Global Strong Solution of a 2D coupled Parabolic-Hyperbolic Magnetohydrodynamic System

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Abstract

The main objective of this paper is to study the global strong solution of the parabolic-hyperbolic incompressible magnetohydrodynamic (MHD) model in two dimensional space. Based on Agmon, Douglis and Nirenberg’s estimates for the stationary Stokes equation and the Solonnikov’s theorem of $L^p$-$L^q$-estimates for the evolution Stokes equation, it is shown that the mixed-type MHD equations exist a global strong solution.

keywords

Global strong solution, Magnetohydrodynamics, Stokes equation, $L^p$-$L^q$-estimates.

1 Introduction

We consider the following 2-D incompressible magnetohydrodynamic (MHD) model, which describes the interaction between moving conductive fluids and electromagnetic fields in [10],

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u &= \nu \Delta u - \frac{1}{\rho_0} \nabla p + \frac{\rho_e}{\rho_0} u \times \text{rot} A + f(x), \quad \text{in } \Omega \times [0,T), \\
\frac{\partial^2 A}{\partial t^2} &= \frac{1}{\epsilon_0 \mu_0} \Delta A + \frac{\rho_e}{\epsilon_0} u - \nabla \Phi, \quad \text{in } \Omega \times [0,T), \quad (1.1) \\
\nabla \cdot u &= 0, \quad \text{in } \Omega \times [0,T), \\
\nabla \cdot A &= 0, \quad \text{in } \Omega \times [0,T).
\end{aligned}
\]

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Here $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $T$ is any fixed time. $u(x,t)$, $A(x,t)$, $p(x,t)$ are the velocity field, the magnetic potential and the pressure function, respectively. $\Phi = \partial A_0 / \partial t$ represents the magnetic pressure with the scalar electromagnetic potential $A_0$. The constants $\nu$, $\rho_0$, $\rho_e$, $\epsilon_0$, $\mu_0$ denote kinetic viscosity, mass density, equivalent charge density, electric permittivity and magnetic permeability of free space.

In this paper, we focus on the system (1.1) with the initial-boundary conditions

$$u(0,x) = \phi(x), \quad A(0,x) = \psi(x), \quad A_t(0,x) = \eta(x), \quad \text{in } \Omega,$$

$$u(t,x) = 0, \quad A(t,x) = 0, \quad \text{on } \partial \Omega \times [0,T].$$

Note that the MHD model (1.1) is established based on the the Newton’s second law and the Maxwell equations for the electromagnetic fields in [10]. In addition, the global weak solutions of the corresponding 3-D MHD model (1.1) with (1.2)-(1.3) has been obtained by using the Galerkin technique and standard energy estimates in [10]. In this paper, what we are concerned is the global strong solution of the 2-D MHD model (1.1) with the initial-boundary conditions (1.2)–(1.3).

It is known that there have been huge mathematical studies on the existence of solutions to the N-dimension ($N \geq 2$) classical MHD model established by Chandrasekhar [4]. In particular, Duvaut and Lions [5] constructed a global weak solution and the local strong solution to the 3-D classical MHD equations the initial boundary value problem, and properties of such solutions have been investigated by Sermange and Temam in [15]. Furthermore, some sufficient conditions for smoothness were presented for the weak solution to the 3-D classical MHD equations in [7] and some sufficient conditions of local regularity of suitable weak solutions to the 3-D classical MHD system for the points belonging to a $C^3$-smooth part of the boundary were obtained in [18]. Also, the global strong solutions for heat conducting 3-D classical magnetohydrodynamic flows with non-negative density were proved in [21].

Moreover, let’s recall some known results for the 2-D classical and generalized MHD equations. It is noticed that the 2D classical MHD equations admits a unique global strong solution in [5, 15]. Furthermore, Ren, Wu, et.al [14] have proved the global existence and the decay estimates of small smooth solution for the 2-D classical MHD equations without magnetic diffusion and Cao, Regmi and Wu [3] have obtained the global regularity for the 2-D classical MHD equations with mixed partial dissipation and magnetic diffusion. Besides, Regmi [13] established the global weak solution for 2-D classical MHD equations with partial dissipation and vertical diffusion. There are also very interesting investigations about the existence of strong solutions to the 2-D classical and generalized MHD equations, see [8, 9, 12, 15, 19, 20, 22] and references therein.
However, it is worth pointing out that the incompressible MHD system (1.1) is a mixed-type differential difference equation, which is combined with the parabolic equation (1.1)$_1$ and the hyperbolic equation (1.1)$_2$. The main challenge in obtaining global strong solution of 2-D MHD model (1.1) with (1.2)-(1.3) is the estimate for $||u \times \text{rot} A||_{L^\infty(0,T;L^2)}$ and $||(u \cdot \nabla)u||_{L^\infty(0,T;L^2)}$. The difficulty is overcome by applying the Solonnikov’s theorem [6, 11, 16] of $L^p - L^q$-estimates for the non-stationary Stokes equations and Agmon, Douglis and Nirenberg’s estimates [1, 2, 11] for the stationary Stokes equations. As is known, Solonnikov [16] first gave the proof of Maximal $L^p$-$L^q$-estimates for the Stokes equation (2.4) using potential theoretic arguments. Recently, Geissert, Hess, Hieber et.al [6] provided a short proof of the corresponding Solonnikov’s theorem in [16].

The rest of this article is organized as follows. In Section 2, we introduce some elementary function spaces, a vital embedding theorem and some regularity results of both the non-stationary Stokes equations and stationary Stokes equations. Section 3 is mainly devoted to the proof of global strong solution of (1.1)—(1.3).

2 Preliminaries
2.1 Notations and definitions
First, we introduce some notations and conventions used throughout this paper.

Let $\Omega \subset \mathbb{R}^2$ be a bounded sufficiently smooth domain. Let $H^r(\Omega)(r = 1, 2)$ be the general Sobolev space on $\Omega$ with the norm $|| \cdot ||_{H^r}$ and $L^2(\Omega)$ be the Hilbert space with the usual norm $|| \cdot ||$. The space $H^r_0(\Omega)$ we mean that the completion of $C_0^\infty(\Omega)$ under the norm $|| \cdot ||_{H^r}$. If $F$ is a Banach space, we denote by $L^p(0,T;F)(1 < p < \infty)$ the Banach space of the $F$-valued functions defined in the interval $(0,T)$ that are $L^p$-integrable.

We also consider the following spaces of divergence-free functions (see Temam [17])

$X = \{ u \in C_0^\infty(\Omega, \mathbb{R}^2) \mid \text{div} u = 0 \text{ in } \Omega \},$

$Y = \text{the closure of } X \text{ in } L^2(\Omega, \mathbb{R}^2)$

$= \{ u \in L^2(\Omega, \mathbb{R}^2) \mid \text{div} u = 0 \text{ in } \Omega \},$

$Z = \text{the closure of } X \text{ in } H^1(\Omega, \mathbb{R}^2)$

$= \{ u \in H^1_0(\Omega, \mathbb{R}^2) \mid \text{div} u = 0 \text{ in } \Omega \}.$

**Definition 2.1**. Suppose that $\phi, \eta \in Y$, $\psi \in Z$. For any $T > 0$, a vector function $(u, A)$ is called a global weak solution of problem (1.1)—(1.3) on $(0, T) \times \Omega$ if it satisfies the following conditions:
1. \( u \in L^2(0, T; Z) \cap L^\infty(0, T; Y) \),

2. \( A \in L^\infty(0, T; Z), \ A_t \in L^\infty(0, T; Y) \),

3. For any function \( v \in X \), there hold

\[
\int_\Omega u \cdot v \, dx + \int_0^t \int_\Omega (u \cdot \nabla) u \cdot v + \nu \nabla u \cdot \nabla v - \frac{\rho_e}{\rho_0} (u \times \text{rot} A) \cdot v \, dx \, dt = \int_0^t \int_\Omega f \cdot v \, dx \, dt + \int_\Omega \phi \cdot v \, dx.
\]

and

\[
\int_\Omega \partial A \cdot v \, dx + \frac{1}{\epsilon_0 \mu_0} \nabla u \cdot \nabla v + \frac{\rho_e}{\epsilon_0} u \cdot v \, dx \, dt = \int_\Omega \eta v \, dx.
\]

Now, we define strong solution of the problem (1.1)–(1.3).

**Definition 2.2.** Suppose that \( \phi, \psi \in H^2(\Omega, \mathbb{R}^2) \cap Z, \eta \in Z, \Phi \in L^2(0, T; H^1_0(\Omega)) \). \((u, A)\) is called a global strong solution to (1.1)–(1.3), if \((u, A)\) satisfy

\[
u_{\infty}(\Omega) \cap Z, \quad u_t \in L^\infty_\text{loc}(0, T; \nabla u \text{ in } L^2) \cap L^2_\text{loc}(0, T; Z), \quad A_t \in L^\infty_\text{loc}(0, T; Y),
\]

Furthermore, both (1.1) and (1.3) hold almost everywhere in \( \Omega \times (0, T) \).

**2.2 Lemmas**

Some more lemmas will be frequently used later. One is the following embedding result in [11].

**Lemma 2.3.** For any \( k \geq 0 \), the following hold

\[
L^p((0, T), W^{k+1,p}(\Omega)) \cap L^\infty((0, T); W^{k,q}(\Omega)) \subset L^q(0, T; W^{k,q}(\Omega)),
\]

where \( q = (r(k + 1)p + np)/(rk + n) \). In the special case of \( k = 0 \), (2.1) equals to

\[
L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty((0, T); W^{1,q}(\Omega)) \subset L^q(\Omega \times (0, T))
\]

provided that \( q = (n + r)p/n \).

**Proof.** From Gagliardo-Nirenberg interpolation inequality, we have

\[
\|u\|_{W^{k,q}} \leq C\|u\|_{W^{m,p}}^{\theta} \|u\|_{W^{2,r}}^{1-\theta}, \quad 0 \leq \theta \leq 1,
\]
provided that
\[
\theta \left( m - \frac{n}{p} \right) + (1 - \theta) \left( j - \frac{n}{r} \right) \geq k - \frac{n}{q},
\]
where \( C \) is a constant independent of \( u \).

Inserting \( j = 0, \quad q \geq p, \quad m = k + 1 \) and \( \theta = \frac{p}{q} \) into (2.3), it is easy to see that
\[
\left( \int_{\Omega} |D^k u|^q \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{\Omega} |D^{k+1} u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |u|^r \, dx \right)^{\frac{1}{r}(1-p/q)},
\]
where \( q = \frac{(r(k+1)p+np)}{rk+n} \).

Then we get
\[
\int_0^T \int_{\Omega} |D^k u|^q \, dx \, dt \leq C \sup_{0 \leq t \leq T} \|u\|_{L^p(\Omega)}^{(q-p)r} \int_0^T \int_{\Omega} |D^{k+1} u|^p \, dx \, dt,
\]
which implies (2.1) and (2.2).

The other lemma is responsible for the estimates for \( u, p, u_t \) and follows from the \( L^r-L^q \)-estimates \([6, 16] \) for non-stationary Stokes equations. For its proof, refer to \([6, 16] \).

Let us consider the following Stokes equations
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \nu \Delta u - \nabla p + f(x, t), \\
\nabla \cdot u &= 0, \\
u|_{\partial \Omega} &= 0, \\
|u(0)| &= u_0,
\end{aligned}
\tag{2.4}
\]
where \( \nu > 0 \) is a constant.

**Lemma 2.4.** Let \( \Omega \subset \mathbb{R}^n(n = 2, 3) \) be a domain with compact \( C^3 \)-boundary, \( 1 < r, p < \infty, \quad 0 < T < \infty \). Then for any \( f \in L^r(0, T; L^q(\Omega, \mathbb{R}^n)) \) and \( u_0 \in W^{2,q}(\Omega, \mathbb{R}^n) \) there exists a unique solution \((u, p)\) of (2.4) satisfying
\[
\begin{aligned}
|u|_{L^r(0, T; W^{2,q})} + |u_t|_{L^r(0, T; L^q)} + |p|_{L^r(0, T; W^{1,q})} \\ \leq C(|f|_{L^r(0, T; L^q)} + |u_0|_{W^{2,q}}),
\end{aligned}
\]
where \( C > 0 \) is a constant.
Finally, we give some regularity results for the stationary Stokes system. For its proof, refer to [1, 2, 11].

**Lemma 2.5.** Assume that \((v, p) \in W^{2,p}(\Omega, \mathbb{R}^n) \times W^{1,p}(\Omega) (1 < p < \infty)\) is a weak solution of the stationary Stokes equations

\[
\begin{cases}
- \nu \Delta v - \nabla p = F(x), & \text{in } \Omega, \\
\nabla \cdot v = 0, & \text{in } \Omega, \\
v|_{\partial \Omega} = 0, & \text{on } \partial \Omega,
\end{cases}
\]

and \(F \in W^{k,q}(\Omega, \mathbb{R}^n)(k \geq 0, 1 < q < \infty)\). Then it holds that

\[(v, p) \in W^{k+2,q}(\Omega, \mathbb{R}^n) \times W^{k+1,q}(\Omega)\]

and

\[||v||_{W^{k+2,q}} + ||p||_{W^{k+1,q}} \leq C(||F||_{W^{k,q}} + ||(u, p)||_{L^q})\]

with some constant \(C\) depending on \(n, \Omega\) and \(q\).

### 3 Main Results

In this section, we state the global weak solution existence theorem and the global strong solution existence one for the problem (1.1)–(1.3), and also prove them.

**Theorem 3.1.** Let the initial value \(\phi, \eta \in Y, \psi \in Z\). If \(f \in Y, \Phi \in L^2(0, T; H^1_0(\Omega))\), then there exists a global weak solution for the problem (1.1)–(1.3).

**Proof.** By the standard Galerkin method and the similar estimates in [10], the existence of global weak solution of (1.1)–(1.3) is also valid, we omit it. \(\square\)

**Theorem 3.2.** Let \(\Omega\) be a bounded domain with compact \(C^3\)-boundary. If \(\phi, \psi \in H^2(\Omega, \mathbb{R}^2) \cap Z, \eta \in Z\), for any \(f \in Y, \Phi \in L^2(0, T; H^1_0(\Omega))\), then there exists a global strong solution for the problem (1.1)–(1.3), i.e., for any \(0 < T < \infty\)

\[
\begin{align*}
u & \in L^\infty(0, T; H^2(\Omega, \mathbb{R}^2) \cap Z), \quad u_t \in L^\infty(0, T; Y) \cap L^2(0, T; Z) \\
p & \in L^\infty(0, T; H^1(\Omega)), \\
A & \in L^\infty(0, T; H^2(\Omega, \mathbb{R}^2) \cap Z), \quad A_t \in L^\infty(0, T; Z), \quad A_{tt} \in L^\infty(0, T; Y).
\end{align*}
\]

**Proof.** The proof can be divided into 3 steps. We will use the same generic constant \(C\) to denote various constants that depend on \(\mu_0, \rho_0, \rho_e, \epsilon_0\) and \(T\) only.

**Step 1 The estimates and regularity for \(A\).**
From Theorem 3.1, for any $0 < T < \infty$, we get the global weak solution
\begin{equation}
\begin{aligned}
u &\in L^2(0, T; Z) \cap L^\infty(0, T; Y), \\
A &\in L^\infty(0, T; Z), \ A_t \in L^\infty(0, T; Y).
\end{aligned}
\tag{3.1}
\end{equation}

Multiplying both sides of (1.1) by $-\Delta A_t$ and integrating over $\Omega$, we have
\begin{equation}
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( \int_\Omega |\nabla A_t|^2 + \frac{1}{\epsilon_0 \mu_0} |\Delta A|^2 \right) &= \frac{\rho_0}{\epsilon_0} \int_\Omega \nabla u \nabla A_t dx \tag{3.2}
\end{aligned}
\end{equation}

since $\text{div} A = 0$ and (1.3).

Using the Hölder inequality, it is easy to see that
\begin{equation}
\begin{aligned}
\frac{d}{dt} \left( ||\nabla A_t||^2_{L^2} + \frac{1}{\epsilon_0 \mu_0} ||\Delta A||^2_{L^2} \right) &\leq 2 \left( ||\nabla A_t||^2_{L^2} + \frac{1}{\epsilon_0 \mu_0} ||\Delta A||^2_{L^2} + \frac{\rho_0^2}{\epsilon_0^2} ||\nabla u||^2_{L^2} \right).
\end{aligned}
\tag{3.3}
\end{equation}

Then, by the Gronwall inequality, (3.3) implies
\begin{equation}
\begin{aligned}
||\nabla A_t||^2_{L^2} + ||\Delta A||^2_{L^2} &\leq e^{2T} \left( ||\Delta \psi||_{L^2} + ||\nabla \eta||_{L^2} + 2 \frac{\rho_0^2}{\epsilon_0^2} \int_0^T ||\nabla u||^2_{L^2} ds \right),
\end{aligned}
\tag{3.4}
\end{equation}

for $\forall \ 0 < T < \infty$.

Therefore, we conclude that
\begin{equation}
\begin{aligned}
\nabla A_t &\in L^\infty(0, T; Y), \ \Delta A \in L^\infty(0, T; Y).
\end{aligned}
\tag{3.5}
\end{equation}

Next, we need to derive an estimate on $||A_{tt}||_{L^\infty(0, T; Y)}$.

Multiplying both sides of Eqs. (1.1) by $A_{tt}$ integrating over $\Omega$ lead to
\begin{equation}
\begin{aligned}
\int_\Omega |A_{tt}|^2 dx &= \frac{1}{\epsilon_0 \mu_0} \int_\Omega \Delta AA_{tt} dx + \frac{\rho_0}{\epsilon_0} \int_\Omega uA_{tt} dx \tag{3.6}
\end{aligned}
\end{equation}

since $-\int_\Omega \nabla \Phi A_{tt} dx = \int_\Omega \Phi \text{div} A_{tt} dx = 0$.

Using the Hölder inequality and Young inequality, we deduce from (3.6) that
\begin{equation}
\begin{aligned}
\int_\Omega |A_{tt}|^2 dx &\leq \frac{1}{\epsilon_0^2 \mu_0^2} \int_\Omega |\Delta A|^2 dx + \frac{\rho_0^2}{\epsilon_0^2} \int_\Omega |u|^2 dx + \frac{1}{2} \int_\Omega |A_{tt}|^2 dx.
\end{aligned}
\tag{3.7}
\end{equation}

It is easy to see that
\begin{equation}
\begin{aligned}
\text{ess sup}_{0 \leq t \leq T} \int_\Omega |A_{tt}|^2 dx &\leq \text{sup}_{0 \leq t \leq T} \frac{2 \rho_0^2}{\epsilon_0^2 \mu_0^2} \int_\Omega |\Delta A|^2 dx + \text{sup}_{0 \leq t \leq T} \frac{2 \rho_0^2}{\epsilon_0^2} \int_\Omega |u|^2 dx.
\end{aligned}
\tag{3.8}
\end{equation}

Putting the estimates (3.1), (3.5) and (3.8) together, we have
\begin{equation}
A_{tt} \in L^\infty(0, T; Y).
\tag{3.9}
\end{equation}

Hence, (3.5) and (3.9) imply the regularity for $A$.

\textbf{Step 2} The $L^3_\sigma$-$L^{\frac{3}{2}}$-estimates for $u \cdot \nabla u$ and $u \times A$. 
From (3.1) and Lemma 2.3 (the case that \( k=0 \)), it is easy to check that
\[ u \in L^4((0, T) \times \Omega). \]  
(3.10)

Note that
\[ \int_0^T \int_\Omega |Du|^4 \frac{1}{3} |u|^4 \, dx \, dt \leq \left( \int_0^T \int_\Omega |Du|^2 \, dx \, dt \right)^{\frac{3}{2}} \left( \int_0^T \int_\Omega |u|^4 \, dx \, dt \right)^{\frac{1}{2}}, \]
which implies that
\[ u \cdot \nabla u \in L^{\frac{4}{3}}(0, T; W^{1, \frac{4}{3}}(\Omega, \mathbb{R}^2)). \]
(3.12)

Combining (3.1) and (3.10), we get
\[ \int_0^T \int_\Omega |u \times \text{rot} A|^4 \frac{1}{3} |u|^4 \, dx \, dt \leq \left( \int_0^T \int_\Omega |u|^4 \, dx \, dt \right)^{\frac{3}{4}} \left( \int_0^T \int_\Omega |\text{rot} A|^2 \, dx \, dt \right)^{\frac{1}{4}} \]
\[ \leq C \left( \int_0^T \int_\Omega |u|^4 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_\Omega |\nabla A|^2 \, dx \, dt \right)^{\frac{1}{2}} < \infty, \]
(3.13)

which in turn implies
\[ u \times \text{rot} A \in L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega, \mathbb{R}^2)). \]
(3.14)

Recall that \((u, p)\) satisfying the following Stokes system
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \nu \Delta u - \frac{1}{\rho_0} \nabla p + F(x, t), \\
\nabla \cdot u &= 0, \\
u|\partial \Omega &= 0, \\
u(0) &= \phi,
\end{aligned}
\]
(3.15)

where \( F(x, t) = f - (u \cdot \nabla)u + \frac{\rho_0}{\rho_0}(u \times \text{rot} A). \)

By (3.12) and (3.14), we get \( F \in L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega, \mathbb{R}^2)) \). Applying this into Lemma 2.4, we obtain that
\[
\begin{aligned}
u \in L^{\frac{4}{3}}(0, T; W^{2, \frac{4}{3}}(\Omega, \mathbb{R}^2)), & \quad u_t \in L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega, \mathbb{R}^2)), \\
p \in L^{\frac{4}{3}}(0, T; W^{1, \frac{4}{3}}(\Omega)).
\end{aligned}
\]
(3.16)

In the next step, the Lemma 2.5 will be used, since (3.15) can be rewritten as the following stationary Stokes equations
\[
\begin{aligned}
- \nu \Delta u + \frac{1}{\rho_0} \nabla p &= \tilde{F}(x, t), \\
\nabla \cdot u &= 0, \\
u|\partial \Omega &= 0, \\
u(0) &= \phi,
\end{aligned}
\]
(3.17)
where \( \tilde{F}(x, t) = f - (u \cdot \nabla)u + \frac{\rho_e}{\rho_0}(u \times \text{rot}A) - u_t \).

**Step 3 The estimate for \( ||\tilde{F}||_{L^\infty(\Omega, L^2(\Omega, \mathbb{R}^2))} \).**

(i) **The estimate for \( ||\nabla u||_{L^\infty(0, T; L^2)} \).**

Multiplying Eq. (1.1) by \( u_t \) and integrating over \( \Omega \), we have

\[
\frac{\mu}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 dx + \int_\Omega |u_t|^2 dx = \int_\Omega -(u \cdot \nabla)u \cdot u_t + \frac{\rho_e}{\rho_0}(u \times \text{rot}A)u_t dx. \tag{3.18}
\]

Note that the following continuous embeddings

\[
W^{2, \frac{4}{3}}(\Omega, \mathbb{R}^2) \hookrightarrow W^{1, 4}(\Omega, \mathbb{R}^2) \hookrightarrow C^{1, 2}_0(\Omega, \mathbb{R}^2) \hookrightarrow C^0(\Omega, \mathbb{R}^2). \tag{3.19}
\]

Combining (3.19), Hölder inequality and \( \epsilon \)-Young inequality, we derive that

\[
\int_\Omega |(u \cdot \nabla)u \cdot u_t| dx \leq C||u_t||_{L^2}||u||_{C^0}||\nabla u||_{L^2} \leq \frac{1}{4}||u_t||_{L^2}^2 + C^2||u||_{C^0}^2||\nabla u||_{L^2}^2 \tag{3.20}
\]

and

\[
\frac{\rho_e}{\rho_0} \int_\Omega |(u \times \text{rot}A)u_t| dx \leq C||u||_{C^0}||\nabla A||_{L^2}||u_t||_{L^2} \leq \frac{1}{4}||u_t||_{L^2}^2 + C^2||u||_{C^0}^2||\nabla A||_{L^2}^2, \tag{3.21}
\]

which together with Gronwall’s inequality implies

\[
\text{ess sup}_{0 < t < T} ||\nabla u||_{L^2} < \infty. \tag{3.22}
\]

(ii) **The estimate for \( ||u_t||_{L^\infty(0, T; L^2)} \).**

Taking \( t \)-derivative of Eq. (1.1), then one gets that

\[
u_{tt} - \mu \Delta u_t = -(u_t \cdot \nabla)u - (u \cdot \nabla)u_t - \frac{1}{\rho_0} \nabla p_t + \frac{\rho_e}{\rho_0} u_t \times \text{rot}A + \frac{\rho_e}{\rho_0} u \times \text{rot}A_t. \tag{3.23}
\]

Multiplying (3.23) by \( u_t \) and integrating over \( \Omega \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_t|^2 dx + \mu \int_\Omega |\nabla u_t|^2 dx = \int_\Omega -(u_t \cdot \nabla)u \cdot u_t + \frac{\rho_e}{\rho_0}(u \times \text{rot}A_t)u_t dx. \tag{3.24}
\]

since

\[(u_t \times \text{rot}A) \cdot u_t = 0, \int_\Omega (u \cdot \nabla)u_t \cdot u_t dx = -\int_\Omega \frac{1}{2} u_t^2 \text{div}u dx = 0.\]
Next, we estimate the two terms on the right hand of (3.24). By (3.19) and integrating by parts yield

\[- \int_{\Omega} (u_t \cdot \nabla)u \cdot u_t \, dx = \int_{\Omega} u_i^j w^j \partial_t u_i^j - u_i^j \partial_t (w^j u_i^j) \, dx\]

(3.25)

And similarly,

\[\frac{\rho_e}{\rho_0} \int_{\Omega} |(u \times \text{rot}A_t)u_t| \, dx \leq \frac{C \rho_e}{\rho_0} \int_{\Omega} |u u_t \nabla A_t| \, dx \leq \frac{C \rho_e}{\rho_0} \|u\|_{C^0} \left( \|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \right).\]

(3.26)

Hence, by (3.24), (3.25) and (3.26), we get that

\[\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_t|^2 \, dx + \mu \int_{\Omega} |\nabla u_t|^2 \, dx \leq \|u\|_{C^0} \left(1 + C \rho_e / \rho_0\right) \|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \]

(3.27)

which together with Gronwall’s inequality completes the estimate

\[\text{ess sup}_{0 < t < T} \|u_t(t)\|_{L^2} < \infty.\]

(3.28)

(iii) **The estimates for** \[\|u \cdot \nabla u\|_{L^\infty(0,T;L^2)}\] **and** \[\|u \times A\|_{L^\infty(0,T;L^2)}.\]

From (3.22), it is easy to see that

\[\nabla u \in L^\infty(0,T;Y).\]

(3.29)

Hence

\[u \in L^\infty(0,T;H^1).\]

It is known that \[H^1 \hookrightarrow L^q(1 < q < \infty)\] when \(n = 2\). Note that

\[\left( \int_{\Omega} |(u \cdot \nabla)u|^r \, dx \right)^{\frac{1}{r}} \leq \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |u|^{\frac{2r}{r-2}} \, dx \right)^{\frac{r-2}{2r}} < \infty\]

(3.30)

provided that \(1 < r < 2\). Hence

\[(u \cdot \nabla)u \in L^\infty(0,T;L^r(\Omega,\mathbb{R}^2)).\]

(3.31)

By using the Hölder inequality and the Sobolev embedding theorem, it follows that

\[\int_{\Omega} |u \times \text{rot}A|^2 \, dx \leq C \int_{\Omega} |u|^2 |\nabla A|^2 \, dx \]

\[\leq C \left( \int_{\Omega} |u|^4 \, dx + \int_{\Omega} |\nabla A|^4 \, dx \right)\]

(3.32)

\[\leq C \left( \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} |\Delta A|^2 \, dx \right).\]
Together (3.5) with (3.32), we have
\[ u \times \text{rot} A \in L^\infty(0, T; L^2(\Omega, \mathbb{R}^2)). \]  
(3.33)

According to (3.28), (3.31), (3.33) and the assumption, \( \tilde{F} \) in (3.17) satisfies
\[ \tilde{F} \in L^r(\Omega, \mathbb{R}^2)(1 < r < 2) \quad \text{for any } 0 < T < \infty. \]  
(3.34)

Applying (3.34) into Lemma 2.5, we get
\[ u \in L^\infty(0, T; W^{2,r}(\Omega, \mathbb{R}^2)), \quad p \in L^\infty(0, T; W^{1,r}(\Omega)). \]  
(3.35)

Using the Sobolev embedding theorem \( W^{2,r} \hookrightarrow C^\alpha \hookrightarrow C^0(0 < \alpha < 1, n = 2, \) ), we deduce from (3.29) and (3.35) that
\[ (u \cdot \nabla)u \in L^2(\Omega, \mathbb{R}^2) \quad \text{for any } 0 < T < \infty. \]  
(3.36)

By (3.28), (3.33) and (3.36), we get that
\[ \tilde{F} = f - (u \cdot \nabla)u + \frac{\rho_e}{\rho_0}(u \times \text{rot} A) - u_t \in L^\infty(\Omega, L^2(\Omega, \mathbb{R}^2)). \]  
(3.37)

Applying (3.37) into Lemma 2.5, we obtain that for any \( T > 0 \)
\[ u \in L^\infty(0, T; W^{2,2}(\Omega, \mathbb{R}^2)), \quad p \in L^\infty(0, T; W^{1,2}(\Omega)). \]  
(3.38)

Therefore, (3.1), (3.5), (3.9), (3.28) and (3.38) complete the proof. \( \square \)
References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. Comm. Pure Appl. Math. (17) 1964, 35-92.

[2] C. Amrouche, V. Girault, Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. Czechoslovak Math. J. 44 (1994), 109-140.

[3] C. Cao, J. Wu, Global regularity for the 2-D MHD equations with mixed partial dissipation and magnetic diffusion. Adv. Math. 226 (2011), 1803-1822.

[4] S. Chandrasekhar, Hydrodynamic and hydromagnetic stability. The International Series of Monographs on Physics Clarendon Press, Oxford, 1961.

[5] G. Duvaut, J. L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, Arch. Ration. Mech. Anal. 46 (1972), 241-279.

[6] M. Geissert, M. Hess, M. Hieber, C. Schwarz, K. Stavrakidis, Maximal $L^p$-$L^q$-estimates for the Stokes equation: a short proof of Solonnikov’s theorem. J. Math. Fluid Mech. 12 (2010), 47-60.

[7] C. He, Z. Xin, On the regularity of solutions to the magnetohydrodynamic equations, J. Differential Equation 213 (2005), 235-254.

[8] X. Huang, Y. Wang, Global strong solution to the 2D nonhomogeneous incompressible MHD system. J. Differential Equations 254 (2013), 511-527.

[9] Q. Jiu, D. Niu, J. Wu, X. Xu and H. Yu, The 2-D magnetohydrodynamic equations with magnetic diffusion. Nonlinearity 28 (2015), 3935-3955.

[10] R. Liu, J. Yang, Magneto-hydrodynamical Model for Plasma, arXiv: 1601.06339.

[11] T. Ma, Theory and method of partial differential equation. (Chineses) Beijing, Science Press, 2011.

[12] N. Masmoudi, Global well posedness for the Maxwell-Navier-Stokes system in 2D. J. Math. Pures Appl. 93 (2010), 559-571.

[13] D. Regmi, Global weak solutions for the two-dimensional magnetohydrodynamic equations with partial dissipation and diffusion. Nonlinear Anal. 144 (2016), 157-164.
[14] X. Ren, J. Wu, Z. Xiang, Z. Zhang, Global existence and decay of 
smooth solution for the 2-D MHD equations without magnetic diffusion. 
J. Funct. Anal. 267 (2014), 503-541.

[15] M. Sermange, R. Temam, Some mathematical questions related to the 
MHD equations, Comm. Pure Appl. Math. 36(5) (1983), 635-664.

[16] V. A. Solonnikov, Estimates for solutions of nonstationary Navier-
Stokes equations, J. Soviet Math. 8 (1977), 467-529.

[17] R. Teman, Navier-Stokes equations Providence RI: AMS, 2000.

[18] V. Vialov, On the regularity of weak solutions to the MHD system near 
the boundary. J. Math. Fluid Mech. 16(4) (2014), 745-769.

[19] T. Wang, A regularity criterion of strong solutions to the 2D com-
pressible magnetohydrodynamic equations. Nonlinear Anal. Real World 
Appl.31 (2016), 100-118.

[20] J. Wu, Generalized MHD equations, J. Differential Equations 195 
(2003), 284-312.

[21] Z. Xin, Global strong solutions for 3D viscous incompressible heat 
conducting magnetohydrodynamic flows with non-negative density. J. 
Math. Anal. Appl. 446 (2017), 707-729.

[22] K. Yamazaki, Global regularity of N-dimensional generalized MHD sys-
tem with anisotropic dissipation and diffusion. Nonlinear Anal. 122 
(2015), 176-191.