Discretized Gabor Frames of Totally Positive Functions
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Abstract
In this paper a large class of universal windows for Gabor frames (Weyl-Heisenberg frames) is constructed.
These windows have the fundamental property that every overcritical rectangular lattice generates a Gabor frame. Likewise, every undercritical rectangular lattice generates a Riesz sequence.

I. Introduction
Gabor frames and Gabor Riesz sequences arise in many engineering contexts. The corresponding series expansions can be interpreted as a sum of local Fourier series, the coefficients carry simultaneous information about time and frequency. Therefore they are an appropriate tool whenever simultaneous time-frequency information is required. Thus Gabor frame expansions are used naturally in speech processing and in the analysis of music signals [1], [2], though often under names as phase vocoder or lapped Fourier transform. One of the principal uses of Gabor expansions arises in wireless digital transmission with OFDM [3]–[5]. A non-exhaustive list of further applications includes antenna analysis [6], the analysis of ultrasound imaging [7], of brain stem responses [8], and even for the (texture) analysis of images [9], [10].

A Gabor family consists of time-frequency shifts of a single window function \( g \) (often called pulse in wireless communications) over a lattice \( \alpha \mathbb{Z} \times \beta \mathbb{Z} \). Formally, let \( g \) be a square-integrable function on \( \mathbb{R} \) and \( \alpha, \beta > 0 \) be the lattice parameters, then the time-frequency shifts are defined as \( M_{\beta n} T_{\alpha m} g(x) = e^{2\pi i \beta nx} g(x - m\alpha), \) \( m,n \in \mathbb{Z} \). In a concrete problem one is asked to design a pulse \( g \) and choose the time-frequency spacing \( \alpha, \beta \) in such a way that the corresponding set of functions possesses required properties.

Basic requirements on the pulse are often explicit formulas and good time-frequency concentration, basic requirements on the set of functions \( G(g, \alpha, \beta) = \{ M_{\beta n} T_{\alpha m} g : m,n \in \mathbb{Z} \} \) are its spanning properties.

If one needs a stable expansion of every signal with respect to the time-frequency shifts \( M_{\beta n} T_{\alpha m} g \), one has to construct a Gabor frame (Weyl-Heisenberg frame). In wireless communications one needs the set \( G(g, \alpha, \beta) \) to be linearly independent and thus has to construct a Gabor Riesz basis for a proper subspace of \( L^2(\mathbb{R}) \). Currently only few explicit pulse shapes are used. The Gaussian function \( e^{-at^2} \) is used in antenna analysis [6] or for the compression of EEG signals [11]. The rectangular pulse is much in favor in orthogonal frequency division multiplexing (OFDM) with cyclic prefix [3], [4]; the raised cosine window is the standard window in speech processing and the analysis of music signals. A representative list of windows for discrete Gabor frames is contained in Sondergaard’s Large Time-Frequency Analysis Toolbox LTFAT [12]. In pulse-shaping OFDM [3], [4], [13], [14] pulses are constructed to satisfy certain optimality criteria, but they are usually not given explicitly.

Once one has chosen a suitable pulse, one needs to determine which shift parameters generate a Gabor frame or basis. The density theory of Gabor frames [15], [16] asserts that the time-frequency plane must be oversampled for \( G(g, \alpha, \beta) \) to be a frame, i.e. \( \alpha \beta \leq 1 \), whereas for a Riesz sequence one needs undersampling \( \alpha \beta \geq 1 \). For the case of critical sampling \( \alpha \beta = 1 \) one can construct orthonormal bases of time-frequency shifts, but the corresponding pulse always lacks time-frequency concentration [17], as can already be seen for the rectangular pulse.

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In many applications it is desirable to operate as close to the critical density as possible. For a large channel capacity a Gabor Riesz sequence should span a “large” subspace of $L^2(\mathbb{R})$ and thus $\alpha\beta = 1 + \epsilon$ for some small $\epsilon > 0$. Likewise, to avoid large redundancy of the frame expansion, the density of a Gabor frame should be $\alpha\beta = 1 - \epsilon$ for small $\epsilon > 0$.

These fundamental requirements pose a difficult and largely unanswered question. Once a pulse shape is chosen, it is not at all clear whether such a choice of lattice parameters close to the critical density is possible. Until recently the complete set of the lattice parameters generating a Gabor frame was understood only for a handful of pulse shapes, namely the Gaussian, the hyperbolic secant, and two exponential functions \cite{18–22}. On the other hand, for the popular rectangular pulse the determination of good lattice parameters is extremely complicated, as is shown by Janssen’s tie \cite{23}.

To this day, it remains a challenging mathematical problem to determine suitable lattice parameters $\alpha, \beta$ for which the corresponding Gabor family $\mathcal{G}(g, \alpha, \beta)$ is a frame or a Riesz sequence. Perhaps this lack of theoretical understanding has prevented the use of other pulses in engineering applications.

In this paper we propose totally positive functions as convenient and universal windows for Gabor frames. The goal is to offer practitioners of time-frequency analysis a new, large class of pulse shapes for which the frame property is easy to determine for all lattice parameters.

Our contribution is based on recent progress in the mathematical analysis of Gabor frames. In \cite{24}, the following result was shown. A function $g$ is said to be totally positive of finite type $m$ (TPFFT), if its Fourier transform $\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-2\pi j x \xi} dx$ factors as follows: let $\delta_i, i = 1, \ldots, m$ be a finite set of nonzero real numbers, then

$$\hat{g}(\xi) = \prod_{i=1}^{m} (1 + 2\pi j \delta_i \xi)^{-1}.$$  

\textbf{Theorem 1 (24).} Let $g$ be a totally positive function of finite type $m \geq 2$. Then the Gabor family $\mathcal{G}(g, \alpha, \beta)$ constitutes a frame if and only if $\alpha\beta < 1$. Moreover the Gabor frame possesses a compactly supported, piecewise continuous dual window $\gamma$.

\textbf{By duality,} the Gabor family $\mathcal{G}(g, \alpha, \beta)$ is a Riesz sequence for the generated subspace, if and only if $\alpha\beta > 1$.

This result provides a family of pulse shapes that is parametrized by a countable number of parameters $\delta_i$. By choosing suitable $\delta_i$ one can finetune the pulse to one’s needs.

We will make this result more useful for applications in signal processing and for numerical use by deriving similar statements for Gabor frames for discrete signals, for continuous periodic signals, and for finite discrete signals. In these cases the Hilbert space is $\ell^2(\mathbb{Z})$, or $L^2(T_K)$, where $K > 0$ and $T_K = [0, K]$ denotes the interval from 0 to $K$ on the real line, or $\mathbb{C}^K$, where $L \in \mathbb{N}$ (instead of $L^2(\mathbb{R})$).

The choice of totally positive functions of finite type as a pulse shape for Gabor frames has several major advantages.

1) Totally positive functions of finite type (TPFFT) are universal pulse shapes that generate a Gabor frame whenever the time frequency plane is oversampled and that generate a Gabor Riesz sequence whenever the time-frequency plane is undersampled.

2) TPFFTs possess exponential decay. Thus these pulse shapes can be approximated with high accuracy by a pulse with compact support. In addition, if the type of $g$ is $m \geq 2$, then $g$ is $m - 2$ times continuously differentiable \cite{25}.

3) The general theory of Gabor frames guarantees that the Gabor frame with a totally positive window function possesses a dual pulse with exponential decay \cite{5}, \cite{26}, \cite{27}.

The construction in \cite{24} yields even a dual pulse with compact support. In this paper we refine this construction and present a simple algorithm that generates a whole family of dual windows with compact support (and increasing smoothness). In contrast to other methods \cite{5}, \cite{28}, \cite{29} this algorithm is exact and requires only the pseudo-inverse of a finite matrix.

We hope that the new theory of Gabor frames offers a new arsenal of convenient windows for many applications where the lattice parameters need to be close to the critical case.
The paper is organized as follows. In Section II we recall the main concepts and necessary results on Gabor frames, we discuss totally positive functions of finite type and their basic properties. In Section III we will state versions of Theorem I for discrete, periodic and finite signals, we give an algorithm to compute a dual window $\gamma$, and provide explicit formulas for totally positive functions of finite type and of their Zak transforms. Furthermore we will study the critical case $\alpha \beta = 1$. Section IV contains numerical examples of totally positive functions and a variety of their dual windows.

II. BACKGROUND

This section is devoted to a brief introduction to Gabor frames and their discretization. For a thorough introduction see [30]–[32].

A. Gabor Frames

Let $g$ be a function in $L^2(\mathbb{R})$. We define the translation operator $T_x$ by $T_x f(t) = f(t-x), t, x \in \mathbb{R}$, and the modulation operator $M_\xi$ by $M_\xi f(t) = e^{2\pi i \xi t} f(t), \xi \in \mathbb{R}$. Their composition is the time-frequency shift operator

$$M_\xi T_x f(t) = f(t-x) e^{2\pi i \xi t}.$$ 

Let $\mathbb{T}_K = [0, K]$ with $K > 0$ and let $L \in \mathbb{N}$. The analogous definitions of the time-frequency shift operator are given for

- discrete signals $f \in l^2(\mathbb{Z})$ by $M_\xi T_k f(t) = f(t-k) e^{2\pi i \xi t}$, where $k, l \in \mathbb{Z}, \xi \in [0, 1)$,
- periodic signals $f \in L^2(\mathbb{T}_K)$ by $M_{m/K} T_x f(t) := f((t-x) \mod K) e^{2\pi i m t/K}$, where $t, x \in [0, K], m \in \mathbb{Z}$,
- finite discrete signals $f \in \mathbb{C}^L$ by $M_{m/L} T_k f(l) := f((l-k) \mod L) e^{2\pi i m l/L}$, where $k, l, m \in \{0, \ldots, L-1\}$.

We first recall the well-known definition of Gabor frames for $L^2(\mathbb{R})$, see [31].

**Definition 2.** Let $\alpha, \beta \in \mathbb{R}^+$ and $g \in L^2(\mathbb{R})$. The set of time-frequency shifts

$$G(g, \alpha, \beta) := \{ M_{\beta k} T_{\alpha l} g \mid k, l \in \mathbb{Z} \}$$

is called a Gabor family. A Gabor family is a Gabor frame, or Weyl-Heisenberg frame, for $L^2(\mathbb{R})$, if there exist constants $A, B > 0$ such that

$$A \| f \|_2^2 \leq \sum_{k, l \in \mathbb{Z}} |\langle f, M_{\beta k} T_{\alpha l} g \rangle|^2 \leq B \| f \|_2^2, \quad \forall f \in L^2(\mathbb{R}).$$

(2)

The set of points $\{(k\alpha, l\beta) \in \mathbb{R}^2 \mid k, l \in \mathbb{Z}\}$ is a lattice with density $(\alpha \beta)^{-1}$. The constants $A, B$ in (2) are called the frame bounds.

The Gabor family $G(g, \alpha, \beta)$ is a (Gabor) Riesz sequence, if there exist constants $A', B' > 0$, such that

$$A' \| c \|_2^2 \leq \left\| \sum_{k, l \in \mathbb{Z}} c_{k,l} M_{\beta k} T_{\alpha l} g \right\|_2^2 \leq B' \| c \|_2^2, \quad \forall c \in \ell^2(\mathbb{Z}^2).$$

(3)

The usefulness of Gabor frames stems from the fact that they allow for a basis like expansion of functions in $L^2(\mathbb{R})$.

**Proposition 3.** ([31]) Let $G(g, \alpha, \beta)$ be a Gabor frame for $L^2(\mathbb{R})$, then there exists a dual window $\gamma \in L^2(\mathbb{R})$, such that every $f \in L^2(\mathbb{R})$ possesses the expansions

$$f = \sum_{k, l \in \mathbb{Z}} \langle f, M_{\beta k} T_{\alpha l} g \rangle M_{\beta k} T_{\alpha l} \gamma$$

$$= \sum_{k, n \in \mathbb{Z}} \langle f, M_{\beta k} T_{\alpha l} \gamma \rangle M_{\beta k} T_{\alpha l} g,$$

where $\alpha, \beta \in \mathbb{R}^+$ and $g \in L^2(\mathbb{R})$. The constants $A, B$ in (2) are called the frame bounds.
and the set $G(\gamma, \alpha, \beta)$ is also a frame for $L^2(\mathbb{R})$. This frame is called a dual frame of $G(g, \alpha, \beta)$.

We note that the dual window is not unique, a formula for all possible dual windows can be found in [30], [31]. The non-uniqueness will offer us some freedom to design a class of windows with compact support in Section III-F.

Gabor frames and Gabor Riesz sequences are dual to each other [33], [34]. In fact, the Gabor family $G(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$, if and only if $G(g, \frac{1}{2}, \frac{1}{\alpha})$ is a Riesz sequence in $L^2(\mathbb{R})$. This connection has been widely used in wireless communications [4].

The definition of a Gabor family easily carries over to the setting of $\ell^2(\mathbb{Z})$, $L^2(\mathbb{T}_K)$ and $C^L$.

- For $g \in \ell^2(\mathbb{Z})$ we fix $\alpha \in \mathbb{N}$, $\beta = \frac{1}{M}$ with $M \in \mathbb{N}$ and let
  $$G(g, \alpha, \beta) = \{M_{\beta}T_{\alpha}g | k \in \mathbb{Z}, l = 0, \ldots, M - 1\}. \quad (4)$$

- For $g \in L^2(\mathbb{T}_K)$ we fix $\alpha = \frac{K}{N}$ with $N \in \mathbb{N}$, $\beta = \frac{p}{K}$ with $p \in \mathbb{N}$ and let
  $$G(g, \alpha, \beta) = \{M_{\beta}T_{\alpha}g | k = 0, \ldots, N - 1, \ l \in \mathbb{Z}\}. \quad (5)$$

- For $g \in C^L$ we fix $\alpha = \frac{1}{N}$ with $N \in \mathbb{N}$, $\beta = \frac{1}{M}$ with $M \in \mathbb{N}$, where we suppose that $\frac{L}{N}, \frac{L}{M} \in \mathbb{N}$ and let
  $$G(g, \alpha, \beta) = \{M_{\beta}T_{\alpha}g | k = 0, \ldots, N - 1, \ l = 0, \ldots, M - 1\}. \quad (6)$$

The analogous definition of Gabor frames for $\ell^2(\mathbb{Z})$, $L^2(\mathbb{T}_K)$ and $C^L$ requires that the inequalities in (2) hold for all $f$ in the respective Hilbert space, and the summation extends over all index pairs $(k, l)$ that are relevant for the Gabor family instead of $\mathbb{Z} \times \mathbb{Z}$. Analogous results to Proposition 3 hold for Gabor frames for $\ell^2(\mathbb{Z}), L^2(\mathbb{T}_K)$ and $C^L$.

### B. Sampling and Periodization of Gabor Frames

First, we introduce the sampling and periodization operators. Let $g \in L^2(\mathbb{R})$ and let $h \in \mathbb{R}$ with $h > 0$. We assume that every point $t = kh$ with $k \in \mathbb{Z}$ is a Lebesgue point of $g$, i.e., evaluation of $g$ at $t$ is defined because

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |g(x + u) - g(x)| \, du = 0.$$  

Then
$$S_hg := (g(hk) : k \in \mathbb{Z}) \quad (7)$$

defines the sampling operator with step-size $h$. It was shown in [35] and [36] that $g \in L^2(\mathbb{R})$ and $S_hg \in \ell^2(\mathbb{Z})$ holds under the slightly stronger condition

$$\lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |g(kh + u) - g(kh)|^2 \, du = 0. \quad (8)$$

This condition is clearly satisfied, if $g$ is piecewise differentiable and its derivative decays exponentially, as will be the case for all $g$ considered in this article. For periodization with period $K \in \mathbb{R}$, $K > 0$, we assume $g \in L^1(\mathbb{R})$ and let

$$P_Kg(x) := \sum_{k \in \mathbb{Z}} g(x - kK), \quad x \in \mathbb{T}_K. \quad (9)$$

Then $P_Kg \in L^1(\mathbb{T}_K)$, and if $g$ is piecewise continuous and decays exponentially, then $P_Kg \in L^2(\mathbb{T}_K)$. We call $P_K$ the periodization operator with period $K$. With $K \in \mathbb{N}$, the periodization operator $P_K$ is defined analogously on $\ell^1(\mathbb{Z})$; its image space is isomorphic to $C^K$.

The family of window functions $g$ considered in this article is invariant under dilation. Therefore, it is no restriction for us to let $h = 1$ in the sequel and write $Sg$ instead of $S_1g$. 

It was shown in [35] and [36] that Gabor frames for $\ell^2(\mathbb{Z})$, $L^2(\mathbb{T}_K)$ and $\mathbb{C}^K$ can be derived from Gabor frames for $L^2(\mathbb{R})$ by sampling and periodization. The precise statements are as follows.

**Proposition 4.** Let $\alpha \in \mathbb{N}$, $\beta = 1/M$ with $M \in \mathbb{N}$, and $g \in L^2(\mathbb{R})$ such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R})$ with frame bounds $A, B > 0$. If $g$ satisfies the condition (8), then

- $\mathcal{G}(Sg, \alpha, \beta)$ is a Gabor frame for $\ell^2(\mathbb{Z})$ with the same frame bounds,
- $\mathcal{G}(P_K g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{T}_K)$ with the same frame bounds, for any period $K \in \mathbb{N}$ with $K \alpha \in \mathbb{N}$ and $K M \in \mathbb{N}$,
- $\mathcal{G}(P_K Sg, \alpha, \beta)$ is a Gabor frame for $\mathbb{C}^K$ with the same frame bounds, for any period $K \in \mathbb{N}$ as above.

Moreover, if $\mathcal{G}(\gamma, \alpha, \beta)$ is a dual Gabor frame of $\mathcal{G}(g, \alpha, \beta)$ in $L^2(\mathbb{R})$ and $\gamma$ satisfies (8), then $S\gamma$, $P_K \gamma$ and $P_K S\gamma$ are window functions of a dual Gabor frame after sampling and/or periodization.

In this proposition, the statement about frame bounds does not mean that the optimal frame bounds remain the same. In many cases, tighter bounds than $A, B$ exist after periodization or sampling.

**C. Density and Zak transform**

The lattice $\alpha \mathbb{Z} \times \beta \mathbb{Z}$ of time-frequency shifts of $g \in L^2(\mathbb{R})$ must satisfy a density criterion, if $\mathcal{G}(g, \alpha, \beta)$ defines a Gabor frame for $L^2(\mathbb{R})$, see [16]. Analogous results for Gabor frames for $\ell^2(\mathbb{Z})$, $L^2(\mathbb{T}_K)$ and $\mathbb{C}^L$ are easier to obtain, see, e.g., [37].

**Theorem 5.** Let $\mathcal{H}$ be one of the spaces $L^2(\mathbb{R}), \ell^2(\mathbb{Z}), L^2(\mathbb{T}_K)$ or $\mathbb{C}^L$, and $g \in \mathcal{H}$. If $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $\mathcal{H}$, then $\alpha \beta \leq 1$.

For the case $\alpha \beta = 1$, which is called the critical density, the frame condition (2) can be checked by inspecting the Zak transform (cf. [31], [38]–[40]) of $g$.

**Definition 6.** Let $f \in L^2(\mathbb{R})$, $\alpha > 0$. The Zak transform $Z_\alpha f$ of $f$ is defined as

$$Z_\alpha f(x, \xi) = \sum_{k \in \mathbb{Z}} f(x - \alpha k) e^{2\pi i \alpha k \xi}, \quad x, \xi \in \mathbb{R}.$$  

(10)

The Zak transform is quasi-periodic, i.e.,

$$Z_\alpha f(x, \xi + n/\alpha) = Z_\alpha f(x, \xi),$$

(11)

$$Z_\alpha f(x + \alpha n, \xi) = e^{2\pi i \alpha n \xi} Z_\alpha f(x, \xi), \quad n \in \mathbb{Z}.$$  

(12)

The quasiperiodicity implies that the Zak transform on $\mathbb{R}^2$ is determined by its values on the rectangle $[0, \alpha) \times [0, 1/\alpha)$.

The subsequent theorem gives a criterion to check whether the frame condition (2) is satisfied at the critical density $\alpha \beta = 1$. In particular, we will see that the frame property of a Gabor set is completely determined by the behaviour of the Zak transform of the window $g$.

**Theorem 7.** (cf. [30], [31], [38], [39]) Let $g \in L^2(\mathbb{R})$ and $\alpha \beta = 1$.

(a) $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A, B$ if and only if

$$0 < A \leq |Z_\alpha g(x, \xi)|^2 \leq B < \infty$$

(13)

for almost all $(x, \xi) \in [0, \alpha) \times [0, 1/\alpha)$.

(b) Assume that $g$ satisfies (8) and $\alpha = M \in \mathbb{N}$, $\beta = 1/M$. Then $\mathcal{G}(Sg, \alpha, \beta)$ is a frame for $\ell^2(\mathbb{Z})$ with frame bounds $A, B$, if and only if

$$0 < A \leq |Z_\alpha g(k, \xi)|^2 \leq B < \infty$$

(14)

for $k \in \{0, \ldots, M - 1\}$ and almost all $\xi \in [0, 1/M)$.
c) Assume that \( g \) satisfies (8) and \( \alpha = M \in \mathbb{N}, \beta = 1/M \). Let \( K \in \mathbb{N} \) such that \( K/M \in \mathbb{N} \). Then \( \mathcal{G}(P_Kg, \alpha, \beta) \) is a frame for \( L^2(\mathbb{T}_K) \) with frame bounds \( A, B \), if and only if
\[
0 < A \leq \left| Z_{\alpha}g \left(x, \frac{l}{K}\right) \right|^2 \leq B < \infty
\]
for almost all \( x \in [0, M) \) and all \( l \in \{0, 1, \ldots, \frac{K}{M} - 1\} \).

d) Assume that \( g \) satisfies (9) and \( \alpha = M \in \mathbb{N}, \beta = 1/M \). Let \( K \in \mathbb{N} \) such that \( K/M \in \mathbb{N} \). Then \( \mathcal{G}(P_KSg, \alpha, \beta) \) is a frame for \( \mathbb{C}^K \) with frame bounds \( A, B \), if and only if
\[
0 < A \leq \left| Z_{\alpha}g \left(k, \frac{l}{K}\right) \right|^2 \leq B < \infty
\]
for all \( k \in \{0, \ldots, M-1\} \) and \( l \in \{0, 1, \ldots, \frac{K}{M} - 1\} \).

The Balian-Low theorem [17] states that windows \( g \in L^2(\mathbb{R}) \) with the property \( g, \hat{g} \in L^1(\mathbb{R}) \) cannot define a Gabor frame for \( L^2(\mathbb{R}) \) at the critical density \( \beta = 1/\alpha \). This was proved in [15] by applying the following result in connection with Theorem 7(a).

**Lemma 8.** Let \( \alpha > 0, g \in L^2(\mathbb{R}) \). If \( Z_\alpha g \) is continuous, then it has a zero in its domain \([0, \alpha) \times [0, 1/\alpha)\).

A more precise statement about the location of some zero of \( Z_\alpha g \) was given in [23] for the case of an even function \( g \).

**Lemma 9.** Let \( \alpha > 0, g \in L^2(\mathbb{R}) \) be even and \( Z_\alpha g \) be continuous. Then \( Z_\alpha g \left(\frac{\alpha}{2}, \frac{1}{2\alpha}\right) = 0 \).

Moreover, assume \( \alpha = M \in \mathbb{N} \) and let \( \beta = 1/M \) and \( K \in \mathbb{N} \) is such that \( K/M \in \mathbb{N} \).

- \( \mathcal{G}(Sg, \alpha, \beta) \) is not a Gabor frame for \( l^2(\mathbb{Z}) \) if \( M \) is even.
- \( \mathcal{G}(P_Kg, \alpha, \beta) \) is not a Gabor frame for \( L^2(\mathbb{T}_K) \) if \( K/M \) is even.
- \( \mathcal{G}(P_KSg, \alpha, \beta) \) is not a Gabor frame for \( \mathbb{C}^K \) if \( M \) and \( K/M \) are even.

**Proof:** The zero at \( (\frac{\alpha}{2}, \frac{1}{2\alpha}) \) of the Zak transform \( Z_\alpha g \) of an even continuous function \( g \) was already described in [23]. If \( M \) is even, the point \( (k, \xi) = (\frac{M}{2}, \frac{1}{2K}) \) is in the domain of \( Z_\alpha g \) in part (b) of Theorem 7. Therefore, no positive lower frame bound exists for the Gabor family \( \mathcal{G}(Sg, \alpha, \beta) \). An analogous argument is used to show that \( \mathcal{G}(P_Kg, \alpha, \beta) \) and \( \mathcal{G}(P_KSg, \alpha, \beta) \) have no positive lower frame bound, if the conditions on \( M \) and \( K \) are satisfied. \( \square \)

### D. Totally Positive Functions

**Definition 10.** A non-constant function \( g \in L^1(\mathbb{R}) \) is said to be totally positive if for every two sequences
\[
x_1 < x_2 < \cdots < x_N, \\
y_1 < y_2 < \cdots < y_N
\]
of real numbers the inequality
\[
\det(g(x_j - y_k))_{1 \leq j, k \leq N} \geq 0
\]
holds.

In [25] Schoenberg showed that the total positivity of a function \( g \in L^1(\mathbb{R}) \) is equivalent to a simple factorization of its Laplace transform. In this paper we will study the special case of totally positive functions of finite type. Instead of the Laplace transform we use the Fourier transform in the following definition.

**Definition 11.** A function \( g \in L^1(\mathbb{R}) \) is totally positive of finite type \( m \in \mathbb{N} \), if its Fourier transform is given by
\[
\hat{g}(\xi) = C \prod_{k=1}^{m} (1 + 2\pi j \delta_k \xi)^{-1}
\]
where \( C > 0 \) and \( \delta_k \neq 0 \) are real numbers.

**Example 12.** In the following examples we let \( C = 1 \) in (18) and we make use of the Heaviside function

\[
h(x) = \begin{cases} 
1, & x > 0, \\
\frac{1}{2}, & x = 0, \\
0, & x < 0.
\end{cases}
\]

The two sided exponential function \( g(x) = \frac{1}{2} e^{-|x|} \) is totally positive of finite type \( m = 2 \), with parameters \( \delta_1 = 1, \delta_2 = -1 \). Also of type \( m = 2 \) are the functions \( g(x) = \frac{ab}{a+b} (e^{ax} h(-x) + e^{bx} h(x)) \) for \( a, b > 0 \), with parameters \( \delta_1 = -1/a \) and \( \delta_2 = 1/b \), and \( g(x) = \frac{ab}{b-a} ((e^{-ax} - e^{-bx})h(x)) \) for \( b > a > 0 \), with \( \delta_1 = 1/a \) and \( \delta_2 = 1/b \). An example of a totally positive function of type \( m = r \in \mathbb{N} \) is \( g(x) = e^{-x} \frac{x^{r-1}}{(r-1)!} h(x) \), with parameters \( \delta_1 = \cdots = \delta_r = 1 \). In Theorem 14 we will give a direct formula for totally positive functions of finite type, without reference to their Fourier transform. This makes their evaluation very simple. By their definition, the dilation and the convolution of totally positive functions of finite type is again totally positive of finite type. Note that the Gaussian \( g(x) = e^{-\pi x^2} \) is totally positive, but since its Fourier transform is again a Gaussian it is not of finite type.

We state two basic properties of totally positive functions which will be needed later on. For a proof see [24] and [25] p.339.

**Proposition 13.** (Properties of totally positive functions)

1) Every totally positive function \( g \in L^1(\mathbb{R}) \) has exponential decay.

2) Every totally positive function \( g \) of finite type \( m \geq 2 \) is continuous.

Consequently, every totally positive function \( g \) of finite type satisfies condition \( \mathcal{S} \), and the Zak transform of every totally positive function of finite type \( m \geq 2 \) is continuous.

## III. Main Results

In this section we will state a version of Theorem 1 for the case of discrete signals \( f \in \ell^2(\mathbb{Z}) \), periodic signals \( f \in L^2(\mathbb{T}_K) \), and the case of finite signals \( f \in \mathbb{C}^K \). We will give explicit formulas for the respective settings and treat the case of the critical density where \( (\alpha, \beta) = (M, 1/M) \) with \( M \in \mathbb{N} \) in detail. This case differs significantly from the setting of Gabor frames of \( L^2(\mathbb{R}) \), as there exist Gabor frames for \( \ell^2(\mathbb{Z}) \) and \( L^2(\mathbb{T}) \) at the critical density which are well localized in time and frequency. We will also describe an efficient algorithm for the computation of a dual window \( \gamma \) in all three cases.

### A. TPFFTs and their Zak transform

First we will present explicit formulas for the computation of totally positive functions of finite type and their Zak transforms in the continuous setting.

**Theorem 14.** Let \( g \) be a totally positive function of finite type \( m \geq 2 \) with Fourier transform

\[
\hat{g}(\xi) = \prod_{k=1}^{m} (1 + 2\pi j \delta_k \xi)^{-1}
\]

and \( \delta_k \in \mathbb{R} \setminus \{0\} \), and suppose \( \delta_i \neq \delta_k \) for \( i \neq k \). Then \( g \) is given by

\[
g(x) = \sum_{i=1}^{m} \left( \frac{1}{|\delta_i|} e^{-\pi |x| h(x) \delta_i} \prod_{k=1, k \neq i}^{m} \left( 1 - \frac{\delta_k}{\delta_i} \right)^{-1} \right).
\]

**Proof:** We use the partial fraction decomposition of \( \hat{g} \),

\[
\hat{g}(\xi) = \prod_{k=1}^{m} (1 + 2\pi j \delta_k \xi)^{-1} = \sum_{k=1}^{m} \frac{C_k}{1 + 2\pi j \delta_k \xi}.
\]
To find $C_i$ for $i \in \{1, \ldots, m\}$, we multiply (21) by $(1 + 2\pi j \delta_i \xi)$ and substitute $\xi = -((2\pi j \delta_i)^{-1})$. then

$$C_i = \prod_{k=1, k \neq i}^{m} \left(1 - \frac{\delta_k}{\delta_i}\right)^{-1}.$$  \hspace{1cm} (22)

The $i$-th summand $\hat{s}_i(\xi)$ in (21) is in $L^2(\mathbb{R})$, and its inverse Fourier transform is

$$s_i(x) = \frac{C_i}{|\delta_i|} e^{-\frac{x}{\delta_i}} h(x\delta_i).$$

This shows that $g(x)$ has the form (20) for all $x \neq 0$, since the summands $s_i$ are continuous in $\mathbb{R} \setminus \{0\}$. Continuity of $g$ in $x = 0$, as stated in Proposition [13] implies

$$g(0) = \lim_{x \searrow 0^+} g(x) = \sum_{i=1, \delta_i > 0}^{m} \frac{C_i}{\delta_i}, \quad g(0) = \lim_{x \nearrow 0^+} g(x) = \sum_{i=1, \delta_i < 0}^{m} \frac{C_i}{|\delta_i|}.$$ 

Therefore, we conclude that

$$g(0) = \frac{1}{2} \left( \sum_{i=1, \delta_i > 0}^{m} \frac{C_i}{\delta_i} + \sum_{i=1, \delta_i < 0}^{m} \frac{C_i}{|\delta_i|} \right) = h(0) \sum_{i=1}^{m} \frac{C_i}{|\delta_i|},$$

which agrees with the explicit form in (20). \hfill \blacksquare

**Remark 15.** Similar expressions for TPFFTs can be derived, if $\hat{g}$ has poles of higher multiplicity, but the formulas quickly become more involved. An extension of the result in Theorem [14] to totally positive functions $g$, where $\hat{g}$ has infinitely many poles, is given in [25, Theorem 4].

Formula (20) can be used to derive an explicit expression for the Zak transform of totally positive functions of finite type.

**Corollary 16.** Let $g$ be a totally positive function of finite type $m \geq 2$ with Fourier transform (19) and $\delta_k$ as in Theorem [14] For $\alpha > 0$ and $x \in [0, \alpha)$, $\xi \in [0, 1/\alpha)$, we have

$$Z_\alpha g(x, \xi) = \sum_{i=1}^{m} \frac{1}{\delta_i} \frac{e^{-x/\delta_i}}{1 - e^{-\alpha(1/\delta_i + 2\pi j \xi)}} \cdot \prod_{k=1, k \neq i}^{m} \left(1 - \frac{\delta_k}{\delta_i}\right)^{-1}. \hspace{1cm} (23)$$

**Proof:** The continuity of $Z_\alpha g$ in Proposition [13] allows us to ignore $x = 0$ and assume $x \in (0, \alpha)$ and $\xi \in \mathbb{R}$. We consider each summand

$$s_i(x) = \frac{C_i}{|\delta_i|} e^{-\frac{x}{\delta_i}} h(x\delta_i)$$

of $g$ in (20) with $C_i$ in (22) separately. For $\delta_i > 0$ we obtain, by the formula for the geometric series,

$$Z_\alpha s_i(x, \xi) = \frac{C_i}{\delta_i} \sum_{l=-\infty}^{\infty} e^{-\alpha l/\delta_i} h((x - \alpha l)\delta_i) e^{2\pi j l \xi}$$

$$= \frac{C_i}{\delta_i} \frac{e^{-x/\delta_i} \sum_{l=0}^{\infty} e^{-\alpha l(1/\delta_i + 2\pi j \xi)}}{1 - e^{-\alpha(1/\delta_i + 2\pi j \xi)}}$$

$$= \frac{C_i}{\delta_i} \frac{e^{-x/\delta_i}}{1 - e^{-\alpha(1/\delta_i + 2\pi j \xi)}}. \hspace{1cm} (24)$$
A similar computation for $\delta_i < 0$ yields

$$Z_{\alpha \delta_i}(x, \xi) = -\frac{C_i}{\delta_i} \sum_{l=-\infty}^{\infty} e^{-(x-\alpha l)/\delta_i} h((x - \alpha l)\delta_i) e^{2\pi j \alpha l \xi}$$

$$= -\frac{C_i}{\delta_i} e^{-x/\delta_i} \sum_{l=1}^{\infty} e^{\alpha l(1/\delta_i + 2\pi j \xi)}$$

$$= -\frac{C_i}{\delta_i} e^{-x/\delta_i} \frac{1}{1 - e^{\alpha(1/\delta_i + 2\pi j \xi)}}$$

$$= C_i \frac{e^{-x/\delta_i}}{1 - e^{-\alpha(1/\delta_i + 2\pi j \xi)}}.$$  \hspace{1cm} (25)

This completes the proof of (23). \hfill \blacksquare

**Remark 17.** The right-hand side in (23) can be interpreted as the expanded form of a divided difference of order $m - 1$ with knots $a_i := 1/\delta_i$, $1 \leq i \leq m$, of the function

$$r_{x, \xi}(y) = (-1)^{m-1} \left( \prod_{i=1}^{m} a_i \right) \frac{e^{-xy}}{1 - e^{-\alpha(y+2\pi j \xi)}}.$$  \hspace{1cm} (26)

Indeed, the divided difference with knots $a_i$ has the expanded form

$$[a_1, \ldots, a_m \mid r_{x, \xi}] = \sum_{i=1}^{m} r_{x, \xi}(a_i) \prod_{k=1, k \neq i}^{m} (a_i - a_k)^{-1},$$

which coincides with the right-hand side in (23) by straightforward computation. This allows us to extend the result of Corollary 16 to totally positive functions of finite type with parameters $\delta_i$ among which several may coincide, by taking the divided difference of $r_{x, \xi}$ with corresponding multiple knots $a_i = 1/\delta_i$, $1 \leq i \leq m$. By this, we obtain the representation of $Z_{\alpha g}$ for every TPFFT

$$Z_{\alpha g}(x, \xi) = [a_1, \ldots, a_m \mid r_{x, \xi}],$$  \hspace{1cm} (27)

where $x \in [0, \alpha)$, $\xi \in [0, 1/\alpha)$ and $r_{x, \xi}$ is given in (26).

**B. Gabor Frames of TPFFTs on $\ell^2(\mathbb{Z})$**

In this section we will state the analog of Theorem 1 for $\ell^2(\mathbb{Z})$.

**Theorem 18.** Let $g$ be a totally positive function of finite type $m \geq 2$, and let $a, M \in \mathbb{N}$. If $\frac{a}{M} < 1$, then $G(Sg, a, 1/M)$ is a frame for $\ell^2(\mathbb{Z})$. Furthermore there exists a finitely supported dual window $\gamma \in \ell^2(\mathbb{Z})$ that can be calculated by Algorithm 24.

By duality, if $\frac{a}{M} > 1$, then $G(Sg, a, 1/M)$ is a Riesz sequence in $\ell^2(\mathbb{Z})$.

**Proof:** Since every totally positive function of finite type has exponential decay and a totally positive function of finite type $m \geq 2$ is continuous by Proposition 13, $g$ satisfies (8). Since $G(g, a, 1/M)$ is a Gabor frame for $L^2(\mathbb{R})$ by Theorem 1, the first statement of Proposition 4 implies that $G(Sg, a, 1/M)$ is a frame for $\ell^2(\mathbb{Z})$. The existence of a dual window $\gamma$ with finite support is proved in the appendix. \hfill \blacksquare
C. Gabor Frames of TPFFTs on $L^2(\mathbb{T}_K)$

We next formulate the main result for continuous periodic signals in $L^2(\mathbb{T}_K)$ where $K > 0$. Note that periodization of the TPFFT $g$ is defined by

$$(\mathcal{P}_K g)(x) = \sum_{k \in \mathbb{Z}} g(x - kK) = Z_K g(x, 0).$$

If all parameters $\delta_i \in \mathbb{R} \setminus \{0\}$ are distinct, then, by Corollary 16,

$$(\mathcal{P}_K g)(x) = \sum_{i=1}^m \frac{1}{\delta_i} e^{-x/\delta_i} \prod_{k=1,k\neq i}^m \left(1 - \frac{\delta_k}{\delta_i}\right)^{-1}$$

for all $x \in \mathbb{T}_K$. If multiple entries $\delta_i$ occur, we can switch to the representation of $Z_K g(x, 0)$ by a divided difference of order $m - 1$ with knots $a_i = 1/\delta_i$, as in Remark 17. This gives

$$(\mathcal{P}_K g)(x) = [a_1, \ldots, a_m \mid r_{x,0}]$$

where

$$r_{x,0}(y) = (-1)^{m-1} \left(\prod_{i=1}^m a_i\right) e^{-xy} \frac{1}{1 - e^{-Ky}}.$$

**Theorem 19.** Let $g$ be a totally positive function of finite type $m \geq 2$ and let $a, K, M \in \mathbb{N}$, such that $K/a \in \mathbb{N}$ and $K/M \in \mathbb{N}$. If $\frac{K}{M} < 1$, then $\mathcal{G}(\mathcal{P}_K g, a, 1/M)$ is a frame for $L^2(\mathbb{T}_K)$. Furthermore there exists a dual window $P_{Kg}$, which is obtained by periodizing the dual window computed by Algorithm 24.

By duality, if $\frac{K}{M} > 1$, then $\mathcal{G}(\mathcal{P}_K g, a, 1/M)$ is a Riesz sequence for $L^2(\mathbb{T}_K)$.

**Proof:** The proof is the same as the proof of Theorem 18 except that we now apply the second statement of Proposition 4.

**D. Gabor Frames of TPFFTs on $\mathbb{C}^K$**

The analog of Theorem 1 for finite, discrete signals in $\mathbb{C}^K$ is as follows.

**Theorem 20.** Let $g$ be a totally positive function of finite type $m \geq 2$ and $a, K, M \in \mathbb{N}$ such that $K/a \in \mathbb{N}$ and $K/M \in \mathbb{N}$. If $\frac{K}{M} < 1$, then $\mathcal{G}(\mathcal{P}_K g, a, 1/M)$ is a frame for $\mathbb{C}^K$. Furthermore a dual window $P_{Kg}$ can be obtained by periodization and sampling the (continuous) dual window computed by Algorithm 24.

**Proof:** Again the proof is the same as the proof of Theorem 18. This time we use the third statement of Proposition 4.

**Remark 21.** In applications of digital signal processing, the sampling step-size $h = 1/N$ (in seconds), where $N \in \mathbb{N}$, is often fixed by physical measurements or devices for transmission. Likewise, the width of the window $g$ (such as the half-band-width of the Gaussian window) is also fixed beforehand. In order to avoid unnecessary scaling of $g$, one uses the periodization of the sampled window at the proper step-size, that is

$$(\mathcal{P}_{K/NG} g)\left(\frac{l}{N}\right) = \sum_{k \in \mathbb{Z}} g\left(\frac{l}{N} - kK\right) = Z_K g\left(\frac{l}{N}, 0\right)$$

with $l \in \{0, \ldots, NK - 1\}$ and period of length $K \in \mathbb{N}$. By additional normalization in $\mathbb{C}^{NK}$, and for a totally positive function $g$ of finite type $m \geq 2$ with distinct nonzero parameters $\delta_k$, we define

$$(\mathcal{Q}_{K,NG})(l) = N^{-1/2} Z_K g\left(\frac{l}{N}, 0\right) = N^{-1/2} \sum_{i=1}^m \frac{1}{\delta_i} e^{-l/(NK\delta_i)} \prod_{k=1,k\neq i}^m \left(1 - \frac{\delta_k}{\delta_i}\right)^{-1}$$

for all $l \in \{0, \ldots, NK - 1\}$. This gives a practical way to quickly generate sampled and periodized windows for Gabor frames. One just needs to choose pairwise distinct values $\delta_i$ to compute a vector $g$ of
length $KN$ according to sampling step-size $h = 1/N$ or $h = 1$ and period of length $K$. Then $G(g, a, 1/M)$ is a Gabor frame for $(\mathbb{C}^{KN})$ for arbitrary $a, M \in \mathbb{N}$ with $a/M < 1$, provided that $K/a$ and $K/M$ are integers.

E. The critical density

In this section we will study Gabor families $G(Sg, \alpha, \beta)$, $G(P_Kg, \alpha, \beta)$, and $G(P_KSg, \alpha, \beta)$ at the critical density $\alpha = M \in \mathbb{N}$, $\beta = 1/M$. Note that for a totally positive function $g$ of finite type $m \geq 2$, Theorem 1 says that $\alpha < 1$ is equivalent to $G(g, \alpha, \beta)$ being a frame for $L^2(\mathbb{R})$. In particular, at the critical density $\alpha/\beta = 1$, the continuity of $Z_{\alpha}g$ in Proposition 13 and Theorem 7(a) show that $G(g, \alpha, \beta)$ has no positive lower frame bound. Yet positive lower frame bounds may exist in the discrete or the periodic setting, as the Zak transform $Z_{\alpha}g$ is restricted to subsets of $[0, \alpha) \times [0, 1/\alpha)$, as in Theorem 7(b)–(d). In Lemma 9 negative conclusions about the lower frame bound were given for the case where $g$ is a continuous even function. Here, we present positive results about Gabor frames at the critical density if $g$ is a TPFFT. We start with the following example.

Example 22. The function $g(x) = \frac{1}{2}e^{-|x|}$ is even and totally positive of finite type $m = 2$. Its Fourier transform has the form (20), with parameters $\delta_{1,2} = \pm 1$ and $\psi_{1,2} = \frac{1}{2}$. Corollary 16 gives the Zak transform

$$Z_{\alpha}g(x, \xi) = \frac{e^{-x}}{2(1 - e^{-\alpha(1+2\pi j \xi)})} - \frac{e^{x}}{2(1 - e^{\alpha(1-2\pi j \xi)})}.$$ 

The only zero of $Z_{\alpha}g$ which lies in the domain $[0, \alpha) \times [0, 1/\alpha)$ is $\left(\frac{\alpha}{2}, \frac{1}{2\alpha}\right)$. Therefore, by Theorem 7 we find that

- $G(Sg, \alpha, \beta)$ is a Gabor frame for $\ell^2(\mathbb{Z})$ if $\alpha$ is odd.
- $G(P_Kg, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{T}_K)$ if $K/M$ is odd.
- $G(P_KSg, \alpha, \beta)$ is a Gabor frame for $\mathbb{C}^K$ if $M$ is odd or $K/M$ is odd.

In a separate work it was shown by T. Kloos and one of the authors [41] that for every totally positive function $g$ of finite type $m \geq 2$ and $\alpha > 0$, the Zak transform $Z_{\alpha}g$ has exactly one zero in the domain $[0, \alpha) \times [0, 1/\alpha)$, and this zero is located at $(x, \frac{1}{2\alpha})$ for some $x \in (0, \alpha)$. Based on this result, we can draw the following conclusion from Theorem 7.

Theorem 23. Let $g$ be a totally positive function of finite type $m \geq 2$. Assume $\alpha = M \in \mathbb{N}$ and let $\beta = 1/M$ and $K \in \mathbb{N}$ such that $K/M \in \mathbb{N}$.

- If $K/M$ is odd, then $G(P_Kg, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{T}_K)$.
- If $K/M$ is odd, then $G(P_KSg, \alpha, \beta)$ is a Gabor frame for $\mathbb{C}^K$.

In addition, assume that now $g$ is even. Then

- If $M$ is odd, then $G(Sg, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{T}_K)$.
- If $M$ is odd, then $G(P_KSg, \alpha, \beta)$ is a Gabor frame for $\mathbb{C}^K$.

Proof: If $K/M$ is odd, then $1/K \neq 1/(2M)$ for the frequency parameter in (15) for all $l \in \{0, 1, \ldots, K/M - 1\}$, hence the infimum in (15) is positive. This shows that $G(P_Kg, \alpha, \beta)$ has a positive lower frame bound. The upper frame bound exists since $|Z_{\alpha}g|^2$ is a continuous function and attains its maximum on the compact set $[0, \alpha) \times [0, 1/\alpha)$. The same argument works for $G(P_KSg, \alpha, \beta)$.

If $g$ is even, then the unique zero of $Z_{\alpha}g$ inside the fundamental domain $[0, \alpha) \times [0, 1/\alpha)$ is at $(n, \frac{1}{2M})$. If $M$ is odd, then $k \neq M/2$ for the time parameter in (14) for all $k \in \{0, \ldots, M - 1\}$. By continuity of $|Z_{\alpha}g|^2$, the infimum in (14) is positive. This shows that $G(Sg, \alpha, \beta)$ is a Gabor frame for $\ell^2(\mathbb{Z})$. The same argument works for $G(P_KSg, \alpha, \beta)$. 


F. Algorithm for the computation of $\gamma$

A special advantage of Gabor frames with TPFFT is the existence of a dual window with compact support. We now describe an algorithm for the computation of many possible dual windows with compact support. This algorithm is an adaptation of the algorithm for the computation of $\gamma$ given in [24]. It is numerically much more stable than the original one. The proof that this algorithm defines a dual window $\gamma$ for the Gabor frame $G(g, \alpha, \beta)$ is given in the appendix.

**Algorithm 24.** Input parameters are the TPFFT $g$, the lattice parameters $\alpha, \beta > 0$ with $\alpha \beta < 1$, a parameter $L \in \mathbb{N}_0$ controlling the support size of the dual window $\gamma$, and a point $x \in [0, \alpha)$. More precisely, $g$ is defined by specifying the vector $\delta = (\delta_1, \ldots, \delta_{m+n})$ of its non-zero parameters $\delta_k$ (see (19)), where $m, n \in \mathbb{N}_0$, $m+n \geq 2$, and $m$ (resp. $n$) is the number of positive (resp. negative) parameters $\delta_k$.

Output parameters are integers $i_1, i_2$ and the vector of values $\gamma(x + \alpha i)$, $i_1 \leq i \leq i_2$, in the support of $\gamma$.

1) Set $r := \left\lfloor \frac{1}{1-\alpha \beta} \right\rfloor$.
2) Set $k_1 = -(r+1)m - L$, $k_2 = (r+1)n + L$.
3) Set $i_1 := \left\lfloor \frac{k_1 + m - 1}{\alpha \beta} - \frac{x}{\alpha} \right\rfloor + 1$, $i_2 := \left\lfloor \frac{k_2 - n + 1}{\alpha \beta} - \frac{x}{\alpha} \right\rfloor - 1$.
4) Set $P = (p_{ik})_{i_1 \leq i \leq i_2, k_1 \leq k \leq k_2}$ with $p_{i,k} = g \left( x + \alpha i - \frac{k}{\beta} \right)$.
5) Compute the pseudoinverse $P^\dagger = (q_{ki})_{k_1 \leq k \leq k_2, i_1 \leq i \leq i_2}$ of $P$.
6) Take the row with index $k = 0$ of $P^\dagger$. Its coefficients define the values of the dual window $\gamma$ at the points $\{x + \alpha i | i_1 \leq i \leq i_2\}$, i.e.

$$\gamma(x + \alpha i) := \begin{cases} \beta q_{0,i}, & \text{if } i_1 \leq i \leq i_2, \\ 0, & \text{if } i < i_1 \text{ or } i > i_2. \end{cases} \quad (30)$$

This algorithm yields a class of dual windows $\gamma$ (depending on $L$) for the Gabor frame $G(g, \alpha, \beta)$ in $L^2(\mathbb{R})$. Note that the function $\gamma$ is piecewise continuous. Indeed, the only discontinuity (with respect to $x$) can occur when the selection of $i_1$ or $i_2$ in step 3 of the algorithm has a jump, since all entries of the matrix $P$ and its pseudoinverse $P^\dagger$ depend continuously on $x$. Moreover, the definitions in the algorithm imply that

$$x + \alpha i_1 > x + \alpha \left( \frac{k_1 + m - 1}{\alpha \beta} - \frac{x}{\alpha} \right) = -rn - L - \frac{1}{\beta},$$

with $r := \left\lfloor \frac{1}{1-\alpha \beta} \right\rfloor$, and likewise $x + \alpha i_2 < \frac{rn + L + 1}{\beta}$. Therefore, the input parameter $L$ is used to control the support of $\gamma$,

$$\text{supp } \gamma \subset \left[ -rn - L - \frac{1}{\beta}, \frac{rn + L + 1}{\beta} \right]. \quad (31)$$

When working with discrete, periodic or finite signals, we need to sample or periodize $\gamma$, or apply both operations. For example, in order to obtain a discrete dual window for the Gabor frame $G(Sg,a,\frac{1}{M})$ of $\ell^2(\mathbb{Z})$, where $a, M \in \mathbb{N}$, we choose $x \in \{0,1,\ldots,a-1\}$ in Step 3 of Algorithm 24 and obtain the dual window $S \gamma$. Additional periodization with period of length $K$ provides the dual window $P_K S \gamma$ of the Gabor frame $G(P_K Sg,a,\frac{1}{M})$ of $\mathbb{C}^K$, provided that $K/a \in \mathbb{N}$ and $K/M \in \mathbb{N}$. 
IV. EXAMPLES

Fig. 1 shows four totally positive functions of finite type. Fig. 1(a) shows the twosided exponential function $g_1(x) = \frac{1}{2}e^{-|x|}$ with parameters $\delta_1 = -1$, $\delta_2 = 1$. This is an even TPFFT of type 2, and $m = 1$, $n = 1$ in Algorithm 24. Fig. 1(b) depicts the function $g_2(x) = (3e^{-3x} - 6e^{-2x} + 3e^{-x})h(x)$ with parameters $\delta_1 = 1$, $\delta_2 = 1/2$, $\delta_3 = 1/3$ and type $m = 3$; here $n = 0$ as all $\delta_k$ are positive, and $\text{supp} g_2 = [0, \infty)$. Fig. 1(c) shows the even function $g_1(x) = \frac{2}{3}e^{-|x|} - \frac{1}{3}e^{-2|x|}$ of type 4 (with $m = 2$, $n = 2$) with parameters $\delta_{1,2} = \pm 1$, $\delta_{3,4} = \pm 1/2$. Finally, Fig. 1(d) depicts the asymmetric exponential $g_4(x) = e^{3x}h(-x) + e^{-2x}h(x)$ with parameters $\delta_1 = -2/3$ and $\delta_2 = 1$, here $m = 1$, $n = 1$.

Fig. 2 shows the function $g(x) = \frac{2}{3}e^{-|x|} - \frac{1}{3}e^{-2|x|}$ and three of its dual windows. The lattice parameters are $\alpha = 2$, $\beta = 1/3$. Fig. 2(b) is the dual window as computed by Algorithm 24 with $L = 0$ giving the smallest support, but obvious discontinuities of $\gamma$. Fig. 2(c) and 2(d) are computed for $L = 1$ and $L = 2$.

The free parameter $L$ in the algorithm not only determines the length of the support of the dual window as in (31), it also seems to parametrize the smoothness of the dual window, see Fig. 3. With increasing $L$ the size of the jumps decreases. As $L$ tends to infinity, the dual window converges to the canonical dual window. At this time we do not have a rigorous proof to confirm these numerical observations.

V. CONCLUSION

We studied totally positive functions as a class of windows for Gabor frames (Weyl-Heisenberg frames). Such windows are universal in two ways:

(a) Oversampling the time-frequency plane, even by a minimal amount, guarantees the frame property. If $\alpha \beta < 1$, the Gabor family $G(g, \alpha, \beta)$ is a frame (without any further restrictions). Likewise, undersampling the time-frequency plane, even by a minimal amount, guarantees the linear independence. If $\alpha \beta > 1$, then the Gabor family $G(g, \alpha, \beta)$ is a Riesz sequence. The case of the critical density is also considered.
Fig. 2: The function \( g(x) = \frac{2}{5} e^{-|x|} - \frac{4}{5} e^{-2|x|} \) and three dual windows obtained by successively increasing \( L \) in Algorithm 24.

Fig. 3: A closeup of the discontinuities of the dual windows in Figure 2.

(b) Totally positive functions can be used as windows for Gabor frames in the setting for time-continuous
signals, for time-discrete signals, for continuous periodic signals, and for discrete periodic signals.

In addition, a totally positive window decays exponentially and thus its numerical support is finite. We give a simple numerical algorithm for the exact computation of dual windows with compact support.

**APPENDIX A**

For the proof that Algorithm 24 computes the values of a dual window $\gamma$ we make use of the following results, see [24], [33], [34] and the references cited.

**Theorem 25** (Characterization of Gabor frames). Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$. Then the following are equivalent.

1) $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$.
2) There exists a so-called dual window $\gamma$, such that $\mathcal{G}(\gamma, \alpha, \beta)$ is a Bessel sequence and satisfies the Wexler-Raz biorthogonality relations

$$
\langle \gamma, M_{1/\alpha} T_{k/\beta} g \rangle = \alpha \beta \delta_{k,0} \delta_{l,0}, \quad \forall k, l \in \mathbb{Z}.
$$

(32)

Analogous characterizations hold for Gabor frames on $L^2(\mathbb{Z})$, on $L^2(\mathbb{T}_K)$ and $\mathbb{C}^L$.

**Theorem 26** (The Schoenberg-Whitney conditions [42]). Let $g \in L^1(\mathbb{R})$ be a TPFFT. In the factorization (18) of $g$ let $m, n \in \mathbb{N}$ denote the number of positive $\delta_\nu$ and $n \in \mathbb{N}$ denote the number of negative $\delta_\nu$ respectively. For two sequences $(x_i)_{i=1,...,N}$ and $(y_k)_{k=1,...,N}$, $N \in \mathbb{N}$, the determinant $\det(g(x_i - y_k)_{i,k=1,...,N})$ is strictly positive, if and only if

$$
y_{i-n} < x_i < y_{i+m} \quad \text{for } 1 \leq i \leq N
$$

(with the interpretation $y_k = -\infty$ for $k \leq 0$ and $y_k = \infty$ for $k > N$.)

Now we prove that Algorithm 24 computes the function $\gamma$ which satisfies the Wexler-Raz biorthogonality relation.

**Proof:** Recall that $\alpha \beta < 1$ and $r = \lceil \frac{1}{1-\alpha \beta} \rceil$, which implies $r + 1 > (1 - \alpha \beta)^{-1}$. We consider the matrix $P$ in Step 4 of Algorithm 24

$$
P := \left(g(x + \alpha i - \frac{k}{\beta})\right)_{i_1 \leq i \leq i_2, \ k_1 \leq k \leq k_2}.
$$

(34)

The number $M := i_2 - i_1 + 1$ of rows of $P$ exceeds the number $N := k_2 - k_1 + 1$ of columns, as by simple computations

$$
i_2 - i_1 \geq \frac{k_2 - n + 1}{\alpha \beta} \frac{1}{\alpha} - 1 - \frac{k_1 + m - 1}{\alpha \beta} \frac{1}{\alpha} + 1
$$

$$
= \frac{k_2 - k_1}{\alpha \beta} \frac{1}{\alpha} - \frac{n + m - 2}{\alpha \beta} - 2
$$

$$
> (k_2 - k_1) + \frac{1}{\alpha \beta} ((k_2 - k_1)(1 - \alpha \beta) - (n + m))
$$

and

$$
(k_2 - k_1)(1 - \alpha \beta) \geq (n + m)(r + 1)(1 - \alpha \beta) > n + m.
$$

**Claim 27.** $P$ has full rank $N$.

**Proof:** Numbering rows of $P$ from 1 to $M$ and columns from 1 to $N$, as usual, the matrix entries are $p_{ik} = g(x_i - y_k)$, where

$$
x_i := x + (i_1 - 1 + i)\alpha, \quad y_k := \frac{k_1 - 1 + k}{\beta},
$$

(35)
for $i = 1, \ldots, M$ and $k = 1, \ldots, N$. Since $\alpha \beta < 1$, each interval $(y_k, y_{k+1})$ contains at least one $x_i$. The definition of $i_1, i_2$ gives

$$y_m < x_1 < y_{m+1}, \quad y_{N-n} < x_M < y_{N-n+1}$$

(with definitions of $y_0, y_{N+1}$ as in (35) if $m = 0$ or $n = 0$). As elaborated in the proof of Theorem 8 in [24], we can select a subsequence $(x_{i_1})_{1 \leq i \leq N}$ of $(x_i)$,

$$x_1 =: x_{i_1} < x_{i_2} < \cdots < x_{i_N} := x_M$$

such that the Schoenberg-Whitney conditions (33) are satisfied for this subsequence intertwined with the sequence $y_1, \ldots, y_N$. Then the corresponding $N \times N$ submatrix consisting of the rows $i_1, \ldots, i_N$ of $P$ has nonzero determinant, and thus $P$ has full rank.

Claim 27 shows that the pseudo-inverse $P^\dagger$ is a left inverse of $P$ satisfying $P^\dagger P = I$. We let $k_0 := 1 - k_1$ be the “central” column index in $P$ associated with $y_{k_0} = 0$. The corresponding row of $P^\dagger$ is denoted by

$$v = (q_{k_0,i})_{1 \leq i \leq M}. \tag{37}$$

Furthermore, define the vector $w_k \in \mathbb{R}^m$ by

$$w_k := \bigg(g(\frac{x_i - k}{\beta})\bigg)_{1 \leq i \leq M}, \quad k \in \mathbb{Z}. \tag{38}$$

**Claim 28.** Every column vector $w_k$ with $k < k_1$ lies in the linear span of the first $m$ columns of $P$. Likewise, $w_k$ with $k > k_2$ lies in the linear span of the last $n$ columns of $P$.

**Proof:** If $m = 0$ and $k < k_1$, the claim means that $w_k = 0$. Indeed, for $m = 0$, the support of $g$ is $(-\infty, 0]$ by (20), and we have $x_1 = x + \alpha i_1 > \frac{k_1 - 1}{\beta}$. Since $(x_i)$ is increasing, we obtain $g(\frac{x_i - k}{\beta}) = 0$ for all $1 \leq i \leq M$.

If $m \geq 1$ and $k < k_1$, we define the $M \times (m+1)$ matrix

$$P_0 = (g(x_i - \frac{k}{\beta}))_{1 \leq i \leq M, \ k = k_1 - m, \ k_1 + m - 1}. \tag{39}$$

The last $m$ columns of $P_0$ are the first $m$ columns of the matrix $P$, therefore they are linearly independent. In order to prove the claim, we show that $\text{rank}(P_0) = m$. Suppose $P_0$ has full rank $m+1$, then there exists an invertible $(m + 1) \times (m + 1)$ submatrix $P_1$ of $P_0$ with row indices $1 \leq s_1 < s_2 < \cdots < s_{m+1} \leq M$. That is, $P_1 = (g(x_{s_i} - \eta_i))_{1 \leq i, k \leq m+1}$ where

$$\eta_1 = \frac{k}{\beta} < \eta_2 = y_1 < \cdots < \eta_{m+1} = y_m.$$ 

Since $P_1$ is invertible, the Schoenberg-Whitney conditions imply that $x_1 \leq x_{s_1} < \eta_{m+1} = y_m$, which contradicts (36). The case $k > k_2$ is analogous.

Now we can complete the proof. The vector $w_0$ is the “central” column of $P$ with column index $k_0 = 1 - k_1$. The identity $P^\dagger P = I$ implies that the row vector $v$ in (37) satisfies $v \cdot w_0 = 1$ and $v \cdot w_k = 0$ for $k_1 \leq k \leq k_2$, $k \neq 0$. Now our choice of $k_1, k_2$ implies $m < k_0 < N - n + 1$ and thus the orthogonality of the vector $v$ to the first $m$ and the last $n$ columns of $P$. By Claim 28 the additional orthogonality relations $v \cdot w_k = 0$ are satisfied for all $k < k_1$, and all $k > k_2$. Recall that the definitions of $P$, $P^\dagger$, $v$ and $w_k$ depend on $x$. The definition of the function values of $\gamma$ in (30) can be written as

$$\gamma(x_i) := \begin{cases} \beta q_{k_0,i}, & \text{if } 1 \leq i \leq M, \\ 0, & \text{otherwise}. \end{cases}$$
where \( x_i = x + (i_1 - 1 + i)\alpha \) for \( i \in \mathbb{Z} \) as in (35). It was proved in [24] that, by letting \( x \in [0, \alpha) \), this defines a measurable function \( \gamma \in L^2(\mathbb{R}) \), and that the Gabor family \( G(\gamma, \alpha, \beta) \) is a Bessel family in \( L^2(\mathbb{R}) \). It remains to check the Wexler-Raz biorthogonality relation. The definition of \( \gamma \) leads to

\[
\langle \gamma, M_{l/\alpha} T_{k/\beta} g \rangle = \int_{\mathbb{R}} \gamma(x) \overline{g(x - \frac{k}{\beta})} e^{-2\pi jlx/\alpha} \, dx
\]

\[
= \int_{\alpha}^{\ell} \sum_{i \in \mathbb{Z}} \gamma(x + i\alpha) \overline{g(x + i\alpha - \frac{k}{\beta})} e^{-2\pi jlx/\alpha} \, dx
\]

\[
= \beta \int_{0}^{\alpha} \delta_{k,0} e^{-2\pi jlx/\alpha} \, dx = \alpha \delta_{k,0} \delta_{l,0}.
\]

The sum in the second integral is finite; it represents the product \( v(x) \cdot w_k(x) \) with \( v(x) \) in (37) and \( w_k(x) \) in (38) where the dependency on \( x \) is now included in the notation. Thus \( \gamma \) is a dual window.

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