Joint invariant sets for non-commutative expanding Markov maps of the circle

Georgios Lamprinakis

Abstract

A long-standing question is what invariant sets can be shared by two maps acting on the same space. A similar question stands for invariant measures. A particular interesting case are expanding Markov maps of the circle. If the two involved maps are commuting the answer is almost complete. However very little is known in the non-commutative case. A first step is to analyze the structure of the invariant sets of a single map. For a mapping of the circle of class $C^\alpha$, $\alpha > 1$, we study the topological structure of the set containing all compact invariant sets. Furthermore for a fixed such mapping we examine locally, in the category sense, how big is the subset of all maps that have at least one non trivial joint invariant compact set. Lastly we show the strong dimensional relation of the maximal invariant set of a given Markov map contained in a subinterval of $[0,1)$ and the set of all right endpoints of its invariants sets contained in the same subinterval as well as the continuity dependence of the dimension on the endpoints of the subinterval.

1 Introduction

One long-standing question is which invariant sets can have two maps of the same space in common. In particular the case when the space is the unit interval or the circle and the maps are expanding Markov maps is interesting. One of the main difficulties here is that those maps are not globally invertible. Therefore (and for other reasons) the methods that helped to give answers in the setting of Anosov diffeomorphisms on the two dimensional torus, used by A. Brown and F.R. Hertz in [1], cannot be applied. On the other hand, A. Johnson and D. Rudolph showed that any two commuting Markov maps can be linearized simultaneously [5]. That reduces the question to joined invariant sets of linear Markov maps that was solved by H. Furstenberg [3]. We will consider the general case at least for a majority of maps. For this we study the structure of the set of invariant sets of a given set. A global result for two generic maps needs a restriction in terms of the Hausdorff dimension of the invariant sets while a local result does not need this restriction.

Let $f : [0,1) \to [0,1)$ be an expanding Markov map of the circle, i.e. there exist finitely many points $x_i \in [0,1)$ such that $f(x_{i-1}, x_i) = [0,1)$, $f \in C^\alpha(x_{i-1}, x_i)$

\footnote{Here by $[0,1)$ we denote is the unit interval where 0 and 1 are identified. The topology on $[0,1)$ is the one that has as a base the intervals $(x - \epsilon, x + \epsilon) \cap [0,1)$, if $x \in (0,1)$ and $[0, \epsilon) \cup (1 - \epsilon, 1)$, if $x = 0$, where $\epsilon > 0$. A metric that is compatible with the topology in $[0,1)$ is $d_{[0,1)}(t,s) := \min \{|t-s|,1-|t-s|\}$, for $t,s \in [0,1)$ and $([0,1),d_{[0,1)})$ is a compact metric space.}
and a $\gamma > 1$ so that $|f'(x)| > \gamma$ for all $x \in [0, 1)$. For such a map we consider $K_f$ to be the set of all compact subsets of $[0, 1)$ that are also invariant under the action of $f$. We endow the set $K_f$ with the well known Hausdorff metric $d_H$. M. Urbanski in [16] and C.C. Conley in [2] present some the topological properties of the this metric space for $C^2$ expanding Markov maps and for flows respectively. Motivated by that, we further study the topological structure of the metric space $(K_f, d_H)$. More precisely, we will show that $(K_f, d_H)$ is a compact and totally disconnected metric space.

Consider the set $E^\alpha$, $\alpha \in (1, +\infty]$, of all $C^\alpha$ expanding Markov maps of the circle. $E^\alpha$ is endowed with the $\| \cdot \|_{C^\alpha}$ norm (the usual norm that is considered in $C^\alpha$). Given a function $f$ in $E^\alpha$ we will show that for a generic function $g$ in $E^\alpha$ and for all $K \in K_f$, with sufficiently small Hausdorff dimension, $K \not\in \mathcal{K}_g$. As a matter of fact we will show something even stronger, namely, for generic $g$ and for all $K$ with sufficiently small dimension, the intersection between $g(K)$ and $K$ is empty. A similar result without restriction on the dimension can be shown locally. More specifically, there is a open neighborhood where $f$ is contained in its closure so that for any $g$ in that neighborhood, there is no joint invariant set. It is worth noting that the case of $C^1$ functions was studied and fully resolved by C.G. Moreira in [9].

In the last part of this paper we give some dimensional results concerning the relation between the Hausdorff dimension of the largest $f$-invariant set contained in $[0, c] \subset [0, 1)$ and of the “right endpoints” contained in the same subinterval, in an attempt to weaken the restriction, we further study the invariant sets by their endpoints. More precisely, if $f$ is in $E^\alpha$ then for $c \in [0, 1)$, we consider the sets $M'_c$ to be the set of all points for which their orbit remains below $c$ and $M_c$ to be the set of all points $x$ that their orbit stays in $[0, c)$ and $f^n(x) < c$ for all $n > 1$. In [10] J. Nilsson shows that that for $f = D$, where $D$ is the doubling map acting on the circle then the sets $M_c$ and $M'_c$ have in fact the same Hausdorff dimension. We will prove that this is true, not only for the doubling map, but for every expanding Markov map in $E^\alpha$, $\alpha \in (1, +\infty]$. In a more general setting, we can show a similar result for the case we have both ways restrictions. Namely, let $M'_{c,d}$ denote the set of all points so that their orbit stays in an interval $[c, d] \subset [0, 1)$. Respectively, $M_{c,d}$ is the set of all points $x$ so that not only their orbit remains in $[c, d]$ but also $f^n(x) < c$ for every $n > 1$. This corresponds to right endpoints of their $\omega$-limit sets. Again we can prove that their respective Hausdorff dimensions in fact coincide, i.e. $\dim_H(M_{c,d}) = \dim_H(M'_{c,d})$.

Finally, we study the behaviour of the dimension of $M'_{c,d}$ as $c$, $d$ change. In [16] M. Urbanski also examines the behaviour of the map $(c, d) \mapsto \dim_H(M_{c,d})$ but for the case of $C^2$ expanding Markov maps. In [10] J. Nilsson examines the behaviour of the map $c \mapsto \dim_H(M_c)$ for the doubling map using more elementary combinatorial methods. Our aim is to extend those results for any $f \in E^\alpha$. In fact, we will show that the Hausdorff dimension of $M_{c,d}$ depends continuously on $(c, d)$, i.e. the map $(c, d) \mapsto \dim_H(M_{c,d})$ is continuous.
2 Preliminaries

2.1 Markov partition and Coding

Let $\Sigma_m$ denote the full shift space corresponding to alphabet $\{0,1,\ldots,m-1\}$, i.e. $\Sigma_m := \{0,1,\ldots,m-1\}^\mathbb{N}$. The topology on $\Sigma_m$ is the product topology and it is a compact, metrizable topological space. Let $\sigma$ be the regular shift operator on $\Sigma_m$, such that $(\sigma(x))_i = x_{i+1}$, for all $x = (x_1,x_2,\ldots) \in \Sigma_m$. The shift operator act continuously on $\Sigma_m$. We also endow $\Sigma_m$ with the lexicographic order, i.e. if $a, b \in \Sigma_m$, then $a < b$ if there exists $i_0 \in \mathbb{N}$ such that $a_i = b_i$ for all $1 \leq i < i_0$ and $a_n < b_n$. For $a \in \{0,1,\ldots,m-1\}$, we consider the cylinder set $C_a := \{x \in \Sigma_m : x_1 = a\}$. The cylinder sets are both closed and open subsets of $\Sigma_m$.

Let $\Sigma$ be a closed subset of $\Sigma_m$. Then $\Sigma$ is a subshift of finite type (SFT) if it is a closed subset of $\Sigma_m$ and the shift operator acts continuously on $\Sigma_m$ so that the forbidden blocks that describe $\Sigma$ consist a finite set. Of course the whole shift space is a SFT. A forbidden block $w = [w_1 \ldots w_d]$ can also be described as the collection of larger blocks

$$\{[w_1 \ldots w_0],[w_1 \ldots w_1],\ldots,[w_1 \ldots w_d(d-1)]\}.$$

Thus we can assume if needed, that all the forbidden words are of the same length, equal to that of the longest forbidden block. Any subshift of finite type can also be represented from a $m^{\ell-1} \times m^{\ell-1}$ matrix, $A = (a_{ij})$, with entries in $\{0,1\}$, where $\ell$ is the length of the longest forbidden word and $a_{ij} = 1$ when it corresponds to an allowed block and $a_{ij} = 0$ otherwise.

Let $f : [0,1] \to [0,1]$ be an expanding Markov map of the circle, i.e. $f$ is a local homeomorphism and there exist finitely many points $x_i \in [0,1)$ such that $f(x_{i-1},x_i) = [0,1)$, $f \in C^\alpha(x_{i-1},x_i)$ and a $\gamma > 1$ so that $|f'(x)| > \gamma$ for all $x \in [0,1)$. We call the intervals $I_i = [x_i,x_{i+1}]$, fundamental intervals.

A Markov partition for an expanding Markov map $f$ is a finite cover $\mathcal{P} = \{P_1,\ldots,P_d\}$ of $[0,1)$ such that,

1. each $P_i$ is the closure of its interior, $\text{int} P_i$
2. $\text{int} P_i \cap \text{int} P_j = \emptyset$
3. each $f(P_i)$ is a union of elements in $\mathcal{P}$.

An expanding Markov map has Markov partition of arbitrary small diameter. If $\mathcal{P} = \{P_1,\ldots,P_d\}$ is a Markov partition then $([0,1),f)$ can be represented, in a natural way, by a subshift of finite type, $\Sigma_A$ in $\Sigma_{d-1}$ so corresponding to the transfer matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{int} P_i \cap f^{-1}(\text{int} P_j) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

This gives a coding map $\chi : \Sigma_A \to [0,1)$, so that $\chi_f \circ \sigma = f \circ \chi_f$. Furthermore $\chi_f$ is Hölder continuous and injective on the set of points whose trajectory never hit the boundary of any $P_i$. These points are at most countable (see [13]).

Now let $f,g$ be two expanding Markov maps of class $C^\alpha$, $\alpha > 1$. If $f,g$ have the same number of fundamental intervals, $\{I_1,\ldots,I_d\}$, $\{J_1,\ldots,J_d\}$, then they are topologically conjugated via a Hölder continuous homeomorphism, $h$, induced
by the respective coding maps of the same coding space. Indeed, we consider the
respective coding maps \( \chi_f : \Sigma_d \to [0, 1) \) and \( \chi_g : \Sigma_d \to [0, 1) \). Define for \( x \in [0, 1) \),
\( h(x) := \chi_g(\chi_f^{-1}(x)) \). Even if \( x \) is a boundary point for some \( I_i \) and \( I_{i+1} \) and thus
it can be represented by two sequences, one that ends with infinite \((i - 1)'s\) and
one that ends with infinite \( i's \), then \( \chi_g(\chi_f^{-1}(x)) \) is again a single point. Then
\( h \) has all the requested properties. For more details and proofs one can see for
example [3, 6, 13].

2.2 Entropy and Dimension

Let \( D \) denote the doubling map of the circle, \( \Sigma_2 = \{0, 1\}^\mathbb{N} \), endowed with the
metric
\[
d_2(a, b) := \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{2^i}
\]
and \( \sigma \) be the shift map in \( \Sigma_2 \). Then, as mentioned above, we can associate the
system \( ([0, 1), D) \) with with the space \( (\Sigma_2, \sigma) \) with an almost one to one
corresponding. The correspondence here is rather natural as we relate the
sequence \((x_1, x_2, \ldots), x_i \in \{0, 1\} \) with the real number, in \([0, 1), x = \sum_{i=1}^{\infty} \frac{x_i}{2^i} \).

**Theorem 1.** Let \( A \) denote a compact invariant set of \( (\Sigma_2, \sigma) \) and \( A^* \) the
(corresponding invariant set on \([0, 1], D) \). Then,
\[
\dim_H(A^*) = \dim_{box}(A^*) = \frac{h_{top}(A^*)}{\log 2}
\]

**PROOF.** See [4, Proposition III.1] ■

Note that every invariant set in \([0, 1], D) \) can be represented as an invariant
set of \( (\Sigma_2, \sigma) \), through the onto map \( h : \Sigma_2 \to [0, 1) \) (coding map). Therefore
Theorem 4 tells us that when it comes down to \( D \)-invariant sets on \([0, 1) \) the
notions upper box-counting dimension, lower box-counting dimension, Hausdorff
dimension and topological entropy coincide.

From now on we will use the above result without any special mention. In
other words, we will use the triple equality \( \dim_H(A^*) = \dim_{box}(A^*) \approx h_{top}(A^*) \)
regularly without saying.

**Proposition 1.** Let \( K_1, K_2 \subset \mathbb{R} \) with upper box-counting dimension \( d_1, d_2 \)
respectively. Assume that \( d_1 + d_2 < 1 \), then the Lebesgue measure of \( K_1 - K_2 \) is
zero.

**PROOF.** See [11, Proposition 1, Chapter 4]. Note that the notion of Cantor set is
not essentially used for the proof of this statement. ■

**Proposition 1** shows the relation between dimension and the possibility for
any perturbation of one set to intersect with the other set. Namely, observe that
\( K_1 - K_2 \) is the set of all \( t \in \mathbb{R} \) so that \( K_1 \cap (K_2 + t) \neq \emptyset \). The result above shows
that almost surely this intersection is in fact empty, given that the two sets have
sufficiently small dimension.

A rather straightforward result is the following Lemma. One shall only use
the definition of the Hausdorff dimension and the Lipschitz continuity to prove it.
A detailed proof as well as some more generally stated results can be found in
the first Chapters of [13].
Lemma 1. Let $X$, $Y$ metric spaces, $A \subseteq X$ and $f : X \to Y$ Lipschitz continuous. Then $\dim_H(f(A)) \leq \dim_H(A)$

Lemma 2. Let $X$ be a compact topological space and $Y$ be a Hausdorff topological space. If $f : X \to Y$ is a continuous function, then $f$ is closed and proper.

Another important fact that will be used extensively throughout this paper is the semicontinuity of the entropy.

Lemma 3. (Semicontinuity of entropy) Let $X_n$ be subshifts (not necessarily of finite type) and assume that $X_{n+1} \subseteq X_n$. If $X = \cap_{n \in \mathbb{N}} X_n$, then

$$h_{top}(X) = \lim_{n \to \infty} h_{top}(X_n)$$

Proof. The limit indeed exists and $\lim_{n \to \infty} h_{top}(X_n) = \inf_{n \in \mathbb{N}} h_{top}(X_n)$, since $X_{n+1} \subseteq X_n$ implies that the sequence $\{h_{top}(X_n)\}_{n \in \mathbb{N}}$ is a decreasing sequence in $\mathbb{R}$. We obviously have that $h_{top}(X) \leq h_{top}(X_n)$ for every $n \in \mathbb{N}$. Thus,

$$h_{top}(X) \leq \lim_{n \to \infty} h_{top}(X_n).$$

Now for every $X_n$ there exists a measure of maximal entropy, $\mu_n$, such that $h_{\mu_n}(X_n) = h_{top}(X_n)$ (see for example [3, Chapter 17, Corollary 2, pp 130]). Any weak-* accumulation point $\mu$ of the sequence $\mu_n$ is supported by $X$. Therefore,

$$h_{\mu}(X) = h_{\mu}(X_n),$$

for every $n \in \mathbb{N}$. By the upper semicontinuity of (measure theoretic) entropy in subshifts, i.e. of the map $\mu \mapsto h_{\mu}(\Sigma)$, where $\Sigma$ is a subshift, we have that for every $n_0 \in \mathbb{N}$,

$$h_{\mu}(X_n) \geq \limsup_{n \to \infty} h_{\mu_n}(X_{n_0}).$$

By the nested property and since each $\mu_n$ is supported by $X_n$, for $n \geq n_0$,

$$h_{\mu_n}(X_{n_0}) = h_{\mu_n}(X_n).$$

Thus,

$$h_{\mu}(X) \geq \limsup_{n \to \infty} h_{\mu_n}(X_n).$$

Finally, since

$$h_{top}(X) = \sup\{h_{\mu}(X) \mid \mu \text{ is a } \sigma\text{-invariant Borel probability measure on } X\},$$

we have that,

$$h_{top}(X) \geq h_{\mu}(X) \geq \limsup_{n \to \infty} h_{\mu_n}(X_n) = \lim_n h_{top}(X_n).$$

Lemma 4. If $K \in K_f$, $f \in E^n$ and $K \neq SFT$ then, $K = \bigcap_{n \in \mathbb{N}} SFT_n$, where $SFT_{n+1} \subseteq SFT_n$, for all $n \in \mathbb{N}$.

Proof. Let $K \in K_f$. Set $\tilde{K} := h^{-1}(K)$, where $h : (\Sigma_d, \sigma) \to ([0,1], f)$ is the respective coding map. Then $\tilde{K}$ is a compact subshift of $\Sigma_d$ and let $L_{\tilde{K}} := \{w_1, w_2, \ldots\}$ be the (countable) set of all the forbidden blocks for $\tilde{K}$. If $L_{\tilde{K}}$ is finite then $\tilde{K}$ is a subshift of finite type. If not, we consider for each $n \in \mathbb{N}$, $L_n := \{w_1, w_2, \ldots, w_n\}$ and we consider the respective subshift of finite type, $SFT_n$. Then clearly $SFT_{n+1} \subseteq SFT_n$ and $\tilde{K} = \cap_{n \geq 1} SFT_n$. By the nesting property of $\{SFT_n\}$ and since every $x \in [0,1]$ has a finite number of $h$-preimages in $\Sigma_d$, we have that $K = \cap_{n \geq 1} S_n^+$, where $S_n^+ = h(SFT_n)$.
By Lemma 3 and Lemma 4, we can deduce immediately the following corollary.

**Corollary 1.** Let \( K \in K_D \). For every \( \varepsilon \geq 0 \), \( \exists \text{SFT} \) such that \( K \subset \text{SFT} \) and \( d(\text{SFT}) - \varepsilon < d(K) \). In particular, if \( d(K) < \frac{1}{2} \), \( \exists \text{SFT} \) such that \( K \subset \text{SFT} \) and \( d(\text{SFT}) < \frac{1}{2} \).

**Note.** Observe that, even if not stated as generally as possible, all the results hold for every symbolic system and thus, not only for the doubling map, but also for any expanding system of the circle. Only the general notion of the shift spaces and subshifts of finite type were used, as well as the corresponding coding mapping of each such system mentioned above.

Let \( f \in \mathcal{E}^\alpha \), \( \alpha > 1 \) and \( \sigma : \Sigma_{m+1} \to \Sigma_{m+1} \) be its corresponding coding space, where \( \Sigma_{m+1} = \{0, 1, \ldots, m\}^\mathbb{N} \). Instead of considering the sets \( M_c, M_c' \) and \( M_{c,d}, M_{c,d}' \), we consider the corresponding sets \( M_c, M_c' \) and \( M_{c,d}, M_{c,d}' \), where \( c, d \in \Sigma_{m+1} \) and \( c < d \). Let \( K \) be a compact \( f \)-invariant set. We consider the corresponding invariant set \( S \) in \( \Sigma_{m+1} \). The right endpoint, \( x_r \), of \( S \), since it is invariant, has the property \( \sigma^n(x_r) \leq x_r \), for all \( n \in \mathbb{N} \). Furthermore \( x_r \) correspond to the right endpoint of \( K, x_r \).

Observe that the set if \( x \) is such that there is an \( n \) so that \( \sigma^n(x) = x \), then for \( n_0 = \min\{n \in \mathbb{N} : \sigma^n(x) = x\} \), we have that \( x \) is of the form,

\[
x = x_1x_2 \ldots x_n x_{n+1} x_1 x_2 \ldots x_n x_1 x_2 \ldots x_n = (x_1 x_2 \ldots x_n)^\infty
\]

Therefore, if \( B_n \) denotes a block of length \( n \) from the alphabet \( \{0, 1, \ldots, m\} \), we have that,

\[
\{x \in \Sigma_{m+1} : \exists n_0 = n_0(x) \text{ such that } \sigma^{n_0}(x) = x = \bigcup_{n=1}^{\infty} \{(B_n)^\infty : B_n \text{ is an n-block}\}
\]

where the last set is countable as countable union of finite sets. Therefore the Hausdorff dimension is not affected if we exclude those points. In particular, for our dimensional related results it suffices to define \( M_{\bar{x}} \) and \( M_{\bar{x} A} \) with strict inequalities.

**Definitions 1.**

1. \( M := \{x \in \Sigma_{m+1} : \sigma^n(x) < x, \forall n > 0\} \).
2. \( M'_c := \{x \in \Sigma_{m+1} : \sigma^n(x) \leq x, \forall n \geq 0\} \).
3. \( M_{c,d} := \{x \in \Sigma_{m+1} : \sigma^n(x) < x, \forall n \geq 0\} \).
4. If \( \bar{x} = (x_1, x_2, \ldots) \in \Sigma_{m+1} \) then, \( \bar{x} := (m - x_1, m - x_2, \ldots) \). If \( A \subset \Sigma_{m+1} \) then, \( \bar{A} := \{x \in \Sigma_{m+1} : \bar{x} \in A\} \).

**Example 2.**

\[
\bar{M}_{c,d}' = \{x \in \Sigma_{m+1} : \bar{x} \in M_{c,d}'\} = \{x \in \Sigma_{m+1} : \sigma^n(\bar{x}) \leq \bar{x}, \forall n \geq 0\}
\]

\[
= \{x \in \Sigma_{m+1} : \sigma^n(\bar{x}) \leq \bar{x} \leq x, \forall n \geq 0\} = \{x \in \Sigma_{m+1} : \sigma^n(\bar{x}) \geq x, \forall n \geq 0\}
\]

The definitions above are related to the compact \( f \)-invariant sets through the coding map. More specifically \( M_{c,d}' \) contains the largest invariant set that lies inside the subinterval \([0, c]\), since the orbit of any \( x \) in there does not escape this interval. From the discussion above, the set \( M_{c,d}' \) contains all the right end points of all the invariant sets that are contained in \( M_{c,d}' \) up to a set of zero Hausdorff dimension.
2.3 \( \beta \)-shift

**Definition 3.** Let \( \beta \in (1, +\infty) \). Then, for every \( x \in [0, 1] \), the \( \beta \)-expansion or the expansion in base \( \beta \) of \( x \) is a sequence of integers out of \( \{0, 1, 2, \ldots, [\beta]\} \), such that \( x_n = [T_\beta^{n-1}(x)] \), where \( T_\beta : [0, 1) \to [0, 1) \) is defined by \( T_\beta(x) := \beta x \mod 1 \).

**Definition 4.** The closure of the set of all \( \beta \)-expansions of \( x \in [0, 1] \) is called the \( \beta \)-shift \( S_\beta \).

**Definition 5.** The \( \beta \)-expansion of 1, for some \( \beta > 1 \), is denoted by \( 1_\beta \).

W. Parry showed in [12] that the \( \beta \)-shift, \( S_\beta \), is completely characterised by its expansions of 1.

**Proposition 2.** If the \( \beta \)-expansion of 1 is \((\beta_1, \beta_2, \ldots)\) and the \( \beta' \)-expansion of 1 is \((\beta'_1, \beta'_2, \ldots)\) then \( \beta > \beta' \) if and only if \((\beta_1, \beta_2, \ldots) > (\beta'_1, \beta'_2, \ldots)\).

**Theorem 2.** (Parry) If \( 1_\beta \) is not finite, then an \( \underline{\beta} \in \{0, 1, 2, \ldots, [\beta]\}^N \) belongs to \( S_\beta \) if and only if
\[
\sigma^n(\underline{\beta}) < 1_\beta, \quad \forall n \geq 1
\]
If \( 1_\beta \) is of the form \( i_1i_2\ldots i_M0^\infty \), then \( \underline{\beta} \in \{0, 1, 2, \ldots, [\beta]\}^N \) belongs to \( S_\beta \) if and only if
\[
\sigma^n(\underline{\beta}) < (i_1i_2\ldots i_{M-1}(i_M - 1))^\infty, \quad \forall n \geq 1
\]

**Theorem 3.** (Parry) A sequence \( \underline{\beta} \in \{0, 1, 2, \ldots, [\beta]\}^N \) is an expansion of 1 for some \( \beta \) if and only if, \( \sigma^n(\underline{\beta}) < \underline{\beta} \), \( \forall n \geq 1 \) and then \( \beta \) is unique. Moreover, the map \( \Xi : \beta \mapsto 1_\beta \) is monotone increasing.

Let \( 1_\beta < 1_\beta' < 1_\beta' \). Then all sequences of expansions of one, \( 1_\beta \), for \( \beta_1 < \beta < \beta_2 \) are at the same time expansions for some \( x \in [0, 1] \) in base \( \beta_2 \). Let us denote the set of those \( x \)'s by \( I(\beta_1, \beta_2) \). If \( \pi : [0, 1) \to S_{\beta_2} \) is the map assigning to each \( x \in [0, 1] \) its \( \beta_2 \)-expansion we define the map, (depending on \( \beta_1 \) and \( \beta_2 \)),
\[
\rho_{\beta_1, \beta_2} : I(\beta_1, \beta_2) \to [\beta_1, \beta_2]
\]
by setting \( \rho_{\beta_1, \beta_2}(x) \) to be the unique \( \beta \in [\beta_1, \beta_2] \) having \( \pi(x) \) as its expansion of 1, i.e. \( \pi(x) = 1_\beta \).

In [14], J. Schmeling showed that the map \( \rho_{\beta_1, \beta_2} \) is Hölder continuous and calculated the Hölder-exponent.

**Theorem 4.** (Schmeling) The map \( \rho = \rho_{\beta_1, \beta_2} : I(\beta_1, \beta_2) \to [\beta_1, \beta_2] \) satisfies the Hölder condition,
\[
|\rho(u) - \rho(v)| \leq C \cdot d(u, v)^{\ln \beta_1 / \ln \beta_2}
\]
where \( d \) is the metric on \( \Sigma_{[\beta_2]} \).

**Definition 6.** The sequence \( x \in \Sigma_{m+1} \) is called kneading if for all \( n > 1 \), \( \sigma^n(x) < x \).

**Remark.** By Theorem 3 a kneading sequence corresponds to a \( \beta \)-expansion of 1 for some \( \beta \in (1, m] \). Therefore, \( M_\beta \) is in fact the set of all points in the interval \([0, \underline{\beta}]\) that are expansion of 1 for some \( \beta \in (1, m] \).
3 Structure of the set of D-invariant sets

Notation. If $A$ is a subset of $[0, 1)$ then we define $U^{(0,1)}(A) := \{ x \in [0, 1) : d(x, A) < \epsilon \}$.

Lemma 5. (Maximality of SFT) If $\Sigma \in \mathcal{K}_D$ is a subshift of finite type, then there exists an $\epsilon_\Sigma > 0$ such that, if $K$ is a $D$-invariant set so that $K \subset U^{(0,1)}(\Sigma)$, then $K \subset \Sigma$.

Remark. If we consider the subshift of finite type $\Sigma$ as a subset of $\Sigma_d$ then it is rather straightforward to show the maximality property. Indeed, let $w_k = [w^k_1 \ldots w^k_\ell]$, $w^\ell_0 \in \{0, 1, \ldots, d - 1\}$, $1 \leq \ell \leq \ell'$, be a forbidden block and assume that there are in total $n$ forbidden blocks. We also assume that the length of all forbidden blocks is the same and equal to $\ell$. Consider $C_k := \{ w \in \Sigma_d : (x_1, \ldots, x_\ell) = (w^k_1, \ldots, w^k_\ell) \} = C_{w^k_1} \cap \sigma^{-1}(C_{w^k_2}) \cap \ldots \cap \sigma^{-\ell+1}(C_{w^k_\ell})$, $1 \leq k \leq n$. Then, each $C_k$ is clopen since the cylinders are clopen and $\sigma$ continuous. Furthermore,

$$\Sigma = \bigcap_{m=0}^{\infty} \bigcap_{k=1}^{n} \sigma^{-m}(\Sigma \setminus C_k)$$

Thus for $\mathcal{U} := \bigcap_{k=1}^{n} (\Sigma \setminus C_k)$ we get the result.

Proof. Let $\Sigma \subset [0, 1)$ be a subset of finite type for $f \in \mathcal{E}$ corresponding to the Markov Partition of the interval $\mathcal{P} = \{ P_1, \ldots, P_d \}$, related to $f$. Consider the derived closed intervals of $[0, 1)$, $P_{\ell_1, \ell_2, \ldots, \ell_r} := P_{\ell_1} \cap f^{-1}(P_{\ell_2}) \cap \ldots \cap f^{-r+1}(P_{\ell_r})$, $\ell_i \in \{1, 2, \ldots, d\}$, $\forall i \in \{1, 2, \ldots, r\}$.

Let $R_0 \in \mathbb{N}$ denote the number of all forbidden words. Then we may assume that all the forbidden words have the same length, i.e. there exists $r_0 \in \mathbb{N}$ such that $w_i = w_{i_1} \ldots w_{i_{r_0}}$, for all $i \in \{1, 2, \ldots, R_0\}$.

Now let $\epsilon_0 = \min\{\text{diam}(P_{\ell_1, \ell_2, \ldots, \ell_{r_0}}) : \ell_1, \ell_2, \ldots, \ell_{r_0} \text{ is not a forbidden word}\}$ and $\ell = \ell_1, \ell_2, \ldots, \ell_r$ be an allowed word. We refer to the previous, with respect to the lexicographic order, word as the left from $\ell$ and for the right word. Define the right word. For the word $0^{r_0}$, the left word is $d^{r_0}$ and for the case $d^{r_0}$ the right word is $0^{r_0}$. We write any fixed $P_\ell$ in the form $[a_\ell, b_\ell]$ and we set,

$$W(P) := \begin{cases} [a_\ell, b_\ell] & \text{left and right word from } \ell \text{ are both allowed} \\ (a_\ell - \epsilon, b_\ell] & \text{only the right word from } \ell \text{ is allowed} \\ (a_\ell, b_\ell + \epsilon) & \text{only the left word from } \ell \text{ is allowed} \\ (a_\ell - \epsilon, b_\ell + \epsilon) & \text{left and right word from } \ell \text{ are both not allowed} \end{cases}$$

Now set,

$$\mathcal{U}(\Sigma) := \bigcup_{\ell_1, \ell_2, \ldots, \ell_n \text{ is an allowed word}} W(P_{\ell_1, \ell_2, \ldots, \ell_n})$$

Then $\mathcal{U}(\Sigma)$ is clearly open and $\bigcap_{n \geq 0} f^{-n}(\mathcal{U}(\Sigma)) \supset \Sigma$. In fact, it has the requested property,

$$\bigcap_{n \geq 0} f^{-n}(\mathcal{U}(\Sigma)) = \Sigma.$$

Indeed, let us assume that there exists an $x \in \bigcap_{n \geq 0} f^{-n}(\mathcal{U}(\Sigma)) \setminus \Sigma$ or equivalently, $x \in f^{-n}(\mathcal{U}(\Sigma)) \setminus \Sigma$, for all $n \geq 0$. This means that, for $n = 0$, there exists a
boundary point of some $P_{t_0} ... t_0$, $s_0$, where either the left or the right word from
$l = t_1 ... t_n$ is not allowed, so that $x \in (s_0 - \epsilon_0, s_0 + \epsilon_0) \setminus \Sigma$, where $\epsilon_0 = \epsilon$. Also,
since boundary points go to boundary points, there exists another boundary point,
t_0, of some $P_{t_1} ... t_0$, where $s_0 \in f^{-1}(t_0)$, so that $x \in f^{-1}((t_0 - \epsilon_0, t_0 + \epsilon_0)) \setminus \Sigma$. In
particular, $(s_0 - \epsilon_0, s_0 + \epsilon_0) \cap f^{-1}((t_0 - \epsilon_0, t_0 + \epsilon_0)) \setminus \Sigma \neq \emptyset$. Since $s_0 \in f^{-1}(t_0)$ and
$\epsilon_0 = \epsilon$ is much smaller that the diameter of the partition $\mathcal{P}$, this intersection is
contained in $(s_0 - \epsilon_1, s_0 + \epsilon_1)$, $\epsilon_1 < \epsilon_0$, since $f^{-1}$ contracts intervals. In the same
manner we can find sequences $(t_n)_{n \geq 1}$ and $(\epsilon_n)_{n \geq 0}$ so that $\epsilon_n \searrow 0$ and

$$x \in ((s_0 - \epsilon_0, s_0 + \epsilon_0) \cap f^{-1}((t_0 - \epsilon_0, t_0 + \epsilon_0)) \cap ... \cap f^{-n}((t_n - \epsilon_0, t_n + \epsilon_0))) \setminus \Sigma$$

But $(s_0 - \epsilon_n, s_0 + \epsilon_n) \to \{s_0\}$, when $n \to \infty$. Thus either the intersection is empty
or $x = s_0$ both of which are contradictions.

**Corollary 2.** Let $\Sigma \in \mathcal{K}_D$ be a subshift of finite type and define

$$V_\Sigma := \{S \in \mathcal{K}_D : S \subseteq \Sigma\}.$$ 

Then $V_\Sigma$ is a clopen subset of $(\mathcal{K}_D, d_H)$.

**Proof.** If $L \in d_H(V_\Sigma)$, then there exist $S \in V_\Sigma$ such that $d_H(S, L) < \epsilon_S$, where
$\epsilon_S$ as in Lemma\[5\]. In particular, $L \subseteq \mathcal{U}_{\Sigma}^{(0, 1)}(S) \subseteq \mathcal{U}_{\Sigma}^{(0, 1)}(\Sigma)$. Follows that $L \subseteq \Sigma$
and thus $L \in V_\Sigma$. Therefore $V_\Sigma$ is a closed set.

The same argument shows that if $S \in V_\Sigma$, then for any $L \in \mathcal{K}_D$ such that
d_H(S, L) < \epsilon_L we have that $L \in V_\Sigma$. Therefore $V_\Sigma$ is an open set. \[\Box\]

**Lemma 6.** Let $K_1, K_2 \in \mathcal{K}_D$ such that $K_1 \cap K_2 = \emptyset$. Then there exist subshifts
of finite type, $\Sigma_1 \supset K_1, \Sigma_2 \supset K_2$, such that $V_{\Sigma_1} \cap V_{\Sigma_2} = \emptyset$.

**Proof.** If $K_1, K_2 \in \mathcal{K}_D$ such that $K_1 \cap K_2 = \emptyset$, then by Lemma\[4\] there exist
subshifts of finite type $\Sigma_1 \supset K_1, \Sigma_2 \supset K_2$ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Follows that
$V_{\Sigma_1} \cap V_{\Sigma_2} = \emptyset$. \[\Box\]

**Theorem 5.** (Disconnectedness of $\mathcal{K}_D$) The space $(\mathcal{K}_D, d_H)$ is compact and
totally disconnected. In particular it is of first category.

**Proof.** Compactness can be easily derived by observing that $\mathcal{K}_D$ is a $d_H$-closed
subset of $\mathcal{M}$.

Let $C$ be the connected component that contains $K \in \mathcal{K}_D$. Then $K$ is a
maximal element of $C$. Indeed, if $S \in C$ such that $K \supsetneq S$, by Lemma\[3\] there
exists a subshift of finite type, $\Sigma$, such that $K \supsetneq \Sigma \supsetneq S$. We then have that
$K \in V_\Sigma$ and $S \notin V_\Sigma$. Since $V_\Sigma$ is clopen (Corollary\[2\]) and $C$ is the connected
component that contains $K$, we have that $V_\Sigma \supset C$ (contradiction since $S \notin V_\Sigma$
and $S \in C$).

Finally, let’s assume that $C$ contains at least two distinct elements, $K_1$ and
$K_2$. Then, since both of them are maximal and distinct, $K_1 \cap K_2 = \emptyset$. By Lemma\[6\] there is a subshift of finite type $\Sigma$ such that $K_1 \in V_\Sigma$ and $K_2 \notin V_\Sigma$. With the
same argument as before we get a contradiction and that completes the proof. \[\Box\]

**Note.** All the results above can be similarly be stated and proven when instead of
the doubling map, $D$, we consider any map $f \in \mathcal{E}_\alpha$. 

4 Global result for subsets of “small” Hausdorff dimension

The following proposition is crucial to show that there is a $\| \cdot \|_{C^0}$-residual set in $\mathcal{E}^\alpha$ so that any $f$ in that set has no joint invariant compact invariant set, of "small" Hausdorff dimension. This dimensional restriction arise from Proposition 4 which is essential in order to acquire a global result for all subshifts of finite type with dimension less that $1/2$. The strong sense of denseness of subshifts of finite type from Lemma 2 allows us to pass to all compact invariant sets with sufficiently small dimension.

**Proposition 3.** For $\| \cdot \|_{C^0}$-generic $f \in \mathcal{E}^\alpha$ and for every SFT with $d(SFT) < \frac{1}{2}$,

$$f(SFT) \cap SFT = \emptyset.$$ 

**Proof.** Let $g \in \mathcal{E}^\alpha$ and SFT such that $g(SFT) \cap SFT \neq \emptyset$. Since $g \in \mathcal{E}^\alpha$, $\alpha > 1$, then $g$ is Lipschitz continuous. Thus, by Lemma 1 $\dim_H(g(SFT)) \leq \dim_H(SFT) < \frac{1}{2}$. By Proposition 1 we have that $\lambda(SFT - g(SFT)) = 0$. Since $SFT - g(SFT) = \{ t \in [0,1] : \text{SFT} \cap (g(SFT) + t) \neq \emptyset \}$, it follows that there exist arbitrary small $\varepsilon > 0$ such that $(g(SFT) + \varepsilon) \cap \text{SFT} = \emptyset$.

Thus, for arbitrarily small $\varepsilon > 0$ as above, by defining $g_\varepsilon = g + \varepsilon$ we have that $g_\varepsilon \in \mathcal{E}^\alpha$, $\|g_\varepsilon - g\|_{C^0} = \|g_\varepsilon - g\|_{\infty} = \varepsilon$ and $g_\varepsilon(SFT) \cap \text{SFT} = \emptyset$.

We have that the set $\{\text{SFT} : \dim_H(SFT) < \frac{1}{2}\}$ is countable. Let $\{SFT_n\}_{n \in \mathbb{N}}$ be an enumeration of this set and define $\mathcal{G}_n = \{ f \in \mathcal{E}^\alpha : f(SFT_n) \cap SFT_n = \emptyset \}$. Then from what discussed above $\mathcal{G}_n$ is $\| \cdot \|_{C^0}$-dense in $\mathcal{E}^\alpha$, $\forall n \in \mathbb{N}$.

By Lemma 2 if $f \in \mathcal{E}^\alpha$ then $f$ is a closed map. Hence, if $f \in \mathcal{G}_n$ then $f(SFT_n) \cap SFT_n = \emptyset$ and since both sets are compact, $\exists \delta = \delta_{n,f} > 0$ s.t. $\text{dist}(f(SFT_n), SFT_n) \geq \delta$. Thus the ball $B_{\mathcal{E}^\alpha}(f, \frac{\delta}{2}) \subset \mathcal{G}_n$. Therefore $\mathcal{G}_n$ is $\| \cdot \|_{C^0}$-open in $\mathcal{E}^\alpha$, $\forall n \in \mathbb{N}$. In particular, it is $\| \cdot \|_{C^0}$-open.

Baire’s Theorem indicates that $\mathcal{G} := \bigcap_{n \in \mathbb{N}} \mathcal{G}_n$ is $\| \cdot \|_{C^0}$-dense in $\mathcal{E}^\alpha$. $lacksquare$

**Remark.** The method used in the proof of Proposition 3 can be used to show the same result for any countable set of compact $f$-invariant subsets of “small” Hausdorff dimension. Moreover, arguing in a similar way, a weaker result can be shown for any countable set, $\mathcal{A}$, of compact $D$-invariant subsets of arbitrary Hausdorff dimension. Namely, for $\| \cdot \|_{C^0}$-generic $f \in \mathcal{E}^\alpha$ and for every $K \in \mathcal{A} \subset \mathcal{K}_D$, $f(K) \neq K$.

**Theorem 6.** For $\| \cdot \|_{C^0}$-generic $f \in \mathcal{E}^\alpha$ and for every $K \in \mathcal{K}_D$ with $d(K) < \frac{1}{2}$,

$$f(K) \cap K = \emptyset.$$ 

**Note.** All the results above can be similarly be stated and proven when instead of the doubling map, $D$, we consider any map $f \in \mathcal{E}^\alpha$.

5 Local result for all subsets

Here we show that locally, i.e. for an open neighborhood in $\mathcal{E}^\alpha$ such that its closure contains $D$, any other map has no joint invariant compact sets. It is a local result in the sense that the density of the open neighborhood, as in Proposition 4 in all $\mathcal{E}^\alpha$ is now replaced by the weaker condition that $D$ is in its closure. That being said, the restrictions in the dimension are lifted.
In other words we may also assume that for our purposes in order to draw results for all monotone expanding Markov maps. Also observe that the investigation of orientation preserving maps is sufficient for \( h \) has to map an endpoint to an endpoint since \( h \) is monotone as a homeomorphism.

The set \( D \) is open. If not, then there exists an \( f \in D \) sequence \( (\epsilon_n) \), where \( \epsilon_n \downarrow 0 \), and, for any \( n \), \( \exists g_n \in B_{C^\infty}(f,\epsilon_n) = \{ f \in \mathcal{E}^\alpha : \| g - f \|_{C^\alpha} < \epsilon_n \} \) and \( K_n \in \mathcal{K}_D \), such that \( g_n(K_n) = K_n \). Since \( \mathcal{K}_D \) is compact, there exist a subsequence \( (K_{n_k})_{k \in \mathbb{N}} \) and a \( D \)-invariant set \( S \), such that \( K_{n_k} \underset{d_H}{\rightarrow} S \). Moreover \( f(S) = S \). Indeed, since \( g_{n_k}(K_{n_k}) = K_{n_k} \),

\[
d_H(S, f(S)) \leq d_H(S,K_{n_k}) + d_H(f(K_{n_k}),g(K_{n_k}))
\]

and \( d_H(f(K_{n_k}),g_{n_k}(K_{n_k})) \to 0 \), since \( g_n \overset{C^\alpha}{\rightarrow} f \), which is a contradiction since we assumed that \( f \in D \).

**Note.** Theorem 6 can be similarly be stated and proven when instead of the doubling map, \( D \), we consider any map \( f \in \mathcal{E}^\alpha \).

### 6 Dimensional results

In this section we will study invariant sets in more detail. In particular we follow the observation that if \( K \) is an invariant set for the orientation preserving maps \( f, g \) that are topologically conjugated \( h \circ f = g \circ h \) via a homeomorphism \( h \), then \( h \) has to map an endpoint to an endpoint since \( h \) is monotone as a homeomorphism.

Also observe that the investigation of orientation preserving maps is sufficient for our purposes in order to draw results for all monotone expanding Markov maps. In other words we may also assume that \( f \) is orientation preserving and thus \( f' > 0 \) and \( \chi_f \) is increasing, otherwise we consider the square of the map. That leads, in a natural way, to the investigation of endpoints of invariant sets and the size of the invariant sets contained between a left and a right endpoint. Of course all the results of this section are not limited to only the case of monotone expanding Markov maps.

The set \( M := \{ x \in \Sigma_m+1 : \sigma^n(x) < x, \forall n > 0 \} \) contains all right endpoints of all the invariant sets and we have that \( \dim_H(M) = 1 \), when we consider the doubling map and its respective usual coding space, \([10]\); in particular when \( m = 2 \). Following the exact same reasoning as in the proof there, we will show that this is true for every \( m \in \mathbb{N} \).

**Lemma 7.** The set \( M := \{ x \in \Sigma_m+1 : \sigma^n(x) < x, \forall n > 0 \} \) has full Hausdorff dimension for every \( m \in \mathbb{N} \).

**Proof.** By Theorem 6 the set \( M \) is the set of \( \beta \) expansions of 1, for \( 1 < \beta < m \). Let \( \beta_k \) be the real number related to \( 1^k0^\infty \), i.e. the unique \( \beta_k \in (1,m) \) such that \( 1^k0^\infty = 1_{\beta_k} \) (Theorem 3). Then we have that \( I(\beta_{k-1},\beta_k) \subset M \) (Proposition 2).
By Theorem 3 the numbers, $\beta_k-1, \beta_k$ can be arbitrarily close for large enough $k$. We also get the following inequalities,

$$\dim_H(M) \geq \dim_H \left( \frac{\ln \beta_k}{\ln \beta_{k-1}} \right) \geq \frac{\ln \beta_k-1}{\ln \beta_k} \dim_H ([\beta_k-1, \beta_k]) = \frac{\ln \beta_k}{\ln \beta_k},$$

where the right hand side of the inequality can grow arbitrarily close to 1, for sufficiently large $k > 1$.

We clearly have that $M_\sigma = M'_\sigma \cap M$ and thus $\dim_H(M_\sigma) \leq \dim_H(M'_\sigma)$. The next result shows that the Hausdorff dimension of those two sets are in fact equal. Observe that for $\sigma = m^\infty$ we have the result from Lemma 7.

**Lemma 8.** Let $\sigma = (c_1, c_2, c_3, \ldots) \in \Sigma_{m+1}$, so that it is not a kneading sequence and $\sigma \neq m^\infty$. Then there exists a kneading sequence $\sigma_1^0, \ldots, \sigma_1^N \in \Sigma_{m+1}$, so that it is not a kneading sequence and $\sigma \neq m^\infty$. Then there exists a kneading sequence $\sigma_1^0, \ldots, \sigma_1^N \in \Sigma_{m+1}$, so that it is not a kneading sequence and $\sigma \neq m^\infty$. Then there exists a kneading sequence $\sigma_1^0, \ldots, \sigma_1^N \in \Sigma_{m+

**Terminology.** Let $x = (x_1, x_2, \ldots) \in \Sigma_{m+1}$. A $x_1$-block $B$, is a block $[x_1 x_2 \ldots x_1]$ that appears in $x$ in the form $[y_1 x_1 \ldots x_1 z] = yBz$, where $y, z \neq x_1$. Of course, if the length of $B$ is denoted by $r \in \mathbb{N}$, then $r \in [1, \infty]$. In particular, the the first $x_1$-block $B_1$, is the first biggest block of the form $[x_1 x_2 \ldots x_1]$ that appears in $x$ and if the length of $B_1$ is denoted by $r_1 \in \mathbb{N}$, then $r_1 \in [1, \infty]$ and $x = (x_1)^\infty$, if $r_1 = \infty$ or $x = B_1 x_{r_1+1} x_{r_1+2} \ldots$, otherwise.

**Proof.** Set $\ell_0 := \min\{\ell \in \mathbb{N} : \sigma^\ell(x) \geq \sigma\}$. Firstly we consider the case $\ell_0 = 1$, i.e. $\sigma(x) \geq \sigma$. Observe that $c_1$ cannot be equal to $m$, otherwise $\sigma = m^\infty$. Indeed, if $c_1 = m$, the relation $\sigma(x) \geq \sigma$ implies that $c_2 \geq c_1 = m$. Follows that $c_3 \geq c_2 = m$, from which follows that $c_4 \geq c_3 = m$ and so on. In other words, we would have that $c = m^\infty$. We set

$$\sigma = (c_1 + 1)^0 \infty$$

Then clearly $d > \sigma$ and it is kneading. Let now $g$ be a kneading sequence so that $\sigma < g$. Then $a_1 \geq c_1$ and in fact we will show that $a_1 > c_1$, which completes the proof for $\ell_0 = 1$. Assume that $a_1 = c_1$. Clearly $\sigma(g) \geq \sigma$ implies that $c_2 \geq c_1$ and since $\sigma > \sigma$, we get that $a_2 \geq c_2 \geq c_1 = a_1$. But, since $\sigma$ is kneading, $a_1 \geq a_2$ and thus $a_1 = a_2 = c_1$. Repeating the same process we get that $a_3 = a_2 = a_1 = c_1$ so that we finally get that $a = c_1^\infty$, which is not kneading. Thus, indeed, $a_1 > c_1$ or equivalently $a_1 \geq c_1 + 1$, which implies that $\sigma \neq d$. Thus in this case, $A$ is the empty set.

Now we consider the case $\ell_0 > 1$. Then $\sigma \neq i^\infty$. Also, from minimality of $\ell_0$, $c_1 \geq c_2, c_3, \ldots, c_{\ell_0}$ and the first $c_1$-block has greater or equal length from any other $c_1$-block that appears in the block $[c_1c_2 \ldots c_{\ell_0}]$.

- Assume that $\sigma^{\ell_0}(x) = \sigma$. Then $\sigma$ is periodic with period $\ell_0$ (from minimality of $\ell_0$). In other words $\sigma$ is of the form $(c_1 c_2 \ldots c_{\ell_0})^{\infty}$ where $c_1 \geq c_2, c_3, \ldots, c_{\ell_0}$ (again from minimality of $\ell_0$). In particular $c_{\ell_0} < c_1$. Indeed, since $\sigma \neq i^\infty$, we have that there exists an $i \in \{2, 3, \ldots, \ell_0\}$ so that $c_i < c_1$. Set $i_0 := \min\{i : c_i < c_1\}$. Then $\sigma = c_1 c_2 \ldots c_{i_0} c_{i_0} \ldots c_{\ell_0} \ldots = (c_1)^{i_0-1} c_{i_0} \ldots c_{\ell_0} \ldots (i_0-1)$-times
If \( c_{\ell_0} \geq c_1 \) we have that,
\[
\sigma^{\ell_0-1}(c) = c_{\ell_0}(c_1)^{h_0-2} \cdot \begin{array}{c}
\ell_0^{\text{position}}
\end{array} c_1 c_{\ell_0} \ldots c_{\ell_0} \ldots
\]
which contradicts the minimality of \( \ell_0 \). We set \( d = c_{1} c_{2} \ldots (c_{\ell_0} + 1) 0^\infty \).
Then clearly \( d > \ell \). Also we cannot have that another kneading sequence in between for it would be of the form
\[
\ell = (c_1 c_2 \ldots c_m)^k c_1 c_2 \ldots c_{i-1} c_i' a_{k \ell_0+i+1} a_{k \ell_0+i+1} \ldots
\]
for some \( k \geq 1 \), since \( \ell < d \) with \( c_i' > c_i \), \( i \in 1, 2, \ldots, \ell_0 \). But then,
\[
\sigma^{k-\ell_0-1}(\ell) = c_1 \ldots c_{i-1} c_i' a_{k \ell_0+i+1} a_{k \ell_0+i+1} \ldots > \ell
\]
Thus, taking for granted that \( d \) is kneading we have that \( A = \emptyset \).
We are left with showing that \( d \) is indeed kneading. It is obvious when \( c_1 > c_i \), for all \( i \in \{2, \ldots, \ell_0\} \). If not, from minimality of \( \ell_0 \), either the first \( c_1 \)-block, \( B_1 \), is of strictly greater length than any other of the \( c_1 \)-blocks, \( B \), that appear before \( \ell_0 \), or there exists another \( c_1 \)-block, \( B_j \), \( j > 1 \), of the same length, where though (from minimality of \( \ell_0 \)), there exists a block \( C \) and \( c' \) \( c'' \) so that they appear in \( \ell \) as blocks of the form \( B_j C c' \) and \( B_j C c'' \).
In both cases we get that \( d \) is kneading.

\[ \bullet \text{ Assume now that } \sigma^{\ell_0}(\ell) > \ell \text{ Then there exists a } k \geq 1 \text{ so that } c_{\ell_0+k} > c_k \text{ and set } k_0 \text{ be the minimum of those } k \text{'s. Then, we have that } c_{k_0} < c_1. \text{ Indeed,}
\]

- if \( k_0 = 1 \), then \( c_{\ell_0+1} > c_1 \) and if we assume that \( c_{k_0} \geq c_1, \)
\[
\sigma^{\ell_0-1}(\ell) = c_{\ell_0} c_{\ell_0+1} \ldots \geq c_1 c_{\ell_0+1} \ldots \geq c_{\ell_0+1} c_{\ell_0} \ldots > c_1 c_2 m^\infty \geq \ell
\]
which contradicts the minimality of \( \ell_0 \).

- if \( k_0 > 1 \), then \( c_{\ell_0+1} = c_1 \), for all \( 1 \leq i < k_0 \). Now, if the length of the first \( c_1 \)-block is \( r \geq 1 \), we have that \( r < \ell_0 + k_0 \). Indeed, since otherwise we would have that \( c_{\ell + k_0} = c_{k_0} = c_1 \), which contradicts the definition of \( k_0 \). In fact we have that \( r < \ell_0 + k_0 - 1 \), for if \( r = \ell_0 + k_0 - 1 \), then \( \ell \) would be of the form
\[
\ell_0 + k_0 - 1 \text{-times}
\]
and \( c_{\ell_0+k_0} \) must be strictly greater than \( c_1 \). But then we would have that \( \sigma(\ell) > \ell \), which contradicts the minimality of \( \ell_0 \).
This means that there exist a \( p < k_0 - 1 \) so that \( c_{k_0 + p} \neq c_1 \) and by the minimality of \( k_0 \), this means that \( c_{k_0 + p} < c_1 \). We set \( p_0 \) to be the first such \( p \). Since \( c_{k_0 + i} = c_i \), for all \( 1 \leq i < k_0 \), we have that \( c_{p_0} = c_{k_0 + p_0} < c_1 \) and also that \( p_0 \) is the first term so that \( c_{p_0} < c_1 \). In other words \( p_0 = r + 1 \) (which also means that \( r < k_0 \)).

If we assume that \( c_{\ell_0} \geq c_1 \), then,

\[
\sigma^{\ell_0-1}(\underline{x}) = c_0c_0c_1\ldots c_{\ell_0-1}\quad \forall i \leq \ell_0\quad \text{with } c_{\ell_0}c_0\ldots c_0 = c_0c_0c_1\ldots c_{\ell_0-1}
\]

which cannot hold from the definition of \( \ell_0 \).

We set

\[
\underline{d} := c_1c_2\ldots (c_{\ell_0} + 1)0^\infty.
\]

Clearly \( \underline{d} > \underline{c} \) and we cannot consider any other sequence in between, otherwise it would have to be of the form

\[
\underline{a} = c_1\ldots c_{\ell_0}a_{\ell_0}+1\ldots
\]

where for some \( i \geq 1 \), \( a_{\ell_0+i} > c_i \) (and \( a_{\ell_0+j} \geq c_j \) for all \( 1 \leq j < i \) ). But, from the inequality, \( \sigma^{\ell_0}(\underline{a}) > \underline{c} \), we would have that \( \sigma^{\ell_0}(\underline{a}) > \underline{a} \) and thus \( \underline{a} \) cannot be kneading.

We are left with showing that \( \underline{d} \) is kneading, where the same argument as above works in this case as well and that completes the proof.

**Proposition 4.** Let \( \underline{c} = (c_1, c_2, c_3, \ldots) \in \Sigma_{m+1} \), then \( \dim_H(M_{\underline{c}}) = \dim_H(M'_{\underline{c}}) \).

**Proof.** We firstly assume that \( \underline{c} \) is a kneading sequence, i.e. \( \sigma^n(\underline{c}) < \underline{c} \) for every \( n > 0 \). Under this assumption we have that \( \dim_H(M') = \dim_H(M'') \), where \( M'' := \{ \underline{x} \in \Sigma_{m+1} : \sigma^n(\underline{x}) < \underline{c} \quad \forall n > 0 \} \). Indeed, let \( \underline{c} \in M_c \) be such that

\[
\exists n_0 = n_0(\underline{c}) \in \mathbb{N} \text{ for which we have } \sigma^{n_0}(\underline{c}) = \underline{c}.
\]

Then \( n_0 \) is unique, since for every \( n > n_0 \), \( \sigma^n(\underline{c}) = \sigma^{n-n_0}(\sigma^{n_0}(\underline{c})) = \sigma^{n-n_0}(\underline{c}) < \underline{c} \), where the last (strict) inequality holds since \( \underline{c} \) is a kneading sequence. Thus such a sequence is of the form \( B_{n_0} : \) where \( B_{n_0} \) is a block of length \( n_0 \) (also called \( n \)-block) and \( B_{n_0} \leq \underline{c} \). Therefore,

\[
\{ \underline{x} \in M_c : \exists n_0 = n_0(\underline{x}) \text{ such that } \sigma^{n_0}(\underline{x}) = \underline{x} \} \subseteq \bigcup_{n=1}^{\infty} \{ B_n : \text{ } B_n \text{ is an } n \text{-block} \}
\]

and the last set is countable as countable union of finite sets. Thus the Hausdorff dimension is not affected by those points, i.e.,

\[
\dim_H(M'_c) = \dim_H(M''_c).
\]
Observe that $M'_{\mathbb{L}} = S_{\beta'_{\mathbb{L}}}$, where $\beta'_{\mathbb{L}}$ is the unique real number in $[1, m]$ such that $c = 1_{\beta'_{\mathbb{L}}}$. Our strategy in order to complete the proof for this case is to consider the encoding of the hole interval by $S_{\beta'_{\mathbb{L}}}$. Following the same reasoning as in Lemma \ref{lemma:2}, since $M'_{\mathbb{L}} \subset S_{\beta'_{\mathbb{L}}}$, we will show that $M'_{\mathbb{L}}$ has full Hausdorff dimension in $S_{\beta'_{\mathbb{L}}}$, i.e. $\dim_H(M'_{\mathbb{L}} \subset S_{\beta'_{\mathbb{L}}}) = 1$. In this setting, $M'_{\mathbb{L}}$ is the set of all $\mathbb{L}$, so that $\dim(dim_{\beta'_{\mathbb{L}}}) \leq 1$ is the set of all $\mathbb{L}$. Following the same reasoning as in Lemma \ref{lemma:8}, we will show that $\mathbb{L}$, i.e. $\mathbb{L}$ has full Hausdorff dimension in $\mathbb{L}$. In particular, $\mathbb{L}$, where $\mathbb{L}$ is the unique real number in $(1 - \beta'_{\mathbb{L}})$ is considered to be kneading and $\mathbb{L}$ is the first position that we hit a non-zero term after $c_1$. Then for any $k \in \mathbb{N}$ we consider the kneading sequence $d_k^k$ as follows,

$$d_k := \begin{cases} (c_1 - 1)^k0^\infty, & c_1 > 1 \\ (10)^k0^\infty, & c_1 = c_2 = 1 (\ell = 2) \\ (10^{\ell-2})0^\infty, & c_1 = 1 \& \ell > 2 \end{cases}$$

Let $\beta_k$ be the real number related to $d_k$, i.e. the unique $\beta_k \in (1, \beta_{\mathbb{L}})$ such that $d_k = 1_{\beta_k}$ (Theorem \ref{theorem:3}). Then we have that $I(\beta_{k-1}, \beta_k) \subset M_{\mathbb{L}}$ (Proposition \ref{proposition:2}).

By Theorem \ref{theorem:4} the numbers, $\beta_{k-1}, \beta_k$ can be arbitrarily close for large enough $k$. We also get the following inequalities,

$$\dim_H(M_{\mathbb{L}}) \geq \dim_H \left( \rho_{\beta_{k-1}, \beta_k}^{-1} \left( [\beta_{k-1}, \beta_k] \right) \right) \geq \frac{\ln \beta_{k-1}}{\ln \beta_k} \dim_H \left( [\beta_{k-1}, \beta_k] \right) = \frac{\ln \beta_{k-1}}{\ln \beta_k}$$

where the right hand side of the inequality can grow arbitrarily close to 1, for sufficiently large $k > 1$.

For the general case, we assume that $\mathbb{L} \in \Sigma_{m+1}$ is not a kneading sequence. If we consider the sequence $d_k$, then by Lemma \ref{lemma:8} we have that,

$$M_{\mathbb{L}} = \{ \mathbb{L} : \sigma^n(\mathbb{L}) < x \leq d_k \ \forall n \geq 1 \} = M_{d_k}.$$  

In particular,  

$$\dim_H(M_{\mathbb{L}}) = \dim_H(M_{d_k}).$$  

Then, as we proved above, $M_{d_k}$ has full Hausdorff dimension in $S_{\beta_{d_k}}$, where $\beta_{d_k}$ is the unique real number in $[1, m]$ such that $d_k = 1_{\beta_{d_k}}$. In particular,  

$$\dim_H(M_{\mathbb{L}}) = \dim_H(M'_{\mathbb{L}}).$$  

Now since $M'_{\mathbb{L}} \supset M'_{d_k} \supset M_{\mathbb{L}}$, we get the result. \hfill $\blacksquare$

A similar approach works for a more general case. Namely, consider the sets

$$M'_{d, \mathbb{L}} := \{ \mathbb{L} \in \Sigma_{m+1} : \mathbb{L} \leq \sigma^n(\mathbb{L}) \leq d_k \ \forall n \geq 0 \}$$

$$M_{d, \mathbb{L}} := \{ \mathbb{L} \in \Sigma_{m+1} : \mathbb{L} \leq \sigma^n(\mathbb{L}) < x \leq d_k \ \forall n \geq 1 \}.$$  

Then $M_{d, \mathbb{L}} \subset M'_{d, \mathbb{L}}$ and thus $\dim_H(M_{d, \mathbb{L}}) \leq \dim_H(M'_{d, \mathbb{L}})$. In particular, we have that they have the same Hausdorff dimension.
Proposition 5. Let \( \underline{x} = (c_1, c_2, c_3, \ldots) \), \( \underline{d} = (d_1, d_2, d_3, \ldots) \) \( \in \Sigma_{m+1} \) with \( \underline{x} < \underline{d} \), then \( \dim_H(M_{\underline{x}, \underline{d}}) = \dim_H(M'_{\underline{x}, \underline{d}}) \).

PROOF. Let us firstly assume that \( \underline{x} \) and \( \underline{d} \) are both kneading sequences. Then, exactly as in Proposition 4, we can show that the sets \( M_{\underline{x}, \underline{d}} \) and \( M'_{\underline{x}, \underline{d}} := \{ \underline{y} \in \Sigma_{m+1} : \underline{y} \leq \sigma^n(\underline{x}) < \underline{d}, \forall n \geq 0 \} \) have the same Hausdorff dimension. Furthermore, \( M_{\underline{x}, \underline{d}} \subset M''_{\underline{x}, \underline{d}} \) and if \( \beta_\underline{x} \) and \( \beta_\underline{d} \) are the real numbers in \( (1, m] \) so that \( 1_\beta_\underline{x} = \underline{x} \) and \( 1_\beta_\underline{d} = \underline{d} \), then \( M''_{\underline{x}, \underline{d}} \subset S_{\beta_\underline{x}} \setminus S_{\beta_\underline{d}} \). Thus it is sufficient to show that \( \dim_H(M_{\underline{x}, \underline{d}} \subset S_{\beta_\underline{x}} \setminus S_{\beta_\underline{d}}) = 1 \).

Observe that \( d_1 > c_1 \), for if \( d_1 = c_1 \), then \( \underline{d} < (d_1)^\infty \). Thus if \( \underline{x} \in M_{\underline{x}, \underline{d}} \) then \( x_i < (d_1)^\infty \) which implies that \( \exists i \in \mathbb{N} \) so that \( x_i < d_1 \) and thus \( \sigma^i(\underline{x}) < \underline{d} \) which is a contradiction (in particular if \( \underline{x} \in M_{\underline{x}, \underline{d}} \) then \( x_1 > c_1 \) and \( x_1 \geq c_1 \) for all \( i \geq 2 \)).

In other words \( M_{\underline{x}, \underline{d}} = \emptyset \) in this case. Firstly, if \( d_1 = c_1 + 2 \), then consider,

\[
\underline{a}_k := (c_1 + 1)^k(c_1)^\infty
\]

for any \( k \in \mathbb{N} \). Then \( (\underline{a}_k) \subset M_{\underline{x}, \underline{d}} \) and it is a Cauchy sequence.

We assume now that \( d_1 = c_1 + 1 \). Since \( \underline{d} \) is assumed to be kneading, we have that \( d_i \leq c_1 + 1 \) for all \( i \geq 2 \) and there exists an \( i_0 \geq 2 \) so that the inequality is strict. If \( \underline{d} = (c_1 + 1)^\infty \) then as above we can show that \( M_{\underline{x}, \underline{d}} = \emptyset \). Therefore there exists an \( \beta_0 \geq 2 \) so that \( d_{\beta_0} = c_1 + 1 \). We consider two cases:

- \( \underline{d} = (c_1 + 1)^{\ell_0}D \), for some \( \ell_0 \geq 2 \) where \( D \) is an infinite block (so that \( \underline{d} \) is kneading). Then consider,

\[
\underline{a}_k := (c_1 + 1)^{\ell_0 - 1}c_1(c_1 + 1)^k(c_1)^\infty
\]

for any \( k \in \mathbb{N} \). Then \( (\underline{a}_k) \subset M_{\underline{x}, \underline{d}} \) and it is a Cauchy sequence.

- \( \underline{d} = (c_1 + 1)d_2d_3 \ldots d_{\ell}d_{\ell + 1}D'' \) where \( d_1, \ldots, d_{\ell} \leq c_1 \) (since and \( D'' \) is an infinite block (so that \( \underline{d} \) is kneading). In particular, from the above,

\[
(c_1 + 1)^{\ell_0}c_1(c_1 + 1)D'' = \underline{d}.
\]

Then consider,

\[
\underline{a}_k := (c_1 + 1)(c_1)^{\ell_0 + k}(c_1 + 1)^k(c_1)^\infty
\]

for any \( k \in \mathbb{N} \). Then \( (\underline{a}_k) \subset M_{\underline{x}, \underline{d}} \) and it is a Cauchy sequence.

Let \( \beta_k \) be the real number related to \( \underline{a}_k \), i.e. the unique \( \beta_k \in (\beta_{\underline{x}}, \beta_{\underline{d}}) \) such that \( \underline{a}_k = 1_{\beta_k} \) (Theorem 3). Then we have that \( I(\beta_{k-1}, \beta_k) \subset M_{\underline{x}, \underline{d}} \) (Proposition 2). Furthermore, observe that \( (\underline{a}_k) \) is Cauchy in \( \Sigma_{m+1} \) and by Theorem 2 the numbers, \( \beta_{k-1}, \beta_k \) can be arbitrarily close for large enough \( k \). We also get the following inequalities,

\[
\dim_H(M_{\underline{x}, \underline{d}}) \geq \dim_H\left(1_{\beta_{k-1}} \cdot 1_{\beta_k} \left(\beta_{k-1}, \beta_k\right)\right)
\]

\[
\geq \frac{\ln \beta_{k-1}}{\ln \beta_k} \dim_H\left[\left(\beta_{k-1}, \beta_k\right)\right] = \frac{\ln \beta_{k-1}}{\ln \beta_k}.
\]

In particular,

\[
\dim_H(M_{\underline{x}, \underline{d}}) \subset S_{\beta_{k-1}} \setminus S_{\beta_k} = 1.
\]

For the general case, i.e. when \( \underline{x} \) and \( \underline{d} \) are not necessarily kneading. In case \( \underline{d} \) is not kneading, by Lemma 6 we consider the kneading sequence \( \underline{d}' \) so that \( \underline{d} < \underline{d}' \) and there is no other kneading sequence in between. Thus,

\[
\dim_H(M_{\underline{x}, \underline{d}}) = \dim_H(M_{\underline{x}, \underline{d}'}).
\]
If \( (c_n) \) is a sequence of kneading sequences that converge to \( c \) then we can choose a subsequence so that \( c_{k_n} \to c \). Then by Theorem 4 and Lemma 3 we have that 
\[
\dim_H(M_{c_{k_n},d'}) \to \dim_H(M_{c,d'}) \quad \text{and} \quad \dim_H(M'_{c_{k_n},d'}) \to \dim_H(M'_{c,d'})
\]
From the above, since \( c_{k_n} \) and \( d' \) are kneading, \( \dim_H(M_{c_{k_n},d'}) = \dim_H(M'_{c_{k_n},d'}) \) for any \( k_n \).

Thus \( \dim_H(M_{c,d'}) = \dim_H(M'_{c,d'}) \) and in particular, \( \dim_H(M_{c,d}) = \dim_H(M'_{c,d}) \).

Since \( M'_{c,d'} \supset M'_{c,d} \), we finally get that,
\[
\dim_H(M_{c,d}) = \dim_H(M'_{c,d'}) .
\]

If on the other hand, there is not such a sequence \( (c_n) \), then there exists a kneading sequence \( d < c \) so that there is no other kneading sequence in between. Then obviously \( M_{c,d} = M_{c,d'} \) and thus \( \dim_H(M_{c,d}) = \dim_H(M'_{c,d}) \). Now since \( M'_{c,d} \subset M'_{c,d'} \), we get the requested, i.e.
\[
\dim_H(M_{c,d}) = \dim_H(M'_{c,d}) .
\]

\[ \square \]

7 Critical Points

Examining a little bit further the behaviour of those sets, we can find some conditions for when the Hausdorff dimension is zero. In fact we can describe the “critical points” so that \( M_{c,d} \) has dimension zero.

Observe that if \( x \in M_{c,d} \), then \( x \leq \sigma^i(x) \leq d \) for all \( i \geq 1 \), which means that,
\[
a_1 \leq x_i \leq b_1 \quad \forall i \geq 1
\]
In other words, the remaining allowed alphabet is \( \{c_1, c_1 + 1, \ldots, d_1 - 1, d_1\} \). In particular, if \( b_1 = a_1 \) then \( M_{c,d} = \emptyset \).

In other words, the remaining allowed alphabet is \( \{c_1, c_1 + 1, \ldots, d_1 - 1, d_1\} \). In particular, if \( c_1 = d_1 \) then \( M'_{c,d} = \emptyset \). We examine now the case where \( c_1 + 1 = d_1 \).

If \( d_2 \leq c_1 \) then \( M_{c,d} = \emptyset \). If \( d_2 > c_1 \), then \( M_{c,d} \) has a positive Hausdorff dimension. Indeed, let \( c = c_1 c_2 \ldots \) and \( d = (c_1 + 1) d_2 \ldots \), with \( d_1 > c_1 \). Since we assume that \( d \) is kneading, then \( d_2 = d_1 = c_1 + 1 \). Then \( M_{c,d} \) contains the subshift of finite type,
\[
S_1 := \{ x \in \{c_1, c_1 + 1\}^\mathbb{N} \mid x_i = d_1 = c_1 + 1 \Rightarrow x_{i+1} = c_1 \}
\]
where \( S_1 \) has positive entropy and thus positive Hausdorff dimension.

In general, if we have that all \( d_i \leq c_1 \), for \( i > 1 \) then \( \dim_H(M_{c,d}) = 0 \). In particular if \( d = (c_1 + 1)c_2 \ldots \), the largest sequence so that \( d_i \leq c_1 \), for \( i > 1 \), then \( \dim_H(M_{c,d}) = 0 \). So we have to look even further in order to have positive dimension. In fact this is exactly the critical point, i.e. for any \( d \geq (c_1 + 1)c_2 \ldots \), we have that \( \dim(M_{c,d}) > 0 \), since it contains a subshift of finite type of the form
\[
S_n := \{ x \in \{c_1, c_1 + 1\}^\mathbb{N} \mid x_i = d_1 = c_1 + 1 \Rightarrow x_{i+1} = x_{i+2} = \ldots = x_{i+n} = c_1 \}
\]
which has a positive entropy.

We sum up the above discussion in the following Proposition.
Proposition 6. Let \( \varrho < \vartheta \). If we assume that \( \vartheta \) is kneading then,
\[
\dim_H(M_{\vartheta, \varphi}) = 0, \quad \text{if } \vartheta \leq (c_1 + 1)c_1^\infty
\]
\[
\dim_H(M_{\vartheta, \varphi}) > 0, \quad \text{if } \vartheta > (c_1 + 1)c_1^\infty
\]

Remark. From Lemma 8, the case of \( \vartheta \) being a kneading sequence is sufficient.

8 Continuity

In [10], J. Nilsson proves that the function \( \phi: \varrho \mapsto \dim_H(\tilde{M}_\varrho') \) (see Example 2) is continuous for the case where \( f(x) = 2x (\text{mod} 1) \). Of course we get it for \( f \in \mathcal{E}^\infty \), where \( f \) is of order 2. We will prove that this is still true for a map in \( \mathcal{E}^\infty \) of any order \( m \in \mathbb{N} \). In fact we will prove the corresponding continuity property for \( M_{\vartheta, \varphi} \), i.e. that \( \dim_H(M_{\vartheta, \varphi}) \) depends continuously on the pair \( (\varrho, \varphi) \), which gives the continuity of \( \varrho \mapsto M_{\vartheta, \varphi} \) as a corollary.

If \( \varrho \) ends with \( m^\infty \), then every term of a sequence of sequences approaching \( \varrho \) from above is eventually equal to \( \varrho \). Thus we will not consider this 'trivial' case as it can be derived in a same way as the more complicated case where \( \varrho \) does not end with \( m^\infty \). Therefore, we can assume that there exists a strictly increasing sequence \( (n_\varphi) \subset \mathbb{N} \) so that, \( c_{n_\varphi} < \mathcal{m} \).

For each \( n \in \mathbb{N} \), consider the following sequences,
\begin{itemize}
  \item \( u_n := c_1 \ldots (c_{n_\varphi} + 1)0^n \)
  \item \( z_n := d_1 \ldots d_{n_\varphi}0^n \)
\end{itemize}

Then for all \( n \in \mathbb{N} \), \( \varrho \leq u_n \) and \( z_n \geq \vartheta \). In particular, \( u_n \searrow \varrho \) and \( z_n \nearrow \vartheta \).

Let \( B_k(S) \) denote the allowed blocks of length \( k \) for the subshift \( S \subset \Sigma_{m+1} \). For any sufficiently large \( n \) so that \( k < j_n \), we have that,
\[
B_k(M_{\varrho, \varphi}) = B_k(u_n, z_n) \tag{8.1}
\]

Indeed, for \( k < j_n \), we have that if the block \( [x_1 \ldots x_k] \) is in \( B_k(M_{\varrho, \varphi}) \), then, as mentioned above, \( x_1, x_2, \ldots, x_k \in \{c_1, c_1 + 1, \ldots, d_1\} \) and it does not contain subblocks of the form
\begin{itemize}
  \item \( [c_1(c_2 - 1)], [c_1(c_2 - 2)], \ldots, [c_1c_1] \), if \( c_2 \geq c_1 \)
  and
  no further restrictions for the subblocks of length greater or equal to 2 other than those that arise from the remaining allowed alphabet, if \( c_2 < c_1 \).
  \item \( [c_1c_2(c_3 - 1)], [c_1c_2(c_3 - 2)], \ldots, [c_1c_2c_1] \), if \( c_3 \geq c_1 \)
  and
  no further restrictions for the subblocks of length greater or equal to 3 other than those that arise from the remaining allowed alphabet, if \( c_3 < c_1 \).
  \quad \vdots
  \item \( [c_1c_2 \ldots c_{k-1}(c_k - 1)], [c_1c_2 \ldots c_{k-1}(c_k - 2)], \ldots, [c_1c_2 \ldots c_{k-1}c_1] \), if \( c_k \geq c_1 \)
  and
  no further restrictions for the subblocks of length greater or equal to \( k \) other than those that arise from the remaining allowed alphabet, if \( c_k < c_1 \).
  \end{itemize}

and
• \([d_1(d_2 + 1)], \ [d_1(d_2 + 2)], \ldots, \ [d_1d_1]\), if \(d_2 \leq d_1\) and no further restrictions for the subblocks of length greater or equal to 2 other than those that arise from the remaining allowed alphabet, if \(d_2 > d_1\).

• \([d_1d_2(d_3 + 1)], \ [d_1d_2(d_3 + 2)], \ldots, \ [d_1d_2d_1]\), if \(d_3 \leq d_1\) and no further restrictions for the subblocks of length greater or equal to 3 other than those that arise from the remaining allowed alphabet, if \(d_3 > d_1\).

\(\vdots\)

• \(d_1d_2\ldots d_{k-1}(d_k + 1), \ d_1d_2\ldots d_{k-1}(d_k + 2), \ldots, \ d_1d_2\ldots d_{k-1}d_1\), if \(d_k \leq d_1\) and no further restrictions for the subblocks of length greater or equal to \(k\) other than those that arise from the remaining allowed alphabet, if \(d_k > d_1\).

Hence, the possible restrictions come from the first \(k\)-block of the sequences \(d\) and \(d\). Since \(k < j_n\), the first \(k\)-blocks of \(y_n\) and \(d_n\) coincide with those of \(c\) and \(d\) respectively, so exactly the same subblocks are excluded for \(B_k(M'_n, z_n)\).

**Lemma 9. (Application of Fatou)** Let \((a_{n,\ell})_{n,\ell}\) be a sequence so that

• \(a_{n,\ell} > 0\), for all \(n, \ell \in \mathbb{N}\),

• it is increasing w.r.t. \(n\), in the sense that \(a_{n,\ell} \leq a_{n,\ell+1}\),

• the limits \(\lim_{\ell \to \infty} a_{n,\ell}\) and \(\lim_{n \to \infty} \lim_{\ell \to \infty} a_{n,\ell}\) exist.

Then,

\[
\lim_{n \to \infty} \lim_{\ell \to \infty} a_{n,\ell} \geq \lim_{\ell \to \infty} \lim_{n \to \infty} a_{n,\ell}
\]

**Proof.** Consider the sequence \(b_{n,1} := a_{n,1}\) and \(b_{n,\ell} := a_{n,\ell} - a_{n,\ell-1}\), for \(\ell > 1\) and the counting measure \(\mu\) on the measurable space \((\mathbb{N}, \mathcal{P}(\mathbb{N}))\). Then by Fatou’s Lemma,

\[
\liminf_n \int_\mathbb{N} b_{n,\ell} \, d\mu(\ell) \geq \int_\mathbb{N} \liminf_n b_{n,\ell} \, d\mu(\ell).
\]

Since we integrate w.r.t. the counting measure and from the definition of \(b_{n,\ell}\)

\[
LHS = \liminf_n \sum_{\ell=1}^L b_{n,\ell} = \liminf_n \lim_{L \to \infty} \sum_{\ell=1}^L b_{n,\ell} = \liminf_n \lim_{L \to \infty} a_{n,L} = \lim_{L \to \infty} \liminf_n a_{n,L}
\]

Furthermore, taking also into account that \(a_{n,\ell}\) is increasing w.r.t. \(n\), we have that,

\[
RHS = \sum_{\ell=1}^\infty \liminf_n b_{n,\ell} = \lim_{L \to \infty} \sum_{\ell=1}^L \liminf_n b_{n,\ell}
\]

\[
= \lim_{L \to \infty} \liminf_n \sum_{\ell=1}^L b_{n,\ell} = \lim_{L \to \infty} \liminf_n a_{n,L} = \lim_{L \to \infty} \liminf_n a_{n,L}.
\]

\[\blacksquare\]
Lemma 10. Let $c, d$ be two sequences in $\Sigma_{m+1}$. Furthermore, consider for every $n \in \mathbb{N}$ the sequences $y_n$ and $z_n$ as above. Then,

$$h_{\text{top}}(M_c', \overline{d}) = \lim_{n \to \infty} h_{\text{top}}(M_{y_n, z_n}').$$

Proof. We set $a_{n, \ell} := \frac{1}{\ell} \log(|B_{\ell}(M_{y_n, z_n}'))|$. Then by Lemma 9 and relation (8.1) we have that,

$$h_{\text{top}}(M_c', \overline{d}) \geq \lim_{n \to \infty} h_{\text{top}}(M_{y_n, z_n}').$$

With the above Lemma in hand, we are now ready to prove that the dimension of $M_{c, d}$ depends continuously on the pair $(c, d)$.

Theorem 8. Let the map defined by $\Phi : (c, d) \mapsto \dim_{H}(M_{c}').$ The map $\Phi$ is continuous.

Proof. It is sufficient to show that there exist sequences $w_n \nearrow c$, $x_n \searrow d$, and $y_n \nearrow c$, $z_n \nearrow d$ such that,

$$\lim_{n \to \infty} \Phi(w_n, x_n) = \Phi(c, d) = \lim_{n \to \infty} \Phi(y_n, z_n).$$

In that case, both of the limits exist from monotonicity. Using Theorem 1 and Lemma 10 we have that the first equality indeed holds for any pair of sequences so that $w_n \nearrow c$ and $z_n \nearrow d$, since then $M_{c, d}' = \bigcap_n M_{w_n, z_n}'$, and $M_{w_n, z_n}' \subset M_{c, d}'$. The second equality is given from Theorem 1 and Lemma 10 and that completes the proof.

Remark. The first inequality in the proof of Theorem 8 can also be derived in a similar manner as the second equality.

8.1 Corollaries of Theorem 8

Theorem 9. The functions $\phi : \xi \mapsto \dim_{H}(M_{\xi}')$ and $\tilde{\phi} : \xi \mapsto \dim_{H}(\tilde{M}_{\xi}')$ are continuous.

Let $N_{\xi} := \{ \xi \in \Sigma_{m+1} : \xi \leq \sigma^n(\xi) < \xi \}$. By Proposition 5 we also have the following.

Theorem 10. The map defined by $\Psi : (c, d) \mapsto \dim_{H}(M_{c, d})$ is continuous. In particular, the functions $\psi : \xi \mapsto \dim_{H}(M_{\xi})$ and $\tilde{\psi} : \xi \mapsto \dim_{H}(N_{\xi})$ are continuous.
9 Further Discussion

Firstly, in view of Lemma 8 we will prove the following Lemma.

Lemma 11. Let \( \bar{c} = (c_1, c_2, c_3, \ldots) \in \Sigma_{m+1} \) be a kneading sequence. Then there exists an \( \epsilon = \epsilon(\bar{c}) > 0 \) so that if \( \bar{d} \) is a sequence so that if \( d_{\Sigma_{m+1}}(\bar{c}, \bar{d}) < \epsilon \), then \( M_{\bar{c}, \bar{d}} = M_{\bar{c}, \bar{d}}^{\epsilon} \) and \( M'_{\bar{c}, \bar{d}} = M'_{\bar{c}, \bar{d}}^{\epsilon} \). In particular, \( M_{\bar{c}, \bar{d}}^{\epsilon} = \{ \bar{x} \in \Sigma_{m+1} : \bar{c} < \sigma^n(\bar{x}) \leq \bar{d}, \forall n \geq 0 \} \) and \( M_{\bar{c}, \bar{d}}' = \{ \bar{x} \in \Sigma_{m+1} : \bar{c} < \sigma^n(\bar{x}) < \bar{d}, \forall n \geq 1 \} \).

Proof. Let \( \bar{c} \) be a kneading sequence. Then we consider \( c_{\min} := \min\{c_i : i \in \mathbb{N}\} \) and \( i_0 \) the first time \( c_{\min} \) appears in the sequence \( \bar{c} \). Since \( \bar{c} \) is kneading, observe that \( c_{\min} < c_1 \) since \( \bar{c} \neq \bar{n}^\infty \), \( n \in \{0, 1, \ldots, m\} \) and of course \( i_0 > 1 \). If \( \bar{c}' \) is such that \( \bar{c}' = c_1 c_2' \ldots \), so that \( c_i' = c_i \), for all \( 1 \leq i \leq i_0 \) and \( \bar{c}' < \bar{c} \), then for every \( \bar{c}'' \) in between we have that \( c_i'' = c_0 < c_1 = c_i' \) and thus \( \sigma^{i_0-1}(\bar{c}'') < \bar{c}' \).

In particular, if we assume that \( \bar{c} \in M_{\bar{c}, \bar{d}} \) then there exists an \( \ell \geq 1 \) so that \( \bar{c} < \sigma^\ell(\bar{c}) < \bar{c} \). But then \( \sigma^{\ell+i_0-1}(\bar{c}'') < \bar{c}' \), which is a contradiction, since we assumed that \( \bar{c} \in M_{\bar{c}, \bar{d}} \).

By symmetry if \( \bar{c} \) is such that \( \bar{c} = c_1 c_2' \ldots \), so that \( c_i' = c_i \), for all \( 1 \leq i \leq i_0 \) and \( \bar{c}' > \bar{c} \), then for every \( \bar{c}'' \) in between we have that \( c_i'' = c_0 < c_1 = c_i' \) and thus \( \sigma^{i_0-1}(\bar{c}'') < \bar{c}' \) and therefore, \( M_{\bar{c}, \bar{d}} = \emptyset \).

Lemma 8 and Lemma 11 give us an idea of what the critical points for the general case should be, i.e. describe the smallest interval so that we hit for the first time any value for the dimension for the first time or the largest interval so that the dimension remains unchanged. For example, if \( \dim_{M_{\bar{c}, \bar{d}}} = a \in [0, 1] \) and if \( \bar{c} \) is a kneading sequence, the dimension remains the same if we perturb a little bit \( \bar{c} \). On the other end, if \( \bar{d} \) is not a kneading sequence, then the dimension is not changed if we enlarge the interval \([\bar{c}, \bar{d}]\) by a little bit from the right end. This is one reason that we are interested in understanding the behaviour of the dimension in the particular case when \( \bar{c} \) is not a kneading sequence and \( \bar{d} \) is a kneading one.

Another reason is that from Theorem 8 and Theorem 10 we can, in principle, approximate the dimension for of \( M_{\bar{c}, \bar{d}} \) for any given interval \([\bar{c}, \bar{d}]\), by calculating the dimension of \( M_{\bar{c}', \bar{d}} \) where \( \bar{c}' \) is a not kneading sequence close to \( \bar{c} \) and \( \bar{d}' \) is a kneading sequence close to \( \bar{d} \). As a matter of fact we can even assume that \( \bar{c}' \) and \( \bar{d}' \) are even finite sequences. Indeed, for \( \bar{c}' \) we have that either it is finite, i.e. ends with infinite 0's or it can be approached from below or above by an increasing sequence of finite non-kneading sequences. On the other hand, either \( \bar{c}' \) is finite or it can be approached, either from below or from above, by a sequence of finite non-kneading sequences. This reduces the problem of estimating the Hausdorff dimension to calculate the dimension of a SFT. That can be easily derived by a similar approach as in the discussion at the beginning of Section 8. Therefore it is sufficient to calculate the dimension of a subshift described above, i.e. find the largest eigenvalue of the adjacency matrix. In principle, this estimation can be arbitrarily sharp.

Acknowledgment. The author would like to express his gratitude to Jörg Schmeling, under whose supervision and guidance this project was conducted and to Tomas Persson for his helpful comments and recommendations.
References

[1] A. Brown, F.R. Hertz: Measure rigidity for random dynamics on surfaces and related skew products, J. Amer. Math. Soc., 30, p. 1055–1132, 2017.

[2] C. C. Conley: Some abstract properties of the set of invariant sets of a flow, Illinois J. Math. 16(4), p. 663-668, 1972.

[3] M. Denker, F. Takens, K. Sigmund: Ergodic Theory on Compacts Sets, Lecture Notes in Mathematics, vol. 527, Springer, 1976.

[4] H. Furstenberg: Disjointness in Ergodic Theory, Minimal Sets, and a Problem in Diophantine Approximation, Math. Systems Theory 1, p. 1–49, 1967.

[5] A. Johnson, D. Rudolph: Commuting endomorphisms of the circle, Ergodic Theory and Dynamical Systems, 12, p. 743 - 748, 1992.

[6] D. Lind, B. Marcus: An Introduction to Symbolic Dynamics and Coding, Cambridge University Press, 1995.

[7] E.A. Michael: Topologies on spaces of subsets, Trans. Amer. Math. Soc., 71, p. 152-182, 1951.

[8] C. G. Moreira Stable intersections of cantor sets and homoclinic bifurcations, Annales de l’Institut Henri Poincaré C, Analyse Non Linéaire, 13(6), p. 741–781, 1996.

[9] C. G. Moreira There are no $C^1$-stable intersections of regular Cantor sets, Acta Math. 206(2), p. 311-323, 2011.

[10] J. Nilsson: On numbers badly approximable by dyadic rationals, Israel Journal of Mathematics, Vol. 171, No. 1, p. 93-110, 2009.

[11] J. Palis and F. Takens: Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations, Cambridge studies in advanced mathematics 35, Cambridge University Press, 1993.

[12] W. Parry: On the $\beta$-expansions of real numbers, Acta Mathematica Acad. Sci. Hung., 11, p. 401-416 1960.

[13] Y. Pesin: Dimension Theory in Dynamical Systems: Conteporary Views and Applications, Chicago Lectures in Mathematics, Chicago University Press, 1997.

[14] J. Schmeling: Symbolic dynamics for $\beta$-shifts and self-normal numbers, Ergodic Theory and Dynamical Systems 17, p. 675-694, 1997.

[15] M. Urbanski: On Hausdorff dimension of invariant sets for expanding maps of a circle, Ergodic Theory and Dynamical Systems 6(2), p. 295–309, 1986.

[16] M. Urbanski: Invariant subsets of expanding mappings of the circle, Ergodic Theory and Dynamical Systems 7(4), p. 627-645, 1987.

Centre for Mathematical Sciences, Lund University, 221 00 Lund, Sweden. E-mail address: georgios.lamprinakis@math.lth.se