Stochastic Wave Equations defined by Fractal Laplacians on Cantor-like Sets

Tim Ehnes

Abstract

We study stochastic wave equations in the sense of Walsh defined by fractal Laplacians on Cantor-like sets. For this purpose, we give an improved estimate on the uniform norm of eigenfunctions and approximate the wave propagator using the resolvent density. Afterwards, we establish existence and uniqueness of mild solutions to stochastic wave equations provided some Lipschitz and linear growth conditions. We prove Hölder continuity in space and time and compute the Hölder exponents. Moreover, we are concerned with the phenomenon of weak intermittency.

1 Introduction

In this paper we study second-order hyperbolic stochastic partial differential equations defined by generalized second order differential operators. To introduce the operator of interest, let $[a, b] \subset \mathbb{R}$ be a finite interval, $\mu$ a finite non-atomic Borel measure on $[a, b]$, $L^2([a, b], \mu)$ the space of measurable functions $f$ such that $\int_a^b f^2 d\mu < \infty$ and $L^2([a, b], \mu)$ the corresponding Hilbert space of equivalence classes with inner product $\langle f, g \rangle_\mu := \int_a^b fg d\mu$. We define

$$D^2_\mu := \{ f \in C^1((a, b)) \cap C^0([a, b]) : \exists (f')^\mu \in L^2([a, b], \mu) : f'(x) = f'(a) + \int_a^x (f')^\mu(y) d\mu(y), \ x \in [a, b] \}.$$ 

The Krein-Feller operator with respect to $\mu$ is given as

$$\Delta_\mu : D^2_\mu \subseteq L^2([a, b], \mu) \to L^2([a, b], \mu), \ f \to (f')^\mu.$$ 

This operator has been introduced, for example, in [11,19,23–25], especially as the infinitesimal generator of a so-called Quasi diffusion. It is a measure-theoretic generalization of the classical second weak derivative $\Delta_{\lambda^1}$, where $\lambda^1$ is the one-dimensional Lebesgue measure.

We recall the well-known physical motivation for Krein-Feller operators (see [1, Section 1.2]): We consider a flexible string of length 1 clamped between two points $x = 0$ and $x = 1$ such that, if we deflect it, a tension force drives it back towards its state of equilibrium. The mass distribution of the bar shall have a density denoted by $\rho : [0, 1] \to \mathbb{R}$. For reasons of simplicity, we assume that for the tangentially acting tension force $F$ it holds $F = 1$. Then, the deviation of the string, the function $u(t, x)$, is determined by the wave equation

$$\kappa \frac{\partial^2 u}{\partial x^2}(t, x) = c\rho(x) \frac{\partial u}{\partial t}(t, x)$$ (1)

with Dirichlet boundary conditions $u(t, 0) = u(t, 1) = 0$ for all $t \geq 0$. We impose Neumann boundary conditions $\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, 1) = 0$ if the ends of the strings are attached to a pair of frictionless tracks.

*Institute of Stochastics and Applications, University of Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany, e-mail: tim.ehnes@mathematik.uni-stuttgart.de
which are free to move up and down. In order to solve this wave equation, we use the separation of variables
and write $u(t, x) = f(x)g(t)$, which yields

$$\kappa f''(x)g(t) = c \rho(x)f(x)g''(t)$$

and by resorting

$$\frac{f''(x)}{\rho(x)f(x)} = \frac{c g''(t)}{\kappa g(t)}$$

for all $t$ and $x$. Consequently, both sides of the equation are constant and we denote the value by $-\lambda$.

We only consider the left-hand side, given by

$$f''(x) = -\lambda \rho(x)f(x).$$

By integration with respect to the Lebesgue measure we get

$$f'(x) - f'(0) = -\lambda \int_0^x f(y) \rho(y) dy,$$

which can be written as

$$f'(x) - f'(0) = -\lambda \int_0^x f(y) d\mu(y),$$

where $\rho$ is the density of the measure $\mu$. By applying the definition of $\Delta_{\mu}$,

$$\Delta_{\mu} f = -\lambda f,$$

which yields

$$\Delta_{\mu} u = \frac{\partial^2 u}{\partial t^2},$$

as a generalization of wave equation (2), since this equation does not involve the density $\rho$. Consequently, we can use it to formulate the problem for measure which possess no density, in particular for fractal measures on $[0, 1]$.

We are interested in the case where $\mu$ is a self-similar measure on a Cantor-like set. More precisely, let $N \geq 2$ and $\{S_1, ..., S_N\}$ be a finite family of affine contractions on $[0, 1]$, i.e.

$$S_i : [0, 1] \rightarrow [0, 1], \ S_i(x) = r_i x + b_i, \ 0 < r_i < 1, \ 0 \leq b_i \leq 1 - r_i, \ i = 1, ..., N,$$

where $S_i(0) = 0 < S_i(1) \leq S_2(0) < S_2(1) \leq ... < S_N(1) = 1$. Further, let $\mu_1, ..., \mu_N$, i.e. $\mu_1, ..., \mu_N \in (0, 1)$ weights and $\sum_{i=1}^N \mu_i = 1$. It is known from [14] that a unique non-empty compact set $F \subseteq [0, 1]$ exists such that

$$F = \bigcup_{i=1}^M S_i(F)$$

and a unique Borel probability measure $\mu$ such that

$$\mu(A) = \sum_{i=1}^N \mu_i \mu \left( S_i^{-1}(A) \right)$$

for any Borel set $A \subseteq [0, 1]$. Further, it holds $\text{supp} \mu = F$. We call the set $F$ Cantor-like set. Wave equations where $\mu$ is defined by an IFS with overlaps and has full support were investigated in [3].
By adding a random external force, more precisely, a space-time white noise $\xi$ on $L^2([0, 1], \mu)$, we are concerned with the hyperbolic stochastic PDE

$$
\frac{\partial^2}{\partial t^2}u(t, x) = \Delta^b_\mu u(t, x) + f(t, u(t, x))\xi(t, x),
$$

$$
u(0, x) = u_0(x),
$$

$$
\frac{\partial}{\partial t}u(0, x) = u_1(x),
$$

where $b \in \{N, D\}$ determines the boundary condition. It is known (see [28]) that the stochastic wave equation defined by the classical one-dimensional weak Laplacian $\Delta^1_\lambda$ has a unique mild solution which is, some regularity conditions provided, essentially $\frac{1}{2}$-Hölder continuous in space and in time. Here, essentially $\alpha$-Hölder continuous means Hölder continuous for every exponent strictly less than $\alpha$. In two space dimensions it turns out that the mild solution is a distribution, no function (see [28]). Hambly and Yang [16] addressed the questions regarding these properties in the setting of a p.c.f. self-similar set (in the sense of [20]) with Hausdorff dimension between one and two. However, the damped wave equation in their paper is a system of first-order SPDEs. According to the knowledge of the author, there are no results regarding these properties in case of second-order Walsh SPDEs defined by a fractal Laplacian. The Krein-Feller operator can be interpreted as a fractal Laplacian on sets with dimension less or equal one.

We prepare the formulation of the main theorem by stating the following regularity conditions, where $\gamma$ is the spectral exponent of $\Delta^b_\mu$ and $\delta := \max_{1 \leq i \leq N} \frac{\log \mu_i}{\log((\nu_0\gamma)^i)}$ is an indicator for the skewness of $\mu$.

**Assumption 1.1:**

(i) $\delta + 1 < \frac{1}{\gamma}$

(ii) $u_0 \in D(\Delta^b_\mu)$, $u_1 \in D\left((-\Delta^b_\mu)^{\frac{1}{2}}\right)$

(iii) There exists $q \geq 2$ such that $f$ is predictable and satisfies the following Lipschitz and linear growth conditions: There exists $L > 0$ and a real predictable process $M : \Omega \times [0, T] \to \mathbb{R}$ with

$$
\sup_{s \in [0,T]}||M(s)||_{L^q(\Omega)} < \infty
$$

such that for all $(w, t, x, y) \in \Omega \times [0, T] \times \mathbb{R}$

$$
|f(\omega, t, x) - f(\omega, t, y)| \leq L|x - y|,
$$

$$
|f(\omega, t, x)| \leq M(w, t) + L|x|.
$$

Note that Condition (i) is satisfied if $\mu$ is the $d_H$-dimensional Hausdorff measure on $F$, where $d_H$ is the Hausdorff dimension of $F$, with the exception of $\lambda^1$ on $[0, 1]$.

We formulate the main result of the present paper, where $d_H$ is the Hausdorff dimension of $F$ and $\nu_{\min} := \min_{1 \leq i \leq N} \frac{\mu_i}{r_i^{d_H}}$.

**Theorem 1.2:** Let $T \geq 0$ and assume Condition 1.1 with $q \geq 2$. Then, there exists a unique mild solution $\{u(t, \cdot) : 0 \leq t \leq T, 0 \leq x \leq 1\}$ to SPDE (5). Furthermore, there exists a version of this solution such that the following holds:

(i) If $q > 2$ and $t \in [0, T]$, $u(t, \cdot)$ is a.s. essentially $\frac{1}{2} - \frac{1}{q}$-Hölder continuous on $[0, 1]$.

(ii) If $q > \left(d_H + 1 + \frac{\log(\nu_{\min})}{\log(\nu_{\max})}\right)^{-1}$ and $x \in [0, 1]$, $u(\cdot, x)$ is a.s. essentially $\frac{1}{d_H + 1 + \frac{\log(\nu_{\min})}{\log(\nu_{\max})}} - \frac{1}{q}$-Hölder continuous on $[0, T]$.

If $\mu$ is chosen as the natural measure, we have $\nu_{\min} = 0$ and thus an increasing temporal Hölder exponent as the Hausdorff dimension of the considered Cantor-like set decreases. In particular, if $q$ can be chosen arbitrarily large, we obtain $\frac{1}{2}$ as ess. spatial and $\frac{1}{d_H - 1}$ as ess. temporal Hölder exponent.
In preparation for proving the main results, we will have a closer look on the wave propagator of $\Delta_b^\mu$, defined by

$$P_D(t, x, y) = \sum_{k \geq 1} \frac{\sin (\sqrt{\lambda_k^D} t)}{\sqrt{\lambda_k^D}} \varphi_k^D(x) \varphi_k^D(y)$$

and

$$P_N(t, x, y) = t + \sum_{k \geq 2} \frac{\sin (\sqrt{\lambda_k^N} t)}{\sqrt{\lambda_k^N}} \varphi_k^N(x) \varphi_k^N(y),$$

respectively. Here, $\lambda_k^b, k \geq 1$ are the eigenvalues and $\varphi_k^b, k \geq 1$ the $L^2(\mu)$-normed eigenfunctions of the Neumann- (or Dirichlet- resp.) Krein-Feller operator $\Delta_b^\mu$. In order to investigate this object, we establish an improved estimate on the uniform norm of $\varphi_k^b$ since the known estimate, which grows exponentially in $k$ (see [1, Lemma 4.1.6]), is too rough for our purposes. Particularly, we prove that a constant $C_2 > 0$ exists such that for all $k \in \mathbb{N}$

$$\|\varphi_k^b\|_\infty \leq C_2 k^{\frac{1}{2}}.$$

A comparable result is known for the eigenfunction of p.c.f. Laplacians (see [20, Theorem 4.5.4]). Afterwards, we approximate the wave propagator by proving that for $x \in F, t \in [0, T]$

$$\int_0^1 (\langle P_b(t, \cdot, y), f_n^x \rangle - P_b(t, x, y))^2 \, d\mu(y) \to 0$$

as $n \to \infty$, where the sequence $(f_n^x)_{n \in \mathbb{N}}$ approximates the Delta functional of $x$. Then, we show that the resulting approximating mild solutions have the desired continuity and that the regularity is preserved upon taking the limit. Next to these continuity properties, we investigate the intermittency of mild solutions to (5). Roughly speaking, an intermittent process develops increasingly high peaks on small space-intervals when the time parameter increases. This is a phenomenon of the mild solution to SPDEs that has found much attention in the last years (see, among many others, [3], [15], [21], [22] for parabolic and [4], [7], [6] for hyperbolic SPDEs). We call a mild solution $u$ weakly intermittent on $[0, 1]$ if the upper moment Lyapunov exponents, which is the function $\bar{\gamma}$ defined by

$$\bar{\gamma}(p, x) := \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[u(t, x)^p], \quad p \in (0, \infty), x \in [0, 1]$$

This is visualized in Figure 1.
satisfies
\[
\bar{\gamma}(2, x) > 0, \quad \bar{\gamma}(p, x) < \infty, \quad p \in [2, \infty), x \in [0, 1]
\] (see [21, Definition 7.5]). We prove this under some conditions on \( g \).

This paper is structured as follows. In Section 2 we give definitions related to Krein-Feller operators and Cantor-like sets, we recall results concerning the spectral asymptotics and establish the explained estimate on the uniform norm of eigenfunctions. Furthermore, we develop a method to approximate the resolvent density and introduce the wave propagator. Section 3 is dedicated to the analysis of SPDE (5), including the proofs of existence, uniqueness and Hölder continuity properties of the mild solution as well as the investigation of weak intermittency.

2 Preliminaries and Preparing Estimates

2.1 Definition of Krein-Feller Operators on Cantor-like Sets

First, we recall the definition and some analytical properties of the operator \( \Delta_b^\mu \), where \( b \in \{ N, D \} \) and \( \mu \) is a self-similar measure on a Cantor-like set according to the definition in Section 1.

We denote the support of the measure \( \mu \) and thus the Cantor-like set by \( F \). If \([0, 1] \setminus F \neq \emptyset\), \([0, 1] \setminus F\) is open in \( \mathbb{R} \) and can be written as
\[
[0, 1] \setminus F = \bigcup_{i=1}^{\infty} (a_i, b_i)
\] (6)
with \( 0 < a_i < b_i < 1 \), \( a_i, b_i \in [0, 1] \) for \( i \geq 1 \). We define
\[
\mathcal{D}_{\lambda^1}^1 := \left\{ f : [0, 1] \to \mathbb{R} : \text{there exists } f' \in L^2([0, 1], \lambda^1) : f(x) = f(0) + \int_0^x f'(y) d\lambda^1(y), \ x \in [0, 1] \right\}
\]
and \( H^1 ([0, 1], \lambda^1) \) as the space of all \( \mathcal{H} := L^2([0, 1], \mu) \)-equivalence classes having a \( \mathcal{D}_{\lambda^1}^1 \)-representative. If \( \mu = \lambda^1 \) on \([0, 1] \), this definition is equivalent to the definition of the Sobolev space \( W_2^1 \).

\( H^1 ([0, 1], \lambda^1) \) is the domain of the non-negative symmetric bilinear form \( \mathcal{E} \) on \( \mathcal{H} \) defined by
\[
\mathcal{E}(u, v) = \int_0^1 u'(x)v'(x)dx, \quad u, v \in \mathcal{F} := H^1 ([0, 1], \lambda^1) .
\]

It is known (see [12, Theorem 4.1]) that \( (\mathcal{E}, \mathcal{F}) \) defines a Dirichlet form on \( \mathcal{H} \). Hence, there exists an associated non-negative, self-adjoint operator \( \Delta_{\mu}^N \) on \( \mathcal{H} \) with \( \mathcal{F} = \mathcal{D} \left( (-\Delta_{\mu}^N)^{\frac{1}{2}} \right) \) such that
\[
\langle -\Delta_{\mu}^N u, v \rangle_\mu = \mathcal{E}(u, v), \quad u \in \mathcal{D} \left( \Delta_{\mu}^N \right), v \in \mathcal{F}
\]
and it holds
\[
\mathcal{D} \left( \Delta_{\mu}^N \right) = \{ f \in \mathcal{H} : f \text{ has a representative } \tilde{f} \text{ with } \tilde{f} \in \mathcal{D}_{\mu}^2 \text{ and } \tilde{f}'(0) = \tilde{f}'(1) = 0 \} .
\]
\( \Delta_{\mu}^N \) is called Neumann Krein-Feller operator w.r.t. \( \mu \). Furthermore, let \( \mathcal{F}_0 := H^1_0 ([0, 1], \lambda^1) \) be the space of all \( \mathcal{H} \)-equivalence classes that have a \( \mathcal{D}_{\lambda^1}^1 \)-representative \( f \) such that \( f(0) = f(1) = 0 \). The bilinear form defined by
\[
\mathcal{E}(u, v) = \int_0^1 u'(x)v'(x)dx, \quad u, v \in \mathcal{F}_0
\]
is a Dirichlet form, too (see [12, Theorem 4.1]). Again, there exists an associated non-negative, self-adjoint operator $\Delta^D_\mu$ on $\mathcal{H}$ with $\mathcal{F}_0 = D\left(\left(-\Delta^D_\mu\right)^{1/2}\right)$ such that
\[\langle -\Delta^D_\mu u, v \rangle_\mu = \mathcal{E}(u, v), \quad u \in (\Delta^D_\mu), \quad v \in \mathcal{F}_0\]
and it holds
\[\mathcal{D}(\Delta^D_\mu) = \{f \in \mathcal{H} : f \text{ has a representative } \tilde{f} \text{ with } \tilde{f}(0) = \tilde{f}(1) = 0\}.\]
$\Delta^D_\mu$ is called Dirichlet Krein-Feller operator w.r.t. $\mu$.

A concept to describe Cantor-like sets is given by the so-called word or code space. Let $I := \{1, \ldots, N\}$, $\mathbb{W}_n = I^n$ be the set of all sequences $\omega$ of length $|\omega| = n$, $\mathbb{W}^* := \cup_{n \in \mathbb{N}} I^n$, the set of all finite sequences and $\mathbb{W} := I^\infty$ the set of all infinite sequences $\theta = \theta_1\theta_2\theta_3\ldots$ with $\theta_i \in I$ for $i \in \mathbb{N}$. Then, $I$ is called alphabet and $\mathbb{W}$, $\mathbb{W}^*$, $\mathbb{W}^*$: $n \in \mathbb{N}$ are called word spaces. We define an ordering on $\mathbb{W}$ by denoting two words $\omega$ and $\sigma$ as equal if $\omega_i = \sigma_i$ for all $i \in \mathbb{N}$ and otherwise, we write $\omega < \sigma :\iff \sigma_k < \omega_k$ or $\omega > \sigma :\iff \sigma_k > \omega_k$, where $k := \inf\{n \in \mathbb{N} : \sigma_n \neq \omega_n\}$. In addition to an ordering we define a metric on the word space by the map $d : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$, $d(\omega, \sigma) = N^{-k}$ with $k$ defined as before. It is known (see e.g. [?, Theorem 2.1]) that for every $x \in [0, 1]$ the map
\[\pi_x : \mathbb{W} \to F, \quad \sigma \mapsto \lim_{n \to \infty} S_{\sigma_1} \circ S_{\sigma_2} \circ \ldots \circ S_{\sigma_n}(x)\]
is well-defined, continuous, surjective and independent of $x \in [0, 1]$, which means $\pi_x(\sigma) = \pi_y(\sigma)$ for all $x, y \in [0, 1], \; \sigma \in \mathbb{W}$. Therefore, for every $x \in [0, 1]$ and every $y \in F$ there exists, at least, one element of $\mathbb{W}$ which is by $\pi_x$ associated to $y$.

### 2.2 Spectral Theory of Krein-Feller Operators

Let $b \in \{N, D\}$ and let $\mu$ be a self-similar measure on a Cantor-like set according to the given conditions. Further, let $\gamma$ be the spectral exponent of $-\Delta^b_\mu$, that is the unique solution of
\[\sum_{i=1}^{N} (\mu_i \gamma_i)^\gamma = 1. \tag{7}\]

It is known from [10, Proposition 6.3, Lemma 6.7, Corollary 6.9] that there exists an orthonormal basis $\{\varphi_k^b : k \in \mathbb{N}\}$ of $L_2([0, 1], \mu)$ consisting of $L_2([0, 1], \mu)$-normed eigenfunctions of $-\Delta^b_\mu$ and that for the related ascending ordered eigenvalues $\{\lambda_k^b : i \in \mathbb{N}\}$ it holds $0 \leq \lambda_1^b \leq \lambda_2^b \leq \ldots$, where $\lambda_1^D > 0$. Furthermore, by [11] there exist constants $C_0, C_1 > 0$ such that for $k \geq 2$
\[C_0 k^{\frac{1}{\gamma}} \leq \lambda_k^b \leq C_1 k^{\frac{1}{\gamma}}. \tag{8}\]

Next, we consider the uniform norm of an eigenfunction $\|\varphi_k^b\|_\infty$ for $k \geq 1$, where the situation is more complicated. The only estimate, established in [13, Section 2] and [2, Lemma 3.6], is easy to derive and grows exponentially in $k$, which is far to rough for later following heat kernel estimates. In the following proposition we establish a better estimate, where we do not use the explicit representation of the eigenfunctions as in [2], but the ideas from [20, Theorem 4.5.4] for a uniform norm estimate for Laplacians on p.c.f. fractals.

**Theorem 2.1:** Let $\delta := \max_{1 \leq i \leq N} \frac{\log \mu_i}{\log((\mu_i \gamma_i)^\gamma)}$. Then, there exists a constant $\bar{C}_2 > 0$ such that for all $k \in \mathbb{N}$
\[\|\varphi_k^b\|_\infty \leq \bar{C}_2 \left(\lambda_k^b\right)^{\frac{2}{\gamma} \delta}.\]
Lemma 2.2: There exists a constant $c_0 > 0$ such that for all $u \in \mathcal{F}_0$
\[ \|u\|_\mu^2 \leq c_0 \mathcal{E}(u). \]

Proof. It holds $\lambda_D^0 > 0$ and therefore (compare [9, Theorem 1.3])
\[ \mathcal{E}(u) \geq \lambda_D^0 \|u\|_\mu^2, \quad u \in \mathcal{F}_0. \]

Lemma 2.3: There is a constant $c_1 > 0$ such that for all $u \in \mathcal{F}$
\[ \|u\|_\mu^2 \leq c_1 \left( \mathcal{E}(u) + \|u\|_1^2 \right), \]
where $\|f\|_1 := \int_0^1 |f(x)| d\mu(x)$.

Proof. Let $u \in \mathcal{F}$ and $u_0$ be the unique harmonic function with $u_0(0) = u(0)$ and $u_0(1) = u(1)$, that is $u_0(x) := u(0)(1-x) + u(1)x$. We have $(u - u_0)(0) = (u - u_0)(1) = 0$ and thus $u - u_0 \in \mathcal{F}_0$. Since the space of harmonic functions on $[0,1]$ with two boundary conditions is two-dimensional, there exists $c_1' > 0$ such that for all harmonic functions $u_0$
\[ \|u_0\|_\mu \leq c_1' \|u_0\|_1 \]
and since $\mu$ is a probability measure we have for all $u \in \mathcal{F}$
\[ \|u\|_1 \leq \|u\|_\mu. \]

Furthermore,
\[ \mathcal{E}(u - u_0) = \mathcal{E}(u) - 2\mathcal{E}(u, u_0) + \mathcal{E}(u_0) \]
\[ = \mathcal{E}(u) - 2 \int_0^1 u'(x)(u(1) - u(0)) dx + (u(1) - u(0))^2 \]
\[ = \mathcal{E}(u) - 2(u(1) - u(0))^2 + (u(1) - u(0))^2 \]
\[ = \mathcal{E}(u) - (u(1) - u(0))^2 \]
and thus
\[ \mathcal{E}(u - u_0) \leq \mathcal{E}(u). \]

By Lemma 2.2 and the above calculations,
\[ \|u\|_\mu^2 \leq \|u_0\|_\mu + \|u - u_0\|_\mu \]
\[ \leq c_1' \|u_0\|_1 + \sqrt{c_0 \mathcal{E}(u - u_0, u - u_0)} \]
\[ \leq c_1' (\|u\|_1 + \|u - u_0\|_1) + \sqrt{c_0 \mathcal{E}(u - u_0, u - u_0)} \]
\[ \leq c_1' (\|u\|_1 + \|u - u_0\|_\mu) + \sqrt{c_0 \mathcal{E}(u - u_0, u - u_0)} \]
\[ \leq c_1' \|u\|_1 + c_1' \sqrt{c_0 \mathcal{E}(u - u_0, u - u_0)} + \sqrt{c_0 \mathcal{E}(u - u_0, u - u_0)} \]
\[ \leq 2c_0^{\frac{1}{2}} c_1'(\|u\|_1 + \sqrt{\mathcal{E}(u)}). \]

The assertion follows from the fact that for positive numbers $a, b, c$ with $a \leq b + c$ it holds $a^2 \leq 2(b^2 + c^2)$. □
Moreover, we need scaling properties for $\mu$ and $\mathcal{E}$. Preliminary, we introduce the notion of a partition (see [20, Definition 1.3.9]).

**Definition 2.4:** For $\omega \in \mathcal{W}^*$ let $\Sigma_\omega := \{ \sigma = \sigma_1 \sigma_2 \cdots \in \mathcal{W} : \sigma_i = \omega_i \text{ for all } 1 \leq i \leq |\omega| \}$. A finite subset $\Lambda \subset \mathcal{W}^*$ is called partition if it holds $\Sigma_\omega \cap \Sigma_\sigma = \emptyset$ for $\omega \neq \sigma \in \Lambda$ and $\mathcal{W} = \bigcup_{\omega \in \Lambda} \Sigma_\omega$.

We introduce some notation for the following lemma. Let $w \in \mathcal{W}^*$. For a function $f$ we define $f_\omega := f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_{|\omega|}}$. Analogously, the pushforward measure $\mu \circ f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_{|\omega|}}$ is denoted by $\mu \circ f_\omega$.

**Lemma 2.5:** Let $\Lambda$ be a partition. It holds

(i) $\mu = \sum_{\omega \in \Lambda} \mu_\omega (\mu \circ S_\omega^{-1})$,

(ii) $\sum_{\omega \in \Lambda} r_\omega^{-1} \mathcal{E}(u \circ S_\omega) \leq \mathcal{E}(u)$ for all $u \in \mathcal{F}$.

We skip the proof of this lemma since it works by standard arguments, as in [1, Section 3.2.1].

**Proof of Theorem 2.1.** Let $u \in \mathcal{F}$ be fixed. Then,

$$\|u\|_\mu^2 = \int_0^1 u^2(x) d\mu(x) = \sum_{\omega \in \Lambda} \mu_\omega \int_0^1 u^2(x) d\mu \circ S_\omega^{-1}(x)$$

$$= \sum_{\omega \in \Lambda} \mu_\omega \int_0^1 u(S_\omega(x))^2 d\mu(x) \leq c_1 \sum_{\omega \in \Lambda} \mu_\omega \left( \mathcal{E}(u \circ S_\omega) + \|u \circ S_\omega\|_1^2 \right)$$

$$\leq c_1 \left( \max_{\omega \in \Lambda} \{ \mu_\omega r_\omega \} \sum_{\omega \in \Lambda} r_\omega^{-1} \mathcal{E}(u \circ S_\omega) + \sum_{\omega \in \Lambda} \mu_\omega^{-1} \left( \mu_\omega \int_0^1 |u \circ S_\omega| d\mu \right)^2 \right)$$

$$\leq c_1 \left( \max_{\omega \in \Lambda} \{ \mu_\omega r_\omega \} \mathcal{E}(u) + \min_{\omega \in \Lambda} \{ \mu_\omega^{-1} \|u\|_1^2 \} \right).$$

Hereby, equation (10) follows from Lemma 2.5(i), inequality (11) from Lemma 2.3 and inequality (13) from Lemma 2.5(ii). Now, let $\nu_i := (\mu_i r_i)^\gamma$, $i = 1, ..., N$. By (7) it holds $\sum_{i=1}^N \nu_i = 1$. Let $\lambda \in (0,1)$ and the partition $\Lambda_\lambda$ defined by

$$\Lambda_\lambda = \{ \omega \in \mathcal{W}^* : \nu_{\omega_1} \cdots \nu_{\omega_{|\omega|-1}} > \lambda \geq \nu_\omega \}.$$

By definition of $\Lambda_\lambda$ we have for $\omega \in \Lambda_\lambda$ $\nu_\omega^{-1} = \mu_\omega r_\omega \leq \lambda^{\frac{1}{\gamma}}$ and from that $\max_{\omega \in \Lambda_\lambda} (\mu_\omega r_\omega) \leq \lambda^{\frac{1}{\gamma}}$. Furthermore, it is known from [20, Proposition 4.5.2] that there exists $C'_2 > 0$, such that $\min_{\omega \in \Lambda_\lambda} \mu_\omega \geq C'_2 \lambda^\delta$, from which it follows $\left( \min_{\omega \in \Lambda_\lambda} \mu_\omega \right)^{-1} \leq \frac{1}{C'_2} \lambda^{-\delta}$. This and (13) yield to the existence of a constant $C''_2 > 0$ such that for all $\lambda \in (0,1)$, $u \in \mathcal{F}$

$$\|u\|_1^2 \leq C''_2 \left( \lambda^{\frac{1}{\gamma}} \mathcal{E}(u) + \lambda^{-\delta} \|u\|_1^2 \right).$$

Let $\theta := 2\gamma \delta$. We assume that $\mathcal{E}(u) > \|u\|^2_1$ and choose $\lambda \in (0,1)$ such that $\lambda^{\frac{1}{\gamma} + \delta} = \frac{\|u\|^2_1}{\mathcal{E}(u)}$. It follows

$$\|u\|_1^2 \leq 2C''_2 \lambda^{-\delta} \|u\|_1^2$$

(14)
and therefore
\[
\|u\|_{\mu}^{\frac{2}{\delta}} \leq (2C''_{2})^{\frac{2}{\delta}} \lambda^{-\frac{2}{\delta}} \|u\|_{1}^{\frac{2}{\delta}} = (2C''_{2})^{\frac{2}{\delta}} \lambda^{-\frac{1}{\delta}} \|u\|_{1}^{\frac{2}{\delta}}.
\]
By combining (14) and (15) we get
\[
\|u\|_{\mu}^{\frac{2}{\delta}} \leq (2C''_{2})^{1+\frac{2}{\delta}} \lambda^{-\frac{1}{\delta}} \|u\|_{1}^{\frac{2}{\delta}} = (2C''_{2})^{1+\frac{2}{\delta}} \|u\|_{1}^{\frac{2}{\delta}} E(u).
\]
If it holds \(E(u) \leq \|u\|_{1}^{2}\), from Lemma 2.3 it follows
\[
\|u\|_{\mu}^{2} \leq 2c_{1} \|u\|_{1}^{2}
\]
and thus
\[
\|u\|_{\mu}^{2+\frac{2}{\delta}} \leq (2c_{1})^{1+\frac{2}{\delta}} \|u\|_{1}^{2} \|u\|_{1}^{\frac{2}{\delta}}.
\]
All in all, there is a \(C''_{2} > 0\) such that for all \(u \in F\) the Nash-type inequality
\[
\|u\|_{\mu}^{2+\frac{2}{\delta}} \leq C''_{2} \left( E(u) + \|u\|_{\mu}^{2} \right) \|u\|_{1}^{\frac{2}{\delta}}
\]
is fulfilled. Let \(\psi : L^{2}([0, 1], \mu) \to L^{2}(F, \mu), f \to f|_{F}\) and \(\Delta_{\mu}^{N} : \psi(D(\Delta_{\mu}^{N})) \to L^{2}(F, \mu), u \to \psi \circ \Delta_{\mu}^{N} \circ \psi^{-1}u\). Then, \(\Delta_{\mu}^{N}\) is self-adjoint, has eigenvalues \(\lambda_{k}^{N}\) with eigenfunctions \(\psi \circ \varphi_{k}^{N}\) for \(k \in \mathbb{N}\) and the Dirichlet form \(\tilde{E}(\tilde{u}, \tilde{v}) := E(\psi^{-1}\tilde{u}, \psi^{-1}\tilde{v}), \tilde{u}, \tilde{v} \in \tilde{F}\) is associated (see Appendix A.1).

Then, for all \(\tilde{u} \in \tilde{F}\) the Nash-type inequality
\[
\|\tilde{u}\|_{\mu}^{2+\frac{2}{\delta}} \leq C''_{2} \left( \tilde{E}(\tilde{u}) + \|\tilde{u}\|_{\mu}^{2} \right) \|\tilde{u}\|_{1}^{\frac{2}{\delta}}
\]
is satisfied. Since it holds \(\mu(O) > 0\) for all open sets \(O \subseteq F\), we can apply [20, Proposition B.3.7] to get the existence of \(C_{2}'' > 0\) such that for all \(k \in \mathbb{N}\)
\[
\left\| \tilde{T}_{t}^{N}\varphi_{k}^{N} \right\|_{\infty} \leq C_{2}'' t^{-\frac{2}{\delta}},
\]
where \(\left( \tilde{T}_{t}^{N} \right)_{t \geq 0}\) is the strongly continuous semigroup associated to \(\Delta_{\mu}^{N}\). With \(\tilde{T}_{t}^{N}\varphi_{k}^{N} = e^{-\lambda_{k}^{N}t}\varphi_{k}^{N}\) for \(t \geq 0\) (see [20, Corollary B.2.7]), \(t := \frac{1}{\lambda_{k}^{N}}\) and \(\tilde{C}_{2} := C_{2}'' e\) we obtain for all \(k \in \mathbb{N}\)
\[
\|\varphi_{k}^{N}\|_{\infty} \leq \tilde{C}_{2} \lambda_{k}^{N},
\]
from which the assertion follows for \(b = N\) since \(\varphi_{k}^{b}\) is linear on the intervals in \(F^{c}\). In case of \(b = D\) the proof works analogously since \(F_{0} \subseteq F\).

2.3 Properties of the Resolvent Operator

For \(\lambda > 0\) and \(b \in \{N, D\}\) let \(\rho_{\lambda}^{b}\) be the resolvent density of \(\Delta_{\mu}^{b}\). That is, with \(R_{b}^{\lambda} : = (\lambda - \Delta_{\mu}^{b})^{-1}\) it holds
\[
R_{b}^{\lambda} f(x) = \int_{0}^{1} \rho_{\lambda}^{b}(x,y) f(y) d\mu(y), \quad f \in \mathcal{H}.
\]
Such a mapping exists and is given by (compare \cite[Theorem 6.1]{10})

\[
\rho^N_\lambda(x,y) = \rho^N_\lambda(y,x) = (B^\lambda_N)^{-1} g^\lambda_{1,N}(x)g^\lambda_{2,N}(y), \quad x, y \in [0,1], x \leq y,
\]

\[
\rho^D_\lambda(x,y) = \rho^D_\lambda(y,x) = (B^\lambda_D)^{-1} g^\lambda_{1,D}(x)g^\lambda_{2,D}(y), \quad x, y \in [0,1], x \leq y,
\]

where \(B^\lambda_N, B^\lambda_D\) are non-vanishing constants and the mappings \(g^\lambda_{1,N}, g^\lambda_{2,N}, g^\lambda_{1,D}, g^\lambda_{2,D}\) are eigenfunctions of \(\Delta_n\) with appropriate boundary conditions (see \cite[Remark 5.2]{10}). We prove that the resolvent density is Lipschitz.

**Proposition 2.6:** Let \(\lambda > 0\). Then, for every \(\lambda > 0\) there exists a constant \(L_\lambda \geq 0\) such that

\[
|\rho^b_\lambda(x,y) - \rho^b_\lambda(x,z)| \leq L_\lambda |y - z|, \quad x, y, z \in [0,1].
\]

**Proof.** Let \(b \in \{N,D\}\). We denote the maximum of the Lipschitz constants of the functions \(g^\lambda_{1,b}, g^\lambda_{2,b}\) (according to the amount) by \(L'_\lambda\) and \(\max \left\{ \|g^\lambda_{1,b}\|_\infty, \|g^\lambda_{2,b}\|_\infty \right\}\) by \(L''_\lambda\). Now, let \(x \in [0,1]\). For \(y, z \in [x,1]\) we have

\[
|\rho^b_\lambda(x,y) - \rho^b_\lambda(x,z)| = \left((B^\lambda_b)^{-1}\right) |g^\lambda_{1,b}(x) (g^\lambda_{2,b}(y) - g^\lambda_{2,b}(z))| \leq \left((B^\lambda_b)^{-1}\right) L'_\lambda L''_\lambda |y - z|.
\]

From the symmetry we get the same for \(y, z \in [0,x]\). For \(0 \leq z \leq x \leq y \leq 1\) we have

\[
|\rho^b_\lambda(x,y) - \rho^b_\lambda(x,z)| = \left((B^\lambda_b)^{-1}\right) \left| g^\lambda_{1,b}(x)g^\lambda_{2,b}(y) - g^\lambda_{1,b}(z)g^\lambda_{2,b}(x) \right| \leq \left((B^\lambda_b)^{-1}\right) \left( L'_\lambda L''_\lambda (|y - x| + |x - z|) \right)
\]

\[
= \left((B^\lambda_b)^{-1}\right) L'_\lambda L''_\lambda |y - z|
\]

and, again, the symmetry implies the same for \(0 \leq y \leq x \leq z \leq 1\). \(\square\)

### 2.4 Approximation of the Resolvent Density

We develop a method to approximate the delta functional on Cantor-like sets, in particular to approximate the just introduced resolvent density, which will then again (dann wiederrum) be used to approximate point evaluations of heat kernels.

For \(n \geq 1\) let \(\Lambda_n\) be the partition of the word space \(\mathbb{W}\) be defined by

\[
\Lambda_n = \{\omega = \omega_1...\omega_m \in \mathbb{W}^n : r_{\omega_1} \cdots r_{\omega_{m-1}} > r_{\text{max}} \geq r_\omega\},
\]

where \(r_{\text{max}} := \max_{i=1,...,N} r_i\). Moreover, let \(\nu_i = \frac{r_{\omega_i}}{r_{\text{H}}^i}, 1 \leq i \leq N\), where \(d_H\) is the Hausdorff dimension of \(F\). Further, for \(\omega \in \mathbb{W}\) we denote \(S_\omega(F)\) by \(F_\omega\).

**Lemma 2.7:** It holds for \(n \in \mathbb{N}\):

(i) \(|\Lambda_n| < \infty \) and \(\bigcup_{\omega \in \Lambda_n} F_\omega = F\).
(ii) For \( \omega \in \Lambda_n \) there exists a subset \( \mathcal{N} \subseteq \Lambda_{n+1} \) such that \( F_\omega = \bigcup_{\nu \in \mathcal{N}} F_\nu \).

(iii) For \( \omega, \nu \in \Lambda_n, \omega \neq \nu \) it holds \( |F_\omega \cap F_\nu| \leq 1 \).

(iv) For \( \omega \in \Lambda_n \) it holds \( \mu(F_\omega) > r_{n_{\max}}^d r_{i_{\min}}^d r_{\min}^d \).

(v) For \( w \in \mathbb{W}^* \) there exists \( n \in \mathbb{N} \) such that \( w \in \Lambda_n \). Consequently, for all \( m \geq n \) there exists \( \Lambda'_m \subseteq \Lambda_m \) such that \( F_w = \bigcup_{\nu \in \Lambda'_m} F_\nu \).

If the measure \( \mu \) is chosen as \( \mu_i = r_i^d \) and thus \( \nu_i = 1, i = 1, \ldots, N \), we get an estimate similar to [16, Lemma 3.5(iv)]. Note that these ideas can be used to generalize the corresponding results in [16].

**Proof.**

(i) The first claim is obvious. For the second we note that \( \bigcup_{w \in \mathbb{W}} F_w = F \) and that \( \bigcup_{w \in \Lambda_n} \Sigma_w = \mathbb{W} \) and thus \( \bigcup_{v \in \Sigma_w, w \in \Lambda_n} F_v = F \). It remains to show that \( F_w = \bigcup_{v \in \Sigma_w} F_v \) for \( w \in \Lambda_n \). This follows from applying \( f_w \) to both sides of the equation \( \bigcup_{v \in \mathbb{W}} F_v = F \).

(ii) Let \( \omega \in \Lambda_n \). We know from part (i) that \( F_w = \bigcup_{v \in \Sigma_w} F_v \). If \( r_w \leq r_{n_{\max}}^{n+1} \), the assertion follows since we can choose \( \Lambda' = \{w\} \). Now, we assume \( r_w > r_{n_{\max}}^{n+1} \). Then it holds for \( i = 1, \ldots, N \) \( r_w r_i \leq r_{n_{\max}}^{n+1} \), since \( r_w \leq r_{n_{\max}}^n \). It follows \( w_i \in \Lambda_{n+1} \) for \( i = 1, \ldots, N \). We get the result by using this and applying \( f_w \) to both sides of equality (3).

(iii) Since \( \omega \neq \nu \), there exists an \( m \leq \min\{|\omega|, |\nu|\} \) such that \( \omega_m \neq \nu_m \). From \( |\text{Im}(f_1) \cap \text{Im}(f_j)| \leq 1 \) for \( 1 \leq i \neq j \leq N \) it follows

\[
|f_{\omega_m} \circ f_{\omega_{m+1}} \circ \cdots \circ f_{\omega_{|\omega|}}(F) \cap f_{\nu_m} \circ f_{\nu_{m+1}} \circ \cdots \circ f_{\nu_{|\nu|}}(F)| \leq 1.
\]

The assertion follows by composing the respective maps \( f_{\omega_{m-1}}, \ldots, f_{\omega_1}, f_{\nu_{m-1}}, f_{\nu_1} \) and using the injectivity if \( \omega_i = \nu_i \) and the disjointness of the images, except at most one point if \( \omega_i \neq \nu_i \) for \( i < m \).

(iv) Let \( \omega \in \Lambda_n \) and \( m := |\omega| \). By definition of \( \Lambda_n \) it holds \( r_{\omega_1} \cdots r_{\omega_{m-1}} > r_{n_{\max}}^m \) and therefore \( r_\omega > r_{n_{\max}}^n r_{\min}^n \). By using that,

\[
\mu_\omega = r_{n}^d \mu_{\omega_1} r_{n}^d \mu_{\omega_2} \cdots r_{n}^d \mu_{\omega_m} \leq r_{n}^d r_{i_{\min}}^d r_{\min}^d = r_{n_{\max}}^d r_{i_{\min}}^d r_{\min}^d = r_{n_{\max}}^d r_{i_{\min}}^d r_{\min}^d,
\]

The last inequality follows from \( m \leq n \) and \( \nu_{\min} \leq 1 \).

(v) Let \( w = w_1 \ldots w_m \in \mathbb{W}^* \). Choose \( n \) such that \( r_w \leq r_{n_{\max}}^n \) and \( r_w > r_{n_{\max}}^{n+1} \). From \( r_{w_1} \cdots r_{w_{m-1}} > r_{n_{\max}}^n \) and \( r_{w_1} \cdots r_{w_m} > r_{n_{\max}}^n r_{\min}^n \) it follows

\[
r_{w_1} \cdots r_{w_{m-1}} > r_{w_1} \cdots r_{w_m} r_{n_{\max}}^{n-1} > r_{n_{\max}}^{n+1} r_{\min}^n = r_{n_{\max}}^n.
\]

Therefore, we can find an \( n \in \mathbb{N} \) such that \( w \in \Lambda_n \). For the second part, we can argue as in (ii) with induction.

\[\square\]

We introduce a sequence of functions approximating the Delta functional. Hereby, we use the notation of [16]. We prepare this definition by defining the \( n \)-neighbourhood of \( x \in F \) for \( n \in \mathbb{N} \) by

\[
D^0_n(x) = \bigcup \{ F_w : w \in \Lambda_n, \ x \in F_w \}.
\]
Note that \( D^0_n(x) \) consists of at least one element of \( \{ F_w, w \in \Lambda_n \} \), which follows from Lemma 2.7(i), and of at most two elements since pairs of these elements intersect in at most one point. From the latter and the definition of \( \Lambda_n \) it follows
\[
|D^0_n(x)| \leq 2 r_{\text{max}}^n. \tag{19}
\]
With that, we can define the approximating functions for \( x \in F \) and \( n \geq 1 \) by
\[
f^*_n = \mu(D^0_n(x))^{-1} \mathbf{1}_{D^0_n(x)}.
\]
From Lemma 2.7(iv) it follows
\[
||f^*_n||^2_{\mu} = \mu(D^0_n(x))^{-1} \leq r_{\text{max}}^{-d_{\text{max}}} r_{\text{min}}^{-d_{\text{min}}} n^{-1}.
\tag{20}
\]
We deduce the following result.

**Lemma 2.8:** Let \( x \in F \). It holds \( \lim_{n \to \infty} \langle f^*_n, g \rangle_{\mu} = g(x) \) for any continuous \( g \in \mathcal{H} \).

**Proof.** For \( n \in \mathbb{N} \) and \( \omega \in \Lambda_n \) it holds \( |F_\omega| \leq r_{\text{max}}^n \) since \( |F| \leq 1 \) and \( r_{\omega} \leq r_{\text{max}}^n \). Therefore, it holds \( |y - x| \leq r_{\text{max}}^n \) for \( x, y \in D^0_n(x) \). Now, let \( x \in F \) and \( \varepsilon > 0 \). Since \( g \) is continuous in \( x \), there exists \( \delta > 0 \) such that \( |g(y) - g(y')| < \varepsilon \) for \( y \in [0, 1] \) with \( |y - x| < \delta \). Choose \( n \in \mathbb{N} \) such that \( r_{\text{max}}^n < \delta \). Then, it follows
\[
|\langle f^*_n, g \rangle_{\mu} - g(x)| = \frac{1}{\mu(D^0_n(x))} \left| \int_{D^0_n(x)} g(y) d\mu(y) - g(x) \right|
\leq \frac{1}{\mu(D^0_n(x))} \int_{D^0_n(x)} |g(y) - g(x)| d\mu(y)
\leq \frac{1}{\mu(D^0_n(x))} \mu(D^0_n(x)) \cdot \varepsilon = \varepsilon.
\]

**Lemma 2.9:** Let \( x_1, x_2 \in F \) and \( m, n \geq 1 \). Then,
\[
\left| \int_0^1 \int_0^1 \rho^l_1(y, z) f^*_{m_1}(y) f^*_{n_2}(z) d\mu(y) d\mu(z) - \rho^l_1(x_1, x_2) \right| \leq 2L_1 (r_{\text{max}}^m + r_{\text{max}}^n),
\]
where \( L_1 \) denotes the Lipschitz constant of \( \rho^l_1 \).

**Proof.** By using the Lipschitz continuity of \( \rho^l_1 \) and (19),
\[
\left| \int_0^1 \int_0^1 \left( \rho^l_1(y, z) - \rho^l_1(x_1, x_2) \right) f^*_{m_1}(y) f^*_{n_2}(z) d\mu(y) d\mu(z) \right|
\leq \int_0^1 \int_0^1 \left| \rho^l_1(y, z) - \rho^l_1(x_1, x_2) \right| + \left| \rho^l_1(x_1, x_2) - \rho^l_1(x_1, x_2) \right|
\int_0^1 \int_0^1 f^*_{m_1}(y) f^*_{n_2}(z) d\mu(y) d\mu(z)
\leq \frac{1}{\mu(D^0_m(x_1)) \mu(D^0_n(x_2))} \left( \int_{D^0_m(x_1)} \int_{D^0_n(x_2)} \left| \rho^l_1(y, z) - \rho^l_1(x_1, x_2) \right|
\leq 2L_1 (r_{\text{max}}^m + r_{\text{max}}^n) \right)
\leq 2L_1 (r_{\text{max}}^m + r_{\text{max}}^n).
\]
\[\square\]
2.5 Wave Equations Defined by Fractal Laplacians

We introduce the notion of a wave propagator in this section which will be used to define the concept of a mild solution to (5). Let $T > 0$, $b \in \{N, D\}$, $u_0 = \sum_{k \geq 1} u_{0,k}^b \varphi_k^b \in \mathcal{D}(\Delta^b_\mu)$ and $u_1 = \sum_{k \geq 1} u_{1,k}^b \varphi_k^b \in \mathcal{D}\left((-\Delta^b_\mu)^{\frac{1}{2}}\right)$, that is $\sum_{k \geq 1} (\lambda_k^b)^2 (u_{0,k}^b)^2 < \infty$ and $\sum_{k \geq 1} \lambda_k^b (u_{1,k}^b)^2 < \infty$. $\Delta^b_\mu$ is a self-adjoint, dissipative operator on $\mathcal{H}$. Hence, it is well-known (compare, e.g., [27]) that the wave equation

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} u(t, x) &= -\Delta^N_\mu u(t, x), \\
u(0, x) &= u_0(x), \\
\frac{\partial u(0, x)}{\partial t} &= u_1(x)
\end{aligned}
\]  

(21)

on $[0, T] \times [0, 1]$ has a unique solution given by

\[
u(t, x) = \int_0^1 P_b(t, x, y)u_1(y)dy + \frac{\partial}{\partial t} \int_0^1 P_b(t, x, y)u_0(y)dy, \quad (t, x) \in [0, T] \times [0, 1],
\]

where $P_b$ is the so-called wave propagator (see [8, Chapter 5]), defined by

\[
P_D(t, x, y) = \sum_{k \geq 1} \frac{\sin \left(\sqrt{\lambda_k^b t}\right)}{\sqrt{\lambda_k^D}} \varphi_k^D(x)\varphi_k^D(y)
\]

and

\[
P_N(t, x, y) = t + \sum_{k \geq 2} \frac{\sin \left(\sqrt{\lambda_k^{N} t}\right)}{\sqrt{\lambda_k^{N}}} \varphi_k^{N}(x)\varphi_k^{N}(y),
\]

respectively. Note that,

\[
\begin{aligned}
\int_0^1 P_N(t, x, y)u_0(y)dy &= tu_{1,k}^N + \sum_{k \geq 2} \frac{\sin \left(\sqrt{\lambda_k^{N} t}\right)}{\sqrt{\lambda_k^{N}}} u_{0,k}^N \varphi_k^N(x), \\
\frac{\partial}{\partial t} \int_0^1 P_N(t, x, y)u_0(y)dy &= u_{1,k}^N + \sum_{k \geq 2} \cos \left(\sqrt{\lambda_k^{N} t}\right) u_{0,k}^N \varphi_k^N(x), \\
\int_0^1 P_D(t, x, y)u_1(y)dy &= \sum_{k \geq 1} \frac{\sin \left(\sqrt{\lambda_k^{D} t}\right)}{\sqrt{\lambda_k^{D}}} u_{1,k}^D \varphi_k^D(x), \\
\frac{\partial}{\partial t} \int_0^1 P_D(t, x, y)u_0(y)dy &= \sum_{k \geq 1} \cos \left(\sqrt{\lambda_k^{D} t}\right) u_{0,k}^D \varphi_k^D(x).
\end{aligned}
\]

Recall that $\gamma$ is the spectral exponent of $-\Delta^b_\mu$ (compare (7)) and $\delta := \max_{1 \leq i \leq N} \frac{\log \mu_i}{\log((\mu_i/r_i)^{\gamma})}$. 

**Lemma 2.10 (Properties of the wave propagator):**

Let $\delta + 1 < \frac{1}{7}$. Then, there exists a constant $C_3 > 0$ such that

\[
\sup_{t \in [0, \infty)} \sup_{x \in [0, 1]} \|P_b(t, \cdot, \cdot)\|_\mu < C_3.
\]
Furthermore, define $R$ and consider the stochastic PDE

$$\sum_{k \geq 2} \frac{\sin \sqrt{\lambda_k}t}{\sqrt{\lambda_k}} \varphi_k(x) \varphi_k(\cdot) \rightarrow^2 t^2 + \sum_{k \geq 2} \frac{\sin^2 \sqrt{\lambda_k}t}{\lambda_k} \varphi_k^2(x) \leq T^2 + C_2 \sum_{k \in \mathbb{N}} \frac{1}{\lambda_k} k^\delta \leq T^2 + C_1 C_2 \sum_{k \in \mathbb{N}} k^{\delta - \frac{1}{2}},$$

which is finite independently of $x \in [0, 1]$ and $t \in [0, T]$ due to the assumption. \qed

3 Analysis of Stochastic Wave Equations

3.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. We consider the stochastic PDE

$$\frac{\partial^2}{\partial t^2} u(t, x) = \Delta u(t, x) + f(t, u(t, x)) \xi(t, x), \quad u(0, x) = u_0(x), \quad \frac{\partial}{\partial t} u(0, x) = u_1(x)$$

for $(t, x) \in [0, T] \times [0, 1]$, where $T > 0$, $b \in \{ N, D \}$ determines the boundary conditions, $u_0, u_1 : [0, 1] \to \mathbb{R}$, $f : \Omega \times [0, T] \times [0, 1] \to \mathbb{R}$ and $\xi$ is a $\mathbb{F}$-space-time white noise on $([0, 1], \mu)$, that is a mean-zero set-indexed Gaussian process on $\mathcal{B}([0, T] \times [0, 1])$ such that $\mathbb{E} [\xi(A)\xi(B)] = |A \cap B|$ (compare [28, Chapter 1]). Moreover, let for a time interval $I \subseteq [0, T]$ and a space interval $J \subseteq [0, \infty)$ $\mathcal{F}_I,J$ be the $\sigma$-algebra generated by simple functions on $\Omega \times I \times J$, where a simple function on $\Omega \times I \times J$ is defined as a finite sum of functions $h : \Omega \times I \times J \to \mathbb{R}$ of the form

$$h(\omega, t, x) = X(\omega) \mathbb{1}_{(a, b]}(t) \mathbb{1}_B(x), \quad (\omega, t, x) \in \Omega \times I \times J$$

with $X$ bounded and $\mathcal{F}_a$-measurable, $a, b \in I, a < b$ and $B \in \mathcal{B}(J)$.

Definition 3.1: Let $q \geq 2, T > 0$ be fixed. Let $\mathcal{S}_{q, T}$ be the space of $[0, T] \times [0, 1]$-indexed processes $v$ being predictable (i.e. measurable from $\mathcal{P}_{[t, T]} \times [t, \infty)$ to $\mathcal{B}(\mathbb{R})$ and satisfying

$$||v||_{q, T} := \sup_{t \in [0, T]} \sup_{x \in [0, 1]} (\mathbb{E}[|v(t, x)|^q])^{\frac{1}{q}} < \infty.$$  

Furthermore, define $\mathcal{S}_{q,T}$ as the space of equivalence classes of processes in $\mathcal{S}_{q,T}$, where two processes $v_1, v_2$ are equivalent if $v_1(t, x) = v_2(t, x)$ almost surely for all $(t, x) \in [0, T] \times [0, 1]$.

Note that $\mathcal{S}_q$ and $\mathcal{S}_{q,T}$ are Banach spaces. The proof works by using standard arguments, so we skip it here.

We define the concept of a solution to (22) which we observe in this paper.

Definition 3.2: A mild solution to the SPDE (22) is defined as a predictable $[0, T] \times [0, 1]$-indexed process such that for every $(t, x) \in [0, T] \times [0, 1]$ it holds almost surely

$$\frac{\partial}{\partial t} u(t, x) = \int_0^1 P_b(t, x, y) u_0(y) d\mu(y) + \int_0^1 \frac{P_b(t, x, y)}{\hat{P}_b(y)} u_1(y) d\mu(y)$$

$$+ \int_0^T \int_0^1 P_b(t-s, x, y) f(s, u(s, y)) \xi(s, y) d\mu(y) ds,$$

for $(t, x) \in [0, T] \times [0, 1]$, where the last term is a stochastic integral in the sense of [28, Chapter 2].
In this chapter, we assume that Condition 1.1 is satisfied for a given $q \geq 2$. Note that since $u_0 \in D(-\Delta^\mu)$, it holds

$$|u_{0,k}| < C_4 k^{\frac{1}{2}}, k \geq 1,$$

where $C_4 := \|(-\Delta^\mu)^{\frac{1}{4}} u_0\|_\mu$. Analogously, since $u_1 \in D\left((\Delta^\mu)^{\frac{1}{2}}\right)$, we have

$$|u_{1,k}| < C_5 k^{\frac{1}{2}}, k \geq 1,$$

where $C_5 := \|(-\Delta^\mu)^{\frac{1}{2}} u_1\|_\mu$.

### 3.2 Existence, Uniqueness and Continuity

Let $b \in \{N, D\}$. In this section, we prove continuity properties of $v_i$, $i = 1, 2, 3$, which are defined as follows for $v_0 \in S_{q,T}$:

$$v_1(t, x) := \int_0^t \int_0^1 P_b(t-s, x, y) f(s, v_0(s, y)) \xi(s, y) d\mu(y) ds, \quad (24)$$

$$v_2(t, x) := \int_0^1 P_b(t, x, y) u_1(y) d\mu(y), \quad (25)$$

$$v_3(t, x) := \frac{\partial}{\partial t} \int_0^1 P_b(t, x, y) u_0(y) d\mu(y). \quad (26)$$

We need some preparing lemmas. The following lemma shows how to find upper estimates of functionals of the wave propagator by using the resolvent density.

**Lemma 3.3:** For all $t \in (0, T]$ and $g \in \mathcal{H}$ it holds

$$\int_0^1 \left( \int_0^1 P_b(t, x, y) g(y) d\mu(y) \right)^2 d\mu(x) \leq 2t^2 \int_0^1 \rho^1_b(x, y) g(x) g(y) d\mu(x) d\mu(y).$$

**Proof.** Let $b = N$ and $g = \sum_{k=0}^{\infty} g_k \varphi_k^N$. Then, since the sequence of eigenvalues is increasing,

$$\int_0^1 \left( \int_0^1 P_N(t-s, x, y) g(y) d\mu(y) \right)^2 d\mu(x) = \left\| t g_0 + \sum_{k=2}^{\infty} \frac{\sin(\sqrt{\lambda_k^N}(t-s))}{\sqrt{\lambda_k^N}} g_k \varphi_k^N \right\|_\mu^2$$

$$= t^2 g_0^2 + \sum_{k=2}^{\infty} \frac{\sin^2(\sqrt{\lambda_k^N}(t-s))}{\lambda_k^N} g_k^2$$

$$\leq t^2 \left( g_0^2 + \sum_{k=2}^{\infty} \frac{1}{\lambda_k^N} g_k^2 \right)$$

$$\leq t^2 \frac{1 + \lambda_2^N}{\lambda_2^N} \sum_{k=1}^{\infty} \frac{1}{1 + \lambda_k^N} g_k^2$$

$$= t^2 \frac{1 + \lambda_2^N}{\lambda_2^N} \left( g, (1 - \Delta_\mu^N)^{-1} g \right)_\mu. \quad (27)$$

By definition of the resolvent density it holds

$$(1 - \Delta_\mu^N)^{-1} g = \int_0^1 \rho^1_N(\cdot, y) g(y) d\mu(y).$$

Plugging this and the fact that $\lambda_2^N > 1$ (see, e.g., [1, Section 3.3.1]) into (27), the assertion for $b = N$ follows. The case $b = D$ works similarly using $\lambda_2^D > 1$ (see, e.g., [26, Lemma 4.9]).
This leads to a useful approximation of $P_b(t, x, \cdot)$ for fixed $(t, x) \in [0, T] \times [0, 1]$.

**Lemma 3.4:** Let $x \in F$. Then, there exists a constant $C_0$ such that for all $t \in (0, \infty)$ and $n \in \mathbb{N}$

\[
\int_0^1 (\langle P_b(t, \cdot, y), f_n^x \rangle - P_b(t, x, y))^2 \, d\mu(y) \leq C_0 l^2 r_n^m.
\]

**Proof.** Let $b = N$. For any $(t, y) \in [0, \infty) \times [0, 1]$ $P_N(t, \cdot, y)$ is an element of $H$ and the inner product on $H$ is continuous in each argument. It thus holds for any $g \in H$

\[
\langle P_N(t, \cdot, y), g \rangle_{\mu} = \left< t + \sum_{k=2}^{\infty} \frac{\sin \left( \sqrt{\lambda_k^N(t)} \right)}{\sqrt{\lambda_k^N}} \varphi_k^N(y) \varphi_k^N, g \right>_{\mu} = \left< \lim_{m \to \infty} t + \sum_{k=2}^{m} \frac{\sin \left( \sqrt{\lambda_k^N(t)} \right)}{\sqrt{\lambda_k^N}} \varphi_k^N(y) \varphi_k^N, g \right>_{\mu} = t \langle 1, g \rangle_{\mu} + \sum_{k=2}^{\infty} \frac{\sin \left( \sqrt{\lambda_k^N(t)} \right)}{\sqrt{\lambda_k^N}} \varphi_k^N(y) \langle \varphi_k^N, g \rangle_{\mu}. \tag{28}
\]

By using this, Lemma 2.8 and Fatou’s Lemma,

\[
\int_0^1 (\langle P_N(t, \cdot, y), f_n^x \rangle - P_N(t, x, y))^2 \, d\mu(y)
\]

\[
= \int_0^1 \left( t \langle 1, f_n^x \rangle_{\mu} + \sum_{k=2}^{\infty} \frac{\sin \left( \sqrt{\lambda_k^N(t)} \right)}{\sqrt{\lambda_k^N}} \varphi_k^N(y) \langle \varphi_k^N, f_n^x \rangle - \langle P_N(t, x, y) \rangle_{\mu} \right)^2 \, d\mu(y)
\]

\[
= \int_0^1 \left( t \langle 1, f_n^x \rangle_{\mu} - 1 + \sum_{k=2}^{\infty} \frac{\sin \left( \sqrt{\lambda_k^N(t)} \right)}{\sqrt{\lambda_k^N}} [(\varphi_k^N, f_n^x) - \varphi_k^N(x)] \varphi_k^N(y) \right)^2 \, d\mu(y)
\]

\[
= \sum_{k=2}^{\infty} \frac{\sin^2 \left( \sqrt{\lambda_k^N(t)} \right)}{\lambda_k^N} [(\varphi_k^N, f_n^x) - \varphi_k^N(x)]^2 \tag{29}
\]

\[
= \sum_{k=2}^{\infty} \frac{\sin^2 \left( \sqrt{\lambda_k^N(t)} \right)}{\lambda_k^N} [\varphi_k^N, f_n^x - \lim_{m \to \infty} \langle \varphi_k^N, f_m^x \rangle]^2
\]

\[
= \sum_{k=2}^{\infty} \lim_{m \to \infty} \frac{\sin^2 \left( \sqrt{\lambda_k^N(t)} \right)}{\lambda_k^N} [(\varphi_k^N, f_n^x) - \langle \varphi_k^N, f_m^x \rangle]^2
\]

\[
\leq \lim_{m \to \infty} \inf \sum_{k=2}^{\infty} \frac{\sin^2 \left( \sqrt{\lambda_k^N(t)} \right)}{\lambda_k^N} [(\varphi_k^N, f_n^x) - \langle \varphi_k^N, f_m^x \rangle]^2
\]

\[
= \lim_{m \to \infty} \inf \int_0^1 \left( \int_0^1 (f_n^x(z) - f_m^x(z)) \, d\mu(z) \right)^2 \, d\mu(y).
\]
Note that \((1, f_n^x)_{\mu} = \int_0^1 f_n^x(y) d\mu(y) = 1\), what we have used in equation (29). By Lemma 3.3,
\[
\int_0^1 \left( \int_0^1 P_N(t, y, z)(f_n^x(z) - f_n^x(z) d\mu(z)) \right)^2 d\mu(y)
\leq 2t^2 \int_0^1 \int_0^1 \rho_1^k(x, y)(f_n^x(y) - f_n^x(y))(f_n^x(z) - f_n^x(z)) d\mu(y) d\mu(y)
\leq 2t^2 \int_0^1 \int_0^1 \rho_1^k(x, y)(f_n^x(z) - f_n^x(z)) d\mu(y) d\mu(y)
\leq 2t^2 \int_0^1 \int_0^1 \rho_1^k(x, y)(f_n^x(z) - f_n^x(z)) d\mu(y) d\mu(y)
\leq 16L_1t^2(r_{\max}^m + r_{\max}^n),
\]
where we have used Lemma (2.9) in the last inequality. We conclude
\[
\int_0^1 (|P_N(t - s, \cdot, y) - f_n^x|)^2 d\mu(y) \leq \liminf_{m \to \infty} 16L_1t^2(r_{\max}^m + r_{\max}^n)
= 16L_1t^2r_{\max}^n.
\]
The case \(b = D\) works similarly.

We need one more estimate to find upper bounds for point evaluations of the wave propagator.

**Lemma 3.5:** Let \(a < 0, b \geq 0\). Then, there exists a constant \(C_{a,b}\) such that for all \(t \in [0, \infty)\)
\[
\sum_{k \in \mathbb{N}} k^{a-1} \wedge t^{b-1} \leq C_{a,b} t^{-\beta a}.
\]

**Proof.** [17, Lemma 5.2]

We are now able to prove Hölder continuity properties of \(v_i\), \(i = 1, 2, 3\). We start with the deterministic ones.

**Proposition 3.6:** Let \(T > 0\) be fixed. Then, there exists a constant \(C_T > 0\) such that for all \(i \in \{2, 3\}\), \(t \in [0, T], x, y \in [0, 1] \) \(v_i(t, x)\) is well-defined and it holds
\[
|v_i(t, x) - v_i(t, y)| \leq C_T|x - y|,
|v_i(s, x) - v_i(t, x)| \leq C_T|s - t|^{(2-(2+\delta)\gamma)\wedge 1}.
\]

**Proof.** First, we consider the spacial continuity, where we use ideas from [18, Proposition 4.1]. Let \(t \in [0, T], x, y \in F\). By Lemma 2.9, 3.3 and 3.4,
\[
\left| \int_0^1 P_b(t, x, z)u_1(z) d\mu(z) - \int_0^1 P_b(t, y, z)u_1(z) d\mu(z) \right|^2
\leq \int_0^1 (P_b(t, x, z) - P_b(t, y, z))^2 u_1^2(z) d\mu(z)
\leq \sup_{z \in [0,1]} (u_1^2(z)) \int_0^1 (P_b(t, x, z) - P_b(t, y, z))^2 d\mu(z)
= \sup_{z \in [0,1]} (u_1^2(z)) \lim_{n \to \infty} \int_0^1 ((P_b(t, \cdot, z) - f_n^x(z))^2 d\mu(z)
\]

17
\[
\leq 2 \sup_{z \in [0,1]} (u_1^2(z)) t^2 \lim_{n \to \infty} \left| \int_0^1 \int_0^1 \rho^N_i(z_1, z_2)(f^N_{i,n}(z_1) - f^N_{i,n}(z_2))f^N_{i,n}(z_2) d\mu(z_1) d\mu(z_2) \right|
\]

\[
= 2 \sup_{z \in [0,1]} (u_1^2(z)) t^2 \left| \rho^N_i(x, x) - 2\rho^N_i(x, y) + \rho^N_i(y, y) \right|
\]

\[
\leq 4L_1 \sup_{z \in [0,1]} (u_1^2(z)) t^2|x - y|.
\]

Recall that \([0, 1] \setminus F = \bigcup_{i=1}^{\infty} (a_i, b_i)\) (see (6)). Now, let \(b = N\) and \(x, y \in F^c\) such that there exists an \(i \in \mathbb{N}\) with \((x, y) \in (a_i, b_i)\), where we assume \(x < y\). Then, since \(a_i, b_i \in F\), the previous calculation implies

\[
\left| \int_0^1 (P_N(t, x, z) - P_N(t, y, z))^2 u_1^2(z) d\mu(z) \right|
\]

\[
\leq \sup_{z \in [0,1]} (u_1^2(z)) \sum_{k=2}^{\infty} \frac{\sin^2(\sqrt{\lambda^N_k} t)}{\lambda^N_k} (\varphi^N_k(x) - \varphi^N_k(y))^2
\]

\[
\leq \sup_{z \in [0,1]} (u_1^2(z)) \left( \frac{x - y}{b_i - a_i} \right)^2 \sum_{k=2}^{\infty} \frac{\sin^2(\sqrt{\lambda^N_k} t)}{\lambda^N_k} (\varphi^N_k(b_i) - \varphi^N_k(a_i))^2
\]

\[
\leq 4L_1 \sup_{z \in [0,1]} (u_1^2(z)) t^2 \left( \frac{x - y}{b_i - a_i} \right)^2 |b_i - a_i|
\]

\[
\leq 4L_1 \sup_{z \in [0,1]} (u_1^2(z)) t^2|y - x|
\]

where we have used that for \(k \in \mathbb{N}\) \(\varphi^N_k\) is linear on \((a_i, b_i), i \in \mathbb{N}\) in (30). The remaining cases for \(x, y \in [0, 1]\) follow by using the triangle inequality for the \(\mathcal{H}\)-norm. Since the Dirichlet case works similarly, we obtain for all \((x, y) \in [0, 1]\)

\[
\left| \int_0^1 P_b(t, x, z) u_1(z) d\mu(z) - \int_0^1 P_b(t, y, z) u_1(z) d\mu(z) \right| \leq 3 \cdot 2^\frac{1}{2} \sup_{z \in [0,1]} (|u_1(z)|)^2 \frac{2^\frac{1}{2} t |x - y|^2}{2^\frac{1}{2} t |x - y|^2}.
\]

We turn to \(v_3\) and define \(\tilde{u}^b_{0,k} = \sqrt{\lambda^N_k} u^b_{0,k}, k \in \mathbb{N}\) for \(k \geq 2\). With that,

\[
\frac{\partial}{\partial t} \int_0^1 P_N(t, x, y) u_0(y) d\mu(y) = u_0^N + \sum_{k \geq 2} \cos \left( \sqrt{\lambda^N_k} t \right) u^N_{0,k} \varphi^N_k(x)
\]

\[
= u_0^N + \sum_{k \geq 2} \cos \left( \sqrt{\lambda^N_k} t \right) u^N_{0,k} \varphi^N_k(x)
\]

and can now argue similar to the proof for \(v_2\) since \(\sum_{k \geq 2} \tilde{u}^N_{0,k} \varphi^N_k \in \mathcal{D} \left( (\Delta^b) + \frac{1}{2} \right)\) as \(u_1\). Again, the proof works analogously for Dirichlet boundary conditions.
For the temporal continuity, let \( s, t \in [0, T] \) with \( s < t \) and \( x \in [0, 1] \). Then,

\[
\left| \int_0^1 (P_b(t, x, y) - P_b(s, x, y)) u_1(y)d\mu(y) \right|
\leq (t-s)|u_{1,0}| + \sum_{k=2}^{\infty} \left( \frac{\sin (\sqrt{\lambda_k}(t)) - \sin (\sqrt{\lambda_k}(s))}{\sqrt{\lambda_k}} \right) |\varphi_k(x)|u_{1,k}|
\leq (t-s)|u_{1,0}| + \sum_{k=2}^{\infty} \left( 2 \wedge (\sqrt{\lambda_k}(t) - \sqrt{\lambda_k}(s)) \right) |\varphi_k(x)|u_{1,k}|
\leq 2TC_0^{-\frac{1}{2}}C_2C_5 \sum_{k=2}^{\infty} \left( k^{\frac{\delta}{2}} \wedge \left( |s-t|k^{\frac{\delta - \frac{1}{2}}{\gamma}} \right) \right).
\]

Choose \( a = \frac{\delta}{2} - \frac{1}{\gamma} + 1 \) and \( b = \frac{\delta}{2} - \frac{1}{2\gamma} + 1 \) in Lemma 3.5 to get

\[
\left| \int_0^1 (P_N(t, x, y) - P_N(s, x, y)) u_1(y)d\mu(y) \right| \leq 2C_0^{-\frac{1}{2}}C_2C_5|s-t|^{(2(\delta - \frac{\delta}{2} + \frac{1}{\gamma})^1).}
\]

With similar methods,

\[
\sum_{k=1}^{\infty} \left| \cos \left( \sqrt{\lambda_k^N} t \right) - \cos \left( \sqrt{\lambda_k^N} s \right) \right| |\varphi_k^N(x)||u_{0,k}| \leq 2C_1^2C_2C_4 \sum_{k \in N} \left( 1 \wedge k^{\frac{1}{\gamma}}(t-s) \right) k^{\frac{\delta}{2}} k^{-\frac{1}{\gamma}}.
\]

Again, choose \( a = \frac{\delta}{2} - \frac{1}{\gamma} + 1 \) and \( b = \frac{\delta}{2} - \frac{1}{2\gamma} + 1 \) to get

\[
\sum_{k \in N^1} \left| \cos \left( \sqrt{\lambda_k^N} t \right) - \cos \left( \sqrt{\lambda_k^N} s \right) \right| |\varphi_k^N(x)||u_{0,k}| \leq 2C_1^2C_2C_4|t-s|^{(2(\delta - \frac{\delta}{2} + \frac{1}{\gamma})^1.)}
\]

The calculation for Dirichlet boundary conditions works similarly. \( \square \)

**Proposition 3.7:** Let \( q \geq 2 \) and \( T > 0 \) be fixed. Then, there exists a constant \( c_8 > 0 \) such that for all \( v_0 \in \mathcal{S}_{q,T} \) \( v_1 \) is well-defined, predictable and it holds for all \( t \in [0, T], x, y \in [0, 1] \)

\[
\mathbb{E}( |v_1(t, x) - v_1(t, y)|^q ) \leq c_8 \left( 1 + \|v_0\|_{q,T}^q \right) |x-y|^{\frac{q}{2}}
\]

\[
\mathbb{E}( |v_1(s, x) - v_1(t, x)|^q ) \leq c_8 \left( 1 + \|v_0\|_{q,T}^q \right) |s-t|^{\frac{1}{2} + \frac{\mu_{\min}}{\mu_{\max}}}.
\]

Proof. For fixed \( x \in [0, 1] \), \( P(\cdot, x, \cdot) \) is measurable and deterministic and therefore predictable and \( f \) and \( v_0 \) are predictable, according to the assumption. Hence, the integrand in (24) is predictable. By Hypothesis (iii) we have

\[
|f(t, v_0(t, x))| \leq M(t) + L|v_0(t, x)|, \quad (t, x) \in [0, T] \times [0, 1].
\]

With that, for \( t \in [0, T], x \in [0, 1] \)

\[
\mathbb{E} \left[ \int_0^T \int_0^1 P_b^2(t-s, x, y)(f(s, v_0(s, y)))^2 d\mu(y)ds \right]
\leq \sup_{s \in [0, T]} \left\| M(s) \right\|_{L^2(\Omega)}^2 + L \|v_0\|_{q,T}^q \int_0^T \int_0^1 P_b^2(t-s, x, y)d\mu(y)ds
\leq \sup_{s \in [0, T]} \left\| M(s) \right\|_{L^2(\Omega)}^2 + LT \|v_0\|_{q,T} \sup_{(s, x) \in [0, T] \times [0, 1] \times [0, 1]} \int_0^1 P_b^2(s, x, y)d\mu(y),
\]

19
which is finite, independently of \(x\) and \(t\), due to Lemma 2.10. Consequently, \(v_1\) is well-defined for \(t, x \in [0, T] \times [0, 1]\). We now prove the spatial estimate for \(v_1\). For that, let \(t \in [0, T], x, y \in [0, 1]\) be fixed. Then, there exists \(C_q > 0\) such that

\[
\mathbb{E} \left( |v_1(t, x) - v_1(t, y)^q \right) \\
= \mathbb{E} \left( \left| \int_0^t \int_0^1 (P_b(t - s, x, z) - P_b(t - s, y, z)) f(s, v_0(s, y)) \xi(s, y) d\mu(z) ds \right|^q \right) \\
\leq C_q \left( \mathbb{E} \left( \left| \int_0^t \int_0^1 (P_b(t - s, x, z) - P_b(t - s, y, z))^2 f(s, v_0(s, y))^2 d\mu(z) ds \right|^{q/2} \right)^{2/2} \right)^{2/2} \\
\leq C_q \left| \int_0^t \int_0^1 (P_b(t - s, x, z) - P_b(t - s, y, z))^2 \mathbb{E} (f(s, v_0(s, y))^q) d\mu(z) ds \right|^{q/2} \\
= C_q \left| \int_0^t \int_0^1 (P_b(t - s, x, z) - P_b(t - s, y, z))^2 \mathbb{E} (f(s, v_0(s, y))^q) d\mu(z) ds \right|^{q/2} \\
\leq 2^{q-1} C_q \left( \|M\|_{q, T} \|L^q\|_{q, T} \right)^{q/2} \left| \int_0^t \int_0^1 (P_b(t - s, x, z) - P_b(t - s, y, z))^2 d\mu(z) ds \right|^{q/2}, \tag{35} \end{align*}

where we have used the Burkholder-Davis-Gundy inequality (see e.g. [21, Theorem B.1]) in (32), which can be used since the considered stochastic integral is a square-integrable martingale (see [28, Theorem 2.5], Minkowski’s integral inequality in (33) and the relation

\[
\mathbb{E} (f(s, v_0(s, y))^q) \leq \mathbb{E} (M(s) + L|v_0(s, y)|)^q \leq 2^{q-1} (\mathbb{E} (M(s))^q + L^q (\mathbb{E} (|v_0(s, y)|)^q)) \tag{36} \]

which follows from (31), in (35). We proceed by estimating the integral term in (35), whereby we first treat the case \(x, y \in F\). Analogously to the proof of Proposition 3.6, we calculate

\[
\int_0^t \int_0^1 (P_b(t - s, x, z) - P_b(t - s, y, z))^2 d\mu(z) ds \\
= \int_0^t \int_0^1 (P_b(s, x, z) - P_b(s, y, z))^2 d\mu(z) ds \\
\leq \int_0^t 4L_1 s^2 |x - y| ds \\
\leq 4L_1 \frac{r^3}{3} |x - y| 
\]

Now, let \(b = N\) and \(x, y \in F^c\) such that there exists an \(i \in \mathbb{N}\) with \((x, y) \in (a_i, b_i)\), where we assume \(x < y\). Again, we can follow the proof of Proposition 3.6 and get

\[
\int_0^t \int_0^1 (P_N(t, x, z) - P_N(t, y, z))^2 d\mu(z) ds \\
\leq \int_0^t 4L_1 s^2 |x - y| ds \\
\leq 4L_1 \frac{r^3}{3} |x - y| 
\]

The remaining cases for \(x, y \in [0, 1]\) follow by using the triangle inequality for the norm \(L^2([0, T] \times [0, 1], \lambda^1 \times \mu)\), whereby this works analogously for \(b = D\). Consequently, for all \((x, y) \in [0, 1]\)

\[
\left| \int_0^t \int_0^1 (P_b(t - s, x, z) - P_b(t - s, y, z))^2 d\mu(z) ds \right|^{q/2} \leq 3 \cdot 2L_1^2 t^2 |x - y|^{1/2}. 
\]
We conclude
\[
\mathbb{E}(|v_1(t, x) - v_1(t, y)|^q) \leq 3^q 2^{2q-1} T^{2q} C_q L_1^{2q} \left( \|M\|_{q,T}^q + L^q \|v_0\|_{q,T}^q \right) |x - y|^{q/2}.
\]
This proves the spacial estimate.

We now turn to the temporal estimate, where we adapt ideas from [18, Proposition 4.3]. Let \(s, t \in [0, T]\) with \(s < t\) and \(x \in [0, 1]\) be fixed. Then, by using the Burkholder-Davis-Gundy inequality, Minkowski’s integral inequality and inequality (36), we get
\[
\begin{align*}
\mathbb{E}(|v_1(t, x) - v_1(s, x)|^q) \\
&\leq C_q \int_0^t \int_0^1 \left| (P_b(t-u,x,y) - P_b(s-u,x,y) 1_{[0,s]}(u))^2 \mathbb{E}(f(s,v_0(s,y))^q) \right|^{\frac{q}{2}} d\mu(y)du \\
&\leq 2^{q-1} C_q \left( \|M\|_{q,T}^q + L^q \|v_0\|_{q,T}^q \right) \int_0^t \int_0^1 \left( P_b(t-u,x,y) - P_b(s-u,x,y) 1_{[0,s]}(u) \right)^2 d\mu(y)du \\
&= \int_0^t \int_0^1 \left( P_b(t-u,x,y) - P_b(s-u,x,y) \right)^2 d\mu(y)du.
\end{align*}
\]
We split the above integral in the time intervals \([0, s]\) and \((s, t]\) and consider the first part,
\[
\int_0^s \int_0^1 \left( P_b(t-u,x,y) - P_b(s-u,x,y) \right)^2 d\mu(y)du
\]
By Lemma 3.4,
\[
\begin{align*}
&\left( \int_0^1 (P_b(t-u,x,y) - P_b(s-u,x,y))^2 d\mu(y) \right)^{\frac{1}{2}} \\
&- \left( \int_0^1 (\langle P_b(t-u,\cdot,y) - P_b(s-u,\cdot,y), f_n^x \rangle)^2 d\mu(y) \right)^{\frac{1}{2}} \\
&\leq \left( \int_0^1 (P_b(t-u,x,y) - P_b(s-u,x,y) - \langle P_b(t-u,\cdot,y) - P_b(s-u,\cdot,y), f_n^x \rangle)^2 d\mu(y) \right)^{\frac{1}{2}} \\
&\leq \left( \int_0^1 (\langle P_b(t-u,\cdot,y), f_n^x \rangle - P_b(t-u,x,y))^2 d\mu(y) \right)^{\frac{1}{2}} \\
&+ \left( \int_0^1 (\langle P_b(s-u,\cdot,y), f_n^x \rangle - P_b(s-u,x,y))^2 d\mu(y) \right)^{\frac{1}{2}} \\
&\leq C_6^\frac{1}{6} (t + s - 2u)^{\frac{2}{3}} r_{\max}^n.
\end{align*}
\]
By resorting and squaring,
\[
\begin{align*}
&\int_0^1 (P_b(t-u,x,y) - P_b(s-u,x,y))^2 d\mu(y) \\
&\leq 2 \int_0^1 (\langle P_b(t-u,\cdot,y) - P_b(s-u,\cdot,y), f_n^x \rangle)^2 d\mu(y) + 2C_6(t + s - 2u)^2 r_{\max}^n
\end{align*}
\]
and by integration,
\[
\begin{align*}
&\int_0^s \int_0^1 (P_b(t-u,x,y) - P_b(s-u,x,y))^2 d\mu(y)ds \\
&\leq 2 \int_0^s \int_0^1 (\langle P_b(t-u,\cdot,y) - P_b(s-u,\cdot,y), f_n^x \rangle)^2 d\mu(y) + C_6(t + s - 2u)^2 r_{\max}^n du \\
&= 2 \int_0^s \int_0^1 (\langle P_b(t-u,\cdot,y) - P_b(s-u,\cdot,y), f_n^x \rangle)^2 d\mu(y)du + \frac{4}{6} C_6 ((t + s)^3 - (t - s)^3) r_{\max}^n.
\end{align*}
\]
21
Now, let \( b = N \). We consider the first term on the right-hand side of the last equality. Applying the Cauchy-Schwarz inequality,

\[
\left| \int_0^1 \int_0^1 ((P_N(t-u,z,y) - P_N(s-u,z,y), f_n^x(z)))^2 d\mu(y)du \right|
\leq \| f_n^x \|_\mu^2 \int_0^1 (t-s)^2 + \sum_{k=2}^{\infty} \left( \frac{\sin(\sqrt{\lambda_k^N}(t-u)) - \sin(\sqrt{\lambda_k^N}(s-u))}{\lambda_k^N} \right)^2 (\varphi_k^N)^2(y) d\mu(y)du.
\]

Since \( \| \varphi_k^N \|_\mu = 1 \),

\[
\int_0^1 (t-s)^2 + \sum_{k=2}^{\infty} \frac{1}{\lambda_k^N} \int_0^s \left( \sin\left( \sqrt{\lambda_k^N}(t-u) \right) - \sin\left( \sqrt{\lambda_k^N}(s-u) \right) \right)^2 \frac{du}{\lambda_k^N}
= s(t-s)^2 + \sum_{k=2}^{\infty} \frac{1}{\lambda_k^N} \int_0^s \left( \sin\left( \sqrt{\lambda_k^N}(t-u) \right) - \sin\left( \sqrt{\lambda_k^N}(s-u) \right) \right)^2 \frac{du}{\lambda_k^N}
= s(t-s)^2 + \sum_{k=2}^{\infty} \frac{1}{\lambda_k^N} \int_0^s \left( \sin\left( \sqrt{\lambda_k^N}(t-s+u) \right) - \sin\left( \sqrt{\lambda_k^N}(u) \right) \right)^2 \frac{du}{\lambda_k^N}
= s(t-s)^2 + \sum_{k=2}^{\infty} \frac{1}{\lambda_k^N} \frac{\sqrt{\lambda_k^Ns}}{(\lambda_k^N)^{\frac{3}{2}}} \left( \sin(t-s+u) - \sin(u) \right)^2 \frac{du}{\lambda_k^N}
= s(t-s)^2 + \sum_{k=2}^{\infty} \frac{1}{(\lambda_k^N)^{\frac{3}{2}}} \frac{\sin^2\left( \frac{t-s}{2} \right)}{\lambda_k^N} \left( \sin(t-s+2\sqrt{\lambda_k^Ns}) - \sin(t-s) + 2\sqrt{\lambda_k^Ns} \right)
\leq (t-s)^2 + (2 + 2T) \sum_{k=2}^{\infty} \frac{1}{\lambda_k^N} \frac{\sin^2\left( \frac{t-s}{2} \right)}{\lambda_k^N}
= T(t-s)^2 + \left( \frac{1}{2} + \frac{1}{2} T \right) \sum_{k=2}^{\infty} \frac{1}{\lambda_k^N} (t-s)^2,
\]

whereby \( \sum_{k=2}^{\infty} \frac{1}{\lambda_k^N} < \infty \) since \( \gamma < \frac{1}{2} \). We turn to the second part and get analogous to the first part

\[
\int_s^t \int_0^1 (P_N(t-u,x,y))^2 d\mu(y)du
\leq 2 \int_s^t \int_0^1 ((P_N(t-u,\cdot,y), f_n^x(\cdot)))^2 d\mu(y)du + 2C_0(t-u)^2 r_{\text{max}}^n du
= 2 \int_s^t \int_0^1 ((P_N(t-u,\cdot,y), f_n^x(\cdot)))^2 d\mu(y)du + \frac{2}{3} C_0(t-3)^3 r_{\text{max}}^n.
\]

Again, we give an upper bound for the integral term.

\[
\int_s^t \int_0^1 ((P_N(t-u,\cdot,y), f_n^x(\cdot)))^2 d\mu(y)du
\leq \| f_n^x \|_\mu^2 \int_s^t (t-u)^2 + \sum_{k=2}^{\infty} \frac{\sin^2\left( \sqrt{\lambda_k^N}(t-u) \right)}{\lambda_k^N} (\varphi_k^N)^2(y) d\mu(y)du.
\]
With similar methods as above,

\[
\int_s^t \int_0^1 (t-u) \, (u) \, \mu(y) \, du = \frac{(t-s)^3}{3} + \sum_{k=2}^{\infty} \int_s^t \sin^2 \left( \sqrt{\lambda_k^N} \frac{t-u}{\lambda_k^N} \right) \, \lambda_k^N \, du.
\]

Consequently, there exists \( C > 0 \) and \( C' > 0 \) such that for all \( t, s \in [0, T] \), \( x \in F \), \( n \in \mathbb{N} \)

\[
\left( \int_0^t \int_0^1 (P_N(t-u,x,y) - P_N(s-u,x,y)1_{[s,u]}(u))^2 \, \mu(y) \, du \right) \leq C(t-s)^2 r_{\min}^{-n} \nu_{\min}^{-n} + C' r_{\max}^{-n}
\]

where \( C'' \) := max \( \{ C', C(t-s)^2 (d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})}) \} \). In order to find the minimum in \( n \), we define

\[
f(y) := C(t-s)^2 e^{y \log \left( \frac{1}{r_{\max}} \right)} \left( d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})} \right) + C'' e^{-\log \left( \frac{1}{r_{\max}} \right)} \nu_{\min}^{-n}.
\]

We differentiate:

\[
f'(y) = C(t-s)^2 e^{y \log \left( \frac{1}{r_{\max}} \right)} \left( d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})} \right) \log \left( \frac{1}{r_{\max}} \right) e^{-\log \left( \frac{1}{r_{\max}} \right)} \nu_{\min}^{-n} - C'' \log \left( \frac{1}{r_{\max}} \right) e^{-\log \left( \frac{1}{r_{\max}} \right)} \nu_{\min}^{-n}.
\]

Setting zero we get

\[
e^{y \log \left( \frac{1}{r_{\max}} \right)} \left( d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})} + 1 \right)
\]

\[
= \frac{C'' \log \left( \frac{1}{r_{\max}} \right)}{C(t-s)^2 \log \left( \frac{1}{r_{\max}} \right) \left( d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})} \right)} = \frac{C''}{C(t-s)^2 \left( d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})} \right)}.
\]

By logarithmising we obtain

\[
y \log \left( \frac{1}{r_{\max}} \right) \left( d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})} + 1 \right) = \log \left( \frac{C''}{C(t-s)^2 \left( d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})} \right)} \right).
\]
Solving this equation for \( y \) we get

\[
y = \log \left( \frac{1}{\tau_{\max}} \right) \left( d_H + \log(\tau_{\min}) \log(\tau_{\max}) + 1 \right) \log \left( \frac{C''}{C(t-s)^2 \left( d_H + \log(\tau_{\min}) \log(\tau_{\max}) \right)} \right),
\]

which we denote by \( y_0 \). This value does not need to be an integer, but there exists an integer \( n \) with \( n \in [y_0, y_0 + 1) \). Since \( y_0 \) is the unique minimum on \( \mathbb{R} \), \( f \) is increasing on \([y_0, \infty)\). Hence, there exists \( C'' \) such that

\[
\int_0^t \int_0^1 (P_N(t-u, x, y) - P_N(s-u, x, y) \mathbb{1}_{[a,s]}(u))^2 d\mu(y) du \\
\leq f(y_0 + 1)
\]

\[
= C(t-s)^2 \left( \frac{1}{\tau_{\max}} \right) \left( d_H + \log(\tau_{\min}) \log(\tau_{\max}) \right) \log \left( \frac{C''}{C(t-s)^2 \left( d_H + \log(\tau_{\min}) \log(\tau_{\max}) \right)} \right) \frac{d_H + \log(\tau_{\min}) \log(\tau_{\max})}{d_H + \log(\tau_{\min}) \log(\tau_{\max})} - 1
\]

\[
+ C'' \left( \frac{1}{\tau_{\max}} \right)^{-\frac{1}{2}} \left( \frac{1}{\tau_{\max}} \right) \log \left( \frac{C''}{C(t-s)^2 \left( d_H + \log(\tau_{\min}) \log(\tau_{\max}) \right)} \right) \frac{d_H + \log(\tau_{\min}) \log(\tau_{\max})}{d_H + \log(\tau_{\min}) \log(\tau_{\max})} - 1
\]

\[
= C(t-s)^2 \left( \frac{1}{\tau_{\max}} \right) \left( d_H + \log(\tau_{\min}) \log(\tau_{\max}) \right) \left( \frac{C''}{C(t-s)^2 \left( d_H + \log(\tau_{\min}) \log(\tau_{\max}) \right)} \right) \frac{d_H + \log(\tau_{\min}) \log(\tau_{\max})}{d_H + \log(\tau_{\min}) \log(\tau_{\max})} - 1
\]

\[
+ C'' \left( \frac{1}{\tau_{\max}} \right)^{-\frac{1}{2}} \left( \frac{1}{\tau_{\max}} \right) \log \left( \frac{C''}{C(t-s)^2 \left( d_H + \log(\tau_{\min}) \log(\tau_{\max}) \right)} \right) \frac{d_H + \log(\tau_{\min}) \log(\tau_{\max})}{d_H + \log(\tau_{\min}) \log(\tau_{\max})} - 1
\]

\[
= C''(t-s)^2 \left( \frac{1}{\tau_{\max}} \right) \left( d_H + \log(\tau_{\min}) \log(\tau_{\max}) \right) \log \left( \frac{C''}{C(t-s)^2 \left( d_H + \log(\tau_{\min}) \log(\tau_{\max}) \right)} \right) \frac{d_H + \log(\tau_{\min}) \log(\tau_{\max})}{d_H + \log(\tau_{\min}) \log(\tau_{\max})} - 1
\]

The case \( b = D \) works similarly.

\[\square\]

**Corollary 3.8:** Let \( q \geq 2 \) and \( v_0 \in S_{q,T} \). Then, \( v_i, i = 1, 2, 3 \), defined as in (24)-(26) are elements of \( S_{q,T} \).

**Proof.** By setting \( s = 0 \) in Proposition 3.7 we obtain \( \|v_i\|_{q,T} < \infty \), \( i = 1, 2 \). We need to show that \( v_1 \) is predictable. For \( n \in \mathbb{N} \) let

\[
v^n(t, x) = \sum_{i,j=0}^{2^n-1} v_1 \left( \frac{i}{2^n} T, \frac{j}{2^n} \right) \mathbb{1}_{\left( \frac{i}{2^n} T, \frac{i+1}{2^n} \right]}(t) \mathbb{1}_{\left( \frac{j}{2^n}, \frac{j+1}{2^n} \right]}(x), \ (t, x) \in [0, T] \times [0, 1].
\]

It holds evidently \( \|v^n\|_{q,T} < \infty \). To prove that \( v^n \) is predictable, we show that \( v^n \) is the \( S_{q,T} \)-limit of a sequence of simple functions. To this end, let for \( N \geq 1 \)

\[
v^n(t, x) = v^n(t, x) \wedge N, \ t \in [0, T], \ x \in [0, 1].
\]

This defines a simple function since \( v_1 \left( \frac{i}{2^n} T, \frac{j}{2^n} \right) \wedge N \) is \( \mathcal{F}_{2^n} \)-measurable and bounded. It converges
in $S_{q,T}$ to $v^n_1$, which can be seen as follows:

$$
\lim_{N \to \infty} \sup_{t \in [0,T]} \sup_{x \in [0,1]} \left\| v^n_1(t, x) - v^n_1(t, x) \right\|_{L^q(\Omega)} \\
\leq \lim_{N \to \infty} \sup_{t \in [0,T]} \sup_{x \in [0,1]} \left( \sum_{i,j=0}^{2^n-1} \left\| v_1 \left( \frac{i}{2^n T}, \frac{j}{2^n} \right) - v_1 \left( \frac{i}{2^n T}, \frac{j}{2^n} \right) \right\|_{L^q(\Omega)} \right) \\
= \lim_{N \to \infty} \sum_{i,j=0}^{2^n-1} \left\| v_1 \left( \frac{i}{2^n T}, \frac{j}{2^n} \right) - v_1 \left( \frac{i}{2^n T}, \frac{j}{2^n} \right) \right\|_{L^q(\Omega)} \\
= 0,
$$

where the last equation follows from the monotone convergence theorem. We conclude that $v^n_1$ is predictable for $n \in \mathbb{N}$. By Proposition 3.7, there exists a constant $C'_8$ such that

$$
\| v_1 - v^n_1 \|_{q,T} \leq \sup_{|s-t| < \frac{1}{n}, |x-y| < \frac{1}{n}} \| v_1(s, x) - v_1(t, y) \|_{L^q(\Omega)} \\
\leq \sup_{|s-t| < \frac{1}{n}, |x-y| < \frac{1}{n}} \| v_1(s, x) - v_1(t, x) \|_{L^q(\Omega)} \\
+ \sup_{|s-t| < \frac{1}{n}, |x-y| < \frac{1}{n}} \| v_1(t, x) - v_1(t, y) \|_{L^q(\Omega)} \\
\leq C'_8 \left( \left( \frac{T}{n} \right)^{\frac{1}{2} - \frac{2d}{4}} + \left( \frac{1}{n} \right)^{\frac{1}{2}} \right) \to 0, \ n \to \infty.
$$

Hence, $v_1$ is predictable. The predictability of $v_2$ and $v_3$ follows from the fact that they are measurable and deterministic.

**Theorem 3.9:** Assume Condition 1.1 with $q \geq 2$. Then the SPDE (22) has a unique mild solution in $S_{q,T}$.

**Proof. Uniqueness:** For that, let $u, \tilde{u} \in S_{q,T}$ be mild solution of (22). Then $v := u - \tilde{u} \in S_{2,T}$. With $G(t) := \sup_{x \in [0,1]} \mathbb{E} \left[ v^2(t, x) \right]$ and by using Walsh’s isometry, we calculate for $(t, x) \in [0, T] \times [0, 1]$

$$
\mathbb{E} \left[ v(t, x)^2 \right] = \mathbb{E} \left[ \left( \int_0^t \int_0^1 P_b(t - s, x, y) (f(s, u(s, y)) - f(s, \tilde{u}(s, y))) \xi(s, y) d\mu(y) ds \right)^2 \right] \\
= \mathbb{E} \left[ \int_0^t \int_0^1 P_b(t - s, x, y)^2 (f(s, u(s, y)) - f(s, \tilde{u}(s, y)))^2 d\mu(y) ds \right] \\
\leq L^2 \mathbb{E} \left[ \int_0^t \int_0^1 v^2(s, y) P_b^2(t - s, x, y) d\mu(y) ds \right] \\
\leq L^2 \left[ \sup_{y \in [0,1]} \mathbb{E} \left[ v^2(s, y) \right] \int_0^1 P_b^2(t - s, x, y) d\mu(y) ds \right] \\
\leq L^2 \sup_{t \in [0, T]} \| P_b(t, x, \cdot) \|_\mu^2 \int_0^t \sup_{y \in [0,1]} \mathbb{E} \left[ v^2(s, x) \right] ds \\
\leq L^2 \sup_{t \in [0, T]} \| P_b(t, x, \cdot) \|_\mu^2 \int_0^t G(s) ds.
$$

It follows

$$
G(t) \leq L^2 \sup_{t \in [0, T]} \| P_b(t, x, \cdot) \|_\mu \int_0^t G(s) ds.
$$
Since $G$ is continuous on $[0, T]$ (use Proposition 3.7 by setting $v_0 = v$), we can use Gronwall’s lemma to conclude $G(s) = 0$ for $s \in [0, T]$ and thus $u(t, x) = \bar{u}(t, x)$ almost surely for every $(t, x) \in [0, T] \times [0, 1]$.

Existence: We follow the methods in [16, Theorem 7.5] and use Picard iteration to find a solution. For that, let $u_2 = 0 \in \mathcal{S}_{q, T}$ and for $n \geq 2$

$$u_{n+1}(t, x) = \int_0^1 \frac{\partial}{\partial t} P_b(t, x, y) u_0(y) d\mu(y) + \int_0^1 P_b(t, x, y) u_n(y) d\mu(y) + \int_0^t \int_0^1 P_b(t-s, x, y) f(s, u_n(s, y)) \xi(s, y) d\mu(y) ds.$$  \hspace{1cm} (37)

$$u_{n+1}(t, x) = \int_0^1 P_b(t, x, y) u_n(y) d\mu(y) + \int_0^t \int_0^1 P_b(t-s, x, y) f(s, u_n(s, y)) \xi(s, y) d\mu(y) ds.$$  \hspace{1cm} (38)

From Proposition 3.6 and 3.7 it follows that $u_n \in \mathcal{S}_{q, T}$ for every $n \geq 2$. We prove that $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_{q, T}$. Let $w_n = u_{n+1} - u_n \in \mathcal{S}_{q, T}$. By using the Burkholder-Davis-Gundy inequality, the Lipschitz property of $f$ as well as Minkowski’s integral inequality we get

$$\mathbb{E}[w_{n+1}(t, x)^q]$$
$$\leq C_q \mathbb{E} \left[ \left| \left( \int_0^1 P_b(t, x, y) f(s, u_{n+1}(s, y)) \xi(s, y) d\mu(y) ds \right)^q \right| \right]$$
$$\leq C_q \mathbb{E} \left[ \left| \left( \int_0^1 P_b(t, x, y) f(s, u_n(s, y)) \xi(s, y) d\mu(y) ds \right)^q \right| \right]$$
$$\leq C_q \mathbb{E} \left[ \left( \int_0^1 P_b(t, x, y) w_n^2(s, y) d\mu(y) ds \right)^{\frac{q}{2}} \right]$$
$$\leq C_q \mathbb{E} \left[ \left( \int_0^1 P_b(t, x, y) \mathbb{E}[|w_n(s, y)|^q] \right)^{\frac{q}{2}} d\mu(y) ds \right]$$
$$\leq C_q^2 \sup_{t \in [0, T]} \mathbb{E} \left[ |w_n(s, y)|^q \right] \left( \int_0^1 \sup_{x \in [0, 1]} \mathbb{E} \left[ |w_n(s, y)|^q \right] d\mu(y) ds \right)^{\frac{q}{2}}.$$

Set $H_n(t) = \sup_{x \in [0, 1]} \mathbb{E} \left[ |w_n(t, y)|^q \right]^{\frac{q}{2}}$ for $n \geq 2$, $t \in [0, T]$. Then for every $n \geq 2$ there exists a constant $\kappa_n$ such that $|H_n(t)| \leq \kappa_n$ for every $t \in [0, T]$. With Proposition 2.10 it follows for $(t, x) \in [0, T] \times [0, 1]$

$$\left( \mathbb{E}[w_{n+1}(t, x)^q] \right)^{\frac{1}{2}} \leq C_q^2 \sup_{t \in [0, T]} \mathbb{E} \left[ |P(t, x, \cdot)|^q \right]^{\frac{2}{3}} \left( \int_0^1 \sup_{x \in [0, 1]} \mathbb{E} \left[ |w_n(s, y)|^q \right] d\mu(y) ds \right)^{\frac{q}{2}}.$$

and thus

$$H_{n+1}(t) \leq C_q^2 \sup_{t \in [0, T]} \mathbb{E} \left[ |P(t, x, \cdot)|^q \right]^{\frac{2}{3}} \left( \int_0^1 \sup_{x \in [0, 1]} \mathbb{E} \left[ |w_n(s, y)|^q \right] d\mu(y) ds \right)^{\frac{q}{2}}.$$

With $\kappa := C_q^2 \sup_{t \in [0, T]} \mathbb{E} \left[ |P(t, x, \cdot)|^q \right]^{\frac{2}{3}}$ we see that $H_2(t) \leq \kappa_2 t$ and deduce inductively

$$H_{n+2}(t) \leq \kappa_2 \frac{(kt)^n}{n!}, \quad n \geq 1.$$

The series $\sum_{n \geq 3} H_{n+1}^\frac{1}{2}(t)$ is uniformly convergent on $[0, T]$, which can be verified by the ratio test using that $\sqrt{\frac{H_{n+1}(t)}{H_n(t)}} \leq \sqrt{\frac{\alpha t}{n+1}}$ for $n \geq 2$. We conclude

$$\sup_{t \in [0, T]} \sqrt{H_n(t)} \to 0, \quad n \to \infty,$$

which implies the same for $\|w_n\|_{q, T}$. Hence, $(u_n)_{n \geq 2}$ is Cauchy in $\mathcal{S}_{q, T}$ and we denote the limit by $u$. To verify that $u$ satisfies (23) we take the limit in $L^2(\Omega)$ for $n \to \infty$ on both sides of (38) for
every \((t, x) \in [0, T] \times [0, 1]\). We get \(u(t, x)\) on the left-hand side for any \((t, x) \in [0, T] \times [0, 1]\). For the right-hand side we note that for \((t, x) \in [0, T] \times [0, 1]\)

\[
\mathbb{E} \left[ \int_0^t \int_0^1 P_b(t-s, x, y) \left( f(s, u(s, y)) - f(s, u_n(s, y)) \right) \xi(s,y) d\mu(y) ds \right]^q
\leq C_q L^q \left( \int_0^t \int_0^1 P_b^2(t-s, x, y) \left( \mathbb{E} \left[ |u(s, y) - u_n(s, y)|^q \right] \right)^{\frac{q}{2}} d\mu(y) ds \right)^{\frac{q}{2}},
\]

which goes to zero as \(n\) tends to infinity with the same argumentation as before. \(\square\)

We have computed different temporal Hölder exponents. The following lemma shows which one is greater.

**Lemma 3.10:** Let \(r_1, \ldots, r_N\) and \(\mu_1, \ldots, \mu_N\) be arbitrary, but chosen according to the conditions given in section 1. Then,

\[
\left( d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}} \right)^{-1} \leq 2 - (2 + \delta) \gamma.
\]

**Proof.** We have

\[
\begin{align*}
\min_{i=1,\ldots,N} & \log \mu_i - \log r_i^{d_H} + d_H + 1 \\
\max_{i=1,\ldots,N} & \log r_i \\
= & \min_{i=1,\ldots,N} \log \mu_i - \log r_i^{d_H} - (1 - d_H) + 2 \\
\geq & \max_{i=1,\ldots,N} \log \mu_i - \log r_i^{d_H} - (1 - d_H) + 2 \\
= & \max_{i=1,\ldots,N} \log \mu_i - \log r_i + (1 - d_H) \log r_i - (1 - d_H) + 2 \\
= & \max_{i=1,\ldots,N} \log \mu_i - \log r_i + 2.
\end{align*}
\]

Using that as well as the fact that \(\gamma < \frac{1}{2}\),

\[
\begin{align*}
\left( d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}} \right)^{-1} & \leq \left( \max_{i=1,\ldots,N} \frac{\log \mu_i - \log r_i}{\log r_i} + 2 \right)^{-1} \\
= & \min_{i=1,\ldots,N} \left( \frac{\log \mu_i - \log r_i}{\log r_i} + 2 \right)^{-1} \\
= & \min_{i=1,\ldots,N} \frac{\log \mu_i + \log r_i}{\log r_i} \\
= & \min_{i=1,\ldots,N} \left( 1 - \frac{\log \mu_i}{\log \mu_i + \log r_i} \right) \\
= & \left( 1 - \max_{i=1,\ldots,N} \frac{\log \mu_i}{\log \mu_i + \log r_i} \right) \\
= & (1 - \gamma \delta) \\
< & 2 - 2\gamma - \gamma \delta.
\end{align*}
\]

\(\square\)

Using this lemma and the established continuity properties (compare Proposition 3.6 and Proposition 3.7), the main result of this paper, Theorem 1.2, is a direct consequence of Kolmogorov’s continuity theorem.
Example 3.11: We discussed the case of $\mu$ being the natural on a given Cantor-like set in Section 1. If $\mu$ is not the natural measure on a given Cantor-like set, then $\nu_{\text{min}}$ does not vanish. As an example, consider the Classical Cantor set with weights $\mu_1, \mu_2 \in (0, 1)$. If $u_0, u_1$ and $f$ satisfy Assumption 1.1 and $f$ is uniformly bounded, $q$ can be chosen arbitrarily large. We then have as ess. temporal Hölder exponent

$$d_H + 1 + \frac{1}{\log(\nu_{\text{min}})} = \frac{1}{\log 2} + 1 - \frac{1}{\log 3} \log(\nu_{\text{min}}) + \frac{1}{\log 3}$$

provided that $\delta + 1 < \frac{1}{\gamma}$. This is satisfied if

$$\max_{i=1,2} \frac{\log \mu_i}{\log 3} + \frac{\log 2}{\log 6} < 1,$n

which holds for $\mu_1, \mu_2$ such that $\min_{i=1,2} \mu_i > 0.18$. The behaviour of the temporal Hölder exponent is visualized on the right-hand side of Figure 2.

3.3 Intermittency

Let $\varepsilon \geq 0$. According to [4] we call the mild solution of a stochastic wave equation $u$ weakly intermittent on $[\varepsilon, 1 - \varepsilon]$ if for the upper moment Lyapunov exponents

$$\bar{\gamma}(p, x) := \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[|u(t, x)|^p]$$

it holds

$$\bar{\gamma}(2, x) > 0, \quad \bar{\gamma}(p, x) < \infty, \quad x \in [\varepsilon, 1 - \varepsilon], \quad p \in [2, \infty).$$

In this section we make the following additional assumption:

Assumption 3.12: We assume Condition 1.1 with $q \geq 2$ and that $f$ fulfills the following Lipschitz and linear growth condition: For all $(w, t, x) \in \Omega \times [0, T] \times \mathbb{R}$ there exists a constant $L > 0$ such that

$$|f(\omega, t, x) - f(\omega, t, y)| \leq L|x - y|,$n

$$|f(\omega, t, y)| \leq L(1 + |x|).$$

Proposition 3.13: Let $p \geq 1$. Then there exists constants $C_9, C_{10} > 0$ such that for $(t, x) \in [0, \infty) \times [0, 1]$

$$\mathbb{E}[|u(t, x)|^p] \leq C_9 e^{C_{10} t^2}.$$
Proof. \( v_2 \) and \( v_3 \) are uniformly bounded on \([0, \infty) \times [0,1]\). This can be verified with the same methods as in the proof of Proposition 3.7. For example, for \( b = D \) and \((t,x) \in [0,\infty) \times [0,1]\)

\[
\sum_{k=1}^{\infty} \sin \left( \sqrt{\lambda_k^D} t \right) \frac{\varphi_k^D (x)}{\sqrt{\lambda_k^D}} u_{1,k}^D \leq \sum_{k=1}^{\infty} C_0^{-\frac{1}{2}} C_2 C_5 k^{-\frac{1}{2}} k^{-\frac{1}{2}} \leq C_0^{-\frac{1}{2}} C_2 C_5 \sum_{k=1}^{\infty} k^{-\frac{1}{2}} \frac{1}{\sqrt{t}} < \infty.
\]

Hence, there exists a constant \( K > 0 \) such that for \( i \in \{2,3\} \), \((t,x) \in [0,\infty) \times [0,1]\) \( v_i(t,x) \leq K, i = 2, 3 \). It follows by using the Burkholder-Davis-Gundy inequality as well as Minkowski’s integral inequality,

\[
e^{-\alpha t} \left( \mathbb{E} \left| u(t,x) \right|^p \right)^{\frac{1}{p}} \leq e^{-\alpha t} 2K + 2\sqrt{p} \left( \mathbb{E} \left[ \left( \int_0^t \int_0^1 e^{-\alpha t} P_b(t-s,x,y) f(s,u(s,y)) \xi(s,y) d\mu(y) ds \right)^p \right] \right)^{\frac{1}{p}} \]

\[
\leq e^{-\alpha t} 2K + 2\sqrt{p} \left( \int_0^t \int_0^1 e^{-2\alpha t} P_b^2(t-s,x,y) \mathbb{E} \left| f(s,u(s,y)) \right|^{p/2} d\mu(y) ds \right)^{\frac{1}{p}} \]

\[
\leq e^{-\alpha t} 2K + e^{-2\alpha t} 2K \left( \int_0^t \int_0^1 e^{-2\alpha(t-s)} P_b^2(t-s,x,y) d\mu(y) ds \right)^{\frac{1}{p}} \]

\[
\leq e^{-\alpha t} 2K + \frac{C_2^2 L^2 \sqrt{p}}{2\alpha} \left( \int_0^t \int_0^1 e^{-2\alpha(t-s)} ds \right)^{\frac{1}{p}} \]

Choose \( \alpha = 8C_3^4 L^2 p \). Then it follows

\[
(\mathbb{E} \left| u(t,x) \right|^p) \frac{1}{p} \leq 4K + e^{\alpha t} = 4K + e^{8C_3^4 L^2 pt}.
\]

For \( p \in [1,2) \) we have for \((t,x) \in [0,\infty) \times [0,1]\)

\[
(\mathbb{E} \left| u(t,x) \right|^p) \frac{1}{p} \leq \left( \mathbb{E} \left( \left| u(t,x) \right|^2 \right)^2 \right)^{\frac{1}{2}} \leq 4K + e^{16C_3^4 L^2 t} \leq 4K + e^{16C_3^4 L^2 pt}.
\]

From the above proposition, it follows immediately for \( p \geq 1 \)

\[
\tilde{\gamma}(p) = \limsup_{t \to \infty} \frac{1}{t} \sup_{x \in [0,1]} \log \mathbb{E} \left| u(t,x) \right|^p \leq \limsup_{t \to \infty} \frac{\log C_0}{t} + C_{10} p^2 = C_{10} p^2.
\]
Proposition 3.14: Assume \( \inf_{x \in [0,1]} |f(x)/x| > 0 \).

1. Let \( b = N, \inf_{x \in [0,1]} u_0(x) > 0 \) and \( \inf_{x \in [0,1]} u_1(x) > 0 \). Then, there exists a constant \( \kappa \) such that \( \gamma(2, x) \geq \kappa \) for all \( x \in [0, 1] \).

2. Let \( b = D, \varepsilon > 0, \inf_{x \in [\varepsilon, 1-\varepsilon]} u_0(x) > 0 \) and \( \inf_{x \in [\varepsilon, 1-\varepsilon]} u_1(x) > 0 \). Then, there exists a constant \( \kappa_\varepsilon \) such that \( \gamma(2, x) \geq \kappa_\varepsilon \) for all \( x \in [\varepsilon, 1-\varepsilon] \).

Proof. Let \( \varepsilon > 0, \inf_{x \in [\varepsilon, 1-\varepsilon]} u_0(x) > 0, \inf_{x \in [\varepsilon, 1-\varepsilon]} u_1(x) > 0 \) and \( x \in [\varepsilon, 1-\varepsilon] \). It suffices to find a constant \( \beta_\varepsilon > 0 \)

\[
\int_0^\infty e^{-\beta t} \mathbb{E} [u(t, x)^2] \, dt = \infty \quad \text{for all} \quad \beta \leq \beta_\varepsilon
\]  

(39)

(see the proof of [4, Theorem 3.3]). By using Walsh’s isometry and the zero-mean property of the stochastic integral we get

\[
\mathbb{E} [u(t, x)^2] \, dt = (v_2(t, x) + v_3(t, x))^2 + \int_0^t \int_0^1 P_b(t-s, x, y) \mathbb{E} [f(u(s, y))^2] \, d\mu(y) \, ds
\]

\[
+ (v_2(t, x) + v_3(t, x))^2 \mathbb{E} \left[ \int_0^t \int_0^1 P_b(t-s, x, y) f(u(s, y)) \xi(s, y) \, d\mu(y) \, ds \right]
\]

\[
= (v_2(t, x) + v_3(t, x))^2 + \int_0^t \int_0^1 P_b^2(t-s, x, y) \mathbb{E} [f(u(s, y))^2] \, d\mu(y) \, ds
\]

and thus, by Laplace transformation,

\[
\int_0^\infty e^{-\beta t} \mathbb{E} [u(t, x)^2] \, dt = \int_0^\infty e^{-\beta t} (v_2(t, x) + v_3(t, x))^2 \, dt + \int_0^\infty e^{-\beta t} \int_0^t \int_0^1 P_b^2(t-s, x, y) \mathbb{E} [f(u(s, y))^2] \, d\mu(y) \, ds \, dt.
\]

In order to bound the first term on the right-hand side from below, we note that \( v_2(0, x) = u_1(x) \geq \inf_{x \in [\varepsilon, 1-\varepsilon]} u_1(x) > 0 \) and \( v_3(0, x) = u_0(x) \geq \inf_{x \in [\varepsilon, 1-\varepsilon]} u_0(x) > 0 \). Using that both functions are Hölder-continuous in \( t \) uniformly for all \( x \in [0, 1] \) (see Proposition 3.6), we obtain the existence of a constant \( t_\varepsilon > 0 \) such that

\[
v_3(t, x) > \frac{u_0}{2}, \quad v_2(t, x) > \frac{u_0}{4}, \quad t \in [0, t_\varepsilon].
\]

We see that \( v_2(t, x) + v_3(t, x) > \frac{u_0}{4} \) for all \( (t, x) \in [0, t_\varepsilon] \times [\varepsilon, 1-\varepsilon] \). It follows that for all \( \beta > 0 \) there exists a constant \( K_{\beta, \varepsilon} \) such that

\[
\int_0^\infty e^{-\beta t} \mathbb{E} [u(t, x)^2] \, dt \geq K_{\beta, \varepsilon} + L_\varepsilon^2 \int_0^\infty e^{-\beta t} \int_0^t \int_0^1 P_b^2(t-s, x, y) \mathbb{E} [(u(s, y))^2] \, d\mu(y) \, ds \, dt,
\]

where \( K_{\beta, \varepsilon} = \frac{u_0^2}{16\varepsilon^2} \). Further, for \( (x, y, t) \in [0, 1]^2 \times [0, \infty) \)

\[
\int_0^t P_b^2(t-s, x, y) \mathbb{E} [(u(s, y))^2] \, d\mu(y) \, ds = (P_b(\cdot, x, y) * \mathbb{E} [u(\cdot, y)^2]) (t),
\]
where ∗ denotes the time convolution. It holds \( \mathcal{L}_\beta(f * g) = \mathcal{L}_\beta f \cdot \mathcal{L}_\beta g \), where \( \mathcal{L} \) denotes the Laplace transformation. We thus see

\[
\int_0^\infty e^{-\beta t} \int_0^t \int_0^1 P^2_b(t-s,x,y) E [(u(s,y)^2)] d\mu(y) ds dt \\
= \int_0^1 \int_0^\infty e^{-\beta t} \int_0^t P^2_b(t-s,x,y) E [(u(s,y)^2)] ds dt d\mu(y) \\
= \int_0^1 \int_0^\infty e^{-\beta t} P^2_b(t,x,y) dt \int_0^\infty e^{-\beta s} E [(u(s,y)^2)] ds d\mu(y).
\]

With \( M_\beta(x) := \int_0^\infty e^{-\beta s} E [(u(s,x)^2)] ds \) it follows

\[
M_\beta(x) \geq K_{\beta,\varepsilon} + L^2 \int_0^1 \int_0^\infty e^{-\beta t} P^2_b(t,x,y) M_\beta(y) dt d\mu(y).
\] (40)

If \( b = N \), it holds for all for all \( t \geq 0 \)

\[
\|P_N(t,x,\cdot)\|^2 = t^2 + \sum_{k \geq 2} \frac{\sin^2 \left( \frac{\sqrt{\lambda^N_k} t}{\lambda^N_k} \right)}{(\varphi^N_k)^2}(x)
\]

and thus

\[
\int_0^\infty \int_0^1 e^{-\beta t} P^2_N(t,x,y) K_{\beta,\varepsilon} d\mu(y) dt = K_{\beta,\varepsilon} \int_0^\infty e^{-\beta t} \|P_N(t,x,\cdot)\|^2 d\mu dt \\
\geq K_{\beta,\varepsilon} \int_0^\infty e^{-\beta t} t^2 dt \\
= 2K_{\beta,\varepsilon}\beta^{-3}.
\]

By iterating this in (40) we obtain

\[
M_\beta(x) \geq K_{\beta,\varepsilon} \sum_{n=0}^\infty (2\beta^{-3})^n.
\]

This sum diverges if and only if \( \beta \leq \sqrt[3]{2} \). Hence, we have shown (39).

If \( b = D \), we define \( c_\varepsilon := \inf_{x \in [\varepsilon,1-\varepsilon]} \varphi^D_1(x) \) and calculate

\[
\int_0^\infty \int_0^1 e^{-\beta t} P^2_D(t,x,y) K_{\beta,\varepsilon} d\mu(y) dt \geq K_{\beta,\varepsilon} \int_0^\infty e^{-\beta t} \sum_{k=1}^\infty \frac{\sin^2 \left( \frac{\sqrt{\lambda^D_k} t}{\lambda^D_k} \right)}{(\varphi^D_k)^2}(x) dt \\
\geq K_{\beta,\varepsilon} \int_0^\infty e^{-\beta t} \frac{\sin^2 \left( \frac{\sqrt{\lambda^D_1} t}{\lambda^D_1} \right)}{(\varphi^D_1)^2}(x) dt \\
\geq K_{\beta,\varepsilon} \int_0^\infty e^{-\beta t} \frac{\sin^2 \left( \frac{\sqrt{\lambda^D_1} t}{\lambda^D_1} \right)}{(\varphi^D_1)^2} c_\varepsilon dt \\
= \frac{K_{\beta,\varepsilon} c_\varepsilon}{(\lambda^D_1)^3} \int_0^\infty e^{-\frac{\beta}{\sqrt{\lambda^D_1}}} \sin^2(t) dt > 0 \\
= \frac{K_{\beta,\varepsilon} c_\varepsilon}{(\lambda^D_1)^3} \left( \frac{\beta}{\sqrt{\lambda^D_1}} \right)^3 + 4 \left( \frac{\beta}{\lambda^D_1} \right) > 0.
\]
The last term is strictly positive, since $\varphi_1^D(x) > 0$ for all $x \in (0, 1)$ (see [13, Proposition 2.5]) and therefore bounded from below for all $x \in [\varepsilon, 1 - \varepsilon]$. By iterating this in (40) we obtain

$$M_\beta(x) \geq K_{\beta, \varepsilon} \sum_{n=0}^{\infty} \left( \frac{2c_\varepsilon \left( \lambda_1^D \right)^{-\frac{3}{2}}}{\left( \frac{\beta}{\sqrt{\lambda_1^D}} \right)^3 + \left( \frac{\beta}{\sqrt{\lambda_1^D}} \right)^2} \right)^n.$$ 

Let $\tilde{\beta} := \frac{\beta}{\sqrt{\lambda_1^D}}$. The above sum is equal to $\infty$ for all $\beta$ such that $\tilde{\beta}^2 + 4\beta \leq 2c_\varepsilon \left( \lambda_1^D \right)^{-\frac{3}{2}}$. This verifies (39). \hfill \Box

A Some Technical Details

Lemma A.1: Let $b \in \{N, D\}$, $\psi : L^2([0, 1], \mu) \to L^2(\text{supp}(\mu), \mu)$, $u \to u|_{\text{supp}(\mu)}$ and

$$\tilde{\Delta}_\mu^b : \psi \left( D \left( \Delta_\mu^b \right) \right) \to L^2(\text{supp}(\mu), \mu), \quad u \to \psi \circ \Delta_\mu^b \circ \psi^{-1} u.$$ 

Then,

(i) $\tilde{\Delta}_\mu^b$ is self-adjoint, dissipative and has eigenvalues $\lambda_k^b$ with eigenfunctions $\psi \varphi_k^b$, $k \in \mathbb{N}$. In particular, $\tilde{\Delta}_\mu^b$ is the generator of a unique strongly continuous semigroup $\left( \tilde{T}_t^b \right)_{t \geq 0}$.

(ii) $\tilde{\mathcal{E}} (\tilde{u}, \tilde{v}) := \mathcal{E} (\psi^{-1} \tilde{u}, \psi^{-1} \tilde{v}), \quad \tilde{u}, \tilde{v} \in \tilde{\mathcal{F}} := \psi(\mathcal{F})$ defines a Dirichlet form which is associated to $\tilde{\Delta}_\mu^N$ and $\tilde{\mathcal{E}} (\tilde{u}, \tilde{v})$, $\tilde{u}, \tilde{v} \in \tilde{\mathcal{F}}_0 := \psi(\mathcal{F}_0)$ defines a Dirichlet form associated to $\tilde{\Delta}_\mu^D$.

Proof. (i) First, we show that $\tilde{\Delta}_\mu^b$ is self-adjoint. We denote the inner product on $L^2(\text{supp}(\mu), \mu)$ also by $(\cdot, \cdot)_{\mu}$. Since $D \left( \Delta_\mu^b \right)$ is dense in $L^2([0, 1], \mu)$, for any $u \in L^2([0, 1], \mu)$ there exists a sequence $(u_n)_{n \in \mathbb{N}}$ with $u_n \in D \left( \Delta_\mu^b \right)$, $n \in \mathbb{N}$ such that $\|u_n - u\|_{\mu} \to 0$ for $n \to \infty$. From $\|u_n - u\|_{\mu} = \|\psi u_n - \tilde{u}\|_{\mu}$ for all $n \in \mathbb{N}$ and $\psi u_n \in D \left( \tilde{\Delta}_\mu^b \right) = \psi \left( D \left( \Delta_\mu^b \right) \right)$ the density of $D \left( \tilde{\Delta}_\mu^b \right)$ in $L^2(\text{supp}(\mu), \mu)$ follows. Now, let $\tilde{u}, \tilde{v} \in D \left( \tilde{\Delta}_\mu^b \right) = \psi \left( D \left( \Delta_\mu^b \right) \right)$, i.e. there exist unique $u, v \in D \left( \Delta_\mu^b \right)$ such that $\tilde{u} = \psi u$, $\tilde{v} = \psi v$. It is straightforward to check that $v \to (u, \Delta_\mu^b v)_{\mu}$ is a linear continuous mapping on $D \left( \Delta_\mu^b \right)$ if and only if $\tilde{v} \to \langle \tilde{u}, \tilde{\Delta}_\mu^b \tilde{v} \rangle_{\mu}$ is linear and continuous on $D \left( \tilde{\Delta}_\mu^b \right)$, which yields $D \left( \tilde{\Delta}_\mu^b \right) = \psi \left( \left( \Delta_\mu^b \right)^* \right)$. Further, for all $\tilde{u}, \tilde{v} \in D \left( \tilde{\Delta}_\mu^b \right)$

$$\langle \tilde{\Delta}_\mu^b \tilde{u}, \tilde{v} \rangle_{\mu} = \langle \psi \circ \Delta_\mu^b \circ \psi^{-1} \circ \psi u, \psi v \rangle_{\mu} = \langle \psi \circ \Delta_\mu^b u, \psi v \rangle_{\mu} = \langle \Delta_\mu^b u, v \rangle_{\mu} = \langle u, \Delta_\mu^b v \rangle_{\mu} = \langle \psi u, \psi \circ \Delta_\mu^b \circ \psi^{-1} \circ \psi v \rangle_{\mu} = \langle \tilde{u}, \tilde{\Delta}_\mu^b \tilde{v} \rangle_{\mu}.$$
The self-adjointness of $\Delta^b_\mu$ follows. For the dissipativity of $\tilde{\Delta}^b_\mu$ we obtain from the dissipativity of $\Delta^b_\mu$
\[ \langle \tilde{\Delta}^b_\mu \tilde{u}, \tilde{u} \rangle_\mu = \langle \Delta^b_\mu u, u \rangle_\mu \leq 0. \]

The self-adjointness along with the dissipativity implies that $\tilde{\Delta}^b_\mu$ generates a strongly continuous semigroup $\left( \tilde{T}_t^\mu \right)_{t \geq 0}$ (see [20, Theorem B.2.2]). It remains to show that eigenvalues and eigenfunctions of $\tilde{\Delta}^b_\mu$ and $\Delta^b_\mu$ coincide. For that, let $\lambda < 0$, $u \in D(\tilde{\Delta}^b_\mu)$. The bijectivity of $\psi$ implies that $(\Delta^b_\mu - \lambda) u = 0$ if and only if $\psi(\Delta^b_\mu - \lambda) u = 0$. The results about eigenvalues and eigenfunctions follow.

(ii) Let $b = N$. Again, let let $\tilde{u}, \tilde{v} \in D(\tilde{\Delta}^N_\mu) = \psi(D(\Delta^N_\mu))$, i.e. there exist $u, v \in D(\Delta^N_\mu)$ with $\tilde{u} = \psi u$, $\tilde{v} = \psi v$. The density of $\tilde{F}$ in $L^2(\text{supp}(\mu), \mu)$ can be checked exactly like the density of $D(\tilde{\Delta}^N_\mu)$ in $\mathcal{H}$. Further, it is obvious that $\tilde{E}$ defines a positive definite, symmetric bilinear form. Moreover, with $\alpha > 0$ and $\tilde{E}_\alpha(\tilde{u}, \tilde{v}) := \tilde{E}(\tilde{u}, \tilde{v}) + \alpha \langle \tilde{u}, \tilde{v} \rangle_\mu$, $(\tilde{F}, \tilde{E}_\alpha)$ is a Hilbert space. To verify this, note that $\tilde{E}_\alpha(\tilde{u}, \tilde{v}) = E_\alpha(u, v)$, which implies that $\tilde{E}_\alpha$ defines an inner product. Now, let $\tilde{u}_n$, $n \in \mathbb{N}$ be a Cauchy sequence in $\tilde{F}$. Then, $u_n = \psi^{-1} \tilde{u}_n$, $n \in \mathbb{N}$ is a Cauchy sequence in $F$ with limit, say $u$. Since $\|\tilde{u}_n - \psi u\| = \|u_n - u\|$ for all $n$, $\psi u$ is the limit of $(\tilde{u}_n)_{n \in \mathbb{N}}$ in $\tilde{F}$. For the Markov property, we calculate
\[ \tilde{E}(0 \lor \tilde{u} \land 1) = E(0 \lor u \land 1) \leq E(u) = \tilde{E}(\tilde{u}). \]

To verify that $\tilde{\Delta}^N_\mu$ is associated to $\tilde{E}$, we apply the correspondence between $\Delta^N_\mu$ and $E$ to get for
\[ -\langle \tilde{\Delta}^N_\mu \tilde{u}, \tilde{v} \rangle_\mu = -\langle \Delta^N_\mu u, v \rangle_\mu = E(u, v) = \tilde{E}(\tilde{u}, \tilde{v}). \]

The case $b = D$ works similarly.

\[ \square \]

References

[1] P. Arzt, Eigenvalues of Measure Theoretic Laplacians on Cantor-like Sets, Dissertation, Universität Siegen, 2014.

[2] P. Arzt, Measure theoretic trigonometric functions, Journal of Fractal Geometry, 2:115-169, 2015.

[3] L. Bertini, N. Cancrini, The Stochastic Heat Equation: Feynman-Kac Formula and Intermittence, Journal of Statistical Physics, 78(5-6):1377-1401, 1995.

[4] D. Conus, M. Joseph, D. Khoshnevisan, S. Shiu, Intermittency and Chaos for a Non-linear Stochastic Wave Equation in Dimension 1, Malliavin Calculus and Stochastic Analysis, Springer Proceedings in Mathematics & Statistics, 34:251-279, Springer, Boston, 2013.

[5] J. F.-C. Chan, S.M. Ngai, A. Teplyaev, One-dimensional wave equations defined by fractal Laplacians, Journal d’Analyse Mathématique, 127(1):219-246, Springer, 2015.

[6] D. Conus D, D. Khoshnevisan, On the existence and position of the farthest peaks of a family of stochastic heat and wave equations, Probab. Theory Relat. Fields, 152(3):681-701, 2012.

[7] R.C. Dalang, C. Mueller, Intermittency properties in a hyperbolic Anderson problem, Annales de l’Institut Henri Poincaré, 45(4):1150-1164, 2009.
[8] K. Dalrymple, R. S. Strichartz, J. P. Vinson Fractal Differential Equations on the Sierpinski Gasket, *The Journal of Fourier Analysis and Applications*, 5(2-3):203-284, 1999.

[9] D. E. Edmunds, W. D. Evans, *Spectral Theory and Differential Operators*, Oxford University Press, Oxford, 1987.

[10] U. Freiberg, Analytical properties of measure geometric Krein-Feller-operators on the real line, *Mathematische Nachrichten*, 260:34-47, 2003.

[11] T. Fujita: A fractional dimension, self similarity and a generalized diffusion operator, *Probabilistic Methods on Mathematical Physics, Proceed. of Taniguchi Int. Symp. (Katata and Kyoto, 1985)*, 83-90, Kinokuniya, 1987.

Wiley and Sons, New York-London-Sydney, 1966.

[12] U. Freiberg, Dirichlet forms on fractal subsets of the real line, *Real Anal. Exchange*, 30(2):589–603, 2004/05.

[13] U. Freiberg, J. Löbus, Zeros of eigenfunctions of a class of generalized second order differential operators on the Cantor set, *Mathematische Nachrichten*, 265:3-14, 2004.

[14] J. E. Hutchinson, Fractals and Self Similarity, *Indiana Univ. Math. J.*, 30(5):713-747, 1981.

[15] Y. Hu, J. Huang, D. Nualart, S. Tindel, Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency, *Electron. J. Probab*, 20(55):1-50, 2015.

[16] B. Hambly, W. Yang, Continuous random field solutions to parabolic SPDEs on p.c.f. fractals, *arXiv:1709.00916v2*, 2018.

[17] B. Hambly, W. Yang, Existence and space-time regularity for stochastic heat equations on p.c.f. fractals, *Electron. J. Probab.*, 23(22):1-30, 2018.

[18] B. Hambly, W. Yang, The damped stochastic wave equation on p.c.f. fractals, *arXiv:1611.04874*, 2017.

[19] K. Itô, H. P. Jr. McKean, *Diffusion Processes and their Sample Paths*, Springer-Verlag, Berlin-Heidelberg-New York, 1965.

[20] J. Kigami, Analysis on Fractals, *Cambridge Tracts in Mathematics 143*, Cambridge University Press, Cambridge, 2001.

[21] D. Khoshnevisan, *Analysis of Stochastic Partial Differential Equations*, American Mathematical Society, 2014.

[22] D. Khoshnevisan, K. Kim, Y. Xiao, Intermittency and Multifractality: A case study via parabolic stochastic PDEs, *Ann. Probab.*, 45(6A):3697-3751, 2017.

[23] U. Küchler, Some Asymptotic Properties of the Transition Densities of One-Dimensional Quasidiffusions, *Publ. RIMS, Kyoto Univ.*, 16:245–268, 1980.

[24] U. Küchler, On sojourn times, excursions and spectral measures connected with quasidiffusions, *J. Math. Kyoto Univ.*, 26(3):403-421, 1986.

[25] J.-U. Löbus, Generalized second order differential operators, *Mathematische Nachrichten*, 152:229-245, 1991.

[26] L. Minorics, Spectral Asymptotics for Krein-Feller-Opeators with respect to V-Variable Cantor Measures, *arXiv:1808.06950*, 2018.
[27] M. Shinbrot, Asymptotic behavior of solutions of abstract wave equations, *Proc. Amer. Math. Soc.*, 19(1968):1403–1406.

[28] J. Walsh, An Introduction to Stochastic Partial Differential Equations, *École d’Été de Probabilités de Saint Flour*, XIV-1984, 1180:265–439, Springer, Berlin, 1986.