An Optimization Method for Measurement Matrix Based on Double Decomposition

Zhaoyang Mao¹, Lan Li*¹

¹School of Science, University of Xi’an Shiyou, Xi’an 710065, China
Zhaoyang Mao; email: zhymao_xsy@163.com.
*Corresponding Author: Lan Li; email: lanli98@126.com.

Abstract: This paper introduces a novel method of measurement matrix in compressed sensing. In order to overcome the difficulties associated with coherence of measurement matrix, we propose double optimization methods by eigenvalue decomposition and singular value decomposition under mild conditions. An efficient algorithm (SVD-EIG) is used to recover sparse inputs from the optimized measurement matrix, based on the adaptation of the optimized matrix by eigenvalue decomposition. Lastly, compared with the other methods as the same sampling rate, we demonstrate through simulations that SVD-EIG algorithm can improve accuracy and probability of the reconstruction.

1. Introduction

Compression sensing (CS) [1-5] is an emerging paradigm in signal/image processing, which senses a sparse signal by taking a set of incomplete measurement.

Measurement matrix is very important to recover CS signal/image. Choosing measurement matrix is mainly divided into two categories. The first is random measurement matrix [3], such as random Gaussian matrix, Bernoulli random matrix and so on. The second is deterministic measurement matrix, such as partial Hadamard matrix. Recent work has explored coherence between perception matrix and representation matrix. These methods construct Gram matrix by product of perception matrix and representation matrix, and research the coherence of Gram. The off-diagonal entries in Gram are the inner products. In [6] off-diagonal entries are minimized by thresholding, but this way always spends more time and produces larger values in the matrix. An optimization method for measurement matrix based on eigenvalue decomposition [7] and singular value decomposition [8], study the relationship between coherence and eigenvalues of Gram matrix, and experiment results show that these methods rely on initial value.

This paper suggests an efficient algorithm (SVD-EIG) for recovering spare input from the optimized measurement matrix, based on an adaptation of the optimized matrix by eigenvalue decomposition. This method reduces the coherence by eigenvalue decomposition and singular value decomposition, therefore it improves the accuracy and probability of the reconstruction.

This paper is organized as follows. In next section we briefly recall the basic concept of compressed sensing. Section 3 shows our main result about optimal measurement matrix by combining singular values and eigenvalues. Section 4 presents SVD-EIG algorithm based on Orthogonal Matching Pursuit (OMP) [9]. Section 5 simply shows the experimental results of the algorithm, compare with several classical algorithm. In section (6), it is concluded that our novel method based on singular values and eigenvalue is very effective in terms of iteration time and computational complexity.
2. Preliminary

2.1. CS
Consider the real signal \( x \in \mathbb{R}^n \), \( \| x \|_0 \) stands for the number of nonzero entries of vector \( x \). If \( \| x \|_0 = k < n \), then \( x \) is \( k \)-sparse. When the vector \( x \) is not sparse, we can always find a sparse representation matrix (sparsity basis) \( \Phi \) such that

\[
x = \Psi s
\]

(1)

where \( s \) is a sparse vector, which contains nonzero coefficients. The linear system is as follows

\[
y = \Phi x = \Phi \Psi s
\]

(2)

where the matrix \( \Phi \in \mathbb{R}^{m \times n} (m \ll n) \) and \( y \) is a set of \( m \) linear measurements. Measurement matrix \( A \) denotes by the \( \Phi \Psi \), then

\[
y = As
\]

(3)

To recover such a signal \( s \), a natural a choice is seek a solution of the \( l_0 \) minimization problem

\[
\min_{s} \| s \|_0 \quad s.t. \quad y = As
\]

(4)

But equation (4) is NP hard problem, so we want to find a way to approximate Equation (4). A general method is

\[
\min_{s} \| s \|_1 \quad s.t. \quad y = As
\]

(5)

Equation (4) is equal to Equation (5) when \( A \) satisfy the RIP (Restricted Isometry Property) condition.

Definition 1. If the matrix \( A \) satisfies the RIP condition of \( k \)-order, If there exists a smallest constant \( \delta_k \in (0,1) \), for all \( k \)-sparse vector \( s \)

\[
(1 - \delta_k) \| s \|_2^2 \leq \| As \|_2^2 \leq (1 + \delta_k) \| s \|_2^2
\]

The matrix \( A \) satisfies the RIP condition of \( k \)-sparse.

In practice it is difficult to verify that a matrix \( A \) satisfies the RIP or calculate the corresponding RIP constant \( \delta_k \). In this respect, results based on coherence are appealing, which is very useful for sparsity. The coherence defined by

\[
\mu(A) = \max_{i,j} \frac{|\langle A_i, A_j \rangle|}{\| A_i \|_2 \| A_j \|_2}
\]

(6)

Here, we denote the \( i \) th column of \( A \) and Euclidean norm of \( A \) by \( A_i \) and \( \| A \|_2 \), respectively.

To guarantee a unique solution to Equation (3), we have the condition based on the coherence of \( A \)

\[
\| x \|_0 \leq \frac{1}{2} \left( 1 + \frac{1}{\mu} \right)
\]

(7)

with \( \| x \|_0 \) being the number of nonzero entries in \( x \).

2.2. Gram Matrix
The different way to obtain the coherence is from the Gram matrix \( G = A^T A \), using the arbitrary dictionaries for normalizing each of its columns. The off-diagonal entries on \( A \) are the inner product
from Equation (6). The mutual coherence is the off-diagonal entry $g_{ij}$ with the largest magnitude, $i,j$ represent the rows and columns of matrix, respectively. Assume

$$\mu(\mathbf{A}) = \max_{\mu_{ij}} |g_{ij}| \quad \text{and} \quad g_{ij} = \mathbf{A}^T \mathbf{A}_{ij}$$

There exist some problems:

- It's impossible that measurement matrix is global incoherent when $\mu$ is small.
- It's impossible that measurement matrix is global coherent when $\mu$ is large.

So we will consider global coherence of $\mathbf{A}$ in next section.

**Lemma 1.** If signal $\mathbf{x}$ is $k$-sparse and satisfied

$$\|\mathbf{A}\|_0 < \frac{1}{2} \left(1 + \frac{1}{\mu(\mathbf{A})}\right)$$

then there is unique signal $\mathbf{x}$ from $\mathbf{y}$ and $\mathbf{y} = \mathbf{A}\mathbf{x}$. Where $\|\mathbf{A}\|_0$ represents the zero norm of a matrix.

### 3. Choosing Measurement Matrix

#### 3.1. Eigenvalue decomposition

In this section, we describe mutual coherence, which reflects average performance. Inspired by [7], we consider global coherence of matrix $\mathbf{A}$ from eigenvalue. The result of reconstruction is associated with coherence of $\Phi$ and $\Psi$. In other words, the smaller global coherence of measurement matrix $\mathbf{A}$, the better effect of reconstruction algorithm.

**Theorem 1.** Suppose $\Phi \in \mathbb{R}^{m \times n}$, $\text{rank}(\Phi) = m$, and sparse representation matrix $\Psi \in \mathbb{R}^{n \times n}$, Let $\mathbf{A} = \Phi\Psi$ and Gram matrix $\mathbf{G} = \mathbf{A}^T \mathbf{A}$. If The $\mathbf{G}$ is semidefinite matrix and $\lambda_s (s = 1, \ldots, m)$ are eigenvalues, then

$$\sum_{s=1}^{m} \lambda_s = n \quad (8)$$

$$\sum_{s=1}^{m} (\lambda_s)^2 = \sum_{i,j=1}^{n} (<\mathbf{A}_i, \mathbf{A}_j>)^2 \quad (9)$$

where $\mathbf{A}_i$ and $\mathbf{A}_j$ represent the $i$ columns and $j$ columns of normalized $\mathbf{A}$, respectively.

The purpose is to minimize the squares sum of the off-diagonal entries of $\mathbf{G}$. According to Equation (8) and Equation (9) we obtain an optimized problem

$$\min \sum_{i,j} (g_{ij})^2 = \sum_{i=1}^{m} (\lambda_s)^2 - \sum_{i,j} (g_{ij})^2 \quad s.t. \quad \sum_{s=1}^{m} \lambda_s = n \quad (10)$$

When $\lambda_s = \frac{n}{m}$, the solution to problem Equation (10) is

$$\sum_{i,j} (g_{ij})^2 = \sum_{i=1}^{m} \left(\frac{n}{m}\right)^2 - n \quad (11)$$

The object function of Equation (10) existsS minimize value.

#### 3.2. Singular value and Condition number

In the previous subsection, we give a method of optimized measurement matrix. But we must have the initial matrix. The general method is to produce Gaussian random matrices or Bernoulli matrix, depends on the initial measurement matrix. Next, we will show that construct the measurement matrix $\mathbf{A}$.
To analyse the condition number we obtain the singular value decomposition (SVD) of measurement matrix $A$:

$$A = U \Sigma V^T$$

(12)

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ is orthogonal matrix. The singular value matrix $\Sigma$ is defined as:

$$\Sigma = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad i \in [1, \ldots, m]$$

(13)

where $\sigma_i = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$ denotes the diagonal matrix.

Suppose $A \in \mathbb{R}^{m \times n}$, $\sigma_1, \ldots, \sigma_n$ are singular value of $A$. The condition number $\text{cond}(A) = \sigma_{\text{max}} / \sigma_{\text{min}}$. $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ represent the maximum singular value and minimum singular value, respectively.

$$\|A\|_2 = \sigma_{\text{max}} \quad \|A\|_2 = \sigma_{\text{min}}$$

(14)

If the condition number $\text{cond}(A)$ is smaller, then the coherence of $A$ becomes smaller.

4. SVD-EIG Algorithm

Firstly, the random Gaussian matrix are optimized by SVD as initial matrix. Secondly we use the eigenvalue decomposition to improve the performance of matrix.

| Algorithm 1 SVD-EIG |
|---------------------|
| **Input:** Observations $y$, Object image $x \in \mathbb{R}^{n \times n}$, Measurement matrix $\Phi \in \mathbb{R}^{m \times n}$, Sparse representation matrix $\Psi \in \mathbb{R}^{n \times n}$, Error $\hat{o}$.
| **Output:** Recovered image $x$
| **Initialize:** $\Phi \leftarrow$ random Gaussian matrix, $\Psi \leftarrow$ DCT matrix, $A = \Phi \Psi$.
| **Svd_Optimize** $A = U \Sigma V^T$, $\Sigma \leftarrow \Sigma = \text{min}(\sigma)$, $A \leftarrow U \Sigma V^T$
| **Function** SVD-EIG($\Phi, \Psi, \hat{o}$)
| While halting criterion false $\hat{o}$ do
| $G = A \hat{A}$ and Eigenvalue decomposition $G = VH V^T$
| $H \leftarrow \text{diag}(H) = \sqrt{\frac{n}{m}}$
| $\hat{D} = VH V^T$
| $\hat{o}_i = \sum \text{off} - \text{diag}(D)$
| If $\|\hat{o}_i - \sum (n / m)^2 - n\| < \text{then}$
| $\Phi = \hat{D} \hat{\Psi}^{-1}$
| Return $\Phi \leftarrow \Phi$
| EndIf
| EndWhile
| EndFunction
| While $i = 1, \ldots$ do
| $x = \text{OMP}(y, A, k)$
| EndWhile

The experimental platform is matlab. In algorithm1, $\hat{A}$ is Estimate value, $\hat{A}$ is normalized by each of its columns, $\text{off} - \text{diag}(\cdot)$ is Off-diagonal element of $(\cdot)$. Since our focus is on optimizing the measurement matrix, we select a simple row as the code of OMP Refer to Section 3 for details of calculations.
5. Implementation

Now we demonstrate the performance of SVD-EIG via simulations. The object image is chosen 512×512 pixel Lena. Initial matrix Φ is random Gaussian matrix, and sparse representation matrix Ψ is discrete cosine transform (DCT) matrix. We select the OMP as reconstruction algorithm, because the main purpose of this paper is to design measurement matrix.

We examine the performance of random Gaussian, eigenvalue decomposition and SVD-EIG in recovering an image x of size 512×512.

![Random Gaussian matrix](image1)

![Eigenvalue decomposition](image2)

Figure 1. Random Gaussian matrix.

Figure 2. Eigenvalue decomposition.
In the simulation in fig.1 and fig.2, we show the recovery result under the random Gaussian matrix, The measurement matrix is optimized by eigenvalue decomposition at different sampling rates(0.3,0.4,0.5), respectively. In the fig.3, we keep the same conditions except for the measurement matrix and we can clearly observed that the recovery are better compared with the other algorithms. Moreover, we can also see from fig.4 that the PSNR value of SVD-EIG algorithm is very high.

6. Conclusion
In this paper, we suggest to improve the performance of image optimization method by singular value decomposition and eigenvalue decomposition. For same rate of measurements, the method of SVD-EIG can lead to better performance than using the random Gaussian and eigenvalue. We also demonstrate that SVD-EIG is able to enhance the quality of recovery as long as Gram matrix is sufficient incoherent. In addition, the experimental result also explains that the double optimization algorithm has less iterations and low computational complexity. There are some drawbacks in this algorithm. Running our code will spent more time compared with other algorithms.

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