A CONVEXITY PROPERTY FOR THE $SO(2,\mathbb{C})$-DOUBLE COSET DECOMPOSITION OF $SL(2,\mathbb{C})$ AND APPLICATIONS TO SPHERICAL FUNCTIONS

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Abstract. We make a fine study of the $SO(2,\mathbb{C})$-double coset decomposition in $SL(2,\mathbb{C})$ and give a full description of the intersection of the various cells with the complex crown $\Xi$ of $SL(2,\mathbb{R})/SO(2)$. A non-linear convexity theorem is proved and applications to analytically continued spherical functions are given.

1. Introduction

With a Riemannian symmetric space $X = G/K$ of the non-compact type comes a natural complexification $\Xi$, the so-called complex crown of $X$. One can define $\Xi$ in various ways. To begin with let $G = NAK$ be an Iwasawa decomposition and let $a = \text{Lie}(A)$ be the Lie algebra of $A$. Studying proper $G$-actions on $X_C = G_C/K_C$ Akhiezer and Gindikin were led to the following definition (cf. [1])

\[(1.1) \quad \Xi = G \exp(i\Omega)K_C/K_C,\]

where $\Omega \subseteq a$ is a certain bounded convex set (cf. Section 2). Observe that $\Xi$ is a $G$-invariant domain in $X_C$ containing $X$. Moreover, the definition of $\Xi$ is independent from the choice of $a$ and hence $\Xi$ is generically defined through $X$.

Equivalently, one can define $\Xi$ by

\[(1.2) \quad \Xi = \left( \bigcap_{g \in G} gN_C A_C K_C/K_C \right)_0,\]

where $(\cdot)_0$ refers to the connected component of $(\cdot)$ containing $X$ (cf. [3, 9, 13], [2, 11, 4, 10, 12] and [13, 17]). Using the complex convexity theorem from [6] one can significantly improve on (1.2), namely

\[(1.3) \quad \Xi = \left( \bigcap_{g \in G} gN_C A \exp(i\Omega)K_C/K_C \right)_0.\]

The various definitions of $\Xi$ are useful. In [14], where the foundations for the $G$-invariant complex geometry of $\Xi$ were laid out (construction of natural $G$-invariant psh exhaustion functions, Kähler metrics etc.), one mainly used the description (1.1). For other purposes such as analytic continuation of eigenfunctions or the heat kernel on $X$ to holomorphic functions on $\Xi$ (cf. [13, 14]) the characterizations in (1.2) and in particular (1.3) are more appropriate.

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One of the main objectives in the study of complex crowns is to achieve a better understanding of harmonic analysis on symmetric spaces. For example it was shown in [5] that all non-compactly causal symmetric spaces \( G/H \) associated to \( G \) appear in the so-called distinguished boundary of \( \Xi \). Subsequently this was used in [7, 8] to construct a Hardy-space for non-compactly causal symmetric spaces \( G/H \) yielding first progress towards a geometric realization of some of the continuous series in \( L^2(G/H) \) (Gelfand-Gindikin program).

The motivation for this article stems from our interest to obtain a first understanding of the various kinds of weighted Bergman-spaces which can be associated with \( \Xi \), most notably Fock-spaces which play a prominent role in the context of the heat kernel transform on \( X \) (cf. [14]). Intimately related to this circle of problems is the growth behavior of analytically continued spherical functions.

A spherical function \( \phi_\lambda \) on \( X \) is left \( K \)-invariant, hence its analytic continuation \( \tilde{\phi}_\lambda \) to \( \Xi \) is necessarily (locally) \( K_C \)-invariant. This suggests that one should study the relation of \( \Xi \) with the complexified polar decomposition \( \Xi_P = K_C A_C K_C / K_C \) in \( X_C \).

Whereas the complex crown \( \Xi \) is contained in the complexified Iwasawa decomposition \( N_C A_C K_C / K_C \) (cf. [12]), the same does not hold for the complexified polar decomposition, i.e., \( \Xi \not\subseteq \Xi_P \). Furthermore, the polar domain \( \Xi_P \) is not even open in \( X_C \). These are disappointing facts. But before giving up, we at least wanted to understand what actually goes “wrong” in the most important case of \( G = SL(2, \mathbb{R}) \).

It turned out that the above mentioned difficulties can be overcome, and, moreover, by settling the technical problems we could make new observations regarding the structure theory of complex crowns.

This paper focuses mostly on \( G = SL(2, \mathbb{R}) \), but we made an effort to present results in a fashion which allows immediate generalization to all semisimple Lie groups. An approach in full generality will be given elsewhere.

Let us now describe some of our results in more detail. First observe that the polar domain \( \Xi_P \) contains a Zariski open subset of \( X_C \); in particular it is dense in \( X_C \). Thus almost all elements in \( \Xi \) are contained in \( \Xi_P \). Our main observation is the following non-linear convexity theorem:

**Theorem 4.1.** Let \( G = SL(2, \mathbb{R}) \) and \( x = g \exp(iY)K_C \in \Xi \cap \Xi_P \) with \( g \in G \) and \( Y \in \Omega \) (cf. (1.1)). Then

\[
x \in K_C A \exp(\text{conv}(WY))K_C
\]

with \( \text{conv}(WY) \) the convex hull of the Weyl group orbit of \( Y \).

We conjecture that Theorem 4.1 holds true for all semisimple Lie groups and we provide additional evidence with a discussion of the Lorentz groups \( G = SO_c(1, n) \).

Using general results of [13] one obtains from Theorem 4.1 the following estimate:

**Theorem 5.2.** Let \( G = SL(2, \mathbb{R}) \). Let \( \phi_\lambda \) be a positive definite spherical function on \( X \) and \( \tilde{\phi}_\lambda \) its analytic continuation to \( \Xi \). Then \( \tilde{\phi}_\lambda \) is bounded.

Finally, as an application of our methods we give an estimate for the \( A_C \)-projection in \( \Xi_P \) for elements in \( \Xi \).

It is our pleasure to thank the referee for his very careful reading and his useful suggestions. In particular the easy proof of Theorem 5.3 is due to him.
2. Notation and general facts

Let \( g \) be a real semisimple Lie algebra and \( g = \mathfrak{k} + \mathfrak{p} \) a Cartan decomposition. For a maximal abelian subspace \( \mathfrak{a} \) of \( \mathfrak{p} \) let \( \Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^* \) be the corresponding restricted root system. Then \( g \) admits a root space decomposition

\[ g = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha, \]

where \( \mathfrak{m} = z_{\mathfrak{k}}(\mathfrak{a}) \) and \( \mathfrak{g}^\alpha = \{ X \in g : (\forall H \in \mathfrak{a}) [H, X] = \alpha(H)X \} \).

For a fixed positive system \( \Sigma^+ \) define \( n := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha \). Then we have the Iwasawa decomposition on the Lie algebra level:

\[ g = \mathfrak{k} \oplus \mathfrak{a} \oplus n. \]

We write \( W = N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \) for the corresponding Weyl group. For an element \( X \in \mathfrak{a} \) we denote by \( \text{conv}(W X) \) the convex hull of the Weyl group orbit of \( X \).

For any real Lie algebra \( l \) we write \( l_C \) for its complexification.

In the sequel \( G_C \) will denote a simply connected Lie group with Lie algebra \( g_C \). We write \( G, K, K_C, A, A_C, N \) and \( N_C \) the analytic subgroups of \( G_C \) corresponding to subalgebras \( \mathfrak{g}, \mathfrak{k}, \mathfrak{k}_C, \mathfrak{a}, \mathfrak{a}_C, \mathfrak{n} \) and \( \mathfrak{n}_C \), respectively.

The following bounded and convex subset of \( \mathfrak{a} \) plays a central role:

\[ \Omega := \{ X \in \mathfrak{a} : |\alpha(X)| < \frac{\pi}{2} \quad \forall \alpha \in \Sigma \}. \]

With \( \Omega \) we define a left \( G \) and right \( K_C \)-invariant domain in \( G_C \) by

\[ \tilde{\Xi} = G \exp(i\Omega)K_C. \]

Also we write

\[ \Xi = \tilde{\Xi}/K_C \]

for the union of right \( K_C \)-cosets of \( \tilde{\Xi} \) in the complex symmetric space \( G_C/K_C \). We refer to \( \Xi \) as the complex crown of the symmetric space \( G/K \). Notice that \( \Xi \) is independent of the choice of \( \mathfrak{a} \) and hence generically defined through \( G/K \).

As every root \( \alpha \in \Sigma \) is analytically integral, it gives rise to a character of \( A_C \) by

\[ \xi_\alpha : A_C \to \mathbb{C}^*, \xi_\alpha(\exp(X)) = e^{\alpha(X)}. \]

Let us define the set of regular elements in \( A_C \) by

\[ A_{C,\text{reg}} = \{ z \in A_C : \xi_\alpha^2(z) \neq 1 \quad \forall \alpha \in \Sigma \}. \]

Notice that \( A_{C,\text{reg}} \) is an algebraic variety (principal open set in \( A_C \)).

We also define

\[ A_{C,\text{sing}} = A_C \setminus A_{C,\text{reg}} \]

and call it the singular set in \( A_C \).

**Lemma 2.1.** The following assertions hold:
The multiplication mapping
\[ m : K_C \times A_{C, \text{reg}} \times K_C \rightarrow G_C, \quad (k_1, a, k_2) \mapsto k_1ak_2, \]
is submersive. In particular, \( K_C A_{C, \text{reg}} K_C \subseteq G_C \) is open.

(2) \( K_C A_{C, \text{reg}} K_C \) is dense in \( G_C \).

Proof. (1) is a standard computation which will not be repeated here.

(2) Notice that \( K_C A_{C, \text{reg}} K_C \subseteq G_C \) is a constructible set as the image under the regular mapping \( m : K_C \times A_{C, \text{reg}} \times K_C \rightarrow G_C \). Thus it follows from (1) that \( K_C A_{C, \text{reg}} K_C \) contains a Zariski-open subset of \( G_C \). This proves (2).

Remark 2.2. Notice that the \( K_C \)-bi-invariant domain \( K_C A_C K_C \) is not open in \( G_C \) (cf. our discussion of \( G = SL(2, \mathbb{R}) \) in Section 3).

3. The structure of the double \( K_C \)-cosets

In this section we will give a detailed analysis of the double \( K_C \)-coset decomposition for \( G_C = SL(2, C) \). Further we will start the investigation of the intersections of \( \tilde{\Xi} \) with the various cells in the double \( K_C \)-coset decomposition.

Let us introduce the necessary notation. Our choice of a maximal compact subgroup of \( G \) is \( K = SO(2, \mathbb{R}) \) and our choice of \( a \) will be
\[ a = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} : t \in \mathbb{R} \right\}. \]

We choose \( \Sigma^+ \) such that \( n = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\}. \)

Notice that \( G_C = SL(2, \mathbb{C}) \) is simply connected and \( K_C = SO(2, \mathbb{C}) \).

In order to study \( K_C \)-double cosets the use of spherical functions is useful.

Definition 3.1. On \( G_C = SL(2, \mathbb{C}) \) we define the elementary spherical function \( \Phi \) by
\[ \Phi : G_C \rightarrow \mathbb{C}, \quad \Phi(g) := \text{tr}(gg^t) \quad \forall \ g \in G_C. \]

(For \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we have \( \Phi(g) = a^2 + b^2 + c^2 + d^2 \).)

Note that the function \( \Phi \) is holomorphic and \( K_C \)-bi-invariant.

The set of singular elements in \( A_C \) is given by
\[ A_{C, \text{sing}} = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \{-i, i, -1, 1\} \right\}. \]

Let \( g \in G_C \setminus K_C A_C K_C \). According to Lemma 2.1 we can find sequences
\[ (k_n) \subseteq K_C, \ (a_n) \subseteq A_C, \ (\tilde{k}_n) \subseteq K_C \] with \( k_na_n\tilde{k}_n \rightarrow g \).

Proposition 3.2. Let \( g \in G_C \setminus K_C A_C K_C \) be the limit of the sequence \( (k_n a_n \tilde{k}_n) \subseteq K_C A_C K_C \). Then \( (a_n) \subseteq A_C \) is bounded and for every convergent subsequence \( (a_{n_k}) \) we have
\[ \lim_{k \rightarrow \infty} a_{n_k} \in A_{C, \text{sing}}. \]
Proof. Write \(a_n = \begin{pmatrix} z_n & 0 \\ 0 & z_n^{-1} \end{pmatrix} \). By continuity of the map \(\Phi\) we get
\[
\Phi(g) = \text{tr}(gg^\dagger) = \lim_{n \to \infty} \text{tr}(a_n a_n^\dagger) = \lim_{n \to \infty} (z_n^2 + z_n^{-2}).
\]
It follows that there exist \(m, M > 0\) such that
\[
0 < m < |z_n| < M < \infty \quad \forall n.
\]
Passing to an appropriate subsequence we can assume that \(z_n \to z_0 \neq 0\).

Suppose \((k_n)\) has a convergent subsequence. W.l.o.g. we may assume that
\[
k_n \to k_0 \in K_C, \quad a_n \to a_0 \in A_C.
\]
But then \(\tilde{k}_n = a_n^{-1} k_n^{-1} (k_n a_n \tilde{k}_n)\) also converges.

We have now
\[
k_n \to k_0 \in K_C, \quad a_n \to a_0 \in A_C, \quad \tilde{k}_n \to \tilde{k}_0 \in K_C.
\]
But this would imply \(g = \lim_{n \to \infty} k_n a_n \tilde{k}_n \in K_C A_C K_C\), contradicting the assumption.

By the same argument we see that \((\tilde{k}_n)\) cannot have a convergent subsequence.

We want to know more about the limits \(z_0\).

Notice that \(k_n a_n^2 k_n^\dagger\) converges in \(G_C\). For simplicity we omit the indices writing \(k = k_n\) and \(a = a_n\). We look at the elements \(k a^2 k^\dagger \in G_C/K_C\).

Due to the \(K_C\)-bi-invariance of \(K_C A_C K_C\) and the compactness of \(K\) we can assume that
\[
k = \begin{pmatrix} \cosh t & i \sinh t \\ -i \sinh t & \cosh t \end{pmatrix} = \begin{pmatrix} \cos it & \sin it \\ -\sin it & \cos it \end{pmatrix},
\]
where \(t \in \mathbb{R}\) and \(|t| \to \infty\). Then with \(a = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\) we have
\[
ka^2 k^\dagger = \begin{pmatrix} \cosh t & i \sinh t \\ -i \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} z^2 & 0 \\ 0 & z^{-2} \end{pmatrix} \begin{pmatrix} \cosh t & -i \sinh t \\ i \sinh t & \cosh t \end{pmatrix} = \begin{pmatrix} z^2 \cosh^2 t - z^{-2} \sinh^2 t \\ (z^{-2} - z^2) i \sinh t \cosh t \end{pmatrix},
\]
for \(t \to \infty\).

We know that \(z \to z_0\) and \(|t| \to \infty\).

The upper left entry in \(3.1\) must converge, therefore
\[
\frac{z^2 \cosh^2 t - z^{-2} \sinh^2 t}{\sinh^2 t} = z^2 \cosh^2 t - z^{-2} \to 0.
\]
But this implies \(z_0^2 = z_0^{-2}\), hence \(z_0^2 = \pm 1\). \(\square\)

The vector space \(p_C\) admits a decomposition \(p_C = p^+ \oplus p^-\) into irreducible \(\mathfrak{t}_C\)-modules with
\[
p^- = \mathbb{C} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \quad \text{and} \quad p^+ = \mathbb{C} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.
\]

The corresponding analytic subgroups of \(G_C\) are given by
\[
P^- = \left\{ \begin{pmatrix} 1 - u & iu \\ iu & 1 + u \end{pmatrix} : u \in \mathbb{C} \right\} \quad \text{and} \quad P^+ = \left\{ \begin{pmatrix} 1 + u & iu \\ iu & 1 - u \end{pmatrix} : u \in \mathbb{C} \right\}.
\]
We introduce punctured discs in $P^−$ and $P^+$ by
\[ P^−_2 = \left\{ \begin{pmatrix} 1-v & iv \\ iv & 1+v \end{pmatrix} : v \in \mathbb{C}, 0 < |v| < \frac{1}{2} \right\} \]
and similarly we define $P^+_2$.

Finally, we set
\[ A'_C := A_C \setminus \{ \pm 1 \} \quad \text{and} \quad P^\pm' := P^\mp \setminus \{ 1 \}, \]
and define an element
\[ y_0 = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} \in A_{\text{C,sing}}. \]

We can now describe how $G_C$ decomposes into a disjoint union of $K_C$-bi-invariant subsets. Furthermore, we will give a complete description of the intersection of $\tilde{\Xi} = G \exp(i\Omega)K_C$ with the lower dimensional cells. The intersection of $\tilde{\Xi}$ with the big cell $K_CA_CK_C$ will be subject of the next section.

**Theorem 3.3.**
(1) The $K_C$-double coset decomposition of $G_C$ is given by:
\[ G_C = K_CA_CK_C \amalg P^−_2K_C \amalg P^+_2K_C \amalg P^−y_0K_C \amalg P^+_yK_C \amalg K_C. \]
(2) The following equality holds:
\[ \tilde{\Xi} \setminus K_CA_CK_C = P^−_2K_C \amalg P^+_2K_C . \]

In order to prove Theorem 3.3 it is useful to adopt a more geometric point of view. We closely follow the suggestions of the referee.

In the sequel we realize $X = SL(2,\mathbb{R})/SO(2,\mathbb{R})$ as the upper sheet of the two-sheeted hyperboloid
\[ x_0^2 - x_1^2 - x_2^2 = 1, \quad x_0 > 0 \quad (x = (x_0,x_1,x_2) \in \mathbb{R}^3) \]
and $X_C = SL(2,\mathbb{C})/SO(2,\mathbb{C})$ as the complex quadric
\[ z_0^2 - z_1^2 - z_2^2 = 1 \quad (z = (z_0,z_1,z_2) \in \mathbb{C}^3). \]

Let $e_0 = (1,0,0)$. Then the polar decomposition in $X$ is given by $X = KA,e_0$,
\[ x_0 = \cosh t, \quad x_1 = \sinh t \cos \varphi, \quad x_2 = \sinh t \sin \varphi \]
with $t,\varphi \in \mathbb{R}$.

**Proof.** (Thm 3.3) Complexifying the real polar decomposition of $X$ it follows that a point $z \in X_C$ belongs to $K_CA_C,e_0$ if and only if there exist $a,b,u,v \in \mathbb{C}$ with
\[ a^2 + b^2 = 1, \quad u^2 - v^2 = 1, \]
such that
\[ z_0 = u, \quad z_1 = va, \quad z_2 = vb. \]

Therefore $X_C \setminus K_CA_C,e_0$ decomposes into four $K_C$-orbits
(1) $z_0 = 1, z_1 = \tau, z_2 = i\tau,$
(2) $z_0 = 1, z_1 = \tau, z_2 = -i\tau,$
(3) $z_0 = -1, z_1 = \tau, z_2 = i\tau,$
(4) $z_0 = -1, z_1 = \tau, z_2 = -i\tau,$
where $\tau \in \mathbb{C}^*$. 

The orbits (1), (2), (3), (4) correspond to the $K_C$-bi-invariant sets $P^-K_C$, $P^+K_C$, $P^-y_0K_C$, $P^+y_0K_C$, respectively.

This proves part (1) of the theorem.

The points $z = x + iy \in \Xi \subset X_C$ are characterized by the property

$$x > 0, \quad x_0^2 - x_1^2 - x_2^2 > 0.$$ 

For an element $z = (1, \tau, i\tau) \in \text{orbit (1)}$ this translates to $|\tau| < 1$. Therefore,

$$\{(1, \tau, i\tau) \in X_C : \tau \in \mathbb{C}^* \} \cap \Xi = \{(1, \tau, i\tau) \in X_C : \tau \in \mathbb{C}^*, \ |\tau| < 1 \} ,$$

and this subset of $\Xi$ corresponds to $P^+K_C \subset \tilde{\Xi}$.

Similarly one relates orbit (2) with $P^+K_C \subset \tilde{\Xi}$.

The intersections of both orbits (3) and (4) with $\tilde{\Xi}$ are empty. \hfill \Box

**Remark 3.4.** (a) Theorem 3.3(1) can be deduced from general results of Matsuki (cf. [15]).

(b) The cells in the decomposition of $G_C$ in Theorem 3.3 are characterized by the values of $\Phi$. The function $\Phi$ attains the value 2 on $K_C$ and on $P^-K_C \sqcup P^+K_C$,

the value $-2$ on $P^-y_0K_C \sqcup P^+y_0K_C$, and $\Phi$ can take any value except 2 on $K_CA^C_kK_C$.

### 4. A Non-linear Convexity Theorem

In this section we will investigate the intersection of $\tilde{\Xi}$ with the big cell $K_CA_CK_C$. An element $g \in G \exp(i\Omega)K_C \cap K_CA_CK_C$ can be written as

$$g = h \begin{pmatrix} e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} \end{pmatrix} k = \tilde{k}_1 \tilde{a} \tilde{k}_2,$$

where $\tilde{k}_1, \tilde{k}_2 \in K_C$ and

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \quad \text{and} \quad \tilde{a} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \in A_C.$$

We want to describe the dependence of $\tilde{a}$ from $\vartheta$ and $h$.

Recall that for $SL(2, \mathbb{R})$ the Weyl group consists of only two elements and for $X \in a$ we have $\text{conv}(W.X) = [-1, 1] \cdot X$.

**Theorem 4.1.** Let $X \in \Omega$ and $h \in G$. Suppose that $g = h \exp(iX) = \tilde{k}_1 \tilde{a} \tilde{k}_2 \in K_CA_CK_C$. Then

$$g \in K_CA_C \exp(i\text{conv}(WX))K_C.$$

**Proof.** Using the Iwasawa decomposition $G = KAN$ we may assume that

$$g = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \in N, \quad a \exp(i\Omega) = \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix} \in K_C, \quad \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \tilde{a} \in A_C, \quad \tilde{k}_2 \in K_C.$$
If we set \( (u_0^{-1} u_0 = a \exp iX \) and \( \theta := \text{Arg}(u) \), then
\[
gg^t = \begin{pmatrix} u^2 + r^2 u^{-2} & ru^{-2} \\ ru^{-2} & u^{-2} \end{pmatrix} = \begin{pmatrix} z^2 \cos^2 \omega + z^{-2} \sin^2 \omega & (z^{-2} - z^2) \sin \omega \cos \omega \\ (z^2 - z^{-2}) \sin \omega \cos \omega & z^2 \sin^2 \omega + z^{-2} \cos^2 \omega \end{pmatrix},
\]
yielding
\[
(4.2) \quad \Phi(g) = \text{tr}(gg^t) = u^2 + (1 + r^2)u^{-2} = z^2 + z^{-2}.
\]
In Figure 1 the set of points of the hyperbola is \( \{le^{2i\theta} + l^{-1}e^{-2i\theta} : l \in \mathbb{R}, l > 0 \} \).
The complex number \( z^2 + z^{-2} \) lies within the shaded region. Therefore, there exists \( w = le^{i\theta} \) such that \( z^2 + z^{-2} = \lambda(w^2 + w^{-2}) \) for some real number \( \lambda > 1 \).

Figure 1

We now obtain the assertion of the theorem with the following Lemma 4.2. □

**Lemma 4.2.** Let \( r_1, r_2 > 0 \) and \( -\frac{\pi}{2} < \varphi_1, \varphi_2 < \frac{\pi}{2} \). Define complex numbers by \( u = r_1 e^{i\varphi_1} \) and \( v = r_2 e^{i\varphi_2} \). Assume that \( v + v^{-1} = \lambda(u + u^{-1}) \), with \( \lambda \in \mathbb{R}, \lambda > 1 \). Then, \( |\varphi_2| \leq |\varphi_1| \).

**Proof.** Clearly we can assume \( \varphi_2 \neq 0 \). Comparing real and imaginary parts of \( v + v^{-1} = \lambda(u + u^{-1}) \), followed by squaring, yields
\[
\lambda^2(r_1 + r_1^{-1})^2 \cos^2 \varphi_1 = (r_2 + r_2^{-1})^2 \cos^2 \varphi_2,
\]
and
\[
(4.3) \quad \lambda^2(r_1 - r_1^{-1})^2 \sin^2 \varphi_1 = (r_2 - r_2^{-1})^2 \sin^2 \varphi_2.
\]
Adding these two equations we get
\[
(4.4) \quad \lambda^2(r_1 - r_1^{-1})^2 + 4\lambda^2 \cos^2 \varphi_1 = (r_2 - r_2^{-1})^2 + 4 \cos^2 \varphi_2.
\]
Combining (4.3) and (4.4) gives
\[
(4.5) \quad \lambda^2(r_1 - r_1^{-1})^2 + 4\lambda^2 \cos^2 \varphi_1 = \lambda^2(r_1 - r_1^{-1})^2 \frac{\sin^2 \varphi_1}{\sin^2 \varphi_2} + 4 \cos^2 \varphi_2.
\]
If we assume \(|\phi_2| > |\phi_1|\), then
\[
\lambda^2(r_1 - r_1^{-1})^2 > \lambda^2(r_1 - r_1^{-1})^2 \frac{\sin^2 \phi_1}{\sin^2 \phi_2},
\]
and
\[
4\lambda^2 \cos^2 \phi_1 > 4 \cos^2 \phi_2,
\]
a contradiction to (4.5). □

**Conjecture 4.3.** Theorem 4.1 holds for any real connected semisimple Lie group \(G\).

**Example 4.4.** To give more evidence for our conjecture we will now discuss the case of \(G = SO_e(1, n), n \geq 2\).

On the Lie algebra level we have the Cartan decomposition \(\mathfrak{g} = \mathfrak{k} + \mathfrak{p}\), where
\[
\mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \in \text{Mat}(n+1, \mathbb{R}) : B \in \mathfrak{so}(n) \right\},
\]
\[
\mathfrak{p} = \left\{ \begin{pmatrix} 0 & v \\ v^t & 0 \end{pmatrix} \in \text{Mat}(n+1, \mathbb{R}) : v \in \mathbb{R}^n \right\}.
\]
Choosing
\[
a = \left\{ \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix} : t \in \mathbb{R} \right\}
\]
as maximal subalgebra of \(\mathfrak{p}\) yields the root system \(\Sigma = \{\alpha, -\alpha\}\) with
\[
\alpha \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 1.
\]

Therefore,
\[
\Omega = \left\{ X \in a : |\alpha(X)| < \frac{\pi}{2} \quad \forall \alpha \in \Sigma \right\} = \left\{ \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix} : t \in \mathbb{R}, |t| < \frac{\pi}{2} \right\}.
\]

We choose \(\alpha\) as the positive root. Then the subgroups in the Iwasawa decomposition \(G = KAN\) are
\[
K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} : B \in SO(n) \right\} \simeq SO(n),
\]
\[
N = \left\{ \begin{pmatrix} 1 + \frac{t}{2} ||v||^2 & v \\ v^t & \text{Id}_{n-1} \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\},
\]
\[
A = \left\{ \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & \text{Id}_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\}.
\]

We consider the standard representation \((\pi, V)\) of \(G\) on \(V = \mathbb{C}^{n+1}\).

Clearly, \(v_0 = e_1 = (1, 0, \cdots, 0)^t\) is a \(K\)-fixed vector. The elementary spherical function \(\Phi\) on \(G_{\mathbb{C}} \simeq SO(n+1, \mathbb{C})\) is given by
\[
\Phi : G_{\mathbb{C}} \longrightarrow \mathbb{C}, \quad \Phi(g) = \langle \pi(g)v_0, v_0 \rangle.
\]
Note that \( \Phi(g) \) is just the upper left entry \( g_{11} \) in the matrix \((g_{ij})_{1 \leq i,j \leq n+1} \).

Now let \( x \in G \exp(i\Omega) \cap K_C A_C K_C \). W.l.o.g. we can write
\[
x = na \exp iX = \tilde{k}_1 \tilde{a} \tilde{k}_2,
\]
for some
\[
n = \begin{pmatrix}
1 + \frac{1}{2}||v||^2 & v & -\frac{1}{2}||v||^2 \\
v^t & \text{Id}_{n-1} & -v^t \\
\frac{1}{2}||v||^2 & v & 1 - \frac{1}{2}||v||^2
\end{pmatrix} \in N,
\]
\[
a \exp iX = \begin{pmatrix}
\cosh z & 0 & \sinh z \\
0 & \text{Id}_{n-1} & 0 \\
\sinh z & 0 & \cosh z
\end{pmatrix} \in A \exp(i\Omega)
\]
and
\[
\tilde{a} = \begin{pmatrix}
\cosh \omega & 0 & \sinh \omega \\
0 & \text{Id}_{n-1} & 0 \\
\sinh \omega & 0 & \cosh \omega
\end{pmatrix} \in A_C, \quad \tilde{k}_{1,2} = \begin{pmatrix}
1 & 0 \\
0 & B_{1,2}
\end{pmatrix} \quad (B_{1,2} \in SO(n, \mathbb{C})).
\]

Applying \( \Phi \) yields
\[
\Phi(x) = (1 + \frac{1}{2}||v||^2) \cosh z - \frac{1}{2}||v||^2 \sinh z = \cosh \omega,
\]
or
\[
e^z + (1 + ||v||^2)e^{-z} = e^{\omega} + e^{-\omega}.
\]

Equation (4.6), Figure 1 (with a slight modification in the notation) and Lemma 4.2 show again that \( \tilde{a} \in A \exp(\text{iconv}(\mathcal{W} \cdot X))K_C \), as claimed.

5. Applications to spherical functions

Let us briefly introduce the necessary notation. Write \( G = NAK \) for the Iwasawa decomposition of \( G = SL(2, \mathbb{R}) \) with \( N, A \) and \( K \) as in Section 3. For \( g \in G \) we define \( a(g) \in A \) by \( g \in Na(g)K \). For \( \lambda \in a_C^\ast \) and \( a \in A \) we set \( a^\lambda = e^{\lambda (\log a)} \).

Following Harish-Chandra we define the spherical function on \( G/K \) with parameter \( \lambda \in a_C^\ast \) by
\[
\phi_\lambda(gK) = \int_K a(kg)^{\rho - \lambda} \, dk \quad (g \in G),
\]
where \( dk \) denotes the normalized Haar-measure on \( K \) and \( \rho = \frac{1}{2} \alpha \) with \( \Sigma^+ = \{ \alpha \} \).

Define
\[
\Pi = \{ \lambda \in a_C^\ast : \phi_\lambda \text{ is positive definite} \}
\]
and recall that \( i\mathfrak{a}^\ast \subseteq \Pi \).

We have to recall some facts from \[13\] §4 on the analytic continuation of the spherical functions (actually valid for all semisimple Lie groups \( G \)).

**Proposition 5.1.** Let \( \lambda \in a_C^\ast \). Then the following assertions hold:

1. The spherical function \( \phi_\lambda \) has a unique holomorphic extension to \( \Xi \).
2. The spherical function \( \phi_\lambda \) has a unique continuation to a \( K_C \)-bi-invariant function on \( K_C A \exp(2i\Omega)K_C/K_C \) such that the restriction to \( A \exp(2i\Omega) \) is holomorphic. Moreover, if \( \lambda \in \Pi \) and \( \Omega_c \subseteq \Omega \) is a compact subset, then the restriction of \( \phi_\lambda \) to \( K_C A \exp(2i\Omega_c)K_C/K_C \) is bounded.
In the sequel we also denote by \( \phi_\lambda \) the holomorphic extension of \( \phi_\lambda \) to \( \Xi \). We denote by \( Z = \{ \pm 1 \} \) the center of \( G \). Our result in this section then is:

**Theorem 5.2.** Let \( \lambda \in \Pi \). Then the spherical function \( \phi_\lambda \) is bounded on \( \Xi \), i.e.,

\[
\| \phi_\lambda \|_{\infty, \Xi} < \infty .
\]

**Proof.** By Theorem 4.1 we have

\[
\Xi \cap K_C A_C K_C / K_C \subseteq K_C A \exp(i\Omega) K_C / K_C .
\]

Thus Proposition 5.1 implies that \( \phi_\lambda \) is bounded on \( \Xi \cap K_C A_C K_C / K_C \).

Let now \( z \in \Xi \) such that \( z \notin K_C A_C K_C / K_C \). By the density of \( K_C A_C K_C \) in \( G_C \) we can find sequences \( (k_n) \subseteq K_C \), \( (a_n) \subseteq A_C \) such that \( k_n a_n K_C \in \Xi \) for all \( n \in \mathbb{N} \) and \( z = \lim_{n \to \infty} k_n a_n K_C \). By Theorem 4.1 we have \( a_n \in Z A \exp(i\Omega) \) and thus

\[
\lim_{n \to \infty} a_n = \pm 1
\]

by Proposition 3.2. By the continuity of \( \phi_\lambda \) we hence obtain that

\[
\phi_\lambda(z) = \lim_{n \to \infty} \phi_\lambda(k_n a_n K_C) = \lim_{n \to \infty} \phi_\lambda(a_n K_C) = \phi_\lambda(\pm 1) = 1 ,
\]

concluding the proof of the theorem. \( \square \)

**Remark 5.3.** (a) Note that Proposition 5.1 holds for all semisimple Lie groups. Thus in order to generalize Theorem 5.2 to all semisimple Lie groups one needs Proposition 3.2 and Theorem 4.1 in full generality (cf. our conjecture in Section 4).

(b) Fix a \( \lambda \in a_C^* \) and consider the following function on \( G/K \times K \)

\[
P_\lambda(g K, k) \mapsto a(k) \rho - \lambda .
\]

Identifying \( G/K \) with the unit disc \( D \) and \( K \) with its boundary, this function is easily seen to be a power of the Poisson kernel on \( D \). According to [13] the function \( P_\lambda \) admits an analytic continuation to \( \Xi \times K \) which is holomorphic in the first variable. Also by [13] we know that this function is unbounded for \( \lambda \in \Pi \setminus \{ \rho, -\rho \} \).

Observe that the spherical function on \( \Xi \) is given by

\[
\phi_\lambda(z) = \int_K P_\lambda(z, k) \, dk \quad (z \in \Xi) .
\]

Now notice that Theorem 5.2 says that \( \phi_\lambda \) stays bounded despite the fact that the integrands become singular towards the boundary of \( \Xi \).

6. A NORM ESTIMATE FOR THE MIDDLE PROJECTION

In this section we will investigate the growth of the middle projection in the \( K_C A_C K_C \)-decomposition for elements in \( \Xi \).

To start with let us introduce a norm on \( G = SL(2, \mathbb{R}) \) by setting

\[
\| g \| = \text{tr}(gg^t) \quad (g \in G) .
\]

Define a compact (here actually finite) group

\[
L = N_{K_C}(A_C) \times Z = N_K(A) \times Z .
\]

We let \( L \) act on \( A_C \) via

\[
(k, z)a = kak^{-1}z \quad (k, z) \in L, \ a \in A_C .
\]

Then for every \( x \in K_C A_C K_C \) we have \( x \in K_C a K_C(x) K_C \) with a unique continuous function

\[
\phi_\lambda(z) = \int_K P_\lambda(z, k) \, dk \quad (z \in \Xi) .
\]
In view of Theorem 5.2 and its proof we also obtain a holomorphic function

\[ a_{K_C} : \Xi \rightarrow \mathbb{A}_C/L, \quad x \mapsto \begin{cases} a_{K_C}(x) & \text{if } x \in K_C A_C K_C / K_C, \\ 1 & \text{if } x \notin K_C A_C K_C / K_C. \end{cases} \]

Finally we define a norm on \( A_C \) by setting

\[ | \cdot | : A_C \rightarrow \mathbb{R}^+, \quad a = \begin{pmatrix} z \\ 0 \\ z^{-1} \end{pmatrix} \mapsto |z|^2 + |z|^{-2}. \]

Notice that \( | \cdot | \) is \( L \)-invariant hence factors to a continuous positive function on \( A_C/L \) also denoted by \( | \cdot | \).

**Proposition 6.1.** For all \( g = h \exp(iX)K_C \in \Xi \), \( h \in G, \ X \in \Omega \), we have

\[ |a_{K_C}(g)| \leq \|h\|. \]

**Proof.** Using \( G = KAK \) we can write

\[ g = \begin{pmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} K_C \]

\[ = h \in KAK \quad \text{and } \quad a_{K_C}(g) = \begin{pmatrix} z \\ 0 \\ z^{-1} \end{pmatrix}. \]

We may assume that \( g \in K_C A_C K_C / K_C \) and write \( a_{K_C}(g) = \begin{pmatrix} z \\ 0 \\ z^{-1} \end{pmatrix} \). Then

\[ \Phi(g) = (s^2 \cos^2 t + s^{-2} \sin^2 t)e^{2i\theta} + (s^2 \sin^2 t + s^{-2} \cos^2 t)e^{-2i\theta} \]

\[ = \cos^2 t(s^2 e^{2i\theta} + s^{-2} e^{-2i\theta}) + \sin^2 t(s^{-2} e^{2i\theta} + s^2 e^{-2i\theta}) \]

\[ = z^2 + z^{-2}. \]

Hence \( z^2 + z^{-2} \) is a convex combination of \( s^2 e^{2i\theta} + s^{-2} e^{-2i\theta} \) and \( s^{-2} e^{2i\theta} + s^2 e^{-2i\theta} \) and therefore lies on the vertical segment shown in Figure 2.

![Figure 2](image-url)
The following Lemma 6.2 applied to \( u = s^2 e^{2i\theta} \) and \( v = z^2 \) completes the proof of the theorem. 

**Lemma 6.2.** Let \( r_1, r_2 > 0 \) and \(-\frac{\pi}{2} < \phi_1, \phi_2 < \frac{\pi}{2}\). Define complex numbers by \( u = r_1 e^{i\phi_1} \) and \( v = r_2 e^{i\phi_2} \). Assume that \( \Re(v + v^{-1}) = \Re(u + u^{-1}) \) and \( \Im(v + v^{-1}) = \mu \Im(u + u^{-1}) \) for some \( \mu \in \mathbb{R}, |\mu| \leq 1 \). Then \( |\phi_2| \leq |\phi_1| \) and \( r_2 + r_2^{-1} \leq r_1 + r_1^{-1} \).

**Proof.** The assumption written out in polar coordinates followed by squaring yields

\[
(6.2) \quad (r_1 + r_1^{-1})^2 \cos^2 \phi_1 = (r_2 + r_2^{-1})^2 \cos^2 \phi_2 ,
\]

\[
\mu^2 (r_1 - r_1^{-1})^2 \sin^2 \phi_1 = (r_2 - r_2^{-1})^2 \sin^2 \phi_2 .
\]

If \( |\phi_2| > |\phi_1| \), then (6.2) implies

\[
(r_1 + r_1^{-1})^2 < (r_2 + r_2^{-1})^2 .
\]

Since \( (r_1 + r_1^{-1})^2 = (r_1 - r_1^{-1})^2 + 4 \) this means

\[
(r_1 - r_1^{-1})^2 < (r_2 - r_2^{-1})^2 .
\]

But now we end up with

\[
\mu^2 (r_1 - r_1^{-1})^2 \sin^2 \phi_1 < (r_2 - r_2^{-1})^2 \sin^2 \phi_2 ,
\]

which contradicts the assumption. Thus \( |\phi_2| \leq |\phi_1| \) proving our first assertion. The second assertion is now immediate from (6.2). 

\[ \Box \]

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