MIYAWAKI TYPE LIFT FOR $G\text{Spin}(2,10)$

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Abstract. Let $\mathbb{T}_2$ (resp. $\mathbb{T}$) be the Hermitian symmetric domain of $\text{Spin}(2,10)$ (resp. $E_{7,3}$). In the previous work [18], we constructed holomorphic cusp forms on $\mathbb{T}_2$ from elliptic cusp forms with respect to $SL_2(\mathbb{Z})$. By using such cusp forms we construct holomorphic cusp forms on $\mathbb{T}_2$ which are similar to Miyawaki lift in symplectic groups established by T. Ikeda [14].

1. Introduction

Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ be the ring of adeles of $\mathbb{Q}$. For a reductive group $G$ over $\mathbb{Q}$ of higher rank with Hermitian symmetric domain $\mathcal{D}$, it is important to construct cuspidal representations of $G(\mathbb{A})$ which give rise to holomorphic cusp forms on $\mathcal{D}$. In general it would be difficult to construct cusp forms directly. One way is to use Langlands functoriality, namely, consider another smaller group $H$ with an $L$-group homomorphism $r : L^H \rightarrow L^G$, and then Langlands functoriality predicts a functorial lift from automorphic representations of $H(\mathbb{A})$ to those of $G(\mathbb{A})$. Some of cases are established by using the trace formula or the theta lift. These are very powerful tools, but the former never gives any explicit construction for classical forms and the latter can be made explicit with a careful choice of test functions, but it usually gives rise to automorphic representations which are generic, away from holomorphic forms, otherwise we need to consider a non-trivial level.

Contrary to these methods, Ikeda [13] gave an explicit construction of cusp forms for the symplectic group $Sp_n$ (with $\mathbb{Q}$-rank $n$) from elliptic cusp forms of $GL_2(\mathbb{A})$ with respect to $SL_2(\mathbb{Z})$. Such a cusp form is called Ikeda lift. In [14] Ikeda studied an integral similar to (1.1) below, obtained by substituting the role of Eisenstein series in the usual pullback formula with the Ikeda lift. Then under the assumption of nonvanishing of the integral, he showed that it gives rise to an essentially new cusp form for symplectic groups which is called Miyawaki lift. The existence

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of both lifts is compatible with the conjectural Arthur’s multiplicity formula which would be a theorem soon \[1\].

In this paper we pursue an analogue of Miyawaki type lift for $GSpin(2,10)$ by using our previous work \[18\]. We now explain the main theorem. We refer the next section for several notations which appear below (or Section 2 of \[18\]).

Let $G = E_{7,3}$ and $G' = GSpin(2,10)$, which split at every prime $p$. Let $\mathcal{T}_2$ (resp. $\mathcal{T}$) be the Hermitian symmetric domain of $PGSpin(2,10)(\mathbb{R})^0$ (resp. $E_{7,3}(\mathbb{R})$). Any elements of $\mathcal{T}$ and $\mathcal{T}_2$ are described in terms of Cayley numbers $C_2$ and we can write $g \in \mathcal{T}_2$ as $g = (Z \ w \ t \ w \ \tau)$ with $Z \in \mathcal{T}_2$, $w \in C_2$, and $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}z > 0\}$. Let $S_{2k}(SL_2(\mathbb{Z}))$ be the space of elliptic cusp forms of weight $2k \geq 12$ with respect to $SL_2(\mathbb{Z})$. For each normalized Hecke eigenform $f = \sum_{n=1}^{\infty} c(n)q^n$, $q = \exp(2\pi \tau \sqrt{-1})$, $\tau \in \mathbb{H}$, in $S_{2k}(SL_2(\mathbb{Z}))$, let $F_f$ be the Ikeda type lift on $\mathcal{T}$ of $f$ which was constructed in \[18\]. This is a Hecke eigen cusp form of weight $2k + 8$ with respect to $G(\mathbb{Z})$.

For a normalized Hecke eigenform $h \in S_{2k+8}(SL_2(\mathbb{Z}))$, consider the integral

$$\mathcal{F}_{f,h}(Z) = \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} F_f \left( \begin{array}{cc} Z & 0 \\ 0 & \tau \end{array} \right) \overline{h(\tau)(\text{Im}\tau)^{2k+6}} \ d\tau.$$  

Note that $\mathcal{F}_{f,h}(Z)$ is a cusp form (possibly zero) of weight $2k + 8$ with respect to $Spin(2,10)(\mathbb{Z})$.

For each prime $p$, let $\{\alpha_p, \alpha_p^{-1}\}$ and $\{\beta_p, \beta_p^{-1}\}$ be the Satake parameters of $f, h$ at $p$, resp. Let $\pi_f, \pi_h$ be the cuspidal representations attached to $f$ and $h$ resp., and let $L(s, \pi_f)$, $L(s, \pi_h)$ be their automorphic $L$-functions.

For a technical reason, we assume the Langlands functorial transfer of automorphic representations of $PGSpin(2,10)(\mathbb{A})$ to $GL_{12}(\mathbb{A})$: Namely, given a cuspidal representation of $PGSpin(2,10)(\mathbb{A})$ which is unramified at every prime $p$, there exists an automorphic representation of $GL_{12}(\mathbb{A})$ which is unramified at every prime $p$, and their Satake parameters correspond under the $L$-group homomorphism $^L GSpin(2,10) = GSO(12,\mathbb{C}) \leftrightarrow GL_{12}(\mathbb{C})$. The transfer is a composition of two transfers: The transfer of automorphic representations of $PGSpin(2,10)(\mathbb{A})$ to the split group $PGSpin(12,\mathbb{A})$ is the Jacquet-Langlands correspondence. Since $PGSpin(12) = PGSO(12)$, we can consider automorphic representations of $PGSO(12,\mathbb{A})$ as automorphic representations of $SO(12,\mathbb{A})$ with the trivial central character. The transfer of automorphic representations of $SO(12,\mathbb{A})$ to $GL_{12}(\mathbb{A})$ is now complete by Arthur \[1\].
We prove

**Theorem 1.1.** Assume that $\mathcal{F}_{f,h}$ is not identically zero. Assume also the existence of the functorial transfer from $\text{PGSpin}(2, 10)(\mathbb{A})$ to $GL_{12}(\mathbb{A})$. Then

1. The cusp form $\mathcal{F}_{f,h}$ is a Hecke eigenform, and hence gives rise to a cuspidal representation $\Pi_{f,h}$ of $G'(\mathbb{A})$ with the trivial central character, which is unramified at every prime $p$.
2. Let $\Pi_{f,h} = \Pi_\infty \otimes \otimes'_p \Pi_p$. For each prime $p$, the Satake parameter of $\Pi_p$ is given by

$$\{(\beta_p \alpha_p)^{\pm 1}, (\beta_p \alpha_p^{-1})^{\pm 1}, 1, 1, p^\pm 1, p^{\pm 2}, p^{\pm 3}\}$$

3. The degree 12 standard $L$-function of the cuspidal representation $\Pi_{f,h}$ is given by

$$L(s, \Pi_{f,h}) = L(s, \pi_f \times \pi_h) \zeta(s) \zeta(s + 1) \zeta(s + 2) \zeta(s + 3),$$

where the first $L$-function is the Rankin-Selberg $L$-function.
4. The transfer of $\Pi_{f,h}$ to $GL_{12}(\mathbb{A})$ is $(\pi_f \boxtimes \pi_h) \boxplus 1_{GL_7} \boxplus 1$, where $1_{GL_7}$ is the trivial representation of $GL_7(\mathbb{A})$.

We first show (Proposition 5.1) that the Satake parameter of $\Pi_p$ is given by

$$(I)_p : \{\varepsilon_p(\beta_p \alpha_p)^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1})^{\pm 1}, b_p^{\pm 1}, (b_p p)^{\pm 1}, (b_p p^2)^{\pm 1}, (b_p p^3)^{\pm 1}\}, \text{ or}$$

$$(II)_p : \{\varepsilon_p(\beta_p \alpha_p)^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1})^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1} p)^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1} p^2)^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1} p^3)^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1} p^4)^{\pm 1}\},$$

where $\varepsilon_p \in \{\pm 1\}$ and $b_p \in \mathbb{C}^\times$. Using the functorial transfer, in Section 6, we prove that only $(I)_p$ occurs, remove the sign ambiguity and $b_p = 1$.

**Remark 1.2.** If we take $h = E_{2k+8}$, the Eisenstein series of weight $2k + 8$, the integral $(1.1)$ still makes sense and defines a cusp form of weight $2k + 8$ with respect to $\text{Spin}(2, 10)(\mathbb{Z})$. If $\mathcal{F}_{f,E_{2k+8}}$ is not zero, then it gives rise to a cuspidal representation $\Pi_{f,E_{2k+8}}$ of $\text{GSpin}(2, 10)$, and the standard $L$-function of $\Pi_{f,E_{2k+8}}$ is

$$L(s, \Pi_{f,E_{2k+8}}) = L(s + \frac{1}{2}, \pi_f) L(s - \frac{1}{2}, \pi_f) \zeta(s) \zeta(s + 1) \zeta(s + 2) \zeta(s + 3).$$

**Remark 1.3.** Here $\Pi_\infty$ is a holomorphic discrete series of the lowest weight $2k + 8$. Since $f$ and $h$ have different weights, they can never be equal. Therefore $L(s, \pi_f \times \pi_h)$ is entire.
Remark 1.4. Note that $L_{\text{Spin}}(2, 10) = PGSO(12, \mathbb{C})$, and $PGSO(12, \mathbb{C})$ does not have a 12-dimensional representation. The minimum dimension among of the algebraic irreducible representations of $PGSO(12, \mathbb{C})$ is 66 by Weyl’s dimension formula and that is given by $\text{Ad} : PGSO(12, \mathbb{C}) \to GL(\text{Lie}(PGSO(12, \mathbb{C}))) \cong GL_{66}(\mathbb{C})$. Therefore, given a cuspidal representation $\pi$ of $\text{Spin}(2, 10)$, we cannot define the degree 12 standard $L$-function of $\pi$. However, $L_{\text{GSpin}}(2, 10) = GSO(12, \mathbb{C})$, and $GSO(12, \mathbb{C})$ has a 12-dimensional representation. Since $L_{\text{GSpin}}(2, 10) = PGSO(2, 10)$, our form $\Pi_{f, h}$ can be considered as a cuspidal representation of $GSpin(2, 10)$ with the trivial central character.

This situation is similar to Siegel cusp forms. Given a Siegel cusp form $F$ on a degree 2 Siegel upper half plane, we need to consider a cuspidal representation $\pi_F$ of $GSp_4$, rather than $Sp_4$ in order to define the degree 4 spin $L$-function.

Remark 1.5. We give a conjectural Arthur parameter of $\Pi_{f, h}$: Let $\phi_f, \phi_h : \mathcal{L} \to SL_2(\mathbb{C})$ be the hypothetical Langlands parameter attached to $f, h$, resp. We have the tensor product map $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to SO_4(\mathbb{C})$. [25], page 88. Use the identification $SL_2(\mathbb{C}) = Sp_1(\mathbb{C})$, and we have a representation of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$. It defines a symmetric, non-degenerate bilinear form on $\mathbb{C}^4$. Then we have $\phi_f \otimes \phi_h : \mathcal{L} \to SO_4(\mathbb{C})$. The distinguished unipotent orbit $(7, 1)$ of $SO_8(\mathbb{C})$ gives rise to a map $SL_2(\mathbb{C}) \to SO_8(\mathbb{C})$. Hence it defines a map $\phi_u : \mathcal{L} \times SL_2(\mathbb{C}) \to SO(8, \mathbb{C})$. Then consider $\phi = (\phi_h \otimes \phi_f) \oplus \phi_u : \mathcal{L} \times SL_2(\mathbb{C}) \to SO_4(\mathbb{C}) \times SO_8(\mathbb{C}) \subset GSO_{12}(\mathbb{C})$.

We expect that $\phi$ parametrizes $\Pi_{f, h}$.

This paper is organized as follows. In Section 2, we recall several facts about the Hermitian symmetric domain of $Spin(2, 10)$ or $PGSpin(2, 10) = PGSO(2, 10)$, and holomorphic modular forms on it. In Section 3, we recall our previous work [18]. In Sections 5 and 6, following Ikeda [14], we study the integral expression (1.1) for $\mathcal{F}_{f, h}$, which gives rise to a cusp form on $\mathfrak{S}_2$. We carry out the essentially same method but we have to rely on roots to describe some double coset space related to this method. The calculation of the double cosets will be devoted in Section 4. In Section 7, we compute $\mathcal{F}_{f, h}$ explicitly using two kinds of Fourier-Jacobi expansions, and indicate that it is most likely nonvanishing.

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2. Preliminaries

2.1. Cayley numbers and the exceptional domain. In this section we refer Section 2 of\footnote{18}. For any field $K$ whose characteristic is different from 2 and 3, the Cayley numbers $\mathfrak{C}_K$ over $K$ is an eight-dimensional vector space over $K$ with basis \( \{ e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7 \} \) satisfying the following rules for multiplication:

1. \( xe_0 = e_0x = x \) for all \( x \in \mathfrak{C}_K \),
2. \( e_i^2 = -e_0 \) for \( i = 1, \ldots, 7 \),
3. \( e_i(e_{i+1}e_{i+3}) = (e_i e_{i+1}) e_{i+3} = -e_0 \) for any \( i \pmod{7} \).

For each \( x = \sum_{i=0}^{7} x_i e_i \in \mathfrak{C}_K \), the map \( x \mapsto \bar{x} = x_0 e_0 - \sum_{i=1}^{7} x_i e_i \) defines an anti-involution on \( \mathfrak{C}_K \).

The trace and the norm on \( \mathfrak{C}_K \) are defined by

\[ \text{Tr}(x) := x + \bar{x} = 2x_0, \quad N(x) := x\bar{x} = \sum_{i=0}^{7} x_i^2. \]

The Cayley numbers \( \mathfrak{C}_K \) is neither commutative nor associative. We denote by \( \mathfrak{o} \), the integral Cayley numbers which is a \( \mathbb{Z} \)-submodule of \( \mathfrak{C}_K \) given by the following basis:

\[ \alpha_0 = e_0, \quad \alpha_1 = e_1, \quad \alpha_2 = e_2, \quad \alpha_3 = -e_4, \quad \alpha_4 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4), \quad \alpha_5 = \frac{1}{2}(-e_0 - e_1 - e_4 + e_5), \]
\[ \alpha_6 = \frac{1}{2}(-e_0 + e_1 - e_2 + e_6), \quad \alpha_7 = \frac{1}{2}(-e_0 + e_2 + e_4 + e_7). \]

It is known that \( \mathfrak{o} \) is stable under the operations of the anti-involution, multiplication, and addition. Further we have \( \text{Tr}(x), \quad N(x) \in \mathbb{Z} \) if \( x \in \mathfrak{o} \). By using this integral structure, for any \( \mathbb{Z} \)-algebra \( R \), one can consider \( \mathfrak{C}_R = \mathfrak{o} \otimes_{\mathbb{Z}} R \).

Let \( \mathfrak{J}_K \) be the exceptional Jordan algebra consisting of the element:

\[ (2.1) \quad X = (x_{ij})_{1 \leq i,j \leq 3} = \begin{pmatrix} a & x & y \\ \bar{x} & b & z \\ \bar{y} & \bar{z} & c \end{pmatrix}, \]

where \( a, b, c \in Ke_0 = K \) and \( x, y, z \in \mathfrak{C}_K \).
By using integral Cayley numbers, we define a lattice

\[ \mathfrak{J}(\mathbb{Z}) := \{ X = (x_{ij}) \in \mathfrak{J}_\mathbb{Q} \mid x_{ii} \in \mathbb{Z}, \text{ and } x_{ij} \in \mathfrak{o} \text{ for } i \neq j \}, \]

and put \( \mathfrak{J}(\mathcal{R}) = \mathfrak{J}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{R} \) for any \( \mathbb{Z} \)-algebra \( \mathcal{R} \).

We define \( R_3^+(K) = \{ X \in \mathfrak{J}_C \mid \det(X) \neq 0 \} \) and define the set \( R_3^+(\mathbb{R}) \) consisting of squares of elements in \( R_3(K) \). It is known that \( R_3^+(\mathbb{R}) \) is an open, convex cone in \( \mathfrak{J}_\mathbb{R} \). We denote by \( \overline{R_3^+(\mathbb{R})} \) the closure of \( R_3^+(\mathbb{R}) \) in \( \mathfrak{J}_\mathbb{R} \simeq \mathbb{R}^{27} \) with respect to Euclidean topology. For any subring \( A \) of \( \mathbb{R} \), set

\[ \mathfrak{J}(A)_{+} := \mathfrak{J}(A) \cap \overline{R_3^+(\mathbb{R})}, \quad \mathfrak{J}(A)_{\geq 0} := \mathfrak{J}(A) \cap \overline{R_3^+(\mathbb{R})}. \]

We define the exceptional domain as follows:

\[ \mathcal{X} := \{ Z = X + Y \sqrt{-1} \in \mathfrak{J}_C \mid X, Y \in \mathfrak{J}_\mathbb{R}, \ Y \in R_3^+(\mathbb{R}) \} \]

which is a complex analytic subspace of \( \mathbb{C}^{27} \).

Let \( G \) be the exceptional Lie group of type \( E_{7,3} \) over \( \mathbb{Q} \) which acts on \( \mathcal{X} \). Then \( G(\mathbb{R}) \) is of real rank 3 (cf. [4]). In loc.cit. Baily constructed an integral model \( G_\mathbb{Z} \) of \( G \) over \( \text{Spec} \mathbb{Z} \) and it follows from this with Proposition 1.1 of [10] that \( G(\mathbb{Q}_p) \) is a split group of type \( E_7 \) for any prime \( p \).

The Satake diagram of \( E_{7,3} \) is

\[ \bullet_{\beta_1} \rightarrow \bullet_{\beta_3} \rightarrow \bullet_{\beta_4} \rightarrow \bullet_{\beta_5} \rightarrow \bullet_{\beta_6} \rightarrow \bullet_{\beta_7} \]

The \( \mathbb{Q} \)-root system is of type \( C_3 \), and the extended Dynkin diagram of \( C_3 \) is

\[ o_{\lambda_0} \rightarrow o_{\lambda_1} \rightarrow o_{\lambda_2} \leftarrow o_{\lambda_3}, \]

where \( \lambda_1 \) corresponds to \( \beta_1 \), \( \lambda_2 \) to \( \beta_6 \), \( \lambda_3 \) to \( \beta_7 \), and \( -\lambda_0 \) is the maximal root in \( C_3 \). Here \( \lambda_1, \lambda_2 \) have multiplicity 8, and \( \lambda_3 \) has multiplicity 1.

Let \( G_1 = SL_2, \ G_2 = Spin(2,10) \). Then \( (G_1, G_2) \) is a dual pair inside \( G = E_{7,3} \) (cf. [6]). They are given as follows: If we remove the root \( \lambda_1 \) in the extended Dynkin diagram, the remaining

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*Since we are not dealing with the exceptional group of type \( G_2 \), we hope that our notation will not cause confusion.
diagram is an almost direct product $G_1G_2$. More precisely, let $\theta = h_{\lambda_0}(-1)$. Then \( \theta \) is an involution whose centralizer $H = C_{E_7}(\theta)$ as an algebraic group is the almost direct product $G_1G_2$. Then $G_1 \cap G_2 = Z = \{(h_{\lambda_0}(-1), h_{\lambda_0}(-1)) \simeq \{\pm 1\}$. Since $G_1$ and $G_2$ are simply connected algebraic groups, one has the following exact sequence

$$1 \longrightarrow \mu_2(k) \longrightarrow G_1(k) \times G_2(k) \longrightarrow H(k) \longrightarrow H^1(\text{Gal}(\overline{k}/k), k^\times) = k^\times/(k^\times)^2 \longrightarrow 1$$

for any local field $k$ of characteristic zero. This means that $H(k)$ is strictly bigger that $G_1(k)G_2(k) \subset E_7(k)$. Furthermore the 2 to 1 isogeny $G_1 \times G_2 \longrightarrow H$ induces a natural inclusion $X^*(T_H) \hookrightarrow X^*(T_{G_1}) \times X^*(T_{G_2})$ of index 2 where $X^*(T)$ stands for the character group of a torus $T$.

We remark that $G_2(k)$ is a split group for any p-adic field $k$. The $\mathbb{Q}$-root system of $G_2$ is of type $C_2$. It is the group $\mathfrak{L}_2$ in [14], page 528, and it acts on the boundary component $\mathfrak{T}_2$ below.

To end this section, we remark on an explicit integral model of $G_2 = \text{Spin}(2,10)$. Since $G_{2\mathbb{Q}} \subset \mathcal{G}_Z$, one can define an integral model $\mathfrak{G}_2$ of $G_2$ as the Zariski closure of $G_{2\mathbb{Q}}$ in $\mathcal{G}_Z$. It follows from Proposition 1.1 of [10] again that $\mathfrak{G}_2$ is a smooth model over $\mathbb{Z}$. Then we have $G_2(\mathbb{Z}) = \text{Spin}(2,10)(\mathbb{Q}) \cap G(\mathbb{Z})$.

We can construct an explicit integral model of $G_2$ up to $\mathbb{Q}$-isomorphism as follows. There is a natural surjective map $\iota : G_2 \longrightarrow G_2/\{h_{\lambda_0}(-1)\} = SO(2,10)$ with kernel $\mu_2$, where $SO(2,10)$ is the special orthogonal group we want to define explicitly. Since $G_2(\mathbb{Q}_p)$ splits, so does $SO(2,10)(\mathbb{Q}_p)$ for any prime $p$. By Hasse principle, there exists a unique $\mathbb{Q}$-isomorphism class of $SO(2,10)$ which splits everywhere (Theorem 4.1.2 of [20]). On the other hand the quadratic space $V = H \perp H \perp (-E_8)$ where $E_8$ is the quadratic form given by the Cartan matrix of the exceptional Lie algebra of type $E_8$ and $H$ is the usual hyperbolic space, defines a special orthogonal group $SO(V)$ with the signature $(2,10)$ which splits at any prime $p$. Hence we have $SO(V) \simeq SO(2,10)$ over $\mathbb{Q}$. Then $\text{Spin}(2,10)$ is defined as the double cover of $SO(V)$ via the isomorphism $SO(V) \simeq SO(2,10)$ as above.

2.2. Hermitian symmetric domain for $GSpin(2,10)$. Define $\mathfrak{J}_2(R)$ as the set of all matrices of forms

$$X = \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix}, \quad a, b \in R, \quad x \in \mathfrak{C}_R.$$

We define the inner product on $\mathfrak{J}_2(R) \times \mathfrak{J}_2(R)$ by $(X,Y) := \frac{1}{2}\text{Tr}(XY + YX)$. For any such $X$, we define $\det(X) := ab - N(x)$. For $X$ as above, $r \in R$, and $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, $\xi_i \in \mathfrak{C}_R$ ($i = 1,2$), we
have \( \begin{pmatrix} X & \xi \\ t\xi & r \end{pmatrix} \in \mathcal{J}(R) \). Define
\[
\mathcal{J}_2(A)_+ = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \in \mathcal{J}_2(A) \mid a, b \in A \cap \mathbb{R}_{>0}, \ ab - N(x) > 0 \right\},
\]
and
\[
\mathcal{J}_2(A)_{\geq 0} = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \in \mathcal{J}_2(A) \mid a, b \in A \cap \mathbb{R}_{\geq 0}, \ ab - N(x) \geq 0 \right\}.
\]

We also define
\[
\mathcal{T}_2 := \{ X + Y \sqrt{-1} \in \mathcal{J}_2(C) \mid X, Y \in \mathcal{J}_2(R), \ Y \in \mathcal{J}_2(R)_+ \}.
\]

It is well-known that \( \mathcal{T}_2 \) is the Hermitian symmetric domain for \( G_2(R) \) which is a tube domain of type (IV). Since \( Spin(2, 10)(R)/\{\pm 1\} \simeq SO(2, 10)(R) \), where \( \{\pm 1\} \) is a subgroup in the center of \( Spin(2, 10)(R) \), \( \mathcal{T}_2 \) is also the symmetric domain for \( SO(2, 10)(R) \) (See Section 6 of Appendix in [28]). For us, it is more convenient to consider \( \tilde{G} = PGSO(2, 10) = PGSpin(2, 10) \). In this case, \( \mathcal{T}_2 \) is also the symmetric domain for \( PGSO(2, 10)(R)^0 \). Then modular forms on \( \mathcal{T}_2 \) can be considered as automorphic forms on \( GSpin(2, 10)(A_\mathbb{Q}) \) with the trivial central character.

2.3. Modular forms on \( \mathcal{T}_2 \). Recall the integral model of \( G_2 = Spin(2, 10) \) over \( \mathbb{Z} \) from Section 2.1. Then one can define the arithmetic group \( \Gamma_2 = G_2(\mathbb{Z}) \) of “level one”. In [3], Eie and Krieg considered an arithmetic subgroup \( \Gamma' \subset \Gamma_2 \), generated by the following. For \( Z \in \mathcal{T}_2 \), let
\[
Z = \begin{pmatrix} z_1 & w \\ \bar{w} & z_2 \end{pmatrix}, \text{ where } z_1, z_2 \in \mathbb{H}, \text{ and } w = x + y\sqrt{-1} \text{ with } x, y \in \mathbb{C}_R, \text{ and } \bar{w} = \bar{x} + \bar{y}\sqrt{-1}. \text{ Let } \det(Z) = z_1z_2 - \bar{w}w:
\]

(1) \( p_B : Z \mapsto Z + B, \ b \in \mathcal{J}_2(\mathbb{Z}) \);
(2) \( t_U : Z \mapsto ZU, \ U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) or \( U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \) for \( u \in \mathfrak{o} \);
(3) \( \iota : Z \mapsto -Z^{-1}, \text{ where } Z^{-1} = \frac{1}{\det(Z)} \begin{pmatrix} z_2 & -w \\ \bar{w} & z_1 \end{pmatrix} \).
If we consider $\Gamma'$ as a subgroup of $G(\mathbb{Z})$, $p_B$ is the element $p_B'$ in $GSpin(2,10)$ with $B' = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ and $B \in \mathfrak{z}_2(\mathbb{Z})$; and $t = te_2e_3$ in $GSpin(2,10)$. Also $tU = m_{ue_23} \in M(\mathbb{Z})$ in $GSpin(2,10)$ for $U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. If $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $tU = m_{e_23}m_{-e_32}m_{e_23}$.

It is likely that $\Gamma' = \Gamma_2$, but we have not shown it yet.

For any $g \in G'(\mathbb{R})$ and $Z \in \mathfrak{z}_2$, one can define a holomorphic automorphic factor $j(g,Z) \in \mathbb{C}$ which satisfies the cocycle condition. More explicitly, $j(p_B,Z) = j(tU,Z) = 1$ and $j(t,Z) = \det(Z)$.

Let $F$ be a holomorphic function on $\mathfrak{z}_2$ which for some integer $k > 0$ satisfies

$$F(\gamma Z) = F(Z)j(\gamma,Z)^k, \quad Z \in \mathfrak{z}_2, \quad \gamma \in \Gamma_2.$$ 

Then $F$ is called a modular form on $\mathfrak{z}_2$ of weight $k$ with respect to $\Gamma_2$. For example, $F$ satisfies

$$F(Z + B) = F(Z), \quad F(tUZU) = F(Z), \quad F(-Z^{-1}) = \det(Z)^k F(Z),$$

for $B \in \mathfrak{z}_2(\mathbb{Z})$, and $U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ for $u \in \mathfrak{o}$.

We denote by $M_k(\Gamma_2)$ the space of such forms. By Koecher principle, we do not need the holomorphy at the cusps. For a holomorphic function $F : \mathfrak{z}_2 \rightarrow \mathbb{C}$, consider, for $\tau \in \mathbb{H}$,

$$\Phi F(\tau) = \lim_{y \rightarrow \infty} F \begin{pmatrix} \tau & * \\ * & iy \end{pmatrix}.$$ 

If $\Phi F = 0$, $F$ is called a cusp form. Let $S_k(\Gamma_2)$ be the space of cusp forms of weight $k$ with respect to $\Gamma_2$.

By [26], Theorem 7.12, the strong approximation theorem holds with respect to $S = \{\infty\}$, namely, $G'(\mathbb{A}) = G'(\mathbb{Q})G'(\mathbb{R})G'(\mathbb{Z})$, and $G_2(\mathbb{Z}) = G'(\mathbb{Q}) \cap G'(\mathbb{R})G'(\mathbb{Z})$. Hence one can associate a Hecke eigen cusp form in $S_k(\Gamma_2)$ with an automorphic form on $G'(\mathbb{A})$ which is fixed by $G'(\mathbb{Z})$, and then we obtain a cuspidal automorphic representation of $G'(\mathbb{A})$ with the trivial central character.

3. Ikeda type lift for $E_{7,3}$

In this section we recall the Ikeda type construction for $E_{7,3}$ in [18]. Let $P = MN$ be the Siegel parabolic subgroup of $G$ where the derived group $M^D = [M,M]$ of the Levi subgroup $M$ is of type $E_6$. Let $\nu : M \rightarrow GL_1$ be the similitude character (see Section 2 of [18]) and it
can be naturally extended to $P$. Let $\Gamma = G(\mathbb{Z})$ be the arithmetic subgroup defined by Baily in \cite{4} which is constructed by using the integral Cayley numbers $\mathfrak{a}$. For a positive integer $k \geq 6$, we constructed in \cite{18} a non-zero Hecke eigen cusp form $F_f(Z)$ in $S_{2k+8}(\Gamma)$ from a Hecke eigen cusp form $f = \sum_{n \geq 1} c(n)q^n \in S_{2k}(SL_2(\mathbb{Z}))$: For a positive integer $k \geq 6$, let $E_{2k+8}$ be the Siegel Eisenstein series on $\mathcal{T}$ of weight $2k + 8$ with respect to $\Gamma$. Then it has the Fourier expansion of form

$$E_{2k+8}(Z) = \sum_{T \in \mathbb{Z}_+} a_{2k+8}(T) \exp(2\pi \sqrt{-1}(T, Z)), \ Z \in \mathcal{T},$$

$$a_{2k+8}(T) = C_{2k+8} \det(T)^{\frac{2k-1}{2}} \prod_{p \mid \det(T)} \tilde{f}_T^p(p^{\frac{2k-1}{2}}),$$

where $C_{2k+8} = 2^{15} \prod_{n=0}^{2} \frac{2k + 8 - 4n}{B_{2k+8-4n}}$, and $\tilde{f}_T^p(X)$ is a Laurent polynomial over $\mathbb{Q}$ in $X$ which is depending only on $T$ and $p$.

Let $S_{2k}(SL_2(\mathbb{Z}))$ be the space of elliptic cusp forms of weight $2k \geq 12$ with respect to $SL_2(\mathbb{Z})$. For each normalized Hecke eigenform $f = \sum_{n = 1}^{\infty} c(n)q^n$, $q = \exp(2\pi \sqrt{-1}\tau)$, $\tau \in \mathbb{H}$ in $S_{2k}(SL_2(\mathbb{Z}))$ and each rational prime $p$, we define the Satake $p$-parameter $\{\alpha_p, \alpha_p^{-1}\}$ by $c(p) = p^{\frac{2k-1}{2}}(\alpha_p + \alpha_p^{-1})$. For such $f$, consider the following formal series on $\mathcal{T}$:

$$F_f(Z) = \sum_{T \in \mathbb{Z}_+} A(T) \exp(2\pi \sqrt{-1}(T, Z)), \ Z \in \mathcal{T}, \ A(T) = \det(T)^{\frac{2k-1}{2}} \prod_{p \mid \det(T)} \tilde{f}_T^p(\alpha_p).$$

Then we showed

**Theorem 3.1.** \cite{18} The function $F_f(Z)$ is a non-zero Hecke eigen cusp form on $\mathcal{T}$ of weight $2k + 8$ with respect to $\Gamma$.

We call $F_f$ the Ikeda type lift of $f$. Then $F = F_f$ gives rise to a cuspidal automorphic representation $\pi_F = \pi_\infty \otimes \otimes'_p \pi_p$ of $G(\mathbb{A})$. Then $\pi_\infty$ is a holomorphic discrete series of the lowest weight $2k + 8$ associated to $-(2k + 8)\varpi_7$ in the notation of \cite{5} (cf. \cite{19}, page 158). For each prime $p$, $\pi_p$ is unramified. In fact, $\pi_p$ turns out to be a degenerate principal series

$$\pi_p \simeq \text{Ind}_{\mathcal{P}(\mathbb{Z})}^{G(\mathbb{Q})} |\nu(g)|^{2s_p},$$

where $p^{s_p} = \alpha_p$ (see Section 11 of \cite{18}). Let $L(s, \pi_f) = \prod_p (1 - \alpha_p p^{-s})(1 - \alpha_p^{-1} p^{-s})$ be the automorphic $L$-function of the cuspidal representation $\pi_f$ attached to $f$. Then
Theorem 3.2. [18] The degree 56 standard $L$-function $L(s, \pi_F, St)$ of $\pi_F$ is given by

$$L(s, \pi_F, St) = L(s, \text{Sym}^3\pi_f)L(s, \pi_f)^2 \prod_{i=1}^{4} L(s \pm i, \pi_f)^2 \prod_{i=5}^{8} L(s \pm i, \pi_f),$$

where $L(s, \text{Sym}^3\pi_f)$ is the symmetric cube $L$-function.

4. Double Coset Decomposition

In order to prove Theorem [14], following [14], we need to compute a suitable representatives of the double coset space over a $p$-adic field related to the unwinding method.

This section is mainly due to R. Lawther. We thank him for a very detailed note [22]. He gave an explicit double coset space related to what we need, but he worked over an algebraically closed field because he relied on the results in [21]. In what follows we modify his argument so that it would work over any $p$-adic field in our case.

Let $p$ be any rational prime, and $G$ to be a simply-connected algebraic group of type $E_7$ over a $p$-adic field $k$, and for simplicity, let $G = G(k), G_1 = G_1(k)$, and $G_2 = G_2(k)$.

Let $T$ to be a fixed maximal torus of $G$. Let $B$ be the standard Borel subgroup containing $T$. Take roots with respect to $T$; let $\{\beta_1, \ldots, \beta_7\}$ be a simple root system, numbered as in Bourbaki [5]. Write roots of $E_7$ as strings of coefficients of simple roots, so that for example, the highest root is 2234321. Let $\Phi$ (resp. $\Phi^+$) be the set of all roots (resp. all positive roots). Let $\gamma_1 = 0112221$, and by adding $\gamma_1$, we get the extended Dynkin diagram of $E_7$:

```
\begin{center}
  o_{\gamma_1} \cdots o_{\beta_1} o_{\beta_3} o_{\beta_4} o_{\beta_5} o_{\beta_6} o_{\beta_7}
  \vdots
  o_{\beta_2}
\end{center}
```

In order to use Lawther’s note [22], we take a different centralizer from Section [2.1]. Let $\theta = h_7(-1)$. Then $\theta$ is an involution whose centralizer $H = C_{E_7}(\theta)$ is of the form $A_1D_6$. Explicitly, the roots whose root subgroups lie in $H$ are those whose $\beta_6$-coefficient is even. The simple roots of the $D_6$ are $\gamma_1$, and $\gamma_2 = \beta_1, \gamma_3 = \beta_3, \gamma_4 = \beta_4, \gamma_5 = \beta_5, \gamma_6 = \beta_2$, and that of the $A_1$ is $\beta_7$. Then $Z := G_1(k) \cap G_2(k) = \{(h_{\beta_2}(-1), h_{\beta_2}(-1))\} \simeq \{\pm 1\}$. Note that $h_{\gamma_1}(-1)h_{\gamma_3}(-1)h_{\gamma_6}(-1) = h_{\beta_7}(-1)$ and $H(k)$ contains $G_1G_2 \simeq G_1(k) \times G_2(k)/Z$.

We set $\gamma_6 = \beta_6$ and $\gamma_7 = \beta_7$. 
4.1. **Double coset space.** For each root $\alpha$ let $x_\alpha(c), c \in k$ be the corresponding root subgroup and put

$$n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1), \quad y_\alpha = x_\alpha(1)n_\alpha x_\alpha(z), \quad h_\alpha(c) = x_\alpha(c)x_{-\alpha}(-c^{-1})x_\alpha(c)n_\alpha^{-1}, \quad c \in k^x.$$ 

Put $h_i = h_{\beta_i}$ for simplicity.

Let $P$ be the Siegel parabolic subgroup of $G$ corresponding to $\{\beta_1, \ldots, \beta_6\}$ which is of type $E_6T_1U_27$ over an algebraic closed field where $T_i$ denotes an $i$-dimensional torus, and $U_j$ is a unipotent group of dimension $j$. For each element $g \in G(k)$, put $Q_g = g^{-1}P(k)g \cap H$. Then we have the following lemma.

**Lemma 4.1.** The double coset space $P(k)\setminus G(k)/H$ is a finite set. For any $g \in G(k)$, there exists $g'$ so that $P(k)gH = P(k)g'H$ and $Q_{g'}$ coincides with $Q_g$ for some $y \in \{1, n, y_{\beta_5}, n, y_{\beta_6}y_{\beta_7}n\}$, where $n = n_{\beta_6} + \beta_7$.

To prove this lemma, we need more arguments which would be a lengthy calculation. Let $C$ be any fixed complete system of the representatives of the Weyl group $W = N/T(k)$.

**Lemma 4.2.** A complete system of representatives of the double coset space $B(k)\setminus G(k)/H$ is a finite set and it consists of the elements of form

$$y_{\alpha_1}\cdots y_{\alpha_r}n', \quad \alpha_1, \ldots, \alpha_r \in \Phi^+, \quad 0 \leq r \leq 7$$

where $\alpha_1, \ldots, \alpha_r$ are mutually orthogonal and $n' \in N = N_{G(k)}(T(k))$ runs over the set $\{1\} \cup \{n' \in C \mid n' \theta := n'\theta n'^{-1} \neq \theta\}$. Furthermore $\alpha_i(n'\theta) = -1$ for any $i$ ($1 \leq i \leq r$).

**Proof.** The proof is almost same as in Section 3 of [21] but we have to take care of the base field because the results in [21] stated for which the base field is an algebraically closed field.

Put $S = \{g\theta(g)^{-1} \mid \theta(g) := \theta g\theta, \ g \in G(k)\}$. Define the action of $G(k)$ on $S$ by

$$g * s = gs\theta(g)^{-1} \quad s \in S, \ g \in G(k).$$

Let us define a bijective map

$$B(k)\setminus G(k)/H \rightarrow \{O_{B(k)}(s) \mid s \in S\}, \ BxK \mapsto O_{B(k)}(x\theta(x)^{-1})$$

where $O_{B(k)}(s)$ stands for the orbit of $s$ for $B(k)$ with respect to the action $*$ as above. By Proposition 6.6 of [12], $O_{B(k)}(s) \cap N \neq \emptyset$, hence there exists $b \in B(k)$ such that $b * s \in N$. Since $\theta \in N$, if $s = x\theta(x)^{-1}$ we also have $(b * s)\theta = bx\theta(bx)^{-1} =: b^x\theta \in N$ which is an involution (hence
$(b_x \theta)^2 = 1)$. Take another $b' \in B(k)$ so that $b' \ast s \in N$ if exists. Put $g = b b'^{-1} \in B(k)$. Then $b_x \theta$ is conjugate by $g$ to $b' \ast \theta$. By using Bruhat decomposition, $b_x \theta$ is conjugate by an element of $T(k)$ to $b' \ast \theta$. Hence $b_x \theta$ is unique up to the conjugate by $T(k)$ and thereby we may denote such a $b$ by $b_x$ with the dependence on $x$. Summing up we have an injective map

$$B(k) \backslash G(k)/H \rightarrow \{ n \in N \mid n^2 = 1 \} / \sim, \quad B x H \mapsto b_x \theta$$

where $\sim$ stands for the equivalence relation of the conjugation by elements in $T$. We now describe the image of this map. Let $g = b_x \theta \in N$ be an involution for some $x \in G(k)$ and $b_x \in B(k)$. Then by the proof of Lemma 2 of [21] (noting that $n_\alpha = n_\alpha^{-1} = n_\alpha t$ for some $t \in T(k)$), there exists $\theta' \in T(k)$ and $t \in T(\mathbb{F})$ ($t = t_2 t_1$ for $t_1$ at line 3, p.119 of [21] and $t_2$ at line 11, p. loc.cit.) such that

$$t g = \theta' n_\alpha \cdots n_\alpha, \quad 0 \leq r \leq 7$$

such that $\alpha_1, \ldots, \alpha_r \in \Phi^+$ are mutually orthogonal and they satisfy $\alpha_i(\theta') = -1$.

We now descent $t$ to an element in $T(k)$. Let $Z_T(\theta'n)$ be the centralizer of $\theta'n$ in $T$ as an algebraic group over $\mathbb{F}$. Put $n = n_\alpha \cdots n_\alpha$. It is easy to see that $Z_T(\theta'n)$ is defined over $k$ and it is also a split torus. We define a one-cocycle on $\text{Gal}(\mathbb{F}/k)$ takes the values in $Z_T(\theta'n)(\mathbb{F})$ by

$$\sigma \mapsto t(t^{-1})^\sigma.$$

Since $H^1(\text{Gal}(\mathbb{F}/k), Z_T(\theta'n)(\mathbb{F})) = 1$ by Hilbert Theorem 90, there exists $s \in Z_T(\theta'n)(\mathbb{F})$ such that $t(t^{-1})^\sigma = s(s^{-1})^\sigma$ for any $\sigma \in \text{Gal}(\mathbb{F}/k)$. This means that $s^{-1} t \in T(k)$ and we have

$$s^{-1} t g = s^{-1} g s = s^{-1} \theta'n s = \theta'n.$$

On the other hand $\theta'$ is conjugate to $\theta$ since $\theta'n = y_{\alpha_1} \cdots y_{\alpha_r} \theta'$. It follows that they have to be conjugate by some $n' \in N$, hence $\theta' = n' \theta$. This gives us the claim. The finiteness is then clear from the above description. \hfill \Box

**Remark 4.3.** The proof of Lemma 4.2 shows that Corollary 3 of [21] holds for any local field $k$ of characteristic zero.

We are ready to prove Lemma 4.1.

**Proof.** The finiteness follows from the natural surjection $B(k) \backslash G(k)/H \rightarrow P(k) \backslash G(k)/H$ and Lemma 4.2.
Henceforth we will make use of the mathematica code implemented by [24]. By direct computation \( n' \) runs over the set \( R = \{1\} \cup \{n_\alpha \mid \alpha \in X\} \) where

\[
X = \{0000010, 0000110, 0000111, 0001110, 0001111, 0101110, 0011110, 0001111, 1011110, 0111110, 0101111, 0011111, 1111110, 1011111, 0112110, 0111111, 1112110, 1112111, 0112210, 0112111, 1122110, 1112210, 1112211, 0112211, 1122210, 1122211, 1123210, 1123211, 1223211\}.
\]

In fact \( n_\alpha \theta = n_\alpha h_7(-1)n_\alpha^{-1} = h_7(-1)h_\alpha((-1)\langle \beta_7, \alpha \rangle) \neq h_7(-1) \) is equivalent to that \( \langle \beta_7, \alpha \rangle \) is odd.

We shall discard extra elements among of \( y_\alpha \ldots y_\alpha, n', r \in R \). Recall that \( E_6(k) \subset P(k) \) (resp. \( H \)) consists of roots generated by \( \beta_1, \ldots, \beta_6 \) (resp. \( \gamma_1, \gamma_2 = \beta_1, \gamma_3 = \beta_3, \gamma_4 = \beta_4, \gamma_5 = \beta_5, \gamma_6 = \beta_2, \beta_7 \)).

Assume \( r = 0 \). We further assume that \( P(k)n_\alpha H \neq P(k)H \) for \( \alpha \in R \). Then by direct computation, there exists \( \beta \in \Phi \) so that \( n_\alpha = n_\beta n_\beta^{-1} \) and \( n_\beta \in P(k) \cap H \) where \( n \) is the element in the statement. Hence we have \( P(k)n_\alpha H = P(k)nH \).

Assume \( r = 1 \). For \( \alpha = \sum_{i=1}^7 a_i \beta_i \in \Phi^+ \), clearly \( y_\alpha \in P(k) \) if \( a_7 = 0 \). Therefore the case \( a_7 > 0 \) will be essential. For each \( n' \in R \) we compute the set \( R_1(n') \) consisting of \( \alpha \) so that \( a_7 > 0 \) and \( \alpha(n' \theta) = -1 \). For example,

\[
R_1(1) = \{000011, 000111, 001111, 010111, 001111, 101111, 011111, 111111, 011211, 111211, 011221, 112211, 112221, 112321, 122321, 122321\}.
\]

By direct calculation for any \( n' \in R \) and \( \alpha \in R_1(n') \) we would check that there exists \( g' \in G(k) \) such that

\[
P(k)g'H = P(k)y_\alpha n'H \text{ and } Q_{g'} = Q_{g'}.
\]

Let us give a few examples. For \( n' = 1 \) and \( \alpha = 0000011 \) we see that

\[
g := y_\alpha = n_6 y_7 n_6^{-1} \equiv y_7 n_6 \equiv y_7 n_6 n_7 \equiv y_7 n_6 \text{ mod } (P(k), H)
\]

where \( n_i = n_\beta_i \) and we use the relation \( n = n_\alpha = n_7 n_6 n_7^{-1} \). Put \( g' = y_7 n_6 \). Then one would be able to check \( Q_{g'} = Q_{y_7 n_6} \). The remaining cases would be done similarly. So it is omitted because it is a routine and lengthy. The case \( r = 2 \) would be checked by using the calculation in case \( r = 1 \). By direct calculation it is easy to check that the case \( r \geq 3 \) never happens because of the orthonormality for simple roots in question. \( \square \)
4.2. **An explicit structure of** $Q_g$. By Lemma 1.1 we may focus on the following four elements to consider $Q_g = g^{-1}P(k)g \cap H, g \in G(k)$. The following table is made by Lawther. Here we put $H(\mathbb{K}) = C_{G(\mathbb{K})}(\theta)$.

| $g \in G(k)$ | $g^{-1}P(\mathbb{K})g \cap H(\mathbb{K})$ |
|--------------|----------------------------------|
| $g_0 = 1$    | $D_5T_2U_{11}$                  |
| $g_1 = n$    | $A_5A_1T_1U_{15}$               |
| $g_2 = y_{37}n$ | $A_4T_2U_{21}$               |
| $g_3 = y_{71}y_{37}n$ | $B_3A_1T_1U_{17}$ |

**Table 1.**

For $i = 0, 1, 2, 3$, put $Q_i = g_i^{-1}Pg_i \cap H$. Let $Q_i = M_iN_i$ be the Levi decomposition and $T_i$ the maximal split torus in $M_i$. We now try to compute $T_i, U_i$, and the values of the modulus character $\delta_{Q_i}$ (resp. the modulus character $\delta_{P(k_i)}$) on $T_i$ (resp. on $g_iT_i g_i^{-1} \subset P(k)$). We first realize $G(k)$ in $GL_{56}(k)$ in terms of roots by using mathematica code implemented by [24]. By using root groups we would know which entries of $P(k) = MU_{27}$ in $GL_{56}(k)$ are always zero (the number of such entries is 379). This can be checked if we look $U_{27} = \{ x_\alpha(c_\alpha) \mid \alpha \in \Phi^-, c_\alpha \in k \}$ because the ($p$-adically) open subgroup $U_{27}^{-1}P(k)$ is Zariski dense in $E_7$ as an algebraic group. This gives rise to a naive criterion for $g \in G(k)$ to be an element of $P(k)$. In what follows we denote by $|\ast|$ the normalized valuation of $k$ so that $|w| = q^{-1}$ for a uniformizer of $k$ where $q$ stands for the cardinality of the residue field of $k$.

4.2.1. **Case $Q_0$.** In this case we have

$$T_0 = \{ h_{71}(t_1)h_{72}(t_2)h_{73}(t_3)h_{74}(t_4)h_{75}(t_5)h_{76}(t_6)h_{77}(t_7) \mid t_1, \ldots, t_7 \in k^\times \}$$

and $N_0 = \langle x_\alpha(c_\alpha) \mid \alpha \in \Phi_0 \rangle$ where

$$\Phi_0 = \{ 0000001, 0112221, 1112221, 1122221, 1123221, 1123321, 1223221, 1223321, $$

$$(1224321, 1234321, 2234321) \}.$$  

Then for $t = h_{71}(t_1)h_{72}(t_2)h_{73}(t_3)h_{74}(t_4)h_{75}(t_5)h_{76}(t_6)h_{77}(t_7)$ one has

$$\delta_{Q_0}(t) = |t_1|^{10}|t_2|^2$$ and

$$\delta_{P(k)}(t) = |t_1t_7|^{18}.$$  

Since $\delta_{P(k)}(t) = |\nu(t)|^{18}$ (see Section 6 of [18]), one concludes $\nu(t) = t_1t_2u$ for some unit $u$ in $\mathcal{O}_k$. In particular $\omega \circ \nu(t) = \omega(t_1t_2)$ for any unramified character $\omega$ of $k^\times$ where $\nu : P \rightarrow GL_1$ is the similitude character.
It is easy to see that $G_1 \cap T_0 = \{ h_{\gamma_7}(t_7) \mid t_7 \in k^\times \}$ and $G_2 \cap T_0 = \{ h_{\gamma_1}(t_1) \cdots h_{\gamma_6}(t_6) \mid t_1, \ldots, t_6 \in k^\times \}$.

4.2.2. Case $Q_1$. In this case we have

$$T_1 = \{ h_{\gamma_1}(t_1)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7) \mid t_1, \ldots, t_7 \in k^\times \}$$

and $N_1 = \langle x_\alpha(c_\alpha) \mid \alpha \in \Phi_1 \rangle$ where

$$\Phi_1 = \{ \langle 0000100, 0001100, 00101100, 01011100, 01111100, 10111100, 101111100, 11111100, 11121100, 11221100, 1123321, 1223321, 1224321, 1234321, 2234321 \},$$

Then for $t = h_{\gamma_1}(t_1)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7)$ one has

$$\delta_{Q_1}(t) = |t_5|^{10} \text{ and } \delta_{P(k)}(g_1t_1g_1^{-1}) = |t_5|^{18}.$$ 

As seen before, $\omega \circ \nu(g_1t_1g_1^{-1}) = \omega(t_5)$ for any unramified character $\omega$ of $k^\times$. We also have $G_1 \cap T_1 = \{ h_{\gamma_7}(t_7) \mid t_7 \in k^\times \}$ and $G_2 \cap T_1 = \{ h_{\gamma_1}(t_1) \cdots h_{\gamma_6}(t_6) \mid t_1, \ldots, t_6 \in k^\times \}$.

4.2.3. Case $Q_2$. In this case we have

$$T_2 = \{ h_{\gamma_1}(t_5t_7)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7) \mid t_2, \ldots, t_7 \in k^\times \}$$

and $N_2 = \langle x_\alpha(c_\alpha) \mid \alpha \in \Phi_2 \rangle$ where

$$\Phi_2 = \{ -0000001, 0000100, 0001100, 00101100, 0011100, 00111100, 10111100, 101111100, 11111100, 11121100, 11221100, 1122211, 1122221, 1122222, 1123221, 1223321, 1224321, 1234321, 2234321 \},$$

Then for $t = h_{\gamma_1}(t_5t_7)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7)$ one has

$$\delta_{Q_2}(t) = |t_5|^{14}|t_7|^4 \text{ and } \delta_{P(k)}(g_2t_2g_2^{-1}) = |t_5|^{18}.$$ 

As seen before, $\omega \circ \nu(g_2t_2g_2^{-1}) = \omega(t_5)$ for any unramified character $\omega$ of $k^\times$. We also have $G_1 \cap T_2 = 1$ and $G_2 \cap T_2 = 1$.

The Levi of $Q_2$ is of type $A_4T_2$ and $A_4$ has simple roots $\beta_1, \gamma_3 = \beta_3, \gamma_4 = \beta_4, \gamma_6 = \beta_2$. One can check that the centralizer $Z_{T_2}(A_4) = \{ t \in T_2 \mid tg = gt \text{ for any } g \in A_4 \}$ is given by

$$T := \{ h_T(a) := h_{\gamma_1}(a^2)h_{\gamma_2}(a^2)h_{\gamma_3}(a^2)h_{\gamma_4}(a^2)h_{\gamma_5}(a)h_{\gamma_6}(a)h_{\gamma_7}(a) \mid a \in k^\times \} \subset G_1G_2 = K.$$ 

We see that $GL_1$ is diagonally embedded in $K = G_1G_2$ via $\Delta : GL_1 \to T, a \mapsto h_T(a)$. Put $T' := \{ h_{\gamma_1}(t_7)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7) \}$ and $T'' := \{ h_{\gamma_7}(t)h_{\gamma_7}(t) \mid t \in k^\times \}$. Then
$T'A_4 = T'' \times A_4$ makes up $GL_5$ and the projection $T'A_4 \to T'' \simeq GL_1$ corresponds to the determinant.

4.2.4. Case $Q_3$. The situation is a little bit more complicated than other cases. Let us first observe that

$$P(k) \cap g_3Hg_3^{-1} = P(k) \cap g_3C_{G(k)}(\theta)g_3^{-1} = C_{P(k)}(g_3\theta g_3^{-1}) = C_{P(k)}(g'),$$

where $g' = h_{\beta_6}(-1)n_{\beta_7}n_{\gamma_1}$. One can easily extend Theorem 6 of [21] to the Siegel parabolic subgroup $P$ and then we get $\dim N_3 = 17$. On the other hand one can consider the unipotent subgroup $U_{17}$ directly in $P \cap gHg^{-1}$ as follows. For the 16 of the 17 root groups in $U_{17}$, there is then a 1-dimensional unipotent group diagonally embedded in the product of the two root groups, of the form

$$\{x_\alpha(t)g'x_\alpha(t)g'^{-1} : t \in k\} = \{x_\alpha(t)x_{g'(\alpha)}(\pm t) : t \in k\},$$

where the sign in the second term is determined by the structure constants. The 17th root subgroup is simply the root subgroup corresponding to the highest root 2234321. The 16 pairs of positive roots $\alpha$, $g'(\alpha)$ interchanged by $g'$ are as follows:

| $\alpha$ | $g'(\alpha)$ | $\alpha$ | $g'(\alpha)$ | $\alpha$ | $g'(\alpha)$ | $\alpha$ | $g'(\alpha)$ |
|----------|--------------|----------|--------------|----------|--------------|----------|--------------|
| 1000000  | 1112221      | 1000000  | 1111111      | 1000000  | 1122221      | 1111110  | 1111111      |
| 1011000  | 1123221      | 1112110  | 1112111      | 1011100  | 1123321      | 1122110  | 1122111      |
| 1111000  | 1223221      | 1122110  | 1112211      | 1111100  | 1223321      | 1122110  | 1122211      |
| 1112100  | 1224321      | 1123210  | 1123211      | 1122100  | 1234321      | 1223210  | 1223211      |

By matching of the dimension we may have $g_3^{-1}U_{17}g_3 = N_3$. On the other hand we have

$$T_3 = \{h_{\gamma_1}(t_5t_7)h_{\gamma_2}(t_5^2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_6}(t_5)h_{\gamma_7}(t_7) : t_3, \ldots, t_7 \in k^\times\}.$$

Then for $t = h_{\gamma_1}(t_5t_7)h_{\gamma_2}(t_5^2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_6}(t_5)h_{\gamma_7}(t_7)$ one has

$$\delta_{Q_3}(t) = |t_5|^{18} \text{ and } \delta_{P(k)}(g_3tg_3^{-1}) = |t_5|^{18}.$$ 

As seen before, $\omega \circ \nu(g_3tg_3^{-1}) = \omega(t_5)$ for any unramified character $\omega$ of $k^\times$. We also have $G_1 \cap T_3 = \{h_{\gamma_7}(t_7) : t_7 \in k^\times\}$ and we also have $G_1 \cap T_2 = 1$ and $G_2 \cap T_2 = 1$. Finally we remark that $G_1 = SL_2$ is common factor of $G_1$ and $G_2$, hence there exists a 2 to 1 homomorphism

$$(4.1) \quad \Delta : SL_2 \to G_1 \times G_2 \to H$$
onto the image. Let \( \iota : SL_2 \rightarrow SL_2 \) be the isomorphism defined by \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} \).

The image of \( \Delta \) is naturally isomorphic to

\[
\{ (\gamma, \iota(\gamma)) \mid \gamma \in SL_2 \}/\{ \pm (I_2, I_2) \}.
\] (4.2)

5. Computation of Satake parameters

In this section, we prove Proposition 5.1 below, which is a key to the proof of Theorem 1.1. It is an analogue of Proposition 3.1 of [14]. Recall \( G_1 = SL_2(\mathbb{Q}_p) \) and \( G_2 = \text{Spin}(12)(\mathbb{Q}_p) \). Let \( \pi'_2 \) be an unramified principal series representation of \( G' = G\text{Spin}(12)(\mathbb{Q}_p) \) with the trivial central character. We compute Satake parameters of \( \pi'_2 \).

Since the group \( G_2 \) appears as a subgroup of \( E_7 \), we need to consider the restriction \( \pi_2 = \pi'_2|_{\text{Spin}(12)} \).

Since \( L_{G\text{Spin}(12)} = G\text{SO}(12, \mathbb{C}) \), the Satake parameter of \( \pi'_2 \) is given by

\[
(b_1, b_2, \ldots, b_6, b_6^{-1}b_0, \ldots, b_2^{-1}b_0, b_1^{-1}b_0) \in G\text{SO}(12, \mathbb{C})
\]

for some \( b_1, \ldots, b_6 \in \mathbb{C}^\times \), and \( b_0 = \omega_{\pi'_2}(p) \). Since the central character is trivial, \( b_0 = 1 \).

Let \( \pi_i \) be an unramified principal series representation of \( G_i \) for \( i = 1, 2 \). Then \( \pi_i = \text{Ind}_{B_i}^{G_i} \chi_i \), where \( B_1, B_2 \) are the standard Borel subgroups of \( G_1, G_2 \), resp. and \( \chi_i : B_i \rightarrow \mathbb{C}^\times \) is an unramified character. The modulus character of each \( B_i \) is given by

\[
\delta_{B_i}(h_{\gamma_i}(t)) = |t_{i,1}^2|, \delta_{B_2}(h_{\gamma_1}(t_1) \cdots h_{\gamma_6}(t_6)) = \prod_{i=1}^{6} |t_i|^2.
\]

Here “Ind” stands for the normalized induction and we will denote by “c-Ind” the compact normalized induction.

Let \( \{ \beta^{\pm 1} \} \) be the Satake parameters of \( \pi_1 \). Then we have

\[
\chi_1(h_{\gamma_i}(p^{-1})) = \beta^2.
\] (5.1)

Also we have

\[
\chi_2(h_{\gamma_i}(p^{-1})) = \frac{b_i}{b_{i+1}}, 1 \leq i \leq 4,
\chi_2(h_{\gamma_6}(p^{-1})) = \frac{b_5}{b_6}, \chi_2(h_{\gamma_5}(p^{-1})) = b_5b_6.
\] (5.2)

Recall that \( H = C_G(\theta) \simeq (G_1 \times G_2)/Z \) where \( Z \simeq \{ \pm 1 \} \) is diagonally embedded in both centers. Let \( \phi : G_1 \times G_2 \rightarrow H \) be the isogeny. As seen before \( \phi(G_1(\mathbb{Q}_p) \times G_2(\mathbb{Q}_p)) \) is a finite index subgroup of \( H(\mathbb{Q}_p) \). Let \( B_H \) be a Borel subgroup of \( H \). Let \( \chi \) be a character of \( B_H \) and let \( \pi(\chi) \) be the spherical subquotient of \( \text{Ind}^{H}_{B_H} \chi \). Let \( \overline{\chi} = \chi \circ \phi \) be a character of \( B_1 \times B_2 \) and let \( \pi(\overline{\chi}) \)
be the spherical subquotient of $\text{Ind}_{B_1 \times B_2}^G \chi$. Then we have a surjective map between unramified $L$-packets:

$$\Pi(H(Q_p)) \longrightarrow \Pi(G_1(Q_p) \times G_2(Q_p)), \quad \pi(\chi) \mapsto \pi(\chi).$$

Given $\chi_1, \chi_2$, unramified characters of $B_1, B_2$, resp., there exist finitely many $\chi$ of $B_H$ such that $\chi_1 \otimes \chi_2^{-1} = \tilde{\chi}$. Let $\pi_H = \pi(\chi)$ for any such $\chi$. Then $\pi_H$ is a subquotient of $\text{Ind}_{\phi(G_1(Q_p) \times G_2(Q_p))} \pi_1 \otimes \pi_2$, and if $\pi_1 \otimes \pi_2$ is unitary, then $\pi_H$ is unitary (Lemma 2.3 of [23]). We call $\pi_H$ a lift of $\pi_1 \otimes \pi_2$ by abuse of notation. Note that for $t \in T = \{\prod_{i=1}^T h_i(t_i) \mid t_i \in Q_p^x\}$, $\pi_H(t)$ acts by

$$(\pi_1 \otimes \pi_2)(t) = \chi_1(t_1) \chi_2^{-1}(\prod_{i=1}^T h_i(t_i)).$$

Let $\omega : Q_p^x \to \mathbb{C}^x$ be an unramified unitary character and let $\alpha = \omega(p^{-1})$.

**Proposition 5.1.** Assume that $\pi_1 \otimes \pi_2$ is unitary. If $\text{Hom}_H(\text{Ind}_{\phi}^G (\omega^{-2} \circ \nu)|_H, \pi_H) \neq 0$ for some lift $\pi_H$ of $\pi_1 \otimes \pi_2$, then as a multiset, $\{b_1 \pm 1, ..., b_6 \pm 1\}$ is equal to one of the followings:

- (I) : $\{\varepsilon(\beta \alpha) \pm 1, \varepsilon(\beta \alpha^{-1}) \pm 1, b \pm 1, (bp)^\pm 1, (bp^2)^\pm 1, (bp^3)^\pm 1\}$, or
- (II) : $\{\varepsilon(\beta \alpha) \pm 1, \varepsilon(\beta \alpha^{-1}) \pm 1, \varepsilon(\beta \alpha^{-1}p)^\pm 1, \varepsilon(\beta \alpha^{-1}p^2)^\pm 1, \varepsilon(\beta \alpha^{-1}p^3)^\pm 1, \varepsilon(\beta \alpha^{-1}p^4)^\pm 1\}$

where $\varepsilon \in \{\pm 1\}$, and $b \in \mathbb{C}^x$.

**Proof.** By Lemma 4.1, one can take the representatives $\{h_n\}_{n=1}^r$ of $P(k) \backslash G(k)/H$ so that $Q_{h_n} \in \{Q_i \mid i = 0, 1, 2, 3\}$. Then in the category of Grothendieck group of admissible representations we have

$$\text{Ind}_{\phi}^G (\omega^{-1} \circ \nu)|_H = \sum_{n=1}^r \text{Ind}_{Q_{h_n}}^H \omega_n \delta_{Q_{h_n}}^{-\frac{i}{4}}$$

where $\omega_n(g) = \delta_{\phi(k)}^H(h_n g h_n^{-1}) \omega^{-2} \circ \nu(h_n g h_n^{-1})$ for $g \in Q_{h_n}$. Put $\omega_n = \omega_1$ if $Q_{h_n} = Q_1$. Then by assumption there exists $i$ ($0 \leq i \leq 3$) such that $Q_{h_n} = Q_i$ and

$$0 \neq \text{Hom}_H(\text{c-Ind}_{Q_i}^H \omega_1 \delta_{Q_i}^{-\frac{1}{4}}, \pi_H)$$

$$= \text{Hom}_H(\pi_H, \text{Ind}_{Q_i}^H \omega_1^{-1} \delta_{Q_i}^\frac{1}{4})$$

$$= \text{Hom}_{Q_i}(\pi_H, \omega_1^{-1}) \text{ (by Frobenius reciprocity)}$$

In the case of $Q_0$, we observe the action of $h_{\gamma_i}(p^{-1}) \in Q_0$ on both spaces. Then one has $\beta^2 = p^{-9}$ which contradicts to the unitarity of $\pi_1$. Similarly we observe the action of $h_{\gamma_i}(p^{-1})$ for $Q_1$. Then it gives a contradiction that $p \beta^2 = 1$.

In the case of $Q_2$, applying [5.1] and [5.2] to the following elements

$$h_{\gamma_2}(p^{-1}), h_{\gamma_3}(p^{-1}), h_{\gamma_4}(p^{-1}), h_{\gamma_6}(p^{-1}), h_{\gamma_7}(p^{-1}) h_{\gamma_5}(p^{-1}), h_{\gamma_1}(p^{-1}) h_{\gamma_7}(p^{-1}) \in T_2$$
respectively, we have
\[
\begin{align*}
(5.3) \quad p \frac{b_2}{b_3} &= 1, \quad p \frac{b_3}{b_4} = 1, \quad p \frac{b_4}{b_5} = 1, \quad p \frac{b_5}{b_6} = 1, \quad (\frac{b_1}{b_2})(pb_5b_6) = p^9\alpha_2^2, \quad (p^{-1}\beta^{-2})(\frac{b_1}{p b_2}) = 1.
\end{align*}
\]

From this, we obtain the Satake parameters
\[
(II) : \{\varepsilon(\beta\alpha)^{\pm 1}, \varepsilon(\beta\alpha^{-1})^{\pm 1}, \varepsilon(\beta\alpha^{-1}p)^{\pm 1}, \varepsilon(\beta\alpha^{-1}p^2)^{\pm 1}, \varepsilon(\beta\alpha^{-1}p^3)^{\pm 1}, \varepsilon(\beta\alpha^{-1}p^4)^{\pm 1}\}
\]
for some \(\varepsilon \in \{\pm 1\}\).

Finally we consider the case of \(Q_3\). For \(t = h_{\gamma_1}(t_5t_7)h_{\gamma_2}(t_5^2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7)\), we see \(\omega_3^{-1}(t) = \omega^2(t_5)|t_5|^{\frac{1}{2}}\delta_{B_2}^\omega(t) = \prod_{i=1}^6 |t_i|\). In this case, applying (5.1) and (5.2) to the following elements
\[
h_{\gamma_3}(p^{-1}), \quad h_{\gamma_4}(p^{-1}), \quad h_{\gamma_6}(p^{-1}), \quad h_{\gamma_1}(p^{-1})h_{\gamma_2}(p^{-2})h_{\gamma_5}(p^{-1})h_{\gamma_6}(p), \quad h_{\gamma_1}(p^{-1})h_{\gamma_7}(p^{-1}) \in T_3
\]
respectively, we have
\[
(5.4) \quad p \frac{b_3}{b_4} = 1, \quad p \frac{b_4}{b_5} = 1, \quad p \frac{b_5}{b_6} = 1, \quad p^3 \frac{b_1b_2b_5^2}{b_3^2} = p^9\alpha_2^2, \quad (p^{-1}\beta^{-2})(\frac{b_1}{p b_2}) = 1.
\]

From the first four equalities, we have \(b_1b_2 = \alpha^2\). From the last equation, \(\frac{b_1}{b_2} = \beta^2\). Hence \(b_1^2 = (\alpha\beta)^2\). Hence \(b_1 = \varepsilon\alpha\beta\) and \(b_2 = \varepsilon\beta\), where \(\varepsilon = \pm 1\).

It follows from (5.4) that
\[
(5.5) \quad b_4 = pb_3, \quad b_5 = p^2b_3, \quad b_6 = p^3b_3,
\]
where \(b_3 \in \mathbb{C}^\times\). Hence the Satake parameters of \(\pi_2\) are
\[
(I) : \{\varepsilon(\alpha\beta)^{\pm 1}, \varepsilon(\alpha\beta^{-1})^{\pm 1}, (bp^3)^{\pm 1}, (bp^2)^{\pm 1}, (bp)^{\pm 1}, b^{\pm 1}\},
\]
where \(\varepsilon \in \{\pm 1\}\) and \(b \in \mathbb{C}^\times\). \(\square\)

6. Proof of Theorem 1.1

Let \(\mathcal{H}(G_i(\mathbb{A}_f)) (i = 1, 2)\) be the Hecke algebra for the finite adele group \(G_i(\mathbb{A}_f)\). Then \(\mathcal{H}(G_1(\mathbb{A}_f)) \cdot h\) and \(\mathcal{H}(G_2(\mathbb{A}_f)) \cdot F_{f,h}\) are the finite part of the cuspidal automorphic representations of \(G_1(\mathbb{A})\) and \(G_2(\mathbb{A})\) generated by \(h\) and \(F_{f,h}\), resp. Here \(\mathcal{H}(G_1(\mathbb{A}_f)) \cdot h\) is an irreducible representation of \(G_1(\mathbb{A}_f)\). Let \(\pi_1\) be the \(p\)-component of \(\mathcal{H}(G_1(\mathbb{A}_f)) \cdot h\). Then \(\pi_1\) is an unramified principal series with the Satake parameter \(\{\beta_p^{\pm 1}\}\). On the other hand, since \(F_{f,h}(Z)\) is a cusp form, the representation \(\mathcal{H}(G'(\mathbb{A}_f)) \cdot F_{f,h}\) of \(G'(\mathbb{A}_f)\) is unitary and of finite length, where \(G' = GSpin(2,10)\). We consider the restriction to \(G_2(\mathbb{A}_f)\), and let \(\pi_2\) be the \(p\)-component of
some irreducible direct summand of that restriction. Then \( \pi_2 \) is also an unramified principal series.

Note that \( \det(\text{Im}Z)^{-10}dZ \) is the invariant measure on \( G'(\mathbb{Z}) \setminus \Sigma_2 \). Then if \( \mathcal{F}_{f,h} \neq 0 \),

\[
\int_{G'(\mathbb{Z}) \setminus \Sigma_2} \int_{SL_2(\mathbb{Z}) \setminus \mathbb{H}} F \left( \begin{array}{c} Z \\ 0 \\ \tau \end{array} \right) h(\tau) \mathcal{F}_{f,h}(Z)(\text{Im} \tau)^{2k+6} \det(\text{Im} Z)^{2k-2} dZd\tau = (\mathcal{F}_{f,h}, \mathcal{F}_{f,h}) \neq 0.
\]

It follows from this that for each prime \( p \),

\[
0 \neq \text{Hom}_{H(\mathbb{Q}_p)}(\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\omega_p^{-2} \circ \det)|_{\phi(G_1(\mathbb{Q}_p) \times G_2(\mathbb{Q}_p))}, \pi_1 \otimes \pi_2)
\]

\[
= \text{Hom}_{H(\mathbb{Q}_p)}(\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\omega_p^{-2} \circ \det)|_{H(\mathbb{Q}_p)}), \text{Ind}_{\phi(G_1(\mathbb{Q}_p) \times G_2(\mathbb{Q}_p))}^{H(\mathbb{Q}_p)}(\pi_1 \otimes \pi_2)
\]

and this implies

\[
\text{Hom}_{H(\mathbb{Q}_p)}(\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\omega_p^{-2} \circ \det)|_{H(\mathbb{Q}_p)}), \pi_H) \neq 0
\]

for some lift \( \pi_H \) to \( H(\mathbb{Q}_p) \) of \( \pi_1 \otimes \pi_2 \) defined as in Proposition 5.1 where \( \omega_p : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times \)

is the unramified character determined by \( \omega_p(p^{-1}) = \alpha_p \). By Proposition 5.1 any irreducible component of \( \mathcal{H}(G'(\mathbb{A}_f)) \cdot \mathcal{F}_{f,h} \) has the Satake \( p \)-parameter

\[
(I)_p : \{ \varepsilon_p(\beta_p \alpha_p)^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1})^{\pm 1}, (b_p \alpha_p^3)^{\pm 1}, (b_p p^2)^{\pm 1}, (b_p^2)^{\pm 1}, (b_p^3)^{\pm 1} \} \quad \text{or}
\]

\[
(II)_p : \{ \varepsilon_p(\beta_p \alpha_p)^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1})^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1} p)^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1} p^2)^{\pm 1}, \varepsilon_p(\beta_p \alpha_p^{-1} p^3)^{\pm 1}, \varepsilon_p(b_p \alpha_p^{-1} p^4)^{\pm 1} \},
\]

where \( \varepsilon_p = \pm 1 \) and \( b_p \in \mathbb{C}^\times \).

Now we assume the Langlands functorial transfer of automorphic representations of \( PGSpin(2, 10)(\mathbb{A}) \) to \( GL_{12}(\mathbb{A}) \) as in the introduction.

Let \( \Pi_{f,h} \) be an irreducible component of the cuspidal representation of \( G'(\mathbb{A}) \) generated by \( \mathcal{F}_{f,h} \). Then it is unramified at every prime \( p \). Let \( \Pi \) be the transfer of \( \Pi_{f,h} \) to \( GL_{12}(\mathbb{A}) \). Then \( \Pi \) is unramified at all \( p \) by the property of Langlands functoriality. By the classification of automorphic representations of \( GL_N \) [13], \( \Pi \) is the Langlands’ quotient of

\[
\sigma_1 | \det | r_1 | \cdots \cdots | \sigma_k | \det | r_k | \cdots | \sigma_{k+1} | \cdots | \sigma_{k+l} \otimes | \sigma_1 \otimes r_1 | \cdots | \sigma_k \otimes \sigma_1 | \det | \sigma_{k+l} \otimes \sigma_1 | \det | r_1,
\]

where \( r_1 \geq r_2 \geq \cdots \geq r_k > 0 \), and \( \sigma_1, \ldots, \sigma_{k+l} \) are unitary (irreducible) cuspidal representations of \( GL_{n_l}(\mathbb{A}) \). Note also that if \( (c_{1p}, \ldots, c_{mp}) \) are Satake parameters of a cuspidal representation \( \pi \) of \( GL_m(\mathbb{A}), p^{-\frac{1}{2}} < |c_{ip}| < p^{-\frac{1}{2}} \) for each \( i \). Hence by comparing the Satake parameters, the Satake parameters should be either \( (I)_p \) for all \( p \), or \( (II)_p \) for all \( p \).

Suppose the Satake parameters are \( (II)_p \) for all \( p \). Then \( \Pi \) is the Langlands’ quotient of

\[
\Pi = \Pi_1 \otimes (\chi| \cdot |)^{\pm 1} \otimes (\chi| \cdot |^2)^{\pm 1} \otimes (\chi| \cdot |^3)^{\pm 1} \otimes (\chi| \cdot |^4)^{\pm 1},
\]
where \( \chi : \mathbb{Q} \times \mathbb{Q}^\times \rightarrow \mathbb{C}^\times \) is a unitary idele class character, and \( \Pi_1 \) is an automorphic representation of \( GL_4(\mathbb{A}) \) whose Satake parameters are \( \{ \varepsilon_p(\beta_p\alpha_p)^{\pm 1}, \varepsilon_p(\beta_p\alpha_p^{-1})^{\pm 1} \} \) at each \( p \). The automorphy of \( \Pi_1 \) is explained as follows: We can see easily that \( \wedge^2 \Pi_1 = \text{Sym}^2(\pi_f) \oplus \text{Sym}^2(\chi) \). It is an automorphic representation of \( GL_6(\mathbb{A}) \). Now the exterior square \( \wedge^2 : GL_4(\mathbb{C}) \rightarrow GL_6(\mathbb{C}) \) is the composition of transfers from \( GL_4(\mathbb{C}) \) to \( GSO_6(\mathbb{C}) \) and \( \varepsilon : GSO_6(\mathbb{C}) \rightarrow GL_6(\mathbb{C}) \), where \( \varepsilon \) is the embedding, and \( \phi : GSpin_6 \rightarrow GL_4 \) is the double covering map. Hence the exterior square transfer is the composition of transfers from \( GL_4(\mathbb{A}) \) to \( GSpin_6(\mathbb{A}) \) and \( GSpin_6(\mathbb{A}) \rightarrow GL_6(\mathbb{A}) \). Since the central character of \( \Pi_1 \) is trivial, it is a representation of \( PGL_4 \simeq PGSpin_6 = PGO_6 \). Hence for representations with the trivial central character, the exterior square transfer is the transfer \( PGSO_6(\mathbb{A}) \rightarrow GL_6(\mathbb{A}) \). Now by the result of Arthur, since \( \wedge^2 \Pi_1 \) is automorphic, \( \Pi_1 \) is an automorphic representation of \( PGSO_6(\mathbb{A}) \simeq PGL_4(\mathbb{A}) \).

Since \( \chi \) is the global unramified character, one must have \( \chi = 1 \), i.e., \( \alpha_p = \beta_p \) and \( \varepsilon_p = 1 \) for all \( p \). Since \( f \) and \( h \) have different weights, they can never be equal. Contradiction.

Hence the Satake parameters should be \( (I)_p \) for all \( p \). Now we recall the classification of spherical unitary representations of \( GL_N(\mathbb{Q}) \). For an unramified unitary character \( \chi \), let \( \chi(\det_n) \) be the representation \( g \mapsto \chi(\det_n(g)) \) of \( GL_n(\mathbb{Q}) \). Let \( \pi(\chi(\det_n), \alpha) \) be the representation of \( GL_{2n}(\mathbb{Q}) \) induced by \( \chi(\det_n)|\det|^{\alpha} \chi(\det_n)|\det|^{-\alpha} \), where \( 0 < \alpha < \frac{1}{2} \). Then any spherical unitary representation of \( GL_N(\mathbb{Q}) \) is induced by

\[
\chi_1(\det_{n_1}) \otimes \cdots \otimes \chi_q(\det_{n_q}) \otimes \pi(\mu_1(\det_{m_1}), \alpha_1) \otimes \cdots \otimes \pi(\mu_r(\det_{m_r}), \alpha_r),
\]

where \( n_1 + \cdots + n_q + 2(m_1 + \cdots + m_r) = N \), \( 0 < \alpha_1, \ldots, \alpha_r < \frac{1}{2} \), and \( \chi_1, \ldots, \chi_q, \mu_1, \ldots, \mu_r \) are unramified unitary characters. Hence by comparing the Satake parameters, we can see that \( |b_p| = 1 \) for all \( p \). Since \( \Pi \) is unramified everywhere, we conclude that \( b_p = 1 \). Hence \( \Pi \) is the Langlands’ quotient of \( \Pi = \Pi_1 \boxplus 1 \boxplus 1 \boxplus | \boxplus 1 \boxplus 1 \boxplus | \boxplus 3 \). Since \( \wedge^2 \Pi_1 = \text{Sym}^2(\pi_f) \oplus \text{Sym}^2(\chi) \), by [2], \( \Pi_1 \) is of the form \( \Pi_1 = \sigma_1 \boxtimes \sigma_2 \) for \( \sigma_1, \sigma_2 \), cuspidal representations of \( GL_2(\mathbb{A}) \).

Since \( \wedge^2(\sigma_1 \boxtimes \sigma_2) = \text{Ad}(\sigma_1) \otimes \omega_1 \omega_2, \text{Ad}(\sigma_2) \otimes \omega_1 \omega_2, \omega_1, \omega_2 = 1, \text{Ad}(\sigma_1) = \text{Ad}(\pi_f) \) and \( \text{Ad}(\sigma_2) = \text{Ad}(\pi_h) \). By [27], \( \sigma_1 = \pi_f \boxtimes \chi_1 \) and \( \sigma_2 = \pi_h \boxtimes \chi_2 \) for some characters \( \chi_1, \chi_2 \). Hence \( \Pi_1 = (\pi_f \boxtimes \pi_h) \boxplus \chi_1 \chi_2 \). However \( \chi_1 \chi_2 \) has to be trivial because \( \Pi_1 \) is unramified everywhere. Therefore, \( \Pi_1 = \pi_f \boxtimes \pi_h \), and \( \varepsilon_p = 1 \) for all \( p \). This shows that \( \Pi = (\pi_f \boxtimes \pi_h) \boxplus 1_{GL_7} \boxplus 1 \), where \( 1_{GL_7} \) is the trivial representation of \( GL_7(\mathbb{A}) \).

The Satake parameters at \( p \) behave uniformly and it follows from this that \( \mathcal{H}(G'_{\mathbb{A}f}) \cdot F_{f,h} \) is isotypic. Since it is generated by the class one vector \( F_{f,h} \), it is irreducible. It follows that \( F_{f,h} \)
is a Hecke eigenform and gives rise to a cuspidal representation \( \Pi_{f,h} \) of \( G'(\mathbb{A}) \). We also showed that the degree 12 standard \( L \)-function is

\[
L(s, \Pi_{f,h}) = L(s, \pi_f \times \pi_h) \zeta(s)^2 \zeta(s+1) \zeta(s+2) \zeta(s+3),
\]

where the first \( L \)-function is the Rankin-Selberg \( L \)-function.

### 7. Remark on non-vanishing hypothesis

Recall

\[
\mathcal{F}_{f,h}(Z) = \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} F_f \begin{pmatrix} Z & 0 \\ w & \tau \end{pmatrix} \frac{h(\tau)(\text{Im}\tau)^{2k+6}}{2^{2k+6} \zeta(s)} d\tau.
\]

We consider the nonvanishing question of \( \mathcal{F}_{f,h} \). We have two Fourier-Jacobi expansions of \( F_f \);

\[
F_f \begin{pmatrix} Z & w \\ t & \tau \end{pmatrix} = \sum_{m=1}^{\infty} \phi_m(Z, w)e^{2\pi i m\tau} = \sum_{S} \mathcal{F}_S(\tau, w)e^{2\pi i T r(ZS)},
\]

where \( \phi_m \) is a Jacobi cusp form of weight \( 2k+8 \) of index \( m \) as in [7]. In the second sum, \( S \in \mathbb{H}/(\mathbb{Z}) \) and \( \mathcal{F}_S \) is a Fourier-Jacobi coefficient of index \( S \) as in [18]. Here

\[
\mathcal{F}_S(\tau, w) = \sum_{\lambda \in \Lambda} \theta_{[\lambda]}(S; \tau, w) F_{S,\lambda}(\tau),
\]

where \( \theta_{[\lambda]}(S; \tau, w) \) is a theta series and \( F_{S,\lambda}(\tau) \) is a vector-valued modular form, which is obtained from the compatible family of Eisenstein series.

**Lemma 7.1.** We have the estimates:

\[
|\phi_m(Z, 0)| \ll \det(Y)^{-(2k+8)m^{2k+8}}, \quad Y = \text{Im}(Z),
\]

\[
|\mathcal{F}_S(\tau, 0)| \ll y^{-(2k+8)Tr(S)^{2(2k+8)}}, \quad y = \text{Im}(\tau).
\]

**Proof.** From the first expansion in (7.1), for any \( y > 0 \),

\[
\phi_m(Z, 0)e^{-2\pi my} = \int_{0}^{1} F_f \begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix} e^{-2\pi i mx} dx.
\]

Here

\[
|F_f \begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix}| \ll (\det(\text{Im}(Z))\text{Im}(\tau))^{-(2k+8)}. \quad \text{Set } y = \frac{1}{m}. \text{ Then}
\]

\[
|\phi_m(Z, 0)| \ll \det(Y)^{-(2k+8)m^{2k+8}}.
\]
From the second expansion in (7.1),
\[ F_S(\tau, 0)e^{-2\pi Tr(Y S)} = \int_X F_f \left( \frac{Z}{X} 0 \right) e^{-2\pi i Tr(X S)} dX, \]
where the integral is over \( \mathcal{T}_2(\mathbb{R})/\mathcal{T}_2(\mathbb{Z}) \). Set \( Y = \frac{1}{Tr(S)} I_2 \). Then
\[ |F_S(\tau, 0)| \ll y^{-(2k+8)} Tr(S)^2(2k+8). \]

Consider the first Fourier-Jacobi expansion. We have
\[ \sum_{m=1}^{\infty} \left| \phi_m(Z, 0)e^{2\pi im\tau} \right| \leq \sum_{m=1}^{\infty} m^{2k+8} \det(Y)^{-(2k+8)} e^{-2\pi m\tau \det(Y)^{-(2k+8)}}. \]
Since \( |h(\tau)| \ll e^{-2\pi y} \),
\[ \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} F_f \left( \frac{Z}{0} 0 \tau \right) \overline{h(\tau)} y^{2k+6} d\tau d\tau \]
converges absolutely. Hence we can interchange the sum and integral. So
\[ F_{f,h}(Z) = \sum_{m=1}^{\infty} \phi_m(Z, 0) \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} e^{2\pi im\tau} \overline{h(\tau)} \operatorname{Im}(\tau)^{2k+6} d\tau. \]
Here \( \phi_m(Z, 0) \) is a linear combination of cusp forms on \( GSpin(2,10) \). So it is unlikely that \( F_{f,h} \) is identically zero.

Next consider the second Fourier-Jacobi expansion of \( F_f \). From the above lemma,
\[ \sum_S \left| F_S(\tau, 0)e^{2\pi i Tr(Z S)} \right| \leq \sum_S y^{-(2k+8)} Tr(S)^{2(2k+8)} e^{-2\pi Tr(Y \tau)} \ll y^{-(2k+8)}. \]
Since \( |h(\tau)| \ll e^{-2\pi y} \),
\[ \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} F_f \left( \frac{Z}{0} 0 \tau \right) \overline{h(\tau)} y^{2k+6} d\tau d\tau \]
converges absolutely. Hence we can interchange the sum and integral. So
\[ F_{f,h}(Z) = \sum_S A_S e^{2\pi i Tr(Z S)}, \]
where
\[ A_S = \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} F_S(\tau, 0) \overline{h(\omega)} \operatorname{Im}(\omega)^{2k+6} d\tau = \sum_{\lambda \in \Lambda} \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} \theta_{\lambda}(S; \tau, 0) F_{S,\lambda}(\tau) \overline{h(\tau)} \operatorname{Im}(\tau)^{2k+6} d\tau. \]
Here $\mathcal{F}_S(\tau, 0)$ is a modular form of weight $2k + 8$. Hence $A_S$ is the Petersson inner product of $\mathcal{F}_S(\tau, 0)$ and $h$. This expression shows that it is very likely that $\mathcal{F}_{f,h}$ is not identically zero.

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