Lattice QCD Calculation of Flavor Singlet Matrix Elements and \( N-N \) Scattering Lengths

Yoshinobu Kuramashi

National Laboratory for High Energy Physics (KEK)
Tsukuba, Ibaraki 305, Japan

Abstract

We report on our recent study of flavor singlet matrix elements calculated with the wall source method without gauge fixing. Results are presented for the mass difference between \( \eta' \) and pseudoscalar octet mesons, the \( \pi-N \sigma \) term and the proton axial vector matrix elements. An exploratory calculation of the \( N-N \) scattering lengths is also discussed including a phenomenological study with one-boson exchange potentials.

1 Introduction

Numerical simulations of lattice QCD have been successfully applied to calculate a wide variety of physically interesting observables. However, the class of observables treated to date are mostly restricted to the flavor non-singlet quantities and little is known for the flavor singlet ones. The reason is the technical difficulty of calculating amplitudes involving disconnected quark loops required in the latter case. We have found, however, that the method of wall source without gauge fixing, previously developed to deal with a similar computational difficulty with full hadron four-point functions[1], allows to overcome the problem. Using this method we have recently calculated the \( \eta' \) meson mass[2, 3], the \( \pi-N \sigma \) term[4] and the flavor singlet axial vector matrix elements of proton[5]. The first aim of this article is to present a summary of our findings on these quantities.

Calculation of hadron scattering lengths is a first step going beyond a study of static properties of hadrons based on lattice QCD. Of particular phenomenological interest is the nucleon-nucleon scattering lengths since, unlike the pion-pion and pion-nucleon cases, their values are not constrained by chiral symmetry. The second aim of this article is to present our study of the subject, including a one-boson exchange model calculation on the quark mass dependence of the scattering lengths[6, 7].

All of our calculations are carried out in quenched QCD at \( \beta = 5.7 \) using the Wilson quark action with the hopping parameters \( K = 0.150 - 0.168 \) on a \( 12^3 \times 20, 16^3 \times 20 \) or \( 20^3 \times 20 \) lattice. Gauge configurations are generated by the pseudo heat-bath method at 1000 sweep intervals. Errors in the physical quantities are estimated by a jackknife procedure. We correct quark field normalization by \( \sqrt{1 - 3K/4K_c} \) with \( K_c = 0.1694 \).
and employ the tadpole-improved one-loop expressions[8] for the renormalization factors of the quark bilinear operators with $\alpha_{MS}(1/a) = 0.2207$ for the coupling constant.

2 $\eta'$ meson mass

The large value of the flavor singlet $\eta'$ meson mass in comparison to the masses of octet pseudoscalar mesons constitutes the well-known U(1) problem[9]. The current view on the resolution of this problem is based on the $1/N_c$ argument[10]: the mass splitting $m_{\eta'}^2 = m_{\eta'}^2 - m_8^2$ is generated through iteration of virtual quark loops in the $\eta'$ propagator, each of which gives a factor $m_8^2/(p^2 + m_8^2)$, and the U(1) anomaly is supposed to give a large value to $m_0$. Our aim is to evaluate $m_0$ directly in lattice QCD and examine the relation between $m_0$ and the topological charge.

In quenched QCD the $\eta'$ propagator only consists of the one-quark-loop amplitude having a single pole $1/(p^2 + m_8^2)$ and the two-quark-loop amplitude with a double pole $m_{\eta'}^2/(p^2 + m_8^2)^2$. This means that the mass splitting $m_0$ can be extracted from the ratio of the two amplitudes defined by

$$R(t) = \frac{\langle \eta'(t)\eta'(0) \rangle_{2\text{-loop}}}{\langle \eta'(t)\eta'(0) \rangle_{1\text{-loop}}} \approx \frac{m_0^2}{2m_8^2}t + \text{const.},$$

where each amplitude is projected onto the zero momentum state.

In order to evaluate the two-loop amplitude, we calculate the quark propagator $G(n, t) = \sum_{(n', t')} G(n, t; n', t'')$ obtained with unit source at every space-time site without gauge fixing. The expression $\sum_n \text{Tr}[G(n, 0)\gamma_5] \sum_n' \text{Tr} [G^\dagger(n', t)\gamma_5]$ equals the two-loop amplitude aside from extra contributions of gauge-variant non-local terms. However, these must cancel out in the ensemble average. This technique has the advantage that only a single quark matrix inversion is required for each gauge configuration, which enables us to achieve high statistics.

Our calculations of the $\eta'$-octet mass splitting are made with four values of the hopping parameter, $K = 0.164, 0.165, 0.1665$ on an $L = 12$ lattice and $K = 0.168$ on an $L = 16$ lattice. For each hopping parameter we use 300, 240, 240 and 43 gauge configurations, respectively, to evaluate one- and two-loop amplitudes. In the last case three exceptional configurations are excluded, for which we fail to find the solution of the quark matrix inversion.

We extract the mass splitting $m_0$ fitting the data of $R(t)$ to the linear form (1) using the value of $m_8$ taken from a single exponential fitting of the pion propagator (see Fig. 2 of Ref. [2]). The results for $m_0$ multiplied by $\sqrt{N_f} = \sqrt{3}$ to obtain the physical value are plotted in Fig. 4 as a function of the quark mass $m_q = (1/K - 1/K_c)/2$ using $K_c = 0.1694$. We observe a linear increase of $m_0$ as $m_q$ decreases toward the chiral limit. Employing a linear extrapolation for $m_0$ to obtain the value at $m_q = 0$, we find $m_0 = 0.518(25)$ or $m_0 = 751(39)$MeV in physical units with $a^{-1} = 1.45(3)$GeV determined from the $\rho$ meson mass.

The value of $m_0$ we obtained may be compared with the empirical mass difference $m_0^2 = m_{\eta'}^2 - (4m_K^2 + 3m_\pi^2 + m_\rho^2)/8 \approx (866\text{MeV})^2$, which is the naive estimation ignoring a small mixing between $\eta$ and $\eta'$. While a precise agreement is not obtained, both values are sufficiently close to conclude that the $\eta'$-octet mass splitting can be understood within quenched QCD.
Figure 1: $m_0$ multiplied by $\sqrt{N_f} = \sqrt{3}$ as a function of $m_q = (1/K - 1/K_c)/2$. Left ordinate and bottom abscissa are in lattice units, while right ordinate and top abscissa are in physical units using $a^{-1} = 1.45\text{GeV}$. Solid line is a linear fit which extrapolates to $m_0 = 0.518(25) \ (751(39)\text{MeV})$ at $m_q = 0$.

Let us next consider the topological charge dependence of the $\eta'$-octet mass splitting $m_0$. The topological charge on the lattice is defined as

$$Q = \frac{1}{32\pi^2} \sum_n \epsilon_{\mu\nu\rho\sigma} \text{ReTr}\{U_{\mu\nu}(n)U_{\rho\sigma}(n)\},$$

(2)

where $U_{\mu\nu}(n)$ is the plaquette in the $\mu\nu$ plane at site $n$. We calculate $Q$ by the cooling procedure\[^{[1]}\] employing 25 cooling sweeps to wash out ultraviolet fluctuations.

To check the anomalous Ward identity relation $m_0^2 = 2N_f\chi/f^2_\pi$ derived by Witten and Veneziano in the large $N_c$ limit\[^{[10]}\], we calculate the topological susceptibility $\chi = \langle Q^2 \rangle / (\text{space-time volume})$ using our 300 pure gauge configurations on an $L = 12$ lattice, and find $\chi = 4.76(39) \times 10^{-4}$. Combining with $f_\pi = 0.0676(24)$ at $m_q = 0$ obtained on the same set of gauge configurations used for the calculation of $m_0$, we find $\sqrt{2N_f\chi/f^2_\pi} = 1146(67)\text{MeV}$ for $N_f = 3$ with the aid of $a^{-1} = 1.45(3)\text{GeV}$. The result is somewhat higher compared to the value of $m_0$ estimated from the direct measurement. It is possible that the disagreement arises from chiral symmetry breaking of the Wilson quark action which induces extra terms in the anomalous Ward identity.

In order to further examine the relation between the $\eta'$-octet mass splitting and the topological charge we classify gauge configurations according to the value of $|\bar{Q}|$, which is the absolute magnitude of $Q$ rounded off to the nearest integer, and evaluate $R(t; |\bar{Q}|)$ on each ensemble. For the one-loop amplitude in the denominator we use the average over the whole ensemble since it shows little variation depending on $|\bar{Q}|$. The mass splitting $m_0$ is extracted by fitting the data of $R(t; |\bar{Q}|)$ to the form of (1).
Figure 2: Representative values of the ratio $R(t;|Q|)$ of two- and single-quark loop contributions to the $\eta'$ propagator for several values of $|Q|$ at $K = 0.1665$ on an $L = 12$ lattice.

In Fig. 2 we present $R(t;|Q|)$ at $K = 0.1665$ for typical values of $|Q|$: $|Q| = 0, 2, 5$. We see that the function $R(t;|Q|)$ is consistent with zero for $|Q| = 0$ and clearly increases as $|Q|$ increases. This fact demonstrates that the excitation of instantons generates the contribution of the two-loop amplitude. The dependence of the $\eta'$-octet mass splitting $m_0$ on $|Q|$ is listed in Table 1. The increase of $m_0$ with $|Q|$ shows that the $\eta'$ meson mass increases with the winding numbers of the instantons.

In full QCD with dynamical quarks topological excitations become suppressed toward vanishing sea quark mass due to the fermionic zero modes. It is an interesting problem to understand how the $\eta'$ meson mass survives the suppression. Full QCD simulations for examining this point have to overcome the difficulty that topological fluctuations are known to exhibit long-range correlations in Monte Carlo time[12].

Table 1: $|Q|$ dependence of $m_0$ in lattice units at $K = 0.1665$ on an $L = 12$ lattice.

| $|Q|$ dependence of $m_0$ in lattice units at $K = 0.1665$ on an $L = 12$ lattice. |
|-----------------|---|---|---|---|---|---|---|---|
| $|Q|$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | $\geq 7$ |
| $\xi_{\text{conf.}}$ | 25 | 47 | 51 | 20 | 35 | 22 | 12 | 28 |
| $m_0(|Q|)$ | 0.19(4) | 0.25(4) | 0.36(2) | 0.34(4) | 0.34(3) | 0.49(4) | 0.45(6) | 0.57(4) |

3 Flavor singlet nucleon matrix elements

3.1 scalar matrix elements

The $\pi-N$ $\sigma$ term is a fundamental quantity measuring the degree of chiral symmetry breaking in the strong interaction and defined as $\sigma = (m_u + m_d)/2 \cdot \langle N|\bar{u}u + \bar{d}d|N\rangle$. 

4
Table 2: Nucleon scalar matrix elements corrected by tadpole-improved $Z$ factor as a function of $K$ for up and down quarks. $\langle N|\bar{s}s|N\rangle$ represents an interpolated value at the physical strange quark mass $K_s = 0.1648$ with up and down quark masses fixed.

| $K$  | $L$ | $\chi_{\text{conf.}}$ | $\langle N|\bar{u}u + \bar{d}d|N\rangle$ | $\langle N|\bar{s}s|N\rangle$ | $F_S$ | $D_S$ |
|------|-----|-----------------------|--------------------------------|-----------------|-------|-------|
|      |     |                       | conn.                        | disc.            |       |       |
| 0.1600 | 12  | 300                   | 2.323(15)                    | 3.56(76)        | 2.09(25) | 0.749(5) | −0.074(2) |
|       | 16  | 260                   | 2.326(17)                    | 3.02(98)        | 1.78(32) | 0.749(7) | −0.075(6) |
| 0.1640 | 12  | 300                   | 2.413(30)                    | 4.58(92)        | 2.36(29) | 0.762(10) | −0.126(6) |
|       | 16  | 260                   | 2.378(30)                    | 4.1(1.2)        | 2.10(40) | 0.748(12) | −0.128(11) |
| 0.1665 | 12  | 400                   | 2.693(81)                    | 5.1(1.1)        | 2.66(35) | 0.831(25) | −0.200(16) |
|       | 16  | 260                   | 2.565(73)                    | 5.6(1.6)        | 2.70(51) | 0.783(26) | −0.204(21) |
| $K_c$ | 12  |                       | 2.615(61)                    | 5.8(1.4)        | 2.84(44) | 0.802(19) | −0.208(11) |
|       | 16  |                       | 2.515(60)                    | 6.2(1.9)        | 2.89(61) | 0.765(23) | −0.222(20) |

Study of the $\sigma$ term has a long history and still shows some controversy due to the difficulty of estimating the sea quark contribution including the strangeness content of the nucleon. A direct calculation of the $\sigma$ term in lattice QCD potentially provides a thorough resolution of this issue. Some previous calculations[13, 14] suggest that the disconnected contribution is as large as the connected one. However, those calculations were made only at quite heavy quark masses and it seems difficult to extrapolate their result to the physical point (see Fig. 1 of Ref. [15]). We have made a systematic calculation of the quark mass dependence of the $\pi$-$N \sigma$ term. The nucleon matrix elements of scalar quark density are extracted from a linear fit in $t$ of the ratio of nucleon three-point function divided by the nucleon propagator[16]. Dirichlet boundary condition is employed in the temporal direction for quark propagators. The source method[17] is used for the connected contribution. The disconnected quark loops are evaluated using quark propagators solved with unit source for every space time site without gauge fixing.

In Table 2 we summarize our results for the nucleon scalar matrix elements for each spatial size $L$ and the hopping parameter $K$. Since the results for $L = 12$ and 16 shows little finite size effects, we give the averages of $L = 12$ and $L = 16$ results in the following.

For the isoscalar matrix element $\langle N|\bar{u}u + \bar{d}d|N\rangle$ we observe that the disconnected contribution is about twice larger than the connected one. In the chiral limit we find for the ratio $\sigma_{\text{disc}}/\sigma_{\text{conn}} = 2.35(46)$.

Calculating the physical value of the $\sigma$ term requires an estimate of the quark mass. For this purpose hadron masses $m_\pi, m_\rho, m_N$ are calculated using the same set of hopping parameters on the same set of gauge configurations. The light quark mass $\hat{m} = (m_u + m_d)/2 = 0.0034(1)$ ($m_q = (1/K - 1/K_c)/2$) is determined using $m_\pi/m_\rho = 0.18$ and the strange quark mass $m_s = 0.0829(19)$ ($K_s = 0.1648$) from $m_K/m_\rho = 0.64$. The lattice scale $a^{-1} = 1.46(2)$GeV is fixed by $m_\rho$ at $K_c$. Here we take the average of results for the $L = 12$ and 16 lattices since we do not detect any finite size effects for hadron masses between the two lattice sizes[13].

We estimate the physical value of $\sigma$ in two ways. The isoscalar matrix element at $K_c$ multiplied by the light quark mass $\hat{m}$ gives $\sigma = 44(6)$MeV. Alternatively, the ratio
$m_N \sigma/m_\pi^2$ extrapolated linearly in $m_q$ to the chiral limit yields $60(9)$MeV with the aid of experimental pion and nucleon masses. The difference between the two values of $\sigma$ originates from the large ratio $m_N/m_\rho = 1.47(3)$ in our simulation carried out at $\beta = 5.7$. Within this uncertainty these estimates show an encouraging agreement with experimental estimates $\sigma = 45$MeV \cite{18}. For the strangeness content of nucleon we find a fairly large value for the $y$ parameter $y = 2 \langle N | s\bar{s} | N \rangle / \langle N | \bar{u}u + \bar{d}d | N \rangle = 0.66(13)$, as compared to a phenomenological estimate $y \approx 0.2$ \cite{18}.

Obvious systematic uncertainties in our results are scaling violation and sea quark effects. In addition we should note the potential problem that the physical light quark mass for the Wilson quark action substantially decreases toward smaller lattice spacing in quenched QCD and that the full QCD values are factor 2 – 3 smaller than those for quenched QCD \cite{19}.

Let us also make a remark concerning the flavor non-singlet $SU(3)$ reduced matrix elements $F_S$ and $D_S$. These may be evaluated from the relations $F_S = \langle p | \bar{u}u - \bar{s}s | p \rangle / 2$ and $D_S = \langle p | \bar{u}u - 2 \bar{d}d + \bar{s}s | p \rangle / 2$ with $|p\rangle$ the proton state, where we may ignore disconnected contributions to leading order in flavor symmetry breaking. For the mass splitting of the baryon octet, our results for the reduced matrix elements yield $m_\Xi - m_N = (m_s - \hat{m}) \cdot 2F_S = 0.182(6)$GeV and $-3/2(m_\Sigma - m_A) = (m_s - \hat{m}) \cdot 2D_S = -0.0498(30)$GeV. The magnitude of these values is roughly a factor two smaller compared to the experimental values $0.379$GeV and $-0.116$GeV, while our result for the ratio $D_S/F_S = -0.275(16)$ is reasonably consistent with the experimental estimate $D_S/F_S \approx -0.32$. A source of uncertainty, besides scaling violations and sea quark effects, is the validity of the first order approximation for the mass splitting. This point should be resolved by a simultaneous calculation of the baryon mass splitting and scalar matrix elements, which we leave for future analysis.

### 3.2 axial vector matrix elements

Interest in the flavor singlet axial vector matrix elements of proton was inspired by the EMC result for the spin dependent structure function $g_1$ of proton \cite{20}, which indicated that the fraction of proton spin carried by quarks has a small value and that the strange quark contribution is unexpectedly large and negative. Recent analyses combining data

| $K$ | $L$ | $\Delta u_{\text{conn}}$ | $\Delta d_{\text{conn}}$ | $\Delta u_{\text{disc}}$ | $\Delta s$ | $F_A$ | $D_A$ |
|-----|-----|----------------|----------------|----------------|----------|-------|-------|
| 0.1600 | 12 | 0.8840(60) | -0.2458(23) | -0.025(10) | -0.0374(89) | 0.4420(30) | 0.6878(46) |
| 0.1640 | 12 | 0.9071(92) | -0.2470(35) | -0.025(10) | -0.0374(89) | 0.4536(46) | 0.7011(37) |
| 0.1665 | 12 | 0.795(11) | -0.2387(51) | -0.025(10) | -0.0374(89) | 0.3974(55) | 0.6358(83) |
| 0.1600 | 16 | 0.839(19) | -0.2382(87) | -0.066(23) | -0.069(16) | 0.4196(93) | 0.6580(71) |
| 0.1640 | 12 | 0.761(20) | -0.232(11) | -0.093(54) | -0.087(33) | 0.409(20) | 0.641(16) |
| 0.1665 | 16 | 0.818(39) | -0.231(23) | -0.093(54) | -0.087(33) | 0.409(20) | 0.641(16) |
| $K_e$ | 12 | 0.694(20) | -0.2284(96) | -0.119(44) | -0.109(30) | 0.347(10) | 0.575(15) |
| 0.1640 | 16 | 0.763(35) | -0.226(17) | -0.119(44) | -0.109(30) | 0.382(18) | 0.607(14) |
Figure 3: Axial coupling $g_A$ for quenched (open) and full QCD (closed) with Wilson quark action. Our results for $L = 16$ (circles) and $L = 12$ (up-triangles) are at $\beta = 5.7$. $\beta = 6.0$ for other quenched results (open squares[13], down-triangles[28], diamonds[29]). Full QCD results (closed squares[13]) are at $\beta = 5.4 - 5.6$. Horizontal lines represent the upper and lower bound of $1\sigma$ for the experimental value $g_A = 1.2573 \pm 0.0028$.

on proton[20, 21, 22], deuteron[22, 23] and neutron[24] targets yield $\Delta \Sigma = \Delta u + \Delta d + \Delta s = +0.83(3) - 0.43(3) - 0.10(3) = 0.31(7)$ at the renormalization point $\mu^2 = 10\text{GeV}^2$[25], or $+0.832(15) - 0.425(15) - 0.097(18) = 0.31(4)$ at $\mu^2 = \infty$[26], where $\Delta q$ is the forward matrix element of $\bar{q}\gamma_i\gamma_5 q$ for a proton polarized in the $i$-th direction averaged over $i = 1, 2, 3$. These results are far from the naive constituent quark model which predict $\Delta u = 4/3, \Delta d = -1/3$ and $\Delta s = 0$.

We have attempted a lattice calculation of $\Delta q$ for each flavor $q$ including both connected and disconnected contributions. We evaluate the proton matrix elements of the axial vector current employing the same calculational technique, the same set of gauge configurations and the same hopping parameters as for the case of the nucleon scalar matrix elements.

Our results are summarized in Table 3. We observe little finite size effects for the connected contributions. For the disconnected piece statistical fluctuations turned out to be significantly larger than those for the scalar matrix elements, to the extent that a reliable signal was not obtained on an $L = 12$ lattice.

Our results for $L = 16$ show that the disconnected contribution, albeit involving a substantial error of 50%, are negative and that their magnitude is small compared to those for the connected ones. Also, the disconnected contributions slightly increase toward the chiral limit while the connected ones exhibit a slight decrease. Adding both contributions for each flavor and making a linear extrapolation to $K = K_c$, we find $\Delta \Sigma = \Delta u + \Delta d + \Delta s = +0.638(54) - 0.347(46) - 0.109(30) = +0.18(10)$ with the disconnected contribution to $\Delta u$ and $\Delta d$ equal to $-0.119(44)$. These values show a reasonable agreement with the phenomenological estimates quoted above[25, 26].
In Table[3] we also list the flavor non-singlet matrix elements $F_A = (\Delta u - \Delta s)/2$ and $D_A = (\Delta u - 2\Delta d + \Delta s)/2$ evaluated without the disconnected contributions under an assumption of exact SU(3) flavor symmetry. In the chiral limit we find $F_A = 0.382(18)$ and $D_A = 0.607(14)$ for $L = 16$, which give the axial charge $g_A = F_A + D_A = 0.985(25)$ and the ratio $F_A/D_A = 0.629(33)$. Compared to the experimental values $g_A = 1.2573(28)$ and $F_A/D_A = 0.586(19)$[27], the ratio shows a good agreement, while $g_A$ is about 25% smaller. Other calculations of $g_A$ for quenched QCD at $\beta = 6.0$[13, 28, 29] and for full QCD[13] at $\beta = 5.4 - 5.6$, with the lattice spacing of $a \approx 0.15 - 0.11$fm, yield similar results if we use the same renormalization factor as we employed. We illustrate this point in Fig. 3. The agreement of results for a variety of simulation parameters indicates that possible systematic errors such as finite size effects, scaling violations and quenching effects are too small to explain the deviation of the lattice estimates from the experimental value. The origin of the discrepancy is an open problem which should be resolved in future work.

4 N-N scattering lengths

Lattice calculation of hadron scattering lengths is based on the Lüschers’s formula[30] which relates the $s$-wave scattering lengths $a_0$ to the energy shift of two particle state at zero relative momentum confined in a finite periodic spatial box of a size $L^3$:

$$E - (m_1 + m_2) = -\frac{2\pi(m_1 + m_2)a_0}{m_1m_2L^3} \left( 1 + c_1 \left( \frac{a_0}{L} \right) + c_2 \left( \frac{a_0}{L} \right)^2 \right) + O(L^{-6})$$

(3)

with $c_1 = -2.837297$, $c_2 = 6.375183$. For the nucleon-nucleon case application of this formula faces the difficulty that the experimental values $a_0(3S_1) = -5.432(5)$fm and $a_0(1S_0) = +20.1(4)$fm[31] are quite large so that a large lattice is required to suppress $O(L^{-6})$ corrections in (3). The negative sign in the $3S_1$ channel due to the presence of the deuteron bound state (Levinson’s theorem) brings an additional complication that the energy of the lowest scattering state orthogonal to the bound-state deuteron has to be computed to apply (3).

To avoid this complication we take a strategy to start our simulation from the heavy quark mass region. In order to have an idea on the behavior of $N-N$ interactions toward large quark masses, we carry out a phenomenological analysis employing a model of one-boson exchange potentials. Since these models include a large number of parameters fine-tuned to reproduce the experimental results for the scattering phase shifts, we make this study not to analyze detailed quark mass dependence of scattering lengths, but to obtain a rough feature of the nuclear force under variation of quark mass around the physical one.

Specifically we employ the model of Ref. [32] and vary the value of $m_\pi$ according to our lattice results for the slope $m_\pi^2/m_q$. The mass of the scalar particle $\sigma$ is also varied in proportion to $m_\pi$ since $\sigma$ represents the contribution of two-pion exchange. For $N$, $\rho$ and $\omega$ masses we assume the linear $m_q$ dependence $m_h = a_h + b_h \cdot m_q$, fixing the slope $b_h$ with the aid of our lattice results. Other meson masses $m_{\eta}$, $m_{\delta}$, $m_{\phi}$ and the meson-nucleon coupling constants are fixed to the the physical value.

The quark mass dependence of the $N-N$ scattering lengths obtained in this way is shown in Fig. 4. We find a divergence of the scattering length in the $3S_1$ channel taking
Figure 4: Quark mass $m_q$ dependence of $N-N$ scattering lengths based on a model of one-boson exchange potentials. Vertical line represents the physical quark mass $m_q = 4.8\text{MeV}$.

Figure 5: $N-N$ scattering lengths in units of fm as compared to $\pi-N$ and $\pi-\pi$ scattering lengths\cite{7} in quenched QCD at $\beta = 5.7$ calculated with the Wilson quark action. Conversion to physical units is made with $a = 0.137(2)\text{fm}$ determined from the $\rho$ meson mass.
place at \( m_q = 6.3 \text{MeV} \) that signals unbinding of the deuteron, which is only 30% larger than the physical point \( m_q = 4.8 \text{MeV} \). On the other hand the \(^1S_0\) channel forms the bound state at a slightly smaller quark mass of \( m_q = 4.6 \text{MeV} \). For both channels the magnitude of the scattering length reduces quickly for heavier quark and becomes close to the order of 1 fm. It is reasonable to expect that the scattering lengths take values similar to the hadron size in the absence of bound-state effects.

These results suggest that the deuteron becomes unbound as the quark mass increases, and hence L"uscher’s formula can be applied to extract the \( N-N \) scattering lengths for heavy quark. Our lattice study of the \( N-N \) scattering lengths is carried out based on this consideration using the same technique as we employed for the \( \pi-\pi \) and \( \pi-N \) cases\(^6,7\). For the calculation of the nucleon four-point function gauge configurations are fixed to the Coulomb gauge over all space-time to enhance signals. We analyzed 20, 30 and 20 gauge configurations for the hopping parameters \( K = 0.150, 0.160 \) and 0.164, respectively, on a \( 20^3 \times 20 \) lattice. In Fig. 5 we compare the results for the \( N-N \) scattering lengths with those for \( \pi-\pi \) and \( \pi-N \) cases. As expected, the \( N-N \) scattering lengths are of the order of 1 fm with a positive sign for both eigenchannels. They are substantially larger than the \( \pi-\pi \) and \( \pi-N \) scattering lengths already for a heavy quark corresponding to \( m_\pi/m_\rho \approx 0.74 \). Also we note the trend, albeit with sizable errors, that the values for the \(^3S_1\) channel are larger than those for the \(^1S_0\) channel. This indicates a stronger attraction in the \(^3S_1\) channel, consistent with the formation of the deuteron bound state at the physical quark mass where a bound state is not yet formed in the \(^1S_0\) channel.

If we reduce the quark mass toward the physical point, we expect an increase of the \( N-N \) scattering lengths in both eigenchannels. For the \(^3S_1\) channel the scattering length should diverge at some quark mass signalling that the deuteron bound state is formed. However, actually observing such a phenomena will be difficult. Large lattice sizes far exceeding the size of \( 2-3 \text{fm} \), which are accessible in current lattice QCD simulations, will be needed. Furthermore increase of statistical fluctuations for large times in the \( N-N \) four-point functions is expected to become quite severe toward the chiral limit as shown in Fig. 23 of Ref. \(^7\).

5 Conclusions

The method of wall source without gauge fixing has proven to be an effective tool for exploring the role of disconnected quark loop amplitudes in the flavor singlet sector of strong interactions. Our results for the \( \eta' \)-octet mass splitting demonstrate that topologically non-trivial gauge configurations boost the \( \eta' \) meson mass to the level observed experimentally. Our application of the method to flavor singlet nucleon matrix elements have shown that disconnected contributions are substantial for the \( \pi-N \) sigma term and the quark content of proton spin. The total contribution we found for these quantities are reasonably consistent with the experimental estimates. On the other hand, the flavor non-singlet pieces calculated from the connected contributions show substantial discrepancies between the lattice estimates and experiments.

We have also described our exploratory calculation of the \( N-N \) scattering lengths at heavy quark masses, for which we found results expected from considerations based on a phenomenological model of one-boson exchange potentials.
References

[1] Y. Kuramashi, M. Fukugita, H. Mino, M. Okawa and A. Ukawa, Phys. Rev. Lett. 71, 2387 (1993).
[2] Y. Kuramashi, M. Fukugita, H. Mino, M. Okawa and A. Ukawa, Phys. Rev. Lett. 72, 3448 (1994).
[3] M. Fukugita, Y. Kuramashi, M. Okawa and A. Ukawa, Phys. Rev. D51, 3952 (1995).
[4] M. Fukugita, Y. Kuramashi, M. Okawa and A. Ukawa, Phys. Rev. D51, 5319 (1995).
[5] M. Fukugita, Y. Kuramashi, M. Okawa and A. Ukawa, Phys. Rev. Lett. 75, 2092 (1995).
[6] M. Fukugita, Y. Kuramashi, H. Mino, M. Okawa and A. Ukawa, Phys. Rev. Lett. 73, 2176 (1994).
[7] M. Fukugita, Y. Kuramashi, M. Okawa, H. Mino, and A. Ukawa, Phys. Rev. D52, 3003 (1995).
[8] G. P. Lepage and P. B. Mackenzie, Phys. Rev. D48, 2250 (1993).
[9] S. Weinberg, Phys. Rev. D11, 3583 (1975).
[10] E. Witten, Nucl. Phys. B156, 269 (1979); G. Veneziano, Nucl. Phys. B159, 213 (1979).
[11] J. Hoek, M. Teper and J. Waterhouse, Nucl. Phys. B288, 589 (1987).
[12] K. M. Bitar et al., Phys. Rev. D44, 2090 (1991); Y. Kuramashi, M. Fukugita, H. Mino, M. Okawa and A. Ukawa, Phys. Lett. B313, 425 (1993).
[13] R. Gupta et al., Phys. Rev. D44, 3272 (1991).
[14] S.-J. Dong and K.-F. Liu, Nucl. Phys. B(Proc. Suppl.) 30, 487 (1993); preprint UK/94-02 [hep-lat/9408007] (revised) (1994). The values quoted in the first reference should be multiplied by 4 (K.-F. Liu, private communication).
[15] M. Fukugita, Y. Kuramashi, M. Okawa and A. Ukawa, Nucl. Phys. B (Proc. Suppl.) 42, 334 (1995).
[16] L. Maiani, G. Martinelli, M. L. Paciello and B. Taglienti, Nucl. Phys. B293, 420 (1987); S. Güsken, K. Schilling, R. Sommer, K.-H. Mütter and A. Patel, Phys. Lett. B212, 216 (1988).
[17] C. Bernard, T. Draper, G. Hockney, and A. Soni, in Lattice Gauge Theory: A Challenge in Large-Scale Computing, eds. B. Bunk et al. (Plenum, New York, 1986); G. W. Kilcup et al., Phys. Lett. B164, 347 (1985).
[18] J. Gasser, H. Leutwyler and M. E. Sainio, Phys. Lett. B253, 252 (1991).
[19] See, e.g., A. Ukawa, Nucl. Phys. B (Proc. Suppl.) 30, 3 (1993).

[20] J. Ashman et al., Phys. Lett. B206, 364 (1988); Nucl. Phys. B328, 1 (1989).

[21] Spin Muon Collaboration, D. Adams et al., Phys. Lett. B329, 399 (1994).

[22] E143 Collaboration, J. McCarthy, in Proc. of the 27th Int. Conf. on High Energy Physics (Glasgow, July 1994).

[23] Spin Muon Collaboration, B. Adeva et al., Phys. Lett. B302, 533 (1993).

[24] E142 Collaboration, P. L. Anthony et al., Phys. Rev. Lett. 71, 959 (1993).

[25] J. Ellis and M. Karliner, Phys. Lett. B341, 397 (1995).

[26] G. Altarelli and G. Ridolfi, preprint CERN-TH.7415/94 (1994).

[27] Particle Data Group, Phys. Rev. D50, 1173 (1994); S. Y. Hsueh et al., Phys. Rev. D38, 2056 (1988).

[28] M. Göckeler et al., preprint DESY 95-128 (hep-lat/9508004) (1995).

[29] K. F. Liu, S. J. Dong, T. Draper, J. M. Wu, and W. Wilcox, Phys. Rev. D49, 4755 (1994).

[30] M. Lüscher, Commun. Math. Phys. 105, 153 (1986); Nucl. Phys. B354, 531 (1991); Nucl. Phys. B364, 237 (1991).

[31] See, e.g., G. A. Miller et al., Phys. Rep. 194, 1 (1990).

[32] K. Holinde, K. Erkelenz and R. Alzetta, Nucl. Phys. A194, 161 (1972); A198, 598 (1972).