The Geometry of Navigation Problems

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Abstract—While many works exploiting an existing Lie group structure have been proposed for state estimation, in particular the invariant extended Kalman filter (IEKF), few papers address the construction of a group structure that allows casting a given system into the framework of invariant filtering. In this article, we introduce a large class of systems encompassing most problems involving a navigating vehicle encountered in practice. For those systems we introduce a novel methodology that systematically provides a group structure for the state space, including vectors of the body frame such as biases. We use it to derive observers having properties akin to those of linear observers or filters. The proposed unifying and versatile framework encompasses all systems, where IEKF has proved successful, improves state-of-the art “imperfect” IEKF for inertial navigation with sensor biases, and allows addressing novel examples, like GNSS antenna lever arm estimation.

Index Terms—Aircraft navigation, autonomous vehicles, geometry, Kalman filters, nonlinear filters, observers, state estimation.

I. INTRODUCTION

The Kalman filter introduced in 1960 was immediately applied to the localization of the manned space capsule going to the Moon and back. Though sixty years have since passed, state estimation for vehicles that navigate still offers challenges. This is because estimating the state of a navigating rigid body inevitably implies estimating the operator that is needed to move the object from a reference placement to its current placement, that is, an operator mapping the fixed reference frame to a frame being attached to the body. This includes a rotation, and rotations do not form a vector space, making the problem inherently nonlinear.

In the 2000s, with the advent of the aerial robotics field, an approach to the problem of estimating attitude revolving around symmetries and equivariance emerged, see [1]–[4] to cite a few, and [5] for a more recent exposition, see also the theory of symmetry preserving or invariant observer design [6], [7]. This body of work dedicated to providing constant-gain observers with convergence properties, and especially the complementary filter of [1], underpinned the first generation of small unmanned aerial vehicles (UAVs) or drones.

Invariant Kalman filtering, namely, the invariant extended Kalman filter (IEKF) [8]–[10], which targets Jacobians that do not depend on the state, has proved successful in various applications. The main theoretical properties of the IEKF having been brought to light so far may be summarized as follows:

1) IEKF possesses convergence properties when used as an observer for systems with group affine dynamics and a specific form of outputs. For these systems error equation is state trajectory independent, and its propagation is actually governed by a linear equation [9], a property called error log-linearity implying no linearization error is made by IEKF at propagation. The discovery that dynamics associated with (unbiased) inertial measurement units (IMU) are group affine in [9] has led to various recent experimental and theoretical successes, e.g., [11]–[16].

2) IEKF possesses consistency properties when the system is not fully observable as in the problem of simultaneous localization and mapping (SLAM) [17]–[19], as exploited in [20]–[25].

3) When the actual state is physically restricted within or near a subspace of the state space, the IEKF’s estimate reflects this information [17], [26], contrary to the EKF, as experimentally confirmed in [12].

4) Group affine dynamics possess the preintegration property [27], [28] that plays key role in modern robotics [29].

The theory is remarkable in that those four properties are a characteristic of the linear case, and are usually lost in the nonlinear case.

The big question when it comes to invariant observers/filtering is: How do we find a group structure for the state space that comes with theoretical results, given a system? The only generic approaches to date revolve around the search for symmetries of the original system, but recent successes of the IEKF turn out not to be a direct application of this idea. The present article makes a leap forward in this respect. We introduce a class of systems modeling rigid bodies in space, called “two-frames systems,” see Fig. 1, covering most practical navigation problems. We use a transformation group encoding the orientation of the vehicle as a building block for a larger general group called the two-frames group (TFG). This second group endows the state space with a group structure, allowing for the application of invariant filtering theory [9], [10], [27]. This leads to a class of observers having striking properties, and to an IEKF based on the TFG. The practitioner only needs to check the system is a two-frames system, and then the methodology is constructive and systematic.
A. Two-Frames State Space

To describe the motion of a rigid mobile body such as a robot or an aircraft, we define two frames: One is attached to the body, and another one is considered as “fixed” with respect to the “world.” The orientation of the body at time step $n$ is generally described by a rotation matrix $R_n$ that maps vectors expressed in the body frame to vectors expressed in the world frame, see Fig. 1. The state of the mobile body may then be described by the rotation $R_n$, which encodes its orientation, a collection of vectors whose expressions are given in the world frame, such as its position and its velocity, and a collection of vectors whose expressions are given in the body frame, like sensor biases or lever arms, to be estimated as well.

**Definition 1 (Two-frames state space):** A two-frames state space is a product space $G \times V \times B$, with $V,B$ two vector spaces of dimensions, respectively, $q, r$, and $G$ a $d$-dimensional Lie group. Elements $\chi$ of this space will be indifferently denoted in lines or columns, with $R \in G, x \in V, x \in B$

$$\chi = \begin{pmatrix} R \\ x \\ X \end{pmatrix} = (R, x, X).$$

$G$ should be viewed as the group of frame changes, $V$ denotes a space of vectors regrouped in the variable $x$, written in “fixed frame” coordinates and $B$ denotes a space of vectors regrouped in the variable $x$, written in “body frame” coordinates. Most often $G$ denotes a group of rotations acting on multivectors of $V$ and $B$. We thus see that Definition 1 is actually the basic setting for any attitude, navigation, or SLAM estimation problem: An attitude $R$, a set of variables written in the fixed frame (the vector $x$) and a set of variables written in the body frame (the vector $X$). More generally, throughout the article lowercase indicates vectors expressed in the fixed frame, and small uppercase vectors expressed in the body frame.

B. Class of Two-Frames Natural Systems

The point of the present article is to show many navigation problems share common properties, which suggests addressing them through a unifying framework. To this end, we are going to define a class of two-frames systems we consider as “natural” in the sense that they show invariance/equivariance with respect to frame changes induced by $G$.

**Definition 2 (Group action):** A (left) group action of $G$ on a set $S$ is a map $(G, S) \rightarrow S$ that we denote as $(R, s) \rightarrow R \ast s$, and which verifies the following conditions.

1) $Id \ast s = s$, where $Id$ denotes the identity element of $G$.
2) $(R_1 R_2) \ast s = R_1 (R_2 \ast s)$ for all $R_1, R_2 \in G, s \in S$.

Throughout the article we assume the mapping $x \rightarrow R \ast x$ is linear (in group theory this is called a “representation” [32]).

The main benefit of Definition 2 is allowing the same element of a group to define different transformations over different sets. From that point onward, we assume $G$ acts both on $V$ and $B$ and shall denote by $\ast$ both of these actions, although different. The following example should be meaningful.

**Example 1 (Term-by-term rotation):** Assume that $G = SO(d)$ is a group of rotations with typically $d = 2$ or $d = 3$ and consider a vector space $W = \mathbb{R}^{Nd}$ whose elements are $N$-tuples of vectors of $\mathbb{R}^d$. Then the following operator $\ast$ defines an action
of \( G \) on \( W \):

\[
R \ast (x^1, \ldots, x^N) := (Rx^1, \ldots, Rx^N).
\]

**Definition 3 (Commuting actions):** Let \( G \) be a group acting on vector spaces \( W_1 \) and \( W_2 \), and let \( H : W_1 \rightarrow W_2 \) be a linear operator (a matrix). The symbol \( \ast \) denotes as previously the action on both spaces \( W_1 \) and \( W_2 \). We say \( H \) commutes with the action of \( G \) if we have for any element \( R \in G \) and \( x \in W_1 \)

\[
R \ast (Hx) = H(R \ast x).
\]

The following result will cover most examples.

**Proposition 1:** Let us consider the term-by-term action (1) of Example 1 on both spaces \( W_1 = \mathbb{R}^{N_d} \) and \( W_2 = \mathbb{R}^{M_d} \), and a block matrix \( H \) of the form

\[
H = \begin{pmatrix}
\alpha_{11}I_d & \cdots & \alpha_{1N}I_d \\
\vdots & \ddots & \vdots \\
\alpha_{M1}I_d & \cdots & \alpha_{MN}I_d
\end{pmatrix}
\]

with \( \alpha_{ij} \)'s real numbers. Then \( H \) commutes with the action of \( G \), as can be easily verified.

**Notation:** Throughout the article, bold mathematical symbols are reserved for linear operators, or more proaically matrices. Elements of matrix Lie groups will be written in bold when their matrix nature is to be emphasized, in particular when they act on vectors through matrix-vector multiplication.

Let us introduce the output space, and a family of output maps that represent observations, i.e., measurements.

**Definition 4 (Natural two-frames output):** Let the output space \( \mathcal{Y} \) be a vector space on which \( G \) acts through an action denoted by \( \ast \). We call a natural output in the fixed frame (or the body frame) a map \( G \times V \times B \rightarrow \mathcal{Y} \) defined by

\[
h(R, x, X) = H^x x + R \ast [H^x X + b]
\]

**Body-frame:** \( \eta(R, x, X) = R^{-1} \ast [b - H^x x] - H^x x \)

**Fixed-frame:** \( \delta(R, x, X) = H^x x + R \ast [H^x X + b] \)

with \( H^x : V \rightarrow \mathcal{Y} \) and \( H^x : B \rightarrow \mathcal{Y} \) two linear maps that commute with the action of \( G \), see Definition 3, and \( b, b \in \mathcal{Y} \).

Note the minus signs in (4) make computations to follow clearer, but are to some extent arbitrary. The purpose of elements \( R \) and \( R^{-1} \) appearing in Definition 4 is to bring variables either from body to fixed coordinates, or the opposite.

**Definition 5 (Natural vector dynamics):** Given a two-frames state space (Definition 1) \( G \times V \times B \) we define discrete-time natural vector dynamics as \( \chi_n = f_n(\chi_{n-1}) \), where

\[
f_n \left( \begin{array}{c}
R \\
x
\end{array} \right) = \left( \begin{array}{c}
F_n x + d_n \\
C_n x + u_n
\end{array} \right) + R \ast \left[ \begin{array}{c}
\Phi_n x + d_n \\
\Delta \Gamma_n x + u_n
\end{array} \right]
\]

and where matrices \( F_n : V \rightarrow V \), \( \Phi_n : B \rightarrow B \), \( C_n : B \rightarrow V \), \( \Gamma_n : V \rightarrow B \) all commute with the action of \( G \), and \( d_n, u_n \) (resp. \( \Delta d_n, \Delta u_n \)) are vectors of \( V \) (resp. \( B \)).

This family of dynamics is ubiquitous in robotics and navigation problems, as soon as some dynamics \( s_n \) allows transformation \( R \) (that encodes a change of frame) to evolve as well. A “natural” class of such frame dynamics is now introduced, also based on a commutation hypothesis.

**Definition 6 (Natural frame dynamics):** We define natural frames dynamics as \( \chi_n = s_n(\chi_{n-1}) \), where

\[
s_n \left( \begin{array}{c}
R \\
x
\end{array} \right) = \left( \begin{array}{c}
R^s_n(\chi) \\
x
\end{array} \right) = \left( \begin{array}{c}
O_n R \Omega_n \\
x
\end{array} \right)
\]

with \( O_n, \Omega_n \in G \) known elements of \( G \) (inputs). Moreover, it is required that the maps \( x \rightarrow O_n \ast x \) and \( x \rightarrow \Omega_n \ast x \) commute with the action of \( G \) on \( V \) and \( B \), respectively.

Each of the preceding notions brings properties. When all are combined, we obtain a two-frames system we call natural.

**Definition 7 (Natural two-frames observed system):** A natural two-frames observed system is defined on a two-frames state space \( (G, V, B) \) as follows:

\[
\chi_n = s_n \circ f_n(\chi_{n-1})
\]

\[
y_n = h_n(\chi_n) \quad \text{or} \quad Y_n = h_n(\chi_n) \quad \text{where}
\]

1) \( h_n \) and \( h_n \) are the natural outputs of Definition 4;
2) \( f_n \) is a natural vector dynamics of Definition 5; and
3) \( s_n \) is a natural frame dynamics of Definition 6.

In practice, some systems of interest are not completely “natural,” so that we will also need the following definition.

**Definition 8 (Generic frame dynamics):** When \( s_n^R(\chi) \) in (6) is generic, that is, either it does not satisfy the commutation requirements of Definition 6 or it is not even of the form \( O_n R \Omega_n \) and possibly depends on state variables \( x \) and \( x \), we speak of generic frame dynamics.

C. Prototypical Example: Navigation on Flat Earth

Consider a mobile body equipped with an inertial measurement unit (IMU) providing gyroscope and accelerometer measurements, and a GNSS receiver providing position measurements \( Y_n \). A simple discretization of the continuous equations [29] yields the discrete-time dynamics

\[
\begin{align*}
R_n &= R_{n-1} \exp_m \left( \Delta t \omega_n + b_n^w \right) \\
v_n &= v_{n-1} + \Delta t \left( g + R_{n-1} \left( a_n + b_n^a \right) \right) \\
p_n &= p_{n-1} + \Delta t v_{n-1} \\
b_n^w &= b_n^w \\
b_n^a &= b_n^a
\end{align*}
\]

with observation \( Y_n = p_n \). In the above \( \Delta t \) is a time step, \( R_n \in G = SO(3) \) which denotes the transformation at time step \( n \) that maps the frame attached to the IMU (body) to the earth-fixed frame, \( p_n \in \mathbb{R}^3 \) denotes the position of the body in space, \( v_n \in \mathbb{R}^3 \) denotes its velocity, \( g \) is the earth gravity vector, \( a_n, \omega_n \in \mathbb{R}^3 \) the accelerometer and gyroscope signals, \( b_n^a \) the accelerometer bias, \( b_n^w \) the gyroscope bias, \( \exp_m() \) denotes the matrix exponential, and for any vector \( \beta \in \mathbb{R}^3 \), the quantity \( \beta \) denotes the skew-symmetric matrix, such that \( \beta \times \gamma = \beta \times \gamma \) for any \( \gamma \in \mathbb{R}^3 \). Let us cast (8) into the framework of two-frames systems. We can define a two-frames state space, where the group \( G = SO(3) \) encodes the orientation \( R_n \), the vector space \( V = \mathbb{R}^3 \times \mathbb{R}^3 \) encodes \( x_n = (p_n, v_n) \), and \( B = \mathbb{R}^3 \times \mathbb{R}^3 \) encodes \( x_n = (b_n^w, b_n^a) \). The group \( G = SO(3) \) acts through the term-by-term action (1) as: \( R \ast x = R \ast (v, p) = (Rx, Rp) \) and \( R \ast x = R \ast (b^w, b^a) = (Rb^w, Rb^a) \). The system matches (7),
where components of \( F_n \) are
\[
F_n = \begin{pmatrix} I_3 & 0_3 \\ \Delta t I_3 & I_3 \end{pmatrix}, \quad C_n = \begin{pmatrix} 0_3 & \Delta t I_3 \\ 0_3 & 0_3 \end{pmatrix},
\]
\[
d_n = \begin{pmatrix} \Delta t \eta \\ 0_{3,1} \end{pmatrix}, \quad \Gamma_n = 0_6, \quad d_n = u_n = 0_{6,1}
\]
and where \( s_n^R \) is easily retrieved from the first line of (8). The position observation \( Y_n = p_n \) matches (3) with
\[
H^x = \begin{pmatrix} 0_3 & I_3 \end{pmatrix}, \quad H^y = 0_{3,6}.
\]
We let \( G \) act on the output through the term-by-term action (1) as \( R \cdot Y_n = R Y_n \). It can then be immediately checked from Proposition 1 that matrices \( F_n, C_n, \Phi_n, \Gamma_n, H^x, H^y \) commute with the action of \( SO(3) \), i.e., with rotations. This proves that we are here dealing with natural outputs and natural vector dynamics indeed. However, we see in this example that the system fails to be entirely “natural,” as frame dynamics \( s_n \) is generic (see Definition 8), owing to \( \Omega_n = \exp\left(\Delta t[\omega_n + b_{\omega_{-1}}]\right) \) containing state element \( b \). In the present article, we show that many systems of interest are natural two-frames systems, and even for the present example, where the frame dynamics fails to be natural, we show in Section V-F the proposed theory brings sensible improvement over existing methods, which makes it a general tool for systems defined on a two-frames state space.

### III. TWO-FRAMES GROUP STRUCTURE

The core of this article is the introduction of a novel group structure on the class of two-frames systems that allows casting them into the invariant filtering framework. In the case of natural two-frames system, it provides the practitioner with a systematic method to design observers or filters that come with theoretical properties akin to linear observers and filters.

#### A. Group Structure for Two-frames Systems

We start by introducing a relevant group structure.

**Definition 9 (Two-frames group structure):** The following operation defines a group structure on the two-frames state space \( G \times V \times B \) of Definition 1:
\[
\begin{pmatrix} R_1 \\ x_1 \end{pmatrix} \cdot \begin{pmatrix} R_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} R_1 R_2 \\ x_2 + R_1^{-1} x_1 \end{pmatrix}.
\]

We call it the TFG law. Its identity element is \((I_d, 0, 0)\) and the inverse is defined as
\[
(R, x, x)^{-1} = (R^{-1}, -R^{-1} x, -R x).
\]

#### B. Natural Two-frames Outputs as TFG Actions

We may now use the action \( * \) of \( G \) on the output space \( \mathcal{Y} \) as a building block to build a group action of the TFG on \( \mathcal{Y} \).

**Lemma 1 (Two-frames group action):** Let \( H^x, H^y \) be two matrices defining linear maps from, respectively, \( V \) and \( B \) to the output space \( \mathcal{Y} \). \( G \) acts on \( \mathcal{Y} \) and we assume its action commutes with \( H^x \) and \( H^y \). Then the following operation, denoted by \( * \), provides a group action on the output space:
\[
G_{\nu, B}^{\top} \times \mathcal{Y} \to \mathcal{Y}
\]
\[
(R, x, x) \ast \beta = H^x x + R \cdot [H^y x + \beta].
\]

**Proof:** We consider \( \chi_1 = (R_1, x_1, 1) \) and \( \chi_2 = (R_2, x_2, 2) \) and prove the identity \( [\chi_1 \ast \chi_2] \ast \beta = \chi_1 \ast [\chi_2 \ast \beta] \). As \( R \) commutes with \( H^x \) and \( H^y \) we have on the one hand
\[
[\chi_1 \ast \chi_2] \ast \beta = (R_1 R_2, x_1 + R_1 x_2 + R_2^{-1} x_1, 1) \ast \beta
\]
\[
= H^x (x_1 + R_1 x_2) + R_1 R_2 \cdot [H^y (x_2 + R_2^{-1} x_1) + \beta]
\]
\[
= H^x x_1 + H^y R_1 x_2 + H^y R_2 x_2 + \beta
\]
and on the other hand
\[
\chi_1 \ast [\chi_2 \ast \beta] = (R_1, x_1, 1) \ast [H^x x_2 + R_2 \ast (H^y x_2 + \beta)]
\]
\[
= H^x x_1 + R_1 \ast [H^y x_1 + H^y x_2 + R_2
\]
\[
\ast (H^y x_2 + \beta)]
\]
\[
= H^x x_1 + H^y R_1 x_1 + H^y R_2 x_2 + \beta
\]
Both expressions match, hence \( * \) is a group action of the TFG.

Lemma 1 allows for our first major result, that is as follows.

**Theorem 1:** Consider a two-frames state space \( G_{\nu, B}^{\top} \) an output space \( \mathcal{Y} \) endowed with an action of \( G \), and let \( * \) be the operation defined by (12). Then the natural output map \( h \) of (3) merely writes
\[
y = h(\chi) = \chi \ast b.
\]

Similarly, the natural output map \( h \) of (4) writes
\[
Y = h(\chi) = \chi^{-1} \ast b
\]
where in both cases \( * \) is a group action, from Lemma 1.

**Proof:** Proving (13) immediately follows from (12). Regarding body frame output (14), \( \chi^{-1} \ast b = (R^{-1}, -R^{-1} x, -R x) \ast b = -H^y R^{-1} x + R^{-1} \ast (b - H^x R x) = R^{-1} \ast [b - H^y x] = H^x x \), which we recognize as (4).
Using our novel group law (10) we have defined a family of group actions of the TFG on the output such that all natural outputs write as group actions. This is a major property of two-frames systems, and one of our main contributions.

C. Natural Vector Dynamics as Group Affine Dynamics

Our second step is to prove that vector dynamics possess a key property of invariant filtering, namely, the group affine property, with respect to the TFG. Let us first recall what this property is. Consider general discrete-time dynamics on the two-frames state space $G^b_{V,B}$ endowed with the TFG structure

$$\chi_n = \phi_n(\chi_{n-1}).$$

(15)

**Definition 10 (Group affine dynamics [9], [10]):** Dynamics (15) is called group affine if $\phi_n : G^b_{V,B} \rightarrow G^b_{V,B}$ satisfies the “group affine property.” For all $\chi_1, \chi_2 \in G^b_{V,B}$, we have

$$\phi_n(\chi_1 \bullet \chi_2) = \phi_n(\chi_1) \bullet \phi_n(Id)^{-1} \bullet \phi_n(\chi_2).$$

(16)

It may be checked that the composition of two group affine maps, i.e., maps that satisfy (16), is also group affine.

Group affine dynamics have proved key to generalize properties of invariant observers related to autonomous estimation error equations. Indeed, the first article wholly dedicated to error autonomy is [7]. This article and others on symmetry-preserving properties of invariant observers related to autonomous estimation autonomy is [7]. This article and others on symmetry-preserving properties of invariant observers related to autonomous estimation autonomy is [7]. This article and others on symmetry-preserving properties of invariant observers related to autonomous estimation autonomy is [7]. This article and others on symmetry-preserving properties of invariant observers related to autonomous estimation autonomy is [7].

**Remark 1:** The property (16) is equivalent to having $\phi_n(\chi) = g_n(\chi) \bullet \phi_n(Id)$, where $g_n$ satisfies the automorphism property

$$g_n(\chi_1 \bullet \chi_2) = g_n(\chi_1) \bullet g_n(\chi_2).$$

(17)

This is shown by multiplying both sides of (16) on the right by a factor $\phi_n(Id)^{-1}$, see [27] for more details.

To date, a major shortcoming of the invariant filtering theory is that the group affine property or, more restrictively, left or right invariance have been studied on a case-by-case basis. We show in this section that the framework of two-frames systems provides a wide range of group affine dynamics, as well as a systematic approach to this property, thanks to the TFG law (10). Our second major result is indeed as follows.

**Theorem 2:** Natural vector dynamics of Definition 5, satisfies the group affine property (16) with respect to the TFG law (10).

**Proof:** We let $\phi_n = f_n$, as defined in (5) and check it is group affine. Letting $g_n(\chi) = \phi_n(\chi) \bullet \phi_n(Id)^{-1}$, all we have to prove is (17). Denoting by $g^R_n(\chi), g^X_n(\chi), g^\Gamma_n(\chi)$ the components of $g_n(\chi)$ and applying the TFG law formula (10), desired identity (17) becomes the following set of identities:

$$g^R_n(\chi_1 \bullet \chi_2) = g^R_n(\chi_1) g^R_n(\chi_2)$$

(18)

$$g^X_n(\chi_1 \bullet \chi_2) = g^X_n(\chi_1) + g^X_n(\chi_1) \bullet g^X_n(\chi_2)$$

(19)

$$g^\Gamma_n(\chi_1 \bullet \chi_2) = g^\Gamma_n(\chi_1) + g^\Gamma_n(\chi_2)^{-1} \bullet g^\Gamma_n(\chi_1).$$

(20)

First, let us write down explicit formulas for $g^R_n, g^X_n, g^\Gamma_n$ using the definition $g_n(\chi) = \phi_n(\chi) \bullet \phi_n(Id)^{-1}$. Setting $R = Id$, $x = 0$, and $x = 0$ in (5) we write $\phi_n(Id)$, then $\phi_n(Id)^{-1}$ using (11)

$$\phi_n(Id) = \begin{pmatrix} \text{Id} \\ d_n + u_n \end{pmatrix}, \quad \phi_n(Id)^{-1} = \begin{pmatrix} \text{Id} \\ -d_n - u_n \end{pmatrix}.$$

(11)

The product $g_n(\chi) = \phi_n(\chi) \bullet \phi_n(Id)^{-1}$ is then computed by combining this expression for $\phi_n(Id)^{-1}$ with (5) and (10), which immediately yields for $g_n(R, x, \chi)$ the components

$$g^R_n(\chi) = R$$

(21)

$$g^X_n(\chi) = F_n x + R \bullet C_n x + (Id - R) \bullet d_n$$

(22)

$$g^\Gamma_n(\chi) = \Phi_n x + R^{-1} \bullet \Gamma_n x + (R^{-1} - Id) \bullet u_n.$$\hspace{1cm}(23)

Now, let us use those to check whether (18)–(20) hold. Checking (18) is easy as both sides are simply $R_1 R_2$, see above. Left-hand side of (19) can be computed using (10) then (22)

$$g_n(\chi_1 \bullet \chi_2) = F_n(x_1 + R_1 x_2)$$

$$+ R_1 R_2 C_n(x_2 + R_2^{-1} x_1) + d_n - R_1 R_2 d_n$$

where we omit symbol $\times$ to alleviate notation. Expanding the parentheses and using the commutation properties $F_n, R_1 = R_1 F_n$ and $C_n, R_2^{-1} = R_2^{-1} C_n$, the latter equation becomes

$$g_n(\chi_1 \bullet \chi_2) = F_n x_1 + R_1 F_n x_2$$

$$+ R_1 R_2 C_n x_2 + R_1 C_n x_1 + d_n - R_1 R_2 d_n.$$\hspace{1cm}(22)

Right-hand side of (19) can be computed from (21) and (22)

$$g^X_n(\chi_1 + g^X_n(\chi_2) = F_n x_1 + R_1 C_n x_1 + d_n - R_1 d_n$$

$$+ R_1 [F_n x_2 + R_2 C_n x_2 + d_n - R_2 d_n]$$

where we recover all terms of $g^X_n(\chi_1 \bullet \chi_2)$ just above, after distributing the factor $R_1$ and noticing cancellation of $R_1 d_n$ terms, which proves (19). Left-hand side of (20) can be computed using (10) then (23)

$$g^\Gamma_n(\chi_1 \bullet \chi_2) = \Phi_n x_2 + R_2^{-1} \Phi_n x_1 + R_2^{-1} R_1^{-1} \Gamma_n(x_1 + R_1 x_2)$$

$$+ R_2^{-1} R_1^{-1} u_n - u_n.$$\hspace{1cm}(23)

Expanding the parentheses and using the commutation properties $\Phi_n, R_2^{-1} = R_2^{-1} \Phi_n$, and $\Gamma_n R_1 = R_1 \Gamma_n$, the latter equation becomes

$$g^\Gamma_n(\chi_1 \bullet \chi_2) = \Phi_n x_2 + R_2^{-1} \Phi_n x_1 + R_2^{-1} R_1^{-1} \Gamma_n x_1$$

$$+ R_2^{-1} \Gamma_n x_2 + R_2^{-1} R_1^{-1} u_n - u_n.$$\hspace{1cm}(23)

Right-hand side of (20) can be computed from (21) and (23)

$$g^\Gamma_n(\chi_2) + g^\Gamma_n(\chi_2)^{-1} \bullet g^\Gamma_n(\chi_1) = \Phi_n x_2 + R_2^{-1} \Gamma_n x_2 + R_2^{-1} u_n$$

$$- u_n + R_2^{-1} \Phi_n x_1 + R_1^{-1} \Gamma_n x_1 + R_1^{-1} u_n - u_n$$

where we recover all terms of $g^\Gamma_n(\chi_1 \bullet \chi_2)$ after distributing the factor $R_2^{-1}$ and noticing cancellation of $R_2^{-1} u_n$ terms, which proves (20), and, thus, (17) and hence Theorem 2.

D. Natural Frame Dynamics as Group Affine Dynamics

The last step of our analysis of two-frames systems regards natural frame dynamics of Definition 6. A counterpart of the result of the previous section is derived, then two cases of specific interest are highlighted.

**Theorem 3:** Natural frame dynamics of Definition 6, satisfies the group affine property (16) with respect to the TFG law (10).
Proof: First step is writing (6) as $s_n = s'_n \circ s''_n$ with

$$s'_n(R, x) = \begin{pmatrix} O_n \cdot (R, x) \cdot (\Omega_n) \\ 0 \cdot (R, x) \cdot 0 \end{pmatrix},$$

$$s''_n(R, x) = \begin{pmatrix} R \cdot (O_n^{-1}x) \\ \Omega_n x \end{pmatrix}.$$ 

This identity can be checked by direct computation based on (10). The operator $s'_n(\chi)$ is mixed-invariant and always satisfies the group-affine property (see [27, Corollary 19]), so $s_n$ is group-affine if $s''_n$ is, as a composition of two group-affine operators. As a second step, note that $s''_n$ matches the definition of vector dynamics, with the maps $x \mapsto O_n^{-1}x$ and $x \mapsto \Omega_n x$ playing the roles of $F_n$ and $\Phi_n$. If they commute with the action of $G$ as required by Definition 5, Theorem 2 then ensures $s''_n$ is group-affine and so is $s_n$. All we have to do is thus show the map $x \mapsto O_n^{-1}x$ commutes with the action of $G$ if the map $x \mapsto \Omega_n x$ does. Let us assume the latter is true. Then for any $R \in G, x \in V$ we have $O_n^{-1}R \cdot x = O_n^{-1}(R) \cdot (O_n^{-1}x) = O_n^{-1}(O_n R) \cdot (O_n^{-1}x) = R O_n^{-1} \cdot x$, proving the last step, and, thus, Theorem 3.

The difficulty here is to figure out when the commutation property required by the definition of natural frame shifts holds. In practice, the following specific cases are meaningful.

**Theorem 4:** Frame dynamics of the shape (6) are natural frame dynamics as soon as we have (a) or (b) or (c):

- a) the state variable boils down to $(R, x)$, and frame dynamics read $s''_n(R, x) = O_n \cdot R$;
- b) the state variable boils down to $(R, x)$, and frame dynamics read $s''_n(R, x) = R \cdot \Omega_n$; and
- c) the frame dynamics read $s''_n(R, x, x) = R$.

Thus, in each of these cases, frame dynamics (6) satisfy the group-affine property (16) with respect to the TFG law (10).

**Proof:** Let us check the commutation hypotheses of Definition 6. Consider, e.g., (a). The map $x \mapsto O_n \cdot x$ is then a trivial action, hence commutes. Besides, $x \mapsto \Omega_n x$ commutes with action of $G$ as $\Omega_n = I_d$. Thus, Theorem 3 applies.

Albeit extremely degenerate with respect to the entire TFG theory, Case (b) of Theorem 4 actually covers all previously discovered group affine dynamics. The TFG structure with $G = SO(d)$ then boils down to the group $SE(d)$.

In particular, coming back to example of Section II-C, we recover immediately from Theorems 2 and 4 case (b) that in the absence of IMU biases (which are encoded by $\chi_n$), the equations of inertial navigation are group affine, which is one of the main discoveries of [9]. Moreover, if we ignore gyroscope bias $\omega^c$, but we want to estimate an accelerometer bias, we let $\chi_n = \omega^a_n$, and we get frame dynamics of the shape (6) with $O_n = I_3, \Omega_n = \exp_m(\Delta \theta [\omega^a_n])$. Case (c) of Theorem 4 shows that frame dynamics become group affine as soon as $\omega_n = 0$ yielding $\Omega_n = I_3$, i.e., attitude $\hat{R}$ is constant but possibly unknown. We thus recover from Theorems 2 and 4 case (c) that inertial navigation dynamics with accelerometer bias become group affine whenever the craft is moving straight forward, which is quite common when navigating.

**Theorem 5:** If $G$ is an abelian group, i.e., for all $R_1, R_2 \in G$ we have $R_1 R_2 = R_2 R_1$, then all frame dynamics of the shape (6) are natural frames dynamics. As a result, they satisfy the group-affine property (16) with respect to the TFG law (10).

**Proof:** They then satisfy commutation hypotheses of Definition 6 as group elements commute with $O_n$ and $\Omega_n$.

This result is of major importance as the group $G = SO(2)$ is abelian, and it models the heading of wheeled robots in 2-D, so that a wide class of systems of practical interest are two-frames systems based on an abelian group $G$.

**E. Two-Frame Systems as Linear Observed Systems on the TFG**

We now have the machinery to meet our end objective: Cast two-frames systems into the framework of invariant filtering.

**Definition 11 (from [27]):** A linear observed system on group is a dynamical system $(\chi_n)_{n \geq 0}$ observed through measurements $y_n$ or $\mathcal{Y}_n$ following equations of the form:

$$\begin{align*}
\chi_n &= \phi_n (\chi_{n-1}) \\
y_n &= \chi_n \ast b \
\mathcal{Y}_n &= \chi_n^{-1} \ast b
\end{align*}$$

(24)

Left action case

Right action case

where $\ast$ is an action on the output vector space $\mathcal{Y}_n$ and $\phi_n$ satisfies the group affine property (16).

The terminology for the right-hand case comes from the map $(\chi, b) \mapsto \chi^{-1} \ast b$ being a right action, see [33].

Linear observed systems on groups are the systems brought up and studied by the invariant filtering theory. They encompass [9], where the studied groups are matrix Lie groups and the action on the output space is then a mere matrix–vector multiplication, and where the group affine property (16) is given in continuous time. In discrete time Definition 11 appears in [27], see also [10]. Gathering our results we get the following.

**Theorem 6:** Using the TFG law (10) and action (12), natural two-frames observed systems of Definition 7 fit the Definition 11 of linear observed systems (24) on the group $G_{V,B}$, with $\phi_n = s_n \circ f_n$.

The introduction of the TFG thus allows for the discovery of new systems that actually fit into the invariant filtering framework. Concrete examples are provided in Section V.

**IV. OBSERVER DESIGN FOR TWO-FRAMES SYSTEMS**

The TFG structure and action enabled us to write natural two frames as in Definition 11. As soon as this can be done, invariant filtering theory [9], [10], [27] automatically brings observers that inherit some key properties of linear observers.

**A. Linear Observers on the Two-Frames Group**

The following Propositions 2–4 are already known from [27], but proofs are given so that the article is self-contained.

For systems that formally write as (24), one may introduce “linear observers on groups,” see [27], by mimicking linear observers, and then readily inherit a number of properties of the
linear case. First, we build alternative innovation (i.e., prediction error) terms using the group action on the output:

$$z_n = \hat{x}_{n-1}^{-1} * y_n - b \quad \text{or} \quad z_n = \hat{x}_{n-1} * Y_n - b. \quad (25)$$

Observers are then readily defined as a copy of the dynamics followed by a “multiplicative” update based on innovation.

**Definition 12** (Linear observer on group [27]): A linear observer on group $G^+_V, B$ for system (24) of Definition 11 consists of estimates $\hat{x}_{n-1}$ and $\hat{x}_n$ defined through a succession of propagation (copy of the dynamics) and update steps

$$\hat{x}_{n-1} = \phi_n(\hat{x}_{n-1}) \quad \text{[Propagation step]}$$

$$\hat{x}_n = L_n(z_n) \quad \text{[Update step: left action case]}$$

$$\hat{x}_n = \hat{x}_n * \hat{x}_{n-1} \quad \text{[Update step: right action case]}$$

with $L_n : Y \rightarrow G^+_V, B$ any mapping and $z_n, z_n$ given by (25).

Linear observers on groups are designed to ensure striking properties of a specific type of error variables called “invariant error variables” and playing a central role in the theory of invariant filtering and equivariant observers

$$E_n(z_n) = \hat{x}_{n-1}^{-1} \cdot \chi_n, \quad e_n = \chi_n \cdot \hat{x}_{n-1}^{-1} \quad (26)$$

and similarly we let $e_{n-1} = \hat{x}_{n-1}^{-1} \cdot \chi_n$ and $e_{n-1} = \chi_n \cdot \hat{x}_{n-1}^{-1}$.

$E$ and $e$ are called, respectively, left- and right-invariant error variables. In previous work they were denoted by $\eta_n^1$ and $\eta_n^R$.

**Definition 13**: The error evolution is said to be state-trajectory independent (or autonomous) when $e_{n-1} (resp. E_{n-1})$ depends only on $e_{n-1} (resp. E_{n-1})$ and $n$, i.e., it does not explicitly depend on the estimated state variables $\hat{x}$.

State-trajectory independence of the error is arguably the most important result of invariant filtering, in that it ensures the mapping $L_n$ can be tuned independently from the actual (unknown) trajectory followed by the state. In this regard, the group-theoretic approach allows for an extension of the properties of the linear case to the nonlinear case. Let us now study this property at each step of the observer’s construction.

**Proposition 2**: Innovation $z_n$ (resp. $Z_n$) is a function of the invariant error $e_{n-1}$ (resp. $E_{n-1}$) only. We have indeed

$$z_n = E_{n-1} * b - b \quad \text{and} \quad z_n = e_{n-1} * b - b. \quad (27)$$

**Proof**: Having cast the outputs as actions in (24) allows writing $\hat{x}_{n-1}^{-1} * y_n = \hat{x}_{n-1}^{-1} * (\chi_n * b) = (\hat{x}_{n-1}^{-1} \cdot \chi_n) * b = E_{n-1} * b$ so $z_n = E_{n-1} * b - b$, and similarly for $z_n$.

The latter feature is a pivotal property that linear observed systems on groups share with conventional linear observers. This allows proving in turn the following.

**Proposition 3**: Error evolution at update step is state-trajectory independent. Indeed, we have

$$E_{n-1} = L_n(z_n) \quad \text{[Update step]}$$

$$e_{n-1} = e_{n-1} \quad \text{[Update step]}$$

$$L_n(z_n) = \hat{x}_{n-1}^{-1} \cdot \chi_n \quad \text{[Update step]}$$

and similarly $e_{n-1} = \chi_n \cdot \hat{x}_{n-1}^{-1} = e_{n-1} \cdot \hat{x}_{n-1}^{-1} \cdot L_n(z_n)$. Using (27) completes the result.

Finally, state-trajectory independence at propagation step is equivalent to group affine dynamics [9, 27]. This implies the following.

**Proposition 4**: If $\phi_n$ satisfies the group affine property (16), then error evolution at propagation step is state-trajectory independent. Indeed, we have

$$E_{n-1} = \phi_n(Id)^{-1} \cdot \phi_n(E_{n-1}) \quad \text{[Update step]}$$

$$e_{n-1} = e_{n-1} \quad \text{[Update step]}$$

**Proof**: Let prove the first equality. The second is proved similarly, see [27]. First, (16) gives $\phi_n(\hat{x}_{n-1}^{-1})\phi_n(Id)^{-1} \phi_n(\hat{x}_{n-1}^{-1}) = \phi_n(Id)^{-1}$, that we rewrite as $\phi_n(\hat{x}_{n-1}^{-1}) = \phi_n(Id)^{-1} \phi_n(\hat{x}_{n-1}^{-1}) \phi_n(Id)$. We use this identity to propagate the error:

$$E_{n-1} = \phi_n(Id)^{-1} \phi_n(\hat{x}_{n-1}) \phi_n(Id) \quad \text{[Update step]}$$

and similarly again we obtain: $E_{n-1} = \phi_n(Id)^{-1} \phi_n(\hat{x}_{n-1}) \phi_n(Id)$.

Gathering the last results, we see linear observers on groups yield state-trajectory independent error evolution for linear systems on groups at all steps. Having picked $\phi$ as the TFG law (10) and $\phi$ as action (12), and recalling Theorem 6, Definition 12 thus allows us to meet our second objective: To manage to build state-trajectory independent observers for two-frames systems.

**Theorem 7**: The evolution of error (26) is state-trajectory independent both at propagation and update steps for the observer of Definition 12 applied to any natural two-frames system.

**B. Invariant Extended Kalman Filter Design**

So far, we have shown that one may derive observers for the class of two-frames systems of Section II that possess properties akin to the linear case. Theses properties facilitate gain design and convergence analysis and open the door to observers possessing strong guarantees. The simplest yet very efficient approach to design meaningful gains is then to follow the extended Kalman filter methodology, which consists in linearizing the error equation and tune the gains on the linearized error system using the linear Kalman filter, i.e., least squares techniques. Indeed, the invariant extended Kalman filter [9, 10, 17] consists of a linear observer on the group, as in Definition 12, where update terms are chosen to be

$$L_n(z_n) = \exp_{G^+_V, B}^+(K_n z_n) \quad (30)$$

$$L_n(z_n) = \exp_{G^+_V, B}^+(K_n z_n) \quad (31)$$

with $\exp_{G^+_V, B}^+(\cdot) \in G^+_V, B$ the exponential map of the TFG, and where the gains $K_n$ are tuned using the Kalman filter applied to the linearized error system. Deriving IEKFs for two-frames systems thus requires 1) computing the exponential map of the TFG, and 2) linearizing the error equations obtained previously, yielding Jacobian matrices. One of the main contributions of the present article is to have derived generic formulas usable for any two-frames systems. Explicit formulas for the exponential map are given in Appendix A. Error equations (28) and (29) in terms of two-frames variables are provided in Appendix B. The main idea to linearize them is to define linearized errors $\bar{z}_{n}^E$ and $\bar{z}_{n}^E$,
see [7], [9], and [10], via
\[ E_{n|n} = \exp_{G_{V,B}}^{+}(\xi_{n|n}), \quad e_{n|n} = \exp_{G_{V,B}}^{+}(\xi_{n|n}). \] (32)
This allows linearizing the error system in \( e_{n|n} \) and \( \xi_{n|n} \) as
\[ \xi_{n|n-1} = A_{n}^{s}A_{n\mid n-1}^{s} \xi_{n-1|n-1} \] (33)
\[ \xi_{n|n} = (I - K_{n}H_{n}) \xi_{n|n-1} \] (34)
with \( A_{n}^{s}, A_{n\mid n-1}^{s}, \) and \( H_{n} \) Jacobian matrices corresponding, respectively, to frame dynamics, vector dynamics, and output map, and where \( K_{n} \) is the Kalman gain. Proofs and explicit generic formulas for the Jacobians are provided in Appendix C.

Equation (33) is the linearization of the propagation of the error (29), while (34) is the linearization of update (28) under IEKF gain design (30), (31). The IEKF does not directly use (33) and (34), though. Instead, the IEKF methodology associates a “noisy” system to the problem, based on realistic sensor noise characteristics and then computes (33) and (34) with the linearized noise terms to tune the gain \( K_{n} \). As a desirable byproduct, it allows the Riccati’s covariance matrix \( P_{n|n} \) to convey the correct extent of statistical uncertainty about the state, and the IEKF equations then provide first-order optimal tuning for the probabilistic problem. The noisy system, and the resulting computation of noise matrices \( \tilde{Q}_{n}, \tilde{N}_{n} \), have been moved to Appendix D. By letting the underlying Lie group in the IEKF methodology of [9] be the TFG, we readily obtain the following.

Definition 14 (TFG-IEKF): The propagation step of TFG-IEKF is a copy of the dynamics. For fixed-frame [resp. body-frame] observations of the form (3) hence (13) [resp. (4) hence (14)], update is defined as the left-action [resp. right-action] case of Definition 12. \( L^{x}, L^{v}, L^{s} \) are extracted from (30) [resp. (31)] using the exponential of the TFG, where the gain is tuned through the following Riccati equation:

\[ P_{n|n-1} = A_{n}^{s}A_{n\mid n-1}^{s}P_{n|n-1}^{s} + \tilde{Q}_{n} \]
\[ S_{n} = H_{n}P_{n|n-1}^{s}H_{n}^{T} + \tilde{N}_{n}, \quad K_{n} = P_{n|n-1}^{s}H_{n}^{T}S_{n}^{s-1} \]
\[ P_{n|n} = (I - K_{n}H_{n}) P_{n|n-1} \] (35)
with \( P_{00} \) the prior covariance of error \( e_{0} \) [resp. \( e_{00} \)].

In Appendix D, a formula allows for retrieving \( P_{00} \) from the initial covariance matrix in the original variables \( (R, x, \dot{x}) \).

C. Properties of the TFG-IEKF

Various properties of the IEKF are recovered. Let us start with one of the most striking properties of invariant filtering.

Theorem 8 (Log-linear property): Natural two-frames dynamics fully possess the log-linear property of the error of [9], as they were shown to be group affine. This means deterministic nonlinear error propagation (29) is exactly equivalent to (33), via the nonlinear correspondence (32).

Invariant Kalman filtering targets state independent Jacobians.

Proposition 5: Each part involved in the definition of a natural two-frames system brings a state-independent Jacobian.
1) The Jacobian matrix \( H_{n} \) is state-trajectory independent as soon as the output is natural.
2) The Jacobian matrix \( A_{n}^{s} \) is state-trajectory independent as soon as the vector dynamics is natural.

V. APPLICATIONS

The TFG has allowed us to cast two-frames observed systems into the framework of invariant filtering, and to readily inherit strong properties. However, to implement the observers in practice, we need to derive explicit formulas in the original two-frames state variables \( \hat{R}, \dot{x}, \) and \( \ddot{x} \). As a byproduct, this provides the reader with a more concrete picture of the derived class of observers. This is the first step of this section devoted to applications. Then, we consider three nontrivial systems of practical interest. The first two examples are novel and have never been shown to fit into the theory of invariant filtering. Applying the theory we derive the form of observers having autonomous error equations, without writing down all their properties, owing to space limitation. The third example shows the methodology allows for significant improvement over state-of-the-art IEKF for inertial navigation.
A. Explicit Formulas for Implementation

Let us first transpose the abstract formulas of Definition 12 by using the definition of the TFG law (10) and action (12), starting with the innovation. Let $y_n$ and $Y_n$ be natural outputs in, respectively, the fixed and body frame as in Definition 4. The innovation variable $z_n$ in (25) associated to fixed-frame output $y_n$, i.e., Observation (3), writes

$$z_n = \hat{R}_{n|n-1}^{-1} \left( y_n - H^T \hat{x}_{n|n-1} \right) - H^T \hat{x}_{n|n-1} - b_n$$  \hspace{1cm} (36)

while the innovation variable $z_n$ in (25) associated to body-frame output $Y_n$, i.e., Observation (4), writes

$$z_n = \hat{R}_{n|n-1}^{-1} \left( Y_n + H^T \hat{x}_{n|n-1} \right) + H^T \hat{x}_{n|n-1} - b_n.$$  \hspace{1cm} (37)

We recognize that we merely have $z_n = \hat{R}_{n|n-1}^{-1} \left( y_n - Y_n \right)$, where $\hat{y}_n = H^T \hat{x}_{n|n-1} - \hat{R}_{n|n-1}^{-1} \left( \hat{H}^T \hat{x}_{n|n-1} + b_n \right)$ denotes the predicted output. Similarly $z_n$ is the usual output error $Y_n - \hat{R}_{n|n-1}^{-1} * b_n + \hat{R}_{n|n-1}^{-1} * H^T \hat{x}_{n|n-1} + H^T \hat{x}_{n|n-1}$, moved to the opposite reference frame by application of $\hat{R}_{n|n-1}$’s action. This modification is characteristic of invariant filtering.

Now, to transpose the observer formulas of Definition 12 into the original variables, let us denote the estimates at time $n$ by

$$\hat{x}_{n|n-1}, \hat{x}_{n|n}, \hat{\theta}_{n|n},$$  respectively, after applying the vector dynamics (Definition 5), after applying the frame dynamics (Definition 6) and after taking into account the observation of Definition 4. Moreover, we split the correction term $L_n\left(\cdot\right)$ in $G_{Y, B}$ as $(L^R_n\left(\cdot\right), L^X_n\left(\cdot\right), L^\theta_n\left(\cdot\right))$.

We find using (10) and Definition 12 that left-invariant observers for System (7) of Definition 7 with fixed-frame observation $y_n$ write

\begin{equation}
\begin{aligned}
\hat{x}_{n|n-1} &= f_n \left( \hat{x}_{n|n-1-1} \right) \\
\hat{x}_{n|n} &= \hat{R}_{n|n-1}^{-1} L^R_n\left( Z_n \right) \\
\hat{\theta}_{n|n} &= \left( \hat{x}_{n|n-1} + \hat{R}_{n|n-1}^{-1} L^X_n\left( Z_n \right) \right) \\
\hat{\chi}_{n|n} &= L^X_n\left( Z_n \right) + L^\theta_n\left( Z_n \right) - \hat{X}_{n|n-1}
\end{aligned}
\end{equation}

where $L_n^R : \mathcal{Y} \to G$, $L_n^X : \mathcal{Y} \to V$, $L_n^\theta : \mathcal{Y} \to B$ can be any functions, and $z_n$ is the innovation variable from (36).

When confronted with body-frame observations, right-invariant observers for System (7) with observation $Y_n$ write

\begin{equation}
\begin{aligned}
\hat{x}_{n|n-1} &= f_n \left( \hat{x}_{n|n-1} \right) \\
\hat{x}_{n|n} &= \hat{R}_{n|n-1}^{-1} L^R_n\left( Y_n \right) \\
\hat{\theta}_{n|n} &= \left( \hat{x}_{n|n-1} + \hat{R}_{n|n-1}^{-1} L^X_n\left( Y_n \right) \right) \\
\hat{\chi}_{n|n} &= L^X_n\left( Z_n \right) + L^\theta_n\left( Z_n \right) - \hat{X}_{n|n-1}
\end{aligned}
\end{equation}

where $L_n^R : \mathcal{Y} \to G$, $L_n^X : \mathcal{Y} \to V$, $L_n^\theta : \mathcal{Y} \to B$ can be any functions, and $z_n$ is the innovation variable from (37).

We see the error variables (26) provide alternative definitions of “discrepancies” between the true state $(R, x, \theta)$ and the estimated state $(\hat{R}, \hat{x}, \hat{\theta})$ of a two-frames system indeed. In the case where observations are performed in the fixed frame (3), one should use the left-invariant error $e_n$ in (26) which writes using the TFG law (10) as

$$\begin{aligned}
\text{Observation (3)} &\Rightarrow e_n = \left( \hat{R}_{n|n}^{-1} R_n \right) \\
&\quad \left( \hat{R}_{n|n}^{-1} R_n x_n - \hat{x}_{n|n} \right)
\end{aligned}$$  \hspace{1cm} (40)

If observations are performed instead in the body frame (4) one should use the right-invariant error $e_n$ in (26), i.e.,

$$\begin{aligned}
\text{Observation (4)} &\Rightarrow e_n = \left( R_n \hat{R}_{n|n}^{-1} \right) \left( x_n - \hat{x}_{n|n} \right)
\end{aligned}$$  \hspace{1cm} (41)

The indexes $n|n$ can be replaced with $n - 1$ (resp. $n-1$), everywhere to define the error variables after vector dynamics, but before frame dynamics (resp. after frame dynamics but before update). Regarding $x_n$ and $X_n$, we see the error variables we consider much differ from the classical linear difference $x_n - \hat{x}_{n|n}$ (resp. $X_n - \hat{X}_{n|n}$).

All the (autonomous) error equations then governing the evolution of $e_n|n$ and $e_n$ are provided in Appendix B.

B. Methodology to Attack Examples

When facing a novel example, the user shall first cast the navigation system into the two-frames systems equations and endow the state with the TFG structure. Derivation of a TFG-IEKF is then automatic following the methodology, and always worth being tested, but is of course especially relevant when Theorem 7 holds. This is guaranteed when 1) commutation properties required by definitions of natural output and vector dynamics (Definitions 4 and 5) hold, which is often checked using Proposition 1 in practice, and 2) frame dynamics is natural. In practice, it suffices that frame dynamics has the shape (6) and that one of the conditions of Theorem 4, or those of Theorem 5, hold.

C. Odmoter-GNSS Navigation With Unknown Lever Arm

The problem of navigating with unknown GNSS antenna lever arm is both very relevant in practice and not addressed until now by the theory of invariant filtering.

Herein, we consider the classical 2-D model of a nonholonomic car, see e.g., [9]. The position of the car in 2-D is described by the middle point of the rear wheels axle $x_n \in \mathbb{R}^2$, and its orientation (heading) denoted by $\theta_n \in \mathbb{R}$ and parameterized by the planar rotation matrix $R_n = \rho(\theta_n)$ of angle $\theta_n$, where $\rho(\theta) := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$. The car is equipped with a GNSS antenna located at unknown position $X_n \in \mathbb{R}^2$ in the car frame with respect to point $x_n$, which provides the world (fixed) frame position measurements

$$y_n = x_n + R_n x_n \in \mathbb{R}^2.$$  \hspace{1cm} (42)

In the schematic diagram below, the triangle is the car and the square is the position measured by the GNSS.
Differential odometers measure linear velocity and angular rate that may be compounded over a time step $dt$ into position and angular shifts $u_n, \omega_n$, while the lever arm $x_n$ between the reference point and the GNSS antenna remains constant, albeit unknown. The dynamics in discrete time write

$$R_n = R_{n-1} \Omega_n, \quad x_n = x_{n-1} + R_{n-1} u_n, \quad X_n = X_{n-1}$$

(43)

with $\Omega_n = \rho(\omega_n)$. This system has been already studied for invariant filtering in [6] and [9], but without the lever arm $x_n$, which makes observer design more difficult. It is pedagogical to first illustrate the results very concretely, without referring to the theory above. Consider the following observer shape:

$$\hat{R}_{n|n-1} = \hat{R}_{n-1|n-1} \Omega_n$$

$$x_{n|n-1} = x_{n-1|n-1} + \hat{R}_{n-1|n-1} u_n, \quad X_{n|n-1} = X_{n-1|n-1}$$

(44)

$$\hat{R}_{n|n} = \hat{R}_{n|n-1} L_n^R(z_n)$$

$$\hat{x}_{n|n} = \hat{x}_{n|n-1} + \hat{R}_{n|n-1} L_n^x(z_n)$$

$$\hat{X}_{n|n} = L_n^x(z_n) + L_n^R(z_n) T \hat{X}_{n|n-1}$$

(45)

with $Z_n = \hat{R}_{n-1|n-1}^T \left(y_n - \hat{x}_{n|n-1}\right) - \hat{X}_{n|n-1}$

where $L_n(z) = \left(L_n^R(z), L_n^x(z), L_n^u(z)\right) \in SO(2) \times \mathbb{R}^2 \times \mathbb{R}^2$ can be any function. Now, let the error variable be defined as

$$E_{n|n} = \left(\hat{R}_{n|n-1} R_n - x_n - \hat{x}_{n|n-1}, X_n - R_n \hat{R}_{n|n} \hat{x}_{n|n}\right).$$

(46)

We invite the reader to first check “manually" Theorem 9.

**Theorem 9:** Observer (44) and (45) ensures the error variable (46) has state-independent evolution.

**Proof:** Using the more concrete variable $\theta$ via the relation $R := \rho(\theta)$ the error (46) rewrites

$$E_{n|n} = \left(\begin{array}{c} \rho^\theta (x_{n|n-1}) \\ \rho^x (X_{n|n-1}) \end{array}\right) = \left(\begin{array}{c} \theta_n - \hat{\theta}_n \\ \hat{X}_{n|n} - \hat{X}_{n|n-1} \end{array}\right).$$

(47)

We readily see the innovation is a function of the error as $Z_n = \hat{R}_{n|n-1}^T \left(y_n - \hat{x}_{n|n-1}\right) + \hat{R}_{n|n-1} R_n X_n - \hat{X}_{n|n-1} = E_{n|n-1}^\theta + \rho \left(E_{n|n-1}^x\right) L_n^X$. Let us study for instance the evolution of $E_{n|n-1}^x$ under update (45). Let us write $L_n^R := \rho(l^R(z_n))$. We find at the update step that $\hat{\theta}$ and $\hat{X}$ are transformed as $\hat{\theta} \rightarrow \hat{\theta} + l^\theta$ and $\hat{X} \rightarrow L^x \hat{X}$ so that error evolves as $E^x \rightarrow X_n - \rho(\hat{\theta} + l^\theta - \theta) (L^x + \rho(-l^\theta) \hat{X}) = E^x - \rho(l^\theta - E^\theta) L^x$, which is a function of the error at previous step indeed.

Of course, checking Theorem 9 is the “easy” part, while finding the observer and error variable is the hard part necessitating the proposed theory. Let us see how it applies indeed. Letting $\star$ be standard product, we readily recognize that:

1) equation (43) formally writes as vector dynamics given by (5) with frame dynamics (6), where $G = SO(2)$ and $F_n = \Phi_n = \Omega_n = I_2, \Gamma_n = C_n = \Omega_n = 0_{2,2}$.

2) observation (42) formally writes as the natural output in the fixed frame (3) with $H_n = H_n^\star = I_2$. $b_n = 0_2$.

Let us apply the method of Section V-B: 1) Commutation properties required by definitions of natural output and natural vector dynamics (Definitions 4 and 5) directly stem from Proposition 1; and 2) frame dynamics has the shape (6) and $G = SO(2)$ is abelian, thus Theorem 5 guarantees that frame dynamics is natural. Besides, the left-invariant error (40) specifies as (46). We have all the conditions of Theorem 6 and know the methodology of previous sections will lead to an observer ensuring state-independent error evolution, as can be seen from Theorem 7.

The present nontrivial example was shown to fit into our theory. Beyond, our approach suggests a novel way to treat lever arms. Looking at (45) we see that despite a simplified car model and an abelian group $G = SO(2)$, we would probably not have managed to come up with an observer guaranteeing state-independent error evolution without the present theory. A last important remark is that this system still fits into the theory if an unknown scale factor affects odometry: $SO(2)$ is then simply replaced with the (abelian) group of scaled rotations.

### D. Inertial 3-D SLAM With Moving Objects Tracking

SLAMMOT is the subject of a rich and vast literature, see, e.g., the landmark paper [39]. In [30], a group structure was discovered to make standard SLAM dynamics left-invariant, later called $S E_k(d)$ in [17] and [18], and SLAM$_M$ in [36] (which designs a geometric observer for SLAM with robot and features having known velocity). For the SLAM problem, TFG boils down to $S E_k(d)$ and the present theory allows recovering those results. However, one could further wonder: *Are features allowed to move, while preserving group affine dynamics?* The present theory provides answers.

Let us consider a robot equipped with an IMU evolving in a 3-D environment containing static unknown features with positions $l_n^k \in \mathbb{R}^3$ and moving features with unknown position $q_n^i \in \mathbb{R}^3$ and unknown velocity $c_n^i \in \mathbb{R}^3$. Attitude, velocity, and position of the robot are denoted by $R_n \in SO(3), v_n \in \mathbb{R}^3,$ and $p_n \in \mathbb{R}^3$, while preintegrated inertial factors [29] are denoted by $\Omega_n \in SO(3), a_n^i \in \mathbb{R}^3, c_n^i \in \mathbb{R}^3$. Dynamics writes

$$R_n = R_{n-1} \Omega_n, \quad v_n = g + R_{n-1} a_n^i, \quad q_n^i = q_{n-1}^i + dt c_n^i$$

$$p_n = p_{n-1} + dt v_n - R_{n-1} a_n^i, \quad c_n^i = c_{n-1}^i.$$  

(48)

A wide range of observations measured in the frame of the vehicle can be considered, such as

1) position of static feature points: $R_n^T (l_n^k - p_n)$;
2) position of moving features: $R_n^T (q_n^i - p_n)$;
3) velocity of moving features: $R_n^T (c_n^i - v_n)$;
4) position of some known landmarks \( r^m_n ; \) \( R^T_n (r^m_n - p_n) ; \)
5) magnetic field \( \beta \) (known); \( R^T_n \beta ; \)
6) magnetic field \( \beta_n \) (to be estimated); \( R^T_n \beta_n . \)

Recalling \( R^T_n = R_n^{-1} \), we immediately see all of them are natural outputs in the body frame as in Definition 4, i.e., Observation (4), which evidences the broad scope of the present theory. Consider, e.g., position of moving features \( q^m_n \) in body frame

\[
Y^u_n = R^T_n (q^m_n - p_n). 
\]

(49)

Here, we recognize the following.

1) The group of frame changes is \( G = SO(3) \), acting term by term on the fixed-frame multivector \( x_n = (v_n, p_n, I_n, q^m_n, c^m_n) \in V \) as in (1).
2) Equation (48) formally writes as the combination of (5) and (6), with no \( B \) (hence no \( x_n, u_n \)), \( O_n = I_3 \), and where

\[
F_n = \begin{pmatrix}
I_3 & 0 & 3 & 3 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 \\
0 & 3 & 3 & 0 & 0 & 0 \\
0 & 0 & 3 & 3 & 0 & 0 \\
0 & 0 & 0 & 3 & I_3 & -I_3 \\
0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}, \quad u_n = \begin{pmatrix}
a^n_v \\
a^n_p \\
a^n_0 \\
a^n_0 \\
a^n_0 \\
a^n_0
\end{pmatrix}.
\]

3) Observation (49) writes as a natural output in the body frame (4) with: \( H^x = \begin{pmatrix} 0 & 3 & 0 & 3 & -I_3 & 0 & 3 \end{pmatrix} \). The minus sign in matrix \( H^x \) looks switched due to Definition 4.

Let us apply the method of Section V-B: 1) Commutations required by Definitions 4 and 5 of \( H^x \) and \( F_n \) with \( R \) acting as a term-by-term rotation stem from Proposition 1, 2) frame dynamics has the shape (6) and we are in the case (b) of Theorem 4 so it is natural. We have all conditions of Theorem 6 and know the methodology of previous sections will lead to an observer ensuring autonomous error evolution, as can be seen from Theorem 7. With the observations in the body frame (4), observers are defined by (37) and (39).

**Theorem 10:** System (48) and (49), where (49) could be replaced with any item of the list above, is a linear observed system on the group \( G_{v,B}^T = SO(3) \times (\mathbb{R}^3)_0 \) with body-frame observations. A family of invariant observers is defined as follows. The propagation step of estimated state \( \hat{\tilde{R}}_n, \hat{v}_n, \hat{p}_n, \hat{I}_n, \hat{q}^m_n, \hat{c}^m_n \) is a copy of (48). Denoting \( z_n = (z^1_n, \ldots, z^J_n) \), update writes

\[
\begin{align*}
\hat{\tilde{R}}_{n+1} &= L^R_n(z_n) \hat{\tilde{R}}_{n|n-1} \\
\hat{\tilde{v}}_{n+1} &= L^v_n(z_n) \hat{v}_{n|n-1} + \hat{p}_{n|n-1} \\
\end{align*}
\]

(50)

where \( \hat{\tilde{R}}_{n|n} = L^R_n(z_n) \hat{\tilde{R}}_{n|n-1} \hat{\tilde{v}}_{n|n-1} + \hat{p}_{n|n-1} \) and \( L^R_n(z) = L^R_n(z) \) in \( SO(3) \times \mathbb{R}^3 \) can be any function. The innovation vector \( z_n \) must be adapted if (49) is replaced by one or various alternative observations proposed above. These observers ensure the right-invariant error (41) has state-trajectory independent evolution.

The reader can alternatively check this result injecting (48)–(50) into (41). A TFG-IEKF is readily built using Definition 14, i.e., tuning the observer via (31) and (35). Jacobians \( A^v_n, A^p_n, H_n \) are retrieved from Proposition 14 and noise matrices from (68) (both in the Appendix). It involves the group affine and error log-linearity properties, and potential consistency properties [18], [19].

In onboard radar tracking contexts, one may refine the moving objects model. Indeed, constant velocity assumption is simplistic, and does not suit targets that perform maneuvers. A celebrated model in the tracking literature and industry is the Singer model [40], that assumes the acceleration of the target is a Gauss–Markov process with autocorrelation time \( \tau = 1/\gamma \). This leads to modifying (48) by letting \( a^v_n = a^v_n + dt.a^v_{n-1} \), \( a^p_n = a^p_n - dt.\gamma a^p_{n-1} + w^n_n \), with \( w^n_n \) a process white noise.

**Proposition 7:** The SLAM equations with features moving according to a Singer tracking model with noise turned off define a linear observed system on group as commutation also stems from Proposition 1.

More generally, the state can be augmented with vectors of the fixed frame, as long as their dynamics commute with \( R \).

### E. Inertial Navigation With IMU Biases

Let us come back to the prototypical example of Section II-C. Inertial navigation without biases has been shown to possess the group affine property and autonomous errors thanks to the introduction of the group \( SE_2(3) \) in [9], which is here generalized by the TFG. To treat the case where IMU biases need be estimated, various publications have built on [9] for models including gyro and accelerometer biases by simply using as group structure the Cartesian product of \( SE_2(3) \) (for attitude, velocity and position) and \( \mathbb{R}^3 \times \mathbb{R}^3 \) for gyroscope and accelerometer bias, see [10]–[12], [17], [20], [21], [23].

However, the TFG structure advocated herein provides an alternative approach and the following experiments prove it should be preferred even if dynamics (8) are not group affine for the TFG. The space state can be cast into a two-frames state space with \( G = SO(3), V = (\mathbb{R}^3)^2, B = (\mathbb{R}^3)^2 \).

Many two-frames natural outputs can be considered for this system, let us stick with the setting of [9] by choosing

\[
Y^m_n = R^T_n (r^m_n - p_n)
\]

(51)

where \( Y^m_n \) for \( m = 1, M \) are landmarks of known position and \( Y^m_n \) is the position of feature point \( m \) at time step \( n \) observed by the IMU carrier in its own reference frame. It corresponds to (4) with \( H^x = \begin{pmatrix} 0 & 3 & I_3 \end{pmatrix}, H^y = 0_{3,6} \).

Let us detail the TFG-IEKF with body-frame observations, using Definition 14 and the Appendix. Propagation step is a copy of dynamics (8). Using (31), where we let \( K^{v,n} = (K^{v,n}, K^{p,n}, K^{L,n}, K^{a^v,n}, K^{a^p,n}, K^{a^g,n}, K^{a^r,n}), \) then extracting \( L^R_n \) in \( SO(3), L^v_n = (L^v_n, L^p_n, L^L_n, L^a^v_n, L^a^p_n, L^a^g_n, L^a^r_n) \) from (54), and applying (39), where we recall \( \ast \) denotes the term-by-term rotation of Example 1, the update writes

\[
\begin{align*}
\hat{\tilde{R}}_{n|n} &= \exp SO_3(K^{v,n}z_n) \hat{\tilde{R}}_{n|n-1} \\
\hat{\tilde{v}}_{n|n} &= \nu_3(K^{p,n}z_n)K^{a^v,n}z_n + \exp SO_3(K^{a^v,n}z_n)\hat{\tilde{v}}_{n|n-1} \\
\hat{\tilde{p}}_{n|n} &= \nu_3(K^{a^p,n}z_n)K^{a^v,n}z_n + \exp SO_3(K^{a^p,n}z_n)\hat{\tilde{p}}_{n|n-1} \\
\hat{\tilde{w}}_{n|n} &= \nu_3(K^{a^g,n}z_n)K^{a^v,n}z_n + \nu_3(K^{a^r,n}z_n)K^{a^v,n}z_n \quad \text{(52)}
\end{align*}
\]
where $\nu_\zeta(\cdot)$ is taken from Proposition 9 of the Appendix, and matrix $K_n$ is obtained from the Riccati equation (35). Matrices $A^n_0$, $H_n$ are retrieved from Proposition 14, referring to (9) and Example 2 proving $(q^m)_n = (\hat{q}^m)_n$, while $\hat{Q}_n, \hat{N}_n$ are read on (68), where in this case $A_{dR} = \hat{R}$. It yields

$$A^n_0 = \begin{pmatrix} I_3 & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\ -\Delta t\left(\Delta R_{g\xi}\right) & I_3 & 0_{3,3} & \Delta t I_3 & 0_{3,3} \\ 0_{3,3} & 0_{3,3} & 0_{3,3} & I_3 & 0_{3,3} \\ 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & I_3 \end{pmatrix}$$

$$H_n = \begin{pmatrix} (r^m)_n & 0_{3,3} & -I_3 & 0_{3,3} & 0_{3,3} \end{pmatrix}.$$ 

They are both state-trajectory independent as could be anticipated from the theory. Matrix $A^n_0$ must be derived manually.

**Proposition 8:** Jacobian for frame dynamics writes

$$A^n_0 = \begin{pmatrix} I_3 & 0_{3,3} & 0_{3,3} & M_1 & 0_{3,3} \\ 0_{3,3} & I_3 & 0_{3,3} & (\hat{\nu}_{n-1})_x & M_1 \\ 0_{3,3} & 0_{3,3} & 0_{3,3} & M_2 & M_2 \\ 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & M_2 \end{pmatrix} (53)$$

with $M_1 = \Delta t\hat{R}_{n|n-1}\hat{J}\hat{R}^T_{n|n-1}$, $M_2 = \hat{R}_{n|n-1}\hat{R}^T_{n|n-1}$, $\hat{J} = I_3 - \frac{1}{\sin|\mu|} (\mu)_x - \frac{\sin|\mu|}{\sin|\mu|} (\mu)_y^2$, $\mu := \omega + \hat{b}^\omega$.

A detailed proof is provided in the supplementary material, see [31]. Let us sketch it here.

**Proof:** We study to the first order the effect of $R_n = R_{n-1}\exp_m(\Delta t[\omega_n + b^\omega_{n-1}]_x)$ and $\hat{R}_{n-1} = \hat{R}_{n-1}\exp_m(\Delta t[\omega + b^\omega]_x)$ on (41), which is linearized as $R_n\hat{R}_n^{-1} \approx Id + (\xi^R)$, $\xi^\zeta \approx x - \hat{x} - (\xi^R)\hat{x}$, $\xi^X \approx \hat{R} \times (x - \hat{x})$, see also the Appendix. To do so we use the right-Jacobian formula [41] $\exp_m(\Delta t[\omega + b^\omega]_x) \approx \exp_m(\Delta t[\omega + b^\omega]_x) \exp_m(\Delta t(\hat{J}[\hat{b}^\omega - \hat{b}^\omega]_x))$, leading to $\xi^R_{n|n-1} = \xi^R_{n|n-1} + M_1\xi^\zeta_{n|n-1}$, which yields the first row of $A^n_0$, and similarly for the other rows.

**F. Simulation Results**

The (TFG)-IEKF of Section V-F is tested on the setting described by Fig. 2, and compared to both conventional EKF, and “imperfect IEKF” as in previous work [10]–[12], [14], [16], [17], [20]–[22], [24]. The name imperfect IEKF was coined in [17], and this filter has led to a high-performance commercial product, see [10]. Initial errors on biases and orientation are Gaussian with respective standard deviations 1 deg/s, 0.1 g, 30°. Results averaged on 100 Monte-Carlo runs are presented on Fig. 3. We see conventional multiplicative EKF based on $SO(3) \times \mathbb{R}^3$ is outperformed by IEKF based on $SE_2(3) \times \mathbb{R}^6$, i.e., imperfect IEKF. The latter is in turn outperformed by TFG-IEKF based on $G_{\nu \gamma B}^+ = SO(3) \times \mathbb{R}^3$.

This example is of major importance as most recent successes of IEKF are related to inertial navigation problems, where gyroscope and accelerometer biases came with a simple additive group structure. We claim this is probably not the most efficient approach and TFG structure should be used from now on when implementing IEKFs. The example shows indeed that even when full-state independence is not achieved, using the TFG structure for body-frame vectors is beneficial in practice. Similarly the treatment of GNSS or camera lever arms should also follow the TFG approach and more generally it should be tested for all systems that involve estimating a change of frame along with variables defined in the two frames. The observed improvements may stem from the drastic reduction of the dependency of the Jacobians and error evolution on the trajectory.

**VI. Conclusion**

In this article a novel class of systems was introduced, and a novel group structure was shown to endow it with strong properties regarding observer design. The obtained versatile and constructive framework unifies a large body of successes of the IEKF to date, and also allows for additional vectors to be estimated such as biases, lever arms, moving landmarks, as long as some commutation relations hold.

We focused on state independence of error, but the four properties of the IEKF displayed in the Introduction apply to natural two-frames systems. Rederiving them all in detail and studying their consequences obviously goes beyond the scope of this article, e.g., consistency of TFG-IEKF SLAM with moving object tracking, see [18], and is left as a perspective.

**APPENDIX**

**A. Exponential Map of the TFG**

Hereinafter, we provide a numerically efficient formula when the actions of $G$ are the term-by-term rotations of Example 1. For a complete theory see [31, Supplementary material].

**Proposition 9 (Exponential map for rotation TFGs):** Assume $G$ is $SO(2)$ or $SO(3)$ and its action is a multivector rotation as in Example 1. Then we have

$$\begin{pmatrix} \xi^R \\ \xi^\zeta \end{pmatrix} = \begin{pmatrix} \exp_{SO(2)}(\xi^R) \\ \nu_{\delta}(\xi^R)\xi^R \end{pmatrix}$$

$$\begin{pmatrix} \xi^N_{N_1} \\ \xi^N_{N_2} \end{pmatrix} = \begin{pmatrix} \nu_{\delta}(\xi^R)\xi^N_{N_1} \\ \nu_{\delta}(-\xi^R)\xi^N_{N_2} \end{pmatrix}$$

$$= \begin{pmatrix} \xi^N_{N_1} \\ \xi^N_{N_2} \end{pmatrix}$$

(54)
where $\mathbf{v}_d$ is given for $d = 3$ and $d = 2$ by

$$
\mathbf{v}_2(\xi) = I_3 + \frac{1 - \cos(||\xi||)}{||\xi||^2} (\xi) \times + \frac{||\xi|| - \sin(||\xi||)}{||\xi||^3} (\xi) \times \times
$$

$$
\mathbf{v}_2(\xi) = \frac{\sin ||\xi||}{||\xi||} I_3 + \frac{1 - \cos ||\xi||}{||\xi||} J \quad \text{with} \quad J := \rho(\pi/2)
$$

where $\rho(\theta)$ denotes the 2×2 rotation matrix of angle $\theta$.

The proof is based on an embedding into a matrix Lie group, where group composition boils down to matrix multiplication

$$
(R, x, X) \rightarrow \begin{pmatrix}
R & x & 0_{q,r} & 0_{1,r} \\
0_{1,q} & 1 & 0_{1,r} & 0 \\
0_{q,r} & 0_{r,1} & R^s & R^s x \\
0_{1,q} & 0 & 0_{1,r} & 1
\end{pmatrix}
$$

(55)

(see Appendix C for notation). It may be checked that by multiplying such two matrices we recover the TFG group law (10).

### B. Error Equations in the Original Variables

To tune the gains of an IEKF, one must linearize the error equations. A first step is to translate the various abstract error-related formulas on the TFG in terms of the original variables.

**Notation:** From that point onward, we may omit $*$ and denote $R \ast x$ as $Rx$ to alleviate calculation and formulas.

Let us first transpose the result of Proposition 2 using the original variables. By using the TFG action (12), (27) proves the innovation (36) or (37) computed from a natural output is given by (56) or (57) as follows:

Observation (3) $\Rightarrow z_n = H^T e_n^x + H^T e_n^R_R e_n^s + e_n^R B_n - B_n$ (56)

Observation (4) $\Rightarrow z_n = - H^T e_n^x - H^T (e_n^R)^{-1} e_n^x$

$+ (e_n^R)^{-1} b_n - b_n$. (57)

This result is of major importance regarding observer design, and should be put in contrast with innovation terms (a.k.a. prediction errors) used for general nonlinear observers and the EKF: $z_n = h(x_n) - h(\hat{x}_{n-1})$, where the innovation is not a function of the error except in the linear case, where innovation has the form $z_n = He_{n-1}$, with $e_{n-1} = x_n - \hat{x}_{n-1}$.

**Remark 3:** In the particular case, where $B = \mathbb{R}^3$ and $X \in B$ denotes a bias or a lever arm in the body frame, (40) and (41) advocate two different errors: $x - R^{-1} \dot{R}x$ or $\dot{R}(x - \dot{x})$. The former was already proposed in [37] using physical arguments, see also [38]. But while the authors of [37] consider this error
variable is intrinsically better, Proposition 10 shows the innovation based on it is better only when observations are performed in the fixed frame, a counterintuitive fact.

By using the TFG law, (28) readily proves the following.

**Proposition 10 (Error evolution at update):** For fixed-frame observations and observer (38) we have

\[
\begin{align*}
E_{n|n}^R &= L_{n}^R(z_n)^{-1}E_{n|n-1}^R \\
E_{n|n}^x &= L_{n}^R(z_n)^{-1} \left( E_{n|n-1}^x - L_n^X(z_n) \right) \\
E_{n|n}^X &= E_{n|n-1}^X - \left( L_{n}^R(z_n)^{-1} L_n^R(z_n) \star L_n^X(z_n) \right)
\end{align*}
\]  

(58)

with \(z_n\) as in (36). On the other hand, for body-frame observations and observer (39) with \(z_n\) as in (37) we have

\[
\begin{align*}
\epsilon_{n|n}^R &= e_{n|n-1}^R L_{n}^R(z_n)^{-1} \\
\epsilon_{n|n}^x &= e_{n|n-1}^x - e_{n|n-1}^R L_{n}^R(z_n)^{-1} L_n^X(z_n) \\
\epsilon_{n|n}^X &= L_{n}^R(z_n) \star \left( e_{n|n-1}^X - L_n^X(z_n) \right).
\end{align*}
\]  

(59)

Recalling (56) and (57), we see that (58) and (59) illustrate our point, which is that state variables \(\hat{X}_n, \hat{\chi}_n, \hat{X}_n^{-1}\) vanish as in the linear case, leading to state independent error evolution at the update step. Then, we proved at Theorem 2 that vector dynamics is group affine. Thus, by letting \(\phi_n = f_n\), the result (29) holds true. Using the TFG law (10), this readily proves the following.

**Proposition 11 (Error via natural vector dynamics):** The left-invariant error through (5) satisfies

\[
\begin{align*}
E_{n|n}^{R|n-1} &= E_{n|n-1}^R \\
E_{n|n}^{x|n-1} &= F_n E_{n|n-1}^x + C_n E_{n|n-1}^R E_{n|n-1}^X + E_{n|n-1}^R U_n - U_n \\
E_{n|n}^{X|n-1} &= \Phi_n E_{n|n-1}^X + D_n - (E_{n|n-1}^R)^{-1} D_n + \Gamma_n (E_{n|n-1}^R)^{-1} E_{n|n-1}^x
\end{align*}
\]  

(60)

and the right-invariant error through (5) satisfies

\[
\begin{align*}
\epsilon_{n|n}^{R|n-1} &= \epsilon_{n|n-1}^R \\
\epsilon_{n|n}^{x|n-1} &= F_n \epsilon_{n|n-1}^x + C_n \epsilon_{n|n-1}^R \epsilon_{n|n-1}^X + d_n - \epsilon_{n|n-1}^R d_n \\
\epsilon_{n|n}^{X|n-1} &= \Phi_n \epsilon_{n|n-1}^X + \Gamma_n (\epsilon_{n|n-1}^R)^{-1} \epsilon_{n|n-1}^x + \left( \epsilon_{n|n-1}^R \right)^{-1} u_n - u_n.
\end{align*}
\]  

(61)

The formulas illustrate more concretely what was already known from Theorem 2 and Proposition 4, that is, state independence of the error evolution systematically occurs at natural vector propagation step.

Finally, error evolution during natural frame dynamics reads as follows in the original variables.

**Proposition 12 (Error via natural frame dynamics):** The left-invariant error through (6) satisfies

\[
\begin{align*}
E_{n|n-1}^R &= \Omega_n^{-1} E_{n|n-1}^R \Omega_n \\
E_{n|n-1}^x &= \Omega_n^{-1} O_n^{-1} E_{n|n-1}^x \Omega_n \\
E_{n|n}^X &= E_{n|n-1}^X
\end{align*}
\]  

(62)

and the right-invariant error through (6) satisfies

\[
\begin{align*}
\epsilon_{n|n-1}^R &= O_n \epsilon_{n|n-1}^R O_n^{-1} \\
\epsilon_{n|n-1}^x &= \epsilon_{n|n-1}^x \\
\epsilon_{n|n}^X &= O_n \Omega_n \epsilon_{n|n}^x \Omega_n^{-1}
\end{align*}
\]  

(63)

Although this was known from Theorem 7, explicit formulas (56)–(61), and, then (62) and (63) concretely illustrate the state-trajectory independence of the error. Note that, all the preceding error equations can also be derived directly, as a good exercise, see [31, Supplementary material].

**C. Linearization of Error Equations and IEKF Jacobians**

Linearizing on groups requires a few technical ingredients. It is useful to be familiar with \(SO(3)\) and think of \(G\) as \(SO(3)\).

**Matrix \(R\):** In the definition of a group action Definition 2 we assumed the mapping \(x \mapsto R \ast x\) is linear on vector space \(V\). In particular, if a basis of \(V\) has been chosen, it can be described by a matrix we denote by \(R\) verifying \((R \ast x) = R \ast x\) for all \(x \in V\). For the term-by-term action (1) of the group \(SO(d)\), writing \(N\)-tuples of vectors as stacked vectors of \(W = \mathbb{R}^{dN}\), we obviously have \(R = \text{diag}(R, \ldots, R)\). Note \(R\) also denotes a (other) matrix verifying \((R \ast x) = R \ast x\) for all \(x \in B\). Albeit a different matrix, confusion is hardly possible and using the same notation is consistent with our choice to use * in both cases hitherto.

**First-order expansion in the \(R\) element:** For any \(\xi = (\xi^R, \xi^x, \xi^X)\) in the Lie algebra of the TFG, by denoting \(R = \exp_{G}^C(\xi^R)\) we may define—see, e.g., [32]—a linear map \((\xi^R)_{\ast}\) on \(V\) through equality \(R_{\ast} = \exp_{m}(\xi^R_{\ast})\), where \(\exp_{m}(M) = I + M + \frac{1}{2} M^2 + \cdots\) denotes the matrix exponential. This yields \((\exp_{G}(\xi^R))_{\ast} = I + (\xi^R)_{\ast} + o(\xi^R)\).

**Linearization w.r.t. \(\xi^R\):** As \((\xi^R)_{\ast}x\) is also linear w.r.t. \(\xi^R\), we define \(g \times d\) matrix \((x)\), via \((x)_{\ast} := (\xi^R)_{\ast}x\), so

\[
\exp_{G}(\xi^R)_{\ast}x \approx x + (\xi^R)_{\ast}x = x - (x)^{\ast} \xi^R. 
\]  

(64)

**Example 2:** Let \(R \in SO(3)\) and \(x \in \mathbb{R}^3\). In this case, we merely have \(R \ast x = Rx\); and, thus, find \(R = R \ast x\). Besides, \((\xi)_x = (\xi)_x \ast x\) where \((\xi)_x\) denotes the skew-symmetric matrix associated with \(\xi\) so the notation is coherent. This ensures in turn that \((x)^{\ast}x = x^T \xi\). In the same way, coming back to Example 1, we have \((\xi)_x = (\xi)_x, x_1, \ldots, (\xi)_x, x_N = -(x)^T \xi\) so that \((\xi)_x = -(x^T x_1, \ldots, x^T x_N) \in \mathbb{R}^{N \times d}\).

**Adjoint Ad on Lie algebra \(g\)** is defined through the relation:

\[R \exp_{G}(\xi^R) R^{-1} = \exp_{G}(Ad G(\xi^R)) R \in G.\]

The following “Rosetta stone” allows for translation of error equations on the TFG, such as formulas (56) up to (63), into linearized vector error equations. The sign \(\approx\) means a quantity may be readily replaced by its linearized counterpart.

**TFG linearization “Rosetta stone”:** The exp on \(G^+\), see [31, Supplementary material], ensures \(\exp_{G} \Rightarrow \exp_{G}(\xi^R)\), and \(E^x \approx \xi^x\) and \(E^x \approx \xi^x\), and naturally \(\Omega \ast E^x \approx \Omega \ast \xi^x\). From (64) we have \(E_{n|n}^R U \approx U + (\xi^R)_{\ast} U = U - (U)^{\ast} \xi^R\) and \(E_{n|n}^R E^X \approx \xi^x\), as \((\xi^R)_{\ast} \xi^x\) is second order, and similarly on \(B\). As \((\xi^R)^{-1} = \exp_{G}(\xi^R)^{-1}\) we have \((\xi^R)^{-1} U \approx U + (U)^{\ast} \xi^R\). The counterparts for the right-invariant error \(e\) are identical.
Proposition 13: For fixed-frame observations, i.e., left-invariant error $E_{n|n}$, the linearized error system writes (33) and (34), and splitting $\xi^e$ as $(\xi^R, \xi^Z, \xi^G)$, Jacobians read
\[
A^v_n = \begin{pmatrix}
I_d & 0_{d,q} & 0_{d,r} \\
(u_n) & F_n & C_n \\
(\Omega^-) & \Gamma_n & \Phi_n
\end{pmatrix}
\]
\[H_n = \begin{pmatrix}
- (b_n) & H^x & H^y
\end{pmatrix}
\]
where we recall that in the (I)EKF theory, $H_n$ always relates the linearized innovation to the linearized error.

Proof: Using the “Rosetta stone” to substitute $\xi$ with $\xi$ in (60) yields $\xi^e_{n|n-1} \approx A^v_n E^e_{n|n-1}$ and similarly with (62) to get $A^v_n$. Doing similarly in (56) proves $Z \approx H^e_n E^e$. There is no need to analyze (58) to get (34). This may be done more concisely. As using (30), the definition (32) and $Z \approx H^e_n E^e$, update (28) rewrites
\[
\exp_{G_n}^e (\xi^e_{n|n-1}) \exp_{G_n}^e (\xi^e_{n|n-1}) \approx \exp_{G_n}^e (\xi^e_{n|n-1}) \exp_{G_n}^e (\xi^e_{n|n-1})
\]
and using the BCH formula. \(\blacksquare\)

Proposition 14: For body-frame observations, i.e., right-invariant error $e_{n|n}$, we get the following Jacobians:
\[
\begin{align*}
A^v_n &= \begin{pmatrix}
1 & 0_{d,q} & 0_{d,r} \\
(d_n) & F_n & C_n \\
(\Omega^-) & \Gamma_n & \Phi_n
\end{pmatrix} \\
A^v_n &= \begin{pmatrix}
(Ad)_{n} & 0_{d,q} & 0_{d,r} \\
0_{q,d} & I_d & 0_{q,r} \\
0_{d,r} & 0_{q,r} & O_{nq} \Omega_n^e
\end{pmatrix} \\
H_n &= \begin{pmatrix}
(b_n) & -H^x & -H^y
\end{pmatrix}
\end{align*}
\]
Proof similarly stems from applying the TFG linearization “Rosetta stone” to the error equations of Appendix B. Note that in the case of generic frame dynamics, $A^v_n$ may be more difficult to obtain, as in the example of Section II-C. But Jacobians $A^v_n, H_n$ can then still be retrieved from the formulas above.

D. Computation of Noise Covariance Matrices

Following the (I)EKF methodology [9], we associate a noisy system with the dynamics. In practice, noises should reflect the magnitude of the sensors’ uncertainty.

Definition 15: Consider system (7) where $s_n$ is generic, with noise turned on, i.e., noisy system $X_n = u_n(X_n-1), y_n = h(X_n) + V_n$ or $Y_n = H(X_n) + V_n$ with $s_n(R, x, x)$ defined as
\[
s_n^R(R, x, x) \exp_Q(W^R)
\]
\[
\begin{pmatrix}
[F_n x + d_n] + R^* [C_n x + u_n] + G_n (\chi^e_n) w^e_n \\
\Phi_n x + d_n] + R^* [\Gamma_n x + u_n] + G_n (\chi^e_n) w^e_n
\end{pmatrix}
\]
where $w^u_n, w^e_n, w^x_n$ are process noises with covariance matrices $Q^R, Q^Z, G_n^e (\chi^e_n), G_n^G (\chi^e_n)$ are state-dependent matrices; $V_n$ is an observation noise with covariance matrix $N_n$.

The IEEF is then fed with the following noise parameters.

Proposition 15: Denote $\hat{G}^x = G^x (\chi_n)$ and $\hat{G}^x = G^x (\chi_n)$ we have regarding the left-invariant error $E$
\[
\begin{pmatrix}
\hat{Q}^R & 0_{d,q} & Q^R (\hat{\chi}_n), T \\
0_{q,d} & \hat{Q}^Z & 0_{q,r} \\
(\hat{\chi}_n)^T, Q^R & 0_{r,q} & \hat{Q}^x
\end{pmatrix}
\]
and regarding the right-invariant error $e$
\[
\begin{pmatrix}
(Ad^R_n)^* & 0_{d,r} & 0_{d,q} \\
(\hat{\chi}_n)^T, Ad^R_n & 0_{q,r} & \hat{Q}^x
\end{pmatrix}
\]

\[
\begin{pmatrix}
(Ad^R_n)^* & 0_{d,r} & 0_{d,q} \\
(\hat{\chi}_n)^T, Ad^R_n & 0_{q,r} & \hat{Q}^x
\end{pmatrix}
\]

The rationale is as follows. At propagation $\xi_{n|n-1}$ becomes $\xi_{n|n}$. If noise is injected it then becomes $\xi_{n|n}^{\text{noisy}} \approx \xi_{n|n} + B_w$. Owing to noise being centered and independent we have $E(\xi_{n|n}^{\text{noisy}} (\xi_{n|n}^{\text{noisy}})^T) = E(\xi_{n|n})^T + B E(w_n w_n)^T = E(\xi_{n|n})^T + \hat{Q}$ with $\hat{Q} := B B^T$. We can now turn to the proof.

Proof: For instance, let us compute $B$ for error $e$.
\[
\begin{pmatrix}
(\xi_{n|n})^T \\
(\xi_{n|n})^T \\
(\xi_{n|n})^T
\end{pmatrix}
\]

\[
\begin{pmatrix}
(\xi_{n|n})^T \\
(\xi_{n|n})^T \\
(\xi_{n|n})^T
\end{pmatrix}
\]

Besides the stone shows $\xi^e \approx e^e_n \approx x_n - \hat{x}_n (\xi_{n|n})^T$. Moreover $\hat{x}_n^{\text{noisy}} = x_n + \hat{G}^x w^x_n$ and $\hat{x}_n^{\text{noisy}}$ was derived, so that $E(\xi_{n|n})^{\text{noisy}} \approx e^e_n + \hat{G}^x w^x_n - (Ad^R_n)^*, \hat{x}_n \approx e^e_n + \hat{G}^x w^x_n + (\hat{x}_n), Ad^R_n w^R_n$.

\[
\begin{pmatrix}
(\xi_n)^T \\
(\xi_n)^T \\
(\xi_n)^T
\end{pmatrix}
\]

Remark 4: If initial covariance matrix is known for the more classical error variable $(\hat{R}_n, x - \hat{x}, x - \hat{x})$ (with value $\hat{P}$), then $P_{00}$ in Definition 14 is given by $P_{00} = L \hat{P} L^T$, with $L = \begin{pmatrix}
I_3 & 0_3 & 0_3 \\
0_3 & \hat{R}_n & 0_3 \\
0_3 & 0_3 & I_3
\end{pmatrix}$ for error $e$. Similarly if $\hat{P}$ denotes the covariance of $(\hat{R}_n, x - \hat{x}, x - \hat{x})$ then $L = \begin{pmatrix}
I_3 & 0_3 & 0_3 \\
0_3 & \hat{R}_n & 0_3 \\
0_3 & 0_3 & I_3
\end{pmatrix}$ for error $e$. 
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