Almost–conservation of the Euler Integral closely to a simple collision

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Abstract

In the three–body problem, after the motions of the fastest body have been averaged out, closely to a simple collision which excludes the most massive one, the level sets of a certain function called Euler Integral vary less than expected in a short time. The proof relies on a normal form theory for fast driven systems, combined with the so–called renormalizable integrability of the Newtonian potential.

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1 Description of the results

We consider a \((n + m + 1)\)-dimensional vector–field \(N\) which, expressed in local coordinates \((I, y, \varphi) \in P = I \times Y \times T^m\) (where \(I \subset \mathbb{R}^n, Y \subset \mathbb{R}\) are open and connected; \(T = \mathbb{R}/(2\pi \mathbb{Z})\) is the standard torus), has the form

\[
N(I, y) = v(I, y)\partial_y + \omega(I, y)\partial_\varphi .
\]

The motion equations of \(N\)

\[
\begin{align*}
\dot{I} &= 0 \\
\dot{y} &= v(I, y) \\
\dot{\varphi} &= \omega(I, y)
\end{align*}
\]

(1)

can be integrated in cascade:

\[
\begin{align*}
I(t) &= I_0 \\
y(t) &= \eta(I_0, t) \\
\varphi(t) &= \varphi_0 + \int_{t_0}^t \omega(I_0, \eta(I_0, t'))dt'
\end{align*}
\]

(2)

with \(\eta(I_0, \cdot)\) being the general solution of the one–dimensional equation \(\dot{y}(t) = v(I_0, y)\). This formula shows that along the solutions of \(N\) the coordinates \(I\) remains constant, while the motion of the coordinates \(\varphi\) in (2) is strongly coupled with the motion of the “driving” coordinate \(y\). It is to be remarked that, in general, \(y\) moves fast (whence the name fast driven system), so such picture of the motion is meaningful only along the time that \(y\) does not leave a given, initially prefixed, domain \(W\). At this respect, it is convenient to define the exit time from \(W\) under \(N\), and denote it as \(t_{ex}^{N,W}\), as as the first time that the solution \(q(t) = (I(t), y(t), \varphi(t))\) of \(N\) in (2) leaves \(W\).

Let us now replace the vector–field \(N(I, y)\) with a new vector–field of the form

\[
X(I, y, \varphi) = N(I, y) + P(I, y, \varphi) .
\]

(3)

where the “perturbation”

\[
P = P_1(I, y, \varphi)dI + P_2(I, y, \varphi)dy + P_3(I, y, \varphi)d\varphi
\]

is more general but, in some sense, “small” (see the next section for precise statements). Let \(t_{ex}^{X,W}\) be the exit time from \(W\) under \(X\), and let \(\epsilon\) be an upper bound for \(|P_1|\) on \(W\). Then, one has a linear–in–time a–priori bound for the variations of \(I\), as follows

\[
|I(t) - I(0)| \leq \epsilon t \quad \forall \ t : |t| < t_{ex}^{X,W}
\]

(4)

In this paper we address the problem of improving the bound (4). To the readers who are familiar with Kolmogorov–Arnold–Moser (KAM) or Nekhorossev theories, this kind of problems is well known, as well as its solution and generalisations \([3, 31, 37, 20]\). Roughly, KAM and Nekhorossev correspond to take \(v \equiv 0\) in (1). It is well known that KAM theory provides more refined bounds (without \(t\) at right hand side of (4)) for the major part of orbits and for all times, while Nekhorossev refines the bound in (4), with \(e^{-t/\epsilon^a}\) replacing \(\epsilon\) and \(e^{t/\epsilon^b}\) replacing \(t_{ex}^{X,W}\), with suitable \(a, b > 0\), for all orbits. It is also worth recalling that the conditions for applying KAM and Nekhorossev are, respectively, that \(\omega\) is “strongly non–resonant” (e.g., “Diophantine”), or “steep” (e.g., it is...
the gradient of a convex function). In this paper we deal with the case that \( v(I, y) \) is a “general” function. In this situation, by no means the motion of \( y \) can be “slown down” (which indeed may be fast already in the unperturbed case yet), or, equivalently, \( t_{ex}^{X,W} \) be increased. However, motivated by an application to celestial mechanics described below, we are interested with replacing \( \epsilon \) in (4) with a smaller number. We shall prove the following result. It is to be remarked that no assumption on \( \omega \) is needed (which might vanish as well).

**Theorem A** let \( X = N + P \) be real–analytic. Under suitable “smallness” assumptions, the bound in (4) holds with \( e^{-t/\epsilon^a} \) replacing \( \epsilon \), with a suitable \( a > 0 \).

A quantitative statement of Theorem A (needed to our purposes) is given in Proposition 2.1 below. Moreover, in view of our application, we also discuss a version to the case when analyticity in \( \varphi \) fails; this is Proposition 2.6.

To describe how we shall use (the most appropriate edition of) Theorem A, we make a digression on the three–body problem and the renormalizable integrability of the simply averaged Newtonian potential discovered in [33].

The Hamiltonian governing the motions of a three–body problem in the plane where the masses are 1, \( \mu \) and \( \kappa \), is (see, e.g., [15])

\[
H_{3b} = \frac{1 + \frac{1}{\kappa}}{2} \|y\|^2 - \frac{\kappa}{\|x\|} + \left( 1 + \frac{1}{\mu} \right) \frac{\|y\|^2}{2} - \frac{\mu}{\|x\|} \|x - x'\| + y \cdot y'
\]

where \( y, y' \in \mathbb{R}^2; x, x' \in \mathbb{R}^2 \), with \( x \neq 0 \neq x' \) and \( x \neq x' \), are impulse–position coordinates; \( \|\cdot\| \) denotes the Euclidean norm and the gravity constant has been chosen equal to 1, by a proper choice of the units system. We rescale \( (y', y) \rightarrow \frac{\kappa^2}{1 + \kappa} (y', y) \), \( (x', x) \rightarrow \frac{1 + \kappa}{\kappa^2} (x', x) \) multiply the Hamiltonian by \( \frac{1 + \kappa}{\kappa^2} \) and obtain

\[
H_{3b}(y', y, x', x) = \frac{\|y\|^2}{2} - \frac{1}{\|x\|} + \delta \left( \frac{\|y\|^2}{2} - \frac{\alpha}{\|x - x'\|} - \frac{\beta}{\|x'\|} \right) + \gamma y \cdot y'
\]

with

\[
\alpha := \frac{\mu^2(1 + \kappa)}{\kappa(1 + \mu)}, \quad \beta := \frac{\mu^2(1 + \kappa)}{\kappa^2(1 + \mu)}, \quad \gamma := \frac{\kappa}{1 + \kappa}, \quad \delta := \frac{\kappa(1 + \mu)}{\mu(1 + \kappa)}
\]

In order to simplify the analysis a little bit, we introduce a main assumption. The Hamiltonian \( H_{3b} \) in (5) includes the Keplerian term

\[
J_0 := \frac{\|y\|^2}{2} - \frac{1}{\|x\|} = -\frac{1}{2\Lambda^2}.
\]

We assume that \( J_0 \) is a “leading” term in such Hamiltonians. By averaging theory, this assumption allows us to replace (at the cost of a small error) \( H_{3b} \) with their \( \ell \)–average

\[
\overline{H} = -\frac{1}{2\Lambda^2} + \delta H
\]

where \( \ell \) is the mean anomaly associated to (65), and\(^1\)

\[
H := \frac{\|y\|^2}{2} - \alpha U - \frac{\beta}{\|x'\|}
\]

\(^1\)Remark that \( y(\ell) \) has vanishing \( \ell \)–average so that the last term in (5) does not survive.
with
\[ U := \frac{1}{2\pi} \int_{0}^{2\pi} \frac{d\ell}{\| \mathbf{x}' - \mathbf{x}(\ell) \|} \]
being the “simply\(^2\) averaged Newtonian potential”. The conditions that allow to switch to the averaged Hamiltonian (8) are well known and hence are not discussed here.

Neglecting the first term in (7), which is an inessential additive constant for \( H_i \) and reabsorbing the constant \( \delta \) with a time change, we are led to look at the Hamiltonian \( H \) in (8). We denote as \( E \) the Keplerian ellipse generated by Hamiltonian (65), for negative values of the energy. Without loss of generality, assume \( E \) is not a circle and \( \Lambda = 1 \). Remark that, as the mean anomaly \( \ell \) is averaged out, we loose any information concerning the position of \( \mathbf{x} \) on \( E \), so we shall only need two couples of coordinates for determining the shape of \( E \) and the vectors \( \mathbf{y}', \mathbf{x}' \). These are:

- the “Delaunay couple” \((G, g)\), where \( G \) is the Euclidean length of \( \mathbf{x} \times \mathbf{y} \) and \( g \) detects the perihelion. We remark that \( g \) is measured with respect to \( \mathbf{x}' \) (instead of with respect to a fixed direction), as the SO(2) reduction we use fixes a rotating frame which moves with \( \mathbf{x}' \) (compare the formulae in (64));
- the “radial–polar couple” \((R, r)\), where \( r := \| \mathbf{x}' \| \) and \( R := \mathbf{y}' \cdot \mathbf{x}' / \| \mathbf{x}' \| \).

Using the coordinates above, the Hamiltonian in (8) becomes
\[ H(R, G, r, g) = \frac{R^2}{2} + \frac{(C - G)^2}{2r^2} - \alpha U(r, G, g) - \frac{\beta}{r} \]
where \( C = \| \mathbf{x} \times \mathbf{y} + \mathbf{x}' \times \mathbf{y}' \| \) is the total angular momentum of the system, and we have assumed \( \mathbf{x} \times \mathbf{y} \parallel \mathbf{x}' \times \mathbf{y}' \).

The Hamiltonian (9) is now wearing 2 degrees–of–freedom, the minimum possible. As the energy is conserved, its motions evolve on the 3–dimensional manifolds \( \{ H = c \} \), labeled by the constant value of \( H \). On each of such manifolds the evolution is associated to 3–dimensional vector–field \( X_c \). As an example, one can take \( X_c \) to be the velocity field of the triple \((r, G, g)\), even though different coordinates will be chosen below. To describe the motions we are looking for, we recall a remarkable property of the function \( U \), pointed out in [33]. First of all, \( U \) is a function of \((r, G, g)\) only, so it is integrable. In addition, there exists a function \( F \) of two arguments such that
\[ U(r, G, g) = F(E(r, G, g), r) \]
where
\[ E(r, G, g) = G^2 + r\sqrt{1 - G^2 \cos g} \]

The function \( E \) is referred to as Euler Integral, and we express (10) by saying that \( U \) is renormalizable integrability via the Euler Integral. This implies that the level sets of \( E \)
\[ G^2 + r\sqrt{1 - G^2 \cos g} = \mathcal{E} \]
are also level sets of \( U \). On the other hand, the phase portrait of (12) keeping \( r \) fixed is completely explicit and has been studied in [34]. We fix the strip \([-\pi, \pi] \times [-1, 1]\). For \( 0 < r < 1 \) or \( 1 < r < 2 \) it includes two minima \((\pm \pi, 0)\) on the \( g \)-axis; two symmetric maxima on the \( G \)-axis and one saddle point at \((0, 0)\). When \( r > 2 \) the saddle point disappears and \((0, 0)\) turns to be a maximum.

\(^2\)Here, “simply” is used as opposed to the more familiar “doubly” averaged Newtonian potential, most often encountered in the literature; e.g. [22, 15, 32, 12, 11].

\(^3\)We can do this as the Hamiltonian \( H_3 b \) rescale by a factor \( \beta^{-2} \) as \((y', y) \rightarrow \beta^{-1}(y', y) \) and \((x', x) \rightarrow \beta^2(x', x) \).
The phase portrait includes two separatrices in the range $0 < r < 1$ or $1 < r < 2$; one separatrix if $r > 2$. These are the level set $S_0(r)$ through the saddle, corresponding to $\mathcal{E} = r$, for $0 < r < 2$, and the level set $S_1(r) = \{ \mathcal{E} = 1 \}$, for any $r$. Rotational motions in between $S_0(r)$ and $S_1(r)$, do exist only for $0 < r < 1$. The minima and the maxima are surrounded by librational motions and different motions (librations about different equilibria or rotations) are separated by $S_0(r)$ and $S_1(r)$. All of this is represented in Figure 1. In Figure 2 the same level sets are drawn in the 3-dimensional space $(r, G, g)$. The spatial visualisation turns out to be useful in the paper, as $r$ moves under (9). We remark that, while $E$ is perfectly defined along the “separating manifold” $S_0 := \bigcup_{0 \leq r \leq 2} \{ S_0(r) \}$, $U$ is not so. Indeed, as

$$S_0(r) = \left\{ (G, g) : \quad G^2 + r \sqrt{1 - G^2} \cos g = r, \quad -1 \leq G \leq 1, \quad g \in \mathbb{T} \right\} \quad 0 \leq r \leq 2 \quad (13)$$

we have\(^4\) $U(r, G, g) = \infty$ for $(G, g) \in S_0(r)$, for all $0 \leq r \leq 2$.

\(^4\)Rewriting (13) as

$$r = \frac{G^2}{1 - \sqrt{1 - G^2} \cos g}$$

Figure 1: Sections, at $r$ fixed, of the level surfaces of $E$. (a): $0 < r < 1$; (b): $1 < r < 2$; (c): $r > 2$.

Figure 2: Logs of the level surfaces of $E$ in the space $(g, G, r)$. (a): $0 < r < 1$; (b): $1 < r < 2$; (c): $r > 2$. 
The natural question now raises whether any of the $E$-levels in Figure 2 is an “approximate” invariant manifold for the Hamiltonian $H$ in (9). In [35] and [13] a positive answer has been given for case $r > 2$, corresponding to panels (c) of Figures 1 and 2. In this paper, we want to focus on motions close to the manifold $\{S_0(r)\}$ with $r$ very close to 2 (corresponding to panels (b)) which, by the discussion above, are to be understood as “quasi-collisional”. This region of phase space is denoted as $C$.

To state our result, we denote as $r_s(A)$ the value of $r$ such that the area encircled by $S_0(r_s(A))$ is $A$. Then the set $\{\exists A : r = r_s(A)\}$ corresponds to $S_0$. We prove:

**Theorem B** Inside the region $C$ there exists an open neighbourhood $W$ such that along any motion with initial datum in $W$, for all $t$ with $|t| \leq t_{ex}^{X,W}$, the Euler integral $E$ afford variations which do not exceed $Ce^{-L/C}t$, provided that the initial value of $r$ is $e^{-L}$ away from $r_s(A)$, with $L > 0$ sufficiently large.

All the details of the proof of Theorem B are given in the technical sections below. Here we limit to mention that it is possible to choose a set of coordinates $(A, y, \psi)$, where $(A, \psi)$ are the action–angle coordinates of $E(r, \cdot, \cdot)$, while $y$ is diffeomorphic to $r$, such that the associated vector–field has the form in (3) with $n = m = 1$. The diffeomorphism $r \rightarrow y$ is a sort of Levi–Civita regularisation, introduced in order that $X_c$ keeps its regularity upon $S_0$.

Before switching to full proofs, we recall how the theme of collisions in $n$–body problems (with $n \geq 3$) has been treated so far. As the literature in the field is countless, by no means we claim completeness. In the late 1890s H. Poincaré [36] conjectured the existence of special solutions in the planar, circular, restricted three-body problem with one of the two primaries having small mass $\mu$, where the infinitesimal body describes an orbit close to two different Keplerian orbits about a large mass primary, glued so as to form a cusp, with the two arcs separated in time by a close encounter between the infinitesimal body and the small primary. These solutions were named by him second species solutions, and their existence has been next proved in [4, 5, 6, 7, 8, 23, 21].

In the early 1900s, J. Chazy classified all the possible final motions of the three-body problem, including the possibility of collisions [9]. The study was reconsidered in [1, 2]. After the advent of KAM theory, the existence of almost-collisional quasi-periodic orbits was proven [10, 14, 40]. The papers [38, 39, 16, 17, 24, 25, 26, 27] deal with rare occurrence of collisions or the existence of chaos in the proximity of collisions. Finally, in [19] it is proved that for the restricted circular planar 3-body problem there exists an open set in phase space of fixed measure, where the set of initial points which lead to collision is $O(\mu^\alpha)$ dense with some $0 < \alpha < 1$.

2 A Normal Form Lemma for fast driven systems

In the next Sections 2.1–2.4 we state and prove a Normal Form Lemma (NFL) for real–analytic systems. For the purpose of the paper, in Section 2.5 we generalise the result, allowing the dependence on the angular coordinate $\psi$ to be just $C^\infty$, rather than holomorphic. In all cases, we limit to the case $n = m = 1$. Generalisations to $n, m \geq 1$ are straightforward.

2.1 Weighted norms

Let us consider a 3-dimensional vector–field

$$(I, y, \psi) \in \mathbb{P}_{r, \sigma, s} := \mathbb{I}_r \times \mathbb{Y}_\sigma \times \mathbb{T}_s \rightarrow X = (X_1, X_2, X_3) \in \mathbb{C}^3$$

tells us that $(G, g) \in S_0(r)$ if and only if $x'$ occupies in the ellipse $E$ the position with true anomaly $\nu = \pi - g$. 


where \( I \subset \mathbb{R}, \forall \subset \mathbb{R} \) are open and connected; \( T = \mathbb{R}/(2\pi \mathbb{Z}) \), which has the form (3). As usual, if \( A \subset \mathbb{R} \) and \( r > 0 \), symbol \( A_r \) denotes the complex \( r \)-neighbourhood of \( A \):

\[
a_r := \bigcup_{x \in A} B_r(x)
\]

with \( B_r(x) \) the complex ball centred at \( x \) with radius \( r \). We assume each \( X_i \) to be holomorphic in \( \mathbb{P}_{r,\sigma,s} \), meaning it has a finite weighted norm defined below. If this holds, we simply write \( X \in \mathcal{O}^{3}_{r,\sigma,s} \).

For functions \( f : (I, y, \psi) \in I_r \times \forall \times \mathbb{T}_s \to \mathbb{C} \), we write \( f \in \mathcal{O}^{3}_{r,\sigma,s} \) if \( f \) is holomorphic in \( \mathbb{P}_{r,\sigma,s} \).

We let

\[
\|f\|_u := \sum_{k \in \mathbb{Z}} \sup_{I_r} \sup_{\forall} |f_k(I, y)| e^{|k|s}
\]

where

\[
f = \sum_{k \in \mathbb{Z}} f_k(I, y)e^{ik\psi}
\]

is the Fourier series associated to \( f \) relatively to the \( \psi \)-coordinate. For \( \psi \)-independent functions or vector–fields we simply write \( \| \cdot \|_{r,\sigma} \).

For vector–fields \( X : (I, y, \psi) \in I_r \times \forall \times \mathbb{T}_s \to X = (X_1, X_2, X_3) \in \mathbb{C}^3 \), we write \( X \in \mathcal{O}^{3}_{r,\sigma,s} \) if \( X_i \in \mathcal{O}^{3}_{r,\sigma,s} \) for \( i = 1, 2, 3 \). We define the weighted norms

\[
\|X\|_w := \sum_i w_i^{-1} \|X_i\|_u
\]

where \( w = (w_1, w_2, w_3) \in \mathbb{R}_+^3 \) are the weights. The weighted norm affords the following properties.

- Monotonicity:

\[
\|X\|_w \leq \|X\|_{u'}, \quad \|X\|_{u'}' \leq \|X\|_w \quad \forall \ u \leq u', \ w \leq w'
\]

where \( u \leq u' \) means \( u_i \leq u_i' \) for \( i = 1, 2, 3 \).

- Homogeneity:

\[
\|X\|_{a,w} = a^{-1} \|X\|_w \quad \forall \ a > 0
\]

### 2.2 The Normal Form Lemma

We shall prove the following result. Observe that the nature of the system does not give rise to any non–resonance condition or ultraviolet cut–off. We name Normal Form Lemma the following

**Proposition 2.1 (NFL)** Let \( u = (r, \sigma, s) \); \( X = N + P \in \mathcal{O}^{3}_{u} \) and let \( w = (\rho, \tau, t) \in \mathbb{R}_+^3 \). Put

\[
Q := 3 \text{diam}(\forall) \left\| \frac{1}{u} \right\|_{r,\sigma}
\]

and assume that for some \( n \in \mathbb{N}, s_2 \in \mathbb{R}_+ \), the following inequalities are satisfied:

\[
0 < \rho < \frac{r}{8}, \quad 0 < \tau < e^{-s_2} \frac{\sigma}{8}, \quad 0 < t < \frac{s}{10}
\]

\(^5\text{diam}(\mathcal{A}) \text{ denotes diameter of the set } \mathcal{A} \).
and

\[ \chi := \frac{\text{diam}(Y_\sigma)}{s_2} \left\| \frac{\partial v}{\partial \sigma} \right\|_{r,\sigma} \leq 1 \]  
(19)

\[ \theta_1 := 2e^{s_2}\text{diam}(Y_\sigma) \left\| \frac{\partial \psi}{\partial \sigma} \right\|_{r,\sigma} \leq 1 \]  
(20)

\[ \theta_2 := 4\text{diam}(Y_\sigma) \left\| \frac{\partial v}{\partial \tau} \right\|_{r,\sigma} \leq 1 \]

\[ \theta_3 := 8\text{diam}(Y_\sigma) \left\| \frac{\partial \psi}{\partial \tau} \right\|_{r,\sigma} \leq 1 \]

\[ \eta^2 := \max \left\{ \frac{\text{diam}(Y_\sigma)}{t} \left\| \frac{\omega}{v} \right\|_{r,\sigma}, 2^7e^{s_2}Q^2(\|P\|^w_u)^2 \right\} < \frac{1}{n}. \]  
(21)

Then, with

\[ u_* = (r_*, \sigma_*, s_*), \quad r_* := r - 8\rho, \quad \sigma_* = \sigma - 8e^{s_2}\tau, \quad s_* = s - 10t \]

there exists a real–analytic change of coordinates \( \Phi_* \) such that \( X_* := \Phi_*X \in \mathcal{O}_u^3 \) and \( X_* = N + P_* \), with

\[ \|P_*\|^w_u \leq 2^{-(n+1)}\|P\|^w_u. \]

### 2.3 The Step Lemma

We denote as

\[ e^{\mathcal{L}_Y} = \sum_{n \geq 0} \frac{\mathcal{L}_Y^n}{n!} \]

the Lie series associated to \( Y \), where, if

\[ [Y, X] = J_XY - J_YX, \quad (J_Z)_{ij} := \partial_jZ_i \]

denotes Lie brackets of two vector–fields,

\[ \mathcal{L}_Y := [Y, \cdot] \]

is the Lie operator.

**Proposition 2.2** Let \( X = N + P \in \mathcal{O}_u^3 \), with \( u = (r, \sigma, s) \), \( N \) as in (33). Assume

\[ \frac{\text{diam}(Y_\sigma)}{s_1} \left\| \frac{\omega}{v} \right\|_{r,\sigma} \leq 1, \quad \frac{\text{diam}(Y_\sigma)}{s_2} \left\| \frac{\partial v}{\partial \sigma} \right\|_{r,\sigma} \leq 1 \]  
(22)

and that \( P \) is so small that

\[ Q\|P\|^w_u < 1 \quad Q := 3\text{diam}(Y_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma}, \quad w = (\rho, \tau, t) \]  
(23)
Let $\rho_*, \tau_*, t_*$ be defined via
\[
\frac{1}{\rho_*} = \frac{1}{\rho} - \text{diam}(\mathcal{Y}_\sigma) \left\| \frac{\partial v}{v} \right\|_{r,\sigma} \left( \frac{1}{\tau} - \frac{e^{s_2} \text{diam}(\mathcal{Y}_\sigma)}{\tau} \right) \left\| \frac{\partial \omega}{v} \right\|_{r,\sigma} \frac{1}{l}
\]
\[
- \text{diam}(\mathcal{Y}_\sigma) \left( \left\| \frac{\partial \omega}{v} \right\|_{r,\sigma} + \frac{e^{s_2} \text{diam}(\mathcal{Y}_\sigma)}{\tau} \right) \left\| \frac{\partial v}{v} \right\|_{r,\sigma} \left\| \frac{\partial \omega}{v} \right\|_{r,\sigma} \frac{1}{l}
\]
\[
\frac{1}{\tau_*} = \frac{e^{-s_2}}{\tau} - \text{diam}(\mathcal{Y}_\sigma) \left\| \frac{\partial \omega}{v} \right\|_{r,\sigma} \frac{1}{l}
\]
\[
t_* = t
\]
and assume
\[
w_* = (\rho_*, \tau_*, t_*) \in \mathbb{R}_+^3, \quad u_* = (r - 2\rho_*, \sigma - 2\tau_*, s - 3s_1 - 2t_*) \in \mathbb{R}_+^3.
\]
Then there exists $Y \in \mathcal{O}_{u_* + w_*}^3$ such that $X_+ := e^{V} X \in \mathcal{O}_{u_*}^3$ and $X_+ = N + P_+$, with
\[
\begin{align*}
\left\| P_+ \right\|_{u_*}^w &\leq \frac{2Q \left( \left\| P \right\|_{u}^w \right)^2}{1 - Q \left\| P \right\|_{u}^w} \\
\end{align*}
\]
In the next section, we shall use Proposition 2.2 in the following “simplified” form.

**Proposition 2.3 (Step Lemma)** If (22), (23) and (25) are replaced with
\[
2e^{s_2} \text{diam}(\mathcal{Y}_\sigma) \left\| \frac{\partial \omega}{v} \right\|_{r,\sigma} \frac{\tau}{l} \leq 1
\]
\[
4 \text{diam}(\mathcal{Y}_\sigma) \left\| \frac{\partial v}{v} \right\|_{r,\sigma} \frac{\rho}{\tau} \leq 1
\]
\[
8 \text{diam}(\mathcal{Y}_\sigma) \left\| \frac{\partial \omega}{v} \right\|_{r,\sigma} \frac{\rho}{\tau} \leq 1
\]
\[
\text{diam}(\mathcal{Y}_\sigma) \left\| \omega \right\|_{v,\sigma} \leq 1, \quad \text{diam}(\mathcal{Y}_\sigma) \left\| \frac{\partial v}{v} \right\|_{r,\sigma} \leq 1
\]
\[
0 < \rho < \frac{r}{4}, \quad 0 < \tau < \frac{\sigma}{4} e^{-s_2}, \quad 0 < t < \frac{s}{5}
\]
\[
2Q \left\| P \right\|_{u}^w < 1
\]
then $X_+ = N + P_+ \in \mathcal{O}_{u_*}^3$ and
\[
\left\| P_+ \right\|_{u_*}^w \leq 8e^{s_2}Q(\left\| P \right\|_{u}^w)^2.
\]

with
\[
u_* := (r - 4\rho, \sigma - 4\tau e^{s_2}, s - 5t).
\]

**Proof** The inequality in (28) guarantees that one can take $s_1 = t$, while the inequalities in (27) and (29) imply
\[
\frac{1}{\rho_*} \geq \frac{1}{2\rho}, \quad \frac{1}{\tau_*} \geq \frac{e^{-s_2}}{2\tau}
\]
whence, as $t_* = t$,
\[
w_* \leq 2e^{s_2}w, \quad u_* \geq u_+ > 0.
\]
Then (31) is implied by (26), monotonicity and homogeneity (16)–(17), and the inequality in (30). \(\square\)
Fix \( y_0 \in Y: v, \omega: I \times Y \to \mathbb{R} \), with \( v \neq 0 \). We define, formally, the operators \( \mathcal{F}_{v,\omega} \) and \( \mathcal{G}_{v,\omega} \) as acting on functions \( g: I \times Y \times T \to \mathbb{R} \) as

\[
\mathcal{F}_{v,\omega}[g](I, y, \psi) := \int_{y_0}^y g(I, \eta, \psi) \frac{\omega(I, \eta')}{v(I, \eta)} d\eta \\
\mathcal{G}_{v,\omega}[g](I, y, \psi) := \int_{y_0}^y g(I, \eta, \psi) \frac{\omega(I, \eta')}{v(I, \eta)} e^{-\int_{y_0}^\eta \frac{\partial v(I, \eta'')}{v(I, \eta'')} d\eta'} d\eta
\]

(32)

Observe that, when existing, \( \mathcal{F}_{v,\omega}, \mathcal{G}_{v,\omega} \) send zero-average functions to zero-average functions. The existence \( \mathcal{F}_{v,\omega}, \mathcal{G}_{v,\omega} \) is established by the following

**Lemma 2.1** If inequalities (22) hold, then

\[
\mathcal{F}_{v,\omega}, \mathcal{G}_{v,\omega}: \mathcal{O}_{r,\sigma,s} \to \mathcal{O}_{r,\sigma,s-s_1}
\]

and

\[
\|\mathcal{F}_{v,\omega}[g]\|_{r,\sigma,s-s_1} \leq \text{diam}(Y_\sigma) \left\| \frac{\partial}{\partial v} \right\|_{r,\sigma,s}, \quad \|\mathcal{G}_{v,\omega}[g]\|_{r,\sigma,s-s_1} \leq e^{s_2} \text{diam}(Y_\sigma) \left\| \frac{\partial}{\partial v} \right\|_{r,\sigma,s}
\]

The proof of Lemma 2.1 is obvious from the definitions (32).

**Proposition 2.4** Let

\[
N = (0, v(I, y), \omega(I, y)), \quad Z = (Z_1(I, y, \psi), Z_2(I, y, \psi), Z_3(I, y, \psi))
\]

belong to \( \mathcal{O}^3_{r,\sigma,s} \) and assume (22). Then the “homological equation”

\[
\mathcal{L}_N[Y] = Z
\]

(34)

has a solution \( Y \in \mathcal{O}_{r,\sigma,s-3s_1} \) verifying

\[
\|Y\|_{r,\sigma,s-3s_1} \leq \text{diam}(Y_\sigma) \left\| \frac{\partial}{\partial v} \right\|_{r,\sigma,s} \|Z\|_{r,\sigma,s}
\]

(35)

with \( \rho_*, \tau_*, t_* \) as in (24).

**Proof** We expand \( Y_j \) and \( Z_j \) along the Fourier basis

\[
Y_j(I, y, \psi) = \sum_{k \in \mathbb{Z}} Y_{j,k}(I, y)e^{ik\psi}, \quad Z_j(I, y, \psi) = \sum_{k \in \mathbb{Z}} Z_{j,k}(I, y)e^{ik\psi}, \quad j = 1, 2, 3
\]

Using

\[
\mathcal{L}_N[Y] = [N, Y] = J_Y N - J_N Y
\]

where \( (J_Z)_{ij} = \partial_j Z_i \) are the Jacobian matrices, we rewrite (34) as

\[
Z_{1,k}(I, y) = v(I, y)\partial_y Y_{1,k} + i\omega(I, y)Y_{1,k} \\
Z_{2,k}(I, y) = v(I, y)\partial_y Y_{2,k} + (ik\omega(I, y) - \partial_y v(I, y))Y_{2,k} - \partial v(I, y)Y_{1,k} \\
Z_{3,k}(I, y) = v(I, y)\partial_y Y_{3,k} + i\omega(I, y)Y_{3,k} - \partial \omega(I, y)Y_{1,k} - \partial_y \omega(I, y)Y_{2,k}
\]

(36)
Regarding (36) as equations for $Y_{j,k}$, we find the solutions

$$Y_{1,k} = \int_{y_0}^y Z_{1,k}(I, \eta) e^{ik f_{\nu}^{{I,\eta} \rightarrow (I,\eta')} d\eta'} d\eta$$

$$Y_{2,k} = \int_{y_0}^y Z_{2,k}(I, \eta) + \partial_{\nu} Y_{1,k} e^{ik f_{\nu}^{\nu (I,\eta) \rightarrow y (I,\eta')} d\eta'} d\eta$$

$$Y_{3,k} = \int_{y_0}^y Z_{3,k}(I, \eta) + \partial_{\nu} Y_{1,k} Y_{1,k} + \partial_{\nu} Y_{1,k} Y_{2,k} e^{ik f_{\nu}^{\nu (I,\eta) \rightarrow y (I,\eta')} d\eta'} d\eta$$

multiplying by $e^{ik\nu}$ and summing over $k \in \mathbb{Z}$ we find

$$Y_1 = \mathcal{F}_{v,}\omega[Z_1]$$

$$Y_2 = G_{v,\omega}[Z_2] + G_{v,\omega}[\partial_{\nu} Y_1]$$

$$Y_3 = \mathcal{F}_{v,}\omega[Z_3] + \mathcal{F}_{v,\omega}[\partial_{\nu} Y_1] + \mathcal{F}_{v,\omega} [\partial_{\nu} Y_2].$$

Then, by Lemma 2.1,

$$\|Y_1\|_{r,\sigma,s-s_1} \leq \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_1\|_{r,\sigma,s}$$

$$\|Y_2\|_{r,\sigma,s-2s_1} \leq e^{s_2} \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_2\|_{r,\sigma,s-s_1} + e^{s_2} \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Y_1\|_{r,\sigma,s-s_1}$$

$$\|Y_3\|_{r,\sigma,s-3s_1} \leq \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_3\|_{r,\sigma,s-2s_1} + \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Y_1\|_{r,\sigma,s-2s_1}$$

$$\leq \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_2\|_{r,\sigma,s-2s_1} + \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_1\|_{r,\sigma,s-s_1}$$

$$\leq \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_3\|_{r,\sigma,s-2s_1} + \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_1\|_{r,\sigma,s-s_1}$$

$$\leq \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_3\|_{r,\sigma,s-2s_1} + \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_1\|_{r,\sigma,s-s_1}$$

$$\leq \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_3\|_{r,\sigma,s-2s_1} + \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_1\|_{r,\sigma,s-s_1}$$

Multiplying the inequalities above by $\rho^{s_1}_{-1}$, $\tau^{s_1}_{-1}$, $t^{s_1}_{-1}$ respectively and taking the sum, we find (35), with

$$\frac{1}{\rho} = \frac{1}{\rho_{s_{-1}}} + e^{s_2} \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_2\|_{r,\sigma,s-s_1}$$

$$\frac{1}{\tau} = e^{s_2} \text{diam}(\mathcal{Y}_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_1\|_{r,\sigma,s-s_1}$$

$$\frac{1}{t} = \frac{1}{t_{s_{-1}}}.$$
We recognise that, under conditions (25), \( \rho_*, \tau_*, \eta_* \) in (24) solve the equations above. \( \square \)

**Lemma 2.2** Let \( u_0 \geq u > v \); \( Y \in \mathcal{O}_u^3, W \in \mathcal{O}_u^3 \). Then

\[
\| \mathcal{L}_Y[W] \|^{u_0-u+w}_{u-w} \leq \| Y \|^{w}_{u-w} \| W \|^{u_0-u+w}_{u-w} \| Y \|^{u_0-u+w}_{u_0}.
\]

**Proof** One has

\[
\| \mathcal{L}_Y[W] \|^{u_0-u+w}_{u-w} = \| J_W Y - J_Y W \|^{u_0-u+w}_{u-w} \leq \| J_W Y \|^{u_0-u+w}_{u-w} + \| J_Y W \|^{u_0-u+w}_{u-w}
\]

Now, \( (J_W Y)_i = \partial_i W_i Y_1 + \partial_{w_i} W_i Y_2 + \partial_{w_2} W_i Y_3 \), so, using Cauchy inequalities,

\[
\|(J_W Y)_i\|^{u-w}_{u-w} \leq \|\partial_i W_i\|^{u-w}_{u-w} \|Y_1\|^{u-w}_{u-w} + \|\partial_{w_i} W_i\|^{u-w}_{u-w} \|Y_2\|^{u-w}_{u-w} + \|\partial_{w_2} W_i\|^{u-w}_{u-w} \|Y_3\|^{u-w}_{u-w}
\]

Similarly,

\[
\|(J_Y W)_i\|^{u-w}_{u-w} \leq \|W\|^{u_0-u+w}_{u-w} \|Y_i\|^{u_0}_{u_0}.
\]

Taking the \( u_0 - u + w \)-weighted norms, the thesis follows. \( \square \)

**Lemma 2.3** Let \( 0 < w < u \in \mathbb{R}^3 \), \( Y \in \mathcal{O}_u^{3}, W \in \mathcal{O}_u^{3} \). Then

\[
\| \mathcal{L}^n_Y[W] \|^{w}_{u-w} \leq 3^n n! \left( \| Y \|^{w}_{u+w} \right)^n \| W \|^{w}_{u-w}.
\]

**Proof** We apply Lemma 2.2 with \( W \) replaced by \( \mathcal{L}^{-1}_Y[W] \), \( u \) replaced by \( u - (i-1)w/n \), \( w \) replaced by \( w/n \) and, finally, \( u_0 = u + w \). With \( \| \cdot \|^{w}_{i} = \| \cdot \|^{w}_{i-w/n} \), \( 0 \leq i \leq n \), so that \( \| \cdot \|^{w}_{0} = \| \cdot \|^{w}_{w} \) and \( \| \cdot \|^{w}_{n} = \| \cdot \|^{w}_{u-w} \).

\[
\| \mathcal{L}^i_Y[W] \|^{w+w/n}_{i} = \left\| \left[ Y, \mathcal{L}^{-1}_Y[W] \right] \right\|^{w+w/n}_{i} \leq \left\| Y \right\|^{w}_{i} \| \mathcal{L}^{-1}_Y[W] \|^{w+w/n}_{i} + \left\| Y \right\|^{w+w/n}_{u+w} \| \mathcal{L}_Y^{-1}[W] \|^{w+w/n}_{i}
\]

Hence, denoting the exponent,

\[
\frac{n}{n+1} \| \mathcal{L}^i_Y[W] \|^{w}_{i} \leq \frac{n}{n+1} \left( \| Y \|^{w}_{i} \| \mathcal{L}^{-1}_Y[W] \|^{w}_{i} + \left( \frac{n^2}{(n+1)^2} \right) \| Y \|^{w}_{u+w} \| \mathcal{L}^{-1}_Y[W] \|^{w}_{i}
\]

Eliminating the common factor \( \frac{n}{n+1} \) and iterating \( n \) times from \( i = n \), by Stirling, we get

\[
\| \mathcal{L}^n_Y[W] \|^{w}_{u-w} \leq \left( 1 + \frac{1}{n} \right)^n \left( \| Y \|^{w}_{u+w} \right)^n \| W \|^{w}_{u-w}.
\]

as claimed. \( \square \)
Proposition 2.5 Let $0 < w < u$, $Y \in \mathcal{O}^3_{u+w}$,

$$q := 3\|Y\|_{u+w}^w < 1.$$  

Then the Lie series $e^{\mathcal{L}_Y}$ defines an operator

$$e^{\mathcal{L}_Y} : \mathcal{O}^3_u \to \mathcal{O}^3_{u-w}$$

and its tails

$$e^{\mathcal{L}_Y}_m = \sum_{n \geq m} \frac{\mathcal{L}_Y^n}{n!}$$

verify

$$\left\| e^{\mathcal{L}_Y}_m W \right\|_{u-w}^w \leq \frac{q^m}{1-q} \|W\|_{u}^w \quad \forall \ W \in \mathcal{O}^3_u.$$  

Proof of Proposition 2.2 We look for $Y$ such that $X_+ := e^{\mathcal{L}_Y} X$ has the desired properties.

$$e^{\mathcal{L}_Y} X = e^{\mathcal{L}_Y} (N + P) = N + P + \mathcal{L}_Y N + e^{\mathcal{L}_Y}_2 N + e^{\mathcal{L}_Y}_1 P$$

$$= N + P - \mathcal{L}_N Y + P_+$$

with $P_+ = e^{\mathcal{L}_Y}_2 N + e^{\mathcal{L}_Y}_1 P$. We choose $Y$ so that the homological equation

$$\mathcal{L}_N Y = P$$

is satisfied. By Proposition 2.4, this equation has a solution $Y \in \mathcal{O}^3_{r,\sigma,s-3s_1}$ verifying

$$q := 3\|Y\|_{r,\sigma,s-3s_1}^{w_*} \leq 3\text{diam}(Y_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma} \|P\|_{u}^w = Q \|P\|_{u}^w < 1.$$  

By Proposition 2.5, the Lie series $e^{\mathcal{L}_Y}$ defines an operator

$$e^{\mathcal{L}_Y} : \ W \in \mathcal{O}_{u+w_*} \to \mathcal{O}_{u_*}$$

and its tails

$$e^{\mathcal{L}_Y}_m = \sum_{n \geq m} \frac{\mathcal{L}_Y^n}{n!}$$

verify

$$\left\| e^{\mathcal{L}_Y}_m W \right\|_{u_*}^{w_*} \leq \frac{q^m}{1-q} \|W\|_{u_*+w_*}^{w_*}$$

$$\leq \frac{(Q\|P\|_{u}^w)_m}{1-Q\|P\|_{u}^w} \|W\|_{u_*+w_*}^{w_*}.$$
for all $W \in \mathcal{O}_{u_0,w}^3$. In particular, $e^{L^*Y}$ is well defined on $\mathcal{O}_{u_0,w}^3 \subset \mathcal{O}_{u_0,w_0}^3$, hence $P_+ \in \mathcal{O}_{u_0}^3$. The bounds on $P_+$ are obtained as follows. Using the homological equation, one finds

$$
\|e^{L^*Y}N\|_{u_+}^w = \left\| \sum_{n=1}^{\infty} \frac{L^{n+1}N}{(n+1)!} \right\|_{u_+}^w \\
\leq \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \left\| L^{\nu+1}N \right\|_{u_+}^w \\
= \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \left\| L^{\nu}P \right\|_{u_+}^w \\
\leq \sum_{n=1}^{\infty} \frac{1}{n!} \left\| L^{\nu}P \right\|_{u_+}^w \\
\leq \frac{Q (\|P\|_{u_0})^2}{1 - Q \|P\|_{u_0}^w} \\
$$

The bound

$$
\|e^{L^*Y}P\|_{u_+}^w \leq \frac{Q (\|P\|_{u_0})^2}{1 - Q \|P\|_{u_0}^w} \\
$$

is even more straightforward. □

### 2.4 Proof of the Normal Form Lemma

The proof of NFL is obtained via iterate applications of the Step Lemma. At the base step, we let

$$
X = X_0 := N + P_0, \quad w = w_0 := (\rho, \tau, t), \quad u = u_0 := (r, \sigma, s)
$$

with $X_0 = N + P_0 \in \mathcal{O}_{u_0}^3$. We let

$$
Q_0 := 3 \text{diam}(\gamma_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma}
$$

Conditions (27)–(30) are implied by the assumptions (20)–(21). We then conjugate $X_0$ to $X_1 = N + P_1 \in \mathcal{O}_{u_1}^3$, where

$$
u_1 = (r - 4\rho, \sigma - 4\tau e^{s_2}, s - 5t) =: (r_1, \sigma_1, s_1).
$$

Then we have

$$
\|P_1\|_{u_1}^{w_0} \leq 8e^{s_2}Q_0 \left(\|P_0\|_{u_0}^{w_0}\right)^2 \leq \frac{1}{2} \|P_0\|_{u_0}^{w_0}.
$$

We assume, inductively, that, for some $1 \leq j \leq n$, we have

$$
X_j = N + P_j \in \mathcal{O}_{u_j}^3, \quad \|P_j\|_{u_j}^{w_0} \leq 2^{-j-1}\|P_1\|_{u_0}^{w_0},
$$

where

$$
u_j = (r_j, \sigma_j, s_j)
$$

with

$$
r_j := r_1 - 4(j - 1)\frac{\rho}{n}, \quad \sigma_j := \sigma_1 - 4e^{s_2}(j - 1)\frac{\tau}{n}, \quad s_j := s_1 - 5(j - 1)\frac{t}{n}.
$$
The case $j = 1$ trivially reduces to the identity $\|P_1\|_{u_1}^{w_0} = \|P_1\|_{u_1}^{w_0}$. We aim to apply Proposition 2.3 with $u = u_j$ as in (42) and

$$w = u_1 := \frac{w_0}{n}, \quad \forall 1 \leq j \leq n.$$  

Conditions (27), (28) and (29) are easily seen to be implied by (20), (19), (18) and the first condition in (21) combined with the inequality $n\eta^2 < 1$, implied by the choice of $n$. We check condition (30). By homogeneity,

$$\|P_j\|_{u_j}^{w_1} = n\|P_j\|_{u_j}^{w_0} \leq n\|P_1\|_{u_1}^{w_0} \leq 8ne^{s_2}Q_0 \left(\|P_0\|_{u_0}^{w_0}\right)^2$$

whence, using

$$Q_j = 3\text{diam}(\mathcal{Y}_{\sigma_j}) \left\|\frac{1}{\nu}\right\|_{\mathcal{Y}_{\sigma_j}} \leq Q_0$$

we see that condition (30) is met:

$$2Q_j\|P_j\|_{u_j}^{w_1} \leq 16ne^{s_2}Q_0^2 \left(\|P_0\|_{u_0}^{w_0}\right)^2 < 1.$$  

Then the Iterative Lemma can be applied and we get $X_{j+1} = N + P_{j+1} \in \mathcal{O}_{u_{j+1}}^3$, with

$$\|P_{j+1}\|_{u_{j+1}}^{w_0} \leq 8e^{s_2}Q_j \left(\|P_j\|_{u_j}^{w_1}\right)^2 \leq 8e^{s_2}Q_0 \left(\|P_j\|_{u_j}^{w_1}\right)^2.$$  

Using homogeneity again to the extreme sides of this inequality and combining it with (41), (40) and (21), we get

$$\|P_{j+1}\|_{u_{j+1}}^{w_0} \leq 8ne^{s_2}Q_0 \left(\|P_j\|_{u_j}^{w_1}\right)^2 \leq 8ne^{s_2}Q_0\|P_1\|_{u_1}^{w_0}\|P_j\|_{u_j}^{w_0}
\leq 64ne^{2s_2}Q_0^2 \left(\|P_0\|_{u_0}^{w_0}\right)^2 \|P_j\|_{u_j}^{w_0} \leq \frac{1}{2}\|P_j\|_{u_j}^{w_0}
\leq 2^{-j}\|P_1\|_{u_1}^{w_0}.$$  

After $n$ iterations,

$$\|P_{n+1}\|_{u_{n+1}}^{w_0} \leq 2^{-n}\|P_1\|_{u_1}^{w_0} \leq 2^{-(n+1)}\|P_0\|_{u_0}^{w_0}$$  

so we can take $X_* = X_{n+1}$, $P_* = P_{n+1}$, $u_* = u_{n+1}$. □

### 2.5 A generalisation when the dependence on $\psi$ is smooth

**Definition 2.1** We denote $\mathcal{C}_{r,\sigma}^3$ the class of vector–fields

$$(I, y, \psi) : \mathbb{P}_{r,\sigma} := \mathbb{I}_r \times \mathbb{Y}_{\sigma} \times \mathbb{T} \to X = (X_1, X_2, X_3) \in \mathbb{C}^3$$

where each $X_i \in \mathcal{C}_{r,\sigma}$, meaning that $X_i$ is $C^\infty$ in $\mathbb{P} := \mathbb{I} \times \mathbb{Y} \times \mathbb{T}$, $X_i(\cdot, \cdot, \psi)$ is holomorphic in $\mathbb{I}_r \times \mathbb{Y}_{\sigma}$ for each fixed $\psi$ in $\mathbb{T}$.

In this section we generalise Proposition 2.1 to the case that $X \in \mathcal{C}_{r,\sigma}^3$. We use techniques going back to J. Nash and J. Moser [30, 28, 29].

First of all, we need a different definition of norms and, especially, *smoothing* operators.

---

6The norm defined in (14)–(15) is in general diverging when $X \in \mathcal{C}_{r,\sigma}^3$. 

1. Generalised weighted norms  We let
\[ \| X \|_{u,\ell}^w := \sum_i w_i^{-1} \| X_i \|_{u,\ell}, \quad \ell \in \{0, 1, \cdots \} \] (43)
where \( w = (w_1, w_2, w_3) \in \mathbb{R}^3_+ \) where, if \( f : \mathbb{P}_{r,\sigma} := \mathbb{I}_r \times \mathbb{Y}_\sigma \times \mathbb{T} \to \mathbb{C} \), then
\[ \| f \|_u := \sup_{I_r} \sup_{\gamma_r \times \mathbb{T}} | f |, \quad \| f \|_{u,\ell} := \max_{0 \leq j \leq \ell} \{ \| \partial^j f \|_u \} \quad u = (r, \sigma) \] (44)
with
\[ f = \sum_{k \in \mathbb{Z}} f_k(I, y) e^{ik\psi}. \]

Clearly, the class \( \mathbb{O}^3_{r,\sigma,s} \) defined in Section 2.1 is a proper subset of \( \mathbb{C}^3_{r,\sigma} \)
Observe that the norms (43) still verify monotonicity and homogeneity in (16) and (17).

2. Smoothing  We call smoothing an operator
\[ T_K : f \in \mathbb{C}_{r,\sigma} \to T_K f \in \mathbb{C}_{r,\sigma}, \quad K \in \{0, 1, \cdots \} \]
verifying the following. Let \( R_K := I - T_K \). There exist \( c_0 > 0, \delta \geq 0 \) such that for all \( f \in \mathbb{C}_{r,\sigma} \), for all \( K, 0 \leq p \leq \ell \),
- \( \| T_K f \|_{u,\ell} \leq c_0 K^{(\ell-p+\delta)} \| f \|_{u,p} \)
- \( \| R_K f \|_{u,p} \leq c_0 K^{-(\ell-p-\delta)} \| f \|_{u,\ell} \)

In this paper, following [3], we shall take
\[ T_K f(I, y, \psi) := \sum_{k \in \mathbb{Z}, |k| \leq K} f_k(I, y) e^{ik\psi} \]
which, with the definitions (43)–(44), verifies the inequalities above with \( \delta = 1 \).
We name Generalised Normal Form Lemma (GNFL) the following

**Proposition 2.6 (GNFL)** Let \( u = (r, \sigma); X = N + P \in \mathbb{C}^3_{u}, n, \ell, K \in \mathbb{N} \) and let \( w_K = \left( \rho, \tau, \frac{1}{c_0 K^{1+\sigma}} \right) \in \mathbb{R}^3_+ \) and assume that for some \( s_1, s_2 \in \mathbb{R}_+ \), the following inequalities are satisfied. Put
\[ Q := 3 e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma} \] (45)
then assume:
\[ 0 < \rho < r, \quad 0 < \tau < e^{-s_2} \frac{\sigma}{8} \] (46)
Let \( X = N + P \in C^3_u \), with \( u = (r, \sigma) \), \( N \) as in (33), \( \ell, K \in \mathbb{N} \). Assume (22) and that \( P \) is so small that

\[
Q \| P \|_{u,K}^{w,K} < 1, \quad Q := 3e^{s_1} \mathrm{diam}(Y_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma}, \quad w_K = \left( \frac{\rho}{c_0 K^{1+\delta}} \right)
\]

Let \( \rho_* \), \( \tau_* \) be defined via

\[
\frac{1}{\rho_*} = 1 - \frac{\mathrm{diam}(Y_\sigma)}{\rho} \left[ \frac{\partial v}{v} \right]_{r,\sigma} \left( e^{s_1} - e^{s_1+2s_2} \frac{\partial \omega}{v} \right) c_0 K^{1+\delta}
\]

\[
- \frac{\mathrm{diam}(Y_\sigma)}{\tau_*} \left( e^{s_1} \left[ \frac{\partial v}{v} \right]_{r,\sigma} + e^{s_1+2s_2} \frac{\partial \omega}{v} \right) c_0 K^{1+\delta}
\]

\[
\frac{1}{\tau_*} = \frac{e^{-s_2}}{\tau} - e^{s_1} \frac{\partial \omega}{v} \right)_{r,\sigma} c_0 K^{1+\delta}
\]

assume

\[
\hat{w}_* = (\rho_*, \tau_*) \in \mathbb{R}^2_+, \quad u_* = (r - 2\rho_*, \sigma - 2\tau_*) \in \mathbb{R}^2_+
\]

and put

\[
w_{*,K} := \left( \hat{w}_*, \frac{1}{c_0 K^{1+\delta}} \right).
\]

Then there exists \( Y \in T_K C^3_{u_*,w} \) such that \( X_* := e^\hat{\nu} X \in C^3_u \), and \( X_* = N + P_* \), with

\[
\| P_* \|_{u,K}^{w,K} \leq \frac{2Q \| P \|_{u,K}^{w,K}^2}{1 - Q \| P \|_{u,K}^{w,K}} + c K^{-\ell+\delta} \| P \|_{u,\ell}^{w,K}
\]
The simplified form of Proposition 2.7, corresponding to Proposition 2.3, is

**Proposition 2.8 (Generalised Step Lemma)** Assume (22) and replace (50) and (52) with

\[
\begin{align*}
2 e^{s_1 + s_2} \text{diam}(Y_\sigma) \left\| \frac{\partial \omega}{v} \right\|_{r,\sigma} &< c_0 K^{1 + \delta} \tau \leq 1 \\
4 e^{s_1} \text{diam}(Y_\sigma) \left\| \frac{\partial v}{v} \right\|_{r,\sigma} &< \frac{\rho}{\tau} \leq 1 \\
8 \text{diam}(Y_\sigma) \left\| \frac{\partial \omega}{v} \right\|_{r,\sigma} &< c_0 K^{1 + \delta} \rho \leq 1 \\
0 < \rho < \frac{r}{4}, \quad 0 < \tau < \frac{\sigma}{4} e^{-s_2} &\leq (54) \\
2Q\|P\|_{u,K}^{u_K} &< 1 (55)
\end{align*}
\]

then \(X_+ = N + P_+ \in C^3_{r,\sigma}\) and

\[
\|P_+\|_{u_+}^{u_K} \leq 8 e^{s_2} Q(\|P\|_{u,K}^{u_K})^2 + c K^{-\ell + \delta} \|P\|_{u,K}^{w_K} (56)
\]

with

\[
u_+ := (r - 4\rho, \sigma - 4\tau e^{s_2}) .
\]

**Proof** The inequalities in (54) guarantee

\[
\frac{1}{\rho_*} \geq \frac{1}{2\rho}, \quad \frac{1}{\tau_*} \geq \frac{e^{-s_2}}{2\tau}
\]

whence

\[
u_* \leq 2e^{s_2} w_K, \quad u_* \geq u_+ > 0 .
\]

Then (57) is implied by (53), monotonicity and homogeneity and the inequality in (56). \(\square\)

Let now \(F_{v,\omega}\) and \(G_{v,\omega}\) be as in (32). First of all, observe that \(F_{v,\omega}, G_{v,\omega}\) take \(T_K C_{r,\sigma}\) to itself. Moreover, generalising Lemma 2.1,

**Lemma 2.4** If inequalities (22) hold, then

\(F_{v,\omega}, G_{v,\omega} : C_{r,\sigma} \rightarrow C_{r,\sigma}\)

and

\[
\|F_{v,\omega}[g]\|_{r,\sigma} \leq e^{s_1} \text{diam}(Y_\sigma) \left\| \frac{g}{v} \right\|_{r,\sigma}, \quad \|G_{v,\omega}[g]\|_{r,\sigma} \leq e^{s_1 + s_2} \text{diam}(Y_\sigma) \left\| \frac{g}{v} \right\|_{r,\sigma} .
\]

**Proposition 2.9** Let

\(N = (0, v(I, y), \omega(I, y))\), \(Z = (Z_1(I, y, \psi), Z_2(I, y, \psi), Z_3(I, y, \psi))\)

belong to \(C^3_{r,\sigma}\) and assume (22). Then the “homological equation”

\[
L_N[Y] = Z
\]

has a solution \(Y \in C_{r,\sigma}\) verifying

\[
\|Y\|_{r,\sigma}^{\rho_*, \tau_*, \ell_*} \leq e^{s_1} \text{diam}(Y_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma} \|Z\|_{r,\sigma}^{\rho_*, \tau_*, \ell} (58)
\]

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with \( \rho_*, \tau_*, t_* \) defined via

\[
\frac{1}{\rho_*} = \frac{1}{\rho} - \diam(Y_\sigma) \left( \frac{\partial v}{v} \right)_{r,\sigma} \left( \frac{\partial \omega}{v} \right)_{r,\sigma} \left( \frac{\partial \omega}{v} \right)_{r,\sigma} \left( \frac{1}{t} \right)
\]

\[
- \diam(Y_\sigma) \left( \frac{e^{s_1}}{t} \frac{\partial \omega}{v} \right)_{r,\sigma} + e^{s_1} + s_2 \diam(Y_\sigma) \left( \frac{\partial v}{v} \right)_{r,\sigma} \left( \frac{\partial \omega}{v} \right)_{r,\sigma} \left( \frac{1}{t} \right)
\]

\[
\frac{1}{\tau_*} = \frac{e^{-s_2}}{t} - e^{s_1} \diam(Y_\sigma) \left( \frac{\partial \omega}{v} \right)_{r,\sigma} \left( \frac{1}{t} \right)
\]

\[
t_* = t
\]

and provided that

\[
(\rho_*, \tau_*) \in \mathbb{R}_+^2.
\]

In particular, if \( Z \in T_K \mathcal{C}_{r,\sigma}^{3} \) for some \( K \in \mathbb{N} \), then also \( Y \in T_K \mathcal{C}_{r,\sigma}^{3} \).

**Proof** The solution (37) satisfies

\[
\|Y_1\|_{r,\sigma} \leq e^{s_1} \diam(Y_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_1\|_{r,\sigma}
\]

\[
\|Y_2\|_{r,\sigma} \leq e^{s_1 + s_2} \diam(Y_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_2\|_{r,\sigma} + e^{s_1 + s_2} \diam(Y_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_1\|_{r,\sigma}
\]

\[
\|Y_3\|_{r,\sigma} \leq e^{s_1} \diam(Y_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_3\|_{r,\sigma} + e^{s_1 + s_2} \diam(Y_\sigma) \left( \frac{1}{v} \right)_{r,\sigma} \|Z_2\|_{r,\sigma}
\]

\[
\|Y_4\|_{r,\sigma} = \left( e^{s_1} \frac{\partial \omega}{v} \right)_{r,\sigma} + e^{s_1 + s_2} \diam(Y_\sigma) \left( \frac{\partial v}{v} \right)_{r,\sigma} \left( \frac{\partial \omega}{v} \right)_{r,\sigma} \left( \frac{1}{t} \right)
\]

Multiplying the inequalities above by \( \rho_*^{-1}, \tau_*^{-1}, t_*^{-1} \) respectively and taking the sum, we find (58), with

\[
\frac{1}{\rho_*} = 1 + e^{s_1 + s_2} \diam(Y_\sigma) \left( \frac{\partial v}{v} \right)_{r,\sigma} \left( \frac{1}{t} \right) + \diam(Y_\sigma) \left( e^{s_1} \frac{\partial \omega}{v} \right)_{r,\sigma} + e^{s_1 + s_2} \diam(Y_\sigma) \left( \frac{\partial v}{v} \right)_{r,\sigma} \left( \frac{\partial \omega}{v} \right)_{r,\sigma} \left( \frac{1}{t} \right)
\]

\[
\frac{1}{\tau_*} = e^{s_2} + e^{s_1 + s_2} \diam(Y_\sigma) \left( \frac{\partial \omega}{v} \right)_{r,\sigma} \left( \frac{1}{t} \right)
\]

\[
\frac{1}{t} = \frac{1}{t_*}.
\]

We recognise that, under conditions (60), \( \rho_*, \tau_*, t_* \) in (59) solve the equations above. Observe that if \( Z \in T_K \mathcal{C}_{r,\sigma}^{3} \), then also \( Y \in T_K \mathcal{C}_{r,\sigma}^{3} \), as \( \mathcal{F}_{v,\omega}, \mathcal{G}_{v,\omega} \) do so. \( \Box \)

**Lemma 2.5** Let \( u_0 \geq u > w \in \mathbb{R}_+ \times \{0\}; \ Y \in T_K \mathcal{C}_{u_0}^{3}, W \in T_K \mathcal{C}_{u_0}^{3} \). Put \( w_K := \left( w_1, w_2, \frac{1}{\sigma^{1+\delta}} \right) \).

Then

\[
\|L_Y [W]\|_{u_0 - u + w_K} \leq \|Y\|_{u_0 - u + w_K} \|W\|_{u_0 - u + w_K} + \|W\|_{u_0 - u + w_K} \|Y\|_{u_0 - u + w_K}.
\]

**Proof** By Cauchy inequalities, the definitions (43)-(44) and the smoothing properties,

\[
\|JW\|_{u - w} \leq \|\partial W_1\|_{u - w} \|Y_1\|_{u - w} + \|\partial W_2\|_{u - w} \|Y_2\|_{u - w} + \|\partial W_3\|_{u - w} \|Y_3\|_{u - w}
\]

\[
\leq w_1^{-1} \|W_1\|_{u} \|Y_1\|_{u - w + w_2} + w_2^{-1} \|W_2\|_{u} \|Y_2\|_{u - w + w_2} + \|W_3\|_{u} \|Y_3\|_{u - w + w_2}
\]

\[
\leq w_1^{-1} \|W_1\|_{u} \|Y_1\|_{u - w} + w_2^{-1} \|W_2\|_{u} \|Y_2\|_{u - w} + w_2 \|W_3\|_{u} \|Y_3\|_{u - w} + \|W_3\|_{u} \|Y_3\|_{u - w + w_2}
\]

\[
= \|Y\|_{u_0 - w} \|W\|_{u}.
\]
Similarly,
\[ \| (J_Y W) i \|_{u-w} \leq \| W \|_{u-w}^{u_0+i+w_K} \| Y_i \|_{u_0} . \]

Taking the \( u_0 - u + w_K \)-weighted norms, the thesis follows. \( \Box \)

**Lemma 2.6** Let \( 0 < w < u \in \mathbb{R}_+ \times \{0\}, w_K := \left( w_1, w_2, \frac{1}{c_0 K^{1+\tau}} \right) \), \( Y \in T_K C^3_{u+w}, W \in T_K C^3_u \).

Then
\[ \| L^n Y \|_{u-w}^{w_K} \leq 3^n n! \left( \| Y \|_{u+w}^{w_K} \| W \|_{u-w}^{w_K} \right) . \]

**Proof** The proof copies the one of Lemma 2.3, up to invoke Lemma 2.5 at the place of Lemma 2.2 and hence replace the \( w \)'s “up” with \( w_K \). \( \Box \)

**Proposition 2.10** Let \( 0 < w < u \in \mathbb{R}_+ \times \{0\}, w_K := \left( w_1, w_2, 1, c_0 K^{1+\delta} \right) \), \( Y \in T_K C^3_{u+w} \), \( q := 3\| Y \|_{u+w}^{w_K} < 1 \).

Then the Lie series \( e^{L_Y} \) defines an operator
\[ e^{L_Y} : T_K C^3_u \rightarrow T_K C^3_{u+w} \]

and its tails
\[ e^{L_Y}_m = \sum_{n \geq m} \frac{L^n Y}{n!} \]

verify
\[ \| e^{L_Y}_m W \|_{u-w}^{w_K} \leq \frac{q^n}{1-q} \| W \|_{u}^{w_K} \quad \forall W \in T_K C^3_u . \]

**Proof of Proposition 2.7** Differently from Proposition 2.2, here we need a “ultraviolet cut-off” of the perturbing term. Namely, we split
\[ e^{L_Y} X = e^{L_Y} (N + P) = N + P + L_Y N + e_2^{L_Y} N + e_1^{L_Y} P \]
\[ = N + T_K P - L_N Y + P_+ \]

with \( P_+ = e_2^{L_Y} N + e_1^{L_Y} P + R_K P \). We choose \( Y \) so that the homological equation
\[ L_N Y = T_K P \]

is satisfied. By Proposition 2.9, this equation has a solution \( Y \in T_K C^3_{r,\sigma} \), verifying
\[ q := 3\| Y \|_{r,\sigma}^{w_*} \leq 3e^{\delta_1} \text{diam}(Y) \left\| \frac{1}{u} \left\| P \right\|_{r,\sigma}^{w_K} = Q \| P \|_{u}^{w_K} < 1 . \]

with \( w_* = (\rho_*, \tau_*, t_*) \) as in (59). As \( t_* = t = \frac{1}{c_0 K^{1+\tau}} \), We let
\[ w_{*,K} := w_* , \quad \tilde{w}_* := (\rho_*, \tau_*) \]

with \( (\rho_*, \tau_*) \) as in (51). By Proposition 2.10, the Lie series \( e^{L_Y} \) defines an operator
\[ e^{L_Y} : W \in T_K C_{u_*+\tilde{w}_*} \rightarrow T_K C_{u_*} . \]
and its tails
\[
e^L_m = \sum_{n \geq m} e^L_n/n!
\]
verify
\[
\left\| e^L_m W \right\|^{u_+, K}_{u_+} \leq \left( \frac{Q \left\| P \right\|^{w, K}_{u}}{1 - Q \left\| P \right\|^{w, K}_{u}} \right)^m \left\| W \right\|^{u_+, +w_+}_{u_+}
\]
for all \( W \in T_K C^3_{u_+, +w_+} \). In particular, \( e^L \) is well defined on \( T_K C^3_{u_+} \subset T_K C^3_{u_+, +w_+} \), hence \( P_+ \in C_{u_+} \).
The bounds on \( P_+ \) are obtained as follows. The terms \( \left\| e^L_2 N \right\|^{u_+, K}_{u_+} \) and \( \left\| e^L_1 P \right\|^{u_+, K}_{u_+} \) are treated quite similarly as (38) and (39):
\[
\left\| e^L_2 N \right\|^{u_+, K}_{u_+} \leq \frac{Q \left\| P \right\|^{w, K}_{u}}{1 - Q \left\| P \right\|^{w, K}_{u}} \left\| N \right\|^{u_+, K}_{u_+}, \quad \left\| e^L_1 P \right\|^{w_+, K}_{u_+} \leq \frac{Q \left\| P \right\|^{w, K}_{u}}{1 - Q \left\| P \right\|^{w, K}_{u}} \left\| P \right\|^{w, K}_{u_+}.
\]
The moreover, here we have the term \( R_K P \), which is obviously bounded as
\[
\left\| R_K P \right\|^{w_+, K}_{u_+} \leq c K^{-\ell + \delta} \left\| P \right\|^{w_+, K}_{u_+, K} \leq c K^{-\ell + \delta} \left\| P \right\|^{w, K}_{u_+}.
\]
We are finally ready for the

**Proof of GNFL** Analogously as in the proof of NFL, we proceed by iterate applications of the Generalised Step Lemma. At the base step, we let
\[
X = X_0 := N + P_0, \quad u_0 := u_{0, K} := \left( \rho, \tau, \frac{1}{cK} \right), \quad u_0 := (r, \sigma)
\]
with \( X_0 = N + P_0 \in C^3_{u_0} \). We let
\[
Q_0 := 3 e^{s_1} \text{diam}(\mathcal{Y}_\sigma) \left\| \frac{1}{v} \right\|_{u_0}
\]
Conditions (54)–(56) are implied by the assumptions (45)–(49). We then conjugate \( X_0 \) to \( X_1 = N + P_1 \in C^3_{u_1} \), where
\[
u_1 = (r - 4\rho, \sigma - 4\tau e^{s_2}) =: (r_1, \sigma_1).
\]
Then we have
\[
\left\| P_1 \right\|^{u_0}_{u_1} \leq 8 e^{s_2} Q_0 \left( \left\| P_0 \right\|^{w_0}_{u_0} \right)^2 + c_0 K^{-\ell + \delta} \left\| P_0 \right\|^{w_0}_{u_0, \ell}.
\]
If \( 8 e^{s_2} Q_0 \left( \left\| P_0 \right\|^{w_0}_{u_0} \right)^2 \leq c_0 K^{-\ell + \delta} \left\| P_0 \right\|^{w_0}_{u_0, \ell} \), the proof finishes here. So, we assume the opposite inequality, which gives
\[
\left\| P_1 \right\|^{w_0}_{u_1} \leq 16 e^{s_2} Q_0 \left( \left\| P_0 \right\|^{w_0}_{u_0} \right)^2 \leq \frac{1}{2} \left\| P_0 \right\|^{w_0}_{u_0}.
\]
We assume, inductively, that, for some \( 1 \leq j \leq n \), we have
\[
X_j = N + P_j \in C^3_{u_j}, \quad \left\| P_j \right\|^{w_0}_{u_j} \leq 2^{-(j-1)} \left\| P_1 \right\|^{w_0}_{u_1}
\]
where
\[
u_j = (r_j, \sigma_j)
\]
with
\[ r_j := r_1 - 4(j - 1) \frac{p}{n}, \quad \sigma_j := \sigma_1 - 4e^{s_2}(j - 1) \frac{\tau}{n}. \]
The case \( j = 1 \) is trivially true because it is the identity \( \|P_1\|_{u_1}^{w_0} = \|P_1\|_{u_1}^{w_0} \). We aim to apply Proposition 2.8 with \( u = u_j \) as in (63) and
\[ w = w_1 := \frac{w_0}{n}, \quad \forall \ 1 \leq j \leq n. \]
Conditions (54) and (55) correspond to (47)–(48), while (56) is implied by (49). We check condition (56). By homogeneity,
\[ \|P_j\|_{u_j}^{w_1} = \|P_j\|_{u_j}^{w_0} \leq \|P_1\|_{u_1}^{w_0} \leq 16ne^{s_2}Q_0 \left( \|P_0\|_{u_0}^{w_0} \right)^2 \]
whence, using
\[ Q_j = 3 \text{diam}(Y_{\sigma_j}) \left\| \frac{1}{v} \right\|_{r_j, \sigma_j} \leq Q_0 \]
we see that condition (30) is met:
\[ 2Q_j\|P_j\|_{u_j}^{w_1} \leq 32ne^{s_2}Q_0^2 \left( \|P_0\|_{u_0}^{w_0} \right)^2 < 1. \]
Then the Iterative Lemma can be applied and we get \( X_{j+1} = N + P_{j+1} \in C_{u_{j+1}}^3 \), with
\[ \|P_{j+1}\|_{u_{j+1}}^{w_0} \leq 8e^{s_2}Q_j \left( \|P_j\|_{u_j}^{w_1} \right)^2 \leq 8e^{s_2}Q_0 \left( \|P_j\|_{u_j}^{w_1} \right)^2 \]
Using homogeneity again to the extreme sides of this inequality and combining it with (62), (61) and (49), we get
\[ \|P_{j+1}\|_{u_{j+1}}^{w_0} \leq 8ne^{s_2}Q_0 \left( \|P_j\|_{u_j}^{w_0} \right)^2 \leq 8ne^{s_2}Q_0 \|P_j\|_{u_j}^{w_0} \|P_j\|_{u_j}^{w_0} \leq \frac{1}{2} \|P_j\|_{u_j}^{w_0} \leq \left( \frac{1}{2} \right)^{j-1} \|P_j\|_{u_j}^{w_0}. \]
After \( n \) iterations,
\[ \|P_n\|_{u_{n+1}}^{w_0} \leq 2^{-n} \|P_1\|_{u_1}^{w_0} \leq 2^{-(n+1)} \|P_0\|_{u_0}^{w_0} \]
so we can take \( X_* = X_{n+1}, \ P_* = P_{n+1}, \ u_* = u_{n+1}. \)

### 3 Symplectic tools

In this section we describe various sets of canonical coordinates that are needed to our application. We remark that during the proof of Theorem B, we shall not use any of such sets completely, but, but rather, a “mix” of action–angle and regularising coordinates, described below.

#### 3.1 Starting coordinates

We begin with the coordinates

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\[
\begin{align*}
C &= \|x \times y + x' \times y'\| \\
G &= \|x \times y\| \\
R &= \frac{y' \cdot x'}{\|x'\|} \\
\Lambda &= \sqrt{a}
\end{align*}
\]
\[
\begin{align*}
\gamma &= \alpha_k(i, x') + \frac{\pi}{2} \\
g &= \alpha_k(x', P) + \pi \\
r &= \|x'\| \\
\ell &= \text{mean anomaly of } x \text{ in } E
\end{align*}
\]

where:

- \(i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\) is a horizontal frame in \(\mathbb{R}^2 \times \{0\}\) and \(k = i \times j\) ("\times" denoting, as usual, the "skew–product");
- after fixing a set of values of \((y, x)\) where the Kepler Hamiltonian
  \[
  \frac{\|y\|^2}{2} - 1 - \frac{1}{\|x\|}
  \]
  takes negative values, \(E\) denotes the elliptic orbit with initial values \((y_0, x_0)\) in such set;
- \(a\) is the semi–major axis of \(E\);
- \(P\), with \(\|P\| = 1\), the direction of the perihelion of \(E\), assuming \(E\) is not a circle;
- \(\ell\) is the mean anomaly of \(x\) on \(E\), defined, mod \(2\pi\), as the area of the elliptic sector spanned from \(P\) to \(x\), normalized to \(2\pi\);
- \(\alpha_w(u, v)\) is the oriented angle from \(u\) to \(v\) relatively to the positive orientation established by \(w\), if \(u, v\) and \(w \in \mathbb{R}^3 \setminus \{0\}\), with \(u, v \perp w\).

The canonical\(^7\) character of the coordinates (64) has been discussed, in a more general setting, in [33].

### 3.2 Energy–time coordinates

We now describe the "energy–time" change of coordinates

\[
\phi_{et} : (R, E, r, \tau) \to (R, G, r, g) = (R + \rho(E, r, \tau), \bar{G}(E, r, \tau), r, \bar{g}(E, r, \tau))
\]

which integrates the function \(E(r, G, g)\) in (11), where \(E\) ("energy") denotes the generic level–set of \(E\), while \(\tau\) is its conjugated ("time") coordinate. The domain of the coordinates (66) is

\[
R \in \mathbb{R}, \quad 0 \leq r \leq 2, \quad -r < E < 1 + \frac{r^2}{4}, \quad \tau \in \mathbb{R}, \quad E \notin \{r, 1\}.
\]

The extremal values of \(E\) are taken to be the minimum and the maximum of the function \(E\) for \(0 \leq r \leq 2\). The values \(r = 1\) and \(r = 1\) have been excluded because they correspond, in the \((g, G)\)–plane, to the curves \(S_0(\tau)\) and \(S_1(\tau)\) in Figure 1, where periodic motions do not exist.

\(^7\)Namely, the change of coordinate (64) satisfies \(\sum_{i=1}^{2} (dy_i \wedge dx_i + dy'_i \wedge dx'_i) = dC \wedge d\gamma + dG \wedge dg + dR \wedge dr + d\Lambda \wedge d\ell\).
The functions \( \bar{G}(\varepsilon, r, \cdot), \bar{g}(\varepsilon, r, \cdot) \) and \( \rho(\varepsilon, r, \cdot) \) appearing in (66) are, respectively, \( 2\tau_p \) periodic, \( 2\tau_p \) periodic, \( 2\tau_p \) quasi–periodic, meaning that they satisfy

\[
P_{cr} : \left\{ \begin{array}{l}
\bar{G}(\varepsilon, r, \tau + 2j\tau_p) = \bar{G}(\varepsilon, r, \tau) \\
\bar{g}(\varepsilon, r, \tau + 2j\tau_p) = \bar{g}(\varepsilon, r, \tau) \\
\rho(\varepsilon, r, \tau + 2j\tau_p) = \rho(\varepsilon, r, \tau) + 2j\rho(\varepsilon, r, \tau_p)
\end{array} \right. \quad \forall \tau \in \mathbb{R}, \forall j \in \mathbb{Z} \quad (68)
\]

with \( \tau_p = \tau_p(\varepsilon, r) \) the period, defined below. Note that one can find a unique splitting

\[
\rho(\varepsilon, r, \tau) = B(\varepsilon, r)\tau + \bar{\rho}(\varepsilon, r, \tau) \quad (69)
\]

such that \( \bar{\rho}(\varepsilon, r, \cdot) \) is \( 2\tau_p \)–periodic. It is obtained taking

\[
B(\varepsilon, r) = \frac{\rho(\varepsilon, r, \tau_p(\varepsilon, r))}{\tau_p(\varepsilon, r)}, \quad \bar{\rho}(\varepsilon, r, \tau) = \rho(\varepsilon, r, \tau) - \frac{\rho(\varepsilon, r, \tau_p(\varepsilon, r))}{\tau_p(\varepsilon, r)} \tau \quad (70)
\]

The transformation (66) turns to satisfy also the following “half–parity” symmetry:

\[
P_{1/2} : \left\{ \begin{array}{l}
\bar{G}(\varepsilon, r, \tau) = \bar{G}(\varepsilon, r, -\tau) \\
\bar{g}(\varepsilon, r, \tau) = 2\pi - \bar{g}(\varepsilon, r, -\tau) \\
\rho(\varepsilon, r, \tau) = -\rho(\varepsilon, r, -\tau)
\end{array} \right. \quad \forall \tau \in \mathbb{R} \quad (71)
\]

In addition, when \( -r < \varepsilon < r \), one has the following “quarter–parity”

\[
P_{1/4} : \left\{ \begin{array}{l}
\bar{G}(\varepsilon, r, \tau) = -G(\varepsilon, r, \tau_p - \tau) \\
\bar{g}(\varepsilon, r, \tau) = \bar{g}(\varepsilon, r, \tau_p - \tau) \\
\rho(\varepsilon, r, \tau) = \rho(\varepsilon, r, \tau_p) - \rho(\varepsilon, r, \tau_p - \tau)
\end{array} \right. \quad \forall 0 \leq \tau \leq \tau_p \quad (72)
\]

The change (66) will be constructed using, as generating function, a solution of the Hamilton–Jacobi equation

\[
E(r, G, \partial_G S_{et}) = G^2 + r\sqrt{1 - G^2} \cos (-\partial_G S_{et}) = \varepsilon \quad (73)
\]

We choose the solution

\[
S_{et}^+(\mathcal{R}, \varepsilon, r, G) = \left\{ \begin{array}{ll}
\pi \sqrt{\alpha_+(\varepsilon, r)} - \int_G^{\sqrt{\alpha_+(\varepsilon, r)}} \cos^{-1} \frac{\varepsilon - \Gamma^2}{r\sqrt{1 - \Gamma^2}} d\Gamma + \mathcal{R}r & \quad -r \leq \varepsilon < 1 \\
\pi - \int_G^{\sqrt{\alpha_+(\varepsilon, r)}} \cos^{-1} \frac{\varepsilon - \Gamma^2}{r\sqrt{1 - \Gamma^2}} d\Gamma + \mathcal{R}r & \quad 1 \leq \varepsilon \leq 1 + \frac{r^2}{4}
\end{array} \right.
\]

where we denote as

\[
\alpha_{\pm}(\varepsilon, r) = \varepsilon - \frac{r^2}{2} \pm r\sqrt{1 + \frac{r^2}{4} - \varepsilon} \quad (74)
\]

the real roots of

\[
x^2 - 2 \left( \varepsilon - \frac{r^2}{2} \right) x + \varepsilon^2 - r^2 = 0 \quad (75)
\]

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Note that the equation in (75) has always a positive real root all \( r, \mathcal{E} \) as in (67), so \( \alpha_+(\mathcal{E}, r) \) is positive. \( S_{et}^+ \) generates the following equations

\[
\begin{align*}
  g &= -\cos^{-1} \frac{\mathcal{E} - G^2}{r\sqrt{1-G^2}} \\
  \tau &= + \int_{\hat{G}(\mathcal{E}, r, \tau)}^{\sqrt{\alpha_+(\mathcal{E}, r)}} \frac{d\Gamma}{\sqrt{(\Gamma^2 - \alpha_-(\mathcal{E}, r))(\alpha_+(\mathcal{E}, r) - \Gamma^2)}} \\
  R &= \mathcal{R} - \frac{1}{r} \int_{\hat{G}(\mathcal{E}, r, \tau)}^{\sqrt{\alpha_+(\mathcal{E}, r)}} \frac{(\mathcal{E} - \Gamma^2)d\Gamma}{\sqrt{(\Gamma^2 - \alpha_-(\mathcal{E}, r))(\alpha_+(\mathcal{E}, r) - \Gamma^2)}} =: \mathcal{R} + \rho(\mathcal{E}, r, \tau) \\
  r &= r
\end{align*}
\]

(76)

The equations for \( g \) and \( r \) are immediate. We check the equation for \( \tau \). Letting, for short, \( \sigma(\mathcal{E}, r) := \sqrt{\alpha_+(\mathcal{E}, r)} \). Then

\[
\begin{align*}
  \tau &= \partial_\mathcal{E} S_{et}^+(\mathcal{R}, \mathcal{E}, r, G) \\
  &= \left\{ \begin{array}{ll}
    \pi \partial_\mathcal{E} \sigma(\mathcal{E}, r) - \partial_\mathcal{E} \sigma(\mathcal{E}, r)g_+(\mathcal{E}, r) - \int_G^{\sigma(\mathcal{E}, r)} \partial_\mathcal{E} \cos^{-1} \frac{\mathcal{E} - \Gamma^2}{r\sqrt{1-G^2}} d\Gamma & -r \leq \mathcal{E} < 1 \\
    -\partial_\mathcal{E} \sigma(\mathcal{E}, r)g_+(\mathcal{E}, r) - \int_G^{\sigma(\mathcal{E}, r)} \partial_\mathcal{E} \cos^{-1} \frac{\mathcal{E} - \Gamma^2}{r\sqrt{1-G^2}} d\Gamma & 1 \leq \mathcal{E} \leq 1 + \frac{r^2}{4}
  \end{array} \right.
\end{align*}
\]

(77)

having let \( g_+(\mathcal{E}, r) := \cos^{-1} \frac{\mathcal{E} - \sigma(\mathcal{E}, r)^2}{r\sqrt{1-\sigma(\mathcal{E}, r)^2}} \) and used, by (74),

\[
  g_+(\mathcal{E}, r) = \cos^{-1} \ \text{sign} \left( \frac{r}{2} - \sqrt{1 + \frac{r^2}{4} - \mathcal{E}} \right) = \begin{cases} 
    \pi & -r \leq \mathcal{E} < 1 \\
    0 & 1 \leq \mathcal{E} \leq 1 + \frac{r^2}{4}
  \end{cases}
\]

Observe that \( (g_+, \sigma) \) are the coordinates of the point where \( \mathcal{E} \) reaches its maximum on each level set (Figure 1). The equation for \( R \) is analogous.

Equations (76) define the segment of the transformation (66) with \( 0 \leq \tau \leq \tau_p \), where

\[
\tau_p(\mathcal{E}, r) := \int_{\beta(\mathcal{E}, r)}^{\alpha_+(\mathcal{E}, r)} \frac{d\Gamma}{\sqrt{(\Gamma^2 - \alpha_-(\mathcal{E}, r))(\alpha_+(\mathcal{E}, r) - \Gamma^2)}}
\]

(78)

is the half-period, with

\[
\beta(\mathcal{E}, r) = \begin{cases} 
    -\sqrt{\alpha_+(\mathcal{E}, r)} & \text{if } \alpha_-(\mathcal{E}, r) < 0 \\
    \sqrt{\alpha_-(\mathcal{E}, r)} & \text{if } \alpha_-(\mathcal{E}, r) > 0
  \end{cases}
\]

(79)
The transformation is prolonged to \(-\tau_p < \tau < 0\) choosing the solution
\[
S_{ct}^- := -2\pi G - S_{ct}^+
\]
of (73). It can be checked that this choice provides the symmetry relation described in (71).
Considering next the functions \(S_k^\pm = S_{ct}^\pm + 2k \Sigma(E, r)\), where \(\Sigma\) solves\(^8\)
\[
\partial \epsilon \Sigma = \tau_p(E, r), \quad \partial r \Sigma = \rho(E, r, \tau_p(E, r))
\]
one obtains the extension of the transformation to \(\tau \in \mathbb{R}\) verifying (68).
Observe that quarter period symmetry (66), holding in the case \(-r < E < r\), is an immediate consequence of the definitions (76).
The coordinates \((R, E, r, \tau)\) are referred to as \textit{energy–time coordinates}.
The regularity of the functions \(\tilde{G}(E, r, \tau), \tilde{\rho}(E, r, \tau), \tilde{B}(E, r)\) and \(\tau_p(E, r)\), which are relevant for the paper, are studied in detail in Section 4. Remark that holomorphy of such functions is not discussed.

### 3.3 Action–angle coordinates

We look at the transformation
\[
\phi_{aa} : (R_*, A_*, r_*, \varphi_*) \rightarrow (R, E, r, \tau)
\]
defined by equations
\[
\left\{
\begin{array}{l}
A_* = A(E, r) \\
\varphi_* = \pi \frac{\tau}{\tau_p(E, r)} \\
r_* = r \\
R_* = R + B(E, r) \tau
\end{array}
\right.
\]
with \(B(E, r)\) as in (70), \(\tau_p(E, r)\) as in (78) and \(A(E, r)\) the “action function”, defined as
\[
A(E, r) := \left\{
\begin{array}{l}
\sqrt{\alpha_+(E, r)} - \frac{1}{\pi} \int_{\beta(E, r)} \cos^{-1} \left( \frac{E - \Gamma^2}{\sqrt{1 - \Gamma^2}} d\Gamma \right) \quad -r \leq E \leq 1 \\
1 - \frac{1}{\pi} \int_{\beta(E, r)} \cos^{-1} \left( \frac{E - \Gamma^2}{\sqrt{1 - \Gamma^2}} d\Gamma \right) \quad 1 < E \leq 1 + \frac{r^2}{4}
\end{array}
\right.
\]
with \(\alpha_+(E, r)\) and \(\beta(E, r)\) being defined in (74), (79).
Geometrically, \(A(E, r)\) represents the area of the region encircled by the level curves of \(E\) in Figure 1 in the former case, the area of its complement in the second case, divided by \(2\pi\).
The canonical character of the transformation (80) is recognised looking at the generating function
\[
S_{na}(R, E, r_*, \varphi_*) = \varphi_* A(E, r_*) + R r_*
\]
\(^8\)The existence of the function \(\Sigma(E, r)\) follows from the arguments of the next section: compare the formula in (82).
and using the following relations (compare the formulae in (76) and (78))

\[
\begin{align*}
A_r(E, r) &= -\frac{1}{\pi} \int_{\beta(E, r)}^{\alpha_+(E, r)} \frac{(E - \Gamma^2)d\Gamma}{\sqrt{(\Gamma^2 - \alpha-(E, r))(\alpha_+(E, r) - \Gamma^2)}} \\
A_r(E, r) &= \frac{1}{\pi}\rho(E, r, \tau_p) \\
A_r(E, r) &= \frac{1}{\pi} \int_{\beta(E, r)}^{\alpha_+(E, r)} \frac{d\Gamma}{\sqrt{(\Gamma^2 - \alpha-(E, r))(\alpha_+(E, r) - \Gamma^2)}} \\
A_r(E, r) &= \frac{1}{\pi}\tau_p(E, r)
\end{align*}
\]

which allow us to rewrite (80) as the transformation generated by (81):

\[
\begin{align*}
A_* &= A(E, r) \\
\varphi_* &= \frac{\tau}{A_r(E, r)} \\
r_* &= r \\
R_* &= R + \frac{A_r(E, r)}{A_r(E, r)}r
\end{align*}
\]

The coordinates \((R_*, A_*, r_*, \varphi_*)\) are referred to as action–angle coordinates.

We conclude this section observing a non–negligible advantage while using action–angle coordinates compared to energy–time – besides the obvious one of dealing with a constant period. It is the law, mentioned in the introduction, that relates \(R\) to \(R_*\), which is (see (66), (69) and (80))

\[
R = R_* + \rho_*(A_*, r_*, \varphi_*) , \quad \text{with} \quad \rho_*(A_*, r_*, \varphi_*) := \tilde{\rho} \circ \phi_{an}(A_*, r_*, \varphi_*)
\]

where \(\tilde{\rho}\) as in (69). Here \(\rho_*(A_*, r_*, \varphi_*)\) is a periodic function because so is the function \(\tilde{\rho}\). This benefit is evident comparing with the corresponding formula with energy–time coordinates:

\[
R = R_* + B(E, r)\tau + \tilde{\rho}(E, r, \tau)
\]

which would include the uncomfortable linear term \(B(E, r)\tau\). Incidentally, such term would unnecessarily complicate the computations we are going to present in the next sections.

### 3.4 Regularising coordinates

In this section we define the the regularising coordinates.

The set (13) written in terms of \((A_*, \varphi_*)\) is

\[
S_0(r_*) = \left\{(A_*, \varphi_*): A_* = A_0(r_*), \varphi_* \in \mathbb{R}\right\} \quad 0 < r_* \leq 2
\]

with \(A_0(r_*)\) being the limiting value of \(A(E, r_*)\) when \(E = r_*:\)

\[
A_0(r_*) = \begin{cases} 
\sqrt{r_*(2-r_*)} - \frac{1}{\pi} \int_0^{\sqrt{r_*(2-r_*)}} \cos^{-1} \frac{r_* - \Gamma^2}{r_* \sqrt{1 - \Gamma^2}} d\Gamma & 0 < r_* < 1 \\
1 - \frac{1}{\pi} \int_0^{\sqrt{r_*(2-r_*)}} \cos^{-1} \frac{r_* - \Gamma^2}{r_* \sqrt{1 - \Gamma^2}} d\Gamma & 1 < r_* \leq 2
\end{cases}
\]

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We observe that the function \( A_\ast(r_\ast) \) is continuous in \([0, 2]\) (in particular, \( A_\ast(1^-) = A_\ast(1^+) \)), with
\[
A_\ast(0) = 0, \quad A_\ast(2) = 1
\]
and increases smoothly between those two values, as it results from the analysis of its derivative. Indeed, letting, for short, \( \sigma_0(r_\ast) := \sqrt{r_\ast(2 - r_\ast)} \) and proceeding analogously as (77), we get
\[
A'_\ast(r_\ast) = -\frac{1}{\pi} \int_0^{\sigma_0(r_\ast)} \frac{r_\ast - \Gamma^2}{r_\ast \sqrt{1 - \Gamma^2}} d\Gamma
= \frac{1}{\pi r_\ast} \int_0^{\sigma_0(r_\ast)} \frac{\Gamma d\Gamma}{\sqrt{\sigma_0(r_\ast)^2 - \Gamma^2}}
= \frac{1}{\pi} \sqrt{2 - r_\ast r_\ast} \quad \forall \ 0 < r_\ast < 2
\]
(85)
We denote as \( A_\ast \to r_\ast(A_\ast) \) the inverse function
\[
r_\ast := A_\ast^{-1}
\]
and we define two different changes of coordinates
\[
\phi^k_{ig} : (Y_k, A_k, y_k, \varphi_k) \to (R_\ast, A_\ast, r_\ast, \varphi_\ast) \quad k = \pm 1
\]
via the formulae
\[
\begin{align*}
R_\ast &= Y_k e^{ky_k} \\
A_\ast &= A_k \\
r_\ast &= -k e^{-ky_k} + r_\ast(A_k) \\
\varphi_\ast &= \varphi_k + Y_k e^{ky_k} \varphi'_\ast(A_k)
\end{align*}
\]
(87)
The transformations (87) are canonical, being generated by
\[
S^k_{ig}(Y_k, A_k, r_\ast, \varphi_\ast) := -\frac{Y_k}{k} \log \left| \frac{r_\ast(A_k) - r_\ast}{k} \right| + A_k \varphi_\ast.
\]
The coordinates \((Y_k, A_k, y_k, \varphi_k)\) with \(k = \pm 1\) are called regularising coordinates.

4 A deeper insight into energy–time coordinates

In this section we study the functions \( \tilde{G}(\mathcal{E}, r, \tau), \tilde{\rho}(\mathcal{E}, r, \tau), B(\mathcal{E}, r) \) and \( \tau_p(\mathcal{E}, r) \), described in Section 3.2. We prove that \( \tilde{G}(\mathcal{E}, r, \tau), \tilde{\rho}(\mathcal{E}, r, \tau) \) are \( C^\infty \) provided that \((\mathcal{E}, r)\) vary in a compact subset set of (67) and we study the behaviour of \( B(\mathcal{E}, r) \) and \( \tau_p(\mathcal{E}, r) \) closely to \( S_0(r) \).

It reveals to be useful to deal with suitable other functions \( \tilde{G}(\kappa, \theta), \tilde{\rho}(\kappa, \theta), A(\kappa) \) and \( T_0(\kappa) \) defined below. We rewrite
\[
\tilde{G}(\mathcal{E}, r, \tau) = \sigma(\mathcal{E}, r) \tilde{G}(\kappa(\mathcal{E}, r), \theta(\mathcal{E}, r, \tau)) \quad \tau_p(\mathcal{E}, r) = \frac{T_p(\kappa(\mathcal{E}, r))}{\sigma(\mathcal{E}, r)}
\]
(88)
and
\[
\tilde{\rho}(\mathcal{E}, r, \tau) = -\frac{\mathcal{E}}{r} + \frac{\sigma(\mathcal{E}, r)}{r} \tilde{\rho}(\kappa(\mathcal{E}, r), \theta(\mathcal{E}, r, \tau)) \quad 0 \leq \theta \leq T_p(\kappa)
\]
(89)
where (changing, in the integrals in (76), the integration variable \( \Gamma = \sigma \xi \)) \( \hat{G}(\kappa, \theta) \) is the unique solution of
\[
\int_{G(\kappa, \theta)}^{1} \frac{d\xi}{\sqrt{(1 - \xi^2)(\xi^2 - \kappa)}} = \theta, \quad 0 \leq \theta \leq T_p(\kappa)
\] (90)

\[
\hat{\rho}(\kappa, \theta) = \int_{G(\kappa, \theta)}^{1} \frac{\xi^2 d\xi}{\sqrt{(1 - \xi^2)(\xi^2 - \kappa)}} \quad 0 \leq \theta \leq T_p(\kappa)
\] (91)

and
\[
T_p(\kappa) = \begin{cases} 
T_0(\kappa) & 0 < \kappa < 1 \\
2T_0(\kappa) & \kappa < 0.
\end{cases}
\] (92)

with
\[
T_0(\kappa) := \int_{G_0(\kappa)}^{1} \frac{d\xi}{\sqrt{(1 - \xi^2)(\xi^2 - \kappa)}}, \quad \text{where} \quad G_0(\kappa) := \begin{cases} 
\sqrt{\kappa} & 0 < \kappa < 1 \\
0 & \kappa < 0.
\end{cases}
\] (93)

The function \( \hat{\rho}(\kappa, \theta) \) in (91) is further split as
\[
\hat{\rho}(\kappa, \theta) = A(\kappa)\theta + \hat{\rho}(\kappa, \theta)
\] (94)

where
\[
A(\kappa) = \frac{\hat{\rho}(\kappa, T_p(\kappa))}{T_p(\kappa)}, \quad \hat{\rho}(\kappa, \theta) = \hat{\rho}(\kappa, \theta) - A(\kappa)\theta
\] (95)

The periodicity of \( \hat{\rho}(\kappa, \cdot) \) (see equation (99) below), the uniqueness of the splitting (69) and the formulae in (89) and (94) imply that \( A(\kappa) \) and \( \hat{\rho}(\kappa, \theta) \) are related to \( B(\epsilon, r) \) and \( \hat{\rho}(\epsilon, r, \tau) \) in (69) via
\[
B(\epsilon, r) = -\frac{\epsilon}{r} + \frac{\sigma(\epsilon, r)^2}{r} A(\kappa), \quad \hat{\rho}(\epsilon, r, \tau) = \frac{\sigma(\epsilon, r)}{r} \hat{\rho}(\kappa(\epsilon, r), \theta(\epsilon, r, \tau))
\] (96)

Finally, \( \sigma(\epsilon, r), \kappa(\epsilon, r) \) and \( \theta(\epsilon, r, \tau) \) are given by
\[
\sigma(\epsilon, r) := \sqrt{\alpha_+(\epsilon, r)} = \sqrt{\epsilon - \frac{r^2}{2} + r\sqrt{1 + \frac{r^2}{4} - \epsilon}}
\]
\[
\kappa(\epsilon, r) := \frac{\alpha_-(\epsilon, r)}{\alpha_+(\epsilon, r)} = \frac{\epsilon^2 - r^2}{(\epsilon - \frac{r^2}{2} + r\sqrt{1 + \frac{r^2}{4} - \epsilon})^2}
\]
\[
\theta(\epsilon, r, \tau) := r\sqrt{\epsilon - \frac{r^2}{2} + r\sqrt{1 + \frac{r^2}{4} - \epsilon}}
\] (97)

In view of relations (88), (92) and (96), we focus on the functions \( \hat{G}(\kappa, \theta), \hat{\rho}(\kappa, \theta), A(\kappa) \) and \( T_0(\kappa) \). The proofs of the following statements are postponed at the end of the section.

Let us denote \( \hat{G}_{ij}(\kappa, \theta) := \partial^{i+j}_{\kappa \theta} \hat{G}(\kappa, \theta), \hat{\rho}_{ij}(\kappa, \theta) := \partial^{i+j}_{\kappa \theta} \hat{\rho}(\kappa, \theta) \).

**Proposition 4.1** Let \( 0 \neq \kappa < 1 \) fixed. The functions \( \hat{G}_{ij}(\kappa, \cdot) \) and \( \hat{\rho}_{ij}(\kappa, \cdot) \) are continuous for all \( \theta \in \mathbb{R} \).
This immediately implies

**Corollary 4.1** Let $K \subset \mathbb{R}$ a compact set, with $0, 1 \notin K$. Then $\tilde{G}, \tilde{\rho}$ are $C^\infty(K \times \mathbb{T})$.

Concerning $T_0(\kappa)$, we have

**Proposition 4.2** Let $0 \neq \kappa < 1$, and let $T_0(\kappa)$ be as in (93). Then one can find two real numbers $C^*, R^*, S^*$ and two functions $\mathcal{R}(\kappa), \mathcal{S}(\kappa)$ verifying

$$
\mathcal{R}(0) = 1 = \mathcal{S}(0), \quad 0 \leq \mathcal{R}(\kappa) \leq R^*, \quad 0 \leq \mathcal{S}(\kappa) \leq S^* \quad \forall \kappa \in (-1, 1)
$$

such that

$$
T_0'(\kappa) = -\frac{\mathcal{R}(\kappa)}{2\kappa}, \quad T_0''(\kappa) = \frac{\mathcal{S}(\kappa)}{4\kappa^2}, \quad \forall \ 0 \neq \kappa < 1
$$

In particular,

$$
|T_0(\kappa)| \leq \frac{R^*}{2} \left| \log |\kappa| \right| + C^*, \quad |T_0'(\kappa)| \leq \frac{R^*}{2} \left| |\kappa|^{-1} \right|, \quad |T_0''(\kappa)| \leq \frac{S^*}{4} \left| |\kappa|^{-2} \right|
$$

Finally, as for $A(\kappa)$, we have

**Proposition 4.3** Let $0 \neq \kappa < 1$, and let $A(\kappa)$ be as in (95). Then one can find $C^* > 0$ such that

$$
|A(\kappa)| \leq C^* \left| \log |\kappa| \right|^{-1}, \quad |A'(\kappa)| \leq C^* \left| |\kappa|^{-1} \right|, \quad |A''(\kappa)| \leq C^* \left| |\kappa|^{-2} \right|
$$

**Proof of Proposition 4.1** Relations (68), (71) and (72) provide

\[
\left\{
\begin{array}{ll}
\tilde{G}(\kappa, \theta + 2jT_p) = \tilde{G}(\kappa, \theta) & \forall \theta \in \mathbb{R}, \ j \in \mathbb{Z} \quad \forall \ 0 \neq \kappa < 1 \\
\tilde{G}(\kappa, -\theta) = \tilde{G}(\kappa, \theta) & \forall \ 0 \leq \theta \leq T_p(\kappa) \quad \forall \ 0 \neq \kappa < 1 \\
\tilde{G}(\kappa, T_p - \theta) = -\tilde{G}(\kappa, \theta) & \forall \ 0 \leq \theta \leq T_0(\kappa) \quad \forall \ \kappa < 0.
\end{array}
\right.
\]

(98)

\[
\left\{
\begin{array}{ll}
\tilde{\rho}(\kappa, \theta + 2jT_p) = \tilde{\rho}(\kappa, \theta) , & \forall \theta \in \mathbb{R}, \ j \in \mathbb{Z} \quad \forall \ 0 \neq \kappa < 1 \\
\tilde{\rho}(\kappa, -\theta) = -\tilde{\rho}(\kappa, \theta) & \forall \ 0 \leq \theta \leq T_p(\kappa) \quad \forall \ 0 \neq \kappa < 1 \\
\tilde{\rho}(\kappa, T_p - \theta) = -\tilde{\rho}(\kappa, \theta) & \forall \ 0 \leq \theta \leq T_0(\kappa) \quad \forall \ \kappa < 0.
\end{array}
\right.
\]

(99)

The following lemmata are obvious

**Lemma 4.1** Let $g(\kappa, \cdot)$ verify (98) with $T_p(\kappa) = \pi$ for all $\kappa$ and $T_0$ as in (92). Then the functions $g_{ij}(\kappa, \theta) := \partial_{\kappa_j \theta_i}^{ij} \tilde{G}(\kappa, \theta)$ are continuous on $\mathbb{R}$ if and only if they are continuous in $[0, T_0]$ and verify

\[
\left\{
\begin{array}{ll}
\text{no further condition} & \text{if } \ j \in 2\mathbb{N}, \quad 0 < \kappa < 1 \\
g_{ij}(\kappa, \frac{\pi}{2}) = 0 & \text{if } \ j \in 2\mathbb{N}, \quad \kappa < 0 \\
g_{ij}(\kappa, 0) = g_{ij}(\kappa, \pi) & \text{if } \ j \in 2\mathbb{N} + 1, \quad 0 < \kappa < 1 \\
g_{ij}(\kappa, 0) = 0 & \text{if } \ j \in 2\mathbb{N} + 1, \quad \kappa < 0
\end{array}
\right.
\]

(100)
Lemma 4.2 Let \( g(\kappa, \cdot) \) verify (99) with \( T_p(\kappa) = \pi \) for all \( \kappa \) and \( T_0 \) as in (92). Then \( g_{ij}(\kappa, \cdot) \), where \( g_{ij}(\kappa, \theta) := \partial^{i+j}_{\kappa, \theta} g(\kappa, \theta) \), are continuous on \( \mathbb{R} \) if and only if they are continuous in \([0, T_0(\kappa)]\) and verify

\[
\begin{align*}
\{ \begin{array}{ll}
g_{ij}(\kappa, 0) = g_{ij}(\kappa, \pi) = 0 & \text{if } j \in 2\mathbb{N}, \quad 0 < \kappa < 1 \\
g_{ij}(\kappa, 0) = g_{ij}(\kappa, \frac{\pi}{2}) = 0 & \text{if } j \in 2\mathbb{N} \quad \kappa < 0 \\
o \text{ further condition} & \text{if } j \in 2\mathbb{N} + 1
\end{array} \}
\tag{101}
\]

We now proceed with the proof of Proposition 4.1. (i) The function \( \tilde{G}(\kappa, \cdot) \) is \( C^\infty(\mathbb{R}) \) for all \( 0 \neq \kappa < 1 \) [18]. Then so is the function \( g(\kappa, \cdot) \), where \( g(\kappa, \theta) := \tilde{G}(\kappa, \frac{T_0(\kappa)}{\pi} \theta) \). Then (100) hold true for \( g(\kappa, \cdot) \) with \( i = 0 \). Hence, the derivatives \( g_{ij}(\kappa, \theta) \), which exist for all \( 0 \neq \kappa < 1 \), also verify (100). Then \( g_{ij}(\kappa, \cdot) \) are continuous for all \( 0 \neq \kappa < 1 \) and so are the \( \tilde{G}_{ij}(\kappa, \cdot) \).

(ii) We check conditions (101) for the function \( g(\kappa, \theta) := \hat{\rho}(\kappa, \frac{T_0(\kappa)}{\pi} \theta) \), in the case \( j = 0 \). Using (91), (90) and (95), we get, for \( 0 < \kappa < 1 \),

\[
g(\kappa, 0) = \hat{\rho}(\kappa, 0) = 0, \quad g(\kappa, \pi) = \hat{\rho}(\kappa, T_p(\kappa)) = \hat{\rho}(\kappa, T_p) - \frac{\hat{\rho}(\kappa, T_0)}{T_p} T_p = 0.
\tag{102}
\]

while, for \( \kappa < 0 \),

\[
g(\kappa, 0) = \hat{\rho}(\kappa, 0) = 0, \quad g\left(\kappa, \frac{\pi}{2}\right) = \hat{\rho}(\kappa, T_0(\kappa)) = \hat{\rho}(\kappa, T_0) - \frac{\hat{\rho}(\kappa, T_0)}{T_0} T_0 = 0.
\tag{103}
\]

The identities (102) and (103) still hold replacing \( g \) with any \( g_{i0}(\kappa, \theta) \), with \( i \in \mathbb{N} \), therefore, any \( g_{i0}(\kappa, \theta) \) satisfies (101). Let us now consider the case \( j \neq 0 \). Again by (91), (90) and (95),

\[
\hat{\rho}_{ij}(\kappa, \theta) = \tilde{G}(\kappa, \theta)^2 - \mathcal{A}(\kappa)
\tag{104}
\]

so, for any \( j \neq 0 \),

\[
\hat{\rho}_{ij}(\kappa, \theta) = \partial^{i+j-1}_{\kappa, \theta} \left( \tilde{G}(\kappa, \theta)^2 \right)
\]

Then the \( \hat{\rho}_{ij}(\kappa, \cdot) \) with \( j \neq 0 \) are continuous because so is \( \tilde{G}_{ij}(\kappa, \cdot) \). \( \Box \)

Proof of Proposition 4.2 The function \( T_0(\kappa) \) in (93) is studied in detail in Appendix A. Combining Lemma A.1 and Proposition A.1 and taking the \( \kappa \)-primitive of such relations, one obtains Proposition 4.2.

Proof of Proposition 4.3

\[
\mathcal{A}(\kappa) = \frac{1}{T_0(\kappa)} \int_{T_0(\kappa)}^{1} \frac{\sqrt{\xi^2 - \kappa}}{\sqrt{1 - \xi^2}} d\xi + \kappa
\]

\[
\mathcal{A}'(\kappa) = \frac{1}{2} + (\kappa - \mathcal{A}(\kappa)) \frac{T_0'(\kappa)}{T_0(\kappa)} = \frac{1}{2} - (\kappa - \mathcal{A}(\kappa)) \frac{\mathcal{R}(\kappa)}{2\kappa T_0(\kappa)}
\]

\[
= \frac{1}{2} - \frac{\mathcal{R}(\kappa)}{2T_0(\kappa)} + \frac{\mathcal{A}(\kappa)\mathcal{R}(\kappa)}{2\kappa T_0(\kappa)}
\]

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and

\[ A''(\kappa) = (1 - A'(\kappa)) \frac{T_0'(\kappa)}{T_0(\kappa)} + (\kappa - A(\kappa)) \left( \frac{T_0''(\kappa)}{T_0(\kappa)} - \frac{(T_0'(\kappa))^2}{(T_0(\kappa))^2} \right) \]

\[ = \frac{T_0'(\kappa)}{2T_0(\kappa)} - 2(\kappa - A(\kappa)) \frac{(T_0'(\kappa))^2}{(T_0(\kappa))^2} + (\kappa - A(\kappa)) \frac{T_0''(\kappa)}{T_0(\kappa)} \]

\[ = -\frac{R(\kappa)}{4\kappa T_0(\kappa)} - 2(\kappa - A(\kappa)) \frac{R(\kappa)^2}{4\kappa^2 T_0(\kappa)} + (\kappa - A(\kappa)) \frac{S(\kappa)}{4\kappa^2 T_0(\kappa)} \]

\[
\square
\]

5 The function \( F(\mathcal{E}, r) \)

In this section we study the function \( F(\mathcal{E}, r) \) in (10). Specifically, we aim to prove the following

**Proposition 5.1** \( F(\mathcal{E}, r) \) is well defined and smooth for all \( (\mathcal{E}, r) \) with \( 0 \leq r \leq 2 \) and \( -r < \mathcal{E} < 1 + \frac{r^2}{4} \), \( \mathcal{E} \neq r \). Moreover, there exists a number \( C > 0 \) and a neighbourhood \( \mathcal{O} \) of \( 0 \in \mathbb{R} \) such that, for all \( 0 \leq r \leq 2 \) and all \( -r < \mathcal{E} < 1 + \frac{r^2}{4} \) such that \( \mathcal{E} - r \in \mathcal{O} \),

\[
|F(\mathcal{E}, r)| \leq C \log |\mathcal{E} - r|^{-1}, \quad |\partial_{\mathcal{E}, r} F(\mathcal{E}, r)| \leq C |\mathcal{E} - r|^{-1}, \quad |\partial_{\mathcal{E}, r}^2 F(\mathcal{E}, r)| \leq C |\mathcal{E} - r|^{-2}.
\]

To prove Proposition 5.1 we need an analytic representation of the function \( F \), which we proceed to provide. In terms of the coordinates (64), the function \( U \) in (9) is given by (recall we have fixed \( \Lambda = 1 \))

\[
U(r, G, g) = -\frac{1}{2\pi} \int_{0}^{2\pi} \frac{(1 - \sqrt{1 - G^2} \cos \xi) d\xi}{\sqrt{(1 - \sqrt{1 - G^2} \cos \xi)^2 + 2r \left( (\cos \xi - \sqrt{1 - G^2} \cos g - G \sin \xi \sin g \right) + r^2}}.
\]

(106)

By [33], \( U \) remains constant along the level curves, at \( r \) fixed, of the function \( E(r, \cdot, \cdot) \) in (11). Therefore, the function \( F(\mathcal{E}, r) \) which realises (10) can be obtained taking \( F(\mathcal{E}, r) \) to be, for the level set \( \mathcal{E} \)-level set of \( E(r, \cdot, \cdot) \), the value that \( U(r, \cdot, \cdot) \) takes at a chosen fixed point belonging such level set. For the purposes\(^9\) of the paper, we choose such point to be the point where the \( \mathcal{E} \)-level curve attains its maximum. It follows from the discussion in Section 3.2 that the coordinates of such point are

\[
\begin{cases}
G_+ = \sqrt{\alpha_+} \\
\begin{cases}
\pi & -r \leq \mathcal{E} < 1 \\
0 & 1 \leq \mathcal{E} \leq 1 + \frac{r^2}{4}
\end{cases}
\end{cases}
\]

(107)

where \( \alpha_+(\mathcal{E}, r) \) is as in (74). Replacing (107) into (106), we obtain

\[
F(\mathcal{E}, r) = -\frac{1}{2\pi} \int_{0}^{2\pi} \frac{(1 - |e(\mathcal{E}, r)| \cos \xi) d\xi}{\sqrt{(1 - |e(\mathcal{E}, r)| \cos \xi)^2 + 2s(\mathcal{E}, r)r(\cos \xi - |e(\mathcal{E}, r)|) + r^2}}
\]

(108)

\^9\text{Compare (108) with the simpler formula proposed in [35], however valid only for values of \( \mathcal{E} \) in the interval \([-r, r]\).}
with
\[ e(\mathcal{E}, r) = \frac{r}{2} - \sqrt{1 + \frac{r^2}{4} - \mathcal{E}}, \quad s(\mathcal{E}, r) := \text{sign}(e(\mathcal{E}, r)) = \begin{cases} -1 & -r \leq \mathcal{E} < 1 \\ +1 & 1 < \mathcal{E} \leq 1 + \frac{r^2}{4} \end{cases} \]

To study the regularity of \( F \), it turns to be useful to rewrite the integral (108) as twice the integral on the half period \([0, \pi] \) and next to make two subsequent changes of variable. The first time, with \( z = s(\mathcal{E}, r) \cos x \). It gives the following formula, which will be used below.

\[ F(\mathcal{E}, r) = -\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1 - z^2}} \frac{(1 - e(\mathcal{E}, r)z)dz}{\sqrt{(1 - e(\mathcal{E}, r)z)^2 + 2r(z - e(\mathcal{E}, r)) + r^2}} \]

We denote as
\[ z_{\pm}(\mathcal{E}, r) := \frac{e(\mathcal{E}, r) - r}{e(\mathcal{E}, r)^2} \pm \sqrt{\frac{1}{2} + \frac{r(1 - e(\mathcal{E}, r)) - \frac{1}{2} e(\mathcal{E}, r)^2}{e(\mathcal{E}, r)^2}} \]

the roots of the polynomial under the square root, which, as we shall see below, are real under conditions (67). As a second change, we let \( z = \frac{1 - \beta \gamma}{1 + \beta \gamma t^2} \). This leads to write \( F(\mathcal{E}, r) \) as

\[ F(\mathcal{E}, r) = -\frac{2(1 - e(\mathcal{E}, r))}{\pi |e(\mathcal{E}, r)| \sqrt{(z_+(\mathcal{E}, r) + 1)(z_+(\mathcal{E}, r) - 1)}} \left( \frac{1 + e(\mathcal{E}, r)}{1 - e(\mathcal{E}, r)} j_0(\kappa(\mathcal{E}, r)) - \frac{2e(\mathcal{E}, r)}{1 - e(\mathcal{E}, r)} j_\beta(\kappa(\mathcal{E}, r)) \right) \]

where \( j_\beta(\kappa) \) is the elliptic integral

\[ j_\beta(\kappa) := \int_0^{+\infty} \frac{1}{1 + \beta t^2} \frac{dt}{\sqrt{(1 + t^2)(1 + \kappa t^2)}} \]

and \( \beta, \gamma \) and \( \kappa \) are taken to be

\[ \beta(\mathcal{E}, r) := \frac{z_-(\mathcal{E}, r) - 1}{1 + z_-(\mathcal{E}, r)}, \quad \kappa(\mathcal{E}, r) := \frac{(1 + z_+(\mathcal{E}, r))(z_-(\mathcal{E}, r) - 1)}{(1 + z_-(\mathcal{E}, r))(z_+(\mathcal{E}, r) - 1)} \]

The elliptic integrals \( j_\beta(\kappa) \) in (112) are studied in Appendix A: compare Proposition A.1.

In terms of \((e, r)\), the inequalities in (67) become

\[ r \in [0, 2], \quad e \in \left[-1, \frac{r}{2}\right] \setminus \{0, r - 1\} \subset [-1, 1]. \]

where \( \{e = -1\} \) corresponds to the minimum level \( \{\mathcal{E} = -r\} \); \( \{e = r - 1\} \) corresponds to the separatrix level \( S_0(r) \); \( \{e = 0\} \) corresponds to the separatrix level \( S_1(r) \) and, finally, \( \{e = \frac{r}{2}\} \) corresponds to maximum level \( \{\mathcal{E} = 1 + \frac{r^2}{4}\} \). It is so evident that the discriminant in (110) is not negative under conditions (113), so \( z_{\pm}(\mathcal{E}, r) \) are real under (67), as claimed. In addition, one can easily verify that, for any \((r, e)\) as (113), it is \( e^2 + e - r \leq 0 \). This implies

\[ z_+ + 1 = \frac{e(\mathcal{E}, r)^2 + e(\mathcal{E}, r) - r}{e(\mathcal{E}, r)^2} + \sqrt{\frac{1}{2} + \frac{r(1 - e(\mathcal{E}, r)) - \frac{1}{2} e(\mathcal{E}, r)^2}{e(\mathcal{E}, r)^2}} < 0 \quad \forall e \neq r - 1 \]

Moreover, since

\[ z_-(\mathcal{E}, r) < z_+(\mathcal{E}, r) \quad \forall r \neq 0, \quad \mathcal{E} \neq 1 + \frac{r^2}{2}, \quad \mathcal{E} \neq -r, \quad (\mathcal{E}, r) \neq (2, 2) \]
we have
\[ \beta(\mathcal{E}, r) > 0 \quad \forall (\mathcal{E}, r) \] as in (114)
and
\[ 0 < \kappa(\mathcal{E}, r) < 1 \quad \forall (\mathcal{E}, r) \] as in (114) and \( \mathcal{E} \neq r - 1 \).

Combining these informations with the representation (111)–(111) and Proposition A.1, we conclude that \( F(\mathcal{E}, r) \) is smooth for all \( r \neq 0 \), \( \mathcal{E} \neq 1 \), \( \mathcal{E} \neq 1 + \frac{\varepsilon}{T} \), \( \mathcal{E} \neq \pm r \), \( (\mathcal{E}, r) \neq (2, 2) \) and that (105) holds. However, the representation in (109) allows to extend regularity for \( F(\mathcal{E}, r) \) to the domain \( 0 \leq r \leq 2, -r \leq \mathcal{E} < 1 + \frac{\varepsilon^2}{4}, \mathcal{E} \neq r \), as claimed. □

6 Proof of Theorem B

In this section we state and prove a more precise statement of Theorem B, which is Theorem 6.1 below.

The framework is as follows:

- fix an energy level \( c \);
- change the time via
  \[ \frac{dt}{dt'} = e^{-2ky} \quad k = \pm 1 \] (115)
  where \( t' \) is the new time and \( t \) the old one. The new time is soon renamed \( t \).
- look at the ODE
  \[ \partial_{t'} q_k = X^{(k)}(q_k; c) \]
  for the triple \( q_k = (A_k, y_k, \psi) \) where \( A_k, y_k \) are as in (87), while \( \psi = \varphi_* \), with \( \varphi_* \) as in (80) in \( \mathcal{P}_k \), where
  \[ \mathcal{P}_k(\varepsilon_-, \varepsilon_+, L_-, L_+, \xi) := \left\{ (A_k, y_k, \psi) : 1 - 2\varepsilon_+ < A_k \leq 1 - 2\varepsilon_-, L_- + 2\xi \leq ky_k \leq L_+ - 2\xi, \psi \in T \right\} \]
  with \( \xi < (L_+ - L_-)/4 \).
- observe that
  - the projection of \( \mathcal{P}_+ \) in the plane \( (g, G) \) in Figure 1 is an inner region of \( S_0(r) \) and \( r \) varies in a \( \varepsilon \)-left neighborhood of 2;
  - the projection of \( \mathcal{P}_- \) in the plane \( (g, G) \) in Figure 1 is an outer region of \( S_0(r) \) and \( r \) varies in a \( \varepsilon \)-left neighborhood of 2;
  - the boundary of \( \mathcal{P}_k \) includes \( S_0 \) if \( L_+ = \infty \); it has a positive distance from it if \( L_+ < +\infty \).

We shall prove

**Theorem 6.1** There exist a graph \( \mathcal{G}_k \subset \mathcal{P}_k(\varepsilon_-, \varepsilon_+, L_-, L_+, \xi) \) and a number \( L_* > 1 \) such that for any \( L_- > L_* \) there exist \( \varepsilon_- , \varepsilon_+ , L_+ , \xi \), a open neighbourhood \( W_k \supset \mathcal{G}_k \) such that along any orbit \( q_k(t) \) such that \( q_k(0) \in W_k \),
\[ |A(q_k(t)) - A(q_k(0))| \leq C_0 \epsilon t e^{-L_* t^2} \quad \forall t : |t| < t_{ex} \]
where \( t_{ex} \) is the first \( t \) such that \( q(t) \notin W_k \) and \( \epsilon \) is an upper bound for \( \|P_t\|_{W_k} \).
Proof For definiteness, from now on we discuss the case \( k = 1 \) (outer orbits). The case \( k = -1 \) (inner orbits) is pretty similar. We neglect to write the sub-fix “+1” everywhere. As the proof is long and technical, we divide it in paragraphs. We shall take

\[
G = \left\{ (A, y, \psi_0(A, y)), \ 1 - 2\varepsilon_+ \leq A \leq 1 - 2\varepsilon_-, \ L_- + 2\xi \leq y \leq L_+ - 2\xi \right\} \subset \mathbb{P}
\]

with \( \varepsilon_-, \varepsilon_+, L_-, L_+ \), \( \psi_0 \) to be chosen below.

Step 1. The vector-field \( X \) As \( \psi \) is one of the action–angle coordinates, while \( A, y \) are two among the regularising coordinates, we need the expressions of the Hamiltonian (9) written in terms of those two sets. The Hamiltonian (9) written in action–angle coordinates is

\[
H_{aa}(\mathcal{R}_s, A, r_s, \varphi_s) = \frac{(\mathcal{R}_s + \rho_e(A_s, r_s, \varphi_s))^2}{2} + \alpha F_e(A_s, r_s) + \frac{(C - G_e(A_s, r_s, \varphi_s))^2}{2r_s^2} - \frac{\beta}{r_s}
\]

where

\[
G_e(A_s, r_s, \varphi_s) := G \circ \phi_{aa}(A_s, r_s, \varphi_s), \quad F_e(A_s, r_s) := F \circ \phi_{aa}(A_s, r_s)
\]

with \( \phi_{aa} \) as in (80), while \( \tilde{G}(E, r, \tau), \tilde{F}(E, r) \) as in (66), (10), respectively, \( \rho_e \) as in (84). The Hamiltonian (9) written in regularising coordinates is

\[
H_{rg}(Y, A, y, \varphi) = \frac{(Ye^y + \rho_e(A, r_o(A, y), \varphi_0(Y, A, y, \varphi))^2}{2} + \alpha F_e(A, r_o(A, y))
\]

\[
+ \frac{(C - G_e(A, r_o(A, y), \varphi_0(Y, A, y, \varphi))^2}{2r_o(A, y)^2} - \frac{\beta}{r_o(A, y)}
\]

where \( r_o(A, y), \varphi_0(Y, A, y, \varphi) \) are the right hand sides of the equations for \( r_s, \varphi_s \) in (87), with \( k = +1 \).

Taking the \( \varphi_s \)-projection of Hamilton equation of \( H_{aa} \), and the \( (A, y) \)-projection of Hamilton equation of \( H_{rg} \), changing the time as prescribed in (115) and reducing the energy via

\[
\mathcal{R}_s + \rho_e(A, r_o(A, y), \psi) = Ye^y + \rho_e(A, r_o(A, y), \psi) = \mathcal{Y}(A, y, \psi; c)
\]

with

\[
\mathcal{Y}(A, y, \psi; c) := \pm \sqrt{2 \left( c - \alpha F_e(A, r_o(A, y)) - \frac{(C - G_e(A, r_o(A, y), \psi))^2}{2r_o(A, y)^2} + \frac{\beta}{r_o(A, y)} \right)}
\]

we find that the evolution for the triple \( q = (A, y, \psi) \) during the time \( t' \) is governed by the vector-field

\[
\begin{align*}
X_1(A, y, \psi; c) &= e^{-2y} \frac{C - G_e(A, r_o(A, y), \psi)}{r_o(A, y)^2} \frac{G_{s, 0}(A, r_o(A, y), \psi)}{r_o(A, y)^2} G_{s, 3}(A, r_o(A, y), \psi) - e^{-2y} \rho_{s, 3}(A, r_o(A, y), \psi) \mathcal{Y}(A, y, \psi; c) \\
X_2(A, y, \psi; c) &= -e^{-y} \frac{C - G_e(A, r_o(A, y), \psi)}{r_o(A, y)^2} \frac{G_{s, 0}(A, r_o(A, y), \psi)}{r_o(A, y)^2} G_{s, 3}(A, r_o(A, y), \psi) \mathcal{Y}(A, y, \psi; c) \\
X_3(A, y, \psi; c) &= \alpha e^{-2y} F_{s, 1}(A, r_o(A, y)) - e^{-2y} \frac{C - G_e(A, r_o(A, y), \psi)}{r_o(A, y)^2} G_{s, 1}(A, r_o(A, y), \psi) \\
&\quad + e^{-2y} \rho_{s, 1}(A, r_o(A, y), \psi) \mathcal{Y}(A, y, \psi; c)
\end{align*}
\]

where we have used the notation, for \( f = \rho_e, G_e, F_e \),

\[
f_1(A, r_s, \psi) := \partial_A f(A, r_s, \psi), \quad f_3(A, r_s, \psi) := \partial_\psi f(A, r_s, \psi).
\]
Step 2. Splitting the vector-field

We write

\[ X(A, y, \psi; c) = N(A, y; c) + P(A, y, \psi; c) \]

with

\[
\begin{align*}
N_1(A, y; c) &= 0 \\
N_2(A, y; c) &= v(A, y; c) := e^{-y} \sqrt{2(c - \alpha F_*(A, r_0(A, y)))} \\
N_3(A, y; c) &= \omega(A, y; c) := \alpha e^{-2y} F_{*,1}(A, r_0(A, y))
\end{align*}
\]

hence,

\[
\begin{align*}
P_1 &= e^{-2y} \frac{C - G_*(A, r_0(A, y), \psi)}{r_0(A, y)^2} G_{*,3}(A, r_0(A, y), \psi) - e^{-2y} \rho_{*,3}(A, r_0(A, y), \psi)Y(A, y; c) \\
P_2 &= -e^{-y} \frac{C - G_*(A, r_0(A, y), \psi)}{r_0(A, y)^2} G_{*,3}(A, r_0(A, y), \psi)Y_{r_0}'(A) + e^{-y} \rho_{*,3}(A, r_0(A, y), \psi)Y_{r_0}'(A) \\
\quad \cdot Y(A, y; c) + e^{-y} \left( Y(A, y; c) - \sqrt{2(c - \alpha F_*(A, r_0(A, y)))} \right) \\
P_3 &= -e^{-2y} \frac{C - G_*(A, r_0(A, y), \psi)}{r_0(A, y)^2} G_{*,1}(A, r_0(A, y), \psi) + e^{-2y} \rho_{*,1}(A, r_0(A, y), \psi)Y(A, y; c)
\end{align*}
\]

(118)

The application of NFL relies on the smallness of the perturbing term \( P \). In the case in point, the “greatest” term of \( P \) is the component \( P_2 \), and precisely \( \rho_{*,3} \). This function is not uniformly small. For this reason, we need to look at its zeroes and localise around them. The localisation (described in detail below) carries the holomorphic perturbation \( P \) to a perturbation \( \bar{P} \), which is smaller, but no longer holomorphic. We shall apply GNFL to the new vector-field \( \bar{X} = N + \bar{P} \).

Step 3. Localisation about non-trivial zeroes of \( \rho_{*,3} \)

The following lemma gives an insight on the term \( \rho_{*,3} \), appearing in (118). It will be proved in Appendix B.

Lemma 6.1 For any \( A \neq r_0(A, r_s) < A < 1 \) \((0 < A < A \neq r_s)\) there exists \( 0 < \psi_*(A, r_s) < \pi \) \((0 < \psi_*(A, r_s) < \pi/2)\) such that \( \rho_{*,3}(A, r_s, \psi_*(A, r_s)) \equiv 0 \) \((\text{and } \rho_{*,3}(A, r_s, \pi - \psi_*(A, r_s)) \equiv 0)\). Moreover, there exists \( C > 0 \) such that, for any \( \delta > 0 \) one can find a neighbourhood \( V_*(A, r_s; \delta) \) of \( \psi_*(A, r_s) \) (and a neighbourhood \( V'(A, r_s; \delta) \) of \( \pi - \psi_*(A, r_s) \)) such that

\[
\begin{align*}
|\rho_{*,3}(A, r_s, \psi)| &\leq C \frac{\sigma_*(A, r_s)}{r_s} \delta \quad \forall \psi \in V_*(A, r_s; \delta) \\
\left( |\rho_{*,3}(A, r_s, \psi)| &\leq C \frac{\sigma_*(A, r_s)}{r_s} \delta \quad \forall \psi \in V_*(A, r_s; \delta) \cup V'(A, r_s; \delta) \right).
\end{align*}
\]

(119)

We now let

\[
\psi_0(A, y) := \psi_*(A, r(A, y)), \quad V_0(A, y; \delta) := V_*(A, r(A, y); \delta).
\]

For definiteness, from now on, we focus on orbits with initial datum \((A_0, y_0, \psi_0)\) such that \( \psi_0 \) is close to \( \psi_0(A_0, y_0) \). The symmetrical cases can be similarly treated.
Let \( W_o(A, y; \delta) \subset V_o(A, y; \delta) \) an open set and let \( g(A, y, \cdot) \) be a \( C^\infty, 2\pi \)-periodic function such that, in each period \([\psi_o(A, y) - \pi, \psi_o(A, y) + \pi)\) satisfies

\[
\begin{align*}
g(A, y, \psi; \delta) &\equiv 1 \quad \forall \psi \in W_o(A, y; \delta) \\
g(A, y, \psi; \delta) &\equiv 0 \quad \forall \psi \in [\psi_o(A, y) - \pi, \psi_o(A, y) + \pi) \setminus V_o(A, y; \delta) \\
\end{align*}
\]

\( \in (0, 1) \quad \forall \psi \in V_o(A, y; \delta) \setminus W_o(A, y; \delta) \) \hspace{1cm} \tag{120}

with

\[
g_0 2^\ell \geq \sup_{\ell \in \mathbb{Z}} \|g\|_{u, \ell} \quad \ell \in \mathbb{Z}
\]

Let \( \tilde{P}(A, y, \psi; \delta) := g(A, y, \psi; \delta)P(A, y, \psi) \). \hspace{1cm} \tag{122}

We let

\[
\tilde{X} := N + \tilde{P}
\]

and

\[
P_{\varepsilon, \xi} = A_{\varepsilon -} \times \mathbb{Y}_{\xi} \times T,
\]

where \( A = [1 - 2\varepsilon_+, 1 - 2\varepsilon_-], \mathbb{Y} = [L_- + 2\xi, L_+ - 2\xi] \) and \( \varepsilon_- < \varepsilon_+, \xi \) are sufficiently small, and \( u = (\varepsilon_-, \xi) \). By construction, \( \tilde{X} \) and \( \tilde{P} \in C^3_0 \).

**Step 4. Bounds** The following uniform bounds follow rather directly from the definitions. Their proof is deferred to Appendix B, in order not to interrupt the flow.

\[
\begin{align*}
\left\| \frac{1}{v} \right\|_u &\leq C \frac{e^{L_+}}{\alpha L_-^2}, \quad \left\| \frac{\partial_A v}{v} \right\|_u \leq C \frac{e^{L_+}}{L_- \sqrt{\varepsilon_-}}, \quad \left\| \frac{\partial_v v}{v} \right\|_u \leq 1 + C \frac{e^{L_+ - L_-}}{L_-^2} \\
\left\| \frac{\omega}{v} \right\|_u &\leq C \frac{e^{L_+ - L_-}}{L_-^{3/2}}, \quad \left\| \frac{\partial_A \omega}{v} \right\|_u \leq C \frac{e^{2L_+ - L_-}}{L_-^{3/2} \varepsilon_-^{1/2}}, \quad \left\| \frac{\partial_v \omega}{v} \right\|_u \leq C \frac{e^{2L_+ - 2L_-}}{L_-^{3/2}} \\
\end{align*}
\]

\( \| \tilde{P}_1 \|_u \leq C e^{-2L_-} \max \left\{ \| \mathcal{C} \|_{L_+ \sqrt{\varepsilon_+}, L_+ \varepsilon_+, \delta \sqrt{\varepsilon_+} \sqrt{\alpha L_+}} \right\} \)

\( \| \tilde{P}_2 \|_u \leq C e^{-L_-} \max \left\{ \| \mathcal{C} \|_{L_+ \sqrt{\varepsilon_+}, L_+ \varepsilon_+, \sqrt{\varepsilon_+ \delta \sqrt{\alpha L_+}, (\alpha L_-)^{-1} \max \{\| \mathcal{C} \|^2, \varepsilon_+, \beta} \left\} \right\} \)

\( \| \tilde{P}_3 \|_u \leq C e^{-2L_-} \max \left\{ \| \mathcal{C} \|_{\sqrt{\varepsilon_+}, \varepsilon_+, \sqrt{\varepsilon_+ \varepsilon_-} \sqrt{\alpha L_+}} \right\} \)

\hspace{1cm} \tag{125}

Here \( C \) is a number not depending on \( L_-, L_+, \xi, \varepsilon_-, \varepsilon_+, c, \| \mathcal{C} \|, \beta, \alpha \) and the norms are meant as in Section 2.5, in the domain \( (123) \). Remark that the validity of \( (125) \) is subject to condition

\[
L_- \geq C \alpha^{-1} \max \{ |c|, \| \mathcal{C} \|^2, \varepsilon_+, \beta \}.
\]

\hspace{1cm} \tag{126}

which will be verified below.
Step 5. Application of GNFL and conclusion  Fix $s_1, s_2 > 0$. Define

$$
\rho := \varepsilon_-, \quad \tau := e^{-s_2} \frac{\xi}{16}, \quad w_K := \left(\frac{\varepsilon_-}{16}, \frac{e^{-s_2} \xi}{16}, \frac{1}{c_0 K^{1+\delta}}\right)
$$

so that (46) are satisfied. With these choices, as a consequence of the bounds in (124)–(125), one has

$$
\chi \leq C(L_+ - L_-) \max \left\{ \frac{e^{L_+ - L_-}}{s_1 L_{-}^{3/2}}, \frac{1}{s_2} \left(1 + C e^{L_+ - L_-} \right) \right\}
$$

$$
\theta_1 \leq C e^{s_1 (L_+ - L_-)} \xi K^{1+\delta} \frac{e^{2L_+ - 2L_-}}{L_{-}^{3/2}}
$$

$$
\theta_2 \leq C e^{s_1 + s_2 (L_+ - L_-)} \sqrt{\varepsilon_-} \frac{e^{L_+ - L_-}}{L_{-}^{3/2}}
$$

$$
\theta_3 \leq C (L_+ - L_-) K^{1+\delta} \sqrt{\varepsilon_-} \frac{e^{2L_+ - L_-}}{L_{-}^{3/2}}
$$

$$
\eta \leq C e^{s_1 + s_2 (L_+ - L_-)} \frac{e^{L_+ - L_-}}{\alpha L_{-}^{1/2}} \max \left\{ e^{-L_-} \varepsilon_-^{-1} \max \left\{ \frac{|C|}{L_+ \sqrt{\varepsilon_-}}, \frac{L_+ \varepsilon_+}{\sqrt{\varepsilon_-}}, \frac{\varepsilon_+}{\varepsilon_-} \sqrt{\alpha L_+}, (\alpha L_-)^{-1/2} \max \{|C|^2, \varepsilon_+^2, \beta\} \right\}, \frac{e^{-L_-} K^{1+\delta}}{\varepsilon_-} \max \left\{ \frac{|C|}{L_+ \sqrt{\varepsilon_-}}, \frac{L_+ \varepsilon_+}{\sqrt{\varepsilon_-}}, \frac{\varepsilon_+}{\varepsilon_-} \sqrt{\alpha L_+} \right\} \right\}
$$

(127)

We now discuss inequalities (46)–(49) and (126). We choose $s_1, L_\pm, \varepsilon_\pm$ and $K$ to be the following functions of $L$ and $\xi$, with $0 < \xi < 1 < L$:

$$
L_- = L, \quad \varepsilon_\pm = c_\pm L^2 e^{-2L}, \quad L_+ = L + 10 \xi, \quad s_1 = C_1 \xi L^{-2}, \quad s_2 = C_1 \xi \quad K^{1+\delta} = \frac{c_1}{\xi \sqrt{L}}
$$

with $0 < c_- < c_+ < 1$ fixed. A more stringent relation between $\xi$ and $L$ will be specified below. We take

$$
|C| < c_1 L^3 e^{-2L}, \quad \beta < c_1 L^4 e^{-4L}, \quad \delta < c_1 L^{3/2} e^{-L}
$$

In view of (127), it is immediate to check that there exist suitable numbers $0 < c_1 < 1 < C_1$ depending only on $c, c_+, c_-$ and $\alpha$ such that inequalities (46)–(48) and (126) are satisfied and

$$
\eta < C_2 L^{-\frac{3}{2}}
$$

An application of GNFL conjugates $\tilde{X} = N + \tilde{P}$ to a new vector–field $\tilde{X}_* = N + \tilde{P}_*$, with

$$
\varepsilon_-^{-1} \|\tilde{P}_{*,1}\|_u \leq \|\tilde{P}_*\|_{u}^{w_K} \leq \max \left\{ 2^{-c_2 L^3} \|\tilde{P}\|_{u}^{w_K}, 2c_0 K^{-\ell + \delta} \|\tilde{P}\|_{u,\ell}^{w_K} \right\}
$$

$$
\leq \max \left\{ 2^{-c_2 L^3}, c_0 g_0 2^\ell \varepsilon^{-\ell + \delta} K^{-\ell + \delta} \right\} \|P\|_{w_K}
$$

$$
\leq C_3 2^{-c_1 L^3} \|P\|_{w_K} \leq C_4 \varepsilon^{-1} 2^{-c_1 L^3} \leq C_4 \varepsilon^{-3} 2^{-c_1 L^3} \varepsilon \quad \varepsilon_3 = c_0 g_0(128)
$$

where $\varepsilon$ is an upper bound for $\|P\|_u$. Here, we have used the definition of $\tilde{P}$ in (122), the chain rule and Cauchy inequalities for $P$, providing

$$
\|\tilde{P}\|_{w_K}^{w_K} \leq 2^{2\ell} g_0 \|\tilde{P}\|_{u,\ell}^{w_K} \leq 2^{2\ell} g_0 \|P\|_{u,\ell}^{w_K} \leq 2^{2\ell} \varepsilon^{-\ell} g_0 \|P\|_{u}^{w_K} \quad \ell \in \mathbb{Z}
$$

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with \( g_0 \) as in (121) and \( \sigma \) the analyticity radius of \( P(A, y, \cdot) \). Moreover, we have chosen

\[
\ell \geq \max\{1, 2\delta, C_2 L^3\}, \quad 0 < \xi < \left( \frac{\sigma^2}{256} \right)^{\frac{\delta+1}{2}} \frac{c_1}{\sqrt{L}}
\]

so that

\[
K = \left( \frac{c_1}{\xi \sqrt{L}} \right)^{\frac{1}{\ell+1}} > \frac{256}{\sigma^2} \quad \Rightarrow \quad 2^{\ell+1} \sigma^{-\ell} K^{-\ell+\delta} < 8^\ell \sigma^{-\ell} K^{-\ell/2} < 2^{-\ell}
\]

Observing now that (analogously to (125))

\[
c L^3 e^{-\frac{4}{L}} \leq \|P_1\|_u \leq c L^3 e^{-\frac{4}{L}}, \quad c L^{-\frac{1}{2}} e^{-\frac{L}{4}} \leq \frac{1}{Q} \leq c L^{-\frac{1}{2}} e^{-\frac{L}{4}}
\]

and using \( \ell \geq C_2 L^3 \), we arrive at (128). □

### A The elliptic integrals \( T_0(\kappa) \) and \( j_\beta(\kappa) \)

The functions \( T_0(\kappa) \) in (93) and \( j_\beta(\kappa) \) in (112) are complete elliptic integrals. We use this appendix to store some useful material concerning such functions.

First of all, in the definition of \( T_0(\kappa) \), we change the integration variable, letting \( \xi \to \frac{1}{\xi} \), so as to rewrite

\[
T_0(\kappa) = \int_1^{\infty} \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa \xi^2)}} \quad 0 \neq \kappa < 1
\]  

with \( G_0(\kappa) \) as in (93). Next, we look at the complex–valued function

\[
g(\kappa) := \int_1^{\infty} \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa \xi^2)}} \quad \kappa \in \mathbb{R} \setminus \{0, 1\}
\]

which is easily related to \( T_0(\kappa) \) and \( j_0(\kappa) \):

**Lemma A.1** let \( 0 \neq \kappa < 1 \). Then

\[
T_0(\kappa) = \begin{cases} 
g(\kappa) & \text{if } \kappa < 0 \\
j_0(\kappa) = \mathfrak{Re} g(\kappa) & \text{if } 0 < \kappa < 1
\end{cases}
\]

**Proof** We have only to prove that \( T_0(\kappa) = j_0(\kappa) \) when \( 0 < \kappa < 1 \), as the other relations are immediate, from (129) and (130). We write

\[
T_0(\kappa) = \left( \int_0^{+\infty} - \int_0^{1} - \int_{\frac{1}{\sqrt{\kappa}}}^{+\infty} \right) \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa \xi^2)}}
\]

We deform the integration path of the first integral at right hand side stretching the real path \( \xi \in [0, +\infty) \) to the purely imaginary line \( z = iy \), with \( y \in [0, +\infty) \), so that

\[
\int_0^{\infty} \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa \xi^2)}} = \int_0^{\infty} \frac{dy}{\sqrt{(y^2 + 1)(1 + \kappa y^2)}} = j_0(\kappa)
\]

Combining this with the observation that, for \( 0 < \kappa < 1 \), \( T_0(\kappa) \) and \( j_0(\kappa) \) are real while the two latter integrals in (132) are purely imaginary, we have \( T_0(\kappa) = j_0(\kappa) \), as claimed. □
Remark A.1 It follows from the proof of Lemma A.1 (compare (132)–(133)) that, in the sense of complex integrals,
\[
\left( \int_0^1 + \int_{\frac{1}{\kappa}}^{+\infty} \right) \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa \xi^2)}} \equiv 0, \quad \forall \ 0 < \kappa < 1. \tag{134}
\]
This identity can be also directly checked, using proper changes of coordinate combined with cuts of the complex plane, in order to make the square roots single–valued in a neighbourhood of the real axis.

The advantage of looking at \( g(\kappa) \) instead of \( T_0(\kappa) \) is that the integration path in (130) is \( \kappa \)–independent, and this turns to be useful when taking \( \kappa \)–derivatives. The main result at this respect in this section is the following

Proposition A.1
- Let \( \kappa \in \mathbb{R} \setminus \{0, 1\} \) and let \( g(\kappa) \) be as in (130). There exists two positive real numbers \( \mathcal{R}_* \), \( \mathcal{S}_* \) and two complex numbers

  \[
  \mathcal{R}(\kappa), \mathcal{S}(\kappa) \in \begin{cases} 
  \mathbb{R}_+ & \text{if } \kappa < 0 \\
  \mathbb{C} & \text{if } 0 < \kappa < 1 \\
  i\mathbb{R}_+ & \text{if } \kappa > 1
  \end{cases}
  \]

  with

  \[
  \Re \mathcal{R}(0) = \Re \mathcal{S}(0) = 1, \quad 0 \leq \Re \mathcal{R}(\kappa) \leq \mathcal{R}_*, \quad 0 \leq \Re \mathcal{S}(\kappa) \leq \mathcal{S}_* \quad \forall \ \kappa \in (-1, 1)
  \]

  such that

  \[
  g'(\kappa) = -\frac{\mathcal{R}(\kappa)}{2\kappa} \quad g''(\kappa) = +\frac{\mathcal{S}(\kappa)}{4\kappa^2} \quad \forall \ \kappa \in \mathbb{R} \setminus \{0, 1\}.
  \]

- Let \( \beta \geq 0; \ 0 < \kappa < 1, \ j_\beta(\kappa) \) as in (112). There exist two positive numbers \( \mathcal{R}_* \), \( \mathcal{S}_* \) \( \in \mathbb{R} \) and two real functions \( \mathcal{R}_\beta(\kappa), \mathcal{S}_\beta(\kappa) \) satisfying

  \[
  \mathcal{R}_\beta(0) = \mathcal{S}_\beta(0) = \begin{cases} 
  1 & \text{if } \beta = 0 \\
  0 & \text{if } \beta > 0
  \end{cases}
  \]

  \[
  0 \leq \mathcal{R}_\beta(\kappa) \leq \mathcal{R}_0, \quad 0 \leq \mathcal{S}_\beta(\kappa) \leq \mathcal{S}_0 \quad \forall \ \beta \geq 0 \quad \forall \ \kappa \in (0, 1) \tag{135}
  \]

  such that

  \[
  j_\beta'(\kappa) = -\frac{\mathcal{R}_\beta(\kappa)}{2\kappa} \quad j_\beta''(\kappa) = +\frac{\mathcal{S}_\beta(\kappa)}{4\kappa^2} \quad \forall \ 0 < \kappa < 1.
  \]

Proof We prove the first statement. We distinguish two cases.

Case 1: \( \kappa < 0 \) or \( \kappa > 1 \). The integral takes real values when \( \kappa < 0 \); purely imaginary ones when \( \kappa > 1 \):

\[
g(\kappa) = \begin{cases} 
  \int_1^{+\infty} \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa \xi^2)}} & \kappa < 0 \\
  -i \int_1^{+\infty} \frac{d\xi}{\sqrt{(\xi^2 - 1)(\kappa \xi^2 - 1)}} & \kappa > 1
  \end{cases}
\]

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The function under the integral is bounded above by \( \frac{1}{\min(1, \sqrt{|\kappa|}) \sqrt{\xi^2 - 1}} \) when \( \kappa < 0 \); by \( \frac{1}{\xi^2 - 1} \) when \( \kappa > 1 \). Both such bounds are integrable. Then it is possible to derive under the integral, and we obtain

\[
g'(\kappa) = \begin{cases} 
\frac{1}{2} \int_1^{+\infty} \frac{\xi^2 d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa \xi^2)^3}} & \kappa < 0 \\
\frac{i}{2} \int_1^{+\infty} \frac{\xi^2 d\xi}{\sqrt{(\xi^2 - 1)(\kappa \xi^2 - 1)^3}} & \kappa > 1 
\end{cases}
\]

and

\[
g''(\kappa) = \begin{cases} 
\frac{3}{4} \int_1^{+\infty} \frac{\xi^4 d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa \xi^2)^5}} & \kappa < 0 \\
-\frac{3}{4} \int_1^{+\infty} \frac{\xi^4 d\xi}{\sqrt{(\xi^2 - 1)(\kappa \xi^2 - 1)^5}} & \kappa > 1 
\end{cases}
\]

We change variable \( 1 - \kappa \xi^2 = \eta \) when \( \kappa < 0 \), \( \kappa \xi^2 - 1 = \eta \) when \( \kappa > 1 \) and rewrite

\[
g'(\kappa) = \begin{cases} 
\frac{1}{4|\kappa|} \int_{1-\kappa}^{+\infty} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa) \eta^3}} d\eta & \kappa < 0 \\
\frac{i}{4|\kappa|} \int_{\kappa - 1}^{+\infty} \sqrt{\frac{\eta + 1}{(\eta + 1 - \kappa) \eta^3}} d\eta & \kappa > 1 
\end{cases}
\]

and

\[
g''(\kappa) = \begin{cases} 
\frac{3}{8|\kappa|^2} \int_{1-\kappa}^{+\infty} (\eta - 1) \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa) \eta^5}} d\eta & \kappa < 0 \\
-\frac{3}{8|\kappa|^2} \int_{1-\kappa}^{+\infty} (\eta + 1) \sqrt{\frac{\eta + 1}{(\eta + 1 - \kappa) \eta^5}} d\eta & \kappa > 1 
\end{cases}
\]

so we take

\[
\Re(\kappa) = \begin{cases} 
\frac{1}{2} \int_{1-\kappa}^{+\infty} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa) \eta^3}} d\eta & \kappa < 0 \\
\frac{i}{2} \int_{\kappa - 1}^{+\infty} \sqrt{\frac{\eta + 1}{(\eta + 1 - \kappa) \eta^3}} d\eta & \kappa > 1 
\end{cases}
\]

and

\[
\Im(\kappa) = \begin{cases} 
\frac{3}{2} \int_{1-\kappa}^{+\infty} (\eta - 1) \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa) \eta^5}} d\eta & \kappa < 0 \\
-\frac{3}{2} \int_{1-\kappa}^{+\infty} (\eta + 1) \sqrt{\frac{\eta + 1}{(\eta + 1 - \kappa) \eta^5}} d\eta & \kappa > 1 
\end{cases}
\]

Observe that, if \(-1 < \kappa < 0\),

\[
\Re(\kappa(0^-)) = 1 = \Re(\Im(0^-))
\]
and

\[ 0 \leq \Re \mathcal{R}(\kappa) = \frac{1}{2} \int_{1-\kappa}^{+\infty} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)^3}} d\eta \leq \frac{1}{2} \int_{1}^{+\infty} \sqrt{\frac{\eta - 1}{(\eta - 2)^3}} d\eta \]

\[ 0 \leq \Re \mathcal{S}(\kappa) \leq \frac{3}{2} \int_{1}^{+\infty} (\eta - 1) \sqrt{\frac{\eta - 1}{(\eta - 2)^3}} d\eta . \]

Case 2: 0 < \kappa < 1. We split \( g(\kappa) \) into its real and imaginary part. Using (131) and (134), we obtain

\[
g(\kappa) = + \int_{1}^{+\infty} \frac{\sqrt{\xi}}{(\xi^2 - 1)(1 - \kappa \xi^2)} + \int_{1}^{+\infty} \frac{\sqrt{\xi}}{(\xi^2 - 1)(1 - \kappa \xi^2)}
\]

\[
= + \int_{0}^{\infty} \frac{d\eta}{\sqrt{(y^2 + 1)(1 + \kappa y^2)}} + \int_{0}^{1} \frac{\sqrt{\xi}}{(1 - \xi^2)(1 - \kappa \xi^2)}
\]

Notice that also in this case, the functions under the integrals may be bounded by integrable functions: \( \frac{1}{\sqrt{\kappa(y^2 + 1)}} \) for the former; \( \frac{1}{\sqrt{1-\xi}} \frac{1}{\sqrt{1-\kappa}} \) in the latter. Again, we can derive under the integral, and obtain

\[
g'(\kappa) = - \frac{1}{2} \int_{0}^{1} \frac{\eta - 1}{(\eta - 1 + \kappa)^3} + \frac{i}{2} \int_{0}^{1} \frac{\xi^2 d\xi}{(1 - \xi^2)(1 - \kappa \xi^2)^3}
\]

and

\[
g''(\kappa) = + \frac{3}{4} \int_{0}^{1} \frac{\eta^4 d\eta}{(y^2 + 1)(1 + \kappa y^2)^5} + \frac{3}{4} i \int_{0}^{1} \frac{\xi^4 d\xi}{(1 - \xi^2)(1 - \kappa \xi^2)^5}
\]

Then, letting \( 1 + \kappa y^2 = \eta \) in the first respective integrals, and \( 1 - \kappa \xi^2 = \eta \) in the second ones,

\[
g'(\kappa) = - \frac{1}{4\kappa} \int_{1}^{+\infty} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)^3}} + \frac{i}{4\kappa} \int_{1-\kappa}^{1} \sqrt{\frac{1 - \eta}{(\eta - 1 + \kappa)^3}} d\eta
\]

and

\[
g''(\kappa) = + \frac{3}{8\kappa^2} \int_{1}^{+\infty} (\eta - 1) \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)^3}} + \frac{3}{8\kappa^2} \int_{1-\kappa}^{1} (1 - \eta) \sqrt{\frac{1 - \eta}{(\eta - 1 + \kappa)^3}} d\eta
\]

and we can take

\[
\mathcal{R}(\kappa) := \frac{1}{2} \int_{1}^{+\infty} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)^3}} - \frac{i}{2} \int_{1-\kappa}^{1} \sqrt{\frac{1 - \eta}{(\eta - 1 + \kappa)^3}} d\eta
\]

and

\[
\mathcal{S}(\kappa) = \frac{3}{2} \int_{1}^{+\infty} (\eta - 1) \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)^3}} + \frac{3}{2} i \int_{1-\kappa}^{1} (1 - \eta) \sqrt{\frac{1 - \eta}{(\eta - 1 + \kappa)^3}} d\eta
\]

Notice now that

\[ \Re \mathcal{R}(0^+) = 1 = \Re \mathcal{S}(0^+) \]
and
\[
0 \leq \Re \mathcal{K}(\kappa) = \frac{1}{2} \int_{1}^{+\infty} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^3}} \leq \frac{1}{2} \int_{1}^{+\infty} \eta^{-\frac{3}{2}} = 1
\]
and
\[
0 \leq \Re \mathcal{S}(\kappa) = \frac{3}{2} \int_{1}^{+\infty} (\eta - 1) \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^3}} \leq \frac{3}{2} \int_{1}^{+\infty} \eta^{-\frac{3}{2}} = 1
\]
for all \(0 < \kappa < 1\).
The proof for \(j_\beta(\kappa)\) is completely analogous to the case 2 above (with the difference that we do not have the imaginary part in that case). One finds
\[
\mathcal{K}_\beta(\kappa) = \frac{1}{2} \int_{1}^{+\infty} \frac{1}{1 + \frac{\beta}{\kappa}(\eta - 1)} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^3}}
\]
and
\[
\mathcal{S}_\beta(\kappa) = \frac{3}{2} \int_{1}^{+\infty} \frac{\eta - 1}{1 + \frac{\beta}{\kappa}(\eta - 1)} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^3}}
\]
which verify (135). \(\square\)

## B Technicalities

In this section of the appendix we prove the bounds in (124), (125) and Proposition 6.1.

### Proof of (124)
We let
\[
\mathcal{E}_\ast(A_s, r_s) := \mathcal{E} \circ \phi_{\alpha\beta}(A_s, r_s) , \quad \mathcal{E}_\circ(A, y) := \mathcal{E} \circ \phi_{\alpha\beta}(A, y) = \mathcal{E}_\ast(A, r_\circ(A, y))
\]
\[
\mathcal{B}_\ast(A_s, r_s) := \mathcal{B} \circ \phi_{\alpha\beta}(A_s, r_s) , \quad \mathcal{B}_\circ(A, y) := \mathcal{B} \circ \phi_{\alpha\beta}(A, y) = \mathcal{B}_\ast(A, r_\circ(A, y))
\]
\[
T_{p, \ast}(A_s, r_s) := T_p \circ \phi_{\alpha\beta}(A_s, r_s) , \quad T_{p, \circ}(A, y) := T_p \circ \phi_{\alpha\beta}(A, y) = T_{p, \ast}(A, r_\circ(A, y))
\]
\[
F_\circ(A, y) := F \circ \phi_{\alpha\beta}(A, y) = F_\ast(A, r_\circ(A, y)) = F(\mathcal{E}_\circ(A, y), r_\circ(A, y))
\]
\[
F_{\ast, 1, \circ}(A, y) := F_{\ast, 1} \circ \phi_{\alpha\beta}(A, y) = F_{\ast, 1}(A, r_\circ(A, y))
\]
(136)

(with \(\mathcal{F}, T_p, \mathcal{B}\) as in (116), (92)–(93), (80)) so as to write, more rapidly,
\[
v(A, y; c) = e^{-\eta} \sqrt{2(c - \alpha F_\circ(A, y))} , \quad \omega(A, y; c) = \alpha e^{-\eta} F_{\ast, 1, \circ}(A, y)
\]
and
\[
\frac{1}{v} = \frac{e^{-\eta}}{\sqrt{2(c - \alpha F_\circ(A, y))}} , \quad \frac{\partial_A v}{v} = -\frac{\alpha}{c - \alpha F_\circ(A, y)} , \quad \frac{\partial_y v}{v} = -\frac{\alpha}{2} \frac{\partial_y F_\circ(A, y)}{c - \alpha F_\circ(A, y)} ,
\]
\[
\omega v = \alpha e^{-\eta} \frac{F_{\ast, 1, \circ}(A, y)}{\sqrt{2(c - \alpha F_\circ(A, y))}} , \quad \frac{\partial_A \omega}{v} = \alpha e^{-\eta} \frac{\partial_A F_{\ast, 1, \circ}(A, y)}{\sqrt{2(c - \alpha F_\circ(A, y))}}
\]
\[
\frac{\partial_y \omega}{v} = -2\alpha e^{-\eta} \frac{F_{\ast, 1, \circ}(A, y)}{\sqrt{2(c - \alpha F_\circ(A, y))}} + \alpha e^{-\eta} \frac{\partial_y F_{\ast, 1, \circ}(A, y)}{\sqrt{2(c - \alpha F_\circ(A, y))}}
\]
(137)
Finally, using (85)–(86), according to which
\[ \kappa \text{ with } E \]
where we have neglected yo write the arguments (e.g., \( F, E \)) again by the chain rule.

\[ \frac{\partial F_{*,1,o}}{T_{p,o}} = \frac{\partial^2 F(E_0, r_o) \partial A \xi_o + \partial^2 F_r(T_r)}{T_{p,o}} - \frac{\partial T_{p} \partial A \xi_o + \partial T_{p} T_r}{T_{p,o}^2} \]

\[ \partial_y F_{*,1,o} = \frac{\partial^2 F(E_0, r_o) \partial y \xi_o - e^{-y} \partial^2 F_r(T_r)}{T_{p,o}} - \frac{\partial T_{p} \partial y \xi_o - e^{-y} \partial T_{p}}{T_{p,o}^2} \]

\[ \partial_A F_o = F_{E}(E_0, r_o) \partial A \xi_o + F_r(E_0, r_o) T_r(A), \quad \partial_y F_o = F_{E}(E_0, r_o) \partial y \xi_o - e^{-y} F_r(E_0, r_o) \]

where we have neglected yo write the arguments (e.g., \( F, E \)) again by the chain rule.

\[ \partial_A \xi_o = \frac{1}{T_{p,o}(A, y)} - r_s'(A)B_o(A, y), \quad \partial_y \xi_o = e^{-y} B_o(A, y) \]

We recall (Sections ?? and 5) that the functions \( F, T_o \) and \( B \) in (136) verify

\[ C' \log |\kappa|^{-1} \leq |F|, \quad |T_o|, \quad |1/B| \leq C \log |\kappa|^{-1}, \quad C'|\kappa|^{-1} \leq |\partial E_r F|, \quad |\partial E_r T_p|, \quad |\partial E_r B| \leq C|\kappa|^{-1} \]

\[ C'|\kappa|^{-2} \leq \|\partial E_r^2 F\|, \quad |\partial E_r T_p|, \quad |\partial E_r B| \leq C|\kappa|^{-2} \]

with \( \kappa = O(\xi - r) = O(e^{-y}) \) so that

\[ C'L_\kappa - \|F_o\|, \quad |T_{p,o}|, \quad |B_o| \leq CL_\kappa \]

\[ C'e^{L_\kappa} - \|\partial E_r F(E_0, r_o)\|, \quad |\partial E_r T_p(E_0, r_o)|, \quad |\partial E_r B(E_0, r_o)| \leq Ce^{L_\kappa} \]

\[ C'e^{2L_\kappa} - \|\partial E_r^2 F(E_0, r_o)\|, \quad |\partial E_r T_p(E_0, r_o)|, \quad |\partial E_r^2 B(E_0, r_o)| \leq Ce^{2L_\kappa} \]

(138)

Finally, using (85)–(86), according to which

\[ r_s'(A) = \frac{1}{\lambda_0(r)} \bigg|_{r = r_s(A)} = \pi \sqrt{\frac{r_s(A)}{2 - r_s(A)}} \]

whence

\[ |r_s'(A)| \leq \frac{C}{\sqrt{\varepsilon^-}} \]

(139)

and collecting the bounds above into (137), we find (124).

**Proof of (125)** We use some results from Section 4. Taking in count (88), (89), (95) and (97) and letting

\[ \sigma_s(A, r_s) := \sigma \circ \phi_{aa}(A, r_s), \quad \kappa_s(A, r_s) := \kappa \circ \phi_{aa}(A, r_s) \]

\[ \tilde{T}_{p,*}(A, r_s) := \sigma_s(A, r_s) \tilde{T}_{p,*}(A, r_s) := \frac{T_p(\kappa_s(A, r_s))}{\pi} \]

we have that

\[ \frac{1}{\sigma_\kappa} \circ \phi_{aa} = \frac{1}{\tilde{T}_{p,*}(A, r_s)} \]

and

\[ \partial_x E_s = -\frac{\partial A}{\partial x} \circ \phi_{aa} = -B_s(A, r_s), \text{ implied by (83).} \]
Similarly, then by

\[
\|G_s(A, r_s, \psi) = \sigma_s(A, r_s)\tilde{G}(\kappa_s(A, r_s), \hat{T}_{p, s}(A, r_s)\psi)
\]

(140)

and

\[
\rho_s(A, r_s, \psi) = \frac{\sigma_s(A, r_s)}{r_s}b(\kappa_s(A, r_s), \hat{T}_{p, s}(A, r_s)\psi)
\]

(141)

By the chain rule

\[
G_{s, 3}(A, r_s, \psi) = \partial_{\psi}G_s(A, r_s, \psi)
\]

\[
= \sigma_s(A, r_s)\partial_{\psi}\tilde{G}(\kappa_s(A, r_s), \hat{T}_{p, s}(A, r_s)\psi)
\]

(142)

Similarly,

\[
\rho_{s, 3}(A, r_s, \psi) = \frac{\sigma_s(A, r_s)}{r_s}\tilde{T}_{p, s}(A, r_s)b(\kappa_s(A, r_s), \hat{T}_{p, s}(A, r_s)\psi)
\]

(143)

By the definitions in (120)–(122), if

\[
\bar{P}_{\varepsilon, \xi} := \bigcup_{(A, y) \in \mathcal{A}_{\varepsilon, \lambda} \times \mathcal{V}_\xi} \{A\} \times \{y\} \times V_\varepsilon(A, y; \delta)
\]

then

\[
\|\bar{P}_i\|_{\mathcal{P}_{\varepsilon, \xi}} \leq \|P_i(A, y; \psi)\|_{\bar{P}_{\varepsilon, \xi}}
\]

so we proceed to uniformly upper bound the \(|P_i|\) in \(\bar{P}_{\varepsilon, \xi}\).

- By Proposition 4.1,

\[
|\tilde{G}(\kappa, \theta)|, |\tilde{G}_3(\kappa, \theta)| \leq C
\]

- By (140), (142) and (138),

\[
|G_s(A, r_0(A, y), \psi)| \leq \sqrt{\varepsilon_+}, \quad |G_{s, 3}(A, r_0(A, y), \psi)| \leq CL_+\sqrt{\varepsilon_+}
\]

(144)

- Both the inequalities in (144) hold (with the same proof) if \(r_0(A, y)\) is replaced by a generic \(r \in 3r_s(A, \cdot)\). Then, if \(r \in 3r_s(A, \cdot)\),

\[
|G_{s, 1}(A, r_0(A, y), \psi)| \leq \sqrt{\varepsilon_+}
\]

- Similarly, by (141), \(|\rho_s(A, r_s, \psi)| \leq \sqrt{\varepsilon_+} \), hence

\[
|\rho_{s, 1}(A, r_s(A, y), \psi)| \leq \sqrt{\varepsilon_+}
\]

- The function \(\gamma(A, y; \psi; c)\) defined in (117) verifies

\[
|\gamma| \leq C\sqrt{\alpha L_+}
\]

having used the simplifying assumption (126).

- By Lemma 6.1,

\[
|\rho_{s, 3}(A, r_0(A, y), \psi)| \leq C\sqrt{\varepsilon_+}\delta
\]

- Recall (139).

- Using the previous bounds into (118) and writing the last term in the definition of \(P_2\) as

\[
e^{-y} \frac{(C - G_s(A, r_0(A, y), \psi))^2}{2r_0(A, y)} - \frac{\delta}{r_0(A, y)} \gamma(A, y; \psi; c) + \sqrt{2(c - \alpha F_s(A, r_0(A, y))})
\]

we obtain, for \(||P_i||_{\bar{P}_{\varepsilon, \xi}}\) \(\), the bounds at the right hand sides of (125).
Proof of Proposition 6.1 Recall (143) and the expression of $\tilde{\rho}_\theta(\kappa, \theta)$ in equation (104). Equation
\[ \tilde{\rho}_\theta(\kappa, \theta) = \tilde{G}(\kappa, \theta)^2 - A(\kappa) = 0 \] (145)
has a unique solution
\[ 0 < \theta^*_\kappa < T_0(\kappa) \]
if and only if
\[ G_0(\kappa)^2 < A(\kappa) < 1. \]
On the other hand, it is immediate to check that such inequality holds for all $0 \neq \kappa < 1$. Indeed, if $0 < \kappa < 1$, then $G_0(\kappa)^2 = \kappa$ and we have
\[ \kappa < A(\kappa) = \frac{\int_1^\kappa \frac{\xi^2 d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}}}{\int_1^\kappa \frac{d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}}} < 1. \]
If $\kappa < 0$, then $G_0(\kappa)^2 = 0$ and we have
\[ 0 < A(\kappa) = \frac{\int_0^1 \frac{\xi^2 d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}}}{\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}}} < 1. \]
As consequence of the formula (145), and continuity of $\tilde{G}(\kappa, \cdot)$, we also find $V(\kappa; \delta) \subset (0, T_0(\kappa))$ (and $V'(\kappa; \delta) \subset (0, T_0(\kappa))$ when $\kappa < 0$) such that
\[ |\tilde{\rho}_3(\kappa, \theta)| \leq \frac{C_\delta}{T_p(\kappa)} \quad \forall \theta \in V(\kappa; \delta) \quad \left( \forall \theta \in V(\kappa; \delta) \cup V'(\kappa; \delta) \right) \]
which implies (119), after using (143). □

References
[1] V. M. Alexeyev. Sur l’allure finale du mouvement dans le problème des trois corps. In Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pages 893–907, 1971.
[2] V. M. Alekseev. Final motions in the three-body problem and symbolic dynamics. Uspekhi Mat. Nauk, 36(4(220)):161–176, 248, 1981.
[3] V. I. Arnold. Small denominators and problems of stability of motion in classical and celestial mechanics. Russian Math. Surveys, 18(6):85–191, 1963.
[4] S. Bolotin. Second species periodic orbits of the elliptic 3 body problem. Celestial Mech. Dynam. Astronom., 93(1-4):343–371, 2005.
[5] S. Bolotin and R. S. MacKay. Nonplanar second species periodic and chaotic trajectories for the circular restricted three-body problem. Celestial Mech. Dynam. Astronom., 94(4):433–449, 2006.
[6] S. Bolotin. Shadowing chains of collision orbits. Discrete Contin. Dyn. Syst., 14(2):235–260, 2006.
[7] S. Bolotin. Symbolic dynamics of almost collision orbits and skew products of symplectic maps. Nonlinearity, 19(9):2041–2063, 2006.
[8] S. Bolotin and Piero Negrini. Variational approach to second species periodic solutions of Poincaré of the 3 body problem. Discrete Contin. Dyn. Syst., 33(3):1009–1032, 2013.
[9] J. Chazy. Sur l’allure du mouvement dans le problème des trois corps quand le temps croît indéfiniment. Ann. Sci. École Norm. Sup. (3), 39:29–130, 1922.
[10] A. Chenciner and J. Llibre. A note on the existence of invariant punctured tori in the planar circular restricted three-body problem. Ergodic Theory Dynam. Systems, 8*(Charles Conley Memorial Issue):63–72, 1988.
[11] L. Chierchia and G. Pinzari. Planetary Birkhoff normal forms. *J. Mod. Dyn.*, 5(4):623–664, 2011.
[12] L. Chierchia and G. Pinzari. The planetary $N$-body problem: symplectic foliation, reductions and invariant tori. *Invent. Math.*, 186(1):1–77, 2011.
[13] S. Di Ruzza, J. Daquin, and G. Pinzari. Symbolic dynamics in a binary asteroid system. *Commun. Nonlinear Sci. Numer. Simul.*, 91:105414, 16, 2020.
[14] J. Féjoz. Quasiperiodic motions in the planar three-body problem. *J. Differential Equations*, 183(2):303–341, 2002.
[15] J. Féjoz. Démonstration du ‘théorème d’Arnold’ sur la stabilité du système planétaire (d’après Herman). *Ergodic Theory Dynam. Systems*, 24(5):1521–1582, 2004.
[16] S. Fleischer and A. Knauf. Improbability of collisions in $n$-body systems. *Arch. Ration. Mech. Anal.*, 234(3):1007–1039, 2019.
[17] S. Fleischer and A. Knauf. Improbability of wandering orbits passing through a sequence of Poincaré surfaces of decreasing size. *Arch. Ration. Mech. Anal.*, 231(3):1781–1800, 2019.
[18] E. Freitag and R. Busam. *Complex analysis*. Universitext. Springer-Verlag, Berlin, 2005. Translated from the 2005 German edition by Dan Fulea.
[19] M. Guardia, V. Kaloshin, and J. Zhang. Asymptotic density of collision orbits in the restricted circular planar 3 body problem. *Arch. Ration. Mech. Anal.*, 233(2):799–836, 2019.
[20] M. Guzzo, L. Chierchia, and G. Benettin. The steep Nekhoroshev’s theorem. *Comm. Math. Phys.*, 342(2):569–601, 2016.
[21] J. Henrard. On Poincaré’s second species solutions. *Celestial Mech.*, 21(1):83–97, 1980.
[22] J. Laskar and P. Robutel. Stability of the planetary three-body problem. I. Expansion of the planetary Hamiltonian. *Celestial Mech. Dynam. Astronom.*, 62(3):193–217, 1995.
[23] J. -Pierre Marco and L. Niederman. Sur la construction des solutions de seconde espèce dans le problème plan restreint des trois corps. *Ann. Inst. H. Poincaré Phys. Théor.*, 62(3):211–249, 1995.
[24] R. B. Moeckel. *Orbits near triple collision in the three-body problem*. ProQuest LLC, Ann Arbor, MI, 1980. Thesis (Ph.D.)–The University of Wisconsin - Madison.
[25] R. Moeckel. Orbits of the three-body problem which pass infinitely close to triple collision. *Amer. J. Math.*, 103(6):1323–1341, 1981.
[26] R. Moeckel. Chaotic dynamics near triple collision. *Arch. Rational Mech. Anal.*, 107(1):37–69, 1989.
[27] R. Moeckel. Symbolic dynamics in the planar three-body problem. *Regul. Chaotic Dyn.*, 12(5):449–475, 2007.
[28] J. Moser. A new technique for the construction of solutions of nonlinear differential equations. *Proc. Nat. Acad. Sci. U.S.A.*, 47:1824–1831, 1961.
[29] J. Moser. On invariant curves of area-preserving mappings of an annulus. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1962:1–20, 1962.
[30] J. Nash. The imbedding problem for Riemannian manifolds. *Ann. of Math.* (2), 63:20–63, 1956.
[31] N. N. Nehorošev. An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. *Uspehi Mat. Nauk*, 32(6(198)):5–66, 287, 1977.
[32] G. Pinzari. *On the Kolmogorov set for many–body problems*. PhD thesis, Università Roma Tre, April 2009.
[33] G. Pinzari. A first integral to the partially averaged newtonian potential of the three-body problem. *Celestial Mechanics and Dynamical Astronomy*, 131(5):22, May 2019.
[34] G. Pinzari. Euler integral and perihelion librations. *Discrete & Continuous Dynamical Systems*, 40(12):6919-6943, 2020.
[35] G. Pinzari. Perihelion librations in the secular three-body problem. *J. Nonlinear Sci.*, 30(4):1771–1808, 2020.
[36] H. Poincaré. *Les méthodes nouvelles de la mécanique céleste*. Gauthier-Villars, Paris, 1892.
[37] J. Pöschel. Nekhoroshev estimates for quasi-convex Hamiltonian systems. *Math. Z.*, 213(2):187–216, 1993.
[38] D. G. Saari. Improbability of collisions in Newtonian gravitational systems. *Trans. Amer. Math. Soc.*, 162:267–271; erratum, ibid. 168 (1972), 521, 1971.
[39] D. G. Saari. Improbability of collisions in Newtonian gravitational systems. II. *Trans. Amer. Math. Soc.*, 181:351–368, 1973.
[40] L. Zhao. Quasi-periodic almost-collision orbits in the spatial three-body problem. *Comm. Pure Appl. Math.*, 68(12):2144–2176, 2015.