MOST ROBUST AND FRAGILE TWO-QUBIT ENTANGLED STATES UNDER DEPOLARIZING CHANNELS

CHAO-QIAN PANG, FU-LIN ZHANG∗
Physics Department, School of Science, Tianjin University, Tianjin 300072, China

YUE JIANG
School of Science, Tianjin Institute of Urban Construction, Tianjin 300384, China

MAI-LIN LIANG
Physics Department, School of Science, Tianjin University, Tianjin 300072, China

JING-LING CHEN
Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin, 300071, China
Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543

Received March 2, 2012
Revised January 25, 2013

For a two-qubit system under local depolarizing channels, the most robust and most fragile states are derived for a given concurrence or negativity. For the one-sided channel, the pure states are proved to be the most robust ones, with the aid of the evolution equation for entanglement given by Konrad et al. [Nat. Phys. 4, 99 (2008)]. Based on a generalization of the evolution equation for entanglement, we classify the ansatz states in our investigation by the amount of robustness, and consequently derive the most fragile states. For the two-sided channel, the pure states are the most robust for a fixed concurrence. Under the uniform channel, the most fragile states have the minimal negativity when the concurrence is given in the region \([1/2, 1]\). For a given negativity, the most robust states are the ones with the maximal concurrence, and the most fragile ones are the pure states with minimum of concurrence. When the entanglement approaches zero, the most fragile states under general nonuniform channels tend to the ones in the uniform channel. Influences on robustness by entanglement, degree of mixture, and asymmetry between the two qubits are discussed through numerical calculations. It turns out that the concurrence and negativity are major factors for the robustness. When they are fixed, the impact of the mixedness becomes obvious. In the nonuniform channels, the most fragile states are closely correlated with the asymmetry, while the most robust ones with the degree of mixture.

Keywords: Entanglement sudden death; Evolution equation of entanglement; Most robust state; Most fragile state

Communicated by: B Kane & C Williams

1 Introduction

Quantum coherent superposition makes quantum information conceptually more powerful than classical information, which is the essential distinction between a quantum system and

∗Corresponding author, email: flzhang@tju.edu.cn
Most robust and fragile two-qubit entangled states under depolarizing channels

Entanglement is a manifestation of the distinction in composite systems [1] and is one of the key resources in the field of quantum information [2]. However, unavoidable coupling between a real quantum system and its environment can cause decoherence, leading to the destruction of entanglement among subsystems simultaneously.

Because of its important role both in fundamental theory and applications in quantum information, dynamics of entanglement in a quantum system under decoherence has attracted wide attention in recent years. And many significant and interesting results have been reported. For instance, the concept of entanglement sudden death (ESD) has been presented by Yu and Eberly [3, 4], which means that entanglement can decay to zero abruptly in a finite time while complete decoherence requires an infinite amount of time. This interesting phenomenon has been recently observed in two sophisticatedly designed experiments with photonic qubits [5] and atomic band [6]. In [7], by utilizing the Jamiołkowski isomorphism, Konrad et al. presented a factorization law for a two-qubit system, which described the evolution of entanglement in a simple and general way, and has been extended in many directions [8, 9, 10, 11].

This paper is concerned with the robustness of entanglement in a quantum system coupled to noise environments. Vidal and Tarrach [12] introduced the robustness of entanglement as a measure of entanglement, corresponding to the minimal amount of mixing with separable states which washed out all entanglement. Subsequently, Simon and Kempe [13] considered the critical amount of depolarization where the entanglement vanishes as a quantitative signature of the robustness of the entanglement, when they studied the robustness of multi-party entanglement under local decoherence, modeled by partially depolarizing channels acting independently on each subsystem. This definition was adopted in the recent work [14], in which an interesting residual effect was pointed out on the robustness of a three-qubit system in an arbitrary superposition of Greenberger-Horne-Zeilinger state and W state.

We notice that, even in the two-qubit system, there are still some questions about the robustness to explore. For instance, the results of [14] show the robustness increases synchronously with the degree of entanglement for a two-qubit pure state. Obviously, it should be very hard to extend this conclusion to mixed states, because the measures of entanglement for mixed states are not unique [15, 16, 17], which would influence the robustness simultaneously. Then, a question arises: For a given value of entanglement, which states are the most robust ones and which are the most fragile ones? In general, the degree of entanglement in a two-qubit system correlates positively with the resource of preparation [15, 16] and its capacity in quantum information [18, 19], therefore it is necessary to compare the stability of states with same entanglement. Many similar investigations have been reported about ten years. For example, the famous maximally entangled mixed states [20, 21] can be considered to have the most residual entanglement under a global noise channel. In [22], Yu and Eberly studied the robustness and fragility of entanglement in some exactly solvable dephasing models. In the recent work of Novotný et al. [23], they present a general and analytically solvable decoherence model without any weak-coupling or Markovian assumption and distinguish the robust and fragile states.

In the present work, we investigate the most robust entangled states (MRES) and the most fragile entangled states (MFES) for a given amount of initial entanglement in the two-qubit system under a local noise channel, and analyze connections between the robustness of entan-
glement and the properties of initial state. Similar analyses have been done for Werner states in dephasing channel [24], X-states of two two-level atoms in the electromagnetic radiation field [25], and dephased pure states in amplitude damping channel [26]. We extend the scope to arbitrary two-qubit entangled states and focus on their properties relating to entanglement, and therefore consider the local depolarizing channels which is invariant under local unitary operations. In our work, the model in [13] and [14] is generalized to the nonuniform case. Namely, the coupling strengths of the two qubits with their environments are different from each other. This generalization is more close to practical cases, in which the entangled two qubits undergo different environments. We adopted the concurrence [15, 16] and the negativity [17] as two measures of entanglement for comparison. In Sec. 2, we review the definitions adopted in this paper and introduce some notations. In Sec. 3, we selectively give the results of the one-sided channel, which is an extreme case of the nonuniform channels. To derive the MFES in this case, a generalization of the evolution equation for quantum entanglement in [7] is presented. In Sec. 4, both the uniform and nonuniform channels are studied. Conclusions and discussions are made in the last section.

2 Notations and definitions

2.1 Channels and robustness

Under local decoherence channels with no interaction between the two qubits, the dynamics of each qubit is governed by a master equation depending on its own environment [27]. The evolution of each reduced density matrix is described by a completely positive trace-preserving map: \( \rho_i(t) = \mathcal{E}_i(t)\rho_i(0) \), \( i = 1, 2 \), and for the whole state \( \rho(t) = \mathcal{E}_1(t) \otimes \mathcal{E}_2(t)\rho(0) \). In the Born-Markov approximation these maps (or channels) can be described using its Kraus representation

\[
\mathcal{E}_i(t)\rho_i(0) = \sum_{j=0}^{M-1} E_{ji}(t)\rho_i(0)E_{ji}^\dagger
\]

where \( E_{ji} \) satisfying \( \sum_{j=0}^{M-1} E_{ji}E_{ji}^\dagger = I \), are the Kraus operators, and \( M \) is the number of operators needed to completely characterize the channel. For the depolarizing channels, the Kraus operators are

\[
E_{0i} = \frac{1}{2}\sqrt{3s_i + 1}I_i, \quad E_{ji} = \frac{1}{2}\sqrt{1-s_i}\sigma_{ji},
\]

where \( \sigma_{ji} \) \( (j = 1, 2, 3) \) is the \( j \)th Pauli matrix for the \( i \)th qubit. We consider the noise parameter \( s_i = \exp(-\kappa_i t) \), where \( \kappa_i \) is the decay constant determined by the strength of the interaction of the \( i \)th qubit with its environment and \( t \) is the interaction time [28]. In the Bloch sphere representation for the qubit density operator, the transformation in Eqs. (1) and (2) is given by \( \rho_i(t) = \frac{1}{2}(I_i + s_i\vec{r}_i \cdot \vec{\sigma}_i) \) with \( \vec{r}_i \) being the the initial Bloch vector.

Following the method in [13] and [14], we adopt the critical noise parameter, which is positively associated with the ESD time, as the definition of the robustness. The asymmetry between the environments interacting with the two qubits is described by the time-independent parameter \( \Delta = (\kappa_1 - \kappa_2)/(\kappa_1 + \kappa_2) \). It is easy to find that the MRES and MFES for a given value of initial entanglement are independent of the value of \( \kappa_1 + \kappa_2 \), when the uniform
parameter $\Delta$ is fixed. Therefore, without loss of generality, we choose the decay constants $\kappa_1 = 1 + \Delta$ and $\kappa_2 = 1 - \Delta$, with the uniform parameter $\Delta \in [0, 1]$. Then, the noise parameters are written as $s_1 = s^{1+\Delta}$ and $s_2 = s^{1-\Delta}$, where $s = e^{-t}$. The robustness for a state $\rho$ under a nonuniform channel is written as

$$R(\rho) = 1 - s_{\text{crit}}(\rho),$$

(3)

where $s_{\text{crit}}(\rho)$ denotes the critical value of the noise parameter $s$, at which ESD of the two-qubit system occurs with the initial state $\rho$. Though the ESD time is irrelevant to assessing the asymptotic behavior of the robustness in multi-qubit systems in the limit of large number of particles [29, 30], it is still practicable to compare the robustness of entangled states in a given two-qubit or three-qubit system. In our recent work [30], we present the speed of disentanglement as a quantitative signature of the robustness of the entanglement and show it is effective to reflect the asymptotic behavior in multi-qubit systems. One can find that, MRES and MFES in subsection 4.2 for the uniform channels and the most robust symmetrical three-qubit pure states in [14] coincide with the results in [30].

2.2 Measures of entanglement

The concurrence of a pure two-qubit state $|\psi\rangle = c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle$ is given by $C(|\psi\rangle) = 2|c_1c_4 - c_2c_3|$. For a mixed state, the concurrence is defined as the average concurrence of the pure states of the decomposition, minimized over all decompositions of $\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|$

$$C(\rho) = \min \sum_j p_j C(|\psi_j\rangle),$$

(4)

It can be expressed explicitly as [15, 16] $C(\rho) = \max \{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, in which $\lambda_1, \ldots, \lambda_4$ are the eigenvalues of the operator $R = \rho(\sigma_y \otimes \sigma_y)\rho^* (\sigma_y \otimes \sigma_y)$ in decreasing order and $\sigma_y$ is the second Pauli matrix.

For a bipartite system described by the density matrix $\rho$, the negativity is defined as [17, 31]

$$N(\rho) = 2 \sum_j |\lambda_j|,$$

(5)

where $\lambda_j$ are the negative eigenvalues of $\rho^{T_2}$ and $T_2$ denotes the partial transpose with respect to the second subsystem. For the two-qubit system, the partially transposed density matrix $\rho^{T_2}$ has at most one negative eigenvalue [21].

2.3 Ansatz states

To derive MRES and MFES for a given value of concurrence or negativity, we adopt the approach in [20, 21]. We randomly generate two-qubit states and derive their degree of entanglement and robustness under a series of nonuniform parameter $\Delta \in \{0, 0.1, 0.2, \ldots, 1\}$. Plotting them in the corresponding robustness-entanglement planes (such as the ones in Fig. 2), we fortunately find that, their regions are always the same as the states

$$\rho_{\text{ansatz}}(r, \theta) = r |\psi(\theta)\rangle \langle \psi(\theta)| + (1 - r)|01\rangle \langle 01|,$$

(6)
where $r \in [0, 1]$ and $|\psi(\theta)\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle$ with $\theta \in [0, \pi/2]$. For each value of $\Delta$, and for both the entanglement measures, this is supported by 150000 random states $\rho_{\text{random}}$ and 150000 weighted random states $\rho_{\delta,\text{random}}$. Here, the random states $\rho_{\text{random}}$ are uniform in the Hilbert space preserving the Haar measure [17, 32], and $\rho_{\delta,\text{random}} = (1 - \delta)\rho_{\text{ansatz}}(r, \theta) + \delta \rho_{\text{random}}$ with random parameters $r$, $\theta$ and $\delta$ with the uniform distributions on the interval $[0, 1]$, $[0, \pi/2]$ and $[0, 0.05]$ respectively. The numerical results indicate that, the family of states (6) contains the entangled states with the maximal or minimal robustness for a given degree of entanglement. Therefore, MRES or MFES can be represented in the form of (6) with a constraint on its parameters $r$ and $\theta$, which we will derive in the following sections. We call the state (6) the \textit{ansatz state} in this paper.

One point to be mentioned is that, the two extreme cases of the states correspond to the bounds in the comparison of negativity and concurrence [33]. As shown in Fig. 1, for a fixed amount of concurrence, the pure state $\rho_{\text{ansatz}}(1, \theta) = |\psi(\theta)\rangle\langle \psi(\theta)|$ achieves the maximum of negativity, and $\rho_{\text{ansatz}}(r, \pi/4)$ has the minimal negativity. Similarly, when the negativity is given, the state $\rho_{\text{ansatz}}(r, \pi/4)$ is at the upper bound of concurrence and $\rho_{\text{ansatz}}(1, \theta)$ at the lower one.

3 One-sided noisy channel

3.1 Robustness vs concurrence in one-sided channel

When $\Delta = 1$, $s_1 = s^2$ and $s_2 = 1$, the channel reduces the one-sided depolarizing channel, where only the first qubit is influenced by its environment. The central result of [7] is the factorization law for an arbitrary pure state $|\psi\rangle$ under a one-sided noisy channel

$$C[(\mathbb{I} \otimes I)|\psi\rangle\langle \psi|] = C[(\mathbb{I} \otimes I)|\psi^+\rangle\langle \psi^+|]C(|\psi\rangle),$$

(7)
where $|\psi^+\rangle = |\psi(\pi/4)\rangle$ and $\$ denotes an arbitrary channel not restricted to a completely positive trace-preserving map. An evident corollary is that, ESD of systems set initially in any entangled pure states occurs at the same time, and consequently all pure states have the same robustness.

An application of (7) in [7] is the inequality for mixed states $\rho_0$,

$$C[(\$ \otimes I)\rho_0] \leq C[(\$ \otimes I)|\psi^+\rangle \langle \psi^+|]C(\rho_0).$$ (8)

From (7) and (8), one can find that under a given one-sided channel, ESD of a mixed state comes not later than the pure state with the same concurrence. Hence, MRES with a fixed concurrence is the pure state. Under the one-sided channel, the state $|\psi^+\rangle$ becomes

$$\rho_s = \frac{1 + s^2}{4} (|00\rangle\langle 00| + |11\rangle\langle 11|) + \frac{1 - s^2}{4} (|01\rangle\langle 01| + |10\rangle\langle 10|) + \frac{s^2}{2} (|00\rangle\langle 11| + |11\rangle\langle 00|).$$

Its concurrence is $C(\rho_s) = \max\{ (3s^2 - 1)/4, 0 \}$. We can derive the value of robustness

$$R_{MRES} = R(|\psi\rangle) = 1 - \frac{1}{\sqrt{3}},$$ (9)

which is a constant and independent of the concurrence.

To obtain MFES, we generalize the factorization law (7) to the arbitrary two-qubit states. Namely, if two states satisfy

$$\rho_1 = \gamma(I \otimes M)\rho_0(I \otimes M^\dagger)$$ (10)

with $1/\gamma = \text{Tr}[(I \otimes M)\rho_0(I \otimes M^\dagger)]$ and $M = I + \vec{a} \cdot \vec{\sigma}$, ($a \in [0, 1]$), the relation

$$C[(\$ \otimes I)\rho_1]C(\rho_0) = C[(\$ \otimes I)\rho_0]C(\rho_1)$$ (11)

holds for all one-sided channels $\$$. We omit the proof of the above relation, which is exactly the same as the process for (7) in [7]. As for the pure states, we can conclude that the robustness of $\rho_0$ and $\rho_1$ in (10) have the same value.
Then, we find that the ansatz states (6) can be classified by the amount of robustness, with the aid of the generalized factorization law (11). Namely, choosing $\rho_0 = \rho_{\text{ansatz}}(c, \pi/4)$ and $\vec{a} = (0, 0, (1 - p)/(1 + p))$, the ansatz states are written as

$$\rho_{\text{ansatz}}(r, \theta) = \rho(c, p) = \gamma(I \otimes M)\rho_0(I \otimes M^\dagger),$$

(12)

where the relations between the two pairs of parameters are

$$c = \frac{r - r \cos 2\theta}{1 - r \cos 2\theta}, \quad p = \tan \theta.
$$

(13)

The robustness of the ansatz state depends only on the parameter $c$ in $\rho_0$ and is given by

$$R[\rho(c, p)] = 1 - \sqrt{\frac{2 - c}{2 + c}},$$

(14)

which becomes the result for the pure states in (9) when $c = 1$. The concurrence is obtained as

$$C[\rho(c, p)] = \frac{2cp}{c + (2 - c)p^2} = r \sin 2\theta.$$

(15)

In Fig. 2 (a), we plot the ansatz states with some discrete values of $c$ or $p$ in the robustness-concurrence plane. One can find that MFES achieve the maximal concurrence when the value of robustness is fixed. It is easy to determine the maximal value of concurrence in (15) with a fixed $c$. Then the constraint on the ansatz states for the MFES can be obtained as

$$c(1 + p^2) - 2p^2 = 0,$$

(16)

where $0 \leq p \leq 1$, or equivalently

$$2r \cos^2 \theta = 1,$$

(17)

with $0 \leq \theta \leq \pi/4$ and $1/2 \leq r \leq 1$. It is interesting that the conditions leading to the length of the Bloch vector for the two qubit are $r_1 = 2(1 - r)$ and $r_2 = 0$, and their difference $\delta_r = r_1 - r_2$ satisfies $\delta_r + N = 1$. Our numerical result shows that, MFES maximize the difference $\delta_r$ with a fixed value of concurrence.

### 3.2 Robustness vs negativity in one-sided channel

It is easy to prove that MRES under an arbitrary one-sided noise channel for a given negativity are also pure states, with the aid of the results in above subsection. This conclusion can be obtained by supposing $\rho$ denotes an arbitrary entangled two-qubit state, $|\psi_1\rangle$ and $|\psi_2\rangle$ are two pure states, and $N(\rho) = N(|\psi_1\rangle)$ and $C(\rho) = C(|\psi_2\rangle)$. Two relations can be obtained as $R(\rho) \leq R(|\psi_2\rangle)$ and $R(|\psi_1\rangle) = R(|\psi_2\rangle)$, from which we deduce the conclusion.

In Fig. 2, one can notice that the region of the ansatz states in the robustness-negativity plane and the one in the robustness-concurrence case are similar. Thus, in order to determine MFES we can just derive the states $\rho(c, p)$ with the maximal negativity for a fixed $c$. The negativity of the ansatz states is

$$N[\rho_{\text{ansatz}}(r, \theta)] = \sqrt{r^2 \sin^2 2\theta + (1 - r)^2} - (1 - r).$$

(18)
Inserting the relations (13) into Eq. (18), one can find that the maximal negativity occurs when

\[ p = \sqrt{\frac{2\sqrt{2} - c(-1 + c)c^{3/2} + 2c^2 - c^3}{-8 + 24c - 20c^3 + 5c^3}}. \]

(19)

The corresponding relation of the parameters \( r \) and \( \theta \) is very close to the numerical results for \( \Delta = 0.99 \) in Fig. 6 (a) in the following section.

In the MFES, when the parameter \( c \to 1 \), the entanglement \( N \to 1 + 2(c - 1) \) and the difference between the lengths of the two Bloch vectors \( \delta_r \) approaches \( 2(1 - c) \). In other words, MFES with negativity close to 1 reaches the maximal \( \delta_r \). At the other extreme, when the parameter \( \theta \to 0 \), and meanwhile entanglement of the entangled composition decreases, its proportion \( r \) drops to 1/2 when \( \theta \to 0 \). Now, MFES have the maximal entropy among the family of the ansatz states.

4 Two-sided noisy channel

![Fig. 3. (color online) The relation between the robustness and the concurrence of a two-qubit system in a pure entangled state. From top to bottom the curves refer to \( \Delta = 1, 0.75, 0.5, 0.25, 0 \).](image)

We fail in generalizing the relations in (10) and (11) to the two-sided channel case. Therefore, we calculate the ESD critical condition of the ansatz states directly

\[ \mathcal{P} = 4r^2 \sin^2 2\theta s_1^2 s_2^2 - [1 + (1 - 2r)s_1 s_2]^2 
+ [r \cos 2\theta(s_1 - s_2) + (1 - r)(s_1 + s_2)]^2 = 0, \]

(20)

which is one of the main bases to determine MRES and MFES in this section. When \( r = 1 \), one can obtain the relation between the entanglement of pure states and their critical noise parameters

\[ \mathcal{C}^2(|\psi\rangle) = \mathcal{N}^2(|\psi\rangle) = \frac{(1 - s_1^2)(1 - s_2^2)}{4s_1^2 s_2^2 - (s_1 - s_2)^2}. \]

(21)

For the Bell state \(|\psi^+\rangle\), the above equation gives the robustness \( \mathcal{R}(|\psi^+\rangle) = 1 - 1/\sqrt{3} \), which is independent of the nonuniform parameter \( \Delta \). In this sense, our generalization of robustness is reasonable and rational. When \( \Delta = 0 \), \( \mathcal{R}(|\psi\rangle) = 1 - 1/\sqrt{2\mathcal{C}(|\psi\rangle)} + 1 \), which is the result of
the uniform channels given in [14]. In Fig. 3, we plot the robustness and the entanglement of the pure state. Under a fixed $\Delta$, the robustness is a monotone increasing function of the entanglement. A non-maximally entangled pure state becomes more fragile as $\Delta$ decreases.

4.1 Robustness vs concurrence in two-sided channel

Because of the invariance of the depolarizing channels under LU transformations, we can show that MRES for concurrence under the two-sided channel are the pure states, which has the maximal negativity for a fixed concurrence as shown in Fig. 1. According to the procedure given by Wootters [16], one can always obtain a decomposition $\{|\phi_i\rangle\}$ minimizing the average concurrence in Eq. (4), $\rho_0 = \sum_i t_i |\phi_i\rangle \langle \phi_i|$, in which $\sum_i t_i = 1$ and all the elements $|\phi_i\rangle \langle \phi_i|$ have the same value of concurrence as the mixed state $\rho_0$. The elements are equivalent under LU transformation to the same state $|\psi\rangle |\phi_i\rangle = U^A_i \otimes U^B_i |\psi\rangle$, (22) with the concurrence $C(|\psi\rangle) = C(\rho_0)$. After passage through a basis-independent local channel $s_1 \otimes s_2$, the concurrence $C[(s_1 \otimes s_2)|\phi_i\rangle \langle \phi_i|] = C[(s_1 \otimes s_2)|\psi\rangle \langle \psi|]$. It then immediately follows, by convexity, that

$$C[(s_1 \otimes s_2)\rho_0] \leq C[(s_1 \otimes s_2)|\psi\rangle \langle \psi|],$$

(23)

which proves that MRES for fixed concurrence are the pure states.

![Fig. 4. (color online) The relation between the robustness and the concurrence of the MRES and MFES under different nonuniform parameters, $\Delta = 0$, 0.25, 0.75, 0.5, 0. The dots on the lines denote the quasi-MFES, whose parameters $\beta = 0.1, 0.2, 0.3, 0.4$, with the values of concurrence increasing.]

Similar to the case of the one-sided depolarizing channels, MFES in the present case are also the states achieving the maximal concurrence for a given value of robustness, which can be noticed in Fig. 4. From (15) and (20), we find that for a given pair of the critical noise parameters $s_1$ and $s_2$, or equivalently the robustness $R$ and the nonuniform parameter $\Delta$, MFES can be obtained by applying the restraint

$$\alpha = \frac{1}{4} \left[ 1 + \sqrt{8\Omega(2\beta-1)\beta + 1} \right]$$

(24)

on the ansatz states (6), where $\alpha = r \cos^2 \theta \in [0, 1/2]$, $\beta = r \sin^2 \theta \in [0, 1/2]$ and $\Omega = \frac{s_2^2(1-s_1^2)}{s_1^2(1-s_2^2)}$. When $\Delta = 1$, the parameter $\Omega = 0$ and the restraint reduces to $\alpha = 1/2$, which is precisely the result of the one-sided channels in (16) and (17).
For the uniform noise channels where $\Delta = 0$ and $\Omega = 1$, the relation in (24) becomes $\alpha = 1/2 - \beta$ when $\beta \leq 1/4$, and $\alpha = \beta$ when $\beta > 1/4$. Consequently, the MFES can be written in two corresponding regions as

$$p_{\text{MFES}} = \begin{cases} \frac{1}{2} |\psi(\theta)\rangle \langle \psi(\theta)| + |01\rangle \langle 01|, & C < 1/2, \\ r |\psi^+\rangle \langle \psi^+| + (1 - r) |01\rangle \langle 01|, & C \geq 1/2. \end{cases}$$ (25)

In the region $C \geq 1/2$, $r \geq 1/2$ and $\theta = \pi/4$, the MFES are the states $p_{\text{ansatz}}(r, \pi/4)$ with the minimal negativity for a fixed concurrence [33] (see Fig. 1). In the other, where $C < 1/2$, $r = 1/2$ and $\theta \in [0, \pi/4)$, MFES have the maximal mixedness among the ansatz states. Here, we adopt the linear entropy $S_L(\sigma) = \frac{1}{2}(1 - \text{Tr}\sigma^2)$ as the measure of mixedness, and for the ansatz states $S_L(p_{\text{ansatz}}) = \frac{1}{2}(1 - r)$. One can notice that, MFES in this region satisfy $S_L(p_{\text{MEMS}}) \geq S_L(p_{\text{MFES}}) \geq S_L[p_{\text{ansatz}}(r, \pi/4)]$ and $N(p_{\text{MEMS}}) \geq N(p_{\text{MFES}}) \geq N[p_{\text{ansatz}}(r, \pi/4)]$, where $p_{\text{MEMS}}$ are the maximally entangled mixed states in [20].

When $0 < \Delta < 1$, MFES for the arbitrary depolarizing channels is given by the solution of equations (20) and (24), which is shown in Fig. 4 for some values of $\Delta$. It can be noticed that, the difference between the robustness of MRES and MFES decreases with the decrease in the nonuniform parameter $\Delta$. When $\beta \to 0$, the concurrence $C \to 0$, the proportion of the entangled state in MFES approaches $1/2$.

Considering the absence of a brief expression of MFES, we present a family of quasi-MFES. Namely, we replace the critical depolarizing parameter $s^2 \in [1/3, 1]$ in (24) by its mean value $2/3$, and obtain $\Omega \to \Omega = \frac{(2/3)^{s^2-1}}{s^2-1}$. Obviously $\Omega|_{\Delta=0} = 1$ and $\Omega|_{\Delta=1} = 0$. The positions of the quasi-MFES in the robustness-concurrence plane are shown in Fig. 4, which are very close to the MFES. Our numerical result shows, for a given $\beta$, the fidelity of the quasi-MFES and the MFES $F(p_{\text{quasi-MFES}}, p_{\text{MFES}}) \geq 1 - 10^{-4}$ with $F(p_1, p_2) = \left| \text{Tr} \left( \sqrt{p_1} \sqrt{p_2} \sqrt{p_1} \right) \right|^2$, which is supported by one million pairs of randomly generated quasi-MFES and MFES.

### 4.2 Robustness vs negativity in two-sided channel

![Fig. 5. (color online) Plots of the states $\rho(r, \theta)$ in the robustness-negativity planes with the MRES and MFES when the nonuniform parameters $\Delta = 0.5$. The parameters $r/\theta$ takes a set of discrete values in (a)/(b), which are labeled nearby the tangent points with the MRES/MFES.](image)

The MRES or MFES for the negativity under the two-sided depolarizing channels are also the states which minimize or maximize the negativity for a fixed robustness. We adopt the method of Lagrange multiplier to derive the extremum of the negativity in (18) under the constraint (20), where $s_1$ and $s_2$ are constants. It is obtained that the parameters $r$ and $\theta$
satisfy the condition,

$$\frac{\partial N}{\partial r} \frac{\partial \rho}{\partial \theta} - \frac{\partial N}{\partial \theta} \frac{\partial \rho}{\partial r} = 0,$$

when the negativity of an ansatz state reaches an extremum for a fixed robustness. Forms of MRES and MFES are given by the physically accepted solutions to the Eqs. (20) and (26).

When $\Delta = 0$, one can check that the two solutions $\theta = \pi/4$ and $r = 1$ correspond to MRES and MFES respectively. In other words, under the uniform channel, MRES are $\rho_{\text{MRES}} = \rho_{\text{ansatz}}(r, \pi/4)$, and MFES $\rho_{\text{MFES}} = \rho_{\text{ansatz}}(1, \theta)$ are the pure states. As shown in Fig. 1, for a given value of negativity, $\rho_{\text{ansatz}}(r, \pi/4)$ has the maximum of concurrence and the pure state $\rho_{\text{ansatz}}(1, \theta)$ has the minimum. This result and the corresponding one for concurrence indicate that, in the uniform channel, when one of the two degrees of entanglement is fixed, the other is the main factor affecting the robustness.

We were not able to derive an explicit expression of the solution to (20) and (26) for arbitrary nonuniform parameter $\Delta \in (0, 1)$. Thus we plot the ansatz states in the robustness negativity plane for various values of the nonuniform parameter $\Delta$, one of which is shown in Fig. 5. One can notice that, MRES can be expressed as the ansatz states with a constrain as $\theta = \theta(r)$, with $r$ varying from 0 to 1. And the constrain for MRES is $r = r(\theta)$, where the domain is $\theta \in [0, \pi/4]$. Then we search the constrains by numerical computation, and plot them in Fig. 6. It is obvious that the proportion of the entangled composition $r$ in MFES is always more than 3/4. The proportion approaches 1 when the entanglement $N \to 0$, and increases with decreasing $\Delta$. Under a fixed $\Delta$, the parameter $\theta$ seldom fluctuates when the robustness varies in a very large region. When $r$ approaches 1, $\theta$ decreases to $\pi/4$ quickly, and the robustness tends to the maximum $1 - 1/\sqrt{3}$. When $\Delta \to 1$, the domain of the robustness becomes a point $\mathcal{R} = 1 - 1/\sqrt{3}$, where $r = 1$ and $\theta$ varies from $\pi/2$ to $\pi/4$.

5 Conclusion and Discussion

In conclusion, we have investigated the robustness of entangled states for two-qubit system under local depolarizing channels. MRES and MFES are derived for a given amount of entanglement measured by concurrence and negativity. With a numerical simulation, we find
a family of ansatz states $\rho_{\text{ansatz}}(r, \theta)$ in (6), which contains MRES and MFES. In Table 1, we list the constrains on the parameters $r$ and $\theta$ corresponding to MRES and MFES in the one-sided channel and the uniform channel.

Table 1. The constrains on the parameters in the ansatz state $\rho_{\text{ansatz}}(r, \theta)$ when it achieves the MRES or MFES for a fixed value of concurrence $C$ or negativity $N$ in the one-sided channel ($\Delta = 1$) and the uniform channel ($\Delta = 0$).

|          | $\Delta = 0$ |          | $\Delta = 1$ |
|----------|--------------|----------|--------------|
| MRES     | MFES         | MRES     | MFES         |
| $C$      | $r = 1$      | $r = \frac{1}{4}$ for $C < \frac{1}{2}$ | $r = 1$ | $2r \cos^2 \theta = 1$ |
| $N$      | $\theta = \frac{\pi}{4}$ | $r = 1$ | $r = 1$ | Defined by Eqs. (19) and (13) |

Under the one-sided channel, for both measures of entanglement the pure states are proved to be MRES by utilizing the factorization law for the evolution of concurrence given by [7]. A generalized factorization law is presented in (10) and (11). Based on this result, we classify the ansatz states by the values of robustness and derive MFES for the two measures of entanglement. In MFES for concurrence, the length of the Bloch vector for the free qubit is zero, and the difference between the lengths of the two Bloch vectors, $\delta_r$, reaches its maximum for a given concurrence. When the value of negativity approaches 1, the corresponding MFES approach the states with the maximal $\delta_r$ for a fixed negativity.

![Figure 7](image)

Fig. 7. (color online) Averages of (a) concurrence, (b) negativity and (c) linear entropy $S_L$ of the states with a given amount of robustness in the case of $\Delta = 0$ (dots) and $\Delta = 1$ (squares).

Under the uniform two-sided channel, MFES for a given negativity are the pure states,
and MRES are \( \rho_{\text{ansatz}}(r, \pi/4) \). In contrast, MRES for concurrence are the pure states, while MFES are \( \rho_{\text{ansatz}}(r, \pi/4) \), when \( C \geq 1/2 \), but the states \( \rho_{\text{ansatz}}(1/2, \theta) \) with \( \theta \in [0, \pi/4) \) when \( C < 1/2 \). The pure states and states \( \rho_{\text{ansatz}}(r, \pi/4) \) are precisely on the boundaries of the region for arbitrary two-qubit entangled states in the concurrence-negativity plane \([33]\), which is shown in Fig. 1. The states \( \rho_{\text{ansatz}}(1/2, \theta) \) have the maximal linear entropy among the ansatz states, and both its negativity and linear entropy are between the ones of \( \rho_{\text{ansatz}}(r, \pi/4) \) and \( \rho_{\text{MEMS}} \), when they have the equal concurrence. Under general two-sided channels, the pure states are proved to be most robust when the concurrence is given. When the entanglement approaches zero, MFES for the two entanglement measures tend to the results in the uniform channel.

To make a interpretation to the characteristics of the MRES and MFES mentioned above, we analyze the influence on the robustness caused by the entanglement properties and other possible factors based on 300000 random entangled states. We adopt the decomposition of the random states in \([17]\) as \( \rho_{\text{random}} = U \rho_D U^\dagger \), and generate the \( 4 \times 4 \) unitary matrices \( U \) uniformly under the Haar measure \([32]\). The difference from \([17]\) is that we generate three independent numbers \( \alpha_j \), \( (j = 1, 2, 3) \) uniformly in the interval \([0, \pi/2]\) and get the nonzero elements of the diagonal density matrices \( \rho_D \) as \( \{ \cos^2 \alpha_1 \cos^2 \alpha_2, \cos^2 \alpha_1 \sin^2 \alpha_2, \sin^2 \alpha_1 \cos^2 \alpha_3, \sin^2 \alpha_1 \sin^2 \alpha_3 \} \). In this scenario, without affecting the qualitative conclusions in the following paragraphs, the probability of the states nearby the MFES and MRES is increased. Based on the random entangled states, the averages of some quantities of the states with the given values of the robustness or normalized robustness are plotted in Figs. 7, 8, and 9. Here, the normalized robustness \( \tilde{R}_C \) and \( \tilde{R}_N \) are defined as

\[
\tilde{R}_{C,N}(\rho) = \frac{R(\rho)}{R(\rho_{\text{MRES}}) - R(\rho_{\text{MFES}})},
\]

where \( \rho_{\text{MRES}} \) and \( \rho_{\text{MFES}} \) are the MRES and MFES with the same concurrence or negativity as \( \rho \), corresponding to the subscripts \( C \) and \( N \) respectively.

Fig. 8. (color online) Averages of linear entropy \( S_L \) of the states a given amount of normalized robustness \( \tilde{R}_C \) (dots) and \( \tilde{R}_N \) (squares) when \( \Delta = 0 \).

In Fig. 7, one can find that, in the statistical sense, the robustness increases with the entanglement, measured by concurrence and negativity, but decreases with the degree of mixture. The trend of the line for \( \tilde{R}_C \) in Fig. 8 is in accord with Fig. 7(c), but the one for \( \tilde{R}_N \) shows an opposite tendency. These indicate two points: (1) When the value of negativity is given, the concurrence is the dominating factor for the robustness in the uniform channel,
and the trend of the linear entropy results from the changes of concurrence; (2) For the case with a fixed concurrence, the degree of mixture may be a influential factor of the robustness besides the negativity. These agree with the MRES and MFES in the uniform channel. For a fixed negativity, MRES maximize the concurrence and MFES minimize it. For a fixed concurrence, MRES maximize the negativity and minimize the linear entropy, and MFES minimize the negativity in the region of $C \geq 1/2$ but has the maximal linear entropy among the ansatz states for $C < 1/2$. And both the negativity and the linear entropy of MFES for $C < 1/2$ are between the MEMS and the states minimizing the negativity.

One can compare the results for $\Delta = 0$ and $\Delta = 1$ in Fig. 7 and find that when the noise channels become nonuniform, the influence of the entanglement on the robustness decrease, and the relationship between the linear entropy and the robustness weakens for $R$ close to zero but strengthens near its maximum. It is obvious that the robustness in the nonuniform channel is affected by the unbalance between the two qubits. Here, we consider $\delta_r$ as a quantitative signature of the asymmetry. It is shown in Fig. 9 that, in the one-sided noise channel, when the value of concurrence or negativity is fixed, the relationship between $\delta_r$ and robustness is significant when the normalized robustness is less than 0.4, but becomes less evident when the robustness approaches its maximum. Therefore, we surmise that, as the channel becomes nonuniform, MFES are closely correlated with $\delta_r$, but MRES with the linear entropy. This point is well represented in the MRES and MFES. When the amount of concurrence is given, MRES under general two-sided channels maximize the negativity and minimize the linear entropy, and MFES in the one-sided channel achieve the maximum of $\delta_r$. When the negativity is fixed, the MRES changes from the states with maximal concurrence to the states minimizing the linear entropy as the nonuniform parameter $\Delta$ increases from 0 to 1. The MFES for negativity in the one-sided channel have the maximal $\delta_r$ when $\mathcal{N} \to 1$.

At last, we make some prospects for further extension of the results in this paper. Based on the classification of the ansatz states under one-sided channel, it is easy to study the relation between robustness and entanglement. It leaves us a question of whether the arbitrary two-qubit states can be classified by robustness? The answer to this question would bring us an explicit expression of robustness under one-sided channel. The evolution equation of entanglement given in [7] has been generalized in many directions [8, 9, 10, 11]. As an application of the generalized evolution equations, it is interesting to study robustness of entanglement in multi-qubit or multi-qudit systems. On the other hand, the capacity of an
entangled qubit pair in quantum information [18, 19] does not depend on the amount of entanglement only. It is necessary to investigate the robustness of the capacities in a specific quantum information process. Our scenario to derive the MRES or MFES when an analytic proof is absent can be applied directly to these topics.

Acknowledgements

The authors are very grateful to the reviewers for helpful comments and criticisms. F.L.Z. is supported by NSF of China (Grant No. 11105097). J.L.C. is supported by National Basic Research Program (973 Program) of China under Grant No. 2012CB921900, NSF of China (Grant Nos. 10975075 and 11175089) and also partly supported by National Research Foundation and Ministry of Education, Singapore (Grant No. WBS: R-710-000-008-271).

References

1. A. Einstein, B. Podolsky, and N. Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Physical Review, 47, pp. 777-780 (1935).
2. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (Cambridge, 2000).
3. T. Yu and J. H. Eberly, *Finite-time disentanglement via spontaneous emission*, Physical Review Letters, 93, 140404 (2004).
4. T. Yu and J. H. Eberly, *Quantum open system theory: bipartite aspects*, Physical Review Letters, 97, 140403 (2006).
5. M. P. Almeida, F. De Melo, M. and Hor-Meyll, A. Salles, S. P. Walborn, P. H. S. Ribeiro and L. Davidovich, *Experimental observation of environment-induced sudden death of entanglement*, Science, 316, 579 (2007).
6. J. Laurat, K. Choi, H. Deng, C. Chou and H. Kimble, *Heralded entanglement between atomic ensembles: preparation, decoherence, and scaling*, Physical Review Letters, 99, 180504 (2007).
7. T. Konrad, F. De Melo, M. Tiersch, C. Kasztelan, A. Aragão and A. Buchleitner, *Evolution equation for quantum entanglement*, Nature Physics, 4, pp. 99-102 (2007).
8. C. Yu, X. X. Yi, and H. S. Song, *Evolution of entanglement for quantum mixed states*, Physical Review A, 78, 062330 (2008).
9. Z. G. Li, S. M. Fei, Z. D. Wang, and W. M. Liu, *Evolution equation of entanglement for bipartite systems*, Physical Review A, 79, 024303 (2009).
10. Z. Liu and H. Fan, *Dynamics of the bounds of squared concurrence*, Physical Review A, 79, 032306 (2009).
11. M. Tiersch, F. De Melo, and A. Buchleitner, *Entanglement evolution in finite dimensions*, Physical Review Letters, 101, 170502 (2008).
12. G. Vidal and R. Tarrach. *Robustness of entanglement*, Physical Review A, 59, pp. 141-155 (1999).
13. C. Simon and J. Kempe, *Robustness of multiparty entanglement*, Physical Review A, 65, 052327 (2002).
14. B.-K. Zhao and F.-G. Deng, *Residual effect on the robustness of multiqubit entanglement*, Physical Review A, 82, 014301 (2010).
15. S. Hill and W. K. Wootters, *Entanglement of a pair of quantum bits*, Physical Review Letters, 78, pp. 5022-5025 (1997).
16. W. K. Wootters, *Entanglement of formation of an arbitrary state of two qubits*, Physical Review Letters, 80, pp. 2245-2248 (1998).
17. K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein(1998), *Volume of the set of separable states*, Physical Review A, 58, pp. 883-892.
18. J. Lee and M. S. Kim, *Entanglement teleportation via Werner states*, Physical Review Letters, 84, pp. 4236-4239 (2000).
Most robust and fragile two-qubit entangled states under depolarizing channels

19. G. Bowen and S. Bose, *Teleportation as a depolarizing quantum channel, relative entropy, and classical capacity*, Physical Review Letters, 87, 267901 (2001).
20. W. J. Munro, D. F. V. James, A. G. White, and P. G. Kwiat, *Maximizing the entanglement of two mixed qubits*, Physical Review A, 64, 030302(R) (2001).
21. T.-C. Wei, K. Nemoto, P. M. Goldbart, P. G. Kwiat, W. J. Munro, and F. Verstraete, *Maximal entanglement versus entropy for mixed quantum states*, Physical Review A, 67, 022110 (2003).
22. T. Yu and J. H. Eberly, *Phonon decoherence of quantum entanglement: Robust and fragile states*, Physical Review B, 66, 193306 (2002).
23. J. Novotný, G. Alber, and I. Jex, *Entanglement and decoherence: fragile and robust entanglement*, Physical Review Letters, 107, 090501 (2011).
24. K. Ann and G. Jaeger, *Local-dephasing-induced entanglement sudden death in two-component finite-dimensional systems*, Physical Review A, 76, 044101 (2007).
25. M. Ali, G. Alber, and A. R. P. Rau, *Manipulating entanglement sudden death of two-qubit X-states in zero-and finite-temperature reservoirs*, Journal of Physics B: Atomic, Molecular and Optical Physics, 42, 025501 (2008).
26. X.-F. Qian and J. Eberly, *Initial conditions and entanglement sudden death*, Physics Letters A, 376, pp. 2931-2934 (2012).
27. A. Borras, A. P. Majtey, A. R. Plastino, M. Casas, and A. Plastino, *Robustness of highly entangled multiqubit states under decoherence*, Physical Review A, 79, 022108 (2009).
28. W. Dür and H. Briegel, *Stability of macroscopic entanglement under decoherence*, Physical Review Letters, 92, 180403 (2004).
29. L. Aolita, R. Chaves, D. Cavalcanti, A. Adán, and L. Davidovich, *Scaling Laws for the Decay of Multiqubit Entanglement*, Physical Review Letters, 100, 080501 (2008).
30. F.-L. Zhang, Y. Jiang, and M.-L. Liang, *Speed of disentanglement in multi-qubit systems under depolarizing channel*, arXiv:1104.5057 (2011).
31. G. Vidal and R. F. Werner, *Computable measure of entanglement*, Physical Review A, 65, 032314 (2002).
32. K. Życzkowski and M. Kuś, *Random unitary matrices*, Journal of Physics A: Mathematical and General, 27, 4235 (1994).
33. F. Verstraete, K. Audenaert, J. Dehaene, and B. Moor, *A comparison of the entanglement measures negativity and concurrence*, Journal of Physics A: Mathematical and General, 34, 10327 (2001).