EXPONENTIAL CONVERGENCE FOR MULTIPOLE EXPANSION AND TRANSLATION TO LOCAL EXPANSIONS FOR SOURCES IN LAYERED MEDIA: 2-D ACOUSTIC WAVE

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Abstract. In this paper, first we will give a derivation of the multipole expansion (ME) for far field from sources in 2-D layered media and the multipole to local translation (M2L) operator using the Bessel generating function. Then, we present a rigorous proof of the exponential convergence of the ME and M2L for 2-D Helmholtz equations in layered media.

1. Introduction

The multipole expansion (ME) and multipole to local (M2L) translation operators form the mathematical foundation of fast multipole method (FMM) for evaluating the integral operators defined by the Green’s function of Helmholtz equations in wave scattering [9]. The ME and M2L formulas for Helmholtz equation in 2-D layered media were first proposed in [1]. There, the ME and M2L were derived by using a Sommerfeld representation in frequency domain of the domain Green’s function [3] where the domain Green’s function is expressed as plane waves in the frequency domain in a similar format as the free-space Green’s function. As a result, the ME and M2L operations for the free space Green’s function can be re-used for the half space and in principle to general layered media. Moreover, in the case of the half space case with impedance boundary condition, an alternative image representation of the domain Green’s function is used to justify the same truncation order of ME and M2L as that for the free space in a proposed heterogeneous FMM (H-FMM) for the half space with impedance boundary condition [2].

In this paper, firstly, by using the generating function of the Bessel function, we give an alternative derivation of the formulas for ME of far field for sources in 2-D layered media for acoustic wave scattering, and also that of the M2L translation operators as proposed in [1]. Secondly, we will give a rigorous proof of the exponential convergence of the ME and M2L translation operator for sources in layered media for 2-D acoustic wave scattering.

The rest of the paper is organized as follows. In Section 2, we first give some technical tools which will be key to the work in this paper, including the generating function of the Bessel functions which relates plane waves to cylindrical waves and the growth condition of the Bessel functions. Section 3 uses the Bessel generating function to derive the analytical formula for the ME expansions for sources in 2-D layered media and also for the M2L translation operator. In section 4, we will give the proof of the exponential convergence of the ME for a special case when the far field location is directly above or below the center of the ME. This proof will show the main technical tools for more general far field cases. In Section 5, we first introduce a special transform such that the the Sommerfeld integrals in Green’s function representation for the general far field location can be related to the integrals in the special far field case in Section 4, however involving a complex domain contour integral. Then, we deform the contour to one parallel to the real axis and carry out the error estimate. Finally, in Section 6, we present the exponential convergence of the ME and M2L translation for general sources and far field locations in layered media. A conclusion is given in Section 7 while Appendices contain some technical lemmas and proofs of two lemmas from the main text.

2. An identity and estimate on Bessel functions

We begin with the identity of the generating function for Bessel functions of the first kind [6] which will be the key to derive the ME expansion for sources in layered media

$$\sum_{p=-\infty}^{\infty} J_p(a)b^p = \exp\left(\frac{a}{2}(b-b^{-1})\right) := g(a,b)$$

for any complex number $a$ and any nonzero complex number $b$.

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The identity (1) expresses a plane wave function in terms of cylindrical functions in contrast to the Sommerfeld integral representation which expresses the Green’s function (cylindrical function) in terms of plane waves. This duality facilitates the derivation of ME and M2L formulas in this paper.

Next, we will give a growth estimate of the Bessel functions, which will be the key to derive the exponential convergence rate of truncated ME and M2L operators for layered media.

**Lemma 1.** Let \( n \in \mathbb{N} \), \( x \in [0, +\infty) \), then

\[
|J_n(2x)| \leq \frac{x^n}{n!},
\]

(2)

**Proof.** First, from

\[
|J_n(2x)| = \left| \frac{1}{2\pi i} \int_0^{2\pi} e^{i(n\phi+2x\cos\phi)} d\phi \right| \leq 1,
\]

(3)

and

\[
\frac{(n+\frac{1}{2})^n}{n!} = \prod_{k=1}^{n} \frac{(\frac{n+1}{2})^2}{k(n+1-k)} \geq 1,
\]

(4)

and by [5], the first positive root of \( J_n \) satisfies

\[
j_{n,1} > \sqrt{(n+1)(n+5)} > n+1,
\]

(5)

here \( j_{n,k} \) is the \( k \)-th positive root of \( J_n \), it suffices to prove the inequality for \( x \in [0, \frac{j_{n,1}}{2}] \).

Second, it is well-known that \( J_n(z) \) is an entire function for \( z \in \mathbb{C} \). By [7], \( J_n(2x) \) has Weierstrass factorization

\[
J_n(2x) = \frac{x^n}{n!} \prod_{k=1}^{\infty} \left( 1 - \frac{4x^2}{j_{n,k}^2} \right).
\]

(6)

For \( x \in [0, \frac{j_{n,1}}{2}] \), each term of the infinite product satisfies

\[
0 \leq 1 - \frac{4x^2}{j_{n,k}^2} \leq 1,
\]

(7)

so

\[
0 \leq J_n(2x) \leq \frac{x^n}{n!}
\]

(8)

for \( x \in [0, \frac{j_{n,1}}{2}] \). \( \square \)

3. **The multipole expansion for sources in layered media and ME to local translation**

We will use the identity (1) to derive both the multipole expansion and the multipole to local translation formulas for the integrals of 2-D layered problems.

3.1. **Multipole expansion in free space.** Consider \( N \) sources with strength \( q_j \) placed at \( x_j = (x_j, y_j) \) in a circle centered at \( x_c = (x_c, y_c) \) with radius \( r \) in free space \( \mathbb{R}^2 \), and a field location at \( x \) due to all the source points given by

\[
u^f(x) = \sum_{j=1}^{N} q_j G(x, x_j),
\]

where the free-space Green’s function

\[
G(x, x') = \frac{1}{4} H_0(k|\mathbf{x} - \mathbf{x}'|).
\]

We say \( x \) is well-separated from the sources if the distance between \( x \) and the source circle center \( x_c \) is at least \( 3r \).

By using Graf’s addition theorem [8], the free-space Green’s function interaction of well-separated sources \( x_j \) and target \( x \) can be compressed as a multipole expansion given by

\[
u^f(x) \approx \frac{i}{4} \sum_{p=-P}^{P} \alpha_p H_p(k|\mathbf{x} - \mathbf{x}_c|) e^{ip\theta_c},
\]

(9)

where

\[
\alpha_p = \sum_{j=1}^{N} q_j J_p(k\rho_j) e^{-ip\theta_j},
\]

(10)

\( \theta_c \) is the polar angle of \( x - x_c \), \((\rho_j, \theta_j)\) are the polar coordinates of the complex number \( x_j - x_c \), and the number of terms \( P \) is a constant independent of the number of the sources \( N \) [9].
3.2. **The multipole expansion in layered media.** First consider a case of two layers media (half space) with 
\( y = d \) as the interface, suppose \( \mathbf{x}, \mathbf{x}_c, \) and each \( \mathbf{x}_j \) are in the upper half space. In this case, the scattering field of all sources \( q_j \) at \( \mathbf{x}_j, \ 1 \leq j \leq N \) are expressed in terms of the integral

\[
u_1(\mathbf{x}) = \sum_{j=1}^{N} \frac{q_j}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2} (y-y_c)+i\lambda (x-x_j)}}{\sqrt{\lambda^2-k^2}} \sigma_1(\lambda) \, d\lambda, \tag{11}
\]

where \( \sigma_1(\lambda) \) is the reflection coefficient in the frequency domain for the layer 1 [4]. We also assume \( \exp(-\epsilon |\lambda|) \sigma_1(\lambda) \rightarrow 0 \) for \( \forall \epsilon > 0 \) as \( \lambda \rightarrow \pm \infty \), because the integral must converge for any pair of source and target points appearing at any places in their corresponding layers.

When we recenter on \( \mathbf{x}_c \), for each \( j \) we use the identity [11] to get

\[
e^{-\sqrt{\lambda^2-k^2} (y-y_c)+i\lambda (x-x_j)} = g \left( k\rho_j, -ie^{i\theta_j} \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right) = \sum_{p=-\infty}^{\infty} J_p(k\rho_j)e^{ip\theta_j}(-i)^p \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \tag{12}
\]

(regardless of the sign choice of the square root), so

\[
\sum_{j=1}^{N} \frac{q_j}{4\pi} e^{-\sqrt{\lambda^2-k^2} (y-y_c)+i\lambda (x-x_j)} = \frac{1}{4\pi} \sum_{p=-\infty}^{\infty} \alpha_p (-i)^p \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p , \tag{13}
\]

where \( \alpha_p \) are exactly the multipole expansion coefficients given in [10] for sources in the free space, and \( \alpha_p \) is the complex conjugate.

Apply equation (13) to the integral \( u_1 \) we get

\[
u_1 = \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2} (y-d)-\sqrt{\lambda^2-k^2} (y_c-d)+i\lambda (x-x_c)}}{\sqrt{\lambda^2-k^2}} \sigma_1(\lambda) \cdot \sum_{j=1}^{N} \frac{q_j}{4\pi} e^{-\sqrt{\lambda^2-k^2} (y-y_c)+i\lambda (x-x_j)} \, d\lambda
\]

\[
= \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2} (y-d)-\sqrt{\lambda^2-k^2} (y_c-d)+i\lambda (x-x_c)}}{\sqrt{\lambda^2-k^2}} \sigma_1(\lambda) \cdot \frac{1}{4\pi} \sum_{p=-\infty}^{\infty} \alpha_p (-i)^p \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \, d\lambda
\]

\[
= \frac{1}{4\pi} \sum_{|p| < \infty} \alpha_p \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2} (y-d)-\sqrt{\lambda^2-k^2} (y_c-d)+i\lambda (x-x_c)}}{\sqrt{\lambda^2-k^2}} (-i)^p \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma_1(\lambda) \, d\lambda. \tag{14}
\]

The proof that guarantees the interchange between the summation and the integration will be given in Theorem [3] with the far-field assumption \((x-x_c)^2+(y-d+y_c-d)^2 > \rho_j^2\).

The truncation of \( u_1 \) will result in a \( P \)-term ME,

\[
u_1 \approx \frac{1}{4\pi} \sum_{|p| < P} \alpha_p \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2} (y-d)-\sqrt{\lambda^2-k^2} (y_c-d)+i\lambda (x-x_c)}}{\sqrt{\lambda^2-k^2}} (-i)^p \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma_1(\lambda) \, d\lambda. \tag{15}
\]

When there are two or more layers, with up to a sign change, we deal with the integrals in the following forms. Consider the field in a layer 2 (any layer, not necessarily adjacent to layer 1 where the sources locate)

\[
u_2^+(\mathbf{x}) = \sum_{j=1}^{N} \frac{q_j}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\lambda^2-k^2} (y-d)+\sqrt{\lambda^2-k^2} (y_c-d_1)+i\lambda (x-x_j)}}{\sqrt{\lambda^2-k^2}} \sigma_2(\lambda) \, d\lambda, \tag{16}
\]

where \( \sigma_2(\lambda) \) is the reflection/transmission coefficient in the frequency domain and we assume that \( \pm (y-d_2) > 0 \), \( y_c - d_1 > 0 \) and each \( y_j - d_1 > 0 \), here \( d_1 \) and \( d_2 \) are interface coordinates closest to the sources and the target, respectively, and the sign of \( \pm \) indicates the direction of the wave propagation. Two other cases can be derived by a \( y \rightarrow -y \) mapping together with \( \lambda \rightarrow -\lambda \). The multipole expansion is given similarly by
\[ u^\pm_2 = \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2-k_1^2} \pm (y-y_c)+i\lambda(x-x_c)} \frac{\sigma_2(\lambda) \cdot \frac{1}{4\pi} \sum_{\alpha=0}^{\infty} \alpha_p(-i)^p \left( \frac{\lambda - \sqrt{\lambda^2-k_1^2}}{k_1} \right)^p d\lambda}{\sqrt{\lambda^2-k_1^2}} \]

\[ \approx \frac{1}{4\pi} \sum_{|p|<P} \alpha_p \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2-k_1^2} \pm (y-y_c)+i\lambda(x-x_c)} \frac{\sigma_2(\lambda) d\lambda}{\sqrt{\lambda^2-k_1^2}} \]

\[ = \frac{1}{4\pi} \sum_{|p|<P} \alpha_p I_p \]

where

\[ I_p = \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2-k_1^2} \pm (y-y_c)+i\lambda(x-x_c)} \frac{\sigma_2(\lambda) d\lambda}{\sqrt{\lambda^2-k_1^2}} (\lambda - \sqrt{\lambda^2-k_1^2})^p \]

and

\[ \alpha_p = \sum_{j=1}^{N} q_j J_p(k_1 \rho_j) e^{-ip\theta_j}, \]

with the wavenumber \( k_1 \) being used inside the Bessel function \( J_p(k_1 \rho) \).

3.3. The M2L translation in layered media. Here we focus on the integral \( I_p \) in the ME (17). When we recenter locally at \((x_c', y_c')\), using (1),

\[ e^{-\sqrt{\lambda^2-k_1^2} \pm (y-y_c)+i\lambda(x-x_c')} = g \left( k_1 \rho, \rho e^{\pm \theta} \frac{\lambda + \sqrt{\lambda^2-k_1^2}}{k_2} \right) \]

\[ = \sum_{m=-\infty}^{\infty} J_m(k_1 \rho) e^{\pm im\theta} m^m \left( \frac{\lambda + \sqrt{\lambda^2-k_1^2}}{k_2} \right)^m, \]

so \( I_p \) can be rewritten as

\[ I_p = \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2-k_1^2} \pm (y-y_c)+i\lambda(x-x_c')} \frac{\sigma_2(\lambda) d\lambda}{\sqrt{\lambda^2-k_1^2}} (\lambda - \sqrt{\lambda^2-k_1^2})^p \cdot e^{-\sqrt{\lambda^2-k_1^2} \pm (y-y_c)+i\lambda(x-x_c')} d\lambda \]

\[ = \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2-k_1^2} \pm (y-y_c)+i\lambda(x-x_c')} \frac{\sigma_2(\lambda) d\lambda}{\sqrt{\lambda^2-k_1^2}} (\lambda - \sqrt{\lambda^2-k_1^2})^p \cdot \sum_{m=-\infty}^{\infty} J_m(k_1 \rho) e^{\pm im\theta} m^m w_2^m d\lambda \]

\[ \approx \sum_{|m|<M} A_{m,p} J_m(k_1 \rho) e^{\pm im\theta}, \]

here

\[ w_1 = \frac{\lambda - \sqrt{\lambda^2-k_1^2}}{k_1}, \quad w_2 = \frac{\lambda - \sqrt{\lambda^2-k_1^2}}{k_2}, \]

and

\[ A_{m,p} = \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2-k_1^2} \pm (y-y_c)+i\lambda(x-x_c')} \frac{\sigma_2(\lambda) d\lambda}{\sqrt{\lambda^2-k_1^2}} (\lambda - \sqrt{\lambda^2-k_1^2})^p \cdot \sum_{m=-\infty}^{\infty} J_m(k_1 \rho) e^{\pm im\theta} m^m w_2^m d\lambda \]

will be the M2L translation matrix for the layered media.

Therefore, if the local expansion is sought in the following form

\[ u^\pm_2 \approx \frac{1}{4} \sum_{|m|<M} \beta_m J_m(k_1 \rho) e^{\pm im\theta}, \]

then, the coefficients are

\[ \beta_m = \frac{1}{k_1} \sum_{|p|<P} A_{m,p} \alpha_p. \]
4. Exponential convergence of ME for far field above the center of ME

In this section, we will give the proof of the exponential convergence for the special case when the far field location is directly above or below the center of the ME. This proof will lay down the main technical approach for general far field locations to be followed.

**Lemma 2.** Let \( k_0 \geq k > 0 \). Let \( r > 0 \), \( x \in \mathbb{R} \), \( y \in \mathbb{R}^+ \), and

\[
E_p^0 = \int_{-k_0}^{k_0} e^{-\sqrt{x^2+y^2+\lambda x}} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma(\lambda) d\lambda, \quad p \in \mathbb{Z},
\]

(27)

here the function \( \sigma(\lambda) \) is assumed to satisfy

\[
\frac{\sigma(\lambda)}{\sqrt{\lambda^2 - k^2}} \in L^1[-k_0,k_0].
\]

(28)

Then, there exists a \( P \in \mathbb{N} \) indepenedy of \( x \) or \( y \) such that for any \( p \) satisfying \( |p| > P \),

\[
|E_p^0| \leq |p|! \left( \frac{kr}{2} \right)^{-|p|},
\]

(29)

where \( r = \sqrt{x^2 + y^2} \).

**Proof.** For any \( \lambda \in [-k_0,k_0] \),

\[
|e^{-\sqrt{x^2+y^2+\lambda x}}| \leq 1
\]

(30)

and

\[
\left| \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right| \leq \frac{2k_0}{k},
\]

(31)

so for each \( p \), \( E_p^0 \) is bounded by

\[
|E_p^0| \leq \int_{-k_0}^{k_0} \left| \frac{\sigma(\lambda)}{\sqrt{\lambda^2 - k^2}} \right| \left( \frac{2k_0}{k} \right)^{|p|} d\lambda = \left( \frac{2k_0}{k} \right)^{|p|} \left\| \frac{\sigma(\lambda)}{\sqrt{\lambda^2 - k^2}} \right\|_{L^1[-k_0,k_0]}.
\]

(32)

Since \( |p| \to +\infty \),

\[
|p|! \left( \frac{kr}{2} \right)^{-|p|} \left( \frac{2k_0}{k} \right)^{-|p|} \to +\infty,
\]

(33)

\( \exists P \in \mathbb{N} \) such that for any integer \( p \) satisfying \( |p| > P \),

\[
|p|! \left( \frac{kr}{2} \right)^{-|p|} \left( \frac{2k_0}{k} \right)^{-|p|} > \left\| \frac{\sigma(\lambda)}{\sqrt{\lambda^2 - k^2}} \right\|_{L^1[-k_0,k_0]},
\]

(34)

then for \( |p| > P \) we have

\[
|E_p^0| \leq \left( \frac{2k_0}{k} \right)^{|p|} \int_{-k_0}^{k_0} \left| \frac{\sigma(\lambda)}{\sqrt{\lambda^2 - k^2}} \right| d\lambda \leq |p|! \left( \frac{kr}{2} \right)^{-|p|}.
\]

(35)

\( \square \)

**Lemma 3.** Let \( k > 0 \). Let \( r > 0 \), \( K \) is a nonnegative integer, and

\[
E_p^+ = \int_k^{\infty} e^{-\sqrt{x^2+y^2}} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma(\lambda) d\lambda, \quad p \in \mathbb{Z},
\]

(36)

here the function \( \sigma(\lambda) \) satisfies

\[
\frac{\sigma(\lambda)}{\lambda k \sqrt{\lambda^2 - k^2}} \in L^1[k, +\infty).
\]

(37)

Then, there exists \( P \in \mathbb{N} \) such that for any \( p \) satisfying \( |p| > P \),

\[
|E_p^+| \leq (|p| + K)! \left( \frac{kr}{2} \right)^{-|p|}.
\]

(38)
Proof. Without loss of generality, assume that \( \sigma(\lambda) \) is real and nonnegative, otherwise replace it with \( |\sigma(\lambda)| \). Since
\[
0 < \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \leq 1 \leq \frac{\lambda + \sqrt{\lambda^2 - k^2}}{k}
\]
for \( \lambda \geq k \), we have \( E^+_{|p|} \leq E^+_{-|p|} \), hence it suffices to consider the \( p < 0 \) case. Notice that the integral with nonnegative integrand
\[
\int_k^\infty e^{\frac{kr \lambda - \sqrt{\lambda^2 - k^2}}{k}} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^K \sigma(\lambda) d\lambda < +\infty,
\]
so
\[
\int_k^\infty e^{\frac{kr \lambda - \sqrt{\lambda^2 - k^2}}{k}} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^K \sigma(\lambda) d\lambda
= \int_k^\infty e^{-\sqrt{\lambda^2 - k^2} r} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^K \sigma(\lambda) \cdot e^{\frac{kr \lambda - \sqrt{\lambda^2 - k^2}}{k} - 1} d\lambda
= \sum_{j=0}^\infty \int_k^\infty e^{-\sqrt{\lambda^2 - k^2} r} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^K \sigma(\lambda) \frac{1}{j!} \left( \frac{kr}{2} \right)^j \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^{-j} d\lambda
= \sum_{p=K}^\infty \frac{1}{(-p + K)!} \left( \frac{kr}{2} \right)^{-p+K} E_p^+ < +\infty.
\]
Therefore, there exists a \( P \in \mathbb{N} \) such that for any \( p < -P \),
\[
\frac{1}{(-p + K)!} \left( \frac{kr}{2} \right)^{-p+K} E_p^+ \leq \left( \frac{kr}{2} \right)^K
\]
which completes the proof. \( \square \)

Lemma 4. Let \( k > 0 \). Let \( r > 0 \), \( K \) is a nonnegative integer, and
\[
E_p^- = \int_{-\infty}^{-k} e^{-\sqrt{\lambda^2 - k^2} r} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma(\lambda) d\lambda, \ p \in \mathbb{Z},
\]
here the function \( \sigma(\lambda) \) satisfies
\[
\frac{\sigma(\lambda)}{\lambda^k \sqrt{\lambda^2 - k^2}} \in L^1(-\infty, -k].
\]
Then, there exists \( P \in \mathbb{N} \) such that for any \( p \) satisfying \( |p| > P \),
\[
|E_p^-| \leq (|p| + K)! \left( \frac{kr}{2} \right)^{-|p|}.
\]

Proof. A \( \lambda \mapsto -\lambda \) transformation reduces \( E_p^- \) to the case in Lemma 3 \( \square \)

Theorem 1. Let \( k > 0 \), \( 0 < \rho_{\min} < \rho_{\max} \). Suppose for some nonnegative integer \( K \), the function \( \sigma(\lambda) \) satisfies the conditions in Lemma 2, Lemma 3 and Lemma 4 namely,
\[
\frac{\sigma(\lambda)}{\sqrt{\lambda^2 - k^2}} \in L^1[-k,k],
\]
\[
\frac{\sigma(\lambda)}{\lambda^K \sqrt{\lambda^2 - k^2}} \in L^1[k, +\infty),
\]
\[
\frac{\sigma(\lambda)}{\lambda^K \sqrt{\lambda^2 - k^2}} \in L^1(-\infty, -k].
\]
Then, for any \( \epsilon > 0 \), there exist \( P \in \mathbb{N} \) and \( D > 0 \), such that for any \( \rho \in (\rho_{\min}, \rho_{\max}) \), any \( \rho_j > 0 \), any \( \theta_j \in [0, 2\pi) \) and any integer \( p \) satisfying \( |p| > P \),
\[
\left| J_p(k \rho_j e^{-ip\theta_j}) \int_{-\infty}^\infty e^{-\sqrt{\lambda^2 - k^2} r} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma(\lambda) d\lambda \right| \leq D|p|^K \left( \frac{\rho_j}{\rho} \right)^{|p|} (1 + \epsilon)^{|p|}.
\]
Proof. First, using Lemma 1 we have

$$|J_p(kp_j)e^{-xp_j}| \leq \frac{1}{|p|!} \left( \frac{kp_j}{2} \right)^{|p|}. \quad (49)$$

Second, let $n_1, n_2 \in \mathbb{Z}$ such that

$$(1 + \varepsilon)^{n_1} < kp_{\min} < kp_{\max} < (1 + \varepsilon)^{n_2}. \quad (50)$$

For each integer $n \in [n_1 - 1, n_2 + 1]$, using Lemma 3 and Lemma 4, there exist $P_n \in \mathbb{N}$ and $D_n > 0$ such that if $|p| > P_n$

$$
\int_{-\infty}^{-k} e^{-\frac{\lambda^2 - k^2}{k}} \left( -\frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p |\sigma(\lambda)| d\lambda \leq D_n |p|^K |p|! \left( \frac{(1 + \varepsilon)^n}{2} \right)^{-|p|} 
$$

and

$$
\int_{k}^{\infty} e^{-\frac{\lambda^2 - k^2}{k}} \left( -\frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p |\sigma(\lambda)| d\lambda \leq D_n |p|^K |p|! \left( \frac{(1 + \varepsilon)^n}{2} \right)^{-|p|}, \quad (52)
$$

here $D_n$ is chosen to ensure

$$D_n |p|^K \geq \frac{(|p| + K)!}{|p|!} = (|p| + 1) \cdots (|p| + K) = |p|^K + O(|p|^{K+1}) \quad (53)$$

for any $|p| > P_n$.

Let

$$P' = \max_{n \in [n_1 - 1, n_2 + 1]} P_n, \quad D' = \max_{n \in [n_1 - 1, n_2 + 1]} D_n, \quad (54)$$

and $n_0 \in \mathbb{Z}$ such that

$$(1 + \varepsilon)^{n_0} \leq kp \leq (1 + \varepsilon)^{n_0 + 1}, \quad (55)$$

then we have the following estimation: for $|p| > P'$,

$$
\left| \int_{-\infty}^{-k} e^{-\frac{\lambda^2 - k^2}{k}} \left( -\frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma(\lambda) d\lambda \right| 
\leq \int_{-\infty}^{-k} e^{-\frac{\lambda^2 - k^2}{k}} \left( -\frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p |\sigma(\lambda)| d\lambda 
\leq \int_{-\infty}^{-k} e^{-\frac{\lambda^2 - k^2}{k}} \left( -\frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p |\sigma(\lambda)| d\lambda 
\leq D'|p|^K |p|! \left( \frac{(1 + \varepsilon)^{n_0}}{2} \right)^{-|p|} \quad (56)
$$

and similarly

$$
\left| \int_{k}^{\infty} e^{-\frac{\lambda^2 - k^2}{k}} \left( -\frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma(\lambda) d\lambda \right| 
\leq \int_{k}^{\infty} e^{-\frac{\lambda^2 - k^2}{k}} \left( -\frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p |\sigma(\lambda)| d\lambda 
\leq D'|p|^K |p|! \left( \frac{(1 + \varepsilon)^{n_0}}{2} \right)^{-|p|}. \quad (57)
$$

For the rest of the integration over interval $[-k, k]$, in Lemma 2 choose $k_0 = k, x = \rho, y = 0$ and $r = (1 + \varepsilon)^{n_2}/k$, we know there exists $P''$ without dependence on $\rho$ such that if $|p| > P''$, 

$$
\left| \int_{-k}^{k} e^{-\frac{\lambda^2 - k^2}{k}} \left( -\frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma(\lambda) d\lambda \right| 
\leq 2 |p|! \left( \frac{(1 + \varepsilon)^{n_2}}{2} \right)^{-|p|} \quad (58)
$$

In sum, let

$$P = \max(P', P''), \quad D = 2D' + 2, \quad (59)$$
we have
\[ \left| \int_{-\infty}^{\infty} e^{-\sqrt{x^2 - k^2} \rho} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma(\lambda) d\lambda \right| \leq D |p|^K |\rho|! \left( \frac{(1 + c) \rho}{2} \right)^{-|p|}. \]  

(60)

Finally, the product of the two parts on the left of (48) can be bounded by
\[ \leq D |p|^K \left( \frac{k \rho_j}{(1 + c) \rho} \right)^{|p|} \]  

(61)
\[ \leq D |p|^K \left( \frac{\rho_j}{\rho} \right)^{|p|} (1 + c)^{|p|}, \]

where the right inequality in (55) is used in the last step. \(\square\)

**Remark 1.** From the proof we can see the constant \(D\) can also be chosen only depending on \(K\), for example,
\[ D = \frac{2K^K}{K!} + 2 \]  

(62)

will work, but the larger \(P_n\) are chosen the smaller \(D\) could be.

**Remark 2.** If for \(C \in \mathbb{R}^+\) and \(K \in \mathbb{N}\), the function \(\sigma(\lambda)\) satisfies \(|\sigma(\lambda)| \leq C (1 + |\lambda/k|^{K-1})\) for any \(\lambda \in \mathbb{R}\), then the conditions in lemmas 2-4 are all met.

**Remark 3.** If \(z_0\) is any complex number such that \(z_0 \neq k\), and \(c \in (0, 1)\), then \(\sigma(\lambda) = |\lambda - z_0|^{-c}\) meets the conditions in the above lemmas.

5. Estimates for Sommerfeld integrals for multilayered media

Theorem 1 can be treated as a special case when \((x, y) = (0, \rho)\). Now we will present a method that converts the general \((x, y)\) case to the \((0, \rho)\) case. Since a number of polynomial bounds will be used, for the sake of convenience, suppose all the variables are already nondimensionalized.

Let \(k_1, k_2, \cdots\) be the wavenumbers in all the layers. They are all the branch points of \(\sigma(\lambda)\) in the layered problem. Let \(k_{\text{max}}\) be a number greater than all the \(k_j\), i.e.,
\[ k_{\text{max}} > \max \{k_j\} \]  

(63)

Suppose the largest box of the FMM algorithm is inside the open rectangle \(\Pi\)
\[ \Pi = (-x_1/2, x_1/2) \times (-y_1/2, y_1/2). \]  

(64)

Let \(y_{\text{min}} > 0\) be a number smaller than any distance between a box center and a nearby interface. These assumptions ensure that when the ME (17) and the M2L (22) cases are described as an \((x, y)\) problem, we always have \(-x_1 < x < x_1\) and \(0 < y_{\text{min}} < y < y_1\). Let
\[ T = \frac{x_1}{y_{\text{min}}} > 0. \]  

(65)

For any open set \(D\), let \(B(D)\) be the collection of all the holomorphic functions \(f(z)\) on \(D\) satisfying the following condition: there exist \(C > 0\) and \(K \in \mathbb{N}\) such that for any \(q \in [-T, T]\) and any \(t > 0\), if \(z = (1 + qt)t \in D\), then
\[ |f(z)| \leq C (1 + |z|^{K-1}). \]  

(66)
i.e. \(f(z)\) is bounded by a given polynomial of \(|z|\) in a fan-shaped region with slope within \([-T, T]\). Note that the coefficients \(C\) and \(K\) could depend on \(T\).

5.1. A transformation. The gist of the conversion from the case of \((x, y)\) to the case of \((0, \rho)\) is to change the exponential component from \(-\sqrt{x^2 - k^2} y + \lambda x\) to \(-\sqrt{x_1^2 - k_0^2} \rho\) by some certain substitution from \(\lambda\) on the real axis to \(\lambda_1 \in \gamma\), where \(\gamma\) is a complex contour defined in (72) (see illustration in Fig 1). Then, we change the contour of integration with respect to \(\lambda_1\) to a horizontal line to prove the convergence. To precisely describe the transformation, some detailed discussion is necessary.

Define open set
\[ \Omega = \{ z \in \mathbb{C} : \mathbb{R}e z > 0, z \notin (0, k_{\text{max}}) \}. \]  

(67)
The analytic continuation of \(\sqrt{x_1^2 - k_0^2}\) (with positive real value) from the real interval \((k_{\text{max}}, +\infty)\) to \(\Omega\) always keeps positive real part.
Consider $\beta \in (0, \pi)$. and a mapping $\phi : \Omega \to \mathbb{C}$ defined by

$$\phi(z) = z \cos \beta + 1 \sqrt{z^2 - k^2 \sin \beta},$$

(68)

here the square root takes positive real part (as a result, the branch cut has no intersection with $\Omega$). Then, $\phi(z)$ is a holomorphic function of $z$ on $\Omega$. $\phi(z)$ satisfies the equation

$$\phi(z)^2 - 2\phi(z)z \cos \beta + z^2 = k^2 \sin^2 \beta.$$

(69)

Define

$$k_0 = \sqrt{3}k_{\text{max}}.$$  

(70)

For positive real $w \in [k_0, +\infty)$, we can find the inverse of $\phi$ in quadrant IV in $\Omega$, which is

$$\phi_1(w) = w \cos \beta - i \sqrt{w^2 - k^2 \sin \beta},$$

(71)

and the image of $[k_0, +\infty)$ as

$$\gamma = \phi_1([k_0, +\infty)),$$

(72)

which a part of the hyperbola in quadrant IV with an asymptote passing the origin with slope $-\tan \beta$, starting from

$$z_0 = k_0 \cos \beta - i \sqrt{k_0^2 - k^2 \sin \beta}$$

(73)

with asymptotic behavior given as

$$\phi_1(w) \sim we^{-i\beta} \rightarrow e^{-i\beta} \cdot \infty$$

(74)

as $w \to +\infty$. As $w$ increases, $\Re \phi_1(w)$ strictly increases, and $\Im \phi_1(w)$ strictly decreases. (Please refer to Fig. 1 for the function $\phi$ and Fig. 2 in Appendix B for an illustration of its mapping of dashed lines).

Let $\Gamma$ be the right branch of the hyperbola where $\phi_1(w)$ defines on, i.e.

$$\Gamma = \left\{ a + bn : a, b \in \mathbb{R}, \frac{a}{\cos \beta} = \sqrt{\frac{b^2}{\sin^2 \beta} + k^2} \right\}.$$  

(75)

Define the open regions to the right of $\Gamma$ in quadrants I and IV, respectively, by

$$D^+ = \{ z + t : z \in \Gamma, \Im z > 0, t > 0 \},$$

(76)

$$D^- = \{ z + t : z \in \Gamma, \Im z < 0, t > 0 \}.$$  

(77)

Lemma 5. $\phi|_{D^-}$ is a bijection from $D^-$ to $D^+$ with inverse

$$\phi_1(w) = w \cos \beta - i \sqrt{w^2 - k^2 \sin \beta},$$

(78)

here the square root adopts positive real part.

Proof. See Appendix B for the proof. \hfill $\square$

Remark 4. The definition of $\phi_1$ can be naturally extended to the whole $\Omega$, here the square root $\sqrt{w^2 - k^2}$ always takes positive real part.

Remark 5. It is easy to check that $\phi(\partial D^-) = \partial D^+$, and $\phi_1 = \phi^{-1}$ on $\overline{D^+}$. Also, the analytic continuation of $\phi$ exists in a neighborhood for any $z \in \partial D^-$ in quadrant IV, and the analytic continuation of $\phi_1$ exists in a neighborhood for any $w \in \partial D^+$ in quadrant I.

The closed set

$$D_\gamma = \{ z + t : z \in \Gamma, t \geq 0 \}$$

(79)

has boundaries $\gamma$ and

$$\ell = \{ z_0 + t : t \geq 0 \}.$$  

(80)

Since $\phi$ maps the interior of $D_\gamma$ to a subset of $D^+$, $\phi(D_\gamma)$ is a closed set bounded by $[k_0, +\infty) = \phi(\gamma)$ and a curve $\phi(\ell)$.

Lemma 6. Let the line $\ell$ be parameterized by $z = z_0 + t, t \geq 0$, then $\Re \phi(z) > \sqrt{k_0^2 - k^2}$ and $\Im \phi(z)$ increases as $t$ increases.
Proof. Let \( a_0, b_0, u, v \in \mathbb{R} \) such that \( z_0 = a_0 + b_0i, \sqrt{z^2 - k^2} = u + vi \). From \( uv = (a_0 + t)b_0 < 0 \) we get \( u > 0, v < 0 \). By simple calculation,

\[
2v^2 = 2b_0^2 \sqrt{(a_0 + t)^2 + b_0^2 - k^2} + 4k^2b_0^2 - ((a_0 + t)^2 + b_0^2 - k^2) \geq 2b_0^2,
\]

so \( -v \geq -b_0 \). Then,

\[
\Re \phi(z) = (a_0 + t) \cos \beta - v \sin \beta \\
\geq a_0 \cos \beta - b_0 \sin \beta \\
= k_0 \cos^2 \beta + \sqrt{k_0^2 - k^2} \sin^2 \beta \geq \sqrt{k_0^2 - k^2}.
\]

As for \( \Im \phi(z) = b_0 \cos \beta + u \sin \beta \), we have \( \partial \Im \phi(z)/\partial t = \sin \beta \cdot \partial u/\partial t \), by simple calculation,

\[
4u \frac{\partial u}{\partial t} = \frac{\partial (2u^2)}{\partial t} = 2(a_0 + t) \left( 1 + \frac{(a_0 + t)^2 + b_0^2 - k^2}{\sqrt{(a_0 + t)^2 + b_0^2 - k^2}^2 + 4b_0^2k^2} \right) > 0,
\]

so \( \Im \phi(z) \) increases as \( t \) increases. \( \square \)

5.2. The Sommerfeld integrals. Let

\[
R_0 = \left\{ \lambda = a + bi : a, b \in \mathbb{R}, a > \sqrt{k_0^2 - k^2}, |b/a| < T \right\}.
\]

Suppose function \( f(\lambda) \) is holomorphic on \( R_0 \). Then, for any \( y \in (y_{\min}, y_1), x \in (-x_1, x_1) \), \( \phi(D_\gamma) \subset R_0 \). For each integer \( p \), the integral

\[
P^+_p = \int_{k_0}^\infty e^{-\sqrt{\lambda^2 - k^2}y + \lambda x} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p f(\lambda) d\lambda
\]

(85)
can be transformed into a integral on $\gamma$, namely, by $\lambda_1 = \phi_1(\lambda)$ (or $\lambda = \phi(\lambda_1)$), the following identities hold

$$-\sqrt{\lambda_1^2 - k^2} \rho = -\sqrt{\lambda^2 - k^2} y + i\lambda x,$$

$$\frac{\lambda_1 - \sqrt{\lambda_1^2 - k^2}}{k} = e^{i\beta} \lambda - \frac{\lambda^2 - k^2}{k},$$

$$\frac{d\lambda_1}{\sqrt{\lambda_1^2 - k^2}} = \frac{d\lambda}{\sqrt{\lambda^2 - k^2}},$$

here

$$y + x = \rho e^{i\beta}, \rho > 0, \beta \in (0, \frac{\pi}{2}).$$

Hence

$$F^+_p = e^{-ip\beta} \int_{\gamma} \frac{e^{-\sqrt{\lambda_1^2 - k^2} \rho}}{\lambda - \sqrt{\lambda_1^2 - k^2}} f_1(\lambda_1) d\lambda_1,$$

and

$$f_1(\lambda_1) = f(\lambda) = f(\phi(\lambda_1)),$$

here $\lambda_1 \in D_\gamma$ and $\lambda = \phi(\lambda_1) \in\phi(D_\gamma) \subset R_0$, so $f_1(\lambda_1)$ is holomorphic in the interior of $D_\gamma$ and has an analytic continuation on the boundary.

If we further suppose $f(\lambda) \in B(R_0)$, since

$$|\phi(\lambda_1)| = |\lambda_1 \cos \beta + i \sqrt{\lambda_1^2 - k^2} \sin \beta| \leq |\lambda_1| + \sqrt{|\lambda_1|^2 + k^2} \leq 2|\lambda_1| + k,$$

$f_1(\lambda_1) = f(\phi(\lambda_1))$ has a polynomial bound in $D_\gamma$, namely, there exists $C_0 \in \mathbb{R}^+$ and $K \in \mathbb{N}$ which do not depend on $x$ or $y$ such that

$$|f_1(\lambda_1)| \leq C_0 \left(1 + |\lambda_1|^{K-1}\right), \quad \text{for} \quad \lambda_1 \in D_\gamma.$$

Next, we claim that the following change of contour of integration holds,

$$F^+_p = e^{-ip\beta} \int_{\gamma} \frac{e^{-\sqrt{\lambda_1^2 - k^2} \rho}}{\lambda - \sqrt{\lambda_1^2 - k^2}} f_1(\lambda_1) d\lambda_1$$

$$= e^{-ip\beta} \int_{\ell} \frac{e^{-\sqrt{\lambda_1^2 - k^2} \rho}}{\lambda - \sqrt{\lambda_1^2 - k^2}} f_1(\lambda_1) d\lambda_1.$$  

To show the equivalence in (94), we again consider a counterclockwise arc $\sigma$ centered at the origin with radius $r$ connecting $\gamma$ and $\ell$. If the arc $\sigma$ is parameterized as $z = re^{i\eta}$, then the range of $\eta$ is a subset of $(\beta, 0)$ (note that the point $re^{-i\beta}$ is always to the left of $\gamma$), and $dz = ire^{i\eta}d\eta$. The integrand decays exponentially with respect to $r$ as $r \to +\infty$, since the real part of the exponent has the asymptotic behavior

$$\Re \left(-\sqrt{(re^{i\eta})^2 - k^2} \rho\right) \sim \Re (re^{-i\eta} \rho) \leq -r y,$$

while the polynomial order contribution from other parts of the integrand can be ignored. Hence the integral along the arc converges to 0 as $r \to +\infty$. Therefore equality holds between the two integrals in (94).

Finally we give an upper bound of the integral on $\ell$. Let $\ell$ be parameterized by $\lambda_1 = a + ib$ with the parameter $a \geq a_0 = k_0 \cos \beta$, and a constant $b = -\sqrt{k^2 - k^2} \sin \beta$. Let

$$K_{1,p}(\rho) = \int_{a_0}^{k_0} \left|e^{-\sqrt{\lambda_1^2 - k^2} \rho} \right| \frac{\lambda_1 - \sqrt{\lambda_1^2 - k^2}}{k} |f_1(\lambda_1)| da,$$

$$K_{2,p}(\rho) = \int_{k_0}^{\infty} \left|e^{-\sqrt{\lambda_1^2 - k^2} \rho} \right| \frac{\lambda_1 - \sqrt{\lambda_1^2 - k^2}}{k} |f_1(\lambda_1)| da,$$

then

$$|F^+_p| \leq K_{1,p}(\rho) + K_{2,p}(\rho).$$
For \( a \in [a_0, k_0) \), by simple calculation,

\[
\left| e^{-\sqrt{\lambda_1^2 - k^2}\rho} \right| \leq 1, \quad \text{(99)}
\]

\[
\sqrt{\lambda_1^2 - k^2} \geq \sqrt{k_0^2 - 2k^2}, \quad \text{(100)}
\]

\[
\left| \frac{\lambda_1 \pm \sqrt{\lambda_1^2 - k^2}}{k} \right| \leq \frac{2\sqrt{2k_0}}{k}, \quad \text{(101)}
\]

\[
|\lambda_1| \leq \sqrt{2k_0}, \quad \text{(102)}
\]

so, with (93) we have

\[
K_{1,p}(\rho) \leq \int_{a_0}^{k_0} \frac{C_0 (1 + (\sqrt{2k_0})^{K-1})}{\sqrt{k_0^2 - 2k^2}} \left( \frac{2\sqrt{2k_0}}{k} \right)^{|p|} \rho \leq C_K \left( \frac{2\sqrt{2k_0}}{k} \right)^{|p|}, \quad \text{(103)}
\]

where

\[
C_K = \frac{k_0 C_0 (1 + (\sqrt{2k_0})^{K-1})}{\sqrt{k_0^2 - 2k^2}}. \quad \text{(104)}
\]

Since as \(|p| \to +\infty\),

\[
\frac{1}{2}(|p| + K)! \left( \frac{k p}{2} \right)^{-|p|} \left( \frac{2\sqrt{2k_0}}{k} \right)^{-|p|} \to +\infty, \quad \text{(105)}
\]

\[\exists P_1 \in \mathbb{N} \text{ such that for any integer } p \text{ satisfying } |p| > P_1,\]

\[
\frac{1}{2}(|p| + K)! \left( \frac{k p}{2} \right)^{-|p|} \left( \frac{2\sqrt{2k_0}}{k} \right)^{-|p|} > C_K, \quad \text{(106)}
\]

which implies that

\[
K_{1,p}(\rho) \leq \frac{1}{2}(|p| + K)! \left( \frac{k p}{2} \right)^{-|p|}. \quad \text{(107)}
\]

For \( a \in [k_0, +\infty) \), by simple calculations,

\[
\left| e^{-\sqrt{\lambda_1^2 - k^2}\rho} \right| \leq e^{-a + \frac{k_0^2}{2k^2}}, \quad \text{(108)}
\]

\[
\sqrt{\lambda_1^2 - k^2} \geq a - \frac{k^2}{k_0}, \quad \text{(109)}
\]

\[
\left| \frac{\lambda_1 \pm \sqrt{\lambda_1^2 - k^2}}{k} \right| \leq \frac{k}{a}, \quad \text{(110)}
\]

\[
\left| \frac{\lambda_1 \pm \sqrt{\lambda_1^2 - k^2}}{k} \right| \leq \frac{2a + k_0}{k}, \quad \text{(111)}
\]

\[
|\lambda_1| \leq a + k_0, \quad \text{(112)}
\]

so

\[
K_{2,p}(\rho) \leq \hat{K}_{2,p}(\rho) = \int_{k_0}^{+\infty} \hat{f}(a) e^{-a + \frac{k_0^2}{2k^2}} \left( \frac{2a + k_0}{k} \right)^{K+|p|} da, \quad \text{(113)}
\]

here

\[
\hat{f}(a) = \frac{C_0 (1 + (a + k_0)^{K-1})}{a - \frac{k_0^2}{k^2}} \left( \frac{k}{a} \right)^{K} \in L^1[k_0, +\infty) \quad \text{(114)}
\]
is nonnegative. Then,
\[
\int_{k_0}^{+\infty} \tilde{f}(a)e^{i\left(\frac{\lambda a}{2} - \frac{a^2}{2}\right)} \rho da \\
= \int_{k_0}^{+\infty} \tilde{f}(a) \sum_{j=0}^{+\infty} e^{-\left(a + \frac{k_j^2}{4\pi^2}\right)\rho} \left(\frac{2a + k_0}{k}\right)^j \frac{1}{j!} \left(\frac{kp}{2}\right)^j da \\
= \sum_{j=0}^{+\infty} \frac{1}{j!} \left(\frac{kp}{2}\right)^j \int_{k_0}^{+\infty} \tilde{f}(a)e^{-\left(a + \frac{k_j^2}{4\pi^2}\right)\rho} \left(\frac{2a + k_0}{k}\right)^j da \\
< + \infty,
\]
so \(\exists P_2 \in \mathbb{N}\) such that for any \(j > P_2\),
\[
\frac{1}{j!} \left(\frac{kp}{2}\right)^j \int_{k_0}^{+\infty} \tilde{f}(a)e^{-\left(a + \frac{k_j^2}{4\pi^2}\right)\rho} \left(\frac{2a + k_0}{k}\right)^j da < \frac{1}{2} \left(\frac{kp}{2}\right)^K.
\]
For any integer \(p\) such that \(|p| > P_2\), let \(j = |p| + K\), we have
\[
K_{2,p}(\rho) \leq \tilde{K}_{2,p}(\rho) \leq \frac{1}{2}(|p| + K)! \left(\frac{kp}{2}\right)^{-|p|}. \tag{116}
\]
In summary, for any integer \(p\) such that \(|p| > \max\{P_1, P_2\}\), we have
\[
|\hat{F}_p| \leq K_{1,p}(\rho) + K_{2,p}(\rho) \leq (|p| + K)! \left(\frac{kp}{2}\right)^{-|p|}. \tag{117}
\]
This can be understood as a \((x, y)\)-variant of Lemma 3.

The above result is immediately generalized to the \(x < 0\) case by a \(\lambda \mapsto -\lambda\) transformation (and thinking about the holomorphic functions of \(\lambda\)), and the \(x = 0\) case coincides the situation discussed by Lemma 3.

Next, the above can be generalized to the interval \((-\infty, -k_0]\) by a \(\lambda \mapsto -\lambda\) transformation, provided \(f(\lambda)\) is an even function.

**Theorem 2.** Let \(k_{\text{max}}\) be a positive real number. Suppose \(\sigma(\lambda)\) is a meromorphic function on \(\mathbb{C}\) excluding branch cuts of \(\sqrt{\lambda^2 - k_j^2}\) starting from branch points \(\pm k_1, \cdots, \pm k_l\), satisfying

1. \(\sigma(-\lambda) = \sigma(\lambda)\);
2. Each pole \(z_j \notin \mathbb{R}\);
3. \(k\) is included in the list \(k_1, \cdots, k_l \in (0, k_{\text{max}}]\), and the branch cuts always ensure \(\sqrt{\lambda^2 - k_j^2} \in \mathbb{R}^+\) for any real \(\lambda\) satisfying \(|\lambda| > k_j\);
4. Let region \(R = \{\lambda = a + bi : a, b \in \mathbb{R}, a > k_{\text{max}}, |b/a| < T\}\) (see Fig. 1 for the cone defined by the slope \(T\)), consider the analytic continuation of \(\sigma(k_{\text{max}}, +\infty)\) to \(R\) satisfying each \(\Re e^{\sqrt{\lambda^2 - k_j^2}} > 0\) for \(\lambda \in R\), denoted by \(f(\lambda)\), so \(f(\lambda)\) is holomorphic on \(R\) and \(f(\lambda) \in \mathcal{H}(R)\).

Then, \(\forall \epsilon > 0, \exists P, K \in \mathbb{N}\) and \(C \in \mathbb{R}^+\) such that \(\forall x \in (-x_1, x_1), \forall y \in (y_{\text{min}}, y_1), \forall p \in \mathbb{Z}\) such that \(|p| > P\), the integral
\[
F_p = \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2 - k_j^2}y + \lambda x} \left(\frac{\lambda - \sqrt{\lambda^2 - k_j^2}}{k}\right)^p \sigma(\lambda) d\lambda \tag{118}
\]
is bounded by
\[
|F_p| \leq C|p|^K |p|! \left(\frac{kp}{2}\right)^{-|p|} (1 + \epsilon)^{|p|}. \tag{119}
\]

**Proof.** The proof is a mixture of the methods in Theorem 1 and Lemma 2. Let
\[
\rho_{\text{min}} = y_{\text{min}}, \rho_{\text{max}} = \sqrt{y_1^2 + x_1^2}, \tag{120}
\]
and \(y + ix = \rho e^{i\beta}, \rho > 0, \beta \in (-\frac{\pi}{2}, \frac{\pi}{2})\). Then \(\rho = \sqrt{x^2 + y^2} \in (\rho_{\text{min}}, \rho_{\text{max}})\).

Let \(k_0 = \sqrt{3}k_{\text{max}}\). The region \(\phi(D_s)\) is on the right of the branch of hyperbola \(\Gamma\) and the right of \(\{\lambda \in \mathbb{C} : \Re e \lambda > \sqrt{k_0^2 - k_{\text{max}}^2}\}\). The asymptotes of \(\Gamma\) with slope \(\pm \tan \beta\), and \(\tan \beta < T\). Also
\[
\sqrt{\frac{k_1^2 - k_{\text{max}}^2}{k_0^2}} > k_{\text{max}}, \tag{121}
\]
so $\phi(D_{\gamma}) \subset R$. Then we can define
\[ f_1(\lambda_1) = f(\lambda) = f(\phi(\lambda_1)) \] (122)
for $\lambda_1 \in D_{\gamma}$ accordingly.

On the interval $(k_0, +\infty)$, by condition (4) we have $\sigma(\lambda) = f(\lambda)$, and
\[
F_p^+ = \int_{k_0}^{\infty} e^{-\sqrt{\lambda^2 - k_0^2 + \lambda x}} \frac{(\lambda - \sqrt{\lambda^2 - k_0^2})^p}{k} \sigma(\lambda) d\lambda \\
= \int_{k_0}^{\infty} e^{-\sqrt{\lambda^2 - k_0^2 + \lambda x}} \frac{(\lambda - \sqrt{\lambda^2 - k_0^2})^p}{k} f(\lambda) d\lambda \\
= e^{-ip\beta} \int_{t}^{\infty} e^{-\sqrt{\lambda^2 - k_0^2 + \lambda \epsilon^2}} \frac{(\lambda - \sqrt{\lambda^2 - k_0^2})^p}{k} f_1(\lambda_1) d\lambda_1,
\] (123)
where the last equation holds due to (14).

Let $n_1, n_2 \in \mathbb{Z}$ such that
\[
(1 + \epsilon)^{n_1 - 1} < k\rho_{\min} < k\rho_{\max} < (1 + \epsilon)^{n_2 + 1}.
\] (124)
By the discussion above, for each integer $n \in [n_1 - 1, n_2 + 1]$, there exists $P_n \in \mathbb{N}$ such that for $\rho^{(n)} = (1 + \epsilon)^n/k$, we have
\[
K_1,p(\rho^{(n)}) \leq \frac{1}{2}(|p| + K)!(\frac{k\rho^{(n)}}{2})^{-|p|} = \frac{1}{2}(|p| + K)!(\frac{(1 + \epsilon)^n}{2})^{-|p|},
\]
\[
\tilde{K}_2,p(\rho^{(n)}) \leq \frac{1}{2}(|p| + K)!(\frac{k\rho^{(n)}}{2})^{-|p|} = \frac{1}{2}(|p| + K)!(\frac{(1 + \epsilon)^n}{2})^{-|p|},
\]
for any $|p| > P_n$. For a general $\rho \in (\rho_{\min}, \rho_{\max})$, we should be able to find a $n_0 \in [n_1 - 1, n_2 + 1]$, such that
\[
(1 + \epsilon)^{n_0} \leq k\rho \leq (1 + \epsilon)^{n_0 + 1}.
\] (125)
Let
\[
P' = \max_{n \in [n_1 - 1, n_2 + 1]} P_n,
\] (126)
then for any $|p| > P'$,
\[
|F_p^+| \leq K_1,p(\rho) + \tilde{K}_2,p(\rho)
\]
\[
= C_K \left( \frac{2\sqrt{2k_0}}{k} \right)^{|p|} + \int_{k_0}^{+\infty} f(a)e^{-(a + \frac{k_0^2}{a})\rho}(\frac{2a + k_0}{k})^{K+|p|} da
\]
\[
\leq C_K \left( \frac{2\sqrt{2k_0}}{k} \right)^{|p|} + \int_{k_0}^{+\infty} f(a)e^{-(a + \frac{k_0^2}{a})^{1 + \epsilon^{n_0}}}(\frac{2a + k_0}{k})^{K+|p|} da
\]
\[
= K_1,p(\rho^{(n_0)}) + \tilde{K}_2,p(\rho^{(n_0)})
\]
\[
\leq (|p| + K)!(\frac{(1 + \epsilon)^{n_0}}{2})^{-|p|}.
\]

Next we consider
\[
F_p^- = \int_{-\infty}^{k_0} e^{-\sqrt{\lambda^2 - k_0^2 + \lambda x}} \frac{(\lambda - \sqrt{\lambda^2 - k_0^2})^p}{k} \sigma(\lambda) d\lambda,
\] (128)
similarly there exists $P'' \in \mathbb{N}$ such that for any integer $p$ satisfying $|p| > P''$,
\[
|F_p^-| \leq (|p| + K)!(\frac{(1 + \epsilon)^{n_0}}{2})^{-|p|}.
\] (129)

Also note that with the physical property of the reflection/transmission coefficient $\sigma(\lambda)$ in layered media, we always have $\sigma(\lambda)/\sqrt{\lambda^2 - k_0^2} \in L^1[-k_0, k_0]$ (regardless of the actual branch cut), so by Lemma 2 there exists $P'''' \in \mathbb{N}$ such that for any $|p| > P''''$,
\[
\int_{-k_0}^{k_0} e^{-\sqrt{\lambda^2 - k_0^2 + \lambda x}} \frac{(\lambda - \sqrt{\lambda^2 - k_0^2})^p}{k} \sigma(\lambda) d\lambda < |p|!(\frac{(1 + \epsilon)^{n_0}}{2})^{-|p|} \leq (|p| + K)!(\frac{(1 + \epsilon)^{n_0}}{2})^{-|p|}.
\] (130)
In summary, let

\[ P = \max\{P', P'', P'''\}, \quad C = 3 \left(1 + \frac{K}{P}\right)^K, \quad \] (131)

then for any integer \( p \) such that \( |p| > P \),

\[ |F_p| \leq |F_p^+| + |F_p^-| + \left| \int_{-k_0}^{k_0} e^{-\sqrt{\lambda^2 - k^2}y + \lambda x} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma(\lambda) d\lambda \right| \leq 3(|p| + K)! \left(\frac{(1 + \epsilon)^n}{2}\right)^{|p|} \leq C|p|^K|p|! \left(\frac{(1 + \epsilon)^n}{2}\right)^{|p|} \leq C|p|^K|p|! \left(\frac{kp}{2}\right)^{|p|} (1 + \epsilon)^{|p|}. \] (132)

**Remark 6.** In physical problems when we assume the media is lossless, there may be a finite number of surface wave poles (of order one) on the real axis. They can be treated as the limit of poles from the upper or the lower half plane as we let the dispersion in the media \( \rightarrow 0 \). Therefore, the integral across such poles can be calculated by the sum of the Cauchy principle value and \( \pm i\pi \) times the residue, here the sign of \( \pm \) depends on the direction the limit is taken in.

For example, consider the integrand with a pole \( \lambda_0 \in (-k_0, k_0) \), namely, consider

\[ \int_{-k_0}^{k_0} A(\lambda) \frac{d\lambda}{\lambda - \lambda_0} \] (133)

assuming \( A(\lambda) \) is analytic in a disk centered at \( \lambda_0 \) with radius \( c < k_0 - |\lambda_0| \). Then

\[ \int_{-k_0}^{k_0} A(\lambda) \frac{d\lambda}{\lambda - \lambda_0} = \int_{-k_0}^{k_0} \frac{A(\lambda) - A(\lambda_0)}{\lambda - \lambda_0} d\lambda + \text{p.v.} \int_{-k_0}^{k_0} \frac{A(\lambda_0)}{\lambda - \lambda_0} d\lambda \pm i\pi A(\lambda_0) \] (134)

so we can think of

\[ \frac{A(\lambda) - A(\lambda_0)}{\lambda - \lambda_0} + \frac{A(\lambda_0) \cdot 1_{\{\lambda - \lambda_0 > c\}}}{\lambda - \lambda_0} \pm i\pi \delta(\lambda - \lambda_0) \] (135)

as a \( L^1 \) function on \([-k_0, k_0]\), provided

\[ \frac{A(\lambda) - A(\lambda_0)}{\lambda - \lambda_0} \in L^1[-k_0, k_0], \] (136)

Therefore, we can apply Lemma 2, which is used to estimate the integral on \([-k_0, k_0]\) in Theorem 2 on this function \( A(\lambda)/(\lambda - \lambda_0) \).

As a result, the condition (2) in the statement of Theorem 2 can be modified to allow real poles of \( \sigma(\lambda) \) (of order one), in the sense of surface wave poles (the total number of real poles must be finite since they are isolated in \([-k_{\text{max}}, k_{\text{max}}]\)).

### 6. Exponential Convergence of ME and M2L Translation Operator

First, we will present a theorem which will be used to justify the interchange of summation and integration in the ME expansion (14) and (15), and the M2L translation (22).

**Theorem 3 (Bessel-type Expansion).** Let \( x, y, x_j, y_j \in \mathbb{R} \) such that \( y > 0, y + y_j > 0, \) and \( \rho = \sqrt{x^2 + y^2} > \rho_j = \sqrt{x_j^2 + y_j^2} \). Let \( \theta_j \in [0, 2\pi) \) such that \( x_j + y_j = \rho_j e^{i\theta_j} \). Suppose function \( \sigma(\lambda) \) satisfies all the conditions in...
Theorem 2 (or the generalized case in Remark 6), then the following series expansion holds:

\[
\int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2 - k^2}(y + y_j) + \lambda(x + x_j)} \frac{\sigma(\lambda)}{\sqrt{\lambda^2 - k^2}} d\lambda = \sum_{p = -\infty}^{\infty} J_p(k\rho_j) e^{-ip\theta_j} F_p,
\]

(137)

here

\[
F_p = \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2 - k^2}(y + y_j) + \lambda(x + x_j)} \left( \frac{\lambda - \sqrt{\lambda^2 - k^2}}{k} \right)^p \sigma(\lambda) d\lambda
\]

(138)
is defined in (118).

Proof. Without loss of generality, suppose \( x \geq 0 \). Let \( \theta \in (0, \frac{\pi}{2}] \) such that \( x = \rho \cos \theta \). Consider the integral

\[
F^+ = \int_{k_0}^{\infty} e^{-\sqrt{\lambda^2 - k^2}(y + y_j) + \lambda(x + x_j)} \frac{\sigma(\lambda)}{\sqrt{\lambda^2 - k^2}} d\lambda,
\]

(139)

here \( k_0, \) etc. are the variables defined in the proof of Theorem 2. By the transformation \( \lambda = \phi(\lambda_1) \), we have

\[
F^+ = \int_{\gamma} e^{-\sqrt{\lambda_1^2 - k^2}p - \sqrt{\lambda_1^2 - k^2}\rho_j \cos(\theta - \theta_j) + i\lambda_1\rho_j \sin(\theta - \theta_j)} \frac{\sigma(\lambda_1)}{\sqrt{\lambda_1^2 - k^2}} f_1(\lambda_1) d\lambda_1,
\]

(140)

where \( f_1(\lambda_1) = \sigma(\lambda) = \sigma(\phi(\lambda_1)) \).

First, we will show that the integral contour of \( F^+ \) can be changed from \( \gamma \) to \( \ell \). Consider the counterclockwise arc \( \sigma \) connecting \( \gamma \) and \( \ell \) centered at the origin with radius \( r \) (see Fig. 1). Let the arc be parameterized by \( \lambda_1 = re^{i\eta} \), \( \eta \) is ranged from a subset of \([\theta - \frac{\pi}{2}, 0]\). As \( r \to +\infty \), the exponent of the integrand

\[
-\sqrt{\lambda_1^2 - k^2}p - \sqrt{\lambda_1^2 - k^2}\rho_j \cos(\theta - \theta_j) + i\lambda_1\rho_j \sin(\theta - \theta_j) \sim -\lambda_1\rho - \lambda_1\rho_j \cos(\theta - \theta_j) + i\lambda_1\rho_j \sin(\theta - \theta_j)
\]

(141)

Thus the contour \( \gamma \) can be changed to \( \ell \), and it becomes

\[
F^+ = \int_{\ell} e^{-\sqrt{\lambda_1^2 - k^2}p - \sqrt{\lambda_1^2 - k^2}\rho_j \cos(\theta - \theta_j) + i\lambda_1\rho_j \sin(\theta - \theta_j)} f_1(\lambda_1) d\lambda_1
\]

\[
= \int_{\ell} e^{-\sqrt{\lambda_1^2 - k^2}p} f_1(\lambda_1) \sum_{p = -\infty}^{\infty} J_p(k\rho_j) e^{ip(\theta - \theta_j)} \left( \frac{\lambda_1 - \sqrt{\lambda_1^2 - k^2}}{k} \right)^p d\lambda_1
\]

\[
= \int_{a_0}^{+\infty} e^{-\sqrt{\lambda_1^2 - k^2}p} f_1(\lambda_1) \sum_{p = -\infty}^{\infty} J_p(k\rho_j) e^{ip(\theta - \theta_j)} \left( \frac{\lambda_1 - \sqrt{\lambda_1^2 - k^2}}{k} \right)^p da,
\]

(143)

here the identity [1] has been used for the exponential term \( e^{-\sqrt{\lambda_1^2 - k^2}\rho_j \cos(\theta - \theta_j) + i\lambda_1\rho_j \sin(\theta - \theta_j)} \) in the first equality above.
Then, recall the estimation \((117)\) on \(K_{1,p}(\rho)\) and \(K_{2,p}(\rho)\), which are defined in [96] and [97], by taking the absolute value of the integrand and using Lemma \([1]\) for the estimate of Bessel functions,
\[
\int_{a_0}^{+\infty} e^{-\sqrt{\lambda^2 - k^2} \rho} \frac{f_1(\lambda_1)}{\sqrt{\lambda^2 - k^2}} \lambda_1 e^{ikp(\theta - \theta_j)} \left( \frac{\lambda_1 - \sqrt{\lambda^2 - k^2}}{k} \right)^p \, da = \sum_{p=-\infty}^{\infty} |J_p(kp_j)| (K_{1,p}(\rho) + K_{2,p}(\rho))
\]
\[
= \sum_{p=-\infty}^{\infty} |J_p(kp_j)| (K_{1,p}(\rho) + K_{2,p}(\rho)) + \sum_{|p|>P} \frac{1}{|p|!} \left( \frac{k \rho_j}{2} \right)^{|p|} + |K| \left( \frac{k \rho_j}{2} \right)^{|p|} \Theta(k \rho_j)
\]
\[
\leq \sum_{p=-P}^{P} |J_p(kp_j)| (K_{1,p}(\rho) + K_{2,p}(\rho)) + \sum_{|p|>P} \frac{(|p| + K)!}{|p|!} \left( \frac{\rho_j}{\rho} \right)^{|p|}
\]
for sufficiently large \(P\). Hence, by Fubini’s theorem,
\[
F^+ = \int_{a_0}^{+\infty} e^{-\sqrt{\lambda^2 - k^2} \rho} \frac{f_1(\lambda_1)}{\sqrt{\lambda^2 - k^2}} \lambda_1 e^{ikp(\theta - \theta_j)} \left( \frac{\lambda_1 - \sqrt{\lambda^2 - k^2}}{k} \right)^p \, da
\]
\[
= \sum_{p=-\infty}^{\infty} J_p(kp_j)^p e^{-\eta p \theta_j} \int_{a_0}^{+\infty} e^{-\sqrt{\lambda^2 - k^2} \rho} \frac{f_1(\lambda_1)}{\sqrt{\lambda^2 - k^2}} \lambda_1 e^{ikp(\theta - \theta_j)} \left( \frac{\lambda_1 - \sqrt{\lambda^2 - k^2}}{k} \right)^p \, da
\]
\[
= \sum_{p=-\infty}^{\infty} J_p(kp_j)^p e^{-\eta p \theta_j} F^+_p.
\]

We can do the same thing for the integral on \((-\infty, -k_0)\). The integral on \([-k_0, k_0]\) is interchangeable with the summation because the integrand is in \(L^1[-k_0, k_0]\). In sum we get the series expansion
\[
\int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2 - k^2} \rho} \frac{f_1(\lambda_1)}{\sqrt{\lambda^2 - k^2}} \sigma(\lambda) d\lambda = \sum_{p=-\infty}^{\infty} J_p(kp_j)^p e^{-\eta p \theta_j} F_p
\]
in terms of Bessel functions of the first kind.

In the ME [17] and the M2L [22] translation formulas, \(\sigma_2(\lambda)\) can be recognized as a rational function of all \(\sqrt{\lambda^2 - k^2}\) and all \(e^{-\sqrt{\lambda^2 - k^2} d_i}\) terms, here numbers \(k_j\) are the wavenumbers in the layers, and \(d_i\) are the \(y\)-coordinates of the interfaces. This is a property decided by the interface conditions. Hence \(\sigma_2(-\lambda) = \sigma_2(\lambda)\). We also know \(\sigma_2(\lambda)\) does not grow exponentially as \(\lambda \to +\infty\), since the formula for the reaction field must be a converging integral, regardless of the coordinates of the sources and the target in their corresponding layers.

To apply Theorem 2 and Theorem 3 to the ME and the M2L, we need to find \(k_{\text{max}}\) to meet the condition (4) in Theorem 2. Let \(k_m > k_j\) for any wavenumbers \(k_j\). Let \(f(\lambda)\) be an analytic continuation of \(\sigma_2\) from \((k_m, +\infty)\) to the region
\[
R_m = \{\lambda \in \mathbb{C} : \Re \lambda > k_m\},
\]
such that \(f(\lambda) = \sigma_2(\lambda)\) when \(\lambda \in (k_m, +\infty)\), and \(f\) is the “same” function consisting of \(\sqrt{\lambda^2 - k_j^2}\) and \(e^{-\sqrt{\lambda^2 - k_j^2} d_i}\) terms but every square root \(\sqrt{\lambda^2 - k_j^2}\) now keeps positive real part in the right half plane \(R_m\).

**Lemma 7.** \(\exists k_{\text{max}} \geq k_m\) such that \(f\) is holomorphic in the region \(R = \{\lambda = a + ib : a, b \in \mathbb{R} : a > k_{\text{max}}, |b/a| < T\}\) and \(f \in \mathcal{B}(R)\).

**Proof.** See Appendix C for the proof. □

With the \(k_{\text{max}}\) specified by this lemma, we can now deal with the ME [17] and the M2L [22].
Theorem 4 (Exponential Convergence of Multipole Expansion). The multipole expansion \(^{(17)}\):

\[
u_2^\pm = \int_{-\infty}^{\infty} e^{-\frac{\sqrt{\lambda^2 - k_1^2} \pm (y-d_2) - \sqrt{\lambda^2 - k_1^2} (y_c-d_1) + i\lambda (x-x_c)}{\sqrt{\lambda^2 - k_1^2}}} \sigma_2(\lambda) \cdot \frac{1}{4\pi} \sum_{|p|<P} \frac{\alpha_p(-i)^p \left( \frac{\lambda - \sqrt{\lambda^2 - k_1^2}}{k_1} \right)^p}{\sqrt{\lambda^2 - k_1^2}} d\lambda
\]

\[
= \frac{1}{4\pi} \sum_{|p|<P} \alpha_p \int_{-\infty}^{\infty} e^{-\frac{\sqrt{\lambda^2 - k_1^2} \pm (y-d_2) - \sqrt{\lambda^2 - k_1^2} (y_c-d_1) + i\lambda (x-x_c)}{\sqrt{\lambda^2 - k_1^2}}} (-i)^p \left( \frac{\lambda - \sqrt{\lambda^2 - k_1^2}}{k_1} \right)^p \sigma_2(\lambda) d\lambda + T_p^{(ME)}
\]

with the far-field condition presented by

\[
\rho_c = \sqrt{(x-x_c)^2 + (\pm (y-d_2) + (y_c-d_1))^2} \geq c \cdot \max_{1 \leq j \leq N} \rho_j, \quad c > 1
\]

has exponential convergence with respect to \(P\). Namely, \(\forall \epsilon_1 \in (0, c-1)\), \(\exists P_1 \in \mathbb{N}\) without dependence on the source particle number \(N\) such that for any \(P > P_1\), if we truncate to \(|p| < P\), then the truncation error for the ME expansion is bounded by

\[
|T_p^{(ME)}| \leq \left( \frac{1 + \epsilon_1}{c} \right)^P.
\]

Proof. Let \(\tilde{x} = x-x_c, \tilde{y} = \pm(y-d_2) + (y_c-d_1)\), then \(|\tilde{x}| < x_1\) and \(y_{\min} < \tilde{y} < y_1\). Let

\[
\tilde{\sigma}(\lambda) = \sigma_2(\lambda) \cdot \frac{\sqrt{\lambda^2 - k_1^2}}{\sqrt{\lambda^2 - k_1^2}} \exp \left( \frac{\left( \sqrt{\lambda^2 - k_1^2} - \sqrt{\lambda^2 - k_1^2} \right)}{\sqrt{\lambda^2 - k_1^2}} \cdot \pm (y-d_2) \right)
\]

with the convention of branch cuts provided by the physical problem, and

\[
\tilde{f}(\lambda) = f(\lambda) \cdot \frac{\sqrt{\lambda^2 - k_1^2}}{\sqrt{\lambda^2 - k_1^2}} \exp \left( \frac{\left( \sqrt{\lambda^2 - k_1^2} - \sqrt{\lambda^2 - k_1^2} \right)}{\sqrt{\lambda^2 - k_1^2}} \cdot \pm (y-d_2) \right)
\]

for \(\Re \lambda > k_{\max}\) (which is found in Lemma \(7\)), with every square root taking positive real part. Using \(\tilde{x}, \tilde{y}, \) etc., in the integrand,

\[
e^{-\frac{\sqrt{\lambda^2 - k_1^2} \pm (y-d_2) - \sqrt{\lambda^2 - k_1^2} (y_c-d_1) + i\lambda (x-x_c)}{\sqrt{\lambda^2 - k_1^2}}} \sigma_2(\lambda) = e^{-\frac{\sqrt{\lambda^2 - k_1^2} \tilde{y} + i\lambda \tilde{x}}{\sqrt{\lambda^2 - k_1^2}}} \tilde{\sigma}(\lambda).
\]

When \(x, y\) are given in the FMM boxes, the product

\[
\frac{\sqrt{\lambda^2 - k_1^2}}{\sqrt{\lambda^2 - k_1^2}} \exp \left( \frac{\left( \sqrt{\lambda^2 - k_1^2} - \sqrt{\lambda^2 - k_1^2} \right)}{\sqrt{\lambda^2 - k_1^2}} \cdot \pm (y-d_2) \right)
\]

in \(\tilde{f}(\lambda)\) does not introduce any new pole, and is bounded by a polynomial of \(|\lambda|\) with constant order and constant coefficients with regard to the box sizes when \(\Re \lambda > k_{\max}\), for example, a rough estimation gives

\[
\left| \frac{\sqrt{\lambda^2 - k_1^2}}{\sqrt{\lambda^2 - k_1^2}} \cdot \exp \left( \frac{\left( \sqrt{\lambda^2 - k_1^2} - \sqrt{\lambda^2 - k_1^2} \right)}{\sqrt{\lambda^2 - k_1^2}} \cdot \pm (y-d_2) \right) \right| \leq \frac{|\lambda| + k_2}{k_{\max} - k_2} \exp \left( \frac{|k_2^2 - k_1^2| y_1}{2k_{\max} - k_1 - k_2} \right).
\]

Since \(f(\lambda) \in \mathcal{B}(R)\), this implies \(\tilde{f}(\lambda) \in \mathcal{B}(R)\). As a result, \(\tilde{\sigma}(\lambda)\) satisfies all the conditions in Remark \(6\). By Theorem \(6\) the multipole expansion converges.

Pick \(\epsilon_2 > 0\) such that \((1 + \epsilon_2)^2 < 1 + \epsilon_1\). Use Theorem \(2\) with \(\tilde{x}, \tilde{y}, \tilde{\sigma}(\lambda)\) and \(\epsilon_2\), there exists \(C \in \mathbb{R}^+, P_2, K \in \mathbb{N}\) such that for any integer \(P\) satisfying \(|p| \geq P_2\),

\[
\left| \int_{-\infty}^{\infty} e^{-\frac{\sqrt{\lambda^2 - k_1^2} \pm (y-d_2) - \sqrt{\lambda^2 - k_1^2} (y_c-d_1) + i\lambda (x-x_c)}{\sqrt{\lambda^2 - k_1^2}}} (-i)^p \left( \frac{\lambda - \sqrt{\lambda^2 - k_1^2}}{k_1} \right)^p \sigma_2(\lambda) d\lambda \right| \leq C|p|^K|p|! \left( \frac{k_1 \rho_c}{2(1 + \epsilon_2)} \right)^{-|p|}.
\]

Then, using Theorem \(1\) the total truncation error is bounded by

\[
\sum_{|p| \geq P_2} \sum_{j=1}^{N} \frac{q_j}{4\pi} |J_p(k_1 \rho_c)| \cdot C|p|^K|p|! \left( \frac{k_1 \rho_c}{2(1 + \epsilon_2)} \right)^{-|p|} \leq \sum_{|p| \geq P_2} \sum_{j=1}^{N} \frac{q_j C}{4\pi} \sum_{|p| \geq P_2} |p|^K \left( \frac{1 + \epsilon_2}{c} \right)^{|p|}.
\]

Since there exists integer \(P_1 \geq P_2\) such that when \(|p| \geq P_1\),

\[
\frac{2 \sum_{j=1}^{N} \frac{q_j C}{4\pi} |p|^K}{1 - \frac{1 + \epsilon_2}{c}} \leq (1 + \epsilon_2)^{|p|},
\]
for any \( P > P_1 \), we further have
\[
\left( \sum_{j=1}^{N} \frac{q_j C}{4\pi} \right) \sum_{|p| \geq P_2} |p|^K \left( \frac{1 + \epsilon_2}{c} \right)^{|p|} \leq \left( 1 - \frac{1 + \epsilon_1}{c} \right) \sum_{p \geq P} \left( \frac{1 + \epsilon_2}{c} \right)^P (1 + \epsilon_2)^P < \left( \frac{1 + \epsilon_1}{c} \right)^P,
\]
so the truncation error is then bounded by \((1 + \epsilon_1)/c)^P\) as we need.
\[
\right.
\]

Theorem 5 (Exponential Convergence of Multipole to Local Translation). In the multipole to local translation \([22]\), for each \( p \) between \(-P + 1\) and \( P - 1\), the series expansion
\[
I_p = \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2 - k_2^2} \pm (y^d - d_2)} (y^d - d_1) d\lambda = \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2 - k_2^2} \pm (y^d - d_2)} (y^d - d_1) d\lambda = \sum_{|m| < M} A_{m,p} J_m(k_2 \rho) e^{\pm i m \theta} + T_M^{(M2L)}
\]
with the far-field condition presented by
\[
\rho_1 = \sqrt{(x^d - x)^2 + (y^d - d_1)^2} \geq c \rho, \quad c > 1
\]
has exponential convergence with respect to \( M \). Namely, \( \forall \epsilon_1 \in (0, c-1), \exists M_1 \in \mathbb{N} \) such that for any \( M > M_1 \), if we truncate to \(|m| < M\), then the truncation error is bounded by
\[
|T_M^{(M2L)}| \leq (2P - 1) \left( \frac{1 + \epsilon_1}{c} \right)^M.
\]

Proof. This is a replica of the ME case with
\[
\tilde{\sigma}(\lambda) = \sigma_2(\lambda) w_1^p \exp \left( -\sqrt{\lambda^2 - k_1^2} + \sqrt{\lambda^2 - k_2^2} \right) (y^d - d_1),
\]
except the bounds of \(|w_1^p|\) for \(|p| < P\) to be found, here \( P \) is supposed to be fixed by the ME. It is obvious that
\[
|w_1|^p \leq \left( 1 + \frac{2}{k_1} |\lambda| \right)^{|p|}
\]
has a polynomial bound for each \( p \) between \(-P + 1\) and \( P - 1\).
\[
\right.
\]

7. Conclusion

ME and M2L translation operators are derived and exponential convergence rate for both are presented. However, we do not have an explicit estimate on the size of the truncation term \( P \) for a given error tolerance. We will extend this result to 3-D Helmholtz equations in layered media in a future work.

Appendix A. Two Lemmas

Lemma 8. Let \( a, b \in \mathbb{R} \) such that \( z = a + bi \in D^+ \), then \( \phi(z) \) has positive real part and positive imaginary part.

Proof. Let \( u, v \in \mathbb{R} \) such that \( u + vi = \sqrt{z^2 - k^2} \), then \( uv = ab < 0 \), so \( u > 0 \) and \( v < 0 \).

We will first show that \( u \sin \beta > -b \cos \beta \). Let
\[
Q_1 = (a^2 - b^2 - k^2)^2 + 4a^2 b^2,
\]
\[
Q_2 = (a^2 - b^2 - k^2) \sin^2 \beta - 2b^2 \cos^2 \beta.
\]
By simple calculation,
\[
2u^2 \sin^2 \beta - 2b^2 \cos^2 \beta = \sqrt{Q_1} \sin^2 \beta + Q_2,
\]
and
\[
Q_1 \sin^4 \beta - Q_2^2 = b^2 \sin^2 (2\beta) \left( \frac{a^2}{\cos^2 \beta} - \frac{b^2}{\sin^2 \beta} - k^2 \right) > 0,
\]
so
\[
\sqrt{Q_1} \sin^2 \beta = \left| \sqrt{Q_1} \sin^2 \beta \right| > |Q_2|,
\]
thus \( \sqrt{Q_1} \sin^2 \beta + Q_2 > 0 \), so \( u \sin \beta > -b \cos \beta \).
Then,
\[
\phi(z) = (a \cos \beta - v \sin \beta) + i(b \cos \beta + u \sin \beta)
\]  \hspace{1cm} (170)

has positive real part and positive imaginary part. \[ \square \]

**Lemma 9.** If \( w \in \Gamma \) and \( \Re w \geq 0 \), then there does not exist any \( z \in D^- \) such that \( \phi(z) = w \).

**Proof.** Suppose for contradiction that \( z \in D^- \), \( \phi(z) = w \). Since \( w \in \Gamma \), there exists a positive real number \( x \geq k \) such that
\[
w = x \cos \beta + i \sqrt{x^2 - k^2} \sin \beta.
\]  \hspace{1cm} (171)

Therefore, \( x \) and \( z \) are distinct roots of the quadratic equation
\[
\lambda^2 - 2\lambda w \cos \beta + w^2 = k^2 \sin^2 \beta
\]  \hspace{1cm} (172)
of \( \lambda \). Thus,
\[
z = 2w \cos \beta - x = x \cos(2\beta) + i \sqrt{x^2 - k^2} \sin(2\beta)
\]  \hspace{1cm} (173)
has nonnegative imaginary part, which contradicts the assumption that \( z \in D^- \). \[ \square \]

**Figure 2.** \( \phi \) on \( D_\gamma \)

**Appendix B.** **Proof of Lemma 5**

**Proof.** First, we will show that \( \phi(D^-) \subset D^+ \). By Lemma 8 and Lemma 9, \( \phi(D^-) \) is a subset of quadrant I, and it has no intersection with \( \Gamma \). If \( w = \phi(z) \) for some \( z \in D^- \) and \( w \notin D^+ \), when we move \( z \) horizontally to the left, eventually \( z \) touches \( \Gamma \) and \( \phi(z) \) approaches the positive real axis, so the trajectory of \( \phi(z) \), which must be continuous because \( \phi \) is holomorphic, crosses \( \Gamma \) in quadrant I, but it contradicts with Lemma 9 since the intersection must has its inverse in \( D^- \).

Second, on \( D^+ \) define the mapping
\[
\phi_1(w) = w \cos \beta - i \sqrt{w^2 - k^2} \sin \beta,
\]  \hspace{1cm} (174)
here the square root adopts positive real part. Similarly \( \phi_1(D^+) \subset D^- \).

Third, we will show that \( \phi \) is bijective on \( D^- \) with inverse \( \phi_1 \). Let \( a, b \in \mathbb{R}^+ \) such that \( z = a + bi \in D^- \), then \( w = \phi(z) \in D^+ \) is one of the roots of the quadratic equation of \( \lambda \)
\[
\lambda^2 - 2\lambda z \cos \beta + z^2 = k^2 \sin^2 \beta.
\]  \hspace{1cm} (175)
Let $u, v \in \mathbb{R}$ such that $\sqrt{z^2 - k^2} = u + vi$, then the pair of roots are given by
\begin{align*}
\lambda_1 &= (a \cos \beta - v \sin \beta) + i(b \cos \beta + u \sin \beta), \quad (176) \\
\lambda_2 &= (a \cos \beta + v \sin \beta) + i(b \cos \beta - u \sin \beta), \quad (177)
\end{align*}
by Lemma 8, exactly one of $\lambda_1$ and $\lambda_2$ has positive imaginary part, so exactly one of them is in $D^+$, and it is $w$. Conversely, $z$ is the only root of the quadratic equation
\[ \lambda^2 - 2w \cos \beta + u^2 = k^2 \sin^2 \beta \]  
(178)
in $D^-$ provided $\phi(z) = w$ by the similar reason, so $\phi$ is injective and $z = \phi_1(w)$. Repeat this step for any $w' \in D^+$ and let $z' = \phi_1(w')$, we have $\phi$ is surjective and $w' = \phi(\phi_1(w'))$. 
\[ \square \]

**Appendix C. Proof of Lemma 7**

**Proof.** Let $\mathcal{G}$ be the collection of all holomorphic functions $g(\lambda)$ in $R_m$
\[ R_m = \{ \lambda \in \mathbb{C} : 9 \Re \lambda > k_m \} \]
such that the Laurent series of $g(\lambda)$ at infinity has finitely many nonzero terms with positive exponent,
\[ \mathcal{G} = \left\{ g(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^{m-n}, m \in \mathbb{Z}, c_n \in \mathbb{C}, g(\lambda) \text{ is holomorphic in } R_m \right\}. \]  
(179)
For example, $\sqrt{\lambda^2 - k_j^2} \in \mathcal{G}$ since
\[ \sqrt{\lambda^2 - k_j^2} = \sum_{n=0}^{\infty} \frac{\sqrt{\pi}(-k_j^2)^n}{2\Gamma(n+1)\Gamma(-n + \frac{1}{2})} \lambda^{1-2n}. \]  
(180)
Let $\mathcal{S}$ be the collection of all holomorphic functions $h(\lambda)$ in $R_m$ in the following form
\[ \mathcal{S} = \left\{ h(\lambda) = \sum_{u=1}^{U} e^{A_u \lambda}g_u(\lambda), U \in \mathbb{Z}^+, A_1 > \cdots > A_U \geq 0, g_u \in \mathcal{G} \right\}. \]  
(181)
For example, each $e^{\sqrt{\lambda^2 - k_j^2}|d_i|} \in \mathcal{S}$. To quickly show this fact, notice that
\[ e^{\sqrt{\lambda^2 - k_j^2}|d_i|} = e^{|d_i| \lambda} e^{\sqrt{\lambda^2 - k_j^2} - \lambda}|d_i|, \]  
(182)
for the second term, let $\mu = \lambda^{-1}$, then
\[ e^{(-k_j^2|d_i|\mu)} = e^{\mu + \sqrt{1-\mu^2k_j^2}}. \]  
(183)
here $0$ becomes a removable singularity in the complex plane of $\mu$. Therefore, the Laurent series in the $\mu$ plane at zero has zero principle part, which immediately implies $e^{\sqrt{\lambda^2 - k_j^2} - \lambda}|d_i| \in \mathcal{G}$ and $e^{\sqrt{\lambda^2 - k_j^2}|d_i|} \in \mathcal{S}$.

It is obvious that $\mathcal{G} \subset \mathcal{S}$, and $\mathcal{S}$ is a ring with function addition and multiplication.

If $h(\lambda) = \sum_{u=1}^{U} e^{A_u \lambda}g_u(\lambda) \in \mathcal{G}$ is not the zero function, and the leading term (the term with highest order in the Laurent series) of $g_1(\lambda)$ is $B\lambda^m$, then
\[ h(\lambda) \sim e^{A_1 \lambda} B \lambda^m \]  
(184)
as $\lambda \to (1 + q) \cdot \infty$ for any $q \in [-T, T]$.

Now consider $f(\lambda)$. By multiplying some necessary exponential terms, we can rewrite $f(\lambda)$ as
\[ f(\lambda) = \frac{I_1}{I_2} = \frac{Q_1 \left( \sqrt{\lambda^2 - k_j^2} \cdots e^{\sqrt{\lambda^2 - k_j^2}|d_i|} \cdots \right)}{Q_2 \left( \sqrt{\lambda^2 - k_j^2} \cdots e^{\sqrt{\lambda^2 - k_j^2}|d_i|} \cdots \right)} \]  
(185)
here $Q_1$ and $Q_2$ are polynomials of the terms in the parentheses. By induction (on the total number of addition, subtraction and multiplication operations required to build up the polynomial), $I_1, I_2 \in \mathcal{S}$. Suppose the numerator
\[ I_1 \sim e^{A_1 \lambda} B \lambda^m \]  
(186)
and the denominator
\[ I_2 \sim e^{A_1' \lambda} B' \lambda^{m'} \]  
(187)
as $\lambda \to (1+q) \cdot \infty$ for any $q \in [-T, T]$. Since $\sigma_2(\lambda)$ does not grow exponentially as $\lambda \to +\infty$, i.e. as $\lambda \to (1+0) \cdot \infty$, we must have

$$A_1 \leq A'_1.$$  \hspace{1cm} (188)

As a result, $|f(\lambda)| \lesssim |\lambda|^{m-m'}$ for $\lambda \in R_m$ as $\lambda \to (1+q) \cdot \infty$ for any $q \in [-T, T]$.

With such an asymptotic bound, the real parts of poles of $f(\lambda)$ must be bounded in the fan-shaped region with boundaries of slope $\pm T$. Let $k_{\text{max}} \geq k_0$ be an upper bound of the real parts of the poles of $f$ in this fan-shaped region, then $f$ becomes holomorphic in $R = \{ \lambda = a + b \cdot i : a, b \in \mathbb{R} : a > k_{\text{max}}, |b/a| < T \}$, and $f(\lambda) \in \mathcal{B}(R)$. □

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