The Thermodynamic Properties of Warped Taub-NUT AdS Black String

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Abstract

When we consider five-dimensional warped Taub-NUT/Bolt AdS black string with minimally coupled massive scalar field, we calculate entropy by using the brick wall method. It is found that they are proportional to being quadratically divergent in a cutoff parameter. In particular, we show that the entropy of warped Taub-NUT AdS black string holds for an area law in the bulk as well as on the brane. Furthermore, when the negative cosmological constant is treated as thermodynamic pressure, we calculate the thermodynamic quantities and investigate their extended thermodynamic properties. Interestingly, we obtain a thermodynamically stable range as a function of the temperature for warped Taub-NUT AdS black string. Finally, we also study a proportional behavior of the thermodynamic quantities along a warp factor, and find that an entropy, a specific heat, a Gibbs free energy, and an action difference increase as a warp factor grows up.
1 Introduction

One can simply obtain $p$-brane solutions by adding a number of $p$ extra dimensions to solutions for Einstein vacuum equations in $d$-dimensional spacetime \[1, 2\]. The simplest case of them can be given as black string solutions by adding just one extra dimension in higher dimensional spacetime. However, one cannot find asymptotically AdS black string solutions through an trivial way in as the above asymptotically flat case. They are able to be found after imposing warped geometry or working numerically \[3, 4\] obtained by solving Einstein equation with an extra dimension. It is of interest to find the black string solution in Taub-NUT AdS background.

The brick wall model was suggested in \[5\] search for black hole with a microscopic quantum states, and as the explanation of the origin of black hole entropy. It was investigated through the extensive application of various black holes \[6\]−\[13\].

In fact, black string forms when the matter field in five-dimensional spacetime is trapped on four-dimensional spacetime and undergoes gravitational collapse towards black hole. Comparing with horizon topology $S^2$ in four-dimensional black hole solution, its horizon topology is $S^2 \times S^1$ since such horizon extends into the extra dimension. Furthermore, the Bekenstein-Hawking entropy with the geometric character comes from the contributions of fields near the horizon. In this context, it was probed with black string \[14\]. In particular, it is shown that the entropy in Taub-NUT metric is quadratically divergent in a cutoff parameter \[15\] even if Taub-NUT metric has special property such as the solutions of the metric are not asymptotically flat (AF) but asymptotically locally flat (ALF) \[16, 17\]. Hence, it is of interest to investigate the corresponding situation in the case of warped Taub-NUT/Bolt AdS black string.

It is well known that black string solution \[18\] is suggested in order to investigate how the physics of black holes is affected from the extra dimension with warped geometry in Randall-Sundrum (RS) brane world models \[19, 20\]. Such solution is noting but RS brane world models with appearing as a Schwarzchild black hole on the brane. In this case, it is found that the entropy is proportional to the area of the event horizon on the brane and in the bulk \[9, 10\]. Therefore, an intriguing question is whether an area law is valid for warped Taub-NUT AdS black string.

On the other hand, it has been recently suggested that considering $(d+1)$-dimensional AdS black holes, the cosmological constant $\Lambda$ can be treated as the thermodynamic pressure $p$

\[
p = \frac{1}{8\pi} \Lambda = \frac{(d-1)d}{16\pi l^2}, \quad (1.1)
\]
in units where $G = c = h = k_B = 1$. Several series of relevant investigations have been performed in this direction \cite{21-43}. Recently, it has been shown that there is a negative thermodynamic volume in the Taub-NUT-AdS case \cite{34} and found that there is the first order phase transition from Taub-NUT-AdS to Taub-Bolt-AdS \cite{38}. The thermodynamic properties have been investigated extensively in higher dimensional NUT/Bolt case \cite{42} and topological NUT/Bolt case \cite{43}. Therefore, it would be interesting to be a similar discussion of the generalizations in warped Taub-NUT/Bolt AdS black string. In this paper, we address these questions.

The paper is organized as follows: In the next section, we will yield warped Taub-NUT AdS black string obtained by solving Einstein equation with a negative cosmological constant in five-dimensional spacetime. In section 3, we will calculate an entropy and show how such entropy is expressed in terms of a cutoff parameter and a black hole’s area. Next, in the case of Taub-Bolt, we will explore the entropy and show it is also proportional to being quadratically divergent in a cutoff parameter. In section 4, when a cosmological constant is treated as a pressure, we will explicitly calculate the thermodynamic quantities such as the entropy, the enthalpy, the specific heat, the temperature, the thermodynamic volume, and the Gibbs free energy. We will discuss their extended thermodynamic properties. In the last section, we will give our conclusion.

2 Warped Taub-NUT AdS Black String

Warped Taub-NUT black string in five-dimensional AdS spacetime is given as

$$ds_5 = a^2(z) \left[ -f(r)(dt + 2n \cos(\theta)d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 + n^2)(d\theta^2 + \sin^2(\theta)d\phi^2) \right] + l_5^2 dz^2,$$

which satisfies the Einstein equation with a negative cosmological constant $\Lambda_5 = -6/l_5^2$

$$G_{AB} - \Lambda_5 g_{AB} = 0. \quad (2.3)$$

Here, $l_5$ is the AdS radius, the warp factor $a(z) = \cosh(z)$, and

$$f(r) = \frac{r^2 - n^2 - 2Mr + l_5^{-2}(r^4 + 6n^2r^2 - 3n^4)}{r^2 + n^2}. $$

Taking $\cosh(z) = 1/\cos(\varphi)$, warped Taub-NUT black string in AdS$_5$ is rewritten as

$$ds_5 = \frac{l_5^2}{\cos^2(\varphi)} \left[ \frac{1}{l_4^2} \left( -f(r)(dt + 2n \cos(\theta)d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 + n^2)(d\theta^2 + \sin^2(\theta)d\phi^2) \right) + d\varphi^2 \right], \quad (2.4)$$
by introducing the relation between the cosmological parameter $l_5$ in the bulk and the cosmological parameter $l_4$ on the brane 

$$l_5 = l_4 \cos(\varphi) \quad (2.5)$$

The metric (2.4) on locally constant $\varphi = \varphi_0$ slice is reduced to AdS Taub-NUT metric localized on a brane

$$ds^4 = - f(r)(dt + 2n \cos(\theta)d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 + n^2)(d\theta^2 + \sin^2(\theta)d\phi^2). \quad (2.6)$$

We consider a scalar field $\Phi$ with mass $m$ propagating in five-dimensional spacetime under the background (2.2), which is described by five-dimensional Klein-Gordon equation as

$$\frac{1}{\sqrt{-\det(g_{\mu\nu})}} \partial_\mu \left( \sqrt{-\det(g_{\mu\nu})} g^{\mu\nu} \partial_\nu \Phi \right) - m^2 \Phi = 0 \quad (2.7)$$

Then we set the wave function

$$\Phi(t, r, \theta, \phi, z) = \Psi(t, r, \theta, \phi)\chi(z), \quad (2.8)$$

which leads to

$$\frac{1}{a^4(z)l_5^2} \partial_z \left( a^4(z) \partial_z \chi(z) \right) - (m^2 - a^2(z)\mu^2)\chi(z) = 0, \quad (2.9)$$

$$\left( \frac{4n^2 \cos^2(\theta)}{(r^2 + n^2)\sin^2(\theta)} - \frac{1}{f(r)} \right) \partial^2_t \Phi - \frac{4n \cos(\theta)}{(r^2 + n^2)\sin^2(\theta)} \partial_t \partial_\theta \Phi$$

$$+ \frac{1}{r^2 + n^2} \partial_t \left( (r^2 + n^2)f(r)\partial_t \Phi \right) + \frac{1}{(r^2 + n^2)\sin(\theta)} \partial_\theta \left( \sin(\theta)\partial_\theta \Phi \right)$$

$$+ \frac{1}{(r^2 + n^2)\sin^2(\theta)} \partial^2_\phi \Phi - \mu^2 \Phi = 0, \quad (2.10)$$

where $\mu$ can be interpreted as the effective mass on the brane by introducing picture borrowed from RS brane world models [19] [20]. Since there are no explicit time dependent terms in Eq.(2.10) one may take the stationary solutions as the following

$$\Psi(t, r, \theta, \phi) = e^{-i\omega t} e^{i\alpha \phi} R(r)Q(\theta), \quad (2.11)$$

and substituting in Eq.(2.10)

$$\frac{1}{\sin(\theta)} \partial_\theta \left( \sin(\theta)\partial_\theta Q(\theta) \right) + \left[ l(l+1) - \frac{\alpha^2 + 2\alpha(2\omega n \cos(\theta)) + 4\omega^2 n^2}{\sin^2(\theta)} \right] Q(\theta) = 0, \quad (2.12)$$
\[
\partial_r \left( (r^2 + n^2) f(r) \partial_r R(r) \right) - \left[ \left( \mu^2 - \frac{\omega^2}{f(r)} \right) (r^2 + n^2) + l(l + 1) - 4\omega^2 n^2 \right] R(r) = 0, \quad (2.13)
\]

where \( l \) is the degree of spherical harmonic. Using Wentzel-Kramers-Brillouin (WKB) approximation, the \( z \)-dependent wave number is given as

\[
k_z^2 = l_5^2 (m^2 - a^2(z) \mu^2), \quad (2.14)
\]

and the \( r \)-dependent wave number

\[
k_r^2 = \frac{h^2(r, l, \omega)}{f^2(r)}, \quad (2.15)
\]

with

\[
h^2(r, l, \omega) = \left[ \left( \frac{4n^2 f(r)}{r^2 + n^2} + 1 \right) \omega^2 - \left( \frac{l(l + 1)}{r^2 + n^2} + \mu^2 \right) f(r) \right]. \quad (2.16)
\]

### 3 Statistical Entropy & Thermal Energy

In this section, we will explicitly calculate the entropy through counting of the quantized modes for the \( z \)-dependent wave number (2.14) and the \( r \)-dependent wave number (2.15).

The extra dimensional degeneracy factor \( n_z \) is obtained through the following the semi-classical quantization condition

\[
\pi n_z = \int_0^{z_c} dz k_z(z, \mu), \quad (3.17)
\]

and the radial degeneracy factor \( n_r \)

\[
\pi n_r = \int_{r_{\text{h}+\epsilon}}^{D} dr k_r(r, l, \omega). \quad (3.18)
\]

From Eqs. (2.14) and (3.17), we obtain

\[
\frac{dn_z}{d\mu} = \frac{ml_5}{\pi \mu} \gamma(\mu), \quad (3.19)
\]

with

\[
\gamma(\mu) = E_2 \left( \sin^{-1} (a(z_c)), \frac{\mu^2}{m^2} \right) - F_1 \left( \sin^{-1} (a(z_c)), \frac{\mu^2}{m^2} \right) - E_1 \left( \frac{\mu^2}{m^2} \right) + K_1 \left( \frac{\mu^2}{m^2} \right), \quad (3.20)
\]

where \( E_1 \) is the complete elliptic integral, \( E_2 \) the elliptic integral of the second kind, \( F_1 \) the elliptic integral of the second kind, and \( K_1 \) the complete elliptic integral of the first kind.
kind. Finally, in the WKB limit, the total number of quantum state with energy not exceeding $\omega$ is expressed as

$$g(\omega) = \int dg(\omega) = \int dl(2l+1) \int d\mu \frac{dn_z}{d\mu} \int \pi dn_r. \quad (3.21)$$

The free energy $F$ of the scalar field at inverse temperature $\beta$ yields

$$\pi \beta F = -\int_0^\infty d\omega \frac{\beta g(\omega)}{e^{\beta \omega} - 1} = -\beta \int_0^\infty d\omega \frac{dg(\omega)}{e^{\beta \omega} - 1} \int dl(2l+1) \int \frac{ml_5 \gamma(\mu)d\mu}{\pi \mu} \times \int_{r_h+\epsilon}^L dr \sqrt{\left(\frac{4n^2 f(r)}{r^2 + n^2} + 1\right) \omega^2 - \left(\frac{l(l+1)}{r^2 + n^2 + \mu^2}\right) f(r)}. \quad (3.22)$$

Here, the reality condition of the free energy leads to the following limits for the remaining integrals:

$$0 \leq l \leq \frac{1}{2} \left[ -1 + \sqrt{1 - 4(n^2 + r^2) \left\{ \left(\frac{4n^2 f(r)}{r^2 + n^2} + 1\right) \frac{\omega^2}{f(r)} - \mu^2 \right\} } \right], \quad (3.23)$$

and

$$m \leq \mu \leq \sqrt{\frac{4n^2 f(r)}{r^2 + n^2} + 1} \frac{\omega}{\sqrt{f(r)}}. \quad (3.24)$$

On the integrating over $l$, the free energy $F$ is given as

$$F = -\frac{2ml_5}{3\pi} \int_0^\infty d\omega \frac{\beta g(\omega)}{e^{\beta \omega} - 1} \int \frac{\gamma(\mu)d\mu}{\mu} \times \int_{r_h+\epsilon}^L dr \sqrt{\left(\frac{4n^2 f(r)}{f^2(r)} + 1\right) \omega^2 - \left(\frac{l(l+1)}{r^2 + n^2 + \mu^2}\right) f(r)}. \quad (3.25)$$

The main contributions to the free energy, on carrying out the integrals over $\mu$, are

$$F \approx -\rho(z_c) \frac{2ml_5}{3\pi} \int_0^\infty d\omega \frac{\omega^3}{e^{\beta \omega} - 1} \int_{r_h+\epsilon}^L dr \left(\frac{r^2 + n^2}{f^2(r)}\right) \quad (3.26)$$

with

$$\rho(z_c) = a(z_c) - 1, \quad (3.27)$$

which in the approximation of $L \gg r_h$ leads to

$$F \approx \left[ -\frac{2\pi^3 m}{45 \epsilon \beta^4} \frac{r_h^2(r_h^2 + n^2)}{(3(r_h^2 + n^2)l_5^2 + 1)^2} \right] \rho(z_c) l_5, \quad (3.28)$$
where the followings are used:

\[
f(r) \approx (r - r_h) \left. \frac{df(r)}{dr} \right|_{r=r_h} = (r - r_h) \left( 3(r_h^2 + n^2)l_5^{-2} + 1 \right),
\]

(3.29)

and

\[
\int_0^\infty d\omega \frac{\omega^3}{e^{\beta \omega} - 1} = \frac{\pi^4}{15 \beta^4}.
\]

(3.30)

The entropy is given as

\[
S = \beta^2 \left( \frac{\partial F}{\partial \beta} \right) = \left[ \frac{8\pi^3 m}{45 \epsilon \beta^3} \left( \frac{r_h^2(r_h^2 + n^2)}{3(r_h^2 + n^2)l_5^{-2} + 1} \right)^2 \right] \rho(z_c)l_5.
\]

(3.31)

The cutoff near the horizon is referred to as the brick wall. The physical distance between the brick wall and the horizon is

\[
\int_{r_h}^{r_h + \epsilon} ds = \int_{r_h}^{r_h + \epsilon} \frac{dr}{\sqrt{f(r)}}.
\]

(3.32)

which leads to

\[
\epsilon = \left[ \frac{3(r_h^2 + n^2)l_5^{-2} + 1}{4r_h} \right] h^2.
\]

(3.33)

In terms of this covariant cutoff parameter \( h \), the ultraviolet divergent part of the scalar field entropy is given by

\[
S = \left[ \frac{32\pi^3 m}{45h^2 \beta^3} \frac{r_h^2(r_h^2 + n^2)}{3(r_h^2 + n^2)l_5^{-2} + 1} \right] \rho(z_c)l_5,
\]

(3.34)

which is written in terms of the parameter \( M \)

\[
S = \frac{4\pi^3 m}{45h^2 \beta^3 M^2} \left[ \frac{(r_h^2 - n^2 + l_5^{-2}(r_h^4 + 6n^2r_h^2 - 3n^4))}{(r_h^2 + n^2)^2((r_h^2 + n^2)^{-1} + 3l_5^{-2})^3} \right] \rho(z_c)l_5.
\]

(3.35)

When taking \( M = n + 4(2 + \sqrt{2})n^3l_5^{-2} \), \( r_h \) is evaluated as \( (\sqrt{2} + 1)n \). Imposing the inverse temperature \( \beta = 8\pi n \), the entropy in warped Tabu-NUT AdS spacetime is obtained as

\[
S_{AdS,NUT} = \frac{0.13n^2 m}{(20.49n^2l_5^{-2} + 1)^3 h^2} \rho(z_c)l_5.
\]

(3.36)
Taking \( M = ((41\sqrt{41} + 365)n^2l_5^{-2} + 40)n/32 \) in warped Tabu-Bolt AdS spacetime, \( r_h \) is given by \((\sqrt{41} + 5)(n/4)\), and the entropy

\[
S_{\text{AdS,Bolt}} = \frac{0.29n^2m}{(27.38n^2l_5^{-2} + 1)^3h^2} \rho(z_c)l_5.
\] (3.37)

They reveal that the entropy in warped Tabu-Taub-NUT/Bolt AdS spacetime has an explicit dependence on the inversely square of the cutoff parameter \( h \).

Since the area of the horizon \( A_h \) on the brane located at \( z = z_c \) is

\[
A_h = \int_0^{2\pi} \int_0^{\pi} \sqrt{g_\theta g_\phi} d\theta d\phi = 4\pi a^2(z_c)(r_h^2 + n^2),
\] (3.38)

the above entropy (3.36) is written as

\[
S_{\text{AdS,NUT}} = \left[ \frac{\rho(z_c)}{16(\sqrt{2} + 1)\pi a^2(z_c)} \left( \frac{0.13m}{(20.49n^2l_5^{-2} + 1)^3} \right) \left( \frac{l_5}{h^2} \right) \right] A_h,
\] (3.39)

which leads to

\[
S_{\text{AdS,NUT}} = \left\{ \left( \frac{\rho(z_c)}{16(\sqrt{2} + 1)\pi a^2(z_c)} \right) \left( \frac{0.13ml_4 \cos(\varphi)}{\cos^3(\varphi)} \right) \left( \frac{1}{h^2} \right) \right\} A_h,
\] (3.40)

by using the relation (2.5). Evidently, the entropy on the brane located at \( z = z_c \) satisfies the black hole area law since the curly-brackets quantity is much more smaller than the cutoff parameter \( h \) if we take \( h \) to be of the order of Planck length.

Introducing the invariant length of the black string along the \( z \)-direction

\[
\int_0^{z_c} dz \sqrt{g_{zz}} = z_c l_5 \equiv \mathcal{R},
\] (3.41)

from the entropy (3.39) we have in the limit of enough large \( \mathcal{R} \)

\[
S_{\text{AdS,NUT}} \approx \left[ \left( \frac{0.13\rho(z_c)m}{16(\sqrt{2} + 1)\pi z_c a^2(z_c)} \right) \left( \frac{1}{h^2} \right) \right] A_{\text{BS}},
\] (3.42)

where the area of the black string horizon \( A_{\text{BS}} \) in the bulk is given as

\[
A_{\text{BS}} = A_h \times \mathcal{R}.
\] (3.43)

It is also shown that the entropy of warped Taub-NUT AdS black string in the bulk holds for an area law from assumption of \( h \) with the order of Planck length.
One the other hand, employing thermal relation \( \mathcal{E} = \partial_{\beta}(\beta F) \), the thermal energy \( \mathcal{E} \) is given as

\[
\mathcal{E} = \left[ \frac{2\pi^3 m}{15\epsilon \beta^4} \frac{r_h^2 (r_h^2 + n^2)}{3(r_h^2 + n^2) l_5^2 + 1} \right] \rho(z_c) l_5.
\]

Taking \( r_h = (\sqrt{2} + 1)n \) and \( \beta = 8\pi n \) and substituting in (3.33), we obtain the thermal energy in the case of Taub-NUT \( \mathcal{E}_{\text{NUT}} \)

\[
\mathcal{E}_{\text{NUT}} = \left[ \frac{(17\sqrt{2} + 24)mn}{3840(6(2 + \sqrt{2})n^2 l_5^{-2} + 1)} \right] \rho(z_c) l_5,
\]

and the thermal energy in the case of Taub-Bolt \( \mathcal{E}_{\text{Bolt}} \)

\[
\mathcal{E}_{\text{Bolt}} = \left[ \frac{(1057\sqrt{41} + 6765)mn}{960(3(5\sqrt{41} + 41)n^2 l_5^{-2} + 8)} \right] \rho(z_c) l_5,
\]

by taking \( r_h = (\sqrt{41} + 5)(n/4) \). It is shown that both thermal energies depend on the bulk parameter \( z_c \) along the \( z \)-direction and diverge much more slowly as \( n \) grows up.

### 4 The Extended Thermodynamic Properties

Employing Wick rotation the time \( (t \rightarrow it) \) and the NUT charge \( (n \rightarrow iN) \), the Euclidean section for warped Taub-NUT AdS black string is obtained as

\[
ds_5 = a^2(z) \left[ g(r)(dt + 2N \cos(\theta)d\phi)^2 + \frac{dr^2}{g(r)} + (r^2 - N^2)(d\theta^2 + \sin^2(\theta)d\phi^2) \right] + l_5^2 dz^2,
\]

with

\[
g(r) = \frac{r^2 + N^2 - 2Mr + l_5^{-2}(r^4 - 6N^2r^2 - 3N^4)}{r^2 - N^2}.
\]

Using calculation of the period of the Euclidean section (4.47), one can get the inverse temperature \( \beta \)

\[
\beta = 8\pi N.
\]
Figure 1: Plot of the entropy $S_4$ (red dotted curve for $a(z = 1)$, red dashed curve for $a(z = 0.7)$, and red solid curve for $a(z = 0)$, respectively), specific heat $C_4$ (blue dotted curve for $a(z = 1)$, blue dashed curve for $a(z = 0.7)$, and blue solid curve for $a(z = 0)$, respectively) and pressure $p$ (the upper black solid curve for $p = 12\pi T^2$, and the lower black solid curve for $p = 4\pi T^2$, respectively) as a function of the temperature $T$ in five dimensions.

Employing counter term subtraction method, the regularized action $I_{\text{NUT}}$ is given as

$$I_{\text{NUT}} = \frac{4a^4(z)\pi(l_5^2 - 2N^2)}{l_5^5}. \quad (4.49)$$

Using the Gibbs-Duhem relation $S = \beta M - I$, the entropy $S_{\text{NUT}}$ is

$$S_{\text{NUT}} = a^4(z)\sqrt{3}(4\pi T^2 - p) \frac{128\sqrt{2\pi^\frac{5}{2}T^4}}{\sqrt{p}}, \quad (4.50)$$

and by thermal relation $C = -\beta\partial_p S$, the specific heat $C_{\text{NUT}}$ is given as

$$C_{\text{NUT}} = a^4(z)\sqrt{3}(p - 2\pi T^2) \frac{32\sqrt{2\pi^\frac{5}{2}T^4}}{\sqrt{p}}, \quad (4.51)$$

and by thermal relation $H = U + pV$, the enthalpy $H_{\text{NUT}}$ is

$$H_{\text{NUT}} = a^4(z)(6\pi T^2 - p) \frac{32\sqrt{6\pi^\frac{5}{2}T^3}}{\sqrt{p}}, \quad (4.52)$$

and the thermodynamic volume $V_{\text{NUT}}$

$$\partial_p H_{\text{NUT}} \bigg|_{S_{\text{NUT}}} = V_{\text{NUT}} = \frac{-a^4(z)(36\pi T^2 + 7p)}{235\sqrt{6\pi^\frac{5}{2}p^\frac{3}{2}T^3}}, \quad (4.53)$$
which shows that the thermodynamic volume of warped Taub-NUT AdS black string is negative like results of Taub-NUT AdS solutions [34, 42, 43]. The above thermodynamic quantities satisfy the generalized Smarr formula [21, 22, 50]

\[ H_{\text{NUT}} - 2TS_{\text{NUT}} + 2pV_{\text{NUT}} = 0, \]  

(4.54)

which is precisely matched with that of five-dimensional Taub-NUT solution with negative cosmological constant [42, 43]. Using an thermal relation \( U = H - pV \), the internal energy \( U_{\text{NUT}} \) is obtained as

\[ U_{\text{NUT}} = \frac{a^4(z)(84\pi T^2 - p)}{256\sqrt{6\pi^2 T^3} \sqrt{p}} \]  

(4.55)

\[ G_{\text{NUT}} = \frac{a^4(z)(12\pi T^2 - p)}{128\sqrt{6\pi^2 T^3} \sqrt{p}} \]  

(4.56)

Requiring the specific heat are positive, one can obtain the following thermally stable range of \( T \) for warped Taub-NUT AdS black string

\[ \sqrt{\frac{p}{4\pi}} < T < \sqrt{\frac{p}{2\pi}}. \]  

(4.57)

Figure 2: Plot of the five-dimensional Gibbs free energy \( G \) as a function of temperature \( T \) for \( p = 1 \) (blue dotted curve for \( a(z = 1) \), green dashed curve for \( a(z = 0.7) \), and red solid curve for \( a(z = 0) \), respectively).

Finally, employing the Legendre transform of enthalpy \( G = H - TS \), the Gibbs free energy is given as
However, since the Gibbs free energy $G_{\text{NUT}}$ is positive, warped Taub-NUT AdS black string in some areas of this region is still unstable for $p < 4\pi T^2$ or for $p > 12\pi T^2$. Finally, requiring the Gibbs free energy $G_{\text{NUT}}$ are negative $4\pi T^2 < p < 12\pi T^2$, warped Taub-NUT AdS black string becomes a thermally stable (such black string in the light-green shaded areas is thermally stable in Fig. 1). A similar result is obtained in [43]. As the warp factor $a(z) = \cosh(z)$ grows up, the entropy $S_{\text{NUT}}$ (4.50), the specific heat $C_{\text{NUT}}$ (4.51), and the Gibbs free energy $G_{\text{NUT}}$ (4.56) increase (as you see in Fig. 1 and Fig. 2).

Let us consider the Bolt case ($r = r_B > N$). Using parallel way as in the case of the NUT case, The inverse of the temperature $\beta$, the action $I_{\text{Bolt}}$, and the enthalpy $H_{\text{Bolt}}$ are respectively

$$\beta = \frac{4\pi}{f'(r)} \bigg|_{r=r_B} = \frac{4\pi l^2 r_B}{kl^2 + (2u+1)(r_B - N^2)},$$

$$I_{\text{Bolt}} = \frac{2a^4(z)\pi N\{-3N^4 - r_B^4 + (N^4 + r_B^2)l_5^2\}}{r_B l_5^2},$$ (4.58)

with the Bolt radius $r_B$

$$r_{B,\pm} = l_5^2 \pm \sqrt{l_5^4 - 48l_5^2 N^2 + 144N^4} \over 12N,$$ (4.60)

where requiring the discriminant of the square root in the Bolt radius $r_{B,\pm}$ is positive, the maximum magnitude of the NUT charge $N_{\text{max}}$ is obtained as

$$N \leq \frac{l_5}{2\sqrt{3(2 + \sqrt{3})}} = N_{\text{max}},$$ (4.61)

The enthalpy $H_{\text{Bolt}}$, the entropy $S_{\text{Bolt}}$, and thermodynamic volume $V_{\text{Bolt}}$ for the Bolt solution are given as

$$H_{\text{Bolt}} = \frac{a^4(z)\{-3p + 24\pi(-16\pi r_B^2 p + 1)T^2 + 512\pi^3 r_B^2 (8\pi r_B^2 p + 3)T^4\}}{2048\sqrt{6}\pi r_B T^4 \sqrt{p}},$$ (4.62)

$$S_{\text{Bolt}} = \frac{a^4(z)\sqrt{3\pi}\{64\pi^2 (\frac{4p}{5x(8\pi r_B^2 p + 1)T^2} + 1) r_B^2 T^2 + 1\}}{128\pi^2 T^2 \sqrt{2p}},$$ (4.63)

$$V_{\text{Bolt}} = \frac{a^4(z)\sqrt{2\pi r_B (64\pi^2 r_B^2 T^2 - 3)}}{64\pi^2 T^2 \sqrt{3p}}.$$ (4.64)
Finally, the specific heat $C_{\text{Bolt}}$, the internal energy $U_{\text{Bolt}}$, and the Gibbs free energy $G_{\text{Bolt}}$ yield respectively

$$C_{\text{Bolt}} = -\frac{a^4(z)\sqrt{3\pi}T^2}{8\sqrt{2}p^2} \pm \frac{a^4(z)\sqrt{3}(p^4 - 6\pi p^3 T^2 + 2\pi^2 p^2 T^4 - 8\pi^3 p T^6 + 8\pi^4 T^8)}{32\pi^2 p^2 T^4 \sqrt{2p^2 - 16\pi p T^2 + 8\pi^2 T^4}}.$$  \hspace{1cm} (4.65)

$$U_{\text{Bolt}} = -p + 8\pi(1 - 8\pi r_B^2 p)T^2 + 512\pi^3(r_B + 2p)r_B T^4 \frac{1024\pi^3 r_B T^4}{1024\pi^3 r_B T^4}, \hspace{1cm} (4.66)$$

$$G_{\text{Bolt}} = \frac{a^4(z)(1 - 32\pi^2 T^2 + 2048\pi^4 T^4)}{256\pi^2 T^2 \sqrt{2} + \sqrt{2}}.$$  \hspace{1cm} (4.67)

![Figure 3: Plot of the action difference as a function of $N$ (blue curve for $a(z = 1)$, green curve for $a(z = 0.7)$, and red curve for $a(z = 0)$, respectively, when $p = 3/8\pi$).](image)

From now on, considering the action difference of warped Taub-NUT AdS black string and warped Taub-Bolt AdS black string, we investigate their instability.

The action difference, $I_{\text{Bolt}} - I_{\text{NUT}} = \mathcal{I}$, is obtained as

$$\mathcal{I} = \frac{\sqrt{\pi}a^4(z)N(N - r_B)^2(3 - 8\pi(3N^2 + 2Nr_B + r_B^2)p)}{r_B \sqrt{6p}}.$$  \hspace{1cm} (4.68)

As shown in Fig. 3, for the pressure $p$ as fixed parameter (isobaric process), the action difference $\mathcal{I}$ becomes negative when $N$ increases beyond a critical NUT charge $N_c$. This implies that the first order phase transition from warped Taub-NUT-AdS black string to warped Taub-Bolt-AdS black string occurs at $N_c$. Then, after getting $N_c$ through
solving the action difference $\mathcal{I} = 0$ and substituting into the inverse of the temperature (4.48), the critical temperature $T_c$ is given as

$$\frac{(\sqrt{5} + \sqrt{2})\sqrt{p}}{\sqrt{6}\pi}.$$ (4.69)

As shown in Fig. 3, the action difference $\mathcal{I}$ (4.68) increases as the warp factor $a(z) = \cosh(z)$ grows up.

5 Conclusion

Previous studies of quantized modes of scalar fields in Taub-NUT background [15] have indicated that the cutoff parameter $h$ is determined by matching the statistical entropy with the gravitational entropy even if it is changed by the relative values of the parameters $M$ and $n$.

Here we found the similar results for warped Taub-NUT AdS black string. Our evaluations of quantized modes of scalar fields in various warped black strings background were shown that the cutoff parameter has the Planck scale only by appropriate choosing the size of the AdS radius $l_5$, in contrast to the results in as [19] [20]. We examined the correction to the statistical entropy due to the extra dimension with warped geometry and found the statistical entropy is still quadratically divergent in the cutoff parameter as in Taub-NUT metric [15].

Next, we showed that the entropy of warped Taub-NUT AdS black satisfies an area law in the bulk as well as on the brane. Furthermore, we investigated the thermal energy in warped Taub-NUT AdS/Bolt black string and found that both thermal energies depend on the bulk parameter $z_c$ along the $z$-direction and diverge much more slowly as $n$ grows up.

Finally, we explicitly calculated their thermal quantities the context of the extended thermodynamics. It was found out that there existed the first order phase transition from warped Taub-NUT-AdS black string to warped Taub-Bolt-AdS black string. Furthermore, we obtained thermodynamically stable region as of a function of the temperature $T$ for NUT case. We also investigated a proportional behavior of the thermodynamic quantities with respect to the warp factor and found that the entropy, the specific heat, the Gibbs free energy and the action difference are proportional to the warp factor.

Recently, an instability issue was covered for all topological spacetime through investigations of quasinormal modes of warped black strings with nontrivial topologies in AdS$_5$ [17]. In this context it would be of interest to study the stability of our warped
black string solution. It was shown that four-dimensional BTZ black string \[48\] still holds for the Gubser-Mitra conjecture \[49\], correlating the classical and thermodynamic instabilities of black strings. Intriguing issue would be to accomplish to test this conjecture for warped Taub-NUT AdS black string.

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