The standing wave model of the mesons and baryons

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Only photons are needed to explain the masses of the $\pi^0$, $\eta$, $\Lambda$, $\Sigma^0$, $\Xi^0$, $\Omega^-$, $\Lambda^+$, $\Sigma^+_c$, $\Xi^0_c$ and $\Omega^0_c$ mesons and baryons with the sum of the energies contained in the frequencies of standing electromagnetic waves in a cubic black body. Only neutrinos are needed to explain the mass of the $\pi^\pm$ mesons with the sum of the energies of standing oscillations of muon and electron neutrinos in a cubic lattice plus the energies contained in the rest masses of the neutrinos. Neutrinos and photons are needed to explain the masses of the K-mesons, the neutron and the D-mesons. Surprisingly the mass of the $\mu^\pm$ mesons can also be explained without an additional assumption by the oscillation energies and rest masses of a neutrino lattice. From the difference of the masses of the $\pi^\pm$ mesons and $\mu^\pm$ mesons we find that the rest mass of the muon neutrino is 47.5 milli-eV/$c^2$. From the difference of the masses of the neutron and proton we find that the rest mass of the electron neutrino is 0.55 meV/$c^2$. The potential of the weak force between the lattice points can be determined from Born’s lattice theory. From the weak force between the lattice points follows automatically the existence of a strong force between the sides of two lattices. The strong nuclear force is the sum of the unsaturated weak forces at the sides of each lattice and is therefore about $10^6$ times stronger than the weak force.

Introduction

A theory giving an equation whose eigenvalues would determine the masses of the elementary particles has sofar not been found. The so-called “Standard Model” of the particles has, until now, not come up with a precise determination of the masses of either the mesons and baryons or of the leptons, which means that neither the mass of the fundamental electron nor the
mass of the fundamental proton have been explained. This is so although
the quarks, the foundation of the standard model, have been introduced by
Gel-Mann [1] nearly forty years ago. The difficulties explaining the masses
of the mesons and baryons with the standard model seem to be related to
the uncertainty of the masses of the quarks for which values are considered
ranging from zero rest mass to values from 2 to 8 MeV for e.g. the u-quark,
according to the Particle Physics Summary [2], to values on the order of
100 MeV. Suppose one has agreed on definite values of the masses of the
various quarks then one stands before the same problem one has faced with
the conventional elementary particles, namely one has to explain why the
quarks have their particular masses and what they are made of. The other
most frequently referred to theory dealing with the elementary particles, the
“String Theory” introduced by Witten [3] about twenty years ago, or its suc-
cessor the superstring theory, have despite their mathematical elegance not
led to experimentally verifiable results. There are very many other attempts
to explain the elementary particles or only one of the particles, too many to
list them here. It appears that there is room for another attempt to explain
the masses of the elementary particles.

1 The spectrum of the masses of the particles

As we have done before [4] we will focus attention on the so-called “stable”
mesons and baryons whose masses are reproduced with other data in Tables
1 and 4. It is obvious that any attempt to explain the masses of the mesons
and baryons should begin with the particles that are affected by the fewest
parameters. These are certainly the particles without isospin (I = 0) and
without spin (J = 0), but also with strangeness S = 0, and charm C = 0.
Looking at the particles with I,J,S,C = 0 it is startling to find that their
masses are quite close to integer multiples of the mass of the \( \pi^0 \) meson. It
is \( m(\eta) = (1.0140 \pm 0.0003) \cdot 4m(\pi^0) \), and the mass of the resonance \( \eta' \)
is \( m(\eta') = (1.0137 \pm 0.00015) \cdot 7m(\pi^0) \). We also note that the average mass
ratios \([m(\eta)/m(\pi^0) + m(\eta)/m(\pi^\pm)]/2 = 3.9892 = 0.9973 \cdot 4 \) and \( [m(\eta')/m(\pi^0) +
m(\eta')/m(\pi^\pm)]/2 = 6.9791 = 0.9970 \cdot 7 \) are good approximations to the
integers 4 and 7. Three particles seem hardly to be sufficient to establish a
rule. However, if we look a little further we find that \( m(\Lambda) = 1.0332 \cdot 8m(\pi^0) \)
or \( m(\Lambda) = 1.0190 \cdot 2m(\eta) \). We note that the \( \Lambda \) particle has spin 1/2, not
spin 0 as the \( \pi^0, \eta \) mesons. Nevertheless, the mass of \( \Lambda \) is close to \( 8m(\pi^0) \).

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Furthermore we have $m(\Sigma^0) = 0.9817 \cdot 9m(\pi^0)$, $m(\Xi^0) = 0.9742 \cdot 10m(\pi^0)$, $m(\Omega^-) = 1.0325 \cdot 12m(\pi^0) = 1.0183 \cdot 3m(\eta)$, $(\Omega^-$ is charged and has spin $3/2$).

Finally the masses of the charmed baryons are $m(\Lambda^+_c) = 0.9958 \cdot 17m(\pi^0) = 1.024 \cdot 2m(\Lambda)$, $m(\Sigma^0_c) = 1.0093 \cdot 18m(\pi^0)$, $m(\Xi^0_c) = 1.0167 \cdot 18m(\pi^0)$, and $m(\Omega^0_c) = 1.0017 \cdot 20m(\pi^0)$.

Now we have enough material to formulate the integer multiple rule, according to which the masses of the $\eta$, $\Lambda$, $\Sigma^0$, $\Omega^-$, $\Lambda^+_c$, $\Sigma^0_c$, $\Xi^0_c$, and $\Omega^0_c$ particles are, in a first approximation, integer multiples of the mass of the $\pi^0$ meson, although some of the particles have spin, and may also have charge as well as strangeness and charm. A consequence of the integer multiple rule must be that the ratio of the mass of any meson or baryon listed above divided by the mass of another meson or baryon listed above is equal to the ratio of two integer numbers. And indeed, for example $m(\eta)/m(\pi^0)$ is practically two times (exactly $0.9950 \cdot 2$) the ratio $m(\Lambda)/m(\eta)$. There is also the ratio $m(\Omega^-)/m(\Lambda) = 0.9993 \cdot 3/2$. We have furthermore e.g. the ratios $m(\Lambda)/m(\eta) = 1.019 \cdot 2$, $m(\Omega^-)/m(\eta) = 1.018 \cdot 3$, $m(\Lambda^+_c)/m(\Lambda) = 1.02399 \cdot 2$, $m(\Sigma^0_c)/m(\Sigma^0) = 1.0281 \cdot 2$, and $m(\Omega^0_c)/m(\Xi^0) = 1.0282 \cdot 2$.

We will call, for reasons to be explained later, the particles discussed above, which follow in a first approximation the integer multiple rule, the $\gamma$-branch of the particle spectrum. The mass ratios of these particles are in Table 1. The deviation of the mass ratios from exact integer multiples of $m(\pi^0)$ is at most $3.3\%$, the average of the factors before the integer multiples of $m(\pi^0)$ of the nine $\gamma$-branch particles in Table 1 is $1.0066 \pm 0.0184$. From a least square analysis follows that the masses of the ten particles on Table 1 lie on a straight line given by the formula

$$m(N)/m(\pi^0) = 1.0065 N - 0.0043 \quad N \geq 1,$$

where $N$ is the integer number nearest to the actual ratio of the particle mass divided by $m(\pi^0)$. The correlation coefficient in equation (1) has the nearly perfect value $r^2 = 0.999$.

The integer multiple rule applies to more than just the stable mesons and baryons. The integer multiple rule applies also to the $\gamma$-branch baryon resonances which are listed in Table 2 and the meson resonances in Table 3. The $\Omega^-$ particle has been omitted from Table 2 because it has spin $3/2$, and the $\Lambda(1810)$ resonance has also been omitted because it differs from $\Lambda(1800)$ only in parity, as have been two $\Xi$ resonances whose spin is uncertain. A least square analysis of the masses in Table 2 yields the formula
Table 1: The $\gamma$-branch of the particle spectrum

| m/m($\pi^0$) | multiples | decays | fraction (%) | spin | mode            |
|--------------|-----------|--------|--------------|------|-----------------|
| $\pi^0$      | 1.0000    | $\gamma$ | 98.798       | 0    | (1.1)           |
|              |           | $e^+e^-\gamma$ | 1.198       |      |                 |
| $\eta$       | 4.0559    | $\gamma\gamma$ | 39.25       | 0    | (2.2)           |
|              |           | $3\pi^0$ | 32.1         |      |                 |
|              |           | $\pi^+\pi^-\pi^0$ | 23.2       |      |                 |
|              |           | $\pi^+\pi^-\gamma$ | 4.78       |      |                 |
| A            | 8.26577   | 1.0332 · $8\pi^0$ | 63.9       | $\frac{1}{2}$ | 2 · (2.2) |
|              |           | 1.0190 · 2$\eta$ | 35.8       |      |                 |
| $\Sigma^0$  | 8.8352    | $\Lambda\gamma$ | 100         | $\frac{1}{2}$ | 2 · (2.2) + (1.1) |
| $\Xi^0$     | 9.7417    | $\Lambda\pi^0$ | 99.54       | $\frac{1}{2}$ | 2 · (2.2) + 2 · (1.1) |
| $\Omega^-$  | 12.390    | $\Lambda K^-$ | 67.8         | $\frac{3}{2}$ | 3 · (2.2) |
|              |           | $\Xi^0\pi^-$ | 23.6         |      |                 |
|              |           | $\Xi^-\pi^0$ | 8.6          |      |                 |
| $\Lambda_c^+$ | 16.928    | many          | $\approx$100 | $\frac{1}{2}$ | 2 · (2.2) + (3.3) |
|              |           | 0.9958 · $17\pi^0$ |           |      |                 |
|              |           | 0.9630 · $17\pi^\pm$ |           |      |                 |
| $\Sigma_c^0$ | 18.167    | $\Lambda_c^+\pi^-$ | $\approx$100 | $\frac{1}{2}$ | $\Lambda_c^+$ + (1.1) |
| $\Xi_c^0$   | 18.302    | seven (seen) | 1           | $\frac{1}{2}$ | 2 · (3.3) |
| $\Omega_c^0$ | 20.033    | four (seen) | 1           | $\frac{1}{2}$ | 2 · (3.3) + 2 · (1.1) |
Table 2: The $\gamma$-branch baryons with $I \leq 1, J = \frac{1}{2}$

| particle | $m/m(\pi^0)$ | I, J | N |
|----------|--------------|-----|---|
| $\Lambda$ | 8.26577 | $0, \frac{1}{2}$ | 8 |
| $\Lambda(1405)$ | 10.42 | $0, \frac{1}{2}$ | 10 |
| $\Lambda(1670)$ | 12.37 | $0, \frac{1}{2}$ | 12 |
| $\Lambda(1800)$ | 13.33 | $0, \frac{1}{2}$ | 13 |
| $\Sigma^0$ | 8.8352 | $1, \frac{1}{2}$ | 9 |
| $\Sigma(1660)$ | 12.298 | $1, \frac{1}{2}$ | 12 |
| $\Sigma(1750)$ | 12.965 | $1, \frac{1}{2}$ | 13 |
| $\Xi^0$ | 9.7417 | $\frac{1}{2}, \frac{1}{2}$ | 10 |
| $\Lambda_c^+(5825)$ | 16.928 | $0, \frac{1}{2}$ | 17 |
| $\Lambda_c(2593)$ | 19.215 | $0, \frac{1}{2}$ | 19 |
| $\Sigma_c^0$ | 18.167 | $1, \frac{1}{2}$ | 18 |
| $\Sigma_c(2593)$ | 19.215 | $0, \frac{1}{2}$ | 20 |

Table 3: The $\gamma$-branch mesons with $I,J=0,0$

| particle | $m/m(\pi^0)$ | N |
|----------|--------------|---|
| $\eta$ | 4.0559 | 4 |
| $\eta'$ | 7.0958 | 7 |
| $f_0$ | 7.261 | 7 |
| $\eta(1295)$ | 9.594 | 10 |
| $\eta(1440)$ | 10.48 | 10 |
| $f_0(1500)$ | 11.135 | 11 |
| $\eta_c$ | 22.017 | 22 |
| $\chi_{c0}$ | 25.301 | 25 |

$$m(N)/m(\pi^0) = 1.0013 N + 0.1259 \quad N > 1,$$  \hspace{1cm} (2)

with the very good correlation coefficient 0.997, and the masses in Table 3 are described by the equation

$$m(N)/m(\pi^0) = 1.0055 N + 0.0592 \quad N > 1,$$  \hspace{1cm} (3)

with $r^2 = 0.999$. If we combine the particles in Tables 1,2,3, that means if we consider all mesons and baryons of the $\gamma$-branch, “stable” or unstable, with $I \leq 1, J \leq 1/2$ then we obtain from a least square analysis the formula

$$m(N)/m(\pi^0) = 1.0056 N + 0.0610 \quad N \geq 1,$$  \hspace{1cm} (4)
with the correlation coefficient 0.9986.

![Graph showing mass of mesons and baryons as a function of integer N]

The line through the points is shown in Fig. 1 which tells that 22 particles of the $\gamma$-branch of different spin and isospin, strangeness and charm; eight $I,J = 0,0$ mesons, thirteen $J = 1/2$ baryons and the $\pi^0$ meson with $I,J = 1,0$, lie on a straight line with slope 1.0056. In other words they approximate the integer multiple rule very well.

Searching for what else the $\pi^0, \eta, \Lambda, \Sigma^0, \Xi^0, \Omega^-$ particles have in common, we find that the principal decays (decays with a fraction $> 1\%$) of these particles, as listed in Table 1, involve primarily $\gamma$-rays, the characteristic case is $\pi^0 \to \gamma\gamma$ (98.8%). We will later on discuss a possible explanation for the 1.198% of the decays of $\pi^0$ which do not follow the $\gamma\gamma$ route. After the $\gamma$-rays the next most frequent decay product of the heavier particles of the $\gamma$-branch are $\pi^0$ mesons which again decay into $\gamma\gamma$. To describe the decays in another way, the principal decays of the particles listed above take place *always without the emission of neutrinos*; see Table 1. There the decays and the fractions of the principal decay modes are given, taken from the Particle Physics Summary. We cannot consider decays with fractions $< 1\%$. We will
refer to the particles whose masses are approximately integer multiples of the mass of the $\pi^0$ meson, and which decay without the emission of neutrinos, as the $\gamma$-branch of the particle spectrum.

To summarize the facts concerning the $\gamma$-branch. Within 0.66% on the average the masses of the particles of the $\gamma$-branch are integer multiples (namely 4, 8, 9, 10, 12, and even 17, 18, 20) of the mass of the $\pi^0$ meson. It is improbable that nine particles have masses so close to integer multiples of $m(\pi^0)$ if there is no correlation between them and the $\pi^0$ meson. It has, on the other hand, been argued that the integer multiple rule is a numerical coincidence. But the probability that the mass ratios of the $\gamma$-branch fall by coincidence on integer numbers between 1 and 20 instead on all possible numbers between 1 and 20 with two decimals after the period is smaller than $10^{-20}$, i.e. nonexistent. The integer multiple rule is not affected by more than 3% by the spin, the isospin, the strangeness, and by charm. The integer multiple rule seems even to apply to the $\Omega^-$ and $\Lambda^+_c$ particles, although they are charged. In order for the integer multiple rule to be valid the deviation of the ratio $m/m(\pi^0)$ from an integer number must be smaller than $1/2N$, where $N$ is the integer number closest to the actual ratio $m/m(\pi^0)$. That means that the permissible deviation decreases rapidly with increased $N$. All particles of the $\gamma$-branch have deviations smaller than $1/2N$.

The remainder of the stable mesons and baryons are the $\pi^\pm$, $K^\pm, 0$, $p$, $n$, $D^\pm, 0$, and $D^*_S$ particles which make up the $\nu$-branch of the particle spectrum. The ratios of the masses are given in Table 4.

These particles are in general charged, exempting the $K^0$ and $D^0$ mesons and the neutron $n$, in contrast to the particles of the $\gamma$-branch, which are in general neutral. It does not make a significant difference whether one considers the mass of a particular charged or neutral particle. After the $\pi$ mesons, the largest mass difference between charged and neutral particles is that of the $K$ mesons (0.81%), and thereafter all mass differences between charged and neutral particles are < 0.5%. The integer multiple rule does not immediately apply to the masses of the $\nu$-branch particles if $m(\pi^\pm)$ (or $m(\pi^0)$) is used as reference, because $m(K^\pm) = 0.8843 \cdot 4m(\pi^\pm)$. 0.8843 \cdot 4 = 3.537 is far from integer. Since the masses of the $\pi^0$ meson and the $\pi^\pm$ mesons differ by only 3.4% it has been argued that the $\pi^\pm$ mesons are, but for the isospin, the same particles as the $\pi^0$ meson, and that therefore the $\pi^\pm$ cannot start another particle branch. However, this argument is not supported by the completely different decays of the $\pi^0$ mesons and the $\pi^\pm$ mesons. The $\pi^0$ meson decays almost exclusively into $\gamma\gamma$ (98.8%), whereas the $\pi^\pm$ mesons
Table 4: The \( \nu \)-branch of the particle spectrum

| \( m/m(\pi^\pm) \) | multiples | decays\(^1\) | fraction (%) | spin | mode |
|---|---|---|---|---|---|
| \( \pi^\pm \) | 1.0000 | \( 1.0000 \cdot \pi^\pm \) | \( \mu^+\nu_\mu \) | 99.9877 | 0 (1.1) |
| \( K^{\pm,0} \) | 3.53713 | 0.8843 \( \cdot 4\pi^\pm \) | \( \mu^+\nu_\mu \) | 63.57 | 0 (2.2) + \( \pi^0 \) |
| | | | \( \pi^+\pi^0 \) | 21.16 |
| | | | \( \pi^+\pi^-\pi^+ \) | 5.59 |
| | | | \( \pi^0e^+\nu_e \) | 4.82 |
| | | | \( \pi^0\mu^+\nu_\mu \) | 3.18 |
| n | 6.73186 | 0.8415 \( \cdot 8\pi^\pm \) | p e\( ^-\nu_e \) | 100. | \( \frac{1}{2} \) | 2\( \cdot ((2.2) + \pi^0) \) |
| | | | 0.9617 \( \cdot 7\pi^\pm \) |
| | | | 0.9516 \( \cdot 2K^\pm \) |
| D^{\pm,0} | 13.393 | 0.8370 \( \cdot 16\pi^\pm \) | e\( ^+\) anything | 17.2 | 0 | 2\( \cdot ((2.2) + \pi^0) \) |
| | | | 0.9466 \( \cdot 4K^\pm \) | K anything | 24.2 |
| | | | 0.9954 \( \cdot (n+p) \) | \( \bar{K}^0 \) anything | 59 |
| | | | +K\( ^0 \) anything |
| | | | \( \eta \) anything | < 13 |
| D^\( \frac{3}{2} \) | 14.104 | 0.8296 \( \cdot 17\pi^\pm \) | K anything | 13 | 0 | 2\( \cdot ((2.2) + \pi^0) \) + \( \pi^0 \) |
| | | | 0.9968 \( \cdot 4K^\pm \) | \( \bar{K}^0 \) anything | 39 |
| | | | +K\( ^0 \) anything | 20 |
| | | | K\( ^+ \) anything |
| | | | e\( ^+\) anything | < 20 |

\(^1\)The particles with negative charges have conjugate charges of the listed decays.
decay practically exclusively into $\mu$-mesons and neutrinos, e.g. $\pi^+ \rightarrow \mu^+ + \nu_\mu$ (99.9877%). Furthermore, the lifetimes of the $\pi^0$ and the $\pi^\pm$ mesons differ by nine orders of magnitude, being $\tau(\pi^0) = 8.4 \cdot 10^{-17}$ sec versus $\tau(\pi^\pm) = 2.6 \cdot 10^{-8}$ sec.

If we make the $\pi^\pm$ mesons the reference particles of the $\nu$-branch, then we must multiply the mass ratios $m/m(\pi^\pm)$ of the above listed particles with an average factor $0.848 \pm 0.025$, as follows from the mass ratios on Table 4. The integer multiple rule may, however, apply directly if one makes $m(K^\pm)$ the reference for masses larger than $m(K^\pm)$. The mass of the neutron is $0.9516 \cdot 2m(K^\pm)$, which is only a fair approximation to an integer multiple. There are, on the other hand, outright integer multiples in $m(D^\pm) = 0.9954 \cdot (m(p) + m(n))$, and in $m(D^\mp_3) = 0.9968 \cdot 4m(K^\pm)$. A least square analysis of the masses of the $\nu$-branch in Table 4 yields the formula

$$m(N)/0.853m(\pi^\pm) = 1.000 N + 0.00575 \quad N \geq 1, \quad (5)$$

with $r^2 = 0.998$. This means that the particles of the $\nu$-branch are integer multiples of $m(\pi^\pm)$ times the factor 0.853. One must, however, consider that the $\pi^\pm$ mesons are not necessarily the perfect reference for all $\nu$-branch particles, because $\pi^\pm$ has $I = 1$, whereas for example $K^\pm$ has $I = 1/2$ and $S = \pm 1$ and the neutron has also $I = 1/2$. Actually the factor 0.853 is only an average. The mass ratios indicate that the factor before $m/m(\pi^\pm)$ decreases slowly with increased $N$. The existence of the factor and its decrease will be explained later.

Contrary to the particles of the $\gamma$-branch, the $\nu$-branch particles decay preferentially with the emission of neutrinos, the foremost example is, e.g., $\pi^+ \rightarrow \mu^+ + \nu_\mu$ with a fraction of 99.988%. Neutrinos characterize the weak interaction. We will refer to the particles in Table 4 as the *neutrino branch* ($\nu$-branch) of the particle spectrum. We emphasize that a weak decay of the particles of the $\nu$-branch is by no means guaranteed. Although the neutron decays via $n \rightarrow p + \bar{\nu}_e$ in 887 sec (100%), the proton is stable. There are, on the other hand, decays as e.g. $K^+ \rightarrow \pi^+\pi^-\pi^+$ (5.59%), but the subsequent decays of the $\pi^\pm$ mesons lead to neutrinos and $e^\pm$. The decays of the particles in the $\nu$-branch follow a mixed rule, either weak or electromagnetic.

To summarize the facts concerning the $\nu$-branch of the mesons and baryons. The masses of these particles seem to follow the integer multiple rule if one uses the $\pi^\pm$ meson as reference, however the mass ratios share a common factor $0.85 \pm 0.025$. 

To summarize what we have learned about the integer multiple rule: In spite of differences in charge, spin, strangeness, and charm the masses of the mesons and baryons of the $\gamma$-branch are integer multiples of the mass of the $\pi^0$ meson within at most 3.3% and on the average within 0.66%. Correspondingly, the masses of the particles of the $\nu$-branch are, after multiplication with a factor $0.85 \pm 0.025$, integer multiples of the mass of the $\pi^\pm$ mesons. The validity of the integer multiple rule can easily be verified with a calculator from the data in the Particle Physics Summary. The integer multiple rule suggests that the particles are the result of superpositions of modes and higher modes of a wave equation.

2 Standing waves in a cubic lattice and the particles of the $\gamma$-branch

In an earlier article [5] we have tried to explain the masses of the elementary particles by monochromatic eigenfrequencies of standing waves in an elastic cube. The explanation of the particles by monochromatic eigenfrequencies does not seem to be tenable because a monochromatic frequency does not accommodate the multitude of frequencies created in a high energy collision of $10^{-23}$ sec duration. We will now study, as we have done in [6], whether the so-called stable particles of the $\gamma$-branch cannot be described by the frequency spectrum of standing waves in a cubic lattice, which can accommodate automatically the Fourier frequency spectrum of an extreme short-time collision. The investigation of the consequences of lattices for particle theory was initiated by Wilson [7] who studied a fermion lattice. This study has developed over time into lattice QCD. The results of such endeavors have culminated in the paper of Weingarten [8] and his colleagues which required elaborate year long numerical calculations. They determined the masses of seven particles ($K^*, p, \phi, \Delta, \Sigma, \Xi, \Omega$), with an uncertainty of up to $\pm 8\%$, agreeing with the observed masses within a few percent, up to 6%. Our theory covers all particles of the $\gamma$-branch, namely the $\pi^0, \eta, \Lambda, \Sigma^0, \Xi^0, \Omega^-, \Lambda^+_c, \Sigma^0_c, \Xi^0_c$ and $\Omega^0_c$ particles and agrees on the average with the measured masses within 0.66%, and covers also the particles of the $\nu$-branch, as we will see later.

It will be necessary to outline the most elementary aspects of the theory of lattice oscillations since we will, in the following, investigate whether the masses of the $\gamma$-branch particles can be explained with the help of the
frequency spectra of standing waves in a cubic lattice. The classic paper describing lattice oscillations is from Born and v.Karman [9], henceforth referred to as B&K. They looked at first at the oscillations of a one-dimensional chain of points with mass \( m \), separated by a constant distance \( a \). This is the monatomic case, all lattice points have the same mass. B&K assume that the forces exerted on each point of the chain originate only from the two neighboring points. These forces are opposed to and proportional to the displacements, as with elastic springs. The equation of motion is in this case

\[
m \ddot{u}_n = \alpha (u_{n+1} - u_n) - \alpha (u_n - u_{n-1}). \tag{6}
\]

The \( u_n \) are the displacements of the mass points from their equilibrium position which are apart by the distance \( a \). The dots signify, as usual, differentiation with respect to time, \( \alpha \) is a constant characterizing the force between the lattice points, and \( n \) is an integer number.

In order to solve (6) B&K set

\[
u_n = Ae^{i(\omega t + n\phi)}, \tag{7}
\]

which is obviously a temporally and spatially periodic solution. \( n \) is an integer, with \( n \leq N \), where \( N \) is the number of points in the chain. \( \phi = 0 \) is the monochromatic case. At \( n\phi = \pi/2 \) there are nodes, where for all times \( t \) the displacements are zero, as with standing waves. If a displacement is repeated after \( n \) points we have \( na = \lambda \), where \( \lambda \) is the wavelength, \( a \) the lattice constant, and it must be \( n\phi = 2\pi \) according to (7). It follows that

\[
\lambda = 2\pi a/\phi. \tag{8}
\]

Inserting (7) into (6) one obtains a continuous frequency spectrum given by Eq.(5) of B&K

\[
\omega = 2\sqrt{\alpha/m} \sin(\phi/2). \tag{9}
\]

B&K point out that there is not only a continuum of frequencies, but also a maximal frequency which is reached when \( \phi = \pi \), or at the minimum of the possible wavelengths \( \lambda = 2a \). B&K then discuss the three-dimensional lattice, with lattice constant \( a \) and masses \( m \). They reduce the complexities of the problem by considering only the 18 points nearest to any point. These are 6 points at distance \( a \), and 12 points at distance \( a\sqrt{2} \). B&K assume that the forces between the points are linear functions of the small displacements,
that the symmetry of the lattice is maintained, and that the equations of motion transform into the equations of motion of continuum mechanics for \( a \to 0 \). We cannot reproduce the lengthy equations of motion of the three-dimensional lattice. In the three-dimensional case we deal with the forces caused by the 6 points at the distance \( a \) which are characterized by the constant \( \alpha \) in the case of central forces. There are also the forces which originate from the 12 points at distance \( a\sqrt{2} \), characterized by the constant \( \gamma \), which we do not need. We investigate the propagation of plane waves in a three-dimensional monatomic lattice with the ansatz

\[
u_{l,m} = u_0 e^{i(\omega t + l\phi_1 + m\phi_2)},
\]

and a similar ansatz for \( v_{l,m} \), with \( l,m \) being integer numbers \( \leq N^\frac{1}{3} \), where \( N^\frac{1}{3} \) is the number of lattice points along a side of the cube. We also consider higher order solutions, with \( i_1 \cdot l \) and \( i_2 \cdot m \), where \( i_1, i_2 \) are integer numbers. The boundary conditions are periodic. The number of normal modes must be equal to the number of particles in the lattice. B&K arrive, in the case of two-dimensional waves, at a secular equation for the frequencies

\[
A(\phi_1, \phi_2) - m\nu^2 \begin{vmatrix} A(\phi_2, \phi_1) & B(\phi_1, \phi_2) \\ B(\phi_1, \phi_2) & A(\phi_2, \phi_1) - m\nu^2 \end{vmatrix} = 0.
\]

The formulas for \( A(\phi_1, \phi_2) \) and \( B(\phi_1, \phi_2) \) are given in equation (17) of B&K.

The theory of lattice oscillations has been pursued in particular by Blackman [10], a summary of his and other studies is in [11]. Comprehensive reviews of the results of linear studies of lattice dynamics have been written by Born and Huang [12], by Maradudin et al. [13], and by Ghatak and Kothari [14].

3 The masses of the particles of the \( \gamma \)-branch

We will now assume, as seems to be quite natural, that the particles of the \( \gamma \)-branch consist of the same particles into which they decay, directly or ultimately. That means that the \( \gamma \)-branch particles consist of photons. We base this assumption on the fact that photons and \( \pi^0 \) mesons are the principal mode of decay of the \( \gamma \)-branch particles, the characteristic example is \( \pi^0 \to \gamma\gamma \) (98.8\%). Table 1 shows that there are decays of the \( \gamma \)-branch particles which lead to particles of the \( \nu \)-branch, in particular to pairs of \( \pi^+ \) and \( \pi^- \) mesons,
already present in 28% of the decays of the $\eta$ meson, as in $\eta \rightarrow \pi^+ \pi^- \pi^0$ (23.2%). It appears that this has to do with pair production in the $\gamma$-branch particles. Pair production is evident in the decay $\pi^0 \rightarrow e^+ + e^- + \gamma$ (1.198%). It requires the presence of electromagnetic waves of high energy. Anyway, the explanation of the $\gamma$-branch particles must begin with the explanation of the most simple example of its kind, the $\pi^0$ meson, which by all means seems to consist of photons. The composition of the particles of the $\gamma$-branch suggested here offers a direct route from the formation of a $\gamma$-branch particle, through its lifetime, to its decay products. Particles that are made of photons are necessarily neutral, as the majority of the particles of the $\gamma$-branch are.

We also base our assumption that the particles of the $\gamma$-branch are made of photons on the circumstances of the formation of the $\gamma$-branch particles. The most simple and straightforward creation of a $\gamma$-branch particle are the reactions $\gamma + p \rightarrow \pi^0 + p$, or in the case that the spins of $\gamma$ and $p$ are parallel $\gamma + p \rightarrow \pi^0 + p + \gamma'$. A photon impinges on a proton and creates a $\pi^0$ meson. In a timespan on the order of $10^{-23}$ sec the pulse of the incoming electromagnetic wave is, according to Fourier analysis, converted into a continuum of electromagnetic waves with frequencies ranging from $10^{23}$ sec$^{-1}$ to $\nu \rightarrow \infty$. The wave packet so created decays, according to experience, after $8.4 \cdot 10^{-17}$ sec into two electromagnetic waves or $\gamma$-rays. It seems to be very unlikely that Fourier analysis does not hold for the case of an electromagnetic wave impinging on a proton. The question then arises of what happens to the electromagnetic waves in the timespan of $10^{-16}$ seconds between the creation of the wave packet and its decay into two $\gamma$-rays? We will investigate whether the electromagnetic waves cannot continue to exist for the $10^{-16}$ seconds until the wave packet decays.

There must be a mechanism which holds the wave packet of the newly created particle together, or else it will disperse. We assume that the very many photons in the new particle are held together in a cubic lattice. Ordinary cubic lattices, such as the NaCl lattice, are held together by attractive forces between particles of opposite polarity. We assume that the photon lattice is held together by weak attractive forces between photons, if they are about $10^{-16}$ cm apart. Electrodynamics does not predict the existence of such a force between two photons. However, electroweak theory says that $e^2 \approx g^2$ on the scale of $10^{-16}$ cm, and we will now assume that there is such a weak force in the photon lattice. The potential of this force is given in section 8, Eq.(45). It is not unprecedented that photons have been considered to be building blocks of the elementary particles. Schwinger [15] has once
studied an exact one-dimensional quantum electrodynamical model in which the photon acquired a mass \( \sim e^2 \).

We will now investigate the standing waves of a cubic photon lattice. We assume that the lattice is held together by a weak force acting from one lattice point to the next. We assume that the range of this force is \( 10^{-16} \) cm, because the range of the weak nuclear force is on the order of \( 10^{-16} \) cm, according to [16]. For the sake of simplicity we set the sidelength of the lattice at \( 10^{-13} \) cm, the exact size of the nucleon is given in [17] and will be used later. With \( a = 10^{-16} \) cm there are then \( 10^9 \) lattice points.

As we will see the ratios of the masses of the particles are independent of the sidelength of the lattice. Because it is the most simple case, we assume that a central force acts between the lattice points. We cannot consider spin, isospin, strangeness or charm of the particles. The frequency equation for the oscillations of an isotropic monatomic cubic lattice with central forces is either given by equation (9), or in the two-dimensional case by

\[
4\pi^2 \nu^2 = \frac{2\alpha}{M}(2 - \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 - \cos \phi_1). \tag{12}
\]

According to Eq.(13) of B&K

\[
\alpha = a(c_{11} - c_{12} - c_{44}), \tag{13}
\]

where \( c_{11}, c_{12} \) and \( c_{44} \) are the elastic constants in continuum mechanics which applies in the limit \( a \rightarrow 0 \). If we consider central forces then \( c_{12} = c_{44} \) which is the classical Cauchy relation. Isotropy requires that \( c_{44} = (c_{11} - c_{12})/2 \). Eq.(12) follows directly from the equation of motion for the displacements in a monatomic lattice, which are given e.g. by Blackman [10]. The minus sign in front of \( \cos \phi_1 \) in (12) means that the waves are longitudinal. Transverse waves in a cubic lattice with concentric forces are not possible according to [14]. All frequencies that solve (12) come with either a plus or a minus sign which is, as we will see, important.

The frequency distribution following from (12) is shown in Fig. 2.
There are some easily verifiable frequencies. For example at $\phi_1, \phi_2 = 0,0$ it is $\nu = 0$, at $\phi_1, \phi_2 = \pi/2, \pi/2$ it is $\nu = \nu_0 \sqrt{6}$, at $\phi_1, \phi_2 = \pi/2, -\pi/2$ we have $\nu = \nu_0 \sqrt{2}$. Furthermore at $\phi_1, \phi_2 = \pi,0$ we have $\nu = \nu_0 \sqrt{8}$, and for all values of $\phi_1$ it is $\nu = 2\nu_0$ at $\phi_2 = \pi$ and $\phi_2 = -\pi$, with

$$\nu_0 = \sqrt{\alpha/4\pi^2 M}, \quad (14)$$

or as we will see, using Eq.(16), $\nu_0 = c_*/2\pi a$.

The limitation of the group velocity in the photon lattice has now to be considered. The group velocity is given by

$$c_g = \frac{d\omega}{dk} = a\sqrt{\frac{\alpha}{M}} \cdot \frac{df(\phi_1, \phi_2)}{d\phi}. \quad (15)$$

The group velocity in the photon lattice has to be equal to the velocity of light $c_*$ throughout the entire frequency spectrum, because photons move with the velocity of light. In order to learn how this requirement affects the frequency distribution we have to know the value of $\sqrt{\alpha/M}$ in a photon lattice. But we do not have information about what either $\alpha$ or $M$ might be in this case. We assume in the following that $a\sqrt{\alpha/M} = c_*$, which means,
since $a = 10^{-16}$ cm, that $\sqrt{\alpha/M} = 3 \cdot 10^{26}$ sec$^{-1}$, or that the corresponding period is $\tau = 1/3 \cdot 10^{-26}$ sec, which is the time it takes for a wave to travel with the velocity of light over one lattice distance. With

$$c_* = a\sqrt{\alpha/M}$$

(16)

the equation for the group velocity is

$$c_g = c_* \cdot \frac{df}{d\phi}.$$  

(17)

For a photon lattice that means, since $c_g$ must then always be equal to $c_*$, that $df/d\phi = 1$. This requirement determines the form of the frequency distribution, whether we deal with an axial oscillation as in Eq.(9) or the two-dimensional oscillation of Eq.(12), regardless of the order of the mode of oscillation. The frequencies of the corrected spectrum must increase from $\nu = 0$ at the origin $\phi_1, \phi_2 = 0,0$ with slope 1 (in units of $\nu_0$) until a maximum is reached, from where the frequency must decrease with slope 1 to $\nu = 0$. The frequency distribution corrected for (17), i.e. with $df/d\phi = 1$, is shown in Fig. 3. The corrected frequency distributions of higher modes are of the same type, but for the area they cover, see e.g. Fig. 4.

The second mode ($i_1, i_2 = 2$) covers 4 times the area of the basic mode, because $2\phi$ ranges from 0 to $2\pi$, (for $\phi > 0$), whereas the basic mode ranges from 0 to $\pi$. Consequently the energy (Eq.18) contained in all frequencies of the second mode is four times larger than the energy of the basic mode, because the energy contained in the lattice oscillations must be proportional to the sum of all frequencies. Adding, by superposition, to the second mode different numbers of basic modes or of second modes will give exact integer multiples of the energy of the basic mode. Now we understand the integer multiple rule of the particles of the $\gamma$-branch. There is, in the framework of this theory, on account of Eq.(17), no alternative but integer multiples of the basic mode for the energy contained in the frequencies of the different modes or for superpositions of different modes. In other words, the masses of the different particles are integer multiples of the mass of the $\pi^0$ meson, assuming that there is no spin, isospin, strangeness or charm.

We remember that the measured masses in Table 1, which incorporate different spins, isospins, strangeness, and charm spell out the integer multiple rule within on the average 0.66% accuracy. It is worth noting that there is no free parameter if one takes the ratio of the energies contained in the frequency distributions of the different modes, because the factor $\sqrt{\alpha/M}$
in Eqs. (9,12) cancels. This means, in particular, that the ratios of the frequency distributions, or the mass ratios, are independent of the mass of the photons at the lattice points, as well as of the magnitude of the force between the lattice points.

It is obvious that the integer multiples of the frequencies are only a first approximation of the theory of lattice oscillations and of the mass ratios of the particles. The equation of motion in the lattice (6) does not apply in the eight corners of the cube, nor does it apply to the twelve edges, nor, in particular, to the six sides of the cube. A cube with $10^9$ lattice points is not correctly described by the periodic boundary condition we have used, but is what is referred to as a microcrystal. A phenomenological theory of the frequency distributions in microcrystals, considering in particular surface waves, can be found in Chapter 6 of Ghatak and Kothari [14]. Surface waves
may account for the small deviations of the mass ratios of the mesons and baryons from the integer multiple rule of the oscillations in a cube. However, it seems to be futile to pursue a more accurate determination of the oscillation frequencies as long as one does not know what the structure of the electron is, whose mass is 0.378\% of the mass of the $\pi^0$ meson and hence is a substantial part of the deviation of the mass ratios from the integer multiple rule.

Let us summarize our findings concerning the particles of the $\gamma$-branch. The $\pi^0$ meson is the basic mode of the photon lattice oscillations. The $\eta$ meson corresponds to the first higher mode ($i_1, i_2 = 2$), as is suggested by $m(\eta) \approx 4m(\pi^0)$. The $\Lambda$ particle corresponds to the superposition of two higher modes ($i_1, i_2 = 2$), as is suggested by $m(\Lambda) \approx 2m(\eta)$. This superposition apparently results in the creation of spin $1/2$. The two modes would then have to be coupled. The $\Sigma^0$ and $\Xi^0$ baryons are superpositions of one or two basic modes on the $\Lambda$ particle. The $\Omega^-$ particle corresponds to the superposition of three coupled higher modes ($i_1, i_2 = 2$) as is suggested by $m(\Omega^-) \approx 3m(\eta)$. This procedure apparently causes spin $3/2$. The charmed $\Lambda^+_c$ particle seems to be the first particle incorporating a $(3.3)$ mode. $\Sigma^0_c$ is apparently the superposition of a basic mode on $\Lambda^+_c$, as is suggested by the decay of $\Sigma^0_c$. The easiest explanation of $\Xi^0_c$ is that it is the superposition of two coupled $(3.3)$ modes. The superposition of two modes of the same type is, as in the case of $\Lambda$, accompanied by spin $1/2$. The $\Omega^0_c$ baryon is apparently the superposition of two basic modes on the $\Xi^0_c$ particle. All neutral particles of the $\gamma$-branch are thus accounted for. We find it interesting that all $\gamma$-branch particles with coupled $2\cdot(2.2)$ modes, or the $\Omega^-$ particle with the coupled $3\cdot(2.2)$ mode, have strangeness. But this rule does not hold in the presence of a $(3.3)$ mode. All $\gamma$-branch particles with a $(3.3)$ mode have charm. The modes of the particles are listed in Table 1.

We have also found the $\gamma$-branch antiparticles, which follow from the negative frequencies which solve Eqs.(9) or (12). Antiparticles have always been associated with negative energies. Following Dirac’s argument for electrons and positrons, we associate the masses with the negative frequency distributions with antiparticles. We emphasize that the existence of antiparticles is an automatic consequence of our theory.

All particles of the $\gamma$-branch are unstable with lifetimes on the order of $10^{-10}$ sec or shorter. Born [18] has shown that the oscillations in cubic lattices held together by central forces are unstable. It also seems to be possible to understand the decay of the $\pi^0$ meson $\pi^0 \rightarrow e^- + e^+ + \gamma$ (1.198\%). Since in our model the $\pi^0$ meson consists of a multitude of electromagnetic waves, some
of them with energies larger than $2m(e^+c^2)$, it seems that pair production takes place within the $\pi^0$ meson, and even more so in the higher modes of the $\gamma$-branch where the electrons and positrons created by pair production tend to settle on mesons, as e.g. in $\eta \rightarrow \pi^+ + \pi^- + \pi^0$ (23.2\%) or in the decay $\eta \rightarrow \pi^+ + \pi^- + \gamma$ (4.78\%), where the origin of the pair of charges is more apparent.

Finally we must explain the reason for which the photon lattice or the $\gamma$-branch particles are limited in size to a particular value, as the experiments tell. Conventional lattice theory using the periodic boundary condition does not limit the size of a crystal, and in fact very large crystals exist. If, however, the lattice consists of standing electromagnetic waves the size of the lattice is limited by the radiation pressure. The lattice will necessarily break up at the latest when the outward directed radiation pressure is equal to the inward directed elastic force which holds the lattice together, in technical terms when the radiation pressure is equal to Young’s modulus of the crystal. The radiation pressure can be calculated and is a function of the side length of the lattice. Young’s modulus of the lattice can be calculated from lattice theory. From the equality of both follows, as shown in [19], that the side length of the photon lattice or of the $\pi^0$ meson must be on the order of $10^{-13}$ cm or about the size of the nucleon, the principal uncertainty being the breaking strength of lattices, for which a satisfactory solution is not known.

### 4 The mass of the $\pi^0$ meson

So far we have studied the ratios of the masses of the particles. We will now determine the mass of the $\pi^0$ meson in order to validate that the mass ratios link with the actual masses of the particles. The energy of the $\pi^0$ meson is

$$E(m(\pi^0)) = 134.9764 \text{ MeV} = 2.1626 \cdot 10^{-4} \text{ erg}.$$ 

For the sum of the energies of all frequencies of the lattice we use the equation

$$E_\nu = \frac{N h \nu_0}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\phi_1, \phi_2) \, d\phi_1 d\phi_2.$$ (18)

This equation originates from B&K. N is the number of all lattice points. The total energy of the frequencies in a cubic lattice is equal to the number of the oscillations times the average of the energy of the individual frequencies. In order to arrive at an exact value of Eq.(18) we have to use the correct
value of the radius of the proton, which is \( r_p = (0.88 \pm 0.015) \cdot 10^{-13} \) cm according to [17], in contrast to the old \( r_p = 0.8 \cdot 10^{-13} \) cm we used in [6]. With \( a = 10^{-16} \) cm it follows that the number of all lattice points in the cubic lattice is

\[
N = 2.854 \cdot 10^9.
\]

The radius of the \( \pi^\pm \) mesons has also been measured [20] and after further analysis [21] was found to be \( 0.83 \cdot 10^{-13} \) cm, which means that within the uncertainty of the radii we have \( r_p = r_\pi \).

If the oscillations are parallel to an axis, the limitation of the group velocity is taken into account, that means if \( \nu = \nu_0 \phi \), and the absolute values of the frequencies are taken, then the value of the double integral in Eq.(18) is \( 2\pi^3 = 62.01 \). If we use for \( f(\phi_1, \phi_2) \) the modified frequency distribution shown in Fig. 3 with the absolute values of the frequencies then it turns out that the numerical value of the double integral in Eq.(18) is 66.9 \((\text{radians}^2)\) for the corrected (1.1) state. With \( N = 2.854 \cdot 10^9 \) and \( \nu_0 = c_s/2\pi a \) it follows from Eq.(18) that the sum of the energy of the frequencies corrected for the group velocity limitation in the case of Eq.(9) is \( E_{\text{corr}}(1.1) = 1.418 \cdot 10^9 \) erg, and for the corrected form of Eq.(12) it is \( E_{\text{corr}}(1.1) = 1.53 \cdot 10^9 \) erg. That means in the first case that the energy is \( 6.56 \cdot 10^{12} \), and in the second case \( 7.07 \cdot 10^{12} \) times larger than \( E(m(\pi^0)) \). This discrepancy is inevitable, because the basic frequency of the Fourier spectrum after a collision on the order of \( 10^{-23} \) sec duration is \( 10^{23} \) sec\(^{-1} \), which means, when \( E = h\nu \), that one basic frequency alone contains an energy of about \( 9m(\pi^0)c^2 \).

To eliminate this discrepancy we use, instead of the simple form \( E = h\nu \), the complete quantum mechanical energy of a linear oscillator as given by Planck

\[
E = \frac{h\nu}{e^{h\nu/kT} - 1}.
\]

This equation was already used by B&K for the determination of the specific heat of cubic crystals or solids. Equation (19) calls into question the value of the temperature \( T \) in the interior of a particle. We determine \( T \) empirically with the formula for the internal energy of solids

\[
u = \frac{R\Theta}{e^{\Theta/T} - 1},
\]

which is from Sommerfeld [22]. In this equation it is now \( R = 2.854 \cdot 10^9 k \), where \( k \) is Boltzmann’s constant, and \( \Theta \) is the characteristic temperature.
introduced by Debye [23] for the explanation of the specific heat of solids. It is \( \Theta = h\nu_m/k \), where \( \nu_m \) is a maximal frequency. In the case of the oscillations making up the \( \pi^0 \) meson the maximal frequency is \( \nu_m = \pi\nu_0 \), see Figs. 3,4, therefore \( \nu_m = 1.5 \cdot 10^{26} \) sec\(^{-1}\), and we find that \( \Theta = 7.2 \cdot 10^{15} \) K.

In order to determine \( T \) we set the internal energy \( u \) equal to \( m(\pi^0)c^2 \). It then follows from Eq.(20) that \( \Theta/T = 30.20 \), or \( T = 2.38 \cdot 10^{14} \) K. That means that Planck’s formula (19) introduces a factor \( 1/(e^{\Theta/T} - 1) \approx 1/e^{30.2} = 1/(1.305 \cdot 10^{13}) \) into Eq.(18). In other words, if we determine the temperature \( T \) of the particle through equation (20), and correct (18) accordingly then we arrive at a sum of the oscillation energies of the \( \pi^0 \) meson which is \( 1.0866 \cdot 10^{-4} \) erg = 67.82 MeV in the case of the corrected Eq.(9) and \( 1.172 \cdot 10^{-4} \) erg in the case of the corrected Eq.(12). That means that the sum of the energies of the oscillations parallel to an axis is \( 0.502E(m(\pi^0)) \), and for the two-dimensional oscillations the sum of the energies in the corrected frequencies is \( 0.542E(m(\pi^0)) \). We have to double this amount because standing waves consist of two waves traveling in opposite direction with the same absolute value of the frequency. The sum of the energy of the lattice oscillations in the \( \pi^0 \) meson is therefore

\[
E_\nu = 2.1732 \cdot 10^{-4} \text{erg} = 135.64 \text{ MeV} = 1.005E(m(\pi^0))(\exp),
\]

in the case that the oscillations are parallel to the \( \phi_1 \) axis, or \( 2.345 \cdot 10^{-4} \) erg = \( 1.084E(m(\pi^0)) \) in the case of the two-dimensional oscillations Eq.(12). The energy in the mass of the \( \pi^0 \) meson and the energy in the sum of the lattice oscillations agree fairly well, considering the uncertainties of the parameters involved. The energy contained in the axial oscillations (Eq.9) matches the measured energy in the \( \pi^0 \) meson much better than the energy contained in the two-dimensional oscillations (Eq.12). The calculations therefore favor the interpretation that the oscillations in the \( \pi^0 \) mesons are purely axial.

To summarize: We find that the energy in the rest mass of the \( \pi^0 \) meson and the other particles of the \( \gamma \)-branch are correctly given by the sum of the energy of the standing electromagnetic waves, if the energy of the oscillations is determined by Planck’s formula for the energy of a linear oscillator. The \( \pi^0 \) meson is like an adiabatic, cubic black body filled with standing electromagnetic waves. We know from Bose’s work [24] that Planck’s formula applies to a photon gas as well. For all \( \gamma \)-branch particles we have found a simple mode of standing waves in a cubic lattice. For the explanation of the mesons and baryons of the \( \gamma \)-branch we use only photons, nothing else. A rather conservative explanation of the \( \pi^0 \) meson, and the \( \gamma \)-branch particles. It is worth noting that in the \( \gamma \)-branch of our model there is a continuous
line leading from the creation of a particle out of photons or electromagnetic waves through the lifetime of the particle as standing electromagnetic waves to the decay products which are electromagnetic waves as well.

From the frequency distributions of the standing waves in the lattice follow the ratios of the masses of the particles which obey the integer multiple rule. It is important to note that in this theory the ratios of the masses of the $\gamma$-branch particles to the mass of the $\pi^0$ meson do not depend on the sidelong of the lattice, and the distance between the lattice points, neither do they depend on the strength of the force between the lattice points nor on the mass of the lattice points. The mass ratios are determined only by the spectra of the frequencies of the standing waves in the lattice. Since the equation determining the frequency of the standing waves is quadratic it follows automatically that for each positive frequency there is also a negative frequency of the same absolute value, that means that for each particle there exists also an antiparticle.

5 The neutrino branch particles

The masses of the neutrino branch, the $\pi^\pm$, $K^{\pm,0}$, n, $D^{\pm,0}$ and $D_S^\pm$ particles, are integer multiples of the mass of the $\pi^\pm$ mesons times a factor $0.85 \pm 0.02$ as we stated before. We have explained the integer multiple rule of the $\gamma$-branch particles with different modes and superpositions of standing electromagnetic waves in a cubic nuclear lattice. For the explanation of the particles of the neutrino branch we follow a similar path. As we have done in [25] we assume, as appears to be quite natural, that the $\pi^\pm$ mesons and the other particles of the $\nu$-branch consist of the same particles into which they decay, that means of neutrinos and electrons or positrons. Since the particles of the $\nu$-branch decay through weak decays, we assume, as appears likewise to be natural, that the weak nuclear force holds the particles of the $\nu$-branch together. Since the range of the weak interaction is only about a thousandth of the diameter of the particles, the weak force can hold particles together only if the particles have a lattice structure, just as macroscopic crystals are held together by microscopic forces between atoms. We will, therefore, investigate the energy which is contained in the oscillations of a cubic lattice consisting of electron and muon neutrinos and in the rest masses of the neutrinos.

Since we will investigate the oscillations of a cubic lattice consisting of muon and electron neutrinos it is necessary to outline the basic aspects of
diatomic lattice oscillations. In diatomic lattices the lattice points have alternately the masses $m$ and $M$. The classic example of a diatomic lattice is the salt crystal with the masses of the Na and Cl atoms in the lattice points. The theory of diatomic lattice oscillations was developed by Born and v. Karman [9]. They first discussed a diatomic chain. The equation of motions in the chain are according to Eq.(22) of B&K

$$m \ddot{u}_{2n} = \alpha(u_{2n+1} + u_{2n-1} - 2u_{2n}) ,$$  

$$M \ddot{u}_{2n+1} = \alpha(u_{2n+2} + u_{2n} - 2u_{2n+1}) ,$$

where the $u_n$ are the displacements, $n$ an integer number and $\alpha$ a constant characterizing the force between the particles. As with any spring the restoring forces in (21,22) increase with increasing distance between the particles. Eqs.(21,22) are solved with

$$u_{2n} = Ae^{i(\omega t + 2n\phi)} ,$$

$$u_{2n+1} = Be^{i(\omega t + (2n+1)\phi)} ,$$

where $A$ and $B$ are constants and $\phi$ is given by $\phi = 2\pi a/\lambda$ as in (8). $a$ is the lattice constant as before and $\lambda$ the wavelength, $\lambda = na$. The solutions of Eqs.(23,24) are obviously periodic in time and space and describe again standing waves. Using (23,24) to solve (21,22) leads to a secular equation from which according to Eq.(24) of B&K the frequencies of the oscillations of the chain follow from

$$4\pi^2 \nu^2 = \alpha/Mm \cdot ((M + m) \pm \sqrt{(M - m)^2 + 4mm\cos^2\phi}) .$$

Longitudinal and transverse waves are distinguished by the minus or plus sign in front of the square root in (25). For a similar equation for the plane waves in an isotropic three-dimensional lattice see [25]. The equations of motion for the oscillations in a three-dimensional diatomic lattice have been developed by Thirring [26].

6 The masses of the $\nu$-branch particles

The particles of the neutrino branch decay primarily by weak decays, see Table 4. The characteristic case are the $\pi^\pm$ mesons which decay via e.g.
\[ \pi^+ \rightarrow \mu^+ + \nu_\mu \ (99.988\%) \] followed by \[ \mu^+ \rightarrow e^+ + \bar{\nu}_\mu + \nu_e \ (\approx 100\%). \] Only \( \mu \) mesons, which decay into neutrinos, and neutrinos result from the decay of the \( \pi^\pm \) mesons, but for \( e^\pm \) which conserve charge. If the particles consist of the particles into which they decay, then the \( \pi^\pm \) mesons and the other particles of the neutrino branch are made of neutrinos and \( e^\pm \). The neutrinos must be held together in some form, otherwise the particles could not exist over a finite lifetime, say \( 10^{-10} \) sec. Since neutrinos interact through the weak force which has a range of \( 10^{-16} \) cm according to p.25 of [16], and since the size of the nucleon is on the order of \( 10^{-13} \) cm, we suggest that the \( \nu \)-branch particles are held together in a cubic lattice. A cubic lattice is held together by central forces, which are the most simple forces to consider. It is not known with certainty that neutrinos actually have a rest mass as was originally suggested by Bethe [27] and Bahcall [28] and what the values of \( m(\nu_e) \) and \( m(\nu_\mu) \) are. However, the results of the Super-Kamiokande [29] and the Sudbury [30] experiments indicate that the neutrinos have rest masses. The neutrino lattice must be diatomic, meaning that the lattice points have alternately larger \( (m(\nu_\mu)) \) and smaller \( (m(\nu_e)) \) masses. We will retain the traditional term diatomic. The lattice we consider is shown in Fig. 5. Since the neutrinos have spin 1/2 this is a four-Fermion lattice as is required for Fermi’s explanation of the \( \beta \)-decay. The first investigation of cubic Fermion lattices in context with the elementary particles was made by Wilson [7]. The entire neutrino lattice is electrically neutral. Since we do not know the structure of the electron we cannot consider lattices with a charge.

A neutrino lattice takes care of the continuum of frequencies which must, according to Fourier analysis, be present after the high energy collision which created the particle. We will, for the sake of simplicity, first set the sidelength of the lattice at \( 10^{-13} \) cm that means approximately equal to the size of the nucleon. The lattice then contains \( 10^9 \) lattice points, since the lattice constant \( a \) is on the order of \( 10^{-16} \) cm. The sidelength of the lattice does not enter the equations, e.g. Eq. (25), for the frequencies of the lattice oscillations. The calculation of the ratios of the masses or of the energy of the lattice oscillations is consequently independent of the size of the lattice, as was the case with the \( \gamma \)-branch. However the size of the lattice can be explained with the pressure which the lattice oscillations exert on a crosssection of the lattice. The pressure cannot exceed Young’s modulus of the lattice. We require that the lattice is isotropic.

From the frequency distribution of the axial diatomic oscillations (Eq. 25), shown in Fig. 6, follows the group velocity \( d\omega/dk = 2\pi a \, dv/d\phi \) at each point.
\[ \phi. \] With \( \nu = \nu_0 f(\phi) \) and \( \nu_0 = c_\ast/2\pi a \) from Eq.(14) we find

\[ c_g = d\omega/dk = a\sqrt{\alpha/M} \cdot d f(\phi)/d\phi. \] (26)

In order to determine the value of \( d\omega/dk \) we have to know the value of \( \sqrt{\alpha/M} \).

From Eq.(13) for \( \alpha \) follows that \( \alpha = a c_{44} \) in the isotropic case with central forces. The group velocity is therefore

\[ c_g = \sqrt{a^3 c_{44}/M} \cdot df/d\phi. \] (27)

We now set \( a\sqrt{\alpha/M} = c_\ast \) as in Eq.(16), where \( c_\ast \) is the velocity of light. It follows that

\[ c_g = c_\ast \cdot df/d\phi, \] (28)

as it was with the \( \gamma \)-branch, only that now on account of the rest masses of the neutrinos the group velocity must be smaller than \( c_\ast \), so the value of \( df/d\phi \) is limited to \(< 1 \), but \( c_g \approx c_\ast \), which is a necessity because the neutrinos in the lattice soon approach the velocity of light as we will see. Equation (28) applies regardless whether we consider \( \nu_+ \) or \( \nu_- \) in Eq.(25). That means that there are no separate transverse oscillations with their theoretically higher frequencies.

From Eq.(27) follows a formula for the mass of the muon neutrino, it is
Fig. 6: The frequency distribution $\nu_-/\nu_0$ of the basic diatomic mode according to Eq.(25) with $M/m = 50$. The dashed line shows the distribution of the frequencies corrected for the group velocity limitation.

$$M = a^3 c_{44}/c_*^2.$$  \hspace{1cm} (29)

The elasticity constant $c_{44} = c_{11}/3$ can be determined theoretically from an exact copy of the determination of $c_{11}$ in Born’s lattice theory, assuming weak nuclear forces between lattice points at distance $a = 10^{-16}$ cm, and replacing $e^2$ by $g^2$, where $g^2$ is the interaction constant of the weak force. This will be discussed in section 8. From $c_{11}$ in Eq.(47) we find that

$$M = m(\nu_{\mu}) = 0.538 g^2 (n - 1)/3ac_*^2,$$ \hspace{1cm} (30)

and with $a = 10^{-16}$ cm, $g^2 = 2.946 \cdot 10^{-17}$ erg.cm (Eq.43) and $(n - 1) = 2.187 \cdot 10^{-12}$ from Eq.(44) we have

$$m(\nu_{\mu}) = 1.28 \cdot 10^{-34} \text{ gr} = 72.1 \cdot 10^{-3} \text{ eV}/c_*^2.$$ \hspace{1cm} (31)
The rest mass of the muon neutrino is 72 milli-electron-Volt/c², within the accuracy of the parameters g², a and (n − 1). We note that (n − 1) depends on the compressibility κ for which values are given in [35] which differ by 33%. We will later on determine the rest mass of the muon neutrino from the difference \( m(\pi^\pm) - m(\mu^\pm) \), which leads to \( m(\nu_\mu) = 47.5 \text{ meV/c}^2 \) (Eq.35). We use, henceforth, only this value because it depends on a single parameter, the number of muon neutrinos in the lattice.

It can be verified easily that the mass \( m(\nu_\mu) = 47.5 \text{ meV/c}^2 \) we have found makes sense. The energy of the rest mass of the \( \pi^\pm \) mesons is 139 MeV, and we have \( N/4 = 0.7135\cdot10^9 \) muon neutrinos and the same number of anti-muon neutrinos. It then follows that the energy in the rest masses of all muon and anti-muon neutrinos is 67.8 MeV, that is 48.5% of the energy of the rest mass of the \( \pi^\pm \) mesons. A small part of \( m(\pi^\pm)c^2 \) goes, as we will see, into the electron neutrino masses, the rest goes into the lattice oscillations.

We can now determine the rest mass of the \( \pi^\pm \) mesons from the sum of the oscillation energies and the sum of the rest masses of the neutrinos. For the sum of the energies of the frequencies we use Eq.(18) with the same \( N \) and \( \nu_0 \) we used for the \( \gamma \)-branch. For the double integral in (18) of the corrected axial diatomic frequencies we find the value \( \pi^3 \) as can be easily derived from the plot of the corrected frequencies in Fig. 6. The value of the double integral in Eq.(18) for the axial diatomic frequencies \( \nu = \nu_0\phi \) is 1/2 of the value \( 2\pi^3 \) of the same integral in the case of axial monatomic frequencies, because in the latter case the increase of the corrected frequencies continues to \( \phi_1 = \pi \), whereas in the diatomic case the increase of the corrected frequencies ends at \( \pi/2 \), see Fig. 6. We consider \( c_g \) to be so close to \( c_* \) that it does not change the value of the double integral significantly. It can be calculated that the time average of the velocity of the electron neutrinos in the \( \pi^\pm \) mesons is \( \bar{v} = 0.99993c_* \), if \( m(\nu_e) = 0.55 \text{ meV/c}^2 \) as in Eq.(36). Consequently we find that the sum of the energies of the corrected diatomic neutrino frequencies is \( 0.5433\cdot10^{-4}\text{erg} = 33.91 \text{ MeV} \). We double this amount because we deal with standing waves and find that the energy of the neutrino oscillations is 67.82 MeV. Adding to that the sum of the energy of the rest masses of the neutrinos \( N/2\cdot(47.5 + 0.55)\text{meV} = 68.57 \text{ MeV} \) we obtain a value for \( m(\pi^\pm)c^2(\text{theor}) = 136.39 \text{ MeV} = 0.977m(\pi^\pm)c^2(\text{exp}) = 0.977\cdot139.57 \text{ MeV} \).

To this theoretical value of the mass we must add \( m(e^\pm)c^2 \) in order to obtain the mass of the \( \pi^\pm \) mesons plus or minus some possible binding energy. We must keep in mind that the relation between the theoretical and experimental
m(\(\pi^\pm\)) still depends on the accuracy of the values of \(N\) and \(\nu_0\) and on a small correction due to the possible presence of surface waves on the lattice.

We have explained the \(\pi^\pm\) mesons with neutrinos. This agrees with what we learn from the decay of the \(\pi^\pm\) mesons, 99.9877\% of which decay with the emission of a muon neutrino and a \(\mu\) meson, which in turn consists of neutrinos as well, as we will see in the next section. The antiparticle of the \(\pi^+\) meson is the particle in which all frequencies of the neutrino lattice oscillations have been replaced by frequencies with the opposite sign, all neutrinos replaced by their antiparticles and the positive charge replaced by the negative charge. Assuming that the antineutrinos have the same rest mass as the neutrinos it follows that the antiparticle of the \(\pi^+\) meson has the same mass as \(\pi^+\) but opposite charge, i.e. is the \(\pi^-\) meson.

The primary decay of the \(K^\pm\) mesons, say, \(K^+ \to \mu^+ + \nu_\mu\) (63.5\%), leads to the same end products as the \(\pi^\pm\) meson decay \(\pi^+ \to \mu^+ + \nu_\mu\). From this and the composition of the \(\mu\) mesons we learn that the \(K\) mesons must, at least partially, be made of the same four neutrino types as the \(\pi^\pm\) mesons namely of muon neutrinos, anti-muon neutrinos, electron neutrinos and anti-electron neutrinos and their oscillation energies. However the \(K^\pm\) mesons cannot be the \((2.2)\) mode of the lattice oscillations of the \(\pi^\pm\) mesons, because the \((2.2)\) mode of the neutrino lattice oscillations has an energy of \(4E_\nu(\pi^\pm) + N/2 \cdot (m(\nu_\mu) + m(\nu_e))c_\star^2 = (271.3 + 68.57)\text{MeV} = 340\text{ MeV}\). The 340 MeV characterize the \((2.2)\) mode of the \(\pi^\pm\) mesons, which fails \(m(K^\pm)c_\star^2 = 493.7\text{ MeV}\) by a wide margin.

Anyway, the concept of the \(K^\pm\) mesons being solely a higher mode of the \(\pi^\pm\) mesons contradicts our point that the particles consist of the particles into which they decay. The decays \(K^\pm \to \pi^\pm + \pi^0\) (21.16\%), as well as \(K^+ \to \pi^0 + e^+ + \nu_e\) (4.82\%) and \(K^+ \to \pi^0 + \mu^+ + \nu_\mu\) (3.18\%) make up 29.16\% of the \(K^\pm\) meson decays. A \(\pi^0\) meson figures in each of these decays. If we add the energy in the rest mass of a \(\pi^0\) meson \(m(\pi^0)c_\star^2 = 134.97\text{ MeV}\) to the 340 MeV in the \((2.2)\) mode of the \(\pi^\pm\) meson then we arrive at an energy of 475 MeV, which is 96.2\% of \(m(K^\pm)c_\star^2\). Therefore we conclude that the \(K^\pm\) mesons consist of the \((2.2)\) mode of the \(\pi^\pm\) mesons plus a \(\pi^0\) meson. Then it is natural that \(\mu\) mesons from the decay of the \((2.2)\) mode of the \(\pi^\pm\) mesons in the \(K^\pm\) mesons as well as \(\pi^0\) mesons from the \(\pi^0\) component in the \(K^\pm\) mesons appear as decay products of the \(K^\pm\) meson. It may be coincidental but is interesting that the ratio 0.273 of the energy in the \(\pi^0\) meson to the energy in the \(K^\pm\) mesons closely resembles the 29.16\% of the decays in which the \(\pi^0\) meson is one of the decay products.
We obtain the $K^0$ meson if we superpose onto the (2.2) mode of the $\pi^\pm$ mesons instead of a $\pi^0$ meson a basic mode of the $\pi^\pm$ mesons with a charge opposite to the charge of the (2.2) mode of the $\pi^\pm$ meson. The $K^0$ mesons, or the state (2.2)$\pi^\pm + \pi^\mp$, is made of neutrinos only without a photon component, because the (2.2) mode of $\pi^\pm$ as well as the basic mode $\pi^\mp$ consist of neutrinos only. Since the mass of a $\pi^\pm$ meson is by 4.59 MeV/c$^2$ larger than the mass of a $\pi^0$ meson the mass of $K^0$ should be larger than $m(K^\pm)$, and indeed $m(K^0) - m(K^\pm) = 3.995$ MeV/c$^2$ according to [2].

The decay $K^0_S \to \pi^+ + \pi^-$ (68.6%) creates directly the $\pi^+$ and $\pi^-$ mesons which are part of the (2.2)$\pi^\pm + \pi^\mp$ structure of $K^0$ we have suggested. The decay $K^0_S \to \pi^0 + \pi^0$ (31.4%) apparently originates from the $2\gamma$ branch of electron positron annihilation. Both decays account for 100% of the decays of $K^0_S$. The decay $K^0_L \to 3\pi^0$ (21.1%) apparently comes from the $3\gamma$ branch of electron positron annihilation. The two decays of $K^0_L$ called $K^0_{\mu 3}$ and $K^0_{e 3}$ which together make up 65.95% of the $K^0_L$ decays apparently originate from the decay of either of the (2.2) mode of the $\pi^\pm$ mesons or the basic mode of $\pi^\mp$ in the $K^0$ structure, either into $\pi^\pm + \mu^\mp + \nu_\mu$ or into $\pi^\pm + e^\mp + \nu_e$. Our rule that the particles consist of the particles into which they decay also holds for the $K^0$ meson.

The neutron is the superposition of a $K^+$ and a $K^-$ meson or of two $K^0$ mesons. The neutrino oscillations in the neutron must be coupled in order to have spin 1/2, just as the $\Lambda$ baryon with spin 1/2 is a superposition of two $\eta$ mesons. With $m(K^\pm)$(theor) = 475 MeV/c$^2$ from above it follows that $m(n)$(theor) = 2$m(K^\pm)$(theor) = 950 MeV/c$^2$ = 1.011$m(n)$(exp). The $D^\pm$ mesons are the superposition of a proton and a neutron of opposite spin and hence have no spin, whereas the superposition of a proton and a neutron with the same spin creates the deuteron with spin 1. In this case the proton and neutron interact with the strong force which will be discussed in the last section. The $D^\pm_S$ mesons seem to be the superposition of a $\pi^0$ meson on the $D^\pm$ mesons.

The proton does not decay and does not tell which particles it is made of. However we learn about the structure of the proton through the decay of the neutron $n \to p + e^- + \bar{\nu}_e$ (100%). One single anti-electron neutrino is emitted when the neutron decays and 1.3 MeV are released. But there is no place for a permanent vacancy of a single missing neutrino and for a small amount of permanently missing oscillation energy in a nuclear lattice. As it appears the entire anti-electron neutrino type is removed from the structure of the neutron in the neutron decay and converted to available energy. This
process will be discussed again in the following section on the \( \mu \) meson. On the other hand, it seems to be certain that the proton contains a cubic lattice consisting of four muon and four anti-muon neutrinos in each lattice cell. This lattice exists already in the superposition of a \( K^+ \) and a \( K^- \) or of two \( K^0 \) mesons which make up the neutron. The muon-anti-muon neutrino lattice is not affected by the decay of the neutron, no muon neutrino of either type is emitted. The neutron from the \( K^+ \) and \( K^- \) mesons also contains the superposition of two basic photon lattice oscillations, which are not affected by the neutron decay either, otherwise the energy released could not be only 1.3 MeV. So the proton contains a cubic muon neutrino-anti-muon neutrino lattice and the superposition of two \( \pi^0 \) mesons. Whether or not \( N/2 \) electron neutrinos are part of the proton structure awaits clarification of the structure of the electron.

The average factor 0.85 \( \pm \) 0.025 in the ratios of the particles of the \( \nu \)-branch to the \( \pi^\pm \) mesons is a consequence of the neutrino rest masses. They make it impossible that the ratios of the particle masses are integer multiples because the particles consist of the energy in the neutrino oscillations plus the neutrino rest masses which are independent of the order of the lattice oscillations. Since the contribution in percent of the neutrino rest masses to the \( \nu \)-branch particle masses decreases with increased particle mass the factor in front of the mass ratios of the \( \nu \)-branch particles must decrease with increased particle mass.

To summarize: We have found that the \( \pi^\pm \) mesons can be described by the oscillations in a cubic lattice consisting of muon neutrinos and electron neutrinos and their antiparticles. The energy in the \( \pi^\pm \) mesons is the sum of the oscillation energies plus the energy in the rest masses of the neutrinos. For the explanation of the \( K^\pm \) mesons, as well as for the explanation of the neutron and the D mesons it is necessary to have also a \( \pi^0 \) meson present in the particles. When the \( \pi^0 \) meson is incorporated in the \( K^\pm \) meson the decay modes of the \( K^\pm \) meson appear to be natural. For the explanation of the neutrino branch particles we need neutrinos and photons.

### 7 The mass of the \( \mu^\pm \) mesons

Surprisingly one can also explain the mass of the \( \mu^\pm \) mesons with the standing wave model as we have shown in [31]. The \( \mu \) mesons are part of the lepton family which is distinguished from the mesons and baryons not so much by
their mass as the name lepton implies, actually the mass of the \( \tau \) meson is about twice the mass of the proton, but rather by the absence of strong interaction with the mesons and baryons. The masses of the leptons are not explained by the standard model of the particles.

The mass of the \( \mu \) mesons is \( m(\mu^\pm) = 105.658389 \pm 3.4 \cdot 10^{-5} \text{ MeV}/c^2 \), according to the Particle Physics Summary [2]. The \( \mu \) mesons are “stable”, their lifetime is \( \tau(\mu^\pm) = 2.19703 \cdot 10^{-6} \pm 4 \cdot 10^{-11} \text{ sec} \), about a hundred times longer than the lifetime of the \( \pi^\pm \) mesons, that means longer than the lifetime of any other elementary particle, but for the electrons, protons and neutrons.

Comparing the mass of the \( \mu \) mesons to the mass of the \( \pi^\pm \) mesons \( m(\pi^\pm) = 139.56995 \text{ MeV}/c^2 \), we find that \( m(\mu^\pm)/m(\pi^\pm) = 0.757028 \approx 3/4 \) or that \( m(\pi^\pm) - m(\mu^\pm) = 33.9116 \text{ MeV}/c^2 = 0.24297 m(\pi^\pm) \) or approximately \( 1/4 \cdot m(\pi^\pm) \). The mass of the electron is approximately \( 1/206 \) of the mass of the muon, the contribution of \( m(e^\pm) \) to \( m(\mu^\pm) \) will therefore be neglected in the following. We assume, as we have done before and as appears to be natural, that the particles, including the muons, consist of the particles into which they decay. The \( \mu^+ \) meson decays via \( \mu^+ \rightarrow e^+ + \bar{\nu}_\mu + \nu_e \) (\( \approx 100\% \)). The muons are apparently composed of some of the neutrinos and their oscillations which are present in the cubic neutrino lattice of the \( \pi^\pm \) mesons according to our standing wave model. The \( \pi^\pm \) meson decays via \( \pi^\pm \rightarrow \mu^\pm + \nu_\mu \) or the conjugate particles in the decay of the \( \pi^- \) meson, with \( 99.988\% \) of the \( \pi^\pm \) decays in this form. The energy \( m(\pi^\pm)c^2 - m(\mu^\pm)c^2 \approx 1/4 \cdot m(\pi^\pm)c^2 \) is lost when a \( \mu^+ \) meson and one muon neutrino \( \nu_\mu \) are emitted by the \( \pi^+ \) mesons. The rest of the energy in the rest mass of the \( \pi^\pm \) mesons passes to the rest mass of the \( \mu^\pm \) mesons.

In the standing wave model of the particles of the neutrino branch the \( \pi^\pm \) mesons are composed of a cubic lattice consisting of \( N/4 = 0.7135 \cdot 10^9 \) muon neutrinos \( \nu_\mu \) and the same number of anti-muon neutrinos \( \bar{\nu}_\mu \), \( m(\nu_\mu) = m(\bar{\nu}_\mu) \), as well as of \( N/4 \) electron neutrinos \( \nu_e \) and the same number of anti-electron neutrinos \( \bar{\nu}_e \), \( m(\nu_e) = m(\bar{\nu}_e) \), plus the oscillation energy of these neutrinos. We say that the mass of a single muon neutrino should be \( 47.5 \text{ milli-eV}/c^2 \) (Eq.(35)), and the mass of a single electron neutrino should be \( 0.55 \text{ meV}/c^2 \) according to Eq.(36). With these values of \( N \) and of the neutrino masses we find that:

(a) The difference of the rest masses of the \( \mu^\pm \) and \( \pi^\pm \) mesons is nearly equal to the sum of the rest masses of all muon, respectively anti-muon, neutrinos in the \( \pi^\pm \) mesons.
m(π±) − m(µ±) = 33.912 MeV/c²

(b) The energy in the oscillations of all neutrinos in the π± mesons is nearly the same as the energy in the oscillations of all ¯νµ, respectively νµ, and the νe and ¯νe neutrinos in the µ± mesons.

\[ E_\nu(\pi^\pm) = m(\pi^\pm)c^2 - N/2 \cdot [m(\nu_\mu) + m(\nu_e)]c^2 = 71.0\text{MeV} \tag{32} \]

versus

\[ E_\nu(\mu^\pm) = m(\mu^\pm)c^2 - N/4 \cdot m(\bar{\nu}_\mu)c^2 - N/2 \cdot m(\nu_e)c^2 = 70.98\text{MeV} \tag{33} \]

Both statements are, of course, valid only within the accuracy with which the number N of all neutrinos in the π± lattice is known, as well as within the accuracy with which the masses m(νµ) = m(¯νµ) and m(νe) = m(¯νe) have been determined. It cannot be expected that this accuracy is better than a few percent, considering in particular the uncertainty of the lattice constant. If \( E_\nu(\pi^\pm) = E_\nu(\mu^\pm) \) then it follows from the difference of Eq.(32) and Eq.(33) that

\[ m(\pi^\pm) - m(\mu^\pm) = N/4 \cdot m(\nu_\mu). \tag{34} \]

We note that according to (a) and Eq.(34) the difference of the rest masses \( m(\pi^\pm) - m(\mu^\pm) \) provides an independent check of the value of the rest mass of the muon neutrino. With \( N/4 = 0.7135 \cdot 10^9 \) and \( m(\pi^\pm) - m(\mu^\pm) = 33.912 \text{MeV}/c^2 \) it follows that the rest mass of the muon neutrino should be

\[ m(\nu_\mu) = 47.5 \text{meV}/c^2, \tag{35} \]

whereas we found \( m(\nu_\mu) = 72 \text{meV}/c^2 \) in Eq.(31) from an entirely theoretical determination. We should note that in the π± decays only one single muon neutrino is emitted, not N/4 of them. The energy in the rest masses of the N/4 − 1 other muon neutrinos is used to supply the kinetic energy in the momentum of the emitted muon neutrino (\( p_{\nu} = 30 \text{MeV} \)) and in the momentum of the emitted µ meson (\( p_{\nu} = 4 \text{MeV} \)).

If the same principle that applies to the decay of the π± mesons, namely that in the decay an entire neutrino type is removed from the neutrino lattice, also applies to the decay of the neutron \( n \rightarrow p + e^- + \bar{\nu}_e \), then the mass of the anti-electron neutrino can be determined from the known difference \( \Delta = m(n) - m(p) = 1.29332 \text{MeV}/c^2 \). Nearly one half of ∆ comes from the energy lost by the emission of the electron, whose mass is 0.510999 MeV/c². N/2
anti-electron neutrinos are in the neutron, which is the superposition of two 
K$^\pm$ or K$^0$ mesons each containing N/4 neutrinos of the four neutrino types.

From $\Delta - m(e^-)c^2 = 0.782321$ MeV follows, after division by N/2, that

$$m(\nu_e) = 0.55 \text{ meV}/c^2,$$

assuming $m(\nu_e) = m(\bar{\nu}_e)$. From the magnitude of $\Delta$ follows also that the
oscillation energy in the neutron must be preserved in the neutron decay, as it is with the oscillation energy in the $\mu^\pm$ decay according to (b).

We have still to account for one type of electron neutrino which, according
to our model of the $\pi^\pm$ mesons, is part of the neutrino lattice of the $\pi^\pm$
mesons but does not appear in the decay of, say, the $\pi^+$ meson $\pi^+ \to \mu^+ +
\nu_\mu$, followed by $\mu^+ \to e^+ + \bar{\nu}_\mu + \nu_e$. The missing anti-electron neutrino $\bar{\nu}_e$ in
the $\pi^+$ decay sequence must go with the emitted positron. Whether or not
this interpretation is correct can be decided only after the explanation of the
structure of the electrons or positrons.

Since according to the statement (a) the decay of the $\pi^\pm$ meson seems to
mean the removal of all $\nu_\mu$, respectively, all $\bar{\nu}_\mu$ neutrinos from the neutrino
lattice of the $\pi^\pm$ mesons, the $\mu$ mesons should contain the remaining neu-
trinos of the original cubic lattice, that means N/4 anti-muon neutrinos $\bar{\nu}_\mu$,
respectively, N/4 muon neutrinos $\nu_\mu$, plus N/4 electron neutrinos $\nu_e$ as well
as N/4 anti-electron neutrinos $\bar{\nu}_e$. If we use the value of $m(\nu_\mu) = 47.5\text{meV}/c^2$
from Eq.(35) we find that $E_\nu(\mu^\pm)/E_\nu(\pi^\pm) = 0.9997$, as in statement (b). We
will show in the following why the energies in the frequencies of $\pi^\pm$ and $\mu^\pm$
mesons are the same.

The neutrinos in the body of the $\mu$ mesons must oscillate because the
collision $e^+ + e^- \to \mu^+ + \mu^-$ tells that a continuum of frequencies must
be present in the $\mu$ mesons, if Fourier analysis holds. The energy of the
oscillations of the neutrinos in the $\mu$ meson lattice is the sum of the energies of
a longitudinal oscillation in a diatomic lattice consisting of N/4 · $\bar{\nu}_\mu$ neutrinos
and N/4 · $\nu_e$ neutrinos, part of the remains of the diatomic neutrino lattice
of the $\pi^\pm$ mesons, plus the energy of the diatomic oscillations of N/4 · $\bar{\nu}_\mu$
and N/4 · $\bar{\nu}_e$ neutrinos which were likewise in the neutrino lattice of the $\pi^\pm$
mesons. The latter oscillations are likely to be perpendicular to the first
mentioned $\bar{\nu}_\mu$-$\nu_\mu$ oscillations because the $\bar{\nu}_e$-$\nu_e$ neutrino pairs are oriented
perpendicular to the $\bar{\nu}_\mu$-$\nu_\mu$ neutrino pairs in the original cubic lattice of the
$\pi^\pm$ mesons, see Fig. 5.

The energy in the frequencies of the lattice is given by
\[ E_\nu = \frac{N\hbar \nu_0}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\phi_1, \phi_2) \, d\phi_1 d\phi_2, \]  

as in Eq.(18) or in the original paper of Born and v.Karman [9], Eq.(50) therein. The function \( f(\phi_1, \phi_2) \) in Eq.(37) describing the frequency spectrum of the oscillations are the same for the diatomic neutrino pairs in the \( \pi^\pm \) mesons and the diatomic neutrino pairs in the \( \mu \) mesons because they must obey the group velocity limitation. When both functions are the same for the \( \pi^\pm \) and the \( \mu^\pm \) mesons the ratio of the energy in the diatomic \( \bar{\nu}_\mu - \nu_e \) lattice oscillations of the \( \mu \) mesons to the energy of the diatomic lattice oscillations in the \( \pi^\pm \) mesons is 1/2 because the number of the pairs \( \bar{\nu}_\mu - \nu_e \) in the \( \mu \) mesons is 1/2 of the combined number \( N \) of the pairs \( \bar{\nu}_\mu - \nu_e \) and \( \nu_\mu - \bar{\nu}_e \) in the \( \pi^\pm \) mesons. But since the same applies for the diatomic oscillations of the \( \bar{\nu}_\mu - \nu_e \) pairs in the \( \mu^\pm \) mesons, the sum of the energies of both oscillations is equal to the oscillation energy of the \( \pi^\pm \) mesons, as (b) says.

Finally we ask why do the \( \mu \) mesons not interact strongly with the mesons and baryons? We will show in the next section that a strong force emanates from the sides of a cubic lattice caused by the unsaturated weak forces of about \( 10^6 \) neutrinos at the surface of the neutrino lattice of the mesons and baryons. This follows from the study of Born and Stern [32] which dealt with the forces between two parts of a cubic lattice cleaved in vacuum. If the \( \mu \) mesons have a lattice consisting of one type of muon neutrinos, say, \( \bar{\nu}_\mu \) and of \( \nu_e \) and \( \bar{\nu}_e \) neutrinos their lattice surface is not the same as the surface of the cubic lattice of the mesons and baryons as described by the standing wave model. Therefore it does not seem likely that the \( \mu \) mesons interact in the same way with the mesons and baryons as the mesons and baryons interact with each other. To put this in another way, the \( \mu \) meson lattice does not bond with the cubic lattice of the mesons and baryons.

To summarize: It has been shown that the mass of the \( \mu^\pm \) mesons can be explained as the sum of the rest masses of \( 0.7 \cdot 10^9 \) muon neutrinos, respectively anti-muon neutrinos, as well as the same number of electron neutrinos and anti-electron neutrinos, plus their oscillation energies. The three neutrino types in the \( \mu \) mesons are the remains of the cubic neutrino lattice of the \( \pi^\pm \) mesons from which the \( \mu \) mesons are formed in the \( \pi^\pm \) decay. The energy contained in the sum of the rest masses of all muon or anti-muon neutrinos in the \( \pi^\pm \) mesons is lost during the \( \pi^\pm \) decay, whereas the oscillation energy of all neutrinos in the \( \pi^\pm \) is preserved during the decay. Hence the
mass of the $\mu^\pm$ mesons differs from the mass of the $\pi^\pm$ mesons by the sum of the energies of the rest masses of $N/4$ muon neutrinos which is $33.9 \text{ MeV} \approx 1/4 \cdot m(\pi^\pm)c_4^2$ or its mass is $m(\mu^\pm) \approx 3/4 \cdot m(\pi^\pm)$. We have also found that the $\mu$ mesons do not interact with the mesons and baryons in the same strong way as the mesons and baryons interact with each other.

8 The weak and strong nuclear force

The potential of the force which holds a cubic ionic lattice together has been determined by Born and Landé (B&L) [33]. They assumed that the static electric potential $\Phi$ of an elementary cube in the lattice has an attractive and repulsive part and is of the form

$$\Phi = -\frac{a}{\delta} + \frac{b}{\delta^n},$$

(38)

where $\delta$ is the distance between two ions of the same type along the edge of the crystal. The constant $b$ is eliminated with the condition that $\Phi$ is a minimum in equilibrium. It follows that $b = a\delta_{0}^{n-1}/n$, where $\delta_{0}$ is the equilibrium distance between the two ions. The interaction constant $a$ follows from the sum of the contributions of all ions of the lattice, using a method originally introduced by Madelung. According to Eq.(5) of B&L it is then

$$a = 13.94\text{e}^2,$$

(39)

with the electron charge $\text{e}$. From (37) we arrive at

$$\Phi = \frac{a}{\delta} \cdot \left(\frac{\delta_0}{\delta}\right)^{n-1} - 1).$$

(40)

The exponent $n$ was determined by B&L with the help of the compression modulus $\kappa$ given by Eq.(4) of B&L

$$\kappa = 9\delta_{0}^{4}/a(n - 1).$$

(41)

In mechanics the value of the compression modulus is known. We follow the same route as B&L and use, as we have done before in [34], for $\kappa$ the value which follows from the compression modulus of the nucleon $K_A = 900$ to $1200 \text{ MeV}$, as determined theoretically by Bhaduri, Dey and Preston [35]. The value of $K_A$ found by them is supported by other theoretical and experimental studies of $K_{NM}$ of nuclei. Setting $K_A = 1000 \text{ MeV}$ and using
the formula $K_A = 9/\rho\kappa$ from [35], where $\rho$ is the number density of nucleons per fm$^3$, we arrive with $r_p = 0.88\text{ fm}$ at

$$\kappa = 1.603 \cdot 10^{-35} \text{ cm}^2/\text{dyn}.$$  

(42)

From Eq.(41) follows $n - 1$ after we have replaced the electrostatic interaction $e^2$ in Eq.(39) with the weak interaction $g^2$, because the lattice is held together by the weak force. According to Perkins [16]

$$g^2 = 1.02 \cdot 10^{-5}(M_W/M_P)^2 \cdot 4\pi\hbar c \cong 2.946 \cdot 10^{-17} \text{ erg cm},$$

(43)

where $M_W$ is the mass of the W boson and $M_P$ is the mass of the proton. It then follows with $\delta_0 = 2r_0$, where $r_0$ is the range of the weak force $r_0 = 10^{-16}$ cm, that

$$n - 1 = \varepsilon = 2.187 \cdot 10^{-12},$$

(44)

which differs slightly from the value of $n - 1$ given in [34] because we now use $r_p = 0.88 \cdot 10^{-13}$ cm, which means that the nucleon number density has decreased. The potential of the weak force which holds the lattice together is then, neglecting a term on the order of $\varepsilon^2$,

$$\Phi = -\frac{13.94g^2}{\delta} \cdot \varepsilon [1 - \ln(\frac{\delta_0}{\delta})].$$

(45)

A graph $\Phi$ versus $\delta$ is in [34].

Born [36] has used lattice theory to explain the elasticity constant $c_{11}$ of cubic crystals which we need to calculate the rest mass of the muon neutrino from Eq.(29). In general $c_{11}$ must be proportional to the inverse of the compressibility $\kappa$ because of the relation

$$\frac{3}{\kappa} = c_{11} + 2c_{12}.$$ 

(46)

Considering the sum of the forces acting between the lattice points Born arrived at an equation for $c_{11}$ which we approximated in Eq.(17) of [34] by

$$c_{11} = \frac{0.538e^2(n - 1)}{r_0^4},$$

(47)

making use of the relation that $n = 1 + \varepsilon$ with $\varepsilon = O(10^{-12})$. For a cubic nuclear lattice we replace $e^2$ by $g^2$, use for $(n - 1)$ the value in Eq.(44), and
\[ r_0 \text{ is equal to } 10^{-16} \text{ cm. It follows that the nuclear elasticity constant } c_{11} \text{ is then} \]

\[ c_{11} = 3.466 \cdot 10^{35} \text{ dyn/cm}^2. \quad (48) \]

The uncertainty of the value of \( c_{11} \) is largely due to the uncertainty of \( n - 1 \) which is caused by the uncertainty of the value of the compression modulus \( \kappa \). Although the mass of the muon neutrino calculated from Eq.(29) with this value of \( c_{11} \) is about 50\% larger than the mass of the muon neutrino determined from the decay of the \( \mu \) meson it is important to be able to determine \( m(\nu_\mu) \) theoretically from lattice theory.

A side of a NaCl monocrystal cleaved in vacuum exerts a strong attractive short-range force on the other side of the cleaved crystal. So does the side of a cubic nuclear lattice exert an attractive force on the side of another cubic nuclear lattice. In the standing wave model the strong force is the sum of the weak forces originating from the lattice points at the sides of the lattice. From the potential of the weak force between the lattice points we can determine the force by which one side of a nuclear lattice attracts the side of another nuclear lattice. As follows from Eq.(45) the force between two cells of a cubic nuclear lattice separated by the distance \( \delta \) is given by

\[ \frac{d\Phi}{d\delta} = -\frac{13.94 g^2}{\delta^2} \varepsilon \ln(\frac{\delta_0}{\delta}). \quad (49) \]

The two cells will certainly not bond when the distance between the two cells exceeds the range of the weak nuclear force, if not at a smaller distance. The range of the weak force is given by \( a = 10^{-16} \text{ cm} \), and \( \delta_0 = 2a \). So the force required to separate the surfaces of two cubic lattice cells is at least

\[ F_w = -\frac{13.94 g^2}{a^2} \varepsilon \ln 2 = -9.66 \frac{g^2 \varepsilon}{a^2}. \quad (50) \]

The force required to separate two sides of an entire cubic nuclear lattice or vice versa the force by which one side of a nuclear lattice attracts the side of another nuclear lattice is then at least

\[ F_s = -4.8 \cdot 10^6 \frac{g^2 \varepsilon}{a^2} \text{ dyn}, \quad (51) \]

if there are \( 2 \cdot 10^6 \) lattice points or \( 0.5 \cdot 10^6 \) elementary cubic cell surfaces at the side of the lattices.
That means that the ratio of the strong nuclear force to the weak nuclear force is on the order of $10^6$. An accurate value of the ratio of the strong and weak nuclear forces is apparently not known from the experiments other than that it is said that the ratio of the strong nuclear interaction constant $\alpha_s$ to the weak interaction constant $\alpha_w$ is $\alpha_s \approx 10^6 \alpha_w$.

**Conclusions**

It is very natural to assume that the mesons and baryons consist of the particles into which they decay. From the well-known decays follows that the particle spectrum consists of a $\gamma$-branch and a neutrino branch. We also take for granted that the elementary integer multiple rule for the ratios of the masses of the mesons and baryons is a consequence of the structure of the particles. We take it for granted that Fourier analysis holds in high energy collisions of the particles, which means that a continuum of high frequencies must be present in the collision products. And we believe that the consensus of opinion that the weak nuclear force has a range on the order of $10^{-16}$ cm is realistic. On these points we base our model of the mesons and baryons.

We have studied cubic nuclear lattices consisting of either photons or neutrinos whose lattice points are $10^{-16}$ cm apart. After a high energy collision the lattice points must oscillate and thereby satisfy the requirement that a continuous spectrum of high frequencies must be present in the mesons and baryons. The oscillations are in the form of standing waves. Standing waves do not have to propagate, so the particles can have a rest mass although the particles of the $\gamma$-branch consist of electromagnetic waves. The frequency spectrum of the waves is determined by the group velocity which cannot exceed the velocity of light. The sum of the energies contained in the frequencies of all standing waves determines the mass of the particles. Because of the group velocity limitation the energy of the different oscillation modes or superpositions of modes are integer multiples of the basic mode. The masses of the particles of the $\gamma$-branch are therefore integer multiples of the basic mode or of the $\pi^0$ meson, in agreement with what the ratios of the particle masses strongly suggest. Each particle has automatically an antiparticle. The size of the particles is limited by the radiation pressure to about $10^{-13}$ cm. The absolute value of the energy of the $\pi^0$ meson is that of a cubic black body filled with standing electromagnetic waves whose energy is determined by Planck’s classical formula for the energy of a linear oscilla-
The masses of the particles of the $\gamma$-branch can thus be explained using photons only. A very conservative explanation which avoids the introduction of any new particle.

As the decay of the neutrino branch particles suggests we assume that they consist of muon and electron neutrinos and their antiparticles. The oscillations of the neutrino lattice contain the continuum of frequencies which must be in the $\nu$-branch particles after their creation. The energy in the $\nu$-branch particles is not only the energy of all lattice oscillations but is the sum of the energy in the neutrino lattice oscillations and the energy in the rest masses of the $10^9$ neutrinos making up the lattice. The $\pi^\pm$ mesons consist of the four types of neutrinos and their oscillation energies, plus electric charge. Their mass so determined is $0.98 m(\pi^\pm)(\text{exp})$. On the other hand, a $\pi^0$ meson has to be superposed on the $(2.2)$ mode of the $\pi^\pm$ meson oscillations in order to explain the mass of the $K^\pm$ mesons. Then it is natural that $\pi^0$ mesons appear in $29\%$ of the $K^\pm$ decays. For the explanation of the higher modes of the particles of the $\nu$-branch it is necessary to consider neutrinos and standing electromagnetic waves and charge.

The mass of the $\mu$ mesons can also be explained with a neutrino lattice. The $\mu$ mesons consist of the neutrinos remaining after the decay of the $\pi^\pm$ neutrino lattice. All muon neutrinos of one type, either the muon neutrinos or the anti-muon neutrinos, are removed from the neutrino lattice of the $\pi^\pm$ mesons in its decay. That releases the sum of the energy of their rest masses which is $\cong 1/4 m(\pi^\pm)c^2$. Since it can be shown that the oscillation energy of all neutrinos in the $\pi^\pm$ mesons is conserved in the decay, the energy of the rest mass of the $\mu^\pm$ mesons is $\cong 3/4 m(\pi^\pm)c^2$, in near agreement with the exact ratio $m(\mu^\pm)/m(\pi^\pm) = 0.757028$. From the decay of the $\pi^\pm$ mesons follows that the rest mass of the muon neutrino is $m(\nu_\mu) = 47.5$ milli-eV/c$^2$. From the decay of the neutron follows that the rest mass of the electron neutrino is $m(\nu_e) = 0.55$ meV/c$^2$.

Born’s lattice theory provides the means to determine the potential of the force which extends from one lattice point to the next and thereby holds the lattice together. Following Born’s approach exactly, replacing only $e^2$ by the weak interaction constant $g^2$, we learn that at the lattice distance $a$ the repulsive term of the potential differs from the attractive term by only $10^{-12}$. We also learn that from the weak force in the lattice follows automatically the existence of the strong nuclear force emanating from the sides of the lattice, caused by the $10^6$ unsaturated weak forces at a side of the lattice.

We conclude that the standing wave model solves a number of problems
for which an answer heretofore has been hard to come by. Only photons and neutrinos and charge are needed to explain the stable mesons and baryons. In a forthcoming paper we will show that the spin of the baryons can also be explained with the standing wave model.

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**References**

[1] Gell-Mann,M. Phys.Lett.B. **111**,1 (1964).
[2] Barnett,R. et al. Rev.Mod.Phys. **68**,611 (1996).
[3] Witten,W. Physics Today **X**,24 (1996).
[4] Koschmieder,E.L. Bull.Acad.Roy.Belgique **X**,281 (1999).
  arXiv: [hep-ph/0002179](https://arxiv.org/abs/hep-ph/0002179) (2000).
[5] Koschmieder,E.L. Nuovo Cim. **99**,555 (1988).
[6] Koschmieder,E.L. and Koschmieder,T.H. Bull.Acad.Roy.Belgique **X**,289 (1999).
  arXiv: [hep-lat/0002016](https://arxiv.org/abs/hep-lat/0002016) (2000).
[7] Wilson,K. Phys.Rev. **D10**,2445 (1974).
[8] Weingarten,D. Scient.Am. **274**,116 (1996).
[9] Born,M. and v.Karman,Th. Phys.Z. **13**,297 (1912).
[10] Blackman,M. Proc.Roy.Soc. **A148**,365;384 (1935).
[11] Blackman,M. Handbuch der Physik VII/1, Sec.12 (1955).
[12] Born,M. and Huang,K. *Dynamical Theory of Crystal Lattices*, (Oxford) (1954).
[13] Maradudin,A. et al. *Theory of Lattice Dynamics in the Harmonic Approximation*, Academic Press, 2nd edition, (1971).
[14] Ghatak,A.K. and Khotari,L.S. *An introduction to Lattice Dynamics*, Addison-Wesley, (1972).
[15] Schwinger,J. Phys.Rev. **128**,2425 (1962).
[16] Perkins,D.H. *Introduction to High-Energy Physics*, Addison Wesley, (1982).
[17] Rosenfelder,R. arXiv: [nucl-th/9912031](https://arxiv.org/abs/nucl-th/9912031) (2000).
[18] Born, M. Proc. Camb. Phil. Soc. 36, 160 (1940).
[19] Koschmieder, E. L. arXiv: hep-lat/0005027 (2000).
[20] Liesenfeld, A. et al. Phys. Lett. B 468, 20 (1999).
[21] Bernard, V., Kaiser, N. and Meissner, U.-G. arXiv: nucl-th/0003062 (2000).
[22] Sommerfeld, A. Vorlesungen über Theoretische Physik, Bd. V, p. 56 (1952).
[23] Debye, P. Ann. d. Phys. 39, 789 (1912).
[24] Bose, S. Zeitschr. f. Phys. 26, 178 (1924).
[25] Koschmieder, E. L. and Koschmieder, T. H. arXiv: hep-lat/0104016 (2001).
[26] Thirring, H. Phys. Z. 15, 127. (1914).
[27] Bethe, H. Phys. Rev. Lett. 58, 2722 (1986).
[28] Bahcall, J. N. Rev. Mod. Phys. 59, 505 (1987).
[29] Fukuda, Y. et al. Phys. Rev. Lett. 81, 1562 (1998).
[30] Ahmad, Q. R. et al. Phys. Rev. Lett. 87, 071301 (2001).
[31] Koschmieder, E. L. arXiv: physics/0110005 (2001).
[32] Born, M. and Stern, O. Sitzungsber. Preuss. Akad. Wiss. 33, 901 (1919).
[33] Born, M. and Landé, A. Verh. Dtsch. Phys. Ges. 20, 210 (1918).
[34] Koschmieder, E. L. Nuovo Cim. 101, 1017 (1989).
[35] Bhaduri, R. K., Dey, J. and Preston, M. A. Phys. Lett. B 136, 289 (1984).
[36] Born, M. Ann. Phys. 61, 87 (1920).