Local Fractional Calculus: a Review

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Abstract

The purpose of this article is to review the developments related to the notion of local fractional derivative introduced in 1996. We consider its definition, properties, implications and possible applications. This involves the local fractional Taylor expansion, Leibnitz rule, chain rule, etc. Among applications we consider the local fractional diffusion equation for fractal time processes and the relation between stress and strain for fractal media. Finally, we indicate a stochastic version of local fractional differential equation.

1 Introduction

In 1970’s Mandelbrot [1] popularized fractals, irregular sets whose appropriately defined dimension is larger than their topological dimension, which were already known in the mathematical literature as pathological examples. He advocated their use in modelling numerous irregular objects and processes found in nature. This has completely changed the outlook and since then several studies have been carried out to determine the fractal dimensions in diverse fields ranging from biology to astrophysics. A natural extension of these developments leads us to related questions which involve fractals, for example, diffusion on fractals, wave equation with fractal boundary condition etc. As a consequence, it becomes necessary to be able to incorporate fractals in usual calculus or its appropriate generalization. However, fractals being generally non-differentiable in some sense or the other, the ordinary calculus fails to apply.

A possible candidate to overcome this impasse is the fractional calculus [2 4 3 5 6 7], a generalization of differentiation and integration to arbitrary orders. The field which existed for quite long but being used in applications only recently. The fractional derivative operators, as defined in the literature using various approaches, turn out to be non-local operators and quite well incorporate and describe long term memory effects and asymptotic scaling. They were also used to describe fractals because of the common scaling property. It is known that fractals
have a local scaling property and in general the scaling exponent can be different at different places. Therefore, in author’s thesis [8] and related publications [9, 10, 11], it was thought to be prudent to suitably modify the definition and make it local. This gave rise to the definition of the local fractional derivative (LFD). Several authors [13, 12, 15, 16, 17, 18, 19, 20] have tried to take further and apply this definition. Some of these works will be reviewed here. In the process, it will also be attempted to clear some possible confusion. Also, quite a few variations of local versions of fractional derivatives were introduced in the course of development. It is planned to restrict to the results related to the original definition of LFD as given in [9] It is not the intent of this review to discuss and compare various versions of LFD though it can be an interesting topic in itself.

2 Definitions

The purpose of this section is to state the prerequisite definitions and also to introduce the LFD. As we have stated earlier, we wish to modify the non-local fractional derivative to make it local. We choose, for this purpose, the Riemann-Liouville derivative since, unlike Weyl derivative, it allows control over the lower limit and unlike Caputo derivative, it does not put extra smoothness conditions on the function to be differentiated.

Definition 2.1 The Riemann-Liouville fractional derivative of a function \( f \) of order \( q (0 < q < 1) \) is defined as:

\[
D^q_x f(x') = \begin{cases} 
D^q_{x,+} f(x'), & x' > x, \\
D^q_{x,-} f(x'), & x' < x.
\end{cases}
\]

\[
D^q_x f(x') = \frac{1}{\Gamma(1-q)} \left\{ \frac{d}{dx'} \int_{x'}^{x} f(t) (x' - t)^{-q} dt, \quad x' > x, \\
-\frac{d}{dx} \int_{x}^{x'} f(t) (t - x')^{-q} dt, \quad x' < x. \right. 
\]  \hspace{1cm} (2.1)

Many a times it becomes necessary to study the local scaling behaviour of a function, especially in the case of a fractal or a multifractal function. In such situations the non-local behaviour of the usual fractional derivatives can be a hurdle in correctly characterizing the local properties of the functions. Therefore, in [9] we defined a local fractional derivative (LFD) as:

Definition 2.2 The local fractional derivative of order \( q (0 < q < 1) \) of a function \( f \in C^0 : \mathbb{R} \rightarrow \mathbb{R} \) is defined as

\[ D^q f(x) = \lim_{x' \to x} D^q_x (f(x') - f(x)) \]

if the limit exists in \( \mathbb{R} \cup \infty \).

Here, we have subtracted the value of the function \( f \) at the point \( x \) which is the point of interest. This washes out the effect of the constant term making the definition independent of the origin or translationally invariant. We have also introduced a limit which makes it explicitly local.
Precursors to this definitions can be found in the work of Hilfer [21] and also in concurrent and independent work of Ben Adda [22].

Let us consider a simplest possible example:

**Example 2.1** \( f(x) = x^p \) where \( 0 < p < 1 \) and \( x \geq 0 \). We would like to find out the LFD of this function at \( x = 0 \). Notice that \( f(0) = 0 \). So we have

\[
\mathcal{D}^q f(0) = \lim_{x \to 0} \frac{d^q (f(x) - f(0))}{d(x - 0)^q} = \lim_{x \to 0} \frac{d^q x^p}{dx^q} = \lim_{x \to 0} \frac{\Gamma(p + 1)}{\Gamma(p - q + 1)} x^{p-q} = \begin{cases} 
0 & q < p \text{ or } q = p + n, \ n = 1, 2, 3, \ldots \\
\Gamma(p + 1) & p = q \\
\infty & \text{otherwise}
\end{cases}
\]

At any other point \( x = x_0 > 0 \), \( f(x) \) is differentiable with non-zero value of the derivative. Hence it can be approximated locally as \( f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0) \). So the LFD becomes

\[
\mathcal{D}^q f(x_0) = \lim_{x \to x_0} \frac{d^q (f(x) - f(x_0))}{d(x - x_0)^q} = f'(x_0) \lim_{x \to x_0} \frac{d^q (x - x_0)}{d(x - x_0)^q} = f'(x_0) \lim_{x \to x_0} \frac{\Gamma(1 + 1)}{\Gamma(1 - q + 1)} (x - x_0)^{1-q} = \begin{cases} 
0 & q < 1 \text{ or } q = 2, 3, 4, \ldots \\
f'(x_0)\Gamma(2) & q = 1 \\
\infty & \text{otherwise}
\end{cases}
\]

One observes in this example that owing to the limiting procedure the LFD has a singular behaviour and the LFD is zero for orders smaller than certain *critical order* and infinite for most of the orders above this order. This leads us to the following definition:

**Definition 2.3** The degree or the critical order of LFD of the continuous function \( f \) at \( x \) is defined as:

\[ q_c(x) = \sup\{q : \mathcal{D}^q f(x) \text{ exists at } x \text{ and is finite}\} \]

In the example [2.1] the critical order is \( p \) at \( x = 0 \) and 1 at other values of \( x \).

In order to generalize the definition [10] to the orders beyond one we have to subtract the Taylor expansion around the point of interest as follows:
Definition 2.4 The LFD of order $q$ ($N < q \leq N + 1$) of a function $f \in C^0 : \mathbb{R} \to \mathbb{R}$ is defined as
\[
D_q f(x) = \lim_{x' \to x} D_x^q \left( f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{\Gamma(n+1)} (x' - x)^n \right)
\]
if the limit exists in $\mathbb{R} \cup \infty$.

With this generalized definition, the critical order in the example 2.1 gets modified. Whereas, it has the same value $p$ at $x = 0$, when $x > 0$ it has value $\infty$ as the function is analytic at all points $x > 0$ and hence has a Taylor series expansion around any point $x > 0$.

A generalization to a multivariable function has also been carried out in the following manner:

Definition 2.5 We first define, for a function $f \in C^0 : \mathbb{R}^n \to \mathbb{R}$,
\[
\Phi(x, t) = f(x + vt) - f(x) \quad v \in \mathbb{R}^n, \ t \in \mathbb{R}.
\]
Then the directional-LFD of $f$ at $x$ of order $q (0 < q < 1)$ in the direction $v$ is defined as
\[
D_q^v f(x) = D_q^t \Phi(y, t)|_{t=0}
\]
if the limit exists in $\mathbb{R} \cup \infty$.

It is interesting to notice that the limit we introduced to make the derivative local in the previous definitions already exists in the definition of the ordinary directional derivative. The generalization of this definition to higher order has been given in [20].

In an interesting development, Chen et al. [16] carried out careful analysis and, in particular, proved the following lemma:

Lemma 2.1 Let $f : (a, b) \to \mathbb{R}$ be continuous such that $D^\alpha_+ f(y)$ exists at some point $y \in (0, 1)$ then
\[
\lim_{h \to 0^+} \int_0^1 (1 - t)^{-\alpha} \frac{f(ht + y) - f(y)}{h^\alpha} dt
\]
exists and
\[
D^\alpha_+ f(y) = \lim_{h \to 0^+} \int_0^1 (1 - t)^{-\alpha} \frac{f(ht + y) - f(y)}{h^\alpha} dt \quad (2.2)
\]

3 Local fractional Taylor expansion

A geometric interpretation is assigned to the LFD when we observe that it naturally appears in a generalization of the Taylor expansion. If one follows the usual steps to derive the Taylor expansion [27] and replaces the ordinary derivative by the Riemann-Liouville fractional derivative then what one ends up with is the local fractional Taylor expansion with a remainder term.
Let
\[ F(x, x' - x; q) = D_x^q(f(x') - f(x)). \] (3.1)

It is clear that
\[ D_q f(x) = F(x, 0; q) \] (3.2)

Now, for \(0 < q \leq 1\),
\[
\begin{align*}
f(x') - f(x) &= D_x^{-q}D_x^q(f(x') - f(x)) \\
&= \frac{1}{\Gamma(q)} \int_0^{x' - x} \frac{F(x, t; q)}{(x' - x - t)^{q+1}} dt \\
&= \frac{1}{\Gamma(q)} \left[ F(x, t; q) \int (x' - x - t)^{q-1} dt \right]^{x' - x}_0 \\
&\quad + \frac{1}{\Gamma(q)} \int_0^{x' - x} \frac{dF(x, t; q)}{dt} (x' - x - t)^q dt \tag{3.3}
\end{align*}
\]
provided the last term exists. Thus
\[
\begin{align*}
f(x') - f(x) &= \frac{D^q f(x)}{\Gamma(q + 1)} (x' - x)^q \\
&\quad + \frac{1}{\Gamma(q + 1)} \int_0^{x' - x} \frac{dF(x, t; q)}{dt} (x' - x - t)^q dt \\
&\tag{3.4}
\end{align*}
\]
i.e.
\[
\begin{align*}
f(x') = f(x) + \frac{D^q f(x)}{\Gamma(q + 1)} (x' - x)^q + R_q(x', x) \\
&\tag{3.5}
\end{align*}
\]
where \(R_q(x', x)\) is a remainder given by
\[
\begin{align*}
R_q(x', x) &= \frac{1}{\Gamma(q + 1)} \int_0^{x' - x} \frac{dF(x, t; q)}{dt} (x' - x - t)^q dt \tag{3.6}
\end{align*}
\]
Equation (3.6) is a fractional Taylor expansion of \(f(x)\) involving only the lowest and the second leading terms. Using the general definition of LFD and following similar steps one arrives at the fractional Taylor expansion for \(N < q \leq N + 1\) (provided \(D^q\) exists and is finite), given by,
\[
\begin{align*}
f(x') &= \sum_{n=0}^{N} \frac{f^{(n)}(x)}{\Gamma(n + 1)} (x' - x)^n + \frac{D^q f(x)}{\Gamma(q + 1)} (x' - x)^q + R_q(x', x) \\
&\tag{3.7}
\end{align*}
\]
where
\[
\begin{align*}
R_q(x', x) &= \frac{1}{\Gamma(q + 1)} \int_0^{x' - x} \frac{dF(x, t; q, N)}{dt} (x' - x - t)^q dt \\
&\tag{3.8}
\end{align*}
\]
We note that the local fractional derivative (not just fractional derivative) as defined above provides the coefficient \( A \) in the approximation of \( f(x') \) by the function \( f(x) + A(x' - x)^q / \Gamma(q+1) \), for \( 0 < q < 1 \), in the vicinity of \( x \). This generalizes the geometric interpretation of derivatives in terms of ‘tangents’. It can be shown [19] that if the LFD in equation (3.8) exists then the remainder term \( R_q(x',x) \) goes to zero as \( x \to x' \).

**Example 3.1** Let us consider a function \( f(x) = ax^\alpha + bx^\beta \) where \( x \geq 0 \) and \( 0 < \alpha < \beta < 1 \) and study its local fractional Taylor expansion at \( x = 0 \). If \( q < \alpha \), then \( D^q f(x) = 0 \) and the remainder term evaluates to \( ax^\alpha + bx^\beta \). But if \( q = \alpha \), the critical order of \( f \), then the second term in the equation (3.8) is finite and equal to \( ax^\alpha \) and the remainder term yields \( bx^\beta \).

The example illustrates the importance of working at the critical order of the function. Osler, in ref. [28], has constructed fractional Taylor series using usual (not local in the sense above) fractional derivatives. His results are, however, applicable to analytic functions and cannot be used for non-differentiable scaling functions directly. Furthermore, Osler’s formulation involves terms with negative \( q \) also and hence is not suitable for approximating schemes. Finally, the local fractional Taylor series has also been extended [20] to multivariable functions.

### 4 Local fractional differentiability

Fractional differentiability of functions can be studied using any definition of the fractional derivative which would signify the existence of that derivative for a given function. Considering *local* fractional differentiability has an added advantage that it preserves the local property of differentiability in the ordinary calculus. Thus it becomes unnecessary to worry about the definition of the function away from the point of interest. We refer the reader to reference [29] for a recent review of different notions of differentiability and their interrelation.

In [9], local fractional differentiability of Weierstrass’ everywhere continuous but nowhere differentiable function was studied. A form of this function is given by

\[
W_\lambda(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin \lambda^k t, \quad \lambda > 1 \quad 1 < s < 2,
\]

where \( \lambda > 1 \) and \( 1 < s < 2 \). Note that \( W_\lambda(0) = 0 \). The graph of this function is fractal with dimension \( s \) and the Hölder exponent at every point of this function is \( 2 - s \). It was shown in [9] that the Weierstrass’ function is locally fractionally differentiable up to order \( 2 - s \) at every point and the LFD does not exist anywhere for orders greater that \( 2 - s \).

In fact, we proved a general result [9] showing the connection between the degree of LFD (critical order) and the Hölder exponent. In [9] we used the definition of the pointwise Hölder exponent but it turns out that the right way is to use the local Hölder exponent. This was pointed out in [26]. The following theorem was proved there:

**Theorem 4.1** Let \( f \) be a continuous function in \( L^2 \). Then \( q_c(f, x_0) = \alpha_l(f, x_0) \) where \( \alpha_l(f, x_0) \) is the local Hölder exponent of \( f \) at \( x_0 \).
The local fractional differentiability of a multifractal function, in which the Hölder exponent varies from point to point, was also studied [9].

As of now, we do not know any example of a continuous everywhere but nowhere differentiable function for which the LFD exist at the critical order. In fact, it was suspected by the author during his thesis that no such function might exist. This is because, a close inspection of such functions shows oscillations of the log-periodic type at every point of these functions which possibly leads to the nonexistence of LFD at the critical point. We have demonstrated such oscillations in the case of devil’s staircase [30]. But this should not be a serious hurdle as the log-periodic oscillations only reflect the lacunarity in the fractal functions. The lacunarity has been discussed at length by Mandelbrot in [1]. But the critical order of the LFD correctly characterizes the Hölder exponent and hence the dimension of the fractal functions. This means, there are two possible future directions of exploration: one is to assume that the LFD exists at the critical order and develop its applications and the other is to try to modify the definition of the LFD. In the following sections we take the first approach.

5 Properties of LFD

In this section, we review some of the properties of LFD as studied by different authors. Papers in questions are mainly [19], [20] and [23]. In [19] Ben Adda and Cresson and in [20], Babakhani and Daftardar-Gejji proved quite a few properties of LFD rigorously. The first one is a Leibnitz rule for a product of two functions one of which is smooth and other could be non-differentiable. The following theorem was proved in [20]:

**Theorem 5.1** Let \( f(x) \) be continuous on \([a, b]\) and \( D^\alpha_+ f(a) \), \( D^\alpha_- f(b) \) and \( D^\alpha_\pm f(x) \) exists for every \( x \in (a, b) \). If further \( \phi(x) \in C^\alpha[a, b] \), then for \( 0 < \alpha < 1 \)

\[
D^\alpha_+ ((\phi f)(a)) = \phi(a)D^\alpha_+ f(a),
\]
\[
D^\alpha_- ((\phi f)(b)) = \phi(b)D^\alpha_- f(b),
\]
\[
D^\alpha_\pm ((\phi f)(x)) = \phi(x)D^\alpha_\pm f(x).
\]

They also extended the result to the case \( n < \alpha < n + 1 \). In [23], Carpinteri et al. used a more general rule for the LFD of a product of two functions. They considered the case when both functions were non-differentiable. If \( f \) and \( g \) are two functions having the same Hölder exponent, say \( \alpha \), then

\[
D^\alpha (f(x)g(x)) = f(x)D^\alpha g(x) + g(x)D^\alpha f(x).
\]

This was already proved by Ben Adda and Cresson in [19]. They also proved the following formula for the LFD of division of two functions when the LFD of individual functions exist:

\[
D^\alpha \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)D^\alpha f(x) + f(x)D^\alpha g(x)}{g^2(x)},
\]

7
where $g(x) \neq 0$.

The next property studied in \[20\] is the chain rule, that is, the LFD of composite function. The following theorem was proved there.

**Theorem 5.2** Let $h : [a, b] \to \mathbb{R}$ be a function of class $C^{n+3}$, $f$ be a function of class $C^n$ on $h[a, b]$, and $\mathcal{D}^{\alpha-n}_\pm \left[ f^{(n)}(h(x)) \right]$ exists. Then

$$
\mathcal{D}^{\alpha}_\pm [f(h(x))] = \left( \frac{dh}{dx} \right)^n \mathcal{D}^{\alpha-n}_\pm \left[ f^{(n)}(h(x)) \right]
$$

where $n < \alpha < n + 1$, $n \in \mathbb{N} \cup \{0\}$.

The reader is referred to [19] for an interesting generalization of the chain rule.

### 6 Local fractional differential equations

Once we have the LFD it is natural to ask if we can write and solve equations in terms of this operator. The simplest such equation will have the form

$$
\mathcal{D}^{\alpha} f(x) = g(x) \quad (6.1)
$$

where $g(x)$ is some known function and $f(x)$ is an unknown function to be found out and $0 < \alpha < 1$. The immediate question that arises is what is the class of function $g(x)$ for which the solution exist. This question will be considered in this section. A formal solution to the above equation can be written down by generalizing the Riemann sum as follows:

$$
f(x) = \lim_{N \to \infty} \sum_{i=0}^{N-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)} g(x^*)
$$

where $x_i \leq x^* \leq x_{i+1}$. This question was first asked in \[31\] where the formal solution in the form above was also discussed. The factor $(x_{i+1} - x_i)^\alpha / \Gamma(\alpha + 1)$ was motivated by the local Taylor expansion. Later, in \[24, 25\], the same factor was written alternatively as an RL integral of constant function 1 of order $\alpha$, that is, $x_i I^\alpha_{x_{i+1}} 1$. This allowed us to define a differential of fractional order and gave another motivation for the factor $\Gamma(\alpha + 1)$ in the denominator of equation (6.2). We denoted the differential of fractional order as $d^\alpha x$, in line with the notation $d^3 x$ for the volume element, and introduced the notation

$$
\int g(x) d^\alpha x \quad (6.3)
$$

for the RHS of equation (6.2) calling it a ”fractal integral” or ”local fractional integral”. So, (6.3) is only a symbol for the RHS of equation (6.2).

It is easy to see that the solution (6.2) does not exist if $g(x)$ is a continuous function.
Proposition 6.1 The solution (6.2) of the local fractional differential equation (LFDE) (6.1) diverges if \( g(x) \) is a continuous function.

Proof: If \( g(x) \) is a continuous function then there exists an interval, say \([\delta, \gamma]\), such that \( g(x) \) is non-zero positive on this interval. That is, there exists \( \epsilon > 0 \) such that \( g(x) > \epsilon \) on this interval. In this interval, we have

\[
f(x) = \lim_{N \to \infty} \sum_{i=0}^{N-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)} g(x^*)
\]

\[
\geq \lim_{N \to \infty} \sum_{i=0}^{N-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)} \epsilon
\]

\[
= \lim_{N \to \infty} N \times \left( \frac{\gamma - \delta}{N} \right)^\alpha \epsilon
\]

\[
= \lim_{N \to \infty} N^{1-\alpha} (\gamma - \delta)^\alpha \epsilon
\]

\[
= \infty.
\]

Hence the result.

This implies that the LFDE (6.1) will have a solution only if \( g(x) \) is a discontinuous function. This also means that any function can not have continuous LFD in any interval as argued in [13]. That is \( g(x) \) can be of two types or their combination: (i) the function \( g(x) \) is an indicator function of some sparse set (like the Cantor set) such that the number of terms in the sum of (6.2) grow sub-linearly and hence the limit is finite, and (ii) the function \( g(x) \) alternates between positive and negative values in any small interval giving rise to cancellations among the terms of the sum in (6.2) leading to convergent limit. There is a classic Conway base 13 function which is an example of a function which is discontinuous in any given interval. Another possible example, perhaps of practical importance, would be white noise. In the following two subsections we discuss each case in more detail.

It is interesting to note that though such integrals arise naturally in our formalism as inverse of LFD, Mandelbrot already in [11] has suggested studying such integrals using nonstandard analysis which extends the real number system to include infinite and infinitesimally small number. However, as is made clear in the following, since we restrict \( g(x) \) to two classes of physically meaningful functions for which this integral is finite it obviates the need to use the nonstandard analysis.

6.1 \( g(x) \) as indicator function of a Cantor set

As discussed above, the solution to the LFDE (6.1) is seen to exist if \( g(x) \) is an indicator function of a fractal set. That is, if \( C \) is a Cantor set then \( g(x) = 1 \) if \( x \in C \) and \( g(x) = 0 \) otherwise. We denote this by \( g(x) = 1_C(x) \). We now proceed to see that the solution with initial condition
\( f(0) = 0 \) exists if \( \alpha = \dim_H C \). In this case the above Riemann sum takes the form

\[
f(x) = \lim_{N \to \infty} \sum_{i=0}^{N-1} \frac{(x_{i+1} - x_i)\alpha}{\Gamma(\alpha + 1)} F_i^C \equiv \frac{P_C(x)}{\Gamma(\alpha + 1)},
\]

where \( x_i \) are subdivision points of the interval \([x_0 = 0, x_N = x]\) and \( F_i^C \) is a flag function which takes value 1 if the interval \([x_i, x_{i+1}]\) contains a point of the set \( C \) and 0 otherwise. Now if we divide the interval \([0, x]\) into equal sub-intervals and denote \( \Delta = \Delta_i = x_{i+1} - x_i \) then we have \( P_C(x) = \Delta^\alpha \sum F_i^C \). But \( \sum F_i^C \) is of the order of \( N^{-\alpha} \). Therefore \( P_C(x) \) satisfies the bounds \( a x^\alpha \leq P_C(x) \leq b x^\alpha \) where \( a \) and \( b \) are suitable positive constants. Note that \( P_C(x) \) is a Lebesgue-Cantor (staircase) function. Since the function \( 1_C(x) \) is zero almost everywhere the function \( P_C(x) \) is constant almost everywhere. As is clear from the equation \( (6.4) \), it rises only at points where \( 1_C(x) \) is non-zero.

Of course, this is the simplest example considered to elucidate the basic principle. One can consider different forms for \( g(x) \), such as any function \( h(x) \) multiplied by the indicator function of the Cantor set \( 1_C(x) \). Such an equation also will have solutions. Complete theory of these equations is still to be worked out. Some more examples of this kind will be considered in the next section discussing applications.

### 6.2 \( g(x) \) as white noise

The second class, which we are going to consider in this subsection, consists of rapidly oscillating functions which oscillate around zero in any small interval. This type of equation was introduced in \[33\] though its mathematical details are still to be worked out. These oscillations then would result in cancellations again giving rise to a finite solution. A realization of the white noise is one example in this class of functions. This immediately prompts us to consider a generalization of the Langevin equation which involves LFD and \( g(x) \) is chosen as white noise.

So we consider a generalization of the Langevin equation \[32\] in high friction limit where one neglects the acceleration term and replaces the first derivative term by the LFD to arrive at

\[
D^\alpha x(t) = \zeta(t),
\]

where \( < \zeta(t) >= 0 \) and \( < \zeta(t)\zeta(t') >= \delta(t-t') \) the Dirac delta function. The solution of the above equation follows from Eq. \( (6.2) \)

\[
x(t) = \lim_{N \to \infty} \sum_{i=0}^{N-1} \frac{\zeta_i (t_{i+1} - t_i)\alpha}{\Gamma(\alpha + 1)}.
\]

Heuristically it can be seen that

\[
< x(t) > = \lim_{N \to \infty} \frac{N^{-\alpha}}{\Gamma(\alpha + 1)} N^{1/2}
\]

and therefore the average is zero if \( \alpha > 1/2 \) and it does not exist if \( \alpha < 1/2 \). This indicates that the above process is of Lévy type with index \( 2\alpha \) for \( \alpha < 1 \).
7 Applications of LFDE

One would immediately wonder about the applications of such equations involving LFDs. We consider two of them over here. The first one was considered in [31] which considers a local fractional diffusion equation and the second one was introduced by Carpinteri and Cornetti [15] in which they generalized the relation between the stress and strain for fractal media using the LFDE.

7.1 Local fractional diffusion equation

In this subsection we consider the local fractional diffusion equation. The equation can be derived systematically starting from the Chapman-Kolmogorov condition and making use of the local fractional Taylor expansion. The reader is referred to [31] for more details. Here we consider the equation and its solution.

\[
D_t^\alpha W(x, t) = \frac{\Gamma(\alpha+1)}{4}\chi_C(t)\frac{\partial^2}{\partial x^2}W(x, t)
\] (7.1)

We note that even though the variable \( t \) is taking all real positive values the actual evolution takes place only for values of \( t \) in the fractal set \( C \). The solution of equation (7.1) can easily be obtained as

\[
W(x, t) = P_{t-t_0}W(x, t_0)
\] (7.2)

where

\[
P_{t-t_0} = \lim_{N \to \infty} \prod_{i=0}^{N-1} \left[ 1 + \frac{1}{4}(t_{i+1} - t_i)^\alpha F_C^i \frac{\partial^2}{\partial x^2} \right].
\] (7.3)

The above product converges because except for terms for which \( F_C^i = 1 \) (which are of order \( N^\alpha \)) all others take value 1. It is clear that for \( t_0 < t' < t \)

\[
W(x, t) = P_{t-t'}P_{t'-t_0}W(x, t_0)
\] (7.4)

and \( P_t \) gives rise to a semigroup evolution. Using equation (6.4) it can be easily seen that

\[
W(x, t) = e^{P_C(t)\frac{\partial^2}{\partial x^2}}W(x, t_0 = 0).
\] (7.5)

Now choosing the initial distribution \( W(x, 0) = \delta(x) \) and using the Fourier representation of delta function, we get the solution

\[
W(x, t) = \frac{1}{\sqrt{\pi P_C(t)}}e^{P_C(t)}
\] (7.6)
Consistency of the equation (7.6) can easily be checked by directly substituting this in Chapman-Kolmogorov equation. We note that this solution satisfies the bounds

\[
\frac{1}{\sqrt{\pi bt^{\alpha}}} e^{-\frac{x^2}{bt}} \leq W(x, t) \leq \frac{1}{\sqrt{\pi at^{\alpha}}} e^{-\frac{x^2}{at}}
\]

(7.7)

for some \(0 < a < b < \infty\). This is a model solution of a subdiffusive behaviour. It is clear that when \(\alpha = 1\) we get back the typical solution of the ordinary diffusion equation, which is \((\pi t)^{-1/2} \exp(-x^2/t)\).

### 7.2 Relation between stress and strain for fractal media

Carpinteri and Cornetti [15] used the concept of LFD to write down a relation between the strain and the displacement when the strain is localized on the fractal set. They proposed

\[
\epsilon^*(x) = D^\alpha u(x)
\]

where \(\epsilon^*(x)\) is a renormalised strain and \(u(x)\) is the displacement. This also offers a physical interpretation to LFD. This formalism is useful in studying the structural properties of concrete-like or disordered materials. In [14], we calculated resultant of a stress distribution and its moment when the stress is distributed over a fractal set. Further use of LFD and the fractal integral for the principle of virtual work and to generalize the constitutive equations of elasticity has been made in [23, 34].

### 8 Conclusions

We have reviewed developments emanating from and related to the notion of local fractional derivative introduced in [9]. The concept has received wide attention and led to rich variety of works. It was not possible to review all of them here. This article was restricted to the developments directly related to the definition of LFD introduced originally in [9]. Also, there seem to be some misconceptions leading to some erroneous results. It is hoped that this review will help to dispel some doubts.

The central idea was to appropriately modify the non-local fractional derivatives so as to be able to characterize local scaling behaviour which gave rise to LFD. The LFD so defined happened to arise naturally in the local Taylor expansion which meant that there was something more to the construction of the LFD though it involved seemingly artificial steps of subtracting the value of the function at the point and an additional limiting procedure. It was important to realize the LFD as a standalone operator even if its action on the function is ”singular” in the sense that the result is zero for most of the orders and hence pursue its properties and applications. Several authors have carried out substantial work in this direction. The LFD gave rise to the notion of local fractional differentiability and the relation between the order of differentiability and the local Hölder exponent. Considering LFD as an operator in its own right.
leads to questions of its inverse and also equations involving LFD leading to local fractional differential equations and local fractional integrals or fractal integrals which is supposed to invert the LFDs. As a result, local fractional diffusion equation, an equation in terms of LFD to describe the relation between stress and strain in fractal media, local fractional stochastic differential equation are some of the outcomes these series of explorations.

Clearly, lot needs to be done. On the one had, putting the formalism on rigorous mathematical footing is something that needs to be pursued vigorously. Also, developing more applications in diverse fields will give impetus to the efforts put in the development of the theory.

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