Recurrence of multiples of composition operators on weighted Dirichlet spaces

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Abstract
A bounded linear operator $T$ acting on a Hilbert space $\mathcal{H}$ is said to be recurrent if for every non-empty open subset $U \subset \mathcal{H}$ there is an integer $n$ such that $T^n(U) \cap U \neq \emptyset$. In this paper, we completely characterize the recurrence of scalar multiples of composition operators, induced by linear fractional self maps of the unit disk, acting on weighted Dirichlet spaces $S_v$; in particular on the Bergman space, the Hardy space, and the Dirichlet space. Consequently, we complete previous work of Costakis, Manoussos, and Parissis on the recurrence of linear fractional composition operators on Hardy space. In this manner, we determine the triples $(k, m, /)$ for which the scalar multiple of composition operator $kC_v$ acting on $S_v$ fails to be recurrent.

Keywords Hypercyclicity · Recurrence · Composition operator · Dirichlet spaces

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1 Introduction and preliminaries

Throughout this paper, $\mathbb{C}$ will represent the complex plane, and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ will be the one-point compactification of $\mathbb{C}$. Moreover, $\mathbb{D}$ will stand for the open unit disk of $\mathbb{C}$, while $\mathbb{T}$ will represent the unit circle of $\mathbb{C}$.

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A bounded linear operator $T$ acting on a Hilbert space $\mathcal{H}$ is said to be hypercyclic if there is a vector $f \in \mathcal{H}$ whose $T$-orbit;

$$\mathcal{O}(f, T) = \{T^nf; \ n \in \mathbb{N}\},$$

is dense in $\mathcal{H}$. In such a case the vector $f$ is called a hypercyclic vector. An operator $T$ is said to be cyclic if there is a vector $f \in \mathcal{H}$ such that the space generated by $\mathcal{O}(f, T)$;

$$\text{span}(\mathcal{O}(f, T)) = \{p(T)f; \ p \text{ a polynomial}\},$$

is dense in $\mathcal{H}$. In this case the vector $f$ is called a cyclic vector. A strong form of cyclicity and weaker than hypercyclicity is supercyclicity. An operator $T$ is said to be supercyclic if there is a vector $f \in \mathcal{H}$ whose projective orbit;

$$\mathbb{C} \cdot \mathcal{O}(f, T) = \{\lambda T^nf; \ n \in \mathbb{N} \text{ and } \lambda \in \mathbb{C}\},$$

is dense in $\mathcal{H}$. The vector $f$ is called a supercyclic vector. In [2–5], it was introduced and studied the dynamics of a set of operators.

Recall [15], that an operator $T$ on a Hilbert space $\mathcal{H}$ is hypercyclic if and only if it is topologically transitive; that is, for any pair of non-empty open subsets $U$, $V$ of $\mathcal{H}$ there exists some $n \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \emptyset.$$ 

Recently, the notions of hypercyclicity and transitivity have been generalized and studied, see [6, 8, 9, 20].

Another important concept in topological dynamics is that of recurrence. This notion has been initiated by Poincaré and Birkhoff, while a systematic study was given by Costakis et al. in [12]. A bounded linear operator on a Hilbert space $\mathcal{H}$ is called recurrent if for every non-empty open subset $U$ of $\mathcal{H}$ there exists some $n \in \mathbb{N}$ such that

$$T^n(U) \cap U \neq \emptyset.$$ 

A vector $f \in \mathcal{H}$ is said to be recurrent for $T$ if there exists a strictly increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ such that

$$T^{n_k}f \to f, \text{ as } k \to \infty.$$ 

In that sense, every hypercyclic operator is recurrent, and every hypercyclic vector is recurrent. According to [12], an operator $T$ is recurrent if and only if the set of recurrent vectors for $T$ is dense. For more information on linear dynamics we refer to [16] and [7].

The study of linear dynamics has become a very active area of research. This work will be devoted to studying the recurrence of composition operators. Recall that if $\mathcal{H}$ is a Hilbert space of analytic functions in the unit disk $\mathbb{D}$, and if $\phi$ is a nonconstant self map of $\mathbb{D}$, then the composition operator $C_{\phi}$ associated to $\phi$ on $\mathcal{H}$ is defined by
In which case, the function \( \phi \) is called a symbol of \( C_\phi \). For general references on the theory of composition operators, see, e.g., Cowen and MacCluer’s book [13], Shapiro’s book [19] and Zhu’s book [21]. A special feature of composition operators is that the properties of \( C_\phi \) depend significantly on the behavior of the symbol \( \phi \). In this paper, we show that the recurrence of the composition operator is influenced by the location of the fixed points of its symbol.

For each real number \( \nu \) the weighted Dirichlet space \( S_\nu \) is the space of functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) analytic on \( D \) such that the following norm

\[
\|f\|_{S_\nu}^2 = \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2\nu}
\]

is finite. Endowed with the inner product

\[
\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} (n+1)^{2\nu},
\]

the spaces \( S_\nu \) are Hilbert spaces, see [13, p. 16] or [14, p. 1]. For some values of \( \nu \) the spaces \( S_\nu \) are very well known classical analytic function spaces: for instance if \( \nu = 1/2 \), \( S_\nu \) is the Dirichlet space \( D \); for \( \nu = 0 \) it is the Hardy space \( H^2 \) and for \( \nu = -1/2 \) it is the Bergman space \( A^2 \). Observe that if \( \nu_1 > \nu_2 \), then the space \( S_{\nu_1} \) is strictly contained in \( S_{\nu_2} \), and that \( \|f\|_{S_{\nu_2}} \leq \|f\|_{S_{\nu_1}} \), for every \( f \in S_{\nu_1} \).

Also, we can define the Dirichlet space as the collection of functions analytic on \( D \) whose first derivatives have square integrable modulus over \( D \). For \( f \in D \) the norm in \( D \) has the representation

\[
\|f\|_D^2 = |f(0)|^2 + \int_D |f'(z)|^2 dA(z),
\]

where here \( dA(z) \) is the Lebesgue area measure on \( D \) normalized to have unit mass. In the Hardy space \( H^2 \) there is also an integral representation of the norm. This representation is the following

\[
\|f\|_{H^2}^2 = \frac{1}{2\pi} \sup_{0<r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta.
\]

The Dirichlet space and the Hardy space will play a central role in our study.

Let \( \nu \in \mathbb{R} \), we define the function on \( D \) as

\[
k(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^{2\nu}},
\]

which is analytic on \( D \). Then for each \( w \in D \), the reproducing kernel is defined by
\[ K_w(z) = k(\bar{w}z). \] (3)

This is easily seen that \( \|K_w\|^2 = k(|w|^2) \) and for every \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{S}_v \) we have that

\[
\langle f, K_w \rangle = \sum_{n=0}^{\infty} a_n w^n = f(w). \tag{4}
\]

By Eqs. (3) and (4), we have that

\[
|f(z) - g(w)| = |(f - g)(w)| = |\langle f - g, K_w \rangle| \leq \|f - g\|_{\mathcal{S}_v} \|K_w\|,
\]

for every \( v \in \mathbb{R}, f, g \in \mathcal{S}_v \) and \( w \in \mathbb{D} \). This shows that convergence in \( \mathcal{S}_v \) implies uniform convergence on compact subsets of \( \mathbb{D} \).

It is known according to a result of Hurst [17] that the composition operator \( C_\phi \) is always bounded on \( \mathcal{S}_v \) when \( \phi \) is a linear fractional map on \( \mathbb{D} \).

Recall (see, e.g., [1, 19, Chapter 0]) that linear fractional maps are those maps of the form

\[
\phi(z) = \frac{az + b}{cz + d},
\]

where \( a, b, c \) and \( d \) are complex numbers satisfying \( ad - bc \neq 0 \). They extend to the extended complex plane \( \mathbb{C} \) by defining \( \phi(\infty) = \frac{a}{c} \), and \( \phi\left(-\frac{d}{c}\right) = \infty \) if \( c \neq 0 \), while \( \phi(\infty) = \infty \) if \( c = 0 \). The linear fractional maps can be classified according to their fixed point, which are at most two fixed points if \( \phi \) is not the identity. Two linear fractional maps \( \phi \) and \( \psi \) are called conjugate if there is another linear fractional map \( T \) such that

\[
\phi = T^{-1}\psi T.
\]

It is called parabolic if \( \phi \) has only one fixed point \( \zeta \) or, equivalently, it is conjugate to a translation \( \psi(z) = z + \tau \), were \( \tau \neq 0 \). If \( \phi \) has two distinct fixed points \( \alpha \) and \( \beta \), then \( \phi \) is conjugate to a dilation \( \psi(z) = \mu z \). In this case, the linear fractional map \( \phi \) is called elliptic if \( |\mu| = 1 \); hyperbolic if \( \mu > 0 \) and loxodromic otherwise, see [1] for more details. The chain rule formula \( (f \circ g)'(z) = f'(g(z)).g'(z) \) proves that the value of the derivative at the fixed point is preserved under conjugation. Therefore, the derivative of a parabolic linear fractional map at its fixed point is 1, while the derivative of a hyperbolic one is strictly less than 1 at its attractive fixed point and greater than 1 at its repulsive point. We will note \( \text{LFM}(\mathbb{D}) \) to refer to the set of all such maps, which are additionally self-maps of the unit disk \( \mathbb{D} \). It is known that if \( \phi \in \text{LFM}(\mathbb{D}) \), then the derivative \( \phi'(\eta) \) exists and is finite for every \( \eta \) in the unit circle \( \mathbb{T} \). The condition \( \phi(\mathbb{D}) \subset \mathbb{D} \) make some exigences on the location of the fixed points of \( \phi \). We have that (see [19, p. 5])

1. If \( \phi \) is parabolic, then its fixed point \( \eta \) is in \( \mathbb{T} \) and it satisfies \( \phi'(\eta) = 1 \).
2. If \( \phi \) is hyperbolic, the attractive fixed point is in \( \mathbb{D} \), and the other fixed point outside of \( \mathbb{D} \) and both fixed points are on \( \mathbb{T} \) if and only if \( \phi \) is an automorphism of \( \mathbb{D} \).

3. If \( \phi \) is loxodromic or elliptic, one fixed point is in \( \mathbb{D} \) and the other fixed point outside of \( \mathbb{D} \). The elliptic ones are always automorphisms of \( \mathbb{D} \). The fixed point in \( \mathbb{D} \) for the loxodromic ones is attractive.

Concerning the dynamics of composition operator on Dirichlet spaces, many works have been realized:

In her dissertation [22], Nina Zorboska studied composition operators induced by non-elliptic disk automorphism and proved that they are all cyclic on the Hardy space. In addition, in [23] she has obtained some results about cyclicity and hypercyclicity on the so-called smooth weighted Hardy spaces, which are the weighted Hardy spaces whose functions have continuous first derivatives on the boundary of the unit disk (basically weighted Hardy spaces strictly smaller than \( S_{3/2} \)).

In [10, 11] Bourdon and Shapiro thoroughly characterized the cyclicity, the supercyclicity, and the hypercyclicity of linear fractional composition operators on the Hardy space \( H^2(\mathbb{D}) \) (see Table 1). Moreover, in [11] they gave a program of transferring the cyclic and hypercyclic properties of linear fractional composition operators to general composition operators on the Hardy space.

In [14] answering some open questions posed by Zorboska [23], Gallardo-Gutiérrez and Montes-Rodríguez obtained a complete characterization of the cyclic and hypercyclic composition operators induced by linear fractional maps on weighted Dirichlet spaces \( S_v \) (see Table 2).

In the present work, we give a complete characterization of recurrence of scalar multiples of composition operator whose symbols are linear fractional maps, acting on weighted Dirichlet spaces \( S_v \); in particular, on the Bergman space \( S_{-1/2} \), the Hardy space \( S_0 \) and the classical Dirichlet space \( S_{1/2} \). To do that, spectral technics will play a significant role.

| Type of \( \phi \)                  | Cyclic | Supercyclic | Hypercyclic |
|------------------------------------|--------|-------------|-------------|
| Hyperbolic automorphism            | \( v < 1/2 \) | \( v < 1/2 \) | \( v < 1/2 \) |
| Parabolic automorphism             | \( v < 1/2 \) | \( v < 1/2 \) | \( v < 1/2 \) |
| Hyperbolic non-automorphism        | Always | \( v \leq 1/2 \) | \( v < 1/2 \) |
| Parabolic non-automorphism         | \( v \leq 3/2 \) | Never | Never |
| Interior and exterior              | Always | Never | Never |
| Interior and boundary              | Never | Never | Never |
| Elliptic irrational rotation       | Always | Never | Never |
| Elliptic rational rotation         | Never | Never | Never |
The paper is organized as follows: in the second section, we characterize the recurrence of linear fractional composition operator $C_\phi$ on $S_v$ by providing a necessary and sufficient condition on $\phi$ and the real number $v$.

The third section is devoted to studying the recurrence of scalar multiples of linear fractional composition operators $\lambda C_\phi$ on the weighted Dirichlet spaces by providing a necessary and sufficient condition on $\phi$, the real $v$, and the complex scalar $\lambda$.

In Tables 1 and 2, we expose the classification of Gallardo-Gutiérrez and Montes-Rodríguez of the dynamics of composition operators and their scalar multiples on $S_v$. In Table 3, we summarize our results about the recurrence of linear fractional composition operator on $S_v$. Finally, in Table 4, we give a summarize of the recurrence of scalar multiples of linear fractional composition operator on $S_v$. In Tables 1 and 3, by saying $\phi$ is of type Interior (Respectively, Exterior or Boundary) if $\phi$ has a fixed point in the unit disk $\mathbb{D}$ (Respectively, at the exterior of $\overline{\mathbb{D}}$ or the boundary of $\overline{\mathbb{D}}$)

Our goal in the next two sections will be completing the previous tables by adding the recurrence of $C_\phi$ and also the recurrence of $\lambda C_\phi$ on $S_v$. All the results about the recurrence of composition operators and its scalar multiples on $S_v$ spaces are summarized in Tables 3 and 4 below.

For the cases not appearing in the Table IV, the operator $\lambda C_\phi$ cannot be recurrent, for any $\lambda \in \mathbb{C}$.
Table 4  Recurrence of $\lambda C_\phi$ on $S_\nu$, $\phi$ linear fractional map and $\eta$ the attractive fixed point of $\phi$

| Type of $\phi$                              | Recurrence                        |
|---------------------------------------------|-----------------------------------|
| Elliptic automorphism                       | $\nu \in \mathbb{R}$ and $|\lambda| = 1$ |
| Hyperbolic automorphism                     | $\nu < 1/2$ and $\phi'(|\eta|^{1-2\nu}) < |\lambda| < \phi'(\nu)^{(2\nu-1)/2}$ |
| Parabolic automorphism                      | $\nu < 1/2$ and $|\lambda| = 1$   |
| Hyperbolic non-automorphism                 | $\nu \leq 1/2$ and $|\lambda| > \phi'(\nu)^{(1-2\nu)/2}$ |

2 Recurrence of $C_\phi$ on $S_\nu$

A useful tool in the study of any of the dynamic properties is the following proposition known as Comparison principle (see [19, p. 111] and [18]).

**Proposition 2.1** Let $E$ be a metric space and $F$ be a dense subspace that itself a linear metric space with a stronger topology. Suppose that $T$ is a continuous linear transformation on $E$ that maps the smaller space $F$ into itself and is also continuous in the topology of this space. If $T$ is cyclic on $F$, then it is also cyclic on $E$ and has an $E$-cyclic vector that belongs to $F$. Furthermore, the same is true for supercyclic and hypercyclic operators.

Observe that the comparison principle is also true for recurrent operators as show the following proposition.

**Proposition 2.2** Let $E$ be a metric space and $F$ be a dense subspace that itself a linear metric space with a stronger topology. Suppose that $T$ is a continuous linear transformation on $E$ that maps the smaller space $F$ into itself and is also continuous in the topology of this space. If $T$ is recurrent on $F$, then it is also recurrent on $E$ and every $F$-recurrent vector of $T$ is $E$-recurrent vector that belongs to $F$.

**Proof** Assume that $T$ is recurrent on $F$ and let $U \subset E$ be an open subset. Since $F$ is dense in $E$, then $U \cap F$ is not empty. Thus, there is an integer $n$ such that

$$\mathcal{T}^n(U \cap F) \cap (U \cap F) \neq \emptyset.$$  

Hence,

$$\mathcal{T}^n U \cap U \neq \emptyset.$$  

Now, if $x$ is an $F$-recurrent, then $x \in F$ and $\mathcal{T}^{n_k}x \to x$ for some sequence $(n_k) \subset \mathbb{N}$. Thus, $x$ is $E$-recurrent for $T$. \hfill $\Box$

If $\nu_1 > \nu_2$, we have that $S_{\nu_1}$ is a dense subspace of the space $S_{\nu_2}$ and that the convergence in $S_{\nu_1}$ implies convergence in $S_{\nu_2}$. Thus we can apply Proposition 2.2 in the sense that, if an operator $T$ acting on $S_{\nu_1}$ is recurrent, then it is recurrent on $S_{\nu_2}$.
Theorem 2.3  Let \( \phi \) be a linear fractional map on \( \mathbb{D} \) with an interior fixed point \( p \in \mathbb{D} \). Then \( C_\phi \) is recurrent on any of the \( S_v \) spaces if and only if \( \phi \) is an elliptic automorphism of \( \mathbb{D} \).

Proof  If \( \phi \) is an elliptic automorphism then \( \phi \) is conjugate to a rotation (see [19, Chapter 0]). Thus, there exists a linear fractional map composition operator \( S \) and a complex number \( k \in \mathbb{T} \) such that \( C_\phi = S^{-1}C_\phi S \), where \( \phi_1(z) = \lambda z \), \( z \in \mathbb{D} \). To prove that \( C_\phi \) is recurrent, we need to prove that \( C_\phi \) is recurrent. Since \( \lambda \in \mathbb{T} \), there exists \( (n_k)_{k \in \mathbb{N}} \) such that \( \lambda^{n_k} \to 1 \). For any \( f(z) = \sum_{m \geq 0} a_m z^m \in S_v \), using equation (1) we have that

\[
\|C_{\phi_{n_k}}(f) - f\|_{S_v} = \sum_{m \geq 0} |a_m|^2 |\lambda^{mn_k} - 1|^2 (m + 1)^{2v} \to 0, \quad k \to \infty.
\]

Thus, every \( f \in S_v \) is a recurrent vector for \( C_{\phi_{n_k}} \), and so \( C_{\phi_{n_k}} \) is recurrent on \( S_v \). Now, if \( \phi \) is not an elliptic automorphism then the Denjoy–Wolff Iteration Theorem, [19, Proposition 1, Chapter 5], implies that \( \phi_n(z) \) converges to \( p \), for every \( z \in \mathbb{D} \). Hence, if \( f \) is a recurrent function of \( C_\phi \) on \( S_v \), then there exists a sequence of positive integers \( (n_k)_{k \in \mathbb{N}} \), such that \( f \circ \phi_{n_k} \to f \) in \( S_v \). Thus, by the continuity of point evaluation functional on \( S_v \), we have that for each \( z \in \mathbb{D} \):

\[
f(z) = \lim_{k \to \infty} f(\phi_{n_k}(z)) = f(p).
\]

We conclude that the only recurrent vectors of \( C_\phi \) are the constant functions. \( \square \)

Theorem 2.4  Let \( \phi \) be a parabolic automorphism of the unit disk. Then \( C_\phi \) is recurrent on \( S_v \) if and only if \( v < 1/2 \).

Proof  First, if \( v < 1/2 \), then \( C_\phi \) is hypercyclic, hence it is recurrent. Now we shall prove that this condition is necessary. Let us start by the case \( v = 1/2 \). Since \( \phi \) is parabolic, it has a fixed point on the unit circle. Without loss of generality we may assume this fixed point is 1. Let \( f \) be a recurrent vector for \( C_\phi \). Then there exists a sequence \( (n_k)_{k \in \mathbb{N}} \) of positive integers such that

\[
C_{\phi_{n_k}}(f) \to f,
\]

in \( \mathcal{D} \). Since \( \phi \) has no fixed point in \( \mathbb{D} \), by the Denjoy–Wolff Iteration Theorem, the fixed point 1 is actually attractive. Thus, \( \phi_{n_k} \) converge to 1 uniformly on compact subsets of \( \mathbb{D} \); in particular, \( \phi_{n_k}(z) \) converge to 1, for every \( z \in \mathbb{D} \). Therefore by Eq. (2) we have that

\[
\|C_{\phi_{n_k}}(f) - f(1)\|_{\mathcal{D}}^2 = \|f \circ \phi_{n_k} - f(1)\|_{\mathcal{D}}^2 = \|f \circ \phi_{n_k} - f(1)\|_{\mathcal{D}}^2 + \int_{\mathbb{D}} |f'(\phi_{n_k}(z))|^2 |\phi_{n_k}'(z)|^2 d\mathcal{A}(z) \to 0,
\]

when \( k \to \infty \) and therefore, by the Maximum Modulus Principle, it follows that \( f \) is a constant function. Thus, all recurrent vectors for \( C_\phi \) are constants, and so \( C_\phi \)
cannot be recurrent in this case. Therefore, $C_\phi$ cannot be recurrent on $D$. Hence, neither on any $S_\nu$ with $\nu > 1/2$, by Proposition 2.2.

**Theorem 2.5** Let $\phi$ be a hyperbolic automorphism of the unit disk. Then $C_\phi$ is recurrent on $S_\nu$ if and only if $\nu < 1/2$.

**Proof** If $\nu < 1/2$, then the operator $C_\phi$ is hypercyclic, hence it is recurrent. For $\nu \geq 1/2$, the non-recurrence of $C_\phi$ follows exactly as in the case of the parabolic automorphism. \hfill $\square$

Now in the sequel let $S^0_\nu$ be the space of functions obtained from the space $S_\nu$ by identifying functions that differ by a constant; that is $f \in S^0_\nu$ if and only if there is $g \in S_\nu$ such that $f - g$ is constant. Then the space $S^0_\nu$ is invariant under the operator $C_\phi$, and so we can consider $C^0_\phi$ the restriction of $C_\phi$ on $S^0_\nu$.

**Lemma 2.6** Let $\phi$ be an analytic self-map of the unit disk. If $C_\phi$ is recurrent on $S_\nu$, then so is $C^0_\phi$ on $S^0_\nu$.

**Proof** We have that for every $f \in S_\nu$ and $k \in \mathbb{N}$,
\[
\left\| C_\phi f - f \right\|_{S_\nu} \geq \left\| C_\phi f - f + f(0) - f(\phi^k(0)) \right\|_{S_\nu} = \left\| C^0_\phi f - f \right\|_{S^0_\nu}.
\]
Thus, $Rec(C^0_\phi) = Rec(C_\phi) \cap S^0_\nu$. Assume that $C_\phi$ is recurrent, then $Rec(C_\phi)$ is dense in $S_\nu$, which implies that $Rec(C^0_\phi)$ is dense in $S^0_\nu$. \hfill $\square$

**Theorem 2.7** Let $\phi$ be a linear fractional map on $D$ without interior fixed point. If $\phi$ is hyperbolic non-automorphism, then $C_\phi$ is recurrent on $S_\nu$ if and if $\nu < 1/2$.

**Proof** If $\nu < 1/2$ then $C_\phi$ is hypercyclic, thus it is recurrent. Now, suppose that $\nu > 1/2$. Since $\phi$ has no interior fixed point, $\phi$ must have an attractive fixed point $p \in \mathbb{T}$. Thus, $\phi$ is hyperbolic non-automorphism with attractive fixed point $p \in \mathbb{T}$. Hence, the spectrum of $C_\phi$ on $S_\nu$ is given by:
\[
\sigma(C_\phi) = \{ \lambda \in \mathbb{C}; \ |\lambda| \leq |\phi'(p)^{-\gamma}| \} \cup \{ \phi'(p)^k; \ k = 0, 1, ... \},
\]
where $\phi'(p)$ is the angular derivation of $\phi$ on $p$ and $\gamma = (1 - 2\nu)/2$, see [17, Corollary 12]. If $\nu > 1/2$, then $\gamma < 0$. In this case the singleton $\{ \phi'(p) \}$ is a component of the spectrum which isn’t intersect $\mathbb{T}$ since $|\phi'(p)| < 1$, and so by [12, Proposition 2.11], $C_\phi$ cannot be recurrent. Now, suppose that $\nu = 1/2$, that is, $S_\nu$ is the Dirichlet space $D$. If $C_\phi$ is recurrent on the Dirichlet space, then, by Lemma 2.6, so is $C^0_\phi$ on $D_0 := S^0_1$. Since $\phi$ is not an automorphism, $\phi(D) \subsetneq D$, and the recurrence of $C_\phi$ implies that $\phi$ is injective, then there exists a non-empty open $U \subsetneq D$, such that $\phi(D) \cap U = \emptyset$. Hence, on $D_0$ we have
\[
\int_{D} |f'(\phi(z))|^2 |\phi'(z)|^2 dA(z) = \int_{\phi(D)} |f'(z)|^2 dA(z) \leq \int_{\phi(D)} |f'(z)|^2 dA(z) + \int_{U} |f'(z)|^2 dA(z) < \int_{D} |f'(z)|^2 dA(z).
\]

Thus, \( ||C_0||_{D_0} < 1 \), and therefore, \( C_0 \) cannot be recurrent on \( D_0 \), a contradiction. Hence, \( C_\phi \) is not recurrent on \( D \) and neither on any \( S_v \) with \( v > 1/2 \), by Proposition 2.2.

**Theorem 2.8** Let \( \phi \) be a parabolic non-automorphism of the unit disk. Then \( C_\phi \) is never recurrent in any of the weighted Dirichlet spaces \( S_v \).

**Proof** Assume that \( \phi \) is parabolic non-automorphism, then by the Denjoy–Wolff Iteration Theorem, \( \phi \) has a boundary fixed point \( p \in \mathbb{T} \) such that \( \phi_n \) converges to \( p \) uniformly on compact subsets of \( \mathbb{D} \). Then, for every \( f \in S_v \), \( f(\phi_n(z)) \rightarrow f(p) \) uniformly on any compact subset of \( \mathbb{D} \). If \( f \) is a recurrent vector for \( C_\phi \), then there exists a sequence \( (n_k)_{k \in \mathbb{N}} \) of positive integers such that

\[
f \circ \phi_{n_k} \rightarrow f,
\]

in \( S_v \). Thus, we have that

\[
f(z) = \lim_{k \to \infty} (f \circ \phi_{n_k})(z) = f(p),
\]

for every \( z \in \mathbb{D} \) and therefore, by the Maximum Modulus Principle, it follows that \( f \) is a constant function. Then the only recurrent vectors \( C_\phi \) are the constant functions. We conclude that \( C_\phi \) is not recurrent on any space \( S_v \).

3 Recurrence of \( \lambda C_\phi \) on \( S_v \)

**Theorem 3.1** Let \( \phi \) be a holomorphic self map of \( \mathbb{D} \) with an interior fixed point. Then the operator \( \lambda C_\phi \) is recurrent on \( S_v \) if and only if \( \phi \) is an elliptic automorphism and \( \lambda \in \mathbb{T} \).

**Proof** Assume that \( \phi \) is an elliptic automorphism and \( |\lambda| = 1 \). Then \( C_\phi \) is recurrent by Theorem 2.3, and since \( |\lambda| = 1 \), then \( \lambda C_\phi \) is recurrent. If \( \phi \) is not an elliptic automorphism, then \( C_\phi \) is not recurrent, which implies that \( \lambda C_\phi \) is not recurrent as well. Now, if \( |\lambda| \neq 1 \) and \( \phi \) is not an elliptic automorphism. Let \( p \in \mathbb{D} \) be the fixed point of \( \phi \). Suppose that \( \lambda C_\phi \) is recurrent and let \( f \) be a recurrent vector. Then there exists a sequence \( (n_k)_{k \in \mathbb{N}} \) of positive integers such that

\[
\lambda^{n_k} f \circ \phi_{n_k} \rightarrow f,
\]

in \( S_v \). Thus,
Now, observe that \( f(\phi_{n_k}(p)) \to f(p) \) as \( k \to \infty \). Therefore, if \(|\lambda| < 1\), then \( f(p) = 0 \) for every \( z \in \mathbb{D} \). If \(|\lambda| > 1\), then \( f(p) \) is not even defined, unless \( f(p) = 0 \). But \( f(p) \) cannot be zero for every recurrent vector, because the set of recurrent vectors is a dense subset.

**Theorem 3.2** Let \( \phi \) be a hyperbolic non-automorphism and \( \eta \) its boundary fixed point. Then \( \lambda \mathcal{C}_\phi \) is recurrent on \( S_\psi \) if and only if \(|\lambda| > \phi'(\eta)^{(1-2\psi)/2} \) and \( \psi \leq 1/2 \). In particular, we have: For the Dirichlet space \((\psi = 1/2)\) the operator \( \lambda \mathcal{C}_\phi \) is recurrent if and only if \(|\lambda| > 1\).

**Proof** First, if \( \psi \leq 1/2 \) and \(|\lambda| > \phi'(\eta)^{(1-2\psi)/2} \), then \( \mathcal{C}_\phi \) is hypercyclic on \( S_\psi \), thus it is recurrent on \( S_\psi \). Now we prove that the conditions are necessary. Since the recurrence is invariant under similarity, we may suppose that the boundary fixed point is 1.

Suppose that \( \psi > 1/2 \). Then the reproducing kernel at 1

\[
K_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^{2\psi}}
\]

is in \( S_\psi \). Furthermore, for any \( f \in S_\psi \), we have

\[
\langle \lambda \mathcal{C}_\phi^* K_1, f \rangle = \lambda \langle K_1, \mathcal{C}_\phi f \rangle = \lambda \langle K_1, f \circ \phi \rangle = \lambda f(\phi(1)) = \lambda f(1) = \langle \lambda K_1, f \rangle.
\]

Then \( \lambda \) is an eigenvalue of \( \lambda \mathcal{C}_\phi^* \). Thus, if \(|\lambda| \neq 1\), the operator \( \lambda \mathcal{C}_\phi \) is not recurrent, since the point spectrum of its adjoint \( \lambda \mathcal{C}_\phi^* \), \( \sigma_p(\lambda \mathcal{C}_\phi^*) \) must be in the unit circle by [12, Proposition 2.15]. If \(|\lambda| = 1\), also in this case \( \lambda \mathcal{C}_\phi \) is not recurrent, since \( \mathcal{C}_\phi \) is not recurrent on \( S_\psi \).

Now, if \( \psi = 1/2 \), in this case \( S_\psi \) will be the Dirichlet space \( D \). Assume that \( \lambda \mathcal{C}_\phi \) is recurrent on \( D \), for some \(|\lambda| \leq 1\), then so is \( \lambda \mathcal{C}_\phi^0 \) on \( D_0 \). But on \( D_0 \) we have

\[
\int_D |\lambda f'(\phi(z))|^2 |\phi'(z)|^2 dA(z) = |\lambda|^2 \int_{\phi(D)} |f'(z)|^2 dA(z) < \int_D |f'(z)|^2 dA(z).
\]

Thus \( \|\lambda \mathcal{C}_\phi^0\|_{D_0} < 1 \), and therefore, \( \lambda \mathcal{C}_\phi^0 \) cannot be recurrent on \( D_0 \), a contradiction.

It remains to prove that if \(|\lambda| \leq \phi'(\eta)^{(1-2\psi)/2} \) and \( \psi < 1/2 \), then the \( \lambda \mathcal{C}_\phi \) cannot be recurrent on \( S_\psi \). We need an easy expression for \( \phi \). We have already assumed that 1 is the boundary fixed point. First, we perform the change of variables

\[
\sigma(z) = \frac{i(1+z)}{1-z}
\]

that sends the unit disk onto the upper half-plane and 1 to \( \infty \) and the exterior fixed point to a point \( p \) in the lower half-plane. Upon conjugating with an appropriate affine map in the upper half-plane, we may suppose that \( p \) is on the imaginary axis.
Finally, coming back to the unit disk, $\sigma^{-1}$ sends $p$ onto a negative number $x < -1$. Therefore, we may suppose that $\phi$ has the expression

$$
\phi(z) = \frac{(\mu x - 1)z + x(1 - \mu)}{(\mu - 1)z + x - \mu} \quad \text{with } 0 < \mu < 1.
$$

Finally, we conjugate one more time with

$$\frac{xz - 1}{x - z}$$

that is an automorphism of the unit disk and fixes 1 and sends $a$ to $1$ and, therefore, we may suppose that

$$\phi(z) = \mu z + 1 - \mu,$$

where $0 < \mu < 1$. (5)

We have that $\phi'(1) = \mu$. The formula (5) above allows us to obtain the following expression for the iterates

$$\phi_n(z) = \mu^n z + 1 - \mu^n.$$ (6)

Let $\lambda$ be a complex number with $|\lambda| \leq (1 - 2\nu)/2$ and let $f \in S_\nu$. Since $\nu < 1/2$, then $(1 - 2\nu)/2 > 0$, and since $\mu < 1$, then $|\lambda| < 1$. Then we have that $\lambda^n f(\phi_n(z)) \to 0$. Thus, the only recurrent vector for $\lambda C_\phi$ is the null function. Hence, $\lambda C_\phi$ cannot be recurrent in this case.

**Theorem 3.3** Let $\phi$ be a parabolic automorphism of the unit disk. Then the operator $\lambda C_\phi$ is recurrent on $S_\nu$ if and only if $\nu < 1/2$ and $|\lambda| = 1$.

**Proof** First if $\nu < 1/2$ and $|\lambda| = 1$, then $\lambda C_\phi$ is hypercyclic on $S_\nu$, which implies that it is recurrent. Now we prove that these conditions are necessary. Let us first examine the case $\nu = 1/2$. Suppose that $|\lambda| \leq 1$. Since $\phi$ is a parabolic automorphism, we may assume that the fixed point is 1. Let $f$ be a recurrent vector for $C_\phi$. Then there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers such that

$$C_{\phi_{n_k}}(f) \to f,$$

in $\mathcal{D}$. Since $\phi$ has no fixed point in $\mathbb{D}$, by the Denjoy–Wolff Iteration Theorem, the fixed point 1 is actually attractive. Thus, $\phi_{n_k}$ converge to 1 uniformly on compact subsets of $\mathbb{D}$; in particular, $\phi_{n_k}(z)$ converge to 1, for every $z \in \mathbb{D}$. Therefore,

$$C_{\phi_{n_k}}(f)(z) \to f(1), \quad \text{for every } z \in \mathbb{D},$$

when $k \to \infty$. Then, if $|\lambda| < 1$, $\lambda^{n_k} C_{\phi_{n_k}}(f) \to 0$, and if $|\lambda| = 1$, $\lambda^{n_k} C_{\phi_{n_k}}(f)(z) \to f(1)$. Thus, all recurrent vectors for $\lambda C_\phi$ are constants, and so $\lambda C_\phi$ cannot be recurrent in this case.

If $|\lambda| > 1$, then, by what we have just proved, $\lambda^{-1} C_{\phi_{\lambda^{-1}}}$ is not recurrent. Now, an invertible operator is recurrent if and only if its inverse is. Consequently, $\lambda C_\phi$ is not...
Now suppose that $v > 1/2$, we distinguish two cases:

**Case 1:** If $|\lambda| \neq 1$, the reproducing kernel at $1$ in $S_v$ is an eigenvalue for the adjoint $\mathcal{Z}C_{\phi}^*$, a contradiction.

**Case 2:** If $|\lambda| = 1$, we know that $C_{\phi}$ is not recurrent on $S_v$, then $\lambda C_{\phi}$ is not recurrent on $S_v$ as well.

Now suppose that $v < 1/2$. Since $\phi$ is parabolic automorphism the fixed point of $\phi$ must be on the boundary of the unit disk. We may suppose that the fixed point is $1$. Let

$$\sigma(w) = \frac{i(1 + w)}{1 - w}, \quad \text{and} \quad \psi = \sigma \circ \phi \circ \sigma^{-1}.$$ 

then $\sigma$ is a linear fractional map of the unit disk onto the upper half plane that takes $1$ to $\infty$, which implies $\infty$ is the only fixed point of $\psi$, and so $\psi(w) = w + a$, with $a \neq 0$ and $\Im a = 0$, where $\Im a$ denote the imaginary part of $a$. The fact that $\Im a = 0$ comes form that $\phi$ corresponds to an automorphism of the upper half plane. Thus $\phi$ satisfies the following formula

$$\phi(z) = \frac{(2 - a)z + a}{-az + 2 + a} \quad \text{with} \quad a \neq 0 \quad \text{and} \quad \Im a = 0. \quad (7)$$

Upon replacing $a$ by $na$ in the formula (7) above we can get a formula for the iterates of $\phi$

$$\phi_n(z) = \frac{(2 - na)z + na}{-naz + 2 + na}. \quad (8)$$

Then, $\phi_n(z) \to 1$ for every $z \in \mathbb{D}$. If $|\lambda| < 1$, then $\lambda^n f(\phi_n(z)) \to 0$, for every $z \in \mathbb{D}$. Thus $\lambda C_{\phi}$ is not recurrent in this case.

If $|\lambda| > 1$, then $\lambda^{-1} C_{\phi,1}$ is not recurrent and, therefore, neither is $\lambda C_{\phi}$. \hfill $\square$

**Theorem 3.4** Let $\phi$ be a hyperbolic automorphism of the unit disk and $\eta$ its attractive fixed point. Then $\lambda C_{\phi}$ is recurrent if and only if $v < 1/2$ and $\phi'(\eta)^{(1-2v)/2} < |\lambda| < \phi'(\eta)^{(2v-1)/2}$.

**Proof** For $v \geq 1/2$, the non-recurrence of $\lambda C_{\phi}$ follows exactly as in the case of the parabolic automorphism.

Now, suppose that $v < 1/2$. We need an expression of $\phi$. Without loss of generality, we may suppose that $\phi$ has $-1$ and $1$ as its fixed points. Moreover, we may assume that $1$ is the attractive fixed point. To compute $\phi$ explicitly, we use the change of variables

$$\sigma(z) = \frac{i(1 - z)}{1 + z}$$

that sends the unit disk onto the upper half-plane, the fixed points $1$ and $-1$ to $0$ and
∞, respectively, and φ to the contraction map \( \phi(w) = \mu w \), where \( 0 < \mu < 1 \). Coming back to the unit disk we have

\[
\phi(z) = \frac{(1 + \mu)z + 1 - \mu}{(1 - \mu)z + 1 + \mu} \quad \text{with } 0 < \mu < 1.
\] (9)

Observe that the derivative at the attractive fixed point is \( \phi'(1) = \mu \).

Upon replacing \( \mu \) by \( \mu^n \) in the formula (9) above we can get a formula for the iterates of \( \phi \)

\[
\phi_n(z) = \frac{(1 + \mu^n)z + 1 - \mu^n}{(1 - \mu^n)z + 1 + \mu^n}.
\] (10)

Now, for any \( f \in S_v \) we have \( f(\phi_n(z)) \to f(-1) \). Also, \( |\lambda| \leq \mu^{(1-2v)/2} < 1 \), then \( \lambda^nf(\phi_n(z)) \to 0 \), which means that \( \lambda C_\phi \) is not recurrent in this case. Therefore, if \( \lambda C_\phi \) is recurrent, then \( |\lambda| > \mu^{(1-2v)/2} \). In addition, the inverse operator \( \lambda^{-1} C_{\phi|} \) must also be recurrent. The attractive fixed point of \( \phi_{-1} \) is \(-1\) and \( \phi_{-1}(-1) = \mu \). Therefore, we must also have \( |\lambda^{-1}| > \mu^{(1-2v)/2} \). Thus the conditions on \( \lambda \) are necessary for \( \lambda C_\phi \) to be recurrent.

**Theorem 3.5** Let \( \phi \) be a parabolic non-automorphism of the unit disk. Then the operator \( \lambda C_\phi \) is never recurrent on any \( S_v \).

**Proof** Assume that \( \phi \) is parabolic non-automorphism, then by the Denjoy–Wolff Iteration Theorem, there is a point \( p \in \mathbb{T} \) such that \( \phi_n \) converges to \( p \) uniformly on compact subsets of \( \mathbb{D} \). Then for every \( f \in S_v \), \( f(\phi_n(z)) \to f(p) \) uniformly on any compact subset of \( \mathbb{D} \).

If \( |\lambda| < 1 \), then \( \lambda^n C_\phi f(z) \to 0 \) and the null function is the only recurrent vector for \( \lambda C_\phi \). Hence, \( \lambda C_\phi \) is not recurrent.

If \( |\lambda| = 1 \), then \( \lambda^n C_\phi f(z) \to f(p) \), and the only recurrent vectors of \( \lambda C_\phi \) are constant functions. Hence \( \lambda C_\phi \) is not recurrent.

If \( |\lambda| > 1 \), \( f(p) \) is not defined unless \( f(p) \) is null, in which case \( \lambda C_\phi \) is not recurrent.

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**References**

1. Ahlfors, L.V.: Complex Analysis. McGraw-Hill, New York (1979)
2. Amouch, M., Benchihib, O.: On linear dynamics of sets of operators. Turk. J. Math. 43(1), 402–411 (2018)
3. Amouch, M., Benchihib, O.: On cyclic sets of operators. Rend. Circ. Mat. Palermo Ser. 2 68(3), 521–529 (2019)
4. Amouch, M., Benchihib, O.: Diskcyclicity of sets of operators and applications. Acta Math. Sin. Engl. Ser. 36(11), 1203–1220 (2020)
5. Amouch, M., Benchihib, O.: Some versions of supercyclicity for a set of operators. Filomat 35(5), 1619–1627 (2021)
6. Amouch, M., Karim, N.: Strong transitivity of composition operators. Acta Math. Hungar. 164, 458–469 (2021)
7. Bayart, F., Matheron, É.: Dynamics of Linear Operators, vol. 179. Cambridge University Press, Cambridge (2009)
8. Bès, J., Menet, Q., Peris, A., Puig, Y.: Recurrence properties of hypercyclic operators. Math. Ann. 366(1), 545–572 (2016)
9. Bonilla, A., Grosse-Erdmann, K.G.: Upper frequent hypercyclicity and related notions. Rev. Mat. Complut. 31(3), 673–711 (2018)
10. Bourdon, P.S., Shapiro, J.H.: Cyclic composition operators on $H^2$. In: Proc. Symp. Pure Math, 51(2), 43–53 (1990)
11. Bourdon, P., Shapiro, J.H.: Cyclic Phenomena for Composition Operators, vol. 596. American Mathematical Soc., Providence (1997)
12. Costakis, G., Manoussos, A., Parissis, I.: Recurrent linear operators. Complex Anal. Oper. Theory 8(8), 1601–1643 (2014)
13. Cowen, C., MacCluer, B.: Composition Operators on Spaces of Analytic Functions. CRC Press, Boca Raton (1995)
14. Gallardo-Gutierrez, E.A., Montes-Rodriguez, A.: The Role of the Spectrum in the Cyclic Behavior of Composition Operators. American Mathematical Soc., Providence (2004)
15. Grosse-Erdmann, K.G.: Universal families and hypercyclic operators. Bull. Am. Math. Soc. 36(3), 345–381 (1999)
16. Grosse-Erdmann, K.G., Manguillot, A.P.: Linear Chaos. Springer Science and Business Media, Berlin (2011)
17. Hurst, P.R.: Relating composition operators on different weighted Hardy spaces. Arch. Math. 68(6), 503–513 (1997)
18. Salas, H.N.: Supercyclicity and weighted shifts. Stud. Math. 135(1), 55–74 (1999)
19. Shapiro, J.H.: Composition Operators: And Classical Function Theory. Springer Science and Business Media, Berlin (2012)
20. Shkarin, S.: On the spectrum of frequently hypercyclic operators. Proc. Am. Math. Soc. 137(1), 123–134 (2009)
21. Zhu, K.: Spaces of Holomorphic Functions in the Unit Ball, vol. 226. Springer, New York (2005)
22. Zorzoska, N.: Composition operators on weighted Hardy spaces. Thesis, Univ. Toronto (1988)
23. Zorzoska, N.: Cyclic composition operators on smooth weighted Hardy spaces. Rocky Mt. J. Math. 29(2), 725–740 (1999)