Abstract. This is the second of two papers introducing and investigating two bivariate zeta functions associated to unipotent group schemes over rings of integers of number fields. In the first part, we proved some of their properties such as rationality and functional equations. Here, we calculate such bivariate zeta functions of three infinite families of nilpotent groups of class 2 generalising the Heisenberg group of $3 \times 3$-unitriangular matrices over rings of integers of number fields. The local factors of these zeta functions are also expressed in terms of sums over finite hyperoctahedral groups, which provides formulae for joint distributions of three statistics on such groups.

1. Introduction and statement of main results

In the first part [7] of this work, we introduced bivariate zeta functions of groups associated to unipotent group schemes over rings of integers of number fields. In this second part, we provide explicit examples of such zeta functions for infinite families of nilpotent groups of class 2, and use these formulae to provide formulae for joint distributions of three statistics on hyperoctahedral groups.

We are interested in understanding the following data of a group $G$.

$$r_n(G) = |\{\text{isomorphism classes of } n\text{-dimensional irreducible complex representations of } G\}|,$$

$$c_n(G) = |\{\text{conjugacy classes of } G \text{ of cardinality } n\}|.$$

If all these numbers are finite—for instance, if $G$ is a finite group—we define the following zeta functions.

**Definition 1.1.** The representation and the conjugacy class zeta functions of the group $G$ are, respectively,

$$\zeta_{irr}^G(s) = \sum_{n=1}^{\infty} r_n(G)n^{-s} \quad \text{and} \quad \zeta_{cc}^G(s) = \sum_{n=1}^{\infty} c_n(G)n^{-s},$$

where $s$ is a complex variable.

Let $K$ denote a number field and $\mathcal{O}$ its ring of integers. Let $G$ be a unipotent group scheme over $\mathcal{O}$. The group $G(\mathcal{O})$ is a finitely generated, torsion-free nilpotent group ($T$-group for short)—see [14] Section 2.1.1—whereas, for a nonzero ideal $I$ of $\mathcal{O}$, the group $G(\mathcal{O}/I)$ is a finite group. For $T$-groups, the numbers $r_n(G)$ and $c_n(G)$ are not all finite, in general. We thus define bivariate zeta functions which count such data for the principal congruence quotients of the group considered.
Definition 1.2. The bivariate representation and the bivariate conjugacy class zeta functions of $G(O)$ are, respectively,

$$Z_{G(O)}^{irr}(s_1, s_2) = \sum_{(0) \neq I \subseteq O} c_{G(O/I)}^{irr}(s_1)|O : I|^{-s_2}$$

and

$$Z_{G(O)}^{cc}(s_1, s_2) = \sum_{(0) \neq I \subseteq O} c_{G(O/I)}^{cc}(s_1)|O : I|^{-s_2},$$

where $s_1$ and $s_2$ are complex variables.

These series converge for $s_1$ and $s_2$ with sufficiently large real parts—cf. [7, Proposition 2.4]—and satisfy the following Euler decompositions:

$$Z_{G(O)}^{s}(s_1, s_2) = \prod_{p \in \text{Spec}(O) \setminus \{(0)\}} Z_{G(O_p)}^{s}(s_1, s_2),$$

where $s \in \{\text{irr}, \text{cc}\}$ and the completion of $O$ at the nonzero prime ideal $p$ is denoted $O_p$. When considering a fixed prime ideal $p$, we write simply $O_p = \mathfrak{o}$ and $G_N := G(\mathfrak{o}/p^N)$. With this notation, the local factor at $p$ is given by

$$Z_{G(\mathfrak{o}_p)}^{s}(s_1, s_2) = Z_{G(\mathfrak{o})}^{s}(s_1, s_2) = \sum_{N=0}^{\infty} \zeta_{G^N}(s_1)|\mathfrak{o} : p|^{-Ns_2}.$$

In this part of the work, we calculate explicitly such local bivariate zeta functions of three infinite families of nilpotent groups of class 2. Such groups are constructed from the following class-2-nilpotent $Z$-Lie lattices, that is, free and finitely generated $O$-modules together with an anti-symmetric bi-additive form $[\cdot, \cdot]$ which satisfies the Jacobi identity.

Definition 1.3. For $n \in \mathbb{N}$ and $\delta \in \{0, 1\}$, consider the nilpotent $Z$-Lie lattices

- $F_{n, \delta} = \langle x_k, y_j \mid [x_i, x_j] = y_{ij}, 1 \leq k \leq 2n + \delta, 1 \leq i < j \leq 2n + \delta\rangle$,
- $G_n = \langle x_k, y_j \mid [x_i, x_{i+j}] = y_{ij}, 1 \leq k \leq 2n, 1 \leq i, j \leq n\rangle$,
- $H_n = \langle x_k, y_j \mid [x_i, x_{i+j}] = y_{ij}, [x_j, x_{i+j}] = y_{ij}, 1 \leq k \leq 2n, 1 \leq i < j \leq n\rangle$.

By convention, relations that do not follow from the given ones are trivial.

Let $\Lambda$ be one of the $Z$-Lie lattices of Definition 1.3. We consider the unipotent group scheme $G^{\Lambda}$ associated to $\Lambda$ obtained by the construction of [14, Section 2.4]. Following [14], these unipotent group schemes are denoted by $F_{n, \delta}, G_n$, and $H_n$, and groups of the form $F_{n, \delta}(O), G_n(O)$, and $H_n(O)$ are called groups of type $F, G$, and $H$, respectively.

The unipotent group schemes $F_{n, \delta}, G_n$, and $H_n$ provide different generalisations of the Heisenberg group scheme $H = F_{1,0} = G_1 = H_1$, where $H(O)$ is the Heisenberg group of upper uni-triangular $3 \times 3$-matrices over $O$. The interest in such $Z$-Lie lattices arises from their very construction. Roughly speaking, their defining relations reflect the reduced, irreducible, prehomogeneous vector spaces of, respectively, complex $n \times n$ antisymmetric matrices, complex $n \times n$-matrices and complex $n \times n$ symmetric matrices—here, the relative invariants are given by Pf, det and det, where Pf$(X)$ denotes the Pfaffian of an antisymmetric matrix $X$. We refer the reader to [14, Section 6] for details.

For the rest of this work, $\Lambda$ is one of the $Z$-Lie lattices of Definition 1.3 and $G = G^{\Lambda}$ denotes the unipotent group schemes associated to $\Lambda$, unless otherwise stated.

1.1. Bivariate conjugacy class and class number zeta functions. Our first result concerns bivariate conjugacy class zeta functions, which leads to similar results for (univariate) class number zeta functions.
Theorem 1.4. Let \( n \in \mathbb{N} \), and \( \delta \in \{0, 1\} \). Then, for each nonzero prime ideal \( p \) of \( \mathcal{O} \) with \( q = [\mathcal{O} : p] \),
\[
Z_{\mathcal{O}, n, \delta}(s_1, s_2) = \frac{1 - q^{(2n+\delta-1)-(2n+\delta-1)s_1-s_2}}{(1 - q^{(2n+\delta-1)s_2})(1 - q^{(2n+\delta-1)+1-(2n+\delta-1)s_1-s_2})}.
\]
Write \( q^{-s_1} = T_1 \) and \( q^{-s_2} = T_2 \). For \( n \geq 2 \),
\[
Z_{\mathcal{O}, n, \delta}(T_1, T_2) = (1 - q^{2}T_1 T_2)(1 - q^{2+1}T_1^{2n-1}T_2 + qn^2T_1^p T_2(1 - q^{-n})(1 - q^{-(n-1)}T_1^{n-1})),
\]
\[
Z_{H_n, \delta}(T_1, T_2) = (1 - q^{2}T_1 T_2)(1 - q^{2+1}T_1^{2n-1}T_2 + qn^2T_1^p T_2(1 - q^{-(n-1)}T_1^{n-1}))(1 - q^{2+1}T_1^{2n-1}T_2).
\]

The proof of Theorem 1.4 is given in Section 3.

Let \( k(G) \) denote the class number of the group \( G \), that is, the number of conjugacy classes or, equivalently, the number of irreducible complex characters of \( G \). In particular, for a finite group \( G \), \( \zeta_G^{(0)}(0) = \zeta_G^{(0)}(0) = k(G) \). We recall from [7, Section 1.2] that, for a unipotent group scheme \( \mathbf{G} \), one may obtain the (univariate) class number zeta function \( \zeta_k(\mathcal{O}; \mathcal{O}) \) via the following specialisation:
\[
Z_{\mathcal{O}, n, \delta}(0, s) = Z_{\mathcal{O}, n, \delta}(0, s) = \sum_{\mathcal{O} : I \neq \mathcal{O}} k(G(\mathcal{O}/I))|I| = : \zeta_k(\mathcal{O}; \mathcal{O}),
\]
where \( s \) is a complex variable. We remark that the term ‘conjugacy class zeta function’ is sometimes used for what we call ‘class number zeta function’; see for instance [1, 9, 12]. Specialisation (1.3) applied to Theorem 1.4 gives the following.

Corollary 1.5. For all \( n \geq 1 \) and \( \delta \in \{0, 1\} \),
\[
\zeta_k^{(0)}(\mathcal{O}; \mathcal{O}) = \zeta_k^{(0)}(s - (2n+\delta) - 1)\zeta_k^{(0)}(s - (2n+\delta - 1)\zeta_k^{(0)}(s - (2n+\delta) - 1)),
\]
where \( \zeta_k(s) \) is the Dedekind zeta function of the number field \( K = \text{Frac}(\mathcal{O}) \). Furthermore, for \( n \geq 2 \), the class number zeta functions of \( G_n(\mathcal{O}) \) and \( H_n(\mathcal{O}) \) are
\[
\zeta_k^{(0)}(\mathcal{O}; \mathcal{O}) = \prod_{p \in \text{Spec}(\mathcal{O}) \setminus \{0\}} \frac{(1 - q_p^{(2) - s})(1 - q_p^{(2+1) - s}) + qn^2(1 - q_p^{-n})(1 - q_p^{n+1})}{(1 - q_p^{(2) - s})(1 - q_p^{(2+1) - s})},
\]
\[
\zeta_k^{(0)}(\mathcal{O}; \mathcal{O}) = \prod_{p \in \text{Spec}(\mathcal{O}) \setminus \{0\}} \frac{(1 - q_p^{(2) - s})(1 - q_p^{(2+1) - s}) + qn^2(1 - q_p^{-n})(1 - q_p^{n+1})}{(1 - q_p^{(2) - s})(1 - q_p^{(2+1) - s})},
\]
where \( q_p = |\mathcal{O} : p| \), for all \( p \in \text{Spec}(\mathcal{O}) \setminus \{0\} \).

In particular, all the local factors of the bivariate conjugacy class zeta functions of groups of type \( F, G, \) and \( H \) are rational in \( q_p, q_p^{-s_1}, \) and \( q_p^{-s_2} \), whilst all local factors of their class number zeta functions are rational in \( q_p \) and \( q_p^{-s} \). Moreover, the local factors of both zeta functions satisfy functional equations. This generalises [7, Theorem 1.4] for these groups.

The formula (1.4) is also shown in [11]; it is a consequence of both [11, Proposition 5.11 and Proposition 6.4]; see Remarks 2.6 and 3.4 respectively. In [7, Section 4.2], we write the bivariate zeta functions of Definition 1.2 in terms of \( p \)-adic integrals. These integrals under specialisation (1.3) coincide with the integrals [11] 3
Theorem 1.6. are defined as follows.

For a 1-dimensional representation \( \chi \) of topological groups, only continuous representations are considered.

Theorem C], namely \( \bar{\zeta}(G) \) under the specialisation of the ask zeta function to the class number zeta function given in [14, Theorem 1.7]; cf. [7, Remark 4.10].

1.2. Bivariate representation and twist representation zeta functions. To state our next result, we introduce some notation.

Let \( X, Y \) denote indeterminates in the field \( \mathbb{Q}(X,Y) \). Given \( n \in \mathbb{N} \), set \( \binom{n}{i} = \frac{1}{i!} \binom{X^n}{i} \). For \( a, b \in \mathbb{N}_0 \) such that \( a \geq b \), the \( X \)-binomial coefficient of \( a \) over \( b \) is

\[
\binom{a}{b}_X = \frac{\binom{a}{b}}{X \binom{a-b}{b}_X}.
\]

Given \( n \in \mathbb{N} \), denote \( [n] = \{1, \ldots, n\} \) and \( [n]_0 = [n] \cup \{0\} \). Given a subset \( \{i_1, \ldots, i_l\} \subset [n]_0 \), we write \( \{i_1, \ldots, i_l\}_< = \{i_1 < i_2 < \cdots < i_l\} \) meaning that \( i_1 < i_2 < \cdots < i_l \). For \( I = \{i_1, \ldots, i_l\}_< \subset [n-1]_0 \), denote \( \mu_j := i_j + 1 - i_j \) for all \( j \in I \), where \( i_0 = 0 \), \( i_{l+1} = n \), and define

\[
\binom{n}{I}_X = \binom{n}{i_1}_X \binom{n}{i_2}_X \cdots \binom{n}{i_l}_X.
\]

The Y-Pochhammer symbol is defined as

\[
(X; Y)_n = \prod_{i=0}^{n-1} (1 - XY^i).
\]

Theorem 1.6. Let \( n \in \mathbb{N} \), and \( \delta \in \{0, 1\} \). Then, for each nonzero prime ideal \( \mathfrak{p} \) of \( \mathcal{O} \) with \( q = [\mathcal{O} : \mathfrak{p}] \),

\[
Z_{G, \mathfrak{p}}^{(s_1, s_2)} = \frac{1}{1 - q^{a(G,n) - s_2}} \sum_{I \subseteq [n-1]_0} f_{G, I}(q^{-1}) \prod_{i \in I} q^{a(G,i) - (n-i)s_1 - s_2} \prod_{i \in [n-1]_0} q^{a(G,i) - (n-i)s_1 - s_2},
\]

where \( f_{G, I}(X) \) and \( \tilde{a}(G, i) \), for all \( I = \{i_1, \ldots, i_l\}_< \subseteq [n-1]_0 \) and for all \( i \in [n]_0 \), are defined as follows.

\[
\begin{array}{c|c|c}
G & f_{G, I}(X) & \tilde{a}(G, i) \\
--- & --- & --- \\
F_{n, \delta} & \binom{n}{i}_X(X^{2(i+\delta)+1}; X^2)_{n-i} & \left(\frac{2n+\delta}{2}\right) - \left(\frac{2i+\delta}{2}\right) + 2i + \delta \\
G_n & \binom{n}{i}_X(X^{i+1}; X)_{n-i} & n^2 - i^2 + 2i \\
H_n & \left(\prod_{j=1}^{l} (X^2; X^2)_{\lceil \mu_j/2 \rceil}\right)(X^{i+1}; X)_{n-i} & \left(\frac{n+1}{2}\right) - \left(\frac{i+1}{2}\right) + 2i \\
\end{array}
\]

The proof of Theorem 1.6 may be found in Section 4.

The numbers \( \tilde{a}(G, i) \) are slight modifications of the numbers \( a(G, i) \) given in [14, Theorem C], namely \( \tilde{a}(F_{n, \delta}, i) = a(F_{n, \delta}, i) + 2i + \delta \) and \( \tilde{a}(G, i) = a(G, i) + 2i \), for \( G \) of type \( G \) and \( H \).

Let \( G \) be a unipotent group scheme over \( \mathcal{O} \) such that \( G(\mathcal{O}) \) has nilpotency class 2. The local factors of the bivariate representation zeta functions of \( G(\mathcal{O}) \) specialise to the local factors of its twist representation zeta function \( \zeta_{G(\mathcal{O})}^{(s)}(s) \). This a variation of the representation zeta function of Definition 1.1 which counts the numbers \( \bar{r}_n(G) \) of \( n \)-dimensional twist-isoclasses of irreducible complex representations of \( G \), that is, the number of classes of the equivalence relation on the set of irreducible complex representations of \( G \) given by \( \rho \sim \sigma \) if and only if there exists a 1-dimensional representation \( \chi \) of \( G \) such that \( \rho \cong \chi \otimes \sigma \). In the context of topological groups, only continuous representations are considered.

Twist representation zeta functions of \( \mathcal{T} \)-groups are investigated, for instance, in [4, 5, 10, 14, 15, 17].
Bivariate zeta functions of $\mathcal{T}$-groups

\[ (1.5) \prod_{p \in \text{Spec}(\mathcal{O}) \setminus \{0\}} \left( 1 - q^{-s} \right)^{Z^\mathcal{O}_{\mathcal{O}}(s_1, s_2) |_{s_2 = r^+}} = \zeta^\mathcal{O}_{\mathcal{O}}(s). \]

Since the groups of type $F$, $G$, and $H$ are $\mathcal{T}$-groups of class 2, we obtain the formulae of [14, Theorem C] describing their twist representation zeta functions by applying specialisation (1.5) to Theorem 1.6.

1.3. Joint distributions on hyperoctahedral groups. The polynomials $f_{G, t}(X)$ appearing in Theorem 1.6 can be expressed in terms of distributions of statistics on Weyl groups of type $B$, also called hyperoctahedral groups $B_n$; see Sections 5.1 and 5.2. These are the groups $B_n$ of the permutations $w$ of the set $[\pm n] = \{ -n, \ldots, n \}$ such that $w(-i) = -w(i)$ for all $i \in [\pm n]_0$.

In Lemma 5.4 we describe the local bivariate representation zeta function of $G(\mathcal{O})$ as a sum over $B_n$ in terms of statistics on such groups. As the local factors of the bivariate representation and the bivariate conjugacy class zeta functions of $G(\mathcal{O})$ specialise to its class number zeta function, the formulae in terms of statistics on hyperoctahedral groups $B_n$ can be compared with the formulae of Corollary 1.5, which leads to formulae for the joint distribution of three functions on Weyl groups of type $B$; see Propositions 5.5 and 5.6.

More precisely, the formulae of Lemma 5.4 under specialisation (1.3) provide a formula for the following form for the class number zeta function of $G(\mathfrak{o})$:

\[ c^k_{G(\mathfrak{o})}(s) = \sum_{w \in B_n} \chi_G(w) q^{-\text{neg}(w) - \text{des}(w)s} \prod_{i=0}^{n} (1 - q^{\ell_G(w_i) - s}), \]

where $\chi_G$ is one of the linear characters $(-1)^{\text{neg}}$ or $(-1)^{\ell}$ of $B_n$, where $\text{neg}(w)$ denotes the number of negative entries of $w$, and $\ell$ is the standard Coxeter length function of $B_n$. Moreover, the functions $\ell_G$ are sums of statistics on $B_n$ for each $G$ and $\text{des}(w)$ is the cardinality of the descent set of $w \in B_n$; see Section 5.1 for definitions.

1.4. Local functional equations. The formulae for the bivariate zeta functions given in Theorems 1.6 and 1.4 allow us to strengthen [7, Theorem 1.4] for groups of type $F$, $G$, and $H$ by showing that its conclusion holds for all local factors:

**Theorem 1.7.** For $s \in \{\text{irr}, \text{cc}\}$ and all nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ with $|\mathcal{O} : \mathfrak{p}| = q$, the local bivariate zeta function $Z^\mathfrak{p}_{G(\mathfrak{o})}(s_1, s_2)$ satisfies the functional equation

\[ Z^\mathfrak{p}_{G(\mathfrak{o})}(s_1, s_2) |_{q = q^h} = -q^{h-s_2} Z^\mathfrak{p}_{G(\mathfrak{o})}(s_1, s_2), \]

where $h$ is the torsion rank of $\Lambda(\mathfrak{o}) = \Lambda \otimes_{\mathfrak{o}} \mathfrak{o}$; see the exact value of $h$ in Table 1.

In fact, [7, Theorem 1.4] states that almost all local factors satisfy functional equations of such form, whilst Theorems 1.4 and 1.6 state that all local factors are given by the same rational functions. We give an alternative proof in Section 5.3.

1.5. Notation. The following list collects frequently used notation.

\[
\begin{array}{c|c}
\mathbb{N} & \{1, 2, \ldots\} \\
\mathbb{N}_0 & \{0, 1, 2, \ldots\} \\
[n] & \{1, \ldots, n\} \\
[n]_0 & \{0, 1, \ldots, n\} \\
[\pm n] & \{-n, \ldots, -1\} \cup [n]_0 \\
\end{array}
\]
Paula Lins

\[ X, Y \] indeterminates in the field \( \mathbb{Q}(X, Y) \)
\[ \binom{n}{X} \] for \( n \in \mathbb{N} \)
\[ \binom{n}{X}! \] for \( n \in \mathbb{N} \)
\[ \binom{a}{b}X \] for \( a \geq b \)
\[ (X; Y)_n \] for \( n \in \mathbb{N} \)
\[ \text{gp}(X) \] for \( n \in \mathbb{N} \)

\[ I = \{ i_1, \ldots, i_t \} < \] set \( I \) of nonnegative integers \( i_1 < i_2 < \cdots < i_t \)
\[ \binom{n}{j} \] for \( n \in \mathbb{N} \)
\[ \binom{n}{i_1}(i_{i_1} \cdots i_{i_t}) \]
\[ i_0 \] for \( n \in \mathbb{N} \)
\[ i_{t+1} \] for \( n \in \mathbb{N} \)
\[ n \] for \( n \in \mathbb{N} \)
\[ \mu \] for \( n \in \mathbb{N} \)
\[ |p| \] for \( n \in \mathbb{N} \)
\[ ||(z_j)_{j \in J}||_p \] for \( n \in \mathbb{N} \)
\[ W_{p, N}^k \] for \( n \in \mathbb{N} \)
\[ W_k \] for \( n \in \mathbb{N} \)

\[ \begin{array}{ll}
K & \text{number field} \\
O & \text{ring of integers of } K \\
p & \text{nonzero prime ideal of } O \\
o = O_p & \text{completion of } O \text{ at } p \\
 & \text{completion of } O \text{ at } p \\
^n & n\text{-fold Cartesian power } o \times \cdots \times o \\
p^m & m\text{-fold Cartesian power } p \times \cdots \times p \\
 & \text{one of the } Z\text{-Lie lattices } F_n, G_n, H_n; \text{ see Definition 1.3} \\
G & \text{one of the unipotent group schemes } F_n, G_n, H_n. \\
A_\Lambda(X) & A\text{-commutator matrix of } \Lambda; \text{ see Definition 2.4} \\
B_\Lambda(Y) & B\text{-commutator matrix of } \Lambda \\
v_p & p\text{-adic valuation} \\
| \cdot |_p & p\text{-adic norm} \\
|| (z_j)_{j \in J} ||_p & \max \{ ||z_j\|_p : j \in J \} = q^{-v_p((z_j)_{j \in J})}, J \text{ finite index set} \\
W_{p, N}^k & \{ x \in (o/pN)^k : v_p(x) = 0 \}, k \in \mathbb{N}, N \in \mathbb{N}_0 \\
W_k & \{ x \in o^k : v_p(x) = 0 \}, k \in \mathbb{N} \\
\end{array} \]

2. Bivariate zeta functions and p-adic integrals

In this section we calculate some generic p-adic integrals which are needed in the current work, and recall from [7] Proposition 4.8 how to write the local bivariate zeta functions of groups of type \( F, G, \text{ and } H \) in terms of p-adic integrals.

For the rest of Section 2, \( p \) is a fixed nonzero prime ideal of \( O \), and \( o \) denotes the completion of \( O \) at \( p \). Denote by \( q \) the cardinality of \( O/p \) and by \( p \) its characteristic.

2.1. Some p-adic integrals. Given an element \( z \in o \) such that \( (z) = p^sp_1^{s_1} \cdots p_r^{s_r} \) is the prime factorisation of the ideal \( (z) \triangleleft O \) with \( p_i \neq p \), for all \( i \in [r] \), the p-adic valuation of \( z \) is given by \( v_p(z) = e \), and its p-adic norm is \( |z|_p = q^{-v_p(z)} \). For a finite index set \( J \) and \( (z_j)_{j \in J} \in o^J \), define \( \|(z_j)_{j \in J}\|_p = \max \{ ||z_j\|_p : j \in J \} \). We denote by \( p^m \) the \( m \)th ideal power \( p \cdots p \) and by \( p^{(m)} \) the \( m \)-fold Cartesian power \( p \times \cdots \times p \).

For now on, \( \mu \) denotes the additive Haar measure on \( o \), normalised so that \( \mu(o) = 1 \). We also denote by \( \mu \) the product measure on \( o^n \), for \( n \in \mathbb{N} \).

The following is well known.

**Proposition 2.1.** Let \( r \) be a complex variable. Then, for each \( k \in \mathbb{N} \),

\[
\int_{w \in p^k} |w|^k d\mu = \frac{q^{-k(r+1)}(1 - q^{-1})}{1 - q^{-k(r+1)}},
\]

if the integral on the left-hand side converges absolutely.
The following lemma is a direct consequence of [11, Lemma 5.8], which assures in particular that, for complex variables \( r \) and \( s \), one has

\[
\int_{(y,z) \in \mathbb{P} \times \mathbb{A}^n} |y|^r |x_1, \ldots, x_n, y|^s \, d\mu = \frac{(1-q^{-1})(1-q^{-r-n})}{(1-q^{-r-s-n})},
\]

if the integral on the left-hand side converges absolutely.

**Lemma 2.2.** Let \( r \) and \( s \) be complex variables. Then, for each \( n \in \mathbb{N}_0 \),

\[
\int_{(y,z) \in \mathbb{P} \times \mathbb{A}^n} |y|^r |x_1, \ldots, x_n, y|^s \, d\mu = \frac{(1-q^{-1})(1-q^{-r-n})}{(1-q^{-r-s-n})},
\]

if the integrals on the left-hand side of each equality converge absolutely.

**Proof.** Since \( \mathbb{P} \times \mathbb{A}^n = \mathbb{P} \times \mathbb{A}^n \setminus W_2^n \times \mathbb{A}^n \) and \( y \in W_2^n \) implies both \( |y|^p = 1 \) and \( |x_1, \ldots, x_n, y|^p = 1 \), it follows that

\[
\int_{(y,z) \in \mathbb{P} \times \mathbb{A}^n} |y|^r |x_1, \ldots, x_n, y|^s \, d\mu = \int_{(y,z) \in \mathbb{P} \times \mathbb{A}^n} |y|^r |x_1, \ldots, x_n, y|^s \, d\mu - \mu(W_2^n \times \mathbb{A}^n).
\]

The first claim then follows from (2.1) and the fact that \( \mu(W_2^n \times \mathbb{A}^n) = 1 - q^{-1} \). Analogously, since \( \mathbb{P} \times \mathbb{A}^n(p) = \mathbb{P} \times \mathbb{A}^n \setminus \mathbb{P} \times W^n_2 \),

\[
\int_{(y,z) \in \mathbb{P} \times \mathbb{A}^n} |y|^r |x_1, \ldots, x_n, y|^s \, d\mu = \int_{(y,z) \in \mathbb{P} \times \mathbb{A}^n} |y|^r |x_1, \ldots, x_n, y|^s \, d\mu - (1-q^{-n}) \int_{y \in \mathbb{P}} |y|^s \, d\mu. \quad \square
\]

The second claim follows from the first part and Lemma 2.1.

Let \( X = (X_1, \ldots, X_{2n}) \) be a vector of variables. In the following, consider the matrix

\[
M(X) = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \end{bmatrix} \in \text{Mat}_{2 \times n}((\mathbb{A}[X]),
\]

and, for \( 1 \leq i < j \leq n \), write \( M_{ij}(X) := X_{ij}X_{2j} - X_{ij}X_{2i} \).

**Proposition 2.3.** For complex variables \( s \) and \( r \), the following holds, provided the integral on the left-hand side converges absolutely.

\[
\int_{(y,z) \in \mathbb{P} \times \mathbb{A}^n} |y|^r \|\{M_{ij}(x) \mid 1 \leq i < j \leq n\} \cup \{y\} \|_p^s \, d\mu
\]

\[
= \frac{(q^n-1)(1-q^{-1})q^{-r-2n-1}}{(1-q^{-r-s-n})^2} \left( (q+1)(1-q^{-r-n})q^{-s} + (q^n-1)(1-q^{-r-s-n}) \right).
\]

**Proof.** Since \( \mathbb{A} = \bigcup_{m=1}^{k} (\pi_m + \mathbb{P}) \), for some representatives \( \pi_m \) of the classes of \( \mathbb{A}/\mathbb{P} \), there exist \( k \in \mathbb{N} \) and representatives \( A_1, \ldots, A_k \) of \( \mathbb{A}^{2n}/\mathbb{P}^{(2n)} \) such that

\[
\mathbb{A}^{2n} = \bigcup_{m=1}^{k} (A_m + \text{Mat}_{2 \times n}(\mathbb{P})) \cup \text{Mat}_{2 \times n}(\mathbb{P}),
\]

where \( \text{Mat}_{2 \times n}(\mathbb{P}) \) denotes the set of all \( 2 \times n \)-matrices over \( \mathbb{P} \). Hence \( W_2^{2n} = \bigcup_{m=1}^{k} (A_m + \text{Mat}_{2 \times n}(\mathbb{P})) \). In the following, we evaluate the integrals

\[
I_{A_m}(s,r) := \int_{(y,z) \in \mathbb{P} \times A_m + \text{Mat}_{2 \times n}(\mathbb{P})} |y|^r \|\{M_{ij}(x) \mid 1 \leq i < j \leq n\} \cup \{y\} \|_p^s \, d\mu.
\]

If \( x \in A_m + \text{Mat}_{2 \times n}(\mathbb{P}) \), then \( \text{rk}(x) = \text{rk}(A_m) \) modulo \( \mathbb{P} \). Let us then consider the two cases \( \text{rk}(A_m) = 1 \) and \( \text{rk}(A_m) = 2 \) modulo \( \mathbb{P} \). For simplicity, assume that
Paula Lins

\( \text{rk}(A_m) = 1 \) for \( 1 \leq m \leq t \), and that \( \text{rk}(A_m) = 2 \) for \( t + 1 \leq m \leq k \), for some \( t \in [k] \).

**Case 1:** Suppose that \( m \in [k] \), that is, \( \text{rk}(A_m) = 1 \). Then, each \( \bar{x} = (x_{ij}) \in A_m + \text{Mat}_{2 \times n}(p) \) has rank 1 modulo \( p \). In particular, \( v_p(M_{ij}(\bar{x})) \geq 1 \) for all \( 1 \leq i < j \leq n \). By making a suitable change of variables, we can consider \( A_m \) to be the matrix with \((1,1)\)-coordinate 1 and 0 elsewhere. Consequently, \( x_{11} = 1 + Q_{11} \) and \( x_{ij} = Q_{ij} \), for \((i,j) \neq (1,1)\), where \( Q_{ij} \in p \). Hence

\[
M_{ij}(\bar{x}) = \begin{cases} 
(1 + Q_{11})Q_{2i} - Q_{2i}Q_{1i}, & \text{for } i = 2, \ldots, n, \\
Q_{11}Q_{2j} - Q_{2j}Q_{1j}, & \text{for } 1 < i < j \leq n,
\end{cases}
\]

so that \( \|\{M_{ij}(\bar{x}) \mid 1 \leq i < j \leq n\}\|_p = \|M_{12}(\bar{x}), \ldots, M_{1n}(\bar{x})\|_p \). Therefore

\[
\mathcal{I}_{A_m}(s,r) = \int_{[y,\bar{x}] \in p \times \text{Mat}_{2 \times n}(p)} |y|^p \|M_{12}(\bar{x}), \ldots, M_{1n}, y\|^s d\mu
\]

\[
= \mu(p^{n+1}) \int_{(y,x_1,\ldots,x_{n-1}) \in p \times p^{n-1}} |y|^p \|x_1, \ldots, x_{n-1}, y\|^s d\mu
\]

\[
= q^{-n-1}(1-q^{-1})(1-q^{-r-n})(q^{-r-s-n})(1-q^{-r-1}),
\]

where the domain of integration of integral in the second equality is justified by the translation invariance of the Haar measure and the last equality is due Proposition [2.3].

**Case 2:** We now assume that \( m \in \{t + 1, \ldots, k\} \), that is, \( \text{rk}(A_m) = 2 \). In this case, each \( \bar{x} \in A_m + \text{Mat}_{2 \times n}(p) \) has rank two modulo \( p \), which means that at least one of the \( M_{ij}(\bar{x}) \) has valuation zero. Consequently,

\[
\mathcal{I}_{A_m}(s,r) = \int_{[y,\bar{x}] \in p \times p^{2n}} |y|^p \|M_{ij}(\bar{x}) \|_p \|y\|^s d\mu = \sum_{m=1}^k \mathcal{I}_{A_m}(s,r)
\]

We conclude the proof using the fact that there are \((q+1)(q^n-1)\) matrices of rank 1 and \(q(q^n-1)(q^{n-1}-1)\) matrices of rank 2 in \( \text{Mat}_{2 \times n}(\mathbb{F}_q) \) and, consequently,

\[
\int_{[y,\bar{x}] \in p \times W_{2n}} |y|^p \|\{M_{ij}(\bar{x}) \mid 1 \leq i < j \leq n\} \cup y\|^s d\mu = \sum_{m=1}^k \mathcal{I}_{A_m}(s,r)
\]

\[
= (q+1)(q^n-1)\mathcal{I}_{A_1}(s,r) + q(q^n-1)(q^{n-1}-1)\mathcal{I}_{A_2}(s,r)
\]

\[
= (q^n-1)(1-q^{-1})q^{-r-2n-1} \left( (q+1)(1-q^{-r-n})q^{-s} + (q^n-1)(1-q^{-r-s-n}) \right),
\]

as desired. \( \square \)

### 2.2. Bivariate zeta functions in terms of \( p \)-adic integrals.

Denote \( g = \Lambda(\mathfrak{o}) = \Lambda \otimes \mathfrak{o} \mathfrak{o} \). Let \( g' \) be the derived Lie sublattice of \( g \), and let \( z \) be its centre. Set

\[
h = \text{rk}(g), \quad a = \text{rk}(g/\mathfrak{j}), \quad b = \text{rk}(g'/\mathfrak{j}), \quad r = \text{rk}(g/g'), \quad z = \text{rk}(\mathfrak{j}).
\]

These numbers are given by the following table for \( \Lambda \in \{\mathcal{F}_{n,\delta}, \mathcal{G}_n, \mathcal{H}_n\} \):

**Table 1.**

| \( \Lambda \) | \( h = \text{rk}(g) \) | \( a = \text{rk}(g/\mathfrak{j}) \) | \( b = \text{rk}(g'/\mathfrak{j}) = \text{rk}(\mathfrak{j}) = z \) |
|----------|-------------------|-------------------|-------------------|
| \( \mathcal{F}_{n,\delta} \) | \( \frac{(2n+\delta+1)}{2} \) | \( 2n + \delta \) | \( \frac{(2n+\delta)}{2} \) |
| \( \mathcal{G}_n \) | \( n^2 + 2n \) | \( 2n \) | \( n^2 \) |
| \( \mathcal{H}_n \) | \( \frac{(n+1)}{2} + 2n \) | \( 2n \) | \( \frac{(n+1)}{2} \) |
We now recall the ordered sets \( e \) and \( f \) of [7, Section 4.1] in the context of the Lie lattice \( \Lambda \in \{ \mathcal{F}_{n,\delta}, \mathcal{G}_n, \mathcal{H}_n \} \). Define the ordered set \( e = (x_1, \ldots, x_n) \) with \( x_i > x_{i+1} \), for each \( i \in [a-1] \), where the \( x_i \) are the elements appearing in the presentation of \( \Lambda \) of Definition 1.3. Then \( \mathfrak{G} = (\mathfrak{G}_1, \ldots, \mathfrak{G}_n) \) is an \( e \)-basis of \( \mathfrak{g} / \mathfrak{z} \), where \( \mathfrak{z} \) denotes the natural surjection \( g \rightarrow \mathfrak{g} / \mathfrak{z} \).

Consider the sets
\[
D_\Lambda = \begin{cases} \{(i, j) \in [2n + \delta]^2 \mid 1 \leq i < j \leq 2n + \delta\}, & \text{if } \Lambda = \mathcal{F}_{n,\delta}, \\ \{i, j\} \in [n]^2, & \text{if } \Lambda = \mathcal{G}_n, \\ \{(i, j) \in [n]^2 \mid 1 \leq i \leq j \leq n\}, & \text{if } \Lambda = \mathcal{H}_n. 
\end{cases}
\]

Let \( y_{ij} \) be the elements appearing in the relations of the presentation of \( \Lambda \) of Definition 1.3. We order the set \( \mathfrak{G} = \{y_{ij} \mid (i, j) \in D_\Lambda\} \), by setting \( y_{ij} > y_{kl} \), whenever either \( i < k \) or \( i = k \) and \( j < l \). We then write \( \mathfrak{G} = (y_{ij})_{(i, j) \in D_\Lambda} = (f_1, \ldots, f_b) \) so that \( f_1 > \cdots > f_b \).

For \( i, j \in [a] \) and \( k \in [b] \), let \( \lambda_{ij}^k \in \mathfrak{g} \) be the structure constants satisfying
\[
[x_i, x_j] = \sum_{k=1}^b \lambda_{ij}^k f_k.
\]

**Definition 2.4.** [8, Definition 2.1] The \( A \)-commutator and the \( B \)-commutator matrices of \( \mathfrak{g} \) with respect to \( e \) and \( f \) are, respectively,
\[
A_\Lambda(X_1, \ldots, X_n) = \left( \sum_{j=1}^a \lambda_{ij}^k X_j \right)_{ik} \in \text{Mat}_a \times b(\mathfrak{g}[X]), \quad \text{and}
\]
\[
B_\Lambda(Y_1, \ldots, Y_b) = \left( \sum_{k=1}^b \lambda_{ij}^k Y_k \right)_{ij} \in \text{Mat}_a \times a(\mathfrak{g}[Y]),
\]
where \( X = (X_1, \ldots, X_a) \) and \( Y = (Y_1, \ldots, Y_b) \) are independent variables.

The \( B \)-commutator matrices of \( \Lambda \) are the following:

- \( B_{\mathcal{F}_{n,\delta}}(Y) \) is the generic antisymmetric \( (2n + \delta) \times (2n + \delta) \)-antisymmetric matrix in the variables \( Y = (Y_1, \ldots, Y_b) \),
- \( B_{\mathcal{G}_n}(Y) = \begin{bmatrix} 0 & M(Y) \\ -M(Y)^\text{trans} & 0 \end{bmatrix} \), where \( M(Y) \) is the generic \( n \times n \)-matrix in the variables \( Y = (Y_1, \ldots, Y_b) \) and \( M(Y)^\text{trans} \) is its transpose,
- \( B_{\mathcal{H}_n}(Y) = \begin{bmatrix} 0 & S(Y) \\ -S(Y) & 0 \end{bmatrix} \), where \( S(Y) \) is the generic symmetric \( n \times n \)-matrix in the variables \( Y = (Y_1, \ldots, Y_b) \).

The precise form of the \( A \)-commutator matrix of each \( \Lambda \in \{ \mathcal{F}_{n,\delta}, \mathcal{G}_n, \mathcal{H}_n \} \) is given in Section 3.1.

**Proposition 2.5.** [7, Proposition 4.8] The bivariate zeta functions of \( \mathcal{G}(\mathfrak{g}) \) can be described by
\[
Z_{\mathcal{G}(\mathfrak{g})}(s, t) = \frac{1}{1 - q^{-s} - q^{-t}} (1 + (1 - q^{-1})^{-1}),
\]
\[
\int_{(w,y) \in \mathfrak{p} \times \mathfrak{w}^\vee} \prod_{k=1}^{u_{\mathcal{G}(\mathfrak{g})} \mathfrak{u} + u_{\mathcal{G}(\mathfrak{g})} \mathfrak{u}} \frac{\|F_{2k}(B_\Lambda(y)) \cup w^2 F_{2(k-1)}(B_\Lambda(y))\|_{\mathfrak{p}}^{\frac{s-1}{2}}}{\|F_{2(k-1)}(B_\Lambda(y))\|_{\mathfrak{p}}^{\frac{s-1}{2}}} d\mu,
\]
\[
9
\]
Proof. For $\Lambda = n,\delta$, when comparing the $p$-adic integral $[11, (4.3)]$ with (2.4), we see that $\zeta_{F_{n,\delta}(a)}(s) = \zeta_{F_{n,\delta}(a)}(q^{-s}+\frac{1}{2})$, and hence $[11, Proposition 5.11]$ shows (1.4).

Remark 2.6. The formula (1.4) for the class number zeta function of $F_{n,\delta}(a)$ is a consequence of $[11, Proposition 5.11]$. This proposition gives a formula for the ask zeta function $\zeta_{F_{n,\delta}(a)}(s)$ of the orthogonal Lie algebra $so_{a}(\theta)$, $d \in \mathbb{N}$; see the definition of this function in $[11, Definition 1.3]$. Since $\{B_{F_{n,\delta}(a)}(z) \mid z \in \mathfrak{o}^{b}\} = so_{a}(\theta)$, when comparing the $p$-adic integral $[11, (4.3)]$ with (2.4), we see that $\zeta_{F_{n,\delta}(a)}(s) = \zeta_{F_{n,\delta}(a)}(q^{-s}+\frac{1}{2})$, and hence $[11, Proposition 5.11]$ shows (1.4).

3. Bivariate conjugacy class zeta functions—proof of Theorem [1.4]

In the following, we provide separate formulae for the local factors of the conjugacy class zeta functions of each type of $G \in \{F_{n,\delta}, G_{n}, H_{n}\}$ by calculating explicitly the corresponding integrals given by expression [2.2] of Proposition [2.5] We first describe the $A$-commutator matrices in each type of $G$.

3.1. $A$-commutator matrices. We describe the $A$-commutator matrix of $g$, given with respect to the ordered bases $e = (x_{1}, \ldots, x_{a})$ and $f = (y_{i})_{i \in D_{A} = \{f_{1}, \ldots, f_{n}\}}$ of Section [2.2]

Lemma 3.1. Let $\omega \Lambda : D_{A} \rightarrow \mathfrak{b}$ denote the map satisfying $y_{ij} = f_{\omega(i,j)}$. Then

$$
\omega \Lambda(i,j) = \begin{cases} 
(i-1)a - \left(\frac{i-1}{2}\right) + j - i, & \text{if } \Lambda = F_{n,\delta}, \\
(i-1)n + j, & \text{if } \Lambda = G_{n}, \\
(i-1)n - \left(\frac{1}{2}\right) + j, & \text{if } \Lambda = H_{n}.
\end{cases}
$$

Proof. For $\Lambda = F_{n,\delta}$, the ordering of the $y_{ij}$ is described by the following identities.

$$
\omega_{F_{n,\delta}}(1,j) = j - 1, \quad \text{for all } j \in \{2, \ldots, a\},
$$

$$
\omega_{F_{n,\delta}}(i+1, i+2) = \omega_{F_{n,\delta}}(i, a) + 1, \quad \text{for all } i \in \{a-2\},
$$

and

$$
\omega_{F_{n,\delta}}(i,j) = \omega_{F_{n,\delta}}(i, i+1) + j - (i+1), \quad \text{for all } i \in \{a-1\}, j \in \{i+1, \ldots, a\}.
$$

It follows by induction that $\omega_{F_{n,\delta}}(i,j) = (i-1)a - \left(\frac{i-1}{2}\right) + j - i$.

The other cases follow from similar arguments.
Let $X = (X_1, \ldots, X_n)$ be a vector of variables and, for $m \in [a]$, set

$$C_{\Lambda,m} = \{\omega_{\Lambda}(m,j) \mid (m,j) \in D_{\Lambda}\}.$$ 

We want to determine the submatrix $A^{(m)}_{\Lambda}(X)$ of $A_{\Lambda}(X)$ composed by the columns of index in $C_{\Lambda,m}$ so that

$$A_{\mathcal{F}_{n,\delta}}(X) = \begin{bmatrix} A^{(1)}_{\mathcal{F}_{n,\delta}}(X) & A^{(2)}_{\mathcal{F}_{n,\delta}}(X) & \cdots & A^{(a-1)}_{\mathcal{F}_{n,\delta}}(X) \end{bmatrix},$$

$$A_{\Lambda}(X) = \begin{bmatrix} A^{(1)}_{\Lambda}(X) & A^{(2)}_{\Lambda}(X) & \cdots & A^{(a)}_{\Lambda}(X) \end{bmatrix},$$

for $\Lambda \in \{\mathcal{G}_n, \mathcal{H}_n\}$. For $\Lambda \in \{\mathcal{F}_{n,\delta}, \mathcal{G}_n, \mathcal{H}_n\}$, the matrices $A_{\Lambda}(X)$ all have size $a \times b$. Denote

$$\nu_{\Lambda,m} = \begin{cases} (m-1)a - \binom{m-1}{2}, & \text{if } \Lambda = \mathcal{F}_{n,\delta} \\ (m-1)n, & \text{if } \Lambda = \mathcal{G}_n \\ (m-1)n - \binom{m}{2} + m - 1, & \text{if } \Lambda = \mathcal{H}_n, \end{cases}$$

so that $C_{\Lambda,m} = \{\nu_{\Lambda,m} + 1, \ldots, \nu_{\Lambda,m} + k_{\Lambda,m}\}$, where

$$k_{\Lambda,m} = \begin{cases} a - m, & \text{if } \Lambda = \mathcal{F}_{n,\delta}, \\ n, & \text{if } \Lambda = \mathcal{G}_n, \\ n - m + 1, & \text{if } \Lambda = \mathcal{H}_n, \end{cases}$$

that is, the $j$th column of $A^{(m)}_{\Lambda}(X)$ is the $(\nu_{\Lambda,m} + j)$th column of $A_{\Lambda}(X)$.

The relations of $\Lambda$ show that, for $(i,j) \in D_{\Lambda}$ and for $k \in C_{\Lambda,m}$, the structure constants involving $(i,j)$ are the ones in the following table:

| $\Lambda$         | structure constants involving $(i,j)$                                                                 |
|-------------------|--------------------------------------------------------------------------------------------------------|
| $\mathcal{F}_{n,\delta}$ | $\lambda^k_{ij} = \begin{cases} 1, & \text{if } k = \omega_{\mathcal{F}_{n,\delta}}(i,j), \\ 0, & \text{otherwise}, \end{cases}$ |
| $\mathcal{G}_n$ | $\lambda^k_{i(n+j)} = \begin{cases} 1, & \text{if } k = \omega_{\mathcal{G}_n}(i,j), \\ 0, & \text{otherwise}, \end{cases}$ |
| $\mathcal{H}_n$ | $\lambda^k_{i(n+j)} = \lambda^k_{j(n+i)} = \begin{cases} 1, & \text{if } k = \omega_{\mathcal{H}_n}(i,j), \\ 0, & \text{otherwise}. \end{cases}$ |

Since $\omega_{\Lambda}(i,j) \in C_{\Lambda,i}$, for each $(i,j) \in D_{\Lambda}$, it is clear that the indices $k \in C_{\Lambda,m}$ of the columns of $A^{(m)}_{\Lambda}(X)$ cannot equal $\omega_{\Lambda}(i,j)$ if $i \neq m$. In particular, $\lambda^k_{ij} = 0$ if $i,j \neq m$. Every $k \in C_{\Lambda,m}$ is of the form $k = \nu_{\Lambda,m} + l$, for some $l \in [k_{\Lambda,m}]$. Recall that the $(i,l)$th entry of $A^{(m)}_{\Lambda}(X)$ is the $(i,\nu_{\Lambda,m} + l)$th entry of $A_{\Lambda}(X)$, that is,

$$A^{(m)}_{\Lambda}(X) = A_{\Lambda}(X)_{i(\nu_{\Lambda,m} + l)},$$

3.1.1. $A$-commutator matrices of groups of type $F$. For $\Lambda = \mathcal{F}_{n,\delta}$, the index $k$ coincides with $\omega_{\mathcal{F}_{n,\delta}}(m,j) = \nu_{\mathcal{F}_{n,\delta},m} + j - m$ if and only if $j = l + m$. It follows that $\lambda^k_{ij} = 1$ if and only if $i = m$ and $j = m + l$. Hence the $(m,l)$th entry of $A^{(m)}_{\mathcal{F}_{n,\delta}}(X)$ is

$$A^{(m)}_{\mathcal{F}_{n,\delta}}(X)_{ml} = \sum_{j=1}^{a} \nu_{\mathcal{F}_{n,\delta},m} + l X_j = X_{m+l},$$

and, for $i \neq m$, its $(i,l)$th entry is

$$A^{(m)}_{\mathcal{F}_{n,\delta}}(X)_{il} = -\sum_{j=1}^{a} \nu_{\mathcal{F}_{n,\delta},i} + m + l X_j = \begin{cases} -X_m, & \text{if } i = m + l, \\ 0, & \text{otherwise}. \end{cases}$$
Given \( s, r \in \mathbb{N} \), let \( \mathbf{0}_{s \times r} \) denote the \((s \times r)\)-zero matrix and let \( \mathbf{1}_s \) denote the \((s \times s)\)-identity matrix, both over \( \mathfrak{g}[X] \). It follows that, for each \( m \in [a-1] \),

\[
A^{(m)}_{\mathcal{K}_n, \delta}(X) = \begin{bmatrix}
0_{(m-1) \times (2n+\delta-m)} & \\
X_{m+1} & X_{m+2} & \ldots & X_{2n+\delta} \\
-X_m & \mathbf{1}_{(2n+\delta-m)}
\end{bmatrix} \in \text{Mat}_{(2n+\delta) \times (2n+\delta-m)}(\mathfrak{g}[X]).
\]

### 3.1.2. A-commutator matrices of groups of type \( G \)

For \( \Lambda = \mathcal{G}_n \), the index \( k \) coincides with \( \omega_{\mathcal{G}_n}(m,j) = \nu_{\mathcal{G}_n,m} + j \) if and only if \( j = l \). It follows that \( \lambda^{k}_{(n+j)} = 1 \) if and only if \( i = m \) and \( j = l \). Hence the \((m,l)\)th entry of \( A^{(m)}_{\mathcal{G}_n}(X) \) is

\[
A^{(m)}_{\mathcal{G}_n}(X)_{ml} = \sum_{j=1}^{a} \lambda^{\nu_{\mathcal{G}_n,m}+l}_{mj} X_j = X_{n+l},
\]

and, for \( i \neq m \), its \((i,l)\)th entry is

\[
A^{(m)}_{\mathcal{G}_n}(X)_{il} = -\sum_{j=1}^{a} \lambda^{\nu_{\mathcal{G}_n,m}+l}_{ji} X_j = \begin{cases} 
-X_m, & \text{if } i = n + l, \\
0, & \text{otherwise.}
\end{cases}
\]

Hence, for each \( m \in [n] \),

\[
(3.1) \quad A^{(m)}_{\mathcal{G}_n}(X) = \begin{bmatrix}
0_{(m-1) \times n} & \\
X_{n+1} & X_{n+2} & \ldots & X_{2n} \\
0_{(n-m) \times n} & \\
-X_m & \mathbf{1}_n
\end{bmatrix} \in \text{Mat}_{2n \times n}(\mathfrak{g}[X]).
\]

### 3.1.3. A-commutator matrices of groups of type \( H \)

For \( \Lambda = \mathcal{H}_n \), the index \( k \) coincides with \( \omega_{\mathcal{H}_n}(m,j) = \nu_{\mathcal{H}_n,m} + j - m + 1 \) if and only if \( j = m + l - 1 \). It follows that \( \lambda^{k}_{(n+j)} = \lambda^{k}_{(n+l+i)} = 1 \) if and only if either \( i = m \) and \( j = m + l - 1 \) or \( j = m \) and \( i = m + l - 1 \). Therefore

\[
A^{(m)}_{\mathcal{H}_n}(X)_{ml} = \sum_{j=1}^{a} \lambda^{\nu_{\mathcal{H}_n,m}+l}_{mj} X_j = X_{n+m+l-1},
\]

\[
A^{(m)}_{\mathcal{H}_n}(X)_{(n+m)l} = -\sum_{j=1}^{a} \lambda^{\nu_{\mathcal{H}_n,m}+l}_{j(n+m)} X_j = -X_{m+l-1}.
\]

For \( i \in [n] \setminus \{m\} \), the \((i,l)\)th entry of \( A^{(m)}_{\mathcal{H}_n}(X) \) is

\[
A^{(m)}_{\mathcal{H}_n}(X)_{il} = \sum_{j=1}^{n} \lambda^{\nu_{\mathcal{H}_n,m}+l}_{j(n+i)} X_{n+j} = \begin{cases} 
X_{n+m}, & \text{if } i = m + l - 1, \\
0, & \text{otherwise.}
\end{cases}
\]

For \( i = n + t \) with \( t \in [n] \setminus \{m\} \), the \((i,l)\)th entry of \( A^{(m)}_{\mathcal{H}_n}(X) \) is

\[
A^{(m)}_{\mathcal{H}_n}(X)_{il} = -\sum_{j=1}^{n} \lambda^{\nu_{\mathcal{H}_n,m}+l}_{j(n+t)} X_j = \begin{cases} 
-X_m, & \text{if } t = m + l - 1, \\
0, & \text{otherwise.}
\end{cases}
\]

12
Hence, for each \( m \in [n] \),

\[
A^{(m)}_{\mathcal{H}_n}(X) = \begin{bmatrix}
X_{n+m} & X_{n+m+1} & \cdots & X_{2n} \\
X_{n+m} & & & \\
\ddots & & & \\
X_{n+m} & & & \\
0_{(m-1)\times(n-m+1)} & & & \\
\end{bmatrix} \in \text{Mat}_{2n \times (n-m+1)}(\mathcal{O}[X]).
\]

**Example 3.2.** The following examples illustrate the form of the commutator matrix for each type of group scheme.

\[
A_{F_2}(X) = \begin{bmatrix}
X_2 & X_3 & X_4 \\
-X_1 & X_3 & X_4 \\
-X_1 & -X_2 & X_3 \\
X_4 & X_5 & X_6 \\
-X_1 & -X_2 & -X_3 \\
\end{bmatrix},
\]

\[
A_{G_3}(X) = \begin{bmatrix}
X_4 & X_5 & X_6 \\
-X_1 & -X_2 & -X_3 \\
-X_1 & -X_2 & -X_3 \\
X_4 & X_5 & X_6 \\
-X_1 & -X_2 & -X_3 \\
\end{bmatrix},
\]

\[
A_{H_3}(X) = \begin{bmatrix}
X_4 & X_5 & X_6 \\
-X_1 & -X_2 & -X_3 \\
-X_1 & -X_2 & -X_3 \\
X_4 & X_5 & X_6 \\
-X_1 & -X_2 & -X_3 \\
\end{bmatrix},
\]

where the omitted entries equal zero.

It is not difficult to see that \( A_{\Lambda}(X) \) has rank \( a-1 \) in all cases, that is, \( u_{A_{\Lambda}} = a-1 \).

We now proceed to a detailed analysis of the \( A \)-commutator matrix in each individual type.

### 3.2. Conjugacy class zeta functions of groups of type \( F \).

**Lemma 3.3.** For \( w \in \mathfrak{p} \) and \( x \in W^\phi_a \), that is, for \( x \in \mathfrak{o}^a \) such that \( v_p(x) = 0 \),

\[
\frac{\|F_k(A_{F_n,s}(x)) \cup wF_{k-1}(A_{F_n,s}(x))\|_p}{\|F_{k-1}(A_{F_n,s}(x))\|_p} = 1, \text{ for all } k \in [a-1].
\]
Proof. The columns of $A_{\mathcal{F}_{n,\delta}}(X)$ are of the form

$$k\text{th row } \{ \begin{bmatrix} X_j \\ -X_k \end{bmatrix}, \text{ for each } j,k \in [a], \tag{3.4}$$

where the nondisplayed entries equal 0.

For each $i \in [a]$, consider the $(a \times (a-1))$-submatrix $K_i(X)$ of $A_{\mathcal{F}_{n,\delta}}(X)$ composed of the columns of expression (3.4) in the following order: the first $i-1$ columns are the ones with $j = i$ and $k \in [i-1]$ being chosen in the increasing order, then the next $a-i$ columns are the ones with $k = i$ and $j \in \{i+1, \ldots, a\}$. The matrix $K_i(X)$ is the matrix with diagonal given by $X_i$ in the first $i-1$ entries and $-X_i$ in the remaining diagonal entries, and the other nontrivial entries are the ones of row $i$, which is given by

$$(-X_1, \ldots, -X_{i-1}, \underbrace{-X_i}_{\text{diagonal term}}, X_{i+1}, \ldots, X_a).$$

Given $x \in W^\delta_n$, it is clear that, for at least one $i_0 \in [a]$, the matrix $K_{i_0}(x)$ has rank $a - 1$. That is, for each $k \in [a-1]$, there exists a $(k \times k)$-minor of $K_{i_0}(x)$ which is a unit. Since the $(k \times k)$-minors of $K_{i_0}(x)$ are elements of $F_k(A_{\mathcal{F}_{n,\delta}}(X))$, expression (3.3) follows.

Lemma 3.3 applied to equality (2.2), and Proposition 2.1 yield

$$\mathcal{Z}_{\mathcal{F}_{n,\delta}(\sigma)}(s_1, s_2) = \frac{1}{1 - q^{(2n+\delta-1) - s_2}} \left( 1 + (1 - q^{1-1})^{-1} \int_{(w, z) \in \mathbb{P} \times W^{2n+\delta}_n} |w|^{(2n+\delta-1)s_1 + s_2 - \left(\frac{2n+\delta}{2}\right) - 2} d\mu \right) \left( 1 - q^{\left(\frac{2n+\delta}{2}\right) - (2n+\delta-1)s_1 - s_2} \right) / \left( 1 - q^{2n+\delta - 2s_2} \right),$$

proving Theorem 1.4 for groups of type $F$.

Remark 3.4. The formula (1.4) reflects the $K$-minimality of $\Lambda = \mathcal{F}_{n,\delta}$; see [11] Lemma 6.2 and Definition 6.3. In fact, the proof of Lemma 3.3 shows in particular that, for $G = \mathcal{F}_{n,\delta}$,

$$\frac{\| F_k(A_{\mathcal{F}_{n,\delta}}(x)) \cup yF_{k-1}(A_{\mathcal{F}_{n,\delta}}(x)) \|_{p}}{\| F_{k-1}(A_{\mathcal{F}_{n,\delta}}(x)) \|_{p}} = \| x, y \|_{p}.$$

The formula for the local factors of the class number zeta function of $F_{n,\delta}(\mathcal{O})$ given in Corollary 1.5 coincides with the formula for the class number zeta function of $\mathcal{F}_{n,\delta}(\sigma)$ given by the specialisation of the formula given in [11] Proposition 6.4 to the corresponding class number zeta function; see [7] Remark 4.10 and Lemma 4.11.

3.3. Conjugacy class zeta functions of groups of type $G$. We first describe the determinant of a square matrix in terms of its 2 × 2-minors, which will be used to describe the minors of $A_{\mathcal{F}_{n}}(X)$. For a matrix $M = (m_{ij})$, denote $M((i,j),(r,s)) = \begin{vmatrix} m_{ij} & m_{is} \\ m_{rj} & m_{rs} \end{vmatrix}$. 

Lemma 3.5. Given \( t \in \mathbb{N} \), let \( G = (g_{ij})_{1 \leq i, j \leq 2t} \) and \( U = (u_{ij})_{1 \leq i, j \leq 2t+1} \) be matrices with \( g_{ij} = g(X)_{ij} \), \( u_{ij} = u(X)_{ij} \in \mathfrak{o}[X] \). Let \( I = \{i_1, \ldots, i_t\} \), \( J = \{j_1, \ldots, j_t\} \subseteq [2t] \). Then, for suitable \( \alpha_{ij}, \beta_{ij} \in \{-1, 1\} \),

\[
\det(G) = \sum_{k,j=[2t]} \alpha_{k,j} \tilde{G}_{(1,i_1), (2,j_2)} \tilde{G}_{(3,i_2), (4,j_2)} \cdots \tilde{G}_{(2t-1,i_t), (2t,j_t)},
\]

\[
\det(U) = \sum_{i=1}^{2t+1} \sum_{k,j=[2t+1]\{i\}} \beta_{k,j} u_{i_1} \tilde{U}_{(1,i_1), (2,j_2)} \tilde{U}_{(3,i_2), (4,j_2)} \cdots \tilde{U}_{(2t-1,i_t), (2t,j_t)}.
\]

Proof. Given two subsets \( I, J \subseteq [2t] \) of equal cardinality \( m \), denote by \( \tilde{G}_{I,J} \) the determinant of the \((2t-m) \times (2t-m)\)-submatrix of \( G \) obtained by excluding the rows of indices in \( I \) and columns of index in \( J \). The entries of the submatrix \( G_{(1), (k)} = (\tilde{g}_{ij})_{ij} \) obtained from \( G \) by excluding its first row and its \( k \)th column are given by

\[
\tilde{g}_{ij} = \begin{cases} 
  g_{(i+1)j}, & \text{if } j \in [k-1], \\
  g_{(i+1)(j+1)}, & \text{if } j \in \{k, \ldots, 2t-1\}.
\end{cases}
\]

Consequently,

\[
\tilde{G}_{(1), (k)} = \sum_{j=1}^{k-1} (-1)^{1+j} g_{2j} \tilde{G}_{(1), (j,k)} + \sum_{j=k}^{2t-1} (-1)^{1+j} g_{2(j+1)} \tilde{G}_{(1), (k,j+1)}.
\]

It follows that

\[
\det(G) = \sum_{k=1}^{2t} (-1)^{1+k} g_{1k} \tilde{G}_{(1), (k)}
\]

\[
= \sum_{k=1}^{2t} \left( \sum_{j=1}^{k-1} (-1)^{1+j} g_{1k} g_{2j} \tilde{G}_{(1), (j,k)} - \sum_{j=k+1}^{2t} (-1)^{1+j} g_{1k} g_{2j} \tilde{G}_{(1), (j,k)} \right)
\]

\[
= \sum_{m=1}^{2t-1} \sum_{i=m}^{2t} \sum_{i=m+1}^{2t} (-1)^{i+m-1} (g_{1m} g_{2i} - g_{1i} g_{2m}) \tilde{G}_{(1), (m, i)}
\]

\[
= \sum_{m=1}^{2t-1} \sum_{i=m}^{2t} \sum_{m+1}^{2t} (-1)^{i+m-1} \tilde{G}_{(1), (m), (i)} \tilde{G}_{(1), (m, i)}.
\]

The relevant claim of Lemma 3.5 for the matrix \( G \) follows by induction on \( t \).

The claim for the matrix \( U \) follows by the first part, since its determinant is

\[
\det(U) = \sum_{i=1}^{2t+1} (-1)^{i+1} u_{i1} \tilde{U}_{(1), (i)}.
\]

Lemma 3.6. For each \( r \in [2n] \), the nonzero elements of \( F_r(A_{G_2}(X)) \) are either of one of the following forms or a sum of these terms.

\[
X_{i_1} \cdots X_{i_r} X_{n+1} \cdots X_{n+j_1} \cdots X_{n+j_2} \text{ or } -X_{i_1} \cdots X_{i_r} X_{n+1} \cdots X_{n+j_1} \cdots X_{n+j_2}.
\]

Proof. Lemma 3.5 describes each element of \( F_r(A_{G_2}(X)) \) in terms of sums of products of \((2 \times 2)\)-minors of \( A_{G_2}(X) \). It then suffices to show that these minors are all either 0 or of the forms \( X_i X_j \) or \(-X_i X_j\), for some \( i, j \in [2n] \). This can be seen from the description of \( A_{G_2}(X) \) in terms of the blocks (4.1).

The main idea of the proof of Theorem 1.4 for groups of type \( G \) is showing the following proposition.
Proposition 3.7. Let $\vec{X} = (X_1, \ldots, X_{2n})$ be a vector of variables. Given $\lambda, \omega \in \{n\}_{0}$ such that $0 < \omega + \lambda \leq 2n - 1$, for all choices of $i_1, \ldots, i_\omega, j_1, \ldots, j_\lambda \in \{n\}$, one of

$$X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda} \quad \text{or} \quad -X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda}$$

is an element of $F_{\omega+\lambda}(A_{\mathcal{G}_n}(\vec{X}))$.

In fact, for $x, y \in \mathfrak{a}$, \(\min\{v_p(x + y), v_p(x), v_p(y)\} = \min\{v_p(x), v_p(y)\}\). Thus, if some term of the form

$$X_{i_1} X_{i_2} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda} - X_{k_1} X_{k_2} \cdots X_{k_\lambda} X_{n+l_1} \cdots X_{n+l_\lambda}$$

is a minor of the commutator matrix $A_{\mathcal{G}_n}(\vec{X})$, then, assuming the claim in Proposition 3.7 holds, both

$$X_{i_1} X_{i_2} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda} \quad \text{and} \quad X_{k_1} X_{k_2} \cdots X_{k_\lambda} X_{n+l_1} \cdots X_{n+l_\lambda}$$

are minors of this commutator matrix (up to sign), and hence, when considering these three terms, only the last two will be relevant in order to determine $\|F_{r}(A_{\mathcal{G}_n}(\vec{X}))\|_p$. In this case, we may then assume that all elements are of the form given in Proposition 3.7 while computing $\|F_{r}(A_{\mathcal{G}_n}(\vec{X}))\|_p$ and $\|F_{r}(A_{\mathcal{G}_n}(\vec{X}))\|_p$.

Firstly we show that Proposition 3.7 holds if $\{i_1, \ldots, i_\omega\}, \{j_1, \ldots, j_\lambda\} \neq \emptyset$.

Lemma 3.8. Let $\omega, \lambda \in \{n\}_{0}$ not both zero and not both $n$. Given $i_1, \ldots, i_\omega, j_1, \ldots, j_\lambda \in \{n\}$ such that $\{i_1, \ldots, i_\omega\}, \{j_1, \ldots, j_\lambda\} < n$, either

$$X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda} \quad \text{or} \quad -X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda}$$

is an element of $F_{\omega+\lambda}(A_{\mathcal{G}_n}(\vec{X}))$.

Proof. For each $(i, j) = (i_1, \ldots, i_\omega, j_1, \ldots, j_\lambda)$ as in the assumption of Lemma 3.8, we construct explicitly a submatrix of $A_{\mathcal{G}_n}(\vec{X})$ which is, up to reordering of rows and columns, of the form

\[
\begin{bmatrix}
X_{n+j_1} & \cdots & X_{n+j_\lambda} \\
\vdots & & \vdots \\
W(\vec{X}) & -X_{i_1} & \cdots & -X_{i_\omega}
\end{bmatrix}
\]

where $T(\vec{X}) = (t(\vec{X})_{ij})$ and $W(\vec{X}) = (w(\vec{X})_{ij})$ are such that $t(\vec{X})_{ij} = 0$ and $w(\vec{X})_{ij} = 0$, if $i \leq j$. It is clear that the determinant of this matrix is one of $\pm X_{i_1} \cdots X_{i_\omega} X_{n+j_1} \cdots X_{n+j_\lambda}$.

The main fact we use is that the columns of $A_{\mathcal{G}_n}(\vec{X})$ are of the form

\[
\begin{bmatrix}
X_{n+j} \\
\vdots \\
-X_i
\end{bmatrix}
\]

where the nondisplayed terms equal zero. For each $i, j \in \{n\}$, there is exactly one column of $A_{\mathcal{G}_n}(\vec{X})$ with $X_{n+j}$ in the $i$th row, and exactly one column with $-X_j$ in the $(n + i)$th row.

Fix $l_1 \in \{n\} \setminus \{i_1, \ldots, i_\omega\}$ and let $c_1$ denote the unique column of $A_{\mathcal{G}_n}(\vec{X})$ with $X_{n+j_1}$ in the $l_1$th row. Inductively, fix $l_k \in \{n\} \setminus \{l_1, \ldots, l_{k-1}, i_k, \ldots, i_\omega\}$, for each $k \in \{\lambda\}$, and let $c_k$ be the unique column of $A_{\mathcal{G}_n}(\vec{X})$ with $X_{n+j_k}$ in the $l_k$th row.
 Analogously, fix $m_1 \in [n] \setminus \{j_1, \ldots, j_\lambda\}$ and let $C_1$ be the index of the unique column of $\mathbb{A}_{\mathcal{G}_n}(X)$ with $-X_{i_k}$ in the $(n + m_1)$th row, and, inductively, fix $m_q \in [n] \setminus \{m_1, \ldots, m_{q-1}, j_q, \ldots, j_\lambda\}$, for each $q \in [\omega]$, and let $C_q$ be the index of the unique column of $\mathbb{A}_{\mathcal{G}_n}(X)$ with $-X_{i_q}$ in the $(n + m_q)$th row.

From expression (3.7), one sees that the columns $c_k$ and $C_q$ are given by

$$
\begin{align*}
&l_k \text{th row } \begin{bmatrix} X_{n+j_k} \end{bmatrix} \quad i_q \text{th row } \begin{bmatrix} X_{n+m_q} \end{bmatrix}, \\
&(n + j_k) \text{ th row } \begin{bmatrix} -X_{i_k} \end{bmatrix} \quad (n + m_q) \text{ th row } \begin{bmatrix} -X_{i_q} \end{bmatrix}.
\end{align*}
$$

By construction, the indices $c_k$ are all distinct, and so are the indices $C_q$. If $c_k = C_q$ for some $k \in [\lambda]$ and some $q \in [\omega]$, then we would obtain $l_k = i_q$. Analogously, the indices $l_1, \ldots, l_\lambda, n + m_1, \ldots, n + m_\omega$ are all distinct.

Consider the matrix $M_{(i,j)}(X)$ composed of columns $c_k$ and $C_q$ and of rows $l_k$ and $n + m_q$, for $k \in [\lambda]$ and $q \in [\omega]$. This matrix is of the form of (3.5) for some matrices $T(X) \in \text{Mat}_{\lambda \times \omega}(\sigma[X])$ and $W(X) \in \text{Mat}_{\omega \times \lambda}(\sigma[X])$. Let us show that, in fact, $t(X)_{ij} = 0$ and $w(X)_{ij} = 0$ for $i \neq j$.

The only nonzero entries of $C_q$ are the ones of indices $i_q$ and $n + m_q$. We chose each $l_k$ so that $l_k \notin \{i_1, \ldots, i_\lambda\}$. Since any of the rows $l_1, \ldots, l_\lambda$ is the $i_q$th row of $\mathbb{A}_{\mathcal{G}_n}(X)$, it follows that $t(X)_{ij} = 0$, for all $i \neq j$. Analogously, since the only nonzero entries of $c_k$ are $l_k$ and $n + j_k$ and $m_q \notin \{j_1, \ldots, j_\lambda\}$, it follows that $w(X)_{ik} = 0$, for all $i \neq k$.

**Proof of Proposition 3.7** Lemma 3.8 shows the claim of Proposition 3.7 for all cases, except for $\omega = n$ and $i_1, \ldots, i_n$ all distinct, and for $\lambda = n$ and $j_1, \ldots, j_n$ all distinct. Let us show the last case, the other one is analogous.

Assume that $j_1, \ldots, j_n$ are all distinct and $\omega \in [n-1]_0$. For $k \in [n]$, we can define $l_k$ as in the proof of Lemma 3.8 since $|\{i_1, \ldots, i_\omega\}| < n$. We also set $c_k$ as in the proof of Lemma 3.8. As $|\{j_1, \ldots, j_n\}| = n$, we cannot choose $m_1 \in [n] \setminus \{j_1, \ldots, j_n\}$.

Instead, we consider the rows $n + j_k$, for $k \in [\omega]$. Denote by $C_0$ the column of $\mathbb{A}_{\mathcal{G}_n}(X)$ with $-X_{i_k}$ in the $(n + j_k)$th row. By construction, the indices $C_0$, for $k \in [n]$, are all distinct, and so are the indices $C_q$, for $q \in [\omega]$. The indices $c_k$ and $C_q$ coincide, for some $k \in [n]$ and $q \in [\omega]$, if and only if $i_q = l_k$. It follows that all $c_k$ and $C_q$ are distinct.

Let $M_{(i,j)}(X)$ be the submatrix of $\mathbb{A}_{\mathcal{G}_n}(X)$ composed by columns $c_k$ and $C_q$ and of rows $l_k$ and $n + j_q$, for each $k \in [n]$ and $q \in [\omega]$, where $i = (i_1, \ldots, i_\lambda)$ and $j = (j_1, \ldots, j_\lambda)$.

Then, as in Lemma 3.8, $B(X)$ is of the form (3.5), but the matrix $W(T)$ is such that $w(T)_{ij} = 0$ if $i \neq j$.

In particular, Proposition 3.7 shows that, for each $r \in [2n]$ and each $k \in [n]$, either $X^k_r$ or $-X^k_r$ is an element of $F_k(\mathbb{A}_{\mathcal{G}_n}(X))$. Hence, if $x \in W_{2n}$, then at least one $(k \times k)$-minor of $\mathbb{A}_{\mathcal{G}_n}(x)$ has valuation zero. This gives

$$
\frac{\|F_k(\mathbb{A}_{\mathcal{G}_n}(x)) \cap wF_{k-1}(\mathbb{A}_{\mathcal{G}_n}(x))\|_p}{\|F_{k-1}(\mathbb{A}_{\mathcal{G}_n}(x))\|_p} = 1, \text{ for all } k \in [n].
$$

For $k \in \{n+1, \ldots, 2n-1\}$, the elements of $F_k(\mathbb{A}_{\mathcal{G}_n}(x))$ can be assumed to be of the form

$$
X_{i_1} \cdots X_{i_{\omega}} X_{n+j_1} \cdots X_{n+j_\lambda},
$$

where $\omega, \lambda \in [n]_0$ satisfy $\omega + \lambda = k$, and $i_1, \ldots, i_\omega, j_1, \ldots, j_\lambda \in [n]$. 

17
Given $x \in W_{2n}^\omega$, denote by $M = v_p(x_1, \ldots, x_n)$ and $N = v_p(x_{n+1}, \ldots, x_{2n})$. Then
\[
\left\| \bigcup_{\omega + \lambda = k \atop 0 \leq \omega, \lambda \leq n} \{X_{i_1} \cdots X_{i_\omega}X_{n+j_1} \cdots X_{n+j_\lambda} \mid i_1, \ldots, i_\omega, j_1, \ldots, j_\lambda \in [n]\} \right\|_p = q^{-n \min\{M,N\} - (k-n) \max\{M,N\}}.
\]

Consequently, for $w \in p$,
\begin{equation}
\frac{\|F_k(A_{G_n}(x)) \cup w F_{k-1}(A_{G_n}(x))\|_p}{\|F_{k-1}(A_{G_n}(x))\|_p} = \begin{cases} \|x_1, \ldots, x_n, w\|_p, & \text{if } 0 = M \leq N, \\ \|x_{n+1}, \ldots, x_{2n}, w\|_p, & \text{if } 0 = M \leq N. \end{cases}
\end{equation}

Combining equations (3.8) and (3.9) yields
\begin{equation}
\prod_{k=1}^{2n-1} \frac{\|F_k(A_{G_n}(x)) \cup w F_{k-1}(A_{G_n}(x))\|_p}{\|F_{k-1}(A_{G_n}(x))\|_p} = \begin{cases} \|x_1, \ldots, x_n, w\|_p^{n-1}, & \text{if } 0 = M \leq N, \\ \|x_{n+1}, \ldots, x_{2n}, w\|_p^{n-1}, & \text{if } 0 = N \leq M. \end{cases}
\end{equation}

Consequently, the $p$-adic integral given in expression (2.2) in this case is
\[
\int_{(w, x) \in p \times W_{2n}^\omega} |w(p^{2n-1})s_1 + s_2 - n^2 - 2 |w(p)\|F_k(A_{G_n}(x)) \cup w F_{k-1}(A_{G_n}(x))\|_p^{1-s_1} - 1| d\mu
\]
\[
= 2 \int_{(w, x_1, \ldots, x_{2n}) \in p \times p^n \times W_{2n}^\omega} |w(p^{2n-1})s_1 + s_2 - n^2 - 2| |x_1, \ldots, x_n, w\|_p^{(n-1)(1+s_1)} d\mu
\]
\[
+ \int_{(w, x_1, \ldots, x_{2n}) \in p \times p^n \times W_{2n}^\omega} |w(p^{2n-1})s_1 + s_2 - n^2 - 2| |x_1, \ldots, x_n, w\|_p^{(n-1)(1+s_1)} d\mu
\]
\[
= (1-q^{-n} + 2q^{-1+(n-1)s_1} - q^{n^2 - n s_1 - s_2} - q^{-1} - q^{-2n}q^{-1} - q^{n^2 - n s_1 - s_2} - q^{n^2 - n s_1 - s_2})
\]
\[
(1-q^{-n} + 2q^{-1+(n-1)s_1} - q^{n^2 - n s_1 - s_2} - q^{-1} - q^{-2n}q^{-1} - q^{n^2 - n s_1 - s_2} - q^{n^2 - n s_1 - s_2}),
\]

where the first and the second integrals of the second equality are calculated, respectively, in Proposition 2.2 and Proposition 2.1 Applying this to formula (2.2), we obtain
\[
Z^{cc}_{G_n}(s_1, s_2) = (1-q^{2(s_1-s_2)} - q^{2(s_2)+1-(2n-1)s_1-s_2} + q^{n^2 - n s_1 - s_2}(1-q^{-n})(1-q^{-(n-1)(1+s_1)}))
\]
\[
(1-q^{n^2 - s_2}) (1-q^{n^2 - n s_1 - s_2})(1-q^{n^2 + 1 - (2n-1)s_1-s_2}),
\]
proving Theorem 1.4 for groups of type $G$.

3.4. Conjugacy class zeta functions of groups of type $H$. In this section, we denote by $A(X)_H$ the $(i,j)$th coordinate of the commutator matrix $A_{H_n}(X)$.

By equality (4.2), each column of $A_{H_n}(X)$ is of one of the following forms:

\[
\text{sth row } \begin{bmatrix} X_{n+s} \end{bmatrix}, \quad \text{sth row } \begin{bmatrix} X_{n+r} \end{bmatrix}, \quad \text{rth row } \begin{bmatrix} X_{r+s} \end{bmatrix}, \quad \text{(n+s)th row } \begin{bmatrix} -X_s \end{bmatrix}, \quad \text{(n+r)th row } \begin{bmatrix} -X_r \end{bmatrix},
\]

(3.10) (3.11)
where the nondisplayed entries equal zero. These columns have the following symmetry:

$$A(X)_{(n+i)k} = \begin{cases} -X_j, & \text{if and only if } A(X)_{ik} = X_{n+j}, \\ 0, & \text{if and only if } A(X)_{ik} = 0. \end{cases}$$

(3.12)

For each $s \in [n]$, there is exactly one column of the form (3.11), and the columns of type (3.11) occur exactly once for each pair $s < r$ of elements of $[n]$.

**Lemma 3.9.** For $w \in \mathfrak{p}$, $x \in W_{2n}^{w}$ and $k \in [n]$, $$\frac{\|F_k(A_{H_n}(x)) \cup wF_{k-1}(A_{H_n}(x))\|_{\mathfrak{p}}}{\|F_{k-1}(A_{H_n}(x))\|_{\mathfrak{p}}} = 1.$$ 

*Proof.* Fix $m \in [n]$. For each $q \in [m-1]$, denote by $C_q$ the index of the unique column of $A_{H_n}(X)$ which has $X_{n+m}$ in the $q$th row. Recall that $A_{H_n}^{(m)}(X)$ is the submatrix of $A_{H_n}(X)$ given in (3.2). The submatrix $L_m(X)$ of $A_{H_n}(X)$ composed of columns $C_1, \ldots, C_{m-1}$ and the columns of $A_{H_n}^{(m)}(X)$ and rows $1, \ldots, n$ is

$$L_m(X) = \begin{bmatrix} C_1 & C_2 & \cdots & C_{m-1} & A_{H_n}^{(m)}(X) \\ X_{n+m} & X_{n+1} & \cdots & X_{n+m} & X_{n+m+1} & \cdots & X_{2n} \\ \cdots & X_{n+1} & \cdots & X_{n+m} & \cdots & \cdots & \cdots \\ X_{n+m} & X_{n+1} & \cdots & X_{2n} \end{bmatrix}.$$ 

If $x \in W_{2n}^{w}$, then there exists $m_0 \in [n]$ such that the matrix $L_m(x)$ has maximal rank $n$, that is, for each $k \in [n]$, at least one of the $(k \times k)$-minors of $L_m(x)$ is a unit. Since the $(k \times k)$-minors of $L_m(x)$ are elements of $F_k(A_{H_n}(x))$, the result follows. \hfill \square

In the next lemma, we show that the sets $F_{w+l}(A_{H_n}(X))$, for $l \in [n-1]$, are given in terms of linear combinations of products of $(i, j)$-minors $M_{ij}(X) := X_iX_{n+j} - X_jX_{n+i}$ of the following matrix

$$M(X_1, \ldots, X_{2n}) = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \\ X_{n+1} & X_{n+2} & \cdots & X_{2n} \end{bmatrix} \in \text{Mat}_{2 \times n}(\mathfrak{o}[X_1, \ldots, X_{2n}]).$$

**Lemma 3.10.** Let $k = n + l$, for some $l \in [n-1]$. Then the nonzero elements of $F_k(A_{H_n}(X))$ are sums of terms of the form

$$X_{f_1} \cdots X_{f_s} M_{i_1j_1}(X) \cdots M_{i_rj_r}(X),$$

for $i_1, \ldots, i_r, j_1, \ldots, j_s \in [n]$, and $f_1, \ldots, f_r \in [2n]$, where $r + 2s = k$ and $s \geq l$.

*Proof.* Lemma 3.5 describes each element $G$ of $F_k(A_{H_n}(X))$ in terms of sums of products of minors of the form $\tilde{G}_{(m_1, n_1), (m_2, n_2)}$. It then suffices to show that these minors are all either 0 or of the forms $X_uX_v - X_uX_v$ or $M_{ij}(X)$, for some $u, v \in [2n]$ and $1 \leq i < j \leq n$. Since $k = n + l$, there are at least $l$ pairs of rows of $G$ whose indices in $A_{H_n}(X)$ are of the form $t$ and $n + t$, for some $t \in [n]$. Denote by $\lambda$ the exact number of such pairs of rows occurring in $G$, and assume that, for $m \in \{1, 3, \ldots, 2\lambda - 1\}$, the $m_{th}$
and the \((m+1)\)th rows of \(G\) correspond, respectively, to rows of indices of the form \(t\) and \(n + t\) in \(A_{\mathcal{H}_n}(X)\), for some \(t \in [n]\). In this case,
\[
A(X)_{ij} = 0 \quad \text{if and only if } \quad A(X)_{(i+1)j} = 0,
\]
for all \(i \in \{1, 3, \ldots, 2\lambda - 1\}\) and \(j \in \binom{n+1}{2}\), because of equality (3.12). Therefore, for \(k_1, k_2 \in [b]\) distinct and \(m \in \{1, 3, \ldots, 2\lambda - 1\}\), the minor \(G_{(m,k_1), (m+1,k_2)}\) is either 0 or \(M_{ij}(X)\), for some \(1 \leq i < j \leq n\), as the columns of this minor are either of the form \((0,0)^T\) or \((X_{n+i}, -X_j)^T\), for some \(i \in [n]\).

For \(i, j \in [n]\) distinct, there is at most one column of \(A_{\mathcal{H}_n}(X)\) whose nonzero rows are the ones of indices in \(\{i, j, n+i, n+j\}\), it follows that each of the remaining minors of \(G\) are either equal to 0 or of one of the forms \(X_iX_j\) or \(-X_iX_j\), for some \(i, j \in [2n]\).

Let \(x = (x_1, \ldots, x_{2n}) \in W^2_n\) with \(v_p(x_{f_0}) = 0\), say. Then
\[
(3.13) \quad v_p(x^r_{f_0}M_{i_1j_1}(x) \cdots M_{i_rj_r}(x)) \leq v_p(x_{f_1} \cdots x_{f_r}M_{i_1j_1}(x) \cdots M_{i_rj_r}(x)),
\]
for all \(r, r' \in \mathbb{N}\), \(f_1, \ldots, f_r \in [2n]\) and \(i_1, \ldots, i_r, j_1, \ldots, j_r \in [n]\).

Furthermore, if \(\|\{M_{i_1j_1}(x) \mid 1 \leq i < j \leq n\}\|_p = \|M_{i_0j_0}(x)\|_p\), for some \(i_0, j_0\), then
\[
(3.14) \quad \|\{M_{i_1j_1}(x) \cdots M_{i_rj_r}(x) \mid 1 \leq i_m < j_m \leq n, \ m \in [k]\}\|_p = \|M_{i_0j_0}(x)\|_p.
\]

Lemma 3.10 states that the \(k \times k\)-minors of \(A_{\mathcal{H}_n}(X)\) are of the form
\[
X_{f_1} \cdots X_{f_r}M_{i_1j_1}(X) \cdots M_{i_rj_r}(X),
\]
or sums of such terms, where \(r + 2s = k\) and \(s \geq l\). The maximal value for \(r\) occurs when \(s = l\). Expressions (3.13) and (3.14) then assure that, for \(m \in [n - 1]\) such that \(k = m + 2l\),
\[
v_p(x^m_{f_0}M_{i_0j_0}(x)^l) \leq v_p(x_{f_1} \cdots x_{f_r}M_{i_1j_1}(x) \cdots M_{i_rj_r}(x)),
\]
for all \(s \geq l\) and \(r \in [n]\) satisfying \(r + 2s = k\), and for all \(i_1, i_2, j_1, j_2 \in [n]\), and \(f_1, \ldots, f_m \in [2n]\).

We now show that, for all \(k = n + l\) with \(l \in [n - 1]\), all terms of the form \(X^m_{f_0}M_{i_0j_0}(X)^l\) are elements of \(F_k(A_{\mathcal{H}_n}(X))\), for \(k = m + 2l\). This implies in particular that, for \(x \in W^2_n\) as above, the term \(x^m_{f_0}M_{i_0j_0}(x)^l\) is an element of \(F_k(A_{\mathcal{H}_n}(X))\) and, therefore
\[
\|F_k(A_{\mathcal{H}_n}(X))\|_p = \|x^m_{f_0}M_{i_0j_0}(X)^l\|_p = \|M_{i_0j_0}(x)\|_p = \|\{M_{i_1j_1}(x) \mid 1 \leq i < j \leq n\}\|_p.
\]

Assuming this holds, the \(\text{integrant of the integral } (2.2)\) can be rewritten as
\[
\|F_{n+l}(A_{\mathcal{H}_n}(X)) \cup wF_{n+l-1}(A_{\mathcal{H}_n}(X))\|_p
\]
\[
= \|\{M_{i_1j_1}(x) \mid 1 \leq i < j \leq n\} \cup \{M_{i_2j_2}(x) \mid 1 \leq i < j \leq n\}\|_p
\]
\[
= \|\{M_{i_1j_1}(x) \mid 1 \leq i < j \leq n\} \cup \{w\}\|_p.
\]

Proposition 3.11. Given \(l \in [n - 1]\), let \(k = n + l\) and \(m = n - l\). Then, for all \(f \in [2n]\) and \(1 \leq i < j \leq n\), either \(X^m_fM_{i_1j_1}(x)^l\) or \(-X^m_fM_{i_1j_1}(x)^l\) is an element of \(F_k(A_{\mathcal{H}_n}(X))\).

Proof. Let \(f \in [n]\) and \(1 \leq i < j \leq n\). We show that, up to sign, both \(X^m_fM_{i_1j_1}(x)^l\) and \(X^m_{f+1}M_{i_1j_1}(x)^l\) lie in \(F_k(A_{\mathcal{H}_n}(X))\). Firstly, we show that \(X^m_{f+1}M_{i_1j_1}(x)^l \in F_k(A_{\mathcal{H}_n}(X))\). We consider the cases \(m \geq 3\), \(m = 2\) and \(m = 1\) separately. In most cases, we do the following: we choose specific
indices \( r_1, \ldots, r_m, R_1, \ldots, R_l \) of rows of \( A_{H_n}(X) \), and then denote by \( c_s \) the index of the unique column of \( A_{H_n}(X) \) having \( X_{n+f} \) in the \( r_s \),th row, by \( C_q \) the index of the unique column having \( X_{n+j} \) in the \( R_q \),th row, and by \( C_q \) the index of the unique column having \( X_{n+j} \) in the \( R_q \),th row. The choices of \( r_s \) and \( R_q \) are made such that the submatrix \( \tilde{A}(X) \) of \( A_{H_n}(X) \) obtained by its rows of indices \( r_1, \ldots, r_m, R_1, n+R_1, \ldots, R_l, n+R_l \) and columns \( c_1, \ldots, c_m, C_1, C_2, \ldots, C_l, C'_l \), in this order, is of the form

\[
\begin{bmatrix}
X_{n+f} & X_{n+r_2} & \cdots & X_{n+r_m} \\
& \ddots & & \\
& & X_{n+f} & \\
0 & & & & \ddots \\
& & & & & X_{n+i} & X_{n+j} \\
& & & & & -X_i & -X_j \\
& & & & & & \ddots \\
& & & & & & & X_{n+i} & X_{n+j} \\
& & & & & & & -X_i & -X_j
\end{bmatrix}
\]

(3.16)

which has determinant \( X_{n+f}^m M_{ij}(X)^l \).

**Case 1.** Assume that \( m \geq 3 \). First, we consider \( f \notin \{i,j\} \). Set \( r_1 = f, r_2 = i, r_3 = j \). Inductively, fix \( r_s \in \{n \setminus \{r_1, \ldots, r_{s-1}\} \) for each \( s \in \{4, \ldots, m\} \). Fix also \( R_1 \in \{n \setminus \{r_1, \ldots, r_m\} \) and, inductively, \( R_q \in \{n \setminus \{r_1, \ldots, r_m, R_1, \ldots, R_{q-1}\} \) for all \( q \in \{1, \ldots, l\} \). Inductively, fix, inductively, \( r_s \in \{n \setminus \{r_1, \ldots, r_{s-1}\} \) for each \( s \in \{3, \ldots, m\} \). The indices \( R_q \) are chosen as in the former case. The matrix \( \tilde{A}(X) \) is in this case of the form (3.16), by similar arguments as the ones for the former case.

**Case 2.** Assume that \( m = 2 \), that is, we want to find a minor of the form \( X^2_{n+f} M_{ij}(X) \). If \( f \notin \{i,j\} \), set \( r_1 = f, r_2 = i, \) and \( R_1 = j \). Then fix, inductively, \( R_q \in \{n \setminus \{r_1, r_2, R_1, \ldots, R_{q-1}\} \) for each \( q \in \{1, \ldots, l\} \). The matrix \( \tilde{A}(X) \) is in this case of the form (3.16), by similar arguments as the ones for the former case.

**Case 3.** Assume that \( m = 1 \). If \( f \notin \{i,j\} \), set \( r_1 = f, R_1 \in \{i,j\} \setminus \{f\} \), \( R_2 \in \{n \setminus \{r_1, R_1\} \) and, inductively, \( R_q \in \{n \setminus \{r_1, R_1, \ldots, R_{q-1}\} \) for all \( q \in \{3, \ldots, l\} \). Denote by \( C_q \) and \( C'_q \) the index of the columns of \( A_{H_n}(X) \) containing, respectively, \( X_{n+i} \) and \( X_{n+j} \) in the \( r_1 \),th row. Then set \( R_1 = i \) and \( R_2 = j \) and, inductively, \( R_q \in \{n \setminus \{r_1, R_1, \ldots, R_{q-1}\} \) for all \( q \in \{3, \ldots, l\} \).
Paula Lins

\[ X_{n+i} \text{ and } X_{n+j} \text{ in the } R_{i} \text{th row. There are only } 2l - 1 \text{ indices } C_{i} \text{ and } C_{j} \text{ in total, since } C_{1} = C_{2}. \]

Similar arguments as the ones of the former cases show that the matrix composed of rows \( r_{1}, R_{1}, n + R_{1}, \ldots, R_{l}, n + R_{l} \) and columns \( c_{1}, c_{1}' \), \( c_{1}' \), \( c_{2}, \ldots, c_{l}' \), \( c_{l}' \), in this order, is

\[
\begin{bmatrix}
X_{n+i} & X_{n+j} & 0 & 0 & 0 & 0 & 0 \\
X_{n+i} & X_{n+j} & 0 & X_{n+R_{3}} & 0 & X_{n+R_{t}} & 0 \\
-X_{f} & 0 & -X_{i} & -X_{j} & 0 & -X_{R_{3}} & 0 \\
0 & X_{n+i} & X_{n+j} & 0 & X_{n+R_{3}} & 0 & X_{n+R_{t}} \\
0 & -X_{f} & 0 & -X_{i} & -X_{j} & 0 & -X_{R_{3}} & 0 \\
0 & 0 & 0 & X_{n+i} & X_{n+j} & 0 \\
0 & 0 & 0 & -X_{i} & -X_{j} & 0 \\
0 & 0 & 0 & 0 & X_{n+i} & X_{n+j} \\
0 & 0 & 0 & 0 & -X_{i} & -X_{j}
\end{bmatrix}
\]

The determinant of such matrix is

\[
M_{ij}(X)^{l-2} \det \left( \begin{bmatrix} X_{n+i} & X_{n+j} & 0 & 0 & 0 \\ X_{n+i} & X_{n+j} & 0 & X_{n+R_{3}} & 0 \\ -X_{f} & 0 & -X_{i} & -X_{j} & 0 \\ 0 & X_{n+i} & X_{n+j} & 0 & X_{n+R_{3}} \\ 0 & -X_{f} & 0 & -X_{i} & -X_{j} \end{bmatrix} \right) = X_{n+f} M_{ij}(X)^{l}.\]

The minors of the form \( X_{n}^{2} M_{ij}(X)^{l} \) (up to sign) are obtained by repeating the constructions above for each case but considering rows \( n + r_{s} \) instead of \( r_{s} \), for all \( s \in [m] \). The determinants of the matrices obtained in this way are of the desired form because of the symmetry of the columns of \( A_{H_{n}}(X) \) given by equality \([3.12]\).

Combining expression \([3.15]\) with Lemma \([3.9]\) we obtain, for each \( x \in W_{p}^{2n} \),

\[
\prod_{k=1}^{2n-1} \|F_{k}(A_{H_{n}}(x)) \cup wF_{k-1}(A_{H_{n}}(x))\|_{p} = \|\{M_{ij}(x) \mid 1 \leq i < j \leq n \} \cup \{w\}\|_{p}^{n-1}.\]

Thus, for groups of the form \( H_{n}(o) \), the \( p \)-adic integral \([2.2]\) is

\[
\mathcal{J}_{H_{n}}(s_{1}, s_{2}) := \int_{(w,x) \in p \times W_{p}^{2n}} \|w\|_{p}^{(2n-2) s_{1} s_{2} - (n+1) s_{2} - 2} \|\{M_{ij}(x) \mid 1 \leq i < j \leq n \} \cup \{w\}\|_{p}^{(n-1)(1+s_{2})} d\mu,
\]

which is a specialisation of the integral given in Proposition \([2.3]\). Combining Proposition \([2.3]\) with Proposition \([2.5]\) yields

\[
Z_{H_{n}(o)}^{\mathcal{J}}(s_{1}, s_{2}) = \frac{1}{1 - q^{(n+1) s_{2}}} \left( 1 + (1 - q^{-1})^{-1} \mathcal{J}_{H_{n}}(s_{1}, s_{2}) \right)
= ZF_{H_{n}}(q, q^{-s_{1}}, q^{-s_{2}}),
\]

proving Theorem \([1.4]\) for groups of type \( H \).

4. Bivariate representation zeta functions—proof of Theorem \([1.6]\)

Recall that \( g := \Lambda(o) \). Consider the \( B \)-commutator matrix \( B_{\Lambda}(Y) \) of \( g \) with respect to \( e \) and \( f \) defined in Section \([2.2]\).

Recall that a matrix \( M \in \text{Mat}_{n \times n}(\mathfrak{o}/p^{N}) \) is said to have elementary divisor type \((m_{1}, \ldots, m_{u})\), denoted \( \nu(M) = (m_{1}, \ldots, m_{u}) \), if it is equivalent to the matrix

\[
\begin{bmatrix}
X_{1} & 0 & \cdots & 0 \\
0 & X_{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & X_{u}
\end{bmatrix}
\]
Diag(\(\pi^m_1, \ldots, \pi^m_n, \mathbf{0}_{n-u}\)), where \(\pi\) denotes a uniformiser of \(\mathfrak{o}\), \(u\) is its rank, \(\mathbf{0}_{n-u} = (0, \ldots, 0) \in \mathbb{Z}^{n-u}\), and \(0 \leq m_1 \leq m_2 \leq \cdots \leq m_u \leq N\).

Denote \(\mathbf{m} = (m_1, \ldots, m_u)_\mathcal{A}\). In [7] Proposition 4.7, it is shown that the bivariate representation zeta function is given by the following sum:

\[
(4.1) \quad (1 - q^{-s_2})Z_{G(\mathfrak{o})}^{irr}(s_1, s_2) = \left(1 + \sum_{N=1}^{\infty} \sum_{\mathbf{m} \in \mathbb{N}^u_{\mathcal{A}}} N_{N, B, \mathbf{m}} q^{-N(u_B, s_1 + s_2 + 2u_B - r) - 2\sum_{j=1}^{N_{\mathfrak{o}}} m_j (-s_1 - 2)}\right),
\]

where \(N_{N, B, \mathbf{m}} = |\{ \mathbf{y} \in W^g_{n,N} \mid \nu(\mathcal{R}(\mathbf{y})) = \mathbf{m} \}|\).

Given a set \(I = \{i_1, \ldots, i_l\} \subseteq [n-1]\), recall that \(\mu_j := i_{j+1} - i_j\) for all \(j \in [l]\), where \(i_0 = 0, i_{l+1} = n\), and choose \(r_I = (r_i)_{i \in I} \in \mathbb{N}^I\) and let \(N = \sum_{i \in I} r_i\). Recall that \(b = \text{rk}(g')\). Following [14] Section 3, we define the following sets, which form a partition of \(W^g_{n,N}\):

\[
N_{I, r_I}(G) = \{ \mathbf{y} \in W^g_{n,N} \mid \nu(B(\mathbf{y})) = \begin{cases} \sum_{i \in I} r_i \in \mathbb{N}^I, & \text{if } G = F_{n, \mathfrak{o}}, \\ \sum_{i \in I} r_i \in \mathbb{N}^I, & \text{if } G \in \{G_n, H_n\}. \end{cases} \}
\]

For \(\Lambda \in \{\mathcal{F}_{n, \mathfrak{o}}, G_n, H_n\}\), \(\mathfrak{o} = \mathfrak{g}'\), so that

\[
r := \text{rk}(g'/g) = a := \text{rk}(g/\mathfrak{g}) = \begin{cases} 2n + \delta, & \text{if } G = F_{n, \mathfrak{o}}, \\ 2n, & \text{if } G \in \{G_n, H_n\}. \end{cases}
\]

For simplicity, consider \(\delta = 0\) when \(G \in \{G_n, H_n\}\), so that we can write \(a = 2n + \delta\) uniformly.

Using these facts, we rewrite equality (4.1) as follows.

\[
Z_{G(\mathfrak{o})}^{irr}(s_1, s_2) = \frac{1}{1 - q^{a(G, n) - s_2}} \sum_{r_I \in [n-1]} \sum_{r_I \in \mathbb{N}^I} \left| N_{I, r_I}(G) \right| q^{-(a(G, n) - s_2 + 2n - r)\sum_{i \in I} r_i - \sum_{i \in I} r_i(-2 - s_1)}\]

(4.2)

\[
= \frac{1}{1 - q^{a(G, n) - s_2}} \sum_{r_I \in [n-1]} \sum_{r_I \in \mathbb{N}^I} \left| N_{I, r_I}(G) \right| q^{\sum_{i \in I} r_i(-a(G, n) - s_1 - 2s_2 + 2)},
\]

where \(a(G, n) = 2n + \delta\), as in Theorem 1.6.

The cardinalities \(\left| N_{I, r_I}(G) \right|\) are described in [14] Proposition 3.4 in terms of the polynomials \(f_{\mathcal{F}, I}\) and the numbers \(b(G, i)\) defined in Theorem 1.6 as follows.

\[
\left| N_{I, r_I}(G) \right| = f_{\mathcal{F}, I}(q^{-1}) q^{\sum_{i \in I} r_i(a(G, i) - 2i - \delta)}.
\]

(4.3)

Combining (4.3) with (4.2) yields

\[
Z_{G(\mathfrak{o})}^{irr}(s_1, s_2) = \frac{1}{1 - q^{a(G, n) - s_2}} \sum_{r_I \in [n-1]} \sum_{r_I \in \mathbb{N}^I} f_{\mathcal{F}, I}(q^{-1}) q^{\sum_{i \in I} r_i(a(G, i) - (n-i)s_1 - s_2)}
\]

\[
= \frac{1}{1 - q^{a(G, n) - s_2}} \sum_{r_I \in [n-1]} f_{\mathcal{F}, I}(q^{-1}) \prod_{i \in I} q^{a(G, i) - (n-i)s_1 - s_2}.
\]

This concludes the proof of Theorem 1.6.
5. Hyperoctahedral groups and functional equations

In this section, we relate the formulae of Theorem 1.6 to statistics on Weyl groups of type $B$, also called hyperoctahedral groups $B_n$. Specialisation (1.3) then provides formulae for the class number zeta functions of groups of type $F$, $G$, and $H$ in terms of such statistics. By comparing these formulae to the ones of Corollary 1.5 we obtain formulae for joint distributions of three functions on such Weyl groups.

We also use the descriptions of the bivariate representation zeta functions in terms of Weyl group statistics in order to prove Theorem 1.7 in Section 5.3.

Some required notation regarding hyperoctahedral groups is given in Section 5.1.

5.1. Hyperoctahedral groups $B_n$. We briefly recall the definition of the hyperoctahedral groups $B_n$, and some statistics associated to them. For further details about Coxeter groups and hyperoctahedral groups we refer the reader to [2].

The Weyl groups of type $B$ are the groups $B_n$, for $n \in \mathbb{N}$, of all bijections $w : [\pm n] \to [\pm n]$ with $w(-a) = -w(a)$, for all $a \in [\pm n]$, with operation given by composition. Given an element $w \in B_n$ write $w = [a_1, \ldots, a_n]$ to denote $w(i) = a_i$.

**Definition 5.1.** For $w \in B_n$, the **inversion number**, the number of negative entries and the number of negative sum pairs of $w$ are defined, respectively, by

\[
\text{inv}(w) = |\{(i, j) \mid i < j, w(i) > w(j)\}|, \\
\text{neg}(w) = |\{i \in [n] \mid w(i) < 0\}|, \\
\text{nsp}(w) = |\{(i, j) \in [n]^2 : i \neq j, w(i) + w(j) < 0\}|.
\]

Let $s_i = [1, \ldots, i-1, i+1, \ldots, n]$ for $i \in [n-1]$ and $s_0 = [-1, 2, \ldots, n]$ be elements of $B_n$. Then $(B_n, S_B)$ is a Coxeter system, where $S_B = \{s_i\}_{i \in [n-1]}$.

In [2] Proposition 8.1.1 it is shown that the **Coxeter length** on $B_n$ with respect to the generating set $S_B$ is given by

\[
\ell(w) = \text{inv}(w) + \text{neg}(w) + \text{nsp}(w), \quad \text{for } w \in B_n.
\]

For simplicity, we identify $S_B$ with $[n-1]_0$ in the obvious way, so that $D(w) \subseteq [n-1]_0$. Moreover, for $I \subseteq S_B$, define

\[
B_n^I = \{w \in B_n \mid D(w) \subseteq I^c = S_B \setminus I\}.
\]

**Example 5.2.** Let $w_0 = [-1, \ldots, -n]$ be the longest element of $B_n$. Then

\[
\text{inv}(w_0) = \binom{n}{2}, \quad \text{neg}(w_0) = n, \quad \ell(w_0) = n^2, \quad D(w_0) = S_B.
\]

Consider $w \in B_n$. The following statistics are used in this work.

\[
L(w) = \frac{1}{2} |\{(i, j) \in [\pm n]^2_0 \mid i < j, w(i) > w(j), i \neq 0 \text{ mod } 2\}|, \\
\text{des}(w) = |D(w)|, \\
\sigma(w) = \sum_{i \in D(w)} n^2 - i^2, \\
\text{maj}(w) = \sum_{i \in D(w)} i, \\
\text{rmaj}(w) = \sum_{i \in D(w)} n - i.
\]

The statistics $\text{des}(w)$, $\text{maj}(w)$, and $\text{rmaj}(w)$ are called, respectively, the **descent number**, the **major index**, and the **reverse major index** of $w$. 

\[24\]
5.2. Bivariate representation zeta functions and statistics of Weyl groups. The following lemma describes the polynomials \( f_{G,I} \) defined in Theorem 1.6 in terms of statistics on the groups \( B_n \), where \( G \in \{ F_{n,\delta}, G_n, H_n \} \).

**Lemma 5.3.** Let \( n \in \mathbb{N} \), \( \delta \in \{0, 1\} \) and \( I \subseteq [n-1]_0 \). Then

1. \( f_{F_{n,\delta},I}(X) = \sum_{w \in B_n^{\text{c}}_{I}} (-1)^{\text{neg}(w)} X^{(2\ell(w)+(2\delta-1)\text{neg}(w))}, \)
2. \( f_{G_n,I}(X) = \sum_{w \in B_n^{\text{c}}_{I}} (-1)^{\text{neg}(w)} X^{\ell(w)}, \)
3. \( f_{H_n,I}(X) = \sum_{w \in B_n^{\text{c}}_{I}} (-1)^{\ell(w)} X^{L(w)}. \)

**Lemma 5.4.** Given \( n \in \mathbb{N} \), \( \delta \in \{0, 1\} \), and a prime ideal \( p \) of \( \mathcal{O} \),

\[
Z_{\mathcal{G}(\mathcal{O})}^{\text{irr}}(s_1, s_2) = \sum_{w \in B_n} \chi_G(w) q^{-\text{hG}(w)} \prod_{i \in D(w)} q^{\bar{a}(G,i) - (n-i)s_1 - s_2} \prod_{i=0}^{n} \left( 1 - q^{\bar{a}(G,i) - (n-i)s_1 - s_2} \right),
\]

where, for each \( w \in B_n \),

\[
\begin{array}{c|c|c|c}
G & \chi_G(w) & h_G(w) \\
\hline
F_{n,\delta} & (-1)^{\text{neg}(w)} & 2\ell(w) + (2\delta - 1)\text{neg}(w) \\
G_n & (-1)^{\text{neg}(w)} & \ell(w) \\
H_n & (-1)^{\ell(w)} & L(w) \\
\end{array}
\]

**Proof.** Applying Lemma 5.3 to the formulæe of Theorem 1.6 one obtains the following expression for \( Z_{\mathcal{G}(\mathcal{O})}^{\text{irr}}(s_1, s_2) \):

\[
\frac{1}{1 - q^{\bar{a}(G,n) - s_2}} \sum_{I \subseteq [n-1]_0} \sum_{w \in B_n^{\text{c}}_{I}} \chi_G(w) q^{-\text{hG}(w)} \prod_{i \in I} q^{\bar{a}(G,i) - (n-i)s_1 - s_2} \prod_{i=0}^{n} \left( 1 - q^{\bar{a}(G,i) - (n-i)s_1 - s_2} \right),
\]

which can be rewritten as the claimed sum because of [14, Lemma 4.4].

**Proposition 5.5.** For \( n \in \mathbb{N} \) and \( \delta \in \{0, 1\} \), the following holds in \( \mathbb{Q}[X, Z] \).

\[
\sum_{w \in B_n} (-1)^{\text{neg}(w)} X^{-(2(\ell - \sigma) + (2\delta - 1)\text{neg} - (2\delta - 3) \text{rmaj} - (2n + \delta) \text{des})(w)} Z^{\text{des}(w)}
\]

\[
= \left( 1 - X^{(2n+\delta-1)} \right) \prod_{i=2}^{n} \left( 1 - X^{(2n+\delta-i) - (2i+\delta)} Z \right)
\]

**Proof.** On the one hand, specialisation 1.3 applied to the formula of Lemma 5.4 for groups of type \( F \) gives

\[
\zeta_{F_{n,\delta}}^k(s) = \sum_{w \in B_n} (-1)^{\text{neg}(w)} q^{-(2\ell + (2\delta - 1)\text{neg})(w)} \prod_{i \in D(w)} q^{\bar{a}(F_{n,\delta},i) - s} \prod_{i=0}^{n} \left( 1 - q^{\bar{a}(F_{n,\delta},i) - s} \right).
\]

But

\[
\bar{a}(F_{n,\delta},i) = \left( \frac{2n + \delta}{2} \right) - \left( \frac{2i + \delta}{2} \right) + 2i + \delta = 2(n^2 - i^2) + (2\delta - 3)(n - i) + 2n + \delta,
\]

so that

\[
\prod_{i \in D(w)} q^{\bar{a}(F_{n,\delta},i) - s} = q^{(2\sigma + (2\delta - 3) \text{rmaj} + (2n + \delta - s) \text{des})(w)}.
\]
On the other hand, Corollary 1.5 asserts that

\[ \zeta_{\mathcal{F}_{n,s}(e)}(s) = \sum_{w \in B_n} (-1)^{\text{neg}(w)} q^{-(2(\ell-\sigma)+(2\delta-1) \text{neg}-(2\delta-3) \text{rmaj}-(2n+\delta-s) \text{des}(w))} \prod_{i=0}^{n} (1 - q^{\mathcal{F}_{n,s}(i)-s})^{-1}. \]

Therefore

\[
\begin{align*}
\sum_{w \in B_n} (-1)^{\text{neg}(w)} q^{-(2(\ell-\sigma)+(2\delta-1) \text{neg}-(2\delta-3) \text{rmaj}-(2n+\delta-s) \text{des}(w))} \\
= \left( 1 - q^{\frac{(2n+\delta-1)}{2} - s} \right)^2 \prod_{i=2}^{n} \left( 1 - q^{\mathcal{F}_{n,s}(i)} q^{-s} \right) \\
= \left( 1 - q^{\frac{(2n+\delta-1)}{2} - s} \right)^2 \prod_{i=2}^{n} \left( 1 - q^{\frac{(2n+\delta)}{2} - \frac{(2i+\delta)}{2} + 2i + \delta - s} q^{-s} \right).
\end{align*}
\]

The formal identity follows as these formulae hold for all prime powers \( q \) and all \( s \in \mathbb{C} \) with sufficiently large real part. \( \square \)

For a geometric interpretation of \( \ell - \sigma \), we refer the reader to [16, Section 2].

It can be easily checked that, for \( n \geq 2 \) and \( w \in B_n \),

\[
\prod_{i \in D(w)} q^{\mathcal{H}_{n,i}-s} = q^{\left(\sigma+2 \text{maj}-(2n-s) \text{des}(w)\right)}, \tag{5.1}
\]

\[
\prod_{i \in D(w)} q^{\mathcal{G}_{n,i}-s} = q^{\left(\sigma-3 \text{rmaj}+2(2n-s) \text{des}(w)\right)}. \tag{5.2}
\]

The following proposition follows from Lemma 5.4, Corollary 1.5, equalities (5.1) and (5.2), and arguments analogous to those given in the proof of Proposition 5.5.

**Proposition 5.6.** For \( n \geq 2 \), the following identities hold in \( \mathbb{Q}[X,Z] \).

\[
\begin{align*}
\sum_{w \in B_n} (-1)^{\text{neg}(w)} X^{-(\ell-\sigma-2 \text{maj})(w)} Z^{\text{des}(w)} &= \left( \prod_{i=3}^{n} 1 - X^{i^2 - i^2 + 2i} Z \right), \\
\left(1 - X^{2(\ell)} Z \right) \left(1 - X^{2(\ell+1)} Z \right) + X^{n^2} Z(1 - X^{-n})(1 - X^{-n+1}), \text{ and}
\end{align*}
\]

\[
\begin{align*}
\sum_{w \in B_n} (-1)^{\ell(w)} X^{-\frac{1}{2}(2L-\sigma+3 \text{rmaj}+4n \text{ des}(w))} Z^{\text{des}(w)} &= \left( \prod_{i=3}^{n} 1 - X^{i^2 - i^2 + 2i} Z \right), \\
\left(1 - X^{\ell(i)} Z \right) \left(1 - X^{\ell(i+1)} Z \right) + X^{n^2} Z(1 - X^{-n+1}), \text{ and}
\end{align*}
\]

**Remark 5.7.** By setting \( X = 1 \) in the equations of Propositions 5.5 and 5.6 we obtain the equalities

\[
\sum_{w \in B_n} (-1)^{\text{neg}(w)} Z^{\text{des}(w)} = \sum_{w \in B_n} (-1)^{\ell(w)} Z^{\text{des}(w)} = (1 - Z)^n,
\]

which were first proven in [9, Theorem 3.2].

### 5.3. Functional equations—proof of Theorem 1.7

We recall that the formulae of Proposition 2.5 of the local factors of the bivariate representation zeta function of groups of type \( F \), \( G \), and \( H \) hold for all nonzero prime ideals \( p \), since we consider the construction of the unipotent group schemes of class 2 given in [14, Section 2.4]. In particular, the descriptions of the local terms of the bivariate representation zeta functions of groups of type \( F \), \( G \), and \( H \) in terms of Weyl statistics
Bivariate zeta functions of $\mathcal{T}$-groups

given in Lemma 5.4 also hold for all nonzero prime ideals. We use Lemma 5.4 to show that all local terms of these bivariate zeta functions satisfy functional equations. Recall that, for each $n \in \mathbb{N}$, $w_0$ denotes the longest element of $B_n$, that is, $w_0 = [-1, -2, \ldots, -n]$.

Theorem 1.7 follows from the same arguments of the proof of [6, Theorem 2.6] applied to the expressions of Lemma 5.4. In fact, although $h_G$ is not one of the statistics $b \cdot l_L$ or $b \cdot l_R$ defined in [6, Theorem 2.6], it satisfies the equations (2.6) of [6], that is,

$$h_G(ww_0) + h_G(w) = h_G(w_0).$$

In fact, one can easily show that $g \in \{\text{inv, neg, } \ell\}$ satisfies $g(ww_0) = g(w_0) - g(w)$, for all $w \in B_n$, and the equation $L(ww_0) = L(w_0w) = L(w_0) - L(w)$ is [13, Corollary 7]. Therefore the conclusion of [6, Theorem 2.6] also holds for the expressions given in Lemma 5.4.

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