On the power of PPT-preserving and non-signalling codes

Debbie Leung, William Matthews

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We derive one-shot upper bounds for quantum noisy channel codes. We do so by regarding a channel code as a bipartite operation with an encoder belonging to the sender and a decoder belonging to the receiver, and imposing constraints on the bipartite operation. We investigate the power of codes whose bipartite operation is non-signalling from Alice to Bob, positive-partial transpose (PPT) preserving, or both, and derive a simple semidefinite program for the achievable entanglement fidelity. Using the semidefinite program, we show that the non-signalling assisted quantum capacity for memoryless channels is equal to the entanglement-assisted capacity. We also relate our PPT-preserving codes and the PPT-preserving entanglement distillation protocols studied by Rains. Applying these results to a concrete example, the 3-dimensional Werner-Holevo channel, we find that codes that are non-signalling and PPT-preserving can be strictly less powerful than codes satisfying either one of the constraints, and therefore provide a tighter bound for unassisted codes. Furthermore, PPT-preserving non-signalling codes can send one qubit perfectly over two uses of the channel, which has no quantum capacity. We discuss whether this can be interpreted as a form of superactivation of quantum capacity.

I. INTRODUCTION

A basic problem in quantum information theory is to determine the ability of a noisy channel to convey quantum information at a given standard of fidelity. The quantum capacity measures the optimal asymptotic rate of transmission (in qubits per channel use) possible for arbitrarily good fidelities (if not perfect fidelity). The LSD (Lloyd [1], Shor [2], Devetak [3]) Theorem shows that the quantum capacity is equal to the regularised coherent information, an optimization that involves unlimited number of copies of the channel. Our understanding of the quantum capacity remains limited — given a simple memoryless channel (such as the qubit depolarizing channel for certain error parameter), determining whether it has a positive quantum capacity is not known to be decidable. To gain insights into the often intractable problem of determining quantum capacities of channels, “assisted capacities” have been studied (see e.g. [4]), where the sender and the receiver are given extra free resources, such as entanglement or classical communication.

In this paper we are interested in the non-asymptotic (or finite blocklength) regime focusing on the trade-off between the dimension of the quantum system to be sent, the number of channel uses made, and the fidelity achieved. In the absence of feedback in the coding protocol, this is also called the ‘one-shot’ regime since we can treat multiple channel uses as a single use of a larger channel. In the one-shot regime, we can remove assumptions such as memoryless channel uses, address questions concerning quantum error correcting codes, and understand how fast the achievable rate converges to the capacity as the number of uses increases. Sometimes, one-shot studies provide results concerning asymptotic capacities. However, the exact trade-off of interest is generally intractable. Even in the classical case, it is not practical to compute the obtainable region of parameters exactly, but quite powerful bounds are known [5]. Parallel to the study of assisted capacities, one can consider assisted codes in the finite blocklength regime.

Mosonyi and Datta [6], Wang and Renner [7] and Renes and Renner [8] have given one-shot converse and achievability bounds for classical data transmission by unassisted codes over classical-quantum channels. In [9] Datta and Hsieh derive converse and achievability results for classical and quantum data transmission by entanglement-assisted codes over general quantum channels in terms of smoothed min- and max-entropies. A drawback of the bounds given in [9] is that no explicit method of computation is given, and it is not clear that an efficient method exists. A one-shot converse bound for entanglement-assisted codes amenable to computation was given in Matthews and Wehner [10] by generalising the hypothesis-testing based ‘meta-converse’ of [5] to quantum channels. In particular, the bound is a semidefinite program (SDP).

An alternative approach to upper bound one-shot performance is to optimize data transmission over a larger class of coding procedures which is mathematically easier to describe. This type of approach is applied to the related task of entanglement distillation in an early paper by Rains [11], which gives one-shot converse bounds for entanglement distillation by local operations and classical communication in the form of an SDP for the performance of the more powerful class of PPT-preserving operations, along with many other insightful results. This was also the approach used in [12], which derives a linear program for the performance of transmitting classical data via classical channels by codes which are non-signalling when the encoder and decoder are considered as a single bipartite operation. The linear program was shown to be equivalent to the meta-converse of of [3].

Our paper follows this approach. We consider quantum data transmission via quantum channels using codes that are non-signalling, PPT-preserving or both, when viewed as bipartite operations. We derive one-shot correspondences that allow our results to be viewed as extensions to results in [10] and [11].
The structure of the paper along with a summary of our results are as follows.

We start with some mathematical and notational preliminaries in Section II. Generally speaking, a “code” refers to a set of operations performed by the sender Alice and the receiver Bob that, when combined with the given channel uses, affects the data transmission. In Section III we define a very general class of codes, the forward-assisted codes, which can be implemented by local operations and forward (i.e. Alice to Bob) quantum communication over an arbitrary auxiliary channel (in addition to the use of the given noisy channel). This class includes a number of important, operationally defined subclasses: the unassisted codes, which only use local operations; the entanglement-assisted codes, where the auxiliary channel is only used to share entanglement between Alice and Bob before the local operations are applied; and the forward-classical-assisted codes, where the auxiliary channel is classical. We use the fact that forward-assisted codes correspond to bipartite operations which are non-signalling from Bob to Alice to define subclasses of forward-assisted code based on constraints on these bipartite operations. The non-signalling codes are those where the bipartite operation is also non-signalling from Alice to Bob. This class includes unassisted and entanglement-assisted codes. The PPT-preserving codes are those for which the bipartite operation is PPT-preserving. This class includes all unassisted and forward-classical-assisted codes, but not all entanglement-assisted codes. Section III provides precise definitions of all these classes and describes the relationships between them.

Section IV contains our main technical contribution. We derive simple semidefinite programs (SDPs) for the optimal channel fidelity of codes which are non-signalling, PPT-preserving, or both.

In section V we present the first application of our SDPs. We compare our optimal channel fidelity for non-signalling codes with an earlier upper bound for entanglement-assisted codes (derived with different techniques in [10] for the success probability of classical data transmission). Surprisingly, our new bound, which applies to a larger class of codes, is at least as tight as the old bound. Furthermore, from the asymptotic analysis of the earlier bound [10], we obtain a new asymptotic result for memoryless noisy channels: that entanglement-assisted and non-signalling codes give the same capacity.

In section VI we study optimal channel fidelity for PPT-preserving codes. We derive connections between PPT-preserving codes and PPT-preserving entanglement distillation scheme studied in by Rains in [11]. We show that Rains’ SDP for the fidelity of PPT-preserving entanglement distillation provides lower bounds on the fidelity of the PPT-preserving codes. We also show that for certain special channels Rains’ SDP coincides with our SDP for the fidelity of PPT-preserving codes.

In section VII we apply our SDPs to a concrete example, computing the fidelity for codes that are PPT-preserving, non-signalling or both, over the Werner-Holevo channels for blocklengths up to 120. The results demonstrate that codes which satisfy both constraints can be strictly less powerful than codes that satisfy either one of the constraints. Thus combining the PPT-preserving and non-signalling constraints provides strictly stronger upper bounds for unassisted communication, at least for finite block-lengths. The results suggest that this improvement may even persist in the asymptotic regime.

Furthermore, the results of section VII and Rains [11] imply that PPT-preserving codes enable zero-error quantum communication (of one qubit) over two uses of three-dimensional Werner-Holevo channel. Surprisingly, the same holds even if the codes are also non-signalling. We discuss the relationship of this phenomenon to the superactivation of quantum capacity [13]. Our result could be considered a form of superactivation, since neither the channel nor the code involved has quantum capacity, yet their combination can communicate quantum data perfectly. However, we do not know whether the code can be implemented by local operations and forward communication over a channel with no quantum capacity. If it could be, then our result would demonstrate a very strong version of superactivation in the sense of [13], where two channels with no quantum capacity could be used together to transmit quantum information perfectly. In this connection, we show, via an example, that not all PPT-preserving and non-signalling codes can be simulated by zero capacity forward quantum channel.

II. PRELIMINARIES

In this section, we summarize mathematical concepts required for the results. We will also define unambiguous conventions concerning our notation for quantum states and operations, which help us avoid a proliferation of brackets and tensor product symbols.

A quantum system $Q$ is associated to a Hilbert space $\mathcal{H}_Q$ of dimension $\text{dim}(Q)$ (in this work we only deal with finite dimensional systems) and is equipped with a real, orthonormal ‘computational basis’ $\{|i\rangle_Q : i = 1, \ldots, d\}$. We will always write linear operators on $\mathcal{H}_Q$ with a subscript identifying the system they act on, for example, $X_Q$.

We assume that there is some fixed underlying order on systems which determines the order in which tensor products are taken. We can write a product of operators acting on disjoint subsystems without the $\otimes$ symbol, by taking it as given that the operators are padded with appropriate identity operators. For example, $X_QY_R = Y_RX_Q = X_Q \otimes Y_R = (X_Q \otimes 1_R)(1_Q \otimes Y_R)$. The same applies to a product of operators acting on different but not necessarily disjoint subsystems, for example, $X_{PQ}Y_{QR} = (X_{PQ} \otimes 1_R)(1_P \otimes Y_{QR})$.

An operation $N_{R-Q}$ (or channel) with input system $Q$ and output system $R$ is a completely positive, trace preserving linear map from the bounded linear operators
on $\mathcal{H}_Q$ to the bounded linear operators on $\mathcal{H}_R$. Since we only deal with finite dimensional systems, all linear operators are bounded. As with operators, we always explicitly write the input and the output systems as subscripts. We write the set of all such operations as $\text{ops}(Q \to R)$. Our subscript convention has one exception: the trace operation on $Q$, $\text{Tr}_Q$, has the trivial, one-dimensional, output system, so we only write the input system.

We denote the transpose map on system $Q$ by $t_{Q \leftarrow Q}$. It is the trace preserving, but not completely positive, linear map such that $t_{Q \leftarrow Q} : |i⟩⟨j|_Q \mapsto |j⟩⟨i|_Q$. We also make use of the conventional notation $X^T_Q$ for $t_{Q \leftarrow Q} X_Q$.

Given two systems $Q$ and $\tilde{Q}$ of equal dimension, we can identify states of $Q$ with states of $\tilde{Q}$ via the identity operation $\text{id}_{Q \leftarrow \tilde{Q}} : |i⟩⟨j|_Q \mapsto |j⟩⟨i|_{\tilde{Q}}$. Furthermore, we denote the isotropic maximally entangled state of $Q \tilde{Q}$ by $\phi_{Q \tilde{Q}} := |φ⟩⟨φ|_{Q \tilde{Q}}$.

$$|φ⟩_{Q \tilde{Q}} := \text{dim}(Q)^{-1/2} \sum_{i=1}^{\text{dim}Q} |i⟩_Q |i⟩_{\tilde{Q}}. \quad (1)$$

A useful fact, sometimes called the ‘transpose trick’, is that for any operator $M_Q$ on $\mathcal{H}_Q$, we have

$$M_Q |φ⟩_{Q \tilde{Q}} = M^T_Q |φ⟩_{Q \tilde{Q}}, \quad (2)$$

where $M_Q := \text{id}_{Q \leftarrow Q} M_Q$. To denote the application of a linear map $\mathcal{N}_{R \leftarrow Q}$ to an operator $X_Q$, we write simply $\mathcal{N}_{R \leftarrow Q} X_Q$, just as we would write the application of a matrix to a vector without parenthesis. Products of operations represent compositions, with a convention similar to that defined for operators above, so that tensor symbols and identity operations are omitted. For example, $\mathcal{N}_{R \leftarrow Q} \mathcal{M}_{T \leftarrow P} X_{QP} = (\mathcal{N}_{R \leftarrow Q} \otimes \mathcal{M}_{T \leftarrow P}) X_{QP}$, and $\mathcal{N}_{R \leftarrow Q} X_{Q\tilde{Q}} = (\mathcal{N}_{R \leftarrow Q} \otimes \text{id}_{\tilde{P} \leftarrow P}) X_{Q\tilde{Q}}$.

We adopt the convention that multiplication of operators takes precedence over the application of linear maps from operators to operators, such as operations or the transpose map. For example $t_{Q \leftarrow Q} X_Q Y_Q = t_{Q \leftarrow Q} (X_Q Y_Q)$, and $\text{Tr}_Q X_{QP} Y_{QR} = \text{Tr}_Q (X_{QP} Y_{QR})$.

To further illustrate these notational conventions, we note a useful fact

$$\text{Tr}_Q X_{QP} t_{Q \leftarrow Q} Y_Q = \text{Tr}_Q (X_{QP} (1_P \otimes (t_{Q \leftarrow Q} Y_Q))) = \text{Tr}_Q ((t_{Q \leftarrow Q} X_{QP}) (1_P \otimes Y_Q)) = \text{Tr}_Q (t_{Q \leftarrow Q} X_{QP}) Y_{QR}. \quad (3)$$

In this paper, we define the Choi matrix $\mathcal{N}_{R \leftarrow Q}$ of an operation $\mathcal{N}_{R \leftarrow Q}$ to be the unique operator on $\mathcal{H}_R \otimes \mathcal{H}_Q$ such that for all operators $X_Q$ on $\mathcal{H}_Q$,

$$\mathcal{N}_{R \leftarrow Q} X_Q = \text{Tr}_Q \mathcal{N}_{RQ} t_{Q \leftarrow Q} X_Q = \text{Tr}_Q (t_{Q \leftarrow Q} N_{RQ}) X_Q \quad (4)$$

where the last equality comes from Eq. (3). Our Choi matrix is equal to the common definition:

$$N_{RQ} = \text{dim}(Q) \text{id}_{Q \leftarrow Q} \mathcal{N}_{R \leftarrow Q} φ_{Q \tilde{Q}}. \quad (5)$$

We adopt the convention that where operations are denoted by a calligraphic letter, the corresponding Choi matrix is the same letter in the regular font.

A bipartite operator $X_{PQ}$ is said to be PPT (positive partial-transpose) if $t_{P \leftarrow P} X_{PQ} \geq 0$. This condition is equivalent to $t_{Q \leftarrow Q} X_{PQ} \geq 0$, and is independent of the basis in which the transpose is taken.

An operation $\mathcal{F}_{B \leftarrow A}$ is called a ‘Horodecki’ channel (or PPT-binding channel) if its Choi matrix $F_{B \leftarrow A}$ is PPT.

Let $\bar{A}$ and $\bar{B}$ be arbitrary systems in the possession of Alice and Bob, respectively. A bipartite operation $Z_{A\bar{B} \leftarrow \bar{A} \bar{B}}$ is PPT-preserving [11][15] if it takes any state which is PPT with respect to the Alice / Bob partition to another PPT state. In other words, $t_{B\bar{B} \leftarrow B\bar{B}} Z_{A\bar{B} \leftarrow \bar{A} \bar{B}} P_{A\bar{B} \leftarrow \bar{A} \bar{B}} \geq 0$ implies $t_{B\bar{B} \leftarrow B\bar{B}} Z_{A\bar{B} \leftarrow \bar{A} \bar{B}} P_{A\bar{B} \leftarrow \bar{A} \bar{B}} \geq 0$. As shown in [11], a bipartite operation $Z_{A\bar{B} \leftarrow \bar{A} \bar{B}}$ is PPT-preserving if and only if its Choi matrix $Z_{A\bar{B} \leftarrow \bar{A} \bar{B}}$ is PPT, that is

$$t_{B\bar{B} \leftarrow B\bar{B}} Z_{A\bar{B} \leftarrow \bar{A} \bar{B}} \geq 0. \quad (6)$$

The PPT-preserving operations include all operations that can be implemented by local operations and arbitrary rounds of two-way classical communication (these are known as ‘LOCC’ operations). In fact, the PPT-preserving operations include even those implemented by local operations and arbitrary rounds of two-way communication over Horodecki channels. To see this, note that a Horodecki channel $\mathcal{F}_{A \rightarrow B}$ is a degenerate PPT-preserving bipartite operation where $\dim A' = \dim B = 1$, and the class of PPT-preserving operations is closed under composition.

A bipartite operation $Z_{A'B' \leftarrow AB}$ is non-signalling from Bob to Alice if $\text{Tr}_B Z_{A'B' \leftarrow AB} = Z_{A' \leftarrow A} \text{Tr}_B$ for some operation $Z_{A' \leftarrow A}$. That is, the marginal state of Alice’s output is given by some fixed operation applied to the marginal state of Alice’s input. The equivalent condition on the Choi matrix $Z_{A'B' \leftarrow AB}$ is

$$\text{Tr}_B Z_{A'B' \leftarrow AB} = Z_{A' \leftarrow A} 1_B. \quad (7)$$

where $Z_{A' \leftarrow A}$ is the Choi matrix for $Z_{A' \leftarrow A}$. As a Choi matrix, $Z_{A' \leftarrow A}$ must satisfy $Z_{A' \leftarrow A} Z_{A' \leftarrow A} = 1_A$, so [7] implies that $Z_{A' \leftarrow A} = \text{Tr}_{B'} Z_{A'B' \leftarrow AB} / \dim(B)$. Similarly, $Z_{A'B' \leftarrow AB}$ is non-signalling from Alice to Bob if

$$\text{Tr}_A Z_{A'B' \leftarrow AB} = Z_{B' \leftarrow B} 1_A, \quad (8)$$

where $Z_{B' \leftarrow B} = \text{Tr}_{A} Z_{A'B' \leftarrow AB} / \dim(A)$. These conditions are quantum generalizations of the classical non-signalling conditions on bipartite conditional probability distributions. One-way non-signalling operations have also been referred to as ‘semi-causal’ in the literature [16][17].

III. CLASSES OF QUANTUM CODES

In this section we define a very general class of codes, the forward-assisted codes, and then various code subclasses with operational or mathematical significance.
We represent the use of the noisy channel connecting Alice to Bob by an operation $N_{B\leftarrow A'}$. A forward-assisted code is one which has the form illustrated in Figure 1. The state to be transmitted by Alice resides on a system $A$ with $\dim(A) = K$. Alice performs an encoding map $E_{A'\leftarrow A}$ and sends the output systems through the noisy channel $N_{B\leftarrow A'}$ and some arbitrary side channel $F_{R\leftarrow Q}$. Then Bob applies a local decoding operation $D_{B'\leftarrow RB}$, where the system $B'$ has $\dim(B') = K$. This results in an overall operation $M_{B'\leftarrow A} = D_{B'\leftarrow RB}F_{R\leftarrow Q}N_{B\leftarrow A'}E_{A'\leftarrow A} \in \text{ops}(A \rightarrow B')$. We call the dimension $K$ the size of the code.

We note that codes for multiple channel uses which make use of some form of feedback between the uses (for example, codes assisted by two-way classical communication) do not necessarily fall into the class of forward-assisted codes.

Given two systems $Q$ and $Q'$ of equal dimension, the entanglement fidelity of a state $\sigma_{QQ'}$ is $\text{Tr}_{QQ'} \phi_{QQ'} \sigma_{QQ'}$. Given $M_{B'\leftarrow A} \in \text{ops}(A \rightarrow B')$ with $\dim(A) = \dim(B')$, we follow [18] in calling

$$F(M_{B'\leftarrow A}) = \text{Tr}_{Q'A} \phi_{Q'A} M_{B'\leftarrow A} \phi_{A'A}$$

the channel fidelity of $M_{B'\leftarrow A}$. When Alice’s input is half of a maximally entangled state $\phi_{A'A}$, the overall effect of the encoded transmission yields a state $\tau_{B'\hat{A}}$, as shown in the figure. The channel fidelity of $M_{B'\leftarrow A}$ is the entanglement fidelity of $\tau_{B'\hat{A}}$, and we call this the channel fidelity of the code.

The encoding procedure results in some average channel input state, which we will denote by $\rho_A := \text{Tr}_{Q'A} \hat{E}_{A'\leftarrow A} \phi_{A'A}$ (also shown in the figure).

Consider the bipartite operation

$$Z_{A'B'\leftarrow AB} := D_{B'\leftarrow RB}F_{R\leftarrow Q}E_{A'\leftarrow A}$$

which is outlined with dashes in Figure 1. Using (4), its Choi matrix $Z_{A'B'\rightarrow AB}$ satisfies

$$Z_{A'B'\rightarrow AB} = \text{Tr}_{QR} D_{B'\leftarrow RB} t_{R\leftarrow Q} F_{R\leftarrow Q} E_{A'\leftarrow A}.$$ (10)

Since this operation is implemented by local operations and one-way quantum communication from Alice to Bob

Such an operation is called “semilocalisable” in [16], it is non-signalling from Bob to Alice.

Such an operation is called “semicausal” in [16] [16]. Conversely, [17] shows that any bipartite operation which is non-signalling from Bob to Alice has an implementation by local operations and one-way quantum communication from Alice to Bob.

In [19], a deterministic supermap $\mathcal{M}$ is defined as a linear map from operations to operations, such that tensoring $\mathcal{M}$ with the identity supermap still takes operations to operations. In this language, the forward-assisted code depicted in Figure 1 constitutes a supermap from $\text{ops}(A' \rightarrow B)$ into $\text{ops}(A \rightarrow B')$

$$M_{B'\leftarrow A} \mapsto M_{B'\leftarrow A} = D_{B'\leftarrow RB}F_{R\leftarrow Q}N_{B\leftarrow A'}E_{A'\leftarrow A}.$$ (11)

In [19], it is shown that any deterministic supermap from $\text{ops}(A' \rightarrow B)$ to $\text{ops}(A \rightarrow B')$ can be implemented as in Figure 1 and eq. (11). By expressing the Choi matrix $M_{B'\rightarrow A}$ in terms of the Choi matrices of constituent operations using Eqs. (4)-(5) and then using Eq. (10), one finds that

$$M_{B'\rightarrow A} = \text{Tr}_{A'B} Z_{A'B'\rightarrow AB} N_{BA'}^T.$$ (12)

Therefore, the action of a forward-assisted code, as a deterministic supermap, is completely determined by the corresponding bipartite operation. In particular, its channel fidelity is

$$K^{-1} \text{Tr} \phi_{B'A} M_{B'\rightarrow A} = K^{-1} \text{Tr} \phi_{B'A} Z_{A'B'\rightarrow AB} N_{BA'}^T.$$ (12)

and its channel input state is

$$\rho_{A'} = \text{Tr}_{ABB'} Z_{A'B'\rightarrow AB} 1_B / \dim(A) \dim(B).$$ (13)

Thus the set of forward-assisted codes of size $K$ for the channel use $N_{B\leftarrow A'}$ corresponds precisely to the set of deterministic supermaps from $\text{ops}(A' \rightarrow B)$ to $\text{ops}(A \rightarrow B')$, where dim$(A) = \dim(B') = K$, and thus to the set of bipartite operations $\text{ops}(A : B \rightarrow A' : B')$ which are non-signalling from Bob to Alice.

While the preceding discussion shows that the class of forward-assisted codes is mathematically natural to define, the class is too powerful to be interesting – perfect performance is trivially achieved for any $K$ and $N_{B\leftarrow A'}$, by choosing $F_{R\leftarrow Q}$ to be a $K$ dimensional quantum identity channel and by using $F_{R\leftarrow Q}$ to transmit $A$ to Bob without even using $N_{B\leftarrow A'}$. We now define several more interesting subclasses of the forward-assisted codes, whose relationships are depicted in Figure 2.

The first three classes are operationally motivated - that is they place further constraints on the way in which
the code can be implemented. A conventional, unassisted quantum error correcting code corresponds to not allowing any forward assistance. Equivalently, the operation \( Z_{A'B'\rightarrow AB} \) must have the product form \( Z_{A'B'\rightarrow AB} = D_{B'\rightarrow B} E_{A'\rightarrow A} \). The operations \( D_{B'\rightarrow B} \) and \( E_{A'\rightarrow A} \) are still arbitrary. We call this subclass unassisted codes (UA).

The strictly larger class of entanglement-assisted codes (EA) corresponds to bipartite operations of the form \( Z_{A'B'\rightarrow AB} = D_{B'\rightarrow B} E_{A'\rightarrow A} \psi_{ab} \), where \( \psi_{ab} \) can be any shared entangled state of arbitrary systems \( a \) and \( b \). The class of forward-classical-assisted codes FCA, is the subclass of forward-assisted codes where we demand that the auxiliary channel \( F_{R\rightarrow Q} \) is classical. This means that \( F_{R\rightarrow Q} C_{Q\rightarrow Q} = F_{R\rightarrow Q} \) and \( C_{R\rightarrow R} F_{R\rightarrow Q} = F_{R\rightarrow Q} \), where \( C_{Q\rightarrow Q} \) denotes the completely dephasing operation in the classical basis on \( Q \).

While the unassisted codes, the entanglement-assisted codes, and the forward-classical-assisted codes possess clear operational interpretations, they are generally difficult to optimise over. Related classes that are more tractable to optimise are often studied instead.

For both entanglement-assisted codes and unassisted codes, the operation \( Z_{A'B'\rightarrow AB} \) is not only non-signalling from Bob to Alice, but also from Alice to Bob. We call the subclass of forward-assisted codes which is non-signalling from Alice to Bob the non-signalling codes (NS). The transmission of classical data using classical channels by non-signalling codes was first studied in [12]. In [12], the performance of non-signalling codes is used to provide a computationally tractable upper bound on non-assisted classical codes over classical channels. The upper bound is equivalent to a powerful bound obtained using different methods in [5].

Unassisted codes and forward-classical-assisted codes satisfy a tractable constraint that \( Z_{A'B'\rightarrow AB} \) is PPT-preserving. We denote the subclass of forward-assisted codes that are PPT-preserving “PPTp.” PPTp also contains forward-Horodecki-assisted codes FHA, consisting of forward-assisted codes where \( F_{R\rightarrow Q} \) is a Horodecki channel. Since classical channels are Horodecki, the class FHA contains FCA. We note that entanglement-assisted codes are generally not PPT-preserving. The relationships between the various classes of codes described above are summarised in Figure 2.

**Definition 1.** Let \( F^\Omega(N, K) \) denote the maximum channel fidelity \( F(M_{B'\rightarrow A}) \) of operations \( M_{B'\rightarrow A} \in \text{ops}(A \rightarrow B') \) with \( \dim A = \dim B' = K \) which can be obtained by applying a forward-assisted code in class \( \Omega \) to \( N_{B'\rightarrow A'} \).

We can now define, for any class of codes \( \Omega \), the asymptotic quantum capacity \( Q(\Omega(N)) \) of the memoryless channel whose operation for \( n \) channel uses is \( N^{\otimes n} \).

**Definition 2.**

\[
Q^\Omega(N) := \sup \{ r : \lim_{n \to \infty} F^\Omega(N^{\otimes n}, [2^r n]) = 1 \}. \quad (14)
\]

FIG. 2: The relationship between various subclasses of forward-assisted codes: PPT-preserving codes PPTp; forward-Horodecki-assisted codes FHA; forward-classical-assisted codes FCA; unassisted codes UA; entanglement-assisted codes EA; non-signalling codes NS.

We also define a corresponding zero-error capacity by

\[
Q^\Omega_0(N) := \sup_n \{ \frac{1}{n} \log_2 K_n : F^\Omega(N^{\otimes n}, K_n) = 1 \}. \quad (15)
\]

Given the results of [18], \( Q^\Omega_0(N) \) is equivalent to other definitions of the (unassisted) quantum capacity \( Q(N) \) of \( N \). No “single-letter” formula for this quantity is known. The best general expression we have for it is the regularised coherent information formula of the LSD Theorem [1–3]. \( Q^\Omega_0(N) \) is the entanglement-assisted capacity of \( N \) for which we have the single-letter formula of Bennett, Shor, Smolin and Thapliyal [21]:

\[
Q^\Omega_{EA}(N_{B'\rightarrow A'}) = \frac{1}{2} \max_{\rho_{RA'}} I(R : B)_{N_{B'\rightarrow A'}^{\rho_{RA'}}}. \quad (16)
\]

where \( \rho_{RA'} \) is a purification of \( \rho_{A'} \) and \( I(R : B)_{\sigma_{RB}} := S(\sigma_B) + S(\sigma_{RB}) - S(\sigma_{RB}) \), where \( S \) is the von Neumann entropy function.

The relationships between the classes of codes described in this section imply the following inequalities:

\[
F^{\Omega_0}(N, K) \leq F^{EA}(N, K) \leq F^{NS}(N, K), \quad (17)
\]

\[
F^{UA}(N, K) \leq F^{FCA}(N, K) \leq F^{FHA}(N, K) \leq F^{PPTp}(N, K), \quad (18)
\]

Similar inequalities hold for the corresponding assisted capacities.

In the next section, we show how the optimal channel fidelity of forward-assisted codes which are non-signalling, PPT-preserving, or both can be formulated as semidefinite programs (SDPs) [22, 23]. SDPs have a
number of attractive qualities: there are efficient algorithms for performing the optimising numerically; feasible points to the dual programs yield upper bounds on the optimal performance; in many cases of interest, strong duality holds, so that dual solutions can certify optimality.

IV. SEMIDEFINITE PROGRAMS FOR PPT-PRESERVING AND NON-SIGNALLING CODES

We have seen that the full set of forward-assisted codes of size $K$ for the channel operation $N_{B \rightarrow A}$ corresponds to those bipartite operations in $\text{ops}(AB \rightarrow A'B')$ which are non-signalling from Bob to Alice, where $\dim(A) = \dim(B') = K$. The corresponding set of Choi matrices are those satisfying

$$Z_{A'B'AB} \geq 0,$$

(20)

$$\text{Tr}_B' Z_{A'B'AB} = I_{AB},$$

(21)

$$\text{Tr}_B' Z_{A'B'AB} = \text{Tr}_B' Z_{A'B'AB} / \dim(B).$$

(22)

Here (20), (21) are equivalent to the operation being completely positive and trace preserving, respectively. The equality (22) is the constraint that the operation is non-signalling from Bob to Alice (see [7]). The code is non-signalling (see [8]) if and only if

$$\text{NS} : \text{Tr}_A' Z_{A'B'AB} = \text{Tr}_A' Z_{A'B'AB} / \dim(A),$$

(23)

and PPT-preserving (see [9]) if and only if

$$\text{PPTp} : t_{BB'\rightarrow BB'} Z_{A'B'AB} \geq 0.$$  

(24)

As noted earlier (eqn. [12]), the channel fidelity is given by

$$f_c = K^{-1} \text{Tr} \phi_{B'A} Z_{A'B'AB} N^T_{BA'}.$$  

(25)

The problem is to maximize $f_c$ subject to (20), (22), with the additional constraints (23), (24) as appropriate.

We begin by showing that we can, without loss of generality, restrict our attention to a highly symmetric form of $Z_{A'B'AB}$. Let $U$ denote the complex conjugate of $U$, and let $p$ denote the unique Haar probability measure on the unitary group $U(K)$. The channel fidelity eq. (25) satisfies

$$K^{-1} \text{Tr} \phi_{B'A} Z_{A'B'AB} N^T_{BA'}$$

$$= K^{-1} \text{Tr} \int dp(U) U'^T_{AB} U'^T_{A'B'} U_{B} U'_{A} Z_{A'B'AB} N^T_{BA'},$$

$$= K^{-1} \text{Tr} \phi_{B'A} Z_{A'B'AB} N^T_{BA'},$$

where

$$Z_{A'B'AB} := \int dp(U) U_{B} U'_{A} Z_{A'B'AB} U'^T_{AB} U'^T_{A'B'}.$$  

(26)

The first equality holds because $U'^T_{A'B'} |\phi_{B'A} = |\phi_{B'A}$ for all unitary operators $U$, by the ‘transpose trick’ (Eq. (2)). The second equality follows from the cyclic property and linearity of the trace. If we define the ‘twirling’ operation

$$T_{B'A \rightarrow B'A} : X_{B'A} \mapsto \int dp(U) U_{B'} U'^T_{A} X_{B'A} U'^T_{AB} U'^T_{A'B'},$$  

(27)

then $Z_{AB'B'AB} = \text{id}_{BB'AB} T_{B'A \rightarrow B'A} Z_{A'B'AB}$.

Consider a general Choi matrix $N_{BQ}$ given by Eq. (5). By the transpose trick, $W_{Q} N_{R} W_{Q}^{T}$ is the Choi matrix of the map that conjugates the input by $W^{T}$ before $N_{R+Q}$ acts. Meanwhile, $W_{R} N_{R} W_{Q}^{T}$ is the Choi matrix of the map that first applies $N_{R+Q}$ before conjugation by $W_{R}$. Therefore, the ‘twirled’ operator in (26) corresponds to the modified bipartite operation $\bar{Z}_{A'B'AB} [\cdot] = \int dp(U) U_{B'} Z_{A'B'AB} (U_{A} \cdot U_{B'}) U'^T_{B'A}$.

The operation $\bar{Z}_{A'B'AB}$ can be implemented as follows: Alice and Bob share a classical random variable identifying a unitary $U$ drawn according to the Haar measure $p$. Alice applies $U_{A}$ to her input system $A$. Alice and Bob then use the forward assisted code corresponding to $\bar{Z}$. Finally, Bob applies $U_{B'}$, inverting Alice’s operation on the input. Since $Z_{AB'B'AB}$ can be transformed to $\bar{Z}_{A'B'AB}$ using local operations and shared randomness, $\bar{Z}_{A'B'AB}$ will be non-signalling from Alice to Bob if $Z_{AB'B'AB}$ is, and will be PPT-preserving if $Z_{AB'B'AB}$ is.

Equation (26) tells us that, for any given $N_{B \rightarrow A'}$, using the $Z_{AB'B'AB}$ will yield the same channel fidelity as using $\bar{Z}_{A'B'AB}$. Therefore, there is no loss of generality in assuming that the Choi matrix lies in the image of the operation $\text{id}_{BB' \rightarrow BA} T_{B'A \rightarrow B'A}$.

As shown in Rains [11], the action of $T_{B'A \rightarrow B'A}$ can also be written

$$T_{B'A \rightarrow B'A} : X_{B'A} \mapsto \phi_{B'A} \text{Tr} \phi_{B'A} X_{B'A} +$$

$$\left(1_{B'A} - \phi_{B'A}\right) \text{Tr} \left(1_{B'A} - \phi_{B'A}\right) X_{B'A}.$$  

(28)

Thus, $\bar{Z}_{A'B'AB}$ lies in the image of $\text{id}_{BB' \rightarrow BA} T_{B'A \rightarrow B'A}$ if and only if

$$\bar{Z}_{A'B'AB} = K(\phi_{B'A} A_{AB} + (1 - \phi_{B'A}) A_{AB}),$$  

(29)

for some operators $A_{AB}$ and $\Gamma_{AB}$. When we write $A$, $\Gamma$ subscripted with only $A'$ or $B$, we refer to the partial traces of the operators, for example, $A_{A'} := T_{B'A} A_{AB}$. From [13], we see that the modified forward-assisted code [20] has channel input state

$$\rho_{\bar{N'}} = (K \dim(B))^{-1} \text{Tr}_{BA} \bar{Z}_{A'B'AB}$$

$$= (A_{A'} + (K^{2} - 1) \Gamma_{A'}) \dim(B)^{-1}.$$  

(30)

Expressing the constraints on $\bar{Z}$ in terms of $A_{AB}$ and $\rho_{\bar{N'}}$, gives the following theorem and corollary.

**Theorem 3.** There is a forward-assisted code (see Figure 1) of size $K$, average channel input $\rho_{\bar{N'}}$ and channel fidelity $f_c$ for $N_{B \rightarrow A'}$, which is PPT preserving and/or
obtains $H$ as constraint (32), and (33) is obtained by using (36) to
constraint (21), we obtain $\text{nel fidelity } (31)$. It follows by substituting (29) into (25)
the optimisation is a semidefinite program.

\[ F_{\text{BB}} = \frac{1}{K^2} \] (34)

\[ \text{PPTp} : \begin{cases} \mathbf{t}_{b'b-b'[A'B']} & \geq -\rho'\mathbb{1}_B/K, \\ \mathbf{t}_{b'b-b'[A'B']} & \leq \rho'\mathbb{1}_B/K. \end{cases} \] (35)

**Corollary 4.** To obtain $F_{\text{NS}}(N,K)$ or $F_{\text{PPTp}}(N,K)$ we maximise the expression (31) subject to either (34) or (35), as appropriate, in addition to the constraints (33), (32), $\rho' \geq 0$, and $\text{Tr} \rho_A = 1$. If we impose both (34) and (35) we obtain $F_{\text{NS/PPTp}}(N,K)$. In all three cases, the optimisation is a semidefinite program.

**Proof.** We begin by deriving the expression for the channel fidelity (31). It follows by substituting (29) into (25) and using (1-\(\phi\))\(_{B'A}\)\(\phi_{B'A} = 0\) and \(\phi_{B'A}\)\(\phi_{B'A} = \phi_{B'A}\).

We next consider the constraints (20)-(22). Using Eqs. (29) and (30), we see that (22) is equivalent to

\[ A_{b'b} + (K^2 - 1)\Gamma_{A'B} = \rho'\mathbb{1}_B. \] (36)

We will use this relation to eliminate $\Gamma_{A'B}$ in the other constraints. Substituting (29) into the ‘trace preserving’ constraint (21), we obtain

\[ A_{b'b} + (K^2 - 1)\Gamma_{A'B} = \mathbb{1}_B. \] (37)

Note that eq. (37) is already implied by (30).

Since (1-\(\phi\))\(_{B'A}\) and \(\phi_{B'A}\) are positive-semidefinite operators supported on orthogonal subspaces, $Z_{A'B'/A'B}$ in eq. (29) satisfies the complete positivity constraint (20) if and only if $A_{A'B} \geq 0$ and $\Gamma_{A'B} \geq 0$. The first of these is constraint (32), and (33) is obtained by using (36) to substitute for $\Gamma_{A'B}$ in the latter.

Now, if we want our forward-assisted code to be non-signalling from Alice to Bob (satisfying (23)) then, by eqs. (29) and (37), this is equivalent to

\[ K(\phi_{B'A} + A_{b'b} + (1 - \phi)\phi_{B'A}\Gamma_{A'B}) = K^{-1}\mathbb{1}_{A'B}. \] (38)

Eliminating $\Gamma_{A'B}$ using (37), the above holds if and only if $A_B = \mathbb{1}_B/K^2$, which is constraint (34) in our Theorem.

Finally, we can show that (29) is PPT-preserving (constraint (24)) if and only if conditions (35) hold, in a way similar to Rains [11]. To see this, apply $t_{b'b''-b''}$ to both sides of (29). Using the fact that $t_{b''-b'r'} = (S_{A'B} - \Lambda_{A'B})/K$ and $S_{A'B} + \Lambda_{A'B} = \mathbb{1}_{A'B}$, where $S_{A'B}$ and $\Lambda_{A'B}$ are the projectors onto the symmetric and the antisymmetric subspaces of $H_A \otimes H_B$, respectively, one obtains

\[ t_{b'b''-b''}[Z_{A'B'/A'B}] = S_{A'B}(t_{b''-b'[A'B'] + (K-1)\Gamma_{A'B}}) + \Lambda_{A'B}(t_{b''-b'[-A'B' + (K+1)\Gamma_{A'B}]}). \]

Using the fact that $S_{A'B}$ and $\Lambda_{A'B}$ are orthogonal projectors, this last expression is positive semidefinite if and only if $t_{b''-b'[A'B'] + (K-1)\Gamma_{A'B}} \geq 0$ and $t_{b''-b'[-A'B' + (K+1)\Gamma_{A'B}]} \geq 0$. Eliminating $\Gamma_{A'B}$ using (36) in these two conditions gives (35).

\[ \square \]

We now derive the dual semidefinite program for the entanglement fidelity achieved by a forward-assisted code that is PPT-preserving and/or non-signalling, using Lagrange multipliers. The weak duality theorem states that the value of the dual program attained at any feasible solution is at least the value of the primal program at any primal feasible solution. Interested readers can consult [22, 23].

**Proposition 5.** The dual semidefinite program for $F_{\text{NS/PPTp}}(N,K)$ is to minimise $\mu + K^{-2} \text{Tr} W_B$ subject to

\[ N_{A'B}^T + t_{b'b''-b''}[\mathbb{1}_{A'B}] \leq X_{A'B} + \mathbb{1}_A, \] (39)

\[ \text{Tr} B(X_{A'B} + K^{-1}\Omega_{A'B}) \leq \mu \mathbb{1}_A', \] (40)

\[ X_{A'B} \geq 0. \] (41)

To remove the PPT constraint, set $\Omega_{A'B} = 0$. To remove the non-signalling constraint, set $W_B = 0$.

**Proof.** We associate a positive-semidefinite Lagrange multiplier for each inequality constraint, and a hermitian Lagrange multiplier to each equality constraint. In particular, we associate the operator $X_{A'B} \geq 0$ to the constraint (32), a hermitian $W_B$ to non-signalling constraint (34), positive semidefinite $Y_{A'B}, V_{A'B}$ to the PPT-preserving constraints (35), and a real multiplier $\mu$ to the constraint that $\text{Tr} \rho_A = 1$. The resulting Lagrangian is

\[ \text{Tr} N_{A'B}^T A_{A'B} + \text{Tr} X_{A'B}(\rho_A - \mathbb{1}_A - A_{A'B}) \]

\[ + \text{Tr} Y_{A'B}(\mathbb{1}_A W_B - \mathbb{1}_A) + \text{Tr} V_{A'B}(\mathbb{1}_A W_B - \mathbb{1}_A) \]

\[ + \text{Tr} \mathbb{1}_A W_B(\text{dim}(A')^{-1} K^{-2} \mathbb{1}_{A'B} - Y_{A'B}) \]

\[ + \mu(1 - \text{Tr} \rho_A') \]

\[ = \text{Tr} L_{A'B}(N_{A'B}^T - X_{A'B} + t_{b'b''}[Y_{A'B} - V_{A'B}] - 1_{A'} W_B) + \text{Tr} \mathbb{1}_A W_B \]

\[ + \mu(1 - \text{Tr} \rho_A') \]

\[ = \text{Tr} A_{A'B}^T N_{A'B} + \text{Tr} A_{A'B}^T(\mathbb{1}_A + V_{A'B} - Y_{A'B}) \]

\[ + \text{Tr} \mathbb{1}_A W_B \mu. \]

The dual SDP is to minimise $\mu + K^{-2} \text{Tr} W_B$ subject to

\[ N_{A'B}^T + t_{b'b''}[Y_{A'B} - V_{A'B}] \leq X_{A'B} + \mathbb{1}_A, \] (42)

\[ \text{Tr} B(X_{A'B} + K^{-1}(Y_{A'B} + V_{A'B})) \leq \mu \mathbb{1}_A', \] (43)

\[ X_{A'B}, Y_{A'B}, V_{A'B} \geq 0. \] (44)

Let $\Omega_{A'B} := Y_{A'B} - V_{A'B}$, then $|\Omega_{A'B}| \leq Y_{A'B} + V_{A'B}$, and this can be made an equality by choosing $Y_{A'B} = (|\Omega_{A'B}| + \Omega_{A'B})/2$ and $V_{A'B} = (|\Omega_{A'B}| - \Omega_{A'B})/2$, without loss of generality.
Finally, to eliminate a constraint from the primal, we impose the additional constraint in the dual that the associated multiplier(s) be set to zero.

An easy consequence of the dual for PPT-preserving codes is that their performance over Horodecki channels is no better than their performance over completely use-

**Proposition 6.** The channel fidelity of a PPT-

preserving code for sending the state of a K-dimensional system over any Horodecki channel $\mathcal{N}^H$ is $1/K$ i.e. $F_{\text{PPT}}(\mathcal{N}^H, K) = 1/K$.

**Proof.** First, the channel fidelity $1/K$ is achieved trivially without even using the Horodecki channel, by choosing $\mathcal{E}_{A^NQ \rightarrow A}$ in Figure 1 to be a measurement in the computational basis, $Q$ to carry the measurement outcome, and $F_{R_{k} \rightarrow q}$ to be a noiseless classical channel of dimension $K$, and $P_{B^{i} \rightarrow RB}$ to be the identity operation.

Second, to see $1/K$ is also an upper bound for the channel fidelity, we exhibit a dual feasible solution whose value in the dual SDP is $1/K$: Since we do not have the Alice to Bob non-signalling constraint, we must set $W_{B} = 0$. For this $W_{B}$, constraint (29) is implied by (11) if we choose $\Omega_{A'B} = -t_{A' \rightarrow A}N_{A'B}$. Furthermore, since $t_{A' \rightarrow A}N_{A'B}$ is a Horodecki channel, $|\Omega_{A'B}| = t_{A' \rightarrow A}N_{A'B}$ and Tr$_{B}$ $|\Omega_{A'B}| = |A'$. Then, choosing $X_{A'B} = 0$ and $\mu = 1/K$ implies (40) and (11). Together, the above gives a dual feasible point with value $\mu = 1/K$.

\[ \square \]

**V. NON-SIGNALLING CODES**

In this section, we compare the performance of entanglement-assisted codes and non-signalling codes. Furthermore, we show that the entanglement-assisted classical capacity of any (memoryless) channel is equal to the non-signalling assisted classical capacity.

First, recall from [17] that our SDP for non-signalling codes in Corollary 3 provides an upper bound on the fidelity of entanglement-assisted codes:

\[ F^{\text{EA}}(\mathcal{N}, K) \leq F^{\text{NS}}(\mathcal{N}, K). \]

Now, given free entanglement, there is a one-to-one correspondence between the performance for transmitting quantum and classical data.

The success probability $P_s(\mathcal{M}_{B'^{i} \rightarrow A})$ of an operation $\mathcal{M}$ is a measure of its ability to send classical data encoded in the computational basis:

\[ P_s(\mathcal{M}_{B'^{i} \rightarrow A}) = \frac{1}{K} \sum_{k=0}^{K-1} \text{Tr}_{B'k}|k\rangle\langle k|\mathcal{M}_{B'^{i} \rightarrow A}|k\rangle\langle k|. \]

**Definition 7.** Let $P^{\text{EA}}(\mathcal{N}, K)$ denote the maximum success probability $P_s(\mathcal{M}_{B'^{i} \rightarrow A})$ of operations $\mathcal{M}_{B'^{i} \rightarrow A} \in \text{ops}(A \rightarrow B')$ with dim $A = \text{dim} B' = K$ which can be obtained by applying a forward-assisted code in class $\Omega$ to $\mathcal{N}_{B'^{i} \rightarrow A'}$.

The correspondence between the performance for transmitting quantum and classical data in the presence of free entanglement is due to superdense coding [24] and teleportation [25]. Using the superdense coding protocol, a $K$-dimensional quantum code of channel fidelity $f$ can be turned to a protocol for sending one out of $K^2$ equiprobable messages with success probability $f$. The reverse holds by means of the teleportation protocol. (See Appendix B for details.) Therefore, for any subclass $\Omega$ of forward-assisted codes that includes the entanglement-assisted codes (that is, $\mathcal{E} \subseteq \Omega$) we have

\[ F^{\text{EA}}(\mathcal{N}, K) = P^{\text{EA}}(\mathcal{N}, K^2). \]

For example, this equation holds for the class NS of non-signalling codes.

We can use the correspondence to obtain an upper bound on $F^{\text{EA}}(\mathcal{N}, K)$ using the results in [10]. There, $P^{\text{EA}}(\mathcal{N}, K)$ is upper bounded by the solution of a semidefinite program which we call $B(\mathcal{N}, K)$. By [46], $B(\mathcal{N}, K)$ provides an SDP upper bound on $F^{\text{EA}}(\mathcal{N}, K)$:

\[ F^{\text{EA}}(\mathcal{N}, K) = \frac{\mu}{K} \leq B(\mathcal{N}, K^2). \]

We now compare the bound (47) due to [10] and our current bound (45) due to Theorem 3. The SDP for $B(\mathcal{N}, K)$ is simply given by relaxing the constraint (34) in the SDP for $F^{\text{NS}}(\mathcal{N}, K)$ to an inequality. Therefore,

\[ F^{\text{NS}}(\mathcal{N}, K) \leq B(\mathcal{N}, K^2). \]

So the expression for $F^{\text{NS}}(\mathcal{N}, K)$ given by Theorem 3 gives an upper bound for entanglement-assisted codes at least as good as $B(\mathcal{N}, K^2)$ (though we do not know if $F^{\text{NS}}(\mathcal{N}, K)$ is strictly better than $B(\mathcal{N}, K^2)$). Furthermore, $F^{\text{NS}}(\mathcal{N}, K)$ is a stronger bound since it applies to the larger class of non-signalling codes.

Regarding the asymptotic performance of non-signalling codes, it is clear that they yield a quantum capacity which is at least as large as the entanglement-assisted capacity.

We now argue that, for memoryless channels, the (asymptotic) capacities for non-signalling codes and entanglement-assisted codes are, in fact, equal. That is,

\[ Q^{\text{EA}}(\mathcal{N}) = Q^{\text{NS}}(\mathcal{N}). \]

Clearly, non-signalling codes yield a quantum capacity no less than the entanglement-assisted capacity. To see the reverse, we start with a result in [10] showing that an asymptotic analysis of $B(\mathcal{N}^\otimes n, K^2 n)$ recovers the single-letter formula (16) as an upper bound on $Q^{\text{EA}}(\mathcal{N})$. From this result and the inequality (48), it follows that (16) is an upper bound even on $Q^{\text{NS}}(\mathcal{N})$. Therefore the entanglement-assisted capacity of a memoryless quantum channel is equal to the quantum capacity attained by non-signalling codes.
VI. PPT PRESERVING CODES AND DISTILLATION PROTOCOLS

The main result of this section, Prop. 8 relates PPT-preserving codes and PPT-preserving entanglement distillation scheme studied in by Rains in [11]. We will use it later to obtain the values of $Q^{PPT}$ and $Q^{PPT}_0$ for the $d$-dimensional Werner-Holevo channel.

In [11], Rains considers entanglement distillation by PPT-preserving operations. He studies the quantity

$$F_t(\rho_{AB}, K) := \max \{ \text{Tr} \phi_{AB} \gamma_{AB' \rightarrow \tilde{A} \tilde{B}} \rho_{AB} : \gamma_{AB' \rightarrow \tilde{A} \tilde{B}} \text{ is PPT-preserving}, \dim \tilde{A} = \dim \tilde{B}' = K \}$$

which is the optimal entanglement fidelity of $K \times K$ states that can be obtained from $\rho_{AB}$ by PPT-preserving operations. (We use these system labels to be consistent with those used later in this section.) He also defines an associated asymptotic rate of distillation

$$D_t(\rho_{AB}) := \sup \{ r : \lim_{n \to \infty} F_t(\rho_{AB} \otimes^n, [2^r]) = 1 \}. \quad (51)$$

In the following, we borrow ideas from [26] relating error correcting codes and entanglement distillation, to relate PPT-preserving distillation of the Choi state of $\mathcal{N}_{B \rightarrow A'}$ to the channel fidelity of PPT-preserving codes over $\mathcal{N}_{B \rightarrow A'}$.

**Proposition 8.** For any channel $\mathcal{N}_{B \rightarrow A'}$, let

$$\nu_{BA'} := \mathcal{N}_{B \rightarrow A'} \phi_{A'\tilde{A} \tilde{A}A} = \text{id}_{A' \tilde{A} \tilde{A}A} \mathcal{N}_{BA'}/\dim(A') \quad (52)$$

denote its Choi state.

(i) If a PPT-preserving operation can distill a $K \times K$ state from $\nu_{BA'}$ with entanglement fidelity $f$, then there is a PPT-preserving code of size $K$ and channel fidelity $f$ for $\mathcal{N}_{B \rightarrow A'}$. Therefore, $F^{PPT}(\mathcal{N}, K) \geq F_t(\nu_{BA'}, K)$ and $Q^{PPT}(\mathcal{N}) \geq D_t(\nu_{BA'})$.

(ii) If $\mathcal{N}_{B \rightarrow A}$ can be implemented exactly using a single copy of its Choi state $\nu_{BA}$ and forward classical communication, then the converse to (i) is also true, and therefore $F^{PPT}(\mathcal{N}, K) = F_t(\nu_{BA'}), K)$ and $Q^{PPT}(\mathcal{N}) = D_t(\nu_{BA'})$.

If the condition for (ii) holds, Rains’ SDP for the PPT fidelity for $\mathcal{N}_{B \rightarrow A'}$ yields a special case of Theorem 3.

**Proof.** (i) Suppose that there is a PPT preserving distillation operation $\gamma_{AB' \rightarrow \tilde{A} \tilde{B}}$ which takes the Choi state $\nu_{BA'}$ to a state with entanglement fidelity $f$. As noted by Rains, this fidelity is unchanged if $\gamma_{AB' \rightarrow \tilde{A} \tilde{B}}$ is followed by the twirling operation $\mathcal{T}_{AB' \rightarrow \tilde{A} \tilde{B}}$ with a definition similar to that in (27). So, the operation $\mathcal{T}_{AB' \rightarrow \tilde{A} \tilde{B}} \gamma_{AB' \rightarrow \tilde{A} \tilde{B}}$ has the same fidelity for input $\nu_{B \tilde{A}}$, and remains PPT preserving, but is also non-signalling in both directions. This is simply because the marginal state of each party’s system after twirling is always a maximally mixed state, independent of the input. Altogether, without loss of generality, $\gamma_{AB' \rightarrow \tilde{A} \tilde{B}}$ can be chosen to be non-signalling in both directions.

**FIG. 3:** Building a PPT-preserving code (the operations in the dotted box) based on a PPT-preserving distillation protocol (the dark grey operations) and on quantum communication from Alice to Bob (represented by the dashed line). We now construct a PPT-preserving code of dimension $K$ that is non-signalling from Bob to Alice using $\mathcal{Y}_{AB' \rightarrow \tilde{A} \tilde{B}}$. Conceptually, the construction is the composition of three operations. First, Alice locally prepares the state $\phi_{A' \tilde{A}}$, and sends $A'$ to Bob using $\mathcal{N}_{B \rightarrow A}$, so they share the Choi state $\mathcal{N}_{B \rightarrow A} \phi_{A' \tilde{A}}$. Second, they apply $\mathcal{Y}_{AB' \rightarrow \tilde{A} \tilde{B}}$ to distill a state $\psi_{AB'}$ with channel fidelity $f$. Finally, Alice teleports a $K$-dimensional system from $A$ to $B'$ using $\psi_{AB'}$ instead of $\phi_{A' \tilde{A}}$. The teleportation has channel fidelity $f$.

These three steps are shown in Fig. 3. Since $\mathcal{Y}_{AB' \rightarrow \tilde{A} \tilde{B}}$ is non-signalling from Bob to Alice, it can be implemented by local operations (the grey boxes in Fig. 3) and quantum communication from Alice to Bob (represented by the dashed line in Fig. 3). This is significant, because it means that Alice can complete all of her local operations before Bob starts his. The teleportation procedure consists of Alice’s local measurement $\mathcal{M}_{C \rightarrow \tilde{A} \tilde{A}}$, forward classical communication of system $C$, and Bob’s locally controlled unitary $\mathcal{U}_{B' \rightarrow BC}$.

The PPT-preserving code is derived from Fig. 3 with the encoder (decoder) being all of Alice’s (Bob’s) local operations combined, and the forward side channel being the communication of $C$ combined with the forward channel in $\mathcal{Y}_{AB' \rightarrow \tilde{A} \tilde{B}}$. Used with $\mathcal{N}_{B \rightarrow A}$, the code effects the same transmission from $A$ to $B'$ as in the conceptual composition described earlier. The forward-assisted code has size $K$ and bipartite operation $\mathcal{Z}_{AB' \rightarrow \tilde{A} \tilde{B}} = \mathcal{U}_{B' \rightarrow BC} \mathcal{M}_{C \rightarrow \tilde{A} \tilde{A}} \mathcal{Y}_{AB' \rightarrow \tilde{A} \tilde{B}}$. Since $\mathcal{Z}_{AB' \rightarrow \tilde{A} \tilde{B}}$ is the composition of the PPT-preserving $\mathcal{Y}_{AB' \rightarrow \tilde{A} \tilde{B}}$ and the (one-way) LOCC operation $\mathcal{U}_{B' \rightarrow BC} \mathcal{M}_{C \rightarrow \tilde{A} \tilde{A}}$, the code is PPT-preserving.

For part (ii), suppose the channel $\mathcal{N}_{B \rightarrow A'}$ can be simulated exactly using a shared copy of its Choi state and forward classical communication. Referring to Figure 4 this means that $\nu_{B \rightarrow BC} \mathcal{M}_{C \rightarrow \tilde{A} \tilde{A}} \nu_{B \tilde{A}} = \mathcal{N}_{B \rightarrow A'}$. Let $\mathcal{Z}_{AB' \rightarrow \tilde{A} \tilde{B}}$ be the bipartite operation corresponding to a forward-assisted code which, transmits a $K$-dimensional state over $\mathcal{N}_{B \rightarrow A'}$ with channel fidelity $f$. If one composes
the channel simulation with $Z_{AB'}\rightarrow AB$ as in figure 4 the operations in the dashed box distills the Choi state with fidelity $f$. Furthermore if $Z_{AB'}\rightarrow AB$ is PPT-preserving and non-signalling from Bob to Alice, so is the distillation operation.

\[ \nu_{BA'} = \mathcal{N}_{BA'} \phi_{AB'} \]

(as shown in Figure 4) where $\mathcal{N}_{BA'} := \mathcal{N}_{BA'} \phi_{AB'}$. Suppose that we choose the measurement operation $\mathcal{M}_{BC} \rightarrow AB'$ and a controlled unitary operation $\mathcal{U}_{AB'} \rightarrow AB'$ so that they comprise a teleportation protocol, such that

\[ \mathcal{U}_{AB'} \mathcal{M}_{BC} \mathcal{U}_{AB'}^\dagger = \phi_{AB'} \]

Here, $\mathcal{U}_{AB'} \rightarrow AB'$ measures system C in the computational basis, obtaining an outcome $i$, and then applies a unitary transformation $\mathcal{U}_{AB'}^{(i)}$ to system $A'$.

Now, suppose that there are unitary operations $\mathcal{V}_{AB}$ for each $i$ such that

\[ \forall i : \mathcal{N}_{BA'} \mathcal{U}_{AB'}^{(i)} = \mathcal{V}_{AB} \mathcal{N}_{BA'} \]

Let $\mathcal{V}_{BC}$ be a controlled unitary which measures $C$ in the computational basis, and applies $\mathcal{V}_{BC}$ on obtaining outcome $i$. Then, using (55) and (54),

\[ \mathcal{V}_{BC} \mathcal{M}_{BC} \mathcal{U}_{AB'} \mathcal{V}_{BC}^\dagger = \mathcal{V}_{BC} \mathcal{M}_{BC} \mathcal{U}_{AB'} \mathcal{V}_{BC}^\dagger = \mathcal{N}_{BA'} \mathcal{U}_{AB'} \mathcal{M}_{BC} \mathcal{U}_{AB'}^\dagger = \mathcal{N}_{BA'} \mathcal{U}_{AB'} \mathcal{V}_{BC}^\dagger \mathcal{U}_{AB'} \mathcal{V}_{BC} \mathcal{N}_{BA'} \]

That is, a use of $\mathcal{N}_{BA'}$ can be implemented by a single copy of its Choi state $\nu_{BA'}$, local operations and forward classical communication.

VII. CODING OVER GENERALISED WERNER-HOLEVO CHANNELS

In this section, we apply the SDPs developed in section IV to investigate the performance of codes which are non-signalling, PPT-preserving or both, over the generalised Werner-Holevo channels [27]. For each dimension $d \geq 2$, consider the one-parameter family of channels

\[ \mathcal{W}_{BA'}^{(d,\alpha)} := (1-\alpha)\mathcal{W}_{BA'}^{(d,0)} + \alpha \mathcal{W}_{BA'}^{(d,1)} \]

where

\[ \mathcal{W}_{BA'}^{(d,1)} : X_{A'} \mapsto \frac{1}{d-1} (1_B \mathrm{Tr} X - \mathrm{id}_B |X_{A'}^T), \quad \text{and} \]

\[ \mathcal{W}_{BA'}^{(d,0)} : X_{A'} \mapsto \frac{1}{d+1} (1_B \mathrm{Tr} X + \mathrm{id}_B |X_{A'}^T). \]

$\mathcal{W}_{BA'}^{(d,1)}$ is often called the d-dimensional Werner-Holevo channel. Recall that $\mathcal{S}_{BA'}$ and $\mathcal{A}_{BA'}$ denote the projectors onto the symmetric and the antisymmetric subspaces of $\mathcal{H}_B \otimes \mathcal{H}_A'$, respectively. The Choi matrices of $\mathcal{W}_{BA'}^{(d,1)}$ and $\mathcal{W}_{BA'}^{(d,0)}$ are proportional to $\mathcal{A}_{BA'}$ and $\mathcal{S}_{BA'}$, respectively.

The three-dimensional Werner-Holevo channel $\mathcal{W}_{BA'}^{(3,1)}$ has a Stinespring representation

\[ \mathcal{W}_{BA'}^{(3,1)} : X_{A'} \mapsto \mathrm{Tr}_E V X_{A'} V^\dagger, \]

\[ V := 2^{-1/2} \sum_{i,j,k=1}^3 \varepsilon_{ijk} |j\rangle_B |k\rangle_E \langle i | \]

where $\varepsilon_{ijk}$ is the three-dimensional Levi-Civita symbol, which is 1 when $ijk$ is an even permutation of 123, −1 when $ijk$ is an odd permutation of 123 and 0 otherwise. From (60), we see that $\mathcal{W}_{BA'}^{(3,1)}$ is symmetric, meaning that

\[ \mathrm{Tr}_E V X_{A'} V^\dagger = \mathrm{id}_{B-E} \mathrm{Tr}_B V X_{A'} V^\dagger. \]

Therefore $\mathcal{W}_{BA'}^{(3,1)}$ is anti-degradable and hence has no unassisted quantum capacity; i.e. $Q^{U_A}(\mathcal{W}_{BA'}^{(3,1)}) = 0$.

The quantum Lovász bound of Duan, Severini and Winter [28] is easily applied to this channel to establish that it has no zero-error classical capacity, even with arbitrary entanglement assistance.

By its definition, the generalised Werner-Holevo channels have the covariance property that, for all unitary operations $\mathcal{U}_{A' \rightarrow A'} : X_{A'} \mapsto U_{A'} X_{A'} U_{A'}^\dagger$ (where $U_{A'}$ is a unitary operator on $\mathcal{H}_{A'}$), we have

\[ \mathcal{W}_{BA'}^{(d,\alpha)} \mathcal{U}_{A' \rightarrow A'} = \mathcal{W}_{BA'}^{(d,\alpha)} \mathcal{U}_{BA'}^{(d,\alpha)} \]

where $\mathcal{V}_{BA} : X_B \mapsto U_B^\dagger X_B U_B$ and $U_B := \mathrm{id}_B \mathcal{U}_{A'}$. By the argument at the end of section VI, $n$ uses of $\mathcal{W}_{BA'}^{(d,\alpha)}$ can be exactly simulated using $n$ copies of the corresponding Choi state and forward classical communication by teleportation. Therefore, by Proposition 8 the
performance of PPT-preservation codes over these channels corresponds exactly to the performance of PPT-preserving distillation protocols on the corresponding Choi states studied by Rains [1].

Corollary 5.6 of Rains [11] shows that PPT-preserving operations can distill entanglement from multiple copies of $W_{B \rightarrow A'}^{(d,1)}$ at an optimal rate of $\log((d + 2)/d)$ ebits per state, asymptotically. Furthermore, this rate is achieved for exact distillation. Thus the quantum capacity and the zero-error quantum capacity of PPT-preserving codes over $W_{B \rightarrow A'}^{(d,1)}$ are both $\log(d + 2)/d$.

$$Q^{\text{PPTp}}(W^{(d,1)}) = Q_0^{\text{PPTp}}(W^{(d,1)}) = \log \frac{d + 2}{d}. \quad (63)$$

On the other hand, using the result of Section V and Eq. (16) one finds

$$Q^{\text{NS}}(W^{(d,1)}) = Q^{\text{EA}}(W^{(d,1)}) = \frac{1}{2} \log \frac{2d}{d - 1}. \quad (64)$$

A. Performance of non-signalling, PPT-preserving codes for $10 \leq n \leq 120$ with fixed rates.

The generalised Werner-Holevo channel has high degree of symmetry. We exploit this symmetry to reduce the semidefinite programs described in Theorem 3 and Corollary 4 to linear programs in $n + 1$ (real) variables, for $n$ uses of the generalised Werner-Holevo channel. (See Appendix A) The resulting linear programs can be stated using rational numbers, and we have evaluated their solutions exactly using Mathematica’s ‘LinearProgramming’ function.

In Figure 5 we plot the log of the fidelity $F^{\text{PPTp}}(W^{(d,1)} \otimes n, [2^{nr}])$ as a function of block-length $n$ for the two rates $r \in \{\log(5/2 - 1/40), \log(5/2 - 1/20)\}$. While the fidelity eventually goes to one at rate $\log(5/2 - 1/40)$, it appears to exhibit an exponential decay at rate log($5/2 - 1/40$).

From Eq. (63), at either rate studied above, the fidelity for PPT-preserving codes is 1. Thus our non-signalling and PPT-preserving codes provide strictly tighter bound for the unassisted performance for strictly block-length.

If the code fidelity $F^{\text{PPTp}}(W^{(d,1)} \otimes n, [2^{nr}])$ at rate $\log(5/2 - 1/40)$ does not eventually increase and approach 1 as $n$ increase, then $Q^{\text{PPTp}}(W^{(d,1)})$ is no more than $\log(5/2 - 1/40)$, which is strictly less than both $Q^{\text{PPTp}}(W^{(d,1)}) = \log(5/2)$ and $Q^{\text{NS}}(W^{(d,1)}) = \frac{1}{2} \log 3$.

Deciding whether there really can be a separation between the asymptotic capacities $Q^{\text{PPTp}}$ and $Q^{\text{PPTp}\cap\text{NS}}$ presents an interesting open problem.

B. Performance of non-signalling, PPT-preserving codes for $n = 2$ with variable rate.

In Figure 6 we plot the channel code fidelities when the channel operation is two uses of the three dimensional Werner-Holevo channel. We consider codes that are non-signalling, PPT-preserving, or both.

C. A bonus observation – superactivation?

Consider the specific data point $K = 2$ in Figure 6. All three curves coincide at this point and have value 1. Thus
zero-error quantum communication of a qubit is possible over the channel even if we demand that the code be both non-signalling and PPT-preserving. While Rain’s work already implies the possibility given PPT-preserving codes, it is somewhat surprising that one can further restrict to non-signalling codes.

**Example 9.** There is a PPT-preserving, non-signalling code which can transmit a qubit with perfect entanglement fidelity over two uses of the three dimensional Werner-Holevo channel: The channel input system is $A' = A_1' A_2'$ and the channel output system is $B = B_1 B_2$, and the channel operation is $W^{(3,1)}_{B_1+ A_1'} W^{(3,1)}_{B_2+ A_2'}$. The code is given by taking the maximally mixed average channel input $\rho_{A'} = 1_{A'}/\dim(A')$ and choosing $\Lambda_{A'B} = \frac{1}{\dim(A') \dim(B')}$ $S_{A_1'B_1} S_{A_2'B_2} + \frac{7}{32} (S_{A_1'B_1} A_{A_2'} B_2 + A_{A_1'} B_1 S_{A_2'} B_2)$ + $A_{A_1'} B_1 A_{A_2'} B_2$ in the expressions (36) and (29).

As discussed in Section III, PPT-preserving codes include codes assisted by arbitrary forward communication over Horodecki channels. Therefore, the results of Smith and Yard on superactivation \cite{13} mean that such codes may yield quantum capacity over symmetric channels. Nevertheless, we were somewhat surprised to find that a PPT-preserving non-signalling code allows the perfection of a single qubit over two uses of a simple example of a symmetric channel.

Since the code operation is non-signalling from Bob to Alice, it can be implemented by forward quantum communication from Alice to Bob. This is the result of Eggeling and Schlingemann and Werner \cite{17}, that “semicausal operations are semilocally realizable”. The use of this forward quantum communication is somehow “hidden” by the local operations performed by Alice and Bob in the implementation so that the resulting bipartite operation is both PPT and non-signalling.

Given the result of Eggeling et al., it might be tempting to guess that a bipartite operation which is non-signalling from Bob to Alice and PPT-preserving, like the forward-assisted code in Example 3 (which is also non-signalling from Alice to Bob), can always be implemented by forward communication over a Horodecki channel. If this were possible for our Example 3 or for some other PPT-preserving code enabling zero-error quantum communication over a channel without quantum capacity then it would constitute a remarkably extreme version of the superactivation phenomenon discovered by Smith and Yard \cite{13}. We leave this question open here. However we can give an example which shows that this kind of implementation is not always possible, even when the bipartite operation is non-signalling in both directions.

The example is a bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ with $\dim A = \dim A' = \dim B = \dim B' = 2$, which we will describe by giving a particular protocol to implement the operation, which is illustrated in the top half of Figure 7. Bob measures his input $B$ in the computational basis and sends the outcome $b$ to Alice. He also generates an unbiased random bit $r$, which he sends to Alice and outputs on $B'$ in the computational basis. If $b = 0$ Alice does nothing, but if $b = 1$ she applies a Hadamard gate to $A$. Then, regardless of the value of $b$, she measures $A$ in the computational basis yielding outcome $a$. She outputs $a \oplus r$ on $A'$ in the computational basis.

Since the operation can be implemented using only classical communication from Bob to Alice, it is certainly a PPT-preserving measurement. The marginal states of $A'$ and $B'$ are both maximally mixed states, independent of the input state, so the operation is non-signalling in both directions.

However, in the implementation just described the communication was in the “backward” direction - from Bob to Alice. We claim that implementing the operation by forward communication only, requires at least one qubit of zero-error quantum communication, which clearly cannot be accomplished by any Horodecki channel. Here is a proof. The most general implementation with only forward communication has the form $\mathcal{Z}_{A'B'\leftarrow AB} = \mathcal{D}_{B'\leftarrow BR} \mathcal{F}_{R\leftarrow Q} \mathcal{E}_{A'\leftarrow A}$ where $\mathcal{E}_{A'\leftarrow A}$ is Alice’s local operation, $\mathcal{F}_{R\leftarrow Q}$ is the channel used for forward communication, $\mathcal{D}_{B'\leftarrow BR}$ is Bob’s local operation. We illustrate this in the bottom half of Figure 7. Now, if Alice sends her bit $a \oplus r$ to Bob with one use of a forward completely dephasing channel $\mathcal{C}_{C\leftarrow A'}$, then Bob can XOR $a \oplus r$ with $r$ to obtain the outcome of Alice’s measurement of the $A$ system. Therefore, by measurement of the output of the operation $\mathcal{G}_{CR\leftarrow A} := \mathcal{C}_{C\leftarrow A'} \mathcal{F}_{R\leftarrow Q} \mathcal{E}_{A'\leftarrow A}$ (outlined by the dotted line in Figure 7), Bob can choose to discriminate perfectly between $|0\rangle_A$ and $|1\rangle_A$ or between $|+\rangle_A$ and $-\rangle_A$ depending on his input. It must therefore be that $\text{Tr}_{CR} (\mathcal{G}_{CR\leftarrow A} |0\rangle_A \langle 0|) (\mathcal{G}_{CR\leftarrow A} |1\rangle_A \langle 1|) = 0,$ $\text{Tr}_{CR} (\mathcal{G}_{CR\leftarrow A} |+\rangle_A \langle +|) (\mathcal{G}_{CR\leftarrow A} |-\rangle_A \langle -|) = 0.$

By Lemma 1 of Cubitt and Smith \cite{29}, this implies that $\mathcal{G}$ is capable of sending a single qubit perfectly. Since the
forward classical communication over $C$ cannot increase the zero-error quantum capacity of $F$, it must be that $F$ itself can send a single qubit perfectly. Clearly, no Horodecki channel can do this.

VIII. CONCLUSION

We have shown how a number of operationally relevant classes of quantum code (such as unassisted codes, entanglement-assisted codes, codes assisted by forward classical communication) can be regarded as sub-classes of the forward-assisted codes, which correspond to deterministic supermaps or, equivalently, to bipartite operations which are non-signalling from Bob to Alice. By requiring additionally that these operations are PPT-preserving, non-signalling (from Alice to Bob), or both, we obtain non-trivial bounds on the performance of the operationally defined classes of codes, in the form of simple semidefinite programs.

The SDP for non-signalling codes gives an upper bound on entanglement-assisted codes which is at least as tight as the one given in [10], and we use this fact to show that the capacity of entanglement-assisted and non-signalling codes is the same for memoryless channels. It would be interesting to find out if the SDP for non-signalling codes is strictly better than the bound in [10].

In the case of codes which are PPT-preserving, we described how these are related to the PPT-preserving entanglement distillation protocols studied by Rains. This gave us a general lower bound on the PPT-preserving code performance and an equality between code performance and distillation fidelity of the Choi state for some special channels. This equality let us use Rains’ results to obtain the PPT-preserving code capacities (even the zero-error capacities) for the $d$-dimensional Werner-Holevo channels. In regarding the conditions for equality, we would be interested to know if the complete characterisation of when a channel can be implemented exactly using a single copy of its Choi state and forward classical communication.

By imposing both non-signalling and PPT-preserving constraints we obtain bounds on the fidelity of unassisted quantum codes. Again using the example of Werner-Holevo channels, we show that this provides a strictly lower bound on the finite block length regime. Numerics suggest that it can may even be stronger asymptotically. It would be interesting to find out whether this is indeed the case; for example, to show a separation of the capacities, it suffices to find feasible solutions for the dual programs for an infinite sequence of block lengths, which yield an upper bound for which an asymptotic separation can be proven.

Even with both constraints on the code we find that zero-error communication of a qubit is possible given two uses of the three-dimensional Werner-Holevo channel. It is not clear to us whether assistance by forward communication over Horodecki channels would allow the same phenomenon via “superactivation” in the sense of Smith and Yard. We have given an example showing that not all non-signalling, PPT-preserving bipartite operations can be implemented by forward communication over Horodecki channels, but this does not settle the question. It would be of interest to do so.

One potential application of our SDP concerns bounds on regular quantum error correcting codes. Consider a family of channels parameterized by some error strength $\epsilon$. For each block length $n$ and codespace dimension $K$, our SDP can be used to evaluate the code fidelity as a function of $\epsilon$. The existence of an unassisted code that corrects for $t$ errors will imply an assisted code fidelity that is at least $1 - O(\epsilon^{t+1})$. Thus, a numerically obtained fidelity worse than $1 - O(\epsilon^{t+1})$ can be viewed as evidence for the non-existence of such unassisted codes.

Finally (as mentioned above) we have determined that $Q^{\text{NS}}$ is equal to $Q^{\text{EA}}$ for which there is a single-letter formula due to [21], but do any of the capacities $Q^{\text{FHA}}$, $Q^{\text{NSPPT}}$ or $Q^{\text{PPTP}}$ have a single-letter formula?

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Appendix A: Linear program for generalised Werner-Holevo channels

We consider $n$ uses of the generalised Werner-Holevo channel [57]. The input system is $A' = A_1' \cdots A_n'$ and the output system is $B = B_1 \cdots B_n$, where $\dim(A_i') = \dim(B_j) = d$. The Choi matrix of the operation is $d^n w(d, \alpha)^{\otimes n}$ where $w(d, \alpha)_{AB}$ is the Werner state $w(d, \alpha)_{AB} = (1 - \alpha)S_{AB}/(\text{Tr} S) + \alpha A' \otimes B/\text{Tr} A'$.

As such, the Choi matrix is invariant under conjugation by $U_A, U_B$, for all unitaries $U$ and $j \in \{1, \ldots, n\}$, and invariant under permutations. Therefore, in the semidefinite program, there is no loss of generality in assuming that the operator $\Lambda_{A'B}$ possesses the same invariance, and that $\rho_{A'}$ is invariant under the restriction of these actions to the input subsystems. Since this means that $\rho_{A'}$ is invariant under an arbitrary unitary transformation of any one of the $n$ input subsystems, $\rho_{A'}$ can only be the maximally mixed state $\rho_{A'} = 1_{A'}/d^n$. As for $\Lambda_{A'B}$, it must be a linear combination of $n+1$ orthogonal projectors

$$\Lambda_{A'B} = \sum_{k=0}^n x_k E^n_k \quad (A1)$$

where $E^n_k$ is the sum of all $n$-fold tensor products of the operators $S$ and $A$ which contain exactly $k$ copies of $A$ (see Example [9] for an example of an $\Lambda_{A'B}$ of this form for $n = 2$). The partial transpose of such an operator is itself given by a sum of orthogonal projectors. Let
The non-signalling constraint, the fact that $\text{Tr}_B \phi$ which contain exactly $k$ copies of $\phi$ e.g. $\phi^2 = (1 - \phi)_{A_1'B_1B_2} + \phi_{A_1B_1}(1 - \phi)_{A_2'B_2}$. Then

$$t_{B\to A'B} = \sum_{i,j=0}^{n} T_{ij}^M x_j,$$

(A2)

where

$$M_{ij}^M := 2^{-n} \min_{i,j} \left( \frac{n-i}{j-k} \right) \frac{k}{(1+d)^{j-1} (1-d)^k}. $$

(A3)

See [10] for the derivation of this formula for $M_{ij}^M$. For the non-signalling constraint, the fact that $\text{Tr}_B \mathbb{I}_{A'B} = \frac{d^{+} + 1}{2} \mathbb{I}_{A'}$ and $\text{Tr}_B \Lambda_{A'B} = \frac{d^{+} - 1}{2} \mathbb{I}_{A'}$ and a little counting show that $\text{Tr}_B E_j = g_j^{(n)} \mathbb{I}_{B}$ where

$$g_j^{(n)} := 2^{-n} \binom{n}{j} (d+1)^{n-j} (d-1)^j. $$

Substituting (A1) and $\rho_{A'} = d^{-n} \mathbb{I}_{A'}$ into the SDP described in Theorem 3 and Corollary 4 and using the facts just established, we obtain

**Proposition 10.** The optimal channel fidelity of a forward-assisted-code of size $K$ for $n$ uses of the $d$-dimensional generalised Werner-Holevo channel $\mathcal{W}(d, \alpha)$ is given by the linear program

$$\max d^n \sum_{j=0}^{n} \binom{n}{j} (1 - \alpha)^{n-j} \alpha^j x_j$$

subject to

for all $i = 0, \ldots, n$

$$0 \leq x_i \leq d^{-n}$$

(A5)

(A6)

with the additional constraint

$$\text{NS}: \sum_{j=0}^{n} g_j^{(n)} x_j = 1/K^2$$

(A7)

if the code is non-signalling, and the constraint

$$\text{PPT}: \begin{cases} \sum_{j=0}^{n} M_{ij}^M x_j \geq -d^{-n}/K, \\ \sum_{j=0}^{n} M_{ij}^M x_j \leq d^{-n}/K, \end{cases}$$

(A8)

if the code is PPT-preserving.

**Appendix B: Teleportation and dense coding**

It will be useful to define a non-signalling channel to be one of the form $\mathcal{R}_{B\to A'} = \sigma_B \text{Tr}_{A'}$. If we apply a non-signalling code (or, more generally, a non-signalling deterministic supermap) to a non-signalling channel, then it is not hard to see that the result is also a non-signalling channel.

The $d$-ary symmetric classical channels $\mathcal{C}^p_{Q^d\to Q}$ in $\text{ops}(Q \to Q')$ with $\dim Q = \dim Q' = d$ can be parameterised by their success probability $p$ such that: $\mathcal{C}^{(1)}_{Q^d\to Q}$ is the classical identity channel $\mathcal{C}^{(1/d)}_{Q^d\to Q}$ := $d^{-1} \mathbb{I}_{Q'} \text{Tr}_Q$, (which is a non-signalling channel); and $\mathcal{C}^{(p)}$ defined so that $p \mapsto \mathcal{C}^{(p)}$ is linear in $p$. This results in $\mathcal{C}^{(p)}$ being a valid operation for the range $p \in [0,1]$. The ‘symmetry’ in their name refers to the fact that the channels commute with an permutation of the computational basis elements. The deterministic supermap $\Psi$ defined by

$$\Psi: \mathcal{M}_{Q^d\to Q} \mapsto \frac{1}{d!} \sum_{\pi} U_{Q^d\to Q}^{\pi} \mathcal{M}_{Q^d\to Q} U_{Q^d\to Q}^{\pi}$$

(B1)

turns any channel $\mathcal{M}_{Q^d\to Q}$ in $\text{ops}(Q \to Q')$ with $P_{\text{success}}(\mathcal{M}_{Q^d\to Q}) = p$ into $\mathcal{C}^{(p)}_{Q^d\to Q}$. Here, $\pi$ ranges over all permutations of the numbers $\{0, \ldots, d-1\}$ and $U^{\pi}$ is the unitary operation which permutes the computational basis vectors according to $\pi$.

Likewise, the $d$-dimensional depolarising channels $\mathcal{X}^{(f)}$ can be parameterised by their channel fidelity $f$, with $\mathcal{X}^{(1)}_{Q^d\to Q} = \text{id}_{Q^d\to Q}$, $\mathcal{X}^{(1/d)}_{Q^d\to Q} = d^{-1} \mathbb{I}_{Q'} \text{Tr}_Q$ and the rest so that $f \mapsto \mathcal{X}^{(f)}$ is linear. Again, this means that $\mathcal{X}^{(f)}$ is a valid operation for all $f \in [0,1]$. Given any channel $\mathcal{M}_{Q^d\to Q}$ in $\text{ops}(Q \to Q')$ with $F(\mathcal{M}_{Q^d\to Q}) = f$, applying the ‘twirling’ deterministic supermap

$$\mathcal{U}: \mathcal{M}_{Q^d\to Q} \mapsto \int d\mu(U) U_{Q^d\to Q}^{\pi} \mathcal{M}_{Q^d\to Q} U_{Q^d\to Q}^{\pi}$$

(B2)

where $\mu$ is the Haar probability measure on $U(d)$ and $\mathcal{U}$ the unitary operation which conjugates by $\mathcal{U}$, will turn it into $\mathcal{X}^{(f)}_{Q^d\to Q}$.

A teleportation protocol is an entanglement-assisted code in $\text{EA}$ taking $\text{ops}(A' \to B)$ to $\text{ops}(B \to A')$ where $\dim A' = \dim B = d^2$ and $\dim A = \dim B = d$. We call the deterministic supermap $\mathcal{T}$. It maps the $d^2$-dimensional classical identity channel to the $d$-dimensional quantum identity channel.

$$\mathcal{T}[\mathcal{C}^{(1)}_{B\to A}] = \text{id}_{B'\to A} = \mathcal{X}^{(1)}_{B'\to A}. $$

(B3)

By twirling, we can assume that the channel produced by the teleportation protocol is a depolarising channel.

The only non-signalling $d$-dimensional depolarising channel is $\mathcal{X}^{1/d^2}$ so

$$\mathcal{U} \circ \mathcal{T}[\mathcal{C}^{(1/d^2)}_{B\to A}] = \mathcal{X}^{(1/d^2)}_{B'\to A}. $$

(B4)

By eqn. (B3), eqn. (B4) and linearity we have

$$\mathcal{U} \circ \mathcal{T}[\mathcal{C}^{(\lambda)}_{B\to A}] = \mathcal{X}^{(\lambda)}_{B'\to A}. $$

(B5)

Therefore, given any operation $\mathcal{M}_{B\to A'}$ with success probability $P_{\text{success}}(\mathcal{M}_{B\to A'}) = \lambda$ we can apply the entanglement-assisted deterministic supermap $\mathcal{U} \circ \mathcal{T} \circ \Psi \in \text{ops}(Q \to Q')$ with $P_{\text{success}}(\mathcal{M}_{B\to A'}) = \lambda$ and success probability $P_{\text{success}}(\mathcal{M}_{B\to A'}) = \lambda$.
to obtain a depolarising channel with channel fidelity $\lambda$:

$$\mathcal{U} \circ \mathcal{D} \circ \mathcal{P}[A_{B \to A}] = \lambda^{(A)}_{B \to A}. \quad (B6)$$

A \textit{dense-coding protocol} is an entanglement-assisted code $\mathcal{D} : \text{ops}(A' \to B) \to \text{ops}(B \to A')$ where $\dim A' = \dim B = d$ and $\dim A = \dim B' = d^2$, such that $\mathcal{D}[\lambda^{(A')}_{B \to A'}] = C^{(1)}_{B' \to A}$. Using a similar argument to the above find that

$$\mathcal{P} \circ \mathcal{D}[\lambda^{(A')}_{B \to A'}] = C^{(A)}_{B' \to A}. \quad (B7)$$

and that, from any operation $\mathcal{N}_{B \to A'}$ with channel fidelity $F(\mathcal{N}_{B \to A'}) = \lambda$ we can obtain a $d^2$-ary symmetric classical channel with success probability $\lambda$:

$$\mathcal{P} \circ \mathcal{D} \circ \mathcal{U}[\mathcal{N}_{B \to A}] = C^{(A)}_{B' \to A}. \quad (B8)$$

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