Charge and Current in the Quantum Hall Matrix Model

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We extend the quantum Hall matrix model to include couplings to external electric and magnetic fields. The associated current suffers from matrix ordering ambiguities even at the classical level. We calculate the linear response at low momenta – this is unambiguously defined. In particular, we obtain the correct fractional quantum Hall conductivity, and the expected density modulations in response to a weak and slowly varying magnetic field. These results show that the classical quantum Hall matrix models describe important aspects of the dynamics of electrons in the lowest Landau level. In the quantum theory the ordering ambiguities are more severe; we discuss possible strategies, but we have not been able to construct a good density operator, satisfying the pertinent lowest Landau level commutator algebra.

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I. INTRODUCTION

It is well established that the low-energy effective Lagrangian for a quantum Hall (QH) system is a Chern-Simons (CS) gauge theory. For the simplest case of the Laughlin states with filling fraction \( \nu = 1/(2m + 1) \) it takes the form,

\[
\mathcal{L} = \frac{k}{4\pi} \epsilon^{\mu \nu \lambda} A_\mu \partial_\nu a_\lambda - \frac{e}{2\pi} \epsilon^{\mu \nu \lambda} A_\mu \partial_\nu a_\lambda, \tag{1}
\]

where \( k = 1/\nu \), and \( A_\mu \) is the external electromagnetic potential. This Lagrangian (and its generalizations to other fractions, multi-component systems \textit{etc.}) correctly describes the quantum Hall conductance, and also the ground state degeneracies on higher genus surfaces and edge excitations\(^{1} \). One can also naturally incorporate quasiparticle currents and edge modes with appropriate quantum numbers.

There are, however, important aspects of the QH liquids that are not incorporated in the CS description. For the Laughlin fractions described by \( \mathbb{I} \) the quantization of the QH conductance corresponds to quantization of \( k \), which does not follow from any general principle but should be regarded as an input.\(^{2,9}\) For hierarchical states, the parameter \( k \) is replaced by a matrix \( K \) with quantized entries which again are input parameters. Furthermore, it is difficult, within the effective CS framework, to account for the successes of the composite fermion model and to single out the experimentally prominent Jain fractions \( \nu = n/(2m \pm 1) \).

There have been several recent attempts to formulate effective theories for the composite fermions, based on various CS mean field theories\(^{10} \) and in these approaches the constraint that keeps the electrons in the lowest Landau level (LLL) is very important. On the level of the effective theories this is manifested in the commutation relation

\[
[\rho_\ell, \rho_{\ell'}] = 2i \sin \left( \frac{\ell^2}{2} (\vec{k} \times \vec{p}) \right) e^{i \frac{\ell^2}{2} (\vec{k} \times \vec{p})} \delta_{\ell, \ell'}, \quad \ell = (\hbar/eB)^{1/2}, \tag{2}
\]

for the Fourier components of the LLL-projected density operator \( \rho_\ell \) and in a dipole form for the quasiparticle current operator. Neither of these properties is captured by effective CS theories like \( \mathbb{I} \). For instance, from \( \mathbb{I} \) we get \( \rho(x) = j^0 = \frac{1}{2\pi} e^{i\phi} \partial_i a_j = \frac{1}{2\pi} b \), hence the commutator \( \mathbb{J} \) vanishes \( [\rho_\ell, \rho_{\ell'}] = -4\pi i \hbar \nu (\vec{p} \times \vec{k}) \delta(\ell + \ell') = 0 \).

In view of the above, the recently proposed matrix model for the QH effect is intriguing\(^{12} \). This model – which can be shown to be a noncommutative version of the effective CS theory \( \mathbb{I} \) – has several appealing features. The quantization of \( k \) follows from basic principles\(^{12} \) and the specific values corresponding to the Laughlin fractions follow by requiring the underlying particles to be fermions, much as in the microscopic Chern-Simons Ginzburg-Landau approach\(^{12} \). The classical ground state of the matrix model is an incompressible homogeneous state at the Laughlin filling fractions, and the model also reproduces several important aspects of fractional QH physics, such as the quasihole charge and statistics, and the presence of edge states.

In spite of all this the matrix models have so far not added anything to our understanding of QH physics – the Laughlin states are very well understood – primarily through the Laughlin wave functions, but also from various Chern-Simons mean field theories. The matrix models will be of real physical interest only if they can be generalized to describe other states. As discussed above, an important challenge is to understand the Jain fractions, for which there are very good wave functions, and where the mean field theory explicitly involves "effective fermions" filling auxiliary higher Landau levels.
We think that a generalized matrix model might be useful, since in the existing matrix model it is enough to specify that the original particles are fermions in order to get the correct Laughlin fractions. In Laughlin’s theory it is the combination of fermi statistics and LLL dynamics that determines the wave functions, so, by analogy, we hope that the matrix model encodes the correct LLL dynamics. There are already some indications that this is the case. For a vanishing non-commutativity parameter $\theta$ it is obviously true by construction, and it has also been shown that a certain class of solutions to the finite-dimensional matrix model describes independent point particles moving in a strong magnetic field at distances larger than $\theta/\ell$ where $\ell$ is the magnetic length. It is, however, far from clear that the matrix model captures the full LLL dynamics in general, and in particular for the incompressible states describing the Laughlin liquids. It is the purpose of this paper to study this question, and in particular to construct a conserved electromagnetic current that will allow us to calculate response functions and to see if the charge commutator algebra is satisfied.

We proceed by constructing, in the classical matrix model, a particle density and a corresponding conserved current density which are then coupled minimally to the external electromagnetic potential. The densities are complicated matrix functions, and furthermore, they are not unique – the noncommutativity of the matrices leads to an ordering ambiguity already at the classical level.

Using the infinite dimensional matrix model, which describes the Laughlin states, we calculate the response to weak perturbing fields using linear response. This amounts to calculating the ground state density, the QH response and the density response to weak and slowly varying magnetic fields. We show that the correct QH results are obtained for any choice of matrix ordering in the definition of the densities.

We then turn to the quantum theory and discuss the construction of the density and current operators. There are several constraints on the correct construction: the density must satisfy the commutation relations, be properly normalized (i.e. reproduce the correct ground state density for homogeneus states), and in the commutative limit it must reproduce the physics of non-interacting particles in the LLL. The problem of constructing a density and a corresponding conserved current density now involves both a matrix and a quantum ordering problem, and is, in general, very involved.

Our aim is to make this paper fairly self-contained, and accessible to the condensed matter community, so we start with a short review of the QH matrix models in the next section. The construction of the classical conserved current and the corresponding Lagrangian is in section 3, and the calculation of the linear response in section 4. The quantization and the discussion of the charge commutation relations is in section 5. We conclude in section 6 with a summary of our results, a guess, and some open problems. Some of the material in this paper was presented in.

II. THE QUANTUM HALL MATRIX MODEL

The original version of the matrix model, proposed by Susskind, describing particles of charge $-e$ in a constant magnetic field $B_z = B_{z0}$ is given by the Lagrangian

$$L_0 = \frac{eB}{2} \text{tr} \left\{ \left( X^a_{\text{m}} - i [X^a_{\text{m}}, \hat{a}_0] \right) \epsilon_{ab} X^b_{\text{m}} + 2 \theta \hat{a}_0 \right\},$$

(3)

where $X^a(t)$, $a = 1, 2, 3$ and $\hat{a}_0(t)$ are Hermitian matrices – the latter is a Lagrange multiplier imposing a constraint. The area $\theta$ that enters $L_0$ is the non-commutativity parameter, see below, and in the classical model the average density is given by $2\pi \theta$. Quantization introduces $\hbar$ and the magnetic length $\ell = (\hbar/eB)^{1/2}$. Since $2\pi \ell^2$ is the area per state in the lowest Landau level, the filling fraction $\nu$ will be related to the dimensionless parameter $\theta/\ell^2$. For quantum Hall states, where $\nu$ is some odd denominator fraction, $\theta$ will be $O(\hbar)$ and sensitive to quantum corrections. In fact, as discussed briefly below, the relation is $\nu^{-1} = \theta/\ell^2 + 1$.

The Lagrangian (3) is formally very similar to the Lagrangian for one particle moving in a strong perpendicular magnetic field $L_{1p} = \frac{eB}{2} \epsilon_{ab} \dot{x}^a x^b$, where $x^a(t)$ is the position of the particle. In fact, when $\theta = 0$, (3) reduces, upon solving the constraint, to $N$ copies of $L_{1p}$ ($N$ being the dimension of the matrices $X^a$). We note that, when including a potential $V(\vec{x})$, the equations of motion obtained from $L_{1p}$ give the drift current $\dot{x}^a = \frac{e}{\hbar} \epsilon^{ab} \partial_b V(\vec{x})$, but not the cyclotron motion. This is as expected since $L_{1p}$ corresponds to infinite cyclotron frequency and hence vanishing cyclotron radius for finite velocity.

As mentioned in the introduction, there is a close connection between the matrix model and the CS theory in that the matrix model is equivalent to a non-commutative generalization of the latter. This is how Susskind arrived at. He noted that in a hydrodynamical description of the QH fluid one obtains an abelian CS theory with additional non-linear terms. He then proposed the non-commutative abelian CS theory – which to lowest nontrivial order in the non-commutativity parameter $\theta$ agrees with the non-linear abelian CS theory – as the effective QH theory.

Varying the Lagrangian (3) with respect to the multiplier $\hat{a}_0$ yields the constraint

$$[X^1, X^2]_{\text{m}} = i\theta,$$

(4)
where the subscript \( "m" \) denotes a matrix, as opposed to a quantum, commutator. Taking the trace of (4) we see that \( \text{Tr}(X^1X^2) \neq \text{Tr}(X^2X^1) \) if \( \theta \neq 0 \). Thus the constraint can only be solved by infinite matrices and, as is well known from quantum mechanics, the solution is essentially unique. The theory is hence rather trivial – there is only one state, corresponding to an incompressible fluid with constant density \( \rho = 1/2\pi \theta \). In order to find other solutions, one must modify the constraint (4). This can be done either by introducing external sources, or by introducing a new dynamical field that couples to \( \hat{a}_\mu \). The latter approach has the advantage of using finite matrices, which means that all the ordinary rules for matrix manipulations, such as the cyclicity of the trace, can be used. The finite matrix model, due to Polychronakos, is obtained by adding the following piece to the Lagrangian (4):

\[
L_b = \Psi^\dagger (i\partial_0 - \hat{a}_0)\Psi ,
\]

where \( \Psi \) is a complex bosonic \( N \)-vector. The \( X^a \)'s in (4) are now Hermitian \( N \times N \) matrices and the constraint (4) is modified into

\[
[X^1, X^2]_m = i\theta - \frac{i}{eB} \Psi \Psi^\dagger .
\]

Taking the trace gives \( \Psi^\dagger \Psi = NeB\theta \), which allows for finite \( N \) solutions also when \( \theta \neq 0 \).

The Lagrangian \( L = L_0 + L_b \) is invariant under the \( U(N) \) gauge transformations

\[
\begin{align*}
\Psi &\to U\Psi \\
X^a &\to UX^aU^\dagger \\
\hat{a}_0 &\to U\hat{a}_0U^\dagger - Ui\partial_0U^\dagger,
\end{align*}
\]

where \( U \) is a \( U(N) \)-matrix.

This classical model was solved in\(^7\). Identifying the eigenvalues of the matrices \( X^1 \) and \( X^2 \) as the positions and momenta of the \( N \) particles respectively, one finds that when the particles are far apart, compared to \( \theta/\ell \), they move as \( N \) independent particles in a strong perpendicular magnetic field (\( i.e. \) governed by \( L_{1p} = \frac{\epsilon B}{2\pi} \varepsilon_{ab}x^ax^b \)). On the other hand, when a potential \( \propto (X^n)^2 \) is added – this attracts the particle to the origin – the ground state is indeed a circular droplet with constant bulk density \( \rho \sim 1/2\pi \theta \). The excitation spectrum is that of the Calogero model – in particular this means that the low lying excitations can be interpreted as surface modes.

The most interesting properties of the matrix models are apparent only after quantization. As usual for gauge theories, there are two ways to handle the constraint. In the first, which is similar to the Gupta-Bleuler quantization of gauge theories, one quantizes the full extended phase space to get the simple canonical commutation relations,

\[
\begin{align*}
[X^1_{mn}, X^2_{rs}] &= \frac{i\hbar}{eB} \delta_{ms} \delta_{nr} \\
[\Psi_m, \Psi_n^\dagger] &= i\hbar \delta_{mn},
\end{align*}
\]

and then implement the constraint (1) or (3) as a projection operator to define physical states. We shall refer to this method as unconstrained quantization.

Alternatively, one proceeds like in Coloumb gauge quantization of a gauge theory, and first solves the “Gauss’ law” constraint (1) or (3) in some suitably chosen gauge (\( e.g. \) \( X^1 \) diagonal). Writing the Lagrangian in terms of the physical variables one can then read off the canonical commutation relations. In this approach, there is no projection – all states are physical – but the commutators between the elements of the matrices become complicated. We shall refer to this method as constrained quantization.

The finite matrix model also gives an unambiguous relation between the filling fraction \( \nu \) and the Chern-Simons level \( \theta/\ell^2 \). This was originally derived by Polychronakos by mapping the matrix model onto the quantum Calogero model with a coupling constant \( \nu^{-1}(\nu^{-1} - 1) \), where \( \nu^{-1} = \theta/\ell^2 + 1 \). From the known ground state of the Calogero model he could then identify \( \nu \) with the filling fraction. The quantum shift in the relation between \( \nu^{-1} \) and the level \( \theta/\ell^2 \) is due to the zero point motion of the particles, as explained by Hellerman and Susskind\(^1\). Assuming the original particles to be fermions, one can furthermore show that the filling fraction is quantized to the Laughlin values \( \nu^{-1} = 2m + 1 \).

We should also mention that invariance under large gauge transformations requires the Chern-Simons level \( k \) to be an integer independent of the physical interpretation of the theory\(^5\). For further discussion of the QH matrix models we refer to\(^6,7,11,12,13\).

### III. A GENERALIZED MATRIX MODEL

We now generalize the matrix model (3) and (5) to include coupling to an external electromagnetic field, \( A_\mu \). We use the more general finite dimensional model, \( L = L_0 + L_b \), which has the additional advantage of allowing...
cyclic permutations in the trace. At the end of the section we comment on the infinite dimensional case – making explicit the modifications this introduces. Since a large constant magnetic field \( B \), perpendicular to the plane, is already incorporated in the model, we let \( \delta B \) denote the magnetic field corresponding to \( A_\mu \), and assume that it has no constant component. To preserve gauge invariance, we must construct a conserved current from the matrix variables. It is useful to remember how to treat a single particle moving along the trajectory \( \vec{x}(t) \). Here the particle and current densities are given by \( \rho(\vec{y}, t) = \delta(\vec{y} - \vec{x}(t)) \) and \( \vec{j}(\vec{y}, t) = \dot{\delta}(\vec{y} - \vec{x}(t)) \), respectively. Current conservation \( \nabla \cdot \vec{j} + \partial_t \rho = 0 \) then follows immediately using the chain rule. Using a \( \delta \)-function to define the particle and current densities corresponds to a point particle – but in a non-relativistic theory, one can in fact use any profile function \( f(\vec{y} - \vec{x}(t)) \), corresponding to a rigid charge distribution moving with velocity \( \dot{\vec{x}}(t) \).

In the matrix model, the diagonal elements of the matrices, \( X_{n,m} \), can, in a suitable gauge, be interpreted as the guiding center coordinates for the particles when these are far apart compared to \( \theta/\vec{x} \). With this in mind it is natural to define the particle density in the matrix model as

\[
\rho(\vec{y}, t) = \text{Tr}[\hat{\delta}(y^a - X^a(t))] ,
\]

where \( \hat{\delta}(y^a - X^a) \) is a matrix-valued kernel. With the same picture in mind, we make the following guess for the current density related to the guiding center motion,

\[
\vec{j}(\vec{y}, t) = \text{Tr}[ (\vec{X} - i\vec{[\vec{X}, \hat{\alpha}_0]}) \hat{\delta}(y^a - X^a(t))] .
\]

(In the presence of an inhomogeneous magnetic field, there is an additional, purely solenoidal, contribution to the current related to the uncancelled cyclotron motion of neighbouring electrons, which will be discussed below.)

The densities (9) and (10) are invariant under the \( U(N) \)-transformations – provided only that the kernel transforms covariantly: \( \hat{\delta} \rightarrow U \hat{\delta} U^\dagger \). However, since \( X^a \) are non-commuting objects there are ordering ambiguities already at the classical level, leading to inequivalent classical theories. For a kernel of the form

\[
\hat{\delta}(y^a - X^a) = \int \frac{d^2 k}{(2\pi)^2} f(k_a(y^a - X^a), \theta k^2) ,
\]

where \( f(z, x) \) is dimensionless and analytic in \( z \), (11) does indeed give a conserved current. The proof is easy:

\[
\partial_t \text{Tr}[k_a(y^a - X^a)]^n = -\text{Tr} \sum_{m=0}^{n-1} [k_b(y^b - X^b)]^m k_a X^a [k_c(y^c - X^c)]^{n-m-1} = -n \text{Tr} k_a X^a [k_b(y^b - X^b)]^{n-1} = -\partial_{y^a} \text{Tr} X^a [k_b(y^b - X^b)]^n
\]

and

\[
\partial_{y^a} \text{Tr} \left( [X^a, \hat{\alpha}_0] [k_b(y^b - X^b)]^n \right) = n \text{Tr} ([k_a X^a, \hat{\alpha}_0] [k_b(y^b - X^b)]^{n-1} = -n \text{Tr} ([k_a X^a, [k_b(y^b - X^b)]^{n-1}, \hat{\alpha}_0]) = 0 ,
\]

where we used the cyclic property of the trace. One kernel of the form (11) is

\[
\hat{\delta}_W(y^a - X^a) = \int \frac{d^2 k}{(2\pi)^2} e^{ik_a(y^a - X^a)} g(\theta k^2) ,
\]

where \( g(0) = 1 \). The special case \( g(x) = 1 \) is known as Weyl-ordering. The corresponding densities become, in the \( \hat{\alpha}_0 = 0 \) gauge,

\[
\rho_k^a = \text{Tr} \left( e^{-ik_a X^a} \right) g(\theta k^2) \quad (14)
\]

\[
\vec{j}_k^a = \text{Tr} \left( \vec{X}^a e^{-ik_a X^a} \right) g(\theta k^2) .
\]

Note the formal similarity to the one-particle densities.

However, more general \( O(2) \)-invariant \( - \) and \( U(N) \)-covariant \( - \) kernels of the form

\[
\hat{\delta}(y^a - X^a) = \int \frac{d^2 k}{(2\pi)^2} f(k_a(y^a - X^a), \epsilon_{abc} k^a(y^b - X^b), \theta k^2) \]

(15)
are possible. (The kernels are assumed to be Hermitian.) For such kernels, \([10]\) does not generally give a conserved current. However, using \([9]\) and current conservation it is straightforward to define a current as an expansion in the matrices \(\hat{X}^a\), \(k^a(y^a - X^a)\) and \(\epsilon_{ab}k^a(y^b - X^b)\) that is conserved. An important example of this type is the anti-ordered kernel
\[
\delta_{\alpha\beta}(z - Z, \bar{z} - Z\dagger) = \int \frac{d^2k}{(2\pi)^2} e^{i\frac{k}{2}(z - Z)} e^{i\frac{\bar{k}}{2}(\bar{z} - Z\dagger)},
\]
(16)
where \(z = y^1 + iy^2\), \(k = k^1 + ik^2\) and \(Z = X^1 + iX^2\). It turns out that the classical limit of the charge density operator in the quantized matrix model discussed below is given by this kernel. The explicit expression for the corresponding conserved current in the \(\hat{a}_0 = 0\) gauge is, using the cyclic property of the trace,
\[
j(z, \bar{z}) = j_x + ij_y = \text{Tr} \left[ \hat{Z} \int \frac{d^2k}{(2\pi)^2} e^{i\frac{k}{2}(z + k\bar{k})} \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \frac{1}{n!} \left(-\frac{i}{2} \bar{Z}\right)^{n-m-1} e^{-\frac{i}{2} \bar{Z}\dagger \left(-\frac{i}{2} \bar{Z}\right)^m} \right].
\]
(17)

Having defined a particle density and a conserved current it is straightforward to couple the matrix model \(L_0 + L_b\) in \([8]\) and \([5]\) to an external electromagnetic field by adding
\[
L_{int} = e \int d^2x \{ \rho(\vec{x}, t)A_0(\vec{x}, t) - j(\vec{x}, t) \cdot \vec{A}(\vec{x}, t) \}.
\]
(18)
We can cast the resulting Lagrangian in a more transparent form by noting that the current always can be written in the form \([10]\) although in general the kernel defining the current will differ from the one defining the charge. For example, the kernel defining the current \(j\) corresponding to the anti-ordered charge kernel, can be read directly from \([17]\) replacing the ordinary time derivative with the corresponding covariant derivative. The resulting Lagrangian is,
\[
L = e \text{Tr}[ \left( \hat{X}^a - i[X^a, \hat{a}_0]_m \right) \left( \frac{B}{2} \epsilon_{ab}X^b - \hat{A}_a \right)] + B\hat{a}_0 + \hat{A}_0 - \frac{\hbar}{2M} \delta \hat{B} + \Psi \dagger (i\hat{\partial}_0 - \hat{a}_0) \Psi.
\]
(19)
Here the matrices \(\hat{A}_\mu(X^a, t), \mu = 0, 1, 2\), are related to the ordinary gauge potential using the kernel \(\delta_{(\mu)}\):
\[
\hat{A}_\mu(X^a, t) = \int d^2x A_\mu(x^a, t)\delta_{(\mu)}(X^a - X^a),
\]
(20)
and the label \(\mu\) on the kernel reminds us that it in general depends on which component of \(A_\mu\) it multiplies. In \([19]\) we have also included a term \(\sim \delta \hat{B}(X^a, t)\), where \(\delta \hat{B}\) is related to the magnetic field \(\delta B(x^a, t)\), caused by \(A_\mu(x^a, t)\), via \(\delta\) as in \([20]\). \(M\) is the effective mass of the electrons. This is the Simon-Stern-Halperin “magnetic moment” interaction that encodes the variation in the Landau energy in a varying magnetic field\([14]\). It gives an additional solenoidal term in the current – due to the non-cancellation of the cyclotron motion currents – that is not included in \([10]\).

Note that the charge and current density operators derived from \([19]\) do not have an explicit dependence on \(\Psi\), but that there is an implicit dependence since \(\Psi\) enters the constraint and thus determines the form of the solutions for the matrices \(X^a\).

It is clear by inspection that the Lagrangian \([19]\) is invariant under the noncommutative gauge transformation \([7]\), and the symmetry under the usual electromagnetic gauge transformation \(\delta A_\mu(x) = \delta_\mu A(x)\) is ensured by construction since the corresponding current is conserved. For general kernels the gauge variation of the matrices \(\hat{A}_\mu\) is rather complicated, but for the special case of Weyl ordering it takes a simple form very reminiscent of the commutative case\([22]\).

In the extended model \([19]\), the constraint corresponding to \([9]\) is modified and depends on \(A_a\):
\[
[X^a, \epsilon_{ab}X^b - \frac{2}{B} \hat{A}_a]_m = 2i\theta - \frac{2i}{eB} \Psi \Psi \dagger,
\]
(21)
and we note that contrary to \([9]\) this does not determine the commutator between \(X^1\) and \(X^2\). However, if we pick the gauge \(\hat{A}_2 = 0\), then we can rewrite the constraint as
\[
[X^1, X^2 - \frac{1}{B} \hat{A}_1]_m = [X^1, \frac{1}{eB} P^1]_m = i\theta - \frac{i}{eB} \Psi \Psi \dagger,
\]
(22)
where $P^1$ is the momentum conjugate to $X^1$ as calculated from the Lagrangian (15). Clearly, the above construction can be carried out in any linear gauge, but we do not know how to handle a general gauge. Note that the simple commutation relation (22) is between $X^1$ and $P^1 = eB X^2 - eA_1$, not between $X^1$ and $X^2$, and this makes it hard to evaluate the densities (9) and (15). However, as long as we are interested only in linear response we can expand the expression for the density as will be shown in the next section. That the constraint becomes simple when expressed in canonically conjugate variables, is in fact very important since it assures that the arguments given in (12) and (17) – which does not hold for infinite matrices. Also, defining the particle density as in the finite dimensional case, $\rho$, one however finds that the constraint still has the form (21) (with $\Psi = 0$), where $\hat{\Psi}(\mu)$ is the same as for finite matrices. The point is that although trace manipulations using cyclicity are not allowed on the level of the Lagrangian, they are permissible in the equations of motion if we (as customary) assume that the variation $\delta \hat{a}_0$ of the multiplier matrix has compact support (15).

We now comment on the case of the infinite dimensional matrix model (12) and its coupling to an electromagnetic field. The above analysis used the cyclicity of the trace – e.g. in (12) and (17) – which does not hold for infinite matrices. Also, defining the particle density as in the finite dimensional case, $\rho$ (with $\Psi = 0$), where $\hat{\rho}$ is given by (20) and the kernel $\hat{\delta}(\mu)$ is the same as for finite matrices. The point is that although trace manipulations using cyclicity are not allowed on the level of the Lagrangian, they are permissible in the equations of motion if we (as customary) assume that the variation $\delta \hat{a}_0$ of the multiplier matrix has compact support (15).

In the simplest case when the constraint takes the form $[X^1, X^2]_{m} = i \theta \gamma$, then (12) – and probably any sensible kernel (15) – can be written as a generalized Weyl kernel (13). Also, if one assumes the Hamiltonian to be a scalar potential, then by using the equations of motion to replace $\hat{X}_i$ in (23) by an expression in $X_j$'s, one can see that all the considered kernels $\hat{\delta}(\mu)$ are in the infinite case related by factors of the type $g(\theta k^2)$ with $g(0) = 1$. This observation will be useful in the next section.

IV. LINEAR RESPONSE IN THE CLASSICAL MATRIX MODEL

In this section we calculate the response of the infinite dimensional classical matrix model to electromagnetic perturbations in the linear approximation. We remind the reader that this model describes only one state with constant density $\rho = 1/2\pi \theta$ – nevertheless we obtain non-trivial results.

We shall consider three quantities, the quantum Hall response to a constant electric field, the ground state density, and the response to a weak, static but slowly modulated magnetic field. The two first examples involve response at $\vec{k} = 0$ and the last at small $\vec{k}$, i.e. $k^2 \theta \ll 1$. From the above discussion of the infinite matrix model it follows that the classical ordering ambiguities will be of no relevance for these quantities since they amount to correction terms $\sim k^2 \theta$. For simplicity, we use the generalized Weyl-kernel (13), but any choice of the form (11) would do.

A. The quantum Hall response

We calculate the current response to an applied constant electric field of the form $A_0 = -E_0 x^a$. To evaluate the current, we first need to determine $\hat{X}^a$ from the equations of motion, a problem very similar to the QH droplet in a quadratic matrix potential considered by Polychronakos (24). Varying the Lagrangian with respect to $X^a$, yields

$$B \hat{X}^a = e^{ab} E_0 1.$$  \hspace{1cm} (25)
Because of $X^a \propto 1$, it follows from expressions like (11) or from the discussion at the end of section III, that the electromagnetic current density is given by

$$J^a_k = -ej^a_k = -e^{ab}E_bB\rho^a_k,$$

and for $\rho = \nu/2\pi\ell^2 = \nu eB/h$ a Fourier transformation yields

$$J^a = -\nu e^2\ell_{ab}E_b = -\sigma_H e^{ab}E_b.$$  

\(\)From refs.6,7 we know that if the particles described by the matrix model are fermions, then $\nu^{-1} = 2m + 1$ and 24 gives the Laughlin values for the quantized Hall conductance $\sigma_H$.

### B. The ground state density

The response to a constant external electric potential $\delta A_0$ will simply give the density of the system

$$\rho(\vec{x}) = \frac{\partial L}{\partial A_0} = \text{Tr} \delta(x^a - X^a).$$

In the finite case this is difficult to calculate because of the presence of $\Psi$ in the constraint, but in the infinite case the calculation can be conveniently done by using coherent states. Defining eigenstates of the lowering operator

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \text{where} \quad \alpha = \frac{1}{\sqrt{2\theta}}(X^1 + iX^2),$$

and using the commutator $[a, a^\dagger] = 1$ appropriate for the infinite matrix model in the absence of external fields, one has

$$\text{Tr} : O(a, a^\dagger): = \int \frac{d\alpha d\bar{\alpha}}{2\pi i} O(\alpha, \bar{\alpha}),$$

where $: :$ indicates normal ordering of the operator $O(a, a^\dagger)$.

For the Weyl-ordered kernel 13 (g = 1) we now have,

$$\text{Tr} e^{-ik_\alpha X^a} = \text{Tr} e^{-i\sqrt{\frac{k}{2}(ka + ka)}} = e^{\frac{ak_\alpha^2}{4}} \text{Tr} e^{-i\sqrt{\frac{k}{2}k^2Z^1} e^{-i\sqrt{\frac{k}{2}k^2Z^2}} =

= e^{\frac{ak_\alpha^2}{4}} \int \frac{d\alpha d\bar{\alpha}}{2\pi i} e^{-i\sqrt{\frac{k}{2}(ka + ka)}} = e^{\frac{ak_\alpha^2}{4}} \int \frac{d^2\alpha}{\pi} e^{-i\sqrt{\frac{k}{2}k^2\alpha^2}} = e^{\frac{ak_\alpha^2}{4}} \frac{2\pi}{\theta} \delta^2(\vec{k}) ,$$

where $k = k_1 + ik_2$. Substituting (21) in (13) and (24) we obtain the constant density $\rho = \frac{1}{2\pi\nu}$, as expected.

Notice that the factor $e^{ak_\alpha^2}$ does not affect the result because of $\delta^2(\vec{k})$. For the same reasons it follows from the discussion at the end of section III that the result is independent of the choice of kernel.

### C. Response to a weak perturbing B field

It is in general hard to calculate the response to an external magnetic field since, as explained above, 22 only gives an implicit relation for the matrix $A^\alpha(X)$. It can however be done for a weak, and slowly varying perturbing field of the form $\delta B(\vec{x}) = \epsilon B \sin(q_\alpha \vec{x})$.

In the $A^2 = 0$ gauge we have $A^1 = \epsilon B \cos(q_\alpha \vec{x})$, and the constraint 22 (with $\Psi = 0$). The annihilation operator then becomes

$$a = \frac{1}{\sqrt{2\theta}}(X^1 + i(X^2 - \frac{1}{B}A^1)),$$

and $k_\alpha X^a = \sqrt{\frac{2}{\theta}(ka + ka)} + \frac{k_2}{B}A^1$. By expanding in $\epsilon$ and $\theta k^2$, we obtain

$$\rho_k = \text{Tr} e^{-ik_\alpha X^a} = \text{Tr} [e^{-i\sqrt{\frac{k}{2}(ka + ka)}}(1 - \frac{ik_2}{B} A^1 + O(\epsilon^2))(1 + \epsilon O(\theta k^2))].$$

(33)
To evaluate
\[ \delta \rho_k = -\text{Tr}[e^{-i\sqrt{\frac{\theta}{2}}(\bar{q}a + qa)}(i\frac{k_2}{B}\hat{A}^1 + \epsilon O(\theta k^2) + O(\epsilon^2))], \] (34)
we first write
\[ \hat{A}_1(X) = \frac{\epsilon B}{2q_2} [e^{iqa.X^a} + e^{-iqa.X^a}] = \frac{\epsilon B}{2q_2} [e^{i\sqrt{\frac{\theta}{2}}(\bar{q}a + qa)} + e^{-i\sqrt{\frac{\theta}{2}}(\bar{q}a + qa)} + O(\epsilon)], \] (35)
and then calculate the trace with the same method as in subsection B above to get
\[ \delta \rho_k = -\frac{i\epsilon k_2}{2q_2} \text{Tr}[e^{-i\sqrt{\frac{\theta}{2}}(\bar{q}a + qa)}(e^{i\sqrt{\frac{\theta}{2}}(\bar{q}a + qa)} + e^{-i\sqrt{\frac{\theta}{2}}(\bar{q}a + qa)} + O(\epsilon)) + O(\theta k^2)] \]
(36)
Taking the Fourier transform, we obtain finally
\[ \delta \rho(\vec{x}) = \frac{\epsilon}{2\pi \theta} \left[ \text{sin}(\vec{q} \cdot \vec{x}) + O(\epsilon + O(\theta k^2)) \right], \] (37)
which is the density response expected in a quantum Hall system.

V. CHARGE AND CURRENT IN THE QUANTUM MATRIX MODEL

As discussed in the introduction, the Fourier components of the density operator projected onto the LLL satisfy the non-trivial commutation relation (2). Since the hope is that the QH matrix model captures the physics of the LLL, we would expect the correctly defined density operator to satisfy (2). First notice that in the infinite matrix model only describes a single state, with density \( \rho \). Simons theory quoted in the introduction, \( \tilde{\theta} \) due to a logical error as will be pointed out.

Below we review an attempt to construct an operator that satisfies the LLL commutator algebra. The basic observation is that for \( N = 2 \), the known result (39) can be expressed in terms of quadratic operators which can be generalized to the \( \theta \neq 0 \) case but satisfy a \( \theta \)-independent algebra. In the first version of this paper we erroneously claimed to have an expression in these quadratic operators that satisfied the LLL algebra (2). This conclusion was due to a logical error as will be pointed out.
A. Constrained quantization

Here we follow Polychronakos\[7\] and use the matrices

\[
\begin{align*}
X^1_{mn} &= x_m\delta_{mn} \\
X^2_{mn} &= y_m\delta_{mn} - \frac{i\theta}{x_m - x_n}(1 - \delta_{mn}),
\end{align*}
\]  

which solve the constraint (10) in the gauge \(\Psi = \sqrt{e^{iB\theta}}(1,1,\ldots,1)\). Quantizing, the diagonal matrix elements become canonically conjugate operators, satisfying (in complex notation),

\[
[z_m, \bar{z}_n] = 2\ell^2\delta_{mn}.
\]

Note that since the particles are identical only even powers of the relative coordinate can appear in (43).

First we decompose the complex \(2 \times 2\) matrix \(Z = X_1 + iX_2\) as

\[
\begin{align*}
Z &= \zeta(1 + z\sigma^3 + i\vartheta\sigma^2) \\
Z^\dagger &= \bar{\zeta}(1 + \bar{z}\sigma^3 - i\vartheta\sigma^2),
\end{align*}
\]

where \(\sigma^i\) are the Pauli matrices, \(\vartheta = \theta/(z + \bar{z})\), and where we have introduced center of mass and relative coordinates,

\[
\begin{align*}
\zeta &= \frac{1}{2}(z_1 + z_2) = \frac{1}{2}\text{Tr}Z \\
z &= \frac{1}{2}(z_1 - z_2),
\end{align*}
\]

satisfying \(|\zeta, \bar{\zeta}| = |z, \bar{z}| = \ell^2\), while all other commutators vanish.

For \(\theta = 0\) we expand the matrix exponentials in (39) using the explicit expressions (41) and the properties of the Pauli matrices to get

\[
\rho_k^{\text{a}}|_{\theta=0} = \text{Tr}e^{-\frac{i}{2}kZ}e^{-\frac{i}{2}kZ^\dagger}|_{\theta=0} = 2 \sum_{m,n=0}^{\infty} \frac{(-i\vartheta)^{2m}(z\vartheta)^{2n}}{(2m)! (2n)!} e^{-\frac{k}{4}\zeta} \left[ (z^2)^m (\bar{z}^2)^n - \frac{|k|^2}{4} \frac{(z^2)^m \bar{z}(\bar{z}^2)^n}{(2m + 1)(2n + 1)} \right] e^{-\frac{k}{4}\bar{\zeta}}. \tag{43}
\]

Note that since the particles are identical only even powers of the relative coordinate can appear in (43).

Next define the quadratic operators,

\[
\begin{align*}
A &= \frac{1}{2}\text{Tr}Z^2 - \left(\frac{1}{4}\text{Tr}Z\right)^2 = z^2 - \vartheta^2 \\
\bar{A} &= \frac{1}{2}\text{Tr}(Z^\dagger)^2 - \frac{1}{4}(\text{Tr}Z^\dagger)^2 = \bar{z}^2 - \vartheta^2 \\
B &= \frac{1}{2}\text{Tr}ZZ^\dagger - \frac{1}{4}\text{Tr}Z\text{Tr}Z^\dagger = z\bar{z} + \vartheta^2.
\end{align*}
\]

It is easy to verify that they commute with \(\zeta\) and satisfy the following \(\theta\)-independent algebra,

\[
\begin{align*}
[\zeta, \bar{\zeta}] &= \ell^2 \\
[A, B] &= 2\ell^2 A \\
[\bar{A}, B] &= -2\ell^2 \bar{A} \\
[A, \bar{A}] &= 4\ell^2 B - 2\ell^4.
\end{align*}
\tag{45}
\]

This allows us to define a \(\theta\)-independent density operator by substituting \(z^2 \rightarrow A, \bar{z}^2 \rightarrow \bar{A}\) and \(z\bar{z} \rightarrow B\) in the series expansion (43) to get,

\[
\rho_k = 2 \sum_{m,n=0}^{\infty} \frac{(-i\vartheta)^{2m}(z\vartheta)^{2n}}{(2m)! (2n)!} e^{-\frac{k}{4}\zeta} \left[ A^m \bar{A}^n - \frac{|k|^2}{4} \frac{A^m \bar{A}^n}{(2m + 1)(2n + 1)} \right] e^{-\frac{k}{4}\bar{\zeta}}. \tag{46}
\]

By construction the commutator algebra of the \(\rho_k\) is \(\theta\)-independent, so the commutator (2) can in a \(\theta\)-independent way be reduced to a sum of terms of the form \(A^m B^k \bar{A}^n\). We can however not reduce these terms to sums of terms of...
the type $A^m B \bar{A}^n$ or $A^m \bar{A}^n$ without using a $\theta$-dependent relation, and thus not conclude that since \ref{eq:13} satisfies the algebra \ref{eq:4}, so does \ref{eq:14}. This was the logical error referred to above.

We now prove that the classical limit of $\rho_k$ in \ref{eq:14} is the anti-ordered classical operator in \ref{eq:16} also for finite $\theta$. A general $2 \times 2$ matrix $Z$ with commuting elements satisfies:

$$0 = Z^2 - Z \text{Tr} Z + \frac{1}{2} (\text{Tr}^2 Z - \text{Tr} Z^2) 1 = (Z - \zeta 1)^2 - A 1$$

(47)

where $\zeta$ and $A$ are defined in \ref{eq:12} and \ref{eq:14}. Using this relation, we can rewrite the classical anti-ordered operator,

$$\rho_k^\theta = \text{Tr} e^{-\frac{i}{\hbar} Z e^{-\frac{i}{\hbar} Z^\dagger}} = \text{Tr} e^{-\frac{i}{\hbar} [Z-\zeta 1]} e^{-\frac{i}{\hbar} (Z^\dagger-\zeta 1)} e^{-\frac{i}{\hbar} \zeta}.$$  

(48)

After expanding the matrix exponentials and using \ref{eq:17} to rewrite all even powers of $(Z - \zeta 1)^m (Z^\dagger - \zeta 1)^n$ in terms of $A$, $\bar{A}$ and $B$, it is a matter of working out a few traces to recover the expansion \ref{eq:16}.

B. Unconstrained quantization

In this approach the matrix elements of $X^\alpha$ satisfy the quantum commutation relations \ref{eq:5}, and the parameter $\theta$ enters only via the $U(1)$ part of the constraint \ref{eq:6} that defines the physical states. Since the density operator is an observable it must be $U(N)$ invariant, and since the commutation relations \ref{eq:5} preserve this invariance, it follows that the commutator of two densities must again be $U(N)$ invariant. (Note that since \ref{eq:10} is expressed only in traces it is a manifestly $U(N)$ invariant expression, although it was derived in a particular gauge.) This would seem to imply that there could be no $\theta$-dependence in the operators, but this conclusion does not necessarily follow since we are considering operators on an extended phase space. The commutator \ref{eq:4} might have additional terms proportional to the constraint \ref{eq:6}, and this would still give the correct algebra on the physical subspace. However, since the gauge constraint explicitly involves this boundary field $\Psi$, the density operators can have a non-trivial $\theta$-dependence only if they also depend explicitly on $\Psi$, and this would bring us out of the class of operators defined by quantum reorderings of \ref{eq:6}. Although we shall not consider this possibility in this paper, it cannot be excluded a priori.

Thus restricting ourselves to $U(N)$ invariant combinations of the matrices $Z$ and $Z^\dagger$, satisfying $[Z_{kl}, Z_{mn}] = 0$, $[Z_{kl}, Z^\dagger_{mn}] = 2\ell^2 \delta_{km} \delta_{ln}$, we note that the classical derivation of \ref{eq:16} based on \ref{eq:18} holds true also in the quantum theory since it did not involve any reordering of $Z_{kl}$’s and $Z^\dagger_{mn}$’s. (In the constrained quantization scheme this derivation brakes down since elements of the matrix $Z$ do not commute with each other.) The operators $A$, $\bar{A}$, $B$ and $\zeta$ are here again defined in terms of the matrices by \ref{eq:12} and \ref{eq:14}, but they do not satisfy the algebra \ref{eq:15}. Remember that the constrained and unconstrained $Z$ are not unitarily equivalent – in that case they would have to satisfy the same algebra – but related via a projection onto the subspace defined by the constraint. The new algebra is however very similar to \ref{eq:15}, the only difference being the commutator

$$[A, \bar{A}] = 4\ell^2 B - 6\ell^4.$$  

(49)

This difference can be absorbed by a quantum reordering of \ref{eq:16} in which $B$ is replaced by

$$\tilde{B} = \frac{1}{4} \text{Tr} (Z Z^\dagger + Z^\dagger Z) - \frac{1}{4} \text{Tr} Z^\dagger \text{Tr} Z = B - \ell^2.$$  

(50)

This gives $[A, \bar{A}] = 4\ell^2 \tilde{B} - 2\ell^4$, while leaving the other commutators unchanged (except for $B \to \tilde{B}$). Since the operator $B$ has the form of a trace of the number operator, it is not surprising that it differs in the two schemes since the number of degrees of freedom are $N$ and $N^2$ respectively. Note, however, that if the gauge-fixed $Z$’s are used, then $\tilde{B} = B$, so \ref{eq:16} with $\tilde{B}$ works in both quantization schemes.

C. The conserved quantum current

Just as in the classical case, the current density can be constructed from the particle density by requiering the current to be conserved,

$$-\frac{i}{\hbar} \partial_k \phi_k - \frac{i\hbar}{2} \partial_k \tilde{\phi}_k = \hat{\rho}_k = \frac{1}{i\hbar} \{\rho_k, H\},$$  

(51)

where $H$ is the full quantum Hamiltonian. From this we can derive two expressions for the current. The first is obtained simply by evaluating the time derivative of \ref{eq:16} and noting that just as in the derivation of a classical
current like \( \tilde{a} \), a \( \tilde{k} \) or a \( k \) can always be factored from the corresponding expressions, thus defining the complex components of the current. The explicit expressions, containing \( \hat{Z} \) and \( \hat{Z}^\dagger \), are not very illuminating. Alternatively, we can assume some particular form for the Hamiltonian, evaluate the commutator, and derive an explicit expression for the current in terms of the basic matrix operators. We exemplify with a Hamiltonian consisting only of an external scalar potential, i.e.

\[
H = V(\rho) = \int \frac{d^2k}{(2\pi)^2} V(k, \tilde{k}) \rho_k.
\]

With this choice, \( \rho_k \) takes the form

\[
\rho_k = \frac{1}{i\hbar} \int \frac{d^2p}{(2\pi)^2} V(p, \tilde{p})[\rho_k, \rho_p] = \frac{i}{\hbar} \int \frac{d^2p}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{(\ell^2/2)^n}{n!} V(p, \tilde{p})[(\tilde{k}p)^n - (kp)^n] \rho_{k+p},
\]

where we used \( \mathbf{\mathbf{\mathbf{1}}} \) for the commutator and expanded the exponential and sine functions in a power series. Comparing with \( \mathbf{\mathbf{\mathbf{1}}} \) we can now directly read off the current,

\[
j_k = -\frac{2}{\hbar} \sum_{n=1}^{\infty} \frac{(\ell^2/2)^n}{n!} \tilde{k}^{n-1} \int \frac{d^2p}{(2\pi)^2} p^n V(p, \tilde{p}) \rho_{k+p},
\]

which can be written more compactly in real space. A Fourier transformation gives,

\[
j(z, \bar{z}) = \frac{i}{\hbar} \sum_{n=1}^{\infty} \frac{(2\ell^2)^n}{n!} \partial_z^{n-1}[\rho(z, \bar{z}) \partial_{\bar{z}} V(z, \bar{z})].
\]

This expression was derived earlier by Martinez and Stone\( \mathbf{\mathbf{\mathbf{1}}} \) using a second quantized formalism. The above derivation is more general in that it applies to any density operator satisfying the algebra \( \mathbf{\mathbf{\mathbf{1}}} \), and in particular to the matrix model density operator \( \mathbf{\mathbf{\mathbf{1}}} \).

VI. SUMMARY AND OPEN PROBLEMS

We have extended the classical quantum Hall matrix models of Susskind and Polychronakos to include couplings to an external field. We gave a general prescription for the construction of a conserved current and noticed that it suffers from ordering ambiguities even at the classical level. Nevertheless, zero and low momentum observables can be reliably calculated using the infinite matrix model and in particular we showed that the ground state density, and the response to a weak and slowly varying magnetic field agree with what is expected for a fractional QH system.

It is much harder to construct the correct charge and current operators in the quantized matrix model. The difficulties are due to the complicated intermingling of matrix and quantum ordering. In this connection, we also notice the difficulties encountered in a recent attempt by Karabali and Sakita to find good particle coordinates in the quantum matrix model\( \mathbf{\mathbf{\mathbf{1}}} \). It turns out that the most natural guess, i.e. defining the coordinates via the coherent states of the annihilation operator \( Z_{ij} \), is not correct in that it does not reproduce the Laughlin wave functions. It is not unlikely that this is related to the failure of the anti-ordered matrix expression \( \mathbf{\mathbf{\mathbf{1}}} \) to satisfy the correct density operator algebra.

Although not conclusive, our study supports the conjecture that the classical quantum Hall matrix model captures important aspects of the LLL physics not present in the usual effective Chern-Simons theory. At the practical level, the conclusion is, however, rather disappointing since local observables like charges and currents are rather complicated and thus cumbersome to work with. This might perhaps have been anticipated, since the gauge invariant objects in non-commutative theories are inherently nonlocal. On the other hand, it is the very fact that a non-commutative field theory can be viewed as a commutative theory with an infinite number of higher derivative terms, that makes it at all possible that the Landau level structure could be incorporated at the level of an effective Lagrangian.

We end with a comment on the relation between the classical and the quantum models. It is striking that the classical matrix model incorporates features of the FQHE which are related to the Landau level structure and not easily described in conventional mean field approaches. The \( \theta \)-dependent repulsion between the particles is a phase space exclusion effect which can be thought of as the classical counterpart of fermion, or more generally anyon, statistics. In this way the classical matrix model can emulate a fermionic system with Haldane type pseudopotentials. In the case of a quadratic confining potential, this idea is made manifest via the mapping onto the Calogero model\( \mathbf{\mathbf{\mathbf{2}}} \),
and it would be interesting if a more general connection could be made between the QH matrix model and classical exclusion statistics. The analogy between the classical matrix model and the QH system goes even further in that the classical counterpart to the density operator algebra can be interpreted as a quantum commutator algebra of fractionally charged quasi-holes. We hope to return to some of these questions in the future.

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16. There are, however, microscopic derivations of 1, relying on various mean field approximations, where the quantization of \( k \) follows from a statistical transmutation. This choice, which we here adhere to, results in the one-particle quantum commutator \([x, y] = +i\ell^2\) and states \( \phi_n \sim \bar{z}^{\frac{1}{2}} \exp\left(-\frac{1}{4\ell^2} |z|^2\right) \). More common in the condensed matter community is to choose \( B_z = -B \), in which case \([x, y] = -i\ell^2\) and \( \phi_n \sim \bar{z}^{\frac{1}{2}} \exp\left(-\frac{1}{4\ell^2} |z|^2\right) \).
17. Using (13), \( \delta\bar{A}_\mu(x) = \partial_\nu \Lambda(x) \) induces the following gauge variations in the matrices \( \delta\bar{A}_m = -\left(\partial\bar{A}/\partial X^a\right)_m = -\partial\bar{\text{Tr}}\bar{\Lambda}/\partial X^a_m \) and \( \delta\Lambda = \partial_0 \Lambda \) where the matrix-valued function \( \bar{\Lambda}(X^a, t) \) is related to \( \Lambda(x^a, t) \) using the Weyl kernel in (13), and the derivative satisfies \( \partial X^a/\partial X^b = -i\partial X^a/\partial X^b = -ik_a e^{-ik_b x^b} \).
18. Here we again the cyclicity of the trace in the equations of motion. This assumption can be avoided for a kernel that behaves like a \( \delta \)-function: \( \int d^2 x x^a \delta(x, X) = X^a \) – then we have \( A_0 = -E_a X^a \). Substituting this into the Lagrangian and varying with respect to \( X^a \) gives.