A POSITIVE-DEFINITE ENERGY FUNCTIONAL FOR AXIALLY SYMMETRIC MAXWELL’S EQUATIONS ON KERR-DE SITTER BLACK HOLE SPACETIMES

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Abstract. We prove that there exists a phase space of canonical variables for the initial value problem for axially symmetric Maxwell fields propagating in Kerr-de Sitter black hole spacetimes such that their motion is restricted to the level sets of a positive-definite Hamiltonian, despite the ergo-region.

1. Kerr-de Sitter Black Holes

Consider the Kerr-de Sitter family of black holes $(\tilde{M}, \tilde{g})$:

\[
\tilde{g} = -\frac{\Delta}{\Sigma} \left( \frac{dt - a \sin^2 \theta d\phi}{1 + \frac{\Lambda}{3} a^2} \right)^2 + \frac{\Sigma}{\Delta} dr^2 + \frac{\Sigma}{\Pi} d\theta^2 + \frac{\sin^2 \theta (1 + \frac{\Lambda}{3} a^2 \cos^2 \theta)}{\Sigma} \left( \frac{adt - (r^2 + a^2) d\phi}{1 + \frac{\Lambda}{3} a^2} \right)^2
\]

where

\[
\Delta = r^2 - 2mr + a^2 - \frac{\Lambda r^2}{3} (r^2 + a^2) \quad (2a)
\]
\[
\Sigma = r^2 + a^2 \cos^2 \theta, \quad (2b)
\]
\[
\Pi = 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \quad (2c)
\]

$\theta \in [0, \phi], \phi \in [0, 2\pi], |a| < m$. The quartic polynomial function $\Delta(r)$ is such that it admits precisely one negative root and three distinct positive roots \( \{r_-, r_+, r_c\}, r_+ < r_c \), which corresponds to the cases of physical interest. In this work we shall restrict to $a \neq 0$ and the regular region $r_+ < r < r_c$. The Kerr-de Sitter family is a solution of Einstein’s equations in 3+1 dimensions with a positive cosmological constant:

\[
\bar{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_{\bar{g}} + \Lambda g_{\mu\nu} = 0, \quad \Lambda > 0 \quad (\tilde{M}, \tilde{g}),
\]

which reduces to the Schwarzschild-de Sitter family if $a = 0$ and de Sitter if $a$ and $m = 0$. The question of stability of the 3+1 de Sitter spacetime has been resolved by Friedrich in a series of landmark works [14, 15, 16]. Existence and stability of even dimensional de Sitter spacetimes in higher dimensions was proved in [1]. In a remarkable recent breakthrough, Hintz and Vasy have resolved the \textit{nonlinear} stability of the Kerr-de Sitter black holes for small angular-momentum [21] (see also [20]).

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The evolution of their methods, developed from Melrose’s $b$–calculus, can be found in the list of references therein. In this context, there were preceding results on the stability of Kerr-de Sitter family for small angular-momentum for various model problems. The local energy decay of the wave equation on Schwarzschild-de Sitter is studied in [6]. Asymptotics and resonances of linear waves on the Kerr-de Sitter metric are studied in [11, 12]. The asymptotic behaviour of the Klein-Gordon equation on the Kerr-de Sitter metric is studied in [17]. Global boundedness for linear waves on Schwarzschild-de Sitter and Kerr-de Sitter cosmologies is proved in [34]. A partial proof of nonlinear stability of Schwarzschild-de Sitter cosmologies is discussed in [35]. The decay of Maxwell’s equations on Schwarzschild-de Sitter metric was proved recently in [25].

The case $\Lambda = 0$ in (1) and (3) corresponds to the Kerr family of black holes, the stability of which is being pursued in a long-standing program that began soon after their discovery. In a remarkable recent development, the linear stability of Schwarzschild black hole spacetimes has been resolved in [7] using the Teukolsky variables, by carrying forward the classic works in [32, 38, 37, 29]. The stability of Schwarzschild using metric coefficients has been resolved in [22] and [23]. A Morawetz estimate for the linearized gravity on Schwarzschild was proved in [4]. The decay of Maxwell’s equations on the Schwarzschild metric was proved in [5]. Model problems for nonlinear stability of Schwarzschild are considered in [27, 26, 30].

In contrast with Schwarzschild and Schwarzschild-de Sitter ($a = 0$), an important obstacle for Kerr and Kerr-de Sitter ($a \neq 0$) is that the energy of even the linear wave equation is not necessarily positive-definite. This is caused by the ergo-region, which always surrounds a Kerr black hole with non-vanishing angular-momentum. A variety of techniques are introduced to cope with this issue for fields propagating on Kerr with small angular-momentum [13, 2, 3, 9, 36, 28, 24].

This leads us naturally to the question of stability of the Kerr-de Sitter family for large, but sub-extremal, angular-momentum ($|a| < m$). It may be noted that the lack of positivity of energy (and the related superradiance effect) makes proving the decay even more subtle for large $|a|$. This is dealt with in the intricate and remarkable work [10] for the linear wave equation on Kerr. However, little is known about the behaviour of nonscalar and coupled fields propagating on Kerr or Kerr-de Sitter for large $|a|.

The special case of axially symmetric linear waves propagating on the Kerr-de Sitter spacetimes, admits a fortuitous simplification as the energy from the energy-momentum tensor is immediately positive-definite and is thus directly amenable to Morawetz and decay estimates (see for e.g., [8, 2] for the Kerr counterpart). The problem becomes much more subtle even for the axially symmetric coupled vector fields (e.g. Maxwell’s equations) propagating on Kerr-de Sitter. Indeed the problem of positivity of total energy of axially symmetric Maxwell’s equations on Kerr spacetimes has been an open problem for decades where, in principle, counter examples for positivity of energy density can be constructed. This has recently been resolved for the full range of sub-extremal Kerr black holes in [19] and separately in [31].

The subject of this paper is to prove equivalent results for Kerr-de Sitter ($|a| < m$). Analogous to [19], these results also hold for the fully coupled axially symmetric Einstein-Maxwell perturbations of the Kerr-Newman-de Sitter spacetimes, which
shall be discussed rigorously in a separate article. Importantly, in the pure Maxwell problem, the positive-definite energy we construct is naturally associated to gauge-invariant quantities. In addition, without the several technicalities of the fully coupled Einstein-Maxwell problem, the pure Maxwell problem is more transparent.

Following [19], we shall use the Hamiltonian formulation as it provides a mechanism to construct a gauge-invariant notion of mass-energy for the perturbative theory of \((\bar{M}, \bar{g})\) for the full \(|a| < m\). Consider a Maxwell 2-form \(F\) and a vector potential \(A\) defined on \((\bar{M}, \bar{g})\), such that \(F := dA\), then Maxwell’s equations are the critical points of the variational principle:

\[
S_M[F] := -\frac{1}{4} \int \|F\|^2 \bar{\mu}_{\bar{g}}
\]

for compactly supported variations. If we perform the ADM decomposition \((\bar{M}, \bar{g}) = (\bar{\Sigma}, \bar{q}) \times \mathbb{R}\) of the metric \(\bar{g}\) and the vector potential \(A\)

\[
\bar{g} = -\bar{N}^2 dt^2 + \bar{q}_{ij}(dx^i + \bar{N}^i dt) \otimes (dx^j + \bar{N}^j dt)
\]

\[
A = A_0 dt + A_i dx^i, \quad i, j = 1, 2, 3
\]

and define

\[
\mathfrak{B}^i := \frac{1}{2} \epsilon^{ijk}(\partial_j A_k - \partial_k A_j),
\]

the ADM variational principle is defined as

\[
I_{ADM}[A_1, \mathcal{E}^i] := \int \left( A_i \partial_t \mathcal{E}^i - \frac{1}{2} \bar{N} \bar{\mu}_{\bar{q}}^{-1} \bar{q}_{ij} (\mathcal{E}^i \mathcal{E}^j + \mathfrak{B}^i \mathfrak{B}^j) + \epsilon_{ijk} \bar{N}^i \mathcal{E}^j \mathfrak{B}^k - A_0 \partial_i \mathcal{E}^i \right) d^4x
\]

for the phase space \(X^{\text{Max}}\),

\[
X^{\text{Max}} := \{(A_i, \mathcal{E}^i), i = 1, 2, 3\},
\]

which results in the Maxwell’s field equations

\[
\partial_t A_i = -\bar{N} \bar{\mu}_{\bar{q}}^{-1} \bar{q}_{ij} \mathcal{E}^j - \epsilon_{ijk} \bar{N}^j \mathfrak{B}^k + \partial_i A_0, \quad (8a)
\]

\[
\partial_t \mathcal{E}^i = -\partial_k (\bar{N} \bar{\mu}_{\bar{q}}^{-1} \bar{q}_{kj} \mathfrak{B}^j \epsilon^{kli}) + \partial_l (\bar{N}^l \mathcal{E}^i - \bar{N}^i \mathcal{E}^l), \quad (8b)
\]

\(\bar{\mu}_{\bar{q}}\) is the metric determinant of \((\bar{\Sigma}, \bar{q})\). The Maxwell constraint equations are

\[
\partial_i \mathcal{E}^i = 0, \quad i = 1, 2, 3.
\]

2. Dynamics with A Positive-Definite Hamiltonian

Let \((\bar{M}, \bar{g})\) is the Kerr-de Sitter spacetime represented in (1), where \(\Delta(r)\)'s positive roots are \(r_\pm\) and \(r_c\) with \((r_\pm < r_c)\). If we consider the ADM decomposition of (1),

\[
\bar{M} = \mathbb{R} \times \bar{\Sigma},
\]

where \(\bar{\Sigma}\) is Riemannian. The \(SO(2)\) group acts on \((\bar{\Sigma}, \bar{q})\) such that \(\partial_\phi\) is the associated Killing vector field. We define

\[
\Sigma := \bar{\Sigma} \setminus SO(2)
\]

and the fixed point set of the \(SO(2)\) action on \(\bar{\Sigma}\) is a union of two disjoint sets, which we represent together as \(\Gamma\) (‘the axes’) for brevity. It may be noted that the
fixed point set $\Gamma$ corresponds to $\|\partial_\phi\|_\bar{g} = 0$ and also a boundary of $\Sigma$. Finally, we define a Lorentzian manifold with boundary $M$ such that,

$$M := \bar{M} \setminus SO(2) = \Sigma \times \mathbb{R}.$$ 

**Proposition 2.1.** Suppose $(\bar{M}, \bar{g})$ is a Kerr-de Sitter spacetime with $\Delta$ as in (1), then the following statements hold

1. The metric $\bar{g}$ can be represented in Weyl-Papapetrou form:

$$\bar{g} = e^{-2\gamma} g + e^{2\gamma} \Phi^2, \tag{11}$$

where $\Phi = d\phi + A_\nu dx^\nu, \nu = 0, 1, 2, 3$, $g$ is the Lorentzian metric of $M$

2. There exists an auxiliary (scalar) potential $\omega$, $\omega : (M, g) \to \mathbb{R}$ such that $(\gamma, \omega)$ satisfies the ‘shifted’ wave maps equation:

$$\Box_g \gamma + \frac{1}{2} e^{-4\gamma} g^{\alpha \beta} \partial_\alpha \gamma \partial_\beta \omega + \Lambda e^{-2\gamma} = 0 \quad (12a)$$

$$\Box_g \omega - 4e^{-4\gamma} g^{\alpha \beta} \partial_\alpha \omega \partial_\beta \gamma = 0, \text{ on } (M, g) \setminus \Gamma, \tag{12b}$$

we shall refer to $\omega$ as the gravitational twist potential.

3. There exists a 3+1 decomposition of $(\bar{M}, \bar{g})$ such that it is smoothly foliated by 3D Riemannian maximal hypersurfaces.

Proof. For proofs of (1) and (2), see [18]; the system (12) is coupled to 2+1 Einstein’s equations. The fact that the expansion parameter $\Lambda$ decouples from the Einstein’s equations and appears as the forcing term of (12a) shall play a crucial role in our problem. For (3), consider the ADM decomposition of (1)

$$\bar{g} = -\bar{N}^2 dt^2 + \bar{q}_{ij}(dx^i + \bar{N}^i dt) \otimes (dx^j + \bar{N}^j dt), \quad i, j = 1, 2, 3. \tag{13}$$

Subsequently, if $\nabla(q)$ is the (intrinsic) covariant derivative of $(\bar{\Sigma}, \bar{q})$, then it follows that

$$\nabla_i(q) \bar{N}^i \equiv 0, \quad (\bar{\Sigma}_t, \bar{q}_t) \tag{14}$$

which holds for all $t$ in view of the $t$–translational symmetry of $(\bar{M}, \bar{g})$ in (1). Furthermore, consider the ADM decomposition of $(M, g)$

$$g = -N^2 dt^2 + q_{ab}(dx^a + N^a dt) \otimes (dx^b + N^b dt), \quad a, b = 1, 2. \tag{15}$$

We also have

$$\nabla_a(q) N^a \equiv 0, \quad (\Sigma_t, q_t), \quad \forall t \in \mathbb{R}. \tag{16}$$

□
Reading off various components of the Weyl-Papapetrou form, we have

\[ e^{2\gamma} = \frac{\sin^2 \theta (-a^2 \sin^2 \theta \Delta + \Pi (r^2 + a^2)^2)}{\Sigma(1 + \frac{2}{3}a^2)^2} \]  

\[ N = \frac{(\Pi \Delta)^{\frac{3}{2}} \sin \theta}{(1 + \frac{2}{3}a^2)^2} \]  

\[ A_0 = \frac{a(-2mr - \frac{2}{3}a^2 r^2 + a^2 (1 + \cos^2 \theta))}{-a^2 \sin^2 \theta \Delta + \Pi (r^2 + a^2)^2} \]  

\[ \bar{\mu}_q^{-1}q_{ab}dx^a \otimes dx^b = \left( \frac{\Pi}{\Delta} \right)^{\frac{1}{2}} dr^2 + \left( \frac{\Delta}{\Pi} \right)^{\frac{1}{2}} d\theta^2 \]  

where \( \bar{\mu}_q \) is the metric determinant of \((\Sigma, q)\). We are interested in the initial value problem of the Maxwell’s equations (8) with axial symmetry. We assume that the axially symmetric \( F \) tensor is derivable from an axially symmetric vector potential \( A \). In view of the fact that the Kerr-de Sitter spacetime is also axially symmetric, let us construct a new phase space \( X \). Firstly, consider a twist potential \( \lambda : (M, g) \rightarrow \mathbb{R} \) such that \( \lambda = A \), so that \( \mathfrak{B}^a = \epsilon^{ab} \partial_b \lambda \) and \( \partial_a \mathfrak{B}^a = 0, a, b = 1, 2 \). It follows from the Maxwell constraint equations \( \partial_q \mathfrak{E}^a = 0 \) and the Poincaré Lemma that there exists a twist potential \( \eta : (M, g) \rightarrow \mathbb{R} \) such that \( \mathfrak{E}^a = \epsilon^{ab} \partial_b \eta \). Likewise, it follows from the variational principle (7) that the conjugate momenta \( u \) and \( v \) defined as

\[ u := \mathfrak{B}^\phi, \quad v := -\mathfrak{E}^\phi \]  

form the dynamical canonical pairs with \( \eta \) and \( \lambda \) respectively. Thus we define the phase space \( X \)

\[ X := \{ (\lambda, u), (\eta, v) \} \].

The Maxwell’s equations (8), can be transformed into the phase space \( X \) and locally represented as

\[ \partial_t \eta = Ne^{2\gamma} \bar{\mu}_q^{-1}u, \quad \partial_t \lambda = Ne^{2\gamma} \bar{\mu}_q^{-1}v \]  

\[ \partial_t u = \partial_b (N \bar{\mu}_q q^{ab} e^{-2\gamma} \partial_a \eta) + N \bar{\mu}_q q^{ab} e^{-2\gamma} \partial_a \omega \partial_b \lambda \]  

\[ \partial_t v = \partial_b (N \bar{\mu}_q q^{ab} e^{-2\gamma} \partial_a \lambda) - N \bar{\mu}_q q^{ab} e^{-2\gamma} \partial_a \omega \partial_b \eta. \]  

For the initial value problem of (19), we assume that the axisymmetric \( F \) tensor has smooth and compactly supported initial data in a \( t = t_0 \) initial data slice \((\Sigma_0, \bar{q}_0)\). Define initial data in \( X \)

\[ ID := \{ (\lambda_0, u_0), (\eta_0, v_0) \}, \quad (\Sigma, q) \].

It may be noted that the global propagation of regularity of Maxwell field \( F \) in the domain of outer communications of the Kerr-de Sitter metric \((\mathcal{M}, \bar{g})\) is standard. As a consequence, we have the following prescribed behaviour on the axes \( \Gamma \) of the quotient space \((\Sigma, q)\) :
\[ \partial_x \lambda = 0, \; \partial_x \eta = 0, \quad \text{on} \; \Gamma \]  
\[ \partial_t \lambda = 0, \; \partial_t \eta = 0, \quad \text{on} \; \Gamma \]  
\[ \partial_t \lambda = 0, \; \partial_t \eta = 0, \quad \text{on} \; \Gamma \; \forall \; t \in \mathbb{R}, \]  
where \( \partial_x \) and \( \partial_t \) are the derivatives tangential and normal to axes \( \Gamma \) respectively. In view of (21) it may be noted that
\[ \mathfrak{C}^t = 0, \quad \mathfrak{B}^t = 0, \quad \text{on} \; \Gamma. \]

It follows from (21) that one can choose \( \lambda, \eta \) such that they are (uniformly) 0 along \( \Gamma \). In principle, our Hamiltonian framework allows for the Coulomb type conserved charges, however, the behaviour at the axes is chosen only for convenience in functional analysis arguments. We now state the main theorem of the paper.

**Theorem 2.2.** Suppose \( F \) is the electromagnetic Faraday tensor with \( \mathcal{L}_{\partial_x} F \equiv 0, \; \mathcal{L}_{\partial_t} A \equiv 0 \), propagating on the Kerr-de Sitter black holes (1) with \((|a| < m)\) and further suppose that \( F \in C^\infty(\Sigma_0, q_0) \) with \( \text{Supp}(F) \subset (\Sigma_0, q_0) \), then the following statements hold for the initial value problem of \( F \) on Kerr-de Sitter \((\bar{M}, \bar{g})\)

1. There exists a positive-definite Hamiltonian \( H^{\text{Alt}} \) for the dynamics of the canonical pairs in the phase space \( X = \{(\lambda, u), (\eta, v)\} \) i.e.,
\[
D_u \cdot H^{\text{Alt}} = \partial_t \eta, \quad D_\eta \cdot H^{\text{Alt}} = -\partial_t u \]  
\[
D_v \cdot H^{\text{Alt}} = \partial_t \lambda, \quad D_\lambda \cdot H^{\text{Alt}} = -\partial_t v
\]

where \( D \) is the (variational) directional derivative in the phase-space \( X \).

2. There exists a divergence-free spacetime vector density \( J \) such that its flux through \( t \)-constant hypersurfaces is positive-definite.

3. There exists canonical transformation \( U : (X, H^{\text{Alt}}) \to (\mathfrak{X}, H^{\text{Reg}}) \), to a 'regularized' phase space
\[ \mathfrak{X} := \{ (\lambda, u), (\eta, v) \} \]

where,
\[ \lambda := e^{-\gamma} \lambda, \quad \eta := e^{-\gamma} \eta, \quad u := e^\gamma u, \quad v := e^\gamma v \]

such that the corresponding Hamiltonian \( H^{\text{Reg}} \) is positive-definite.

**Proof.** The ADM Hamiltonian energy, reexpressed in the phase space \( X \), consecutively transforms as follows
\[
H = \int_{\Sigma} \left( \frac{1}{2} N \bar{\mu}^{-1} e^{2\gamma} (u^2 + v^2) + \frac{1}{2} N \bar{\mu} q^{ab} e^{-2\gamma} (\partial_a \eta \partial_b \eta + \partial_a \lambda \partial_b \lambda) - A_0 e^{ab} \partial_a \eta \partial_b \lambda \right) d^2x
\]
\[
= \int_{\Sigma} \left( \frac{1}{2} N \bar{\mu}^{-1} e^{2\gamma} (u^2 + v^2) + \frac{1}{2} N \bar{\mu} q^{ab} e^{-2\gamma} (\partial_a \eta \partial_b \eta + \partial_a \lambda \partial_b \lambda) + N e^{-4\gamma} \bar{\mu} q^{ab} \partial_a \omega \partial_b \eta \lambda \right) d^2x
\]  
(23)
where the \(-v\epsilon^{ab}\partial_b \eta + u\epsilon^{ab}\partial_b \lambda\) terms drop out. Now consider the difference

\[
I := \frac{1}{4} Ne^{-2\gamma} \tilde{\mu}_q q^{ab} (\partial_a \lambda - 2\lambda \partial_a \gamma)(\partial_b \lambda - 2\lambda \partial_b \gamma) + (\partial_a \eta - 2\eta \partial_a \gamma)(\partial_b \eta - 2\eta \partial_b \gamma))
+ \frac{1}{4} Ne^{-2\gamma} \tilde{\mu}_q q^{ab} ((\partial_a \eta + \lambda \epsilon^{ab}\partial_a \omega)(\partial_b \eta + \lambda \epsilon^{ab}\partial_b \omega)) + (\partial_a \lambda - \eta \epsilon^{ab}\partial_a \omega)(\partial_b \lambda - \eta \epsilon^{ab}\partial_b \omega))
- \frac{1}{2} Ne^{-2\gamma} \tilde{\mu}_q q^{ab}(\partial_a \lambda \partial_b \lambda + \partial_a \eta \partial_b \eta)
\]

We have,

\[
I = Ne^{-2\gamma} \tilde{\mu}_q q^{ab} (\partial_a \gamma \partial_b \gamma) + \frac{1}{4} e^{-4\gamma} \partial_a \omega \partial_b \omega)(\lambda^2 + \eta^2)
- Ne^{-2\gamma} \tilde{\mu}_q q^{ab}(\lambda \partial_a \gamma \partial_b \lambda + \eta \partial_a \gamma \partial_b \eta - \frac{1}{2} Ne^{-2\gamma} \partial_a \omega \partial_b \eta + \frac{1}{2} Ne^{-2\gamma} \partial_a \omega \partial_b \lambda) \quad (24)
\]

Recall the wave map system satisfied by \((\gamma, \omega)\):

\[
\partial_b (N \tilde{\mu}_q q^{ab} \partial_a \gamma) + \frac{1}{2} N \tilde{\mu}_q e^{-4\gamma} q^{ab} \partial_a \omega \partial_b \omega + N \tilde{\mu}_q \Lambda e^{-2\gamma} = 0 \quad (25a)
\]

\[
\partial_b \tilde{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega = 0 \quad (25b)
\]

Now consider the quantity:

\[
II := \frac{1}{2} \partial_b (N \tilde{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \eta \lambda - N \tilde{\mu}_q q^{ab} e^{-2\gamma} \partial_a \gamma (\eta^2 + \lambda^2)) \quad (26)
\]

we have

\[
II = - \frac{1}{2} N \tilde{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \eta \lambda + \lambda \partial_b (N \tilde{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \eta) + N \tilde{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \lambda \eta)
- \frac{1}{2} (e^{-2\gamma} \partial_b (N \tilde{\mu}_q q^{ab} \partial_a \gamma)(\lambda^2 + \eta^2) - 2e^{-2\gamma} N \tilde{\mu}_q q^{ab} \partial_a \gamma \partial_b \lambda (\lambda^2 + \eta^2) - 2e^{-2\gamma} N \tilde{\mu}_q q^{ab} \partial_a \gamma (\lambda \partial_b \lambda + \eta \partial_b \eta))
- \frac{1}{2} (N \tilde{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \eta \lambda + N \tilde{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \lambda \eta) - \frac{1}{2} (e^{-2\gamma} (N \tilde{\mu}_q q^{ab} \partial_a \gamma \partial_b \lambda - N \tilde{\mu}_q q^{ab} \partial_a \gamma \partial_b \lambda))
- 2e^{-2\gamma} N \tilde{\mu}_q q^{ab} \partial_a \gamma \partial_b \gamma (\lambda^2 + \eta^2) - e^{-2\gamma} N \tilde{\mu}_q q^{ab} \partial_a \gamma (\lambda \partial_b \lambda + \eta \partial_b \eta)
\]

Consequently,

\[
I - II = - \frac{1}{2} (\lambda^2 + \eta^2) \Lambda e^{-4\gamma} N \tilde{\mu}_q + N \tilde{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \eta \lambda \quad (27)
\]

as the \(\partial \omega \partial \eta\) term occurs in both \(I\) and \(II\) with opposite signs. Therefore, using \(I - II\), we can transform the Hamiltonian energy \((23)\) into the following manifestly positive form:

\[
H^{Alt} := \int_{\Sigma} \left( \frac{1}{2} N \tilde{\epsilon} e^{2\gamma} \tilde{\mu}_q^{-1} (u^2 + v^2) + \frac{1}{2} \Lambda N \tilde{\mu}_q e^{-4\gamma}(\lambda^2 + \eta^2)
+ \frac{1}{4} N \tilde{\mu}_q q^{ab} (\partial_a \gamma - 2\lambda \partial_a \lambda)(\partial_b \lambda - 2\lambda \partial_b \gamma) + (\partial_a \gamma - 2\lambda \partial_a \lambda)(\partial_b \gamma - 2\lambda \partial_b \lambda))
+ \frac{1}{4} N \tilde{\mu}_q q^{ab} (\partial_a \gamma + \lambda \epsilon^{ab}\partial_a \omega)(\partial_b \gamma + \lambda \epsilon^{ab}\partial_b \omega)
+ (\partial_a \lambda - \eta \epsilon^{ab}\partial_a \omega)(\partial_b \lambda - \eta \epsilon^{ab}\partial_b \omega)) \right) d^2 x \quad (28)
\]

The aforementioned transformations of the original ADM Hamiltonian into a positive form in \((28)\) are motivated by the construction of the Robinson’s identity \([33]\), but now adapted to our problem. Crucially, we need to prove that the Hamiltonian structure of the equations is retained.
Consider a (variational) 1-parameter flow of a generic phase point \( P \) in the phase space \( X \), parametrized by \( s \). We shall denote the components of the variation at \( P \) with respect to this flow as

\[
u' := D \cdot u(P), \quad v' := D \cdot v(P), \quad \lambda' := D \cdot \lambda(P), \quad \eta' := D \cdot \eta(P).
\]

Consider \( D_u \cdot H^{Alt} \) and \( D_v \cdot H^{Alt} \), we have

\[
D_u \cdot H^{Alt} = Ne^{2\gamma} \tilde{\mu}_q^{-1} u \quad \text{and} \quad D_v \cdot H^{Alt} = Ne^{2\gamma} \tilde{\mu}_q^{-1} v
\]

respectively. The quantities \( D_\lambda \cdot H^{Alt} \) and \( D_\eta \cdot H^{Alt} \) are more difficult. From (28), \( D_\lambda \cdot H^{Alt} \) and \( D_\eta \cdot H^{Alt} \) have the following types of terms:

- 1st order \( \partial \eta \partial \eta' \) and \( \partial \lambda \partial \lambda' \):
  \[
  Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \partial_a \eta \partial_b \eta' = \partial_b (Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \partial_a \eta) \eta' - \partial_b (Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \partial_b \eta) \eta'
  \]
  and
  \[
  Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \partial_a \lambda \partial_b \lambda' = \partial_b (Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \partial_a \lambda) \lambda' - \partial_b (Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \partial_b \lambda) \lambda'
  \]

- mixed type \( \eta' \partial \eta \), \( \eta \partial \eta' \) and \( \lambda' \partial \lambda \), \( \lambda \partial \lambda' \):
  \[
  -Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \partial_a \eta \partial_b \gamma \eta' - Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \eta \partial_a \eta' \partial_b \gamma
  \]
  \[
  = -Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \partial_a \eta \partial_b \gamma \eta' - \partial_b (Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \eta \partial_a \gamma \eta') + \partial_b (Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \eta \partial_b \gamma) \eta'
  \]
  and
  \[
  -Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \lambda' \partial_a \lambda \partial_b \gamma - Ne^{2\gamma} \tilde{\mu}_q q^{ab} \lambda \partial_a \lambda' \partial_b \gamma
  \]
  \[
  = -Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \lambda' \partial_a \lambda \partial_b \gamma - \partial_b (e^{-2\gamma} N \tilde{\mu}_q q^{ab} \partial_b \gamma \lambda') + \partial_b (e^{-2\gamma} N \tilde{\mu}_q q^{ab} \partial_a \gamma) \lambda'
  \]

- 0th order:
  \[
  2(N \tilde{\mu}_q e^{-2\gamma} q^{ab} \partial_a \gamma \partial_b \gamma + \frac{1}{4} N \tilde{\mu}_q e^{-6\gamma} q^{ab} \partial_a \omega \partial_b \omega + \frac{1}{2} \Lambda N \tilde{\mu}_q e^{-4\gamma}) \eta' \]
  \[
  \quad + \frac{1}{2} \Lambda N \tilde{\mu}_q e^{-4\gamma} \lambda' \lambda
  \]

Combining all the above, while using the system (25) again, we recover the full set of the field equations:

\[
D_\eta \cdot H^{Alt} = -\partial_b (Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \partial_b \eta) - N \tilde{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \lambda = -\partial_t u
\]

and

\[
D_\lambda \cdot H^{Alt} = -\partial_b (Ne^{-2\gamma} \tilde{\mu}_q q^{ab} \partial_b \lambda) + N \tilde{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \eta = -\partial_t v.
\]
Now let us turn to the part 2), define the energy density $e^{Alt}$ of the Hamiltonian $H^{Alt}$ such that

$$H^{Alt} = \int_{\Sigma} e^{Alt} d^2 x.$$  \hfill (34)

We shall construct the divergence-free vector density from the time derivative of the density $e^{Alt}$ of the Hamiltonian $H^{Alt}$: defining $\bar{v} := N\bar{\mu}_q^{-1}v$ and $\bar{u} := N\bar{\mu}_q^{-1}u$ and plugging in the field equations for $\lambda$ and $\eta$ and their conjugate momenta $u, v$, we have the following terms in $\frac{\partial}{\partial t} e^{Alt}$

$$\partial_t \bar{v}(\frac{1}{2} N\bar{\mu}_q q^{ab}(2\partial_t \lambda - \eta e^{-2\gamma} \partial_t \omega - 2\lambda \partial_t \lambda))$$  \hfill (35)

and

$$\partial_t \bar{u}(\frac{1}{2} N\bar{\mu}_q q^{ab}(2\partial_t \eta + \lambda e^{-2\gamma} \partial_t \omega - 2\eta \partial_t \gamma))$$  \hfill (36)

the terms with $\bar{v}$ and $\bar{u}$ are

$$\bar{u}(\frac{1}{2} Ne^{-2\gamma} \bar{\mu}_q q^{ab}(-\partial_\omega (\partial_t \lambda - \eta e^{-2\gamma} \partial_t \omega) + 2e^{2\gamma} \partial_a \gamma(\partial_t \eta + \lambda e^{-2\gamma} \partial_t \omega)))$$

$$+ \bar{u} e^{2\gamma} (\partial_t (N\bar{\mu}_q q^{ab} e^{-2\gamma} \partial_a \eta)) + N\bar{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \lambda + N\lambda e^{-4\gamma} \bar{\mu}_q \Lambda)$$  \hfill (37)

and

$$\bar{v}(\frac{1}{2} Ne^{-2\gamma} \bar{\mu}_q q^{ab} (\partial_\omega (\partial_t \eta + \lambda e^{-2\gamma} \partial_t \omega) + 2e^{2\gamma} \partial_a \gamma(\partial_t \lambda - \eta e^{-2\gamma} \partial_t \omega)))$$

$$+ \bar{v} e^{2\gamma} (\partial_t (N\bar{\mu}_q q^{ab} e^{-2\gamma} \partial_a \lambda)) - N\bar{\mu}_q q^{ab} e^{-4\gamma} \partial_a \omega \partial_b \eta + N\eta e^{-4\gamma} \bar{\mu}_q \Lambda)$$  \hfill (38)

which, in view of the system (25), can be transformed to

$$\bar{v} \partial_\omega (N\bar{\mu}_q q^{ab} (\partial_a \eta - \frac{1}{2} \eta e^{-2\gamma} \partial_a \omega - \lambda \partial_a \gamma))$$  \hfill (39)

and

$$\bar{u} \partial_\omega (N\bar{\mu}_q q^{ab} (\partial_a \eta + \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega - \eta \partial_a \gamma))$$  \hfill (40)

respectively. Therefore, the time derivative of $e^{Alt}$ can be transformed into a pure spatial divergence:

$$\frac{\partial}{\partial t} e^{Alt} = \frac{\partial}{\partial x^b}(\bar{u} N\bar{\mu}_q q^{ab}(\partial_a \eta + \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega - \eta \partial_a \gamma)) + \bar{v} N\bar{\mu}_q q^{ab}(\partial_a \lambda - \frac{1}{2} e^{-2\gamma} \partial_a \omega - \lambda \partial_a \gamma))$$

$$= \frac{\partial}{\partial x^b}(u N^2 q^{ab}(\partial_a \eta + \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega - \eta \partial_a \gamma)) + v N^2 q^{ab}(\partial_a \lambda - \frac{1}{2} e^{-2\gamma} \partial_a \omega - \lambda \partial_a \gamma))$$  \hfill (41)

which can be transformed into a divergence-free vector density

$$J := J^t \partial_t + J^b \partial_b$$  \hfill (42)

where

$$J^t := e^{Alt}$$

$$J^b := u N^2 q^{ab}(\partial_a \eta + \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega - \eta \partial_a \gamma) + v N^2 q^{ab}(\partial_a \lambda - \frac{1}{2} e^{-2\gamma} \partial_a \omega - \lambda \partial_a \gamma).$$
For the part 3), consider the regularized phase space $\mathcal{X} := \{(\lambda, \nu), (\eta, \mu)\}$
\[
\gamma := e^{-\gamma} \lambda, \quad \eta := e^{-\gamma} \eta, \quad \mu = e^{\gamma} u, \quad \nu = e^{\gamma} \nu
\]

To construct a regularized Hamiltonian $H^{\text{Reg}}$, we shall use further identities:
\[
\frac{1}{4} N e^{-2\gamma} \bar{\mu}_q a^b ((\partial_a \lambda - 2 \lambda \partial_a \gamma)(\partial_b \lambda - 2 \lambda \partial_b \gamma) + (\partial_a \eta - 2 \eta \partial_a \gamma)(\partial_b \eta - 2 \eta \partial_b \gamma))
\]
\[
+ \frac{1}{4} N e^{-2\gamma} \bar{\mu}_q a^b ((\partial_a \eta + \lambda e^{-2\gamma} \partial_a \omega)(\partial_b \eta + \lambda e^{-2\gamma} \partial_b \omega) + (\partial_a \lambda - \eta e^{-2\gamma} \partial_a \omega)(\partial_b \lambda - \eta e^{-2\gamma} \partial_b \omega))
\]
\[
= \frac{1}{4} N \bar{\mu}_q a^b ((\partial_a \lambda - \lambda \partial_a \gamma)(\partial_b \lambda - \lambda \partial_b \gamma) + (\partial_a \eta - \eta \partial_a \gamma)(\partial_b \eta - \eta \partial_b \gamma))
\]
\[
+ \frac{1}{4} N \bar{\mu}_q a^b ((\partial_a \eta + \lambda e^{-2\gamma} \partial_a \omega)(\partial_b \eta + \lambda e^{-2\gamma} \partial_b \omega) + (\partial_a \lambda - \eta e^{-2\gamma} \partial_a \omega)(\partial_b \lambda - \eta e^{-2\gamma} \partial_b \omega))
\]
\[
+ \frac{1}{4} N \bar{\mu}_q a^b ((\partial_a \lambda + \lambda \partial_a \gamma - \frac{\lambda}{2} e^{-2\gamma} \partial_a \omega)(\partial_b \lambda + \lambda \partial_b \gamma - \frac{\lambda}{2} e^{-2\gamma} \partial_b \omega))
\]
\[
= \frac{1}{4} N \bar{\mu}_q a^b ((\partial_a \lambda - \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega)(\partial_b \lambda - \frac{1}{2} \lambda e^{-2\gamma} \partial_b \omega) + (\partial_a \eta + \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega)(\partial_b \eta + \frac{1}{2} \lambda e^{-2\gamma} \partial_b \omega))
\]
\[
+ \frac{1}{4} N \bar{\mu}_q a^b (\partial_a \gamma \partial_b \gamma + \frac{1}{4} e^{-4\gamma} \partial_a \omega \partial_b \omega)(\lambda^2 + \eta^2)
\]

where the $\partial_\omega \partial_\gamma$ terms cancel. Thus, the energy $H^{\text{Alt}}$ can be transformed into
\[
H^{\text{Reg}} := \int \left( \frac{1}{2} N \bar{\mu}_q (\bar{w}^2 + \bar{v}^2) + \frac{1}{2} N \bar{\lambda} \bar{\mu}_q e^{-2\gamma} (\lambda^2 + \eta^2) + \frac{1}{2} N \bar{\mu}_q a^b ((\partial_a \gamma \partial_b \gamma + \frac{1}{4} e^{-4\gamma} \partial_a \omega \partial_b \omega)(\lambda^2 + \eta^2))
\]
\[
\right) d^2 x.
\]

Analogous to calculations for $H^{\text{Alt}}$, we recover the field equations for the regularized phase space $\mathcal{X}$
\[
\begin{align}
D_\lambda \cdot H^{\text{Reg}} &= \partial \eta, \\
D_\lambda \cdot H^{\text{Reg}} &= -\partial \mu
\end{align}
\]
(44a)
\[
\begin{align}
D_u \cdot H^{\text{Reg}} &= \partial \lambda, \\
D_u \cdot H^{\text{Reg}} &= -\partial \nu
\end{align}
\]
(44b)

where $D$ is the usual (variational) directional derivative in $\mathcal{X}$. Likewise, if we define the energy density $e^{\text{Reg}}$ such that
\[
H^{\text{Reg}} = \int_{\Sigma} e^{\text{Reg}} \ d^2 x
\]
(45)

and time differentiate, using the system (44) we get:
\[
\frac{\partial}{\partial t} e^{\text{Reg}} = \frac{\partial}{\partial x^5} (N^2 \bar{q}^{ab} u (\partial_a \eta + \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega) + N^2 \bar{q}^{ab} u (\partial_a \lambda - \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega))
\]
(46)

which results in a vector field density
\[
J^{\text{Reg}} := (J^{\text{Reg}})^t \partial_t + (J^{\text{Reg}})^b \partial_b
\]
(47)

where
\[
(J^{\text{Reg}})^t = e^{\text{Reg}}
\]
\[
(J^{\text{Reg}})^b = N^2 \bar{q}^{ab} u (\partial_a \eta + \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega) + N^2 \bar{q}^{ab} u (\partial_a \lambda - \frac{1}{2} \lambda e^{-2\gamma} \partial_a \omega)
\]
(48)
which is also divergence-free. The advantage of recasting in the \((X, H^{\text{Reg}})\) framework is that it has better behaviour on the axes and the horizon than \((X, H^{\text{Alt}})\). □

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