The Hodge filtration and the contact-order filtration of derivations of Coxeter arrangements

Hiroaki Terao
Tokyo Metropolitan University, Mathematics Department
Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan

Abstract
The Hodge filtration of the module of derivations on the orbit space of a finite real reflection group acting on an $\ell$-dimensional Euclidean space was introduced and studied by K. Saito [4] [5]. The filtration is equivalent data to the flat structure or the Frobenius manifold structure. We will show that the Hodge filtration coincides with the filtration by the order of contacts to the reflecting hyperplanes. Moreover, a standard basis for the Hodge filtration is explicitly given. Mathematics Subject Classification (2000): 32S22

1 Introduction and main results
Let $V$ be an $\ell$-dimensional Euclidean vector space with inner product $I$. Let $W$ be a finite irreducible orthogonal reflection group (a Coxeter group) acting on $V$. Let $A$ be the Coxeter arrangement: $A$ is the set of reflecting hyperplanes. Denote the dual vector space of $V$ by $V^*$. Then $V^*$ is equipped with the inner product $I^*$ which is induced from $I$. Let $S$ be the symmetric algebra of $V^*$ over $\mathbb{R}$, which is identified with the algebra of polynomial functions on $V$. The algebra $S$ is naturally graded by $S = \oplus_{q \geq 0} S_q$ where $S_q$ is the space of homogeneous polynomials of degree $q$. (Then $S_1 = V^*$.) Let $\text{Der}_S$ be the $S$-module of $\mathbb{R}$-derivations of $S$. We say that $\theta \in \text{Der}_S$ is homogeneous of degree $q$ if $\theta(S_1) \subseteq S_q$. Choose for each hyperplane $H \in A$ a linear form $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. Let

$$D^{(m)}(A) = \{ \theta \in \text{Der}_S \mid \theta(\alpha_H) \in S_{nH}^m \text{ for any } H \in A \}$$

for each nonnegative integer $m$. Elements of $D^{(m)}(A)$ are called $m$-derivations which were introduced by G. Ziegler [8]. Then one has the contact-order

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filtration of \( \text{Der}_S \):

\[
\text{Der}_S = \mathcal{D}^{(0)}(A) \supset \mathcal{D}^{(1)}(A) \supset \mathcal{D}^{(2)}(A) \supset \cdots.
\]

In \([7]\), we proved that each \( \mathcal{D}^{(m)}(A) \) is a free \( S \)-module of rank \( \ell \).

The Coxeter group \( W \) naturally acts on \( V^* \), \( S \) and \( \text{Der}_S \). The \( W \)-invariant subring of \( S \) is denoted by \( R \). Then it is classically known \([7\text{, V.5.3, Theorem 3}]\) that there exist algebraically independent homogeneous polynomials \( P_1, \ldots, P_\ell \in R \) with \( \deg P_1 \leq \cdots \leq \deg P_\ell \), which are called basic invariants, such that \( R = \mathbb{R}[P_1, \ldots, P_\ell] \). The primitive derivation \( D \in \text{Der}_R \) is characterized by

\[
DP_i = \begin{cases} 1 & \text{for } i = \ell, \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( \Delta \) be an anti-invariant, in other words, a constant multiple of \( Q := \prod_{H \in A} \alpha_H \). Then \( \Delta^2 \in R \). Let

\[
I : \text{Der}_R \times \text{Der}_R \to \frac{1}{\Delta^2} R
\]

be the symmetric \( R \)-bilinear form induced from \( I \). Let

\[
\nabla : \text{Der}_R \times \text{Der}_R \to \frac{1}{\Delta^2} \text{Der}_R
\]

\( (X, Y) \mapsto \nabla_X Y \)

be the Levi-Civita connection with respect to \( I \). Define

\[
T := \{ f \in R \mid Df = 0 \} = \mathbb{R}[P_1, \ldots, P_{\ell-1}].
\]

Then \( \nabla_D : \text{Der}_R \to \frac{\partial}{\partial f} \text{Der}_R \) is a \( T \)-linear covariant derivative.

In \([7]\), we introduced an \( S \)-basis \( \xi^{(m)}_1, \ldots, \xi^{(m)}_\ell \in \text{Der}_S \) for \( \mathcal{D}^{(m)}(A) \) (Definition 2.4) for \( m \geq 0 \). Recall, also from \([7]\), the matrices \( B^{(k)} \ (k \geq 1) \) with entries in \( T \) whose determinants are nonzero constants (Lemma 2.1). Then

**Theorem 1.1.** For \( k \geq 1 \),

1. \((1) \ \nabla_{D \xi^{(2k+1)}_1} \cdots, \nabla_{D \xi^{(2k+1)}_\ell} = (\xi^{(2k-1)}_1, \ldots, \xi^{(2k-1)}_\ell)(B^{(k)})^{-1}B^{(k+1)},
2. \((2) \ \nabla_{D \xi^{(2k-1)}_1} \cdots, \nabla_{D \xi^{(2k-1)}_\ell} = (-1)^{k-1}(\frac{\partial}{\partial f_1}, \ldots, \frac{\partial}{\partial f_\ell}) B^{(k)}.

Define \( G_0 := \{ \delta \in \text{Der}_R \mid [D, \delta] = 0 \} \) and \( G_k := \{ \delta \in \text{Der}_R \mid \nabla^{(k)}_D \delta \in G_0 \} \) for \( k \geq 1 \). Let \( p \geq 0 \). Put \( \mathcal{H}^{(p)} := \bigoplus_{k \geq p} G_k \). The decreasing filtration

\[
\mathcal{H}^{(0)} \supset \mathcal{H}^{(1)} \supset \mathcal{H}^{(2)} \supset \ldots
\]

of \( \mathcal{H}^{(0)} = \text{Der}_R \) is called the Hodge filtration, which was introduced by K. Saito \([3, 4]\) to study the primitive integrals. The following theorem asserts that the Hodge filtration is equal to the filtration induced from the contact-order filtration.

**Theorem 1.2.** (1) \( G_k = \bigoplus_{j=1}^\ell T^{(2k-1)}_j \) \((k \geq 1)\),

(2) \( \mathcal{H}^{(p)} = \mathcal{D}^{(2p-1)}(A) \cap \text{Der}_R = \bigoplus_{j=1}^\ell R^{(2p-1)}_j \) \((p \geq 1)\).
2 Proofs

We will prove Theorems 1.1 and 1.2 in this section. In what follows we use the following notation:

- \( X_1, \ldots, X_\ell \): a basis for \( V^* \)
- \( \partial_i := \partial/\partial X_i \): the partial derivation with respect to \( X_i \) \( (i = 1, \ldots, \ell) \)
- \( A := (a_{ij}) = (I^*(X_i, X_j)) \)
- \( X := (X_1, \ldots, X_\ell) \)
- \( P := (P_1, \ldots, P_\ell) \)
- \( m_j := \deg P_j - 1 \) \( (j = 1, 2, \ldots, \ell) \)
- \( h := m_\ell + 1 = \deg P_\ell \): the Coxeter number
- \( J(g) := (\partial g_j/\partial X_i) \): the Jacobian matrix for \( g = (g_1, \ldots, g_\ell) \)
- \( \delta[M] := (\delta(M_{ij})) \) for a matrix \( M = (m_{ij}) \) and any mapping \( \delta \)
- \( D^j := D \circ D \circ \cdots \circ D \) \( (j \text{ times}) \)
- \( \nabla^j_D := \nabla_D \circ \nabla_D \circ \cdots \circ \nabla_D \) \( (j \text{ times}) \).

Let \( k \geq 1 \). Define

\[
B^{(k)} := -J(P)^T AJ(D^k[X])J(D^{k-1}[X])^{-1}J(P)
\]
as in [7]. Then we have

Lemma 2.1. (1) Every entry of \( B^{(k)} \) lies in \( T \); \( D[B^{(k)}] = 0 \),
(2) \( \det B^{(k)} \in \mathbb{R}^* \),
(3) \( \deg B^{(k)}_{ij} = m_i + m_j - h \),
(4) \( B^{(k+1)} - B^{(k)} = B^{(1)} + (B^{(1)})^T \).

Proof. [7, Lemma 3.2, Lemma 3.4].

The Levi-Civita connection with respect to \( I \)

\[
\nabla : \text{Der}_R \times \text{Der}_R \longrightarrow \frac{1}{\Delta^2} \text{Der}_R
\]

\[
(X, Y) \longmapsto \nabla_X Y
\]
is characterized by the following two properties:

(A) \( X(I(Y, Z)) = I(\nabla_X Y, Z) + I(Y, \nabla_X Z) \) (compatibility),
(B) \( \nabla_X Y - \nabla_Y X = [X, Y] \) (torsion-freeness).

Define the Christoffel symbol \( \{\Gamma^k_{ij}\} \) by

\[
\nabla_{\partial_i} \partial_j = \sum_{k=1}^\ell \Gamma^k_{ij} \partial_k.
\]
Denote the $R$-module of Kähler differentials by $\Omega^1_R$. Let

$$I^*: \Omega^1_R \times \Omega^1_R \rightarrow R$$

be the symmetric $R$-bilinear form induced from $I^*$. Let $g^{ij} := I^*(dP_i, dP_j)$ and $G := (g^{ij})$, which is an $\ell \times \ell$-matrix with entries in $R$. Note

$$G = J(P)^T A J(P).$$

Define the contravariant Christoffel symbol \{\Gamma^{ij}_k\} by

$$\Gamma^{ij}_k := -\ell \sum_{s=1}^\ell g^{is} \Gamma^{ij}_{sk}$$

as in [3, 3.25]. Define two $\ell \times \ell$-matrices

$$\Gamma^*_k := (\Gamma^{ij}_k), \quad \Gamma_k := (\Gamma^{ij}_{ik}).$$

**Lemma 2.2.** Let $1 \leq k \leq \ell$.

1. $\Gamma^*_k = -G \Gamma_k$,
2. $\frac{\partial}{\partial P_k} [G] = \Gamma^*_k + (\Gamma^*_k)^T$, in particular, $D[G] = \Gamma^*_\ell + (\Gamma^*_\ell)^T$,
3. $\Gamma_k = J(P)^T A \frac{\partial}{\partial P_k} [J(P)]$,
4. $\Gamma^*_\ell = B^{(1)}$.

**Proof.**

1: By definition.

2: Apply [3, 3.26].

3: Let $S_k = (S^{ij}_k)$ be an $\ell \times \ell$-matrix defined by

$$S_k := J(P)^T A \frac{\partial}{\partial P_k} [J(P)].$$

It is enough to prove the (contravariant) compatibility (A) and the torsion-freeness (B):

(A) $\frac{\partial}{\partial P_k} [G] = S_k + S_k^T$,

(B) $\sum_t g^{kt} S^{ij}_t = \sum_t g^{it} S^k_{ij}$,

because of the uniqueness of the Levi-Civita connection. We can verify (A) and (B) as follows:

\[
\frac{\partial}{\partial P_k} [G] = \frac{\partial}{\partial P_k} [J(P)^T A J(P)] = \frac{\partial}{\partial P_k} [J(P)^T] AJ(P) + J(P)^T A \frac{\partial}{\partial P_k} [J(P)] = S_k^T + S_k,
\]
\[
\sum_t g^{kt} S_t^{ij} = \sum_{t,p,q,r,s} \frac{\partial P_k}{\partial X_r} a^{rs} \frac{\partial P_t}{\partial X_p} a^{pq} \frac{\partial}{\partial X_q} \left( \frac{\partial P_j}{\partial X_t} \right)
\]
\[
= \sum_{p,q,r,s} \frac{\partial P_k}{\partial X_r} a^{rs} \frac{\partial P_t}{\partial X_p} a^{pq} \frac{\partial P_i}{\partial X_p} a^{rq} \frac{\partial P_j}{\partial X_q}.
\]

which is symmetric with respect to \(i\) and \(k\). Thus \(\sum_t g^{kt} S_t^{ij} = \sum_t g^{it} S_t^{kj}\).

(4) is easy, e.g., [6, (2.17) (2.18)].

Remark 2.3. If \(P_1, \ldots, P_\ell\) are chosen so that they satisfy the equality
\[
D[g^{ij}] = \delta_{i+j,\ell+1},
\]

it is known (e.g., [2, pp. 275]) that
\[
\Gamma^{ij}_k / m_j = \Gamma^{ji}_k / m_i.
\]

By Lemma 2.2 (4) and (2), in this case, one has
\[
B_{ij}^{(1)} = \frac{m_j}{h} \delta_{i+j,\ell+1}.
\]

By Lemma 2.4 (4),
\[
B_{ij}^{(k)} = \left\{ (k-1) + \frac{m_j}{h} \right\} \delta_{i+j,\ell+1}.
\]

Definition 2.4. Let \(m \geq 0\). Define \(\xi_1^{(m)}, \ldots, \xi_\ell^{(m)} \in \text{Der}_S\) by:
\[
(\xi_1^{(m)}, \ldots, \xi_\ell^{(m)}) := \left\{ \left( \frac{\partial}{\partial P_1}, \ldots, \frac{\partial}{\partial P_{\ell}} \right) J(P)^T A J(D^k [X])^{-1} \right\} \quad \text{if } m = 2k,
\[
(\xi_1^{(m)}, \ldots, \xi_\ell^{(m)}) := \left\{ \left( \frac{\partial}{\partial P_1}, \ldots, \frac{\partial}{\partial P_{\ell}} \right) J(P)^T A J(D^k [X])^{-1} J(P) \right\} \quad \text{if } m = 2k + 1.
\]

It is easy to see that

Proposition 2.5. [7, Proposition 3.9] For \(k \geq 1\),
\[
(\xi_1^{(2k+1)}, \ldots, \xi_\ell^{(2k+1)}) = - (\xi_1^{(2k-1)}, \ldots, \xi_\ell^{(2k-1)}) (B^{(k)})^{-1} G. \square
\]

The following result is the main theorem of [7]:

Theorem 2.6. [7, Theorem 1.1] Let \(m \geq 0\).

(1) The derivations \(\xi_1^{(m)}, \ldots, \xi_\ell^{(m)}\) form a basis for \(D^{(m)}(A)\). In particular, \(D^{(m)}(A)\) is a free \(S\)-module of rank \(\ell\),

(2)
\[
\text{deg } \xi_j^{(m)} = \begin{cases} 
kh & (m = 2k) 
\kh + m_j & (m = 2k + 1)
\end{cases}
\]

for \(1 \leq j \leq \ell\). \square

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Proof of Theorem 2.4. It is easy to see that each ξ_j^{(2k+1)} lies in Der_R by Definition 2.3 and Proposition 2.5. We have

\[ \left( \nabla D \xi_1^{(1)}, \ldots, \nabla D \xi_\ell^{(1)} \right) \]

\[ = \nabla_D \left( (\partial/\partial P_1, \ldots, \partial/\partial P_\ell) J(P)^T AJ(P) \right) = \nabla_D ((\partial/\partial P_1, \ldots, \partial/\partial P_\ell) G) \]

\[ = (\nabla_D (\partial/\partial P_1), \ldots, \nabla_D (\partial/\partial P_\ell)) G + (\partial/\partial P_1, \ldots, \partial/\partial P_\ell) D[G] \]

\[ = (\partial/\partial P_1, \ldots, \partial/\partial P_\ell) \Gamma_1^T G + (\partial/\partial P_1, \ldots, \partial/\partial P_\ell) (\Gamma_1^T + (\Gamma_1^*)^T) \]

\[ = (\partial/\partial P_1, \ldots, \partial/\partial P_\ell) \Gamma_1^* = (\partial/\partial P_1, \ldots, \partial/\partial P_\ell) B^{(1)} \]

by Lemma 2.2. This shows (2) for k = 1.

Next we will show (1) by induction on k. By Proposition 2.3 and Lemma 2.1, one computes

\[ \left( \nabla D \xi_1^{(2k+1)}, \ldots, \nabla D \xi_\ell^{(2k+1)} \right) = -\nabla_D \left( \left( \xi_1^{(2k-1)}, \ldots, \xi_\ell^{(2k-1)} \right) (B^{(k)})^{-1} G \right) \]

\[ = -\left( \nabla D \xi_1^{(2k-1)}, \ldots, \nabla D \xi_\ell^{(2k-1)} \right) (B^{(k)})^{-1} G \]

\[ - \left( \xi_1^{(2k-1)}, \ldots, \xi_\ell^{(2k-1)} \right) (B^{(k)})^{-1} D[G] \]

\[ = \left( \xi_1^{(2k-3)}, \ldots, \xi_\ell^{(2k-3)} \right) (B^{(k-1)})^{-1} B^{(k)} (B^{(k)})^{-1} G \]

\[ - \left( \xi_1^{(2k-3)}, \ldots, \xi_\ell^{(2k-3)} \right) (B^{(k-1)})^{-1} B^{(1)} + (B^{(1)})^T \]

\[ = \left( \xi_1^{(2k-3)}, \ldots, \xi_\ell^{(2k-3)} \right) (B^{(k-1)})^{-1} G \]

\[ - \left( \xi_1^{(2k-3)}, \ldots, \xi_\ell^{(2k-3)} \right) (B^{(k-1)})^{-1} (B^{(k)} - B^{(k-1)}) \]

\[ = \left( \xi_1^{(2k-1)}, \ldots, \xi_\ell^{(2k-1)} \right) (-I_\ell - (B^{(k-1)})^{-1} B^{(k)} + I_\ell) \]

\[ = - \left( \xi_1^{(2k-1)}, \ldots, \xi_\ell^{(2k-1)} \right) (B^{(k-1)})^{-1} B^{(k)}. \]

Here, when k = 1, read ξ_j^{(-1)} = ∂/∂P_j (j = 1, ..., ℓ) and B^{(0)} = I_ℓ. Then this computation proves (1). The assertion (2) for k > 1 easily follows from (1) and (2) for k = 1. □

Lemma 2.7. Let \( T_k := \bigoplus_{j=1}^\ell T_{k,j}^{(2k-1)} \) for \( k \geq 1 \). Then, for \( p \geq 1 \),

\[ \bigoplus_{k \geq p} T_k = \bigoplus_{j=1}^\ell R \xi_j^{(2p-1)} = D^{(2p-1)}(A) \cap \text{Der}_R. \]

Proof. For the second equality, it is easy to see

\[ \bigoplus_{j=1}^\ell R \xi_j^{(2p-1)} = \left( \bigoplus_{j=1}^\ell S \xi_j^{(2p-1)} \right) \cap \text{Der}_R. \]
by averaging over $W$.

Next we will prove the first equality. Define a set $S := \{\xi_j^{(2k-1)} | 1 \leq j \leq \ell, 1 \leq k \leq k\}$. First we will show that $S$ is linearly independent over $T$. Define matrices $H_k$ ($k \geq 0$) by

$$H_0 := I_\ell \quad \text{and} \quad H_k := (-1)^k (B^{(1)})^{-1} G (B^{(2)})^{-1} G \cdots (B^{(k)})^{-1} G \quad (k \geq 1).$$

By applying Proposition 2.3 inductively, we have

$$(\xi_1^{(2k+1)}, \ldots, \xi_{\ell}^{(2k+1)}) = (\xi_1^{(1)}, \ldots, \xi_{\ell}^{(1)}) H_k$$

for $k \geq 0$. Since $D[B^{(k)}] = 0$ and $D^2[G] = D[B^{(1)} + (B^{(1)})^T] = 0$ by Lemmas 2.1 and 2.2, we obtain $D^j[H_k] = 0$ for $j > k$. Suppose that $S$ is linearly dependent over $T$. Then there exist vectors $x_0, x_1, \ldots, x_m \in T^\ell$ satisfying $x_m \neq 0$ and

$$\sum_{k=0}^{m} (\xi_1^{(2k+1)}, \ldots, \xi_{\ell}^{(2k+1)}) x_k = 0.$$ 

Thus we have

$$\sum_{k=0}^{m} H_k x_k = 0.$$

Apply $D^m$ to both sides, and we get $D^m[H_m] x_m = 0$. Note that $\det D[G]$ is a nonzero constant by K. Saito [4, 5.1][2, Corollary 4.1]. So

$$D^m[H_m] = (m!)[B^{(1)}]^{-1} D[G](B^{(2)})^{-1} D[G] \cdots (B^{(k)})^{-1} D[G]$$

is invertible. Thus $x_m = 0$, which is a contradiction. This shows that the set $S$ is linearly independent over $T$. Therefore the sum $\sum_{k \geq p} T_k$ is a direct sum.

For $k \geq p$,

$$T_k \subseteq D^{(2k-1)}(A) \cap \text{Der}_R \subseteq D^{(2p-1)}(A) \cap \text{Der}_R = \bigoplus_{j=1}^{\ell} R \xi_j^{(2p-1)}.$$

Therefore we have

$$\bigoplus_{k \geq p} T_k \subseteq \bigoplus_{j=1}^{\ell} R \xi_j^{(2p-1)}.$$

Compare the Poincaré series of both sides:

$$\text{Poin}(\bigoplus_{k \geq p} T_k, t) = \left( \prod_{j=1}^{p-1} (1 - t^{m_j+1}) \right)^{-1} \sum_{k \geq p} (t^{(k-1)h+m_1} + \ldots + t^{(k-1)h+m_\ell})$$

$$= \left( \prod_{j=1}^{\ell} (1 - t^{m_j+1}) \right)^{-1} (t^{(p-1)h+m_1} + \ldots + t^{(p-1)h+m_\ell})$$

$$= \text{Poin}(\bigoplus_{j=1}^{\ell} R \xi_j^{(2p-1)}, t).$$
This implies the first equality. □

Proof of Theorem 1.2. It is enough to show $G_k = T_k$ for each $k \geq 1$ because of Lemma 2.7. Since Theorem 1.1 (2) asserts
\[
(\nabla_D^k \xi_{2^k-1}, \ldots, \nabla_D^k \xi_{2^k-1}) = (-1)^{k-1}(\frac{\partial}{\partial P_1}, \ldots, \frac{\partial}{\partial P_\ell})B^{(k)},
\]
one has
\[
\nabla_D^k \xi_{2^k-1} \in G_0 \ (1 \leq j \leq \ell, k \geq 1)
\]
because $D[B^{(k)}] = 0$. Therefore we have
\[
G_k = \nabla_D^{-k} G_0 \supseteq \bigoplus_{j=1}^\ell T \xi_{2^k-1} = T_k
\]
for each $k \geq 1$. In [4, 6.3] [5, 5.7], K. Saito showed
\[
\bigoplus_{k \geq 1} G_k = D_R(\Delta^2) := \{ \theta \in \text{Der}_R \mid \theta(\Delta^2) \in \Delta^2 R \}.
\]
On the other hand, it is known [3, Theorem 6.60] that
\[
D_R(\Delta^2) = D^{(1)}(A) \cap \text{Der}_R.
\]
Therefore, by Lemma 2.7, we obtain
\[
\bigoplus_{k \geq 1} G_k = \bigoplus_{k \geq 1} T_k
\]
and thus $G_k = T_k$ for each $k \geq 1$. □

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