Branes and Quantization for an A–Model
Complexification of Einstein Gravity
in Almost Kähler Variables

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Abstract
The general relativity theory is redefined equivalently in almost
Kähler variables: symplectic form, $\theta[g]$, and canonical symplectic con-
nection, $\hat{D}[g]$ (distorted from the Levi–Civita connection by a tensor
constructed only from metric coefficients and their derivatives). The
fundamental geometric and physical objects are uniquely determined
in metric compatible form by a (pseudo) Riemannian metric $g$ on a
manifold $V$ enabled with a necessary type nonholonomic $2 + 2$ distri-
bution. Such nonholonomic symplectic variables allow us to formulate
the problem of quantizing Einstein gravity in terms of the A–model
complexification of almost complex structures on $V$, generalizing the
Gukov–Witten method [1]. Quantizing $(V, \theta[g], \hat{D}[g])$, we derive a
Hilbert space as a space of strings with two A–branes which for the
Einstein gravity theory are nonholonomic because of induced nonlin-
ear connection structures. Finally, we speculate on relation of such a
method of quantization to curve flows and solitonic hierarchies defined
by Einstein metrics on (pseudo) Riemannian spacetimes.

Keywords: quantum gravity, Einstein gravity, nonholonomic man-
ifolds, symplectic variables, nonlinear connections, strings and A–branes

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In this paper we address the question of quantization of an A–model complexification of spacetime in general relativity following a new perspective on symplectic geometry, branes and strings proposed in Ref. [1] (the Gukov–Witten approach). Symplectic techniques has a long history in physics (see,
for instance, [2]) and, last two decades, has gained more and more interest in the theory of deformation quantization [3, 4, 5, 6, 7]. Recently, an approach based on Fedosov quantization of Einstein gravity [8] and Lagrange–Finsler and Hamilton–Cartan spaces [9, 10] was elaborated in terms of almost Kähler variables on nonholonomic manifolds (see [11, 12], for a review of methods applied to standard theories of physics, and [13, 10, 14], for alternative geometrizations of nonholonomic mechanics on manifolds and/or bundle spaces).

The problem of quantization of nonlinear physical theories involves a great amount of ambiguity because the quantum world requests a more refined and sophisticate description of physical systems than the classical approach. Different types of quantization result in very different mathematical constructions which lead to inequivalent quantizations of the same classical theories (the most important approaches to quantization are discussed in [1, 15, 16], see also references therein).

Deformation quantization was concluded to be a systematic mathematical procedure but considered not standard in a standard manner [1]. This is because a deformation quantization of the ring of holomorphic functions on a complex symplectic/Poisson manifold requires not arbitrary choices but quantization does. It does not use deformations with a complex parameter, works with deformations over rings of formal power series and does not lead to a natural Hilbert space on which the deformed algebra acts. Nevertheless, different methods and results obtained in deformation quantization play a very important role in all approaches to quantization, as a consequent geometric formalism in nonlinear functional analysis. Perhaps, the most important results in deformation quantization can be redefined in the language of other approaches to quantization which is very useful for developing new methods of quantization.

In this paper we consider a new perspective on quantization of Einstein gravity based on two–dimensional sigma–models, following the A–model quantization via branes proposed in [1]. Our purpose is to get closer to a systematic theory of quantum gravity in symplectic variables related to

\[ (V, \mathcal{N}), \text{ where } V \text{ is a manifold and } \mathcal{N} \text{ is a nonintegrable distribution on } V, \text{ is called a nonholonomic manifold. We emphasize that in this paper we shall not work with classical physical theories on (co) tangent bundles but only apply in classical and quantum gravity certain methods formally elaborated in the geometry of Lagrange–Finsler spaces and nonholonomic manifolds. Readers may find in Appendix and Section 2 the most important definitions and formulas from the geometry of nonholonomic manifolds and a subclass of such spaces defined by nonlinear connection, N–connection, structure (i.e. N–anholonomic manifolds).} \]
an almost Kähler formulation of general relativity. The novel results in this paper are those that we propose an explicit application of the Gukov–Witten quantization method to gravity and construct a Hilbert space for Einstein spaces parametrizing it as the spaces of two nonholonomic A–brane strings. We relate the constructions to group symmetries of curve flows, bi–Hamilton structures and solitonic hierarchies defined by (pseudo) Riemannian/ Einstein metrics.

The new results have a strong relationship to our former results on nonholonomic Fedosov quantization of gravity [8] and Lagrange–Finser/Hamilton–Cartan systems [9, 10]. Geometrically, such relations follow from the fact that in all cases the deformation quantization of a nonholonomic complex manifold, constructed following the Gukov–Witten approach, produces a so–called distinguished algebra (adapted to a nonlinear connection structure) that then acts in the quantization of a real almost Kähler manifold. It is obvious that different attempts and procedures to quantize gravity theories are not equivalent. For such generic nonlinear quantum models, it is possible only to investigate the conditions when the variables from one approach can be re–defined into variables for another one. Then, a more detailed analysis allows us to state the conditions when physical results for one quantization are equivalent to certain ones for another quantum formalism.

The paper is organized as follows: In Section 2, we provide an introduction in the almost Kähler model of Einstein gravity. Section 3 is devoted to formulation of quantization method for the A–model with nonholonomic branes. In section 4, an approach to Gukov–Witten quantization of the almost Kähler model of Einstein gravity is developed. Finally, we present conclusions in section 5. In Appendix, we summarize some important component formulas necessary for the almost Kähler formulation of gravity.

Readers may consult additionally the Refs. [17, 18, 12, 11] on conventions for our system of denotations and reviews of the geometric formalism for nonholonomic manifolds, and various applications in standard theories of physics.

2 Almost Kähler Variables in Gravity

The standard formulation of the Einstein gravity theory is in variables (g, ɡ∇), for ɡ∇ = ∇|g| = { ɡΓαβγ, ɡΓαβ, ɡ} being the Levi–Cevita connection completely defined by a metric g = {gμν} on a spacetime manifold V and constrained to satisfy the conditions ɡ∇g = 0 and ɡΓαβγ = 0, where
For different approaches in classical and quantum gravity, there are considered tetradic, or spinor, variables and 3+1 spacetime decompositions (for instance, in the so-called Arnowit–Deser–Misner, ADM, formalism, Ashtekar variables and loop quantum gravity), or nonholonomic 2+2 splittings, see a discussion and references in [16].

2.1 Nonholonomic distributions and alternative connections

For any (pseudo) Riemannian metric \( g \), we can construct an infinite number of linear connections \( gD \) which are metric compatible, \( gDg = 0 \), and completely defined by coefficients \( g = \{g_{\mu\nu}\} \). Of course, in general, the torsion \( gT = \triangledown T[g] \) of a \( gD \) is not zero. Nevertheless, we can work equivalently both with \( g\triangledown \) and any \( gD \), because the distorsion tensor \( gZ = Z[g] \) from the corresponding connection deformation,

\[
\tag{1}
g\triangledown = gD + gZ,
\]

(in the metric compatible cases, \( gZ \) is proportional to \( gT \)) is also completely defined by the metric structure \( g \). In Appendix, we provide an explicit example of two metric compatible linear connections completely defined by the same metric structure, see formula (A.27). Such torsions induced by nonholonomic deformations of geometric objects are not similar to those from the Einstein–Cartan and/or string/gauge gravity theories, where certain additional field equations (to the Einstein equations) are considered for physical definition of torsion fields.

Even the Einstein equations are usually formulated for the Ricci tensor and scalar curvature defined by data \( (g, g\triangledown) \), the fundamental equations and physical objects and conservation laws can be re–written equivalently in terms of any data \( (g, gD) \). This may result in a more sophisticated structure of equations but for well defined conditions may help, for instance, in constructing new classes of exact solutions or to define alternative methods of quantization (like in the Ashtekar approach to gravity) [17, 11, 12, 16, 8].

In order to apply the A–model quantization via branes proposed in Ref. [1], and relevant methods of deformation/geometric quantization, it is convenient to select from the set of linear connections \{ \( gD \) \} such a symplectic

\[\text{for a general linear connection, we do not use boldface symbols if such a geometric object is not adapted to a prescribed nonholonomic distribution}\]

\[\text{We follow our conventions from [12] [11] when "boldfaced" symbols are used for spaces and geometric objects enabled with (or adapted to) a nonholonomic distribution/ nonlinear connection/ frame structure; we also use left "up" and "low" indices as additional labels for geometric/physical objects, for instance, in order to emphasize that } g\triangledown = \triangledown[g] \text{ is defined by a metric } g; \text{ the right indices are usual abstract or coordinate tensor ones.}\]
one which is "canonically" defined by the coefficients of an Einstein metric $g = \{g_{\mu\nu}\}$, being compatible to a well defined almost complex and symplectic structure, and for an associated complex manifold. In section 2 and Appendix of Ref. [19], there are presented all details on the so-called almost Kähler model of general relativity (see also the constructions and applications to Fedosov quantization of gravity in Refs. [8, 18, 16]). For convenience, we summarize in Appendix A some most important definitions and component formulas on almost Kähler redefinition of gravity.

Let us remember how almost Kähler variables can be introduced in classical and quantum gravity: Having prescribed on a (pseudo) Riemannian manifold $V$ a generating function $L(u)$ (this can be any function, for certain models of analogous gravity [11] [12], considered as a formal regular pseudo–Lagrangian $L(x,y)$ with nondegenerate $L_{,\mu\nu} = \frac{1}{2} \frac{\partial^2 L}{\partial y^\mu \partial y^\nu}$, we construct a canonical almost complex structure $J = L\mathbf{J}$ (when $J \circ J = -\mathbf{I}$ for $\mathbf{I}$ being the unity matrix) adapted to a canonical nonlinear connection (N–connection) structure $N = {}^L N$ defined as a nonholonomic distribution on $TV$. For simplicity, in this work we shall omit left labels like $\alpha$,$\beta$, $\gamma$, $\delta$, etc., and write $\mathbf{D}$ and $\mathbf{D} g = 0$. The variables ($g\theta$, $\mathbf{D}$) define an almost Kähler model of general relativity, with distortion of connection $g\mathbf{D} \rightarrow g\mathbf{Z}$, for $g\nabla = g\mathbf{D} + g\mathbf{Z}$, see formula (I).

Explicit coordinate formulas for $g\nabla$, $g\mathbf{D}$ and $g\mathbf{Z}$ are given by (A.27)

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4We use the word "pseudo" because a spacetime in general relativity is considered as a real four dimensional (pseudo) Riemannian spacetime manifold $V$ of necessary smooth class and signature $(-,+,+,+)$. For a conventional $2+2$ splitting, the local coordinates $u = (x,y)$ on a open region $U \subset V$ are labelled in the form $u^\alpha = (x^i, y^a)$, where indices of type $i,j,k,... = 1,2$ and $a,b,c,... = 3,4$, for tensor like objects, will be considered with respect to a general (non-coordinate) local basis $e_\alpha = (e_i, e_a)$. One says that $x^i$ and $y^a$ are respectively the conventional horizontal/ holonomic (h) and vertical / nonholonomic (y) coordinates (both types of such coordinates can be time– or space–like ones). Primed indices of type $i',a'$ will be used for labeling coordinates with respect to a different local basis $e_{\alpha'} = (e_{i'}, e_{a'})$ for instance, for an orthonormalized basis. For the local tangent Minkowski space, we can chose $e_{\alpha'} = i \partial / \partial u^\alpha$, where $i$ is the imaginary unity, $i^2 = -1$, and write $e_{\alpha'} = (i \partial / \partial u^\alpha, \partial / \partial u^i, \partial / \partial u^{a'}, \partial / \partial u^{a'})$. 
and (A.28), see proofs in Refs. [19][11][13]. It should be noted here that for an arbitrary nonholonomic 2+2 splitting we can construct a Hermitian model of Einstein gravity with \( d \theta \neq 0 \). In such a case, to perform a deformation, or other, quantization is a more difficult problem. Choosing nonholonomic distributions generated by regular pseudo–Lagrangians, we can work only with almost Kähler variables which simplifies substantially the procedure of quantization, see discussions from Refs. [8][9][12][18].

2.2 Einstein gravity in almost Kähler variables

In the canonical approach to the general relativity theory, one works with the Levi–Civita connection \( \nabla = \{ \Gamma^\gamma_{\alpha \beta} \} \) which is uniquely derived following the conditions \( \mathcal{T} = 0 \) and \( \nabla g = 0 \). This is a linear connection but not a distinguished connection (d–connection) because \( \nabla \) does not preserve the nonholonomic splitting (see in Appendix the discussions relevant to (A.8)) under parallelism. Both linear connections \( \nabla \) and \( \hat{\mathcal{D}} \equiv \theta \hat{\mathcal{D}} \) are uniquely defined in metric compatible forms by the same metric structure \( g \) (A.1). The second one contains nontrivial d–torsion components \( \hat{T}^\alpha_{\beta \gamma} \) (A.26), induced effectively by an equivalent Lagrange metric \( g = Lg \) (A.5) and adapted both to the N–connection \( L\mathcal{N} \), see (A.4) and (A.8), and almost symplectic \( L\theta \) (A.4) structures \( \mathcal{L} \).

Having chosen a canonical almost symplectic d–connection, we compute the Ricci d–tensor \( \hat{R}^\alpha_{\beta \gamma} \) (A.31) and the scalar curvature \( L \hat{R} \) (A.32)). Then, we can postulate in a straightforward form the field equations

\[
\frac{\hat{R}^\alpha_{\beta \gamma}}{2} - \frac{1}{2} (L \hat{R} + \lambda) e^\alpha_{\beta} = 8\pi G T^\alpha_{\beta},
\]

where \( \hat{R}^\alpha_{\beta} = e^\alpha_{\gamma} \hat{R}^\gamma_{\beta \gamma}, \) \( T^\alpha_{\beta} \) is the effective energy–momentum tensor, \( \lambda \) is the cosmological constant, \( G \) is the Newton constant in the units when the light velocity \( c = 1 \), and the coefficients \( e^\alpha_{\beta} \) of vierbein decomposition \( e = e^\alpha_{\beta} \partial / \partial u^\alpha \) are defined by the N–coefficients of the N–elongated operator of partial derivation, see (A.10). But the equations (2) for the canonical \( \hat{\Gamma}^\gamma_{\alpha \beta}(\theta) \) are not equivalent to the Einstein equations in general relativity written for the Levi–Civita connection \( \Gamma^\gamma_{\alpha \beta}(\theta) \) if the tensor \( T^\alpha_{\beta} \) does not include contributions of \( \mathcal{Z}'^\alpha_{\alpha \beta}(\theta) \) in a necessary form.

Introducing the absolute antisymmetric tensor \( \epsilon_{\alpha \beta \gamma \delta} \) and the effective source 3–form

\[
T^\alpha_{\beta} = T^\alpha_{\beta \delta} \epsilon_{\alpha \beta \gamma \delta} du^\gamma \wedge du^\delta \wedge du^\delta
\]
and expressing the curvature tensor $\hat{R}_\gamma^\tau = \hat{R}_\gamma^\tau e^\alpha \wedge e^\beta$ of $\hat{\Gamma}_\alpha^\beta\gamma = \Gamma^\alpha_{\beta\gamma} - \hat{Z}_\alpha^\beta\gamma$, as $\hat{R}_\gamma^\tau = \hat{R}_\gamma^\tau e^\alpha \wedge e^\beta$ is the curvature 2–form of the Levi–Civita connection $\nabla$ and the distorsion of curvature 2–form $\hat{Z}_\alpha^\beta\gamma$ is defined by $\hat{Z}_\alpha^\beta\gamma$ (A.28), we derive the equations (2) (varying the action on components of $e_\beta$, see details in Ref. [16]). The gravitational field equations are represented as 3–form equations,

$$
\epsilon_{\alpha\beta\gamma\tau} \left( e^\alpha \wedge \hat{R}^{\beta\gamma} + \lambda e^\alpha \wedge e^\beta \wedge e^\gamma \right) = 8\pi GT_\tau, \tag{3}
$$

when

$$
\begin{align*}
T_\tau &= mT_\tau + Z\hat{T}_\tau, \\
mT_\tau &= mT^\alpha_\tau \epsilon_{\alpha\beta\gamma\delta} du^\beta \wedge du^\gamma \wedge du^\delta, \\
ZT_\tau &= (8\pi G)^{-1} \hat{Z}^\alpha_\tau \epsilon_{\alpha\beta\gamma\delta} du^\beta \wedge du^\gamma \wedge du^\delta,
\end{align*}
$$

where $mT^\alpha_\tau$ is the matter tensor field. The above mentioned equations are equivalent to the usual Einstein equations for the Levi–Civita connection $\nabla$,

$$
\hat{R}^\alpha_\beta\gamma - \frac{1}{2} \left( \hat{R} + \lambda \right) e^\alpha \epsilon \beta \gamma = 8\pi G mT^\alpha_\beta.
$$

For $\hat{D} \equiv \delta\hat{D}$, the equations (3) define the so–called almost Kähler model of Einstein gravity.

Such formulas expressed in terms of canonical almost symplectic form $\theta$ (A.18) and normal d–connection $\hat{D} \equiv \delta\hat{D}$ (A.20) are necessary for encoding the vacuum field equations into cohomological structure of quantum (in the sense of Fedosov quantization) almost Kähler models of the Einstein gravity, see [8, 18, 19, 16].

We conclude that all geometric and classical physical information for any data 1 $(g, \hat{\Gamma})$, for Einstein gravity, can be transformed equivalently into canonical constructions with 2 $(\theta, \hat{D})$, for an almost Kähler model of general relativity. A formal scheme for general relativity sketching a mathematical physics ”dictionary” between two equivalent geometric ”languages” (the Levi–Civita and almost Kähler ones) is presented in Figure 1.

### 3 The A–Model, Quantization, and Nonholonomic Branes

The goal of this section is to generalize the A–model approach to quantization [1] for the case when branes are nonholonomic and the symplectic structure is induced by variables $(\theta, \hat{D})$ in Einstein gravity.
(pseudo) Riemannian metric $g$

arbitrary frames/coordinates and/or 3+1 splitting

any 2+2 nonholonomic splitting with generating function $L(x, y)$

$$g = g_{\alpha\beta} e^\alpha \otimes e^\beta$$

Levi–Civita connection $g\nabla$:

$$g\nabla g = 0, \quad \Theta^\alpha_{\beta\gamma} = 0,$$

Levi–Civita variables:

$$(g, g\nabla) = (g_{\mu\nu}, \Gamma^\alpha_{\beta\gamma})$$

canonical symplectic connection

$$\hat{D} = \{ \hat{\Gamma}^\gamma_{\alpha\beta} \}; \quad \hat{D} \theta = \hat{D} g = 0,$$

$$g\nabla = \hat{D} + \hat{Z},$$

almost Kähler variables:

$$(\theta, \hat{D}) = (\theta_{\alpha\beta}, \hat{\Gamma}^\gamma_{\alpha\beta})$$

Classical Einstein Spaces

Brane A–model quantization;
Deformation quantization;
Nonholonomic Ashtekar variables

Quantum almost Kähler Einstein Spaces

Figure 1: Levi–Civita and almost Kähler Variables in Gravity
3.1 On quantization and nonholonomic branes

We start with an almost Kähler model of a (pseudo) Riemannian manifold \( V \) (which is a nonholonomic manifold, or, more exactly, \( N \)-anholonomic manifold [11, 12], see also definitions in Appendix) endowed with structures \((g_\theta, \hat{D})\), which we wish to quantize. Our goal is to develop the method of quantization [1] and to apply it to the case of nonholonomic manifolds provided with gravitational symplectic variables. We consider a complex line bundle \( \mathcal{L} \to V \) with a unitary connection of curvature \( R \), like in geometric quantization [20, 21, 22, 23].

In this work, we shall use an affine variety \( Y \) which, by definition, is a complexification of \( V \) such that: 1) it is a complex manifold with an antiholomorphic involution \( \tau : Y \to Y \), when \( V \) is a component of the fixed point set of \( \tau \); 2) there is a nondegenerate holomorphic 2–form \( \Theta \) on \( Y \) such that its restriction, \( \tau^\ast(\Theta) \), to \( V \) is just \( g_\theta \); 3) the unitary line bundle \( \mathcal{V} \to V \) can be extended to a unitary line bundle \( \mathcal{Y} \to Y \) enabled with a connection of curvature \( \text{Re}(\Theta) \), when the action of \( \tau \) on \( Y \) results to an action on \( \mathcal{Y} \), restricting to an identity on \( \mathcal{V} \). In brief, the approach to quantization is based on a "good" \( A \)-model associated with the real symplectic form \( Y_\theta = \text{Im}(\Theta) \). Such a choice determines a canonical coisotropic brane (in our case, in general, it is a nonholonomic manifold, because \( V \) is nonholonomic) in the \( A \)-model of \( Y \) [24, 25].

Similarly to [1], we will make use of ordinary Lagrangian \( A \)-branes when \( V \) is also modeled as a Lagrangian submanifold and we can define a rank 1 \( A \)-brane supported on such a (nonholonomic) manifold. We denote by \( gB' \) such a brane (one could be inequivalent choices for a not simply–connected). It will be written \( gB_{cc} \) for the canonical coisotropic \( A \)-brane and \( gB \) for an \( A \)-brane of unspecified type. We can construct a quantum model (i. e. quantization of \( V \) enabled with variables \((g_\theta, \hat{D})\)) postulating that the Hilbert space associated to \( V \) is the space \( \mathcal{H} \) of \((gB_{cc}, gB')\). This is related to the geometry of a vector bundle provided with nonlinear connection structure (in different contexts such spaces and applications to modern physics, mechanics and Finsler geometry and generalizations were studied in Refs. [11, 12, 13]) associated to the choice of \( A \)-brane \( gB' \). The first explicit constructions of such vector spaces (without \( N \)-connection structure) were

\[5\] For such a good \( A \)-model, the relevant correlation functions and observables are complex–valued rather than formal power series depending on a formal deformation parameter; such series converge to complex–valued functions; following [1], this is possible if the supersymmetric sigma–model with target \( Y \) can be twisted to give the \( A \)-model. Here, we also note that we follow our system of denotations relevant to nonholonomic manifolds and corresponding geometric constructions.
originally considered in Refs. [26, 1].

In order to quantize almost Kähler models of Einstein spaces, we need a more sophisticated techniques from the so-called locally anisotropic string theory [27, 28] and noncommutative generalizations of Lagrange–Finsler and nonholonomic branes and gauge theories [29, 30, 31, 32]. This will be discussed in further sections.

3.2 The canonical coisotropic nonholonomic brane and A–model

Let us consider a complex symplectic manifold $Y$ endowed with a nondegenerate holomorphic 2–form $\Theta$ of type $(2, 0)$ splitting into respective real, $J\theta$, and imaginary, $K\theta$, parts, i.e.

$$\Theta = J\theta + iK\theta, \tag{4}$$

where

$$I^t\Theta = i\Theta, \text{ or } I^tJ\theta = -K\theta \text{ and } I^tK\theta = J\theta, \tag{5}$$

for $I$ being the complex structure on $Y$, which may be regarded as a linear transformation of tangent vectors, $I^t$ denoting the transpose map acting on 1–forms; $\Theta$ and $I^t\Theta$ are regarded as maps from tangent vectors to 1–forms. 

In this work, we view $Y$ as a real symplectic manifold with symplectic structure $Y\theta = Im\Theta = K\theta$ and study the associated A–model as a case in [24], when such A–models are Lagrangian branes; for our purposes, it is enough to take a rank 1 coisotropic A–brane whose support is just the manifold $Y$.

Any rank 1 brane can be endowed with a unitary line bundle $V$ with a connection for which we denote the curvature by $F$. If for $I = Y\theta^{-1}F$, we have $I^2 = -1$, i.e. this is an integrable complex structure; we call such a 1 brane to be an A–brane. We obey these conditions if we set $F = J\theta$, when $Y\theta^{-1}F = K\theta^{-1}J\theta$ coincides with $I$ from (5). So, we can construct

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6In low energy limits, we proved that from string/brane and/or gauge gravity models one generates various versions of (non) commutative Lagrange–Finsler spaces. Such constructions were considered for a long time to be very "exotic" and far from scopes of standard physics theories. But some years latter, it was proven that Finsler like structures can be modelled even as exact solutions in Einstein gravity if generic off-diagonal metrics and nonholonomic frames are introduced into consideration [17, 12]. More than that, the N–connection formalism formally developed in Finsler geometry, happen to be very useful for various geometric purposes and applications to modern gravity and quantum field theory.

7Even we consider such notations, we do not impose the condition that $Y$ has a hyper–Kähler structure.
always an A–brane in the A–model of symplectic structure \( Y \theta \) starting with a complex symplectic manifold \((Y, \Theta)\), for any choice of a unitary line bundle \( V \) enabled with a connection of curvature \( J \theta = \text{Re}(\Theta) \). Following [1], we call this A–brane the canonical coisotropic brane and denote it as \( g\mathcal{B}_c \) (we put a left up label \( g \) because for the almost Kähler models of Einstein spaces there are induced by metric canonical decompositions of fundamental geometric objects related to almost complex and symplectic structures, see respective formulas \((A.1)\) with \((A.5)\) and \((A.6)\); \((A.14)\) and \((A.18)\)).

We emphasize that the constructions leading to an A–model depend only on \( Y \theta \) and do not depend on the chosen almost complex structure; there is no need for the almost complex structure to be integrable. So, we can always chose \( Y \theta \) to be, let say a complexification, or proportional to the gravitational symplectic 1–form \( g \theta \). Together with the almost complex structure \( L^J \) \((A.1)\), this would make the A–model more concrete (we shall consider details in section 4). Here we adapt some key constructions from [1] to the case of nonholonomic A–models, when \( Y \theta = K \theta \) for an almost complex structure \( K \) (in a particular case, \( Y \theta = g \theta \) and \( K = L^J \)). So, in this section, we consider that \( Y \) is a nonholonomic complex manifold enabled with a \( N \)–connection structure defined by a distribution of type \((A.8)\). For our purposes, it is enough to consider that such a distribution is related to a decomposition of tangent spaces of type

\[
TY = TV \oplus \mathbb{I}(TV),
\]

when we chose such a \( K \) that \( IK = -KI \) implying that \( J = KI \) is also an almost complex structure.

In general, \( J \) and \( K \) are not integrable but \( K \) always can be defined to satisfy the properties that \( K \theta \) is of type \((1,1)\) and \( IK = -KI \), for any nonholonomic \( V \), and the space of choices for \( K \) is contractable. This can be verified for any \( Y \) of dimension \( 4k \), for \( k = 1, 2, \ldots \), as it was considered in section 2.1 of [1]. For nonholonomic manifolds related to general relativity this is encoded not just in properties of compact form symplectic groups, \( Sp(2k) \), acting on \( \mathbb{C}^{2k} \), and their complexification, \( Sp(2k)_c \). The nonholonomic 2 + 2 decomposition (similarly, we can consider \( n + n \)), results not in groups and Lie algebras acting on some real/complex vector spaces, but into distinguished similar geometric objects, adapted to the \( N \)–connection structure. In brief, they are called d–algebras, d–groups and d–vectors. The geometry of d–groups and d–spinors, and their applications in physics and noncommutative geometry, is considered in details in Refs. [33, 34, 35], see also Part III of [11], and references therein (we refer readers to those works).
Here, we note that for our constructions in quantum gravity, it is enough to take $Y$ of dimension 8, for $k = 2$, when $V$ is enabled with a nonholonomic splitting $2 + 2$ and the $\text{d}$-group $^d\text{Sp}(4)$ is modelled as $^d\text{Sp}(4) = \text{Sp}(2) \oplus \text{Sp}(2)$, which is adapted to both type decompositions (A.8) and (A.8). We can now write a sigma–model action using an associated metric $\kappa g = - \kappa \theta K$, when $J = KI$ will be used for quantization. The first nonholonomic sigma– and (super) string models where considered in works [27, 28] for the so–called Finsler–Lagrange (super) strings and (super) spaces, but in those works $V = E$, for $E$ being a vector (supervector bundle). In this section, we work with complex geometric structures on $T\mathcal{Y} = h\mathcal{Y} \oplus v\mathcal{Y}$, when such a nonholonomic splitting is induces both by (6) and (A.8), i.e. $\mathcal{Y}$ is also enabled with $N$–connection structure and its symplectic and complex forms are related to the corresponding symplectic and almost complex structures on $\mathcal{V}$ (in particular, those for the almost Kähler model of Einstein gravity).

### 3.3 Space of nonholonomic ($^gB_{cc}$, $^gB'$) strings

We can consider the space of ($^gB_{cc}$, $^gB_{cc}$) strings in a nonholonomic A–model as the space of operators that can be inserted in the A–model on a boundary of a nonholonomic string world–sheet $\Sigma$, with a splitting $T\Sigma = h\Sigma \oplus v\Sigma$, similarly to (A.8), that ends on the brane $^g\mathcal{B}_{cc}$. In general, the constructions should be performed for a sigma–model with nonholonomic target $\mathcal{Y}$, bosonic fields $U$ and $N$–adapted fermionic d–fields $\psi = (h\psi, v\psi)$, for left–moving d–spinors, and $\bar{\psi} = (h\bar{\psi}, v\bar{\psi})$, where the $h$– and $v$–components are defined with respect to splitting (7).

An example of local d–operator $f(U)$ is that corresponding to a complex–valued function $f : \mathcal{Y} \rightarrow \mathbb{C}$. This d–operator inserted at an interior point of $\Sigma$ is invariant under supersymmetry transforms (on nonholonomic supersymmetric spaces, see [28]) of the A–model,

$$\delta U = (1 - iK)(\bar{\psi}) + (1 + iK)(\psi),$$

if $f$ is constant. The $N$–connection structure results in similar transforms of the $h$– and $v$–projections of the fermionic d–fields. Boundary d–operators must be invariant under transforms (8) of the A–model. In sigma models, one works with boundary (d-) operators, rather than bulk (d-) operators,
when the boundary conditions are nonholonomic ones obeyed by fermionic (d-) fields,
\[(\kappa g - F)(\psi) = (\kappa g + F)(-\psi),\]
where, for \(\kappa g = -\kappa \theta K\) and \(F = J\theta\), we have \((\kappa g - F)^{-1}(\kappa g + F) = J = KI\).

The boundary conditions (8) can be written in equivalent form
\[
\delta U = ((1 - iK)J + (1 + iK))( -\psi),
\]
\[
= (1 + iI)(1 + iJ)( -\psi).
\]
This implies a topological symmetry of the nonholonomic A–model,
\[
\delta^{1,0}U = 0 \text{ and } \delta^{0,1}U = \rho, \tag{9}
\]
\[
\delta \rho = 0,
\]
where \(\rho = (1 + iI)(1 + iJ)( -\psi)\) and decomposition \(\delta U = \delta^{1,0}U + \delta^{0,1}U\) for decompositions of the two parts of respective types (1,0) and (0,1) with respect to the complex structure \(I\). The N–splitting (7) results in \(\rho = (h\rho, v\rho)\), induced by \(h–\), \(v–\)splitting \( -\psi = (h\psi, v\psi)\).

It follows from (9) that the topological supercharges of the nonholonomic A–model corresponds to the \(\partial\) operator of \(Y\). We wrote "supercharges" because one of them is for the \(h–\)decomposition and the second one is for the \(v–\)decomposition. The observables of the nonholonomic A–model correspond additively to the graded d–vector space \(H^{0,*}_\theta(Y)\), where \(Y\) is viewed both as a complex manifold with complex structure \(I\) and as a nonholonomic manifold enabled with N–connection structure. We shall work with the ghost number zero part of the ring of observables which corresponds to the set of holomorphic functions on \(Y\).

For instance, all boundary observables of the nonholonomic A–model can be constructed from \(U\) and \(\rho = (h\rho, v\rho)\). Let us fix the local complex coordinates on \(Y\) to correspond to complex fields \(U^a(\tau, \sigma) = (X^i(\tau, \sigma), Y^{\alpha}(\tau, \sigma))\), for string parameters \((\tau, \sigma)\), see also beginning of section 4 on coordinate parametrizations on \(Y\). This allows us to construct general d–operators:

- of \(q–\)th order in \(\rho\), having d–charge \(q\) under the ghost number symmetry of the A–model, \(\rho^{\pi_1^{\cdot\cdot\cdot} \pi_2^{\cdot\cdot\cdot} \cdot \rho^{\pi_q} f_{\pi_{1q}^{\cdot\cdot\cdot} \pi_{pq}^{\cdot\cdot\cdot}}(U, U)\), which is an d–operator as a \((0, q)–\)form on \(Y\);
- of \(p–\)th order in \(h\rho\), having h–charge \(p\) under the ghost number symmetry of the A–model, \(\rho^{\pi_1^{\cdot\cdot\cdot} \pi_2^{\cdot\cdot\cdot} \cdot \rho^{\pi_p} f_{\pi_{1p}^{\cdot\cdot\cdot} \pi_{pq}^{\cdot\cdot\cdot}}(X, X)\), which is an h–operator as a \((0, p)–\)form on \(Y\);
• of s-th order in \( v^\rho \), having v–charge \( s \) under the ghost number symmetry of the A–model, \( \rho^\pi_1 \rho^\pi_2 \ldots \rho^\pi_s \mathfrak{J}_{\pi_1 \pi_2 \ldots \pi_s} (Y, \overline{Y}) \), which is an v–operator as a \((0, s)\)–form on \( Y \).

For a series of consequent h- and v–operators, it is important to consider the order of such operators.

We conclude that the constructions in this subsection determine the d–algebra \( \mathcal{A} = (h, v, A) \) of \((g^{B_{cc}}, g^{B'})\) strings.

### 3.4 Quantization for Lagrangian nonholonomic branes

Our first purpose, in this section, is to find something (other than itself) that the d–algebra \( \mathcal{A} = (h, v, A) \) can act on. The simplest construction is to introduce a second nonholonomic A–brane \( g^{B'} \), which allows us to define a natural action of \( \mathcal{A} \) on the space of \((g^{B_{cc}}, g^{B'})\) strings. In this paper, we consider \( g^{B'} \) to be a Lagrangian A–brane of rank 1, enabled with a nonholonomic distribution, i.e. \( g^{B'} \) is supported on a Lagrangian nonholonomic submanifold \( V \) also endowed with a flat line bundle \( V' \).

We assume that \( J^\theta \) is nondegenerate when restricted to \( V \) and consider \((V, J^\theta)\) as a symplectic manifold to be quantized. For a given \( V \), it is convenient to further constrain \( K \) such that \( TV \) is \( J \)–invariant, when the values \( I, K \) and \( J = KI \) obey the algebra of quaternions.

The quantization of \((g^{B_{cc}}, g^{B'})\) strings leads to quantization of the symplectic manifold \((V, J^\theta)\), and in a particular case of the almost Kähler model of Einstein gravity. We do not present a proof of this result because it is similar to that presented in section 2.3 of [1] by using holonomic manifolds and strings. For nonholonomic constructions, we have only a formal h– and v–component dubbing of geometric objects because of the N–connection structure.

There are also necessary some additional constructions with the action of a string ending on a nonholonomic brane with a Chan–Paton connection \( A \), which is given by a boundary term \( \int_{\partial \Sigma} A_\mu dU^\mu \). For \((g^{B_{cc}}, g^{B'})\) strings, we can take that the Chan–Paton bundle \( V' \), with a connection \( A' \), of the brane \( g^{B'} \) is flat but the Chan–Paton bundle of \( g^{B_{cc}} \) is the unitary line bundle \( V \), with connection \( A \) of curvature \( J^\theta \). Next step, we consider a line bundle \( 8 \)

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8 The proposed interpretation of \( V' \) is oversimplified because of relation of branes to K–theory, possible contributions of the the so-called B–field, disc instanton effects etc, see discussion in Ref. [1]. For the quantum gravity model to be elaborated in this paper, this is the simplest approach. Here we also note that our system of denotations is quite different from that by Gukov and Witten because we work with nonholonomic spaces and distinguished geometric objects.
$g\mathcal{V} = \mathcal{V} \otimes \mathcal{V}'$ over $\mathbf{V}$ which is a unitary bundle with a connection $\mathbf{B} = \mathbf{A} - \mathbf{A}'$ of curvature $\mathbf{J}\theta$. This corresponds, with a corresponding approximation, to a classical action for the zero modes

$$\int d\tau (A_\mu - A'_\mu) \frac{du^\mu}{d\tau} \approx \int d\tau B_\mu \frac{du^\mu}{d\tau}.$$ 

To quantize the zero modes with this action, and related "quantum" corrections, is a quantization of $\mathbf{V}$ with prequantum line bundle $\mathbf{V}$. We have not yet provided a solution of the problem of quantization for the almost Kähler model of Einstein gravity. It was only solved the second aim to understand that if the A–model of a nonholonomic $\mathbf{Y}$ exists, then the space of $(\mathbf{g}B_{cc}, \mathbf{g}B')$ strings can be modelled as a result of quantizing $\mathbf{V}$ with prequantum line bundle $\mathbf{g}\mathcal{V}$. It is still difficult to describe such spaces explicitly, in a general case, but the constructions became well defined for $(\mathbf{g}B_{cc}, \mathbf{g}D)$ on $\mathbf{V}$. Using such information from the geometry of almost Kähler spaces, induced by a generating function with nonholonomic 2+2 splitting, we can establish certain important general properties of the A–model to learn general properties of quantization.

### 3.4.1 Properties of distinguished Hilbert spaces for nonholonomic A–models

There are two interpretations of the space of $(\mathbf{g}B_{cc}, \mathbf{g}B')$ strings: The first one is to say that there is a Hilbert space $\mathbf{g}\mathcal{H}$ in the space of such strings in the nonholonomic A–model of $\mathbf{Y}\theta = \mathbf{K}\theta$. The second one is to consider a Hilbert space $\mathbf{g}\mathcal{H}$ of such strings in the B–model of complex structure $\mathbf{J}$. For a compact $\mathbf{V}$, or for wave functions required to vanish sufficiently rapidly at infinity, both spaces $\mathcal{H}$ and $\mathcal{H}$ are the same: they can be described as the space of zero energy states of the sigma–model with target $\mathbf{Y}$. The model is compactified on an interval with boundary conditions at the ends determined by nonholonomic $\mathbf{g}B_{cc}$ and $\mathbf{g}B'$. It should be noted here that both $\mathbf{g}\mathcal{H}$ and $\mathbf{g}\mathcal{H}$ are $\mathbf{Z}$–graded by the "ghost number" but differently, such grading being conjugate, because the A—model is of type $\mathbf{K}\theta$ and the B–model is of type $\mathbf{Y}\theta$.

We can use a sigma–model of target $\mathbf{Y}$ when the boundary conditions are set by two branes of type $(\mathbf{A}, \mathbf{B}, \mathbf{A})$. In this case, the model has an $SU(2)$ group of so–called R–symmetries, called $SU(2)_R$, with two ghost number symmetries being conjugate but different $U(1)$ subgroups of $SU(2)_R$. For nonholonomic models, we have to dub the groups and subgroups correspondingly for the h– and v–subspaces. In the simplest case, we can per-
form quantization for trivial grading, i.e. when $g\mathcal{V}$ is very “ample” as a line bundle in complex structure $J$.

For the B–model, we can chose any Kähler metric, for instance, to rescale the metric of $Y$ as to have a valid sigma–model perturbation theory. In such a limit, it is possible to describe $g\mathcal{H}$ by a $\mathcal{J}$ cohomology,

$$g\mathcal{H} = \bigoplus_{j=0}^{\dimc V} H^2_j(V, K^{1/2} \otimes g\mathcal{V}),$$

with a similar decomposition for $g\mathcal{H} \cong g\mathcal{\tilde{H}}$, i.e.

$$g\mathcal{H} = \bigoplus_{j=0}^{\dimc V} H^2_j(V, K^{1/2} \otimes g\mathcal{V}).$$

(10)

The value $K^{1/2}$ is the square root (this is a rough approximation) of the canonical line bundle $K$ on $V$. This is because, in general, $V$ may not be a spin manifold. Such nonholonomic configurations related to gerbes are discussed in [36, 37]. In relation to $K$–theory, details are given in [1] (nonholonomic configurations existing in our constructions do not change those conclusions). Here, we note that for very ample $g\mathcal{V}$ the cohomology vanishes except for $j = 0$ and its $\mathbb{Z}$–grading is trivial. This is one problem. The second limitation is that $g\mathcal{H}$ is described as a vector space which does not lead to a natural description as a Hilbert space with a hermitian inner product. This description of $g\mathcal{H}$ has certain resemblance to constructions in geometric quantization because the above cohomology defines quantization with a complex polarization. Nevertheless, this paper is based on the Gukov–Witten approach to quantization and does not provide a variant of geometric quantization, see in [1] why this is not geometric quantization.

### 3.4.2 Topological restrictions and unitarity

There is a topological obstruction to having a $Spin_c$ structure and there are further obstructions to having a flat $Spin_c$ structure on $V$. Such nonholonomic spinor constructions were analyzed firstly in relation to definition of Finsler–Lagrange spinors [33], further developments are outlined in Refs. [34, 35]. The problem of definition of spinors and Dirac operators for nonholonomic manifolds can be solved by the same N–connection methods with that difference that instead of vector/tangent bundles we have to work with manifolds enabled with nonholonomic distributions.

In general, $Spin_c$ structures on $V$ are classified topologically: we have to chose a way of lifting the second Steiffl–Whitney class $w_2(V) \in H^2(V, \mathbb{Z}_2)$ to an integral cohomology class $\zeta \in H^2(V, \mathbb{Z}_2)$. In their turn, flat $Spin_c$ structures are parametrized and classified by a choice of a lift $\zeta$ as a torsion
element of $H^2(V,\mathbb{Z}_2)$. We emphasize that a symplectic manifold that does not admit a flat $\text{Spin}_c$ structure cannot be quantized in the sense [1]. Perhaps, the cohomological analysis used in Fedosov quantization for almost Kähler models of gravity [8, 18], can be re-defined for the Gukov–Witten approach. In this work, we shall consider such gravitational fields and their quantization when the flat $\text{Spin}_c$ structure exists and they are distinguished into h– and v–components adapted to N–connection structures defined by certain generating functions.

Our next purpose is to define a Hermitian inner product on $g\mathcal{H}$ (10). For our further application in quantum gravity, we chose a nonholonomic A–model as a twisted version of a standard physical model, unitary, defined also as a supersymmetric field theory. Such a theory has an antilinear CPT symmetry, in our approach denoted $\Xi$. This operator maps any $(B_1, B_2)$ string into a $(B_2, B_1)$ which also defines an antilinear map from $(gB_{cc}, gB')$ strings into $(gB', gB_{cc})$ strings, but this is not a symmetry of an A–model. In explicit form, the definition of an A–model depends on a choice of a differential $Q$ as a complex linear combination of supercharges. For nonholonomic configurations, we work with couples of h– and v–supercharges $Q=(hQ, vQ)$. The maps with CPT symmetry transform $Q$ into its Hermitian adjoint $Q^+=(hQ^+, vQ^+)$ being the differential of a complex conjugate nonholonomic A–model.

Let us suppose that the nonholonomic complex manifold $Y$ admits an involution $\tau$, i.e. a N–adapted diffeomorphism obeying $\tau^2 = 1$, with the (odd) property:

$$\tau^*(K\theta) = -K\theta. \quad (11)$$

Such an operator can be always introduced on $Y$ by construction. This $\tau$ maps a nonholonomic A–model into a conjugate nonholonomic A–model and $\Xi_{\tau} = \tau\Xi$ is a N–adapted antilinear map from the nonholonomic A–model to itself. This is a general property which holds both for holonomic and nonholonomic geometrical models of certain underlying physical theory; one may be a twisting of the underlying model and in such a case the structure $K$ can be chosen to be integrable.

We consider explicitly the antiholomorphic involution $\tau$ for a nonholonomic complex symplectic manifold defined by data $(Y, I, \Theta)$, with a lift to $V$, where $\Theta = J\theta + i K\theta, I = J\theta^{-1} K\theta$, for $J$ being the curvature of the Chan–Paton bundle of the (τ–invariant) brane $gB_{cc}$, when

$$\tau^*(J\theta) = J\theta, \quad \tau^*(I) = -I, \quad \tau^*(\Theta) = \overline{\Theta}.$$ 

Using the topological inner product $(, )$ as the pairing between (in general, nonholonomic) $(B_1, B_2)$ strings and $(B_2, B_1)$, we can introduce the inner
product on $gH$ as

$$<\psi,\psi'> = <\Xi_{\tau}\psi,\psi'>.$$  

So, we conclude that if $B_1$ and $B_2$ are $\tau$–invariant nonholonomic A–branes, we can use $\Xi_{\tau}$ to define a Hermitian inner product on the (already Hilbert) space $gH$ of $(B_1, B_2)$ strings.

### 3.4.3 A Hermitian inner product on $gH$ of $(gB_{cc}, gB')$ strings

We suppose that the nonholonomic spacetime manifold $V$ is $\tau$–invariant and that there is a lift of $\tau$ to act on the Chan–Paton line bundle $V'$ on $V$. In such cases, the corresponding nonholonomic Lagrangian A–brane $gB'$ is also $\tau$–invariant. This allows us to construct a Hermitian form, and corresponding inner product $<,>$, on the space $gH$ of $(gB_{cc}, gB')$ strings if $\tau$ maps $V$ to itself.

Nearly the classical limit, the norm of a state $\psi \in gH$ is approximated

$$<\psi,\psi> = \int_V \overline{\psi}(\tau u)\psi(u) du;$$

this form is positive–definite only if $\tau$ acts trivially on $V$. In general, $<\psi,\psi>$ is nondegenerate but not necessarily positive definite. These are some consequences from nondegeneracy of topological inner product ($,$) and the property that $\Xi_{\tau}^2 = 1$. We can chose any $\psi_0$ such that $(\psi_0, \psi) \neq 0$ and set $\psi_1 = \Xi_{\tau}\psi_0$ which results in $<\psi_1, \psi> \neq 0$, i.e. nondegenerate property, but this does not constrain that $<\psi,\psi> > 0$ for all nonzero $\psi \in gH$. For the inverse construction, we consider $V$ to be a component of the fixed point set of map $\tau$. Because of property (11), we get that $V$ is Lagrangian for a $\Lambda$ and that $J\theta$ is nondegenerate when restricted to $V$. Such properties are automatically satisfied for the almost Kähler model of Einstein gravity with splitting (7). The map $\tau$ acts as 1 and -1 on summands $TV$ and $ITV$ and $J\theta$ is the sum of nondegenerate 2–forms on respective spaces.

There is a construction leading to the classical limit [11], even in our case we have certain additional nonholonomic distributions. For this, we can take the space of $(gB_{cc}, gB_{cc})$ strings and perform the deformation quantization of the complex nonholonomic symplectic manifold $Y$ (such constructions are presented in detail for almost complex models of gravity [8, 9, 18]). We get an associative d–algebra $gA$. Then we chose an antiholomorphic involution $\tau$, with a lift to the line bundle $Y \to Y$, and a component $V$ of the fixed point set supporting a $\tau$–invariant nonholonomic A–brane $gB'$. Now, we can say that $gA$ acts on $gH$ defined as the space of $(gB_{cc}, gB')$ strings. To consider the action $\Xi_{\tau}$ we model such a map acts on a function on $Y$, defining a $(gB_{cc}, gB_{cc})$ string as the composition of $\tau$ with complex conjugation. For an operator $O_f : gH \to gH$ associated to a function $f$, we
define the Hermitian adjoint of $\mathcal{O}_f$ to be associated with the function $\tau(\mathcal{F})$. Working with a real function $f$ when restricted to $V$ and if $\tau$ leaves $V$ fixed pointwise, we get $\tau(\mathcal{F}) = f$ with a Hermitian $\mathcal{O}_f$.

Finally, in this section, we conclude that the Gukov–Witten method \cite{1} really allows us to construct a physical viable Hilbert space for quantum almost Kähler models of Einstein gravity related to the Fedosov quantization of a corresponding complex nonholonomic symplectic manifold $Y$.

4 Quantization of the Almost Kähler Model of Einstein Gravity

In this section, we provide explicit constructions for quantum physical states of the almost Kähler model of Einstein gravity.

4.1 Coordinate parametrizations for almost Kähler gravitational A–models

The local coordinates on a nonholonomic complex manifold $Y$ are denoted $u^\alpha = (u^\alpha, i u^\dot{\alpha})$, for $i^2 = -1$, where $u^\alpha = (x^i, y^a)$ are local coordinates on $V$, and $\tilde{u}^\alpha = u^\dot{\alpha} = (\tilde{x}^i = x^i, \tilde{y}^a = y^a)$ are real local (pseudo) Euclidean coordinates of a conventional ”left–primed” nonholonomic manifold $\tilde{V}$. We emphasize that ”nonprimed” indices cannot be contracted with primed indices, because they label objects on different spaces. For (non)holonomic Einstein spaces, the coordinate indices will run values $i, j, ... = 1, 2; a, b, ... = 3, 4$ and $\dot{i}, \dot{j}, ... = 1, 2; \dot{a}, \dot{b}, ... = 3, 4$. We can also treat $\tilde{u}^\alpha = u^\dot{\alpha} = (\tilde{x}^i = x^j + i \dot{x}^j, \tilde{y}^a = y^a + i \dot{y}^a)$ as complex coordinates on $Y$, when, for instance, $\tilde{u}^1 = u^1 + i \dot{u}^1$.

In brief, such coordinates are labelled respectively $\tilde{u} = (\tilde{x}, \tilde{y}), u = (x, y)$ and $\dot{u} = (\dot{x}, \dot{y})$.

In general, such real manifolds $V, \tilde{V}$ and complex manifold $Y$ are enabled with respective $N$–connection structures, see formula (A.9), $\mathbf{N} = \{N^\alpha_i(u)\}, \tilde{\mathbf{N}} = \{\tilde{N}^\dot{\alpha}_i(\tilde{u})\}$ and $\mathbf{N} = \{\tilde{N}^\dot{\alpha}_i(\tilde{u})\} = \{N^\alpha_i(\tilde{u}) + i \dot{N}^\alpha_i(\tilde{u})\}$.

For $\tilde{V}$ and $Y$, the corresponding $N$–adapted bases (A.10) are

\[
\tilde{\mathbf{e}}_\nu = (\tilde{\mathbf{e}}_1 = \frac{\partial}{\partial \tilde{x}^i} - \tilde{N}^\dot{\alpha}_i(\tilde{u}) \frac{\partial}{\partial \tilde{y}^a}, \tilde{\mathbf{e}}_a = \frac{\partial}{\partial \tilde{y}^a}) = \\
\mathbf{e}_\nu = (\mathbf{e}_1 = \frac{\partial}{\partial x^i} - N^\alpha_i(u) \frac{\partial}{\partial y^a}, \mathbf{e}_a = \frac{\partial}{\partial y^a})
\]

and

\[
\tilde{\mathbf{e}}_\nu = (\tilde{\mathbf{e}}_1 = \frac{\partial}{\partial \tilde{x}^i} - \tilde{N}^\dot{\alpha}_i(\tilde{u}) \frac{\partial}{\partial \tilde{y}^a}, \tilde{\mathbf{e}}_a = \frac{\partial}{\partial \tilde{y}^a})
\]
and dual bases (A.11) are

\[ \hat{e}^\mu = \left( \hat{e}^i = dx^i, \hat{e}^a = dy^a + \hat{N}_i^a(\hat{u})dx^i \right) = \]

\[ e^{\hat{\mu}} = \left( e^i = dx^i, e^b = dy^b + \hat{N}_i^b(\hat{u})dx^i \right) \]

and

\[ \tilde{e}^\mu = \left( \tilde{e}^i = d\tilde{x}^i, \tilde{e}^a = d\tilde{y}^a + \tilde{N}_i^a(\tilde{u})d\tilde{x}^i \right). \]

We can also use parametrizations

\[ \tilde{e}_\nu = e_\nu = \left( e_\nu, -i\tilde{e}_\nu \right) \in T^*Y \]

and

\[ \tilde{e}_\mu = e_\mu = \left( e_\mu, -i\tilde{e}_\mu \right) \in T^*Y. \]

The above presented formulas for N–adapted (co) bases are derived following splitting (6) and (7).

The almost complex structure $K$ on $\hat{V}$ is defined similarly to (A.14)

\[ K = K^\alpha_\beta e_\alpha \otimes e^\beta = K^\alpha_\beta \frac{\partial}{\partial \hat{u}^\alpha} \otimes d\hat{u}^\beta = K^\alpha_\beta e_\alpha \otimes e^{\hat{\beta}} = -\hat{e}_{2+i} \otimes \hat{e}^i + \hat{e}_i \otimes \hat{e}^{2+i}. \]

The almost symplectic structure on a manifold $V$ is defined by

\[ J_\theta = g_\theta = \theta = L_\theta, \text{ see formula (A.18).} \]

A similar construction can be defined on $\hat{V}$ as $K_\theta = \hat{k_\theta} = \hat{\theta} = i\hat{\theta}$.

\[ \hat{\theta} = i\hat{\theta} = \frac{1}{2} \hat{L}_{ij}(\hat{u})\hat{\theta}^i \wedge \hat{\theta}^j + \frac{1}{2} \hat{L}_{ab}(\hat{u})\hat{\theta}^a \wedge \hat{\theta}^b \]

\[ = \hat{g}_{ij}(\hat{x}, \hat{y}) \left[ d\hat{y}^i + \hat{N}_i^k(\hat{x}, \hat{y})d\hat{x}^k \right] \wedge d\hat{x}^j. \]

Here, it should be noted that, in general, we can consider different geometric objects like the generation function $L(x, y)$, and metric $g_{ij}(x, y)$ on $V$ and, respectively, $\hat{L}(\hat{x}, \hat{y})$ and $\hat{g}_{ij}(\hat{x}, \hat{y})$ on $\hat{V}$, (as some particular cases, we can take two different exact solutions of classical Einstein equations). But the constructions from this section hold true also if we dub identically the geometric constructions on $V$ and $\hat{V}$.

A canonical holomorphic 2–form $\Theta$ of type $(2, 0)$ on $Y$, see [1], induced from the almost Kähler model of Einstein gravity, is computed

\[ \Theta = \Theta_{\bar{\mu}\nu} e^{\bar{\mu}} \wedge e^\nu, \]

for $\Theta_{\bar{\mu}\nu} = \tilde{\Theta}_{\mu\nu} = \left( \begin{array}{cc} \theta_{\mu\nu}(\hat{u}) & 0 \\ 0 & -i \theta_{\mu\nu}(\hat{u}) \end{array} \right)$ and general complex coordinates $\hat{u}^\mu$. This form is holomorphic by construction, $\overline{\partial} \Theta = 0$, for $\overline{\partial}$ defined by complex adjoints of $\hat{u}^\mu$. 

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The above presented formulations allow us a straightforward redefinition of the component tensor calculus adapted to N–connections structures on real nonholonomic manifolds and their almost Kähler models (see Appendix and details in Refs. [11, 12, 18, 32, 39]) to similar constructions with complex nonholonomic manifolds and geometric objects on such complex spaces.

4.2 N–adapted symmetries for nonholonomic curve flows and bi–Hamilton structures

An explicit construction of a Hilber space $^g\mathcal{H}$ with a Hermitian inner product on ($^g\mathcal{B}_{cc}$, $^g\mathcal{B}'$) strings for the almost Kähler model of Einstein gravity is possible if we prescribe in the theory certain generic groups of symmetries. There are proofs that metric structures on a (pseudo) Riemannian manifold [38, 39] can be decomposed into solitonic data with corresponding hierarchies of nonlinear waves. Such constructions hold true for more general types of Finsler–Lagrange–Hamilton geometries and/or their Ricci flows [40, 41] and related to the geometry of curve flows adapted to a N–connection structure on a (pseudo) Riemannian (in general, nonholonomic) manifold $V$. Our idea is to consider in quantum gravity models the same symmetries as for the ”solitonic” encoding of classical gravitational interactions.

A well known class of Riemannian manifolds for which the frame curvature matrix constant consists of the symmetric spaces $M = G/H$ for compact semisimple Lie groups $G \supset H$. A complete classification and summary of main results on such spaces are given in Refs. [42, 43]. The class of nonholonomic manifolds enabled with N–connection structure are characterized by conventional nonholonomic splitting of dimensions. For explicit constructions, we suppose that the ”horizontal” distribution is a symmetric space $hV = hG/SO(n)$ with the isotropy subgroup $hH = SO(n) \supset O(n)$ and the typical fiber space is a symmetric space $F = vG/SO(m)$ with the isotropy subgroup $vH = SO(m) \supset O(m)$. This means that $hG = SO(n+1)$ and $vG = SO(m+1)$ which is enough for a study of real holonomic and nonholonomic manifolds and geometric mechanics models.

\footnote{A non–stretching curve $\gamma(\tau, l)$ on $V$, where $\tau$ is a real parameter and $l$ is the arclength of the curve on $V$, is defined with such evolution d–vector $Y = \gamma_\tau$ and tangent d–vector $X = \gamma_l$ that $g(X, X) = 1$. Such a curve $\gamma(\tau, l)$ swept out a two–dimensional surface in $T_{\gamma(\tau, l)}V \subset TV$.}

\footnote{we can consider $hG = SU(n)$ and $vG = SU(m)$ for geometric models with spinor and gauge fields and in quantum gravity}
The Riemannian curvature and the metric tensors for \( M = G/H \) are covariantly constant and \( G \)-invariant resulting in constant curvature matrices. Such constructions are related to the formalism of bi–Hamiltonian operators, originally investigated for symmetric spaces with \( M = G/SO(n) \) with \( H = SO(n) \supset O(n-1) \) and when \( G = SO(n+1), SU(n) \), see [44] and references in [40, 41].

For nonholonomic manifolds, our aim was to solder in a canonical way (like in the \( N \)-connection geometry) the horizontal and vertical symmetric Riemannian spaces of dimension \( n \) and \( m \) with a (total) symmetric Riemannian space \( V \) of dimension \( n + m \), when \( V = G/SO(n+m) \) with the isotropy group \( H = SO(n+m) \supset O(n+m) \) and \( G = SO(n+m+1) \). There are natural settings to Klein geometry of the just mentioned horizontal, vertical and total symmetric Riemannian spaces: The metric tensor \( hg = \{ \hat{g}_{ij} \} \) on \( hV \) is defined by the Cartan–Killing inner product \( \langle \cdot, \cdot \rangle_V \) restricted to the Lie algebra quotient spaces \( hp = hg/hh \), with \( T_x hG \simeq hV \), where \( hg = h\mathfrak{h} \oplus hp \) is stated such that there is an involutive automorphism of \( hG \) under \( hH \) is fixed, i.e. \( \{ h\mathfrak{h}, hp \} \subseteq hp \) and \( \{ hp, hp \} \subseteq h\mathfrak{h} \).

Any metric structure with constant coefficients on \( V = G/SO(n+m) \) can be parametrized in the form

\[
\hat{g} = \hat{g}_{\alpha\beta} du^\alpha \otimes du^\beta,
\]

where \( u^\alpha \) are local coordinates and

\[
\hat{g}_{\alpha\beta} = \begin{bmatrix}
\hat{g}_{ij} + \hat{N}^a_i \hat{N}^b_j \hat{h}_{ab} & \hat{N}^e_i \hat{h}_{ae} \\
\hat{N}^c_i \hat{h}_{be} & \hat{h}_{ab}
\end{bmatrix}.
\]

The constant (trivial) \( N \)-connection coefficients in [13] are computed \( \hat{N}^e_i = \hat{h}^{eb} \hat{g}_{jb} \) for any given sets \( h^{eb} \) and \( \hat{g}_{jb} \), i.e. from the inverse metrics coefficients defined respectively on \( hG = SO(n+1) \) and by off–blocks \((n \times n)–\) and \((m \times m)–\)terms of the metric \( \hat{g}_{\alpha\beta} \). This way, we can define an equivalent
d–metric structure of type (A.1)

\[ \hat{g} = \hat{g}_{ij} \hat{e}^i \otimes \hat{e}^j + \hat{h}_{ab} \hat{e}^a \otimes \hat{e}^b, \]  
\[ e^i = dx^i, \quad \hat{e}^a = dy^a + N^a_i dx^i \]  
\[ e^i = \hat{e}^a = dy^a + N^a_i dx^i. \]  

(14)

defining a trivial \((n + m)\)–splitting \(\hat{g} = \hat{g} \oplus \hat{N} \hat{h}\) because all nonholonomy coefficients \(\hat{W}_{\alpha \beta}^\gamma\) and N–connection curvature coefficients \(\hat{\Omega}_{ij}^a\) are zero.

It is possible to consider any covariant coordinate transforms of (14) preserving the \((n + m)\)–splitting resulting in \(w_{\alpha \beta}^\gamma = 0\), see (A.12) and \(\Omega_{ij}^a = 0\) (A.13). Such trivial parametrizations define algebraic classifications of symmetric Riemannian spaces of dimension \(n + m\) with constant matrix curvature admitting splitting (by certain algebraic constraints) into symmetric Riemannian subspaces of dimension \(n\) and \(m\), also both with constant matrix curvature. This way, we get the simplest example of nonholonomic Riemannian space \(V = hG = SO(n + 1), vG = SO(m + 1), N_i^a\) possessing a Lie d–algebra symmetry \(\mathfrak{so}_N(n + m) \doteq \mathfrak{so}(n) \oplus \mathfrak{so}(m)\).

We can generalize the constructions if we consider nonholonomic distributions on \(V = G/\text{SO}(n + m)\) defined locally by arbitrary N–connection coefficients \(N_i^a(x, y)\), with nonvanishing \(w_{\alpha \beta}^\gamma\) and \(\Omega_{ij}^a\) but with constant d–metric coefficients when

\[ \hat{g} = g = \hat{g}_{ij} e^i \otimes e^j + \hat{h}_{ab} e^a \otimes e^b, \]  
\[ e^i = dx^i, \quad e^a = dy^a + N_i^a(x, y) dx^i. \]  

(15)

This metric is equivalent to a d–metric \(g_{\alpha \beta} = [g_{\alpha', \beta'}, h_{\alpha', \beta'}]\) (A.1) with constant coefficients induced by the corresponding Lie d–algebra structure \(\mathfrak{so}_N(n + m)\). Such spaces transform into nonholonomic manifolds \(V_N = [hG = SO(n + 1), vG = SO(m + 1), N_i^a]\) with nontrivial N–connection curvature and induced d–torsion coefficients of the d–connection (A.19). One has zero curvature for this d–connection (in general, such spaces are curved ones with generic off–diagonal metric (15) and nonzero curvature tensor for the Levi–Civita connection). So, such nonholonomic manifolds posses the same group and algebraic structures of couples of symmetric Riemannian spaces of dimension \(n\) and \(m\) but nonholonomically soldered to the symmetric Riemannian space of dimension \(n + m\). With respect to N–adapted orthonormal bases, with distinguished h– and v–subspaces, we obtain the same inner products and group and Lie algebra spaces as in (12).

The bi–Hamiltonian and solitonic constructions are based on an extrinsic approach soldering the Riemannian symmetric–space geometry to the Klein geometry [44]. For the N–anholonomic spaces of dimension \(n + m\), with a
constant d–curvature, similar constructions hold true but we have to adapt them to the N–connection structure. In Ref. [39], we proved that any (pseudo) Riemannian metric \( \mathbf{g} \) on \( \mathbf{V} \) defines a set of metric compatible d–connections of type

\[
\hat{\Gamma}^{\prime\prime}_{\alpha\beta\gamma} = \begin{pmatrix}
\hat{L}^{\prime\prime}_{j,k} = 0, \hat{L}^{\prime\prime\prime}_{i,j} = \text{const}, \hat{C}^{\prime\prime\prime}_{j,k} = 0, \hat{C}^{\prime\prime}_{j,k} = 0 \end{pmatrix}
\]

(16)

with respect to N–adapted frames \([A,10]\) and \([A,11]\) for any \( \mathbf{N} = \{N^\alpha(x, y)\} \) being a nontrivial solution of the system of equations

\[
2 \hat{a}^{\alpha\prime\prime\prime}_{j,k} = \frac{\partial N^\alpha_{k'}}{\partial y'^j} - \hat{h}^\alpha_{c'} \frac{\partial N^d_{k'}}{\partial y'^c} \quad (17)
\]

for any nondegenerate constant–coefficients symmetric matrix \( \hat{h}_{c'd'} \) and its inverse \( \hat{h}^\alpha_{c'} \). Here, we emphasize that the coefficients \( \hat{\Gamma}^{\prime\prime}_{\alpha\beta\gamma} \) of the corresponding to \( \mathbf{g} \) Levi–Civita connection \( \mathbf{g}\nabla \) are not constant with respect to N–adapted frames.

By straightforward computations, we get that the curvature d–tensor of a d–connection \( \hat{\Gamma}^{\prime\prime}_{\alpha\beta\gamma} \) defined by a metric \( \mathbf{g} \) has constant coefficients

\[
\hat{\mathbf{R}}^{\prime\prime}_{\beta\gamma\delta\alpha} = \begin{pmatrix}
\hat{R}^{\prime\prime}_{h,j,k} = 0, \hat{R}^{\prime\prime\prime}_{b,j'k'} = \hat{L}^{\prime\prime\prime}_{b,j'} \hat{L}^{\prime\prime\prime}_{c,k'} - \hat{L}^{\prime\prime\prime}_{b,k'} \hat{L}^{\prime\prime\prime}_{c,j'} = \text{cons.} \\
\end{pmatrix}
\]

(16)

(17)

with respect to N–adapted frames \( \mathbf{e}_\alpha = \{e_\alpha, e_\alpha'\} \) and \( \mathbf{e}' = \{e', e_0', \hat{h}_{d'c'} \} \) when \( N^d_{k'} \) are subjected to the conditions \( (17) \). Using a deformation relation of type \( (A,27) \), we can compute the corresponding Ricci tensor \( \hat{R}^{\prime\prime\prime}_{\alpha\beta\gamma} \) for the Levi–Civita connection \( \mathbf{g}\nabla \), which is a general one with 'non-constant' coefficients with respect to any local frames.

A d–connection \( \hat{\Gamma}^{\prime\prime}_{\alpha\beta\gamma} \) defined by a metric \( \mathbf{g} \) has constant scalar curvature,

\[
\hat{\tilde{R}} = \hat{g}^{\alpha\beta\gamma} \hat{R}^{\alpha\beta\gamma} = \hat{g}^{\alpha\beta\gamma} \hat{R}^{\alpha\beta\gamma} + \hat{h}_{d'c'} \hat{S}_{d'c'} = \hat{\tilde{R}} + \hat{\tilde{S}} = \text{const.}
\]

Nevertheless, the scalar curvature \( \nabla \tilde{R} \) of \( \mathbf{g}\nabla \), for the same metric, is not constant.

The constructions with different types of metric compatible connections generated by a metric structure are summarized in Table 1.

We conclude that the algebraic structure of nonholonomic spaces enabled with N–connection structure is defined by a conventional splitting of dimensions with certain holonomic and nonholonomic variables (defining a distribution of horizontal and vertical subspaces). Such subspaces are
Table 1: Some metric connections generated by a \( d \)-metric \( g = \{ g_{\alpha\beta} \} \)

| Geometric objects for: | Levi-Civita connection | normal \( d \)-connection | constant coefficients \( d \)-connection |
|------------------------|-------------------------|-----------------------------|------------------------------------------|
| Co-frames \( g_{\alpha\beta} = A^\dagger (u) du_A \) | \( e^\alpha = [e^i = dx^i, \]
\( e^a = dy^a - N^a_j dx^j] \) \( \) | \( e^\alpha' = [e^i = dx^i', \]
\( e^a' = dy^a - N^a'_j dx^j'] \) | \( \) |
| Metric decomp. \( \) | \( g_{\alpha\beta} = [g_{ij}, h_{ab}] \)
\( g = g_{ij} e^i \otimes e^j + h_{ab} e^a \otimes e^b \) \( \) | \( \) | \( \) |
| Connections and distortions | \( \Gamma^\gamma_{\alpha\beta} \)
\( \Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + \mathcal{Z}^\gamma_{\alpha\beta} \) \( \) | \( \) | \( \) |
| Riemannian (\( d \))-tensors | \( R^\alpha_{\beta\gamma\delta} \)
\( R^\alpha_{\beta\gamma\delta} = \) \( \) | \( \) | \( \) |

modelled locally as Riemannian symmetric manifolds and their properties are exhausted by the geometry of distinguished Lie groups \( G = GO(n) \oplus GO(m) \) and \( G = SU(n) \oplus SU(m) \) and the geometry of \( N \)-connections on a conventional vector bundle with base manifold \( M, \) \( \dim M = n, \) and typical fiber \( F, \) \( \dim F = m. \) For constructions related to Einstein gravity, we have to consider \( n = 2 \) and \( m = 2. \) This can be formulated equivalently in terms of geometric objects on couples of Klein spaces. The bi–Hamiltonian and related solitonic (of type mKdV and SG) hierarchies are generated naturally by wave map equations and recursion operators associated to the horizontal and vertical flows of curves on such spaces [38, 39, 40, 41].

The approach allowed us to elaborate a ”solitonic” formalism when the geometry of (semi) Riemannian / Einstein manifolds is encoded into nonholonomic hierarchies of bi–Hamiltonian structures and related solitonic equations derived for curve flows on spaces with conventional splitting of dimensions. The same distinguished group (\( d \)-group) formalism may be applied for quantum string models and nonholonomic A–branes for the almost Kähler model of Einstein gravity.

### 4.3 Nonholonomic deformations and quantization of distinguished Chern–Simons gravity theories

Let us consider a metric field \( g \) which is a solution of usual Einstein equations [3]. For different purposes, we can work equivalently with any

26
linear connection $\mathcal{g}\nabla = \{ \mathcal{g}\Gamma^\alpha_{\beta\gamma} \}$, $\mathcal{D} = \{ \nabla^\gamma_{\alpha\beta} \}$, or $\mathcal{O}\mathcal{D} = \{ \mathcal{O}\nabla^\gamma_{\alpha\beta} \}$, for $\mathcal{g}$ and/or $\mathcal{g}\theta(X, Y) \equiv \mathcal{g}(JX, Y)$. For any $\mathcal{g}\theta = (h\theta, v\theta)$, we can associate two symplectic forms related formally to a Chern–Simons theory with a compact gauge d–group $\mathcal{G} = GO(2) \oplus GO(2)$, or $\mathcal{G} = SU(2) \oplus SU(2)$ (for simplicity, hereafter we shall consider the case of unitary d–groups, one for the h–part, $h\mathcal{G} = SU(2)$, and another for the v–part, $v\mathcal{G} = SU(2)$).

We chose two two–manifolds without boundary, denoted $h\mathcal{C}$ and $v\mathcal{C}$ and consider $\mathcal{G}\mathcal{V} = (h\mathcal{V}, v\mathcal{V})$ to be defined by a couple of moduli spaces of homomorphisms (up to conjugation) from $\pi_1(h\mathcal{C})$ into $h\mathcal{G}$ and, respectively, $\pi_1(v\mathcal{C})$ into $v\mathcal{G}$, of given topological types. We can consider the same local coordinate parametrization for $\mathcal{G}\mathcal{V}$ and open regions of a nonholonomic spacetime $\mathcal{V}$. The further geometric constructions are related to a symplectic d–structure on the infinite–dimensional linear d–spaces $\mathcal{G}\mathcal{A} = (h\mathcal{A}, v\mathcal{A})$ as all couples of d–connections on a distinguished $\mathcal{G}$–bundle $\mathcal{E} \rightarrow C$, for $\mathcal{C} = h\mathcal{C} \oplus v\mathcal{C}$.

We fix such a parametrization of coefficients of a gravitation symplectic d–form $\mathcal{g}\theta = (h\theta, v\theta)$ which in a point of $\mathcal{V}$ is proportional to the coefficients

$$h\theta = \frac{1}{4\pi} \int_{h\mathcal{C}} Tr \delta h A \wedge \delta h A \quad \text{and} \quad v\theta = \frac{1}{4\pi} \int_{v\mathcal{C}} Tr \delta v A \wedge \delta v A,$$

when, in brief, in “boldfaced” form, $\frac{h\theta}{4\pi} = \int_{h\mathcal{C}} Tr \delta h A \wedge \delta A$. The trace symbol $Tr$ is considered respectively for the $h$– and $v$–forms, as invariant ones on the Lie algebras $h\mathcal{g}$, of $h\mathcal{G}$, and $v\mathcal{g}$, of $v\mathcal{G}$, in our case, in the 2–dimensional representation. Such $h\theta$, or $v\theta$, are normalized to $\theta/2\pi$ being the image in de Rham cohomology of a generator of $H^2(h\mathcal{V}, \mathbb{Z}) \cong \mathbb{Z}$, or $H^2(v\mathcal{V}, \mathbb{Z}) \cong \mathbb{Z}$. Here, it should be emphasized that such local identifications of the gravitational almost Kähler symplectic structures with couples of symplectic structure of respective Chern–Simons theories (with $\mathcal{G}\mathcal{V}$ modelling the classical phase d–spaces for such models) do not impose elaboration of classical and/or gravitational models with structures d–groups of type $\mathcal{G}$. We only fixed an explicit common parametrization for a “background” curve flow network the chosen method of quantization and generating gravitational solitonic hierarchies, like in [39]. Real classical/quantum Einstein gravitational interactions can be generated by deformations of connections $\mathcal{g}\nabla = \mathcal{g}\nabla + \mathcal{g}Z$, where $\mathcal{g}\nabla$ is any necessary type connection, for instance, parametrized as a gauge one in a Chern–Simons d–model, $\mathcal{A} = (h\mathcal{A}, v\mathcal{A})$ with coefficients determined by a d–connection (A.22), or any its nonholonomic transform, but the related $\mathcal{g}Z$ is such a distorsion tensor which nonholonomically deforms $\mathcal{g}\nabla$ into $\mathcal{g}\nabla$ defined by an Einstein solution, in the
classical limit. Any such schemes with equivalent geometric objects defined by a d–metric $g$ but for suitable nonholonomic and topologic structures correspond to a N–connection splitting and adapted frames as it is described in Figure 2.

Levi–Civita variables:
$$(g, g\nabla) = (g_{\mu\nu}, g^{\alpha}_{\beta\gamma})$$

$2+2$ nonholonomic splitting; generating function $L(x, y)$

almost Kähler variables:
$$(g\theta, \tilde{D}, \tilde{\mathcal{N}}) = (\theta_{\alpha\beta}, \tilde{\Gamma}_{\alpha\beta}^\gamma)$$

metricity: $\tilde{D}g = 0$, $g^{\alpha}_{\beta\gamma} = \tilde{\Gamma}_{\beta\gamma}^\alpha + g^{\alpha}_{\beta\gamma}$

constant coefficient variables:
$$(\circ g_{\alpha\beta}, \circ D, N)$$

metricity: $\circ D \circ g = 0$, $g^{\alpha}_{\beta\gamma} = \circ \Gamma_{\beta\gamma}^\alpha + \circ Z_{\beta\gamma}^\alpha$

Classical Einstein Spaces

Quantum almost Kähler Einstein Spaces

Figure 2: The Levi–Civita, normal and constant coefficients connections in Einstein gravity

Next we introduce a distinguished line bundle (line d–bundle).\(^{11}\) Fixing two integers $k = (h_k, v_k)$, we can quantize our nonholonomic model $G^V$ as in \(^{11}\) but using a symplectic d–form $\ast \theta = (h_k h_\theta, v_k v_\theta)$ with a prequantum

\(^{11}\)This can be constructed as in usual Chern–Simons theory \(^{45}^{46}\) but using d–groups.
line d–bundle \( * \mathcal{V} = h^k \mathcal{V} \oplus v^k \mathcal{V} \). Taking \( ^G Y \) to be the distinguished moduli space of homomorphysms (preserving the splitting by a prescribed \( \mathbb{N} \)–connection structure) from \( \pi_1(\mathcal{C}) \) to \( ^h G_C \oplus ^v G_C \), for \( \mathcal{C} = ^h C \oplus ^v C \) and \(^h G_C\) and \(^v G_C\) being the respective complexifications of the respective \( h \)– and \( v \)–groups (up to conjugation), we define \( ^G Y \) as a natural nonholonomic complex manifold.

We denote by \( r = ( ^h r, ^v r ) \) certain finite–dimensional representations of complex Lie d–group \( ^h G_C \oplus ^v G_C \) and consider two oriented closed loops \( s = ( ^h s, ^v s ) \) on \(^h C \oplus ^v C\) and the holonomies of respective two flat connections (defining a d–connection) around \( s \), denoted respectively \( \text{Hol}(s) = \text{Hol}( ^h s ) \oplus \text{Hol}( ^v s ) \). This way, we define

\[
H_r(s) = Tr_{^h} \text{Hol}( ^h s ) + Tr_{^v} \text{Hol}( ^v s )
\]

(19)

is a holomorphic function on \( ^G Y \).

For a gauge \( ^g \mathcal{C} \)–valued d–connection \( A = ( ^h A, ^v A ) \), the function (19) can be written using an oriented exponential product \( P \) on both \( h \)– and \( v \)–subspaces,

\[
H_r(s) = Tr \ P \exp \left( - \oint_{^h s} ^h A \right) + Tr \ P \exp \left( - \oint_{^v s} ^v A \right).
\]

The restrictions of such holomorphic functions on \( ^G Y \) define a dense set of functions on \( ^G V \) associated to \( V \). Using nonholonomic transforms preserving the \( \mathbb{N} \)–connection structure, and corresponding deformations of d–connections, we can relate such distinguished group constructions to those with a gravitationally induced symplectic form on \( Y \).

The \( ^g \mathcal{C} \)–valued d–connection \( A \) also generates a nondegenerate holomorphic distinguished 2–form \( ^* \Theta = ( ^h \Theta, ^v \Theta ) \); we use the complexified formulas (19). We can consider \( ^G Y \) as a complex symplectic manifold enabled by symplectic d–form \( ^G \Theta = \left( h^k, v^k \right) \Theta \), with a restriction of \( \Theta \) to \( ^G V \) coinciding with \( \Theta \). This also allows us to construct a nonholonomic A–model of \( ^G Y \) with symplectic structure \( ^G \theta = \text{Im}^G \Theta \) like we have done at the beginning of section 3.2. Such an A–model dubs the constructions from \[1\] (see there sections 1.3 and 2.3) and also can be endowed with a complete hyper–Kähler metric (consisting from \( h \)– and \( v \)–parts) extending its structure as a complex symplex manifold. It is a ”very good” A–model,
which allows us to pick up complex structures on $C = hC \oplus vC$ and define complex structures on $G Y$ in a natural way (requiring no structures on $hC$ and $vC$ except corresponding orientations).

There is also a natural antiholomorphic involution $G \tau : G Y \to G Y$ as a complex conjugation of all monodromies (preserving $h$– and $v$–components), were $G V$ is the component of the fixed point set of $G \tau$ (i.e. is is the locus in $G Y$ of all monodromy with values in $G = SU(2) \oplus SU(2)$).

Having introduced two branes in the A–model of $G Y$, we can perform the Gukov–Witten quantization of $G V$. The first brane is a distinguished one consisting from two canonical isotropic branes, $G B_{cc} = (h_{B_{cc}}, v_{B_{cc}})$, with curvature form $Re G \Theta$ and support by all $G Y$. We want to quantize the symplectic d–form $\hat{\theta} = (h \hat{k}, v \hat{k})$ which is the restriction of $Re G \Theta$ to $G V$. The second brane $G B'$ is defined as a unique one up to an isomorphism preserving $h$– and $v$–splitting when $G V$ is a simply–connected spin manifold; this is also a rank 1 A–brane supported on $G V$.

The $N$–adapted diffeomorphisms of $C = hC \oplus vC$ that are continuously connected to the identity on $h$–part and identity on $v$–part act trivially both on the A–model of $G Y$ and on $G Y$. Such diffeomorphisms do not preserve hyper–Kähler metrics on $G Y$, but this is not a problem because the A–model observables do not depend on a fixed hyper–Kähler metric, see more details in [10], on nonholonomic manifolds and Hamilton–Cartan spaces and their deformation quantization. Following [1], we can consider the Teichmüller space $T$ of $C$ when any point $\zeta \in T$ determines (in a unique way, up to isotopy) a complex structure on $C$ and (as a result) a hyper–Kähler polarization of $(G Y, G V)$. The interesting thing is that the space $G \zeta H$ of $(G B_{cc}, G B')$ strings constructed with such a polarization does not depend on $\zeta$. This follows from the fact that the A–model is invariant under a local change in the hyper–Kähler polarization and we can also define $G \zeta H$ as a typical fiber of a flat d–vector bundle $G H$ over $T$.

In general, it is a difficult task to compute exactly the (Hilbert) space $G H$ for certain general nonholonomic manifolds (this would imply the index theorem for a family of nonholonomic Dirac operators, see [36, 37, 11]). Nevertheless, in the case relevant to Einstein gravity and solitonic hierarchies, i.e. for $G = SU(2) \oplus SU(2)$, one holds a standard (N–adapted) algebro–geometric description of a physical Hilbert space of a nonholonomic Chern–Simons theory at levels $\hat{k} = (h \hat{k}, v \hat{k})$, when $G H = H^0(G V, (h \hat{k}, v \hat{k}) \otimes (h \hat{k}, v \hat{k}))$. Here one should be noted that in the nonholonomic A–model the (classical) commutative algebra of holonomy functions $H_{r}(s)$ [19] is nonholonomically deformed to a noncommutative d–algebra $A = (h A, vA)$, i.e. the
space of \((G_Bcc, G_Bcc)\) strings, which acts on \(G\mathcal{H}\). When we work with a nonholonomic Chern–Simons gauge theory, the quantization transforms Wilson loops on a distinguished \(C = hC \oplus vC\) into operators that acts on the quantum Hilbert space.

The Gukov–Witten quantization method is very powerful because it allows us to consider nonholonomic deformations of geometric classical and quantum structures and the very same \(d\)-algebra \(A = (h, v)\) acts on the spaces of any nonholonomic strings for any other choice of nonholonomic A-brane \(\mathcal{B}\). Let us explain these new applications to gravity and nonholonomic geometries which were not considered in [1]:

The coefficients of the gauge \(g_{C}\)-valued \(d\)-connection \(A = (h, v)\) used for constructing our nonholonomic Chern–Simons theory can be identified with the coefficients of a \(d\)-connection of type \(\tilde{\Gamma}^i_j = \tilde{L}^i_j e^k + \tilde{C}^i_{jk} e^k\), but with constant \(d\)-connection coefficients, i.e. \(\tilde{\Gamma}^\gamma_{\alpha'\beta'}\), when via corresponding distortion tensor, see Figure 2, \(\tilde{\Gamma}^i_j = g_{ij} \tilde{L}^i_j e^k\) with \(hA^i_j = \tilde{L}^i_j\) and \(vA = 0\) (but this may be for an explicit local parametrization, in general \(vA\) is not zero). Further nonholonomic transforms from \(\tilde{\Gamma}\) to \(\Gamma\) change the nonholonomic structure of geometric objects but does not affect the defined above hyper–Kähler polarization. This means that the former \(g_{C}\)-valued \(d\)-connection structure \(A = (h, v)\) relevant to a chosen parametrization of curve flows, for spaces \((G_Y, G_V)\), deforms nonholonomically, but equivalently, into a couple of nonholonomic manifolds \((g_Y, g_V)\) if the metric structure of \(d\)-group \(G = SU(2) \oplus SU(2)\) maps nonholonomically into a \(d\)-metric \(g\). For such constructions, the corresponding (classical) Levi–Civita connection \(\gamma_i\Gamma\) is constrained to be a solution of the Einstein equations [3]. Using vierbein transforms of type \(A.0\) relating \((L_g, L_N)\), see \(A.5\) and \(A.7\), to an "Einstein solution" \((g, N)\), and similar transforms to \((\tilde{g}, \tilde{N})\), we nonholonomically deform the space of \((G_Bcc, G_Bcc)\) strings into that of \((g_Bcc, g_Bcc)\) strings. This obviously result in equivalent transforms of \(G\mathcal{H} = H^0(\mathcal{G}\mathcal{V}, *_k\mathcal{V} \oplus *_k\mathcal{V})\) into the Hilbert space \(g\mathcal{H} = \oplus_{j=0}^{\dim_c} H^j(\mathcal{V}, K^{1/2} \otimes g\mathcal{V})\) which is a good approximation for both holonomic and nonholonomic quantum Einstein spaces.

Finally, we conclude that crucial for such a quantization with nonholonomic A–branes and strings relevant to Einstein gravity are the constructions when we define the almost Kähler variables in general relativity, in Section 2 and coordinate parametrizations for almost Kähler gravitational A–models, in Section 4.1. The associated constructions with a nonholonomic Chern–Simons theory for a gauge \(d\)-group \(G = SU(2) \oplus SU(2)\) are also important because they allow us to apply both a computation tech-
niques formally elaborated for topological gauge models and relate the constructions to further developments for quantum curve flows, nonholonomic bi–Hamilton structures and derived solitonic hierarchies.

5 Final Remarks

In the present paper we have applied the Gukov–Witten formalism [1] to quantize the almost Kähler model of Einstein gravity. We have used some our former results on deformation and loop quantization of gravity following ideas and methods from the geometry of nonholonomic manifolds and (non) commutative spaces enabled with nonholonomic distributions and associated nonlinear connections structures [8, 9, 16, 18]. It was shown that the approach with nonholonomic A–branes endowed with geometric structures induced by (pseudo) Riemannian metrics serves to quantize standard gravity theories and redefine previous geometric results on the language of string theories and branes subjected to different types of nonholonomic constrains.

The A–model approach to quantization [1, 24, 25, 46] seems to be efficient for elaborating quantum versions of (non) commutative gauge models of gravity [29, 32], nonholonomic Clifford–Lie algebroid systems [35, 34], gerbes and Clifford modules [36, 37] etc when a synthesis with the methods of geometric [20, 21, 22, 23] and deformation [3, 4, 5] quantization is considered. Further perspectives are related to nonholonomic Ricci flows and almost Kähler models of spaces with symmetric and nonsymmetric metrics [19, 40].

It was shown that deformation quantization of the relativistic particles gives the same results as the canonical quantization and path integral methods and the direction was developed for systems with second class constraints. On such results, we cite a series of works on commutative and noncommutative physical models of particles and strings [47, 48, 49, 50, 51, 52, 53, 6, 7] and emphasize that the Stratonovich–Weyl quantizer, Weyl correspondence, Moyal product and the Wigner function were obtained for all the analyzed systems which allows a straightforward generalization to nonholonomic spaces and related models of gravity, gauge and spinor interactions and strings [8, 9, 29, 32, 27, 28]. Introducing almost Kähler variables for gravity theories, such constructions and generalizations can be deformed nonholonomically to relate (for certain well defined limits) the Gukov–Witten quantization to deformation and geometric quantization,

\[\text{and other more general nonholonomic geometric and physical models, for instance, various types Finsler–Lagrange and Hamilton–Cartan spaces etc}\]
loop configurations and noncommutative geometry.

One of the main results of this work is that we have shown that it is possible to construct in explicit form a Hilbert space with a Hermitian inner product in quantum gravity on certain couples of strings for the almost Kähler model of Einstein gravity. This is completely different from the loop quantum gravity philosophy and methods (see critical remarks, discussions and references in [54, 55, 16]) when the background field method is not legitimate for non-perturbative considerations. Working with almost Kähler variables, it is obvious how the constructions for deformation quantization of gravity, loop quantum gravity (in our case with nonholonomic Ashtekar variables) and other directions can be related to be equivalent for certain quantum and/or classical configurations. Here, it should be noted that different methods of quantization of nonlinear field/string theories, in general, result in very different quantum theories.

Nevertheless, we have not analyzed the Gukov–Witten method and the almost Kähler approach to gravity (relevant also to the proposed Fedosov quantization of the Einstein theory and Lagrange–Finsler systems) in connection to one– and two–loops computations in perturbative gravity [56, 57]. We also have not concerned the problem of non-renormalizability of Einstein gravity (see recent reviews of results in Refs. [58, 59]) and have not discussed the ideas on a possible asymptotic safety scenario in quantum gravity [60, 61] in connection to the A–model formalism and deformation quantization.

Our approach with compatible multi–connections defined by a metric structure (in particular, by a solution of the Einstein equation) allows us to put the above mentioned problems of non–renormalizability and safety of gravitational interactions in a different manner, when the background constructions are re–defined for an alternative distinguished connection for which the one–, two– and certain higher order loops contributions can be formally renormalized (such constructions with necessary types of background distinguished connections are under elaboration). But even in such cases, the dimension of gravitational constant states explicitly that a standard renormalization similar to gauge models is not possible for the Einstein gravity.

Perhaps, a variant of modified by nonholonomic distorsions of gravitational connections resulting in a gauge like model of gravity with an additional effective constant may present interest for applications in modern cosmology and high energy physics. To work with almost Kähler variables and using the constructions for Hilbert spaces and nonholonomic methods developed in this paper is to provide a beginning for future investigations on
effective and modified perturbative gravity models and more general non–
perturbative nonholonomic quantum geometries.

The extension of the nonlinear connection formalism to methods of quan-
tization and application to more general supergravity and superstring theory [62, 63, 64] and spaces enabled with nonholonomic (super) distributions [11, 28] consist a set of open problems that may be pursued in the near future. It is interesting also to apply the matters of almost Kähler variables and nonholonomic classical and quantum interactions and geometries to more complicated second class constrained systems as the BRST quantization in gauge and gravity theories and Batalin–Wilkovisky quantization (see original results and reviews in [65, 66, 67]). We are going to address such topics in the future.

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A Almost Kähler Variables in Component Form

We parametrize a general (pseudo) Riemannian metric \( g \) on a spacetime \( V \) in the form:

\[
  g = g_{ij'}(u)e^i \otimes e^{j'} + h_{\alpha'\beta'}(u)e^{\alpha'} \otimes e^{\beta'},
\]

(A.1)

\[
e^{\alpha'} = e^{\alpha'} - N^{a'}_{\alpha'}(u) e^{a'},
\]

where the vierbein coefficients \( e^{\alpha'}_{\alpha} \) of the dual basis

\[
  e^{\alpha'} = (e^{i'},e^{a'}) = e^{\alpha'}_{\alpha}(u) du^\alpha,
\]

(A.2)

define a formal \( 2 + 2 \) splitting.

Let us consider any generating function \( L(u) = L(x^i,y^a) \) on \( V \) (it is a formal pseudo–Lagrangian if an effective continuous mechanical model of general relativity is elaborated, see Refs. [12, 11] with nondegenerate Hessian

\[
  L h_{ab} = \frac{\partial^2 L}{2 \partial y^a \partial y^b},
\]

(A.3)

when \( \det | L h_{ab} | \neq 0 \). We define

\[
  L N^a_i = \frac{\partial G^a}{\partial y^{2+i}},
\]

(A.4)

\[
  G^a = \frac{1}{4} L h_{a 2+i} \left( \frac{\partial^2 L}{\partial y^{2+i} \partial x^k} y^{2+k} - \frac{\partial L}{\partial x^i} \right),
\]

(A.5)
where $L_{h}^{ab}$ is inverse to $L_{h}^{ab}$ and respective contractions of $h$– and $v$– indices, $i,j,$ ... and $a,b,$... are performed following the rule: we can write, for instance, an up $v$–index $a$ as $a = 2 + i$ and contract it with a low index $i = 1,2$. Briefly, we shall write $y^i$ instead of $y^{2+i}$, or $y^a$. The values (A.3) and (A.4) allow us to consider

$$\Lambda g = L_{g}^{ij}dx^i \otimes dx^j + L_{h}^{ab}L e^a \otimes L e^b,$$

(A.5)

$$L e^a = dy^a + L N_i^a dx^i, \quad L g_{ij} = L h_{2+i 2+j}.$$  

A metric $g$ (A.1) with coefficients $g_{\alpha' \beta'} = [g_{i'j'}, h_{a'b'}]$ computed with respect to a dual basis $e_{\alpha'} = (e_{i'}, e_{a'})$ can be related to the metric $L g_{\alpha \beta} = [L g_{ij}, L h_{ab}]$ (A.5) with coefficients defined with respect to a N–adapted dual basis $L e^\alpha = (dx^i, L e^a)$ if there are satisfied the conditions

$$g_{\alpha' \beta'} e_{\alpha'}^i e_{\beta'}^a = L g_{\alpha \beta}. \quad (A.6)$$

Considering any given values $g_{\alpha' \beta'}$ and $L g_{\alpha \beta}$, we have to solve a system of quadratic algebraic equations with unknown variables $e_{\alpha'}$, see details in Ref. [16]. Usually, for given values $[g_{i'j'}, h_{a'b'}, N_i^a]$ and $[L g_{ij}, L h_{ab}, L N_i^a]$, we can write

$$N_i^a = e_{i'}^i e_{a'}^a L N_i^a \quad (A.7)$$

for $e_{i'}$ being inverse to $e_{i'}$.

A nonlinear connection (N–connection) structure $N$ on $V$ can be introduced as a nonholonomic distribution (a Whitney sum)

$$TV = hV \oplus vV \quad (A.8)$$

into conventional horizontal (h) and vertical (v) subspaces. In local form, a N–connection is given by its coefficients $N_i^a(u)$, when

$$N = N_i^a(u)dx^i \otimes \frac{\partial}{\partial y^a}. \quad (A.9)$$

A N–connection on $V^{n+n}$ induces a (N–adapted) frame (vielbein) structure

$$e_{\nu} = \left( e_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a} \right), \quad (A.10)$$

and a dual frame (coframe) structure

$$e^{\mu} = (e^{i} = dx^{i}, e^{a} = dy^{a} + N_i^a(u)dx^{i}). \quad (A.11)$$
The vielbeins (A.11) satisfy the nonholonomy relations

\[ [e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = w^\gamma_{\alpha\beta} e_\gamma \]  

(A.12)

with (antisymmetric) nontrivial anholonomy coefficients \( w^b_{ia} = \partial_a N^b_i \) and \( w^a_{ji} = \Omega^a_{ij} \), where

\[ \Omega^a_{ij} = e_j (N^a_i) - e_i (N^a_j) \]  

(A.13)

are the coefficients of N–connection curvature (defined as the Neijenhuis tensor on \( V^{n+n} \)). The particular holonomic/ integrable case is selected by the integrability conditions \( w^\gamma_{\alpha\beta} = 0 \).

A N–anholonomic manifold is a (nonholonomic) manifold enabled with N–connection structure (A.8). The geometric properties of a N–anholonomic manifold are distinguished by some N–adapted bases (A.10) and (A.11). A geometric object is N–adapted (equivalently, distinguished), i.e. a d–object, if it can be defined by components adapted to the splitting (A.8) (one uses terms d–vector, d–form, d–tensor). For instance, a d–vector \( X = X^\alpha e_\alpha = X^i e_i + X^a e_a \) and a one d–form \( \tilde{X} \) (dual to \( X \)) is \( \tilde{X} = X_\alpha e^\alpha = X_i e^i + X_a e^a \).

We introduce a linear operator \( J \) acting on vectors on \( V \) following formulas \( J(e_i) = -e_{2+i} \) and \( J(e_{2+i}) = e_i \), where and \( J \circ J = -I \), for \( I \) being the unity matrix, and construct a tensor field on \( V \),

\[ J = J^\alpha_\beta e_\alpha \otimes e^\beta = J^\alpha_\beta \frac{\partial}{\partial u^\alpha} \otimes du^\beta \]

(A.14)

\[ = J^\alpha_\beta e_\alpha^\prime \otimes e^\beta = -e_{2+i} \otimes e^i + e_i \otimes e^{2+i} \]

\[ = -\frac{\partial}{\partial y^i} \otimes dx^i + \left( \frac{\partial}{\partial x^i} - L_{N^{2+j}_i} \frac{\partial}{\partial y^i} \right) \otimes \left( dy^i + L_{N^{2+i}_k} dx^k \right), \]

defining globally an almost complex structure on \( V \) completely determined by a fixed \( L(x,y) \). In this work we consider only structures \( J = L J \) induced by a \( L N^{2+j}_i \), i.e. one should be written \( L J \), but, for simplicity, we shall omit left label \( L \), because the constructions hold true for any regular generating function \( L(x,y) \). Using vielbeins \( e_\alpha \) and their duals \( e^\alpha \), defined by \( e^\alpha_{\beta} \) solving (A.6), we can compute the coefficients of tensor \( J \) with respect to any local basis \( e_\alpha \) and \( e^\alpha \) on \( V \),

\[ J^\alpha_\beta e_\alpha \otimes e^\beta = J^\alpha_\beta e_\alpha \otimes J^\alpha_\beta e^\beta. \]  

In general, we can define

\[ ^{14} \text{we use boldface symbols for spaces (and geometric objects on such spaces) enabled with N–connection structure} \]

\[ ^{15} \text{We can redefine equivalently the geometric constructions for arbitrary frame and coordinate systems; the N–adapted constructions allow us to preserve the conventional h– and v–splitting.} \]
an almost complex structure $J$ for an arbitrary $N$–connection $N$, stating a
nonholonomic $2 + 2$ splitting, by using $N$–adapted bases (A.10) and (A.11).

The Neijenhuis tensor field for any almost complex structure $J$ defined
by a $N$–connection (equivalently, the curvature of $N$–connection) is

\[ J\Omega(X, Y) = -[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY], \tag{A.15} \]

for any $d$–vectors $X$ and $Y$. With respect to $N$–adapted bases (A.10) and
(A.11), a subset of the coefficients of the Neijenhuis tensor defines the $N$–
connection curvature,

\[ \Omega^a_{ij} = \frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b}. \tag{A.16} \]

A $N$–anholonomic manifold $V$ is integrable if $\Omega^a_{ij} = 0$. We get a complex
structure if and only if both the $h$– and $v$–distributions are integrable, i.e.
if and only if $\Omega^a_{ij} = 0$ and

\[ \frac{\partial N^a_i}{\partial y^j} - \frac{\partial N^a_j}{\partial y^i} = 0. \]

One calls an almost symplectic structure on a manifold $V$ a nondegen-
erate 2–form

\[ \theta = \frac{1}{2} \theta_{\alpha\beta}(u) e^\alpha \wedge e^\beta = \frac{1}{2} \theta_{ij}(u) e^i \wedge e^j + \frac{1}{2} \theta_{ab}(u) e^a \wedge e^b. \]

An almost Hermitian model of a (pseudo) Riemannian space $V$ equipped
with a $N$–connection structure $N$ is defined by a triple $H_2+2 = (V, \theta, J)$,
where $\theta(X, Y) = g(JX, Y)$ for any $g$ (A.1). A space $H_2+2$ is almost Kähler,
denoted $K_2+2$, if and only if $d\theta = 0$.

For $g = Lg$ (A.5) and structures $LN$ (A.4) and $J$ canonically defined
by $L$, we define $L\theta(X, Y) = Lg(JX, Y)$ for any $d$–vectors $X$ and $Y$. In
local $N$–adapted form form, we have

\[ L\theta = \frac{1}{2} L\theta_{\alpha\beta}(u) e^\alpha \wedge e^\beta = \frac{1}{2} L\theta_{ij}(u) du^i \wedge du^j \tag{A.17} \]

\[ = Lg_{ij}(x, y) e^{2+i} \wedge dx^j = Lg_{ij}(x, y)(dy^{2+i} + N^i_k(x, y)dx^k) \wedge dx^j. \]

Let us consider the form $L\omega = \frac{1}{2} \partial L\omega / \partial y^i dx^i$. A straightforward computation shows that $L\theta = d L\omega$, which means that $d L\theta = dd L\omega = 0$, i.e. the
canonical effective Lagrange structures $g = Lg$, $LN$ and $J$ induce an
almost Kähler geometry. We can express the 2–form (A.17) as

\[ \theta = \frac{1}{2} L\theta_{ij}(u) e^i \wedge e^j + \frac{1}{2} L\theta_{ab}(u) e^a \wedge e^b \tag{A.18} \]

\[ = g_{ij}(x, y) \left[ dy^i + N^i_k(x, y)dx^k \right] \wedge dx^j, \]

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where the coefficients $L_{ab} = L_{2+i 2+j}$ are equal respectively to the coefficients $L_{ij}$. It should be noted that for a general 2–form $\theta$ constructed for any metric $g$ and almost complex $J$ structures on $V$ one holds $d\theta \neq 0$. But for any $2 + 2$ splitting induced by an effective Lagrange generating function, we have $d L \theta = 0$. We have also $d \theta = 0$ for any set of 2–form coefficients $\theta_{\alpha'\beta'}e_{\alpha'}^i e_{\beta'}^j = L_{\alpha'\beta'}$ (such a 2–form $\theta$ will be called to be a canonical one).

We conclude that having chosen a regular generating function $L(x, y)$ on a (pseudo) Riemannian spacetime $V$, we can always model this spacetime equivalently as an almost Kähler manifold.

A distinguished connection (in brief, d–connection) on a spacetime $V$,

$$D = (hD; vD) = \{\Gamma^\alpha_{\beta\gamma} = (L^i_{jk}, vL^i_{bk}; C^i_{jc}, vC^i_{bc})\}, \quad (A.19)$$

is a linear connection which preserves under parallel transports the distribution (A.8). In explicit form, the coefficients $\Gamma^\alpha_{\beta\gamma}$ are computed with respect to a $N$–adapted basis (A.10) and (A.11). A d–connection $D$ is metric compatible with a d–metric $g$ if $D_X g = 0$ for any d–vector field $X$.

If an almost symplectic structure $\theta$ is considered on a N–anholonomic manifold, an almost symplectic d–connection $\theta D$ on $V$ is defined by the conditions that it is N–adapted, i.e. it is a d–connection, and $\theta D_X \theta = 0$, for any d–vector $X$. From the set of metric and/or almost symplectic compatible d–connections on a (pseudo) Riemannian manifold $V$, we can select those which are completely defined by a metric $g = L g$ (A.5) and an effective Lagrange structure $L(x, y)$:

There is a unique normal d–connection

$$\hat{D} = \{h\hat{D} = (\hat{D}_k, v\hat{D}_k = \hat{D}_k); v\hat{D} = (\hat{D}_c, v\hat{D}_c = \hat{D}_c)\} \quad (A.20)$$

which is metric compatible, $\hat{D}_k L g_{ij} = 0$ and $\hat{D}_c L g_{ij} = 0$, and completely defined by a couple of h– and v–components $D_\alpha = (\hat{D}_k, \hat{D}_c)$, with N–adapted coefficients $\hat{\Gamma}^\alpha_{\beta\gamma} = (\hat{L}^i_{jk}, \hat{C}^i_{jc})$, where

$$\hat{L}^i_{jk} = \frac{1}{2} L g^{ih} (e_k L g_{jh} + e_j L g_{hk} - e_h L g_{jk}) \quad (A.21)$$

$$\hat{C}^i_{jk} = \frac{1}{2} L g^{ih} \left( \frac{\partial L g_{jh}}{\partial y^k} + \frac{\partial L g_{hk}}{\partial y^j} - \frac{\partial L g_{jk}}{\partial y^h} \right).$$

In general, we can omit label $L$ and work with arbitrary $g^\alpha_{\alpha'} e^\alpha_{\alpha'}$ and $\hat{\Gamma}^\alpha_{\beta\gamma'}$ with the coefficients recomputed by frame transforms (A.2).
Introducing the normal d–connection 1–form
\[ \hat{T}^i = \hat{I}^i_{jk} e^k + \hat{C}^i_{jk} e^k, \quad (A.22) \]
we prove that the Cartan structure equations are satisfied,
\[ de^k - \varepsilon^j \wedge \hat{T}^k = -\hat{T}^i, \quad de^k - \varepsilon^j \wedge \hat{\Gamma}^k_j = -v^i\hat{T}^i, \quad (A.23) \]
and
\[ d\hat{\Gamma}^i_j - \hat{\Gamma}^h_j \wedge \hat{T}^i_h = -\hat{\mathcal{R}}^i_j. \quad (A.24) \]
The h– and v–components of the torsion 2–form \( \hat{T}^\alpha = (\hat{T}^i, v\hat{T}^i) = \hat{T}^\alpha_{\gamma \beta} e^\gamma \wedge e^\beta \) from \( (A.23) \) are computed
\[ \hat{T}^i = \hat{C}^i_{jk} e^j \wedge e^k, \quad v\hat{T}^i = \frac{1}{2} L\Omega^i_{kj} e^k \wedge e^j + (\frac{\partial L\Omega^i_{kj}}{\partial y^j} - \hat{\mathcal{L}}^i_{kj}) e^k \wedge e^j, \quad (A.25) \]
where \( L\Omega^i_{kj} \) are coefficients of the curvature of the canonical N–connection \( N^i_k \) defined by formulas similar to \( (A.16) \). The formulas \( (A.25) \) parametrize the h– and v–components of torsion \( \hat{T}^\alpha_{\beta \gamma} \) in the form
\[ \hat{T}^i_{jk} = 0, \hat{T}^i_{jc} = \hat{C}^i_{jc}, \hat{T}^a_{ij} = L\Omega^a_{ij}, \hat{T}^a_{ib} = \varepsilon_b (L N^a_k) - \hat{\mathcal{L}}^a_{bi}, \hat{T}^a_{bc} = 0. \quad (A.26) \]
It should be noted that \( \hat{T} \) vanishes on h– and v–subspaces, i.e. \( \hat{T}^i_{jk} = 0 \) and \( \hat{T}^a_{bc} = 0 \), but certain nontrivial h–v–components induced by the nonholonomic structure are defined canonically by \( g = Lg \) \( (A.5) \) and \( L \).

Similar formulas holds true, for instance, for the Levi–Civita linear connection \( \nabla = \{ L\Omega^\alpha_{\beta \gamma} \} \) which is uniquely defined by a metric structure by conditions \( \nabla T = 0 \) and \( \nabla g = 0 \). It should be noted that this connection is not adapted to the distribution \( \{ L \} \) because it does not preserve under parallelism the h– and v–distribution. Any geometric construction for the canonical d–connection \( \hat{\mathbf{D}} \) can be re–defined by the Levi–Civita connection by using the formula
\[ \hat{\Gamma}^\gamma_{\alpha \beta} = \hat{\mathbf{D}}^\gamma_{\alpha \beta} + Z^\gamma_{\alpha \beta}, \quad (A.27) \]
where the both connections \( \hat{\Gamma}^\gamma_{\alpha \beta} \) and \( \hat{\mathbf{D}}^\gamma_{\alpha \beta} \) and the distortion tensor \( Z^\gamma_{\alpha \beta} \) with N–adapted coefficients where
\[ Z^a_{jk} = -C^i_{jk} g^h_{ab} - \frac{1}{2} L\Omega^i_{kj}, Z^i_{bk} = \frac{1}{2} L\Omega^c_{jk} h^g_{ij}, L\Omega^i_{jk} = \Xi^i_{jk}, \Xi^i_{jk} = C^i_{jk}, \Xi^i_{jk} = 0, \quad (A.28) \]
for $\Sigma_{jk}^i = \frac{1}{2}(\delta_j^i \delta_k^h - g_{jk} g^{ih})$, $\pm \Sigma_{cd}^{ab} = \frac{1}{2}(\delta_c^a \delta_d^b + h_{cd} h^{ab})$ and $\circ L^c_{a_j} = L^c_{a_j} - e_a(N^c_j)$.

If we work with nonholonomic constraints on the dynamics/geometry of gravity fields, it is more convenient to use a $N$–adapted approach. For other purposes, it is preferred to use only the Levi–Civita connection.

We compute also the curvature 2–form from (A.24),

$$\hat{R}^{ij}_{\gamma} = \hat{R}^{ij}_{\alpha\beta} e^\alpha \wedge e^\beta$$

$$= \frac{1}{2} \hat{R}^{ij}_{jkh} e^k \wedge e^h + \hat{P}^{ij}_{jka} e^k \wedge e^a + \frac{1}{2} \hat{S}^{ij}_{jcd} e^c \wedge e^d,$$

where the nontrivial $N$–adapted coefficients of curvature $\hat{R}^\alpha_{\beta\gamma\tau}$ of $\hat{D}$ are

$$\hat{R}^i_{hjk} = e_k \hat{L}^i_{hj} - e_j \hat{L}^i_{hk} + \hat{L}^m_{hk} \hat{L}^i_{mj} - \hat{L}^{m}_{hj} \hat{L}^i_{mk} - \hat{C}^i_{ha} L^a_{kj}$$

$$\hat{P}^{i}_{jka} = e_a \hat{L}^i_{jk} - \hat{D}_k \hat{C}^i_{ja}, \quad \hat{S}^a_{bcd} = e_d \hat{C}^a_{bc} - e_c \hat{C}^a_{bd} + \hat{C}^a_{bd} \hat{C}^a_{cd} - \hat{C}^a_{bd} \hat{C}^a_{cd}.$$

Contracting the first and forth indices $\hat{R}^i_{\beta\gamma} = \hat{R}^\alpha_{\beta\gamma\alpha}$, we get the $N$–adapted coefficients for the Ricci tensor

$$\hat{R}_{\beta\gamma} = \left(\hat{R}_{ij}, \hat{R}_{ia}, \hat{R}_{ai}, \hat{R}_{ab}\right).$$

The scalar curvature $L^R = \hat{R}$ of $\hat{D}$ is

$$L^R = L g^{\beta\gamma} \hat{R}_{\beta\gamma} = g^{\beta\gamma} \hat{R}_{\beta\gamma}.$$

The normal $d$–connection $\hat{D}$ (A.20) defines a canonical almost symplectic $d$–connection, $\hat{D} = \theta \hat{D}$, which is $N$–adapted to the effective Lagrange and, related, almost symplectic structures, i.e. it preserves under parallelism the splitting (A.8), $\hat{D}_X L^R = L^R X = 0$ and its torsion is constrained to satisfy the conditions $\hat{T}^i_{jk} = \hat{T}^a_{bc} = 0$.

References

[1] S. Gukov and E. Witten, Branes and quantization, arXiv: 0809.0305 [hep-th]

[2] V. Guillemin and S. Sternberg, Symplectic Techniques in Physics (Cambridge University Press, Cambridge, 1984)

[3] B. Fedosov, Deformation Quantization and Index Theory (Akademie-Verlag, Berlin, 1996)
[4] M. Kontsevich, Deformation quantization of Poisson manifolds, I, Lett. Math. Phys. 66 (2003) 157–216; q-alg/9709040

[5] A. V. Karabegov and M. Schlichenmaier, Almost Kähler Deformation Quantization, Lett. Math. Phys. 57 (2001) 135–148

[6] C. Castro, W–geometry from Fedosov’s deformation quantization, J. Geom. Phys. 33 (2000) 173–190

[7] C. Castro, On Born’s deformed reciprocal complex gravitational theory and noncommutative gravity, Phys. Lett. B 668 (2008) 442–446

[8] S. Vacaru, Deformation quantization of nonholonomic almost Kähler models and Einstein gravity, Phys. Lett. A 372 (2008) 2949-2955

[9] S. Vacaru, Deformation quantization of almost Kähler models and Lagrange-Finsler spaces, J. Math. Phys. 48 (2007) 123509

[10] M. Anastasiei and S. Vacaru, Fedosov quantizaion of Lagrange–Finsler and Hamilton–Cartan spaces and Einstein gravity lifts on (co) tangent bundles, J. Math. Phys. 50 (2009) 13510

[11] S. Vacaru, P. Stavrinos, E. Gaburov and D. Gonţa, Clifford and Riemann- Finsler Structures in Geometric Mechanics and Gravity, Selected Works, Differential Geometry – Dynamical Systems, Monograph 7 (Geometry Balkan Press, 2006); www.mathem.pub.ro/dgds/mono/va-t.pdf and gr-qc/0508023

[12] S. Vacaru, Finsler and Lagrange geometries in Einstein and string gravity, Int. J. Geom. Methods. Mod. Phys. (IJGMMP) 5 (2008) 473-511

[13] R. Miron and M. Anastasiei, The Geometry of Lagrange Spaces: Theory and Applications, FTPH no. 59 (Kluwer Academic Publishers, Dordrecht, Boston, London, 1994)

[14] A. Bejancu and H. R. Farran, Foliations and Geometric Structures (Springer, 2005)

[15] S. Twareque Ali and M. Engliš, Quantization methods: A guide for physicists and analysts, Rev. Math. Phys. 17 (2005) 391-490

[16] S. Vacaru, Loop quantum gravity in Ashtekar and Lagrange-Finsler variables and Fedosov quantization of general relativity, arXiv: 0801.4942 [gr-qc]
[17] S. Vacaru, Parametric Nonholonomic Frame Transforms and Exact Solutions in Gravity, Int. J. Geom. Methods Mod. Phys. (IJGMMM) 4 (2007) 1285-1334

[18] S. Vacaru, Einstein gravity as a nonholonomic almost Kähler geometry, Lagrange–Finser variables, and deformation quantization, arXiv: 0709.3609 [math-ph]

[19] S. Vacaru, Einstein gravity in almost Kähler variables and stability of gravity with nonholonomic distributions and nonsymmetric metrics, in press: Int. J. Theory. Phys. 48 (2009), arXiv: 0806.3808 [gr-qc]

[20] B. Kostant, Quantization and Unitary Representations, Lecture Notes in Mathematics, Vol. 170 (Springer, Berlin, 1970)

[21] J.-M. Souriau, Structures des Systems Dynamiques (Oxford Univ. Press, Oxford, 1980)

[22] N. Woodhouse, Geometric Quantization (Oxford Univ. Press, Oxford, 1980)

[23] J. Sniatycki, Geometric Quantization and Quantum Mechanics (Springer, New York, 1980)

[24] A. Kapustin and D. Orlov, Remarks on A-branes, mirror symmetry, and the Fukaya category, J. Geom. Phys. 48 (2003) 84 - 99

[25] A. Kapustin and E. Witten, Electric–magnetic duality and the geometric langlands program, Comm. Number Theory and Physics 1 (2007) 1 - 236

[26] M. Aldi and E. Zaslow, Coisotropic branes, noncommutativity, and the mirror correspondence, JHEP 0506 (2005) 019

[27] S. Vacaru, Locally anisotropic gravity and strings, Ann. Phys. (NY), 256 (1997) 39-61

[28] S. Vacaru, Superstrings in higher order extensions of Finsler super-spaces, Nucl. Phys. B, 434 (1997) 590 -656

[29] S. Vacaru, Gauge and Einstein gravity from non–Abelian gauge models on noncommutative spaces, Phys. Lett. B 498 (2001) 74-82

[30] S. Vacaru, Noncommutative Finsler Geometry, Gauge Fields and Gravity, math-ph/0205023; Chapter 13 in [11]
[31] S. Vacaru, (Non) Commutative Finsler Geometry from String/M-Theory, hep-th/0211068; Chapter 14 in [11]

[32] S. Vacaru, Exact solutions with noncommutative symmetries in Einstein and gauge gravity, J. Math. Phys. 46 (2005) 042503

[33] S. Vacaru, Spinor structures and nonlinear connections in vector bundles, generalized Lagrange and Finsler spaces, J. Math. Phys. 37 (1996) 508-523

[34] S. Vacaru and N. Vicol, Nonlinear connections and spinor geometry, Int. J. Math. and Math. Sciences (IJMMS), 23 (2004) 1189-1237

[35] S. Vacaru, Clifford–Finsler algebroids and nonholonomic Einstein-Dirac structures, J. Math. Phys. 47 (2006) 093504

[36] S. Vacaru, Nonholonomic Gerbes, Riemann–Lagrange Spaces, and the Atiyah–Singer Theorems, math-ph/0507068

[37] S. Vacaru and J. F. Gonzalez–Hernandez, Nonlinear connections on gerbes, Clifford modules, and the index theorems, Indian J. Math. 50 (2008) 572–606

[38] S. Vacaru, Curve Flows and Solitonic Hierarchies Generated by (Semi) Riemannian Metrics, math-ph/0608024

[39] S. Vacaru, Curve Flows and Solitonic Hierarchies Generated by Einstein Metrics, in press: Acta Applicandae Mathematicae [ACAP] (2009); arXiv: 0810.0707 [math-ph]

[40] S. Vacaru, Nonholonomic Ricci Flows: III. Curve Flows and Solitonic Hierarchies, arXiv: 0704.2062 [math.DG]

[41] S. Anco and S. Vacaru, Curve flows in Lagrange–Finsler geometry, bi–Hamiltonian structures and solitons, J. Geom. Phys. 59 (2009) 79-103

[42] S. Helgason, Differential geometry, Lie groups, and Symmetric Spaces (Providence, Amer. Math. Soc., 2001)

[43] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vols. I and II (Wiley, 1969)

[44] R. W. Sharpe, Differential Geometry (New York, Springer–Verlag, 1997)
[45] T. R. Ramadas, I. M. Singer and J. Weitsman, Some comments on Chern-Simons gauge theory, Commun. Math. Phys. 126 (1989) 409–420

[46] D. Bar-Natan and E. Witten, Perturbative expansion of Chern-Simons gauge theory with non–compact gauge group, Commun. Math. Phys. 141 (1991) 423-440

[47] L. Sánchez, I. Galaviz and H. García–Compeán, Deformation quantization of relativistic particles in electromagnetic fields, Int. J. Mod. Phys. A 23 (2008) 1757–1790

[48] H. García–Compeán, J. F. Plebański, M. Przanowski and F. J. Turriiates, Deformation quantization of bosonic strings, J. Phys. A: Math. Gen. 33 (2000) 7935–7954

[49] F. Antonsen, Deformation quantization of constrained systems, Phys. Rev. D 56 (1997) 920–935

[50] D. J. Louis–Martínez, Weyl–Wigner–Moyal formulation of a Dirac quantized constrained system, Phys. Lett. A 269 (2000) 277–280

[51] M. I. Krivoruchenko, A. A. Raduta and A. Faessler, Quantum deformation of the Dirac Bracket, Phys. Rev. D 73 (2006) 025008

[52] A. A. Deriglazov, Noncommutative relativistic particle on the electromagnetic background, Phys. Lett. B 555 (2003) 83–88

[53] T. Hori, T. Koikawa and T. Maki, Moyal quantization for constrained system, Progr. Theor. Phys. 108 (2002) 1123–1141

[54] H. Nicolai, K. Peeters and M. Zamaklar, Loop quantum gravity: an outside view. Class. Quant. Grav. 22 (2005) R193 – R247

[55] T. Thiemann, Loop quantum gravity: an inside view. Lect. Notes. Phys. 721 (2007) 185 – 263

[56] M. H. Goroff and A. Sagnotti, The ultraviolet behavior of Einstein gravity, Nucl. Phys. B 266 (1986) 709–736

[57] A. E. M. van de Ven, Two loop quantum gravity, Nucl. Phys. B 378 (1992) 309–366
[58] Z. Bern, Perturbative quantum gravity and its relation to gauge theory, Living Rev. Rel. 5 (2002) 5; www.livingreviews.org/Articles/Volume5/2002-5bern

[59] J. Gomis and S. Weinberg, Are nonrenormalizable gauge theories renormalizable? Nucl. Phys. B 469 (1996) 473–487

[60] M. Niedermaier and M. Reuter, The asymptotic safety scenario in quantum gravity, Living Rev. Rel. 9 (2006) 5

[61] O. Lauscher and M. Reuter, Asymptotic Safety in Quantum Einstein Gravity: Nonperturbative Renormalizibility and Fractal Spacetime Structure, hep-th/0511260

[62] M. B. Green, J. H. Schwarz and E. Witten, Superstring Theory, vols. 1 & 2 (Cambridge University Press, Cambridge, 1986)

[63] J. Polchinski, String Theory, vols. 1 & 2 (Cambridge University Press, Cambridge, 1998)

[64] P. Deligne, P. Etingof P., D. S. Freed et all (eds.) Quantum Fields and Strings: A Course for Mathematicians, Vols. 1 & 2, Institute for Advanced Study (American Mathematical Society, 1994)

[65] P. O. Kazinski, S. L. Lyakhovich and A. A. Sharapov, Lagrange structure and quantization, JHEP 0507 (2005) 076

[66] S. L. Lyakhovich and A. A. Sharapov, BRST theory without Hamiltonian and Lagrangian, JHEP 0503 (2005) 011

[67] J. Gomis, J. Paris and S. Samuel, Antibracket, antifields and gauge theory quantization, Phys. Rept. 259 (1995) 1–145