The Camassa-Holm equations as a closure model for turbulent channel flow

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We propose the viscous Camassa-Holm equations as a closure approximation for the Reynolds-averaged equations of the incompressible Navier-Stokes fluid. This approximation is tested on turbulent channel flows with steady mean. Analytical solutions for the mean velocity and the Reynolds shear stress across the entire channel are obtained, showing good agreement with experimental measurements and direct numerical simulations. As Reynolds number varies, these analytical mean velocity profiles form a family of curves whose envelopes are shown to have either power law, or logarithmic behavior, depending on the choice of drag law.

Laminar Poiseuille flow occurs when a fluid in a straight channel, or duct, is driven by a constant upstream pressure gradient, yielding a parabolic streamwise velocity profile which is symmetric about the midplane of the channel. In turbulent states, the mean streamwise velocity profile remains symmetric, but is flattened in the center because of the increase of the velocity fluctuation. Although a lot of research has been carried out for turbulent channel flow, e.g., \cite{1–6}, accurate measurement of the mean velocity and the Reynolds stress profiles, in particular for flows at high Reynolds numbers, is still an experimental challenge and the fundamental understanding of how these profiles change as functions of Reynolds number is still missing.

In turbulent channel flows it is customary to define a characteristic velocity \( u_c \) and Reynolds number \( R_0 \) by \( u_c = \sqrt{\tau_0}/\rho \) and \( R_0 = du_c/\nu \), where \( \tau_0 \) is the boundary shear stress, we take the density \( \rho \) to be unity, \( d \) is the channel half-width and \( \nu \) is the molecular viscosity of the fluid. Based on experimental observation and numerical simulation, a piecewise expression of the mean velocity across channel has been proposed, for which the nondimensional mean streamwise velocity, \( \phi = U/u_c \), is assumed to depend on \( \eta = u_c z/\nu \) and have three types of behavior depending on the distance away from the wall boundary, \( z \): a viscous sublayer, in which \( \phi \sim \eta \); the von Kármán-Prandtl logarithmic “law of the wall,” in which \( \phi(\eta) = \kappa^{-1}\text{ln}\eta + A \), where \( \kappa \approx 0.41 \) and \( A \approx 5.5 \); and a power law region, in which \( \phi \sim \eta^p \), \( 0 < p < 1 \).

In this paper, we propose the viscous Camassa-Holm equations (VCHE) in \textsuperscript{4} as a closure approximation for the Reynolds equations. The analytical solutions of the steady VCHE depend on the three parameters: the flux Reynolds number \( R = d\tau/
u \) (where \( \tau \) is the streamwise velocity, averaged across the channel); the drag law for wall friction \( D = 2\tau_0/\nu^2 = 2R_0^2/R^2 \) as a function of \( R \); and a shape factor \( c \in (0, 1) \) which specifies the flattening of the velocity profile and is given by the ratio \( u_{max}/u = (3-c)/2 \) of the maximum streamwise velocity \( u_{max} \) (occurring at the center of the channel) to its average \( \bar{u} \). This family of analytical solutions of the steady VCHE provides profiles of the mean velocity and the Reynolds shear stress depending on the three parameters \( R, D \) and \( c \). (Giving these three parameters is equivalent to giving the Reynolds numbers \( R, R_0 \) and \( R_{max} \), corresponding to the channel flux, boundary stress and velocity at the midplane, respectively.) The VCHE profiles agree well with data obtained from turbulent channel flow measurements and simulations across the entire channel.

As the Reynolds number is varied in these solutions, the velocity profiles \( \phi(\eta, R) \) form a family of curves with upper and lower envelopes. The Blasius \( R^{-1/4} \) law for the drag coefficient of the wall boundary leads to \( \eta^{1/7} \) power law behavior for these envelopes, while the von Kármán drag law leads to logarithmic envelopes.

We begin our theoretical treatment by recalling the Reynolds-averaged Navier-Stokes equations \textsuperscript{4}

\[
\frac{\partial\langle u \rangle}{\partial t} + \langle u \rangle \cdot \nabla \langle u \rangle = \text{div} \langle \mathbf{T} \rangle, \quad \text{div} \langle u \rangle = 0, \tag{1}
\]

where \( \langle u \rangle \) denotes the ensemble average of the velocity and \( \langle \mathbf{T} \rangle = -(\rho)\mathbf{I} - (u \otimes u) + \nu(\nabla \langle u \rangle + \nabla \langle u \rangle^T) \). For turbulent channel flow, the mean velocity is of the form \( \langle u \rangle = (\mathbf{U}(z), 0, 0)^T \), with \( \langle p \rangle = \mathbf{P}(x, y, z) \) and the Reynolds equations \textsuperscript{4} reduce to \( \text{div} \langle \mathbf{T} \rangle = 0 \), or equivalently,

\[
-\nu \mathbf{U}'' + \partial_z \langle uu \rangle = -\partial_z \mathbf{P}, \quad \partial_z \langle uw \rangle = -\partial_y \mathbf{P}, \quad \partial_z \langle w^2 \rangle = -\partial_z \mathbf{P}, \tag{2}
\]

where \((u, v, w)^T \) is the fluctuation velocity in the infinite channel \( \{(x, y, z) \in \mathbb{R}^3, -d \leq z \leq d \} \). The (1,3) component of the averaged stress tensor \( \langle \mathbf{T} \rangle \) is given by \( \langle T_{13} \rangle = -\nu \mathbf{U}'(z) + \langle uu \rangle \). On the boundary, the velocity components all vanish and one has the stress condition

\[
\pm \tau_0 \big|_{z=\pm d} = \nu \mathbf{U}'(z)|_{z=\pm d}, \tag{3}
\]
upon using \( \langle wu \rangle = 0 \) at \( z = \pm d \). Hence, the Reynolds equations imply \( \langle wu \rangle(z) \equiv 0 \) and \( \mathbf{P} = P_0 - \tau_0 x/d - \langle w^2 \rangle(z) \), with integration constant \( P_0 \).

The viscous Camassa-Holm equations (VCHE) are

\[
\frac{dv}{dt} + v \nabla u^2 + v \left( p - \frac{1}{2} |u|^2 - \frac{\alpha^2}{2} |\nabla u|^2 \right) = \nu \Delta v, \tag{4}
\]

with \( \nabla \cdot v = 0 = \nabla \cdot u \), where \( v = (1 - \alpha^2 \Delta) u \), \( d/dt = \partial/\partial t + u \cdot \nabla \) is the material derivative. Also, \( \alpha \) is a constant lengthscale, \( |\nabla u|^2 = \text{Tr}(\nabla u \cdot \nabla u^T) \) and \( \Delta \) is the Laplacian. Equation (4) with \( \nu = 0 \) is derived in [1] by decomposing Lagrangian parcel trajectories into mean and fluctuating parts, then applying asymptotic expansions, Lagrangian means, and assuming isotropy of fluctuations in Hamilton’s principle for an ideal incompressible fluid. That derivation generalizes a one-dimensional integrable dispersive shallow water model studied in [10] to \( n \)-dimensions and provides the interpretation of \( \alpha \) as the typical amplitude of the rapid fluctuations over whose phase the Lagrangian mean is taken in Hamilton’s principle. Moreover, that derivation makes it clear the solutions of VCHE are mean quantities.

Before comparing VCHE with Reynolds averaged equations, we rewrite (4) in the equivalent ‘constitutive’ form

\[
\frac{du}{dt} = \text{div} \mathbf{T}, \quad \mathbf{T} = -p\mathbf{I} + 2\nu(1 - \alpha^2 \Delta) \mathbf{D} + 2\alpha^2 \mathbf{D}, \tag{5}
\]

with \( \nabla \cdot u = 0 \), \( \mathbf{D} = (1/2)(\nabla u + \nabla u^T), \) \( \Omega = (1/2)(\nabla u - \nabla u^T) \), and co-rotational (Jaumann) derivative given by \( \mathbf{D} = \mathbf{dD}/dt + \mathbf{D} \Omega - \Omega \mathbf{D}. \) In this form, one recognizes the constitutive relation for VCHE as a variant of the rate-dependent incompressible homogeneous fluid of second grade [11, 12], whose viscous dissipation, however, is modified by the Helmholtz operator \( (1 - \alpha^2 \Delta) \). There is a tradition at least since Rivlin [13] of modeling turbulence by using continuum mechanics principles such as objectivity and material frame indifference. For example, this sort of approach is taken in deriving Reynolds stress algebraic equation models [14]. Rate-dependent closure models of mean turbulence have also been obtained by the two-scale DIA approach [15] and by the renormalization group method [16]. Since VCHE describe mean quantities, we propose to use (5), or equivalently (4), as a turbulence closure model and test this ansatz by applying it to turbulent channel flow. The corresponding closure for turbulent pipe flows will be treated elsewhere [17].

We denote the velocity \( \mathbf{u} \) in (4) by \( \mathbf{U} \) and seek its steady state solutions in the form \( \mathbf{U} = (U(z), 0, 0)^T \) subject to the boundary condition \( U(z) = 0 \) and the symmetry condition \( U(z) = U(-z) \). In this particular case, the steady VCHE reduces to,

\[
\begin{align*}
-\nu U'' + \nu \alpha^2 U''' &= -\partial_z p, \quad 0 = -\partial_y p, \\
0 &= -\partial_z \left( p - \alpha^2 (U')^2 \right). \tag{6}
\end{align*}
\]

That is, \( dv/dt \) vanishes in (4) and the remaining \( \alpha^2 \) terms modify the pressure and dissipation. Comparing (4) and (5), we identify counterparts as,

\[
\mathbf{U} = U, \quad \partial_z \langle wu \rangle = \nu \alpha^2 U''' + p_0, \quad \partial_z \langle uw \rangle = 0, \quad \nabla(\mathbf{P} + \langle w^2 \rangle) = \nabla(p - p_0 x - \alpha^2 (U')^2),
\]

for a constant \( p_0 \). This identification gives

\[
\langle wu \rangle(z) = 0, \quad \langle uw \rangle(z) = -p_0 z - \nu \alpha^2 U'''(z), \tag{8}
\]

and leaves \( \langle w^2 \rangle \) undetermined up to an arbitrary function of \( z \). A closure relation for \( -\langle wu \rangle \) involving \( U''(z) \) also appears in Yoshizawa [15], cf. also equation (8) of [4].

The solution of the steady VCHE (6) in a channel subject to these boundary and symmetry conditions [18] is

\[
U(z) = a \left( 1 - \frac{\cosh(z/\alpha)}{\cosh(d/\alpha)} \right) + b \left( 1 - \frac{z^2}{d^2} \right) \tag{9}
\]

with constants \( a, b \). Any time dependent solution of (6) such that \( u(z,t) = (U(z,t), 0, 0)^T \) with \( U(z,t) = U(-z,t) \) and \( U(\pm d,t) = 0 \) converges exponentially in time to the solution (9) with the same mean flow and boundary shear stresses [14].

We consider channel flows for \( R >> 1 \). Our basic assumption is that \( \xi = d/\alpha \rightarrow \infty \) as \( R \rightarrow \infty \). The solution (9) must satisfy the stress condition (5), which imposes

\[
\tau_0 = \nu U'(z)|_{z=-d} = \frac{\alpha}{\alpha} \frac{\nu}{\alpha} \tanh \xi + \frac{2b\nu}{d}. \tag{10}
\]

Substituting the definitions,

\[
\tau = \frac{1}{2d} \int_{-d}^{d} U(z)dz = a \left( 1 - \frac{\xi^{-1} \tanh \xi}{\tanh \xi} \right) + \frac{2b}{3}, \tag{11}
\]

and \( R = \frac{\pi d}{\nu}, \quad R_0 = \frac{\alpha^{1/2}}{\nu}, \quad \theta = \frac{2b\nu}{dR_0} \),

with \( 0 < \theta < 1 \), into relation (10) re-expresses it as

\[
R_0^2 = \left( R - \frac{\theta R_0^2}{3} \right) \left( \frac{\xi \tanh \xi}{1 - \xi^{-1} \tanh \xi} \right) + \theta R_0^2. \tag{12}
\]

Upon setting \( \xi = \delta R_0 \) and taking \( \xi >> 1 \), this simplifies to \( O(1/\xi) \) into the basic relation

\[
\frac{1 - \theta}{\delta} = \frac{R}{R_0} - \frac{\theta R_0}{3}. \tag{13}
\]

In terms of the parameters \( \theta \) and \( \xi \), equations (8) and (9) imply the Reynolds shear stress,

\[
- \langle wu \rangle(z) = \tau_0 (1 - \theta) \left( \frac{\sinh(z/\alpha)}{\sinh(\xi)} - \frac{z}{d} \right). \tag{14}
\]

Thus, the solution (9) implies \( -\langle wu \rangle \geq 0 \) for \( -d \leq z \leq 0 \), as seen empirically [4]. In the lower half of the channel,
the symmetric solution \[ \phi(\eta) = U(z)/u_\ast = (z + d)/\ell_\ast \], with \( \ell_\ast = \nu/u_\ast = d/R_0 \), to order \( O(\xi e^{-\xi}) \) as

\[
\phi(\eta) = \frac{1 - \theta}{\delta} \left( 1 - e^{-\delta \eta} \right) + \theta \eta \left( 1 - \frac{\eta}{2 R_0} \right), \tag{15}
\]

for \( 0 \leq \eta \leq R_0 \). In this notation, we have \( \alpha = d/\xi = \ell_\ast /\delta \) for the lengthscale in \( \eta \). The velocity profile \( U(z) \) in \( \eta \) has its maximum at the center of the channel (\( \eta = R_0 \)). At this point \( \phi_{\text{max}} \approx \phi(R_0) \) is given to leading order by

\[
\phi_{\text{max}} = \frac{1 - \theta}{\delta} + \frac{\theta R_0}{2}. \tag{16}
\]

Hence, from \( \phi_{\text{max}} \), \( \theta R_0 \) and \( \sqrt{2/D} = R/R_0 \) we have

\[
\theta R_0 = 6 \left( \phi_{\text{max}} - \sqrt{2/D} \right), \quad \frac{1 - \theta}{\delta} = 3 \sqrt{2/D} - 2 \phi_{\text{max}}. \tag{17}
\]

Since \( 0 < \theta < 1 \), relations \( \theta R_0 \) imply the inequalities \( 3/2 > \phi_{\text{max}} \sqrt{D/2} > 1 \), and we may write

\[
\theta R_0 = 3(1 - c) \sqrt{2/D}, \quad \text{and} \quad \frac{1 - \theta}{\delta} = c \sqrt{2/D}. \tag{18}
\]

by introducing the quantity \( c \in (0, 1) \) defined in terms of the velocity profile flatness or midplane velocity ratio \( u_{\text{max}}/\bar{u} \) as

\[
\frac{3 - c}{2} = \frac{u_{\text{max}}}{\bar{u}} = \phi_{\text{max}} \sqrt{D/2} \quad \text{with} \quad 0 < c < 1. \tag{19}
\]

Comparison with data will show that \( c \) is nearly independent of \( R \) and bounded away from its endpoint values. The first equation in \( \phi_{\text{max}} \) and the basic relation \( \theta R_0 \) then imply

\[
\theta = \left( 1 + \frac{c \xi}{3(1 - c)} \right)^{-1} = O(1/\xi). \tag{20}
\]

Substituting this into the second equation in \( \phi_{\text{max}} \) gives,

\[
R_0 = c\delta R \left( 1 + \frac{3(1 - c)}{c \xi} \right) = c\delta R + O(1/\xi). \tag{21}
\]

Thus, to leading order, \( \delta = c^{-1} R_0 / R = c^{-1} \sqrt{D/2} \) and the velocity profile \( \phi(\eta) \) is given by

\[
\phi(\eta) = \frac{R}{R_0} \left[ c \left( 1 - e^{-\frac{R}{R_0}} \right) + 3(1 - c) \frac{\eta}{R_0} \left( 1 - \frac{\eta}{2 R_0} \right) \right]. \tag{22}
\]

Thus, for \( \xi = d/\alpha \gg 1 \), the drag \( \sqrt{D/2} = R_0/R \) and the constant \( c \) determine the steady velocity profile of VCHE \( \phi(\eta) \) at each \( R \). The lengthscale \( \alpha \) is given to leading order by \( \alpha/d = 1/(\delta R_0) = 2c/(DR) \). For the Blasius drag law, \( D = \lambda R^{-1/4}, \lambda = \text{const} \), this implies

\[
\frac{1}{\xi} = \frac{\alpha}{d} = \frac{2c}{\lambda} R^{-3/4} = c \left( \frac{2}{\lambda} \right)^{4/7} R_0^{6/7}. \tag{23}
\]

This scaling in \( R \) agrees with the scaling law for the Kolmogorov fluctuation dissipation length, \( \ell_d \). Thus, in this case, \( \alpha \) is proportional to \( \ell_d \). For the Blasius drag law, we also have from \( \phi(\eta) \)

\[
R_0 = \frac{\lambda}{2} R^{7/8}, \quad \delta = \frac{\lambda^{1/2}}{c} R^{-1/8}, \quad \theta = \frac{3(1 - c)}{\lambda/2} R^{-3/4}, \quad 1 - \frac{\theta}{\delta} = c \sqrt{\lambda/2} R^{1/8}. \tag{24}
\]

\[
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The mean-velocity profile \( \phi \) as a function of \( \log(\eta) \). Experimental data \( \square \), DNS data \( \bullet \). \( \phi_v \) and \( \phi_u \) represent the viscous relation and von Kármán logarithm law respectively.}
\end{figure}

\[
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{The Reynolds-stress \( \langle wu \rangle /u^2 \). Experimental data \( \square \), DNS data \( \bullet \).}
\end{figure}

Figures 1–3 compare our theoretical solution \( \phi(\eta) \) with experimental data from \( \square \), and direct numerical simulation (DNS) data from \( \bullet \). We assume the Blasius drag law, \( D = \lambda R^{-1/4}, \lambda = 0.06 \). We vary \( c \) slightly with \( R \) in order to best fit the data \( (c \in [0.728, 0.77]) \). Note,
the curves can be brought into even better agreement by allowing $\lambda$ to also vary slightly with $R$.

After the standard Blasius drag law is chosen and $c$ is determined from the midplane velocity data, no free parameters remain in the model. Figure 2 compares the theoretical prediction with the measured data for the Reynolds-stress $-\langle w u \rangle / u_*^2$. Figure 3 compares the predicted and measured turbulent kinetic energy production profiles. Except for the region nearest the wall, $\log_{10}(\eta) \leq 1$ in Figure 2 and Figure 3, the theoretical model shows excellent agreement with the data. Using the fundamental relation (22) yields a family of velocity profiles $\phi(\eta; R)$, plotted in Figure 4 at various values of $R$ for the Blasius drag law. Remarkably, the upper and lower envelopes of this family are found analytically to be $\eta^{1/7}$ power laws. If instead of the Blasius drag law, one uses the von Kármán drag law, $\sqrt{2/\ell D} = \lambda_1 \log(R_0) + \lambda_2$, for constants $\lambda_1$ and $\lambda_2$, then (22) yields a family of velocity profiles $\phi(\eta; R)$ whose upper and lower envelopes are nearly linear in $\log_{10} \eta$ with different slopes.

FIG. 3. The turbulent kinetic energy production profile. Experimental data is from Wei & Willmarth [4].

FIG. 4. The upper and lower envelopes of the velocity profile for the Blasius drag law.

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[1] H. Reichhardt, Naturwissenschaften 24/25 (1938), 404.
[2] H. Eckelmann, Mitteilungen aus dem Max Planck Institut, Göttingen, no. 48 (1970).
[3] H. Eckelmann, J. Fluid Mech. 65 (1974), 439–459.
[4] T. Wei and W.W. Willmarth, J. Fluid Mech. 204 (1989), 57–95.
[5] R.A. Antonia, M. Teitel, J. Kim and L.W.B. Browne, J. Fluid Mech. 236 (1992), 579–605.
[6] J. Kim, P. Moin and R. Moser, J. Fluid Mech. 177 (1987), 133–166.
[7] J.O. Hinze, Turbulence, Mc-Graw-Hill: New York, 2nd edition (1975).
[8] A.A. Townsend, The Structure of Turbulent Flow, Cambridge University Press (1967).
[9] D.D. Holm, J.E. Marsden, T.S Ratiu, Phys. Rev. Lett., to appear.
[10] R. Camassa and D.D. Holm, Phys. Rev. Lett. 71 (1993) 1661–1664.
[11] J.E. Dunn and R.L. Fosdick, Arch. Rat. Mech. Anal. 56 (1974) 191–252.
[12] J.E. Dunn and K.R. Rajagopal, Int. J. Engng. Sci. 33 (1995), 689–729.
[13] R.S. Rivlin, Q. Appl. Math. 15 (1957), 212–215.
[14] T.H. Shih, J. Zhu, and J.L. Lumley, Comput. Methods Appl. Mech. Engrg. 125 (1995), 287–302.
[15] A. Yoshizawa, Phys. Fluids 27 (1984), 1377–1387.
[16] R. Rubinstein and J.M. Barton, Phys. Fluids A, 2 (1990), 1472–1476.
[17] See, e.g., G.I. Barenblatt, A.J. Chorin and V.M. Prostokishin, Appl. Mech. Rev., 50 (1997), 413–429, for a recent survey of pipe flows.
[18] With the antisymmetric steady solution, one may address turbulent states of Couette flow by a similar analysis.
[19] In fact, the time-dependent VCHE in a periodic box has unique classical solutions and a global attractor, whose fractal dimension is finite and scales according to Kolmogorov’s estimate, $N \sim (L/\ell_d)^3$, where $\ell_d = (\nu^3/\epsilon)^{1/4}$ is the Kolmogorov dissipation length.
[20] C. Foias, D.D. Holm and E.S. Titi, in preparation.
[21] Detailed analysis of (3) and use of Kolmogorov scaling also yields an estimate of the energy-containing macroscale in terms of $\ell_d$ and $d$. 

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