Detecting large-scale deviations from FRW geometry with future CMB measurements

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ABSTRACT
We discuss the question to what degree the geometrical structure and the matter content of the universe at scales exceeding the present Hubble horizon is constrained by cosmological observations, in particular by measurements of the cosmic microwave background radiation. For an answer, a simple formalism is described, which goes back to a paper by Kristian and Sachs in 1966.

KEYWORDS: Cosmology; Large-scale gravitational fields; Cosmic microwave background.

1. INTRODUCTION
It is usually assumed that the observable universe can be described by one of the homogeneous-isotropic Friedman-Robertson-Walker models, at least up to scales of the order of the horizon $\lambda_H$. The structure of the universe at larger scales is not known. The inflationary scenario suggests, that the observable universe is part of an inflated region of size $\lambda_I$, possibly much larger than $\lambda_H$. Outside the inflated and hence nearly homogeneous region the spacetime geometry might be inhomogeneous and anisotropic. Could structures outside the horizon by means of their tidal fields have some imprint on the observationally accessible part of the universe (shortly: the local universe), in particular, may they influence the cosmic microwave background radiation?

Twenty years ago L. Grishchuk and Ya. B. Zeldovich (1978) (see also Turner 1991, Grishchuk 1992, Kashlinsky et al. 1994, Lyth 1995, Garcia-ellido et al. 1995) have discussed a similar question. They found that the CMB perturbations $\delta T/T$ resulting from the growing mode of density or gravitational wave perturbations on a scale $\lambda$ and corresponding to a metric perturbation $h$ are of the order $\delta T/T \sim h(\lambda_H/\lambda)^2$. Since one expects $h \sim 1$ for regions $\lambda \geq \lambda_I$, the COBE observations lead to $\lambda_I > 10^3 \lambda_H$, thus the ”homogeneous region” must be much larger than the Hubble distance, independent of any inflationary hypothesis. This argument does not exclude structures exceeding $\lambda_H$ only slightly, if they correspond to metric amplitudes $h$ much smaller than 1. Also other deviations from FRW models may be allowed to a certain degree. For instance, if one restricts the geometry to homogeneous Bianchi models, one is able to obtain from the CMB measurements...
upper limits for the vorticity and shear of these models (Hawking 1969, Collins and Hawking 1973, Bunn et al. 1996, Kogut et al. 1997). But constraining the geometry to a small class of simple models may be misleading, since the real universe might have a much more complicated geometrical structure and matter content at very large distances. In view of the remarkable improvements expected for future CMB observations with the MAP and PLANCK missions it is of interest to see how much of this structure could be detected or constrained. A first step is a mathematical framework dealing with sufficiently general cosmological models. We describe a very simple proposal, which tackles the problem, but needs further elaboration.

3. KRISTIAN-SACHS APPROACH

J. Kristian and R.K. Sachs (1966) made in their classical paper only few assumptions: (i) The universe is described by a Riemannian spacetime with slowly varying metric tensor; (ii) light travels along null geodesics and obeys the usual area-intensity law; (iii) the gravitational field is related to the matter by the Einstein field equation for dust; (iv) all quantities of interest can be expanded in a power series around here-and-now as origin. In their treatment it not necessary to adopt a specific cosmological model a priori, the results are complete and general (apart from the assumption of dust, which can quickly be generalized). Ignoring global properties (or rather assuming that the scale of a possible compactification of the universe is larger than all other scales) made the calculation manageable, and last not least all introduced quantities are measurable in principle.

With a power series expansion around here-and-now one can hardly treat evolutionary problems in cosmology. In particular, the CMB fits not easily into the framework. We remove this shortcoming by treating space and time coordinates on a different footing. Temporal variations are seen to appear with much higher amplitude than spatial variations - at least if one averages over the inevitable small-scale fluctuations as origin of galactic structures. This leads to the idea to carry out the power-law expansion only within spatial hyper-surfaces orthogonally intersecting the observers world line. The Kristian-Sachs expansion coefficients then become time-dependent. Actually, we reduce the generality by adopting the (anthropomorphic) assumption, that space-time reduces along the observers world line to the homogeneous-isotropic FRW models. In geometrical terms, the observers world line \( L \) is assumed as geodesic and shear-free, generally in contrast to the world-lines of neighbouring observers. The method remembers to some degree the covariant approach put forward by G. Ellis and his coworkers in numerous publications, but contrary to their work we try to fix the coordinate system as much as possible in order to reduce the number of coefficients needed to describe deviations from FRW models.

3. COORDINATE SYSTEM

Basic to the method is the construction of a coordinate system, which is adapted
to the matter distribution. We represent the cosmic matter as relativistic fluid flow with \( V^\mu \) as the four-velocity vector tangent to the flow lines (\( V_\mu V^\mu = -1 \), our notation follows the book by Misner, Thorne and Wheeler 1973). Comoving coordinates are introduced such that \( V^\mu = \delta^\mu_0 \). The flow lines are given by \( x^i = \text{const} \ (i = 1, 2, 3) \). The time coordinate \( t \) is not completely fixed by its coincidence with the proper time on the flow lines, and also the spatial coordinates are subject to arbitrary time-independent changes, \( \tilde{t} = t + f(x^i) \), \( \tilde{x}^i = \mathbf{x}^i(x^k) \).

Let the particular flow line \( x^i = 0 \) be the observers world line \( L \). We demand that on and near \( L \) a FRW solution (with zero spatial curvature, see below) is a good approximation. It is sufficient to assume that \( L \) (and only this world line) is geodesic and shear-free. This is a weak form of reducing to FRW along \( L \), since it is compatible with a non-vanishing Weyl tensor and a nonzero vorticity on \( L \). We may then represent the metric tensor near \( L \) as a Taylor series in the spatial coordinates \( x^i \):

\[
\begin{align*}
g_{00} &= -1 \\
g_{0i} &= l_{ik} x^k + l_{ikl} x^k x^l x^m + \ldots \\
g_{ik} &= a^2 \delta_{ik} + h_{ikl} x^l + h_{iklm} x^l x^m + h_{iklmn} x^l x^m x^n + \ldots
\end{align*}
\]

where \( a(t) \) is the Friedman scale factor, the other expansion coefficients are as well time functions in general. The coordinate transformations preserving this form of the metric may also be expanded with constant coefficients, which later serve to reduce initial values of the metric expansion coefficient:

\[
\begin{align*}
\tilde{t} &= t + a_{kl} x^k x^l + a_{klm} x^k x^l x^m + \ldots, \\
\tilde{x}^i &= x^i + b_{ikl} x^k x^l + b_{iklm} x^k x^l x^m + \ldots
\end{align*}
\]

Only in a linear approximation the terms added to the FRW metric can uniquely be separated into scalar, vector and tensor (gravitational wave) perturbations, we shall not attempt this here. - We have assumed a flat FRW metric along \( L \), since the spatial curvature of the FRW models is hidden in the expansion coefficients of Eqn (3).

4. MATTER CONGRUENCE

Using the decomposition of the velocity gradient

\[
V_{\mu \nu} = \omega_{\mu \nu} + \sigma_{\mu \nu} + \Theta (g_{\mu \nu} + V_\mu V_\nu) / 3 - V_\nu A_\mu,
\]

we can expand acceleration, shear, expansion and vorticity around \( L \):

\[
\begin{align*}
A_i &= (i_{ik} + i_{ikl} x^l) x^k + o(3), \\
\sigma_{kl} &= \frac{1}{2} (h_{klr} x^r - \delta_{kl} h_{ssr}) x^r + o(2), \\
\Theta &= \frac{3}{a} \frac{\dot{a}}{a} + \frac{1}{2} \frac{d}{dt} \left( \frac{h_{kl}}{a^2} \right) x^l + o(2)
\end{align*}
\]
\[ \omega_{kl} = \frac{1}{2}(l_{kl} - l_{lk}) + (l_{klr} - l_{lkr})x^r + o(2), \]  
\tag{10}

Together with \( A_0 = 0, \sigma_{0\mu} = 0, \omega_{0k} = 0. \) The lowest terms in this expansion transform under (3),(4) as

\[ \tilde{l}_{kl}(\mathbf{r}) = l_{kl} + 2a_{kl}, \]  
\tag{11}
\[ T_{klm}(\mathbf{r}) = h_{klm}(t) - 2a^2(b_{klm} + b_{lkm}). \]  
\tag{12}

Since \( a_{kl} = a_{lk}, \) the vorticity \( \omega_{kl} \) on \( L \) is not affected by coordinate transformations. Similar simple transformation rules hold for the higher-order coefficients. For the matter tensor we assume a perfect fluid

\[ T_{\mu\nu} = V_\mu V_\nu (\mu + p) + pg_{\mu\nu} \]  
\tag{13}

with an equation of state \( p = w\mu \) (w = const). The conservation equations \( T_{\mu\nu}^{\mu\nu} = 0 \) can partly be integrated without expanding \( \mu \) and \( p \) in a Taylor series. One obtains

\[ \mu = m(x^i)((1 + g_{0k}g_{0l}\gamma^{kl})\det(g_{ik}))^{-(1+w)/2}, \]  
\tag{14}
\[ 0 = \dot{g}_{0k} + \frac{w}{1+w}(\frac{\mu_{,k}}{\mu} + g_{0k}\frac{\dot{\mu}}{\mu}), \]  
\tag{15}

\( \gamma^{ik} \) is the inverse matrix to \( g_{ik}. \) We are here particularly interested in the case of dust \( w = 0, \) which is an approximation accurate enough to illustrate the method, when we integrate the null geodesics back to the surface of last scattering of the CMB photons. In the dust case the fluid is not accelerated, and \( g_{0i} \) is time-independent. \( g_{0i} \) is not a gradient, when vorticity (the antisymmetric part in \( l_{ik} \)) is present. We may however use Eqn (11) to transform the symmetric part of \( l_{ik} \) to zero. For more information we have to look at the field equations.

5. FIELD EQUATIONS

The Einstein field equations cannot be integrated as easily as the matter conservation equations, so we use Taylor expansion again. It is straightforward to write them down for the metric (1)-(3) and the matter tensor (13). To zero order one obtains, again assuming dust and with \( \rho_0 \) as matter density on \( L: \)

\[ 3\frac{\dot{a}}{a} + \frac{1}{2}\kappa\rho_0 = \frac{\omega_{kl}\omega_{kl}}{a^2}, \]  
\tag{16}
\[ 0 = \frac{1}{2}\frac{d}{dt}(\frac{h_{ilk} - h_{klr}}{a^2}) + \frac{2}{a^2}(l_{ilk} - l_{lkl}) + \frac{1}{a^4}(\omega_{lr}(h_{lkr} - h_{rkl}) + \omega_{lk}(-2h_{lrr} + h_{rrl})) \]  
\tag{17}
\[ \delta_{kl}(\frac{\dot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{1}{2}\kappa\rho_0) = \]
\[
\frac{1}{a^4}\omega_{kr} \omega_{lr} + \frac{1}{a^4}(h_{klrr} + h_{rrkl} - h_{rlkr} - h_{krlr})
+ \frac{1}{4a^6}(2h_{srr} - h_{rrs})(h_{skl} + h_{skk} - h_{kls})
- \frac{1}{4a^6}(h_{srl} + h_{srk} - h_{rls})(h_{skr} + h_{srk} - h_{krs}).
\]

(18)

We have written the FRW terms on the lhs and the terms representing external sources on the rhs. We have also kept all nonlinear terms to see the structure of the equations, even if these terms can be neglected for a first calculation of the effects of external gravitational fields. The equations show that corrections to the Friedman equations occur already at zero order, i.e., on the observer world-line \( L \).

The time dependence of the expansion coefficients is only partly determined by the field equations at the given level, thus one has to take higher levels into account. Some equations become algebraic relations between the coefficients. We note that the restrictions for the expansion coefficients in (1)-(3) found in this way are only necessary conditions for \( g_{\mu\nu} \) to represent a solution of the field equation. Our expansion procedure is as yet not based on a well-founded initial value problem, and it is therefore not sure that every approximate expressions for \( g_{\mu\nu} \) can be extended to a full solution of the Einstein field equations.

6. THE CMB: ORIGIN OF A COSMOLOGICAL DIPOLE COMPONENT

The CMB anisotropy in our local cosmological model can be calculated using a well-known relation derived by Panek (1986), Russ et al. (1993) and Dunsby (1997). We write the equation as

\[
\frac{\delta T}{T} = -\int_{t_1}^{t_0} \left( \frac{1}{4\mu(r)E^0} (p^\mu \mu(r),p - \dot{\mu}(r)E + \sigma_{\mu\nu}p^\mu p^\nu E + \frac{V_\mu p^\mu}{E}) \right) dt,
\]

(19)

where \( p^\mu = \frac{dx^\mu}{dt} \) is the tangential vector of a null geodesic connecting the emission (at time \( t_1 \)) of the radiation with the recording event (at time \( t_0 \)), \( \mu(r) \) the density of the radiation field and \( E = -p^\mu V_\mu \) the photon energy relative to \( V_\mu \). Using this relation requires integration of null-geodesics, i.e., solution of the geodesic equation \( p^\mu_{\mu} = 0 \). This can easily be done within the formalism, if one considers the additional external terms as perturbations to the known photon motion in a FRW geometry. The external or super-horizon modes under discussion affect the lowest-order multipoles. As discussed by Bunn et al. (1996), they must be distinguished from the familiar statistical fluctuations in \( T \), which correspond to an isotropic Gaussian random field and are believed to be generated by quantum processes in an inflationary stage. We treat as example the possible presence of an intrinsic dipole component \( \frac{\delta T}{T}_{dipole} = \int n^i \frac{\delta T}{T} d\Omega \) of the CMB, which is independent of the usual kinematic component due to observer motion relative to the CMB rest frame (Langlois 1966, Lineweaver 1966). In principle, all terms in (19) may contain dipole
contributions. The first term results from a spatial gradient of the radiation energy density along the past null geodesics and requires the use of perturbed geodesics to obtain a nonzero result. The second term is produced by large-scale gravitational potential variations and is calculated using the FRW approximation for the null geodesics. Its dipole contribution can be written

\[
\left( \frac{\delta T}{T} \right)_k = \frac{1}{15} \int_{t_1}^{t_0} \frac{A_1}{a^2} \left( \dot{h}_{kl} - \frac{1}{3} \dot{h}_{kk} \right)
\]

with \( A_1 = \int_{t_1}^{t_0} dt/a \). The last term proportional to the acceleration of the fluid is absent in the case of noninteracting dust.

7. CONCLUDING REMARK

The MAP and PLANCK SURVEYOR satellite missions are expected to measure - together with other observations - many cosmological parameters such as the amplitudes and spectral indices of scalar and tensor fluctuations and the densities of various mass components of the universe. We emphasize that to these fundamental cosmological quantities one has to add further geometrical and kinematical parameters - the spatial curvature of the FRW models is only the simplest one.

REFERENCES

Bunn, E.F., Ferreira, P., Silk, J. 1996, Phys. Rev. Lett. 77, 2883
Collins, C., Hawking, S. 1973, M.N.R.A.S. 162, 307
Dunsby, P.K.S. 1997, Class. Quant. Grav. 14, 3391.
Dunsby, P.K.S., Bruni, M., Ellis, G.F.R. 1992, ApJ 395, 54
Ellis, G.F.R., Bruni, M. 1989, Phys. Rev. D 40, 1804
Ellis, G.F.R., Hwang, J., Bruni, M. 1989, Phys. Rev. D 40, 1819
Garcia-Bellido, J., Liddle, A.R., Lyth, D.H., Wand, D. 1995, Phys.Rev. D 52, 6750
Grishchuk, L.P., Zeldovich, Ya.B. 1978, Astron. Zh. 55, 209 [Sov. Astron. 1978 22, 125].
Grishchuk, L.P. 1992, Phys. Rev. D 45, 4717
Hawking, S. 1969, M.N.R.A.S. 142, 129
Kashlinsky, A., Tkachev, I.I., Frieman, J. 1994, Phys. Rev. Lett. 73, 1582
Kristian, J., Sachs, R. K. 1966, ApJ 143, 379
Kogut, A., Hinshaw, G., Banday, A.J. 1997, Phys.Rev. D 55, 1901
Langlois, D. 1996 Phys. Rev. D 53, 2908, Phys. Rev. D 54, 2447, Helv. Phys. Acta 69, 229
Lineweaver, C.H., Tenario, L., Smoot, G.F., Keegstra, P., Banday, A.J., Lubin, P. 1996, astro-ph/9601151
Lyth, D.H. 1995, Ann. N.Y. Acad. Science 759, 701
Misner, C.W., Thorne, K.S., Wheeler, J.A. 1973, Gravitation (W.H. Freeman and Company, San Francisco)
Panek, M. 1986, Phys.Rev. D 34, 416
Russ, H., Soffel, M., Xu, C.M., Dunsby, P.K.S. 1993, Phys. Rev., D 48, 4552
Turner, M.S. 1991, Phys. Rev., D 44, 3737