Experiments in $\mathcal{PT}$–symmetric quantum mechanics

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Abstract

Extended quantum mechanics using non-Hermitian (pseudo-Hermitian) Hamiltonians $H = H^\dagger$ is briefly reviewed. A few related mathematical experiments concerning supersymmetric regularizations, solvable simulations and large-$N$ expansion techniques are summarized. We suggest that they could initiate a deeper study of nonlocalized structures (quasi-particles) and/or of their unstable and many-particle generalizations. Using the Klein-Gordon example for illustration we show how the $\mathcal{PT}$ symmetry of its Feshbach-Villars Hamiltonian $H^{(FV)}$ might clarify experimental aspects of relativistic quantum mechanics.

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1 Introduction

The formulation of $\mathcal{PT}$–symmetric quantum mechanics (PTSQM, [1]) is not completed yet. The great portion of its intensive recent development (sampled by contributions [2]-[20] in this volume as well as by many further references cited therein) is attracted by the related open questions in mathematics. New methods are being developed using perturbation series and their resummations [4, 21], sophisticated changes of variables [5, 9, 11], semiclassical analysis [14, 22], pseudo-orthogonal random matrix ensembles [23] as well as efficient approximations [6], updated numerical techniques [8, 15] and abstract representation theory [16, 20].
During all the similar speculations one must keep in mind that physics is, undoubtedly, an experimental science. All its predictions must be verified or falsified by measurements. Although the related arguments are scattered over the literature, they support the natural expectation that a “being-the-real-physics” test will be passed by at least some of the PTSQM-type theories. Still, the applicability and applications of $\mathcal{PT}$-symmetric Hamiltonians in phenomenology may seem, temporarily, less impressive, even though several models already proved useful, reflecting, e.g., the supersymmetry of a system [17, 18] or its finite-dimensional Hilbert-space description [19]. Their not quite usual properties inspired intensive old as well as new developments in field theory [3, 12], in nuclear physics [7, 24], in many-body theory [2, 6, 9] and, last but not least, in condensed matter physics [25] and, recently, in cosmology [13] and magneto-hydrodynamics [26].

The need of the necessary full balance between the steady progress in mathematics and its fructification in experimental physics inspired our present short review. In fact, there exist several reasons for changing an overall reluctance of our colleagues in experimental physics to weaken their reliance on the comfort provided by the Hermitian phenomenological models. We mention just a few.

(1) In the general non-Hermitian context with the conservation of the mere pseudonorm [27], new types of observables should be introduced and studied, having an indeterminate sign. One might, for example, change the current habit of characterizing a point particle by its mass (which is positive semidefinite) and try to measure, e.g., the (conserved) charge in the fields or systems of charged particles and antiparticles the mass of which could be neglected.

(2) One should note that in the PTSQM context, the laboratory of solvable models is being enriched by many new completely as well as partially solvable items; in particular, the superintegrable or Calogerian models of many-body spectra become significantly modified within the complexified, $\mathcal{PT}$-symmetric framework [2]. In all these exactly solvable models, a transition to their non-Hermitian versions enables one to specify the charge itself anew, as an operator of the so-called quasi-parity, which was introduced in ref. [28] and given new emphasis in [29].

(3) New horizons may be revealed when one tries to revisit relativistic quantum theory. Recently, A. Mostafazadeh has noticed that the free Klein-Gordon operator of coordinate acquires certain interesting and quite subtle “minimally nonlocal” properties [30]. This might add new interpretations to the famous EPR paradox, one of the most influential “Gedankenexperimente” which made the nonlocality in quantum mechanics explicitly exposed to a broad public even in a non-relativistic setting.

(4) A similar paradox connected much more directly to the relativistic kinematics re-emerges in Bethe-Salpeter equation which offers a “minimal” description of a (relativistic) two-body problem in which only one time coordinate should remain observable. In such an “operator of time” context, a return to the nonrelativistic limit
leads to many non-particle concepts like, e.g., the quantum clock \[31\]. It is obvious that during the consistent physical interpretation of all the similar systems one may significantly deviate from the current versions and applications of the correspondence principle.

In parallel to the new possibilities to describe physical reality one must also solve numerous open questions in the formalism itself. Giving a purely personal selection I would like to list, e.g., the puzzles emerging within the framework of the \(PT\)–symmetric versions of the popular large-\(N\) expansion techniques \[32, 33\], of the more-than-one-body exactly solvable models \[33, 34\] and of the use of the simplifying discretizations \[34, 35\].

The common purpose of all these (and similar) studies should be a verification of our understanding of the subtleties of the PTSQM formalism of ref. \[1\].

2 Less usual properties of pseudo-Hermitian Hamiltonians

The difference between textbook Quantum Mechanics and the apparently nonstandard formalism of PTSQM may be illustrated in a schematic two-dimensional Hilbert space for which we may compare the real and time-dependent Hermitian toy Hamiltonian

\[
H_{(+)}(t) = \begin{pmatrix} a & b(t) \\ b(t) & -a \end{pmatrix} \equiv H^\dagger_{(+)}(t)
\]

(1)

with its non-Hermitian, \(PT\)–symmetric analogue

\[
H_{(-)}(t) = \begin{pmatrix} a & b(t) \\ -b(t) & -a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H^\dagger_{(-)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv H^\dagger_{(-)}(t).
\]

(2)

The pertaining spectra are both given as roots of their secular determinants,

\[
E_{(\pm)}(t) = (-1)^n \sqrt{a^2 \pm b^2(t)}, \quad n = 1, 2.
\]

(3)

This indicates that the energies may be real not only in the common Hermitian case, with interesting consequences in random matrix theories \[23\] etc.

In the \(PT\)–symmetric example \(H_{(-)}(t)\) we have to distinguish between three regimes, with \(|a| > |b|\) (all energies are real), with \(|a| < |b|\) (all energies occur in complex conjugate pairs) and with \(|a| = |b|\) ("intermediate" domain). According to a number of authors \[36]-[41], this is a generic situation where one may simplify the discussion by changing the scalar product in Hilbert space. In particular, in the regime where all the energies remain real (i.e., at \(|a| > |b|\) in our illustrative \(H_{(-)}(t)\)), one can reinterpret the non-Hermitian Hamiltonian as an operator which is Hermitian in the new metric.
Although the connection between the old and new metric may be complicated in general, the puzzle is solved at least partially in principle. In the new models with real spectra one need not modify the foundations of quantum theory at all. A new territory is merely found there for an innovative applications of the classical – quantum correspondence.

In our toy model (2) one can easily visualize, in addition, the confluence of the two eigenvalues and, possibly, their subsequent complexification at $t > t_0$. This may mean a decay of our system. Within such an unorthodox quantum model based on the weakened mathematical assumptions, the matrix form of the non-Hermitian Hamiltonian would be permitted to contain the so-called Jordan blocks. The occurrence of such a situation need not be exceptional at all, finding its elementary illustration not only in the spiked harmonic oscillator [28] but also in some much more sophisticated models in field theory [42].

In the next step of time evolution of any generic model, the complex-conjugate pairs of the energies may emerge as generated by a smooth change of the coupling strengths. Beyond the point of the decay of the system, there may still exist an indirect access to the spectra and wave functions in a non-causal domain. A few preliminary steps in this promising direction have already been made in field theory [43] as well as within supersymmetric quantum mechanics [44].

### 3 The Klein-Gordon equation as a generic illustrative example

The use of non-Hermitian Hamiltonians $H$ seems incompatible with Stone’s theorem which relates the Hermiticity of $H$ to the unitarity of the time evolution. A counterexample against such a current belief is provided by Klein-Gordon equation for a spinless particle of mass $m$ and charge $e$. In an external electromagnetic four-field denoted by $A_\mu$ with index $\mu = 0, 1, 2, 3$ such a particle may be assigned the Klein-Gordon operator [45]

$$H^{(KG)} = \sum_\mu \left( p_\mu - \frac{e}{c} A_\mu \right)^2 + m^2 c^2 \psi, \quad p_\mu = -i\hbar \frac{\partial}{\partial x_\mu}$$

in a way compatible with the requirements of relativistic covariance.

The core of the idea is that one does not need the standard form $H^{(KG)} \psi = 0$ of the Klein-Gordon equation itself but rather a definition of the (instantaneous, i.e., in general, time-dependent) Feshbach-Villars’ [37] generator $H^{(FV)}$ of the time evolution which happens to be different from $H^{(KG)}$ of course [see, e.g., eq. (XII.13) in [40]]. The reason is that the differential operator $H^{(KG)}$ is of the second order in time.
The explicit definition of $H^{(FV)}$ may be copied from any textbook [see, e.g., eq. (7.7) in [40]]. We only have to keep in mind that this operator is given, by definition, just in a fixed inertial frame. In this sense one has to reconsider the meaning of the concept of observable quantities (like energy or coordinate) and of the “preparable” wave functions and of their feasible “filtration” under the relativistic kinematics.

An enormous advantage stems from the fact that all the non-Hermitian operators $H^{(FV)}$ prove manifestly $\mathcal{PT}$-symmetric after one defines the operator of “parity” in terms of the third Pauli matrix, $\mathcal{P} \equiv \sigma_3$. We recommend the recently refreshed and updated presentation of the latter set of problems by A. Mostafazadeh [30] who considers the free motion in more detail, working with the simplified partitioned matrix operator

$$H^{(FV)} = -\frac{1}{2}(\sigma_3 + i\sigma_2)\nabla^2 + \sigma_3,$$

where the speed of light $c$, Planck constant $\hbar$ and mass $m$ have all been put equal to one [cf. also eq. (2e) in [40]].

Many questions become re-opened in the light of the latter new development. *Pars pro toto*, let us consider the relativistic version of repeated measurements of a system in a state which has been fixed (“prepared”, projected on a ket $|\psi\rangle$) in a distant past and which is measured again in a distant future (“filtered”, projected on the same ket $|\psi\rangle$). By the postulates of quantum mechanics, the new measurement does not change the wave function even if the “past” and “future” frames move with respect to each other. In such an arrangement one arrives at a new version of the old EPR paradox since one can hardly imagine a consistent experimental setup in which the measurement would obey the standard causality requirements.

4 Summary and outlook

Let us re-emphasize that the time development in PTSQM may move us to a point $t = t_0$ at which our systems start living in an “intermediate” regime. As long as the new metric in Hilbert space becomes manifestly singular at this point, the corresponding $\mathcal{PT}$-symmetric Hamiltonians cease to be tractable as equivalent to any Hermitian “physical” partner. In such a limiting case of the theory, its genuine quantum meaning must be modified (or rejected).

Such a speculative idea moves us already beyond the scope of this short review, with a useful intuitive guidance provided by the mere toy example (2), the singularity of the new metric of which is easily verified by immediate calculation. Such an $a = b(t_0)$ model could still retain some information about the collapse of the system at the critical time $t = t_0$.

In the same schematic example the third regime with $|a| < |b|$ exhibits even more mind-boggling properties. Still the time-development remains pseudounitary [27] and the smoothness of transition offers a certain guidance in the not yet well-explored
domain where even causality may be put under question-mark. As we mentioned, the explicit interest in this possibility characterizes not only the older models in field theory [42] but also some of their innovated versions [43].

In summary, simplified calculations open the path towards new phenomenological models. In particular, one may work with models using mere bosons (in place of more complicated fermions) in nuclear physics [38] or speak about a generalized form of the $\mathcal{CPT}$–symmetry within relativistic quantum field theory [29] etc. In the future, perhaps, we shall be able to construct the models where the pseudo-Hermiticity will prove necessary for a deeper or more consistent treatment of the dynamical symmetries and/or laws of evolution in quantized systems.

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References

[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243; C. M. Bender and S. Boettcher and P. N. Meisinger, J. Math. Phys. 40 (1999) 2201.

[2] B. Basu-Mallick: Czech. J. Phys. 54 (2004) ...

[3] Carl M. Bender: Czech. J. Phys. 54 (2004) ...

[4] Emanuela Caliceti: Czech. J. Phys. 54 (2004) ...

[5] Clare Dunning: Czech. J. Phys. 54 (2004) ...

[6] Shao-Ming Fei: Czech. J. Phys. 54 (2004) ...

[7] Hendrik B. Geyer: Czech. J. Phys. 54 (2004) ...

[8] Carlos Handy: Czech. J. Phys. 54 (2004) ...

[9] Vít Jakubský: Czech. J. Phys. 54 (2004) ...

[10] Ralph Kretschmer: Czech. J. Phys. 54 (2004) ...

[11] Geza Levai: Czech. J. Phys. 54 (2004) ...

[12] Kim Milton: Czech. J. Phys. 54 (2004) ...

[13] Ali Mostafazadeh: Czech. J. Phys. 54 (2004) ...
[14] Asiri Nayakkara: Czech. J. Phys. 54 (2004) ...
[15] Van M. Savage: Czech. J. Phys. 54 (2004) ...
[16] Giuseppe Scolarici: Czech. J. Phys. 54 (2004) ...
[17] Anjana Sinha: Czech. J. Phys. 54 (2004) ...
[18] Izak Snyman: Czech. J. Phys. 54 (2004) ...
[19] Qinghai Wang: Czech. J. Phys. 54 (2004) ...
[20] Stefan Weigert: Czech. J. Phys. 54 (2004) ...
[21] E. Caliceti, S. Graffi and M. Maioli: Commun.Math.Phys. 75 (1980) 51.
[22] G. Alvarez: J. Phys. A: Math. Gen 27 (1995) 4589.
[23] Z. Ahmed and S. R. Jain, Phys. Rev. E 67 (2003) R 045106 and J. Phys. A: Math. Gen 34 (2003) 3349.
[24] F. Kleefeld, in Nuclear Dynamics: From Quarks to Nuclei, ed. M. T. Peña et al., Few Body Systems Suppl. 15, Springer, Wien, 2003, p. 201.
[25] N. Hatano and D. R. Nelson: Phys. Rev. Lett. 77 (1996) 570 and Phys. Rev. B 56 (1997) 8651.
[26] U. Günther and F. Stefani: J. Math. Phys. 44 (2003) 3097.
[27] M. Znojil: preprint math-ph/0104012, Rendiconti Circ. Mat. di Palermo, to appear.
[28] M. Znojil: Phys. Lett. A 259 (1999) 220.
[29] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. 89 (2002) 270401.
[30] A. Mostafazadeh: preprint quant-ph/0307059.
[31] J. Hilgevoord: Am. J. Phys. 70 (2002) 301.
[32] M. Znojil, D. Yanovich and V. P. Gerdt: J. Phys. A: Math. Gen. 36 (2003) 6531.
[33] M. Znojil: preprint quant-ph/0307239, J. Phys. A: Math. Gen., to appear.
[34] M. Znojil: J. Phys. A: Math. Gen. 36 (2003) 7825.
[35] M. Znojil: J. Phys. A: Math. Gen. 36 (2003) 7639.
[36] P. A. M. Dirac, Proc. Roy. Soc. London A 180, 1 (1942); W. Pauli, Rev. Mod. Phys., 15, 175 (1943).

[37] H. Feshbach and F. Villars, Rev. Mod. Phys., 30, 24 (1958).

[38] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. 213 (1992) 74.

[39] J. Bognár, Indefinite Inner Product Spaces (Springer, Berlin, 1974).

[40] F. Constantinescu and E. Magyari, Problems in Quantum Mechanics. Pergamon, Oxford, 1971.

[41] N. Nakanishi: Prog. Theor. Phys. Suppl. 51 (1972) 1; A. Mostafazadeh, J. Math. Phys. 43 (2002) 3944.

[42] T. D. Lee: Phys. Rev. 95 (1954) 1329; N. Nakanishi: Prog. Theor. Phys. 19 (1958) 607.

[43] F. Kleefeld: in “Hadron Physics, Effective Theories of Low Energy QCD”, AIP Conf.Proc. 660 (2003) 325.

[44] M. Znojil: Nucl. Phys. B 662 (2003) 554 and in GROUP 24: Physical and Mathematical Aspects of Symmetries (proc. 24th Int. Colloquium on Group Theoretical Methods in Physics, Paris, 15-20 July 2002), ed. J-P Gazeau et al, Inst. of Phys. Conf. Series 173; hep-th/0209062.

[45] W. Greiner, Relativistic Quantum Mechanics, Springer, Berlin, 2000.