Memory-Optimality for Non-Blocking Containers

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A bounded container maintains a collection of elements that can be inserted and extracted as long as the number of stored elements does not exceed the predefined capacity. We consider the concurrent implementations of a bounded container more or less memory-friendly depending on how much memory they use in addition to storing the elements. This way, memory-optimal implementation employs the minimal possible memory overhead and ensures that data and metadata are stored most compactly, reducing cache misses and unnecessary memory reclamation.

In this paper, we show that non-blocking implementations of a large class of element-independent bounded containers, including queues, stacks, and pools, must incur linear in the number of concurrent processes memory overhead. In the special cases when the ABA problem is excluded, e.g., by assuming that the hardware supports LL/SC instructions or by storing distinct elements, we show that the lower bound can be circumvented, by presenting lock-free bounded queues with constant memory overhead. For the general case, we present a CAS-based bounded queue implementation that asymptotically matches our lower bound. We believe that these results open a new research avenue devoted to the memory-optimality phenomenon.

1 INTRODUCTION

A typical dynamic data structure, e.g., a queue, a stack, or a pool, maintains a collection of elements that can grow and shrink over time. Implementing such a data structure in a concurrent environment inevitably incurs memory overhead, in addition to the shared memory used for storing the elements itself. Intuitively, the overhead consists of memory locations that are necessary to ensure that the elements are accessed consistently by concurrent processes, for example, descriptors, special flags, or locks. Depending on the amount of memory used for this and the access patterns to this memory, we can qualify implementations as more or less memory-friendly. Maintaining the data and the metadata in a compact manner and decreasing the number of memory allocations allows to reduce cache misses and reclamation expenses. The latter is particularly important for languages with garbage collection (e.g., Java, Scala, Kotlin, Go, C#, etc.), where such memory-friendly data structures might exhibit more predictable performance. Therefore, memory-friendly implementations are typically faster in both high and low contention scenarios, which is vital for general-purpose solutions used in standard libraries of programming languages and popular frameworks. For example, in queue implementations, performance can be gained by allocating memory for the data elements in contiguous chunks [5] or (for a bounded queue) maintaining a fixed-size array for the data and only allocate memory for descriptors.

However, this paper is not primarily about the relation between performance and “memory-friendlyness”. We address here the question of memory-optimality: what is the minimal memory overhead a concurrent implementation may incur?

To answer this question, we need a way to define the memory overhead formally. We restrict our attention to a bounded dynamic data structure, parameterized with a fixed capacity: the maximal number of elements that can be put...
into it. Bounded data structures are particularly interesting in our context, as they enable an intuitive definition of the memory overhead as the amount of memory that needs to be allocated on top of the bounded memory needed to store the data elements.

Let us consider a prominent example of a bounded queue, widely used for task and resource management systems, schedulers, and other modules of high-load applications. Its simple sequential implementation (Figure 1) maintains an array $a$ of $C$ memory locations ($C$ is the capacity of the queue) and two integers, enqueues and dequeues, representing the total numbers of effective enqueue and dequeue operations, respectively. Intuitively, when the enqueues and dequeues counters coincide, the queue is empty, and when enqueues − dequeues = $C$, the queue is full, while the counter value modulo $C$ references the cell for the next corresponding operation. The algorithm uses exactly $C$ memory cells for the contents of the queue, where each slot is used exclusively to store a single element of the queue, and two additional memory locations for the counters. Thus, its memory overhead is $\Theta(1)$, while the total used memory is $C + \Theta(1)$.

Can one think of a concurrent bounded queue implementation with the same (even asymptotically), memory footprint? A trivial algorithm based on a single global lock incurs constant overhead, but does not scale and is inefficient in highly concurrent scenarios. Therefore, it appears more interesting (and appealing in practice) to study non-blocking implementations.

Classical non-blocking queue implementations (starting from the celebrated Michel-Scott queue [12]) maintain a linked list, where each list node stores a queue element and a reference to the next node, employing $\Omega(C)$ additional memory, where $C$ is the number of queue elements. In more recent non-blocking solutions [5, 13, 20], a list node stores a fixed-size number of elements in the form of a segment, which reduces the memory overhead. In the bounded case, one might want to minimize the overhead by storing all the queue elements in a single segment. This may allow us to prevent the queue size from playing a role in the memory overhead (the memory occupied by the elements does not count into that), but requires a solution to re-use this single segment. Can we organize the algorithm in a way so it does not depend on contention, i.e., the number of processes $n$ that can concurrently access the queue? In general, can the overhead be constant (independent of $n$)?
In this paper, we show that the answer is no. We prove that a large class of obstruction-free implementations of bounded containers (e.g., queues, stacks, and pools) must allocate $\Omega(n)$ memory, in addition to $C$ memory cells used to store $C$ elements that can be put in the container.\footnote{Obstruction-freedom \cite{7} is believed to be the weakest non-blocking progress condition. The lower bound trivially extends to lock-free or wait-free implementations.}

To warm up before facing the challenge of memory-optimality, we consider a simple unbounded lock-free queue algorithm and calculate its overhead which appears to be linear in both the number of processes and the queue capacity (Section 2). Though not optimal, the algorithm is interesting, as its “memory-friendliness” can be easily adjusted by tuning the size of continuous memory segments it uses.

After presenting our lower bound (Section 3), we then discuss (Section 4) how the lower bound can be circumvented under various restrictions, either on the application (assuming that the queue contains distinct elements) or on the model (LL/SC or DCSS primitives are available).

Finally, we show (Section 5) that our lower bound is asymptotically tight by presenting an algorithm for the model that exports only read, write, and CAS primitives, and both “bit stealing” and storing metadata and values in the same memory locations is forbidden.

Overall, we hope that this work opens a wide research avenue of memory-friendly concurrent computing. This study has a value from the theoretical perspective, establishing tight bounds on the memory overhead in dynamic data structures, as well as in practice, enabling memory-friendly implementations with high and predictable performance.

2 MEMORY-FRIENDLY BOUNDED QUEUE ALGORITHM

We start with a simple memory-friendly bounded queue algorithm to provide some intuition on the memory-friendliness. While this algorithm is not memory-optimal, which will be shown in Section 3, it can be tuned to be more or less memory-friendly. The main idea is to build a bounded queue on an infinite array (so enqueues and dequeues counters directly point to the array cells) and simulating it with a list of fixed-size segments.

Listing 1 presents the basic implementation, assuming that the array of elements $a$ is infinite; we discuss how to simulate such an array below. Since counters point directly to the cells, it is guaranteed that each cell can be modified by at most one enqueue via CAS from $\perp$ to the inserting element (line 9), and by at most one dequeue via an opposite modification (lines 21–22).

```
1 func enqueue(x: T) Bool:
2   while (true):
3     // Read the counters snapshot
4     e := enqueues // read `enqueues` at first
5     d := dequeues // then read `dequeues`
6     // Re-start if `enqueues` has been changed
7     if e != enqueues: continue
8     if e == d + C: return false // is queue full?
9     done := CAS(&a[e], ⊥, x) // put the element
10    if done: return true // finish on success
11
12 func dequeue() T?:
13   while (true):
14     // Read the counters snapshot
15     d := dequeues
16     e := enqueues
17     if d != dequeues: continue
18     if e == d: return ⊥ // is queue empty?
19     x := a[d] // read the current cell state
20     done := x != ⊥ && CAS(&a[d], x, ⊥)
21     if done: return x // finish on success
22     CAS(&dequeues, d, d + 1) // inc the counter
23
24```

Fig. 1. Memory-friendly (but not memory-optimal) bounded queue algorithms which uses an infinite array simulated by a concurrent linked list of fixed size segments (see Figure 2). The difference from the baseline algorithm in Listing 2 is highlighted with yellow.
Infinite array. To simulate an infinite array, we can use the approach used for synchronous queues in [11], maintaining a Michael-Scott linked list of numbered segments. Each segment contains $K$ element cells and has a unique id (thus, $\text{segm.next.id} = \text{segm.id} + 1$), and the counter value $i$ corresponds to cell $i$ modulo $K$ in the segment with id equals to $i/K$. We illustrate such a structure in Figure 2. Similarly to descriptors, it is possible to recycle segments, so at most $n$ additional segments are required.

Memory-friendliness. The resulting memory overhead required for segments is then bounded by $\Theta(C/K + Kn)$ in addition to $C$ memory cells for elements: $\Theta(C/K)$ comes from the pointers of linked list and $\Theta(Kn)$ from the implementation of the infinite array where each thread can allocate one new segment. Such algorithm might be attractive for small-contention scenarios. At the same time, the total utilized memory can be smaller if the number of stored elements does not exceed the capacity, which adds an extra point to memory-friendliness. Though not memory-optimal, this simple algorithm can be tuned to be more or less memory-friendly by adjusting the constant $K$.

However, in case of high contention and when the capacity $C$ is relatively small, it is better, in our opinion, to use the bounded queue algorithm presented by Nikolaev [14].

3 LINEAR OVERHEAD LOWER BOUND

Intuitively, memory overhead incurred by a concurrent implementation of a bounded container is the amount of memory required in addition to the memory used to store the elements. To grasp this notion formally, we describe the model and make several natural simplifying assumptions on concurrent implementations.

3.1 Model

We consider a system of $n$ processes $p_1, \ldots, p_n$ communicating via invocations of primitives on a shared memory. Besides conventional reads and writes, we assume Compare-and-Set primitive, denoted as $\text{CAS}(a, \text{old}, \text{new})$, that atomically checks the value at address $a$ and, if it equals $\text{old}$, replaces it with $\text{new}$ and returns true, and returns false, otherwise.

A data type is a tuple $(\Phi, \Gamma, Q, q_0, \theta)$, where $\Phi$ is a set of operations, $\Gamma$ is a set of responses, $Q$ is a set of states, $q_0$ is the initial state, and $\theta$ is the transition function that determines, for each state and each operation, the set of possible resulting states and produced responses, $\theta \subseteq (Q \times Q) \times (\Gamma \times \Gamma)$.

The data type Bounded Container (BC), parameterized by type $T$ and capacity $C$, maintains a multi-set of elements of type $T$ and exports two operations:

- $\text{put}(T \ x)$: if the size of the multi-set is less than $C$, the operation adds value $x$ to it and returns true; otherwise, the operation returns false.
- $\text{extract}()$ retrieves some element from the multi-set, or returns $\bot$ if the set is empty.

We further omit type $T$ when we talk about BC.

Depending on the order of elements to be returned by $\text{extract}()$ operation, we can define multiple subtypes of BC: stack extracts the most recent inserted element, FIFO queue extracts the oldest one, pool extracts any, and priority queue extracts the smallest.
Informally, an implementation of a high-level object $O$ is a distributed algorithm $A$ consisting of automata $A_1, \ldots, A_n$. $A_i$ specifies the sequence of primitives $p_i$ needs to execute in order to return a response to an invoked operation on $O$. An execution of an implementation of a high-level object $O$ is a sequence of events: invocations and responses of high-level operations on $O$, as well as primitives applied by the processes on the shared-memory locations and the responses they return, so that the automata $A_i$ and the primitive specifications are respected. An operation is complete in a given execution if its invocation is followed by a matching response. An incomplete operation can be completed by inserting a matching response after its invocation. We only consider well-formed executions: a process never invokes a new high-level operation before its previous operation returns. An operation $o_1$ precedes operation $o_2$ in execution $\alpha$ (we write $o_1 \preceq_\alpha o_2$) if the response of $o_1$ precedes the invocation of $o_2$ in $\alpha$.

For simplicity, we only consider deterministic implementations: all shared-memory primitives and automata $A_1, \ldots, A_n$ exhibit deterministic behavior. Our lower bound can be, however, easily extended to randomized implementations.

A concurrent execution $\alpha$ is linearizable (with respect to a given data type) [2, 9] if all complete operations in it and a subset of incomplete ones can be put in a total order $S$ such that (1) $S$ respects the sequential specification of the data type, and (2) $\preceq_\alpha \subseteq \preceq_S$, i.e., the total order respects the real-time order on operation in $\alpha$. Informally, every operation $o$ of $\alpha$ can be associated with a linearization point put within the operation’s interval, i.e., the fragment of $\alpha$ between the invocation of $o$ and the response of $o$. An implementation is linearizable if its every execution is linearizable.

In this paper, we focus on non-blocking implementations that, intuitively, do not involve any form of locking: a failure of a process does not prevent other process from making progress. We can also talk about non-blocking implementation of specific operations. Popular non-blocking liveness criteria are lock-freedom and obstruction-freedom.

An implementation of an operation is lock-free if in every execution, at least one process is guaranteed to complete its call of this operation in a finite number of its own steps. An implementation of an operation is obstruction-free if it guarantees that whenever a process calls this operation and takes sufficiently many steps in isolation, i.e., without contending with other processes, then the operation eventually competes. We say that an implementation is obstruction-free (resp., lock-free) if its methods are obstruction-free (resp., lock-free). Lock-freedom is a strictly stronger liveness criterion than obstruction-freedom: any lock-free implementation is also obstruction-free, but not vice versa [8].

### 3.2 Algorithmic Restrictions

First, we restrict our attention to algorithms that partition the shared memory into value-locations, used exclusively to store values added to the container, and metadata-locations, used to store everything else except for values. A memory location can be used to store at most one value. The assumption is reasonable if we do not have any possibility to distinguish values from the metadata. State-of-the-art algorithms typically maintain a one-bit marker values specifying the type of a memory location (value or metadata) which incurs $\Omega(1)$ memory overhead for each value, thus giving at least $\Omega(C)$ overhead in total.

Furthermore, we assume implementations that do not account for the values of the items added to and extracted from the container in choosing value-locations for storing and locating them. We call such implementations element-independent. This notion consists of two parts: put-element-independence and extract-element-independence.

**Definition 1.** Let $S$ be a memory state reached by an execution of a BC implementation $A$. Let $a_1, \ldots, a_k$ and $v$ be distinct values that do not appear in value-locations in $S$. Let $S_1$ be the state of $A$ reached from $S$ by sequentially performing $\text{put}(a_1), \ldots, \text{put}(a_1), \ldots, \text{put}(a_k)$ and $S_2$ be the state of $A$ reached from $S$ by sequentially performing...
put \((a_1), \ldots, \) put \((v), \ldots, \) put \((a_k)\) (where, instead of \(a_i, v\) is inserted). \(A\) is \textit{put-element-independent} if \(S_1\) is identical to \(S_2\), except that every value-location storing \(a_i\) in \(S_1\) stores \(v\) in \(S_2\).

\textbf{Definition 2.} Let \(S\) be a memory state reached by an execution of a BC implementation \(A\). Let \(x\) be a value that appears only in one value-location in \(S\) and it does not appear in the local memory of a process \(p\). Let process \(p\) sequentially apply \textit{extract} operations as long as they are successful, i.e., until the container becomes empty. Suppose that exactly one \textit{extract} operation returns \(x\). \(A\) is \textit{extract-element-independent} if when we replace \(x\) with \(y\) in \(S\) and replay \textit{extract} operations by process \(p\), we obtain an execution of \(A\) in which the \textit{extract} operation that earlier returned \(x\) now returns \(y\).

\textbf{Definition 3.} A BC implementation \(A\) is \textit{element-independent} if it is \textit{put-element-independent} and \textit{extract-element-independent}.

Many subtypes of BC, e.g., bounded queues, stacks and pools, have natural element-independent implementations. Typical implementations of these subtypes store elements in the order in which they are added to the container, regardless of their values. In Sections 4 and 5, we present several element-independent bounded queue implementations. However, implementations of certain subtypes of BC, such as bounded priority queues, are often not element-independent, as they store added elements in a sorted way, based on their values.

### 3.3 Lower Bound

\textbf{Theorem 4.} Any obstruction-free linearizable value-independent implementation of Bounded Container with capacity \(C\) uses \(C + \Omega(n)\) value-locations, assuming that there exists at least \(n \cdot C\) different values, the number of processes \(n \in \left[2, \frac{C}{T}\right]\), and \(\perp\) is not of type \(T\), i.e., it cannot be stored in value-locations.

\textbf{Proof.} Without loss of generality, assume that our bounded-container implementation can store values from the range \(1, \ldots, n \cdot C\). We are going to show that the implementation cannot use less than \(C + \lceil n/2 \rceil - 1\) value-locations. By contradiction, suppose that an implementation \(A\) of BC with capacity \(C\) uses only \(C + X\) value-locations where \(X = \lceil n/2 \rceil - 2\).

To establish a contradiction, we construct a non-linearizable execution of \(A\). At a high-level, the construction consists of three phases. In the first phase, process \(p_1\) sequentially puts values \(1, \ldots, C\) into the empty container. Note that all the operations succeed. Also, each of these values should be stored in some value-location. Otherwise, no other process will be able to extract a missing value, violating linearizability.

In the second phase, we extract all the values to make the container empty, i.e., we perform \textit{extract} operations as long as they are successful, and then we put values \(C + 1, \ldots, 2C\) into the empty container. During these extracts and puts, we “catch” \(n – 2\) processes in a special state: the next primitive of operation of each of them is write or CAS that changes the value of a distinct value-location to a distinct non-\(\perp\) value. In the resulting, the container is possibly not full, as some of these operations are not finished (the corresponding processes are “caught”).

In the third phase, we again operate in a sequential manner. Process \(p_{n-1}\) puts new values \(2C + 1, \ldots, 3C\) and process \(n\) performs \textit{extract} operations, as long as they are successful. Notice that if some of these put operations fail and there are more than \(C\) successful extracts then some previously suspended operations “took effect” and, thus, should appear in any linearization. After that process \(p_{n-1}\) puts next new \(C\) values and process \(p_n\) performs \textit{extract} operations until the container is empty. If some put operation is unsuccessful, we start over. Note, that there can only be \(n – 2\) such unsuccessful attempts, since after each attempt we get rid of one earlier suspended operation. Finally, we make sure
that exactly \( C \) values are successfully put and successfully extracted. Notice that just before these extract operations each of these \( C \) values should be stored in distinct value-locations, otherwise, process \( n \) cannot extract them.

We show that at least one of these values is stored in exactly one value-location that is “covered” by a pending write or CAS operation. If it is a CAS that is about change the value at the location from \( x \) to \( y \), using \textit{put-value-independence}, we can replace this value with the expected value \( x \). By \textit{extract-value-independence}, the corresponding extract would then return \( x \). Also, if we replace \( x \) with \( y \) in the memory state after the put operations, the corresponding extract would return \( y \). Thus, is we perform the pending write \( y \) or CAS(\( x, y \)) before extractions, the extract operations see the same memory state and the corresponding extract returns now \( y \), which is against linearizability — it should have returned \( x \).

We now present the three phases of the construction in detail. Recall that our goal is to reach a state in which \( n - 2 \) processes are going to perform \textit{non-trivial} (write or successful CAS) operations on distinct value-locations. We then schedule one of these processes to perform its pending primitive operation which will alter one value returned by extract, violating linearizability.

**Phase 1.** Starting with the empty container, we let \( p_1 \) sequentially perform \( C \) operations \texttt{put(1)}, \texttt{put(2)}, \ldots, \texttt{put(\( C \))}. All these operations successfully finish because \( p_1 \) works in isolation. Since any other process should then be able to extract all these values, the reached state should have at least \( C \) value-locations, \( a_1, \ldots, a_C \), storing distinct elements \( 1, \ldots, C \), respectively.

**Phase 2.** In this phase, our goal is to reach a state in which \( n - 2 \) processes, \( p_1, \ldots, p_{n-2} \), are concurrently accessing the container, where each process \( p_i \), \( i = 1, \ldots, n - 2 \), is about to execute a write or a successful CAS primitive on a distinct value-location \( a_j \in \{ a_1, \ldots, a_C \} \):

- write \( y_i \), where \( y_i \) is some value different from \( j \) (the value stored in \( a_j \) after Phase 1);
- CAS from value \( j \) to \( y_i \neq j \).

We call these \( n - 2 \) processes \textit{cover-processes} and we say that process \( p_i \) covers value-location \( a_j \).

We let process \( p_1 \) perform \texttt{extract(\( \))} operations, one after another, until either some \texttt{extract(\( \))} returns \( \perp \), i.e., the container becomes empty, or the process is poised to perform a write or CAS primitive that is about to modify a value-location \( a_j \in \{ a_1, \ldots, a_C \} \), i.e., \( p_1 \) covers \( a_j \). If \( p_1 \) did not get \( \perp \), we repeat the procedure with process \( p_2 \), until it is poised to cover a not yet covered value-location in \( a_j \notin \{ a_1, \ldots, a_C \} \) or the container becomes empty. If the container is still non-empty, we proceed with processes \( p_3 \) up to \( p_{n-2} \) etc., until either (1) the container is empty, or (2) all processes \( p_1, \ldots, p_{n-2} \) become cover-processes. In case (2), we let \( p_{n-1} \) execute \texttt{extract(\( \))} operations until the container becomes empty.

As a result, we make processes \( p_1, \ldots, p_t \) (\( 0 \leq t \leq n - 2 \)) to cover distinct locations \( a_{j_1}, \ldots, a_{j_t} \), storing distinct values \( j_1, \ldots, j_t \).

As the container is now empty, we can now put values \([C + 1, 2C]\). Assuming \( \ell < n - 2 \), we let process \( p_{t+1} \) perform \texttt{put(C + 1)}, \ldots, \texttt{put(2C)} until it is going to cover a not yet covered value-location \( a_{j_{t+1}} \) (storing a distinct value \( j_{t+1} \)) or we complete operation \texttt{put(2C)}. Recall that the implementation is obstruction-free, so \( p_{t+1} \) is guaranteed either to be poised to cover a new value-location or to complete \texttt{put(2C)}. Suppose that \( p_{t+1} \) was executing \texttt{put(z_{t+1})} when it got poised to cover \( a_{j_{t+1}} \). Assuming that \( \ell + 1 < n - 2 \), we now let process \( p_{t+2} \) perform \texttt{put(z_{t+1} + 1)}, \ldots, \texttt{put(2C)}, etc.

As a result, we either engage all processes \( p_1, \ldots, p_{n-2} \) to cover distinct value-locations in \( \{ a_1, \ldots, a_C \} \) or complete \texttt{put(2C)}. In the former case, we are done and we can directly go to Phase 3. Now we are going to show that the latter case is not possible.
Indeed, suppose, by contradiction, that put(2C) is completed. Suppose that exactly \( \ell + z < n - 2 \) processes are covering \( \ell + z \) distinct value-locations in \( \{a_1, \ldots, a_C\} \) at this moment. Thus, out of total number of C launched put operations, exactly \( C - z \) are completed. Therefore, at least \( C - z \) value-locations should be used for the corresponding \( C - z \) values from \( [C + 1, 2C] \). Otherwise, no subsequent extract() operation by process \( p_{n-1} \) will return one of these values, violating linearizability. Moreover, none of these value-locations belongs to non-covered \( C - \ell - z \) value-locations in \( \{a_1, \ldots, a_C\} \). Otherwise, the process that modifies a non-covered value-location in \( \{a_1, \ldots, a_C\} \) would become one of the covering processes and we would have at least \( \ell + z + 1 \) covering processes in total.

As only \( X + \ell + z \) value-locations can be used for storing the \( C - z \) values, we have \( X + \ell + z \geq C - z \). By our assumption \( \ell + z < n - 2 \), we obtain \( X + n - 2 + z > C \). Recall that \( C \geq 3n \) and \( z \leq n - 2 \) we establish a contradiction, as \( \lfloor n/2 \rfloor + 2n - 6 < 3n \).

Thus, at the end of the phase, processes \( p_1, \ldots, p_{n-2} \) cover distinct value-locations \( \{a_{j_1}, \ldots, a_{j_{n-2}}\} \).

**Phase 3.** Now we let process \( p_{n-1} \) perform extract() operations until the container is empty, i.e., \( \bot \) is returned. Then we let \( p_{n-1} \) perform operations put(2C + 1), ..., put(3C). And, finally, we let \( p_n \) perform extract operations until the container is empty.

Notice that if some put operation is not successful and there are not exactly \( C \) successful extract operations, then some of the suspended operations “took effect” and must be linearized before the failed put operation. In this case, we let \( p_{n-1} \) and \( p_n \) perform the same series of operations with values \( [3 \cdot C + 1, 4 \cdot C] \) (recall that it is applied to the empty container). If some put operation is not successful and there are not exactly \( C \) successful extract operations, we continue with \( [4 \cdot C + 1.5 \cdot C] \) and so on, until all 2C operations succeed in a round—put operations and extract operations. Note, that there can be at most \( n - 2 \) such rounds, since after each failed round at least one of up to \( n - 2 \) suspended operations becomes linearized.

Finally, we complete a round in which all 2C operations succeed and \( p_{n-1} \) put the values \( [I \cdot C + 1, (I + 1) \cdot C] \). At least \( C \) memory locations should now contain distinct values in \( [I \cdot C + 1, (I + 1) \cdot C] \). Otherwise, subsequent extract() operations performed by \( p_n \) will not be able to return one of these values, violating linearizability.

Let \( S \) denote the memory state after put operations performed by process \( p_{n-1} \). We are going to show that in \( S \), there exist at least one covering process \( p_i \), such that (1) \( p_i \) is covering a value-location \( a_{j_i} \) that contains a unique value \( v \in [I \cdot C + 1, (I + 1) \cdot C] \), i.e., a value that is not stored in any other value-location; (2) \( j_i \) is not stored in any value-location.

Recall that we have \( C + X = C + ([n/2] - 2) \) value-locations in total. As at least \( C \) out of them are used to store values \( [I \cdot C + 1, (I + 1) \cdot C] \), at least \( n - 2 - X \geq [n/2] \) value-locations in \( \{a_{j_1}, \ldots, a_{j_{n-2}}\} \) must store values from \( [I \cdot C + 1, (I + 1) \cdot C] \). On the other hand at most \( X \) out of them can be stored more than twice (we only have \( C + X \) value-locations), thus at least one value-location \( a_{j_i} \), \( ([n/2] - X + 1) \) stores a unique value \( v \in [I \cdot C + 1, (I + 1) \cdot C] \).

As the implementation is put-element-independent, we can replace, in Phase 3, put(\( v \)) with put(\( j_i \)). Thus, the resulting state \( S_1 \) after put operations will only differ from \( S \) in that the value-location \( a_{j_i} \) now stores \( j_i \). Also, by extract-value-independence, the corresponding extract operation by \( p_n \) returns \( j_i \) instead of \( v \).

Suppose that the suspended primitive operation by \( p_{j_i} \) on \( a_{j_i} \) is \( \text{CAS}(a_{j_i}, j_i, y_{j_i}) \). Note that if we replace put(\( v \)) with put(\( j_i \)), then the resulting state \( S_2 \) after put operations differs from \( S \) and \( S_1 \) in that the value-location \( a_{j_i} \) now stores \( y_{j_i} \). Consequently, for the extract operations by \( p_n \) the corresponding extract returns \( y_{j_i} \) by extract-value-independence.
Thus, we replace \( \text{put}(v) \) with \( \text{put}(j_i) \) to get to state \( S_1 \), then we run the suspended CAS and get the system in state \( S_2 \), and, finally, we run extract operations by \( p_n \) and the corresponding extract returns \( y_i \). This violates linearizability, as that extract must have returned \( j_i \).

The contradiction implies that at least \( C + X = C + \lfloor n/2 \rfloor - 1 \) are necessary to implement Bounded Container with the desired properties.

**Remark 1.** We can reduce the number of used values from \( n \cdot C \) to \( 5 \cdot C \). Specifically for that we reduce \( X \) in three times and catch \( \frac{n}{3} - 1 \) processes instead of \( n - 2 \). Thus, we will have \( 2 \cdot \frac{n}{3} - 2 \) untouched processes for Phase 3 that can load and unload only \( 5 \cdot C \) values while not storing the information in local memory. Thus, for this proof \( X \) should be approximately \( \frac{n}{6} \).

The other way to reduce the number of used values is to bound the total number of values that a process can store in its local memory. Thus, we re-use values on the tries of Phase 3 and insert always only values that are not present in the shared memory and local memory of any process.

**Remark 2.** In the theorem above we relaxed the model: the special \( \perp \) value is not of type \( T \). However, we can tweak a proof a little bit to allow that. For that, instead of one process per value-location we catch either one process that changes from ordinary value to ordinary value or two processes: one from ordinary value \( v \in [1, C] \) to \( \perp \) and one from \( \perp \) to ordinary value \( u \in [C + 1, 2C] \). By that, we have to reduce \( X \) in two times since we can now cover only \( \frac{n}{2} - 1 \) value-locations with \( n - 2 \) process, i.e., the proof holds for \( X = \frac{n}{4} - 1 \).

**Remark 3.** Also, note that in Theorem we used an assumption that \( 3 \cdot n \leq C \). Note, that if we do not have such an assumption, we can prove that the memory overhead should be \( \Omega(\min(n, C)) \) which is satisfied by the state-of-the-art algorithms.

### 4 ALGORITHMS UNDER RESTRICTIONS

Before presenting our memory-optimal algorithm in Section 5, we describe several simple lock-free algorithms under additional restrictions, either on the application or on the model, exhibiting constant or \( \Omega(n) \) memory overhead. We believe that these algorithms are useful, as the restrictions are quite common in practice. Moreover, they gradually bring us closer to our memory-optimal algorithm.

#### 4.1 Distinct Elements Assumption

The first lock-free algorithm we consider uses only \( O(1) \) additional memory under a few reasonable assumptions. At first, it requires all the elements to be distinct; thus, the ABA problem \([8]\) is excluded. It is common to use queues for storing uniquely identified tasks or identifiers themselves. The second assumption is that the system is provided with an unlimited supply of versioned \( \perp \) (null) values, so we can replace unique elements with unique \( \perp \)-s on extractions. This condition is also practical and can be achieved by stealing one bit from addresses (values) to mark them as \( \perp \) values and use the rest of the address (value) for storing a version.

Note that the latter assumption precludes the use of value-locations, exclusively dedicated to storing values, and makes it harder to implement the algorithm in a languages with garbage collection (e.g., it is almost impossible to steal a bit in JVM-based languages, though achievable in Go). We believe that the algorithm is an interesting first step towards a generic solution with constant memory overhead.

The pseudo-code of the algorithm is presented in Figure 2. Essentially, both enqueue and dequeue: 1) read the counters in a snapshot manner (dequeue also reads the element to be extracted); 2) try to perform a “round-valid”
update (enqueue replaces \( \perp_{\text{round}} \) with the element while dequeue does the opposite, where \( \text{round} = \text{counter} / C \)), the operation restarts if the attempt fails; and 3) increase the operation counter by CAS.

```c
func enqueue(x: T) Bool:
  while (true): // CAS loop
    // Read the counters snapshot
    e := enqueues
    d := dequeues
    if e != enqueues: continue
    // Is the queue full?
    if e == d + C: return false
    // Try to perform the enqueue
    round := e / C
    done := CAS(&a[e % C], \( \perp \), e)
    if done: return true // finish on success

func dequeue(): T?:
  while (true): // CAS loop
    // Read a snapshot of the counters and the element
    e := enqueues
    d := dequeues
    x := a[d % C]
    if e != enqueues: continue
    if e == d: return \( \perp \) // is queue empty?
    // Try to retrieve the element
    round := d / C + 1 // next round
    done := x != \( \perp \) && CAS(&a[d % C], x, \( \perp \))
    if done: return x // finish on success
```

Fig. 2. Bounded queue algorithm with \( O(1) \) additional memory that requires elements to be distinct and an unlimited supply of versioned \( \perp \) (null) values.

Adding a new element. At first, enqueue atomically snapshots the monotonically increasing enqueues and dequeues counters using the double-collect technique (lines 5–6). After that, it checks whether the queue is full by looking at the difference between the total number of enqueues and dequeues (line 8). Note that there can be a concurrent dequeue invocation that already retrieved the element at position \( d \% C \) but has not increased dequeues counter yet — we can linearize the fullness detection before this dequeue.

As the next step, the algorithm tries to put the element into the cell using CAS from the null-value of the current round \( \perp_{\text{round}} \) to the element (line 11). This CAS synchronizes concurrent enqueue operations so only one of them succeeds at this cell. It is possible for enqueue to suspend and skip its round: another element can be inserted to this cell and further retrieved by some future dequeue. In this case, we need a mechanism to detect that the round is missed, and fail the element insertion CAS. For this purpose, we use different \( \perp \) values for each round (the array \( a \) is filled with \( \perp_0 \) initially).

After the insertion attempt, we guarantee that either the current operation or a concurrent one succeeded. Therefore, the algorithm increments the number of completed enqueue invocations (line 12), and returns true if the algorithm successfully inserted the element, retrying the whole operation otherwise. The linearization point of a successful enqueue operation is the corresponding counter increment (line 12), which can be performed by another thread as well.

Retrieving an element. The algorithm for dequeue is similar to enqueue. We start with the double-collect technique to get an atomic snapshot of the counters and the element to be extracted (lines 18–20). After that, the algorithm checks whether the queue is empty (line 21). It might be the case that the counters coincide, but the array contains one element—a concurrent enqueue successfully inserted its element, but has not updated the counter yet; we linearize the dequeue operation in line 21, just before before this enqueue.

Since our dequeue algorithm increments the corresponding counter at the end (line 25), the element can be already taken at the point of getting the snapshot, before the counter is updated. Thus, the algorithm checks whether the read element is not \( \perp_{\text{round}} \) (line 24) and tries to replace it with the one (line 24). As we guarantee that all the elements are distinct, the extraction cannot miss its round. At the end, the algorithm increments the counter (line 25). Similarly to enqueue, we can linearize successful retrievals at the point of the counter is incremented, which can be performed by another thread.
4.2 Synchronization via LL/SC

Another way to avoid the ABA problem is to use LL/SC (load-link/store-conditional) primitive. The primitive exports two operations: (1) LL(&a) reads a value located by address (field) a; and (2) SC(&a, x) stores the value x into the memory by address (field) a if the value has not been changed since the last LL invocation on this location; returns true if succeeds and false otherwise. LL/SC primitives are available on popular platforms like ARM or PowerPC. However, programming languages rarely support them.

The algorithm, presented in Listing 3, is very close to the previous one, important differences are highlighted with yellow. At first, both enqueue and dequeue now read the current states of the cells additionally to the counters snapshots (lines 4–6 and 12–18, respectively). As we use double-collect here, we guarantee that the read values correspond to the current round. In order to perform a cell update, we use SC instead of CAS. As we read values using LL, an update fails if and only if the cell has been changed. Notice that enqueue and dequeue are almost identical.

```
1 func enqueue(x: T) Bool:
2 while (true):
3     // Read a "counters + cell" snapshot
4     e := enqueues; d := dequeues
5     state := LL(&a[e % C])
6     if e != enqueues: continue
7     if e == d + C: return false // is queue full?
8     // Try to perform the enqueue
9     done := state == ⊥ && SC(&a[e % C], x)
10    // Increment the counter
11    if LL(&enqueues) == e: SC(&enqueues, e + 1)
12    if done: return true // finish on success

13 func dequeue() T?:
14    while(true):
15        // Read a "counters + cell" snapshot
16        d := dequeues; e := enqueues
17        x := LL(&a[d % C])
18        if d != dequeues: continue
19        if e == d: return ⊥ // is queue empty?
20        // Try to retrieve the element
21        done := x != ⊥ && SC(&a[d % C], ⊥)
22        // Increment the counter
23        if LL(&dequeues) == d: SC(&dequeues, d + 1)
24        if done: return x // finish on success
```

Fig. 3. Bounded queue algorithm with \(O(1)\) additional memory via LL/SC. This is a modification of the algorithm from Figure 2, the changes are highlighted.

4.3 Synchronization via DCSS

Finally, we describe a simple and straightforward algorithm based on the Double-Compare-Single-Set (DCSS) synchronization primitive: DCSS(&a, expectedA, updateA, &b, expectedB) checks that the values located at addresses a and b are equal to expectedA and expectedB, respectively, and if the check succeeds it stores updateA to the memory location a and returns true; otherwise, it returns false.

Listing 4 shows a pseudo-code of the algorithm. We use DCSS to atomically update the cell and check that the corresponding counter has not been changed. If DCSS fails, the algorithm helps to increment the counter and restarts the operation. The rest is the same as in the previous algorithms: getting an atomic snapshot of the counters (and the cell state for dequeue), checking whether the operation is legal (fullness check for enqueue and emptiness one for dequeue), trying to update the cell, and incrementing the corresponding counter at the end.

The straightforward DCSS [6] implementation uses descriptors, which can be recycled in a way so that only 2n of them are required [1], thus, incurring \(\Theta(n)\) additional memory. In a few words, a descriptor is a special object that differs from the values and announces an operation, in this case DCSS, to get help from other processes. Assuming that the queue stores references, the distinction between values and descriptors can be implemented by either stealing a marker bit from addresses or using a language construction like instanceof in Java to check whether the object is a descriptor or not.
Fig. 4. Bounded queue algorithm with $\Theta(n)$ overhead via recyclable DCSS descriptors. This is a modification of the algorithm from Figure 2, the changes are highlighted.

However, the descriptor-based implementation of DCSS requires values and descriptors to be stored in the same array cells, which undermines our assumption that value-locations do not store metadata (Section 3). Nevertheless, in practice, this solution can be used for a bounded queue of references in most of the popular languages. In the next section, we describe $\Theta(n)$ overhead algorithm that uses only read, write and CAS primitives, at the expense of being more complicated.

5 MEMORY-OPTIMAL BOUNDED QUEUE

The solutions in Section 4 show that handling the ABA problem without additional requirements is nontrivial. The algorithm with recyclable DCSS descriptors matches our linear lower bound, but requires the ability to store descriptors and values in the same array cells, violating our restriction that value-locations do not store metadata. Also, it makes it practically impossible to store primitive elements, such as integers. In this section, we present an algorithm (Listing 5) that asymptotically matches the lower bound (Theorem 4) under the same assumptions. To avoid the ABA problem, we use descriptors for enqueue operations stored in $n$ pre-allocated metadata locations. The algorithm requires $2n$ descriptors to be recycled [1] (each takes $\Theta(1)$ memory) and $n$ additional locations to store the references to the descriptors, some kind of an announcement array, resulting in $\Theta(n)$ memory overhead.

High-level overview. As in the previous algorithms, both dequeue and enqueue start with taking a snapshot of the counters (lines 58–60 and 65–66) and check if the queue is empty (lines 61 and 67). However, the implementation of dequeue slightly differs from those in the algorithms above. To read the element to be retrieved during the snapshot (line 59), it uses a special readElem function. Then it tries to increment the dequeues counter and returns the element if the corresponding CAS succeeds (line 62).

As for enqueue, it creates a special EnqOp descriptor that tries to atomically apply the operation (lines 68–68). Then the operation increments the enqueues counter, possibly helping a concurrent operation (line 69). If the descriptor is successfully applied, the operation completes. Otherwise, the operation is restarted.

Additionally, the algorithms maintain an array ops of EnqOp descriptors (line 22), which specifies "in-progress" enqueue invocations; see the EnqOp declaration in lines 1–20. Intuitively, EnqOp descriptor is an intention to perform an enqueue operation if the enqueues counter has not been changed; that is very similar to our DCSS-based solution above. This way, the descriptor stores an operation status in successful field (line 9). The main idea here is that only one descriptor in the successful state can "cover" a given cell in the elements array $a$, no other successful descriptor can point to the same cell (see putOp function at lines 28–40).

Thus, only the thread that started covering the cell is eligible to update it, and there is no conflict on updates here. When the thread with an exclusive access for modifications finishes the operation, it "uncovers" the cell (see function completeOp in lines 52–56), so another operation is able to cover it. However, when an enqueue operation finds a cell
Adding a new element. As discussed above, enqueue creates a new EnqOp descriptor (line 68), which tries to apply the operation if the enqueue counter has not been changed (line 68). The corresponding apply function is described in lines 73–89. At first, it tries to find an operation which already covers the same cell in a (line 74). If no one covers the cell, it tries to put EnqOp into ops; this attempt can fail if a concurrent enqueue does the same—only one enqueue should succeed. If the current EnqOp is successfully inserted into ops (putOp returns a valid insertion slot number at line 77), the cell is covered and the current thread is eligible to update it, thus, the operation is completed (line 79). Otherwise, if another operation covers the cell, it checks that the descriptor belongs to the previous round (line 82) and tries to replace it with the current one (lines 86–89). However, since the current thread can be suspended, the found descriptor might belong to the current or a subsequent round, or the replacement in line 74 fails because a concurrent enqueue succeeds before us; in this case, the enqueue attempt fails.

An intuitive state diagram for ops slots is presented in Figure 3. Starting from the initial empty state (⊥), an operation that reads the counter value \( e_i \) and intends to cover cell \( a[e_i \% C] \) occupies the slot (the state changes to the “yellow” one). After that, it (or another helping thread) checks whether there exists an operation descriptor in other slots that already covers the same cell, and fails if so (moving to the “red” state), or successfully applies the operation and covers the slot (moving to the “green” state). At last, the operation writes the value to the cell and frees the slot (moving to the initial state). If a next-round enqueue arrives while the cell is still covered, it replaces the old EnqOp descriptor with a new one (moving to the next “green” state). In this case, the operation is completed by the thread that covers the cell.

Atomic EnqOp put. Finally, we describe how to atomically change the descriptor status depending on whether the corresponding cell is covered. In our algorithm, we perform all such puts sequentially using the activeOp field to define the current active EnqOp (line 24). For the sake of lock-freedom, other threads can perform helping while only the

Reading an element. Once dequeue reads the first element during the snapshot, it cannot simply read cell \( a[d \% C] \), as there can be a successful EnqOp in ops array which covers the cell but has not written the element to the cell yet. Thus, we use a special readElem function (lines 93–96), which goes through array ops looking for a successful descriptor that covers the cell, and returns the corresponding element if one is found. If there are no such descriptors, readElem returns the value stored in the elements array.

Note that we invoke readElem between two dequeues counter reads, checking if they coincide. Thus, we guarantee that the current dequeue has not been missed the round during the readElem invocation, and, therefore, cannot return a value inserted during one of the next rounds. Concurrently, the latest successful enqueue invocation to the corresponding cell might either be “stuck” in the ops array or have successfully written its value to the cell and completed; readElem finds the correct element in both scenarios.
// Descriptor for enqueues

class EnqOp<T>(enqueues: Long, element: T) {
    val e = enqueues // 'enqueues' value
    val x = element // the inserting element
    val i = e % C // cell index in 'a'
    // Op status: true, false, or ⊥;
    // we consider ⊥ as the second 'false' in logical expressions.
    var successful: Bool? = ⊥
    fun tryPut(): // performs the logical put
        // Is there an operation which
        // already covers cell 'i'?
        (op, opSlot) := findOp(i)
        if op != ⊥ & op != ⊥:
            | CAS(&successful, ⊥, false)
    // Has 'enqueues' been changed?
    eValid := e == enqueues
    CAS(&successful, ⊥, eValid)
    }
}

// Currently running enqueues
var ops: EnqOp?[n] = new EnqOp?[n]
// The next operation to be applied
var activeOp: EnqOp? = ⊥

// Puts 'op' into 'ops' and returns
// its location, or -1 on failure.
func putOp(op: EnqOp): Int {
    for j in 0..n-1 {
        | opSlot := j % n
        | if CAS(&ops[opSlot], ⊥, op):
            | continue // occupied
        startPutOp(op)
        tryPut() // logical addition
        // Finished, free 'activeOp'.
        CAS(&activeOp, op, ⊥)
        if !op.successful:
            | ops[opSlot] = ⊥ // clean the slot
        return -1
    }
    return opSlot
}

// Starts the 'op' addition.
func startPutOp(op: EnqOp) = while (true):
    cur := activeOp
    if cur != ⊥: // need to help
        cur.trryPut()
    CAS(&activeOp, cur, ⊥)
    if CAS(&activeOp, ⊥, op): return
    // Only the thread that covers the
    // cell is eligible to invoke this.
    func completeOp(opSlot: Int) = while (true):
        op := readOp(opSlot) // not ⊥
        | opSlot := j % n
        CAS(&enqueues, op.e, op.e + 1)
        if CAS(&enqueues, d, d + C): return

func dequeue() T? = while (true):
    d := enqueues; e := enqueues
    x := readElem(d % C)
    if d != enqueues: continue
    if e == d: return ⊥
    if CAS(&enqueues, d, d + 1): return x

func enqueue(x: T) Bool = while (true):
    e := enqueues; d := enqueues
    if e != enqueues: continue
    if e == d + C: return false
    op := new EnqOp(x); apply(op)
    CAS(&enqueues, e, e + 1)
    if op.successful: return

// Tries to apply 'op'.
func apply(op: EnqOp):
    | cur, opSlot := findOp(op.i)
    // Try to cover the cell by 'op'.
    if cur == ⊥:
        opSlot = putOp(op)
        // Complete 'op' if the cell is covered
        if opSlot != -1: completeOp(opSlot)
        return
        // 'cur' already covers the cell.
        if cur.e >= e: // is 'op' outdated?
            op.successful = false
            return
        // Try to replace 'cur' with 'op'.
        op.successful = true
        if CAS(&ops[opSlot], cur, op): return
        // The replacement failed.
        op.successful = false
        return
    // Looks for 'EnqOp' that covers cell
    // 'i', returns 'a[i]' if not found.
    func readElem(i: Int): T {
        (op, _) := findOp(i)
        if op != ⊥: return op.elem
        return a[i] // EnqOp is not found
    }

    // Returns the operation located at
    // 'ops[opSlot]' if it is successful.
    func readOp(opSlot: Int) EnqOp?:
        op := ops[opSlot]
        if op != ⊥ & & op.successful: return op
        return ⊥

// Returns a successful operation that
// covers cell 'i', ⊥ otherwise.
func findOp(i: Int) (EnqOp?, Int):
    for opSlot in 0..n-1 {
        op := readOp(opSlot)
        if op != ⊥ & & op.i == i:
            return (op.value, opSlot)
        return ⊥ // not found

Fig. 5. Bounded queue algorithm with Θ(n) overhead.
“fastest” one modifies the status of the operation. Hence, putOp function (lines 28–40) circularly goes through the ops slots, trying to find an empty one and to insert the current operation descriptor into it (line 31). At this point, the status of this EnqOp is not defined. Then the operation should be placed to activeOp via startPutOp, which tries to atomically change the field from ⊥ to the descriptor (line 48) and helps other operations if needed (lines 44–47). Afterwards, the status of the operation is examined in tryPut method in EnqOp class — it checks that the cell is not covered (lines 14–14) and the enqueues counter has not been changed (lines 18–19). At the end, activeOp is cleaned up (line 36) so other descriptors can be processed. If the put attempt is successful, putOp returns the corresponding slot number; otherwise, it cleans the slot (line 38) and returns −1.

Correctness. Detailed proofs of lock-freedom and linearizability are presented in Appendices A and B. Here we sketch an intuition.

For dequeue, lock-freedom guarantee is immediate, as the only case when the operation fails and has to retry is when the dequeues counter was concurrently incremented, which indicates that a concurrent dequeue has made progress.

As for enqueue, all the CAS failures except for when the enqueues counter increments (line 69 in enqueue and 55 in completeOp) or the ops slot occupation (line 31) fail intuitively indicate the system’s progress. An enqueue fails only due to helping, which not cause retries. The only non-trivial situation is when a thread is stuck while trying to occupy a slot in putOp. Since the ops size equals the number of processes, it is guaranteed that when a thread intends to put an operation descriptor into it, there is at least one free slot. Thus, if no slot is occupied during the traversal through the whole ops array, enqueues by other processes successfully occupy slots, and the system as a whole still makes progress.

The linearization points of the operations can be assigned as follows. A successful dequeue operation linearizes at successful CAS in line 62. A failed dequeue operation linearizes in line 58. For a successful enqueue operation, we consider the descriptor op (created by that operation) that appears in ops array and has its successful field set. The linearization point of the operation is in CAS that changes enqueues counter from op.e to op.e + 1. A failed enqueue operation linearizes in line 65. One can easily check that queue operations ordered according to their linearization points constitute a correct sequential history.

6 RELATED WORK

Memory efficiency has always been one of the central concerns in concurrent computing, many theoretical bounds have been established on memory requirements of various concurrent abstractions, such as mutual exclusion [3], perturbable objects [10], or consensus [21]. However, it appears that minimizing memory overhead in dynamic concurrent data structures has not been in the highlight until recently. A standard way to implement a lock-free bounded queue, the major running example of this paper, is to use descriptors [15, 18] or additional meta-information per each element [4, 16, 17, 19]. The overhead of resulting solutions is proportional to the queue size: a descriptor contains an additional Ω(1) data to distinguish it from a value, while an additional meta-information is Ω(1) memory appended to the value by the definition.

The fastest queues, however, store multiple elements in nodes so the memory overhead per element is relatively small from the practical point of view [13, 20], and we consider them as memory-friendly, though not memory-optimal.

A notable exception is the work by Tsigas et al. [17] that tries to answer our question: whether there exists a lock-free concurrent bounded queue with O(1) additional memory. The solution proposed in [17] is still a subject to the ABA problem even when all the elements are different: it uses only two null-values, and if one process becomes asleep for two “rounds” (i.e., the pointers for enqueue and dequeue has made two traversals through all the elements), waking
up it can incorrectly place the element into the queue. Besides resolving the issue, our algorithm, under the same assumptions, is shorter and easier to understand (see Subsection 4.1).

The tightest algorithm we found is the recent work by Nikolaev [14], that proposes a lock-free bounded queue with capacity $C$ implemented on top of an array with $2C$ memory cells. While the algorithm manipulates the counters via Fetch-And-Add, it still requires descriptors, one per each ongoing operation. This leads to the additional overhead linear in $n$. Thus, the total memory overhead is $\Omega(C + n)$, while our algorithm simply needs $\Theta(n)$.

7 DISCUSSION AND OPEN QUESTIONS

In this paper, we show that a wide class of non-blocking implementations of bounded containers incur $\Omega(n)$ memory overhead, and show that the bound is tight for bounded queues by presenting a matching algorithm. While we believe that the algorithm can be easily extended to pools, the existence of a stack algorithm with the same overhead is open and interesting for further research.

In Section 4, we also present a series of algorithms that work under several practical restrictions on the system or/and the application with constant memory overhead, and a very simple DCSS-based algorithm that matches our lower bound. Other important assumptions yet to consider include (1) single-producer/single-consumer application restriction, (2) ability to store descriptors in value locations (free in JVM and Go) or “steal” a couple of bits from addresses (free in C++) for containers of references, and (3) relaxation of the object semantics, including the probabilistic ones. Each of these assumptions corresponds to a popular class of applications, and determining the optimal memory overhead in these scenarios is very appealing in practice.

The current design trend to make concurrent container more scalable is based on using Fetch-And-Add (FAA) on the contended path and CAS for remaining synchronization. As for queues, the fastest algorithms we know use FAA to increment the counters, so at most one enqueue and dequeue manipulate with the same value-location [13, 20]. Following this pattern, it would be interesting to restrict the number of parallel accesses to value- or, even, metadata-locations, adjusting the lower bound and providing a matching algorithm. It would also be interesting to consider the problem of memory overhead incurred by wait-free containers since such solutions are typically more complicated.

Finally, in this paper we only focused on bounded containers, as they allow a natural definition of memory overhead. One may try to extend the notion to the case of unbounded containers by considering the ratio between the amount of memory allocated for the metadata with respect to the memory allocated for data elements. Defining the bounds on this ratio for various data structures remains an intriguing open question.

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A PROOF OF CORRECTNESS: LOCK-FREEDOM

In this section, we prove the lock-freedom of our bounded queue algorithm which matches the lower bound and is presented in Section 5. We examine dequeue and enqueue operations separately.

The dequeue operation. Following the code, the only place where the operation can restart is an unsuccessful dequeue counter increment (line 62). However, each increment failure indicates that a concurrent dequeue has successfully performed the same increment and completed. Thus, the lock-freedom for dequeues is trivial.

The enqueue operation. It is easy to notice that a successful enqueues counter increment indicates the system’s progress. This way, if we will show that the following assumption is incorrect: the counter is never changing and all the in-progress enqueues are stuck in a live-lock, we automatically show the enqueue lock-freedom.

Let’s denote the current enqueues value, that cannot be changed, as $e_{\text{stuck}}$. Our main idea is proving that there should appear a successful descriptor with field $e$ equals $e_{\text{stuck}}$. In this case, following the code, if the enqueues are stuck in a live-lock, there should exist an infinite number of restarts in enqueue operations. By that these enqueue operations should find the successful descriptor in ops array, if there exists one, and, thus, the counter enqueues should be incremented at some point. Thus, the assumption is correct only if a descriptor with successful field set to true cannot be put into ops.

Consider the function apply which is called by enqueue operation (line 68). There are two possible scenarios: if $\text{findOp}$ in line 74 finds a descriptor or not. Consider the first case. If it finds the descriptor $\text{op}$ with $\text{op}.e \geq e_{\text{stuck}}$ then either we found the required descriptor or enqueues descriptor was incremented. Thus the found descriptor is from the previous round, the operation tries to replace it with a new one, already in the successful state (line 87). The corresponding CAS failure indicate that either the previous-round operation is completed and the slot becomes ⊥, or a concurrent enqueue successfully replaced the descriptor with a new one with $\text{op}.e = e_{\text{stuck}}$. Since the second case breaks our assumption, we consider that the previous-round operation put ⊥ into the slot, which leads to the second scenario, when we do not find the descriptor since the cell is not covered.
In the second case, findOp in apply function (line 74) did not find a descriptor and tries to put its own. Let’s assume now that each thread successfully finds and occupies an empty slot in a bounded number of steps in putOp function (lines 29–32). Thus, some descriptor with op.e = \( e_{\text{stuck}} \) should be successfully set into activeOp (see function startPutOp), and the following tryPut invocation sets the status to the successful one since there is no other successful descriptor that covers the cell according to the main assumption. This means, that the only way not to break this assumption is that is to never occupy a slot in putOp. However, since the ops.size equals to the number of processes, the only way not to cover a slot during an array traversal, is that there exist an infinite amount of EnqOp descriptors which are put into ops. Obviously, \( e \) field on all of them is equal to \( e_{\text{stuck}} \) and at least one of them passes to activeOp and become successful by using the same argument as earlier. As a result, we show that there should occur a successful descriptor with op.e = \( e_{\text{stuck}} \) in ops array in any possible scenario, which breaks the assumption and, thus, provides the lock-freedom guarantee for enqueue.

B PROOF OF CORRECTNESS: LINEARIZABILITY

This section is devoted for the linearizability proof of our algorithm that matches the lower bound (see Section 5). At first, we suppose that all operations during execution are successful so the checks for emptiness and fullness never satisfy. This way, we are provided with an arbitrary finite history \( H \), and we need to construct a linearization \( S \) of \( H \) by assigning a linearization point to each completed operation \( \delta \) in \( H \). We prove several lemmas before providing these linearization points.

**Lemma 1.** At any time during the whole execution it is guaranteed that dequeues \( \leq \) enqueues \( \leq \) dequeues + \( C \).

**Proof.** We prove this by contradiction. Let dequeues exceed enqueues. Since the counters are always increased by one, we choose the first moment when dequeues becomes equal enqueues + 1 which, obviously, happens at the corresponding CAS invocation \( \delta_1 \) in dequeue (line 62). Consider the situation right before this CAS \( \delta_1 \): enqueues and dequeues were equal. Consider the previous successful CAS \( \delta_0 \) that incremented enqueues or dequeues counter. Obviously, it was a CAS on dequeues since otherwise we chose not the first moment in the beginning of the proof. Thus, between \( \delta_0 \) and \( \delta_1 \) enqueues and dequeues did not change, and the emptiness check at line 61 should succeed during the dequeue invocation that has performed \( \delta_1 \), so the CAS \( \delta_1 \) cannot be executed.

The situation when enqueues exceed dequeues + \( C \) can be shown to be unreachable in a similar manner. \( \square \)

Further, we say that a descriptor EnqOp is successful if at some moment its successful field was set to true and it was in ops array.

**Lemma 2.** During the execution, for each value \( \text{pos} \in [0, \text{enqueues}] \) there existed exactly one successful EnqOp object with the field \( e \) set to the specified \( \text{pos} \).

**Proof.** At first, we show that there exists at least one EnqOp descriptor. The enqueues counter can never increment if there does not exist the corresponding EnqOp object: CAS at line 69 is successful only if we put new object or the object exists, the same goes for line 55.

This object is unique, since we add objects EnqOp into array ops one by one using startPutOp function. In other words, the checks in Lines 14-19 are performed in the serialized manner. Thus, the object cannot become successful if it finds another object with the same value in field \( i \) in line 14. \( \square \)

We denote the last EnqOp created by an enqueue operation in line 68 as the last descriptor of the operation.
LEMMA 3. The last descriptor of enqueue(x) either becomes successful or the operation never finishes. Also, exactly one EnqOp descriptor created by the operation becomes successful.

Proof. We note that if the descriptor does not become successful then the operation restarts at line 70. Note that the flag successful cannot be true at line 70 if the object was not added to ops array. Thus, it should become successful. □

Since by Lemma 2 for each value pos there exists exactly one EnqOp descriptor with the specified pos and by Lemma 3 enqueue operation creates only one successful description, we can make a bijection between enqueue operations and the positions pos: that are chosen as the value of the field e field in the last descriptor of the operation. (Note, that there is an obvious bijection between operations and descriptors.) Thus, we can say "the position of the enqueue" and "the enqueue of the position". Also, we say that EnqOp descriptor op covers a cell id if op.i = id, or e.c % C = id.

The linearization points are defined straightforwardly. For dequeue() the linearization point is at the successful CAS performed in line 62. The linearization point for \( \pi = \text{enqueue}(x) \) is the successful CAS performed on enqueues counter in line 55 or line 69 from \( E \) to \( E + 1 \) where \( E \) is the position of \( \pi \).

Let \( S_\sigma \) be the linearization of the prefix \( \sigma \) of the execution \( H \). Also, we map the state of our concurrent queue after the prefix \( \sigma \) of \( H \) to the state of the sequential queue \( Q_\sigma \) provided by the algorithm in Listing 1. This map is defined as follows: the values of the counters enqueues and dequeues are taken as they are, and for each \( pos \in \{ \text{dequeues, enqueues} \} \) the value in the position \( pos \ % C \) contains \( op.x \) of successful EnqOp op with \( op.e = pos \) from ops array or, if such op does not exist, simply \( a[pos \ % C] \).

LEMMA 4. The sequential history \( S_H \) complies with the queue specification.

Proof. We use induction on \( \sigma \) to show that for every prefix \( \sigma \) of \( H \), \( S_\sigma \) is a queue history and the state of the sequential queue \( L_\sigma \) from Listing 1 after executing \( S_\sigma \) is equal to \( Q_\sigma \). The claim is clearly true for \( \sigma = \epsilon \). For the inductive step we have \( \sigma = \sigma_1 \circ \delta \) and \( L_\sigma \), after \( S_\sigma \), coincides with \( Q_\sigma \).

Suppose, \( \delta \) is the successful CAS in line 62 of operation \( \pi = \text{dequeue}() \). Consider the last two reads \( d := \text{dequeues} \), \( \delta_1 \), and \( e := \text{enqueues} \), \( \delta_2 \), in Lines 58 and 58 in \( \pi \). Suppose that \( \delta_1 \) reads \( D \) as \( d \) and \( \delta_2 \) reads \( E \) as \( e \). Let \( \sigma_2 \) be the prefix of \( H \) that ends on \( \delta_2 \). Since we forbid \( \pi \) to satisfy the emptiness property we could totally say that the linearization point of the enqueue of position \( D \) already passed since \( E > D \), otherwise, \( \pi \) should have been restarted in line 60. Thus, by induction \( Q_{\sigma_2} \) and \( L_{\sigma_2} \) both contains same \( v \) at position \( d \ % C \). The same holds for \( Q_\sigma \) and \( L_\sigma \). Thus, between \( \delta_2 \) and \( \delta \) queue \( Q \) always contains \( v \) in \( d \ % C \) such as queue \( L \). By the definition of \( Q \) this means that during that interval there is always exist either successful EnqOp op with \( op.e = D \) and \( op.x = v \) or \( v \) is stored in \( a[D \ % C] \), thus \( \text{readElem}(d \ % C) \) in line 59 would correctly read \( v \). This means, that the result of \( \text{dequeue}() \) in both, the concurrent and the sequential, queues coincides.

Suppose, \( \delta \) is the successful CAS in line 69 of operation \( \pi = \text{enqueue}(x) \). Suppose that during \( e := \text{enqueues} \) in line 65, \( \delta_1 \), we read \( E \). We want to prove that three things: 1) there already existed successful EnqOp op with \( op.e = E \); 2) there is no EnqOp op with \( op.e < E \) and \( op.i = E \ % C \) in ops array; and 3) none of EnqOp descriptor \( op_2 \) with \( op_2.e > E \) and \( op_2.i = E \ % C \) exist before \( \delta \). It will be enough since by our mapping the cell \( E \ % C \) will contain the same value in both queues \( Q_\sigma \) and \( L_\sigma \), because if there is no successful descriptor \( op \) in ops with \( op.e = E \) then it already dumps its value to array \( a \) in line 54 and nobody could have overwritten in.

Note, that the second and the third part can be proved very simply. As for the second part, if we find an outdated descriptor with the same \( i \) we replace it to the new one in line 87. As for the third part, since enqueues does not exceed \( E \) before \( \delta \), then no process can even create EnqOp descriptor \( op \) with \( op.e > E \).
This means, that we are left to prove the first part. Since the CAS in line 69, δ, is successful then enqueues counter did not change in-between δ₁ and δ. Thus, during apply operation in line 68 enqueues was constant. Let us look on what this operation is doing. At first, it tries to find a successful descriptor EnqOp op for which \(op = E \mod C\) in line 74. If it does not exist (the check in line 76 succeeds) it tries to put its own descriptor by using the function putOp in line 77. As the first step, it tries to put not yet successful EnqOp into array ops (line 31). The operation did not become successful in tryPut() function only if either there exists EnqOp op with \(op = E \mod C\) or enqueues changes. Since, enqueues cannot change as we discussed prior, thus in line 14 finds EnqOp op with \(op.i = E \mod C\). Note that since enqueues did not change and findOp in line 74 did not find an operation, then \(op.e\) has to be equal to \(E\), thus, providing with what we desired. In the other case, apply finds EnqOp op with \(op.i = E \mod C\). Note that since enqueues does not change \(op.i\) cannot exceed \(e\). Thus, if the check in line 82 succeeds, \(op.e\) should be equal to \(E\) and we are done. Otherwise, this means that we found some old \(op\) and we try to replace it by CAS in line 87. Due to the fact that enqueues does not change, CAS can fail only if another EnqOp descriptor \(op\) with \(op.e = E\) is successfully placed. By that, there existed EnqOp object that we desired. Thus, after we add \(\delta\) to \(\sigma_1\), \(Q_\sigma\) and \(L_\sigma\) coincides.

Suppose, \(\delta\) is the successful CAS in line 55 of operation \(\pi = \text{enqueue}(v)\). As in the previous case, it is enough to prove two things: 1) there already exists corresponding EnqOp; 2) there are no other EnqOp that cover the same cell in ops array. The second statement is easy to prove as before. For the first one, we note that we apply \(\delta\) only if we find a successful descriptor by readOp(opSlot) in line 53.

Finally, suppose that \(\delta\) is none of the successful CAS in Lines 62, 69 and 55. Thus, \(L_\sigma\) does not change between \(\sigma_1\) and \(\sigma\). At the same time, \(Q_\sigma\) can change only due to the removal of EnqOp descriptor from array ops. This can happen in two places: successful CAS operations in line 87 and in line 56. In the first case, nothing happens since we simply replace old EnqOp descriptor with the new one, thus \(Q_\sigma\) remains the same. In the second case, the successful CAS means that no new descriptor is stored in array ops with the same value in field \(i\) and enqueues does not yet come to the next round, otherwise, \(op\) read in line 53 should have been replaced by 87. Thus, the value stored in array \(a\) is exactly what was stored by \(op\) until dequeues passes through that position.

Note that the proof of the previous lemma is enough to show that the implementation is linearizable if we throw away operations for which check for emptiness or fullness is satisfied. However, such operations are very easy to linearize. Unsuccessful dequeue() is linearized at line 58 while unsuccessful enqueue() is linearized at line 65. Note that the correctness of the choice of the linearization points can be proved the same way as above, since enqueues and dequeues counters of \(Q_\sigma\) match enqueues and dequeues counters of \(L_\sigma\).