Inverse source problems in elastodynamics

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Abstract

We are concerned with time-dependent inverse source problems in elastodynamics. The source term is supposed to be the product of a spatial function and a temporal function with compact support. We present frequency-domain and time-domain approaches to show uniqueness in determining the spatial function from wave fields on a large sphere over a finite interval. Stability estimate of the temporal function from the data of one receiver and uniqueness result using partial boundary data are proved. Our arguments rely heavily on the use of the Fourier transform, which motivated inversion schemes that can be easily implemented. A Landweber iterative algorithm for recovering the spatial function and a non- iterative inversion scheme based on the uniqueness proof for recovering the temporal function are proposed. Numerical examples are demonstrated in both two and three dimensions.

Keywords: Inverse source problems, Lamé system, uniqueness, Landweber iteration, Fourier transform.

1 Introduction

Consider the radiation of elastic (seismic) waves from a time-varying source term $F(x,t)$, $x \in \mathbb{R}^3$, embedded in an infinite and homogeneous elastic medium. The real-valued radiated field is governed by the inhomogeneous Lamé system:

$$\rho \partial_t u(x,t) = \nabla \cdot \sigma(x,t) + F(x,t), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0$$

(1.1)

together with the initial conditions

$$U(x,0) = \partial_t U(x,0) = 0, \quad x \in \mathbb{R}^3.$$  

(1.2)

Here, $\rho > 0$ denotes the density, $U = (u_1, u_2, u_3)^T$ is the displacement vector, $\sigma = \sigma(U)$ is the stress tensor and $F$ is the source term which causes the elastic vibration in $\mathbb{R}^3$. By Hooke’s law, the stress tensor
relates to the stiffness tensor \( \tilde{C} = (C_{ijkl})_{i,j,k,l=1}^3 \) via the identity \( \sigma(U) := \tilde{C} : \nabla U \), where the action of \( \tilde{C} \) on a matrix \( A = (a_{ij})_{i,j=1}^3 \) is defined as
\[
\tilde{C} : A = (\tilde{C} : A)_{ij} = \sum_{k,l=1}^3 C_{ijkl} a_{kl}.
\]

In an isotropic and homogeneous elastic medium, the stiffness tensor is characterized by
\[
C_{ijkl}(x) = \lambda \delta_{i,j} \delta_{k,l} + \mu (\delta_{i,k} \delta_{j,l} + \delta_{i,l} \delta_{j,k}).
\]  

(1.3)

where the Lamé constants satisfy \( \mu > 0, 3\lambda + 2\mu > 0 \). Hence, the Lamé system (1.1) can be rewritten as
\[
\begin{align*}
\rho \partial_t u(x,t) &= \mathcal{L}_{\lambda,\mu} u(x,t) + f(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+ , \\
\mathcal{L}_{\lambda,\mu} u &= -\mu \nabla \times \nabla \times u + (\lambda + 2\mu) \nabla \cdot u = \mu \Delta u + (\lambda + \mu) \nabla \nabla \cdot u.
\end{align*}
\]

Note that the above equation has a more complex form than the scalar wave equation, because it accounts for both longitudinal and transverse motions. Throughout this paper it is supposed that \( \rho \), \( \lambda \), \( \mu \) are given as a prior data and that the dependence of the source term on time and space variables are separated, that is,
\[
F(x,t) = f(x) g(t). \tag{1.4}
\]

In other words, the source term is a product of the spatial function \( f \) and the temporal function \( g \). Moreover, we suppose that \( f \) is compactly supported in the space region \( B_{R_0} := \{ x : |x| < R_0 \} \) and the source radiates only over a finite time period \([0,T_0] \) for some \( T_0 > 0 \). This implies that \( g(t) = 0 \) for \( t \geq T_0 \) and \( t \leq 0 \). The source term (1.4) can be regarded as an approximation of the elastic pulse and are commonly used in modeling vibration phenomena in seismology.

Inverse hyperbolic problems have attracted considerable attention over the last years. Most of the existing works treated scalar acoustic wave equations. We refer to Bukhgaim & Klibanov [13], Klibanov [25], Yamamoto [32,33], Khaǐdarov [24], Isakov [20,21], Imanuvilov & Yamamoto [18,19], Choulli and Yamamoto [17], Kian, Sambou & Soccorsi [23] and the recent work by Jiang, Liu & Yamamoto [22] for uniqueness and stability of inverse source problems using Carleman estimates, and refer also to Fujishiro & Kian [17] for results of recovery of a time-dependent source. In addition, it is worth to mention the work of Rakesh & Symes [29], dealing with coefficient determination problems based on the construction of appropriate geometric optic solutions. There are also rich references on inverse problems arising in the context of linear elasticity. Many investigations are devoted to mathematical and numerical techniques for the identification of elastic coefficients and buried objects of a geometrical nature (such as cracks, cavities and inclusions) in the time-harmonic regime; see e.g., the review article [12] by Bonnet & Constantinescu , the monograph [5] by Ammari et.al. and references therein. Due to our limited knowledge, we have found only a few mathematical works on inverse source problems for the time-dependent Lamé system. In [4], a time-reversal imaging algorithm based on a weighted Helmholtz decomposition was proposed for reconstructing \( f \) in a homogeneous isotropic medium, where the temporal function takes the special form \( g(t) = d\delta(t)/dt \).

This paper concerns uniqueness and numerical reconstructions of \( f \) or \( g \) from radiated elastodynamic fields over a finite time interval. Such kind of inverse problems have many significant applications in biomedical engineering (see, e.g., [5]) and geophysics (see, e.g., [2]). The Uniqueness issue is important in inverse scattering theory, while it provides insight into whether the measurement data are sufficient for recovering the unknowns and ensures uniqueness of global minimizers in iterative schemes. Being different from previously mentioned existing works, our uniqueness proofs rely heavily on the use of the Fourier transform, which motivated novel inversion schemes that can be easily implemented. Our arguments carry over the scalar wave equations without any additional difficulties. We believe that the Fourier-transform-based approach explored in this paper would also lead to stability estimates of our inverse problems, which deserves to be further investigated in future. We shall address the following inverse issues:
(i) Uniqueness in recovering $f$ from emitted waves on a closed surface surrounding the source. We present frequency-domain and time-domain approaches for recovering the spatial function. The frequency-domain approach is of independent interest, while it reduces the time-dependent inverse problem to an inverse scattering problem in the Fourier domain with multi-frequency data. Our arguments are motivated by recent studies on inverse source problems for the time-harmonic Helmholtz equation with multi-frequency data (see e.g., [3,9,11,16]). The time-domain approach is inspired by the Lipschitz stability estimate of source terms for the scalar acoustic wave equation with additional a priori assumptions; see, e.g., [18,19,32,33]. A Landweber iterative algorithm is proposed for recovering $f$ in 2D and numerical tests are presented to show validity and effectiveness of the proposed inversion scheme; see Section 5.1.

(ii) Stability estimate of $g$ from measured data of one receiver. Under the assumption that the spatial function does not vanish at the position of the receiver, we estimate a vector-valued temporal functions in Section 3. The stability estimate relies on an explicit expression of the solution in terms of $f$ and $g$. Such an idea seems well-known in the case of scalar acoustic wave equations, but to the best of our knowledge not available for time-dependent Lamé systems.

(iii) Unique determination of $g$ from partial boundary measurement data. If the spatial function $f$ is known to be not a non-radiating source (see Definition 4.3), we prove that the temporal function $g$ can be uniquely determined by the time-domain data on any subboundary of a large sphere; see Theorem 4.4. The uniqueness proof is based on the Fourier transform and yields a non-iterative inversion scheme in subsection 5.2. Numerical examples are demonstrated to verify our theory.

The remaining part is organized as following. In Section 2, preliminary studies of the time-dependent Lamé system are carried out. Unique determination of spatial and temporal functions will be presented in Sections 3 and 4, respectively. In particular, as a bi-product of the Fourier-domain approach presented in subsection 3.1, we show uniqueness in recovering a source term of the time-dependent Schrödinger equation. Finally, Numerical tests are reported in Section 5 and proofs of several lemmas are postponed to the appendix in Section 6.

2 Preliminaries

For all $r > 0$, we denote by $B_r$ the open ball of $\mathbb{R}^3$ defined by $B_r := \{x \in \mathbb{R}^3 : |x| < r\}$. By Helmholtz decomposition, the function $f \in (L^2(\mathbb{R}^3))^3$ supported in $B_{R_0}$ admits a unique decomposition of the form (see Lemma 6.1 in the Appendix)

$$f(x) = \nabla f_p(x) + \nabla \times f_s(x), \quad \nabla \cdot f_s = 0,$$

(2.1)

where $f_p \in H^1(B_{R_0})$, $f_s \in H_{\text{curl}}(B_{R_0}) := \{u : u \in (L^2(B_{R_0}))^3, \text{curl} u \in (L^2(B_{R_0}))^3\}$ also have compact support in $B_{R_0}$. We choose also $g \in C(\mathbb{R})$ supported in $[0, T_0]$. By the completeness theorem (see [1] Theorem 3.3 or [2] Chapter 4.1.1), there exist vector-valued functions $U_p(x, t)$ and $U_s(x, t)$ such that $U(x, t)$ can be expressed as

$$U = U_p + U_s, \quad U_p = \nabla u_p, \quad U_s = \nabla \times u_s, \quad \nabla \cdot u_s = 0.$$  

(2.2)

Moreover, the scalar function $u_p$ and the vector function $u_s$ satisfy the inhomogeneous wave equations

$$\frac{1}{c^2} \partial_{tt} u_\alpha - \Delta u_\alpha = \frac{1}{\gamma_\alpha} f_\alpha(x)g(t) \quad \text{in} \quad \mathbb{R}^3 \times (0, +\infty), \quad \alpha = p, s,$$

(2.3)

together with the initial conditions

$$u_\alpha|_{t=0} = \partial_t u_\alpha|_{t=0} = 0 \quad \text{in} \quad \mathbb{R}^3.$$
Note that
\[ c_p := \sqrt{(\lambda + 2\mu)/\rho}, \quad c_s := \sqrt{\mu/\rho}, \quad \gamma_p := \lambda + 2\mu, \quad \gamma_s := \mu, \] (2.4)
and that \( \lambda + 2\mu > 0 \) since \( \mu > 0 \), \( 3\lambda + 2\mu > 0 \). This implies that \( U_p \) and \( U_s \) propagate at different wave speeds, which will be referred to as compressional waves (or simply P-waves) and shear waves (or simply S-waves), respectively.

It is well-known that the electrodynamic Green’s tensor \( G(x,t) = (G_{ij}(x,t))_{i,j=1}^3 \in \mathbb{C}^{3 \times 3} \), which satisfies
\[
\rho \partial_t G(x,t) e_j - \nabla \cdot \sigma(x,t) = -\delta(x)\delta(t) e_j, \quad j = 1, 2, 3,
\]
is given by (see e.g., [14])
\[
G_{i,j}(x,t) = \frac{1}{4\pi \rho |x|^3} \left\{ t^2 \left( \frac{x_j x_k}{|x|^2} \delta(t - |x|/c_p) + (\delta_{jk} - \frac{x_j x_k}{|x|^2})\delta(t - |x|/c_s) \right) \right\} + \frac{1}{4\pi \rho |x|^3} \left\{ t \left( \frac{3x_i x_k}{|x|^2} - \delta_{jk} \right) \left( \Theta(t - |x|/c_p) - \Theta(t - |x|/c_s) \right) \right\}.
\] (2.5)

Here, \( \delta_{ij} \) is the Kronecker symbol, \( \delta \) is the Dirac distribution, \( \Theta \) is the Heaviside function and \( e_j \) \( (j = 1, 2, 3) \) are the unit vectors in \( \mathbb{R}^3 \). Physically, the Green’s tensor \( G(x,t) \) is the response of the Lamé system to a point body force at the origin that emits an impulse at time \( t = 0 \). Using the above Green’s tensor, the solution \( U \) to the inhomogeneous Lamé system \( \|\| \) can be represented as
\[
U(x,t) = \int_0^\infty \int_{\mathbb{R}^3} G(x-y,t-s) f(y) g(s) \, dx ds, \quad x \in \mathbb{R}^3, \ t \in \mathbb{R}.
\] (2.6)

Note that, since \( \text{supp}(g) \subset [0, +\infty) \), for every \( t \in (-\infty, 0) \) and \( x \in \mathbb{R}^3 \), we have \( U(x,t) = 0 \). Throughout the paper we define
\[
T_p := T_0 + (R + R_0)/c_p, \quad T_s := T_0 + (R + R_0)/c_s,
\] (2.7)
for some \( R > R_0 \). Obviously, it holds that \( T_s > T_p \), since \( c_p > c_s \) by (2.4). The following lemma states that the wave fields over \( B_R \) must vanish after a finite time that depends on \( R \) and the support of \( f \) and \( g \).

**Lemma 2.1.** We have \( U(x,t) = 0 \) for all \( x \in B_R \) and \( t > T_s \).

**Proof.** For \( x = (x_1, x_2, x_3)^T, \ y = (y_1, y_2, y_3)^T \in \mathbb{R}^3 \), write \( x \otimes y = xy^T \in \mathbb{R}^{3 \times 3} \) and \( \hat{x} = x/|x| \) for simplicity. Introduce
\[
V(x,t) = \Theta(t - |x|/c_p) - \Theta(t - |x|/c_s).
\]
Combining (2.6) and (2.5), we have
\[
U(x,t) = \int_0^\infty \int_{\mathbb{R}^3} \frac{(t-s)^2 (\hat{x} - \hat{y}) \otimes (\hat{x} - \hat{y})}{4\pi \rho |x-y|^3} \delta(t-s - \frac{|x-y|}{c_p}) f(y) g(s) \, dx ds
\]
\[
+ \int_0^\infty \int_{\mathbb{R}^3} \frac{(t-s)^2 [I - (\hat{x} - \hat{y}) \otimes (\hat{x} - \hat{y})]}{4\pi \rho |x-y|^3} \delta(t-s - \frac{|x-y|}{c_s}) f(y) g(s) \, dx ds
\]
\[
+ \int_0^\infty \int_{\mathbb{R}^3} \frac{(t-s)[3(\hat{x} - \hat{y}) \otimes (\hat{x} - \hat{y}) - I]}{4\pi \rho |x-y|^3} V(x-y,t-s) f(y) g(s) \, dx ds
\]
\[
= \int_{|y-x| \leq c_p(t+T_0)} \frac{(\hat{x} - \hat{y}) \otimes (\hat{x} - \hat{y})}{4\pi (\lambda + 2\mu) |x-y|} g(t - \frac{|x-y|}{c_p}) f(y) dy ds
\]
\[
+ \int_{|y-x| < c_p(t+T_0)} \frac{I - (\hat{x} - \hat{y}) \otimes (\hat{x} - \hat{y})}{4\pi \mu |x-y|} g(t - \frac{|x-y|}{c_s}) f(y) dy ds
\]
\[
+ \int_{B_R} \int_{|y-x| \leq c_p(t+T_0)} \frac{(t-s)[3(\hat{x} - \hat{y}) \otimes (\hat{x} - \hat{y}) - I]}{4\pi \rho |x-y|^3} V(x-y,t-s) f(y) g(s) \, dx ds,
\]
Theorem 3.1. transform. Below we present a frequency-domain approach and a time-domain approach to the proof of 3 Unique determination of spatial functions

Since \( g(x,\omega) \) is given. We suppose that \( g(x,\omega) \) is real valued. Consequently, it holds in the frequency domain that

\[
\hat{G}(\alpha, \omega)e_j + (\lambda + \mu)\nabla (\cdot \hat{G}(\alpha, \omega)e_j) + \omega^2 \rho \hat{G}(\alpha, \omega)e_j = -\delta(\cdot)e_j, \quad j = 1, 2, 3
\]

and

\[
\hat{G}(x - y, \omega) = \frac{1}{\mu} \Phi_{k_p}(x, y)I + \frac{1}{\rho \omega^2} \nabla \cdot \nabla (\Phi_{k_s}(x, y) - \Phi_{k_p}(x, y)), \quad x \neq y. \tag{2.8}
\]

Here \( \Phi_k(x, y) = e^{ik|x-y|/(4\pi|x-y|)} \) \( k = k_p, k_s \) is the fundamental solution to the Helmholtz equation \( (\Delta + k^2)u = 0 \) in \( \mathbb{R}^3 \). By Lemma 2.1, we may take the Fourier transform of \( U(x, t) \) with respect to \( t \). Consequently, it holds in the frequency domain that

\[
\mu \Delta \hat{U}(x, \omega) + (\lambda + \mu)\nabla (\cdot \hat{U}(x, \omega)) + \omega^2 \rho \hat{U}(x, \omega) = -f(x)\hat{g}(\omega), \quad \omega \in \mathbb{R}. \tag{2.9}
\]

Corresponding to the representation of \( U(x, t) \) in the time domain, we have in the Fourier domain that

\[
\hat{U}(x, \omega) = \int_{\mathbb{R}^3} \hat{F}(G(x - y, \cdot) \ast g(\cdot))f(y)dy = \hat{g}(\omega) \int_{\mathbb{R}^3} \hat{G}(x - y, \omega)f(y)dy, \quad x \in \mathbb{R}^3, \quad \omega \in \mathbb{R}^+. \tag{2.10}
\]

Here \( \ast \) denotes the convolution product with respect to the time variable. Note that \( \hat{U}(x, -\omega) = \hat{U}(x, \omega) \), since \( U(x, t) \) is real valued.

3 Unique determination of spatial functions

In this section we are interested in the inverse source problem of recovering \( f \) from the radiated wave field \( \{U(x, t) : |x| = R, t > T\} \) for some \( R > R_0 \) and \( T > T_0 \) under the a prior assumption that \( g \) is given. We suppose that \( f \in (L^2(\mathbb{R}^3))^3, \supp(f) \subset B_{R_0}, \; g \in C_0([0, T_0]) \). Since \( f \) and \( g \) have compact support, the initial boundary value problem (1.1), (1.2) and (1.4) admits a unique solution \( U \in C(\mathbb{R}, H^1(B_R))^3 \cap C(\mathbb{R}, L^2(B_R))^3 \) for any \( R > 0 \). Let \( f_\alpha \) and \( U_\alpha \) \( (\alpha = p, s) \) be specified as in (2.1) and (2.2), respectively. Our uniqueness results are stated as following.

Theorem 3.1. (i) The data set \( \{U(x, t) : |x| = R, t \in (0, T_s)\} \) uniquely determines the spatial function \( f \). (ii) The data set of pure P- and S-waves, \( \{U_\alpha(x, t) : |x| = R, t \in (0, T_\alpha)\} \), uniquely determines \( f_\alpha \) \( (\alpha = p, s) \).

We remark that, since the measurement surface is spherical, the compressional and shear components \( U_\alpha(x, t) \) \( (\alpha = p, s) \) can be decoupled from the whole wave fields \( U(x, t) \) on \( |x| = R \). In fact, in the Fourier domain, \( \hat{U}_\alpha(x, \omega) \) can be decoupled from \( \hat{U}(x, \omega) \) on \( |x| = R \) for every fixed \( \omega \in \mathbb{R}^+ \); see e.g., [8] or Section 5.1 in the 2D case. Hence the decoupling in the time domain can be achieved via Fourier transform. Below we present a frequency-domain approach and a time-domain approach to the proof of Theorem 3.1.
3.1 Frequency-domain approach

**Proof of Theorem 3.1** (i) Assuming that \( U(x,t) = 0 \) for all \( |x| = R \) and \( t \in (0,T) \), we need to prove that \( f \equiv 0 \) in \( B_{R_0} \). Recalling Lemma 2.1, we have \( U_\alpha(x,t) = 0 \) for all \( |x| = R, t \in \mathbb{R}^+ \). Combining this with the fact that \( U(x,t) = 0, (x,t) \in \mathbb{R}^2 \times (-\infty,0] \), we deduce that \( U_\alpha(x,t) = 0 \) for all \( |x| = R, t \in \mathbb{R} \). Then, applying the Fourier transform in time to \( U_\alpha(x,\cdot) \) gives

\[
\hat{U}_\alpha(x,\omega) = \int_{\mathbb{R}} U_\alpha(x,t) e^{i\omega t} dt = 0, \quad \text{for all} \quad |x| = R, \omega \in \mathbb{R}^+.
\]

Introduce the functions

\[
v_p(x,\omega) := de^{-ik_x d \cdot x}, \quad v_s(x,\omega) := d^\perp e^{-ik_s d \cdot x}, \quad d \in S^2 := \{ x \in \mathbb{R}^3 : |x| = 1 \},
\]

where \( k_\alpha = k_\alpha(\omega) \) (\( \alpha = p, s \)) are the compressional and shear wave numbers, respectively, and \( d^\perp \in S^2 \) stands for a unit vector that orthogonal to \( d \). Physically, \( v_p \) and \( v_s \) denote the compressional and shear plane waves propagating along the direction \( d \), respectively. They fulfill the time-harmonic Navier equation as follows

\[
\mu \Delta v_\alpha + (\lambda + \mu) \nabla (\nabla \cdot v_\alpha) + \omega^2 \rho v_\alpha = 0, \quad \alpha = p, s.
\]

Multiplying \( v_\alpha \) to (2.9) and applying Betti’s formula to \( \hat{U} \) and \( v_\alpha \) in \( B_R \), we obtain

\[
-\hat{g}(\omega) \int_{B_R} f(x) \cdot v_\alpha(x,\omega) \ dx = \int_{|x|=R} \left[ T_\nu \hat{U}(x,\omega) \cdot v_\alpha(x,\omega) - T_\nu v_\alpha(x,\omega) \cdot \hat{U}(x,\omega) \right] ds,
\]

where \( \nu = (\nu_1, \nu_2, \nu_3)^\top \in S^2 \) is the normal direction on \( |x| = R \) pointing into \( |x| > R \) and \( T_\nu = T_\nu^{(\lambda,\mu)} \) is the traction operator defined by

\[
T_\nu \hat{U} := 2\mu \partial_\nu \hat{U} + \lambda \nu \text{div} \hat{U} + \mu \nu \times \text{curl} \hat{U}.
\]

(3.1)

It follows from (2.10) that \( \hat{U}(x,\omega) \) satisfies the Kupradze radiation when \( |x| \to \infty \). By well-posedness of the Dirichlet boundary value problem for the time-harmonic Navier system in \( |x| > R \), we obtain \( T_\nu \hat{U}(x,\omega) \equiv 0 \) for all \( |x| = R, \omega \in \mathbb{R}^+ \). This also follows from the well-defined Dirichlet-to-Neumann operator applied to \( \hat{U}|_{|x|=R} \) for fixed \( \omega \in \mathbb{R}^+ \); see e.g., [8]. Hence,

\[
\hat{g}(\omega) \int_{B_R} f(x) \cdot v_\alpha(x,\omega) \ dx = 0, \quad \text{for all} \quad \omega \in \mathbb{R}^+.
\]

(3.2)

On the other hand, applying integration by parts we get

\[
\int_{\mathbb{R}^3} \nabla \times f_s(x) \cdot de^{-ik_p x \cdot d} dx = -\int_{\mathbb{R}^3} f_s(x) \cdot \nabla \times (de^{-ik_p x \cdot d}) dx = 0,
\]

in which the boundary integral over \( \partial B_R \) vanish due to the compact support of \( f \) in \( B_{R_0} \subset B_R \). It then follows

\[
\int_{B_R} f(x) \cdot v_p(x,\omega) \ dx
\]

\[
= \int_{\mathbb{R}^3} \nabla f_p(x) \cdot d e^{-ik_x x \cdot d} dx + \int_{\mathbb{R}^3} \nabla \times f_s(x) \cdot d e^{-ik_s x \cdot d} dx
\]

\[
= ik_s(2\pi)^\frac{1}{2} \hat{f}_p(k_p d).
\]

Note that here \( \hat{f}_p \) refers to the Fourier transform of \( f_p \) with respect to spatial variables, given by

\[
\hat{f}_p(\xi) := \int_{\mathbb{R}^3} f(x) e^{ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^3.
\]
In the same way, we have
\[
\int_{\mathbb{R}^3} \nabla f_p(x) \cdot d^4 e^{-ik_s x} dx = - \int_{\mathbb{R}^3} f_p(x) \nabla \cdot (d^4 e^{-ik_s x} d) dx = i k_s (d \cdot d^1) \int_{\mathbb{R}^3} f_p(x) e^{-ik_s x} dx = 0
\]
and we find
\[
\int_{B_R} f(x) \cdot v_s(x, \omega) dx = \int_{\mathbb{R}^3} \nabla \times f_s(x) \cdot d^1 e^{-ik_s x} dx = i k_s (2\pi)^2 \hat{f}_s(k_s d) \cdot (d \times d^1).
\]
Therefore, it follows from (3.2) that
\[
\hat{f}_p(k_s d) = \hat{f}_s(k_s d) \cdot (d \times d^1) = 0
\]
for all \(d \in \mathbb{S}^2\) and for all \(\omega \in \{ \omega \in \mathbb{R}^+ : \hat{g}(\omega) \neq 0 \}\). Since \(g \neq 0\), one can always find an interval \((a, b) \subset \mathbb{R}^+\) such that \(\hat{g}(\omega) \neq 0\) for \(\omega \in (a, b)\). By the analyticity of \(\hat{f}_\alpha (\alpha = p, s)\) and the arbitrariness of \(d \in \mathbb{S}^2\), we finally obtain \(\hat{f}_s = 0\). Applying inverse Fourier transform we get \(f_s = 0\), implying that \(f = 0\).

(ii) By (2.3), the P- and S-waves fulfill the wave equations
\[
\frac{1}{c_p^2} \partial_{tt} U_p(x, t) - \Delta U_p(x, t) = \frac{1}{\gamma_p} \nabla f_p(x) g(t),
\]
\[
\frac{1}{c_s^2} \partial_{tt} U_s(x, t) - \Delta U_s(x, t) = \frac{1}{\gamma_s} \nabla \times f_s(x) g(t),
\]
in \(\mathbb{R}^3 \times (0, +\infty)\) together with the zero initial conditions at \(t = 0\), where \(c_\alpha\) and \(\gamma_\alpha\) (\(\alpha = p, s\)) are given in (2.4). Applying Duhalme’s principle and Kirchhoff’s formula for wave equations, we can represent these P- and S-waves as
\[
U_p(x, t) = \frac{1}{4\pi \gamma_p} \int_{|y-x| \leq c_p(t+T_0)} \nabla f_p(y) g(t-|y-x|/c_p) dy,
\]
\[
U_s(x, t) = \frac{1}{4\pi \gamma_s} \int_{|y-x| \leq c_s(t+T_0)} \nabla \times f_s(y) g(t-|y-x|/c_s) dy,
\]
for \(x \in \mathbb{R}^3, t > 0\). As done for the Navier equation in the proof of Lemma 2.1, one can show that \(U_\alpha(x, t) = 0\) for all \(x \in B_R, t > T_\alpha\) (\(\alpha = p, s\)). Hence, the relation \(U_\alpha(x, t) = 0\) for \(x \in B_R, t \in (0, T_\alpha)\) would imply the vanishing of \(U_s(x, t)\) over \(B_R\) for all \(t \in \mathbb{R}^+\). Now, repeating the argument in the proof of the first assertion we deduce that
\[
\nabla f_p = 0, \quad \nabla \times f_s = 0, \quad \text{div} f_s = 0 \quad \text{in} \quad B_R.
\]
This implies that \(f_p \equiv 0\), since \(f_p \in H^1(B_R)\) and \(f_p = 0 \in B_R \setminus B_{R_0}\). To prove the vanishing of \(f_s \in (L^2(B_R))^3\), we apply the Helmholtz decomposition to \(f_s\), i.e., \(f_s = \nabla h_p + \text{curl} h_s\), where \(h_p \in H^1(B_R)\) and \(h_s \in H_{\text{curl}}(B_R)\) are compactly supported in \(B_R\). Then it follows that \(h_p = h_s \equiv 0 \in B_R\) and thus \(f_s = 0 \in B_R\); see the proof of Lemma 6.1 in the Appendix. \(\square\)

Remark 3.2. The above proof of Theorem 3.1 by using Fourier transform is valid in odd dimensions only. The vanishing of the wavefields on \(|x| = R\) for \(t > T_s\) can be physically interpreted by Huygens’ Principle, which however does not hold when the number of spatial dimensions is even. The frequency-domain approach applies to two dimensions if we know the time-domain data for all \(0 < t < \infty\).

As a bi-product of the frequency-domain approach to the proof of Theorem 3.1 we show uniqueness in recovering the source term of the time-dependent Schrödinger equation:
\[
\begin{cases}
    i\hbar \partial_t W(x, t) = -\frac{\hbar^2}{2m} \Delta + q(x) W(x, t) + f_0(x) g_0(t) & \text{in} \quad \mathbb{R}^3 \times (0, +\infty) \\
    W(x, 0) = 0 & \text{on} \quad \mathbb{R}^3,
\end{cases}
\]
(3.3)
where $\hbar$ is the reduced Planck constant, $\mu$ is the particle’s reduced mass and $q$ is the particle’s potential energy which is assumed to be time-independent. Similar to the Lamé system, we shall assume that $f_0 \in L^2(\mathbb{R}^3)$, support ($f_0$) $\subset B_{R_0}$, $g_0 \in H^1_0(0, T_0)$. The potential is supposed to be a real-valued nonnegative function with compact support on $\overline{B}_R$ for some $R > R_0$. The number $\omega \in \mathbb{C}$ is called a Dirichlet eigenvalue of the operator $L_q := \Delta - q$ with $\tilde{q}(x) := 2\mu/\hbar^2 q(x)$ if there exists a non-trivial function $V \in (H^1_0(B_R))^2$ such that $(L_q + \omega)V = 0$ in $B_R$.

It can be easily proved that the set of Dirichlet eigenvalues is discrete, which we denote by $\{\omega_n\}_{n=1}^{\infty}$, and that each eigenvalue is positive. According to [27, Theorem 10.1, Chapter 3] and [27, Remark 10.2, Chapter 3], the initial problem (3.3) admits a unique solution $W \in \mathcal{C}([0, +\infty); H^1(\mathbb{R}^3)) \cap \mathcal{C}^1([0, +\infty); H^{-1}(\mathbb{R}^3))$. Therefore, we can introduce the data $\{W(x, t) : |x| = R, t \in \mathbb{R}^+\}$ for $W$ the unique solution of (3.3). The following result extends the uniqueness proof of the inverse source problem for the Helmholtz equation [16] to the case of time-dependent Lamé system with an inhomogeneous time-independent potential function.

**Corollary 3.3.** Assume that $q \in C_0(B_R)$ is known and that $\tilde{g}_0(\omega_n') \neq 0$, $\omega_n' = \omega_n \hbar/(2\mu)$, for all $n = 1, 2, \cdots$. Then the data set $\{W(x, t) : |x| = R, t \in \mathbb{R}^+\}$ uniquely determines $f_0$.

**Proof.** We assume that $W(x, t) = 0$, $|x| = R$, $t \in [0, +\infty)$. Since $g \in H^1_0(0, T)$, the extension of $W$ by 0 on $\mathbb{R}^3 \times (-\infty, 0]$, is the unique solution of

$$
\begin{cases}
    i\hbar \partial_t W(x, t) = [-\nabla^2 + q(x)]W(x, t) + f_0(x)g_0(t) & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\
    W(x, 0) = 0 & \text{on } \mathbb{R}^3.
\end{cases}
$$

Thus, without loss of generality we can assume that the solution of (3.3) is the solution of the problem on $\mathbb{R}^3 \times \mathbb{R}$. Then, condition (3.4) implies that $W(x, t) = 0$, $|x| = R$, $t \in \mathbb{R}$.

According to the estimate (10.14) in the proof of [27, Theorem 10.1, Chapter 3] we have

$$
\|W(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 \leq C \int_0^{+\infty} (|g_0(s)|^2 + |d g_0(s)/ds|^2) \|f_0\|_{L^2(\mathbb{R}^3)}^2 ds
\leq C \|g_0\|_{L^2(0, T_0)}^2 \|f_0\|_{L^2(\mathbb{R}^3)}^2,
$$

for $t \in [0, +\infty)$, where $C > 0$ is a constant independent of $t$. In particular, this estimate and the fact that $W(x, t) = 0$ for $(x, t) \in \mathbb{R}^3 \times (-\infty, 0]$, proves that $W \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^3)) \subset S'(\mathbb{R}; H^1(\mathbb{R}^3))$. Therefore, we can apply the Fourier transform $\mathcal{F}_{t \rightarrow \omega}$ to $W$ and deduce from (3.3) that $\hat{W} = \mathcal{F}_{t \rightarrow \omega} W \in S'(\mathbb{R}; H^1(\mathbb{R}^3))$ satisfies

$$
L_q \hat{W}(x, \omega) + \eta_1 \omega \hat{W}(x, \omega) = \eta_2 f_0(x) \hat{g}_0(\omega), \quad x \in \mathbb{R}^3, \ \omega \in \mathbb{R}^+,
$$

with $\eta_1 = 2\mu/\hbar$, $\eta_2 = 2\mu/\hbar^2$. Note that the identity (3.6) is considered in the sense of distribution with respect to $(x, \omega) \in \mathbb{R}^3 \times \mathbb{R}^+$. In view of (3.6), we have $\Delta \hat{W} \in S'(\mathbb{R}; L^2(\mathbb{R}^3))$ which implies that $\hat{W} \in S'(\mathbb{R}; H^2(\mathbb{R}^3))$. The equation (3.6) can be rewritten as

$$
\Delta \hat{W}(x, \omega) + k^2 \hat{W}(x, \omega) = \eta_2 f_0(x) \hat{g}(\omega) + \tilde{q}(x) \hat{W}(x, \omega), \quad k := \sqrt{\eta_1 \omega}.
$$

Recalling Green’s formula, for any $R_1 > R$ we may represent $\hat{W}$ as the integral equation

$$
\hat{W}(x, \omega) = \int_{\partial B_{R_1+1}} \left[ \partial_y \hat{W}(y, \omega) \Phi_k(x - y) - \partial_y \Phi_k(x - y) \hat{W}(y, \omega) \right] ds(y) - \int_{\mathbb{R}^3} \Phi_k(x - y) \hat{q}(y) \hat{W}(y, \omega) dy - \eta_2 \hat{g}_0(\omega) \int_{\mathbb{R}^3} \Phi_k(x - y) f_0(y) dy.
$$
for \( x \in B_{R_1} \), where \( \Phi_k \) is the fundamental solution to the Helmholtz equation \((\Delta + k^2)u = 0\). On the other hand, we have

\[
\left| \int_{\partial B_{R_1+1}} \left[ \partial_n \hat{W}(y, \omega) \Phi_k(x-y) - \partial_n \Phi_k(x-y) \hat{W}(y, \omega) \right] \, ds \right|
\leq C(\| \partial_n \hat{W}(y, \omega) \|_{L^2(\partial B_{R_1+1})} + \| \hat{W}(y, \omega) \|_{L^2(\partial B_{R_1+1})}) \left( \int_{\partial B_{R_1+1}} |x-y|^{-2} \, ds(y) \right)^{\frac{1}{2}}
\leq C(\| \partial_n \hat{W}(y, \omega) \|_{L^2(\partial B_{R_1+1})} + \| \hat{W}(y, \omega) \|_{L^2(\partial B_{R_1+1})})
\]

and, since \( \hat{W}(\cdot, \omega) \in H^2(\mathbb{R}^3) \), by density we deduce that

\[
\lim_{R_1 \to +\infty} (\| \partial_n \hat{W}(y, \omega) \|_{L^2(\partial B_{R_1+1})} + \| \hat{W}(y, \omega) \|_{L^2(\partial B_{R_1+1})}) = 0.
\]

Therefore, sending \( R_1 \to +\infty \), we get

\[
\hat{W}(x, \omega) = - \int_{\mathbb{R}^3} \Phi_k(x-y) \hat{g}(y) \hat{W}(y, \omega) \, dy - \eta_2 \hat{g}_0(\omega) \int_{\mathbb{R}^3} \Phi_k(x-y) f_0(y) \, dy, \quad x \in \mathbb{R}^3.
\]

This implies that \( \hat{W}(\cdot, \omega) \) is the unique solution of (3.6) satisfying the Sommerfeld radiation condition when \( |x| \to \infty \). Let \( V_n \in (H^1_0(B_R))^2 \) be an eigenfunction that corresponds to the Dirichlet eigenvalue \( \omega_n \). Using the fact that \( \hat{W}(\cdot, \omega) \in \{ S \in H^1(B_R) : \Delta S \in L^2(B_R) \} \) and multiplying \( V_n \) to both sides of (3.6) with \( \omega = \omega_n \) and applying integral by parts, we obtain

\[
\eta_1 \hat{g}_0(w_n') \int_{B_R} f_0(x)V_n(x) \, dx = \left\langle \partial_n \hat{W}_n(\cdot; \omega_n'), V_n \right\rangle_{H^{-\frac{1}{2}}(\partial B_R), H^{\frac{1}{2}}(\partial B_R)} - \int_{\partial B_R} \partial_n V_n(x) \hat{W}_n(x; \omega_n') \, ds(x)
\]

\[
= \left\langle T_n[\hat{W}(\cdot; \omega_n')], V_n \right\rangle_{H^{-\frac{1}{2}}(\partial B_R), H^{\frac{1}{2}}(\partial B_R)} - \int_{\partial B_R} \partial_n V_n(x) \hat{W}_n(x; \omega_n') \, ds(x),
\]

where \( T_n : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R) \) is the Dirichlet-to-Neumann map for radiating solutions to the Helmholtz equation \((\Delta + (\omega_n')^2)u = 0\) which satisfies the Sommerfeld radiation condition at infinity. In view of (3.3), the fact that \( \hat{g}_0(w_n') \neq 0 \) and the fact that \( T_n \) is a linear bounded map, for every \( n = 1, 2, \cdots \), we deduce, from the previous identity that

\[
\int_{B_R} f_0(x)V_n(x) \, dx = 0, \quad n = 1, 2, \cdots.
\]

Since the set of the Dirichlet eigenfunctions is complete over \((L^2(B_R))^2\), we conclude that the temporal function \( f_0 \) can be uniquely determined by the data. This finishes the uniqueness proof. \( \square \)

### 3.2 Time-domain approach

In this subsection we present a time-domain proof of Theorem 3.1. Note that this demonstration can be extended to dimension two provided that we replace the data \( \{ U(x, t) : |x| = R, \ t \in (0, T_s) \} \) by \( \{ U(x, t) : |x| = R, \ t \in (0, +\infty) \} \), since Lemma 2.1 does not hold in dimension two.

**Proof of Theorem 3.1** (i) Again we assume that \( U(x, t) = 0 \) for all \( |x| = R, \ t \in (0, T_s) \) and, in view of Lemma 2.1, this implies that \( U(x, t) = 0 \) for all \( |x| = R, \ t \in (0, +\infty) \). Our aim is to deduce that \( f \equiv 0 \). Since the temporal function \( g \) is known, we apply Duhalme’s principle to \( U \) by setting

\[
U(x, t) = \int_0^t V(t-s, x)g(s) \, ds, \quad x \in \mathbb{R}^3, t > 0.
\]

(3.7)

The function \( V \) then fulfills the homogeneous Lamé equation with non-zero initial conditions

\[
\partial_t V(x, t) = -c_p^2 \nabla \times \nabla \times V(x, t) + c_s^2 \nabla (\nabla \cdot V(x, t)),
\]

\[
V(x, 0) = 0, \quad \partial_t V(x, 0) = f(x).
\]
Further, we can continue $V$ onto $\mathbb{R}^3 \times (-\infty, 0)$ preserving the Lamé equation and the initial conditions. Since $g(t) = 0$ for $t < 0$, the function $t \to U(x, t)$, given by (3.7), can be regarded as the convolution of $V(x, \cdot)\chi(\cdot)$ and $g(\cdot)\chi(\cdot)$, i.e.,

$$U(x, t) = [V(x, t)\chi(t)] * [g(t)\chi(t)], \quad (3.8)$$

where $\chi$ is the characteristic function of $(0, \infty)$. By Lemma 2.1, $U(x, t) = 0$ for $|x| = R$ and $t \in \mathbb{R}$. Taking the Fourier transform to (3.8), we see

$$0 = F_{t \to \omega}[V(x, t)\chi(t)]F_{t \to \omega}[g(t)\chi(t)] = F_{t \to \omega}[V(x, t)\chi(t)]\hat{g}(\omega), \quad |x| = R.$$ 

Making use of the analyticity of $F_{t \to \omega}[V(x, t)\chi(t)]$ with respect to $\omega$ and that the fact that $g$ does not vanish identically, we deduce that $V(x, t) = 0$ for $|x| = R$ and $t \in \mathbb{R}$.

We decouple $V$ into the sum of the compressional part $V_p$ and shear part $V_s$:

$$V = V_p + V_s, \quad V_p = \nabla v_p + \nabla \times v_s, \quad \nabla \cdot v_s = 0 \quad \text{in} \quad \mathbb{R}^3,$$

where $V_\alpha$ ($\alpha = p, s$) fulfills the homogeneous wave equation

$$\partial_t^2 V_\alpha(x, t) - c^2_\alpha \Delta V_\alpha \quad \text{in} \quad B_R$$

and the initial conditions

$$V_\alpha(x, 0) = 0, \quad \partial_t V_\alpha(x, 0) = \nabla f_\alpha(x), \quad \partial_t V_\alpha(x, 0) = \nabla \times f_\alpha(x).$$

Since $\text{Supp}(f) \subset B_{R_0} \subset B_R$, $V(x, t)$ has zero initial conditions in the unbounded domain $|x| > R$. Consequently, we get $V \equiv 0$ for all $|x| > R$ and $t \in \mathbb{R}$, due to the unique solvability of the hyperbolic system in $|x| > R$ with the Dirichlet boundary condition at $|x| = R$ for all $t > 0$. By uniqueness of the Helmholtz decomposition, it follows that $V_\alpha = 0$ in $|x| > R$ for all $t \in \mathbb{R}$. In view of the unique continuation for the homogeneous wave equation (see e.g., [30,31]), it can be deduced that $V_\alpha(x, t) = 0$ in $B_R \times \mathbb{R}$, implying that $V = 0$ for $x \in B_R$ and $t \in \mathbb{R}$. In particular, $\partial_t V(x, 0) = f(x) = 0$ for $x \in B_R$.

(ii) If $U_\alpha(x, t) = 0$ for $|x| = R$, $t \in (0, T_\alpha)$, then we have $V_\alpha = 0$ on $\{|x| = R\} \times \mathbb{R}$. Repeating the arguments above, it follows that $V_\alpha(x, 0) = 0$ in $|x| > R$ for $t \in \mathbb{R}$. As a consequence of the unique continuation we get $V_\alpha(x, 0) = 0$ in $B_R \times \mathbb{R}$. Setting $t = 0$ we obtain $f_\alpha = 0$ for $\alpha = p, s$. \hfill \qed

**Remark 3.4.** We think that the frequency-domain and time-domain approaches presented above could also yield stability estimate of the spatial function in terms of the time-domain data $\{U(x, t) : |x| = R, 0 < t < T_\alpha\}$. The terminal time $T_\alpha$ ($\alpha = p, s$) in Theorem 3.1 are optimal. Non-uniqueness examples can be readily reconstructed if the terminal time is less than $T_\alpha$.

### 4 Unique determination of temporal functions

Given some $T > 0$, we suppose that $g \in (L^2(0, T))^3$ is an unknown vector-valued temporal function and that the spatial function $f$ is known to be compactly supported in $B_{R_0}$ for some $R_0 > 0$. We consider the inverse problem of determining $g$ from observations of the solution of

$$\begin{cases}
\rho \partial_t U(x, t) = \nabla \cdot \sigma(x, t) + f(x)g(t), & (x, t) \in \mathbb{R}^3 \times (0, T), \\
U(x, 0) = \partial_t U(x, 0) = 0, & x \in \mathbb{R}^3,
\end{cases} \quad (4.1)$$

at one fixed point $x_0 \in \text{supp}(f)$ (i.e., interior observations) or at the subboundary $\Gamma \subset \partial B_R$ (i.e., partial boundary observations). In order to state rigorously our problem, we start by considering the regularity of this initial value problem (4.1).

**Lemma 4.1.** Let $g \in (L^2(0, T))^3$ and let $f \in H^p(\mathbb{R}^3)$, with $p > 5/2$ be supported on $B_R$ for some $R > R_0$. Then problem (4.1) admits a unique solution $U \in C([0, T]; H^{p+1}(\mathbb{R}^3))^3 \cap H^2((0, T); H^{p-1}(\mathbb{R}^3))^3$ satisfying

$$\|U\|_{C([0, T]; H^{p+1}(\mathbb{R}^3))^3} + \|U\|_{H^2((0, T); H^{p-1}(\mathbb{R}^3))^3} \leq C\|g\|_{L^2(0, T)^3} \|f\|_{H^p(\mathbb{R}^3)}, \quad (4.2)$$

with $C > 0$ depending on $\rho, \lambda, \mu, R$. 

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Proof. Applying Fourier transform to $U(\cdot, t)$ with respect to spatial variables, denoted by $\hat{U}$, we find
\[
\partial_t \hat{U}(\xi, t) + A(\xi)\hat{U}(\xi, t) = \frac{g(t)\hat{f}(\xi)}{\rho} \quad \text{in} \quad \mathbb{R}^3 \times (0, T),
\]
where the matrix $A(\xi) \in \mathbb{R}^{3 \times 3}$ is defined by
\[
A(\xi) := \frac{\mu}{\rho}[\xi]^2 I + \left(\frac{\lambda + \mu}{\rho}\right)\xi \otimes \xi, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.
\]
Evidently, $A(\xi)$ is a real-valued symmetric matrix, with the eigenvalues given by $\frac{(\lambda + 2\mu)[\xi]^2}{\rho}$, $\frac{\rho[\xi]^2}{\rho}$, $\frac{\rho[\xi]^2}{\rho}$, see Lemma 6.2 in the Appendix. Denote by $A^{1/2}(\xi)$ the square roof of $A(\xi)$ and by $A^{-1/2}(\xi)$ the inverse of $A^{1/2}(\xi)$. Then the unique solution to (4.3) takes the form
\[
\hat{U}(s, \xi, t) = \int_0^t g_T(s) A^{-1/2}(\xi) \sin \left( A^{1/2}(\xi)(t - s) \right) \frac{\hat{f}(\xi)}{\rho} ds.
\]
On the other hand, for all $t \in [0, T]$ and $s \in [0, t]$, fixing
\[
H(t - s, \cdot) := \xi \mapsto A^{-1/2}(\xi) \sin \left( A^{1/2}(\xi)(t - s) \right) \frac{\hat{f}(\xi)}{\rho},
\]
we have
\[
\|H(t - s, \cdot)\|^2_{L^2(\mathbb{R}^3)^3} \leq \mu^{-1} \rho^{-1} \|\hat{f}\|^2_{L^\infty(\mathbb{R}^3)} \int_{B_1} |\xi|^{-2} d\xi + 4\mu^{-1} \rho^{-1} \int_{\mathbb{R}^3 \setminus B_1} (1 + |\xi|^2)^{-1} \hat{f}(\xi)^2 d\xi
\]
\[
\leq C \mu^{-1} \rho^{-1} |B_R| \|\hat{f}\|^2_{L^2(\mathbb{R}^3)} + 4\mu^{-1} \rho^{-1} \|f\|^2_{L^2(\mathbb{R}^3)},
\]
with $C$ a constant. Note that here we use the fact that $\xi \mapsto |\xi|^{-2} \in L^1(B_1)$, since $2 < 3$, and the fact that supp$(f) \subset B_R$. Moreover, we apply the fact that $\lambda + \mu > 0$ to deduce that $|A^{-1/2}(\xi)| \leq \frac{1}{2} \mu^{-1/2}|\xi|^{-1}$
In the same way, we have
\[
\|\xi|^{\rho + 1} H(t - s, \cdot)\|^2_{L^2(\mathbb{R}^3)^3} \leq \mu^{-1} \rho^{-1} \int_{\mathbb{R}^3} |\xi|^{2\rho} \hat{f}(\xi)^2 d\xi \leq \mu^{-1} \rho^{-1} \|f\|^2_{H^\rho(\mathbb{R}^3)}.
\]
Combining estimates (4.5) and (4.6), one can easily deduce that $U \in C([0, T]; H^{\rho + 1}(\mathbb{R}^3))^3$. In the same way, we have
\[
\|(1 + |\xi|^2)^{\frac{\rho - 1}{2}} \partial_t H(t - s, \cdot)\|_{L^2(\mathbb{R}^3)^3} + \|(1 + |\xi|^2)^{\frac{\rho - 1}{2}} \partial_t^2 H(t - s, \cdot)\|_{L^2(\mathbb{R}^3)^3} \leq C \|f\|_{H^\rho(\mathbb{R}^3)},
\]
where $C$ depends on $\rho$, $\lambda$, $\mu$, $R$. Moreover, for almost every $\xi \in \mathbb{R}^3$, we have
\[
\hat{U}(\xi, \cdot) : t \mapsto \hat{U}(\xi, t) \in H^2(0, T),
\]
with
\[
\partial_t \hat{U}(\xi, t) = \int_0^t g(s)T \partial_t H(t - s, \xi) ds,
\]
\[
\partial_t \hat{U}(\xi, t) = \frac{g(t)\hat{f}(\xi)}{\rho} + \int_0^t g(s)T \partial_t H(t - s, \xi) ds.
\]
Combining this with (4.7), we deduce that $U \in H^2((0, T); H^{\rho - 1}(\mathbb{R}^3))^3$ and we deduce (4.2) from the previous estimates.

\[
\square
\]
Lemma 6.3, for almost every
Since
and applying Fubini’s theorem we find
Theorem 4.2
interior observation of the wave fields
and (4.8) that

Proof. According to (4.4), the solution \(U\) of (4.1) is given by

\[
U(x,t)^T = (2\pi)^{-3} \int_{\mathbb{R}^3} \left( \int_0^t g(s)^T A^{-1/2}(\xi) \sin \left( A^{1/2}(\xi)(t-s) \right) \frac{\hat{f}(\xi)}{\rho} \right) e^{i\xi \cdot x} d\xi, \quad (x,t) \in \mathbb{R}^3 \times [0,T]
\]

and applying Fubini’s theorem we find

\[
U(x,t)^T = (2\pi)^{-3} \int_0^t g(s)^T \left( \int_{\mathbb{R}^3} A^{-1/2}(\xi) \sin \left( A^{1/2}(\xi)(t-s) \right) \frac{\hat{f}(\xi)}{\rho} e^{i\xi \cdot x} d\xi \right) ds, \quad (x,t) \in \mathbb{R}^3 \times [0,T].
\]

In particular, in view of Lemma 4.1, \(U \in C([0,T]; H^{p+1}(\mathbb{R}^3))^3 \cap H^2((0,T); H^{P-1}(\mathbb{R}^3))^3\) satisfies (4.2). Further, direct calculations show that

\[
\mathcal{L}_{\lambda,\mu} U(x,t)^T = -(2\pi)^{-3} \int_0^t g(s)^T \left( \int_{\mathbb{R}^3} A^{1/2}(\xi) \sin \left( A^{1/2}(\xi)(t-s) \right) \frac{\hat{f}(\xi)}{\rho} e^{i\xi \cdot x} d\xi \right) ds.
\]

Since \(|A^{1/2}(\xi) a| \leq \frac{\sqrt{\lambda + 2\mu}}{\sqrt{\rho}} \|\xi\| \|a\|\) for all \(a \in \mathbb{C}^3\), the previous identity can be estimated by

\[
|\mathcal{L}_{\lambda,\mu} U(x,t) | \leq \frac{\sqrt{\lambda + 2\mu}}{\sqrt{\rho}} \int_0^t |g(s)| ds \int_{\mathbb{R}^3} |\hat{f}(\xi)||\xi| d\xi
\]

\[
\leq \frac{\sqrt{\lambda + 2\mu}}{\sqrt{\rho}} \int_0^t |g(s)| ds \|\hat{f}(\xi)(1 + |\xi|^2)^{P/2}\|_{L^2(\mathbb{R}^3)} \|\xi\|^{(1-2P)/2}\|_{L^2(\mathbb{R}^3)}
\]

\[
\leq M_0 \frac{\sqrt{\lambda + 2\mu}}{\sqrt{\rho}} \|f\|_{H^p(\mathbb{R}^3)} \int_0^t |g(s)| ds
\]

(4.8)

where \(M_0 = \|(1 + |\xi|^2)^{(1-P)/2}\|_{L^2(\mathbb{R}^3)} < \infty\). Since \(|f(x_0)| \geq \delta\), we derive from the governing equation of \(U\) and (4.8) that

\[
|g(t)| = \frac{1}{|f(x_0)|}|\rho \partial_t U(x_0, t) - \mathcal{L}_{\lambda,\mu} U(x_0, t)|
\]

\[
\leq M_1 |\partial_t U(x_0, t)| + M_2 \int_0^t |g(s)| ds
\]

for all \(t \in (0,T)\), where \(M_1 = \rho/\delta, M_2 = M_0 \frac{\sqrt{\lambda + 2\mu}}{\sqrt{\rho}} M/\delta\). Applying the Gronwall inequality stated in Lemma 6.3 for almost every \(t \in (0,T)\), we find

\[
|g(t)| \leq M_1 |\partial_t U(x_0, t)| + M_1 M_2 \int_0^t |\partial_t U(x_0, s) e^{M_2(t-s)} ds
\]

\[
\leq M_1 |\partial_t U(x_0, t)| + M_1 M_2 T^2 e^{M_2 T} \|\partial_t U(x_0, \cdot)\|_{L^2(0,T)^3}.
\]
Therefore, taking the norm $L^2(0,T)$ on both sides of the inequality, implies that
\[
\|g\|_{L^2(0,T)^3} \leq (M_1 + M_1 M_2 T e^{M_2 T}) \|\rho_{ij} U(x_0,\cdot)\|_{L^2(0,T)^3}.
\]
This completes the proof. \qed

To state uniqueness with partial boundary measurement data, we need the concept of non-radiating source.

**Definition 4.3.** The compactly supported function $f$ is called a non-radiating source at the frequency $\omega \in \mathbb{R}^+$ to the Lamé system if the unique radiating solution to the inhomogeneous Lamé system
\[
\mathcal{L}_{\lambda,\mu} u(x) + \omega^2 \rho u(x) = f(x) P, \quad j = 1, 2, 3,
\]
does not vanish identically in $\mathbb{R}^3 \setminus \text{supp}(f)$ for any $P \in \mathbb{C}^3$.

**Theorem 4.4 (Uniqueness with partial boundary data).** Suppose that $f \in L^2(B_R)$ is known to be a compacted supported function over $B_{R_0}$ for some $R_0 < R$ and that $f$ is not a non-radiating source for all $\omega \in \mathbb{R}^+$. Then the temporal function $g \in C_0^\infty([0,T_0])^3$ can be uniquely determined by the partial boundary measurement data $\{U(x,t) : x \in \Gamma, t \in (0,T_s)\}$ where $\Gamma \subset \partial B_R$ is an arbitrary subboundary with positive Lebesgue measure and $T_s$ is defined in (2.7).

**Proof.** Let $w_j = w_j(x,\omega)$ ($j = 1, 2, 3$) be the unique radiating solution to the inhomogeneous Lamé system
\[
\mathcal{L}_{\lambda,\mu} w_j(x) + \omega^2 \rho w_j(x) = f(x) e_j, \quad j = 1, 2, 3,
\]
does not vanish identically in $|x| \geq R$ by our assumption. Set the matrix $W := (w_1, w_2, w_3) \in \mathbb{C}^{3 \times 3}$. Then $W(\cdot,\cdot)$ solves the matrix equation
\[
\mathcal{L}_{\lambda,\mu} W(x,\omega) + \omega^2 \rho W(x,\omega) = f(x) I \quad \text{in} \quad \mathbb{R}^3 \times (0,\infty).
\]
Note that here the action of the differential operator is understood column-wisely, and $W$ can be represented as
\[
W(x,\omega) = \int_{\mathbb{R}^3} \hat{G}(x-y,\omega) f(y) dy, \quad x \in \mathbb{R}^3,
\]
where $\hat{G}$ is the Green’s tensor to the time-harmonic Lamé system. In view of (4.9), the Fourier transform $\hat{U}(x,\omega)$ of $U(x,t)$ can be written as
\[
\hat{U}(x,\omega) = W(x,\omega) \hat{g}(\omega) \quad \text{for all} \quad \omega \in \mathbb{R}^+, \ |x| = R.
\]
We claim that for each $\omega_0 \in \mathbb{R}^+$, there always exists $x_0 \in \Gamma \subset \partial B_R$ such that $\text{Det}(W(x_0,\omega_0)) \neq 0$. Suppose on the contrary that $\text{Det}(W(x,\omega_0)) = 0$ for all $x \in \Gamma$. This implies that there exist $c_j \in \mathbb{C}^3$ such that
\[
V(x) := c_1 w_1(x,\omega_0) + c_2 w_2(x,\omega_0) + c_3 w_3(x,\omega_0) = 0 \quad \text{on} \quad \Gamma.
\]
By the analyticity of $w_j$ in a neighborhood of $|x| = R$ and the analyticity of the surface $\Gamma \subset \partial B_R$, we conclude that (4.11) holds on $|x| = R$. By uniqueness of the exterior Dirichlet boundary value problem, we have $V(x) = 0$ in $|x| > R$, and by unique continuation it holds that $V(x) = 0$ for all $x$ lying outside of the support of $f$. However, it is easy to observe that $V$ satisfies the inhomogeneous equation (4.9) with $P = c_1 e_1 + c_2 e_2 + c_3 e_3$, which contradicts the fact that $f$ is not a non-radiating source. Therefore, by (4.10) we get
\[
\hat{g}(\omega_0) = [W(x_0,\omega_0)]^{-1} \hat{U}(x_0,\omega_0) \in \mathbb{C}^{3 \times 1} \quad \text{for some} \quad x_0 \in \Gamma.
\]
Note that $\omega_0$ is arbitrary and the point $x_0$ depends on $\omega_0$. Hence, if $U(x,t) = 0$ for all $x \in \Gamma$ and $t \in (0,T_s)$, then $\hat{U}(x,\omega) = 0$ for all $x \in \Gamma$ and $\omega \in \mathbb{R}^+$. This implies that $\hat{g}(\omega) = 0$ for all $\omega \in \mathbb{R}$ and thus $g \equiv 0$. \qed
5 Numerical experiments

In this section, we propose a Landweber iterative method for reconstructing the spatial function \( f \) in 2D and a non-iterative inversion scheme based on the proof of Theorem 4.4 for recovering the temporal function \( g \) in 3D. Several numerical examples will be illustrated to examine the effectiveness of the proposed methods.

5.1 Reconstruction of spatial functions

We consider the inverse source problem presented in Section 4. Our aim is to reconstruct the spatial function in two dimensions, relying on the Landweber iterative method for solving linear algebraic equations. Assume that the time-dependent data \( U(x,t), x \in \partial B_R \) \((R > R_0)\) is measured over the time interval \([0,T]\) where \( T > 0 \) is sufficiently large such that the integral

\[
\int_0^T U(x,t) \exp(i\omega t) \, dt
\]

can be used to approximate the Fourier transform \( \hat{U}(x,\omega) \) for any \( \omega \in \mathbb{R}^+ \). In the time-harmonic regime, it is supposed that the multi-frequency data \( \hat{U}(x,\omega_k), x \in \partial B_R \) for \( k = 1, \cdots, K \) are available. Hence, the time-dependent inverse source problem can be transformed to a problem in the Fourier domain with near-field data of multi frequencies. In 2D, the Helmholtz decomposition of \( \hat{U} \) takes the form \( \hat{U} = \hat{U}_p + \hat{U}_s \), where the compressional part \( \hat{U}_p \) and shear part \( \hat{U}_s \) are given by

\[
\hat{U}_p = -\frac{1}{k_p^2} \text{grad} \text{ div} \hat{U}, \quad \hat{U}_s = \frac{1}{k_s^2} \text{curl} \text{ curl} \hat{U}. \tag{5.1}
\]

Here the two-dimensional operators \text{curl} and \text{curl} are defined respectively by

\[
\text{curl} \, v = \partial_1 v_2 - \partial_2 v_1, \quad v = (v_1, v_2)^\top, \quad \text{curl} \, h := (\partial_2 h, -\partial_1 h)^\top.
\]

Writing \( \hat{u}_p := -1/k_p^2 \text{div} \hat{U} \) and \( \hat{u}_s = 1/k_s^2 \text{curl} \hat{U} \), we have \( \hat{U} = \hat{u}_p + \text{curl} \hat{u}_s \) and the scalar functions \( \hat{u}_\alpha \) \((\alpha = p, s)\) satisfy the Sommerfeld radiation condition

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial \hat{u}_\alpha}{\partial r} - ik_\alpha \hat{u}_\alpha \right) = 0, \quad r = |x|, \quad \alpha = p, s
\]

uniformly with respect to all \( \hat{x} = x/|x| \in \mathbb{S}^1 \).

For \( |x| \geq R \), the radiation solutions \( \hat{u}_\alpha \) can be expressed in terms of Hankel functions of the first kind,

\[
\hat{u}_\alpha(|x|, \theta) = \sum_{n \in \mathbb{Z}} \hat{u}_{\alpha,n} H^{(1)}_n(k_\alpha |x|) \exp(\iota n \theta), \quad x = |x|(\cos \theta, \sin \theta), \quad |x| \geq R. \tag{5.2}
\]

For every fixed \( \omega \in \mathbb{R}^+ \), the coefficients \( \hat{u}_{\alpha,n} \in \mathbb{C} \) are uniquely determined by \( \hat{U}(x,\omega)|_{x=R} \) as follows (see e.g., [3])

\[
\begin{pmatrix}
\hat{u}_{p,n} \\
\hat{u}_{s,n}
\end{pmatrix} = \frac{1}{2\pi R} [A_n(R)]^{-1} \int_0^{2\pi} \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \hat{U}(R, \theta; \omega) \, d\theta, \tag{5.3}
\]

where

\[
A_n(R) = \begin{pmatrix}
t_p H^{(1)}_n(t_p) & \iota n H^{(1)}_n(t_s) \\
n H^{(1)}_n(t_p) & -t_s H^{(1)}_n(t_s)
\end{pmatrix}, \quad t_\alpha = k_\alpha R, \quad \alpha = p, s. \tag{5.4}
\]

This means that, in the Fourier domain, the \( P \) and \( S \)-waves can be decoupled from the whole wave field \( \hat{U} \) on \( |x| = R \) for every fixed frequency \( \omega \).
Below we shall consider the inverse problems of reconstructing $f_p$, $f_s$ and $f$ from the wave fields $\hat{u}_p(x, \omega)|_{\partial B_R}$, $\hat{u}_s(x, \omega)|_{\partial B_R}$ and $u(x, \omega)|_{\partial B_R}$ at a finite number of frequencies $\omega = \omega_k$, $k = 1, \cdots, K$, respectively. Recall from (2.10) that

$$\hat{U}(x, \omega)/\hat{g}(\omega) = \int_{B_R} \hat{G}(x - y) f(y)dy, \quad |x| = R, \quad \hat{g}(\omega) \neq 0,$$

(5.5)

where $\hat{G}$ is the fundamental displacement tensor of the Navier equation of the form (2.8) with the fundamental solution of the two-dimensional Helmholtz equation given by

$$\Phi_k(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|), \quad x \neq y, \quad x, y \in \mathbb{R}^2.$$

Analogously, the compressional and shear components of $\hat{U}$ can be represented by (cf. (2.3))

$$\hat{u}_\alpha(x, \omega)/\hat{g}(\omega) = \frac{1}{\gamma_\alpha} \int_{B_R} \Phi_{k\alpha}(x, y) f_\alpha(y)dy, \quad \alpha = p, s.$$

(5.6)

Our numerical scheme relies on solvability of the ill-posed integral equations (5.5) and (5.6) for finding $f$ and $f_\alpha$. Since $f(x)$ is real-valued, it is more convenient to consider real-valued integral equations from numerical point of view. Taking the real and imaginary parts of (5.5) gives

$$\text{Re}\{\hat{U}(x, \omega)/\hat{g}(\omega)\} = \int_{B_R} \text{Re}\{\hat{G}(x - y, \omega)\} f(y)dy, \quad |x| = R,$$

(5.7)

$$\text{Im}\{\hat{U}(x, \omega)/\hat{g}(\omega)\} = \int_{B_R} \text{Im}\{\hat{G}(x - y, \omega)\} f(y)dy, \quad |x| = R.$$

(5.8)

Furthermore, for the pressure part $\hat{u}_p$ and shear part $\hat{u}_s$, we have

$$\text{Re}\{\hat{u}_\alpha(x, \omega)/\hat{g}(\omega)\} = \frac{1}{\gamma_\alpha} \int_{B_R} \text{Re}\{\Phi_{k\alpha}(x, y)\} f_\alpha(y)dy, \quad |x| = R,$$

(5.9)

$$\text{Im}\{\hat{u}_\alpha(x, \omega)/\hat{g}(\omega)\} = \frac{1}{\gamma_\alpha} \int_{B_R} \text{Im}\{\Phi_{k\alpha}(x, y)\} f_\alpha(y)dy, \quad |x| = R.$$

(5.10)

The equations (5.7)-(5.10) are Fredholm integral equations of the first kind. These equations are ill-posed, since the singular values of the matrix resulting from the discretized integral kernel are rapidly decaying. Now, we describe a Landweber iterative method to solve the ill-posed integral equations (5.7)-(5.10). Consider the linear operator equations

$$V_k(S) = v_k, \quad k = 1, \cdots, K, \quad S = f, f_p, f_s,$$

(5.11)

where $v_k = \hat{U}(x, \omega_k)$ or $v_k = \hat{u}_\alpha(x, \omega_k)$ denotes the measurement data at the frequency $\omega_k$. We denote by $S_{l,k}$ the inverse solution obtained at the l-th iteration step reconstructed from the data set at the frequency $\omega_k$. Due to the linearity of (5.11), a straightforward Landweber iteration (see, e.g., [11]) can be applied as a regularization scheme for solving (5.11). For clarity We summarize the inversion process in Table 1.

Below we present several numerical examples to demonstrate the validity and effectiveness of the proposed method. In the following we always choose

$$\hat{g}(\omega) = \int_0^T g(t) \exp(i\omega t)dt, \quad g(t) = \begin{cases} \cos(1.5\pi(t - t_0)) \exp(-\pi(t - t_0)^2), & t \leq T, \\ 0, & t > T, \end{cases}$$

where $T = 5$, $t_0 = 2$. The functions $\hat{g}$ and $g$ are plotted in Figure 1 which shows that $\hat{g}$ is nonzero in $(0, 20)$. The source function $f$ in $B_R$ with $R = 1$ is defined by

$$f = (f_1, f_2)^T = \nabla f_p + \text{curl} f_s,$$
Table 1: Landweber iterative method for reconstructing spatial functions.

**Step 1** Set an initial guess $S_{0,0}$

**Step 2** Update the source function $S$ by the iterative formula

$$S_{l,k} = S_{l-1,k} + \epsilon V_k^* (v_k - V_k S_{l-1,k}), \quad l = 1, \ldots, L,$$

where $\epsilon$ and $L$ are the step length and total number of iterations, respectively.

**Step 3** Set $S_{0,k+1} = S_{L,k}$ and repeat Step 2 until the highest frequency $\omega_K$ is reached.

Figure 1: The exact pulse function $g(t)$ and its Fourier transformation $\hat{g}(\omega)$.

Figure 2: The exact spatial source function $f = (f_1, f_2)$ and its compressional component $f_p$ and shear component $f_s$. 

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where
\[
\begin{align*}
    f_p(x) &= 0.3(1 - 3x_1)^2 \exp(-9x_1^2 - (3x_2 + 1)^2) - (0.6x_1 - 27x_1^3 - 3^5x_2^2) \exp(-9x_1^2 - 9x_2^2) \\
    f_s(x) &= 135x_1^2x_2 \exp(-9x_1^2 - 9x_2^2); \\
\end{align*}
\]
see Figure 2. We choose \( \mu = 1, \lambda = 2, \rho = 1 \) and \( R = 2 \). The scattering data is collected at 64 uniformly distributed points on the circle \( \partial B_R \). The total number of iterations is set to be \( L = 10 \).

In the static case, we simulate the data \( \hat{U}(x, \omega) \) by solving the inhomogeneous time-harmonic Navier equation using finite element method coupled with an exact transparent boundary condition. Then the compressional and shear parts, \( \hat{u}_p \) and \( \hat{u}_s \), are decoupled from \( \hat{U}(x, \omega) \) via (5.2)-(5.4). The near-field data of twenty equally spaced frequencies from 1 to 20 are calculated. Figure 3 shows the reconstructed \( S_1 \) and \( S_2 \) from \( \{\hat{U}(x, \omega_k) : \|x\| = R, k = 1, 2, \cdots, 20\} \), while Figure 4 presents the reconstructed \( f_p \) and \( f_s \) from the counterpart of compressional and shear waves, respectively.

![Reconstructed f1](image)

![Reconstructed f1](image)

![Reconstructed f2](image)

![Reconstructed f2](image)

Figure 3: Reconstructions of \( f = (f_1, f_2) \) from time-harmonic data at multi frequencies. Figures (b) and (e) are reconstructed from (5.7), whereas (c),(f) are from (5.8).

In the time-dependent case, we first consider the numerical solution of the acoustic wave equation
\[
\begin{align*}
    \frac{1}{c_\alpha^2} \partial_t u_\alpha(x, t) - \Delta u_\alpha(x, t) &= g(t) f_\alpha(x), \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}^+, \\
    u_\alpha|_{t=0} &= \partial_t u_\alpha|_{t=0} = 0, \quad \text{in} \quad \mathbb{R}^2, \quad \alpha = p, s. 
\end{align*}
\]  
(5.12) 
(5.13)

To reduce the unbounded solution domain to a bounded computational domain, we use the local absorbing boundary condition
\[
\partial_n u_\alpha + \frac{1}{c_\alpha} \partial_t u_\alpha + \frac{1}{2R} u_\alpha = 0 \quad \text{on} \quad \partial B_R. 
\]
Figure 4: Reconstructions of the compressional and shear components of $f$. Figures (b) and (e) are reconstructed from (5.9), whereas (c) and (f) are from (5.10).
Then the solutions to the acoustic scattering problem (5.12)-(5.13) are computed over \( B_R \) by using interior penalty discontinuous Galerking method in space and Newmark method in time. Consequently, the data \( U(x, t) \) of the Lamé system are obtained through

\[
U(x, t) = \frac{1}{\gamma_p} \text{grad} \ u_p(x, t) + \frac{1}{\gamma_s} \overrightarrow{\text{curl}} \ u_s(x, t).
\]

In our numerical examples, we collect the scattering data \( U(x, t) \big|_{\partial B_R} \) for \( t \in [0, T] \) with \( T = 20 > T_0 + (R + R_1)/c_s = 8 \). In Figure 5, we compare the scattering data \( \hat{u}(x, \omega) \big|_{\partial B_R} \) at frequencies \( \omega = 3 \) and \( \omega = 10 \) obtained by solving the time-harmonic Lamé system and that by applying Fourier transform (denoted by \( \hat{u}'(x, \omega) \big|_{\partial B_R} \)) to the time-dependent data \( U(x, t) \big|_{\partial B_R} \). It can be seen that the data set via Fourier transformation slightly differs from those time-harmonic data, possibly due to numerical errors in the Fourier transform and in the numerical scheme for solving time-dependent Lamé systems as well. To Fourier transform the time domain data, we use fifteen equally spaced frequencies from 1 to 15. Numerical solutions for reconstructing \( f \) and \( f_\alpha, \alpha = p, s \) are presented in Figures 6 and 7, respectively. We conclude from Figures 3-7 that satisfactory reconstructions are obtained through the proposed Landweber iterative algorithm.

Figure 5: Comparison of the scattering data \( \hat{u}(x, \omega)/\hat{g}(\omega) \) and \( \hat{u}'(x, \omega)/\hat{g}(\omega) \) at \( \omega = 3, 10 \) obtained respectively by solving the time-harmonic Navier equation (blue) and by applying Fourier transform to the time-domain data (red).

5.2 Reconstruction of temporal functions

We consider the inverse problem of reconstructing \( g \) from the wave fields \( \{U(x, t) : x \in \Gamma \subset \partial B_R, t \in (0, T) \} \) for some \( T > 0 \) in three dimensions. For simplicity we choose the scalar spatial function to be the delta function, i.e., \( f(x) = \delta(x) \). Then the function \( W \) (see the proof of Theorem 4.4) takes the form

\[
W(x, \omega) = \int_{\mathbb{R}^3} \hat{G}(x - y, \omega) f(y) \, dy = \hat{G}(x, \omega)
\]

\[
= \frac{1}{\mu} \Phi_{k_s}(x) I + \frac{1}{\rho \omega^2} \text{grad}_x \text{grad}_x^T [\Phi_{k_s}(x) - \Phi_{k_p}(x)],
\]
Figure 6: Reconstructions of $f = (f_1, f_2)$ from Fourier-transformed time-domain scattering data. Figures (b) and (e) are reconstructed from (5.7), whereas (c) and (f) are from (5.8).
Figure 7: Reconstructions of the compressional and shear components of \( f \) from Fourier-transformed time-domain scattering data. Figures (b) and (e) are reconstructed from (5.9), whereas (c) and (f) are from (5.10).
where \( \Phi_k(x) = e^{ik|x|/(4\pi|x|)}(k = k_p, k_s) \). Hence, \( f \) is indeed not a non-radiation source for all \( \omega \in \mathbb{R}^+ \).

In our example, we set the vector temporal function \( g(t) \) to be

\[
g(t) = (g_1, g_2, g_3)^T,
\]

\[
g_1(t) = \begin{cases} 
\cos(1.5\pi(t - t_1)) \exp(-\pi(t - t_1)^2), & t \leq T_1, \\
0, & t > T_1,
\end{cases}
\]

\[
g_2(t) = \begin{cases} 
\sin(2\pi(t - t_2)) \exp(-\pi(t - t_2)^2), & t \leq T_2, \\
0, & t > T_2,
\end{cases}
\]

\[
g_3(t) = \begin{cases} 
\sin(\pi(t - t_3)) \exp(-\pi(t - t_3)^2), & t \leq T_3, \\
0, & t > T_3,
\end{cases}
\]

where \( T_1 = 5, T_2 = 4, T_3 = 3, t_1 = 2, t_2 = 3 \) and \( t_3 = 2 \). The function pairs \( (g_1, \hat{g}_1), (g_2, \hat{g}_2) \) and \( (g_3, \hat{g}_3) \) are plotted in Figures 1, 8 and 9, respectively. Moreover, we set \( g(t) = 0 \) for \( t < 0 \). With the choice of \( f \) and \( g \), the forward time-domain scattering data can be expressed as \( U = (u_1, u_2, u_3) \), where

\[
u_i(x, t) = \sum_{j=1}^{3} \int_{0}^{\infty} \int_{\mathbb{R}^3} G_{i,j}(x - y, t - s) f(y) g_j(s) \, dx \, ds 
\]

\[
= \sum_{j=1}^{3} \int_{0}^{\infty} G_{i,j}(x, t - s) g_j(s) \, ds 
\]

\[
= \frac{1}{4\pi \rho |x|^3} \sum_{j=1}^{3} \left( \frac{x_i x_j}{c_p} g_j(t - |x| / c_p) + \frac{1}{c_s} (\delta_{ij} |x|^2 - x_i x_j) g_j(t - |x| / c_s) \right) 
\]

\[
+ \frac{1}{4\pi \rho |x|^3} \sum_{j=1}^{3} (3x_i x_j - \delta_{ij} |x|^2) \int_{1/c_p}^{1/c_s} s g_j(t - s) \, ds.
\]

Taking the Fourier transform gives the data \( \hat{U}(x, \omega) \) in the Fourier domain. The sampling frequencies are chosen as

\[
\omega_j = 1 + (j - 1)h, \quad h = 19/49, \quad j = 1, \ldots, K, \quad K = 50.
\]

Fixing \( \omega_i \in \mathbb{R}^+ (i = 1, 2, \cdots, K) \), we can always find \( x_{0,i} \in \Gamma \) such that \( W(x_{0,i}, \omega) \) exists and the value of the indicator

\[
I_1(\omega_i) = \left[ W(x_{0,i}, \omega_i) \right]^{-1} \hat{U}(x_{0,i}, \omega_i)
\]

is identical to \( \hat{g}(\omega) \). Taking the inverse Fourier transform of the indicator function \( I_1(\omega) \) enables us to plot the function \( t \rightarrow g_i(t) \) for \( i = 1, 2, 3 \). In our tests we choose \( x_{0,i} = (1,1,0)^T \) uniformly in all \( i = 1, 2, \cdots, K \). Numerical reconstructions of \( \hat{g}_i, i = 1, 2, 3 \) from the indicator \( I_1 \) are presented in Figure 10.

One can readily observe that the choice of \( x_{0,i} \) is not unique. Our numerics show that \( \text{Det}(W(x, \omega_i)) \) does not vanish for almost all \( x \in \partial \Omega_R \). For \( \omega_i \in \mathbb{R}^+ \), we denote by \( \{x_{j,i} : j = 1, 2, \cdots, M\} \) a set of points lying on \( |x| = R \) such that \( W(x_{j,i}, \omega_i) \) is invertible for each \( j \). To make our inversion scheme computationally stable, we can calculate \( I_1(\omega_i) \) using each \( x_{j,i} \) (\( j = 1, 2, \cdots, M \)) and then take the average as the value of \( \hat{g}(\omega_i) \). Hence, we propose another indicator function in the Fourier domain as following

\[
I_2(\omega_i) := \frac{1}{M} \sum_{j=1}^{M} \left[ W(x_{j,i}, \omega_i) \right]^{-1} \hat{U}(x_{j,i}, \omega_i), \quad i = 1, 2, \cdots, K,
\]

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where the time domain data \( \{U(x_{j,i}, t): j = 1, 2, \cdots, M, i = 1, 2, \cdots, K\} \) are used. In our experiments, we make use of the boundary data equivalently distributed on \( |x| = R \) and set
\[
x_{j,i} = x_j = (\cos((j - 1)d\theta), 1, \sin((j - 1)h))^T, \quad h = 2\pi/M, j = 1, 2, \cdots, M,
\]
uniformly in all \( i = 1, 2, \cdots, K \). Numerics show that such kind of boundary data are adequate for the choice of \( f \) and \( g \). Next we consider reconstructions from the noised data
\[
U_\delta(x, t) = (1 + \delta\epsilon(x, t))U(x, t)
\]
where \( \epsilon(x, t) \) is a function whose value is random between -1 and 1, and the noise level \( \delta \) is set to be 30%. We present the reconstructions of \( \hat{g}_j \) \( (j = 1, 2, 3) \) based on the indicators \( I_1 \) and \( I_2 \) in Figures 11 and 12 respectively. Reconstructions from the inverse Fourier transform of \( I_j \) (that is, the temporal function \( g(t) \)) are illustrated in Figures 13 and 14 where the time-domain data with 30% noise are again used. Comparing Figures 11, 12, 13 and 14 one may conclude that the inversion scheme using \( I_2 \) is indeed more computationally stable than \( I_1 \).

![Figure 8](image1.png)

Figure 8: The exact pulse function \( g_2(t) \) and its Fourier transformation \( \hat{g}_2(\omega) \).

![Figure 9](image2.png)

Figure 9: The exact pulse function \( g_3(t) \) and its Fourier transformation \( \hat{g}_3(\omega) \).

6 Appendix

Lemma 6.1. Suppose that \( S \in (L^2(\mathbb{R}^3))^3 \) has a compact support in \( B_R \) for some \( R > 0 \), then the Helmholtz decomposition of \( S \) is unique.
Figure 10: Reconstruction of temporal functions from $I_1$ without noise.

Figure 11: Reconstruction of temporal functions from $I_1$ with 30% noise.

Figure 12: Reconstruction of temporal functions from $I_2$ with 30% noise.

Figure 13: Reconstruction of temporal functions from $\tilde{I}_1$ with 30% noise.
Figure 14: Reconstruction of temporal functions from $\tilde{I}_2$ with 30% noise.

**Proof.** Due to the Helmholtz decomposition, every $S \in (L^2(\mathbb{R}^3))^3$ admits a decomposition:

$$S = \nabla S_p + \nabla \times S_s, \quad \nabla \cdot S_s = 0,$$

where

$$S_p \in H^1(B_R), \quad S_s \in H_{\text{curl}}(B_R) := \{U : U \in L^2(B_R)^3, \text{curl} U \in L^2(B_R)^3\}$$

also have compact support in $B_R$. Suppose that $S$ admits another orthogonal decomposition $S = \nabla S'_p + \nabla \times S'_s$, $\nabla \cdot S'_s = 0$. Then we have

$$\nabla (S_p - S'_p) + \nabla \times (S_s - S'_s) = 0. \quad (6.1)$$

Taking the divergence of both sides of (6.1) gives $\Delta (S_p - S'_p) = 0$ in $B_R$, i.e., $S_p - S'_p$ is harmonic over $B_R$. Note that $S_p - S'_p = 0$ on $\partial B_R$. Applying the maximum principle for harmonic functions yields $S_p = S'_p$ in $B_R$. On the other hand, applying $\nabla \times$ to the both sides of (6.1) we obtain

$$0 = \nabla \times (\nabla \times (S_s - S'_s)) = \nabla (\nabla \cdot (S_s - S'_s)) - \Delta (S_s - S'_s) = -\Delta (S_s - S'_s).$$

Then the relation $S_s = S'_s$ in $B_R$ can be proved analogously. This completes the proof.

In the following lemma, the notation $I_{n \times n}$ denotes the unit matrix in $\mathbb{R}^{n \times n}$ for $n \geq 2$.

**Lemma 6.2.** Let $\xi = (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^{n \times 1}$ and $A(\xi) = \mu|\xi|^2 I_{n \times n} + (\lambda + \mu)\xi \otimes \xi \in \mathbb{R}^{n \times n}$. Then the eigenvalues $\tau_j$ ($j = 1, 2, \ldots, n$) of $A(\xi)$ are given by

$$\tau_1 = (\lambda + 2\mu)|\xi|^2, \quad \tau_2 = \cdots = \tau_n = \mu|\xi|^2.$$

**Proof.** Set $\tilde{A} = A - \tau I_{n \times n}$. We may rewrite $\tilde{A}$ in the form $\tilde{A} = B + VV^T$, where

$$B = (\mu|\xi|^2 - \tau) I_{n \times n}, \quad V = \sqrt{\lambda + \mu} \xi.$$

Straightforward calculations show that

$$\text{Det}(\tilde{A}) = \text{Det}(B + VV^T)$$

$$= (1 + V^T B^{-1} V) \text{Det}(B)$$

$$= \left(1 + \frac{(\lambda + \mu)|\xi|^2}{\mu|\xi|^2 - \tau}\right) (\mu|\xi|^2 - \tau)^n$$

$$= [(\lambda + 2\mu)|\xi|^2 - \tau](\mu|\xi|^2 - \tau)^{n-1},$$

which implies the eigenvalues of $A$. \qed

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Lemma 6.3. (Gronwall-type inequality) Let $T > 0$ and $u \in L^2(0,T)$ be nonnegative and fulfill, for almost every $t \in (0,T)$, the inequality

$$u(t) \leq a(t) + \int_{0}^{t} b(s) u(s) \, ds,$$

where $a \in L^2(0,T)$ and $b \in C([0,T])$ are two nonnegative functions. Then, for almost every $t \in (0,T)$, we have

$$u(t) \leq a(t) + \int_{0}^{t} a(s) b(s) e^{\int_{s}^{t} b(\tau) \, d\tau} \, ds.$$

Proof. We consider $Y$ defined, for almost every $t \in (0,T)$, by

$$Y(t) := e^{-\int_{0}^{t} b(s) \, ds} \int_{0}^{t} b(s) u(s) \, ds$$

and we remark that $Y \in H^1(0,T)$ and satisfies $Y(0) = 0$. Then, for almost every $t \in (0,T)$, we find

$$Y'(t) = b(t) u(t) e^{-\int_{0}^{t} b(s) \, ds} - b(t) e^{-\int_{0}^{t} b(s) \, ds} \int_{0}^{t} b(s) u(s) \, ds.$$

On the other hand, in view of (6.2), for almost every $t \in (0,T)$, we get

$$\int_{0}^{t} b(s) u(s) \, ds \geq u(t) - a(t)$$

and we deduce that

$$Y''(t) \leq a(t) b(t) e^{-\int_{0}^{t} b(s) \, ds}.$$

Integrating on both side of this inequality we get

$$\int_{0}^{t} Y''(s) \, ds \leq \int_{0}^{t} a(s) b(s) e^{-\int_{0}^{s} b(\tau) \, d\tau} \, ds, \quad t \in (0,T).$$

On the other hand, since $Y \in H^1(0,T)$ and satisfies $Y(0) = 0$, by density one can check that $Y(t) = \int_{0}^{t} Y'(s) \, ds$ which implies that

$$e^{-\int_{0}^{t} b(s) \, ds} \int_{0}^{t} b(s) u(s) \, ds \leq \int_{0}^{t} a(s) b(s) e^{-\int_{0}^{s} b(\tau) \, d\tau} \, ds$$

and by the same way, for almost every $t \in (0,T)$, the following inequality

$$\int_{0}^{t} b(s) u(s) \, ds \leq e^{\int_{0}^{t} b(s) \, ds} \left( \int_{0}^{t} a(s) b(s) e^{-\int_{0}^{s} b(\tau) \, d\tau} \, ds \right) = \int_{0}^{t} a(s) b(s) e^{\int_{s}^{t} b(\tau) \, d\tau} \, ds.$$

Finally, applying again (6.2), for almost every $t \in (0,T)$, we find

$$u(t) \leq a(t) + \int_{0}^{t} b(s) u(s) \, ds \leq a(t) + \int_{0}^{t} a(s) b(s) e^{\int_{s}^{t} b(\tau) \, d\tau} \, ds.$$

This proves (6.3). \quad \Box

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