A class of quasi-linear equations in coframe gravity

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Abstract

We have shown recently that the gravity field phenomena can be described by a traceless part of the wave-type field equation. This is an essentially non-Einsteinian gravity model. It has an exact spherically-symmetric static solution, that yields to the Yilmaz-Rosen metric. This metric is very close to the Schwarzchild metric. The wave-type field equation can not be derived from a suitable variational principle by free variations, as it was shown by Hehl and his collaborators. In the present work we are seeking for another field equation having the same exact spherically-symmetric static solution. The differential-geometric structure on the manifold endowed with a smooth orthonormal coframe field is described by the scalar objects of anholonomy and its exterior derivative. We construct a list of the first and second order $SO(1,3)$-covariants (one- and two-indexed quantities) and a quasi-linear field equation with free parameters. We fix a part of the parameters by a condition that the field equation is satisfied by a quasi-conformal coframe with a harmonic conformal function. Thus we obtain a wide class of field equations with a solution that yields the Majumdar-Papapetrou metric and, in particularly, the Yilmaz-Rosen metric, that is viable in the framework of three classical tests.

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1 Introduction

The teleparallel space was introduced for the first time by Einstein [1], [2] in a certain variant of a unified theory of gravity and electromagnetism. The work of Weitzenböck [3] was the first devoted to the investigation of the geometric structure of teleparallel spaces. The theories based on this geometrical structure appear in physics for time to time in order to give an alternative model of gravity or to describe the spin properties of matter. For the recent investigations in this area see Refs. [4] to [8], and [9] to [15]. The investigations in Poincaré gauge theory and in the metric-affine theory of gravity opened a new wide field for applications of the teleparallel geometric structure. The status of the teleparallel space in the metric-affine framework is described in Refs. [10], [11].

In [22] a certain type of an alternative gravity model based on the teleparallel space was proposed. We consider a differential $4D$-manifold $M$ endowed with a smooth coframe field $\{\vartheta^a(x), \, a = 0, 1, 2, 3\}$. The coframe is declared to be pseudo-orthonormal in every point of $M$ and this condition defines uniquely the following geometrical objects:

1. a metric on the manifold:

$$ g = \eta_{ab} \vartheta^a \otimes \vartheta^b, \tag{1.1} $$

where $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ is the Minkowskian metric tensor,

2. a Hodge dual map $*\Omega^p \rightarrow \Omega^{4-p}$

$$ *\vartheta^a = \epsilon^{abcd} \vartheta_b \wedge \vartheta_c \wedge \vartheta_d, \tag{1.2} $$

where $\epsilon^{abcd}$ is the completely antisymmetric pseudo-tensor and 1-forms with lowered index is defined to be $\vartheta_a = \eta_{am} \vartheta^m$,

3. in additional to the derivative operator $d : \Omega^p \rightarrow \Omega^{p+1}$ a dual operator - coderivative is defined as

$$ d^\dagger = - * d*, \tag{1.3} $$

which acts as $d^\dagger : \Omega^p \rightarrow \Omega^{p-1}$.

4. The Hodge - de Rham Laplacian on differential forms (d’Alembertian in the case of a manifold with the Lorentzian signature)

$$ \Box = d^\dagger d + dd^\dagger \tag{1.4} $$
In [22] it was proposed a wave-type field equation:

$$\square \vartheta^a = \lambda(x) \vartheta^a,$$  \hspace{1cm} (1.5)

where $\lambda(x)$ is a function of the coframe field $\vartheta^a$ and its derivatives. Note that for simplicity we use the definitions of the Hodge dual and the Hodge de Rham Laplacian without any sign factor. So we restrict ourselves to the case of a fixed dimension and signature. For the convenient definitions see Ref. [10]. It can be checked that the properties of the coderivative operator and the Laplacian hold also for our definition.

The equation (1.5) exhibits a system of hyperbolic second order nonlinear PDE. It is a covariant equation relative to the group of the smooth transformations of the local coordinate system on the manifold $x^\mu \rightarrow \tilde{x}^\mu(x^\mu)$.

The additional symmetry is an invariance relative to the group of the global (rigid) $SO(1,3)$ transformations of the coframe field: $\vartheta^a \rightarrow \tilde{\vartheta}^a = A^a_b \vartheta^b$, where $A^a_b \in SO(1,3)$. Thus, we are interested in a representative class of $SO(1,3)$-connected coframes, not in a specific coframe. All coframes from the same class lead to a unique metric and for the test particles to a unique equation of motion. Note that we accept the geodesic principle in exactly the same form as it is done in the Einsteinian gravity. The work of deriving this principle from the field equations of type (1.5) in the Einstein-Infeld-Hoffmann manner is in progress.

In the case of a static spherically-symmetric space-time the field equation (1.5) has a unique spherically-symmetric asymptotically flat static solution:

$$\vartheta^0 = e^{-m r} dt \quad \vartheta^i = e^m dx^i \quad i = 1, 2, 3,$$  \hspace{1cm} (1.6)

where $m$ is an arbitrary parameter, which can be interpreted as a mass of a central body that produces the field. This coframe field leads by (1.1) to the Yilmaz-Rosen metric (in isotropic coordinates)

$$ds^2 = e^{-2m r} dt^2 - e^{2m r} (dx^2 + dy^2 + dz^2).$$  \hspace{1cm} (1.7)

The metric (1.7) appeared for the first time in the Yilmaz scalar model of gravity [17] (see also [18]) and after a few years in a different framework of Rosen’s bimetric theory of gravity (see [19], [20]). The metric (1.7) possesses a following fine analytic property: it has the same leading terms in the long-distance Taylor expansion as the canonical Schwarzschild metric. This fact leads to a good accordance of the Yilmaz-Rosen metric with the observation data (at least in the three classical tests).

In [24] it was shown that a quasi-conformal coframe conformed to (1.6)

$$\vartheta^0 = e^{-f} dt \quad \vartheta^i = e^f dx^i \quad i = 1, 2, 3,$$  \hspace{1cm} (1.8)
where \( f = f(x, y, z) \) is an arbitrary scalar function, solves the field equation (1.5) if and only if \( f \) is a (spatial) harmonic function

\[
\Delta f = f_{xx} + f_{yy} + f_{zz} = 0. \tag{1.9}
\]

By (1.1) this coframe leads to the metric

\[
ds^2 = e^{-2f}dt^2 - e^{2f}(dx^2 + dy^2 + dz^2), \tag{1.10}
\]

that is known in the classical theory of GR as the Majumdar-Papapetrou metric. Thus, we have for the field equation (1.5) a wide class of solutions that can be obtained by a specific choice of a harmonic function \( f \). Some of physical interesting solutions of this type are exhibited in [24].

In the present work we construct a free-parametric class of quasi-linear field equations that involves in the covariant and \( SO(1,3) \)-invariant form the coframe \( \vartheta^a \) and its first and second derivatives.

One of the possibility to fix these numerical free parameters is to require the equation to have an exact solution of the form (1.8) with a harmonic function \( f \).

The outline of the work is as following. In the section 2 we develop mathematical tools that we need to preserve the covariance and \( SO(1,3) \)-invariance. We consider the first order “derivatives” of the coframe - objects of anholonomity \( C^a_{\ bca} \) and introduce the second order “derivatives” - \( B \)-objects \( B^a_{bcd} \). We reduce the problem of obtaining the diffeomorphic covariant, quasi-linear, rigid \( SO(1,3) \) invariant equation to purely algebraic problem of finding a maximal set of two-indexed quantities that involve the \( B \)-type and the \( C \)-type objects.

We restrict ourselves to the case of a quasi-linear field equation with free parameters that is linear in \( B \)-type objects and quadratic in the \( C \)-type objects.

In the section 3 we study which conditions on the free parameters in the field equation should be required in order to preserve the solution of the form (1.8). Using the calculations of the second order invariants (Appendix A) and considering the leading part of the field equation, we obtain a necessary conditions on the free parameters in order to obtain the harmonic equation for the scalar function \( f \). These conditions are not sufficient because the leading part includes, together with the harmonic terms \( \Delta f \), the gradient terms \( \nabla^2 f \). In order to eliminate these gradient terms we consider a quadratic part of the field equation. We calculate the quadratic invariants for a pseudo-conformal coframe (1.8) in Appendix B. Using these calculations we obtain a necessary conditions on the parameters that should be accepted in order to eliminate the gradient terms in the leading part. Thus we obtain a complete system of conditions on the free parameters of our quasi-linear
field equations. This system gives a necessary and sufficient condition for the field equation to have a pseudo-conformal solution (1.8) with a harmonic function $f$.

In the section 5 we show that the traceless wave-type field equation (1.5) contains within our general class of the quasi-linear field equations. We obtain the expression of the Hodge-de Rham Laplacian via the $C$- and $B$-objects. The numerical parameters for the field equation (1.5) are calculated. These parameters satisfy our system of harmonic conditions.

## 2 Quasi-linear equations

In order to construct some appropriate field equation one need to apply some general principle. The principle of the quasi-linearity was formulated for the first time by Hehl in the Ref. [21]. This principle holds in most of the viable field theories and it can be used as a heuristic tool for construction of new physical models.

The action and accordingly the field equations should be linear in the higher (second order) derivatives. We accommodate this principle in the following restricted form: the field equation should be linear in the second order derivatives and quadratic in the first order derivatives.

This restriction can be motivated by the following facts

- It is necessary to consider the second order derivatives terms in the linear form in order to obtain the wave equation as a first approximation of the field equation.

- The quadratic first order derivatives terms are already involved in the field equation (1.5) as well as in the more general equation derived from the Rumpf Lagrangian, see for instant [23].

- Considering the field variable $\vartheta^a$ to be dimensionless one obtain the same dimension for the linear and quadratic pieces. As a result all free parameters considering later are dimensionless.

In order to construct the covariant and rigid $SO(1,3)$ invariant set of first derivatives quantities we begin with the exterior derivative of the coframe field $\vartheta^a$. This 2-form can be written uniquely as

$$d\vartheta^a = \frac{1}{2} C^a_{\ bc} \vartheta^b.$$

(2.1)
We will refer to the coefficients $C^a_{bc}$ as objects of anholonomity. These coefficients are antisymmetric in the down indices:

$$C^a_{bc} = -C^a_{cb}. \quad (2.2)$$

Using twice the interior product with basis vectors $e_m$ we obtain

$$C^a_{bc} = e_c \mathcal{J}(e_b \mathcal{J} d \vartheta^a). \quad (2.3)$$

The 3-indexed objects $C^a_{bc}$ can be contracted in order to obtain 1-indexed quantities. Because of the antisymmetry in the down indices only one contraction (up to a sign coefficient) of the object of anholonomity is possible:

$$C_a = C^m_{am} = e_m \mathcal{J}(e_a \mathcal{J} d \vartheta^m). \quad (2.4)$$

In order to describe the second derivative terms we consider the exterior derivative of the object of anholonomity. This 1-form can be expressed by its basic components as

$$dC^a_{mn} := B^a_{mnp} \partial^p. \quad (2.5)$$

We will refer to the coefficients $B^a_{mnp}$ as $B$-objects. This is a set of scalar quantities that transforms in as a tensor under the rigid $SO(1,3)$ pseudo-rotations of the coframe $\vartheta^a$. The explicit expression of the 4-indexed $B$-objects can be written as

$$B^a_{mnp} = e_p \mathcal{J} dC^a_{mn} = e_p \mathcal{J} \left( e_n \mathcal{J} (e_m \mathcal{J} d \vartheta^a) \right). \quad (2.6)$$

Note, that these coefficients are antisymmetric in the middle indices:

$$B^a_{mnp} = -B^a_{nmp}. \quad (2.7)$$

Contracting two indices in the object $B^a_{mnp}$ we obtain the following two-indexed $B$-objects

$$(1)B_{ab} := B^m_{mab} = e^m \mathcal{J} \left( e_n \mathcal{J} (e_b \mathcal{J} d \vartheta_a) \right), \quad (2.8)$$

$$(2)B_{ab} := B^m_{mab} = e_b \mathcal{J} \left( e_a \mathcal{J} (e_m \mathcal{J} d \vartheta^m) \right), \quad (2.9)$$

$$(3)B_{ab} := B^m_{mab} = e_m \mathcal{J} \left( e_b \mathcal{J} (e_a \mathcal{J} d \vartheta^m) \right). \quad (2.10)$$

Note, that

$$dC_a = dC^m_{am} = B^m_{amp} \partial^p = -(2)B_{ap} \partial^p. \quad (2.11)$$

Let us apply the contraction of the indices for the 2-indexed $B$-objects. Note, that $(3)B_{ab}$ is antisymmetric and its contraction is zero. We have only one (up to a sign) full contraction of the quantities $B^a_{mnp}$

$$B := B^a_{ab} \mathcal{J} = (1)B^a_a = (2)B^a_a. \quad (2.12)$$
We will refer to the quantity $B$ as a scalar $B$-object.
Now we are ready to construct a general field equation. The coframe field $\vartheta^a$ has 16 independent components, these components in our approach are the independent dynamical variables. In order to have a good defined dynamical system we need to construct a $SO(1, 3)$ tensorial field equation that is a system of 16 independent hyperbolic partial differential equations of the second order. Recall that we are looking for the field equations, that are linear in the $B$-objects and quadratic in the $C$-objects. The leading (second order) part of the equation will be a linear combination of two-indexed $B$-objects:

$$L_{ab} = \beta_1 B_{(ab)} + \beta_2 B_{(ab)} + \beta_3 B_{ab} + \beta_4 \eta_{ab} B + \beta_5 B_{[ab]} + \beta_6 B_{[ab]}.$$ (2.13)

The quadratic part of the equation can be constructed as a linear combination of two-indexed terms of the type $C \times C$ contracted by the Minkowskian metric $\eta_{ab}$. Considering all the possible combination of the indices and taking in account the antisymmetry of the objects $C$ we obtain the following list of independent two-indexed terms:

\[
\begin{align*}
(1) A_{ab} & := C_{abm}^m \\ (2) A_{ab} & := C_{mab}^m \\ (3) A_{ab} & := C_{amn}^n C_{b}^m \\ (4) A_{ab} & := C_{amn}^n C_{b}^m \\ (5) A_{ab} & := C_{man}^n C_{b}^m \\ (6) A_{ab} & := C_{man}^n C_{b}^m \\ (7) A_{ab} & := C_a C_b \\
\end{align*}
\]

Taking the traces of these matrices we obtain the following list of scalar type quadratic invariants:

\[
\begin{align*}
(1) A & := (1) A_{a}^a = -(7) A_{a}^a \\ (2) A & := (3) A_{a}^a = (6) A_{a}^a \\ (3) A & := (4) A_{a}^a = (5) A_{a}^a \\
\end{align*}
\]

The trace of the antisymmetric matrix $(2) A_{ab}$ is zero. Note, that each one of the objects $\text{[2.14, 2.23]}$ is a covariant and a $SO(1, 3)$ invariant value.

The quadratic part of the equation can be expressed as a linear combination of the values $\text{[2.14, 2.23]}$, namely

\[
Q_{ab} = \alpha_1 (1) A_{(ab)} + \alpha_2 (2) A_{ab} + \alpha_3 (3) A_{ab} + \alpha_4 (4) A_{(ab)} + \alpha_5 (5) A_{ab} + \alpha_6 (6) A_{ab} + \alpha_7 (7) A_{ab} + \alpha_8 (1) A_{[ab]} + \alpha_9 (4) A_{[ab]} + \eta_{ab} \left(\alpha_{10} (1) A + \alpha_{11} (2) A + \alpha_{12} (3) A\right) \\ (2.24)
\]
The general field equation that satisfied the requirement described above can be expressed as

\[
(L_{ab} + Q_{ab}) \ast \vartheta^b = \kappa \Sigma_a, \tag{2.25}
\]

where \(\Sigma_a\) is the energy-momentum current of material fields and \(\kappa\) is a couple constant. We will use the field equation (2.25) only in vacuum thus it is enough to consider the equation

\[
L_{ab} + Q_{ab} = 0, \tag{2.26}
\]

where the leading part defined by (2.13) and the quadratic part by (2.24).

### 3 Harmonic conditions

As we have showed in [22] the traceless wave-type field equation (1.5) has a unique spherically-symmetric asymptotically flat static solution (1.6). This solution includes a scalar function \(f = \frac{m}{r}\), which is a 3D-harmonic function. In [24] we prove that the equation (1.5) is satisfied by a general quasi-conformal coframe

\[
\vartheta^0 = e^{-f} dt, \quad \vartheta^i = e^f dx^i, \quad i = 1, 2, 3,
\]

where \(f = f(x, y, z)\) is an arbitrary 3D-harmonic function i.e. \(\triangle f = 0\).

In this section we are looking for a general field equation of the type (2.26) which have the same solution: quasi-conformal coframe with an arbitrary 3D-harmonic function. Let us begin with the leading part of the equation (2.26). The computations in the Appendix A for the quasi-conformal coframe (1.8) yield

\[
L_{ab} = \beta_1 e^{-2f} \left( \begin{array}{cccc}
-\tilde{\Delta} \varphi & 0 & 0 & 0 \\
0 & -\varphi_{22} - \varphi_{33} & \varphi_{12} & \varphi_{13} \\
0 & \varphi_{12} & -\varphi_{11} - \varphi_{33} & \varphi_{23} \\
0 & \varphi_{13} & -\varphi_{23} & -\varphi_{11} - \varphi_{22}
\end{array} \right) 
- \beta_2 e^{-2f} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & \varphi_{11} & \varphi_{12} & \varphi_{13} \\
0 & \varphi_{12} & \varphi_{22} & \varphi_{23} \\
0 & \varphi_{13} & \varphi_{23} & \varphi_{33}
\end{array} \right) + \beta_4 \eta_{ab} e^{-2f} \tilde{\Delta} \varphi, \tag{3.1}
\]

where we use the following notations

\[
\varphi_{ab} := f_{ab} - f_a f_b, \quad \tilde{\Delta} \varphi := \sum_{a=1}^{3} \varphi_{aa}, \tag{3.2}
\]
Note that the expression in the right hand side of (3.4) does not involve the parameters $\beta_3, \beta_5$ and $\beta_6$. These parameters remain free. In order to eliminate the mixed second order derivative terms $(f_{ij} \text{ with } i \neq j)$ in the expression (3.1), we should take

$$\beta_1 = \beta_2.$$ (3.3)

Then the leading part of (2.26) takes the form:

$$L_{ab} = -e^{-2f} \tilde{\Delta} \varphi \left( \begin{array}{cccc}
(\beta_1 - \beta_4) & 0 & 0 & 0 \\
0 & (\beta_1 + \beta_4) & 0 & 0 \\
0 & 0 & (\beta_1 + \beta_4) & 0 \\
0 & 0 & 0 & (\beta_1 + \beta_4)
\end{array} \right).$$ (3.4)

Note that this form does not yet yield a harmonic equation because

$$\tilde{\Delta} \varphi = \Delta f - \nabla^2 f.$$ (3.5)

In order to compensate the gradient terms $\nabla^2 f$, we need to add in the field equation the quadratic part.

The computations of the quadratic invariants in the Appendix A for the quasi-conformal coframe (1.8) yield

$$Q_{ab} = -\alpha_1 e^{-2f} \left( \begin{array}{cccc}
\nabla^2 f & 0 & 0 & 0 \\
0 & \nabla^2 - f_1^2 & -f_1 f_2 & -f_1 f_3 \\
0 & -f_1 f_2 & \nabla^2 - f_2^2 & -f_2 f_3 \\
0 & -f_1 f_3 & -f_2 f_3 & \nabla^2 - f_3^2
\end{array} \right)
+ \alpha_3 e^{-2f} \left( \begin{array}{cccc}
-2\nabla^2 f & 0 & 0 & 0 \\
0 & 2(\nabla^2 - f_1^2) & -2 f_1 f_2 & -2 f_1 f_3 \\
0 & -2 f_1 f_2 & 2(\nabla^2 - f_2^2) & -2 f_2 f_3 \\
0 & -2 f_1 f_3 & -2 f_2 f_3 & 2(\nabla^2 - f_3^2)
\end{array} \right)
- \alpha_4 e^{-2f} \left( \begin{array}{cccc}
\nabla^2 f & 0 & 0 & 0 \\
0 & -(\nabla^2 - f_1^2) & f_1 f_2 & f_1 f_3 \\
0 & f_1 f_2 & -(\nabla^2 - f_2^2) & f_2 f_3 \\
0 & f_1 f_3 & f_2 f_3 & -(\nabla^2 - f_3^2)
\end{array} \right)
+ \alpha_5 e^{-2f} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 3 f_1^2 & 3 f_1 f_2 & 3 f_1 f_3 \\
0 & 3 f_1 f_2 & 3 f_2^2 & 3 f_2 f_3 \\
0 & 3 f_1 f_3 & 3 f_2 f_3 & 3 f_3^2
\end{array} \right)
+ \alpha_6 e^{-2f} \left( \begin{array}{cccc}
\nabla^2 f & 0 & 0 & 0 \\
0 & 2 f_1^2 + \nabla^2 f & 2 f_1 f_2 & 2 f_1 f_3 \\
0 & 2 f_1 f_2 & 2 f_2^2 + \nabla^2 f & 2 f_2 f_3 \\
0 & 2 f_1 f_3 & 2 f_2 f_3 & 2 f_3^2 + \nabla^2 f
\end{array} \right) + \text{...}$$
Recall that the matrix $L_{ab}$ is diagonal after our choice (3.3) of the coefficients $\alpha$. Thus, our first task is to remove all the off-diagonal terms in the matrix $Q_{ab}$. Then we have to take

$$\alpha_1 - 2\alpha_3 - \alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = 0. \quad (3.7)$$

In order to compensate the quadratic terms $\nabla^2 f$ on the diagonal of the matrix $L_{ab}$, we should require

$$\beta_1 - \beta_4 = -\alpha_1 - 2\alpha_3 - \alpha_4 - \alpha_6 - \alpha_{10} - 6\alpha_{11} - 6\alpha_{12},$$
$$\beta_1 + \beta_4 = -\alpha_1 + 2\alpha_3 + \alpha_4 + \alpha_6 + \alpha_{10} + 6\alpha_{11} + 6\alpha_{12}.$$

Thus we have two additional conditions

$$\beta_1 = -\alpha_1, \quad (3.8)$$
$$\beta_4 = 2\alpha_3 + \alpha_4 + \alpha_6 + \alpha_{10} + 6\alpha_{11} + 6\alpha_{12}. \quad (3.9)$$

If these two conditions are met, the field equation (2.26) takes on the following form:

$$L_{ab} + Q_{ab} = -e^{-2f} \Delta f \begin{pmatrix}
(\beta_1 - \beta_4) & 0 & 0 & 0 \\
0 & (\beta_1 + \beta_4) & 0 & 0 \\
0 & 0 & (\beta_1 + \beta_4) & 0 \\
0 & 0 & 0 & (\beta_1 + \beta_4)
\end{pmatrix} = 0. \quad (3.10)$$

Therefore, in the case when the coefficients $\beta_1$ and $\beta_4$ do not vanish simultaneously, the field equation transforms to the harmonic equation $\Delta f = 0$. The result can be stated in the form of the following proposition:

**Theorem 3.1:** The field equation

$$L_{ab} + Q_{ab} = 0, \quad (3.11)$$

where

$$L_{ab} = \beta_1^{(1)}B_{(ab)} + \beta_2^{(2)}B_{(ab)} + \beta_3^{(3)}B_{ab} + \beta_4\eta_{ab}B + \beta_5^{(1)}B_{[ab]} + \beta_6^{(2)}B_{[ab]} \quad (3.12)$$
and
\[ Q_{ab} = \alpha_1 A_{(ab)} + \alpha_2 A_{ab} + \alpha_3 A_{(ab)} + \alpha_4 A_{(ab)} + \alpha_5 A_{ab} + \alpha_6 A_{ab} + \alpha_7 A_{ab} + \alpha_8 A_{(ab)} + \alpha_9 A_{(ab)} + \eta_{ab} \left( \alpha_{10} A + \alpha_{11} A + \alpha_{12} A + \alpha_{13} A \right), \]

(3.13)
is satisfied by a coframe
\[ \vartheta^0 = e^{-f} dx^0, \quad \vartheta^i = e^f dx^i, \quad i = 1, 2, 3, \]

(3.14)
with a harmonic function \( f = f(x, y, z) \) if and only if the following conditions hold
\[ \beta_1 = \beta_2, \]  
(3.15)
\[ \beta_1 = -\alpha_1, \]  
(3.16)
\[ \alpha_1 - 2\alpha_3 - \alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = 0, \]  
(3.17)
\[ \beta_4 = 2\alpha_3 + \alpha_4 + \alpha_6 - \alpha_{10} + 6\alpha_{11} + 6\alpha_{12}. \]  
(3.18)

## 4 The Hodge-de Rham Laplacian

Let us show that the field equation (1.5) satisfies the conditions of the theorem 3.1. In order to express the Hodge-de Rham Laplacian via the \( A^- \) and \( B^- \) objects we need the expressions for the coderivatives of the basis 1-forms. Recall two formulas that are deal with the interior product
\[ \vartheta^a \land (e_a \land \alpha) = p\alpha, \]

(4.1)
\[ e_a \land \vartheta^a \land \alpha = (4 - p)\alpha, \]

(4.2)
where \( p = \text{deg}(\alpha) \).

**Proposition 4.1:**
\[ d^\dagger(\vartheta^{a_1 \ldots a_p}) = \frac{1}{p-1} \left( \sum_{i=1}^p (-1)^i \vartheta^{a_i} d^\dagger(\vartheta^{a_1 \ldots \hat{a}_i \ldots a_p}) + (-1)^p \vartheta^m \land \ast(d \vartheta^m \land \ast \vartheta^{a_1 \ldots a_p}) \right) \]

(4.3)
Proof: Consider the interior product
\[
e_mJ[d^l(\vartheta^{a_1...a_p})] = *(\vartheta_m \wedge \ast^2 d \ast \vartheta^{a_1...a_p}) = (-1)^p \ast (\vartheta_m \wedge d \ast \vartheta^{a_1...a_p})
\]
\[= (-1)^p \left[ -d(\vartheta_m \wedge \ast \vartheta^{a_1...a_p}) + d\vartheta_m \wedge \ast \vartheta^{a_1...a_p} \right]
\]
\[= (-1)^p \left[ d^l[\ast^2 (\vartheta_m J \vartheta^{a_1...a_p})] + \ast (d\vartheta_m \wedge \ast \vartheta^{a_1...a_p}) \right]
\]
\[= \sum_{i=1}^p (-1)^i \delta^a_m d^l \vartheta^{a_1...\hat{a}_i...a_p} + (-1)^p \ast (d\vartheta_m \wedge \ast \vartheta^{a_1...a_p}).
\]

Taking now the exterior product of this expression with the form \(\vartheta^m\) and using the formula (4.1) we obtain the relation (4.3)
\[\tag{4.4}
(p - 1)d^l(\vartheta^{a_1...a_p}) = \sum_{i=1}^p (-1)^i \vartheta^a_i d^l \vartheta^{a_1...\hat{a}_i...a_p} + (-1)^p \vartheta^m \wedge \ast (d\vartheta_m \wedge \ast \vartheta^{a_1...a_p}).
\]

Using the formula (4.3) we express the coderivative of the basis forms \(\vartheta^a\) via the object anholonomity
\[
d^l(\vartheta^a) = C^a,
\]
\[
d^l(\vartheta^{ab}) = C^a \vartheta^b - C^b \vartheta^a - C^{ab} \vartheta^m,
\]
\[
d^l(\vartheta^{abc}) = C^a \vartheta^{bc} - C^b \vartheta^{ac} + C^{bc} \vartheta^a + C^{m} \vartheta^{am} - C^{ac} \vartheta^{bm} + C^{ab} \vartheta^{cm},
\]
\[
d^l(\vartheta^{abcd}) = 0.
\]

The Hodge-de Rham Laplacian of the basis 1-form
\[\vartheta^a = (dd^l + d^l d) \vartheta^a\]
can now be expressed as
\[\vartheta^a = -\left( (1) B^a_b + (2) B^a_{bk} C^k - \frac{1}{2} C^a_{mn} C^m_{bn} \right) \vartheta^b. \quad \tag{4.9}\]

Indeed, using (4.3) we obtain
\[dd^l \vartheta^a = dC^a = d(C^a_k) = B^a_{kk} \vartheta^b = -(2) B^a_{bk} \vartheta^b.
\]

As for the second part of the Laplacian we obtain using (4.6)
\[
d^l d^l \vartheta^a = \frac{1}{2} d^l (C^a_{mn} \vartheta^{mn}) = \frac{1}{2} d^l (C^a_{mn} \ast \vartheta^{mn})
\]
\[= \frac{1}{2} \left( B^a_{mnk} \ast (\vartheta^k \wedge \ast \vartheta^{mn} C^a_{mn} d^l \vartheta^{mn}) \right)
\]

\[= 12\]
\[
\begin{align*}
&= \frac{1}{2} \left( B^a_{mn} k^p (\delta^m_k \vartheta^n - \delta^n_k \vartheta^m) + C^a_{mn}(C^m \vartheta^n - C^n \vartheta^m - C^p_{mn} \vartheta^p) \right) \\
&= B^a_{kn} \vartheta^n + C^a_{mn} C^m \vartheta^n - \frac{1}{2} C^a_{mn} C^p_{mn} \vartheta^p \\
&= (-1) B^a_{b} + C^a_{bn} C^m - \frac{1}{2} C^a_{mn} C^p_{mn} \vartheta^b.
\end{align*}
\]

Let the 1-form of Laplacian \( \Box \vartheta^a \) will be written as
\[
\Box \vartheta^a = M^a_{m} \vartheta^m. \tag{4.10}
\]

The field equation \( \Box \vartheta^a = \lambda(x) \vartheta^a \) can be rewritten as
\[
M^a_{m} \vartheta^m = \lambda(x) \vartheta^a.
\]

Taking in two sides of this equation the interior product with the basis vector \( e_m \) we obtain
\[
M^a_{b} = \lambda(x) \delta^a_{b}.
\]

Taking the trace of matrices we get
\[
\lambda = \frac{1}{4} M^a_{a}
\]
and the field equation takes the form
\[
M_{ab} - \frac{1}{4} \eta_{ab} M^p_{p} = 0. \tag{4.11}
\]

Using the expression \( \Box \vartheta^a = M^a_{m} \vartheta^m \) we obtain
\[
(1) B^a_{b} + (2) B^a_{b} - C^a_{bn} C^m - \frac{1}{2} C^a_{mn} C^m_{bn} - \frac{1}{4} \eta_{ab} ((1) B^p_{p} + (2) B^p_{p} - C^p_{pm} C^m - \frac{1}{2} C^p_{mn} C^m_{p}) = 0. \tag{4.12}
\]

By the definitions of the matrices \( A_{ab} \) we can rewrite this equation as
\[
(1) B^a_{b} + (2) B^a_{b} - (1) A_{ab} - \frac{1}{2} (3) A_{ab} - \frac{1}{4} \eta_{ab} (2B - (1) A - \frac{1}{2} (2) A) = 0. \tag{4.13}
\]

Thus the non-vanishing coefficients of the general quasi-linear equation (2.26) in this special case take the values
\[
\beta_1 = \beta_2 = 1, \quad \beta_4 = -\frac{1}{2}, \quad \alpha_1 = -1, \quad \alpha_8 = 1, \\
\alpha_3 = -\frac{1}{2}, \quad \alpha_{10} = \frac{1}{4}, \quad \alpha_{11} = \frac{1}{8}. \tag{4.14}
\]

These coefficients satisfy the conditions of the theorem. Thus we restate from this more general point of view our result [24] that the traceless wave-type field equation (1.5) has the solution (1.6) with an arbitrary 3D-harmonic function.
5 Concluding remarks

The field equation we used in [22] and [24] is not a viable field equation mostly because it can not be derived from a certain action principle without additional constraints. In the same time the spherically-symmetric solution of this equation yields to a metric which is in a good accordance with three classical tests.

We are looking for another field equation that has the same static spherically-symmetric solution.

We have constructed a diffeomorphic covariant quasi-linear field equation that involves only the coframe field $\vartheta^a$ and its first and second order derivatives and that is invariant relative to the group of rigid $SO(1,3)$ transformations of the coframe field. This equation involves a set of free dimensionless numerical parameters. We fix a part of these free parameters in this field equation in order to obtain a quasi-conformal solution of the equation with a harmonic scalar function.

This general solution involves a unique spherically-symmetric solution, that leads to the viable (in the framework of three classical tests) Yilmaz-Rosen metric. Another conditions for fixing the free parameters is a requirement for the field equation to be derivable from a suitable action principle. We study this condition in [25].

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A Calculation of objects of curvature for a pseudo-conformal coframe

Let us calculate the objects of anholonomity for a pseudo-conformal coframe:

$$\vartheta^0 = e^{-f} dt, \quad \vartheta^i = e^f dx^i, \quad i = 1, 2, 3, \quad f = f(x, y, z). \quad (A.1)$$

The exterior derivatives of the coframe forms are

$$d\vartheta^0 = e^{-f} (f_1 \vartheta^{01} + f_2 \vartheta^{02} + f_3 \vartheta^{03}), \quad d\vartheta^1 = e^{-f} (-f_2 \vartheta^{12} - f_3 \vartheta^{13}), \quad (A.2)$$

$$d\vartheta^2 = e^{-f} (f_1 \vartheta^{12} - f_3 \vartheta^{23}), \quad d\vartheta^3 = e^{-f} (f_1 \vartheta^{13} + f_2 \vartheta^{23}), \quad (A.3)$$
Thus the non-vanishing coefficients $B_i = \frac{\partial f_i}{\partial x}$. Accordingly taking the exterior derivatives of holonomity (2.1) take the following forms

\[
C^0_{mn} = e^{-f} \begin{pmatrix} 0 & f_1 & f_2 & f_3 \\ -f_1 & 0 & 0 & 0 \\ -f_2 & 0 & 0 & 0 \\ -f_3 & 0 & 0 & 0 \end{pmatrix}, \quad C^1_{mn} = e^{-f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -f_2 & -f_3 \\ 0 & f_2 & 0 & 0 \\ 0 & f_3 & 0 & 0 \end{pmatrix} \\
C^2_{mn} = e^{-f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & f_1 & 0 \\ 0 & -f_1 & 0 & -f_3 \\ 0 & 0 & f_3 & 0 \end{pmatrix}, \quad C^3_{mn} = e^{-f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_1 \\ 0 & 0 & 0 & f_2 \\ 0 & -f_1 & -f_2 & 0 \end{pmatrix}
\]

(A.4)

Let us calculate the quantities $C^a := C^a_{mn}$

\[
C_0 = 0, \quad C_1 = e^{-f} f_1, \quad C_2 = e^{-f} f_2, \quad C_3 = e^{-f} f_3.
\]

(A.5)

The $B$-objects are defined (2.3) by the exterior derivative of the objects of anholonomity. Denoting $\varphi_{ab} := f_{ab} - f_a f_b$ we have

\[
dC^0_{mn} = e^{-2f} \begin{pmatrix} 0 & \varphi_{11} & \varphi_{12} & \varphi_{13} \\ -\varphi_{11} & 0 & 0 & 0 \\ -\varphi_{12} & 0 & 0 & 0 \\ -\varphi_{13} & 0 & 0 & 0 \end{pmatrix} \ d\theta^1 + e^{-2f} \begin{pmatrix} 0 & \varphi_{12} & \varphi_{22} & \varphi_{23} \\ -\varphi_{12} & 0 & 0 & 0 \\ -\varphi_{22} & 0 & 0 & 0 \\ -\varphi_{23} & 0 & 0 & 0 \end{pmatrix} \ d\theta^2
\]

\[
+ e^{-2f} \begin{pmatrix} 0 & \varphi_{13} & \varphi_{23} & \varphi_{33} \\ -\varphi_{13} & 0 & 0 & 0 \\ -\varphi_{23} & 0 & 0 & 0 \\ -\varphi_{33} & 0 & 0 & 0 \end{pmatrix} \ d\theta^3
\]

Thus the non-vanishing coefficients $B^0_{mn}$ are

\[
B^0_{m1} = e^{-2f} \begin{pmatrix} 0 & \varphi_{11} & \varphi_{12} & \varphi_{13} \\ -\varphi_{11} & 0 & 0 & 0 \\ -\varphi_{12} & 0 & 0 & 0 \\ -\varphi_{13} & 0 & 0 & 0 \end{pmatrix}, \quad B^0_{mn2} = e^{-2f} \begin{pmatrix} 0 & \varphi_{12} & \varphi_{22} & \varphi_{23} \\ -\varphi_{12} & 0 & 0 & 0 \\ -\varphi_{22} & 0 & 0 & 0 \\ -\varphi_{23} & 0 & 0 & 0 \end{pmatrix},
\]

\[
B^0_{mn3} = e^{-2f} \begin{pmatrix} 0 & \varphi_{13} & \varphi_{23} & \varphi_{33} \\ -\varphi_{13} & 0 & 0 & 0 \\ -\varphi_{23} & 0 & 0 & 0 \\ -\varphi_{33} & 0 & 0 & 0 \end{pmatrix}
\]

Accordingly taking the exterior derivatives of $C^1_{mn}, C^1_{mn}, C^1_{mn}$ we obtain:

\[
B^1_{mn1} = e^{-2f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi_{12} & -\varphi_{13} \\ 0 & \varphi_{12} & 0 & 0 \\ 0 & \varphi_{13} & 0 & 0 \end{pmatrix}, \quad B^1_{mn2} = e^{-2f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi_{22} & -\varphi_{23} \\ 0 & \varphi_{22} & 0 & 0 \\ 0 & \varphi_{23} & 0 & 0 \end{pmatrix}
\]
Let us now calculate the two-indexed $B$-objects $B_{ab}$. 

$B^1_{mn3} = e^{-2f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi_{23} & -\varphi_{33} \\ 0 & \varphi_{23} & 0 & 0 \\ 0 & \varphi_{33} & 0 & 0 \end{pmatrix}$.

$B^2_{mn1} = e^{-2f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{11} & 0 \\ -\varphi_{11} & 0 & -\varphi_{13} \\ 0 & \varphi_{13} & 0 \end{pmatrix}$, $B^2_{mn2} = e^{-2f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varphi_{12} & 0 & 0 \\ -\varphi_{12} & 0 & -\varphi_{23} \\ 0 & \varphi_{23} & 0 \end{pmatrix}$, $B^2_{mn3} = e^{-2f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{13} & 0 \\ -\varphi_{13} & 0 & -\varphi_{33} \\ 0 & \varphi_{33} & 0 \end{pmatrix}$

$B^3_{mn1} = e^{-2f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{11} & 0 \\ -\varphi_{11} & -\varphi_{12} & 0 \end{pmatrix}$, $B^3_{mn2} = e^{-2f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{12} & 0 \\ -\varphi_{12} & -\varphi_{22} & 0 \end{pmatrix}$, $B^3_{mn3} = e^{-2f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_{13} & 0 \\ -\varphi_{13} & -\varphi_{23} & 0 \end{pmatrix}$

Thus we obtain

$$B_{ab} = e^{-2f} \begin{pmatrix} -\Delta \varphi & 0 & 0 & 0 \\ 0 & -\varphi_{22} & -\varphi_{33} & \varphi_{12} \\ 0 & \varphi_{12} & -\varphi_{11} & -\varphi_{33} \\ 0 & \varphi_{13} & -\varphi_{23} & -\varphi_{11} - \varphi_{22} \end{pmatrix} \quad (A.6)$$

For the object $B_{ab}$ we obtain

$$B_{ab} = B^m_{mab} = B^0_{0ab} + B^1_{1ab} + B^2_{2ab} + B^3_{3ab}$$
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
\varphi_{11} & \varphi_{12} & \varphi_{13} \\
0 & \varphi_{12} & \varphi_{22} & \varphi_{23} \\
0 & \varphi_{13} & \varphi_{23} & \varphi_{33}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\varphi_{12} & -\varphi_{22} & -\varphi_{23} \\
0 & -\varphi_{13} & -\varphi_{23} & -\varphi_{33} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\varphi_{11} & -\varphi_{12} & -\varphi_{13} \\
0 & 0 & 0 & 0 \\
0 & -\varphi_{13} & -\varphi_{23} & -\varphi_{33}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\varphi_{11} & -\varphi_{12} & -\varphi_{13} \\
0 & -\varphi_{12} & -\varphi_{22} & -\varphi_{23} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
\varphi_{11} & \varphi_{12} & \varphi_{13} \\
\varphi_{12} & \varphi_{22} & \varphi_{23} \\
\varphi_{13} & \varphi_{23} & \varphi_{33}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\varphi_{11} & -\varphi_{12} & -\varphi_{13} \\
0 & -\varphi_{12} & -\varphi_{22} & -\varphi_{23} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Therefore

\[
(2)B_{ab} = -e^{-2f} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{12} & \varphi_{22} & \varphi_{23} \\ 0 & \varphi_{13} & \varphi_{23} & \varphi_{33} \end{pmatrix}
\] (A.7)

The antisymmetric matrix \( (2)B_{ab} \) is vanished:

\[
(3)B_{ab} = B_{mab} = B_{0ab} + B_{1ab} + B_{2ab} + B_{3ab} = 0
\] (A.8)

For the scalar B-object we obtain

\[
B = (1)B_{m} = (2)B_{m} = e^{-2f} \tilde{\Delta} \varphi
\] (A.9)

**B Calculation of the quadratic invariants for a pseudo-conformal coframe**

**B.1 Calculation of \((1)A_{ab}\)**

Using the values \((A.5)\) we can rewrite \((2.14)\) as follows

\[
(1)A_{ab} := C_{abm}C^{m} = -e^{-f}(C_{ab1}f_{1} + C_{ab3}f_{2} + C_{ab3}f_{3})
\]

Using the matrices \((A.4)\) we have

\[
(1)A_{ab} = -e^{-2f} \begin{pmatrix} \nabla^{2}f & 0 & 0 & 0 \\ 0 & f_{2}^{2} + f_{3}^{2} & -f_{1}f_{2} & -f_{1}f_{3} \\ 0 & -f_{1}f_{2} & f_{1}^{2} + f_{3}^{2} & -f_{2}f_{3} \\ 0 & -f_{1}f_{3} & -f_{2}f_{3} & f_{1}^{2} + f_{2}^{2} \end{pmatrix}
\] (B.1)

where \( \nabla^{2}f = f_{1}^{2} + f_{2}^{2} + f_{3}^{2} \).

Note, that for a pseudo-conformal coframe the matrix \((1)A_{ab}\) is symmetric, its trace takes the value

\[
(1)A = (1)A^{a}a = e^{-2f} \nabla^{2}f
\] (B.2)
B.2 Calculation of $^{(2)}A_{ab}$

The antisymmetric matrix $^{(2)}A_{ab}$ vanishes for the pseudo-conformal coframe (1.8) identical. Indeed

$$^{(2)}A_{ab} = C^{m}_{a b} C_m = e^{-f} (f_1 C^{1}_{a b} + f_2 C^{2}_{a b} + f_3 C^{3}_{a b})$$

The antisymmetric matrix $^{(2)}A_{ab}$ can be direct calculated as following

$$^{(2)}A_{ab} = e^{-2f} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -f_1 f_2 & -f_1 f_3 \\
f_1 f_2 & 0 & 0 \\
f_1 f_3 & 0 & 0 \\
-2 f_1 f_3 & -f_2 f_3 & 0
\end{pmatrix} + e^{-2f} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & f_1 f_2 & 0 \\
0 & f_1 f_2 & 0 & 0 \\
0 & f_1 f_3 & 0 & 0 \\
0 & f_2 f_3 & 0 & 0
\end{pmatrix}$$

$$+ e^{-2f} \begin{pmatrix}
0 & 0 & 0 & f_1 f_3 \\
0 & 0 & 0 & f_2 f_3 \\
0 & -f_1 f_3 & 0 & 0 \\
0 & -f_2 f_3 & -f_1 f_3 & 0
\end{pmatrix} = 0$$

B.3 Calculation of $^{(3)}A_{ab}$

The symmetric matrix $^{(3)}A_{ab}$ can be direct calculated as following

$$^{(3)}A_{ab} = C_{mn} C_b^{mn} = C_{a00} C_b^{00} + C_{a01} C_b^{01} + \ldots + C_{a33} C_b^{33}$$

We obtain

$$^{(3)}A_{ab} = e^{-2f} \begin{pmatrix}
-2 \nabla^2 f & 0 & 0 & 0 \\
0 & 2(f_1^2 + f_2^2) & -2 f_1 f_2 & -2 f_1 f_3 \\
0 & -2 f_1 f_2 & 2(f_1^2 + f_2^2) & -2 f_2 f_3 \\
0 & -2 f_1 f_3 & -2 f_2 f_3 & 2(f_1^2 + f_2^2)
\end{pmatrix}$$ (B.3)

The trace of this matrix takes the form

$$(^{2)}A = ^{(3)}A^a_a = -6 e^{-2f} \nabla^2 f$$ (B.4)

B.4 Calculation of $^{(4)}A_{ab}$

The general matrix $^{(4)}A_{ab}$ can be rewritten as follows

$$^{(4)}A_{ab} = C_{mn} C^{m}_{a b} - C_{a00} (C^0_{b1} + C^1_{b0}) - C_{a02} (C^0_{b2} + C^2_{b0}) - C_{a03} (C^0_{b3} + C^3_{b0}) - C_{a12} (C^1_{b2} + C^2_{b1}) - C_{a13} (C^1_{b3} + C^3_{b1}) - C_{a23} (C^2_{b3} + C^3_{b2})$$

The direct calculations yield

$$^{(4)}A_{ab} = -e^{-2f} \begin{pmatrix}
\nabla^2 f & 0 & 0 & 0 \\
0 & -(f_1^2 + f_2^2) & f_1 f_2 & f_1 f_3 \\
0 & f_1 f_2 & -(f_1^2 + f_2^2) & f_2 f_3 \\
0 & f_1 f_3 & f_2 f_3 & -(f_1^2 + f_2^2)
\end{pmatrix}$$ (B.5)
Note, that for the coframe (1.8) the matrix is symmetric and its trace takes the value
\[
(3)A = (4)A^a_a = -3e^{-2f}\nabla^2 f. \tag{B.6}
\]

**B.5 Calculation of (5)\(A_{ab}\)**

The matrix \((5)A_{ab} = C^m_{an}C^n_{bm}\) is symmetric. Indeed
\[
(5)A_{ba} = C^m_{bn}C^n_{am} = C^m_{bm}C^n_{am} = (5)A_{ab}.
\]

By straightforward calculations we obtain
\[
(5)A_{ab} = e^{-2f}\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 3f_1^2 & 3f_1f_2 & 3f_1f_3 \\
0 & 3f_1f_2 & 3f_2^2 & 3f_2f_3 \\
0 & 3f_1f_3 & 3f_2f_3 & 3f_3^2
\end{pmatrix} \tag{B.7}
\]

The trace of this matrix is
\[
(4)A = (5)A^a_a = -3e^{-2f}\nabla^2 f. \tag{B.8}
\]

**B.6 Calculation of (6)\(A_{ab}\)**

The matrix \((6)A_{ab} = C_{man}C^n_{bm}\) is symmetric. It can be rewritten explicitly as
\[
(6)A_{ab} = C^0_{a0}C^0_{b0} - C^0_{a1}C^0_{b1} - C^0_{a2}C^0_{b2} - C^0_{a3}C^0_{b3} - C^1_{a0}C^1_{b0} + C^1_{a1}C^1_{b1} + C^1_{a2}C^1_{b2} + C^1_{a3}C^1_{b3} - C^2_{a0}C^2_{b0} + C^2_{a1}C^2_{b1} + C^2_{a2}C^2_{b2} + C^2_{a3}C^2_{b3} - C^3_{a0}C^3_{b0} + C^3_{a1}C^3_{b1} + C^3_{a2}C^3_{b2} + C^3_{a3}C^3_{b3}
\]

The direct calculations give
\[
(6)A_{ab} = e^{-2f}\begin{pmatrix}
-\nabla^2 f & 0 & 0 & 0 \\
0 & 2f_1^2 + \nabla^2 f & 2f_1f_2 & 2f_1f_3 \\
0 & 2f_1f_2 & 2f_2^2 + \nabla^2 f & 2f_2f_3 \\
0 & 2f_1f_3 & 2f_2f_3 & 2f_3^2 + \nabla^2 f
\end{pmatrix} \tag{B.9}
\]

The trace of the matrix is
\[
(6)A^a_a = (2)A = -6e^{-2f}\nabla^2 f. \tag{B.10}
\]
B.7 Calculation of \(^{(7)}A_{ab}\)

The matrix \(^{(7)}A_{ab} = C_a C_b\) is symmetric and its explicit form is

\[
^{(7)}A_{ab} = e^{-2f} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & f_1^2 & f_1 f_2 & f_1 f_3 \\
0 & f_1 f_2 & f_2^2 & f_2 f_3 \\
0 & f_1 f_3 & f_2 f_3 & f_3^2
\end{pmatrix}
\] (B.11)

The trace of this matrix is

\[
^{(6)}A = (^{(7)}A^a_a) = -(^{(1)}A = -e^{-2f} \nabla^2 f.
\] (B.12)

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