Galilean noncommutative gauge theory: symmetries & vortices

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Abstract

Noncommutative Chern-Simons gauge theory coupled to nonrelativistic scalars or spinors is shown to admit the “exotic” two-parameter-centrally extended Galilean symmetry, realized in a unique way consistent with the Seiberg-Witten map. Nontopological spinor vortices and topological external-field vortices are constructed by reducing the problem to previously solved self-dual equations.

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1 Introduction

In [1] Hagen suggested to consider a nonrelativistic scalar field minimally coupled to a gauge field with Chern-Simons dynamics. When a suitable self-interaction potential is added, the system admits exact self-dual vortex solutions [2].

Noncommutative field theory has attracted much recent attention [3]. It was found, for example, that the free scalar theory in $2+1$ dimensions is symmetric not only w. r. t. the usual one-parameter centrally extended Galilei and Schrödinger groups, but also with respect to their “exotic” two-parameter central extensions [4]. The hallmark of exotic symmetry is that the components of the conserved boost generators do not commute,

\[ \{G_i, G_j\} = \epsilon_{ij} \theta \int |\psi|^2 d^2x \]  

(1.1)

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where $\theta$ is the noncommutative parameter [5].

Some potentials break the Galilean symmetry [6], while others do not [4].

Commutative gauge theory can accommodate also a fourth-order self-interaction potential so that it remains invariant w. r. t. the conformal (Schrödinger) extension of the Galilei group. This allows one to prove, e. g., that, for the critical coupling, all finite-energy solutions of the second order field equations are selfdual [2]. The scalar field can also be replaced by a spinor so that the modified theory still supports self-dual vortices [4].

The noncommutative version of the nonrelativistic scalar field + Chern-Simons gauge field model was considered by Lozano, Moreno and Schaposnik [8], and by Bak, Kim, Soh, and Yee [9], who also find exact, nontopological vortex-like solutions which generalize those in the commutative theory.

Our paper consists of two parts. First we extend the symmetry investigations of [4] to noncommutative Chern-Simons gauge theory coupled to scalars and spinors. The second part is devoted to the study of various vortex solutions. Such theories are physically important for the Fractional Quantum Hall Effect [10]. These vortices correspond in particular to Laughlin’s quasiparticles and quasiholes.

Our paper is organized as follows. With hindsight to the noncommutative generalization to come, in Section 2 we review, following the gauge independent approach of [11], some aspects of commutative Chern Simons gauge theory.

Sections 3-4-5 deal with the symmetry properties of the noncommutative theory. We argue that boosts should act from the right, and show that this is the only possibility which is consistent with the Seiberg-Witten map [12]. Scale invariance is broken.

The NC vortices of [8, 9] and of [13, 14, 15] are shortly discussed in Section 6.

In Section 7 we extend our results to fermions. Spinors were studied in [7] in a Kaluza-Klein-type framework; here we present a rather more direct approach that is readily generalized to the noncommutative theory.

Topological vortices in a constant (electro)magnetic background, relevant for the FQHE, are discussed in our final Section 8.

In both cases, the new vortices are constructed by reducing the problem to the self-duality equation solved before by others, namely [8, 9] in the nontopological case, and [13, 14, 15] in the topological case.

## 2 Galilean symmetry of commutative gauge theory

In the following we review briefly the main results of the gauge-independent analysis presented in [11]. One considers the Lagrangian

$$L = L_{\text{matter}} + L_{\text{field}} = i\bar{\psi}D_{\mu}\psi - \frac{i}{2}|\vec{D}\psi|^2 + \kappa \left( \frac{i}{2} \epsilon_{ij} \partial_{t} A_{i} A_{j} + A_{t} B \right)$$

(2.1)

where $D_{\mu} = \partial_{\mu} - ieA_{\mu}$ is the covariant derivative, $B = \vec{\nabla} \times \vec{A}$ and $E_{i} = \partial_{t} A_{i} - \partial_{i} A_{t}$. The associated Euler-Lagrange equations consist of a gauged Schrödinger equation, \(2.2\), of a field-current identity (FCI), \(2.3\), and of the constraint \(2.4\) which replaces the Gauss’ law of Maxwell’s electromagnetism:

$$iD_{t}\psi + \frac{1}{2}\vec{D}^{2}\psi = 0$$

(2.2)
\[ \kappa E_i - e \epsilon_{ik} j_k = 0 \]  
\[ \kappa B + e \rho = 0 \]  

The FCI is particularly relevant as it is precisely the Hall law. It follows that the density and the current,

\[ \rho = |\psi|^2 \quad \text{and} \quad j_k = \frac{1}{2i} \left( (D_k \psi) \bar{\psi} - \psi (\overline{D_k \psi}) \right) \]

respectively, satisfy the continuity equation \( \partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0 \).

Let us implement an infinitesimal Galilean boost with parameter \( \vec{b} \) as

\[ \delta^0 \psi = i \vec{b} \cdot \vec{x} \psi - t \vec{b} \cdot \vec{\nabla} \psi \]  
\[ \delta^0 A_i = -t \vec{b} \cdot \vec{\nabla} A_i \]  
\[ \delta^0 A_t = -\vec{b} \cdot \vec{A} - t \vec{b} \cdot \vec{\nabla} A_t. \]

Then, using the relations

\[ \delta^0 \rho = -t \vec{b} \cdot \vec{\nabla} \rho, \quad \delta^0 j = -t \vec{b} \cdot \vec{\nabla} j + \vec{b} \rho, \]
\[ \delta^0 B = -t \vec{b} \cdot \vec{\nabla} B \quad \delta^0 E_i = -t \vec{b} \cdot \vec{\nabla} E_i - \epsilon_{ij} b_j B \]

one proves readily that the system (2.2-2.3-2.4) is form-invariant w. r. t. boosts. Positing the fundamental Poisson brackets

\[ \{ \psi(\vec{x}, t), \bar{\psi}(\vec{x}', t') \} = -i \delta(\vec{x} - \vec{x}') \]  
\[ \{ A_i(\vec{x}, t), A_j(\vec{x}', t') \} = \frac{\epsilon_{ij}}{\kappa} \delta(\vec{x} - \vec{x}') \]

the field equations (2.2-2.3) can be recast into a Hamiltonian form \( \dot{Y} = \{ Y, H \} \), \( Y = \psi, \bar{\psi}, A_i \) with the Hamiltonian

\[ H = \frac{1}{2} \int |\vec{D} \psi|^2 d^2 \vec{x} - \int A_t (e \rho + \kappa B) d^2 \vec{x}. \]

In restricted phase space defined by \( A_t = \pi_t = 0 \), the momentum, the angular momentum, and the boosts have the form

\[ P_i = \int \frac{1}{2i} (\bar{\psi} \partial_i \psi - (\overline{\partial_i \psi}) \psi) d^2 \vec{x} - \kappa \int \epsilon_{jk} A_k \partial_i A_j d^2 \vec{x} \]
\[ J = \int \left\{ \epsilon_{ij} x_i \left( \frac{1}{2i} (\bar{\psi} \partial_j \psi - (\overline{\partial_j \psi}) \psi) + \kappa \epsilon_{mn} A_m \partial_j A_n \right) - \kappa A_j^2 \right\} d^2 \vec{x} \]
\[ \vec{G}^0 = t \vec{P} - \int \vec{x} \rho d^2 \vec{x}. \]

They are also constants of the motion. When constrained to the surface defined by the Gauss law (2.4), they assume more familiar forms,

\[ H = \frac{1}{2} \int |\vec{D} \psi|^2, \quad \vec{P} = \int \vec{j}, \quad J = \int \vec{x} \times \vec{j}. \]

Conversely, an infinitesimal coordinate change \( \delta \vec{x} \) is a symmetry if it changes the Lagrangian by a surface term, \( \delta \mathcal{L} = \partial_\alpha K^\alpha \). Then Noether’s theorem yields the constant of the motion

\[ \int \left( \frac{\delta \mathcal{L}}{\delta (\partial_\xi \psi)} \delta \psi + \delta \bar{\psi} \frac{\delta \mathcal{L}}{\delta (\partial_\xi \bar{\psi})} - K^\xi \right) d^2 \vec{x} \]
Using the P.B. (2.10) and (2.11) these quantities generate translations, rotations, and boosts for the matter field and the gauge field, respectively, according to

$$-\partial_i Y = \{ Y, P_i \}$$ etc. The quantities above provide us with the usual 1-parameter centrally extended Galilei (“Bargmann”) algebra with the particle number [mass], $M = \int \rho$, as central term. In particular, the boost components commute,

$$\{ G_1^0, G_2^0 \} = 0. \quad (2.18)$$

3 Noncommutative gauge theory

Let us now turn to the noncommutative version of the above theory [8, 9]. The Lagrangian $L^I = L^I_{\text{matter}} + L^I_{\text{field}}$ is formally the same as in the commutative case, Eq. (2.1); the noncommutative structure is hidden in the definition of the covariant derivative and the field strength,

$$D_\mu \psi = \partial_\mu \psi - ieA_\mu \psi, \quad (3.1)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie(A_\mu \star A_\nu - A_\nu \star A_\mu), \quad (3.2)$$

respectively, where the Moyal “star” product is associated with the non-commutative parameter $\theta$,

$$( f \star g)(x_1, x_2) = \exp \left( i\frac{\theta}{2}(\partial_{x_1} \partial_{y_2} - \partial_{x_2} \partial_{y_1}) \right) f(x_1, x_2) g(y_1, y_2) \big|_{\vec{x} = \vec{y}}. \quad (3.3)$$

According to (3.1) the matter field $\psi$ is in the fundamental representation of the gauge group $U(1)^*$, i.e. $A_\mu$ acts from the left. Hence $D_\mu \psi = \partial_\mu \bar{\psi} + \bar{\psi} \star (ieA)$. Note also that $L^I_{\text{field}}$ is equivalent to the Moyal-star Chern-Simons three-form

$$\kappa/2 \varepsilon_{\mu\nu\sigma}(A_\mu \star \partial_\nu A_\sigma - \frac{2ie}{3}A_\mu \star A_\nu \star A_\sigma). \quad (3.4)$$

A remarkable feature of the NC Chern-Simons theory is that gauge invariance requires the coefficient $\kappa$ to be quantized even for the gauge group $U(1)^*$ [15],

$$\kappa = \frac{n}{2\pi}, \quad n = 0, \pm 1, \ldots \quad (3.5)$$

Apart from subtle differences, the field equations look as in the commutative case,

$$iD_t \psi + \frac{1}{2} \bar{D}^2 \psi = 0 \quad (3.6)$$

$$\kappa E_i - e \epsilon_{ikj} j^l_k = 0 \quad (3.7)$$

$$\kappa B + e \rho^l = 0 \quad (3.8)$$

where $B = \epsilon_{ij} F_{ij}$, $E_i = F_{i0}$, and $\rho^l$ and $j^l$ denote the left density and left current, respectively,

$$\rho^l = \psi \star \bar{\psi}, \quad j^l = \frac{1}{2i} \left( \bar{D} \psi \star \bar{\psi} - \psi \star (\bar{D} \bar{\psi}) \right). \quad (3.9)$$

The condition (3.5) implies that the Hall conductance is quantized in units of $(2\pi e)^{-1}$.

The continuity equation only holds for the right quantities

$$\rho^r = \bar{\psi} \star \psi, \quad j^r = \frac{1}{2i} \left( \bar{\psi} \star \bar{D} \bar{\psi} - (\bar{D} \bar{\psi}) \star \psi \right), \quad (3.10)$$

but not for the left-quantities (3.9). These latter satisfy in fact a covariant version, namely $D_t \rho^l + \bar{D} \cdot j^l = 0$. The integral property implies $\int (\rho^l + \bar{D} \cdot j^l) = 0$, though.
Owing to the “handedness”, the noncommutative system behaves somewhat unusually under a Galilean boost. Firstly, unlike in a pure scalar theory, the field equations (3.6)- (3.7)-(3.8) are not invariant w. r. t. the conventional boost implementation (2.6)-(2.7)-(2.8). In fact, (2.9) and # (2.1) of [3] imply that

$$\delta^0 B = -t b \cdot \nabla B \quad \text{but} \quad \delta^0 \rho^l = - \frac{\theta}{2} b \times \nabla \rho^l. \quad (3.11)$$

The Gauss constraint is hence not form-invariant.

In [3] we proposed another implementation which takes into account the Moyal structure, i.e., to replace the above formulae by the fundamental Moyal representation

$$\delta^l \psi = (i b \cdot \vec{x}) \star \psi - t b \cdot \vec{\nabla} \psi = (i b \cdot \vec{x})\psi - (\theta/2) b \times \vec{\nabla} \psi - t b \cdot \vec{\nabla} \psi. \quad (3.12)$$

This still leaves the free theory invariant, and can also accomodate “pure interactions” of the form $V(\rho^0)$ a = $l$, $r$.

But how to implement a boost on the gauge field? According to Eqn. (2.4) of [17], a coordinate transformation $f = f^\alpha$ which is at most linear in $\vec{x}$ should be implemented as

$$\delta f A_\mu = \frac{1}{2}(f^\alpha \star \partial_\alpha A_\mu + \partial_\alpha A_\mu \star f^\alpha) + \partial_\mu f^\alpha A_\alpha. \quad (3.13)$$

But $f_i = 0$, $f_i = -tb_i$ for a boost, so this simply reduces to the standard Lie derivative, $L_f A_\mu$, the same as in the commutative case. In conclusion, the standard implementation (2.7)- (2.8) is retained, $\delta^l A_\mu = \delta^0 A_\mu$. Hence $\delta^l B = B$ as before.

However, as $\delta^l \psi$ is in fact $\delta^0 \psi - (\theta/2) b \times \vec{\nabla} \psi$ the $\theta$ terms add up making things “even worse”,

$$\delta^l \psi = \delta^0 \psi - t b \cdot \vec{\nabla} \psi. \quad (3.14)$$

cf. (3.11). Hence, also the Moyal implementation $\delta^l$ breaks down for the gauged system! Galilean symmetry is restored, though, if we consider instead the antifundamental representation

$$\delta^r \psi = \psi \star (i b \cdot \vec{x}) - t b \cdot \vec{\nabla} \psi = (i b \cdot \vec{x})\psi + \frac{\theta}{2} b \times \vec{\nabla} \psi - t b \cdot \vec{\nabla} \psi \quad (3.15)$$

which is (3.12) with the sign of $\theta$ reversed. Observing that

$$\delta^r \psi = \delta^0 \psi + \frac{\theta}{2} b \times \vec{\nabla} \psi \quad (3.16)$$

we find that the $\theta$-terms cancel in $\delta^r \psi$, leaving us with the homogeneous transformation law

$$\delta^r \rho^l = - t b \cdot \vec{\nabla} \rho^l. \quad (3.16)$$

Putting $\delta^r A_\mu = \delta^0 A_\mu$, so that $\delta^r B = \delta^0 B$, the Gauss constraint (3.8) is right-invariant, just like the remaining equations. For Eqn. (3.17) this follows from

$$\delta^r f^l_k = - t b \cdot \vec{\nabla} f^l_k + b_k \rho^l \quad \text{and} \quad \delta^r E_i = - t b \cdot \vec{\nabla} E_i - \epsilon_{ij} b_j B, \quad (3.18)$$

while for (3.18) this comes from

$$\delta^r (i D_t \psi + 4 \vec{D}^2 \psi) = - t b \cdot \vec{\nabla} (i D_t \psi + 4 \vec{D}^2 \psi) + (i D_t \psi + 4 \vec{D}^2 \psi) \star (i b \cdot \vec{x}). \quad (3.19)$$

In conclusion, the antifundamental implementation (3.14) allows us to restore the Galilean symmetry of the model.
The field equations can still be put into a Hamiltonian form, using the same Poisson structure \((2.10-2.11)\) as before. When restricted to the surface in phase space defined by the Gauss’ law \((3.8)\), the momentum, \((2.13)\), remains a constant of the motion. For the boost generator we get, instead of \((2.15)\),
\[
\vec{G}^r = t\vec{P} - \int x^r \rho \, d^2\vec{x} \tag{3.17}
\]
whose conservation can also be checked directly, using the continuity equation satisfied by \(\rho^r\).

For the sake of comparison, we also present the second term here as
\[
- \int x_i |\psi|^2 \, d^2\vec{x} - \frac{\theta}{2} \epsilon_{ij} \int \frac{1}{2i} \left( \bar{\psi} \partial_j \psi - (\partial_j \bar{\psi}) \psi \right) \, d^2\vec{x} \tag{3.18}
\]
which differs in the sign of \(\theta\) from the analogous expression for noncommutative scalar field theory, \# (2.10) of \[4\]. This is due to our using the antifundamental, rather than the fundamental representation. Finally, for the commutator of the boost components we find
\[
\{G_i, G_j\} = \epsilon_{ij} k, \quad k \equiv -\theta \int |\psi|^2 \, d^2\vec{x} \tag{3.19}
\]
which is \((1.1)\) with the sign of \(\theta\) reversed. Apart from this, the two-parameter “exotic” central extension \[5\] is recovered. Our results extend those obtained in \[4\] to noncommutative CS gauge theory.

### 4 Family of boost generators and the Seiberg-Witten (SW) map

One may wonder whether the boost generator given by \((3.17)\) is unique. In the free case it has been noted by Hagen \[1\] that the conventional boost generator may be redefined by adding \((\kappa/2)\epsilon_{ij} P_j\), which leads to a trivial second central extension of the planar Galilei group. At first sight, the same seems to hold also in noncommutative theory. One can indeed define a whole family of generalizations of the boost generator \((3.17)\) depending on a real parameter \(\alpha\),
\[
G_i^\alpha = G_i^r + \frac{\alpha}{2} \epsilon_{ij} P_j. \tag{4.1}
\]

Being a combination of two separately conserved quantities, \(G_i^\alpha\) is plainly conserved, and leads to the new transformation rules
\[
\delta^\alpha \psi = \vec{b} \cdot \{\psi, G^\alpha\} = i\vec{b} \cdot \vec{x} \psi - \bar{t}\vec{\bar{b}} \cdot \vec{\nabla} \psi + \frac{i}{2} (\theta - \alpha) \bar{\vec{b}} \times \vec{\nabla} \psi, \tag{4.2}
\]
\[
\delta^\alpha \vec{A} = b_i \{\vec{A}, G_i^\alpha\} = -t\vec{b} \cdot \vec{\nabla} \vec{A} - \frac{\alpha}{2} \bar{\vec{b}} \times \vec{\nabla} \vec{A}. \tag{4.3}
\]

Then the Poisson brackets of the boost components would change from \((3.19)\) to
\[
\{G_i^\alpha, G_j^\alpha\} = \epsilon_{ij} (\alpha - \theta) \int |\psi|^2 \, d^2\vec{x}. \tag{4.4}
\]

For \(\alpha = 0\) we recover the right-implementation \((3.14)\); for \(\alpha = \theta\) instead, we act on the matter field as in the commutative case, \((2.6)\), but non-conventionally on the gauge potential; we get a vanishing second central charge. The question arises, however, if the generalization \((4.1)\) is allowed if we insist, in the spirit of Seiberg and Witten \[12\] to recover the conventional implementation \((2.6) - (2.7)\) in the commutative limit. The matter and gauge fields in the noncommutative \((\theta \neq 0)\) and commutative \((\theta = 0)\) theory must be indeed related to each other by the
Seiberg-Witten (SW) map \[12\]. In particular, \( \vec{A}(\theta) \) must satisfy a differential equation \[12\], Eq. (3.8) namely
\[
\frac{\partial}{\partial \theta} A_i(\theta) = -\frac{1}{4} \epsilon_{kl} \left( A_k \star (\partial_l A_i + F_{li}) + (\partial_l A_i + F_{li}) \star A_k \right).
\]
(4.5)

Eqn. (4.5) is manifestly form-invariant w. r. t. the boost transformations (4.3), provided \( \alpha \) does not depend on \( \theta \), \( \frac{\partial}{\partial \theta} \alpha = 0 \). In the limit \( \theta \rightarrow 0 \), this is consistent with the boost transformation (2.7) only for \( \alpha = 0 \). In conclusion, the boost generator (3.17) is the only allowed one if we require to recover the conventional implementation in the commutative limit. The nontrivial second charge, (3.19), is hence dynamically defined.

5 Potentials and the breaking of the scale invariance

At this stage, we can add a potential. As explained in \[4\], the mixed expression \( V(\rho^r \rho^l) \), favored, e. g., by Bak et al. \[6\], breaks the Galilean symmetry whereas “pure” expressions of the form
\[
V(\rho^r) \equiv V(\bar{\psi} \psi) \quad \text{or} \quad V(\rho^l) \equiv V(\psi \bar{\psi})
\]
(5.1)
are invariant w. r. t. both the conventional and the “left-exotic” \{fundamental\} implementations, (2.6) and \( \delta^l \), (3.12), respectively. Using (3.16) it is straightforward to prove that the same statement holds for the antifundamental representation. In conclusion, noncommutative CS theory augmented with a pure potential \( V(\rho^a) a = r, l \) is consistent with (right-) Galilean symmetry.

As we said already, commutative Chern-Simons theory is consistent with the “Schrödinger” (conformal) extension of the Galilei group \[2\], but any potential breaks the scale invariance \[6,4\]. Let us now show that this also what happens in NC-CS gauge theory. Our proof relies on the non form-invariant behaviour of the Moyal product under a dilatation. To see this, let consider a generic star product \( K_1 \star K_2 \) where the \( K_i \) transform w. r. t. an infinitesimal dilatation as
\[
\delta_\Delta K_i = \Delta \cdot (k_i - \vec{x} \cdot \vec{\nabla} - 2t\partial_t) K_i
\]
(5.2)
where \( k_i \) is the scaling dimension and \( \Delta \cdot \) means multiplication with the parameter \( \Delta > 0 \). Then it follows from Eqn. (2.4) of \[4\] that
\[
\delta_\Delta \cdot (K_1 \star K_2) = \Delta \cdot (k_1 + k_2 - \vec{x} \cdot \vec{\nabla} - 2t\partial_t)(K_1 \star K_2) - i\theta \Delta \cdot \epsilon_{ij} \partial_i K_1 \star \partial_j K_2
\]
(5.3)
where the term behind \( \theta \) breaks the form invariance. This formula generalizes the one proved for the (right) density in \[4\].

Let us now turn to the NC Gauss law \[3,3]\ which contains two Moyal products, namely the left-density, \( \rho^l = \psi \star \bar{\psi} \), and the magnetic field, \( B \), which involves \( i\epsilon_{ij} A_i \star A_j \). As the individual factors transform as in (5.2), the inhomogeneous terms clearly break the scaling symmetry.

The same statement is readily seen to hold also for expansions.

6 NC scalar vortices

6.1 Nontopological Chern-Simons vortices

In order to have a reference for the fermionic case (Section 7), we review briefly the main results of Lozano, Moreno, and Schaposnik \[8\], and of Bak, Kim, Soh, and Yee \[9\], respectively. These
authors consider the previous NC-CS gauge theory to which they add a quartic “left-potential” $V = \left(\frac{\lambda}{4}\right)\psi \star \psi = \left(\frac{\lambda}{4}\right)\rho^l$. This only changes (3.3) into

$$iD_t\psi + \frac{1}{2}D^2\psi + \frac{\lambda}{2}\rho^l \star \psi = 0$$  \hspace{1cm} (6.1)

The conserved quantities are routinely obtained. For the energy we recover in particular their

$$H = \int \left(\frac{1}{2}D\bar{D}\psi + \frac{\lambda}{4}(\rho^l)^2\right) d^2\vec{x}.$$  \hspace{1cm} (6.2)

Then, using the Bogomolny trick, for $\lambda = \pm 2e^2/\kappa$ this becomes

$$\int |D_\pm \psi|^2 d^2\vec{x} \geq 0,$$  \hspace{1cm} (6.3)

where $D_\pm = D_1 \pm iD_2$. The Bogomolny bound, (namely zero) is therefore saturated when the fields are self-dual or antiself-dual, respectively, i.e. when

$$D_\pm \psi = 0 \hspace{1cm} (6.4)$$

$$\kappa B \pm e\rho^l = 0 \hspace{1cm} (6.5)$$

In the commutative case, the upper equation (6.4) could be solved for the vector potential; inserting the result into the lower one would yield the Liouville equation with its known solutions [2]. In the NC case, vortices are in turn constructed by solving these equations [8, 9] using a rather involved technique we do not reproduce here. We note for further reference that their SD ($D_+ \psi = 0$) solution is regular for $\kappa < 0$, and their ASD ($D_- \psi = 0$) solution is regular for $\kappa > 0$, respectively. Their vortices are purely magnetic as it can be seen from the second order field equations. They are also nontopological in that the density vanishes at infinity.

### 6.2 NC Nielsen-Olesen vortices

For the sake of their use in Section 8 we briefly review also the noncommutative generalization of the Nielsen-Olesen vortices examined in [13, 14, 15]. As this theory is relativistic, we will not review it in detail, merely contend ourselves with mentioning that the static energy functional can again be written using the Bogomolny trick as

$$H = \int \left(\frac{1}{2}(B \mp (\rho_0 - \rho^l))^2 + (D_\pm \psi \overline{D_\mp \psi})\right) d^2\vec{x} \pm \int \rho_0 B d^2\vec{x}$$  \hspace{1cm} (6.6)

where $\rho_0 > 0$ is a constant. The absolute minimum of the energy, namely $\rho_0 |(\text{magnetic flux})|$ is therefore attained when the self-dual or the antiself-dual equations,

$$D_+ \Phi = 0 \hspace{1cm} B = \rho_0 - \rho^l \hspace{1cm} B > 0$$

$$D_- \Phi = 0 \hspace{1cm} -B = \rho_0 - \rho^l \hspace{1cm} B < 0$$  \hspace{1cm} (6.7) \hspace{1cm} (6.8)

hold, respectively. In the commutative case the above procedure would yield a “Liouville-type” (but not explicitly soluble) equation; in the NC case specific techniques were used [14, 15].

It is worth pointing out that here we follow, together with Lozano et al. [8, 15], the sign convention (3.3). Some people including Bak et al. [14] use the opposite sign for $\theta$. Their results are, therefore, translated by interchanging the words “self-dual” and “antiself-dual” [15]. This statement is not entirely obvious but can be proved using the properties of the Moyal product.
7 Fermions

7.1 The gauged Lévy-Leblond + Chern-Simons equations

A Galilean covariant “non-relativistic Dirac equation” was constructed by Lévy-Leblond [18]. A Dirac spinor in $3 + 1$ dimensions has four-components, but in the plane it only has two components; there are instead two sets of “Dirac” matrices, appropriate to accommodate spin $+\frac{1}{2}$ and spin $-\frac{1}{2}$. (In [7] the same two systems were obtained as the chiral components of the 4-component theory). We consider, for definiteness, the spin $+\frac{1}{2}$ theory. Let hence $\Psi$ denote a two-component spinor and consider the fermionic matter Lagrange density

$$L_{\text{matter}} = i \left\{ \Psi^\dagger (\Sigma_t D_t - \Sigma \cdot \vec{D} - i \Sigma_s) \Psi \right\} \quad (7.1)$$

where the covariant derivatives have the same meaning as before, and $\Sigma_t = \frac{1}{2}(1 + \sigma_3), \Sigma_i = \sigma_i (i = 1, 2), \Sigma_s = (1 - \sigma_3)$ denote the “Dirac” matrices. Observe that $\Sigma_t$ and $\Sigma_s$ are singular matrices. Setting $\Psi = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}$ yields the first-order “Lévy-Leblond” (LL) equations

$$D_+ \Phi + 2i \chi = 0$$
$$D_t \Phi - D_- \chi = 0 \quad (7.2)$$

where $D_\pm = D_1 \pm i D_2$. Choosing the same field Lagrangian as above, we find that the Chern-Simons field equations retain their form (2.3) and (2.4) up to the definition of the density and current,

$$\rho = |\Phi|^2, \quad j_1 = -(\bar{\Phi} \chi + \bar{\chi} \Phi)$$
$$j_2 = i(\bar{\Phi} \chi - \bar{\chi} \Phi) \quad (7.3)$$

In particular, the density only involves the upper component, since $\Sigma_t$ is a projector.

Implementing an infinitesimal boost as [18, 7]

$$\delta^0 \Phi = (i \vec{b} \cdot \vec{x})\Phi - t \vec{b} \cdot \vec{\nabla} \Phi$$
$$\delta^0 \chi = -\frac{1}{2}(b_1 + ib_2)\Phi + (i \vec{b} \cdot \vec{x})\chi - t \vec{b} \cdot \vec{\nabla} \chi \quad (7.4)$$

while maintaining the previous implementation on the gauge fields, we find that the LL equations (7.2) vary as

$$i \vec{b} \cdot \vec{x} \left( D_+ \Phi + 2i \chi \right) \quad \text{and} \quad i \vec{b} \cdot \vec{x} \left( D_t \Phi - D_- \chi \right) - \frac{i}{2}(b_1 - ib_2) \left( D_+ \Phi + 2i \chi \right),$$

respectively, which both vanish together with the LL equations (7.2). This establishes the Galilean symmetry for the LL equations. To extend this statement to the coupled system, we observe that the spinor density and current in (7.3) change precisely as $\rho$ and $\vec{j}$ in (2.9). The invariance of the Chern-Simons equations is hence retained.

For a solution of the Chern-Simons field equations the associated conserved quantities, derived using Noether’s theorem, can be expressed in terms of the upper component alone. They

---

1 Due to time translational symmetry, it is enough to vary at $t = 0$. 
are the mass [particle number], \( M = \int |\Phi|^2 \), and

\[
\vec{P} = \int \vec{P} d^2\vec{x} \equiv \int \left( \frac{1}{2i} \left( \Phi \vec{D} \Phi - (\vec{D} \Phi) \Phi \right) \right) d^2\vec{x} \quad \text{linear momentum}
\]

\[
J = J_{\text{orbital}} + J_{\text{spin}} = \int \vec{x} \times \vec{P} d^2\vec{x} + \frac{1}{2} M \quad \text{angular momentum}
\]

\[
\vec{G} = t\vec{P} - \int \vec{x} |\Phi|^2 d^2\vec{x} \quad \text{boost}
\]

\[
H = \int \left\{ \frac{1}{2} |\vec{D} \Phi|^2 + \frac{e^2}{2\kappa} |\Phi|^4 \right\} d^2\vec{x} \quad \text{energy}
\]

which shows clearly that the spin is indeed \( \frac{1}{2} \). The components of the boost plainly commute, (2.18); (7.5) provides us with the usual one-parameter centrally extended Galilean relations [18].

Then the usual trick applied to the energy yields the Bogolyubov bound

\[
H = \frac{1}{2} \int |D_+ \Phi|^2 \geq 0.
\]

The absolute minimum of the energy is attained therefore when the self-duality condition, \( D_+ \Phi = 0 \) holds. (Note that antiself-duality, \( D_- \Phi = 0 \), does not qualify here; it would be appropriate for spin \( -\frac{1}{2} \)). Then exact, purely magnetic solution can be constructed by solving again the Liouville equation [7]. Normalizable solutions arise provided \( \kappa < 0 \). An alternative, more detailed discussion will be presented below in the NC context.

On the surface defined by the Gauss constraint (2.4) \( H \) acts as a Hamiltonian.

### 7.2 Noncommutative fermions

The noncommutative generalization of these results is quite straightforward. Both the matter and the field Lagrangian, (7.1) and (2.1), retain their form, but the covariant derivative and the field strength assume their NC meaning, cf. Section 3. The associated field equations are still (7.2) with the Moyal structure hidden in the covariant derivative, augmented with the NC Chern-Simons equations (3.7) and (3.8) with left-density and current,

\[
\rho^j = \Phi \star \bar{\Phi}, \quad j_1 = -(\chi \star \bar{\Phi} + \Phi \star \bar{\chi})
\]

\[
\frac{1}{2} \vec{b} \times \vec{\nabla} \Psi (7.8)
\]

Galilean boosts have to be implemented by combining the right Moyal action of the \( \vec{x} \)-dependent imaginary factor, (3.14), with the spinor term in (7.4). This yields simply

\[
\delta^x \Psi = \delta^0 \Psi + \frac{\theta}{2} \vec{b} \times \vec{\nabla} \Psi
\]

\[
(7.8)
\]

cf. (3.15). It follows that (7.8) is again a symmetry. Firstly, the (NC) LL equation merely changes by \( (\theta/2) \vec{b} \times \vec{\nabla} (\text{LL eqn}) \). As the density and current change once again as before, the boost invariance of the NC spinor system is established.

It follows from (7.8) that the associated conserved quantities are obtained by combining those in the NC scalar case with the commutative spinorial expressions in (7.5). For a boost, e. g., Eq. (3.17) is still valid, when \( \psi \) is replaced by the upper component \( \Phi \). The boost components satisfy the exotic relation (3.19). Similarly, the energy is

\[
H = \int \left( \frac{1}{2} \overline{\vec{D} \Phi} \vec{D} \Phi + \frac{e^2}{2\kappa} (\rho^j)^2 \right) d^2\vec{x},
\]

(7.9)
which is precisely the same as in the NC scalar theory studied by Lozano et al. [8], and could be used therefore to construct vortex solutions.

We prefer, however, to follow another procedure which is peculiar to the first-order setting. Let us observe indeed that the static and purely magnetic Ansatz

\[ D_+ \Phi = 0, \quad \chi = 0, \quad \partial_t \Phi = 0, \quad A_t = 0 \]  

(7.10)

plainly solves the static version of the LL equations (7.2). Then both the electric field, \( \vec{E} \), and the current, \( \vec{j} \), vanish cf. (7.3), so that the FCI (8.7) holds identically, and we are left with the Gauss law (3.8). As \( \rho^j = \Phi \bar{\Phi} \), we arrive at the problem solved before by Lozano et al., and by Bak et al. [8, 9]. The only difference is that our spinor vortices are purely magnetic, while those in [8] carry also an electric field – as do their commutative limit [2]. This “coincidence” is explained by eliminating the “lower” component \( \chi \) from the LL equation by using the identity

\[ D - D^+ = \vec{D}^2 + eB \Phi. \]  

Then the “upper” component satisfies the gauged Schrödinger equation with a Pauli term ²,

\[ \left[iD_t + \frac{1}{2} \vec{D}^2 + \frac{e}{2} B \Phi, \right] \Phi = 0 \quad \text{i.e.} \quad \left[iD_t + \frac{1}{2} \vec{D}^2 + \frac{e^2}{2K} \rho^j \Phi = 0 \right. \]  

(7.11)

where the Gauss law (3.8) has been used. This is precisely the field equation of Lozano et al. [8], (6.1), with the specific SD value of the coupling constant. It can be solved using the solutions of the SD equations (6.4)-(6.5). Note that the quartic “left-potential” with the “SD” coupling coefficient here was not put in by hand but came rather from the (“left”) Gauss law (3.8).

The spinor theory is, in this sense, automatically self-dual.

8 Vortices in the Landau-Ginzburg theory of the QHE

A “Landau-Ginzburg” theory for the QHE has been proposed by Zhang et al. [19]. They consider a scalar field \( \psi \) coupled to a gauge field \( a_\mu \) and an external electromagnetic potential \( \bar{A}^\text{ext}_\mu \), described by the Lagrangian

\[ \frac{K}{2} \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma + \bar{\psi} i \partial_\mu - e(a_t + \bar{A}^\text{ext}_t) \right] \psi - \bar{\psi} \left[ - i \vec{\nabla} - e(\tilde{a} + \bar{A}) \right]\psi - U(\psi) \]  

(8.1)

where \( U(\psi) \) is a (quartic) self-interaction potential. The gauge field \( a_\mu \) obeys hence the Chern-Simons dynamics, but the matter field \( \psi \) moves under the joint influence of \( a_\mu \) and the external field. Let us assume that the background field is constant. Putting \( A_t = -a_t - \bar{A}^\text{ext}_t \) and \( \bar{A} = \bar{a} + \bar{A}^\text{ext} \), performing some partial integrations and dropping surface terms, this is further written as

\[ \frac{K}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - e \left(D^\text{ext}_t A_t - \bar{E}^\text{ext} \times \bar{A} \right) + i \bar{\psi} D_t \psi - \frac{1}{4} |D \psi|^2 - U(\psi) \]  

(8.2)

where \( D_\mu = \partial_\mu - ieA_\mu \psi \) and the external field has been redefined as \( F^\mu_\nu = (\kappa/e) \bar{F}^\text{ext}_\mu_\nu \). Augmented with the natural magnetic term \( B^2/2 \), the system (8.2) was introduced by Manton [20]. Note also that the second, “external” term here is also equivalent to \( -e\epsilon^{\mu\nu\lambda} A^\text{ext}_\mu \partial_\nu A_\lambda \) considered in [10].

²For spin \( -\frac{1}{2} \) one merely changes the sign of \( \Sigma_2 \) which results in changing the sign of the magnetic field in \( (8.2) \).
In [20] Manton found in particular exact purely magnetic vortices as imbedded Bogomolny-Nielsen-Olesen solutions. His results were generalized to vortices with an electric field in [21].

Recently, it was argued that to describe the Fractional Quantum Hall Effect the commutative Chern-Simons term should be replaced by the noncommutative expression (3.4) [10]. Here we propose therefore to consider the noncommutative model given by

\[
L_{\text{ext}} = -\frac{i}{2} B^2 + \bar{\psi} D_t \psi - \frac{i}{2} |\bar{D} \psi|^2 - \frac{1}{4} (\rho_0 - \psi \psi)^2 + \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} \left( A_\mu \partial_\nu A_\lambda - \frac{2ie}{3} A_\mu \star A_\nu \star A_\lambda \right) - e \left( B^{\text{ext}} A_t - \bar{E}^{\text{ext}} \times \bar{A} \right)
\]  

(8.3)

where \( \rho_0 \) is a constant. The covariant derivative and the field strength are again as in (3.1)-(3.2). The absence of an electric Maxwell term is dictated by Galilean, rather than Lorentz covariance [21]. If the external fields are constant, the system is translational invariant. The “naked” external-field term \(-eB^{\text{ext}} A_t\) modifies the quantization condition (3.5) as [10]

\[
\kappa + eB^{\text{ext}} \theta = \frac{n}{2\pi}, \quad n = 0, \pm 1, \ldots
\]  

(8.4)

\( A_t \) is a Lagrange multiplier and variation w. r. t. it yields the modified Gauss constraint

\[
\kappa B = e(B^{\text{ext}} - \rho^l).
\]  

(8.5)

Before considering the remaining field equations let us inquire about finite-energy configurations. If the external electric field is also constant that we assume henceforth, it can be eliminated by a Galilean boost. In the frame where \( \bar{E}^{\text{ext}} = 0 \) the energy can be expressed as

\[
\int \left( \frac{1}{2} |\bar{D} \psi|^2 + \frac{1}{2} B^2 + \frac{\lambda}{4} (\rho_0 - \rho^l)^2 \right) d^2 \vec{x}.
\]  

(8.6)

This expression can be justified, e.g., by considering the Hamiltonian associated with (8.3) and using the Gauss constraint. Let us call attention to that both the electric term \( \frac{1}{2} \bar{E}^2 \) and the time-derivative of the matter field are absent here. This follows from the nonrelativistic form (8.3) of the action. This is in contrast with the relativistic setting of Nielsen-Olesen, where these terms are eliminated by a static, purely-magnetic Ansatz.

In order to make the energy integral converge, we require

\[
\bar{D} \psi \to 0, \quad B \to 0 \quad \rho^l \to \rho_0
\]  

(8.7)

sufficiently rapidly as \( r = |\vec{x}| \to \infty \). Comparision with (8.5) shows that necessarily \( B^{\text{ext}} = \rho_0 > 0 \).

Let us now turn to the field equations,

\[
i D_t \psi + \frac{1}{2} \bar{D}^2 \psi + \frac{\lambda}{2} (B^{\text{ext}} - \rho^l) \star \psi = 0
\]  

(8.8)

\[
\kappa \epsilon_{ik} E_k + e (\rho^j_i + \epsilon_{ik} E^{\text{ext}}_k) - \epsilon_{ik} D_k B = 0
\]  

(8.9)

\[
\kappa B = e(B^{\text{ext}} - \rho^l)
\]  

(8.10)

The first equation here is a non-linear Schrödinger equation which involves the left-density \( \rho^l \); the second one combines Ampère’s law with the FCI appropriate for the Hall effect. (The current \( j^i_i \) here is (3.9)). The last is a modified Gauss law.
Now we show that for a suitable $\lambda$ this system admits self-dual vortex solutions. Let us namely combine the self-duality Ansatz (8.4) with the modified Gauss law (8.10)

$$D_{\pm}\psi = 0$$
$$\kappa B = e(B^{\text{ext}} - \rho^l).$$

(8.11)

In the frame $\vec{E}^{\text{ext}} = 0$, the static version of the upper two field equations require, using $\vec{D}^2\psi = \mp eB \star \psi$,

$$[eA_t + 1/2 eB + \lambda/2 (B^{\text{ext}} - \rho^l)] \star \psi = 0$$
$$\kappa \epsilon_{ik} E_k + e_j^l i - \epsilon_{ik} D_k B = 0.$$

Then the first equation is satisfied with $A_t = \mu(B^{\text{ext}} - \rho^l)$, where $\mu = 1/2(\frac{\mp e}{\kappa} - \frac{\lambda}{e})$. Using self-duality (8.11), we find also

$$\vec{J}^l = \mp \frac{1}{2} \vec{D} \times \rho^l, \quad \vec{E} = -\mu \vec{D} \rho^l, \quad \text{and} \quad \vec{D} \times B = -\frac{e}{\kappa} \vec{D} \times \rho^l.$$

[Here $\vec{D}$ acts on $\rho^l$ in the adjoint representation]. Then Ampère’s law fixes the coefficient of the self-interaction potential and hence the electric potential as

$$\lambda = \frac{2e^2}{\kappa^2} (\pm \kappa - 1) \quad \implies \quad A_t = \frac{e}{2\kappa^2} (2 \mp \kappa) (B^{\text{ext}} - \rho^l).$$

(8.12)

The self-interaction potential is physically admissible (attractive) when $\lambda \geq 0$. For the upper and lower signs this requires $\kappa \geq 1$ and $\kappa \leq -1$, respectively. Interestingly, for $\kappa = \pm 1$, the SD equations work with $\lambda = 0$, i.e., without a self-interaction potential. The asymptotic behaviour $\rho^l \to B^{\text{ext}}$ is still mandatory, owing to $B \to 0$ and the Gauss law.

The self-duality equations (8.11) could also have been derived using the Bogomolny trick. Using the Gauss constraint and the identity

$$|\vec{D}\psi|^2 \sim |D_{\pm}\psi|^2 \pm eB \star \rho^l$$

where “$\sim$” means up to surface terms, the energy can be further written as

$$\int \left( \frac{1}{4}|D_{\pm}\psi|^2 + \frac{1}{4}(\lambda + \frac{2e^2}{\kappa^2} (1 \mp \kappa) (B^{\text{ext}} - \rho^l))^2 \right) d^2\vec{x} \pm \frac{1}{2} B^{\text{ext}} \int eB d^2\vec{x}. \quad (8.13)$$

The last integral here is the magnetic flux, $\int B d^2\vec{x}$. For the specific choice (8.12) of $\lambda$ the middle term vanishes and, choosing the upper/lower sign depending on the sign of $eB$ yields the usual Bogomolny inequality for the energy

$$H \geq \frac{1}{2} B^{\text{ext}} |e \times (\text{magnetic flux})|.$$

(8.14)

Equality is achieved here for the SD/ASD equations (8.11).

In the commutative context, the flux is quantized [23],

$$\text{magnetic flux} = \int B d^2\vec{x} = \frac{2\pi}{e} \times n. \quad (8.15)$$

In the noncommutative theory the situation is less clear. For topological vortices of the Nielsen-Olesen type, circumstantial evidence [13, 15] indicates that (8.15) likely remains true, even if no
general proof is available as yet. For comparison, for the non-topological vortices that appear in non-relativistic Chern-Simons theory without an external field the flux does not appear to be quantized \[8\].

At this stage, self-dual solutions can be constructed using those results in \[13\] \[14\] \[15\]. Comparison with the Nielsen-Olesen SD equations (6.7-6.8) reveals, however, a subtle difference: we don’t have the freedom to choose the sign in the second equation: our (8.11) is in fact one of the field equations. There is instead the freedom in choosing the sign of \(\kappa/e\). Redefining \(B\) as \(\tilde{B} = |\kappa/e|B\) brings indeed (8.11) to the SD form in (6.7) for \(\frac{\kappa}{e} \geq 0\), and to the ASD form (6.8) for \(\frac{\kappa}{e} \leq 0\).

Then the SD solutions constructed in \[13\] \[14\] \[15\] provide us with non-relativistic, external-field vortices. Our vortices carry a (statistical) electric field except for \(\kappa = \pm 2\) while those, relativistic, considered in \[13\] \[14\] \[15\] are purely magnetic. Our NC external-field vortices also differ from those Maxwell-Chern-Simons objects in \[22\] as these latter are fully relativistic.

As a final example, let us combine the models in Sections 7 and 8, i.e., consider a spin \(\frac{1}{2}\) field in a constant external field. In the frame where \(E^{ext} = 0\), we study hence the static system

\[
e A_t \star \Phi + \frac{1}{2} \tilde{D}^2 \Phi + \frac{e}{2} B \star \Phi = 0
\]

\[
\kappa e_{ik} E_k + e j_i \rho - e_{ik} D_k B = 0
\]

\[
\kappa B = e (B^{ext} - \rho)
\]

cf. (7.11-8.9-8.10), where \(\Phi\) denotes the upper component of the Pauli spinor \((\Phi \chi)\). Eliminating the \(\chi\) component, the current reads

\[
j_1 = \frac{1}{2i} (D_+ \Phi \star \Phi - \Phi \star D_+ \Phi), \quad j_2 = -\frac{1}{2} (D_+ \Phi \star \Phi + \Phi \star D_+ \Phi).
\]

The electric field is \(E = \tilde{D} A_t\).

Let us now search for solutions. As a first attempt, try the self-duality, \(D_+ \Phi = 0\). Then \(\tilde{D}^2 \Phi = -e B \star \Phi\) so that (8.16) requires \(A_t = 0\). But then \(E = 0\) and as plainly \(j = 0\), Ampère’s law (8.17), only allows for a trivial magnetic field, \(\tilde{D} B = 0\). No SD solution is hence obtained.

Somewhat surprisingly, *antiselfdual* solutions may exist, however \[21\]. For \(D \Phi = 0\) we have instead \(\tilde{D}^2 \Phi = +e B \star \Phi\) so that the static Schrödinger-Pauli equation (8.16) [as well as its “square root, the gauged Lévy-Leblond equation (7.2)] can be satisfied with a nontrivial scalar potential, \(A_t = -B\). As now \(D_+ \Phi = 2D_1 \Phi = 2iD_2 \Phi\), the currents do not vanish but are rather expressed as \(j_i = e_{ik} D_k \rho\). Ampère’s law (8.17) requires therefore \((2\kappa + 1)\tilde{D} \times B = 0\). In conclusion, the field equations are satisfied by the ASD Ansatz, provided

\[
\kappa = -\frac{1}{2}.
\]

Note that this solution, obtained again by using the results in \[14\] \[15\], has nonvanishing electric field and also a nonvanishing lower component, namely \(\chi = (i/2)D_+ \Phi = iD_1 \Phi = -D_2 \Phi\).

\[\text{The fact that the latter investigations concern the noncommutative relativistic Abelian Higgs model are without importance here: we are interested by solving the same equations with the same boundary conditions, and we can ignore their origin.}\]
9 Conclusion

In our previous paper [4], we found that a scalar field theory augmented with a “pure” potential $V(\rho_a)$, $a = l, r$ admitted both the conventional and implementing the boosts from the left, also the “exotic” Galilean symmetry. When a $U(1)$ gauge field with Chern-Simons is added, both these implementations are broken, but Galilean symmetry can be restored by having the boost act from the right. Then we recover the exotic symmetry up to changing the sign of the noncommutative parameter $\theta$. This is, furthermore, the unique implementation consistent with the Seiberg-Witten map. It is remarkable that the interactions determine the way Galilean symmetry should act.

Interestingly, the theory can be modified so that the fundamental representation of the boosts, $\delta^l$, acts as a symmetry. Switching from the covariant derivative $D^l_\mu \equiv D_\mu$ in (3.1) to

$$D^r_\mu \psi = \partial_\mu \psi + \psi \ast (ieA_\mu)$$

merely results in replacing the left-quantities $\vec{j}$ and $\rho^l$ in (3.9) by minus the corresponding right-quantities (3.10). As this latter transforms homogeneously under the left-boost $\delta^l$ (and inhomogeneously under the right-boost $\delta^r$),

$$\delta^l \rho^r = -t\vec{b} \cdot \vec{\nabla} \rho^r \quad \text{and} \quad \delta^r \rho^r = \theta \vec{b} \times \vec{\nabla} \rho^r - t \vec{b} \cdot \vec{\nabla} \rho^r,$$

(cf. 3.13-3.16), the new Gauss law with $\rho^r$ is form-invariant w. r. t. the fundamental representation $\delta^l$. The invariance of the remaining equations can be shown readily. Then the (right)boost (3.17) becomes left-boost,

$$\vec{G}^l = t\vec{P} - \int d^2 \vec{x} \bar{\psi} \gamma^l \psi = t\vec{P} - \int x_i |\psi|^2 d^2 \vec{x} + \frac{\theta}{2} \epsilon_{ij} \int \frac{1}{2i} \left( \bar{\psi} \partial_j \psi - \partial_j (\bar{\psi}) \psi \right) d^2 \vec{x}.$$  

Switching form the right-handed expressions to the left-handed one amounts hence to changing the sign of $\theta$.

The second part of this paper is devoted to a discussion of various vortex solutions. First we studied spin $\frac{1}{2}$ particles, described by the Lévy-Leblond equation. After demonstrating the (exotic) Galilean symmetry, we have shown how spinning nontopological vortices can be constructed using previous results. Due to the breaking of the scale invariance cf. Section 5, we can not guarantee, however, that all solutions would be self-dual, even for the critical value of the coupling. Finally, we presented topological scalar vortices in a constant (electro)magnetic background.

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