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Abstract. The general PDE governing linear, adiabatic, nonradial oscillations in a spherical, differentially and slowly rotating non-magnetic star is derived. This equation describes mainly low-frequency and high-degree \( g \)-modes, convective \( g \)-modes, and rotational Rossby-like vorticity modes and their mutual interaction for arbitrarily given radial and latitudinal gradients of the rotation rate. Applying to this equation the ‘traditional approximation’ of geophysics results in a separation into radial- and angular-dependent parts of the physical variables, each of which is described by an ODE. The condition for the applicability of the traditional approximation is discussed. The angular parts of the eigenfunctions are described by Laplace’s tidal equation generalized here to take into account differential rotation. From a qualitative analysis of Laplace’s tidal equation the sufficient condition for the formation of the dynamic shear latitudinal Kelvin-Helmholtz instability (LKHI) is obtained. A small rotation gradient causes LKHI of prograde waves (seen in the rotating frame), while strong gradients are responsible for retrograde LKHI. The value of the latitudinal rotation gradient has a lower limit, below which LKHI disappears.

The LKHI result is applied to real solar helioseismology rotation data. It is shown that the \( m = 1 \) mode (\( m = \) azimuthal wave number) instability can develop. This global instability takes place in the whole envelope of the Sun, including the greatest part of the tachocline, in radial direction and at almost all latitudes in horizontal direction. The exact solutions of Laplace’s equation for low frequencies and rigid rotation are obtained. There exists only a retrograde wave spectrum in this ideal case. The modes are subdivided into two branches: fast and slow modes. The long fast waves carry energy opposite to the rotation direction, while the shorter slow-mode group velocity is in the azimuthal plane along the direction of rotation. The eigenfunctions are expressed by Jacobi’s polynomials which are polynomials of higher order than the Legendre’s for spherical harmonics. The solar 22-year mode spectrum is calculated. It is shown that the slow 22-year modes are concentrated around the equator, while the fast modes are around the poles. The band of latitude where the mode energy is concentrated is narrow, and the spatial place of these band depends on the wave numbers (\( l, m \)).

Key words. hydrodynamics – Sun: activity – Sun: interior – Sun: oscillations – Sun: rotation – Stars: oscillations

1. Introduction

In a recent paper Dzhalilov et al. (2001; paper 1) investigated which lowest-frequency eigenoscillations can occur in the real Sun, moreover, which role they play in redistributing angular momentum and causing solar activity. We found that such waves could only be differential rotation Rossby-like vorticity modes. However, the general nonradial pulsation theory adopted from stellar rotation has some difficulties. For slow rotation, when the sphericity of the star is violated not seriously, the degeneracy of the high-frequency spherical \( p \)- and \( g \)-modes with respect to the azimuthal number \( m \) is abandoned by rotation (Unno et al. 1989). Independent of the spherical modes non-rotating toroidal flows (called ‘trivial’ modes with a zero frequency) become quasi-toroidal with rotation (called \( r \)-modes with a nonzero frequency; Ledoux 1951; Papaloizou & Pringle 1978; Provost et al. 1981; Smeyers et al. 1981; Wolff 1998). Although rotation abandons the degeneracy of the modes, it also couples the modes with the same azimuthal order, and this makes the problem more difficult. For the high-frequency modes \( \varepsilon_R = \omega / 2 \Omega \geq 1 \), where \( \omega \) and \( \Omega \) are the angular frequencies of oscillations and of stellar rotation, respectively) this difficulty is resolved more or less successfully. For this case the small perturbation rotation theory is applied, in which the eigenfunctions are represented by power series, the an-
gular parts of which are expressed by spherical harmonic functions $Y^m_l$ (Unno et al. 1989). These power series are well truncated, unless $\varepsilon_R < 1$, when the role of Coriolis force is increasing.

Namely the low-frequency instabilities are discovered in most pulsating stars (Cox 1980; Unno et al. 1989). Rotation couples strongly together the high-order $g$, the convective $g$, and the $\tau$-modes with $\varepsilon_R < 1$ and with the same $m$, but different $l$ (Lee & Saio 1986). Generally the matrix of the coupling coefficients to be determined is singular (e.g. Townsend 1997). In all papers on the eigenvalue problem of nonradially pulsating stars, there exists a ‘truncation problem’ for the serial eigenfunctions, the angular parts of which are represented by spherical harmonics (e.g. Lee & Saio 1997; Clement 1998).

The governing partial differential equations (PDEs) of the eigenoscillations of rotating stars are complicated from the point of view of the mathematical treatment, even if the motions are adiabatic. This difficulty arises because in spherical geometry an eigenvalue problem with a singular boundary condition has to be solved. These equations are simplified considerably to neglect the tangential components of the angular velocity $\Omega$ in the low-frequency case $\varepsilon_R < 1$ (this means that the motion caused by the Coriolis force is primarily horizontal). This limitation widely used in geophysical hydrodynamics (e.g. Eckart 1960) is called ‘traditional approximation’ and has been used first by Laplace (1778) to study tidal waves (Lindzen & Chapman 1969). Laplace’s equation (or the traditional approximation) for $\varepsilon_R < 1$ is applicable to the stellar case too. The main advantage of this approximation is that it decomposes the initial system of equations into a pair of ordinary differential equations (ODEs) (e.g. Lindzen & Chapman 1969; Berthomieu et al. 1978; Bildsten et al. 1996; Lee & Saio 1997). The angular parts of the eigenfunctions are described by Laplace’s tidal equation. Solving this equation numerically by using a relaxation method, Lee and Saio (1997) first avoided the representation of the solutions by $Y^m_l(\cos \theta)$ functions for the $\Omega =$const case, and they had no problem with the truncation of the series.

In the present work for the non-magnetic and non-convective cases we receive one PDE in spherical geometry for the adiabatic pressure oscillations in the differentially rotating star ($\Omega = \Omega(r, \theta)$) with arbitrary spatial gradients of rotation (Sect. 2). This general equation is split into the $\theta$- and $r$-component ODEs, if the traditional approximation is applied (Sect. 3). The $\theta$-component equation is Laplace’s tidal equation generalized for the differentially rotating case. In Sect. 4 we analyse more qualitatively this equation. We find the general condition for the shear instability due to differential rotation in latitude. We find that the smallest rotation gradient is responsible for the prograde (seen in the rotating frame) vorticity wave instability, while a stronger gradient causes the retrograde wave instability. For solar data (small rotation gradients) the $m = 1$ prograde mode instability is possible (Sect. 4.4). The possible existence of such a global horizontal shear instability on the Sun has been investigated by Watson (1981) and Gilman & Fox (1997), that of shear and other dynamic instabilities and of thermal-type instabilities in stars as well by Knobloch & Spruit (1982) and others. Laplace’s tidal equation for low frequencies in the rigid-rotation case is investigated in detail in Sect. 5. It is shown that the eigenfunctions are defined by Jacobi’s polynomials which are of higher order than the Legendre’s.

2. Basic equations

The fluid motion in a self-gravitating star, neglecting a magnetic field and viscosity, may be described in an inertial frame by the hydrodynamic equations. These equations in conventional definition are written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (1)$$

$$\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = -\nabla p - \rho \nabla \phi, \quad (2)$$

$$\rho T \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \right) s = \rho \varepsilon_N - \nabla \cdot \mathbf{F}, \quad (3)$$

$$\nabla^2 \phi = 4\pi G \rho. \quad (4)$$

2.1. Equilibrium state

We suppose that the equilibrium state (variables with zero indices) of the star is stationary and that its differential rotation is axially symmetric:

$$\mathbf{V}_0(r, \theta) = \Omega \times r = \Omega r \sin \theta \mathbf{e}_\phi, \quad (5)$$

where $\Omega(r, \theta) = \Omega_r e_r + \Omega_\theta e_\theta$, with the components $\Omega_r = \Omega \cos \theta$, $\Omega_\theta = -\Omega \sin \theta$, and $\Omega_\phi \equiv 0$, is the stellar angular velocity of rotation described in spherical polar coordinates, $(r, \theta, \phi)$. Here $\mathbf{e}_i$ with $i = r, \theta, \phi$ are the unit vectors. We will not include convective motion and meridional flows into the initial steady state. In that case we may obtain, in particular from the Eq. (2) of motion, the hydrostatic equilibrium relation

$$-\nabla \frac{p_0}{\rho_0} = \nabla \left( \phi_0 - \frac{1}{2} \Omega \times r \right)^2 + \Omega^2 \sin^2 \theta \nabla \Omega = \bar{g}. \quad (6)$$

It follows that the effective gravity $\bar{g}$ cannot be a potential field if differential rotation $\nabla \Omega \neq 0$ is present. This is important for rapidly rotating stars where the configuration is deformed by the centrifugal force as well as by differential rotation. For slowly rotating stars (the Sun as well) we may assume that the initial state is only marginally disturbed by rotation and $\bar{g} \approx g = \nabla \phi_0$ can be applied. That is, non-sphericity is not essential for the generation of waves (Unno et al. 1989).

2.2. Equations of oscillation

Small amplitude deviations from the basic state of the star may be investigated by linearizing Eqs. (1)-(6). For
Eulerian perturbations (variables with a prime) the equation of motion becomes (Unno et al. 1989)

\[ \begin{align*}
\rho \frac{\partial \mathbf{V}}{\partial t} + 2 \Omega \times \mathbf{V} + \mathbf{e}_\phi \rho \sin \theta (\mathbf{V} \cdot \nabla \Omega) &= \nabla \rho_0 \rho \frac{\rho'}{\rho_0} - \frac{1}{\rho_0} \nabla \rho' - \nabla \Phi', \\
\end{align*} \]

(7)

Here the operator

\[ \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \]

(8)

represents the temporal derivative referring to a local frame rotating with an angular velocity \( \Omega = \Omega(r, \theta) \). For low-frequency waves Saio (1982) has shown numerically in detail, that the Cowling approximation is good enough in most cases. Thus we will neglect perturbation of the gravitational potential in Eq. (8), \( \Phi' = 0 \). We are interested in very slow motions such that \( v_{\phi \text{ph}} \ll c_s \), where \( v_{\phi \text{ph}} \) is the phase velocity of the waves and \( c_s \) is the sound speed. Then the incompressible fluid motion limit, \( c_s^2 \rightarrow \infty \) (it is within the adiabatic approximation) may be applied, and instead of the Eq. (6) of mass conservation we use

\[ \nabla \cdot \mathbf{V} = \frac{1}{r^2} \frac{\partial (r^2 V_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (V_r \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial V_\theta}{\partial \phi} = 0. \]

(9)

We have shown in Paper 1, that nonadiabatic effects are of great importance for the dynamics of low-frequency rotation modes. However, here we shall restrict ourselves to the adiabatic case only because the mathematical treatment of wave equations in spherical geometry is rather difficult. For adiabatic waves we receive from Eq. (8)

\[ \frac{\partial \rho'}{\partial t'} - \rho_0 \frac{N^2}{g} V_r = \frac{1}{c_s^2} \frac{\partial \rho'}{\partial t'}, \]

(10)

where the squared Brunt-Väisälä frequency

\[ N^2 = g \left( \frac{1}{\Gamma_1} \frac{1}{p_0} \frac{dp_0}{dr} - \frac{1}{\rho_0} \frac{d\rho_0}{dr} \right) \]

(11)

is written for the slow rotation case where \( \tilde{g} \approx g \). For rapid rotation \( N^2 = N_r^2(r, \theta) \) and \( g \) should be changed here to \( \tilde{g} \). In the incompressible limit \( (c_s^2 \rightarrow \infty) \) Eq. (10) reads

\[ \frac{\partial \rho'}{\partial t'} = \rho_0 \frac{N^2}{g} V_r. \]

(12)

Thus we have a complete set of Eqs. (1, 8, 12) to describe adiabatic, low-frequency, non-radial oscillations in a differentially rotating star. For our axisymmetric stationary initial state we may represent all the perturbed variables \( \mathbf{V}, \rho', \) and \( p' \) in the inertial frame as

\[ \mathbf{V}(r, \theta, \phi; t) \rightarrow \mathbf{V}(r, \theta) e^{i(m_\phi - \omega_\text{rot})t}. \]

(13)

Considering \( \partial /\partial t' = -i\omega \) and \( \partial /\partial \phi = im \) we find from Eq. (8) the relation between the frequencies in the inertial and rotating frames

\[ \omega = \omega_0 - m \Omega(r, \theta), \]

(14)

from where we get \( \nabla \omega = -m \nabla \Omega \). If we separate the variable part of the rotation frequency, \( \Omega(r, \theta) = \Omega_0 + \Omega(r, \theta) \), then \( \omega = \omega_0 - m \Omega_0 - m \Omega = \omega_0 - k_\phi v_{\phi \text{ph}} \) \((k_\phi \) and \( v_{\phi \text{ph}} \) are the local azimuthal wave number and the phase velocity, respectively). We will study the case \( \Omega \ll \Omega_0 \). From now \( \Omega \approx \Omega_0 \) and \( \nabla \Omega \neq \nabla \Omega_0 \) will be used. Low frequencies seen in the rotating frame mean that we are close to the resonant frequencies in the inertial frame \( (\omega_0 \approx k_\phi v_{\phi \text{ph}}) \).

Now excluding \( \rho' \) from Eq. (7) by using Eq. (12), taking the projection of this equation onto the rotation axis \( \Omega \) and two tangential components as well, and adding Eq. (7) we get

\[ \begin{align*}
\left( 1 - \frac{N^2}{\omega^2} \right) V_r \cos \theta - j V_\theta \sin \theta &= \frac{1}{\rho_0 \omega} \left( \cos \frac{\partial}{\partial r} - j \sin \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \right) p', \\
V_\theta + \frac{2 \Omega \cos \theta}{\omega} V_r &= \frac{1}{r \rho_0 \omega} \frac{\partial p'}{\partial \phi}, \\
- i \omega V_\phi + j \sin \theta \left( 2 \Omega + r \frac{\partial \Omega}{\partial r} \right) V_r + \\
&+ \left( 2 \Omega \cos \theta + \sin \theta \frac{\partial \Omega}{\partial \theta} \right) V_\theta = - \frac{i m}{r \rho_0 \sin \theta} p', \\
im V_\phi + \frac{\sin \theta}{r} \frac{\partial}{\partial r} (r^2 V_r) + \frac{\partial}{\partial \phi} (\sin \theta V_\theta) &= 0.
\end{align*} \]

(15–18)

Here we have introduced the special parameter \( j \) to switch to the traditional approximation (\( \Omega_0 = 0, \Omega_r \neq 0 \)). In the general case \( j = 1 \), and for switching to the traditional approximation we put \( j = 0 \). Further we will obtain one additional equation for \( p' \).

Let \( \mu = \sin \theta \) be a new independent variable and

\[ \varepsilon_R = \frac{\omega}{2 \Omega}, \quad \beta_r = \frac{r \partial \Omega}{2 \Omega}, \quad \beta_\mu = \frac{\mu \partial \Omega}{2 \Omega}, \]

\[ \alpha = \varepsilon_R^2 - (1 - \mu^2)(1 + \beta_\mu), \]

\[ a_1 = \frac{1 + \beta_r}{\varepsilon_R}, \quad a_2 = \frac{1 + \beta_\mu}{\varepsilon_R}, \quad a_3 = \frac{\alpha}{\varepsilon_R}, \]

\[ a_4 = \mu \frac{1 + \beta_r}{\alpha}, \quad a_5 = \frac{m}{\varepsilon_R} (1 + \beta_\mu) - 1. \]

\( \varepsilon_R \) is the Rossby number; we are interested in \( \varepsilon_R \leq 1 \).

Using

\[ i V_\phi = j a_1 \mu V_r + a_2 V_\theta \cos \theta - \frac{m}{\rho_0 \omega} \frac{p'}{r \mu}, \]

(20)

we get three equations from the Eqs. (15–18) for \( V_r, V_\theta \), and \( p' \). From those \( V_\theta \) is excluded by

\[ \frac{V_\theta}{\cos \theta} = j a_4 V_r - \frac{1}{a_3} \left( a_5 - \mu \frac{\partial}{\partial \mu} \right) \tilde{p}, \]

(21)

where

\[ \tilde{p}' = \mu \rho_0 \omega \tilde{p}, \quad V_r = \mu \tilde{V}_r, \]

(22)
and we get two equations for \( \dot{V}_r \) and \( \dot{P} \):

\[
\left(1 - \frac{N^2}{\omega^2} - ja_4\right) \dot{V}_r = \\
\left[b_1 + r \frac{\partial}{\partial r} - ja_6 - j \left(1 - \frac{1}{a_3} \right) \mu \frac{\partial}{\partial \mu}\right] \dot{P},
\]

\[
\left[2 + r \frac{\partial}{\partial r} + j \left( a_8 + a_6 \mu \frac{\partial}{\partial \mu}\right) \right] \ddot{V}_r = \ddot{A}(\mu) \ddot{P},
\]

\[
\ddot{A}(\mu) = \frac{1}{\mu^2} \left[ a_7 + (1-\mu^2) \frac{\partial}{\partial \mu} \right] \left[ a_5 - \frac{\mu}{a_3} \frac{\partial}{\partial \mu} + \frac{m^2}{\mu^2} \right].
\]  

(23)\( \) (24)

Here the dimensionless coefficients are defined as

\[
a_6 = \frac{1}{\alpha} \left[ m \epsilon_R + (1 - \mu^2)(1 + \beta_\mu) \frac{m}{\epsilon_R} \beta_\mu - 1 \right],
\]

\[
a_7 = \frac{m}{\epsilon_R} (1 + \beta_\mu)(1 - \mu^2) + 1 - 2\mu^2,
\]

\[
a_8 = \frac{a_7}{\epsilon_R^2}; a_9 = \frac{1}{\alpha} (1 - \mu^2)(1 + \beta_\mu),
\]

\[
a_8^* = (1 + \beta_\mu \left[ a_3(3 - 4\mu^2 + 2m) - \right. \right.
\]

\[
\left. - (1 - \mu^2) \delta_\mu \alpha - 2\alpha(1 - \mu^2) \delta_\mu \beta_\mu, \right]
\]

\[
b_1 = 1 + \omega \rho - \frac{m}{\epsilon_R} \beta_\mu, \quad \omega = \frac{\rho_0 \frac{d \rho}{dr}}{\rho_0}.
\]

Deriving these equations the second derivatives \( \Omega''(\theta) \) and \( \Omega''(r) \) have been omitted as very small quantities. Eq. (23) allows to obtain one equation for \( \dot{p}' \):

\[
\left[ \psi_1 \delta_\theta^2 + \psi_2 \delta_\mu + \psi_3 \delta_\tau + \psi_4 \delta_\mu + \psi_5 \delta_\tau + \psi_6 \right] \ddot{P} = 0.
\]

(26)

This is the main singular PDE for nonradial rotation-gravity waves in a differentially rotating star. The coefficients \( \psi_1 - \psi_6 \) of Eq. (26) are rather complicated, we present them in Appendix A. The operators are defined as

\[
\delta_\tau = r \frac{\partial}{\partial r}, \quad \delta_\mu = \mu \frac{\partial}{\partial \mu}, \quad \text{and} \quad \delta_\theta^2 = \delta_\theta \delta_\tau.
\]

(27)

First we will study this equation in different simplified approximations. The most popular case is the traditional approximation.

### 3. The ‘traditional approximation’: \( j \equiv 0 \)

Strictly speaking, the condition \( j = 0 \) in Eq. (23) is not applicable in two points: (1) at \( \omega^2 = N^2(r) \), that is in the turning points in radial direction; (2) at \( \alpha = 0 \) or \( \epsilon_R^2 \approx \cos^2 \theta \), that is in the latitudinal turning point. The last one is more important, because the traditional approximation filters out such important and interesting phenomena as the trapping of Rossby-like waves around the equator. In geophysics this phenomenon is investigated separately (Pedlosky 1982; Gill 1982). The applicability of the traditional approximation for rigid rotation has been checked by numerical modelling as well as by experimental verification. For minor differential rotation of the star (small \( \beta_\mu \) and \( \beta_\mu \)) these examinations are also valid. Thus for \( j = 0 \) Eq. (23) becomes:

\[
\left[ \alpha \epsilon_R (1 - \mu^2) \mu^2 \frac{\partial^2}{\partial \mu^2} + q_3 \frac{\partial}{\partial \mu} + q_4 \right] \theta' = \\
- \frac{1}{\psi} \left[ r^2 \frac{\partial^2}{\partial r^2} + b_4 r \frac{\partial}{\partial r} + b_5 \right] \theta'
\]

(28)

with the parameters

\[
q_1 = 2 - \mu^2 - \epsilon_R^2 \frac{2 - 3\mu^2}{1 - \mu^2} + \]

\[
+ \beta_\mu \left[ 2\mu^2 - 2(1 - \mu^2)(1 + \beta_\mu) (m/\epsilon_R + 1) \right],
\]

\[
q_2 = h_0 + h_1 \beta_\mu + h_2 \beta_\mu^2,
\]

\[
h_0 = \epsilon_R (m^2 - 1)(\epsilon_R^2 + \mu^2 - 1) - \\
\mu^2 m(1 - \mu^2) + \epsilon_R^2 (m - 2\epsilon_R),
\]

\[
h_1 = (1 - \mu^2) \left[-2m\mu^2 + \epsilon_R (m^2 - 4 + 5\mu^2) + \\
+ 4m\epsilon_R^2 \right] - m\epsilon_R^2,
\]

\[
h_2 = (1 - \mu^2)^2 (3m - 2\epsilon_R + 2m\beta_\mu) - \\
- (1 - \mu^2) m(1 - 2m\beta_\mu),
\]

\[
b_2 = 3 + \omega \rho - \beta_\mu \frac{m}{\epsilon_R} \delta_\mu \theta,
\]

\[
b_3 = 2 + 3\omega \rho - \beta_\mu \frac{m}{\epsilon_R} \left( 3 + \beta_\mu \frac{m}{\epsilon_R} \right) - \\
- \left[ 1 + \omega \rho - \beta_\mu \frac{m}{\epsilon_R} \right] \theta,
\]

\[
\psi = 1 - \frac{N^2}{\omega^2}, \quad \delta_\mu \psi = - \frac{N^2 \epsilon_R m}{\omega^2} \beta_\mu r - \frac{N^2 \epsilon_R}{\omega^2} \frac{dN^2}{dr}
\]

(29)

This means that \( \Omega(\tau, \theta) \ll \Omega \), the left-hand side of Eq. (28) is a function of \( \mu = \sin \theta \), while the right-hand side is a function of \( r \) only. In that way we may separate the variables

\[
p'(r, \mu) = \Theta(\mu) Q(r).
\]

(30)

Now putting Eq. (24) into (28) we receive the ‘\( \theta \)- and \( r \)-equations:

\[
\left[ \alpha \epsilon_R (1 - \mu^2) \mu^2 \frac{d^2}{d\mu^2} + q_3 \frac{d}{d\mu} + q_4 + \Lambda \epsilon_R^2 \right] \Theta(\mu) = 0,
\]

(31)

\[
\left( r^2 \frac{d^2}{dr^2} + b_4 \frac{d}{dr} + b_5 - \Lambda \psi \right) Q(r) = 0.
\]

(32)

These two equations are coupled to each other by two common spectral parameters: \( \omega \) – the oscillation frequency...
and $\Lambda$ – the separation parameter. Both must be searched for as a solution of the boundary value problem. So far the logarithmic gradients of the rotation rate are arbitrary functions, $\beta_{\mu} = \beta_{\mu}(\mu)$, $\beta_r = \beta_r(r)$.

4. Generalized Laplace’s tidal equation

Eq. (31) is the generalized Laplace equation if differential rotation is present, $\beta_{\mu} \neq 0$. For rigid rotation, $\beta_{\mu} = 0$, Eq. (31) becomes the standard Laplace equation:

$$\left( \frac{1 - \mu^2}{\varepsilon_R^2 - \mu^2} \frac{d^2}{d\mu^2} + \frac{1 - \varepsilon_R^2}{2\varepsilon_R^2 \mu} \right) 2\mu_s \frac{d}{d\mu_s} - \frac{1}{\varepsilon_R^2 - \mu^2} \left( \frac{m^2}{1 - \mu^2} \frac{\varepsilon_R^2 + \mu^2}{\varepsilon_R^2 - \mu^2} + \frac{\Lambda}{\varepsilon_R^2} \right) \Theta = 0,$$

where $\mu_s = \cos \theta$. A peculiarity of this equation is the presence of three singular points: at the pole, at the equator, and between both if $\varepsilon_R^2 \leq 1$ (our case). Therefore it is hard to solve such an equation analytically or numerically to find the eigenvalues. Since the time of Laplace in geophysics investigations were focused on almost two-dimensional (horizontal) motions in strongly stratified fluids with $V_r \approx 0$. In this situation the $r$-component Eq. (32) does not appear, and one is looking for the eigenvalues $\Lambda$ in Laplace’s Eq. (33) for a given $\varepsilon_R$ (it is expressed through the thickness of the fluid layer, e.g., in a shallow water-wave system). For the special cases and for the general case too, when the eigenfunctions are expressed through the Hough functions (essentially these are the same as the infinite series of $Y_m^\pm$ harmonics) references could be found in a paper by Lindzen & Chapman (1969).

In astrophysics the $r$-component Eq. (32) appears, and two equations must be solved together to find both spectral parameters. In the rigid-rotation case Lee and Saio (1997) looked for $\Lambda$ numerically in an approach similar to that in geophysics, fixing $\varepsilon_R$ in Eq. (41). We offer here another approach, where we will find $\Lambda$ from the $r$-equation for a given $\varepsilon_R \leq 1$.

It is convenient to introduce into Eq. (31) the new variable $x = \mu^2 = \sin^2 \theta$:

$$4(1+\beta_{\mu})(1-x)(x-a)x^2 \frac{d^2\Theta}{dx^2} - A_2 x \frac{d\Theta}{dx} + A_2 \Theta = 0,$$

where

$$A_1 = (1-x) \left[ 2 - 3\varepsilon_R^2 + \beta_{\mu} (4x - 3 - 2m\varepsilon_R) \right] - 2\beta_{\mu} (1-x)^2 + \varepsilon_R^2,$$

$$A_2 = (1-x) \left[ m^2 (1-\beta_{\mu}) + \frac{m}{\varepsilon_R^2} x + \beta_{\mu} \frac{m}{\varepsilon_R^2} (4x - 3 - 2\varepsilon_R^2) \right] - m\varepsilon_R (1-x) + m\varepsilon_R (1-m\varepsilon_R) + \frac{\Lambda}{\varepsilon_R^2} x (x-a)^2 (1+\beta_{\mu})^2,$$

$$a = 1 - \frac{\varepsilon_R^2}{1 + \beta_{\mu}}.$$

(36)

This equation determines the tangential structure of the eigenfunctions, while Eq. (32) is responsible for the radial behavior. A detailed investigation of Eq. (32) is not included in this work. A similar equation for a realistically stratified model of the Sun has been investigated by Oraevsky & Dzhalilov (1997). We remind that the radial structure of the eigenfunctions depends on the sign of $\Lambda$ (either radiative or convective modes). Now we can already estimate an approximate value of $\Lambda$:

$$\frac{\Lambda}{\varepsilon_R^2} \sim \frac{n^2 \pi^2}{N_m^2 \mu / \omega^2} = \frac{n^2 \pi^2}{N_m^2 / 4\Omega^2} \sim n^2 \times 10^{-6},$$

where $n$ is the radial harmonic number, $N_m$ is mean value of the Brunt-Väisälä frequency in the radiative interior or in the convection zone, and $N_m / 2\Omega$ is the Prandtl number. An estimate of Eq. (37) is done for a solar model, but we think similar values of the Prandtl number are valid for most other stars too. Thus we can omit from $A_2$ in Eq. (33) the last term, if the radial number $n$ is not too large.

4.1. Fluid velocities

From Eqs. (31), (32), (33), we can derive in the traditional approximation the following formulae for the components of the fluid velocity:

$$V_r = 1 \frac{1}{\rho_0 \omega \psi} \frac{\partial p'}{\partial r},$$

$$V_\theta = \frac{\pm 1}{r \rho_0 \omega} \frac{1}{1 + \beta_{\mu}} \frac{1}{a - x} \frac{1}{\sqrt{1-x}} \left[ \frac{m}{\varepsilon_R^2} - 2x \frac{\partial}{\partial x} \right] p',$$

$$V_\phi = \frac{1}{r \rho_0 \omega} \frac{1}{a - x} \left[ \frac{1}{\sqrt{1-x}} \left( 2\varepsilon_R^2 \frac{\partial}{\partial x} - m \right) + m \right] p'.$$

Here the different signs $\pm$ of $V_\theta$ correspond to the northern and southern hemispheres, so that $\cos \theta = \pm \sqrt{1-x}$. Our further aim is to find such solutions for $p' = \Theta(x)Q(r)$ that all the components of the velocity remain limited at the pole ($x = 0$), at the equator ($x = 1$), and in both turning points, where $x = a$ and $\psi(r) = 0$.

4.2. Heun’s equation

Now we will impose a restriction to $2\beta_{\mu} = \partial \ln \Omega / \partial \ln \mu \approx const$, the logarithmic latitudinal gradient of the rotation frequency. We might take the linear dependence $\beta \sim x$, but for such a profile the structure of the solutions is not changed qualitatively. Let us introduce the new dependent variable

$$\Theta = x^2 Y(x),$$

where

$$2\sigma = \beta_{\mu} S_1 \pm \sqrt{\beta_{\mu}^2 S_1^2 - S_2} = 2\sigma_{1,2},$$

$$S_1 = \frac{5 + 2m\varepsilon_R + 2\beta_{\mu}}{2(1 + \beta_{\mu} - \varepsilon_R^2)},$$

$$S_2 = \frac{\beta_{\mu} m \left( 3 + 2\varepsilon_R^2 \right) - m^2 (1 - \beta_{\mu} - \varepsilon_R^2)}{1 + \beta_{\mu} - \varepsilon_R^2}.$$
Then for $Y(x)$ we get a new equation from Eq. (43):

$$x(1-x)(x-a)Y'' + \frac{1}{2} \left[ 4\sigma(1-x)(x-a) - \frac{A_1}{1+\beta_\mu} \right] Y' - \left[ (x-1)\nu_0 + \varepsilon_R\nu_1 \right] Y = 0,$$

(43)

with

$$\nu_0 = \frac{1+4\beta_\mu}{4(1+\beta_\mu)} \left( \frac{m}{\varepsilon_R} - 2\sigma \right) + \sigma \beta_\mu(S_1-1) - \frac{S_2}{4},$$

$$\nu_1 = \frac{m(m\varepsilon_R-1) + 2\sigma\varepsilon_R}{4(1+\beta_\mu)}.$$  (44)

Eq. (13) is Heun’s equation (Heun 1889) in standard form

$$x(1-x)(x-a)Y'' + \left[ \gamma(x-1)(x-a) + \delta x(x-a) + \varepsilon x(x-1) \right] Y' + \hat{\alpha}\hat{\beta}(x-h)y = 0.$$  (45)

The Riemann scheme for this equation is

$$p \begin{cases} 0 & 1 & a \infty \\ 0 & 0 & 0 \hat{\alpha} x \\ 1 - \gamma & 1 - \delta & 1 - \varepsilon \hat{\beta} \end{cases},$$

where the exponents are connected by Riemann’s relation

$$\hat{\alpha} + \hat{\beta} - \gamma - \delta - \varepsilon + 1 = 0.$$

In the Riemann scheme the first row defines the singular points of Heun’s equation, while the corresponding exponents are placed in the second and third rows. These exponents are

$$1 - \gamma = \beta_\mu S_1 - 2\sigma, \quad 1 - \delta = \frac{1}{\beta},$$

$$1 - \varepsilon = 2 + \beta_\mu \left( 1 - S_1 + \frac{3/2}{1+\beta_\mu} \right),$$

$$2\hat{\alpha} = S + q, \quad 2\hat{\beta} = S - q, \quad q = \sqrt{S^2 - 4\nu_0},$$

$$h = 1 - \varepsilon_R \frac{v_0}{\nu_0}, \quad S = 2\sigma - 2 - \beta_\mu + \frac{3/2}{1+\beta_\mu}.$$  (47)

Note that the second exponent at $x = a$ is $1 - \varepsilon \to 2$ if $\varepsilon_R \to 0$ or if $\beta_\mu \to 0$. If $\beta_\mu \to \infty$ then $1 - \varepsilon \to 7/2$. That means, the second independent solution of Eq. (13) with the exponent $1 - \varepsilon$ is regular at the singular point $x = a$ for all variables, which follows from Eqs. (13)–(40).

The exponent $1 - \delta = 1/2$ also provides limited $\nu_0$ and $\nu_1$ at the equator ($x = 1$). The singularity $x = \infty$ in the Riemann scheme does not occur in our task. The situation is more complicated around the pole $x = 0$ with the exponent $(1 - \gamma)$. Let us consider this point in detail.

### 4.3. Condition for the latitudinal Kelvin-Helmholtz instability

If we put the solutions with the exponents 0 and $(1 - \gamma)$ into Eq. (13), we get $p' \sim \Theta \sim x^{\sigma_1,2}$. Then $V_{\theta,\phi} \sim x^{(2\sigma - 1)/2}$ means that for the regularity of the solutions at the pole the condition $Re(2\sigma) \geq 1$ must be obeyed. On the other hand, an instability is possible when the eigenfrequencies are complex, that means complex $\sigma$. For the latter it follows from Eq. (14), that the necessary condition is $S_2 > 0$. It is clear that the axially-symmetric mode with $m = 0$ is excluded. For lower values of the rotation gradient $|\beta_\mu| < 1$ the necessary condition $S_2 > 0$ demands for the prograde waves ($m\varepsilon_R > 0$) the condition $\beta_\mu > 0$, which is more realistic for stellar situations (equatorward spinning up at the surface with radius $r$). Rayleigh’s necessary condition for instability (Rayleigh 1880; Watson 1981) says that the function $RI = \partial^2(\Omega \sin^2 \theta)/\partial \cos^2 \theta$ (gradient of vorticity) must change its sign in the flow. Rewriting this function in our definitions we get that

$$RI = \frac{2\Omega}{\mu^2}[3\beta_\mu - \mu^2(1 + 4\beta_\mu)]$$

(48)

may change its sign if $\beta_\mu > 0$. There are instability possibilities for negative $\beta_\mu$ which are not considered in this work. However, all formulas are valid for this case too.

The sufficient condition for instability is obtained from Eq. (12) and reads $\beta^2_\mu S_1^2 < S_2$. The regularity condition at the pole $\beta_\mu S_1 \geq 1$ can be rewritten as

$$\frac{\varepsilon_R}{m} \leq \frac{\beta_\mu \varepsilon_R^2}{1 - \varepsilon_R^2 - \beta_\mu(\beta_\mu + 3/2)} = \chi_3.$$  (49)

By this condition the phase space $\{\varepsilon_R/m, \varepsilon_R^2 \}$ is divided into three parts, depending on the values of $\beta_\mu$. For $0 \leq \beta_\mu < 1/2$ we have the following situation: if $\varepsilon_R^2 < 1 - \beta_\mu(\beta_\mu + 3/2)$ the condition Eq. (13) is fulfilled for prograde waves $\varepsilon_R/m \geq 0$ (region I); in the opposite case when $\varepsilon_R^2 > 1 - \beta_\mu(\beta_\mu + 3/2)$ the condition (49) is fulfilled for retrograde waves with $\varepsilon_R/m < -1/2$ (region III); for $\varepsilon_R^2 = 1 - \beta_\mu(\beta_\mu + 3/2)$ these regions are separated by the asymptote $\chi_3 = \infty$.

For $\beta_\mu > 1/2$ (strong gradients) the condition Eq. (49) is met only for retrograde waves in the range $-1/2 < \varepsilon_R/m \leq 0$ (region II). For $\beta_\mu = 1/2$ we get for any $\varepsilon_R$ that $\chi_3 = -1/2$. This is the line between the regions II and III. All three regions are shown in Fig. 1a. It is seen that the regularity condition is working for $|m| \geq 1$ and $|\varepsilon_R| \leq 1$ if $\beta_\mu \neq 0$. For very small $\beta_\mu$ only modes with large $m$ are possible. The smallest $m \lesssim 1$ modes may appear in the limit $\varepsilon_R \approx 1$. These conclusions are correct only if the instability occurs.

Now let us consider the second condition, the complex frequency condition $\beta^2_\mu S_1^2 < S_2$. This inequality may be rewritten as

$$\chi_2 < \frac{\varepsilon_R}{m} < \chi_1,$$

(50)

$$\chi_{1,2} = \frac{2}{\beta_\mu(5 + 2\beta_\mu)^2} \left[ b_\mu \pm \sqrt{b_\mu^2 - a_\mu \varepsilon_R^2(5 + 2\beta_\mu)^2} \right],$$

$$a_\mu = (1 - \varepsilon_R^2)(1 - \varepsilon_R^2 - \beta_\mu^2),$$

$$b_\mu = (3 + 2\varepsilon_R^2)(1 - \varepsilon_R^2 + \beta_\mu^2) - \varepsilon_R^2 \beta_\mu(5 + 2\beta_\mu).$$

In the limiting cases we have

$$\chi_1 \approx \frac{12(1 + \beta_\mu)}{\beta_\mu(5 + 2\beta_\mu)^2}, \quad \chi_2 \approx 0$$

for $\varepsilon_R \to 0$,  (51)
The domains of validity of the solution regularity and of the instability conditions in the phase space \( \{ \varepsilon_R/m, \varepsilon^2_R \} \) for given values of the rotation gradient \( \beta_\mu \). (a) shows the values \( \varepsilon_R/m = \chi_3 \), Eq. (49). In the area \( \varepsilon_R/m \leq \chi_3 \) the solutions are limited at the pole. (b) shows the behavior of \( \chi_1 \) (dashes) and \( \chi_2 \) (solid). Between the solid and dashed lines with the same labels (values of \( \beta_\mu \)) LKHI is possible.

\[
\begin{align*}
\chi_1 &\approx 0, \quad \chi_2 \approx -\frac{8\beta_\mu}{(5 + 2\beta_\mu)^2} \quad \text{for } \varepsilon_R \rightarrow 1. \\
\end{align*}
\]

In Fig. 1b we plot for \( |m| \geq 1 \) and \( \varepsilon_R \leq 1 \) the curves \( \chi_1 \) and \( \chi_2 \) versus \( \varepsilon^2_R \) for a wide range of \( \beta_\mu \). A comparison of Figs. 1a and 1b shows that in the region III with \( \varepsilon_R/m < -1/2 \) LKHI will never appear. In the region I LKHI is possible for prograde waves, if \( \beta_\mu < 1/2 \) and \( \varepsilon^2_R < 1 - \beta_\mu (\beta_\mu + 3/2) \). With decreasing \( \beta_\mu \) the solid and dashed curves for the same \( \beta_\mu \) are close to \( \varepsilon_R \approx 1 \). For retrograde waves LKHI is possible for strong gradients of \( \beta_\mu > 1/2 \) only in the range \(-0.2 \leq \varepsilon_R/m < 0 \). Here \( (\chi_2)_{\min} = -0.2 \) is valid for \( \varepsilon^2_R = 1 \) and \( \beta_\mu = 5/2 \), which follows from Eq. (52).

The total condition for the existence of spatially stable but temporarily unstable waves reads as follows:

\[
\begin{align*}
\chi_2 < \frac{\varepsilon_R}{m} < \min(\chi_1, \chi_3). \\
\end{align*}
\]

Figs. 2a, b show the validity ranges of this condition for some typical values of \( \beta_\mu \). The hatched areas are places where LKHI is possible. These figures are obtained by overlapping the Figs. 1a and 1b. For prograde waves on both sides these hatched areas become very narrow: with decreasing \( \beta_\mu \) the extent of the hatched area decreases and tends to the point \( \{ \varepsilon_R, m \} = \{ 1, 1 \} \).
the solar case. The hatched areas of LKHI disappear with decreasing \( \beta_\mu \). This means we have a lower limit \(( \beta_\mu )_{\text{min}} \).

For retrograde waves it is sufficient to write the condition as \( \chi_2 < \varepsilon_R/m < \chi_3 \). In Fig. 2a \( \chi_2 \) is the dashed curve and \( \chi_3 \) is the solid curve. LKHI is possible if \( \beta_\mu > 3.5 \) for \( 0 \geq \varepsilon_R/m \geq -0.2 \).

In Fig. 2 the hatched areas means that only these hatched areas for given \( \beta_\mu \) are possible, if LKHI takes place. Outside these hatched areas regular solutions are impossible. The case without LKHI (neutral oscillations) must be investigated separately.

### 4.4. Solar rotation profile

Let us consider at which places we might expect LKHI in the Sun. Unfortunately, it is not clear how the core rotates. Nevertheless some rotation gradients close to Sun’s center might exist, and we could expect LKHI there. It is known from helioseismology that the radiative interior has a very small \( \beta_\mu \), but the exact value is unknown. We have better information on the rotation profile of the solar envelope, including the tachocline. Helioseismology data may be described by different approximate formulae. One of these is (Charbonneau et al. 1998)

\[
\Omega(r, \theta) = \Omega_c + \frac{1}{2} [1 + \text{erf}(\Delta)] (\Omega_s(\theta) - \Omega_c),
\]

\[
\Omega_s = \Omega_{eq} + c_1 \cos^2 \theta + c_2 \cos^4 \theta, \quad \Delta = \frac{r - r_c}{w},
\]

\[
\Omega_c/2\pi = 432.8 \text{ nHz}, \quad \Omega_{eq}/2\pi = 460.7 \text{ nHz},
\]

\[
c_1 = -62.69 \text{ nHz}, \quad c_2 = -67.13 \text{ nHz},
\]

where \( r_c = 0.713R_\odot \) is the radius at the bottom of the convective zone, and \( w = 0.025R_\odot \) is the tachocline thickness.

We can easily check that our approximation \( \Omega/\Omega_c \ll 1 \) is always applicable, if Eq. (54) is represented by \( \Omega = \Omega_c + \Omega(r, \theta) \). The maximum value \( (\Omega/\Omega_c)_{\text{max}} \approx 0.06 \) for \( \theta = \pi/2 \) (equator) is in the convection zone. From Eq. (54) we find the latitudinal gradient of rotation

\[
\beta_\mu = -\frac{1 + \text{erf}(\Delta)}{2\Omega} \sin^2 \theta (c_1 + 2c_2 \cos^2 \theta).
\]

Our supposition about \( \beta_\mu \approx \text{const} \) and \( \beta_c \approx \text{const} \) is based on the presentation \( \Omega \approx \Omega_1(r) + \Omega_2(\theta) \). We could receive such an approximate formula instead of Eq. (54), but we need only local values of the gradients for which Eq. (55) is acceptable. Using this formula we show in Fig. 3 the \( \beta_\mu(\theta) \) dependence for different \( r/R_\odot \). We see that in the Sun \( \beta_\mu \leq 0.02 \), and the maximum is in the photosphere. From Fig. 2 follows that the LKHI of retrograde waves is not present in the solar case. LKHI of prograde waves in the Sun occurs in the upper right corner of Fig. 2a which is enlarged in Fig. 4. Here we see that the LKHI area disappears when \( \beta_\mu < 3 \times 10^{-4} \). This boundary is located at the bold horizon in Fig. 3. Thus the prograde waves become unstable in the Sun at those places where \( 3 \times 10^{-4} \leq \beta_\mu \leq 2 \times 10^{-2} \). It means that LKHI is possible in the area \( r/R_\odot > 0.6725 \) which includes the greatest part of the tachocline, the convective zone, and the photosphere. With increasing \( r \) the LKHI zone expands from middle to high latitudes. Figs. 2a and 4 show that LKHI is occurs at high frequencies (\( \varepsilon_R \sim 1 \)) and in global scales \( (\varepsilon_R/m > 0.5) \). Considering that \( m \neq 0 \) is an integer we get \( m = 1 \).

However, our quantitative analysis of LKHI is based on the general Riemann scheme of Hein’s equation, which

---

**Fig. 3.** The local estimate of the logarithmic gradient of the solar rotation frequency \( \beta_\mu \) for real solar data from helioseismology depending on the co-latitude and the radial distance (the labels are values of \( r/R_\odot \)). The bold horizon is \( \beta_\mu = 3 \times 10^{-4} \), above which the prograde waves become unstable (see next picture).

**Fig. 4.** Enlarged part of Fig. 2a for the smallest gradients of rotation. The labels 1, 2, ..., 6 correspond to \( \beta_\mu = 3 \times 10^{-4}, 5 \times 10^{-4}, 1 \times 10^{-3}, 2 \times 10^{-3}, 3 \times 10^{-3}, 4 \times 10^{-3} \). Areas of instability exist only if \( \beta_\mu \geq 3 \times 10^{-4} \).
is valid only if the middle singular point $x = a$ is far from the other edges at $x = 0$ and $x = 1$. Thus, the limiting cases $\varepsilon_R \to 0$ and $\varepsilon_R \to 1$ (the latter is more important for solar LKHI) should be considered separately. In these limiting cases the regularity condition Eq. (48) may be changed, and the curve in Fig. 4 limiting the instability areas from below may be shifted. In this case LKHI with higher $m$-modes should be expected.

5. The low-frequency waves

After the qualitative analysis of Heun's Eq. (35) we can start a quantitative analysis. Note that the qualitative conclusions drawn above are valid for the more general Eq. (34) with $\Lambda$ term. Heun's equation with four singularities in the general case is solved by a series of hypergeometric Gauss functions. A similar task has been considered for the damping of MHD waves at resonance levels by Dzhalilov & Zhuzhzhda (1990). We will start to study Eq. (35) for some simple limiting cases. At high frequencies ($\varepsilon_R \approx 1$, when LKHI is acting in the Sun) and at low frequencies ($\varepsilon_R^2 \ll 1$, when the waves are stable against LKHI in the Sun) Heun's equation is strongly simplified. In these cases the singular level $x = a$ is shifted either to the pole or to the equator. For both cases solutions are expressed by one hypergeometric function.

In the present work we consider particularly the second case. Let $\varepsilon_R^2 \ll 1 + \beta_\mu$. Then we have $a \approx 1$ and $h \approx 1$. Eq. (35) is now the hypergeometric equation:

$$x(1 - x)Y'' + [\gamma - (\tilde{\alpha} + \tilde{\beta} + 1) x] Y' - \tilde{\alpha}\tilde{\beta} Y = 0,$$

(56)

where all parameters are defined by Eqs. (12, 14, 13). In these definitions $\varepsilon_R^2$ should set to zero. Then the $\Theta$-part of the pressure perturbations, Eq. (11), is expressed by two Gaussian hypergeometric functions:

$$\Theta = C_1 x^{\sigma_1} Y_1(x) + C_2 x^{\sigma_2} Y_2(x).$$

(57)

Here $Y_{1,2}(x) = F(\tilde{\alpha}; \tilde{\beta}; \gamma; x)|_{\sigma = \sigma_{1,2}},$

(58)

and $C_{1,2}$ are arbitrary constants. This general solution includes LKHI for larger $\beta_\mu$ too. This could be realized perhaps in other, younger stars. For the Sun we have $\beta_\mu \ll 1$. We will finish this paper by considering in detail the more popular case when rotation is uniform, $\beta_\mu = 0$.

5.1. Rigid rotation case

Using the conditions $\beta_\chi = 0$ and $\varepsilon_R^2 \ll 1$ the parameters in the solution Eq. (57) are strongly simplified. Because $2\sigma = \pm |m|$ only a regular solution at the pole ($x = 0$) will be left. In the standard definitions of hypergeometric functions (Abramowitz & Stegun 1984) we have

$$\Theta = C x^{|m|/2} F(a, b; c; x),$$

(59)

$$a = \frac{1}{2} \left( -\frac{1}{2} + |m| + q \right), \quad b = \frac{1}{2} \left( -\frac{1}{2} + |m| - q \right),$$

$$c = 1 + |m|, \quad q = \sqrt{\frac{1}{4} - \frac{m}{\varepsilon_R}}, \quad \text{and} \quad C = \text{const}.$$  

Note that $c - a - b = 3/2$. The analytical continuation of the solution Eq. (59) to the equator, $x \to 1$, gives

$$\Theta = C x^{|m|/2} \left[ A F(a, b; -\frac{1}{2}; 1 - x) + B (1 - x)^{3/2} F(c - a, c - b; \frac{5}{2}; 1 - x) \right].$$

(60)

Here the continuation coefficients are (Abramowitz & Stegun 1984)

$$A = \frac{\Gamma(c)\Gamma(3/2)}{\Gamma(c - a)\Gamma(c - b)} = F(a, b; c; 1),$$

(61)

$$B = \frac{\Gamma(c)\Gamma(-3/2)}{\Gamma(a)\Gamma(b)}.$$  

(62)

Eqs. (59, 60) mean that pressure is limited everywhere in the hemisphere. Now let us consider the velocity components. Putting Eq. (59) into Eq. (39) we get

$$V_\theta = \pm \frac{C Q \varepsilon_R^2}{r \rho_0 i \omega} x^{|m|/2} \left[ \left( \frac{m}{\varepsilon_R} - |m| \right) F(a, b; c; x) - x^2 \frac{2ab}{c} F(a + 1, b + 1; c + 1; x) \right].$$

(63)

Taking into account that $F(a, b; c; 0) = 1$, we receive from the regularity of $V_\theta$ at $x = 0$ that $|m| \geq 1$. Axially-symmetric waves $m = 0$ cannot be formed. The continuation of the solution Eq. (63) to the equator, $x \to 1$, gives

$$V_\theta = \pm \frac{C Q \varepsilon_R^2}{r \rho_0 i \omega} x^{|m|/2} \left( \frac{A}{\sqrt{1 - x}} L_1 + B L_2 \right).$$

(64)
The new dispersion relation Eq. (66) completely differs from the dispersion relation of the almost toroidal r-modes. For the r-modes the spectrum Eq. (66) has a maximum at \( m^2 = m_0^2 = 2(1 + l)(3 + 2l) \). If \( l \gg 1 \) we have \( |\varepsilon_R|_{\text{max}} \approx 1/8l \). The spectrum of retrograde waves is shown in Fig. 5. Rossby waves in geophysics (Pedlosky 1982) have a similar spectrum. Using Eq. (66) we may define the local phase and group velocities (at fixed latitude \( \theta_0 \) and radial distance \( r_0 \)) in the azimuthal plane

\[
\begin{align*}
v_{\text{ph}} &= -\frac{2\Omega_0 \sin \theta_0}{(2 + 2l + |m|)(3 + 2l + |m|)} \varepsilon_R, \\
v_{\text{gr}} &= \frac{m^2 - 2(1 + l)(3 + 2l)}{(2 + 2l + |m|)(3 + 2l + |m|)^2}.
\end{align*}
\]

We see that the group velocity changes its sign at the maximum of the spectrum, where \( m^2 = m_0^2 \). Fig. 6 shows that the \( l = 0 \) mode has a maximum group velocity. Long waves carry energy opposite to the rotation direction, while a packet of short waves is carried in rotation direction. The facts that the frequencies \( \omega(m) \) have a maximum (two different \( |m| \) correspond to the same \( \omega \)) and at the maximum of \( \omega = \omega_{\text{max}} \) the group velocity changes the direction hints at the existence of two branches of oscillations. Solving Eq. (66) for \( |m| \) we get these branches. Let \( \omega > 0 \) and \( m = -|m| \). Then

\[
|m| = \frac{1}{2\varepsilon_R}(w_1 \mp \sqrt{w_2}) = m_{1,2},
\]

\[
w_1 = 1 - \varepsilon_R - 4\varepsilon_R(1 + l),
\]

\[
w_2 = (1 - \varepsilon_R)^2 - 8\varepsilon_R(1 + l).
\]

From here we get an upper limit for \( l \) if \( \varepsilon_R \) is given:

\[
l \leq l_{\text{max}} = \left( 1 - \varepsilon_R \right)^2/8\varepsilon_R - 1.
\]

For such degrees of \( l \) the azimuthal numbers are also limited: \( m_1 \leq |m| \leq m_2 \). For \( \varepsilon_R \to 0 \) we have \( l_{\text{max}} \to 1/8\varepsilon_R \), \( m_2 \to 1/\varepsilon_R \), and \( m_1 \to 0 \). However, considering the regularity of the solutions Eq. (66), we must take \( m_1 \geq 1 \). From here we get the lower limit of \( l \):

\[
l \geq l_{\text{min}} = \frac{1}{4} \left( \sqrt{1 + \frac{4}{\varepsilon_R^2} - 1} \right).
\]

Since \( l \geq 0 \), we get \( \varepsilon_R \leq 1/12 \) from Eq. (71). For \( \varepsilon_R = 1/12 \) we have \( l_{\text{min}} = 0 \). Thus the eigenmodes exist for \( \varepsilon_R \leq 1/12 \) if \( l_{\text{min}} \leq l \leq l_{\text{max}} \) and \( m_1 \leq |m| \leq m_2 \). For \( l = l_{\text{max}} \) we have \( m_1 = m_2 \), and \( m_1 = 1 \) for \( l = l_{\text{min}} \). This situation is shown in Fig. 7 for different values of \( \varepsilon_R \). A decrease of the frequency decreases the domain of the existence of the modes.

Setting Eq. (71) into Eqs. (68), (69) gives

\[
\begin{align*}
v_{\text{ph}}^{+} &= \frac{w_1 \pm \sqrt{w_2}}{4(1 + l)(3 + 2l)}, \\
v_{\text{gr}}^{+} &= \frac{w_2 \pm w_1 \sqrt{w_2}}{4(1 + l)(3 + 2l)}.
\end{align*}
\]

Here \( v_{\text{ph}}^{+} \) is the phase velocity of the fast modes and \( v_{\text{ph}}^{-} \) that of the slow modes. For \( l = l_{\text{max}} \) we have \( v_{\text{ph}}^{+} = 0 \)

\[
L_1 = \left( \frac{m}{\varepsilon_R} - |m| \right) F(a, b; -\frac{1}{2}; 1 - x) - xabF(a + 1, b + 1; \frac{1}{2}; 1 - x),
\]

\[
L_2 = \left( \frac{m}{\varepsilon_R} - |m| \right) (1 - x) + 3x F(c - a, c - b; \frac{5}{2}; 1 - x) + \frac{4}{5} x (1 - x)(c - a)(c - b) F(c - a + 1, c - b + 1; \frac{7}{2}; 1 - x).
\]

As for \( x = 1 \) the functions \( L_1 \) and \( L_2 \) are limited, we have the relation \( A = 0 \) from the regularity requirement. Using a property of the gamma-functions (its presentation as an infinite product) in Eq. (61) we get the condition of quantization

\[
q^2 = 2l(|m| + \frac{5}{2})^2, \quad l = 0, 1, 2, ...
\]

From here we get the simple dispersion relation

\[
\varepsilon_R = \frac{\omega}{m 2\Omega} = \frac{1}{(2 + 2l + |m|)(3 + 2l + |m|)}.
\]

Since \( \varepsilon_R/m < 0 \) for any \( l \geq 0 \), only retrograde modes (as seen in the rotating frame) are possible.

5.1.1. Spectrum of retrograde modes

The new dispersion relation Eq. (66) completely differs from the dispersion relation of the almost toroidal r-modes. Their dispersion relation can be derived from Eq. (66) if we formally set \( l_s = 2(1 + l) + |m| \). Then

\[
\omega = -2\Omega \frac{m}{l_s(1 + l_s)}.
\]

However, here the degrees \( l_s \) are functions of \( m \). Due to the coupling of the modes in our case the eigenfunctions can never be expressed by the associated Legendre functions \( P_{l_s}^m \).
curves). For example, the case \( \varepsilon = 0.002 \) (close to the 22-year modes) is emphasized. All possible values of the azimuthal numbers \( (m = m_1 \text{ and } m = m_2) \) are on this curve for given discrete \( l \) in the range \( l_{\text{min}} \leq l \leq l_{\text{max}} \) (see text). If \( \varepsilon > 1/12 \), the eigenmodes disappear.

Fig. 7. The possible domain of the existence of eigen-modes for given Rossby numbers \( \varepsilon_R \) (the numbers at the curves). For example, the case \( \varepsilon_R = 0.002 \) (close to the 22-year modes) is emphasized. All possible values of the azimuthal numbers \( (m = m_1 \text{ and } m = m_2) \) are on this curve for given discrete \( l \) in the range \( l_{\text{min}} \leq l \leq l_{\text{max}} \) (see text). If \( \varepsilon > 1/12 \), the eigenmodes disappear.

and \( v^+_{\text{ph}} = v^-_{\text{ph}} \). In Eq. (71) and in Fig. 7 the \( m_1 \) branch corresponds to the fast mode, but \( m_2 \) to the slow modes, since \( m_2 \geq m_1 \). In Fig. 8 the normalized phase velocities (with inverse sign) for the selected values of \( \varepsilon_R \) in Fig. 7 are shown versus \( l \). Both branches are retrograde modes \( (v^+_{\text{ph}} < 0) \). Using \( \Omega R_\odot \approx 2 \text{ km/s} \) for the Sun, we get from Fig. 8 very slow phase velocities. The fast wave velocity (solid lines) depends more strongly on \( l \). With increasing \( \varepsilon_R \) both branches are accelerated.

In Fig. 9 the group velocities are presented in the same way. For fast waves the group velocity is always parallel to the phase velocity \( (v^+_{\text{gr}} < 0) \), while for the slow waves we have the opposite behavior \( v^-_{\text{gr}} > 0 \). Slow modes packets carry off energy in the rotation direction. Always \( |v^+_{\text{gr}}| > |v^-_{\text{gr}}| \) is valid. With decreasing \( \varepsilon_R \) the range \( [l_{\text{min}}, l_{\text{max}}] \) is shifted to the right-hand side, and it is seen in Fig. 9 that \( v^-_{\text{gr}} \) for such low \( \varepsilon_R \) is almost zero.

Note that \( m = l \) modes are always fast modes.

5.1.2. The eigenfunctions

Taking into account the quantization condition Eq. (65) in the solutions Eqs. (60, 64, 38, and 40) we obtain the eigenfunctions. Turning from complex velocities into the real displacements, \( V = -i \omega \xi \) (recall that \( V \) is the velocity seen in the rotating frame), we get

\[
p' = Q(r)\Theta(\theta) \cos \left[ m \left( \omega - \frac{\Omega R_\odot}{m} \right) t \right],
\]

\[
\xi_\theta = \xi_\theta^* \cos \left[ m \left( \omega - \frac{\Omega R_\odot}{m} \right) t \right],
\]

\[
\xi_\phi = \xi_\phi^* \sin \left[ m \left( \omega - \frac{\Omega R_\odot}{m} \right) t \right],
\]

\[
\xi_r = \xi_r^* \cos \left[ m \left( \omega - \frac{\Omega R_\odot}{m} \right) t \right].
\]

Here the amplitude functions are

\[
\xi_r^* = \frac{1}{\omega^2 - \frac{\Omega^2 R_\odot}{m^2}} \Theta \frac{dQ(r)}{dr},
\]

\[
\xi_\theta^* = \frac{Q(r)}{4 \Omega^2 R_\odot} \xi_{\theta A}(\theta),
\]

\[
\xi_\phi^* = \frac{Q(r)}{4 \Omega^2 R_\odot} \xi_{\phi A}(\theta).
\]

The amplitudes are expressed by Jacobi’s polynomials:

\[
\Theta = B(\sin \theta)^{|m|} \cos^3 \theta F_1(\theta),
\]

\[
\xi_{\theta A} = B(\sin \theta)^{|m|-1} \cos^2 \theta \left( \frac{\varepsilon R}{F_1 - \frac{4lk}{5} \sin^2 \theta F_2} \right),
\]

\[
\xi_{\phi A} = -\frac{B 2lk}{5} \sin(2\theta) \cos^2 \theta \sin^{|m|}\theta F_2(\theta),
\]

\[
F_1 = F \left( -l, k; \frac{5}{2}; \cos^2 \theta \right)
\]

\[
= (-1)^l \frac{l! \Gamma(\frac{5}{2})}{\Gamma(l + \frac{5}{2})} P^{(|m|,3/2)}_{l} (\cos 2\theta) =
\]

\[
= (-1)^l \frac{l! \Gamma(\frac{5}{2})}{\Gamma(l + \frac{5}{2})} \left[ \sum_{j=0}^{l} \frac{(-l)_j (j)_j}{(5/2)_j (5/2)_j} (\cos \theta)^{2j} \right],
\]

\[
F_2 = F \left( 1 - l, 1 + k; \frac{7}{2}; \cos^2 \theta \right)
\]

\[
= (-1)^{l-1} \frac{(l-1)! \Gamma(\frac{7}{2})}{\Gamma(l + \frac{7}{2})} P^{(1|m|,5/2)}_{l-1} (\cos 2\theta) =
\]
The latter case is important because the range \( l_{\min} \leq l \leq l_{\max} \) is large for small \( \varepsilon_{R}^{2} \). Remember that for \( \varepsilon_{R} = 1/12 \) we have \( l_{\min} = 0 \). Using the asymptotic formula for Jacobi’s polynomial (or the hypergeometric function) of large degree we have

\[
\Theta \simeq \frac{|m|! \cos \lambda \cos \theta}{\sqrt{\pi l} \Gamma(|m|) \cos \theta} \left( \frac{m}{\varepsilon_{R}^{2}} + \frac{5}{3} \right), \tag{87}
\]

\[
\xi_{\theta A} \simeq \frac{|m|! \cos \lambda}{\sqrt{\pi l} \Gamma(|m|) \cos \theta} \left( \frac{m}{\varepsilon_{R}^{2}} + \frac{5}{3} \right), \tag{88}
\]

\[
\xi_{\phi A} \simeq \frac{14 |m|! \cos \lambda}{5 \varepsilon_{R} \sqrt{\pi l} \Gamma(|m|) \cos \theta} \left( \frac{m}{\varepsilon_{R}^{2}} + \frac{5}{3} \right). \tag{89}
\]

These extremely simple asymptotic formulae for the eigenfunctions might be used in most cases. The formula for the case of large \( l \) with large \( |m| \) also could be found, e.g. in the book of Bateman & Erdélyi (1953).

In Fig. 10 for some typical selected \((l, m)\) pairs the amplitude functions, Eqs. \((74-83)\), are shown as function of \( \theta \). The first row is the pressure \( \Theta(\theta) \) function normalized to its maximum. The first \( l = 1 \) window represents \( \Theta \) for different \( |m| \) (increasing \(|m|\) from left to right). As the eigenfunctions are multiplied by a \( |m| \) \( \theta \) factor, the amplitudes are strongly suppressed around the pole. Increasing \( |m| \) for a given \( l \) shifts the maxima towards the equator. \( l = |m| \) is the equilibrium case. In the third window of pressure the balance latitude with a maximum amplitude is defined by \( \theta = 20^\circ \) for all \( l = |m| \) modes.

The second and third rows of Fig. 10 are the latitudinal (\( \xi_{\theta A} \)) and azimuthal (\( \xi_{\phi A} \)) eigenfunction amplitudes, respectively, normalized to the maximum of \( \xi_{\theta A} \), see Eqs. \((74-83)\). The latitudinal amplitude behavior is similar to that of the pressure. The azimuthal amplitudes are smaller than the latitudinal amplitudes, but with a change of \( l \) a redistribution of the amplitudes will not take place. \( \xi_{\phi A} \) has practically the same amplitude at all latitudes and for all \( l \).

From Fig. 10 follows that we can expect an interesting behavior of the eigenfunction amplitudes, when both \( l \) and \( |m| \) are large. A suppression from two sides may evoke a concentration of wave energy in narrow latitudinal bands. For example, this is the case for the 22-year solar mode.

For the 22-year modes we take \( \omega_{22} = 2\pi(1.441 \, \text{Hz}) \), for which \( \varepsilon_{R} \approx 0.0016 \). Then we derive from Eq. \((74)\) the limiting values of the integer \( l \), \( 11 \leq l \leq 76 \). For all \( l \) in this range we find from Eq. \((74)\), rounding off, integer azimuthal numbers \( m_{1} \) and \( m_{2} \) for the fast and slow modes, respectively. Putting these integer numbers into Eq. \((74)\), we find the deviation \( \delta = \omega_{22} - \omega \) from the central frequency due to the integer azimuthal numbers. The results are given in Table 1 in Appendix A. It is seen that the

![Fig. 9. Absolute values of the group velocities of fast (solid) and slow (dashed curves) modes normalized to \( 2\Omega_{0} \sin \theta_{0} \) versus \( l \) for selected \( \varepsilon_{R} \). For fast waves \( v_{g}^{+} < 0 \), for slow modes \( v_{g}^{-} > 0 \). For \( l = l_{\max} \) we have \( v_{g}^{+} = v_{g}^{-} = 0 \).](image)
Fig. 10. The amplitudes of eigenfunctions versus latitude for given pairs \((l, |m|)\), Eqs. (81)–(83). The pressure row shows the \(\Theta(\theta)\) function normalized to its own maximum. From left to right all curves are related to the wave numbers given above. The middle and last rows are similar to the first row, but show now the latitudinal \(\xi_{\theta A}\) and azimuthal \(\xi_{\phi A}\) amplitude functions, normalized to the maximum of \(\xi_{\theta A}\).

fast modes with low \(l\) have larger deviations. This table includes all possible \((l, |m|)\) pairs which correspond to the 22-year period. For some example pairs of \((l, |m|)\) we plot in Fig. 11 the latitude dependence of the the quantity \((\xi_{\theta}^2 + \xi_{\phi}^2)^{1/2}\), averaged over the wave period, which characterizes the energy density of the modes. The hemisphere is divided into two equal parts: slow modes are located around the equator (solid lines), fast modes are concentrated around the pole. Each \((l, |m|)\) pair is located in a narrow latitudinal band. Note that the slow modes (the group velocity of which is in the rotation direction) with sunspot-like spatial scales are at latitudes of 30\(^\circ\)–40\(^\circ\) from the equator.

5.1.3. Flow patterns

The eigenfunctions Eqs. (74–77) allow us to discuss the flow character produced by the waves, even if the solution of the radial equation \(Q(r)\) is unknown. Excluding from these equations the time-dependent phase we can receive the trajectory equations of the fluid elements. In the meridional \((r, \theta)\) plane we have

\[
\frac{\xi_{\theta}}{\xi_{r}} = \frac{\xi_{r} Q(r) \psi(r)}{r Q'(r)} \Theta = \tan(\alpha_r).
\]

(90)

It follows from here that the motion in the meridional plane is linear. The poloidal displacement vector in the meridional plane is inclined to the radius by an angle \(\alpha_r\). For \(\alpha_r \approx \frac{\pi}{2}\) we have meridional flows and for \(\alpha_r \approx 0\) radial flows. However, since \(\xi_{\theta}/\xi_{r} \approx 1/\sin(2\theta)\), for any \(r\)
the motions around the pole and around the equator will be almost meridional (|ξ_φ| ≫ |ξ_θ|). We see from Eq. (92) also that the direction of the flow vector will be changed to π/2 if we pass along the radius from the node of the Q(r)-function to Q'(r). Note that every node of Q'(r) is located between the neighbouring two nodes of Q(r). For large radial numbers (such orders are expected) these nodes are located close to each other. Then we have a complicated motion in the meridional plane.

At the surface of the cone over (r, φ) the trajectory of each fluid element is an ellipse around the equilibrium point. The cone displacement equation is

$$\frac{\xi_\phi^2}{\xi_\theta^2} + \frac{\xi_\theta^2}{\xi_r^2} = 1.$$  \hspace{1cm} (91)

The ratio of the semiaxes ξ_θ and ξ_r

$$\frac{\xi_\theta}{\xi_r} = -\frac{Q(r)}{rQ'(r)} \frac{4lkF_2}{F_1} \sin \theta$$  \hspace{1cm} (92)

means that close to the axis of rotation the flow is directed along this axis. On the bigger cone through the equator, where F_1 = F_2 = 1, the motion is mainly azimuthal since l/k > 1. An exception are in this case the nodes of the Q(r)-function at the radius. For large l ≫ 1 the ratio ξ_θ/ξ_r ∼ tan(2θ)/cos θ means that the character of the flows is changing at middle latitudes.

At the surface of any sphere with a radius r the motion of the fluid elements is on trajectories of the ellipse

$$\frac{\xi_\phi^2}{\xi_\theta^2} + \frac{\xi_\theta^2}{\xi_\phi^2} = 1.$$  \hspace{1cm} (93)

Here the ratio of the semiaxes is independent of the radius: ξ_θ/ξ_r = 4lk sin^2 θ cos θ m/F_1 - 4lk F_2 sin^2 θ/ε_R ξ_r^2.

It is seen that near the equator and near the pole the motion is mainly in e_θ direction. For large l ≫ 1 this ratio is

$$\frac{\xi_\phi}{\xi_\theta} \sim \frac{14k}{5\varepsilon_R m} \sin \theta \tan \lambda \cos \theta + \frac{14k}{5} \tan \theta \tan \lambda.$$  \hspace{1cm} (95)

The flow pattern considered here drifts opposite to the sense of rotation with a speed v_ph, if it is observed in a frame rotating with the star.

6. Conclusions

In the present work we have derived the general PDE governing non-radial, adiabatic, long-period (with respect to the rotation period), linear oscillations of a slowly and differentially rotating star. This general equation includes all the high-order g-modes and all possible hybrids of rotation modes as well as their mutual interaction. The geophysical ‘traditional approximation’ considerably simplifies this general equation, and we get two ODEs for the r- and θ-components instead of one with arbitrary gradients of rotation Ω(r, θ). We have received a more stringent condition for the applicability of this approximation to the pulsation of stars. Only for very low frequencies this restriction is the same as that of the standard case.

The θ-equation is Laplace’s equation generalized to the latitudinal differential rotation. Without solving this equation qualitatively we found the exact condition for the appearance of a global instability. This instability is driven by the latitudinal shear, it is not influenced by buoyancy. We call that a ‘latitudinal Kelvin-Helmholtz instability’ (LKHI). The appearance of LKHI strongly depends on the Rossby number (the ratio of rotation period and period of motion), on the azimuthal wave numbers and on the latitudinal rotation gradients. Very large gradients produce retrograde waves (seen in the rotating frame), while a slower rotation gradient is responsible for prograde mode LKHI. The rotation gradient has a lower boundary below which LKHI is not possible for any Rossby number or azimuthal number m.

We have applied the LKHI condition to the helioseismological data of the Sun. Here a global LKHI is possible for the m = 1 mode at practically all latitudes. Radially the LKHI is extended from the greatest part of the tachocline up to the photosphere. The LKHI for the Sun was first obtained by Watson (1981). According to his results the instability is possible only at photosphere layers. Later Gilman & Fox (1997) have shown that such an instability is possible in the tachocline too, if strong toroidal magnetic fields are included. Our results show that the instability of the m = 1 modes and other modes is possible without magnetic fields, in contradiction to Gilman & Fox (1997). This difference is probably connected with the incompleteness of the equations used by

Fig. 11. The normalized energy density of the 22-year fast (dashed) and slow (solid) modes versus co-latitude. The numbers at the curves are (l, |m|) pairs taken from Table 1.
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Watson (1981) and by Gilman & Fox (1997); their equations are two-dimensional only.

The exact solutions of Laplace’s tidal equation for lower frequencies are expressed by Jacobi’s polynomials. Just for lower frequencies the numerical calculations of stellar pulsation analyses meet great problems, when looking for the eigenfunction as infinite series of Legendre functions. The eigenfunctions, defined by higher-order polynomials of Jacobi, cannot be expressed by convergent series of associated Legendre functions. Every Legendre function is a particular case of a Jacobi polynomial.

It has been shown here that the retrograde (slow and fast) modes with high surface wave numbers (l, m) are energetically concentrated in narrow bands of latitudes. This analysis was done for the 22-year modes as an example. Such a concentration of mode energy in a narrow spatial area makes such modes vulnerable to different instability mechanisms such as the ε–mechanism considered in Paper 1.

Appendix A: Coefficients of the main equation

In Section 2.2 our main Eq. (26) for the pressure perturbations has been derived:

$$[\psi_1 \ddot{\delta}^2 + \psi_2 \ddot{\delta}^2 + \psi_3 \dot{\delta} + \psi_4 \dot{\delta} + \psi_5 \dot{\delta} + \psi_6] \dot{P} = 0. \quad (A.1)$$

This singular PDE has the following coefficients:

$$\psi_1 = 1 - \frac{N^2}{\omega^2} - j \mu \frac{1 + \beta_r}{\alpha},$$

$$\psi_2 = \psi_1 \frac{1 - \mu^2}{\alpha} \left[ \frac{\kappa^2}{\alpha} \left( 1 - \frac{N^2}{\omega^2} \right) - j (1 + \beta_r) \right],$$

$$\psi_3 = f_1 + \psi_1 f_2,$$

$$\psi_4 = \frac{\psi^2}{\mu^2} \left[ \frac{7}{\alpha^3} + (1 - \mu^2) \left( \frac{\beta_r}{a_3} - \frac{a_5}{a_3^2} \right) \right] + j \left[ a_3 \psi_1 (f_2 - \beta_r a_5) - f_1 \alpha \beta \right] - \psi_1 \frac{1 - \mu^2}{\alpha} \left( 2 + \beta_r + \beta_r \right),$$

$$\psi_5 = j \psi_1 \frac{1 - \mu^2}{\alpha} \left( 2 + \beta_r + \beta_r \right),$$

$$\psi_6 = f_1 f_2 + \psi_1 (\delta_f f_2) + j \psi_1 a_3 (\delta_r f_2) - \psi^2 \frac{1}{\mu^2} \left[ m^2 + \frac{a_5}{a_3} \left( \frac{\beta_r}{a_3^2} \right) \right].$$

where

$$f_1 = \psi_1 (2 + ja_8) - j a_9 \delta \psi_1,$$

$$f_2 = 1 + \omega_r - \frac{m}{\epsilon_r} \beta_r - j a_6,$$

$$a_3^* = (\mu^2 - 1)(1 + \beta_r) / \alpha = 1 - 1/a_3.$$

Taking into account $\nabla \omega = -m \nabla \Omega$ we can easily obtain all the required derivations of the parameters.

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### Table 1. Results for the 22-year period: frequency deviations $\delta = \omega_{22} - \omega$ for permitted quantum numbers $(l, |m|)$

| $l$ | $m$ | $\delta_{\text{fast}}$ (nHz) | $m$ | $\delta_{\text{slow}}$ (nHz) | $l$ | $m$ | $\delta_{\text{fast}}$ (nHz) | $m$ | $\delta_{\text{slow}}$ (nHz) | $l$ | $m$ | $\delta_{\text{fast}}$ (nHz) | $m$ | $\delta_{\text{slow}}$ (nHz) |
|-----|-----|-------------------------------|-----|-------------------------------|-----|-----|-------------------------------|-----|-------------------------------|-----|-----|-------------------------------|-----|-------------------------------|
| 11  | 1   | 0.055                         | 575 | 0.000                         | 33  | 10  | -0.021                       | 478 | 0.000                         | 55  | 35  | -0.008                       | 365 | -0.001                        |
| 12  | 1   | 0.250                         | 571 | 0.001                         | 34  | 10  | 0.051                        | 474 | 0.001                         | 56  | 36  | 0.010                        | 360 | 0.001                         |
| 13  | 1   | 0.406                         | 567 | 0.001                         | 35  | 11  | 0.020                        | 469 | 0.000                         | 57  | 38  | 0.007                        | 354 | 0.001                         |
| 14  | 2   | -0.265                        | 562 | -0.001                        | 36  | 12  | -0.003                       | 464 | 0.000                         | 58  | 40  | 0.007                        | 348 | 0.001                         |
| 15  | 2   | -0.073                        | 558 | 0.000                         | 37  | 13  | -0.021                       | 459 | -0.001                        | 59  | 43  | -0.008                       | 341 | -0.001                        |
| 16  | 2   | 0.089                         | 554 | 0.000                         | 38  | 14  | -0.033                       | 454 | -0.001                        | 60  | 45  | -0.004                       | 335 | 0.000                         |
| 17  | 2   | 0.226                         | 550 | 0.001                         | 39  | 14  | 0.029                        | 450 | 0.001                         | 61  | 47  | 0.002                        | 329 | 0.000                         |
| 18  | 3   | -0.128                        | 545 | -0.001                        | 40  | 15  | 0.020                        | 445 | 0.001                         | 62  | 50  | -0.005                       | 322 | -0.001                        |
| 19  | 3   | 0.013                         | 541 | 0.000                         | 41  | 16  | 0.014                        | 440 | 0.001                         | 63  | 52  | 0.004                        | 316 | 0.001                         |
| 20  | 3   | 0.136                         | 537 | 0.001                         | 42  | 17  | 0.012                        | 435 | 0.000                         | 64  | 55  | 0.001                        | 309 | 0.000                         |
| 21  | 4   | -0.091                        | 532 | -0.001                        | 43  | 18  | 0.012                        | 430 | 0.000                         | 65  | 58  | 0.002                        | 302 | 0.000                         |
| 22  | 4   | 0.028                         | 528 | 0.000                         | 44  | 19  | 0.014                        | 425 | 0.001                         | 66  | 61  | 0.004                        | 295 | 0.001                         |
| 23  | 4   | 0.134                         | 524 | 0.001                         | 45  | 20  | 0.018                        | 420 | 0.001                         | 67  | 65  | -0.001                       | 287 | 0.000                         |
| 24  | 5   | -0.021                        | 519 | 0.000                         | 46  | 22  | -0.019                       | 414 | -0.001                        | 68  | 69  | -0.002                       | 279 | -0.001                        |
| 25  | 5   | 0.079                         | 515 | 0.001                         | 47  | 23  | 0.010                        | 409 | -0.001                        | 69  | 73  | -0.001                       | 271 | 0.000                         |
| 26  | 6   | -0.035                        | 510 | 0.000                         | 48  | 24  | 0.001                        | 404 | 0.000                         | 70  | 77  | 0.002                        | 263 | 0.000                         |
| 27  | 6   | 0.058                         | 506 | 0.001                         | 49  | 25  | 0.011                        | 399 | 0.001                         | 71  | 82  | 0.001                        | 254 | 0.000                         |
| 28  | 7   | -0.029                        | 501 | 0.000                         | 50  | 27  | -0.009                       | 393 | -0.001                        | 72  | 88  | 0.000                        | 244 | 0.000                         |
| 29  | 7   | 0.057                         | 497 | 0.001                         | 51  | 28  | 0.005                        | 388 | 0.000                         | 73  | 94  | 0.001                        | 234 | 0.001                         |
| 30  | 8   | -0.009                        | 492 | 0.000                         | 52  | 30  | -0.009                       | 382 | -0.001                        | 74  | 102 | 0.000                        | 222 | 0.000                         |
| 31  | 9   | -0.060                        | 487 | -0.001                        | 53  | 31  | 0.006                        | 377 | 0.001                         | 75  | 112 | -0.001                       | 208 | 0.000                         |
| 32  | 9   | 0.019                         | 483 | 0.000                         | 54  | 33  | -0.002                       | 371 | 0.000                         | 76  | 125 | 0.000                        | 191 | 0.000                         |