frac{\Lambda}{\mu} \right) + \tilde{C} \right], \tag{13}
\]

where \(\mu\) is of order \(p\) and could be set equal to either \(m_\mu\) or \(m_\pi\). Recall that \(\Lambda \sim 4\pi F_\pi \sim 1\ \text{GeV}\). The function \(f(r)\), with \(r = m_\mu^2/m_\pi^2\), arises from the one loop diagram with a coupling proportional to \(\chi(\mu)\) (Fig. 1b) and is given by\(^2\)

\[
f(r) = \ln \left( \frac{m_\mu^2}{\mu^2} \right) + \frac{1}{6} r^2 \ln r - \frac{1}{6} (2r + 13) + \frac{1}{3} (2 + r) \sqrt{r(4 - r)} \cos^{-1} \left( \frac{\sqrt{r}}{2} \right). \tag{14}
\]

Note that we have absorbed all the remaining terms, including those proportional to \(C(\Lambda)\) and \(\chi(\Lambda)\) not enhanced by large logarithms, into \(\tilde{C}\).

The logarithmically enhanced hadronic light-by-light contributions to \(a_\mu\) are renormalization scheme-independent. However, the values of \(\chi(\Lambda), f(r), \) and the constant appearing in the renormalization group equation for \(C\) (leading to the factor of 1/2 in the ln term in Eq. (13) ) depend on one’s choice of scheme\(^3\). This scheme-dependence cancels in the sum of their contributions to \(a^{LL}_\mu(\text{had})\). For our calculations, we adopted a scheme in which the loop integrals were evaluated in \(d\)-dimensions with \(d > 4\) (corresponding to \(\epsilon < 0\)), while the Dirac matrices and photon polarization indices were taken as four-dimensional. For this choice we have \(\eta_{\mu\nu} \text{Tr}(\gamma^\mu\gamma^\nu) = 16\) instead of 4\(d\). Moreover, the value of \(\chi(\Lambda)\) in this scheme is the same as that in \([14]\) but is four less than the \(\chi(\Lambda)\) used in \([17]\). An alternative, equally convenient scheme is again to treat the Dirac matrices, epsilon tensors, and photon polarizations as four dimensional; take \(d < 4\) (\(\epsilon > 0\)); and rewrite (3) as

\[
i \frac{N_C \alpha^2}{48\pi^2 F_\pi} \left( \frac{\chi_1 + \chi_2}{6} \right) e^{\mu\alpha\nu\lambda} \bar{\mu} \gamma_\mu \gamma_\beta \gamma_\mu \gamma_\nu \eta_{\alpha\beta} \partial_\lambda \left( \pi^0 + \frac{1}{\sqrt{3}} \eta \right) + \cdots, \tag{15}
\]

where the metric tensor \(\eta_{\alpha\beta}\) is \(d\)-dimensional. In this scheme the value of \(\chi(\Lambda)\) is still the same as in \([14]\), but \(f(r) \to f(r) + 3/2\) and the \(-3\) in the renormalization group equation for \(C\) becomes \(-12\). Notice that the total value for the logarithmically enhanced contribution to \(a^{LL}_\mu(\text{had})\) is unchanged.

\(^2\)Since we only compute the terms enhanced by large logarithms to get \(f\) we replace \(\chi(\mu)\) by 12\(\ln(\Lambda/\mu)\).

\(^3\)We will discuss the issue of scheme dependence and give more details of the calculation in a future publication.
As a check on the result in Eq. (13), one may compute the one- and two-loop amplitudes with the insertion of \( \chi(\Lambda) \) in Fig. 1b and \( C(\Lambda) \) in Fig. 1c. In this case, all of the large logarithms arise from the Feynman amplitudes and not from the operator coefficients. Using an explicit calculation, we have verified in the limit \( m_\rho \to 0 \) that this procedure exactly reproduces the expression in Eq. (13). We note that the \( \ln^2 \) term and the term proportional to \( \chi \) agree with the expression in Ref. [4].

Chiral perturbation theory can be used for the \( \eta \to \mu^+\mu^- \) amplitude [4], and the LEC \( \chi(\Lambda) \) can be deduced from the measured \( \eta \to \mu^+\mu^- \) branching ratio [10]. This yields \( \chi(1\text{GeV}) = -14^{+4}_{-8} \) or \(-39^{+5}_{-4}\), where we have scaled the results in Ref. [7] from \( \Lambda = m_\rho \) up to \( \Lambda = 1 \text{ GeV} \) and subtracted four. Note that because the \( \eta \to \mu^+\mu^- \) branching ratio is a quadratic function of \( \chi \), two different values for this LEC may be extracted from experiment.

Our calculation of the logarithmically enhanced contributions to \( a_{\mu}^{LL}(\text{had}) \) only involved the use of chiral \( SU(2)_L \times SU(2)_R \) while this extraction of the LEC involves the use of chiral \( SU(3)_L \times SU(3)_R \). Since one expects chiral perturbation theory to work better in the case where only the pions are treated as light it is desirable to have an extraction of \( \chi \) that only relies on chiral \( SU(2)_L \times SU(2)_R \). This can be done using the measured \( \pi^0 \to e^+e^- \) branching ratio which yields \( \chi(1\text{GeV}) = -29^{+25}_{-16} \) or \(+74^{+16}_{-25}\). Unfortunately, the errors on the extracted \( \chi(1\text{GeV}) \) are very large in this case. A more precise determination of the \( \pi^0 \to e^+e^- \) branching ratio could reduce the theoretical uncertainty in \( \chi(1\text{GeV}) \).

Model calculations for \( a_{\mu}^{LL}(\text{had}) \) differ from our analysis typically through insertion of form factors at the \( P\gamma^*\gamma^* \) vertices obtained from the WZW interaction in Eq. (1). For example, one widely-followed model employs form factors based on a vector meson dominance picture. This approach – known as resonance saturation – may also be used to obtain \( \chi \), giving \( \chi(1\text{GeV})_{\text{res sat}} \approx -17 \). In a similar vein, one may analyze this LEC at leading order in \( N_C \), where it depends on a sum over an infinite tower of vector meson resonances \( [18] \). Using a model-dependent form factor for the sum over vector resonances that at short distances is consistent with the properties of QCD gives \( \chi(1\text{GeV})_{N_C, \text{res sat}} = -16 \pm 5 \). This result provides some quantitative support for phenomenological models since it is close to the value \( \chi(1\text{GeV}) \simeq -14 \) obtained from experiment. However, one would not want to draw conclusions about the validity of the Standard Model using such a model-dependent approach.

Using \( \chi(1\text{GeV}) = -14^{+4}_{-8} \) as input, setting \( \mu = m_\mu \), and adding the large \( \ln^2 \) and \( \ln \) terms in Eq. (13), we obtain

\[
a_{\mu}^{LL}(\text{had})_{\text{log}} = \left( 57^{+50}_{-60} \right) \times 10^{-11} \hspace{1cm} (16)
\]

We emphasize that inclusion of both the \(-f(r) + 1/2\) and \(\chi/6\) parts of the \( \ln(\Lambda/\mu) \) term is crucial to obtaining an accurate numerical result for \( a_{\mu}^{LL}(\text{had})_{\text{log}} \). If, for example, one were to keep only the dependence on \( \chi(\Lambda) \), one would find substantial cancelations between the \( \ln^2 \) and \( \ln \) contributions. The presence of the calculable \(-f(r) + 1/2\) term, however, substantially mitigates these cancelations.

We observe that the central value in Eq. (14) is roughly a factor of two larger than obtained in model calculations for the \( \pi^0 \) contribution, and that the uncertainty is about a third the size of the present experimental error in \( a_{\mu} \). After the full E821 data set is analyzed, the uncertainty in Eq. (14) will be comparable to the anticipated experimental error. As noted above, improved measurements of the \( \pi^0 \to e^+e^- \) branching ratio could
reduce the theoretical uncertainty in the large logarithmic contributions to $a_\mu^{LT}(\text{had})$. Using the other value of $\chi$ obtained from the $\eta \to \mu^+ \mu^-$ branching ratio, $\chi(1\,\text{GeV}) = -39^{+5}_{-6}$, leads to $a_\mu^{LT}(\text{had})_{\log} = -190^{+60}_{-50} \times 10^{-11}$. Although there exists a strong theoretical prejudice in favor of the first solution [Eq. (16)] based on both the resonance saturation model for $\chi$ and $a_\mu^{LT}(\text{had})$ as well as consistency between the values of $\chi$ obtained from the $\eta \to \mu^+ \mu^-$ and $\pi^0 \to e^+ e^-$ branching ratios, one cannot rule out the second value for $a_\mu^{LT}(\text{had})_{\log}$.

Adding in the $O(N_C^0 \alpha^3)$ charged pion loop contribution [8,9], $a_\mu^{LT}(\text{had})_{\text{i.o.}} = -44.6 \times 10^{-11}$, to that in Eq. (16) gives the following $\chi$PT expression for $a_\mu^{LT}(\text{had})$,

$$a_\mu^{LT}(\text{had}) = \left(13^{+50}_{-60} + 31 \tilde{C}\right) \times 10^{-11} \, .$$

(17)

The largest uncertainty in the expression for $a_\mu^{LT}(\text{had})$ above arises from the subdominant terms that have not been calculated and are parameterized by the LEC $\tilde{C}$. As noted above, this constant includes the effects of non-logarithmically enhanced two-loop contributions, heavy mesons such as the $\eta$ and $\eta'$ which have been integrated out, and other non-perturbative dynamics. On general grounds, one could expect its natural size to be of order unity. A comparison of Eq. (17) with the results of model calculations is consistent with this expectation. For example, the model calculation of Ref. [11] corresponds roughly to $\tilde{C} \approx 1$. Rigorously speaking, however, the precise value – as well as the sign – of $\tilde{C}$ is unknown. An uncertainty $\Delta \tilde{C} = \pm 1$ corresponds to $\Delta a_\mu^{LT}(\text{had}) = \pm 31 \times 10^{-11}$, which is roughly one fifth of a standard deviation for the published Brookhaven measurement [4]. One should not, however, treat this as an estimate of the theoretical uncertainty in $a_\mu^{LT}(\text{had})$.

A value of $\tilde{C}$ equal to $+3$ or $-3$, for example, would not be unusual.

Alternatively, one may use the experimental result for $a_\mu$ to determine $\tilde{C}$. To that end, we use the updated results for hadronic vacuum polarization contributions [2], the QED and electroweak loop contributions in Ref. [19] and the value for $a_\mu^{LT}(\text{had})_{\text{chiral}}$ given in Eq. (17). From the E821 result for $a_\mu$ we obtain

$$\tilde{C} = 7 \pm 5 \pm 3 \pm 2 \, ,$$

(18)

where the first uncertainty arises from the experimental error in $a_\mu$, the second corresponds to the theoretical QED, electroweak, and hadronic vacuum polarization errors, and the final uncertainty arises from the error in $\chi$. In the future, the first uncertainty will be considerably reduced upon complete analysis of the full E821 data set. The value of $\tilde{C}$ is consistent with unity, though it could be considerably larger, given the other experimental and theoretical inputs into Eq. (18). Using the second solution for $\chi$ and $a_\mu^{LT}(\text{had})$ gives $\tilde{C} = 16 \pm 5 \pm 3 \pm 2$.

At present there is no indication that the hadronic LEC $\tilde{C}$ differs substantially from its natural size and, thus, no reason to discern effects of new physics, such as loops containing supersymmetric particles [19], from the $a_\mu$ result. In principle, a systematic calculation of some of the effects arising at higher order – such as terms of $O(\alpha^2 N_C^0 p^2 / \Lambda^2)$ enhanced by large logarithms – could modify this conclusion [20]. Similarly, should the full E821 data imply a value for $\tilde{C}$ which differs significantly from $\pm 1$ (e.g., by an order of magnitude), one might argue that there is evidence of new physics. Such a conclusion would presumably require considerable disagreement between the published E821 result [4] and the analysis of the full data set. The most convincing analysis, however, would rely on a first principles QCD calculation of $a_\mu^{LT}(\text{had})$, a prospect which seems to lie well into the future.
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FIGURES

(a)

(b)

(c)

FIG. 1. Hadronic light-by-light contributions to muon anomalous magnetic moment, $a_{\mu}^{LL}(\text{had})$, obtained from insertion of the WZW interaction at the $\pi^0\gamma\gamma$ vertices. The $\times$ in (b) indicates the insertion of the the low-energy constant $\chi$ plus its counterterm. The $\times$ in (c) denotes the magnetic moment coupling $C$ plus its counter term. The solid, dashed, and wavey lines denote the $\mu$, $\pi^0$, and $\gamma$, respectively.

(a)

(b)

FIG. 2. Some hadronic light-by-light contributions to muon anomalous magnetic moment that are not enhanced by large logarithms.