The characteristic initial-boundary value problem for the Einstein–massless Vlasov system in spherical symmetry

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Abstract

In this paper, we initiate the study of the asymptotically AdS initial-boundary value problem for the Einstein–massless Vlasov system with $\Lambda < 0$ in spherical symmetry. We will establish the existence and uniqueness of a maximal future development for the characteristic initial-boundary value problem in the case when smooth initial data are prescribed on a future light cone $C^+$ emanating from a point on the center of symmetry $\{r = 0\}$ and a reflecting boundary condition is imposed on conformal infinity $I$. We will then prove a number of continuation criteria for smooth solutions of the spherically symmetric Einstein–massless Vlasov system, under the condition that the ratio $2m/r$ remains small in a neighborhood of $\{r = 0\}$. Finally, we will establish a Cauchy stability statement for Anti-de Sitter spacetime as a solution of the spherically symmetric Einstein–massless Vlasov system under initial perturbations which are small only with respect to a low regularity, scale invariant norm $\| \cdot \|$. This result will imply, in particular, a long time of existence statement for $\| \cdot \|$-small initial data.

This paper provides the necessary tools for addressing the AdS instability conjecture in the setting of the spherically symmetric Einstein–massless Vlasov system, a task which is carried out in our companion paper [16]. However, the results of this paper are also of independent interest.

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1 Introduction
In recent years, the study of the geometry and dynamics of asymptotically Anti-de Sitter solutions \((M^{3+1}, g)\) to
the vacuum Einstein equations

\[ \text{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \]

with a negative cosmological constant \( \Lambda \) has been the subject of intense ongoing research. In the high energy physics
literature, this surge of interest was mainly motivated by the AdS/CFT correspondence conjecture proposed by
Maldacena, Gubser–Klebanov–Polyakov and Witten [15, 9, 21]; see [1, 10, 2] and references therein.

A distinctive feature of any asymptotically AdS spacetime \((M, g)\) is the presence of a conformal boundary \( \mathcal{I} \) at
infinity, which has the conformal structure of a timelike hypersurface (see [11]). In view of the timelike character
of \( \mathcal{I} \), the appropriate framework for the study of asymptotically AdS solutions \((M, g)\) of (1.1) is that of an initial-
boundary value problem, roughly formulated as follows:

- Initial data for \( g \) are prescribed in the form of Cauchy or characteristic data, satisfying, in each case, the
  associated constraint equations for (1.1).

- Boundary conditions are imposed asymptotically on \( \mathcal{I} \), with the requirement that the initial data and the
  boundary conditions satisfy a certain set of compatibility conditions asymptotically on the initial data hyper-
surface.

The well-posedness of the asymptotically AdS initial-boundary value problem for (1.1) was first addressed by
Friedrich [7]. In particular, [7] showed that, for any suitably regular, asymptotically AdS Cauchy data set \((\Sigma^t; \bar{g}, \bar{k})\)
and any smooth Lorentzian conformal structure on \( \mathcal{I} \) (such that a set of compatibility conditions are satisfied, in
an adapted gauge, at \( \mathcal{I} \cap \Sigma \)), there exists, at least locally in time, a unique corresponding solution \( g \) of (1.1), which
is conformally regular up to \( \mathcal{I} \). The broad class of boundary conditions on \( \mathcal{I} \) encoded in terms of the prescribed
conformal structure on \( \mathcal{I} \) contains examples both of reflecting and of dissipative conditions (see also the discussion
in [8] [12]). For an extension of this result in higher dimensions (providing also an alternative proof using wave
coordinates), see [8].

In the presence of matter fields that are not conformally regular up to \( \mathcal{I} \), analogous well-posedness results for the
associated Einstein–matter systems are only known under symmetry assumptions. The study of the well-posedness
of the initial-boundary value problem for the spherically symmetric Einstein–Klein Gordon system

\[ \begin{cases} 
\text{Ric}(g)_{\mu\nu} - \frac{1}{2} R[g] g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}[\varphi], \\
\nabla_{\mu} \varphi - \mu \varphi = 0, \\
T_{\mu\nu}[\varphi] \cdot \partial_{\mu} \varphi \partial_{\nu} \varphi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \varphi \partial^\alpha \varphi,
\end{cases} \tag{1.2} \]

was initiated by Holzegel–Smulevici \[13\], who showed that, when the Klein–Gordon mass \( \mu \) satisfies the Breitenlohner–Freedman bound \( \mu > -\frac{3}{2} |\Lambda| \), the characteristic initial-boundary value problem for \( (1.2) \) with \textit{Dirichlet} conditions on \( I \) is well-posed. In \[14\], this result was extended to a more general class of boundary conditions on \( I \), including conditions which require the initial energy of the scalar field \( \varphi \) to be \textit{infinite}\[1\].

In this paper, we initiate the study of the asymptotically AdS initial-boundary value problem for the Einstein–massless Vlasov system in spherical symmetry. We will consider the case when initial data are prescribed on a future light cone emanating from a point in the center of symmetry and a reflecting boundary condition is imposed on \( I \). In this setting, we will establish the well-posedness of the initial-boundary value problem for \textit{smooth} initial data. We will also obtain a number of extension principles for smooth developments, presented in terms of scale invariant quantities associated to the evolution.

Our final result will be a Cauchy stability statement for Anti-de Sitter spacetime as a solution of the Einstein–massless Vlasov system, under spherically symmetric perturbations which are initially small with respect to a low-regularity, scale-invariant norm. This result provides the necessary first step in the direction of addressing the AdS instability conjecture in the setting of the spherically symmetric Einstein–massless Vlasov system, with respect to a low regularity initial data topology; this task is carried out in our companion paper \[16\].

1.1 Statement of the main results

The Einstein–massless Vlasov system for a \( 3+1 \) dimensional Lorentzian manifold \( (\mathcal{M}, g) \) and a non-negative measure \( f \) on \( T\mathcal{M} \) takes the form

\[ \begin{cases} 
\text{Ric}(g)_{\mu\nu} - \frac{1}{2} R[g] g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}[f], \\
\mathcal{L}(g) f = 0, \\
\text{supp}(f) \subset N^+ \end{cases} \tag{1.3} \]

where \( T_{\mu\nu}[f] \) is the Vlasov energy momentum tensor (defined by the relation \( (2.23) \)), \( \mathcal{L}(g) \in \Gamma(TTM) \) is the geodesic spray of \( g \) and \( N^+ \subset T\mathcal{M} \) is the set of future directed null vectors. The trivial solution of \( (1.3) \) is \textit{Anti-de Sitter} spacetime \( (\mathcal{M}_{AdS}; g_{AdS}; 0) \), where \( \mathcal{M}_{AdS} \simeq \mathbb{R}^{3+1} \) and, in the standard double null coordinate chart \((u, v, \theta, \varphi)\) on \( \mathcal{M}_{AdS} \):

\[ g_{AdS} = -\Omega^2_{AdS}(u, v) dv du + r^2_{AdS}(u, v) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \]

where

\[ \begin{align*}
\Omega_{AdS}(u, v) &= \sqrt{-\frac{3}{\Lambda} \tan \left( \frac{1}{2} \sqrt{-\frac{\Lambda}{3}} (v - u) \right)}, \\
r_{AdS}(u, v) &= 1 - \frac{1}{3} \Lambda r^2_{AdS}(u, v). \end{align*} \tag{1.4} \]

In this paper, we will only consider the case when \( \Lambda < 0 \) and \((\mathcal{M}, g; f)\) is \textit{spherically symmetric} and \textit{asymptotically Anti-de Sitter}, with regular axis of symmetry \( Z \neq \emptyset \) and regular conformal infinity \( I \); for the relevant definitions, see Section \[2\]. On this class of spacetimes, a \textit{reflecting} boundary condition for \( (1.3) \) can be naturally defined by the requirement that \( f \) is conserved along the reflection of null geodesics off \( I \) (see Section \[2.5\]).

We will establish a number of results related to the dynamics of spherically symmetric and asymptotically AdS solutions of the system \( (1.3) \), including:

\[1\] In the case of the \textit{linear} Klein–Gordon equation on general asymptotically AdS backgrounds, well-posedness under Dirichlet boundary conditions with no symmetry assumptions was established earlier by Vasy \[19\]; well-posedness for a broader class of boundary conditions (again, without any symmetry conditions) was shown Warnick \[20\].

3
• A fundamental well-posedness result for the spherically symmetric, asymptotically AdS characteristic initial-boundary value problem for (1.3), with initial data prescribed on a future light cone,

• A number of continuation criteria for smooth solutions of (1.3) in spherical symmetry and

• A Cauchy stability statement for the trivial solution of (1.3) in a low regularity, scale invariant topology in spherical symmetry.

We will now proceed to briefly present the main results of this paper.

1.1.1 Well-posedness for smoothly compatible characteristic initial data

In this paper, we will consider the spherically symmetric characteristic initial-boundary value problem for (1.3), obtained by prescribing characteristic initial data for \(g, f\) on a future light cone \(C^+\) emanating from a point \(p \in \mathcal{Z}\) and imposing the reflecting boundary condition on \(\mathcal{I}\). In terms of a spherically symmetric, double null coordinate chart \((u, v, \theta, \varphi)\) on \((\mathcal{M}, g)\), where the level sets of the optical functions \(u\) and \(v\) are future light cones and past light cones, respectively, of points lying on \(\mathcal{Z}\), a characteristic initial data set for \(g, f\) on \(C^+\) consists of a triplet \((r, \Omega^2; f)|_{u=0}\) defined along \(u = 0\) and satisfying the characteristic constraint equations for (1.3), where \((r, \Omega^2)|_{u=0}\) are the restrictions on \(u = 0\) of the components of a spherically symmetric metric

\[ g = -\Omega^2(u, v)du dv + r^2(u, v)(d\theta^2 + \sin^2 \theta d\varphi^2). \]

We will obtain the following fundamental well-posedness result:

**Theorem 1.1.** Let \((r, \Omega^2; f)|_{u=0}\) be a regular, spherically symmetric and asymptotically AdS characteristic initial data set for (1.3) on \(\{u = 0\}\), admitting a smooth expression in a smoothly compatible gauge. Assume, in addition, \(f|_{u=0}\) has bounded support in phase space. Then, there exists a unique, maximal solution \((\mathcal{M}, g; f)\) of (1.3) inducing the given initial data on \(\{u = 0\} \subset \mathcal{M}\) and satisfying the reflecting boundary condition on \(\mathcal{I}\).

For a more detailed statement of Theorem 1.1, see Theorem 4.1 and Corollary 4.2. For the definition of the smooth compatibility gauge condition for \((r, \Omega^2; f)|_{u=0}\), as well as for a detailed discussion on the gauge choices involved in the statement of Theorem 1.1, see Section 3.2.

1.1.2 Continuation criteria for smooth solutions

Having established the well-posedness of the characteristic initial-boundary value problem for (1.3), we will proceed to obtain a number of continuation criteria for smooth solutions \((\mathcal{M}, g; f)\) of (1.3). Our most technically involved continuation criterion, providing an extension principle in a neighborhood of the axis \(\mathcal{Z}\), will be the following:

**Theorem 1.2.** Let \((\mathcal{M}, g; f)\) be a smooth spherically symmetric solution of (1.3) with a non-empty axis \(\mathcal{Z}\), and let \(J^- (\mathcal{Z}) \subset \mathcal{M}\) be the past domain of influence of \(\mathcal{Z}\). Let also \((u, v)\) be a double null coordinate pair on \((\mathcal{M}, g)\) as in Section 1.1.1. Assume that there exists a point \(p \in \mathcal{Z}\) such that the following conditions are satisfied on the light cone \(\{u = u(p)\}\):

- \(\{u = u(p)\} \cap J^- (\mathcal{Z})\) has compact closure in \(\{u = u(p)\}\),
- \(f|_{\{u = u(p)\} \cap J^- (\mathcal{Z})}\) has bounded support in phase space

Assume, moreover, that \((\mathcal{M}, g; f)\) satisfies the scale invariant condition

\[ \sup_{J^- (\mathcal{Z}) \cap \{u = u(p)\}} \frac{2m}{r} \ll 1, \]

where the Hawking mass \(m\) is defined by (2.10). Then, there exists a smooth, spherically symmetric solution \((\mathcal{M}, g; f)\) of (1.3) which strictly extends \((\mathcal{M}, g; f)\) to the future along \(\mathcal{Z}\).
For a more detailed statement of Theorem 1.2 see Theorem 5.1.

As a Corollary of Proposition 1.2 (and a number of related extension principles), we will infer the following general continuation criterion for the domain of outer communications of \((\mathcal{M}, g; f)\):

**Corollary 1.1.** Let \((\mathcal{M}, g; f)\) be a smooth, spherically symmetric and asymptotically AdS solution of (1.3) satisfying the reflecting boundary condition on conformal infinity \(\mathcal{I}\), such that \((\mathcal{M}, g; f)\) is the maximal future development of a characteristic initial data set \((r, \Omega^2; f)|_{\{u=0\}}\) with \(f|_{\{u=0\}}\) of bounded support in phase space, as in Theorem 1.1.

Let us also fix a double null pair \(u, v\) satisfying the additional gauge condition

\[
\begin{align*}
  u &= v \text{ on } Z, \\
  u &= v - v_T \text{ on } \mathcal{I}
\end{align*}
\]

for some given \(v_T > 0\).

Assume that, for some \(u_* > 0\), the projection of \(\mathcal{M}\) in the \((u, v)\)-plane contains the domain

\[
\mathcal{U}_{u_*, v_T} \triangleq \{ 0 \leq u < u_* \} \cap \{ u \leq v < u + v_T \}.
\]

If \((\mathcal{M}, g; f)\) satisfies the scale invariant conditions

\[
(1.5) \quad \sup_{\mathcal{U}_{u_*, v_T}} \frac{2m}{r} < 1
\]

and

\[
(1.7) \quad \limsup_{(u,v) \to (u_*, u_*)} \frac{2m}{r} \ll 1,
\]

where the Hawking mass \(m\) is defined by (2.10), then there exists a \(\bar{u}_* > u_*\) such that the projection of \(\mathcal{M}\) in the \((u, v)\)-plane contains the larger domain \(\mathcal{U}_{\bar{u}_*, v_T} \supset \mathcal{U}_{u_*, v_T}\).

For a more detailed statement of Corollary 1.1, see Corollary 5.1.

### 1.1.3 Cauchy stability of AdS in a scale invariant initial data topology

Our final result will be a low-regularity Cauchy stability statement for the trivial solution \((\mathcal{M}_{\text{AdS}}, g_{\text{AdS}}; 0)\) of (1.3) in spherical symmetry. In particular, we will consider the following norm on the space of spherically symmetric characteristic initial data \((r, \Omega^2; f)|_{\{u=0\}}\) for (1.3) satisfying the conditions of Theorem 1.1.

**Definition.** For any characteristic initial data set \((r, \Omega^2; f)|_{\{u=0\}}\) as in Theorem 1.1 let \(f^{(\text{AdS})}\) be the solution of the (free) Vlasov field equation

\[
\mathcal{L}^{(g_{\text{AdS}})} f^{(\text{AdS})} = 0
\]

on Anti-de Sitter spacetime \((\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})\) arising from initial data \(f_{\text{AdS}}|_{C_{\text{AdS}}^+}\) on a future light cone \(C_{\text{AdS}}^+ \subset \mathcal{M}_{\text{AdS}}\) which are obtained from \(f|_{\{u=0\}}\) through the choice of a suitable normalising gauge condition along \(\{ u = 0 \} \). We will define the “norm” \(\| (r, \Omega^2; f)|_{\{u=0\}} \|\) of \((r, \Omega^2; f)|_{\{u=0\}}\) by the following relation measuring the concentration of energy occurring in the evolution of the free Vlasov field \(f^{(\text{AdS})}\) in the region \(\{ u \geq 0 \} \) of \((\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})\):

\[
\begin{align*}
  \| (r, \Omega^2; f)|_{\{u=0\}} \| &\triangleq \sup_{U \geq 0} \int_U - \frac{\sqrt{-\chi}}{r} \left( T^{\nu \rho}_{\nu \rho}[f^{(\text{AdS})}](U, v) + T^{\nu \rho}_{\nu \rho}[f^{(\text{AdS})}](U, v) \right) dv^+ \\
  &\quad + \sup_{V, \lambda \geq 0} \int_{\max(0, V - \sqrt{-\chi})}^V \left( T^{\nu \rho}_{\nu \rho}[f^{(\text{AdS})}](U, V) + T^{\nu \rho}_{\nu \rho}[f^{(\text{AdS})}](U, V) \right) du^+ \\
  &\quad + \sqrt{-\Lambda \tilde{m}_I}|_{\mathcal{I}},
\end{align*}
\]

where \(T^{\nu \rho}_{\nu \rho}[f^{(\text{AdS})}]\) are the components of the energy momentum tensor associated to the free Vlasov \(f^{(\text{AdS})}\) on \((\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})\) and \(\tilde{m}_I|_{\mathcal{I}}\) is the value of the (renormalised) Hawking mass of the initial data set \((r, \Omega^2; f)|_{\{u=0\}}\) at infinity.
For the precise definition of the functional $\| \cdot \|$, as well as for a discussion on the properties of the resulting initial data topology, see Definition 6.3 in Section 6.

We will establish the following Cauchy stability statement of the trivial solution with respect to the initial data topology defined by the norm $\| \cdot \|$:

**Theorem 1.3.** For any $U > 0$, there exists an $\varepsilon > 0$, such that, for every characteristic initial data set $(r, \Omega^2; f)\{u=0\}$ as in Theorem 1.1 satisfying the smallness condition

$$\|(r, \Omega^2; f)\{u=0\}\| < \varepsilon,$$

the corresponding maximal future development $(\mathcal{M}, g; f)$ solving (1.3) with the reflecting boundary condition on $\mathcal{I}$ satisfies the following conditions:

- **Long time of existence.** Fixing a double null coordinate pair $(u, v)$ on $\mathcal{M}$ satisfying the gauge condition (1.5) with $v = \sqrt{-\frac{3}{\Lambda}} \pi$, the projection of $\mathcal{M}$ in the $(u, v)$-plane contains the domain

$$U_U := \{0 \leq u \leq U\} \cap \{u < v < u + \sqrt{-\frac{3}{\Lambda}} \pi\}.$$

- **Cauchy stability estimates.** In the region $\{0 \leq u \leq U\}$, the solution $(\mathcal{M}, g; f)$ satisfies the scale invariant bounds

$$\sup_{u, \epsilon \in (0, U)} \|(r, \Omega^2; f)\{u=\epsilon\}\| \leq C \varepsilon \tag{1.10}$$

and

$$\sup_{0 \leq u \leq U} \left( \frac{2 \tilde{m}}{r} (u, v) \right) < C \varepsilon, \tag{1.11}$$

where $C > 0$ is an absolute constant, $(r, \Omega^2; f)\{u=\epsilon\}$ are the initial data induced by $(\mathcal{M}, g; f)$ on the cone $\{u = \epsilon\}$ and $\tilde{m}(u, v)$ is the renormalised Hawking mass associated to the sphere $\{u, v = \text{const}\}$. In particular, $(g; f)$ remains close to $(g_{\text{AdS}}; 0)$ in the region $\{0 \leq u \leq U\}$ with respect to the topology defined by (1.8) on the slices $\{u = \epsilon\}, \epsilon \in [0, U]$.

For a more detailed statement of Theorem 1.3, see Theorem 6.1.

### 1.2 Outline of the paper

This paper is organised as follows:

- In Section 2, we will introduce the class of spherically symmetric and asymptotically AdS solutions of the Einstein–massless Vlasov system (1.3), expressed in a double null coordinate chart. We will also define the notion of a reflecting boundary condition for (1.3) at conformal infinity, and we will present some fundamental identities for the null geodesic flow on spherically symmetric solutions of (1.3).

- In Section 3, we will introduce the notion of a regular solution of (1.3) with smooth axis and smooth conformal infinity. We will then proceed to set up the asymptotically AdS, characteristic initial-boundary value problem for (1.3) with a reflecting boundary condition at infinity. In particular, we will define the class of smoothly compatible initial data sets for (1.3), and we will inspect the properties of gauge transformations mapping smoothly compatible data to data satisfying a certain gauge normalisation condition, which allows one to uniquely determine an initial data set in terms of a freely prescribed initial datum for the Vlasov field; the loss of smooth compatibility under gauge normalisation will be also discussed.
• In Section 4, we will establish the well-posedness of the characteristic initial-boundary value problem for (1.3) in spherical symmetry, in the class of smoothly compatible initial data. In particular, we will prove Theorem 1.1.

• In Section 5, we will establish a number of continuation criteria for smooth solutions of the system (1.3) in spherical symmetry. Among the results obtained in Section 5, we will prove Theorem 1.2 and Corollary 1.1.

• In Section 6, we will introduce the scale invariant norm \( \| \cdot \| \) on the space of smoothly compatible initial data sets for (1.3). In the topology defined by \( \| \cdot \| \), we will establish the Cauchy stability statement of Theorem 1.3.

• In Section A of the Appendix, we will review some fundamental identities related to the geodesic flow on Anti-de Sitter spacetime.

• Finally, in Section B of the Appendix, we will establish that, in the presence of a black hole region, any maximally extended, spherically symmetric and asymptotically AdS solution of (1.3) possesses a future complete conformal infinity.

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2 Spherically symmetric spacetimes and the Einstein–massless Vlasov system

This section will be mainly devoted to reviewing the properties of the class of spherically symmetric and asymptotically AdS solutions to the Einstein–massless Vlasov system. We will follow similar conventions as those adopted by Dafermos–Rendall in [4], expressing the Einstein–massless Vlasov system system in a double null coordinate chart. In particular, after reviewing the general properties of spherically symmetric and asymptotically AdS spacetimes, we will proceed to fix our notations regarding the Vlasov field equation and the Einstein–Vlasov system, listing a number of fundamental identities that will be useful later in this paper. We will conclude this section by introducing the reflecting boundary condition for a massless Vlasov field along conformal infinity.

2.1 Spherically symmetric spacetimes in double null coordinates and Anti-de Sitter spacetime

In this paper, we will follow similar conventions as [18, 17] regarding double null coordinate charts on spherically symmetric spacetimes.

Let \((M^{3+1}, g)\) be a connected, time oriented, smooth Lorentzian manifold which is spherically symmetric, i.e. there exists a smooth isometric action \(A : M \times SO(3) \to M\) of \(SO(3)\) on \((M, g)\) such that, for each \(p \in M\), \(\text{Orb}(p) \approx S^2\) or \(\text{Orb}(p) = \{p\}\), and at least one point \(p \in M\) has a non-trivial orbit. We will define the axis \(Z\) of \((M, g)\) to be the set of fixed points of \(A\), i.e.

\[
Z = \{ p \in M : A(p, s) = p \, \forall s \in SO(3) \}.
\]

Using the fact that \(M\) is connected, \(g\) is a Lorentzian metric and \(A\) maps geodesics of \((M, g)\) to geodesics, it can be readily shown that \(Z\) consists of a disjoint union of timelike geodesics. For the rest of this paper, we will only consider the case when \(Z\) consists of a single timelike geodesic and \(M \setminus Z\) splits diffeomorphically under the action \(A\) as

\[
M \setminus Z \approx U \times S^2,
\]

(2.1)
where $\mathcal{U}$ is a smooth 2 dimensional manifold. Furthermore, we will restrict to spacetimes $(\mathcal{M}, g)$ such that

$$\bigcup_{p \in \mathbb{Z}} C^+(p) \cup C^-(p) = \mathcal{M},$$

where $C^+(p), C^-(p)$ denote the future and past light cones emanating from $p$, respectively.

Under the above assumptions on $(\mathcal{M}, g)$, it can be readily deduced that the two families of spherically symmetric null hypersurfaces $\mathcal{H} = \{ C^+(p) : p \in \mathbb{Z} \}$ and $\overline{\mathcal{H}} = \{ C^-(p) : p \in \mathbb{Z} \}$ foliate regularly the region $\mathcal{M}\setminus\mathcal{Z}$. A pair of continuous functions $u, v : \mathcal{M} \to \mathbb{R}$ which are a smooth parametrization of the foliations $\mathcal{H}, \overline{\mathcal{H}}$, respectively, on $\mathcal{M}\setminus\mathcal{Z}$ will be called a **double null coordinate pair**. Note that any double null coordinate pair $(u, v)$ on $(\mathcal{M}, g)$ can be naturally viewed as a smooth coordinate chart on $\mathcal{U}$.

**Remark.** We will only consider double null coordinate pairs $(u, v)$ which are compatible with the chosen time orientation of $(\mathcal{M}, g)$, i.e. coordinate pairs such that both $u, v$ are increasing functions along any future directed timelike curve. In this case, the vector field $\partial_u + \partial_v$ on $\mathcal{M}\setminus\mathcal{Z} \cong \mathcal{U} \times \mathbb{S}^2$ is timelike and future directed.

Given any double null coordinate pair on $(\mathcal{M}, g)$, it readily follows that the metric $g$ takes the following form on $\mathcal{M}\setminus\mathcal{Z}$:

$$g = -\Omega^2(u, v)dudv + r^2(u, v)g_{\mathbb{S}^2},$$

where $g_{\mathbb{S}^2}$ is the standard round metric on $\mathbb{S}^2$ and $\Omega, r : \mathcal{U} \to (0, +\infty)$ are smooth functions.

**Remark.** Viewed as a function on $\mathcal{M}\setminus\mathcal{Z}$, $r$ is expressed geometrically as

$$r(p) = \sqrt{\frac{\text{Area}(\text{Orb}(p))}{4\pi}}.$$

As a result, $r$ extends continuously to 0 on $\mathcal{Z}$.

For any pair of smooth functions $U, V : \mathbb{R} \to \mathbb{R}$ with $U', V' \neq 0$, in the new double null coordinate pair

$$(\bar{u}, \bar{v}) = (U(u), V(v)),$$

the metric $g$ takes the form

$$g = -\bar{\Omega}^2(\bar{u}, \bar{v})d\bar{u}d\bar{v} + r^2(\bar{u}, \bar{v})g_{\mathbb{S}^2},$$

where

$$\bar{\Omega}^2(\bar{u}, \bar{v}) = \frac{1}{U'V'}\Omega^2(U^{-1}(\bar{u}), V^{-1}(\bar{v})),
$$

$$r(\bar{u}, \bar{v}) = r(U^{-1}(\bar{u}), V^{-1}(\bar{v})).$$

We will also define the **Hawking mass** $m : \mathcal{M} \to \mathbb{R}$ by the expression

$$m = \frac{r}{2} \left(1 - g(\nabla r, \nabla r)\right).$$

Viewed as a function on $\mathcal{U}$, $m$ is related to $\Omega$ and $r$ by the following formula

$$m = \frac{r}{2} \left(1 + 4\Omega^{-2}\partial_u r \partial_v r\right) \Leftrightarrow \Omega^2 = \frac{4\partial_u r (-\partial_v r)}{1 - \frac{2m}{r}}.$$

In any local coordinate chart $(y^1, y^2)$ on $\mathbb{S}^2$, the non-zero Christoffel symbols of \(2.3\) in the $(u, v, y^1, y^2)$ local coordinate chart on $\mathcal{M}\setminus\mathcal{Z}$ are computed as follows:

$$\Gamma^u_{uu} = \partial_u \log(\Omega^2), \quad \Gamma^v_{uv} = \partial_v \log(\Omega^2),$$

$$\Gamma^u_{AB} = \Omega^{-2} \partial_u (r^2)(g_{\mathbb{S}^2})_{AB}, \quad \Gamma^v_{AB} = \Omega^{-2} \partial_v (r^2)(g_{\mathbb{S}^2})_{AB},$$

$$\Gamma^A_{uB} = r^{-1} \partial_u r \delta^A_B, \quad \Gamma^A_{vB} = r^{-1} \partial_v r \delta^A_B,$$

$$\Gamma^A_{BC} = (\delta^A_{\mathbb{S}^2})^A_{BC}.$$
where the latin indices $A, B, C$ are associated to the spherical coordinates $y^1, y^2$, $\delta^{ij}$ is Kronecker delta and $\Gamma_{ij}^k$ are the Christoffel symbols of the round sphere in the $(y^1, y^2)$ coordinate chart.

A fundamental example of a family of spherically symmetric spacetimes admitting a globally defined double null coordinate pair is the family of Anti-de Sitter spacetimes $(M_{AdS}, g_{AdS})$, parametrised by the value $\Lambda < 0$ of the cosmological constant. In polar coordinates $(t, \bar{r}, \theta, \varphi)$ on $M_{AdS} \simeq \mathbb{R}^{3+1}$, $g_{AdS}$ takes the form

$$
(2.12) \quad g_{AdS}^{(A)} = -(1 - \frac{1}{3}\Lambda\bar{r}^2)dt^2 + \left(1 - \frac{1}{3}\Lambda\bar{r}^2\right)^{-1}d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\varphi^2).
$$

In this paper, we will usually drop the superscript $\Lambda$ in (2.12). The standard polar coordinate pair $(u, v)$ on $(M_{AdS}, g_{AdS})$, defined by

$$
\begin{align*}
    du &= dt - \frac{d\bar{r}}{1 - \frac{1}{3}\Lambda\bar{r}^2}, \\
    dv &= dt + \frac{d\bar{r}}{1 - \frac{1}{3}\Lambda\bar{r}^2},
\end{align*}
$$

maps the manifold $M_{AdS}\backslash \mathcal{Z} \simeq \mathbb{R}^{3+1}\backslash \{r = 0\}$ to the planar domain

$$
\mathcal{U}_{AdS} = \left\{u < v < u + \sqrt{\frac{3}{\Lambda}}\right\}.
$$

In these coordinates, the metric coefficients (2.3) associated to $g_{AdS}$ take the form [14].

### 2.2 Asymptotically AdS spacetimes

Let $(M, g)$ be a spherically symmetric spacetime as in Section 2.1. Recall that $M\backslash \mathcal{Z}$ splits topologically as the product

$$
M\backslash \mathcal{Z} \simeq \mathcal{U} \times S^2
$$

and, in any spherically symmetric double null coordinate chart on $M\backslash \mathcal{Z}$, the metric $g$ splits as (2.3). Note also that any choice of double null coordinate pair $(u, v)$ on $M$ fixes a smooth embedding $(u, v) : \mathcal{U} \rightarrow \mathbb{R}^2$. In this section, we will identify $\mathcal{U}$ with its image in $\mathbb{R}^2$ associated to a given null coordinate pair.

**Definition 2.1.** Let $(M, g)$ be a spacetime as above with $\sup_M r = +\infty$. We will call $(M, g)$ *asymptotically AdS* if, for some $R_0 \gg 1$, there exists a spherically symmetric double null coordinate pair $(u, v)$ on $M$ covering the whole region $\mathcal{V}_{as} = \{r \geq R_0\}$, with the following properties:

1. The region $\mathcal{V}_{as}$ has the form

$$
\mathcal{V}_{as} = \{u_1 < u < u_2\} \cap \{u + v_{R_0}(u) \leq v < u + v_{\Gamma}\}
$$

   for some $u_1 < u_2 \in \mathbb{R} \cup \{-\infty\}$, $v_{\Gamma} \in \mathbb{R}$ and $v_{R_0} : (u_1, u_2) \rightarrow \mathbb{R}$ with $v(u) < v_{\Gamma}$ (see Figure 2.1).

2. The function $\frac{1}{r}$ on $\mathcal{U}$ extends smoothly on

$$
\mathcal{I} \doteq \{u_1 < u < u_2\} \cap \{v = u + v_{\Gamma}\} \subset \text{ clos}(\mathcal{U})
$$

   with

$$
\frac{1}{r} \big|_{\mathcal{I}} = 0.
$$

3. The renormalised metric coefficient $r^{-2}\Omega^2$ extends smoothly on $\mathcal{I}$, with

$$
r^{-2}\Omega^2 \big|_{\mathcal{I}} \neq 0.
$$
Figure 2.1: Schematic depiction of the asymptotic region $\mathcal{V}_{as} = \{ r \geq R_0 \gg 1 \}$ of an asymptotically AdS spacetime. The function $r$ extends to $+\infty$ on conformal infinity $I = \{ v = u + v_I \}$.

Remark. The boundary condition (2.14) implies that

\[ \partial_v \left( \frac{1}{r} \right) \bigg|_I = -\partial_u \left( \frac{1}{r} \right) \bigg|_I. \]

In the class of spacetimes introduced in Section 2.1, Definition 2.1 coincides with the standard definition of asymptotically AdS spacetimes (see e.g. [7]). In particular, for an asymptotically AdS spacetime $(M, g)$ as above, the conformal metric

\[ \tilde{g} = r^{-2} g \]

in the region $\{ r \geq R_0 \}$ admits a smooth extension across $r = +\infty$, with $I^{(2+1)} = \{ r = +\infty \}$ corresponding to a timelike conformal boundary of $(M, g)$. The isometric action of $SO(3)$ on $(M, g)$ by rotations extends to a smooth isometric action on the manifold with boundary $(\tilde{M}, \tilde{g})$, where

\[ \tilde{M} = M \cup I^{(2+1)}. \]

In this extension, $I$ defined by (2.13) corresponds to the spherical quotient of $I^{(2+1)}$. We will use the term conformal infinity for both $I^{(2+1)}$ and $I$.

2.3 The massless Vlasov equation in spherical symmetry

Let $(M, g)$ be as in Section 2.1. In any local coordinate chart $(x^0, x^1, x^2, x^3)$ on $M$ with associated momentum coordinates $(p^0, p^1, p^2, p^3)$ on the fibers of $TM$, the geodesic flow takes the form

\[ \frac{dx^\alpha}{ds} = p^\alpha, \quad \frac{dp^\alpha}{ds} + \Gamma_\beta^\alpha p^\beta p^\gamma = 0, \]

where $\Gamma_\beta^\alpha$ are the Christoffel symbols of $g$ in the coordinates $(x^0, x^1, x^2, x^3)$. The set $P^+ \subset TM$ of future directed null vectors, i.e. the subset of $TM$ where

\[ g_{\beta\gamma}(x)p^\beta p^\gamma = 0 \text{ and } g_{\beta\gamma}(x)p^\gamma Q^\beta(x) \leq 0, \]
where $Q$ is a fixed, non-vanishing future directed vector field on $\mathcal{M}$, is preserved under (2.17).

In view of the spherical symmetry of $(\mathcal{M}, g)$, the angular momentum function $l: T\mathcal{M} \to [0, +\infty)$ defined (in a local coordinate chart $(u, v, y^1, y^2)$ as in Section 2.1) by

$$ l^2 \equiv r^2 g_{AB} p^A p^B = r^4 (g_{\tilde{B}2})_{AB} p^A p^B $$

is a constant of motion for the geodesic flow (2.17). Thus, the geodesic flow equations (2.17) can be reduced to a system depending only on the parameters $u$, $v$, $p^u$, $p^v$ and $l$. In terms of these parameters, condition (2.18) is expressed as

$$ \Omega^2 p^u p^v = \frac{l^2}{r^2} \quad \text{and} \quad p^n \geq 0 $$

while (2.17) for null geodesics is reduced to

$$ \begin{aligned}
\frac{du}{ds} &= p^u, \\
\frac{dv}{ds} &= p^v, \\
\frac{d}{ds} (\Omega^2 p^u) &= \left( \partial_v \log(\Omega^2) - 2 \frac{\partial r}{r} \right) \frac{l^2}{r^2}, \\
\frac{d}{ds} (\Omega^2 p^v) &= \left( \partial_u \log(\Omega^2) - 2 \frac{\partial r}{r} \right) \frac{l^2}{r^2}, \\
\frac{dl}{ds} &= 0.
\end{aligned} $$

Note that the relations (2.20) and (2.21) imply that, on a smooth spacetime $(\mathcal{M}, g)$ as above, a geodesic $\gamma$ with angular momentum $l > 0$ cannot reach the axis $r = 0$.

Remark. We will frequently identify a geodesic $\gamma$ in $\mathcal{M}$ with its image in $\mathcal{U}$. For this reason, we will frequently refer to (2.21) simply as the equations of motion for a geodesic in $\mathcal{U}$.

A Vlasov field $f$ on $(\mathcal{M}, g)$ is a non-negative measure on $T\mathcal{M}$ which is invariant under the geodesic flow (2.17). In particular, in any coordinate chart $(x^\alpha; p^n)$ on $T\mathcal{M}$, a Vlasov field $f$ satisfies (as a distribution) the equation

$$ p^n \partial_{x^n} f - \Gamma^\alpha_{\beta\gamma} p^\beta \partial_{p^n} f = 0. $$

A Vlasov field $f$ supported on (2.18) will be called a massless Vlasov field.

Associated to any Vlasov field $f$ is a symmetric $(0, 2)$-form on $\mathcal{M}$, the energy momentum tensor of $f$, given (formally) by the expression

$$ T_{\alpha\beta}(x) = \int_{\pi^{-1}(x)} p_\alpha p_\beta f \sqrt{-\det(g(x))} dp^0 ... dp^3, $$

where $\pi^{-1}(x)$ denotes the fiber of $T\mathcal{M}$ over $x \in \mathcal{M}$ and the indices of the momentum coordinates are lowered with the use of the metric $g$, i.e.

$$ p_\gamma = g_{\gamma\delta}(x) p^\delta. $$

Equation (2.22) implies that $T_{\alpha\beta}$ is conserved, i.e.:

$$ \nabla^\alpha T_{\alpha\beta} = 0. $$

Furthermore, for any Vlasov field $f$ on $(\mathcal{M}, g)$, associated to each open set $\mathcal{V} \subseteq T\mathcal{M}$ is a 1-form $N^{(\mathcal{V})}$ called the the particle current, expressed in any coordinate chart $(x^\alpha; p^n)$ on $T\mathcal{M}$ as

$$ N^{(\mathcal{V})}_\alpha(x) = \int_{T_x\mathcal{M} \cap \mathcal{V}} p_\alpha f \sqrt{-\det(g(x))} dp^0 ... dp^3. $$

Note that (2.19) is coordinate independent.
In the case when $\mathcal{V}$ is invariant under the geodesic flow, \((2.26)\) is conserved:

\[ \nabla^\alpha N_a^{(\mathcal{V})} = 0. \]

We will denote

\[ N = N^{(TM)}. \]

**Remark.** We will only consider smooth Vlasov fields $f$ which decay sufficiently fast as $p^a \to +\infty$ (for any fixed $x$). In this case, the expressions \((2.25)\) and \((2.28)\) are finite and depend smoothly on $x \in \mathcal{M}$.

It can be readily inferred that a spherically symmetric Vlasov field $f$, i.e. a Vlasov field which is invariant under the induced action of $SO(3)$ on $TM$, only depends on $u$, $v$, $p^u$, $p^v$ and $l$, and, in the massless case, is conserved along the flow lines of the reduced system \((2.21)\). In particular, a smooth spherically symmetric massless Vlasov field $f$ will necessarily be of the form

\[ f(u, v; p^u, p^v, l) = \bar{f}(u, v; p^u, p^v, l) \cdot \delta\left(\Omega^2 p^u p^v - \frac{l^2}{r^2}\right), \]

where $\bar{f}$ is smooth in its variables and $\delta$ is Dirac’s delta function. For a spherically symmetric Vlasov field $f$, equation \((2.22)\) takes the form:

\[ p^u \partial_u f + p^v \partial_v f = \left(\partial_u \log(\Omega^2)(p^v)^2 + \frac{2}{r} \Omega^2 \partial_v r \frac{l^2}{r^2}\right) \partial_{p^v} f + \left(\partial_v \log(\Omega^2)(p^u)^2 + \frac{2}{r} \Omega^2 \partial_u r \frac{l^2}{r^2}\right) \partial_{p^u} f. \]

**Remark.** Given a smooth, spherically symmetric massless Vlasov field $f$, we will frequently denote with $\bar{f}$ any smooth function for which \((2.29)\) holds. Note that $\bar{f}$ is uniquely determined only along the future null set $\mathcal{P}^+$.

For a smooth, spherically symmetric massless Vlasov field, the energy-momentum tensor \((2.23)\) is of the form

\[ T = T_{uu}(u, v) du^2 + 2T_{uv}(u, v) du dv + T_{vv}(u, v) dv^2 + T_{AB}(u, v) dy^A dy^B \]

and the components of \((2.31)\) can be expressed in terms of the variables $p^u, p^v, l$ as

\[ T_{uu} = \frac{\pi}{2} r^{-2} \int_0^\infty \int_0^\infty \left(\Omega^2 p^v\right)^2 \bar{f}(u, v; p^u, p^v, l) \left|_{\Omega^2 p^u p^v + \frac{l^2}{r^2}} \frac{dp^u}{p^u} \right| dldl, \]

\[ T_{vv} = \frac{\pi}{2} r^{-2} \int_0^\infty \int_0^\infty \left(\Omega^2 p^u\right)^2 \bar{f}(u, v; p^u, p^v, l) \left|_{\Omega^2 p^u p^v + \frac{l^2}{r^2}} \frac{dp^v}{p^v} \right| dldl, \]

\[ T_{uv} = \frac{\pi}{2} r^{-2} \int_0^\infty \int_0^\infty \left(\Omega^2 p^u\right) \left(\Omega^2 p^v\right) \bar{f}(u, v; p^u, p^v, l) \left|_{\Omega^2 p^u p^v + \frac{l^2}{r^2}} \frac{dp^u}{p^u} \right| dldl. \]

Similarly, in this case, provided $\mathcal{V} \subseteq TM$ is invariant under the action of $SO(3)$ on $TM$, \((2.26)\) takes the form

\[ N^{(\mathcal{V})} = N^{(\mathcal{V})}_u du + N^{(\mathcal{V})}_v dv, \]

with

\[ N^{(\mathcal{V})}_u = \frac{\pi}{2} r^{-2} \int_{\mathcal{V}(\mathcal{U}, T \mathcal{M}) \cap \{\Omega^2 p^u p^v + \frac{l^2}{r^2} \geq 0\}} \left(\Omega^2 p^v\right) \bar{f}(u, v; p^u, p^v, l) \left|_{\Omega^2 p^u p^v + \frac{l^2}{r^2}} \frac{dp^u}{p^u} \right| dldl, \]

\[ N^{(\mathcal{V})}_v = \frac{\pi}{2} r^{-2} \int_{\mathcal{V}(\mathcal{U}, T \mathcal{M}) \cap \{\Omega^2 p^u p^v + \frac{l^2}{r^2} \geq 0\}} \left(\Omega^2 p^u\right) \bar{f}(u, v; p^u, p^v, l) \left|_{\Omega^2 p^u p^v + \frac{l^2}{r^2}} \frac{dp^v}{p^v} \right| dldl. \]

For any smooth and spherically symmetric massless Vlasov field $f$, any open set $\mathcal{V} \subseteq TM$ which is invariant under the action of $SO(3)$ on $TM$ and any $l \geq 0$, we will also introduce the quantities

\[ N^{(\mathcal{V})}_\mu = 2\pi r^{-2} \int_{\mathcal{V}(\mathcal{U}, T \mathcal{M}) \cap \{\Omega^2 p^u p^v + \frac{l^2}{r^2} \geq 0\}} p_\mu \bar{f}(u, v; p^u, p^v, l) \left|_{\Omega^2 p^u p^v + \frac{l^2}{r^2}} \frac{dp^u}{p^u} \right| dldl. \]
We will also denote

\[
T^{(l)}_{\mu\nu} = 2\pi r^{-2}\int_0^{+\infty} p_{\mu} p_{\nu} \tilde{f}(u, v; p^\mu, p^\nu, l) \left|_{\Omega^2 p^\mu p^\nu + \frac{\Lambda}{\pi}} \frac{dp^\mu}{p^\mu}.
\]

Note that, when \( V \) is invariant under the geodesic flow, (2.35) is conserved, i.e.

\[
\nabla^\mu N^V_\mu = 0.
\]

Note also the relations

\[
N^V_\mu = \int_0^{+\infty} N^V_\mu \ d\ell, \quad T_{\mu\nu} = \int_0^{+\infty} T^{(l)}_{\mu\nu} \ d\ell.
\]

We will also denote

\[
N^{(l)}_\mu = N^{(TM;l)}_\mu.
\]

Combining the expressions (2.10), (2.35), (2.36), we can readily estimate for any \( l > 0 \):

\[
1 - \frac{2m}{r} T_{\mu\nu}^{(l)}(u, v) + \frac{1 - \frac{2m}{r}}{\partial_u r} T_{\mu\nu}^{(l)}(u, v) \leq 2 \sup_{\text{supp}(f(u, v; \cdot; \cdot; \cdot))} \left( \partial_v r(u, v) p^\nu - \partial_u r(u, v) p^u \right) \cdot N^{(l)}_v(u, v)
\]

and

\[
1 - \frac{2m}{r} T_{\mu\nu}^{(l)}(u, v) + \frac{1 - \frac{2m}{r}}{\partial_u r} T_{\mu\nu}^{(l)}(u, v) \leq 2 \sup_{\text{supp}(f(u, v; \cdot; \cdot; \cdot))} \left( \partial_v r(u, v) p^\nu - \partial_u r(u, v) p^u \right) \cdot N^{(l)}_u(u, v).
\]

### 2.4 The spherically symmetric Einstein–massless Vlasov system

Let \((M^{3+1}, g)\) be a smooth Lorentzian manifold and let \(f\) be a non-negative measure on \(TM\). The Einstein–Vlasov system for \((M, g; f)\) with a cosmological constant \(\Lambda < 0\) is

\[
\begin{align*}
Ric_{\mu\nu}(g) - \frac{1}{2} R(g) g_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi T_{\mu\nu}, \\
p^\mu \partial_{\mu\nu} f - \Gamma^\nu_{\beta\gamma} p^\beta \partial_{\nu\gamma} f &= 0,
\end{align*}
\]

where \(T_{\mu\nu}\) is expressed in terms of \(f\) by (2.23) (see also [4, 13, 17]).

In the case when \((M, g)\) is a spherically symmetric spacetime as in Section 2.1 and \(f\) is a spherically symmetric massless Vlasov field (see Section 2.3), (2.42) is equivalent to the following system for \((r, \Omega^2, f)\):

\[
\begin{align*}
\partial_\nu \partial_\nu (r^2) &= -\frac{1}{2}(1 - \Lambda r^2) \Omega^2 + 8\pi r^2 T_{uv}, \\
\partial_\nu \partial_\nu \log(\Omega^2) &= \Omega^2 \frac{2}{2r^2} \left( 1 + 4\Omega^2 \partial_\nu r \partial_\nu r \right) - 8\pi T_{uv} - 2\pi \Omega^2 g^{AB} T_{AB}, \\
\partial_\nu (\Omega^2 \partial_\nu r) &= -4\pi \Omega^2 T_{uv} \Omega^2, \\
\partial_\nu (\Omega^2 \partial_\nu r) &= -4\pi \Omega^2 T_{uv} \Omega^2, \\
p^\nu \partial_\nu f + p^\nu \partial_\nu f = & \left( \partial_\nu \log(\Omega^2) (p^\nu)^2 + 2 \frac{\Omega^2 \partial_\nu r l^2}{r^2} \right) \partial_{p^\nu} f + \\
& + \left( \partial_\nu \log(\Omega^2) (p^\nu)^2 + 2 \frac{\Omega^2 \partial_\nu r l^2}{r^2} \right) \partial_{p^\nu} f, \\
\text{supp}(f) \subseteq & \{ \Omega^2 p^\mu p^\nu - \frac{l^2}{r^2} = 0, \ p^\mu > 0 \}.
\end{align*}
\]
Note that \((r, \Omega^2, f) = (r_{AdS}, \Omega^2_{AdS}, 0)\) (where \(r_{AdS}, \Omega^2_{AdS}\) are given by (1.4)) is a trivial solution for the system (2.43)–(2.48).

Defining the renormalised Hawking mass as

\[
\tilde{m} = m - \frac{1}{6} \Lambda r^3,
\]

and using the relation (2.40), the constraint equations (2.45)–(2.46) are equivalent (in the region of \(\mathcal{M}\) where \(\partial_u r > 0, \partial_u r < 0\) and \(1 - \frac{2m}{r} > 0\)) to

\[
\partial_u \log \left( \frac{\partial_u r}{1 - \frac{2m}{r}} \right) = - \frac{4\pi r}{1 - \frac{2m}{r}} T_{uu} - \frac{4\pi r}{1 - \frac{2m}{r}} T_{uv},
\]

(2.50)

\[
\partial_u \log \left( \frac{-\partial_u r}{1 - \frac{2m}{r}} \right) = \frac{4\pi r}{1 - \frac{2m}{r}} T_{uv}.
\]

Equations (2.43)–(2.47) also formally give rise to the following set of equations for \(r, \tilde{m}\):

\[
\partial_u \partial_v r = - \frac{2\tilde{m} - \frac{2}{3} \Lambda r^3 (-\partial_u r) \partial_v r}{1 - \frac{2m}{r}} + 4\pi r T_{uv},
\]

(2.52)

\[
\partial_u \tilde{m} = - \frac{2\tilde{m}}{r} \left( 1 - \frac{2m}{r} \right) \left( \frac{1}{\partial_u r} T_{uu} - \frac{1}{\partial_v r} T_{uv} \right),
\]

(2.53)

\[
\partial_v \tilde{m} = \frac{2\pi}{r} \left( 1 - \frac{2m}{r} \right) \left( \frac{1}{\partial_u r} T_{uu} - \frac{1}{\partial_v r} T_{uv} \right),
\]

(2.54)

Note that, in view of the relation \(4\Omega^{-2} T_{uv} = g^{AB} T_{AB}\), (2.44) is equivalent to

\[
\partial_u \partial_v \log (\Omega^2) = \frac{4\pi r}{1 - \frac{2m}{r}} \left( \frac{1}{\partial_u r} T_{uu} - \frac{1}{\partial_v r} T_{uv} \right).
\]

(2.55)

When considering asymptotically AdS solutions of the system (2.43)–(2.48), it will be useful to consider the following renormalised quantities near \(\mathcal{I}\):

\[
\tilde{\Omega}^2 = \frac{\Omega^2}{1 - \frac{2}{3} \Lambda r^2},
\]

\[
\rho = \tan^{-1} \left( \sqrt{\frac{\Lambda}{3}} r \right),
\]

\[
\tau_{\mu \nu} = \Omega^2 T_{\mu \nu}.
\]

The quantities \((\tilde{\Omega}^2, \rho, \tau_{\mu \nu})\) satisfy the following renormalised equations (readily obtained from (2.52) and (2.55)):

\[
\partial_u \partial_v \log (\tilde{\Omega}^2) = \frac{\tilde{m}}{r} \left( \frac{1}{r^2} + \frac{1}{3} \frac{\Lambda}{1 - \frac{2}{3} \Lambda r^2} \right) \tilde{\Omega}^2 - \frac{1}{1 - \frac{2}{3} \Lambda r^2} T_{uu} - \frac{1}{1 - \frac{2}{3} \Lambda r^2} T_{uv},
\]

(2.57)

\[
\partial_u \partial_v \rho = - \frac{1}{2} \sqrt{\frac{\Lambda}{3}} \tilde{m} \left( \frac{1}{3} r^2 \frac{1}{1 - \frac{2}{3} \Lambda r^2} \tilde{\Omega}^2 + 4\pi \right) - \frac{1}{3} \frac{\Lambda}{r - \frac{2}{3} \Lambda r^2} \tau_{uu}.
\]

The following relation will be useful throughout this paper: Let \(u_1(v)\) be a given function of \(v\) and let \(\gamma : [0, a) \to \mathcal{M}\) be a null geodesic contained in the region \(\{ u \geq u_1(v) \}\) and having non-zero angular momentum \(l\), such that \(\gamma(0)\) lies on the curve \(\{ u = u_1(v) \}\). Then, the projection of \(\gamma\) on \(\mathcal{U}\) will be a strictly timelike curve with respect to the reference metric

\[
g_{\text{ref}} = -dudv
\]

(2.58)
on $\mathcal{U}$, and the equations of motion (2.21) (combined with the relation (2.20) for null vectors) imply that, for any $s \in [0,a)$:

\[
\log(\Omega^2\dot{\gamma}^u)(s) - \log(\Omega^2\dot{\gamma}^u)(0) = \int_{\gamma([0,s])} \left( \partial_u \log(\Omega^2) - 2 \frac{\partial_v r}{r} \right) dv = \\
= \int_{\gamma(0)}^{\gamma(s)} \left( \partial_u \log(\Omega^2) - 2 \frac{\partial_v r}{r} \right) du dv + \\
+ \int_{\gamma(0)}^{\gamma(s)} \left( \partial_v \log(\Omega^2) - 2 \frac{\partial_r r}{r} \right)(u_1(v), v) dv,
\]

where, for any $\bar{v} \in (v(\gamma(0)), v(\gamma(s)))$, $s_{\bar{v}}$ denotes the value of the parameter $s$ for which

\[
v(\gamma(s_{\bar{v}})) = \bar{v}.
\]

In view of the evolution equations (2.52), (2.55) for $r, \Omega^2$ and the definition (2.49) of $\tilde{m}$, the relation (2.59) can be expressed as

\[
\log(\Omega^2\dot{\gamma}^u)(s) - \log(\Omega^2\dot{\gamma}^u)(0) = \int_{\gamma(0)}^{\gamma(s)} \left( \frac{6m}{r^2} - \frac{1}{2} \right) \left( \frac{1}{2} \Omega^2 - 24\pi T uv \right) du dv + \\
+ \int_{\gamma(0)}^{\gamma(s)} \left( \partial_v \log(\Omega^2) - 2 \frac{\partial_r r}{r} \right)(u_1(v), v) dv
\]

(see also Figure 2.2).

Figure 2.2: For a null geodesic $\gamma$, the domain of integration appearing in the right hand side of (2.61) is as depicted above.

Remark. In this paper, we will be mainly interested in the case when $u_1(v)$ is of the form

\[
u_1(v) = \begin{cases} u_0, & v \leq v_c \\
v - v_\Sigma, & v \geq v_c \end{cases}
\]

for some constants $u_0$, $v_\Sigma$ and $v_c$. 

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Similarly, inverting the roles of the $u,v$ variables, for any null geodesic $\gamma : [0,a) \to \{ v \geq v_1(u) \}$ with $\gamma(0) \in \{ v = v_1(u) \}$ we calculate:

\[
\begin{align*}
\log (\Omega^2 \dot{\gamma}^v)(s) - \log (\Omega^2 \dot{\gamma}^v)(0) &= \int_{u(\gamma(0))}^{u(\gamma(s))} \left( \frac{6m}{r^2} - \frac{1}{r^2} \Omega^2 - 24\pi T_{uv} \right) dvdu + \\
&\quad + \int_{u(\gamma(0))}^{u(\gamma(s))} (\partial_u \log(\Omega^2) - 2\frac{\partial_u r}{r})(u,v_1(u)) du,
\end{align*}
\]

where $s_u$ is defined by

\[ u(\gamma(s_u)) = \bar{u}. \]

2.5 Reflection of null geodesics and the reflecting boundary condition on $\mathcal{I}$

Let $(\mathcal{M},g)$ be a spacetime as in Section 2.2. Let $\gamma : [0,+\infty) \to \mathcal{M}$ be a future directed null geodesic with respect to $g$, which is future inextendible and satisfies

\[ \lim_{s \to +\infty} r \circ \gamma(s) = +\infty \]

and

\[ \limsup_{s \to +\infty} u \circ \gamma(s) < \sup u. \]

From the expression (2.11) of the Christoffel symbols and the fact that $r^{-2} \Omega^2$ extends smoothly on $\mathcal{I}$ and satisfies (2.15), it can be readily deduced that the curve

\[
(\gamma,r^{2}\dot{\gamma}) : [0,+\infty) \to T\bar{M}
\]

has a regular limit on $\mathcal{I}^{(2+1)}$:

\[ \lim_{s \to +\infty} (\gamma(s),r^{2}\dot{\gamma}(s)) = (q,v) \in T|_{\mathcal{I}^{(2+1)}} \bar{M}, \]

with $v \in T_q\bar{M}$ being a non-zero null vector with respect to $\bar{g}$, satisfying

\[ \bar{g}(v,n_{\mathcal{I}}) > 0, \]

where $n_{\mathcal{I}} \in \Gamma(T|_{\mathcal{I}^{(2+1)}} \bar{M})$ is the (spacelike) unit normal to $\mathcal{I}^{(2+1)}$ pointing outwards. An analogous statement also holds for past inextendible, future directed null geodesics $\gamma : (-\infty,0] \to \mathcal{M}$ with past limit on $\mathcal{I}^{(2+1)}$, in which case (2.67) holds with the opposite sign. Conversely, for any point $q \in \mathcal{I}^{(2+1)}$ and any non-zero vector $v \in T_q\bar{M}$ which is null and future directed with respect to $\bar{g}$:

- In the case $\bar{g}(v,n_{\mathcal{I}}) > 0$, there exists a unique future directed, future inextendible null geodesic
  \[
  \gamma : (\alpha,+\infty) \to (\mathcal{M},g)
  \]
  for which (2.66) holds.
- In the case $\bar{g}(v,n_{\mathcal{I}}) < 0$, there exists a unique future directed, past inextendible null geodesic
  \[
  \gamma : (-\infty,a] \to (\mathcal{M},g)
  \]
  for which (2.66) holds with $-\infty$ in place of $+\infty$.

We can therefore define the reflection of null geodesics on conformal infinity as follows:
Definition 2.2. Let $(\mathcal{M}, g)$ be as in Definition 2.1 and let $\gamma : (a, +\infty) \rightarrow (\mathcal{M}, g)$ be a future directed, future inextendible null geodesic, satisfying (2.64) and (2.65), and let $q \in \mathcal{I}^{(2+1)}$, $v \in T_q \tilde{\mathcal{M}}$ be defined by (2.66). Let also $v_\text{m} \in T_q \tilde{\mathcal{M}}$ be the reflection of $v$ across $T_q \mathcal{I}^{(2+1)}$, i.e. the unique future directed, null vector satisfying $v_\text{m} \neq v$ and

$$(v_\text{m} - v) \parallel n_\mathcal{I}.$$  

We define the reflection of $\gamma$ off $\mathcal{I}^{(2+1)}$ to be the unique future directed, past inextendible null geodesic $\gamma_\text{m} : (-\infty, b) \rightarrow (\mathcal{M}, g)$ such that

$$\lim_{s \rightarrow -\infty} (\gamma_\text{m}(s), r^2 \dot{\gamma}_\text{m}(s)) = (q, v_\text{m}).$$  

Remark. Note that the angular momenta of $\gamma$ and $\gamma_\text{m}$ (defined by (2.19)) are necessarily the same, since $n_\mathcal{I}$ points in the radial direction. In particular, the projections of $\gamma$, $\gamma_\text{m}$ on $\mathcal{U}$ will satisfy on $\mathcal{I}$ (in the notation of Section 2.3):

$$\begin{cases} r^2 \dot{\gamma}_\text{m}^u |_{\mathcal{I}} = r^2 \dot{\gamma}^u |_{\mathcal{I}}, \\ r^2 \dot{\gamma}_\text{m}^n |_{\mathcal{I}} = r^2 \dot{\gamma}^n |_{\mathcal{I}}, \\ r^2 \dot{\gamma}_\text{m}^A |_{\mathcal{I}} = r^2 \dot{\gamma}^A |_{\mathcal{I}}. \end{cases}$$

(2.69)

By successively extending a null geodesic $\gamma$ through its reflections off $\mathcal{I}^{(2+1)}$, we can construct its maximal extension through reflections:

Definition 2.3. Let $(\mathcal{M}, g)$ be as in Definition 2.1 and let $\gamma = \bigcup_{n=0}^N \gamma_n$, $N \in \mathbb{N} \cup \{\infty\}$, be the union of a future directed, future inextendible, affinely parametrised null geodesics $\gamma_n : (a_n, b_n) \rightarrow \mathcal{M}$, $-\infty \leq a_n < b_n \leq +\infty$. We will say that $\gamma$ is an affinely parametrised, maximally extended geodesic through reflections if all of the following conditions hold:

1. $a_n = -\infty$ for any $n > 0$ and $b_n = +\infty$ for any $n < N$,
2. For any $0 < n \leq N$, $\gamma_n$ is the reflection of $\gamma_{n-1}$ off $\mathcal{I}^{(2+1)}$, in accordance with Definition 2.2,
3. If $N = \infty$, then $\lim_{s \rightarrow -N} \gamma_N(s)$ does not exist in $\tilde{\mathcal{M}}$.

Remark. The reflecting condition (2.68) uniquely determines the affine parametrisation of $\gamma_n$ in terms of the parametrisation of $\gamma_{n-1}$. Thus, the affine parametrisation of $\gamma_0$ uniquely determines the parametrisation of all the $\gamma_n$’s.

Having defined the reflection of null geodesics on $\mathcal{I}^{(2+1)}$, we can now introduce the notion of the reflecting boundary condition for the massless Vlasov equation on $\mathcal{I}^{(2+1)}$:

Definition 2.4. Let $(\mathcal{M}, g)$ be as in Definition 2.1 and let $f$ be a smooth massless Vlasov field on $T\mathcal{M}$ (see Section 2.3 for the relevant definition). We will say that $f$ satisfies the reflecting boundary condition on conformal infinity if, for any pair of future directed null geodesics $\gamma : (a, +\infty) \rightarrow \mathcal{M}$ and $\gamma_\text{m} : (-\infty, b) \rightarrow \mathcal{M}$ such that $\gamma_\text{m}$ is the reflection of $\gamma$ on $\mathcal{I}^{(2+1)}$ according to Definition 2.2, $f$ satisfies

$$f|_{(\gamma, \dot{\gamma})} = f|_{(\gamma_\text{m}, \dot{\gamma}_\text{m})},$$

where $f|_{(\gamma, \dot{\gamma})}$ is the (constant) value of $f$ along the curve $(\gamma, \dot{\gamma})$ in $T\mathcal{M}$.

Remark. Equivalently, $f$ satisfies the reflecting condition on $\mathcal{I}^{(2+1)}$ if $f$ is constant along the trajectory of $(\gamma, \dot{\gamma})$ for any future directed, affinely parametrised null geodesic $\gamma$ which is maximally extended through reflections.

The following Lemma is a straightforward consequence of the relations (2.53)–(2.54), the condition (2.14) on $\mathcal{I}$ and the reflecting boundary condition (2.69):

Lemma 2.1. Let $(r, \Omega^2, f)$ be an asymptotically AdS solution of the spherically symmetric Einstein–massless Vlasov system (2.43)–(2.48), such that $f$ satisfies the reflecting boundary condition on conformal infinity. Then,

$$\left(\partial_u + \partial_\nu\right)\tilde{m}|_{\mathcal{I}} = 0,$$

i.e. $\tilde{m}$ is conserved along $\mathcal{I}$. In particular, if $\tilde{m}$ has a finite limit on some point $q \in \mathcal{I}$, then it has a finite limit everywhere on $\mathcal{I}$.
3 The initial-boundary value problem for the spherically symmetric Einstein–massless Vlasov system

The aim of this section is to introduce the characteristic initial-boundary value problem for (2.43)–(2.48), expressed in terms of characteristic initial data sets at \( u = 0 \). We will focus on the class of *smoothly compatible* initial data sets, which are precisely those initial data sets which, formally at least, are induced on \( u = 0 \) by smooth solutions \((\mathcal{M}, g; f)\) of (2.42) satisfying the reflecting boundary condition on \( I \) (the fact that any such initial data set indeed admits a smooth development will be established in Section 6). To this end, we will first introduce the notion of a smooth, asymptotically AdS solution \((r, \Omega^2, f)\) to (2.43)–(2.48) with a regular axis of symmetry.

3.1 The class of smooth solutions of (2.43)–(2.48)

Before formulating the characteristic initial-boundary value problem for (2.43)–(2.48), we will introduce the regularity conditions that will define the class of solutions to (2.43)–(2.48) which will be of interest to us in the following sections. This class will consist of precisely those solutions \((r, \Omega^2, f)\) to (2.43)–(2.48) which arise from *smooth* spherically symmetric solutions \((\mathcal{M}, g; f)\) of (2.42).

Checking whether a solution of (2.43)–(2.48) corresponds to a smooth solution of (2.42) requires taking into consideration the coordinate singularities introduced along the axis when switching to a spherically symmetric double null coordinate chart. We will therefore adopt the following definition for the smoothness of a solution \((r, \Omega^2, f)\) of (2.43)–(2.48) in the presence of an axis-type boundary, which guarantees the existence of a corresponding smooth solution \((\mathcal{M}, g; f)\) of (2.42) with a non-trivial axis:

**Definition 3.1.** Let \( \mathcal{U} \) be a domain in the \((u, v)\)-plane satisfying \( \mathcal{U} \subseteq \{ u < v \} \), such that

\[
\bar{\gamma}_Z \equiv \{ u = v \} \cap \partial \mathcal{U}
\]
is a non-empty connected curve of the form \( \{ u = v \} \cap \{ u_1 \leq u \leq u_2 \} \) for some \( u_1 < u_2 \). Let us also set
\[
\gamma_Z = \{ u = v \} \cap \{ u_1 < u < u_2 \}.
\]
A solution \((r, \Omega^2, f)\) of (2.43)–(2.48) on \( \mathcal{U} \) will possess a smooth axis \( \gamma_Z \) if
- The functions \((r, \Omega^2)\) are smooth and positive on \( \mathcal{U} \), and \( r \) extends continuously to 0 on \( \gamma_Z \).
- The Vlasov field \( f \) is of the form (2.29), where \( \bar{f} \) is smooth in its variables.
- The coefficients of the metric (2.3) on \( \mathcal{U} \times \mathbb{S}^2 \), when expressed in the Cartesian coordinate chart
\[
(x^0, x^1, x^2, x^3) : \mathcal{U} \times \mathbb{S}^2 \to \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})
\]
defined by the relations
\[
(x^0, x^1, x^2, x^3) = \left( \frac{1}{2} (u + v), \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = r, \cos^{-1} \left( \frac{x^3}{r} \right) = \theta, \frac{x^2}{x^1} = \tan \varphi \right)
\]
(where \((\varphi, \theta)\) are the standard angular coordinates on \( \mathbb{S}^2 \)), can be smoothly extended on \((x^1)^2 + (x^2)^2 + (x^3)^2 = 0\). Similarly, expressed as a function of the Cartesian coordinates \( x^a \) and their conjugate momentum coordinates \( p^a \), the function \( \bar{f}(x^a, p^a) \) can be smoothly extended on \((x^1)^2 + (x^2)^2 + (x^3)^2 = 0\).

**Remark.** In the Cartesian coordinates \( x^a \) defined by (3.1) the metric (2.3) is expressed as:
\[
g = -\Omega^2 \left( 1 - \frac{r^2}{(r')^2} \right) (dx^0)^2 - 2 \frac{r \Omega^2}{(r')^2} \frac{x^k \delta_{ki}}{r} dx^i dx^0 + \left( \delta_{ij} + \left( \frac{2m + \lambda - 1}{r^2} + \frac{\lambda - 1}{2m} \right) \frac{x^k x^l}{r^2} \delta_{kl} \delta_{ij} \right) dx^i dx^j,
\]
where latin indices range over the index set \( \{1, 2, 3\} \) and
\[
\dot{r} = \partial_r r + \partial_u r, \quad r' = \partial_r r - \partial_u r, \quad \lambda = -4 \partial_u r \partial_u r \left( \partial_r r - \partial_u r \right)^2.
\]
The fact that \( r|_{\gamma_Z} = 0 \) also implies that
\[
\partial_r r|_{\gamma_Z} = -\partial_u r|_{\gamma_Z}.
\]
For asymptotically AdS solutions, smoothness along conformal infinity will be defined in accordance to Definition 2.1.

**Definition 3.2.** Let $\mathcal{U}$ be a domain in the $(u,v)$-plane satisfying $\mathcal{U} \subset \{ v < u + v_I \}$ for some $v_I \in \mathbb{R}$, such that

$$I \doteq \{ v = u + v_I \} \cap \partial \mathcal{U}$$

is a non-empty connected curve of the form $\{ v = u + v_I \} \cap \{ u_1 \leq u \leq u_2 \}$ for some $u_1 < u_2$. Let us also set

$$(3.4) \quad I \doteq \{ v = u + v_I \} \cap \{ u_1 < u < u_2 \}.$$  

A solution $(r, \Omega^2, f)$ of (2.43)–(2.48) on $\mathcal{U}$ will be said to possess a smooth conformal infinity $I$ if

- The functions $(r, \Omega^2)$ extend on the whole of the domain

$$\mathcal{U}^* \doteq \{ u + v_*(u) \leq v < u + v_I \} \cap \{ u_1 < u < u_2 \}$$

(for some smooth $v_*(u) < v_I$), so that $(\mathcal{U}^*; r, \Omega^2)$ is asymptotically AdS in accordance with Definition 2.1;

- The distribution $f$ is of the form (2.29), for some smooth function $\bar{f}$ which extends smoothly on $\{ v = u + v_I \}$ as a function of $\bar{p}_u = \Omega^2 p_u$, $\bar{p}_v = \Omega^2 p_v$ and $l$.

![Figure 3.2: Schematic depiction of a domain $\mathcal{U}$ satisfying the conditions of Definition 3.2](image)

Away from the axis and conformal infinity, smoothness for solutions of (2.43)–(2.48) will be defined in the usual way:

**Definition 3.3.** Let $\mathcal{U}$ be a domain in the $(u,v)$-plane. A solution $(r, \Omega^2, f)$ of (2.43)–(2.48) on $\mathcal{U}$ will be called smooth if the functions $(r, \Omega^2)$ are smooth and positive on $\mathcal{U}$, while $f$ is of the form (2.29), for some smooth function $\bar{f}$.

**Remark.** More generally, a solution $(r, \Omega^2, f)$ of (2.43)–(2.48) will be called smooth on a subset $\mathcal{F}$ of the $(u,v)$ plane of $(r, \Omega^2, f)$ is smooth on an open domain $\mathcal{U} \supseteq \mathcal{F}$ (with a similar generalization applying to the definition of a smooth axis and a smooth conformal infinity).

### 3.2 Asymptotically AdS characteristic initial data sets for (2.43)–(2.48)

In this paper, as well as in the companion paper [16], we will be mainly interested in solutions to (2.43)–(2.48) arising from asymptotically AdS characteristic initial data sets prescribed at $u = 0$. In this section, we will introduce the class of initial data under consideration. We will also present a gauge normalisation condition, that will later allow us to compare different initial data sets with the trivial one, as well as a condition related to the compatibility of initial data sets with the reflecting boundary condition at $I$.

We will adopt the following fundamental definition:
Definition 3.4. Let \( v_\Omega > 0 \) and let \( r_j, \Omega_j : [0,v_\Omega) \to [0, +\infty) \), \( \bar{f}_j : (0, v_\Omega) \times [0, +\infty)^2 \to [0, +\infty) \) be smooth functions. We will call \( (r_j, \Omega_j^2, \bar{f}_j ; v_\Omega) \) a smooth, asymptotically AdS initial data set for \( (2.43) \to (2.48) \) if:

1. On \((0, v_\Omega), \) the functions \( (r_j, \Omega_j^2, \bar{f}_j) \) satisfy the constraint equation \((2.45), \) where the energy momentum component \( T_{\nu \mu} \) is defined on \((0, v_\Omega)\) using the relation \((2.32) \) (with \( \bar{f}_j(v; p^\nu, l) \) in place of \( \bar{f}(u, v; p^\nu, p^\tau, l) \) \( \Omega^2 p^\nu p^\tau \)).

2. At \( v = 0, \) the functions \( r_j, \Omega_j^2 \) satisfy

\[
(3.5) \quad r_j(0) = 0,
\]

\[
(3.6) \quad \Omega_j^2(0) > 0.
\]

3. At \( v = v_\Omega, \) the functions \( 1/r_j, r_j^{-2} \Omega_j^2 \) extend smoothly and satisfy

\[
(3.7) \quad 1/r_j(v_\Omega) = 0,
\]

\[
(3.8) \quad \partial_v (1/r_j)(v_\Omega) < 0,
\]

\[
(3.9) \quad \frac{\Omega_j^2}{r_j^2}(v_\Omega) > 0.
\]

Furthermore, for any \( p \geq 0 \) and \( l \geq 0, \) \( \bar{f}(v, \Omega_j^{-2}(v)p, l) \) extends smoothly on \( v = v_\Omega. \)

We will also say that an initial data set \( (r_j, \Omega_j^2, \bar{f}_j; v_\Omega) \) is of bounded support in phase space if there exists some \( C > 0 \) such that, for every \( v \in (0, v_\Omega) \) and \( l \geq 0: \)

\[
(3.10) \quad \sup_{p^\mu \in \text{supp}(f_j(v; v))} \left( \Omega_j^2(p^\mu + \frac{l^2}{\Omega_j^2 p^\nu}) \right) \leq C.
\]

We should make the following remarks regarding the class of asymptotically AdS initial data sets:

Remark 1. The inequality \( \partial_v (\Omega_j^{-2} \partial_v r_j) \leq 0 \) (following from the constraint equation \((2.45) \)) and the fact that

\[
r_j(v_\Omega) = +\infty, \quad \frac{\Omega_j^2}{r_j^2}(v_\Omega) > 0 \text{ and } \Omega_j^2(0) > 0,
\]

imply that, for any smooth asymptotically AdS initial data set \( (r_j, \Omega_j^2, \bar{f}_j; v_\Omega), \)

\[
(3.11) \quad \inf_{v \in [0, v_\Omega]} \partial_v r_j(v) > 0 \text{ and } \inf_{v \in [0, v_\Omega]} \Omega_j^2(v) > 0.
\]

In particular, a smooth asymptotically AdS initial data set does not contain trapped spheres.

Remark 2. For any smooth asymptotically AdS initial data set \( (r_j, \Omega_j^2, \bar{f}_j; v_\Omega), \) we can formally define the initial renormalised Hawking mass \( \tilde{m}_j \) in terms of \( (r_j, \Omega_j^2, \bar{f}_j) \) by the relation \((2.54), \) i.e.:

\[
(3.12) \quad \begin{cases}
\partial_v \tilde{m}_j = 2\pi \left( 1 - \frac{2\tilde{m}_j}{r_j} - \frac{1}{r_j^2} \left[ \frac{\partial^2(T_j)_{v v}}{\partial_v r_j} \right] - 4 \frac{\partial^2(T_j)_{u v}}{r_j^2} \right), \\
\tilde{m}_j(0) = 0, \\
r_j^2(T_j)_{v v}(v) \approx \frac{\pi}{2} \int_0^{v_\Omega} \int_0^{v_\Omega} (\Omega_j^2(v)p^2) \bar{f}_j(v; p, l) \frac{dp}{p} dl, \\
r_j^2(T_j)_{u v}(v) \approx \frac{\pi}{2} \Omega_j^2(v) \int_0^{v_\Omega} \int_0^{v_\Omega} \frac{l^2}{r_j^2} \bar{f}_j(v; p, l) \frac{dp}{p} dl.
\end{cases}
\]
Arguing as in the proof of Proposition 5.2, the condition (3.10) implies that, for any smooth asymptotically AdS initial data set \((r_j, \Omega^2_j, f_j; v_T)\) with bounded support in phase space, we have:

\[
\lim_{v \to v_T^-} \hat{m}(v) < +\infty.
\]

**Remark 3.** Let \((r_j, \Omega^2_j, f_j; v_T)\) be a smooth asymptotically AdS initial data set as in Definition 3.4 satisfying, moreover, the gauge condition condition

\[
\frac{\partial_v r_j}{1 - \frac{2}{3} \Lambda r_j^2}(0) = \frac{\Omega_j^2}{4 \partial_v r_j}(0)
\]

at the axis, as well as the regularity condition

\[
\limsup_{v \to 0} \left| \partial_v^m \left( \frac{\hat{m}_j}{r_j^3} \right) \right| < +\infty \quad \text{for all} \; n \in \mathbb{N},
\]

where \(\hat{m}_j\) is defined in terms of \((r_j, \Omega^2_j, f_j)\) by (3.12). Let also \((r, \Omega^2, f)\) be a solution of the Einstein–massless Vlasov \((2.43)–(2.48)\) which satisfies at \(u = 0\)

\[
(r, \Omega^2)(0, v) = (r_j, \Omega^2_j)(v)
\]

and

\[
f(0, v; p^u, p^v, l) = \tilde{f}_j(v; p^u, l) \cdot \delta(\Omega_j^2 p^u p^v - \frac{l^2}{r_j^2}),
\]

as well as the boundary condition

\[
(\partial_u + \partial_v)^n r(0, 0) = 0 \quad \text{for all} \; n \in \mathbb{N}
\]

(note that (3.18) is necessary for \((r, \Omega^2, f)\) to satisfy \(r = 0\) along \(u = v\); the condition (3.14) is a necessary condition for (3.18) to hold when \(n = 1\).

For any \(k \in \mathbb{N}\), we can formally determine the values of the higher order transversal derivatives \(\partial_u^k r|_{u=0}, \partial_u^k \Omega|_{u=0}\) and \(\partial_u^k f|_{u=0}\) in terms of \((r_j, \Omega^2_j, f_j)\) using equations (2.43), (2.44) and (2.47). In particular, by defining \(f\) in terms of \(f\) through the relation

\[
f(u, v; p^u, p^v, l) = \hat{f}(v; p^u, l) \cdot \delta(\Omega^2(u, v) p^u p^v - \frac{l^2}{r^2(u, v)})
\]

differentiating (2.43), (2.44) and (2.30), using also (2.48), we formally obtain:

\[
\partial_u^{k+1} r(0, v) = \partial_u^{k+1} r(0, 0) + \int_0^v \partial_u^k \left( - \frac{\hat{m} - \frac{2}{3} \Lambda r^3}{r^2} \Omega^2 + 2 \pi^2 \Omega^2 r^{-3} \int_0^{+\infty} \int_0^{+\infty} l^2 \hat{f}(\cdot; p, l) \, dl \frac{dp}{p} \right)(v) \, dv,
\]

\[
\partial_u^{k+1} \log \Omega^2(0, v) = \partial_u^{k+1} \log \Omega^2(0, 0) + \int_0^v \partial_u^k \left( 2 \hat{m} + \frac{2}{3} \Lambda \right) - 8 \pi^2 \Omega^2 r^{-4} \int_0^{+\infty} \int_0^{+\infty} l^2 \hat{f}(\cdot; p, l) \, dl \frac{dp}{p} \right)(v) \, dv
\]

and

\[
\partial_u^{k+1} \hat{f}(0, v; p, l) = \partial_u^k \left( - \frac{l^2}{\Omega^2 p^2} \partial_v \hat{f} + \frac{1}{p} \left( \partial_u \log(\Omega^2) p^2 + \frac{2}{r} \Omega^{-2} \partial_v r \frac{l^2}{r^2} \partial_p \hat{f} \right) \right)(0, v; p, l).
\]
Arguing inductively in \( k \geq 0 \) and assuming that \( \partial^k r|_{u=0} \), \( \partial^k \Omega|_{u=0} \) and \( \partial^k \bar{f}|_{u=0} \) have been computed for \( 0 \leq \bar{k} \leq k \) in terms of \( (r_j, \Omega^2_j, f_j) \), the relations (3.20)–(3.22) (combined with the expressions (2.53)–(2.54) for the derivatives of \( \tilde{n} \)) uniquely determine, successively, \( \partial^{k+1} r|_{u=0} \), \( \partial^{k+1} \Omega|_{u=0} \) and \( \partial^{k+1} \bar{f}|_{u=0} \) in terms of \( (r_j, \Omega^2_j, f_j) \), using at \( v = 0 \) the boundary conditions

\[
(3.23) \quad \partial^{k+1} r(0, 0) = \sum_{k=0}^{k} (-1)^{k-k+1} \left( \frac{k+1}{k} \right) \partial^{k-k+1} r(0, 0)
\]

and

\[
(3.24) \quad \partial^{k+1} \Omega^2(0, 0) = 4 \left( \sum_{k=0}^{k} \left\{ (-1)^{k-k} \left( \frac{k+1}{k} \right) \partial^{k-k} \Omega^2(0, 0) \right\} \right) \cdot \frac{\partial_v r}{1 - \frac{2m}{r} - \frac{1}{3} \Lambda r^2}(0, 0) - 4 \sum_{k=0}^{k} \left\{ \partial^{k-k} r(0, 0) \cdot \partial^{k+1} r(0, 0) \right\} \left( \frac{\partial_v r}{1 - \frac{2m}{r} - \frac{1}{3} \Lambda r^2}(0, 0) \right)
\]

(following from (3.18) and the relation (2.10)).

It is natural to consider asymptotically AdS initial data sets related by a gauge transformation as equivalent. For characteristic initial data as those introduced in Definition 3.4, a general gauge transformation will consist of a change of coordinates \( v \to V(v) \) and a parameter \( \frac{dU}{du}(0) \) at \( u = 0 \), satisfying

\[
(3.25) \quad v \to V(v), \quad \frac{dV}{dv}(v) > 0 \text{ for all } v, \quad V(0) = 0, \quad \frac{dU}{du}(0) > 0.
\]

At a spacetime level, this corresponds to a coordinate transformation of the form

\[
(3.26) \quad (u, v) \to (u', v') = \left( U(u), V(v) \right),
\]

where \( U(u) \) is a function with \( U(0) = 0 \) and \( \frac{dU}{du}(0) \) equal to the given parameter. Under such a gauge transformation, an asymptotically AdS initial data set \( (r_j, \Omega^2_j, f_j; v_2) \) transforms as \( (r_j, \Omega^2_j, f_j; v_2) \to (r_j', \Omega^2_j', f_j'; v_2') \), where

\[
(3.27) \quad r_j'(v') = r_j(v),
\]

\[
(\Omega^2_j')(v') = \frac{1}{\frac{dV}{dv}(v)} \Omega^2_j(v),
\]

\[
f_j'(v'; \frac{dU}{du}(0) \cdot p, l) = f_j(v; p, l).
\]

**Remark.** Under the gauge transformation (3.27), the higher order transversal derivatives computed along \( u = 0 \) through the relations (3.20)–(3.22) transform analogously, i.e. by assuming that \( (r, \Omega^2, f) \) is transformed under the gauge transformation

\[
(3.28) \quad r'(u', v') = r(u, v),
\]

\[
(\Omega^2)'(u', v') = \frac{1}{\frac{dV}{dv}(u)} \cdot \frac{dV}{dv}(v) \Omega^2(u, v),
\]

\[
f'(u', v'; \frac{dU}{du}(u) p, \frac{dV}{dv}(v) p, l) = f(u, v; p, p, l).
\]

provided \( (u, v) \to (u', v') \) fixes the straight line \( \{ u = v \} \) (recall that we have assumed that the axis \( r = 0 \) lies on \( \{ u = v \} \) in the computations (3.20)–(3.22)); the latter condition necessitates that \( U(x) = V(x) \) in a neighborhood of \( x = 0 \), hence fixing germ of \( U \) at \( u = 0 \) by the condition

\[
\frac{d^k U}{(du)^k}(0) = \frac{d^k V}{(du)^k}(0) \text{ for all } k \in \mathbb{N}.
\]

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In this paper, we will be mainly interested in initial data sets \((r_j, \Omega^2_j, \tilde{f}_j; v_x)\) on \(\{u = 0\}\) which give rise to solutions \((r, \Omega^2, f)\) of (2.43)–(2.48) which are smooth, with smooth axis \(\{u = v\}\) and smooth conformal infinity \(\{u = v - v_x\}\), in accordance with Definitions 3.1–3.3. To this end, we will have to ensure that the class of initial data under consideration is compatible with smoothness both at the axis and at conformal infinity; in the latter case, additional compatibility issues arise in regard to the reflecting boundary condition (2.70). The class of smoothly compatible initial data, consisting of data which are consistent with the aforementioned regularity conditions, will be defined as follows:

**Definition 3.5.** Let \((r_j, \Omega^2_j, \tilde{f}_j; v_x)\) be a smooth asymptotically AdS initial data set for (2.43)–(2.48), in accordance with Definition 3.4. We will say that \((r_j, \Omega^2_j, \tilde{f}_j; v_x)\) is smoothly compatible if there exists a \(T > 0\), a pair of smooth functions \(r, \Omega^2 > 0\) on the domain

\[
\mathcal{U}_{T,v_x} = \{0 \leq u < T\} \cap \{u < v < u - v_x\}
\]

in the \((u, v)\) plane and a smooth function \(\tilde{f} : \mathcal{U}_{T,v_x} \times [0, +\infty)^3 \to [0, +\infty)\), such that, defining

\[
f(u, v; p^u, p^v, l) = \tilde{f}(u, v; \Omega^2(u, v)p^u, \Omega^2(u, v)p^v, l) \cdot \delta(\Omega^2(u, v)p^u p^v - \frac{l^2}{r^2(u, v)}),
\]

the following conditions are satisfied by the triplet \((r, \Omega^2, \tilde{f})\):

1. **Smooth extendibility of the data.** Along \(u = 0\)

\[
r(0, v) = r_j(v), \quad \Omega^2(0, v) = \Omega^2_j(0, v), \quad \tilde{f}(0, v; \Omega^2(0, v)p^u, \Omega^2(0, v)p^v, l) = \tilde{f}_j(v, p^u, l)
\]

and, for every integer \(k \geq 1\), the quantities \(\partial^k_u r|_{u=0}, \partial^k_u \Omega^2|_{u=0}\) and \(\partial^k_u \tilde{f}|_{u=0}\) are equal to the values determined by the initial data set \((r_j, \Omega^2_j, \tilde{f}_j; v_x)\) according to the process described in Remark 3 below Definition 3.4.

2. **Axis regularity.** The functions \(r, \Omega^2\) and \(\tilde{f}\) extend smoothly on \(\{u = v\} \cap \{0 \leq u < T\}\) and satisfy

\[
r|_{u=v} = 0, \quad \Omega^2|_{u=v} > 0.
\]

Moreover, the distribution \(f\) satisfies the Vlasov equation (2.30) with respect to \((r, \Omega^2)\) in a neighborhood of \((0, 0)\). Furthermore, switching to the Cartesian coordinate chart \((x^0, x^1, x^2, x^3)\) defined by (3.1), the Cartesian components of the metric \(g_{\alpha\beta}\) extend smoothly on the axis \(\sum_{i=1}^3 (x^i)^2 = 0\).

3. **Compatibility at infinity.** The functions \(\frac{1}{r}\) and \(\frac{\Omega^2}{r^2}\) extend smoothly on \(\{u = v - v_x\} \cap \{0 \leq u < T\}\) and satisfy

\[
\frac{1}{r}|_{u=v-v_x} = 0, \quad \partial_r \left(\frac{1}{r}\right)|_{u=v-v_x} > 0,
\]

and

\[
\frac{\Omega^2}{r^2}|_{u=v-v_x} > 0.
\]

The functions \(\frac{1}{r}\) and \(\frac{\Omega^2}{r^2}\) extend smoothly in an open neighborhood of the point \((0, v_x)\) in the \((u, v)\)-plane, while \(\tilde{f}\) extends smoothly in a neighborhood of \(\{(0, v_x)\} \times [0, +\infty)^3\) in \(\mathbb{R}^2 \times [0, +\infty)^3\). Finally, the distribution \(f\) solves the Vlasov equation (2.30) with respect to \((r, \Omega^2)\) in a left neighborhood of \(\{u = v - v_x\} \cap \{0 \leq u < T\}\) in \(\mathcal{U}_{T,v_x}\), satisfying moreover the reflecting boundary condition (2.70) on \(\{u = v - v_x\}\).

Given a smoothly compatible, asymptotically AdS initial data set \((r_j, \Omega^2_j, \tilde{f}_j; v_x)\), any gauge transformation of the form (3.25)–(3.27) that preserves the Conditions 1–3 above will be called a smoothly compatible transformation for \((r_j, \Omega^2_j, \tilde{f}_j; v_x)\).

The following remarks should be noted regarding Definition 3.5.
where (3.32) The gauge condition (3.31) is equivalent to

initial data sets introduced by Definition 3.4 that uniquely chooses a representative in each equivalence class under

It will be useful for us in this paper, as well as in our companion paper [16], to fix a suitable gauge condition on

\[ \partial_r \] say that Definition 3.6. Let \((r, \Omega^2, f)\) be a smooth, asymptotically AdS solution of (2.43)–(2.48), such that \((r, \Omega^2, f)\) has smooth axis \(\gamma_{\Sigma}\) and smooth conformal infinity \(\mathcal{I}\) and \(f\) satisfies the reflecting boundary condition on \(\mathcal{I}\). In this case, it follows trivially from Definition 3.5 that the initial data set induced by \((r, \Omega^2, f)\) on any slice of the form \(\{u = u_*\}\) is smoothly compatible; in particular, Condition 1 of Definition 3.5 follows trivially from the fact that \((r, \Omega^2, f)\) satisfies the Einstein-equations (2.43)–(2.48) in a whole neighborhood of \(\{u = u_*\}\), while Conditions 2 and 3 are a consequence of the smoothness of \((r, \Omega^2, f)\) on \(\gamma_{\Sigma}\) and \(\mathcal{I}\).

While the triplet \((r, \Omega^2, f)\) in Definition 3.5 is not assumed to satisfy any particular set of equations (apart from the Vlasov equation satisfied by \(f\) near \(v = 0\) and \(v = v_Z\)), the condition that the higher order transversal derivatives \(\partial_r^2 r_{u=0}\) and \(\partial_r^2 \Omega^2_{u=0}\) coincide with the values determined by the process described in Remark 3 below Definition 3.4 is equivalent to the statement that, along \(u = 0\), \((r, \Omega^2, f)\) satisfies the system (2.43)–(2.48) at all orders.

Given a smoothly compatible, asymptotically AdS initial data set \((r_j, \Omega^2_j, \tilde{f}_j; v_Z)\), for a gauge transformation of the form (3.27) to be smoothly compatible, it is necessary that

\begin{equation}
\frac{dU}{du}(0) = \frac{dV}{dv}(0) \quad \text{and} \quad V \in C^\infty([0, v_Z]).
\end{equation}

However, (3.30) is not sufficient for a gauge transformation to be smoothly compatible, since Conditions 1–3 of Definition 3.5 imply an additional relation between \(d^k V / (du)^k(0)\) and \(d^k V / (dv)^k(v_Z)\) for all \(k \in \mathbb{N}\).

It will be useful for us in this paper, as well as in our companion paper [16], to fix a suitable gauge condition on initial data sets introduced by Definition 3.4 that uniquely chooses a representative in each equivalence class under gauge transformations (at least up to rescalings \((u, v) \rightarrow (\lambda u, \lambda v)\)), with the additional property that initial data sets can be uniquely constructed by freely prescribing the initial Vlasov field \(\tilde{f}_j\) in this gauge. This will be achieved at the expense of choosing a gauge in which smooth compatibility is, in general, not preserved.

In particular, we will introduce the following condition:

**Definition 3.6.** Let \((r_j, \Omega^2_j, \tilde{f}_j; v_Z)\) be a smooth, asymptotically AdS initial data set as in Definition 3.4. We will say that \((r_j, \Omega^2_j, \tilde{f}_j; v_Z)\) is gauge normalised if it satisfies the condition

\begin{equation}
\frac{\partial_v r_j}{1 - \frac{1}{3} \Delta r_j^2} = \frac{\Omega^2_j}{4 \partial_v r_j} \quad \text{on} \quad [0, v_Z).
\end{equation}

The gauge condition (3.31) is equivalent to

\begin{equation}
\frac{\partial_v r_j}{1 - \frac{1}{3} \Delta r_j^2} = \frac{\partial_v r_j}{1 - \frac{2 n_j}{r_j} - \frac{1}{3} \Delta r_j^2},
\end{equation}

where \((\partial_v r)_j = \partial_v r|_{u=0}\) is determined in terms of \((r_j, \Omega^2_j, \tilde{f}_j; v_Z)\) by (3.20) and (3.23) for \(k = 0\).
Remark. The condition (3.31) fixes a unique representation of the trivial initial data (i.e. for \( \bar{f}_j \equiv 0 \)) for each value of the endpoint parameter \( v_x \). For the standard choice \( v_x = \sqrt{-\frac{3}{\Lambda} \pi} \), the trivial initial data set \((r_{AdS}, \Omega_{AdS}^2, 0; \sqrt{-\frac{3}{\Lambda} \pi})\) is expressed as:

\[
(3.33) \quad r_{AdS}(v) = \sqrt{\frac{3}{\Lambda}} \tan \left( \frac{1}{2} \sqrt{-\frac{3}{\Lambda}} v \right), \\
\Omega_{AdS}^2(v) = 1 - \frac{1}{3} \Lambda r_{AdS}^2(v).
\]

For different values of \( v_x \), we obtain by rescaling:

\[
(3.34) \quad r^{(v_x)}_{AdS}(v) = r_{AdS}(\sqrt{\frac{3}{\Lambda}} v/v_x), \\
(\Omega^{(v_x)}_{AdS})^2(v) = \frac{3 \pi^2}{\Lambda v_x^2} \Omega_{AdS}^2(\sqrt{\frac{3}{\Lambda}} v/v_x).
\]

Let us also point out that, in view of the constraint equation (2.45), the gauge condition (3.31) can be alternatively expressed as

\[
(3.35) \quad \frac{\partial_v r_f}{1 - \frac{4}{3} \Lambda r_f^2}(v) = \frac{\Omega_f^2}{4 \partial_v r_f}(0) \cdot \exp \left( 4\pi \int_0^v \frac{r_f(T_j)_{vv}(\partial_v r_f)^2(\partial_v r_f)\,dv}{(\partial_v r_f)^2(\partial_v r_f)\,dv} \right)
\]
or, equivalently (by noting that the left hand side of (3.35) integrates to \( \sqrt{-\frac{3}{\Lambda} \pi} \) over \([0, v_x]\)):

\[
(3.36) \quad \frac{\partial_v r_f}{1 - \frac{4}{3} \Lambda r_f^2}(v) = \frac{1}{2a} \exp \left( 4\pi \int_0^v \frac{r_f(T_j)_{vv}(\partial_v r_f)^2(\partial_v r_f)\,dv}{(\partial_v r_f)^2(\partial_v r_f)\,dv} \right),
\]

where the constant

\[
(3.37) \quad a \doteq \sqrt{-\frac{3}{\Lambda} \pi} \frac{1}{3} \pi \int_0^{v_x} \exp \left( 4\pi \int_0^v \frac{r_f(T_j)_{vv}(\partial_v r_f)^2(\partial_v r_f)\,dv}{(\partial_v r_f)^2(\partial_v r_f)\,dv} \right) dv
\]
is determined by the condition that \( r_f(0) = 0, r_f(v_x) = \infty \).

Finally, we should highlight that, in general, an initial data set \((r_f, \Omega_f^2, \bar{f}_j; v_x)\) satisfying the gauge condition (3.31) will *not* be smoothly compatible (although smooth compatibility might be feasible through a suitable gauge transformation). This follows from the observation that, at \( v = v_x \), (3.31) is not consistent with \((\partial_u + \partial_v)^k \frac{\Omega_f^2}{\pi} \) vanishing at \( u = v - v_k \) for \( k \geq 3 \) when \( \bar{m} \neq 0 \). In the case of the trivial AdS initial data set (6.1), the renormalised gauge is trivially a smoothly compatible gauge.

The gauge condition (3.31) allows us to construct initial data sets \((r_f, \Omega_f^2, \bar{f}_j; v_x)\) by freely prescribing the initial Vlasov field \( \bar{f}_j \) under a few regularity conditions. In particular, the following lemma will be useful for the constructions in [16]:

**Lemma 3.1.** Let \( v_x > 0 \) and let \( F : [0, v_x] \times [0, +\infty)^2 \rightarrow [0, +\infty) \) be a smooth function which is compactly supported in \((0, v_x) \times (0, +\infty)^2 \). Then there exists a unique smooth, asymptotically AdS initial data set \((r_f, \Omega_f^2, \bar{f}_j; v_x)\) for (2.45)–(2.48) (as in Definition 3.4) satisfying the gauge condition (3.31), such that

\[
(3.38) \quad \bar{f}_j(v; p^u, l) = F(v; \partial_v r_f(v) p^u, l).
\]

Furthermore, in the case when \( F \) satisfies the smallness condition

\[
(3.39) \quad \mathcal{M}[F] \doteq \int_0^{v_x} \frac{r_{AdS}(T_{AdS}(F))_{ww}}{\partial_v r_{AdS}^2}(\bar{v}) \, d\bar{v} < c_0 \ll 1,
\]

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where \( c_0 > 0 \) is an absolute constant, \((T^{(v_2)}_{AdS}[F])_{vv}\) is defined by

\[
(T^{(v_2)}_{AdS}[F])_{vv}(v) \doteq \frac{\pi}{2} \frac{1}{(r^{(v_2)}_{AdS})^2(v)} \int_0^{+\infty} \int_0^{+\infty} p^2 F(v; p, l) \frac{dp}{p} \, dl
\]

and \( r^{(v_2)}_{AdS} \), \( (\Omega^{(v_2)}_{AdS})^2 \) are given by (3.34), the following estimates hold for some absolute constant \( C > 0 \):

\[
\sup_{v \in (0, v_2)} \left| \frac{\partial_v r_f}{1 - \frac{1}{3} \Lambda r_f^2}(v) - \frac{\partial_v r^{(v_2)}_{AdS}}{1 - \frac{1}{3} \Lambda (r^{(v_2)}_{AdS})^2}(v) \right| \leq CM[F]
\]

and

\[
\int_0^{v_2} r_f(T_f)^{vv}(\tilde{v}) \, d\tilde{v} \leq CM[F].
\]

Remark. As noted below Definition 3.6, the initial data set \((r_f, \Omega_f^2, f; v_2)\) constructed in Lemma 3.1 will not, in general, satisfy the last condition of Definition 3.5.

Proof. Let \( v_2 > 0 \) and \( F : (0, v_2) \times (0, +\infty)^2 \to [0, +\infty) \) be a smooth function of compact support. Let us define the function \( G \in C_\infty((0, v_2)) \) by the relation

\[
G(v) \doteq 32\pi^2 \int_0^{+\infty} \int_0^{+\infty} p^2 F(v; p, l) \frac{dp}{p} \, dl \, dv.
\]

For any \( \bar{a} \geq 0 \), we will consider the following integral equation for the function \( \rho \bar{v}(v), v \geq 0 \):

\[
\rho \bar{v}(v) = \frac{1}{2\bar{a}} \exp \left( \int_0^v \bar{r}(x) \left( \frac{1}{r(x)} (1 - \frac{1}{3} \Lambda r^2(x)) \right) \rho \bar{v}(x) G(x) \, dx \right),
\]

where

\[
\bar{r}(x) \doteq \sqrt{-\frac{3}{\Lambda}} \tan \left( \sqrt{-\frac{\Lambda}{3}} \int_0^x \rho \bar{v}(y) \, dy \right).
\]

The standard theory of ordinary differential equations (and the fact that \( G(x) \) is compactly supported away from \( x = 0 \), as a consequence of our assumptions for \( F \)) implies that, for any \( \bar{a} > 0 \), there exists a maximal \( v^{(a)} \in (0, +\infty] \) such that (3.43) determines a unique pair \( \rho, \bar{r} \in C_\infty([0, v^{(a)}) \) on the interval \([0, v^{(a)}) \) and, if \( v^{(a)} < +\infty \), then

\[
\limsup_{v \to v^{(a)}} |\rho \bar{v}(v)| = +\infty,
\]

\[
\liminf_{v \to v^{(a)}} |\bar{r}(v)| = 0 \quad \text{and} \quad v^{(a)} \in \text{supp}(G)
\]

or

\[
\limsup_{v \to v^{(a)}} |\bar{r}(v)| = +\infty.
\]

We will show that, for any \( \bar{a} > 0 \), the endpoint \( v^{(a)} \) is finite, i.e.

\[
v^{(a)} < +\infty
\]

and

\[
\lim_{v \to v^{(a)}} \bar{r}(v) = +\infty.
\]
Proof of (3.48) and (3.49). The fact that (3.48) holds follows readily from the assumption that $F$ is compactly supported, and hence there exists a $v_0 > 0$ such that $G(v) = 0$ for $v \geq v_0$: Assuming, for the sake of contradiction that 

$$v^{(a)} = +\infty$$

from (3.43) we infer that

$$\rho(v) = \text{const} = c_1 > 0 \text{ for } v \geq v_0.$$ 

Thus, there exists a $v_1 > v_0$ such that

$$\sqrt{-\frac{\Lambda}{3}} \int_{0}^{v_1} \rho(y) dy = \frac{\pi}{2}.$$ 

From (3.44), we therefore infer that

$$\bar{r}(v_1) = +\infty,$$

which implies that $v^{(a)} < v_1$; hence, assuming that (3.48) does not hold, we reach a contradiction.

We will now proceed to establish (3.49). The relation (3.43) implies that $\rho > 0$ and, hence, $\bar{r}(v)$ defined by (3.44) is strictly increasing and positive for $v > 0$; hence, if (3.47) holds, then (3.49) would follow. Moreover, since $G(v)$ is compactly supported away from $v = 0$ and $\bar{r}(v) \geq 0$ is strictly increasing in $v$, (3.46) cannot hold. In order to establish (3.49), it thus suffices to rule the case where

$$\limsup_{v \to v^{(a)}} |\bar{r}(v)| = +\infty \text{ and } \limsup_{v \to v^{(a)}} |\bar{r}(v)| < +\infty.$$ 

Assume, for the sake of contradiction, that (3.52) holds. Since $\rho > 0$, the upper bound (3.52) for $\bar{r}$ and the relation (3.44) between $\bar{r}$ and $\rho$ imply that

$$\int_{0}^{\bar{r}^{(a)}} \rho(v) dv < +\infty.$$ 

From (3.43), using the bound (3.52) for $\bar{r}$, the fact that $G \in C^{\infty}_{c}((0, v_{T}))$ and that $\bar{r} > 0$ on the support of $G$, we can readily estimate for some $C_1 > 0$ depending on $\bar{r}$, $G$:

$$\limsup_{v \to v^{(a)}} |\bar{r}(v)| = \frac{1}{2\bar{a}} \exp \left( \int_{0}^{v^{(a)}} \frac{1}{\bar{r}(x)} \left( 1 - \frac{1}{2} \Lambda \bar{r}^{2}(x) \right) \rho(x) G(x) dx \right) \leq \frac{1}{2\bar{a}} \exp \left( C_1 \int_{0}^{v^{(a)}} \rho(v) dv \right) < +\infty.$$ 

Therefore, we deduce that (3.52) does not hold, reaching a contradiction. As a consequence, (3.49) holds.

It can be readily shown that $v^{(a)}$ depends continuously on $\bar{a} > 0$. Furthermore, the relations (3.43) and (3.44) imply that, for any fixed value $v_0 > 0$, the functions $\rho(v_0)$ and $\bar{r}(v_0)$ are strictly decreasing in $\bar{a}$, provided $v_0 < v^{(a)}$, and we have

$$\lim_{\bar{a} \to 0^{+}} v^{(a)} = 0 \text{ and } \lim_{\bar{a} \to +\infty} v^{(a)} = +\infty.$$ 

Therefore, there exists a unique $\bar{a}_0 > 0$ such that

$$v^{(\bar{a}_0)} = v_{T}.$$ 

Let us fix $\bar{a} = \bar{a}_0$ and let us define the function $r_f : [0, v_{T}) \to [0, +\infty)$ by the relation

$$r_f(v) = \bar{r}(v)$$

(3.55)
(note that this is possible in view of (3.54). The relations (3.43)–(3.44) and (3.49) then imply that \( \frac{1}{r_f} \) extends smoothly at \( v = v_\ell \), and we have

\[
(3.56) \quad r_f(0) = 0, \quad \frac{1}{r_f(v_\ell)} = 0,
\]

\[
\partial_v r_f(0) > 0, \quad \partial_v \left( \frac{1}{r_f(v_\ell)} \right) < 0.
\]

Let us also define \( \bar{f}_f: (0, v_\ell) \times [0, +\infty) \to [0, +\infty) \) by (3.38). The relations (3.43)–(3.44) can be reexpressed in terms of \( r_f \) and \( \bar{f}_f \) as follows:

\[
(3.57) \quad \frac{\partial_v r_f}{1 - \frac{1}{3} \Lambda r_f^2}(v) = \frac{1}{2a_0} \exp \left\{ 32\pi^2 \int_0^v \frac{(\partial_v r_f(x))^3}{r_f(x)} \left( \int_0^{+\infty} \int_0^{+\infty} (p^n)^2 \bar{f}_f(x; p^n, l) \frac{dp^n}{p^n} \right) dx \right\}.
\]

Note that (3.49) and (3.57) imply that

\[
(3.58) \quad a_0 = \sqrt{-\frac{\Lambda}{3\pi}} \int_0^{v_\ell} \exp \left\{ 32\pi^2 \int_0^v \frac{(\partial_v r_f(x))^3}{r_f(x)} \left( \int_0^{+\infty} \int_0^{+\infty} (p^n)^2 \bar{f}_f(x; p^n, l) \frac{dp^n}{p^n} \right) dx \right\}.
\]

Using (3.31) to define \( \Omega_f^2 \), i.e.

\[
(3.59) \quad \Omega_f^2(v) = \frac{4(\partial_v r_f)^2}{1 - \frac{1}{3} \Lambda r_f^2}(v),
\]

we can readily calculate that the right hand side of (3.57) is the same as the right hand side of (3.36). Furthermore, (3.59) and (3.56) imply that \( r_f^{-2} \Omega_f^2 \) extends smoothly on \( v = v_\ell \) and satisfies

\[
\Omega_f^2(0) > 0, \quad r_f^{-2} \Omega_f^2(v_\ell) > 0.
\]

As a result (in view also of the fact that \( \bar{f}_f \in C^\infty_c((0, v_\ell) \times (0, +\infty)) \), the quadruplet \( (r_f, \Omega_f^2, \bar{f}_f; v_\ell) \) satisfies all conditions of Definition 3.4.

It remains to show that, when \( F \) satisfies the smallness condition (3.39), then the bounds (3.40) and (3.41) hold. Let us introduce the continuity parameter \( \varepsilon \in [0, 1] \), and let us consider the family of initial data sets \( (r_f^{(\varepsilon)}, \Omega_f^{(\varepsilon)}^2, \bar{f}_f^{(\varepsilon)}; v_\ell) \) corresponding to

\[
F^{(\varepsilon)} = \varepsilon F.
\]

Note that the standard theory for odes applied to (3.43)–(3.44) implies that, for any \( v \in (0, v_\ell) \), the quantities \( r_f^{(\varepsilon)}(v), \Omega_f^{(\varepsilon)}(v) \) depend continuously on \( \varepsilon \). Note also that, when \( \varepsilon = 0 \), (3.43)–(3.44) imply that

\[
(3.60) \quad \sup_{v \in (0, v_\ell)} \left| \frac{\partial_v r_f^{(\varepsilon)}}{1 - \frac{1}{3} \Lambda r_f^{(\varepsilon)}(v)} - \frac{\partial_v r_f^{(0)}}{1 - \frac{1}{3} \Lambda r_f^{(0)}} \right| (v) \leq 2C_1 \varepsilon M[F]
\]

and

\[
(3.61) \quad \int_0^{v_\ell} \frac{r_f^{(\varepsilon)}(T_f^{(\varepsilon)})(v)}{\partial_v r_f^{(\varepsilon)}}(v) \, dv \leq 2C_1 \varepsilon M[F]
\]

\[
(3.62) \quad \int_0^{v_\ell} \frac{\partial_v r_f^{(\varepsilon)}}{\partial_v r_f^{(\varepsilon)}}(v) \, dv \leq 2C_1 \varepsilon M[F]
\]
then, in fact, the following stronger estimates hold for all \( \epsilon \in [0, \epsilon_0] \):

\[
\left(3.62\right) \quad \sup_{v \in (0, v_2)} \left| \frac{\partial v f^{(\epsilon)}(v)}{1 - \frac{1}{3} \Lambda r^{(\epsilon)}_j} - \frac{\partial v f^{(\epsilon)}_{\text{AdS}}(v)}{1 - \frac{1}{3} \Lambda r_{\text{AdS}}^{(\epsilon)}} \right| \leq C_1 \epsilon \mathcal{M}[F]
\]

and

\[
\left(3.63\right) \quad \int_0^{v_2} \frac{\partial v f^{(\epsilon)}(v)}{\partial v r^{(\epsilon)}_j} \cdot (\hat{v}) \, dv \leq C_1 \epsilon \mathcal{M}[F]
\]

Differentiating (3.43)–(3.44) for \( F^{(\epsilon)} \) with respect to \( \epsilon \) (noting that \( \frac{1}{\rho_0} = \bar{\rho}^{(\epsilon)}(0) \)), we obtain

\[
\left(3.64\right) \quad \frac{d}{d \epsilon} \left( \bar{\rho}^{(\epsilon)}(v) - \bar{\rho}^{(\epsilon)}(0) \right) = \bar{\rho}^{(\epsilon)}(v) \cdot \int_0^v \frac{d}{d \epsilon} \left( \frac{1}{\bar{r}^{(\epsilon)}(1 - \frac{1}{3} \Lambda \bar{r}^{(\epsilon)})^2} \right) \bar{\rho}^{(\epsilon)}(x) G^{(\epsilon)}(x) \, dx + \bar{\rho}^{(\epsilon)}(v) \cdot \int_0^v \frac{1}{\bar{r}^{(\epsilon)}(1 - \frac{1}{3} \Lambda \bar{r}^{(\epsilon)})^2} \bar{\rho}^{(\epsilon)}(x) G(x) \, dx
\]

and

\[
\left(3.65\right) \quad \frac{d}{d \epsilon} \bar{r}^{(\epsilon)}(v) = \left( 1 - \frac{1}{3} \Lambda \bar{r}^{(\epsilon)} \right)^2 \int_0^v \frac{d}{d \epsilon} \bar{\rho}^{(\epsilon)}(y) \, dy.
\]

Setting for convenience

\[
R_1^{(\epsilon)} \doteq \frac{d}{d \epsilon} \log \bar{\rho}^{(\epsilon)}, \quad R_2^{(\epsilon)} \doteq \frac{\partial v r^{(\epsilon)}_j}{1 - \frac{1}{3} \Lambda r^{(\epsilon)}_j(v)},
\]

and using the fact that

\[
\bar{r}^{(\epsilon)} = r^{(\epsilon)}_j, \quad \bar{\rho}^{(\epsilon)} = \frac{\partial v r^{(\epsilon)}_j}{1 - \frac{1}{3} \Lambda r^{(\epsilon)}_j(v)},
\]

(see (3.55)), the relations (3.64)–(3.65) and the bounds (3.39) and (3.66)–(3.67) readily yield:

\[
\left(3.66\right) \quad |R_1^{(\epsilon)}(v) - R_1^{(\epsilon)}(0)| \leq C_0 \int_0^v H_1^{(\epsilon)}(\hat{v}) \cdot \left( |R_1^{(\epsilon)}(\hat{v})| + \frac{|R_2^{(\epsilon)}(\hat{v})|}{\hat{v}} \right) \hat{v} \, d\hat{v} + C_0 \mathcal{M}[F]
\]

and

\[
\left(3.67\right) \quad |R_2^{(\epsilon)}(v)| \leq C_0 \int_0^v |R_1^{(\epsilon)}(\hat{v})| \, d\hat{v}.
\]

for some absolute constant \( C_0 > 0 \), where \( H_1^{(\epsilon)}(v) \geq 0 \) satisfies the bound

\[
\int_0^v H_1^{(\epsilon)}(\hat{v}) \, d\hat{v} \leq C(C_1) \epsilon \mathcal{M}[F]
\]

for some \( C(C_1) > 0 \) depending only on \( C_1 \).

Plugging (3.67) in (3.66) and applying Gronwall’s inequality for the function

\[
\bar{R}_1^{(\epsilon)}(v) = \max_{\hat{v} \in [0, v]} |R_1^{(\epsilon)}(\hat{v}) - R_1^{(\epsilon)}(0)|,
\]

we readily obtain that

\[
\left(3.68\right) \quad \sup_{v \in (0, v_2)} |R_1^{(\epsilon)}(v) - R_1^{(\epsilon)}(0)| \leq C_0 \exp \left( C(C_1) \mathcal{M}[F] \right) \mathcal{M}[F] \cdot \left( 1 + |R_1^{(\epsilon)}(0)| \right).
\]
The identity
\[ \frac{d}{d\varepsilon} \left( \int_0^{v_\varepsilon} \hat{\rho}^{(e)}(v) \, dv \right) = \frac{d}{d\varepsilon} \left( \sqrt{-\frac{3}{2} \Lambda} \right) = 0, \]
which holds for all \( \varepsilon \in [0, \varepsilon_0] \) (as a consequence of (3.44) and the fact that \( \hat{\rho}^{(e)}(v) = +\infty \), in view of (3.54)), implies, in view of the bound (3.68) and the estimate (3.62) that
\[ (3.69) \quad 0 = \int_0^{v_\varepsilon} \hat{\rho}^{(e)}(v) \hat{R}^{(e)}(v) \, dv = \int_0^{v_\varepsilon} \left( \frac{\partial_v \hat{\rho}^{(e)}(v)_{\text{AdS}}}{1 - \frac{1}{3} \Lambda r^{(e)}_{\text{AdS}} v} \right) + O(C_1 M[F]) \cdot \left( \hat{R}^{(e)}(0) + O(\exp(C_1 M[F])) \right) \hat{R}^{(e)}(v) \, dv. \]
The identity (3.69) implies (since \( \hat{R}^{(e)}(0) \) is independent of \( v \) and hence can be moved outside of the integral) that, after possibly choosing a larger absolute constant \( C_0 > 0 \), and assuming that the constant \( c_0 \) in (3.39) has been chosen small enough in terms of \( C_1 \):
\[ (3.70) \quad |\hat{R}^{(e)}(0)| \leq C_0 \exp(C_1 M[F]) M[F] \leq C_0 M[F]. \]
From (3.68) and (3.70) we therefore infer that, provided \( C_1 > C_0 \) and \( c_0 \) is small enough in terms of \( C_1 \), we can estimate for for any \( \varepsilon \in [0, \varepsilon_0] \):
\[ (3.71) \quad \sup_{v \in (0,v_\varepsilon)} \left| \frac{d}{d\varepsilon} \left( \frac{\partial_v \hat{\rho}^{(e)}(v)_{\text{AdS}}}{1 - \frac{1}{3} \Lambda r^{(e)}_{\text{AdS}} v} \right) \right| \leq 2 C_0 M[F] \leq C_1 M[F]. \]
Integrating (3.71) in \( \varepsilon \in [0,\varepsilon_0] \), we readily infer (3.62). Using (3.62), the bound (3.63) follows readily from (3.39) and the relation (3.38) between \( f_j \) and \( F \). Therefore, the proof of the lemma is complete. \( \square \)

### 3.3 Transforms between gauge normalised and smoothly compatible initial data sets

In this section, we will investigate the relationship between smoothly compatible and gauge normalised initial data sets, as introduced by Definitions 3.5 and 3.6 respectively.

The following lemma shows that, for every smoothly compatible, asymptotically AdS initial data set \((r_j, \Omega_j^2, \tilde{f}_j; v_\varepsilon)\), there exists a unique gauge transformation \( v \mapsto \varepsilon = V(v) \) of the form (3.25) satisfying (3.30) and
\[ V(v_\varepsilon) = v_\varepsilon, \]
such that the transformed initial data set \((r'_j, \Omega'^2_j, \tilde{f}_j; v_\varepsilon)\) (see the relation (3.27)) satisfies the gauge condition (3.31).

**Lemma 3.2.** Let \((r_j, \Omega_j^2, \tilde{f}_j; v_\varepsilon)\) be a smoothly compatible asymptotically AdS initial data set, in accordance with Definition 3.5 with bounded support in phase space (i.e. satisfying the condition (3.10)). Then, there exists a gauge transformation \( v \mapsto \varepsilon = V(v) \) of the form (3.25) satisfying (3.30) and
\[ V(v_\varepsilon) = v_\varepsilon, \]
such that the transformed initial data set \((r'_j, \Omega'^2_j, \tilde{f}_j; v_\varepsilon)\) (see the relation (3.27)) satisfies the gauge condition (3.31).

**Remark.** As noted below Definition 3.6 (see also the remark below Definition 3.5), the aforementioned gauge transformation will not be, in general, smoothly compatible, despite satisfying (3.30). See also Lemma 4.1.

**Proof.** In order to construct the function \( \varepsilon = V(v) \), we will make use of the equivalent form (3.35) of the gauge condition (3.31). To this end, let us note that the function
\[ F(v) \triangleq \exp \left( 4\pi \int_0^v r_j(T_j)_{vv} (\partial_v r_j)(\partial_v v) \, dv \right). \]
(appearing in the right hand side of (3.35)) is gauge independent, i.e. transforms under a gauge transformation of the form (3.27) as
\[ F'(V(v)) = F(v). \]

As a result, the function
\[ G(v) \equiv \frac{\Omega^2_1}{4\partial_v r_j}(0) \cdot F(v) \]
transforms under such a gauge transformation with \( \frac{dU}{dv}(0) \) as:
\[ G'(V(v)) = \frac{1}{\frac{dV}{dv}(0)} \cdot G(v). \]

The gauge condition (3.25) for the gauge transformed initial data set \((r'_j, (\Omega'_j)^2, \bar{f}_j; v_I)\) is equivalent to the relation
\[ \frac{dV}{dv}(v) = \frac{dV}{dv}(0) \cdot \frac{\partial_v r_j}{1 - \frac{1}{3} \Lambda r_j^2}(v) \cdot \frac{1}{G(v)} \quad \text{for all } v \in [0, v_I]. \]

Note that the relation (3.72) is trivially satisfied at \( v = 0 \) for any choice of the function \( V(v) \), as a consequence of the fact that (3.29) holds for any smoothly compatible initial data set.

Considering
\[ b = \frac{dV}{dv}(0) \]
as a positive parameter, we infer that the proof of the lemma will conclude by showing that there exists a unique \( b > 0 \) such that the smooth function \( V(v) \) determined by (3.72) and the condition \( V(0) = 0 \), i.e.
\[ V(v) = b \int_0^v \frac{\partial_v r_j}{1 - \frac{1}{3} \Lambda r_j^2}(\bar{v}) \cdot \frac{1}{G(\bar{v})} \, d\bar{v}, \]
satisfies in addition
\[ V(v_I) = v_I. \]

It can be readily verified that (3.73) uniquely fixes \( b \) by the relation
\[ b = \int_0^{v_I} \frac{\partial_v r_j}{1 - \frac{1}{3} \Lambda r_j^2}(\bar{v}) \cdot \frac{1}{G(\bar{v})} \, d\bar{v}, \]
noting that the finiteness of the denominator in (3.74) follows from the fact that \( f_j \) was assumed to be of bounded support in phase space.

Let us now turn to the opposite question of that addressed by Lemma 3.2, namely that of determining whether an initial data set \((r_j, \Omega_j^2, \bar{f}_j; v_I)\) given in the normalised gauge of Definition 3.6 is gauge-equivalent to a smoothly compatible initial data set, as in Definition 3.5. The following lemma (which will not be used again in this paper, but which will be useful for our companion paper [16]) provides a broad class of normalised initial data sets for which such a transformation always exists; this class contains, in particular, the initial data sets considered in our companion paper [16]. The proof of this result will in make use of Proposition 4.2 for double characterisitc initial value problems for (2.43)–(2.48), established later in Section 4.

The advantage of working in the normalised gauge condition of Definition 3.6 lies in the flexibility it provides to uniquely determine an initial data set \((r_j, \Omega_j^2, \bar{f}_j; v_I)\) by freely prescribing the value of \( f_j \).
Lemma 3.3. Let \((r_f, \Omega_f^2, \bar{f}_f; v_I)\) be a smooth asymptotically AdS initial data set with bounded support in phase space, in accordance with Definition 3.4, satisfying the normalised gauge condition (3.31). Assume that \(\bar{f}_f\) is supported away from \(v = 0, v_I\) and \(l = 0\), i.e. there exists some \(\varepsilon > 0\), such that \(\bar{f}_f\) satisfies

\[
\bar{f}_f(v; p, l) = 0 \quad \text{for} \quad v \in (0, \varepsilon] \cup [v_I - \varepsilon, v_I)
\]

and

\[
\bar{f}_f(v; p, l) = 0 \quad \text{for} \quad l \in [0, \varepsilon].
\]

Then, there exists a gauge transformation of the form (3.25)-(3.27) with \(\frac{dU}{du}(0) = 1\) and

\[
u'(v) = v \quad \text{for} \quad v \leq v_I - \frac{1}{2} \varepsilon \quad \text{and} \quad \nu'(v_I) = v_I
\]

such that the transformed initial data set \((r'_f, (\Omega_f')^2, \bar{f}_f'; v_I)\) is smoothly compatible, in accordance with Definition 3.5. Furthermore, for any \(\eta_0 \in (0, 1)\), the gauge transformation can be chosen so that

\[
1 - \eta_0 \leq \frac{dv'}{dv}(v) \leq 1 + \eta_0 \quad \text{for} \quad v \in [0, v_I]
\]

and

\[
\max_{v \in [0, v_I]} \left| \frac{d^2 v'}{d v^2} (v) \right| \leq \int_0^{v_I} \left( \frac{1 - \Lambda v^2_f}{1 - \frac{1}{3} \Lambda v^2_f} (T_f)_{vv} + 3 (T_f)_{uv} \right) (v) \, dv + \frac{\eta_0}{v_I}
\]

where \((T_f)_{vv}, (T_f)_{uv}\) are defined in terms of \((r_f, \Omega_f^2, \bar{f}_f)\) by the corresponding relations in (3.12).

Remark. At a spacetime level, the gauge transformation in the statement of Lemma 3.3 is of the form \((u, v) \rightarrow (u, v')\).

Proof. The proof of Lemma 3.3 will proceed by constructing an asymptotically AdS solution \((r, \Omega^2, f)\) of (2.43)-(2.48) on the domain \(\mathcal{U}_{v_f, v_I}\) for some \(T > 0\), satisfying the reflecting boundary condition on \(\{u = v - v_I\}\), such that:

- \((r, \Omega^2, f)\) induces on \(u = 0\) the initial data set \((r_f, \Omega_f^2, \bar{f}_f; v_I)\),

- \((r, \Omega^2, f)\) can be transformed into a solution with smooth axis \(\{u = v\}\) and smooth conformal infinity \(\{u = v - v_I\}\) after applying a gauge transformation \((u, v) \rightarrow (u, v')\) with \(v'\) satisfying (3.77).

The construction of \((r, \Omega^2, f)\) will be performed in three steps.

1. In view of the assumption (3.75) for \(\bar{f}_f\) and the fact that \(r_f, \Omega_f^2\) satisfy the normalised gauge condition (3.31), it follows readily that \((r_f, \Omega_f^2, \bar{f}_f)\) coincides, for \(v \in (0, \varepsilon]\), with the normalised (rescaled) trivial initial data set \((r^{(v_0)}_{AdS}), (\Omega^{(v_0)}_{AdS})^2, 0\) , given by (3.34) for some \(v_0 > 0\). Therefore, if we define the triplet \((r, \Omega^2, f)\) on the domain

\[
D_0^\varepsilon = \{ 0 \leq u \leq \varepsilon \} \cap \{ u < v < \varepsilon \}
\]

(see Figure 3.3) as

\[
(r, \Omega^2, f)|_{D_0^\varepsilon} \doteq (r^{(v_0)}_{AdS}, (\Omega^{(v_0)}_{AdS})^2, 0),
\]

where

\[
r^{(v_0)}_{AdS}(u, v) = r_{AdS} \left( \sqrt{\frac{3}{\Lambda}} \frac{u}{v_0}, \sqrt{\frac{3}{\Lambda}} \frac{v}{v_0} \right),
\]

\[
(\Omega^{(v_0)}_{AdS})^2(u, v) = -\frac{3}{\Lambda} \frac{\pi^2}{v_0} \Omega^2_{AdS} \left( \sqrt{\frac{3}{\Lambda}} \frac{u}{v_0}, \sqrt{\frac{3}{\Lambda}} \frac{v}{v_0} \right),
\]

then \((r, \Omega^2, f)|_{D_0^\varepsilon}\) is (trivially) a smooth solution of (2.43)-(2.48), with smooth axis \(\{u = v\}\), inducing on \(\{u = 0\} \cap D_0^\varepsilon\) the initial data \((r_f, \Omega_f^2, \bar{f}_f)\).
2. For any \( u_* \in (0, \frac{1}{2} \varepsilon) \), let us consider the double characteristic initial value problem for the system (2.43)–(2.48) on the domain
\[
\mathcal{W}_* \doteq [0, u_*) \times \left[ \frac{1}{2} \varepsilon, v_* - \frac{1}{4} \varepsilon \right]
\]
(see Figure 3.3) with characteristic initial data on \([0, u_*) \times \{ \frac{1}{2} \varepsilon \} \) and \([0] \times \left[ \frac{1}{2} \varepsilon, v_* - \frac{1}{4} \varepsilon \right]\) given, respectively, by
\[
(r, \Omega^2, \tilde{f})(u) = (r^{(v_0)}(u), \frac{1}{2} \varepsilon), (\Omega^{(v_0)}(u), \frac{1}{2} \varepsilon), 0) \text{ for } u \in [0, u_*]
\]
and
\[
(r, \Omega^2, \tilde{f})|_{\frac{1}{2} \varepsilon, v_* - \frac{1}{4} \varepsilon}.
\]
In view of the fact that \( u_* < \frac{1}{4} \varepsilon \), there exists a constant \( c_0 \) (depending on \( \varepsilon \)) such that
\[
\min_{u \in [0, u_*]} r(u), \min_{v \in [\frac{1}{2} \varepsilon, v_* - \frac{1}{4} \varepsilon]} r_f(v) > c_0
\]
and
\[
\max_{u \in [0, u_*]} r(u), \max_{v \in [\frac{1}{2} \varepsilon, v_* - \frac{1}{4} \varepsilon]} r_f(v) < c_0^{-1}.
\]
As a result, by applying Proposition 4.2, we infer that, provided \( u_* \) is chosen small enough in terms of \((r_f, \Omega^2, \tilde{f})\) and \( \varepsilon \), there exists a unique smooth solution \((r, \Omega^2, f)\) of (2.43)–(2.48) on \( \mathcal{W}_* \) inducing on \([0, u_*] \times \{ \frac{1}{2} \varepsilon \} \) and \([0] \times \left[ \frac{1}{2} \varepsilon, v_* - \frac{1}{4} \varepsilon \right]\) the initial data (3.82) and (3.83), respectively.

Furthermore, since \((r_f, \Omega^2, \tilde{f}) \equiv (r^{(v_0)}(u), (\Omega^{(v_0)}(u), 0)\text{ for } v \in [\frac{1}{2} \varepsilon, \varepsilon], \) the solution \((r, \Omega^2, f)\) satisfies
\[
(r, \Omega^2, f)|_{[0, u_*] \times [\frac{1}{2} \varepsilon, \varepsilon]} \equiv (r^{(v_0)}(u), (\Omega^{(v_0)}(u), 0).
\]

Therefore, the solution \((r, \Omega^2, f)\) constructed in this step on \( \mathcal{W}_* \) coincides on \( \mathcal{D}_0^\varepsilon \cap \mathcal{W}_* \) with the solution \((r, \Omega^2, f)\) constructed in the previous step on \( \mathcal{D}_0^\varepsilon \) (see Figure 3.3); as a result, by gluing the two solutions, we obtain a single smooth solution \((r, \Omega^2, f)\) of (2.43)–(2.48) on \( \mathcal{D}_0^\varepsilon \cup \mathcal{W}_* \).
3. In view of the assumption that \( \bar{f}_j \) has bounded support in phase space and satisfies (3.76), we infer that there exists constant \( c > 0 \) such that, for all \( v \in [0, v_T - \frac{1}{4} \varepsilon] \):

\[
(3.85) \quad \bar{f}_j(v, p, l) = 0 \text{ when } p \leq c \text{ or } \frac{1}{p} \leq c \text{ or } l \geq c^{-1} v_T.
\]

The condition (3.85) and the smoothness of the solution \( (r, \Omega^2, f) \) constructed in the previous step on \( \mathcal{D}_0^p \cup \mathcal{W}_* \) implies there exists a \( u' > 0 \) sufficiently small and a \( C > 0 \) sufficiently large such that \( f \) satisfies on \( \{0 \leq u \leq u'\} \):

\[
(3.86) \quad \text{supp}(f) \cap \{0 \leq u \leq u'\} \subset \{ \frac{p^v}{p^u} \leq C \},
\]

i.e. that the phase space support of \( f \) restricted on the physical space domain \( \{0 \leq u \leq u'\} \cap \mathcal{W}_* \) contains null geodesics which project on the \((u, v)\)-plane as timelike curves \( \gamma \) of slope satisfying

\[
\frac{\dot{v}}{\dot{u}} \leq C.
\]

In particular, since \( \bar{f}_j \) satisfies (3.75) (and hence \( \bar{f}_j \equiv 0 \) for \( v \geq v_T - \varepsilon \)), every null geodesic \( \gamma \) in the support of \( f \) satisfies

\[
\gamma \cap \{0 \leq u \leq u'\} \subset \{ v \leq v_T - \varepsilon + Cu \},
\]

that is to say:

\[
(3.87) \quad f \equiv 0 \text{ on } \mathcal{W}_* \cap (\{0, u'\} \times [v_T - \varepsilon + Cu', +\infty]).
\]

The constraint equations (2.45)–(2.46) then imply that, on \( \mathcal{W}_* \cap (\{0, u'\} \times [v_T - \varepsilon + Cu', +\infty]) \), \( (r, \Omega^2) \) is locally isometric to a member of the Schwarzschild–AdS family of metrics (this is a consequence of the extension of Birkhoff’s theorem to the case \( \Lambda < 0 \); see [5]). In particular, setting

\[
M \doteq m_j(v_T - \frac{1}{4} \varepsilon) = \bar{m}_j(v_T)
\]

(the last equality following from the assumption (3.75) on \( \bar{f}_j \)) and assuming that \( u' \) has been fixed small enough so that

\[
Cu' < \frac{1}{2} \varepsilon,
\]

the relations (2.10), (2.45), (2.46) and (3.87) imply that, on \( \mathcal{W}_* \cap (\{0, u'\} \times [v_T - \frac{1}{2} \varepsilon, +\infty]) \):

\[
(3.88) \quad \Omega^2 = 4 \frac{\partial_v r (-\partial_u r)}{1 - \frac{2M}{r} - \frac{1}{3} \Lambda r^2}
\]

and

\[
(3.89) \quad \partial_u \left( \frac{\partial_v r}{1 - \frac{2M}{r} - \frac{1}{3} \Lambda r^2} \right) = \partial_v \left( \frac{-\partial_u r}{1 - \frac{2M}{r} - \frac{1}{3} \Lambda r^2} \right) = 0.
\]

Remark. By considering a possibly smaller value of \( \varepsilon \) (depending on \( (r_j, \Omega_j^2, \bar{f}_j) \)), we can arrange so that \( \frac{2m_j(v_T)}{r(v_T)} < 1 \) for \( v \geq v_T - \frac{1}{2} \varepsilon \). In this case, assuming that \( u' \) is even smaller, if necessary, we can bound \( 1 - \frac{2M}{r} - \frac{1}{3} \Lambda r^2 > 0 \) on \([0, u'] \times [v_T - \frac{1}{2} \varepsilon, v_T - \frac{1}{4} \varepsilon] \). Hence, for the rest of the proof, dividing with \( 1 - \frac{2M}{r} - \frac{1}{3} \Lambda r^2 \) will not pose a concern.
Let us define the functions $K : [v_\mathcal{I} - \frac{1}{2} \varepsilon, v_\mathcal{I}] \to (0, +\infty)$ and $\tilde{K} : [0, u'] \to (0, +\infty)$ by

$$K(v) = \frac{\partial_v r_I}{1 - \frac{2 \hat{m}_I}{r_I} - \frac{4}{3} \Lambda r_I^2} (v)$$

and

$$\tilde{K}(u) = \frac{-\partial_u r}{1 - \frac{2 \hat{m}}{r} - \frac{4}{3} \Lambda r^2} (u, v_\mathcal{I} - \frac{1}{2} \varepsilon).$$

Since the functions $\frac{1}{r_I}$, $\Omega_I^2$ and $\hat{f}_I(\cdot; \Omega_I^2 p, l)$ extend smoothly on $v = v_\mathcal{I}$ for any $p, l \geq 0$ (as a consequence of the conditions imposed by Definition 3.4), the function $K$ can be extended on the whole of the interval $[v_\mathcal{I} - \frac{1}{2} \varepsilon, v_\mathcal{I} + \delta]$ for some $\delta > 0$ such that

$$K \in C^\infty([v_\mathcal{I} - \frac{1}{2} \varepsilon, v_\mathcal{I} + \delta]).$$

From now on, we will assume that we have fixed such a smooth extension of $K$. Note also that, in view of the fact that $\partial_v \hat{m}_I = 0$ on $[v_\mathcal{I} - \frac{1}{2} \varepsilon, v_\mathcal{I}]$ (and hence $\hat{m}_I = M$ on that interval), the normalised gauge condition (3.31) implies that

$$\frac{\tilde{K}(0)}{\tilde{K}(v)} = \frac{1 - \frac{2 M}{r_I(v)} - \frac{4}{3} \Lambda r_I^2(v)}{1 - \frac{4}{3} \Lambda r_I^2(v)} \text{ for } v \in [v_\mathcal{I} - \frac{1}{2} \varepsilon, v_\mathcal{I}].$$

The relation (3.89) implies that on $[0, u'] \times [v_\mathcal{I} - \frac{1}{2} \varepsilon, v_\mathcal{I} - \frac{1}{4} \varepsilon]$, we have

$$\frac{\partial_v r}{1 - \frac{2 M}{r} - \frac{4}{3} \Lambda r^2} (u, v) = K(v) \text{ and } \frac{-\partial_u r}{1 - \frac{2 \hat{m}}{r} - \frac{4}{3} \Lambda r^2} (u, v) = \tilde{K}(u) \text{ for all } (u, v) \in [0, u'] \times [v_\mathcal{I} - \frac{1}{2} \varepsilon, v_\mathcal{I} - \frac{1}{4} \varepsilon],$$

in which case (3.88) can be reexpressed as

$$\frac{\Omega^2}{1 - \frac{2 M}{r} - \frac{4}{3} \Lambda r^2} (u, v) = 4K(v)\tilde{K}(u) \text{ for all } (u, v) \in [0, u'] \times [v_\mathcal{I} - \frac{1}{2} \varepsilon, v_\mathcal{I} - \frac{1}{4} \varepsilon].$$

Let us fix a smooth function $v' : \mathbb{R} \to \mathbb{R}$ with the following properties:

(a) $v'(v) = v$ for $v \leq v_\mathcal{I} - \frac{1}{2} \varepsilon$,

(b) $\frac{dv'}{d\varepsilon} > 0$ for $v \in [0, v_\mathcal{I}]$,

(c) $v'(v_\mathcal{I}) = v_\mathcal{I}$ and

(d) There exists some $\delta' \in (0, \delta)$ such that $\frac{dv'}{d\varepsilon}(v)$ satisfies

$$\frac{dv'}{d\varepsilon}(v) = \frac{K(v)}{K(v'(v) - v_\mathcal{I})} \text{ for all } v \in [v_\mathcal{I}, v_\mathcal{I} + \delta']$$

(recall that we have extended $K(v)$ smoothly for $v \in [v_\mathcal{I}, v_\mathcal{I} + \delta]$).

(e) The function $v'(v)$ satisfies the following $C^2$ bounds on $[0, v_\mathcal{I}]$:

$$1 - \eta_0 \leq \frac{dv'}{d\varepsilon}(v) \leq 1 + \eta_0 \text{ for } v \in [0, v_\mathcal{I}] \text{ and }$$

$$\max_{v \in [0, v_\mathcal{I}]} \left| \frac{d^2 v'}{(dv')^2}(v) \right| \leq 2 \left| \frac{d}{dv} \frac{K(v)}{K(v'(v) - v_\mathcal{I})} \right|_{v = v_\mathcal{I}} + \frac{\eta_0}{v_\mathcal{I}}.$$
Figure 3.4: Schematic depiction of the domains $\mathcal{D}_0^\varepsilon$, $\mathcal{W}_*^\varepsilon$, and $\mathcal{V}$ in the $(u,v')$-plane, i.e., after applying the transformation $T: (u,v) \mapsto T(u,v) = (u,v'(v))$ in the $(u,v)$ plane. The assumption that $\frac{dv'}{dv}(v) = \frac{K(v)}{K(v'(v) - v)}$ for $v \in [v_T, v_T + \delta]$ guarantees that, in the $(u,v')$ coordinates, the conformal infinity of the vacuum Schwarzschild–AdS region $\mathcal{V}$ lies on the vertical line $u = v' - v_T$.

Remark. It is trivial to verify that a function $v': \mathbb{R} \to \mathbb{R}$ satisfying Conditions 1–4 exists. The fact that $v'$ can be chosen in the interval $[v_T - \frac{1}{2} \varepsilon, v_T]$ so that it satisfies, in addition, the bounds (3.94) follows from the observation that Conditions 1–4 and the relation (3.90) imply that

$$\frac{dv'}{dv}(v) = 1 \text{ for } v \leq v_T - \frac{1}{2} \varepsilon \text{ and } \frac{dv'}{dv}(v_T) = \frac{K(v_T)}{K(0)} = 1$$

and

$$\frac{d^2v'}{(dv)^2}(v) = 0 \text{ for } v \leq v_T - \frac{1}{2} \varepsilon \text{ and } \frac{d^2v'}{(dv)^2}(v_T) = \frac{d}{dv} \frac{K(v)}{K(v'(v) - v)} \bigg|_{v = v_T}.$$

Let us consider the global change of coordinates, $T: \mathbb{R}^2 \to \mathbb{R}^2$, $(u,v) \mapsto (u,v')$ (note that $T$ is equal to the identity when restricted on $\{ v \leq v_T - \frac{1}{2} \varepsilon \}$) and let us define

$$\mathcal{D}_0^\varepsilon = T(\mathcal{D}_0^\varepsilon) = \mathcal{D}_\varepsilon^\varepsilon,$$

$$\mathcal{W}_*^\varepsilon = T(\mathcal{W}_*) = [0,u_*] \times \left[ \frac{1}{2}\varepsilon, v'(v_T - \frac{1}{4}\varepsilon) \right]$$

(see Figure 3.4). Under this change of coordinates, the initial data set $(r_I, \Omega_I^2, \tilde{f}_I; v_T)$ is mapped to a new initial data set $(r'_I, (\Omega'_I)^2, \tilde{f}'_I; v_T)$ determined by (3.27), while the solution $(r, \Omega^2, f)$ on $\mathcal{D}_0^\varepsilon \cup \mathcal{W}_*$ is mapped to a new solution $(r', (\Omega')^2, f')$ on $\mathcal{D}_0^\varepsilon \cup \mathcal{W}_*^\varepsilon$ under the transformation

$$r'(u,v'(v)) \equiv r(u,v),$$

$$(\Omega')^2(u,v'(v)) \equiv \left( \frac{dv'}{dv}(v) \right)^{-1} \Omega^2(u,v),$$

$$f'(u,v'(v);p^u, \frac{dv'}{dv}p^v,l) \equiv f(u,v;p^u,p^v,l).$$

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Since $v'(v) = v$ for $v \leq v_\Sigma - \frac{1}{2} \varepsilon$, we have

$$(r', (\Omega')^2, f') \equiv (r, \Omega^2, f) \text{ on } (D_0^\varepsilon \cup W') \cap \{v \leq v_\Sigma - \frac{1}{2} \varepsilon\}.$$ 

On the domain

$$[0, u'] \times [v_\Sigma - \frac{1}{2} \varepsilon, v'(v_\Sigma - \frac{1}{4} \varepsilon)] = T([0, u'] \times [v_\Sigma - \frac{1}{2} \varepsilon, v_\Sigma - \frac{1}{4} \varepsilon]),$$

the relations (3.91) and (3.92) become:

$$(3.96) \quad \frac{\partial_v r'}{1 - 2M/r^2 - \frac{1}{3} \Lambda (r')^2} (u, v') = \left(\frac{dv}{dv'}(v')\right) K(v(v')),$$

and

$$(3.97) \quad \frac{(\Omega')^2}{1 - 2M/r^2 - \frac{1}{3} \Lambda (r')^2} (u, v') = 4\left(\frac{dv}{dv'}(v')\right) K(v(v')) K(u).$$

In view of the fact that $v'$ was chosen so that (3.93) holds, we infer that, by choosing some $T \in (0, \min\{u', \delta'\})$ sufficiently small, the pair $(r', (\Omega')^2)$ can be extended by the formulas (3.96) and (3.97) on the whole of the domain

$$\mathcal{V} \ni \{0 \leq u \leq T\} \cap \{v_\Sigma - \frac{1}{2} \varepsilon < v < u + v_\Sigma\},$$

such that $(r', (\Omega')^2, 0)$ is a smooth asymptotically AdS solution of (2.43)–(2.48) with smooth conformal infinity \( \{u = v' - v_\Sigma\} \), inducing on $\mathcal{V} \cap \{u = 0\}$ the initial data set $(r', (\Omega')^2, f')|_{[v_\Sigma - \frac{1}{2} \varepsilon, v_\Sigma]} = (r', (\Omega')^2, 0)$

**Remark.** The functions $r', (\Omega')^2$ are simply the metric components of the Schwarzschild–AdS metric with mass $M$ in the $(u, v')$ coordinates, with (3.93) ensuring that \( \{r = \infty\} \) coincides with the straight line \( \{u = v' - v_\Sigma\} \).

By gluing the extension $(r', (\Omega')^2, 0)$ on $\mathcal{V}$ with $(r', (\Omega')^2, f')|_{D_0^\varepsilon \cup W'}$ along $(D_0^\varepsilon \cup W') \cap \mathcal{V} = [0, T] \times [v_\Sigma - \frac{1}{2} \varepsilon, v'(v_\Sigma - \frac{1}{4} \varepsilon)]$ (see Figure 3.4), we therefore obtain a smooth solution $(r', (\Omega')^2, f')$ of the system (2.43)–(2.48) on

$$D_0^\varepsilon \cup W' \cup \mathcal{V} \supset \mathcal{U}_{T, v_\Sigma}$$

such that $(r', (\Omega')^2, f')$ has smooth axis \( \{u = v'\} \) and smooth conformal infinity \( \{u = v' - v_\Sigma\} \) and induces on \( \{u = 0\} \) the initial data set $(r', (\Omega')^2, f'; v_\Sigma)$. Therefore, we conclude that $(r', (\Omega')^2, f'; v_\Sigma)$ is smoothly compatible, in accordance with Definition 3.5.

Furthermore, using the bounds (3.94) for $\frac{d^2 v'}{dv'^2}$ and the relations (3.90),

$$(3.98) \quad \frac{d \log(K)}{du}(0) = \partial_u \log \left(\frac{-\partial_u r}{1 - 2M/r^2 - \frac{1}{3} \Lambda r^2}\right)(0, v_\Sigma) = -4\pi \int_0^{v_\Sigma} \partial_u \left(r\frac{T_{uv}}{\partial_u r}\right)(0, v) dv$$

(which is obtained by integrating (2.51) in $v$ for $u = 0$),

$$\partial_u (rT_{uv}) = -\Omega^2 \partial_v (\Omega^2 r T_{uv}) - \partial_u r T_{vv} - 3 \partial_v r T_{uv}.$$

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Integrating by parts in Definition 4.1.

\( \partial \) piecewise Lipschitz boundary

4.1 Developments of characteristic initial data sets

boundary conditions on \( I \), well-posedness results related to the characteristic initial-boundary value problem for (2.43)–(2.48) with reflecting initial data set for (2.43)–(2.48) with \( \Omega^2 \).

The bounds (3.78) and (3.79) follow immediately from (3.94) (for \( \frac{dv}{dt} \)) and using the fact that \( T_{uv}(0,v) \equiv 0 \) for \( v \in [0,\varepsilon] \cup [\varepsilon,v_\Omega] \), the constraint equation (2.45), the relation (2.52) for \( \partial_u \partial_v r, \) the gauge condition (3.31) and the relation (2.10) for \( \Omega^2 \), we obtain from (3.99) that:

\[
\max \frac{d^2 v'}{(dv)^2} (v) \leq \frac{K(v)}{K'(v)} \left| \frac{d}{dv} K'(v) \right|_{v=v_\Omega} + \frac{\eta_0}{v_\Omega} = \frac{dK(v)}{K(0)} \left| \frac{dK(0)}{dv} \right|_{v=v_\Omega} + \frac{\eta_0}{v_\Omega} = 2 \frac{d}{dv} \log(K)(0) + \frac{\eta_0}{v_\Omega} = 8\pi \int_0^{v_\Omega} \partial_u \left( \frac{rT_{uv}}{\partial_v r} \right)(0,v) dv + \frac{\eta_0}{v_\Omega} = 8\pi \int_0^{v_\Omega} \left( - \frac{1}{\Omega^2} \partial_v (\Omega^{-2} rT_{uv}) - \partial_u \partial_v rT_{uv} - 3T_{uv} - \partial_u \partial_v rT_{uv} \right)(0,v) dv + \frac{\eta_0}{v_\Omega}.
\]

Integrating by parts in \( v \) in the first term in the right hand side of (3.99) and using the fact that \( T_{uv}(0,v) \equiv 0 \) for \( v \in [0,\varepsilon] \cup [\varepsilon,v_\Omega] \), the constraint equation (2.45), the relation (2.52) for \( \partial_u \partial_v r, \) the gauge condition (3.31) and the relation (2.10) for \( \Omega^2 \), we readily calculate:

\[
\max \frac{d^2 v'}{(dv)^2} (v) \leq 2 \frac{dK(v)}{K(0)} \left| \frac{dK(0)}{dv} \right|_{v=v_\Omega} + \frac{\eta_0}{v_\Omega} = 2 \frac{d}{dv} \log(K)(0) + \frac{\eta_0}{v_\Omega} = 8\pi \int_0^{v_\Omega} \partial_u \left( \frac{rT_{uv}}{\partial_v r} \right)(0,v) dv + \frac{\eta_0}{v_\Omega} = 8\pi \int_0^{v_\Omega} \left( - \frac{1}{\Omega^2} \partial_v (\Omega^{-2} rT_{uv}) - \partial_u \partial_v rT_{uv} - 3T_{uv} - \partial_u \partial_v rT_{uv} \right)(0,v) dv + \frac{\eta_0}{v_\Omega}.
\]

The bounds (3.78) and (3.79) follow immediately from (3.94) (for \( \frac{dv}{dt} \)) and (3.100).

4 Well-posedness of the smooth characteristic initial-boundary value problem and properties of the maximal development

In this section, we will introduce the notion of a development of a smoothly compatible, asymptotically AdS initial data set for (2.43)–(2.48) with reflecting boundary conditions on \( \mathcal{I} \). We will then present some fundamental well-posedness results related to the characteristic initial-boundary value problem for (2.43)–(2.48) with reflecting boundary conditions on \( \mathcal{I} \), and show the existence and uniqueness of a maximal smooth development for any smoothly compatible, asymptotically AdS initial data set for (2.43)–(2.48).

4.1 Developments of characteristic initial data sets

We will fix a class of domains in the \((u,v)\)-plane which will naturally arise as domains of definition for solutions \((r,\Omega^2,f)\) to the characteristic initial-boundary value problem for (2.43)–(2.48); this class of domains has been also considered in [12,17], in the context of the study of the Einstein–null dust system.

**Definition 4.1.** For any \( v_\Omega \), let \( \mathcal{U}_{v_\Omega} \) be the set of all connected open domains \( \mathcal{U} \) of the \((u,v)\)-plane with piecewise Lipschitz boundary \( \partial \mathcal{U} \), having the property that \( \partial \mathcal{U} \) can be expressed as

\[
\partial \mathcal{U} = \gamma_\partial \cup \mathcal{I} \cup S_{v_\Omega} \cup \text{clos}(\zeta),
\]

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Figure 4.1: Depicted above is a typical domain $\mathcal{U} \in \mathcal{W}_{v_I}$. In the case when the boundary set $\zeta$ is empty, it is necessary that both $\gamma_Z$ and $\mathcal{I}$ are unbounded (i.e. $u_{\gamma_Z} = u_\mathcal{I} = +\infty$).

where, for some $u_{\gamma_Z}, u_\mathcal{I} \in (0, +\infty]$,

\begin{align}
\gamma_Z &= \{u = v\} \cap \{0 \leq u < u_{\gamma_Z}\}, \\
\mathcal{I} &= \{u = v - v_\mathcal{I}\} \cap \{0 \leq u < u_\mathcal{I}\}, \\
\mathcal{S}_{v_I} &= \{0\} \times [0, v_\mathcal{I}]
\end{align}

and the Lipschitz curve $\zeta \subset \mathbb{R}^2$ is achronal with respect to the reference Lorentzian metric

\begin{equation}
g_{\text{ref}} := -dudv
\end{equation}
on the $(u, v)$-plane (the closure $\text{clos}(\zeta)$ of $\zeta$ in (4.1) is considered with respect to the standard topology of $\mathbb{R}^2$). In particular, $\zeta$ is allowed to be empty.

**Remark.** It follows readily from Definition 4.1 that any $\mathcal{U} \in \mathcal{W}_{v_I}$ is necessarily contained in the future domain of dependence of $\mathcal{S}_{v_I} \cup \gamma_Z \cup \mathcal{I}$ (with respect to the comparison metric (4.5)). In the case when $\zeta = \emptyset$ in (4.1), it is necessary that both $\gamma_Z$ and $\mathcal{I}$ extend all the way to $u + v = +\infty$.

A development of an asymptotically AdS initial data set for (2.43)–(2.48) with reflecting boundary conditions on $\mathcal{I}$ can be naturally defined as follows:

**Definition 4.2.** For any $v_\gamma > 0$, let $(r_j, \Omega_j^2, f_j; v_\gamma)$ be a smoothly compatible, asymptotically AdS initial data set for the system (2.43)–(2.48), according to Definition 3.5. A future development of $(r_j, \Omega_j^2, f_j; v_\gamma)$ for (2.43)–(2.48) with reflecting boundary conditions on $\mathcal{I}$ consists of an open set $\mathcal{U} \in \mathcal{W}_{v_I}$ (see Definition 4.1) and a smooth solution $(r, \Omega^2, f)$ of (2.43)–(2.48) on $\mathcal{U}$ satisfying the following conditions:

1. Using the notations of Definition 4.1, $(\mathcal{U}; r, \Omega^2, f)$ has smooth axis $\gamma_Z$ and smooth conformal infinity $\mathcal{I}$, in accordance with Definitions 3.1, 3.2.
2. The solution \((r, \Omega^2, f)\) coincides with \((r_j, \Omega^2_j, \bar{f}_j; v_{I})\) at \(u = 0\), i.e.

\[
(4.6) \quad (r, \Omega^2)(0, v) = (r_j, \Omega^2_j)(v)
\]

and

\[
(4.7) \quad f(0, v; p^u, p^v, l) = \bar{f}(v; p^u, l) \cdot \delta\left(\frac{\Omega_j^2(v)p^u p^v}{r_j(v)} - \frac{l^2}{r_j(v)}\right).
\]

3. The Vlasov field \(f\) satisfies the reflecting boundary condition \((2.70)\) on \(I\).

If \(\mathcal{D} = (U; r, \Omega^2, f)\) and \(\mathcal{D}' = (U'; r', (\Omega')^2, f')\) are two future developments of the same initial data \((r_j, \Omega^2_j, \bar{f}_j; v_{I})\), we will say that \(\mathcal{D}'\) is an extension of \(\mathcal{D}\), writing \(\mathcal{D} \subseteq \mathcal{D}'\), if \(U \subseteq U'\) and the restriction of \((r', (\Omega')^2, f')\) on \(U\) coincides with \((r, \Omega^2, f)\).

In Section 6, we will need to perform gauge transformations on developments \((U; r, \Omega^2, f)\) of smoothly compatible initial data sets \((r_j, \Omega^2_j, \bar{f}_j; v_{I})\) that normalise the initial data according to the gauge condition \((3.6)\). The following lemma will be useful for this procedure:

**Lemma 4.1.** Let \((r_j, \Omega^2_j, \bar{f}_j; v_{I})\) be a smoothly compatible, asymptotically AdS initial data set for the system \((2.43) - (2.48)\) of bounded support in phase space and let \(\mathcal{D} = (U; r, \Omega^2, f)\) be a development of \((r_j, \Omega^2_j, \bar{f}_j; v_{I})\), as in Definition \(1.2\). Let also \((r_j, \Omega^2_j, \bar{f}_j; v_{I}) \rightarrow (r_j', (\Omega')^2, \bar{f}_j'; v_{I})\) be the gauge normalising transformation provided by Lemma \(3.2\) with associated coordinate transformation \(T_j : [0, v_I] \rightarrow [0, v], v \rightarrow \bar{V}(v)\).

There exists a unique spacetime gauge transformation \(T : \mathcal{D} \rightarrow \mathbb{R}^2, (u, v) \rightarrow (u', v') = (U(u), V(v)), (U; r, \Omega^2, f) \rightarrow (T(U); r', (\Omega')^2, f')\) (see \(3.25\)), such that

- The lines \(\{u = v\}\) and \(\{u = v - v_I\}\) remain invariant under \(T\), i.e.

\[
(4.8) \quad U(v) = V(v) \text{ for all } v \in [0, \sup v),
\]

\[
U(v - v_I) = V(v) - v_I \text{ for all } v \in [0, \sup v).
\]

- \(T\) is an extension of the initial data transformation \(T_j\), i.e.

\[
(4.9) \quad (U(0), V(v)) = (0, \bar{V}(v)) \text{ for all } v \in [0, v_I).
\]

Furthermore, \(T\) is piecewise smooth on \(\mathcal{D}\) and smooth on \(\mathcal{D}\setminus \cup_{k=1}^{\infty} \{\{u = kv_I\} \cup \{v = kv_I\}\}\) and satisfies the Lipschitz estimate

\[
(4.10) \quad \left| \log\left(\frac{dU}{dv}\right) \right| + \left| \log\left(\frac{dV}{dv}\right) \right| \leq C_{v_{I}} \left(\sup_{v \in [0, v_I]} \log\left(\frac{\partial_v r_j}{1 - \frac{4}{3} \Lambda r_j^2}\right) - \log(\partial_v r_j(0)) \right) + 4\pi \int_{0}^{v} \frac{r_j(T)}{\partial_v r_j} \partial_v(\bar{U}) \partial_v v_I dv_I,
\]

for some constant \(C_{v_{I}} > 0\) depending only on \(v_{I}\).

**Proof.** The conditions \((4.8)\) and \((4.9)\) provide an explicit formula for \(T(u, v) = (U(u), V(v))\):

\[
(4.11) \quad U(u) = \bar{V}\left(u - \left[\frac{u}{v_I}\right] \cdot v_I\right) + \left[\frac{u}{v_I}\right] \cdot v_I,
\]

\[
V(v) = \bar{V}\left(v - \left[\frac{v}{v_I}\right] \cdot v_I\right) + \left[\frac{v}{v_I}\right] \cdot v_I,
\]

where \(\left[ x \right]\) denotes the integral part of \(x\). The formula \((4.11)\) and the fact that \(\bar{V} \in C^\infty([0, +\infty))\) readily imply that \(T\) is piecewise smooth on \(\mathcal{D}\) and smooth on \(\mathcal{D}\setminus \cup_{k=1}^{\infty} \{\{u = kv_I\} \cup \{v = kv_I\}\}\) (the continuity of \(T\) along the lines
According to Definition 3.5, satisfying (3.10). Then, for some Proposition 4.1.

Let us define for any $I$, $v_{\bar{z}}>0$ the domain

$$U_{v_{\bar{z}}} = \{0 \leq u < U\} \cap \{u < v < u + v_{\bar{z}}\}$$

and the boundary curves

$$\gamma_{u_{1}} = \{u = v\} \cap \{0 \leq u < u_{1}\}$$

and

$$\gamma_{v_{1}} = \{v = u + v_{\bar{z}}\} \cap \{0 \leq u < u_{1}\}.$$ 

The main result of this section is the following:

**Theorem 4.1.** Let $(r, \Omega_{r}^{2}, \bar{f}_{r}; v_{\bar{z}})$ be any smoothly compatible asymptotically AdS initial data set for (2.43)–(2.48) with reflecting boundary conditions on $I$ in the class of smoothly compatible, asymptotically AdS initial data sets with bounded support in phase space, introduced by Definition 3.5. As a byproduct of the proof of the main result, we will also establish the well-posedness of the characteristic initial value problem restricted in bounded support in phase space and show that it is unique (see Corollary 4.2).

Then, for some $u_{0} > 0$ sufficiently small in terms of $(r_{0}, \Omega_{r_{0}}^{2}, \bar{f}_{r_{0}}; v_{\bar{z}_{0}})$, there exists a unique smooth solution $(r, \Omega^{2} f)$ of (2.43)–(2.48) on the domain $U_{u_{0}, v_{\bar{z}}}$ (defined by (4.13)) such that $(U_{u_{0}, v_{\bar{z}}}; r, \Omega^{2} f)$ is a future development of $(r_{0}, \Omega_{r_{0}}^{2}, \bar{f}_{r_{0}}; v_{\bar{z}_{0}})$ with reflecting boundary conditions on $I$ (see Definition 4.3).

For the proof of Theorem 4.1, see Section 4.3.

As a corollary of the proof of Theorem 4.1, we can also obtain the following well-posedness result for characteristic initial data sets with a smooth axis, restricted to the region where $r \leq R + \infty$:

**Proposition 4.1.** For any $v_{0} > 0$, let $r_{0}, \Omega_{r_{0}}^{2}, \bar{f}_{r_{0}} : (0, v_{0}) \to (0, +\infty)$ and $\bar{f}_{r} : (0, v_{0}) \times (0, +\infty)^{2} \to [0, +\infty)$ be smooth functions satisfying Conditions 1 and 2 of Definition 3.4, as well as Conditions 1 and 2 of Definition 3.5. Assume, moreover, that the support of $\bar{f}_{r}$ satisfies the bound (3.10) for some $C > 0$. Then, for some $u_{0} > 0$ sufficiently small in terms of $(r_{0}, \Omega_{r_{0}}^{2}, \bar{f}_{r_{0}})$, there exists a unique smooth solution $(r, \Omega^{2} f)$ of (2.43)–(2.48) on the domain

$$D_{u_{0}} = \{(0, u_{0}) \times (0, v_{0})\} \cap \{u < v\}$$

such that $(D_{u_{0}}; r, \Omega^{2} f)$ has smooth axis $\gamma_{Z} = \{u = v\} \cap \{0 < u < u_{0}\}$ (see Definition 3.4) and $(r, \Omega^{2} f)$ satisfy the initial conditions (4.6)–(4.7) at $u = 0$. 

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The analogous statement for the domain the form (4.3); it will thus be omitted. For a similar result in the case of the Einstein–massive Vlasov system on domains of Proposition 4.2.

Proposition 4.2. For any \( u_1 < u_2 \) and \( v_1 < v_2 \), let \( r_1, \Omega_1 : [u_1, u_2] \rightarrow (0, +\infty) \), \( \bar{f}_1 : [u_1, u_2] \times (0, +\infty)^2 \rightarrow (0, +\infty) \), \( r_2, \Omega_2 : [v_1, v_2] \rightarrow (0, +\infty) \) and \( \bar{f}_2 : [v_1, v_2] \times (0, +\infty)^2 \rightarrow (0, +\infty) \) be smooth functions, such that \( (r_1, \Omega_1, \bar{f}_1) \) and \( (r_2, \Omega_2, \bar{f}_2) \) satisfy the constraint equations (2.46) and (2.45) respectively, as well as the compatibility conditions

\[
\begin{align*}
\Omega_1^2(u_1) &= \Omega_2^2(v_1), \\
\bar{f}_1(u_1, p^v, l) &= \bar{f}_2(v_1, \frac{l^2}{r^2\Omega_1^2(u_1)p^v}, l).
\end{align*}
\]

Assume, moreover, that the supports of \( \bar{f}_1, \bar{f}_2 \) satisfy the bound (3.10). Then, for some \( \delta > 0 \) sufficiently small in terms of \( (r_1, \Omega_1^2, \bar{f}_1) \) and \( (r_2, \Omega_2^2, \bar{f}_2) \), there exists a unique smooth solution \( (r, \Omega^2, f) \) of (2.43)–(2.48) on the domain \([u_1, u_1 + \delta] \times [v_1, v_2]\), such that

\[
\begin{align*}
(r, \Omega^2)(u_1, v_1) &= (r_1, \Omega_1^2)(u), \\
(r, \Omega^2)(u_1, v) &= (r_2, \Omega_2^2)(v)
\end{align*}
\]

and

\[
\begin{align*}
f(u, v_1; p^u, p^v, l) &= \bar{f}_1(u; p^u, l) \cdot \delta\left(\Omega_1^2(u)p^v - \frac{l^2}{r_1(u)}\right), \\
f(u_1, v; p^u, p^v, l) &= \bar{f}_2(v; p^v, l) \cdot \delta\left(\Omega_2^2(v)p^u - \frac{l^2}{r_2(v)}\right).
\end{align*}
\]

The analogous statement for the domain \([u_1, u_2] \times [v_1, v_1 + \delta]\) also holds.

The proof of Proposition 4.2 follows by exactly the same arguments as the proof of Theorem 4.1 (see Step 2 in Section 4.3); it will thus be omitted. For a similar result in the case of the Einstein–massive Vlasov system on domains of the form \([u_1, u_1 + \delta] \times [v_1, v_1 + \delta]\), see [3].

As a straightforward corollary of the local uniqueness statements of Propositions 4.1–4.2, we infer the following global uniqueness result

Corollary 4.1. If \( \mathcal{D} = (\mathcal{U}; r, \Omega^2, f) \) and \( \mathcal{D}' = (\mathcal{U}'; r', (\Omega')^2, f') \) are two future developments with reflecting boundary conditions on \( I \) of the same smoothly compatible, asymptotically AdS initial data set \( (r_1, \Omega_1^2, \bar{f}_1; v_1) \) with bounded support in phase space, then

\[
(r, \Omega^2, f)|_{\mathcal{U}\times\mathcal{U}'} = (r', (\Omega')^2, f')|_{\mathcal{U}\times\mathcal{U}'}.
\]

Remark. Our choice of gauge (fixing the axis \( \gamma_z \) and conformal infinity \( I \) of any development to be straight, vertical lines) does not provide any freedom in performing gauge transformations on the development that fix the gauge of the initial data at \( u = 0 \). This is the reason why Corollary 4.1 can be stated as a uniqueness statement without appealing to equivalence classes of developments under gauge transformations.

Using standard arguments, the well-posedness results of Theorem 4.1 and Propositions 4.1–4.2 also allow us to assign to each smoothly compatible, asymptotically AdS initial data set for (2.43)–(2.48) a unique maximal future development:

Corollary 4.2. Let \( (r_1, \Omega_1^2, \bar{f}_1; v_1) \) be any smoothly compatible, asymptotically AdS initial data set for (2.43)–(2.48) with bounded support in phase space. Then there exists a unique future development \( (\mathcal{U}_{\text{max}}; r, \Omega^2, f) \) of \( (r_1, \Omega_1^2, \bar{f}_1; v_1) \)
with reflecting boundary conditions on \( I \) having the following property: If \((U_*; r_*, \Omega_*^2, f_*)\) is any other future development of \((r_j, \Omega_j^2, \bar{f}_j; v_I)\) with reflecting boundary conditions on \( I \), then

\[
(4.20) \quad U_* \subseteq U_{max}
\]

and

\[
(4.21) \quad (r, \Omega^2, f)|_{\partial I} = (r_*, \Omega_*^2, f_*).
\]

We will call \((U_{max}; r, \Omega^2, f)\) the maximal future development of \((r_j, \Omega_j^2, \bar{f}_j; v_I)\) with reflecting boundary conditions on \( I \).

The maximal future development of an initial data set \((r_j, \Omega_j^2, \bar{f}_j; v_I)\) is the main object of study related to questions on the global dynamics of \((2.43)\)–\((2.48)\), such that the ones addressed in our companion paper \([16]\).

### 4.3 Local existence and uniqueness: Proof of Theorem 4.1

In this section, we will establish Theorem 4.1. To this end, let \((r_j, \Omega_j^2, \bar{f}_j; v_I)\) be as in the statement of Theorem 4.1 and let \(C_0 > 0\) be a constant for which the bounded support condition \((3.10)\) is satisfied by \((r_j, \Omega_j^2, \bar{f}_j; v_I)\) with \(C_0\) in place of \(C\). The construction of a (unique) smooth development \((U_{u_0, v_I}; r, \Omega^2, f)\) of \((r_j, \Omega_j^2, \bar{f}_j; v_I)\) will be separated into three steps, covering, successively, the region near the axis, the intermediate region and the region near conformal infinity.

**Step 1: The region \( D_0^{u_0} \)** Let us fix a constant \( R_0 \) satisfying

\[
(4.22) \quad \sqrt{-\Lambda} R_0 \ll 1,
\]

and let \( V_* \in (0, v_I) \) be such that

\[
(4.23) \quad r_j(V_*) = R_0.
\]

We will assume that \( R_0 \) is sufficiently small, so that

\[
(4.24) \quad \sup_{v \in (0, V_*)} \partial_u r_j < 0
\]

(this is possible, in view of \((3.32)\) and the fact that \( \tilde{m}_j = O(r_j^4) \)). We will show that, for some \( u_0 > 0 \) sufficiently small in terms of \( R_0 \) and \( V_* \), there exists a unique smooth solution \((r, \Omega^2; f)\) of \((2.43)\)–\((2.48)\) on the domain

\[
(4.25) \quad D_0^{u_0} \equiv \{(0, u_0] \times [0, u_0]\} \cap \{u < v\} \subset \mathbb{R}^2
\]

with smooth axis

\[
(4.26) \quad \gamma_0^{u_0} = \{(0, u_0] \times [0, u_0]\} \cap \{u = v\} \subset \partial D_{u_1}^{u_2},
\]

satisfying \((4.6)\)–\((4.7)\).

Existence. The existence of a smooth solution \((r, \Omega^2; f)\) to \((2.43)\)–\((2.48)\) on \( D_0^{u_0} \) will follow by applying a standard iteration procedure. For any integer \( n \in \mathbb{N} \), let us define the functions \( r_n : D_0^{u_0} \to [0, +\infty) \), \( \Omega_n^2 : D_0^{u_0} \to (0, +\infty) \) and \( f_n : D_0^{u_0} \times [0, +\infty)^2 \to [0, +\infty) \) by solving recursively (for a definition of the relevant parameters and functions, see below):

\[
(4.27) \quad \partial_u \partial_v r_{n+1} = -\frac{1}{2} \frac{\tilde{m}_n - \frac{\Lambda}{r_n} \Omega_n^2}{r_n^2} + 4\pi r_n (T_{uv})_n,
\]

\[
(4.28) \quad \partial_u \partial_v (\Omega_{n+1}^2) = \left( \frac{\tilde{m}_n}{r_n^2} + \frac{\Lambda}{6} \right) \Omega_n^2 - 16\pi (T_{uv})_n,
\]

\[
(4.29) \quad \left[ p^\alpha \partial_{\alpha n} - (\Gamma_{\alpha\beta})_n p^\beta \partial_{p^\beta} \right] f_{n+1} = 0,
\]

where \( p \) and \( \Omega \) are defined in (4.29)–(4.31) on \( D_0^{u_0} \).
Figure 4.2: Schematic depiction of the domain $D_{\gamma}^{u_0}$.

with initial conditions on $u = 0$:

\begin{equation}
(r_{n+1}, \Omega_{n+1}^2; \tilde{f}_{n+1}'')|_{u=0} = (r_{f'}, \Omega_{f'}^2; \tilde{f}_f')
\end{equation}

and boundary conditions on the axis $\gamma_{0}^{u_0}$:

\begin{equation}
\begin{array}{l}
r_{n+1}(u, u) = 0, \\
\partial_{u} \Omega_{n+1}^2(u, u) = \partial_{u} \Omega_{n+1}^2(u, u).
\end{array}
\end{equation}

In the above, we have adopted the following notational conventions for $n \geq 1$:

- The Vlasov field $f_n$ in (4.29) is defined in terms of $\tilde{f}_n'$ for any $l > 0$ by

\begin{equation}
\tilde{f}_{n+1}(u, v; p^u, p^v, l) \doteq \tilde{f}_n'(u, v; p^u, l) \cdot \delta\left(\Omega_{n+1}^2(u, v) p^u p^v - \frac{l^2}{r_{n+1}(u, v)}\right),
\end{equation}

while its extension to $l = 0$ is uniquely fixed by the condition that $f_{n+1}$ is a continuous multiple of $\delta\left(\Omega_{n+1}^2(u, v) p^u p^v - \frac{l^2}{r_{n+1}(u, v)}\right)$.

- The terms $\tilde{m}_n$ in (4.27)–(4.28) are defined by

\begin{equation}
\tilde{m}_n(u, v) \doteq 8\pi \int_u^v r_n^2 \Omega_n^-2 \left( - \partial_u r_n \cdot (T_{uv})_n + \partial_v r_n \cdot (T_{uv})_n \right)(u, \tilde{v}) \, d\tilde{v},
\end{equation}

where $(T_{\alpha\beta})_n$ are defined in terms of $r_n$, $\Omega_n^2$ and $f_n$ by (2.32).

- The terms $(\Gamma^n_{\alpha\beta})_n$ in (4.29) are the Christoffel symbols of the metric $g_n$ on $(D_{\gamma}^{u_0} \setminus \gamma_0^{u_0}) \times S^2$ in the $(u, v, \theta, \varphi)$ coordinate chart, where

\begin{equation}
g_n \doteq -\Omega_n^2 \, du dv + r_n^2 g_{S^2}.
\end{equation}

When $n = 0$, we will set

\begin{equation}
(r_0, \Omega_{0}^2; f_0) \doteq (r_{AdS}, \Omega_{AdS}^2; 0).
\end{equation}

Remark. Note that the boundary condition (4.31) implies that

\begin{equation}
\partial_{u} r_{n+1}(u, u) = -\partial_{u} r_{n+1}(u, u).
\end{equation}
Let $C_1 \gg 1$ be a large constant depending only on the initial data $(r_j, \Omega_j^2, f_j)$. We will establish the following inductive bounds for $(r_n, \Omega_n^2, f_n)$: Assuming that, for any $0 \leq k \leq n$:

\begin{equation}
\sup_{D_0^u} \left( \sum_{j=0}^{2} \sum_{j_1+j_2=j} R_0^j \left[ |\partial_u^{j_1} \partial_v^{j_2} \log(\Omega_k^2)| + |\partial_u^{j_1} \partial_v^{j_2} \log(\partial_v r_k)| + |\partial_u^{j_1} \partial_v^{j_2} \log(-\partial_u r_k)| \right] \right) + \\
+ \sum_{j=0}^{1} \sum_{j_1+j_2=j} R_0^{j_1} \left[ |\partial_u^{j_1} \partial_v^{j_2} (T_{uv})_k| + |\partial_u^{j_1} \partial_v^{j_2} (T_{uv})_k| + |\partial_u^{j_1} \partial_v^{j_2} (T_{vv})_k| \right] \right) < C_1,
\end{equation}

we will show that (4.37) also holds for $k = n + 1$ and, moreover (in the case $n \geq 2$):

\begin{equation}
D_{n+1} \leq \frac{1}{2} D_n,
\end{equation}

where

\begin{equation}
D_k = \sup_{D_0^u} \left\{ \sum_{j=0}^{2} \sum_{j_1+j_2=j} R_0^j \left[ |\partial_u^{j_1} \partial_v^{j_2} r_k - \partial_u^{j_1} \partial_v^{j_2} r_{k-1}| + \sum_{j=0}^{1} \sum_{j_1+j_2=j} R_0^{j_1} \left[ |\partial_u^{j_1} \partial_v^{j_2} \log(\Omega_k^2 - \partial_v^{j_1} \partial_v^{j_2} \log(\Omega_{k-1})| \right] \\
+ R_0^{j_1+j_2} \left[ |\partial_u^{j_1} \partial_v^{j_2} (T_{uv})_k| + |\partial_u^{j_1} \partial_v^{j_2} (T_{uv})_k| + |\partial_u^{j_1} \partial_v^{j_2} (T_{vv})_k| \right] \right) \right\}.
\end{equation}

Remark. The fact that the left hand side of (4.37) is indeed bounded on $\{0\} \times [0, u_0]$ when $u_0$ is sufficiently small follows from (3.11) and (4.24).

Let us first infer from the inductive bound (4.37) a useful estimate for $\hat{\tau}_k$. For any $0 \leq k \leq n$ and any $(u, v) \in D_0^u$, the definition (4.33) of $\hat{\tau}_k$ yields

\begin{equation}
\frac{\hat{\tau}_k}{r_k} (u, v) = 8\pi \int_u^v r_k^2 \Omega_k^{-2} \left( - \partial_u r_k \cdot (T_{vv})_k + \partial_v r_k \cdot (T_{uv})_k \right) (u, \bar{v}) \, d\bar{v}.
\end{equation}

Differentiating (4.40) with respect to $\partial_v$, we readily calculate that:

\begin{equation}
\partial_v \left( \frac{\hat{\tau}_k}{r_k^3} \right) (u, v) = 8\pi \partial_v \left\{ r_k^3 (u, v) \int_u^v r_k^2 \Omega_k^{-2} \left( - \partial_u r_k \cdot (T_{vv})_k + \partial_v r_k \cdot (T_{uv})_k \right) (u, \bar{v}) \, d\bar{v} \right\} = \\
= 8\pi \left\{ -3 (r_k^4 \partial_v r_k) (u, v) \int_u^v r_k^2 \Omega_k^{-2} \left( - \partial_u r_k \cdot (T_{vv})_k + \partial_v r_k \cdot (T_{uv})_k \right) (u, \bar{v}) \, d\bar{v} + \\
+ r_k^{-1} \Omega_k^2 \left( - \partial_u r_k \cdot (T_{vv})_k + \partial_v r_k \cdot (T_{uv})_k \right) (u, v) \right\} = \\
= 8\pi \left\{ -3 (r_k^4 \partial_v r_k) (u, v) \int_u^v \frac{1}{3} \partial_v (r_k^2) \Omega_k^{-2} \left( - \partial_u r_k \cdot (T_{vv})_k + \partial_v r_k \cdot (T_{uv})_k \right) (u, \bar{v}) \, d\bar{v} + \\
+ r_k^{-1} \Omega_k^2 \left( - \partial_u r_k \cdot (T_{vv})_k + \partial_v r_k \cdot (T_{uv})_k \right) (u, v) \right\} = \\
= 8\pi (r_k^4 \partial_v r_k) (u, v) \left\{ \int_u^v r_k^3 \left[ \partial_v (r_k^2 \Omega_k^{-2} \partial_u r_k) \cdot (T_{vv})_k + \partial_v (T_{uv})_k \right] (u, \bar{v}) \, d\bar{v} + \\
+ \int_u^v r_k^3 \Omega_k^2 \partial_v r_k \cdot (T_{vv})_k + \partial_v r_k \cdot (T_{uv})_k \right\} (u, v) \, d\bar{v} \right. .
\end{equation}
where, in passing from the fifth to the sixth line of (4.41), we integrated by parts once. Differentiating (4.40) (and using the fact that \( r_k(u, u) = 0 \), we also calculate that:

\[
\partial_u \left( \frac{\bar{m}_k}{r_k^2} \right)(u, v) = 8\pi \left\{ -3\left( r_k^{-4} \partial_u r\right)(u, v) \cdot \int_u^v r_k^2 \partial_u^2 \left( -\partial_u r_k \cdot (Tv) + \partial_v r_k \cdot (Tv) \right)(u, \bar{v}) d\bar{v} + \right.
\]

\[
+ r_k^{-3}(u, v) \int_u^v \partial_u \left[ r_k^2 \partial_u^2 \left( -\partial_u r_k \cdot (Tv) + \partial_v r_k \cdot (Tv) \right)(u, \bar{v}) d\bar{v} \right].
\]

Using the relations

\[
(4.43)
\]

\[
\left( r_k^{-4} \partial_u r\right)(u, v) \cdot \int_u^v r_k^2 \partial_u^2 \left( -\partial_u r_k \cdot (Tv) + \partial_v r_k \cdot (Tv) \right)(u, \bar{v}) d\bar{v} =
\]

\[
= \frac{1}{3} \left( r_k^{-4} \partial_u r\right)(u, v) \cdot \int_u^v \left( \partial_u r_k^3 \partial_u^2 \left( -\partial_u r_k \cdot (Tv) + \partial_v r_k \cdot (Tv) \right)(u, \bar{v}) d\bar{v} =
\]

\[
= \frac{1}{3} \left( r_k^{-4} \partial_u r\right)(u, v) \cdot \left\{ r_k^3 \partial_u^2 \left( -\partial_u r_k \cdot (Tv) + \partial_v r_k \cdot (Tv) \right)(u, \bar{v}) d\bar{v} -
\right.
\]

\[
- \int_u^v \partial_u \left( \Omega_k^{-2} \partial_u r_k \cdot (Tv) + \Omega_k^{-2} \partial_v r_k \cdot (Tv) \right)(u, \bar{v}) d\bar{v} -
\]

\[
- \int_u^v \partial_u \left[ \Omega_k^{-2} \partial_u r_k \cdot (Tv) + \Omega_k^{-2} \partial_v r_k \cdot (Tv) \right](u, \bar{v}) d\bar{v} \right\}
\]

and

\[
(4.44)
\]

\[
\int_u^v \partial_u \left[ r_k^2 \partial_u^2 \left( -\partial_u r_k \cdot (Tv) + \partial_v r_k \cdot (Tv) \right)(u, \bar{v}) d\bar{v} =
\]

\[
= 2 \int_u^v r_k \partial_u r_k \Omega_k^{-2} \left( -\partial_u r_k \cdot (Tv) + \partial_v r_k \cdot (Tv) \right)(u, \bar{v}) d\bar{v} -
\]

\[
- \int_u^v r_k^2 \left[ -\partial_u \left( \Omega_k^2 \partial_u r_k \right) \cdot (Tv) + \partial_u \left( \Omega_k^2 \partial_v r_k \right) \cdot (Tv) \right](u, \bar{v}) d\bar{v} -
\]

\[
- \int_u^v r_k^2 \left[ -\Omega_k^2 \partial_u r_k \cdot \partial_u (Tv) + \Omega_k^2 \partial_v r_k \cdot \partial_u (Tv) \right](u, \bar{v}) d\bar{v} =
\]

\[
= \int_u^v \partial_u \left( r_k^{-2} \partial_u r_k \Omega_k^{-2} \partial_u \left( \Omega_k^{-2} \partial_u r_k \cdot (Tv) + \Omega_k^{-2} \partial_v r_k \cdot (Tv) \right)(u, \bar{v}) d\bar{v} -
\]

\[
- \int_u^v r_k^2 \left[ -\partial_u \left( \Omega_k^{-2} \partial_u r_k \right) + \partial_v \left( \Omega_k^{-2} \partial_v r_k \right) \right] \cdot (Tv) k +
\]

\[
+ \left\{ \partial_u \left( \Omega_k^2 \partial_v r_k \right) + \partial_v \left( \Omega_k^2 \partial_u r_k \right) \right\} \cdot (Tv) k \right\}(u, \bar{v}) d\bar{v} -
\]

\[
- \int_u^v r_k^2 \partial_u r_k \Omega_k^{-2} \partial_u \left( \Omega_k^{-2} \partial_u r_k \cdot (Tv) + \Omega_k^{-2} \partial_v r_k \cdot (Tv) \right)(u, \bar{v}) d\bar{v} -
\]

\[
- \int_u^v r_k^2 \left[ -\Omega_k^{-2} \partial_u r_k \cdot \partial_u (Tv) + \Omega_k^{-2} \partial_v r_k \cdot \partial_u (Tv) \right](u, \bar{v}) d\bar{v} \right\},
\]
we deduce from (4.42) that

\begin{align}
(4.45) \quad \partial_u \left( \frac{\bar{m}_k}{r_k^2} \right)(u, v) &= -8\pi \left\{ \frac{1}{r_k^3(u, v)} \cdot \left( \int_u^v r^2 \left( A_1^{(u)}(T_{uv})_k + A_1^{(v)}(T_{uv})_k + A_2^{(v)} \partial_v (T_{uv})_k + A_2^{(u)} \partial_u (T_{uv})_k \right) \right) + \right. \\
& \quad + A_2^{(u)} \partial_u (T_{uv})_k + A_3^{(v)} \partial_u (T_{uv})_k + A_3^{(u)} \partial_u (T_{uv})_k \right\}(u, \bar{v}) \, d\bar{v} + \\
& \quad + \partial_u r(u, v) \frac{1}{r_k^3(u, v)} \cdot \left( \int_u^v r^3 \left( B_1^{(u)}(T_{uv})_k + B_1^{(v)}(T_{uv})_k + B_2^{(v)} \partial_v (T_{uv})_k + B_2^{(u)} \partial_u (T_{uv})_k \right) \right) \right) \bigg) \\
& \quad + \partial_u r(u, v) \frac{1}{r_k^3(u, v)} \cdot \left( \int_u^v r^3 \left( B_1^{(u)}(T_{uv})_k + B_1^{(v)}(T_{uv})_k + B_2^{(v)} \partial_v (T_{uv})_k + B_2^{(u)} \partial_u (T_{uv})_k \right) \right) \bigg( u, \bar{v} \bigg) \, d\bar{v} \\
& \quad + \partial_u r(u, v) \frac{1}{r_k^3(u, v)} \cdot \left( \int_u^v r^3 \left( B_1^{(u)}(T_{uv})_k + B_1^{(v)}(T_{uv})_k + B_2^{(v)} \partial_v (T_{uv})_k + B_2^{(u)} \partial_u (T_{uv})_k \right) \right) \bigg( u, \bar{v} \bigg) \, d\bar{v} \bigg) \bigg),
\end{align}

where

\begin{align*}
A_1^{(u)} &= - \partial_u \left( \Omega_k^{-2} \partial_u r_k \right) - \partial_u \left( \Omega_k^{-2} \left( \frac{\partial_u r_k}{\partial_r r_k} \right) \right), \\
A_1^{(v)} &= - \partial_v \left( \Omega_k^{-2} \partial_v r_k \right) + \partial_v \left( \Omega_k^{-2} \partial_u r_k \right), \\
A_2^{(v)} &= - \Omega_k^{-2} \left( \frac{\partial_u r_k}{\partial_r r_k} \right), \\
A_2^{(u)} &= - \Omega_k^{-2} \partial_u r_k, \\
A_3^{(v)} &= - \Omega_k^{-2} \partial_v r_k, \\
A_3^{(u)} &= - \Omega_k^{-2} \partial_u r_k,
\end{align*}

and

\begin{align*}
B_1^{(u)} &= - \partial_v \left( \Omega_k^{-2} \partial_u r_k \right), \\
B_1^{(v)} &= - \left( \frac{\partial_u r_k}{\partial_r r_k} \right), \\
B_2^{(u)} &= - \Omega_k^{-2} \partial_u r_k, \\
B_2^{(v)} &= - \Omega_k^{-2} \partial_v r_k.
\end{align*}

(note the cancellation of the “bare” terms $r_k^2 \partial_u r_k \Omega_k^{-2} \left( \frac{\partial_u r_k}{\partial_r r_k} \left( T_{uv} \right)_k + \left( T_{uv} \right)_k \right)(u, v)$ from (4.43) and (4.44) in (4.45)).

Using the inductive bound (4.37) (and the fact that $v - u \leq e^{C_1} r_k(u, v)$, as a consequence of (4.37)), we immediately infer from the relations (4.41) and (4.45) that there exists some constant $C(C_1) > 0$ depending only on $C_1$ such that, for any $0 \leq k \leq n$, we can estimate:

\begin{equation}
(4.46) \quad \sup_{\mathcal{D}_{r_k}^{(n)} j=0, j_1+j_2=j} \sum \left| R_{0}^{2+j_1} \partial_{j_1}^j \partial_{j_2}^j \left( \frac{\bar{m}_k}{r_k^2} \right) \right| \leq C(C_1).
\end{equation}

Differentiating the boundary condition (4.31) on $\gamma_0^{(u)}$ in the direction $\partial_u + \partial_v$ (which is tangential to $\gamma_0^{(u)}$) and using the relations (4.33) and (4.27) (combined with the bounds (4.37) and (4.46)), we infer the following higher order relations for $r_{n+1}$ and $\Omega_{n+1}^2$ on $\gamma_0^{(u)}$:

\begin{align}
(4.47) \quad & \partial_u^2 r_{n+1}(u, u) + \partial_v^2 r_{n+1}(u, u) = 0, \\
& \partial_u^3 r_{n+1}(u, u) + \partial_v^3 r_{n+1}(u, u) = 0
\end{align}

and

\begin{equation}
(4.48) \quad \partial_u^2 \Omega_{n+1}^2(u, u) = \partial_u^2 \Omega_{n+1}^2(u, u).
\end{equation}
Integrating (4.27)–(4.28) in \( u, v \) and using the initial condition (4.30) at \( u = 0 \), the boundary conditions (4.31) (note also (4.36)) on the axis \( \gamma_0^{(0)} \), as well as the bounds (4.37) and (4.46) for \( k = n \), we can readily bound:

\[
(4.49) \quad \sup_{D_0^{n_1}} \left( |\log(-\partial_u r_n)| + |\log \partial_v r_n| + |\log \Omega_{n+1}^2 + R_0 |\partial_u \log \Omega_{n+1}^2 + R_0 |\partial_v \log \Omega_{n+1}^2| \right) \\
\leq C(C_1) R_0^{-1} u_0 + 10 \sup_{v(0,u_0)} \left( |\log \partial_u r_j| + R_0 |\partial_v \log \Omega_j^2(v)| + |\log \Omega_j^2(v)| \right)
\]

for some \( C(C_1) > 0 \) depending only on \( C_1 \). Repeating the same procedure after commuting (4.27) with \( \partial_u, \partial_v \) and using (4.47)–(4.48), we also infer:

\[
(4.50) \quad \sup_{D_0^{n_1}} \left( \sum_{j_1+j_2=1}^2 R_0^{-1+j_1+j_2} |\partial_u^{j_1} \partial_v^{j_2} r_n| + \sum_{j_1+j_2=1}^2 R_0^{-1+j_1+j_2} |\partial_u^{j_1} \partial_v^{j_2} \Omega_j^2| \right) \\
\leq C(C_1) R_0^{-1} u_0 + 10 \sup_{v(0,u_0)} \left( \sum_{j=0}^2 R_0^j |\partial_u^{j} \log \partial_v r_j| + \sum_{j=0}^2 R_0^j |\partial_v^{j} \log \Omega_j^2| \right).
\]

On the other hand, after subtracting from (4.27)–(4.28) the same equations with \( n - 1 \) in place of \( n \) (assuming that \( n \geq 2 \)), and similarly integrating in \( u, v \) using (4.30), (4.31) and the bounds (4.37) and (4.46) for \( k = n, n - 1 \), we infer (for a possibly larger \( C(C_1) \)):

\[
(4.51) \quad \overline{\Omega}_{n+1} \leq C(C_1) u_0 \overline{\Omega}_n,
\]

where we have set:

\[
(4.52) \quad \overline{\Omega}_k \equiv \sup_{D_0^{n_1}} \left( |\log(-\partial_u r_k) - (\partial_u r_{k-1})| + |\log \partial_v r_k - \log \partial_v r_{k-1}| + |\log \Omega_k^2 - \log \Omega_{k-1}^2| + \right. \\
+ \left. R_0 |\partial_u \log \Omega_k^2 - \partial_u \log \Omega_{k-1}^2| + R_0 |\partial_v \log \Omega_k^2 - \partial_v \log \Omega_{k-1}^2| \right).
\]

Since \( f_n \) solves the Vlasov equation (4.32), the conservation of the energy momentum tensor \( (T_{\alpha\beta})_n \) of \( f_n \) (i.e. (2.25)) implies that

\[
(4.53) \quad \partial_u (r_n(T_{uu})) = -\partial_u (r_n(T_{uu})) + \partial_u (r_n(T_{uv})) - (\partial_u r_n)(T_{uv}) = + \partial_u (r_n(T_{vv})) - (\partial_u r_n)(T_{uv}) + (r_n \partial_u \log \Omega_n^2 - 2 \partial_u r_n)(T_{uv}),
\]

\[
(4.54) \quad \partial_v (r_n(T_{uv})) = -\partial_v (r_n(T_{uv})) - \partial_v (r_n(T_{uv})) = + \partial_v (r_n(T_{vv})) - \partial_v (r_n)(T_{uv}) + (r_n \partial_v \log \Omega_n^2 - 2 \partial_v r_n)(T_{uv}) +.
\]

After commuting (4.27) with \( \partial_u^2 \) and \( \partial_u \partial_v \), replacing the derivatives of \( \partial_v (r(T_{uv})) \) with the corresponding derivatives of the expression (4.54) and integrating in \( u \), we infer, using (4.37), (4.46) and the trivial bounds

\[
\sup_{D_0^{n_1}} \overline{\Omega}_n \leq \sup_{v(0,u_0)} \int_u^{u_0} \partial_v r_n (u,v) \, dv \leq C(C_1) u_0,
\]

(following from (4.37), noting also that \( \partial_u [r_n \partial_v (T_{uv})] \) gives only boundary terms after integration in \( u \), we can readily bound:

\[
(4.55) \quad \sup_{D_0^{n_1}} \left( \sum_{j_1+j_2=2}^2 R_0^{j_1+j_2} |\partial_u^{j_1} \partial_v^{j_2} r_n| \right) \leq C(C_1) R_0^{-1} u_0 + 10 \sup_{v(0,u_0)} \sum_{j=0}^2 R_0^j \left( |\partial_u^{j} \log \partial_v r_j| + |\partial_v^{j} \log \Omega_j^2| \right).
\]

Performing the same procedure after commuting (4.27) with \( \partial_v^2 \) and \( \partial_u \partial_v \) and integrating in \( v \) starting from the axis \( \gamma_0^{(0)} \) (using the boundary conditions (4.47), (4.48) and the bounds (4.50), (4.55)), adding the resulting estimate to (4.55), we finally obtain:

\[
(4.56) \quad \sup_{D_0^{n_1}} \left( \sum_{j_1+j_2=3}^2 R_0^{j_1+j_2} |\partial_u^{j_1} \partial_v^{j_2} r_n| \right) \leq C(C_1) R_0^{-1} u_0 + 10 \sup_{v(0,u_0)} \sum_{j=0}^2 R_0^j \left( |\partial_u^{j} \log \partial_v r_j| + |\partial_v^{j} \log \Omega_j^2| \right).
\]
Similarly, subtracting from (4.27) the same equation with \(n - 1\) in place of \(n\) (assuming that \(n \geq 2\) after commuting once with \(\partial_u, \partial_v\), we infer after integrating in \(u, v\) (using the bound (4.37) and the boundary condition (4.47)):

\[
\text{sup} \sum_{j_1 + j_2 = 2} R_0^{j_1 + j_2} \left| \partial_u^{j_1} \partial_v^{j_2} r_{n+1} - \partial_u^{j_1} \partial_v^{j_2} r_n \right| \leq C(C_1) R_0^{1} u_0 \mathcal{D}_n.
\]

Let \(\gamma : [0, a) \rightarrow \mathcal{P}_0, \gamma(0) \in \{u = 0\}\), be a future directed, null geodesic of the metric \(g_n\) (defined by 4.34), such that \((\gamma; \bar{\gamma})\) lies in the support of the Vlasov field \(f_{n+1}\) and has angular momentum \(l > 0\). The relations (2.21) for \(\gamma\) yield that, with respect to the affine parameter \(s\):

\[
d\left(\Omega_n^2(\gamma(s))(\gamma^u(s) + \dot{\gamma}^u(s))\right) = \left((\partial_u + \partial_v) \log \Omega_n^2 - 2 \frac{\partial_u r_n + \partial_v r_n}{r_n}\right) \frac{l^2}{r_n^2(\gamma(s))}.
\]

In view of the relation (2.20) for \(\dot{\gamma}\), we can bound:

\[
\frac{l^2}{r_n^2(\gamma(s))} = \Omega_n^2(\gamma(s))(\gamma^u(s) + \dot{\gamma}^u(s)) \leq \frac{1}{2} \Omega_n^{-2}(\gamma(s)) \cdot \left(\Omega_n^2(\gamma(s))(\dot{\gamma}^u(s) + \dot{\gamma}^v(s))\right)^2.
\]

Moreover, in view of the boundary condition (4.30) on \(\gamma^{u_0}\), we can bound for some constant \(C\) depending only on \(V_n\):

\[
\text{sup}_{\mathcal{P}_0^0} \left| \partial_v r_n + \partial_u r_n \right| \leq C \text{sup}_{\mathcal{P}_0^0} \left(\partial_v - \partial_u\right)(\partial_v r_n + \partial_u r_n).
\]

Therefore, using (4.37), (4.59) and (4.60), the relation (4.58) yields the estimate

\[
\frac{d}{ds}\left(\Omega_n^2(\gamma(s))(\gamma^u(s) + \delta^u(s))\right) \leq C(C_1) \left(\Omega_n^2(\gamma(s))(\dot{\gamma}^u(s) + \dot{\gamma}^v(s))\right)^2.
\]

Using the fact that, along \(\gamma\), we have

\[
(\gamma_u(s) + \dot{\gamma}_v(s))ds \sim (du + dv)_{\gamma},
\]

we deduce from (4.61), after applying Gronwall’s inequality and using (4.37), that:

\[
\text{sup}_{\gamma} \left(\Omega_n^2(\gamma^u + \dot{\gamma}^u)\right) \leq \Omega_n^2(\gamma^u + \dot{\gamma}^u)(0) \cdot \exp\left(C(C_1) R_0^{-1} u_0\right).
\]

The bound (4.62) implies, in view of the bound (3.10) (with \(C_0\) in place of \(C\)) for the support of the initial data \(\tilde{f}_1\) (using also (2.20)) that:

\[
\text{sup}(f_{n+1}) \in \left\{\Omega_n^2(p^u + p^v) \leq C_0 \exp\left(C(C_1) R_0^{-1} u_0\right)\right\} \cap \left\{ \frac{l}{r_n} \leq C_0 \exp\left(C(C_1) R_0^{-1} u_0\right)\right\}.
\]

Thus, from (2.32) and (4.63) (using also the fact that \(\|f_{n+1}\|_{\infty} = \|\tilde{f}_1\|_{\infty}\)) we deduce that, for some constant \(C > 0\) depending only on \(V_n\):

\[
\text{sup}_{\mathcal{P}_0^0} \left(\left(T_{uu}\right)_{n+1} + \left(T_{uv}\right)_{n+1} + \left(T_{vv}\right)_{n+1}\right) \leq C \cdot C_0^2 \|\tilde{f}_1\|_{\infty} \exp\left(C(C_1) R_0^{-1} u_0\right).
\]

Let \((x_0^0, \ldots, x_3^0)\) be the Cartesian coordinates associated to the metric \(g_n\), defined by the relations (3.1). The bound (4.37) implies that the coordinate transformation \((u, v, \theta, \varphi) \rightarrow (x_0^0, \ldots, x_3^0)\) is of \(C^{2,1}\) regularity. Furthermore, in view of the expression (3.2) of the metric \(g_n\) in the Cartesian coordinates, the bound (4.37) implies that the Cartesian Christoffel symbols of \(g_n\) satisfy

\[
\|\Gamma^\text{Cart}_n\|_{C^{0,1}} \leq C(C_1).
\]
Differentiating the Vlasov equation (4.29) for \( f_{n+1} \) with respect to the Cartesian coordinates and using (4.65) and (4.63), we can therefore readily estimate:

\[
(4.66) \quad \left\| \partial_{x_n} \bar{f}_{n+1} \right\|_{\infty} \leq \left\| \partial_{x_n} \bar{f}_j \right\|_{\infty} + C(C_1) R_0^{-2} u_0.
\]

Using (4.63) and (4.66), as well as (4.50) and (4.64) for \( n \) in place of \( n + 1 \), we can estimate for the Cartesian components of the energy momentum tensor of \( f_{n+1} \) for some constant \( C > 0 \) depending only on \( \Lambda \):

\[
(4.67) \quad R_0 \left\| \partial_{x_n} (T_{\beta\gamma})_{n+1} \right\|_{\infty} \leq CC_0^5 \left\{ 1 + \left\{ \sum_{j=0}^n R_0^2 \left\| \partial_{x_n} \partial_p \gamma_r \right\|_{\infty} + \sum_{j=0}^2 \left( \partial_{x_n} \partial_p \log \Omega^2 \right) \right\} + C(C_1) R_0^{-1} u_0 \right\} \times \exp \left( C(C_1) R_0^{-1} u_0 \right).
\]

Using and (4.64) for \( n \) in place of \( n + 1 \), as well as the relations (3.1) defining the Cartesian coordinates, from (4.67) we obtain the following bound for the first order derivatives of \( T_{n+1} \) in the double null coordinate system (for a possibly larger constant \( C \)):

\[
(4.68) \quad \sup_{\mathcal{D}_0^0} \sum_{j=0}^n \sum_{j_1+j_2=j} R_0^{j_2} \left( \left\| \partial_{n_1} \partial_{n_2} \gamma_n \right\|_{\infty} + \left\| \partial_{n_1} \partial_{n_2} \partial_{x_{n+1}} \right\|_{\infty} + \left\| \partial_{n_1} \partial_{x_{n+1}} \right\|_{\infty} \right) \leq CC_0^5 \left\{ 1 + \left\{ \sum_{j=0}^n R_0^2 \left\| \partial_{x_n} \partial_p \gamma_r \right\|_{\infty} + \sum_{j=0}^2 \left( \partial_{x_n} \partial_p \log \Omega^2 \right) \right\} + C(C_1) R_0^{-1} u_0 \right\} \times \exp \left( C(C_1) R_0^{-1} u_0 \right).
\]

For \( n \geq 2 \), let \( \gamma(n), \gamma(n-1) : (-s_1, s_1) \to \mathcal{D}_0^0 \) be two future directed curves such that:

- \( \gamma(i) \) is a null geodesic for \( g_i, i = n, n - 1 \),
- \( \gamma(n) = \gamma(n-1)_{(0)} \) and \( \dot{\gamma}(n) = \dot{\gamma}(n-1)_{(0)} \),
- \( \gamma(i) : \gamma(i) \) lies in the support of the Vlasov field \( f_{i+1}, i = n, n - 1 \).

By subtracting from equation (2.17) for \( \gamma(n) \) the same equation for \( \gamma(n-1) \), working in the Cartesian coordinate systems \( (x_n^0, \ldots, x_n^{3n}) \), \( (x_{n-1}^0, \ldots, x_{n-1}^{3n}) \) for \( g_n, g_{n-1} \), we can estimate for some absolute constant \( C > 0 \):

\[
(4.69) \quad \frac{d}{ds} |\gamma(n) - \gamma(n-1)| \leq |\dot{\gamma}(n) - \dot{\gamma}(n-1)|,
\]

\[
(4.70) \quad \frac{d}{ds} \left| \dot{\gamma}(n) - \dot{\gamma}(n-1) \right| \leq CC \left( 1 + \left| \dot{\gamma}(n) \right|_{\infty} + \left| \dot{\gamma}(n-1) \right|_{\infty} \right)^2 \left( \left| T_{\gamma\beta} \right|_{\infty} + \left| \Gamma^\delta \right|_{\infty} \right) \leq CC_0^5 \left\{ 1 + \left\{ \sum_{j=0}^n R_0^2 \left\| \partial_{x_n} \partial_p \gamma_r \right\|_{\infty} + \sum_{j=0}^2 \left( \partial_{x_n} \partial_p \log \Omega^2 \right) \right\} + C(C_1) R_0^{-1} u_0 \right\} \times \exp \left( C(C_1) R_0^{-1} u_0 \right).
\]

where \( |\cdot| \) denotes the Euclidean norm in the Cartesian coordinates. Using (4.37) and (4.63) for \( n, n - 1 \) in place of \( n + 1 \), as well as (4.62) and the expression (3.2) for \( g_n, g_{n-1} \) in Cartesian coordinates, from (4.69)–(4.70) (and the definition (4.39) of \( \mathcal{D}_n \)) we infer that:

\[
(4.72) \quad \frac{\left\| \gamma(n) - \gamma(n-1) \right\| + \left\| R_0^{-1} \gamma(n) - \gamma(n-1) \right\|_{\infty}}{\left\| \dot{\gamma}(n) \right\|_{\infty} + \left\| \dot{\gamma}(n-1) \right\|_{\infty}} \leq C(C_1) R_0^{-1} u_0 \mathcal{D}_n.
\]

The bound (4.72) readily implies the following estimate for the Vlasov fields \( f_{n+1}, f_n \) (using also the mean value theorem for \( f_n \), as well as the condition (3.10)):

\[
(4.73) \quad \frac{R_0^2 \sup_{\mathcal{D}_0^0} \left( 1 + \right)^2 \left\| f_{n+1} - f_n \right\|_{\infty}}{\left\| \nabla \nabla \right\|_{\infty} + \left\| \nabla \right\|_{\infty}} \leq C(C_1) R_0^{-1} u_0 \mathcal{D}_n \cdot \left( \left\| \partial_{x_n} \bar{f}_j \right\|_{\infty} + \left\| \partial_{x_n} \bar{f}_j \right\|_{\infty} \right).
\]
Note that the boundedness of the right hand side of (4.73) follows from the assumption that \( (r_{f}, \Omega_{f}^{2}, \tilde{f}_{f}) \) satisfy Condition 2 of Definition 3.5.

Provided \( C_{1} > 0 \) has been fixed large in terms of the initial data \( (r_{f}, \Omega_{f}^{2}, \tilde{f}_{f}) \) and \( u_{0} \) has been chosen small enough in terms of \( C_{1} \), the bounds (4.49)–(4.50), (4.56)–(4.64) and (4.68) readily yield (4.37) for \( k = n + 1 \). Similarly, the bounds (4.51), (4.57) and (4.73) yield (4.38).

Having established that (4.37) and (4.38) hold for all \( n \geq 2 \), it readily follows (using standard arguments) that (4.39) converges in the \( C^{1}(D_{0}^{0}) \times C^{1}(D_{0}^{0}) \times C^{1}(r_{p,l}^{1}, L_{p,l}^{1}) \) topology (where \( L_{p,l}^{1} \) is a weighted \( L^{1} \) norm in the momentum variables for \( \tilde{f} \)) to a solution \( (r, \Omega^{2}, f) \) of (2.43)–(2.48) on \( D_{0}^{0} \) (the fact that the constraint equations (2.45)–(2.46) are also satisfied follows readily from the fact that \( (r_{f}, \Omega_{f}^{2}, \tilde{f}_{f}) \) was assumed to satisfy (2.45)). In view of (4.37), \( (r, \Omega^{2}, f) \) satisfies the estimate

\[
\| \log \Omega^{2} \|_{C^{1.1}(D_{0}^{0})} + \| \log(\partial_{r}r) \|_{C^{1.1}(D_{0}^{0})} + \| \log(-\partial_{u}u) \|_{C^{1.1}(D_{0}^{0})} + \| T_{\alpha\beta} \|_{C^{0.1}(D_{0}^{0})} \leq C_{1}.
\]

Let us also remark that the bound (4.63) also implies the following bound for the support of the Vlasov field \( f \):

\[
supp(f) \subseteq \{ \Omega^{2}(p^{2} + \gamma^{2}) \leq C_{0} \exp(C_{1}R_{0}^{-1}u_{0}) \} \cap \left\{ \frac{1}{r} \leq C_{0} \exp(C_{1}R_{0}^{-1}u_{0}) \right\}.
\]

Using the relations (2.53) and (2.54) for \( \partial_{\alpha}\bar{m} \) and \( \partial_{\alpha}\bar{n} \), respectively, as well as the bound (4.74), we can readily estimate for any \( (u, v) \in D_{0}^{0} \):

\[
\left| \partial_{\alpha}\left( \frac{\bar{m}}{r^{3}} \right) \right| (u, v) = 8\pi \left| \partial_{\alpha} \left( \frac{\int_{u}^{v} r^{-2} \Omega^{-2} \left( (-\partial_{u}r)T_{uv} + \partial_{v}rT_{uv} \right)(u, v) \cdot d\bar{v}}{r^{3}(u, v)} \right) \right| =
\]

\[
= 8\pi \left| \int_{u}^{v} r^{-2} \Omega^{-2} \left( (-\partial_{u}r)T_{uv} + \partial_{v}rT_{uv} \right) d\bar{v} \cdot \frac{r^{-3}(u, v)}{r^{3}(u, v)} \right|
\]

\[
- 3 \left| \int_{u}^{v} r^{-2} \Omega^{-2} \left( (-\partial_{u}r)T_{uv} + \partial_{v}rT_{uv} \right) d\bar{v} \cdot \partial_{r}r(u, v) \right|
\]

\[
= 8\pi \left| \int_{u}^{v} r^{-2} \Omega^{-2} \left( (-\partial_{u}r)T_{uv} + \partial_{v}rT_{uv} \right) \frac{d\bar{v}}{r^{4}(u, v)} \right|
\]

\[
- \int_{u}^{v} \partial_{\alpha}(r^{3}) \Omega^{-2} \left( (-\partial_{u}r)T_{uv} + T_{uv} \right) \frac{d\bar{v}}{r^{4}(u, v)} \cdot \partial_{r}r(u, v)
\]

\[
\leq C(C_{1}).
\]

Similarly,

\[
\left| \partial_{\alpha}\left( \frac{\bar{n}}{r^{3}} \right) \right| (u, v) \leq C(C_{1}).
\]

The boundary condition \( r_{\mid\gamma_{0}} = 0 \) on the axis \( \gamma_{0} \) implies that

\[
(\partial_{u} + \partial_{\alpha})r_{\mid\gamma_{0}} = 0.
\]

In view of (4.78), the bound (4.74) implies, through an application of the mean value theorem, that, for any \( (u, v) \in D_{0}^{0} \):

\[
\left\| \frac{\partial_{\alpha}r}{r} \right\|_{C^{0.1}(D_{0}^{0})} \leq C(C_{1}) \left( 1 + \sup_{\gamma_{0}} \left| \partial_{\alpha}^{3}r \right| (u, v) \right) \leq C(C_{1}).
\]
Uniqueness. We will establish that a smooth solution of (4.38) (with equations (2.52) and (2.55) in place of (4.27) and (4.28)),

\[ \text{Combining (4.74), (4.76), (4.77), (4.79) and (4.80), we infer that:} \]

\[ \| \log \Omega \|_{C^{1,1}(D_{0}^{u})} + \| \log(\partial r) \|_{C^{1,1}(D_{0}^{u})} + \| \log(-\partial u) \|_{C^{1,1}(D_{0}^{u})} + \| T_{\alpha\beta} \|_{C^{0,1}(D_{0}^{u})} + \]

\[ + \| \bar{m}_{j} \|_{C^{0,1}(D_{0}^{u})} + \| \frac{\partial_{r}u + \partial_{u}r}{r} \|_{C^{0,1}(D_{0}^{u})} + \| \frac{r^{-2}(-4\partial_{u}r\partial_{r}r)(\partial_{r}u - \partial_{u}r)^{2} - 1)}{r} \|_{C^{0,1}(D_{0}^{u})} \leq C(1). \]

Let us switch to the Cartesian coordinate system (3.1) on \( \mathcal{N}_{u_{0}} = (D_{0}^{u} \times S^{2}) \cup \mathcal{Z} \). The expression (3.2) for the Cartesian components \( g_{\alpha\beta} \) of \( g \) implies, in view of (4.81), that

\[ \| g_{\alpha\beta} \|_{C^{1,1}(\mathcal{N}_{u_{0}})} < +\infty. \]

Combining (2.43)–(2.48) with \( \partial_{u}, \partial_{r} \), and arguing inductively in the number of commutations, using also the fact that the initial data set \( (r_{1}, \Omega_{1}^{2}, f_{1}) \) satisfies Condition 1 of Definition 3.5, we can readily infer that, for any \( k \in \mathbb{N} \), the Cartesian components \( g_{\alpha\beta} \) in fact satisfy

\[ \| g_{\alpha\beta} \|_{C^{0,1}(\mathcal{N}_{u_{0}})} < +\infty. \]

Thus, using the fact that \( f \) satisfies the Vlasov equation on the background \( (\mathcal{N}_{u_{0}}, g) \), we readily infer the smoothness of \( (r, \Omega^{2}, f) \) in accordance with Definition 3.1

Uniqueness. We will establish that a smooth solution \( (r, \Omega^{2}, f) \) of (2.43)–(2.48) on \( D_{0}^{u} \) with initial data (4.6)–(4.7) on \( u = 0 \) is unique through a contradiction argument. Let us assume that \( (r_{*}, \Omega_{*}^{2}, f_{*}) \) is another smooth solution of (2.43)–(2.48) on \( D_{0}^{u} \) with the same initial data. Since \( (r, \Omega^{2}, f), (r_{*}, \Omega_{*}^{2}, f_{*}) \) are smooth, there exists some \( C_{1} > 0 \) so that the bound (4.37) is satisfied with \( (r, \Omega^{2}, f), (r_{*}, \Omega_{*}^{2}, f_{*}) \) in place of \( (r_{k}, \Omega_{k}^{2}, f_{k}) \). Then, by repeating exactly the same arguments that led to the proof of (4.38) (with equations (2.52) and (2.55) in place of (4.27) and (4.28)), we infer that, provided \( u_{0} \) is sufficiently small with respect to \( C_{1} \):

\[ \mathfrak{D} \leq \frac{1}{2} \mathcal{D}, \]

where

\[ \mathfrak{D} \geq \sup_{D_{0}^{u}} \left\{ \sum_{j=1}^{2} \sum_{j_{1}, j_{2}, j_{3}, j_{4}} R_{0}^{j_{1}-1} \partial_{j_{1}}^{j_{2}} \partial_{j_{3}}^{j_{4}} r - \partial_{j_{1}}^{j_{2}} \partial_{j_{3}}^{j_{4}} r_{*} + \sum_{j=0}^{1} \sum_{j_{1}, j_{2}, j_{3}, j_{4}} R_{0}^{j_{1}} \partial_{j_{1}}^{j_{2}} \partial_{j_{3}}^{j_{4}} \log \Omega^{2} - \partial_{u}^{j_{1}} \partial_{v}^{j_{2}} \log \Omega_{*}^{2} \right\} \]

\[ + r^{-2} \int_{0}^{+\infty} \int_{0}^{+\infty} \left\{ \Omega^{2}(p^{u} + p^{v}) \right\}^{2} \left( \bar{f}_{*} - \bar{f}_{*} \right) (\cdot ; p^{u}, l) \frac{dp^{u}}{p^{u}} \frac{dl}{l}, \]

The bound (4.83) implies that \( \mathfrak{D} = 0 \), and thus \( (r, \Omega^{2}, f) \) and \( (r_{*}, \Omega_{*}^{2}, f_{*}) \) coincide.

**Step 2: the region** \([0, u_{1}] \times [u_{0}, v_{2}]\). For some \( u_{1} \in (0, \frac{1}{2} u_{0}) \) to be determined later, let \( (r_{1}, \Omega_{1}^{2}, \bar{f}_{1}) \) be the data induced by the solution \( (D_{0}^{u_{0}}; r, \Omega^{2} f) \) (constructed in the previous step) on \( \{ v = u_{0} \} \cap \{ 0 \leq u \leq u_{1} \} \), i.e., for any \( u \in [0, u_{1}] \):

\[ (r_{1}, \Omega_{1}^{2})(u) = (r, \Omega^{2})(u, u_{0}) \]

and

\[ \bar{f}_{1}(u; p^{u}, l) \cdot \delta \left( \Omega_{1}^{2}(u) p^{u} p^{v} - \frac{r^{2}}{r_{1}(u)} \right) = f(u, u_{0}; p^{u}, p^{v}, l). \]
We will show that, provided $u_1$ is sufficiently small, there exists a unique smooth solution $(r, \Omega^2; f)$ of (2.43)–(2.48) on $W \equiv [0, u_1] \times [u_0, v_x]$ satisfying (4.6)–(4.7) on $\{0\} \times [u_0, v_x]$ and (4.85)–(4.86) on $[0, u_1] \times \{u_0\}$.

Remark. As a consequence of the bound (4.75), we infer that

$$\text{supp}(\tilde{f}) \subset \left\{ \Omega_0^2 p^v + \frac{l^2}{r^v} p^v \leq 2C_0 \exp \left( C(C_1) R_0^{-1} u_0 \right) \right\} \cap \left\{ \frac{l}{\lambda} \leq C_0 \exp \left( C(C_1) R_0^{-1} u_0 \right) \right\}.$$

Our proof will be very similar to the one carried out in the previous step. In particular, for any integer $n \in \mathbb{N}$, we will define the functions $r_n : W \rightarrow [0, +\infty)$, $\Omega_n^2 : W \rightarrow (0, +\infty)$ and $f_n' : W \times [0, +\infty)^2 \rightarrow [0, +\infty)$ recursively by the following conditions:

1. The functions $\Omega_n^2$, $r_n$ satisfy the following system, which is a recursive analogue of the renormalised equations (2.57) (see (4.97) for the definition of $\tilde{m}_n$)

$$\partial_u \partial_v \log \left( \frac{\Omega_{n+1}^2}{1 - \frac{1}{3} \Lambda r_{n+1}^2} \right) = \tilde{m}_n \left( \frac{1}{r_n^2} + \frac{1}{3} \Lambda r_n^2 - 1 \right) - \frac{\Omega_n^2}{1 - \frac{1}{3} \Lambda r_n^2} - 16\pi \frac{1}{3} \Lambda r_n^2 (T_{uv})_n,$$

$$\partial_u \partial_v \left( \tan^{-1} \left( \sqrt{\frac{\Lambda}{3} r_{n+1}^2} \right) \right) = -\frac{1}{2} \sqrt{\frac{\Lambda}{3} \tilde{m}_n} \frac{1 - \frac{2}{3} \Lambda r_n^2}{1 - \frac{1}{3} \Lambda r_n^2} \frac{\Omega_n^2}{1 - \frac{1}{3} \Lambda r_n^2} + 4\pi \sqrt{-\frac{\Lambda}{3} r_n - \frac{1}{3} \Lambda r_n^3} (T_{uv})_n,$$

with characteristic initial conditions

$$\left( r_n, \Omega_n^2 \right)_{[0, u_1] \times \{u_0\}} = \left( r_1, \Omega_1^2 \right)$$

and

$$\left( r_{n+1}, \Omega_{n+1}^2 \right)_{\{0\} \times [u_0, v_x]} = \left( r_f, \Omega_f^2 \right).$$

2. For any $l > 0$, the function $\tilde{f}'$ satisfies the following relations:

(a) For any future directed, causal (with respect to the reference metric (2.58)) curve $\gamma : [0, a) \rightarrow W$ satisfying

$$\Omega_n^2 \gamma^u \gamma^v = \frac{l^2}{r_n^2}.$$
with $\gamma(0) \in \{0\} \times [u_0, v_L)$ and solving

$$
(4.93) \quad \log \left( \Omega_n^{2 \tilde{r}^u} \right)(s) - \log \left( \Omega_n^{2 \tilde{r}^v} \right)(0) = \int_{v(\gamma(0))}^{v(\gamma(s_+))} \int_0^{u(\gamma(s_+))} \left( \frac{\delta_m n - 1}{2} \frac{r_n}{r_n^2} \Omega_n^2 - 24\pi (T_{uv})_n \right) dudv + \int_{v(\gamma(0))}^{v(\gamma(s_+))} \left( \partial_v \log(\Omega_j^n) - 2 \frac{\partial_u r_j}{r_j} \right)(v) \, dv
$$

(where $s_+$ is defined by \[2.60\]), $\tilde{f}_{n+1}'$ satisfies

$$
(4.94) \quad \tilde{f}_{n+1}'(u(\gamma(s)), v(\gamma(s)); \dot{\gamma}^u(s), l) = \tilde{f}_{j}(u(\gamma(0)); \dot{\gamma}^u(0), l)
$$

(b) For any future directed, causal (with respect to the reference metric \[2.58\]) curve $\gamma : [0, a) \to \mathcal{W}$ satisfying \[4.92\] with $\gamma(0) \in [0, u_1) \times \{u_0\}$ and solving

$$
(4.95) \quad \log \left( \Omega_n^{2 \tilde{r}^u} \right)(s) - \log \left( \Omega_n^{2 \tilde{r}^v} \right)(0) = \int_{u(\gamma(0))}^{u(\gamma(s_+))} \int_0^{v(\gamma(s_+))} \left( \frac{\delta_m n - 1}{2} \frac{r_n}{r_n^2} \Omega_n^2 - 24\pi (T_{uv})_n \right) dvdu + \int_{u(\gamma(0))}^{u(\gamma(s_+))} \left( \partial_u \log(\Omega_j^n) - 2 \frac{\partial_v r_j}{r_j} \right)(u) \, du
$$

(where $s_+$ is defined by \[2.63\]), the function $\tilde{f}_{n+1}'$ satisfies

$$
(4.96) \quad \tilde{f}_{n+1}''(u(\gamma(s)), v(\gamma(s)); \dot{\gamma}^v(s), l) = \tilde{f}_{j}(u(\gamma(0)); \dot{\gamma}^v(0), l)
$$

In the above, the function $\tilde{m}_n$ is defined in terms of $\Omega_n^2$, $r_n$ and $\tilde{f}_{n}'$ by the implicit relation

$$
(4.97) \quad \tilde{m}_n(u, v) = \tilde{m}_j(v) - 8\pi \int_{0}^{u} \left( 1 - \frac{2\delta_m n}{r_n} - \frac{1}{3} \Lambda r_n \right) (T_{uv})_{n-1} + \Omega_{n-1}^2 \partial_v r_{n-1} \cdot (T_{uv})_{n-1} \, du,
$$

where $(T_{uv})_n$ is defined in terms of $\Omega_n^2$, $r_n$ and $\tilde{f}_{n}'$ as in the previous step of the proof. When $n = 0$, we will adopt the convention that

$$
(4.98) \quad (r_0, \Omega_0^2, f_0) = (r_{AdS}^{(vz)}, (\Omega_{AdS}^{(vz)}))^2(0, \cdot) ; 0
$$

and

$$
(4.99) \quad \tilde{m}_0 = 0,
$$

where the rescaled AdS metric coefficients $r_{AdS}^{(vz)}, (\Omega_{AdS}^{(vz)})^2$ are given by

$$
(4.100) \quad r_{AdS}^{(vz)}(u, v) = r_{AdS} \left( \sqrt{\frac{3}{\Lambda} \pi^2 v^2} \right), \quad (\Omega_{AdS}^{(vz)})^2(u, v) = \frac{3}{\Lambda} \frac{\pi^2}{\Lambda v^2} \Omega_{AdS} \left( \sqrt{\frac{3}{\Lambda} \pi v^2} \right).
$$

Let $C_1 \gg 1$ be a large constant depending only on the initial data $(r_n, \Omega_n^2, \tilde{f}_j)$ and $(r_j, \Omega_j^2, \tilde{f}_j)$. We will establish the following inductive bounds for $(r_n, \Omega_n^2; f_n)$: Assuming that, for any $0 \leq k \leq n$,

$$
(4.101) \sup_{\mathcal{W}} \left| \frac{\Omega_k^2}{1 - \frac{1}{3} \Lambda r_k^2} \right| + \left| \log \left( \frac{\Omega_k^2}{1 - \frac{1}{3} \Lambda r_k^2} \right) \right| + \left| \log \left( \frac{\partial_u r_k}{1 - \frac{1}{3} \Lambda r_k^2} \right) \right| + \sqrt{-\Lambda |\tilde{m}_k| + \epsilon_k^2 (T_{uu})_k + (T_{uv})_k + (T_{ve})_k)} < C_1,
$$

55
we will show that (4.101) also holds for $k = n + 1$ and, moreover (in the case $n \geq 2$):

(4.102) \[ \mathcal{D}'_{n+1} \leq \frac{1}{2} \mathcal{D}'_n, \]

where

(4.103) \[ \mathcal{D}'_k \equiv \sup_{W} \left\{ v_2 \left[ \partial_{v} \log \left( \frac{\Omega_n^2}{1 - \frac{3}{2} \Lambda r_k^2} \right) - \partial_{v} \log \left( \frac{\Omega_{n+1}^2}{1 - \frac{3}{2} \Lambda r_k^2} \right) \right] + \left| \partial_{v} \log \left( \frac{\Omega_k^2}{1 - \frac{3}{2} \Lambda r_{k-1}^2} \right) - \log \left( \frac{\Omega_{k-1}^2}{1 - \frac{3}{2} \Lambda r_{k-1}^2} \right) \right| + \right. \]

\[ + \left| \partial_{v} \log \left( \frac{\partial_{v}r_k}{1 - \frac{3}{2} \Lambda r_k^2} \right) - \log \left( \frac{\partial_{v}r_{k-1}}{1 - \frac{3}{2} \Lambda r_{k-1}^2} \right) \right| + \sqrt{\Lambda} |\tilde{m}_k - \tilde{m}_{k-1}| + \]

\[ + (-\Lambda)^{-\frac{1}{2}} \int_0^{+\infty} \int_0^{+\infty} \left( \Omega_k^2(p^u + v^u) \right)^2 \left( f_k - f_{k-1} \right) (\cdot: p^u, l) \frac{dp^u}{d\mu} \, dt \right\}. \]

Remark. Notice that, in the definition (4.97) of $\tilde{m}_k$ and the left hand sides of (4.101), (4.103), (4.93) and (4.95), only $\partial_v$ derivatives of $\Omega_k^2$, $r_k$ appear. The reason that we took special care to arrange those expressions in this way is a small parameter (necessary for the iteration procedure to be successful) can only be obtained from the equations (4.88)–(4.89) by integrating in the $u$ direction.

Let $r_{\min}$ be defined in terms of the initial data as

\[ r_{\min} \equiv r \backslash (u_1) > 0. \]

Notice that, because $\partial_v r \backslash < 0$ (following from the properties of the solution $(r, \Omega^2, f)$ on $\mathcal{D}'_k$ constructed in the previous step), for any $k \in \mathbb{N}$ for which $\inf_{W} \partial_v r_k \geq 0$ (which is necessarily true if (4.101) holds), then

(4.104) \[ \inf_{W} r_k = r_{\min} > 0. \]

We will assume that $C_1$ in (4.101) has been chosen large enough so that

(4.105) \[ v_2 r_{\min}^{-1} \ll C_1. \]

For any $n \geq 0$, using the bound (4.101) and (4.104)–(4.105) for $k = n$ and integrating equations (4.88)–(4.89) in the $u$ direction, we immediately infer that

(4.106) \[ \sup_{W} \left\{ v_2 \left| \partial_{v} \log \left( \frac{\Omega_{n+1}^2}{1 - \frac{3}{2} \Lambda r_{n+1}^2} \right) \right| + \left| \partial_{v} \log \left( \frac{\partial_{v}r_{n+1}}{1 - \frac{3}{2} \Lambda r_{n+1}^2} \right) \right| \right\} \leq C(C_1) u_1 + \sup_{v \in \{u_0, v_2\}} \left\{ v_2 \left| \partial_{v} \log \left( \frac{\Omega_f^2}{1 - \frac{3}{2} \Lambda r_f^2} \right) \right| + \left| \partial_{v} \log \left( \frac{\partial_{v}r_f}{1 - \frac{3}{2} \Lambda r_f^2} \right) \right| \right\} \]

Integrating in $v$ the bound for $\partial_v \log \left( \frac{\Omega_{n+1}^2}{1 - \frac{3}{2} \Lambda r_{n+1}^2} \right)$ provided by (4.106) and adding the resulting estimate for $\log \left( \frac{\Omega_{n+1}^2}{1 - \frac{3}{2} \Lambda r_{n+1}^2} \right)$ to (4.106), we therefore infer (provided $C_1$ has been chosen large enough in terms of $(r_k, \Omega^2_k)$ and $(r_f, \Omega^2_f)$):

(4.107) \[ \sup_{W} \left\{ v_2 \left| \partial_{v} \log \left( \frac{\Omega_{n+1}^2}{1 - \frac{3}{2} \Lambda r_{n+1}^2} \right) \right| + \right. \]

\[ \left. + \left| \partial_{v} \log \left( \frac{\Omega_{n+1}^2}{1 - \frac{3}{2} \Lambda r_{n+1}^2} \right) \right| \right\} \leq C(C_1) v_2^{-1} u_1 + \frac{1}{10} C_1. \]

Similarly, for $n \geq 1$, subtracting from (4.88)–(4.89) the same equations with $n - 1$ in place of $n$ and using (4.101) and (4.104)–(4.105) for $k = n, n - 1$, we can readily bound:

(4.108) \[ \sup_{W} \left\{ v_2 \left| \partial_{v} \log \left( \frac{\Omega_{n+1}^2}{1 - \frac{3}{2} \Lambda r_{n+1}^2} \right) - \partial_{v} \log \left( \frac{\Omega_{n}^2}{1 - \frac{3}{2} \Lambda r_{n}^2} \right) \right| \right. \]

\[ + \left. \left| \partial_{v} \log \left( \frac{\partial_{v}r_{n+1}}{1 - \frac{3}{2} \Lambda r_{n+1}^2} \right) - \log \left( \frac{\partial_{v}r_{n}}{1 - \frac{3}{2} \Lambda r_{n}^2} \right) \right| \right\} \leq C(C_1) v_2^{-1} u_1 \mathcal{D}'_n. \]
We will show that:

\[(4.116)\]

\[
\Omega_{n+1}^2 \leq C(C_1).
\]

Similarly, subtracting from (4.97) for \(r_{n+1}^\text{max}\) from which we readily infer that the following useful estimates for the \(\partial_u\) derivatives of \(\Omega_{n+1}^2\): \(\Omega_{n+1}^2\):

\[(4.109)\]

\[
\sup_{Ω} \left\{ v_{\sharp} \partial_u \log \left( \frac{\Omega_{n+1}^2}{1 - \frac{3}{4} \Lambda v^2_{n+1}} \right) + \left| \frac{\partial_u r_{n+1}}{1 - \frac{3}{4} \Lambda v^2_{n+1}} \right| \right\} \leq C(C_1).
\]

Using the bound (4.101) for \(k = n\), we can readily infer from the relation (4.97) for \(\tilde{m}_{n+1}\) (using a simple application of Gronwall’s inequality) that

\[(4.110)\]

\[
\sup_{Ω} \sqrt{-\Lambda} |\tilde{m}_{n+1}| \leq \exp \left( C(C_1) v_{\sharp}^{-1} u_{1} \right) \left( C(C_1) v_{\sharp}^{-1} u_{1} + \sup_{v \in [u_0, v_{\sharp}]} \sqrt{-\Lambda} \tilde{m}_n(v) \right).
\]

Similarly, subtracting from (4.97) for \(\tilde{m}_{n+1}\) the same relation for \(\tilde{m}_n\) and using (4.101) for \(k = n, n - 1\), we can similarly estimate:

\[(4.111)\]

\[
\sup_{Ω} \sqrt{-\Lambda} |\tilde{m}_{n+1} - \tilde{m}_n| \leq C(C_1) v_{\sharp}^{-1} u_{1} \mathcal{D}_n'.
\]

Let \(\gamma : [0, a) \to \mathcal{W}\) be a future directed, causal curve (with respect to the reference metric (2.58)) satisfying (4.92) and (4.93) for some \(l > 0\), such that initially

\[(4.112)\]

\[
\gamma(0) \in \{0\} \times [u_0, v_{\sharp}).
\]

Using the bound (4.101) (for \(k = n\)) for the right hand side of (4.93), as well as the lower bound (4.104) for \(r_n\), we trivially infer from (4.92) and (4.93) that (provided \(C_1\) is sufficiently large in terms of \((r_f, \Omega_f^2))\):

\[(4.113)\]

\[
\frac{1}{C(C_1)} \leq \sup_{s \in [0, a)} \left( \frac{\Omega_n^2(\gamma(s))(\dot{\gamma}^u + \dot{\gamma}^v)(s)}{\Omega_n^2(\gamma(0))(\dot{\gamma}^u + \dot{\gamma}^v)(0)} \right) \leq C(C_1).
\]

Arguing in exactly the same way, we also infer that the bound (4.113) also holds for future directed, causal curve \(\gamma : [0, a) \to \mathcal{W}\) satisfying (4.92) and (4.95) for some \(l > 0\), such that initially

\[(4.114)\]

\[
\gamma(0) \in \{0, u_1\} \times \{0\}.
\]

We will now proceed to obtain a more refined energy bound for curves \(\gamma\) as above which moreover satisfy a quantitative lower bound on their angular momentum \(l\). In particular, let \(\gamma : [0, a) \to \mathcal{W}\) be a future directed, causal curve satisfying (4.92), (4.93) and (4.112), for some \(l > 0\) such that

\[(4.115)\]

\[
\frac{l}{\Omega_n^2(\gamma(0))(\dot{\gamma}^u + \dot{\gamma}^v)(0)} > v_{\sharp} \left( \frac{u_1}{v_{\sharp}} \right)^{\frac{1}{3}}.
\]

The relation (4.112), combined with the bounds (4.101) and (4.104) (for \(k = n\)) imply that

\[(4.116)\]

\[
\frac{\Omega_n^2(\gamma(0))(\dot{\gamma}^u(0) \cdot \Omega_n^2(\gamma(0))(\dot{\gamma}^v(0) \cdot (\dot{\gamma}^u + \dot{\gamma}^v)(0))^2 \geq c(C_1) \left( \frac{u_1}{v_{\sharp}} \right)^{\frac{2}{3}},
\]

from which we readily infer that

\[(4.117)\]

\[
\max \left\{ \frac{\dot{\gamma}^u(0)}{\dot{\gamma}^v(0)}, \frac{\dot{\gamma}^v(0)}{\dot{\gamma}^u(0)} \right\} \leq C(C_1) \left( \frac{u_1}{v_{\sharp}} \right)^{\frac{2}{3}}.
\]

We will show that:

\[(4.118)\]

\[
\exp \left( - C(C_1) \left( \frac{u_1}{v_{\sharp}} \right)^{\frac{2}{3}} \right) \leq \sup_{s \in [0, a)} \left( \frac{\Omega_n^2(\gamma(s))(\dot{\gamma}^u + \dot{\gamma}^v)(s)}{\Omega_n^2(\gamma(0))(\dot{\gamma}^u + \dot{\gamma}^v)(0)} \right) \leq \exp \left( C(C_1) \left( \frac{u_1}{v_{\sharp}} \right)^{\frac{2}{3}} \right)
\]
From (4.123), the relation (4.92) between for all (4.120) (4.119) sup and (4.123) in place of.

In particular, (4.121) will imply by continuity that (4.122) c = a

is defined by (2.63). Using the upper bound for s ∈ [0, c], we also obtain that (4.121)

\[ \sup_{s \in [0, c]} \left( \max \left\{ \frac{\dot{\gamma}^u}{\gamma^v}(s), \frac{\dot{\gamma}^v}{\gamma^u}(s) \right\} - \max \left\{ \frac{\dot{\gamma}^u}{\gamma^v}(0), \frac{\dot{\gamma}^v}{\gamma^u}(0) \right\} \right) \leq 1 \]

for all s ∈ [0, min{a, s_c + δ}) for some small δ > 0, which yields a contradiction in view of the definition of s_c, unless s_c = a.

The relation (4.93) for \( \dot{\gamma}^u \) yields, in view of (4.101) and (4.104) for k = n, that for any s ∈ [0, s_c):

\[ \left| \log \left( \Omega^2_n \dot{\gamma}^u \right) (s) - \log \left( \Omega^2_n \dot{\gamma}^u \right) (0) \right| \leq C(C_1) v^{-1} u_1 + C(C_1) v^{-1} u_1 \int_0^{\gamma(s)} \dot{\gamma}^v(s) d\tilde{s} \leq C(C_1) v^{-1} u_1 + C(C_1) v^{-1} u_1 \int_{u_1(\gamma(s))}^{u_1(s_a)} \frac{\dot{\gamma}^v(s_a)}{\dot{\gamma}^u(s_a)} du, \]

where s_a is defined by (2.63). Using the upper bound for \( \frac{\dot{\gamma}^v(s_a)}{\dot{\gamma}^u(s_a)} \) provided by the definition (4.120) of s_c, we therefore infer from (4.122) that:

\[ \sup_{s \in [0, s_c]} \left( \log \left( \Omega^2_n \dot{\gamma}^u \right) (s) - \log \left( \Omega^2_n \dot{\gamma}^u \right) (0) \right) \leq C(C_1) v^{-1} u_1 + C(C_1) v^{-1} u_1 \cdot \left( \frac{u_1}{v_2} \right)^{1/4} \leq C(C_1) \left( \frac{u_1}{v_2} \right)^{1/2}. \]

From (4.123), the relation (4.92) between \( \dot{\gamma}^v \) and \( \dot{\gamma}^u \), the bound (4.101) for k = n, as well as the bound (4.109) (for n in place of n + 1) on the \( \partial_u \) derivatives of \( \Omega^2_n \), r_n and the upper bound for \( \frac{\dot{\gamma}^v(s_a)}{\dot{\gamma}^u(s_a)} \) provided by the definition (4.120) of s_c, we also obtain that

\[ \sup_{s \in [0, s_c]} \left( \log \left( \Omega^2_n \dot{\gamma}^v \right) (s) - \log \left( \Omega^2_n \dot{\gamma}^v \right) (0) \right) \leq \]

\[ \leq \sup_{s \in [0, s_c]} \left( \log \left( \Omega^2_n \dot{\gamma}^u \right) (s) - \log \left( \Omega^2_n \dot{\gamma}^u \right) (0) \right) + \sup_{s \in [0, s_c]} \left( \log \left( \frac{\Omega^2_n r_n^2}{\Omega^2_n r_n^2} \right) \right) \gamma(s) - \log \left( \frac{\Omega^2_n r_n^2}{\Omega^2_n r_n^2} \right) \gamma(0) \left| \gamma(0) \right| \leq C(C_1) \left( \frac{u_1}{v_2} \right)^{1/2} + C(C_1) v^{-1} u_1 \int_{u_1(\gamma(s))}^{u_1(s_a)} \frac{\dot{\gamma}^v(s_a)}{\dot{\gamma}^u(s_a)} du + u_1 \leq C(C_1) \left( \frac{u_1}{v_2} \right)^{1/2}.
\]
Combining the bounds \((4.123)\) and \((4.124)\), we readily infer that

\[
\left(\sup_{s \in [0, s_c)} \log \left( \frac{\Omega^2_r \hat{\gamma}^u(s)}{\Omega^2_n \hat{\gamma}^n(s)} + \frac{\Omega^2_r \hat{\gamma}^v(s)}{\Omega^2_n \hat{\gamma}^n(0)} \right) \right) \leq C_1 \left( \frac{u_1}{v_L} \right)^\frac{3}{2}
\]

and

\[
\left(\sup_{s \in [0, s_c)} \log \left( \frac{\Omega^2_r \hat{\gamma}^u(s)}{\Omega^2_n \hat{\gamma}^n(s)} - \frac{\Omega^2_r \hat{\gamma}^v(0)}{\Omega^2_n \hat{\gamma}^n(0)} \right) \right) \leq C_1 \left( \frac{u_1}{v_L} \right)^\frac{3}{2},
\]

from which it follows that \((4.118)-(4.119)\) (with the sup considered over \([0, s_c)\)) and \((4.121)\) hold.

In exactly the same way, it can be shown that the bounds \((4.118)\) and \((4.119)\) also hold for any future directed, causal curve \(\gamma : [0, a) \rightarrow W\) satisfying \((4.92), (4.95)\) and \((4.114)\), for some \(t > 0\) such that \((4.115)\) holds.

At any point \((u, v) \in W\), we can decompose the components \((T_{\mu \nu})_{n+1}\) of the energy momentum tensor \(T_{n+1}\) as

\[
(T_{\mu \nu})_{n+1}(u, v) = (T_{\mu \nu})^+_{n+1}(u, v) + (T_{\mu \nu})^0_{n+1}(u, v)
\]

with

\[
(T_{\mu \nu})^+_{n+1}(u, v) \doteq 2\pi r_n^{-2} \int_{l[u_1]}^{+\infty} \int_{0}^{+\infty} p_{\mu} p_{\nu} f_{n+1}^{l'}(u, v; p^u, p^v, l) \frac{dp^u}{p^u} ld\ll
\]

and

\[
(T_{\mu \nu})^0_{n+1}(u, v) \doteq 2\pi r_n^{-2} \int_{0}^{l[u_1]} \int_{0}^{+\infty} p_{\mu} p_{\nu} f_{n+1}^{l'}(u, v; p^u, l) \frac{dp^u}{p^u} ld\ll,
\]

where \(l[u_1]\) is defined in terms of \(u_1\) by

\[
l[u_1] \doteq 2v_L \left( \frac{u_1}{v_L} \right)^\frac{3}{2} C_0
\]

with \(C_0\) being the constant appearing in the right hand side of \((4.87)\) (which is also the constant for which \((r_1, \Omega^2_r, \hat{f})\) were assumed to satisfy \((3.10)\)) and

\[
p_{\mu} = (g_{\mu \nu})_{n+1} p^\nu.
\]

As a consequence of the transport relations for \((4.94)\) and \((4.96)\) for \(f_{n+1}\), the bound \((4.113)\) for the characteristic curves \(\gamma\) on which \(f_{n+1}\) is conserved, combined with the bounds \((4.87)\) and \((3.10)\) (with \(C_0\) in place of \(C\)) on the initial data, readily imply that

\[
(T_{uu})_{n+1}^0 + (T_{uw})_{n+1}^0 + (T_{vw})_{n+1}^0 \leq C(C_1) C_0^2 \, \left( \|f_l\|_{l^\infty} + \|\hat{f}\|_{l^\infty} \right) \left( l[u_1] \right)^2.
\]

On the other hand, the more refined estimate \((4.118)\) for the characteristic curves \(\gamma\) which satisfy the additional assumption \((4.115)\) (which implies that \(l > l[u_1]\) when \(u_1\) is sufficiently small in terms of \(C_1, C_0\)) yields:

\[
(T_{uu})_{n+1}^+ + (T_{uw})_{n+1}^+ + (T_{vw})_{n+1}^+ \leq 3\|T_{data}\|_{l^\infty} \exp \left( C(C_1) \left( \frac{u_1}{v_L} \right)^\frac{3}{2} \right),
\]

where

\[
\|T_{data}\|_{l^\infty} \doteq \left( \|T_{uu}\|_{l^\infty} + \|T_{uw}\|_{l^\infty} + \|T_{vw}\|_{l^\infty} + \|T_{uu}\|_{l^\infty} + \|T_{uw}\|_{l^\infty} + \|T_{vw}\|_{l^\infty} \right)^\frac{1}{3}.
\]
Let \( \gamma_{n+1}, \gamma_n : [0, a] \to \mathcal{W} \) be two future directed, causal curves such that \( \gamma_{n+1} \) satisfies (4.93) and (4.92), \( \gamma_n \) satisfies (4.93) and (4.92) with \( n-1 \) in place of \( n \), and

\[
\gamma_{n+1}(0) = \gamma_n(0) \in \{0\} \times [u_0, v], \\
\dot{\gamma}_{n+1}(0) = \dot{\gamma}_n(0).
\]

Let us also consider the parametrization of \( \gamma_{n+1}, \gamma_n \) by

\[
\tau = u + v - u(\gamma_{n+1}(0)) - v(\gamma_{n+1}(0)),
\]

with corresponding parameter domains \([0, \tau_{n+1}]\) and \([0, \tau_n]\).

**Remark.** We will only denote by \( \dot{} \) differentiation with respect to \( s \).

Subtracting from (4.93) for \( \gamma_{n+1} \) the same equation (with \( n-1 \) in place of \( n \)) for \( \gamma_n \) and similarly for (4.92), we can readily estimate using (4.101) (and (4.104)) for \( k = n, n-1 \) as well as the initial conditions (4.133) that, for any \( \tau \in [0, \min\{\tau_{n+1}, \tau_n\}] \):

\[
|u(\gamma_{n+1}(\tau)) - u(\gamma_n(\tau))| + |v(\gamma_{n+1}(\tau)) - v(\gamma_n(\tau))| \\
\leq \int_0^\tau \left( |\Omega_n^2 \gamma_{n+1}(\tau) - \Omega_n^2 \gamma_n(\tau)_n(\tau)| + |\Omega_n^2 \gamma_{n+1}(\tau) - \Omega_n^2 \gamma_n(\tau)_n(\tau)| \right) d\tau,
\]

\[
|\Omega_n^2 \gamma_{n+1}(\tau) - \Omega_n^2 \gamma_n(\tau)_n(\tau)| + |\Omega_n^2 \gamma_{n+1}(\tau) - \Omega_n^2 \gamma_n(\tau)_n(\tau)| \leq C(C_1) v_\tau^{-1} u_1 + C(C_1) v_\tau^2 \int_0^\tau \left( |u(\gamma_{n+1}(\tau)) - u(\gamma_n(\tau))| + |v(\gamma_{n+1}(\tau)) - v(\gamma_n(\tau))| \right) d\tau.
\]

Applying Gronwall’s inequality on (4.134), we therefore infer that, for any \( \tau \in [0, \min\{\tau_{n+1}, \tau_n\}] \):

\[
v_\tau^{-1} |u(\gamma_{n+1}(\tau)) - u(\gamma_n(\tau))| + v_\tau^{-1} |v(\gamma_{n+1}(\tau)) - v(\gamma_n(\tau))| + \]

\[
|\Omega_n^2 \gamma_{n+1}(\tau) - \Omega_n^2 \gamma_n(\tau)_n(\tau)| + |\Omega_n^2 \gamma_{n+1}(\tau) - \Omega_n^2 \gamma_n(\tau)_n(\tau)| \leq C(C_1) v_\tau^{-1} u_1.
\]

In exactly the same way, the estimate (4.135) can be also established in the case when \( \gamma_{n+1} \) and \( \gamma_n \) satisfy (4.95) in place of (4.93) (with \( n-1 \) in place of \( n \) in the case of \( \gamma_n \)) and

\[
\gamma_{n+1}(0) = \gamma_n(0) \in [0, u_1], \{u_0\} ,
\]

\[
\dot{\gamma}_{n+1}(0) = \dot{\gamma}_n(0).
\]

As a consequence of the estimate (4.134) and the bounds (4.101) (for \( k = n, n-1 \)), (3.10) and (4.87), the transport relations (4.94) and (4.96) for \( f_n, f'_n \) readily imply that, provided \( C_1 \) has been chosen large enough in terms of the initial data (in particular in terms of \( \partial_{x} f, \partial_{y} f \) and \( \partial_{x} f', \partial_{y} f' \)):

\[
(-\Lambda)^{-2} r^{-2} \int_0^\tau \int_0^\infty \left( \Omega_{k-1}(p^u + p^v) \right)^2 |\tilde{f}_n - \tilde{f}'_n| (\cdot; p^u, l) \frac{dp^u}{p^u} d\ell \leq C(C_1) v_\tau^{-1} u_1 \mathcal{O}_n.
\]

The bound (4.101) for \( k = n + 1 \) now readily follows from (4.107), (4.110), (4.131) and (4.132), provided \( u_1 \) has been chosen small enough and \( C_1 \) large enough, both depending only on the initial data \( (r_1, \Omega_1, f_1) \) and \( (r, \Omega, f) \).

The bound (4.102) for \( k = n + 1 \) follows from (4.108), (4.111) and (4.137). Therefore, by induction, (4.101) and (4.102) hold for all \( k \in \mathbb{N} \).

Having established that the bounds (4.101) and (4.102) hold for all \( k \in \mathbb{N} \), we infer that \( (r_k, \Omega_k, f_k) \) converge in the topology defined by the right hand side of (4.103) to some limit functions \( (r, \Omega, f) \), for which the associated norm defined by the left hand side of (4.101) is finite. In particular, (4.92), (4.93), (4.94), (4.95) and (4.96) imply that

\[
f(u, v, p^u, p^v, l) \pm f(u, v, p^u, p^v, l) \cdot \delta(\Omega^2(u, v)p^u p^v - \frac{l^2}{r^2(u, v)})
\]
is a massless Vlasov field for the metric
\[ g = -\Omega^2 du dv + r^2 g_{S^2} \]
on \mathcal{W} \times S^2, since it is transported along its geodesic flow. As a consequence of \((4.88)-(4.89)\) and the fact that \((4.103)\) controls the \(C^0\) norm of \(r, \Omega^2\) and \(T_{\mu\nu}\), we infer that \((r, \Omega^2, f)\) is a \(C^0\) distributional solution of \((2.43)-(2.48)\).

Commuting \((2.43)-(2.48)\) with \(\partial_u, \partial_v\) and arguing inductively treating the equations as linear in the highest order terms (using the estimates provided by \((4.101)\) for the lower order terms at the first step), we infer the higher order regularity of \((r, \Omega^2; f)\); we will omit the details of this standard procedure.

The uniqueness of the solution \((r, \Omega^2, f)\) on \(\mathcal{W}\) follows as in the previous step, by repeating the arguments leading to the proof of the difference estimate \((4.102)\); we will omit the details.

**Step 3: The region** \(\{u \leq u_2\} \cap \{v \leq v < u + v\}\)

By a slight abuse of notation, let us denote at this step by \((r, \Omega^2, \bar{f}, \bar{f})\) the characteristic initial data induced by the solution \((r, \Omega^2; f)\) on \(\mathcal{W}\) (constructed in the previous step) on \(\{v = v\} \cap \{0 < u < u_1\}\). We will show that, for some \(u_2 \in (0, u_1]\) sufficiently small in terms of \((r, \Omega^2; \bar{f}, \bar{f})\), there exists a unique smooth solution \((r, \Omega^2; f)\) of \((2.43)-(2.48)\) on
\[ \mathcal{V} \doteq \{u \leq u_2\} \cap \{v \leq v < u + v\} \]
(see Figure 4.4) with characteristic initial data \((r, \Omega^2, \bar{f})\) on \((0, u_2) \times \{v\}\) such that \(r\) satisfies the gauge condition
\[ \left. \frac{1}{r} \right|_{u=v=v} = 0 \]
and \(f\) satisfies the reflecting boundary condition stated in Definition 2.4.

**Figure 4.4:** Schematic depiction of the domain \(\mathcal{V}\).

The proof proceeds by repeating essentially the same arguments as in the previous step (using again an iteration scheme for the renormalised equations \(2.57\) for \(\Omega^2/(1 - \frac{1}{3} \Lambda r^2)\) and \(\tan^{-1}(\sqrt{-\frac{2}{3}}r)\) instead of the standard equations \(2.43 \text{ and } 2.44\)\), the propagation of the constraint equations \((2.45)-(2.46)\) under the reflecting boundary condition for \(f\) on \(\mathcal{I}\) is also inferred readily, using the bound on the support of \(f\) in phase space). We will therefore omit the relevant details.

Let us define the solution \((r, \Omega^2, f)\) of \((2.43)-(2.48)\) on the domain \(\mathcal{U}_{u_1, v}\) for \(u_1 = u_2\), so that \((r, \Omega^2, f)\) coincides with the solutions constructed in the previous three steps on \(D_{0, v_{\bar{z}}}^{u_0} \cap \mathcal{U}_{u_2, v}\) and \(\mathcal{W}\) on \(\mathcal{U}_{u_2, v}\). In order to conclude the proof of Theorem 4.1, it only remains to verify that \((r, \Omega^2, f)\) is smooth across the “gluing” boundaries \(\{v = u_0\} \cap \mathcal{U}_{u, v}\) and \(\{v = v\} \cap \mathcal{U}_{u, v}\); since \(f\) solves the Vlasov equation in terms of \(r, \Omega^2\), it suffices to establish the smoothness of \(r, \Omega^2\).
• Along \( \{v = u_0\} \cap \mathcal{U}_{u_0,v_2} \), the fact that
\[
\lim_{v \to u_0} \partial^k_r |_{v = \bar{v}} = \lim_{v \to u_0} \partial^k_r |_{v = \bar{v}} \quad \text{and} \quad \lim_{v \to u_0} \partial^k_r \Omega^2 |_{v = \bar{v}} = \lim_{v \to u_0} \partial^k_r \Omega^2 |_{v = \bar{v}}
\]
follows by arguing inductively on \( k \) and integrating equations (2.43) and (2.44) in \( u \) (after differentiating sufficiently many times with respect to \( \partial_r \)), using also the smoothness of the initial data set \((r_f, \Omega_f^2, f_{v_2}; v_2)\) at \( v = u_0 \).

• Similarly, along \( \{v = v_2\} \cap \mathcal{U}_{u_0,v_2} \), the fact that
\[
\lim_{v \to v_2} \partial^k_r |_{v = \bar{v}} = \lim_{v \to v_2} \partial^k_r |_{v = \bar{v}} \quad \text{and} \quad \lim_{v \to v_2} \partial^k_r \Omega^2 |_{v = \bar{v}} = \lim_{v \to v_2} \partial^k_r \Omega^2 |_{v = \bar{v}}
\]
follows by integrating the renormalised equations (2.57) in \( u \) (again after differentiating sufficiently many times with respect to \( \partial_r \)) and using the condition that the initial data set \((r_f, \Omega_f^2, f_{v_2}; v_2)\) is smoothly compatible, in accordance with Definition 3.5.

Thus, the proof of Theorem 4.1 is completed. \( \square \)

5 Extension principles for smooth solutions of (2.43)–(2.48)

In this section, we will establish a number of sufficient conditions for smooth solutions of (2.43)–(2.48) to admit a smooth extension beyond their original domain of definition; in this discussion, we will adopt the notions of smoothness for solutions to (2.43)–(2.48) introduced in Section 3.1. The extension principles established in this section will be used in obtaining a long-time existence result in Section 6. The results of this section will also be useful for the proof of the main theorem of our companion paper [16].

5.1 Smooth extension along \( r = 0 \) when \( \frac{2m}{r} \ll 1 \)

For any \( u_1 < u_2 \in \mathbb{R} \), let us define the domain
\[
\mathcal{D}_{u_1}^{u_2} := ([u_1, u_2] \times [u_1, u_2]) \cap \{ u \leq v \} \subset \mathbb{R}^2
\]
and the axis component \( \gamma_{u_1}^{u_2} \) of the boundary of \( \mathcal{D}_{u_1}^{u_2} \):
\[
\gamma_{u_1}^{u_2} := ([u_1, u_2] \times [u_1, u_2]) \cap \{ u = v \} \subset \partial \mathcal{D}_{u_1}^{u_2}.
\]

Our first (and most technically involved) extension principle concerns the smooth extendibility of solutions to (2.43)–(2.48) in neighborhoods of the axis \( r = 0 \) of the form \( \mathcal{D}_{u_1}^{u_2} \):

**Theorem 5.1.** There exists a constant \( \delta_0 \in (0, \frac{1}{4}] \) with the following property: For any \( u_1 < u_2 \) and any \( \Lambda \in \mathbb{R} \), let \((r, \Omega^2, f)\) be any solution of (2.43)–(2.48) on \( \mathcal{D}_{u_1}^{u_2} \cap \{ u < v \} \), where \( \mathcal{D}_{u_1}^{u_2} \) is an open neighborhood of \( \mathcal{D}_{u_1}^{u_2} \setminus \{(u_2, u_2)\} \), such that \((r, \Omega^2, f)\) is smooth with smooth axis \( \gamma_{u_1}^{u_2} \setminus \{(u_2, u_2)\} \), in accordance with Definition 3.1 (see Figure 5.1). Assume, moreover, that \((r, \Omega^2, f)\) satisfies the following conditions:

1. The function \( r \) satisfies at \( u = u_1 \) the one sided bound
\[
\partial_u r |_{u = u_1} < 0.
\]
2. There exist some \( C > 0 \), so that, at \( u = u_1 \), the support of \( f \) in the \( p^u \) variable is bounded from above:
\[
\text{supp}(f(u_1, \cdot, \cdot)) \subseteq \{ \partial_u r \cdot p^u - \partial_u r \cdot p^u \leq C \}.
\]
3. The solution \((r, \Omega^2, f)\) satisfies

\[
\limsup_{(u,v) \to (u_2, u_2)} \frac{2\tilde{m}}{r} < \delta_0
\]

and, in the case \(\Lambda > 0\):

\[
\limsup_{(u,v) \to (u_2, u_2)} \frac{2m}{r} < \delta_0.
\]

Then \((r, \Omega^2, f)\) extends on a neighborhood of \(\{(u_1, u_2)\}\) as a smooth solution with smooth axis \(\gamma_{u_1}^{u_2}\), according to Definition 3.1.

In the case when the closure of the support of \(f\) does not contain geodesics of vanishing angular momentum, i.e. when \(f\) satisfies

\[
\inf_{\text{supp} f(u_1, \cdot)} l > 0,
\]

the constant \(\delta_0\) can be chosen to be equal to 1/3.

![Figure 5.1: Schematic depiction of the domains \(D_{u_1}^{u_2}\) and \(D_* \cap \{u < v\}\) appearing in the statement of Theorem 5.1.](image)

**Proof.** The proof of Theorem 5.1 will be obtained in a number of steps, with each successive step improving the regularity estimates for \(r, \Omega^2, f\) in a neighborhood of the point \((u_2, u_2)\) which were obtained in the previous step. At the final step, we will show that

\[
\limsup_{(u,v) \to (u_2, u_2)} \frac{\tilde{m}}{r^3} < +\infty,
\]

from which the smooth extension of \((r, \Omega^2, f)\) on \((u_2, u_2)\) will then follow by a simple argument.

Gauge fixing and some initial bounds. In view of the initial condition (5.3) and the one sided bound

\[
\partial_u (\Omega^{-2} \partial_u r) \leq 0
\]

(following from (2.46)), we readily infer that

\[
\partial_u r < 0 \text{ on } D_{u_1}^{u_2} \setminus \{(u_2, u_2)\}.
\]
In view of (5.5), we deduce that there exists some 
\( u^*_1 \in [u_1, u_2) \) such that, on 
(5.11) 
\[ \mathcal{D}^{u_2}_{u_1} = ([u_1^*, u_2] \times [u_1^*, u_2]) \cap \{ u \leq v \} \]
we have:
(5.12) 
\[ \sup_{\mathcal{D}^{u_2}_{u_1}} \frac{2\tilde{m}}{r} \leq \delta_0. \]

In the case \( \Lambda \leq 0 \), in view of the relation (2.49) for \( \tilde{m}, m \), the bound (5.12) immediately implies that 
(5.13) 
\[ \sup_{\mathcal{D}^{u_2}_{u_1}} \frac{2m}{r} \leq \delta_0. \]

Provided \( u_1^* \) is sufficiently close to \( u_2 \), the relation (5.13) also holds in the case \( \Lambda > 0 \), in view of the assumption (5.6).

Remark. From now on, we will not need to distinguish between the cases \( \Lambda \leq 0 \) and \( \Lambda > 0 \).

Since \( \Omega \) is smooth on \( \mathcal{D}^{u_2}_{u_1} \times [u_1, u_2] \), from (5.10), (2.10) and (5.13) we infer that 
(5.14) 
\[ \partial_v r > 0 \text{ on } \mathcal{D}^{u_2}_{u_1} \times \{ u \leq u_1^* \}. \]

Moreover, the smoothness of \( (r, \Omega^2, f) \) on \( \mathcal{D}^{u_2}_{u_1} \cap \{ u \leq u_1^* \} \), combined with the initial bound (5.4) on the support of \( f \) imply that, for some \( C_* > 0 \):
(5.15) 
\[ \text{supp} \left( f(u_1^*, \cdot ; \cdot) \right) \subseteq \{ \partial_v r \cdot p - \partial_u r \cdot p^u \leq C_* \}. \]

Let us define 
(5.16) 
\[ R_* = \sup_{\mathcal{D}^{u_2}_{u_1}} r. \]

In view of (5.10) and the fact that \( r \) is bounded on \( \{ u_1 \} \times [u_1, u_2] \), we can readily infer that 
(5.17) 
\[ R_* = \sup_{\{ u_1 \} \times [u_1, u_2]} r < +\infty. \]

Furthermore, the constraint equation (2.50) readily implies that 
(5.18) 
\[ \partial_u \left( \frac{\partial_v r}{1 - \frac{2m}{r}} \right) \leq 0. \]

By integrating (5.18) on rectangles of the form \( [u_1^*, u] \times [u, v] \) for any \( u \in [u_1^*, u_2) \) and \( v \in [u, u_2] \) and using (5.13), (5.14) and the fact that 
\[ r(u, u) = 0, \]
we readily infer that 
(5.19) 
\[ r(u, v) \leq \frac{1 + \frac{1}{2} |\Lambda| R_*^2}{1 - \delta_0} \left( r(u_1, u_2) - r(u_1, v) \right). \]

In view of the smoothness of \( r \) on \( \{ u_1 \} \times [u_1, u_2] \), from (5.19) we obtain that 
(5.20) 
\[ \lim_{(u, v) \to (u_2, u_2)} r(u, v) = 0. \]
In view of (5.20), we can assume without loss of generality that $u_1$ has been fixed sufficiently close to $u_2$, so that

\begin{equation}
(5.21) \quad \sqrt{\Lambda} \sup_{\Omega(\gamma, \dot{\gamma})} r = \sqrt{\Lambda} R_s \leq \delta_0.
\end{equation}

By possibly applying a smooth coordinate transformation of the form (2.5), we will also assume without loss of generality that

\begin{equation}
(5.22) \quad \Omega^2 |_{u = u_1} = 1.
\end{equation}

Note that such a transformation does not affect the relations (5.10), (5.14) and (5.15), which are gauge invariant.

In view of (5.7) is satisfied, i.e. when there exists some $c_0 > 0$ sufficiently small

\begin{equation}
(5.23) \quad \inf \sup_{\Omega(\gamma, \dot{\gamma})} \Omega^2 (-\partial_u r) \geq c_0 > 0.
\end{equation}

From (2.10), (5.13) and (5.23), we therefore deduce that, for some $C_0 > 0$:

\begin{equation}
(5.24) \quad \sup \sup_{\Omega(\gamma, \dot{\gamma})} \partial_v r \leq C_0.
\end{equation}

A first bound for $r$ on geodesics in the support of $f$. We will show that there exists some $c_1 > 0$ such that, for any null geodesic $\gamma$ lying in the support of the Vlasov field $f$ and having angular momentum $l > 0$,

\begin{equation}
(5.25) \quad \inf_{\gamma \in f} \frac{r}{l} \geq c_1 \min \left( \left( \frac{l}{R_s} \right)^{\frac{1}{\delta_0} \Lambda^{\frac{1}{2}}}, 1 \right).
\end{equation}

Remark. In the case when (5.7) is satisfied, i.e. when there exists some $l_0 > 0$ such that $f$ is supported on $\{ l \geq l_0 \}$, the bound (5.25) would immediately imply the statement of Theorem 5.1. In view of (5.24), the bound (5.25) would yield in this case that there exists some $\delta_0 > 0$, so that no geodesic in the support of $f$ reaches the region $\{ v \geq v_0 - \nu_0 \} \cap D_{u_1}^2 \{ (u_2, u_2) \}$ and, therefore, $(r, \Omega^2, f)$ is isometric to the trivial solution on $\{ v \geq v_0 \} \cap D_{u_2}^2 \{ (u_2, u_2) \}$. The smooth extension of $(r, \Omega^2, f)$ on $\{ (u_2, u_2) \}$ would then follow trivially. The argument for the proof of (5.25) only requires that $\delta_0 \leq \frac{1}{3}$, and thus one can choose $\delta_0 = \frac{1}{3}$ in the case when (5.7) holds.

Let $\gamma : [0, b) \to D_{u_2}^1 \{ (u_2, u_2) \}$ be a future directed, null geodesic with $(\gamma, \dot{\gamma})$ lying in the support of $f$, with its parametrization normalised so that

\begin{equation}
(5.26) \quad \gamma(0) \in \{ u = u_1^* \}.
\end{equation}

Since

\begin{equation}
(5.27) \quad \sup_{s \in (0, b)} \log \left( \Omega^2 \dot{\gamma}^u \right)(s) \leq \log \left( \Omega^2 \dot{\gamma}^u \right)(0).
\end{equation}

Thus, in view of (5.15), the bound (5.27) implies (using also (2.10), (5.10) and (5.14)):

\begin{equation}
(5.28) \quad - \partial_u r \cdot \dot{\gamma}^u \leq \frac{1 - 2m}{4\partial_v r} C_1.
\end{equation}
for some constant \( C_1 \) depending only on \( C_* \) in \((5.24)\). Using \((5.28)\), the null shell relation \((2.20)\) and the formula \((2.10)\), we also deduce that

\[
(5.29) \hspace{1cm} \partial_v r \cdot \dot{\gamma}^u \geq \frac{l^2}{r^2} \partial_v r \cdot C_1^{-1}.
\]

Integrating \((2.50)\) in \( u \) and using \((2.53)\) and the bounds \((5.12)\), \((5.13)\), we infer that, for any \((u, v) \in \mathcal{D}_{u_1^*}^{v_1^*} \setminus \{(u_2, u_2)\}:

\[
(5.30) \hspace{1cm} \left| \log \left( \frac{\partial_v r}{1 - \frac{2m}{r}} \right)_{(u,v)} - \log \left( \frac{\partial_v r}{1 - \frac{2m}{r}} \right)_{(u_1^*, v_1^*)} \right| \leq \int_{u_1^*}^u \frac{2}{r} \left( -\partial_u \tilde{m} \right)(\tilde{u}, v) \, d\tilde{u} \leq \frac{2}{1 - \delta_0} \int_{u_1^*}^u \frac{-\partial_u \tilde{m}}{r}(\tilde{u}, v) \, d\tilde{u} = \frac{2}{1 - \delta_0} \left( -\int_{u_1^*}^u \frac{m}{r^2} (\tilde{u}, v) (-\partial_u r) d\tilde{u} + \left[ \frac{m}{r} \right]_{u=u_1^*} \right) \leq \frac{\delta_0}{1 - \delta_0} \log \frac{r(u_1^*, v)}{r(u, v)} + 1.
\]

The bound \((5.30)\) implies that for some \( C_2 > 0 \) depending only on \( \inf_{u=u_1^*} \partial_v r \):

\[
(5.31) \hspace{1cm} \frac{\partial_v r}{1 - \frac{2m}{r}} \geq C_2 \left( \frac{r}{R_*} \right)^{\frac{\delta_0}{1 - \delta_0}}.
\]

Thus, in view of \((5.13)\), the bounds \((5.28)\) and \((5.29)\) yield

\[
(5.32) \hspace{1cm} -\frac{\partial_u r \cdot \dot{\gamma}^u}{\partial_v r \cdot \dot{\gamma}^u} \leq C_1' \left( \frac{r}{R_*} \right)^{-\frac{2\delta_0}{1 - \delta_0}}.
\]

for some \( C_1' > 0 \) depending only on \( C_1, C_2 \).

Along the geodesic \( \gamma \) we calculate

\[
(5.33) \hspace{1cm} \dot{r} = \partial_v r \cdot \dot{\gamma}^u + \partial_u r \cdot \dot{\gamma}^u.
\]

As a consequence, the upper bound \((5.32)\) implies that \( \dot{r} \leq 0 \) can only be achieved in the region where

\[
(5.34) \hspace{1cm} C_1' \left( \frac{r}{R_*} \right)^{-\frac{2\delta_0}{1 - \delta_0}} \geq 1.
\]

Therefore, \((5.25)\) holds (for some \( c_1 \) possibly depending on \( R_* \)).

Bounds for \( \dot{\gamma} \) for geodesics \( \gamma \) in the support of \( f \). Let \( \gamma : [0, b) \to \mathcal{D}_{u_1^*}^{v_1^*} \setminus \{(u_2, u_2)\} \) be a future directed, null geodesic in the support of \( f \) as before, satisfying \((5.26)\). For any \( s \in [0, b) \) such that

\[
(5.35) \hspace{1cm} \partial_v r \cdot \dot{\gamma}^u(s) \leq 4\delta_0^{-1} (-\partial_u r) \cdot \dot{\gamma}^u(s),
\]

we can estimate, in view of \((5.28)\) and \((5.31)\) (using also \((5.13)\)), that:

\[
(5.36) \hspace{1cm} \partial_v r \cdot \dot{\gamma}^u(s) \leq 4\delta_0^{-1} (-\partial_u r) \cdot \dot{\gamma}^u(s) \leq C_3\delta_0^{-1} \left( \frac{2m}{r} - \frac{2m}{r} \right) \dot{\gamma}^u(s) \leq C_3\delta_0^{-1} \left( \frac{r}{R_*} \right)^{\frac{\delta_0}{1 - \delta_0}}
\]

for some \( C_3 > 0 \) depending only on \( C_1, C_2 \).

We will now establish a bound for \( \dot{\gamma}^u \) in the region where \((5.35)\) does not hold. Let \( s_1 \leq s_2 \in [0, b) \) be such that, for any \( s \in [s_1, s_2] \), we can bound:

\[
(5.37) \hspace{1cm} \partial_v r \cdot \dot{\gamma}^u(s) \geq 2\delta_0^{-1} (-\partial_u r) \cdot \dot{\gamma}^u(s).
\]
Note that, in view of the relation \((5.33), (5.37)\) implies that, along \(\gamma([s_1, s_2])\):

\[
(5.38) \quad \left. \frac{d v}{d s} \right|_{\gamma([s_1, s_2])} = \frac{\dot{s}^u}{\dot{s}^v} \left|_{\gamma([s_1, s_2])} \leq \left(1 - \frac{1}{2} \delta_0 \right)^{-1} \left(\varepsilon \right)^{-1} \left. d r \right|_{\gamma([s_1, s_2])}.
\]

Equation \((2.21)\) for \(\dot{s}^u\), in view of \((2.20), (2.55), (5.22)\) and \((5.37)\), yields the following bound for any \(s \in [s_1, s_2]\):

\[
(5.39) \quad -\frac{1}{\dot{s}^v} \frac{d}{ds} \log \left( r^2 \Omega^2 \gamma^u \right)(s) = -\partial_v \log \Omega^2 \left|_{\gamma(s)} \right. + 2 \frac{\dot{s}^u}{\dot{s}^v} \left(-\partial_u r \right) \left|_{\gamma(s)} \leq \right.
\]

\[
\leq -\partial_v \log \Omega^2 |_{\gamma(s)} + \delta_0 \frac{\partial_v r}{r} |_{\gamma(s)} =
\]

\[
= -\int_{u_1^*}^{u_1} \partial_u \partial_v \log \Omega^2(u, v(\gamma(s))) du \left. - \partial_v \log \Omega^2 \left( u_1^*, v(\gamma(s)) \right) + \delta_0 \frac{\partial_v r}{r} |_{\gamma(s)} =
\]

\[
= \int_{u_1}^{u_1} \left( -4 \left( \frac{\dot{m}}{r^3} + \frac{\Lambda}{6} \right) \left( -\partial_u r \right) \partial_v r \right) \left( u, v(\gamma(s)) \right) du + 16\pi T_{uv} \left( u, v(\gamma(s)) \right) du + \delta_0 \frac{\partial_v r}{r} |_{\gamma(s)}.
\]

Using \((2.53), (5.12), (2.21), (5.13)\) and the fact that the right hand side of \((2.50)\) is non-positive, we can estimate

\[
(5.40) \quad \int_{u_1^*}^{u_1} \left( -4 \left( \frac{\dot{m}}{r^3} + \frac{\Lambda}{6} \right) \left( -\partial_u r \right) \partial_v r \right) \left( u, v(\gamma(s)) \right) du \leq
\]

\[
\leq \int_{u_1^*}^{u_1} \left( -4 \left( \frac{\dot{m}}{r^3} + \frac{\Lambda}{6} \right) \left( -\partial_u r \right) \partial_v r \right) \left( u, v(\gamma(s)) \right) du =
\]

\[
= \int_{u_1}^{u_1} \left( -4 \left( \frac{\dot{m}}{r^3} + \frac{\Lambda}{6} \right) \left( -\partial_u r \right) \partial_v r \right) \left( \partial_v \log \left( \frac{\partial_v r}{r} \right) + \frac{2}{r} \right) \left( u, v(\gamma(s)) \right) du + 16\pi T_{uv} \left( u, v(\gamma(s)) \right) du + \delta_0 \frac{\partial_v r}{r} |_{\gamma(s)}.
\]

Substituting in \((5.39)\), we therefore obtain for any \(s \in [s_1, s_2]\):

\[
(5.41) \quad -\frac{1}{\dot{s}^v} \frac{d}{ds} \log \left( r^2 \Omega^2 \gamma^u \right)(s) \leq \left(1 - \frac{1}{2} \delta_0 \right)^{-1} \left(\varepsilon \right)^{-1} \left. d r \right|_{\gamma(s)}.
\]

Integrating \((5.41)\) in \(s\) and using the relation

\[
\gamma^v = \frac{l^2}{\left( r^2 \Omega^2 \gamma^u \right)^{\frac{1}{2}}}
\]

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Integrating (2.37) and using (5.46), we can therefore bound for any $s_1 \leq s_2$ for which (5.37) holds on $[s_1, s_2]$:

$$\hat{\gamma}^v(s_2) \leq \frac{r|\gamma(s_2)|}{r|\gamma(s_1)|} \frac{\tilde{\gamma}_{\delta_0}^2}{r^2}.$$  

Using the fact that (5.36) holds for all $s$ for which (5.35) is true, while (5.42) holds for all $s_1 \leq s_2$ for which (5.37) is true on $[s_1, s_2]$, we can estimate for all $s \in [0, b)$, in view of the bounds (5.24) and (5.31) for $\partial \nu r$, the bound (5.13) for $\frac{2m}{r}$ and the lower bound (5.25) for $r$ along $\gamma$:

$$\partial \nu r \cdot \hat{\gamma}^v(s) \leq C_4 \delta_0^{-1} \max \left\{ \left( \frac{l}{R_s} \right)^{-\frac{\gamma_{\delta_0}^2}{(1-2\delta_0)^2}}, 1 \right\}$$

for some $C_4 > 0$ depending on $c_1, C_1, C_2$.

From (5.28), (5.31) and (5.25), we also obtain (using the formula (2.10) for $\Omega^2$ in $\Omega^2 \hat{\gamma}^v$)

$$-\partial_\nu r \cdot \hat{\gamma}^v(s) \leq C_4 \max \left\{ \left( \frac{l}{R_s} \right)^{-\frac{\gamma_{\delta_0}^2}{(1-2\delta_0)^2}}, 1 \right\}.$$  

First improved bound for $\frac{2m}{r}$. We will now proceed to establish an improved bound for $\frac{2m}{r}$ in terms of $r$.

In view of the fact that $f$ satisfies the transport equation (2.22), we can trivially bound for $\tilde{f}|_{\text{supp}(f)}$ (defined by the relation (2.29))

$$\sup_{\text{supp}(f)} \tilde{f} \leq \sup_{(u)=u^*_1} \tilde{f} < +\infty.$$  

In view of (5.15), the definitions (2.35) and (2.39) imply that there exists some $C_5 > 0$ such that:

$$\sup_{l \geq 0, v \epsilon \lbrack u^*_1, u_2 \rbrack} r^2 N_v^{(l)}(u^*_1, v) \leq C_5.$$  

Integrating (2.37) and using (5.46), we can therefore bound for any $l \geq 0$:

$$\sup_{u_1^*_1 \leq u_2 \leq u_1^*_2} \int_{u}^{u_2} r^2 N_v^{(l)}(u, v) \, dv \leq \int_{u_1^*_1}^{u_2} r^2 N_v^{(l)}(u_1^*, v) \, dv \leq C_5 (u_2 - u_1^*_1).$$  

Integrating equation (2.54) in $v$ (using the fact that $\tilde{m}_v \vert \gamma = 0$), the bound (2.40) implies that, for any $u \geq u_1^*_1$:

$$\tilde{m}(u, v) = 2\pi \int_{u}^{v} \left( \frac{1 - 2m}{r} r^2 T_{v,v}(u, \bar{v}) + \frac{1 - 2m}{r} \frac{r^2}{\partial v \partial r} T_{u,v}(u, \bar{v}) \right) d\bar{v} =$$

$$= 2\pi \int_{u}^{v} \left( \frac{1 - 2m}{r} r^2 T_{v,v}(u, \bar{v}) + \frac{1 - 2m}{r} \frac{r^2}{\partial v \partial r} T_{u,v}(u, \bar{v}) \right) d\bar{v} =$$

$$\leq 4\pi \int_{0}^{\infty} \int_{\text{supp}(f(u, v))] \partial_\nu r (u, \bar{v}) p^\nu - \partial_\nu r (u, \bar{v}) p^u \cdot r^2 N_v^{(l)}(u, \bar{v}) d\nu d\bar{v}.$$  

In view of the bounds (5.43) and (5.44) for geodesics in the support of $f$, as well as the particle current bound (5.47), the estimate (5.48) yields for any $u \geq u_1^*_1$:

$$\tilde{m}(u, v) \leq 4\pi C_4 \delta_0^{-1} \int_{0}^{\infty} \max \left\{ \left( \frac{l}{R_s} \right)^{-\frac{\gamma_{\delta_0}^2}{(1-2\delta_0)^2}}, 1 \right\} \left( \int_{u}^{v} r^2 N_v^{(l)}(u, \bar{v}) \, d\bar{v} \right) dld \leq$$

$$\leq C_4 \delta_0 \int_{0}^{l^*(u, v)} l \cdot \max \left\{ \left( \frac{l}{R_s} \right)^{-\frac{\gamma_{\delta_0}^2}{(1-2\delta_0)^2}}, 1 \right\} dl \leq$$

$$\leq C_4 \delta_0 (l^*(u, v))^2 \max \left\{ \left( \frac{l^*(u, v)}{R_s} \right)^{-\frac{\gamma_{\delta_0}^2}{(1-2\delta_0)^2}}, 1 \right\},$$
where
\[ l^*(u, v) = \sup_{\bar{v} \in [u, v]} \left( \sup_{\text{supp} \left( f(u, \bar{v}, \cdot, \cdot, \cdot) \right)} l \right). \]

Notice that (5.25) (in view also of (5.14)) implies that
\[ l^*(u, v) \leq C r(u, v) \max \left\{ \left( \frac{r(u, v)}{R_*} \right)^{-\frac{m_0}{1-\delta_0}}, 1 \right\}. \]

Hence, from (5.49) and (5.51) we obtain:
\[ \sup_{(u, v) \in \mathcal{D}_{u_2}^{u_1} \setminus \{(u_2, u_2)\}} \frac{\tilde{m}}{r^{2-100\delta_0}} < +\infty. \]

**Improved bounds for the geometry.** Assuming that
\[ \delta_0 \leq 10^{-3}, \]
returning to the proof of (5.25) and (5.31) and using the stronger bound (5.52) in place of the weaker initial bound (5.12), we can readily improve (5.25) and (5.31) as follows: There exists some \( C > 0 \), such that:

- For any \((u, v) \in \mathcal{D}_{u_2}^{u_1} \setminus \{(u_2, u_2)\}\)
  \[ \partial_v r(u, v) \geq c. \]

- For any future directed null geodesic \( \gamma : [0, b) \to \mathcal{D}_{u_2}^{u_1} \setminus \{(u_2, u_2)\} \) in the support of \( f \):
  \[ \inf_{\gamma} r \leq c \cdot l \]
  and (in view of (5.28) and (5.54))
  \[ \sup_{s \in [0, b)} \left( -\partial_u r \cdot \dot{\gamma}^u (s) \right) \leq c^{-1}. \]

Integrating (2.51) in \( v \) and using the boundary condition
\[ \partial_v r|_{\gamma} = -\partial_u r|_{\gamma}, \]
the bounds (5.13), (5.52), (5.24) and (5.54) imply that there exists some \( C > 0 \) such that, for any \((u, v) \in \mathcal{D}_{u_2}^{u_1} \setminus \{(u_2, u_2)\}\):
\[ C^{-1} \leq -\partial_u r(u, v) \leq C. \]

Note also that (5.24), (5.54) and (5.58) imply (in view of (2.10))
\[ \sup_{\mathcal{D}_{u_2}^{u_1} \setminus \{(u_2, u_2)\}} \left| \log \Omega^2 \right| < +\infty. \]

We will now show that there exists some \( 0 < c < 1 \) such that, for any \( v_* \in (u_1^*, u_2] \) and any future directed null geodesic \( \gamma : [0, b) \to \mathcal{D}_{u_2}^{u_1} \setminus \{(u_2, u_2)\} \) in the support of \( f \) satisfying \( u(\gamma(0)) = u_1^* \) and \( v(\gamma(0)) \leq v_* \), we have
\[ v_* - \sup_{\gamma \in \{v \leq v_*\}} \geq c \left( v_* - v(\gamma(0)) \right) \]
Figure 5.2: The bound (5.60) implies that a null geodesic $\gamma$ cannot remain in the region close to the axis for a long period of retarded time compared to its initial separation $u_2 - v(\gamma(0))$ from $v = u_2$.

(see also Figure 5.2).

**Proof of (5.60).** By possibly restricting the domain of $\gamma$ to a subdomain of the form $[0, b_*] \subset [0, b)$, if necessary, we will assume that, without loss of generality,

$$
(5.61) \quad \sup_{s \in [0, b)} v(\gamma(s)) \leq v_*.
$$

Differentiating (2.61) (with $u_1(v) = u_1^*$) and using (5.22), (5.12), (5.58) and (5.59), we obtain for any $s \in [0, b)$:

$$
(5.62) \quad \frac{1}{\gamma^u} \frac{d}{ds} \log \left( \Omega^2 \gamma^u \right)(s) = \left( \int_{u_1^*}^{u(\gamma(s))} \left( \frac{1}{2} \frac{\delta m}{r^2} - \frac{1}{2} \Omega^2 - 24 \pi T_{uv} \right)(u, v(\gamma(s))) \, du \right) - 2 \frac{\partial_r r}{r}(u_1^*, v(\gamma(s))) \leq
$$

$$
\leq -c \left( \frac{1}{r}(u(\gamma(s)), v(\gamma(s))) - \frac{1}{r}(u_1^*, v(\gamma(s))) \right).
$$

Let $0 < \delta_1 \ll 1$ be a sufficiently small parameter that will be fixed later, and let $s_0 \in [0, b)$ be the minimum value of $s$ for which

$$
(5.63) \quad v_* - u(\gamma(s_0)) = \delta_1 \cdot (v_* - v(\gamma(0))
$$

(note that if (5.63) does not hold for any $s_0 \in [0, b)$, then (5.60) trivially holds for $c = \delta_1$; see Figure 5.3). Since $\gamma$ is future directed and null, $u(\gamma(s))$ is increasing in $s$ and, hence, (5.63) implies that, for any $s \in [s_0, b)$:

$$
(5.64) \quad v_* - u(\gamma(s)) \leq \delta_1 \cdot (v_* - v(\gamma(0))).
$$

Using the bound (5.64) for $s \geq s_0$ and the fact that $u \leq v$ on $D^{u_2}_{u_1^*}$, we infer that, for any $s \in [s_0, b)$:

$$
(5.65) \quad v(\gamma(s)) - u_1^* \geq (1 - \delta_1)(v_* - u_1^*).
$$

Hence, from (5.54) and (5.65) (in view also of the fact that $r(\gamma(0)) = 0$) we obtain that, for any $s \in [s_0, b)$:

$$
(5.66) \quad r(u_1^*, v(\gamma(s))) \geq c(1 - \delta_1)(v_* - u_1^*).
$$

Similarly, in view of (5.58) and the bounds (5.64) and (5.61), we can readily estimate for any $s \in [s_0, b)$:

$$
(5.67) \quad r(u(\gamma(s)), v(\gamma(s))) \leq C(v(\gamma(s)) - u(\gamma(s))) \leq C\delta_1 \cdot (v_* - v(\gamma(0))).
$$
Note that, since \( v(\gamma(0)) \geq u(\gamma(0)) \geq u^*_1 \), the bounds (5.66) and (5.67) imply (provided \( \delta_1 \) is small depending on \( C, c \)):

\[
\frac{1}{r(u(\gamma(s)), v(\gamma(s)))} \geq \frac{1}{C \delta_1^{-1}} \frac{1}{r(u^*_1, v(\gamma(s)))}.
\]

Plugging (5.66) and (5.67) in (5.62) (using also (5.68)) and integrating the resulting expression in \( s \) starting from \( s = s_0 \), we infer that, for any \( s \in [s_0, b) \) (provided \( \delta_1 \) is small depending on \( C, c \)):

\[
\log (\Omega^2 \dot{\gamma}^u)(s) - \log (\Omega^2 \dot{\gamma}^u)(s_0) \leq -\delta_1^{-1/2} (v_* - v(\gamma(0)))^{-1}.
\]

Using (5.59), (5.27) and (5.15), and provided \( \delta_1 \) is small depending on \( C, c, u_2 - u^*_1 \), we infer that, for any \( s \in [s_0, b) \):

\[
\dot{\gamma}^u(s) \leq C \exp \left( -c \delta_1^{-1/2}(v_* - v(\gamma(0)))^{-1} \right).
\]

In view of the relation (2.18) for \( \dot{\gamma}^u, \dot{\gamma}^v \) and the bounds (5.56) and (5.59), we infer from (5.70) that, for any \( s \in [s_0, b) \) (provided \( \delta_1 \) is small depending on \( C, c, u_2 - u^*_1 \)):

\[
\frac{\dot{\gamma}^u(s)}{\dot{\gamma}^v(s)} \leq \exp \left( -\delta_1^{-1/4}(v_* - v(\gamma(0)))^{-1} \right)
\]

and, thus:

\[
\dot{\gamma}^u(s) \leq \exp \left( -\delta_1^{-1/4}(v_* - v(\gamma(0)))^{-1} \right) \dot{\gamma}^v(s).
\]

Integrating (5.72) in \( s \) for \( s \geq s_0 \), we therefore obtain for any \( s \in [s_0, b) \):

\[
u(\gamma(s)) - u(\gamma(s_0)) \leq \int_{s_0}^{s} \exp \left( -\delta_1^{-1/4}(v_* - v(\gamma(0)))^{-1} \right) \dot{\gamma}^v(s) \, ds = \\
\geq \left( v(\gamma(s)) - v(\gamma(s_0)) \right) \exp \left( -\delta_1^{-1/4}(v_* - v(\gamma(0)))^{-1} \right) \leq \\
\leq (v_* - v(\gamma(0))) \exp \left( -\delta_1^{-1/4}(v_* - u^*_1)^{-1} \right).
\]
In view of (5.63), we obtain from (5.73), provided \( \delta_1 \) is small enough depending on \( C, c \) and \( u_2 - u_1^* \):

\[
(5.74) \quad u_2 - \sup_{s \in [s_0, b]} u(\gamma(s)) \geq \frac{1}{2}(v_s - v(0)).
\]

In view of the fact that the definition (5.63) of \( s_0 \) and the fact that \( u_2 \geq v_s \) imply that

\[
(5.75) \quad u_2 - \sup_{s \in [0, s_0]} u(\gamma(s)) \geq \delta_1(v_s - v(0)).
\]

from (5.74) and (5.75) we finally infer (5.60).

**Further improved bounds for \( \widetilde{m}_\gamma \).** For any \( \tilde{u} \in [u_1^*, u_2] \) and any \( \tilde{v} \in (\tilde{u}, u_2) \), let us define

\[
\mathcal{V}_{\tilde{u}\tilde{v}}^* = \left\{ (u, v; p^u, p^v, l) \in \text{supp}(f) \cap \{ \{ \tilde{u} \leq u \leq v \} \cap \{ v \leq \tilde{v} \} \} \times [0, +\infty)^3 \right\}
\]

and let

\[
(5.76) \quad \mathcal{V}_{\tilde{u}\tilde{v}} = \left( \cup_{s \in \mathbb{R}} \Phi_s(\mathcal{V}_{\tilde{u}\tilde{v}}^*) \right) \cap \left( \{ \{ \tilde{u} \leq u \leq v \} \cap \{ v \leq \tilde{v} \} \} \times [0, +\infty)^3 \right),
\]

where \( \Phi_s \) denotes the image of the geodesic flow (2.21) after time \( s \). Note that \( \mathcal{V}_{\tilde{u}\tilde{v}} \) is invariant under the geodesic flow and consists exactly to the region in phase space traced out by those geodesics in the support of \( f \) that intersect the physical space domain \( \{ u \geq \tilde{u} \} \cap \{ v \leq \tilde{v} \} \).

The estimate (5.60) is equivalent to the following statement: There exists some \( C > 0 \) such that, for any \( \tilde{u} \in [u_1^*, u_2] \) and any \( \tilde{v} \in (\tilde{u}, u_2) \), every future directed null geodesic \( \gamma \) in the support of \( f \) with \( \gamma(0) \in \{ u = u_1^* \} \) that intersects the region \( \{ u \geq \tilde{u} \} \cap \{ v \leq \tilde{v} \} \) satisfies

\[
(5.77) \quad \gamma(0) \in \mathcal{C}_{\tilde{u}\tilde{v}} \doteq \{ u = u_1^* \} \cap \{ \max\{ u_1^*, \tilde{v} - C(\tilde{v} - \tilde{u}) \} \leq v \leq \tilde{v} \}.
\]

Note that, if \( v(\gamma(0)) > \tilde{v} \), then \( v(\gamma(s)) > \tilde{v} \) for all \( s \), since \( \gamma \) is causal. In view of the conservation law (2.37), the above statement implies that, for any \( \tilde{u} \in [u_1^*, u_2] \), any \( \tilde{v} \in (\tilde{u}, u_2) \) and any \( l \geq 0 \):

\[
(5.78) \quad \int_{\tilde{u}}^{\tilde{v}} \int_{\tilde{u}}^{\tilde{v}} r^2 N_\psi(u, v) du dv = \int_{\tilde{u}}^{\tilde{v}} \int_{u_1^*}^{\tilde{v}} r^2 N_\psi(u, v) du dv \leq
\]

\[
\leq \int_{u_1^*}^{\tilde{v}} \int_{u_1^*}^{\tilde{v}} r^2 N_\psi(u, v) du dv = \int_{\max(u_1^*, \tilde{v} - C(\tilde{v} - \tilde{u}))}^{\tilde{v}} r^2 N_\psi(u_1^*, v) dv.
\]

Using (5.58), we can estimate

\[
(5.79) \quad \sup_{\mathcal{V}_{\tilde{u}\tilde{v}}} r \leq C(\tilde{v} - \tilde{u}).
\]

Therefore, in view of (5.55), the bound (5.79) implies that any geodesic \( \gamma \) in the support of \( f \) intersecting the region \( \{ u \geq \tilde{u} \} \cap \{ v \leq \tilde{v} \} \) must necessarily have angular momentum \( l \) bounded as follows (for a possibly larger constant \( C \)):

\[
(5.80) \quad l \leq C(\tilde{v} - \tilde{u}).
\]

Hence,

\[
(5.81) \quad \sup_{\mathcal{V}_{\tilde{u}\tilde{v}}} l \leq C(\tilde{v} - \tilde{u}).
\]
In view of (5.48) and the bounds (5.43) and (5.56), we can estimate for any \( \tilde{u} \in [u_1^*, u_2^*] \), \( \tilde{v} \in (\tilde{u}, u_2^*) \):

\[
\tilde{m}(\tilde{u}, \tilde{v}) \leq 4\pi \int_0^{+\infty} \int_{\bar{u}}^{\bar{v}} \sup_{\text{supp}\(f(\tilde{u}, \tilde{v}, \cdot, \cdot)\)} \left( \partial_{\tilde{v}} r(\tilde{u}, v) p^v - \partial_{\tilde{u}} r(\tilde{u}, v) p^u \right) \cdot r^2 N^{(f)}(\tilde{u}, v) \, dv dl \leq C_{\delta_0} \int_0^{+\infty} \max \left\{ \left( \frac{l}{R_*} \right)^{-\frac{7\delta_0}{(1-2\delta_0)^2}}, 1 \right\} \left( \int_{\tilde{u}}^{\tilde{v}} r^2 N^{(f)}(\tilde{u}, v) \, dv \right) dl.
\]

In view of (5.46), (5.24), (5.58) (5.78) and (5.81), we readily deduce from (5.82) that, for any \( \tilde{u} \in [u_1^*, u_2^*] \), \( \tilde{v} \in (\tilde{u}, u_2^*) \):

\[
\tilde{m}(\tilde{u}, \tilde{v}) \leq C_{\delta_0} \int_0^{\sup_{v \in [\tilde{u}, u_2^*]} l} \max \left\{ \left( \frac{l}{R_*} \right)^{-\frac{7\delta_0}{(1-2\delta_0)^2}}, 1 \right\} \left( \int_{\max\{u_1^*, \tilde{v} - C(\tilde{v} - \tilde{u})\}}^{\tilde{v}} r^2 N^{(f)}(\tilde{v}, \tilde{v}; l)(u_1^*, v) \, dv \right) dl \leq C_{\delta_0} (\tilde{v} - \tilde{u}) \int_0^{\sup_{v \in [\tilde{u}, u_2^*]} C(\tilde{v} - \tilde{u})} \max \left\{ \left( \frac{l}{R_*} \right)^{-\frac{7\delta_0}{(1-2\delta_0)^2}}, 1 \right\} dl \leq C_{\delta_0} (\tilde{v} - \tilde{u})^3 \left( \frac{\tilde{v} - \tilde{u}}{R_*} \right)^{-\frac{7\delta_0}{(1-2\delta_0)^2}}.
\]

The estimate (5.83) yields, in view of (5.24) and (5.54), the following improvement of (5.52):

\[
\sup_{\mathcal{D}^u_{u_1^*}(u_2^* u_2^*)} \frac{\tilde{m}}{r^{3 - 10\delta_0}} < +\infty.
\]

**Bounds in \( L^\infty \) for \( \frac{\tilde{m}}{u^2} \) and \( T_{\mu \nu} \).** In view of (5.84), we can estimate using the formula (2.53), the bounds (5.24), (5.54) and (5.58):

\[
\sup_{v \in [u_1^*, u_2^*]} \int_0^{u_2^*} T_{uu}(u, v) \, du \leq C \sup_{v \in [u_1^*, u_2^*]} \int_0^{u_2^*} \frac{-\partial_u \tilde{m}(u, v)}{r^2} \, du \leq C \sup_{v \in [u_1^*, u_2^*]} \left( \int_0^{u_2^*} \frac{\tilde{m}}{r^3}(u, v) \, du + \frac{\tilde{m}}{r^2}(u_1^*, v) \right) < +\infty.
\]

Similarly, using (2.54) and the bounds (5.24), (5.54), as well as the condition

\[
\frac{\tilde{m}}{r^2} \bigg|_{\gamma_{u_1^*}^u \setminus \{(u_2, u_2)\}} = 0
\]

(following from the smoothness of \( (r, \Omega^2, f) \) on \( \mathcal{D}^u_{u_1^*}(u_2^* u_2^*) \)), we can estimate:

\[
\sup_{v \in [u_1^*, u_2^*]} \int_0^{u_2^*} T_{uu}(u, v) \, dv < +\infty.
\]

In view of (2.45), (2.46), (2.52) and the fact that \( r|_{\gamma_{u_1^*}^u} = 0 \) (and that \( (r, \Omega^2, f) \) is smooth on \( \mathcal{D}^u_{u_1^*}(u_2^* u_2^*) \)), we calculate on \( \gamma_{u_1^*}^u \setminus \{(u_2, u_2)\} \):

\[
\partial^2_r r|_{\gamma_{u_1^*}^u \setminus \{(u_2, u_2)\}} = -\partial_r \Omega^{-2}|_{\gamma_{u_1^*}^u \setminus \{(u_2, u_2)\}} \partial_r r|_{\gamma_{u_1^*}^u \setminus \{(u_2, u_2)\}},
\]

\[
\partial^2_r r|_{\gamma_{u_1^*}^u \setminus \{(u_2, u_2)\}} = -\partial_u \Omega^{-2}|_{\gamma_{u_1^*}^u \setminus \{(u_2, u_2)\}} \partial_u r|_{\gamma_{u_1^*}^u \setminus \{(u_2, u_2)\}},
\]

\[
\partial_u \partial_r r|_{\gamma_{u_1^*}^u \setminus \{(u_2, u_2)\}} = 0.
\]

Thus, in view of the fact that \( r|_{\gamma_{u_1^*}^u} = 0 \), we readily infer that

\[
\partial_u \Omega^2|_{\gamma_{u_1^*}^u \setminus \{(u_2, u_2)\}} = \partial_r \Omega^2|_{\gamma_{u_1^*}^u \setminus \{(u_2, u_2)\}}.
\]
By plugging (5.84), (5.85) and (5.86) in (2.55) and using (5.24), (5.54), (5.58) and the boundary condition (5.87), we therefore obtain after integration in \( u, v \), respectively:

\[
\text{(5.88)} \quad \sup_{D_{u_1}^{u_2} \setminus (u_2, u_2)} \left( \left| \partial_\nu \log \Omega^2 \right| + \left| \partial_\nu \log \Omega^2 \right| \right) < +\infty.
\]

In view of the bounds (5.54) and (5.58), there exists some \( \bar{C} > 0 \) such that, for any \((u, v) \in D_{u_1}^{u_2} \setminus (u_2, u_2)\):

\[
\text{(5.89)} \quad \bar{C} \partial_\nu r + \partial_\nu r > 0.
\]

Let \( \gamma : [0, b) \to D_{u_1}^{u_2} \setminus (u_2, u_2) \) be a future directed null geodesic in the support of \( f \). We will establish that there exists some \( C > 0 \) independent of \( \gamma \), so that

\[
\text{(5.90)} \quad \sup_{s \in [0, b)} \hat{\gamma}^v(s) \leq C.
\]

It suffices to establish that, for any \( s_1 < s_2 \in [0, b) \) such that

\[
\text{(5.91)} \quad \hat{\gamma}^v(s) \geq 1 \text{ for all } s \in [s_1, s_2),
\]

\[
\text{(5.90)} \text{ holds on } [s_1, s_2], \text{ i.e.}
\]

\[
\text{(5.92)} \quad \sup_{s \in [s_1, s_2]} \hat{\gamma}^v(s) \leq C.
\]

(note that if no such \( s_1, s_2 \) exist, then (5.90) automatically holds). Note that (5.91) implies the bound

\[
\text{(5.93)} \quad s_2 - s_1 \leq \int_{s_1}^{s_2} \hat{\gamma}^v ds \leq v(s_2) - v(s_1) \leq u_2 - u_1^*.
\]

Using equations (2.21), we calculate for \( \bar{C} \) as in (5.89):

\[
\text{(5.94)} \quad \frac{d}{ds} \left( \bar{C} \Omega^2 \hat{\gamma}^u + \Omega^2 \hat{\gamma}^v \right)(s) = \left( \bar{C} \partial_\nu \log(\Omega^2) + \partial_\nu \log(\Omega^2) - 2 \bar{C} \partial_\nu r + \partial_\nu r \right) \frac{r^2}{r^2} \hat{\gamma}(s).
\]

Using (5.55), (5.59) and (5.89), we deduce that there exists some \( C' > 0 \) independent of \( \gamma \) such that

\[
\text{(5.95)} \quad \frac{d}{ds} \left( \bar{C} \Omega^2 \hat{\gamma}^u + \Omega^2 \hat{\gamma}^v \right)(s) \leq C'.
\]

Integrating (5.95) and using (5.59) and (5.93), we readily infer (5.92).

It therefore follows from (5.24), (5.58), (5.56) and (5.95) that there exists some \( C_b > 0 \) such that, for any \((u, v) \in D_{u_1}^{u_2} \setminus (u_2, u_2)\):

\[
\text{(5.96)} \quad \text{supp} \left( f(u, v; \cdot, \cdot) \right) \subseteq \left\{ - \partial_\nu r \cdot p^u + \partial_\nu r \cdot p^v \leq C_b \right\}.
\]

Since \( f \) is supported on \( 2.20 \) and any geodesic \( \gamma \) in the support of \( f \) satisfies (5.55), the bound (5.96) yields in view of (2.10) and (5.13):

\[
\text{(5.97)} \quad \text{supp} \left( f(u, v; \cdot, \cdot) \right) \subseteq \left\{ - \partial_\nu r \cdot p^u + \partial_\nu r \cdot p^v \leq C_b \right\} \cap \left\{ \frac{l}{r} \leq 4C_b \right\}.
\]

It can be readily deduced from the estimate (5.45) combined with the bound (5.97) for the support of \( f \) that

\[
\text{(5.98)} \quad \sup_{D_{u_1}^{u_2} \setminus (u_2, u_2)} \left( T_{uu} + T_{uv} + T_{vv} \right) < +\infty.
\]
In view of (2.54), from (5.98) we also infer that

$$\sup_{D^u_1(u_2, u_2)} \frac{\tilde{m}}{r^3} < +\infty.$$  

Note also that, in view of the relations (2.4.4), (2.4.6) and (2.2.2), the bounds (5.24), (5.54), (5.58), (5.59), (5.88), (5.101) and (5.100) imply that

$$\sup_{D^u_1(u_2, u_2)} \left( |\partial_c^2 r| + |\partial_u \partial_c r| + |\partial^2_{uv} r| \right) < +\infty.$$  

**Proof of the smooth extendibility of \((r, \Omega^2, f)\).** Let

$$x : (D^u_1(u_2, u_2)) \times S^2 \rightarrow V \subset \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$$

be the diffeomorphism associated to the Cartesian coordinate chart \((x^0, \ldots, x^3)\) defined by (3.1), and let \(\mathcal{N}\) denote the closure of \(V\) in \(\mathbb{R} \times \mathbb{R}^3\). Note that, in view of (5.2.1), (5.2.4), (5.54), (5.58), (5.59), (5.88), (5.101) and (5.100) and the assumption that \((r, \Omega^2, f)\) is a smooth solution of (2.4.4)–(2.4.8) on \(D^u_1(u_2, u_2)\) with smooth axis \(\gamma^u_{u_1} \setminus \{(u_2, u_2)\}\), we infer that, for any \(\sigma \in S^2\), \(x(\cdot, \sigma)\) extends as a \(C^\infty\) embedding on \(\gamma^u_{u_1} \setminus \{(u_2, u_2)\}\) and as a \(C^1\) embedding on \(\gamma^u_{u_1}\), with

$$x(\gamma^u_{u_1}, \sigma) = (\mathbb{R} \times \{0\}) \cap \mathcal{N},$$

and the matrix \(\mathcal{T}\) of the frame transformation

$$\mathcal{T} : \{\partial_u, \partial_v, r^{-1} \partial_\theta, r^{-1}(\sin \theta)^{-1} \partial_\varphi\} \rightarrow \{\partial_{x^0}, \partial_{x^1}, \partial_{x^2}, \partial_{x^3}\}$$

satisfies

$$\sup_{V \setminus \{0, 0, 0\}} (||\mathcal{T}|| + ||\mathcal{T}^{-1}||) < +\infty,$$

while the matrix \(\mathcal{T}_r\) of the frame transformation

$$\mathcal{T}_r : \{\partial_u, \partial_v\} \rightarrow \{\partial_{x^0}, \frac{x^3}{\sqrt{\delta_{ij} x^i x^j}} \partial_i\}$$

satisfies

$$||\mathcal{T}_r||_{C^1(\mathcal{N})} + ||\mathcal{T}_r^{-1}||_{C^1(\mathcal{N})} < +\infty.$$  

By a slight abuse of notation, we will also denote \((\mathbb{R} \times \{0\}) \cap \mathcal{N}\) by \(\gamma^u_{u_1}\). Let also define \(q \in \mathcal{N}\) to be the point corresponding to \(\{(u_2, u_2)\}\).

Let \(g\) be the metric (2.3) on \(\mathcal{N} \setminus \gamma^u_{u_1}\). Our assumption that \((r, \Omega^2, f)\) is a smooth solution of (2.4.4)–(2.4.8) on \(D^u_1(u_2, u_2)\) with smooth axis \(\gamma^u_{u_1} \setminus \{(u_2, u_2)\}\) implies that \(g\) extends as a smooth metric on \(\mathcal{N} \setminus \{\}\), i.e. in the Cartesian coordinates

$$g_{\alpha\beta} \in C^\infty(\mathcal{N}\setminus \{\})$$

Furthermore, the bounds (5.24), (5.54), (5.58), (5.59), (5.88), (5.98), (5.99) and (5.100) imply, after integrating equations (2.52) and (2.55) and using the boundary condition (5.87) (as well as the relations (2.4.5), (2.4.6) to express \(\partial_c^2 r, \partial^2_{uv} r\) in terms of \(\partial \Omega^2\)), that the Cartesian components of \(g\) extend as \(C^1\) functions in a neighborhood of \(q\), i.e.

$$||g_{\alpha\beta}||_{C^1(\mathcal{N})} < +\infty.$$  

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In view of the fact that $r, \Omega^2$ satisfy (2.43)-(2.46) and

$$g^{\mu\nu}T_{\mu\nu} = 0$$

(since $f$ is supported on (2.20)), we readily calculate that for the Riemann curvature tensor of $g$ in the $(u, v, \theta, \phi)$ coordinate chart:

\begin{align*}
(5.104) & \quad R_{uAAB} = 4\pi g_{AB} T_{uu}, \\
(5.105) & \quad R_{vAAB} = 4\pi g_{AB} T_{vv}, \\
(5.106) & \quad R_{uvuv} = \Omega^2 \partial_u \partial_v \log(\Omega^2) = \left(\frac{\tilde{m}}{r^2} + \frac{1}{6} \Lambda \right) \Omega^4 - 16\pi \Omega^2 T_{uv}, \\
(5.107) & \quad R_{uAxB} = \left(- \partial_u \partial_v \log(\Omega^2) + 4\pi T_{uv}\right) g_{AB} = \\
& \quad \left(\left(\frac{\tilde{m}}{r^2} + \frac{1}{6} \Lambda \right) \Omega^4 + 20\pi T_{uv}\right) g_{AB}, \\
(5.108) & \quad R_{ABCD} = 2\Omega^2 \partial_u \partial_v \log(\Omega^2) \left(g_{ACBD} - g_{ADBC}\right) = \\
& \quad \left(\left(\frac{2\tilde{m}}{r^2} + \frac{1}{3} \Lambda \right) \Omega^4 - 32\pi \Omega^2 T_{uv}\right) \left(g_{ACBD} - g_{ADBC}\right),
\end{align*}

while all the other components are either identically 0 or can be expressed as a suitable linear combination of (5.104)-(5.108). Thus, in view of the bounds (5.59) and (5.98)-(5.99), we infer, after switching to the Cartesian coordinates $x^\alpha$ and using (5.101), that the components $R_{\alpha\beta\gamma\delta}$ of the Riemann curvature tensor satisfy

\begin{equation}(5.109) \sup_{\mathcal{N} \setminus \mathcal{Q}} |R_{\alpha\beta\gamma\delta}| < +\infty.\end{equation}

Let $\mathcal{S} \subset \mathcal{N} \setminus \mathcal{Q}$ be the smooth spacelike hypersurface defined by

$$\mathcal{S} \equiv \{ x^0 = \frac{1}{2}(u_1 + u_2) \} \cap \mathcal{N}.$$ 

Note that $\mathcal{S}$ is a Cauchy hypersurface of $(\mathcal{N}, g)$. Let us also define the map $\mathcal{P} : C^*_n(\mathcal{N} \setminus \mathcal{Q}) \times \mathbb{R}^{3+1} \to \mathcal{S} \times (\mathbb{R}^{3+1})^3$ (where $C^*_n(\mathcal{N} \setminus \mathcal{Q}) \subset T(\mathcal{N} \setminus \mathcal{Q})$ is the subset of future directed null vectors), so that, for each triad $(y, u^\alpha, z^\alpha)$ with $y \in \mathcal{N} \setminus \mathcal{Q}$, $U = u^\alpha \partial_{x^\alpha}$ being a future directed null vector in $T_y \mathcal{N}$ and $Z = z^\alpha \partial_{x^\alpha}$, we have

\begin{equation}(5.110) \mathcal{P}(y, u^\alpha, z^\alpha) = (s, \tilde{u}^\alpha, v^\alpha, w^\alpha),\end{equation}

where, in the standard coordinates $(x^\alpha, p^\alpha)$ on $\mathcal{T} \mathcal{N}$ corresponding to the Cartesian coordinates $x^\alpha$:

- $s \equiv \mathcal{S} \cap \gamma_U$, where $\gamma_U$ is the unique inextendible null geodesic emanating from $y$ in the direction of $U$.
- $\tilde{u}^\alpha = \tilde{\gamma}_U^\alpha(t_f)$, where, after normalising the affine parametrization of $\gamma_U$ so that $\gamma_U(0) = y$ and $\dot{\gamma}_U^\alpha(0) = -u^\alpha$, $t_f$ is defined so that $\gamma_U(t_f) = s$.
- $V = v^\alpha \partial_{x^\alpha} + w^\alpha \partial_{p^\alpha}$ is the unique vector in the fiber of $\mathcal{T} \mathcal{N}$ over $s$ such that

\begin{equation}(v^\alpha, w^\alpha + \Gamma^\beta_{\beta\gamma} v^\beta \dot{\gamma}_U^\gamma) = (J^\alpha(t_f), \frac{\nabla}{\partial t} J^\alpha(t_f)),\end{equation}

where the vector field $J : [0, t_f] \to T_{\gamma_U} \mathcal{N}$ is a Jacobi field along $\gamma_U$, i.e. satisfies

\begin{equation}(5.111) \frac{d^2}{dt^2} J^\alpha + R^\alpha_{\beta\gamma\delta} \dot{\gamma}_U^\beta \dot{\gamma}_U^\gamma J^\delta = 0,\end{equation}

with initial conditions at $s$:

\begin{equation}(5.112) J^\alpha(0) = z^\alpha, \quad \frac{\nabla}{\partial t} J^\alpha(0) = 0.\end{equation}
The Jacobi field $J$ with initial conditions (5.112) appearing in the definition (5.110) of $P$ corresponds to the infinitesimal variation of the geodesic $\gamma_U$ obtained by varying the basepoint $\gamma_U(0)$ in the direction of $Z$, while parallel translating the initial direction vector $\dot{\gamma}_U(0)$. Thus, since the Vlasov field $f$ is conserved along the geodesic flow, the following relation holds whenever $P(y, u^\alpha, z^\alpha) = (s, v^\alpha, w^\alpha)$:

\[(5.113) \quad \left(\frac{\partial^\alpha}{\partial x^\alpha} + \Gamma_{\alpha\beta\gamma}^\alpha x^\beta \frac{\partial}{\partial x^\gamma} \right) f(y, u^\alpha) = \left(\frac{\partial^\alpha}{\partial x^\alpha} + w^\alpha \frac{\partial}{\partial p^\alpha} \right) f(s, u^\alpha).
\]

In view of (5.102), we infer from (5.117) for a possibly different constant $C$.

Using the boundary condition (3.3) for $g^\alpha$ using (5.29) of the Vlasov field and the bound (5.97) for the support of $f$, implies that there exists some $C > 0$ such that, for any $y \in \mathcal{N}\setminus q$,

\[(5.114) \quad \|\dot{u}\| \leq C\|u\|,
\]

\[(5.115) \quad \|v\| \leq C\|z\| \cdot \|u\|,
\]

where $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^{3+1}$.

The relation (5.115), combined with (5.103), the bounds (5.114), the form (2.29) of the Vlasov field $f$ and the bound (5.97) for the support of $f$, implies that there exists some $C > 0$ such that, for any $y \in \mathcal{N}\setminus q$, we can estimate in the $(\tilde{x}^\alpha, p^\alpha)$ coordinate chart:

\[(5.116) \quad \sup_{y \in \mathcal{N}\setminus q} \left( \sum_{\alpha, \beta, \gamma = 0}^3 \left| \partial^\alpha_{\alpha} T_{\beta\gamma}(y) \right| \right) < +\infty.
\]

In view of the expression (3.2) for the Cartesian components of $g$, using (2.53), (2.54), (4.24), (5.54), (5.58), (5.59), (5.88) and (5.102), we can readily bound for some $C > 0$ and any $y \in \mathcal{V}$:

\[(5.117) \quad \sum_{\alpha, \beta, \gamma, \delta = 0}^3 \left| \partial^\alpha_{\alpha} \partial^\beta_{\beta} g_{\gamma\delta} \right|(y) \leq C \sum_{\alpha, \beta, \gamma = 0}^3 \left| \partial^\alpha_{\alpha} \partial^\beta_{\beta} \log \Omega^2 \right| + \left| \partial^\alpha_{\alpha} \partial^\beta_{\beta} \partial^\delta_{\delta r} \right| + \left| \partial^\alpha_{\alpha} \partial^\beta_{\beta} \partial^\delta_{\delta r} \right| + \left| \partial^\alpha_{\alpha} \left( \frac{\partial^\beta_{\beta} \partial^\delta_{\delta r} + \partial^\delta_{\delta r}}{r} \right) \right| + r^{-2} \left( \frac{-4 \partial^\beta_{\beta} \partial^\delta_{\delta r} + \partial^\delta_{\delta r}}{(\partial^\beta_{\beta} \partial^\delta_{\delta r} - \partial^\delta_{\delta r})^2} - 1 \right) + \frac{\overline{m}}{r^3} + |T_{\alpha\beta}| + |\partial^\alpha_{\alpha} T_{\alpha\beta}| + 1(y),
\]

where $\partial^k_{u,v}$ denotes a sum over all combinations of $k$ derivatives of the form $\partial_u$ or $\partial_v$. Using the boundary condition (3.3) for $r$ on the axis $\gamma_{u_1}^{u_2}$, we readily infer that

\[(5.118) \quad \sum_{\alpha, \beta, \gamma, \delta = 0}^3 \left| \partial^\alpha_{\alpha} \partial^\beta_{\beta} g_{\gamma\delta} \right|(y) \leq C \left( |\partial^2_{u,v} \Omega^2| + |\partial^3_{u,v} r| + \left| \partial^1_{u,v} \left( \frac{\partial^\beta_{\beta} \partial^\delta_{\delta r} + \partial^\delta_{\delta r}}{r} \right) \right| + r^{-2} \left( \frac{-4 \partial^\beta_{\beta} \partial^\delta_{\delta r} + \partial^\delta_{\delta r}}{(\partial^\beta_{\beta} \partial^\delta_{\delta r} - \partial^\delta_{\delta r})^2} - 1 \right) \right) + \frac{\overline{m}}{r^3} + \sum_{\alpha, \beta, \gamma = 0}^3 \left( |T_{\alpha\beta}| + |\partial^\alpha_{\alpha} T_{\alpha\beta}| + 1(y) \right),
\]

(5.118) where $\partial^k_{u,v}$ denotes a sum over all combinations of $k$ derivatives of the form $\partial_u$ or $\partial_v$. Using the boundary condition (3.3) for $r$ on the axis $\gamma_{u_1}^{u_2}$, we readily infer that

\[(5.119) \quad (\partial_u + \partial_v) r\big|_{u_1}^{u_2} \setminus \{u_2, u_3\} = 0.
\]

In view of (5.119), the bounds (5.24), (5.54) for $\partial^2 r$ and (5.100) for $\partial^3 r$ imply, through an application of the mean value theorem, that there exists some $C > 0$ such that, for any $(u, v) \in D_{u_1}^{u_2} \setminus \{u_2, u_3\}$:

\[(5.120) \quad \left| \partial^1_{u,v} \left( \frac{\partial^\beta_{\beta} \partial^\delta_{\delta r} + \partial^\delta_{\delta r}}{r} \right) \right|(u, v) \leq C \left( 1 + \sup_{u < v} \left| \partial^3_{u,v} r \right|(u, \bar{v}) \right)
\]

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Similarly, (5.125) \[ r^{-2} \left( \frac{-4 \partial_r r \partial_r r}{\partial_r r - \partial_u r} \right)^2 - 1 \right) (u, v) = \frac{1}{(\partial_r r - \partial_u r)^2} \left| \frac{\partial_r r + \partial_u r}{r} \right|^2 (u, v) \leq C \left( 1 + \sup_{u \in \mathbb{R}^d} |\partial^2_{u,v} r|^2 (u, \bar{v}) \right) \leq C.

In view of (5.120) and (5.121), the bound (5.118) can be simplified as follows for any \( y \in \mathcal{N} \setminus q \):

(5.122) \[ \sum_{\alpha, \beta, \gamma, \delta = 0} |\partial_{x \nu} \partial_{x \nu} g_{\alpha \beta \gamma \delta}(y) | \leq C \sup_{\vec{y} \in \mathcal{N}(y)} \left( |\partial^3_{u,v} \Omega^2| + |\partial^3_{u,v} r| + \frac{\bar{m}}{r^3} + \frac{3}{\alpha, \beta, \gamma = 0} \left( |T_{\alpha \beta}| + |\partial_{x \nu} T_{\alpha \beta}| \right) + 1 \right)(\bar{y}). \]

By differentiating (2.45), (2.46) and (2.52) with respect to \( v, u \), we infer that:

(5.123) \[ \partial^3_{v} r = \left( -4 \pi \partial_r (r T_{v u}) + \partial_v \log \Omega^2 \partial^2 r + \partial_r \partial_v \partial^2 \log \Omega^2 \right), \]

and similarly with \( u \leftrightarrow v \). Hence, in view of the bounds (5.24), (5.54), (5.58), (5.59), (5.88) and (6.102) and the relations (5.123), the estimate (5.122) yields for any \( y \in \mathcal{N} \setminus q \):

(5.124) \[ \sum_{\alpha, \beta, \gamma, \delta = 0} |\partial_{x \nu} \partial_{x \nu} g_{\alpha \beta \gamma \delta}(y) | \leq C \sup_{\vec{y} \in \mathcal{N}(y)} \left( |\partial^3_{u,v} \Omega^2| + |\partial^3_{u,v} r| + \frac{\bar{m}}{r^3} + \frac{3}{\alpha, \beta, \gamma = 0} \left( |T_{\alpha \beta}| + |\partial_{x \nu} T_{\alpha \beta}| \right) + 1 \right)(\bar{y}). \]

Using the relations (2.53) and (2.54) for \( \partial_u \bar{m} \) and \( \partial_v \bar{m} \), respectively, as well as the bounds (5.24), (5.54), (5.58), (5.59) and (5.88), we can readily estimate for any \( (u, v) \in D_{u_1}(\{(u_2, u_2)\}) \):

(5.125) \[ \left| \partial_u \left( \frac{\bar{m}}{r^3} \right) \right|(u, v) = 8 \pi \left| \partial_u \left( \int_{u}^{v} \frac{2 \Omega^2 ((-\partial_r r) T_{v u} + \partial_v r T_{u v}) (u, \bar{v}) d\bar{v}}{r^3(u, v)} \right) \right| = 8 \pi \left| \int_{u}^{v} \frac{2 \Omega^2 ((-\partial_r r) T_{v u} + \partial_v r T_{u v}) (u, \bar{v}) d\bar{v}}{r^3(u, v)} - 3 \int_{u}^{v} \frac{2 \Omega^2 ((-\partial_r r) T_{v u} + \partial_v r T_{u v}) (u, v) d\bar{v}}{r^3(u, v)} \right| = 8 \pi \left| \int_{u}^{v} \frac{2 \Omega^2 ((-\partial_r r) T_{v u} + \partial_v r T_{u v}) (u, \bar{v}) d\bar{v}}{r^3(u, v)} - \int_{u}^{v} \partial_v (r^3 \Omega^2 \frac{(-\partial_r r) T_{v u} + T_{u v}}{r^3(u, v)}) (u, v) d\bar{v} \right| \leq \leq \left| \int_{u}^{v} \frac{2 \Omega^2 ((-\partial_r r) T_{v u} + \partial_v r T_{u v}) (u, \bar{v}) d\bar{v}}{r^3(u, v)} - \partial_v (r^3 \Omega^2 \frac{(-\partial_r r) T_{v u} + T_{u v}}{r^3(u, v)}) (u, v) \right| \leq C \sup_{u \leq \bar{v} \leq v} \left( |\partial_u T_{v u}| + |\partial_u T_{u u}| + 1 \right).

Similarly,

(5.126) \[ \left| \partial_u \left( \frac{\bar{m}}{r^3} \right) \right|(u, v) \leq C \sup_{u \leq \bar{v} \leq v} \left( |\partial_u T_{v u}| + |\partial_u T_{u u}| + 1 \right).

By differentiating (2.55) in \( u, v \) and then integrating the resulting relation in \( v, u \), using the boundary relation

(5.127) \[ \partial^2_{r \nu} \Omega^2 \bigg|_{\gamma_2} \setminus \{(u_2, u_2)\} = \partial^2_{r \nu} \Omega^2 \bigg|_{\gamma_1} \setminus \{(u_2, u_2)\}\]

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(following by differentiating (5.87) in the direction tangent to \( \gamma_{u_2}^{u_2} \)) and the bounds (5.124), (5.125), (5.126), (5.127), (5.128) and (5.129), we obtain from (5.124):

\[
(5.128) \quad \sup_{y \in \mathcal{N} \setminus q} \left( \sum_{\alpha, \beta, \gamma, \delta = 0}^{k-1} |\partial_x \alpha \partial_x \beta g_{\gamma \delta}(y)| \right) < +\infty.
\]

In particular, in the Cartesian coordinates \( x^\alpha \) on \( \mathcal{N} \setminus q \), \( g \) extends as a \( C^{1,1} \) metric in a neighborhood of \( q \), i.e. the following improvement of (5.103) holds:

\[
(5.129) \quad \|g_{\alpha \beta}\|_{C^{1,1}(\mathcal{N})} < +\infty.
\]

Arguing inductively, we can similarly show that, for any \( k \geq 2 \), if

\[
(5.130) \quad \sup_{\mathcal{D}^{u_2}_{u_1} \setminus \{(u_2, u_2)\}} \left( \sum_{j=1}^{k-1} \left( |\partial^{j+3}_{u, v} r| + |\partial^{j+1}_{u, v} T_{u u}| + |\partial^j_{u, v} T_{uv}| + |\partial^j_{u, v} T_{uu}| \right) < +\infty
\]

and

\[
(5.131) \quad \|g_{\alpha \beta}\|_{C^{k-1,1}(\mathcal{N})} < +\infty,
\]

then

\[
(5.132) \quad \sup_{\mathcal{D}^{u_2}_{u_1} \setminus \{(u_2, u_2)\}} \left( |\partial^{k+3}_{u, v} r| + |\partial^{k+1}_{u, v} \Omega^2| + |\partial^k_{u, v} T_{uv}| + |\partial^k_{u, v} T_{uu}| \right) < +\infty
\]

and

\[
(5.133) \quad \|g_{\alpha \beta}\|_{C^{k,1}(\mathcal{N})} < +\infty
\]

(note that, for \( k = 2 \), (5.130), (5.131) follow immediately from the (5.129) and our proof that the right hand side of (5.117) is bounded). The proof of (5.132), (5.133) can be achieved as follows:

1. From the expressions (5.104), (5.105), (5.106), (5.107), (5.108) for the Riemann curvature components, the bounds (5.124), (5.125), (5.126), (5.127), (5.128), (5.129), (5.130), (5.131), (5.132), (5.133), and the Taylor expansion formula), we can readily bound in the Cartesian coordinates \( x^\alpha \):

\[
(5.134) \quad \sup_{\mathcal{D}^{u_2}_{u_1} \setminus \{(u_2, u_2)\}} \left( \sum_{j=0}^{k-1} |\partial^j_{u, v} t_1(y)| \right) \leq C_k \sup_{\mathcal{D}^{u_2}_{u_1} \setminus \{(u_2, u_2)\}} \left( \sum_{j=0}^{k-1} |\partial^j_{u, v} T_{uv}| + |\partial^j_{u, v} T_{uu}| + 1 \right)
\]

(5.135)

\[
(5.135) \quad \sup_{\mathcal{D}^{u_2}_{u_1} \setminus \{(u_2, u_2)\}} \left( \sum_{j=0}^{k-1} |\partial^j_{u, v} t_2(y)| \right) \leq C_k \sup_{\mathcal{D}^{u_2}_{u_1} \setminus \{(u_2, u_2)\}} \left( \sum_{j=0}^{k-1} |\partial^j_{u, v} T_{uv}| + |\partial^j_{u, v} T_{uu}| + 1 \right)
\]

(5.136)

\[
(5.136) \quad \sup_{\mathcal{D}^{u_2}_{u_1} \setminus \{(u_2, u_2)\}} \left( \sum_{j=0}^{k-1} |\partial^j_{u, v} T_{uv}| \right) < +\infty.
\]

2. By differentiating the Jacobi field equation (5.111) and using (5.131) and (5.135), we can readily obtain the following higher order analogue of (5.116) (inferred by a similar argument):

\[
(5.137) \quad \sup_{\mathcal{D}^{u_2}_{u_1} \setminus \{(u_2, u_2)\}} \left( \sum_{\alpha, \beta, \gamma, \delta = 0}^{k-1} |\partial^j_{u, v} T_{uv}| \right) < +\infty.
\]

3. By differentiating equations (2.45), (2.46), (2.52) and (2.55) and arguing similarly as for the proof of (5.128) (using (5.130), (5.131), (5.132), (5.133), (5.134) and (5.135)), we readily infer (5.132), (5.133).

We therefore deduce that the metric \( g \) on \( \mathcal{N} \setminus q \) admits a \( C^\infty \) extension on \( q \). The smooth extendibility of the whole solution \( (\mathcal{N} \setminus q, g; f) \) in a neighborhood of \( q \) then follows readily.
5.2 Extension principles away from $r = 0$

In this section, we will establish two extension principles for smooth solutions of (2.43)–(2.48): One which is valid in the region where $r$ is bounded away from 0, $\infty$, and one along $I$.

The next extension principle is a straightforward modification of a more general extension principle for solutions to the Einstein–massive Vlasov system obtained in [4] (see Proposition 3.1 in [4]):

**Proposition 5.1** (Smooth extension away from $r = 0$, $\infty$). For any $u_1 < u_2$, $v_1 < v_2$ and $\Lambda \in \mathbb{R}$, let $(r, \Omega^2, f)$ be any solution of (2.43)–(2.48) on an open neighborhood $\mathcal{V}$ of

$$
\mathcal{R} \triangleq [u_1, u_2] \times [v_1, v_2] \backslash \{(u_2, v_2)\}
$$

(see Figure 5.4) which is smooth according to Definition 3.3 and satisfies

\begin{align}
\inf_{\mathcal{V}} r &> 0, \\
\sup_{\mathcal{V}} r &< +\infty, \\
\sup_{\mathcal{V}} \tilde{\rho} &< +\infty, \\
\sup_{\{(u_1) \times [v_1, v_2] \cup (\{u_1, u_2\} \times \{v_1\})}} \partial_u r &< 0,
\end{align}

and, for some $C < +\infty$:

\begin{equation}
\text{supp}(f(u_1, :, :), \text{supp}(f(\cdot, v_1, :))) \subseteq \{\Omega^2(p^v + p^u) \leq C\}.
\end{equation}

Then, $(r, \Omega^2, f)$ extends smoothly in a neighborhood of $\{(u_2, v_2)\}$.

![Figure 5.4: Schematic depiction of the domains $\mathcal{R}$ and $\mathcal{V}$ appearing in the statement of Proposition 5.1](image)

**Proof.** In view of the smoothness of $r, \Omega^2$ in a neighborhood of $\{u_1\} \times [v_1, v_2]$, by integrating in the $u$ direction the following inequality:

$$
\partial_u \left( \frac{\Omega^2}{-\partial_u r} \right) \leq 0
$$

(which follows readily from (2.46) and the fact that $T_{uu} \geq 0$), we can bound

\begin{equation}
\sup_{\mathcal{R}} \frac{\Omega^2}{-\partial_u r} < +\infty.
\end{equation}
Note that, in particular, in view of the smoothness of $r, \Omega^2$ in $\mathcal{R}$ and the initial sign condition (5.10), the bound (5.142) implies that
\begin{equation}
(5.143) \qquad \partial_u r < 0 \text{ on } \mathcal{R}.
\end{equation}
Moreover, in view of the smoothness of $r, \Omega^2$ in a neighborhood of $[u_1, u_2] \times \{v_1\}$, the initial sign condition (5.140) and the given bounds (5.137)–(5.139) on $\mathcal{R}$, by integrating in the $v$ direction the following inequality:
\begin{equation}
(5.144) \quad \partial_v \log (-\partial_u r) \leq \tilde{m} - \frac{1}{2} \Lambda r^3 \frac{\Omega^2}{2r^2} - \partial_u r
\end{equation}
(which follows readily from (2.43), (2.10), (2.49) and the fact that $T_{uv} \geq 0$), using also the upper bound (5.142), we estimate:
\begin{equation}
(5.145) \quad \sup_{\mathcal{R}} (-\partial_u r) < +\infty.
\end{equation}
From (5.142) and (5.145), we therefore deduce that
\begin{equation}
(5.146) \quad \sup_{\mathcal{R}} \Omega^2 < +\infty.
\end{equation}

The bounds (5.137), (5.138), (5.141) and (5.146) allow us to apply the proof of Proposition 3.1 in [4] without any change (except for replacing the massive mass shell relation (26) in [4] with its massless analogue (2.20), which does not affect the proof). As a result, we obtain the required smooth extendibility of $(r, \Omega^2, f)$ in a neighborhood of $\{(u_2, v_2)\}$ (see also the comment at the end of the proof of Proposition 3.1 in [4] on how to obtain upgrade $C^2$ bounds on the extension into $C^\infty$ bounds).

For any $v_1 < v_2 \in \mathbb{R}$, let us set
\begin{equation}
(5.147) \quad \mathcal{V}^{v_2}_{v_1} \hat{=} \left( [v_1, v_2] \times [v_1, v_2] \right) \cap \{u \geq v\} \subset \mathbb{R}^2
\end{equation}
and
\begin{equation}
(5.148) \quad \mathcal{I}^{v_2}_{v_1} \hat{=} \left( [v_1, v_2] \times [v_1, v_2] \right) \cap \{u = v\} \subset \partial \mathcal{V}^{v_2}_{v_1}.
\end{equation}
The next extension principle will concern the smooth extendibility of solutions to (2.43)–(2.48) in neighborhoods of conformal infinity $\mathcal{I}$:

**Proposition 5.2** (Smooth extension along $r = \infty$). For any $v_1 < v_2$ and $\Lambda < 0$, let $(r, \Omega^2, f)$ be any solution of (2.43)–(2.48) on $\mathcal{V} \cap \{u > v\}$, where $\mathcal{V}$ is an open neighborhood of $\mathcal{V}^{v_2}_{v_1} \setminus \{(v_2, v_2)\}$ (see Figure 5.5), such that $(r, \Omega^2, f)$ is smooth with smooth conformal infinity $\mathcal{I}^{v_2}_{v_1} \setminus \{(v_2, v_2)\}$, according to Definition 3.3, and in addition, $f$ satisfies the reflecting boundary condition on $\mathcal{I}^{v_2}_{v_1} \setminus \{(v_2, v_2)\}$, according to Definition 2.4. Assume, moreover, that
\begin{equation}
(5.149) \quad \sup_{\mathcal{V}^{v_2}_{v_1} \setminus \mathcal{I}^{v_2}_{v_1}} \frac{2m}{r} < 1
\end{equation}
and, for some $C < +\infty$,
\begin{equation}
(5.150) \quad \operatorname{supp}(f(\cdot, v_1; \cdot)) \subset \{\Omega^2(p^v + p^u) \leq C\}.
\end{equation}
Then, $(r, \Omega^2, f)$ extends as a smooth solution of (2.43)–(2.48) with smooth conformal infinity to a larger open set $\hat{\mathcal{V}} \cap \{u > v\}$ such that $(v_2, v_2) \in \hat{\mathcal{V}}$. 

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Proof. Since \((r, \Omega^2, f)\) is smooth with smooth conformal infinity \(\mathcal{I}_{v_1}^{v_2}\), it is necessary that

\[ \partial_t \left( \frac{1}{r} \right)|_{\mathcal{I}_{v_1}^{v_2}\setminus\{(v_2, v_2)\}} < 0 \quad \text{and} \quad \partial_u \left( \frac{1}{r} \right)|_{\mathcal{I}_{v_1}^{v_2}\setminus\{(v_2, v_2)\}} > 0. \]

Since \(\Omega^2\) is smooth on \(\mathcal{V}_{v_1}^{v_2}\), the relation \((2.10)\), assumption \((5.149)\) and the inequalities \((5.151)\) on \(\mathcal{I}_{v_1}^{v_2}\setminus\{(v_2, v_2)\}\) imply that, everywhere on \(\mathcal{V}_{v_1}^{v_2}\): \(\partial_t r < 0 < \partial_u r\).

The sign condition \((5.152)\) implies that \(r\) is strictly positive in the interior of \(\mathcal{V}_{v_1}^{v_2}\). By possibly choosing a slightly larger \(v_1\), we will assume without loss of generality that

\[ \inf_{\mathcal{V}_{v_1}^{v_2}\setminus\mathcal{I}_{v_1}^{v_2}} r > 0. \]

The bound \((5.150)\) implies, in view of the relation \((2.20)\), the fact that \(r^{-2}\Omega^2\) extends smoothly on \(\mathcal{I}_{v_1}^{v_2}\setminus\{(v_2, v_2)\}\) and the conservation of angular momentum for null geodesics, that there exists some \(L_0 > 0\) such that, for any \((u, v) \in \mathcal{V}_{v_1}^{v_2}\setminus\mathcal{I}_{v_1}^{v_2}\)

\[ \sup_{[v_1, v_2] \times \{v_1\}} \left( r^2 (T_{uu} + T_{uv} + T_{vv}) \right) < +\infty. \]

In view of \((5.150)\) and \((5.154)\), the relations \((2.32)\) imply that

\[ \limsup_{u \to v_1} \tilde{m}(u, v_1) < +\infty. \]

The relations \((5.152)\) and \((5.149)\) imply, in view of \((2.53)\)–\((2.54)\), that

\[ \partial_u \tilde{m} \leq 0 \leq \partial_v \tilde{m} \]

and hence, in view of \((2.71)\), \((5.156)\) and the fact that \(r, \Omega^2\) are smooth on \([v_1, v_2] \times \{v_1\}\), we infer that

\[ \sup_{\mathcal{V}_{v_1}^{v_2}\setminus\mathcal{I}_{v_1}^{v_2}} \tilde{m} < +\infty, \]

\[ \inf_{\mathcal{V}_{v_1}^{v_2}\setminus\mathcal{I}_{v_1}^{v_2}} \tilde{m} > -\infty. \]
Furthermore, in view of (2.49), (5.149) and (5.157), there exists some $C > 0$ such that
\begin{equation}
\sup_{\nu_i^0دلانتیلا | x_i^p |} \left| \log \left( \frac{1 - \frac{1}{3} \Lambda r^2}{1 - \frac{2m}{r}} \right) \right| < +\infty.
\end{equation}

We can readily estimate in view of (2.53)–(2.54), (5.149)–(5.150) and (5.157):
\begin{equation}
\sup_{\bar{u} \in \{v_1, v_2\}} \int_{\{u = \bar{u}\} \cap \mathcal{V}_{\nu_i^0}^2 \setminus \mathcal{I}_{\nu_i^0}^2} \frac{4\pi r T_{uu}}{\partial_r r} dv \leq \sup_{\bar{u} \in \{v_1, v_2\}} \int_{\{u = \bar{u}\} \cap \mathcal{V}_{\nu_i^0}^2 \setminus \mathcal{I}_{\nu_i^0}^2} \frac{2\partial_r \hat{m}}{r \cdot (1 - \frac{2m}{r})} dv \leq C \sup_{\bar{u} \in \{v_1, v_2\}} \int_{\{u = \bar{u}\} \cap \mathcal{V}_{\nu_i^0}^2 \setminus \mathcal{I}_{\nu_i^0}^2} \partial_r \hat{m} dv < +\infty.
\end{equation}

and, similarly
\begin{equation}
\sup_{\bar{v} \in \{v_1, v_2\}} \int_{\{v = \bar{v}\} \cap \mathcal{V}_{\nu_i^0}^2 \setminus \mathcal{I}_{\nu_i^0}^2} \frac{4\pi r T_{vv}}{-\partial_r r} du < +\infty.
\end{equation}

Integrating (2.50), (2.51) in $u, v$ respectively, and using (5.159)–(5.160), as well as (2.49), (5.13) and the boundary condition (2.16) on $\mathcal{I}_{\nu_i^0}^2 \setminus \{(v_2, v_2)\}$, we infer that
\begin{equation}
\sup_{\nu_i^0دلانتیلا | x_i^p |} \left( \left| \log \left( \frac{\partial_r r}{1 - \frac{2m}{r}} \right) \right| + \left| \log \left( -\frac{-\partial_r r}{1 - \frac{2m}{r}} \right) \right| \right) < +\infty.
\end{equation}

The bound (5.161) implies, in view of (2.53)–(2.54), that there exists some $C > 0$ such that for any $(u, v) \in \mathcal{V}_{\nu_i^0}^2 \setminus \mathcal{I}_{\nu_i^0}^2$:
\begin{equation}
r^2 T_{uv}(u, v) \leq C \min \left\{ \partial_r \hat{m}, -\partial_r \hat{m} \right\}
\end{equation}

and hence, after integrating over curves of the form $u = const, v = const$ and using (5.157):
\begin{equation}
\sup_{\bar{u} \in \{v_1, v_2\}} \int_{\{u = \bar{u}\}} r^2 T_{uv} dv + \sup_{\bar{v} \in \{v_1, v_2\}} \int_{\{v = \bar{v}\}} r^2 T_{uv} dv < +\infty.
\end{equation}

Let us consider the renormalised quantities $(\rho, \hat{\Omega}^2, \tau_{\mu\nu})$, defined by (2.56) and satisfying (2.57). In view of (5.150) and the fact that $(r, \Omega^2, f)$ is smooth on $\mathcal{V}_{\nu_i^0}^2 \setminus \mathcal{I}_{\nu_i^0}^2$ with smooth conformal infinity $\mathcal{I}_{\nu_i^0}^2 \setminus \{(v_2, v_2)\}$, the quantities $(\rho, \hat{\Omega}^2, \tau_{\mu\nu})$ extend smoothly on $\mathcal{I}_{\nu_i^0}^2 \setminus \{(v_2, v_2)\}$, with $\hat{\Omega}^2 \big|_{\mathcal{I}_{\nu_i^0}^2 \setminus \{(v_2, v_2)\}} > 0$. The proof of Proposition 5.2 will thus follow by showing that the renormalised quantities also extend smoothly on $(v_2, v_2)$, with
\begin{equation}
\hat{\Omega}^2(v_2, v_2) > 0.
\end{equation}

In view of (2.10), (5.158) and (5.161), we immediately infer that
\begin{equation}
\sup_{\nu_i^0دلانتیلا \setminus \{(v_2, v_2)\}} \left| \log(\hat{\Omega}^2) \right| < +\infty,
\end{equation}
as well as
\begin{equation}
\sup_{\nu_i^0دلانتیلا \setminus \{(v_2, v_2)\}} \left( \left| \log \left( \partial_v \rho \right) \right| + \left| \log \left( -\partial_u \rho \right) \right| \right) < +\infty.
\end{equation}

Integrating (2.57) in $u, v$ and using (5.157), (5.163) and (5.165), we also obtain:
\begin{equation}
\sup_{\nu_i^0دلانتیلا \setminus \{(v_2, v_2)\}} \left( \left| \partial_v \log(\hat{\Omega}^2) \right| + \left| \partial_u \log(\hat{\Omega}^2) \right| \right).
\end{equation}
The bounds \([5.165]–[5.167]\) imply that \(\rho, \bar{\Omega}^2\) admit a \(C^{0,1}\) extension on \((v_2, v_2)\), satisfying \(5.164\).

Using the identity appearing in the first line of \([2.59]\) (as well as the analogous identity obtained after switching the roles of \(u, v\)) for any null geodesic \(\gamma: I \to \mathcal{V}_{v_1}^{\rho} \setminus \mathcal{I}_{v_1}^{\rho}\) in the support of \(f\) with non-vanishing angular momentum, possibly extended to its reflection off \(\mathcal{I}_{v_1}^{\rho}\) (according to Definition \(2.2\)), we infer, in view of the assumption \(5.150\) for the support of \(f\) initially, that, for some absolute constant \(C > 0\):

\[
\sup_{x \in X} \Omega^2(\gamma^v + \gamma^u)(s) \leq C \exp \left( C \sup_{v^1 \in \mathcal{V}_{v_1}^{\rho}} \left( |\partial_v \log \Omega^2 - \frac{2}{r} \frac{\partial_r \rho}{r}| + |\partial_u \log \Omega^2 - \frac{2}{r} \frac{\partial_r \rho}{r}| \right) \right).
\]

Since the definition \([2.56]\) of \(\bar{\Omega}^2\) implies that

\[
\partial_v \log \Omega^2 - \frac{2}{r} \frac{\partial_r \rho}{r} = \partial_u \log \bar{\Omega}^2 + O(r^{-3}) \partial_r \rho,
\]

the estimate \(5.168\) yields (in view of the bounds \(5.166\)–\(5.167\)) for a possibly different absolute constant \(C > 0\):

\[
\sup_{x \in X} \Omega^2(\gamma^v + \gamma^u)(s) \leq C.
\]

Hence, there exists some \(C_\ast > 0\) such that, for any \((u, v) \in \mathcal{V}_{v_1}^{\rho} \setminus \mathcal{I}_{v_1}^{\rho}\):

\[
\sup_{v^1 \in \mathcal{V}_{v_1}^{\rho}} \left( \Omega^2(p^u + p^v) \right) \leq C_\ast.
\]

Arguing similarly as for the proof of \([5.155]\), from \(5.171\) we therefore infer that:

\[
\sup_{v^1 \in \mathcal{V}_{v_1}^{\rho} \setminus ((v_2, v_2))} \left( \tau_{uu} + \tau_{uv} + \tau_{vv} \right) < +\infty.
\]

Commuting equations \([2.57]\) and \([2.47]\) with \(\partial_u, \partial_v\) and arguing inductively (using \(5.163\), \(5.166\), \(5.167\), \(5.166\), \(5.167\) and \(5.172\)) as a basis for the induction), we similarly obtain the following higher order analogues of \(5.165\), \(5.166\), \(5.167\) and \(5.172\) for any \(k \in \mathbb{N}\):

\[
\sup_{v^1 \in \mathcal{V}_{v_1}^{\rho} \setminus ((v_2, v_2))} \left( |\partial_{u,v}^k \log(\bar{\Omega}^2)| + |\partial_{u,v}^{k-1} \rho| + |\partial_{u,v}^{k-1} \tau| \right) < +\infty.
\]

Hence, the smooth extension of \((\rho, \bar{\Omega}^2, \tau)\) on \(((v_2, v_2))\) follows readily.

\[\square\]

### 5.3 A general extension principle for domains of outer communications

In this section, we will obtain, as a corollary of Theorem \(5.1\) and Propositions \(5.1\) and \(5.2\), a general extension principle for asymptotically AdS, smooth solutions of \([2.43]–[2.48]\) which coincide with their domain of outer communications.

**Corollary 5.1.** For any \(v_2, u_1 > 0\), let \((r, \Omega^2, f)\) be a smooth solution of \([2.43]–[2.48]\) on \(\mathcal{U}_{u_1; v_2}\), with smooth axis \(\gamma_{u_2}\) and smooth conformal infinity \(\mathcal{I}_{u_1}\) (for the relevant notation, see \([4.13]–[4.15]\) and Definitions \(3.1–3.3\)). Assume, moreover, that \((r, \Omega^2, f)\) satisfies

\[
\sup_{\mathcal{U}_{u_1; v_2}} \frac{2m}{r} < 1,
\]

\[
\limsup_{(u,v) \to (u_1,u_1)} \frac{2m}{r} \leq \delta_0,
\]

where \(\delta_0\) is the constant appearing in the statement of Theorem \(5.1\) as well as the initial bound

\[
\sup_{v_1} \left( |\Omega^2(p^u + p^v)| \right) \leq C
\]

(for some \(C < +\infty\)). Then, there exists some \(u_1 > u_1\), such that \((r, \Omega^2, f)\) extends on the whole of \(\mathcal{U}_{u_1; v_2}\) as a smooth solution of \([2.43]–[2.48]\) with smooth axis \(\gamma_{u_1}\) and smooth conformal infinity \(\mathcal{I}_{u_1}\).
Proof. Using the condition
\[ \partial_r (\Omega^{-2} \partial_r r) \leq 0 \]
(following readily from the constraint equation (2.43)) and the fact that \( r|_{r_*} = +\infty \), we readily infer that
\[ \partial_r r > 0 \text{ on } \mathcal{U}_{u_1:v_x}. \]
As a consequence of the assumption (5.174) and the relation (2.10) between \( \Omega^2, \partial_r r \) and \( \partial_u r \), we therefore infer that
\[ \partial_u r < 0 \text{ on } \mathcal{U}_{u_1:v_x} \]
and hence the condition (5.152) holds everywhere on \( \mathcal{U}_{u_1:v_x} \). By arguing exactly as in the proof of (5.157) in Proposition 5.2 using (5.176) and the fact that \( (r, \Omega^2, f) \) is smooth on the axis \( \gamma_{u_1} \), we infer that
\[ 0 \leq \inf_{\mathcal{U}_{u_1:v_x}} \tilde{m} \leq \sup_{\mathcal{U}_{u_1:v_x}} \tilde{m} < +\infty. \]

By applying Theorem 5.1, we immediately obtain that \( (r, \Omega^2, f) \) extends as a smooth solution of (2.43)–(2.48) with smooth axis on a neighborhood of \( (u_1, \tilde{u}_1) \). Then, applying Proposition 5.1 combined with a simple continuity argument, we infer that \( (r, \Omega^2, f) \) extends as a smooth solution of (2.43)–(2.48) in an open neighborhood of \( \{ u_1 \} \times \{ u_1, u_1 + v_x \} \). Finally, by applying Proposition 5.2 we deduce that \( (r, \Omega^2, f) \) extends as a smooth solution of (2.43)–(2.48) with smooth conformal infinity on \( V \cap \{ v < u + v_x \} \), where \( V \) is an open neighborhood of \( (u_1, u_1 + v_x) \). Hence, for some \( \tilde{u}_1 > u_1 \) sufficiently close to \( u_1 \), \( (r, \Omega^2, f) \) extends as a smooth solution of (2.43)–(2.48) on the whole of \( \mathcal{U}_{u_1:v_x} \), with smooth axis \( \gamma_{u_1} \) and smooth conformal infinity \( \mathcal{I}_{u_1} \). \( \Box \)

6 A Cauchy stability statement in a low regularity topology

In this section, we will introduce a low regularity, scale invariant norm \( \| \cdot \| \) on the space of smoothly compatible, asymptotically AdS initial data sets \( (r_j, \Omega_j^2, f_j; v_x) \), as introduced in Section 3.2. This norm will measure the “concentration of energy” along the evolution of a free Vlasov field \( f \) on \( (\mathcal{M}_{AdS}, g_{AdS}) \) determined from \( f_1 \) by an explicit formula in a renormalised gauge. We will then proceed to establish a Cauchy stability statement for the trivial solution \( (r_{AdS}, \Omega_{AdS}^2, 0) \) of (2.43)–(2.48) in the initial data topology defined by \( \| \cdot \| \): We will show that, for any fixed retarded time \( U_* \geq 0 \) and for any initial data set \( S \) with \( ||S|| \) sufficiently small, the geometry of the corresponding maximal development \( (\mathcal{M}, g; f) \) with a reflecting boundary condition on \( \mathcal{I} \) exists for sufficiently long time and remains close to that of \( (\mathcal{M}_{AdS}, g_{AdS}; 0) \) in the fixed retarded time interval \( u \in [0, U_*] \).

The results of this section allow addressing the AdS instability conjecture for the system (2.43)–(2.48) in the low regularity setting of \( || \cdot || \) and will be crucial for the results of our companion paper [16].

6.1 A low regularity norm on the space of initial data

In this section, we will introduce a low regularity norm on the space of smoothly compatible, asymptotically AdS initial data sets for (2.43)–(2.48). In order to simplify our notations, we will adopt the following definition:

Definition 6.1. We will denote with \( \mathcal{B} \) the set of smoothly compatible, asymptotically AdS initial data sets \( (r_j, \Omega_j^2, f_j; v_x) \) for (2.43)–(2.48) which are bounded in phase space, in accordance with Definitions 3.4 and 3.5. For any \( (r_j, \Omega_j^2, f_j; v_x) \in \mathcal{B} \), we will denote with \( (V(v); \frac{dV}{dv}(0)) \) the parameters of the unique gauge normalising gauge transformation provided by Lemma 3.2, i.e. a gauge transformation \( (r_j, \Omega_j^2, f_j; v_x) \rightarrow (r'_j, (\Omega'_j)^2, f'_j; v_x) \) defined by the relations 3.27 such that the transformed initial data set \( (r'_j, (\Omega'_j)^2, f'_j; v_x) \) satisfies the normalisation condition 3.31.

We will define a map from \( \mathcal{B} \) to the space of smooth solution of the (free) massless Vlasov equation (2.22) on AdS spacetime as follows:
Definition 6.2. For any \((r_j, \Omega_j^2, \bar{f}_j; v \bar{z}) \in \mathcal{B}\), let \(\bar{f}_j^{(AdS)} : [0, \sqrt{-\frac{\pi}{A}}] \times [0, +\infty) \to [0, +\infty)\) be given by the expression

\[
(6.1) \quad \bar{f}_j^{(AdS)}(v; p^u, l) = \bar{f}_j\left(\frac{\sqrt{-\frac{3}{A}}}{v \bar{z}}, v; p^u, l\right),
\]

where \(\bar{f}_j\) is the initial Vlasov field in the gauge normalised expression \((r_j', (\Omega_j')^2, \bar{f}_j'; v \bar{z})\) of \((r_j, \Omega_j^2, \bar{f}_j; v \bar{z})\) provided by Lemma 3.2 (see Definition 6.1).

We will define \(f^{(AdS)} : T \mathcal{M}_{AdS} \to [0, +\infty)\) to be the unique solution of the massless Vlasov equation \((2.22)\) on \((\mathcal{M}_{AdS}, g_{AdS})\) with initial conditions on \(u = 0\) corresponding to \(\bar{f}_j^{(AdS)}\). In particular, denoting with \(\Omega^2_{AdS}(u, v), r_{AdS}(u, v)\) the coefficients of the AdS metric \((1.4)\), the Vlasov field \(f^{(AdS)}\) is expressed as

\[
f^{(AdS)}(u, v; p^u, p^v, l) = \bar{f}_j^{(AdS)}(u, v; p^u, p^v, l) \cdot \delta\left(\Omega^2_{AdS}(u, v)p^u p^v - \frac{l^2}{r^2_{AdS}(u, v)}\right)
\]

for some smooth function \(f^{(AdS)}\) satisfying the initial condition

\[
(6.2) \quad \bar{f}_j^{(AdS)}(0; p^u, l) = \bar{f}_j^{(AdS)}(v; p^u, l).
\]

For any \(\bar{u} \geq 0\) and \(\bar{v} \in (\bar{u}, \bar{u} + \sqrt{-\frac{\pi}{A}})\), we will also set

\[
(6.3) \quad f^{(AdS)}_{\bar{u}, \bar{v}}(u, v; p^u, l) = f^{(AdS)}(u, v; p^u, l) \cdot \delta\left(\Omega^2_{AdS}(u, v)p^u p^v - \frac{l^2}{r^2_{AdS}(u, v)}\right)
\]

(and similarly for \(f^{(AdS)}_{\bar{u}, \bar{v}}\), \(f^{(AdS)}_{\bar{u}, \bar{v}}\), and \(f^{(AdS)}_{\bar{u}, \bar{v}}\), where the energy momentum components \(T_{\alpha\beta}[f^{(AdS)}]\) are defined using the relations \((2.32)\) (with \(\Omega^2_{AdS}, r_{AdS}\) in place of \(\Omega^2, r\)).

Remark. For \(u \geq 0, v \in (u, u + \sqrt{-\frac{\pi}{A}})\), \(p^u, p^v \geq 0\) and \(l \geq 0\), the free Vlasov field \(f^{(AdS)}(u, v; p^u, p^v, l)\) is expressed in terms of \(\bar{f}_j^{(AdS)}\) through the explicit relations \((A.11)\)–\((A.12)\) \((A.12)\) and \(f^{(AdS)}\) in place of \(F\) in \((A.12)\) (for various useful identities regarding the geodesic flow on AdS spacetime, see Section 3A of the Appendix).

The following definition introduces a non-negative functional \(||.|.||\) on the space \(\mathcal{B}\) which can be used to measure the distance from the trivial initial data set \(\mathcal{S}_{AdS}\):

Definition 6.3. For any \((r_j, \Omega_j^2, \bar{f}_j; v \bar{z}) \in \mathcal{B}\), we will define:

\[
(6.4) \quad ||(r_j, \Omega_j^2, \bar{f}_j; v \bar{z})|| = \sup_{U_+ \geq 0, U_*} \int_{U_* - \sqrt{-\frac{\pi}{A}}}^{U_* + \sqrt{-\frac{\pi}{A}}} \left(\left[r T_{uv}\right]^{(AdS)}(u, v) + \left[r T_{uv}\right]^{(AdS)}(u, v)\right) dv + \sup_{V_* \geq 0, V_* - \sqrt{-\frac{\pi}{A}}} \left(\left[r T_{uv}\right]^{(AdS)}(u, V_*) + \left[r T_{uv}\right]^{(AdS)}(u, V_*)\right) du + \sqrt{-\Lambda m_{\mathcal{S}_{AdS}}}(v \bar{z}).
\]

Remark. In view of the periodicity of the null geodesic flow on the spherical quotient of AdS spacetime (see Section 3A of the Appendix), the value of \((6.4)\) does not change if one restricts to the supremum over \(U_* \in [0, \sqrt{-\frac{\pi}{A}}]\) and \(V_* \in [0, 2\sqrt{-\frac{\pi}{A}}]\) in the right hand side of \((6.4)\). Therefore, the condition that the elements of \(\mathcal{B}\) have bounded support in phase space (i.e. satisfy \((3.10)\)) implies that \(||S||\) is finite for any \(S \in \mathcal{B}\).

The norm \(||(r_j, \Omega_j^2, \bar{f}_j; v \bar{z})||\) vanishes if and only if \(f^{(AdS)} \equiv 0\), i.e. if \(\bar{f}_j \equiv 0\). In this case, \((r_j, \Omega_j^2, 0; v \bar{z})\) is mapped through the gauge transformation provided by Lemma 3.2 to the rescaled normalised trivial data \((r_{AdS}\bar{z}^2, (\Omega_{AdS})^2, 0; v \bar{z})\), given by \((3.34)\).
We should also point out that the quantity (6.4) is both gauge invariant (i.e. invariant under coordinate transformations of the form (3.25)–(3.27)) and scale invariant, i.e. invariant under transformations of \((r_j, \Omega_j^2; f_j)\) of the form
\[
\begin{align*}
  r_j(v) &\to \lambda^{-1} r_j(\lambda v), \\
  \Omega_j^2(v) &\to \Omega_j^2(\lambda v), \\
  f_j(v; p^v, l) &\to \lambda^2 (\lambda')^l f_j(\lambda v; \lambda' p^v, \lambda \cdot \lambda'), \\
  \Lambda &\to \lambda^2 \Lambda,
\end{align*}
\]
for any \(\lambda, \lambda' > 0\). The scale invariance of \(\| \cdot \|\) is used in a fundamental way in the constructions of our companion paper [16].

Any smooth solution \((r, \Omega^2; f)\) of \((2.43)–(2.48)\) on a domain \(\mathcal{U}_{u,1;v_2}\) of the form \((4.13)\), with smooth axis \(\{ u = v \}\) and smooth conformal infinity \(\{ u = v - v_2 \}\), induces a smoothly compatible initial data set \((r_{/u_0}, \Omega^2_{/u_0}, f_{/u_0}; v_2)\) on slices of the form \(\{ u = u_0 \} \cap \mathcal{U}_{u,1;v_2}\) for any \(u_0 \in (0, 1)\), where
\[
(r_{/u_0}, \Omega^2_{/u_0})(\bar{v}) \simeq (r, \Omega^2)(u_0 + \bar{v})
\]
and
\[
(f_{/u_0}(\bar{v}; p, l) = \bar{f}(u_0, u_0 + \bar{v}; p, \frac{l^2}{2p^2\| (u_0, u_0 + \bar{v}) \|^2}).
\]

As a result, \(\| \cdot \|\) can be used to measure the “size” of a solution \((r, \Omega^2; f)\) at time \(u = u_0\):

**Definition 6.4.** Let \((r, \Omega^2; f), \mathcal{U}_{u,1;v_2}\) and \(u_0 \in (0, 1)\) be as above, with \(f\) of bounded support in phase space. We will define the norm on the initial data induced by \((r, \Omega^2; f)\) on the slice \(\{ u = u_0 \} \cap \mathcal{U}_{u,1;v_2}\) by the relation
\[
\|(r, \Omega^2; f)|_{u = u_0}\| \equiv \|(r_{/u_0}, \Omega^2_{/u_0}, f_{/u_0}; v_2)\|
\]
where \(\| \cdot \|\) is defined by \((6.4)\) and \(r_{/u_0}, \Omega^2_{/u_0}, f_{/u_0}\) are given by \((6.5)–(6.6)\).  

### 6.2 Cauchy stability of AdS in the low regularity topology

In this section, we will establish a Cauchy stability statement for AdS spacetime \((\mathcal{M}_{AdS}, g_{AdS})\) as a solution of \((2.43)–(2.48)\), with respect to the initial data topology defined by \((6.4)\). This result will also provide us with some a priori control on the geometry of solutions of \((2.43)–(2.48)\) arising as small perturbations of \((\mathcal{M}_{AdS}, g_{AdS})\) with respect to \((6.4)\), which will be useful for the results in our companion paper [16].

In particular, we will prove the following result:

**Theorem 6.1.** For any \(v_2 > 0\), any \(U = n \cdot v_2 > 0\) (where \(n \in \mathbb{N}\)) and any \(C_0 > 0\), there exist \(\varepsilon_0 > 0\) and \(C_1 > 0\) such that the following statement holds: For any \(0 \leq \varepsilon < \varepsilon_0\) and any smooth initial data set \((r_j, \Omega_j^2; f_j; v_2) \in \mathcal{B}\) satisfying the smallness condition
\[
\|(r_j, \Omega_j^2, f_j; v_2)\| < \varepsilon
\]
(where \(\| \cdot \|\) is defined by \((6.4)\)), as well as the bound \((7.10)\) with \(C_0\) in place of \(C\), the maximal future development \((\mathcal{U}_{\max}; r, \Omega^2, f)\) of \((r_j, \Omega_j^2, f_j; v_2)\) with reflecting boundary conditions on \(\mathcal{I}\) (see Corollary 4.2) is defined on the whole of the the domain \(\mathcal{U}_{u;v_2}\) (defined by \((4.13)\)), i.e.
\[
\mathcal{U}_{u;v_2} \subset \mathcal{U}_{\max}.
\]

In addition, \((r, \Omega^2; f)\) satisfies the estimates:
\[
\sup_{u, \epsilon \in (0, U)} \|(r, \Omega^2; f)|_{u = u_0}\| \leq C_1 \varepsilon
\]
(where the notation \(||(r, \Omega^2; f))_{u=v}||\) was introduced in Definition \(6.4\),

\[
(6.11) \sup_{(u,v)\in U_{U,v_2}} \left( \sup_{p^n, p^\eta \in \text{supp}(f(u,v,\cdot,\cdot))} \left( \left( (-\partial_u r)p^n + (\partial_v r)p^\eta \right) \right) \right) \leq (1 + C_1 \varepsilon) C_0,
\]

\[
(6.12) \sup_{u(0,U)} \int_{u}^{u+v} r \left( \frac{T_{uv}}{\partial_v r} + \frac{T_{uu}}{\partial_u r} \right)(u,v) \, dv + \sup_{v(0,U-v_2)} \int_{\min(v,U)}^{\max(v,U)} r \left( \frac{T_{uv}}{\partial_v r} + \frac{T_{uu}}{\partial_u r} \right)(u,v) \, du \leq C_1 \varepsilon,
\]

and

\[
(6.13) \sup_{U_{U,v_2}} \frac{2\tilde{m}}{r} < C_1 \varepsilon.
\]

Remark. Let us note that the bounds \((6.10) \sim (6.13)\) are gauge invariant. Furthermore, any coordinate transformation \((u,v) \rightarrow (U(u), V(v))\) with \(U(0) = 0\) for which the lines \(\{u = v\}\) and \(\{v = v_2\}\) remain invariant necessarily satisfies \(U(v_2) = nU(v_2)\) for any \(n \in \mathbb{N}\); for this reason, the inclusion \((6.9)\) is also a gauge invariant statement.

Proof. In order to establish the gauge invariant estimates \((6.9) \sim (6.13)\), it will be convenient for us to work in a gauge which normalises the initial data. For this reason, we will assume that

\[
(6.14) \sup_{v(0,U)} \frac{\partial_v r_j}{1 - \frac{1}{3} \Lambda r_j^2(v)} - \frac{\partial_v r_j^{(v)}(v)}{1 - \frac{1}{3} \Lambda (r_j^{(v)}(v))^2} \leq C \varepsilon
\]

for some absolute constant \(C > 0\).

Let us introduce some shorthand notation for various geometric objects that will appear in the proof of Theorem \(6.1\). Let \(U_{u,v_2} \subset \mathcal{D}\) be a domain of the form \((4.13)\); given any \(v_0 \in (0, v_T), p_0 \in (0, +\infty)\) and \(l \in (0, +\infty)\), we will denote with \(\gamma[\cdot; v_0, p_0, l] : [0, a) \rightarrow U_{u,v_2}\) the affinely parametrised null geodesic of \((r, \Omega^2)\) which is uniquely determined by the condition that it has angular momentum \(l\) and satisfies initially:

\[
(6.15) \left\{ \begin{array}{l} 
\gamma[0; v_0, p_0, l] = (0, v_0), \\
\dot{\gamma}[0; v_0, p_0, l] = p_0.
\end{array} \right.
\]

Remark. Since the gauge transformation \(T\) is piecewise smooth on \(\mathcal{D}\) and smooth on \(\mathcal{D} \setminus \cup_{k=1}^\infty \left\{ \{u = kv_2\} \cup \{v = kv_2\} \right\}\) (see \(4.1\)), the initial condition \((6.15)\) on \(u = 0\) is regularly transformed under \(T^{-1}\), i.e. \((D_{0,v_0})T^{-1}\) \(\gamma\) is well defined; hence the existence and uniqueness of a null geodesic \(\gamma\) satisfying \((6.15)\) can be shown readily by applying the inverse transformation \(T^{-1}\) on \((\mathcal{D}; r, \Omega^2, f)\) and appealing to the standard theory.

We will denote the derivative of \(\gamma\) with respect to the affine parametrization by \(\dot{\gamma}\); while \(\tau\) will denote the parameter defined by the function \(u + v\). As a function of \(\tau\), we will assume that \(\gamma[\cdot; v_0, p_0, l]\) is maximally extended in \(U_{u,v_2}\) through reflections off \(\mathcal{I}\), according to Definition \(2.3\). Furthermore, we will set

\[
(6.16) E[\tau; v_0, p_0, l] \overset{\text{def}}{=} \frac{1}{2} \Omega^2(\gamma[\tau; v_0, p_0, l])(\dot{\gamma}[\tau; v_0, p_0, l] + \dot{\gamma}[\tau; v_0, p_0, l]).
\]

We will also define \(\gamma_{\text{AdS}}[\cdot; v_0, p_0, l] : [0, a') \rightarrow U_{u,v_2}\) to be the unique maximally extended (through reflections) null geodesic of the normalised rescaled AdS metric \((r_{\text{AdS}}^{(v)})^2(\Omega_{\text{AdS}}^{(v)}))^2\) on \(U_{u,v_2}\) (the coefficients of which are given by
satisfying the initial conditions \((6.15)\) and having angular momentum \(l\). We will adopt the same notational conventions for \(\gamma_{\text{AdS}}\) as for \(\gamma\).

Let \(C_2 = C_2(U) \gg 1\) be a fixed large constant depending only on \(U\). In order to establish Theorem 6.1, we will first assume that, for some \(U_* \in (0, U)\),

\[
\mathcal{U}_{U_*, v \xi} \equiv \{0 < u < U_*\} \cap \{u < v < u + v \xi\} \subset \mathcal{D}
\]

and that the following bootstrap assumptions are satisfied:

\[
\sup_{u_* \in (0, U_*)} \|(r, \Omega^2; f)\|_{u = u_*} \leq C_2 \varepsilon, \tag{6.17}
\]

\[
\sup_{u \in (0, U_*)} \int_0^{u+v\xi} r\left(\frac{T_{uv}}{\partial_v r} + \frac{T_{uu}}{\partial_u r}\right)(u, v) \, dv + \sup_{v \in (0, U_*)} \int_0^{\min\{v, U_*\}} r\left(\frac{T_{uv}}{\partial_v r} + \frac{T_{uu}}{\partial_u r}\right)(u, v) \, du \leq C_2 \varepsilon, \tag{6.18}
\]

\[
\sup_{\mathcal{U}_{U_*, v \xi}} \frac{2\tilde{m}}{r} < C_2 \varepsilon \tag{6.19}
\]

and, for any \(v_0 \in (0, v_\xi)\), \(p_0 > 0\), \(l > 0\):

\[
\left|u(\gamma[\tau; v_0, p_0, l]) - u(\gamma_{\text{AdS}}[\tau; v_0, p_0, l])\right| + \left|v(\gamma[\tau; v_0, p_0, l]) - v(\gamma_{\text{AdS}}[\tau; v_0, p_0, l])\right| \leq C_2 \frac{l}{E[0; v_0, p_0, l]} \varepsilon, \tag{6.20}
\]

\[
\left|\Omega^2 \dot{\gamma}^{u}[\tau; v_0, p_0, l] - \Omega^2 \dot{\gamma}_{\text{AdS}}^{u}[\tau; v_0, p_0, l]\right| + \left|\Omega^2 \dot{\gamma}^v[\tau; v_0, p_0, l] - \Omega^2 \dot{\gamma}_{\text{AdS}}^v[\tau; v_0, p_0, l]\right| \leq C_2 \varepsilon E[0; v_0, p_0, l]. \tag{6.21}
\]

We will then show that the following improvement of \((6.17) - (6.21)\) actually holds on \(\mathcal{U}_{U_*, v \xi}\):

\[
\sup_{u_* \in (0, U_*)} \|(r, \Omega^2; f)\|_{u = u_*} \leq 4 \varepsilon, \tag{6.22}
\]

\[
\sup_{u \in (0, U_*)} \int_0^{u+v\xi} r\left(\frac{T_{uv}}{\partial_v r} + \frac{T_{uu}}{\partial_u r}\right)(u, v) \, dv + \sup_{v \in (0, U_*)} \int_0^{\min\{v, U_*\}} r\left(\frac{T_{uv}}{\partial_v r} + \frac{T_{uu}}{\partial_u r}\right)(u, v) \, du \leq 4 \varepsilon, \tag{6.23}
\]

\[
\sup_{\mathcal{U}_{U_*, v \xi}} \frac{2\tilde{m}}{r} < \frac{1}{2} C_2 \varepsilon \tag{6.24}
\]

and, for any \(v_0 \in (0, v_\xi)\), \(p_0 > 0\), \(l > 0\):

\[
\left|u(\gamma[\tau; v_0, p_0, l]) - u(\gamma_{\text{AdS}}[\tau; v_0, p_0, l])\right| + \left|v(\gamma[\tau; v_0, p_0, l]) - v(\gamma_{\text{AdS}}[\tau; v_0, p_0, l])\right| \leq \frac{1}{2} C_2 \frac{l}{E[0; v_0, p_0, l]} \varepsilon, \tag{6.25}
\]

\[
\left|\Omega^2 \dot{\gamma}^{u}[\tau; v_0, p_0, l] - \Omega^2 \dot{\gamma}_{\text{AdS}}^{u}[\tau; v_0, p_0, l]\right| + \left|\Omega^2 \dot{\gamma}^v[\tau; v_0, p_0, l] - \Omega^2 \dot{\gamma}_{\text{AdS}}^v[\tau; v_0, p_0, l]\right| \leq \frac{1}{2} C_2 \varepsilon E[0; v_0, p_0, l]. \tag{6.26}
\]

The proof of Theorem 6.1 will then follow immediately through a standard continuity argument (using also the gauge dependent bounds \((6.29) - (6.30)\) and \((6.39)\) to compare \(\Omega^2 \dot{\gamma}^v, \Omega^2 \dot{\gamma}^{u}\) with \(\partial_\nu r \dot{\gamma}^v, -\partial_u r \dot{\gamma}^{u}\),) by applying the inverse transformation \(T^{-1}\) on \(\mathcal{D}\) and using the extension principle of Corollary 5.1 (which guarantees that, given a smooth solution \((r, \Omega^2, f)\) on \(\mathcal{U}_{U_*, v \xi}\) satisfying \(6.24\), it can be smoothly extended on \(\mathcal{U}_{U_*, v \xi + c v \xi}\) for some \(c > 0\)).

We will now proceed to establish \((6.22) - (6.26)\) in two steps, assuming that \((6.17) - (6.21)\) hold.
Step 1: Proof of (6.22)–(6.24). Integrating (2.50) and (2.51) using the initial conditions
\[(r, \Omega^2)(0, v) = (r_f, \Omega_f^2)(v),\]
the initial bound (6.14), the bounds (6.18) and (6.19), as well as the boundary conditions
\[(6.28)\]
\[\left\{ \begin{aligned}
\frac{\partial_u r}{u = v} &= -\frac{\partial_v r}{u = v}, \\
\frac{\partial_u r}{u = v} &= \frac{\partial_v r}{u = v - v_f}
\end{aligned} \right. \]
on the axis and conformal infinity and the expression (3.36) for \(r_f\) (combined with the initial bound (6.8)), we can readily estimate for some absolute constant \(C > 0\), noting that \(U_s \leq U\):
\[(6.29)\]
\[\sup_{u_s, v \leq 0} \left| \frac{\partial_v r}{1 - \frac{1}{3} \Lambda r^2} - \frac{\partial_v r_{\text{AdS}}}{1 - \frac{1}{3} \Lambda (r_{\text{AdS}})^2} \right| \leq CC_2 \varepsilon \cdot (1 + \sqrt{-\Lambda} U)
\]
and
\[(6.30)\]
\[\sup_{u_s, v \leq 0} \left| \frac{\partial_v r}{1 - \frac{1}{3} \Lambda r^2} - \frac{\partial_v r_{\text{AdS}}}{1 - \frac{1}{3} \Lambda (r_{\text{AdS}})^2} \right| \leq CC_2 \varepsilon \cdot (1 + \sqrt{-\Lambda} U).
\]
Integrating (6.29) and using the boundary conditions \(r|_{u = v} = 0\) and \(r|_{u = v = v_f} = +\infty\), we infer that
\[(6.31)\]
\[\sup_{u_s, v \leq 0} \left| \frac{r - r_{\text{AdS}}}{r} \right| \leq CC_2 \varepsilon \cdot (1 + \sqrt{-\Lambda} U).
\]
As a consequence of the initial bound (6.8), Definition (6.4) of \(\|(r, \Omega^2; f)|_{u = u_s}\|\), the bounds (6.29)–(6.31), the bootstrap estimates (6.20)–(6.21) for the difference of the dynamics of the geodesic flow of \((r, \Omega^2)\) and \((r_{\text{AdS}}(\Omega_{\text{AdS}})^2)\) and the explicit description (A.3)–(A.6) of the geodesic flow in AdS spacetime (in particular, the bound (A.2)), we can readily obtain the following improvement of (6.17) and (6.18) (assuming \(\varepsilon_0\) is small enough in terms of \(C_2\) and \(U\)):
\[(6.32)\]
\[\sup_{u_s, u \in (0, U_s)} \|(r, \Omega^2; f)|_{u = u_s}\| \leq (1 + CC_1 \varepsilon)^5 \varepsilon \leq 2 \varepsilon
\]
and
\[(6.33)\]
\[\sup_{u \in (0, U_s)} \int_{u^{+}} \frac{T_{uv}}{u} + \frac{T_{uv}}{u} (u, v) dv + \sup_{v \in (0, U_s)} \int_{v^{+}} \frac{T_{uv}}{u} + \frac{T_{uv}}{u} (u, v) du \leq 2 \varepsilon.
\]
Therefore, we infer (6.22)–(6.23). Returning to the proof of (6.29)–(6.31) and using (6.33) in place of (6.18), we infer that
\[(6.34)\]
\[\sup_{u_s, v \leq 0} \left| \frac{\partial_v r}{1 - \frac{1}{3} \Lambda r^2} - \frac{\partial_v r_{\text{AdS}}}{1 - \frac{1}{3} \Lambda (r_{\text{AdS}})^2} \right| \leq C(1 + \sqrt{-\Lambda} U) \varepsilon,
\]
\[(6.35)\]
\[\sup_{u_s, v \leq 0} \left| \frac{\partial_v r}{1 - \frac{1}{3} \Lambda r^2} - \frac{\partial_v r_{\text{AdS}}}{1 - \frac{1}{3} \Lambda (r_{\text{AdS}})^2} \right| \leq C(1 + \sqrt{-\Lambda} U) \varepsilon
\]
and
\[(6.36)\]
\[\sup_{u_s, v \leq 0} \left| \frac{r - r_{\text{AdS}}}{r} \right| \leq C(1 + \sqrt{-\Lambda} U) \varepsilon.
\]
Integrating the relation (2.54) for $\tilde{m}$ and using the boundary condition $\tilde{m}(u=\nu) = 0$ (following from the smoothness of $T^{-1}(r, \Omega^2; f)$ on the axis) and using the bounds (6.19) and (6.33)–(6.36), we can readily estimate provided $\varepsilon_0$ is sufficiently small with respect to $C_2, U$:

$$\sup_{U_{u,v_Z}} \left\{ \tilde{m} \right\} \leq C \varepsilon. \tag{6.37}$$

On the other hand, in view of the initial bound (6.8), the fact that $\sqrt{-\Lambda} \tilde{m}(u=\nu) \leq \| (r_j, \Omega^2, \bar{f}_j; v_Z) \|$ and the conservation of $\tilde{m}$ along conformal infinity $\{ u = v - v_Z \}$, we can immediately bound:

$$\sup_{U_{u,v_Z}} \left\{ \tilde{m} \right\} \leq C \varepsilon. \tag{6.38}$$

In particular, combining (6.37) and (6.38), we infer (6.24). As a consequence of (2.10) and (2.49), from (6.34)–(6.38) we deduce that

$$\sup_{U_{u,v_Z}} \left\{ \tilde{m} \right\} \leq C(1 + \sqrt{-\Lambda}) \varepsilon. \tag{6.39}$$

Step 2: Proof of (6.25)–(6.26). As a consequence of the bound (A.2) for geodesics on AdS and the bootstrap assumption (6.20) (using also (6.34)–(6.36)), we infer (provided $\varepsilon_0$ is small enough in terms of $C_2(U)$) that, for any $v_0 \in (0, v_Z)$, $p_0 > 0, l > 0$:

$$\inf_{\gamma \in \gamma[v_0, p_0, l]} r = r_{\min}[v_0, p_0, l] \geq \frac{1}{2} \left( \frac{E^2[\gamma[0; v_0, p_0, l]]}{l^2} + \frac{1}{3} \Lambda \right)^{\frac{1}{2}}. \tag{6.40}$$

Let us set for any $k \in \mathbb{N}$ and any $v_0 \in (0, v_Z)$, $p_0 > 0$ and $l > 0$:

$$\tau_k[v_0] \doteq \min \left\{ (2k - 1)v_Z + 2v_0, U_* \right\} \tag{6.41}$$

and

$$\tau'_k[v_0, p_0, l] \doteq \min \left\{ U_*, \text{value of } \tau \text{ when } \gamma[\tau; v_0, p_0, l] \text{ reaches } \{ u = v - v_Z \} \text{ for the } k\text{-th time} \right\}. \tag{6.41}$$

We will also set for convenience

$$\tau_0[v_0] = \tau'_0[v_0, p_0, l] = v_0. \tag{6.42}$$

Note that, as a consequence of the bootstrap assumption (6.20) and the explicit description (A.3)–(A.6) of the geodesic flow on AdS, we have for any $k \in \mathbb{N}$

$$|\tau_k[v_0] - \tau'_k[v_0, p_0, l]| \leq C_2 \varepsilon \frac{l}{E[0; v_0, p_0, l]]. \tag{6.43}$$

Remark: In order to simplify our notation, we will frequently drop the parameters $v_0, p_0, l$ from the notation for $\tau_k, \tau'_k$ when no confusion arises.

In view of the bootstrap assumptions (6.20)–(6.21) the bounds (6.34)–(6.36) and the explicit description (A.3)–(A.6) of the geodesic flow on AdS (using, in particular, the fact that the energy (A.1) of null geodesics in AdS is conserved), we can readily show that, for some constant $C > 0$ depending only on $\Lambda$, we have for any $k \geq 0$ (provided $\varepsilon_0$ is small enough in terms of $C_2(U)$):

$$\bigcup_{\tau \in \tau_k \cap \tau_{k+1}} \gamma[\tau; v_0, p_0, l] \subset \left\{ r \geq r_{\min}[v_0, p_0, l] \right\} \cap \left( U^{(k)}_{in} \cup U^{(k)}_{out} \right) \cap U^*_{u,v_Z}. \tag{6.44}$$
where

\begin{align}
U_{\text{in}}^{(k)} &= [(k-1)v_L + v_0 - C \frac{l}{E_0}, kv_L + v_0 + C \frac{l}{E_0}] \times [kv_L + v_0 - C \frac{l}{E_0}, kv_L + v_0 + C \frac{l}{E_0}], \\
U_{\text{out}}^{(k)} &= [kv_L + v_0 - C \frac{l}{E_0}, kv_L + v_0 + C \frac{l}{E_0}] \times [kv_L + v_0 - C \frac{l}{E_0}, (k+1)v_L + v_0 + C \frac{l}{E_0}]
\end{align}

and we have denoted for simplicity

\[ E_0 = E[0; v_0, p_0, l] \]

(see Figure 6.1). Moreover,

\begin{align}
\frac{1}{2} \leq \frac{E[\tau; v_0, p_0, l]}{E_0} \leq \frac{3}{2}.
\end{align}

Remark. The bound (6.44) becomes non-trivial only when \( l/E_0 \ll 1 \), since when \( l/E_0 \gg 1 \) the regions in the right hand side of (6.44) might contain all of \( \{ r \geq r_{\min}[v_0, p_0, l] \} \).

Let us define the \textit{approximately ingoing} and \textit{approximately outgoing} intervals for \( \gamma[\cdot; v_0, p_0, l] \) as follows: For any integer \( k \geq 0 \), we will set (for some fixed \( C \gg 1 \) depending on \( A \))

\begin{align}
I_{\text{in}}^{(k)} &= [\tau'_k, \tau'_{k+1} - v_L + \frac{C l}{E_0}], \\
I_{\text{out}}^{(k)} &= [\tau'_{k+1} - v_L - \frac{C l}{E_0}, \tau'_{k+1}].
\end{align}

Note that \( I_{\text{in}}^{(k)} \cap I_{\text{out}}^{(k)} \neq \emptyset \). Notice also that, as a consequence of the bootstrap assumptions \[ 6.20 \text{--} 6.21 \] \( l/E_0 \ll 1 \), the bounds \[ 6.34 \text{--} 6.36 \] and the explicit description \[ \text{(A.3) -- (A.6)} \] of the geodesic flow on \( \text{AdS} \), we have for any \( k \geq 0 \):

\begin{align}
\bigcup_{\tau \in I_{\text{in}}^{(k)}} \gamma[\tau; v_0, p_0, l] \subset \left\{ r \geq r_{\min}[v_0, p_0, l] \right\} \cap U_{\text{in}}^{(k)}, \\
\bigcup_{\tau \in I_{\text{out}}^{(k)}} \gamma[\tau; v_0, p_0, l] \subset \left\{ r \geq r_{\min}[v_0, p_0, l] \right\} \cap U_{\text{out}}^{(k)}.
\end{align}

Figure 6.1: Schematic depiction of the domains \( U_{\text{in}}^{(k)} \) and \( U_{\text{out}}^{(k)} \) for \( k \geq 1 \) when \( l/E_0 \ll 1 \).
and, for some fixed \(c > 0\) depending on \(\Lambda\):

\[
\Omega^2 \gamma^\nu[\tau; v_0, p_0, l] \geq cE_0 \text{ for } \tau \in \mathcal{I}_{\text{in}}^{(k)},
\]

\[
\Omega^2 \gamma^\nu[\tau; v_0, p_0, l] \geq cE_0 \text{ for } \tau \in \mathcal{I}^{(k)}_{\text{out}}.
\]

**Remark.** In view of (2.20) and the conservation of (A.1) on pure AdS spacetime, the bounds (6.51)–(6.52) become non-trivial only when \(l/E_0 \ll 1\).

For any \(k \in \mathbb{N}, v_0 \in (0, v_Z), p_0 > 0, l > 0\), let \(\mathcal{W}\) denote the region between \(\gamma[\cdot; v_0, p_0, l]\) and \(\gamma_{\text{AdS}}[\cdot; v_0, p_0, l]\) in \(\mathcal{U}_{U_*, v_Z}\), i.e.:

\[
\mathcal{W} = \mathcal{W}[v_0, p_0, l] = \{ (\bar{u}, \bar{v}) \in \mathcal{U}_{U_*, v_Z} : \min \left\{ (v - u) \gamma[\bar{u} + \bar{v}; v_0, p_0, l], (v - u) \gamma_{\text{AdS}}[\bar{u} + \bar{v}; v_0, p_0, l] \right\} \leq \bar{v} - \bar{u} \leq \\
\leq \max \left\{ (v - u) \gamma[\bar{u} + \bar{v}; v_0, p_0, l], (v - u) \gamma_{\text{AdS}}[\bar{u} + \bar{v}; v_0, p_0, l] \right\} \}
\]

Figure 6.2: Schematic depiction of the region \(\mathcal{W}_\tau\) bounded by \(\gamma[\cdot; v_0, p_0, l]\), \(\gamma_{\text{AdS}}[\cdot; v_0, p_0, l]\) and \(\{u + v \leq \tau\}\).

Moreover, for any \(\tau \in (v_0, U_* + v_Z)\), we will set:

\[
\mathcal{W}_\tau = \mathcal{W}_\tau[v_0, p_0, l] = \mathcal{W} \cap \{ u + v \leq \tau \}
\]

and

\[
|\mathcal{W}_\tau| = \frac{1}{r_{\min}[v_0, p_0, l]} \left( \sup \left\{ |u_1 - u_1| : (u_1, v_1), (u_2, v_2) \in \mathcal{W}_\tau \text{ with } u_1 + v_1 = u_2 + v_2 \right\} \right)
\]

(see Figure 6.2). Using the geodesic equation (2.21) and the fact that \(\tau\) corresponds to the parametrization of \(\gamma, \gamma_{\text{AdS}}\) with respect to \(u + v\), we can readily estimate:

\[
\frac{d}{d\tau} |\mathcal{W}_\tau| \leq \frac{1}{r_{\min}[v_0, p_0, l]} \left[ \frac{\Omega^2 \gamma^\nu - \Omega^2 \gamma_{\text{AdS}}^\nu}{\Omega^2 \gamma^\nu + \Omega^2 \gamma_{\text{AdS}}^\nu} [\tau; v_0, p_0, l] - \frac{(\Omega_{\text{AdS}}^\nu)^2 \gamma_{\text{AdS}}^\nu}{(\Omega_{\text{AdS}}^\nu)^2 \gamma_{\text{AdS}}^\nu + (\Omega_{\text{AdS}}^\nu)^2 \gamma_{\text{AdS}}^\nu} [\tau; v_0, p_0, l] \right].
\]

Using (2.20), (6.47) and the fact that

\[
\frac{1}{2} \leq \frac{\Omega^2 \gamma^\nu}{\Omega^2 \gamma_{\text{AdS}}^\nu} \left( \frac{(\Omega_{\text{AdS}}^\nu)^2 \gamma_{\text{AdS}}^\nu}{(\Omega_{\text{AdS}}^\nu)^2 \gamma_{\text{AdS}}^\nu + (\Omega_{\text{AdS}}^\nu)^2 \gamma_{\text{AdS}}^\nu} [\tau; v_0, p_0, l] \right) \leq 2
\]

(following from the bootstrap assumption (6.21)), from (6.56) we readily infer for some absolute constant \(C > 0\):

\[
\frac{d}{d\tau} |\mathcal{W}_\tau| \leq \frac{C}{r_{\min}[v_0, p_0, l]} \frac{l^2 \Omega^2}{r^2} |\gamma_{\text{AdS}}[\tau; v_0, p_0, l]| \left( \frac{1}{E_0^2} E_0 [\tau; v_0, p_0, l] \right).
\]
where the energy difference \( E_D[\tau; v_0, p_0, l] \) is defined as
\[
(6.58) \quad E_D[\tau; v_0, p_0, l] = \left| \left( \Omega^2 z^u + \Omega^2 z^v \right)[\tau; v_0, p_0, l] - \left( \Omega^2_{AdS} z^u + \Omega^2_{AdS} z^v \right)[\tau; v_0, p_0, l] \right|.
\]

Remark. Note that the factor in front of \( E_D \) in the right hand side of (6.57) is expected to behave like \( \sim \frac{m}{r^2} \) along \( \gamma \). Thus, its integral over the geodesic is independent of the value of \( r_{\min} \) (that would not have been the case if the factor was merely \( \frac{m}{\gamma} \)). This fact will be crucial for an application of Gronwall’s inequality later in the proof.

In view of (6.34)–(6.36), (6.39), (6.40) for \( r_{\min} \), and the definition (6.45), (6.46) of \( U_{in}^{(k)}, U_{out}^{(k)} \), we can readily estimate for some constant \( C > 0 \) depending only on \( \Lambda \):
\[
(6.59) \quad \int_{U_{in}^{(k)} \cap \{ \tau \geq r_{\min}[v_0, p_0, l] \}} \frac{\Omega^2}{r^2} \, dudv + \int_{U_{out}^{(k)} \cap \{ \tau \geq r_{\min}[v_0, p_0, l] \}} \frac{\Omega^2}{r^2} \, dudv \leq C.
\]
We can also bound for any \( \tau \in (v_0, U_\ast + v_\mathcal{I}) \):
\[
(6.60) \quad \int_{W_\tau \cap U_{in}^{(k)}} \frac{\Omega^2}{r^2} \, dudv + \int_{W_\tau \cap U_{out}^{(k)}} \frac{\Omega^2}{r^2} \, dudv \leq C|\mathcal{W}_\tau|.
\]
Setting
\[
S^{(k)} = \begin{cases} U_{in}^{(0)} \cap \{ u = 0 \}, & \text{if } k = 0, \\ U_{in}^{(k)} \cap \{ u = v - v_\mathcal{I} \}, & \text{if } k > 0, \end{cases}
\]
using the relations (2.53), (2.54) and the bounds (6.34)–(6.39), we can also estimate
\[
(6.61) \quad \int_{U_{in}^{(k)} \cap \{ \tau \geq r_{\min}[v_0, p_0, l] \}} \left( \frac{\tilde{m}}{r^3} \Omega^2 + T_{uv} \right) dudv \leq \]
\[
\leq C \left( \int_{U_{in}^{(k)} \cap \{ \tau \geq r_{\min}[v_0, p_0, l] \}} \left( \frac{\tilde{m}}{r^3} \Omega^2 + \frac{-\partial_u \tilde{m}}{r^2 \Lambda^2} \right) dudv \right) \leq \]
\[
\leq C \\left( \int_{U_{in}^{(k)} \cap \{ \tau \geq r_{\min}[v_0, p_0, l] \}} \frac{\tilde{m}}{r^3} (1 - \frac{1}{3} \Lambda r^2) dudv + C \int_{S^{(k)}} \frac{\tilde{m}}{r^3} (1 - \frac{1}{3} \Lambda r^2) dv \right) \leq \]
\[
\leq C \left( \int_{U_{in}^{(k)} \cap \{ \tau \geq r_{\min}[v_0, p_0, l] \}} \frac{\tilde{m}}{r^3} \Omega^2 dudv + \sup_{U_{in}^{(k)} \cap \{ \tau \geq r_{\min}[v_0, p_0, l] }} \left( \frac{\tilde{m}}{r^3} (1 - \frac{1}{3} \Lambda r^2) \right) dmin[U_0, p_0, l] \right) \leq \]
\[
\leq C \varepsilon.
\]
where, in passing from the first to the second line in (6.61), we have integrated by parts in \( u \) and used the fact that \( \frac{\tilde{m}}{r^3} (1 - \frac{1}{3} \Lambda r^2) \) is non-negative. Similarly,
\[
(6.62) \quad \int_{U_{out}^{(k)} \cap \{ \tau \geq r_{\min}[v_0, p_0, l] \}} \left( \frac{\tilde{m}}{r^3} \Omega^2 + T_{uv} \right) dudv \leq C \varepsilon.
\]

In view of the boundary condition \( 1/r|_{u = v - v_\mathcal{I}} = 0 \), the finiteness of \( \tilde{m}|_{u = v - v_\mathcal{I}}, v^2 T_{uv}|_{u = v - v_\mathcal{I}} \), formula (2.10) and the renormalised equation (2.57) for \( \tan^{-1}(\sqrt{-\Lambda} r) \), the renormalised quantity \( \Omega^2/(1 - \frac{1}{3} \Lambda r^2) \) satisfies the following boundary condition on conformal infinity:
\[
(6.63) \quad \left( \partial_v - \partial_u \right) \frac{\Omega^2}{1 - \frac{1}{3} \Lambda r^2} |_{u = v - v_\mathcal{I}} = 0.
\]
Setting
\[
u_1(v) = \begin{cases} 0, & v \leq v_\mathcal{I}, \\ v - v_\mathcal{I}, & v \geq v_\mathcal{I} \end{cases}
\]
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and subtracting from the relation \((2.61)\) for \(\gamma[v; v_0, p_0, l_0]\) the same relation for \(\gamma_{\text{AdS}}[\tau; v_0, p_0, l_0]\) (thus replacing \(r, \Omega\) in \((2.61)\) with \(r_{\text{AdS}}, \Omega_{\text{AdS}}\) and setting \(\tilde{m} = 0, T_{uv} = 0\), using also the relation

\[
(6.64) \quad \partial_t \left( \frac{\Omega^2}{1 - \frac{2}{3}\Lambda r^2} \right)_{\{u,v=\tau\}} = \frac{1}{2} \left( \partial_u + \partial_v \right) \left( \frac{\Omega^2}{1 - \frac{2}{3}\Lambda r^2} \right)_{\{u,v=\tau\}}
\]

(and the analogous relation for \(r_{\text{AdS}}, (\Omega_{\text{AdS}})^2\) following readily from \((6.63)\), we infer after using the bounds \((6.34) - (6.39)\) and \((6.59) - (6.62)\) that, for some \(C > 0\) depending only on \(\Lambda\), the following bound holds for any \(\tau \in \mathcal{I}_{in}^{(0)}\):

\[
(6.65) \quad \left| \log \left( \Omega_{\text{AdS}}^2 \gamma_{\tau} \left[ \tau; v_0, p_0, l_0 \right] \right) - \log \left( \Omega_{\text{AdS}}^2 \gamma_{\text{AdS}} \left[ \tau; v_0, p_0, l_0 \right] \right) \right| \leq \frac{C}{r^2} \frac{1 + \sqrt{-\Lambda} U}{\tau} \leq \frac{C}{r^2} \frac{1 + \sqrt{-\Lambda} U}{\tau}
\]

From \((6.65)\), using \((2.20), (6.51), (6.47), (6.57)\) and the bootstrap assumption \((6.20)\), we infer that, for any \(\tau \in \mathcal{I}_{in}^{(0)}\) (recalling the definition \((6.58)\) of the energy difference \(E_D\)):

\[
(6.66) \quad E_D[\tau; v_0, p_0, l_0] \leq C \left( 1 + \sqrt{-\Lambda} U \right) E_D \left( \varepsilon + \frac{l^2}{r_{\text{min}}[v_0, p_0, l]} E_D^2 \right) \int_{\tau}^{\tau_0} \frac{(\Omega_{\text{AdS}})^2}{(r_{\text{AdS}})^2} \gamma_{\text{AdS}}[\tau; v_0, p_0, l] E_D[\tau; v_0, p_0, l] d\tau.
\]

Repeating the same procedure for \(\tau \in \mathcal{I}_{out}^{(0)}\) using \((2.62)\) with \(v_1(\tau) = v_0 - C \frac{l}{E_D}\) (replacing \((6.51)\) with \((6.52)\) and using \((6.66)\) to obtain an initial estimate for \(E_D\) on \(\mathcal{I}_{in}^{(0)} \cap \mathcal{I}_{out}^{(0)}\), we deduce that \((6.65)\) also holds for \(\tau \in \mathcal{I}_{out}^{(0)}\). Furthermore, the same argument yields that \((6.65)\) also holds for \(\tau \in \mathcal{I}_{in}^{(k)} \cap \mathcal{I}_{out}^{(k)}\) (for a possibly larger \(C > 0\)) for any \(k > 0\). Therefore, since \((6.66)\) holds for all \(\tau \in [v_0, u_\ast + \varepsilon \mathcal{I}_{\ast}]\), an application of Gronwall’s inequality, in combination with the trivial estimate

\[
(6.67) \quad \frac{l^2}{r_{\text{min}}[v_0, p_0, l]} E_D \int_{v_0}^{\tau} \frac{(\Omega_{\text{AdS}})^2}{(r_{\text{AdS}})^2} \gamma_{\text{AdS}}[\tau; v_0, p_0, l] d\tau \leq C \left( \frac{\tau}{2v_{\mathcal{I}}^2} \right)
\]

(coming from the lower bound \((6.40)\) for \(r_{\text{min}}[v_0, p_0, l]\) and the explicit relations \((A.3) - (A.6)\) for the geodesic flow on \(\text{AdS}\)), yield that

\[
(6.68) \quad \sup_{\tau} E_D[\tau; v_0, p_0, l_0] \leq C E_D \exp(C(-\Lambda) U^2) \cdot \varepsilon.
\]

Using the relation \((2.20)\) and the bounds \((6.34) - (6.39)\) and \((6.40)\), from \((6.68)\) we infer \((6.26)\), provided \(C_2(U) \gg \exp(C(-\Lambda) U^2)\). Integrating the bound \((6.55)\) and recalling the definition \((6.55)\) of \(|\mathcal{W}|\), we then obtain \((6.25)\). Hence, the proof of Theorem \(6.1\) is complete.

### A Geodesic flow on AdS spacetime

In this section, we will collect a few useful relations regarding the geodesic flow on AdS spacetime \((\mathcal{M}^{5+1}_{\text{AdS}}, g_{\text{AdS}})\). Let us fix a spherically symmetric double null coordinate pair \((u, v)\) on \(\mathcal{M}_{\text{AdS}}\) by the condition

\[
\frac{\partial_u r}{1 - \frac{3}{2}\Lambda r^2} = \frac{1}{2}.
\]
(see Section 2.1 for the relevant definitions). In this coordinate chart, $\mathcal{M}_{AdS}\backslash \mathcal{I}$ is mapped to the coordinate domain

$$
\mathcal{U}_{AdS} = \{ u < v < u + \sqrt{\frac{3}{\Lambda}} \},
$$

while the metric $g_{AdS}$ is expressed as

$$
g_{AdS} = -\Omega^2_{AdS} ds^2 + r^2 g_{S^2},
$$

where $\Omega^2_{AdS}$, $r$ are given by (1.4).

In view of the fact that the vector field $T = \partial_u + \partial_v$ on $(\mathcal{M}_{AdS}, g_{AdS})$ is Killing, the quantity

$$
E[\gamma] = \frac{1}{2} \Omega^2 (\dot{\gamma}^u + \dot{\gamma}^v)
$$

is constant along any affinely parametrised geodesic $\gamma$. Given an initial point $\gamma(0)$ for a null geodesic, the values of the conserved quantities $E[\gamma]$ and $I[\gamma]$ (see Section 2.3), together with the sign of $\dot{\gamma}^v - \dot{\gamma}^u$, determine $\gamma(0)$ (and thus the whole geodesic $\gamma$) uniquely up to a rotation of $\mathcal{M}_{AdS}$. For a future directed null geodesic $\gamma$, the relation (2.20) yields (after multiplication with $\Omega^2_{AdS}$) that, at any point $\gamma(s)$ on $\gamma$:

$$
\frac{E^2[\gamma]}{I^2[\gamma]} \geq \frac{1}{r^2(\gamma(s))} - \frac{1}{3}\Lambda.
$$

Furthermore, using the relation (2.20), we can determine the minimum value of $r$ along an inextendible, future directed null geodesic $\gamma$ (attained at the point where $\dot{\gamma}^v = \dot{\gamma}^u$) only in terms of $E[\gamma], I[\gamma]$; in particular:

$$
\min_{\gamma} r = \left( \frac{E^2[\gamma]}{I^2[\gamma]} + \frac{1}{3}\Lambda \right)^{\frac{1}{2}}.
$$

Note that only radial null geodesics (i.e. those with $l = 0$) pass through the center $r = 0$.

Let $\gamma$ be any future inextendible, future directed null geodesic in $(\mathcal{M}_{AdS}, g_{AdS})$ with affine parameter $s$, such that $\gamma(0)$ lies on $u = 0$ and $\gamma(\infty)$ (i.e. the intersection point of $\gamma$ with $\mathcal{I}$) lies at $u + v = \tau_\infty[\gamma]$. Integrating the geodesic equation for (1.4), we obtain the following useful relations for $\gamma$: Denoting with $\text{sgn}(\cdot)$ the sign function on $\mathbb{R}$, we calculate for any $\tau \in [v(0), \tau_\infty[\gamma])]$

$$
\begin{align*}
\upsilon|_{\gamma(0)u+e^*} &= U\{ \tau; v(0), E[\gamma], I[\gamma], (\dot{\gamma}^v - \dot{\gamma}^u)(0) \}, \\
\upsilon|_{\gamma(0)u+e^*} &= V\{ \tau; v(0), E[\gamma], I[\gamma], (\dot{\gamma}^v - \dot{\gamma}^u)(0) \}, \\
\frac{d}{ds} \gamma^u|_{\gamma(0)u+e^*} &= \Omega^2_{AdS}|_{\gamma(0)u+e^*} \cdot \Gamma^u\{ \tau; v(0), E[\gamma], I[\gamma], \text{sgn}(\dot{\gamma}^v - \dot{\gamma}^u)(0) \}, \\
\frac{d}{ds} \gamma^v|_{\gamma(0)u+e^*} &= \Omega^2_{AdS}|_{\gamma(0)u+e^*} \cdot \Gamma^v\{ \tau; v(0), E[\gamma], I[\gamma], \text{sgn}(\dot{\gamma}^v - \dot{\gamma}^u)(0) \},
\end{align*}
$$

where, setting

$$
\rho_{\min}\{ E, l \} = \tan^{-1}\left( \frac{1}{3} \frac{E^2}{I^2} \right)^{\frac{1}{2}},
$$

and

$$
\omega_0\{ v_0, E, l, \sigma \} = -\sigma \cdot \cos^{-1}\left( \frac{\cos\left( \frac{1}{2} \sqrt{\frac{4}{3}} v_0 \right)}{\cos \rho_{\min}} \right),
$$

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the functions $U, V, G^u, G^v$ appearing in (A.3) are defined by the relations

\begin{align}
(A.6) \quad & U\{\tau; v_0, E, l, \sigma\} = \frac{1}{2} \tau - \sqrt{-\frac{3}{\Lambda}} \cos^{-1}\left\{ \cos \rho_{\min} \cdot \cos \left[ \omega_0 - \frac{1}{2} \sqrt{-\frac{3}{\Lambda}} (\tau - v_0) \right] \right\}, \\
& V\{\tau; v_0, E, l, \sigma\} = \frac{1}{2} \tau + \sqrt{-\frac{3}{\Lambda}} \cos^{-1}\left\{ \cos \rho_{\min} \cdot \cos \left[ \omega_0 - \frac{1}{2} \sqrt{-\frac{3}{\Lambda}} (\tau - v_0) \right] \right\}, \\
& G^u\{\tau; v_0, E, l, \sigma\} = 2E \cdot \frac{dU}{d\tau} (\tau; v_0, E, l, \sigma), \\
& G^v\{\tau; v_0, E, l, \sigma\} = 2E \cdot \frac{dV}{d\tau} (\tau; v_0, E, l, \sigma).
\end{align}

Note that $\tau_{\infty}[\gamma]$ can be explicitly expressed as

$$
\tau_{\infty}[\gamma] = v[0] + \sqrt{-\frac{3}{\Lambda}} (2\omega_0 + \pi).
$$

**Remark.** In the expressions (A.5)–(A.6), we use the convention that $\cos^{-1} x \geq 0$. In particular, for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have that $\cos^{-1}(\cos \theta) = |\theta|$. Note also that, when $(\dot{v}^v - \dot{v}^u)(0) = 0$, the relation (A.5) yields $\omega_0 = 0$ independently of the convention for $\text{sgn}(0)$ (and, thus, the expressions (A.3) are smooth along variations of $(\dot{v}^v - \dot{v}^u)(0)$ with non-constant sign).

By extending the geodesic $\gamma$ through its reflection off $\mathcal{I}$ (according to Definition 2.2), we can extend the functions $U, V, G^u$ and $G^v$ (defined by (A.6)) for $\tau$ belonging to the whole interval $[v_0, +\infty)$. After this extension, the functions $U, V$ become continuous and piecewise smooth in $\tau$, while $G^u, G^v$ have a jump discontinuity at $\tau = \tau_{\infty}[\gamma] + 2k\sqrt{-\frac{3}{\Lambda}} \pi$, $k \in \mathbb{N}$. In particular, $U, V, G^u$ and $G^v$ satisfy for any $\tau \in [v_0, +\infty)$

\begin{align}
(A.7) \quad & (U, V, G^u, G^v)\{\tau + 2\sqrt{-\frac{3}{\Lambda}} \pi; v_0, E, l, \sigma\} = (U, V, G^u, G^v)\{\tau; v_0, E, l, \sigma\}
\end{align}

and

\begin{align}
(A.8) \quad & \lim_{\tau \to (\tau_{\infty}[\gamma])^-} G^u\{\tau; v_0, E, l, \sigma\} = \lim_{\tau \to (\tau_{\infty}[\gamma])^+} G^u\{\tau; v_0, E, l, \sigma\}, \\
& \lim_{\tau \to (\tau_{\infty}[\gamma])^-} G^v\{\tau; v_0, E, l, \sigma\} = \lim_{\tau \to (\tau_{\infty}[\gamma])^+} G^v\{\tau; v_0, E, l, \sigma\}.
\end{align}

**Remark.** For a null geodesic $\gamma$ such that

$$
\varepsilon \equiv \frac{d[\gamma]}{E[\gamma]} \ll \sqrt{-\frac{3}{\Lambda}},
$$

the relations (A.6) simplify as follows (for $\tau \in [v(0), \tau_{\infty}[\gamma])$):

\begin{align}
(A.9) \quad & U\{\tau; v_0, E, l, \sigma\} = \frac{1}{2} \tau - \sqrt{\frac{\varepsilon^2 + \frac{1}{4} (\tau - v_0 - 2\sqrt{-\frac{3}{\Lambda} \omega_0})^2}{2}} + O(\varepsilon^2), \\
& V\{\tau; v_0, E, l, \sigma\} = \frac{1}{2} \tau + \sqrt{\frac{\varepsilon^2 + \frac{1}{4} (\tau - v_0 - 2\sqrt{-\frac{3}{\Lambda} \omega_0})^2}{2}} + O(\varepsilon^2), \\
& G^u\{\tau; v_0, E, l, \sigma\} = E \cdot \left\{ 1 - \frac{1}{2} \frac{(\tau - v_0 - 2\sqrt{-\frac{3}{\Lambda} \omega_0})}{\sqrt{\frac{\varepsilon^2 + \frac{1}{4} (\tau - v_0 - 2\sqrt{-\frac{3}{\Lambda} \omega_0})^2}{2}}} + O(\varepsilon^2) \right\}, \\
& G^v\{\tau; v_0, E, l, \sigma\} = E \cdot \left\{ 1 - \frac{1}{2} \frac{(\tau - v_0 - 2\sqrt{-\frac{3}{\Lambda} \omega_0})}{\sqrt{\frac{\varepsilon^2 + \frac{1}{4} (\tau - v_0 - 2\sqrt{-\frac{3}{\Lambda} \omega_0})^2}{2}}} + O(\varepsilon^2) \right\},
\end{align}
i.e. $\gamma$ approximately traces a hyperbola in the $(u, v)$ plane (when $\varepsilon > 0$), with its vertex at a point where $r \approx \varepsilon$.

![Figure A.1: Schematic depiction of the projection onto the $(u, v)$-plane of two null geodesics in AdS spacetime emanating from the same point on $\{u = 0\}$, with $l = 0$ and $l = \epsilon E_0$, respectively, where $\epsilon \ll \sqrt{-\frac{3}{\Lambda}}$. Due to the special form of the AdS metric, null geodesics emanating, instead, from the same point on $I$ will also terminate on the same point on $I$.](image)

The relations (A.3) allow us to obtain some simple expressions for solutions of the massless Vlasov equation on $(M_{\text{AdS}}, g_{\text{AdS}})$. In particular, for any smooth function $F : [0, \sqrt{-\frac{3}{\Lambda}} \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$, the unique solution $f$ of the massless Vlasov equation (2.22) satisfying at $u = 0$ the initial condition

\begin{equation}
 f(0, v; p^u, p^v, l) = F(v; p^u, l) \cdot \delta\left(\Omega^2_{\text{AdS}} p^u p^v - \frac{l^2}{r^2}\right)
 \end{equation}

can be expressed as

\begin{equation}
 f(u, v; p^u, p^v, l) = \tilde{f}(u, v; p^u, p^v, l) \cdot \delta\left(\Omega^2_{\text{AdS}} p^u p^v - \frac{l^2}{r^2}\right),
 \end{equation}

where the smooth function $\tilde{f}(u, v; p^u, p^v, l)$ satisfies for any $\tau \geq 0$ and any $v \in [0, \sqrt{-\frac{3}{\Lambda}} \pi)$, $E > 0$ and $l \geq 0$ the relation

\begin{equation}
 \tilde{f}\left(U\{\tau\}, V\{\tau\}; \Omega^2_{\text{AdS}} (U\{\tau\}, V\{\tau\}) G^u\{\tau\}, \Omega^2_{\text{AdS}} (U\{\tau\}, V\{\tau\}) G^v\{\tau\}, l\right) = F(v, p^u[E, v], l).
 \end{equation}

where $U\{\tau\}$ is shorthand for $U\{\tau; v, E, l, sgn(p^u[E, v] - \frac{l^2}{r^2(\Omega^2_{\text{AdS}}(0, v)p^u[E, v])})\}$ (and similarly for $V\{\tau\}$, $G^u\{\tau\}$ and $G^v\{\tau\}$), while $p^u[E, v]$ (corresponding to $p^u$ along $u = 0$ at energy level $E$) is determined by the relation

\[\Omega^2_{\text{AdS}}(0, v)p^u + \frac{l^2}{r^2(0, v)p^u} = 2E.\]

**B Completeness of $I$ in the presence of a trapped sphere**

In this section, we will address the question of completeness of conformal infinity $I$ for the maximal future development of characteristic, asymptotically AdS initial data sets. In general, we will not be able to show that the
maximal future development of a smoothly compatible initial data set \( (r, \Omega_j^2, \bar{f}_j; v_I) \) (in accordance with Definition 3.5) has future complete conformal infinity \( \mathcal{I} \) i.e. satisfies

\[
\int_0^{u_\mathcal{X}} \frac{\Omega}{(1 - \frac{1}{3} \Lambda r^2)^2} (u, u + v_I) \, du = +\infty.
\]

However, we will be able to infer future completeness for \( \mathcal{I} \) in the presence of a trapped sphere; this result will be useful in our companion paper [16].

**Lemma B.1.** Let \( (U_{\text{max}}; r, \Omega^2, f) \) be the maximal future development of a smoothly compatible, asymptotically AdS initial data set \( (r, \Omega_j^2, \bar{f}_j; v_I) \) for \((2.43)-\(2.48)) with bounded support in phase space, in accordance with Definitions 3.4 and 3.5. Assume that there exists some \( (\bar{u}, \bar{v}) \in U_{\text{max}} \) such that

\[
2m r(\bar{u}, \bar{v}) > 1.
\]

Then the conformal infinity \( \mathcal{I} \) of \((U_{\text{max}}; r, \Omega^2, f) \) is future complete, i.e. (B.1) is satisfied.

**Proof.** Let \( u_{\gamma_z}, u_\mathcal{I} \in (0, +\infty] \) be the endpoint parameters of \( \gamma_z, \mathcal{I} \) and let \( \zeta \) be the (possibly empty) achronal future boundary of \( U_{\text{max}} \), defined as in Definition 4.1. As a consequence of the relation (2.10) and the fact that \( \partial_u r < 0 \) everywhere on \( \mathcal{U} \) (following from (2.46) and the fact that \( \partial_u r < 0 \) on \( \{ u = 0 \} \cup \mathcal{I} \)), for any point \((u_*, v_*) \in \mathcal{U} \) satisfying

\[
2m r(u_*, v_*) \geq 1,
\]

we can bound from above

\[
\partial_v r(u_*, v_*) < 0.
\]

Thus, integrating (2.45) along \( u = u_* \) starting from \((u_*, v_*) \), we infer that:

\[
\sup_{v \in v_*} \partial_v r(u_*, v) < 0.
\]

Therefore, the function \( r(u_*, \cdot) \) is bounded from above, which implies that

\[
\{ u = u_* \} \cap \mathcal{I} = \emptyset
\]

(since \( \mathcal{I} = \{ r = \infty \} \)) or, equivalently:

\[
u_* \geq u_\mathcal{I}.
\]

Since (B.5) holds for any \( u_* \) for which (B.3) is satisfied for some \( v_\mathcal{I} \), we infer that:

\[
2m r(u_*, v_*) < 1 \text{ for all } (u, v) \in \{ u < u_\mathcal{I} \} \cap \mathcal{U}.
\]

Since \((\bar{u}, \bar{v})\) satisfies (B.3) (in view of the assumption (B.2)), it also satisfies (B.5), i.e.

\[
\bar{u} \geq u_\mathcal{I}.
\]

In view of the fact that \( \zeta \) is achronal (yielding that \( u|\zeta \leq \max\{ u_\mathcal{I}, u_{\gamma_z} \} \)) and \((\bar{u}, \bar{v})\) is an interior point of \( U_{\text{max}} \), (B.7) also implies that

\[
\bar{u} < u_{\gamma_z}.
\]

\(^4\)The statement that for generic initial data, \( \mathcal{I} \) is future complete, is of course equivalent to the statement of the weak cosmic censorship conjecture in the asymptotically AdS settings for \((2.43)\) in spherical symmetry.
As a consequence of (B.7) and (B.8) and the fact that $\zeta$ is achronal, we therefore infer that, for any $\varepsilon > 0$, there exists some $\delta(\varepsilon) > 0$ such that

$$\left( (u_{\text{I}} - \delta(\varepsilon), u_{\text{I}} + \delta(\varepsilon)) \times (u_{\text{I}}, v_{\text{I}} + u_{\text{I}} - \varepsilon) \right) \cap \{ u < v \} \subset \mathcal{U}_{\text{max}}.$$  

(B.9)

The maximality of $(\mathcal{U}_{\text{max}}; r, \Omega^2, f)$ implies that $(\mathcal{U}_{\text{max}}; r, \Omega^2, f)$ cannot be extended as a smooth solution of (2.43)–(2.48) with smooth conformal infinity $\mathcal{I}$ beyond $u = u_{\text{I}}$ to any open set of the form $\{ u \leq u_{\text{I}} + \delta \} \cap \{ u < v < u + v_{\text{I}} \}$ for $\delta > 0$. This fact, together with the fact that (B.9) holds for any $\varepsilon > 0$, implies that the conditions of the extension principle of Proposition 5.2 fail to hold for $(\mathcal{U}_{\text{max}}; r, \Omega^2, f)$ at the boundary point at $(u_{\text{I}}, u_{\text{I}} + v_{\text{I}})$, i.e. there exists a sequence $\{(u_n, v_n)\}_{n \in \mathbb{N}} \subset \mathcal{U}_{\text{max}}$ with $u_n < u_{\text{I}}$ and $(u_n, v_n) \to (u_{\text{I}}, u_{\text{I}} + v_{\text{I}})$, such that

$$\lim_{n \to \infty} \frac{2m}{r} (u_n, v_n) \geq 1.$$  

(B.10)

In view of (B.6), the inequality (B.10) trivially reduces to the equality

$$\lim_{n \to \infty} \frac{2m}{r} (u_n, v_n) = 1.$$  

(B.11)

The relation (2.51) (which is well defined on $\{ u < u_{\text{I}} \} \cap \mathcal{U}$ in view of (B.6)) implies that

$$\partial_n \left( -\frac{\partial_n r}{1 - \frac{2m}{r}} \right) \geq 0.$$  

(B.12)

Integrating (B.12) over the triangle $\{ u \leq u_n \} \cap \{ v \geq v_n \} \cap \{ u > v - v_{\text{I}} \}$, we infer that

$$\int_{\max\{v_n - v_{\text{I}}, 0\}}^{u_n} -\frac{\partial_n r}{1 - \frac{2m}{r}} (u, u + v_{\text{I}}) du \geq \int_{\max\{v_n - v_{\text{I}}, 0\}}^{u_n} -\frac{\partial_n r}{1 - \frac{2m}{r}} (u, v_n) du$$  

(B.13)

for all $n \in \mathbb{N}$. Using the relation (2.49) between $m$ and $\tilde{m}$ and the fact that $\partial_n \tilde{m} \leq 0$ on $\{ u < u_{\text{I}} \} \cap \mathcal{U}$ (following from (2.53) and (B.6)), we can trivially estimate from below

$$\int_{v_n - v_{\text{I}}}^{u_{\text{I}}} -\frac{\partial_n r}{1 - \frac{2m}{r}} (u, v_n) du \geq \int_{v_n - v_{\text{I}}}^{u_{\text{I}}} -\frac{\partial_n r}{1 - \frac{2m}{r}} (u, v_n) du =$$  

$$= \int_{r(u_n, v_n)}^{+\infty} \frac{1}{1 - \frac{2\tilde{m}(u_n, v_n)}{r} - \frac{1}{3}\Lambda r^2} dr \geq$$  

$$\geq -\log \left( 1 - \frac{2m}{r}(u_n, v_n) \right) - C$$  

for some $C > 0$ independent of $n$. On the other hand, in view of the relations (2.10), (2.49) and the bound $\tilde{m}|_{\mathcal{I}} < \infty$ (following from the fact that the initial data $(r_I, \Omega^2, f; u_{\text{I}})$ were assumed to be of bounded support in phase space), we readily infer using the boundary condition (2.16) that

$$\left. -\frac{\partial_n r}{1 - \frac{2m}{r}} \right|_{\mathcal{I}} = \frac{1}{2} \frac{\Omega}{(1 - \frac{1}{3}\Lambda r^2)^{\frac{3}{2}}}.$$  

(B.15)

Using (B.14) and (B.15), the inequality (B.13) yields for any $n \in \mathbb{N}$

$$\int_{v_n - v_{\text{I}}}^{u_{\text{I}}} \frac{\Omega}{(1 - \frac{1}{3}\Lambda r^2)^{\frac{3}{2}}} (u, u + v_{\text{I}}) du \geq -\log \left( 1 - \frac{2m}{r}(u_n, v_n) \right) - C.$$  

(B.16)

Considering the limit $n \to \infty$ for (B.16) and using (B.11), we therefore infer (B.1).
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