DATA APPROXIMATION WITH TIME-FREQUENCY INVARIANT SYSTEMS

DAVIDE BARBIERI, C. CABRELLI, E. HERNÁNDEZ, AND U. MOLTER

Abstract. In this paper we prove the existence of a time-frequency space that best approximates a given finite set of data. Here best approximation is in the least square sense, among all time-frequency spaces with no more than a prescribed number of generators. We provide a formula to construct the generators from the data and give the exact error of approximation. The setting is in the space of square integrable functions defined on a second countable LCA group and we use the Zak transform as the main tool.

1. Introduction and main result

Time-frequency systems, also called Gabor or Weyl-Heisenberg systems in the literature, are used extensively in the theory of communication, to analyze continuous signals, and to process digital data such as sampled audio or images.

Time-frequency spaces try to represent features of both a function and its frequencies by decomposing the signal into time-frequency atoms given by modulations and translations of a finite number of functions \[9\]. If one looks at a musical score, on the horizontal axis the composer represents the time, and on the vertical axis the “frequency” given by the amplitude of the signal at that instant. Finding \textit{sparse representations} (i.e. spaces generated by a small set of functions) will be useful for example in classification tasks.

In numerical applications to time-dependent phenomena, one often encounters uniformly sampled signals of finite length, i.e. vectors of \(d\) elements, such as audio signals with a constant sampling frequency. In this case the most direct approach is to consider Fourier analysis on the cyclic group \(\mathbb{Z}_d\).

To include a large variety of situations, our setting will be that of a locally compact abelian (LCA) group. The general construction developed in this paper will be specialised to the cyclic group \(\mathbb{Z}_d\) in Example 2.2.

In this paper \(G = (G,+)\) will be a second countable LCA group, that is, an abelian group endowed with a locally compact and second countable Hausdorff topology for which \((x,y) \mapsto x - y\) is continuous from \(G \times G\) into \(G\). We denote by \(\hat{G}\) the dual group of \(G\), formed by the characters of \(G\): an element \(\alpha \in \hat{G}\) is a continuous homomorphism from \(G\) into \(T = \{z \in \mathbb{C} : |z| = 1\}\). The action of \(\alpha\) on \(x \in G\) will be denoted by \((x,\alpha) := \alpha(x)\), to reflect the fact that the dual of \(\hat{G}\) is isomorphic to \(G\), and therefore \(x\) can also act on \(\alpha\). For \(\alpha_1,\alpha_2 \in \hat{G}\) the group law is denoted by \(\alpha_1 \cdot \alpha_2\), so that \((x,\alpha_1 \cdot \alpha_2) = (x,\alpha_1)(x,\alpha_2)\).

A uniform lattice, \(L \subset G\), is a subgroup of \(G\) whose relative topology is the discrete one and for which \(G/L\) is compact in the quotient topology. The annihilator

\[\text{Key words and phrases.} \quad \text{Time-frequency space; Eckart-Young theorem; LCA groups; Zak transform.}\]
of $L$ is $L^\perp = \{ \alpha \in \hat{G} : \langle \ell, \alpha \rangle = 1 \; \forall \ell \in L \}$. Since $L^\perp \approx \widehat{(G/L)}$ ([1], Theorem 2.1.2) and $G/L$ is compact, $L^\perp$ is discrete ([1], Theorem 1.2.5). In particular, since $G$ is second countable, $\hat{G}$ is also second countable, so both discrete groups $L$ and $L^\perp$ are countable.

Let $L$ be a uniform lattice in the LCA group $G$ and $B \subset L^\perp$ be a uniform lattice in the dual group $\hat{G}$. For $f \in L^2(G)$, $\ell \in L$, and $\beta \in B$ let $T_\ell f(x) = f(x - \ell), x \in G$, be the translation operator, and $M_\beta f(x) = (x, \beta) f(x), x \in G$, be the modulation operator. The collection

$$\{T_\ell M_\beta f : \ell \in L, \beta \in B\},$$

is the time-frequency system generated by $f \in L^2(G)$.

Since $B \subset L^\perp$, we have $T_\ell M_\beta f = M_\beta T_\ell f$ for all $f \in L^2(G), \ell \in L$, and $\beta \in B$. Thus $\Pi(\ell, \beta) := T_\ell M_\beta$ is a unitary representation of the abelian group $\Gamma := L \times B$, with operation $(\ell_1, \beta_1) \cdot (\ell_2, \beta_2) = (\ell_1 + \ell_2, \beta_1 \cdot \beta_2)$, in $L^2(G)$.

A closed subspace $V$ of $L^2(G)$ is said to be $\Gamma$-invariant (or time-frequency invariant) if for every $f \in V$, $\Pi(\ell, \beta) f \in V$ for every $(\ell, \beta) \in \Gamma$. All $\Gamma$-invariant subspaces $V$ of $L^2(G)$ are of the form

$$V = S_\Gamma(A) := \text{span}\{T_\ell M_\beta \phi : \phi \in A, (\ell, \beta) \in \Gamma\}$$

for some countable collection $A$ of elements of $L^2(G)$. If $A$ is a finite collection we say that $V = S_\Gamma(A)$ has finite length, and $A$ is a set of generators of $V$. We call the length of $V$, denoted length($V$), the minimum positive integer $n$ such that $V$ has a set of generators with $n$ elements.

We now state our approximation problem. Let $F = \{f_1, f_2, ..., f_m\} \subset L^2(G)$ be a set of functional data. Given a closed subspace $V$ of $L^2(G)$ define

$$\mathcal{E}(F; V) := \sum_{j=1}^m \|f_j - P_V f_j\|_{L^2(G)}^2$$

as the error of approximation of $F$ by $V$, where $P_V$ denotes the orthogonal projection of $L^2(G)$ onto $V$.

Is it possible to find a $\Gamma$-invariant space of length at most $n < m$ that best approximates our functions, in the sense that

$$\mathcal{E}(F; S_\Gamma\{\psi_1, ..., \psi_n\}) \leq \mathcal{E}(F; V)$$

for all $\Gamma$-invariant subspaces $V$ of $L^2(G)$ with length($V$) $\leq n$?

This question is relevant in applications. For example, if $\{f_1, \ldots, f_m\}$ are audio signals, the best $\Gamma$-invariant space provides a time-frequency optimal model to represent these signals.

The answer to this question is affirmative, and is given by the main theorem of this work.

**Theorem 1.1.** Let $G$ be a second countable LCA group, $L$ and $B$ be uniform lattices in $G$ and $\hat{G}$ respectively, with $B \subset L^\perp$. For each set of functional data $F = \{f_1, f_2, ..., f_m\} \subset L^2(G)$ and each $n \in \mathbb{N}, n < m$, there exists $\{\psi_1, ..., \psi_n\} \subset L^2(G)$ such that

$$\mathcal{E}(F; S_\Gamma\{\psi_1, ..., \psi_n\}) \leq \mathcal{E}(F; V)$$

for all $\Gamma$-invariant subspaces $V$ of $L^2(G)$ with length($V$) $\leq n$. 


Remark 1.1. Observe that, in the previous statement, some of the generators \( \{ \psi_1, \ldots, \psi_n \} \) may be zero. In this case, the length of \( S_\Gamma \{ \psi_1, \ldots, \psi_n \} \) would be strictly smaller than \( n \).

The proof of Theorem 1.1 will follow the ideas originally developed in [1] for approximating data in \( L^2(\mathbb{R}^d) \) by shift-invariant subspaces of finite length, and which have also been used in [6, 3].

We reduce the problem of finding the collection \( \{ \psi_1, \ldots, \psi_n \} \), whose existence is asserted in Theorem 1.1, to solve infinitely many approximation problems for data in a particular Hilbert space of sequences. This is accomplished with the help of an isometric isomorphism \( H_\Gamma \) that intertwines the unitary representation \( \Pi \) with the characters of \( \Gamma \). This isometry \( H_\Gamma \) generalizes the fiberization map of [4] used in [1], and has the properties of a Helson map as defined in [2] (Definition 7). The definition and properties of \( H_\Gamma \) are given in Section 2.

The reduced problems are then solved by using Eckart-Young theorem as stated and proved in [1] (Theorem 4.1). The solutions of all of these reduced problems are patched together to finally obtain the proof of Theorem 1.1 in Section 3.

2. An isometric isomorphism

Let \( G \) be a second countable LCA group, \( L \) a uniform lattice in \( G \), and \( B \subset L^\perp \) a uniform lattice in \( \hat{G} \) (see definitions in Section 1). With \( \Gamma = L \times B \), each \( \Gamma \)-invariant subspace \( V \) of \( L^2(G) \) is of the form

\[
V = S_\Gamma(\mathcal{A}) := \text{span}\{ T_\ell M_\beta : \varphi \in \mathcal{A}, (\ell, \beta) \in \Gamma \}^{L^2(G)}
\]

for some countable set \( \mathcal{A} \subset L^2(G) \). Therefore

\[
V = S_L(\{ M_\beta \varphi : \varphi \in \mathcal{A}, \beta \in B \})
\]

is also an \( L \)-invariant subspace, that is \( T_\ell f \in V \) for all \( \ell \in L \) whenever \( f \in V \). The theory of shift-invariant spaces on LCA groups, as developed in [7], can be applied to this situation.

Let \( T_{L^\perp} \subset \hat{G} \) be a measurable cross-section of \( \hat{G}/L^\perp \). The set \( T_{L^\perp} \) is in one to one correspondence with the elements of \( \hat{G}/L^\perp \), and \( \{ T_{L^\perp} + \lambda : \lambda \in L^\perp \} \) is a tiling of \( \hat{G} \).

Let \( \hat{f}(\omega) := \int_G f(x)(x, \omega)dx \) denote the unitary Fourier transform of \( f \in L^2(G) \cap L^1(G) \) and extended to \( L^2(G) \) by density. By Proposition 3.3 in [7] the mapping \( \mathcal{F} : L^2(G) \rightarrow L^2(T_{L^\perp}, \ell^2(L^\perp)) \) given by

\[
\mathcal{F} f(\omega) = \{ \hat{f}(\omega + \lambda) \}_{\lambda \in L^\perp}, \; f \in L^2(G),
\]

is an isometric isomorphism. Moreover, since \( V \subset L^2(G) \) is an \( L \)-invariant space, it has an associated measurable range function

\[
J : T_{L^\perp} \longrightarrow \{ \text{closed subspaces of } \ell^2(L^\perp) \}
\]

such that (See Theorem 3.10 in [7])

\[
J(\omega) = \text{span}\{ \mathcal{F}(M_\beta \varphi)(\omega) : \beta \in B, \varphi \in \mathcal{A} \}^{\ell^2(L^\perp)}, \; \text{a.e } \omega \in T_{L^\perp}.
\]

Using the definition of \( \mathcal{F} \) given in (2.1), for each \( \beta \in B \) and each \( \varphi \in L^2(G) \) we have

\[
\mathcal{F}(M_\beta \varphi)(\omega) = \{ \hat{M_\beta \varphi}(\omega + \lambda) \}_{\lambda \in L^\perp} = \{ \hat{\varphi}(\omega + \lambda - \beta) \}_{\lambda \in L^\perp} = t_\beta(\mathcal{F} \varphi(\omega))
\]
where \( t_\beta : \ell^2(L^+) \to \ell^2(L^+) \) is the translation of sequences in \( \ell^2(L^+) \) by elements of \( \beta \in B \), that is \( t_\beta(\{a(\lambda)\}_{\lambda \in L^+}) = \{a(\lambda - \beta)\}_{\lambda \in L^+} \). Therefore, \( \mathcal{F} \) intertwines the modulations \( \{M_\beta\}_{\beta \in B} \) with the translations by \( B \) on \( \ell^2(L^+) \).

By equations (2.2) and (2.3), for a.e. \( \omega \in T_{L^+} \),

\[
J(\omega) = \text{span} \{t_\beta(\mathcal{F}(\varphi(\omega)) : \beta \in B, \varphi \in \mathcal{A})\}^{\ell^2(L^+)}.\]

Therefore, \( J(\omega) \) is a \( B \)-invariant subspace of \( L^2(L^+) \). We can apply the theory of shift-invariant spaces as developed in [7] to the discrete LCA group \( L^+ \) and its uniform lattice \( B \).

Let \( B^\perp \) be the annihilator of \( B \) in the compact group \( \hat{L}^+ \subset G \), that is

\[
(4.4) \quad B^\perp = \{b \in \hat{L}^+ : (b, \beta) = 1 \forall \beta \in B\}. \]

Observe that \( B^\perp \) is finite, because it is a discrete subgroup of a compact group.

Let \( T_{B^\perp} \subset \hat{L}^+ \) be a measurable cross-section of \( \hat{L}^+/B^\perp \). The set \( T_{B^\perp} \) is in one to one correspondence with the elements of \( \hat{L}^+/B^\perp \) and \( \{T_{B^\perp} + b : b \in B^\perp\} \) is a tiling of \( L^+ \).

**Example 2.1.** Let \( G = \mathbb{R}/L = \mathbb{Z} \) and \( B = n\mathbb{Z} \subset L^+ = \mathbb{Z} \subset \hat{\mathbb{R}} \). Since \( L^+ = \hat{\mathbb{Z}} \approx [0,1) \), \( \ell \in B^\perp \) if and only if \( \ell \in \{0,1,\ldots,n-1\} \) and \( e^{2\pi i\ell n/k} = 1 \) for all \( k \in \mathbb{Z} \).

Hence

\[
B^\perp = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}. 
\]

We can take \( T_{B^\perp} = [0, \frac{1}{n}) \). Notice that as a subgroup of \( \hat{\mathbb{R}} \) the annihilator of \( B \) is \( \mathbb{Z}/n \mathbb{Z} \).

**Example 2.2.** Let \( p, q \in \mathbb{N}, d = pq \), and \( G = \mathbb{Z}_d = \{0,1,\ldots,d-1\} \). Let \( L = \{0,p,2p,\ldots,p(q-1)\} = \{np : n = 0,\ldots,q-1\} \approx \mathbb{Z}_q \). Its annihilator lattice is

\[
L^+ = \{\lambda \in \{0,1,\ldots,d-1\} : e^{2\pi i\frac{\lambda n}{q}} = 1 \forall n = 0,\ldots,q-1\}
\]

\[
= \{0,2q,\ldots,q(p-1)\} = \{kq : k = 0,\ldots,p-1\} \approx \mathbb{Z}_p. 
\]

A fundamental set \( T_{L^+} \) for \( L^+ \) in \( \hat{G} \approx \mathbb{Z}_d \) is \( T_{L^+} = \{0,\ldots,q-1\} \approx \mathbb{Z}_q \). The characters \( \omega \in \hat{L}^+ \) of this group are of the form (see e.g. [8] Lemma 5.1.3) \( \omega_\nu(\lambda) = e^{2\pi i\frac{\lambda \nu}{q}}, \lambda \in L^+ \) for \( \nu \in \{\frac{\lambda}{q} : \ell = 0,\ldots,p-1\} \approx \mathbb{Z}_p \).

Suppose now that \( p = rs \) for some \( r, s \in \mathbb{N} \), and let \( B \subset L^+ \) be

\[
B = \{0, rq, 2rq, \ldots, (s-1)rq\} = \{jqr : j = 0,\ldots,s-1\} \approx \mathbb{Z}_s. \]

The annihilator of \( B \) in \( \hat{L}^+ \) thus reads

\[
B^\perp = \{b \in \frac{\ell}{q} : \ell = 0,\ldots,p-1\} : e^{2\pi i\frac{b j r q}{q}} = 1 \forall j = 0,\ldots,s-1\}
\]

\[
= \{0, \frac{s}{q}, \frac{2s}{q}, \ldots, \frac{s(r-1)}{q}\} = \{\frac{h s}{q} : h = 0,\ldots,r-1\} \approx \mathbb{Z}_r. 
\]

A fundamental set in \( \hat{L}^+ = \{\frac{\ell}{q} : \ell = 0,\ldots,p-1\} \) for \( B^\perp \) is

\[
T_{B^\perp} = \left\{0, \frac{1}{q}, \ldots, \frac{s-1}{q}\right\} \approx \mathbb{Z}_s. 
\]
By Proposition 3.3 in [7], the mapping \( \mathcal{H} : \ell^2(L^+) \to L^2(T_{B^+}, \ell^2(B^+)) \) given by
\[
\mathcal{H}((a(\lambda))_{\lambda \in L^+})(t) = \left\{ \left( (a(\lambda))_{\lambda \in L^+} \right)^\wedge (t + b) \right\}_{b \in B^+}
\]
(2.5)
is an isometric isomorphism. Moreover, each \( B \)-invariant subspace \( J(\omega), \omega \in T_{L^+}, \)
has an associated measurable range function
\[
J(\omega, \cdot) : T_{B^+} \to \{ \text{closed subspaces of } \ell^2(B^+) \},
\]
such that for almost every \( t \in T_{B^+}, J(\omega, t) = \text{span} \left\{ \mathcal{H}(\mathcal{F}\varphi)(\omega))(t) : \varphi \in A \right\}^{\ell^2(B^+)}.
\]
From the definition of \( \mathcal{F} \) given in (2.1) and the definition of \( \mathcal{H} \) given in (2.5) we obtain
\[
\mathcal{H}(\mathcal{F}\varphi)(\omega))(t) = \left\{ \sum_{\lambda \in L^+} \hat{f}(\omega + \lambda)(t + b, \lambda) \right\}_{b \in B^+},
\]
(2.6)
when \( f \in L^2(G), \omega \in T_{L^+}, \) and \( t \in T_{B^+} \).

For \( f \in L^2(G), \omega \in \hat{G}, \) and \( t \in G \) define
\[
Zf(\omega, t) := \sum_{\lambda \in L^+} \hat{f}(\omega + \lambda)(t, \lambda),
\]
(2.7)
the Zak transform of \( \hat{f} \) with respect to the lattice \( L^+ \). Observe that in terms of this map, \( \mathcal{H}(\mathcal{F}\varphi)(\omega))(t) = \{ Zf(\omega, t + b) \}_{b \in B^+} \).

To simplify the statement of the next theorem we write \( X_\beta \) for the character on \( G \) associated to \( \beta \in B \), that is \( X_\beta : G \to \mathbb{T} \) with \( X_\beta(x) = (x, \beta) \) for all \( x \in G \). Similarly \( X_\ell \) will denote the character on \( \hat{G} \) associated to \( \ell \in L \), that is \( X_\ell : \hat{G} \to \mathbb{T} \) with \( X_\ell(\omega) = (\ell, \omega) \) for all \( \omega \in \hat{G} \).

**Theorem 2.1.** Let \( G \) be a second countable LCA group, \( L \) and \( B \) be uniform lattices in \( G \) and \( \hat{G} \) respectively, with \( B \subset L^+ \). Let \( \Gamma = L \times B \) and for \( f \in L^2(G), \omega \in T_{L^+}, \)
and \( t \in T_{B^+} \) define
\[
H_\Gamma f(\omega, t) = \{ Zf(\omega, t + b) \}_{b \in B^+}.
\]
(2.8)
Then
1) The map \( H_\Gamma \) intertwines \( \Pi \) with the characters of \( \Gamma \), that is \( H_\Gamma \Pi(\ell, \beta)f = \Pi_X X_\ell X_\beta H_\Gamma f \) for all \( f \in L^2(G), \ell \in L, \beta \in B \).
2) The map \( H_\Gamma \) defined in (2.8) is an isometric isomorphism from \( L^2(G) \) onto \( L^2(T_{L^+} \times T_{B^+}, \ell^2(B^+)) \).

**Proof.** For each \( b \in B^+ \), the definition of \( Z \) given in (2.7) and the properties of the Fourier transform give
\[
Z\Pi(\ell, \beta)f(\omega, t + b) = \sum_{\lambda \in A^l} \overline{T_\ell \mathcal{M}_\beta \hat{f}(\omega + \lambda)(t + b, \lambda)}
\]
\[
= \sum_{\lambda \in A^l} (\ell, \omega + \lambda) \overline{\hat{f}(\omega + \lambda - \beta)(t + b, \lambda)}.
\]
Using that \((\ell, \lambda) = 1\) and the change of variables \(\lambda - \beta = \lambda' \in L^+\) yields
\[
2\Pi(\ell, \beta) f(\omega, t + b) = (\ell, \omega) \sum_{\lambda' \in \Lambda^+} \hat{f}(\omega + \lambda')(t + b, \lambda' + \beta).
\]
Using that \((t + b, \beta) = (t, \beta) \cdot (b, \beta) = (t, \beta)\) we obtain
\[
2\Pi(\ell, \beta) f(\omega, t + b) = (\ell, \omega) \sum_{\lambda' \in \Lambda^+} \hat{f}(\omega + \lambda')(t + b, \lambda') = X_{-\ell}(\omega)X_{-\beta}(t)Z f(\omega, t + b).
\]
This proves 1). To prove 2) observe that by the definition of \(H_T\) given in (2.3) together with (2.6) and (2.7) we have
\[
H_T f(\omega, t) = \mathcal{H}(\mathcal{T} f(\omega))(t).
\]
That \(H_T\) is an isometry now follows from the fact that \(\mathcal{S}\) and \(\mathcal{H}\) are isometries in their respective spaces.

We need to prove that \(H_T\) is onto. Since \(\mathcal{H} : \ell^2(L^+) \to L^2(T_{B^+}, \ell^2(B^+))\) is an isometric isomorphism between Hilbert spaces, by Lemma 4.1 in the Appendix, the isomorphism is given by \(\Phi(\omega, t) = \hat{f}(\omega)\). Moreover, by Fubini's theorem, the Hilbert spaces \(L^2(T_{\Lambda^+}, \ell^2(L^+))\) and \(L^2(T_{\Lambda^+} \times T_{B^+}, \ell^2(B^+))\) are also isomorphic and the isomorphism is given by \(\Phi(f)(\omega, t) = f(\omega)(t)\), for \(f \in L^2(T_{\Lambda^+}, \ell^2(L^+))\).

Let now \(F \in L^2(T_{\Lambda^+} \times T_{B^+}, L^2(\ell^2(B^+)))\). Choose \(g \in L^2(T_{\Lambda^+}, \ell^2(L^+))\) such that \(\Phi \circ Q_{\mathcal{H}}(g) = F\). Hence
\[
F(\omega, t) = \Phi \circ Q_{\mathcal{H}}(g)(\omega, t) = Q_{\mathcal{H}}(g)(\omega)(t) = \mathcal{H}(g(\omega))(t).
\]
Choose now \(f \in L^2(G)\) such that \(\mathcal{T}(f) = g\). Then
\[
H_T f(\omega, t) = \mathcal{H}(\mathcal{T} f(\omega))(t) = F(\omega, t).
\]
This finishes the proof of the theorem. \(\square\)

**Example 2.3.** For the cyclic group of Example 2.2 recall that, for \(f \in \mathbb{C}^d\)
\[
\hat{f}(\omega) = \frac{1}{\sqrt{d}} \sum_{g=0}^{d-1} f(g)e^{-2\pi i \frac{g \omega}{d}}, \quad \omega \in \{0, \ldots, d - 1\}.
\]
For \(t \in T_{B^+} = \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{d-1}{q}\right\}\), the Zak transform (2.7) thus reads
\[
\mathcal{Z} f(\omega, t) = \sum_{k=0}^{p-1} \hat{f}(\omega + kq)e^{-2\pi i \frac{kq |t|}{d}} = \sum_{k=0}^{p-1} \frac{1}{\sqrt{d}} \sum_{g=0}^{d-1} f(g)e^{-2\pi i \frac{g (\omega + kq)}{d}} e^{-2\pi i \frac{kq}{d}} = \frac{1}{\sqrt{d}} \sum_{g=0}^{d-1} f(g)e^{-2\pi i \frac{g}{d}} K(g + qt) = \frac{e^{2\pi i \frac{\omega}{d}}}{\sqrt{d}} \sum_{g=0}^{d-1} f(g - qt)e^{-2\pi i \frac{g}{d}} K(g)\]
Proof. Observe first that, since \( H \) spaces, then \( D \) the set \( T \) Indeed, this is Fourier uniqueness theorem since \( K \) given by 

\[
Z f(\omega, t) = \sqrt{|p|}^{2\pi |m|} \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} f(pn - qt) e^{-2\pi \frac{imn}{q}} .
\]

Before embarking in the proof of Theorem 1.1 which will be accomplished in Section 3 we need an additional result.

Let \( V = S_\Gamma(A) \) be a \( \Gamma \) invariant subspace of \( L^2(G) \), where \( A \subset L^2(G) \). For each \((\omega, t) \in T_{L^1} \times T_{B^1} \), consider the range function 

\[
J_V : \mathcal{F}_{L^1} \times T_{B^1} \rightarrow \{ \text{closed subspaces of } \ell^2(B^1) \}
\]

given by 

\[
(2.9) \quad J_V(\omega, t) := \overline{\text{span} \{ H_\Gamma \varphi(\omega, t) : \varphi \in A \}}^{\ell^2(B^1)}.
\]

**Proposition 2.1.** With \( V = S_\Gamma(A) \) as above, let \( \mathcal{P}_{J_V(\omega, t)} \) be the orthogonal projection of \( \ell^2(B^1) \) onto \( J_V(\omega, t) \). Then, for all \( f \in L^2(G) \) and \((\omega, t) \in T_{L^1} \times T_{B^1} \),

\[
H_\Gamma \mathcal{P}_{S_\Gamma(A)} f(\omega, t) = \mathcal{P}_{J_V(\omega, t)}(H_\Gamma f(\omega, t)).
\]

**Proof.** Observe first that, since \( H_\Gamma \) is an isometric isomorphism between Hilbert spaces, then 

\[
(2.10) \quad H_\Gamma \mathcal{P}_{S_\Gamma(A)} = \mathcal{P}_{H_\Gamma(S_\Gamma(A))} H_\Gamma .
\]

The set \( D := \{ X_\ell X_\beta : (\ell, \beta) \in \Gamma \} \) of characters of \( \Gamma \) is a determining set for \( L^1(T_{L^1} \times T_{B^1}) \) in the sense of Definition 2.2 in [5], because

\[
\int_{T_{L^1} \times T_{B^1}} f(\omega, t) X_\ell(\omega) X_\beta(t) d\omega dt = 0 \Rightarrow f = 0 \quad \forall \ f \in L^1(T_{L^1} \times T_{B^1}) .
\]

Indeed, this is Fourier uniqueness theorem since \( T_{L^1} \) and \( T_{B^1} \) are relatively compact.

By 1) of Theorem 2.1 for all \( f \in L^2(G) \), \( H_\Gamma(T_{\ell} M_\beta f) = X_{-\ell} X_{-\beta}(H_\Gamma f) \). Thus, \( H_\Gamma(S_\Gamma(A)) \) is \( D \)-multiplicative invariant in the sense of Definition 2.3 in [5]. Indeed, if \( X_\ell X_\beta \in D \), \( F \in H_\Gamma(S_\Gamma(A)) \) writing \( H_\Gamma F = F \) we have

\[
X_\ell X_\beta F = X_\ell X_\beta(H_\Gamma f) = H_\Gamma(T_{-\ell} M_{-\beta} f) \in H_\Gamma(S_\Gamma(A)) .
\]

By Theorem 2.4 in [5], \( J_V \) is a measurable range function. By Proposition 2.2 in [5],

\[
\mathcal{P}_{H_\Gamma(S_\Gamma(A))}(H_\Gamma f)(\omega, t) = \mathcal{P}_{J_V(\omega, t)}(H_\Gamma f(\omega, t)) .
\]

The result now follows from (2.10). \( \square \)

### 3. Solution to the Approximation Problem

This section is dedicated to the proof of Theorem 1.1. Let \( F = \{ f_1, \ldots, f_m \} \subset L^2(G) \) be a collection of functional data. With the notation of Theorem 1.1, for each \( n < m \) we need to find \( \{ \psi_1, \ldots, \psi_n \} \subset L^2(G) \) such that \( \mathcal{E}(F; S_\Gamma(\psi_1, \ldots, \psi_n)) \leq \mathcal{E}(F; V) \) for any \( \Gamma \) invariant subspace \( V \) of \( L^2(G) \) of length less than or equal \( n \). The definition of \( \mathcal{E}(F; V) \) is given in (1.1) and for convenience of the reader we recall it here.

\[
\mathcal{E}(F; V) := \sum_{j=1}^{m} \| f_j - \mathcal{P}_V f_j \|^2_{L^2(G)} .
\]
Proposition 3.1. For any $Q(w,t) := \{H_\Gamma f_1(\omega,t), \ldots, H_\Gamma f_m(\omega,t)\}$.

Let $G_{\Gamma,F}(w,t)$ be the $m \times m$ $\mathbb{C}$-valued matrix whose $(i,j)$ entry is given by

$$[G_{\Gamma,F}(w,t)]_{i,j} = \langle H_\Gamma f_i(\omega,t), H_\Gamma f_j(\omega,t) \rangle_{\ell^2(B^\perp)}.$$

The matrix $G_{\Gamma,F}(w,t)$ is hermitian and its entries are measurable functions defined on $T_{L^+} \times T_{G^+}$. Write

$$\lambda_1(\omega,t) \geq \lambda_2(\omega,t) \geq \ldots \geq \lambda_m(\omega,t) \geq 0$$

for the eigenvalues of $G_{\Gamma,F}(w,t)$. By Lemma 2.3.5 in [11] the eigenvalues $\lambda_i(\omega,t)$, $i = 1, \ldots, m$, are measurable and there exist corresponding measurable vectors $y_i(\omega,t) = (y_{i,1}(\omega,t), \ldots, y_{i,m}(\omega,t))$ that are orthonormal left eigenvectors of the matrix $G_{\Gamma,F}(w,t)$. That is,

$$y_i(\omega,t) G_{\Gamma,F}(w,t) = \lambda_i(\omega,t) y_i(\omega,t), \quad i = 1, \ldots, m.$$

For $n \leq m$, define $q_1(\omega,t), \ldots, q_n(\omega,t) \in \ell^2(B^\perp)$ by

$$q_i(\omega,t) = \bar{\sigma}_i(\omega,t) \sum_{j=1}^m y_{i,j}(\omega,t) H_\Gamma f_j(\omega,t), \quad i = 1, \ldots, n,$$

where

$$\bar{\sigma}_i(\omega,t) = \begin{cases} 1 \sqrt{\lambda_i(\omega,t)} & \text{if } \lambda_i(\omega,t) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

By the Eckart-Young Theorem (see the version stated and proved in Theorem 4.1 of [11]), $\{q_1(\omega,t), \ldots, q_n(\omega,t)\}$ is a Parseval frame for the space it generates $Q(\omega,t) := \text{span}\{q_1(\omega,t), \ldots, q_n(\omega,t)\}$ and $Q(\omega,t)$ is optimal in the sense that

$$E(H_\Gamma(F)(w,t); Q(\omega,t)) := \sum_{i=1}^m \|H_\Gamma f_i(\omega,t) - P_{Q(\omega,t)}H_\Gamma f_i(\omega,t)\|_{\ell^2(B^\perp)}^2$$

$$\leq \sum_{i=1}^m \|H_\Gamma f_i(\omega,t) - P_{Q'}H_\Gamma(F)(w,t)\|_{\ell^2(B^\perp)}^2 := E(H_\Gamma(f_i)(w,t); Q')$$

for any $Q'$ subspace of $\ell^2(B^\perp)$ of dimension less than or equal to $n$. Moreover,

$$E(H_\Gamma(F)(w,t); Q(\omega,t)) = \sum_{i=n+1}^m \lambda_i(\omega,t).$$

Before continuing with the proof, let us relate the pointwise errors that appear in (3.3) to the error defined in (3.1) for $\Gamma$-invariant subspaces.

**Proposition 3.1.** For $V = S_\Gamma(A)$ as in Proposition 2.3

$$\mathcal{E}(F; V) = \int_{T_{L^+}} \int_{T_{G^+}} E(H_\Gamma(F)(w,t); J_V(\omega,t)) \, dt \, d\omega,$$

where $J_V(\omega,t)$ is defined in (2.10).
Proof. By 2) of Theorem 2.1, $H_{\Gamma}$ is an isometry from $L^2(G)$ onto the space $L^2(T_{L\perp} \times T_{B\perp}, \ell^2(B^\perp))$. Therefore,

$$
\mathcal{E}(F; V) = \sum_{j=1}^{m} \|f_j - \mathcal{P}_V f_j\|_{L^2(G)}^2
$$

$$
= \sum_{j=1}^{m} \|H_{\Gamma} f_j - H_{\Gamma} \mathcal{P}_V f_j\|_{L^2(T_{L\perp} \times T_{B\perp}, \ell^2(B^\perp))}^2
$$

$$
= \sum_{j=1}^{m} \int_{T_{L\perp}} \int_{B_{L\perp}} \|H_{\Gamma} f_j(\omega, t) - H_{\Gamma} \mathcal{P}_V f_j(\omega, t)\|_{\ell^2(B^\perp)}^2 \, dt \, d\omega.
$$

By Proposition 2.1,

$$
\mathcal{E}(F; V) = \int_{T_{L\perp}} \int_{B_{L\perp}} \sum_{j=1}^{m} \|H_{\Gamma} f_j(\omega, t) - \mathcal{P}_J(\omega, t)(H_{\Gamma} f_j(\omega, t))\|_{\ell^2(B^\perp)}^2 \, dt \, d\omega
$$

$$
= \int_{T_{L\perp}} \int_{B_{L\perp}} E(H_{\Gamma}(F)(\omega, t); J_V(\omega, t)) \, dt \, d\omega. \quad \square
$$

Let us now continue with the proof of Theorem 1.1. By definition 3.2, each $q_i(\omega, t)$ is measurable and defined on $T_{L\perp} \times T_{B\perp}$ with values in $\ell^2(B^\perp)$. Moreover,

$$
\|q_i(\omega, t)\|_{\ell^2(B^\perp)}^2 = \langle q_i(\omega, t), q_i(\omega, t) \rangle_{\ell^2(B^\perp)}
$$

$$
= \bar{\sigma}_i(\omega, t)^2 \sum_{b \in B^\perp} \sum_{j=1}^{m} \sum_{s=1}^{m} y_{i,j}(\omega, t) Z f_j(\omega, t + b) Z f_s(\omega, t + b) \bar{y}_{i,s}(\omega, t)
$$

$$
= \bar{\sigma}_i(\omega, t)^2 \sum_{j=1}^{m} y_{i,j}(\omega, t) \sum_{s=1}^{m} \langle Z f_j(\omega, t), Z f_s(\omega, t) \rangle_{\ell^2(B^\perp)} y_{i,s}(\omega, t).
$$

In matrix form,

$$
\|q_i(\omega, t)\|_{\ell^2(B^\perp)}^2 = \bar{\sigma}_i(\omega, t)^2 \bar{y}_i(\omega, t) G_F(\omega, t) \bar{y}_i(\omega, t)\, dt \, d\omega.
$$

By 3.3, the orthonormality of the vectors $y_i(\omega, t)$, and the definition of $\bar{\sigma}_i(\omega, t)$, we have

$$
\|q_i(\omega, t)\|_{\ell^2(B^\perp)}^2 = \bar{\sigma}_i(\omega, t)^2 \lambda_i(\omega, t) \|y_i(\omega, t)\|^2 \leq 1.
$$

Since $T_{L\perp}$ and $T_{B\perp}$ have finite measure, we conclude that for $i = 1, \ldots, n$, $q_i \in L^2(T_{L\perp} \times T_{B\perp}, \ell^2(B^\perp))$. The mapping $H_{\Gamma}$ is onto by part 2) of Theorem 2.1. Therefore there exist $\psi_i \in L^2(G)$ such that

$$
H_{\Gamma}(\psi_i) = q_i, \quad i = 1, \ldots, n.
$$

It remains to show that the space $W := S_{\Gamma}(\psi_1, \ldots, \psi_n)$ is the optimal one as required in the statement of Theorem 1.1.

By Proposition 3.1,

$$
\mathcal{E}(F; W) = \int_{T_{L\perp}} \int_{T_{B\perp}} E(H_{\Gamma}(F)(\omega, t); J_W(\omega, t)) \, dt \, d\omega.
$$

By 3.3 and the definitions of $\psi_i$, $J_W(\omega, t) = Q(\omega, t)$. Therefore, we can write,

$$
(3.5) \quad \mathcal{E}(F; W) = \int_{T_{L\perp}} \int_{T_{B\perp}} E(H_{\Gamma}(F)(\omega, t); Q(\omega, t)) \, dt \, d\omega.
$$
Let now $V = S_\Gamma(\varphi_1, \ldots, \varphi_r)$, $r \leq n$, be any $\Gamma$-invariant subspace of length less than or equal to $n$. Since $J_V(\omega, t)$ has dimension less than or equal to $n$, (3.3) gives
\[ E(\mathcal{F}; W) \leq \int_{T_{\Gamma}} \int_{T_{H}} E(H_\Gamma(\mathcal{F})(w, t); J_V(\omega, t)) \, dt \, \omega = E(\mathcal{F}; V), \]
where the last equality is due to Proposition 3.1. Moreover, by (3.5) and (3.4)
\[ E(\mathcal{F}; W) = \sum_{i=n+1}^{m} \int_{T_{\Gamma}} \int_{T_{H}} \lambda_i(\omega, t) \, d\omega \, dt. \]
This finishes the proof of Theorem 1.1. \(\square\)

4. Appendix

We give the proof of the following Lemma that has been used in Section 2 to prove part 2) of Theorem 2.1.

Lemma 4.1. Let $\sigma : \mathbb{H}_1 \rightarrow \mathbb{H}_2$ be an isometric isomorphism between the Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$. For a measure spaces $(X, d\mu)$ the map $Q_\sigma : L^2(X, \mathbb{H}_1) \rightarrow L^2(X, \mathbb{H}_2)$ given by $(Q_\sigma f)(x) = \sigma(f(x))$ is also an isometric isomorphism.

Proof. Let $f$ be a measurable vector function in $L^2(X, \mathbb{H}_1)$, that is, for every $y \in \mathbb{H}_1$ the scalar function $x \rightarrow \langle f(x), y \rangle_{\mathbb{H}_1}$ is measurable. We must prove that $Qf$ is also a measurable vector function in $L^2(X, \mathbb{H}_2)$. For $z \in \mathbb{H}_2$ we have
\[ \langle Qf(x), z \rangle_{\mathbb{H}_2} = \langle \sigma(f(x)), z \rangle_{\mathbb{H}_2} = \langle f(x), \sigma^*(z) \rangle_{\mathbb{H}_1}. \]
Since $\sigma^*(z) = \sigma^{-1}(z)$ is a general element of $\mathbb{H}_1$, this shows that $Qf$ is measurable. Moreover, for $f, g \in L^2(X, \mathbb{H}_1)$,
\[ \langle Qf, Qg \rangle_{L^2(X, \mathbb{H}_2)} = \int_X \langle \sigma(f(x)), \sigma(g(x)) \rangle_{\mathbb{H}_2} \, d\mu(x) \]
\[ = \int_X \langle f(x), g(x) \rangle_{\mathbb{H}_1} \, d\mu(x) = \langle f, g \rangle_{L^2(X, \mathbb{H}_1)}. \]
This shows that if $f \in L^2(X, \mathbb{H}_1)$, $Q_\sigma f \in L^2(X, \mathbb{H}_2)$ and that $Q_\sigma$ is isometry.

Finally, it is easy to see that $R : L^2(X, \mathbb{H}_2) \rightarrow L^2(X, \mathbb{H}_1)$ defined by $Rg(x) = \sigma^{-1}(g(x))$ is the inverse and the adjoint of $Q$. Therefore, $Q_\sigma$ is onto. \(\square\)

5. Acknowledgements

This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 777822.

In addition, D. Barbieri and E. Hernández were supported by Grant MTM2016-76566-P (Ministerio de Ciencia, Innovación y Universidades, Spain). C. Cabrelli and U. Molter were supported by Grants UBACyT 20020170100430BA (University of Buenos Aires), PIP11220150100355 (CONICET) and PICT 2014-1480 (Ministerio de Ciencia, Tecnología e Innovación, Argentina).
References

[1] A. Aldroubi, C. Cabrelli, D. Hardin, and U. Molter, Optimal shift invariant spaces and their Parseval frame generators. Appl. Comput. Harmon. Anal. 23 (2007), pp. 273-283.

[2] D. Barbieri, E. Hernández, V. Paternostro, Spaces invariant under unitary representations of discrete groups. Preprint, https://arxiv.org/abs/1811.02993

[3] D. Barbieri, C. Cabrelli, E. Hernández, U. Molter, Approximation by group invariant subspaces. Preprint, https://arxiv.org/abs/1907.08300

[4] C. de Boor, R. A. DeVore, and A. Ron, Approximation from shift-invariant subspaces of $L^2(\mathbb{R}^d)$. Trans. Amer. Math. Soc. 341 (1994), pp. 787-806.

[5] M. Bownik, K. Ross, The structure of translation-invariant spaces on locally compact abelian groups. J. Fourier Anal. Appl. 21 (2015), pp. 849-884.

[6] C. Cabrelli, C. Mosquera, V. Paternostro, An approximation problem in multiplicatively invariant spaces. In “Functional Analysis, Harmonic Analysis, and Image Processing: A Collection of Papers in Honor of Björn Jawerth”, M. Cwikel and M. Milman (eds.). Contemp. Math. 693 (2017), pp. 143-166.

[7] C. Cabrelli, V. Paternostro, Shift-invariant spaces on LCA groups. J. Funct. Anal. 258 (2010), pp. 2034-2059.

[8] A. Deitmar, A first course in harmonic analysis. Springer, 2nd ed. 2005.

[9] K.-H. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, (2001).

[10] A. Ron, Z. Shen, Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$, Canad. J. Math., 47, (1995), no. 5, 1051-1094.

[11] W. Rudin, Fourier Analysis on Groups, John Wiley, (1992).

Davide Barbieri and Eugenio Hernández at Universidad Autónoma de Madrid, Campus Cantoblanco, 28049 Madrid, España
E-mail address: davide.barbieri@uam.es, eugenio.hernandez@uam.es

Carlos Cabrelli and Ursula Molter at Departamento de Matemática, Universidad de Buenos Aires, and Instituto de Matemática “Luis Santaló” (IMAS-CONICET-UBA), 1428 Buenos Aires, Argentina
E-mail address: carlos.cabrelli@gmail.com, umolter@gmail.com