First-order impulsive differential systems: sufficient and necessary conditions for oscillatory or asymptotic behavior

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Abstract

In this paper, we study the oscillatory and asymptotic behavior of a class of first-order neutral delay impulsive differential systems and establish some new sufficient conditions for oscillation and sufficient and necessary conditions for the asymptotic behavior of the same impulsive differential system. To prove the necessary part of the theorem for asymptotic behavior, we use the Banach fixed point theorem and the Knaster–Tarski fixed point theorem. In the conclusion section, we mention the future scope of this study. Finally, two examples are provided to show the defectiveness and feasibility of the main results.

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1 Introduction

Nowadays impulsive differential systems are attracting a lot of attention. They appear in several real world problems (see, for instance, [1–3]). In general, it is well known that several natural phenomena are driven by differential equations. However, the description of some real world problems requires studies on impulsive differential systems, a subject very interesting from the mathematical point of view. Examples of the aforementioned phenomena are related to theoretical physics, pharmacokinetics, population dynamics, biotechnology processes, biological systems, mechanical systems, control theory, chemistry, engineering (we also stress that the modeling of these phenomena is suitably formulated by evolutive partial differential equations and, moreover, moment problem approaches appear also as a natural instrument in control theory of neutral type systems; see [4–6] and [7–9], respectively).

The literature related to impulsive differential equations is very wide. Here we mention some recent developments in this field.
In [10], Shen and Wang considered impulsive differential systems of the following form:

\[
\begin{cases}
  u'(\xi) + b_1(\xi)u(\xi - \mu_1) = 0, & \xi \neq \alpha_i, \xi \geq \xi_0, \\
  u(\alpha_i^+) - u(\alpha_i^-) = I_i(u(\alpha_i)), & i \in \mathbb{N},
\end{cases}
\]

(1)

where \( b \in C(\mathbb{R}, \mathbb{R}) \) and \( I_i \in C(\mathbb{R}, \mathbb{R}) \) for \( i \in \mathbb{N} \), and established some sufficient conditions that ensure the oscillatory and asymptotic behavior of (1).

In [11], Graef et al. studied the impulsive system

\[
\begin{cases}
  (u(\xi) - b(\xi)u(\xi - \mu))^\prime + b_1(\xi)u(\xi - \mu_1) - b_2(\xi)u(\xi - \mu_2) = 0, & \xi \geq \xi_0, \\
  u(\alpha_i^+) = b_1u(\alpha_i), & i \in \mathbb{N},
\end{cases}
\]

(2)

assuming that \( b(\xi) \in PC([\xi_0, \infty), \mathbb{R}_+) \) (that is, \( b(\xi) \) is piecewise continuous in \([\xi_0, \infty)\)), established some new sufficient conditions for the oscillation of (2).

In [12], the authors established some new oscillation criteria for first order impulsive neutral delay differential systems of the form

\[
\begin{cases}
  (u(\xi) - b(\xi)u(\xi - \mu))^\prime + b_1(\xi)u(\xi - \mu_1) - b_2(\xi)u(\xi - \mu_2) = 0, & \mu_1 \geq \mu_2 > 0, \\
  u(\alpha_i^+) = I_i(u(\alpha_i)), & i \in \mathbb{N},
\end{cases}
\]

(3)

under the assumptions that \( b(\xi) \in PC([\xi_0, \infty), \mathbb{R}_+) \) and \( b_1 \leq \frac{b(\xi)}{\alpha_i} \leq 1 \).

Karpuz et al. in [13] extended the results contained in [12] by taking the non-homogeneous counterpart of system (3) with variable delays.

Oscillation and non-oscillation properties of second-order linear neutral differential equations with impulses were studied by Tripathy and Santra in [14], where the authors considered the problem

\[
\begin{cases}
  (u(\xi) - b_1u(\xi - \mu))'' + b_1(\xi)u(\xi - \mu_1) = 0, & \xi \neq \alpha_i, i \in \mathbb{N}, \\
  \Delta(u(\alpha_i) - b_1u(\alpha_i - \mu)) + b_2u(\alpha_i - \mu_1) = 0, & i \in \mathbb{N},
\end{cases}
\]

(4)

where all coefficients and delays are constant. Other sufficient and necessary conditions for the oscillatory or asymptotic behavior of second-order neutral delay differential equations with impulses were obtained in [15], where Tripathy and Santra studied systems of the form

\[
\begin{cases}
  (p(\xi)(u(\xi) + b(\xi)u(\xi - \mu)))' + b_1(\xi)g(u(\xi - \mu_1)) = 0, & \xi \neq \alpha_i, i \in \mathbb{N}, \\
  \Delta(p(\alpha_i)(u(\alpha_i) + b(\alpha_i)u(\alpha_i - \mu))) + b_2(\alpha_i)g(u(\alpha_i - \mu_1)) = 0, & i \in \mathbb{N}.
\end{cases}
\]

(5)

In [15], in particular, the authors were interested in oscillating systems that, after a perturbation by instantaneous change of state, remain oscillating.
In [16], Santra and Tripathy investigated the oscillatory or asymptotic behavior of the solutions for first-order neutral delay differential system

\[
\begin{aligned}
&\left( v(\xi) - b(\xi) v(\xi - \mu) \right)' + b_1(\xi) g(\nu(\xi - \mu_1)) = 0, \quad \xi \neq \alpha_i, \xi \geq \xi_0, \\
&v(\alpha_i^+) = I_1(\nu(\alpha_i)), \quad i \in \mathbb{N}, \\
&v(\alpha_i^+ - \mu_1) = I_1(\nu(\alpha_i - \mu_1)), \quad i \in \mathbb{N},
\end{aligned}
\] (6)

for different values of the neutral coefficient \(b\).

We also mention the paper [17] in which Santra and Dix, using the Lebesgue dominated convergence theorem, obtained sufficient and necessary conditions for the oscillation of the following second-order neutral differential equations with impulses:

\[
\begin{aligned}
&\left( p(\xi)(h'(\xi))'' + \sum_{j=1}^{m} r_j(\xi) g(\nu(\mu_j(\xi))) = 0, \quad \xi \geq \xi_0, \xi \neq \alpha_i, i \in \mathbb{N}, \\
&\Delta(p(\alpha_i)(h'(\alpha_i))'' + \sum_{j=1}^{m} r_j(\alpha_i) g(\nu(\mu_j(\alpha_i))) = 0,
\end{aligned}
\] (7)

where

\[
h(\xi) = v(\xi) + b(\xi) v(\mu(\xi)), \quad \Delta v(\xi) = \lim_{\eta \rightarrow \xi^-} v(\eta) - \lim_{\eta \rightarrow \xi^+} v(\eta), \quad -1 \leq b(\xi) \leq 0.
\]

In line with the contents of [17], Tripathy and Santra in [18] examined oscillation and non-oscillation properties for the solutions of the following forced nonlinear neutral impulsive differential system:

\[
\begin{aligned}
&\left( p(\xi)(u(\xi) + b(\xi) v(\xi - \mu))'' + r(\xi) g(u(\xi - \mu_1)) = f(\xi), \quad \xi \neq \alpha_i, i \in \mathbb{N}, \\
&\Delta(p(\alpha_i)(u(\alpha_i) + b(\alpha_i) v(\alpha_i - \mu))'' + r(\alpha_i) g(u(\alpha_i - \mu_1)) = f(\alpha_i), \quad i \in \mathbb{N},
\end{aligned}
\] (8)

for different values of \(b(\xi)\) and established sufficient conditions for the existence of positive bounded solutions of system (8).

Finally we mention the recent work [19] in which Tripathy and Santra studied the characterizations for the oscillation of second-order neutral delay impulsive differential system

\[
\begin{aligned}
&\left( p(\xi)(h'(\xi))'' + \sum_{j=1}^{m} \tilde{b}_j(\xi) v^{(\gamma_j)}(\mu_j(\xi)) = 0, \quad \xi \geq \xi_0, \xi \neq \alpha_i, \\
&\Delta(p(\alpha_i)(h'(\alpha_i))'' + \sum_{j=1}^{m} \tilde{b}_j(\alpha_i) v^{(\gamma_j)}(\mu_j(\alpha_i)) = 0, \quad i \in \mathbb{N},
\end{aligned}
\] (9)

where \(h(\xi) = v(\xi) + b(\xi) v(\mu(\xi))\) and \(-1 < b(\xi) \leq 0\).

For further details on neutral impulsive differential equations and for recent results related to the oscillation theory for delay differential equations, we refer the reader to the papers [20–53] and to the references therein. In particular, the study of oscillation of half-linear/Emden–Fowler (neutral) differential equations with deviating arguments (delayed or advanced arguments or mixed arguments) has numerous applications in physics and engineering (e.g., half-linear/Emden–Fowler differential equations arise in a variety of real world problems such as in the study of \(p\)-Laplace equations, chemotaxis models, and so forth); see, e.g., the papers [4, 44–47, 49, 50, 52, 53] for more details. In particular, by using different methods, the following papers were concerned with the oscillation of various classes of half-linear/Emden–Fowler differential equations and half-linear/Emden–Fowler differential equations with different neutral coefficients (e.g., the paper [43] was
concerned with neutral differential equations assuming that \(0 \leq b(\xi) < 1\) and \(b(\xi) > 1\); in [44] the authors studied neutral differential equations assuming that \(0 \leq b(\xi) < 1\); in [46], the authors considered neutral differential equations assuming that \(b(\xi)\) is nonpositive; in [47, 51] the author considered neutral differential equations in the case where \(b(\xi) > 1\); the paper [50] was concerned with neutral differential equations assuming that \(0 \leq b(\xi) \leq q_0 < \infty\) and \(b(\xi) > 1\); in [52] the authors considered neutral differential equations in the case where \(0 \leq b(\xi) \leq q_0 < \infty\); in [53] the author studied neutral differential equations in the case when \(0 \leq b(\xi) = b_0 \neq 1\); whereas the paper [49] was concerned with differential equations with a nonlinear neutral term assuming that \(0 \leq b(\xi) \leq a < 1\), which is the same research topic as that of this paper.

Motivated by the aforementioned findings, in this paper we prove sufficient and necessary conditions for oscillatory or asymptotic behavior of solutions to a first-order nonlinear impulsive differential system in the form

\[
\begin{cases}
(u(\xi) + b(\xi)u(\xi - \mu))' + b_1(\xi)G(u(\xi - \mu)) = f(\xi), & \xi \neq \alpha_i, i \in \mathbb{N}, \\
\Delta(u(\alpha_i) + b(\alpha_i)u(\alpha_i - \mu)) + b_2(\alpha_i)G(u(\alpha_i - \mu)) = g(\alpha_i), & i \in \mathbb{N},
\end{cases}
\]  

(E)

where

(a) \(\mu > 0, \mu_1 \geq 0\) are real constants; \(b_1, b_2 \in C(\mathbb{R}_+, \mathbb{R}_+), b \in PC(\mathbb{R}_+, \mathbb{R})\);
(b) \(f, g \in C(\mathbb{R}, \mathbb{R}), G \in C(\mathbb{R}, \mathbb{R})\) is nondecreasing with \(uG(u) > 0\) for \(u \neq 0\);
(c) \(\alpha_i\) for \(i \in \mathbb{N}\) with \(\alpha_1 < \alpha_2 < \cdots < \alpha_i < \cdots\) and \(\lim_{i \to \infty} \alpha_i = \infty\) are fixed moments of impulsive effect;
(d) \(\Delta\) is the difference operator defined by

\[
\Delta u(\alpha_i) = \lim_{\eta \to \alpha_i^+} u(\eta) - \lim_{\eta \to \alpha_i^-} u(\eta);
\]

(e) there exists \(F \in C(\mathbb{R}, \mathbb{R})\) such that \(f(\xi) = F(\xi)\) and \(g(\alpha_i) = \Delta F(\alpha_i), i \in \mathbb{N}\).

Next, we are listing all the assumptions/conditions which we need to study the oscillation and non-oscillation properties of the solution of system (E).

(A1) \(\lim_{\xi \to \infty} F(\xi) = M, |M| < \infty\).

(A2) \(F(\xi)\) changes sign with \(-\infty < \lim\inf_{\xi \to \infty} F(\xi) < 0 < \lim\sup_{\xi \to \infty} F(\xi) < \infty\).

(A3) \(F(\xi)\) changes sign with \(F^+(\xi) = \max\{F(\xi), 0\}, F^-(\xi) = \max\{-F(\xi), 0\}\).

(A4) \(G\) is odd with \(G(uv) = G(u)G(v)\), and \(G(u) + G(v) \geq \lambda G(u + v)\) for \(u, v, \lambda > 0\).

(A5) \(\int_\xi^\infty B_1(\xi)G(F^+(\xi - \mu_1))\,d\xi + \sum_{i=1}^{\infty} B_2(\alpha_i)G(F^+(\alpha_i - \mu_1)) = \infty\), where \(\zeta > 0, \xi > \mu, B_1(\xi) = \min\{b_1(\xi), b_1(\xi - \mu)\}\) and \(B_2(\alpha_i) = \min\{b_2(\alpha_i), b_2(\alpha_i - \mu)\}\).

(A6) \(\int_\xi^\infty B_1(\xi)G(F^-(\xi - \mu_1))\,d\xi + \sum_{i=1}^{\infty} B_2(\alpha_i)G(F^-(\alpha_i - \mu_1)) = \infty\), where \(\zeta > 0\).

(A7) \(\int_\xi^\infty b_1(\xi)G(F^+(\xi - \mu_1))\,d\xi + \sum_{i=1}^{\infty} b_2(\alpha_i)G(F^+(\alpha_i - \mu_1)) = \infty\), where \(\zeta > 0\).

(A8) \(\int_\xi^\infty b_1(\xi)G(F^-(\xi - \mu_1))\,d\xi + \sum_{i=1}^{\infty} b_2(\alpha_i)G(F^-(\alpha_i - \mu_1)) = \infty\), where \(\zeta > 0\).

(A9) \(\int_\xi^\infty b_1(\xi)G(F^+(\xi + \mu - \mu_1))\,d\xi + \sum_{i=1}^{\infty} b_2(\alpha_i)G(F^+(\alpha_i + \mu - \mu_1)) = \infty\), where \(\zeta > 0\).

(A10) \(\int_\xi^\infty b_1(\xi)G(F^-(\xi + \mu - \mu_1))\,d\xi + \sum_{i=1}^{\infty} b_2(\alpha_i)G(F^-(\alpha_i + \mu - \mu_1)) = \infty\), where \(\zeta > 0\).

(A11) \(G\) is superlinear and \(\frac{G(\eta)}{\eta^p} \geq \frac{G(\nu)}{\nu^p}\) for \(u \geq \nu\) and \(p > 1\).

(A12) \(\int_\xi^\infty b_1(\xi)G(F^+(\xi + \mu - \mu_1))\,d\xi + \sum_{i=1}^{\infty} b_2(\alpha_i)G(F^+(\alpha_i + \mu - \mu_1)) = \infty\), where \(\zeta > 0, \xi \geq 0\) and \(c \leq 1\).

(A13) \(\int_\xi^\infty b_1(\xi)G(F^-(\xi + \mu - \mu_1))\,d\xi + \sum_{i=1}^{\infty} b_2(\alpha_i)G(F^-(\alpha_i + \mu - \mu_1)) = \infty\), where \(\zeta > 0, \xi \geq 0\) and \(c \leq 1\).

(A14) \(\int_0^\infty b_1(\xi)\,d\xi + \sum_{i=1}^{\infty} b_2(\alpha_i) = \infty\).
2 Sufficient conditions for oscillation

In this section, we establish sufficient conditions for the oscillation of the impulsive system (E).

**Theorem 2.1** Under the assumptions $0 \leq b(\xi) \leq a < \infty$ for $\xi \in \mathbb{R}_+$ and (A2)-(A6), every solution of system (E) is oscillatory.

**Proof** Let $\nu(\xi)$ be a solution of (E). For the sake of contradiction, let the solution be non-oscillatory. So, there exists $\xi_0 > \rho = \max\{\mu, \mu_1\}$ such that $\nu(\xi) > 0$, $\nu(\xi - \mu) > 0$ and $\nu(\xi - \mu_1) > 0$ for $\xi \geq \xi_0$. Setting

$$h(\xi) = \nu(\xi) + b(\xi)\nu(\xi - \mu), \quad \xi \neq \alpha_i, i \in \mathbb{N}$$

$$h(\alpha_i) = \nu(\alpha_i) + b(\alpha_i)\nu(\alpha_i - \mu), \quad i \in \mathbb{N}, (10)$$

and

$$H(\xi) = h(\xi) - F(\xi), \quad H(\alpha_i) = h(\alpha_i) - F(\alpha_i), (11)$$

it follows from (E) that

$$H'(\xi) = -b_1(\xi)G(\nu(\xi - \mu_1)) \leq 0, \quad \xi \neq \alpha_i, i \in \mathbb{N}, (12)$$

$$\Delta H(\alpha_i) = -b_2(\alpha_i)G(\nu(\alpha_i - \mu_1)) \leq 0, \quad i \in \mathbb{N}, (13)$$

for $\xi \geq \xi_1 > \xi_0 + \mu_1$. Consequently, $H(\xi)$ is nonincreasing and monotonic on $[\xi_2, \infty)$, where $\xi_2 > \xi_1$. So, we have the following two possible cases.

**Case (i).** Let $H(\xi) < 0$ for $\xi \geq \xi_2$. Since $h(\xi) > 0$, then $F(\xi) > 0$ for $\xi \geq \xi_2$, which is a contradiction.

**Case (ii).** Let $H(\xi) > 0$ for $\xi \geq \xi_2$. Ultimately, $h(\xi) > F(\xi)$ and hence $h(\xi) > \max\{0, F'(\xi)\} = F''(\xi)$ for $\xi \geq \xi_2$. Using (11), the first equation of system (E) becomes

$$0 = H'(\xi) + b_1(\xi)G(\nu(\xi - \mu_1)) + G(\alpha_1)H'(\xi - \mu) + b_1(\xi - \mu)G(\nu(\xi - \mu - \mu_1)) \quad (14)$$

for $\xi \geq \xi_2$. Using (A4), (14) becomes

$$0 \geq H'(\xi) + G(\alpha_1)H'(\xi - \mu) + B_1(\xi) \left[ G(\nu(\xi - \mu_1)) + G(\alpha_1) \nu(\xi - \mu - \mu_1) \right]$$

$$\geq H'(\xi) + G(\alpha_1)H'(\xi - \mu) + \lambda B_1(\xi)G(\nu(\xi - \mu_1)) \quad (15)$$

for $\xi \geq \xi_3 > \xi_2 + \mu_1$. Similarly, from the second equation of system (E), we get

$$0 \geq \Delta H(\alpha_i) + G(\alpha_i)\Delta H(\alpha_i - \mu) + \lambda B_2(\alpha_i)G(\nu(\alpha_i - \mu_1)) \quad (16)$$
for \( i \in \mathbb{N} \). Integrating (15) from \( \xi_3 \) to +\( \infty \), we have
\[
\lambda \int_{\xi_3}^{\infty} B_1(\xi) G(h(\xi - \mu_1)) \, d\xi \\
\leq -\left[ H(\xi) + G(a)H(\xi - \mu) \right]_{\xi_3}^{\infty} + \sum_{\xi_3 \leq \alpha_i < \infty} \Delta H(\alpha_i) + G(a) \Delta H(\alpha_i - \mu) \\
\leq -\left[ H(\xi) + G(a)H(\xi - \mu) \right]_{\xi_3}^{\infty} - \lambda \sum_{\xi_3 \leq \alpha_i < \infty} B_2(\alpha_i) G(h(\alpha_i - \mu_1)).
\]

Since \( \lim_{\xi \to \infty} H(\xi) \) exists, then the above inequality becomes
\[
\lambda \int_{\xi_3}^{\infty} B_1(\xi) G(h(\xi - \mu_1)) \, d\xi + \sum_{\xi_3 \leq \alpha_i < \infty} B_2(\alpha_i) G(h(\alpha_i - \mu_1)) < \infty.
\]
Consequently,
\[
\lambda \left[ \int_{\xi_3}^{\infty} B_1(\xi) G(F^i(\xi - \mu_1)) \, d\xi + \sum_{\xi_3 \leq \alpha_i < \infty} B_2(\alpha_i) G(F^i(\alpha_i - \mu_1)) \right] < \infty,
\]
which contradicts (A5).

If \( \nu(\xi) < 0 \) for \( \xi \geq \xi_0 \), then we set \( \nu_1(\xi) = -\nu(\xi) \) for \( \xi \geq \xi_3 \) in (E), and we find
\[
\begin{cases}
(\nu_1(\xi) + b(\xi)\nu_1(\xi - \mu))' + b_1(\xi)G(\nu_1(\xi - \mu_1)) = \widetilde{f}(\xi), & \xi \neq \mu_i, i \in \mathbb{N}, \\
\Delta (\nu_1(\alpha_i) + b(\alpha_i)\nu_1(\alpha_i - \mu)) + b_1(\alpha_i)G(\nu_1(\alpha_i - \mu_1)) = \widetilde{g}(\alpha_i), & i \in \mathbb{N},
\end{cases}
\]
where \( \widetilde{f}(\xi) = -f(\xi), \widetilde{g}(\alpha_i) = -g(\alpha_i) \) because of (A4). Letting \( \widetilde{F}(\xi) = -F(\xi) \), we have that
\[
-\infty < \lim_{i \to \infty} \inf \widetilde{F}(\xi) < 0 < \lim_{i \to \infty} \sup \widetilde{F}(\xi) < \infty
\]
and \( \widetilde{F}(\xi) = \widetilde{f}(\xi), \Delta \widetilde{F}(\alpha_i) = \widetilde{g}(\alpha_i) \) hold. Hence, proceeding as in the positive solution, we find a contradiction to (A6).

Thus, the theorem is proved. \( \square \)

**Theorem 2.2** Under the assumptions \(-1 \leq b(\xi) \leq 0 \) for \( \xi \in \mathbb{R}_+ \), (A2)–(A4), and (A7)–(A10), every solution of (E) oscillates.

**Proof** To prove by contradiction, we follow the proof of Theorem 2.1 to get that \( H(\xi) \) is monotonic on \( [\xi_2, \infty) \). So, we have the following two possible cases.

**Case (i).** Let \( H(\xi) < 0 \) for \( \xi \geq \xi_2 \). Then, for \( \xi \geq \xi_3 \),
\[
-\nu(\xi - \mu) \leq b(\xi)\nu(\xi - \mu) \leq h(\xi) < F(\xi),
\]
we have \( \nu(\xi - \mu_1) < -F(\xi + \mu - \mu_1), \xi \geq \xi_3 \geq \xi_2 \) and hence \( \nu(\xi - \mu_1) > F^-(\xi + \mu - \mu_1), \xi \geq \xi_3 \geq \xi_2 \). Thus (12) and (13) are reduced to
\[
H'(\xi) + b_1(\xi)G(F^-(\xi + \mu - \mu_1)) \leq 0, \quad \xi \neq \alpha_i, i \in \mathbb{N},
\]
\[
\Delta H(\alpha_i) + b_2(\alpha_i)G(F^-(\alpha_i + \mu - \mu_1)) \leq 0, \quad i \in \mathbb{N},
\]
(17)
for \( \xi \geq \xi_4 \). Next, we are going to prove \(-\infty < \lim_{\xi \to \infty} H(\xi) < 0 \). If not, letting \( \lim_{\xi \to \infty} H(\xi) = \infty \) for \( \xi \geq \xi_4 \). For \( 0 < \epsilon < \lambda - \gamma \), where \( \lambda > \gamma > 0 \), there exists \( \xi_5 > \xi_4 \) such that \( F(\xi) < \gamma + \epsilon \) when \( \xi \geq \xi_5 \). Further, there is \( \xi_6 > \xi_5 \) such that \( \xi \geq \xi_6 \) implies that \( H(\xi) < -\lambda \), that is, \( \nu(\xi) \leq \nu(\xi - \mu) - \lambda + \gamma + \epsilon \). For \( \xi \geq \xi_6 + j\mu \), we have \( \nu(\xi) \leq \nu(\xi - j\mu) + j(\gamma + \epsilon - \lambda) \). In particular, \( \nu(\xi_6 + j\mu) \leq \nu(\xi_6) + j(\gamma + \epsilon - \lambda) < 0 \) for large \( \xi \), a contradiction to the fact \( \nu(\xi) > 0 \) for \( \xi \geq \xi_5 \). Hence \(-\infty < I < 0 \). Integrating (17) from \( \xi_6 \) to \( +\infty \), we get

\[
\int_{\xi_6}^{\infty} b_1(\xi) G(F^-(\xi + \mu - \mu_1)) \, d\xi + \sum_{\xi_6 \leq \alpha_i < \infty} b_2(\alpha_i) G(F^-(\alpha_i + \mu - \mu_1)) < \infty,
\]

which contradicts (A10).

**Case (ii).** Let \( H(\xi) > 0 \) for \( \xi \geq \xi_2 \). We note that \( \nu(\xi) \geq h(\xi) > F(\xi) \) for \( \xi \geq \xi_3 > \xi_2 \). In this case, \( \lim_{\xi \to \infty} H(\xi) \) exists. Because it happens that \( \nu(\xi) > F^+(\xi) \) for \( \xi \geq \xi_3 \), then (12) and (13) can be viewed as

\[
H'(\xi) + b_1(\xi) G(F^+(\xi - \mu_1)) \leq 0, \quad \xi \neq \alpha_i, i \in \mathbb{N}, \\
\Delta H(\alpha_i) + b_2(\alpha_i) G(F^+(\alpha_i - \mu_1)) \leq 0, \quad i \in \mathbb{N}.
\]

Integrating (18) from \( \xi_3 \) to \( +\infty \), we have

\[
\int_{\xi_3}^{\infty} b_1(\xi) G(F^+(\xi - \mu_1)) \, d\xi + \sum_{\xi_3 \leq \alpha_i < \infty} b_2(\alpha_i) G(F^+(\alpha_i - \mu_1)) < \infty,
\]

which is a contradiction to (A7).

The case \( \nu(\xi) < 0 \) for \( \xi \geq \xi_0 \) is similar. Hence the details are omitted.

Thus, the theorem is proved. \( \square \)

**Theorem 2.3** Under the assumption \(-\infty < -b \leq b(\xi) \leq -1 \) for \( \xi \in \mathbb{R}_+ \) and \( b > 0 \), and all the conditions of Theorem 2.2, every bounded solution of (E) oscillates.

**Proof** The proof of the theorem can be obtained from that of Theorem 2.2. Hence the details are omitted. \( \square \)

**Remark 2.1** In Theorems 2.1–2.3, \( G \) could be linear, sublinear, or superlinear.

**Remark 2.2** In Theorem 2.3, we are restricted on the solution of (E) to ensure the oscillation of (E). If we do not want to restrict on the solution, then \( G \) should be superlinear. Hence, we have the following result.

**Theorem 2.4** Under the assumptions \(-\infty < -b \leq b(\xi) \leq -1 \) for \( \xi \in \mathbb{R}_+ \), \( b > 0 \), and \( \mu \geq \mu_1 \), and (A2), (A3), (A4), (A7), (A8), (A11)–(A13), every solution of (E) oscillates.

**Proof** The proof of the theorem follows from the proof of Theorem 2.2 except for the case when \( H(\xi) < 0, h(\xi) < 0 \) for \( \xi \geq \xi_3 \). Since \( h(\xi) \geq b(\xi) \nu(\xi - \mu) \), then

\[
H(\xi) = h(\xi) - F(\xi) \geq b(\xi) \nu(\xi - \mu) - F(\xi) \quad \text{for} \quad \xi \geq \xi_3
\]
 Consequently, using the inequality that is, for due to (12) and (20). We may note from (19) that implies that \( H(\xi) - b(\xi) v(\xi - \mu) \geq -F(\xi) \) for \( \xi \geq \xi_4 \). Clearly, \( H(\xi) - b(\xi) v(\xi - \mu) < 0 \) is not possible due to (A3) and the fact that \( H(\xi) - b(\xi) v(\xi - \mu) = v(\xi) - F(\xi) \geq -F(\xi) \) if and only if \( v(\xi) > 0 \) for \( \xi \geq \xi_3 \). Ultimately, \( H(\xi) - b(\xi) v(\xi - \mu) > 0 \) and hence

\[
H(\xi) - b(\xi) v(\xi - \mu) \geq \max\{0, -F(\xi)\} = F^-(\xi),
\]

that is,

\[
H(\xi) \geq b(\xi) v(\xi - \mu) + F^-(\xi) \geq -bv(\xi - \mu) + F^-(\xi) > -bv(\xi - \mu)
\] (19)

for \( \xi \geq \xi_4 > \xi_3 \). Since \( H(\xi) \) is decreasing and \( \mu \geq \mu_1 \), then it follows that

\[-H(\xi) \leq -H(\xi + \mu - \mu_1) < bv(\xi - \mu_1) \quad \text{for} \quad \xi \geq \xi_4.\]

Therefore,

\[
\frac{G(v(\xi - \mu_1))}{-[H(\xi)]^\gamma} \geq \frac{G(v(\xi - \mu_1))}{b^\gamma v(\xi - \mu_1)} \quad \text{for} \quad \xi \geq \xi_4.
\] (20)

Consequently,

\[
-\frac{d}{d\xi} \left(-H(\xi)\right)^{1-\gamma} = -(1-\gamma)(-H(\xi))^{-\gamma}(-H'(\xi))
\]

\[
= (\gamma - 1)(-H(\xi))^{-\gamma} b_1(\xi) G(v(\xi - \mu_1))
\]

\[
\geq (\gamma - 1)b_1(\xi) \frac{G(v(\xi - \mu_1))}{b^\gamma v(\xi - \mu_1)} \quad \text{for} \quad \xi \geq \xi_4
\]

due to (12) and (20). We may note from (19) that \( 0 > H(\xi) > -bv(\xi - \mu) + F^-(\xi) \) implies that \( v(\xi - \mu_1) > b^{-1}F^-(\xi + \mu - \mu_1) \), and hence

\[
-\frac{d}{d\xi} \left(-H(\xi)\right)^{1-\gamma} \geq (\gamma - 1)b_1(\xi) \frac{G(b^{-1}F^-(\xi + \mu - \mu_1))}{b^\gamma [b^{-1}F^-(\xi + \mu - \mu_1)]^{\gamma}}
\]

for \( \xi \geq \xi_4 \) due to (A11). Consequently,

\[
(\gamma - 1) \int_{\xi_4}^{\xi} b_1(s) \frac{G(b^{-1}F^-(s + \mu - \mu_1))}{[F^-(s + \mu - \mu_1)]^{\gamma}} \, ds
\]

\[
\leq -[w^{1-\gamma}(s)]_{\xi_4}^{\xi} + \sum_{\xi_4 < a_i < \xi} \Delta [-w^{1-\gamma}(a_i)].
\] (21)

Using the inequality

\[
\delta_2^{1-\gamma} - \delta_1^{1-\gamma} \leq (1-\gamma)b^{1-\gamma}(\delta_2 - \delta_1) \quad \text{for} \quad \delta_1 < \delta_2 \text{ and } \gamma > 1,
\]

it follows that

\[
\Delta [-H(a_i)]^{1-\gamma} = [-H(a_i + 0)]^{1-\gamma} - [-H(a_i - 0)]^{1-\gamma}
\]

\[
\leq (1-\gamma)[-H(a_i + 0)]^{1-\gamma} \Delta [-H(a_i)].
\]
due to (13) and (20). From (13), it follows that \( \Delta H(\alpha_i) = H(\alpha_i + 0) - H(\alpha_i - 0) \leq 0 \), and hence
\[
q_k = \frac{[-H(\alpha_i - 0)]}{[-H(\alpha_i + 0)]} = \frac{[-H(\alpha_i)]}{[-H(\alpha_i + 0)]} \leq 1, \quad i \in \mathbb{N},
\]
that is, \( \{q_k\} \) is a bounded sequence. Let \( c = \min_{i \in \mathbb{N}}(q_k) \). Then (22) becomes
\[
\Delta[-H(\alpha_i)]^{1-\gamma} \leq (1 - \gamma)c^2 b(\alpha_i) \frac{G(b^{-1}F^{-}(\alpha_i + \mu - \mu_1))}{[F^{-}(\alpha_i + \mu - \mu_1)]^{\gamma}}.
\]
Using (23) in (21), we obtain
\[
(\gamma - 1) \left[ \int_{\xi_4}^{\xi} \left( b_1(s) \frac{G(b^{-1}F^{-}(s + \mu - \mu_1))}{[F^{-}(s + \mu - \mu_1)]^{\gamma}} + c'' \sum_{\xi_4 < s < \xi} b_2(\alpha_i) \frac{G(b^{-1}F^{-}(\alpha_i + \mu - \mu_1))}{[F^{-}(\alpha_i + \mu - \mu_1)]^{\gamma}} \right) \right] ds \leq \left[ w^{1-\gamma}(s) \right]_{\xi_4}^{\xi}
\]
that is,
\[
(\gamma - 1)G(b^{-1}) \left[ \int_{\xi_4}^{\xi} \left( b_1(s) \frac{G(F^{-}(s + \mu - \mu_1))}{[F^{-}(s + \mu - \mu_1)]^{\gamma}} + c'' \sum_{\xi_4 < s < \xi} b_2(\alpha_i) \frac{G(F^{-}(\alpha_i + \mu - \mu_1))}{[F^{-}(\alpha_i + \mu - \mu_1)]^{\gamma}} \right) \right] ds \leq \left[ w^{1-\gamma}(s) \right]_{\xi_4}^{\xi}
\]
due to (A4). Taking limit as \( \xi \to \infty \), we get a contradiction to (A13). This completes the proof of the theorem. \( \square \)

3 Sufficient and necessary conditions for oscillation

In this section, we are going to present the sufficient and necessary condition for oscillatory or asymptotic behavior of system (E).

Lemma 3.1 ([4]) Considering \( b, \xi, h \in C([0, \infty), \mathbb{R}) \) such that \( h(\xi) = v(\xi) + b(\xi)v(\xi - \mu) \), \( \xi \geq 0 \), \( v(\xi) > 0 \) for \( \xi > \xi_1 \), \( \lim_{\xi \to -\infty} v(\xi) = 0 \) and \( \lim_{\xi \to -\infty} h(\xi) = L \) exist. If \( b(\xi) \) satisfies one of the following conditions:
(i) \( 0 \leq a_1 \leq b(\xi) \leq a_2 < 1 \),
(ii) \( 1 < a_3 \leq b(\xi) \leq a_4 < \infty \),
(iii) \( -\infty < -a_5 \leq b(\xi) \leq 0 \),
where \( a_i > 0 \), \( 1 \leq i \leq 5 \), then \( L = 0 \).

Theorem 3.1 Under the assumptions \( 0 \leq a_1 \leq b(\xi) \leq a_2 < 1 \) for \( \xi \in \mathbb{R} \), (A1), and \( G \) is Lipschitzian on \([c, d]\), where \( 0 < c < d < \infty \), every solution of (E) either oscillates or \( \lim_{\xi \to -\infty} v(\xi) = 0 \) if and only if (A14) holds.
Proof. To prove sufficiency, we assume that (A14) holds and \( \nu(\xi) \) is a solution of (E) on \([\xi_0, \infty)\) where \( \xi_0 \geq 0 \). If \( \nu(\xi) \) is oscillatory, then there is nothing to prove. Let the solution \( \nu(\xi) > 0 \) for \( \xi \geq \xi_0 \). Then, proceeding as in Theorem 2.1, we have obtained (12) and (13) for \( \xi \geq \xi_1 > \xi_0 + \mu_1 \), where \( \xi_0 > \rho > \xi_1 \). Hence, \( H(\xi) \) is monotonic on \([\xi_2, \infty)\), where \( \xi_2 > \xi_1 \).

So, we have the following two possible cases.

Case (i). Let \( H(\xi) > 0 \) for \( \xi \geq \xi_2 \). So, \( \lim_{\xi \to \infty} H(\xi) \) exists and \( \lim_{\xi \to \infty} H(\omega_i) \) exists. Now, we are going to prove that \( \nu(\xi) \) is bounded. If not, there exists \( \{\eta_n\} \) such that \( \eta_n \to \infty \) as \( n \to \infty \), \( \nu(\eta_n) \to \infty \) as \( n \to \infty \) and

\[
\nu(\eta_n) = \max \{ \nu(s) : \xi_2 \leq s \leq \eta_n \}.
\]

Therefore,

\[
H(\eta_n) = \nu(\eta_n) + b(\eta_n) \nu(\eta_n - \mu) - F(\eta_n) \geq (1 + a_1) \nu(\eta_n) - F(\eta_n) \to +\infty, \quad \text{as } n \to \infty,
\]

which is a contradiction. So, \( \nu(\xi) \) is bounded. The same contradiction holds when \( H(\xi) < 0 \) for \( \xi \geq \xi_2 \). Consequently, \( H(\xi) \) is bounded and \( \lim_{\xi \to \infty} H(\xi) \) exists. Our aim is to show that \( \lim_{\xi \to \infty} \nu(\xi) = 0 \). For this, we need to show that \( \liminf_{\xi \to \infty} \nu(\xi) = 0 \) and \( \limsup_{\xi \to \infty} \nu(\xi) = 0 \) for every \( \xi \) and \( \mu_i \). First, we are going to prove \( \liminf_{\xi \to \infty} \nu(\xi) = 0 \). To prove this by contradiction, letting \( \liminf_{\xi \to \infty} \nu(\xi) \neq 0 \), then for \( \xi_3 > \xi_2 \) and \( \gamma > 0 \), we have \( \nu(\xi - \mu_i) \geq \gamma > 0 \) for \( \xi \geq \xi_3 \). Ultimately,

\[
\int_{\xi_3}^{\xi} b_1(\xi) G(\nu(\xi - \mu_1)) \, d\xi + \sum_{\xi_3 \leq \alpha_i < \xi} b_2(\alpha_i) G(\nu(\alpha_i - \mu_1)) \\
\geq G(\gamma) \left[ \int_{\xi_3}^{\xi} b_1(\xi) \, d\xi + \sum_{\xi_3 \leq \alpha_i < \xi} b_2(\alpha_i) \right] \to +\infty, \quad \text{as } \xi \to \infty,
\]

(24)

due to (A14). Again, if we integrate (12) from \( \xi_3 \) to \( \xi \), we get

\[
\left[ H(s) \right]_{\xi_3}^{\xi} + \int_{\xi_3}^{\xi} b_1(\xi) G(\nu(\xi - \mu_1)) \, ds - \sum_{\xi_3 \leq \alpha_i < \xi} \Delta H(\alpha_i) = 0,
\]

and hence, using (13), it follows that

\[
\int_{\xi_3}^{\xi} b_1(\xi) G(\nu(\xi - \mu_1)) \, d\xi + \sum_{\xi_3 \leq \alpha_i < \xi} b_2(\alpha_i) G(\nu(\alpha_i - \mu_1)) \\
= -\left[ H(\xi) \right]_{\xi_3}^{\xi} < +\infty, \quad \text{as } \xi \to \infty.
\]

(25)

Using (24) and (25), we have a contradiction. So, \( \liminf_{\xi \to \infty} \nu(\xi) = 0 \) for \( \xi \geq \xi_3 \). Since \( \lim_{\xi \to \infty} H(\xi) \) exists, then \( \lim_{\xi \to \infty} h(\xi) \) exists due to (A1). Therefore, by Lemma 3.1, we conclude that \( \lim_{\xi \to \infty} h(\xi) = 0 \). Consequently,

\[
0 = \lim_{\xi \to \infty} h(\xi) = \limsup_{\xi \to \infty} \left( \nu(\xi) + b(\xi) \nu(\xi - \mu) \right) \geq \limsup_{\xi \to \infty} \nu(\xi)
\]
implies that \( \limsup_{\xi \to \infty} \upsilon(\xi) = 0 \). Ultimately, \( \lim_{\xi \to \infty} \upsilon(\xi) = 0 \) for \( \xi \neq \alpha_i \), where \( i \in \mathbb{N} \). Note that \( \{\upsilon(\alpha_i - 0)\} \) and \( \{\upsilon(\alpha_i + 0)\} \) are sequences of real numbers and \( \upsilon \) is continuous. So, we have \( \lim_{\xi \to \infty} \upsilon(\alpha_i - 0) = 0 \) and \( \lim_{\xi \to \infty} \upsilon(\alpha_i + 0) = 0 \) due to \( \liminf_{\xi \to \infty} \upsilon(\xi) = 0 \) and \( \limsup_{\xi \to \infty} \upsilon(\xi) = 0 \) respectively. Hence, for all \( \xi \) and \( \alpha_i \), where \( i \in \mathbb{N} \), we have \( \lim_{\xi \to \infty} \upsilon(\xi) = 0 \).

The similar procedure can be followed when \( \upsilon(\xi) < 0 \) for \( \xi \geq \xi_0 \) to show that \( \lim_{\xi \to \infty} \upsilon(\xi) = 0 \).

Next, to prove the necessary part, we assume that

\[
\int_0^\infty b_1(\xi) \, d\xi + \sum_{i=1}^\infty b_2(\alpha_i) < \infty, \tag{26}
\]

and we must have to prove that the impulsive system (E) has a non-oscillatory solution and \( \lim_{\xi \to \infty} \upsilon(\xi) \neq 0 \). If possible, let there exist \( \xi_1 > 0 \) such that

\[
\int_{\xi_1}^\infty b_1(\kappa) \, d\kappa + \sum_{i=1}^\infty b_2(\alpha_i) < \frac{1-a_2}{5L},
\]

where \( L = \max\{L_1, G(1)\} \) and \( L_1 \) is the Lipschitz constant on \([\frac{1-a_2}{10}, 1]\). By \((A_1)\), let \( \lim_{\xi \to \infty} F(\xi) = M \). Then we can find \( \xi_2 > \xi_1 \) so that \( |F(\xi) - M| < \frac{1-a_2}{10} \) for \( \xi \geq \xi_2 \). For \( \xi_3 > \max(\xi_1, \xi_2) \), we assume \( X = \text{BC}(\xi_3, \infty, \mathbb{R}) \), (that is, the space bounded continuous real-valued functions on \( [\xi_3, \infty) \)). Therefore, \( X \) is a Banach space with respect to sup norm defined by

\[
\|x\| = \sup\{|\upsilon_1(\xi)| : \xi \geq \xi_3\}.
\]

Let us define

\[
S = \left\{ \upsilon \in X : \frac{1-a_2}{10} \leq \upsilon(\xi) \leq 1, \xi \geq \xi_3 \right\}.
\]

Clearly, \( S \) is a convex and closed subspace of \( X \). Let \( \Omega : S \to S \) be an operator defined by

\[
(\Omega \upsilon)(\xi) = \begin{cases} 
(\Omega \upsilon)(\xi_3 + \rho), & \xi \in [\xi_3, \xi_3 + \rho], \\
-b(\xi)\upsilon(\xi - \mu) + \frac{1+4a_2}{5} + (F(\xi) - M) + \int_\xi^\infty b_1(\kappa) G(\upsilon(\kappa - \mu_1)) \, d\kappa \\
+ \sum_{\xi_3 \leq \alpha_i < \xi} b_2(\alpha_i) G(\upsilon(\alpha_i - \mu_1)), & \xi \geq \xi_3 + \rho.
\end{cases}
\]

For every \( \xi \in S \),

\[
(\Omega \upsilon)(\xi) \leq \frac{1-a_2}{10} + \frac{1+4a_2}{5} + G(1) \left[ \int_\xi^\infty b_1(\kappa) \, d\kappa + \sum_{\xi_3 \leq \alpha_i < \xi} b_2(\alpha_i) \right] \\
< \frac{1-a_2}{10} + \frac{1+4a_2}{5} + \frac{1-a_2}{5} = \frac{1+a_2}{2} < 1.
\]
Theorem 3.2
Under assumptions that $\xi$ is a non-oscillatory solution of system (E) on $[1–\xi, 1)$ and $\Omega_1\xi$ has a unique fixed point on $[1–\xi, 1)$, where $\xi \geq 1 > 0$ we have

$$(\Omega\nu)(\xi) \geq -b(\xi)v(\xi - \mu) + \frac{1 + 4a_2}{5} + (F(\xi) - M)$$

and

$$\geq -a_2 + \frac{1 + 4a_2}{5} - \frac{1 - a_2}{10} = \frac{1 - a_2}{10}$$

imply that $(\Omega\nu) \in S$. Again, for $\nu_1, \nu_2 \in S$,

$$| (\Omega\nu_1)(\xi) - (\Omega\nu_2)(\xi) |$$

$$\leq | b(\xi) | | \nu_1(\xi - \mu) - \nu_2(\xi - \mu) | + L_1 \int_\xi^\infty b_1(\kappa) | \nu_1(\kappa - \mu_1) - \nu_2(\kappa - \mu_1) | d\kappa$$

$$+ L_1 \sum_{\xi_1 \leq \kappa < \xi} b_2(\alpha_i) | \nu_1(\alpha_i - \mu_1) - \nu_2(\alpha_i - \mu_1) |,$$

that is,

$$| (\Omega\nu_1)(\xi) - (\Omega\nu_2)(\xi) | \leq a_2 \| \nu_1 - \nu_2 \| + L_1 \| \nu_1 - \nu_2 \| \left[ \int_\xi^\infty b_1(\kappa) d\kappa + \sum_{\xi_1 \leq \kappa < \xi} b_2(\alpha_i) \right]$$

$$< \left( a_2 + \frac{1 - a_2}{5} \right) \| \nu_1 - \nu_2 \|,$$

that is, $\Omega$ is a contraction mapping with the contraction $(a_2 + \frac{1 - a_2}{5}) = \frac{1 + 4a_2}{5} < 1$. Note that $\Omega$ is a contraction on $S$ and $S$ is complete. Then, by using the Banach fixed point theorem, $\xi$ has a unique fixed point on $[\frac{1 - a_2}{10}, 1]$. Hence $\Omega\nu = \nu$ and

$$v(\xi) = \begin{cases} 
\nu(\xi_3 + \rho), & \xi \in [\xi_3, \xi_3 + \rho], \\
-b(\xi)v(\xi - \mu) + \frac{1 + 4a_2}{5} + (F(\xi) - M) + \int_\xi^\infty b_1(\kappa)G(\nu(\kappa - \mu_1)) d\kappa \\
+ \sum_{\xi_1 \leq \kappa < \xi} b_2(\alpha_i)G(\nu(\alpha_i - \mu_1)), & \xi \geq \xi_3 + \rho,
\end{cases}$$

is a non-oscillatory solution of system (E) on $[\frac{1 - a_2}{10}, 1]$ such that $\lim_{\xi \to \infty} v(\xi) = 0$. Thus, the theorem is proved. $\square$

Theorem 3.2
Under assumptions $1 < a_3 \leq b(\xi) \leq a_4 < \infty$ for $\xi \in \mathbb{R}_+, a_3^2 > a_4$, (A1), and $G$ is Lipschitzian on $[c, d]$, where $0 < c < d < \infty$, every solution of system (E) either oscillates or $\lim_{\xi \to \infty} v(\xi) = 0$ if and only if (A14) holds.

Proof
The proof of the sufficient part is the same as in the proof of Theorem 3.1.

To prove necessity, we assume that (26) holds. So, $\xi_1 > 0$ we have

$$\int_{\xi_1}^\infty b_1(\xi) d\xi + \sum_{i=1}^\infty b_2(\alpha_i) < \frac{a_4 - 1}{2L},$$

where $L = \max\{L_1, L_2\}$ and $L_1$ is the Lipschitz constant of $G$ on $[c, d]$, where $L_2 = G(d)$ such that

$$a = \frac{2(a_3^2 - a_4)}{2a_4^2a_4}, \quad b = \frac{a_3 - 1 + c}{a_3}, \quad c > \frac{a_4(a_3 + a_3^2 - 2)}{2(a_3^2 - a_4)} > 0.$$
Also, we can find $\xi_2 > 0$ such that $|F(\xi) - M| < \frac{1}{2}(a_3 - 1)$ for $\xi \geq \xi_2 > \xi_1$. Next, we define a Banach space $X$ as in the proof of Theorem 3.1 with respect to the sup norm

$$\|x\| = \sup \{ |v_1(\xi)| : \xi \geq \xi_2 \}.$$

Define

$$S = \{ v \in X : a \leq v(\xi) \leq b, \xi \geq \xi_2 \}.$$

Clearly, $S$ is a convex and closed subspace of $X$. Let $\Omega : S \to S$ be an operator defined by

$$(\Omega v)(\xi) = \begin{cases} 
(\Omega v)\bigl(\xi_2 + \rho\bigr), & \xi \in [\xi_2, \xi_2 + \rho], \\
\frac{v(\xi + \mu)}{b(\xi + \mu)} + \frac{F(\xi + \mu) - M}{b(\xi + \mu)} + \frac{c}{b(\xi + \mu)} + \frac{1}{\prod_{j=1}^{\infty} b_1(k)} \int_{\xi + \mu}^{\infty} b_1(k) G(v(\kappa - \mu_1)) \, d\kappa \\
+ \sum_{\xi_2 < \alpha_i < \xi_2 + \mu} b_2(\alpha_i) G(v(\alpha_i - \mu_1)), & \xi \geq \xi_2 + \rho.
\end{cases}$$

For every $v \in S$,

$$(\Omega v)(\xi) \leq \frac{G(d)}{b(\xi + \mu)} \left[ \int_{\xi + \mu}^{\infty} b_1(k) \, d\kappa + \sum_{\xi_2 < \alpha_i < \xi_2 + \mu} b_2(\alpha_i) \right] + \frac{a_3 - 1}{2b(\xi + \mu)} + \frac{c}{b(\xi + \mu)}$$

and

$$(\Omega v)(\xi) \geq \frac{v(\xi + \mu)}{b(\xi + \mu)} + \frac{F(\xi + \mu) - M}{b(\xi + \mu)} + \frac{c}{b(\xi + \mu)} - \frac{b}{a_3} \frac{a_3 - 1}{2a_3} + \frac{c}{a_4} + \frac{2c(a_3^2 - a_4) - a_4(a_3 - 2 + a_3^2)}{2a_3^2 a_4} = a$$

implies that $\Omega \in S$. For $v_1, v_2 \in S$,

$$\left| (\Omega v_1)(\xi) - (\Omega v_2)(\xi) \right| \leq \frac{1}{b(\xi + \mu)} \left| v_1(\xi + \mu) - v_2(\xi + \mu) \right|$$

$$+ \frac{G(b)}{b(\xi + \mu)} \left[ \int_{\xi + \mu}^{\infty} b_1(k) \left| v_1(\kappa - \mu_1) - v_2(\kappa - \mu_1) \right| \, d\kappa \\
+ \sum_{\xi_2 < \alpha_i < \xi_2 + \mu} b_2(\alpha_i) \left| v_1(\alpha_i - \mu_1) - v_2(\alpha_i - \mu_1) \right| \right],$$

that is,

$$\left| (\Omega v_1)(\xi) - (\Omega v_2)(\xi) \right|$$

$$\leq \frac{1}{a_3} \|v_1 - v_2\| + \frac{G(b)}{a_3} \|v_1 - v_2\| \left[ \int_{\xi + \mu}^{\infty} b_1(k) \, d\kappa + \sum_{\xi_2 < \alpha_i < \xi_2 + \mu} b_2(\alpha_i) \right]$$

$$< \left( \frac{1}{a_3} + \frac{a_3 - 1}{2a_3} \right) \|v_1 - v_2\|,$$
that is, \( \Omega \) is a contraction mapping with the contraction \((\frac{1}{a_3} + \frac{a_3-1}{2a_3}) < 1\). Hence, by the Banach fixed point theorem, \( \Omega \) has a unique fixed point on \([a, b]\) which is a non-oscillatory (specially positive) solution of system (E).

Thus, the theorem is proved. \( \square \)

**Theorem 3.3** Under the assumptions \(-1 < -a_5 \leq b(\xi) \leq 0 \) for \( \xi \in \mathbb{R}^+, \ a_5 > 0 \) and (A1), every solution of system (E) either oscillates or \( \lim_{\xi \to \infty} \nu(\xi) = 0 \) if and only if (A14) holds.

**Proof** For the sufficient part, we follow the proof of Theorem 3.1 to show that \( \nu(\xi) \) is bounded when \( H(\xi) > 0 \) for \( \xi \geq \xi_2 \). Also, \( \nu(\xi) \) is bounded when \( H(\xi) < 0 \) for \( \xi \geq \xi_2 \). Consequently, \( \lim_{\xi \to \infty} H(\xi) \) exists and hence \( \lim_{\xi \to \infty} h(\xi) \) exists. By Theorem 3.1, it is easy to prove that \( \lim_{\xi \to \infty} \nu(\xi) = 0 \), and by Lemma 3.1, \( \lim_{\xi \to \infty} h(\xi) = 0 \). So,

\[
0 = \lim_{\xi \to \infty} h(\xi) = \lim_{\xi \to \infty} \left( \nu(\xi) + b(\xi)\nu(\xi - \mu) \right)
\geq \lim_{\xi \to \infty} \nu(\xi) + \limsup_{\xi \to \infty} (-a_5\nu(\xi - \mu))
= (1 - a_5)\limsup_{\xi \to \infty} \nu(\xi)
\]

implies that \( \limsup_{\xi \to \infty} \nu(\xi) = 0 \). The rest of the sufficient part comes from the proof of Theorem 3.1.

Next, we suppose that (26) holds. Then there exist \( \xi_1, \xi_2 > 0 \) such that

\[
\int_{\xi_1}^{\infty} b_1(\xi) \, d\xi + \sum_{i=1}^{\infty} b_2(\alpha_i) < \frac{1-a_5}{10G(1)} \quad \text{for} \quad \xi \geq \xi_1
\]

and \( |F(\xi) - M| < \frac{1-a_5}{20} \) for \( \xi \geq \xi_2 \). Next, we define a Banach space \( X \) as in the proof of Theorem 3.1 with respect to the sup norm

\[
\|x\| = \sup\{ \nu(\xi) : \xi \geq \xi_2 \}.
\]

Let \( K = \{ x \in X : \nu_1(\xi) \geq 0, \xi \geq \xi_3 \} \). Then \( X \) is a partially ordered Banach space (see, for instance, [54], p. 30). For \( x, y \in X \), we define \( x \leq y \) if and only if \( x - y \in K \). Let

\[
S = \left\{ \nu \in X : \frac{1-a_5}{20} \leq \nu(\xi) \leq 1, \xi \geq \xi_3 \right\}.
\]

If \( \nu_0(\xi) = \frac{1-a_5}{20} \), then \( \nu_0 \in S \) and \( \nu_0 = \text{g.l.b} S \). Further, if \( \Phi \subset S^* \subset S \), then

\[
S^* = \left\{ \nu \in X : l_1 \leq \nu(\xi) \leq l_2, \frac{1-a_5}{20} \leq l_1, l_2 \leq 1 \right\}.
\]
Let \( v_0(\xi) = l'_2, \xi \geq \xi_4 \), where \( l'_2 = \sup\{l_2 : \frac{1-a_5}{20} \leq l_2 \leq 1\} \). Then \( v_0 \in S \) and \( v_0 = \text{l.u.b} \ S^* \). For \( \xi_4 = \xi_3 + \rho \), define \( \xi : S \to S \) by

\[
(\Omega \nu)(\xi) = \begin{cases} 
(\Omega \nu)(\xi_4), & \xi \in [\xi_3, \xi_4], \\
-b(\xi)\nu(\xi - \mu) + \frac{1-a_5}{10} + (F(\xi) - M) \\
+ \int_{\xi}^{\infty} b_1(\kappa)G(\nu(\kappa - \mu_1)) \, d\kappa \\
+ \sum_{\xi_4 \leq \alpha_i < \xi} b_2(\alpha_i)G(\nu(\alpha_i - \mu_1)), & \xi \geq \xi_4.
\end{cases}
\]

For every \( \nu \in S \),

\[
(\Omega \nu)(\xi) \leq a_5 + G(1) \left[ \int_{\xi}^{\infty} b_1(\kappa) \, d\kappa + \sum_{\xi_4 \leq \alpha_i < \xi} b_2(\alpha_i) \right] + \frac{1-a_5}{20} + \frac{1-a_5}{10} < \frac{1+3a_5}{4} < 1
\]

and

\[
(\Omega \nu)(\xi) \geq \frac{1-a_5}{10} + (F(\xi) - M) \frac{1-a_5}{10} - \frac{1-a_5}{20} = \frac{1-a_5}{20}
\]

implies that \( \Omega \nu \in S \). Now, for \( \nu_1, \nu_2 \in S \), it is easy to verify that \( \nu_1 \leq \nu_2 \) implies that \( (\Omega \nu_1) \leq (\Omega \nu_2) \). Hence, by the Knaster–Tarski fixed point theorem (see, e.g., [54], Theorem 1.7.3), \( \Omega \) has a unique fixed point such that \( \lim_{\xi \to \infty} \nu(\xi) \neq 0 \).

This completes the proof of the theorem. \( \square \)

**Theorem 3.4** Under the assumptions \(-\infty < -a_5 \leq b(\xi) \leq -a_6 < -1 \) for \( \xi \in \mathbb{R}_+ \), \( a_5, a_6 > 0 \), (A1), and \( G \) is Lipschitzian on the intervals of the form \([c, d]\), where \( 0 < c < d < \infty \), every bounded solution of (E) either oscillates or \( \lim_{\xi \to \infty} \nu(\xi) = 0 \) if and only if (A14) holds.

**Proof** The proof is totally the same as in the proof of Theorem 3.2, but, for the necessary part, we provide the following settings:

\[
\int_{\xi_1}^{\infty} b_1(\kappa) \, d\kappa + \sum_{i=1}^{\infty} b_2(\alpha_i) < \frac{a_6 - 1}{2L} \quad \text{and} \quad |F(\xi) - M| > \frac{1}{2}(a_6 - 1),
\]

where \( L = \max\{L_1, L_2\} \) and \( L_1 \) is the Lipschitz constant of \( G \) on \([c, d]\), where \( L_2 = G(d) \) such that

\[
a = \frac{ca_6 - a_5(a_6 - 1)}{a_5a_6},
\]

\[
b = \frac{1 + \frac{c}{a_6 - 1}}{2},
\]

\[
c > \frac{a_5(a_6 - 1)}{a_6} > 0,
\]
and
\[
(\Omega v)(\xi) = \begin{cases} 
\{\Omega v\}(\xi_2 + \rho), & \xi \in [\xi_2, \xi_2 + \rho] \\
\frac{\nu(\xi_1 + \mu)}{b(\xi + \mu)} + \frac{\nu(s) - M}{b(s)} - \frac{c}{b(s)} \\
+ \frac{1}{b(\xi + \mu)} \left( \int_{\xi_2 + \mu}^{\infty} b_1(s) G(\nu(s - \mu_1)) \, ds \right) \\
+ \sum_{\xi_2 + \mu \leq \alpha_i < \xi_2} b_2(\alpha_i) G(\nu(\alpha_i - \mu_1)), & \xi \geq \xi_2 + \rho.
\end{cases}
\]

This completes the proof of the theorem. □

**Remark 3.1** In Theorems 3.1–3.4, we do not have any restriction on \(G\) (that is, \(G\) could be linear, sublinear, or superlinear).

### 4 Conclusion

In [20], the author studied the oscillatory behavior of solutions of the impulsive system

\[
\begin{cases} 
(\nu(\xi) + b(\xi)\nu(\xi - \mu))' + b_1(\xi) G(\nu(\xi - \mu_1)) = 0, & \xi \neq \alpha_i, i \in \mathbb{N}, \\
\Delta(\nu(\alpha_i) + B_2(\alpha_i)\nu(\alpha_i - \mu)) + q_i G(\nu(\alpha_i - \mu_1)) = 0, & i \in \mathbb{N},
\end{cases}

(E1)
\]

under the sufficient condition

\[
\int_{0}^{\infty} b_1(\xi) \, d\xi + \sum_{i=1}^{\infty} b_1(\alpha_i) = \infty. \tag{27}
\]

Because of Theorem 3.1 [20], (27) could be a sufficient and necessary condition for the oscillatory and asymptotic behavior of solutions of system (E1) for different ranges of the neutral coefficient \(b(\xi)\). We guess that (27) could be a sufficient and necessary condition for the oscillation of a non-homogeneous counterpart of (E). In this work, we have obtained sufficient conditions for the oscillation of (E), which is presented in Sect. 2, and in Sect. 3 we have established sufficient and necessary conditions for the oscillatory or asymptotic behavior of (E).

It would be of interest to examine the oscillation of (E) with a different neutral coefficient; see, e.g., the papers [43, 46, 47, 50–53] for more details. Furthermore, it is also interesting to analyze the oscillation of (E) with a nonlinear neutral term; see, e.g., the paper [49] for more details.

**Remark 4.1** Theorems 3.1–3.4 hold true for \(M = 0\).

**Remark 4.2** Lemma 3.1 does not hold when \(b(\xi) \equiv 1\) for all \(\xi\) (see, e.g., [54]), and the present study does not allow us when \(b(\xi) \equiv -1\) for all \(\xi\). Thus, in this paper, we have obtained necessary and sufficient conditions for the oscillatory or asymptotic behavior of (E) except \(b(\xi) = \pm 1\) for all \(\xi\). Hence, it is clear that a different method is necessary to study the oscillatory or asymptotic behavior of (E) when \(b(\xi) = \pm 1\). However, we have established sufficient conditions for \(b(\xi) = \pm 1\) in Sect. 2.

### 5 Examples

In this section, we provide two examples to validate our main results.
Example 5.1 Consider the impulsive system

\[
\begin{align*}
\left\{
\begin{array}{l}
(v(\xi) + 2v(\xi - 1))' + 4(\xi - 1)^3 v^3(\xi - 1) = -\frac{2}{\xi^2}, \quad \xi > 1, \\
\Delta (v(\alpha_i) + 2v(\alpha_i - 1)) + b_2(\alpha_i) v^3(\alpha_i - 1) = - \frac{4b_2^2}{(\nu - h_i)^2},
\end{array}
\right.
\end{align*}
\]

(E1)

where \( b_2(\alpha_i) = \frac{8b_2^2(2^i - 1)^2(2^i - h_i)^6}{((2^i - 1)^2(2^i - h_i)^2)^2} \), \( \alpha_i = 2^i, \ i \in \mathbb{N}, \) and \( G(\xi) = \xi^3. \) If we choose \( F(\xi) = \frac{1}{\xi^2} \), then \( F'(\xi) = -\frac{2}{\xi^3} \) and

\[
\Delta F(\alpha_i) = F(\alpha_i + h) - F(\alpha_i - h) \\
= F(2^i + h) - F(2^i - h) \\
= - \frac{4b_2^2}{(\nu - h_i)^2} = g(\alpha_i), \quad i \in \mathbb{N}.
\]

Clearly, \( (A14) \) holds. Since the conditions of Theorem 3.2 are true for \( (E_1) \), then every solution of \( (E_1) \) either oscillates or tends to zero as \( \xi \to \infty. \) In particular, \( v(\xi) = \frac{1}{\xi^2} \) is a solution of the impulsive system \( (E_1). \)

Example 5.2 Consider the impulsive system

\[
\begin{align*}
\left\{
\begin{array}{l}
(v(\xi) + v(\xi - \pi))' + v(\xi - \pi) = \cos(\xi - \frac{\pi}{4}), \quad \xi > \frac{\pi}{4}, \\
\Delta (v(\alpha_i) + v(\alpha_i - \pi)) + b_2(\alpha_i)v(\alpha_i - \pi) = 2\sin(h) \cos(k - \frac{\pi}{4}),
\end{array}
\right.
\end{align*}
\]

(E2)

where \( b_2(\alpha_i) = \frac{2}{\cos(\alpha_i)}, \alpha_i = i, \ i \in \mathbb{N}, \) and \( G(\xi) = \xi, f(\xi) = \cos(\xi - \frac{\pi}{4}). \) Indeed, if we choose \( F(\xi) = \sin(\xi - \frac{\pi}{4}) \), then \( F'(\xi) = f(\xi) \) and

\[
\Delta F(\alpha_i) = F(\alpha_i + h) - F(\alpha_i - h) \\
= F(i + h) - F(i - h) \\
= \sqrt{2} \sin(h)(\sin(i) + \cos(i)) = g(\alpha_i), \quad i \in \mathbb{N}.
\]

Clearly,

\[
F'(\xi) = \begin{cases} 
\sin(\xi - \frac{\pi}{4}), & 2n\pi + \frac{\pi}{4} \leq \xi < 2n\pi + \frac{5\pi}{4}, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
F^-(\xi) = \begin{cases} 
-\sin(\xi - \frac{\pi}{4}), & 2n\pi + \frac{5\pi}{4} \leq \xi < 2n\pi + \frac{3\pi}{4}, \\
0, & \text{otherwise},
\end{cases}
\]

imply that

\[
F^+(\xi - \frac{\pi}{4}) = \begin{cases} 
-\cos(\xi), & 2n\pi + \frac{\pi}{4} \leq \xi < 2n\pi + \frac{3\pi}{4}, \\
0, & \text{otherwise},
\end{cases}
\]
and

\[ F^{-}\left(\xi - \frac{\pi}{2}\right) = \begin{cases} \sin(\xi), & 2n\pi + \frac{3\pi}{2} \leq \xi \leq 2n\pi + \frac{5\pi}{2}, \\ 0, & \text{otherwise}. \end{cases} \]

Since

\[
\int_{\frac{\pi}{2}}^{\infty} F^{+}\left(\xi - \frac{\pi}{4}\right) d\xi = \sum_{n=0}^{\infty} \int_{2n\pi + \frac{3\pi}{2}}^{2n\pi + \frac{5\pi}{2}} \left[-\cos(\xi)\right] d\xi = \infty,
\]

then, for \( n = 0, 1, 2, \ldots \), we get

\[
\int_{\frac{\pi}{2}}^{\infty} F^{+}\left(\xi - \frac{\pi}{4}\right) d\xi + \sum_{i=1}^{\infty} \left(\frac{2}{1 + \cot(h)}\right) F^{+}\left(i - \frac{\pi}{4}\right) = \infty.
\]

Clearly, (A2)--(A6) are satisfied. Hence, by Theorem 2.1, every solution of \((E_2)\) is oscillatory. In particular, \(u_1(\xi) = \cos(\xi)\) is a solution of \((E_2)\).

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