Overcompleteness and unlike closure relations

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Abstract. One says that a set \{\ket{\nu}\} is an overcomplete “basis” of a system’s Hilbert space if a proper subset \{\ket{\nu}\} \subset \{\ket{v}\} suffices to represent an arbitrary ket. Although less explored, the fact that \langle v | v' \rangle is usually finite and non-zero for any pair \nu, \nu' may be equally relevant in the context of coherent states \{\ket{z}\}. First we illustrate this point with a simple, but non-trivial example in \mathbb{R}^2. In the Hilbert space of a quantum particle the standard coherent-state resolution of unity is written in terms of a phase-space integration of the outer product \ket{z}\bra{z}. Because no pair of coherent states is orthogonal, one can represent the closure relation in non-standard ways, in terms of a single phase-space integration of the “unlike” outer product \ket{z'}\bra{z}, \nu' \neq \nu. This makes it possible, for instance, to write a formal expression for a phase-space path integral in terms of weak energy values.

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1. Introduction: Different representations and a single basis

In quantum mechanics the term overcompleteness is used to designate a redundant set of vectors that spans a system’s Hilbert space. We refer to such a set as an overcomplete basis, keeping in mind that it is not a minimal generating set. The use of a basis of this type gives rise to an infinity of representations in the sense that a single ket can be decomposed in different ways in terms of the same set of vectors. In this regard the terms “choosing a basis” and “choosing a representation”, which are routinely used interchangeably, are no longer equivalent.

At first, employing an overcomplete basis may seem pointless. Why would one bother to represent a vector in an intricate and potentially ambiguous way? It turns out that the vectors belonging to certain overcomplete basis have a relevant physical meaning and mathematical properties that have proved to be useful in many situations. As an example, the use of canonical coherent states \{\ket{z}\} , arguably the most important overcomplete set of quantum theory, in quantum optics [1, 2] is more than sufficient to illustrate the physical relevance of overcomplete sets.

For a basis to be fully operational, one should be able to write a closure relation in terms of it. The standard way to do that with coherent states is to represent the unit operator by

\[ \hat{I} = \int \frac{d^2z}{\pi} |z\rangle\langle z| , \]

where \( z \) is a complex label and the integration is over its real and imaginary parts.

The overcompleteness of \{\ket{z}\} is a direct consequence of the analyticity of the Bargmann function \( \psi(z^*) = \exp\{+|z|^2/2\} \langle z|\psi \} \). For example, \{\ket{z_j}\} with \{\{z_j\}_{j\in\mathbb{N}}\} being a convergent sequence on the complex plane, has been shown to constitute a basis [4]. As a corollary, the coherent states associated to any curve with finite length in the complex plane also forms a basis. Works can be found in the literature in which different subsets of \{\ket{z}\} are used to construct alternative representations. An interesting example is the circle decomposition, in which an arbitrary ket can be written as

\[ |\psi\rangle = \frac{e^{R^2/2}}{2\pi i} \oint_{|z|=R} dz \; g(z) |z\rangle , \]

where only coherent states on the circle of radius \( R \) are used [5]. In a related representation, only coherent states of vanishing momentum are employed (corresponding to the real axis in the \( z \)-plane) [6, 7]. These and other examples make it clear the redundant character of coherent states.

There is, however, another property that is also relevant: two arbitrary coherent states are never orthogonal. This apparently innocent feature has an immediate consequence, namely, the unit operator can also be expressed as

\[ \hat{I} = \hat{I}^2 = \int \int \frac{d^2z}{\pi} \frac{d^2z'}{\pi} |z\rangle\langle z| |z'\rangle\langle z'| , \]
which is not trivially equivalent to (1). Note that in the \{ |x\rangle \} representation, e.g., the analogous of (3) would be identical to that corresponding to (1), since \langle x | x' \rangle = \delta (x - x')

The significance of relations (1) and (3) is described by Klauder and Sudarshan as “two manifestly different decompositions for the same operator in terms of one set of states” [9]. They also describe (1) as involving a superposition of “like outer products” in opposition to (3) which is a composition of “unlike outer products” [9], involving a double phase-space integration.

Until recently, despite the many alternative representations, the only existing expressions for the unit operator in terms of coherent states were (1) and its immediate consequences, e.g., (3). It has been shown that alternative forms of the resolution of unit employing “unlike outer products” can be constructed via a single phase-space integration. They are given by

\[ \hat{I} = \int \frac{d^2z}{\pi} \lambda e^{\frac{1}{2}(1-\lambda^2)|z|^2} |\lambda z\rangle \langle z| , \]  

with \( \lambda \) being a positive real number [10]. The above identity can be pictorially understood as follows. Coherent-state overcompleteness not only imply in redundancy, but also in non-orthogonality, \( \langle z | z' \rangle \neq 0 \). Therefore, we can take the component of an arbitrary ket \( |\psi\rangle \), \( \langle z | \psi \rangle \), and force it to be the coefficient of \( |\psi\rangle \) in the “wrong direction” \( |\lambda z\rangle \) provided that we correct this by an appropriate measure in phase space. We will return to this point shortly.

The manuscript is organized as follows. In the next two sections we give some basic properties of coherent states and define the concept of frame, which makes the notion of overcompleteness more precise. In section 4 we build an overcomplete set in \( \mathbb{R}^2 \) and give an explicit example of what we mean by unlike closure relation. Section 5 contains the derivation of a coherent-state unlike closure relation and some direct consequences. In section 6 we derive a phase-space path integral for which the role of the Hamiltonian is played by a weak energy value, leading to the concept of quasi-classical weak values. Finally, in section 7, we call attention to some recurrences in quantum phase-space transforms, and summarize our conclusions.

### 2. Selected properties of canonical coherent states

There are entire books dedicated to the properties and applications of coherent states. Classical examples are those by Perelomov [11] and by Klauder and Skagerstam [12]. A more recent account is the book by Gazeau [13]. In this section we list a few properties of canonical coherent states that will be directly used in the remainder of this work.

Given an harmonic oscillator with mass \( m \) and angular frequency \( \omega \), a canonical coherent state \( |z\rangle \) is defined by the eigenvalue equation \( \hat{a} |z\rangle = z |z\rangle \), where

\[ \hat{a} = \frac{1}{\sqrt{2}} \left( \frac{\hat{q}}{b} + \frac{ib\hat{p}}{\hbar} \right) \]  

is the bosonic annihilation operator and \( b = \sqrt{\hbar/m\omega} \). An equivalent and insightful
definition is $|z\rangle = \hat{D}(z)|0\rangle$, where $|0\rangle$ is the ground state of the harmonic oscillator, and the displacement operator is given by

$$\hat{D}(z) = \exp\{z\hat{a}^{\dagger} - z^*\hat{a}\} = e^{-\frac{z}{2}|z|^2} e^{z\hat{a}^{\dagger}} e^{-z^*\hat{a}} ,$$

(6)

where, in the second equality we used the Baker-Haussdorf formula. It is easy to show that coherent states are Gaussian wave packets in the position representation, satisfying the minimum uncertainty relation:

$$\langle x|z\rangle = \frac{1}{\pi^{1/4} b^{1/2}} \exp\left\{-\frac{1}{2} \left(\frac{x}{b} - \sqrt{2}z\right)^2 + \frac{1}{2} z(z - z^*)\right\} .$$

(7)

An analogous expression holds for the momentum representation $\langle p|z\rangle$. The overlap between two arbitrary coherent states associated to the same mode is simply

$$\langle z|z'\rangle = \exp\left\{-\frac{1}{2} |z|^2 + z^*z' - \frac{1}{2} |z'|^2\right\} ,$$

(8)

which is always finite and non-zero. The inner product between coherent states $|z\rangle$ and $|z'\rangle$ related to distinct frequencies (or modes) reads

$$\langle z|z'\rangle = \sqrt{\frac{2bb'}{b^2 + b'^2}} \exp\left\{-\frac{1}{2} |z|^2 - \frac{1}{2} |z'|^2 + \frac{1}{2} \left(\frac{b^2 - b'^2}{b^2 + b'^2}\right) (z'^2 - z^2) + 2\frac{bb'}{b^2 + b'^2} z^*z'\right\} ,$$

(9)

where $z = (q/b + ibp/h)/\sqrt{2}$ and $z' = (q'/b' + ib'p'/h)/\sqrt{2}$. Note that expression (8) is recovered for $b = b'$.

3. Frames and bases

We proceed by giving a little mathematical context and terminology related to our next discussions. The central idea here is that of frame, which, roughly speaking, is a generalization of the concept of basis [14]. A discrete set of non-vanishing kets $\{|u_i\rangle\}$ belonging to a Hilbert space $\mathbb{H}$ is a frame if there exist $A, B > 0$ such that $A||\psi||^2 \leq \sum_i |\langle v_i|\psi\rangle|^2 \leq B||\psi||^2$ for every ket $|\psi\rangle$ in $\mathbb{H}$. If $A = B$ the frame is tight. Note that the number of elements of a frame may be greater than the dimension of $\mathbb{H}$. If, by removing any of its elements, it ceases to be a frame, then it is called exact. Thus, in particular, every orthonormal basis is exact and tight, with $A = B = 1$. If we have a continuous set $\{|v\rangle\} \in \mathbb{H}$, $v \in M$, then it is a frame if

$$A||\psi||^2 \leq \int_M d\mu \langle v|\psi\rangle^2 \leq B||\psi||^2 ,$$

(10)

where $\mu$ is a measure over $M$. For our purposes it is sufficient to consider $M$ to be either $\mathbb{R}^n$ or $\mathbb{C}^n$ and that the measure $\mu$ is uniform over $M$, i. e., does not depend on $|\psi\rangle$.

Another important concept is as follows. Given a set of vectors $\{|u_i\rangle\}$, if there exists an invertible linear transform $\hat{T}$ and an orthonormal basis $|e_i\rangle$, such that $|u_i\rangle = \hat{T}|e_i\rangle$, then $\{|u_i\rangle\}$ is a Riesz basis. A frame that is not a Riesz basis is called an overcomplete frame.
From identity (11), where $M = C$, it is immediate that the coherent states $\{|z\rangle\}$ constitute a tight overcomplete frame with $A = B = 1$. More generally, for a frame $\{|v\rangle\}$, whenever one is able to express the resolution of unity in terms of the projectors $|v\rangle\langle v|$, the frame is tight with $A = B = 1$. This is evident because if
\[ \hat{I} = \int_M d\mu |v\rangle\langle v| , \]
then
\[ \langle \psi|\psi \rangle = \int_M d\mu \langle \psi|v\rangle\langle v|\psi \rangle = \int_M d\mu |\langle v|\psi \rangle|^2 = |\langle \psi |\psi \rangle|^2 . \]
Finally, in this work, we say that $\{|v\rangle\}$ is a mutually overlapping (or mutually non-orthogonal) frame if $\langle v|v' \rangle \neq 0$ for all possible pairs $v$ and $v'$.

4. Mimicking coherent-state overcompleteness in $\mathbb{R}^2$

Before going into infinite dimensional Hilbert spaces, we discuss some aspects of overcompleteness with a toy construction in the Euclidean plane. A similar example can be found in [13], but here we go further to ensure the mutual overlapping property.

Consider an arbitrary basis that we denote by $\{|U\rangle, |V\rangle\}$,
\[ |U\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |V\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]
whose transposition is represented by $\langle U| \equiv (1, 0)$ and $\langle V| \equiv (0, 1)$. The identity operator reads
\[ \hat{I} = |U\rangle\langle U| + |V\rangle\langle V| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \]
Now take the frame composed of $N$ normalized vectors given by
\[ |Z_n\rangle \equiv \begin{bmatrix} \cos \left(\frac{n\Delta \theta}{N}\right) \\ \sin \left(\frac{n\Delta \theta}{N}\right) \end{bmatrix} = \cos \left(\frac{n\Delta \theta}{N}\right) |U\rangle + \sin \left(\frac{n\Delta \theta}{N}\right) |V\rangle , \]
with $n = 1, 2, ..., N$ and define the operator
\[ \hat{A}_N = \frac{2}{N} \sum_{n=1}^{N} |Z_n\rangle\langle Z_n| . \]
Let us initially assume that $\Delta \theta = 2\pi$, so that, the $N$ vectors have directions uniformly distributed over $(0, 2\pi]$, with step $2\pi/N$. From (13) it follows that
\[ \hat{A}_N = \frac{2}{N} \sum_{n=1}^{N} \cos^2 \left(\frac{2\pi n}{N}\right) |U\rangle\langle U| + \frac{2}{N} \sum_{n=1}^{N} \sin^2 \left(\frac{2\pi n}{N}\right) |V\rangle\langle V| + \frac{2}{N} \sum_{n=1}^{N} \cos \left(\frac{2\pi n}{N}\right) \sin \left(\frac{2\pi n}{N}\right) (|U\rangle\langle V| + |V\rangle\langle U|) . \]
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Figure 1. Set \( \{ |Z_n\rangle \} \) of \( N = 33 \) mutually non-orthogonal unit vectors separated by a constant angle \( \Delta \theta = (2 - \epsilon)\pi / (N - 1) \), \( \epsilon = \sqrt{2}/35 \approx 0.04 \). The angle between \( |Z_N\rangle \) and \( |Z_1\rangle \) is not \( \Delta \theta \). This asymmetry becomes less relevant for increasing values of \( N \).

Now take the limit \( N \to \infty \). In this case we have \( 2\pi / N \to d\theta \), \( 2\pi n / N \to \theta \), such that
\[
\frac{1}{N} \sum_{n=1}^{N} |Z_n\rangle \langle Z_n| = |U\rangle \langle U| + |V\rangle \langle V| = \hat{I},
\]
(18)
which resembles the quantum coherent state relation (1) in its form, and in the sense that a redundant set of vectors is employed to represent the resolution of unity with uniform weight (or measure). Here the usual feature of overcompleteness appears: any pair of non-degenerate vectors \( \{ |Z_r\rangle, |Z_s\rangle \} \) suffices to generate arbitrary vectors in \( \mathbb{R}^2 \). The same is valid for an arbitrary subset with \( 2 < k \leq N \) vectors. This fact is the direct analogous of quantum representations using subsets of the z-plane [5, 6]. This is basically the construction given by Gazeau in [13].

For our purposes, however, the previous analogy is still unsatisfactory because it disregards the essential fact that \( \langle z'|z \rangle \neq 0 \). In the present case \( \langle Z_n|Z_m \rangle \) does vanish if \( (n - m)/N = 1/4, 3/4 \). This difficulty can be avoided if we assume \( \Delta \theta = (2 - \epsilon)\pi \), where \( \epsilon \) is an irrational number that can be made \( \text{arbitrarily small} \). In this case the condition of orthogonality reads \( 2 - \epsilon = N/[2(n - m)] \) or \( 2 - \epsilon = 3N/[2(n - m)] \). In both cases we have an irrational number in the left-hand side and a rational number in the right-hand side, thus, ensuring that any pair of vectors in the frame is non-orthogonal. The price to be paid is that there will be a residual anisotropy in the set \( \{ |Z_n\rangle \} \) [see figure 1]. With this, the operator \( \hat{A}_\infty \), (18), becomes \( \hat{I} + O(\epsilon) \). This apparently futile detail is important due to the nature of the unusual closure relation we find in what follows.

So far we are trying to imitate some features of coherent states with the simplest vectors. Taking a step forward we ask whether one can derive a property in the simpler structure that is unsuspected, but true, for coherent states. Specifically, let us investigate if it is possible to express the resolution of unity in terms of single sums of unlike outer products \( |Z_k\rangle \langle Z_n|, k = k(n) \neq n \). We intend to write \( \hat{I} \propto \sum_n \mu(n) |Z_{k(n)}\rangle \langle Z_n| \), were \( \mu(n) \) is a correction due to the projection of the nth component in the distinct direction.
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\[ |Z_{k(n)}\rangle \]. Our guess is

\[ \mu(n) = \langle Z_n| Z_{k(n)}\rangle^{-1}, \tag{19} \]

which is well defined because of the mutual overlapping property and will be better justified soon. As an example take \( k(n) = N - n \) and define the operator

\[ \hat{B}_N = \frac{2}{N} \sum_{n=1}^{N} \frac{|Z_{N-n}\rangle \langle Z_n|}{\langle Z_n| Z_{N-n}\rangle} = \frac{2}{N} \sum_{n=1}^{N} \frac{|Z_{N-n}\rangle \langle Z_n|}{\cos(\Delta \theta(1 - 2n/N))}. \tag{20} \]

It can be easily shown that in the limit \( N \to \infty \) we get

\[ \hat{B}_\infty = (1 + L)|U\rangle \langle U| + (1 - L)|V\rangle \langle V| + J_+|U\rangle \langle V| + J_-|V\rangle \langle U|, \tag{21} \]

where

\[ L = \cos \Delta \theta \int_0^{\Delta \theta} \sec(2\theta - \Delta \theta) \, d\theta = O(\epsilon), \tag{22} \]

\[ J_\pm = \frac{1}{\Delta \theta} \int_0^{\Delta \theta} \left[ \sin \Delta \theta \sec(2\theta - \Delta \theta) \pm \tan(2\theta - \Delta \theta) \right] \, d\theta = O(\epsilon^2). \tag{23} \]

This leads to

\[ \hat{I} = \lim_{N \to \infty} \frac{2}{N} \sum_{n=1}^{N} \frac{|Z_{N-n}\rangle \langle Z_n|}{\langle Z_n| Z_{N-n}\rangle} + O(\epsilon), \tag{24} \]

where we used \( \Delta \theta = (2 - \epsilon)\pi \), with \( \epsilon \) being an arbitrary irrational that can be made as small as needed from the beginning. Note that this kind of closure relation would be ill defined if one deals with an orthonormal basis, since all terms \( \langle e_i| e_j\rangle^{-1} \) would diverge for \( i \neq j \).

5. Unlike coherent-state closure relation

With (24) in mind we go back to quantum mechanics and realize that the unlike, or off-center, closure relation (4) derived in [10] can be written in a more suggestive way as

\[ \hat{I} = \int \frac{\lambda d^2z}{\pi} \frac{|\lambda z\rangle \langle z|}{\langle z| \lambda z\rangle}, \tag{25} \]

where \( d^2z/\pi \) is replaced by \( \frac{\lambda d^2z}{\pi} \), since \( \lambda \) is the Jacobian determinant associated to the linear transformation \( z \to \lambda z \) and \( z^* \to g(z^*) \), assuming that \( z \) and \( z^* \) are independent variables. This is an usual procedure that is justified by the analytical extension of \( \Re(z) \propto q \) and \( \Im(z) \propto p \) into the complex plane (see [13]).

At this point it is almost mandatory to ask whether or not there are other resolutions of unity in the form

\[ \int \frac{|J| d^2z}{\pi} \frac{|f(z)\rangle \langle g(z)|}{\langle g(z)| f(z)\rangle}, \tag{26} \]

where \( |J| \) is the modulus of the Jacobian determinant associated to the transformation \( z \to f(z) \) and \( z^* \to g(z^*) \). Given the generality of the question, it is hard to devise
an answer. We may start by noticing that the continuous counterpart of (24) would be \( \int \frac{d^2z}{\pi} \frac{|z - \zeta\rangle\langle z + \zeta|}{\langle z + \zeta|z - \zeta\rangle} \). Nevertheless, this expression, far from being a resolution of unity, yields \( \int \frac{d^2z}{\pi} \frac{|n\rangle\langle n|}{\langle z|z\rangle} \rightarrow \infty \), for an arbitrary Fock state \(|n\rangle\). We must, then, recognize the limitations of our analogy. There is, however, at least one more pair of functions \( f \) and \( g \) for which the answer is affirmative, as we will show. Let us define the following operator in terms of what we may call Weyl-like outer products:

\[
\hat{B} \equiv \int \frac{d^2z}{\pi} \frac{|z - \zeta\rangle\langle z + \zeta|}{\langle z + \zeta|z - \zeta\rangle} = \int \frac{d^2z}{\pi} e^{-\frac{1}{2}|\zeta - z + \zeta^*|} |z - \zeta\rangle\langle z + \zeta| ,
\]

where \( \zeta \) is an arbitrary complex number, representing a point in phase space. Note that, while (11) is similar in form to \( \int dx |x\rangle\langle x| \), a definition analogous to (27) involving \(|x\rangle\} \) would be ill defined due to the presence of \( (x'|x)^{-1} \).

We now proceed to prove that the previous operator is a genuine representation of the unit operator. A more usual, but weaker, proof is given in the appendix. Here we take an algebraic shortcut by, somehow, disassembling (27) in terms of its constituent displacement operators [see (6)]. Note that

\[
\hat{D}(z + \zeta) = \exp\{\hat{z}\hat{a}^\dagger - \hat{z}^*\hat{a} + \zeta \hat{a}^\dagger - \zeta^* \hat{a}\}
= \exp \left\{ -\frac{1}{2} \zeta^*z + \frac{1}{2} \zeta z \right\} \hat{D}(\zeta) \hat{D}(z) ,
\]

which readily leads to

\[
|z - \zeta\rangle = \exp \left\{ +\frac{1}{2} \zeta^* - \frac{1}{2} \zeta \right\} \hat{D}(-\zeta)|z\rangle ,
\]

\[
\langle z + \zeta| = \langle z|\hat{D}^\dagger(\zeta) \exp \left\{ +\frac{1}{2} \zeta^* - \frac{1}{2} \zeta \right\}
\]

so that \( |z - \zeta\rangle\langle z + \zeta| = \exp \{+\zeta^* - \zeta \zeta^* \} \hat{D}(-\zeta)|z\rangle\langle z|\hat{D}^\dagger(\zeta) \). Replacing this relation in (27) we obtain

\[
\hat{B} = \hat{D}(-\zeta) \int \frac{d^2z}{\pi} e^{-\zeta^*z} \langle z|e^{\zeta^*} \hat{D}^\dagger(\zeta)\]

\[
= e^{2\zeta^*z} \hat{D}(-\zeta) \int \frac{d^2z}{\pi} e^{-2\zeta^*z} \langle z|e^{2\zeta^*} \hat{D}^\dagger(\zeta)\]

\[
= e^{2\zeta^*z} \hat{D}(-\zeta)e^{-2\zeta^*\hat{a}} \left\{ \int \frac{d^2z}{\pi} \langle z|\langle z| \right\} e^{2\zeta^*\hat{a}} \hat{D}^\dagger(\zeta)\]

\[
= e^{2\zeta^*z} \hat{D}(-\zeta)e^{-2\zeta^*\hat{a}} 2\zeta^*\hat{a} \hat{D}^\dagger(\zeta) = \hat{D}(-\zeta)\hat{D}(2\zeta)\hat{D}(-\zeta) = \hat{I} ,
\]

where we used \( \hat{D}^\dagger(\zeta) = \hat{D}(-\zeta) \) and the Baker-Hausdorff formula. Thus we can write our main result:

\[
\hat{I} = \int \frac{d^2z}{\pi} \frac{|z - \zeta\rangle\langle z + \zeta|}{\langle z + \zeta|z - \zeta \rangle} ,
\]

which is a strong operator identity by the very nature of its derivation. The reader might want to check that the inner products (7) and (9), for instance, are correctly
Figure 2. Contour plots of the real part of the coefficients in Eq. (38) for \( n = 4 \) as functions of the dimensionless phase-space coordinates \( Q \equiv q/b \) and \( P \equiv bp/\hbar \). In (a) \( \zeta = 0 \) and in (b) \( \zeta = (1 + i)/2\sqrt{2} \).

reproduced by:

\[
\langle x | w \rangle = \int \frac{d^2 z}{\pi} \frac{\langle x | z - \zeta \rangle \langle z + \zeta | w \rangle}{\langle z + \zeta | z - \zeta \rangle}, \tag{33}
\]

\[
\langle w | w' \rangle = \int \frac{d^2 z}{\pi} \frac{\langle w | z - \zeta \rangle \langle z + \zeta | w' \rangle}{\langle z + \zeta | z - \zeta \rangle}, \tag{34}
\]

where \( |w\rangle \) and \( |w'\rangle \) are arbitrary coherent states, and also

\[
\langle x | p \rangle = \int \frac{d^2 z}{\pi} \frac{\langle x | z - \zeta \rangle \langle z + \zeta | p \rangle}{\langle z + \zeta | z - \zeta \rangle} = \frac{1}{\sqrt{2\pi \hbar}} \exp \left\{ \frac{i}{\hbar} px \right\}, \tag{35}
\]

with \( |x\rangle \) and \( |p\rangle \) being the position and momentum eigenstates, respectively. In these expressions no vestige of \( \zeta \) is left in the final results, as it should be. Therefore, an arbitrary ket \( |\psi\rangle \) can be written as:

\[
|\psi\rangle = \int \frac{d^2 z}{\pi} \frac{\langle z + \zeta | \psi \rangle}{\langle z + \zeta | z - \zeta \rangle} |z - \zeta\rangle. \tag{36}
\]

In fact (32) can be expressed, via a global phase-space coordinate change, as \( \hat{I} = \int \frac{d^2 z}{\pi} \frac{|z\rangle \langle z + 2\zeta|}{\langle z + 2\zeta | z \rangle} \), implying

\[
|\psi\rangle = \int \frac{d^2 z}{\pi} \frac{[\langle z | \psi \rangle] |z\rangle}{\langle z + 2\zeta | z \rangle} \equiv \int \frac{d^2 z}{\pi} \left[ \frac{\langle z + 2\zeta | \psi \rangle}{\langle z + 2\zeta | z \rangle} \right] |z\rangle, \tag{37}
\]

which makes the ambiguity in the coherent state representation absolutely evident. In the left hand side equality we used the standard closure relation (11) and in the right hand side we applied (32). The same state is decomposed, with distinct coefficients, in terms of the same set \( \{|z\rangle\} \). To better illustrate this point let us take a Fock state \( |\psi\rangle = |n\rangle \). The coefficients between square brackets in (37) become

\[
\langle z | n \rangle = \frac{1}{\sqrt{n!}} (z^*)^n e^{-|z|^2/2}, \quad \frac{\langle z + 2\zeta | n \rangle}{\langle z + 2\zeta | z \rangle} = \frac{1}{\sqrt{n!}} (z^* + 2\zeta^*)^n e^{-|z|^2/2 - 2\zeta^* z}. \tag{38}
\]
In figure 2 we show a phase-space plot of the real part of these expressions for \( n = 4 \) and \( \zeta = 0 \) (a), \( \zeta = (1 + i)/2\sqrt{2} \) (b).

6. Path integrals and weak energy values

In this section we apply identity (32) to build a phase-space path integral in which the role of the Hamiltonian is played by a weak energy value. Below we give the analogous of Klauder’s first form of the path integral [12]. We intend to evaluate the propagator

\[
K(z', z'', T) = \langle z''| \exp\left\{ -\frac{i\hat{T}\hat{H}}{\hbar} \right\} |z'\rangle,
\]

where \( \hat{H} \) is the Hamiltonian operator. Defining \( z'' \equiv z_{N+1} + \zeta_{N+1}, \ z' \equiv z_0 - \zeta_0 \), and \( \tau \equiv T/(N + 1) \) one can write

\[
K(z', z'', T) = \lim_{N\to\infty} \langle z_{N+1} + \zeta_{N+1}|(\hat{I} - i\tau\hat{H}/\hbar)|z_0 - \zeta_0\rangle.
\]

By inserting \( N \) unit operators (32) between the products we obtain

\[
K(z', z'', T) = \lim_{N\to\infty} \int \frac{d^2z_1}{\pi} \ldots \frac{d^2z_N}{\pi} \frac{\langle z_{N+1} + \zeta_{N+1}|(\hat{I} - i\tau\hat{H}/\hbar)|z_N - \zeta_N\rangle}{\langle z_j + \zeta_j|z_j - \zeta_j\rangle} \ldots \frac{\langle z_2 + \zeta_2|(\hat{I} - i\tau\hat{H}/\hbar)|z_1 - \zeta_1\rangle}{\langle z_1 + \zeta_1|z_1 - \zeta_1\rangle} \langle z_1 + \zeta_1|(\hat{I} - i\tau\hat{H}/\hbar)|z_0 - \zeta_0\rangle.
\]

We, thus, get an inconvenient asymmetry, since there is no \( \langle z_0 + \zeta_0|z_0 - \zeta_0\rangle \) in the denominator of the last term. This difficulty can be circumvented at the cost of an extra constraint, namely, \( \zeta_0 = 0 \), implying \( z_0 = z' \). In the continuum \( \zeta \) becomes a function of time, so that the previous condition reads \( \zeta(0) = 0 \). With this, it is harmless to write

\[
K(z', z'', T) = \lim_{N\to\infty} \int \prod_{n=0}^{N} \frac{\langle z_{n+1} + \zeta_{n+1}|(\hat{I} - i\tau\hat{H}/\hbar)|z_n - \zeta_n\rangle}{\langle z_n + \zeta_n|z_n - \zeta_n\rangle} \prod_{n=1}^{N} \frac{d^2z_n}{\pi}.
\]

Let us recast the numerator in this expression as

\[
\langle z_{n+1} + \zeta_{n+1}|z_n - \zeta_n\rangle - i\tau/\hbar \langle z_{n+1} + \zeta_{n+1}|\hat{H}|z_n - \zeta_n\rangle = \langle z_{n+1} + \zeta_{n+1}|z_n - \zeta_n\rangle \left( 1 - \frac{i\tau}{\hbar} \hat{H}_n \right) = \langle z_{n+1} + \zeta_{n+1}|z_n - \zeta_n\rangle \exp\left\{ -\frac{i\tau}{\hbar} \hat{H}_n \right\},
\]

where

\[
\hat{H}_n = \frac{\langle z_{n+1} + \zeta_{n+1}|\hat{H}|z_n - \zeta_n\rangle}{\langle z_{n+1} + \zeta_{n+1}|z_n - \zeta_n\rangle}.
\]
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Therefore

\[ K(z', z'', T) = \lim_{N \to \infty} \int \prod_{n=0}^{N} F_n \exp \left\{ -\frac{iT}{\hbar} \mathcal{H}_n \right\} \prod_{n=1}^{N} \frac{q^2 z_n}{\pi}, \]

with

\[ F_n = \frac{\langle z_{n+1} + \zeta_{n+1} | z_n - \zeta_n \rangle}{\langle z_n + \zeta_n | z_n - \zeta_n \rangle}. \]

Expression (45) represents a valid discrete version of a phase-space path integral. Typically, the paths that enter in the evaluation of (45) are nowhere continuous. However, it is helpful, although not rigorously justifiable, to imagine the paths to be continuous and differentiable and take the limit \( N \to \infty \) before proceeding to the integrations. This assumption becomes more reasonable in the semiclassical regime, since in this limit we expect that the contributing paths are in the vicinity of the classical (continuous) trajectory. The key point is that, in this case, one can write \( z_{n+1} + \zeta_{n+1} \equiv z_n + \zeta_n + \delta z_n + \delta \zeta_n \), where \( |\delta z_n + \delta \zeta_n| \to 0 \) for \( \tau \to 0 \). To first order in \( \delta z_n \) and \( \delta \zeta_n \) we get

\[ F_n = \exp \left\{ -\frac{1}{2} (\delta z_n + \delta \zeta_n)^*(z_n + \zeta_n) \right\} + \frac{1}{2} (\delta z_n + \delta \zeta_n)(z_n + \zeta_n) + (\delta z_n + \delta \zeta_n)^*(z_n - \zeta_n) \].

Exchanging the ordering of integrations and products and taking the limit \( N \to \infty \), we get

\[ K(z', z'', T) = \int \exp \left\{ \int_{0}^{T} dt F(t) - \frac{i}{\hbar} \int_{0}^{T} dt \mathcal{H}_\zeta \right\} Dz, \]

where \( Dz \equiv \lim_{N \to \infty} \prod_{n=1}^{N} \frac{d^2 z_n}{\pi} \), and

\[ \mathcal{H}_\zeta = \frac{\langle z + \zeta | \hat{H} | z - \zeta \rangle}{\langle z + \zeta | z - \zeta \rangle} \]

is the continuous counterpart of (14). The discrete quantity \( F_n \) becomes

\[ F(t) = -\frac{1}{2} \frac{d}{dt} |z + \zeta|^2 + (z - \zeta) \frac{d}{dt} (z + \zeta)^* \]

\[ = -\frac{d}{dt} (\zeta (z^* + \zeta^*)) + \frac{1}{2} (z - \zeta) \frac{d}{dt} (z + \zeta)^* - \frac{1}{2} (z + \zeta)^* \frac{d}{dt} (z - \zeta). \]

Finally one can write the formal expression for the path integral as

\[ K(z', z'', T) = \int \exp \left\{ -\zeta (T)[z''^* + \zeta^* (T)] + \frac{i}{\hbar} S_\zeta \right\} Dz, \]

where the first expression in the argument of the exponential is a surface term for which we already employed the condition \( \zeta (0) = 0 \). The last term is a generalized action

\[ S_\zeta = \int_{0}^{T} \left[ \frac{i\hbar}{2} (z + \zeta)^*(\dot{z} - \dot{\zeta}) - \frac{i\hbar}{2} (z - \zeta)(\dot{z} + \dot{\zeta})^* - \mathcal{H}_\zeta \right] dt, \]

where the dot denotes time derivative. For \( \zeta \equiv 0 \) the surface term vanishes and we get

\[ S_0 = \int_{0}^{T} dt [i\hbar (z^* \dot{z} - z \dot{z}^*) / 2 - \mathcal{H}_0] = \int_{0}^{T} dt [(pq - qp) / 2 - \mathcal{H}_0], \quad \mathcal{H}_0 = \langle z | \hat{H} | z \rangle, \]

as expected.
While $H_0 = \langle z|\hat{H}|z \rangle$ is a real function of the phase-space coordinates $q$ and $p$, the weak value $\mathcal{H}_\zeta$ is, in general, complex valued. This might seem a strong disadvantage of expression (51), but, in fact, it is not. The functions $H_0$ and $\mathcal{H}_\zeta$ fully assume the role of Hamiltonians into classical equations of motion only in the semiclassical limit. It is well known, however, that in this regime, even for $H_0$ the classical trajectories are, so to speak, overloaded with boundary conditions $[z^\ast(0) = z'^\ast$ and $z(T) = z'\rangle$, which can be satisfied only by extending both $q(t)$ and $p(t)$ to the complex plane. In this context, a complex function as the effective Hamiltonian is fairly natural.

### 6.1. Weak values in the quasi-classical domain

It is perhaps worth to note that (49) is a weak energy value as originally defined by Aharonov, Albert, and Vaidman [16]:

$$\mathcal{H}_{\text{weak}} = \frac{\langle \psi_f|\hat{H}|\psi_0 \rangle}{\langle \psi_f|\psi_0 \rangle},$$

for $\langle \psi_f|\psi_0 \rangle \neq 0$. In fact, in addition, we should have $\langle \psi_f|\psi_0 \rangle \approx 1$, which would demand $|\zeta| \ll 1$ in (49). Thus, in this regime, the complex number $\mathcal{H}_\zeta$, is a weak value of energy related to the states $|\psi_0 \rangle = |z - \zeta \rangle$ and $|\psi_f \rangle = |z + \zeta \rangle$. For small $|\zeta|$ one can write

$$\mathcal{H}_\zeta = \frac{\langle z|\hat{D}^\dagger(\zeta)\hat{H}\hat{D}(-\zeta)|z \rangle}{\langle z|\hat{D}^\dagger(\zeta)\hat{D}(-\zeta)|z \rangle} \approx \mathcal{H}_0 - \zeta \langle z|\{\hat{H}, \hat{a}^\dagger\}|z \rangle + \zeta^* \langle z|\{\hat{H}, \hat{a}\}|z \rangle \approx \mathcal{H}_0 - \zeta \langle z|\{\hat{H}, \hat{a}^\dagger\}|z \rangle + \zeta^* \langle z|\{\hat{H}, \hat{a}\}|z \rangle + 2(\zeta^* - \zeta^*|z \rangle \mathcal{H}_0,$$

where $\{, \}$ stands for the anticommutator, $\mathcal{H}_0 = \langle z|\hat{H}|z \rangle$ and we used $\hat{D}(-\zeta) \approx \hat{I} - \zeta \hat{a}^\dagger + \zeta \hat{a}$. After reordering operators, the previous expression can be written as

$$\mathcal{H}_\zeta = \mathcal{H}_0 + \zeta^* \langle z|\{\hat{a}, \hat{H}\}|z \rangle + \zeta \langle z|\{\hat{a}^\dagger, \hat{H}\}|z \rangle + O(|\zeta|^2).$$

Here we must be careful in handling the expectation values by noting that $\langle z(t)|[\hat{a}, \hat{H}]|z(t)\rangle \neq i\hbar \dot{z}(t)$, since $|z(t)\rangle$ is not, in general, a solution of the time-dependent Schrödinger equation. Nonetheless, when the system scales (size, energy, etc) are such that $S_{\text{class}} \gg \hbar$, given that $|\psi(0)\rangle$ is a coherent state, it will remain so for a time which is longer for larger ratios $S_{\text{class}}/\hbar$, with the center of the packet following, to first order, the classical trajectory. Therefore, in this quasi-classical regime $|\psi(t)\rangle \approx |z_{\text{class}}(t)\rangle$. It is also a well known result that $\mathcal{H}_0 = H_{\text{class}} + O(\hbar)$ [9, 15]. In this limit one can write the quasi-classical weak energy value as

$$\mathcal{H}_\zeta \approx H_{\text{class}} + i\hbar(\zeta^* \dot{z}_{\text{class}} + \zeta \dot{z}_{\text{class}}^*) = H_{\text{class}} + i\left[\alpha X \dot{q}_{\text{class}} + \frac{\Pi_{\text{class}}}{\alpha} \right].$$

with $\zeta \equiv X/\sqrt{2b} + ib\Pi/\sqrt{2\hbar}$ and recalling that $b = \sqrt{\hbar/\alpha}$, where $\alpha$ is a constant with dimension of mass/time (for the harmonic oscillator $\alpha = m\omega$). Therefore, the real part of the quasi-classical weak energy is the Hamiltonian itself, while the imaginary part (first order in $|\zeta|$ and zeroth order in $\hbar$) depends on the tangent field in phase space. By replacing this into (52), we get, with a slightly abusive language, the associated weak action integral.
7. Discussion and Conclusion

About ninety years after Schrödinger discovered coherent states as quasi-classical minimum uncertainty wave functions \[17\, 18\], they can still offer us some surprise. The structure behind overcomplete frames is richer if they fulfill the mutual non-orthogonality property. Under this condition we showed that, at least in some cases, it is possible to represent the closure relation via unlike outer products:

\[
\hat{I} \propto \int \text{d}V \frac{|v'angle \langle v|}{\langle v|v'\rangle},
\]

where \(\text{d}V\) is a volume element \((\text{d}\mu = \langle v|v'\rangle^{-1} \text{d}V)\). The fact that \(|v\rangle\) has a non-zero projection onto every \(|v'\rangle\) in the set, enlarges the notion of component or coefficient of a vector, such that \(\langle v|\psi \rangle\) can be made the component associated to \(|v'\rangle\), provided that this is accompanied by a suitable correction by an amplifying measure, \(\langle v|v'\rangle^{-1}\), where the amplification \(1 \leq |\langle v|v'\rangle|^{-1} < \infty\) is larger for larger Euclidean distances \(||v| - |v'\||\).

It is curious to realize that the appearance of this kind of weighting factors in quantum mechanical integrals is not unusual, although, sometimes concealed. As an example consider the Weyl symbol of an arbitrary operator \(\hat{A}\) \[19\], given by

\[
A_W = \int \text{d}x \langle q + x/2|\hat{A}|q - x/2\rangle e^{-ipx/\hbar},
\]

and note that it can be expressed as

\[
A_W = \frac{1}{2\pi\hbar} \int \text{d}x \frac{\langle q + x/2|\hat{A}|q - x/2\rangle}{\langle q + x/2|p\rangle\langle p|q - x/2\rangle}.
\]

Another case, belonging to a slightly different category, is the recently defined dual representation to the Bargmann function \(\psi(z^*)\) \[20\, 21\], that has found some application in semiclassical physics \[20\, 22\] and in quantum gravity \[23\, 24\]. It is given by

\[
f_\psi(w) = \int_\gamma \text{d}z^* \psi(z^*) e^{-z^*w} = \int_\gamma \text{d}z^* \frac{\langle z|\psi \rangle}{\langle z|w\rangle},
\]

where \(\gamma\) is a curve in the complex plane. The previous definition can be seen as an extension of a Fourier transform connecting the wave function \(\psi(x) = \langle x|\psi \rangle\) to the momentum representation

\[
\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \text{d}x \psi(x)e^{-ixp/\hbar} = \frac{1}{\sqrt{2\pi\hbar}} \int \text{d}x \frac{\langle x|\psi \rangle}{\langle x|p\rangle}.
\]

These transformations can be mnemonically seen as though the bra’s \(|z|\) in (59) and \(|x|\) in (60) take part in some sort of cancelation.

The question raised about operators of the form (26), or more specifically, on which further functions \(f(z)\) and \(g(z)\), if any, would lead to resolutions of unity, seems to be one worth of some thought. The failure of (24) to have a valid counterpart in the Hilbert space seems to indicate that analyticity is a necessary ingredient, that is, \(\exp\{\frac{|z|^2}{2}\langle \psi|f(z)\rangle\} \quad \text{and} \quad \exp\{\frac{|z|^2}{2}\langle g(z)|\psi \rangle\} \quad \text{should be analytic functions of} \quad z \quad \text{and} \quad z^*,\) respectively.
As for the alternative form of the coherent-state path integral we presented here, it is hoped that it may be useful in, e. g., attenuating root-search problems in the semiclassical dynamics [25, 26], since we have the free parameter $\zeta$ to deal with mixed ($t = 0$ and $t = T$) boundary conditions. A completely analogous procedure can be adopted to derive yet another form of the path integral starting from (4). However, this does not seem to bring any relevant new information, unless a specific application arises.

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Appendix A. Alternative derivation of (32)

We start, as usual in many texts, by employing Fock closure relations in (27)

$$\hat{I} \hat{B} \hat{I} = \hat{B} = \sum_n \sum_k \int \frac{d^2z}{\pi} e^{-2|\zeta|^2 - \zeta^* z + \zeta z^*} \langle n | z - \zeta \rangle \langle z + \zeta | k \rangle \langle k |$$

$$= \sum_n \sum_k \langle n | k \rangle \int \frac{d^2z}{\pi} e^{-2|\zeta|^2 - \zeta^* z + \zeta z^*} \langle n | z - \zeta \rangle \langle z + \zeta | k \rangle$$

$$= \sum_n \sum_k \langle n | k | B_{nk} \rangle . \quad (A.1)$$

We, thus, focus on the quantity

$$B_{nk} = \frac{1}{\sqrt{n!k!}} \int \frac{d^2z}{\pi} (z - C)^n (z^* + \zeta^*)^k e^{-2|\zeta|^2 - \zeta^* z + \zeta z^* - \frac{1}{2}|z - \zeta|^2 - \frac{1}{2}|z^* + \zeta|^2}$$

$$= \frac{1}{\sqrt{n!k!}} \int \frac{d^2z}{\pi} (z - C)^n (z^* + \zeta^*)^k e^{-(z-C)(z^*+\zeta^*)} . \quad (A.2)$$

Suppose initially that $k \leq n$. The integral can be written as

$$B_{nk} = \frac{1}{\sqrt{n!k!}} \int \frac{d^2z}{\pi} (z - C)^n (z^* + \zeta^*)^k e^{-\alpha(z-C)(z^*+\zeta^*)} |_{\alpha = 1}$$

$$= \frac{(-1)^k}{\sqrt{n!k!}} \int \frac{d^2z}{\pi} (z - C)^n (z^* + \zeta^*)^k e^{-\alpha(z-C)(z^*+\zeta^*)} |_{\alpha = 1}$$

$$= \frac{(-1)^k}{\sqrt{n!k!}} \frac{\partial^{n-k}}{\partial \alpha^{n-k} \partial \zeta^{*n-k}} \left( \frac{1}{\alpha} \right)^{n-k} \int \frac{d^2z}{\pi} e^{-\alpha(z-C)(z^*+\zeta^*)} |_{\alpha = 1} . \quad (A.3)$$

The Gaussian integration leads to

$$B_{nk} = \frac{(-1)^n}{\sqrt{n!k!}} \frac{\partial^{n-k}}{\partial \alpha^{k} \partial \zeta^{*n-k}} \left( \frac{1}{\alpha} \right)^{n-k} (\alpha^{k-n-1}) |_{\alpha = 1} . \quad (A.4)$$
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Since the result does not depend on $\zeta^*$, if $n \neq k$, then $B_{nk} = 0$. For $n = k$ we get

$$B_{nn} = \frac{(-1)^n}{n!} \frac{\partial^k}{\partial \alpha^k} \left( \frac{1}{\alpha} \right)_{\alpha=1} = 1 ,$$

therefore, $B_{nk} = \delta_{nk}$. The case $k > n$ can be dealt with in a completely analogous way. Thus, we can write

$$\hat{B} = \sum_n \sum_k |n\rangle \langle k| B_{nk} = \sum_n |n\rangle \langle n| = \hat{I} ,$$

in the weak sense that $\langle \psi| \hat{B}| \phi \rangle = \langle \psi| \phi \rangle$ for arbitrary kets $|\psi\rangle$ and $|\phi\rangle$. 
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