ON THE M-XX EQUATION

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Abstract

Some exact solutions of the (2+1)-dimensional integrable classical continuous isotropic Heisenberg spin chain (the M-XX equation) are obtained by using Hirota’s method. These solutions are characterized by an integer topological charge.
1 Introduction

Classical continuum Heisenberg ferromagnet models (CCHFM) exhibit a rich variety of nonlinear behaviour. In particular, over the past two decades, several integrable nonlinear ferromagnetic models have been identified [1-3]. So, some number integrable spin systems in 2+1 dimensions are found [4-8]. The underlying nonlinear spin excitations in such spin systems represented by solitons, domain walls, vortecies, lumps and dromions [4-11]. Theirs study are of considerable intrinsic interest, especially from the points of view of both mathematics and physics. Integrable spin systems in 2+1 dimensions as their (1+1)-dimensional counterparts, display fascinating geometrical aspects: they are gauge and L-equivalent to the nonlinear Schrödinger - type equations (the Davey-Stewartson equation, the Zakharov equations (ZE), the Strachan equation and so on) [5, 9-10, 12]. Generally speaking, between spin systems and the differential geometry take places the deep connection [12, 13-17, 20]. An important feature of two dimensional spin systems is the existence the topological invariant

\[ Q = \frac{1}{4\pi} \int \int dx dy \textbf{S} \cdot (\textbf{S}_x \wedge \textbf{S}_y) \] (1)

such that the solutions of them are classified by the topological charge (1). In (9) \( \textbf{S} = (S_1, S_2, S_3) \) is the three-dimensional unit vector (the spin vector). The class of exact solutions of (2+1)-dimensional spin systems is a very rich one. The solitons, vortices, dromions, lumps are among them. These solutions have the important physical significance. So, for example, vortecies play an active role in the dynamics and termodynamics of quasi-two-dimensional magnets [18-20].
The present paper is devoted to the study of the following CCHFM in two space and one time dimensions

\[
S_t + S \wedge \{(b + 1)S_{\xi \xi} - bS_{\eta \eta}\} + bu_\eta S_\eta + (b + 1)u_\xi S_\xi = 0 \tag{2a}
\]

\[
u_{\xi \eta} = S(S_\xi \wedge S_\eta) \tag{2b}
\]

with

\[
\xi = \frac{x}{2} + \frac{a + 1}{\alpha} y, \quad \eta = -\frac{x}{2} - \frac{a}{\alpha} y. \tag{3}
\]

where \(a, b\) are real constants, \(\alpha^2 = \pm 1\). It is the Myrzakulov XX (M-XX) equation [5]. Hereafter, for convenience, we use the conditional notations, e.g. equation (2) we denote by the M-XX equation. Equation (2) is integrable. We will distinguish the two integrable cases: the M-XXA equation as \(\alpha^2 = 1\) and the M-XXB equation as \(\alpha^2 = -1\). Also, equation (2) contains several integrable cases:

(i) \(b = 0\), yields the M-VIII equation

\[
S_t + S \wedge S_{\xi \xi} + wS_\xi = 0 \tag{4a}
\]

\[
w_\eta = S(S_\xi \wedge S_\eta) \tag{4b}
\]

where \(w = u_\xi\).

(ii) \(a = b = -\frac{1}{2}\), yields the celebrated Ishimori equation [4]

\[
S_t + \frac{1}{2}S \wedge \{S_{\xi \xi} + S_{\eta \eta}\} - \frac{1}{2}u_\eta S_\eta + \frac{1}{2}u_\xi S_\xi = 0 \tag{5a}
\]

\[
u_{\xi \eta} = S(S_\xi \wedge S_\eta) \tag{5b}
\]

where \(\xi = \frac{1}{2}(x + \frac{1}{\alpha} y), \quad \eta = -\frac{1}{2}(x - \frac{1}{\alpha} y).\) The Ishimori equation (5) is the first integrable spin system in plan, which can be solved by the inverse scattering method (IST). This equation were studied by the different authors from variety points of view (see, e.g. [4,6-7,9,11,21]).

(iii) Let now we put \(b = 0, \eta = t\), then equation (2) reduces to the (1+1)-dimensional M-XXXIV equation

\[
S_t + S \wedge S_{\xi \xi} + wS_\xi = 0 \tag{6a}
\]

\[
w_t + \frac{1}{2}(S_\xi^2)_{\xi} = 0 \tag{6b}
\]

This integrable equation was introduced in [5] to describe nonlinear dynamics of compressible magnets.

Equation (2) is the (2+1)-dimensional integrable generalization of the (1+1)-dimensional CCHFM or the isotropic Landau-Lifshitz equation (LLE)

\[
S_t = S \wedge S_{xx} \tag{7}
\]

and in 1+1 dimensions reduces to it. Here, it should be mentioned that the M-XX equation (2) is not the only integrable generalization of the LLE (7) in
2+1 dimensions. There exist several another integrable generalizations, e.g the following one,
\[ S_x = (S \wedge S_y + uS)_x \] (8a)
\[ u_x = -S(S_x \wedge S_y) \] (8b)
This equation, which is known as the Myrzakulov I (M-I) equation, is again completely integrable [5,10,12].

As integrable, equation (2) can be solved by the IST method. The applicability of the IST method to the M-XX equation (2) is based on its equivalence to the compatibility condition of the following linear equations [5]
\[ \Phi_t = S \Phi_Z - W \Phi_Z - \frac{1}{2} S S_Z - \frac{1}{2} S S_Z^+ \] (9a)
where \( Z^\pm = \xi \pm \eta \) and
\[ W = 2i((2b+1)(F^+ + F^-) + (F^+ S + F^-) + (2b+1)SS_Z^+ + \frac{1}{2} S Z^- + \frac{1}{2} S S Z^+) \],
\[ S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^\pm = S_1 \pm iS_2 \quad S^2 = EI, \quad E = \pm 1, \quad r^2 = \pm 1, \quad F^+ = 2iuZ^-, \quad F^- = 2iuZ^+. \]
In fact, from the condition \( \Phi_{Z^+ t} = \Phi tZ^+ \), we get the equation
\[ iS_t + \frac{1}{2}[S, (b+1)S_{\xi\xi} - bS_{\eta\eta}] + ibu_{\eta}S_{\eta} + i(b + 1) u_{\xi} S_{\xi} = 0 \] (10a)
\[ u_{\xi\eta} = \frac{1}{4i} tr(S[S_{\xi}, S_{\eta}]) \] (10b)
which is the matrix form of equation (2).

2 Bilinearization

It could be of interest to study the equation (2) by the IST method. But to look for the some special solutions, it is convenient use the other may be more practical method - the Hirota bilinear method. Remaining the use of the IST method in future, in this paper, we work with the Hirota method. To this purpose, we construct the bilinear form of (2) as \( E = r = 1 \). Let us now introduce the following transformation for the components of spin vector \( S \) and for the derivatives of scalar potential \( u \)
\[ S^+ = \frac{2fg}{ff + gg}, \quad S_3 = \frac{ff - gg}{ff + gg} \] (11a)
\[ u_{\xi} = -2iD_{\xi}\left(\frac{f \circ f + g \circ g}{ff + gg}\right), \quad u_{\eta} = 2iD_{\eta}\left(\frac{f \circ f + g \circ g}{ff + gg}\right) \] (11b)
Here the Hirota operators $D_x, D_y$ and $D_t$ are defined by

$$D^l_\xi D^m_\eta D^n_t f(\xi, \eta, t) \circ g(\xi, \eta, t) = (\partial_\xi - \partial_\xi')^l (\partial_\eta - \partial_\eta')^m (\partial_t - \partial_t')^n f(\xi, \eta, t) \circ g(\xi', \eta', t') \mid_{\xi = \xi', \eta = \eta', t = t'}.$$  

(12)

Substituting the formulae (11) into the M-XX equation (2), we obtain the bilinear equations

$$[iD_t - (b + 1)D^2_\xi + bD^2_\eta](\bar{f} \circ g) = 0 \quad (13a)$$
$$[iD_t - (b + 1)D^2_\xi + bD^2_\eta](\bar{f} \circ f - \bar{g} \circ g) = 0 \quad (13b)$$
$$\{D_\xi D_\eta + D_\eta D_\xi\}(\bar{f}f + \bar{g}g) \circ (ff + gg) = 0 \quad (13c)$$

Note that equation (13c) coincide with the compatibility condition $u_{\xi\eta} = u_{\eta\xi}$. Equations (13) is the desired Hirota bilinear form of equation (2).

3 Solutions

Now we can construct some special solutions of equation (2). As examples, we find simplest soliton, domain wall and vortex solutions.

3.1 Soliton solution

FIND THE SOLITON SOLUTIONS.

3.2 Domain wall solution

In order to obtain a domain wall solutions, we make the choice

$$f = 1.$$  

(14)

Then, equations (13a,b) reduce to

$$ig_t + (b + 1)g_{\xi\xi} - bg_{\eta\eta} = 0 \quad (15a)$$
$$\quad (b + 1)\bar{g}_{\xi\xi}g_{\eta\eta} - b\bar{g}_{\eta\eta}g_{\eta\eta} = 0 \quad (15b)$$

Let us we consider the case, when $\alpha^2 = -1$, i.e. the M-XXB equation. In this case, equation (13c) is identically satisfied by any analytical function $g = g(\xi, \eta, t)$. For example, the simplest non-trivial solution of equation (2) is

$$g = \exp \chi_1$$  

(16)

where

$$\chi_1 = m_1\xi + n_1\eta + i[(b + 1)m^2 - bnt^2]t + \chi_{10} = \chi_{1R} + i\chi_{1I}.$$  

(17)

Thus, the spin components and the potential field are given by

$$S^+ = e^{i\chi_{1I}}sech\chi_{1R}, \quad S_3 = -th\chi_{1R}, \quad u = 2\ln(1 + e^{2\chi_{1R}}).$$  

(18)
3.3 **Vortex solution**

To construct vortex solution, we use the equation (13) and assume that its solution has the form

\[ f = f(\xi, t), \quad g = g(\xi, t) \]  (19)

Then the condition (13c) is satisfied automatically. At the same time, equations (13a,b) are satisfy if

\[ if_t + (b + 1)f_{\xi\xi} = 0 \quad ig_t + (b + 1)g_{\xi\xi} = 0 \]  (20)

Hence, we obtain the following multi-vortex solutions of the M-XXB equation (2)

\[ g_N = \sum_{j=0}^{N} \sum_{m+2n=j} \frac{a_j}{m!n!} (\frac{2}{b + 1})^n \xi^m (2it)^n \]  (21a)

\[ f_N = \sum_{j=0}^{N-1} \sum_{m+2n=j} \frac{b_j}{m!n!} (\frac{2}{b + 1})^n \xi^m (2it)^n \]  (21b)

where \( a_j \) and \( b_j \) are arbitrary complex constants, \( m, n \) are the non-negative integer numbers. In particular, the 1-vortex solution is given by

\[ f = b_0, \quad g = a_1^\prime \xi + a_0 \]  (22)

where \( a_1^\prime = a_1 \left( \frac{2}{b + 1} \right)^{\frac{1}{2}} \). The corresponding solution of equation (2) is given by

\[ S^+ = \frac{2b_0(a_1^\prime \xi + a_0)}{|b_0|^2 + |a_1^\prime \xi + a_0|^2} \]  (23a)

\[ S_3 = \frac{|b_0|^2 - |a_1^\prime \xi + a_0|^2}{|b_0|^2 + |a_1^\prime \xi + a_0|^2} \]  (23b)

\[ u = 2 \ln(|b_0|^2 + |a_1^\prime \xi + a_0|^2) \]  (23c)

So, the 1-vortex solution is static. To find the dynamic solution, we consider the 2-vortex solution, which has the form

\[ f = b_0^\prime \xi + b_0, \quad g = \frac{a_2}{b + 1} \xi^2 + \frac{a_2}{2} 2it + a_1^\prime \xi + a_0, \quad b_1^\prime = b_1 \left( \frac{2}{b + 1} \right)^{\frac{1}{2}} \]  (24)

The interesting question is the dynamics of these vortices. Let us rewrite the solution (21) in the following factorized form

\[ f(\xi, t) = b_0 \prod_{j=1}^{N} [\xi - p_j(t)] \]  (25a)

\[ g(\xi, t) = a_0 \prod_{j=1}^{N} [\xi - q_j(t)] \]  (25b)
where $p_j$ and $q_j$ denote the positions of the zeros for $f$ and $g$, and $a_0, b_0$ are constants. Substituting (25) into (20), we get the evolutions of $p_j$ and $q_j$ as

$$p_{jt} = -i(b + 1) \sum_{k \neq j}^N \frac{1}{p_j - p_k}$$

(26a)

$$q_{jt} = -i(b + 1) \sum_{k \neq j}^N \frac{1}{q_j - q_k}$$

(26b)

where $j, k = 1, 2, ..., N$. Hence, we get the Calogero-Moser type system

$$p_{jtt} = 2(b + 1)^2 \sum_{k \neq j}^N \frac{1}{(p_j - p_k)^3}$$

(27a)

$$q_{jtt} = 2(b + 1)^2 \sum_{k \neq j}^N \frac{1}{(q_j - q_k)^3}$$

(27b)

with the following Hamiltonian

$$H = \frac{1}{2} \sum (p_j^2 + q_j^2) + (b + 1)^2 \sum [(p_j - p_k)^{-2} + (q_j - q_k)^{-2}]$$.

(28)

### 3.4 Dromion solution

In this subsection we would like get the dromion [23] solution of equation (2), but please

**FIND THE DROMION SOLUTIONS.**

### 4 Gauge equivalent equation

Finally, let us we present the gauge equivalent counterpart of equation (2). It has the form

$$iq_t + (1 + b)q_{\xi \xi} - bq_{\eta \eta} + vq = 0$$

(29a)

$$ip_t - (1 + b)p_{\xi \xi} + bp_{\eta \eta} - vp = 0$$

(29b)

$$v_{\xi \eta} = -2\{ (1 + b)(pq)_{\xi \xi} - b(pq)_{\eta \eta} \}$$

(29c)

where $p, q$ are some complex functions. Equation (29) is related with the Zakharov equations [22]. To prove gauge equivalence between equations (2) and (29), let us perform the gauge transformation $\Psi = g\Phi$, where the function $\Phi$ is the solution of equations (9) and $g$ is a 2x2 matrix such that

$$S = g^{-1}\sigma_3g$$

(30)

and

$$g_Zg^{-1} - \sigma_3g_Z^{-1}g^{-1} = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$$

(31)
Under this transformation the function $\Psi$ satisfies the following set of linear equations

$$
\Psi^+ = \sigma_3 \Psi^- + B_0 \Psi 
$$

$$
\Psi_t = 4iC_2 \Psi_{Z^-} Z^- + 2C_1 \Psi_{Z^-} + C_0 \Psi.
$$

(32a)

(32b)

where $B_0$, $C_j$ are given by

$$
B_0 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} b + 1 & 0 \\ 0 & b \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & iq \\ ip & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}
$$

$$
c_{12} = i[(4b + 3)q_{Z^-} + q_{Z^+}] \quad c_{21} = -i[(4b + 1)p_{Z^-} + p_{Z^+}]
$$

and $v = i(c_{22} - c_{11})$. Here $c_{jj}$ are the solution of the following equations

$$
c_{11Z^-} - c_{11Z^+} = i[(4b + 3)(pq)_{Z^-} + (pq)_{Z^+}]
$$

$$
c_{22Z^-} + c_{22Z^+} = i[(4b + 1)(pq)_{Z^-} + (pq)_{Z^+}].
$$

The compatibility condition of equations (32) gives the equation (29). Therefore the M-XX equation (2) and the equation (29) are gauge equivalent to each other. Now let us proceed to the M-XX equation (10). It is not difficult to check that if $g$ obeys equations (31) then the $S$ in the form (30) obeys the M-XX equation (10) with

$$
u = -2i\alpha \ln \det g
$$

(33)

5 Conclusion

To conclude, in this paper we have found some exact solutions, namely domain wall and vortex solutions of the (2+1)-dimensional CCHFM - the M-XX equation. We have shown that the dynamics of vortices are governed by the Calogero-Moser type systems. Also we have presented the gauge equivalent counterpart of this equation.

6 Particular open problems

Finally, we also would like to pose the following particular problems:

**Problem N1:** Find solutions of the M-XX equation by the $\bar{\partial}$-dressing method.

**Problem N2:** Find solutions of the M-XX equation using the nonlocal Riemann-Hilbert problems method.

**Problem N3:** Find solutions of the M-XX equation by means of the Darboux transformation as in [21].

**Problem N4:** Find the other solutions of the M-XX equation (solitons, dromions and so on) by the Hirota bilinear method.

If you have some results in these directions, please, inform me by E-mail: cnlpmyra@satsun.sci.kz. We are ready to interaction.
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