ON ARITHMETIC ZARISKI PAIRS IN DEGREE 6

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Dedicated to Professor Mutsuo Oka for his sixtieth birthday.

Abstract. We define a topological invariant of complex projective plane curves. As an application, we present new examples of arithmetic Zariski pairs.

1. Introduction

In this paper, we mean by a plane curve a complex reduced (possibly reducible) projective plane curve. The following definition is due to Artal-Bartoło [3]:

Definition 1.1. A pair \((C, C')\) of plane curves of the same degree is called a Zariski pair if there exist tubular neighborhoods \(T \subset \mathbb{P}^2\) of \(C\) and \(T' \subset \mathbb{P}^2\) of \(C'\) such that \((T, C)\) and \((T', C')\) are diffeomorphic, while \((\mathbb{P}^2, C)\) and \((\mathbb{P}^2, C')\) are not homeomorphic.

The first example of Zariski pairs was studied by Zariski [33] in order to show that an equisingular family of plane curves need not be connected. Zariski considered a six-cuspidal sextic curve \(C\) with the six cusps lying on a conic, and proved that \(\pi_1(\mathbb{P}^2 \setminus C)\) is isomorphic to the free product of cyclic groups of order 2 and 3. He then showed that, if there exists a six-cuspidal sextic curve \(C'\) with the six cusps not lying on a conic, then \(\pi_1(\mathbb{P}^2 \setminus C')\) is not isomorphic to \(\pi_1(\mathbb{P}^2 \setminus C)\). Oka [17] completed Zariski’s work by constructing explicitly a non-conical six-cuspidal sextic curve \(C'\), and showed that \(\pi_1(\mathbb{P}^2 \setminus C')\) is a cyclic group of order 6. Therefore the moduli space \(\mathcal{M}(6A_2)\) of plane sextics possessing six cusps as their only singularities has at least two connected components that are distinguished by the fundamental groups of the complements. (See [21] for a simple construction of the pair \((C, C')\).) Recently, Degtyarev [11] showed that \(\mathcal{M}(6A_2)\) has exactly two connected components.

Many examples of Zariski pairs have been known now. The standard method to distinguish \((\mathbb{P}^2, C)\) and \((\mathbb{P}^2, C')\) topologically is to compare the fundamental groups of the complements. The fundamental groups are calculated directly by Zariski-van Kampen theorem (see [23]), or they are compared by some indirect methods: for example, by means of Alexander polynomials, or by proving (non-)existence of finite étale Galois coverings of the complements with a given Galois group.

Definition 1.2. Plane curves \(C\) and \(C'\) are said to be conjugate if there exist a homogeneous polynomial \(\Phi(x_0, x_1, x_2)\) of complex coefficients and an automorphism \(\sigma\) of the field \(\mathbb{C}\) such that we have

\[C = \{\Phi = 0\}\quad \text{and} \quad C' = \{\Phi^\sigma = 0\},\]

where \(\Phi^\sigma\) is the polynomial obtained from \(\Phi\) by applying \(\sigma\) to the coefficients of \(\Phi\).

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Definition 1.3. A Zariski pair \((C, C')\) is called an arithmetic Zariski pair if \(C\) and \(C'\) are conjugate.

A difficulty in constructing examples of arithmetic Zariski pairs comes from the fact that, if \(C\) and \(C'\) are conjugate, then \(\pi_1(\mathbb{P}^2 \setminus C)\) and \(\pi_1(\mathbb{P}^2 \setminus C')\) have the same pro-finite completions.

Artal Bartolo, Carmona Ruber, and Cogolludo Agustín ([5], [6]) constructed an arithmetic Zariski pair in degree 12. They distinguished \((\mathbb{P}^2, C)\) and \((\mathbb{P}^2, C')\) by means of the braid monodromy.

In this paper, we introduce an invariant \(N_C\) of the homeomorphism type of \((\mathbb{P}^2, C)\) for plane curves \(C\) of even degree. By means of this invariant, we present some examples of arithmetic Zariski pairs in degree 6.

In order to explain our examples more precisely, we introduce some terminologies. A Dynkin type is a finite formal sum

\[
R = \sum_{l \geq 1} a_l A_l + \sum_{m \geq 4} d_m D_m + \sum_{n=6}^8 e_n E_n,
\]

where \(a_l, d_m\) and \(e_n\) are non-negative integers, almost all of which are zero. The rank of the Dynkin type \(R\) is defined by

\[
\text{rank}(R) = \sum a_l l + \sum d_m m + \sum e_n n.
\]

An ADE-sextic is a plane curve of degree 6 with only simple singularities. The type \(R\) of an ADE-sextic \(C\) is the Dynkin type of the singularities of \(C\). Then \(\text{rank}(R)\) is equal to the total Milnor number of \(C\), and hence it is at most 19. We say that an ADE-sextic is a maximizing sextic if the total Milnor number is 19 (see Persson [18]). If \(C\) is an ADE-sextic, then the minimal resolution \(X_C\) of the double covering \(Y_C \rightarrow \mathbb{P}^2\) that branches exactly along \(C\) is a \(K3\) surface. When \(C\) is a maximizing sextic, our invariant \(N_C\) of \((\mathbb{P}^2, C)\) coincides with the transcendental lattice of \(X_C\).

Combining our main result with the results of Artal-Bartolo, Carmona-Ruber, Cogolludo-Agustín [4], Degtyarev [11] and the result in [26], we obtain the following:

Theorem 1.4. There exists an arithmetic Zariski pair of maximizing sextics for each of the following Dynkin types:

(i) \(A_{16} + A_2 + A_1\),  (ii) \(A_{16} + A_3\),  (iii) \(A_{18} + A_1\),  (iv) \(A_{10} + A_9\).

The plan of this paper is as follows. In [2] we define an invariant \(N_C\) for curves \(C\) on a smooth projective surface \(S\) satisfying certain conditions, and show that \(N_C\) is in fact an invariant of the \(\Gamma\)-equivalence class of \((S, C)\) (see Definition 2.3). The \(\Gamma\)-equivalence is an equivalence relation coarser than the homeomorphism type of \((S, C)\), and finer than the homeomorphism type of \(S \setminus C\). Applying the main result (Theorem 2.4) to the case \(S = \mathbb{P}^2\), we obtain an invariant of the homeomorphism type of \((\mathbb{P}^2, C)\) for plane curves \(C\) of even degree. In [3] we review the theory of Degtyarev [11] on the connected components of the moduli space \(M(R)\) of ADE-sextics with a given Dynkin type \(R\). In [4] we calculate the connected components of \(M(R)\) for some \(R\) with \(\text{rank}(R) = 19\). Combining this calculation with the result of [3] Theorem 5.8 and using our invariant, we show that some pairs of conjugate maximizing sextics obtained in [4] yield examples of arithmetic Zariski
pairs (the examples (i)-(iii) above). In §5 we present another example of arithmetic Zariski pairs constructed by means of the theory of Hilbert class fields of imaginary quadratic fields (the example (iv) above).

The first example of non-homeomorphic conjugate complex varieties was given by Serre [20]. Since then, only few examples seem to have been treated (e.g., Abelson [1]). The argument of this paper provides us with a new method to construct examples of non-trivial effects of $\text{Aut}(\mathbb{C})$ on the topology of complex varieties.

2. The invariant $N_C$

First we fix some notation and terminologies.

Let $A$ be a finitely generated $\mathbb{Z}$-module. We denote by $A_{\text{tor}}$ the torsion subgroup of $A$, and by $A^{\text{tf}} := A/A_{\text{tor}}$ the torsion-free quotient of $A$. If $b : A \times A \to \mathbb{Z}$ is a symmetric bilinear form on $A$, then $b$ induces a symmetric bilinear form on $A^{\text{tf}}$ in the natural way.

Let $A$ be a free $\mathbb{Z}$-module of finite rank, and $A'$ a submodule of $A$. The primitive closure of $A'$ in $A$ is defined to be the intersection of $A' \otimes \mathbb{Q}$ and $A$ in $A \otimes \mathbb{Q}$. We say that $A'$ is primitive in $A$ if the primitive closure of $A'$ is equal to $A'$.

A lattice is a free $\mathbb{Z}$-module $A$ of finite rank equipped with a non-degenerate symmetric bilinear form $A \times A \to \mathbb{Z}$. Two lattices $A$ and $A'$ are isomorphic if there exists an isomorphism $A \cong A'$ of $\mathbb{Z}$-modules that preserves the symmetric bilinear forms. The automorphism group of a lattice $A$ is denoted by $O(A)$. If $A$ and $A'$ are lattices, then $A \perp A'$ denotes the orthogonal direct-sum of $A$ and $A'$.

For a topological space $Z$, we denote by $H_2(Z)$ the homology group $H_2(Z, \mathbb{Z})$. When $Z$ is an oriented $C^\infty$-manifold with $\dim_{\mathbb{R}}(Z) = 4$, we have the intersection pairing $b_Z : H_2(Z) \times H_2(Z) \to \mathbb{Z}$. If we further assume that $Z$ is compact, then $H_2(Z)^{\text{tf}}$ becomes a lattice by $b_Z$.

Let $S$ be a smooth complex projective surface such that $\text{Pic}(S) \cong \mathbb{Z}$ and $\pi_1(S) = \{1\}$, and let $H$ be the line bundle on $S$ such that its class is the positive generator of $\text{Pic}(S)$. Let $d$ be a positive even integer, and put

$$\mathcal{L} := H^{\otimes d} \quad \text{and} \quad \mathcal{M} := H^{\otimes d/2}.$$  

An $\mathcal{L}$-curve is a reduced (possibly reducible) member of the complete linear system $|\mathcal{L}|$. Let $C$ be an $\mathcal{L}$-curve given by $s = 0$, where $s$ is a global section of $\mathcal{L}$, and let $\pi : Y \to S$ be the finite double covering that branches exactly along $C$, where $Y$ is the pull-back of the image of the global section $s$ by the squaring morphism $\mathcal{M} \to \mathcal{M}^{\otimes 2} = \mathcal{L}$ over $S$. Note that $Y$ is normal, because $Y$ is a hypersurface in the total space of the line bundle $\mathcal{M}$ with only isolated singular points (Altman and Kleiman [2, Chapter VII, Corollary (2.14)])]. Let $\rho : X \to Y$ be a proper birational morphism from a smooth surface $X$ that induces an isomorphism $\rho^{-1}(Y \setminus \pi^{-1}(C)) \cong Y \setminus \pi^{-1}(C)$. We put $\phi := \pi \circ \rho : X \to S$.  


Then $\phi$ is an étale double covering over $S \setminus C$. We denote by

$$\tilde{M}_C \subset H_2(X)$$

the submodule generated by the homology classes of the integral components of $\phi^{-1}(C) \subset X$. We then put

$$\tilde{N}_C := \{ x \in H_2(X) \mid b_X(x, y) = 0 \text{ for any } y \in \tilde{M}_C \} \quad \text{and} \quad N_C := (\tilde{N}_C)^{tf} \subset H_2(X)^{tf}.$$ 

Note that $N_C$ is primitive in $H_2(X)^{tf}$.

**Lemma 2.1.** The restriction of $b_X$ to $N_C$ is non-degenerate.

**Proof.** Since $H_2(X)^{tf}$ is a lattice by $b_X$, and $N_C$ is the orthogonal complement of $(\tilde{M}_C)^{tf}$ in $H_2(X)^{tf}$, it is enough to show that the restriction of $b_X$ to $(\tilde{M}_C)^{tf}$ is non-degenerate. Let $h \in H_2(X)$ be the first Chern class of the line bundle $\phi^*(\mathcal{H})$. (We have a canonical isomorphism $H_2(X) \cong H^2(X)$.). Since $b_X(h, h) > 0$, the $\mathbb{Z}$-module $\langle h \rangle$ generated by $h$ is a positive-definite lattice of rank 1 by $b_X$. Let $p_1, \ldots, p_t$ be the singular points of $Y$. For each $p_i$, we denote by $\Sigma_i \subset H_2(X)$ the submodule generated by the homology classes of integral curves on $X$ that are contracted to $p_i$ by $\rho$. By the theorem of Mumford [15], the $\mathbb{Z}$-module $\Sigma$ is a negative-definite lattice by $b_X$. The lattice $\Sigma$ is perpendicular to $\langle h \rangle$ and $\Sigma_j (j \neq i)$ with respect to $b_X$. Therefore the submodule

$$M_C^0 := \langle h \rangle \perp \Sigma_1 \perp \cdots \perp \Sigma_t$$

of $H_2(X)$ is a lattice by $b_X$. Let $C_i$ be an irreducible component of $C$, and let $\tilde{C}_i$ be the integral curve on $X$ such that $\phi(\tilde{C}_i) = C_i$. Since Pic($S$) $\cong \mathbb{Z}$ is generated by the class of $\mathcal{H}$, there exists an integer $d_i$ such that $C_i$ is linearly equivalent to $\mathcal{H} \otimes d_i$ on $S$. Then there exists $\gamma \in \Sigma_1 \perp \cdots \perp \Sigma_t$ such that $2[C_i] = d_i + \gamma$ holds in $H_2(X)$. Therefore $M_C \otimes \mathbb{Q}$ is equal to $M_C^0 \otimes \mathbb{Q}$ in $H_2(X) \otimes \mathbb{Q}$. \hfill $\square$

From now on, we consider $N_C$ as a lattice by $b_X$.

**Lemma 2.2.** The isomorphism class of the lattice $N_C$ does not depend on the choice of the morphism $\rho : X \rightarrow Y$.

**Proof.** Let $X' \rightarrow X$ be the blowing up at a point $P$ on $\phi^{-1}(C)$, and let $E$ be the $(-1)$-curve on $X'$ contracted to $P$. Then we have a natural isomorphism

$$H_2(X') = H_2(X) \perp \langle [E] \rangle.$$ 

Hence the lattice $N_C' \subset H_2(X)^{tf}$ constructed from $X'$ by the method described above is isomorphic to $N_C$. \hfill $\square$

Therefore we can consider the isomorphism class of the lattice $N_C$ as an invariant of the $\mathcal{L}$-curve $C$.

**Definition 2.3.** Let $C_1, \ldots, C_m$ be the irreducible components of an $\mathcal{L}$-curve $C$. We denote by $\Gamma_i \subset \pi_1(S \setminus C)$ the conjugacy class of simple loops around $C_i$, and put $\Gamma(C) := \{ \Gamma_1, \ldots, \Gamma_m \}$. Let $C$ and $C'$ be $\mathcal{L}$-curves. We say that $(S, C)$ and $(S, C')$ are $\Gamma$-equivalent if there exists a homeomorphism $S \setminus C \cong S \setminus C'$ such that the induced isomorphism $\pi_1(S \setminus C) \cong \pi_1(S \setminus C')$ gives rise to a bijection from $\Gamma(C)$ to $\Gamma(C')$. 

\[\]
It is obvious that, for \( L \)-curves \( C \) and \( C' \), if \( (S, C) \) and \( (S, C') \) are homeomorphic, then they are \( \Gamma \)-equivalent.

The following is the main result of this paper:

**Theorem 2.4.** The isomorphism class of the lattice \( N_C \) is an invariant of the \( \Gamma \)-equivalence class of \( (S, C) \).

**Proof.** Since \( \pi_1(S) \) is assumed to be trivial, \( \pi_1(S \setminus C) \) is generated by simple loops around irreducible components of \( C \). Therefore a homomorphism \( \pi_1(S \setminus C) \to \mathbb{Z}/2\mathbb{Z} \) that maps every element of \( \Gamma_1 \cup \cdots \cup \Gamma_m \) to the non-trivial element of \( \mathbb{Z}/2\mathbb{Z} \) is unique. Consequently, the homeomorphism type of

\[ U := \phi^{-1}(S \setminus C) \subset X \]

is uniquely determined by the \( \Gamma \)-equivalence class of \( (S, C) \). For a compact subset \( K \) of \( U \), we denote by \( B^K \) the image of the natural homomorphism \( H_2(U \setminus K) \to H_2(U) \). We then put

\[ B_\infty := \bigcap B^K, \]

where \( K \) runs through the set of all compact subsets of \( U \), and set

\[ \tilde{B}_U := H_2(U)/B_\infty \quad \text{and} \quad B_U := (\tilde{B}_U)^{tf}. \]

Since every topological cycle is compact, the intersection pairing \( b_U \) on \( H_2(U) \) sets up a symmetric bilinear form

\[ \beta_U : \tilde{B}_U \times \tilde{B}_U \to \mathbb{Z}. \]

By construction, the isomorphism class of \( (B_U, \beta_U) \) is determined by the homomorphism type of \( U \), and hence by the \( \Gamma \)-equivalence class of \( (S, C) \). Therefore it is enough to show that the lattice \( N_C \) is isomorphic to \( (B_U, \beta_U) \).

We put \( D := \phi^{-1}(C) \), and equip \( D \) with the reduced structure. Let \( T \subset X \) be a tubular neighborhood of \( D \). We put

\[ T^\times := T \setminus D = T \cap U, \]

and consider the Mayer-Vietoris sequence

\[
\begin{array}{cccccc}
\rightarrow & H_2(T^\times) & \overset{i}{\rightarrow} & H_2(T) \oplus H_2(U) & \overset{j}{\rightarrow} & H_2(X) & \rightarrow \\
\rightarrow & x & \overset{(i_T(x), i_U(x))}{\rightarrow} & (y, z) & \overset{j_T(y) - j_U(z)}{\rightarrow} & .
\end{array}
\]

First we prove

\[ (2.1) \quad \text{Im}(j_U) = \tilde{N}_C. \]

It is obvious that \( \text{Im}(j_U) \subseteq \tilde{N}_C \). Let \( [W] \) be an element of \( \tilde{N}_C \) represented by a 2-dimensional topological cycle \( W \subset X \). We can assume the following:

(i) \( W \cap \text{Sing}(D) = \emptyset \),
(ii) \( W \cap D \) consists of a finite number of points, and
(iii) locally around each point \( P \) of \( W \cap D \), \( W \) is a \( C^\infty \)-manifold intersecting \( D \) at \( P \) transversely.
Let $D^{(1)}, \ldots, D^{(n)}$ be the irreducible components of $D$. For each $\nu = 1, \ldots, n$, let $P_{+,1}^{(\nu)}, \ldots, P_{+,k(\nu)}^{(\nu)}$ (resp. $P_{-,1}^{(\nu)}, \ldots, P_{-,l(\nu)}^{(\nu)}$) be the intersection points of $W$ and $D^{(\nu)}$ with local intersection number 1 (resp. $-1$). Since $b_X([W], [D^{(\nu)}]) = 0$, we have $k(\nu) = l(\nu)$.

Let $\Delta \subset \mathbb{C}$ be the closed unit disk, and let $I \subset \mathbb{R}$ be the closed interval $[0, 1]$. Since $D^{(\nu)} \setminus \text{Sing}(D)$ is path-connected for each $\nu$, we have continuous maps

$$\xi^{(\nu)}_i : \Delta \times I \to X$$

for $\nu = 1, \ldots, n$ and $i = 1, \ldots, k(\nu)$ with the following properties:

(i) $\xi^{(\nu)}_i(\Delta \times I) \cap D \subset D^{(\nu)} \setminus \text{Sing}(D)$,

(ii) $(\xi^{(\nu)}_i)^{-1}(D) = \{0\} \times I$,

(iii) $\xi^{(\nu)}_i(0, 0) = P_{+,i}^{(\nu)}$, and $\xi^{(\nu)}_i$ induces a homeomorphism from $\Delta \times \{0\}$ to a closed neighborhood $\Delta^{(\nu)}_{+,i} \subset W$ of $P_{+,i}^{(\nu)}$ in $W$, and

(iv) $\xi^{(\nu)}_i(0, 1) = P_{-,i}^{(\nu)}$, and $\xi^{(\nu)}_i$ induces a homeomorphism from $\Delta \times \{1\}$ to a closed neighborhood $\Delta^{(\nu)}_{-,i} \subset W$ of $P_{-,i}^{(\nu)}$ in $W$.

We then put

$$W' := \left( W \setminus \bigcup_{\nu,i} (\Delta^{(\nu)}_{+,i} \cup \Delta^{(\nu)}_{-,i}) \right) \cup \bigcup_{\nu,i} \xi^{(\nu)}_i(\partial \Delta \times I).$$

That is, we cut out discs $\Delta^{(\nu)}_{+,i}$ and $\Delta^{(\nu)}_{-,i}$ from $W$, and put tubes $\xi^{(\nu)}_i(\partial \Delta \times I)$ that connect pairs of circles $\partial \Delta^{(\nu)}_{+,i}$ and $\partial \Delta^{(\nu)}_{-,i}$ around the paths $\xi^{(\nu)}_i(\{0\} \times I)$ on $D^{(\nu)}$ from $P^{(\nu)}_{+,i}$ to $P^{(\nu)}_{-,i}$. Since the local intersection numbers of $W$ and $D^{(\nu)}$ at $P^{(\nu)}_{+,i}$ and $P^{(\nu)}_{-,i}$ have opposite signs, we can put an orientation on each $\xi^{(\nu)}_i(\Delta \times I)$ in such a way that $\Delta^{(\nu)}_{+,i} \subset W$ and $\xi^{(\nu)}_i(\Delta \times \{0\}) \subset \partial(\xi^{(\nu)}_i(\Delta \times I))$ (resp. $\Delta^{(\nu)}_{-,i} \subset W$ and $\xi^{(\nu)}_i(\Delta \times \{1\}) \subset \partial(\xi^{(\nu)}_i(\Delta \times I))$) have the opposite orientations. Then, with the orientation on the tubes $\xi^{(\nu)}_i(\partial \Delta \times I)$ induced from the orientation of $\xi^{(\nu)}_i(\Delta \times I)$, the space $W'$ becomes a topological cycle. Note that $W$ and $W'$ are homologous in $X$, because $W - W'$ is the boundary of the 3-dimensional topological chain $\bigcup \xi^{(\nu)}_i(\Delta \times I)$. Moreover $W'$ is disjoint from $D$. Therefore $[W] = [W']$ is contained in $\text{Im}(j_U)$, and hence (2.1) is proved.

Since $T$ is a tubular neighborhood of $D$, we have

$$\text{Im}(i_U) = B_{\infty}.$$  (2.2)

If $j_U(z) = 0$, then $(0, z) \in \text{Ker}(j) = \text{Im}(i)$, and hence $z \in \text{Im}(i_U)$. Therefore we have $\text{Ker}(j_U) \subseteq \text{Im}(i_U)$. Consider the following commutative diagram: @ @

$$\begin{array}{ccc}
0 & \longrightarrow & \text{Ker}(j_U) & \longrightarrow & H_2(U) & \xrightarrow{j_U} & \widetilde{N}_C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Im}(i_U) & \longrightarrow & H_2(U) & \xrightarrow{\lambda} & \widetilde{B}_U & \longrightarrow & 0.
\end{array}$$  (2.3)

The upper sequence is exact by (2.2), and the lower sequence is exact by (2.3). Remark that, by the definition of the intersection pairing, we have

$$b_U(z, z') = b_X(j_U(z), j_U(z'))$$
for any $z, z' \in H_2(U)$. Hence, for any $\zeta, \zeta' \in \bar{N}_C \subset H_2(X)$, we have
\[ b_\lambda(\zeta, \zeta') = \beta_U(\lambda(\zeta), \lambda(\zeta')) \]
where $\lambda : \bar{N}_C \rightarrow \bar{B}_U$ is the vertical surjection in the diagram (2.3). Therefore, in order to show that $N_C$ is isomorphic to $(B_U, \beta_U)$, it is enough to prove that $\lambda$ induces an isomorphism $N_C \cong B_U$ on the torsion-free quotients, or equivalently, $\lambda^{-1}((\bar{B}_U)_\text{tor}) = (N_C)_\text{tor}$ holds. It is obvious that $\lambda((N_C)_\text{tor}) \subseteq (\bar{B}_U)_\text{tor}$. Suppose that $\zeta \in N_C$ satisfies $\lambda(\zeta) \in (\bar{B}_U)_\text{tor}$. By (2.4), we have $b_\lambda(\zeta, \zeta') = 0$ for any $\zeta' \in \bar{N}_C$. Since $N_C$ is a lattice, we have $\zeta \in (N_C)_\text{tor}$. \hfill \square

**Corollary 2.5.** Let $C$ and $C'$ be plane curves of the same degree. Suppose that $\deg C = \deg C'$ is even. If $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are homeomorphic, then $N_C$ and $N_{C'}$ are isomorphic.

### 3. Sextics with only simple singularities

Let $\mathbb{P}_s(H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(6)))$ be the projective space of one-dimensional subspaces of the vector space $H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(6))$ of homogeneous polynomials of degree 6 on $\mathbb{P}^2$. For a Dynkin type $R$ of rank $\leq 19$, we denote by
\[ \mathcal{M}(R) \subset \mathbb{P}_s(H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(6))) \]
the space of $ADE$-sextics of type $R$. Using Urabe’s idea [31], Yang [32] made the complete list of Dynkin types $R$ such that $\mathcal{M}(R) \neq \emptyset$. Degtyarev [11] refined Yang’s argument, and gave a method to calculate the connected components of $\mathcal{M}(R)$ for a given $R$. In this section, we expound Degtyarev’s theory.

We fix some notation and terminologies about lattices.

Let $\Lambda$ be a lattice of rank $n = 2 + s_-$ and signature $(2, s_-)$. For a non-zero vector $\omega \in \Lambda \otimes \mathbb{C}$, we denote by $[\omega] \in \mathbb{P}_s(\Lambda \otimes \mathbb{C})$ the one-dimensional vector space spanned by $\omega$. We put
\[ \Omega_\Lambda := \{ [\omega] \in \mathbb{P}_s(\Lambda \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \} . \]
It is easy to verify that $\Omega_\Lambda$ is a complex manifold of dimension $s_- = n-2$ consisting of two connected components.

The **dual lattice** $\Lambda^\vee$ of a lattice $\Lambda$ is defined by
\[ \Lambda^\vee := \{ v \in \Lambda \otimes \mathbb{Q} \mid (x, v) \in \mathbb{Z} \quad \text{for all} \quad x \in \Lambda \} . \]
We have $\Lambda \subset \Lambda^\vee$. An **overlattice** of $\Lambda$ is a submodule $\Lambda'$ of $\Lambda^\vee$ containing $\Lambda$ such that the natural $\mathbb{Q}$-valued symmetric bilinear form on $\Lambda \otimes \mathbb{Q}$ takes values in $\mathbb{Z}$ on $\Lambda'$. The **discriminant group** $G_\Lambda$ of $\Lambda$ is defined by
\[ G_\Lambda := \Lambda^\vee/\Lambda. \]
A lattice is called **unimodular** if $\Lambda^\vee = \Lambda$. A lattice $\Lambda$ is said to be **even** if $(v, v) \in 2\mathbb{Z}$ holds for every $v \in \Lambda$. If $\Lambda$ is an even lattice, we can define a quadratic form
\[ q_\Lambda : G_\Lambda \rightarrow \mathbb{Q}/2\mathbb{Z} \]
by $q_\Lambda(v) := (v, v) \mod 2\mathbb{Z}$, where $v \in \Lambda^\vee$ and $\bar{v} := v \mod \Lambda$. This quadratic form is called the **discriminant form** of $\Lambda$. See Nikulin [16] for the basic properties of discriminant forms.

Let $\Lambda$ be a negative-definite even lattice. A vector $d \in \Lambda$ is called a **root** if $(d, d) = -2$ holds. We say that $\Lambda$ is a **root lattice** if $\Lambda$ is generated by the roots in
The isomorphism classes of root lattices are in one-to-one correspondence with the Dynkin types (see, for example, Ebeling [12, Section 1.4]). We denote by $\Sigma_R^-$ the negative-definite root lattice of Dynkin type $R$. A subset $F$ of $\Sigma_R^-$ is called a fundamental system of roots if every element of $F$ is a root, $F$ is a basis of $\Sigma_R^-$, and each root $v \in \Sigma_R^-$ is written as a linear combination $v = \sum_{d \in F} k_d d$ of roots $d \in F$ with integer coefficients $k_d$ all non-positive or all non-negative. A fundamental system $F$ of roots exists (see [12, Section 1.4]). The intersection matrix of $\Sigma_R^-$ with respect to the basis $F$ is the Cartan matrix of type $R$ multiplied by $-1$.

A lattice is called a K3 lattice if it is even, unimodular, of rank 22 and with signature $(3, 19)$. By the structure theorem of unimodular lattices, a K3 lattice is unique up to isomorphisms (see, for example, Serre [19, Chapter V]).

We now start explaining Degtyarev’s theory. Let $R$ be a Dynkin type with

$$r := \text{rank}(R) \leq 19.$$ 

First, we define a set $Q(R)$ and an equivalence relation $\sim$ on it. We denote by $\langle h \rangle$ the lattice of rank 1 generated by a vector $h$ with $(h, h) = 2$. We put

$$M^0 := \Sigma_R^- \perp \langle h \rangle,$$

which is an even lattice of signature $(1, r)$. We choose a fundamental system of roots $F \subset \Sigma_R^-$ once and for all, and put

$$O_{F, h}(M^0) := \{ g \in O(M^0) \mid g(F) = F, g(h) = h \}.$$

We denote by $Ms$ the set of even overlattices $M$ of $M^0$ satisfying the following two conditions:

1. $\{ v \in M \mid (v, h) = 1, (v, v) = 0 \} = \emptyset$, and
2. $\{ v \in M \mid (v, h) = 0, (v, v) = -2 \} = \{ v \in \Sigma_R^- \mid (v, v) = -2 \}$.

(These conditions correspond to the conditions (a) and (b) in [32, Theorem 2.3].)

For $M \in Ms$, we denote by $Ns(M)$ a complete set of representatives of isomorphism classes of even lattices $N$ of rank $21 - r$ satisfying the following two conditions:

1. $N$ is of signature $(2, 19 - r)$, and
2. the discriminant form $(G_N, q_N)$ of $N$ is isomorphic to $(G_M, -q_M)$.

By Nikulin [16, Proposition 1.6.1], the conditions (n1) and (n2) are equivalent to the following condition:

1. there exists an even unimodular overlattice $L$ of $M \perp N$ with signature $(3, 19)$ such that $M$ and $N$ are primitive in $L$.

Let $N$ be an element of $Ns(M)$. We denote by $Ls(M, N)$ the set of even unimodular overlattices $L$ of $M \perp N$ such that $M$ and $N$ are primitive in $L$. Note that every $L \in Ls(M, N)$ is a K3 lattice. We also denote by $c\Omega s(N)$ the set of connected components of the complex manifold $\Omega_N$. Remark that we have $|c\Omega s(N)| = 2$.

We define $Q(R)$ to be the set of quartets $(M, N, L, c\Omega)$ such that $M \in Ms$, $N \in Ns(M)$, $L \in Ls(M, N)$, and $c\Omega \in c\Omega s(N)$. For quartets $(M, N, L, c\Omega)$ and $(M', N', L', c\Omega')$ in $Q(R)$, we write

$$(M, N, L, c\Omega) \sim (M', N', L', c\Omega')$$

if the following hold.
(i) There exists \( g^0 \in O_{F,h}(M^0) \subset O(M^0) \) such that the induced action of \( g^0 \) on \( M_s \) maps \( M \in M_s \) to \( M' \in M_s \). We denote by \( g_M : M \sim M' \) the unique isomorphism satisfying \( g_M | M^0 = g^0 \).

(ii) Since \( (G_M, -q_M) \) and \( (G_M', -q_M') \) are isomorphic, there exists a canonical bijection between \( N_s(M) \) and \( N_s(M') \). The elements \( N \in N_s(M) \) and \( N' \in N_s(M') \) are corresponding by this bijection; that is, \( N \) and \( N' \) are isomorphic.

(iii) There exists an isomorphism \( g_N : N \sim N' \) of lattices such that the bijection \( Ls(M, N) \sim Ls(M', N') \) induced by the isomorphism \( g_M \oplus g_N \) from \( M \perp N \) to \( M' \perp N' \) maps \( L \in Ls(M, N) \) to \( L' \in Ls(M', N') \), and that the induced isomorphism \( \Omega_N \sim \Omega_N' \) maps \( c \Omega \) to \( c \Omega' \).

For \( (M, N, L, c \Omega) \in Q(R) \), we denote by \([M, N, L, c \Omega] \in Q(R)/\sim \) the equivalence class of the relation \( \sim \) containing \((M, N, L, c \Omega)\).

**Remark 3.1.** If \((M, N, L, c \Omega) \in Q(R)\), then \( M^0 \) is the sublattice of \( L \) generated by \( F \subset L \) and \( h \in L \), \( M \) is the primitive closure of \( M^0 \) in \( L \), and \( N \) is the orthogonal complement of \( M \) in \( L \). Hence \([M, N, L, c \Omega] = [M', N', L', c \Omega']\) holds if and only if there exists an isomorphism \( L \sim L' \) that maps \( F \) to \( F \), \( h \) to \( h \), and such that the induced isomorphism \( \Omega_L \sim \Omega_{L'} \) maps the connected component \( c \Omega \) of \( \Omega_N \subset \Omega_L \) to the connected component \( c \Omega' \) of \( \Omega_{N'} \subset \Omega_{L'} \).

Next we define a map \( \rho \) from the space \( M(R) \) to the set \( Q(R)/\sim \). Let \( C \) be an \( ADE \)-sextic of type \( R \), and let \( X \) be the minimal resolution of the double covering \( Y \to \mathbb{P}^2 \) that branches exactly along \( C \). We denote by \( \mathcal{L} \) the line bundle on \( X \) corresponding to the pull-back of the invertible sheaf \( \mathcal{O}_{\mathbb{P}^2}(1) \). We have \( ([\mathcal{L}],[\mathcal{L}]) = 2 \). We then put

\[ L_X := H^2(X, \mathbb{Z}), \]

which is a \( K3 \) lattice. Let \( F_{(X, \mathcal{L})} \subset L_X \) be the set of cohomology classes of \((-2)\)-curves that are contracted by the desingularization morphism \( X \to Y \), and let \( \Sigma_{(X, \mathcal{L})} \subset L_X \) be the sublattice of \( L_X \) generated by \( F_{(X, \mathcal{L})} \). Then \( \Sigma_{(X, \mathcal{L})} \) is a negative-definite root lattice of type \( R \). It is known that \( F_{(X, \mathcal{L})} \) is a fundamental system of roots in \( \Sigma_{(X, \mathcal{L})} \) (see [25, Proposition 2.4]). In particular, there exists an isomorphism of lattices from \( \Sigma_{(X, \mathcal{L})} \) to \( \Sigma_{R}^{\sim} \) that maps \( F_{(X, \mathcal{L})} \) to the fixed fundamental system of roots \( F \subset \Sigma_{R}^{\sim} \) bijectively. We put

\[ M^0_{(X, \mathcal{L})} := \Sigma_{(X, \mathcal{L})} \perp ([\mathcal{L}]), \]

and choose an isomorphism

\[ \gamma^0_M : M^0_{(X, \mathcal{L})} \sim M^0 \]

satisfying \( \gamma^0_M(F_{(X, \mathcal{L})}) = F \) and \( \gamma^0_M([\mathcal{L}]) = h \). Let \( M_{(X, \mathcal{L})} \) be the primitive closure of \( M^0_{(X, \mathcal{L})} \) in \( L_X \), and \( M \) the even overlattice of \( M^0 \) corresponding to the even overlattice \( M_{(X, \mathcal{L})} \) of \( M^0_{(X, \mathcal{L})} \) by \( \gamma^0_M \). Then \( M \) satisfies the conditions (m1) and (m2) (see [25, Proposition 2.1]). Hence \( M \in M_S \). We denote by

\[ \gamma_M : M_{(X, \mathcal{L})} \sim M \]

the isomorphism induced by \( \gamma^0_M \). Let \( N_{(X, \mathcal{L})} \) be the orthogonal complement of \( M_{(X, \mathcal{L})} \) in \( L_X \). Since the \( K3 \) lattice \( L_X \) is an even unimodular overlattice of \( M_{(X, \mathcal{L})} \perp N_{(X, \mathcal{L})} \) in which \( M_{(X, \mathcal{L})} \) and \( N_{(X, \mathcal{L})} \) are primitive, the lattice \( N_{(X, \mathcal{L})} \)
satisfies the condition (n). Hence there exists a unique element \( N \) of \( N_s(M) \) that is isomorphic to \( N_{(X,\mathcal{L})} \). We choose an isomorphism 
\[
\gamma_N : N_{(X,\mathcal{L})} \cong N.
\]

By the isomorphism
\[
\gamma_M \oplus \gamma_N : M_{(X,\mathcal{L})} \perp N_{(X,\mathcal{L})} \cong M \perp N,
\]
the even unimodular overlattice \( L_X \) of \( M_{(X,\mathcal{L})} \perp N_{(X,\mathcal{L})} \) corresponds to an element \( L \) of \( Ls(M,N) \). We denote by 
\[
\omega_X \in H^{2,0}(X) \subset L_X \otimes \mathbb{C}
\]
the cohomology class of a non-zero holomorphic 2-form on \( X \). Since \( M_{(X,\mathcal{L})} \subset H^{1,1}(X) \), the vector \( \omega_X \) defines a point \([\omega_X]\) of \( \Omega_{N_{(X,\mathcal{L})}} \). Let \( c\Omega \) be the connected component of \( \Omega_N \) that contains the point \([\gamma_N(\omega_X)]\). Thus we obtain a quartet \((M,N,L,c\Omega) \in Q(R)\). The choices we have made during the process of finding \((M,N,L,c\Omega)\) are only on \( \gamma_M^0 \) and \( \gamma_N \). Since \( \gamma_M^0 \) is unique up to \( O_{F,h}(M^0) \) and \( \gamma_N \) is unique up to \( O(N) \), the equivalence class \([M,N,L,c\Omega]\) does not depend on these choices. We thus can put 
\[
\rho(C) := [M,N,L,c\Omega].
\]

**Remark 3.2.** By definition, we have \( \rho(C) = [M,N,L,c\Omega] \) if and only if there exists an isomorphism \( L_X \cong L \) that maps \( F_{(X,\mathcal{L})} \) to \( F, \mathcal{L} \) to \( h \), and such that the induced isomorphism \( \Omega_{L_X} \cong \Omega_L \) maps the point \([\omega_X]\) to \( \Omega_{N_{(X,\mathcal{L})}} \subset \Omega_{L_X} \) to a point of the connected component \( c\Omega \) of \( \Omega_N \subset \Omega_L \).

We now have all the ingredients that are needed to state the main theorem of Degtyarev [11]:

**Theorem 3.3.** The map \( \rho \) induces a bijection from the set of connected components of the space \( \mathcal{M}(R) \) to the set \( Q(R)/\sim \).

The main tool of the proof is the Torelli theorem for the refined period map of marked \( K3 \) surfaces. See the book by Barth, Hulek, Peters and Van de Ven [7, Theorems 12.3 and 14.1 in Chapter VIII].

By definition, we have the following:

**Corollary 3.4.** Let \( C \) be an \( ADE \)-sextic such that \( \rho(C) = [M,N,L,c\Omega] \). Then the lattice \( N \) is isomorphic to the invariant \( N_C \) of the \( \Gamma \)-equivalence class of \((\mathbb{P}^2, C)\).

We explain how to calculate the set \( Q(R)/\sim \). By [16, Proposition 1.4.1], the even overlattices of \( M^0 = \Sigma_R \perp \langle h \rangle \) are in one-to-one correspondence with the totally isotropic subgroups of the discriminant form \((G_{M^0}, q_{M^0})\). For an even overlattice \( M \) of \( M^0 \), we can determine whether \( M \) satisfies the conditions (m1) and (m2) by the method described in [24]. Since \( G_{M^0} \) is finite, we obtain the set \( M_s \). The group \( O_{F,h}(M^0) \) is isomorphic to the automorphism group of the Dynkin diagram of type \( R \), and hence it is finite. Therefore the image of the natural homomorphism
\[
O_{F,h}(M^0) \twoheadrightarrow O(M^0) \rightarrow O(q_{M^0})
\]
is easy to calculate, where \( O(q_{M^0}) \) is the automorphism group of the finite quadratic form \((G_{M^0}, q_{M^0})\) (see [22, Section 6.2]). Consequently we obtain the set
\[
\overline{M_s} := O_{F,h}(M^0) \backslash M_s
\]
of the orbits of the action of $O_{F,h}(M^0)$ on $Ms$. For an element $M$ of $Ms$, let $[M] \in \overline{Ms}$ denote the orbit containing $M$. We also put

$$O_{F,h,M}(M^0) := \{ g \in O_{F,h}(M^0) \mid g \text{ fixes } M \in Ms \}.$$  

We have a natural map

$$pr : Q(R)/\sim \to \overline{Ms}$$

that maps $[M, N, L, cΩ]$ to $[M]$. We denote by $\overline{Ms^2} \subset \overline{Ms}$ the image of the map $pr : Q(R)/\sim \to \overline{Ms}$; that is, we put

$$\overline{Ms^2} := \{ [M] \in \overline{Ms} \mid Ns(M) \neq \emptyset \}.$$  

For $[M] \in \overline{Ms}$, we can determine whether $Ns(M)$ is empty or not by the criterion of Nikulin [10, Theorem 1.10.1]. Hence $\overline{Ms^2}$ is calculated.

Suppose that $[M] \in \overline{Ms^2}$. By [10, Corollary 1.9.4], the set $Ns(M)$ forms a genus. If $r := \text{rank}(R) < 19$, then the isomorphism class of an indefinite lattice $N$ of signature $(2, 19 - r)$ is determined by the spinor genus by Eichler’s theorem (see, for example, Cassels [8]). The method of enumeration of spinor genera in a given genus is described in Conway and Sloane [9, Chapter 15]. When $\text{rank}(R) = 19$, the set $Ns(M)$ is easily calculated by Corollary 3.3 below.

For each $[M] \in \overline{Ms^2}$, we have a natural map

$$pr_{[M]} : pr^{-1}([M]) \to Ns(M)$$

that maps $[M', N', L', cΩ'] \in pr^{-1}([M])$ to the lattice $N \in Ns(M)$ isomorphic to $N' \in Ns(M')$. (Note that, if $[M] = [M']$, then $M$ and $M'$ are isomorphic, and hence $Ns(M)$ and $Ns(M')$ are canonically identified.) Let $N$ be an element of $Ns(M)$. We put

$$F([M], N) := pr^{-1}_{[M]}(N).$$  

We can regard $O_{F,h,M}(M^0)$ as a subgroup of $O(M)$:

$$O_{F,h,M}(M^0) = \{ g \in O(M) \mid g(F) = F, g(h) = h \}.$$  

Then the group $O_{F,h,M}(M^0) \times O(N)$ acts on the set $Ls(M, N) \times cΩs(N)$ in the natural way. The fiber $F([M], N)$ of $pr_{[M]}$ over $N$ is, by definition, equal to the set of orbits of this action:

$$F([M], N) = (O_{F,h,M}(M^0) \times O(N)) \backslash (Ls(M, N) \times cΩs(N)).$$  

By [10, Proposition 1.6.1], there exists a natural bijection between the set $Ls(M, N)$ and the set of isomorphisms of finite quadratic forms from $(G_M, -q_M)$ to $(G_N, q_N)$. Since $G_M \cong G_N$ is a finite abelian group, we obtain the set $Ls(M, N)$. Hence the set $F([M], N)$ can be calculated, provided that the group $O(N)$ and its actions on $(G_N, q_N)$ and on $cΩs(N)$ are calculated.

**Remark 3.5.** When $\text{rank}(R) = 19$, the lattice $N$ is positive-definite. Hence $O(N)$ is finite, and we can easily make the list of elements of $O(N)$. The actions of $O(N)$ on $(G_N, q_N)$ and on $cΩs(N)$ are then readily calculated.

We use the following terminology in [41 and 45]

**Definition 3.6.** Let $τ : cΩs(N) \cong cΩs(N)$ be the transposition of the two connected components of $Ω_N$. An orbit $U \subset Ls(M, N) \times cΩs(N)$ of the action of $O_{F,h,M}(M^0) \times O(N)$ is called real if $U$ is stable under $τ$. 
We review the classical theory of binary forms due to Gauss (see Edwards [13] or Conway and Sloane [9, Chapter 15]). For integers \(a, b, c\), we denote by \(Q[a,b,c]\) the matrix

\[
\begin{bmatrix}
  a & b \\
  b & c \\
\end{bmatrix}
\]

For a positive integer \(d\), we put

\[
Q_d := \{ Q[a,b,c] \mid a \equiv c \equiv 0 \pmod{2}, \ a > 0, \ c > 0, \ ac - b^2 = d \},
\]

on which \(GL_2(\mathbb{Z})\) acts from right by \((Q,g) \mapsto T_g Q g\). The set of isomorphism classes of even positive-definite lattices of rank 2 with discriminant \(d\) is canonically identified with the set \(Q_d/GL_2(\mathbb{Z})\) of \(GL_2(\mathbb{Z})\)-orbits in \(Q_d\).

**Definition 3.7.** We call an \(SL_2(\mathbb{Z})\)-orbit in \(Q_d\) an isomorphism class of even positive-definite oriented lattices of rank 2 with discriminant \(d\).

For \(Q[a,b,c] \in Q_d\), we denote by \(\Lambda[a,b,c]\) (resp. \(\tilde{\Lambda}[a,b,c]\)) the lattice (resp. the oriented lattice) expressed by \(Q[a,b,c]\).

**Proposition 3.8.** Let \(d\) be a positive integer. Then the set

\[
\{ \tilde{\Lambda}[a,b,c] \mid Q[a,b,c] \in Q_d, \ -a < 2b \leq a \leq c \text{ with } b \geq 0 \text{ if } a = c \}
\]

is a complete set of representatives of isomorphism classes of even positive-definite oriented lattices of rank 2 with discriminant \(d\).

**Corollary 3.9.** Let \(d\) be a positive integer. Then the set

\[
(3.1) \quad \{ \Lambda[a,b,c] \mid Q[a,b,c] \in Q_d, \ 0 \leq 2b \leq a \leq c \}
\]

is a complete set of representatives of isomorphism classes of even positive-definite lattices of rank 2 with discriminant \(d\).

**Remark 3.10.** Let \(\Lambda[a,b,c]\) be an element of the set \((3.1)\), and let \([\Lambda[a,b,c]] \in Q_d/GL_2(\mathbb{Z})\) be the \(GL_2(\mathbb{Z})\)-orbit containing \(\Lambda[a,b,c]\). Then the fiber of the natural map \(Q_d/SL_2(\mathbb{Z}) \to Q_d/GL_2(\mathbb{Z})\) over \([\Lambda[a,b,c]]\) consists of two elements if

\[
0 < 2b < a < c,
\]

while it consists of a single element otherwise.

4. **Examples of arithmetic Zariski pairs**

Let \(f \in \mathbb{Q}[t]\) be an irreducible polynomial. We denote by \(F_f\) the field \(\mathbb{Q}[t]/(f)\). Suppose that a homogeneous polynomial \(\Phi(x_0, x_1, x_2)\) with coefficients in \(F_f\) is given. For a complex root \(\alpha\) of \(f\), we denote by \(\sigma_\alpha : F_f \hookrightarrow \mathbb{C}\) the embedding given by \(t \mapsto \alpha\), and by \(\Phi^\alpha\) the homogeneous polynomial obtained by applying \(\sigma_\alpha\) to the coefficients of \(\Phi\). We say that two plane curves \(C\) and \(C'\) are conjugate in \(F_f\) if there exist a homogeneous polynomial \(\Phi\) with coefficients in \(F_f\) and distinct complex roots \(\alpha\) and \(\alpha'\) of \(f\) such that \(C = \{ \Phi^\alpha = 0 \}\) and \(C' = \{ \Phi^{\alpha'} = 0 \}\).

Suppose that \(C\) and \(C'\) are conjugate \(ADE\)-sextics. Then the configurations of \(C\) and \(C'\) are the same. (See Yang [32, §3] for the precise definition of the configuration of an \(ADE\)-sextic.) In particular, there exist tubular neighborhoods \(T \subset \mathbb{F}^2\) of \(C\) and \(T' \subset \mathbb{F}^2\) of \(C'\) such that \((T,C)\) and \((T',C')\) are diffeomorphic.

Combining this fact with Corollaries 2.5 and 3.4, we see that the following pairs of conjugate maximizing sextics discovered by Artal-Bartolo, Carmona-Ruber and Cogolludo-Agustín [4] Theorem 5.8] are in fact arithmetic Zariski pairs.
Example 4.1. Consider the Dynkin type $R = A_{16} + A_2 + A_1$. We put

$$f := 17t^3 - 18t^2 - 228t + 536,$$

which has two non-real roots $\alpha, \bar{\alpha}$ and a real root $\beta$. In [4], it is shown that $\mathcal{M}(R)$ consists of three connected components $\mathcal{M}(R)_\alpha$, $\mathcal{M}(R)_{\bar{\alpha}}$ and $\mathcal{M}(R)_\beta$, and that there exists a homogeneous polynomial $\Phi(x_0, x_1, x_2)$ of degree 6 with coefficients in $F_f$ such that the conjugate sextics

$$C_\alpha = \{ \Phi^\alpha = 0 \}, \quad C_{\bar{\alpha}} = \{ \Phi^{\bar{\alpha}} = 0 \}, \quad C_\beta = \{ \Phi^\beta = 0 \}$$

are members of $\mathcal{M}(R)_\alpha$, $\mathcal{M}(R)_{\bar{\alpha}}$ and $\mathcal{M}(R)_\beta$, respectively. On the other hand, by the method described in the previous section, we calculate that $\overline{\mathcal{M}_S} = \{ [M^0] \}$ and $\text{Ns}(M^0) = \{ N^1, N^2 \}$, where

$$N^1 = \Lambda[10, 4, 22] \quad \text{and} \quad N^2 = \Lambda[6, 0, 34].$$

The set $F([M^0], N^1)$ consists of two non-real orbits, while the set $F([M^0], N^2)$ consists of a single real orbit. Since the complex conjugation induces a homeomorphism $(\mathbb{P}^2, C_\alpha) \cong (\mathbb{P}^2, C_{\bar{\alpha}})$, the invariants $N_{C_\alpha}$ and $N_{C_{\bar{\alpha}}}$ must be equal. Hence $N_{C_\alpha}$ and $N_{C_{\bar{\alpha}}}$ are isomorphic to $N^1$, while $N_{C_\beta}$ is isomorphic to $N^2$. Since $N^1$ and $N^2$ are not isomorphic, we conclude that $(C_\alpha, C_\beta)$ is an arithmetic Zariski pair.

Example 4.2. Consider the Dynkin type $R = A_{16} + A_3$. In [4], it is shown that $\mathcal{M}(R)$ consists of two connected components $\mathcal{M}_+$ and $\mathcal{M}_-$, and that there exist members $C_+$ of $\mathcal{M}_+$ and $C_-$ of $\mathcal{M}_-$ that are conjugate in $\mathbb{Q}(\sqrt{17})$. On the other hand, we calculate that $\overline{\mathcal{M}_S} = \{ [M^0] \}$ and $\text{Ns}(M^0) = \{ N^1, N^2 \}$, where

$$N^1 = \Lambda[4, 0, 34] \quad \text{and} \quad N^2 = \Lambda[2, 0, 68].$$

Each of $F([M^0], N^1)$ and $F([M^0], N^2)$ consists of a single real orbit. Therefore $(C_+, C_-)$ is an arithmetic Zariski pair.

Example 4.3. Suppose that $R = A_{18} + A_1$. We put

$$f := 19t^3 + 50t^2 + 36t + 8,$$

which has two non-real roots $\alpha, \bar{\alpha}$ and a real root $\beta$. Again by [4], the moduli space $\mathcal{M}(R)$ consists of three connected components $\mathcal{M}(R)_\alpha$, $\mathcal{M}(R)_{\bar{\alpha}}$, and $\mathcal{M}(R)_\beta$ that have members $C_\alpha = \{ \Phi^\alpha = 0 \}$, $C_{\bar{\alpha}} = \{ \Phi^{\bar{\alpha}} = 0 \}$, $C_\beta = \{ \Phi^\beta = 0 \}$, respectively, for some homogeneous polynomial $\Phi$ with coefficients in $F_f$. On the other hand, we have $\overline{\mathcal{M}_S} = \{ [M^0] \}$ and $\text{Ns}(M^0) = \{ N^1, N^2 \}$, where

$$N^1 = \Lambda[8, 2, 10] \quad \text{and} \quad N^2 = \Lambda[2, 0, 38].$$

The set $F([M^0], N^1)$ consists of two non-real orbits, while the set $F([M^0], N^2)$ consists of a single real orbit. Hence $(C_\alpha, C_\beta)$ is an arithmetic Zariski pair.

For the cases $R = A_{15} + A_4$ and $R = A_{19}$ that are also treated in [4] Theorem 5.8, the situation is as follows.

Example 4.4. Suppose that $R = A_{15} + A_4$. We have $\overline{\mathcal{M}_S} = \{ [M^0], [M^1] \}$, where $M^1$ is an overlattice of $M^0$ with index 2. We have $\text{Ns}([M^0]) = \{ N^0 \}$ with $N^0 = \Lambda[8, 4, 22]$, and $F([M^0], N^0)$ consists of two non-real orbits, while we have $\text{Ns}([M^1]) = \{ N^1 \}$ with $N^1 = \Lambda[2, 0, 20]$, and $F([M^1], N^1)$ consists of a single real orbit. According to Yang’s list [22], there exist two configurations of maximizing sextics of type $A_{15} + A_4$. By [4], there exist members $C$ and $\overline{C}$ of distinct connected
components of $\mathcal{M}(R)$ that are conjugate in $\mathbb{Q}(\sqrt{-1})$. The complex conjugation yields a homeomorphism $(\mathbb{P}^2, C) \cong (\overline{\mathbb{P}^2}, \overline{C})$. Hence we must have $N_C \cong N_{\overline{C}} \cong N^0$.

**Example 4.5.** Suppose that $R = \mathbb{A}_{19}$. We have $\overline{\text{Ms}}^i = \{ [M^0] \}$ and $\text{Ns}(\overline{M^0}) = \{ N^0 \}$ with $N^0 = \Lambda[2, 0, 20]$. The set $F([M^0], N^0)$ consists of two real orbits. According to [3], there exist members $C_+$ and $C_-$ of $\mathcal{M}(R)$ belonging to the distinct connected components that are conjugate in $\mathbb{Q}(\sqrt{5})$. Our invariant fails to distinguish $(\mathbb{P}^2, C_+)$ and $(\mathbb{P}^2, C_-)$ topologically, because we have $N_{C_+} \cong N_{C_-} \cong N^0$. It would be an interesting problem to determine whether $(\mathbb{P}^2, C_+)$ and $(\mathbb{P}^2, C_-)$ are homeomorphic or not.

5. A singular $K3$ surface defined over a number field

Let $Y$ be a complex $K3$ surface or a complex abelian surface such that the transcendental lattice $T(Y)$ is of rank 2. Then $T(Y)$ is an even positive-definite lattice. Moreover the Hodge structure

$$T(Y) \otimes \mathbb{C} = H^{2,0}(Y) \oplus \overline{H^{2,0}(Y)}$$

of $T(Y)$ defines a canonical orientation on $T(Y)$; namely, an ordered basis $e_1, e_2$ of $T(Y)$ is positive if the imaginary part of the complex number $(e_1, \omega_Y)/(e_2, \omega_Y)$ is positive, where $\omega_Y \in H^{2,0}(Y)$ is the cohomology class of a non-zero holomorphic 2-form of $Y$. We denote by $\overline{T}(Y)$ the oriented transcendental lattice of $Y$.

**Definition 5.1.** A (smooth) $K3$ surface $X$ defined over a field $k$ of characteristic 0 is called *singular* if the Picard number of $X \otimes \bar{k}$ is 20.

If $X$ is a complex singular $K3$ surface, then we have the oriented transcendental lattice $\overline{T}(X)$. We have the following important theorem due to Shioda and Inose [27]:

**Theorem 5.2.** The correspondence $X \mapsto \overline{T}(X)$ yields a bijection from the set of isomorphism classes of complex singular $K3$ surfaces to the set of isomorphism classes of even positive-definite oriented lattices of rank 2.

Notice that, if $C$ is a complex maximizing sextic, then the minimal resolution $X_C$ of the double covering $Y_C : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ that branches exactly along $C$ is a complex singular $K3$ surface, and $T(X_C)$ is isomorphic to $N_C$.

Let $X$ be a singular $K3$ surface defined over a number field $F$. For an embedding $\sigma$ of $F$ into $\mathbb{C}$, we denote by $X^\sigma$ the complex $K3$ surface obtained from $X$ by $\sigma$.

The following is a special case of [26, Theorem 3].

**Proposition 5.3.** There exist a singular $K3$ surface $X$ defined over a number field $F$ and two embeddings $\tau$ and $\tau'$ of $F$ into $\mathbb{C}$ such that

$$\overline{T}(X^\tau) \cong \Lambda[2, 1, 28] \quad \text{and} \quad \overline{T}(X^{\tau'}) \cong \Lambda[8, 3, 8].$$

**Proof.** We put

$$K := \mathbb{Q}(\sqrt{-55}) \subset \mathbb{C},$$

and denote by $\mathbb{Z}_K$ the ring of integers of $K$. For a number field $L$ containing $K$, we denote by $\text{Emb}(L/K)$ the set of embeddings of $L$ into $\mathbb{C}$ whose restrictions to $K$
are the identity of $K$. We define fractional ideals $I_0, \ldots, I_3$ of $\mathcal{K}$ by the following:

\[
\begin{align*}
I_0 & := \mathcal{K} = \mathbb{Z} + \mathbb{Z}\tau_0 \quad \text{where} \quad \tau_0 := (1 + \sqrt{-55})/2, \\
I_1 & := \mathcal{K} = \mathbb{Z} + \mathbb{Z}\tau_1 \quad \text{where} \quad \tau_1 := (3 + \sqrt{-55})/4, \\
I_2 & := \mathcal{K} = \mathbb{Z} + \mathbb{Z}\tau_2 \quad \text{where} \quad \tau_2 := (5 + \sqrt{-55})/8, \\
I_3 & := \mathcal{K} = \mathbb{Z} + \mathbb{Z}\tau_3 \quad \text{where} \quad \tau_3 := (1 + \sqrt{-55})/4.
\end{align*}
\]

The ideal class group $\text{Cl}_K$ of $\mathcal{K}$ is a cyclic group of order 4 generated by the class $[I_1]$, and we have $[I_2] = [I_1]^2$ and $[I_3] = [I_1]^3$. We consider the Hilbert class polynomial

\[
\mathcal{H}(t) := (t - j(\tau_0))(t - j(\tau_1))(t - j(\tau_2))(t - j(\tau_3))
\]

\[
= t^4 + 13136684625 t^3 - 20948398473375 t^2 + 18577989025032784359375 t - 18577989025032784359375
\]

of $\mathcal{K}$, and the Hilbert class field

\[
H := K[t]/(\mathcal{H}(t))
\]

of $K$ (see Cox [10]). We put

\[
\gamma := t \text{ mod } (\mathcal{H}) \in H.
\]

Then we have $\text{Emb}(H/K) = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_i$ is the embedding defined by $\sigma_i(\gamma) = j(\tau_i) \in \mathbb{C}$. Consider the elliptic curve

\[
y^2 + xy = x^3 - \frac{36}{\gamma - 1728} x - \frac{1}{\gamma - 1728}
\]

defined over $H$ (see Silverman [30, page 52]). Then we have $j(E) = \gamma \in H$, and hence $j(E^{\sigma_i}) = j(\tau_i)$ holds for $i = 0, \ldots, 3$, where $E^{\sigma_i}$ is the complex elliptic curve defined by (5.1) with $\gamma$ replaced by $\sigma_i(\gamma) = j(\tau_i)$. Therefore we have an isomorphism of Riemann surfaces

\[
E^{\sigma_i} \cong \mathbb{C}/I_i
\]

for $i = 0, \ldots, 3$. We then put

\[
A := E \times E.
\]

Note that $T(A)$ is of rank 2. By means of a double covering of the Kummer surface $\text{Km}(A)$ of $A$, Shioda and Inose [27] constructed a singular $K3$ surface $X$ defined over a finite extension $F$ of $H$ with the following properties (see also [26, Propositions 6.1 and 6.4]).

For any $\sigma \in \text{Emb}(F/K)$, the oriented transcendental lattice $\tilde{T}(X^\sigma)$ is isomorphic to the oriented transcendental lattice $\tilde{T}(A^\sigma)$ of the complex abelian surface $A^\sigma = E^\sigma \times E^\sigma$.

See Inose [14] and Shioda [28] for an explicit defining equation of $X$.

The oriented lattice $\tilde{T}(A^\sigma)$ is calculated by Shioda and Mitani [29]. Suppose that the restriction of $\sigma \in \text{Emb}(F/K)$ to $H$ is $\sigma_i$. Then we have

\[
A^\sigma \cong \mathbb{C}/I_i \times \mathbb{C}/I_i \cong \mathbb{C}/I_i^2 \times \mathbb{C}/I_0 \cong \begin{cases} 
\mathbb{C}/I_0 \times \mathbb{C}/I_0 & \text{if } i = 0 \text{ or } i = 2, \\
\mathbb{C}/I_2 \times \mathbb{C}/I_0 & \text{if } i = 1 \text{ or } i = 3,
\end{cases}
\]
by \cite{5.2} and \cite{29} (4.14). Hence, by \cite{29} Section 3, we have
\[
\overline{T}(A^\sigma) \cong \begin{cases} \overline{A}[2,1,28] & \text{if } \sigma|H \text{ is } \sigma_0 \text{ or } \sigma_2, \\ \overline{A}[8,3,8] & \text{if } \sigma|H \text{ is } \sigma_1 \text{ or } \sigma_3, \end{cases}
\]
(see also \cite{26} §6.3.) Thus we obtain the hoped-for \(X\) and \(\tau, \tau'\).

**Remark 5.4.** Note that the orientation reversing does not change the isomorphism classes of the oriented lattices \(\overline{A}[2,1,28]\) and \(\overline{A}[8,3,8]\) (see Remark \(\ref{5.4}\)). Hence, by Theorem \(\ref{5.2}\) if a complex singular \(K3\) surface \(Y\) satisfies \(T(Y) \cong \overline{A}[2,1,28]\) (resp. \(T(Y) \cong \overline{A}[8,3,8]\)), then \(Y\) is isomorphic to the complex \(K3\) surface \(X^\tau\) (resp. to the complex \(K3\) surface \(X^{\tau'}\)) in Proposition \(\ref{5.3}\).

Using Proposition \(\ref{5.3}\) and Remark \(\ref{5.4}\) we obtain the following example of arithmetic Zariski pairs.

**Example 5.5.** Consider the Dynkin type \(R = A_{10} + A_9\). We have
\[
\overline{M}^2 = \{[M^0], [M^1]\},
\]
where \(M^1\) is an overlattice of \(M^0\) with index 2. We then have
\[
\begin{align*}
\text{Ns}([M^0]) &= \{ \Lambda[10,0,22], \Lambda[2,0,110] \} \quad \text{and} \\
\text{Ns}([M^1]) &= \{ \Lambda[2,1,28], \Lambda[8,3,8] \},
\end{align*}
\]
and each of the sets
\[
\begin{align*}
\text{F}([M^0],\Lambda[10,0,22]), & \quad \text{F}([M^0],\Lambda[2,0,110]), \\
\text{F}([M^1],\Lambda[2,1,28]), & \quad \text{F}([M^1],\Lambda[8,3,8])
\end{align*}
\]
consists of a single real orbit. In particular, the number of the connected components of \(\mathcal{M}(R)\) is four. Let \(C\) and \(C'\) be members of the connected components of \(\mathcal{M}(R)\) corresponding to \(F([M^1],\Lambda[2,1,28])\) and \(F([M^1],\Lambda[8,3,8])\), respectively. Note that we have
\[
\text{N}_C \cong T(X_C) \cong \Lambda[2,1,28] \quad \text{and} \quad \text{N}_{C'} \cong T(X_{C'}) \cong \Lambda[8,3,8].
\]
By Remark \(\ref{5.4}\) we see that \(X_C\) is isomorphic to \(X^\tau\) and \(X_{C'}\) is isomorphic to \(X^{\tau'}\). Consider the composites
\[
\phi_C : X_C \rightarrow Y_C \xrightarrow{\pi_C} \mathbb{P}^2 \quad \text{and} \quad \phi_{C'} : X_{C'} \rightarrow Y_{C'} \xrightarrow{\pi_{C'}} \mathbb{P}^2
\]
of the finite double coverings branching along \(C\) and \(C'\) and the minimal desingularizations. Since \(X_C \cong X^\tau\), there exists a morphism \(\phi_L : X \otimes L \rightarrow \mathbb{P}^2\) with the Stein factorization
\[
\phi_L : X \otimes L \rightarrow Y_L \xrightarrow{\pi_L} \mathbb{P}^2
\]
defined over a finite extension \(L\) of \(F\) such that, for some embedding \(\theta\) of \(L\) into \(\mathbb{C}\) satisfying \(\theta|F = \tau\), the morphism
\[
\phi_L^\theta : (X \otimes L)^\theta = X^\tau \rightarrow Y_L^\theta \xrightarrow{\pi_L^\theta} \mathbb{P}^2
\]
is isomorphic to \(\phi_C\). In particular, the branch curve \(B\) of the finite double covering \(\pi_L^\theta\) is isomorphic to \(C\) as a complex plane curve. Let \(\theta'\) be an embedding of \(L\) into \(\mathbb{C}\) such that \(\theta'|F = \tau'\), and consider the morphism
\[
\phi_L^{\theta'} : (X \otimes L)^{\theta'} = X^{\tau'} \rightarrow Y_L^{\theta'} \xrightarrow{\pi_L^{\theta'}} \mathbb{P}^2.
\]
Since the branch curve \(B'\) of \(\pi_L^{\theta'}\) is conjugate to the branch curve \(B\) of \(\pi_L^\theta\), it is a maximizing sextic of type \(A_{10} + A_9\). Since \((X \otimes L)^{\theta'} = X^{\tau'}\) is isomorphic to
$X_{C'}$, the morphism $\phi'$ must be isomorphic to $\phi_{C'}$, and hence $B'$ is isomorphic to $C'$ as a complex plane curve. Therefore the conjugate pair $(B, B')$ of plane curves is isomorphic to the pair $(C, C')$ with $N_C \not\cong N_{C'}$, and thus yields an example of arithmetic Zariski pairs.

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