Large N Spin Quantum Hall Effect.

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Abstract

We introduce a large $N$ version of the spin quantum Hall transition problem. It is formulated as a problem of Dirac fermions coupled to disorder, whose Hamiltonian belong to the symmetry class $C$. The fermions carry spin degrees of freedom valued in the algebra $sp(2N)$, the spin quantum Hall effect corresponding to $N = 1$. Arguments based on renormalization group transformations as well as on a sigma model formulation, valid in the large $N$ limit, indicate the existence of a crossover as $N$ varies. Contrary to the $N = 1$ case, the large $N$ models are shown to lead to localized states at zero energy. We also present a sigma model analysis for the system of Dirac fermions coupled to only $sp(2N)$ random gauge potentials, which reproduces known exact results.

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1 Introduction.

The landscape of delocalization transitions is wider in two dimensions than in higher dimensions, paralleling the classification of the new random ensembles of ref. [1]. Many of these transitions may be modeled by Dirac fermions coupled to various disordered variables. When formulated as field theories these transitions are usually mapped into difficult strong coupled systems requiring the identification of non-trivial infrared fixed points. In this respect, the recently introduced $su(2)$ spin quantum Hall transition [2], which corresponds to random Hamiltonians belonging to class C of ref. [1], appears as an exception. Indeed, using a network formulation, it has been argued [3] that the critical properties of the latter model are potentially described by percolation in two dimensions. This opened the possibility of exact computations of characteristic critical exponents [3] or of the mean conductance [4]. See also ref. [5] for a spin chain formulation of this model. However, these results still resist to a field theory derivation, see eg. [6]. Such alternative approach would be useful for the related, but yet unsolved, problem of the quantum Hall transition.

The aim of this paper is to study a large $N$ version of the spin quantum Hall transition. We shall formulate it as Dirac fermions with $sp(2N)$ spin degrees of freedom whose Hamiltonians belong to the class C. This requirement, which forces us to choose the algebra $sp(2N)$, is compatible with the introduction of four types of disordered potentials. These potentials are all generated under renormalization group transformations. Some of the results valid in the $N = 1$ spin quantum Hall effect extend to the large $N$ generalization. This holds true for instance for the spin-charge separation for particular fine tuned versions of the models. However the following analysis indicates that there exists a crossover as $N$ is increased. Namely, we will find that, contrary to the $N = 1$ case, zero energy states are localized at large $N$.

We analyse these models in two steps. First, renormalization group transformations allow us to identify the field theory describing the universality class of such models. It corresponds to disordered variables isotropically distributed among the four possible types of disordered potentials. We then use supersymmetric sigma model techniques to analyse the universal model, which for class C Dirac fermions is a sigma model on the Riemannian symmetric superspace $OSp(2|2)/GL(1|1)$, of type $DIII|CI$ according to ref. [7]. This sigma model, characteristic for the symmetry class C, also appeared in the context of disordered superconductors with spin rotation symmetry.
and no time reversal invariance \cite{8,9}. The resulting sigma model turns out to be a massive theory. As a consequence the low energy limit is non critical. This is in contrast with class $D$ Dirac fermions which were described in refs. \cite{10,11} by a massless sigma model of type $\text{CI}$/$\text{DIII}$. In both cases, the derivation of the sigma model is justified only at large $N$.

For completeness, we have also included a sigma model description of Dirac fermions coupled to $sp(2N)$ random gauge potentials. This corresponds to a particular fine tuning of the previously discussed models enlarging the symmetry of the Hamiltonians which then belongs to class $\text{CI}$. Surprisingly, large $N$ sigma model analysis reproduces the exact result \cite{1} that the critical theory is an $osp(2|2)_{k=-2N}$ WZW model. The fixed point of class $\text{CI}$, including the WZW term, was also found in \cite{12,13} using replica and in \cite{14} by direct means.

The paper is organized as follows. The models as well as their supersymmetric formulations are introduced in Section 2 and 3. There, we also introduce the appropriate orthosymplectic transformations and the effective actions. Symmetries of the pure systems are identified in Section 4. The beta functions computed in Section 5 show that the model with isotropic randomness is universal and attractive at large $N$. The sigma model formulations of the latter, which is valid in the large $N$ limit, is analysed in Section 6. We show that, at large scale, this model is driven to a strongly coupled massive phase, due to the absence of non-trivial topological $\theta$-term. This leads to our main result concerning localization of zero energy eigenstates. Spin-charge separation in the pure system and sigma model formalism for perturbations by random gauge potentials, corresponding to random Hamiltonians in class $\text{CI}$, are presented in Section 7 and 8.

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2 The models.

We consider $2N$ species of Dirac fermions, coupled by disorder in class C [1]. The Dirac Hamiltonian for generic disorder in this class is

$$H = \begin{pmatrix} \alpha + M + m & -2i\partial + A \\ -2i\bar{\partial} + \bar{A} & \alpha - M - m \end{pmatrix}$$

(1)

where $\partial = (\partial_x - i\partial_y)/2$, $\bar{\partial} = (\partial_x + i\partial_y)/2$, $A = A_x - iA_y$, $\bar{A} = A_x + iA_y$. Here, $A_\mu = \sum_a A^a_\mu(x,y)\tau^a$ are random $sp(2N)$ gauge potentials, $\alpha = \sum_a \alpha_a(x,y)\tau^a$ is a random “spin” potential, and $M = \sum_i M_i(x,y)T^i$ and $m = m(x,y)\mathbf{1}$ are random mass terms. We consider generators of $sp(2N)$ defined by the relation

$$\tau^a = -\Sigma (\tau^a)^T \Sigma^{-1},$$

where the superscript $T$ denotes usual transposition, and $\Sigma$ is the symplectic unit

$$\Sigma = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}.$$  

(2)

The generators $T^i$, with the property $T^i = \Sigma(T^i)^T \Sigma^{-1}$, form the complement of $sp(2N)$ with respect to $sl(2N)$. The generators $\tau^a$ and $T^i$, together with the identity $\mathbf{1}$, span the algebra $gl(2N)$. Remark that for $N = 1$ there is no generator $T^i$, so that $sp(2) \simeq su(2)$.

The Hamiltonian (1) enjoys the following particle-hole symmetry defining symmetry class C (spin rotation invariance and no time reversal symmetry) in the classification of [1]

$$H = -C H^T C^{-1},$$

(3)

with $C = \sigma_1 \otimes \Sigma$ an antisymmetric matrix. This relation implies that the eigenvalues occur in pairs with opposite signs, and it relates the advanced and retarded Green functions

$$G_R(E) = -C G_A(-E)^T C^{-1}.$$  

We consider centered gaussian distributions for the four types of disorder, with strengths $g_A$, $g_\alpha$, $g_M$ and $g_m$ (positive real numbers for real disorder). The disorder variances are:

$$\langle m(x)m(y) \rangle = \frac{g_m}{N}\delta^{(2)}(x - y), \quad \langle M(x)M(y) \rangle = \frac{g_M}{N}\delta^{(2)}(x - y)$$

$$\langle \alpha^a(x)\alpha^b(y) \rangle = \frac{g_\alpha}{N}\delta^{ab}\delta^{(2)}(x - y), \quad \langle A^a(x)\bar{A}^b(y) \rangle = \frac{2g_A}{N}\delta^{ab}\delta^{(2)}(x - y)$$
The reason to consider the four types of disorder is that they are generated by renormalization of the effective action. As we will see in the following sections, there are particular cases where the symmetry of the Hamiltonian (1) is greater than (3) and the renormalization group flow closes on a subset of the disorder coupling constants; this is the case for example for the random vector potential alone, when the symmetry class changes to $CI$.

The single particle Green functions are defined by the functional integral $Z^{-1} \int D\Psi^* D\Psi \exp(-S)$ with $Z$ the partition function and

$$S = \int \frac{d^2 x}{2\pi} \Psi^*(x) i (H - \mathcal{E}) \Psi(x)$$

where $\mathcal{E} = E + i\varepsilon$. For $\varepsilon = 0^+$, this defines the retarded Green function

$$G_R(x, x'; E) = \lim_{\varepsilon \to 0^+} \frac{1}{E - (E + i\varepsilon)} \langle x|\Psi(x)\Psi^*(x')\rangle$$

Letting

$$\Psi = \begin{pmatrix} \psi_+ \\ \bar{\psi}_+ \end{pmatrix}, \quad \Psi^* = (\bar{\psi}_-, \psi_-)$$

where $\psi_+$ is a $2N$-component fermion $\psi^+_i, i = 1, \ldots, 2N$, and similarly for $\bar{\psi}_\pm$, one finds

$$S = \int \frac{d^2 x}{2\pi} \left( \psi_-(2\partial + iA)\psi_+ + \bar{\psi}_-(2\partial + iA)\bar{\psi}_+ + i(\psi_-\alpha\bar{\psi}_+ + \bar{\psi}_-\psi_+) 
+ \im \left( \psi_-\bar{\psi}_+ - \bar{\psi}_-\psi_+ \right) + i(\psi_-M\bar{\psi}_+ - \bar{\psi}_-M\psi_+) - i\mathcal{E}\Phi_E \right)$$

where $\Phi_E = \psi_-\bar{\psi}_+ + \bar{\psi}_-\psi_+$.

### 3 Supersymmetric effective action.

Since the disorder is gaussian and the deterministic part of the hamiltonian is free, we can use the supersymmetric method to compute disorder averages of Green functions. For each fermion field, we introduce bosonic partners, $\beta^i_\pm, \bar{\beta}^i_\pm, i = 1, \ldots, 2N$, so that the inverse of the partition function for fermions become a partition function for bosons

$$Z(\alpha, m, M, A)^{-1} = \int D\beta \ e^{-S(\psi \to \beta)}$$
In order to simplify the notation and to take advantage of the symmetries of the problem, we are going to introduce supermultiplets, each containing $4N$ fermion fields and $4N$ boson fields

$$\phi = \begin{pmatrix} \psi_+ \\ \Sigma \psi_T \\ \beta_+ \\ \Sigma \beta_T \end{pmatrix}, \quad \bar{\phi} = \begin{pmatrix} \bar{\psi}_+ \\ \bar{\Sigma} \bar{\psi}_T \\ \bar{\beta}_+ \\ \bar{\Sigma} \bar{\beta}_T \end{pmatrix},$$

with $\Sigma$ defined in eq. (2). Here the index $i = 1, \ldots, 2N$ was omitted, and the superscript $T$ represents the usual transposition. $\psi_+$ is a column vector and $\psi_-$ a row vector. We define also orthosymplectic transposes $\phi^t$ of $\phi$ and $\bar{\phi}^t$ of $\bar{\phi}$ by

$$\phi^t \equiv (\psi_-, \psi_T \Sigma^{-1}, \beta_-, -\beta_T \Sigma^{-1}), \quad \bar{\phi}^t \equiv (\bar{\psi}_-, \bar{\psi}_T \Sigma^{-1}, \bar{\beta}_-, -\bar{\beta}_T \Sigma^{-1}). \quad (9)$$

The inner product associated to this transposition is skew

$$\bar{\phi}^t \phi = -\phi^t \bar{\phi}.$$

Similarly,

$$\bar{\phi}^t \tau^a \phi = \phi^t \tau^a \bar{\phi} ; \quad \bar{\phi}^t T^i \phi = -\phi^t T^i \bar{\phi}. \quad (10)$$

One can define a transpose of $8N \times 8N$ supermatrices $A$ by $(A\phi)^t \equiv \phi^t A^t$. Its explicit relation to the usual supertranspose (we use same symbol $T$ to denote the transposition for the usual matrices and the supertransposition for the supermatrices), is given by

$$A^t = \gamma A^T \gamma^{-1}, \quad \gamma = \begin{pmatrix} 0 & \Sigma^{-1} \\ \Sigma & 0 \end{pmatrix} \otimes E_{FF} - \begin{pmatrix} 0 & \Sigma^{-1} \\ -\Sigma & 0 \end{pmatrix} \otimes E_{BB}, \quad (11)$$

$E_{FF}$ and $E_{BB}$ being the projectors on the fermion-fermion space and boson-boson space respectively. The present definition of the supermultiplets $\phi$ and $\phi^t$, as well as of the orthosymplectic transpose is different from the one used in [11], and it is adapted to the present type of disorder. To be able to compare the symmetry properties of supervectors and supermatrices in the two cases it useful to retain the following relations

$$\phi_{\text{class } C} = \begin{pmatrix} 1 & 0 \\ 0 & \Sigma \end{pmatrix} \otimes 1_{\text{susy}} \phi_{\text{class } D}, \quad \phi^t_{\text{class } C} = \phi^t_{\text{class } D} \begin{pmatrix} 1 & 0 \\ 0 & \Sigma^{-1} \end{pmatrix} \otimes 1_{\text{susy}}, \quad (12)$$
and similarly for \( \bar{\phi} \) and \( \bar{\phi} \).

The free theory of fermions plus the bosonic ghosts is conformal, with Virasoro central charge \( c = 0 \) and with action

\[
S_{\text{cft}} = 2 \int \frac{d^2x}{2\pi} \left( \psi_- \partial_z \psi_+ + \bar{\psi}_- \partial_{\bar{z}} \bar{\psi}_+ + \beta_- \partial_z \beta_+ + \bar{\beta}_- \partial_{\bar{z}} \bar{\beta}_+ \right)
\]

\[
\equiv \int \frac{d^2x}{2\pi} \left( \phi^t \partial_z \phi + \bar{\phi}^t \partial_{\bar{z}} \bar{\phi} \right).
\]

The short distance singularity of the holomorphic fields in the supermultiplet \( \phi \) can be written as

\[
\phi(z) \phi^t(w) \simeq \frac{1}{z - w}
\]

and similarly for the supermultiplet \( \bar{\phi} \).

After performing the integrals over the disorder, we obtain the following effective action

\[
S_{\text{eff}} = S_{\text{cft}} + \frac{1}{2N} \int \frac{d^2x}{2\pi} \left( g_{\alpha} O_{\alpha} + g_m O_m + g_M O_M + g_A O_A \right),
\]

with the following operators perturbing away from the conformal field theory

\[
O_{\alpha} = \langle \bar{\phi}^t \tau^a \phi \rangle \langle \bar{\phi}^t \tau^a \phi \rangle,
\]

\[
O_m = \frac{1}{2N} \langle \bar{\phi}^t \phi \rangle \langle \bar{\phi}^t \phi \rangle,
\]

\[
O_M = \langle \bar{\phi}^t T^i \phi \rangle \langle \bar{\phi}^t T^i \phi \rangle,
\]

\[
O_A = \langle \phi^t \tau^a \phi \rangle \langle \bar{\phi}^t \tau^a \bar{\phi} \rangle.
\]

It will turn out to be useful to know alternative ways of writing these operators. Using the antisymmetry properties (11) as well as the cyclicity of the supertrace, one has:

\[
O_{\alpha} = \text{STr} \left( \bar{\phi} \phi^t \tau^a \phi \phi^t \tau^a \right) = \text{STr} \left( \bar{\phi} \phi^t \tau^a \phi \phi^t \tau^a \right)
\]

\[
O_m = \frac{1}{2N} \text{STr} \left( \bar{\phi} \phi^t \phi \phi^t \phi \right) = -\frac{1}{2N} \text{STr} \left( \bar{\phi} \phi^t \phi \phi^t \phi \right)
\]

\[
O_M = \text{STr} \left( \bar{\phi} \phi^t T^i \phi \phi^t T^i \right) = -\text{STr} \left( \bar{\phi} \phi^t T^i \phi \phi^t T^i \right)
\]

\[
O_A = \text{STr} \left( \bar{\phi} \phi^t \tau^a \phi \phi^t \tau^a \right).
\]
Finally, one can consider the energy term, which can be written in the present notations as
\[ \Phi_E = (\bar{\phi}^t \Sigma_3 \phi), \] (18)
where \( \Sigma_3 = 1_{2N} \otimes \sigma_3 \otimes 1_{\text{susy}} \). In order for the integrals over the bosonic field to converge, one has to relate the bosonic fields to each other by complex conjugation
\[ \beta_+^\dagger = \tilde{\beta}_+, \quad \beta_-^\dagger = \tilde{\beta}_- . \] (19)
Then, a positive imaginary part of the energy insures convergence of the bosonic functional integrals.

4 Symmetries.

Before perturbation by the disorder operators \([10]\), the action \([13]\) possesses conformal invariance with current algebra symmetry \([23]\). Bilinears in the chiral supermultiplets \((\phi \phi^t)\) generate an \(osp(4N|4N)\) algebra.

Two natural subalgebras of \(osp(4N|4N)\) are \(sp(2N)\) and \(osp(2|2)\). The \(sp(2N)\) algebra, which we refer to as the spin algebra, is generated by the currents,
\[ J^a(z) = (\phi^t \tau^a \phi)(z) \] (20)
They satisfy
\[ J^a(z) J^b(0) \simeq f^{ab}_{\cdot\cdot} z J^c(0) + \text{reg}. \]
with \( f^{ab}_{\cdot\cdot} \) the \( sp(2N) \) structure constants, showing that they form a representation of the affine algebra \( sp(2N) \) at level zero.

The \(osp(2|2)\) algebra, which we refer to as the charge algebra, is generated by the \(sp(2N)\) scalar
\[ K(z) = \text{Tr}_{gl(2N)}(\phi \phi^t)(z) = \text{Tr}_{gl(2N)}(E_I (\phi \phi^t)(z) E_I) \] (21)
where the trace \( Tr_{gl(2N)} \) is over the spin indices and \( E_I \) is any orthonormalized basis of \( gl(2N) \). A convenient basis is be made of \( \tau^a \), \( \mathcal{T}^i \) and the identity \( 1 \). The currents \( K \) may be viewed as a \( 4 \times 4 \) supermatrix, \( K_{\alpha\beta} = \sum_i \phi_{i\alpha} \phi^t_{i\beta} \), with components:
\[ J = \psi^i_+ \psi^i_- , \quad J_\pm = \Sigma_{ij} \psi^i_+ \psi^j_\pm \] (22)
\[ H = \beta^i_+ \beta^i_- , \quad S_\pm = \psi^i_+ \beta^i_\pm , \quad \hat{S}_\pm = \Sigma_{ij} \beta^i_\pm \beta^j_\mp \]
The even part of $K$ is made of two blocks $K_{FF}$ and $K_{BB}$ with:

$$K_{FF} = \begin{pmatrix} J & J_+ \\ J_- & -J \end{pmatrix}, \quad K_{BB} = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}.$$

The currents (22) generate an $osp(2|2)_k$ current algebra at level $k = -2N$, cf ref.[6]. In particular, the $K_{BB}$ block generates a $so(2)$ algebra and the $K_{FF}$ block generates a $sp(2)$ algebra.

The $osp(2|2)$ generators commute with the $sp(2N)$ currents:

$$[J^a, K] = 0.$$

The supermultiplets $\phi$ and $\phi^t$ transform as affine primary fields with value in the tensor product of the defining vector representation of $sp(2N)$ by the 4-dimensional representation of $osp(2|2)$. One may alternatively [15] view the field $\phi$ as a rectangular supermatrix with components $\phi_{i\alpha}$, $i = 1, \cdots, 2N$, $\alpha = 1, \cdots, 4$, on which the algebra $sp(2N)$ acts by multiplication on the left, while $osp(2|2)$ acts by multiplication on the right. These actions of course commute.

## 5 One loop beta functions.

The first step in studying the effect of the disorder perturbation in (15) is to analyse the renormalisation group flow for the coupling constants. Even at one loop, beta functions code for important properties of the system. They can be deduced from the operator product expansions (OPE) of marginal perturbing operators $O_i$, where $i$ stands for one of the indices of the disorder operators (16). Using the following notation for OPE

$$O_i(z, \bar{z})O_j(0) \simeq \frac{1}{z\bar{z}} C_{ij}^k O_k(0) + \text{reg.},$$

the one loop beta functions are given by [15, 17]

$$\beta_k \equiv l\partial_l g_k = -\frac{1}{2N} \sum_{i,j} C_{ij}^k g_i g_j.$$

After some careful calculations, we obtain the following results

$$\beta_m = \frac{(2N+1)}{N} \left( 2g_A(g_m + g_\alpha) - g_m g_\alpha - \frac{(N-1)}{N} g_m g_M \right) - \frac{1}{N^2} g_m^2.$$
\[
\beta_A = \frac{1}{2}(g_M + g_a)^2 + \frac{1}{2N}(g_a^2 - g_M^2) + \frac{2(N+1)}{N}g_A^2 + \frac{1}{N^2}g_a(g_m - g_M) \\
\beta_a = 2g_A(g_a + g_M) + \frac{1}{N}g_a(g_m - g_M) + \frac{1}{N^2}\left(g_a + 2g_A\right)(g_m - g_M) \\
\beta_M = \frac{2(N+1)}{N}g_A(g_a + g_M) + \frac{1}{N}g_M(g_m - g_M) + \frac{1}{N^2}g_M(g_m - g_M) 
\]

(23)

Readers who are interested in details can find some hints in the Appendices.

From now on, we shall be interested in the large $N$ limit. The beta functions (23) then simplify dramatically

\[
\beta_m = -2g_m(g_a + g_M) + 4g_A(g_a + g_M) , \quad \beta_A = \frac{1}{2}(g_a + g_M)^2 + 2g_A^2 \\
\beta_a = 2g_A(g_a + g_M) , \quad \beta_M = 2g_A(g_a + g_M) 
\]

(24)

Of course the large $N$ limit only applies for $g \ll N$.

After a large number of RG iterations, RG trajectories are asymptotic to directions which are preserved by the RG flow and which pass through the origin. One can show that there exist six stable directions. But only one is attractive in the region where coupling constants are positive. This direction, which we call the “isotropic direction”, corresponds to the case where all coupling constants are equal: $g_m = g_a = g_M = g_A = g$. The associated beta function is

\[
\beta_g = 4g^2 
\]

(25)

To show the isotropic direction is attractive, we use the same method as in [3]. We project RG flow on the sphere and parameterize coupling constants with a radial coordinate $\rho$ and three angle coordinates $\theta_i$. We rewrite the RG equations as

\[
\dot{\theta}_i = \rho\beta_i(\theta_1, \theta_2, \theta_3) \quad \text{and} \quad \dot{\rho} = \rho^2\beta_\rho(\theta_1, \theta_2, \theta_3)
\]

We develop the $\beta_i$ around the isotropic direction, and compute the corresponding eigenvalues. These eigenvalues are all negative, proving that the direction is attractive. Furthermore, it can be shown that $\dot{\rho} > 0$. These results are corroborated by the all order computations of Appendix C.

Thus, the isotropic direction describes the universality class in the region where coupling constants are all positive. It corresponds to a strong coupled system that we will study in details in the following section.
6 The sigma model approach.

When the disorder couplings are all positive (and \( N \) is large), the renormalization group flow is attracted towards the line \( g_\alpha = g_m = g_M = g_A = g \) ("isotropic disorder"). On this line, preserved by the flow, the low energy physics can be described by a non-linear sigma model with a topological term. In this section, we derive the effective action for this sigma model. Its target space is the Riemannian symmetric superspace \( OSp(2|2)/GL(1|1) \), denoted DIII|CI in [7] (note that there exists another \( OSp(2|2)/GL(1|1) \) symmetric superspace, denoted CI|DIII, appearing in the context of the symmetry class \( D \) [11]).

When all the couplings are equal, the perturbation term in the Lagrangian, quartic in the Dirac fields can be decoupled by a Hubbard-Stratonovich transformation involving a unique supermatrix field \( Q \). Using the fact that \( (\phi^t \phi) = (\phi^t T^i \phi) = 0 \) and denoting

\[
B = \bar{\phi} \phi^t + \phi \bar{\phi}^t
\]

the perturbation term becomes

\[
L_{\text{pert}} = \frac{g}{2N} \left( (\bar{\phi}^t \tau^a \phi)(\bar{\phi}^t \tau^a \phi) + \frac{1}{2N}(\bar{\phi}^t \phi)(\bar{\phi}^t \phi) \right)
+ \left( (\bar{\phi}^t T^i \phi)(\bar{\phi}^t \tau^a \phi) + (\bar{\phi}^t \tau^a \phi)(\phi^t \tau^a \phi) \right)
= \frac{g}{4N} \text{STr} B \left( \tau^a B \tau^a + \frac{1}{2N} B + \tau^i B \tau^i \right).
\]

The supermatrix \( B \) obeys the relation \( B = -B^t \), relation which defines an element of the algebra \( osp(4N|4N) \). Conjugation by the generators of the subalgebra \( gl(2N) \) projects \( B \) on the identity on this subalgebra

\[
\tau^a B \tau^a + \frac{1}{2N} B + \tau^i B \tau^i = (\text{Tr}_{gl(2N)} B) \otimes 1_{2N} ,
\]

so that the perturbation term reads

\[
L_{\text{pert}} = \frac{g}{4N} \text{STr} \left( \text{Tr}_{gl(2N)} (\bar{\phi} \phi^t + \phi \bar{\phi}^t) \right)^2 .
\]

This interaction can be decoupled using a supermatrix belonging to the \( osp(2|2) \) algebra, \( Q \sim \text{Tr}_{gl(2N)} (\bar{\phi} \phi^t + \phi \bar{\phi}^t) \). The resulting effective lagrangian
The effective action is
\[
L_{\text{eff}} = \bar{\phi} \partial \phi + \phi \partial \bar{\phi} + \text{STr} \left( Q \text{Tr}_{gl(2N)} (\phi \bar{\phi} \phi \bar{\phi}) \right) - \frac{N}{g} \text{STr} Q^2
\]
\[
= (\bar{\phi} \phi') \left( \frac{Q}{\bar{\phi}} \frac{\partial}{Q} \right) \left( \frac{\phi}{\bar{\phi}} \right) - \frac{N}{g} \text{STr} Q^2 .
\]  
(27)

The next step is to perform the gaussian integrals over the Dirac fields, resulting in the following effective action
\[
S[Q] = -\frac{N}{g} \int \frac{d^2 x}{2\pi} \text{STr} Q^2 + N \text{STr} \ln \left( \frac{\Sigma_3 Q}{\partial Q \Sigma_3} \right) ,
\]  
(28)

where \text{STr} combines the operations of taking the supertrace STr over the matrix indices and integrating over position space. The factors \Sigma_3 under the logarithm appear after the transformation \bar{\phi} \mapsto \Sigma_3 \bar{\phi} and \phi' \mapsto \phi' \Sigma_3, correcting for the fact that the complex conjugate of \bar{\phi} is not \phi' but \phi' \Sigma_3, see eq.(19). The number \(N\) appears now as a factor in the action, suggesting to treat the integral in the saddle point approximation. The saddle point equation for the action (28) is given by
\[
2Q(x) = \text{Tr}_{gl(2)} \langle x | D^{-1} | x \rangle , \quad \text{with} \quad D = \left( \frac{Q}{\bar{\phi}} \frac{\partial}{Q} \right) .
\]  
(29)

We look for a spatially homogeneous solution of the form \(Q(x) = \mu \Sigma_3\). The saddle-point equation then reduces to
\[
g^{-1} = \int \frac{d^2 k}{2\pi} (\mu^2 + k^2/4)^{-1}.
\]
Cutting off the integral in the ultraviolet by \(|k| < 1/\ell_0\) yields the equation
\[
1/2g = \ln \left( 1 + (2\mu \ell_0)^{-2} \right) \approx -2 \ln(2\mu \ell_0) ,
\]  
(30)

and by inversion,
\[
\mu = (2\ell_0)^{-1}/\sqrt{e^{1/2g} - 1} \approx (2\ell_0)^{-1} e^{-1/4g} ,
\]  
(31)

As the dynamically generated mass \(\mu\) is a renormalization group invariant, eq.(31) corresponds to a beta function \(\beta_g = 4g^2\) in agreement with eq.(25).
Symmetry of $Q$ and the saddle point manifold. The bilinear in the Dirac fields $B = \bar{\phi}\phi^t + \phi\bar{\phi}^t$ belongs to the complex algebra $osp(4N|4N)$ defined by the relation $B = -B^t$. The even part of this algebra is $sp(4N) \otimes so(4N)$, with $so(4N)$ in the fermion-fermion ($FF$) block and $sp(4N)$ in the boson-boson ($BB$) block (this is due to the fact that the matrix $\gamma$ in (11) is symmetric in the $FF$ sector and antisymmetric in the $BB$ sector). Taking the trace over $gl(2N)$ in $B$ leaves us with an object belonging to the $osp(2|2)$ algebra, but now with the $so(2)$ part in the $BB$ sector and the $sp(2)$ part in the $FF$ sector. To see the way it happens, let us look at the $FF$ part in a $osp(4N|4N)$ matrix, defined by $M^t = -M$. It is easily verified that this block has the structure

$$M_{FF} = \begin{pmatrix} M_{11} & \Sigma M_{12} \\ M_{21} \Sigma^{-1} & -\Sigma M_{11}^T \Sigma^{-1} \end{pmatrix},$$

(32)

where $M_{11}$, $M_{12}$ and $M_{21}$ are $2N \times 2N$ ordinary matrices obeying $M_{12}^T = -M_{12}$ and $M_{21}^T = -M_{21}$. This implies that $M_{FF}$ belongs to $so(4N)$. Let us now take the trace over the colour indices

$$\tilde{M}_{FF} \equiv Tr_{gl(2N)} M_{FF} = \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & -\tilde{M}_{11}^T \end{pmatrix}.$$  

(33)

Using the cyclicity of the trace and the antisymmetry of the matrix $\Sigma$, we can show that $\tilde{M}_{12}^T = \tilde{M}_{12}$ and $\tilde{M}_{21}^T = -\tilde{M}_{21}$, which means that $\tilde{M}_{FF}$ belongs to $sp(2)$ algebra. Similarly, $\tilde{M}_{BB}$ can be shown to belong to $so(2)$. The matrix $Q$ inherits this symmetry from the object to which it couples, $\tilde{B} = Tr_{gl(2N)} B$, therefore it belongs to $osp(2|2)$.

When decoupling the interaction part with the help of the supermatrix $Q$, one of the question that has to be addressed is the choice of the contour of integration. In particular, solving this question allows to choose the acceptable solutions for the saddle point equation, that is the ones which lie on the contour of integration or which can be attained from it by analytical continuation. These questions have been addressed in detail in [7] for the class $C$. There, it was shown that the dominant diagonal saddle point is of the form

$$Q_0 = \mu \Sigma_3,$$

with $\Sigma_3 = \sigma_3 \otimes 1_{\text{suss}}$ being an element of $osp(2|2)$. Due to the global $OSp(2|2)$ symmetry of the effective action, this saddle point extends to a saddle point manifold

$$Q_0 \to \mu T \Sigma_3 T^{-1},$$

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where $T$ is a constant element of $OSp(2|2)$. Since the stabilizer of $\Sigma_3$ is $GL(1|1)$, the saddle point manifold is the coset space $OSp(2|2)/GL(1|1)$. This coset space can be parameterized by $T = \exp X$, with $\{X, \Sigma_3\} = 0$.

The convergence conditions for the integrals over $Q$ restrict the saddle point manifold to a real submanifold of the complex space $OSp(2|2)/GL(1|1)$. In the $FF$ sector, convergence of the integrals over $Q$ can be insured by choosing $Q^\dagger_{FF} = Q_{FF}$. At the level of the saddle point, this translates into $X^\dagger_{FF} = -X_{FF}$. Therefore, the fermion-fermion sector of the saddle point manifold is isomorphic to the compact symmetric space $Sp(2)/U(1)$. The convergence conditions on $Q$ in the boson-boson sector are more involved \[7\]; on the saddle point manifold they can be reduced to $X^\dagger_{BB} = X_{BB}$, showing that the bosonic part of the saddle point manifold is non-compact. When averages of $n$ Green functions are considered, it is isomorphic to the non-compact symmetric space $SO^*(2n)/U(n)$, with $SO^*(2n)$ some real form of $SO(2n)$. For $n = 1$, the boson-boson sector is empty. The two symmetric spaces form the base manifold of a Riemannian symmetric superspace of type $DIII|CI$ \[7\].

Gradient expansion. The next step in the derivation of an effective action is to perform a gradient expansion of the action (28). The low energy configurations are given by the slowly varying field

$$q(x) \equiv Q(x)/\mu = T(x)\Sigma_3 T(x)^{-1},$$

(34)

where $T(x)$ is a (slowly varying) element of $OSp(2|2)$. Note that $q(x)$ satisfies the nonlinear constraint $q(x)^2 = 1$. The degrees of freedom $q(x)$ correspond to the Goldstone modes of the broken symmetry $OSp(2|2) \rightarrow GL(1|1)$. Fluctuations transverse to the saddle point manifold are massive and can be neglected at this stage.

The effective action for the Goldstone modes is a non-linear sigma model on the symmetric superspace $OSp(2|2)/GL(1|1)$ described previously. This sigma model may support a topological term, since $\Pi_2(Sp(2)/U(1)) = \mathbb{Z}$. The easiest way to extract the coupling constants of the kinetic and topological term is by using the non-abelian bosonisation \[18\]. In the supersymmetric setting, this method was used and explained in detail for class D in \[11\] and it can be applied with minimal changes to the present case. We want to evaluate the action (28) on configurations of the type (34). Due to the nonlinear constraint $q(x)^2 = 1$, the first term in (28) vanishes. The second term can
be written, by undoing the integral over the Dirac fields,

\[ e^{-S[q]} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \exp - \int d^2 x \left( \bar{\phi}^i \partial^i \phi + \phi^i \partial^i \bar{\phi} + \mu \bar{\phi}^i q \phi + \mu \phi^i q \bar{\phi} \right). \tag{35} \]

Here \( q \) has to be understood as acting like the identity on the spin indices \( i = 1, \cdots, 2N \). The free Dirac theory plus the bosonic ghosts is equivalent to a WZW model with action

\[ W_{osp(4N|4N)}[M] = \frac{1}{16\pi} \int_S d^2 x \text{STr} \left( M^{-1} \partial_\mu M \right)^2 + \frac{i \Gamma[M]}{24\pi}, \tag{36} \]

where the matrix \( M \) takes values in a subspace of the complex supergroup \( OSp(4N|4N) \), and the topological term is expressed by assuming some extension \( \tilde{M} \) of \( M \) to a 3-ball \( \mathcal{B} \) that has position space \( S \) for its boundary \((\partial \mathcal{B} = S)\)

\[ \Gamma[M] = \int_{\mathcal{B}} d^3 x \epsilon_{\mu\nu\lambda} \text{STr} \tilde{M}^{-1} \partial_\mu \tilde{M} \tilde{M}^{-1} \partial_\nu \tilde{M} \tilde{M}^{-1} \partial_\lambda \tilde{M}. \tag{37} \]

The rules of bosonisation for the last two terms in the exponent in (35) are to replace the bilinears \( \phi^i \Sigma_3 \) and \( \Sigma_3 \phi^i \) by \( \ell^{-1} M \) resp. \( \ell^{-1} M^{-1} \), where the factor \( \ell^{-1} \) is a large mass scale, of the order of \( \ell_0^{-1} \), which enters for dimensional reasons. Up to a conjugation with the matrix \( \text{diag}(1, \Sigma) \), these are the same bosonisation rules as in (33). The term \( \text{STr} \left( M \Sigma_3 q + q \Sigma_3 M^{-1} \right) \) can be viewed as a kind of mass term. At large \( \mu/\ell \) it forces the field \( M \) to follow \( q \Sigma_3 \). This approximation is valid at momentum scale \( k \ll (\mu/\ell)^{1/2} \).

Neglecting the fluctuations we can set \( M \Sigma_3 q = 1 \), which yields

\[ S[q] \bigg|_{q = \Sigma_3 T^{-1}} = 2N W[q \Sigma_3]. \]

where the factor \( 2N \) appears from taking the trace over the spin indices. Here \( W[q \Sigma_3] \) is the WZW action on \( OSp(2|2) \). Recall that \( \Sigma_3 \in osp(2|2) \) so that \( \pm i \Sigma_3 = \exp(\pm i \pi \Sigma_3/2) \) and \( q \Sigma_3 \) belong to \( OSp(2|2) \). However, \( q \Sigma_3 \) does not explore all this group as \( q \) is only a function on the coset space \( OSp(2|2)/GL(1|1) \). Evaluating the topological term for this configuration can be done by making a smooth extension of \( M = q \Sigma_3 \) to the ball \( \mathcal{B} \) with radial coordinate \( 0 \leq s \leq 1 \), for example

\[ \tilde{M}(x, s) = T(x) \exp(\pm is \pi \Sigma_3/2) T(x)^{-1}(\mp i \Sigma_3). \]
At $s = 1$ we have $\tilde{M}(x, 1) = q(x)\Sigma_3$, while for $s = 0$ we get $\tilde{M}(x, 0) = \mp i\Sigma_3$, independent of $x$. Inserting this extension into the expression (37) for $\Gamma[M]$, and converting the integral over $B$ into an integral over $S = \partial B$, we find a theta term

$$\frac{i}{24\pi} \Gamma[q\Sigma_3] = \pm \frac{1}{32} \int d^2x \epsilon_{\mu\nu} \text{Str} q \partial_\mu q \partial_\nu q \equiv \pm S_{\text{top}}(q).$$

Since the value of the WZW topological term does not depend on the extension, the two opposite expressions for $i\Gamma[q\Sigma_3]/24\pi$ are equivalent and $S_{\text{top}}(q) \in i\pi\mathbb{Z}$. Gathering the kinetic and topological term, we obtain the following effective action

$$S[q] \big|_{q=T\Sigma_3 T^{-1}} = -\frac{2N}{16\pi} \int d^2x \text{Str} \partial_\mu q \partial_\mu q \pm \frac{2N}{32} \int d^2x \epsilon_{\mu\nu} \text{Str} q \partial_\mu q \partial_\nu q .$$

(38)

The angle of the theta term is $\theta = \pm 2N\pi$. It contributes trivially to the path integral as the topological action is multiplied by $2N$ so that $2NS_{\text{top}} \in 2i\pi\mathbb{Z}$. The effective action is thus:

$$S_{\text{eff}}[q] = -\frac{2N}{16\pi} \int d^2x \text{Str} \partial_\mu q \partial_\mu q .$$

(39)

The natural ultraviolet cut-off for this effective action is $\mu^{-1} \simeq 2\ell_0 e^{1/4g}$ since in deriving it we neglected transverse modes of effective mass $\mu$.

Action (39) is conjectured to be a massive theory as it is a sigma model on a symmetric space with positive curvature. Recall [19] the one loop renormalization group equations for sigma model metrics $G_{ab}$:

$$l\partial_l G_{ab} = -R_{ab}$$

with $R_{ab}$ the Ricci curvature. For symmetric spaces the Ricci tensor is proportional to the metric. We need to compute this proportionality coefficient in our case. Since $q = T\Sigma_3 T^{-1}$, the tangent space at the point $q = \Sigma_3$ is spanned by elements of the form $[X, \Sigma_3]$ with $X \in \text{osp}(2|2)/\text{gl}(1|1)$. By construction we can choose $X$ such that $\{X, \Sigma_3\} = 0$. The metric is then:

$$G(X, X) = -\frac{1}{8\pi\lambda} \text{Str}([X, \Sigma_3])^2 = \frac{1}{2\pi\lambda} \text{Str}(X^2)$$

where the supertrace is understood in the defining 4 dimensional representation, and $\lambda = 1/2N$. Similarly, the Ricci tensor at $q = \Sigma_3$ is defined by [20]:

$$R(X, X) = -\text{Str}(\text{ad} X)^2$$

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Here the supertrace is in the adjoint representation. Recall that Ricci tensor are invariant under metric dilatations. To compute the proportionality coefficient we pick a particular element of $osp(2|2)$ anticommuting with $\Sigma_3$, e.g. $X = E_{FF} \otimes \sigma_1$. We have $\text{STr}(X^2) = -2$. Diagonalizing the adjoint action, we get a set of eigenvalues zero, two bosonic non-degenerate eigenvalues $\pm 2$ and two fermionic eigenvalues $\pm 1$ with multiplicity two. Hence, $\text{STr}(\text{ad}X)^2 = 2(2^2 - 2) = 4$. Thus $G = R/4\pi \lambda$. The RG equation then becomes:

$$l \partial_l \lambda = +4\pi \lambda^2$$

At large distance, the model is driven to strong coupling, and it presumably becomes massive because there is no contribution from the topological term.

The generated mass scale is of order $m_N \simeq \mu e^{-N/2\pi} \ll \mu$ as the coupling constant is equal to $N$ at the ultraviolet cut-off $\mu$. This is the energy scale at which the coupling constant $\lambda$ becomes of order one.

As a consequence, the infrared fixed point is trivial and the zero energy states are localized with localization length of order $1/m_N$. In the localized regime, the behavior of the density of states is expected to be governed by the class $C$ matrix ensemble [21].

7 Spin-charge separation.

The conformal field theory with action (13) admits a spin-charge separation. Its stress tensor can be decomposed into the sum of the Sugawara stress tensors associated to the spin and charge current algebras:

$$T_{cft} = T_{sp(2N)_0} + T_{osp(2|2)_{-2N}}$$

Both Sugawara stress tensors have Virasoro central charge zero and are bilinear in the corresponding currents:

$$T_{sp(2N)_0} \equiv \frac{1}{8(N+1)} \lim_{w \to z} J^a(z) J^a(w)$$

$$T_{osp(2|2)_{-2N}} \equiv -\frac{1}{8(N+1)} \lim_{w \to z} \text{STr}(K(z)K(w))$$

The normalization of $T_{osp(2|2)_{-2N}}$ may be found in ref. [22]. Eq.(40) is proved in Appendix B.
This may be checked by computing the dimension of the supermultiplet \( \phi \). In the free theory, its conformal dimension is \( 1/2 \). The dimensions in the spin sector are \( \Delta_{sp}(2N)_0 = \frac{\text{Cas}}{2(N+1)} \) with \( \text{Cas} \) the casimir of the corresponding representation of \( sp(2N) \), cf. ref. [23]. For the vector representation this gives \( \Delta_{sp}(2N)_0 = \frac{2N+1}{4(N+1)} \). For the charge sector, regular representations of \( osp(2|2) \) are labeled by two integers \( j, b \) and their conformal dimensions are \( \Delta_{osp}(2|2) = \frac{1}{4(N+1)} \). As it should, the spin and charge conformal dimensions add up to \( 1/2 \).

For \( N = 1 \), it was shown in ref. [24] that the four point correlation function of the supermultiplet \( \phi \) may be factorized as the product of correlation functions in the \( sp(2N)_0 \) and \( osp(2|2)_{-2N} \) conformal theories. However, this spin-charge factorization possesses peculiar properties inherited from indecomposability properties of representations of \( osp(2|2) \). In particular, \( osp(4N|4N) \) decomposes as:

\[
osp(4N|4N) = osp(2|2) \otimes [1] + [8] \otimes sp(2N) + [8] \otimes [R]
\]

with \([R]\) a \((2N+1)(N-1)\) dimensional representation of \( sp(2N) \) and \([8]\) isomorphic to the adjoint representation of \( osp(2|2) \). The spin currents \( J^a = \phi^t \tau^a \phi \) belong to \([8] \otimes sp(2N)\) with \([8]\) an eight dimensional indecomposable representation of \( osp(2|2) \). Thus, although these currents commute with the \( osp(2|2) \) charge generators they do not belong to a trivial representation of the charge algebra.

This separation between spin and charge degree of freedoms still holds in the perturbed theory provided one fine tunes the coupling constants such that \( g_\alpha + g_m = g_\alpha + g_M = 0 \). In this case:

\[
L_{\text{charge}} \equiv g_\alpha (\mathcal{O}_\alpha - \mathcal{O}_m - \mathcal{O}_M) = g_\alpha \text{STr}[(\tau^a \phi \phi^t \tau^b + \frac{1}{2N} \phi \phi^t + T^i \phi \phi^t T^i) \phi \phi^t]
\]

where we used again the antisymmetry property (10). The sum in the above r.h.s. projects out the \( sp(2N) \) colour indices leaving only the \( osp(2|2)_{-2N} \) currents (21). Thus

\[
L_{\text{charge}} = g_\alpha \text{STr}(K \bar{K})
\]

The remaining perturbing operator,

\[
L_{\text{spin}} \equiv g_A \mathcal{O}_A = g_A J^a \bar{J}^a
\]

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describes a current interaction involving only the $sp(2N)_0$ generators. Hence the total perturbation,

$$L_{\text{pert}} = L_{\text{spin}} + L_{\text{charge}}$$

is then the sum of the two commuting current-current interactions.

The spin-charge separation can also be seen on the beta functions. When fine tuning coupling constants such that $g_\alpha + g_m = g_\alpha + g_M = 0$, the beta functions (23) decouple: $\beta_A = \frac{2(N+1)}{N} g_A^2$ and $\beta_\alpha = -\beta_m = -\beta_M = -\frac{2}{N} g_\alpha^2$. In the fine tuned regime, disorder in the gauge potential $A$ is marginally relevant while disorder in the spin potential $\alpha$ is marginally irrelevant.

8 Sigma model approach for the spin-charge separated system.

The line $g_\alpha = g_m = g_M = 0, g_A = g > 0$ is stable for the renormalization group flow. It is attractive in the fine tuned regime $g_\alpha + g_m = g_\alpha + g_M = 0, g_\alpha > 0$, and along it the flow is towards strong coupling, $g_A \to \infty$. This model was formulated and analysed in [12] using replica and in [14] by direct means or with supersymmetry. Comparison of the various methods was done in [25]. Based on the spin-charge separation, it can be deduced [3] that the low energy physics on this line is given by an $osp(2|2)_{k=-2N}$ theory. Let us derive the same result by using the sigma model approach. We shall obtain that the effective action is a sigma model on the supergroup $OSp(2|2)$, endowed with a WZW term, the coupling constants being such that $k = -2N$. Strictly speaking [11], the resulting WZW model is defined on a submanifold of the complex supergroup which is a Riemannian symmetric superspace of type $D|C$ (meaning that the bosons have an orthogonal structure and the fermions are symplectic). Let us remind that by bosonising the free Dirac fermions/bosons, one obtains a WZW model on a Riemannian symmetric superspace of type $C|D$, at level $k = 1$. Since the metric changes sign when passing from a space of type $C|D$ to one of type $D|C$, the WZW model on the space of type $D|C$ is well defined for negative values of the level $k$.

On the fixed line $g_\alpha = g_m = g_M = 0, g_A \neq 0$ the symmetry is not that of class $C$ any more but that of class $CI$. The reason is that the Hamiltonian has now an extra symmetry

$$H = \mathcal{T} H^T \mathcal{T}^{-1} \quad \text{with} \quad \mathcal{T} = i\sigma_2 \otimes \Sigma,$$
which can be interpreted as a time reversal symmetry. The disorder perturbation is now simply the $sp(2N)_0$ current-current perturbation. Adding formally terms which are zero, we obtain

$$L_{\text{spin}} = \frac{g_A}{2N} \left( \bar{\phi}^t \tau^a \phi \right) \left( \phi^t \tau^a \phi \right) = \frac{g_A}{2N} \text{STr} \left( \bar{\phi} \phi^t \right) \left( \phi \bar{\phi}^t \right)$$

with $E_I$ the (orthonormal) generators of $gl(2N)$. Since $(\bar{\phi} \phi)^t = -\phi \bar{\phi}$, we can decouple this interaction by introducing a supermatrix $Q \sim \text{Tr} gl(2N) \bar{\phi} \phi^t$. We can define an orthosymplectic transposition for $Q$ by $Q^t \equiv \text{Tr} gl(2N) (1 \otimes Q)^t$. Remark that in contrast to the preceding section $Q$ has no specific symmetry properties. In particular $Q \neq -Q^t$, so the supermatrix $Q$ belongs to a space larger than $osp(2|2)$. After decoupling of the interaction term, the effective lagrangian becomes

$$L_{\text{eff}} = \bar{\phi}^t \bar{\phi} + \phi^t \phi + 2 \text{STr} \left( Q \text{Tr}_{gl(2N)} (\bar{\phi} \phi^t) \right) + \frac{2N}{g_A} \text{STr} QQ^t$$

$$= (\phi^t \phi) \left( \bar{\phi} \bar{\phi} \right) - N \frac{2}{g_A} \text{STr} \left( \begin{array}{cc} 0 & Q^t \\ -Q^t & 0 \end{array} \right)^2.$$  

(44)

Here, we have embedded $Q$ in the $osp(4|4)$ algebra represented by matrices of the form

$$\hat{A} = \begin{pmatrix} A & B \\ -B^t & D \end{pmatrix}, \quad \text{with} \quad A^t = -A, \quad D^t = -D.$$ 

It is easy to see that the diagonal blocks of $\hat{A}$ span two commuting $osp(2|2)$ algebras, so we conclude that the space to which $Q$ belongs is isomorphic to the complement of $osp(2|2) \oplus osp(2|2)$ in $osp(4|4)$.

In the absence of the energy term, the lagrangian (44) is invariant under the holomorphic/antiholomorphic transformations

$$\phi \rightarrow g_L(z) \phi, \quad \phi^t \rightarrow g_L^{-1}(z) \phi^t, \quad (45)$$

$$\bar{\phi} \rightarrow g_R(\bar{\bar{z}}) \bar{\phi}, \quad \bar{\phi}^t \rightarrow \bar{\phi}^t g_R^{-1}(\bar{\bar{z}}),$$

$$Q \rightarrow g_L(z) Q g_R^{-1}(\bar{\bar{z}}), \quad Q^t \rightarrow g_R(\bar{\bar{z}}) Q^t g_L^{-1}(z),$$

with $g_L(z)$ and $g_R(\bar{\bar{z}})$ elements of the $OSp(2|2)$ group: $g_L^t = g_L^{-1}$ and $g_R^t = g_R^{-1}$. It defines an action of the group $G \equiv OSp(2|2) \otimes OSp(2|2)$ on $Q$. We
therefore expect conformal invariance of the \(osp(2|2)\) (or charge) part of the theory. This is not surprising, in view of the spin-charge separation, since the charge sector is unperturbed by the disorder.

Integrating out the Dirac fields, we obtain the effective action

\[
S_{\text{eff}} = -\frac{N}{g_A} \int \frac{d^2 x}{2\pi} \text{STr} \left( \begin{array}{cc} 0 & Q \\ -Q^t & 0 \end{array} \right)^2 + N \text{STr} \ln \left( \frac{Q \Sigma_3}{\partial} - \Sigma_3 Q^t \right). \tag{46}
\]

Again, for \(N\) large, the integral over the \(Q\) field can be treated in the saddle point approximation. The diagonal saddle point solution is also given by a constant matrix of the form \(Q = \mu_A \Sigma_3\). The saddle point equation fixes the value of \(\mu_A\),

\[
\mu_A \approx (2\ell_0)^{-1} e^{-1/2g_A}.
\]

Remark that the factor in the exponent has changed compared to (31). Due to the fact that, at large \(N\), \(\beta_A = 2g_A^2\), \(\mu_A\) is also invariant under the RG flow and it has the properties of a dynamically generated mass.

As usually, the saddle point solution extends to a saddle point manifold, due to the invariance of the effective action. This time, the saddle point manifold is generated by

\[
Q = \mu_A g_L(z) \Sigma_3 g_R^{-1}(\bar{z}),
\]

which is a much larger manifold than that of constant matrices. \(Q\) satisfies the non linear constraint \(QQ^t = -1\). Remember that \(\Sigma_3^t = -\Sigma_3\). The stabilizer \(H\) of \(\Sigma_3\) is constituted of elements \(\text{diag}(g_L, g_R)\) obeying

\[
g_L \Sigma_3 g_R^{-1} = \Sigma_3 \Rightarrow g_L = \Sigma_3 g_R \Sigma_3.
\]

This is compatible with the multiplication law as \(g_L h_L = \Sigma_3 g_R h_R \Sigma_3\) for any pairs \((g_L, g_R)\) and \((h_L, h_R)\). This means that \(H \simeq OSp(2|2)\). The fluctuations around the saddle point are described by a sigma model on the coset space \(G/H = OSp(2|2) \otimes OSp(2|2)/OSp(2|2)\). Elements of \(G/H\) are pairs \((g_L, g_R)\) with the identification \((g_L, g_R) \sim (g_L \Sigma_3 h \Sigma_3, g_R h)\) with \(h \in OSp(2|2)\). Convergence of the integrals on the saddle point manifold is insured, in the \(FF\) sector, by \(Q_{FF}^\dagger = -Q_{FF}^t\). In the \(BB\) sector, convergence requires \(g_{L,B}^\dagger = \Sigma_3 g_{R,B}^{-1} \Sigma_3\), or \(Q_{BB} = g_{L,B} g_{L,B}^\dagger \Sigma_3\). This is equivalent to demand that the hermiticity condition \([19]\) on the bosonic component, \(\phi_B^\dagger = \bar{\phi}_B \Sigma_3\), is preserved by the transformation \([13]\). The even part (or the base) of the saddle point manifold is then equivalent to the product of two symmetric
spaces, the fermionic sector being compact and the bosonic one being non-compact. The full saddle point manifold is a subspace of the complex group $OSp(2|2)$, being a Riemannian symmetric superspace of type $D|C$.

In order to perform the gradient expansion, we are making use again of the non-abelian bosonisation. First, we redefine the supermatrix field $Q(x)$

$$Q(x) = \mu_A g(x) \Sigma_3$$

with $g = g_L \Sigma_3 g_R \Sigma_3$.

The nonlinear constraint on the new field is $g^t(x) = g(x)^{-1}$ which is nothing else that the defining relation for an element of the (complex) supergroup $OSp(2|2)$. This shows that the coset space $G/H$ is diffeomorphic to $OSp(2|2)$. When inserting this expression in the effective lagrangian (44) the last term vanishes. The other terms can be bosonized using the same rules as in the preceding section. The free Dirac part gives the WZW action (36), while the “mass” term can be bosonised replacing again $\bar{\phi} \phi^t \Sigma_3$ and $\Sigma_3 \bar{\phi} \phi$ by $\ell^{-1} M$ resp. $\ell^{-1} M^{-1} = \ell^{-1} M^t$. The effective action becomes

$$S[Q] = W_{osp(4N|4N)}[M] + \frac{\mu_A}{\ell} \int \frac{d^2x}{2\pi} \text{STr} \left( g M^{-1} + M g^{-1} \right).$$

The mass term forces $M$ to follow $g$. Neglecting the fluctuations of $M$ around the minimum $M = g$, we obtain

$$S[Q] = 2N W_{osp(2|2)}[g],$$

(47)

where the factor $2N$ appears from taking the trace over the spin indices, on which $g$ acts as the identity. The topological WZW term survives to this reduction from $OSp(4N|4N)$ to $OSp(2|2)$. The reality conditions discussed above mean that the restrictions $g_F$ and $g_B$ of $g$ in the FF and BB sectors satisfy $g_F^{-1} = \bar{g}_F$ and $g_B = \bar{g}_B$, respectively. This ensures the stability of the action (47).

As already mentioned, $W_{osp(4N|4N)}[M]$ is defined on a Riemannian symmetric space of type $C|D$, while $W_{osp(2|2)}[g]$ is defined on a Riemannian symmetric space of type $D|C$. The quadratic form defining the metric changes sign between the two types of spaces, since $\text{STr} \equiv \text{Tr}_{BB} - \text{Tr}_{FF} = \text{Tr}_{Sp} - \text{Tr}_{SO}$ in the first case and $\text{STr} = \text{Tr}_{SO} - \text{Tr}_{Sp}$ in the second case. In order to have a well defined functional integral, the level $k$ of the WZW action has to change sign between the two cases. We conclude that the action (47) corresponds to an $osp(2|2)_{k=-2N}$ theory. It is surprising that the saddle point approximation is able to reproduce this exact result.
9 Appendix A: Algebraic coefficients

The aim of this appendix is to define the algebraic conventions used for the calculations. We will also gather various algebraic formulas helpful for reproducing our results for the beta functions.

The ensemble of $sl(2N)$ generators are denoted by $T^A$, and they can be divided in generators of the subalgebra $sp(2N)$, $\tau^a$, and generators of the complement of $sp(2N)$ in $sl(2N)$, $T^i$. The normalization we choose is

$$Tr(T^AT^B) = \delta^{AB}, \quad Tr(T^iT^j) = \delta^{ij}, \quad Tr(\tau^a\tau^b) = \delta^{ab}$$

The various structure constants are defined by

$$[T^i, T^j] = if^{ijk} T^k, \quad [\tau^a, \tau^b] = if^{abc} \tau^c$$

$$\{\tau^a, \tau^b\} = ib^{ab} \tau^i + x^{ab} I, \quad [\tau^a, T^i] = if^{aij} \tau^j, \quad \{\tau^a, T^i\} = ib^{abi} \tau^b$$

$$[T^A, T^B] = i\delta^{ABC} T^C$$

where $[\ldots]$ stands for commutators and $\{\ldots\}$ anticommutators.

When evaluating beta-functions, a certain number of algebraic identities are needed to simplify expressions. Here is a list of algebraic coefficients that are needed:

| Coefficient | Identity | Value |
|-------------|----------|-------|
| $C$         | $T^a T^A = C$ | $(4N^2 - 1)/2N$ |
| $C'$        | $\tau^a \tau^a = C'$ | $(2N + 1)/2$ |
| $x_{TTT}$   | $T^A T^B T^A = x_{TTT} T^B$ | $-1/2N$ |
| $x_{T\tau}$ | $\tau^a T^a = x_{T\tau} T^a$ | $(N - 1)/2N$ |
| $x_{T\tau}$ | $\tau^a T^a = x_{T\tau} T^a$ | $1/2$ |
| $B_{TTT}$   | $f^{CDA} f^{CDB} = -D_{TT} \delta^{AB}$ | $-4N$ |
| $D_{TT}$    | $f^{cda} f^{cab} = -D_{TT} \delta^{ab}$ | $2(N + 1)$ |
| $D_{T\tau}$ | $f^{ija} f^{i\beta} = -D_{T\tau} \delta^{ab}$ | $2(N - 1)$ |
| $D_{T\tau}$ | $f^{akl} f^{akl} = -D_{T\tau} \delta^{ij}$ | $2N$ |
| $B_{TT}$    | $b^{abc} b^{abc} = -B_{TT} \delta^{ab}$ | $-2(N + 1)$ |
| $B_{T\tau}$ | $b^{abc} b^{abc} = -B_{T\tau} \delta^{ab}$ | $-2(N^2 - 1)/N$ |
| $x$         | $x^{ab} x^{ab} = x$ | $(2N + 1)/N$ |
10 Appendix B: Stress tensor factorization.

The decomposition (40) may be verified directly by computing the operator products defining the stress tensors using Wick theorem with the normalization (14). For the \( sp(2N) \) sector we get:

\[
T_{sp(2N)0} = \frac{1}{(N+1)} \lim_{w \to z} \left[ \frac{1}{2(z-w)} (\phi^I(z) \tau^a \phi(w)) + \frac{1}{8} (\phi^I \tau^a \phi)^2 \right]
\]

\[
= -\frac{2N+1}{4(N+1)} (\phi^I \partial \phi) + \frac{1}{8(N+1)} (\phi^I \tau^a \phi)^2
\]

where we used \( \phi^I = 0 \), by antisymmetry, and the value of the \( sp(2N) \) Casimir in the defining representation, \( \tau^a \tau^a = (2N+1)/2 \). Similarly for the \( osp(2|2) \) sector we first introduce the appropriate basis of \( gl(2N) \) to decompose \( T_{osp(2|2) - 2N} \) as follows:

\[
T_{osp(2|2) - 2N} = -\frac{1}{8(N+1)} \lim_{w \to z} \text{STr} \left( (\phi^I(z) E_I) (\phi^I(w) E_I) \right)
\]

\[
= -\frac{1}{8(N+1)} \lim_{w \to z} \left[ (\phi^I(z) \tau^a \phi(w)) (\phi^I(w) \tau^a \phi(z)) \right]
\]

\[
+ \frac{1}{2N} (\phi^I(z) \phi(w)) (\phi^I(w) \phi(z)) + (\phi^I(z) T^i \phi(w)) (\phi^I(w) T^i \phi(z)) \right]
\]

Because of the antisymmetry property, \( \phi^I \phi = \phi^I T^i \phi = 0 \), eq. (10), the two last terms are proportional to \( (\phi^I \partial \phi) \). The first term gives a contribution proportional to \( (\phi^I \partial \phi) \) but also to \( (\phi^I \tau^a \phi)^2 \). Gathering these contributions and using again the value of the \( sp(2N) \) Casimir as well as \( T^i T^i = (2N + 1)(N - 1)/2N \), we get:

\[
T_{osp(2|2) - 2N} = -\frac{1}{4(N+1)} (\phi^I \partial \phi) \quad \text{and} \quad \frac{1}{8(N+1)} (\phi^I \tau^a \phi)^2
\]

Hence:

\[
T_{sp(2N)0} + T_{osp(2|2) - 2N} = -\frac{1}{2} (\phi^I \partial \phi) \equiv T_{cft}
\]

which proves the claimed factorization.
11 Appendix C: All order beta functions.

In this appendix we derive expressions for the beta functions using formula suggested in ref. [26]. These formula apply to current-current perturbations of WZW models of the form:

\[ S = S_{wzw} + \sum_K h_K \int \frac{dx^2}{2\pi} \mathcal{O}^K , \quad \mathcal{O}^K = d^K_{a\bar{a}} J^a J^{\bar{a}} \]

where \( J^a \) and \( J^{\bar{a}} \) are the left and right conserved currents of the WZW models generating an affine (super) algebra at some level \( k \). In the case of a superalgebra the currents \( J^a \) possess a bosonic or fermionic character depending whether their degree \( |a| \) are zero or one. In our case, the underlying algebra is the affine \( osp(4N|4N) \) at level \( k = 1 \). The currents are bilinear in the supermultiplet \( J^a = \phi^I X^a \phi \), with \( X^a \) generators of \( osp(4N|4N) \).

For the theory to be perturbatively renormalizable, one needs to choose the tensors \( d^K_{a\bar{a}} \) such that:

\[
(-)^{|b||c|} d^K_{a\bar{a}} d^L_{cd} f^{ac}_{ij} f^{bd}_{j} = C^K_L d^I_{ij} \\
\eta^{ij} d^K_{ai} d^L_{bj} = D^K_L d^I_{ab} \\
d^K_{ij} f^{ia}_K f^{ik}_L = R^K_L \eta^{ac} d^L_{cb}
\]

where \( \eta^{ab} \) is the Killing invariant bilinear form of the superalgebra and \( f^{ab}_c \) its structure constants. These conditions are satisfied by the four operators \( \mathcal{O}_a, \mathcal{O}_A, \mathcal{O}_m \) and \( \mathcal{O}_M \) defined in eq. (16).

With an appropriate renormalization prescription, the proposed beta functions [26] read for \( k = 1 \),

\[ \beta_h = -C(h', h')(1 + D(h)^2) + 2C(h' D(h), h'D(h)) - 2h'D(h)RD(h) \quad (48) \]

where \( D(h) \) is the matrix \( D(h)^K_L = D^K_L h_I, C(x, y) \) is the row vector with components \( C(x, y)_K = C^K_L x_I y_I \), and finally

\[ h' = h(1 - D(h)^2)^{-1} \]

We shall assume that these formula are correct and capture perturbative contributions to all orders.

With the normalization of eq.(13), \( h_K = g_K / 2N \).
One has to compute all the tensors $C^{KL}_I$, $D^{KL}_I$ and $R^K_I$. The first one codes the operator product expansion of the perturbing operators:

$$O^K(z)O^L(0) \simeq \frac{1}{|z|^2} \sum C^{KL}_I O^I(0) + \text{reg}$$

The coefficients $C^{KL}_I$ have been determined when computing the one-loop beta functions, see eq.(23).

The tensor $D^{KL}_I$ may be computed using a variation on the free field representation of the currents. Namely introduce auxiliary copies $\phi^\alpha$, $\alpha = 0, 1$ or 2 of the supermultiplets with two point function $\langle \phi_\alpha \phi^\alpha_\alpha \rangle = 1$. Here $\langle \cdots \rangle_\alpha$ denotes the expectation value in the auxiliary Fock space associated the the auxiliary field $\phi_\alpha$. Let

$$O^K_{\alpha\beta} \equiv d^K_{ab}(\phi^a_\alpha X^a \phi_\beta)(\phi^b_\beta X^b \phi_\beta)$$

Then one has:

$$\langle O^K_{01} O^L_{02} \rangle_0 = D^{KL}_I O^I_{12}$$

Hence, $D^{KL}_I$ is computable simply using Wick theorem.

The last tensor $R^K_I$ is computable by looking at the following operator product expansion:

$$T^K(z)O^L(0) \simeq \frac{1}{z^2}(2D^{KL}_I + R^K_N D^{LN}_I)O^I(0) + \text{reg}$$

with $T^K(z) = \frac{d^K_{ab}}{2} J^a(z) J^b(z)$.

The result is summarized in the following tables:

| $D^{mm}_m$ | $D^{Am}_a$ | $D^{am}_a$ | $D^{Mm}_M$ | $D^{MM}_M$ |
|------------|------------|------------|------------|------------|
| $D^{mm}_m = D^{Am}_a = D^{am}_a = D^{Mm}_M = 1/N$ | $D^{MM}_M = (2N + 1)(N - 1)/N$ |
| $D^{AA}_A = 1$ | $D^\alpha_\alpha = -D^{\alpha\alpha}_\alpha = N + 1$ | $D^{MM}_M = (N + 1)(N - 2)/N$ |
| $D^{m}_m = 2N + 1, D^{M}_M = -N + 1$ |
| $D^{MA}_A = (N - 1)/N, D^{M}_M = -N, D^{M}_M = (N^2 - 1)/N$ |

The non-vanishing $F^{KL}_I = 2D^{KL}_I + R^K_N D^{LN}_I$ are:

| $F^{AA}_A$ | $F^{AA}_M$ | $F^{AM}_M$ | $F^{AM}_A$ | $F^{MM}_M$ |
|------------|------------|------------|------------|------------|
| $F^{AA}_A = 2F^{AM}_A = 2F^{AM}_M = 2F^{MM}_M = 8(N + 1)$ |
| $F^{AM}_m = F^{MM}_m = -2F^{mm}_m = 4(2N + 1)$ |
| $F^{Am}/2 = F^{am}/2 = -F^{ma} = -F^{mm} = -F^{mM} = -F^{MM} = 2/2N + 1)$ |
| $F^{Am}_A = 2, F^{MA}_A = F^{Mm}_m = F^{Ma}_a = F^{MM}_M = -2(2N + 1)(N - 1)/N$ |
| $F^{MM}_M = 4(N^2 - 1)/N$ |

The non-vanishing $F^{KL}_I = 2D^{KL}_I + R^K_N D^{LN}_I$ are:

| $F^{AA}_A$ | $F^{AA}_M$ | $F^{AM}_M$ | $F^{AM}_A$ | $F^{MM}_M$ |
|------------|------------|------------|------------|------------|
| $F^{AA}_A = 2F^{AM}_A = 2F^{AM}_M = 2F^{MM}_M = 8(N + 1)$ |
| $F^{AM}_m = F^{MM}_m = -2F^{mm}_m = 4(2N + 1)$ |
| $F^{Am}/2 = F^{am}/2 = -F^{ma} = -F^{mm} = -F^{mM} = -F^{MM} = 2/2N + 1)$ |
| $F^{Am}_A = 2, F^{MA}_A = F^{Mm}_m = F^{Ma}_a = F^{MM}_M = -2(2N + 1)(N - 1)/N$ |
| $F^{MM}_M = 4(N^2 - 1)/N$ |

with $T^K(z) = \frac{d^K_{ab}}{2} J^a(z) J^b(z)$.
For arbitrary $N$, the beta functions are then derived from eq.(48). In the $N = \infty$ limit, they reduce to:

\begin{align}
\beta_M &= \beta_\alpha = 2(g_\alpha + g_M)g_A \\
\beta_A &= 2g_A^2 + (g_\alpha + g_M)^2/2 \\
\beta_m &= 4g_A(g_\alpha + g_m) - 2g_m(g_\alpha + g_M) - \frac{(g_\alpha + g_M)(g_\alpha - g_M)^2}{(g_\alpha - g_M - 2)}
\end{align}

These are easy to integrate. In particular $(g_\alpha - g_M)$ is a RG invariant in the large $N$ limit while the beta functions for $2g_A \pm (g_\alpha + g_M)$ are separated and quadratic.

It is then simple to verify that the isotropic line $g_\alpha = g_m = g_M = g_A \equiv g$ is stable and attractive to all orders and $\beta_g = 4g^2$. The fact that it is quadratic is in agreement with eq.(30) and it provides a tiny check of the all order beta functions. The isotropic coupling grows indefinitely with the scale. Of course the large $N$ approximation remains valid only for $g \ll N$. It is possible to verify that all RG trajectories starting in the domain of positive couplings sufficiently close to the origin are asymptotic to the isotropic line at large distances. This confirms the one loop computation.

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