Competitive Routing in Hybrid Communication Networks

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Abstract
Routing is a challenging problem for wireless ad hoc networks, especially when the nodes are mobile and spread so widely that in most cases multiple hops are needed to route a message from one node to another. In fact, it is known that any online routing protocol has a poor performance in the worst case, in a sense that there is a distribution of nodes resulting in bad routing paths for that protocol, even if the nodes know their geographic positions and the geographic position of the destination of a message is known. The reason for that is that radio holes in the ad hoc network may require messages to take long detours in order to get to a destination, which are hard to find in an online fashion.

In this paper, we assume that the wireless ad hoc network can make limited use of long-range links provided by a global communication infrastructure like a cellular infrastructure or a satellite in order to compute an abstraction of the wireless ad hoc network that allows the messages to be sent along near-shortest paths in the ad hoc network. We present distributed algorithms that compute an abstraction of the ad hoc network in \(O(\log^2 n)\) time using long-range links, which results in \(c\)-competitive routing paths between any two nodes of the ad hoc network for some constant \(c\) if the convex hulls of the radio holes do not intersect. We also show that the storage needed for the abstraction just depends on the number and size of the radio holes in the wireless ad hoc network and is independent on the number of nodes, and this information just has to be known to a few nodes for the routing to work.

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1 Introduction

Nowadays almost every person has a cell phone. Hence, in a city center the density of cell phones would, in principle, be sufficiently high to set up a well-connected wireless ad hoc network spanning the entire city center, which could then be used for many interesting applications in the area of social networks. Wireless ad hoc networks have the advantage that there is no limit (other than the bandwidth and battery constraints) on the amount of data that can be exchanged while the amount of data that can be transferred at a reasonable rate via long-range links using the cellular infrastructure or satellite is limited (by some data plan) or costly. However, routing in a mobile ad hoc network is challenging, even if the geographic position of the destination is known, since buildings or other obstacles like rivers may create radio holes that make it non-trivial to find a near-shortest routing path. So the question we address in this paper is:

Can long-range links be used effectively to find near-shortest routing paths in the ad hoc network?

A simple solution to that problem would be that all nodes regularly post their geographic position and the nodes within their communication range to a server in the Internet. This would allow the server to compute optimal routing paths so that whenever a node wants to forward a message to a certain destination, the server can tell it which of the neighbors to send it to. While this would in principle work since WhatsApp, for example, is already offering a service to track friends, there are severe privacy concerns with this approach. An alternative approach that we are pursuing in this paper is a purely peer-to-peer based approach in which no other equipment other than the cell phones (and an infrastructure for the long-range links) needs to be used. To the best of our knowledge, our approach is the first one that is making use of a global communication infrastructure in a peer-to-peer manner in order to efficiently determine short routing paths for an ad hoc network. Wireless ad hoc networks have been considered before that utilize base stations in order to exchange messages more effectively, but there message will be sent via long-range links to bridge long distances while we will only allow messages to be sent via ad hoc links.

1.1 Model

Throughout this paper, we consider $V \subset \mathbb{R}^2$ to be a static set of nodes with fixed positions in the Euclidean plane and $|V| = n$. Assuming fixed positions is reasonable since the computation of our abstraction of the ad hoc network is fast enough so that the positions of the nodes shouldn’t have changed much in the meanwhile. For any given pair of nodes $u = (x_1, y_1)$, $v = (x_2, y_2)$, we denote by $||uv|| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ the Euclidean distance between $u$ and $v$.

We model our cell phone network as a hybrid directed graph $H = (V, E, E_{AH})$ where the node set $V$ represents the set of cell phones, an edge $(v, w)$ is in $E$ whenever $v$ knows the phone number (or simply ID) of $w$, and an edge $(v, w) \in E$ is also in the ad hoc edge set $E_{AH}$ whenever $v$ can send a message to $w$ using its Wifi interface. For all edges $(v, w) \in E \setminus E_{AH}$, $v$ can only use a long-range link to directly send a message to $w$. For simplicity, we adopt the unit disk graph model for the edges in $E_L$. However, our approach would also work for arbitrary convex communication ranges, as long as local connections are (only established if they are) bi-directed.

\textbf{Definition 1.} For any point set $V \subseteq \mathbb{R}^2$ the Unit Disk Graph of $V$, $\text{UDG}(V)$, is a bi-directed graph that contains all edges $(u, v)$ with $||uv|| \leq 1$. 
When assuming the Unit Disk Graph model, we can simplify the definition of our hybrid graph to \( H = (V, E) \), since the ad hoc edges are implied by the positions of \( V \). We assume \( \text{UDG}(V) \) to be strongly connected so that a message can be sent from every node to every other node in \( V \) by just using ad hoc edges. While the ad hoc edges are fixed, the nodes can nevertheless change \( E \) over time: If a node \( v \) knows the IDs of nodes \( w \) and \( w' \) (because they are neighbors in \( H \)), then it can send the ID of \( w \) to \( w' \), which adds \((w, w')\) to \( E \). Alternatively, if \( v \) deletes the address of some node \( w \) with \((v, w) \in E\), then \((v, w)\) is removed from \( E \). There are no other means of changing \( E \), i.e., a node \( v \) cannot learn about an ID of a node \( w \) unless \( w \) is in \( v \)'s UDG-neighborhood or the ID of \( w \) is sent to \( v \) by some other node.

We consider the classical \( \text{LOCAL} \)-model for distributed computation [18, 21] in which nodes communicate via message passing. The \( \text{LOCAL} \)-model assumes synchronous message passing where the time is divided into communication rounds. Every message initiated in round \( i \) is delivered at the beginning of round \( i + 1 \), and a node can process all messages in a round that have been delivered at the beginning of that round.

1.2 Objective

Our objective is to design an efficient routing algorithm for ad hoc networks, where the source \( s \) of a message knows the ID of the destination \( t \), or in other words, \((s, t) \in E\). This is a reasonable constraint since cell phone users normally wouldn’t call cell phones whose users are unknown to them. Thus, whenever a message needs to be sent from a source \( s \) to some destination \( t \), we can assume that the geographic position of \( t \) is known, since \( s \) can ask \( t \) for that before sending the message towards \( t \) using the ad hoc network.

Our routing algorithm consists of two parts: (1) computing an abstraction of the wireless ad hoc network, and (2) using that abstraction in order to route messages along short paths. In order to measure the quality of the selected routing paths, we will use a competitive approach. We call a routing strategy \( c \)-competitive if for all node pairs \((s, t)\), the routing path \( \text{RP}(s, t) \) from \( s \) to \( t \) satisfies \( ||\text{RP}(s, t)|| \leq c \cdot d(s, t) \), where \( ||\text{RP}(s, t)|| \) denotes the Euclidean length of \( \text{RP}(s, t) \) and \( d(s, t) \) denotes the shortest Euclidean length of a path in \( \text{UDG}(V) \) from \( s \) to \( t \).

As proven by Kuhn et al., we cannot find competitive paths in ad hoc networks that achieve a better than quadratic competitiveness by only considering local information at the nodes [16]. In order to find better routing paths, we need additional knowledge about the ad hoc network. The naive approach would be to disseminate the geographic positions of all nodes to all other nodes in the network, but that would be far too costly. Instead, we will focus on computing suitable abstractions of radio holes in the ad hoc network. The intuition behind that is simple: if there are no radio holes, then simple greedy routing (i.e., always take the neighbor that is closest to the destination) would already give us short routing paths to arbitrary destinations. Radio holes can be specified by specifying the nodes along its boundary, but there can be many such nodes. Therefore, we will also look at more compact representations of radio holes like the (nodes forming the) convex hull of its boundary. Considering convex hulls as radio hole abstractions makes sense because in huge cities like New York City the shape of obstacles like buildings is in many cases convex or close to a convex shape, and these shapes do not overlap. In order to obtain the desired abstraction, we will make use ID-introductions in order to form an overlay network that allows us to compute these abstractions in a distributed manner using the long-range links. Since sending messages via long-range links is costly (in terms of money), our goal is to keep the long-range communication work of the nodes as low as possible.
1.3 Our Contributions

In this paper, we achieve the following results for the objectives above:

1. Restricted Delaunay Graph (Section 3): In order to detect a hole, we assume that there is an ad hoc network based on Delaunay Graphs, called restricted Delaunay Graphs. After reaching a stable state, each node is able to detect locally if it is located on the border of a hole.

2. We use convex hulls as abstractions for holes. We assume that no pair of intersecting convex hulls exists and initially we also assume that every node pair \( s \) and \( t \) which wants to find a routing path to each other, where the geometric positions of \( s \) and \( t \) are outside of any convex hull. We develop a distributed algorithm to compute each convex hull with \( n \) nodes in time \( O(\log^2 n) \). Afterwards, we drop our initial assumption and describe a procedure to build an overlay to find short paths between any pair of nodes \( s \) and \( t \) which geographic positions are inside of convex hulls. Therefore we introduce an overlay network called InCOver that allows us to find shortest paths between any pair of nodes \( s \) and \( t \). We prove that InCOver can be obtained in \( O(\log^2 n) \) communication rounds with high probability.

3. To construct an overlay, the cellular infrastructure, we build a clique of nodes of all different convex hulls.

4. We introduce Skip Delaunay Graphs which allow us to distribute broadcasts in time \( O(\log n) \) with high probability.

1.4 Related Work

Wireless ad hoc networks and appropriate routing schemes have been extensively studied in the past decades. An often considered network topology for ad hoc networks is the so called Delaunay Graph [9]. Delaunay Graphs are \( c \)-spanners which means that they contain a path for any pair of nodes of length at most \( c \) times their Euclidean distance. The currently best known bound for \( c \) is 1.998 and was proven by Xia [26]. In general, Delaunay Graphs are not appropriate for the purpose of modelling wireless ad hoc networks since edges can become arbitrarily large. This contradicts the nature of the wireless channel because wireless communication is only applicable for limited distances. Hence, topologies that take the limited distance into account are preferred. Unit Disk Graphs are a topology in which each node is assumed to have a transmission range of 1 and is connected to each node within its transmission range. This results in a very high degree for nodes that are in densely connected regions. A way of reducing the degree is the notion of Restricted Delaunay Graphs, proposed by Yang Li et al. [17] in which the Delaunay Graph is built and only edges are kept that do not exceed the transmission range of nodes. Note that Delaunay Graphs in general do not have a constant degree. Nevertheless, the average degree of a node is 6 and scenarios in which a high degree is obtained are rather theoretical. Restricted Delaunay Graphs can be constructed in a self-stabilizing manner by using the algorithm of Jacob et al. [13]. The advantage of self-stabilizing algorithms is that they are fault-tolerant and able to recover from topology changes. A protocol is called to be self-stabilizing if it fulfills the properties Convergence and Closure. Convergence demands that the protocol is able to transform an arbitrary initial network state into a legal state. Closure demands that the protocol only produces legal succeeding states after once reaching a legal state. The definition of legal states has to be defined for each application individually. Although we are considering a static scenario in this paper, we are interested in extending our ideas to a dynamic scenario, and self-stabilization fits well to the nature of moving nodes. Hence, the algorithm of Jacob
et al. may also be used in further research when we are considering a dynamic setting.

Another often considered topic is routing in ad hoc networks. Optimal paths can in general only be found if knowledge about the whole network is available. Since in ad hoc networks only local information about the neighbors of a node can be used, so called online routing strategies can in general not find optimal paths. For Delaunay Graphs, competitive online routing has attracted attention of several research groups. The best known algorithm finds paths of length at most 5.9 times the Euclidean distance between any pair of nodes \(5\). However, this routing strategy is not applicable in Restricted Delaunay Graphs, because the ad hoc network may contain radio holes. Rührup and Schindelhauer considered routing strategies for grids that contain failed nodes \(24\). This is similar to our scenario as failed nodes behave as routing holes in an ad hoc network. Their procedure uses a strategic search which distributes a message over multiple paths. They proved that their procedure is asymptotically optimal for their setting. However, it is not clear how the strategy can be generalized to arbitrary node distributions. Additionally, we are interested in direct routing paths in order to reduce the congestion in the network. Kuhn et al. considered routing strategies for ad hoc networks containing radio holes and introduced GOAFR, a routing strategy that uses a combination of greedy and perimeter routing, which finds paths with quadratic competitiveness \(16\). They have also proven that this is worst-case optimal. Hence, it is not possible to design routing strategies which use only local knowledge and achieve a better competitiveness than quadratic. We are interested in improving the competitiveness by using additional communication mode that is not distance limited. The idea of designing networks that combine different communication modes – so called Hybrid Communication Networks – is not new and has already been introduced in different contexts \(6, 20\). To the best of our knowledge, we are the first ones that consider these types of networks for the purpose of finding paths in ad hoc networks. Our idea is to establish an overlay network via the long-range communication mode that can collect information about radio holes in the network. In order to reduce the complexity of the overlay network, holes are represented by their convex hulls. Computing the convex hull of a set of points is one of the most considered problems in computational geometry. However, the number of publications dealing with distributed computation of convex hulls is relatively small. The distributed QuickHull-algorithm of Bez and Edwards achieves a runtime of \(O(n \cdot h)\) for our scenario if \(n\) denotes the number of nodes and \(h\) the number of convex hull edges \(4\). Rajsbaum and Urrutia introduced a distributed algorithm for computing a convex hull of a geometric ring which achieves a runtime of \(O(n \cdot \log^2 n)\) \(22\). Miller et al. proposed a parallel algorithm which computes the convex hull in time \(O(\log n)\) \(19\). The algorithm makes use of a hypercube. In this paper, we describe a preprocessing strategy that allows us to use the algorithm of Miller in a distributed setting and obtain a polylogarithmic runtime. Preprocessing includes the construction of a hypercube by pointer doubling. This technique has been mentioned by Wyllie for the first time \(25\). Further, Batcher’s Bitonic Sorting algorithm \(2\) is included in the preprocessing process.

Another often considered problem in distributed systems is the computation of an approximation of the Dominating Set. In our scenario, a Dominating Set is useful to get routing information in the area inside of convex hulls. For the distributed computation of Dominating Sets, several algorithms have been introduced in the literature. A popular algorithm has been introduced by Jia et al. which computes a \(O(\log \Delta)\) approximation of the smallest possible dominating set \(14\). Note that the computation of smallest dominating sets is proven to be NP complete. The algorithm requires \(O(\log n \cdot \log \Delta)\) communication rounds with high probability.
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Figure 1 The Delaunay Graph of a set of nodes $V \subset \mathbb{R}^2$.

2 Delaunay Graphs

In this work, we aim at forming a Restricted Delaunay Graph with the ad hoc links. Before we give the formal definition of this topology, we first introduce the Delaunay Graph.

Definition 2. Let $\bigcirc(u, v, w)$ be the unique circle through the nodes $u, v$ and $w$. For any $V \subseteq \mathbb{R}^2$, the Delaunay Graph $\text{Del}(V)$ of $V$ consists of all edges $(u, v)$ that have a node $w$ of $V$ for which $\bigcirc(u, v, w)$ does not contain any other node of $V$.

The Restricted Delaunay Graph contains all edges of the Delaunay Graph that do not exceed the transmission range of a node. More formally:

Definition 3. The Restricted Delaunay Graph $\text{RDG}(V) = (V, RD)$ of $V$ is defined to be the graph of node set $V$ and the set of edges $RD$ such that $\{u, v\} \in RD$ if and only if $\{u, v\}$ is of $\text{UDG}(V)$ and $\{u, v\}$ is of $\text{Del}(V)$.

The Restricted Delaunay Graph can be computed with the self-stabilizing algorithm introduced by Jacob et al. by using long-range edges. After the Delaunay Graph has reached a stable state, each node can detect locally which edges of its direct neighborhood are edges of the Restricted Delaunay Graph and which edges are part of the Delaunay Graph and have length larger than 1.

Since Restricted Delaunay Graphs do not contain all edges of a corresponding Delaunay Graph, one cannot simply use routing strategies for Delaunay Graphs in our scenario. Missing edges in the Delaunay Graph create radio holes. We distinguish between inner and outer radio holes (or simply holes).

Definition 4 (Inner Hole). Let $V \in \mathbb{R}^2$. An inner hole is a face of $\text{RDG}(V)$ with at least 4 nodes.

Definition 5 (Outer Hole). Let $V \in \mathbb{R}^2$. Furthermore, let $\text{CH}(V)$ be the set of all edges of the convex hull of $V$. Define an outer hole to be a face in $\text{RDG}' = (V, RD \cup \text{CH}(V))$ with at least 3 nodes, that contains an edge $e \in \text{CH}(V)$ with $\|e\| > 1$.

Note that the nodes on the perimeter of a hole form a ring, i.e. each node on the perimeter is adjacent to exactly two other nodes on the perimeter of the hole.

The choice of the restricted Delaunay Graph $\text{RDG}(V, RD)$ as network topology is motivated by a useful property of Delaunay Graphs, i.e. the Delaunay Graph $\text{Del}(V)$ contains paths between every pair of nodes $v$ and $w$ of $V$ which is not longer than $c$ times their Euclidean distance $\|vw\|$. We call these paths as follows:

Definition 6. A path $p(w, v)$ between two nodes $w$ and $v$ in $G$ is a geometric $c$-spanning path between $w$ and $v$, if its length is at most $c$ times the Euclidean distance between $w$ and $v$. The length of a path $p(w, v)$ is denoted as $\|p(w, v)\|$.
Classes of graphs that contain such paths for every pair of nodes are called geometric $c$–spanners.

Definition 7. A graph $G = (V, E)$ is called a geometric $c$-spanner of $G$, if for all $u, v \in V$ there is a geometric $c$-spanning path $p(u, v)$.

Delaunay Graphs are $c$–spanners. The currently best known bound on $c$ is 1.998 and was proven by Xia [26].

Theorem 8. There exists a path in a Delaunay Graph from node $a$ to $b$ of length less than $1.998 \cdot ||ab||$.

The so far mentioned bounds are only achievable if knowledge about the entire graph is available. To the best of our knowledge, the currently best known online routing strategy is due to Bonichon et al. [5]:

Theorem 9. There exists an online routing strategy for Delaunay Graphs which finds a path between any source $s$ and target $t$ with length at most $5.9 \cdot ||st||$.

The routing strategy which achieves the mentioned bound is called Chew’s algorithm and is based on a routing strategy for Delaunay Graphs in the $L_1$-norm developed by Chew [7]. In Chew’s algorithm, all triangles intersected by $st$ are ordered from left to right. Let $C_i$ be the circumcircle of $T_i$, $T_i$ is part of that intersects $st$. Afterwards, the circumcircle $C_i$ of $T_i$ is computed. $C_i$ is split into an upper and a lower arc by a line connecting the leftmost point of $C_i$ and the rightmost intersection point between $C_i$ and $st$. In case $p_i$ is located on the upper arc, the next node clockwise around $C_i$ is chosen. Otherwise, the succeeding counter-clockwise node of $C_i$ is chosen as next node.

Unfortunately, this routing strategy is not applicable to all pairs of nodes in Restricted Delaunay Graphs as Chew’s algorithm would choose edges of length larger than 1. Also, recall that Kuhn and Wattenhofer have proven that there exists no online routing algorithm which achieves better competitiveness than quadratic when considering ad hoc networks containing holes [15]. Hence, additional knowledge about the shape and locations of holes is needed. Shapes of holes are polygons and hence our model is related to routing in polygonal domains. Routing in polygonal domains is one of the most studied problems in Computational Geometry. It deals with finding a path between any pair of nodes in the presence of polygonal obstacles. De Berg et al. showed that it is enough to consider nodes of polygons for path finding in polygonal domains [3]:

Lemma 10. Any shortest path between $s$ and $t$ among a set $S$ of disjoint polygonal obstacles is a polygonal path whose inner nodes are nodes of $S$. 
Figure 3 represents an example of a shortest path between a start node and a goal node according to the lemma above.

To apply the hybrid communication model, we gave formal definitions of the considered ad hoc network (Restricted Delaunay Graph) as well as a definition of holes. In the following sections of this work, we present a routing protocol for Restricted Delaunay Graphs (Section 3) and distributed protocols for
1. Construction of the Restricted Delaunay Graph and detection of holes (Section 4)
2. Computation of convex hulls of holes (Section 5)
3. Dissemination of convex hull information (Section 5)

3 Routing in Restricted Delaunay Graphs

In this section, we introduce a general routing strategy for Restricted Delaunay Graphs. As described in Section 2, we can find the shortest path in a domain with polygonal obstacles by only taking the corners of the polygons into account. The usual procedure for finding shortest paths in polygonal domains is the computation of a Visibility Graph and applying a single source shortest path algorithm (e.g., the algorithm of Dijkstra) [3]. In the Visibility Graph Vis(V) of a set of polygons, V represents the set of corners of the polygon, and there is an edge \{v, w\} in Vis(V) if and only if a line can be drawn from v to w without crossing any polygon, i.e., v is visible from w. Suppose we know the Visibility Graph of all hole nodes. The results of Xia [26] indicate that the Restricted Delaunay Graph contains paths which are 1.998-competitive to paths which can be found in the Visibility Graph of all hole nodes.

For the rest of the section, we assume that every node which is located on the perimeter of a hole stores the Visibility Graph of all hole nodes and every node of the ad hoc network stores a reference to a node on the perimeter of a hole. This scenario allows us to use the following routing strategy.

A source node s that wants to send data to a target node t sends a path query containing s and t via a long-range link to the node on the perimeter of a hole known to it. The hole node will then insert s and t into its Visibility Graph and apply a shortest path algorithm. The resulting shortest path in the Visibility Graph (h₀ = s, h₁, h₂, ..., hₖ = t) is sent back (via a long-range link) to s. Afterwards, s initiates a message with destination h₁ including the entire path and the transmitted data. By applying Chew’s algorithm, a path of length 5.9 · ∥sh₁∥ is obtained. After reaching h₁, the procedure is repeated until the message finally reaches t. Let ∥pₛₜ∥ be the shortest path between s and t in the Visibility Graph. The resulting path in the Restricted Delaunay Graph has length at most 5.9 · ∥pₛₜ∥.
Unfortunately, the nodes on the perimeter of a radio hole potentially have to store a huge Visibility Graph. In fact, it is possible to have a radio hole in $\text{RDG}(V)$ covering just a constant-size area with $\Theta(n)$ nodes on its perimeter. Also, if $h$ is the number of nodes on the perimeter of a hole, then the Visibility Graphs may contain up to $\Theta(h^2)$ edges. An idea to reduce the number of edges to $O(h)$ is to not compute the entire Visibility Graph but only a Delaunay Graph of all nodes lying on the perimeter of a hole. As Delaunay Graphs are planar graphs, this reduces the number of edges to $O(h)$. However, this also affects the obtained length of the paths. Delaunay Graphs do not contain the shortest geometric connection between two nodes in general but only a path which is 1.998-competitive to such a path. Hence, we obtain a path length of $1.998 \cdot 5.9 \cdot \|p_x\| \leq 11.79 \cdot \|p_x\|$. A natural question is how to reduce the number of nodes in the Visibility Graph even further while still being able to compute competitive paths.

We can obtain a further space reduction if we focus on locally convex hulls of the radio holes.

**Definition 11.** Let $(v_1, v_2, \ldots, v_k, v_1)$ be a cycle of nodes in $\text{RDG}(V)$ at the perimeter of some hole. We call $(v_{i_1}, v_{i_2}, \ldots, v_{i_ℓ}, v_{i_1})$ for some $1 \leq i_1 < i_2 < \ldots, i_ℓ \leq k$ a locally convex hull of that hole if (1) $\|v_{i_j}v_{i_{j+1}}\| \leq 1$ for all $j \in \{1, \ldots, ℓ\}$ (where $v_{i_{ℓ+1}} = v_{i_1}$), and (2) there are no 3 consecutive nodes $u, v, w$ in that sequence where $\angle(u, v, w) \geq 180^\circ$ and $\|uw\| \leq 1$.

For the locally convex hulls it can be shown:

**Lemma 12.** For any cycle $(v_1, v_2, \ldots, v_k, v_1)$ of nodes in $\text{RDG}(V)$ at the perimeter of some hole of the $\text{RDG}(V)$ that covers an area of $A$, any locally convex hull of that cycle contains $O(A)$ nodes.

**Proof of Lemma 12** Consider any locally convex hull $(v_{i_1}, v_{i_2}, \ldots, v_{i_ℓ}, v_{i_1})$, and let $u, v, w$ by 3 consecutive nodes in that sequence. If $\angle(u, v, w) \geq 180^\circ$, then we know from the definition of the locally convex hull that $\|uw\| > 1$. If $\angle(u, v, w) < 180^\circ$, then $\|uw\| > 1$ as well since otherwise $v$ would not be on the perimeter of the hole. This implies for the predecessor $p$ of $x$ and the successor $s$ of $w$ that $\|pv\| > 1$ and $\|sw\| > 1$. Also, there cannot be any other node $x \in \{v_{i_1}, \ldots, v_{i_ℓ}\}$ with $\|ex\| \leq 1$ as otherwise we had a shortcut in the perimeter, meaning that $(v_1, v_2, \ldots, v_k, v_1)$ cannot be the perimeter of a hole. Hence, the unit cycle around each $v_{i_j}$ can contain at most 2 other nodes of the locally convex hull, which implies that $ℓ = O(A)$.

Hence, locally convex hulls contain a number of nodes that is independent of the total number of nodes in the system but only depends on the area covered by the hole.

A further reduction in the number of nodes can be achieved when only looking at the convex hull of a hole.

**Lemma 13.** For any cycle $(v_1, v_2, \ldots, v_k, v_1)$ of nodes in $\text{RDG}(V)$ at the perimeter of some hole of the $\text{RDG}(V)$ with a bounding box (i.e., the box of minimum size containing $v_1, \ldots, v_k$) of circumference $L$, the convex hull $(v_{i_1}, v_{i_2}, \ldots, v_{i_ℓ}, v_{i_1})$ of the cycle contains $O(L)$ nodes.

**Proof of Lemma 13** Let $B$ be the bounding box of the cycle and $x$ be its center point. Let the points $w_{i_1}, \ldots, w_{i_ℓ}$ be the projections of $v_1, v_2, \ldots, v_k$ from $x$ onto the boundary of $B$, i.e., the points where the ray from $x$ in the direction of $v_j$ intersects the boundary of $B$. As is easy to check, the $l_1$-distance of $w_{i_1}$ and $w_{i_{j+1}}$ on $B$ is at least as large as $\|v_{i_1}v_{i_{j+1}}\|$ for all $j$. Moreover, for any 3 consecutive points $u, v, w$ on the convex hull it must hold that...
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||uw|| > 1. Hence, for any 3 consecutive points \(u', v', w'\) on the projection of the convex hull onto \(B\) it must hold that the \(\ell_1\)-distance of \(u'\) and \(w'\) is more than 1, which implies that the convex hull contains only \(O(L)\) nodes.

Since a bounding box of circumference \(L\) may cover an area of size \(\Theta(L^2)\), we may get another significant reduction in the number of nodes when switching from a locally convex hull to a convex hull.

4 Routing in Restricted Delaunay Graphs with Convex Hulls as Hole Abstractions

Recall that we need information about shapes and positions of convex hulls to obtain routing paths between a source \(s\) and a target \(t\) that achieve a better competitiveness than quadratic. As computing the entire visibility graph of the hole structure is expensive, we consider a scenario in which convex hulls of holes do not intersect. When considering routing in Restricted Delaunay Graphs without intersecting convex hulls of holes, we have to distinguish several cases for the locations of source and target nodes \(s\) and \(t\). These nodes can lie inside and outside of convex hulls. To be precise, however, we have to consider more detailed cases concerning the positions of \(s\) and \(t\). To investigate all different cases of the different geographical positions, we introduce bay areas. A bay area \(H_A\) of a hole consists of the nodes and edges of the Restricted Delaunay Graph that are inside the convex hull and between two adjacent convex hull nodes. After getting familiar with bay areas, we are able to present all cases of geographical positions of \(s\) and \(t\) which have to be distinguished.

1. \(s\) and \(t\) are outside of convex hulls
2. \(s\) or \(t\) is inside of a convex hull
3. \(s\) and \(t\) are inside different convex hulls
4. \(s\) and \(t\) are inside the same convex hull but in different bay areas
5. \(s\) and \(t\) are inside the same convex hull and in the same bay area.

Our goal is to find \(c\)-competitive paths in the Restricted Delaunay Graph via the design of the Overlay Network. This overlay consists of a clique between all convex hull nodes. Afterwards, each convex hull node is able to locally compute a Delaunay Graph between all convex hull nodes without the edges of this graph that cross a hole. The routing itself takes place in the ad hoc network, the Restricted Delaunay Graph. The following image gives a visual impression of the Delaunay Graph of convex hull nodes.

The general routing algorithm in Restricted Delaunay Graphs works as follows: Whenever \(s\) wants to send a message to \(t\), \(s\) initiates a request \(\text{request}(id_t)\) with \(t\)'s id \(id_t\) to \(t\). In return, \(s\) receives \(t\)'s geographic coordinates \((x_t, y_t)\). Then \(s\) contacts the convex hull node \(c_s\), which has the smallest distance of all convex hull nodes to \(s\) and makes a request \(\text{request}(c_s, (x_t, y_t))\) for the shortest path over edges in \(\text{Del}(\text{CH}(V))\) to \(t\). In return \(s\) receives this shortest path \((s, c_s = c_1, c_2, \ldots, c_l = c_t, t)\). This path is visualized in Figure 5. After preprocessing the routing of the message \(m\) that \(s\) wants to send to \(t\) is forwarded with Chew’s algorithm [5]. In particular, \(s\) sends \((m, (c_s = c_1, c_2, \ldots, c_l = c_t, (x_t, y_t)))\) by using Chew’s algorithm in the Restricted Delaunay Graph to \(c_s\). Then each convex hull node \(c_i\) that is a node incident to edges of the shortest path \((c_s = c_1, c_2, \ldots, c_l = c_t)\), forwards the message triple \((m, (c_s = c_1, c_2, \ldots, c_l = c_t), (x_t, y_t))\) in direction to the next convex hull node \(c_{i+1}\) by using Chew’s algorithm in the Restricted Delaunay Graph. For example, \(c_s\) forwards \((m, (c_s = c_1, c_2, \ldots, c_l = c_t), (x_t, y_t))\) to \(c_2\) by routing in the direction of \(c_2\) and using...
Chew’s algorithm in the Restricted Delaunay Graph. Afterwards, \( c_2 \) uses the same protocol for routing \((m, (c_s = c_1, c_2, \ldots, c_l = c_t), (x_t, y_t))\) to \( c_3 \) and so on. Finally, message \( m \) is sent from \( c_l \) to \((x_t, y_t)\). See Figure 6 for a visualization of the path taken in the ad hoc network.

After the reader is now familiar with the routing in Restricted Delaunay Graphs, we prove that it makes sense to consider only convex hull nodes in order to find shortest paths, provided no pair of intersecting convex hulls exists, for cases 1 – 4. Case 5 will be discussed in Section 5. In case the shortest path between two nodes \( s \) and \( t \) in the Restricted Delaunay Graph \( RDG(V) \) is not identical to the shortest path between \( s \) and \( t \) in the Delaunay Graph \( Del(V) \), we make use of the following lemma

▶ **Lemma 14.** The shortest path between any pair of nodes of the Restricted Delaunay Graph consists of convex hull nodes.

**Proof of Lemma 14.** Let \( s, t \) be two nodes of the Restricted Delaunay Graph, whose direct Euclidean connection (line segment) is blocked by a hole. Starting from \( s \), let \( \ell \) be the first intersected line segment of the boundary of the intersected convex hull with endpoints \( v, w \). We assume that the shortest path contains points of \((v, \ldots, w)\). Else, the argumentation must be repeated with the neighboring edges of the convex hull.

By contradiction, we assume that the shortest path from \( s \) to \( t \) contains a point \( p \in (v, \ldots, w) \) from the interior of the convex hull, i.e., excluding \( v, w \). Wlog. we assume that the shortest path furthermore contains the point \( w \) (The same holds for \( v \)).

Because of the triangle inequality, the following holds: \( \|sw\| \leq \|sp\| + \|pw\| \) And we know that \( \|(x, y)\| \leq 1.998 \cdot \|xy\| \) holds for any two nodes of a Delaunay triangulation.

Then:

\[
\frac{\|(s, w)\|}{1.998} \leq \|sw\| \leq \|sp\| + \|pw\| \leq \|(s, p)\| + \|(p, w)\| = \|(s, \ldots, p, \ldots, w)\|
\]

We continue with considering case 1 in which \( s \) and \( t \) are not inside of any convex hull. This following theorem is a conclusion of the so far mentioned properties:
Theorem 15. Let $s$ and $t$ be two nodes of a Restricted Delaunay Graph that do not lie inside of any convex hull. Further, let $(s, c_1, c_2, ..., c_{l-1}, c_l, t)$ be the shortest path in the overlay Delaunay Graph of the cellular infrastructure, where $c_1$ is the convex hull node with the shortest distance from source $s$ to $c_1$ and $c_l$ the convex hull node with the shortest distance from target $t$ to $c_l$. Then we have

1. There is a $1.998 \cdot (\|sc_1\| + \|c_l t\| + \sum_{m=1}^{l-1} d_m)$-path in the Restricted Delaunay Graph from $s$ to $t$, where $d_m := \|c_m c_{m+1}\|$.

2. By applying Chew’s algorithm, we obtain a $5.9 \cdot (\|sc_1\| + \|c_l t\| + \sum_{m=1}^{l-1} d_m)$-path in the Restricted Delaunay Graph from $s$ to $t$, where $d_m := \|c_m c_{m+1}\|$.

Hence, we argue that this approach finds $c$-competitive paths from source $s$ to target $t$ in the Restricted Delaunay Graph. We will prove the theorem with the following two lemmata:

Lemma 16. Let $a$ and $b$ be nodes of different convex hulls. Then there is a $1.998 \cdot \|ab\|$-spanning path between them in the Restricted Delaunay Graph.

and

Lemma 17. Let $a$ and $b$ be adjacent nodes on a convex hull, where $a \neq b$. Then there is a $1.998 \cdot \|ab\|$-spanning path in the restricted Delaunay Graph between $a$ and $b$.

And to prove the second sub-item of the theorem, we consider the following lemmata:

Lemma 18. Let $a$ and $b$ be nodes of different convex hulls. Then there is a $5.9 \cdot \|ab\|$-routing path between them in the Restricted Delaunay Graph.

and

Lemma 19. Let $a$ and $b$ be adjacent node on a convex hull, where $a \neq b$. Then there is a $5.9 \cdot \|ab\|$-routing path in the Restricted Delaunay Graph between $a$ and $b$.

We apply this theorem to the holes, where we define the set $S$ to be all holes of the Restricted Delaunay Graph, i.e. the shortest path between $s$ and $t$ is a polygonal path whose inner nodes are nodes of the holes. In Section 5 we present the algorithm for calculating these convex hulls. Above we explained the reasons of considering convex hulls in our setting. Next, we concentrate on the first sub-item of the theorem. Therefore, we consider the first two lemmas above. We start with Lemma 16.
Proof. Since the Delaunay Graph is a 1.998-spanner of the complete Euclidean graph [26] and the Restricted Delaunay Graph under the lines of nodes of different convex hull nodes contains all edges of the original Delaunay Graph, there always exists a $1.998 \cdot ||ab||$ path in the bay area between two nodes $a$ and $b$ of two different convex hulls. A bay area consists of all nodes and edges that are enclosed by a polygon that consists of the edge between two adjacent convex hull nodes $a$ and $b$ and the line of hole edges and nodes that are positioned between $a$ and $b$. This proves Lemma 16. ◀

We continue with the proof of Lemma 17.

Proof of Lemma 17. We use the observation of Xia that a Restricted Delaunay Graph is a 1.998-spanner of the Unit Disk Graph. Thus there is a $1.998$-competitive path between the to convex hull nodes of a bay area. This proves Lemma 17. ◀

Since Lemma 17 holds for every pair of different convex hull nodes $c_1, ..., c_l$ that belong to the shortest path of the Overlay Delaunay Graph from $s$ to $t$, we have a path of length $1.998 \cdot \sum_{m=1}^{l-1} d_m$, where $d_m := ||c_i c_j||$. We add the shortest path from source $s$ to its closest convex hull node $c_1$ and the shortest path from target node $t$ to its closest convex hull node $c_l$, which is $1.998 \cdot ||sc_1|| + 1.998 \cdot ||ct||$. Thus, we obtain the first sub-item of Theorem 15.

We continue with the second sub-item of Theorem 15.

Proof. Recall that there exists an online routing strategy for Delaunay Graphs which finds a path between any source $s$ and target $t$ with length at most $5.9 \cdot ||st||$. Furthermore, recall that the Restricted Delaunay Graph under the lines of nodes of different convex hull nodes contains all edges of the original Delaunay Graph. Thus, there is a routing strategy from any convex hull node $a$ to any other convex hull node $b$ in cases $a$ and $b$ are nodes of different convex hulls with length at most $5.9 \cdot ||ab||$. This routing strategy can be applied to route in the Restricted Delaunay Graph between to adjacent convex hull nodes $a$ and $b$ as well. This is due to the fact, that the routing strategy chooses the path along triangles in the Restricted Delaunay Graph that are intersected by the line from $a$ to $b$. All nodes that are visited by the routing strategy of this path are in the bay area between $a$ and $b$ or are not inside convex hulls. In case the routing strategy chooses nodes on the bay area, they are connected via edges of the Restricted Delaunay Graph. Thus, the routing strategy applied on the hybrid communication model gives a $5.9 \cdot (||sc_1|| + ||ct|| + \sum_{m=1}^{l-1} d_m)$. All in all, we obtain Theorem 15. ◀

So far, we have considered routing between two nodes under the assumption that both, the source node $s$ and the target node $t$ are nodes of the convex hulls or in the areas outside of them. In later sections, we propose strategies to find routing paths for every other case. The most interesting case is introduced in Section 5, where we consider InCOver which allows us to find competitive paths between nodes of the same bay area.

5 Convex Hulls

In this section, we first describe the distributed preprocessing algorithm that then allows us to apply the parallel convex hull algorithm by [19] to calculate the convex hulls of the Restricted Delaunay Graph’s holes and then present the recursive algorithm InCOver, which computes convex hulls of bay areas recursively. The algorithm by [19] uses a data structure called hypercube for the convex hull calculation. A hypercube is defined as follows
Definition 20. A $d$-dimensional hypercube consists of $n$ nodes, where $n = 2^d$, such that each node has a unique bitstring $(x_1, ..., x_d) \in \{0, 1\}^d$ and there is an edge between two nodes if and only if their bitstring differs in only one bit.

The advantage of this hypercube data structure is that the diameter as well as the degree of a hypercube is $d$ and $d = \log n$ since $\log n$ bits are needed to represent $n$ nodes.

Given only the hole ring, we are interested in calculating hypercubes on these hole nodes. For the construction of the hypercube we use the technique of pointer jumping. On the one hand, this technique enables us to calculate the overlay edges such that the result is the hole ring added with those overlay edges. This builds the hypercube. On the other hand, the technique of pointer jumping can be used for leader election, which is used to determine hypercube id’s for the hole nodes. The leader of the ring is the node with the minimal $x-$coordinate. In case two nodes share the same $x-$coordinate, the $y-$coordinate is used as tie breaker since we assume that no two nodes share exactly the same coordinates. In the following, we give the idea of designing a hypercube of a hole by using pointer jumping and argue that this design leads to the minimal id leader election. The pseudocode-description of the algorithm can be found in Appendix A.1 (Algorithm 1). A hole is a ring, thus each node of this ring has exactly two adjacent nodes that are hole nodes as well, i.e. a predecessor and a successor with distance at most 1. Hence, each node of the hole ring is able to communicate ad hoc and for free with those two nodes via the edges that connect them. To communicate directly not only with its predecessor and its successor of a hole node $v$, we are interested in adding overlay edges, i.e. edges of length more than one, to the hole ring. Therefore, we build a hypercube on all nodes of this hole ring as follows: Let $v$ be a node of the hole ring and let $\text{pred}_0$ be its predecessor and $\text{succ}_0$ be the successor of $v$ on the hole ring. In round 1 of the protocol, $v$ introduces $\text{succ}_0$ to $\text{pred}_0$ to each other. Thus $\text{succ}_0$ and $\text{pred}_0$ become adjacent nodes and an overlay edge is established to connect the two nodes. In addition, the id’s of $v$, $\text{succ}_0$, and $\text{pred}_0$ are compared and the $\min\{id_v, id_{\text{succ}_0}, id_{\text{pred}_0}\}$ is added to the new overlay edge.

Since there is this new overlay edge between $\text{succ}_0$ and $\text{pred}_0$, $\text{pred}_0$ becomes the new predecessor $\text{pred}_1$ of $\text{succ}_0$ and $\text{succ}_0$ is adjacent to a hole node $\text{succ}_1$ via an overlay edge, $\text{succ}_0$ introduces $\text{succ}_1$ to $\text{pred}_1$ and $\text{pred}_1$ to $\text{succ}_1$. Thus a new overlay edge is established between them and the minimal id of nodes skipped by the edge is exchanged. In particular, in each round $t$, each node $v$ of the hole ring introduces its predecessor $\text{pred}_t$ to its successor $\text{succ}_t$ and $\text{succ}_t$ to $\text{pred}_{t+1}$ and adds a new overlay edge $e_t$ to the ring. Since the hop distance between neighbors doubles from round to round, this protocol stops after $O(\log k)$ rounds, where $k$ is the number of hole ring nodes. While establishing each overlay edge $e_t$ in round $t$, we also choose the minimal id of $v$, $\text{pred}_t$, and $\text{succ}_t$ and compare it in the next round $t + 1$ with the corresponding id’s of the $\text{pred}_{t+1}$ and $\text{succ}_{t+1}$. Finally, the last round ends with $\text{min}_{id}$. This is the node with the minimal id of all id’s of the hole ring nodes and it is denoted by $\text{min}_{id}$.

Consider the case that $k$ is not a power of 2 and $k_1$ is the largest power of 2 such that $k_1 < k$. Let $k_1, ..., k_m$, be the indices of the nodes that are in the interval from $k_2$ to $k$, where $k_1 < k_2 < ... < k_m$. Then the node with index $k_1$ represents $\{k_1, k_2, ..., k_m\}$.

With the above protocol of pointer jumping, we built hypercube edges. Recall that the node id’s of the hypercube are bitstrings of length $O(\log k)$. To broadcast the hypercube id’s to the corresponding hole node, we select a leader of the hole node ring which is $\text{min}_{id}$.

After the leader election of $\text{min}_{id}$, $\text{min}_{id}$ broadcasts the hypercube id’s to the corresponding ring node. Obviously, that takes $O(\log k)$ rounds.
To sort all ring hole nodes \( \{r_1, \ldots, r_k\} \) such that the x-coordinate of \( r_1 \) is less than the x-coordinate of \( r_k \), we use Batcher’s Bitonic Sort on the hypercube of the hole ring with parallel runtime \( \mathcal{O}(\log^2 n) \) \[2\]. Finally, we are ready to apply the parallel convex hull algorithm by \[19\], which takes in \( \mathcal{O}(\log n) \) rounds. More formally, we present the following pseudocode of the described algorithm above.

\[\begin{align*}
\textbf{Theorem 21.} & \quad \text{Given a hole ring with } k \text{ nodes, the convex hull of this hole ring can be calculated in } \mathcal{O}(\log^2 k). \\
\textbf{Corollary 22.} & \quad \text{The message complexity for computing a convex hull of a hole ring with } k \text{ nodes is } \mathcal{O}(k \log k).
\end{align*}\]

\[\textbf{Proof of Corollary 22.} \quad \text{Each node receives } \mathcal{O}(\log k) \text{ messages during the execution of Batcher’s Bitonic Sort and the parallel convex hull algorithm of Miller which results in a total message complexity of } \mathcal{O}(k \log k). \text{ It remains to argue that the construction of the hypercube has the same message complexity. Each node of a hole ring introduces in every round two of its neighbors to each other. Hence, a node sends two messages per round. As the number of rounds is } \mathcal{O}(\log k), \text{ the total costs are } \mathcal{O}(k \log k). \]

To conclude the convex hull algorithm, we analyze the storage demands for each node lying on the perimeter of a hole.

\[\begin{align*}
\textbf{Corollary 23.} & \quad \text{For computing a convex hull of } k \text{ nodes, } \mathcal{O}(\log k) \text{ references have to be stored at each node.} \\
\textbf{Proof of Corollary 23.} & \quad \text{Observe that a hypercube is structure with logarithmic degree. Hence, each node has to store } \mathcal{O}(\log k) \text{ references to its hypercube neighbors. Note that the pointer doubling technique for computing the hypercube does not require more storage, as each node gets introduced two neighbors per round.}
\end{align*}\]

Moreover, we take a look at the storage demands for the computation of a convex hull. Initially, Batcher’s Bitonic Sort is applied which does not incur more costs as each node compares values of hypercube neighbors per round. Miller’s algorithm requires \( \mathcal{O}(\log k) \) storage per node \[19\]. Since each nodes has at most two convex hull neighbors, the total space requirements are \( \mathcal{O}(\log k) \).

Recall that we so far completely ignored getting information about the inside of convex hulls. In the cases 1-4 of Section 3, the only information about the graph that \( s \) and \( t \) have, are the position of their closest convex hull nodes and a shortest path to them. In cases 1, 2 and 3, the routing strategy is the one we provided in the previous sections and we do not need any additional information about bay areas. In case 4, where \( s \) and \( t \) are inside the same convex hull and in the same bay area, it is not sufficient to use the mentioned strategy. Thus, we are interested into getting information about the structure this bay area has. To obtain this information, we calculate an overlay called Inner Convex Hull Overlay (InCOver): We consider each pair of adjacent nodes of the same convex hull. Recall that the nodes on the perimeter of a hole form ring. For each pair of adjacent nodes of a convex hull, this ring is separated into lists from each convex hull node to its neighbor. To obtain additional graph information about shortest paths inside of convex hulls, two neighbored
convex hull nodes store the following additional information: A Dominating Set of all nodes at distance at most $\frac{1}{2}$ from nodes lying on the hole ring in the bay area in between. The calculation of the Dominating Set can be done with the algorithm of Jia et al., achieving a $O(\log \Delta)$ approximation of a smallest possible Dominating Set, where $\Delta$ denotes the degree of the network [14]. This approximation requires $O(\log \Delta \cdot \log \Delta)$ communication rounds with high probability. This approach is based on two main observations. **Observation 1:** The neighbors of the convex hull are aware of the mentioned Dominating Set of their bay area. With this knowledge, they are able to compute competitive paths as follows. They compute a Delaunay Graph of all nodes of the mentioned Dominating Set and apply afterwards a standard single source shortest path algorithm like the algorithm of Dijkstra [10]. Note that this approach also finds $c$-competitive paths. Although not every node of the area at distance $\frac{1}{2}$ from the perimeter of the hole is contained in the Dominating Set, the convex hull nodes have a sufficient approximation of the area, as every missing node can be found with only one additional hop. This observation in combination with the observation that shortest paths would use nodes lying on the perimeter of a hole (Section 2) implies that we can still find $c$-competitive paths with this reduction. **Observation 2:** Additionally, we can conclude that the storage demands of the affected convex hull nodes is small. Based on the observations in Section 3, the size of a hole is independent of the network size. Let $H$ be the set of all holes in the network and $P(h)$ the perimeter of a hole $h \in H$. We obtain the following storage bound: $O \left( \max_{h \in H} P(h) + \sum_{h \in H} P(\text{Bounding Box of } H) \right)$.

After computing convex hulls, we are interested in connecting all convex hulls in an Overlay-structure. The chosen Overlay is a clique of all convex hull nodes. Since a convex hull node can locally not decide in which direction it can find other convex hull nodes, we decided to use broadcast mechanisms in order to built a clique of all convex hull nodes. Since broadcast need at least as many rounds as the diameter of the network, this step can take up to $O(n)$ communication rounds. Afterwards, each convex hull node is able to compute a Delaunay Graph of all convex hull nodes locally. Therefore, the algorithm of Guibas et al. can be applied [11].

Furthermore, we need a second broadcast-mechanism that ensures that each node of the ad hoc network gets to know its closest convex hull node such that shortest path-queries can be directed to these nodes. In this approach, we only use local broadcasts. This means that not every broadcast is flooded through the whole system. Instead, a node that receives two broadcast messages is able to decide which of these message has to forwarded to which neighbor (since each node knows the geographical positions of its neighbors).

Finally, nodes of the ad hoc network are able to send shortest path queries to their closest convex hull nodes. The shortest path query contains the coordinates of a source node $s$ and a target node $t$. The convex hull node that receives the query can insert $s$ and $t$ into the Delaunay Graph of convex hull nodes and compute a shortest path between $s$ and $t$ afterwards. The integration into the Delaunay Graph takes only constant time on expectation when applying the algorithm of Guibas et al. [11]. A shortest path is computed with the variation of Dijkstra’s algorithm using Fibonacci Heaps [10] with runtime $O(|V| \log |V| + |E|)$. In our scenario, we obtain a runtime of $O(k \log k)$ where $k$ denotes the number of all convex hull nodes, since $E \in O(V)$ for planar graphs.
6 Skip Delaunay Graphs

Dissemination of convex hull information needs up to $O(n)$ communication rounds. We are interested in reducing the number of communication rounds needed for broadcasting. This can only be achieved by reducing the diameter of the Overlay Network. Therefore, we propose an additional Overlay Network which creates a Skip Delaunay Graph out of all participants. Skip Delaunay Graphs are formally introduced in Section 6.1. Skip Delaunay Graphs have a diameter of $O(\log n)$ with high probability. The protocol for establishing a Skip Delaunay Graph is an extension of the protocol proposed by Jacob et al. [13] and is introduced in Section 6.2. By using Skip Delaunay Graphs and broadcasts, the following theorems can be proven easily:

▶ Theorem 24. All convex hull nodes form a clique after $O(\log n)$ communication rounds with high probability.

▶ Theorem 25. Each node of the ad hoc network gets to know its closest convex hull node after $O(\log n)$ communication rounds with high probability.

6.1 Skip Delaunay Graphs

Our protocol is highly dependent on broadcast mechanisms. Since broadcasts need up to $O(n)$ communication rounds in Restricted Delaunay Graphs, we propose an idea that reduces the number of communication rounds to $O(\log n)$ with high probability. In the following section, we introduce Skip Delaunay Graphs which are an extension of Skip Graphs [1] for the two-dimensional case. The main idea is to establish a hierarchical Delaunay Graph of a point set $V \subset \mathbb{R}^2$. The lowest level (level 0) of the hierarchy contains the standard Delaunay Graph of $V$. Level 1 consists of two Delaunay Graphs, Level 2 comprises 4 Delaunay Graphs and so on. Which Delaunay Graphs a node $v$ belongs to is controlled by a membership vector $r(v)$. $r(v)$ is an infinite bit string which is chosen uniformly at random. The first $i$ bits of any bit string $r(v)$ are denoted as $\text{prefix}_i(v)$. Consider a fixed point set $V \subset \mathbb{R}^2$. For $u, v, w \in V$, $\triangle_{uvw}$ denotes the triangle with endpoints $u, v$ and $w$ for which $\text{prefix}_i(u) = \text{prefix}_i(v) = \text{prefix}_i(w)$.

▶ Definition 26 (Skip Delaunay Graph). A geometric graph $G = (V, E)$ is called Skip Delaunay Graph if $V \subset \mathbb{R}^2$ and a triangle $\triangle_{uvw}$ is contained in $E$ if and only if $\text{prefix}_i(u) = \text{prefix}_i(v) = \text{prefix}_i(w)$ and $\bigcup_{i \geq 1} \text{prefix}_i(v)$ does not contain any node $z$ with $\text{prefix}_i(z) = \text{prefix}_i(u) = \text{prefix}_i(v) = \text{prefix}_i(w)$ for all $i \in \mathbb{N}_{0}$. Additionally, each $v \in V$ stores a node to the closest node $u \in V$ with $\text{prefix}_{i-1}(v) = \text{prefix}_{i-1}(v)$ and $\text{prefix}_i(v) \neq \text{prefix}_i(v)$ for all $i \in \mathbb{N}_{0}$. These nodes are denoted as $\text{broadcastNeighbor}_i(v)$.

The following proofs are equivalent to proofs for Skip Graphs proving similar properties [1].

▶ Lemma 27. The number of non empty levels in Skip Delaunay Graphs is $O(\log n)$ with high probability.

Proof of Lemma 27 We prove Lemma 27 by proving that there is no pair of nodes which shares a prefix of length larger than $3 \cdot \log n$ with high probability. The probability that two nodes $u$ and $v$ share a prefix of length $i$ is $\frac{1}{2}$ since the bit strings are chosen uniformly at random. Formally, we analyze the probability that there is a pair of nodes which share a prefix of length $i$. Consider the binary random variable $X_i$, which is true if and only if there
is a pair of nodes $u, v$ with $\text{prefix}_i(u) = \text{prefix}_i(v)$. Further $X^i_{u,v}$ denotes a second binary random variable which is true if and only of $\text{prefix}_i(u) = \text{prefix}_i(v)$.

\[
\Pr[X_i = 1] \leq \sum_{u,v} \Pr[X^i_{uv} = 1] = \sum_{u,v} \frac{1}{2^i} = \frac{n^2}{2i}.
\]

Hence, for $i \geq 3 \log n$, it holds that

\[
\Pr[X_i = 1] \leq \frac{n^2}{2i} = \frac{n^2}{2^3 \log n} = \frac{n^2}{n^3} = \frac{1}{n}.
\]

Consequently, the probability that there is no pair of nodes which share a prefix of length at least $3 \log n$ is at least $1 - \frac{1}{n}$. Thus, the number of non empty levels in Skip Delaunay Graphs is $O(\log n)$ with high probability.

In the following, we analyze the diameter of Skip Delaunay Graph and prove that the diameter is $O(\log n)$ with high probability.

**Theorem 28.** The diameter of Skip Delaunay Graphs is $O(\log n)$ with high probability.

**Proof of Theorem 28.** The proof idea is to construct a path between any pair of nodes which has length $O(\log n)$ with high probability. Consider a fixed node $u$. We construct a path from $u$ to $v_0$ which is the node with longest prefix that only comprises 0’s. We start at $u$ and switch to a node $u_1$ with $\text{prefix}_1(u_1) = 0$. Afterward, we walk to $u_2$ which is a node with $\text{prefix}_2(u_2) = 00$ and so on until finally reaching $v_0$. Observe that the $\text{broadcastNeighbor}$-references allow us to switch from $u_i$ to $u_{i+1}$ within a single hop. Due to Lemma 27 we are aware of the existence of at most $3 \log n$ levels with high probability and thus the length of a path from $u$ to $v_0$ is $O(3 \log n) = O(\log n)$ with high probability. To construct a shortest path between any pair of nodes $u, w$ we initially route from $u$ to $v_0$ and afterwards from $v_0$ to $w$. The total path length is $O(\log n) + O(\log n) = O(\log n)$ with high probability. This proves Theorem 28.

### 6.2 Skip Delaunay Protocol

In order to use Skip Delaunay Graphs in our setting, we have to extend the self-stabilizing Delaunay protocol [13] to a protocol which is able to build a self-stabilizing Skip Delaunay Graph. When considering self-stabilizing protocols for Skip Delaunay Graphs, we are faced with the same challenges as for self-stabilizing protocols for Skip Graphs. Skip Delaunay Graphs (similar to Skip Graphs [12]) cannot locally check the correctness of their structure (see Figure 7). In Skip+ Graphs, researchers introduced ranges of nodes in order to allow nodes to locally check correctness of the structure. It is not clear how ranges of a Skip+ Graph can be mapped to the Euclidean metric. Therefore, we use a broadcast mechanism that guarantees that all nodes sharing the same prefix finally build a connected component such that the Delaunay protocol can build a Delaunay Graph for each prefix.

To prevent the system from getting congested by too many broadcasts, we introduce a mechanism that guarantees at least in stable states that each broadcast is only forwarded twice on expectation. The idea is called local broadcasting. In order to do so, each node stores an additional edge per level. For a fixed node $v$, we denote the geographically closest node $w$ with $\text{prefix}_i(v) = \text{prefix}_i(v)$ and $\text{prefix}_{i+1}(w) \neq \text{prefix}_{i+1}(v)$ by $\text{broadcastNeighbor}_i(v)$ for all $i$.

Our focus is to introduce a protocol that establishes connected components for each prefix.
Figure 7 A (non stable) Skip Delaunay Graph. Edges of level 1 are marked in green. In the local views of nodes $v$ and $u$, the Skip Delaunay Graph is correct.

$i$ such that the Delaunay-protocol can be executed for each prefix in order to build a Skip Delaunay Graph. The protocol comprises four subroutines, which are introduced now. The concrete pseudocode-descriptions can be found in Appendix A.2.

The first subroutine is the timeout-procedure which is called $\text{onTimeOutSDStab}$ (Algorithm 2). Initially, each node $v$ checks whether there is any node in its neighborhood that is closer to any $\text{broadcastNeighbor}_i(v)$-reference and has the same prefix. In case such a node exists, the local reference is updated. Afterwards, a node $v$ introduces itself to all $\text{broadcastNeighbor}_i(v)$-references it stores in order to detect all wrong references. Finally, it starts a broadcast on all levels such that it gets to know nodes which share the same prefix.

$\text{introduceToBroadcastNeighbor}(w, i)$ (Algorithm 3) is called after any node $v$ receives an introduction message. After receiving the message $v$ checks whether $\text{prefix}_i(v) = \text{prefix}_i(w)$. In cases the prefixes do not coincide up to the $i$-th bit, $v$ responds by sending a rejection message back, because $v$ cannot be a broadcast-neighbor of $w$ for level $i$.

The third subroutine is called $\text{rejectBroadCastNeighbor}(w)$ (Algorithm 4) and is executed after a rejection-message is received. The receiving node checks whether the initiator $w$ is stored as a reference locally. In case a reference to $w$ is stored locally, the reference is deleted.

The last procedure is called $\text{broadcastSkipDelaunay}(w, i)$ (Algorithm 5) and is called after a node of the Skip Delaunay graph $v$ receives a broadcast message of any node. Broadcast-messages are initiated in $\text{onTimeOutSDStab}$ in order to find neighbors on level $i + 1$. Hence, a node checks locally whether it could be a neighbor on level $i + 1$. In case $\text{prefix}_{i+1}(v) = \text{prefix}_{i+1}(w)$, $v$ responds by sending an introduction message back, such that the Delaunay protocol builds a correct Delaunay structure on level $i + 1$. Provided that $v$ and $w$ differ in the $i + 1 - th$ bit, $v$ checks whether it has stored a reference to $\text{broadcastNeighbor}_i(v)$. Due to the definition of $\text{broadcastNeighbor}_i(v)$, there has to be stored a node that shares the first $i$ bits with $v$. There are still two cases to consider. Either $\text{broadcastNeighbor}_i(v) = w$ or not. In case $\text{broadcastNeighbor}_i(v) = w$, the broadcast message has to be further forwarded such that $w$ can get to know further neighbors on level $i + 1$. Otherwise, $w$ is introduced to $\text{broadcastNeighbor}_i(v)$.

The section is continued with the proof of Theorem 29 which states that the proposed Skip Delaunay Protocol is a self-stabilizing protocol.

- **Theorem 29.** $\text{SDStab}$ is able to build a Skip Delaunay Graph out of any weakly connected
state after expected $O(n^3)$ communication rounds.

Theorem 29 is split into three lemmas. Lemma 30 states that all nodes which share a common prefix are in a connected component after a finite amount of protocol executions. Lemma 31 states that the protocol converges after $O(n^3)$ communications rounds on expectation. Lemma 32 finally deals with the Closure-property of the protocol.

Lemma 30. After a Delaunay Graph of level $i$ has stabilized, it needs expected $O(n^2)$ communication rounds until all nodes with same prefix of length $i+1$ build a connected component.

Proof of Lemma 30. We prove Lemma 30 by induction on the number of levels $i$.

Induction Base: $i = 0$

Due to the proof of Jacob et. al [13], we know that the Delaunay Graph of level 0 stabilizes after a finite amount of protocol executions. We take a look at the Voronoi Diagram of all nodes $v$ with $\text{prefix}_1(v) = 0$. The Voronoi cell of a node $v$ is denoted as $C_v$. Due to the Voronoi property, all nodes $w \in C_v$ with $\text{prefix}_1(w) = 1$ have to store a reference to $v$ in $\text{broadcastNeighbor}_0(w)$. The broadcast guarantees that a broadcast of node $v$ reaches after $O(n)$ communication rounds all nodes in $C_v$. Hence, all nodes have a correct reference to $\text{broadcastNeighbor}_i$ after $O(n)$ communication rounds. Each succeeding broadcast of $v$ is forwarded until it reaches a neighboring Voronoi cell. In each neighboring Voronoi Cell, all nodes have stored a node $x$ with $\text{prefix}_{i+1}(x) = \text{prefix}_{i+1}(v)$ in their $\text{broadcastNeighbor}_i$-reference. Hence, $v$ gets to know a potential neighbor of level $i + 1$. Since the Voronoi diagram is a connected graph, we finally obtain a connected component of all nodes with prefix 0. The same arguments can be applied for all nodes with prefix 1. Thus, the Delaunay protocol builds a Delaunay Graph of all nodes with first bit 0 and an additional Delaunay Graph of all nodes with first bit 1 after a finite amount of protocol executions.

Induction Hypothesis: After all Delaunay Graphs on level $i - 1$ are built, connected components of all nodes which share a prefix of length $i$ are obtained.

Induction Step: $i \rightarrow i + 1$:

Based on the induction hypothesis, we know that the Delaunay protocol finally builds all Delaunay Graphs of level $i$. Hence, we can apply the same argumentation we have already used for the Induction base. Consider all nodes with a fixed prefix $i$ and their Voronoi Diagram. For a single node $v$ it holds, that all nodes $w \in C_v$ with $\text{prefix}_{i+1}(w) \neq \text{prefix}_{i+1}(v)$ have to store $v$ in $\text{broadcastNeighbor}_i(w)$. Since the expected number of nodes sharing a prefix of length $i$ is $O(\frac{n}{2})$, we know that these references are correctly set after an expected number of $O(\frac{n}{2})$ communication rounds. Finally, connected components of all nodes that share a prefix of length $i + 1$ are obtained after an expected number of $O(\frac{n}{2})$ communication rounds.

Lemma 31 (Convergence of SDStab). The Skip Delaunay Protocol builds a Skip Delaunay Graph out of any weakly connected initial network state after expected $O(n^3)$ communication rounds.

Proof of Lemma 31. The correctness of Lemma 30 and the Delaunay protocol [13] helps us to conclude that the Skip Delaunay Graph finally stabilizes. It remains to prove the
complexity of the protocol. For each level we have to consider the number of communication rounds it takes until all connected components are built and additionally the number of communication rounds the Delaunay-protocol needs to finally obtain each Delaunay Graph. Note that the expected number of nodes in a connected component of level \( i \) in a Skip Delaunay Graph is \( O(n^{2i}) \) and there are not more than \( 3\log n \) levels with high probability (Lemma 27). First of all, we calculate the expected number of communication rounds which are needed to obtain all connected components. The expected number is upper bounded by:

\[
\sum_{i=0}^{3\log n} \frac{n}{2^i} = n \sum_{i=0}^{3\log n} \frac{1}{2^i} = 2n - \frac{1}{n^2}
\]

Hence, the expected amount of communication rounds is upper bounded by \( O(n) \). Further we have a bound on the stabilization work for each prefix:

\[
\sum_{i=0}^{3\log n} 2^i \cdot \left(\frac{n}{2^i}\right)^3 = n^3 \cdot \sum_{i=0}^{3\log n} \left(\frac{1}{4}\right)^i = \frac{4}{3} n^3 - \frac{1}{3n^2}
\]

Thus, the total number of communication rounds needed by the Delaunay protocol is \( O(n^3) \).

Finally, we have proven that the Skip Delaunay Protocol is able to build a Skip Delaunay Graph out of any weakly connected initial state after not more than \( O(n^3) \) communication rounds on expectation.

\[\boxed{\text{Lemma 32. SDStab fulfills Closure.}}\]

Finally, we have introduced a protocol which is able to build a Skip Delaunay Graph out of any initial weakly connected network state. By using the Skip Delaunay Graph instead of a Restricted Delaunay Graph as basis for our protocols, we can obtain a speed-up for all broadcasts. Nevertheless, we increased the costs for the Overlay Network since the broadcasts cannot be sent through the ad hoc network any more. Since the concrete message complexity for DStab has not been analyzed in [13], we can not give a sharp bound on the costs at this point. Nonetheless, costs are polynomial as each node communicate only with its local neighbors in each communication round and there are \( O(n^3) \) communication rounds until stabilization is achieved.

\section{Further Work}

In this work we introduced a new model to find competitive routing paths in ad hoc networks when only geographical coordinates of the source and the target are known. It turned out that the runtime of our distributed algorithm for computing convex hulls is highly dependent on initial sorting of coordinates. Batcher’s bitonic sort is able to sort \( n \) numbers in time \( O(\log^2 n) \). There are several parallel sorting algorithms (deterministic or randomized) which achieve sorting of \( n \) numbers in (expected) time \( O(\log n) \), for example the algorithms of Reif and Valiant [23] or Cole [8]. One of our next research goals could be to analyze if any of these solutions is applicable in our scenario. Moreover, the current solution for finding paths
between nodes inside of convex hulls. INCOVER needs $O(n^3)$ communication rounds. An interesting open question is whether an Overlay can be found that contains competitive paths for nodes inside of convex hulls and can be built in polylogarithmic runtime. Furthermore, we considered non intersecting convex hulls and a static setting. Following research could analyze how competitive paths can be obtained in cases where convex hulls intersect. Additionally, our setting naturally contains movements of participants. Further research could analyze how to maintain the Overlay network efficiently if participants are able to change their positions and additionally considering joining and leaving nodes.

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## A Omitted Pseudocode Descriptions

In the following sections, we provide the omitted pseudocode descriptions of the convex hull algorithm and the descriptions of the Skip Delaunay protocol.

### A.1 Convex Hull Algorithm

**Algorithm 1** The Algorithm for the distributed computation of a convex hull

```plaintext
1: procedure ConvexHull ⋆ executed in node v
2: currentPred := id(leftHoleNeighbor)
3: currentSucc := id(rightHoleNeighbor)
4: leftMin := ∞
5: rightMin := ∞ − 1
6: while leftMin != rightMin do
7:   currentPred ← introduce(currentSucc, min(leftMin, id(v)))
8:   currentSucc ← introduce(currentPred, min(rightMin, id(v)))
9:   update currentPred, currentSucc, leftMin
    and rightMin upon Introduction
10: if id(leftMin) = id(v) then
11:   distribute Hypercube IDs
12: else
13:   wait for Hypercube ID
14: Apply Batcher’s Bitonic Sort
15: Apply Miller’s Parallel Hypercube Algorithm
```

### A.2 Skip Delaunay Protocol

In the following section, the omitted pseudocode descriptions of the Skip Delaunay protocol can be found.

**Algorithm 2** The Timeout-Procedure of the Skip Delaunay Protocol(executed in node v)

```plaintext
1: procedure onTimeoutSDStab( )
2: for all u ∈ N(v) do
3:   for all i do
4:     if prefixi(u) = prefixi(v) and prefixi+1(u) ≠ prefixi+1(v) then
5:       if ∥uv∥ ≤ ∥v broadcastNeighbori(v)∥ then
6:         broadcastNeighbori(v) := u
7:   for all i do
8:     broadcastNeighbori(v) ← introduceToBroadcastNeighbor(v, i)
9: for all i do
10:   for all u ∈ N(v) do
11:     if prefixi(u) = prefixi(v) then
12:       u ← broadcastSkipDelaunay(v, i)
```
Algorithm 3 The procedure which is called after a node introduces itself to a broadcast-neighbor-reference (executed in node $v$)

1: procedure introduceToBroadcastNeighbor($w$, $i$)
2: \textbf{if} $\text{prefix}_i(w) \neq \text{prefix}_i(v)$ \textbf{then}
3: $w \leftarrow \text{rejectBroadcastNeighbor}(v)$

Algorithm 4 The procedure which is executed after a node receives a rejection message (executed in node $v$)

1: procedure rejectBroadcastNeighbor($w$)
2: for all $i$ do
3: \textbf{if} $\text{broadcastNeighbor}_i(v) = w$ \textbf{then}
4: $\text{broadcastNeighbor}_i(v) := \text{NIL}$

Algorithm 5 The broadcast-procedure of the Skip Delaunay protocol (executed in node $v$)

1: procedure broadcastSkipDelaunay($w$, $i$)
2: \textbf{if} $\text{prefix}_{i+1}(v) = \text{prefix}_{i+1}(w)$ \textbf{then}
3: $w \leftarrow \text{introduceDelaunay}(v)$
4: else
5: \textbf{if} $\text{prefix}_i(v) = \text{prefix}_i(w)$ and $\text{prefix}_{i+1}(v) \neq \text{prefix}_{i+1}(w)$ \textbf{then}
6: \textbf{if} $\|vw\| \leq \|v \text{broadcastNeighbor}_i(v)\|$ \textbf{then}
7: $\text{broadcastNeighbor}_i(v) := w$
8: \textbf{if} $\text{broadcastNeighbor}_i(v) = w$ \textbf{then}
9: for all $u \in N(v)$ do
10: \textbf{if} $\text{prefix}_i(u) = \text{prefix}_i(w)$ \textbf{then}
11: $u \leftarrow \text{broadcastSkipDelaunay}(v, i)$
12: else
13: $\text{broadcastNghbor}_i(v) \leftarrow \text{introduceDelaunay}(w)$