Distances Release with Differential Privacy in Tree and Grid Graph

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Abstract

Data about individuals may contain private and sensitive information. The differential privacy (DP) was proposed to address the problem of protecting the privacy of each individual while keeping useful information about a population. Sealfon (2016) introduced a private graph model in which the graph topology is assumed to be public while the weight information is assumed to be private. That model can express hidden congestion patterns in a known transportation system. In this paper, we revisit the problem of privately releasing approximate distances between all pairs of vertices in Sealfon (2016). Our goal is to minimize the additive error, namely the difference between the released distance and actual distance under private setting. We propose improved solutions to that problem for several cases.

For the problem of privately releasing all-pairs distances, we show that for tree with depth $h$, we can release all-pairs distances with additive error $O(\log^{1.5} h \cdot \log^{1.5} V)$ for fixed privacy parameter where $V$ the number of vertices in the tree, which improves the previous error bound $O(\log^{2.5} V)$, since the size of $h$ can be as small as $O(\log V)$. Our result implies that a log $V$ factor is saved, and the additive error in tree can be smaller than the error on array/path. Additionally, for the grid graph with arbitrary edge weights, we also propose a method to release all-pairs distances with additive error $\tilde{O}(V^{3/4})$ for fixed privacy parameters. On the application side, many cities like Manhattan are composed of horizontal streets and vertical avenues, which can be modeled as a grid graph.

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1 Introduction

It has been a popular topic of research that machine learning practitioners hope to protect user privacy while effectively building machine learning models from the data. The motivation for differential privacy (DP) (Blum et al., 2005; Chawla et al., 2005; Dwork, 2006) is to keep useful information for model learning while protecting the privacy for individuals. The formulation of DP provides a rigorous guarantee that an adversary could learn very little about an individual.

For the graph problem under private setting, there exists a line of works in the last decade or so (Hay et al., 2009; Rastogi et al., 2009; Gupta et al., 2010; Karwa et al., 2011; Gupta et al., 2012; Blocki et al., 2013; Kasiviswanathan et al., 2013; Bun et al., 2015; Sealfon, 2016; Ullman and Sealfon, 2019; Borgs et al., 2018; Arora and Upadhyay, 2019) including node privacy, edge privacy, and weight privacy. In this paper, we study differential privacy for the “weight private graph model” (Sealfon, 2016), particularly in tree and grid graph. As the name suggests, in the weight private graph model, the topology of the graph is public but the weights are private. The weight private model can be well-suited, for example, for modeling the traffic navigation system (Sealfon, 2016).

For two neighboring input graphs with weight functions differing by one unit, it is obvious that the single pair shortest distance can only differ by at most one for two neighboring inputs, since the short path is a simple path with edges appearing at most once. To achieve privacy for single pairs, a popular strategy is by adding Laplace noise according to the $O(1/\epsilon)$ based Laplace mechanism (Dwork, 2006). Since it is a trivial task to achieve privacy for a single pair, Sealfon (2016) focused on the more difficult task for releasing all-pairs distances privately. The author showed that one can release all-pairs distances with additive error $O(\log^{4.5} V)$ on trees, where $V$ is the number of vertices in the tree. For general graphs, the author proposed a simple approach that achieves $O(V)$ error, which was then improved when the weights are all bounded.

In this paper, we revisit the problem of releasing all pairwise distances in the private graph model. Here we summarize our new results, compared with the previous results obtained in Sealfon (2016). The additive error is the largest absolute difference between the released distance and the actual distance among all node pairs, which applies to both this paper and reference (Sealfon, 2016).

- For a tree with depth $h$, we propose a new algorithm to release all-pairs distances each with error $O((\log^{1.5} h) \cdot (\log^{1.5} V))$ for fixed privacy parameters, which is a significant improvement to previous additive error $O(\log^{2.5} V)$ (Sealfon, 2016). Our method is based on heavy path decomposition (Harel and Tarjan, 1984): We divide a tree into disjoint heavy paths and light paths (the definition about heavy path decomposition is provided later in the paper). The unique path between any pair crosses at most log $V$ heavy paths. Each heavy path is a path graph, where the releasing of approximate all-pairs distances is equivalent to query release of threshold functions. The results of Dwork et al. (2010a) yield the same error bound as the error bound in computing distances on the path graph in Sealfon (2016). We use their method as a subroutine to deal with each heavy path after heavy path decomposition (Harel and Tarjan, 1984). General graph in metric space can be embedded into a tree with expected distortion $O(\log V)$ and bounded depth $O(\log V)$ by “Padded Decomposition” (Bartal, 1996), where distortion is the factor between distance/length in graph and distances in tree. Hence the transportation network can be embedded into a depth bounded tree network. Our private algorithm on tree cases likely results in better private algorithms on a general graph later. On the practical side, many internet networks are tree networks, or star-bus networks, which can be modeled as a tree. Hence trees are a natural case that deserve to be studied.

- For a weight bounded graph $G(V,E)$, the previous work (Sealfon, 2016) picked a subset $Z \in V$ of vertices to form the so called “$k$-covering set”. A $k$-covering set $S$ guarantees that any vertex in $G$ has at most $k$ hops to its closest hub in $S$. One can use the $k$-covering set to approximate the original graph. Each vertex $u$ can map to its closest vertex $z_u$ in the covering set. For any pair $u,v$, the distance between $u$ and $v$ can be approximated by the distance between $z_u$ and $z_v$ with additional error $O(kM)$. Thus, the solution, in general, is to use $O(|Z|^2)$ pair distances to represent/approximate the $O(V^2)$ pair distances with additional error $kM$, where $M$ is the upper bound of edge weight. As a special case, for bounded grid graphs, the authors also gave an error bound where $k = V^{1/3}$ approximately. However, it is unknown how to generalize this approach to general graphs (with arbitrary weights). To shed
light on this problem, in this paper, we first consider the grid graph with general positive weights. We divide the grid graph into blocks and then we separate the distances into several types: 1) the distances between those vertices in each block; 2) the distances between pairs of vertices on the boundary of blocks; and 3) the distances not included in neither type 1 nor type 2, but composed of type 1 and type 2 distances with concatenation. Our method could release all-pairs distances on general grid graphs with additive error \( \hat{O}(V^{3/4}) \) for fixed privacy parameters; more details are given later in the paper. We believe that our idea can be extended to more general graphs.

2 Background: Private Graph Model

For readability, we adopt the same notions as used in Sealfon (2016). For a general graph \( G = (V, E) \), throughout the paper we use \( w(e) \) to denote the original weight of any edge \( e \in E \), and for any subset \( E' \subseteq E \) we denote \( w(E') = \sum_{e \in E'} w(e) \). Let \( V = |V|, E = |E| \), and for simplicity we assume \( G \) is connected and hence we always have \( V - 1 \leq E \).

Let \( P_{xy} \) denote the set of simple paths between a pair of vertices \( x, y \in V \). For any path \( P \in P_{xy} \), the weight \( w(P) \) is the summation \( \sum_{e \in P} w(e) \) of the edge weights in \( P \). The distance \( d(x, y) \) from \( x \) to \( y \) denote the weighted distance \( \min_{P \in P_{xy}} w(P) \). We first introduce the definition of differential privacy (DP) in the private edge weight model (Sealfon, 2016). The following notion of neighboring graphs will be used.

**Definition 2.1** (Neighboring graphs). For any edge set \( E \), two weight functions \( w, w' : E \rightarrow \mathbb{R}^+ \) are neighboring, denoted \( w \sim w' \) if

\[
||w - w'||_1 = \sum_{e \in E} |w(e) - w'(e)| \leq 1.
\]

**Definition 2.2** (Differential Privacy in graph model (Sealfon, 2016)). For any graph \( G = (V, E) \) let \( A \) be an algorithm that takes as input a weight function \( w : E \rightarrow \mathbb{R}^+ \). If for all pairs of neighboring graphs with weight \( w, w' \) and for all set of outcomes \( O \subseteq \text{Range}(A) \) such that

\[
Pr[A(w) \in O] \leq e^\epsilon Pr[A(w') \in O] + \delta,
\]

algorithm \( A \) is said to be \((\epsilon, \delta)\)-differentially private, and \( \epsilon \)-differentially private on \( G \) if \( \delta = 0 \).

The privacy guarantees may be achieved through the introduction of noise to the output. In order to achieve \((\epsilon, 0)\)-differential privacy, for example, the noise added typically comes from the Laplace distribution (the so-called Laplace mechanism will be introduced formally later). Intuitively, differential privacy requires that after removing any observation, the output of \( w' \) should not be too different from that of the original data set \( w \). Smaller \( \epsilon \) and \( \delta \) indicate stronger privacy, which, however, usually sacrifices utility. Thus, one of the central topics in the differential privacy literature is to balance the utility-privacy trade-off.

Statistically, one merit of differential privacy is that, different DP algorithms can be integrated together with provable privacy guarantee.

**Lemma 2.1** (Composition of DP (Dwork et al., 2010b)). For any \( \epsilon, \delta, \delta' \geq 0 \), the adaptive composition of \( k \) times \((\epsilon, \delta)\)-differentially private mechanisms is \((\epsilon', k\delta + \delta')\)-differentially private for

\[
\epsilon' = \sqrt{2k \log(1/\delta')} \cdot \epsilon + k \cdot (\epsilon' - 1),
\]

which is \( O(\sqrt{k \log(1/\delta')} \cdot \epsilon) \) when \( k \leq 1/\epsilon^2 \). In particular, if \( \epsilon' \in (0, 1) \), \( \delta' > 0 \), the composition of \( k \) times \((\epsilon, 0)\)-differentially private mechanism is \((\epsilon', \delta')\)-differentially private for

\[
\epsilon = \epsilon' / (\sqrt{8k \log(1/\delta')}).
\]
Private graph distance release. In this paper, we consider the approximate distances release problem on graphs, where the goal is to publish all the pair-wise distances (i.e., distance matrix) between all node pairs. The error is evaluated by the absolute difference between the released/estimated distance \( d_e(x, y) \) between a pair of vertices \( x, y \) and the actual distance \( d(x, y) \). For each pair of vertices \( (x, y) \), we call \( |d_e(x, y) - d(x, y)| \) the additive error of that pair. The objective is to minimize the largest additive error among all pairs, namely, minimizing \( \{\max\{|d_e(x, y) - d(x, y)|, (x, y) \in E\}\} \) under the constraints that all \( \{d_e(x, y), (x, y) \in E\} \) are differentially private achieving Definition 2.2.

Technical tools. We now introduce a few technical tools which will be used throughout the remainder of this paper. A number of differential privacy techniques incorporate noise sampled according to the Laplace distribution. The so-called Laplace mechanism will be frequently used in our algorithm design and analysis.

Lemma 2.2 (Laplace mechanism (Dwork, 2006)). For a function \( f : G \to \mathbb{R} \) with \( G \) the input space of graphs, define the \( l_1 \) sensitivity as
\[
\Delta_f = \max_{G \sim G'} ||f(G) - f(G')||_1,
\]
where \( G, G' \) are two neighboring graphs as in Definition 2.1. Let \( \xi \) be a random noise drawn from \( \text{Lap}(0, \Delta_f/\epsilon) \). The Laplace mechanism outputs
\[
M_{f, \epsilon}(G) = f(G) + \xi,
\]
and the approach is \( \epsilon \)-differentially private.

We state a concentration bound for the summation of Laplace random variables. Even though these results are already well known (and used heavily in the prior work (Sealfon, 2016)), we just include them for completeness.

Lemma 2.3 (Concentration of Laplace RV (Chan et al., 2010)). Let \( Z \) be the sum of \( n \) i.i.d. random variables \( Z_1, Z_2, ..., Z_n \) following \( \text{Lap}(b) \), and denote \( Z = \sum_i Z_i \). For \( 0 < t < 2\sqrt{2}b n \), we have
\[
\Pr[|Z| > t] \leq 2 \exp\left(-\frac{t^2}{8nb^2}\right).
\]
For \( \gamma \in (2/e^n, 1) \), with probability at least \( 1 - \gamma \),
\[
|Z| < 2\sqrt{2}b\sqrt{n}\log(2/\gamma) = O(b\sqrt{n}\log(1/\gamma)).
\]

3 Distances Release in Trees

The notion of depth bounded tree is widely used in computer science. For example, general graphs can be embedded into a tree with bounded depth \( O(\log V) \) (Bartal, 1996), and improvements on private trees would likely lead to better private algorithms on general graphs. Since the previous paper (Sealfon, 2016) could deal with the tree case with additive error \( O(\log^{2.5} V) \) for fixed privacy parameters, it is natural to ask whether that bound can be improved. The answer is “yes”, and this paper provides a method which achieves better performance for the depth bounded tree.
Figure 1: The input tree $T$ is partitioned into paths by classic heavy path decomposition. Each heavy path is marked with a distinct color. The shortest path between $s$ and $t$ can be decomposed into sub paths like $(s_1, t_1), (s_2, t_2), (s_3, t_3)$ inside disjoint heavy paths and light edge between them.

We can decompose the tree into paths using the heavy path decomposition (Harel and Tarjan, 1984), which is also called heavy-light decomposition and is a technique for decomposing a rooted tree into a set of paths as long as possible; see Algorithm 1. In a heavy path decomposition, each non-leaf node selects one branch, the edge to the child that has the largest depth (breaking ties arbitrarily). The selected edges form the paths of the decomposition, an example is given in Figure 1. Since the tree is decomposed into a set of paths, some edges may not be included in any one of the heavy paths produced during the decomposition process. We call those edges the “light edge”, in comparison to those heavy edges included in the heavy paths.

**Lemma 3.1** (Tree Decomposition (Harel and Tarjan, 1984)). For any root-to-leaf path of a tree with $V$ nodes, there can be at most $\log V$ light edges. Equivalently, the path tree has height at most $\log V$.

We now present the general idea of our algorithm. We first decompose the tree into heavy paths. Each heavy path is a path graph, which we can use the classic private algorithm in Sealfon (2016); Dwork et al. (2010a) to deal with. Also, these heavy paths are all disjoint to each other. Thus each of them can be handled separately. For those light edges, each of them can be added with a random Laplace noise according to the random variable $\text{Lap}(1/\epsilon)$.

The query process: for a pair $(s, t)$ inside one heavy path, which is similar to the query process of path graphs. For another pair $(s, t)$ crossing several heavy paths, the path distance can be released by summing several subqueries of heavy paths and light edges.
Algorithm 1: Tree-Decomposition($T$): To partition a tree $T$ into a set of edge disjoint paths.

Input: Tree $T$.
Output: A set of edge disjoint paths by heavy path decomposition.
Let $R$ denote a set of roots of subtrees in $T$.
Let $P$ denote a path (a set of consecutive nodes in $T$).
if $T$ is a leaf then
    Add $T$ to $P$.
Let node pointer $p$ point to the root of $T$.
while $p$ is not null do
    Add $p$ to $P$.
    Compute the depth of each children of $T$.
    Let $p$ point to the children of $T$ with maximum depth.
    Add another children nodes of $T$ than $p$ to $R$.
for node $v$ in $R$ do
    Call recursive Tree-Decomposition($v$).
    Add the obtained paths into $S$.
Let $S = \{P_1, P_2, ..., P_k\}$ be a set of heavy paths produced in the decomposition above.
Return $S$.

Algorithm 2: Private algorithm to release all pairwise distances for tree.

Input: Length bounded Tree $T$, private parameter $\epsilon$.
Output: Private all pairwise distances of vertices in $T$.
Call Tree-decomposition algorithm in Algorithm 1 on $T$.
Let $S = \{P_1, P_2, ..., P_k\}$ be a set of heavy paths produced in the decomposition above.
for each heavy path $P_i$ in $S$ do
    Call the Algorithm 1 in Sealfon (2016) to release private all pairwise distances of path $P_i$ with privacy parameter $\epsilon$.
for each light edge $(u,v)$ do
    $d_e(u,v) := d(u,v) + \text{Lap}(1/\epsilon)$.
for each pair $(s,t)$ do
    Find the path $P(s,t)$ from vertex $s$ to $t$.
The shortest path between $s$ and $t$ can be decomposed into sub paths like $(s_1,t_1), (s_2,t_2), (s_3,t_3), ...$ into disjoint heavy paths.
    for each heavy path intersecting with $P(s,t)$ do
        Add the released distance $d_e(s_i,t_i)$ to $d_e(s,t)$.
    for each light edge intersecting with $P(s,t)$ do
        Add the release distance of that light edge to $d_e(s,t)$.
Return all pairwise distances released above.
Theorem 3.2 (All Pairwise Shortest Path Distances on Rooted Trees). Let \( T = (V, E) \) be a tree with \( V \) vertices, \( \epsilon > 0 \), Algorithm 2 is \( \epsilon \)-differentially private on \( T \) and releases all-pairs distances such that with probability \( 1 - \gamma \), all released distances have additive error bounded by

\[
O((\log^{1.5} V \cdot \log^{1.5} h) \log(1/\gamma))/\epsilon,
\]

which is

\[
O(\log^{1.5} V \cdot \log \log^{1.5} V \cdot \log(1/\gamma))/\epsilon
\]

when \( h = O(\log V) \).

Proof. Since each of heavy paths and light edges are disjoint to each other, we can assign \( \epsilon \) privacy to each of them, which gives \( \epsilon \)-DP in total. Let \( S \) be a set of disjoint heavy paths of \( T \). Let \( D_i \) denote the length of \( i \)-th heavy path of \( S \) (\( |S| \leq \log V \)), and \( d_e(s_i, t_i) \) be the sub query inside the \( i \)-th path needed for the pair \((u, v)\). The length of each heavy path is bounded by the depth \( h \) of the tree. The additive error of each pair \( d_e(s_i, t_i) \) inside path \((s_i, t_i)\) is the sum of \( \log L(s_i, t_i) \) Laplace random variable according to \( \text{Lap}(\log h/\epsilon) \) based on Sealfon (2016) and the link length of each heavy path is bounded by \( h \), and \( L(s_i, t_i) \) denotes the link length between \( s_i \) and \( t_i \).

For those heavy paths between \( s \) and \( t \), the additive error of each of them is decided by sum of \( \log L(s_i, t_i) \) Laplace random variables each following \( \text{Lap}(\log h/\epsilon) \). Hence, the sum of additive error of those heavy paths is determined by the sum of \( \sum \log L(s_i, t_i) \) Laplace random variables each according to \( \text{Lap}(\log h/\epsilon) \). We know that \( \sum L(s_i, t_i) \leq 2h \) because the link length of any path is less or equal than \( 2h \). With probability \( 1 - \gamma \) this error is bounded by

\[
O(\sqrt{\sum \log L(s_i, t_i)(\log h) \log(1/\gamma)})/\epsilon
\]

based on Lemma 2.3. As the path between \((s, t)\) passes at most \( \log V \) light edges based on Lemma 3.1, with probability \( 1 - \gamma \), the error of this part is bounded by

\[
O(\sqrt{\log V \log(1/\gamma)})/\epsilon
\]

based on Lemma 2.3. Therefore, the total sum of additive error between \((s, t)\) is bounded by

\[
O(\sqrt{\sum (\log L(s_i, t_i)) \log h + \sqrt{\log V} \frac{\log(1/\gamma)}{\epsilon}}).
\]

On the other hand, \( \sum \log L(s_i, t_i) \) can be bounded by \( O((\log V) \log h) \) as \( L(s_i, t_i) \leq h \) and the path from \( s \) to \( t \) passes at most \( O(\log V) \) heavy paths. Combining parts together, the additive error is bounded by

\[
O(\log^{1.5} V \log^{1.5} h + \log^{0.5} V) \frac{\log(1/\gamma)}{\epsilon}.
\]

By a union bound, for any \( \gamma \in (0, 1) \), with probability at least \( 1 - \gamma \), each error among the \( O(V^2) \) all-pairs distances released is at most \( O(\log^{1.5} V \cdot \log^{1.5} h \frac{\log(1/\gamma)}{\epsilon}) \).

\[\square\]

4 Distances Release in Grid Graph

For a general graph \( G(V, E) \), one can pick a subset \( U \subseteq V \) of vertices, the so called "\( k \)-covering set", to approximate the graph when the weight is bounded. That approach could not be extended to more general settings, whereas in this paper we consider the distance release in grid graph with arbitrary weights.

The general idea is as follows. Let \( G \) be the \( \sqrt{V} \times \sqrt{V} \) grid. A path from \( s \) to \( t \) can be divided into three parts by two immediate vertices \((u, v): (s, ..., u), (u, ..., v), (v, ..., t)\). In order to obtain the immediate vertex set, we divide the grid into \( \sqrt{V} \) blocks with size \( V^{1/4} \times V^{1/4} \). Let \( B_1, B_2, ..., B_{\sqrt{V}} \) denote the \( \sqrt{V} \) blocks respectively. Let set \( A_i \) denote the set of vertices located on the boundary of \( B_i \), and the size of \( A_i \) is \( O(V^{1/4}) \). Let set \( A = \bigcup A_i \), then \( A \) is the immediate vertex set for grid graph. The additive error of \( d_e(s, u) \) is decided by the number of edges from \( s \) to \( u \), and similar analysis for \( d_e(v, t) \). The additive error of \( d_e(u, v) \) is the noise added to \( d(u, v) \), which depends on the size of the immediate vertex set \( A \), specified next.
For each pair vertices \((u, v)\) in \(A\), we add Laplace random noise \(\text{Lap}(V^{3/4}(\sqrt{8\log 1/\delta})/\epsilon)\) to the released distance, namely \(d_e(u, v) := d(u, v) + \text{Lap}(V^{3/4}(\sqrt{8\log 1/\delta})/\epsilon)\), recall that \(d(u, v)\) is the exact distance between \(u\) and \(v\). For each pair vertices \((x, y)\) in each \(B_i\), add Laplace random noise \(\text{Lap}(V^{3/4}(\sqrt{8\log 1/\delta})/\epsilon)\) to the released distance, namely \(d_e(x, y) := d(u, v) + \text{Lap}(V^{3/4}(\sqrt{8\log 1/\delta})/\epsilon)\). For a pair \((s, t)\) such that \(s\) is located in \(B_i\) while \(t\) is located in some \(B_j\). We release \(d_e(s, t)\) as follows.

\[
d_e(s, t) := \min_{u \in A_i, v \in A_j} \{d_e(s, u) + d_e(u, v) + d_e(v, t)\}.
\]

An illustration is given in Figure 2. In Theorem 4.1, we show that for any \(\epsilon, \gamma \in (0, 1)\) and \(\delta > 0\), one can release with probability \(1 - \gamma\) all-pairs distances each with additive error \(O(V^{3/4} \log(V/\gamma)\sqrt{\log 1/\delta})/\epsilon\).

![Figure 2: Illustration of released distance between \((s, t)\) in two different blocks.](image)

**Algorithm 3:** Private Algorithm for Grid Graph.

**Theorem 4.1.** Given any input \(G(\sqrt{V} \times \sqrt{V})\), and parameter \(\epsilon, \delta > 0\), Algorithm 3 is \((\epsilon, \delta)\)-differentially private. With probability \(1 - \gamma\), Algorithm 3 releases all pairwise distances, for each distance, with probability \(1 - \gamma\) the additive error is \(O(V^{3/4}(\sqrt{\log 1/\delta}) \log(V/\gamma))/\epsilon\).
Proof. For the privacy part, only $O(V^{3/2})$ pairs of distances are computed directly, other pairs are composed of by them. Hence adding noise according to $\text{Lap}(V^{3/4}(\sqrt{8 \log 1/\delta})/\epsilon)$ suffice to guarantee privacy based on the Lemma 2.1 and Laplace Mechanism.

For the utility part, the pair $(s, t)$ in two distinct blocks $B_i, B_j$, the shortest path between $B_i, B_j$ has to pass two points $u, v$ such that $u \in A_i$ and $v \in A_j$. Hence $d_e(s, t) = \min_{u \in A_i, v \in A_j} \{d_e(s, u) + d_e(u, v) + d_e(v, t)\}$ can get the shortest path distance between $s$ and $t$. With probability $1 - \gamma$, each of the total $V^{3/2}$ Laplace random variables according to $O(V^{3/4}(\sqrt{\log 1/\delta})/\epsilon)$ can be bounded by $O(V^{3/4}(\sqrt{\log 1/\delta})\log(V/\gamma)/\epsilon)$. The sum of three variables increase the error by at most three times, which can be ignored. Hence, for any $\gamma \in (0, 1)$, with probability at least $1 - \gamma$, the additive error of $d_e(s, t)$ is within $O(V^{3/4}\log(V/\gamma)\sqrt{\log 1/\delta})/\epsilon$. \hfill \qed

5 Conclusion

In this paper, we study the problem of releasing all pairwise distances in the private graph model as studied in Sealfon (2016). For the problem of privately releasing all-pairs distances in trees, the author proposed a solution to achieve additive error $O(\log^{2.5} V)$ for fixed privacy parameters, where $V$ is the number of vertices in the tree. In this paper, we propose a new algorithm which can release all-pairs distances with error $O(\log^{1.5} h \cdot \log^{1.5} V)$ for fixed privacy parameters. Our method is based on heavy path decomposition (Harel and Tarjan, 1984), and $h$ is small in many applications. Additionally, in this paper, we also consider the grid graph with general positive weights. Our approach is based on dividing the graph into blocks and selecting an intermediate set of vertices in distance computation. Our proposed method releases all-pairs distances with additive error $\tilde{O}(V^{3/4})$ for fixed privacy parameter on general grid graphs.

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