A LEFSCHETZ FIBRATION STRUCTURE ON MINIMAL SYMPLECTIC FILLINGS OF A QUOTIENT SURFACE SINGULARITY

HAKHO CHOI AND JONGIL PARK

Abstract. In this article we construct a genus-0 or genus-1 positive allowable Lefschetz fibration structure on any minimal symplectic filling of the link of non-cyclic quotient surface singularities. As by-product, we also show that any minimal symplectic filling of the link of quotient surface singularities can be obtained by a sequence of rational blowdowns from its minimal resolution.

1. INTRODUCTION

Since it was known due to S. Donaldson [D] that any closed symplectic 4-manifold admits a Lefschetz pencil and that a Lefschetz fibration structure can be obtained from a Lefschetz pencil by blowing-up the base loci, the study of Lefschetz fibrations has become an important theme for understanding symplectic 4-manifolds topologically. In fact, Lefschetz pencils and Lefschetz fibrations have long been studied extensively by algebraic geometers and topologists in the complex category, and these notions can be extended to the symplectic category. It was also known that an isomorphism class of Lefschetz fibrations is characterized by monodromy factorization, an ordered sequence of right-handed Dehn twists, up to Hurwitz equivalence and global conjugation equivalence.

On the other hand, it is also a main research topic in symplectic 4-manifolds topology to classify symplectic fillings of certain 3-manifolds equipped with a contact structure. Among them, people have classified symplectic fillings of the link of a quotient surface singularity. Note that the link of a quotient surface singularity admits a natural contact structure, called the Milnor fillable contact structure. For example, P. Lisca [L] classified symplectic fillings of cyclic quotient singularities whose corresponding link is Lens space, and M. Bhupal and K. Ono [BO] listed all possible symplectic fillings of non-cyclic quotient surface singularities. Furthermore, the second author together with H. Park, D. Shin and G. Urzúa [PPSU] constructed an explicit one-to-one correspondence between the minimal symplectic fillings and the Milnor fibers of non-cyclic quotient surface singularities. Note that the last result above implies that every minimal symplectic filling of a quotient surface singularity is in fact a Stein filling.

Date: February 8, 2018.

2010 Mathematics Subject Classification. 57R17, 53D05, 14E15, 14J17.

Key words and phrases. Lefschetz fibration, quotient surface singularity, symplectic filling.
Although the existence of a (positive allowable) Lefschetz fibration, called briefly PALF, structure on a Stein filling is well known in general \cite{AOz, LoP}, it is somewhat a different problem to find an explicit monodromy description for the Lefschetz fibration structure on a given Stein filling. In this article, we investigate the problem for minimal symplectic fillings of the link of quotient surface singularities. Regarding it, M. Bhupal and B. Ozbagci \cite{BOz} found an algorithm to present each minimal symplectic filling of a cyclic quotient surface singularity as an explicit genus-0 positive allowable Lefschetz fibration structure over the disk. Furthermore, they also showed that such a PALF structure can be obtained from the minimal resolution by monodromy substitutions corresponding to rational blowdowns topologically. The main goal of this article is to generalize their result for the non-cyclic quotient surface singularity cases. That is, we obtain the following result.

**Theorem 1.1.** Every minimal symplectic filling of the link of non-cyclic quotient surface singularities admits a genus-0 or genus-1 positive allowable Lefschetz fibration structure over the disk. Furthermore, each such a symplectic filling can be also obtained by rational blowdowns from the minimal resolution of its singularity.

**Remark 1.1.** Note that a genus of the PALF structure in Theorem 1.1 above is determined only by the existence of a bad vertex (refer to Section 2.1 for definition) in the minimal resolution graph of the corresponding singularity. Explicitly, if the minimal resolution graph of a quotient surface singularity has no bad vertex, a genus of the PALF structure is 0, and the genus is 1 otherwise.

In order to prove Theorem 1.1 above, we first construct a PALF structure on the minimal resolution graph of a non-cyclic quotient surface singularity: If there is no bad vertex in the minimal resolution graph, we follow the idea of D. Gay and T. Mark in \cite{GaM}, where they initially constructed a genus-0 PALF structure on the minimal resolution graph. If there is a bad vertex, then we construct a genus-1 PALF structure, which is in turn a special case of open book decompositions on the boundary of plumbings obtained by J. Etnyre and B. Ozbagci \cite{EtOz1}. Next, we show that the induced contact structure on the boundary is the Milnor fillable contact structure by computing the first Chern class in terms of vanishing cycles and the rotation number of these vanishing cycles. And then, we construct a PALF structure on any minimal symplectic filling via the corresponding P-resolution. Since every Milnor fiber, hence every minimal symplectic filling, of a quotient surface singularity can be obtained topologically by rationally blowing down the corresponding P-resolution, it is enough to construct a PALF structure on the general fiber of P-resolutions. Finally we show that the Lefschetz fibration structures of any minimal symplectic filling can be obtained by monodromy substitutions from the minimal resolution of the corresponding singularity by adapting the same technique that H. Endo, T. Mark and J. Van Horn-Morris used in \cite{EnMV}.

This article is organized as follows: We briefly review some generalities on quotient surface singularities including minimal resolutions and P-resolutions, and the relation between monodromy substitutions and rational blowdowns in Section 2.
And we introduce Lisca’s classification result on symplectic fillings and Bhupal-Ozbagci’s algorithm for finding a PALF structure on the cyclic cases in Section 3. And then we explain how to construct a genus-0 or genus-1 Lefschetz fibration structure on the minimal resolutions and we show that the induced contact structure on the boundary is indeed Milnor fillable in Section 4. Finally, we give an explicit algorithm for a PALF structure on any minimal symplectic filling by investigating PALF structures on each \( P \)-resolution in Section 5.

Acknowledgements. Jongil Park is supported by Samsung Science and Technology Foundation under Project Number SSTF-BA1602-02. He also holds a joint appointment at KIAS and in the Research Institute of Mathematics, SNU.

2. Generalities on quotient surface singularities

In this section we recall briefly some basics on quotient surface singularities (refer to [PPSU] for details). Let \((X, 0) = (\mathbb{C}^2/G, 0)\) be a germ of quotient surface singularity, where \(G\) is a finite subgroup of \(GL(2, \mathbb{C})\) without reflections. Since \((\mathbb{C}^2/G_1, 0)\) is analytically isomorphic to \((\mathbb{C}^2/G_2, 0)\) if and only if \(G_1\) is conjugate to \(G_2\), it is enough to classify finite subgroups of \(GL(2, \mathbb{C})\) without reflections up to conjugation for classifying quotient surface singularities \((\mathbb{C}^2/G, 0)\). We may assume that \(G \subset U(2)\) because \(G\) is finite. Then the action of \(G\) on \(\mathbb{C}^2\) lifts to an action on the blowing-up of \(\mathbb{C}^2\) at the origin. So \(G\) acts on the exceptional divisor \(E \sim \mathbb{CP}^1\), where the action is induced by the double covering \(G \subset U(2) \to PU(2) \cong SO(3)\). The image of \(G\) in \(SO(3)\) is either a (finite) cyclic subgroup, a dihedral group, the tetrahedral group, the octahedral group, or the icosahedral group. Therefore quotient surface singularities are divided into five classes: cyclic quotient surface singularities, dihedral singularities, tetrahedral singularities, octahedral singularities, and icosahedral singularities. We call the last four cases non-cyclic quotient surface singularities.

2.1. Symplectic fillings and Milnor fibers. Let \((X, 0) = (\mathbb{C}^2/G, 0)\) be a germ of a quotient surface singularity, where \(G\) is a finite subgroup of \(U(2)\) without reflections. Assume that \((X, 0) \subset (\mathbb{C}^N, 0)\), which is always possible. If \(B \subset \mathbb{C}^N\) is a small ball centered at the origin, then the small neighborhood \(X \cap B\) of the singularity is homeomorphic to the cone over its boundary \(L := X \cap \partial B\). The smooth compact 3-manifold \(L\) is called the link of the singularity. It is well known that the topology of the germ \((X, 0)\) is completely determined by its link \(L\) and the link \(L\) admits a natural contact structure \(\xi_{st}\), so-called Milnor fillable contact structure \(\xi_{st} = TL \cap JTL\), where \(J\) is an induced complex structure along \(L\). A (strong) symplectic filling of \((X, 0)\) is a symplectic 4-manifold \((W, \omega)\) with the boundary \(\partial W = L\) satisfying the compatibility condition \(\omega = d\alpha_{st}\) near \(L\), where \(\alpha_{st}\) is a 1-form defining the contact structure \(\xi_{st} = \ker \alpha_{st}\) on \(L\). One may also define a so-called weak symplectic filling. But it is known that two notions of symplectic fillings coincide in our case because the link \(L\) is a rational homology sphere. So we simply call them symplectic fillings.
Next, we call $W$ a Stein filling of $(X, 0)$ if it is a Stein manifold $W$ with $L$ as its strictly pseudoconvex boundary and $\xi_{st}$ is the set of complex tangencies to $L$. It is clear that Stein fillings are minimal symplectic fillings of the link $L$ of $(X, 0)$.

Third, we call a proper flat map $\pi : X \rightarrow \Delta$ with $\Delta = \{ t \in \mathbb{C} : |t| < \epsilon \}$ a smoothing of $(X, 0)$ if it satisfies $\pi^{-1}(0) = X$ and $\pi^{-1}(t)$ is smooth for all $t \neq 0$. The Milnor fiber $M$ of a smoothing $\pi$ of $(X, 0)$ is defined to be a general fiber $\pi^{-1}(t)$ ($0 < t < \epsilon$). It is known that the Milnor fiber $M$ is a compact 4-manifold with the link $L$ as its boundary and the diffeomorphism type of $M$ depends only on the smoothing $\pi$. Furthermore, $M$ has a natural Stein (hence symplectic) structure, and so it provides an example of a Stein (and minimal symplectic) filling of $(L, \xi_{st})$. Recall that, as mentioned in the Introduction, H. Park, J. Park, D. Shin and G. Urzúa [PPSU] constructed an explicit one-to-one correspondence between the minimal symplectic fillings and the Milnor fibers of quotient surface singularities. Hence it is now a well-known fact that every minimal symplectic filling of a quotient surface singularity is a Stein filling and a Milnor fiber of the singularity.

2.2. Minimal resolutions. We first denote the Hirzebruch-Jung continued fraction by $[c_1, \ldots, c_t](c_i \geq 1)$, which is defined recursively as follows:

$$[c_1] = c_t, \quad \text{and} \quad [c_i, c_{i+1}, \ldots, c_t] = c_i - \frac{1}{[c_{i+1}, \ldots, c_t]}.$$ 

Since a continued fraction $[c_1, c_2, \ldots, c_t]$ often describes a chain of smooth rational curves on a complex surface whose dual graph is given by

$$\begin{array}{cccccc}
- & - & \cdots & - & - \\
\bullet & \bullet & \cdots & \bullet & \bullet
\end{array},$$

we use by analogy the term ‘blowing up’ for the following operations and the term ‘blowing down’ for their inverses

$$[c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_t] \rightarrow [c_1, \ldots, c_{i-1} + 1, 1, c_{i+1} + 1, \ldots, c_t]$$

$$[c_1, \ldots, c_{i-1}] \rightarrow [c_1, \ldots, c_{i-1} + 1, 1].$$

Now we describe the (dual graph of) minimal resolution of quotient surface singularities. In the resolution graph, note that a vertex $v$ corresponds to the irreducible component $E_v$ of the exceptional divisor $E$ and the edges correspond to the intersections of the irreducible components $E_v$’s. We call the number of edges connected to the vertex $v$ the valence of $v$ and the self-intersection of $E_v$ the degree of $v$. If the absolute value of the degree of $v$ is strictly less than the valence of $v$, we call the vertex $v$ a bad vertex.

Example 2.1. The following figures show the cases of minimal resolution graphs with no bad vertex and with a bad vertex, respectively. A central vertex (vertex with a valence 3) in the right-handed figure is a bad vertex.
SYMPLECTIC FILLINGS OF QUOTIENT SINGULARITIES AS LEFSCHETZ FIBRATIONS

Cyclic singularities $A_{n,q}$. A cyclic quotient surface singularity $(X,0)$ of type $\frac{1}{n}(1,q)$ with $1 \leq q < n$ and $(n,q) = 1$ is a quotient surface singularity, where a cyclic group $\mathbb{Z}_n$ acts by $\zeta \cdot (x,y) = (\zeta x, \zeta^q y)$. Then the minimal resolution graph of $(X,0)$ is given by

```
-2 -b_2 \cdots -b_r -3
```

where $\frac{n}{q} = [b_1, b_2, \ldots, b_r]$ with $b_i \geq 2$ for all $i$.

Dihedral singularities $D_{n,q}$. Let $(X,0)$ be a dihedral singularity of type $D_{n,q}$, where $1 < q < n$ and $(n,q) = 1$. The minimal resolution graph of $(X,0)$ is given by

```
-2 -b -b_1 \cdots -b_{r-1} -b_r
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where $\frac{n}{q} = [b, b_1, \ldots, b_{r-1}, b_r]$ with $b \geq 2$ and $b_i \geq 2$ for all $i$.

Other cases. For a tetrahedral, octahedral, or icosahedral singularity, the minimal resolution has a central curve $C_0$ with $C_0 \cdot C_0 = -b$ ($b \geq 2$) and three arms, which can be divided into the following two types - type $(3,1)$ and type $(3,2)$:

```
-3 \cdots -b_r
```

```
-2 -2 -b -b_1 \cdots -b_r
```

(a) type $(3,1)$  (b) type $(3,2)$

2.3. $P$-resolutions. We first define a normal surface singularity of class $T$ which appears in the definition of $P$-resolution of a quotient surface singularity.

**Definition 2.1.** A normal surface singularity is of class $T$ if it is a quotient surface singularity and it admits a $Q$-Gorenstein one-parameter smoothing. Equivalently, it is a rational double point singularity or a cyclic quotient surface singularity of type $\frac{1}{na}(1, dna - 1)$ with $d \geq 1$, $n \geq 2$, $1 \leq a < n$, and $(n,a) = 1$.

Note that the one-parameter $\mathbb{Q}$-Gorenstein smoothing of a singularity of class $T$ is interpreted topologically as a rational blowdown surgery defined by R. Fintushel and R. Stern [FS] and extended by J. Park [P]. Furthermore, due essentially to J. Wahl [W], a cyclic quotient surface singularity of class $T$ can be recognized from its minimal resolution as follows:
Proposition 2.2.  

1. The singularities $-4 \quad -3 \quad -2 \quad \ldots \quad -2 \quad -3$ are of type $T$.

2. If $-b_1 \quad -b_2 \quad \ldots \quad -b_{r-1} \quad -b_r$ is of type $T$, so are $-2 \quad -b_1 \quad \ldots \quad -b_{r-1} \quad -(b_r + 1)$ and $-(b_1 + 1) \quad -b_2 \quad \ldots \quad -b_r \quad -2$

3. Every singularity of class $T$ that is not a rational double point can be obtained by starting with one of the singularities described in (1) and iterating the steps described in (2) above.

Definition 2.3. A $P$-resolution $f : (Y, E) \to (X, 0)$ of a quotient surface singularity $(X, 0)$ is a partial resolution such that $Y$ has at most rational double points or singularities of type $T$ and $K_Y$ is ample relative to $f$.

We usually describe a $P$-resolution $Y \to X$ by indicating the $\pi$-exceptional divisors on the minimal resolution $\pi : Z \to Y$ of $Y$. Note that the ampleness condition in the definition of a $P$-resolution can be checked on $Z$: Every $(-1)$ curve on $Z$ must intersect two curves $E_1$ and $E_2$, which are exceptional for singularities of type $T$ on $Y$ and the sum of the coefficients $k_i$ of $E_i$ in the canonical divisor $K_Z$ is less than $-1$.

By the way, according to J. Kollar and N. Shepherd-Barron [KSB], there is a one-to-one correspondence between the set of all irreducible components of the versal deformation space of a quotient surface singularity $(X, 0)$ and the set of all $P$-resolutions of $(X, 0)$. Hence, since the Milnor fibers are invariants of the irreducible components of the versal deformation space of $(X, 0)$, the Milnor fibers are in one-to-one correspondence with $P$-resolutions. Furthermore, it is also known by J. Stevens [Ste2] how to find all $P$-resolutions of quotient surface singularities.

Example 2.2. Let $(X, 0)$ be a dihedral singularity of type $D_{9.2}$. Since $9/2 = [5, 2]$, the minimal resolution of $(X, 0)$ is given by

There are four $P$-resolutions of $(X, 0)$ as follows: Here a linear chain of vertices decorated by a rectangle $\square$ denotes curves on the minimal resolution of a $P$-resolution which are contracted to a singularity of class $T$ on the $P$-resolution. Note that
there are certain symmetries in the list of $P$-resolutions.

\[ \begin{array}{cccc}
-2 & -5 & -2 \\
\hline
-2 & -5 & -2
\end{array} \]

\[ \begin{array}{cccc}
-2 & -5 & -2 \\
\hline
-2 & -5 & -2
\end{array} \]

\[ \begin{array}{cccc}
-2 & -5 & -2 \\
\hline
-2 & -5 & -2
\end{array} \]

\[ \begin{array}{cccc}
-2 & -5 & -2 \\
\hline
-2 & -5 & -2
\end{array} \]

2.4. Monodromy substitutions and rational blowdowns. In [FS], R. Fintushel and R. Stern introduced a rational blowdown surgery: Let $C_p$ be a smooth 4-manifold obtained by plumbing disk bundles over 2-sphere according to the following linear diagram:

\[ -(p+2) \quad -2 \quad \ldots \quad -2 \quad -2 \]

Then the boundary of $C_p$ is a Lens space $L(p^2, p - 1)$, which bounds a rational ball $B_p$, i.e., $H_*(B_p; \mathbb{Q}) = H_*(D^4; \mathbb{Q})$. So, if there is an embedding of $C_p$ in a smooth 4-manifold $X$, one can construct a new smooth 4-manifold $X_p$ by replacing $C_p$ with $B_p$. This procedure is called a rational blowdown surgery and we say that $X_p$ is obtained by rationally blowing down $X$. Furthermore, M. Symington proved that a rational blowdown manifold $X_p$ admits a symplectic structure in some cases.

For example, if $X$ is a symplectic 4-manifold containing a configuration $C_p$ such that all 2-spheres in $C_p$ are symplectically embedded and intersect positively, then the rational blowdown manifold $X_p$ also admits a symplectic structure. Later, the Fintushel-Stern’s rational blowdown surgery is generalized by J. Park [P] using a configuration $C_{p,q}$ obtained by plumbing disk bundles over 2-sphere according to the dual resolution graph of $L(p^2, pq - 1)$ which also bounds a rational ball $B_{p,q}$ as follows:

**Definition 2.4.** Suppose $X$ is a smooth 4-manifold containing a configuration $C_{p,q}$. Then one can construct a new smooth 4-manifold $X_{p,q}$, called a (generalized) rational blowdown of $X$, by replacing $C_{p,q}$ with the rational ball $B_{p,q}$. We also call this a (generalized) rational blowdown surgery.

Next, we introduce a notion of monodromy substitution which is closely related to a rational blowdown surgery. That is, we briefly explain how to replace a rational blowdown surgery by a monodromy substitution in some cases.

Suppose that a symplectic 4-manifold $X$ with a possibly non-empty boundary admits a Lefschetz fibration structure characterized by a monodromy factorization $\mathcal{W}_X$. Assume that $W$ and $W'$ are distinct products of right-handed Dehn twists
which give the same element as a global monodromy in the mapping class group of the fiber. If there is a partial monodromy factorization equal to $W$ in the monodromy factorization $\mathcal{W}_X$ of $X$, then we can obtain a Lefschetz fibration structure on a new symplectic 4-manifold $X'$ whose monodromy factorization $\mathcal{W}_{X'}$ is obtained by replacing $W$ with $W'$ partially. Note that the diffeomorphism types and the induced contact structures of $\partial X$ and $\partial X'$ are the same. We call this procedure a monodromy substitution. For example, a famous lantern relation gives a rational blowdown surgery involving the Lens space $L(4,1)$ [EnGu]: The PALF with a monodromy $abcd$ gives a configuration $C_2$ while the PALF with a monodromy $xyz$ gives a rational ball $B_2$. As another example, the daisy relation, introduced in [EnMV], gives the monodromy substitution for a configuration $C_p$ and a rational ball $B_p$. One can also find the monodromy substitution for a (generalized) rational blowdown surgery in [EnMV].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{lantern_relation.png}
\caption{Lantern relation}
\end{figure}

3. Review for the cyclic singularity cases

In this section we briefly review P. Lisca’s classification of minimal symplectic fillings and Bhupal-Ozbagci’s algorithm of positive allowable Lefschetz fibration structures for the minimal symplectic fillings of cyclic quotient surface singularities.

We first review Lisca’s classification (refer to [L] for details): Let $(X, 0)$ be a cyclic quotient surface singularity of type $\frac{1}{n}(1, q)$ with $(n, q) = 1$ whose link is the Lens space $L(n, q)$. P. Lisca [L] parametrized all minimal symplectic fillings of $(X, 0)$ by a set $\mathcal{Z}_e(\frac{n}{n-q})$ of certain sequences of integers $n = (n_1, \ldots, n_e) \in \mathbb{N}^e$ (see Definition 3.1 below). That is, by surgery diagrams, he constructed compact oriented symplectic 4-manifolds $W_{n,q}(n)$ with boundary $L(n, q)$ which are parametrized by $n \in \mathcal{Z}_e(\frac{n}{n-q})$ and he showed that $W_{n,q}(n)$ is in fact a Stein filling of $L(n, q)$. Finally he proved that any symplectic filling of $L(n, q)$ is orientation-preserving diffeomorphic to a manifold obtained by blow-ups from one of $W_{n,q}(n)$’s. Hence every minimal symplectic filling is diffeomorphic to one of $W_{n,q}(n)$’s. On the other hand, J. Christophersen [C] and J. Stevens [Ste1] parametrized all reduced irreducible components of the versal deformation space of $(X, 0)$ by the same set $\mathcal{Z}_e(\frac{n}{n-q})$ but with different methods. So it was a natural conjecture that every Milnor fiber of $(X, 0)$ is diffeomorphic to one of $W_{n,q}(n)$’s which are parametrized by the same
element in $\mathbb{Z}(\frac{n}{n-q})$. And the conjecture was proved affirmatively by A. Nemethi and P. Popescu-Pampu \[NPo\].

**Definition 3.1.** An $e$-tuple of nonnegative integers $(n_1, \ldots, n_e)$ is called admissible if every denominator in the continued fraction $[n_1, \ldots, n_e]$ is positive. It is easy to see that an admissible $e$-tuple of nonnegative integers is either 0 or consisting only of positive integers. Let $Z_e$ be the set of all admissible $e$-tuples such that $[n_1, \ldots, n_e] = 0$, or equivalently, $Z_e$ be the set of all $e$-tuples of integers which can be obtained by a sequence of blow-ups from $(0)$. For $\frac{n}{n-q} = [a_1, \ldots, a_e]$, we define

$$Z_e(\frac{n}{n-q}) := \{(n_1, \ldots, n_e) \in Z_e | \ 0 \leq n_i \leq a_i, \ \text{for} \ i = 1, \ldots, e\}.$$ 

For each $e$-tuple $n \in Z_e(\frac{n}{n-q})$, P. Lisca constructed a smooth 4-manifold $W_{n,q}(n)$ whose boundary is diffeomorphic to the link of a cyclic singularity of type $A_{n,q}$, also known as the Lens space $L(n, q)$: First consider a linear chain consisting of $e$ number of unknots in $S^3$ with framings $n_1, \ldots, n_e$ respectively. Let $N(n)$ be a 3-manifold obtained by Dehn surgery on this framed link. Since $[n_1, \ldots, n_e] = 0$, it is clear that $N(n)$ is diffeomorphic to $S^1 \times S^2$. And then, using a framed link $L$ in $N(n)$ as in Figure 2 one can obtain a cobordism $C_{n,q}(n)$ by attaching 4-dimensional 2-handles to the $L \subset S^1 \times S^2 \times \{1\} \subset S^1 \times S^2 \times I$. Finally, choosing a diffeomorphism $\varphi : N(n) \to S^1 \times S^2$ again, one can construct a desired smooth (in fact symplectic) 4-manifold

$$W_{n,q}(n) := C_{n,q}(n) \cup \varphi S^1 \times D^3.$$ 

Note that, since any self-diffeomorphism $\varphi$ of $S^1 \times S^2$ extends to $S^1 \times D^3$, the diffeomorphism type of $W_{n,q}(n)$ is independent of the choice of $\varphi$. According to P. Lisca [L], any symplectic filling of $(L(n, q), \xi_{st})$ is orientation-preserving diffeomorphic to a blowing up of $W_{n,q}(n)$ for some $n \in Z_e(\frac{n}{n-q})$.

Next, for each $n \in Z_e(\frac{n}{n-q})$, M. Bhupal and B. Ozbagci constructed a genus-0 PALF structure on $S^1 \times D^3$ so that the attaching circles of $(-1)$-framed 2-handles in $W_{n,q}(n)$ lie on a generic fiber (refer to [BOz] for details). 

**Figure 2.** The framed link $L \subset N(n)$
One can construct a PALF structure on $S^1 \times D^3$ over the disk corresponding to each $n \in \mathbb{Z}_e\left(\frac{n}{n-q}\right)$ as follows: Note that it depends on a blowing-up sequence from (0). For each $n \in \mathbb{Z}_e\left(\frac{n}{n-q}\right)$, a generic fiber $F_n$ is the disk with $e$ holes. We may assume the holes in the disk are ordered linearly from left to right as in Figure 3. If $n \in \mathbb{Z}_e\left(\frac{n}{n-q}\right)$ is obtained from $n' \in \mathbb{Z}_{e-1}$ by blowing up at $j^{th}$ term $(1 \leq j \leq e-2)$, we make a generic fiber $F_n$ to be a surface obtained from $F_{n'}$ by splitting the $(j+1)^{th}$ hole so that vanishing cycles $\{x_i | i = 1, 2, \ldots, e-2\}$ for $n'$ are naturally extended to $\{\tilde{x}_i | i = 1, 2, \ldots, e\}$ in $F_n$. Then the monodromy factorization changes from $x_1 x_2 \cdots x_{e-2}$ to $\tilde{x}_1 \tilde{x}_2 \cdots \tilde{x}_{e-2} \beta_j$, where $\beta_j$ is a curve on $F_n$ so that it encircles the 1-, $\ldots$, $j$-, $(j+2)$-labelled holes but skipping the $(j+1)$-labelled hole. For a blowing up at $(e-1)^{th}$ term, we just add the $e^{th}$ hole to $F_{n'}$ at the right of the $(e-1)^{th}$ hole and we add a Dehn twist on a curve encircling the $e^{th}$ hole. In this way we get a genus-0 PALF structure on $W_{n,q}(n)$ so that, if the attaching circle of a $(-1)$-framed 2-handle $h$ in $W_{n,q}(n)$ is the meridian of a $n_i$-framed unknot, the 2-handle $h$ corresponds to a Dehn twist on a curve $\gamma_i$ encircling first $i$ holes. For example, we refer to Figure 4 below.

M. Bhupal and B. Ozbagci [BOz] also showed that the monodromy factorization for each minimal symplectic filling of the Lens space $L(n, q)$ can be obtained by a sequence of monodromy substitutions from the minimal resolution of the corresponding singularity inductively. Recall that each $i^{th}$ step of the sequence consists of a monodromy substitution of the form

$$W_i \cdot \gamma_1^{m_{i,1}} \cdots \gamma_e^{m_{i,e}} = W_{i+1} \cdot \gamma_{a_{i+1}},$$

where $W_{i+1}$ is a monodromy factorization for some $n \in \mathbb{Z}_e\left(\frac{n}{n-q}\right)$ whose the $a_{i+1}^{th}$ component is 1. Then it is easy to check that the PALF with a monodromy factorization $W_{i+1} \cdot \gamma_{a_{i+1}}$ is a rational 4-ball.
In this section, as the first step for the proof of our main theorem (Theorem 1.1), we construct a genus-0 or genus-1 positive allowable Lefschetz fibration (PALF) structure on each minimal resolution of non-cyclic quotient surface singularities. Note that a genus of the PALF structure is determined only by the existence of a bad vertex in the minimal resolution graph of the corresponding singularity. That is, if the minimal resolution graph has no bad vertex, a genus of the PALF structure is 0, and the genus is 1 otherwise. And then, we check that a contact structure on the boundary induced from the PALF structure is the Milnor fillable contact structure, so that every PALF structure obtained by monodromy substitutions is also a Stein filling of \((L, \xi_{st})\).

4.1. No bad vertex cases. If the minimal resolution graph \(\Gamma\) of a quotient surface singularity does not have a bad vertex, then there is a well-known genus-0 PALF structure on the minimal resolution \(\Gamma\) due to D. Gay and T. Mark [GaM]: For each vertex \(v_i\) with a degree \(-b_i\), we consider the 2-sphere \(\Sigma_i\) with \(b_i\) holes. Then the
fiber surface $\Sigma$ is obtained by gluing $\Sigma_i$ along their boundaries according to $\Gamma$ and the vanishing cycles are the set of curves which are parallel to the boundary of each $\Sigma_i$. Note that we end up with only one right-handed Dehn twist on the connecting neck. For example, we refer to Figure 5 below. Note that this PALF structure is compatible with the symplectic structure $\omega$ given by a convex plumbing $X_\Gamma$ of symplectic surfaces, where each vertex represents a symplectic surface with self-intersection $-b_i$ and they intersect each other $\omega$-orthogonally according to $\Gamma$. Then the induced contact structure $\xi$ on the boundary $\partial X_\Gamma$ is compatible with the open book decomposition coming from the PALF structure above. It was shown that $\xi$ is indeed the Milnor fillable contact structure by H. Park and A. Stipsicz [PS]. In fact, their argument holds for any negative-definite intersection matrix of $\Gamma$.

![Figure 5. A genus-0 PALF on minimal resolution of $D_{8,3}$](image)

4.2. Bad vertex cases. If the minimal resolution graph $\Gamma$ of a non-cyclic quotient surface singularity has a bad vertex, we now construct a genus-1 PALF structure on the minimal resolution $\Gamma$ as follows: First we construct a PALF structure on $X_L$, where $X_L$ is a convex plumbing given by a maximal linear subgraph $\Gamma_L$ of $\Gamma$. And we consider a 4-dimensional Kirby diagram of $X_L$, which can be easily obtained from the PALF structure of $X_L$. Then we could get a Kirby diagram of $X_\Gamma$ by adding a 2-handle $h$ or two 2-handles $\{h_1, h_2\}$ to that of $X_L$ depending on the type of arm which is not in $\Gamma_L$. After introducing a cancelling 1-handle/2-handle pair, the 2-handles not coming from the Kirby diagram of $X_L$ can be thought of as vanishing cycles of a new fiber $F$ which is obtained by attaching a 1-handle to the
surface $F_L$. Note that the new fiber $F$ is a genus-1 surface with holes. For example, we refer to Figure 6 below.
Next, we check that a contact structure on the boundary induced from the PALF structure constructed above is the Milnor fillable contact structure. First, recall that, for a contact 3-manifold \((L, \xi)\), the 2-plane field \(\xi\) induces a Spin\(^c\) structure \(t_\xi\) on \(L\). Furthermore, if \((W, J)\) is a Stein filling of \((L, \xi)\), then \(t_\xi\) is a restriction of Spin\(^c\) structure \(S\) on \(W\) to \(\partial W = L\) induced by its complex structure \(J\) on \(W\). On the other hand, there is a theorem of Gay-Stipsicz [GaS] which characterizes the contact structure on the link of a quotient surface singularity.

**Theorem 4.1** ([GaS]). Suppose that a small Seifert 3-manifold \(M = M(s_0; r_1, r_2, r_3)\) satisfies \(s_0 \leq -2\) and \(M\) is an L-space. Then two tight contact structures \(\xi_1, \xi_2\) on \(M\) are isotopic if and only if \(t_{\xi_1} = t_{\xi_2}\).

**Theorem 4.2.** The contact structure on the link of non-cyclic quotient surface singularities induced by the PALF structure constructed above is Milnor fillable.

**Proof.** First note that, since a convex plumbing \(X_\Gamma\) of the minimal resolution graph \(\Gamma\) of a quotient singularity is simply connected, the Spin\(^c\) structure \(S\) on \(X_\Gamma\) is determined by the first Chern class \(c_1(S)\). On the other hand, \(t_{\xi_\omega}\) is a restriction of \(S\) whose first Chern class \(c_1(S)\) satisfies the adjunction equality on each vertex in \(\Gamma\). Hence, by Theorem 4.1 above, a PALF structure on \(X_\Gamma\) induces the Milnor fillable contact structure on the boundary if and only if \(c_1(J)\) satisfies the adjunction equality for each vertex in \(\Gamma\), where \(J\) is a complex structure coming from the Stein structure of the PALF structure. From the PALF structure on \(X_\Gamma\) constructed above, we can compute the first Chern class \(c_1(J)\) in terms of vanishing cycles \(C_i\): \(c_1(J)\) is represented by a cocycle whose value on the 2-handle corresponding to \(C_i\) is the rotation number \(r(C_i)\) [G], which can be computed once we fix a trivialization of the tangent bundle of a page [EtOz2]. For the vertices in \(\Gamma_L\), it satisfies the adjunction equality because the PALF structure for no bad vertex cases induces the Milnor fillable contact structure [PS]. The homology classes of the vertices not in \(\Gamma_L\) can be represented by new vanishing cycles together with some vanishing cycles in \(\Gamma_L\), so that we can check whether they satisfy the adjunction equality by computing the rotation number of the vanishing cycles. For example, the \((-2)\)-sphere not in \(\Gamma_L\) in Figure 6 is obtained by blue, red and two purple vanishing cycles.
And the rotation numbers of blue, red and purple vanishing cycles could be $-1$, $+1$, $+1$ respectively while the homology class is represented by $C_{\text{blue}} - C_{\text{red}} + 2C_{\text{purple}}$. Hence $c_1(J)$ satisfies the adjunction equality for all vertices in $\Gamma$, so that the PALF structure $(X_\Gamma, J)$ we constructed in this section is a Stein filling of $(L, \xi_{\text{st}})$. □

5. Lefschetz fibrations on minimal symplectic fillings

As mentioned in the Introduction, M. Bhupal and K. Ono \cite{BOn} listed all possible minimal symplectic fillings for non-cyclic quotient surface singularities. In fact, they showed that each minimal symplectic filling of a non-cyclic singularity $(X, 0)$ is orientation-preserving diffeomorphic to $Z - \nu(E_{\infty})$, where $E_{\infty}$ is the compactifying divisor of $X$ embedded in a rational symplectic 4-manifold $Z$. And they also found all possible pairs of $(Z, E_{\infty})$ for non-cyclic quotient singularities. On the other hand, H. Park, J. Park, D. Shin and G. Urzúa \cite{PPSU} observed that the number of $P$-resolutions in J. Stevens \cite{Ste2} and that of minimal symplectic fillings in Bhupal-Ono’s list \cite{BOn} are almost same. Since there is a one-to-one correspondence between Milnor fibers and $P$-resolutions for quotient singularities \cite{KSB}, it is natural to ask whether every minimal symplectic filling of non-cyclic quotient singularities is a Milnor fiber. In \cite{PPSU}, they proved it using the corresponding complex model. In fact, they even constructed an explicit one-to-one correspondence between the minimal symplectic fillings and the Milnor fibers of non-cyclic quotient surface singularities.

A strategy for the proof of our main theorem (Theorem 1.1) is following: For a given $P$-resolution $Y$ of a non-cyclic quotient singularity $X$, we construct a PALF structure on the minimal resolution of $X$ so that, after appropriate monodromy substitutions, the underlying 4-manifold is diffeomorphic to a 4-manifold obtained by rationally blowing down all singularities of class $T$ in the minimal resolution $Z$ of $Y$. Then, since any Milnor fiber of quotient surface singularities can be obtained by rationally blowing down all singularities of class $T$ in the minimal resolution of the corresponding $P$-resolution topologically, we would be done.

Now we construct a PALF structure on each minimal symplectic filling of a non-cyclic quotient singularity $X$ via the corresponding $P$-resolution $Y$. That is, we first construct a PALF structure on the minimal resolution of $X$ and then we perform a monodromy substitution method to get a PALF structure on the 4-manifold obtained by rationally blowing down all singularities of class $T$ in $Z$, which are following: Let $\Gamma_Z$ be the dual graph of the minimal resolution $Z$ of $Y$. For the sake of convenience, we divide all $P$-resolutions into the following two cases - Those which have a maximal linear subgraph $\Gamma_L$ of $\Gamma_Z$ containing all the singularities of class $T$ and those which do not have such a maximal linear subgraph.

5.1. **Case 1.** Let $Y$ be a $P$-resolution of a non-cyclic quotient singularity $X$ whose minimal resolution graph $\Gamma_Z$ has a maximal linear subgraph $\Gamma_L$ containing all singularities of class $T$ in $Y$. Note that the subgraph $\Gamma_L$ becomes the minimal resolution graph of a $P$-resolution $Y'$ for some cyclic quotient singularity $X'$. Then, by combining a PALF structure on the minimal resolution of $X'$ and a technique
developed in Section 4, we can construct a PALF structure on the minimal resolution of \(X\) based on the PALF structure of the minimal resolution of \(X'\). Explicitly, starting from a PALF structure \((F_X', y_1y_2 \cdots y_m)\) on the minimal resolution of \(X'\), we get a PALF structure \((F_X, x_1 \cdots x_n\tilde{y}_1\tilde{y}_2 \cdots \tilde{y}_m)\) on the minimal resolution of \(X\), where the vanishing cycles \(\tilde{y}_i's\) in \(F_X\) are natural extensions of the corresponding vanishing cycles \(y_i's\) in \(F_{X'}\), and the vanishing cycles \(\{x_1, \ldots, x_n\}\) come from the corresponding vertices in the arm which is not contained in \(\Gamma_L\). Note that a genus of the generic fiber \(F_X\) depends on the existence of a bad vertex in the minimal resolution of \(X\). And then, we perform a monodromy substitution method to get a PALF structure on \(Y\): Since \(Y'\) contains all singularities of class \(T\) lying in \(Y\), if a monodromy substitution of the form \(y_1y_2 \cdots y_m = z_1z_2 \cdots z_l\) gives a PALF structure on \(Y'\), then \((F_X, x_1 \cdots x_n\tilde{z}_1\tilde{z}_2 \cdots \tilde{z}_l)\) gives a desired PALF structure on \(Y\), where the vanishing cycles \(\tilde{z}_i's\) in \(F_X\) are natural extensions of the corresponding vanishing cycles \(z_i's\) in \(F_{X'}\). In some cases, even if \(y_1y_2 \cdots y_m = z_1z_2 \cdots z_l\) in \(F_X\) can be interpreted as a rational blowdown surgery, a monodromy substitution of the form \(\tilde{y}_1\tilde{y}_2 \cdots \tilde{y}_m = \tilde{z}_1\tilde{z}_2 \cdots \tilde{z}_l\) in the generic fiber \(F_X\) does not correspond to a rational blowdown surgery. Nevertheless, after adding some right-handed Dehn twists \(x_i\) to both sides, we can interpret it as a rational blowdown surgery. See Example 5.1 below.

**Example 5.1.** Let \((X, 0)\) be a tetrahedral singularity of type \(T_{6(5-2)+5}\) which has the following \(P\)-resolution \(Y\):

```
-3
-2
-5
```

Then a PALF structure on \(P\)-resolution \(Y' = \square \square \square \square \square \) of \(\bullet \bullet \bullet \bullet \bullet \) can be obtained by a monodromy substitution of the form

\[
\alpha_2^2\alpha_3\alpha_4\alpha_5\gamma_4\gamma_5 = xyzw\gamma_3
\]

which is depicted in Figure 7 below. Hence we get a PALF structure \(\beta_1\beta_2\tilde{x}\tilde{y}\tilde{z}\tilde{w}\tilde{\gamma}_3\) on the \(P\)-resolution \(Y\) from a PALF structure \(\beta_1\beta_2\tilde{\alpha}_2\tilde{\alpha}_3\tilde{\alpha}_4\tilde{\alpha}_5\tilde{\gamma}_4\tilde{\gamma}_5\) on the minimal resolution of \(X\) by using a monodromy substitution of the form

\[
\beta_2\tilde{\alpha}_2\tilde{\alpha}_3\tilde{\alpha}_4\tilde{\alpha}_5\tilde{\gamma}_4\tilde{\gamma}_5 = \beta_2\tilde{xy}\tilde{zw}\tilde{\gamma}_3,
\]

which can be interpreted as a rational blowdown surgery topologically. See Figure 8.

5.2. **Case 2.** In this subsection we deal with a \(P\)-resolution \(Y\) of \(X\) such that any maximal linear subgraph \(\Gamma_L\) of the minimal resolution of \(Y\) cannot contain all singularities of class \(T\) lying in \(Y\). Hence we may assume that such a \(P\)-resolution \(Y\) contains one of the subgraphs \(\Gamma_i\) depicted in Figure 9 below except for two cases, which will be discussed in the next subsection later.

Note that each subgraph \(\Gamma_i\) in Figure 9 represents a \(P\)-resolution \(Y_i\) of another quotient surface singularity, say \(X_i\). Since the subgraphs in Figure 9 contain all
Recall that the minimal symplectic filling of $X_i$ corresponding to the $P$-resolution $Y_i$ can be obtained as follows: First we rationally blow down all singularities of class $T$ lying in $Y_i$ except the one containing a central vertex. This yields a 4-manifold diffeomorphic to the minimal resolution of $X_i$ which can be also obtained from $X\Gamma_i$ by blowing down all $(-1)$-spheres until there is no $(-1)$-sphere in the resulting plumbing graph. Since the blowing-ups and blowing-downs can be done in symplectic category, the above argument implies that there is a convex plumbing of a symplectic submanifold of codimension 0 in the minimal resolution of $X_i$ according to the dual graph of singularities of class $T$ containing the central vertex of $\Gamma_i$ and the desired minimal symplectic filling of $X_i$ is obtained by rationally blowing down the convex plumbing. Hence, in order to get a PALF structure on $Y_i$, we only need to find a subword representing the convex plumbing from the monodromy factorization of the minimal resolution of $X_i$. And, it is always possible because we know explicitly how vanishing cycles (i.e. 2-handles) in the PALF structure on the minimal resolution of $X_i$ are corresponding to vertices (i.e. embedded 2-spheres) in the minimal resolution graph of $X_i$. 

Figure 7. A PALF structure on $P$-resolution $Y'$

Figure 8. A PALF structure on $P$-resolution $Y$
Example 5.2. In Figure 10 we have a PALF structure on the minimal resolution of $X_1$ whose monodromy factorization is given by

$$\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5\beta\gamma_4\gamma_5.$$
Note that the monodromy factorization without $\beta$ above represents a 4-manifold diffeomorphic to a convex plumbing of a subgraph

$$\begin{array}{ccc}
-2 & -5 & -3
\end{array}$$

lying in $\Gamma_1$. Hence the right-handed side of Figure 11 above gives a desired PAL structure $\beta xyzw\gamma z_3$ on $Y_1$.

5.3. Exceptional cases. There are two exceptional cases in types of $P$-resolutions we cannot cover in Case 1 and Case 2 above, which are $P$-resolutions coming from a tetrahedral singularity of type $T_{30(2-2)+29}$ and an octahedral singularity of type $O_{12(2-2)+11}$. See Figure 11.

As in Case 2, we first rationally blow down all singularities of class $T$ except the one containing a central vertex so that the result 4-manifold is diffeomorphic to the minimal resolution. Since there is a bad vertex in the minimal resolution graph, we have to consider a genus-1 PALF structure on it.

For example, using the same technique as before, we obtain a monodromy factorization of the following form for the minimal resolution of $T_{30(2-2)+29}$

$$xy\gamma_2^2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\gamma_5,$$

where $\alpha_i$ and $\gamma_i$ are curves encircling $i^{th}$ hole and first $i$ holes respectively (refer to Figure 12). Note that, using Hurwitz moves and $y = t_{\alpha_2}(t_{\gamma_2}(x))$, we can change the monodromy factorization as follows:

$$xy\gamma_2^2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\gamma_5$$

$$\sim x\alpha_2^2\gamma_2\alpha_3\alpha_4\alpha_5\gamma_2\gamma_5$$

$$\sim t_x(\alpha_2)t_x(\gamma_2)x^2\alpha_3\alpha_4\alpha_5\gamma_2\gamma_5$$

$$\sim t_x(\alpha_2)^2t_x(\gamma_2)^2(\alpha_3)x^2\alpha_4\alpha_5\gamma_2\gamma_5$$

$$\sim t_x^2(\alpha_3)(t_x^2\cdot t_{\alpha_3}^{-1}\cdot t_x^{-1})(\alpha_2)(t_x^2\cdot t_{\alpha_3}^{-1}\cdot t_x^{-1})(\gamma_2)x^2\alpha_1\alpha_4\alpha_5\gamma_2\gamma_5.$$
factorization becomes
\[ t_x(\alpha_3)\alpha_2\gamma_2\alpha_2^3\alpha_1\alpha_4\alpha_5 f(\gamma_2)\gamma_5. \]

Since the subword \( \alpha_2\gamma_2\alpha_2^3\alpha_1\alpha_4\alpha_5\gamma_5 \) in the monodromy factorization above corresponds to \(-3 \quad -5 \quad -2\) lying in the minimal resolution graph, we get a PALF structure on \( Y_1 \) by rationally blowing down it. In the same way, one could also get a PALF structure on \( Y_2 \).

Hence, summarizing all the arguments in this section, we conclude that

**Theorem 5.1.** There is an explicit algorithm for a genus-0 or genus-1 PALF structure on any minimal symplectic filling of the link of non-cyclic quotient surface singularities.

Recall that we divided all \( P \)-resolutions of \( X \) into two families in the construction of PALF structures on each \( P \)-resolution \( Y \): Those with a maximal subgraph \( \Gamma_L \) containing all singularities of class \( T \) in \( Y \) and those without such a maximal linear subgraph. The algorithm of PALF structures for the first family is essentially the same algorithm as for cyclic cases, which means that the Milnor fiber corresponding to \( P \)-resolution \( Y \) is obtained by rational blowdowns from the minimal resolution of \( X \) topologically [BOZ]. On the other hand, we found a subword diffeomorphic to a convex neighborhood of a linear chain of 2-spheres in a smooth 4-manifold whose boundary is \( L(p^2, pq - 1) \) for the second family, which also can be rationally blowdown [EnMV]. Hence we have

**Corollary 5.2.** Any Milnor fiber of the link of quotient surface singularities can be obtained, up to diffeomorphism, by a sequence of rational blowdowns from the minimal resolution of the singularity.
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