Multi-intersection Traffic Light Control Using Infinitesimal Perturbation Analysis

Yanfeng Geng* Christos G. Cassandras*

* Division of Systems Engineering and Center for Information and Systems Engineering, Boston University, Brookline, MA, 02446
(e-mail: gengyf@bu.edu, cgc@bu.edu)

Abstract: We address the traffic light control problem for multiple intersections in tandem by viewing it as a stochastic hybrid system and developing a Stochastic Flow Model (SFM) for it. Using Infinitesimal Perturbation Analysis (IPA), we derive on-line gradient estimates of a cost metric with respect to the controllable green and red cycle lengths. The IPA estimators obtained require counting traffic light switchings and estimating car flow rates only when specific events occur. The estimators are used to iteratively adjust light cycle lengths to improve performance and, in conjunction with a standard gradient-based algorithm, to obtain optimal values which adapt to changing traffic conditions. Simulation results are included to illustrate the approach.

Keywords: Traffic Light Control, SFM, IPA.

1. INTRODUCTION

The Traffic Light Control (TLC) problem aims at dynamically controlling the flow of traffic at an intersection through the timing of green/red light cycles with the objective of reducing congestion, hence also the delays incurred by drivers. The more general problem involves a set of intersections and traffic lights with the objective of reducing overall congestion over an area covering multiple urban blocks. Since the control of one intersection influences the traffic flow from or towards others, this is a complex problem further complicated by the fact that traffic flows constantly change depending on the time of day, accidents, weather conditions, etc. Recent technological developments involving better, inexpensive sensors and wireless sensor networks have enabled the collection of data (e.g., counting vehicles in a specific road section) which can be used to optimally select traffic light cycles over specific time intervals in a day or even to dynamically control them based on real-time data. Thus, methodologies that would not be possible to implement not long ago are now becoming feasible. The approach proposed in this paper to the TLC problem is specifically intended to exploit these recent developments.

Several different approaches have been proposed to solve the TLC problem. It is formulated as a Mixed Integer Linear Programming (MILP) problem in Dujardin et al. [2011], and as an Extended Linear Complementary Problem (ELCP) in DeSchutter [1999]. A Markov Decision Process (MDP) approach has been proposed in Yu and Recker [2006] and Reinforcement Learning (RL) was used in Thorpe [1997], with several extensions found in Prashanth and Bhatnagar [2011], Wiering et al. [2004]. A game theoretic viewpoint is given in Alvarez and Poznyak [2010], while a hybrid system formulation is presented in Zhao and Chen [2003]. Due to its complexity when viewed as an optimization problem, fuzzy logic is often used in both a single (isolated) junction Murat and Gedizlioglu [2005] and multiple junctions Choi et al. [2002]. Expert systems Findler and Stapp [1992] and evolutionary algorithms Taale et al. [1998] have also been applied to develop a traffic light controller for a single intersection. Perturbation analysis techniques were used in Head et al. [1996] and a formal approach using Infinitesimal Perturbation Analysis (IPA) to solve the TLC problem was presented in Panayiotou et al. [2005] for a single intersection.

In Geng and Cassandras [2012], we study the TLC problem for a single intersection using a Stochastic Flow Model (SFM) and Infinitesimal Perturbation Analysis (IPA). In this paper, we extend our analysis to two tandem intersections. We still adopt a stochastic hybrid system modeling framework (see Cassandras and Lygeros [2006]), since the problem involves both event-driven dynamics in the switching of traffic lights and time-driven dynamics that capture the flow of vehicles through an intersection. Although one can also view this as a purely Discrete Event System (DES) with the intersection area as a “server” processing “users” (vehicles), the fact that a vehicle does not exclusively occupy this area makes a flow-based viewpoint a more accurate way to model such a process. While in most traditional flow models the flow rates involved are treated as deterministic parameters, a SFM as introduced in Cassandras et al. [2002] treats them as stochastic processes. In the TLC problem, this is consistent with continuously and randomly varying traffic flows, especially in heavy traffic conditions where the problem is most interesting. With only minor technical assumptions imposed on the properties of such processes, a general IPA theory for stochastic hybrid systems was recently presented in Wardi et al. [2010], Cassand-
In Section 2, we formulate the TLC problem for two intersections and construct a SFM. In Section 3, we derive an IPA estimator for a cost function gradient with respect to a controllable parameter vector defined by green and red cycle lengths. This is then used to iteratively adjust these cycle lengths to improve performance and, under proper conditions, obtain optimal parameter values. Simulation-based examples are given in Section 4 and we conclude with Section 5.

2. PROBLEM FORMULATION

In this paper, we concentrate on solving the TLC problem for two coupled intersections, as shown in Fig. 1. There are four roads and four traffic lights, with each traffic light controlling the associated incoming traffic flow. The traffic in road 1 of intersection $I_1$ flows into road 3 of $I_2$. For simplicity, we make the following assumptions: (i) Left-turn and right-turn traffic flows are not considered, i.e., $\alpha$ where

\[ x(t) = \begin{cases} 1 & \text{if } x_n(t) < \theta_n \text{ or } z_n(t) = \theta_n \\ 0 & \text{otherwise} \end{cases} \]  

where $x_n(t)$ is not empty. According to assumption (iii) $\beta_1(t)$ is independent of $x_3(t)$. However, $\alpha_3(t)$ depends on the departure process $\beta_1(t)$ of queue 1; in particular, $\alpha_3(t) = \beta_1(t)$.

Combining (2) through (4), we have the dynamics of queue 3:

\[ \dot{x}_3(t) = \begin{cases} h_3(t) & \text{if } B_3(z, \theta) = 0, B_1(z, \theta) = 1, \;
\text{and } x_3(t) > 0 \\
\alpha_3(t) & \text{if } B_3(z, \theta) = 0, B_1(z, \theta) = 1, \;
\text{and } x_3(t) = 0 \\
0 & \text{if } B_3(z, \theta) = 0, B_1(z, \theta) = 0, \;
or B_3(z, \theta) = 1, x_3(t) = 0 \\
\alpha_1(t) - h_3(t) & \text{if } B_1(z, \theta) = 1, B_1(z, \theta) = 1, \;
\text{and } x_3(t) > 0, x_1(t) > 0 \\
\alpha_1(t) - h_3(t) & \text{if } B_1(z, \theta) = 1, B_1(z, \theta) = 1, \;
\text{and } x_3(t) > 0, x_1(t) = 0 \\
-h_3(t) & \text{if } B_1(z, \theta) = 1, B_1(z, \theta) = 0, \;
\text{and } x_3(t) > 0 \end{cases} \]

The operation of the intersection can be viewed as a hybrid system with the time-driven dynamics described by (2)-(5) and event-driven dynamics dictated by GREEN-RED light switches and by events causing some $x_n(t)$ to switch from positive to zero or vice versa.
is \( \Phi \) a switch in the sign of \( \alpha \) of \( x \) and EPs. The event set that affects any queue entire sample path consists of a series of alternating NEPs consists of intervals over which path of any one of the queue contents (as shown in Fig. 3) we concentrate on the SHA for the operation of queue 3 as shown in Fig. 2. This reflects the fact that a typical sample path of any one of the queue contents (as shown in Fig. 3) consists of intervals over which \( x_n(t) > 0 \), which we call Non-Empty Periods (NEPs), followed by intervals where \( x_n(t) = 0 \), which we call Empty Periods (EPs). Thus, the entire sample path consists of a series of alternating NEPs and EPs. The event set that affects any queue \( n = 1, 2, 4 \) is \( \Phi_n = \{e_1, e_2, e_3, e_4, e_5\} \) where \( e_1 \) is a switch in the sign of \( \alpha_n(t) - \beta_n(t) \) from non-positive to strictly positive, \( e_2 \) is a switch in the sign of \( \alpha_n(t) \) from 0 to strictly positive, \( e_3 \) is the queue content becoming empty, i.e., \( x_1 = 0 \), which terminates a NEP (and initiates an EP), \( e_4 \) switches a light from RED to GREEN, and \( e_5 \) switches a light from GREEN to RED. For easier reference, we label \( \Phi_3 \) as “\( E_3 \)” for the end of NEP events, \( e_4 \) as “\( R2G_e \)” and \( e_5 \) as “\( G2R_e \)” for the light switching events. The resulting start of a NEP is an event “induced” by either \( e_5 \) or \( e_2 \) or \( e_1 \) which we will refer to as an “\( S_3 \)” event. For queue 3, the event set includes all those events that cause a jump in the value of \( \dot{x}_3(t) \) in (5). As we can see from Fig. 2, every event of \( \Phi_1 \) also affects the dynamics of queue 3. Thus, we have \( \Phi_3 = \{S_1, E_1, R2G_1, G2R_1, S_3, E_3, R2G_3, G2R_3\} \).

Returning to Fig. 3, the \( m \)th NEP in a sample path of queue 3, \( m = 1, 2, \ldots \), is denoted by \( (\xi_{3,m}, \eta_{3,m}) \), i.e., \( \xi_{3,m}, \eta_{3,m} \) are the occurrence times of the \( m \)th \( S_3 \) and \( E_3 \) event respectively at this queue. During the \( m \)th NEP, \( t_{3,j}^m, j = 1, \ldots, J_{3,m} \), denotes the time when an event occurs.

![Fig. 2. The Stochastic Hybrid Automaton model](image)

Using the standard definition of a Stochastic Hybrid Automaton (SHA) (e.g., see Cassandras and Lafortune [2008]), we may obtain a SHA model for queue 1, 2 and 4 which is similar to Geng and Cassandras [2012]. Here, we concentrate on the SHA for the operation of queue 3 as shown in Fig. 2. This reflects the fact that a typical sample path of any one of the queue contents (as shown in Fig. 3) consists of intervals over which \( x_n(t) > 0 \), which we call Non-Empty Periods (NEPs), followed by intervals where \( x_n(t) = 0 \), which we call Empty Periods (EPs). Thus, the entire sample path consists of a series of alternating NEPs and EPs. The event set that affects any queue \( n = 1, 2, 4 \) is \( \Phi_n = \{e_1, e_2, e_3, e_4, e_5\} \) where \( e_1 \) is a switch in the sign of \( \alpha_n(t) - \beta_n(t) \) from non-positive to strictly positive, \( e_2 \) is a switch in the sign of \( \alpha_n(t) \) from 0 to strictly positive, \( e_3 \) is the queue content becoming empty, i.e., \( x_1 = 0 \), which terminates a NEP (and initiates an EP), \( e_4 \) switches a light from RED to GREEN, and \( e_5 \) switches a light from GREEN to RED. For easier reference, we label \( \Phi_3 \) as “\( E_3 \)” for the end of NEP events, \( e_4 \) as “\( R2G_e \)” and \( e_5 \) as “\( G2R_e \)” for the light switching events. The resulting start of a NEP is an event “induced” by either \( e_5 \) or \( e_2 \) or \( e_1 \) which we will refer to as an “\( S_3 \)” event. For queue 3, the event set includes all those events that cause a jump in the value of \( \dot{x}_3(t) \) in (5). As we can see from Fig. 2, every event of \( \Phi_1 \) also affects the dynamics of queue 3. Thus, we have \( \Phi_3 = \{S_1, E_1, R2G_1, G2R_1, S_3, E_3, R2G_3, G2R_3\} \).

Returning to Fig. 3, the \( m \)th NEP in a sample path of queue 3, \( m = 1, 2, \ldots \), is denoted by \( (\xi_{3,m}, \eta_{3,m}) \), i.e., \( \xi_{3,m}, \eta_{3,m} \) are the occurrence times of the \( m \)th \( S_3 \) and \( E_3 \) event respectively at this queue. During the \( m \)th NEP, \( t_{3,j}^m, j = 1, \ldots, J_{3,m} \), denotes the time when an event occurs.

![Fig. 3. A typical sample path of traffic light queue 3](image)

Our objective is to select \( \theta \) so as to minimize a cost function that measures a weighted mean of the queue lengths over a fixed time interval \([0, T]\). In particular, we define the sample function

\[
L(\theta; x(0), z(0), T) = \frac{1}{T} \sum_{n=1}^{M_n} \int_0^T w_n x_n(\theta, t) dt \tag{6}
\]

where \( w_n \) is a cost weight associated with queue \( n \) and \( x(0), z(0) \) are given initial conditions. It is obvious that since \( x_n(t) = 0 \) during EPs of queue \( n \), we can rewrite (6) in the form

\[
L(\theta; x(0), z(0), T) = \frac{1}{T} \sum_{n=1}^{M_n} \sum_{m=1}^{J_n} \int_{\xi_{n,m}}^{\eta_{n,m}} w_n x_n(\theta, t) dt \tag{7}
\]

where \( M_n \) is the total number of NEPs during the sample path of queue \( n \). For convenience, we also define

\[
L_{n,m}(\theta) = \int_{\xi_{n,m}}^{\eta_{n,m}} x_n(\theta, t) dt \tag{8}
\]

to be the sample cost associated with the \( m \)th NEP of queue \( n \). We can now define our overall performance metric as

\[
J(\theta; x(0), z(0), T) = E[L(\theta; x(0), z(0), T)] \tag{9}
\]

Since we do not impose any limitations on the processes \( \{\alpha_n(t)\} \) and \( \{\beta_n(t)\} \), it is infeasible to obtain a closed-form expression of \( J(\theta; x(0), z(0), T) \). The only assumption we make is that \( \alpha_n(t), \beta_n(t) \) are piecewise continuous w.p. 1. The value of IPA as developed for general stochastic hybrid systems in Cassandras et al. [2010] is in providing the means to estimate the performance metric gradient \( \nabla J(\theta) \), by evaluating the sample gradient \( \nabla L(\theta) \). As shown elsewhere (e.g., see Cassandras et al. [2010]), these estimates are unbiased under mild technical conditions. Moreover, an important property of IPA estimates is that they are often independent of the unknown processes \( \{\alpha_n(t)\} \) and \( \{\beta_n(t)\} \) or they depend on values of \( \alpha_n(t) \) or \( \beta_n(t) \) at specific event times only. Such robustness properties of IPA (formally established in Yao and Cassandras [2011]) make it attractive for estimating on line performance sensitivities with respect to controllable parameters such as \( \theta \) in our case. One can then use this information to either improve performance or, under appropriate conditions, solve an optimization problem and determine an optimal \( \theta^* \) through an iterative scheme:

\[
\theta_{i,k+1} = \theta_{i,k} - \gamma_k H_{i,k}(\theta_{i,k}, x(0), T, \omega_k), k = 0, 1, \ldots \tag{10}
\]

where \( H_{i,k}(\theta_{i,k}, x(0), T, \omega_k) \) is an estimate of \( dJ/d\theta \), based on the information obtained from the sample path denoted by \( \omega_k \), and \( \gamma_k \) is the stepsize at the \( k \)th iteration. Next
we will focus on how to obtain $dL/d\theta_i$, $i = 1, 2, 3, 4$. We may then also obtain $\theta^*$ through (10), provided that the random processes $\{x_n(t)\}$ and $\{\beta_n(t)\}$ are stationary over $[0, T]$. We will assume that the derivatives $dL/d\theta_i$ exist for all $\theta_i \in \mathbb{R}^+$ w.p. 1 (if this is violated, IPA is still possible by considering one-sided derivatives; see Cassandras et al. [2002].)

### 3. INFINITESIMAL PERTURBATION ANALYSIS (IPA)

Consider a sample path of the system as modeled in Fig. 2 over $[0, T]$ and let $\tau_k(\theta)$ denote the occurrence time of the $k$th event (of any type), where we stress its dependence on $\theta$. To simplify notation, we define the derivatives of the states $x_n(t, \theta)$ and $z_n(t, \theta)$ and event times $\tau_k(\theta)$ with respect to $\theta_i$, $i = 1, \ldots, 4$, as follows:

$$
x'_{n,i}(t) \equiv \frac{\partial x_n(t, \theta)}{\partial \theta_i}, \quad z'_{n,i}(t, \theta) \equiv \frac{\partial z_n(t, \theta)}{\partial \theta_i}, \quad \tau_k' \equiv \frac{\partial \tau_k(\theta)}{\partial \theta_i}.
$$

Taking derivatives with respect to $\theta_i$ in (7), we obtain

$$
dL(\theta) = \frac{1}{T} \sum_{n=1}^{N} \left[ \int_{\xi_{n,m}} w_n x'_{n,i}(t) dt + w_n x_n(\xi_{n,m}) \frac{\partial \xi_{n,m}}{\partial \theta_i} \right]
$$

where the last equality follows from the definition (8).

By assumption (iii), $\beta_i(t)$ is independent of $x_i(t)$. Therefore, $dL_{n,m}/d\theta_i = 0$ for $n = 1, 2$ and $i = 3, 4$. It follows that $\tau'_i(t)$ for $n = 1, 2$ and $i = 1, 2$ can be obtained by the analysis of a single isolated intersection in Geng and Cassandras [2012]. Since equation (2) can still be applied for $x_4(t)$, we can obtain $x'_4(t)$, $i = 1, \ldots, 4$, similar to a queue in an isolated intersection. Therefore, in what follows, we focus on obtaining $x'_{3,i}(t)$ and hence $dL_{3,m}/d\theta_i$.

#### 3.1 State Derivatives

Observe that the determination of the sample derivatives in (12) depends on the state derivatives $x'_{3,i}(t)$. The purpose of IPA is to evaluate these derivatives as functions of observable sample path quantities. We pursue this next, using the framework established in Cassandras et al. [2010] where, for arbitrary stochastic hybrid systems, it is shown that the state and event time derivatives in (11) can be obtained from three fundamental “IPA equations”. For the sake of self-sufficiency, these equations are rederived here as they pertain to our specific SFM. Looking at (2) and Fig. 2, note that the dynamics of $x_n(t)$ are fixed over any interevent interval $[\tau_k, \tau_{k+1}]$ and we write $x_n(t) = f_{n,k}(t)$ to represent the appropriate expression on the right-hand side of (2) over this interval. We have

$$
x_n(t) = x_n(\tau_k) + \int_{\tau_k}^{t} f_{n,k}(\tau) d\tau
$$

and taking derivatives with respect to $\theta_i$, we get

$$
x'_n(t) = x'_n(\tau_k) \frac{\partial x_n(\tau_k)}{\partial \theta_i} + \frac{\partial f_{n,k}(\tau)}{\partial \theta_i} 
$$

and

$$
x'_n(t) = x'_n(\tau_k) \frac{\partial x_n(\tau_k)}{\partial \theta_i} + \frac{\partial f_{n,k}(\tau)}{\partial \theta_i} dt - f_{n,k}(\tau) \frac{\partial x'_n(\tau)}{\partial \theta_i} t_n, i = 1, \ldots, 4
$$

Letting $t = \tau_k^+$ and since $\frac{\partial x_n(\tau_k)}{\partial \theta_i} = f_{n,k-1}(\tau_k^-)$ and $\frac{\partial f_{n,k}(\tau)}{\partial \theta_i} = 0$ from (2), we obtain

$$
x'_n(t) = x'_n(\tau_k^-) + f_{n,k-1}(\tau_k^-) - f_{n,k}(\tau_k^+) \frac{\partial x'_n(\tau)}{\partial \theta_i} t_n, i = 1, \ldots, 4
$$

Thus, focusing on a NEP of $x_n(t)$, the queue content derivative is piecewise constant with jumps occurring according to (14). The final step is to obtain the event time derivatives $\tau'_k$, appearing in (14), which we do next.

#### 3.2 Event Time Derivatives

Clearly $\tau'_k$ depends on the type of event occurring at time $\tau_k$. Following the framework in Cassandras et al. [2010], there are three types of events for a general stochastic hybrid system. For the purpose of these definitions, let the continuous state component of the hybrid system be $X \subseteq \mathbb{R}^N$, $x = [x_1, \ldots, x_N]$, and let $\theta \in \Theta \subseteq \mathbb{R}^M$.

1. **Exogenous Events.** An event is *exogenous* if it causes a discrete state transition at time $\tau_k$ independent of the controllable parameter $\theta$. Thus, it satisfies $\tau'_k = 0$.

2. **Endogenous Events.** An event is occurring at time $\tau_k$ is *endogenous* if there exists a continuously differentiable function $g_k : \mathbb{R}^N \times \Theta \rightarrow \mathbb{R}$ such that

$$
\tau_k = \min \{ t > \tau_{k-1} : g_k(x(\theta, t), \theta) = 0 \}
$$

where the function $g_k$ normally corresponds to a guard condition in a hybrid automaton. Taking derivatives with respect to $\theta_i$, $i = 1, \ldots, m$, it is straightforward to obtain

$$
\tau'_k = -\frac{\partial g_k}{\partial \theta_i} + \sum_{j=1}^{N} \frac{\partial g_k}{\partial \theta_i} x'_j(\tau_k)
$$

3. **Induced Events.** An event at time $\tau_k$ is *induced* if it is triggered by the occurrence of another event at time $\tau_m \leq \tau_k$. In this case, $\tau'_k$ depends on the derivative $\tau'_m$ (details can be found in Cassandras et al. [2010].)

In the following, we consider each of the event types at queue 3 that were identified in the previous section and derive the corresponding event time derivatives. Based on these, we can then also derive the state derivatives through (14) and (16).
(1) Event $E_1$ ends a NEP of queue $1$. This is an endogenous event that occurs when $x_1(\theta, t) = 0$. Thus, when such an event occurs at $\tau_k$, let $g_1(x(\theta, t), \theta) = x_1(\theta, t) = 0$.

Using (19), we get $\tau'_{k,i} = -x'_{1,i}(\tau_k) f_{k-1}(\tau_k)$. Looking at (5), we have either $f_{3,k-1}(\tau_k) = h_1(\tau_k) - h_3(\tau_k)^+ \alpha_1(\tau_k) f_{k-1}(\tau_k) = \alpha_1(\tau_k) f_{k-1}(\tau_k)$ when $B_3(z, \theta) = 1$, or $f_{3,k-1}(\tau_k) = h_1(\tau_k)$ and $f_{3,k-1}(\tau_k) = \alpha_1(\tau_k)$ when $B_3(z, \theta) = 0$. In both cases, $f_{3,k-1}(\tau_k) = h_1(\tau_k) - \alpha_1(\tau_k)$. Using these values in (14) along with $\tau'_{k,i}$ above we get

$$x'_{3,i}(\tau_k) = x'_{3,i}(\tau_k) - \frac{h_1(\tau_k) - \alpha_1(\tau_k)}{\alpha_1(\tau_k) f_{k-1}(\tau_k)} x'_{1,i}(\tau_k) = 0$$

Thus, at the end of a NEP $[\tau_{1,m}, \tau_{3,m}]$ of queue 3 we have

$$x_{3,i}(\tau_{3,m}^+) = 0, \quad i = 1, \ldots, 4 \quad (21)$$

indicating that these state derivatives are always reset to 0 upon ending a NEP.

(2) Event $E_3$ ends a NEP of queue 3. This is an endogenous event that occurs when $x_3(\theta, t) = 0$. Thus, when such an event occurs at $\tau_k$, let $g_3(x(\theta, t), \theta) = x_3(\theta, t) = 0$.

Using (19), we get $\tau'_{k,i} = -x'_{1,i}(\tau_k) f_{k-1}(\tau_k)$. According to (5), we have $f_{3,k}(\tau_k^+) = 0$. Using these values in (14) along with $\tau'_{k,i}$ above we get

$$x'_{3,i}(\tau_k) = x'_{3,i}(\tau_k) - (f_{3,k-1}(\tau_k) - 0) \frac{x'_{3,i}(\tau_k)}{f_{3,k-1}(\tau_k)} = 0$$

Thus, the end of a NEP $[\tau_{1,m}, \tau_{3,m}]$ of queue 3 we have

$$x_{3,i}(\tau_{3,m}^+) = 0, \quad i = 1, \ldots, 4 \quad (21)$$

indicating that these state derivatives are always reset to 0 upon ending a NEP.

(3) Event $G2R_1$, i.e., the GREEN light of queue 1 switches to RED. This is an endogenous event that occurs when $g_2(x(\theta, t), \theta) = z_1(\tau_k) = \theta_1, \tau'_{k,i}$ is determined by the following lemma.

Lemma 1. Let $\zeta_{1,k}$ be the total number of $G2R_1$ events that have occurred before or at $\tau_k$, and $\rho_{1,k}$ be the total number of $G2R_1$ events that have occurred before or at $\tau_k$. Then, $\tau_{1,1} = \zeta_{1,k}, \tau_{1,2} = \rho_{1,k}, \tau_{1,3} = 0$ and $\tau_{1,4} = 0$.

The proof of this lemma can be found in Geng and Cassandras [2012]. According to (5), we have either $f_{3,k-1}(\tau_k) - f_{3,k}(\tau_k^+) = h_1(\tau_k)$ \( (5.1)-(5.3) \) or $\alpha_1(\tau_k)$ \( (5.5)-(5.6) \), or $f_{3,k-1}(\tau_k^+) = \alpha_1(\tau_k)$ \( (5.2)-(5.3) \) or $\alpha_1(\tau_k)$ \( (5.5)-(5.6) \). From (3), we can combine these two situations and simply so that $f_{3,k-1}(\tau_k) - f_{3,k}(\tau_k^+) = \beta_1(\tau_k)$. According to (14), we get

$$x'_{3,i}(\tau_k) = \begin{cases} x'_{3,i}(\tau_k) + \beta_1(\tau_k) \zeta_{1,k} & i = 1 \\ x'_{3,i}(\tau_k) + \beta_1(\tau_k) \rho_{1,k} & i = 2 \\ x'_{3,i}(\tau_k) & i = 3, 4 \end{cases}$$

(4) Event $G2R_3$, i.e., the GREEN light of queue 3 switches to RED. This is an endogenous event that occurs when $g_2(x(\theta, t), \theta) = z_3(\tau_k) = \theta_3, \tau'_{k,i}$ is determined by the following lemma.

Lemma 2. Let $\zeta_{3,k}$ be the total number of $G2R_3$ events that have occurred before or at $\tau_k$, and $\rho_{3,k}$ be the total number of $G2R_3$ events that have occurred before or at $\tau_k$. Then, $\tau_{3,1} = \zeta_{3,k}, \tau_{3,2} = \rho_{3,k}, \tau_{3,3} = \rho_{4,k}, \tau_{3,4} = 0$.

From (5), if $x_3(\tau_k^+) > 0$, we have $f_{3,k-1}(\tau_k^+) - f_{3,k}(\tau_k^+) = -h_3(\tau_k^+)$. Thus, when $(5.4)-(5.1), (5.5)-(5.2), (5.6)-(5.3)$.

According to (14), the state derivative is

$$x'_{3,i}(\tau_k) = \begin{cases} \beta_1(\tau_k) \zeta_{3,k} & i = 1, 2 \\ \beta_1(\tau_k) \rho_{3,k} & i = 3, 4 \end{cases}$$

(5) Event $R2G_1$, i.e., the RED light of queue 1 switches to GREEN. This is an endogenous event that occurs when $g_3(x(\theta, t), \theta) = z_2(\tau_k) = \theta_2, \tau'_{k,i}$ is determined by Lemma 1. Similar to the analysis of a $G2R_1$ event, we have either $f_{3,k-1}(\tau_k) - f_{3,k}(\tau_k^+) = -\beta_1(\tau_k^+)$ \( (5.3)-(5.1) \), or $\alpha_1(\tau_k)$ \( (5.2)-(5.4), (5.3)-(5.5), (5.5)-(5.6) \). Thus the state derivative is

$$x'_{3,i}(\tau_k) = \begin{cases} \beta_1(\tau_k) \zeta_{3,k} & i = 1 \\ \beta_1(\tau_k) \rho_{1,k} & i = 2 \\ \beta_1(\tau_k) \rho_{3,k} & i = 3, 4 \end{cases}$$

(6) Event $R2G_3$, i.e., the RED light of queue 3 switches to GREEN. This is an endogenous event that occurs when $g_3(x(\theta, t), \theta) = z_2(\tau_k) = \theta_2, \tau'_{k,i}$ is determined by Lemma 2. From (5), we have either $f_{3,k-1}(\tau_k) - f_{3,k}(\tau_k^+) = h_3(\tau_k^+)$ \( (5.1)-(5.4), (5.2)-(5.5), (5.5)-(5.6) \). The state derivative is

$$x'_{3,i}(\tau_k) = \begin{cases} \beta_1(\tau_k) \zeta_{3,k} & i = 1, 2 \\ \beta_1(\tau_k) \rho_{3,k} & i = 3, 4 \end{cases}$$

(7) Event $S_1$ starts a NEP of queue 1. As already mentioned, this is an event induced by a $G2R_1$ event or a switch of $\alpha_1(t)$ from zero to a strictly positive value \( (5.2) \) occurring during a RED cycle, or a switch of $\alpha_1(t) - \beta_1(t)$ from a non-positive to a strictly positive value \( (5.3) \) occurring during a GREEN cycle (see Fig. 4). Consequently, there are three possible cases to consider as follows.

Case (7a): A NEP of queue 1 starts right after a $G2R_1$ event. This is an endogenous event and was analyzed in Case (3). Since $x_1(\tau_k^+) = 0, f_{3,k-1}(\tau_k^+) = f_{3,k}(\tau_k^+) = \alpha_1(\tau_k)$ \( (5.2)-(5.3) \) or $\alpha_1(\tau_k)$ \( (5.5)-(5.6) \). We get

$$x'_{3,i}(\tau_k) = \begin{cases} \beta_1(\tau_k) \zeta_{3,k} + \alpha_1(\tau_k) \rho_{3,k} & i = 1 \\ \beta_1(\tau_k) \rho_{3,k} & i = 2 \\ \beta_1(\tau_k) \rho_{3,k} & i = 3, 4 \end{cases}$$

Case (7b): A NEP of queue 1 starts while $z_1(\tau_k) = 0, z_2(\tau_k) > 0$. This is an exogenous event occurring during a RED cycle for queue 1 and is due to a change in $\alpha_1(\tau_k)$ from zero to a strictly positive value. Therefore, $\tau_{3,1} = 0$. We then have

$$x'_{3,i}(\tau_k) = \begin{cases} \beta_1(\tau_k) \zeta_{3,k} & i = 1, 2, 3, 4 \end{cases}$$

Case (7c): A NEP of queue 1 starts while $z_2(\tau_k) = 0, z_1(\tau_k) > 0$. This is an exogenous event occurring during
a GREEN cycle for queue 1 due to a change in \( \alpha_1(\tau_k) \) or \( \beta_1(\tau_k) \) that results in \( \alpha_1(\tau_k) - \beta_1(\tau_k) \) switching from a non-positive to a strictly positive value. The analysis is exactly the same as Case (7b) above and (28) applies.

(8) Event \( S_3 \) starts a NEP of queue 3. This is similar to Case (7), and there are also three possible cases to consider.

Case (8a): A NEP of queue 3 starts right after a \( G2R_3 \) event. This is an endogenous event and was analyzed in Case (4). Since \( x_3'(\tau_k) = 0 \), we have \( f_{3,k-1}(\tau_k) - f_{3,k}(\tau_k) = -\beta_1(\tau_k) \), and (24) applies. Suppose that this is the \( n \)th NEP, i.e., \( \tau_k = \xi_{3,m} \). We have already shown in (21) that \( x_{3,i}(\eta_{n,m-1}^{\tau_k}) = 0 \). In addition, we have \( x_3'(t) = 0 \) over the interval \( [\eta_{3,m-1}, \xi_{3,m}] \), thus \( x_{3,i}'(t) = 0 \) for all \( t \in [\eta_{3,m-1}, \xi_{3,m}] \) and we get \( x_{3,i}'(\tau_k^-) = x_{3,i}'(\xi_k^-) = 0 \), the state derivative in (24) becomes

\[
x_{3,i}'(\tau_k^-) = \begin{cases} 0 & i = 1, 2, 3, 4 \\ \end{cases} \quad (29)
\]

Case (8b): A NEP of queue 3 starts while \( z_3(\tau_k) = 0 \), \( z_1(\tau_k) > 0 \). This is due to a change in \( \alpha_3(\tau_k) \) from a zero to a strictly positive value. It also happens in two ways. First, \( \alpha_3(\tau_k) = \beta_1(\tau_k) \) becomes positive because a \( G2R_1 \) event occurs. Then (22) applies, where \( x_{3,i}'(\tau_k) = 0 \). Second, \( \beta_1(\tau_k) \) becomes positive because either \( h_1(\tau_k) \) or \( \alpha_1(\tau_k) \) switches from 0 to a strictly positive value, which is an exogenous event. Therefore, the state derivative is

\[
x_{3,i}(\tau_k^+) = x_{3,i}(\tau_k^-) = 0, \quad i = 1, 2, 3, 4 \quad (30)
\]

Case (8c): A NEP of queue 3 starts while \( z_4(\tau_k) = 0 \). This is due to a change in \( \alpha_3(\tau_k) - \beta_3(\tau_k) \) which may happen in two ways. First, it becomes positive because a \( G2R_1 \) event occurs, which makes \( \alpha_3(t) \) larger. Then (22) applies, where \( x_{3,i}'(\tau_k) = 0 \). Second, it is due to a change of value in either \( h_1(\tau_k) \) or \( \alpha_1(\tau_k) \) or \( \beta_2(\tau_k) \), which are all exogenous events. The state derivative is the same as (30).

This completes the derivation of all state and event time derivatives required to evaluate the sample performance derivative in (12). Using the definition of \( L_{n,m}(\theta) \) in (8), note that we can decompose (12) into its NEPs and evaluate the derivatives \( dL_{n,m}(\theta)/d\theta_i \) as shown next.

### 3.3 Cost Derivatives

By virtue of (16), \( x_{n,i}'(t) \) is piecewise constant during a NEP and its value changes only at an event point \( t_{n,m}^j \), \( j = 1, ..., J_{n,m} \). Therefore, we have

\[
\frac{dL_{n,m}(\theta)}{d\theta_i} = \int_{\xi_{n,m}(\theta)}^{\eta_{n,m}(\theta)} x_{n,i}'(t)dt
\]

\[
= x_{n,i}'(t_{n,m}^-) - x_{n,i}'(t_{n,m}^+) + \sum_{j=2}^{J_{n,m}} \left[ x_{n,i}'(t_{n,m}^-) - x_{n,i}'(t_{n,m}^-) \right] + (\eta_{n,m} - t_{n,m}^-) x_{n,i}'(t_{n,m}^-) + (t_{n,m}^- - \eta_{n,m}) x_{n,i}'(t_{n,m}^-) + \sum_{j=2}^{J_{n,m}} \left[ x_{n,i}'(t_{n,m}^-) - x_{n,i}'(t_{n,m}^-) \right] (t_{n,m}^- - \eta_{n,m}) + (t_{n,m}^- - \eta_{n,m}) x_{n,i}'(t_{n,m}^-)
\]

\[
= \sum_{j=1}^{J_{n,m}} x_{n,i}'(t_{n,m}^-) (t_{n,m}^- - \eta_{n,m}) + (\eta_{n,m} - t_{n,m}^-) x_{n,i}'(t_{n,m}^-)
\]

\[
(31)
\]

Clearly, the state derivative at each event point is determined by (14) which in turn depends on the event type at \( t_{n,m}^j, j = 1, ..., J_{n,m} \) and is given by the corresponding expression in (20) through (30). An explicit closed-form expression of \( dL_{n,m}(\theta)/d\theta_i \) may be obtained in this manner but becomes complicated. A simple algorithm that updates \( dL_{n,m}(\theta)/d\theta_i \) after every observed event is simple to implement. More importantly, note that this IPA derivative depends on: (i) the number of events in each NEP \( J_{n,m} \), (ii) the number of total \( G2R_1 \) events \( \xi_{n,k} \), (iii) the number of total \( R2G_2 \) events \( \rho_{n,k} \), (iv) the event times \( \xi_{n,m}, \eta_{n,m} \) and \( t_{n,m}^j \), and (v) the arrival and departure rates \( \alpha_n(\tau_k), \beta_n(\tau_k) \) at an event time only. The quantities in (i) - (iv) are easily observed through counters and timers. The rates in (v) may be obtained through simple estimators, emphasizing that they are only needed at a finite number of observed event times.

### 4. SIMULATION RESULTS

We describe how the IPA estimator derived for the SFM can be used to determine optimal light cycles for two intersections modeled as a DES. We apply the IPA estimator using actual data from an observed sample path of this DES (in this case, by simulating as a pure DES).

We assume cars arrive according to a Poisson process with rate \( \alpha_n, n = 1, 2, 4 \) (as already emphasized, our results are independent of this distribution.). We also assume cars depart at a rate \( h_n(t) \) which we fix to be a constant \( H_n \) when the road is not empty. We also constrain \( \theta, i = 1, ..., 4 \), to take values in \( [\theta_{\min}, \theta_{\max}] \).

For the simulated DES model, we use a brute-force (BF) method to find an optimal \( \theta_{IP}^* \); we discretize all real values of \( \theta \) and for \( \theta, i = 1, ..., 4 \) combinations we run 10 sample paths to obtain the average total cost. The value of \( \theta_{IP}^* \) is the one generating the least average cost, to be compared to \( \theta_{IP}^* \), the IPA-based method. In our simulations, we estimate \( \alpha_n(\tau_k) \) through \( N_n/t_{w} \) by counting car arrivals \( N_n \) over a time window \( t_{w} \) before or after \( \xi_{n,m} ; \beta_n(\tau_k) \) is similarly estimated.

In the results reported here, we set \( \alpha_n = 1/4, n = 1, 2, 4 \), \( H_n = 1, n = 1, 2, 3, 4 \), \( \theta_{\min} = 15\sec \), \( \theta_{\max} = 40\sec \) and the sample length \( T = 1000\sec \). Fig. 5 shows the trajectories of \( J \) and \( \theta \) using the IPA-based method where \( w = [10, 1, 1, 1] \) and initial \( \theta_0 = [25, 30, 30, 25] \). More results are shown in Table 1. As we can see, \( \theta_{IP}^* \) is approaching the optimal value obtained by the BF method. Notice that BF method becomes impractical when there are more controlling parameters, or when the range of the parameter is large. However, the IPA method is still effective in such situations. Moreover, we notice that the value of \( \theta_3 + \theta_4 \) is similar to \( \theta_1 + \theta_2 \). This indicates that the two intersections tend to have the same traffic light switching cycle to balance traffic flows.

### Table 1. IPA vs BF method result I

| w     | BF   | IPA   |
|-------|------|-------|
| \[1,1,1]\ | 15,15,15,15 | 5.4  | 15,15,15,15 | 5.4  |
| \[10,1,1]\ | 27,15,15,29 | 16.6 | 28.8,15,15,27.8 | 17.5 |
| \[1.5,5,1]\ | 35,23,7,12 | 12.6 | 15,1.5,16,18,18.5 | 13.2 |
| \[2,1,1,1]\ | 25,15,15,25 | 22.0 | 22.1,18,18,22.9 | 22.5 |
| \[1,10,1,1]\ | 15,29,15,29 | 16.3 | 15,31,12,18,12.6,16.3 | 17.2 |
Based on this observation, we also do simulations by setting $\theta_1 + \theta_2 = T_1$ and $\theta_3 + \theta_4 = T_2$, which indicates that we set the “GREEN plus RED” cycle to be fixed $T_1$ for each intersection. With this constraint, we only need to find optimal $\theta_1^*$ and $\theta_3^*$, since $\theta_2^* = T_1 - \theta_1^*$ and $\theta_4^* = T_2 - \theta_3^*$. We first let $T_1 = T_2$, which restricts the two intersections to have the same traffic light switching cycle. Table 2 shows the simulation results. $T_1$ and $T_2$ are set to be the value obtained from Table 1. For example, when $w = [10, 1, 1, 1]$, $\theta_1 + \theta_2 = 42$ and $\theta_3 + \theta_4 = 44$ in Table 1. We then set $T_1 = T_2 = 44$ in Table 2, and restrict $\theta_{\min} = 15$. Comparing the results in Table 2 with Table 1, we find that it supports the results where we allow independent $\theta_i, i = 1, 2, ..., 4$.

In Fig. 6, we set $w = [1, 10, 1, 1]$ and change $T_2$ to obtain $J_{iPA}$ while keeping $T_1 = 44$. As we can see, the minimum $J_{iPA}$ is achieved when $T_1 = T_2$, which also matches the observations under independent $\theta_i$.

We are also interested to see the optimal control parameters under different traffic intensities. We set $w = [1, 10, 1, 1]$, $T_1 = T_2 = 44$ and $\alpha = [1/r, 1/4, 1/4]$, i.e., we operate under different arrival rate of queue 1. Fig. 7 shows the optimal cost and optimal $\theta_1$ and $\theta_3$ while $r$ varies. It is clear to see that $\theta_1^*$ increases as $r$ decreases. This indicates more GREEN light duration is assigned to queue 1 as more cars are accumulated in queue 1 because of the fast arrival rate. $\theta_3^*$ also increases because more cars flow into queue 3.

5. CONCLUSIONS AND FUTURE WORK

We have developed a SFM for a traffic light control problem with two coupled intersections, based on which we derive an IPA gradient estimator of a cost metric with respect to the controllable green and red cycle lengths. The estimators are used to iteratively adjust light cycle lengths to improve performance and, under proper conditions, obtain optimal values which adapt to changing traffic conditions. The analysis in the paper can be readily extended to $N$ intersections in tandem. Future work will extend our method to solving the TLC problem over multiple junctions without assumption (iii), i.e., allowing a finite car capacity between intersections to cause blocking effects.

REFERENCES

Alvarez, I. and Poznyak, A. (2010). Game theory applied to urban traffic control problem. Intentional Conference on Control, Automation and Systems, 2164–2169.

Cassandras, C.G. and Lafortune, S. (2008). Introduction to Discrete Event Systems, 2nd ed. Springer.

Cassandras, C.G. and Lygeros, J. (2006). Stochastic Hybrid Systems. Taylor and Francis.

Cassandras, C.G., Wardi, Y., Melamed, B., Sun, G., and Panayiotou, C.G. (2002). Perturbation analysis for online control and optimization of stochastic fluid models. IEEE Trans. Automat. Control, 47(8), 1234–1248.

Cassandras, C.G., Wardi, Y., Panayiotou, C.G., and Yao, C. (2010). Perturbation analysis and optimization of stochastic hybrid systems. Europ. J. of Control, 16(6), 642–664.

Choi, W., Yoon, H., Kim, K., Chung, I., and Lee, S. (2002). A traffic light controlling flic considering the traffic congestion. AFSS 2002, International Conference on Fuzzy Systems, 69–75.

DeSchutter, B. (1999). Optimal traffic light control for a single intersection. Proceedings of the IEEE American Control Conference, 3, 2195–2199.

Dujardin, Y., Boillot, F., Vanderpooten, D., and Vinant, P. (2011). Multiobjective and multimodal adaptive traffic
light control on single junctions. *14th IEEE Conference on Intelligent Transportation Systems*, 1361–1368.

Fidler, N. and Stapp, J. (1992). A distributed approach to optimized control of street traffic signals. *Journal of Transportation Engineering*, 118.

Geng, Y. and Cassandras, C.G. (2012). Traffic light control using infinitesimal perturbation analysis. Technical report.

Head, L., Ciarallo, F., and an V. Kaduwela, D.L. (1996). A perturbation analysis approach to traffic signal optimization. *INFORMS National Meeting*.

Murat, Y.S. and Gedizlioglu, E. (2005). A fuzzy logic multi-phased signal control model for isolated junctions. *Transportation Research Part C*, 18, 19–36.

Panayiotou, C.G., Howell, W.C., and Fu, M. (2005). Online traffic light control through gradient estimation using stochastic fluid models. *Proceedings of the IFAC 16th Triennial World Congress*.

Prashanth, L. and Bhatnagar, S. (2011). Reinforcement learning with average cost for adaptive control of traffic lights at intersections. *14th IEEE Conference on Intelligent Transportation Systems*, 1640–1645.

Taale, H., Back, T., Preub, M., Eiben, A., Graaf, J., and Schippers, C. (1998). Optimizing traffic light controllers by means of evolutionary algorithms. *EUFIT*.

Thorpe, T. (1997). *Vehicle traffic light control using sarsa*. Master thesis, Dept. of Comp. Sci., Colorado State Univ.

Wardi, Y., Adams, R., and Melamed, B. (2010). A unified approach to infinitesimal perturbation analysis in stochastic flow models: the single-stage case. *IEEE Trans. Automat. Control*, 55(1), 89–103.

Wiering, M., Veenen, J., Vreeken, J., and Koopman, A. (2004). Intelligent traffic light control. *Technical Report UU-CS-2004*.

Yao, C. and Cassandras, C.G. (2011a). Resource contention games in multiclass stochastic flow models. *Nonlinear Analysis: Hybrid Systems*, 5(2), 301–319.

Yao, C. and Cassandras, C. (2011b). Perturbation analysis of stochastic hybrid systems and applications to resource contention games. *Frontiers of Electrical and Electronic Engineering in China*, 6(3), 453–467.

Yu, X. and Recker, W. (2006). Stochastic adaptive control model for traffic signal systems. *Transportation Research Part C: Emerging Technology*, 14(4), 263–282.

Zhao, X. and Chen, Y. (2003). Traffic light control method for a single intersection based on hybrid systems. *Proc. of the IEEE Intelligent Transp. Systems*, 1105–1109.