A Bayesian Nonparametric Estimation to Entropy

Luai Al-Labadi∗1, Vishakh Patel†1, Kasra Vakiloroayaei‡1, and Clement Wan§1

1Department of Mathematical and Computational Sciences, University of Toronto Mississauga, Mississauga, Ontario L5L 1C6, Canada.

Abstract

A Bayesian nonparametric estimator to entropy is proposed. The derivation of the new estimator relies on using the Dirichlet process and adapting the well-known frequentist estimators of Vasicek (1976) and Ebrahimi, Pflugheft and Soofi (1994). Several theoretical properties, such as consistency, of the proposed estimator are obtained. The quality of the proposed estimator has been investigated through several examples, in which it exhibits excellent performance.

KEYWORDS: Bayesian non-parametric, Dirichlet process, Entropy, Kullback-Leibler divergence, Model checking.

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∗Corresponding author: luai.allabadi@utoronto.ca
†vishakh.patel@mail.utoronto.ca
‡k.vakiloroayaei@mail.utoronto.ca
§cmclement.wan@mail.utoronto.ca
1 Introduction

The concept of (differential) entropy was introduced in Shannon (1948). Since then, entropy has been one of the most interesting areas with endless applications in many fields such as thermodynamics, communication theory, computer science, biology, economic, mathematics and statistics (Cover and Thomas, 2006). The entropy of a continuous cumulative distribution function (cdf) $P$ with a probability density function $p$ (with respect to Lebesgue measure) is defined as

$$H(P) = - \int_{-\infty}^{\infty} p(x) \log p(x) dx = -E_P \left[ \log p(x) \right]. \quad (1)$$

From practical perspective, one must estimate (1) from the data, which is not a trivial task. Various frequentist procedures for the estimation of entropy are offered in the literature. Among several estimators, due to its simplicity, Vasicek’s (1976) estimator has been the most common and the widely used one. Vasicek (1976) noticed that (1) can be written as

$$H(P) = - \int_{0}^{1} \log \left( \frac{d}{dt} P^{-1}(t) \right) dt.$$ 

Thus, $H(P)$ is estimated by using estimates of the derivative of inverse of the distribution function on the sample points. Specifically, if $x = (x_1, \ldots, x_n)$ is a sample from a distribution $P$, then, at each sample point $x_i$, the derivative of $P^{-1}(t)$ is estimated by the slope defined by

$$\frac{x(i+m) - x(i-m)}{F_n(x(i+m)) - F_n(x(i-m))} = \frac{x(i+m) - x(i-m)}{\frac{i+m}{n} - \frac{i-m}{n}} = \frac{x(i+m) - x(i-m)}{2m/n}, \quad (2)$$

where $F_n$ is the empirical distribution function. Consequently, Vasicek (1976)
estimator is given by

$$H_{m,n}^V = n^{-1} \sum_{i=1}^{n} \log \left( \frac{x_{(i+m)} - x_{(i-m)}}{2m/n} \right),$$

(3)

where $m$, called the window size, is a positive integer smaller than $n/2$ and $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ are the order statistics of $x_1, x_2, \ldots, x_n$ with $x_{(i)} = x_{(1)}$ if $i < 1$, $x_{(i)} = x_{(n)}$ if $i > n$. Vasicek (1976) showed that $H_{m,n}^V \overset{P}{\to} H(P)$, where $\overset{P}{\to}$ denotes convergence in probability. Ebrahimi, Pflughoft and Soofi (1994) noticed that (3) does not give the correct formula for the slope when $i \leq m$ or $i \geq n - m + 1$. They proposed the following modification to (3):

$$H_{m,n}^{EPS} = n^{-1} \sum_{i=1}^{n} \log \left( \frac{x_{(i+m)} - x_{(i-m)}}{c_i m/n} \right),$$

(4)

where

$$c_i = \begin{cases} \frac{m+i-1}{m} & 1 \leq i \leq m \\ 2 & m+1 \leq i \leq N - m \\ \frac{N+m-i}{m} & N - m + 1 \leq i \leq N \end{cases}.$$  

(5)

They also showed that $H_{m,N}^{EPS} \overset{P}{\to} H(P)$.

Other nonparametric frequentist estimators of entropy includes, among others, the work of van Es (1992), Correa (1995), Wieczorkowski and Grzegorzewski (1999), Alizadeh Noughabi (2010), Alizadeh Noughabi and Arghami (2010), Bouzebda, Elhattab, Keziou and Louinis (2013) and Al-Omari (2014, 2016). We refer the reader for the work of Beirlant, Dudewicz, Györi and van der Meulen (1997) for a comprehensive review for nonparametric entropy estimators.

On the other hand, Bayesian estimation of entropy has not received much attention. Exceptions include the work of Mazzuchi, Soofi and Soyer (2000, 2008), who develop a Bayes estimate of $H(P)$ based on the Dirichlet process.
(Furguson, 1973) and provided a computational algorithm for their procedure. The main goal of this paper is to derive an efficient and easy-to-implement Bayesian nonparametric estimator of \((\text{1})\). The anticipated estimator may be viewed as the Bayesian nonparametric counterpart of the estimator of Ebrahimi, Pfughoeft and Soofi (1994). A main motive of having this estimator, among others, is to use it in Bayesian methods such as model checking as discussed, for instance, in Al-Labadi and Evans (2018). Therefore, it can be worthwhile to have such an estimator in practice.

The remainder of this paper is organized as follows. In Section 2, the Dirichlet process prior is briefly reviewed. In Section 3, a Bayesian non-parametric estimator of the entropy is obtained and several of its properties are derived. Section 4 develops a computational algorithm of the approach, where particular choices of \(m\) and the hyperparameters of the Dirichlet process should be used. Section 5 presents a number of examples where the behavior of the estimator is examined in some detail. A comparison between the new estimator, the estimator of Vasicek’s estimator and the estimator of Ebrahimi, Pfughoeft and Soofi is also considered. Section 6 ends with a brief summary of the results. Proofs are placed in the Appendix.

1.1 Dirichlet process

A relevant summary of the Dirichlet process is presented in this section. The Dirichlet process, formally introduced in Ferguson (1973), is considered the most well-known and widely used prior in Bayesian nonparametric inference. Let \(\mathcal{X}\) be a space and \(\mathcal{A}\) be a \(\sigma\)-algebra of subsets of \(\mathcal{X}\). Let \(G\) be a fixed probability measure on \((\mathcal{X}, \mathcal{A})\), called the base measure, and \(a\) be a positive number, called the concentration parameter. Following Ferguson (1973), a random probability measure \(P = \{P(A)\}_{A \in \mathcal{A}}\) is called a Dirichlet process on \((\mathcal{X}, \mathcal{A})\) with
parameters $a$ and $G$, denoted by $DP(a,G)$, if for any finite measurable partition $\{A_1, \ldots, A_k\}$ of $X$ with $k \geq 2$, $(P(A_1), \ldots, P(A_k)) \sim \text{Dirichlet}(aG(A_1), \ldots, aG(A_k))$. It is assumed that if $G(A_j) = 0$, then $P(A_j) = 0$ with a probability one. For any $A \in \mathcal{A}$, $P(A) \sim \text{Beta}(aG(A), a(1-G(A)))$ and so $E(P(A)) = G(A)$ and $\text{Var}(P(A)) = G(A)(1-G(A))/(1+a)$. Thus, $G$ can be viewed as the center of the process. On the other hand, $a$ controls concentration, as the larger value of $a$, the more likely that $P$ will be close to $G$.

An important feature of the Dirichlet process is the conjugacy property. Specifically, if $x = (x_1, \ldots, x_n)$ is a sample from $P \sim DP(a,G)$, then the posterior distribution of $P$ is $P|x = P_x \sim DP(a+n,G_x)$ where

$$G_x = a(a+n)^{-1}G + n(a+n)^{-1}F_n,$$  \hspace{1cm} (6)

with $F_n = n^{-1} \sum_{i=1}^n \delta_{x_i}$, and $\delta_{x_i}$, the Dirac measure at $x_i$. Notice that, $G_x$ is a convex combination of the prior base distribution and the empirical distribution. Clearly, $G_x \to G$ as $a \to \infty$ while $G_x \to F_n$ as $a \to 0$. On the other hand, by Glivenko-Cantelli theorem, when $a/n \to 0$ (i.e., $a$ is small comparable to $n$), $G_x$ converges to true distribution function. We refer the reader to Al-Labadi and Zarepour (2013a,b; 2014a) and Al-Labadi and Abdelrazeq (2017) for other asymptotic properties of the Dirichlet process.

Following Ferguson (1973), $P \sim DP(a,G)$ has the following series representation

$$P = \sum_{i=1}^{\infty} J_i \delta_{Y_i},$$  \hspace{1cm} (7)

where $\Gamma_i = E_1 + \cdots + E_i$, $E_i \overset{i.i.d.}{\sim} \text{exponential}(1)$, $Y_i \overset{i.i.d.}{\sim} G$ independent of $\Gamma_i$, $L(x) = a \int_{x}^{\infty} t^{-1}e^{-t}dt, x > 0$, $L^{-1}(y) = \inf\{x > 0 : L(x) \geq y\}$ and $J_i = L^{-1}(\Gamma_i)/\sum_{i=1}^{\infty} L^{-1}(\Gamma_i)$. It follows clearly from (7) that a realization of the Dirichlet process is a discrete probability measure. This is correct even when $G$
is absolutely continuous. We refer to $(Y_i)_{i \geq 1}$ and $(J_i)_{i \geq 1}$ as the atoms and the weights, respectively. Note that, one could resemble the discreteness of $P$ with the discreteness of $F_n$. Since data is always measured to finite accuracy, the true distribution being sampled from is discrete. This makes the discreteness property of $P$ with no practical significant limitation.

Because there is no closed form for the inverse of Lévy measure $L(x)$, using Ferguson (1973) representation of the Dirichlet process is difficult in practice. As an alternative, Sethuraman (1994) uses the stick-breaking approach to define the Dirichlet Process. Let $(\beta_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with a Beta$(1, \alpha)$ distribution. In (7), set

$$J_1 = \beta_1, \ J_i = \beta_i \prod_{k=1}^{i-1} (1 - \beta_k), \ i \geq 2.$$  

(8)

and $(Y_i)_{i \geq 1}$ independent of $(\beta_i)_{i \geq 1}$. Unlike Ferguson’s approach, the stick-breaking construction does not need normalization. By truncating the higher order terms in the sum to simulate Dirichlet process, we can approximate the Sethuraman stick breaking representation by

$$P_N = \sum_{i=1}^{N} J_{i,N} \delta_{Y_i}.$$  

In here, $(\beta_i)_{i \geq 1}$, $(J_{i,N})_{i \geq 1}$, and $(Y_i)_{i \geq 1}$ are as defined in (8) with $\beta_N = 1$. The assumption that $\beta_N = 1$ is necessary to make the weights add up to 1 almost surely (Ishwaran and James, 2001).

The Dirichlet process can also be obtained from the following finite mixture models developed by Ishwaran and Zarepour (2002). Let $P_N$ has the from given in (7) with $(J_{1,N}, \ldots, J_{N,N}) \sim \text{Dirichlet}(a/N, \ldots, a/N)$. Then $E_{P_N}(g) \to E_P(g)$ in distribution as $N \to \infty$, for any measurable function $g : \mathbb{R} \to \mathbb{R}$ with $\int_{\mathbb{R}} |g(x)|H(dx) < \infty$ and $P \sim DP(a, G)$. In particular, $(P_N)_{N \geq 1}$ converges
in distribution to \( P \), where \( P_N \) and \( P \) are random values in the space \( M_1(\mathbb{R}) \) of probability measures on \( \mathbb{R} \) endowed with the topology of weak convergence.

To generate \( (J_{i,N})_{1 \leq i \leq N} \) put \( J_{i,N} = G_{i,N} / \sum_{i=1}^{N} G_{i,N} \), where \( (G_{i,N})_{1 \leq i \leq N} \) is a sequence of i.i.d. gamma\((a/N,1)\) random variables independent of \((Y_i)_{1 \leq i \leq N}\).

For other simulation methods for the Dirichlet process, see Bondesson (1982), Wolpert and Ickstadt (1998) and Zarepour and Al-Labadi (2012), Al-Labadi and Zarepour (2014b).

## 2 Bayesian Estimation of the Entropy

Let \( P_N = \sum_{i=1}^{N} J_{i,N} \delta_{Y_i} \). Similar to (2), the slope of the straight line that joins the two points \( \left( P_N(Y_{i-m}) = \sum_{k=1}^{i-m} J_{k,N}, Y_{(i-m)} \right) \) and \( \left( P_N(Y_{(i+m)}) = \sum_{k=1}^{i+m} J_{k,N}, Y_{(i+m)} \right) \) is

\[
\frac{Y_{(i+m)} - Y_{(i-m)}}{P_N(Y_{(i+m)}) - P_N(Y_{(i-m)})} = \frac{Y_{(i+m)} - Y_{(i-m)}}{c_{i,a}},
\]

where

\[
c_{i,a} = \begin{cases} 
\sum_{k=2}^{i+m} J_{k,N} & 1 \leq i \leq m \\
\sum_{k=i-m+1}^{i+m} J_{k,N} & m + 1 \leq i \leq N - m \\
\sum_{k=i-m+1}^{N} J_{k,N} & N - m + 1 \leq i \leq N 
\end{cases}
\]

(9)

Note that, from the properties of the Dirichlet distribution, we have \( J_{i,N} \sim \text{Beta}(a/N,a(1-1/N)) \). Thus, \( E(J_{i,N}) = N^{-1} \). Hence,

\[
E[c_{i,a}] = \begin{cases} 
\frac{m+i-1}{N} & 1 \leq i \leq m \\
\frac{2m}{N} & m + 1 \leq i \leq N - m \\
\frac{N+i-m}{N} & N - m + 1 \leq i \leq N 
\end{cases}
\]

\[
= \frac{m}{N} c_i,
\]
where \( c_i \) is defined in (5). The next proposition underlines a direct connection between \( c_{i,a} \) and \( c_i \). Its proof is given in the Appendix.

**Proposition 1** Let \( (J_{i,N})_{1 \leq i \leq N} \sim \text{Dirichlet}(a/N, \ldots, a/N) \). As \( N \to \infty \),

1. \( J_{i,N} - 1/N \overset{p}{\to} 0 \)

2. \( c_{i,a} - \frac{m}{N} c_i \overset{p}{\to} 0 \), where \( c_{i,a} \) and \( c_i \) are defined in (9) and (4), respectively.

Proposition 1 motivates the possibility of constructing a Bayesian non-parametric version of (5) based on the Dirichlet process. The precise form of the anticipated estimator to (1) is presented in the next lemma. The proof is placed in the Appendix.

**Lemma 2** Let \( P_N = \sum_{i=1}^{N} J_{i,N} \delta_{Y_i} \) as defined in Section 1.1, where \( Y_1, Y_2, \ldots, Y_N \overset{i.i.d.}{\sim} G \). Let \( m \) be a positive integer smaller than \( N/2 \), \( Y_{(i)} = Y_{(1)} \) if \( i < 1 \), \( Y_{(i)} = Y_{(N)} \) if \( i > N \) and \( Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(N)} \) are the order statistics of \( Y_1, Y_2, \ldots, Y_N \). Let

\[
H_{m,N,a} = \frac{1}{N} \sum_{i=1}^{N} \log \left( \frac{Y_{(i+m)} - Y_{(i-m)}}{c_{i,a}} \right),
\]

where \( c_{i,a} \) is defined in (9). As \( N \to \infty \), \( m \to \infty \), \( m/N \to 0 \) and \( a \to \infty \), we have

\[
E[H_{m,N,a}] - E[H_{m,N}^{EPS}] \to 0,
\]

where \( H_{m,N}^{EPS} \) is defined in (4).

The next lemma shows that the estimator defined in (10) is consistent. The defined a Bayesian nonparametric prior for the entropy. A formal proof is given in the Appendix.
Lemma 3 Let \( H_{m,N,a} \), \( N, m, a \) and \( G \) be as defined in Lemma 2. Then as \( N \to \infty, m \to \infty, m/N \to 0 \) and \( a \to \infty \), we have

\[
H_{m,N,a} \overset{P}{\to} H(G) = -\int_{-\infty}^{\infty} g(x) \log g(x) \, dx,
\]

where \( G'(x) = g(x) \).

3 Computations and the Choices of \( m, a \) and \( G \)

Let \( x = (x_1, \ldots, x_n) \) be a sample from a continuous distribution \( P \). The aim is to approximate \( H(P) \) as defined in (1). We will use the prior \( P \sim DP(a, G) \) for some choice of \( a \) and \( G \) so \( P|x \sim DP(a + n, G_x) \). See Section 2.

To fully implement the approximation \( H_{m,N,a} \) as in Lemma 3, it is necessary to discuss the choices for \( m, a \) and \( G \). We start by the choice of \( m \), where its optimal value is still an open problem in entropy estimation. However, as discussed in Vasicek (1976), with increasing \( N \), the best value of \( m \) increases while the ratio \( m/N \) tends to zero. For example, for \( N = 10, 20, 50 \), Vasicek (1976) recommended using \( m = 2, 3, 4 \), respectively. On the other hand, Grzegorzewski and Wieczorkowski (1999) proposed the following formula for optimal values of \( m \):

\[
m = \lfloor \sqrt{N} + 0.5 \rfloor,
\]

where \( \lfloor y \rfloor \) is the largest integer less than or equal to \( y \). Thus, by (11), for \( N = 10, 20, 50 \), the best choices of \( m \) are 2, 3, 7, respectively. In this paper, we will use the rule (11). Note that, the value of \( m \) in (11) is the value that will be used for the prior. For the posterior, one should replace \( N \) by the number of distinct atoms in \( P_N|x \), an approximation of \( P|x \). Observe that, it follows from
(6) that if $a/n$ is close to zero, then the number of distinct atoms in $P_N | x$ will typically be $n$.

As for hyperparameters $a$ and $G$, their choices depend on the application of interest. For instance, for model checking, to detect small deviations, $a$ and $G$ should be selected so that there is a good concentration about the prior (Al-Labadi and Evans, 2018). Further, they recommended that $a$ should be chosen so that its value does not exceed $0.5n$ as otherwise the prior may become too influential. In light of this under the context of entropy estimation, any choice of $a$ such that $a/n$ is close to zero should be compatible with any choice of $G$. This follows from (6) as the sample will dominate the prior guess $G$. For example, setting $a = 0.05$ and $n = 10$ in (6) gives

$$G_x = 0.005G + 0.995F_n,$$

which means the chance to draw a sample from the collected data is $99.5\%$ over a new sample from $G$. For simplicity, we suggest to set $G = N(0, 1)$ and $a = 0.05$, although other choices are certainly possible. An example studying the sensitivity of the approach to the choice of $G$ is covered in Section 4.

The following result shows that, as the sample size increases (i.e., the concentration parameter $a$ is small comparable to the sample size $n$), then the posterior of $H_{m,N,a}$ (i.e., the proposed estimator) converges in probability to (1). The proof follows from (6), Glivenko-Cantelli theorem and Lemma 3.

**Lemma 4** Let $x = (x_1, \ldots, x_n)$ be a sample from $P \sim DP(a, G)$. Let $H_{m,N,a}$ be as defined in Lemma 2. Then as $N \to \infty$, $m \to \infty$, $n \to \infty$, $m/N \to 0$ and $a/n \to 0$

$$H_{m,N,a}|x \xrightarrow{p} H(P) = - \int_{-\infty}^{\infty} p(x) \log p(x) dx.$$
Now, based on Lemma 4, the following gives a computational algorithm for estimating (1).

**Algorithm A (Nonparametric Estimation of Entropy):**

(i) Let \( P \sim DP(a, G) \) and \( P_N \) be an approximation of \( P \). Set \( a = 0.05 \) and \( G = N(0, 1) \).

(ii) Generate a sample from \( P_N|x \), where \( P_N|x \) is an approximation of \( P|x \sim DP(a + n, G_x) \). See Section 2.

(iii) Compute \( H_{m,N,a}|x \) as in Lemma 4.

(iv) Repeat steps (i) and (iii) to obtain a sample of \( r \) values from \( H_{m,N,a}|x \).

For large \( r \), the empirical distribution of these values is an approximation to the distribution of \( H_{m,N,a}|x \).

(v) The average of the \( r \) values generated in step (iv) will be the estimator of the entropy.

Note that, for estimation purposes, the prior has no significant role. This is not necessarily will be the case for other applications such as model checking.

4 Examples

In this section, we study the behaviour of the proposed estimator in terms of efficiency and robustness. The proposed estimator is compared with its non-Bayesian counterpart estimators of Vasicek (1976) and Ebrahimi, Pflughoeft and Soofi (1994). Additionally, for a comprehensive comparison, we included the (weighted) KozachenkoLeonenko entropy estimator (Kozachenko and Leonenko, 1987; Berrett, Samworth and Yuan, 2018), which is based on the \( k \)-nearest neighbour distances of the sample. We set the value of \( k \) to equal to
This value of \( k \) (square root of the sample size) is recommended, for instance, by Mitra, Murthy and Pal (2002) and Bhattacharya, Ghosh, and Chowdhur (2012). For each sample size \( (n = 10, 20, 50) \), 1000 samples were generated. We have considered four distributions: exponential with mean 1 (exact entropy is 1), Uniform on \((0,1)\) (exact entropy is 0), \( N(0,1) \) (exact entropy is \( 0.5 \log(2\pi e) \)) and Weibull distribution with shape parameter equal to 2 and scale parameter equal to 0.5 (exact value of \(-0.0977\)). The estimators and their mean squared errors are computed. Here each sample of the 1000 samples gives an estimate. The reported value of the estimator (Est) is the average of the 1000 estimates. On the other hand, the mean squared error (MSE) is computed as follows: \( (\text{estimated value for each sample} - \text{true value})^2/1000 \). The computing program codes were implemented in the programming language \( R \). For the KL entropy estimator, we used the package \textbf{IndepTest} (Berrett, Grose and Samworth, 2018). In all cases, the prior was taken to be \( DP(a, N(0,1)) \). In Algorithm A, we set \( r = 1000 \) and \( N = 200 \). The sensitivity to the choice of \( a \) is investigated and we record only a few values in the tables.

| \( n \) | \( m \) | \( a \) | \( H_{m,N,a|x} \) \( \text{Est}(\text{MSE}) \) | \( H_{m,n,a}^1 \) \( \text{Est}(\text{MSE}) \) | \( H_{m,n,a}^{EP} \) \( \text{Est}(\text{MSE}) \) | KL Entropy \( \text{st}(\text{MSE}) \) |
|---|---|---|---|---|---|---|
| 10 | 3 | 0.05 | -0.017(0.017) | -0.410(0.193) | -0.154(0.048) | 0.073(0.080) |
| 1 | | | 0.517(0.275) | | | |
| 5 | | | 0.767(0.593) | | | |
| 20 | 4 | 0.05 | -0.050(0.010) | -0.260(0.077) | -0.102(0.017) | 0.038(0.035) |
| 1 | | | 0.418(0.179) | | | |
| 5 | | | 0.657(0.435) | | | |
| 50 | 7 | 0.05 | -0.051(0.004) | -0.150(0.024) | -0.051(0.004) | 0.014(0.013) |
| 1 | | | 0.230(0.055) | | | |
| 5 | | | 0.496(0.247) | | | |

It follows clearly from Table 1 - Table 4 that, when \( a = 0.05 \), the new approximation of entropy has the lowest mean squared error for most cases. As illustrated in Section 3, the choice of \( a \) is extremely important and for the
| Table 2: Exponential with mean 1. | $H_{m,N,a}$ | $H_{m,n,a}$ | $H_{m,n,a}^{EPS}$ | KL Entropy |
|---|---|---|---|---|
| $n$ | $m$ | $a$ | Est(MSE) | Est(MSE) | Est(MSE) | Est(MSE) |
| 10 | 3 | 0.05 | 0.891(0.114) | 0.575(0.298) | 0.831(0.146) | 0.947(0.149) |
| 1 | 0.105(0.073) | 0.575(0.068) |
| 5 | 0.126(0.075) |
| 20 | 4 | 0.05 | 0.937(0.052) | 0.752(0.112) | 0.910(0.059) | 0.967(0.069) |
| 1 | 0.148(0.066) | 0.75(0.061) |
| 5 | 0.128(0.075) |
| 50 | 7 | 0.05 | 0.956(0.022) | 0.864(0.039) | 0.964(0.022) | 0.967(0.026) |
| 1 | 0.137(0.039) | 0.864(0.039) |
| 5 | 0.122(0.068) |

| Table 3: $N(0,1)$. | $H_{m,N,a}$ | $H_{m,n,a}$ | $H_{m,n,a}^{EPS}$ | KL Entropy |
|---|---|---|---|---|
| $n$ | $m$ | $a$ | Est(MSE) | Est(MSE) | Est(MSE) | Est(MSE) |
| 10 | 3 | 0.05 | 1.112(0.159) | 0.869(0.374) | 1.125(0.158) | 1.293(0.124) |
| 1 | 0.137(0.087) | 0.869(0.087) |
| 5 | 0.120(0.061) |
| 20 | 4 | 0.05 | 1.223(0.069) | 1.092(0.138) | 1.251(0.060) | 1.344(0.058) |
| 1 | 0.251(0.049) | 1.092(0.049) |
| 5 | 0.282(0.031) |
| 50 | 7 | 0.05 | 1.331(0.020) | 1.258(0.038) | 1.358(0.016) | 1.388(0.022) |
| 1 | 0.332(0.018) | 1.258(0.038) |
| 5 | 0.340(0.014) |

| Table 4: Weibull with shape parameter 2 and scale parameter 0.5. | $H_{m,N,a}$ | $H_{m,n,a}$ | $H_{m,n,a}^{EPS}$ | KL Entropy |
|---|---|---|---|---|
| $n$ | $m$ | $a$ | Est(MSE) | Est(MSE) | Est(MSE) | Est(MSE) |
| 10 | 3 | 0.05 | -0.200(0.056) | -0.641(0.363) | -0.385(0.150) | -0.209(0.123) |
| 1 | 0.379(0.242) | -0.641(0.363) |
| 5 | 0.652(0.572) |
| 20 | 4 | 0.05 | -0.195(0.033) | -0.417(0.129) | -0.258(0.053) | -0.148(0.055) |
| 1 | 0.278(0.150) | -0.417(0.129) |
| 5 | 0.509(0.376) |
| 50 | 7 | 0.05 | -0.170(0.016) | -0.269(0.040) | -0.169(0.016) | -0.131(0.023) |
| 1 | 0.094(0.043) | -0.269(0.040) |
| 5 | 0.330(0.187) |
case of estimation it should be chosen so that $a/n$ is close to zero. The choice $a = 0.05$ is found to be satisfactory in all the cases considered in the paper.

It is also interesting to consider the effect of using different base measures $G$ on the methodology. We fix $a$ at 0.05 and 5. We used several values of $G$. To this end, the next data set is generated from the exponential distribution with mean 20.

1.884, 5.289, 20.890, 20.093, 21.007, 15.261, 7.716, 18.979, 27.537, 10.291, 31.048, 1.215, 13.564, 14.966, 24.896, 10.849

The results of the estimated entropy for the previous data set are reported in Table 5. Clearly, using different $G$ with $a = 0.05$ has no impact on the estimated value. However, when $a = 5$, the estimated value depends on the choice of $G$.

| $G$      | Estimate: $a = 0.05$ | Estimate: $a = 5$ |
|----------|----------------------|-------------------|
| $N(0, 1)$| 3.402                | 3.352             |
| $N(3, 9)$| 3.407                | 3.393             |
| $t_1$    | 3.407                | 3.735             |
| $E(1)$   | 3.402                | 3.143             |
| $U[0, 1]$| 3.398                | 3.118             |

Table 5: Study of the effect of the proposed estimator using different base measures $G$ of $P \sim DP(a = 0.05, G)$. Here $N(\mu, \sigma^2)$ is the normal distribution with mean $\mu$ and standard deviation $\sigma$, $t_1$ is the $t$ distribution with 1 degrees of freedom, $E(1)$ is the exponential distribution with mean 1 and $U[0, 1]$ is the uniform distribution over $[0, 1]$.

5 Conclusion

In this paper, an efficient yet simple Bayesian nonparametric estimator of entropy is proposed. The proposed estimator is considered an analogous Bayesian estimator to the estimator of Ebrahimi, Pflughoeft and Soofi (1994). Through several examples, it has been shown that the approach performs extremely well where a smaller mean squared error is obtained. A foremost motive of having
this estimator to use it in applications such as model checking as discussed, for instance, in Al-Labadi and Evans (2018). We have left this critical avenue of research to future work.

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A Proofs

A.1 Proof of Proposition 1

1. Note that, for each $1 \leq i \leq N$, from the properties of the Dirichlet distribution, $J_{i,N} \sim \text{Beta}(a/N, a(1 - 1/N))$. It follows that $E[J_{i,N}] = 1/N$ and

$$V[J_{i,N}] = \frac{1/N(1 - 1/N)}{a + 1},$$

where $V$ stands for the variance. Since, as $N \to \infty$, $V(J_{i,N}) \to 0$, we conclude the result.

2. By (??), $E[c_{i,a}] = \frac{m}{N} c_i$. Thus, to prove the proposition, by Chebyshev’s inequality, it is sufficient to show that $V(c_{i,a}) \to 0$. We consider three cases.

Case I (for $1 \leq i \leq m$): From the aggregation property of the Dirichlet distribution,

$$\sum_{k=2}^{i+m} J_{k,N} \sim \text{Beta} \left( \frac{a}{N}, a - \sum_{k=2}^{i+m} \frac{a}{N} \right). \quad (A.1)$$
Hence,

$$V\left( \sum_{k=2}^{i+m} J_{k,N} \right) = \sum_{k=2}^{i+m} \frac{a}{\hat{N}} \left( a - \sum_{k=2}^{i+m} \frac{a}{\hat{N}} \right)$$

$$= \frac{(i + m - 1)(N - i - m + 1)}{N^2(a + 1)} \to 0,$$

as $N \to \infty$.

**Case II** (for $m + 1 \leq i \leq N - m$): similar to Case I,

$$\sum_{k=i-m+1}^{i+m} J_{k,N} \sim \text{Beta} \left( \sum_{k=i-m+1}^{i+m} \frac{a}{N}, a - \sum_{k=i-m+1}^{i+m} \frac{a}{N} \right). \hspace{1cm} (A.2)$$

Hence,

$$V\left( \sum_{k=i-m+1}^{i+m} J_{k,N} \right) = \frac{\sum_{k=i-m+1}^{i+m} \frac{a}{\hat{N}} \left( a - \sum_{k=i-m+1}^{i+m} \frac{a}{\hat{N}} \right)}{a^2(1 + a)}$$

$$= \frac{2m(N - 2m)}{N^2(a + 1)} \to 0,$$

as $N \to \infty$.

**Case III** (for $N - m + 1 \leq i \leq N$): As in the previous cases,

$$\sum_{k=i-m+1}^{N} J_{k,N} \sim \text{Beta} \left( \sum_{k=i-m+1}^{N} \frac{a}{N}, a - \sum_{k=i-m+1}^{N} \frac{a}{N} \right). \hspace{1cm} (A.3)$$

Therefore,

$$V\left( \sum_{k=i-m+1}^{N} J_{k,N} \right) = \frac{\sum_{k=i-m+1}^{N} \frac{a}{\hat{N}} \left( a - \sum_{k=i-m+1}^{N} \frac{a}{\hat{N}} \right)}{a^2(1 + a)}$$

$$= \frac{(N - i + m)(i - m)}{N^2(a + 1)} \to 0,$$

as $N \to \infty$. This complete the proof of the proposition.
A.2 Proof of Lemma 2

Recall that,

\[ H_{m,N,a} = \frac{1}{N} \sum_{i=1}^{N} \log \left( \frac{Y_{(i+m)} - Y_{(i-m)}}{c_{i,a}} \right) \]

and

\[ H_{m,N}^{EPS} = \frac{1}{N} \sum_{i=1}^{N} \log \left( \frac{Y_{(i+m)} - Y_{(i-m)}}{mc_i/N} \right), \]

where \( c_{i,a} \) and \( c_i \) are defined, respectively, on (9) and (5). Thus,

\[ E[H_{m,N,a}] - E[H_{m,N}^{EPS}] = \frac{1}{N} \sum_{i=1}^{N} E \left[ \log \left( \frac{mc_i/N}{c_{i,a}} \right) \right] = \frac{1}{N} \sum_{i=1}^{N} \log (c_im/N) - \frac{1}{N} \sum_{i=1}^{N} E[\log c_{i,a}]. \]  

(A.4)

We want to show that, as \( N \to \infty, m \to \infty, m/N \to 0 \) and \( a \to \infty \), (A.4) \to 0.

We consider three cases.

Case I (for \( 1 \leq i \leq m \)): notice that, with \( \alpha_k = aN^{-1} \) and \( \alpha_0 = \sum_{k=1}^{N} \alpha_k = a \),

\[ E[J_{i,N} \log c_{i,a}] = E \left[ J_{i,N} \log \left( \sum_{k=2}^{i+m} Z_{k,N} \right) \right] = \]

\[ \int \cdots \int z_i \log \left( \sum_{k=2}^{i+m} z_k \right) \frac{\Gamma(\alpha_0)}{\prod_{k=1}^{N} \Gamma(\alpha_k)} z_1^{\alpha_1-1} \cdots z_i^{\alpha_i-1} \cdots z_N^{\alpha_N-1} dz_1 \cdots dz_i \cdots dz_N, \]

\[ = \int \cdots \int \log \left( \sum_{k=2}^{i+m} z_k \right) \frac{\Gamma(\alpha_0)}{\prod_{k=1}^{N} \Gamma(\alpha_k)} z_1^{\alpha_1-1} \cdots z_i^{(\alpha_i+1)-1} \cdots z_N^{\alpha_N-1} dz_1 \cdots dz_i \cdots dz_N \]

\[ = \frac{\alpha_i}{\alpha_0} E \left[ \log \left( \sum_{k=2}^{i+m} Z_{k,N} \right) \right]. \]  

(A.5)

where \( \sum_{k=2}^{i+m} Z_{k,N} \sim \text{Beta} \left( \sum_{k=2}^{i+m} \alpha_k + 1, (a + 1) - \left( \sum_{k=2}^{i+m} \alpha_k + 1 \right) \right) \). For \( \alpha_k = \)
\[ aN^{-1}, \sum_{k=2}^{i+m} Z_{k,N} \sim \text{Beta}(a(m + i - 1)N^{-1} + 1, a - a(m + i - 1)N^{-1}). \]

From the properties of the beta distribution, we have

\[
(A.5) = \frac{\alpha_i}{\alpha_0} \left( \psi \left( \sum_{k=1}^{m} \alpha_k + 1 \right) - \psi(\alpha_0 + 1) \right) \\
= \frac{1}{N} \left( \psi \left( \frac{a(m + i - 1)}{N} + 1 \right) - \psi(a + 1) \right),
\]

where \( \psi(x) = \Gamma'(x) / \Gamma(x) \) is the digamma function. Therefore, by (A.6) and for \( 1 \leq i \leq m \), we obtain

\[
(A.4) = \frac{1}{N} \sum_{i=1}^{m} \log \left( \frac{m + i - 1}{N} \right) \\
- \frac{1}{N} \sum_{i=1}^{m} \left( \psi \left( \frac{a(m + i - 1)}{N} + 1 \right) - \psi(a + 1) \right).
\]

Using the facts that \( \psi(x + 1) = \log(x) + O(x^{-1}) \) and \( \sum_{i=0}^{L-1} \frac{1}{x+i} = \psi(x + L) - \psi(x) = \log \left( \frac{x+L}{x} \right) + O(x^{-1}) \) (Abramowitz and Stegun, 1972), we have

\[
(A.4) = -\frac{1}{N} \sum_{i=1}^{m} O \left( \frac{N}{a(m + i - 1)} \right) + \frac{1}{N} \sum_{i=1}^{m} O \left( \frac{1}{a} \right) \\
= O \left( \sum_{i=1}^{m} \frac{1}{a(m + i - 1)} \right) + O \left( \frac{m}{Na} \right) \\
= \frac{1}{a} O \left( \psi(2m) - \psi(m - 1) \right) + O \left( \frac{m}{Na} \right) \\
= \frac{1}{a} O \left( \log \left( \frac{2m}{m - 1} \right) + \frac{1}{2m} \right) + O \left( \frac{m}{Na} \right) \rightarrow 0
\]

as \( N \rightarrow \infty, m \rightarrow \infty, m/N \rightarrow 0 \) and \( a \rightarrow \infty \).

Case II (for \( m + 1 \leq i \leq N - m \)): similar to Case I,

\[
E [J_{i,N} \log c_{i,a}] = \frac{\alpha_i}{\alpha_0} E \left[ \log \left( \sum_{k=i-m+1}^{i+m} Z_{k,N} \right) \right],
\]
where $\sum_{k=i-m+1}^{i+m} Z_{k,N} \sim Beta\left(2amN^{-1} + 1, a - 2amN^{-1}\right)$. Thus, from the properties of the beta distribution, we have

\begin{equation}
(A.7) = \frac{1}{N} \left( \psi \left( \frac{2am}{N} + 1 \right) - \psi(a + 1) \right). \tag{A.8}
\end{equation}

Therefore, by (A.8), we have

\begin{equation}
(A.4) = \sum_{i=m+1}^{N-m} \log \left( \frac{2m}{N} \right) - \frac{1}{N} \sum_{i=m+1}^{N-m} \left( \psi \left( \frac{2am}{N} + 1 \right) + \psi(a + 1) \right)
= O \left( \frac{N}{2am} \right) + O \left( \frac{1}{a} \right) \to 0
\end{equation}

as $N \to \infty$, $m \to \infty$, $m/N \to 0$ and $a \to \infty$.

*Case III (for $N - m + 1 \leq i \leq N$):* similar to the previous cases,

\begin{equation}
E[J_{i,N} \log c_{i,a}] = \frac{\alpha_i}{\alpha_0} \left. E \left[ \log \left( \frac{N}{N-m+i} \right) \right] \right|_{k=i-m+1}^{N}, \tag{A.9}
\end{equation}

where $\sum_{k=i-m+1}^{N} Z_{k,N} \sim Beta\left(a(N + m - i)N^{-1} + 1, a - a(N + m - i)N^{-1}\right)$. Thus, from the properties of the beta distribution, we have

\begin{equation}
(A.9) = \frac{1}{N} \left( \psi \left( \frac{a(N + m - i)}{N} + 1 \right) - \psi(a + 1) \right). \tag{A.10}
\end{equation}
Therefore, by (A.10), we have

\[
(A.4) = \frac{1}{N} \sum_{i=N+m+1}^{N} \log \left( \frac{N + m - i}{N} \right) \\
- \frac{1}{N} \sum_{i=N-m+1}^{N} \psi \left( \frac{a(N + m - i)}{N} + 1 \right) + \psi(a + 1) \\
= O \left( \sum_{i=N-m+1}^{N} \frac{1}{a(N + m - i)} \right) + O \left( \frac{m}{aN} \right) \\
= O \left( \frac{\psi(1-2m)}{a} - \frac{\psi(1-m)}{a} \right) + O \left( \frac{m}{aN} \right) \\
\rightarrow 0
\]

as \( N \rightarrow \infty, m \rightarrow \infty, m/N \rightarrow 0 \) and \( a \rightarrow \infty \). Thus, in all cases, as \( N \rightarrow \infty, m \rightarrow \infty, m/N \rightarrow 0 \) and \( a \rightarrow \infty \), (A.4) \( \rightarrow 0 \). This complete the proof of Lemma 2.  

A.3 Proof of Lemma 3

Note that,

\[
H_{m,N,a} = (H_{m,N,a} - H_{m,N,a}^{EPS}) + H_{m,N,a}^{EPS},
\]

where \( H_{m,N,a}^{EPS} \) is the approximation given in (4). It follows that,

\[
H_{m,N,a} - H_{m,N,a}^{EPS} = \frac{1}{N} \sum_{i=1}^{N} \log \left( \frac{mc_{i}}{c_{i,a}} \right),
\]

Since \( (J_{i,N})_{1 \leq i \leq N} \) is a sequence of pairwise negative associated identically distributed random variables with finite expectations, by Theorem 4.2.8 of Atkinson (2017), the weak law of large numbers holds for the sequence \( \left( \frac{mc_{i}/N}{c_{i,a}} \right)_{1 \leq i \leq N} \).
Thus we have

\[ H_{m,N,a} - H_{m,N}^{EPS} - E\left(\log \left( \frac{mc_i/N}{c_{i,a}} \right) \right) \to 0. \]

We show that \( E\left(\log \left( \frac{mc_i/N}{c_{i,a}} \right) \right) \to 0. \) We consider three cases.

**Case I (for \( 1 \leq i \leq m \)):** From (A.1) and the well-known property of the beta distribution, we have

\[
E\left(\log \left( \frac{mc_i/N}{c_{i,a}} \right) \right) = \log \left( \frac{mc_i}{N} \right) - E\left(\log (c_{i,a}) \right)
= \log \left( \frac{i + m - 1}{N} \right) - \psi \left( \frac{a(i + m - 1)}{N} \right) + \psi (a)
= \log \left( \frac{i + m - 1}{N} \right) - \log \left( \frac{a(i + m - 1)}{N} \right) - O \left( \frac{N}{a(i + m - 1)} \right)
+ \log(a) + O \left( \frac{1}{a} \right)
= -O \left( \frac{N}{a(i + m - 1)} \right) + O \left( \frac{1}{a} \right) \to 0.
\]

as \( N \to \infty, m \to \infty, m/N \to 0 \) and \( a \to \infty \).

**Case II (for \( m + 1 \leq i \leq N - m \)):** From (A.2) and the well-known property of the beta distribution, we have

\[
E\left(\log \left( \frac{mc_i/N}{c_{i,a}} \right) \right) = \log \left( \frac{mc_i}{N} \right) - E\left(\log (c_{i,a}) \right)
= \log \left( \frac{2m}{N} \right) - \psi \left( \frac{2am}{N} \right) + \psi (a)
= \log \left( \frac{2m}{N} \right) - \log \left( \frac{2am}{N} \right) - O \left( \frac{N}{2am} \right)
+ \log(a) + O \left( \frac{1}{a} \right)
= -O \left( \frac{N}{2am} \right) + O \left( \frac{1}{a} \right) \to 0.
\]

as \( N \to \infty, m \to \infty, m/N \to 0 \) and \( a \to \infty \).
Case III (for $N - m + 1 \leq i \leq N$): As in the previous cases, from (A.3), we have

$$E \left( \log \left( \frac{mc_i}{N} \right) \right) = \log \left( \frac{mc_i}{N} \right) - E \left( \log (c_{i,a}) \right)$$

$$= \log \left( \frac{n + m - i}{N} \right) - \psi \left( \frac{a(n + m - i)}{N} \right) + \psi(a)$$

$$= \log \left( \frac{n + m - i}{N} \right) - \log \left( \frac{a(n + m - i)}{N} \right) - O \left( \frac{N}{a(n + m - i)} \right)$$

$$+ \log(a) + O \left( \frac{1}{a} \right)$$

$$= -O \left( \frac{N}{a(n + m - i)} \right) + O \left( \frac{1}{a} \right) \rightarrow 0.$$  

as $N \rightarrow \infty$, $m \rightarrow \infty$, $m/N \rightarrow 0$ and $a \rightarrow \infty$. Thus, in all cases, $H_{m,N,a} - H_{m,N}^{EPS} \xrightarrow{p} 0$. Also, by Ebrahimi, Pflughoeft and Soofi (1994), as $N \rightarrow \infty$, $m \rightarrow \infty$ and $m/N \rightarrow 0$ we have $H_{m,N}^{EPS} \xrightarrow{p} H(G)$. Now, applying Slutsky’s theorem (Ferguson, 1996) completes the proof. ■