Well-posedness of a fourth order evolution equation
Modeling MEMS *

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Abstract
We consider a fourth order evolution equation involving a singular nonlinear term \( \frac{-\lambda}{1-u} \) in a bounded domain \( \Omega \subset \mathbb{R}^n \). This equation arises in the modeling of microelectromechnical systems. We first investigate the well-posedness of a fourth order parabolic equation which has been studied in [17], where the authors, by the semigroup argument, obtained the well-posedness of this equation for \( n \leq 2 \). Instead of semigroup method, we use the Faedo-Galerkin technique to construct a unique solution of the fourth order parabolic equation for \( n \leq 7 \), which improves and completes the result of [17]. Besides, the well-posedness of the corresponding fourth order hyperbolic equation is obtained by the similar argument for \( n \leq 7 \).

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1 Introduction
Electrostatically actuated microelectromechanical systems (MEMS) are microscopic devices which combine mechanical and electrostatic effects. MEMS devices have therefore become key components of many commercial systems, including accelerometers for airbag deployment in automobiles, ink jet printer heads, optical switches, chemical sensors, and so on (see, for example, [23]). A typical MEMS device is made of a rigid conducting ground plate above which a clamped deformable plate (or membrane) coated with a thin conducting film is suspended. An applied voltage difference between the two plates results in the deflection of the elastic plate, and a consequent change in the MEMS capacitance, and thus transforms electrostatic energy into mechanical energy. The applied voltage potential has an upper limit, beyond which the electrostatic Coulomb force is not balanced by the elastic restoring force in the deformable plate, the two plates snap together and the MEMS collapses. This phenomenon, called pull-in instability, was simultaneously observed experimentally by Taylor [25], and Nathanson et al. [22]. The critical displacement and the critical voltage potential associated with this instability are called pull-in displacement and pull-in voltage potential, respectively. Their accurate evaluation is crucial in the design of electrostatically actuated MEMS.

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Mathematical models have been derived, see, for example, [11, 19, 23], to describe the dynamics of the displacement \( u = u(x, t) \) of the membrane \( \Omega \subset \mathbb{R}^n \). Let us sketch the derivation of this model for the sake of completeness. Indeed, according to the Newton’s second law and the narrow gap asymptotic analysis, we see

\[
\gamma \frac{\partial^2 u}{\partial t^2} = \text{electrostatic force} + \text{elastic force} + \text{damping force}
\]

where \( \gamma \) is a constant denoted by the mass of membrane. Since we consider here the idealized situation where the applied voltage and the permittivity of the membrane are constant (normalized to one), then

\[
\text{electrostatic force} = \lambda (\epsilon^2 |\nabla_x \psi(x, z, t)|^2 + |\partial_z \psi(x, z, t)|^2), \quad x \in \Omega, \quad z > 0,
\]

where \( \lambda \) is proportional to the square of the applied voltage, \( \psi \) is the electrostatic potential and \( \epsilon \) denote the aspect ratio of the device. Under the small aspect ratio condition (\( \epsilon \approx 0 \)), the \( \psi \) is solved by

\[
\psi = \frac{(1 - z)}{1 - u},
\]

for details, see [4]. Besides, we note that the damping force is linearly proportional to the velocity, that is

\[
\text{damping force} = -a \frac{\partial u}{\partial t},
\]

where \( a \) is damping intensity, and

\[
\text{elastic force} = \tau \Delta u - \beta \Delta^2 u
\]

where \( \tau \) is the tension constant in the stretching component of the energy, \( \beta \) accounts for the bending energy. According to the above discussion, the dimensionless dynamic deflection \( u(x, t) \) of the membrane on a bounded domain \( \Omega \subset \mathbb{R}^n \), under the small aspect ratio assumption, satisfies the following dynamic problem

\[
\begin{cases}
  \gamma u_{tt} + au_t + \beta \Delta^2 u - \tau \Delta u = \frac{\lambda}{(1-u)^2}, & x \in \Omega, \quad t > 0, \\
  u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), & x \in \Omega, \\
  \text{boundary conditions}, & x \in \partial \Omega, \quad t > 0.
\end{cases}
\]

(1.1)

Observe that the right-hand side of equation features a singularity when \( u = 1 \), which corresponds to the touchdown phenomenon already mentioned above.

The initial values \( u^0(x), u^1(x) \) are assumed to belong to some Sobolev space. Usually, one considers the following sets of boundary conditions

\[
u = \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial \Omega,
\]

which we will refer to as Dirichlet boundary conditions, and

\[
u = \Delta u = 0, \quad x \in \partial \Omega,
\]

which we will refer to as Navier boundary conditions. The Dirichlet boundary condition is also called clamped boundary condition, which corresponds to the case where the capacitive actuator at the boundary is clamped, giving rise to zero vertical displacement and
zero slope. Physically, the Navier boundary condition, usually referred to as the pinned boundary condition, gives rise to a device which is ideally hinged along all its edges so that it is free to rotate and does not experience any torque or bending moment about its edges.

For the stationary case, (1.1) has been studied extensively, see, for example, \[3,12,15,16\]. For the non-stationary case, due to the lack of the maximum principle, little is known in the literature about the well-posedness of (1.1) for \(\beta > 0\) so far. The author in [9] established the local and global well-posedness of (1.1) for pinned boundary conditions, \(\gamma > 0\) and the lower-dimensional case where \(1 \leq n \leq 3\). Later, The authors in [17] used the semigroup approach to obtain the existence of the strong solutions of (1.1) for \(n \leq 2, \gamma > 0\). However, for the higher-dimensional case, the well-posedness of (1.1) is open.

In the damping dominated limit \(\gamma \ll 1\) when viscous forces dominate over inertial forces, (1.1) reduces to the following fourth order initial-boundary value parabolic problem

\[
\begin{aligned}
\begin{cases}
 u_t + \beta\Delta^2 u - \tau\Delta u &= \frac{\lambda}{(1-u)^2}, & x \in \Omega, t > 0, \\
 u(x,0) &= u^0(x), & x \in \Omega, \\
 \text{boundary conditions}, & x \in \partial\Omega, t > 0.
\end{cases}
\end{aligned}
\]

Here we let \(a = 1\) for simplicity.

In the present paper, we first investigate the local and global well-posedness of the parabolic problem (1.2). When bending is neglected, that is, when \(\beta = 0\), this problem reduces to a second-order parabolic problem that has been studied extensively in the recent past, see, for example, \[5,8\] and the references therein. Due to lack of the maximum principle, which plays an important role in studying the corresponding stationary problems, only the references \[17,20,21\], to the best our knowledge, give some partial results to this problem (1.2) with \(\beta > 0\) so far. To be more precise, the authors in [17] use the semigroup argument to obtain the well-posedness of (1.2) for any bounded domain \(\Omega \subset \mathbb{R}^n\) and the lower-dimensional case \(n \leq 2\); the authors in \[20,21\], by use of numerical methods and asymptotic analysis, considered the quenching phenomenon on a onedimensional strip and the unit disc. In the present paper, we, instead of semigroup theory, will use the Faedo-Galerkin method to construct a solution of (1.2) for \(n \leq 7\), which improves and completes results of [17].

**Theorem 1.1.** Let \(\Omega \subset \mathbb{R}^n\) be an arbitrary bounded smooth domain for \(n \leq 7\) and \(\beta > 0, \tau \geq 0, \lambda > 0\). Let \(u^0 \in W^{4,2}(\Omega) \cap W^{2,2}_0(\Omega)\) be such that \(\|u^0\|_{W^{4,2}(\Omega) \cap W^{2,2}_0(\Omega)} \leq \rho\) for some small \(\rho \in (0,1)\). Then (1.2) with Dirichlet boundary conditions admits a unique solution \(u(x,t)\) in

\[
\mathcal{X}_T := C^0([0,T]; W^{4,2}(\Omega)) \cap W^{1,2}(0,T; W^{2,2}_0(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega))
\]

with \(\|u\|_{L^\infty(\Omega)} < 1\), provided one of the following conditions holds.

(i) \(\lambda \in \mathbb{R}^+\) and \(T > 0\) is sufficiently small;

(ii) \(T = \infty\) and \(\lambda \in \mathbb{R}^+\) is sufficiently small.

An identical result holds for the Navier problem but this time the solution belongs to the space

\[
\mathcal{X}_T := C^0([0,T]; W^{4,2}(\Omega)) \cap W^{1,2}(0,T; W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega)).
\]
(iii) If $\lambda$ is sufficiently large, then $T_m < \infty$ for Dirichlet boundary conditions with $\Omega = B_1$ (or Navier boundary conditions with any smooth domain $\Omega$). Here $B_1$ is the unit ball and $T_m$ is the maximal existence time.

**Remark 1.1.** It is worth pointing out that the outcome of this Theorem complies with the physical viewpoint. More precisely, a “pull-in” instability occurs for high voltage values. Accordingly, for large values of $\lambda$ solutions cease to exist globally, while solutions corresponding to small $\lambda$ values exist globally in time.

**Remark 1.2.** For the third result on Dirichlet boundary conditions of the above Theorem, a restriction with $\Omega = B$ is needed. The essential reason of this is the lack of maximum principle in general domain.

When inertial forces dominate over viscous forces in (1.1), i.e., $a \ll 1$, then (1.1) reduces to the following hyperbolic problem (set $\gamma = 1$ for simplicity)

$$
\begin{cases}
    u_{tt} + \beta \Delta^2 u - \tau \Delta u = \frac{\lambda}{(1-u)^2}, & x \in \Omega, t > 0, \\
    u(x, 0) = u^0(x), u_t(x, 0) = u^1(x) & x \in \Omega, \\
    u = \frac{\partial u}{\partial n} = 0 \ (\text{or } u = \Delta u = 0) & x \in \partial \Omega, t > 0.
\end{cases}
$$

(1.3)

When $\beta = 0$, this problem reduces to the second hyperbolic problem which has been studied in [13, 18]. For $\beta > 0$ this problem, to our knowledge, has not been investigated so far. For this reason, we will give a result on its well-posedness though its argument is similar to the parabolic case. To state our results precisely, we first introduce

**Definition 1.1.** We call a function $u$ is weakly continuous from $[0, T]$ into the Banach space $Y$, if

$$
\forall \ v \in Y', \ \text{the function } t \to <u(t), v> \text{ is continuous},
$$

the set of all such functions will be denoted as $C_w([0, T]; Y)$.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded smooth domain for $n \leq 7$ and $\beta > 0, \tau \geq 0, \lambda > 0$. Let $u^0 \in W^{4,2}(\Omega) \cap W_0^{2,2}(\Omega), u^1 \in W_0^{2,2}(\Omega)$ such that

$$
\|u^0\|_{W^{4,2}(\Omega)} + \|u^1\|_{W^{2,2}(\Omega)} \leq \rho
$$

(1.4)

for some small $\rho \in (0, 1)$.

(i) $\forall \ T > 0, \exists \bar{\lambda}(T, \rho) > 0, \text{ if } 0 < \lambda < \bar{\lambda}(T, \rho)$, then (1.3) with Dirichlet boundary conditions admits a unique solution such that

$$
\begin{align*}
    u \in C_w(0, T; W^{4,2}(\Omega)), \\
    u' \in C_w(0, T; W_0^{2,2}(\Omega)), \\
    u'' \in L^\infty(0, T; L^2(\Omega)).
\end{align*}
$$

(1.5)

For the Navier problem: if the initial values

$$
\begin{align*}
    u^0 \in W^{4,2}(\Omega) \cap W_0^{4,2}(\Omega), \\
    u^1 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)
\end{align*}
$$

satisfies (1.4), then an identical result holds but this time the solution satisfies

$$
\begin{align*}
    u \in C_w(0, T; W^{4,2}(\Omega)), \\
    u' \in C_w(0, T; W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)), \\
    u'' \in L^\infty(0, T; L^2(\Omega)).
\end{align*}
$$

(1.6)

(ii) If $\lambda$ is sufficiently large, then the maximal existence time $T_m < \infty$ for Dirichlet boundary conditions with $\Omega = B_1$ (or Navier boundary conditions with any smooth domain $\Omega$).
Let us conclude this section with organization of the present paper as follows:
- in section 2 we recall some preliminary tools;
- in section 3 we will consider the well-posedness of the parabolic problem (1.2). To this end, we first study the well-posedness of the corresponding linear parabolic problem which is of independent interested;
- in section 4 we will study the well-posedness of the hyperbolic problem (1.3) by the same argument as that of section 3.

2 Preliminaries

Throughout the paper, we always suppose $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. We denote by $\| \cdot \|_p$ the $L^p(\Omega)$ norm for $1 \leq p \leq \infty$ and by $\| \cdot \|_{W^{s,p}}$ the $W^{s,p}(\Omega)$ norm. Define

$$(u, v)_2 := \int_{\Omega} uv dx \quad \text{for all } u, v \in L^2(\Omega).$$

On the space $W_0^{2,2}(\Omega)$, the bilinear form

$$(u, v) \mapsto (u, v)_{W_0^{2,2}} := \beta(\Delta u, \Delta v)_2 + \tau(\nabla u, \nabla v)_2 \quad \text{for all } u, v \in W_0^{2,2}(\Omega) \quad (2.1)$$

define a scalar product over $W_0^{2,2}(\Omega)$ which induces a norm equal to $\| \cdot \|_{W^{2,2}(\Omega)}$. The space $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ becomes a Hilbert space when endowed with the scalar product (2.1); please see [7] for details. Without loss of generality, we let $\mathcal{V}'$ denote the dual space of $\mathcal{V}$.

Lemma 2.1. (i) Each eigenvalue of $\mathcal{L}$ is real.

(ii) Furthermore, if we repeat each eigenvalue according to its (finite) multiplicity, all the eigenvalues is given by

$$\Sigma = \{\lambda_k\}_{k=1}^{\infty},$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots,$$

and

$$\lambda_k \to \infty \quad \text{as } k \to \infty.$$

(iii) Finally, there exists an orthonormal basis $\{\omega_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, where $\omega_k \in W_0^{2,2}(\Omega)$ (or $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$) is an eigenfunction corresponding to $\lambda_k$:

$$\begin{cases}
\mathcal{L}\omega_k = \lambda_k \omega_k \\
\omega_k = \frac{\partial \omega_k}{\partial n} = 0 \quad (\text{or } \omega_k = \Delta \omega_k = 0)
\end{cases} \quad \text{in } \Omega, \quad \text{on } \partial \Omega,$$

for $k = 1, 2, \ldots$ Here $\mathcal{L} := \beta \Delta^2 - \tau \Delta$.

Remark 2.1. By the regularity theory of the elliptic operator, $\omega_k \in C^\infty(\Omega)$ (and $\omega_k \in C^\infty(\bar{\Omega})$ if $\partial \Omega$ is smooth), for $k = 1, 2, \ldots$.

Proof. By the Lax-Milgram theorem, $L^2$ theory of the elliptic operator and compact embedding theorem, we have

$$\mathcal{S} := \mathcal{L}^{-1}$$
is bounded, linear, compact operator mapping $L^2(\Omega)$ into itself. Integrating by parts leads to
\[
\int_{\Omega} g(Sf)dx = \int_{\Omega} L(Sg)(Sf)dx = \int_{\Omega} L(Sg)(Sf)dx = \int_{\Omega} f(Sg)dx, \quad \forall f, g \in L^2(\Omega),
\]
which means that the operator $S$ is self-adjoint. Therefore, by Hilbert-Schmidt’s theorem, there exists a standard orthogonal basis $\{\omega_k\}_{k=1}^\infty$ such that $S\omega_k = \eta_k \omega_k$, $\eta_k \to 0$ as $k \to \infty$. Notice also
\[
(Sf, f)_2 = (\beta(\Delta u, \Delta u)_2 + \tau(\nabla u, \nabla u)_2 \geq 0 \quad (f \in L^2(\Omega))
\]
and
\[
\begin{cases}
L u = 0 & \text{in } \Omega, \\
u = \frac{\partial u}{\partial n} = 0 \quad (u = \Delta u = 0) & \text{on } \partial \Omega,
\end{cases}
\]
admits only a trivial solution, it is certainly $\eta_k > 0$ and hence
\[
\begin{cases}
L \omega_k = \frac{1}{\eta_k} \omega_k & \text{in } \Omega, \\
\omega_k = \frac{\partial \omega_k}{\partial n} = 0 \quad (\omega_k = \Delta \omega_k = 0) & \text{on } \partial \Omega.
\end{cases}
\]
The lemma follows. □

The following Lemma is about the interpolation between $L^2(0, T; W^{m+4,2}(\Omega))$ and $W^{1,2}(0, T; W^{m,2}(\Omega))$.

**Lemma 2.2.** Assume that $\Omega$ is open, bounded, and $\partial \Omega$ is smooth. Take $m$ to be a nonnegative integer. Suppose
\[
u \in L^2(0, T; W^{m+4,2}(\Omega)), \quad \text{with } u' \in L^2(0, T; W^{m,2}(\Omega)),
\]
then
\[
u \in C([0, T]; W^{m+2,2}(\Omega)),
\]
and
\[
\max_{0 \leq t \leq T} \|u\|_{W^{m+2,2}(\Omega)} \leq C(\|u\|_{L^2(0, T; W^{m+4,2}(\Omega))} + \|u'\|_{L^2(0, T; W^{m,2}(\Omega))}). \quad (2.2)
\]

**Proof.** The proof is standard, here we give a sketch of the proof. Suppose first that $m = 0$, in which case
\[
u \in L^2(0, T; W^{4,2}(\Omega)), \quad u' \in L^2(0, T; L^2(\Omega)).
\]
We select a bounded open set $\tilde{\Omega} \supset \Omega$, and define a corresponding extension operator $E$ as follows: $Eu = \tilde{u}$ a.e. in $\tilde{\Omega}$ and $Eu$ has support within $\tilde{\Omega}$. We denote $Eu$ by $\tilde{u}$ for simplicity. By the extension theorem of Sobolev space, we have
\[
\|\tilde{u}\|_{L^2(0, T; W^{4,2}(\Omega))} \leq C \|u\|_{L^2(0, T; W^{4,2}(\Omega))}; \quad \|\tilde{u}'\|_{L^2(0, T; L^2(\tilde{\Omega}))} \leq C \|u'\|_{L^2(0, T; L^2(\Omega))}. \quad (2.3)
\]
We first claim that
\[
u \in C([0, T]; W^{2,2}(\tilde{\Omega})),
\]
from which we have
\[
u \in C([0, T]; W^{2,2}(\Omega)).
\]
Indeed, we check as in the proof of Theorem 2 in section 5.3.2 of [6] that there exist functions \( \tilde{u}^\epsilon(t) \in C^\infty(0, T; W^{4,2}(\tilde{\Omega})) \) such that as \( \epsilon \to 0 \),
\[
\tilde{u}^\epsilon(t) \to \tilde{u} \quad \text{in} \quad L^2(0, T; W^{4,2}(\tilde{\Omega})) \cap W^{1,2}(0, T; L^2(\tilde{\Omega})).
\]
Now for \( \epsilon, \delta > 0 \), we see that
\[
\frac{d}{dt}\|\Delta(\tilde{u}^\epsilon(t) - \tilde{u}^\delta(t))\|_2^2 = 2(\Delta(\tilde{u}^\epsilon(t) - \tilde{u}^\delta(t))', \Delta(\tilde{u}^\epsilon(t) - \tilde{u}^\delta(t)))
= 2((\tilde{u}^\epsilon(t) - \tilde{u}^\delta(t))', \Delta^2(\tilde{u}^\epsilon(t) - \tilde{u}^\delta(t)))
\]
Thus
\[
\|\tilde{u}^\epsilon(t) - \tilde{u}^\delta(t)\|_{W^{2,2}(\tilde{\Omega})}^2 = \|\tilde{u}^\epsilon(s) - \tilde{u}^\delta(s)\|_{W^{2,2}(\tilde{\Omega})}^2
+ 2 \int_s^t ((\tilde{u}^\epsilon(\tau) - \tilde{u}^\delta(\tau))', \Delta^2(\tilde{u}^\epsilon(\tau) - \tilde{u}^\delta(\tau)))d\tau
\tag{2.4}
\]
for all \( 0 \leq s, t \leq T \). Fix any point \( s \in (0, T) \) for which
\[
\tilde{u}^\epsilon(s) \to \tilde{u}(s) \quad \text{in} \quad W^{2,2}(\tilde{\Omega}).
\]
Then we have from (2.4)
\[
\lim_{\epsilon, \delta \to 0} \sup_{0 \leq t \leq T} \|\tilde{u}^\epsilon(t) - \tilde{u}^\delta(t)\|_{W^{2,2}(\tilde{\Omega})}^2
\leq \lim_{\epsilon, \delta \to 0} \int_0^T \left(\|((\tilde{u}^\epsilon)'(\tau) - (\tilde{u}^\delta)'(\tau))\|_2^2 + \|\tilde{u}^\epsilon(\tau) - \tilde{u}^\delta(\tau)\|_{W^{4,2}(\tilde{\Omega})}^2\right)d\tau
= 0.
\]
i.e.,
\[
\tilde{u}^\epsilon \to \tilde{u} \quad \text{in} \quad C([0, t]; W^{2,2}(\tilde{\Omega})).
\]
Besides, we also know that
\[
\tilde{u}^\epsilon \to \tilde{u}(t) \quad \text{for a.e. } t
\]
and then \( \tilde{u}(t) = v(t) \) a.e. \( t \). The claim follows.

Now we prove (2.2). Assume for the moment that \( \tilde{u} \) is smooth. We then compute
\[
\frac{d}{dt} \int_\Omega |\Delta \tilde{u}|^2 dx = 2 \int_\Omega \Delta \tilde{u} \Delta \tilde{u}' dx = 2 \int_\Omega \Delta^2 \tilde{u} \tilde{u}' dx \leq (\|\tilde{u}\|_{W^{4,2}(\tilde{\Omega})}^2 + \|\tilde{u}'\|_2^2).
\]
Thus
\[
\int_\Omega |\Delta \tilde{u}(t)|^2 dx \leq \int_\Omega |\Delta \tilde{u}(s)|^2 dx + C(\|\tilde{u}\|_{L^2(0, T; W^{4,2}(\tilde{\Omega}))}^2 + \|\tilde{u}'\|_2^2)
\tag{2.5}
\]
for all \( 0 \leq s, t \leq T \). We integrate (2.5) with respect to \( s \) and recall (2.3) to obtain
\[
\max_{0 \leq t \leq T} \|u\|_{W^{2,2}(\tilde{\Omega})} \leq C(\|u\|_{L^2(0, T; W^{4,2}(\tilde{\Omega}))} + \|u\|_{L^2(0, T; L^2(\tilde{\Omega}))}).
\tag{2.6}
\]
In the general case that \( m > 1 \), we let \( \alpha \) be a multiindex of order \( |\alpha| \leq m \), and set \( v := D^\alpha u \). Then
\[
v \in L^2(0, T; W^{4,2}(\tilde{\Omega})), v' \in L^2(0, T; L^2(\tilde{\Omega})).
\]
We apply estimate (2.6), with \( v \) replacing \( u \), and sum over all indices \( |\alpha| \leq m \) to obtain (2.2). We obtain the same estimate if \( u \) is not smooth, upon approximating by a smooth sequence \( u^\epsilon \), as before.
Lemma 2.3. Let $X$ and $Y$ be two Banach Spaces, such that $X \subset Y$ with a continuous injection. If a function $\phi$ belongs to $L^\infty(0,T;X)$ and is weakly continuous with values in $Y$, then $\phi$ is weakly continuous with values in $X$.

For proof, please see [25], here we omit it.

3 Well-posedness for the parabolic problem

This section is devoted to the study of the parabolic problem (1.2). We first consider the well-posedness of the following associated linear parabolic problem

\[
\begin{aligned}
\begin{cases}
\frac{\partial u}{\partial t} + \beta \Delta^2 u - \tau \Delta u = f(x,t) & \text{in } \Omega \times (0,T), \\
 u(x,0) = u^0 & \text{in } \Omega, \\
 u = \frac{\partial u}{\partial n} = 0 \text{ (or } u = \Delta u = 0) & \text{on } \Omega \times (0,T).
\end{cases}
\end{aligned}
\]

(3.1)

Theorem 3.1. Let $0 < \beta, 0 < T \leq \infty$ and $f \in L^2(\Omega \times (0,T))$. The Dirichlet problem for the linear fourth order parabolic equation (3.1) with initial datum $u^0 \in W_0^{2,2}(\Omega)$ admits a unique weak solution in the space

\[
C([0,T]; W_0^{2,2}(\Omega)) \cap L^2(0,T; W^{4,2}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)).
\]

The corresponding Navier problem with initial datum $u^0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ admits a unique weak solution in the space

\[
C([0,T]; W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \cap L^2(0,T; W^{4,2}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)).
\]

Furthermore, both cases admit the estimate

\[
\max_{0 \leq t \leq T} \|u(t)\|_{W^{2,2}(\Omega)}^2 + \int_0^T \|u(t)\|_{W^{4,2}(\Omega)}^2 + \int_0^T \|u_t\|_2^2 \leq C(\|\Delta u^0\|_2^2 + \int_0^T \|f\|_2^2) \tag{3.2}
\]

with the constant $C$ depending only on $\Omega, \beta, \tau$.

Definition 3.1. We say a function

\[
u \in L^2(0,T; W_0^{2,2}(\Omega)) \text{ (or } L^2(0,T; W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)))
\]

with

\[
u' \in L^2(0,T; W^{-2,2}(\Omega)) \text{ (or } L^2(0,T; (W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega))'))
\]

is a weak solution of the parabolic initial/ boundary-value problem (3.1) provided

(i) \quad <u',v> + \beta(\Delta u, \Delta v)_2 + \tau(Du, Dv)_2 = (f,v)_2

for each $v \in W_0^{2,2}(\Omega)$ (or $W_0^{1,2} \cap W^{2,2}(\Omega)$) and a.e. $0 \leq t \leq T$, and

(ii) $u(0) = u^0$. Here <$,>$ denotes the pairing between $W_0^{2,2}(\Omega)$ and $W^{-2,2}(\Omega)$ (or $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ and $(W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega))'$).

Remark 3.1. In view of Lemma 2.2 we see $u \in C([0,T]; L^2(\Omega))$, and thus the equality (ii) makes sense.
Proof Theorem 3.1. We will focus on Dirichlet boundary conditions, the proof for the Navier problem follows with obvious modifications. Let \( u^0 \in W_0^{2,2}(\Omega) \) and consider the following linear problem

\[
\begin{aligned}
\begin{cases}
  u_t + \beta \Delta^2 u - \tau \Delta u = f, & x \in \Omega, t > 0, \\
  u(x, 0) = u^0(x), & x \in \Omega, \\
  u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial \Omega, t > 0.
\end{cases}
\end{aligned}
\] (3.3)

We intend to build a weak solution of (3.3) by the so called “Faedo-Galerkin” method. More precisely, let \( \{\omega_k\}_{k=1}^{\infty} \subset W_0^{2,2}(\Omega) \) be an orthogonal complete system of eigenfunctions of \( \beta \Delta^2 - \tau \Delta \) under Dirichlet boundary conditions normalized by \( \|\omega_k\|_2 = 1 \). By Lemma 2.1

\( \{\omega_k\}_{k=1}^{\infty} \) is an orthonormal basis of \( L^2(\Omega) \).

Denote by \( \{\lambda_k\}_{k=1}^{\infty} \) the unbounded sequence of corresponding eigenvalues. For any \( k \geq 1 \) let

\[
u^k_0 := \sum_{i=1}^{k} (u^0, \omega_i) \omega_i
\]

so that \( \nu^k_0 \rightarrow u^0 \) in \( W_0^{2,2}(\Omega) \) as \( k \rightarrow +\infty \). For each \( k \geq 1 \) we define an approximate solution \( u_k : [0, T] \rightarrow W_0^{2,2}(\Omega) \) of (3.3) as follows:

\[
u_k(t) = \sum_{i=1}^{k} g^k_i(t) \omega_i
\]

and

\[
\begin{aligned}
\begin{cases}
  (u_k'(t), \omega_j)_2 + \beta (\Delta u_k, \Delta \omega_j)_2 + \tau (\nabla u_k, \nabla \omega_j)_2 = (f(t), \omega_j)_2, \\
  u_k(x, 0) = \nu^k_0(x), x \in \Omega.
\end{cases}
\end{aligned}
\] (3.4)

So that for any \( 1 \leq i \leq k \) the function \( g^k_i(t) \) solves the Cauchy problem

\[
\begin{aligned}
\begin{cases}
  (g^k_i(t))' + \sum_{i=1}^{k} \lambda_i g^k_i(t) = (f(t), \omega_i)_2, \\
  g^k_i(0) = (u^0_0, \omega_i)_2.
\end{cases}
\end{aligned}
\] (3.5)

According to the standard existence theory for ordinary differential equations, the linear ordinary differential equation (3.5) admits a unique solution \( g^k_i \) such that \( g^k_i \in W^{1,2}(0, T) \), and hence also (3.4) admits \( u_k \in W^{1,2}(0, T; W_0^{2,2}(\Omega)) \) as a unique solution.

We will obtain the a priori estimates independent of \( k \) for the approximate solution \( u_k \) and then pass to limit.

Step 1. A priori estimates. Indeed, we multiply the first equation of (3.4) by \( g^k_j(t) \) and sum on \( j \) from 1 up to \( k \). We get

\[
\frac{1}{2} \frac{d}{dt} \|u_k(t)\|_2^2 + \beta \|\Delta u_k\|_2^2 + \tau \|\nabla u_k\|_2^2 = (f(t), u_k)_2.
\] (3.6)

Integrating over \( (0, t) \) and using Cauchy’s inequality with \( \epsilon \), we are led to

\[
\|u_k(t)\|_2^2 - \|u^k_0(t)\|_2^2 + 2\|u_k(t)\|_{L^2[0,T;W^{2,2}(\Omega)]}^2 \leq \int_0^T \left( 4\|f(s)\|_2^2 + \|u_k(s)\|_{W^{2,2}(\Omega)}^2 \right) ds.
\]
And therefore
\[ \|u_k(t)\|_{L^\infty(0,T;L^2(\Omega))} + \|u_k(t)\|_{L^2(0,T;W^{2,2}(\Omega))} \leq \|u_0(t)\|_{L^2(0,T;L^2(\Omega))}^2 + 4\|f\|_{L^2(0,T;L^2(\Omega))} \]
\[ \leq 4(\|u^0\|_{L^2(0,T;L^2(\Omega))}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}). \]

(3.7)

Next we multiply both sides of (3.4) by \((g_j^k(t))'\) and sum on \(j\) to obtain
\[ (u_k(t), u_k(t)) + \beta(\Delta u_k, \Delta u_k(t)) + \tau(\nabla u_k, \nabla u_k(t)) = (f(t), u_k(t)) \]

Integrating over \((0, T)\) and using Cauchy’s inequality with \(\epsilon\), we see that
\[ \int_0^T \int_\Omega |u_k'(t)|^2 dxdt + \frac{\beta}{2} \|\Delta u_k(\cdot, T)\|^2_{L^2(\Omega)} + \frac{\tau}{2} \|\nabla u_k(\cdot, T)\|^2_{L^2(\Omega)} \]
\[ \leq \frac{\beta}{2} \|\Delta u_k(\cdot, 0)\|^2_{L^2(\Omega)} + \frac{\tau}{2} \|\nabla u_k(\cdot, 0)\|^2_{L^2(\Omega)} + \int_0^T \int_\Omega |f|^2 dxdt \]
\[ \leq \|u^0\|_{W^{2,2}(\Omega)}^2 + \int_0^T \int_\Omega |f|^2 dxdt. \]

(3.8)

Step 2. Passage to limit. From (3.7) and (3.8), we may extract a subsequence, still denoted by \(\{u_k\}\) such that
\[ u_k \rightharpoonup u \quad \text{in} \quad L^2(0, T; W^{2,2}(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)). \]

We expect that the limit function \(u\) is a weak solution of (3.3). To this end, we introduce a function \(h \in C^1([0, T]; C^2_0(\Omega))\) and take an approximate sequence of \(h\)
\[ h_j(x, t) = \sum_{m=1}^{j} \alpha_{j,m}(t) \omega_m(x) \]

such that \(\|h_j - h\|_{L^2(0,T;W^{2,2}_0(\Omega))} \to 0\) as \(j \to \infty\). Here \(\{\alpha_{j,m}(t)\}_{j=1}^{m}\) are given smooth functions. Now multiplying the first equation of (3.4) by \(\alpha_{j,m}\) and summing on \(m\) from 1 up to \(j\), we, by taking the limit for \(k \to \infty\), see that
\[ \int_0^T (u'(t), h_j) + \int_0^T \beta(\Delta u, \Delta h_j) + \int_0^T \tau(\nabla u, \nabla h_j) = \int_0^T (f(t), h_j). \]

Letting \(j \to \infty\) then we are led to
\[ \int_0^T (u'(t), h) + \int_0^T \beta(\Delta u, \Delta h) + \int_0^T \tau(\nabla u, \nabla h) = \int_0^T (f(t), h). \]

(3.9)

Since \(C^1([0, T]; C^2_0(\Omega))\) is dense in \(L^2(0, T; W^{2,2}_0(\Omega))\), we conclude equality (3.9) is valid for any \(h \in L^2(0, T; W^{2,2}_0(\Omega))\), which further implies
\[ (u'(t), h) + \beta(\Delta u, \Delta h) + \tau(\nabla u, \nabla h) = (f(t), h) \]

for all \(h \in W^{2,2}_0(\Omega)\) and a.e. \(0 \leq t \leq T\).

Now we claim \(u(0) = u^0\). Indeed, from (3.9) we deduce that
\[ \int_0^T -(v', u)_2 + \beta(\Delta u, \Delta v)_2 + \tau(\nabla u, \nabla v)_2 dt = \int_0^T (f, v)_2 dt + (u(0), v(0))_2 \]

(3.10)
for each \( v \in C^1([0, T]; W_0^{2,2}(\Omega)) \) with \( v(T) = 0 \). Similar, from (3.4) we also have
\[
\int_0^T -(v', u_k)_2 + \beta(\Delta u_k, \Delta v)_2 + \tau(\nabla u_k, \nabla v)_2 dt = \int_0^T (f, v)_2 dt + (u_k(0), v(0))_2
\]
Let \( k \to \infty \), we deduce that
\[
\int_0^T -(v', u)_2 + \beta(\Delta u, \Delta v)_2 + \tau(\nabla u, \nabla v)_2 dt = \int_0^T (f, v)_2 dt + (u, v(0))_2 \tag{3.11}
\]
Here we have used the fact \( u_k(0) \to u^0 \) in \( L^2(\Omega) \). As \( v(0) \) is arbitrary, comparing (3.10) and (3.11), we conclude \( u(0) = u^0 \). From this and (3.9), we conclude
\[
u \in L^2(0, T; W^{2,2}(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega))
\]
is a weak solution of (3.3) which satisfies (3.2). Uniqueness follows from the contradiction argument: if \( v, w \) were two solutions of (3.3) which share the same initial date, by subtracting the equations and (3.2) we would get
\[
\max_{0 \leq t \leq T} \|\Delta(v - w)\|_2^2 + \int_0^T \|\Delta^2(v - w)\|_2^2 + \int_0^T \|(v - w)_t\|_2^2 \leq 0,
\]
which immediately yields \( v \equiv w \).

Step 3. Ends of proof. Since
\[
\begin{cases}
\beta \Delta^2 u = f - u_t + \tau \Delta u \in L^2(\Omega \times (0, T)), & x \in \Omega, t > 0, \\
u = \frac{\partial u}{\partial t} = 0, & x \in \partial \Omega, t > 0,
\end{cases}
\]
then we have \( u \in L^2(0, T; W^{4,2}(\Omega)) \) by the regularity theorem of elliptic operator. Taking advantaging of interpolation between \( L^2(0, t; W^{4,2}(\Omega)) \) and \( W^{1,2}(0, T; L^2(\Omega)) \), we obtain \( u \in C(0, T; W^{2,2}(\Omega)) \).

**Theorem 3.2.** (Improved regularity). If \( u^0 \in W^{4,2}(\Omega) \cap W^{2,2}_0(\Omega) \) (or \( W^{4,2}(\Omega) \cap W^{1,2}_0(\Omega) \)), \( f' \in L^2(0, T; L^2(\Omega)) \), then
\[
u \in L^\infty(0, T; W^{4,2}(\Omega)), u' \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{2,2}_0(\Omega)),
\]
with the estimate
\[
\text{ess sup}_{0 \leq t \leq T} (\|u'(t)\|_2^2 + \|u\|_{W^{4,2}(\Omega)}^2) + \int_0^T \|u'(t)\|_{W^{2,2}(\Omega)}^2 dt \leq C(\|f\|_{W^{1,2}(0, T; L^2(\Omega))} + \|u^0\|_{W^{4,2}(\Omega)}).
\tag{3.12}
\]
Here the constant \( C \) depends only on \( \Omega, \beta, \tau \).

**Proof.** Fix \( k \geq 1 \) and differentiate equation (3.4) with respect to \( t \), we find
\[
(\tilde{u}'_k, \omega)_2 + \beta(\Delta \tilde{u}_k, \Delta \omega)_2 + \tau(\nabla \tilde{u}_k, \nabla \omega)_2 = (f', \omega)_2, \quad (j = 1, \cdots, k) \tag{3.13}
\]
where \( \tilde{u}_k := u_k' \). Multiply (3.13) by \( \frac{d}{dt} g_j^i(t) \) and sum \( j = 1, \cdots, k \), we see
\[
(u'_{\tilde{k}}, u_k)_2 + \beta(\Delta u_k, \Delta \tilde{u}_k)_2 + \tau(\nabla u_k, \nabla \tilde{u}_k)_2 = (f', \tilde{u}_k)_2.
\]
Integrating over \((0, T)\) and using Cauchy’s inequality with \(\epsilon\), we deduce

\[
\sup_{0 \leq t \leq T} \| u_k'(t) \|_2^2 + 2\beta \int_0^T \| \Delta u_k'(t) \|_2^2 dt + 2\tau \int_0^T \| \nabla u_k'(t) \|_2^2 dt \\
\leq C(\| u_k'(0) \|_2^2 + \| f' \|_{L^2(0,T;L^2(\Omega))}^2) \\
\leq C(\| f \|_{W^{1,2}(0,T;L^2(\Omega))}^2 + \| u_k(0) \|_{W^{4,2}(\Omega)}^2).
\]  

(3.14)

Here, we employed the first equation of (3.3) in the last inequality.

Remember that we have taken \(\{\omega_j\}\) to be the complete collection of (smooth) eigenfunctions for \(\beta \Delta^2 - \tau \Delta\) on \(W_0^{2,2}(\Omega)\). In particular \((\beta \Delta^2 - \tau \Delta)u_k = 0\) on \(\partial \Omega\). Thus

\[
\| u_k(0) \|_{W^{4,2}(\Omega)}^2 \leq C\|(\beta \Delta^2 - \tau \Delta)u_k(0)\|_{L^2}^2 = C(u_k(0), (\beta \Delta^2 - \tau \Delta)^2 u_k(0)).
\]

Since \((\beta \Delta^2 - \tau \Delta)^2 u_k(0) \in \text{span}\{\omega_j\}_{j=1}^k\) and \((u_k(0), \omega_j)_2 = (u_0, \omega_j)_2\) for \(j = 1, \ldots, k\), we have

\[
\| u_k(0) \|_{W^{4,2}(\Omega)}^2 \leq C(u_0, (\beta \Delta^2 - \tau \Delta)^2 u_k(0))_2 = C((\beta \Delta^2 - \tau \Delta)u_k(0))_2 \\
\leq \frac{1}{2} \| u_k(0) \|_{W^{4,2}(\Omega)}^2 + C\| u_0 \|_{W^{4,2}(\Omega)}^2.
\]

Therefore, combining with (3.14), we have

\[
\sup_{0 \leq t \leq T} \| u_k'(t) \|_{2}^2 + \int_0^T \| u_k'(t) \|_{W^{2,2}(\Omega)}^2 dt \leq C(\| f \|_{W^{1,2}(0,T;L^2(\Omega))}^2 + \| u_0 \|_{W^{4,2}(\Omega)}^2).
\]

(3.15)

Now

\[
((\beta \Delta^2 - \tau \Delta)u_k, \omega_j)_2 = (f - u_k', \omega_j)_2 \quad (j = 1, \ldots, k).
\]

And multiplying this identity by \(\lambda_j g_k^j(t)\) and summing \(j = 1, \ldots, k\), we deduce for \(0 \leq t \leq T\) that

\[
((\beta \Delta^2 - \tau \Delta)u_k, (\beta \Delta^2 - \tau \Delta)u_k)_2 = (f - u_k', (\beta \Delta^2 - \tau \Delta)u_k)_2,
\]

By the Hölder inequality, we see that

\[
\|\Delta^2 u_k\|_2^2 \leq (f - u_k'(t), (\beta \Delta^2 - \tau \Delta)u_k)_2 + C\| u_k \|_{W^{2,2}(\Omega)}^2 \\
\leq C\| f \|_2^2 + \frac{1}{2}\|\Delta^2 u_k\|_2^2 + C\| u_k'(t) \|_2^2 + C\| u_k \|_{W^{2,2}(\Omega)}^2,
\]

and then passing to limits as \(k = k_t \to \infty\) and combining (3.2) and (3.15), we deduce

\[
\sup_{0 \leq t \leq T} (\| u'(t) \|_2^2 + \| u \|_{W^{4,2}(\Omega)}^2) + \int_0^T \| u'(t) \|_{W^{2,2}(\Omega)}^2 dt \leq C(\| f \|_{W^{1,2}(0,T;L^2(\Omega))}^2 + \| u_0 \|_{W^{4,2}(\Omega)}^2).
\]

**Proof of Theorem 1.1** We only consider Dirichlet boundary conditions, the proof for the Navier problem is similar. Since we consider the case \(1 \leq N \leq 7\), then by the Sobolev embedding theorem, we deduce

\[
\| u \|_{L^\infty(\Omega)} \leq C(\Omega)\| u \|_{W^{4,2}(\Omega)}.
\]

(3.16)

Now define

\[
\mathcal{X}_T := C^0([0, T]; W^{4,2}(\Omega)) \cap W^{1,2}(0, T; W^{2,2}_0(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega))
\]
with norm
\[ \|v\|_{X_T}^2 := \int_0^T (\|v_t\|_{W^{2,2}(\Omega)}^2 + \|v\|_{W^{2,2}(\Omega)}^2)dt + \max_{0 \leq t \leq T} (\|v\|_{W^{4,2}(\Omega)}^2 + \|v_t\|_2^2). \]

And define
\[ \bar{M}(R, T) := \{v \in X_T : \|v\|_{X_T} \leq R\} \]
with \( R \) satisfying \( C(\Omega)R < 1 \). Here \( C(\Omega) \) is defined in (3.16). Let \( 0 < r < R \), we also define the set
\[ M(r, T) := \{v \in X_T : \|v\|_{X_T} < r\}. \]

From (3.16), we have
\[ u(t) \in \bar{M}(R, T) \Rightarrow \|u\|_{L^\infty(\Omega \times (0, T))} \leq C(\Omega)R < 1. \] (3.17)

Now let \( r \in (0, R) \) be fixed and
\[ u_i(t) \in \bar{M}(r, T), \]
for \( i = 1, 2 \), then by the Theorem 3.1 the initial-Dirichlet linear problem
\[
\begin{cases}
    v_t + \beta \Delta^2 v - \tau \Delta v = \frac{\lambda}{(1-u_i)^2}, & x \in \Omega, t > 0, \\
    v(x, 0) = u_0^0(x), v_t(x, 0) = u_0^1(x) & x \in \Omega, \\
    v = \frac{\partial v}{\partial n} = 0, & x \in \partial \Omega, t > 0,
\end{cases}
\] (3.18)
has a unique solution
\[ v_i(t) := \mathcal{F}(u_i) \in C([0, T]; W^{2,2}_0(\Omega)) \cap L^2(0, T; W^{4,2}(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)) \]
for \( i = 1, 2 \).

Now we claim
\[ v_i \in C([0, T]; W^{4,2}(\Omega)), \]
from which we have \( v_i \in X_T \). Indeed, since \( \|u_i\|_{L^\infty(\Omega)} < 1 \) and \( u_i \in W^{1,2}(0, T; L^2(\Omega)) \), we have
\[ \frac{\lambda}{(1-u_i)^2} \in W^{1,2}(0, T; L^2(\Omega)). \]

And then by Theorem 3.2 we see
\[ v_i \in W^{1,\infty}(0, T; L^2(\Omega)) \cap W^{1,2}(0, T; W^{2,2}_0(\Omega)). \]

From this, it is easy to see that
\[ \beta \Delta^2 v_i = \frac{\lambda}{(1-u_i)^2} + \tau \Delta v_i - \frac{dv_i(t)}{dt} \in W^{2,2}(\Omega), \text{ a.e. } t. \]

Then by the regularity theory of the elliptic operator, we are led to
\[ v_i \in L^2(0, T; W^{6,2}(\Omega)). \]
Combining \( v_i \in W^{1,2}(0, T; W^{2,2}_0(\Omega)) \) with Lemma 2.2, we have
\[
v_i \in C([0, T]; W^{4,2}(\Omega)).
\]

Using the Theorem 3.2 again, we see that
\[
\|v_1 - v_2\|_{\mathcal{X}_T} \leq \lambda C \|\frac{1}{(1 - u_1)^2} - \frac{1}{(1 - u_2)^2}\|_{W^{1,2}(0, T; L^2(\Omega))}
\]
\[
= \lambda C \left( \int_0^T \int_\Omega \frac{1}{(1 - u_1)^2} - \frac{1}{(1 - u_2)^2} \right)^{\frac{1}{2}} dxdt
\]
\[
+ 2\lambda C \left( \int_0^T \int_\Omega \frac{u_1'}{(1 - u_1)^3} - \frac{u_2'}{(1 - u_2)^3} \right)^{\frac{1}{2}} dxdt
\]
\[
=: I + II.
\]

For \( I \), we have
\[
I = 2\lambda C \left( \int_0^T \int_\Omega \frac{(u_1 - u_2)^2}{(1 - (\theta u_1 + (1 - \theta) u_2))^6} \right)^{\frac{1}{2}} dxdt
\]
\[
\leq 2\lambda C k(r) \left( \int_0^T \int_\Omega (u_1 - u_2)^2 \right)^{\frac{1}{2}} dxdt
\]
\[
\leq 2\lambda C k(r) \|u_1 - u_2\|_{\mathcal{X}_T}. \tag{3.19}
\]

or
\[
I \leq 2\lambda C k(r) T^{\frac{1}{2}} \|u_1 - u_2\|_{C([0, T]; W^{4,2}(\Omega))} \leq 2\lambda C k(r) T^{\frac{1}{2}} \|u_1 - u_2\|_{\mathcal{X}_T}. \tag{3.20}
\]

For \( II \), we have
\[
II \leq 2\lambda C \left( \int_0^T \int_\Omega \frac{(u_1' - u_2')^2}{(1 - u_1)^6} \right)^{\frac{1}{2}} dxdt
\]
\[
+ 2\lambda C \left( \int_0^T \int_\Omega |u_2'|^2 \frac{1}{(1 - u_1)^3} - \frac{1}{(1 - u_2)^3} \right)^{\frac{1}{2}} dxdt
\]
\[
\leq k(r)2\lambda C \left[ \|u_1 - u_2\|_{\mathcal{X}_T} + \|u_1 - u_2\|_{L^\infty([0, T] \times \Omega)} \left( \int_0^T \int_\Omega |u_2'|^2 \right)^{\frac{1}{2}} \right]
\]
\[
\leq 2\lambda C (r + k(r)) \|u_1 - u_2\|_{\mathcal{X}_T}.
\]

or
\[
II \leq 2\lambda C T^{\frac{1}{2}} k(r) \left( \|u_1 - u_2\|_{W^{1,\infty}(0, T; L^2(\Omega))} + \|u_1 - u_2\|_{L^\infty([0, T] \times \Omega)} \|u_2\|_{W^{1,\infty}(0, T; L^2(\Omega))} \right)
\]
\[
\leq 2\lambda C (r + k(r)) T^{\frac{1}{2}} \|u_1 - u_2\|_{\mathcal{X}_T}. \tag{3.21}
\]

Here and in what follows \( k(r) \) is a positive nondecreasing function for \( r \in [0, R_0] \) and \( C \) depends only on \( \Omega \). From (3.19) and (3.21), we have
\[
\|v_1 - v_2\|_{\mathcal{X}_T} \leq 2\lambda C (r + k(r)) \|u_1 - u_2\|_{\mathcal{X}_T}, \tag{3.22}
\]
and from \eqref{3.20} and \eqref{3.22}
\[\|v_1 - v_2\|_{X_T} \leq 2\lambda T^2 C(r + k(r))\|u_1 - u_2\|_{X_T}.\] 

Now consider the unique solution $w(t)$ to the linear problem
\[w_t + \beta \Delta^2 w - \tau \Delta w = 0, \quad x \in \Omega, t > 0,
\]
with the same boundary and initial conditions as \eqref{3.3}. By the Theorem \ref{thm:existence}, we have $w(t) \in C([0, T]; W^{2,2}_0(\Omega)) \cap L^2(0, T; W^{4,2}(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega))$
such that
\[\|w\|_{X_T} \leq C\|u^0\|_{W^{4,2}(\Omega)} := C\rho.\] 

Define the ball
\[B_r^2 = \{u \in X_T : \|u - w\|_{X_T} \leq \frac{r}{2}\}.
\]
Choosing $\rho$ small enough such that
\[C\rho + \frac{r}{2} < r,
\]
we then have, if $u(t) \in B_r^2$,
\[\|u(t)\|_{X_T} \leq \|w(t)\|_{X_T} + \frac{r}{2} < r, \quad \forall 0 \leq t \leq T,
\]
and then $B_r^2 \subset M(r, T)$.

Case 1. Global existence for small $\lambda$. Now using estimate \eqref{3.23}, we find
\[\|v_i - w\|_{X_T} \leq 2C\lambda(k(r) + r)\] 
for $i = 1, 2$. Now choosing $\lambda$ so small that
\[\lambda \leq \lambda(r) := \frac{1}{4C(k(r) + r)},
\]
we have from \eqref{3.23} and \eqref{3.26} that
\[\|v_1 - v_2\|_{X_T} \leq 2\|u_1 - u_2\|_{X_T};
\]
\[\|v_i - w\|_{X_T} \leq \frac{r}{2}.
\]
Hence the map
\[\mathcal{F} : B_r^2 \to B_r^2
\]
\[u_i \to v_i \quad (i = 1, 2)
\]
is a contraction map and it has a unique fixed point $u = \mathcal{F}(u)$ in $B_r^2$ for $0 < \lambda \leq \lambda(r)$
and arbitrary $T > 0$, which is a global weak solution of \eqref{1.2} with Dirichlet boundary conditions.
Case 2. Local existence in time. Similarly, using estimate (3.2), we are led to
\[\|v_i - w\|_{X_T} \leq C\lambda T^{\frac{1}{2}} (k(r) + r)\|u_i\|_{X_T} \leq \lambda T^{\frac{1}{2}} C(k(r) + r)r \tag{3.28}\]
for \(i = 1, 2\). Let \(T\) small enough such that
\[0 < T^{\frac{1}{2}} \leq \bar{T}(\lambda, \rho, r) := \frac{1}{2C\lambda(k(r) + r)},\]
we then have from (3.24) and (3.28) that
\[\|v_1 - v_2\|_{X_T} \leq \frac{1}{2}\|u_1 - u_2\|_{X_T};\]
\[\|v_i - w\|_{X_T} \leq r.\]
The existence of a unique solution to (1.2) over \([0, T]\) for all \(T \leq \bar{T}(\lambda, r)\) follows from the application of the Banach fixed point Theorem to the map.

Now we give the proof of (iii) of Theorem 1.1 as follows. To this end, we will use the eigenfunction method which comes from, for example, [2, 14, 17]. Indeed, from [7], there exists a pair \((\lambda_1, \phi_1)\) such that \(0 < \lambda_1, 0 < \phi_1 \in C^4(\mathbb{B}) \cap W^{2,2}_0(\mathbb{B}), \|\phi_1\|_1 = 1\) and
\[\begin{cases}
\beta \Delta^2 \phi_1 - \tau \Delta \phi_1 = \lambda_1 \phi_1, & x \in \mathbb{B}, \\
\phi_1 = \frac{\partial \phi_1}{\partial n} = 0, & x \in \partial \mathbb{B}.
\end{cases}\]
Let \(u(x, t)\) be the solution on \([0, T_m]\) to (1.2) and define for \(t \in [0, T_m]\)
\[M(t) := \int_{\mathbb{B}} \phi_1(x)u(x, t)dx \leq \int_{\mathbb{B}} \phi_1 dx = 1.\]
Now we multiply (1.2) by \(\phi_1\), integrate over \(\mathbb{B}\), and use the properties of \(\phi_1\) and Jensen’s inequality to obtain
\[\frac{dM}{dt} = -\int_{\mathbb{B}} (\beta \Delta^2 \phi_1 - \tau \Delta \phi_1) u dx + \lambda \int_{\mathbb{B}} \frac{\phi_1}{(1 - u)^2} dx \geq -\lambda_1 \int_{\mathbb{B}} \phi_1 u dx + \lambda \frac{\lambda}{(1 - \int_{\mathbb{B}} \phi_1 u dx)^2} \tag{3.29}\]
\[= -\lambda_1 M(0) + \lambda \frac{\lambda}{(1 - M)^2} := g(M)\]
By a simple calculation, we have \(g(M) > c_0 > 0\) if we choose \(\lambda > \frac{\lambda_1}{27}\). From (3.29), we immediately have
\[1 - M(0) \geq M(t) - M(0) \geq c_0 t,\]
consequently, \(T_m \leq \frac{1-M(0)}{c_0} < \infty\).

4 Well-posedness for the hyperbolic problem

In this section, we will consider the well-posedness of the dynamic problem (1.3). As in Section 3, we first study the well-posedness of the corresponding the linear hyperbolic problem
\[\begin{cases}
\frac{\partial^2 u}{\partial t^2} + \beta \Delta^2 u - \tau \Delta u = f(x, t) & x \in \Omega, t > 0, \\
u(x, 0) = u^0(x), u_t(x, 0) = u^1(x) & x \in \Omega, \\
\frac{\partial u}{\partial n} = 0 \quad \text{(or } u = \Delta u = 0) & x \in \partial \Omega, t > 0,
\end{cases}\tag{4.1}\]
where \(u^0(x), u^1(x)\) are assumed to belong to some Sobolev space, \(f(x, t) \in L^2(\Omega \times (0, T))\).
Definition 4.1. We say a function

\[ u \in L^2(0, T; W^{2,2}_0(\Omega)) \]  

(or \( L^2(0, T; W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)) \))

with

\[ u'' \in L^2(0, T; W^{-2,2}(\Omega)) \]  

(or \( L^2(0, T; (W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega))^\prime) \))

is a weak solution of the hyperbolic initial/ boundary-value problem \((4.1)\) provided

(i) \[ < u'', v > + \beta(\Delta u, \Delta v) + \tau(\nabla u, \nabla v) = (f, v) \]

for each \( v \in W^{2,2}_0(\Omega) \) (or \( W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega) \)) and a.e. time \( 0 \leq t \leq T \), and

(ii) \( u(0) = u^0, u'(0) = u^1 \). Here \(<,>\) denotes the pairing between \( W^{2,2}_0(\Omega) \) and \( W^{-2,2}(\Omega) \) (or \( W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega) \) and \((W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega))^\prime\)).

Theorem 4.1. Let \( 0 < T < \infty \) and \( f \in L^2(\Omega \times (0, T)) \). The Dirichlet problem for the linear fourth order hyperbolic equation \((4.1)\) with initial datums \( u^0 \in W^{2,2}_0(\Omega), u^1 \in L^2(\Omega) \) admits a unique weak solution such that

\[ u \in C_w([0, T]; W^{2,2}_0(\Omega)), u'(t) \in C_w([0, T]; L^2(\Omega)), u''(t) \in L^2(0, T; W^{-2,2}(\Omega)). \]

And the corresponding Navier problem with initial datums \( u^0 \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega), u^1 \in L^2(\Omega) \) admits a unique weak solution such that

\[ u \in C_w([0, T]; W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)), u'(t) \in C_w([0, T]; L^2(\Omega)), u''(t) \in L^2(0, T; (W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega))'). \]

Furthermore, both admit the estimate

\[ \text{ess sup}_{0 \leq t \leq T} (\|u\|_{W^{2,2}_0(\Omega)}^2 + \|u'(t)\|_2^2) + \int_0^T \|u''\|_{W^{-2,2}(\Omega)}^2 \, dt \leq C \left( \|\Delta u^0\|_2^2 + \|u^1\|_2^2 + \int_0^T \|f\|_2^2 \, dt \right). \]

(4.2)

Here the constant \( C \) depends only on \( \Omega, T, \beta, \tau \).

Proof of Theorem 4.1. As in Theorem 3.1 we only consider the Dirichlet boundary condition case, the proof for the Navier problem follows with the obvious modification. Similar to Theorem 3.1 we will once more employ “Faedo-Galerkin” method to construct our weak solutions. To this end, we, exactly as in the proof of Theorem 3.1 define an approximate solution \( u_k : [0, T] \to W^{2,2}_0(\Omega) \) of \((4.1)\) as follows:

\[ u_k(x, t) = \sum_{i=1}^k g^k_i(t) \omega_i(x), \quad k \geq 1, \]

where \( \omega_i(x) \) is defined as in Lemma 2.1 and the function \( g^k_i(t) \) (1 \( \leq i \leq k \)) solves the Cauchy problem

\[ \begin{cases} 
(g^k_i(t))'' + \lambda_i g^k_i(t) = (f(t), \omega_i)_2, \\
g^k_i(0) = (u^0_k, \omega_i)_2, \quad \frac{d}{dt}g^k_i(0) = (u^1_k, \omega_i)_2, 
\end{cases} \]

(4.3)
with
\[ u_0^k(x) := \sum_{i=1}^k (u^0, \omega_i) \omega_i(x); \quad u_1^k(x) := \sum_{i=1}^k (u^1, \omega_i) \omega_i(x). \]

According to the standard theory for ordinary differential equations, there exists a unique function \( g_k(t) \in W^{2,2}(0, T) \) solving (4.3) for \( 0 \leq t \leq T \).

As in proof of Theorem 3.1 we first study \textit{a priori estimates} of the approximate solution \( u_k \). Indeed,
\[ (u_k''(t), \omega_j)_2 + \beta(\Delta u_k, \Delta \omega_j)_2 + \tau(\nabla u_k, \nabla \omega_j)_2 = (f, \omega_j)_2, \quad (4.4) \]
multiply this equality by \( \frac{d}{dt} g_j^k(t) \), sum \( j = 1, \ldots, k \), we see
\[ (u_k''(t), u_k'(t))_2 + \beta(\Delta u_k, \Delta u_k')_2 + \tau(\nabla u_k, \nabla u_k')_2 = (f, u_k')_2. \]

From this, we immediately have
\[ \frac{d}{dt} \left( \|u_k'\|_{L^2}^2 + \beta\|\Delta u_k\|_{L^2}^2 + \tau\|\nabla u_k\|_{L^2}^2 \right) \leq C(\|u_k'\|_{L^2}^2 + \|f\|_{L^2}^2). \quad (4.5) \]

Now write
\[ \eta(t) := \|u_k'\|_{L^2(\Omega)}^2 + \beta\|\Delta u_k\|_{L^2(\Omega)}^2 + \tau\|\nabla u_k\|_{L^2(\Omega)}^2. \]

Then inequality (4.5) reads
\[ \eta'(t) \leq C(\eta(t) + \|f\|_{L^2}^2) \]
for \( 0 \leq t \leq T \). Thus Grownwall’s inequality yields the estimate
\[ \eta(t) \leq e^{Ct} \left( \eta(0) + \int_0^t \|f(s)\|_{L^2}^2 ds \right), \]
where
\[ \eta(0) = \|u_k'(0)\|_{L^2(\Omega)}^2 + \beta\|\Delta u_k(0)\|_{L^2(\Omega)}^2 + \tau\|\nabla u_k(0)\|_{L^2(\Omega)}^2 \]
\[ \leq C(\|u_1\|_{L^2(\Omega)}^2 + \|u^0\|_{W^{2,2}(\Omega)}^2). \]

Thus, we are led to
\[ \max_{0 \leq t \leq T} \left( \|u_k'\|_{L^2(\Omega)}^2 + \|u_k\|_{W^{2,2}(\Omega)}^2 \right) \leq C \left( \|u_1\|_{L^2(\Omega)}^2 + \|u^0\|_{W^{2,2}(\Omega)}^2 + \int_0^T \|f(s)\|_{L^2}^2 ds \right). \quad (4.6) \]

Fix any \( v \in W^{2,2}_0(\Omega), \|v\|_{W^{2,2}_0(\Omega)} \leq 1 \), and write \( v = v^1 + v^2 \), where \( v^1 \in \text{span}\{\omega_k\} \) and \( (v^2, \omega_k) = 0 \ (k = 1, \ldots, m) \). Then from (4.4), we see
\[ < u_m'', v > = (u_m'', v)_2 = (u_m', v)_2 = (f, v)_2 - \tau(\Delta u_m, \Delta v^1)_2 - \beta(\nabla u_m, \nabla v^1)_2. \]

Thus
\[ | < u_m'', v > | \leq C(\|f\|_{L^2(\Omega)} + \|u_m\|_{W^{2,2}(\Omega)}), \]
here we have used the fact that \( \|v^1\|_{W^{2,2}(\Omega)} \leq 1 \). Consequently
\[ \int_0^T \|u_m''\|_{W^{2,2}(\Omega)} dt \leq C \int_0^T (\|f\|_{L^2(\Omega)} + \|u_m\|_{W^{2,2}(\Omega)}) dt \]
\[ \leq C(\|u^0\|_{W^{2,2}(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^2(0, T; L^2(\Omega))}). \quad (4.7) \]
Now from (4.4) and (4.7), we see that there exist a subsequence \( \{u_k\}_{k=1}^{\infty} \) and \( u \in L^2(0, T; W^{2,2}_0(\Omega)) \), with \( u' \in L^2(0, T; L^2(\Omega)) \), \( u'' \in L^2(0, T; W^{-2,2}_0(\Omega)) \), such that

\[
u_k \to u \quad \text{in} \quad L^\infty(0, T; W^{2,2}_0(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega)) \cap W^{2,2}(0, T; W^{-2,2}_0(\Omega)).
\]

Now as in Theorem 3.1 we choose a function of the form

\[
h_j(x, t) = \sum_{m=1}^{j} \alpha_j(m)\omega_m(x)
\]

such that

\[
\|h_j - h\|_{L^2(0, T; W^{2,2}_0(\Omega))} \to 0 \quad \text{as} \quad j \to \infty
\]

for some \( h \in C^1([0, T]; C^2_0(\Omega)) \). Here \( \{\alpha_j(m)\}_{m=1}^{k} \) are given smooth functions. Now multiplying the first equation of (4.4) by \( \alpha_j, m \) and summing on \( m \) from 1 up to \( j \), we, by taking the limit for \( k \to \infty \), see that

\[
\int_0^T < u''(t), h_j > + \int_0^T \beta(\Delta u, \Delta h_j)_2 + \int_0^T \tau(\nabla u, \nabla h_j)_2 = \int_0^T (f(t), h_j)_2.
\]

Letting \( j \to \infty \) then we are led to

\[
\int_0^T < u''(t), h > + \int_0^T \beta(\Delta u, \Delta h)_2 + \int_0^T \tau(\nabla u, \nabla h)_2 = \int_0^T (f(t), h)_2. \quad (4.8)
\]

Since \( C^1([0, T]; C^2_0(\Omega)) \) is dense in \( L^2(0, T; W^{2,2}_0(\Omega)) \), we conclude equality (4.8) is valid for any \( h \in L^2(0, T; W^{2,2}_0(\Omega)) \), which further implies

\[
< u''(t), h > + \beta(\Delta u, \Delta h)_2 + \tau(\nabla u, \nabla h)_2 = (f(t), h)_2
\]

for all \( h \in W^{2,2}_0(\Omega) \) and a.e. \( 0 \leq t \leq T \). Using the same argument as Theorem 3.1 we can also prove \( u(0) = u^0, u'(0) = u^1 \), here we omit its details. Hence

\[
u \in L^\infty(0, T; W^{2,2}(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega)) \cap W^{2,2}(0, T; W^{-2,2}(\Omega))
\]

is a weak solution of (4.1). Besides, we note that

\[
u'(t) \in C([0, T]; W^{-2,2}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))
\]

and \( L^2(\Omega) \subset W^{-2,2}(\Omega) \) with a continuous injection, we, by Lemma 2.3 have \( u'(t) \in C_w([0, T]; L^2(\Omega)) \). Similar, we also have \( u \in C_w(0, T; W^{2,2}(\Omega)) \). The uniqueness follows from the standard contradiction argument.

**Theorem 4.2.** (Improved regularity). If \( u^0 \in W^{4,2}(\Omega) \cap W^{2,2}_0(\Omega) \) (or \( W^{4,2}(\Omega) \cap W^{1,2}_0(\Omega) \)), \( u^1 \in W^{2,2}_0(\Omega) \) (or \( W^{2,2}_0(\Omega) \cap W^{1,2}_0(\Omega) \)), \( f' \in L^2(0, T; L^2(\Omega)) \), then

\[
u \in C_w(0, T; W^{4,2}(\Omega)), \quad u' \in C_w(0, T; W^{2,2}_0(\Omega)) \quad \text{(or} \quad C_w(0, T; W^{2,2}_0(\Omega) \cap W^{1,2}_0(\Omega)) \quad \text{))}
\]

\[
u'' \in L^\infty(0, T; L^2(\Omega)).
\]

with the estimate

\[
es\sup_{0 \leq t \leq T} (\|u'(t)\|_{W^{2,2}(\Omega)}^2 + \|u''(t)\|_2^2 + \|u(t)\|_{W^{4,2}(\Omega)}^2)
\]

\[
\leq C(\|f\|_{L^2(0, T; L^2(\Omega))}^2 + \|u^0\|_{W^{4,2}(\Omega)}^2 + \|u^1\|_{W^{2,2}(\Omega)}^2).
\]

Here the constant \( C \) depends only on \( \Omega, T, \beta, \tau \).
Proof. Fix a positive integer $m$ and write $\tilde{u}_m := u'_m$, we obtain by differentiating the identity (4.4) with respect to $t$,

$$(u''_m, \omega_k)_2 + (\Delta u''_m, \Delta \omega_k)_2 + \tau (\nabla u''_m, \nabla \omega_k)_2 = (f', \omega_k)_2.$$ 

Multiplying by $\frac{d^2}{dt^2}g^k_m$ and adding for $k = 1, \ldots, m$, we discover

$$(\tilde{u}''_m, \tilde{u}'_m)_2 + (\Delta \tilde{u}''_m, \Delta \tilde{u}'_m)_2 + \tau (\nabla \tilde{u}''_m, \nabla \tilde{u}'_m)_2 = (f', \tilde{u}'_m)_2.$$ 

and then

$$\frac{d}{dt} \left( \|\tilde{u}'_m\|^2_2 + \beta \|\Delta \tilde{u}_m\|^2_2 + \tau \|\nabla \tilde{u}_m\|^2_2 \right) \leq C(\|\tilde{u}'_m\|^2_2 + \|f'\|^2_2). \quad (4.11)$$

Now write

$$\eta(t) := \|\tilde{u}'_m\|^2_{L^2(\Omega)} + \beta \|\Delta \tilde{u}_m\|^2_{L^2(\Omega)} + \tau \|\nabla \tilde{u}_m\|^2_{L^2(\Omega)}.$$ 

Then inequality (4.11) reads

$$\eta'(t) \leq C(\eta(t) + \|f'\|^2_{L^2})$$

for $0 \leq t \leq T$.

Besides, we note

$$(f - u''_m(t), \omega_k)_2 = \beta (\Delta^2 u_m, \omega_k)_2 - \tau (\Delta u_m, \omega_k)_2. \quad (4.12)$$

Multiplying (4.12) by $\lambda_k g^k_m(t)$ and summing $k = 1, \ldots, m$, we deduce

$$\|\Delta^2 u_m\|^2_2 \leq (f - u''_m, (\beta \Delta^2 - \tau \Delta) u_m)_2 + C \|u_k\|_{W^{2,2}(\Omega)}$$

$$\leq C(\|f\|^2_2 + \|u'_m(t)\|^2_2 + \|u''_m\|^2_{W^{2,2}(\Omega)}). \quad (4.13)$$

Applying Grownwall’s inequality, we have

$$\eta(t) \leq e^{Ct} \left( \eta(0) + \int_0^t \|f'(s)\|^2_{L^2} ds \right), \quad (4.14)$$

where

$$\eta(0) = \|u''_m(0)\|^2_{L^2} + \beta \|\Delta u'_m(0)\|^2_{L^2} + \tau \|\nabla u'_m(0)\|^2_{L^2}$$

Employing (4.4) and the fact

$$\|u_k(0)\|^2_{W^{4,2}(\Omega)} \leq C \|u^0\|^2_{W^{4,2}(\Omega)}.$$

we have

$$\eta(0) \leq C(\|u^0\|^2_{W^{4,2}(\Omega)} + \|u^1\|^2_{W^{2,2}(\Omega)}). \quad (4.15)$$

Combining (4.13)-(4.15), we, by passing to limits as $m = m_l \to \infty$, obtain (4.10). Finally, we deduce (4.9) by Lemma 2.2.
Proof of Theorem 1.2. As in the proof of Theorem 1.1, we only consider the Dirichlet boundary condition. Now define

\[ X_T := L^\infty(0, T; W^{4,2}(\Omega)) \cap W^{1,\infty}(0, T; W_0^{2,2}(\Omega)) \cap W^{2,\infty}(0, T; L^2(\Omega)) \]

with norm

\[ \|v\|_{X_T}^2 := \text{ess sup}_{0 \leq t \leq T} (\|v\|_{W^{4,2}(\Omega)}^2 + \|v_t\|_{W^{2,2}(\Omega)}^2 + \|vu\|_2^2). \]

And define

\[ \bar{M}(R, T) := \{ v \in X_T : \|v\|_{X_T} \leq R \} \]

with \( R \) satisfying \( C(\Omega)R < 1 \). Here \( C(\Omega) \) is defined in (3.16). From (3.16), we have

\[ u(t) \in \bar{M}(R, T) \Rightarrow \|u\|_{L^\infty(\Omega \times (0, T))} \leq C(\Omega)R < 1, \] (4.16)

and further implies \( \frac{1}{(1-u)^2} \in W^{1,2}(0, T; L^2(\Omega)) \).

Now let \( r \in (0, R) \) be fixed and

\[ u_i(t) \in \bar{M}(r, T), \]

for \( i = 1, 2 \), then by the Theorem 4.1 and Theorem 4.2, the initial-Dirichlet linear problem

\[
\begin{aligned}
&v_{tt} + \beta \Delta^2 v - \tau \Delta v = \frac{\lambda}{(1-u_i)^2}, \quad x \in \Omega, t > 0, \\
v(x, 0) = u_0^i(x), v_t(x, 0) = u_1^i(x) \quad x \in \Omega, \\
v = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0,
\end{aligned}
\] (4.17)

has a unique solution

\[ v_i(t) := F(u_i) \in L^\infty(0, T; W^{4,2}(\Omega)) \cap W^{1,\infty}(0, T; W_0^{2,2}(\Omega)) \cap W^{2,\infty}(0, T; L^2(\Omega)) \]

for \( i = 1, 2 \).

Using the Theorem 4.2 again, we see that

\[
\|v_1 - v_2\|_{X_T} \leq \lambda C \left( \frac{1}{(1-u_1)^2} - \frac{1}{(1-u_2)^2} \right) \|v\|_{W^{1,2}(0,T;L^2(\Omega))} \\
= \lambda C \left( \int_0^T \int_\Omega \left( \frac{1}{(1-u_1)^2} - \frac{1}{(1-u_2)^2} \right)^2 dxdt \right)^{\frac{1}{2}} \\
+ 2\lambda C \left( \int_0^T \int_\Omega \left( \frac{u_1'}{(1-u_1)^3} - \frac{u_2'}{(1-u_2)^3} \right)^2 dxdt \right)^{\frac{1}{2}} \\
=: I + II.
\]

For \( I \), we have

\[
I \leq 2\lambda C \left( \int_0^T \int_\Omega \frac{(u_1 - u_2)^2}{(1-(\theta u_1 + (1-\theta)u_2))^6} dxdt \right)^{\frac{1}{2}} \\
\leq 2\lambda C k(r) \left( \int_0^T \int_\Omega (u_1 - u_2)^2 dxdt \right)^{\frac{1}{2}}, \quad (4.18)
\]

\[
\leq 2\lambda T^{\frac{1}{2}} C k(r) \|u_1 - u_2\|_{X_T}.
\]
For $II$, we have
\[
II \leq 2\lambda C \left( \int_0^T \int_{\Omega} \frac{(u'_1 - u'_2)^2}{(1 - u_1)^6} \, dx \, dt \right)^{\frac{1}{2}} + 2\lambda C \left( \int_0^T \int_{\Omega} \left| \frac{1}{(1 - u_1)^3} - \frac{1}{(1 - u_2)^3} \right|^2 \, dx \, dt \right)^{\frac{1}{2}} \\
\leq 2\lambda CT^\frac{1}{2} k(r) \left( \|u_1 - u_2\|_{W^{1,\infty}(0,T;L^2(\Omega))} + \|u_1 - u_2\|_{L^{\infty}((0,T) \times \Omega)} \|u_2\|_{W^{1,\infty}(0,T;L^2(\Omega))} \right) \\
\leq 2\lambda C (r + k(r)) T^\frac{1}{2} \|u_1 - u_2\|_{X_T}. \tag{4.19}
\]

Here and in what follows $k(r)$ is a positive nondecreasing function for $r \in [0, R_0]$ and $C$ depends only on $\Omega, T, \beta, \tau$. From (4.18) and (4.19), we have
\[
\|v_1 - v_2\|_{X_T} \leq 2\lambda T^\frac{1}{2} C (r + k(r)) \|u_1 - u_2\|_{X_T}. \tag{4.20}
\]

Now consider the unique solution $w(t)$ to the linear problem
\[
w_{tt} + \beta \Delta^2 w - \tau \Delta w = 0, \quad x \in \Omega, t > 0,
\]
with the same boundary and initial conditions as (4.1). Obviously we have, by the Theorem 4.2
\[
w(t) \in L^\infty(0,T;W^{4,2}(\Omega)) \cap W^{1,\infty}(0,T;W^{2,2}_0(\Omega)) \cap W^{2,\infty}(0,T;L^2(\Omega))
\]
such that
\[
\|w\|_{X_T} \leq C(\|u^0\|_{W^{4,2}(\Omega)} + \|u^1\|_{W^{2,2}(\Omega)}) = C\rho \tag{4.21}
\]

Define the ball
\[
B_{\frac{r}{2}} = \{u \in X_T : \|u - w\|_{X_T} \leq \frac{r}{2} \}.
\]
Choosing $\rho$ small enough such that
\[
C\rho + \frac{r}{2} < r,
\]
we then have, if $u(t) \in B_{\frac{r}{2}}$,
\[
\|u(t)\|_{X_T} \leq \|w(t)\|_{X_T} + \frac{r}{2} < r, \quad \forall \ 0 \leq t \leq T,
\]
and then $B_{\frac{r}{2}} \subset M(r, T)$.

Now using estimate (4.20), we find
\[
\|v_i - w\|_{X_T} \leq 2\lambda T^\frac{1}{2} C (r + k(r)) r \tag{4.22}
\]
for $i = 1, 2$. Now choosing $\lambda$ so small that
\[
\lambda \leq \lambda(r, T) := \frac{1}{4T^\frac{1}{2} C (k(r) + r)}.
\]

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we have from (4.20) and (4.22) that
\[ \| v_1 - v_2 \|_{X^T} \leq \frac{1}{2} \| u_1 - u_2 \|_{X^T}; \]
\[ \| v_i - w \|_{X^T} \leq \frac{r}{2}. \]
Hence the map
\[ F : B_{r^2} \rightarrow B_{r^2} \]
\[ u_i \rightarrow v_i \quad (i = 1, 2) \quad (4.23) \]
is a contraction map and it has a unique fixed point \( u = F(u) \) in \( B_{r^2} \) for \( 0 < \lambda \leq \lambda(r, T) \).
Finally, we claim that this solution satisfies
\[ u(t) \in C_w([0, T]; W^{4,2}(\Omega)), u'(t) \in C_w([0, T]; W^{2,2}_0(\Omega)). \]
Indeed, since \( u'(t) \in C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; W^{2,2}_0(\Omega)) \) and \( W^{2,2}_0(\Omega) \subset L^2(\Omega) \) with a continuous injection, and then \( u'(t) \in C_w([0, T]; W^{2,2}_0(\Omega)) \) by Lemma 2.2. Similar, we have \( u(t) \in C_w([0, T]; W^{4,2}(\Omega)). \)

The proof of (ii) is similar with (iii) of Theorem 1.1, we omit it here.

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