Local Efficiency Corrections to Higher Order Cumulants

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In this brief note we derive and present the formulas necessary to correct measurements of cumulants for detection efficiency. In particular we consider the case where the efficiency may depend on the phase-space, such as transverse momentum, rapidity etc.

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I. INTRODUCTION

The study of the phase structure of the strong interaction has been a central topic in strong interaction physics for many years. Lattice quantum chromodynamics (QCD) calculations have meanwhile established that the transition at vanishing net-baryon density is an analytic cross-over transition [1]. The situation at finite net-baryon density, however, is not yet settled. Indeed many effective models for the strong interaction (see Ref. 2 for an overview) predict a first order phase-coexistence line which ends in a critical point, the location of which being rather model dependent.

In order to explore the QCD phase diagram in experiment, a beam energy scan has been carried out at the Relativistic Heavy Ion Collider (RHIC). By varying the beam energy the temperature and the net-baryon density of the system created in a heavy-ion reaction can be changed, with collisions at lower energies leading to system at higher net-baryon density. Among many observables the cumulants of the proton number distribution has received particular attention, since it can be considered to be a measure for net-baryon number fluctuations [3], which in turn are sensitive to the presence of any structure in the QCD phase diagram at finite net-baryon density [4, 5]. The analysis and interpretation of these cumulants, however, need to be carried out with some care. Effects such as baryon number conservation [6], the non-observation of neutrons [7], and efficiency corrections [8, 9] need to be taken into account before any conclusions on possible phase changes can be drawn from the data.

The purpose of this short note is to revisit the corrections due to finite detection efficiency, ϵ, on particle number cumulants, as discussed in [8], and extend the formalism to include a possible dependence of the efficiency on the phase space, such as transverse momentum, rapidity or azimuthal angle.

Before we start let us recall why the detection efficiency ϵ leads to corrections of any fluctuation observable, such as particle number cumulants of order n ≥ 2. Suppose in each event we have exactly N particles, i.e., the variance of the number of produced particles vanishes. Given a detection efficiency ϵ, the mean number of observed particles, ni, is then ⟨ni⟩ = ϵN. However, this does not imply that in each event i the number of observed particles is ni = ϵN. Instead, ni fluctuates around the mean ⟨ni⟩ so that the distribution of observed particles has a finite variance. More formally, assume that the number of produced particles N is distributed according to P(N) then the number of observed particles follows a distribution p(ni) = \( \sum_{N} w(n|N)P(N) \) where w(n|N) denotes the probability to observe n particles given N produced particles. Clearly, in general, the cumulants of P(N) and p(n) are different. Typically w(n|N) is modeled by a binomial distribution, and a recent analysis of net-proton cumulants by the STAR collaboration employed this approach [3].

Efficiency corrections to variances of the net-charge and net-baryon distributions have been studied in [8, 9, 10] and have been extended to higher order cumulants in [8, 9]. In all these studies a constant efficiency ϵ has been assumed. In reality, however, the efficiency may very well depend on the kinematics of the particles, such as their transverse momentum, rapidity and azimuthal angle. This will likely result in further corrections especially for higher order cumulants. It is the purpose of this short note to derive the necessary formulae that relate the moments and cumulants of the distribution of produced particles to those of the observed particles for the situation of phase-space...
dependent efficiencies. As in Ref. 8, we will assume that the (phase-space dependent) detection probabilities may be modeled by binomial distributions.

II. LOCAL EFFICIENCY CORRECTIONS

Let us start by introducing some notation. In general we will use upper case characters to refer to the produced particles and lower case for the observed particles. Since we are interested in cumulants and moments of the net-baryon number, net-proton number, or net-charge, we will have two types of particles such as baryon/anti-baryon, proton/anti-proton, positive/negative charge. Thus we will denote the number of produced particles and “anti”-particles by \( N \) and \( \bar{N} \), respectively, and the observed ones by \( n \) and \( \bar{n} \). Next, we assume that the phase-space will be partitioned into bins of potentially varying size. The location of these bins in rapidity, transverse momentum and azimuthal angle, \([y, p_{\perp}, \phi]\), will be denoted by \( x \) and \( \bar{x} \) for particles and anti-particles, respectively. Thus \( N(x) \) denotes the number of produced particles in the phase-space bin located at \( x \), while \( n(x) \) is the number of observed particles at \( x \), etc. The event averaged number of produced and observed particles in the phase-space bin located at \( x \) are then given by \( \langle N(x) \rangle \) and \( \langle n(x) \rangle \), respectively. Similarly \( \langle N(x) \bar{N}(\bar{x}) \rangle \) is the event average of the product of the number of particles at \( x \) and anti-particles at \( \bar{x} \). In order to obtain the event average over all the particles in the considered phase-space we need to sum over all phase-space bins

\[
\begin{align*}
\langle N \rangle &= \sum_x \langle N(x) \rangle \\
\langle N \bar{N} \rangle &= \sum_x \sum_{\bar{x}} \langle N(x) \bar{N}(\bar{x}) \rangle \\
\langle N^n \bar{N}^k \rangle &= \sum_{x_1,\ldots,x_n} \sum_{\bar{x}_1,\ldots,\bar{x}_k} \langle N(x_1) \ldots N(x_n) \bar{N}(\bar{x}_1) \ldots \bar{N}(\bar{x}_k) \rangle
\end{align*}
\]

and similarly for the observed particles.

Next we introduce the probability \( w(n(x)|N(x); \epsilon(x)) \) to observe \( n(x) \) particles in the phase-space bin at \( x \) given \( N(x) \) produced particles and a phase-space dependent detection efficiency of \( \epsilon(x) \). The efficiencies for anti-particles are correspondingly \( \bar{\epsilon}(\bar{x}) \). As already discussed we will model \( w \) as a binomial distribution, where the binomial probability is given by the efficiency \( \epsilon(x) \):

\[
w(n(x)|N(x); \epsilon(x)) = \frac{N(x)!}{n(x)! (N(x) - n(x))!} \epsilon(x)^{n(x)} (1 - \epsilon(x))^{N(x) - n(x)}. \tag{4}
\]

Given \( w(n(x)|N(x); \epsilon(x)) \) we can relate the probability to observe a given number of particles \( n(x_i) \) at the various phase space points \( x_i \) to the probability for producing particles at these points,

\[
p(n(x_1),\ldots,n(x_n); \bar{n}(\bar{x}_1),\ldots,\bar{n}(\bar{x}_m)) = \sum_{N(x_1)=n(x_1)}^{\infty} \ldots \sum_{N(x_m)=n(x_m)}^{\infty} w(n(x_1)|N(x_1); \epsilon(x_1)) \ldots w(\bar{n}(\bar{x}_m)|\bar{N}(\bar{x}_m); \bar{\epsilon}(\bar{x}_m)) P(N(x_1),\ldots,N(x_n); \bar{N}(\bar{x}_1),\ldots,\bar{N}(\bar{x}_m)). \tag{5}
\]

Although the above expression looks rather involved, the relations between the various moments of the observed and produced particle distributions are straightforward to write down. The reason is that the binomial distributions for the various bins in phase-space and between particles and anti-particles are independent from each other. For the lowest moments we get

\[
\begin{align*}
\langle n(x) \rangle &= \epsilon(x) \langle N(x) \rangle \\
\langle n(x) \bar{n}(\bar{x}) \rangle &= \epsilon(x) \bar{\epsilon}(\bar{x}) \langle N(x) \bar{N}(\bar{x}) \rangle \\
\langle n(x_1) n(x_2) \rangle &= \epsilon(x_1) \epsilon(x_2) \langle N(x_1) N(x_2) \rangle \quad x_1 \neq x_2 \\
\langle n(x)(n(x) - 1) \rangle &= \epsilon(x)^2 \langle N(x)(N(x) - 1) \rangle.
\end{align*}
\]

\(^1\) We use the term anti-particles quite generally. For example, in case of net-charge, “particles” refer to positively charged particles and “anti-particles” refer to negatively charged particles.
The last two equations, Eqs. (8), (9), can be conveniently written as
\[ \langle n(x_1) n(x_2) - \delta_{x_1,x_2} \rangle = \epsilon(x_1) \epsilon(x_2) \langle N(x_1)(N(x_2) - \delta_{x_1,x_2}) \rangle \] (10)
where \( \delta_{x_1,x_2} = 1 \) if \( x_1 = x_2 \) and zero otherwise. In order to proceed and to arrive at a general relation between the cumulants of the observed and produced particle distributions we follow the same strategy as in our previous work \[8\]. There we expressed the cumulants in terms of factorial moments and used a general relation between the factorial moments of the distribution of observed and produced particles. The factorial moments are given by
\[ F_{i,k} = \frac{N!}{(N-i)!(N-k)!} \] (11)
\[ f_{i,k} = \frac{n!}{(n-i)!(n-k)!} \]
for the distribution of produced and observed particles, respectively. For constant, phase-space independent, efficiency correction \[8\]
\[ f_{i,k} = e^{\epsilon_k} F_{i,k}. \] (13)
In order to allow for phase space dependent efficiency corrections we introduce the “local” factorial moments
\[ A_{i,k} (x_1, \ldots, x_i; \bar{x}_1, \ldots, \bar{x}_k) = \langle N(x_1)[N(x_2) - \delta_{x_1,x_2}] \ldots [N(x_i) - \delta_{x_1,x_i} - \ldots - \delta_{x_{i-1},x_i}] \rangle \]
\[ \bar{N}(\bar{x}_1)[\bar{N}(\bar{x}_2) - \delta_{\bar{x}_1,\bar{x}_2}] \ldots [\bar{N}(\bar{x}_k) - \delta_{\bar{x}_1,\bar{x}_k} - \ldots - \delta_{\bar{x}_{k-1},\bar{x}_k}] \] (14)
\[ a_{i,k} (x_1, \ldots, x_i; \bar{x}_1, \ldots, \bar{x}_k) = \langle n(x_1)[n(x_2) - \delta_{x_1,x_2}] \ldots [n(x_i) - \delta_{x_1,x_i} - \ldots - \delta_{x_{i-1},x_i}] \rangle \]
\[ \bar{n}(\bar{x}_1)[\bar{n}(\bar{x}_2) - \delta_{\bar{x}_1,\bar{x}_2}] \ldots [\bar{n}(\bar{x}_k) - \delta_{\bar{x}_1,\bar{x}_k} - \ldots - \delta_{\bar{x}_{k-1},\bar{x}_k}] \] (15)
Using Eq. (8), it is straightforward to show that the “local” factorial moments, \( A_{i,k} \) and \( a_{i,k} \), are related to the factorial moments \( F_{i,k} \) and \( f_{i,k} \) by summation over the phase-space bins
\[ F_{i,k} = \sum_{x_1, \ldots, x_i; \bar{x}_1, \ldots, \bar{x}_k} A_{i,k} (x_1, \ldots, x_i; \bar{x}_1, \ldots, \bar{x}_k) \] (16)
\[ f_{i,k} = \sum_{x_1, \ldots, x_i; \bar{x}_1, \ldots, \bar{x}_k} a_{i,k} (x_1, \ldots, x_i; \bar{x}_1, \ldots, \bar{x}_k) \] (17)
Analogous to Eq. (13) the local factorial moments of the observed particle distribution are related to that of the produced particles by
\[ a_{i,k} = \epsilon(x_1) \ldots \epsilon(x_i) \epsilon(\bar{x}_1) \ldots \epsilon(\bar{x}_k) A_{i,k}. \] (18)
This relation follows from Eq. (13) and the fact, that the binomial efficiency corrections for different phase-space bins are independent from each other. Clearly our generalized relation, Eq. (18), gives the correct results for the second order moments, Eqs. (10,11).

By virtue of Eqs. (16) and (18), the factorial moments of the produced particle distribution can be extracted from the measured local particle distribution via
\[ F_{i,k} = \sum_{x_1, \ldots, x_i; \bar{x}_1, \ldots, \bar{x}_k} \frac{a_{i,k} (x_1, \ldots, x_i; \bar{x}_1, \ldots, \bar{x}_k)}{\epsilon(x_1) \ldots \epsilon(x_i) \epsilon(\bar{x}_1) \ldots \epsilon(\bar{x}_k)}. \] (19)
For example for \( F_{2,2} \) we obtain
\[ F_{2,2} = \sum_{x_1, x_2; \bar{x}_1, \bar{x}_2} \frac{\langle n(x_1)[n(x_2) - \delta_{x_1,x_2}] \bar{n}(\bar{x}_1)[\bar{n}(\bar{x}_2) - \delta_{\bar{x}_1,\bar{x}_2}] \rangle}{\epsilon(x_1) \epsilon(x_2) \epsilon(\bar{x}_1) \epsilon(\bar{x}_2)} \]
\[ = \sum_{x_1, x_2; \bar{x}_1, \bar{x}_2} \frac{\langle n(x_1)n(x_2)\bar{n}(\bar{x}_1)[\bar{n}(\bar{x}_2)] \rangle}{\epsilon(x_1) \epsilon(x_2) \epsilon(\bar{x}_1) \epsilon(\bar{x}_2)} - \sum_{x_1, x_2; \bar{x}_1} \frac{\langle n(x_1)n(x_2)\bar{n}(\bar{x}_1) \rangle}{\epsilon(x_1)^2 \epsilon(\bar{x}_1)^2} \]
\[ - \sum_{x_1, \bar{x}_1; \bar{x}_2} \frac{\langle n(x_1)\bar{n}(\bar{x}_1) \bar{n}(\bar{x}_2) \rangle}{\epsilon(x_1)^2 \epsilon(\bar{x}_1)} + \sum_{x_1, \bar{x}_1; \bar{x}_2} \frac{\langle n(x_1)\bar{n}(\bar{x}_1) \rangle}{\epsilon(x_1)^2 \epsilon(\bar{x}_1)^2}. \] (20)
The relation, Eq. (18), between the factorial moments of the distribution of produced particles, \( F_{i,k} \), and the local factorial moments of the observed particles, \( a_{i,k} \), is the main result of this paper. To extract cumulants from the factorial moments \( F_{i,k} \) is straightforward, and it has been discussed in \[8\], where the relevant formulas for cumulants up to the sixth order are provided.
III. CONCLUDING REMARKS

Let us conclude with a few remarks.

1. We note that the above result, Eq. (19), does not require the phase-space bins to be of equal size. Thus Eq. (19) applies to any choice of binning most suitable for a given experiment.

2. In the limit of constant efficiency over the entire phase-space under consideration, our result, Eq. (19), reduces to Eq. (13), the consequences of which were subject of our previous paper [8].

3. As already pointed out at the beginning of the paper, the above results assume a binomial distribution as a model for particle detection efficiencies. Thus the above expressions need to be suitably modified if this is not the case in a given experiment.

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