Two Stages in Evolution of Binary Alkali BEC Mixtures towards Phase Segregation

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Abstract

The quantum analogy of the usual spinodal decomposition is explored for the recently achieved binary alkali Bose-Einstein condensates mixture. We conclude that an analogy is possible within the formulation of coupled non-linear Schrödinger equations, and find that the quantum spinodal decomposition consists of two stages. The non-equilibrium stage I is dominated by the fastest growth mode, associated with a characteristic length. Expressions for both time and length scales in the stage I are obtained. The state II is a relaxation process of approaching equilibrium, dominated by the slowest mode. We propose that for this state the slow evolution towards the phase segregation is due to the Josephson effect between different domains of same condensate, and its time scale is estimated.

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I. INTRODUCTION

Spinodal decomposition in a binary-solution system is a typical example of phase ordering dynamics: the growth of order through domain coarsening when a system is quenched from the homogeneous phase into a broken-symmetry phase [1]. Systems quenched from a disordered phase into an ordered phase do not order instantaneously. Instead, different length scales set in as the domains form and grows with time, and different broken symmetry phases compete to select the equilibrium state. In the dynamical equation, the Cahn-Hilliard equation, for the spinodal decomposition, the diffusion constant of the material plays a decisive role and determines both the length and time scales. It appears that a system without the diffusion constant may not show all the main features of the spinodal decomposition. With the recent realization of the binary Alkali Bose-Einstein condensates (BEC’s) [2] we demonstrate in the present article that indeed it is possible to have a spinodal decomposition without the diffusion constant and that this may have been realized experimentally. The binary alkali BEC mixtures provide new systems to non-equilibrium phenomena in new parameter regimes. Its mathematical description is simple enough that a theoretical description of the whole process is feasible. A direct comparison between theoretical calculations and experimental observations can be made. Here, we shall study the gross features in the dynamical evolution process, starting from the homogeneous unstable state. To differentiate the present situation from the usual ones, we shall call the present one the quantum spinodal decomposition, and the previous ones the classical spinodal decomposition.

In a classical spinodal decomposition process, the particle number for each specie is conserved separately. The process can be classified into two stages, the initial stage of fast growth from the homogeneous unstable state and the late stage towards to the equilibrium of true ordering. The initial stage is dominated by the fastest growth mode determined by the dynamical equation. It is a highly non-equilibrium process. There is a length scale associated with this time scale, which gives the characteristic domain size after this initial stage. In the late stage the domains grow and
merge, and their numbers becomes smaller and smaller. This is a relaxation process towards the equilibrium, dominated by the slowest time scale. For an infinitely large system, the time scale may be infinity. Hence the system may never achieve true equilibrium. We shall show below that the quantum spinodal decomposition occurring in the binary BEC mixtures shares all the features of the classical spinodal decomposition. The main difference is that the size of binary BEC mixtures is finite. Therefore it is possible to achieve true equilibrium within a finite time scale. To further differentiate the present process from the classical one, we shall call the initial stage in the dynamical evolution of the binary BEC mixtures the stage I, and call the second stage towards equilibrium the stage II.

In the following, we first formulate the problem in Sec. II. The coupled nonlinear Schrödinger equations and the parameter regime to realize the quantum spinodal decomposition will be specified. In Sec. III the stage I is studied in detail. The fastest growth mode is explicitly given. Interspersed regions or domains of coexisting condensate 1 and 2 are formed at this stage, characterized by a length scale. Stage II is studied in Sec. IV. We first demonstrate that the scenarios in classical spinodal decompositions are not possible here. We propose and analyze a new mechanism for the approach towards equilibrium: the Josephson effect between different domains of same condensate. This is in accordance with the present quantum spinodal decomposition concept. A comparison to experimental situations is discussed in Sec. V. We conclude there that the quantum spinodal decomposition can be realized. We summarize in Sec. VI.

II. FORMULATION OF PROBLEM

We start from the Hamiltonian formulation of a binary BEC mixture at zero temperature:

\[
H = \int d^3x \left[ \psi_1^*(x) \left( -\frac{i\hbar^2\nabla^2}{2m_1} \right) \psi_1(x) + \psi_1^*(x)U_1(x)\psi_1(x) \right] \\
+ \int d^3x \left[ \psi_2^*(x) \left( -\frac{i\hbar^2\nabla^2}{2m_2} \right) \psi_2(x) + \psi_2^*(x)U_2(x)\psi_2(x) \right]
\]

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Here $\psi_j$, $m_j$, $U_j$ with $j = 1, 2$ are the effective wave function, the mass, and the trapping potential of the $j$th condensate. The interaction between the $j$th condensate atoms is specified by $G_{jj}$, and that between 1 and 2 by $G_{12}$. In the present paper all $G$'s will be taken to be positive.

The corresponding time dependent equations of motion are the well-known non-linear Schrödinger equations \[\text{(3)},\] obtained here by minimization of the action, \[S = \int dt \left\{ \sum_{j=1,2} \psi_j^* \hbar \frac{\partial}{\partial x} \psi_j - H \right\},\]

\[
i \hbar \frac{\partial}{\partial t} \psi_1(x,t) = -\frac{\hbar^2}{2m_1} \nabla^2 \psi_1(x,t) + (U_1(x) - \mu_1)\psi_1(x,t) + G_{11} |\psi_1(x,t)|^2 \psi_1(x,t)
+ G_{12} |\psi_2(x,t)|^2 \psi_1(x,t),\]

\[
i \hbar \frac{\partial}{\partial t} \psi_2(x,t) = -\frac{\hbar^2}{2m_2} \nabla^2 \psi_2(x,t) + (U_2(x) - \mu_2)\psi_2(x,t) + G_{22} |\psi_2(x,t)|^2 \psi_2(x,t)
+ G_{12} |\psi_1(x,t)|^2 \psi_2(x,t).\]

The Lagrangian multipliers, the chemical potentials $\mu_1$ and $\mu_2$, are fixed by the relations \[\int d^3x |\psi_j(x,t)|^2 = N_j , j = 1, 2,\] with $N_j$ the number of the $j$th condensate atoms. Eq. (2) and (3) are mean field equations, since we treat the effective wave functions as $c$ numbers, corresponding to the Hartree-Fock-Bogoliubov approximation.

Experimentally, the trapping potentials \{U_j\} are simple harmonic in nature. For the sake of simplicity and to illustrate the physics we shall consider a square well trapping potential $U_j = U$: zero inside and large (infinite) outside, unless otherwise explicitly specified. We will come back to this question in the discussion of experimental feasibility.

We shall consider the strong mutual repulsive regime

\[G_{12} > \sqrt{G_{11} G_{22}}.\]
In this regime the equilibrium state for two Bose-Einstein condensates is a spatial segregation of two condensates, where two phases, the weakly and strong segregated phases, characterized by the healing length and the penetration depth, have been predicted [4]. We shall use Eq. (2) and (3) under the condition (4) to study a highly non-linear dynamical process: The two condensates are initially in a homogeneously mixed state, then eventually approach to the phase segregated state. In the same mean field manner as in Ref. [4], we find that this dynamic process can be classified into two main stages: The initially high non-equilibrium dynamical growth in stage I, where the dynamics is governed by the fastest growth mode, and the stage II of approaching to equilibrium where the dynamics is governed by the slowest mode. The stage II is the typical of a relaxation process near equilibrium. However, we shall show again it is governed by a quantum effect, namely, the Josephson effect.

III. STAGE I: FASTEST GROWTH MODE

With the square well trapping potential specified in Sec. II, the coupled non-linear Schrödinger equations have an obvious homogeneous solution: Inside the trap the condensate densities $|\psi_j|^2 = \rho_{j0}, \rho_{j0} = N_j/V$, with $V$ the volume of the square well potential trap, and the chemical potentials $\mu_1 = G_{11}\rho_{10} + G_{12}\rho_{20}$ and $\mu_2 = G_{22}\rho_{20} + G_{12}\rho_{10}$. This is the initial condition of the present problem. It is known that for a large enough mutual repulsive interaction, that is, if $G_{12} > \sqrt{G_{11}G_{22}}$, this initial state is not the ground state. Rich physics has been displayed by various theoretical studies [5,6]. Among many unstable and growing modes in this parameter regime, we will find the fastest growth mode in the initial stage of the process towards to the equilibrium state.

To look for the fastest growth mode out of the homogeneous state, we start with small fluctuations from the homogeneous state. This is consistent with the usual stability analysis [5]. Our approach here is to emphasize the connection with the physics of the classical spinodal decomposition and the role played by the Josephson relationships. Define
\[ \psi_1(x, t) = \sqrt{\rho_1(x, t)} e^{i\theta_1(x, t)}, \]  

(5)

and

\[ \psi_2(x, t) = \sqrt{\rho_2(x, t)} e^{i\theta_2(x, t)}. \]  

(6)

and define the density fluctuations \( \delta \rho_j = \rho_j - \rho_{j0} \) and the phase fluctuations \( \theta_j \), and assume they are small: \( |\delta \rho_j|/\rho_j, |\theta_j| << 1 \). The definition of the phase fluctuations here has made use of the implicit assumption that there is no average current. Then, to the linear order, we have from Eq. (2,3), for condensate 1,

\[ \dot{\delta \rho}_1 + \frac{\hbar}{m_1} \nabla^2 \theta_1 = 0, \]  

(7)

\[ \hbar \dot{\theta}_1 = \frac{\hbar^2}{4m_1} \nabla^2 \delta \rho_1 - G_{11} \delta \rho_1 - G_{12} \delta \rho_2. \]  

(8)

In terms of hydrodynamics, Eq.(7) is the continuity equation, and Eq. (8) is the Bernoulli equation, with the first term at the right hand side as the so-called ‘quantum pressure’. Similarly, for condensate 2 we have

\[ \dot{\delta \rho}_2 + \frac{\hbar}{m_2} \nabla^2 \theta_2 = 0, \]  

(9)

\[ \hbar \dot{\theta}_2 = \frac{\hbar^2}{4m_2} \nabla^2 \delta \rho_2 - G_{22} \delta \rho_2 - G_{12} \delta \rho_1. \]  

(10)

Eliminating the phase variables from Eqs. (7-10), we have

\[
\frac{\partial^2}{\partial t^2} \begin{pmatrix} \delta \rho_1 \\ \delta \rho_2 \end{pmatrix} = \begin{pmatrix} -\frac{\hbar^2}{4m_1^2} \nabla^4 + \frac{\rho_{10}}{m_1} G_{11} \nabla^2 & \frac{\rho_{10}}{m_1} G_{12} \nabla^2 \\ \frac{\rho_{20}}{m_2} G_{12} \nabla^2 & -\frac{\hbar^2}{4m_2^2} \nabla^4 + \frac{\rho_{20}}{m_2} G_{22} \nabla^2 \end{pmatrix} \begin{pmatrix} \delta \rho_1 \\ \delta \rho_2 \end{pmatrix}.
\]  

(11)

We look for the solution of the form

\[
\begin{pmatrix} \delta \rho_1 \\ \delta \rho_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i(q \cdot r - \omega t)},
\]

with \( A, B \) constants. Eq. (11) then becomes
\[
\begin{pmatrix}
\frac{\hbar^2}{4m_1^2}q^4 + \frac{\rho_{10}}{m_1}G_{11}q^2 - \omega^2 & \frac{\rho_{10}}{m_1}G_{12}q^2 \\
\frac{\rho_{20}}{m_2}G_{12}q^2 & \frac{\hbar^2}{4m_2^2}q^4 + \frac{\rho_{20}}{m_2}G_{22}q^2 - \omega^2
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix} = 0 .
\]

In order to have a non-zero solution for \(A\) and \(B\), the determinant must be zero:

\[
\det \begin{pmatrix}
\frac{\hbar^2}{4m_1^2}q^4 + \frac{\rho_{10}}{m_1}G_{11}q^2 - \omega^2 & \frac{\rho_{10}}{m_1}G_{12}q^2 \\
\frac{\rho_{20}}{m_2}G_{12}q^2 & \frac{\hbar^2}{4m_2^2}q^4 + \frac{\rho_{20}}{m_2}G_{22}q^2 - \omega^2
\end{pmatrix} = 0 ,
\]

which determines the dispersion relation between the frequency and the wave number. One can easily check that for the one component condensate case, by putting \(G_{12} = 0\), Eq.(12) indeed gives the usual phonon spectrum. In the present binary BEC mixture case, we have

\[
\omega_\pm^2 = \frac{q^2}{2} \left[ b_{11} + b_{22} \pm \sqrt{(b_{11} - b_{22})^2 + 4b_{12}b_{21}} \right] ,
\]

with

\[
b_{11} = \frac{\hbar^2}{4m_1^2}q^2 + \frac{\rho_{10}}{m_1}G_{11} , \quad b_{12} = \frac{\rho_{10}}{m_1}G_{12} , \quad b_{21} = \frac{\rho_{20}}{m_2}G_{12} , \quad b_{22} = \frac{\hbar^2}{4m_2^2}q^2 + \frac{\rho_{20}}{m_2}G_{22} .
\]

Obviously, both \(\omega_+^2\) and \(\omega_-^2\) are real as given by Eq.(13). They give the two phonon velocities in the binary BEC mixture:

\[
c_\pm = \sqrt{\left[ \rho_{10}G_{11}/m_1 + \rho_{20}G_{22}/m_2 \pm \sqrt{(\rho_{10}G_{11}/m_1 - \rho_{20}G_{22}/m_2)^2 + 4\rho_{10}\rho_{20}G_{12}^2/(m_1m_2)} \right]}/2 .
\]

However, if

\[
b_{11}b_{22} - b_{12}b_{21} < 0 ,
\]

the branch \(\omega_-^2\) can be negative. This implies an imaginary frequency \(\omega_-\) and an imaginary phonon velocity, which shows that the initial homogeneously mixed state is unstable. One can verify that the sufficient condition for the inequality Eq.(14) to hold for small enough wave number \(q\) is the validity of the inequality Eq.(4). The modes defined by Eq. (13) with imaginary frequencies will
then grow exponentially with time. Unlike the usual situation near equilibrium, this growth from the present non-equilibrium homogeneously mixed state will be dominated by the fastest growth mode. This is precisely the same case as in the initial stage of the classical spinodal decomposition.

Now we look for the fastest growth mode. It is equivalent to find the most negative value of $\omega^2$. The condition $\partial \omega^2 / \partial (q^2) = 0$ leads to

$$
\frac{[(b_{11} - b_{22})^2 + (b_{11} - b_{22}) \left( \frac{\hbar^2}{4m_1} - \frac{\hbar^2}{4m_2} \right) q_{\text{max}}^2 + 4 \frac{\rho_{10} \rho_{20}}{m_1 m_2} G_{12}^2]^2}{(b_{11} - b_{22})^2 + 4 \frac{\rho_{10} \rho_{20}}{m_1 m_2} G_{12}^2}
= 2q_{\text{max}}^2 \left( \frac{\hbar^2}{4m_1} + \frac{\hbar^2}{4m_2} \right) + \frac{\rho_{10}}{m_1} G_{11} + \frac{\rho_{20}}{m_2} G_{22} \right]^2,
$$

(15)

with

$$
b_{11} - b_{22} = \left( \frac{\hbar^2}{4m_1} - \frac{\hbar^2}{4m_2} \right) q_{\text{max}}^2 + \frac{\rho_{10}}{m_1} G_{11} - \frac{\rho_{20}}{m_2} G_{22}.
$$

Eq.(15) is actually a quartic equation for $q_{\text{max}}^2$, whose algebraic solutions can be found by the standard way. The corresponding imaginary frequency branch is

$$
\omega_{-\text{max}}^2 = -\frac{\hbar^4}{2} \left( \frac{\hbar^2}{4m_1} + \frac{\hbar^2}{4m_2} \right) - \left( b_{11} - b_{22} \right) \left( \frac{\hbar^2}{4m_1} - \frac{\hbar^2}{4m_2} \right) \sqrt{ \left( b_{11} - b_{22} \right)^2 + 4 \frac{\rho_{10} \rho_{20}}{m_1 m_2} G_{12}^2 }.
$$

(16)

The detailed general expressions for both $q_{\text{max}}$ and $\omega_{-\text{max}}$ are not physically transparent and particularly illustrative. We will not explore them here.

To get a better understanding of the physical implications of Eqs. (15,16), we consider a special case relevant to recent experiments where particles of the two condensates have the same mass, $m_1 = m_2 = m$. In this case we find the wavenumber corresponding to the most negative $\omega_{-\text{max}}^2$ is

$$
q_{\text{max}} = \frac{m}{\hbar^2} \left[ \sqrt{ \left( \rho_{10} G_{11} + \rho_{20} G_{22} \right)^2 + 4 \rho_{10} \rho_{20} G_{11} G_{22} \left( \frac{G_{12}^2}{G_{11} G_{22}} - 1 \right) - \left( \rho_{10} G_{11} + \rho_{20} G_{22} \right) } \right],
$$

(17)

and

$$
\omega_{-\text{max}} = -i \frac{\hbar}{2m} q_{\text{max}}^2.
$$

(18)
The physics implied in Eqs. (15,16) or Eqs. (17,18) is as follows. Starting from the initial homogeneous mixture of the two condensates, on the time scale given by

\[ t_I = \frac{1}{|\omega_{\max}|}, \]

domain patterns of the phase segregation with the characteristic length

\[ l_I = \frac{1}{q_{\max}} \]

will appear. Particularly, for the weakly segregated phase of \( \frac{G_{12}^2}{G_{11}G_{22}} - 1 \to 0 \), we have the length scale

\[ l_{I}^{-1} = \frac{\sqrt{m}}{\hbar} \left( \frac{2\rho_{10}\rho_{20}G_{11}G_{22}}{(\rho_{10}G_{11} + \rho_{20}G_{22})} \left( \frac{G_{12}^2}{G_{11}G_{22}} - 1 \right) \right) = \frac{\sqrt{2}}{\sqrt{\Lambda_1^2 + \Lambda_2^2}}, \]

and for the strongly segregated phase of \( \frac{G_{12}^2}{G_{11}G_{22}} - 1 \to \infty \), we have

\[ l_{I}^{-1} = \frac{2m^{1/2}}{\hbar} \left( \rho_{10}\rho_{20}G_{11}G_{22} \left( \frac{G_{12}^2}{G_{11}G_{22}} - 1 \right) \right)^{1/4} = \frac{\sqrt{2}}{\sqrt{\Lambda_1\Lambda_2}}. \]

Here \( \Lambda_j = \xi_j/\sqrt{G_{12}/\sqrt{G_{11}G_{22}} - 1} \) and \( \xi_j = \sqrt{\hbar^2/2m_j} 1/\rho_{j0}G_{jj} \) are the penetration and healing lengths in the binary BEC mixture [4]. Those length and times scales can be measured experimentally. We will come back to the experimental situation below.

After the stage I of fast growth into the domain pattern characterized by the length scale \( l_I \), the system will gradually approach to the true ground state of the complete phase segregation: one condensate in one region and the second condensate in another region. This stage is slow, dominated by the slowest mode, and is the subject of next section.

It is now evident that the stage I of the growth of binary BEC’s shares the same phenomenology of the initial stage of the classical spinodal decomposition: the domination of the fastest growth mode, the appearance of domains of segregated phases, and the conservation of particle numbers. There are, however, two important differences. First, the dynamical evolution of the binary BEC’s is governed by a coupled nonlinear time dependent Schrödinger equations, not by a nonlinear
diffusion equation supplemented with the continuity equation, the Cahn-Hilliard equation. There is no external relaxation process for the present wave functions. Secondly, the energy of the binary BEC’s is conserved during the growth process, not as in the case of classical spinodal decomposition where the system energy always decreases.

We wish to point out that the initial stage of spinodal decomposition without diffusion constant has been explored in heavy ion collisions for Fermi systems, based on the kinetic equation approach \[7\]. Qualitatively similar results have been obtained there. The study in alkali BEC mixtures should shed light on this important process, because the observation of real time evolution is in principle attainable experimentally.

IV. STAGE II: MERGING AND OSCILLATING BETWEEN DOMAINS

The BEC binary mixture occurs in a trap. This finite size effect of the droplet leads to the broken symmetry of the condensate profiles \[6\], which tends to separate the condensates in mutually isolated regions. This implies that there is no contact between different domains of the same condensates formed in the stage I. The classical spinodal process involves diffusion. An estimate of the diffusion constant for the BEC system can be made from kinetics theory. The ratio of the time scales for quantum and classical particle transport is of the order of the ratio of the BEC cloud size to the de Broglie wavelength. This is much smaller than one for the experimental systems of interest. The classical diffusion process is thus not important. Because of the domains are not connected and because the diffusion for the BEC mixture is extremely low, all the mechanisms for the late stage classical spinodal decomposition are not applicable. We propose that it is the Josephson effect that is responsible for the approach to equilibrium in stage II. Two models for the Josephson effect, the ‘rigid pendulum’ model \[8\] and the ‘soft pendulum’ model \[9\] will be discussed. They both give the same time scale when the ‘Rabi’ frequency is small. Those model studies are performed for a single BEC condensate separated by an external potential barrier. We adapt their analysis to the present
situation of one condensate sitting in the potential wells formed by another condensate during the phase segregation process.

For a classical spinodal decomposition in a binary fluid mixture, there are several scenarios of growth in stage II. (1) For the domains to grow by the diffusion of materials between them, a finite diffusion constant is needed. For the binary BEC mixture, the temperature is too low for the diffusion to occur. The diffusion constant is practically zero. Furthermore, particles of one condensate have to diffuse into the domains of the other condensate because domains of the same condensate are not connected. This requires a finite activation energy. Hence this process is not possible at low enough temperatures. (2) The same argument also rules out the applicability of the diffusion-enhanced collisions by which the diffusion field around the domains leads to an attraction between them. (3) Another mechanism is the noise-induced growth. However, there is no external noise in the BEC described by the time dependent nonlinear Schrödinger equations. This mechanism is thus irrelevant here. (4) A seemingly relevant one is the hydrodynamic growth that is driven by the pressure difference between the points of different curvatures. This would be a rather fast process. However, this also requires a connected phase of one of the species. Hence, at zero temperature and for a finite system, classical scenarios for the later stage will not work here. We need to find a scenario working under the present conditions, and at the same time that is consistent with the time dependent nonlinear Schrödinger equations.

The only scenario which can transport particles across a forbidden region at zero temperature is the tunneling process. Let us consider the specific case of two domains of condensate 1 separated by a domain with width \( d \) of condensate 2. The ability of condensate 1 to tunnel through condensate 2 is described by the penetration depth \( \Lambda \), as discussed in Ref. [4]. Hence the probability of condensate 1 to tunnel through condensate 2 can be estimated as

\[
p = e^{-\frac{2d}{\Lambda}},
\]

when \( p \) is much smaller than 1. The finiteness, though small, of the tunneling probability suggests
that it is the Josephson effect responsible for the relaxation process in the stage II. The Josephson effect leads to the merging of two domains of the same condensate at sufficiently low temperatures. The dynamics of such motion may be governed by the ‘rigid pendulum’ Hamiltonian for a Josephson junction [8]:

$$H(\phi, n) = E_J(1 - \cos \phi) + \frac{1}{2} E_C n^2 ,$$

where $E_J$ is the Josephson coupling energy determined by the tunneling probability, $n = (n_x - n_y)/2$ is the particle number difference between the numbers of particles, $n_x$ and $n_y$, in the two domains, and $E_C \equiv \partial \mu / \partial n$ is the ‘capacitive’ energy due to interactions. In the absence of external constraints, $\mu = E_C n$. The phase difference $\phi$ between the two domains is conjugated to $n$, as in usual Josephson junctions. Under the appropriate condition, such as low temperature and smallness of the capacitive energy, there may be an oscillation between the two domains of condensate 1 separated by condensate 2. In such a case, we may estimate that the oscillation period $\tau_{II} = 2\pi / \omega_{JP}$, with the so-called Josephson plasma frequency [8]

$$\omega_{JP} = \frac{\sqrt{E_C E_J}}{\hbar} .$$

(20)

For small tunneling probability, the Josephson junction energy may be estimated as [8]

$$E_J = n_T^{1/3} \hbar \omega_0 e^{-2\phi} ,$$

and the capacitive energy as

$$E_C = \frac{2}{5} \left( \frac{n_T}{2} \right)^{-0.6} \left( \frac{15a_{11}}{a_0} \right)^{0.4} \hbar \omega_0 .$$

Here $\omega_0$ is the harmonic oscillator frequency for condensate 1 in a harmonic trap, $a_0 = \sqrt{\hbar/m_1 \omega_0}$ is the corresponding oscillator length, the $a_{11}$ is the scattering length of condensate 1, and $n_T = n_x + n_y$ is the total number of particles in domain $x$ and $y$. Then the oscillatory time scale between the domains determines by the Josephson plasma frequency
\[ \tau_{II}^{-1} = \left(\frac{2 a_0}{15 a_{11} n_T}\right)^{2/15} \frac{\omega_0}{2\pi} e^{-\frac{4}{45}}. \]  

(21)

The rigid pendulum Hamiltonian would give a good description when \( n << n_T \).

Another description of the Josephson effect uses the ‘soft pendulum’ Hamiltonian proposed in Ref. \[9\]:

\[ H(\phi, n) = -E_J(n) \cos \phi + \frac{1}{2} E_C n^2, \]

with the number dependent Josephson coupling energy \( E_J(n) = n_T \hbar \omega_R \sqrt{1 - (n/n_T)^2} \). Here \( \omega_R \approx E_J/\hbar \) is the Rabi frequency determined by the overlap integral between the two wave functions in the absence of the mutual repulsive interaction \( G_{12} \). The appropriate frequency scale in this case is

\[ \omega_{sp}^2 = \omega_{JP}^2 + \omega_R^2, \]

(22)

which determines the time scale for the two domains to merge. The Josephson plasma frequency in Eq.(22) of the soft pendulum model is \( \omega_{JP} = \sqrt{n_T E_C \omega_R / \hbar} \). For the experimental situation of interest, \( \omega_R << \omega_{JP} \). Thus the two approaches give essentially the same answer. The detailed derivation of \( E_J \) was given in Ref. \[8\].

Given the Josephson effect is the dominant mechanism in the stage II, the time scale to arrive at the ground state will be determined by the Josephson effect at the final two domains, in which the two domains of condensate 1 is separated by the domain of condensate 2, in the manner of 1 2 1 2 spatial configuration for the case of equal numbers of the condensates. The width of each domain is then \( D/4 \), with \( D \) the size of the trap. According to the above analysis, the largest time scale determined by Eq.(22), the slowest mode, in the stage II is:

\[ t_{II} = 2\pi / \omega_{sp} . \]

(23)

The arrangement of 1 2 1 spatial configuration may also occur here, in which it is more likely for condensate 2 to tunnel through 1 to the edge of the trap, because of the larger tunneling probability.
V. DISCUSSIONS

The first question is that the quantum spinodal decomposition discussed above can happen or not. In terms of the atomic scattering lengths of condensate atoms $a_{jj}$, the interactions are $G_{jj} = 4\pi\hbar^2a_{jj}/m_j$. The typical value of $a_{jj}$ for $^{87}\text{Rb}$ is about 50 Å. The typical density realized for the binary BEC mixture is about $\rho_{i0} \sim 10^{14}/\text{cm}^3$. Hence the healing length is $\xi = \sqrt{(\hbar^2/2m)(1/G_{jj}\rho_{i0})} = \sqrt{1/(8\pi a_{jj}\rho_{i0})} \sim 3000\text{Å}$. For the different hyperfine states of $^{87}\text{Rb}$, it is known now [10,6,4] that $G_{12}/\sqrt{G_{11}G_{22}} > 1$. Hence the ground state of the phase segregated phase can be realizable. Therefore, the quantum spinodal decomposition can happen.

Experimentally, starting from the initially homogeneous state, after a short period of time a domain pattern does appear. Then a damped oscillation between the domain pattern has been observed. Eventually the binary BEC mixture sets to the segregated phase [10]. If we take $G_{12}/\sqrt{G_{11}G_{22}} = 1.04$, the penetration depth is $\Lambda = \xi/\sqrt{G_{12}/\sqrt{G_{11}G_{22}} - 1} \sim 1.5\mu\text{m}$. The length scale $l_I$ determined by Eq.(17) in the stage I is 1.5 µm, which is the same order of magnitude in comparison with experimental data [10]. This length is also comparable to the domain wall width seen experimentally. The corresponding time scale $t_I$ (Eq. (18)) is then 6 ms, again the same order of magnitude, though both estimated $l_I$ and $t_I$ are somewhat smaller. If $G_{12}/\sqrt{G_{11}G_{22}} = 1.0004$, the corresponding length and time scales in stage I according to the present analysis are $l_I = 15\mu\text{m}$ and $t_I = 600\text{ms}$, respectively. Time and length scales calculated in this way are larger than the numbers extracted from the experiment [10]. If we assume the damped oscillation to equilibrium observed experimentally is the stage II discussed here, taking the Rabi frequency $\omega_R = 0.2nK = 4\text{ Hz}$ [8] and the total number of particle $N_1 = 10^6$, we find the period according to Eq. (23) is 30 ms, comparable to the experimental value. The estimation from Ref. [8] gives a larger value of about 200 ms. We think this estimate of the larger time scale is due to the use of a larger estimate of $d/\Lambda$. In our view, $d$ and $\Lambda$ are comparable in stage I. Thus for stage II, $d/\Lambda$ is of the order of $n_d$, the number of domains formed in stage I. For the experiment of interest to use, $n_d \approx 2$. Thus
for the case at hand, we think the first estimate is more reasonable.

The present analysis shows that the stage I is insensitive to damping. Similar conclusion has been reached in the study of heavy ion collisions for fermions \cite{1}. At this moment we do not have a reliable estimation of the damping in the Josephson oscillation, as indicated experimentally. Nevertheless, given the uncertainty in the value of $G_{12}$ or $a_{12}$, we conclude that the stage II of the quantum spinodal decomposition may have been observed.

\section*{VI. CONCLUSION}

In the present paper we have studied the problem of the dynamical evolution of the binary BEC mixtures starting from the homogeneously mixed unstable state. We have found a parallel analogy to the usual spinodal decomposition process by calling the present one the quantum spinodal decomposition. The quantum spinodal decomposition consists of two stages: the stage I of initial fast growth of domains, characterized by a time and a length scale, and the stage II of a slow relaxation towards equilibrium dominated by the Josephson effect. The coupled non-linear Schrödinger equations provide a good theoretical description, which enable us to calculate the time and length scales. In comparison with recent experiments, we conclude that the quantum spinodal decomposition can be, and may have been, realized.

The length and time scales for Rb mixtures is controlled by the difference of $r = G_{12}/\sqrt{G_{11}G_{22}}$ and 1. Since $r$ is close to 1 experimentally, the data can provide for a very sensitive estimate of r-1. Since this parameter determines \textbf{both} the time and length scale of stage I, it is a self-consistent check on the physical picture provided here.

Periodic-like structures have also been observed in the phase segregation of spin 1 Na mixtures \cite{11}. We think a similar picture of quantum spinodal decomposition applies to that case as well.

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