Abstract: This paper studies exact linearization methods for stochastic SISO affine controlled dynamical systems. The systems are defined as vectorfield triplets in Euclidean space. The goal is to find, for a given nonlinear stochastic system, a combination of invertible transformations which transform the system into a controllable linear form. Of course, for most nonlinear systems such transformation does not exist.

We are focused on linearization by state coordinate transformation combined with feedback. The difference between Itô and Stratonovich systems is emphasized. Moreover, we define three types of linearity of stochastic systems — $g$-linearity, $\sigma$-linearity, and $g\sigma$-linearity.

Six variants of the stochastic exact linearization problem are studied. The most useful problem — the Itô - $g\sigma$ linearization is solved using the correcting term, which proved to be a very useful tool for Itô systems. The results are illustrated on a numerical example solved with help of symbolic algebra.

Keywords: exact linearization, feedback linearization, nonlinear dynamical system, Itô integral, Stratonovich integral, correcting term

MCS classification: 93B18, 93E03
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1 Introduction

The theory of exact linearization of deterministic dynamical systems has been thoroughly studied since seventies. This paper attempts to apply some of the results to the stochastic area. We emphasize the exact linearization by state coordinate transformation combined with feedback (further abbreviated as SFB linearization). Our main goal is to identify the main difficulties of this approach and to consider applicability of the methods known from the deterministic systems.

The task of SFB linearization is following: given a dynamical systems $\Theta$ we are looking for a combination of coordinate transformation $T_T$ and feedback $F_{\alpha,\beta}$ which will make the resulting system $F_{\alpha,\beta} \circ T_T(\Theta)$ linear and controllable. One can also define the feedback-less linearization by coordinate transformation only (here abbreviated as SCT) or several variants of the input-output linearization. These variants are not considered here.

Remark 1. The notation for composition of mappings sometimes differs; right-to-left convention is used here: $f \circ g(x) := f(g(x))$.

The subject of exact linearization of stochastic controlled dynamical systems lies on the intersection of three branches of science: differential geometry, control theory, and the theory of stochastic processes. Each of them is very broad and it is virtually impossible to cover all details of their combination. Hence it is necessary to choose a minimalistic simplified model for our problem and to refrain from most technical details. We decided to represent dynamical systems under investigation by triplets of smooth vectorfields and to concentrate on transformation rules for these triplets. The detailed interpretation of the vectorfield systems (i.e. solvability of underlying differential equations, properties of flows and trajectories) will be considered only on an informal, motivational level.

For simplicity, we shall confine all the definitions of geometrical object to the Euclidean space; we will work in a fixed coordinate system using explicit local coordinates, which may be considered to be local coordinates of some manifold. This is mainly because we are unable to capture all consequences of the modern, coordinate-free differential geometry to the stochastic calculus (see eg. Kendall [3], Malliavin [12], Émery [2]). We believe that this approach is quite satisfactory for the majority of practical applications.
1.1 Dynamical systems

Definition 2. In this paper, a stochastic dynamical system \( \Theta := (f, g, \sigma, U, x_0) \) is defined to be a triplet of smooth and bounded vectorfields \( f, g, \) and \( \sigma \) defined on an open neighborhood \( U \) of a point \( x_0 \in \mathbb{R}^n \). We usually call \( U \in \mathbb{R}^n \) the state space, \( f \) the drift vectorfield, \( g \) the control vectorfield, and \( \sigma \) the dispersion vectorfield.

From now on, let’s assume that all functions, vectorfields, forms, and distributions are smooth and bounded on \( U \).

In this paper, we will study almost only SISO systems, but in the case of stochastic MIMO systems with \( m \) control inputs and \( k \)-dimensional noise the symbols \( g \) and \( \sigma \) stand for \( n \times m \) (\( n \times k \) respectively) matrix of smooth vectorfields having its rank equal to \( m \) (\( k \) respectively). The class of all deterministic \( n \)-dimensional dynamical systems with \( m \) inputs will be called \( \mathcal{X}_D(n,m) \) and the class of stochastic systems with \( k \)-dimensional noise will be denoted with \( \mathcal{X}(n,m,k) \).

Similarly, autonomous deterministic dynamical system corresponds to a single vectorfield and a controlled deterministic dynamical system corresponds to a vectorfield pair. It is obvious that this approach is limited to time invariant, affine systems.

Remark 3. The acronyms SISO and MIMO are used in the usual meaning even for systems without outputs, where the wording “scalar-input” and “vector-input” will be appropriate. Stochastic systems with \( m = 1 \) and \( k = 1 \) will be considered SISO.

The definition may be interpreted as follows: there is a stochastic process \( x_t \) defined on \( \mathbb{R}^n \) which is a strong solution of the stochastic differential equation \( dx_t = f(x_t)\,dt + g(x_t)u(t)\,dt + \sigma(x_t)\,dw_t \), with initial condition \( x_0 \), where \( u(t) \) is a smooth function with bounded derivatives and \( w_t \) is a one-dimensional Brownian motion. The differential \( dw_t \) is just a notational shortcut for the stochastic integral.

Details of the theory of stochastic processes are beyond the scope of this article. The reader is referred to Wong and Hajek [20], Øksendal [14], Sage and Melsa [17], Malliavin [12], Kendall [1], Karatzas and Shreve [8]; the text of Kohlmann [11] is freely available on the Internet.

Theory of stochastic processes offers several alternative definitions of the stochastic integral, among them the Itô integral and the Stratonovich integral; each of them is used to model different physical problems. Consequently there are two classes of differential equations and two alternative definitions of a stochastic dynamical system — Itô dynamical systems...
defined by Itô integrals and Stratonovich systems defined by Stratonovich integrals.

Serious differences between these integrals exists but from out point of view there is a single important one: the rules for coordinate transformations of dynamical systems defined by Itô stochastic integral are quite different from the transformation rules which are valid for Stratonovich systems.

The definition of the Itô dynamical system used by us is formally equivalent to the definition of the Stratonovich system; the only difference will be in the corresponding coordinate transformation.

If necessary, Itô and Stratonovich dynamical systems will be distinguished by a subscript: \( \Theta_I \in \mathcal{X}_I(n, m, k) \) and \( \Theta_S \in \mathcal{X}_S(n, m, k) \).

Remark 4. In this paper we will use the adjectives Itô and Stratonovich rather freely. For example we will speak of 'Stratonovich linearization' instead of 'exact linearization of stochastic dynamical system defined by Stratonovich integral'.

\[1.2\] Transformations

Furthermore, we will study two transformations of dynamical systems: the coordinate transformation \( T_T \) and the feedback \( F_{\alpha, \beta} \). The definition of these transformation should be in accord with their common interpretation. This can be illustrated on the definition of the coordinate transformation of a deterministic dynamical system \( T_T : \mathcal{X}_D(n, m) \rightarrow \mathcal{X}_D(n, m) \) which is induced by a diffeomorphism \( T : U \rightarrow \mathbb{R}^n \) between two coordinate systems on an open set \( U \subset \mathbb{R}^n \). The mapping \( T_T \) is defined by:

\[
T_T(f, g, U, x_0) := (T*f, T*g, T(U), T(x_0)).
\]

(1)

Recall that the symbol \( T_s \) stands for the contravariant transformation \( (T_s f)_i = \sum_{j=0}^n f_j \frac{\partial T_i}{\partial x_j} \). Moreover, we will require that the coordinate transformation \( T \) preserves the equilibrium state of the system i.e. \( T(x_0) = 0 \).

The definition captures the contravariant transformation rules for differential equations known from the basic calculus.

Note that the words “coordinate transformation” are used in two different meanings: first as the diffeomorphism \( T : U \rightarrow \mathbb{R}^n \) between coordinates; second as the mapping between systems \( T_T : \mathcal{X}_D(n, m) \rightarrow \mathcal{X}_D(n, m) \).

Coordinate transformation of stochastic systems distinguish between Itô and Stratonovich systems. One of the major complications of the linearization problems for Itô systems is the second-order term in the transformation rules for Itô systems:
Definition 5. Let $U \in \mathbb{R}^n$ be an open set and let $T : U \to \mathbb{R}^n$ be a diffeomorphism from $U$ to $\mathbb{R}^n$ with bounded first derivative on $U$ such that $T(x_0) = 0$. The mapping $T_T : \mathbb{X}_I(n, m, k) \to \mathbb{X}_I(n, m, k)$ will be called a coordinate transformation of an Itô dynamical system induced by diffeomorphism $T$ if the systems $\Theta_1 := (f, g, \sigma, U, x_0)$ and $\Theta_2 := (\tilde{f}, \tilde{g}, \tilde{\sigma}, T(U), x_0)$; $\Theta_2 = T_T(\Theta_1)$ are related by:

$$\tilde{f} = T^*f + P_\sigma T$$

$$\tilde{g}_i = T^*g_i$$ \text{ for } 1 \leq i \leq m \quad (3)$$

$$\tilde{\sigma}_i = T^*\sigma_i$$ \text{ for } 1 \leq i \leq k. \quad (4)$$

The symbol $P_\sigma T$ stands for the Itô term which is a second order linear operator defined by the following relation for the $m$-th component of $P_\sigma T$, $1 \leq m \leq n$

$$P_\sigma T_m := \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} \sigma_{ij}. \quad (5)$$

The transformation rules for Stratonovich systems $T_T : \mathbb{X}_S(n, m, k) \to \mathbb{X}_S(n, m, k)$, $(f, g, \sigma, U, x_0) \mapsto (T^*f, T^*g, T^*\sigma, T(U), T(x_0))$ are equivalent to rules valid for the deterministic systems; only the rule (4) for the drift vectorfield must be added.

The difference between the coordinate transformation of Itô and Stratonovich systems should be emphasized: in the Stratonovich case all the vectorfields transform contravariantly; on the other hand, in the Itô case, the Itô term $P_\sigma T$ is added to the drift vectorfield of the resulting system.

Another important transformation of dynamical systems is the regular feedback transformation. A feedback transformation is determined by two smooth nonlinear functions $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ and $\beta : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^m$ defined on $U$ with $\beta$ nonsingular for every $x \in U$ (see Figure 1). Usually, $\alpha$ is written as a column $m \times 1$ matrix; $\beta$ as a square $m \times m$ matrix.
Definition 6. Let $\Theta = (f, g, \sigma, U, x_0) \in X(n, m, k)$ be a stochastic dynamical system. A **regular state feedback** is the transformation $F_{\alpha, \beta} : X(n, m, k) \to X(n, m, k)$, $(f, g, \sigma, U, x_0) \mapsto (f + g\alpha, g\beta, \sigma, U, x_0)$. ■

A new input variable $v$ is introduced by the relation $u = \alpha + \beta v$. Given the feedback $F_{\alpha, \beta}$ with nonsingular $\beta$, we can always construct an inverse relation $F_{a, b} := F_{\alpha, \beta}^{-1}$ such that $F_{\alpha, \beta} \circ F_{a, b} = F_{a, b} \circ F_{\alpha, \beta}$ is the identity. The coefficients are related as follows: $\beta = b^{-1}$, $\alpha = -b^{-1}a$, and $a = -\beta^{-1} \alpha$.

This definition of feedback transformation can be used also for deterministic systems provided that the drift vectorfield $\sigma$ is assumed to be zero.

The symbol $J_{T, \alpha, \beta}$ is used to indicate the combination of coordinate transformation with feedback $J_{T, \alpha, \beta} := F_{\alpha, \beta} \circ T$ can be interchanged.

Remark 7. Observe that the order of feedback and coordinate transformation in the composed transformation $J_{T, \alpha, \beta} := F_{\alpha, \beta} \circ T$ can be interchanged.

$$J_{T, \alpha, \beta} = T \circ F_{\alpha, \beta} (f, g, U, x_0) = T(f + g\alpha, g\beta, U, x_0) = (T_s f + T_s g\alpha, T_s g\beta, U, z_0) = (T_s f + (T_s g)\alpha, (T_s g)\beta, U, z_0)$$

$$= F_{\alpha', \beta'} \circ T. \tag{6}$$

The functions $\alpha(z)'$, $\beta(z)'$ are equal to $\alpha(x)$ and $\beta(x)$ written in the $z$ coordinates $\alpha(z)' = \alpha(x) \circ T^{-1}(z)$, $\beta(z)' = \beta(x) \circ T^{-1}(z)$. ▽

1.3 Linearity

The definition of linearity is straightforward in the deterministic case. In contrast, the stochastic case is more complex, because there are two “input” vectorfields and thereby several degrees of linearity can be specified.

Definition 8. The deterministic dynamical system $\Theta_D = (f, g, U, 0) \in X_D(n, m)$ is **linear** if the vectorfield $f$ is a linear mapping without no constant term and the vectorfields $g_i$ are constant; that is they can be written as $f(x) = Ax$, $g(x) = B$ with $A$ a square $n \times n$ matrix and $B$ an $n \times m$ matrix. The matrices must be constant on whole $U$. ■

Definition 9. The stochastic dynamical system $\Theta = (f, g, \sigma, U, 0)$ is:

- **$g$-linear** if the mapping $f(x) = Ax$ is linear without constant term and $g(x) = B$ is constant on $U$.

- **$\sigma$-linear** if the mapping $f(x) = Ax$ is linear without constant term and $\sigma(x) = S$ is constant on $U$. 

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• **$g\sigma$-linear** if it is both $g$-linear and $\sigma$-linear.

The matrices $A$ and $B$ have the same dimensions as in Definition 8. $S$ is an $n \times k$ matrix.

Remark 10. The vectorfield $g$ is **constant** on $U$ if the value of $g(x)$ is the same for every $x \in U$. The vectorfield $f$ on $U$ is **linear without constant term** if $f(x_0) = 0$ and the superposition principle $f(x_1 + x_2) = f(x_1) + f(x_2)$ holds for every $x_1, x_2 \in U$.

We study systems at equilibrium i.e. we require that $f(x_0) = 0$ and that all transformations preserve the equilibrium: $T(x_0) = 0$, $\alpha(x_0) = 0$, and $\beta(x_0)$ is nonsingular. The Itô systems require an additional condition $f(x_0) + \text{corr}_\sigma(x_0) = 0$. The non-equilibrium case can be easily handled by extending the linear model with a constant term.

Moreover we require that the resulting linear systems are controllable. A controlled deterministic dynamical linear system $\Theta_D = (Ax, B, \mathbb{R}^n, 0) \in \mathcal{X}_D(n, m)$ is **controllable** if its first $n$ repeated brackets form an $n$-dimensional space

$$\dim \left\{ A^kB, 0 \leq k \leq n-1 \right\} = n. \quad (7)$$

Other definitions of controllability of linear systems exist. For example Theorem 3.1 of Zhou et al. [21] gives six definitions with proofs of equivalence.

The controllability property deserves some attention in the stochastic case. The linear stochastic dynamical system is characterized by two input vectorfields $g(x) = B$ and $\sigma(x) = S$.

1. The definition of the controllability for the control vectorfield $g$ is identical to the deterministic case; i.e. (7) must be satisfied. This property will be called **$g$-controllability**.

2. We will also define **$\sigma$-controllability** as the requirement that the repeated brackets $S, AS, A^2S, \ldots, A^{n-1}S$ form an $n$-dimensional space.

3. Finally, the linear system is **$g\sigma$-controllable** if

$$\dim \left( \left\{ A^kB, 0 \leq k \leq n-1 \right\} \bigcup \left\{ A^kS, 0 \leq k \leq n-1 \right\} \right) = n. \quad (8)$$

In this paper, we do not deal with the reachability, controllability, accessibility, observability, and similar properties of nonlinear systems.
Figure 2: Dynamical System with a Prefilter.

**Definition 11.** Let $\Theta = (f, g, \sigma, U, x_0) \in \mathcal{X}(n, m, k)$ be a dynamical system such that $f(x_0) = 0$. We call the combination of a coordinate transformation $T_T$ and a regular feedback $F_{\alpha, \beta}$ such that $T(x_0) = 0$, $\alpha(x_0) = 0$, and $\beta(x_0)$ is nonsingular the **linearizing transformation** of $\Theta$ if the transformation $F_{\alpha, \beta} \circ T_T$ converts $\Theta$ into a **controllable** linear system.

For stochastic system we distinguish:

(i) **$g$-linearizing transformation** which transforms $\Theta$ into a $g$-linear and $g$-controllable system

(ii) **$\sigma$-linearizing transformation** which transforms $\Theta$ into a $\sigma$-linear and $\sigma$-controllable system

(iii) **$g\sigma$-linearizing transformation** which transforms $\Theta$ into a $g\sigma$-linear and $g$-controllable system

Note that for $g\sigma$-linearization we require $g$-controllability. This is slightly stricter requirement than $g\sigma$ controllability but it should be naturally fulfilled by the majority of practical control systems. This requirement cancels many “uncomfortable” linear forms. Consider for example the system with prefilter of Figure 2 which is $g\sigma$-controllable but $g$-uncontrollable.

The system $\Theta$ is **linearizable** if there exists linearizing transformation of $\Theta$. ■
1.4 Computational Issues

In most practical circumstances, computational issues are the limiting factor of any application of differential geometric methods in control.

The equations of exact linearization algorithms must be dealt in a symbolic form. Even the simplest exact linearization problems are extremely complex from the computational point of view. Therefore, the computer algebra tools are often employed. The results presented in this paper were tested by the author on few simulations of control systems in the symbolic system Mathematica.

Of course, the computer algebra has apparently serious limitations and drawbacks. Viability of the symbolic computational approach to the problems of the nonlinear control is studied by Jager [6]. Some very useful theoretical notes about the symbolic computation can be found in Winkler [19]. Unfortunately, the limited scope of this article does not allow deeper discussion of these subjects.

1.5 Applications

We propose, very briefly, two applications of the theory presented here:

(i) **Control** — a dynamical systems $\Theta$ obtained by exact linearization will be controlled using the linear feedback law:

$$v = K z + \kappa \nu,$$

where $K$ is a row matrix of feedback gains, $\kappa$ is an input gain, $z$ is the state vector, $v$ is the original control input, and $\nu$ is a new control input.

Two approaches can be studied — classical linear control methods and the more sophisticated stochastic optimal control approach studied for example by Øksendal [14].

The $g\sigma$-linear systems are natural candidates for such approach because the other linear forms leave certain part of the resulting system nonlinear.

(ii) **Filtering** — the filtering problem is probably the most useful application of the theory of stochastic processes. We want to give the best estimate to the state of a dynamical system defined by the stochastic differential equation:

$$dx_t = f(x_t) \, dt + \sigma f(x_t) \, dw_t;$$

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based on observations of the from:

\[ y_t = h(x_t) + \sigma_h(x_t) dw_t. \]

\( x_t \) is an \( n \)-dimensional stochastic process, \( f, \sigma_f, \sigma_h \) are smooth vectorfields and \( h \) is a smooth function.

It would be interesting to use exact linearization of the nonlinear system to design an exact linear filter. Unfortunately, this idea has no direct association with the linearization results presented below, because it requires output exact linearization or linearization of an autonomous system. Therefore, it would be helpful to extend our results to these cases in the future.

1.6 Previous Work

The problem of SFB \( g \)-linearization of SISO dynamical system defined in the Itô formalism has been studied by Lahdhiri and Alouani [11]. The authors derive equations corresponding to (108), (109) (eq 14, 15, 16 in [11]). These equations are combined and then reduced to a set of PDEs of a single unknown function \( T_1 \). Because there is no commuting relation similar to (16) the equations contain partial derivatives of \( T_1 \) up to the \( 2n \)-th order (eq 23 in [11]). Next, the authors propose a lemma (Lemma 1) that identifies the linearity conditions with non-singularity and involutiveness of \( \{ \text{ad} f^i g, 0 \leq i \leq n - 2 \} \). Unfortunately, we disagree with this result.

It can be easily verified that for \( \sigma = 0 \) this statement does not correspond to the deterministic conditions (Proposition 14), because the deterministic case requires non-singularity up to the \((n - 1)\)-th bracket, not only up to the \((n - 2)\)-th one. Second, although the method of finding \( T_1 \) was given (solving PDE), we do not think that the existence of \( T_1 \) was proved.

After this paper was finished, we discovered recent works of Pan [15] and [16]. In the article [14] Pan defines and solves the problem of feedback complete linearization of stochastic nonlinear systems. In our terminology, this problem is equivalent to SFB MIMO input-output Itô \( g \sigma \) linearization which was not studied by us.

In [16] Pan declares and proves so called invariance under transformation rule which is exactly equivalent to our Theorem 18 which is probably the most important result of our paper.

Although the problems solved by Pan were slightly different, he uses the same equivalence — Theorem 18. This proves that our conclusions about applicability of the correcting term are perfectly valid.
In [16] Pan consider three other canonical forms of stochastic nonlinear systems, namely the noise-prone strict feedback form, zero dynamics canonical form and observer canonical form also not studied by us.

2 Deterministic Case

In this section we recapitulate the results of the SFB and SCT exact linearization theory for SISO systems. For detailed treatment and proofs we refer to existing literature, above all the classical monographs of Isidori [4] and Nijmeijer and van der Schaft [13]. For a very readable introduction to the field we refer to the seventh chapter of Vidyasagar [18]. The books also contain extensive bibliography. The monograph of Isidori [4] builds mainly on the concept of relative degree. In contrast we will emphasize the approach of Vidyasagar [18] because the method is more suitable for the stochastic case.

2.1 Useful Relations

The solution of the SFB linearization problem as presented here uses the Leibniz rule

\[ \mathcal{L}_{[f,g]}^\alpha = \mathcal{L}_f (\mathcal{L}_g^\alpha) - \mathcal{L}_g (\mathcal{L}_f^\alpha) \]  

(12)

with \( f, g \) smooth vectorfields on \( U \); \( \alpha : U \to \mathbb{R} \) is a smooth function. The recursive form of the Leibniz rule allows to simplify the chains of differential equations for the transformation \( T \). This can be expressed in the form of the following statement:

For all \( x \in U, \ k \geq 0 \) these two sets of conditions are equivalent:

(i) \[ \mathcal{L}_g^\alpha = \mathcal{L}_g \mathcal{L}_f^\alpha = \cdots = \mathcal{L}_g \mathcal{L}_f^k \alpha = 0 \]  

(13)

(ii) \[ \mathcal{L}_g^\alpha = \mathcal{L}_{\text{ad} f} g^\alpha = \cdots = \mathcal{L}_{\text{ad} f}^k g^\alpha = 0. \]  

(14)

Recall that the symbol \( \mathcal{L}_f h \) stands for the Lie derivative defined by \( \mathcal{L}_f h = \langle f, \text{grad} \ h \rangle = \sum_{i=1}^n f_i(x) \frac{\partial h}{\partial x_i}(x) \). Higher order Lie derivatives can be defined recursively \( \mathcal{L}_f^0 h = h, \mathcal{L}_f^{k+1} h = \mathcal{L}_f \mathcal{L}_f^k h \) for \( k \geq 0 \). The Lie Bracket is defined as \( [f, g] := \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \); there is also a recursive definition:

\[ \text{ad}^0 f := g; \quad \text{ad}^{k+1} f := [f, \text{ad}^k f] \quad \text{for} \ k \geq 0. \]  

(15)
Another very important result of the differential geometry is invariance of the Lie bracket under the tangent transformation $T_*$ (see Nijmeijer and van der Schaft [13] Proposition 2.30 p. 50):

Let $T: U \to \mathbb{R}^n$ be a diffeomorphic coordinate transformation, and $f$ and $g$ be smooth vector fields. Then

$$T_*[f, g] = [T_*f, T_*g].$$

(16)

### 2.2 SFB Linearization

Every controllable linear system may be, by a linear coordinate transformation, transformed to the controllable canonical form (Kalman et al. [7]). Furthermore, this controllable canonical form can be always transformed into the integrator chain by a linear regular feedback. Therefore, the integrator chain is a canonical form for all feedback linearizable systems. (See Vidyasagar [18] section 7.4). Consequently, the equations of the integrator chain can be compared with the equations of the nonlinear systems and the following proposition can be proved:

**Proposition 12.** There is a SFB linearizing transformation $J_{T,\alpha,\beta}$ of a SISO deterministic dynamical system $\Theta_D = (f, g, U, x_0) \in \mathbb{X}_D(n, 1)$ into a controllable linear system if and only if there is a solution $T_1, T_2, \ldots, T_n: \mathbb{R}^n \to \mathbb{R}$ to the set of partial differential equations defined on $U$

$$\mathcal{L}_f T_i = T_{i+1} \quad \text{for} \quad 1 \leq i \leq n - 1$$

(17)

$$\mathcal{L}_g T_i = 0 \quad \text{for} \quad 1 \leq i \leq n - 1$$

(18)

$$\mathcal{L}_g T_n \neq 0.$$  

(19)

Then the feedback is defined as follows:

$$\alpha = -\frac{\mathcal{L}_f T_n}{\mathcal{L}_g T_n}, \quad \beta = \frac{1}{\mathcal{L}_g T_n}.$$  

(20)

**Proof.** See Vidyasagar [18] equations 7.4.20–21.

**Proposition 13.** The SFB linearizing transformation $J_{T,\alpha,\beta}$ of a SISO deterministic dynamical system $\Theta_D = (f, g, U, x_0) \in \mathbb{X}_D(n, 1)$ into a controllable linear system exists if and only if there is a solution $\lambda: \mathbb{R}^n \to \mathbb{R}$ to the set of partial differential equations:

$$\langle d\lambda, \text{ad}^i_f g \rangle = 0 \quad \text{for} \quad 0 \leq i \leq n - 2$$

(21)

$$\langle d\lambda, \text{ad}^{n-1}_f g \rangle \neq 0.$$  

(22)
The linearizing transformation $T(x)$ is given by:

$$T_i = \mathcal{L}_f^{i-1}\lambda \quad \text{for} \quad 1 \leq i \leq n \quad (23)$$

$$\alpha = -\mathcal{L}_g^n\frac{\lambda}{\mathcal{L}_f^{n-1}\lambda} \quad \beta = \frac{1}{\mathcal{L}_g^n\mathcal{L}_f^{n-1}\lambda}. \quad (24)$$

**Proof.** See Vidyasagar [18] equations 7.4.23–33 and Nijmeijer and van der Schaft [13] Corollary 6.16. □

Finally, the geometrical conditions for the existence of the linearizing transformation are studied.

**Theorem 14.** A deterministic SFB linearizing transformation of \( \Theta_D = (f, g, U, x_0) \in \mathbb{X}_D(n, 1) \) into a controllable linear system exists if and only if the distribution \( \Delta_n := \text{span} \{ \text{ad}_f^i g, 0 \leq i \leq n-1 \} \) is nonsingular on \( U \) and the distribution \( \Delta_{n-1} := \text{span} \{ \text{ad}_f^i g, 0 \leq i \leq n-2 \} \) is involutive on \( U \).

**Proof.** See Nijmeijer and van der Schaft [13] Corollary 6.17, Vidyasagar [18] Theorem 7.4.16, Isidori [5] Theorem 4.2.3 . □

### 2.3 SCT Linearization

**Theorem 15.** There is a SCT linearizing transformation \( T_T \) of a deterministic MIMO system \( \Theta_D = (f, g, U, x_0) \in \mathbb{X}_D(n, m) \) into a controllable linear system if and only if there exists a reordering of the vectorfields \( g_1 \ldots g_m \) and an \( m \)-tuple of integers \( \kappa_1 \leq \kappa_2 \leq \ldots \kappa_m \) with \( \sum_{i=1}^m \kappa_i = n \) called the controllability indexes such that the following conditions are satisfied for all \( x \in U \):

\[
\begin{align*}
(i) & \quad \dim \left( \text{span} \left\{ \text{ad}_f^i g_i(x), 0 \leq i \leq m, 0 \leq j \leq \kappa_i - 1 \right\} \right) = n \\
(ii) & \quad [\text{ad}_f^k g_i, \text{ad}_f^l g_j] = 0 \quad \text{for} \quad 0 \leq k + l \leq \kappa_i + \kappa_j - 1, \ 0 \leq i, j \leq m. 
\end{align*}
\]

**Proof.** See Nijmeijer and van der Schaft [13] Theorem 5.3 and Corollary 5.6. □

The following corollary can be verified for SISO systems:

**Corollary 16.** For a SISO system with \( m = 1 \) the condition (ii) of [13] can be simplified as follows:

\[
[g, \text{ad}_f^l g] = 0, \quad i = 1, 3, 5, \ldots, 2n - 1, \forall x \in U. \quad (27)
\]
Proof. See Nijmeijer and van der Schaft [13] Corollary 5.6 and the text which follows.

\section{Transformations of Itô Dynamical Systems}

The transformation rules of Itô systems are motivated by the Itô differential rule (see eg. Wong and Hajek [20] Section 3.3), which defines the influence of nonlinear coordinate transformations on Itô stochastic processes.

The Itô differential rule applies to the situation where a scalar valued stochastic process $x_t$ defined by a stochastic differential equation 
\[ \frac{dx_t}{dt} = f(x_t) dt + \sigma(x_t) dw_t \]
with $f: \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ smooth real functions and $w_t$ a Brownian motion, is transformed by a a diffeomorphic coordinate transformation $T: \mathbb{R} \rightarrow \mathbb{R}$. Then the stochastic process $z_t := T(x_t)$ exists and is an Itô process. Further, the process $z_t$ is the solution of the stochastic differential equation
\[
\frac{dz_t}{dt} = \frac{\partial T}{\partial x} f(x_t) dt + \frac{\partial T}{\partial x} \sigma(x_t) dw_t + \frac{1}{2} \sigma^2 \frac{\partial^2 T}{\partial x^2} dt.
\]
(28)

All details together with a proof are available for example in Karatzas and Shreve [8].

The Itô rule can be also derived for the multidimensional case: for the $m$-th component of an $n$-dimensional stochastic process the Itô rule can be expressed as follows:
\[
dz_m = \sum_{i=1}^{n} \frac{\partial T_m}{\partial x_i} f_i dt + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{\partial^2 T_m}{\partial x_i \partial x_j} \sigma_{ij} dw_j + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{l=1}^{k} \sigma_{il} \sigma_{jl} dt
\]
\[
= \mathcal{L}_f T_m dt + \sum_{j=1}^{k} \mathcal{L}_{\sigma_j} T_m dw_j + P_\sigma T_m dt. \quad (29)
\]

For the most common case with scalar noise $k = 1$ the equation can be further simplified to:
\[
dz_m = \mathcal{L}_f T_m dt + \mathcal{L}_{\sigma} T_m dw + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 T_m}{\partial x_i \partial x_j} \sigma_i \sigma_j dt.
\]
(30)

The operator $P_\sigma T_m$ is sometimes written using matrix notation as:
\[
P_\sigma T_m = \frac{1}{2} \text{trace} \left( \sigma^T \sigma \frac{\partial^2 T_m}{\partial x^2} \right). \quad (31)
\]

Generally, $P_\sigma$ vanishes for linear $T$ or zero $\sigma$. 

3.1 The Correcting Term

In this section we introduce an extremely useful equivalence between Itô and Stratonovich systems, which allows to use some Stratonovich linearization techniques for Itô problems. The motivation is following: let \( \Theta_I = (f, g, \sigma, U, x_0) \in \mathcal{X}_I(n, m, k) \) be an Itô system. We are looking for a Stratonovich system \( \Theta_S = (\tilde{f}, \tilde{g}, \tilde{\sigma}, U, x_0) \) such that the trajectories of \( \Theta_I \) and \( \Theta_S \) are identical. The aim is to find equations relating the quantities \( \tilde{f}, \tilde{g}, \) and \( \tilde{\sigma} \) with \( f, g, \) and \( \sigma. \)

**Definition 17.** Let \( \Theta_{1I} = (f, g, \sigma, U, x_0) \in \mathcal{X}_I(n, m, k) \) be an \( n \)-dimensional Itô dynamical system with \( k \)-dimensional Brownian motion \( w. \) The vector-field corr\(_\sigma(x)\) whose \( r \)-th coordinate is equal to

\[
(corr_\sigma(x))_r = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{k} \partial_{x_i} \sigma_{ij} \quad \text{for} \quad 1 \leq r \leq n \tag{32}
\]

is called the **correcting term**. Note that the derivative is always evaluated in the corresponding coordinate system. Further, let us define the **correcting mapping** \( \text{Corr}_\sigma : \mathcal{X}_I(n, m, k) \to \mathcal{X}_S(n, m, k) \) by

\[
\text{Corr}_\sigma(f, g, \sigma, U, x_0) := (f + corr_\sigma(x), g, \sigma, U, x_0). \tag{33}
\]

The general treatment of the subject can be found for example in Wong and Hajek [20] p. 160 or in Sage and Melsa [17]. The following theorem describes the behavior of the correcting term under the coordinate transformation.

**Theorem 18.** Let \( \Theta_I = (f, g, \sigma, U, x_0) \in \mathcal{X}_I(n, 1, 1) \) be a one-dimensional Itô dynamical system. Let \( T \) be a diffeomorphism defined on \( U \) and the symbols \( \tau^I_T \) and \( \tau^S_T \) denote a Itô coordinate transformation and a Stratonovich coordinate transformation induced by the same diffeomorphism \( T \) and \( \tilde{\sigma} = T_\ast \sigma. \) Then the following diagram commutes:

\[
\begin{array}{ccc}
\Theta_{1I} & \xrightarrow{\tau^I_T} & \Theta_{2I} \\
\downarrow_{\text{Corr}_\sigma} & & \uparrow_{\text{Corr}_\sigma^{-1}} \\
\Theta_{1S} & \xrightarrow{\tau^S_T} & \Theta_{2S}
\end{array}
\]
In other words:

\[ T^I_T = \text{Corr}^{-1}_\sigma \circ T^S_T \circ \text{Corr}_\sigma \quad \text{and} \]
\[ T^S_T = \text{Corr}_\sigma^{-1} \circ T^I_T \circ \text{Corr}_\bar{\sigma}. \]  

(35)  

(36)

The notation \( \text{Corr}^{-1}_\sigma \) is used to denote the inverse mapping

\[ \text{Corr}_\sigma^{-1}(f, g, \sigma, U, x_0) := (f - \text{corr}_\sigma(x), g, \sigma, U, x_0). \]

(37)

Proof. The correcting term \( \text{corr}_\sigma(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is equal to

\[ \text{corr}_\sigma(x) = -\frac{1}{2} \frac{\partial \sigma}{\partial x} \sigma. \]

(38)

The transformations specified in the diagram (34) will be evaluated in the following order:

\[ \Theta_{1I} \xrightarrow{(a)} \Theta_{2I} \]
\[ \Downarrow \]
\[ \Theta_{1S} \xrightarrow{(c)} \Theta_{2S} \xrightarrow{(d)} \Theta_{3I} \]

(39)

We want to prove the equivalence of \( \Theta_{2I} \) and \( \Theta_{3I} \). The symbols (a) and (c) denote Itô coordinate transformations; the symbols (b) (d) stand for the correcting mapping and its inverse. Note that the systems \( \Theta_{2I}, \Theta_{2S}, \) and \( \Theta_{3I} \) are defined in the \( z \)-coordinate systems. Further, let

\[ \bar{\sigma} := \frac{\partial T}{\partial x} \sigma \]
\[ \kappa := T\sigma \text{corr}_\sigma(x) = -\frac{\partial T}{\partial x} \left( \frac{1}{2} \frac{\partial \sigma}{\partial x} \sigma \right) \]
\[ z_0 := T(x_0). \]

(40)  

(41)  

(42)

Then

\[ \Theta_{2I} = \left( \frac{\partial T}{\partial x} f + \frac{1}{2} \sigma^2 \frac{\partial^2 T}{\partial x^2}, \frac{\partial T}{\partial x} g, \bar{\sigma}, U, z_0 \right) \]
\[ \Theta_{1S} = \left( f - \frac{1}{2} \frac{\partial \sigma}{\partial x} \sigma, g, \sigma, U, x_0 \right) \]
\[ \Theta_{2S} = \left( \frac{\partial T}{\partial x} \left( f - \frac{1}{2} \frac{\partial \sigma}{\partial x} \sigma \right), \frac{\partial T}{\partial x} g, \frac{\partial T}{\partial x} \sigma, U, z_0 \right) \]
\[ \Theta_{3I} = \left( \frac{\partial T}{\partial x} f + \kappa + \frac{1}{2} \frac{\partial \bar{\sigma}}{\partial z} \bar{\sigma}, \frac{\partial T}{\partial x} g, \bar{\sigma}, U, z_0 \right). \]

(43)  

(44)  

(45)  

(46)
All the terms in (a) are equivalent to the respective terms in (d) except for the drift terms containing functions of $\sigma$. Therefore, we continue comparing these terms only. For (a):

$$L := \frac{1}{2} \sigma^2 \frac{\partial^2 T}{\partial x^2}. \quad (47)$$

For (d):

$$R := \kappa + \frac{1}{2} \left( \frac{\partial T}{\partial x} \sigma \right) \frac{\partial T}{\partial x} = \kappa + \frac{1}{2} \left( \frac{\partial T}{\partial x} \sigma \right) \frac{\partial T}{\partial x} =$$

$$\kappa + \frac{1}{2} \left( \frac{\partial^2 T}{\partial x^2} \sigma + \frac{\partial T}{\partial x} \right) \frac{\partial T}{\partial x} \sigma = \kappa + \frac{1}{2} \left( \frac{\partial^2 T}{\partial x^2} \sigma^2 \right) - \kappa = \frac{1}{2} \sigma^2 \frac{\partial^2 T}{\partial x^2}. \quad (48)$$

Thus $L = R$ and $\Theta_{2I} = \Theta_{3I}$.

Theorem 18 is valid also for combined transformations:

**Corollary 19.** Let $\Theta_I = (f, g, \sigma, U, x_0) \in \mathcal{X}_I(n, 1, 1)$, $T, T^I_I$, and $T^S_I$ have the same meaning as in Theorem 18. Then the following diagram commutes for arbitrary regular feedback $F_{\alpha,\beta}$:

$$\begin{array}{ccc}
\Theta_{1I} & \xrightarrow{T^I_I} & \Theta_{2I} \\
\downarrow \text{Corr}_{\sigma} & & \uparrow \text{Corr}_{T,\sigma}^{-1} \\
\Theta_{1S} & \xrightarrow{T^S_I} & \Theta_{2S}
\end{array} \quad \text{(49)}$$

**Proof.** We want to prove equivalence of $\Theta_{4I}$ and $\text{Corr}_{T,\sigma}^{-1} \Theta_{4S}$.

The control and dispersion vectorfields of $\Theta_{4I}$ and $\Theta_{4S}$ are identical and they are not influenced by the correcting mapping.

Using the notation of Theorem 18 we can express the drift term of $\Theta_{4I}$ as $T_\ast f + L + g \alpha$. The drift term of $\text{Corr}_{T,\sigma}^{-1} \Theta_{4S}$ is $T_\ast f + R + g \alpha$. The effect of feedback is purely additive and both the systems are equal.

At first glance the correcting term is rather surprising. How can the second derivative of $T$ in (47) be compensated by the correcting term, which does not contain the $T$ at all? The answer is quite simple: the second derivative is hidden in the correcting term implicitly because the correcting term depends on the coordinate system in which the system $\Theta_{1I}$ is defined. The derivative $\frac{\partial \sigma}{\partial x}$ contained in the correcting term is always taken in the appropriate coordinate system. To emphasize the dependence of the correcting
term on the coordinate system, we will never omit the independent variable (e.g. \(x\) or \(z\)) from the symbol \(\text{corr}_\sigma(x)\).

Proposition 18 is valid for general multidimensional systems \(\Theta_I = (f, g, \sigma, U, x_0) \in \mathcal{X}_I(n, m, k)\); the proof is purely mechanical and is not presented here.

Let us now turn our attention to several special cases of the correcting mapping.

**Corollary 20.** Let \(\Theta_I = (f, g, \sigma, U, x_0) \in \mathcal{X}_I(n, m, 1)\) be an \(n\)-dimensional stochastic dynamical system with an one-dimensional Brownian motion \(w\). The \(r\)-th coordinate of the correcting term \((\text{corr}_\sigma(x))_r\) is equal to

\[
(\text{corr}_\sigma(x))_r = \frac{1}{2} \sum_{i=1}^{n} \sigma_i \frac{\partial \sigma_r}{\partial x_i} \text{ for } 1 \leq r \leq n.
\]

**Proof.** Substitute \(k = 1\) into (32).

**Corollary 21.** For systems with one-dimensional noise \((k = 1)\) define the matrix valued Itô term \(P_\sigma T\) for \(T : \mathbb{R}^n \to \mathbb{R}^n\) with components \(T_i, 1 \leq i \leq n\), as a column \(n \times 1\) matrix

\[
P_\sigma T := [P_\sigma T_1, P_\sigma T_2, \ldots, P_\sigma T_n]^{T}.
\]

Then the relations (50) can be expressed as

\[
P_\sigma T = T_\ast (\text{corr}_\sigma(x)) - \text{corr}_\sigma(z).
\]

**Proof.** The proof is almost identical to that of the multidimensional variant of Corollary (18). The symbols can be identified as follows:

\[
L_i = (P_\sigma T)_i \quad \text{for } 1 \leq i \leq n \quad (52)
\]

\[
\kappa_i = (T_\ast (\text{corr}_\sigma(x)))_i \quad (53)
\]

\[
R_i = (T_\ast (\text{corr}_\sigma(x)))_i - (\text{corr}_\sigma(z))_i. \quad (54)
\]

**Corollary 22.** Assume that the conditions of Proposition 21 hold. The relation (51) can be written as:

\[
\frac{1}{2} L_\sigma L_\sigma T_i = P_\sigma T_i - L_{\text{corr}_\sigma(x)} T_i \quad \text{for } 1 \leq i \leq n.
\]

**Proof.** The formula can be expressed as:

\[
\frac{1}{2} L_\sigma L_\sigma T_i = \frac{1}{2} L_\sigma \left( \sum_{j=1}^{n} \frac{\partial T_i}{\partial x_j} \sigma_j \right) = \frac{1}{2} \sum_{l=1}^{n} \sigma_k \frac{\partial}{\partial x_k} \left( \sum_{j=1}^{n} \frac{\partial T_i}{\partial x_j} \sigma_j \right) = \frac{1}{2} \sum_{k,j=1}^{n} \left( \sigma_j \sigma_k \frac{\partial^2 T_i}{\partial x_k \partial x_j} + \frac{\partial T_i}{\partial x_j} \frac{\partial \sigma_j}{\partial x_k} \sigma_k \right) = P_\sigma T_i - L_{\text{corr}_\sigma(x)} T_i. \quad (56)
\]
3.2 Composition of Coordinate Transformations of Itô Systems

The set of all deterministic coordinate transformations $T$ together with composition $T_{RS} := T_S \circ T_R$ forms a group. Obviously, this fact is a straightforward result of the behavior of the contravariant transformation and therefore an analogous statement must hold for Stratonovich systems. Surprisingly, this is valid also for Itô systems as will be shown here. This has an important consequence: we may always find the inverse transformation to a given coordinate transformation of Itô systems. We will prove the following assertion:

**Theorem 23.** Let $T^I_R, T^I_S \in \mathcal{X}_I(n,1,1)$ be coordinate transformations of one-dimensional Itô systems induced by diffeomorphisms $R$ and $S$. Then

$$T^I_S \circ T^I_R = T^I_{S \circ R}. \quad (57)$$

**Proof.** We will transform the system in two different ways and show that the results are equal.

1. In the first method the system $A = (0,0,a,U,x_0)$ which corresponds to a differential equation $dx = a(x) \, dw$ will be transformed twice:

   (a) first, by $y = R(x)$ to $y$ coordinates

   (b) and then the result $B = (g,0,b,U,x_0)$ which corresponds to $dy = g(y) \, dt + b(y) \, dw$ by $z = S(y)$ to $z$ coordinates.

2. The other method transforms the system $A$ only once by $z = T(x) = S(R(x)) = (S \circ R)(x)$.

Without loss of generality, the equation $dx = a(x) \, dw$ is assumed to have no drift term because the drift term transforms in the contravariant fashion. The derivatives will be denoted by $\frac{\partial T(x)}{\partial x} =: T', \frac{\partial R(x)}{\partial x} =: R'$ and similarly for $T'', R''$ and $S''$. Note that the prime is always used to denote derivatives by the argument of the function.

The transformation by $R$ gives:

$$dx = a \, dw \quad (58)$$

$$dy = R' a \, dw + \frac{1}{2} a^2 R'' \, dt. \quad (59)$$
Thus the coefficients of the second SDE are defined by

\[ b(y) := (R' a) \circ R^{-1}(y) \]  
\[ g(y) := \left( \frac{1}{2} a^2 R'' \right) \circ R^{-1}(y). \]  

The second transformation (by \( S \)) gives

\[
dz = \left( S' g + \frac{1}{2} b^2 S'' \right) dt + S' b \, dw = \]
\[
= \left( \frac{1}{2} a^2 S' R'' + \frac{1}{2} (R')^2 a^2 S'' \right) dt + S' R' a \, dw = \]
\[
= \frac{1}{2} a^2 (S' R'' + (R')^2 S'') dt + S' R' a \, dw = \]
\[
= \frac{1}{2} a^2 T'' dt + T' a \, dw. \]

The last equality follows from the fact that \( T'' = S'' (R')^2 + S' R'' \). \[ \square \]

One can verify the multidimensional case in the same spirit.

3.3 Invariants

In the deterministic case, some useful propositions about the invariant properties for example the Leibniz rule (12) and the relation (13) were employed. Unfortunately, we have not found any analogy for the Itô systems yet. To point out the main complications, we will analyze the Itô equivalent of the Leibniz rule (12), which is essential for reducing the order of partial differential equations in the deterministic exact linearization.

If the Lie derivative \( \mathcal{L}_g \) is interpreted as a general first order operator

\[ \mathcal{L}_g = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \]  

then the commutator of two such first order operators \( \mathcal{L}_f \mathcal{L}_g - \mathcal{L}_g \mathcal{L}_f \) is also a first order operator \( \mathcal{L}_{[f,g]} \) (see (12)).

Similarly, define the general second order operator as

\[ O(g, G) := \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} + \sum_{i,j=0}^{n} G_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \]  

20
where $g_i, G_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}$ for $1 \leq i, j \leq n$. We can compute the commutator

$$C(f, F, g, G) := O(f, F)O(g, G) - O(g, G)O(f, F)$$

(68)

of such second order operators. If this commutator was also a second order operator (i.e. there were $\varphi$ and $\Phi$ such that the operator $C(f, F, g, G) = O(\varphi, \Phi)$), then we would be able to simplify any PDEs of stochastic transformations. (See Proposition 29).

Because the operator $O$ is linear, i.e. $O(f, F) = O(f, 0) + O(0, F)$, we can split the computation into four independent, reusable parts:

$$C(f, F, g, G) = O(f, F)O(g, G) - O(g, G)O(f, F) =$$

$$= \mathcal{L}_{[f, g]} + C(0, F, 0, G) + C(f, 0, 0, G) + C(0, F, g, 0).$$

(69)

The first term is already a first order operator. Only the second and the third terms need to be computed because the fourth term can be obtained from the third one by formal substitution. For the third term:

$$O(f, 0)O(0, G) = \sum_{i=1}^{n} f_i \partial \left( \sum_{k,l=1}^{n} G_{kl} \frac{\partial^2}{\partial x_k \partial x_l} \right) =$$

$$= \sum_{i,k,l=1}^{n} G_{kl} \left( f_i \frac{\partial G_{kl}}{\partial x_i} \frac{\partial^2}{\partial x_k \partial x_l} + f_i \frac{\partial^3}{\partial x_i \partial x_k \partial x_l} \right).$$

(70)

Further,

$$O(0, G)O(f, 0) = \sum_{k,l=1}^{n} G_{kl} \frac{\partial^2}{\partial x_k \partial x_l} \left( \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \right) =$$

$$= \sum_{i,k,l=1}^{n} G_{kl} \left( \frac{\partial^2 f_i}{\partial x_k \partial x_l} \frac{\partial}{\partial x_i} + 2 \frac{\partial f_i}{\partial x_i} \frac{\partial^2}{\partial x_k \partial x_l} + f_i \frac{\partial^3}{\partial x_i \partial x_k \partial x_l} \right).$$

(71)

The intermediate results for the third and the fourth terms can be combined into

$$O(f, 0)O(0, G) + O(0, F)O(g, 0) - O(g, 0)O(0, F) - O(0, G)O(f, 0) =$$

$$= \sum_{i,k,l=1}^{n} \left( f_i \frac{\partial G_{kl}}{\partial x_i} - g_i \frac{\partial F_{kl}}{\partial x_i} + 2 F_{kl} \frac{\partial g_i}{\partial x_i} - 2 G_{kl} \frac{\partial f_i}{\partial x_i} \right) \frac{\partial^2}{\partial x_k \partial x_l}$$

$$+ \left( F_{kl} \frac{\partial^2 f_i}{\partial x_k \partial x_l} - G_{kl} \frac{\partial^2 g_i}{\partial x_k \partial x_l} \right) \frac{\partial}{\partial x_i}.$$
All of them are first and second order operators. Now let’s evaluate the second term

\[ O(0, F)O(0, G) = \sum_{i,j=1}^{n} F_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left( \sum_{k,l=1}^{n} G_{kl} \frac{\partial^2}{\partial x_k \partial x_l} \right) = \sum_{i,j,k,l=1}^{n} \left( F_{ij} \frac{\partial^2 G_{kl}}{\partial x_i \partial x_j \partial x_k \partial x_l} + (F_{ij} + F_{ji}) \frac{\partial G_{kl}}{\partial x_j \partial x_i \partial x_k \partial x_l} + F_{ij} G_{kl} \frac{\partial^4}{\partial x_i \partial x_j \partial x_k \partial x_l} \right). \] (73)

Unfortunately the last term

\[ \left( (F_{ij} + F_{ji}) \frac{\partial G_{kl}}{\partial x_j} - (G_{ij} + G_{ji}) \frac{\partial F_{kl}}{\partial x_j} \right) \frac{\partial^3}{\partial x_i \partial x_k \partial x_l} \] (75)

is of third order and, in general, it does not vanish. Thus we have shown that the commutator of two general second order operators is of third order. Consequently, the Leibniz rule simplifications used in the deterministic case cannot be applied to the general stochastic linearization problem.

4 Stochastic Case

Since there are two definitions of coordinate transformation of stochastic differential equations (Itô, Stratonovich) and three definitions of linearity \((g, \sigma, g\sigma)\), we face at least six stochastic problems per a deterministic one. In this section we will discuss all of them giving at least partial solutions to the feedback linearization problem. We consider mainly the SISO problem except for cases where the MIMO extension is trivial.

4.1 Stratonovich \(g\)-linearization

We show that the method known for deterministic systems can be applied without modifications.
Proposition 24. The Stratonovich dynamical system \( \Theta_S = (f, g, \sigma, U, x_0) \in \mathcal{X}_S(n, m, k) \) is \( g \)-linearizable if and only if the deterministic system \( \Theta_D = (f, g, U, x_0) \in \mathcal{X}_D(n, m) \) is linearizable. These two linearizing transformations are equal. This holds both for SISO and MIMO systems and both for SFB and SCT linearization.

Proof. The comparison of transformation laws for deterministic and Stratonovich systems shows that the coefficients of \( f \) and \( g \) transform in the same way. The controllability conditions and the definition of linearity are also identical (compare Definition 8 with Definition 9). Identical problems have identical solutions.

4.2 Stratonovich \( g\sigma \)-linearization

The Stratonovich problems are not complicated by the second order Itô term. The transformation laws for Stratonovich systems are the same as the deterministic transformation laws, therefore many results of the deterministic linearization theory can be used.

For example, the stochastic SCT \( g\sigma \)-linearization of a Stratonovich system is equivalent to the linearization of a deterministic, non-square MIMO system with two inputs and a single output. The SFB problem, which is studied in this section, is not as simple as the SCT one because the feedback influences only the control input \( u \) (Figure 3). The “dispersion input” is not a part of the feedback. Consequently, in order to solve the Stratonovich SFB \( g\sigma \)-linearization, we have to deal with a combined deterministic SFB-SCT problem.

4.2.1 Canonical Form

Recall that we require \( g \)-controllability of the resulting system.
Since this is a Stratonovich problem, the transformed vectorfields \( \tilde{f} \) and \( \tilde{g} \) do not depend on the dispersion vectorfield \( \sigma \). Therefore, the control part and the dispersion part can be studied independently.

Any \( g \)-linear system can be transformed into integrator chain by a combination of a linear coordinate transformation and linear feedback. Therefore, if we set \( \sigma = 0 \), the canonical form is the integrator chain.

In general, the dispersion vectorfield \( \tilde{\sigma} \) is assumed to be arbitrary constant vectorfield \( \tilde{\sigma}(x) = s_i, \ 1 \leq i \leq n \) (See Definition 9) and this form is preserved by arbitrary linear transformations. Therefore the canonical form can be written as:

\[
\begin{align*}
\tilde{f}_i(x) &= x_{i+1} \quad \text{for} \quad 1 \leq i \leq n-1 \\
\tilde{f}_n(x) &= 0 \\
\tilde{g}_i(x) &= 0 \quad \text{for} \quad 1 \leq i \leq n-1 \\
\tilde{g}_n(x) &= 1 \\
\tilde{\sigma}_i(x) &= s_i \quad \text{for} \quad 1 \leq i \leq n.
\end{align*}
\]

We can compare this canonical form with the equations which define the transformed system \( \tilde{\Theta} \).

**Proposition 25.** There is a SFB \( g_{\sigma} \)-linearizing transformation of the SISO Stratonovich system \( \Theta_S = (f, g, \sigma, U, x_0) \in \mathbb{X}_S(n, 1, 1) \) into a \( g \)-controllable linear system if and only if there is a solution \( \lambda: \mathbb{R}^n \rightarrow \mathbb{R} \) of the set of partial differential equations:

\[
\begin{align*}
\langle d\lambda, \text{ad}_f^i g \rangle &= 0 \quad \text{for} \quad 0 \leq i \leq n-2 \\
\langle d\lambda, \text{ad}_f^{n-1} g \rangle &\neq 0 \\
\langle d\lambda, \text{ad}_f \sigma \rangle &= s'_{i+1} \quad \text{for} \quad 0 \leq i \leq n-1
\end{align*}
\]

such that \( s'_i \in \mathbb{R} \) are constants on \( U \) for \( 1 \leq i \leq n \). Then the linearizing transformation is given by:

\[
\begin{align*}
T_i &= \mathcal{L}_f^{-1} \lambda \quad \text{for} \quad 1 \leq i \leq n \\
\alpha &= \frac{-\mathcal{L}_f^{n} \lambda}{\mathcal{L}_g \mathcal{L}_f^{n-1} \lambda} \quad \beta = \frac{1}{\mathcal{L}_g \mathcal{L}_f^{n-1} \lambda}.
\end{align*}
\]

**Proof.** Assume that \( \Theta_S \) is transformed by \( \mathcal{J}_{T,\alpha,\beta} \) into \( \tilde{\Theta} := (\tilde{f}, \tilde{g}, \tilde{\sigma}, T(U), T(x_0)) \) where the \( i \)-th components of \( f, g, \) and \( \sigma \) can be expressed as: \( \tilde{f}_i = \mathcal{L}_f T_i, \ \tilde{g}_i = \mathcal{L}_g T_i, \ \tilde{\sigma}_i = \mathcal{L}_\sigma T_i \). Moreover, the feedback is defined by \( u = \alpha + \beta v \). The
equations of $\Theta$ can be compared to the equation of the canonical form (74)-(80).

\[
\mathcal{L}_f T_i = T_{i+1} \quad \text{for} \quad 1 \leq i \leq n - 1 \tag{86}
\]
\[
\mathcal{L}_g T_i = 0 \quad \text{for} \quad 1 \leq i \leq n - 1 \tag{87}
\]
\[
\mathcal{L}_g T_n = 1/\beta \neq 0 \tag{88}
\]
\[
\mathcal{L}_f T_n = -\alpha/\beta. \tag{89}
\]

The relations (81), (82), (84), and (85) are equivalent to relations (21)-(24) from Proposition 13. The relation (83) can be verified in a similar way:

\[
\mathcal{L}_\sigma T_i = s_i \quad \text{for} \quad 1 \leq i \leq n \tag{90}
\]

thus by (86)

\[
\mathcal{L}_\sigma \mathcal{L}_f T_i = s_{i+1} \quad \text{for} \quad 1 \leq i \leq n - 1 \tag{91}
\]

and by (12)

\[
\mathcal{L}_\sigma \mathcal{L}_f T_i = \mathcal{L}_f \mathcal{L}_\sigma T_i - \mathcal{L}_{[f,\sigma]} T_i \quad \text{for} \quad 1 \leq i \leq n - 1 \tag{92}
\]

since the Lie derivative of a constant is zero:

\[
\mathcal{L}_f \mathcal{L}_\sigma T_i = \mathcal{L}_f s_i = 0 \tag{93}
\]

\[
s_{i+1} := \mathcal{L}_\sigma \mathcal{L}_f T_i = -\mathcal{L}_{[f,\sigma]} T_i \quad \text{for} \quad 1 \leq i \leq n - 1. \tag{94}
\]

The equations (83) are obtained by successive application of this relation. The symbols $s_i$ are equal to $s'_i$ except for the signs. $\square$

### 4.2.2 Conditions for the Control Part

The necessary conditions for linearizability of the control part of $\Theta$ (i.e. the system $(f, g, 0, U, x_0)$) can be expressed in geometrical form. We intentionally omit the dispersion part, using the fact that the resulting system must be linear when the noise is zero.

Further, the class of all solutions of this subproblem will be called $C$. This class can be studied to find if some member of $C$ linearizes the dispersion part of the system.

We can find a geometrical criterion similar to the conditions of Proposition 14. In this case these conditions are necessary but not sufficient since also (83) must be satisfied.
Proposition 26. SFB $g\sigma$-linearizing transformation of the Stratonovich system $\Theta_S$ into a $g\sigma$-controllable system linear system exists only if the distribution $\{\text{ad}_i^f g, 0 \leq i \leq n - 2\}$ is involutive and the distribution $\{\text{ad}_i^f g, 0 \leq i \leq n - 1\}$ is $n$-dimensional.

Proof. This theorem is equivalent to Proposition 14 which is a direct consequence of Proposition 12 which corresponds to Proposition 25.

4.2.3 Condition for the Dispersion Part

The conditions of Proposition 26 can be written in matrix form. We are looking for $T_1 = \lambda: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

\[
\begin{bmatrix}
\text{ad}_0^f g \\
\text{ad}_1^f g \\
\vdots \\
\text{ad}_{n-2}^f g \\
\end{bmatrix} \begin{bmatrix}
\frac{\partial \lambda}{\partial x_1} \\
\frac{\partial \lambda}{\partial x_2} \\
\vdots \\
\frac{\partial \lambda}{\partial x_n} \\
\end{bmatrix} = [0].
\]

(95)

The vectors $\text{ad}_0^f g, \ldots, \text{ad}_{n-2}^f g$ are written in coordinates as $1 \times n$ rows. The first matrix is $(n - 1) \times n$. Moreover it is required that

\[
\langle d\lambda, \text{ad}_f^{n-1} g \rangle
\]

(96)
is nonzero.

We will use the algorithm for SFB deterministic linearization (see Section 4) to find such a transformation $\lambda$. Then we will verify if the conditions for linearity of the dispersion part of the system (83) are also valid. There are $n$ additional linearity conditions ($s_i$ are constants):

\[
\begin{bmatrix}
\text{ad}_0^f \sigma \\
\text{ad}_1^f \sigma \\
\vdots \\
\text{ad}_{n-1}^f \sigma \\
\end{bmatrix} \begin{bmatrix}
\frac{\partial \lambda}{\partial x_1} \\
\frac{\partial \lambda}{\partial x_2} \\
\vdots \\
\frac{\partial \lambda}{\partial x_n} \\
\end{bmatrix} = \begin{bmatrix}
s_1 \\
s_2 \\
\vdots \\
s_n \\
\end{bmatrix}.
\]

(97)

In the deterministic case we were satisfied with arbitrary solution $\lambda$ to the equations (95), and (96). In this stochastic case we must find the class of all solutions and then check if this class contains the solution for the $\sigma$ part (97). Details depend on the methods used for solving the set of PDEs.

This result is summarized in the following algorithm:

- Find $\Delta_k := \text{ad}_f^i g$ for $0 \leq i \leq k - 1$. 

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• Verify that \( \dim(\Delta_n) \) is \( n \).

• Verify that \( \Delta_{n-1} \) is involutive (see Nijmeijer and van der Schaft [13] Remark following Definition 2.39) otherwise no linearizing transformation exists.

• Find all \( \lambda \) satisfying (100) by solving PDEs (100); denote \( C \) the set of all such functions.

• Verify that there is a \( \lambda_1 \in C \) such that the conditions (97) are satisfied, otherwise no linearizing transformation exists.

• Compute \( T, \alpha, \beta \) from (86) – (89).

Now, we can illustrate one possible practical approach which worked for several simple problems solved by us (see the example in Section 5).

First we can compute the kernel of the matrix \( M_g \) to find the form \( \omega = [\omega_1, \omega_2, \ldots, \omega_n]^T \) which satisfies \( M_g \omega = 0 \), i.e. \( \omega \) is perpendicular to \( M_g \). In modern computer algebra systems there is a single command for this.

Proposition 26 assumes that \( n \) vectorfields \( \Delta_n := \{ \text{ad}_{f \cdot j} g, 0 \leq i \leq n - 1 \} \) form an \( n \) dimensional space. The vectorfields \( \Delta_{n-1} := \{ \text{ad}_{f \cdot j} g, 0 \leq i \leq n - 2 \} \) are chosen from them and consequently must form an \((n - 1)\)-dimensional space. Thus their kernel \( d\lambda \) is exactly one dimensional and arbitrary \( \omega' = c(x)\omega(x) \) also belongs to the kernel (\( c(x) \) is a scalar).

But not every \( \omega' \) that is perpendicular to \( M_g \) is a solution to the original linearization problem. The function \( \omega' \) must be an exact one-form i.e. there must be a scalar function \( \lambda \) such that \( d\lambda = c(x)\omega(x) \). The Frobenius theorem guarantees that if \( \Delta_{n-1} \) is involutive, then there is always \( c(x) \in \mathbb{R} \) such that \( c(x)\omega(x) \) is the exact one-form.

A necessary condition for a one-form \( \omega = \sum_{i=1}^{n} \omega_i \) to be exact is

\[
\frac{\partial \omega_i}{\partial x_j} = \frac{\partial \omega_j}{\partial x_i} \quad \text{for} \quad 1 \leq i, j \leq n. \tag{98}
\]

Hence for every \( 1 \leq i, j \leq n \)

\[
\frac{\partial}{\partial x_j} (c(x)\omega_i) = \frac{\partial}{\partial x_i} (c(x)\omega_j), \tag{99}
\]

thus for every \( 1 \leq i, j \leq n \)

\[
\frac{\partial c(x)}{\partial x_i} \omega_j - \frac{\partial c(x)}{\partial x_j} \omega_i + c(x) \left( \frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j} \right) = 0. \tag{100}
\]
The later condition is a set of linear PDEs, with unknown \( c(x) \), which are guaranteed to have a solution by the involutivness of \( \Delta_{n-1} \) (the Frobenius theorem).

In our computations the equation (100) was in a simple form which allowed to determine all the solutions easily. More complicated cases will require more sophisticated analysis.

4.3 Itô \( g\sigma \)-linearization

In the previous subsection we tried to find \( g\sigma \)-linearizations for Stratonovich dynamical systems. Once this is done, the correcting mapping can be used to construct Itô \( g\sigma \)-linearizing transformation. This method works for both the SFB and the SCT case.

Given an Itô system \( \Theta_I \), the corresponding Stratonovich system \( \Theta_S \) can be obtained using the correcting mapping \( \Theta_S = \text{Corr}_\sigma (\Theta_I) \). Afterward, the Stratonovich \( g\sigma \)-linearization algorithm can be applied giving a linear system \( \Theta_{2S} \). Due to linearity of the drift vectorfield \( \tilde{\sigma} \) of \( \Theta_{2S} \), the correcting term \( \text{corr}_\tilde{\sigma}(z) \) of the backward transformation \( \text{Corr}_{\tilde{\sigma}}^{-1} \) vanishes.

**Theorem 27.** The SFB \( g\sigma \)-linearizing transformation \( J_I \) of the Itô dynamical system \( \Theta_I = (f, g, \sigma, U, x_0) \in \mathcal{X}_I(n, m, k) \), \( f(x_0) = 0 \), \( \text{corr}_\sigma(x_0) = 0 \), into a \( g\sigma \)-controllable linear system exists if and only if there is a SFB \( g\sigma \)-linearizing transformation \( J_S \) of the Stratonovich dynamical system

\[
\Theta_S = \left( \tilde{f}, g, \sigma, U, x_0 \right) = \text{Corr}_\sigma (\Theta_I) \tag{101}
\]

\[
\tilde{f} = f + \text{corr}_\sigma(x). \tag{102}
\]

Moreover \( J_I = J_S \circ \text{Corr}_\sigma \).

*Proof (sufficiency).* We use the properties of the correcting term (Subsection [3.1]). Assume that there is a mapping \( J_S \) which transform \( \Theta_S \) into a linear \( g \)-controllable system \( (Ax, B, S, U, 0) \). By (35)

\[
J_I = \text{Corr}_{\tilde{\sigma}}^{-1} \circ J_S \circ \text{Corr}_\sigma \tag{103}
\]

The backward correcting transformation \( \text{Corr}_{\tilde{\sigma}}^{-1} \) is identity because the correcting term of a linear mapping \( \text{corr}_\tilde{\sigma}(x) \) is zero. Thus \( J_I = J_S \circ \text{Corr}_\sigma \) and \( J_I(\Theta_I) \) equals \( (Ax, B, S, U, 0) \), which is linear and \( g \)-controllable by assumption. ∎
Proof (necessity). Assume that there is the Itô transformation $J_I$ which linearizes $\Theta_I$ and by (101) $\Theta_I = \text{Corr}_\sigma^{-1}(\Theta_S)$. Construct Stratonovich linearization by $J_S = J_I \circ \text{Corr}_\sigma^{-1}$. Hence $J_I$ linearizes $\text{Corr}_\sigma^{-1}(\Theta_S)$ and $J_S$ linearizes $\Theta_S$ into the same linear and controllable system as $J_I$.  

4.4 Itô $g$-linearization

The Itô $g$-linearization problem is probably the most complicated variant of exact linearization studied in this paper. The dispersion vectorfield of an Itô dynamical system transformed by a coordinate transformation $T$ consists of two terms: the transformed vectorfield $T_*f$ and the Itô term $P_\sigma$. We require that the sum of these terms is linear, thus the nonlinearity of the drift $T_*f$ must compensate for the Itô term. Since the Itô term behaves to $T$ as a second order differential operator, this problems generates a set of second order partial differential equations. One can attempt to use simplifications as in the deterministic linearization, namely, the recursive Leibniz rule (14). Unfortunately, this approach does not work for the stochastic case. In general, the Itô equations cannot be easily simplified by commutators because the commutator of second order operators is not a second order operator but a third order operator (see Subsection 3.3).

Nevertheless, there are special cases for which simpler conditions can be found. The most important special case (commuting $g$ and $\sigma$) will be studied here.

4.4.1 Canonical Form — $n$ unknowns

The canonical form for the $g$-linearization is the integrator chain with a nonlinear drift

$$f_i(x) = x_{i+1} \quad \text{for} \quad 1 \leq i \leq n - 1 \quad (104)$$

$$f_n(x) = 0 \quad (105)$$

$$g_i(x) = 0 \quad \text{for} \quad 1 \leq i \leq n - 1 \quad (106)$$

$$g_n(x) = 1 \quad (107)$$

Assume that there is a $g$-linear system $\Theta_I = (Ax, B, \sigma(x), U, x_0)$. Then the drift part of $\Theta_I$ can be transformed by a linear transformation into the integrator chain. This is because the Itô term of a linear transformation vanishes.

The equations which define $T$ can be obtained by comparing this canonical form with the equations of $\tilde{\Theta}$.  

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Proposition 28. Let \( \Theta_I = (f, g, \sigma, U, x_0) \in \mathbb{X}_I(n, 1, 1) \) be an Itô dynamical system with \( f(x_0) = 0 \) such that \( \text{corr}_\sigma(x_0) = 0 \). There is a SFB \( g \)-linearizing transformation \( J_{T,\alpha,\beta} \) of the system \( \Theta_I \) into a \( g \)-controllable linear system if and only if there is a solution \( T_i : \mathbb{R}^n \to \mathbb{R}, 1 \leq i \leq n \), to the set of partial differential equations defined on \( U \):

\[
T_{i+1} = \mathcal{L}_f T_i + P_\sigma T_i \quad \text{for} \quad 1 \leq i \leq n - 1 \tag{108}
\]
\[
\mathcal{L}_g T_i = 0 \quad \text{for} \quad 1 \leq i \leq n - 1 \tag{109}
\]
\[
\mathcal{L}_g T_n \neq 0. \tag{110}
\]

The symbol \( P_\sigma \) denotes the Itô operator (see Definition 3). The feedback can be constructed as:

\[
\alpha = -\frac{(\mathcal{L}_f T_n + P_\sigma T_n)}{\mathcal{L}_g T_n}, \quad \beta = \frac{1}{\mathcal{L}_g T_n}. \tag{111}
\]

Proof. The \( i \)-th components of \( f, g \), and \( \sigma \) are: \( \tilde{f}_i = \mathcal{L}_f T_i + P_\sigma T_i, \tilde{g}_i = \mathcal{L}_g T_i, \)
\( \tilde{\sigma}_i = \mathcal{L}_\sigma T_i \). The partial differential equations (108), (109) and (110) are obtained by comparison of (1)-(4) with the equations (104)-(107).

4.4.2 PDEs of single unknown

One can attempt to reduce the equations (108), (109), and (110), to a set of equations of a single unknown, similarly to the results of Proposition 13.

Corollary 29. Define the general second order operator \( O(f, F) \) as in (67). The exponential notation for \( O \) will be defined recursively: \( O^0(f, F)T : = T \) and \( O^{l+1}(f, F)T := O(f, F)O^l(f, F)T \) for \( l \geq 0 \). Next, define

\[
F_{ij} := \frac{1}{2} \sigma_i \sigma_j. \tag{112}
\]

Then the set of partial differential equations (108)-(111) of \( n \) unknowns has a solution if and only if there is a solution \( \lambda : \mathbb{R}^n \to \mathbb{R} \) defined on \( U \) to the set of PDEs of single unknown:

\[
O(g, 0)O^i(f, F)\lambda = 0 \quad \text{for} \quad 0 \leq i \leq n - 2 \tag{113}
\]
\[
O(g, 0)O^{n-1}(f, F)\lambda \neq 0. \tag{114}
\]

The original solution and the feedback can be found as

\[
T_i = O^{i-1}(f, F)\lambda \quad \text{for} \quad 1 \leq i \leq n \tag{115}
\]
\[
\alpha = -\frac{O(f, F)^{n-1}\lambda}{O(g, 0)^{n-1}\lambda}, \quad \beta = \frac{1}{O(g, 0)^{n-1}\lambda}. \tag{116}
\]
Proof. Since \( T_1 = \lambda \) and by definition of \( P_\sigma \) (1) and \( O \) (67):

\[
O(f,F)T_i = \mathcal{L}_f T_i + P_\sigma T_i \quad \text{for} \quad 1 \leq i \leq n \quad (117)
\]
\[
O(g,0)T_i = \mathcal{L}_g T_i \quad \text{for} \quad 1 \leq i \leq n; \quad (118)
\]

then \( T_{i+1} = O(f,F)T_i \) by (108) and \( T_i = O^{i-1}(f,F)T_1 = O^{i-1}(f,F)\lambda \).

Similarly the equation (111) implies Proposition 114.

Note, that for the deterministic case \( \sigma = 0 \), the operators \( O(g,0) \) and \( O(g,0)O^i(f,F) \) degenerate to \( \mathcal{L}_g \) and to \( \mathcal{L}_g \mathcal{L}_f^i \) respectively, thus the result is the same as that of Proposition 13.

In general, the equations of the system are of an order up to \( 2n \) and cannot be reduced to a lower order. The commutator of two second order operators is of third order as was pointed out by (75). In particular, for \( i = 1 \) we have to evaluate \( C(g,0,f,F) =: O(a,A) \), which is of second order due to the fact that \( G = 0 \). But starting from \( i = 2 \) the commutator \( C(f,F,a,A) = C(f,F,C(g,0,f,F)) \) is of third order.

4.4.3 Correcting Term

The same problem can be reformulated using conversion to the Stratonovich formalism. One can compute the drift vectorfield \( \tilde{f} \) of the equivalent Stratonovich system by applying the correcting term \( \tilde{f} := f + \text{corr}_\sigma(x) \). Then the Stratonovich system \( \left( \tilde{f}, g, \sigma, U, x_0 \right) \) may be transformed, by a suitable transformation, to such a form \( \left( \tilde{f}, \tilde{g}, \tilde{\sigma}, T(U), 0 \right) \) that after applying the backward correcting term \( -\text{corr}_\tilde{\sigma}(z) \) the resulting Itô system will be linear.

Compare this formulation with the \( g\sigma \)-linearization where the backward correcting term vanished due to linearity of \( \tilde{\sigma} \). This does not happen with the \( g \)-linearization, and the backward correcting term is a part of the equations.

In general, it may be difficult to solve these equations. Nevertheless there are special cases when the solution can be obtained. See for example Section 5. Another important case (commuting \( g \) and \( \sigma \)) is studied below.

Corollary 30. The equations of Proposition 28 are equivalent to \( n \) partial
differential equations:

\[ T_{i+1} = L_f T_i - \text{corr}_\sigma(z) = \]

\[ L_f T_i + \frac{1}{2} L_\sigma L_\sigma T_i \quad \text{for} \quad 1 \leq i \leq n - 1 \]  

\[ L_g T_i = 0 \quad \text{for} \quad 1 \leq i \leq n - 1 \]  

\[ L_g T_n \neq 0. \]

Then the feedback can be constructed as

\[ \alpha = - \frac{L_f T_n + \frac{1}{2} L_\sigma L_\sigma T_n}{L_g T_n} \quad \beta = \frac{1}{L_g T_n}. \]  

Where \( \bar{f} := f + \text{corr}_\sigma(x) \).

Proof. The equations (119) can be obtained from (108) by applying (51) and (55):

\[ T_{i+1} = L_f T_i + P_\sigma T_i = L_{(\bar{f} - \text{corr}_\sigma(x))} T_i + P_\sigma T_i = \]

\[ L_f T_i - L_{\text{corr}_\sigma(x)} T_i + P_\sigma T_i = L_f T_i + \frac{1}{2} L_\sigma L_\sigma T_i. \]  

The other equations are adopted from Proposition 28. The set of PDEs of \( n \) unknowns can be transformed into a set of PDEs of a single unknown \( \lambda = T_1 \), but the order of the equation will be \( 2n - 1 \).

Remark 31. Observe that the set of \( n \) second order partial differential equations (119)- (122) defined in Proposition 30 can be transformed, by introducing new variables \( S_i = L_\sigma T_i \), to the following system of \( 2n - 1 \) first order partial differential equations for \( 1 \leq i \leq n - 1 \):

\[ L_g T_i = 0 \]  

\[ L_\sigma T_i - S_i = 0 \]  

\[ L_f T_i + \frac{1}{2} L_\sigma S_i = T_{i+1} \]  

\[ L_g T_n \neq 0. \]

\( T_i \) and \( S_i \): \( \mathbb{R}^n \rightarrow \mathbb{R} \) are unknown real valued functions defined on \( U \).  

\[ \Box \]
4.4.4 Systems with Commuting $g$ and $\sigma$

There is a special case of Itô dynamical systems for which the solution is completely known and can be computed using only first order PDEs.

**Theorem 32.** Let $\Theta_I = (f, g, \sigma, U, x_0) \in X_I(n, 1, 1)$ be a SISO Itô dynamical system. If the vectorfield $\sigma$ commutes with all vectorfields $\text{ad}^i f$ for $0 \leq i \leq n - 1$, i.e. $[\text{ad}^i f, \sigma] = 0$, where $\vec{f} := f + \text{corr}_\sigma(x)$, then the Itô system is $g$-linearizable if and only if the distribution

$$\tilde{\Delta}_n := \text{span} \left\{ \text{ad}^i g, 0 \leq i \leq n - 1 \right\}$$

is nonsingular on $U$ and the distribution

$$\tilde{\Delta}_{n-1} := \text{span} \left\{ \text{ad}^i g, 0 \leq i \leq n - 2 \right\}$$

is involutive on $U$. If these conditions hold, then a solution $\lambda : \mathbb{R}^n \to \mathbb{R}$ to the set of partial differential equations exists

$$\langle d\lambda, \text{ad}^i f \rangle = 0 \quad \text{for} \quad 0 \leq i \leq n - 2$$

$$\langle d\lambda, \text{ad}^{n-1} f \rangle \neq 0.$$ (131)

the linearizing transformation is given by:

$$T_1 = \lambda$$

$$T_{i+1} = \mathcal{L}_f T_i + \frac{1}{2} \mathcal{L}_\sigma \mathcal{L}_\sigma T_i$$

$$\alpha = \frac{-\mathcal{L}_\sigma \lambda}{\mathcal{L}_g \mathcal{L}_f^{n-1} \lambda}$$

$$\beta = \frac{1}{\mathcal{L}_g \mathcal{L}_f^{n-1} \lambda}.$$ (133)

**Proof.** We will apply the Leibniz rule to the relation of (119) Corollary 30 to expand the term $\mathcal{L}_g T_{i+1}$ for $1 \leq i \leq n - 1$

$$\mathcal{L}_g T_{i+1} = \mathcal{L}_g \left( \mathcal{L}_f T_i + \frac{1}{2} \mathcal{L}_\sigma \mathcal{L}_\sigma T_i \right) = \mathcal{L}_g \mathcal{L}_f T_i + \frac{1}{2} \mathcal{L}_g \mathcal{L}_\sigma \mathcal{L}_\sigma T_i =$$

$$\mathcal{L}_f \mathcal{L}_g T_i - \mathcal{L}_{[f,g]} T_i + \frac{1}{2} \mathcal{L}_\sigma \mathcal{L}_g(\mathcal{L}_\sigma T_i) - \frac{1}{2} \mathcal{L}_{[\sigma,g]}(\mathcal{L}_\sigma T_i) =$$

$$0 - \mathcal{L}_{[f,g]} T_i + \frac{1}{2} \mathcal{L}_\sigma \mathcal{L}_g T_i - \mathcal{L}_{[\sigma,g]} T_i - \frac{1}{2} \mathcal{L}_\sigma \mathcal{L}_{[\sigma,g]} T_i =$$

$$- \mathcal{L}_{[f,g]} T_i + 0 - \frac{1}{2} \mathcal{L}_\sigma \mathcal{L}_{[\sigma,g]} T_i + \frac{1}{2} \mathcal{L}_{[\sigma,[\sigma,g]]} T_i.$$ (136)
If the vectorfields $g$ and $\sigma$ commute, then the second and third terms vanish. If, moreover, $[\sigma, [\vec{f}, g]] = 0$ then
\begin{equation}
L_g T_{i+2} = -L_{[\vec{f}, g]} \left( L_{\vec{f}} T_i + \frac{1}{2} L_\sigma L_\sigma T_i \right) = L_{[\vec{f}, [\vec{f}, g]]} T_i.
\end{equation}
(137)

In general if $[\sigma, \text{ad}^i_{\vec{f}} g] = 0$ for $0 \leq i \leq n - 1$ then
\begin{equation}
L_g T_k = (-1)^k L_{[\text{ad}^i_{\vec{f}} g]} T_1.
\end{equation}
(138)

Thus the equations (119) and (121) will be equivalent to
\begin{align}
\langle d\lambda, \text{ad}^i_{\vec{f}} g \rangle &= 0 \quad \text{for} \quad 0 \leq i \leq n - 2 \\
\langle d\lambda, \text{ad}^{n-1}_{\vec{f}} g \rangle &\neq 0,
\end{align}
(139) (140)
which are of the same form as the equations of Proposition 13 and consequently the conditions from Proposition 14 can be used.

4.5 Itô and Stratonovich $\sigma$-linearization

The stochastic SFB $\sigma$-linearization problem is similar to deterministic SCT linearization. The dispersion vectorfield $\sigma$ transforms in the same way as deterministic drift vectorfields do. Consequently, no Itô term complicates the transformation. Moreover, the Itô and Stratonovich cases are equivalent.

On the other hand, in the SFB $\sigma$-linearization we are free to choose the feedback $\mathcal{F}_{\alpha,\beta}$ that perturbs the drift vectorfield $f$ into $\tilde{f} = f + g\alpha$.

**Proposition 33.** Let $\Theta$ be a SISO stochastic system $\Theta = (f, g, \sigma, U, x_0)$. There is a SFB $\sigma$-linearizing transformation $\mathcal{J}_{T,\alpha,\beta}$ into a $\sigma$-controllable linear system if and only if there is a smooth feedback function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that the deterministic system $(f + g\alpha, \sigma, U, x_0)$ has a SCT linearizing
transformation $T$. Equivalently, there must be such $\alpha$ that the modified odd bracket condition:

$$[\sigma, \text{ad}^{l}_{f + ga} \sigma] = 0 \quad \text{for} \quad l = 1, \ldots, 2n - 1 \quad (141)$$

is satisfied (see (27)). The resulting combined transformation consists of composition of the coordinate transformation $T$ and the feedback $F_{\alpha, \beta}$ where $\beta$ is arbitrary function of $x$; for instance $\beta = 1$.

**Proof.** Compare definition of linearity of a deterministic system with definition of $\sigma$-linearity. The system is is $\sigma$-linearizable if and only if the deterministic systems $(f + ga, \sigma, U, x_{0})$ is SCT linearizable (see Corollary 16).

It is evident that the function $\beta$ (see Figure 4) has no effect on the dispersion part and can be chosen arbitrarily. (Probably nonzero for otherwise the system will be $g$-uncontrollable).

The condition (141) can be expressed in terms of derivatives of $\alpha$ using bracket relations known from differential geometry. For example, for $l = 1$:

$$[\sigma, [f + ga, \sigma]] = [\sigma, [f, \sigma] + [ga, \sigma]] = [\sigma, [f, \sigma]] + [\sigma, [ga, \sigma]] = [\sigma, g + [f + g, \sigma]] + g\mathcal{L}_\sigma \mathcal{L}_\sigma \alpha. \quad (142)$$

The other conditions for $k = 3, 5, 7, \ldots$ can be expressed in a similar way giving the set of $n$ partial differential equations of the order up to $2n$ for example by a computer using symbolic algebra tools. The problem is not very interesting from the practical point of view.

## 5 Example—Crane

In this section the methods of stochastic exact linearization are demonstrated on an example — control of a crane under the influence of random disturbances. The description of the plant was adopted from Ackermann et al. [1] where the model of a crane linearized by approximative methods was studied. Unlike Ackermann, we control the same system using the exact model. Moreover the influence of random disturbances is added.

Consider the crane of Figure 4, which can be used for example for loading containers into a ship. The hook must be automatically placed to a given position. Feedback control is needed in order to dampen the motion before the hook is lowered into the ship. The input signal is the force $u$ that
accelerates the crab. The crab mass is $m_C$, the mass of the load $m_L$, the rope length is $l$, and the gravity acceleration $g$.

We assume that the driving motor has no nonlinearities, there is no friction or slip, no elasticity of the rope and no damping of the pendulum (e.g. from air drag). We will define four state variables: the rope angle $x_1$ (in radian), the angular velocity $x_2 = \dot{x}_1$, the position of the crab $x_3$, and the velocity of the crab $x_4 = \dot{x}_3$. As shown in [1], the plant is described by two second order differential equations:

$$u = (m_L + m_C)\ddot{x}_3 + m_L l(\ddot{x}_1 \cos x_1 - \dot{x}_1^2 \sin x_1)$$  \hspace{1cm} (144)

$$0 = m_L \ddot{x}_3 \cos x_1 + m_L \ddot{x}_1 + m_L g \sin x_1.$$  \hspace{1cm} (145)

Additionally, we assume that the load is under influence of random disturbance, which can be modeled as a white noise process. The disturbance (wind) is horizontal, has zero mean and can be described by the Itô differential $dw$:

$$dx_2 = \frac{F \cos x_1}{m_L l} \, dw,$$  \hspace{1cm} (146)

where $F$ is a constant having the physical unit of force.

We used symbolic algebraic system Mathematica to handle the computations. The complete Mathematica worksheet can be downloaded from the web page of the author \texttt{http://www.tenzor.cz/sladecek}.

Mathematica was used to solve the equations of the system for unknown values $\ddot{x}_2$ and $\ddot{x}_4$ (angular and positional acceleration). Values of vector-fields $f$, $g$, and $\sigma$ were derived as follows:
Figure 6: The State Space Model of Crane

\[
f = \begin{bmatrix} x_2, -\frac{\sin x_1 (g(m_L + m_C) + lm_Lx_2 \cos x_1)}{l(m_C + m_L - m_L \cos^2 x_1)}, x_4, 0 \end{bmatrix}^T
\]

(147)

\[
g = \begin{bmatrix} 0, \frac{-\cos x_1}{l(m_C + m_L - m_L \cos^2 x_1)}, 0, u \end{bmatrix}^T
\]

(148)

\[
\sigma = \begin{bmatrix} 0, \frac{F \cos x_1}{m_L l}, 0, 0 \end{bmatrix}^T.
\]

(149)

The state space model is shown in Figure 6. We can see that the positional state variables \( x_3 \) and \( x_4 \) are isolated from the angular state variables \( x_1 \) and \( x_2 \). Later, we will concentrate on the angular variables pretending that the load will be stabilized no matter where the crane is. Consequently, we obtain only two-dimensional system on which the exact linearization techniques can be demonstrated.

Next, consider the random disturbances. Because the correcting term \( \text{corr}_x(x) \) is zero, there is no difference in using either the Itô or the Stratonovich integral. In case of more “nonlinear” noise, one of the integrals must be selected. If the Itô model is chosen, Theorem 27 must be applied.

Now we evaluate the conditions of Proposition 26 to check that the system is linearizable. In fact, we must only evaluate the non-singularity condition because every one-dimensional distribution is involutive, and the integrability is satisfied automatically. To this end, we will compute the null space (kernel) of the matrix \([[f, g], g]]\), which is empty and therefore the matrix is nonsingular. We conclude, that the deterministic SFB problem is solvable.
Notice, that the system is already in the integrator chain form and hence \( \lambda = x_1 \) satisfies this condition. Therefore, the deterministic system is linearizable by feedback only, with no state space transformation at all, i.e. \( z = T(x) = x \).

This choice of the output function \( \lambda \) is natural but does not cancel the nonlinearity in the dispersion coefficient \( \sigma \). For this purpose, we must use the algorithm of Section 4.2.3 to construct another nontrivial coordinate transformation \( T \).

To obtain this transformation, we must find the space of all functions \( \lambda \) satisfying conditions for feedback linearity (81). Observe that \( \mathcal{L}_g \lambda \) must be zero hence

\[
\frac{\partial \lambda}{\partial x_1} g_1 + \frac{\partial \lambda}{\partial x_2} g_2 = 0.
\]

(150)

Since \( g_1 = 0 \) and \( g_2 \neq 0 \) in neighborhood of \( x_0 \), then \( \frac{\partial \lambda}{\partial x_2} = 0 \) and \( \lambda = c_1(x_1) \) is a function of \( x_1 \) only (i.e. without \( x_2 \)). The coordinate transformation is \( T = [\lambda, \mathcal{L}_f \lambda]^T \). We want to select such \( c_1(x_1) \) that the dispersion vectorfield \( \tilde{\sigma} := T_\ast \sigma \) in the new coordinate system \( z = T(x) \) will be constant:

\[
\frac{\partial c_1(x_1)}{\partial x_1} \frac{F \cos x_1}{m_L l} = \text{constant}.
\]

(151)

We decided to define the constant as \( F/(m_L l) \), therefore

\[
\frac{\partial c_1}{\partial x_1} = \frac{1}{\cos x_1}
\]

(152)

and

\[
T_1 = \lambda = c_1(x_1) = \int \frac{1}{\cos x_1} \, dx_1 = - \ln \left( \cos \frac{1}{2} x_1 - \sin \frac{1}{2} x_1 \right) + \ln \left( \cos \frac{1}{2} x_1 + \sin \frac{1}{2} x_1 \right).
\]

(153)

\[
T_2 = \mathcal{L}_f \lambda = x_2 \sec x_1.
\]

(154)

Finally, we can compute the feedback from (85). In the Mathematica worksheet we validate the results by computing \( \Theta = J_{T,\alpha,\beta} (f, g, \sigma, U, x_0) \). The computation showed that the system \( \left( \hat{f}, \hat{g}, \hat{\sigma}, U, x_0 \right) \) is in the integrator chain form in the \( z \) coordinate chart.
\[ b = \frac{1}{l \left( m_c + l \left( \sin(x_1) \right)^2 \right)} \]
\[ a = \tan(x_1) \left( \sec(x_1) x_2^2 - b g \left( m_c + m_l \right) + l m_l x_2 \cos(x_1) \right) \]  
(155)

6 Conclusion

6.1 Main Results

(i) The structure of the stochastic linearization problem is much richer than the structure of the deterministic one. Two definitions of coordinate transformation exist and there are differences between the \( g \), \( \sigma \), and \( g\sigma \)-linearization.

(ii) In the case of Itô integrals, the coordinate transformation laws are of second order (the Itô rule).

There is a large difference between \( g\sigma \)-linearization and \( g \)-linearization. In the former case the effect of the Itô term can be reduced to the first order operator and consequently the problem is solvable by differential geometry. On the other hand, in the later case there is no easy method to eliminate the Itô term and the a set of second order partial differential equations must be solved to get the linearizing transformation.

(iii) We have given (at least partial) solutions to all SISO SFB problems. The results are listed in Table 1.

(iv) Itô linearization problems can be approached by means of the correcting term. The Itô differential equation can be converted to the Stratonovich equation whose behavior under coordinate transformations is simpler. This method is only partially applicable to the \( g \)-linearization.

| Linearization | \( g \) | \( g\sigma \) | \( \sigma \) |
|---------------|------|-------|-------|
| Deterministic | Theorem p. 14 | p. 13 |
| Stratonovich | Propos. p. 24 | p. 23 | Theorem p. 26 | p. 26 | Propos. p. 33 | p. 34 |
| Itô           | Corollary p. 29 | p. 30 | Theorem p. 27 | p. 28 | Propos. p. 33 | p. 34 |

Table 1: Overview of results — SISO SFB case
(v) An important special case was identified for the Itô $g$-linearization. The case is characterized by commuting control vectorfields $g$ and dispersion vectorfields $\sigma$. Solutions can be found using first order methods.

(vi) Computer algebra proved to be a useful tool for solving exact linearization problems.

(vii) Industrial applications of the exact linearization in general are still unlikely, mainly due to complexity, sensitivity, and limited robustness of the control laws designed by the method.

6.2 Future Research

(i) Find a solution to the Itô $g$-linearization problem in general case, including geometric criteria, using second order geometry.

(ii) Analyze the computability issues; implement a universal symbolic algebra toolbox for the problem.

(iii) Solve the SCT problem.

(iv) Extend the results to the MIMO systems.

(v) Extend the results to the input-output problems and linearization of autonomous systems. Work out the applications of nonlinear filtering.

(vi) Perhaps, some of the results can be used as a starting point for approaching more general class of problems as the problems of disturbance decoupling, input invariance of stochastic non-linear systems, or problems of reachability and observability.

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