On $C^r$–closing for flows on 2-manifolds.

Carlos Gutierrez
e-mail: gutp@impa.br
IMPA. Estrada Dona Castorina 110, J. Botânico,
22460-320, Rio de Janeiro, R.J., Brazil.

Abstract

For some full measure subset $B$ of the set of iet’s (i.e. interval exchange transformations) the following is satisfied:

Let $X$ be a $C^r$, $1 \leq r \leq \infty$, vector field, with finitely many singularities, on a compact orientable surface $M$. Given a nontrivial recurrent point $p \in M$ of $X$, the holonomy map around $p$ is semi-conjugate to an iet $E : [0,1) \to [0,1)$. If $E \in B$ then there exists a $C^r$ vector field $Y$, arbitrarily close to $X$, in the $C^r$–topology, such that $Y$ has a closed trajectory passing through $p$.

1 Introduction

The open problem “$C^r$-closing lemma” is stated as follows:

“Let $M$ be a smooth compact manifold, $r \geq 2$ be an integer, $f \in \text{Diff}^r(M)$ (resp. $X \in \mathcal{X}^r(M)$) and $p$ be a nonwandering point of $f$ (resp. of $X$). There exists $g \in \text{Diff}^r(M)$ (resp. $Y \in \mathcal{X}^r(M)$) arbitrarily close to $f$ (resp. to $X$) in the $C^r$–topology so that $p$ is a periodic point of $g$ (resp. of $Y$)”.

C. Pugh proved the $C^1$–closing lemma [Pg1]. There are few previous results when $r \geq 2$ : Gutierrez [Gu1] showed similar results to this paper when the manifold is the torus $T^2$. There are negative answers: Gutierrez [Gu3] proved that if the perturbation is localized in a small neighborhood of the nontrivial recurrent point, then $C^2$–closing is not always possible. C. Carroll’s [Car] proved that, even for flows with finitely many singularities, $C^2$–closing
by a twist-perturbation (supported in a cylinder) is not always possible. Concerning hamiltonian flows, M. Herman [Her] has remarkable counter-examples to the $C^r$—closing lemma. Within the context of geodesic laminations, S. Aranson and E. Zhuzhoma announced in 1988 [A-Z] the $C^r$—closing lemma for a class of flows on surfaces; however, their proofs have not been published yet. For basic definition the reader may consult [K-H].

2 Statement of the results

Throughout this article, $M$ will be a smooth, orientable, compact, two manifold and $\chi$ will be its Euler characteristic. We shall denote by $\mathcal{X}^r(M)$ the space of vector field of class $C^r$, $1 \leq r \leq \infty$, with the $C^r$—topology. The trajectory of $X \in \mathcal{X}^r(M)$ passing through $p \in M$ will be denoted by $\gamma_p$. The domain of definition of a map $S$ will be denoted by $\text{DOM}(S)$. Smooth segments on $M$ will be denoted and referred as (open, half-open, closed) intervals.

A bijective map $E : [0,1) \to [0,1)$ is said to be an iet, i.e. an Interval Exchange Transformation (with $m$ intervals) if there exists a finite sequence $0 = a_1 < a_2 < \cdots < a_m < a_{m+1} = 1$ such that, for all $i \in \{1,2,\cdots,m\}$ and for all $x \in [a_i,a_{i+1})$, $E(x) = E(a_i) + x - a_i$, and moreover, $E$ is discontinuous at exactly $a_2,a_3,\cdots,a_m$. This $E$ will be identified with the pair $(\lambda,\pi) \in \Delta_m \times \mathfrak{S}_m$ made up of the positive probability vector $\lambda = \{|a_{i+1} - a_i|\}^m_{i=1}$ and the permutation $\pi$ on the symbols $1,2,\cdots,m$, defined by $\pi(i) = \#\{j : E(a_j) \leq E(a_i)\}$. The space of iet’s, with $m$ intervals, defined in $[0,1)$, will be identified with the measurable space $\Delta_m \times \mathfrak{S}_m$ endowed with the product measure, where $\Delta_m$ is the simplex of positive probability vectors of $\mathbb{R}^m$, with Lebesgue measure, and $\mathfrak{S}_m$ is the finite set of permutations on $m$ symbols with counting measure. Let $E : [a,b) \to [a,b)$ be an iet. We say that $[s,t) \subset [a,b)$ is a virtual orthogonal edge for $E$, if $E$ restricted to $[s,t]$ is continuous and $s < E(s) < E^2(s) = t$. Given $k \in \mathbb{N}$, let $\mathcal{B}_k$ be the set of iet’s $E : [a,b) \to [a,b)$ such that for some sequence $b_n \to a$ of points of $(a,b)$, and for every $n \in \mathbb{N}$, the iet $E_n : [a,b_n) \to [a,b_n)$, induced by $E$, has at least $\chi + k + 3$, pairwise disjoint, virtual orthogonal edges. Denote $\mathcal{B} = \bigcap_{k \geq 1} \mathcal{B}_k$. It will be seen that, as a direct consequence of the work of W. A. Veech [Vee] and H. Masur [Mas],

**Theorem 2.1.** For all $m \geq 2$, $\Delta_m \times \mathfrak{S}_m \setminus \mathcal{B}$ is a measure zero set.

By transporting information along flow boxes, Item (a2) below follows from the definition of $\mathcal{B}_k$. 

2
Theorem 2.2. ([Gu2, Structure Theorem, Section 3]) Let \( X \in \mathfrak{X}^1(M) \). There are finitely many nontrivial recurrent trajectories \( \gamma_{p_1}, \gamma_{p_2}, \cdots, \gamma_{p_\ell} \) of \( X \) such that if \( \gamma_p \) is any nontrivial recurrent trajectory of \( X \), then \( \gamma_p = \gamma_{p_i} \), for some \( i = 1, 2, \cdots, \ell \).

Suppose that \( X \) has exactly \( K \in \mathbb{N} \) singularities (\( K=0 \) is allowed). Let \( p \in M \) be a nontrivial recurrent point of \( X \). Take a half-open interval \( [p, q) \subset M \) transversal to \( X \), such that \( p \) is a cluster point of \( \gamma_p \cap [p, q) \). Denote by \( P_X : [p, q) \to [p, q) \) the forward Poincaré map induced by \( X \). If \( [p, q) \) is small enough, it can be associated to \( (p, [p, q)) \), an iet \( E = E_{(p, [p, q))} : [0, 1) \to [0, 1) \) and a continuous monotone surjective map \( h : [p, q) \to [0, 1) \) such that \( h(p) = 0 \), \( h \) restricted to any given orbit of \( P_X \) is injective and, for all \( x \in \text{DOM}(P_X) \), \( E \circ h(x) = h \circ P_X(x) \); moreover,

(a1) there exists a subset \( S \subset [0, 1) \) of at most \( \chi + K + 2 \) elements such that if \( A \) is a connected component of \( [0, 1) \setminus S \), then \( h^{-1}(A) \) is contained in \( \text{DOM}(T) \);

(a2) Let \( \overline{\eta} \in \overline{\gamma}_p \) be a nontrivial recurrent point of \( X \) and \( (\eta, [\overline{\eta}, \overline{\eta}]) \) be a pair satisfying the same conditions as those of \( [p, [p, q)) \) above. Then the property that the iet \( E_{(\overline{\eta}, [\overline{\eta}, \overline{\eta}])} \) belongs to \( \mathcal{B}_K \) does not depend on \( (\overline{\eta}, [\overline{\eta}, \overline{\eta}) \).

Under conditions of theorem above and if \( E \in \mathcal{B}_K \), any nontrivial recurrent point of \( \overline{\gamma}_p \) is said to be of \( \mathcal{B}_K \)-type. Our result is the combination of Theorems 2.1 - 2.3.

Theorem 2.3. Let \( X \in \mathfrak{X}^r(M), \ 1 \leq r \leq \infty \), have \( K \geq 0 \) singularities. Let \( p \in M \) be a \( \mathcal{B}_K \)-type nontrivial recurrent point of \( X \). Then there exists \( Y \in \mathfrak{X}^r(M) \), arbitrarily close to \( X \), having a closed trajectory passing through \( p \).

Related to this theorem (see [Gu2]), we have that: For any \( E \in \mathcal{B} \), it can be constructed \( Y \in \mathfrak{X}^\infty(S) \), for some surface \( S \), having a nontrivial recurrent point \( p_0 \) such that item (a1) is satisfied for some \( h : [p_0, q_0) \to [0, 1) \), and \( P_Y : [p_0, q_0) \to [p_0, q_0) \). Here, \( P_Y \) can be obtained to be injective or not.

3 Proof of the results

Suppose that \( M \) is endowed with an orientation and with a smooth riemannian metric \( <,> \). Given a \( X \in \mathfrak{X}^r(M), \ 1 \leq r \leq \infty \), we define \( X^\perp \in \mathfrak{X}^r(M) \)
by the following conditions: (a) \( < X, X > = < X^\perp, X^\perp > \); and (b) when \( p \in M \) is regular point of \( X \), the ordered pair \((X(p), X^\perp(p))\) is an orthogonal positive basis of \( T_p(M) \) (according to the given orientation of \( M \)).

Let \( \Sigma \) be an arc of trajectory of \( X^\perp \). A \( \Sigma - \)flow-box (for \( X \)) is a compact subset \( F \subset M \) whose interior is a flow box of \( X \) and whose boundary \( \partial F \) is a graph, homeomorphic to the figure “8”, which is the union of arcs of trajectory \([\overline{a}, \overline{c}] \) and \([\overline{c}, \overline{a}] \) (connecting \( \overline{a} \) and \( \overline{c} \)) of \( X \) and \( X^\perp \), respectively. We shall refer to \([\overline{a}, \overline{c}] \) (resp. \([\overline{c}, \overline{a}] \)) as the orthogonal (resp. tangent) edge of either \( F \) or \( \partial F \).

Let \( X \in \mathcal{X}^r(M), 1 \leq r \leq \infty \), and let \( p \in M \) be a nontrivial recurrent point of \( X \). We say that \( X \) is T-closable at \( p \) (i.e. twist-closable at \( p \)) if there exists a half-open interval \( \Sigma = [p, q) \) tangent to \( X^\perp \), such that, for any neighborhood \( V \) of \( p \), there exists a \( \Sigma - \)flow-box for \( X \) having its orthogonal edge contained in \( \Sigma \cap V \).

**Proposition 3.1.** Let \( X \in \mathcal{X}^r(M), 1 \leq r \leq \infty \), and let \( p \in M \) be a nontrivial recurrent point of \( X \). Suppose that \( X \) is T-closable at \( p \). Then there are sequences \( t_n \to 0 \), of real numbers, and \( p_n \to p \), of points of \( M \), such that \( X + t_n X^\perp \) has a closed trajectory through \( p_n \)

**Proof:** As \( X \) is T-closable at \( p \), there exists a half-open interval \( \Sigma = [p, q) \) tangent to \( X^\perp \), such that, given neighborhoods \( V \) of \( X \) and \( V \) of \( p \), we may choose a \( \Sigma - \)flow-box \( F \subset M \) (for \( X \)) and \( \sigma > 0 \) such that if \([\overline{a}, \overline{c}] \) and \([\overline{c}, \overline{a}] \) are the tangent and orthogonal edges, respectively, of \( \partial F \), and \( \overline{b} \) is the vertex of \( \partial F \), then:

1. \( [\overline{a}, \overline{c}] \subset V \) and the flow of \( X \) enters into \( F \) through the closed subinterval \([\overline{b}, \overline{c}] \) of \( \Sigma \); moreover, for all \( t \in [-\sigma, \sigma] \), \( X(t) := X + t X^\perp \in V \);
(b2) both $X(\sigma)$ and $X(-\sigma)$ have an arc of trajectory contained in $F$, which is a global cross section for $X|_F$.

We shall continue considering only the case in which the flow of $X^\perp$ goes from $\overline{a}$ to $\overline{b}$. Let $\Gamma$ be the set of real numbers $s \in [0, \sigma]$ such that when $t \in [0, s]$ there is an arc of trajectory $[\overline{b}, \overline{a}(t)]_{X(t)}$ of $X(t)$, joining $\overline{b}$ with $\overline{a}(t) \in [\overline{a}, \overline{b}]$, contained in $F$, with $\overline{a}(0) = \overline{a}$, and such that $\overline{a}(t)$ depends continuously on $t$. When $t \in \Gamma$, these conditions determine $\overline{a}(t)$ and also that $[\overline{b}, \overline{a}(t)]_{X(t)}$ is transversal to $X$. Therefore, by (b2), $\Gamma = [0, \sigma_1]$ is a closed interval, $\overline{a}(\sigma_1) = b$ and $[\overline{b}, \overline{a}(\sigma_1)]_{X(\sigma_1)}$ is a closed trajectory of $X(\sigma_1)$. See Fig. 1.b

Under the assumptions and conclusions of this proposition, there exists a sequence $F_n : M \to M$ of $C^r$–diffeomorphisms, taking $p_n$ to $p$. We may assume that $F_n$ converges to the identity diffeomorphism in the $C^{r+1}$–topology. Therefore, the sequence of vector fields $(F_n)_*(X + t_nX^\perp) \to X$ in the $C^r$–topology and each $(F_n)_*(X + t_nX^\perp)$ has a closed trajectory passing through $p$. This proves the following

**Theorem 3.2.** Let $X \in \mathcal{X}^r(M)$, $1 \leq r \leq \infty$. Let $p \in M$ be a nontrivial recurrent point of $X$. Suppose that that $X$ is $T$-closable at $p$. Then there exists $Y \in \mathcal{X}^r(M)$ arbitrarily close to $X$ having a closed trajectory through $p$.

**Proof of Theorem 2.1:** We shall prove that: For all $m \geq 2$, $\Delta_m \times \mathcal{G}_m \setminus \mathcal{B}$ is a measure zero set. It was proved by W. A. Veech [Vee] and H. Masur [Mas] that the Rauzy operator $\mathcal{R} : \mathcal{M} \to \mathcal{M}$, defined in a full measure subset $\mathcal{M}$ of $\Delta_m \times \mathcal{G}_m$, is ergodic and has the following property:

(c) Given $E \in \mathcal{M}$, there exists a sequence $\{[0, a_n]\}$ of subintervals of $[0, 1)$ such that $a_n \to 0$ and, if $\bar{E}_n : [0, a_n) \to [0, a_n)$ denote the iet induced by $E$, then, up to re-scaling $\mathcal{R}^n(E)$ coincides with $\bar{E}_n$; more precisely, $\mathcal{R}^n(E)(z) = (1/a_n)\bar{E}_n(a_nz)$, for all $z \in [0, 1)$.

Given $k \geq 1$, let $A_k$ be the set of $E \in \Delta_m \times \mathcal{G}_m$ such that for some $a \in (16^{-k} - 32^{-k}, 16^{-k} + 32^{-k})$, $E(x) = a + x$, for all $x \in [0, 1/2]$. We observe that $A_k$ is open and so it has positive measure. Let $\mathcal{B}_k$ be the set of $E \in \mathcal{M}$ such that the positive $\mathcal{R}$–orbit of $E$ visits $A_k$ infinitely many often. As $A_k$ has positive measure and $\mathcal{R}$ is ergodic, the complement of $\mathcal{B}_k$ has measure zero. Therefore, the complement of $\mathcal{B} = \cap_{k \geq 2} \mathcal{B}_k$ has measure zero. Observe that if and iet $E \in A_k$, then $E$ has more than $k$, pairwise disjoint, virtual orthogonal edges. Therefore, as $\mathcal{R}$ satisfy (c) right above and since the positive $\mathcal{R}$–orbit
of any given $E \in \tilde{B}$ visits every $A_k$ infinitely many often, we obtain that $\tilde{B} \subset B$. This proves the theorem.

**Proof of Theorem 2.3:** This theorem is stated as follows: Let $p \in M$ be a $B_K$-type nontrivial recurrent point of $X \in \mathcal{X}^r(M)$, $1 \leq r \leq \infty$. Suppose that $X$ has $K \geq 0$ singularities. Then there exists a $Y \in \mathcal{X}^r(M)$ arbitrarily close to $X$, having a closed trajectory passing through $p$.

By theorem 3.2, it is enough to prove that $X$ is $T$-closable at $p$. Let $\Sigma = [p, q)$, $T : [p, q) \to [p, q)$, $E : [0, 1) \to [0, 1)$, $h : [p, q) \to [0, 1)$ be as in Theorem 2.2. As $E \in B_K$, given a neighborhood $V$ of $p$, there exist $b \in (0, 1)$ and an $\text{tet } E_V : [0, b) \to [0, b)$, such that:

(e) $E_V$ has at least $\chi + K + 3$ pairwise disjoint virtual orthogonal edges contained in $[0, b)$; moreover, the interval $\Sigma_V = h^{-1}([0, b))$ is contained in $V$.

Let $T_V : \Sigma_V \to \Sigma_V$ be the map induced by $T$. As $X$ has $K$ singularities, (e) and Theorem 2.2 imply that $E_V$ has a virtual orthogonal edge $[a, E_V(a)] \subset [0, b)$ such that, for some $\overline{a} \in \text{DOM}((T_{\Sigma_V}))$, $[\overline{a}, T_V(\overline{a})] = h^{-1}([a, E(a)]) \subset \text{DOM}(T|_{\Sigma_V})$. Therefore, there exists a $\Sigma$-flow-box bounded by $[\overline{a}, T_V^2(\overline{a})] \cup [\overline{a}, T_V^2(\overline{a})]_X$. As $V$ is arbitrary, this proves that $X$ is $T$-closable at $p$. □

**References**

[A-Z] S. Aranson and E. Zhuzhoma. *On the $C^r$—closing lemma on surfaces*. Russian Math. Surv., 43, 1988, 5, 209-210.

[Car] C. Carroll. *Rokhlin towers and $C^r$ closing for flows on $T^2$*. Erg. Th. and Dynam. Sys., 12, 1992, 683-706.

[Gu1] C. Gutierrez. *On the $C^r$—closing lemma for flows on the torus $T^2$*. Erg. Th. and Dyn. Sys. (1986), 6, 45-56.

[Gu2] C. Gutierrez. *Smoothing continuous flows on two-manifolds and recurrences*. Erg. Th. and Dyn. Sys. (1986), 6, 17-44.

[Gu3] C. Gutierrez. *A counter-example to a $C^2$—closing lemma*. Erg. Th. and Dyn. Sys. (1987), 7, 509-530.
M. Herman. Exemples de flots hamiltoniens dont aucune perturbation en topologie $C^\infty$ n’a d’orbites périodiques sur un ouvert de surfaces d’énergies. C. R. Acad. Sci. Paris, t. 312, Série I (1991) 989-994.

A. Katok and B. Hasselblatt Introduction to the Modern Theory of Dynamical Systems. Cambridge University Press, New York (1995).

H. Masur. Interval exchange transformations and measured foliations. Ann. Math. 115 (1982), 169-200.

C. Pugh. An improved closing lemma and a general density theorem. Amer. Jour. math., 89 (1967), 1010-1021.

W. Veech. Gauss measures for transformations on the space of interval exchange maps. Ann. Math. 115 (1982), 201-242.