THE $q$-MULTIPLE GAMMA FUNCTIONS OF BARNES-MILNOR TYPE

HANAMICHI KAWAMURA

Abstract. The multiple gamma functions of BM (Barnes-Milnor) type and the $q$-multiple gamma functions have been studied independently. In this paper, we introduce a new generalization of the multiple gamma functions called the $q$-BM multiple gamma function including those functions and prove some properties the BM multiple gamma functions satisfy for them.

1. Introduction

The $q$-gamma function has been studied since Barnes [2]. Jackson defined it as the reciprocal of the $q$-Pochhammer symbol, which satisfies the periodicity and some transformation properties like the usual gamma functions. Almost 90 years later, Kurokawa [3] introduced the $q$-multiple gamma and sine function. This generalization is also derived from the multiple gamma functions defined by Barnes [1]. In [3], some properties, such as the multiplication formula and the periodicity, were proved. A bit later, Tanaka [8] pointed out that the $q$-multiple gamma functions can be regarded as Appell’s O-function.

Let $w, \omega_1, \omega_2, \cdots, \omega_r$ be complex numbers with positive real parts. Barnes’ multiple gamma functions are defined by

$$\Gamma_r(w; \omega) = \exp \left( \frac{\partial}{\partial s} \zeta_r(s, w; \omega) \right|_{s=0},$$

where $\zeta_r$ are the multiple Hurwitz’ zeta functions

$$\zeta_r(s, w; \omega) = \sum_{n \geq 0} (n \cdot \omega + w)^{-s} \quad (\text{Re}(s) > r),$$

$0 = (0, \cdots, 0)$, $n = (n_1, \cdots, n_r)$, $\omega = (\omega_1, \cdots, \omega_r)$, $n \geq 0 \overset{\text{def}}{\iff} n_i \geq 0$ ($i = 1, \cdots, r$) and $n \cdot \omega = n_1 \omega_1 + \cdots + n_r \omega_r$. We use similar notation for other vectors. Kurokawa’s definition of the $q$-multiple gamma functions is

$$\Gamma_q^r(w; \omega) = \frac{\Gamma_{r+1}(w; (\omega, \tau')) \Gamma_{r+1}(w; (\omega, -\tau'))}{\Gamma_r(w; \omega)},$$

where $q = \exp(-2\pi i/\tau')$. 

In 2007, Kurokawa and Ochiai [4] constructed the theory of the multiple gamma functions of BM (Barnes-Milnor) type:

\[ \Gamma_{r,k}(w; \omega) = \exp \left( \frac{\partial}{\partial s} \zeta_r(s, w; \omega) \bigg|_{s=-k} \right). \]

The case of \( k = 0 \) is the Barnes’ multiple gamma functions \( \Gamma_r \). The parameters \( r \) and \( k \) are usually called “order” and “depth” respectively. Their generalization of Kinkelin’s formula enables the function \( \Gamma_{r,k} \) to bring down its depth by integration:

**Theorem 1.1** ([4], Theorem 4). For \( k \geq 1 \), we have

\[
\int_0^w \left( \log \Gamma_{r,k-1}(t; \omega) - \frac{1}{k} \zeta_r(1 - k, t; \omega) \right) dt = \frac{1}{k} \log \frac{\Gamma_{r,k}(w; \omega)}{\Gamma_{r,k}(0; \omega)}.
\]

Moreover, Kurokawa and Wakayama [5] investigated period deformations and a generalization of Raabe’s integral formula for the BM multiple gamma functions. These results can be written as

\[
\lim_{\mu_j \to \infty} \prod_{1 \leq j \leq l} |\alpha|_\mu \log \Gamma_{r,l+k}(w; (\omega, \alpha \mu)) = (-1)^l k! \binom{l + k}{k} \zeta_r(-k - l, w; \omega),
\]

where \( |\omega|_\times = \prod_{i=1}^l \omega_i \). This theorem allows adjusting the order and the depth of \( \Gamma_{r,k} \) by removing some parts of periods. Our purpose in this paper is finding the \( q \)-analogue of the above theorems and its applications. To prove them, we define the \( q \)-BM multiple gamma functions as follows:

\[ \Gamma_{r,k}(w; \omega) = \frac{\Gamma_{r+1,k}(w; (\omega, \tau')) \Gamma_{r+1,k}(w; (\omega, -\tau'))}{\Gamma_{r,k}(w; \omega)}. \]

At the first onset, we generalize Shibukawa’s expression ([7], Corollary 4.9):

**Theorem 1.2.** Let \( \text{Li}_s(x) = \sum_{n \geq 1} x^n/n^s \) be the usual polylogarithm function. For \( k \geq 0 \), we have

\[ \Gamma_{r,k}^q(w; \omega) = \exp \left( \frac{k!}{(\log q) l} \sum_{n \geq 0} \text{Li}_{k+1}(q^{n+1}w) \right). \]

Additionally, we show that the \( q \)-multiple gamma functions satisfies the result of Kurokawa and Ochiai more concisely than the multiple gamma functions.
Theorem 1.3. We have
\[
\lim_{\mu_j \to \infty} \Gamma_{r+l,k}^q \left( w; \left( \omega, \frac{\alpha}{\mu} \right) \right)^{|\alpha| \times |\mu|} = \Gamma_{r,l+k}^q (w; \omega)^{(-1)^{l+k}},
\]
where \( \alpha / \mu = \left( \frac{\alpha_1}{\mu_1}, \ldots, \frac{\alpha_l}{\mu_l} \right) \).

2. Product expression of \( \Gamma_{r,k}^q \)

In this section, we prove Theorem 1.2.

Proof. Let
\[
\zeta_{r}^q (s, w; \omega) = \zeta_{r+1} (s, w; (\omega, \tau')) + \zeta_{r+1} (s, w; (\omega, -\tau')) - \zeta_r (s, w; \omega).
\]

Then we show that
\[
\zeta_{r}^q (s, w; \omega) = \sum_{n \geq 0} \sum_{n \in \mathbb{Z}} (n \cdot \omega - n \tau' + w)^{-s}
= (-\tau')^{-s} \sum_{n \geq 0} \sum_{n \in \mathbb{Z}} \left( n - \frac{1}{\tau} (n \cdot \omega + w) \right)^{-s}
= \left( -\log q \right)^s \Gamma(s) \sum_{n \geq 0} \sum_{n=1}^{\infty} n^{s-1} q^n (n \omega + w),
\]
where we used Lipschitz' formula
\[
\sum_{n \in \mathbb{Z}} (n + z)^{-s} = \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} e^{2\pi inz}.
\]

Hence we get
\[
\Gamma_{r,k}^q (w; \omega) = \exp \left( \frac{\partial}{\partial s} \left( \frac{-\log q}{s} \right) \sum_{n \geq 0} \sum_{n=1}^{\infty} n^{s-1} \right.
\times q^n (n \omega + w) \bigg |_{s=-k}
\]
\[
= \exp \left( \left( \frac{\partial}{\partial s} \frac{(-\log q)^s}{\Gamma(s)} \right) \bigg |_{s=-k}
\times \sum_{n \geq 0} \sum_{n=1}^{\infty} n^{k-1} q^n (n \omega + w) \right),
\]
and we can easily get
\[
\lim_{s \to -k} \frac{d}{ds} \frac{a^s}{\Gamma(s)} = \frac{k!}{(-a)^k}.
\]
\[\square\]
Corollary 2.1 ([8], Theorem 1). We have
\[
\Gamma^q_r(w; \omega) = \prod_{n \geq 0} (1 - q^n \omega + w)^{-1}.
\]

3. Period deformation and Raabe’s formula

We prove period deformation and Raabe’s formula of \(\Gamma^q_{r,k}\) in the similar form of [5].

Proposition 3.1 (Period deformation). We have
\[
\int_0^1 \cdots \int_0^1 \log \Gamma^q_{r+l,k}(w + t \cdot \alpha; (\omega, \alpha)) \, dt
= \lim_{\mu_j \to \infty} \frac{1}{|\mu|} \log \Gamma_{r+l,k} \left( w; \left( \frac{\alpha}{\mu} \right) \right).
\]

Proof. We obtain
\[
\int_0^1 \cdots \int_0^1 \zeta^q_{r+l,k}(w + t \cdot \alpha; (\omega, \alpha)) \, dt
= \lim_{\mu_j \to \infty} \frac{1}{|\mu|} \sum_{n \geq 0} \sum_{m \geq 0} \sum_{n \in \mathbb{Z}} \left( n \cdot \omega + m \cdot \alpha - \frac{n \tau'}{n} + w \right)^{-s}.
\]

Here, by substituting \(\mu_j m + k_j = \nu_j\), we have
\[
\int_0^1 \cdots \int_0^1 \zeta^q_{r+l,k}(w + t \cdot \alpha; (\omega, \alpha)) \, dt
= \lim_{\mu_j \to \infty} \frac{1}{|\mu|} \sum_{n \geq 0} \sum_{m \geq 0} \sum_{n \in \mathbb{Z}} \left( n \cdot \omega + \frac{\nu \cdot \alpha}{\mu} - \frac{n \tau'}{n} + w \right)^{-s}
= \lim_{\mu_j \to \infty} \frac{1}{|\mu|} \zeta_{r+l} \left( s, w; \left( \frac{\alpha}{\mu} \right) \right).
\]

Thus the statement is given by the differentiation at \(s = -k\). \(\square\)
Theorem 3.2 (Raabe’s formula). We have

\[ \int_0^1 \cdots \int_0^1 \log \Gamma_{r+l,k}(w + t \cdot \alpha; (\omega, \alpha)) \, dt = \frac{(-1)^l k!}{|\alpha| (l + k)!} \left( \log \Gamma_{r,l+k}(w; \omega) + \left( \frac{1}{k+1} + \cdots + \frac{1}{k+l} \right) \zeta_q((-k-l,w; \omega)) \right). \]

Proof. We plan to prove is similar to that of [5], Theorem 4. We only have to show

\[ \int_0^{\alpha_1} \cdots \int_0^{\alpha_l} \zeta_q^{r+l}(s, w + |t|; (\omega, \alpha)) \, dt = \frac{\zeta_q(s-l, w; \omega)}{\prod_{i=1}^{l}(s-i)} \]

when Re(s) > −r − l because both sides are meromorphically extendable on the whole plane \( \mathbb{C} \). Moreover, it is sufficient that we get

\[ \int_0^{\alpha_l} \zeta_q^{r+l}(s, w + |t|; (\omega, \alpha)) \, dt = \frac{\zeta_q^{r+l-1}(s-1, w + |t(l)|; (\omega; \alpha(l)))}{s-1} \]

where \( \alpha(l) \) means \((\alpha_1, \cdots, \alpha_{l-1})\). Thus we show this. By substitution \( w + |t| \mapsto z \), we get

\[ \int_0^{\alpha_l} \zeta_q^{r+l}(s, w + |t|; (\omega, \alpha)) \, dt = \sum_{n \geq 0} \sum_{m \geq 0} \sum_{n \in \mathbb{Z}} \int_{w + |t(l)|}^{\infty} (n \cdot \omega + m \langle l \rangle \cdot \alpha(l)) \]

\[ \cdot (\omega + n \tau' + w)^{-s} = \frac{\zeta_q^{r+l-1}(s-1, w + |t(l)|; (\omega; \alpha(l)))}{s-1}. \]
From the above, we get the desired result inductively. □

4. Proof of Theorem 1.3 and corollaries

In this section, we prove Theorem 1.3 and show its applications.

Proof. It follows that

\[
\lim_{\mu_j \to \infty} \frac{1}{|\mu| \times \log \Gamma_{r+l,k} \left( w; \left( \omega, \frac{\alpha}{\mu} \right) \right)} = \frac{k!}{|\mu| \times (\log q)^k} \times \sum_{n=1}^{\infty} \frac{q^{nw}}{n^{k+1} \prod_{i=1}^{r} (1 - q^{\omega_i n}) \prod_{j=1}^{l} (1 - q^{\alpha_j n/\mu_j})} = \frac{(-1)^l k!}{|\alpha| \times (\log q)^{l+k}} \sum_{n=1}^{\infty} \frac{q^{nw}}{n^{k+l+1} \prod_{i=1}^{r} (1 - q^{\omega_i n})},
\]

since

\[
\lim_{\mu_j \to \infty} \mu_j \left( 1 - q^{\alpha_j n/\mu_j} \right) = \lim_{\mu_j \to \infty} \mu_j \left( 1 - \sum_{\nu=0}^{\infty} \frac{(\alpha_j n \log q)^\nu}{\nu!} \right) = -\alpha_j n \log q.
\]

Therefore we obtain the statement. □

This theorem has some applications. In the following corollaries, we see two of them. The first is a proof of the transformation property (or the modular property) of Dedekind’s eta function:

\[
\eta \left( -\frac{1}{\tau} \right) = \sqrt{\frac{-i}{\tau}} \eta(\tau).
\]

This can be derived from the following corollary which is found in [8].

Corollary 4.1 ([8], Theorem 2). We have

\[
\rho^p_q(\omega) = -\log q \prod_{n \geq 0, n \neq 0} (1 - q^{n\omega}).
\]

This is the generalization of Shintani’s result [6], Proposition 2 (1).

The second is the vanishing property of \(\zeta^p(s, w; \omega)\). By comparing Theorem 1.3 and Theorem 3.2 we get the following:

Corollary 4.2 ([7], Corollary 4.8 (2)). For \(n \geq 0\), we have

\[
\zeta^p(-n, w; \omega) = 0.
\]
Proof. It is easy to see that
\[
\frac{(-1)^k}{|\alpha_n| (k+l)!} \left( \frac{1}{k+1} + \cdots + \frac{1}{k+l} \right) \neq 0.
\]

\[\square\]

References

[1] E. W. Barnes, *On the theory of the multiple gamma functions*, Trans. Cambridge Philos. Soc. 19 (1904), 374–425.

[2] F. H. Jackson, *The basic gamma function and the elliptic functions*, Proc. Roy. Soc. London. A 76 (1905), 127–144.

[3] N. Kurokawa, *Multiple Sine Functions*, Lecture Notes in Japanese. Lectures delivered at Tokyo University, notes taken by S. Koyama (1991).

[4] N. Kurokawa and H. Ochiai, *Generalized Kinkelin’s formula*, Kodai Math. J. 30 (2007), 195–212.

[5] N. Kurokawa, M. Wakayama, *Period deformations and Raabe’s formulas for generalized gamma and sine functions*, Kyushu J. Math. 62 (2008), 171–187.

[6] T. Shintani, *A proof of the classical Kronecker limit formula*, Tokyo J. Math. 3 (1980), 191–199.

[7] G. Shibukawa, *Bilateral zeta functions and their applications*, Kyushu J. Math. 67 (2013), 429–451.

[8] H. Tanaka, *Multiple gamma functions, multiple sine functions, and Appell’s O-functions*, Ramanujan J. 24 (2011), 33–60.