An explicit model for the homotopy theory of finite type Lie $n$-algebras

Christopher L. Rogers

Abstract

Lie $n$-algebras are the $L_\infty$ analogs of chain Lie algebras from rational homotopy theory. Henriques showed that finite type Lie $n$-algebras can be integrated to produce certain simplicial Banach manifolds, known as Lie $\infty$-groups, via a smooth analog of Sullivan’s realization functor. In this paper, we provide an explicit proof that the category of finite type Lie $n$-algebras and (weak) $L_\infty$-morphisms admits the structure of a category of fibrant objects (CFO) for a homotopy theory. Roughly speaking, this CFO structure can be thought of as the transfer of the classical projective CFO structure on non-negatively graded chain complexes via the tangent functor. In particular, the weak equivalences are precisely the $L_\infty$ quasi-isomorphisms. Along the way, we give explicit constructions for pullbacks and factorizations of $L_\infty$-morphisms between finite type Lie $n$-algebras. We also analyze Postnikov towers and Maurer–Cartan/deformation functors associated to such Lie $n$-algebras. The main application of this work is our joint paper [16] with C. Zhu which characterizes the compatibility of Henriques’ integration functor with the homotopy theory of Lie $n$-algebras and that of Lie $\infty$-groups.

Contents

1 Introduction 2

2 Preliminaries and notation 5
  2.1 Graded linear algebra ........................................... 5
  2.2 Notation ........................................................... 5
  2.3 Conilpotent cocommutative coalgebras ................................ 6
  2.4 Cofree conilpotent coalgebras and their morphisms .................. 7

3 Lie $n$-algebras 9
  3.1 $L_\infty$-algebras and $L_\infty$-morphisms ................................ 9
  3.2 The category of Lie $n$-algebras .................................... 11
  3.3 Factoring Lie $n$-algebra morphisms .................................... 12

4 Pullbacks in $\text{Lie}_n\text{Alg}$ 16
  4.1 Strict fibrations ................................................... 17
  4.2 Pullbacks of fibrations and acyclic fibrations ...................... 22

5 $\text{Lie}_n\text{Alg}^{\text{fin}}$ as a category of fibrant objects 23

Department of Mathematics & Statistics, University of Nevada, Reno. 1664 N. Virginia Street Reno, NV 89557-0084 USA. | chrisrogers@unr.edu, chris.rogers.math@gmail.com
1 Introduction

The motivation for this paper lies within the various algebraic formalisms developed decades ago for classifying rational homotopy types. In [15], Quillen established the relationship between the rational homotopy theory of simply-connected spaces, and the homotopy theory of connected chain Lie algebras over \( \mathbb{Q} \). By a chain Lie algebra, we mean a chain complex \( L \) of vector spaces concentrated in non-negative degrees equipped with a differential graded Lie algebra (dgla) structure. A chain Lie algebra \( L \) is connected if it is strictly positively graded, i.e. \( L_0 = 0 \). Quillen formalized the homotopy theory of connected chain Lie algebras using a model structure in which a weak equivalence is defined to be a dgla morphism whose underlying chain map induces an isomorphism in homology, and in which a fibration is defined to be a dgla morphism whose underlying chain map is surjective in all degrees.

From here, we can make a quick leap to the approach developed by Sullivan in [18], provided we restrict our attention to “finite type” chain Lie algebras, i.e. those algebras whose underlying chain complex is finite-dimensional in each degree. First, recall that the Chevalley-Eilenberg algebra \( CE(L) \) associated to a chain Lie algebra \( L \) is the commutative dg algebra (cdga) obtained by taking the linear dual of the bar construction of \( L \). If \( L \) is finite type, then \( CE(L) \) admits the structure of a simply connected Sullivan algebra. Moreover, \( CE(L) \) is a model for the rational homotopy type of its realization, i.e. the simplicial set \( \langle CE(L) \rangle := \text{hom}_{\text{cdga}}(CE(L), \Omega^\ast_{\text{poly}}(\Delta^\ast)) \), where \( \Omega^\ast_{\text{poly}}(\Delta^\ast) \) is the cdga of polynomial de Rham forms on the geometric \( n \)-simplex.

A more direct path from chain Lie algebras to Kan complexes is via deformation theory. Associated to any \( \mathbb{Z} \)-graded dgla \( (L, d, [\cdot, \cdot]) \) is its set of Maurer-Cartan elements \( MC(L) := \{ a \in L_{-1} \mid da + \frac{1}{2}[a, a] = 0 \} \). Furthermore, the dgla structure on \( L \) induces a natural simplicial dgla structure on the tensor product \( L \otimes_{\mathbb{Q}} \Omega^\ast_{\text{poly}}(\Delta^\ast) \). As noted by Getzler [8], if \( L \) is a finite type, then there is a natural isomorphism of simplicial sets:

\[ \langle CE(L) \rangle \cong \int L \]

where \( (\int L)_n := MC(L \otimes \Omega^\ast_{\text{poly}}(\Delta^n)) \). Hence, if \( L \) is finite type connected, then \( L \) is a Lie model for the rational homotopy type of the space \( \int L \). Moreover, the “simplicial Maurer-Cartan functor” \( \int \) is compatible with the respective homotopy theories. Indeed, by combining Quillen’s work [15] with the results of Bousfield and Gugenheim [4], it follows that \( \int \) preserves both weak equivalences and fibrations. In particular, \( \int L \) is a Kan complex for every finite type connected chain Lie algebra \( L \).

This leads to an obvious question: What is the analog of this story for arbitrary chain Lie algebras over a field of characteristic zero? That is, suppose there are no constraints imposed on degree zero elements and no assumptions involving nilpotency or completeness. To begin with, Quillen’s model for the homotopy theory of connected rational chain Lie algebras extends in a straightforward way to the more general case over any field of characteristic zero. It follows from the work of Getzler and Jones [7, Thm. 4.4] that the category of chain Lie algebras over such a field admits a model structure induced by the projective model structure on non-negatively graded chain complexes. Hence, a weak equivalence is, as before, a dgla morphism whose underlying chain map is a quasi-isomorphism, and a fibration is defined to be a dgla morphism whose under-
lying chain map is surjective in positive degrees. This is a natural generalization of Quillen’s model structure for the rational connected case.

However, the spatial realization of finite type chain Lie algebras with no constraints on connectivity is a more subtle endeavor. Indeed, all finite dimensional non-nilpotent Lie algebras are examples of such chain Lie algebras. Consequently, as demonstrated by Sullivan [18, “Theorem 8.1”], the realization of chain Lie algebras over \( \mathbb{R} \) necessarily involves the diffeo-geometric integration of Lie algebras to Lie groups (i.e. Lie’s Third Theorem).

The existence of a smooth realization functor for such chain Lie algebras is a special case of the more general problem addressed by Henriques’ in his work [9] on the integration of Lie \( n \)-algebras. Lie \( n \)-algebras are \( L_\infty \)-algebras (or “strong homotopy Lie algebras”) whose underlying chain complexes are non-negatively graded. Thus they include chain Lie algebras as a special case. However, the “correct” notion of morphism between Lie \( n \)-algebras is significantly weaker than just a linear map which preserves the \( L_\infty \)-structure on the nose. This implies that the category of Lie \( n \)-algebras and weak \( L_\infty \)-morphisms is not a category of algebras over the \( L_\infty \) operad. Hence, a model for the homotopy theory of Lie \( n \)-algebras does not follow from the aforementioned result of Getzler and Jones. The main result (Thm. 5.2) of this paper resolves this issue by explicitly providing such a model.

Henriques’ integration procedure for finite type Lie \( n \)-algebras involves replacing the polynomial de Rham forms in Sullivan’s realization functor with the dg Banach algebra of \( C^r \)-differential forms. The output of this procedure is a group-like simplicial Banach manifold, or “Lie \( \infty \)-group”, which satisfies a diffeo-geometric analog of the horn filling condition for Kan simplicial sets. Simplicial manifolds of this kind have been used as geometric models for the higher stages of the Whitehead tower of the orthogonal group. The most famous example of such a model is called the “String Lie 2–group”, which Henriques showed can be obtained by integrating its infinitesimal analog, the “string Lie 2–algebra”.

The results of this paper, when combined with our joint work with Zhu in the companion paper [16], address the compatibility of Henriques’ integration functor with the homotopy theories of Lie \( n \)-algebras and Lie \( \infty \)-groups. This can be understood as the smooth analog of the aforementioned results of Quillen and Bousfield-Gugenheim which characterize the homotopical properties of the realization functor for connected Lie models for rational homotopy types.

Overview and main results

After reviewing standard facts concerning \( L_\infty \)-algebras in the sections leading up to Sec. 3.2, we consider in Def. 3.6 two classes of morphisms in the category \( \text{Lie}_n\text{Alg} \) of Lie \( n \)-algebras. We say a \( L_\infty \)-morphism \((f_1, f_2, \ldots) : (L, \ell_1, \ell_2, \ell_3, \ldots) \rightarrow (L', \ell'_1, \ell'_2, \ell'_3, \ldots)\) is a weak equivalence iff the chain map \( f_1 : (L, \ell_1) \rightarrow (L', \ell'_1)\) is a quasi-isomorphism, and we say it is a fibration iff the chain map \( f_1\) is a surjection in all positive degrees. Thus weak equivalences coincide with \( L_\infty \) quasi-isomorphisms. Although every weak equivalence between Lie \( n \)-algebra induces a quasi-isomorphism between their associated Chevalley-Eilenberg (co)algebras, the converse is not true, in contrast with the simply connected case mentioned in the above introduction.

In Sec. 3.3, we prove several useful technical results concerning morphisms in \( \text{Lie}_n\text{Alg} \). We show in Prop. 3.8 that every strict \( L_\infty \)-morphism (in the sense of Sec. 3.1.1) can be factored into a fibration followed by a weak equivalence. In particular, the diagonal map \( L \rightarrow L \oplus L \) in \( \text{Lie}_n\text{Alg} \) admits such a factorization. Therefore, it follows from our main theorem (see below) and Brown’s Factorization Lemma (see Lemma 5.4) that every weak \( L_\infty \)-morphism in \( \text{Lie}_n\text{Alg} \) admits such a factorization.

Next, in Lemma 3.11, we show that every fibration can be factored into an isomorphism followed by a strict fibration. The proof is a simple modification of a result of Valette [20] concerning the factorization of “\( \infty \)-epimorphisms”. (See, for example, Def. 3.3). This “strictification of fibrations” is a very useful tool which we use repeatedly throughout this paper and in the companion paper [16].
In Prop. 4.1 and Cor. 4.4, we explicitly construct pullbacks of fibrations and acyclic fibrations along arbitrary morphisms in LieₐAlg. Moreover, the pullback of a (acyclic) fibration is again a (acyclic) fibration. The proof of these facts involves a clever use of certain coalgebra endomorphisms, which we learned from studying Vallette’s proof of his Thm. 4.1 in [20].

Since our main application is integration, in Sec. 5, we restrict our attention to the full subcategory LieₐAlg^{fin} of finite type Lie n-algebras. The category LieₐAlg^{fin} does not admit a model structure, since it does not have all limits and colimits. So instead, we work within Brown’s framework [2] of a category of fibrant objects, or “CFO”, for a homotopy theory (Def. 5.1). The main result (Thm. 5.2) of the paper is:

**Theorem.** Let \( n \in \mathbb{N} \cup \{\infty\} \). The category LieₐAlg^{fin} of finite type Lie n-algebras over a field of characteristic zero and weak \( L_{\infty} \)-morphisms has the structure of a category of fibrant objects, in which a morphism

\[
(f_1, f_2, \ldots) : (L, \ell_1, \ell_2, \ell_3, \ldots) \to (L', \ell'_1, \ell'_2, \ell'_3, \ldots)
\]

is:

- a weak equivalence if the chain map \( f_1 : (L, \ell_1) \to (L', \ell'_1) \) is a quasi-isomorphism of chain complexes, and

- a fibration if the chain map \( f_1 : (L, \ell_1) \to (L', \ell'_1) \) is a surjection in all positive degrees.

Hence, the homotopy theory that we consider for Lie n-algebras is inherited from the projective model structure for non-negatively graded chain complexes (Sec. 2), via the tangent functor which assigns to a \( L_{\infty} \)-morphism \((f_1, f_2, \ldots)\) as above, the chain map \( f_1 : (L, \ell_1) \to (L', \ell'_1) \). In particular, our results imply that the tangent functor is an exact functor between categories of fibrant objects (Cor. 5.8). We also note that this CFO structure is compatible with the one induced on the category of chain Lie algebras by the aforementioned Getzler-Jones/Quillen model structure. (Recall that chain Lie algebras form a non-full subcategory of Lie_{\infty}Alg.)

Also in Sec. 5, we compare the category of fibrant objects structure on finite type Lie n-algebras with Vallette’s CFO structure (Thm. 5.9) on \( \mathbb{Z} \)-graded \( L_{\infty} \)-algebras.

In Sec. 6, we analyze Maurer-Cartan (MC) sets of certain \( \mathbb{Z} \)-graded \( L_{\infty} \)-algebras which are constructed by tensoring Lie n-algebras with bounded commutative dg algebras. This is a familiar procedure used in deformation theory for constructing deformation functors out of pronilpotent \( L_{\infty} \)-algebras. Maurer-Cartan sets are also used to define Henriques’ integration functor. We prove that this construction sends pullback diagrams of fibrations in LieₐAlg to pullback diagrams of MC sets. In Cor. 6.7, we show that an analogous statement holds in the smooth category when the MC sets are equipped with a Banach manifold structure. The proofs crucially depend on the explicit description of pullbacks given in Sec. 4.

Finally, in Sec. 7, we analyze a very useful Postnikov tower construction for Lie n-algebras which was introduced Henriques in [9]. They play a key role in his proof that Lie n-algebras integrate to fibrant simplicial manifolds. We also introduce an important class of fibrations in LieₐAlg called “quasi-split fibrations” (Def. 7.1). Such fibrations naturally arise in applications, e.g. string extensions. We show that a morphism of Postnikov towers induced by a quasi-split fibration admits a convenient functorial decomposition (Prop. 7.5 and Prop. 7.6). We use this result and the aforementioned results concerning MC sets in [16] to show that Henriques’ integration functor is an exact functor with respect to the class of quasi-fibrations. (See Thm. 9.16 in [16].)

In this paper, we work with \( L_{\infty} \)-algebras within the context of dg cocommutative coalgebras, rather than commutative dg algebras, even though we are ultimately interested in finite type objects. This is because our main result Thm. 5.2, as well as many of the auxiliary results, also hold for infinite dimensional Lie n-algebras. (See Remark 5.3.) Furthermore, many of the technical tools we develop in order to prove our main results are the “non-negatively graded” variations of Vallette’s work [20] on the homotopy theory of
Acknowledgments

The author thanks: Aydin Ozbek for very helpful discussions during the preparation of this manuscript, Chen-chang Zhu for careful proofreading, and Ezra Getzler for several informative conversations on the homotopy theory of $L_\infty$-algebras. The author is also indebted to Bruno Vallette for writing the inspiring preprint [20]. This work was supported by a grant from the Simons Foundation/SFARI (585631,CR).

2 Preliminaries and notation

2.1 Graded linear algebra

Throughout, $\mathbb{k}$ denotes a field of characteristic zero. For a $\mathbb{Z}$-graded vector space $V$ we denote by $sV$ (resp. by $s^{-1}V$) the suspension (resp. the desuspension) of $V$. In other words,

$$sV_i := V[-1]_i = V_{i-1} \quad s^{-1}V_i := V[1]_i = V_{i+1}$$

We denote by $|x|$ the degree of a homogeneous element $x \in V$. Similarly, $|f|$ denotes the degree of a linear map $f : V \to V'$ between graded vector spaces $V$ and $V'$.

Let $x_1, \ldots, x_n$ be elements of $V$ and $\sigma \in S_n$ a permutation. The notation $\epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n)$ is reserved for the Koszul sign, which is defined by the equality

$$x_1 \lor \cdots \lor x_n := \epsilon(\sigma) x_{\sigma(1)} \lor \cdots \lor x_{\sigma(n)} \in S(V)$$

which holds in the free graded commutative algebra $S(V)$ generated by $V$. Note that $\epsilon(\sigma)$ does not include the sign $(-1)^{\sigma}$ of the permutation $\sigma$.

The notation $\sigma \cdot (x_1 \otimes x_2 \otimes \cdots \otimes x_n)$ is reserved for the left action of $S_n$ on $V^\otimes n$, i.e.

$$\sigma \cdot x_1 \otimes x_2 \otimes \cdots \otimes x_n := \epsilon(\sigma) x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} \text{.} \quad (2.1)$$

We denote by $\text{Sh}_{p_1, \ldots, p_k} \subseteq S_n$ the subset of $(p_1, \ldots, p_k)$-shuffles in $S_n$, i.e. $\text{Sh}_{p_1, \ldots, p_k}$ consists of elements $\sigma \in S_n, n = p_1 + p_2 + \cdots + p_k$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p_1),$$

$$\sigma(p_1 + 1) < \sigma(p_1 + 2) < \cdots < \sigma(p_1 + p_2),$$

$$\cdots$$

$$\sigma(n - p_k + 1) < \sigma(n - p_k + 2) < \cdots < \sigma(n) \text{.}$$

We use homological conventions for all differential graded (dg) structures except for the cochain algebras that appear in Sec. 6.

2.2 Notation

2.2.1 The model category $\text{Ch}_{\geq 0}^{\text{proj}}$

Throughout the paper, $\text{Ch}_{\geq 0}$ denotes the category of chain complexes over $\mathbb{k}$ concentrated in non-negative degrees. The notation $\text{Ch}_{\geq 0}^{\text{proj}}$ is reserved for category $\text{Ch}_{\geq 0}$ equipped with the projective model structure.
Weak equivalences are the quasi-isomorphisms, and fibrations are those chain maps which are surjective in all positive degrees. Since we work over a field, the cofibrations are those chain maps which are injective in all degrees.

Since all objects in $\text{Ch}^\text{proj}_{\geq 0}$ are (bi)fibrant, we will also consider $\text{Ch}^\text{proj}_{\geq 0}$ as a category of fibrant objects (Def. 5.1) whenever it is convenient to do so.

2.2.2 Notation for categories

We will consider several closely related categories. We list them here as a convenient reference for the reader as they traverse through the paper.

- $\text{Ch}$ denotes the category of $\mathbb{Z}$-graded (i.e., unbounded) chain complexes over $k$.
- $\text{CoCom}$ (resp. $\text{dgCoCom}$) denotes the category of $\mathbb{Z}$-graded (resp. dg) conilpotent cocommutative coalgebras (Sec. 2.3).
- $L_\infty\text{Alg}$ is the category whose objects are $\mathbb{Z}$-graded $L_\infty$-algebras and whose morphisms are weak $L_\infty$-morphisms (Sec. 3.1). We will tacitly identify $L_\infty\text{Alg}$ as the full subcategory of $\text{dgCoCom}$ consisting of those dg coalgebras whose underlying graded coalgebras are cofree (Sec. 2.4).
- $\text{CoCom}_{\geq 0}$ (resp. $\text{dgCoCom}_{\geq 0}$) denotes the full subcategory of $\text{CoCom}$ (resp. $\text{dgCoCom}$) consisting of those graded (resp. dg) coalgebras whose underlying graded vector spaces are concentrated in non-negative degrees.
- Fix $n \in \mathbb{N} \cup \{\infty\}$. We denote by $\text{Lie}_n\text{Alg}$ the category of Lie $n$-algebras: the full subcategory of $L_\infty\text{Alg}$ consisting of those $L_\infty$-algebras whose underlying chain complex is concentrated in degrees $0, 1, \ldots, n - 1$. (Sec. 3.2). Hence, the morphisms in $\text{Lie}_n\text{Alg}$ are always taken to be weak $L_\infty$-morphisms. Again we will usually identify $\text{Lie}_n\text{Alg}$ as a full subcategory of $\text{dgCoCom}_{\geq 0}$.

The notation $\text{Lie}_n\text{Alg}^{\text{fin}}$ is reserved for the full subcategory of $\text{Lie}_n\text{Alg}$ consisting of finite type Lie $n$-algebra (Sec. 3.2.)

- We denote by $\text{cdga}_{\geq 0}^{\text{bnd}}$ the category of bounded cochain algebras. (Sec. 6.) Its objects are cohomologically graded unital commutative dg $k$-algebras whose underlying graded vector spaces are concentrated in non-negative degrees and bounded from above. The morphisms in $\text{cdga}_{\geq 0}^{\text{bnd}}$ are unit preserving cdga morphisms.

2.3 Conilpotent cocommutative coalgebras

The following facts and notational conventions concerning dg coalgebras are standard, and the reader already familiar with treatments of $L_\infty$-algebras as dg coalgebras, e.g. [12] or [14, Sec. 10.1.6] can likely skip this section.

For a basic introduction to dg coalgebras and their morphisms, we suggest Sections 3d and 22a of [6], Appendix B of [15], and Section 2 of [10]. We recall that a graded counital cocommutative coalgebra $(C, \Delta, \epsilon)$ is coaugmented if it is equipped with a distinguished degree zero element $1 \in C_0$ satisfying $\Delta 1 = 1 \otimes 1$, and $\epsilon(1) = 1$. For such a coalgebra, we have a decomposition of vector spaces

$$C \cong k \cdot 1 \oplus \bar{C}, \quad \bar{C} := \ker \epsilon.$$

Let $\bar{\Delta} : \bar{C} \to \bar{C} \otimes \bar{C}$ denote the reduced comultiplication defined as $\bar{\Delta}(c) := \bar{\Delta}c = \Delta c - c \otimes 1 - 1 \otimes c$.

We call the non-counital cocommutative coalgebra $(\bar{C}, \bar{\Delta})$ the associated reduced coalgebra. The reduced
diagonal $\bar{\Delta}^{(n)}$ is recursively defined by the formulas:

$$\bar{\Delta}^{(0)} := \text{id}, \quad \bar{\Delta}^{(1)} := \Delta, \quad \bar{\Delta}^{(n)} := (\Delta \otimes \text{id}^{\otimes (n-1)}) \circ \bar{\Delta}^{(n-1)}; \ C \to C^{\otimes (n+1)}.$$ 

A coaugmented counital cocommutative coalgebra $(C, \Delta, \epsilon, 1)$ is conilpotent iff $\bar{\Delta} = \bigcup_n \ker \bar{\Delta}^{(n)}$. We define CoCom to be the category whose objects are $\mathbb{Z}$-graded conilpotent coaugmented counital cocommutative coalgebras. Similarly, we define dgCoCom to be the category whose objects are coalgebras $(C, \Delta, \epsilon, 1)$ in CoCom equipped with a degree $-1$ codifferential $\delta$ satisfying $\delta(1) = 0$.

The full subcategories $\text{CoCom}_{\geq 0} \subseteq \text{CoCom}$ and $\text{dgCoCom}_{\geq 0} \subseteq \text{dgCoCom}$ are defined analogously.

### Reduced dg coalgebras $(\bar{C}, \bar{\Delta}, \delta)$ and their morphisms

The codifferential $\delta$ of any conilpotent dg coalgebra $(C, \Delta, \epsilon, 1, \delta) \in \text{dgCoCom}$ is uniquely determined by its restriction to the corresponding reduced coalgebra $(\bar{C}, \bar{\Delta})$. We will use the same notation for $\delta$ and its restriction to $\bar{C}$. Similarly, since we are over a field, morphisms in $\text{dgCoCom}$ are uniquely determined by their restriction to the associated reduced dg coalgebras [10, Sec. 2.1]. From now on, when dealing with the categories $\text{dgCoCom}_{\geq 0}$, $\text{dgCoCom}$, etc., we will tacitly work with the associated reduced dg coalgebras and their morphisms, and make no mention of counits or coaugmentations.

#### 2.4 Cofree conilpotent coalgebras and their morphisms

Let $V$ be a graded vector space. The symmetric algebra generated by $V$

$$S(V) = k \oplus \bar{S}(V), \quad \bar{S}(V) := V \oplus S^2(V) \oplus S^3(V) \oplus \ldots$$

is naturally a graded conilpotent cocommutative coalgebra with comultiplication $\Delta$ defined as the unique morphism of algebras such that $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$. The comultiplication for the corresponding reduced coalgebra $(\bar{S}(V), \bar{\Delta})$ is explicitly:

$$\bar{\Delta}(v_1, v_2, \ldots, v_m) = \sum_{1 \leq p \leq m} \sum_{1 \in S_{p-1}^{(m-p)}} \epsilon(\sigma) \left( v_{\sigma(1)} \lor v_{\sigma(2)} \lor \cdots \lor v_{\sigma(p)} \right) \otimes \left( v_{\sigma(p+1)} \lor v_{\sigma(p+2)} \lor \cdots \lor v_{\sigma(m)} \right).$$

Let $V$ and $V'$ be graded vector spaces. Let $\Phi: \bar{S}(V) \to \bar{S}(V')$ be a linear map. For $p, m \geq 1$ the notation $\Phi^p_m$ is reserved for the restriction-projections

$$\Phi^p_m: \bar{S}^m(V) \to \bar{S}^p(V') \quad \Phi^p_m := \text{pr}_{\bar{S}^p(V')} \circ \Phi|_{\bar{S}^m(V)} \quad (2.2)$$

$\Phi$ is obviously completely determined by its restriction-projection maps $\{\Phi^p_m\}$. Furthermore, we denote by $\Phi^1: \bar{S}(V) \to V'$ the linear map

$$\Phi^1 := \Phi^1_1 + \Phi^1_2 + \cdots.$$

We recall that $(\bar{S}(V), \bar{\Delta})$ is cofree over $V$ in the category CoCom (cf. Lemma 22.1 in [6]). In particular, a degree zero linear map $F^1: \bar{S}(V) \to V'$ uniquely determines a coalgebra morphism $F: \bar{S}(V) \to \bar{S}(V')$ via the following formula

$$F(v_1, \ldots, v_m) = F^1_m(v_1, \ldots, v_m) + \sum_{p=1}^{m-1} \sum_{k_1+k_2+\cdots+k_{p+1} = m} \sum_{\sigma \in S_{k_1,k_2,\ldots,k_{p+1}}} \epsilon(\sigma) \left( \prod_{i=1}^{p+1} F^1_{k_i}(v_{\sigma(k_i)}, \ldots, v_{\sigma(k_{i+1})}) \lor \cdots \lor \prod_{i=1}^{m-k_{p+1}} F^1_{k_{p+1}}(v_{\sigma(m-k_{p+1}+1)}, \ldots, v_{\sigma(m)}) \right).$$

(2.3)
This gives explicit formulas for the restriction-projections $F^p_m$:
\[
F^p_m(v_1, \ldots, v_m) = \sum_{k_1+k_2+\ldots+k_p=m} \sum_{k_1, k_2, \ldots, k_p \geq 1} \sum_{\sigma \in Sh(k_1, k_2, \ldots, k_p)} \frac{e(\sigma)}{p!} F^1_{k_1}(v_{\sigma(1)}, \ldots, v_{\sigma(k_1)}) \vee \cdots \vee F^1_{k_p}(v_{\sigma(m-k_p+1)}, \ldots, v_{\sigma(m)}).
\]
(2.4)

In particular, we have
\[
F^m_m(v_1, \ldots, v_m) = F^1_1(v_1) \vee \cdots \vee F^1_m(v_m), \quad F^p_m(v_1, \ldots, v_m) = 0 \quad \forall p > m.
\]
(2.5)

Hence, a coalgebra morphism between cofree conilpotent coalgebras $F: \bar{S}(V) \to \bar{S}(V')$ is uniquely determined by its structure maps $F^1_k: \bar{S}^k(V) \to V'$.

**Composition**

Let $F: \bar{S}(V) \to \bar{S}(V')$ and $G: \bar{S}(V') \to \bar{S}(V'')$ be coalgebra morphisms. It follows from Eqs. 2.3–2.5 that the composition $GF: \bar{S}(V) \to \bar{S}(V'')$ is the unique coalgebra morphism whose structure maps $(GF)^1_m: \bar{S}^m(V) \to V''$ are
\[
(GF)^1_m(v_1, \ldots, v_m) = \sum_{p=1}^{m} G^1_p F^p_m(v_1, \ldots, v_m)
\]
(2.6)

**Coderivations**

Analogously, we recall that a degree $-1$ linear map $\delta^1: \bar{S}(V) \to V$ uniquely determines a degree $-1$ coderivation $\delta: \bar{S}(V) \to \bar{S}(V)$ via the following formula
\[
\delta_m(v_1, \ldots, v_m) = \delta^1_m(v_1, \ldots, v_m) + \sum_{i=1}^{m-1} \sum_{\sigma \in Sh(i, m-i)} e(\sigma) \delta^1_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)} \lor v_{\sigma(i+1)} \lor \cdots \lor v_{\sigma(m)}).
\]
(2.7)

(See, for example, Lemma 2.4 in [12]). This gives explicit formulas for the restriction-projections:
\[
\delta^p_m(v_1, \ldots, v_m) = \sum_{\sigma \in Sh(m-p+1, p-1)} e(\sigma) \delta^1_{m-p+1}(v_{\sigma(1)}, \ldots, v_{\sigma(i)} \lor v_{\sigma(i+1)} \lor \cdots \lor v_{\sigma(m)}).
\]
(2.8)

In particular, we have
\[
\delta^m_m(v_1, \ldots, v_m) = (\delta^1)^{\otimes}(v_1 \lor v_2 \lor \cdots \lor v_m), \quad \delta^p_m(v_1, \ldots, v_m) = 0 \quad \forall p > m,
\]
(2.9)

where $(\delta^1)^{\otimes}$ denotes the usual derivation on $S(V)$ induced by the linear map $\delta^1: V \to V$. Hence, a coderivation on a cofree conilpotent coalgebra is uniquely determined by its structure maps $\delta^1_m: \bar{S}^m(V) \to V$.

It follows from Eqs. 2.7–2.9 that a degree $-1$ coderivation $\delta$ on $\bar{S}(V)$ is a **codifferential**, i.e., $\delta^2 = 0$, if and only if
\[
\sum_{k=1}^{m} \delta^1_k \delta^k_m(v_1, \ldots, v_m) = 0 \quad \forall m \geq 1.
\]
(2.10)

Analogously, a coalgebra morphism of the form $F: \bar{S}(V) \to \bar{S}(V')$ lifts to a dg coalgebra morphism $F: (\bar{S}(V), \delta) \to (\bar{S}(V'), \delta')$ if and only if
\[
\sum_{k=1}^{m} \delta^1_k F^k_m(v_1, \ldots, v_m) = \sum_{k=1}^{m} F^1_k \delta^k_m(v_1, \ldots, v_m) \quad \forall m \geq 1.
\]
(2.11)
3 Lie \(n\)-algebras

3.1 \(L_\infty\)-algebras and \(L_\infty\)-morphisms

An \(L_\infty\)-algebra \((L,\ell)\) is a \(\mathbb{Z}\)-graded vector space \(L\) equipped with a collection \(\ell = \{\ell_1, \ell_2, \ell_3, \ldots\}\) of graded skew-symmetric linear maps, or “brackets”,

\[ \ell_k : \Lambda^k L \to L, \quad 1 \leq k < \infty \] (3.1)

with \(|\ell_k| = k - 2\), satisfying an infinite sequence of Jacobi-like identities of the form:

\[ \sum_{i+j=m+1, \sigma \in \text{Sh}(i,m-i)} (-1)^{\sigma} \epsilon(\sigma) (-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(m)}) = 0. \] (3.2)

for all \(m \geq 1\) [12, Def. 2.1]. In particular, Eq. 3.2 implies that \((L, \ell_1) \in \text{Ch}\).

Equivalently, a \(L_\infty\)-structure on a graded vector space \(L\) is a degree \(-1\) codifferential \(\delta\) on the coalgebra \(\bar{S}(sL)\). (See, for example, Thm. 2.4 in [12].) The correspondence between the structure maps \(\delta^1_m : \bar{S}^m(sL) \to sL\) and the brackets is given by the formula

\[ \delta^1_m = (-1)^{\frac{m(m-1)}{2}} s \circ \ell_m \circ (s^{-1})^\otimes m. \] (3.3)

Let \(L\) and \(L'\) be graded vector spaces. We recall that there is a one-to-one correspondence between collections \(f = \{f_1, f_2, \ldots\}\) of skew-symmetric linear maps

\[ f_k : \Lambda^k L \to L', \quad 1 \leq k < \infty \] (3.4)

with \(|f_k| = k - 1\), and coalgebra morphisms

\[ F : \bar{S}(sL) \to \bar{S}(sL') \]

whose degree 0 structure maps \(F^1_k : \bar{S}^k(sL) \to sL\) are given by the formula

\[ F^1_k = (-1)^{\frac{k(k-1)}{2}} s \circ f_k \circ (s^{-1})^\otimes k. \] (3.5)

A morphism (i.e. a weak \(L_\infty\)-morphism) of \(L_\infty\)-algebras

\[ f : (L, \ell) \to (L', \ell') \]

is a collection \(f = \{f_1, f_2, \ldots\}\) of skew-symmetric linear maps as in (3.4) whose corresponding coalgebra morphism (3.5) satisfies Eq. 2.11 and therefore lifts to a morphism of dg coalgebras

\[ F : (\bar{S}(sL), \delta) \to (\bar{S}(sL'), \delta') \]

We treat the category \(L_\infty\text{Alg}\) of \(L_\infty\)-algebras as a full subcategory of \(\text{dgCoCom}\). Hence, composition of morphisms in \(L_\infty\text{Alg}\) is given by Eq. 2.6.

A morphism \(f : (L, \ell) \to (L', \ell')\) is an \(L_\infty\)-isomorphism iff the linear map \(f_1 : L \to L'\) is an isomorphism in \(\text{Ch}\). It is easy to show that this condition implies that \(f\) corresponds to an actual isomorphism in the category \(\text{dgCoCom}\).
Notation 3.1. In contrast with some other conventions found in the literature, we will write weak $L_\infty$-morphisms in $L_\infty$Alg using a single lower-case (Roman or Greek) letter, e.g.

$$f : (L, \ell) \to (L', \ell'),$$

and the $k$-ary map in the collection $f$ will always be denoted by $f_k$. The dg coalgebra morphism encoded by the collection $f$ will always be written using the corresponding upper-case letter, e.g. $F : \bar{S}(sL) \to \bar{S}(sL')$.

Remark 3.2. By definition, if $f : (L, \ell) \to (L', \ell')$ is an $L_\infty$-morphism, then the coalgebra morphism $F : \bar{S}(sL) \to \bar{S}(sL')$ satisfies $F\delta = \delta'F$. Therefore, by setting $m = 1$ in Eq. 2.11, we observe that degree 0 map $f_1$ is necessarily a chain map:

$$f_1 : (L, \ell_1) \to (L', \ell'_1).$$

Furthermore, if $x, y \in L$, then by setting $m = 2$ in Eq. 2.11 we see that

$$f_1(\ell_2(x, y)) - \ell'_2(f_1(x), f_1(y)) = \ell'_1f_2(x, y).$$

(3.6)

Since the bilinear bracket $\ell_2$ on a $L_\infty$-algebra $(L, \ell)$ induces a Lie algebra structure on $H_0(L)$, it follows from Eq. 3.6 that $H_0(f_1) : H_0(L) \to H_0(L')$ is a morphism of Lie algebras.

Next, we recall three useful classes of $L_\infty$-morphisms.

Definition 3.3. Let $f : (L, \ell) \to (L', \ell')$ be a morphism of $L_\infty$-algebras.

1. We say $f$ is a $L_\infty$-quasi-isomorphism iff the chain map $f_1 : (L, \ell_1) \to (L', \ell'_1)$ is a quasi-isomorphism, i.e. the induced map on homology

$$H(f_1) : H(L) \to H(L')$$

is an isomorphism.

2. We say $f$ is a $L_\infty$-epimorphism [20, Def. 4.1] iff the chain map $f_1 : (L, \ell_1) \to (L', \ell'_1)$ is a surjection in degree $n$ for all $n \in \mathbb{Z}$.

3.1.1 Strict $L_\infty$-morphisms

Finally, recall that a morphism $f : (L, \ell) \to (L', \ell')$ in $L_\infty$Alg is a strict $L_\infty$-morphism iff $f_k = 0$ for all $k \geq 2$. If

$$f = f_1 : (L, \ell) \to (L', \ell)$$

is a strict morphism, then it follows from the definition that the restriction-projections (2.4) of the coalgebra morphism $F$ satisfy

$$F^k_m = 0, \quad \text{if } k \neq m.$$

Combining this with Eq. 2.11, it follows that every $k$-ary bracket $\ell_k$ is preserved by the chain map $f_1$:

$$\ell'_k \circ f_1^{\otimes k} = f_1 \circ \ell_k \quad \text{for all } k \geq 1$$
3.2 The category of Lie $n$-algebras

Let $(L, \ell)$ be a $L_\infty$-algebra. If the underlying graded vector space $L$ is concentrated in the first $n - 1$ non-negative degrees, i.e.

$$L = \bigoplus_{i \geq 0} L_i$$

then $L$ is called a Lie $n$-algebra [3, Def. 4.3.2].

For a fixed $n \in \mathbb{N} \cup \{\infty\}$, we denote by $\text{Lie}_n\text{Alg}$ the full subcategory of $L_\infty\text{Alg}$ whose objects are Lie $n$-algebras. Note that if $n < \infty$ then, for degree reasons, $\ell_k = 0$ for all $k > n + 1$. Similarly, if $f: (L, \ell) \to (L', \ell')$ is a morphism in $\text{Lie}_n\text{Alg}$, then $f_k = 0$ for all $k > n$.

Example 3.4. We recall a few elementary, but important, examples.

1. A Lie 1-algebra is just a Lie algebra. This gives a full and faithful embedding of the category of Lie algebras over $k$ into $\text{Lie}_n\text{Alg}$ for any $n$.

2. We say a Lie $n$-algebra $(L, \ell)$ is abelian iff $\ell_k = 0$ for all $k \geq 2$. Hence, an abelian Lie $n$-algebra is the same thing as a chain complex concentrated in degrees $0, \ldots, n - 1$.

3. Let $\text{dgl}_{\geq 0}^n$ denote the category of chain Lie algebras whose underlying chain complex is concentrated in degrees $0, \ldots, n - 1$. There is a functor $\text{dgl}_{\geq 0}^n \to \text{Lie}_n\text{Alg}$ which sends a chain Lie algebra $(L, d, [\cdot, \cdot])$ to the Lie $n$-algebra $(L, \ell)$, where $\ell_1 = d$, $\ell_2 = [\cdot, \cdot]$ and $\ell_k = 0$ for all $k > 2$. Under this embedding, dgla morphisms are mapped to strict Lie $n$-algebra morphisms.

A simple but non-trivial example of a Lie 2-algebra is the “string Lie 2-algebra” [9, Def. 8.1]. See Sec. 7 for further discussion.

Proposition 3.5. The category $\text{Lie}_n\text{Alg}$ is closed under finite products. Moreover, the forgetful functor $\text{Lie}_n\text{Alg} \to \text{dgCoCom}_{\geq 0}$ creates products.

Proof. Indeed, the categorical product of any two Lie $n$-algebras $(L, \ell_k)$ and $(L', \ell'_k)$ is the Lie $n$-algebra $(L \oplus L', \ell_k \oplus \ell'_k)$ where

$$l_k \oplus \ell'_k((x_1, x'_1), \ldots, (x_k, x'_k)) := (\ell_k(x_1, \ldots, x_k), \ell'_k(x'_1, \ldots, x'_k)).$$  (3.7)

Furthermore, the usual projections $\text{pr}: L \oplus L' \to L$, $\text{pr}' : L \oplus L' \to L'$ lift to strict $L_\infty$-epimorphisms

$$(L, \ell) \xrightarrow{\text{pr}} (L \oplus L', \ell \oplus \ell') \xrightarrow{\text{pr}'} (L', \ell').$$

The product of Lie $n$-algebras then coincides with the product in $\text{dgCoCom}_{\geq 0}$ via the natural isomorphism of graded vector spaces $S(V \oplus V') \cong S(V) \otimes S(V')$.

Definition 3.6. Let $f: (L, \ell) \to (L', \ell')$ be a morphism of Lie $n$-algebras.

1. We say $f$ is a weak equivalence iff the chain map $f_1 : (L, \ell_1) \to (L', \ell'_1)$ is a quasi-isomorphism.

2. We say $f$ is a fibration iff the chain map $f_1 : (L, \ell_1) \to (L', \ell'_1)$ is surjective in all positive degrees.

3. We say $f$ is an acyclic fibration iff $f$ is a weak equivalence and a fibration.

Remark 3.7. Clearly, a morphism $f$ in $\text{Lie}_n\text{Alg}$ is a weak equivalence iff it is an $L_\infty$-quasi-isomorphism. Furthermore, $f$ is an acyclic fibration if and only if it is both a $L_\infty$-quasi-isomorphism and a $L_\infty$-epimorphism. Indeed, if $(L, \ell_k)$ is a Lie $n$-algebra, then $H_0(L) = L_0/\text{im} \ell_1$.  

11
Following the standard terminology from deformation theory, let

\[ \tan_{\geq 0} : \text{Lie}_n \text{Alg} \to \text{Ch}_{\geq 0}^{\text{proj}} \]

denote the tangent functor, which is defined by the assignments:

\[ (L, \ell) \mapsto (L, \ell_1) \]
\[ (L, \ell) \xrightarrow{f} (L', \ell') \mapsto (L, \ell_1) \xrightarrow{f_1} (L'_1, \ell'_1) \]

(3.8)

Then it follows from Sec. 2.2.1 that \( f : (L, \ell) \to (L', \ell') \) is a weak equivalence (resp. fibration) of Lie \( n \)-algebras if and only if \( \tan_{\geq 0} f \) is a weak equivalence (resp. fibration) in \( \text{Ch}_{\geq 0}^{\text{proj}} \). We show later in Cor. 5.8 that the tangent functor is an exact functor.

### 3.3 Factoring Lie \( n \)-algebra morphisms

We now give an explicit factorization for morphisms in \( \text{Lie}_n \text{Alg} \), which we will use to show the existence of path objects.

**Factoring chain maps in \( \text{Ch}_{\geq 0}^{\text{proj}} \)**

Suppose \( f : (V, d_V) \to (W, d_W) \) is a morphism of chain complexes. We will factor \( f \) explicitly as \( f = pfj \), where \( j \) is an acyclic cofibration and \( pf \) is a fibration as defined in Sec. 2.2.1. Let \( (P(W), d_{P(W)}) \in \text{Ch}_{\geq 0} \) denote the chain complex with underlying graded vector space

\[ P(W)_0 := \{0\} \oplus W_1, \quad P(W)_i := W_i \oplus W_{i+1}, \quad \text{for } i \geq 1, \]

(3.9)

with differential

\[ d_{P(W)}(x, y) := (0, x). \]

(3.10)

Note that the complex \( (P(W), d_{P(W)}) \) is acyclic. In particular, the linear map

\[ h : P(W) \to P(W)[1], \quad h(x, y) := (y, 0) \]

(3.11)

is a contracting chain homotopy. Let \( \pi : P(W) \to W \) denote the degree zero linear map

\[ \pi(x, y) := x + d_W y. \]

(3.12)

It is easy to verify that \( \pi \) is a chain map and that it is surjective in all positive degrees. We can therefore factor \( f \) into an acyclic cofibration followed by a fibration:

\[ V \xrightarrow{j} V \oplus P(W) \xrightarrow{pf} W, \]
\[ j(v) := (v, (0, 0)), \quad pf(v, (x, y)) := f(v) + \pi(x, y) \]

(3.13)

**Factoring strict maps in \( \text{Lie}_n \text{Alg} \)**

Next we extend the above factorization in \( \text{Ch}_{\geq 0}^{\text{proj}} \) to strict morphisms in \( \text{Lie}_n \text{Alg} \).

**Proposition 3.8.** Let \( f = f_1 : (L, \ell) \to (L', \ell') \) be a strict morphism of Lie \( n \)-algebras. Then \( f \) can be factored in the category \( \text{Lie}_n \text{Alg} \) as

\[ (L, \ell) \xrightarrow{j} (\tilde{L}, \tilde{\ell}) \xrightarrow{\phi} (L', \ell') \]

where \( j \) is a weak equivalence, and \( \phi \) is a fibration in \( \text{Lie}_n \text{Alg} \).
For the proof, we’ll use the following lemma:

**Lemma 3.9** (see Theorem A.1 [20]). Let $(\tilde{L}, \tilde{\ell})$ and $(L', \ell')$ be Lie $n$-algebras. Let $m > 1$ and suppose \{\(\Phi_i^1 : \tilde{S}^i(s\tilde{L}) \to sL'\)\}_{1 \leq i \leq m - 1} is a collection of linear maps satisfying

\[
\sum_{i=1}^{k} \delta^i \Phi_k^i = \sum_{i=1}^{k} \Phi_i^1 \tilde{\delta}^i_k \quad \forall k \leq m - 1,
\]

where \(\tilde{\delta}\) and \(\delta^i\) are the codifferentials encoding the \(L_\infty\)-structures on \(\tilde{L}\) and \(L'\), respectively. Then:

1. The linear map \(c_m : \tilde{S}^m(s\tilde{L}) \to sL'\) defined as

\[
c_m := \sum_{k=1}^{m-1} \Phi_k^1 \tilde{\delta}^k_m - \sum_{k=2}^{m} \delta^k_k \Phi_m^k
\]

is a degree \(-1\) cycle in the chain complex \(\text{Hom}(S(s\tilde{L}), sL'), \partial\), where

\[
\partial c_m = \delta^1 \circ c_m - (-1)^{\ell_m} |c_m| \circ \tilde{\delta}^m_m.
\]

2. There exists a linear map \(\Phi_1^1 : \tilde{S}^m(s\tilde{L}) \to sL'\) such that the collection \{\(\Phi_1^1, \ldots, \Phi_{m-1}^1, \Phi_m^1\)\} satisfies

\[
\sum_{i=1}^{m} \delta^i \Phi_m^i = \sum_{i=1}^{m} \Phi_1^1 \tilde{\delta}^i_m
\]

if and only the homology class \([c_m]\) is trivial.

The lemma can be verified by direct computation, which we leave to the reader. Alternatively, both the lemma and Prop. 3.8 follow from applying the obstruction theory developed in [20] to weak \(L_\infty\)-morphisms between algebras over the Lie\(_\infty\) operad.

**Proof of Prop. 3.8.** Let \(f = f_1 : (L, \ell) \to (L', \ell')\) be a strict morphism of Lie \(n\)-algebras. Via (3.13), we factor the chain map \(f : (L, \ell_1) \to (L', \ell'_1)\) in \(\text{Ch}_{\geq 0}^{\text{proj}}\) as

\[
\begin{align*}
L &\to \tilde{L} \xrightarrow{P} L',
\end{align*}
\]

where, for brevity, we denote by \(\tilde{L}\) the chain complex

\[
(\tilde{L}, \tilde{\ell}_1) := (L \oplus P(L'), \ell_1 \oplus d_{P(L')}).
\]

We then extend the differential \(\tilde{\ell}_1\) to the following \(L_\infty\)-structure:

\[
\tilde{\ell}_k((v_1, \bar{v}_1), \ldots, (v_k, \bar{v}_k)) := (\ell_k(v_1, \ldots, v_k), (0, 0)), \quad \forall k \geq 2,
\]

for all \(v_i \in L\) and \(\bar{v}_i = (x_i', y_i') \in P(L')\). Note that, by construction, if \(L\) and \(L'\) are concentrated in degrees \(0, \ldots, n - 1\), then \(\tilde{L}\) is as well. Hence, \((\tilde{L}, \tilde{\ell}) \in \text{Lie}_n\text{Alg}\).

It is easy to see that the inclusion of complexes \(j : L \to \tilde{L}\) lifts to a strict \(L_\infty\)-morphism \(j : (L, \ell) \to (\tilde{L}, \tilde{\ell})\) which is, by construction, a weak equivalence in \(\text{Lie}_n\text{Alg}\).

Switching to the coalgebra picture, let \(\delta\) be the codifferential on \(S(s\tilde{L}) \cong S(sL) \otimes S(sP(L'))\) that encodes the \(L_\infty\)-structure on \(\tilde{L}\). We have the equality

\[
\tilde{\delta} = \delta \otimes \text{id} + \text{id} \otimes \hat{\delta},
\]
where \( \hat{\delta} \) denotes the extension of the differential \( d_{P(L')} \) on \( P(L') \) to \( \tilde{S}(sP(L')) \), i.e., \( \hat{\delta}^{1} = sd_{P(L')}s^{-1} \). Let \( J \) denote the coalgebra map corresponding to the strict \( L_{\infty} \)-morphism \( j \). Our goal is to construct a morphism in \( \text{dgCoCom}_{\geq 0} \)

\[
\Phi : (\tilde{S}(sL), \hat{\delta}) \to (\tilde{S}(sL'), \delta')
\]

that has the following properties:

1. \( \Phi^{1}_{1} = s \circ p_{f} \circ s^{-1} \),

2. \( \Phi J = F \), and hence \( \Phi^{1}_{k}J_{k}^{k} = 0 \) for all \( k \geq 2 \), since \( f \) is strict.

The above implies that the morphism \( \Phi \), combined with \( J \), will give us the desired factorization of \( f \) in \( L_{\infty}\text{Alg} \).

We construct \( \Phi \) by induction. Let \( \Phi^{1}_{1} := s \circ p_{f} \circ s^{-1} \). Let \( m > 1 \) and suppose \( \{ \Phi^{1}_{i} : \tilde{S}^{i}(sL) \to sL' \mid 1 \leq i \leq m - 1 \} \) is a collection of linear maps satisfying

\[
\sum_{i=1}^{k} \delta^{i}_{i} \Phi_{k}^{i} = \sum_{i=1}^{k} \Phi_{i}^{i} \tilde{\delta}_{k}^{i} \quad \forall k \leq m - 1,
\]

and

\[
\Phi^{1}_{k}J_{k}^{k} = 0, \quad 2 \leq k \leq m - 1.
\]

Part 1 of Lemma 3.9 implies that the degree \(-1\) linear map

\[
c_{m} : \tilde{S}^{m}(sL) \to sL', \quad c_{m} := \sum_{k=1}^{m-1} \Phi_{k}^{1} \tilde{\delta}_{k}^{1} - \sum_{k=2}^{m} \delta^{i}_{i} \Phi_{m}^{i},
\]

as an element of the chain complex \( \text{Hom}_{k}(\tilde{S}(sL), sL') \), satisfies

\[
\partial c_{m} = \delta^{i}_{1} \circ c_{m} - (-1)^{|c_{m}|} c_{m} \circ \hat{\delta}^{m}_{m} = 0.
\]

Recalling Eq. (2.6), which describes the composition of morphisms in \( \text{Lie}_{\mu}\text{Alg} \), we observe that Eq. 3.15 implies that \( c_{m} \) vanishes when restricted to the subspace \( \text{im} J_{m}^{m} \). Hence, \( c_{m} \) descends to cocycle \( \bar{c}_{m} \) in the subcomplex \( (\text{Hom}_{k}(\text{coker} J_{m}^{m}, sL'), \partial) \) where:

\[
\text{coker} J_{m}^{m} \cong \bigoplus_{i=0}^{m-1} S^{i}(sL) \otimes S^{m-i}(sP(L')) = \bigoplus_{i+j=m} S^{i}(sL) \otimes \tilde{S}^{j}(sP(L')),
\]

and where \( \partial \) denotes, by slight abuse of notation, the restriction of the differential on the ambient complex \( \text{Hom}_{k}(\tilde{S}(sL), sL') \).

Now let \( h : P(L') \to P(L')[1] \) be the contracting chain homotopy defined in Eq. 3.11. By the symmetric version of the “tensor trick” (e.g. [14, Sec. 10.3.6]), we extend \( h \) to a contracting chain homotopy \( H \) on the complex \( (\tilde{S}(sP(L')) , \hat{\delta}) \). Explicitly, the restriction of \( H \) to length \( k \) tensors \( H_{k} : \tilde{S}^{k}(sP(L')) \to \tilde{S}^{k}(sP(L'))[1] \) is defined as:

\[
H_{k}(\bar{v}_{1}', \ldots, \bar{v}_{k}') := \sum_{\sigma \in S_{k}} \sigma^{-1} \cdot (\text{id} \otimes_{k-1} s h s^{-1}) \sigma \cdot (\bar{v}_{1}', \ldots, \bar{v}_{k}'), \quad \forall \bar{v}_{i}' \in sP(L'),
\]

where \( \sigma \cdot \) denotes the left action defined in Eq. 2.1. A direct calculation shows that indeed \( \text{id} = \hat{\delta}H + H\hat{\delta} \).

Since \( \hat{\delta}^{m}_{m} = \sum_{i+j=m}(\delta^{i}_{i} \otimes \text{id} + \text{id} \otimes \hat{\delta}^{j}_{j}) \), it follows that the map

\[
\text{id}_{\tilde{S}(sL)} \otimes H : \text{coker} J_{m}^{m} \to \text{coker} J_{m}^{m}[1]
\]
is a contracting chain homotopy for the complex \((coker J^m_m, \tilde{\delta}_m^m)\). Therefore, the linear map 
\[ \tilde{c}_m \circ (id \otimes H) : coker J^m_m \to sL' \] satisfies 
\[ \tilde{c}_m = -\partial (\tilde{c}_m \circ (id \otimes H)). \] (3.17)

Finally, we extend \(H\) to all of \(S(sP(L'))\) by declaring \(H(1_k) := 0\), and we let \(\Phi^1_m : S(L) \to sL'\) be the linear map 
\[ \Phi^1_m ((v_1, \tilde{\sigma}^1_1), \ldots, (v_k, \tilde{\sigma}^1_m)) := -c_m \circ (id \otimes H)((v_1, \tilde{\sigma}^1_1), \ldots, (v_k, \tilde{\sigma}^1_m)). \]

Hence Eq. 3.17 implies that \(c_m = \partial \Phi^1_k\). It then follows from part 2 of Lemma 3.9 that the collection \(\{\Phi^1_1, \ldots, \Phi^1_{m-1}, \Phi^1_m\}\) satisfies 
\[ \sum_{i=1}^k \delta_i \Phi^i_k = \sum_{i=1}^k \Phi^i_1 \tilde{\delta}_k \quad \forall k \leq m, \]
and \(\Phi^1_k \delta^k_1 = 0\) for \(2 \leq k \leq m\). This completes the inductive step, and therefore the proof the proposition. \(\square\)

**Remark 3.10.** The factorization recalled in the beginning of this section of a chain map \(f : (V, d_V) \to (W, d_W)\) in \(Ch^\text{proj}_{\geq 0}\) is functorial. Indeed, a commutative diagram in \(Ch_{\geq 0}\) of the form:

\[
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\alpha \downarrow & & \downarrow \beta \\
V' & \xrightarrow{f'} & W'
\end{array}
\]

(3.18)

factors as

\[
\begin{array}{ccc}
V & \xrightarrow{j} & V \oplus P(W) & \xrightarrow{\pi_f} & W \\
\alpha \downarrow & & \downarrow \gamma & & \downarrow \beta \\
V' & \xrightarrow{f'} & V' \oplus P(W') & \xrightarrow{\pi_{f'}} & W'
\end{array}
\]

(3.19)

where \(\gamma : V \oplus P(W) \to V' \oplus P(W')\) is the chain map

\[ \gamma(v, (x, y)) := (\alpha(v), (\beta(x), \beta(y))). \]

It would be convenient if the factorization in Prop. 3.8 could be made functorial in a similar way, perhaps using the symmetrized homotopy \(H : S(sP(L')) \to S(sP(L'))[1]\). Then, Thm. 5.2 in Sec. 5 would imply that \(\text{Lie}_n\text{Alg}_{\text{fin}}\) is equipped with a functorial path object. We leave this as an open question.

**Factoring arbitrary maps in \(\text{Lie}_n\text{Alg}\)**

Let \(f : (L, \ell) \to (L', \ell')\) be an arbitrary (not necessarily strict) \(L_\infty\)-morphism in \(\text{Lie}_n\text{Alg}\). Then \(f\) can be factored into a weak equivalence composed with a fibration in the following way. First, we observe that the diagonal map \(\text{diag} : (L', \ell') \to (L' \oplus L', \ell' \oplus \ell')\) is a strict \(L_\infty\)-morphism. Hence, Prop. 3.8 gives us an explicit factorization of the diagonal into a weak equivalence followed by a fibration

\[ (L', \ell') \xrightarrow{\phi} (L', \ell') \xrightarrow{\psi} (L' \oplus L', \ell' \oplus \ell'). \]

The Lie \(n\)-algebra \((L', \ell')\) is a path object for \((L', \ell')\), in the sense of Def. 5.1. When combined with Cor. 4.4, which concerns the existence of pullbacks of (acyclic) fibrations, the existence of a path object for \((L', \ell')\) implies the existence of a factorization\(^1\) of \(f\). We provide more details later in Cor. 5.5 for morphisms in \(\text{Lie}_n\text{Alg}_{\text{fin}}\), our main category of interest.

\(^1\)This is Brown’s Factorization Lemma. (See Lemma 5.4)
3.3.1 **Strictification of fibrations**

In the last part of this section, we show that every fibration in \( \text{Lie}_n\text{Alg} \) can be factored into an isomorphism followed by a strict fibration. We will use this fact repeatedly throughout the paper. The proof is a modification of a result of Vallette [20] concerning the factorization of "\( \infty \)-epimorphisms" between \( \mathbb{Z} \)-graded homotopy algebras.

**Lemma 3.11.** Let \( f : (L, \ell) \to (L', \ell') \) be a fibration between \( \text{Lie} \ n \)-algebras. Then there exists a \( \text{Lie} \ n \)-algebra \( (L, \tilde{\ell}) \) and an isomorphism \( \phi : (L, \tilde{\ell}) \xrightarrow{\cong} (L, \ell) \) such that

\[
f \circ \phi : (L, \tilde{\ell}) \to (L', \ell)
\]

is a strict fibration with \( f \circ \phi = (f \circ \phi)_1 = f_1 \).

**Proof.** Let \( F : \tilde{S}(sL) \to \tilde{S}(sL') \) be the coalgebra morphism corresponding to the fibration \( f \). Then the linear map \( F_1^1 : \bigoplus_{i \geq 1} sL_i \to \bigoplus_{i \geq 1} sL'_i \) is surjective in all degrees \( i \geq 2 \). Let \( \sigma : \bigoplus_{i \geq 1} sL'_i \to \bigoplus_{i \geq 1} sL_i \) be a map of graded vector space whose restrictions satisfy the equalities

\[
\sigma|_{sL'_i} = 0, \quad F_1^1 \circ \sigma|_{\bigoplus_{i \geq 2} sL'_i} = \text{id}.
\]

Let \( \Phi^1_1 := \text{id} : sL \to sL \). For \( m \geq 2 \) and suppose we have defined a sequence of degree 0 linear maps \( \{\Phi^1_k : \tilde{S}^k(sL) \to sL\}_{k=1}^{m-1} \). It follows from Eq. 2.4 that this sequence gives us well defined linear maps \( \Phi^k_m : \tilde{S}^m(sL) \to \tilde{S}^k(sL) \) for \( 2 \leq k \leq m \). Now define \( \Phi^1_m : \tilde{S}^m(sL) \to sL \) as:

\[
\Phi^1_m = -\sum_{k \geq 2} \sigma F_1^1 \Phi^k_m.
\]

This construction inductively yields a coalgebra isomorphism \( \Phi : \tilde{S}(sL) \to \tilde{S}(sL) \). Now let \( \delta \) be the codifferential on \( \tilde{S}(sL) \) corresponding to the \( \text{Lie} \ n \)-algebra structure on \( L \). We define a new codifferential \( \tilde{\delta} := \Phi^{-1} \delta \Phi \). Clearly, we can promote \( \Phi \) to an isomorphism of dg coalgebras \( \Phi : (\tilde{S}(sL), \tilde{\delta}) \to (\tilde{S}(sL), \delta) \).

And since \( \Phi^1_1 = \text{id} \), it follows from Eq. 2.6 that we have

\[
(F \Phi)^1_1 = F_1^1 : sL \to sL'.
\]

It remains to show that \( F \Phi \) is strict, i.e. \( (F \Phi)^1_m = 0 \) for all \( m \geq 2 \). Eq. 2.6 and the construction of \( \Phi \) imply that

\[
(F \Phi)^1_m = -F_1^1 \left( \sum_{k \geq 2} \sigma F_1^1 \Phi^k_m \right) + \sum_{k \geq 2} F_1^1 \Phi^k_m.
\]

If \( m \geq 2 \), then for any homogeneous element \( y \in \tilde{S}^m(sL) \), we have \( |y| > 1 \). Hence, \( |F_1^1 \Phi^k_m(y)| > 1 \) for any \( k \geq 1 \). It then follows from the definition of \( \sigma \) that \( F_1^1 \left( \sigma F_1^1 \Phi^k_m(y) \right) = \Phi^k_{m+1}(y) \). Therefore, \( (F \Phi)^1_m = 0 \) for all \( m \geq 2 \).

\[\square\]

4 Pullbacks in Lie\(_n\)Alg

In this section, we give an explicit description of pullbacks of fibrations and acyclic fibrations in the category \( \text{Lie}_n\text{Alg} \). After recalling the construction of pullbacks in \( \text{dgCoCom}_{\geq 0} \), we focus on the special case of pulling back strict fibrations in \( \text{Lie}_n\text{Alg} \). We then use Lemma 3.11 to address the more general non-strict case.
Consider a diagram of Lie $n$-algebras of the form $(L', \ell') \xrightarrow{f} (L'', \ell'') \xleftarrow{g} (L, \ell)$. The category $\text{dgCoCom}_{\geq 0}$ is complete, so the pullback of the corresponding diagram of dg coalgebras exists:

$$
\begin{array}{ccc}
(\bar{P}, \delta_P) & \xrightarrow{\text{Pr}} & (\bar{S}(sL), \delta) \\
\text{Pr'} & \downarrow \text{F} & \downarrow \text{G} \\
(\bar{S}(sL'), \delta') & \xrightarrow{\text{GPr'}} & (\bar{S}(sL''), \delta'') \\
\end{array}
$$

The graded coalgebra $\bar{P}$ is the equalizer of the diagram $\bar{S}(sL \oplus sL') \xrightarrow{F \text{Pr}} \bar{S}(sL'')$ which can be characterized as the largest sub-coalgebra of $\bar{S}(sL \oplus sL')$ contained in the vector space $\ker(F \text{Pr} - G \text{Pr'})$. By generalizing a construction of Sweedler [19, Sec. 16.1], we have the following explicit description of $\bar{P}$:

$$
\bar{P} = \left\{ v \in \ker(F \text{Pr} - G \text{Pr'}) \mid (\text{id} \otimes F \text{Pr})\bar{\Delta}(v) = (\text{id} \otimes G \text{Pr'})\bar{\Delta}(v) \in \bar{S}(sL \oplus sL') \otimes \bar{S}(sL'') \right\}.
$$

Above $\bar{\Delta}$ is the reduced comultiplication on $\bar{S}(sL \oplus sL')$. Using arguments analogous to those in the proof of Lemma 16.1.1 in [19], it is not difficult to show that the restriction $\bar{\Delta}|_P$ gives $\bar{P}$ the structure of a cocommutative coalgebra. The codifferential $\delta_P$ is, of course, the restriction of the codifferential $\delta_{\oplus}$ on $\bar{S}(sL \oplus sL')$.

### 4.1 Strict fibrations

The pullback of a strict fibration in $\text{Lie}_n\text{Alg}$ has a convenient explicit description.

**Proposition 4.1.** Suppose $f = f_1: (L, \ell) \rightarrow (L'', \ell'')$ is a strict fibration in $\text{Lie}_n\text{Alg}$ and $g: (L', \ell') \rightarrow (L'', \ell'')$ is an arbitrary morphism between Lie $n$-algebras. Let $(\bar{L}, \bar{\ell}_1) \in \text{Ch}_{\geq 0}$ denote the pullback of the diagram of chain maps $(L'_1, \ell_1) \xrightarrow{g_1} (L'', \ell'') \xleftarrow{f_1} (L, \ell_1)$. Then:

1. The pullback square in $\text{Ch}_{\geq 0}$ containing $f_1$ and $g_1$ lifts to a commutative diagram in $\text{Lie}_n\text{Alg}$:

$$
\begin{array}{ccc}
(\bar{L}, \bar{\ell}) & \xrightarrow{f} & (L, \ell) \\
\downarrow \text{F} & & \downarrow \text{G} \\
(L', \ell'), \delta' & \xrightarrow{g} & (L'', \ell'') \\
\end{array}
$$

2. Let $(\bar{P}, \delta_P)$ denote the pullback of the diagram $(\bar{S}(sL'), \delta') \xrightarrow{\bar{G}} (\bar{S}(sL''), \delta'') \xleftarrow{\bar{F}} (\bar{S}(sL), \delta)$, where $F$ and $G$ are the dg coalgebra morphisms corresponding to $f$ and $g$, respectively. Then there exists an isomorphism of dg coalgebras

$$
(\bar{P}, \delta_P) \xrightarrow{\text{iso}} (\bar{S}(sL), \delta)
$$

which induces an isomorphism of cones over commuting squares in $\text{dgCoCom}_{\geq 0}$. Hence, the diagram (4.3) is a pullback diagram in $\text{Lie}_n\text{Alg}$.

Before we prove Prop. 4.1, we will need to discuss a few technical constructions.
Useful endomorphisms of the product coalgebra

As above, consider a diagram of the form \((L', \ell') \xrightarrow{\sigma} (L''', \ell''') \xleftarrow{\ell'' = f_1} (L, \ell)\) in \(\text{Lie}_c \text{Alg}\), in which \(f\) is a strict fibration. In order to give an explicit description of the \(L_\infty\)-structure on the pullback, we first construct two auxiliary endomorphisms of the graded coalgebra \(S(sL' \oplus sL)\).

Throughout this section, we denote elements of the direct sum \(sL' \oplus sL\) as vectors \(\vec{v} := (v', v)\). We first give a convenient description of \(sL\), the suspension of the pullback of \(f_1\), as a graded vector space. Since \(f\) is a fibration, the linear map \(F^1_1: \bigoplus_{i \geq 1} sL_i \to \bigoplus_{i \geq 1} sL''_i\) is surjective in all degrees \(i \geq 2\). Let \(\sigma: \bigoplus_{i \geq 1} sL''_i \to \bigoplus_{i \geq 1} sL_i\) be a map of graded vector space whose restrictions satisfy the equalities

\[
\sigma|_{sL''} = 0, \quad F^1_1 \circ \sigma|_{\bigoplus_{i \geq 2} sL''_i} = \text{id}.
\]

Then we have

\[
sL_1 = sL' \times_{sL''} sL_1, \quad sL_{i \geq 2} = sL'_{i \geq 2} \oplus \ker F_1
\]

and a pullback diagram of graded vector spaces

\[
\begin{array}{ccc}
sL & \xrightarrow{\text{pr} + \sigma G_{1} \text{pr}'} & sL' \\
pr' & \downarrow & \downarrow F_1 \\
sL'' & \xrightarrow{G_1} & sL''
\end{array}
\]

Clearly, \(\bar{S}(sL) \subseteq \bar{S}(sL' \oplus sL)\) as graded coalgebras. We define the linear maps \(H^1_k: \bar{S}^k(sL' \oplus sL) \to sL' \oplus sL\) to be

\[
H^1_k(\vec{v}) := (v', \sigma G_1(v') + v), \quad H^1_k(\vec{v}_1, \ldots, \vec{v}_k) := (0, \sigma G^1_k(v'_1, \ldots, v'_k)).
\]

Similarly, let \(J^1_k: \bar{S}^k(sL' \oplus sL) \to sL' \oplus sL\) denote the linear maps

\[
J^1_k(\vec{v}) := (v', -\sigma G_1(v') + v), \quad J^1_k(\vec{v}_1, \ldots, \vec{v}_k) := (0, -\sigma G^1_k(v'_1, \ldots, v'_k)).
\]

Claim 4.2. We have the equality \(HJ = \text{id}_{\bar{S}(sL') \oplus sL'}\).

Proof. Indeed, using the definition of \(\sigma\), a direct computation verifies that \((HJ)^1_1 = H^1_1 J^1_1 = \text{id}_{sL' \oplus sL}^\text{pr}.\) Now suppose \(m \geq 2\). It remains to show \((HJ)^1_m = 0\). It follows from Eq. 2.6 that \((HJ)^1_m = \sum_{k=1}^m H^1_k J^1_k\). From Eq. 2.4, we see that the formula for \(J^1_k\) involves a summation of tensor products of linear maps of the form

\[
\sum_{j_1 + j_2 + \ldots + j_k = m} J^1_{j_1} \otimes J^1_{j_2} \otimes \cdots \otimes J^1_{j_k}.
\]

Hence, if \(k < m\), then in each term of above sum, there exists a \(j_i > 1\), and so it follows from the definition (4.6) of \(J\) that \(\text{pr'} J^1_{j_i} = 0\). Combining this observation with the definition (4.5) of \(H\), we deduce that if \(2 \leq k < m\), then \(H^1_k J^1_k = 0\). Therefore,

\[
(HJ)^1_m(\vec{v}_1, \ldots, \vec{v}_m) = H^1_1 J^1_m(\vec{v}_1, \ldots, \vec{v}_m) + H^1_m J^1_m(\vec{v}_1, \ldots, \vec{v}_m)
\]

\[
= H^1_1(0, -\sigma G^1_m(v'_1, \ldots, v'_m)) + H^1_m((v'_1, -\sigma G_1(v'_1) + v_1), (v'_2, -\sigma G_1(v'_2) + v_2), \ldots, (v'_m, -\sigma G_1(v'_m) + v_m))
\]

\[
= (0, -\sigma G^1_m(v'_1, \ldots, v'_m)) + (0, \sigma G^1_m(v'_1, \ldots, v'_m)) = 0.
\]

Thus, the claim has been verified. \(\square\)

\(^{2}\)To the best of our knowledge, this construction is due to Bruno Vallette. It is implicit in his proof of Thm. 4.1 in [20].
Now let $\delta_{\otimes}$ denote the codifferential on the product $\tilde{S}(sL' \oplus sL)$, and define

$$\tilde{\delta} := J \circ \delta_{\otimes} \circ H|_{\tilde{S}(sL')}.$$  

Claim 4.3. $\tilde{\delta}$ induces a well-defined codifferential on $\tilde{S}(s\tilde{L})$.

Proof. Note that if $\text{im} \tilde{\delta} \subseteq \tilde{S}(s\tilde{L})$, then Claim 4.2 implies that $\tilde{\delta}^2 = 0$, and hence $\tilde{\delta}$ is a codifferential. Therefore, all we need to show is that $\text{im} \delta_{\otimes}^m \subseteq sL$ for $m \geq 1$. We proceed by considering a few cases.

**Case $m = 1$:** If $\tilde{v} \in sL_{i \geq 2}$, then it follows from Eq. 2.6 and the definitions of $H$ and $J$ that:

$$\tilde{\delta}_{\otimes}^1(\tilde{v}) = J_1(\delta_{\otimes})_1H_1^1(\tilde{v}) = (\delta_{\otimes}^1(v'), -\sigma G_1(\delta_{\otimes}^1(v')) + \delta_{\otimes}^1\sigma G_1(v') + \delta_{\otimes}^1(v)).$$

If $|\tilde{v}| = 2$, then $\tilde{v} \in sL'_1 \times sL''_1 sL_1$. Therefore, $\sigma G_1(v') = 0$ and so:

$$F_1(-\sigma G_1(\delta_{\otimes}^1(v')) + \delta_{\otimes}^1\sigma G_1(v') + \delta_{\otimes}^1(v)) = F_1(\delta_{\otimes}^1(v)) = \delta_{\otimes}^1(G_1(v')) = G_1(\delta_{\otimes}^1(v')).$$

Hence, $\tilde{\delta}_{\otimes}^1(\tilde{v}) \in s\tilde{L}_1$. If $|\tilde{v}| > 2$, then $v \in \ker F_1$ and so:

$$F_1(-\sigma G_1(\delta_{\otimes}^1(v')) + \delta_{\otimes}^1\sigma G_1(v') + \delta_{\otimes}^1(v)) = -G_1(\delta_{\otimes}^1(v') + \delta_{\otimes}^1(F \sigma G_1(v') + F(v)))$$

$$= -G_1(\delta_{\otimes}^1(v') + \delta_{\otimes}^1 G_1(v') = 0.$$ 

Hence, $\tilde{\delta}_{\otimes}^1(\tilde{v}) \in s\tilde{L}_{i > 1}$.

**Case $m \geq 2$:** Let $\tilde{v}_1, \ldots, \tilde{v}_m \in s\tilde{L}$. It follows from Eq. 2.6 that

$$\tilde{\delta}_{\otimes}^m(\tilde{v}_1, \ldots, \tilde{v}_m) = \sum_{k \geq 1} \sum_{l \geq 1} J_1(\delta_{\otimes})_k H_k^m(\tilde{v}_1, \ldots, \tilde{v}_m).$$

(4.8)

First suppose that $|(\delta_{\otimes})_k H_k^m(\tilde{v}_1, \ldots, \tilde{v}_m)| = 1$. Then, for degree reasons it must be the case that $m = 2$, and $\tilde{v}_1, \tilde{v}_2 \in s\tilde{L}_1$ are in lowest degree. This implies that $(\delta_{\otimes})_2 H_2^2(\tilde{v}_1, \tilde{v}_2) = (\delta_{\otimes})_2 H_2^2((\tilde{v}_1) \mathbin{\lor} H_1^1(\tilde{v}_2)) = 0$. Therefore, by expanding Eq. 4.8 we obtain the equality

$$\tilde{\delta}_{\otimes}^2(\tilde{v}_1, \tilde{v}_2) = J_1((\delta_{\otimes})_1 H_1^2(\tilde{v}_1, \tilde{v}_2) + (\delta_{\otimes})_1 H_1^2(\tilde{v}_1, \tilde{v}_2))$$

$$= (0, \delta_{\otimes}^1\sigma G_1(v', v'_2) + \delta_{\otimes}^1(\tilde{v}_1, v_2)).$$

Since $F$ corresponds to a strict $L_\infty$-morphism, we have $F_1(\delta_{\otimes}^2(v_1, v_2)) = \delta_{\otimes}^2(F_1(v_1), F_1(v_2))$. Furthermore, since $\tilde{v}_1, \tilde{v}_2 \in s\tilde{L}_1$, we have $(F_1(v_1), F_1(v_2)) = (G_1(v'_1), G_1(v'_2))$. From these two equalities, we deduce that

$$F_1(\delta_{\otimes}^1\sigma G_1(v', v'_2) + \delta_{\otimes}^1(v_1, v_2)) = (\delta_{\otimes}^2 G_2^2 + \delta_{\otimes}^1 G_2^1)(\tilde{v}_1, \tilde{v}_2).$$

(4.9)

Since $G$ is a morphism of dg-coalgebras, we have

$$(\delta_{\otimes}^2 G_2^2 + \delta_{\otimes}^1 G_2^1)(\tilde{v}_1, \tilde{v}_2) = (G_1^1 \delta_{\otimes}^1 + G_2^1 \delta_{\otimes}^2)(\tilde{v}_1, \tilde{v}_2) = G_1^1 \delta_{\otimes}^1(\tilde{v}_1, \tilde{v}_2).$$

By substituting this last equality into Eq. 4.9, we conclude that

$$G_1(\delta_{\otimes}^1(v'_1, v'_2)) = F_1(\delta_{\otimes}^1\sigma G_1^1(v'_1, v'_2) + \delta_{\otimes}^1(v_1, v_2)),$$

and hence $\delta_{\otimes}^1(\tilde{v}_1, \tilde{v}_2) \in s\tilde{L}_1$. 

19
Now we consider the remaining sub-case and suppose $|\vec{v}_1 + \ldots + \vec{v}_m| > 2$. From Eq. 2.4, we see that the formula for $H^k_m$ consists of a sum of tensor products of linear maps $H^1_{j_1} \otimes H^1_{j_2} \otimes \cdots \otimes H^1_{j_n}$, just as $J^k_m$ involved the summation (4.7). Hence, we can apply the argument used in the verification of Claim 4.2 to the present case and deduce that if $2 \leq l \leq k < m$, then

$$J^l_1(\delta_\otimes) H^k_m(\vec{v}_1, \ldots, \vec{v}_m) = 0.$$ 

Consequently, we have

$$\tilde{\delta}_m^1(\vec{v}_1, \ldots, \vec{v}_m) = J^1_1 \left( \sum_{k=1}^{m-1} (\delta_\otimes)_k H^k_m(\vec{v}_1, \ldots, \vec{v}_m) \right) + \sum_{l=1}^{m} J^1_1(\delta_\otimes)_m H^m_m(\vec{v}_1, \ldots, \vec{v}_m). \quad (4.10)$$

Note that in order to show $\tilde{\delta}_m^1(\vec{v}_1, \ldots, \vec{v}_m) \in s\tilde{L}$, it suffices to verify that

$$\text{pr} \, \delta^1_m(\vec{v}_1, \ldots, \vec{v}_m) \in \ker F_1,$$

since $|\vec{v}_1 + \ldots + \vec{v}_m| > 2$. Let us focus on the first summation on the right hand side of Eq. 4.10. It follows from the definition of $\delta_\otimes$ that we have $(\delta_\otimes)_k H^k_m = (\delta^1_k \text{pr} \otimes^k H^k_m, \tilde{\delta}^1_k \text{pr} \otimes^k H^k_m)$. Since $k < m$, we have $\text{pr} \otimes^k H^k_m = 0$, and hence

$$\text{pr} \, J^1_1 \left( \sum_{k=1}^{m-1} (\delta_\otimes)_k H^k_m(\vec{v}_1, \ldots, \vec{v}_m) \right) = \sum_{k=1}^{m-1} \delta^1_k \text{pr} \otimes^k H^k_m(\vec{v}_1, \ldots, \vec{v}_m).$$

Since $F$ corresponds to a strict $\varphi_\infty$-morphism, applying $F_1$ to the right hand side of the above equality gives

$$\sum_{k=1}^{m-1} F_1 \delta^1_k \text{pr} \otimes^k H^k_m(\vec{v}_1, \ldots, \vec{v}_m) = \sum_{k=1}^{m-1} \delta''_k \text{pr} \otimes^k H^k_m(\vec{v}_1, \ldots, \vec{v}_m). \quad (4.11)$$

The expansion of $(F_1 \text{pr}) \otimes^k H^k_m$ using Eq. 2.4 involves sums of the following form

$$\sum_{j_1 + j_2 + \cdots + j_k = m} (F_1 \text{pr}) H^1_{j_1} \otimes (F_1 \text{pr}) H^1_{j_2} \otimes \cdots \otimes (F_1 \text{pr}) H^1_{j_k}. \quad (4.12)$$

If $j_r > 1$ and $\vec{w}_1, \ldots, \vec{w}_{j_r} \in \{\vec{v}_1, \ldots, \vec{v}_m\} \subseteq s\tilde{L}$, then

$$(F_1 \text{pr}) H^1_{j_r}(\vec{w}_1, \ldots, \vec{w}_{j_r}) = F_1 \sigma G^1_{j_r}(w'_1, \ldots, w'_{j_r}) = G^1_{j_r}(w'_1, \ldots, w'_{j_r}) \quad (4.13)$$

where the last equality above follows from the fact that $|\vec{w}_1, \ldots, \vec{w}_{j_r}| > 1$. For the $j_r = 1$ case, if $\vec{v}_i \in s\tilde{L}_1$, then the definition of $H^1_{1}$ implies that

$$F_1 \text{pr} \, H^1_{1}(\vec{v}_i) = F_1(\sigma G^1_{1}(v'_i) + v_i) = F_1(v_i) = G^1_{1}(v_i). \quad (4.14)$$

And if $\vec{v}_i \in s\tilde{L}_{j \geq 2}$, then $v_i \in \ker F$, which implies that

$$F_1 \text{pr} \, H^1_{1}(\vec{v}_i) = F_1(\sigma G^1_{1}(v'_i)) = G^1_{1}(v'_i). \quad (4.15)$$

Therefore, by combining Eqs. 4.11 – 4.15, we conclude that

$$F_1 \text{pr} \, J^1_1 \left( \sum_{k=1}^{m-1} (\delta_\otimes)_k H^k_m(\vec{v}_1, \ldots, \vec{v}_m) \right) = \sum_{k=1}^{m-1} \delta''_k G^k_m(v'_1, \ldots, v'_m). \quad (4.16)$$
We apply $\text{pr} \sum_{l=2}^{m} J_{1}^{l} (\delta_{\oplus})_{m} H_{m}^{l} = - \sum_{l=2}^{m} \sigma G_{1} \delta_{m}^{l} \text{pr} \otimes m H_{m}^{l}$.

Hence,

$$\text{pr} \sum_{l=2}^{m} J_{1}^{l} (\delta_{\oplus})_{m} H_{m}^{l} (\bar{v}_{1}, \ldots, \bar{v}_{m}) = - \sum_{l=2}^{m} \sigma G_{1} \delta_{m}^{l} (v_{1}', \ldots, v_{m}')$$

Furthermore, it follows directly from the definition of $J_{1}^{l}$ that

$$\text{pr} J_{1}^{1} (\delta_{\oplus})_{m} H_{m}^{1} (\bar{v}_{1}, \ldots, \bar{v}_{m}) = \left( - \sigma G_{1} \delta_{1}^{1} \text{pr} \otimes m H_{m}^{1} + \delta_{m}^{1} \text{pr} \otimes m H_{m}^{1} \right) (\bar{v}_{1}, \ldots, \bar{v}_{m})$$

$$= - \sigma G_{1} \delta_{m}^{1} (v_{1}', \ldots, v_{m}') + \delta_{m}^{1} (\sigma G_{1} (v_{1}') + v_{1}, \ldots, \sigma G_{1} (v_{m}') + v_{m}).$$

We apply $F_{1}$, and by using the above equalities, we obtain the following:

$$F_{1} \text{pr} \left( \sum_{l=1}^{m} J_{1}^{l} (\delta_{\oplus})_{m} H_{m}^{l} (\bar{v}_{1}, \ldots, \bar{v}_{m}) \right) = - \sum_{l=2}^{m} F_{1} \sigma G_{1} \delta_{m}^{l} (v_{1}', \ldots, v_{m}') - F_{1} \sigma G_{1} \delta_{m}^{l} (v_{1}', \ldots, v_{m}')$$

$$+ F_{1} \delta_{m}^{1} (\sigma G_{1} (v_{1}') + v_{1}, \ldots, \sigma G_{1} (v_{m}') + v_{m})$$

$$= - \sum_{l=2}^{m} G_{1} \delta_{m}^{l} (v_{1}', \ldots, v_{m}') - G_{1} \delta_{m}^{l} (v_{1}', \ldots, v_{m}')$$

$$+ \delta_{m}^{1} (F_{1} \otimes m (\sigma G_{1} (v_{1}') + v_{1}, \ldots, G_{1} (v_{m}') + v_{m})$$

$$= - \sum_{l=1}^{m} G_{1} \delta_{m}^{l} (v_{1}', \ldots, v_{m}') + \delta_{m}^{1} G_{m} (v_{1}', \ldots, v_{m}')$$

To obtain the last two lines above, we used the fact that $F$ is a strict $L_{\infty}$-morphism, as well as the definition of $\sigma$, and the fact that either $v_{i} \in \ker F_{1}$ or $G_{1} (v_{i}') = F_{1} (v_{i})$ for all $i = 1, \ldots, m$. Finally, we combine Eq. 4.16 with Eq. 4.17 to obtain

$$F_{1} \text{pr} \tilde{\delta}_{m}^{1} (v_{1}', \ldots, v_{m}') = \sum_{k=1}^{m} \delta_{k}^{1} \delta_{m}^{k} (\bar{v}_{1}, \ldots, \bar{v}_{m}) - \sum_{l=1}^{m} G_{1} \delta_{m}^{l} (v_{1}', \ldots, v_{m}').$$

Therefore, since $G$ is a dg coalgebra morphism, we conclude that $F_{1} \text{pr} \tilde{\delta}_{m}^{1} (v_{1}', \ldots, v_{m}') = 0$. This completes the verification of the claim. \[\square\]

**The proof of Proposition 4.1**

**Proof of statement (1).** Claims 4.2 and 4.3 above imply that $(\tilde{S}(sL), \tilde{\delta})$ is a dg coalgebra, and that $H : (\tilde{S}(sL), \tilde{\delta}) \rightarrow (\tilde{S}(sL' \oplus sL), \delta_{\oplus})$ is a dg-coalgebra morphism. A straightforward calculation shows that the following diagram in Lie$_{n}$Alg $\subseteq$ dgCoCom$_{\geq 0}$ commutes:

$$\begin{array}{ccc}
(\tilde{S}(sL'), \tilde{\delta}') & \xrightarrow{\text{pr} H} & (\tilde{S}(sL), \tilde{\delta}) \\
\text{pr} H & & \downarrow F \\
(\tilde{S}(sL'), \tilde{\delta}') & \xrightarrow{G} & (\tilde{S}(sL''), \tilde{\delta}'')
\end{array}$$

Furthermore, by construction, the above diagram lifts the pullback square in Ch$_{\geq 0}$ to a commutative diagram in the category Lie$_{n}$Alg. \[\square\]
Proof of statement (2). Let \((\tilde{P}, \delta_{\tilde{P}})\) denote the pullback of diagram (4.1) in \(dgCoCom_{\geq 0}\) with \(F: (\tilde{S}(sL), \delta) \to (\tilde{S}(sL''), \delta'')\) corresponding to a strict fibration in \(Lie_n\Alg\).

Recall from (4.2) that \(\tilde{P}\) is a subcoalgebra of \(\tilde{S}(sL' \oplus sL)\). Let

\[ J|_{\tilde{P}} : \tilde{P} \to \tilde{S}(sL' \oplus sL) \]

denote the restriction of the coalgebra morphism \(J\) defined in Eq. 4.6. We claim that \(\text{im} J|_{\tilde{P}} \subseteq \tilde{S}(s\tilde{L})\). Since \(\tilde{S}(sL' \oplus sL)\) is cofree in \(CoCom_{\geq 0}\), it suffices to check that the image of the linear map \(J^1|_{\tilde{P}} := J|_{\tilde{P}} : \tilde{P} \to sL' \oplus sL\) is contained in \(s\tilde{L}\).

Let \(\bar{y} \in \tilde{P}\) be an element of homogeneous degree. Write \(\bar{y}\) as sum of elements of increasing word length, i.e., \(\bar{y} = \bar{y}_1 + \bar{y}_2 + \cdots + \bar{y}_r\), with \(\bar{y}_i \in \tilde{S}^{n_i}(sL' \oplus sL)\) and \(1 = n_1 < n_2 < \cdots < n_r\). Suppose \(n_r = 1\). Then \(\bar{y} = \bar{y}_1 = (v', v) \in sL' \oplus sL\), and \(\bar{y} \in \ker(F \Pr - G \Pr')\) implies that \((v', v) \in s\tilde{L}\). Therefore \(J(\bar{y}) = (v', -\sigma G_1(v') + v) \in s\tilde{L}\).

For the higher arity case, suppose \(n_r > 1\). The equality \(F \Pr(\bar{y}) = G \Pr'(\bar{y})\) implies that \(F^1 \Pr(\bar{y}) = G^1 \Pr'(\bar{y})\). Since \(F\) is a strict \(L_\infty\)-morphism, \(F^1_{n \geq 2} = 0\). Therefore, we deduce that

\[ F_1 \Pr(\bar{y}_1) = \sum_{i=1}^r G^1_{n_i} \Pr^{\otimes n_i}(\bar{y}_i). \]  

(4.19)

Note that \(|\bar{y}| > 1\), since \(n_r > 1\). Hence, it suffices to show that \(\Pr J^1(\bar{y}) \in \ker F_1\). It follows directly from the definition of \(J\) that

\[ \Pr J^1(\bar{y}) = \sum_{i=1}^r \Pr J^1_{n_i}(\bar{y}_i) = \Pr \bar{y}_1 - \sum_{i=1}^r \sigma G^1_{n_i} \Pr^{\otimes n_i}(\bar{y}_i). \]

Applying \(F_1\) to the above gives

\[ F_1 \Pr J^1(\bar{y}) = F_1 \Pr \bar{y}_1 - \sum_{i=1}^r G^1_{n_i} \Pr^{\otimes n_i}(\bar{y}_i). \]

It then follows from Eq. 4.19 that \(F_1 \Pr J^1(\bar{y}) = 0\).

Thus, \(J|_{\tilde{P}} : \tilde{P} \to \tilde{S}(s\tilde{L})\) is a well-defined coalgebra morphism. Since \(\delta_{\tilde{P}} = \delta_{\tilde{P}}|_{\tilde{P}}\) and \(HJ = id\), the morphism \(J|_{\tilde{P}}\) is compatible with the differentials. Furthermore, the following diagram in \(dgCoCom_{\geq 0}\) commutes:

\[ \begin{array}{cccc}
(P, \delta_P) & \xrightarrow{\Pr} & (\tilde{S}(s\tilde{L}), \delta) & \xrightarrow{\Pr H} & (\tilde{S}(sL), \delta) \\
(P, \delta_P) & \xrightarrow{\Pr'} & (\tilde{S}(s\tilde{L}'), \delta') & \xrightarrow{G} & (\tilde{S}(sL''), \delta'')
\end{array} \]

Hence, we conclude that \((\tilde{S}(s\tilde{L}), \delta)\) is a pullback in \(dgCoCom_{\geq 0}\) and therefore a pullback in \(Lie_n\Alg\). This completes the proof of statement (2) of the proposition. \(\square\)

### 4.2 Pullbacks of fibrations and acyclic fibrations

**Corollary 4.4.** Suppose \(f: (L, \ell) \to (L'', \ell'')\) is a (acyclic) fibration in \(Lie_n\Alg\) and \(g: (L', \ell') \to (L'', \ell'')\) is an arbitrary morphism between \(Lie\ n\)-algebras. Then the pullback of the diagram

\[ (L', \ell'_k) \xrightarrow{g} (L'', \ell''_k) \xleftarrow{f} (L, \ell_k) \]

exists in \(Lie_n\Alg\), and the morphism induced by the pullback of \(f\) along \(g\) is a (acyclic) fibration.
Proof. Suppose \( f \) is a (acyclic) fibration. Lemma 3.11 implies that there exists a Lie \( n \)-algebra \((L, \ell)\) and an isomorphism \( \phi: (L, \ell) \xrightarrow{\cong} (L, \ell)\) such that \( f\phi: (L, \ell) \to (L'', \ell'')\) is a strict (acyclic) fibration with \( f\phi = (f\phi)_1 = f_1\). It follows from Prop. 4.1, that there exists a pullback diagram in \( \text{Lie}_n\text{Alg} \) of the form

\[
\begin{array}{ccc}
(L, \ell) & \xrightarrow{\phi} & (L', \ell') \\
\downarrow{\tilde{f}\phi} & & \downarrow{\tilde{g}} \\
(L, \ell) & \xrightarrow{f} & (L'', \ell'')
\end{array}
\]

Moreover, Prop. 4.1 implies that the image of above diagram under the tangent functor (3.8) is the pullback diagram of \((f\phi)_1\) along \(g_1\) in \( \text{Ch}_{\geq 0} \). Hence, \( \text{tan}_{\geq 0}(\tilde{f}\phi) \) is a (acyclic) fibration in \( \text{Ch}_{\geq 0} \), and therefore \( \tilde{f}\phi \) is a (acyclic) fibration in \( \text{Lie}_n\text{Alg} \).

Now let \( F, G, \Phi \) denote the dg coalgebra morphisms corresponding to \( f, g, \) and \( \phi \), respectively. Let \((\tilde{C}, \delta_C)\) denote the dg coalgebra witnessing the pullback of \( F \) along \( G \). The second statement of Prop. 4.1, combined with the pasting lemma for pullbacks, implies that the following diagram in \( \text{dgCoCom}_{\geq 0} \) commutes:

\[
\begin{array}{ccc}
(S(sL), \delta) & \xrightarrow{\Phi} & (S(s\tilde{L}), \delta) \\
\downarrow{\tilde{\Phi}} & & \downarrow{\Phi} \\
(C, \delta_C) & \xrightarrow{F} & (S(sL), \delta)
\end{array}
\]

Since \( \tilde{\Phi} \) is an isomorphism, \( \tilde{C} \) is the cofree coalgebra in \( \text{CoCom}_{\geq 0} \) cogenerated by \( \Phi(s\tilde{L}) \). Hence, \((\tilde{C}, \delta_C)\) is also Lie \( n \)-algebra and \( \tilde{F} \) is an (acyclic) fibration. \( \square \)

We end this section with the corollary below, whose proof follows immediately from the construction of the pullbacks in Prop. 4.1 and Cor. 4.4.

**Corollary 4.5.** Let \( f: (L, \ell) \to (L', \ell') \) be a fibration in \( \text{Lie}_n\text{Alg} \). Then the tangent functor (3.8) maps pullback squares in \( \text{Lie}_n\text{Alg} \) of the form

\[
\begin{array}{ccc}
(L, \ell) & \xrightarrow{f} & (L', \ell') \\
\downarrow{\tilde{f}} & & \downarrow{g} \\
(L, \ell) & \xrightarrow{\phi} & (L'', \ell'')
\end{array}
\]

to pullback squares in \( \text{Ch}_{\geq 0} \).

### 5 Lie\(_n\)Alg\(_{\text{fin}}\) as a category of fibrant objects

Let \((L, \ell)\) be a Lie \( n \)-algebra with underlying graded vector space \( L = \bigoplus_{i \geq 0} L_i \). If each \( L_i \) is finite-dimensional, we say \((L, \ell)\) is **finite type**. For a fixed \( n \in \mathbb{N} \cup \{\infty\} \), we denote by \( \text{Lie}_n\text{Alg}_{\text{fin}} \) the full subcategory of \( \text{L}_{\infty}\text{Alg} \) whose objects are finite type Lie \( n \)-algebras.

23
**Definition 5.1** (Sec. 1 [2]). Let $C$ be a category with finite products, with terminal object $\ast \in C$, and equipped with two classes of morphisms called *weak equivalences* and *fibrations*. A morphism which is both a weak equivalence and a fibration is called an *acyclic fibration*. Then $C$ is a category of fibrant objects (CFO) for a homotopy theory iff:

1. Every isomorphism in $C$ is an acyclic fibration.
2. The class of weak equivalences satisfies “2 out of 3”. That is, if $f$ and $g$ are composable morphisms in $C$ and any two of $f, g, g \circ f$ are weak equivalences, then so is the third.
3. The composition of two fibrations is a fibration.
4. The pullback of a fibration exists, and is a fibration. That is, if $Y \xrightarrow{g} Z \xleftarrow{f} X$ is a diagram in $C$ with $f$ a fibration, then the pullback $X \times_Z Y$ exists, and the induced projection $X \times_Z Y \rightarrow Y$ is a fibration.
5. The pullback of an acyclic fibration exists, and is an acyclic fibration.
6. For any object $X \in C$ there exists a (not necessarily functorial) path object, that is, an object $X^I$ equipped with morphisms

$$X \xrightarrow{s} X^I \xrightarrow{(d_0, d_1)} X \times X,$$

such that $s$ is a weak equivalence, $(d_0, d_1)$ is a fibration, and their composite is the diagonal map.
7. All objects of $C$ are fibrant. That is, for any $X \in C$ the unique map $X \rightarrow \ast$ is a fibration.

We now prove the main result of the paper.

**Theorem 5.2.** Let $n \in \mathbb{N} \cup \{\infty\}$. The category $\text{Lie}_n \text{Alg}^{\text{fin}}$ of finite type Lie $n$-algebras and weak $L_\infty$-morphisms has the structure of a category of fibrant objects, in which the weak equivalences and fibrations are those morphisms that satisfy the defining criteria given in Def. 3.6. That is, a morphism $f : (L, \ell) \rightarrow (L', \ell')$ is

- a weak equivalence iff $f$ is a $L_\infty$ quasi-isomorphism,
- a fibration iff the associated chain map $f_1 : (L, \ell_1) \rightarrow (L', \ell'_1)$ is a surjection in all positive degrees.

**Proof.** We begin by noting that Prop. 3.5 implies that $\text{Lie}_n \text{Alg}^{\text{fin}}$ has finite products. Next, it follows immediately from the definition of weak equivalences and fibrations that: every isomorphism is an acyclic fibration, the weak equivalences satisfy “2 out of 3”, the composition of two fibrations is again a fibration, and that the trivial map $(L, \ell) \rightarrow 0$ is a fibration for any $(L, \ell) \in \text{Lie}_n \text{Alg}^{\text{fin}}$. Hence axioms 1, 2, 3, and 7 in Def. 5.1 for a CFO are satisfied.

To verify axiom 4, suppose $(L', \ell') \xrightarrow{g} (L'', \ell'') \xrightarrow{f} (L, \ell)$ is a diagram in $\text{Lie}_n \text{Alg}^{\text{fin}}$ and $f$ is a fibration. It follows from Cor. 4.4 that the pullback $(\tilde{L}, \tilde{\ell})$ of the diagram exists in $\text{Lie}_n \text{Alg}$ and that the morphism induced by the pullback is a fibration. Cor. 4.4 also implies that the underlying complex of $(\tilde{L}, \tilde{\ell})$ is the pullback of the diagram $\tan_{\geq 0}((L', \ell') \xrightarrow{g} (L'', \ell'') \xrightarrow{f} (L, \ell))$ in $\text{Ch}_{\geq 0}$. Hence, $(\tilde{L}, \tilde{\ell})$ is clearly finite type, and axiom 4 is satisfied. The same argument also verifies axiom 5.

Finally, recall that any diagonal map $\text{diag} : (L, \ell) \rightarrow (L \oplus L, \ell \oplus \ell)$ is a strict morphism in $\text{Lie}_n \text{Alg}^{\text{fin}}$. Hence, Prop. 3.8 implies that $\text{diag}$ has a factorization $(L, \ell) \xrightarrow{j} (\tilde{L}, \tilde{\ell}) \xrightarrow{\phi} (L \oplus L, \ell \oplus \ell)$ in the category $\text{Lie}_n \text{Alg}$, in which $j$ is a weak equivalence and $\phi$ is a fibration. Recall from the proof of Prop. 3.8 that $\tilde{L} = (L \oplus P(L \oplus L))$, where $P(L \oplus L)$ is the graded vector space defined in (3.9). Therefore, $\tilde{L}$ is finite type, and $(\tilde{L}, \tilde{\ell})$ is a path object for $(L, \ell)$ in $\text{Lie}_n \text{Alg}^{\text{fin}}$. Hence, axiom 6 in Def. 5.1 is satisfied, and this completes the proof. \hfill $\square$
Remark 5.3. Clearly, the same proof shows that the category $\mathbf{Lie}_n\mathbf{Alg}$ is also a category of fibrant objects with the same weak equivalences, fibrations, and path objects as $\mathbf{Lie}_n\mathbf{Alg}^{\text{fin}}$.

We can now complete the discussion that we started in Sec. 3.3 concerning the factorization of arbitrary weak $L_\infty$-morphisms in $\mathbf{Lie}_n\mathbf{Alg}^{\text{fin}}$. Let us recall Brown’s Factorization Lemma:

**Lemma 5.4** (Sec. 1 [2]). Let $C$ be a category of fibrant objects. Let $f : X \to Y$ be a morphism in $C$, and let $Y^I$ be a path object for $Y$. Then $f$ can be factored as

$$X \xrightarrow{j} X \times_Y Y^I \xrightarrow{\phi_f} Y$$

where $\phi_f$ is a fibration, and $j$ is a weak equivalence which is a section (right inverse) of an acyclic fibration.

The morphisms $j$ and $\phi_f$ in the lemma can easily be expressed in terms of the maps $Y \xrightarrow{\Delta} Y^I \xrightarrow{(d_0, d_1)} Y \times Y$ that appear in factorization of the diagonal. See, for example, Sec. 2.1 of [16] for details.

Hence, Thm. 5.2 and the factorization lemma imply the following:

**Corollary 5.5.** Let $f : (L, \ell) \to (L', \ell')$ be a weak $L_\infty$-morphism between finite type Lie $n$-algebras. Then $f$ can be factored in the category $\mathbf{Lie}_n\mathbf{Alg}^{\text{fin}}$ as

$$(L, \ell) \xrightarrow{j} (\tilde{L}, \tilde{\ell}) \xrightarrow{p_f} (L', \ell')$$

where $j$ is a weak equivalence, and $p_f$ is a fibration in $\mathbf{Lie}_n\mathbf{Alg}$.

The CFO structure on Lie $n$-algebras can be thought of as a lift of the projective CFO structure (Sec. 2.2.1) on $\mathbf{Ch}_{\geq 0}$ via the tangent functor (3.8) in the following sense:

**Definition 5.6** (Def. 2.3.3 [1]). A functor $F : C \to D$ between categories of fibrant objects is a (left) exact functor iff

1. $F$ preserves the terminal object, fibrations, and acyclic fibrations.
2. Any pullback square in $C$ of the form

$$
\begin{array}{ccc}
P & \rightarrow & X \\
\downarrow & & \downarrow f \\
Z & \rightarrow & Y
\end{array}
$$

in which $f : X \to Y$ is a fibration in $C$, is mapped by $F$ to a pullback square in $D$.

**Remark 5.7.** Note that axiom 1 in the above definition combined with Lemma 5.4 implies that an exact functor $F : C \to D$ between CFOs sends weak equivalences to weak equivalences.

**Corollary 5.8.** The tangent functor $\text{tan}_{\geq 0} : \mathbf{Lie}_n\mathbf{Alg}^{\text{fin}} \to \mathbf{Ch}_{\geq 0}^{\text{proj}}$ is an exact functor between categories of fibrant objects.

**Proof.** Theorem 5.2 and Cor. 4.5.
Comparison with the Vallette CFO structure on $L_\infty\text{Alg}$

In [20], Vallette showed that the category $L_\infty\text{Alg}$ of $\mathbb{Z}$-graded $L_\infty$-algebras is exactly the subcategory of bifibrant objects in the Hinich model structure [10] on dgCoCom. This result implies the following theorem:

**Theorem 5.9** (Thm. 2.1 [20]).

1. The category $L_\infty\text{Alg}$ of $\mathbb{Z}$-graded $L_\infty$-algebras and weak $L_\infty$-morphisms has the structure of a category of fibrant objects in which a morphism

   \[ f : (L, \ell) \rightarrow (L', \ell') \]

   is a weak equivalence iff it is a $L_\infty$-quasi-isomorphism, and a fibration iff it is a $L_\infty$-epimorphism (Def. 3.3).

2. Every $L_\infty$-algebra $(L, \ell)$ has a functorial path object whose underlying graded vector space is

   \[ (L \otimes \Omega^*_\text{poly}(\Delta^1))_m \cong L_m[t] \oplus L_{m+1}[t]dt \]

   where $\Omega^*_\text{poly}(\Delta^1)$ denotes the commutative dg algebra of polynomial de Rham forms on the geometric 1-simplex.

Let us make a few comparisons between the CFO structure on $\text{Lie}_n\text{Alg}^{\text{fin}}$ given in Thm. 5.2 and Vallette’s CFO structure on $L_\infty\text{Alg}$.

First, the weak equivalences in $\text{Lie}_n\text{Alg}^{\text{fin}}$ and $L_\infty\text{Alg}$ obviously coincide. Furthermore, it follows from Remark 3.7 that the acyclic fibrations also coincide. If $(L, \ell) \in \text{Lie}_n\text{Alg}$, then the path object $(L \otimes \Omega^*_\text{poly}(\Delta^1), \ell\Omega)$ for $(L, \ell) \in L_\infty\text{Alg}$ is not a Lie $n$-algebra. And the degree zero truncation of $(L \otimes \Omega^*_\text{poly}(\Delta^1))$ is certainly not finite type. However, Vallette’s Prop. 3.3 in [20] implies that $L_\infty\text{Alg}$ is also equipped with a functorial cylinder object. If $(L, \ell)$ is a finite type Lie $n$-algebra, then the degree zero truncation of the cylinder object $(L \otimes \mathcal{J}, \ell)$ is also finite type. It would be interesting to work out further details and show that the CFO structure on $\text{Lie}_n\text{Alg}^{\text{fin}}$ satisfies Brown’s additional axioms (F) and (G) in [2, Sec. 6]. This would imply that $\text{Lie}_n\text{Alg}^{\text{fin}}$ is “almost” a model category, in the sense of Vallette [20, Sec. 4.1].

6 Maurer–Cartan sets

In this section, we analyze Maurer-Cartan (MC) sets of $\mathbb{Z}$-graded $L_\infty$-algebras constructed by tensoring Lie $n$-algebras with bounded commutative dg algebras. This construction naturally arises when studying formal deformation problems in characteristic zero. It also appears, as mentioned in the introduction, in the definition of the spatial realization functor for chain Lie algebras. The smooth analog of the Maurer-Cartan set is featured in the definition of Henriques’ integration functor [9] for Lie $n$-algebras.

We begin by recalling some basic facts about Maurer-Cartan elements from [5, Sec. 2] and references within. We note that the $L_\infty$-algebras in [5] are assumed to be complete and filtered, which is not the case in this paper. However, the particular results that we recall below will still hold since all $L_\infty$-algebras involved are sufficiently “tame” in the following sense:

**Definition 6.1.** A $\mathbb{Z}$-graded $L_\infty$-algebra $(L, \ell)$ is tame iff there exists an $N \geq 1$ such that for all $k \geq N$

\[ \ell_k(x_1, \ldots, x_k) = 0 \quad \forall x_1, \ldots x_k \in L_{-1}. \]

A (weak) $L_\infty$-morphism $f : (L, \ell) \rightarrow (L', \ell')$ between tame $L_\infty$-algebras is a tame morphism iff there exists an $N \geq 1$ such that for all $k \geq N$

\[ f_k(x_1, \ldots, x_k) = 0 \quad \forall x_1, \ldots x_k \in L_{-1}. \]
For trivial reasons, every Lie $n$-algebra is tame and every morphism between Lie $n$-algebra is tame for any $n \in \mathbb{N} \cup \{\infty\}$. Given a tame $\mathbb{Z}$-graded $L_{\infty}$-algebra $(L, \ell)$, the curvature $\text{curv} : L_{-1} \to L_{-2}$ is the function

$$\text{curv}(a) := s^{-1} \sum_{k \geq 1} \frac{1}{k!} \delta_k^1(sa, sa, \ldots, sa)$$

$$= \ell_1(a) + \sum_{k \geq 2} (-1)^{k(k-1)/2} \frac{1}{k!} \ell_k(a, a, \ldots, a) \in L_{-2}. \quad (6.1)$$

where $\delta$ is the corresponding codifferential on $\bar{S}(sL)$. For any $a \in L_{-1}$, the expression $\exp(sa) - 1$ is a well-defined element of the completion of $\bar{S}(sL)$ defined by the corresponding formal power series. By extending $\delta$ in the natural way, a straightforward calculation shows that

$$\delta(\exp(sa) - 1) = \exp(sa)(s \text{curv}(a)) \quad (6.2)$$

and hence

$$s \text{curv}(a) = \text{pr}_{sL} \circ \delta(\exp(sa) - 1), \quad (6.3)$$

where $\text{pr}_{sL}$ denotes the canonical projection to the vector space $sL$. The Maurer-Cartan elements of $L$ are the elements of the subset

$$\text{MC}(L) := \{x \in L_{-1} \mid \text{curv}(x) = 0\}.$$

Similarly, let $f : (L, \ell_k) \to (L', \ell'_k)$ be a tame morphism. Such a morphism induces a function $f_* : L_{-1} \to L'_{-1}$ defined as:

$$f_*(a) := s^{-1} \sum_{k \geq 1} \frac{1}{k!} F_k^1(sa, sa, \ldots, sa)$$

$$= f_1(a) + \sum_{k \geq 2} (-1)^{k(k-1)/2} \frac{1}{k!} f_k(a, a, \ldots, a). \quad (6.4)$$

As in Eq. 6.2, we extend $F$ to the completions of $\bar{S}(sL)$ and $\bar{S}(sL')$, and a straightforward calculation shows that

$$F(\exp(sa) - 1) = \exp(s f_*(a)) - 1. \quad (6.5)$$

Remark 6.2.

1. If $f : (L, \ell) \to (L', \ell')$ and $g : (L', \ell') \to (L'', \ell'')$ are tame morphisms, then it follows from the composition formula (2.6) for $L_{\infty}$-morphisms that $gf : (L, \ell) \to (L'', \ell'')$ is also tame. Indeed, if $f_k$ vanishes on $(L_{-1})^{\otimes k}$ for all $k \geq N_L$ and $g_k$ vanishes on $(L'_{-1})^{\otimes k}$ for all $k \geq N_{L'}$ then $(gf)_k$ vanishes on $(L_{-1})^{\otimes k}$ for all $k \geq N_LN_{L'}$.

2. Note that the function $f_*$ in Eq. 6.4 is also well-defined for any coalgebra morphism $F : \bar{S}(sL) \to \bar{S}(sL')$ satisfying

$$F_k^1(sx_1, \ldots, sx_k) = 0 \quad \forall x_1, \ldots x_k \in L_{-1}.$$

for $k \gg 1$. Compatibility of $F$ with the codifferentials is obviously not necessary. We will use this in the proof of Prop. 6.5.

We note that the first statement in Remark 6.2 along with Eqs. 6.2, 6.3, and 6.5 imply the following result:
Proposition 6.3 (cf. Prop. 2.2 [5]). Let $f : (L, \ell) \to (L', \ell')$ be a tame $L_\infty$-morphism between tame $\mathbb{Z}$-graded $L_\infty$-algebras. Then the function (6.4) restricts to a well-defined function $f_* : \text{MC}(L) \to \text{MC}(L')$ between the corresponding Maurer-Cartan sets. Moreover, the assignment

$$(L, \ell) \mapsto (L', \ell') \quad \mapsto \quad \text{MC}(L) \xrightarrow{f_*} \text{MC}(L')$$

is functorial.

6.1 “Deformation functors”

The following construction provides important examples of tame $L_\infty$-algebras. We denote by $\text{cdga}_{\geq 0}^{\text{bnd}}$ the category whose objects are unital, non-negatively and cohomologically graded commutative dg $k$-algebras which are bounded from above. Morphisms in $\text{cdga}_{\geq 0}^{\text{bnd}}$ are unit preserving cdga morphisms. Let $n \in \mathbb{N} \cup \{\infty\}$. Let $(L, \ell) \in \text{Lie}_n\text{Alg}^{\text{fin}}$ be a finite type Lie $n$-algebra and let $(B, d_B) \in \text{cdga}_{\geq 0}^{\text{bnd}}$. Denote by $(L \otimes B, \ell^B)$ the $\mathbb{Z}$-graded $L_\infty$-algebra whose underlying chain complex is $(L \otimes B, \ell^B_1)$ where

$$(L \otimes B)_m := \bigoplus_{i+j=m} L_i \otimes B^{-j}$$

$$\ell^B_1(x \otimes b) := \ell_1 x \otimes b \pm x \otimes d_B b$$

and whose higher brackets are defined as:

$$\ell^B_k(x_1 \otimes b_1, \ldots, x_k \otimes b_k) := \pm \ell_k(x_1, \ldots, x_k) \otimes b_1 b_2 \cdots b_k. \quad (6.6)$$

If $f : (L, \ell) \to (L', \ell')$ is a morphism of Lie $n$-algebras, then it is easy to verify that the maps $f^B_k : \Lambda^k(L \otimes B) \to L' \otimes B$ defined as

$$f^B_k(x_1 \otimes b_1, \ldots, x_k \otimes b_k) := \pm f_k(x_1, \ldots, x_k) \otimes b_1 b_2 \cdots b_k. \quad (6.7)$$

assemble together to give a $L_\infty$-morphism $f^B : (L \otimes B, \ell^B) \to (L' \otimes B, \ell'^B)$ in $\text{L}_\infty\text{Alg}$.

Lemma 6.4. Let $(L, \ell) \in \text{Lie}_n\text{Alg}^{\text{fin}}$ be a finite type Lie $n$-algebra and $(B, d_B) \in \text{cdga}_{\geq 0}^{\text{bnd}}$ a bounded cdga.

1. If $f : (L, \ell) \to (L', \ell')$ is a morphism of Lie $n$-algebras, then the induced $L_\infty$-morphism

$$f^B : (L \otimes B, \ell^B) \to (L' \otimes B, \ell'^B)$$

is a tame morphism between tame $L_\infty$-algebras.

2. The assignment

$$(L \otimes B, \ell^B_k) \xrightarrow{f^B_k} (L' \otimes B, \ell'^B_k) \quad \mapsto \quad \text{MC}(L \otimes B) \xrightarrow{f^B_k} \text{MC}(L' \otimes B).$$

defines a functor

$$\text{MC}(- \otimes B) : \text{Lie}_n\text{Alg}^{\text{fin}} \to \text{Set} \quad (6.8)$$

natural in $B \in \text{cdga}_{\geq 0}^{\text{bnd}}$. 

28
Proof. Statement (2) is straightforward. For statement (1), note that since the underlying cochain complex of $B$ is bounded, there exists an $N \geq 0$ such that $B = \bigoplus_{i \geq 0} B^i$. Since the underlying chain complex of $L$ is concentrated in non-negative degrees we have $(L \otimes B)^{-1} = \bigoplus_{i \geq -1} L_i \otimes B^{i+1}$. It follows from Eq. 6.6 and Eq. 6.7 that for all $k \geq N$, and any $a_1, \ldots, a_k \in (L \otimes B)^{-1}$, we have $\ell_k^B(a_1, \ldots, a_k) = 0$ and $f_k^B(a_1, \ldots, a_k) = 0$. A similar argument shows that $(L' \otimes B, \ell'^B)$ is also tame. \hfill \square

We end this section by showing that the functor $\text{MC}(- \otimes B)$ defined in (6.8) preserves certain pullback diagrams. We provide a detailed proof of this rather straightforward fact in order to have the analogous statement in the category of Banach manifolds follow as a simple corollary (Cor. 6.7).

**Proposition 6.5.** Let $B \in \text{cdga}_{\geq 0}^{bnd}$ be a bounded cdga. Let $f: (L, \ell) \to (L'', \ell'')$ be a fibration and $g: (L', \ell') \to (L'', \ell'')$ be a morphism in $\text{Lie}_n \text{Alg}^{\text{fin}}$. Let $(\tilde{L}, \tilde{\ell})$ be the pullback of the diagram $(L', \ell') \xrightarrow{g} (L'', \ell'') \xleftarrow{f} (L, \ell)$. Then the induced commutative diagram of sets

$$
\begin{array}{ccc}
\text{MC}(\tilde{L} \otimes B) & \longrightarrow & \text{MC}(L \otimes B) \\
\downarrow & & \downarrow^{\ell''}
\text{MC}(L' \otimes B) & \xrightarrow{g^B} & \text{MC}(L'' \otimes B)
\end{array}
$$

(6.9)

is a pullback square.

Proof. We only consider here the special case when $f: (L, \ell) \to (L'', \ell'')$ is a strict fibration, since the general case will then follow from Lemma 3.11.

Let $f = f_1: (L, \ell) \to (L'', \ell'')$ be strict fibration and $g: (L', \ell') \to (L'', \ell'')$ be a morphism in $\text{Lie}_n \text{Alg}^{\text{fin}}$. As in the proof of Prop. 4.1, we have the following pullback diagram in $\text{Lie}_n \text{Alg}^{\text{fin}} \subseteq \text{dgCoCom}_{\geq 0}$:

$$
\begin{array}{c}
(\tilde{S}(s\tilde{L}), \tilde{\delta}) \\
\text{Pr'} H
\end{array}
\xrightarrow{	ext{Pr} H}
\begin{array}{c}
(\tilde{S}(sL), \delta)
\end{array}

\begin{array}{c}
(\tilde{S}(sL'), \delta')
\end{array}
\xrightarrow{G}
\begin{array}{c}
(\tilde{S}(sL''), \delta'')
\end{array}
$$

where $\tilde{L}$ is the pullback of diagram (4.4) in $\text{Ch}_{\geq 0}$, and $H: \tilde{S}(sL' \oplus sL) \to \tilde{S}(sL' \oplus sL)$ is the coalgebra endomorphism (4.5). Let $B \in \text{cdga}_{\geq 0}^{bnd}$. For brevity, denote the pullback of diagram (6.9) as

$$
E := \text{MC}(L' \otimes B) \times_{\text{MC}(L'' \otimes B)} \text{MC}(L \otimes B)
$$

(6.10)

Then $E$ is the equalizer of the following diagram of sets

$$
(L' \otimes B)^{-1} \times (L'' \otimes B)^{-1} \xrightarrow{\text{curv}^B \times \text{curv}^B} (L' \otimes B)^{-2} \times (L' \otimes B)^{-2}
$$

(6.11)

where $\text{curv}^B$ and $\text{curv}^B$ are the curvature functions (6.1) for the tame $L_\infty$-algebras $L \otimes B$ and $L' \otimes B$, respectively. There is also a similar equalizer diagram involving $\tilde{L} \otimes B$:
Let \( J : \tilde{S}(sL' \oplus sL) \to \tilde{S}(sL' \oplus sL) \) denote the right inverse of the coalgebra morphism \( H \) given by Eq. 4.6. It follows from Remark 6.2 and Lemma 6.4 that the function \( j^B_s : (L' \otimes B \oplus L \otimes B)^{-1} \to (L' \otimes B \oplus L \otimes B)^{-1} \) is well-defined. Now let

\[
\varphi : (L' \otimes B)^{-1} \times (L'' \otimes B)^{-1} \to ((L' \oplus L) \otimes B)^{-1}
\]

denote the function

\[
\varphi(a', a) := j^B_s(a', a) = (a', a - (\sigma \otimes \text{id}_B)g^B_s(a')).
\]

First, we observe that \( \text{im} \varphi \subseteq \tilde{L} \otimes B \). Indeed, we have \( f^B_s(a) = g^B_s(a') \), and since \( f \) is strict, \( f^B_s = f_1 \otimes \text{id}_B \).

Since the functor \( - \otimes_k B \) is exact, it follows from diagram (4.4) that

\[
\begin{align*}
\tilde{L} \otimes B & \xrightarrow{\text{Pr} \otimes \text{id}_B + \sigma G_1 \text{ Pr}' \otimes \text{id}_B} sL \otimes B \\
& \quad \downarrow \\
\text{Pr}' \otimes \text{id}_B \\
\end{align*}
\]

\[
\begin{align*}
\text{Pr}' \otimes \text{id}_B & \xrightarrow{G_1 \otimes \text{id}_B} sL' \otimes B \\
& \quad \downarrow \\
\text{Pr} \otimes \text{id}_B \\
\end{align*}
\]

is a pullback diagram. A straightforward calculation then shows that \( \varphi(a', a) \in \tilde{L} \otimes B \).

Next, we claim that the restriction of \( \varphi \) to the pullback (6.10) induces a function

\[
\varphi|_E : E \to \text{MC}(\tilde{L} \otimes B).
\]

So suppose \( (\text{curv}^B(a'), \text{curv}^B(a)) = 0 \). We will show that \( \text{curv}^B(\varphi(a', a)) = 0 \). Let \( \delta_\oplus \) denote the codifferential that encodes the product \( L_\infty \) -structure on \( L' \oplus L \). It then follows that

\[
\text{curv}^B_\oplus(a', a) = (\text{curv}^B(a'), \text{curv}^B(a)) = 0.
\]

Let \( \tilde{\delta} = J \circ \delta_\oplus \circ H \) denote the codifferential encoding the \( L_\infty \) structure on \( \tilde{L} \), and let \( \tilde{\delta}^B \) denote the codifferential for the induced structure on \( \tilde{L} \otimes B \). Passing to the completions, Eq. 6.2 implies that

\[
\tilde{\delta}^B(\exp(s\varphi(a', a)) - 1) = \exp(s\varphi(a', a))(s\tilde{\text{curv}}^B(\varphi(a', a))),(6.14)
\]

while Eq. 6.5 gives us

\[
\exp(s\varphi(a', a)) - 1 = \exp(sj^B_s(a', a)) - 1 = J^B(\exp(s(a', a)) - 1).
\]

A straightforward calculation shows that \( \tilde{\delta}^B = J^B \circ \delta^B_\oplus \circ H^B \). Therefore, by combining Claim 4.2 with Eq. 6.15 we obtain:

\[
\begin{align*}
\tilde{\delta}^B(\exp(s\varphi(a', a)) - 1) & = J^B \tilde{\delta}^B_\oplus((\exp(s(a', a)) - 1) \\
& = J^B((\exp(s(a', a)))(s\text{curv}^B_\oplus((a', a)))).
\end{align*}
\]

It then follows from the above equality and Eq. 6.13 that \( \tilde{\delta}^B(\exp(s\varphi(a', a)) - 1) = 0 \). Hence, Eq. 6.14 implies that \( \text{curv}^B(\varphi(a', a)) = 0 \), and so the function \( \varphi|_E \) fits into the following commutative diagram

\[
\begin{diagram}
E & \xrightarrow{\varphi|_E} & \text{MC}(\tilde{L} \otimes B) & \xrightarrow{(pr_\oplus)^B} & \text{MC}(L \otimes B) \\
\downarrow{(pr'_\oplus)^B} & & \downarrow{J^B} & & \\
\text{MC}(L' \otimes B) & \xrightarrow{g^B} & \text{MC}(L'' \otimes B)
\end{diagram}
\]

We therefore conclude that \( \text{MC}(\tilde{L} \otimes B) \) is indeed the pullback of diagram (6.9).
6.2 Maurer-Cartan sets with differentiable structure

We now consider the scenario in which all of the Maurer-Cartan sets in Prop. 6.5 have geometric structure. Specifically, we are interested in the case when the geometry arises from a dg Banach algebra structure on \((B, d_B)\). Examples relevant to our applications in [16] include the cdgas \(\Omega(\Delta^n)\) and \(\Omega(\Lambda^k)\): the dg Banach algebras of \(r\)-times continuously differentiable forms on the geometric \(n\)-simplex \(\Delta^n\) and the geometric horn \(\Lambda^k \subseteq \Delta^k\), respectively. (See Sec. 5.1 of [9]).

So let \(k = \mathbb{R}\) and suppose \((B, d_B) \in \operatorname{cdga}_{\geq 0}^{\text{bind}}\) is a fixed cdga equipped with the structure of a dg Banach algebra. If \((L, \ell) \in \operatorname{Lie}_n\operatorname{Alg}^{\text{fin}}\), then since \(L\) is finite type, the structure on \(B\) naturally makes \((L \otimes B)\) into a graded Banach space. From Eqs. 6.1 and 6.6, we see that the curvature \(\text{curv}^B: (L \otimes B)_{-1} \to (L \otimes B)_{-2}\) is a polynomial and hence a smooth function between Banach manifolds, in the sense of [13, Ch. I,3]. Similarly, if \(f: (L, \ell) \to (L', \ell')\) is a morphism in \(\operatorname{Lie}_n\operatorname{Alg}^{\text{fin}}\), then it follows from Eq. 6.4 and Eq. 6.7 that \(f^B: (L \otimes B)_{-1} \to (L' \otimes B)_{-1}\) is a smooth function.

We now restrict our focus to those \((L, \ell) \in \operatorname{Lie}_n\operatorname{Alg}^{\text{fin}}\) which satisfy the following assumption:

**Assumption 6.6.** The Maurer-Cartan set \(\text{MC}(L \otimes B) \subseteq (L \otimes B)_{-1}\) is a Banach submanifold [13, Ch. II, 2] and therefore

\[
\begin{array}{c}
\text{MC}(L \otimes B) \\ \downarrow \text{curv}^B \\
(L \otimes B)_{-1} \\
\hline
0 \\
(L \otimes B)_{-2}
\end{array}
\]

is an equalizer diagram in the category of Banach manifolds.

**Corollary 6.7.** Suppose \(B \in \operatorname{cdga}_{\geq 0}^{\text{bind}}\) has the structure of a dg Banach algebra. Let \(f: (L, \ell) \to (L'', \ell'')\) be a fibration and \(g: (L', \ell') \to (L'', \ell'')\) be a morphism in \(\operatorname{Lie}_n\operatorname{Alg}^{\text{fin}}\), and let \((\tilde{L}, \tilde{\ell})\) be the pullback of \(f\) the pullback of the diagram \((L', \ell') \xrightarrow{\alpha} (L'', \ell'') \xleftarrow{\beta} (L, \ell)\).

Assume that all of the aforementioned Lie \(n\)-algebra satisfy assumption (6.6). If the pullback of the diagram

\[
\begin{array}{cc}
\text{MC}(L' \otimes B) & \xrightarrow{g^B} \text{MC}(L'' \otimes B) \\
\xrightarrow{f^B} & \xleftarrow{f^B} \text{MC}(L \otimes B)
\end{array}
\]

exists as a Banach manifold, then the induced commutative diagram

\[
\begin{array}{ccc}
\text{MC}(\tilde{L} \otimes B) & \xrightarrow{\text{MC}(\alpha)} & \text{MC}(L \otimes B) \\
\downarrow & & \downarrow f^B \\
\text{MC}(L' \otimes B) & \xrightarrow{g^B} & \text{MC}(L'' \otimes B)
\end{array}
\]

is a pullback square in the category of Banach manifolds.

**Proof.** As in the proof of Prop. 6.5, we only consider the case when \(f\) is a strict fibration, and leave the general case as an exercise. First we note that the underlying set of the pullback in the category of Banach manifolds is the usual fiber product in sets [13, Ch.II.2]. Therefore, by hypothesis, \(E := \text{MC}(L' \otimes B) \times_{\text{MC}(L'' \otimes B)} \text{MC}(L \otimes B)\) admits the structure of a manifold. In the proof of Prop. 6.5, we showed that the function \(\varphi\) defined in Eq. 6.12 induces an isomorphism

\[
\varphi|_E: E \to \text{MC}(\tilde{L} \otimes B)
\]

between cones over the diagram consisting of the functions \(f_\ast^B\) and \(g_\ast^B\) in the category of sets. Using Eq. 6.1 and Eq. 6.4, we observe that all functions appearing in the proof of Prop. 6.5 are either polynomial functions between Banach spaces, or maps between equalizers of such polynomial functions. The equalizers themselves are Banach submanifolds by assumption. Hence, the morphism \(\varphi|_E\) respects the differentiable structures, and therefore is a diffeomorphism via the universal property.

\[\square\]
7 Postnikov towers for Lie $n$-algebras

In this last section, we analyze the functorial aspects of Henriques’ Postnikov construction for Lie $n$-algebras [9]. We show that in certain cases the Postnikov tower admits a convenient functorial decomposition. We use this in [16] to prove that the integration functor sends a certain distinguished class of fibrations in $\text{Lie}_n\text{Alg}^{\text{fin}}$ to fibrations between simplicial Banach manifolds. We call these distinguished fibrations “quasi-split”.

**Definition 7.1.** A fibration of Lie $n$-algebras $f: (L, \ell) \to (L', \ell')$ is a quasi-split fibration iff:

1. the induced map in homology $H(f_1): H(L) \to H(L')$ is surjective in all positive degrees and,
2. in degree zero, the map $H(f_1): (H_0(L), [\cdot, \cdot]) \to (H_0(L'), [\cdot, \cdot'])$ is a split epimorphism of Lie algebras.

Note that every acyclic fibration in $\text{Lie}_n\text{Alg}^{\text{fin}}$ is a quasi-split fibration. More generally, $f: (L, \ell) \to (L', \ell')$ is a quasi-split fibration if $H(f_1): H(L) \to H(L')$ is a split epimorphism in the category of $H_0(L)$-modules.

The string Lie 2-algebra $(\mathfrak{g} \oplus \mathbb{R}[-1], \{\ell_1, \ell_2, \ell_3\})$ associated to a simple Lie algebra $\mathfrak{g}$ of compact type was the original motivation for Henriques’ work in [9]. It sits in a quasi-split fiber sequence of the form

$$
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\text{id}} & \mathbb{R} \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\ell_1 = 0} & \mathfrak{g} \\
0 & \xrightarrow{} & \mathfrak{g}
\end{array}
$$

This is a special case of a more general fiber sequence called a "central $n$-extension". (See [16, Sec. 6].)

7.1 Morphisms between towers

Let $(L, \ell)$ be a Lie $n$-algebra. Following [9, Def. 5.6], we consider two different truncations of the underlying chain complex $(L, d = \ell_1)$. For any $m \geq 0$, denote by $\tau_{\leq m}L$ and $\tau_{< m}L$ the following $(m + 1)$-term complexes:

$$(\tau_{\leq m}L)_i = \begin{cases} 
L_i & \text{if } i < m, \\
\text{coker}(d_{m+1}) & \text{if } i = m, \\
0 & \text{if } i > m,
\end{cases}$$

$$(\tau_{< m}L)_i = \begin{cases} 
L_i & \text{if } i < m, \\
im(d_m) & \text{if } i = m, \\
0 & \text{if } i > m.
\end{cases}$$

(7.1)

In degree $m$, the differentials for $\tau_{\leq m}L$ and $\tau_{< m}L$ are $d_m: L_m/\text{im}(d_{m+1}) \to L_{m-1}$, and the inclusion $\text{im}(d_m) \hookrightarrow L_{m-1}$, respectively. The homology complexes of $\tau_{\leq m}L$ and $\tau_{< m}L$ are

$$H_i(\tau_{\leq m}L) = \begin{cases} 
H_i(L) & \text{if } i \leq m, \\
0 & \text{if } i > m,
\end{cases}$$

$$H_i(\tau_{< m}L) = \begin{cases} 
H_i(L) & \text{if } i < m, \\
0 & \text{if } i \geq m.
\end{cases}$$
We have the following surjective chain maps
\[ p_{\leq m}: L \to \tau_{\leq m}L \quad p_{< m}: L \to \tau_{< m}L \] (7.2)
where in degree \( m \), the map \( p_{\leq m} \) is the surjection \( L_m \to \text{coker}(d_{m+1}) \), and \( p_{< m} \) is the differential \( d_m: L_m \to \text{im}(d_m) \). There are also the similarly defined surjective chain maps
\[ q_{\leq m}: \tau_{\leq m}L \to \tau_{< m}L, \quad q_{< m+1}: \tau_{< m+1}L \to \tau_{\leq m}L. \] (7.3)
The map \( q_{\leq m} \) in degree \( m \) is the differential \( d_m: \text{coker} d_{m+1} \to \text{im} d_m \), and the identity in all other degrees. The map \( q_{< m+1} \) is the projection \( L_m \to \text{coker} d_{m+1} \) in degree \( m \), the identity in all degrees \( < m \), and the zero map in degree \( m + 1 \). We note that \( q_{< m+1} \) is a quasi-isomorphism of complexes.

Proposition 7.2. Let \((L, \ell)\) be a Lie \( n \)-algebra.

1. The Lie \( n \)-algebra structure on \((L, \ell)\) induces Lie \((m + 1)\)-structures on the complexes \( \tau_{\leq m}L \) and \( \tau_{< m}L \) whose brackets are given by
\[ \tau_{\leq m}L^{\ell}(x_1, \ldots, x_k) := p_{\leq m}L^{\ell}(x_1, \ldots, x_k), \quad \tau_{< m}L^{\ell}(y_1, \ldots, y_k) := p_{< m}L^{\ell}(y_1, \ldots, y_k), \]
where \( \bar{x}_i = p_{\leq m}(x_i) \) and \( \bar{y}_i = p_{< m}(y_i) \).

2. The assignments \((L, \ell) \mapsto (\tau_{\leq m}L, \tau_{< m}L)\) and \((L, \ell) \mapsto (\tau_{\leq m}L, \tau_{< \ell}L)\) are functorial.

3. An \( L_\infty \)-morphism \( \phi: (L, \ell) \to (L', \ell') \) induces a morphism of towers of Lie \( n \)-algebras
\[ \cdots \xrightarrow{\tau_{\leq m-1}} \xrightarrow{q_{\leq m-1}} \xrightarrow{q_{\leq m-2}} \cdots \xrightarrow{\tau_{\leq 1}} \xrightarrow{q_{\leq 1}} \xrightarrow{q_{\leq 0}} \xrightarrow{\tau_{\leq 0}} \] (7.4)
in which the horizontal arrows are the strict \( L_\infty \)-morphisms induced by the surjective chain maps (7.3).

Proof. We prove (1) and (2) for \( \tau_{\leq m}L \). The same arguments apply for \( \tau_{< m}L \). First, we verify that the brackets \( \tau_{\leq m}L^{\ell} \) are well-defined. For degree reasons, the only non-trivial case to check is \( \tau_{\leq m}L^{\ell}(z, x, x) \) when \( \bar{x} \) is in degree \( m \) and \( \bar{z} \) is in degree \( 0 \). Suppose \( x = x_1 + d_{m+1}z \), where \( d_{m+1} \) is the differential \( \ell_1 \) in degree \( m+1 \). The Jacobi-like identities (3.2) for the \( L_\infty \)-structure imply that the degree 0 bracket \( \ell_2 \) satisfies \( \ell_1 \ell_2(x_1, x_2) = \ell_2(x_1, x_2) = \ell_1(x_1, x_2) \). Hence, \( \ell_2(x_1, x_2) = \ell_1(x_1, x_2) + d_{m+1} \ell_2(z, x_2) \), and so \( \tau_{\leq m}L^{\ell} \) is well-defined. The fact that the brackets \( \ell_k \) satisfy the identities (3.2) immediately implies that the brackets \( \tau_{\leq m}L^{\ell} \) satisfy them as well.

Next, let \( \phi: (L, \ell) \to (L', \ell') \) be a morphism in \( \text{Lie}_n \text{Alg} \). Define maps \( \tau_{\leq m} \phi_k \): \( \Lambda^k \tau_{\leq m}L \to \tau_{\leq m}L' \) by
\[ \tau_{\leq m} \phi_k(\bar{x}_1, \ldots, \bar{x}_k) := p_{\leq m} \phi_k(x_1, \ldots, x_k), \]
where \( p_{\leq m}: L' \to \tau_{\leq m}L' \) is the projection (7.2). We verify that these are well-defined. Again, for degree reasons, the only non-trivial case to check is \( \tau_{\leq m} \phi_1(\bar{x}) \) with \( \bar{x} \) in degree \( m \). Recall that \( \phi_1 \) is a chain map (Remark 3.2). Hence, if \( x = x_1 + d_{m+1}z \), then \( \tau_{\leq m} \phi_1(\bar{x}) = \tau_{\leq m} \phi_1(x) \). The fact that the maps \( \phi_k \) satisfy the defining equations (2.11) immediately implies that the maps \( \tau_{\leq m} \phi_k \) form an \( L_\infty \)-morphism \( \tau_{\leq m} \phi: (\tau_{\leq m}L, \tau_{\leq m} \ell) \to (\tau_{\leq m}L', \tau_{\leq m} \ell') \).

For statement (3), since the \( L_\infty \)-brackets for \( \tau_{\leq m}L \) and \( \tau_{< m}L \) are defined using the projection maps (7.2), it is easy to see that the horizontal projections in the diagram (7.4) are strict \( L_\infty \)-morphisms. Since the vertical morphisms \( \tau_{\leq m} \phi \) and \( \tau_{< m} \phi \) are also defined using the projection maps (7.2), the diagram indeed commutes. \( \square \)
Remark 7.3. The Lie $n$-algebra $\tau_{\leq 0}L$ is just the Lie algebra $H_0(L)$ concentrated in degree zero. Given a morphism of Lie $n$-algebras $f: (L, \ell) \to (L', \ell')$, the induced morphism $\tau_{\leq 0}f: H_0(L) \to H_0(L')$ of Lie algebras is the morphism $H_0(f_1)$ from Remark 3.2.

7.2 A functorial decomposition of towers

Let $f: (L, \ell) \to (L', \ell')$ be a quasi-split fibration (7.1). Our goal is to decompose the induced morphism between the Postnikov towers associated to $L$ and $L'$. Let us make two simple initial observations. First, recall that every fibration in $\text{Lie}_n\text{Alg}$ can be factored into an isomorphism followed by a strict fibration (Prop. 3.8). Hence, we restrict our discussion here to strict quasi-split fibrations. Second, it follows directly from the definition that every quasi-split fibration is an $L_\infty$-epimorphism (Def. 3.3). Therefore, we present our results below in a slightly more general context for the case when $f$ is a strict $L_\infty$-epimorphism.

We begin with the following useful lemma. A variation of this result arises in the construction of minimal models for $L_\infty$-algebras.

**Lemma 7.4.** Let $f: (L, \ell) \to (L', \ell')$ be an acyclic fibration in $\text{Lie}_n\text{Alg}$. Let $(\ker f_1, \ell_1)$ denote the kernel of the chain map $f_1: (L, \ell_1) \to (L', \ell'_1)$ considered as an abelian Lie $n$-algebra (Example 3.4). Then there exists a $L_\infty$-morphism

$$r: (L, \ell_k) \to (\ker f_1, \ell_1)$$

such that the morphism induced via the universal property of the product:

$$(f, r): (L, \ell) \to (L' \oplus \ker f_1, \ell' \oplus \ell_{k}f)$$

(7.5)

is an isomorphism of Lie $n$-algebras.

**Proof.** Since $f$ is an acyclic fibration, the chain map $f_1: (L, \ell_1) \to (L', \ell'_1)$ is an acyclic fibration in $\text{Ch}^{\text{pro}}_{\geq 0}$. Therefore, there exists a chain map $\sigma: (L', \ell'_1) \to (L, \ell_1)$ such that $f_1 \sigma = \text{id}_{L'}$. Moreover, since the complexes $(L, \ell_1)$ and $(L', \ell'_1)$ are bifibrant in $\text{Ch}^{\text{pro}}_{\geq 0}$, there exists a chain homotopy $h: L \to L$ such that $\text{id}_L - \sigma f_1 = \ell_1 h + h \ell_1$. We consider the following chain map:

$$g_1: L \to \ker f_1, \quad g_1 := \text{id}_L - \sigma f_1$$

(7.6)

and for $k \geq 2$, define the degree $k - 1$ multi-linear maps

$$g_k: L^k \to \ker f_1, \quad g_k := g_1 \circ h \circ \ell_k.$$  

(7.7)

Let $G: \bar{S}(sL) \to \bar{S}(s \ker f_1)$ denote the coalgebra map associated to the maps $g_k$. Let $\delta$ and $\delta_{\ker f_1}$ denote the codifferentials corresponding to the $L_\infty$-structures on $L$ and $\ker f_1$, respectively. We verify that $(\delta_{\ker f_1} \circ G)^1_m$ equals $(G \circ \delta)^1_m$ for all $m \geq 1$. Indeed, since $g_1$ is a chain map, we have $(\delta_{\ker f_1} \circ G)^1_1 = (G \circ \delta)^1_1$. Now let $m \geq 2$. Equations 3.3 and 3.5 imply that

$$(\delta_{\ker f_1} \circ G)^1_m = (-1)^{m(m-1)/2} \bar{s}(\ell_1 g_1 \ell_m)(s^{-1})^{\otimes m}$$

(7.8)

and

$$(G \circ \delta)^1_m = G^1_1 \delta^1_m + \sum_{k \geq 2} G^1_k \delta^k_m = G^1_1 \delta^1_m + \sum_{k \geq 2} g_1 \circ h \delta^k_m.$$  

(7.9)

Since $\delta \circ \delta = 0$, we use Eq. 2.10 to rewrite the last term on the right hand side:

$$(G \circ \delta)^1_m = G^1_1 \delta^1_m - g_1 h s^{-1} \delta^1_1 \delta^1_m = (-1)^{m(m-1)/2} \bar{s}(g_1 - g_1 h \ell_1) \ell_m(s^{-1})^{\otimes m}.$$  

34
By comparing the above equality with Eq. 7.8, and using the fact that \( g_1 - g_1 h \ell_1 = \ell_1 g h \), we conclude that 
\[
(\delta_{\ker f_1} \circ G)_m^j = (G \circ \delta)_m^j.
\]
Hence, \( g: (L, \ell_k) \to (\ker f_1, \ell_1) \) is an \( L_{\infty} \)-morphism. Finally, since the linear map \((f_1, g_1): L \to L' \oplus \ker f_1\) is an isomorphism of complexes, it follows that the induced morphism

\[
(F, G): L \to L' \oplus \ker f_1
\]
is an \( L_{\infty} \)-isomorphism.

\[\square\]

Our first decomposition result involves the commuting squares in (7.4) whose top edges are the strict acyclic fibrations \( q_{<m+1}: (\tau_{<m+1}L, \tau_{<m+1}\ell) \to (\tau_{\leq m}L, \tau_{\leq m}\ell) \) defined (7.3). Let \( \ker q_{<m+1} \) denote the kernel of the strict acyclic fibration of the chain map \( q_{<m+1} \). Then \( \ker q_{<m+1} \) is an abelian Lie \( n \)-algebra concentrated in degrees \( m \) and \( m + 1 \) with

\[
(\ker q_{<m+1})_m = \im d_{m+1}, \quad (\ker q_{<m+1})_{m+1} = \im d_{m+1}[-1].
\]

(7.10)
The induced differential \( \ell^\ker = \ell_1 \) on \( \ker q_{<m+1} \) is simply the desuspension isomorphism.

**Proposition 7.5.** Let \( f: (L, \ell) \to (L', \ell') \) be a strict \( L_{\infty} \)-epimorphism between Lie \( n \)-algebras. Then there exists morphisms in \( \text{Lie}_n\text{Alg} \)

\[
r: (\tau_{<m+1}L, \tau_{<m+1}\ell) \to (\ker q_{<m+1}, \ell^\ker), \quad r': (\tau_{<m+1}L', \tau_{<m+1}\ell') \to (\ker q'_{<m+1}, \ell'^\ker)
\]

inducing \( L_{\infty} \)-isomorphisms

\[
(q_{<m+1}, r): (\tau_{<m+1}L, \tau_{<m+1}\ell) \xrightarrow{\cong} (\tau_{\leq m}L \oplus \ker q_{<m+1}, \tau_{\leq m}\ell \oplus \ell^\ker)
\]

\[
(q'_{<m+1}, r'): (\tau_{<m+1}L', \tau_{<m+1}\ell') \xrightarrow{\cong} (\tau_{\leq m}L' \oplus \ker q'_{<m+1}, \tau_{\leq m}\ell' \oplus \ell'^\ker)
\]

such that the following diagram commutes in \( \text{Lie}_n\text{Alg} \):

\[
\begin{array}{ccc}
\tau_{<m+1}L & \xrightarrow{(q_{<m+1}, r)} & \tau_{\leq m}L \oplus \ker q_{<m+1} \\
\tau_{<m+1}f & \cong & \tau_{\leq m}f \oplus \ker q_{<m+1} \\
\tau_{<m+1}L' & \xrightarrow{(q'_{<m+1}, r')} & \tau_{\leq m}L' \oplus \ker q'_{<m+1}
\end{array}
\]

(7.11)

**Proof.** Since \( f \) is strict, we have \( f = f_1 \) and so Prop. 7.2 implies that we have the following commutative diagram in \( \text{Lie}_n\text{Alg} \):

\[
\begin{array}{ccc}
\tau_{<m+1}L & \xrightarrow{q_{<m+1}} & \tau_{\leq m}L \\
\tau_{<m+1}f_1 & \cong & \tau_{\leq m}f_1 \\
\tau_{<m+1}L' & \xrightarrow{q'_{<m+1}} & \tau_{\leq m}L'
\end{array}
\]

(7.12)

For the sake of brevity, let \( V \) and \( V' \) denote the abelian Lie \( n \)-algebras \( \ker q_{<m+1} \) and \( \ker q'_{<m+1} \), respectively.

Since \( q_{<m+1} \) and \( q'_{<m+1} \) are acyclic fibrations, Lemma 7.4 provides us with \( L_{\infty} \)-isomorphisms 
\( (q_{<m+1}, r): \tau_{<m+1}L \xrightarrow{\cong} \tau_{\leq m}L \oplus V \) and 
\( (q'_{<m+1}, r'): \tau_{<m+1}L' \xrightarrow{\cong} \tau_{\leq m}L' \oplus V' \). We will show that in the proof of Lemma 7.4, we can choose the morphisms \( r \) and \( r' \) such that diagram (7.11) commutes.
In degree $m$, (7.12) corresponds to the following commutative diagram between short exact sequences of vector spaces:

\[
\begin{array}{cccc}
\text{im } d_{m+1} & \rightarrow & L_m & \rightarrow & \text{coker } d_{m+1} \\
\tau_{<m+1} f_1 |_{\text{im } d} & \downarrow & \tau_{<m+1} f_1 & \downarrow & \tau_{<m} f_1 \\
\text{im } d_{m+1}' & \rightarrow & L_m' & \rightarrow & \text{coker } d_{m+1}'
\end{array}
\]

Since $f$ is an $L_\infty$-epimorphism, the maps $\tau_{<m+1} f_1 |_{\text{im } d}$ and $\tau_{<m} f_1$ are surjections. This, along with the fact that the rows are exact, implies that there exists sections $s: \text{coker } d_{m+1} \rightarrow L_m$, and $s': \text{coker } d_{m+1}' \rightarrow L_m'$, of $\pi$ and $\pi'$, respectively, such that

\[\tau_{<m+1} f_1 \circ s = s' \circ \tau_{<m} f_1.\]

The linear maps $s$ and $s'$ induce sections $\sigma: \tau_{\leq m} L \rightarrow \tau_{<m+1} L$ and $\sigma': \tau_{\leq m} L' \rightarrow \tau_{<m+1} L'$ in $\text{Ch}_{\geq 0}$ of the chain maps $q_{<m+1}$ and $q'_{m+1}$, respectively. Explicitly, we have

\[\sigma(x) := \begin{cases} 
  x, & \text{if } |x| < m \\
  0, & \text{if } |x| > m \\
  s(x), & \text{if } |x| = m
\end{cases}\]

with an analogous formula for $\sigma'$. Moreover, it follows that

\[\tau_{<m+1} f_1 \circ \sigma = \sigma' \circ \tau_{\leq m} f_1. \quad (7.13)\]

We then construct chain homotopies $h: \tau_{<m+1} L \rightarrow \tau_{<m+1} L[1]$ and $h': \tau_{<m+1} L' \rightarrow \tau_{<m+1} L'[1]$, as in the proof of Lemma 7.4. Explicitly, we have

\[h(x) := \begin{cases} 
  0, & \text{if } |x| < m \text{ or } |x| > m \\
  s(x - s\pi(x)) \in \text{im}(d_{m+1})[1], & \text{if } |x| = m
\end{cases}\]

with an analogous formula for $h'$. Hence, the homotopies satisfy

\[\tau_{<m+1} f_1 \circ h = h' \circ \tau_{<m+1} f_1. \quad (7.14)\]

We then use the chain maps $\sigma$, $\sigma'$, the homotopies $h$ and $h'$, and the $L_\infty$-structures $\tau_{<m+1} \ell$ and $\tau_{<m+1} \ell'$ to construct $L_\infty$-morphisms

\[r: \tau_{<m+1} L \rightarrow V, \quad r': \tau_{<m+1} L' \rightarrow V', \quad \text{via formulas } (7.6) \text{ and } (7.7) \text{ in the proof of Lemma 7.4.}
\]

Using (7.13), (7.14), and the fact that $\tau_{<m+1} f_1$ is an $L_\infty$-morphism, a direct computation verifies that for all $k \geq 1$, we have

\[r'_k \circ (\tau_{<m+1} f_1)^{\otimes k} = \tau_{<m+1} f_1 |_V \circ r_k.\]

Hence, the diagram (7.11) commutes. \qed

Now let $m \geq 1$. We focus on those commuting squares in (7.4) whose top edges are the strict quasi-split fibrations $q_{\leq m}: (\tau_{\leq m} L, \tau_{\leq m} \ell) \rightarrow (\tau_{<m} L, \tau_{<m} \ell)$ defined (7.3). As a chain map, $q_{\leq m}$ induces a short exact sequence of chain complexes

\[H_m \rightarrow \tau_{\leq m} L \rightarrow \tau_{<m} L \rightarrow \text{coker } q_{\leq m}, \quad \text{where } H_n \text{ is the homology group } H_n(L) \text{ concentrated in degree } n \text{ with trivial differential. Our second decomposition result is the following:}\]
Proposition 7.6. Let $m \geq 1$ and let $f: (L, \ell) \to (L', \ell')$ be a strict $L_\infty$-epimorphism in $\operatorname{Lie}_n\text{Alg}$ such that the induced map in homology

$$
H(f_1): H_m \to H'_m
$$

is surjective in degree $m$. Then there exists $L_\infty$-structures $\ell$ and $\ell'$ on the graded vector spaces $\tau_{\leq m} L \oplus H_m$ and $\tau_{\leq m} L' \oplus H'_m$, respectively, and $L_\infty$-isomorphisms

$$
\hat{q}: (\tau_{\leq m} L, \tau_{\leq m} \ell) \cong (\tau_{\leq m} L \oplus H_m, \hat{\ell})
$$

$$
\hat{q}': (\tau_{\leq m} L', \tau_{\leq m} \ell') \cong (\tau_{\leq m} L' \oplus H'_m, \hat{\ell'})
$$

such that the following diagram of strict $L_\infty$-morphisms commutes

$$
\begin{array}{ccc}
(\tau_{\leq m} L, \tau_{\leq m} \ell) & \xrightarrow{\hat{q}} & (\tau_{\leq m} L \oplus H_m, \hat{\ell}) \\
\downarrow_{\tau_{\leq m} f} & & \downarrow_{\tau_{\leq m} f \oplus H(f)} \\
(\tau_{\leq m} L', \tau_{\leq m} \ell') & \xrightarrow{\hat{q}'} & (\tau_{\leq m} L' \oplus H'_m, \hat{\ell}')
\end{array}
$$

(7.15)

Proof. Since $f$ is strict, we have $f = f_1$ and so Prop. 7.2 implies that we have the following commutative diagram in $\operatorname{Lie}_n\text{Alg}$:

$$
\begin{array}{ccc}
\tau_{\leq m} L & \xrightarrow{q_{\leq m}} & \tau_{\leq m} L \\
\downarrow_{\tau_{\leq m} f_1} & & \downarrow_{\tau_{\leq m} f_1} \\
\tau_{\leq m} L' & \xrightarrow{q'_{\leq m}} & \tau_{\leq m} L'
\end{array}
$$

(7.16)

In degree $m$, this corresponds to the following commutative diagram between short exact sequences of vector spaces:

$$
\begin{array}{ccc}
H_m & \xrightarrow{\psi} & \operatorname{coker} d_{m+1} \xrightarrow{d_m} \operatorname{im} d_m[-1] \\
\downarrow_{H(f_1)} & & \downarrow_{\tau_{\leq m} f_1} \\
H'_m & \xrightarrow{s'} & \operatorname{coker} d'_{m+1} \xrightarrow{d'_m} \operatorname{im} d'_m[-1]
\end{array}
$$

By hypothesis, the vertical maps are surjections. Let

$$
\mu: H'_m \to H_m, \quad \nu: \operatorname{im} d'_m[-1] \to \operatorname{im} d_m[-1], \quad \psi: \operatorname{im} d_m[-1] \to \operatorname{coker} d_{m+1},
$$

be sections of $H(f_1)$, $\tau_{\leq m} f_1$, and $d_m$, respectively. Denote by $s': \operatorname{im} d'_m[-1] \to \operatorname{coker} d'_{m+1}$ the composition $s' := \tau_{\leq m} f_1 \circ \psi \circ \nu$. In general, the linear map $\tau_{\leq m} f_1 \psi - s' \tau_{\leq m} f_1$ will not equal zero. So let $s: \operatorname{im} d_m[-1] \to \operatorname{coker} d_{m+1}$ be the composition $s := \psi - \mu \circ (\tau_{\leq m} f_1 \psi - s' \tau_{\leq m} f_1)$. Then $s$ and $s'$ are sections of $d_m$ and $d'_{m}$, respectively, and

$$
\tau_{\leq m} f_1 \circ s = s' \circ \tau_{\leq m} f_1.
$$

We use $s$ and $s'$ to define chain maps. Let $t: \tau_{\leq m} L \to \tau_{\leq m} L$ and $\hat{t}: \tau_{\leq m} L \to H_m$ be the linear maps

$$
t(x) := \begin{cases}
s(x), & \text{if } |x| = m \\
x, & \text{if } |x| < m,
\end{cases}
\hat{t} := \operatorname{id} - t q_{\leq m}
$$

respectively. Then the isomorphism $\hat{q}: \tau_{\leq m} L \to \tau_{\leq m} L \oplus H_m$ is defined to be

$$
\hat{q}(x) := \left(q_{\leq m}(z), \hat{r}(z)\right).
$$
The map \(q\)' is defined in the analogous way, using the section \(s'\) instead of \(s\). Hence (7.15) commutes as a diagram in the category \(\text{Ch}_{\geq 0}\).

Finally, we construct compatible \(L_\infty\)-structures on \(\tau_{<m}L \oplus H_m\) and \(\tau_{<m}L' \oplus H'_m\) via transfer across the isomorphisms \(\hat{q}\) and \(\hat{q}'\). For each \(k \geq 1\), we define

\[
\hat{\ell}_k := \hat{q} \circ \tau_{\leq m} \ell_k \circ (\hat{q}^{-1})^\otimes k, \quad \hat{\ell}'_k := \hat{q}' \circ \tau_{\leq m} \ell_k \circ (\hat{q}'^{-1})^\otimes k.
\]

Hence, by construction, the chain maps \(\hat{q}\) and \(\hat{q}'\) lift to \(L_\infty\)-isomorphisms.

\[\square\]

### 7.2.1 The structure of the “twisted product” \((\tau_{<m}L \oplus H_m, \hat{\ell})\)

We emphasize that, in general, the Lie \(n\)-algebra \((\tau_{<m}L \oplus H_m, \hat{\ell})\) defined in the above proof of Cor. 7.6 is not the categorical product of the Lie \(n\)-algebra \((\tau_{<m}L, \tau_{<m}L)\) with the abelian Lie \(n\)-algebra \(H_m\). This is in contrast with the decomposition \(\tau_{<m+1}L \cong \tau_{<m}L \oplus \ker q_{<m+1}\) given in Prop. 7.5.

Indeed, it follows from the definition of \(\hat{q}\) given in the above proof that the \(L_\infty\)-structure maps \(\hat{\ell}_k : \Lambda^k(\tau_{<m}L) \rightarrow \tau_{<m}L \oplus H_m\) defined in Eq. 7.17 can be written as

\[
\hat{\ell}_k((x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)) = \\
(\tau_{<m} \ell_k(x_1, x_2, \ldots, x_k), \hat{\ell}_k(tx_1 + y_1, tx_2 + y_2, \ldots, tx_k + y_k)).
\]

(7.18)

Note that there is only one non-trivial structure map \(\hat{\ell}_k\) that involves non-zero inputs from \(H_m\), since \(H_m\) is concentrated in top degree \(m\). Namely:

\[
\hat{\ell}_2((x, 0), (0, y)) = (0, \ell_2(x, y)),
\]

where \(x \in L_0 = \tau_{<m}L_0\) is an element of degree 0 and \(y \in H_m\). This simple observation plays a key role in our study [16] of integrated quasi-split fibrations.

### References

1. I. Barnea, Y. Harpaz, and G. Horel, Pro-categories in homotopy theory, *Algebr. Geom. Topol.* **17** (2017), 179–189.

2. K. S. Brown, Abstract homotopy theory and generalized sheaf cohomology, *Trans. Amer. Math. Soc.* **186** (1973), 419–458.

3. J. C. Baez and A. S. Crans, Higher-dimensional algebra. VI. Lie 2-algebras, *Theory Appl. Categ.* **12** (2004), 492–538.

4. A. K. Bousfield and V. K. A. M. Gugenheim, On PL de Rham theory and rational homotopy type, *Mem. Amer. Math. Soc.* **8** (1976), no. 179, ix+94 pp.

5. V. A. Dolgushev and C. L. Rogers, On an enhancement of the category of shifted \(L_\infty\)-algebras, *Appl. Categ. Structures* **25** (2017), no. 4, 489–503.

6. Y. Félix, S. Halperin, and J.-C. Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

7. E. Getzler, J.D.S. Jones, “Operads, homotopy algebra and iterated integrals for double loop spaces”. Available as arXiv:hep-th/9403055.
[8] E. Getzler, Lie theory for nilpotent $L_\infty$-algebras, Ann. of Math. (2) 170, 1 (2009) 271–301;

[9] A. Henriques, Integrating $L_\infty$-algebras, Compos. Math. 144 (2008), no. 4, 1017–1045. arXiv:math/0603563.

[10] V. Hinich. DG coalgebras as formal stacks. J. Pure Appl. Algebra, 162(2-3):209–250, 2001. Also available as arXiv:9812034.

[11] G. Horel, Brown categories and bicategories. Available as arXiv:1506.02851.

[12] T. Lada and M. Markl, Strongly homotopy Lie algebras, Comm. Algebra 23 (1995), no. 6, 2147–2161. arXiv:hep-th/9406095.

[13] S. Lang, Differential and Riemannian manifolds, third edition, Graduate Texts in Mathematics, 160, Springer, New York, 1995.

[14] J–L Loday and B. Vallette, Algebraic operads, Grundlehren Math. Wiss. 346, Springer, Heidelberg, 2012.

[15] D. Quillen. Rational homotopy theory. Ann. of Math. (2), 90:205–295, 1969.

[16] C. Rogers and C. Zhu, On the homotopy theory for Lie $\infty$-groupoids, with an application to integrating $L_\infty$-algebras. arXiv:1609.01394.

[17] P. Ševera and M. Širaň, Integration of differential graded manifolds. Available as arXiv:1506.04898.

[18] D. Sullivan, Infinitesimal computations in topology, Inst. Hautes Études Sci. Publ. Math. No. 47 (1977), 269–331 (1978).

[19] M. E. Sweedler, Hopf algebras, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.

[20] B. Vallette, Homotopy theory of homotopy algebras. Available as arXiv:1411.5533.