On the determinations of the size and shape of the interaction region from Bose-Einstein correlations

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Abstract

Determinations of the size and shape of the interaction region from $k$-particle ($k = 1, 2, \ldots$) momentum distributions of identical particles are analyzed. The full group of transformations changing the single particle density matrix without affecting any of the measurable momentum distributions is identified. The corresponding uncertainties in the inferred parameters of the interaction region are discussed.

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1 Introduction

The study of Bose-Einstein correlations, or more generally of single particle, two particle etc. momentum distributions for identical particles, is the standard tool used to learn about the size and shape of the interaction region

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where the hadrons are produced. We will explicitly discuss spin zero bosons, because in practice this is the most important case. Much of the argument, however, can be extended to bosons with non-zero spins and to fermions.

Most published analyses are variants of the following approach which we will also follow. One defines a single particle density matrix in the momentum representation \( \rho(p; p') \). The diagonal elements of the (symmetrized) \( n \)-particle density matrix are taken as

\[
\rho(p_1, \ldots, p_n; p_1, \ldots, p_n) = \sum_{P} \prod_{j=1}^{n} \rho(p_j; p_{Pj}),
\]

where the summation is over all the permutations \( j \rightarrow Pj \) of the indices \( j \). The \( n \)-particle momentum distribution is given by the diagonal elements of this \( n \)-particle density matrix\(^1\). Thus, all the information about the system is contained in the single-particle density matrix \( \rho(p; p') \). The question is: how well this function can be determined from the experimental data?

The single particle momentum distribution is

\[
P(p) = \rho(p; p).
\]

Thus the diagonal elements are measurable. The two-particle distribution is

\[
P(p_1, p_2) = \rho(p_1; p_1)\rho(p_2; p_2) + \left| \rho(p_1; p_2) \right|^2,
\]

where the hermiticity of the density matrix

\[
\rho(p_1; p_2) = \rho^*(p_2; p_1)
\]

has been used. Thus, the absolute value of every matrix element is also measurable. For the three-particle momentum distribution formula \( \rho(p; p') \) yields

\[
P(p_1, p_2, p_3) = \ldots + 2\Re \left[ \rho(p_1; p_2)\rho(p_2; p_3)\rho(p_3; p_1) \right],
\]

where the dots denote terms which can be determined from the single particle and two-particle momentum distributions and \( \Re \) stands for real part

\(^1\)One can do somewhat better by including more-particle density matrices, but these corrections, known as multiparticle corrections, are usually neglected, because they are not very important (cf. e.g. the review \[2\]).
of. In general, since every permutation can be decomposed into cycles, the $n$-particle momentum distribution is

$$P(p_1, \ldots, p_n) = \ldots + 2\Re \left[ \rho(p_1; p_2)\rho(p_2; p_3) \ldots \rho(p_n; p_1) \right],$$

where the dots denote terms which can be determined by measuring momentum distributions for less than $n$ particle.

In a previous paper \cite{3} it has been pointed out that the measurable quantities do not change when the substitution

$$\rho(p; p') \to e^{i(f(p_1) - f(p_2))} \rho(p; p')$$

is made\textsuperscript{2}. In the present paper, we show that this group of transformations supplemented by the transformation

$$\rho(p; p') \to \rho^*(p; p') = \rho(p'; p)$$

generates the full group of transformations which leave the measurable distributions invariant.

We will discuss the following problem. Somebody found that a density matrix $\rho_0(p; p')$ gives the best fit to all the measured momentum distributions. What other density matrices $\rho(p; p')$ give exactly the same fit and how the conclusions concerning the interaction region are affected when $\rho_0(p; p')$ is replaced by $\rho(p; p')$? This is the problem of the theoretical uncertainty in the determination of the parameters of the interaction region from the measured momentum distributions.

\section{What can and what cannot be measured}

Let us write the single particle density matrix in the form

$$\rho(p_1; p_2) = |\rho(p_1; p_2)| e^{i\chi(p_1; p_2)}.$$  \hfill (9)

The modulus is measurable, thus all the ambiguities result from the phase. It is convenient to choose for the phase the convention

$$-\pi \leq \chi(p, p') \leq \pi.$$  \hfill (10)

\textsuperscript{2}This remains valid also when the multiparticle corrections are included.
The hermiticity of the single particle density matrix implies that
\[ \chi(p, p') = -\chi(p', p). \] (11)
Moreover, the phases \( \chi(p_j, p_k) \) are constrained by the measurable three-particle momentum distribution which yields
\[ \cos [\chi(p_1; p_2) + \chi(p_2; p_3) + \chi(p_3, p_1)]. \] (12)
The sign of the argument of the cosine is not constrained. This implies that \( \rho(p; p') \) can be replaced by its complex conjugate without affecting the fit to the data.

Once the sign is fixed, it is possible to determine from the data the function
\[ \Psi(p_1, p_2, p_3) = \chi(p_1; p_2) + \chi(p_2; p_3) + \chi(p_3, p_1). \] (13)
Formally, this is an inhomogeneous, linear functional equation for the function \( \chi(p, p') \). Let us assume that some function \( \chi_0(p, p') \) is a solution of this equation. If the model agrees with experiment such a function must exist. Let us define further a function \( \chi_h(p, p') \) which is a solution of the homogenous equation
\[ \chi_h(p_1; p_2) + \chi_h(p_2; p_3) + \chi_h(p_3, p_1) = 0. \] (14)
Function \( \chi_0(p, p') + \chi_h(p, p') \) is then also a solution of equation (13). Thus, the question what is the uncertainty of solution \( \chi_0 \) reduces to the question what is the full set of solutions of equation (14).

Differentiating both sides of this equation with respect to \( p_{1j} \) and \( p_{2k} \) we get the equations
\[ \frac{\partial \chi_h(p, p')}{\partial p_{j} \partial p'_{k}} = 0, \quad j, k = x, y, z. \] (15)
The general solution of this equation system can be written in the form
\[ \chi(p, p') = f(p) - g(p'), \] (16)
where \( f(p) \) and \( g(p) \) are arbitrary functions. Every solution of equations (14) is a solution of equation (15), but not the other way round. Substituting
solution (16) into equation (14) one easily finds that the equation is satisfied if and only if

\[ g(p) = f(p). \] (17)

The same conclusion follows from the symmetry (11). Thus, the most general single particle density matrix giving the same fit to the data as matrix \( \rho_0(p, p') \) can be written in the form

\[ \rho(p; p') = e^{i(f(p_1) - f(p_2))} \rho_0(p; p') \quad \text{or} \quad e^{-i(f(p_1) - f(p_2))} \rho_0^*(p; p'), \] (18)

where \( f(p) \) is an arbitrary real function of the momentum \( p \). The four- and more particle distributions do not introduce any more constraints.

To see how this ambiguity can be eliminated by a theoretical assumption let us suppose that there are no position-momentum correlations. Then the Wigner function (cf. (36)) is a product of a function of \( x \) only and a function of \( K \) only. Consequently, each of the density matrices \( \rho(p; p') \) and \( \rho_0(p; p') \) is a product of a function of \( p + p' \) only and a function of \( p - p' \) only. This implies that

\[ f(p) - f(p') = a \cdot (p - p'), \] (19)

where \( a \) is an arbitrary constant vector. The ambiguities in the interaction region, corresponding to the transformations (18), reduce to rigid translations by the vector \( a \) and/or space inversion. Since these do not affect the size and shape of the interaction region, one may claim that there is no significant ambiguity left.

We conclude that, if somebody fitting the experimental data and using some model (this is necessary to get a well-defined result), finds the effective single particle density matrix \( \rho_0(p; p') \), another model leading to any of the density matrices (18) agrees with the data just as well. Of course, a priori some model may be more plausible than another, but these are arguments on a different level – unrelated to the fitting of momentum distributions.

The effects of this uncertainty on the interpretation of the data can be spectacular. It is plausible, for instance, that the momenta and positions at birth of the hadrons are correlated so that

\[ p \approx \lambda x, \] (20)
where $\lambda$ is a positive constant. Using the relation between the density matrix and the Wigner function one easily finds that the complex conjugation of the density matrix corresponds to the change of sign of the coordinate in the Wigner function. Using further the interpretation of the Wigner function as an approximation to the phase space distribution one finds that models where

$$p \approx -\lambda x$$

fit the data exactly as well as the previous ones.

## 3 Moments of the particle density distribution in the interaction region

The density of particles in the interaction region is given by the diagonal elements of the effective single particle density matrix in the coordinate representation:

$$\tilde{\rho}(x; x) \equiv \tilde{\rho}(x) = \int \frac{dK dq}{(2\pi)^3} e^{i qx} \rho(K, q),$$

where

$$K = \frac{1}{2} (p + p'), \quad q = p - p'$$

and $\rho(K, q)$ stands for $\rho(p, p')$. Integrating $n$ times by parts over the components of $q$ and assuming that $\rho(K, q)$ and all its derivatives with respect to components of $q$ tend to zero for $|q| \to \infty$ one finds in particular

$$\tilde{\rho}(x) = \int \frac{i^n}{x^n} \frac{dK dq}{(2\pi)^3} e^{i qx} \left( \frac{\partial^n}{\partial q^n} \rho(K, q) \right), \quad j = x, y, z.$$  

Multiplying both sides of this identity by $x^n_j$ and integrating over $x$ one gets the moments

$$\int dx \tilde{\rho}(x)x^n_j \equiv \langle x^n_j \rangle = i^n \int dK \left( \frac{\partial^n}{\partial q^n_j} \rho(K, q) \right)_{q=0}.$$  

Usually only the first and second order moments are considered.
Let us introduce the following notation for the moments corresponding to the density matrix \( \rho_0(p, p') \). Putting

\[
    r_{0j} = i \int dK \left( \frac{\partial}{\partial q_j} \rho_0(K, q) \right)_{q=0}, \quad j = x, y, z; \tag{26}
\]

\[
    R_{0jk}^2 = - \int dK \left( \frac{\partial^2}{\partial q_j \partial q_k} \rho_0(K, q) \right)_{q=0} \quad j, k = x, y, z \tag{27}
\]

and assuming that the effective single particle density matrix \( \rho_0(p_1, ; p_2) \) is the correct one, the moments are

\[
    \langle x \rangle = r_0 \tag{28}
\]

\[
    \langle x_j x_k \rangle = R_{0jk}^2. \tag{29}
\]

It has been known for a long time (cf. e.g. [2]) that it is not possible to determine either \( \langle x \rangle \) or \( \langle x_j x_k \rangle \) from momentum measurements. The usual way out (cf. e.g. [2]) is to calculate instead the variances

\[
    \sigma_0^2(x_j) = \langle x_j^2 \rangle - \langle x_j \rangle^2. \tag{30}
\]

Another set of much discussed quantities is related to the integrands of the integrals given above

\[
    r_{0j}(K) = i \left( \frac{\partial}{\partial q_j} \rho_0(K, q) \right)_{q=0} \tag{31}
\]

\[
    R_{0jk}^2(K) = - \left( \frac{\partial^2}{\partial q_j \partial q_k} \rho_0(K, q) \right)_{q=0} \tag{32}
\]

The corresponding variances and correlation functions are

\[
    R_{0jk}^2(K) - r_{0j}(K)r_{0k}(K) = - \left( \frac{\partial^2}{\partial q_j \partial q_k} \log \rho_0(K, q) \right)_{q=0}. \tag{33}
\]

Denoting the left-hand side of this equality by \( R_{HBT,jk}^2 \), and noting that according to (11) all the second derivatives of the phase with respect to components of \( q \) vanish at \( q = 0 \), we get

\[
    R_{HBT,jk}^2(K) = - \left( \frac{\partial^2}{\partial q_j \partial q_k} \log |\rho_0(K, q)| \right)_{q=0}. \tag{34}
\]
It is seen that these functions are measurable and do not depend on the specific choice of the function \( \rho_0(K, x) \) among the functions giving the same fits. They are just the well-known HBT radii. We have kept the traditional notation with the squares (cf. e.g. [2]) in spite of the well-known fact that \( R_{HBT}^2 \) for \( j \neq k \) may be negative (cf. e.g. [2]). An important point is that, they are not sufficient to calculate the overall variances and correlation functions because in general

\[
\langle r_{0j}(K)r_{0k}(K) \rangle \neq \langle r_{0j}(K) \rangle \langle r_{0k}(K) \rangle.
\] (35)

The equations for the moments become more suggestive when written in terms of the Wigner function, related to the density matrix \( \rho_0(K, q) \) by

\[
\rho_0(K, q) = \int dx \; W_0(K, x)e^{-iqx}.
\] (36)

Then

\[
\begin{align*}
 r_{0j}(K) &= \frac{\int dx \; x_j W_0(K, x)}{\int dx \; W_0(K, x)}, \\
 R_{0jk}^2(K) &= \frac{\int dx \; x_j x_k W_0(K, x)}{\int dx \; W_0(K, x)}.
\end{align*}
\] (37)

Interpreting the Wigner function as the phase space density, as often done, one sees that \( r_{0j}(K) \) and \( R_{0jk}^2(K) \) are the moments of the components of \( x \) evaluated at fixed \( K \). This point of views is supported by the fact that when averaged over \( K \), i.e. multiplied by the distribution of \( K \) \( (\rho_0(K) \equiv \rho_0(K, 0)) \) and integrated over \( K \), they reproduce correctly the overall moments. However, according to Heisenberg’s uncertainty principle, when the variance of \( K \) tends to zero the variance of \( x \) tends to infinity. Only when the variance of \( K \) grows from zero to finite values, destructive interference between the contributions corresponding to different \( K \) values limits the \( x \)-range. Thus, the interpretation of \( r_{0j}(K) \) and \( R_{0jk}^2(K) \) as moments of \( x \) at given \( K \) has heuristic value, but should be used with care. Note also that the average over \( K \) of the variance \( \sigma_k^2(x_j) \) does not give \( \sigma^2(x_j) \).

In the following section we will discuss how the moments introduced here are affected by the uncertainty in the phase of the density matrix \( \rho(p, p') \).
4 Uncertainties in the determination of the moments

Let us see how the first and second order moments of the coordinates change under transformations (13). The terms higher order than second in the components of $q$ do not contribute and, therefore, the factors multiplying $\rho_0(K, q)$ and $\rho^*_0(K, q)$ can be written respectively as

$$F_{\pm}(K, q) = 1 \pm i \sum_j q_j \frac{\partial f(K)}{\partial K_j} - \frac{1}{2} \sum_{j,k} q_j q_k \frac{\partial f(K)}{\partial K_j} \frac{\partial f(K)}{\partial K_k}.\tag{39}$$

Using the formulae from the preceding section one finds

$$r_j(K) = \pm \left( r_{0j}(K) - \frac{\partial f(K)}{\partial K_j} \right),\tag{40}$$

$$R_{ij}^2(K) = R_{0ij}^2(K) - \left( r_{0j}(K) \frac{\partial f(K)}{\partial K_j} + r_{0k}(K) \frac{\partial f(K)}{\partial K_k} \right) + \frac{\partial f(K)}{\partial K_j} \frac{\partial f(K)}{\partial K_k}.\tag{41}$$

The HBT radii remain, of course, unchanged.

The formulae for the overall moments are

$$r_j = r_{0j} - \left\langle \frac{\partial f(K)}{\partial K_j} \right\rangle_K,\tag{42}$$

$$R_{jk}^2 = R_{0jk}^2 - \left\langle r_{0j}(K) \frac{\partial f(K)}{\partial K_k} + r_{0k}(K) \frac{\partial f(K)}{\partial K_j} \right\rangle_K + \left\langle \frac{\partial f(K)}{\partial K_j} \frac{\partial f(K)}{\partial K_k} \right\rangle_K,\tag{43}$$

where $\langle \ldots \rangle_K$ denotes averaging over $K$. The overall variances and correlation functions are
\[ R_{jk}^2 - r_j r_k = \left( R_{HBT,jk}^2(K) \right)_K + \left( \frac{\partial f(K)}{\partial K_j} \frac{\partial f(K)}{\partial K_k} \right)_K - \left( \frac{\partial f(K)}{\partial K_j} \right)_K \left( \frac{\partial f(K)}{\partial K_k} \right)_K \\
- \left( r_{0j}(K) \frac{\partial f(K)}{\partial K_k} \right)_K + r_{0j} \left( \frac{\partial f(K)}{\partial K_j} \right)_K \\
- \left( r_{0k}(K) \frac{\partial f(K)}{\partial K_j} \right)_K + r_{0k} \left( \frac{\partial f(K)}{\partial K_k} \right)_K. \] (44)

They are not measurable without further assumptions.

Let us discuss some illustrative examples. For the models where

\[ r_0(K) \equiv 0, \] (45)

one gets

\[ R_{jj}^2 = R_{HBT, jj}^2 + \sigma_f^2(K_j), \quad j = x, y, z, \] (46)

where the variance

\[ \sigma_f^2(K_j) \equiv \left( \left( \frac{\partial f(K)}{\partial K_j} \right)^2 \right)_K - \left( \frac{\partial f(K)}{\partial K_j} \right)_K \left( \frac{\partial f(K)}{\partial K_j} \right)_K. \] (47)

It is seen that the HBT radii are a rigorous lower bounds for the true radii, while there is no upper bound. The interaction region can be assumed to be the size of a football and agreement with experiment remains as good as before. In practice, of course, so large interactions regions are excluded by what is known about the detector, but this is information going beyond the measurements of momentum distributions.

If a model giving exactly the same fit as the models satisfying (45) exists, it must be possible to find a function \( f(K) \) satisfying

\[ \nabla f(K) = r_0(K). \] (48)

Unless there are singularities, this equation can be solved when the vector field \( r_0(K) \) is rotationless:

\[ \nabla \times r_0(K) = 0. \] (49)
Consider the following model discussed by Akkelin and Sinyukov [6]. A classical gas of noninteracting particles has at time $t = 0$ a Gaussian distribution in coordinate space with $\langle x \rangle = 0$ and a Boltzmann distribution in momentum space; there are no position-momentum correlations. It is obvious how this gas will evolve. The momentum distribution will not change. In coordinate space each group of particle having identical momenta $K$ will propagate rigidly with velocity $Km$, where $m$ is the particle mass. This picture can be confirmed by solving the corresponding Boltzmann equation [6]. For the particles with momentum $K$

\[
\langle x \rangle(K) = \frac{K}{m}t = r_0(K),
\]

where the second equality corresponds to the choice $f(K) = 0$, and corresponds to

\[
R^2_{jk} = R^2_{HBT, jk}.
\]

We can choose, however, just as well

\[
f(K) = \frac{K^2}{2mt}.
\]

The corresponding density matrix satisfies (45) and consequently yields

\[
R^2_{jj} = R^2_{HBT, jj} + \left\langle \frac{K^2_j}{m^2t^2} \right\rangle_K.
\]

From the description of the model, which contains more information than just about the momentum distributions, we know that the first description is wrong and the second is correct. In this example the HBT radii can be interpreted as follows. For particles with any given value of $K$ there is some distribution of $x$. Shifting rigidly all these $x$ distributions on top of each other so that their averages $\langle x \rangle(K)$ coincide one gets a reduced distribution. The HBT radii are the radii of this reduced distribution.

In [5] a model was proposed, where for the transverse degrees of freedom

\[
\langle x_j \rangle(K_j) = \lambda K_j
\]
and $\lambda$ does not depend on the transverse momenta. Again, using the description of the model one finds

$$ f(K_j) = \frac{\lambda}{2} K_j^2 $$

(55)

and

$$ R_{jj}^2 = R_{HBT, jj}^2 + \lambda^2 \left< K_j^2 \right>_{K}. $$

(56)

This was used to explain why experimentally the $HBT$ radii decrease when the particle mass increases. In this model the true radii $R_{jj}^2$ do not depend on the particle mass and the observed decrease of the $HBT$ radii with increasing particle mass just reflects the well-known mass dependence of the second term on the right-hand side.

## 5 Conclusions

We have considered the most popular class of models, where all the information about the size and shape of the interaction region is contained in a single-particle density matrix $\rho(p; p')$. The aim is to determine this matrix from momentum distributions measured for sets of identical particles. We show that, without further assumptions, this is not possible. For each single particle density matrix there is an equivalence class of matrices which give exactly the same fits to all the momentum distributions. All these matrices are related by the transformations (18). Thus, the full group of transformations leaving the momentum distributions invariant has been identified.

The $HBT$ radii, which can be unambiguously determined from the momentum distributions, are in general different from the true radii of the interaction region. Sometimes, but not always, they are lower bounds for the true radii. In order to determine the true radii it is necessary to determine a function $\nabla f(K)$. If only the results of momentum measurements are available function $f(K)$ is completely arbitrary. Models, however, usually contain information going beyond that obtainable from momentum measurements and then function $\nabla f(K)$ may be possible to determine. It is recommended, when proposing a model, to formulate clearly and justify the assumptions which yield $f(K) = 0$, because these assumptions cannot be justified even by the best fits to the measured momentum distributions.
We have discussed a comparatively simple version of the standard approach. It is easy to extend our analysis to include the multiparticle corrections. The final state interactions and, more generally, interparticle correlations, however, would make the discussion much more complicated. It is most unlikely, nevertheless, that they would make the result simpler. Ambiguities analogous to those described here are expected to occur also in these more complicated models.

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