Criteria of Spectral Gap for Markov Operators

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Abstract

Let \((E, \mathcal{F}, \mu)\) be a probability space, and let \(P\) be a symmetric Markov operator on \(L^2(\mu)\) with 1 a simple eigenvalue. Then \(P\) has a spectral gap, i.e. 1 is isolated in the spectrum of \(P\), if and only if

\[
\|P\|_\tau := \lim_{R \to \infty} \sup_{\mu(f^2) \leq 1} \mu(f(Pf - R)^+) < 1.
\]

This strengthens a conjecture of Simon and Høegh-Krohn on the spectral gap for hyperbounded operators solved recently by L. Miclo in [10]. Consequently, for a symmetric, conservative, irreducible Dirichlet form on \(L^2(\mu)\), a Poincaré/log-Sobolev type inequality holds if and only if so does the corresponding defective inequality. Extensions to sub-Markov operators and non-conservative Dirichlet forms are also presented.

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1 Introduction

Let \((E, \mathcal{F}, \mu)\) be a probability space. Let \(P\) be a symmetric Markov operator on \(L^2(\mu)\); i.e. \(P\) is a contraction linear operator on \(L^2(\mu)\) such that \(P1 = 1\) and \(f \geq 0\)

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implies $Pf \geq 0$. It is well known that $1$ is a simple eigenvalue of $P$ if and only if $P$ is ergodic, i.e. for $f \in L^2(\mu)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P^k f = \mu(f) \quad \text{holds in } L^2(\mu).$$

The ergodicity is also equivalent to the $\mu$-essential irreducibility (or resolvent-positive-improving property) of $P$:

$$\sum_{n=1}^{\infty} \mu(A^n B) > 0, \quad \mu(A), \mu(B) > 0. \quad (1.1)$$

Obviously, the spectrum $\sigma(P)$ of $P$ is contained in $[-1, 1]$. $P$ is said to have a spectral gap if $P$ is ergodic and $\sigma(P) \subset \{1\} \cup [-1, \theta]$ for some $\theta \in [-1, 1)$; or equivalently, the Poincaré inequality

$$\mu(f^2) \leq \frac{1}{1-\theta} \mu((f(f - Pf))) + \mu(f)^2, \quad f \in L^2(\mu) \quad (1.2)$$

holds.

Recall that $P$ is called hyperbounded if for some $p > 2$

$$\|P\|_{2\to p} := \sup_{\mu(f^2) \leq 1} \mu(|Pf|^p)^{\frac{1}{p}} < \infty.$$ 

It was conjectured by Simon and Høegh-Krohn [14] that if $P$ is ergodic and hyperbounded then it has a spectral gap. Although numerous papers aiming to solve this problem or to construct counterexamples have been published, see e.g. [1, 2, 5, 18, 21, 3] where some weaker notions such as the uniform integrability and a tail norm condition have been used to replace the hyperboundedness, the conjecture has been open for more than 40 years until Miclo found a complete proof in his recent paper [10].

On the other hand, there are a lot of non-hyperbounded Markov operators having spectral gap. So, in the spirit of [3, 3], we shall prove a stronger statement by using a tail norm condition to replace the hyperboundedness. The tail norm we will use is the following:

$$\|P\|_\tau := \lim_{R \to \infty} \sup_{\mu(f^2) \leq 1} \mu((Pf)^R \geq R) = \lim_{R \to \infty} \sup_{f \geq 0, \mu(f^2) \leq 1} \mu((Pf - R)^+).$$

According to the Schwartz inequality, $\|P\|_\tau$ is smaller than the following one used in [3, 3]:

$$\|P\|_{\text{tail}} := \lim_{R \to \infty} \sup_{\mu(f^2) \leq 1} \mu((|Pf| - R)^+)^{\frac{1}{2}}.$$ 

Moreover, it is easy to see from the Jensen inequality that $\|P^m\|_{\text{tail}}$ and $\|P^{2m-1}\|_\tau$ are decreasing in $m \in \mathbb{N}$.

According to [21], $P$ is called uniformly integrable if $\|P\|_{\text{tail}} = 0$. Thus, the uniformly integrability implies $\|P\|_\tau = 0$. In particular, $\|P\|_\tau = 0$ holds for hyperbounded $P$. We will then strengthen the above conjecture by replacing the
hyperboundedness with \( \|P\|_\tau < 1 \), which is also necessary for the existence of the spectral gap as shown in our following main result. See Theorem 2.1 below for one more equivalent statement on isoperimetric constants for the existence of spectral gap.

**Theorem 1.1.** Let \( P \) be a symmetric ergodic Markov operator on \( L^2(\mu) \). Then the following statements are equivalent:

1. \( P \) has a spectral gap, i.e. \( \|P\|_\tau < 1 \).
2. \( \inf_{m \in \mathbb{N}} \|P^{2m-1}\|_\tau < 1 \).
3. \( \inf_{m \in \mathbb{N}} \|P^m\|_{\text{tail}} < 1 \).
4. \( \inf_{m \in \mathbb{N}} \|P^m\|_{\text{tail}} < 1 \).

Note that as an improvement of an earlier result in [5], the equivalence of the existence of spectral gap in \( L^p(\mu)(1 < p < \infty) \) and \( \inf_{m \in \mathbb{N}} \|P^m\|_{\text{tail}} < 1 \) has been proved in [3] for resolvent-uniform-positive-improving Markov operators (see [3, Lemma 3.6, Theorem 4.1]): for any \( \varepsilon > 0 \), there exists \( m \in \mathbb{N} \) such that

\[
\inf \left\{ \sum_{k=1}^m \mu(A P^k 1_B) : \mu(A), \mu(B) \geq \varepsilon \right\} > 0.
\]

As mentioned above that \( \inf_{m \in \mathbb{N}} \|P^m\|_{\text{tail}} < 1 \) implies \( \inf_{m \in \mathbb{N}} \|P^{2m-1}\|_\tau < 1 \). Moreover, the resolvent-uniform-positive-improving property is strictly stronger than the ergodicity, which is equivalent to the resolvent-positivity-improving property (1.1).

Here, we would like to mention links of the uniform-positive-improving property of a symmetric Markov semigroup \( P_t \) and the weak Poincaré inequality of the associated Dirichlet form \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \). It is well known that \( P_t \) (for some/all \( t > 0 \)) is ergodic if and only if the Dirichlet form is irreducible, i.e. \( \mathcal{E}(f, f) = 0 \) implies \( f \) is constant. Next, according to [6] (see also [1]), the uniform-positive-improving property of the semigroup implies the weak spectral gap property, which is equivalent to the validity of the weak Poincaré inequality (see [13]): for some \( \alpha : (0, \infty) \to (0, \infty) \)

\[
\mu(f^2) \leq \alpha(r) \mathcal{E}(f, f) + r \|f\|_{\infty}^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}). \tag{1.3}
\]

Moreover, it is shown in [13, §7] that there are conservative irreducible Dirichlet forms which do not satisfy the weak Poincaré inequality. Therefore, the uniform-positive-improving property is strictly stronger than the ergodicity.

Finally, although Theorem 1.1 is only stated for symmetric operators, it can be applied to the non-symmetric setting as well. Indeed, for a non-symmetric Markov operator on \( L^2(\mu) \) such that \( \mu P = \mu \) (i.e. \( \mu \) is invariant for \( P \)), \( \hat{P} := \frac{1}{2}(P + P^*) \) is a symmetric Markov operator, where \( P^* \) is the adjoint of \( P \). Obviously, \( P \) satisfies (1.1) if and only if so does \( \hat{P} \), and \( \|\hat{P}\|_\tau < 1 \) implies \( \|\hat{P}\|_\tau \leq \frac{1}{2}(\|P\|_\tau + \|P^*\|_\tau) < 1 \). Therefore, by Theorem 1.1 if \( P \) satisfies (1.1) and \( \|P\|_\tau < 1 \), then there exists a constant \( \delta < 1 \) such that \( \text{Re} \sigma(P) = \sigma(\hat{P}) \subset \{1\} \cup [-1, \delta] \).

As applications of Theorem 1.1 we consider functional inequalities conservative symmetric Dirichlet forms. A simple consequence of the equivalence of (1) and (3) is that the defective Poincaré inequality implies the tight one.
**Corollary 1.2.** Let \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) be a conservative, irreducible, symmetric Dirichlet form on \(L^2(\mu)\). Then the Poincaré inequality
\[ \mu(f^2) \leq C \mathcal{E}(f, f) + \mu(f)^2, \quad f \in \mathcal{D}(\mathcal{E}) \tag{1.4} \]
holds for some constant \(C > 0\) if and only if the defective Poincaré inequality
\[ \mu(f^2) \leq C_1 \mathcal{E}(f, f) + C_2 \mu(|f|)^2, \quad f \in \mathcal{D}(\mathcal{E}) \tag{1.5} \]
holds for some constants \(C_1, C_2 > 0\).

This result improves \([13, \text{Proposition 1.3}]\) where the weak Poincaré inequality \([1.3]\) is used to replace the irreducibility of the Dirichlet form. Basing on Corollary 1.2, we are able to prove the equivalence of the defective version and the tight version for more general functional inequalities. Here, we consider a family of functional inequalities introduced in \([7]\), which interpolate the Poincaré inequality \((1.4)\) and the Gross \([4]\) log-Sobolev inequality
\[ \mu(f^2 \log f^2) \leq C \mathcal{E}(f, f), \quad f \in \mathcal{D}(\mathcal{E}), \mu(f^2) = 1 \tag{1.6} \]
for some constant \(C > 0\). Let \(\phi \in C([1, 2])\) such that \(\phi > 0\) on \([1, 2)\) and \(\phi(2) = 0\) with
\[ c_\phi := \sup_{p \in [1, 2]} \frac{2 - p}{\phi(p)} < \infty. \]
Consider the functional inequality
\[ \text{Var}_{\mu, \phi}(f) := \sup_{p \in [1, 2]} \frac{\mu(f^2) - \mu(|f|^p)^{\frac{2}{p}}}{\phi(p)} \leq C \mathcal{E}(f, f), \quad f \in \mathcal{D}(\mathcal{E}) \tag{1.7} \]
and its defective version
\[ \text{Var}_{\mu, \phi}(f) \leq C_1 \mathcal{E}(f, f) + C_2 \mu(f^2), \quad f \in \mathcal{D}(\mathcal{E}). \tag{1.8} \]
When \(\phi(p) = 2 - p\) the inequality \((1.6)\) is equivalent to the log-Sobolev inequality, and when \(\phi\) reduces to a positive constant it becomes the Poincaré inequality. See \([19]\) for detailed discussions on properties and applications of the inequality \((1.7)\).

**Corollary 1.3.** Let \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) be a conservative, irreducible, symmetric Dirichlet form on \(L^2(\mu)\). Then \((1.7)\) holds for some constant \(C > 0\) if and only if \((1.8)\) holds for some constants \(C_1, C_2 > 0\).

The remainder of the paper is organized as follows. In Section 2, by using an approximation argument introduced in \([10]\), we extend a known Cheeger type inequality for high order eigenvalues of finite-state Markov chains to the abstract setting, then use this estimate to characterize the existence of spectral gap with high-order isoperimetric constants. This characterization is then used in Section 3 to prove Theorem 1.1. Proofs of Corollaries 1.2 and 1.3 are also addressed in Section 3. Finally, in Section 4 we extend Theorem 1.1 and Corollary 1.2 to the sub-Markov setting.
2 Essential spectrum and isoperimetric constants

For any \( n \in \mathbb{N} \), let
\[
D_n = \{ (A_1, \ldots, A_n) : A_1, \ldots, A_n \text{ are disjoint, } \mu(A_k) > 0, 1 \leq k \leq n \}.
\]

We define the \( n \)-th isoperimetric constant by
\[
\kappa_n = \inf_{(A_1, \ldots, A_n) \in D_n} \max_{1 \leq k \leq n} \frac{\mu(1_{A_k} P 1_{A_k}^c)}{\mu(A_k)}.
\]

Obviously, \( \kappa_1 = 0 \) and if \( L^2(\mu) \) is infinite-dimensional then \( D_n \neq \emptyset \) for all \( n \geq 2 \).

We will only consider the case that \( L^2(\mu) \) is infinite-dimensional, since otherwise the spectrum of \( P \) is finite so that the existence of spectral gap becomes trivial. By the Cheeger inequality we know that \( P \) has a spectral gap if and only if \( \kappa_2 > 0 \), see e.g. [8] by noting that in (1.2) we have
\[
\mu(f(f - Pf)) = \frac{1}{2} \int_{E \times E} (f(x) - f(y))^2 J(dx, dy)
\]
for the symmetric measure \( J \) on \( E \times E \) determined by \( J(A \times B) := \mu(1_A P 1_B), A, B \in \mathcal{F} \).

In this section we consider \( \lambda_{\text{ess}}(P) := \sup_{P \in \mathcal{P}} \sigma_{\text{ess}}(P) \), where \( \sigma_{\text{ess}}(P) \) is the essential spectrum of \( P \). Obviously, \( P \) has a spectral gap if and only if it is ergodic and \( \lambda_{\text{ess}}(P) < 1 \).

**Theorem 2.1.** Let \( P \) be a symmetric Markov operator on \( L^2(\mu) \). Then \( \lambda_{\text{ess}}(P) < 1 \) if and only if \( \sup_{n \geq 1} \kappa_n > 0 \).

As mentioned at the end of Introduction, to prove this result we will extend a known estimate on the high order eigenvalues using \( \kappa_n \) for finite-state Markov chains. So, below we first consider Markov operators on a finite set.

Let \( n \geq 1 \) be fixed, and let \( \hat{E} \) be a finite set with \( |\hat{E}| \geq n \), where \( |\hat{E}| \) denotes the number of elements in \( \hat{E} \). Let \( \tilde{\mu} \) be a strictly positive probability measure on \( \hat{E} \) equipped with the largest \( \sigma \)-field \( \mathcal{B}(\hat{E}) \), i.e. \( \mathcal{B}(\hat{E}) \) is the class of all subsets of \( \hat{E} \) and \( \tilde{\mu}(\{x\}) > 0 \) for any \( x \in \hat{E} \). For a symmetric Markov operator \( \tilde{P} \) on \( \hat{E} \), let
\[
0 = \tilde{\lambda}_1 \leq \cdots \leq \tilde{\lambda}_{|\hat{E}|} \leq 2
\]
be all eigenvalues of \( 1 - \tilde{P} \). According to [9], there exists a constant \( c(n) \) depends only on \( n \) such that
\[
\sqrt{\tilde{\lambda}_n} \geq c(n) \inf_{(\hat{A}_1, \ldots, \hat{A}_n) \in \tilde{D}_n} \max_{1 \leq k \leq n} \frac{\tilde{\mu}(1_{\hat{A}_k} \tilde{P} 1_{\hat{A}_k}^c)}{\tilde{\mu}(\hat{A}_k)},
\]
(2.1)
where
\[
\tilde{D}_n = \{ (\hat{A}_1, \ldots, \hat{A}_n) : \hat{A}_1, \cdots, \hat{A}_n \text{ are disjoint non-empty subsets of } \hat{E} \}.\]
As shown in [10] Theorem 2, we may take \( c(n) = \frac{c_0}{n^2} \) for a universal constant \( c_0 > 0 \).

Below we aim to extend this estimate to our abstract setting by using an approximation argument introduced in [10]. For any \( n \geq 1 \), let

\[
\lambda_n = \sup_{f_1, \ldots, f_{n-1} \in L^2(\mu)} \inf_{\mu(f^2) = 1} \mu(f(1-P)f).
\]  

(2.2)

To see that this quantity can be regarded as the \( n \)-th eigenvalue of \( L := 1-P \), let \( \lambda_{\text{ess}}(L) = \inf \sigma_{\text{ess}}(L) \) be the bottom of the essential spectrum of \( L \). Then, see e.g. [11] Theorem XIII.2, \( \lambda_n \) is the \( n \)-th eigenvalue of \( L \) if \( \lambda_n < \lambda_{\text{ess}}(L) \), and \( \lambda_n = \lambda_{\text{ess}}(L) \) otherwise. The following result was stated as Proposition 5 in [10], we include here a proof for completeness.

**Lemma 2.2 (10).** Let \( c(n) \) be in (2.1) for \( n \geq 1 \). Then \( \kappa_n \geq \lambda_n \geq c(n)^2 \kappa_n^2 \).

**Proof.** Let \( L = 1-P \). Since \( \lambda_1 = \kappa_1 = 0 \), we only prove for \( n \geq 2 \).

(a) Upper bound estimate of \( \lambda_n \). For any \( f_1, \ldots, f_{n-1} \in L^2(\mu) \) and any \( (A_1, \ldots, A_n) \in D_n \), there exists a function \( f := \sum_{i=1}^{n} a_i 1_{A_i} \) such that \( \sum_{i=1}^{n} a_i^2 = 1 \) and \( \mu(f_i) = 0, 1 \leq i \leq n \). Let

\[
\kappa = \max_{1 \leq i \leq n} \frac{\mu(1_{A_i} P1_{A_i'})}{\mu(A-i)}.
\]

Then

\[
\mu(Lf) = \sum_{i,j=1}^{n} a_i a_j \mu(1_{A_i} L1_{A_j}) \leq \sum_{i,j=1}^{n} |a_i| \cdot |a_j| \sqrt{\mu(1_{A_i} L1_{A_i}) \mu(1_{A_j} L1_{A_j})}
\]

\[
\leq \kappa \sum_{i,j=1}^{n} |a_i| \cdot |a_j| \sqrt{\mu(A_i) \mu(A_j)} \leq \kappa \sum_{i,j=1}^{n} a_i^2 \mu(A_j) \leq \kappa.
\]

Therefore, by the definition of \( \lambda_n \) and \( \kappa_n \), we have \( \lambda_n \leq \kappa_n \).

(b) Lower bound estimate of \( \lambda_n \). Assume that \( \lambda_n < c(n)^2 \kappa_n^2 \). Then there exist \( f_1, \ldots, f_n \in L^2(\mu) \) such that

\[
\mu(f_i f_j) = \delta_{ij}, \quad \mu(f_i L f_i) \leq c(n)^2 \kappa_n^2 - \frac{\delta_n}{2},
\]

\[
\max_{1 \leq i \leq n} \sum_{j \neq i, 1 \leq j \leq n} |\mu(f_i L f_j)| \leq \frac{\delta_n}{4}, \quad 1 \leq i, j \leq n.
\]  

(2.3)

where \( \delta_n := c(n)^2 \kappa_n^2 - \lambda_n > 0 \). Indeed, for any \( \varepsilon > 0 \) we may find an orthonormal family \( \{f_1, \ldots, f_n\} \subset L^2(\mu) \) such that \( \mu(|Lf_i - \lambda_i f_i|^2) \leq \varepsilon, 1 \leq i \leq n \). Since \( \max_{1 \leq i \leq n} \lambda_i = \lambda_n = c(n)^2 \kappa_n^2 - \delta_n \) and \( \mu(f_i f_j) = \delta_{ij} \), (2.3) holds for small enough \( \varepsilon > 0 \).

Let \( \mathcal{F}_\infty = \sigma(f_1, \ldots, f_n) \). Since \( \mathcal{F}_\infty \) is separable, we may find an increasing sequence of \( \sigma \)-fields \( \{\mathcal{F}_N\}_{N \geq 1} \) such that

\[
\bigvee_{N \geq 1} \mathcal{F}_N = \mathcal{F}_\infty.
\]
Let $\mu_N$ be the restriction of $\mu$ on $\mathcal{F}_N$, and let $E^{F_N}$ be the conditional expectation under $\mu$ given $\mathcal{F}_N$. Let
\[ P_N g = E^{F_N} P g, \quad g \in L^2(\mu_N). \]
Then $P_N$ is a symmetric Markov operator on $L^2(\mu_N)$.

To identify $P_N$ with a Markov operator on a finite set, let
\[ \tilde{E}_N = \{ A : A \text{ is atom of } \mathcal{F}_N, \mu_N(A) > 0 \}, \quad \tilde{\mu}_N(\{ A \}) = \mu_N(A) = \mu(A), \quad A \in \tilde{E}_N. \]
Then $\tilde{E}_N$ is a finite set with $\tilde{\mu}_N$ a strictly probability measure. It is easy to see that the map
\[ \phi_N : \tilde{g} \mapsto \sum_{A \in \tilde{E}_N} \tilde{g}(A) 1_A \]
is isometric from $L^2(\tilde{\mu}_N)$ to $L^2(\mu_N)$. Moreover, the inverse of $\phi_N$ is given by (note that $g$ is constant on atoms of $\mathcal{F}_N$)
\[ \phi_N^{-1}(g)(A) = g|_A, \quad A \in \tilde{E}_N, \quad g \in L^2(\mu_N). \]
Define
\[ \tilde{P}_N \tilde{g} = \phi_N^{-1}(P_N \phi_N(g)), \quad \tilde{g} \in L^2(\tilde{\mu}_N). \]
Then $\tilde{P}_N$ is a symmetric Markov operator on $L^2(\tilde{\mu}_N)$ and having the same spectral information of $P_N$. Therefore, (2.1) is valid for $P_N$, i.e. letting
\[ 0 = \lambda_{N,1} \leq \cdots \leq \lambda_{N,\ell_N} \leq 2 \]
be all eigenvalues of $L := 1 - P_N$, where $\ell_N = |\tilde{E}_N| = \dim L^2(\mu_N)$, if $\ell_N \geq n$ then
\[ \sqrt{\lambda_{N,n}} \geq c(n) \inf_{(A_1, \cdots, A_n) \in D_{N,n}} \max_{1 \leq k \leq n} \frac{\mu_N(1_{A_k} P_N 1_{A_k})}{\mu_N(A_k)} \]
holds for $D_{N,n} := \{(A_1, \cdots, A_n) \in D_n : A_k \in \mathcal{F}_N, 1 \leq k \leq n\}$. Since for $A_k \in \mathcal{F}_N$ we have $\mu_N(A_k) = \mu(A_k)$ and
\[ \mu_N(1_{A_k} P_N 1_{A_k}) = \mu(1_{A_k} E^{F_N} P 1_{A_k}) = \mu(1_{A_k} P 1_{A_k}), \]
this implies
\[ \lambda_{N,n} \geq c(n) 2^2 \kappa_n^2, \quad n \leq \ell_N. \quad (2.4) \]

Now, let $f_{N,k} = E^{F_N} f_k, 1 \leq k \leq n$. Then by the martingale convergence theorem
\[ \lim_{N \to \infty} f_{N,k} = f_k \text{ holds in } L^2(\mu), 1 \leq k \leq n. \]
Let $L_N = 1 - P_N$. Since $P$ is continuous in $L^2(\mu)$, combining this with (2.3) we obtain
\[ \lim_{N \to \infty} \mu_N( f_{N,i} f_{N,j} ) = \lim_{N \to \infty} \mu( f_{N,i} f_{N,j} ) = \delta_{ij}, \quad 1 \leq i, j \leq n, \]
\[ \lim_{N \to \infty} \mu( f_{N,i} L_N f_{N,i} ) = \lim_{N \to \infty} \mu( f_{N,i} (1 - P) f_{N,i} ) \leq c(n) 2^2 \kappa_n^2 - \frac{\delta_n}{2}, \quad 1 \leq i \leq n, \]
\[ \lim_{N \to \infty} \sup_{1 \leq i \leq j} \sum_{j \neq i} |\mu( f_{N,i} L_N f_{N,j} )| \leq \frac{\delta_n}{4}. \]
This is contradictory to (2.4). Indeed, from this we may find orthonormal \( \{ \tilde{f}_{N,i} : 1 \leq i \leq n \} \subset L^2(\mu_N) \) such that
\[
\varepsilon_N := \max_{1 \leq i \leq n} \mu(|f_{N,i} - \tilde{f}_{N,i}|^2)^{\frac{1}{2}} \to 0 \text{ as } N \to \infty,
\]
and thus,
\[
\mu(\tilde{f}_{N,i}L_N \tilde{f}_{N,i}) \leq \mu(f_{N,i}L_N f_{N,i}) + 4\varepsilon_N \leq c(n)^2 \kappa_n^2 - \frac{\delta_n}{2} + 4\varepsilon_N, \quad 1 \leq i \leq n,
\]
\[
\max_{1 \leq i \leq n} \sum_{j \neq i} |\mu(\tilde{f}_{N,i}L_N \tilde{f}_{N,j})| \leq \frac{\delta_n}{4} + 4(n-1)\varepsilon_N.
\]
Therefore, for \( f := \sum_{i=1}^n a_i \tilde{f}_{N,i} \) with \( \sum_{i=1}^n a_i^2 = 1 \),
\[
\mu_N(fL_N f) = \sum_{i,j=1}^n \mu(\tilde{f}_{N,i}L_N \tilde{f}_{N,j})a_i a_j \leq c(n)^2 \kappa_n^2 - \frac{\delta_n}{2} + 4\varepsilon_N + \frac{\delta_n}{4} + 4(n-1)\varepsilon_N = c(n)^2 \kappa_n^2 - \frac{\delta_n}{4} + 4n\varepsilon_N.
\]
Since for any \( g_1, \cdots, g_{n-1} \in L^2(\mu_N) \) there exists \( f \in \text{span}\{ \tilde{f}_{N,i} : 1 \leq i \leq n \} \) with \( \mu(f^2) = 1 \) such that \( \mu_N(f g_i) = 0, 1 \leq i \leq n-1 \), combining this with (2.2) for \( P_N \) in place of \( P \), we arrive at
\[
\lambda_{N,n} \leq c(n)^2 \kappa_n^2 - \frac{\delta_n}{4} + 4n\varepsilon_N,
\]
which contradicts (2.4) for large \( N \) such that \( \varepsilon_N < \frac{\delta_n}{16n} \).

Proof of Theorem 2.1 If \( \lambda_{ess}(P) < 1 \), then \( \sigma(L) \cap [0,1 - \lambda_{ess}(P)) \) is discrete and each eigenvalue in this set is of finite multiplicity. So, in this case \( \lambda_n > 0 \) for \( n \) larger than the multiplicity of the first eigenvalue \( \lambda_1 = 0 \), and hence by Lemma 2.2 \( \kappa_n > 0 \) for large \( n \). On the other hand, we aim to prove that if \( \lambda_{ess}(P) = 1 \) then \( \kappa_n = 0 \) for all \( n \geq 1 \). Since \( \lambda_{ess}(P) = 1 \), i.e. \( 0 \in \sigma_{ess}(L) \), we have \( \lambda_n = 0 \) for all \( n \). Combining this with Lemma 2.2 we prove \( \kappa_n = 0 \) for all \( n \geq 1 \).

3 Proofs of Theorem 1.1 and corollaries

Proof of Theorem 1.1 By [3] Theorem 1.4, (1) implies (4). Since \( \|P^m\|_\tau \leq \|P^m\|_{\text{tail}} \) and by the Jensen inequality the latter is decreasing in \( m \), (4) implies (3). Moreover, (3) implies (2). So, we only need to prove the equivalence of (1) and (2), from which we conclude that (3) is also equivalent to (1) since the existence of spectral gap for \( P \) is equivalent to that for \( P^{2m-1} \).

(1) implies (2). If \( P \) has a spectral gap, then there exists a constant \( \theta \in (0,1) \) such that (1.2) holds. So, for any \( f \geq 0 \) with \( \mu(f^2) \leq 1 \), we have
\[
\mu(f^2) \leq \frac{1}{1 - \theta} \mu(f(f - Pf)) + \mu(f)^2 = \frac{\mu(f^2)}{1 - \theta} - \frac{\mu(f Pf)}{1 - \theta} + \mu(f)^2.
\]
This implies that
\[ \mu(fPf) \leq \theta \mu(f^2) + (1-\theta)\mu(f)^2 \leq \theta + (1-\theta)\mu(f)^2. \]
Replacing \( f \) by \( (f - \sqrt{R})^+ \) for \( R > 1 \), and noting that
\[ \mu((f-s)^+) \leq \mu((f-s)^2) \mu(f > s) \leq \frac{1}{s^2}, \quad s > 0, \tag{3.1} \]
we obtain
\[ \mu\left( (f - \sqrt{R})^+ P(f - \sqrt{R})^+ \right) \leq \theta + \frac{1-\theta}{R}. \]
Since by the Jensen inequality \( P(f - R)^+ \leq P(f - R)^+ \), combining this with (3.1) we arrive at
\[ \mu(f(Pf - R)^+) \leq \mu(fP(f - R)^+) \]
\[ \leq \mu( (f - \sqrt{R})^+ P(f - R)^+) + \sqrt{R} \mu(P(f - R)^+) \]
\[ \leq \mu( (f - \sqrt{R})^+ P(f - \sqrt{R})^+) + \sqrt{R} \mu((f - R)^+) \]
\[ \leq \theta + \frac{1-\theta}{R} + \frac{1}{\sqrt{R}}, \quad R > 1. \]
Therefore,
\[ \|P\|_\tau = \lim_{R \to \infty} \sup_{0 \leq f, \mu(f^2) \leq 1} \mu(f(Pf - R)^+) \leq \theta < 1. \tag{2} \implies (1) \]

(2) implies (1). It suffices to prove for the case that \( L^2(\mu) \) is infinite-dimensional since in the finite dimensional case the existence of spectral gap is trivial. Assume that \( \|P\|_\tau < 1 \). Then there exists \( \delta \in (0,1) \) and \( R > 0 \) such that
\[ \sup_{f \geq 0, \mu(f^2) \leq 1} \mu(f(Pf - R)^+) \leq \delta. \tag{3.2} \]
On the other hand, if \( P \) does not have spectral gap, by Theorem 2.1 for any \( \varepsilon \in (0,1) \) and \( n \geq 2 \), there exists \( (A_1, \cdots, A_n) \in D_n \) such that
\[ \max_{1 \leq k \leq n} \frac{\mu(1_{A_k}P1_{A_k})}{\mu(A_k)} \leq \varepsilon. \]
Letting \( B_k = A_k \cap \{ P1_{A_k} \geq 1 - \sqrt{\varepsilon} \} \), this implies
\[ \varepsilon \mu(A_k) \geq \mu(1_{A_k}P1_{A_k}) = \mu(A_k) - \mu(1_{A_k}P1_{A_k}) \]
\[ \geq \mu(A_k) - \mu(B_k) - (1 - \sqrt{\varepsilon}) (\mu(A_k) - \mu(B_k)). \]
Thus,
\[ \mu(A_k) \geq \mu(B_k) \geq (1 - \sqrt{\varepsilon}) \mu(A_k). \tag{3.3} \]
Since \( A_1, \cdots, A_n \) are disjoint with positive \( \mu \)-mass and \( \mu \) is a probability measure, there exists \( 1 \leq k \leq n \) such that \( \mu(A_k) \in (0, \frac{1}{\varepsilon}) \). Take
\[ f = \frac{1_{A_k}}{\sqrt{\mu(A_k)}}. \]
We have
\[ Pf = \frac{P 1_{A_k}}{\sqrt{\mu(A_k)}} \geq \frac{1 - \sqrt{\varepsilon}}{\sqrt{\mu(A_k)}} 1_{B_k}. \]

Combining this with (3.2), (3.3) and \( \mu(A_k) \in (0, \frac{1}{n}) \), we arrive at
\[ \delta \geq \mu(f(Pf - R)^+) \geq \frac{\mu(B_k)(1 - \sqrt{\varepsilon} - R\sqrt{\mu(A_k)})^+}{\mu(A_k)} \geq (1 - \sqrt{\varepsilon}) \left(1 - \sqrt{\varepsilon} - \frac{R}{\sqrt{n}} \right)^+. \]

Since \( \varepsilon \in (0, 1) \) and \( n \geq 2 \) are arbitrary, by letting \( \varepsilon \to 0 \) and \( n \to \infty \) we obtain \( \delta \geq 1 \) which contradicts \( \delta \in (0, 1) \).

**Proof of Corollary 1.2**
It suffices to prove (1.4) from (1.5). Let \( P_t \) be the associated Markov semigroup. Then the irreducibility of the Dirichlet form implies that of \( P_t \) for \( t > 0 \). Next, by [16, Theorem 3.3] with \( \phi \equiv 1 \), (1.5) implies that \( \|P_t\|_r \leq e^{-t/C_1} < 1 \) for \( t > 0 \). Then due to Theorem 1.1 we conclude that \( P_t \) has a spectral gap, equivalently, the Poincaré inequality (1.4) holds for some constant \( C > 0 \).

**Proof of Corollary 1.3** By [19] Proposition 2.1, for any \( f \in L^2(\mu) \) we have
\[ \mu(f^2) - \mu(|f|^2) = \mu(f^2) + (1 - p)\mu(|\hat{f}|^2)^\frac{2}{p} \]
\[ = (2 - p)\mu(f^2) + (p - 1)(\mu(f^2) - \mu(|\hat{f}|^2)^\frac{2}{p}). \]

Then it follows from (3.4) that
\[ \text{Var}_{\mu, \phi}(f) \leq c_2 \mu(f^2) + \text{Var}_{\mu, \phi}(\hat{f}) \leq (v_\phi + C_2)\mu(f^2) + C_1\mathcal{E}(f, f). \quad (3.4) \]

Next, since \( \phi(2) = 0 \), from the proof of [19] Theorem 1.2(2)] we see that the \( F \)-Sobolev inequality (1.7) in [19] holds for some nonnegative function \( F \) with \( F(r) \uparrow \infty \) as \( r \uparrow \infty \). According to [15] (see also [20]), this inequality is equivalent to the super Poincaré inequality
\[ \mu(f^2) \leq r\mathcal{E}(f, f) + \beta(r)\mu(|f|^2), \quad r > 0, f \in \mathcal{D}(\mathcal{E}) \]

for some function \( \beta : (0, \infty) \to (0, \infty) \). In particular, the defective Poincaré inequality holds. Thus, by Corollary 1.2 we have the Poincaré inequality
\[ \mu(f^2) \leq \tilde{C}\mathcal{E}(f, f), \quad f \in \mathcal{D}(\mathcal{E}) \]

for some constant \( \tilde{C} > 0 \). Combining this with (3.4) we prove (1.7) for \( C = C_1 + \tilde{C}(c_\phi + C_2) \).

4 Extensions to the sub-Markov setting

In this section we let \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) be a non-conservative Dirichlet form, for which either \( 1 \notin \mathcal{D}(\mathcal{E}) \) or \( 1 \in \mathcal{D}(\mathcal{E}) \) but \( \mathcal{E}(1, 1) > 0 \). In this case we call the Dirichlet form irreducible if for any \( f \in \mathcal{D}(\mathcal{E}), \mathcal{E}(f, f) = 0 \) implies \( f = 0 \). Let \( \|P\|_p \) be the norm in \( L^p(\mu) \) for \( p \geq 1 \). Below is an extension of Corollary 1.2 to the present situation.
Theorem 4.1. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a non-conservative irreducible Dirichlet form on $L^2(\mu)$, where $\mu$ is a probability measure on $E$. Then the Poincaré inequality
\[ \mu(f^2) \leq C \mathcal{E}(f, f), \quad f \in \mathcal{D}(\mathcal{E}) \] (4.1)
holds for some $C > 0$ if and only if the defective Poincaré inequality
\[ \mu(f^2) \leq C_1 \mathcal{E}(f, f) + C_2 \mu(|f|)^2, \quad f \in \mathcal{D}(\mathcal{E}) \] (4.2)
holds for some $C_1, C_2 > 0$.

Proof. Assume that (4.2) holds. We aim to prove (4.1) for some constant $C > 0$. According to [17, Proposition 3.2] we need only to prove the weak Poincaré inequality
\[ \mu(f^2) \leq \alpha(r) \mathcal{E}(f, f) + r \|f\|_\infty^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}) \]
for some function $\alpha : (0, \infty) \to (0, \infty)$. This follows from the following more general result Theorem 4.3 where $\mu$ since in the present situation we have $\|\cdot\|_1 \leq \|\cdot\|_\infty$.

Before introducing Theorem 4.3, we present an application of Theorem 4.1 to sub-Markov operators.

Corollary 4.2. Let $(E, \mathcal{F}, \mu)$ be a probability space. Let $P$ be a sub-Markov operator on $L^2(\mu)$; i.e. $P$ is a contraction linear operator on $L^2(\mu)$ with $P1 \leq 1$ such that $f \geq 0$ implies $Pf \geq 0$. Let $P^*$ be the adjoint operator of $P$. Assume that Ker$(1 - P^*P) = 0$. Then
\[ \|P\|_2 := \sup_{\mu(f^2) \leq 1} \mu((Pf)^2)^{\frac{1}{2}} < 1 \]
if and only if
\[ \|P\|_{\text{tail}} := \lim_{R \to \infty} \sup_{\mu(f^2) \leq 1} \mu(\{|Pf| - R\}^2) < 1. \]

Proof. It suffices to prove $\|P\|_2 < 1$ from $\|P\|_{\text{tail}} < 1$. To apply Theorem 4.1 let
\[ \mathcal{E}(f, g) = \mu(f(1 - P^*P)g)), \quad \mathcal{D}(\mathcal{E}) = L^2(\mu). \]
Then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a symmetric Dirichlet form. Since Ker$(1 - P^*P) = 0$ and
\[ \mu((f - P^*P)^2) = \mu(f^2) - \mu((Pf)^2) + \mu((P^*Pf)^2) - \mu((Pf)^2) = \mathcal{E}(f, f), \]
this Dirichlet form is non-conservative and irreducible. By $\|P\|_{\text{tail}} < 1$, there exist $R > 0$ and $\varepsilon \in (0, 1)$ such that
\[ \mu((|Pf| - R)^2) \leq \varepsilon^2, \quad \mu(f^2) \leq 1. \]
So, for any $f$ with $\mu(f^2) = 1$,
\[ \mu((Pf)^2) \leq \mu(|Pf||Pf| - R^2) + R \mu(|Pf|) \leq \varepsilon + R \mu(|Pf|). \]
This implies

\[ \mathcal{E}(f, f) := \mu(f^2) - \mu((P f)^2) \geq 1 - \varepsilon - R\mu(|f|) \geq \frac{1 - \varepsilon}{2} - \frac{R^2}{2(1 - \varepsilon)} \mu(|f|)^2. \]

Therefore,

\[ \mu(f^2) = 1 \leq \frac{2}{1 - \varepsilon} \mathcal{E}(f, f) + \frac{R^2}{(1 - \varepsilon)^2} \mu(|f|)^2, \quad f \in L^2(\mu). \]

Thus, \((4.2)\) holds for \(C_1 = \frac{2}{1 - \varepsilon}, C_2 = \frac{R^2}{(1 - \varepsilon)^2}\). By Theorem 4.1 there exists \(C > 0\) such that

\[ \mu(f^2) \leq C\mathcal{E}(f, f) = C\mu(f(1 - P^* P)f) = C\mu(f^2) - C\mu((P f)^2). \]

This implies that \(\|P\|^2 \leq \frac{C-1}{C} < 1. \) \(\Box\)

From now on, we assume that \(\mu\) is a \(\sigma\)-finite measure, for which the framework of non-conservative Dirichlet operators includes Schrödinger operators of type \(\Delta - V\) for nonnegative measurable function \(V\) on \(\mathbb{R}^d\) as typical examples.

**Theorem 4.3.** Let \((\mathcal{E}, \mathcal{F}, \mu)\) be a \(\sigma\)-finite measure space. A non-conservative Dirichlet form \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) on \(L^2(\mu)\) is irreducible if and only if there exists \(\alpha : (0, \infty) \to (0, \infty)\) such that the weak Poincaré inequality

\[ \mu(f^2) \leq \alpha(r)\mathcal{E}(f, f) + r(|f|_\infty \lor |f|_1)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}). \] (4.3)

holds. Consequently, for any symmetric (sub-) Markov semigroup \(P_t\) on \(L^2(\mu)\),

\[ \lim_{t \to \infty} \sup_{\|f\|_1 \lor \|f\|_\infty \leq 1} \|P_t f\|_2 = 0 \quad \text{for any} \quad f \in L^2(\mu) \quad \text{as} \quad t \to \infty \] if and only if

\[ \lim_{t \to \infty} \sup_{\|f\|_1 \lor \|f\|_\infty \leq 1} \mu((P_t f)^2) = 0. \] (4.4)

**Proof.** (a) Let \(P_t\) be the associated semigroup of \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\). Then \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is irreducible if and only if \(\mu((P_t f)^2) \to 0\) as \(t \to \infty\) for any \(f \in L^2(\mu)\). On the other hand, by [13] Theorem 2.1 with \(\Phi(f) = |f|_1^2 \lor |f|_\infty^2\), \((4.3)\) holds for some \(\alpha\) if and only if

\[ \lim_{t \to \infty} \sup_{\|f\|_1 \lor \|f\|_\infty \leq 1} \mu((P_t f)^2) = 0. \]

So, the second assertion follows from the first one.

(b) Let \(f \in \mathcal{D}(\mathcal{E})\) with \(\mathcal{E}(f, f) = 0\). For any \(\varepsilon > 0\) let \(f_\varepsilon = (|f| - \varepsilon)^+ \land 1\). We have \(\mathcal{E}(f_\varepsilon, f_\varepsilon) = 0\) and by the Schwartz inequality

\[ \|f_\varepsilon\|_1^2 \leq \mu(f^2)\mu(|f| > \varepsilon) \leq \frac{\mu(f^2)^2}{\varepsilon^2}. \]

So, applying \((4.3)\) to \(f_\varepsilon\) we obtain \(\mu(f_\varepsilon) \leq r(1 + \varepsilon^{-2}\mu(f^2)^2)\) for all \(r > 0\). This implies \(f_\varepsilon = 0\) for all \(\varepsilon > 0\) and thus, \(f = 0\).
(c) Now, let \((\mathcal{E}, D(\mathcal{E}))\) be irreducible. We claim that (4.3) holds for some function \(\alpha : (0, \infty) \to (0, \infty)\). Otherwise, there exist some \(r > 0\) and a sequence \(\{f_n\} \subset D(\mathcal{E})\) such that

\[
1 = \mu(f_n^2) > n\mathcal{E}(f_n, f_n) + r(||f_n||_1 \vee ||f_n||_\infty)^2, \quad n \geq 1. \tag{4.5}
\]

Since \(\mathcal{E}(|f_n|, |f_n|) \leq \mathcal{E}(f_n, f_n)\), we may and do assume that \(f_n \geq 0\) for all \(n \geq 1\). Since \(\{f_n\}\) is bounded both in \(L^2(\mu)\) and \(L^1(\mu)\), there exist two functions \(f \in L^2(\mu), \tilde{f} \in L^1(\mu)\) and a subsequence \(\{f_{n_k}\}\) such that \(f_{n_k}\) converges weakly to \(f\) in \(L^2(\mu)\) and \(\tilde{f}\) in \(L^1(\mu)\) respectively. Then \(\mu(fg) = \mu(\tilde{f}g)\) for all \(g \in L^2(\mu) \cap L^\infty(\mu)\), and hence \(f = \tilde{f}\).

Let \(P_t\) be the (sub-) Markov semigroup and \((L, D(L))\) the generator associated to \((\mathcal{E}, D(\mathcal{E}))\). Then \(P_tf \in D(L)\) for any \(t > 0\). By the symmetry of \(P_t\) and the weak convergence of \(\{f_{n_k}\}\) to \(f\) in \(L^2(\mu)\), we have

\[
\lim_{k \to \infty} \mu((P_t f_{n_k})g) = \lim_{k \to \infty} \mu(f_{n_k} P_t g) = \mu(f P_t g) = \mu((P_t f)g), \quad g \in L^2(\mu).
\]

This implies

\[
\lim_{k \to \infty} \mathcal{E}(P_t f_{n_k}, g) = - \lim_{k \to \infty} \mu((P_t f_{n_k})Lg) = - \mu((P_t f)Lg) = \mathcal{E}(P_t f, g), \quad g \in D(L). \tag{4.6}
\]

Moreover, due to (4.5) and the symmetry of \(\mathcal{E}\),

\[
\lim_{k \to \infty} \mathcal{E}(P_t f_{n_k}, g)^2 \leq \lim_{k \to \infty} \mathcal{E}(P_t f_{n_k}, P_t f_{n_k})\mathcal{E}(g, g) \leq \lim_{k \to \infty} \mathcal{E}(f_{n_k}, f_{n_k})\mathcal{E}(g, g) = 0.
\]

Combining this with (4.6) we conclude that \(\mathcal{E}(P_t f, P_t f) = 0\) for all \(t > 0\). Thus, by the irreducibility, \(P_t f = 0\) holds for all \(t > 0\). This implies \(f = 0\) by the strong continuity of \(P_t\) in \(L^2(\mu)\). Since (4.5) implies \(f_n \leq r^{-1/2}\), by the weak convergence of \(\{f_{n_k}\}\) to \(f = 0\) in \(L^2(\mu)\) we obtain

\[
\lim_{k \to \infty} \mu(f_{n_k}^2) \leq r^{-1/2} \lim_{k \to \infty} \mu(f_{n_k}) = 0.
\]

This contradicts the assumption that \(\mu(f_n^2) = 1\) for all \(n \geq 1\). Therefore, (4.3) holds for some function \(\alpha : (0, \infty) \to (0, \infty)\).

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\textbf{References}

[1] S. Aida, Uniformly positivity improving property, Sobolev inequalities and spectral gap, J. Funct. Anal. 158 (1998), 152–185.

[2] P. Cattiaux, A pathwise approach of some classical inequalities, Pot. Anal. 20(2004), 361–394.
[3] F. Gong and L. Wu, *Spectral gap of positive operators and applications*, J. Math. Pures Appl. 85(2006), 151–191.

[4] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. 97 (1975) 1061–1083.

[5] M. Hino, *Exponential decay of positivity preserving semigroups on $L^p$*, Osaka J. Math. 37 (2000), 603–624.

[6] S. Kusuoka, *Analysis on Wiener spaces II: differential forms*, J. Funct. Anal. 103 (1992), 229–274.

[7] R. Latała and K. Oleszkiewicz, *Between Sobolev and Poincaré*, Lecture Notes in Math. Vol. 1745, 147–168, 2000.

[8] G. F. Lawler and A. D. Sokal, *Bounds on the $L^2$ spectrum for Markov chains and Markov processes: a generalization of Cheeger’s inequality*, Trans. Amer. Math. Soc. 309(1988), 557–580.

[9] J. R. Lee, S. Oveis Gharan, L. Trevisan, *Multi-way spectral partitioning and higher-order Cheeger inequalities*, arXiv: 1111.1055v4. STOC’12 Proceedings of the 44th Symposium on Theory of Computing, pp. 1117–1130, New York, NY, USA, 2012.

[10] L. Miclo, *On hyperboundedness and spectrum of Markov operators*, http://hal.archives-ouvertes.fr/hal-00777146.

[11] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, New York, 1978.

[12] M. Reed and B. Simon, *Method of Modern Mathematical Physics, I: Functional Analysis*, Academic Press, 1980.

[13] M. Röckner and F.-Y. Wang, *Weak Poincaré inequalities and $L^2$-convergence rates of Markov semigroups*, J. Funct. Anal. 185(2001), 564–603.

[14] B. Simon and R. Høegh-Krohn, *Hypercontractive semigroups and two dimensional self-coupled Bose fields*, J. Funct. Anal. 9(1972), 121–180.

[15] F.-Y. Wang, *Functional inequalities, semigroup properties and spectrum estimates*, Infin. Dimer. Anal. Quant. Probab. Relat. Topics, 3(2000), 263–295.

[16] F.-Y. Wang, *Functional inequalities and spectrum estimates: the infinite measure case*, J. Funct. Anal. 194(2002), 288–310.

[17] F.-Y. Wang, *Functional inequalities for the decay of sub-Markov semigroups*, Pot. Anal. 18(2003), 1–23.

[18] F.-Y. Wang, *Spectral gap for hyperbounded operators*, Proc. Amer. Math. Soc. 132(2004), 2629–2638.

[19] F.-Y. Wang, *A generalization of Poincaré and log-Sobolev inequalities*, Potential Analysis 22(2005), 1–15.

[20] F.-Y. Wang, *Functional Inequalities, Markov Processes and Spectral Theory*, Science Press, Beijing, 2005.

[21] L. Wu, *Uniformly integrable operators and large deviations for Markov processes*, J. Funct. Anal. 172(2000), 301–376.