POLARIMETRIC STUDIES OF MAGNETIC TURBULENCE WITH AN INTERFEROMETER

HYESEUNG LEE\(^1\), A. LAZARIAN\(^2\), AND JUNGYEON CHO\(^{1,2}\)

\(^1\) Department of Astronomy and Space Science, Chungnam National University, Daejeon, Korea
\(^2\) Department of Astronomy, University of Wisconsin, Madison, USA

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ABSTRACT

We study statistical properties of synchrotron polarization emitted from media with magnetohydrodynamic (MHD) turbulence. We use both synthetic and MHD turbulence simulation data for our studies. We obtain the spatial spectrum and its derivative with respect to the wavelength of synchrotron polarization arising from both synchrotron radiation and Faraday rotation fluctuations. In particular, we investigate how the spectrum changes with frequency. We find that our simulations agree with the theoretical prediction in Lazarian & Pogosyan. We conclude that the spectrum of synchrotron polarization and its derivative can be very informative tools to obtain detailed information about the statistical properties of MHD turbulence from radio observations of diffuse synchrotron polarization. They are especially useful for recovering the statistics of a turbulent magnetic field as well as the turbulent density of electrons. We also simulate interferometric observations that incorporate the effects of noise and finite telescope beam size, and demonstrate how we recover statistics of underlying MHD turbulence.

Key words: galaxies: clusters: intracluster medium – magnetohydrodynamics (MHD) – radio continuum: ISM – turbulence

1. INTRODUCTION

Turbulence with an embedded magnetic field is almost everywhere in the universe on a wide variety of scales, such as the interstellar medium (Elmegreen & Falgarone 1996; Elmegreen & Scalo 2004) and the intracluster medium (Emslilin & Vogt 2006; Lazarian 2006). Substantial theoretical progress in relating MHD turbulence with astrophysical processes has been made. The areas include star formation (Larson 1981; Elmegreen 2002; McKee & Tan 2002; Mac Low & Klessen 2004, Ballesteros-Paredes et al. 2007; McKee & Ostriker 2007), accretion disks (Balbus & Hawley 2002), the solar wind (Hartman & McGregor 1980; Podesta 2006; Wicks et al. 2012), magnetic reconnection (Lazarian & Vishniac 1999; Lazarian et al. 2015), and cosmic rays (Schlickeiser 2002).

Turbulence is a chaotic phenomenon. However, it allows for a very simple statistical description. The landmark achievements include the famous Kolmogorov statistical theory (Kolmogorov 1941) as well as Goldreich & Sridhar MHD turbulence theory (Goldreich & Sridhar 1995). The latter is the theory relevant to most magnetized astrophysical fluids, including magnetic fields responsible for most of the Galactic and extragalactic synchrotron emission.\(^3\)

Because no in situ measurements of turbulence are possible beyond the very limited volume of the interplanetary medium and the solar wind, it is challenging to obtain turbulence statistics from observations. This area has been a focus of intensive theoretical and observational research for a number of decades with significant progress recently achieved (Munch 1958; Munch & Wheelon 1958; Chepurnov et al. 2010, 2015; Burkhardt et al. 2012; Brunt & Heyer 2013).

\(^3\) We note parenthetically that the first attempts to formulate the MHD turbulence theory can be traced back to the classical works of Iroshnikov (1964) and Kraichnan (1968). Later advances include Montgomery & Turner (1981), Shebalin et al. (1983), and Higdon (1984). For further advancements of the theory and its testing, one can refer to a number of papers that include Lazarian & Vishniac (1999), Cho & Vishniac (2000), Maron & Goldreich (2001), and Cho et al. (2002). The extension of MHD turbulence theory for compressible turbulence can be found in Lithwick & Goldreich (2001), Cho & Lazarian (2002a, 2002b, 2003), and Kowal & Lazarian (2007). Recent reviews on the subject include Brandenburg & Lazarian (2013) and Beresnyak & Lazarian (2015, p. 163).
of the Milky Way galaxy as well as for other galaxies and even clusters of galaxies. This research can help us to constrain the driving and understand how energy is injected on large scales and transferred to smaller scales. Furthermore, our understanding of the spectrum can make it possible to produce a precise polarization map arising from magnetized turbulence and remove the synchrotron foreground in future CMB polarization observations.

We expect that the high sensitivity of new generation telescopes, e.g., the Square Kilometer Array (SKA) and the LOw Frequency ARray (LOFAR), will carry numerous information to map synchrotron polarization (Beck & Wielebinski 2013). For example, cosmic-ray electrons with relatively low energies (~GeV) that originated from supernova remnants in the Galactic disk generate synchrotron emission at low frequencies. LOFAR can detect their propagation and evolution process at low frequencies, which result in fluctuations of polarization, and their relation to the properties of turbulent magnetic field. Moreover, the observational facility, such as the VLA and Australian Square Kilometre Array Pathfinder (ASKAP), can produce considerable observations for making sensitive polarization images of the entire sky. The Polarization Sky Survey of the universe’s Magnetism (POSSIM; Gaensler et al. 2010) conducted with ASKAP is an ongoing project focused on Faraday structure determination, which covers a frequency range from 1100 MHz to 1400 MHz (Sun et al. 2015). In the future, we will be able to reconstruct the polarization spectra of synchrotron emission and Faraday rotation by comparing with those observations, and obtain detailed magnetic field statistics from polarization observations with SKA at both lower frequencies and higher frequencies in the Milky Way and intracluster medium.

In this paper, we investigate spectral behavior of polarized synchrotron fluctuations in the presence of Faraday rotation. Since the effects of Faraday rotation are proportional to $\chi^2$, where $\lambda$ is the wavelength, we show how the spectrum changes as the wavelength increases. We describe numerical methods in Section 2. We present results in Section 3, discussions in Section 4, and a summary in Section 5. In Section 3, we include calculations for interferometric observations.

2. NUMERICAL METHODS

2.1. Numerical Simulations

The use of synthetic data is the simplest approach to simulate turbulence. We generate 3D synthetic data cubes of magnetic field and electron density in a periodic box of size $2\pi$ in the Cartesian coordinate system $(x, y, z)$, where $z$ is along the LOS. The numerical resolution is $512^3$ for all synthetic data. In general, the magnetic field ($B$) consists of a uniform background field ($B_0$) and a fluctuating field ($b$). However, for synthetic magnetic fields, we assume $B_0 = 0$, so that $B = b$. The generation of the synthetic data is actually done in Fourier space:

$$B(x) = B_0 + \sum_{k} A(k) \tilde{e}(k),$$  

(1)

where $k_{\text{max}} = 512/2$, $A(k)$ is a real vector that is perpendicular to the wavevector $k$, $\chi = k \cdot x + \phi(k)$, $x = (x, y, z)$, and $\phi(k)$ is a random number in the range of 0 and $2\pi$. We enforce $\phi(-k) = -\phi(k)$ for a reality condition. The amplitude of each Fourier component randomly fluctuates, but on average the

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Footnote:

4 The numerical study in Herron et al. (2016) successfully tested the predictions in LP12 for fluctuating synchrotron intensities.
amplitude follows $|A(k)|^2 = Ck^m$, where $m$ is a constant (e.g., $m = -11/3$ for Kolmogorov spectrum) and $C$ is a normalization constant that makes the rms magnetic field fluctuation of the order of unity. Fluctuations of magnetic field reflect the power-law spectra of the underlying magnetohydrodynamic (MHD) turbulence (Cho & Lazarian 2010). Since it is the best-known spectrum for turbulence, we start with a Kolmogorov spectrum but we also use other spectral indices. For magnetic field and density, the peak of the spectrum appears at $k = 2$. Density consists of a uniform density and a fluctuating density. The uniform density is set to one and the rms density is of the order of unity. When the power-law index for density spectrum is $-11/3$, the product of the uniform density ($\rho_0$) and the fluctuating magnetic field ($B$) is slightly larger than that of the uniform magnetic field ($B_0$) and the fluctuating density ($\delta \rho$), i.e., $\rho_0 B > B_0 \delta \rho$. On the other hand, when the power-law index for the density spectrum is $-3$, $\rho_0 b$ is still larger than $B_0 \delta \rho$ on large scales. However, since the density spectrum is shallower than the magnetic spectrum, the fluctuation of density decreases relatively slowly as the scale decreases. As a result, $\rho_0 b$ becomes smaller than $B_0 \delta \rho$ on small scales. The transition happens at near $k \sim 60$ (see Figure 1). In some runs, we only use the uniform density.

We also simulate ideal isotropic MHD turbulence using a code based on a third-order accurate hybrid essentially non-oscillatory (ENO) scheme in a periodic box of size $2\pi$:

$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$

$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \rho^{-1} \nabla (\rho \mathbf{a}^2 \mathbf{v}) - (\nabla \times \mathbf{B}) \times \mathbf{B} / 4\pi \rho = \mathbf{f}$

$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0,$

with $\mathbf{v} \cdot \mathbf{B} = 0$, where $\rho$ is density, $a$ is the sound speed, $f$ is a random driving vector, and other variables have their usual meanings. The rms velocity is maintained to be approximately unity, so that $v$ can be viewed as the velocity measured in units of the rms velocity of the system. The simulation was performed with a resolution of $512^3$ grid points. We drive turbulence solenoidally in Fourier space. Both the Alfvén mach number ($M_A = \nu_{rms} / V_A$, where $\nu_{rms}$ is the rms velocity and $V_A$ is the Alfvén speed) and the sonic Mach number ($M_s = \nu_{rms} / a$) of the turbulence are approximately 0.7. For the calculation of synchrotron polarization (in Section 3.4.3), we take the magnetic field directly from the simulation data and assume the electron number density is proportional to $\rho$. In actual calculations, we use $n_e = \rho$.

### 2.2. Polarization from Synchrotron Radiation

The magnetic field, $\mathbf{B}$, and the electron energy distribution determine synchrotron emission. We assume an isotropic pitch angle distribution and a power-law energy distribution of electron population characterized by the power-law index of electron energy distribution, $p$:

$$N(E) dE = N_0 E^{-p} dE,$$

where $E$ is the electron energy, $N$ is the number of electrons per $E$ per unit volume, and $N_0$ represents the homogeneous distribution of relativistic electrons.

The combination of the Stokes parameters provides a valuable description of synchrotron polarization. In this paper, we focus only on linear polarization defined by the Stokes parameters $Q$ and $U$:

$$P = Q + iU.$$  \hfill (3)

We can write polarized intensity observed at a two-dimensional position $X$ on the plane of the sky at wavelength $\lambda$ as follows.

$$P(X, \lambda^2) = \int_0^L dz P_j(X, z) e^{2i\lambda \Phi(X, z)},$$  \hfill (4)

where $X = (x, y)$, $P_j$ is the intrinsic polarized intensity density, $L$ is the extent of the source along the LOS, and the exponential factor describes Faraday rotation from the source point at $z$ along the line of sight to the observer. The Faraday rotation measure, $\Phi(X, z)$, is given by

$$\Phi(X, z) = \int_0^L \left( \frac{n_e(z)}{0.01 \text{ cm}^{-3}} \right) \left( \frac{B_z(z)}{1.23 \mu \text{G}} \right) \left( \frac{dz}{100 \text{ pc}} \right) \text{ rad m}^{-2},$$  \hfill (5)

where $n_e$ is the number density of electrons, and $B_z$ is the strength of the parallel component to the LOS component of the magnetic field. Here, we assume that the rotation measure is only attributed to the source region but not any intervening material between the observer and the source region along the LOS. This normalization is equivalent to the assumption that polarized synchrotron radiation is emitted from a region (e.g., Galactic halo) that has a size of 100 pc and is located at 1 kpc from the observer, the electron number density in this region is 0.01 cm$^{-3}$, and the magnitude of the magnetic field is 1.23 $\mu$G. Using this normalization, we can convert polarization in computational units (Sections 3.1–3.3) to real units (Section 3.4).
2.3. Data Set for the Interferometer

The interferometric observational data can be directly used to obtain the spectra of turbulence (see LP16). In Sections 3.3 and 3.4, we simulate realistic interferometric observations with a finite beam size and a noise. The procedure of simulating an interferometric observation that we employ is as follows.

1. Using the whole 3D magnetic field and density data, we calculate synchrotron polarization map projected on the plane of the sky for all LOSs.
2. To reflect a finite beam size of telescopes, we smooth the map using a Gaussian kernel:

\[ g(x, y) \propto \exp\left(-\frac{(x^2 + y^2)}{2\sigma_{\text{beam}}^2}\right), \]

(6)

where \( \sigma_{\text{beam}} (= \theta_{\text{FWHM}}/2\ln 2) \) is the resolution of the telescopes.
3. We obtain the complete 2D power spectrum of the smoothed polarization map.
4. To mimic an interferometric observation, we randomly select the wave-vectors corresponding to the baselines of a telescope array in Fourier space.
5. We identify the 2D power spectrum at each selected wave-vector, \( \mathbf{K} \).
6. We add a Gaussian random noise to the 2D power spectrum at each selected wave-vector. The amplitude of the noise is independent of the wavenumber.

3. RESULTS

3.1. Statistics: Power Spectrum

3.1.1. Statistics without Faraday rotation

The synchrotron emissivity depends on \( \mathbf{B}_i \), where \( \mathbf{B}_i \) is the plane-of-the-sky magnetic field. It was predicted in LP12 that the variations of the spectral index (\( \gamma \)) of relativistic electron energy distribution change the amplitude of the fluctuations, but not the spectral slope of the synchrotron power spectrum. The polarized synchrotron emissivity also depends on \( \mathbf{B}_i \). Therefore, we expect a similar behavior for polarized synchrotron emission. We test this below.

The spectrum of polarized synchrotron radiation moves upward as \( \gamma \) increases as shown in Figure 2(a). We explore the dependence of the synchrotron polarization statistics, i.e., power spectrum, on the power-law index \( \gamma \). We consider \( \gamma \) ranging from 1.5 to 4, which cover all possible important cases of astrophysical power-law indices. To see the effect of \( \gamma \), we fix the spectrum of turbulence: we use Kolmogorov spectra for magnetic field and density. As we can see in Figure 2(a), the spectra exhibit extended power laws at large wavenumbers for all values of \( \gamma \) shown in the figure. The measured power-law slopes at large wavenumbers are very close to that of a Kolmogorov spectrum, i.e., \(-8/3\) for ring-integrated 1D spectrum \( E_{2D}(\mathbf{K}) \) (compare the spectra with the black solid straight line in the figure). We can also see that the compensated spectra have plateaus at large wavenumbers, which means that the ring-integrated 1D spectra \( E_{2D}(\mathbf{K}) \) at large wavenumbers are proportional to \( K^{-8/3} \).

To see the dependence of the spectrum on \( \gamma \) more quantitatively, we plot the amplitude of polarized synchrotron emission spectra for different spectral indices in Figure 2(b). The red solid line is obtained from Equation (37) in LP12 and shows the dependence of amplitude on spectral index, and the black plus signs represent measured amplitudes of polarized synchrotron emission normalized by \( E_{2D,\gamma=2} \). We can clearly see that the measured spectral amplitude is in agreement with the prediction of LP12.

3.1.2. Effects of Faraday Rotation

Faraday rotation depends on wavelength as well as the electron number density and the strength of magnetic field parallel to the LOS (Equation (14)). To see the effects of Faraday rotation only, we calculate synchrotron polarization with uniform intrinsic polarized emission. That is, we use \( Q/I = 1 \) and \( U/I = 0 \) for intrinsic synchrotron emission at all points in space and calculate Faraday rotation. We, however, use fluctuating LOS magnetic field and electron number density. The average electron number density is set to one. We plot the results in Figure 3. In Figure 3(a), we can see that the spectrum goes up as the wavelength increases, which means that the polarization deviates more and more from \( Q/I = 1 \) and \( U/I = 0 \) at longer wavelengths. When \( \lambda \gtrsim 1 \), the small-\( K \) part
of the spectrum no longer moves up with increasing wavelength, while the large-K part of the spectrum continues to move up.

The reason why the small-K part of the spectrum does not move up when $\lambda \gtrsim 1$ is Faraday depolarization effect. Figure 3(a) suggests that Faraday depolarization happens on the large scale first. In fact, Faraday depolarization is significant when $K \lesssim 2\pi \lambda^2 \langle n_e | B_\parallel | \rangle$, and it is insignificant when $K \gg 2\pi \lambda^2 \langle n_e | B_\parallel | \rangle$ (in the synthetic data, $\langle n_e | B_\parallel | \rangle \sim 0.5$).5

The reason why the small-K part of the spectrum moves up is that the depolarization effect is significant becomes smaller and smaller. When $\lambda^2 \approx \frac{K_{\text{min}}}{2\pi \langle n_e | B_\parallel | \rangle}$, the polarized emission from each grid point becomes completely uncorrelated and the emission from each grid point contributes randomly,6 which makes the spectrum proportional to $K$.

In the previous example in this paragraph, we calculated Faraday rotation, assuming a constant intrinsic synchrotron polarization. When we use a turbulence spectrum for the plane-of-the-sky magnetic field, which is responsible for synchrotron emission, we may be able to see the effect of Faraday depolarization more clearly. For simplicity, let us assume that the density and the LOS component of the magnetic field are constants. This is definitely a toy model to probe the effects of Faraday rotation induced by the simplest realization of the Faraday rotation effect. In Figure 3(b), we present the resulting spectrum of synchrotron polarization in which we use a Kolmogorov spectrum for the plane-of-the-sky magnetic field. When $\lambda$ is very small (e.g., see the black solid curve), the Faraday rotation is small and the spectrum does not suffer from the Faraday depolarization effect. As $\lambda$ increases, the spectrum at small $K$ decreases due to the Faraday depolarization effect. The fact that the spectrum at large $K$ does not show much change implies that the depolarization effect is negligible at large $K$.

A similar effect may be present also in the case when the Faraday rotation effect is induced by a regular magnetic field $B_0$ and fluctuations of density. In this case, we may also see a distortion of the spectrum at small $K$ and the preservation of the spectral slope at large $K$. The effect of additional fluctuations arising from the Faraday fluctuations may increase the amplitude of the measured spectrum.

Dependence of the spectrum on wavelength is very different for Faraday rotation fluctuations (Figure 3(a)) and for synchrotron polarization fluctuations (Figure 3(b)). When Faraday rotation fluctuations are much larger than synchrotron polarization fluctuations, the spectrum will show a change with wavelength, as plotted in Figure 3(a). When synchrotron polarization fluctuations are dominant, the spectrum will be changed as wavelength increases as plotted in Figure 3(b). We can use these behaviors to distinguish two cases.

### 3.1.3. Polarization by Synchrotron Radiation and Faraday Rotation

From Equation (5), we expect that fluctuations of electron density also introduce fluctuations of Faraday rotation, which will change the shape of the spectrum at long wavelengths. In this subsection, we take into account fluctuations of electron density as well as magnetic field and calculate synchrotron polarization.

We first assume that both density and magnetic field follow Kolmogorov spectra. Figure 4(a) shows the resulting spectrum of polarization. When the wavelength $\lambda$ is very small (e.g., see the black solid curve), Faraday rotation is small and the spectrum reflects that of pure polarized synchrotron emission. As $\lambda$ increases, the effect of Faraday rotation begins to change the shape of the spectrum. It is interesting that not only the small-$K$ part but also the large-$K$ part of the spectrum is affected by Faraday rotation. The small-$K$ part of the spectrum goes down due to the depolarization effect. The rise of the spectrum at large-$K$ might be due to additional fluctuations in the Faraday rotation measure arising from fluctuations in electron number density. According to Figure 4(a), the power-law slope of the polarization spectrum does not change

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5 In this paper, we assume that the size of the computational domain is $2\pi$. If the size of the system is $L$, then the expression becomes $K \lesssim L^2 \langle n_e | B_\parallel | \rangle$, where $K$ is the number of waves that exist over the size $L$.

6 When this happens, all Fourier modes have similar powers. In this case, the ring-integrated energy spectrum, $E_{2D}(K)$, becomes proportional to the number of Fourier modes in $K - 0.5 < |K| < K + 0.5$, which makes the spectrum proportional to $K$. 

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Figure 3. (a) Ring-integrated 1D spectrum $E_{2D}(K)$ arising only from fluctuations of Faraday rotation at different wavelengths in code units. We use fluctuating electron number density and $B_\parallel$, but uniform intrinsic polarized emission. The asterisks mark the wavenumber $K = 2\pi \lambda^2 \langle n_e | B_\parallel | \rangle$. When $K$ is smaller than this wavenumber, Faraday depolarization is significant. (b) Ring-integrated 1D spectrum $E_{2D}(K)$ arising only from fluctuations of synchrotron emission. We use a fluctuating plane-of-the-sky magnetic field, but uniform electron density and magnetic field along the LOS.
much with increased wavelength at large $K$. As $\lambda$ increases, Faraday depolarization becomes more and more important and the portion of the spectrum that shows a power-law dependence on $K$ gets limited. Therefore, if we are interested in measuring the power-law slope, we will observe synchrotron polarization at small wavelengths.

In the case of Figure 4(a), we have assumed that both the density and the magnetic field have the same spectral indices, i.e., both of them have Kolmogorov spectra. However, they can have different spectral slopes in real astrophysical environments. Therefore, we perform a calculation in which the density and the magnetic field have different spectral slopes. To be specific, we assume that the 3D spectral index of the magnetic field is $-11/3$ (i.e., the same as Kolmogorov) and that of density is $-3$ (i.e., shallower than Kolmogorov; see Figure 1). Figure 4(b) shows the resulting spectra of polarization at different wavelengths. When the wavelength $\lambda$ is very small (e.g., see the black solid curve), Faraday rotation is small and the spectrum reflects that of pure polarized synchrotron emission. The spectral slope at large wavenumbers is compatible with $-8/3$. As $\lambda$ increases, the effect of Faraday rotation begins to change the shape of the spectrum. The power at small wavenumbers goes down due to the Faraday depolarization effect, and the power at large wavenumbers goes up due to increased fluctuations in Faraday rotation. The spectral slope at large wavenumbers becomes shallower as $\lambda$ increases, implying that shallow density spectrum influences the small-scale polarization spectrum. The overall behavior of the spectra is in agreement with the prediction in LP16.

3.3. Effect of Telescope Resolution

The resolution of telescopes affects the spectrum of synchrotron polarization because features smaller than the telescope resolution are smoothed out. To reflect a finite beam size of telescopes, we smooth the polarization map using a Gaussian kernel as described in Section 2.3. In Figure 6, we present power spectra of polarized emission smoothed with different telescope beam resolutions $\theta_{\text{FWHM}}$, which is given in units of grid points. The black solid curve represents the power spectrum without smoothing, which shows a well-defined power law at large wavenumbers. As $\theta_{\text{FWHM}}$ increases, the spectrum gradually deviates from the power law. In fact, we expect the functional form of the spectrum to be proportional to

$$K^m \exp(-K^2/2\sigma_K^2),$$

where $\sigma_K = 1/(\sqrt{2\pi}\sigma_{\text{beam}})$.

Figure 4. Ring-integrated 1D spectrum $E_{\text{pol}}(K)$ arising from fluctuations of synchrotron radiation and Faraday rotation. We use fluctuating electron number density and fluctuating B of synthetic data to calculate the spectrum of polarized synchrotron emission. (a) The 3D power spectrum $P_{\text{pol}}(k)$ is proportional to $k^{-11/3}$ for both electron number density and B. (b) The 3D power spectrum $P_{\text{pol}}(k)$ is proportional to $k^{-9/3}$ for electron number density and $k^{-11/3}$ for B. Different curves correspond to different wavelengths. The asterisks mark the wavenumber $K = 2\pi\lambda^2/(\nu_0 |B|)$. When $K$ is smaller than this wavenumber, Faraday depolarization is significant.
As $\theta_{FWHM}$ becomes larger than $\sim 20$, which corresponds to $\sim 1/25$ of the size of the computational domain, the spectra show virtually no power-law inertial range. Obviously, it is advantageous to perform high-resolution observations to reveal the true power-law spectrum. Therefore, one should employ high-resolution observations with ground-based interferometry to get the true power-law spectrum.

### 3.4. Study of Turbulence with Interferometer

The advantage of the technique suggested in LP16 is that one can directly use interferometric data to get spectra of turbulence. Therefore, there is no need for restoring the polarization image of the turbulent object. We numerically explore this possibility by using interferometric measurements with just a few baselines. The ability to measure the synchrotron spectrum this way stems from the fact that the statistics of turbulence is a rather simple object with a high degree of symmetries (see more details in LP16). In what follows, we take a few interferometric measurements of synthetic observations of the polarized synchrotron emission and add noise to them to simulate realistic observations. We explore how many baselines we require to recover the information about the underlying spectrum of turbulence.

#### 3.4.1. Effect of the Number of Baselines

If we use an interferometer that consists of $N$ telescopes for a short amount of time, we can obtain 2D power spectrum for $N_{base} = N(N - 1)/2$ wave-vectors that correspond to the baselines. On the other hand, if we make use of Earth’s rotation as in typical interferometric observations, we can increase the number of baselines $N_{base}$. Since we do not cover the whole wave-vector space, it may be difficult to recover the true power spectrum if $N_{base}$ is small. Then, what will be a reasonable value for $N_{base}$? In Figure 7, we demonstrate that $N_{base,30} = 30 \times 29/2$ or $N_{base,60} = 60 \times 59/2$ is good enough to reconstruct the true power spectrum. The black solid curves correspond to the true power spectrum, which is calculated from complete Fourier modes without any missing components. In order to mimic an interferometric observation with $N_{base}$ baselines, we randomly select $N_{base}$ wave vectors and calculate an average power spectrum based on the power at the selected wave vectors:

$$P_{2D}(K) = \langle |\tilde{P}(K)|^2 \rangle = \frac{1}{N_K} \sum_{i=1}^{N_K} |\tilde{P}(K)|^2,$$

where the summation is taken over $K - 0.5 < |K| < K + 0.5$, $\tilde{P}(K)$ is the Fourier transform of $P(k)$, and $N_{K}$ is the number of observed wave vectors in the range of $K - 0.5 < |K| < K + 0.5$. The red symbols in Figure 7 represent the simulated average 2D power spectrum. In Figures 7(a) and (b), we assume that $N_{base} = N_{base,30}$ and $N_{base,60}$, respectively. The simulated average spectrum for $N_{base,30}$ shows a larger scatter than that for $N_{base,60}$. Nevertheless, both $N_{base,30}$ and $N_{base,60}$ can reproduce the true spectrum reasonably well.

#### 3.4.2. Effects of Noise and Finite Beam Size: Simulations with Synthetic Data

First, we use synthetic data to see how the power-law slope is affected by noise. We follow the procedure of simulating a realistic interferometric observation described in Section 2.3. We assume $N_{base} = 30 \cdot 29/2 (= N_{base,30})$. Figure 8 shows the...
2D power spectra at $\lambda \sim 1$ cm, at which the effect of Faraday rotation is non-negligible but still not significant. The black solid curve corresponds to the true spectrum without noise, which is calculated from the complete Fourier modes without any missing components. The blue asterisk, the green diamond, and the red square symbols denote average 2D spectra from interferometric observations with random Gaussian noise. Different symbols represent different noise levels: the noise levels for the blue, the green, and the red symbols are 1%, 10%, and 20% of the standard deviation of the true signal, respectively. The blue symbols follow the black solid curve quite well because the noise level is very low. Therefore, as long as the noise level is sufficiently low, an interferometric observation with $N_{\text{base},30}$ baselines is good enough to recover the underlying turbulence spectrum. When the noise level increases, the power spectrum deviates from the black solid curve at large wavenumbers. The wavenumber at which the spectrum begins to deviate depends on the noise level: the turn-off wavenumber decreases as the noise level increases, which makes the power-law inertial range narrower.

Figure 9(a) shows more clearly the spectrum denoted by the red symbols (i.e., noise level of 20%) in Figure 8. Due to the noise, the spectrum shows a break near $K \sim 70$. Before the break, the spectrum follows the true turbulence spectrum (see the red line) quite well. However, after the break, the spectrum becomes flat. Note that the spectrum shown in Figure 9(a) is an average 2D spectrum (see Equation (9)). Since the amplitude of noise is independent of wavenumber, the spectrum after the break is flat. As we can see in the figure, we still have approximately one decade of power-law inertial range for the noise level of $\sim 20\%$. Therefore, we can recover the true turbulence spectrum via proper fitting in the inertial range if the beam size is negligibly small and the noise level is not substantially large.

Even with a finite beam size and a noise, we may be able to recover the true turbulence spectrum. Figure 9(b) demonstrates this possibility. The spectrum in the figure is obtained from the following procedure. We assume that the angular size of the observed patch in the sky is $\sim 6^\circ \times 6^\circ$ and generate a synthetic polarization map at $\lambda = 1$ cm on a grid of $512^2$. We also assume an underlying turbulence with a Kolmogorov spectrum (i.e., $k^{-11/3}$). We then smooth the map using a Gaussian beam with $\theta_{\text{FWHM}} = 3'$. Finally, we add a noise with a level that is 20% of the true polarization signal. Other observational setup is similar to the one described earlier in this subsection. The resulting spectrum is shown in Figure 9(b). Due to a finite beam size, the spectrum does not follow a power law. Instead, it declines fast at large wavenumbers. Although the spectrum does not show a power law, we can estimate the turbulence spectrum with a fitting function of the form

$$K^me^{-K^2/2\sigma_K^2}, \quad (10)$$

where $\sigma_K$ is the standard deviation in Fourier space defined as $1/(\sqrt{2\pi} \sigma_{\text{beam}})$. By changing the power-law index, $m$, we can find the turbulence spectral index. The red solid line in Figure 9(a) is for $m = -11/3$ and fits the observed spectrum of the synthetic data better than other curves corresponding to different $m$ values. Therefore, it is possible to recover the turbulence spectral index ($m = -11/3$) using the fitting function even in the case of finite beam size.
3.4.3. Effects of the Noise and Finite Beam Size: Simulations with MHD Turbulence Data

So far, we have used synthetic data for calculations. To check whether or not the fitting procedure also works for more realistic turbulence data, we use actual MHD turbulence simulation data and repeat the fitting procedure described in the previous subsection. The numerical method for generating the turbulence data is described in Section 2.1. Both the sonic and the Alfvén Mach number for the simulation are ∼0.7. As in the case of synthetic data, we calculate a synchrotron polarization map, add a random white noise, smooth the signal with a Gaussian beam, and obtain the complete 2D power spectrum in wave-vector space. Then, we randomly select \( N_{\text{base,30}} = 30 \cdot 29/2 \) wave-vectors and calculate the average 2D power spectrum based on the power at the selected wave-vectors (see Equation (9)) to mimic an interferometric observation with \( N_{\text{base,30}} \) baselines. Figure 10 shows the resulting average 2D power spectra. Each panel in Figure 10 has different noise levels. The levels of noise in the left and the right panels are 1% and 20% of the true signal, respectively.

The solid curve in the left panel represents the complete 2D power spectrum without noise and without smoothing. The spectrum shows about one decade of inertial range for \( K < 40 \) and the spectral index in the inertial range is compatible with \(-11/3\). The filled squares in the left panel denote the simulated average 2D power spectrum. The red solid line in Figure 10(b) is the function \( S = K^{-11/3} \) with \( m = -11/3 \) and fits the observed 2D power spectrum.

Figure 9. Average 2D power spectrum from a simulated interferometric observation with noise (black asterisks). The level of noise is 20% and \( N_{\text{base}} = N_{\text{base,30}} \). The synthetic data are used. (a) Spectrum from an observation with a negligible beam size. The straight red line is proportional to \( K^{-11/3} \). (b) Spectrum from an observation with a finite beam size (\( \theta_{\text{FWHM}} = 3' \)). The red solid line is proportional to the fitting function in Equation (10).

Figure 10. Average 2D power spectra using trans-Alfvénic (\( M_A \sim 0.73 \)) turbulence data with two different noise levels. (a) The noise level is 1%. (b) The noise level is 20%. Lines denote the fitting function in Equation (10) with different \( m \) values: \( m = -11/3 \) (red solid line), \( m = -12/3 \) (green dotted line), and \( m = -10/3 \) (blue dashed line).
spectrum fairly well. Therefore, it is possible to recover the turbulence spectral index \( m \approx -11/3 \) using the fitting function even in the case of real MHD data.

4. DISCUSSION

4.1. Numerical Tests of Theory

4.1.1. Polarization from Spatially Coincident Synchrotron Emission and Faraday Rotation Regions

In the paper, we successfully tested the predictions of LP16 that by using synchrotron polarization fluctuations it is possible to recover both magnetic field and density statistics. We tested both the predictions for the spectrum of polarization \( P \) and its derivative \( dP/d\lambda^2 \) and showed that these statistics are complimentary. In particular, the latter is focused more on the fluctuations of Faraday rotation.

For our testing, we used synthetic data, the spectrum of which we varied to test the theoretical predictions. We explored the effects of varying the wavelength of measurements on our studies of turbulence statistics. We found a number of numerical effects related to the finite numerical resolution of our synthetic turbulent data sets. In particular, we found that when \( \lambda^2 \sim \frac{K_{\max}}{2\pi(n_i |B|)} \), structures are decorrelated with small scales and derive depolarization showing that the spectrum of polarization is getting proportional to \( K \).

We tested theoretical predictions in the most complicated case considered in LP16, namely, when the volume that is emitting polarized synchrotron emission is also providing Faraday rotation due to the presence of electrons. The cases when only one effect is present in the turbulent volume are the limiting cases of the condition that we tested.

In addition, we tested the prediction in an earlier paper by LP12 that the slope of the synchrotron spectrum does not depend on the spectral index \( \gamma \) of relativistic electrons. Similar studies for synchrotron intensities were performed in Herron et al. (2016), while our study is dealing with the fluctuations of synchrotron polarization. This testing is important because \( \gamma \) varies in astrophysical objects. Our results show that, using expressions in LP12, one can study turbulence in synchrotron emitting volumes for arbitrary \( \gamma \).

4.1.2. Polarization from Spatially Separated Synchrotron Emission and Faraday Rotation Regions

In this paper, we did not consider the case in which the synchrotron emitting region and Faraday rotation region are spatially separated. However, there can be a situation when synchrotron originates in one distinct region while Faraday rotation acts on synchrotron radiation in another region. Figure 11 shows the spectra arising from separated regions of polarized synchrotron emission and Faraday rotation. We assume that background polarized synchrotron radiation is wavelength-independent and passes through a foreground medium that produces Faraday rotation. To obtain the background polarized radiation, we first generate 3D density and magnetic field on a grid of \( 512^3 \) points (see Section 2.1 for details) and calculate polarized synchrotron emission without Faraday rotation. The result of the calculation is polarized radiation on a grid of \( 512^3 \), which is used as background polarized synchrotron radiation. The spectrum of the radiation follows Kolmogorov one (see the black solid curve). We also generate foreground density and magnetic field on a grid of \( 512^3 \) points using different seeds for random numbers. We set \( B_0 = 1 \) along the LOS in the foreground medium and Faraday rotation is dominated by the mean field. The spectra of foreground magnetic field and density also follow Kolmogorov ones. We can clearly see depolarization in Figure 11. In fact, the overall behavior of spectra in Figure 11 is very similar to that in Figure 4(a). Therefore, roughly speaking, the results for the spatially coincident case can be applicable to the case in which synchrotron emitting region and Faraday rotation region are spatially separated. Note, however, that the observed spectrum of polarized synchrotron emission arising from the spatially separated regions may be more complicated. For example, the width of the Faraday rotation region may also affect the observed spectrum (see LP16).

4.2. Importance of Synchrotron Studies

Magnetic turbulence is essential for key astrophysical processes. Thus a number of techniques have been suggested to study it. While some of them are purely empirical, i.e., based on the comparison of the synthetic observations and the underlying turbulence within the numerical simulations (see Rosolowsky et al. 1999; Padoan et al. 2001; Brunt et al. 2003; Heyer et al. 2008; Gaensler et al. 2011; Tofflemire et al. 2011; Burkhart et al. 2012; Brunt & Heyer 2013; Burkhart & Lazarian 2015), others are based on the theoretical description of turbulence statistics. For instance, in a series of papers by Lazarian & Pogosyan (2000), Lazarian & Pogosyan (2004), and Lazarian & Pogosyan (2006, 2008), the statistics of intensity fluctuations within Position–Position–Velocity data cutes was described for the Doppler shifted spectral lines from...
turbulent volumes. These studies provide the way to recover the statistics of density and velocity (see Lazarian 2009 for a review, as well as Padoan et al. 2009, Chepurnov et al. 2010, and Chepurnov et al. 2015). The magnetic field statistics provides the complimentary essential piece of information and the studies in LP16 were aimed at obtaining a theory-motivated way to study magnetic turbulence from observations. Our successful testing of some of the suggested techniques paves the way to the application of these techniques to observational data.

Our study is complementary to that in a recent paper by Zhang et al. (2016), where the other analytical expressions from LP16 were tested. Indeed, Zhang et al. (2016) tested the fluctuations of the polarization variance with the wavelength. The input data for such studies is the polarized synchrotron radiation collected along a single line of sight, but with the wavelength being changed. In contrast, in this paper, we studied the statistics of two point correlations for the same wavelength.

Studying synchrotron variations is not only important for astrophysical applications, e.g., for understanding better processes of cosmic-ray propagation, transport of heat, star formation, etc., but also for observational cosmology. Indeed, a synchrotron polarization fluctuation presents an important foreground for the search of enigmatic B-modes of cosmological origin. Thus our testing of how these fluctuation varies with the wavelength as well as with the variations of γ are of prime applicability to such a search.

4.3. Complementary Ways of Studying Turbulence

Our present work opens ways for studying turbulence using synchrotron polarization. Such studies are complementary to the spectroscopic Doppler-shift studies of fluctuations. Those studies can be performed using Doppler shifted lines using Velocity Channel Analysis (Lazarian & Pogosyan 2000, 2004; Kandel et al. 2016a), Velocity Coordinate Spectrum (Lazarian & Pogosyan 2006, 2008), and Velocity Centroids (Lazarian & Esquivel 2003; Esquivel & Lazarian 2005; Esquivel & Lazarian 2010; Burkhart et al. 2014; Kandel et al. 2016b; Cho & Yoo 2016).

Combining the techniques, it is possible to see how the turbulence varies from one media, which can be sampled by spectral lines to another and can be sampled by synchrotron fluctuations. This can answer important questions related to whether the turbulence in the interstellar medium presents one big cascade or whether different phases of the ISM maintain their individual turbulent cascades.

5. SUMMARY

We successfully tested analytical expressions in LP16 and proved that the techniques suggested there can be used to analyze the observed fluctuation of polarized synchrotron radiation in order to restore the statistics of underlying magnetic turbulence. Our numerical results testify that such a study can be performed

1. for arbitrary spectral index of relativistic electrons,
2. in the presence of Faraday rotation and depolarization caused by turbulent magnetic field,
3. in the settings when only Faraday rotation is responsible for the polarization fluctuations,
4. in the presence of effects of finite beam size and noise, and
5. in the case in which the data are obtained with an interferometer with the measurement performed for just a few baselines.

We believe that our present study paves the way for the successful use of LP16 techniques with observational data.

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APPENDIX A

SYNCHROTRON EMISSION AND THE STOKES PARAMETERS

The study of intensity and linear polarization of the synchrotron emission provides a valuable information on the magnetic field. The observed intensity and polarized emission can be described by

\[ I = \int d\Omega I, \quad P = Q + iU = \int d\Omega P(X, \chi^2), \]  

where \( I \) is the specific intensity and \( P(X, \chi^2) \) is observed polarized intensity given by

\[ I = \int_0^L (j_\perp(\omega, x) + j_\parallel(\omega, x)) dz \]
\[ P(X, \chi^2) = \int_0^L (j_\perp(\omega, x) - j_\parallel(\omega, x)) e^{2\chi(x, z)} dz, \]

where \( x = (X, z) \), \( j_\perp \) and \( j_\parallel \) are the synchrotron emissivity perpendicular and parallel to \( B_\perp \), respectively (Waelkens et al. 2009):

\[ j_\perp(\omega, x) = (F(p) + G(p)) \omega^{\frac{\gamma-2}{\gamma}} |B_\perp(x)| \]
\[ j_\parallel(\omega, x) = (F(p) - G(p)) \omega^{\frac{\gamma-2}{\gamma}} |B_\perp(x)|. \]

The polarization angle with respect to the plane of the sky is

\[ \chi(X, z) = \chi_0 + \chi^2 \Phi(X, z), \]

where \( \chi_0 \) is the intrinsic polarization angle:

\[ \chi_0 = \tan^{-1}\left(\frac{B_\parallel}{B_\perp}\right) \]

and \( \Phi(X, z) \) is the Faraday rotation measure (see Equation (5)).

The polarization can be represented by the Stokes vector, \( S = (I, Q, U) \), where I is the total intensity, and Q and U describe the linear polarization. The Stokes parameters neglecting Faraday rotation (\( \Phi(X, z) = 0 \)) are defined as
follows.

\[
I = 2F(p) \omega e^{2} \int dz \int dz (B_{x}^{2} + B_{y}^{2})^{\frac{1}{2}} (B_{x}^{2} + B_{y}^{2})
\]

\[
Q = -2G(p) \omega e^{2} \int dz \int dz (B_{x}^{2} + B_{y}^{2})^{\frac{1}{2}} (B_{x}^{2} - B_{y}^{2})
\]

\[
U = -2G(p) \omega e^{2} \int dz \int dz (B_{x}^{2} + B_{y}^{2})^{\frac{1}{2}} 2(B_{x}B_{y}),
\]

where \( \omega = 2\pi c/\lambda \), \( \lambda \) is the observation wavelength, \( p \) is the spectral index of the electron energy distribution, and

\[
F(p) = \frac{\sqrt{3\pi} e^{2} N_{0} (2 m_{e} c / 3 e)^{1/2}}{64 \pi^{2} c^{2} m_{e}} \times \frac{2^{p+1}}{p+1} \Gamma\left(\frac{p}{2} + \frac{7}{12}\right) \Gamma\left(\frac{p}{2} - \frac{1}{12}\right) \Gamma\left(\frac{p}{2} + \frac{5}{4}\right)
\]

\[
G(p) = \frac{\sqrt{3\pi} e^{2} N_{0} (2 m_{e} c / 3 e)^{1/2}}{64 \pi^{2} c^{2} m_{e}} \times \frac{2^{p-1}}{p+1} \Gamma\left(\frac{p}{2} + \frac{7}{12}\right) \Gamma\left(\frac{p}{2} - \frac{1}{12}\right) \Gamma\left(\frac{p}{2} + \frac{5}{4}\right)
\]

(16)

Here \( m_{e} \) is the electron mass, \( e \) is its charge, and \( N_{0} \) is the pre-factor of the electron distribution.

**APPENDIX B**

**COMPARISON OF TWO POINT STATISTICS: POWER SPECTRA AND STRUCTURE FUNCTION**

Since the power spectrum can provide information on energy transfer processes in turbulence, the accurate measurement of the power spectrum is essential for our understanding of astrophysical turbulence. The power spectrum can be obtained from the Fourier transform of the correlation function, \( CF(r) = \langle P(x)P(x + r)\rangle \):

\[
P_{3D}(k) = \left(\frac{1}{2\pi}\right)^{2} \int \langle P(x)P(x + r)\rangle e^{-ikr}dr,
\]

where \( \langle...\rangle \) denotes the average over \( x \). Here, \( P_{3D}(k) \) is the 3D power spectrum. In this paper, we use the following definitions for different types of spectrum.

1. \( P_{3D}(k) \): the 3D power spectrum. \( P_{3D}(k) = |\tilde{\nu}_{k}|^{2} \), where \( \tilde{\nu}_{k} \) is the 3D Fourier transform of a real space variable \( \nu(x) \).
2. In the case of a 2D observable, we also use \( P_{2D}(K) \) for the 2D power spectrum. In this case, \( P_{2D}(K) = |\tilde{S}_{K}|^{2} \), where \( \tilde{S}_{K} \) is the 2D Fourier transform of a 2D real space variable \( S(x) \). If \( S = \int sdl \), where \( s \) is a variable in 3D space and the integration is done along the LOS, the 2D power spectrum of \( S \) is proportional to the 3D power spectrum of \( s \) if the domain is periodic in the direction of the LOS.
3. \( E_{3D}(k) \): the shell-integrated 1D spectrum for a 3D variable. \( E_{3D}(k) = \int_{k^{-0.5}}^{k+0.5} P_{3D}(k) dk \), where the integration is done over a shell of radius \( k \) in 3D Fourier space and \( P_{3D}(k) \) is the 3D power spectrum. If turbulence is isotropic and \( P_{2D}(k) \propto k^{\alpha} \), then the shell-integrated 1D spectrum \( E_{3D}(k) \) is proportional to \( k^{\alpha+2} \).
4. \( E_{2D}(K) \): the ring-integrated 1D spectrum for a 2D variable. \( E_{2D}(K) = \int_{K^{-0.5}}^{K+0.5} P_{2D}(K) dK \), where the integration is done over a ring of radius \( K \) in 2D Fourier space and \( P_{2D}(K) \) is the 2D power spectrum. If the 2D spectrum is isotropic and \( P_{2D}(k) \propto k^{\alpha} \), then the next \( E_{2D}(K) \) is proportional to \( K^{\alpha+1} \).

In some cases, it is easier to obtain the second-order structure function from observations than the spectrum. The power-law indices for the former and the latter are related as follows. For both 2D and 3D cases, if the spectral index of the 1D spectrum is \( \alpha \) (i.e., \( E(k) \propto k^{\alpha} \)), then the power-law index for the second-order structure function is \( -\alpha - 1 \) (i.e., \( SF_{2}(r) \propto r^{-\alpha-1} \)):

\[
E(k) \propto k^{-\alpha} \rightarrow SF_{2}(r) \propto r^{\alpha-1} \ (\alpha < 3)
\]

(19)

(see, for example, Cho & Lazarian 2009). Here, the 1D spectrum can be either the shell-integrated 1D spectrum for 3D data or the ring-integrated 1D spectrum for 2D data. Therefore, we can indirectly obtain the spectral index of the spectrum using the second-order structure function. The second-order structure function in either 2D or 3D can be directly obtained by

\[
SF_{2}(r) = \langle (P(x) - P(x + r))^{2} \rangle,
\]

(20)

where \( r \) is the separation vector in either 2D or 3D. Or, given a spectrum (or a correlation function \( CF(r) = \langle P(x)P(x + r) \rangle \)), the second-order structure function can be calculated by

\[
SF_{2}(r) = 2[CF(0) - CF(r)].
\]

(21)

**APPENDIX C**

**LIMITATION OF THE SECOND-ORDER STRUCTURE FUNCTION**

Note that the use of the second-order structure function to obtain the true turbulence spectrum requires a very high numerical resolution. In this appendix, we demonstrate this limitation of the structure function.

We first generate an isotropic 2D power spectrum for a scalar variable in Fourier space, which follows a \( K^{-11/3} \) spectrum between \( K = 2 \) and \( K = K_{max} - 2 \), where \( K_{max} \) is equal to half of the numerical resolution \( N \). We use \( N = 512, 1024, 2048, 4096, \) and 8192. Then, we calculate the second-order structure function using Equation (21). Since the 2D power spectrum is isotropic, the second-order structure function depends on the scalar separation \( r \).

We plot the resulting second-order structure function in Figure 12(a). Since the ring-integrated 1D spectrum \( E_{2D}(K) \) is proportional to \( k^{-5/3} \), we expect that \( SF_{2}(r) \propto r^{5/3} \). However, according to the figure, the second-order structure function does not show the expected scaling relation. When \( N = 512 \), the power-law slope of the second-order structure function lies between 4/3 and 5/3. As numerical resolution \( N \) increases, the slope becomes steeper. However, even with 8192\(^{2} \) resolution, the slope is still less than 5/3. Therefore, it is not easy to reveal the true turbulence spectrum using the second-order structure function unless the resolution is very high.

Indeed, if we plot the second-order structure function of synchrotron polarization for the case in which both density and magnetic field have Kolmogorov spectra (see Figure 4(a)), we get a slope that is less than the expected one. In Figure 12(b), we plot the corresponding second-order structure function for different wavelengths, \( \lambda \). We expect that the second-order structure functions are proportional to \( r^{5/3} \) for small spatial separation, since the spectral index of the ring-integrated 1D
spectrum $E_{2\text{D}}(K)$ is $-8/3$ at large $K$. However, Figure 12(b) shows that the slopes of second-order functions do not follow the expectation.

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