Heat-kernel approach for scattering

Wen-Du Li\textsuperscript{a} and Wu-Sheng Dai\textsuperscript{a,b,1}

\textsuperscript{a}Department of Physics, Tianjin University, Tianjin 300072, P.R. China
\textsuperscript{b}LiuHui Center for Applied Mathematics, Nankai University & Tianjin University, Tianjin 300072, P.R. China

ABSTRACT: An approach for solving scattering problems, based on two quantum field theory methods, the heat kernel method and the scattering spectral method, is constructed. This approach has a special advantage: it is not only one single approach; it is indeed a set of approaches for solving scattering problems. Concretely, we build a bridge between a scattering problem and the heat kernel method, so that each method of calculating heat kernels can be converted into a method of solving a scattering problem. As applications, we construct two approaches for solving scattering problems based on two heat-kernel expansions: the Seeley-DeWitt expansion and the covariant perturbation theory. In order to apply the heat kernel method to scattering problems, we also calculate two off-diagonal heat-kernel expansions in the frames of the Seeley-DeWitt expansion and the covariant perturbation theory, respectively. Moreover, as an alternative application of the relation between heat kernels and partial-wave phase shifts presented in this paper, we give an example on how to calculate a global heat kernel from a known scattering phase shift.

\textsuperscript{1}daiwusheng@tju.edu.cn.
1 Introduction

In this paper, based on two quantum field theory methods, heat kernel method [1] and scattering spectral method [2], we present a new approach to solve scattering problems. This approach is not only one single approach; it is a series of approaches for scatterings. Concretely, our key result is a direct relation between partial-wave scattering phase shifts

\[ \delta_l^{(1)}(k) \]

and heat kernel.
and heat kernels. By this result, each method of calculating heat kernels leads to an approach of calculating phase shifts. In a word, the approach provided in this paper converts a heat kernel method into a method of solving scattering problems. Many methods for scattering problems can be constructed, since the heat kernel theory is well-studied in both mathematics and physics and there are many mature methods for the calculation of heat kernels.

**Phase shift.** All information of an elastic scattering process are embedded in a scattering phaseshift. This can be seen by directly observing the asymptotic solution of the radial wave equation. For spherically symmetric cases, the asymptotic solution of the free radial wave equation,

$$
\left[ -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} \right] R_l = k^2 R_l,
$$

is

$$
R_l (r) \quad r \to \infty = \frac{1}{kr} \sin \left[ kr - \frac{l\pi}{2} + \delta_l (k) \right].
$$

This defines the partial-wave phase shift $\delta_l (k)$, which is the only effect on the radial wave function at asymptotic distances [3]. Therefore, all we need to do in solving a scattering problem is to solve the phase shift $\delta_l (k)$.

**Heat kernel.** The information embedded in an operator $D$ can be extracted from a heat kernel $K(t; r, r')$ which is the Green function of the initial-value problem of the heat-type equation $(\partial_t + D) \phi = 0$, determined by

$$
(\partial_t + D) K(t; r, r') = 0, \quad K(0; r, r') = \delta (r - r').
$$

The global heat kernel $K(t)$ is the trace of the local heat kernel $K(t; r, r')$: $K(t) = \int \! dr' K(t; r, r') = \sum_{n,l} e^{-\lambda_{nl}t}$, where $\lambda_{nl}$ is the eigenvalue of the operator $D$.

The main aim of the present paper is to seek a relation between the partial-wave phase shift $\delta_l (k)$ and the heat kernel $K(t; r, r')$. By this relation, we can explicitly express a partial-wave phase shift by a given heat kernel. There are many studies on the approximate calculation of heat kernels [1, 4–13] and each approximate method of heat kernels gives us an approximate method for calculating partial-wave phase shifts.

The present work is based on our preceding work given in Ref. [14], which reveals a relation between two quantum field theory methods, heat kernel method [1] and scattering spectral method [2]. In Ref. [14], using the relation between spectral counting functions and heat kernels given by Ref. [15] and the relation between phase shifts and state densities given by Ref. [2], we provide a relation between the global heat kernel and the total scattering phase shift (the total scattering phase shift is the summation of all partial-wave phase shifts, $\delta (k) = \sum_l (2l + 1) \delta_l (k)$).

Nevertheless, the result given by Ref. [14] — the relation between total scattering phase shifts and heat kernels — can hardly be applied to scattering problems, since the total phase shift has no clear physical meaning. To apply the heat kernel method to scattering problems, we in fact need a relation between partial-wave phase shifts (rather than total phase shifts) and heat kernels. In the present paper, we find such a relation. This relation allows us to express a partial-wave
phase shift by a known heat kernel. Then, all physical quantities of a scattering process, such as scattering amplitudes and cross sections, can be expressed by a heat kernel.

To find the relation between partial-wave phase shifts and heat kernels, we will first prove a relation between heat kernels and partial-wave heat kernels. The heat kernel \( K(t; \mathbf{r}, \mathbf{r}') \) is the Green function of initial-value problem of the heat equation (1.2) with the operator \( D = -\nabla^2 + V(r) \) and the partial-wave heat kernel \( K_l(t; r, r') \) is the Green function of initial-value problem of the heat equation (1.2) with the radial operator \( D_l = -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} + V(r) \). By this relation, we can calculate a partial-wave heat kernel \( K_l(t; r, r') \) from a heat kernel \( K(t; \mathbf{r}, \mathbf{r}') \) directly.

The main aim of this paper is to explicitly express the partial-wave phase shift and scattering amplitude by a given heat kernel. As mentioned above, by our result, each method of calculating heat kernels can be converted to a method of calculating scattering problems.

In order to calculate a scattering phase shift from a heat kernel, we need off-diagonal heat kernels (i.e., heat kernels). For this purpose, in the following, we first calculate two kinds of off-diagonal heat-kernel expansions in the frames of the Seeley-DeWitt expansion and the covariant perturbation theory, respectively. It should be pointed out that many methods on the calculation of diagonal heat-kernel expansions in literature can be directly apply to the calculation of off diagonal heat kernels.

Moreover, we compare the scattering method established in the present paper with the Born approximation. By comparing the approximation results given by the three methods, the Seeley-DeWitt expansion, the covariant perturbation theory, and the Born approximation through an exactly solvable potential, we show that the method established based on the covariant perturbation theory provided in the present paper is the best approximation.

On the other hand, besides applying the heat kernel method to scattering problems, by the method suggested in the present paper, we can also apply the scattering method to the heat kernel theory. In this paper, we provide a simple example for illustrating how to calculate a heat kernel from a known scattering result; more details on this subject will be given in a subsequent work. The value of developing such a method, for example, is that though it is relatively easy to obtain a high-energy expansion of heat kernels, it is difficult to obtain a low-energy heat-kernel expansion. With the help of scattering theory, we can calculate a low-energy heat-kernel expansion from a low-energy scattering theory.

The starting point of this work, as mentioned above, is a relation between the heat kernel method and the scattering spectral method in quantum field theory. The heat kernel method is important both in physics and mathematics. In physics, the heat kernel method has important applications in, e.g., Euclidean field theory, gravitation theory, and statistical mechanics \[1, 13, 16–18\]. In mathematics, the heat kernel method is an important basis of the spectral geometry \[1, 19\]. There are many researches on the calculation of heat kernels. Besides exact solutions, there are many systematic studies on the asymptotic expansion of heat kernels, such as the Seeley-DeWitt expansion \[1\] and the covariant perturbation theory \[10–13\]. With various heat-kernel expansion techniques, one can obtain many approximate solutions of heat kernels. Scattering spectral method is an important quantum theory method which can be used to solve a variety of problems in quantum field theory, e.g., to
characterize the spectrum of energy eigenstates in a potential background [2] and to solve the Casimir energy [20–24]. The method particular focuses on the property of the quantum vacuum.

In Sec. 2, we find a relation between partial-wave phase shifts and heat kernels. As a key step, we give a relation between partial-wave heat kernels and heat kernels. In Secs. 3 and 4, based on the relation between partial-wave phase shifts and heat kernels given in Sec. 2, we establish two approaches for the calculation of partial-wave phase shifts, based on two heat-kernel expansions, the Seeley-DeWitt expansion and the covariant perturbation theory. In Sec. 5, a comparison of the two approaches for partial-wave phase shifts established in the present paper and the Born approximation is given; especially, we compare these three methods through an exactly solvable potential. In Sec. 6, we give an example for calculating a heat kernel from a given phase shift. Conclusions and outlook are given in Sec. 7. Moreover, an integral formula and two integral representations are given in appendices A, B, and C.

2 Relation between partial-wave phase shift and heat kernel: calculating scattering phase shift from heat kernel

The main result of the present paper is the following theorem which reveals a relation between partial-wave scattering phase shifts and heat kernels. This relation allows us to obtain a partial-wave phase shift from a known heat kernel directly. By this relation, what we can obtain is not only one method for scattering problems. It is in fact a series of methods for scattering problems: each heat kernel method leads to a method for solving scattering problems.

**Theorem 1** The relation between the partial-wave scattering phase shift, $\delta_l(k)$, and the heat kernel, $K(t; r, r') = K(t; r, \theta, r', \theta', \varphi')$, is

$$\delta_l(k) = 2\pi \int_0^\infty r^2dr \int_{-1}^{1} d\cos \gamma P_l(\cos \gamma) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \frac{e^{kt}}{t} K^s(t; r, \theta, r, \theta', \varphi') + \delta_l(0),$$

(2.1)

where $K^s(t; r, r')$ is the scattering part of a heat kernel, $P_l(\cos \gamma)$ is the Legendre polynomial, and $\gamma$ is the angle between $r$ and $r'$ with $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi)$.

Notice that only the radial diagonal heat kernel, $K^s(t; r, \theta, \varphi, r, \theta', \varphi')$, appears in Eq. (2.1). Here the heat kernel $K(t; r, r')$ is split into three parts: $K(t; r, r') = K^s(t; r, r') + K^b(t; r, r') + K^f(t; r, r')$, where $K^s(t; r, r')$ is the scattering part of the heat kernel, $K^b(t; r, r')$ the bound part, and $K^f(t; r, r')$ the free part [14]. Note that $\delta_l(0) = \pi/2$ if there is a half-bound state and $\delta_l(0) = 0$ if there is no half-bound state [14].

The remaining task of this section is to prove this theorem. In order to prove the theorem, we need to first find a relation between partial-wave heat kernels and heat kernels.

2.1 Relation between partial-wave heat kernel and heat kernel

As mentioned above, the heat kernel $K(t; r, r')$ of an operator $D$ is determined by the heat equation (1.2) [1]. For a spherically symmetric operator $D$, the heat kernel $K(t; r, r') =...
$K(t; r, \theta, \varphi, r', \theta', \varphi')$ can be expressed as

$$K(t; r, r') = \sum_{n,l,m} e^{-\lambda_{nl}t} \psi_{nlm}(r) \psi_{nlm}^*(r'),$$  \hspace{1cm} (2.2)

where $\lambda_{nl}$ and $\psi_{nlm}(r) = R_{nl}(r) Y_{lm}(\theta, \varphi)$ are the eigenvalue and eigenfunction of $D$, determined by the eigen equation $D\psi_{nlm} = \lambda_{nl}\psi_{nlm}$, where $R_{nl}(r)$ is the radial wave function and $Y_{lm}(\theta, \varphi)$ is the spherical harmonics. The global heat kernel is the trace of the local heat kernel $K(t; r, r')$:

$$K(t) = \int dr K(t; r, r) = \sum_n e^{-\lambda_{nt}}.$$  \hspace{1cm} (2.3)

The local partial-wave heat kernel

$$K_l(t; r, r') = \sum_n e^{-\lambda_{nt}} R_{nl}(r) R_{nl}(r')$$  \hspace{1cm} (2.4)

of the operator $D$ is the heat kernel of the $l$-th partial-wave radial operator [14]

$$D_l = -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} + V(r)$$  \hspace{1cm} (2.5)

which determines the radial equation $D_l R_{nl} = \lambda_{nl} R_{nl}$. The global partial-wave heat kernel is the trace of the local partial-wave heat kernel $K_l(t; r, r')$,

$$K_l(t) = \int_0^\infty r^2 dr K_l(t; r, r) = \sum_n e^{-\lambda_{nt}}.$$  \hspace{1cm} (2.6)

Now we prove that the relation between $K_l(t; r, r')$ and $K(t; r, r')$ can be expressed as follows.

**Lemma 2** The relation between the partial-wave heat kernel $K_l(t; r, r')$ and the heat kernel $K(t; r, r')$ is

$$K_l(t; r, r') = 2\pi \int_{-1}^1 d\cos \gamma P_l(\cos \gamma) K(t; r, \theta, \varphi, r', \theta', \varphi')$$  \hspace{1cm} (2.7)

and

$$K(t; r, \theta, \varphi, r', \theta', \varphi') = \frac{1}{4\pi} \sum_l (2l + 1) P_l(\cos \gamma) K_l(t; r, r').$$  \hspace{1cm} (2.8)

**Proof.** In a scattering with a spherically symmetric potential, the scattering wave function $\psi_{nlm}(r, \theta, \varphi) = R_{nl}(r) Y_{lm}(\theta, \varphi)$. Then, by Eq. (2.2), the heat kernel can be expressed as

$$K(t; r, \theta, \varphi, r', \theta', \varphi') = \sum_{n,l} e^{-\lambda_{nl}t} R_{nl}(r) R_{nl}(r') \sum_{m=-l}^l Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi').$$  \hspace{1cm} (2.9)

Using the relation [25]

$$\sum_{m=-l}^l Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') = \frac{2l+1}{4\pi} P_l(\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi' - \varphi)),$$  \hspace{1cm} (2.10)
we obtain

\[ K(t; r, \theta, \varphi, r', \theta', \varphi') = \frac{1}{4\pi} \sum_l (2l + 1) P_l(\cos \gamma) \sum_n e^{-\lambda n t} R_{nl}(r) R_{nl}(r'). \]  

(2.11)

Then, by Eq. (2.4), we prove the relation (2.8).

Multiplying on the both sides of Eq. (2.8) by \( P_l'(\cos \gamma) \) and then integrating \( \cos \gamma \) from \(-1\) to \(1\) give

\[ \int_{-1}^{1} d \cos \gamma P_l'(\cos \gamma) K(t; r, \theta, r', \theta', \varphi') = \frac{1}{4\pi} \sum_l (2l + 1) \left[ \int_{-1}^{1} d \cos \gamma P_l'(\cos \gamma) P_l(\cos \gamma) \right] K_l(t; r, r'). \]  

(2.12)

Using the orthogonality of the Legendre polynomials [25]

\[ \int_{-1}^{1} d \cos \gamma P_l'(\cos \gamma) P_l(\cos \gamma) = \frac{2}{2l' + 1} \delta_{ll'}, \]  

(2.13)

we obtain

\[ \int_{-1}^{1} d \cos \gamma P_l'(\cos \gamma) K(t; r, \theta, r', \theta', \varphi') = \frac{1}{2\pi} K_{l'}(t; r, r'). \]  

(2.14)

This proves the relation (2.7).

2.2 Proof of Theorem 1

Now, with Lemma 2, we can prove Theorem 1.

Proof. In Ref. [14], we prove a relation between total phase shifts and global heat kernels,

\[ \delta(k) = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} K^s(t) e^{k^2 t} dt + \delta(0), \]  

(2.15)

and a relation between partial-wave phase shifts and partial-wave global heat kernels,

\[ \delta_l(k) = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{K^s_l(t)}{t} e^{k^2 t} dt + \delta_l(0). \]  

(2.16)

Here the global heat kernel and the global partial-wave heat kernel are split into the scattering part, the bound part, and the free part: \( K(t) = K^s(t) + K^b(t) + K^f(t) \) and \( K_l(t) = K^s_l(t) + K^b_l(t) + K^f_l(t) \) [14].

Starting from the global partial-wave heat kernel given by Eq. (2.6) and using the relation between partial-wave heat kernels and heat kernels given by Lemma 2, Eq. (2.7), we have

\[ K_l(t) = \int_{0}^{\infty} r^2 dr K_l(t; r, r) \]

\[ = 2\pi \int_{0}^{\infty} r^2 dr \int_{-1}^{1} d \cos \gamma P_l(\cos \gamma) K(t; r, \theta, r, \theta', \varphi'). \]  

(2.17)
Substituting Eq. (2.17) into Eq. (2.16) proves Theorem 1. ■

It should be noted here that the relation given by Ref. [14], Eqs. (2.15) and (2.16), only allows one to calculate the total phase shift \( \delta (k) \) from a heat kernel \( K (t) \) or to calculate the partial-wave phase shift \( \delta_l (k) \) from a partial-wave heat kernel \( K_l (t) \). Such results, however, are not useful in scattering problems, because the total phase shift \( \delta (k) \) is not physical meaningful and the partial-wave heat kernel \( K_l (t) \) is often difficult to obtain.

Nevertheless, the result given by Theorem 1, Eq. (2.1), allows one to calculate the partial-wave phase shift \( \delta_l (k) \) form a heat kernel \( K (t) \) rather than a partial-wave heat kernel \( K_l (t) \). The heat kernel has been fully studied and there are many known results [1].

3 Heat-kernel approach for phase shift: Seeley-DeWitt expansion

In this section, we present an asymptotic expansion method for phase shifts based on the Seeley-DeWitt expansion of heat kernels.

The Seeley-DeWitt asymptotic expansion of heat kernels is an important method in the heat kernel theory. In the following, by Theorem 1, we convert the Seeley-DeWitt heat-kernel expansion into an expansion for partial-wave phase shifts.

The Seeley-DeWitt type expansion for a partial-wave scattering phase shift is \( \delta_l (k) = \delta_l^{(1)} (k) + \delta_l^{(2)} (k) + \cdots \) with

\[
\delta_l^{(1)} (k) = - \frac{\pi}{2} \int_0^\infty r dr J_{l+1/2}^2 (kr) V (r) - \frac{\pi}{24} \int_0^\infty dr \frac{d^2 J_{l+1/2}^2 (kr)}{dk^2} \frac{dV (r)}{dr},
\]

\[
\delta_l^{(2)} (k) = \frac{\pi}{8k} \int_0^\infty r dr \frac{dJ_{l+1/2}^2 (kr)}{dk} \left[ V^2 (r) - \frac{1}{3} \nabla^2 V (r) \right]
- \frac{\pi}{240k} \int_0^\infty dr \frac{d^3 J_{l+1/2}^2 (kr)}{dk^3} \left[ \frac{d}{dr} \nabla^2 V (r) - \frac{7}{2} V (r) \frac{dV (r)}{dr} \right],
\]

where \( J_l (z) \) is the Bessel function of the first kind [25].

A detailed calculation is as follows.

3.1 Seeley-DeWitt expansion of heat kernel

The Seeley-DeWitt expansion is an asymptotic expansion of heat kernels in powers of the proper time \( t \) [13]. For a Laplace-type operator \( D = - \nabla^2 + V \), the Seeley-DeWitt expansion of heat kernels reads [26]

\[
K (t; r, r') = \frac{1}{(4\pi t)^{3/2}} \exp \left( - \frac{\sigma (r, r')}{2t} \right) \sum_{j=0}^\infty a_j (r, r') t^j,
\]

where \( a_j (r, r') \) is the heat-kernel coefficient and \( \sigma (r, r') = d^2 (r, r') / 2 \) with \( d (r, r') \) the geodesic distance between \( r \) and \( r' \). Note that \( \nabla^2 \) here is the Laplace-Beltrami operator which reduces to the Laplace operator in flat space. For the Seeley-DeWitt expansion, the heat-kernel coefficient \( a_j (r, r') \) satisfies the recurrence identity [26]

\[
\frac{a_j+1 (r, r')}{\triangle_{VV}^{1/2} (r, r')} = - \int_0^1 \lambda \frac{d \lambda}{\triangle_{VV}^{1/2} (r, r')} \left[ \triangle_{VV}^{1/2} D_r a_j (r (\lambda), r') \right] (\lambda, r')
\]

\[
+ \int_0^1 \lambda \frac{d \lambda}{\triangle_{VV}^{1/2} (r, r')} \left[ \triangle_{VV}^{1/2} D_r a_j - \frac{\sigma (r, r')}{2t} \right] (r (\lambda), r')
\]

\[
= - \int_0^1 \lambda \frac{d \lambda}{\triangle_{VV}^{1/2} (r, r')} \left[ \triangle_{VV}^{1/2} D_r a_j - \frac{\sigma (r, r')}{2t} \right] (r (\lambda), r')
\]

\[
+ \int_0^1 \lambda \frac{d \lambda}{\triangle_{VV}^{1/2} (r, r')} \left[ \triangle_{VV}^{1/2} D_r a_j - \frac{\sigma (r, r')}{2t} \right] (r (\lambda), r')
\]

\[
= - \int_0^1 \lambda \frac{d \lambda}{\triangle_{VV}^{1/2} (r, r')} \left[ \triangle_{VV}^{1/2} D_r a_j - \frac{\sigma (r, r')}{2t} \right] (r (\lambda), r')
\]

\[
+ \int_0^1 \lambda \frac{d \lambda}{\triangle_{VV}^{1/2} (r, r')} \left[ \triangle_{VV}^{1/2} D_r a_j - \frac{\sigma (r, r')}{2t} \right] (r (\lambda), r')
\]

\[
= - \int_0^1 \lambda \frac{d \lambda}{\triangle_{VV}^{1/2} (r, r')} \left[ \triangle_{VV}^{1/2} D_r a_j - \frac{\sigma (r, r')}{2t} \right] (r (\lambda), r')
\]

\[
+ \int_0^1 \lambda \frac{d \lambda}{\triangle_{VV}^{1/2} (r, r')} \left[ \triangle_{VV}^{1/2} D_r a_j - \frac{\sigma (r, r')}{2t} \right] (r (\lambda), r')
\]
with \( a_0 (r, r') = \triangle_{VVM}^{1/2} (r, r') \), where \( r (\lambda) \) describes a geodesic segment with \( r (0) = r' \) and \( r (1) = r \) and \( \triangle_{VVM} (r, r') \) is the van Vleck-Morette determinant \([26]\),

\[
\triangle_{VVM} (r, r') = (-1)^d \frac{\operatorname{sgn}(g(r))}{\sqrt{g(r) g(r')}} \det \left[ \frac{\partial^2 \sigma(r, r')}{\partial r^a \partial r'^b} \right] \quad (3.5)
\]

with \( g(r) = \det g_{\mu \nu} (r) \) and \( d \) the dimension of space.

For flat space, the Seeley-DeWitt expansion, Eq. (3.3), reduces to

\[
K (t; r, r') = \frac{1}{(4\pi t)^{d/2}} \exp \left( -\frac{|r-r'|^2}{4t} \right) \sum_{j=0}^{\infty} a_j (r, r') t^j. \quad (3.6)
\]

In flat space, \( \triangle_{VVM} (r, r') = 1 \), \( r (\lambda) = \lambda r + (1 - \lambda) r' \), and the recurrence identity, Eq. (3.4), becomes

\[
a_{j+1} (r, r') = - \int_0^1 \lambda^j d\lambda \left[ (-\nabla^2_r + V) a_j \right] (r (\lambda), r')
\]

\[
= - \int_0^1 \lambda^j d\lambda \left[ (-\nabla^2_r + V) a_j \right] (r' + \lambda (r - r'), r') \quad (3.7)
\]

with

\[
a_0 (r, r') = 1. \quad (3.8)
\]

### 3.2 Expressing phase shift by heat-kernel coefficient

In this section, in terms of the Seeley-DeWitt heat-kernel coefficients, we present an asymptotic expansion for partial-wave scattering phase shifts.

It should be noted that only the radial diagonal heat kernel, \( K (t; r, \theta, \varphi, r', \theta', \varphi') \), appears in Eq. (2.1) rather than the heat kernel, \( K (t; r, \theta, r', \theta', \varphi, \varphi') \), so only the radial diagonal heat-kernel coefficient is needed in the calculation of phase shifts. For spherical potentials, by taking \( r' = r \) in Eq. (3.6), we achieve a radial diagonal Seeley-DeWitt expansion for heat kernels:

\[
K (t; r, \gamma) = \frac{1}{(4\pi t)^{d/2}} \exp \left( -\frac{r^2}{2t} (1 - \cos \gamma) \right) \sum_{j=0}^{\infty} a_j (r, \gamma) t^j, \quad (3.9)
\]

where \( a_j (r, \gamma) = a_j (r, r') \big|_{r=r'} \) is the radial diagonal heat kernel coefficient.

The scattering phase shift can be obtained by substituting Eq. (3.9) into the relation between partial-wave phase shifts and heat kernels, Eq. (2.1),

\[
\delta_\ell (k) = 2\pi^2 \int_0^\infty r^2 dr \int_{-1}^{1} d\cos \gamma \Phi_\ell (\cos \gamma) \sum_{j=1}^{\infty} a_j (r, \gamma) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt e^{kt} \frac{1}{(4\pi t)^{d/2}} \exp \left( -\frac{r^2}{2t} (1 - \cos \gamma) \right) t^j \quad (3.10)
\]

where the free part has been subtracted. The inverse Laplace transformation here can be worked out,

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\alpha t} e^{-\beta/t} t^\nu = \alpha^{(\nu-1)/2} \beta^{(\nu+1)/2} J_{-\nu} \left( 2\sqrt{\alpha\beta} \right), \quad (3.11)
\]
and the scattering phase shift then reads
\[ \delta_l (k) = \frac{\sqrt{\pi}}{4} \int_0^\infty r^2 dr \int_{-1}^1 d \cos \gamma P_l (\cos \gamma) \sum_{j=1}^\infty (2k^2)^{3/2-j} \frac{J_{3/2-j} (\sqrt{2kr} \sqrt{1 - \cos \gamma})}{(\sqrt{2kr} \sqrt{1 - \cos \gamma})^{3/2-j}} a_j (r, \gamma). \] (3.12)

The first two contributions are
\[ \delta_l^{(1)} (k) = \frac{k}{2} \int_0^\infty r^2 dr \int_{-1}^1 d \cos \gamma P_l (\cos \gamma) \frac{\sin (\sqrt{2kr} \sqrt{1 - \cos \gamma})}{\sqrt{2kr} \sqrt{1 - \cos \gamma}} a_1 (r, \gamma), \] (3.13)
\[ \delta_l^{(2)} (k) = \int_0^\infty r^2 dr \int_{-1}^1 d \cos \gamma P_l (\cos \gamma) \frac{\cos (\sqrt{2kr} \sqrt{1 - \cos \gamma})}{4k} a_2 (r, \gamma). \] (3.14)

3.3 First-order phase shift \( \delta_l^{(1)} (k) \)

In this section, we calculate the first-order phase shift in the frame of the Seeley-DeWitt expansion.

First-order heat-kernel coefficient \( a_1 (r, r') \): In order to calculate the phase shift \( \delta_l (k) \), we need to first calculate the heat-kernel coefficient \( a_1 (r, r') \).

The first-order Seeley-DeWitt heat-kernel coefficient can be obtained by the recurrence identity (3.7):
\[ a_1 (r, r') = - \int_0^1 d \lambda V (R), \] (3.15)
where \( R = r' + \lambda (r - r') \). As mentioned above, in our case, only the radial diagonal heat-kernel coefficient, \( a_1 (r, r') |_{r'=r} \), is needed in the calculation of phase shifts. Therefore, for spherically symmetric cases, the first-order radial diagonal heat-kernel coefficient, from Eq. (3.15), reads
\[ a_1 (r, \gamma) = - \int_0^1 d \lambda V (R), \] (3.16)
where \( R = |R|_{r'=r} = r \sqrt{1 - 2\lambda (1 - \lambda) (1 - \cos \gamma)}. \)

Fourier transforming \( V (R) \) and working out the integral of the angle give
\[ V (R) = \frac{1}{(2\pi)^{3/2}} \int d^3 p V (p) e^{ipR \cos \gamma_{pR}} = \frac{1}{(2\pi)^{1/2}} \int_0^\infty p^2 dp V (p) \frac{2 \sin pR}{pR} \] (3.17)
where \( \gamma_{pR} \) is the angle between \( p \) and \( R \). Substituting Eq. (3.17) into Eq. (3.16), we have
\[ a_1 (r, \gamma) = - \frac{1}{(2\pi)^{1/2}} \int_0^1 d \lambda \int_0^\infty p^2 dp V (p) h, \] (3.18)
where \( h = 2 \sin \left[ pr \sqrt{1 - 2\lambda (1 - \lambda) (1 - x)} \right] / \left[ pr \sqrt{1 - 2\lambda (1 - \lambda) (1 - x)} \right] \) with \( x = \cos \gamma \).

Expanding \( h \) around \( x = 1 \),
\[ h = 2 \frac{\sin pr}{pr} - 2\lambda (1 - \lambda) (1 - x) \left( \cos pr - \frac{\sin pr}{pr} \right) + \cdots, \] (3.19)
substituting the expansion (3.19) into Eq. (3.18), and integrating with respect to \( \lambda \), give

\[
a_1 (r, \gamma) = -\frac{1}{(2\pi)^{1/2}} \int_0^\infty p^2 dpV (p) \left[ \frac{2 \sin pr}{pr} - \frac{1}{3} (1 - x) \left( \cos pr - \frac{\sin pr}{pr} \right) + \cdots \right].
\]

(3.20)

By constructing a derivative representation

\[
\cos pr - \frac{\sin pr}{pr} = r \frac{d}{dr} \frac{\sin pr}{pr},
\]

(3.21)

we rewritten \( a_1 (r, \gamma) \) as

\[
a_1 (r, \gamma) = -\frac{1}{(2\pi)^{1/2}} \int_0^\infty p^2 dpV (p) \left[ \frac{2 \sin pr}{pr} - \frac{1}{3} (1 - x) r \frac{d}{dr} \frac{\sin pr}{pr} + \cdots \right]
\]

\[
= - \left[ 1 - \frac{1}{6} (1 - x) r \frac{d}{dr} \right] \frac{1}{(2\pi)^{1/2}} \int_0^\infty p^2 dpV (p) \frac{2 \sin pr}{pr} + \cdots
\]

\[
= -V (r) + \frac{1}{6} (1 - x) r \frac{dV (r)}{dr} + \cdots,
\]

(3.22)

where Eq. (3.17) is used.

First-order phase shift \( \delta_1^{(1)} (k) \): The first-order phase shift \( \delta_1^{(1)} (k) \) can be obtained by substituting the first-order radial diagonal heat-kernel coefficient, Eq. (3.22), into Eq. (3.13):

\[
\delta_1^{(1)} (k) = -\frac{1}{2} \int_0^\infty r drV (r) \int_{-1}^1 dx P_1 (x) \frac{\sin (\sqrt{2kr} \sqrt{1 - x})}{\sqrt{2r} \sqrt{1 - x}}
\]

\[
+ \frac{1}{12} \int_0^\infty r^3 dr \frac{dV (r)}{dr} \int_{-1}^1 dx P_1 (x) \frac{\sin (\sqrt{2kr} \sqrt{1 - x})}{\sqrt{2r} \sqrt{1 - x}}.
\]

(3.23)

By constructing a derivative representation

\[
\frac{\sin (\sqrt{2kr} \sqrt{1 - x})}{\sqrt{2r} \sqrt{1 - x}} = -\frac{1}{2r^3} \frac{d^2}{dk^2} \frac{\sin (\sqrt{2kr} \sqrt{1 - x})}{\sqrt{2} \sqrt{1 - x}},
\]

we rewritten Eq. (3.23) as

\[
\delta_1^{(1)} (k) = \left[ -\frac{1}{2} \int_0^\infty r drV (r) - \frac{1}{24} \int_0^\infty \frac{dV (r)}{dr} \frac{d^2}{dk^2} \right] \int_{-1}^1 dx P_1 (x) \frac{\sin (\sqrt{2kr} \sqrt{1 - x})}{\sqrt{2r} \sqrt{1 - x}}.
\]

(3.24)

By using the integral formula (B.1) given in Appendix B, we have

\[
J_{l+1/2}^2 (kr) = \frac{1}{\pi} \int_{-1}^1 dx P_1 (x) \frac{\sin (\sqrt{2kr} \sqrt{1 - x})}{\sqrt{2} \sqrt{1 - x}}.
\]

(3.25)

Finally, we arrive at Eq. (3.1).
3.4 Second-order phase shift $\delta_i^{(2)}(k)$

In this section, we calculate the second-order phase shift in the frame of the Seeley-DeWitt expansion.

**Second-order heat-kernel coefficient $a_2(r, r')$:** The second-order Seeley-DeWitt heat-kernel coefficient can be obtained by the recurrence identity (3.7):

$$a_2(r, r') = -\int_0^1 \lambda d\lambda \left[ (-\nabla^2_{r(r)} + V) a_1(r, r') \right] + \int_0^1 \lambda d\lambda \int_0^1 \lambda d\lambda' \left[ \Delta V + V (r, r') \right] a_1(r, r'), \quad (3.26)$$

where $a_1(r, r')$ is the first-order heat-kernel coefficient.

For spherically symmetric cases, substituting Eq. (3.15) into Eq. (3.26) gives

$$a_2(r, r') = -\int_0^1 \lambda d\lambda \int_0^1 \lambda d\lambda' \left[ (-\nabla^2_{r(r)} + V) a_1(r, r') \right]$$

$$+ \int_0^1 \lambda d\lambda \int_0^1 \lambda d\lambda' \left[ \Delta V + V (r, r') \right] a_1(r, r'), \quad (3.27)$$

Fourier transforming $V(|\lambda r + (1 - \lambda) r'|)$ and $V(r(r))$,

$$V(|\lambda r + (1 - \lambda) r'|) = \frac{1}{(2\pi)^3} \int d^3 p V(|p|) e^{ip(\lambda r + (1 - \lambda) r'|)}$$

$$V(r(r)) = \frac{1}{(2\pi)^3} \int d^3 q V(|q|) e^{iq r(r)}, \quad (3.28)$$

we have

$$a_2(r, r') = -\int_0^1 \lambda d\lambda \int_0^1 \lambda d\lambda' \frac{1}{(2\pi)^3} \int d^3 p V(|p|) \left( -\lambda^2 p^2 \right) e^{ip(\lambda r + (1 - \lambda) r')}$$

$$+ \int_0^1 \lambda d\lambda \int_0^1 \lambda d\lambda' \frac{1}{(2\pi)^3} \int d^3 q V(|q|) e^{iq r(r)} e^{ip(\lambda r + (1 - \lambda) r')}.$$

The angle integrals in Eq. (3.31) can be worked out directly,

$$a_2(r, r') = -\int_0^1 \lambda d\lambda \int_0^1 \lambda d\lambda' \frac{1}{(2\pi)^3} \int_0^\infty p^2 dp V(p) \left( -\lambda^2 p^2 \right) \frac{2 \sin (p |\lambda r + (1 - \lambda) r'|)}{p |\lambda r + (1 - \lambda) r'|}$$

$$+ \int_0^1 \lambda d\lambda \int_0^1 \lambda d\lambda' \frac{1}{(2\pi)^3} \int_0^\infty q^2 dq \int_0^\infty p^2 dp V(q) V(p) \frac{2 \sin (q |r(r)|)}{q |r(r)|} \frac{2 \sin (p |\lambda r + (1 - \lambda) r'|)}{p |\lambda r + (1 - \lambda) r'|}.$$  

Notice that only the radial diagonal heat kernel is needed in our case. For the radial diagonal case, we have

$$|r(r)|_{r'=r} = |\lambda r + (1 - \lambda) r'|_{r'=r}$$

$$= \sqrt{\lambda^2 r^2 + (1 - \lambda)^2 r'^2 + 2\lambda (1 - \lambda) rr' \cos \gamma}$$

$$= r \sqrt{1 - 2\lambda (1 - \lambda) (1 - \cos \gamma)} \quad (3.33)$$
and
\[ |\lambda_1 r (\lambda) + (1 - \lambda_1) r'|_{r'=r} = |\lambda_1 [\lambda r + (1 - \lambda) r'] + (1 - \lambda_1) r'|_{r'=r} \]
\[ = \sqrt{\lambda_1^2 r^2 + (1 - \lambda \lambda_1)^2 r'^2} + 2 \lambda_1 \lambda (1 - \lambda \lambda_1) \frac{rr' \cos \gamma}{2} \]
\[ = r \sqrt{1 - 2 \lambda \lambda_1 (1 - \lambda \lambda_1) (1 - \cos \gamma)}. \quad (3.34) \]

Then, the second-order radial diagonal heat-kernel coefficient becomes
\[ a_2 (r, \gamma) = a_{2-1} (r, \gamma) + a_{2-2} (r, \gamma) \quad (3.35) \]

with
\[ a_{2-1} (r, \gamma) = - \int_0^1 \lambda d\lambda \int_0^1 d\lambda_1 \frac{1}{2\pi} \int_0^\infty p^2 dpV (p) \left( -\frac{\lambda_1 p^2}{pr} \right) 2 \sin \left( pr \sqrt{1 - 2 \lambda \lambda_1 (1 - \lambda \lambda_1) (1 - x)} \right), \]
\[ a_{2-2} (r, \gamma) = \int_0^1 \lambda d\lambda \int_0^1 d\lambda_1 \frac{1}{2\pi} \int_0^\infty q^2 dq \int_0^\infty p^2 dpV (q) V (p) \]
\[ \times 2 \sin \left( qr \sqrt{1 - 2 \lambda (1 - \lambda) (1 - x)} \right) \frac{2 \sin \left( pr \sqrt{1 - 2 \lambda \lambda_1 (1 - \lambda \lambda_1) (1 - x)} \right)}{qr \sqrt{1 - 2 \lambda (1 - \lambda) (1 - x)}}, \quad (3.36) \]
\[ \quad \times \frac{2 \sin \left( qr \sqrt{1 - 2 \lambda (1 - \lambda) (1 - x)} \right) 2 \sin \left( pr \sqrt{1 - 2 \lambda \lambda_1 (1 - \lambda \lambda_1) (1 - x)} \right)}{qr \sqrt{1 - 2 \lambda (1 - \lambda) (1 - x)}}, \quad (3.37) \]

where \( x = \cos \gamma \).

Expanding \( a_{2-1} (r, \gamma) \) around \( x = 1 \), we have
\[ a_{2-1} (r, \gamma) = a_{2-1}^{(1)} (r, \gamma) + a_{2-1}^{(2)} (r, \gamma) + \cdots \quad (3.39) \]

with
\[ a_{2-1}^{(1)} (r, \gamma) = - \int_0^1 \lambda d\lambda \int_0^1 \lambda_1^2 d\lambda_1 \frac{1}{2\pi^{1/2}} \int_0^\infty p^2 dpV (p) \left( -p^2 \right) \frac{2 \sin pr}{pr}, \]
\[ a_{2-1}^{(2)} (r, \gamma) = \int_0^1 \lambda d\lambda \int_0^1 \lambda_1 \frac{1}{2\pi^{1/2}} \int_0^\infty p^2 dpV (p) \left( -\lambda_1 p^2 \right) 2 \lambda \lambda_1 (1 - \lambda \lambda_1) \left( \cos pr - \frac{\sin pr}{pr} \right). \quad (3.40) \]

Performing the integrals with respect to \( \lambda \) and \( \lambda_1 \) in \( a_{2-1}^{(1)} (r, \gamma) \) gives
\[ a_{2-1}^{(1)} (r, \gamma) = - \frac{1}{6} \frac{1}{(2\pi)^{1/2}} \int_0^\infty p^2 dpV (p) \left( -p^2 \right) \frac{2 \sin pr}{pr} \]
\[ = - \frac{1}{6} \nabla_r^2 V (r), \quad (3.42) \]

where \( \nabla_r^2 = - \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) \). Performing the integrals with respect to \( \lambda \) and \( \lambda_1 \) in \( a_{2-1}^{(2)} (r, \gamma) \) gives
\[ a_{2-1}^{(2)} (r, \gamma) = \frac{1}{15} (1 - x) \frac{1}{(2\pi)^{1/2}} \int_0^\infty p^2 dpV (p) \left( -p^2 \right) \left( \cos pr - \frac{\sin pr}{pr} \right). \quad (3.43) \]
By use of the derivative representation (3.21), $a_{2-1}^{(2)} (r, \gamma)$ can be rewritten as

$$a_{2-1}^{(2)} (r, \gamma) = \frac{1}{15} r \frac{d}{dr} (1 - x) \left( \frac{1}{(2\pi)^{1/2}} \int_0^\infty dp V (p) \left( -p^2 \frac{\sin pr}{pr} \right) \right)$$

$$= \frac{1}{30} r \frac{d}{dr} (1 - x) \left( \frac{1}{(2\pi)^{1/2}} \int_0^\infty dp V (p) \left( 2 \frac{\sin pr}{pr} \right) \right)$$

$$= \frac{1}{30} (1 - x) r \frac{d}{dr} \nabla_r^2 V (r). \quad (3.44)$$

Substituting Eqs. (3.42) and (3.44) into Eq. (3.39), we have

$$a_{2-1} (r, \gamma) = -\frac{1}{6} \nabla_r^2 V (r) + \frac{1}{30} (1 - x) r \frac{d}{dr} \nabla_r^2 V (r) + \cdots. \quad (3.45)$$

Similarly, expanding $a_{2-2} (r, \gamma)$ around $x = 1$, we have

$$a_{2-2} (r, \gamma) = a_{2-2}^{(1)} (r, \gamma) + a_{2-2}^{(2)} (r, \gamma) + \cdots, \quad (3.46)$$

where

$$a_{2-2}^{(1)} (r, \gamma) = \int_0^1 d\lambda \int_0^1 d\lambda_1 \frac{1}{2\pi} \int_0^\infty q^2 dq \int_0^\infty p^2 dp V (q) V (p) \frac{2 \sin qr}{qr} \frac{2 \sin pr}{pr},$$

$$a_{2-2}^{(2)} (r, \gamma) = -\int_0^1 d\lambda \int_0^1 d\lambda_1 \frac{1}{2\pi} \int_0^\infty q^2 dq \int_0^\infty p^2 dp V (q) V (p)$$

$$\times 2\lambda (1 - \lambda) (1 - x) \left( \cos pr - \frac{\sin pr}{pr} \right) \frac{2 \sin pr}{pr}$$

$$-\int_0^1 d\lambda \int_0^1 d\lambda_1 \frac{1}{2\pi} \int_0^\infty q^2 dq \int_0^\infty p^2 dp V (q) V (p)$$

$$\times \frac{2 \sin qr}{qr} 2 \lambda_1 (1 - \lambda_1) (1 - x) \left( \cos pr - \frac{\sin pr}{pr} \right) + \cdots. \quad (3.47)$$

Performing the integrals with respect to $\lambda$ and $\lambda_1$ in $a_{2-2}^{(2)} (r, \gamma)$ gives

$$a_{2-2}^{(1)} (r, \gamma) = \frac{1}{2} \frac{1}{(2\pi)^{1/2}} \int_0^\infty q^2 dq V (q) \frac{2 \sin qr}{qr} \frac{1}{(2\pi)^{1/2}} \int_0^\infty p^2 dp V (p) \frac{2 \sin pr}{pr}$$

$$= \frac{1}{2} V^2 (r). \quad (3.48)$$

Performing the integrals with respect to $\lambda$ and $\lambda_1$ in $a_{2-2}^{(2)} (r, \gamma)$ gives

$$a_{2-2}^{(2)} (r, \gamma) = -\frac{1}{6} (1 - x) \frac{1}{2\pi} \int_0^\infty q^2 dq V (q) \left( \cos qr - \frac{\sin qr}{qr} \right) \int_0^\infty p^2 dp V (p) \frac{2 \sin pr}{pr}$$

$$-\frac{1}{15} (1 - x) \frac{1}{2\pi} \int_0^\infty q^2 dq V (q) \frac{2 \sin qr}{qr} \int_0^\infty p^2 dp V (p) \left( \cos pr - \frac{\sin pr}{pr} \right) + \cdots. \quad (3.49)$$

By use of the derivative representation (3.21), $a_{2-2}^{(2)} (r, \gamma)$ can be rewritten as

$$a_{2-2}^{(2)} (r, \gamma) = -\frac{7}{60} (1 - x) V (r) \frac{dV (r)}{dr} + \cdots. \quad (3.50)$$
Substituting Eqs. (3.48) and (3.50) into Eq. (3.46), we have

$$a_{2-2} (r, \gamma) = \frac{1}{2} V^2 (r) - \frac{7}{60} (1 - x) V (r) r \frac{dV (r)}{dr} + \cdots. \quad (3.51)$$

Then, the second-order radial diagonal heat-kernel coefficient, by substituting Eqs. (3.45) and (3.51) into Eq. (3.35), reads

$$a_2 (r, \gamma) = \frac{1}{2} V^2 (r) - \frac{1}{6} \nabla_r^2 V (r) + (1 - x) \left[ \frac{1}{30} r \frac{d}{dr} \nabla_r^2 V (r) - \frac{7}{60} V (r) r \frac{dV (r)}{dr} \right] + \cdots. \quad (3.52)$$

**Second-order phase shift:** The second-order phase shift $\delta^{(2)}_l (k)$ can be obtained by substituting the second-order radial diagonal heat-kernel coefficient, Eq. (3.52), into Eq. (3.14):

$$\delta^{(2)}_l (k) = \frac{1}{4k} \int_0^\infty r^2 dr \int_{-1}^1 dx P_l (x) \cos \left( \sqrt{2kr} \sqrt{1 - x} \right) \left[ \frac{1}{2} V^2 (r) - \frac{1}{6} \nabla_r^2 V (r) \right]$$

$$+ \frac{1}{4k} \int_0^\infty r^2 dr \int_{-1}^1 dx P_l (x) \cos \left( \sqrt{2kr} \sqrt{1 - x} \right) (1 - x) \left[ \frac{1}{30} r \frac{d}{dr} \nabla_r^2 V (r) - \frac{7}{60} V (r) r \frac{dV (r)}{dr} \right] + \cdots. \quad (3.53)$$

By constructing the derivative representations

$$\cos \left( \sqrt{2kr} \sqrt{1 - x} \right) = \frac{\pi}{r} \frac{d}{dr} \frac{\sin \left( \sqrt{2kr} \sqrt{1 - x} \right)}{\sqrt{2\pi \sqrt{1 - x}}}, \quad (3.54)$$

and

$$(1 - x) \cos \left( \sqrt{2kr} \sqrt{1 - x} \right) = - \frac{\pi}{2r^3} \frac{d^3}{dk^3} \frac{\sin \left( \sqrt{2kr} \sqrt{1 - x} \right)}{\sqrt{2\pi \sqrt{1 - x}}}, \quad (3.55)$$

we have

$$\delta^{(2)}_l (k) = \frac{1}{8k} \int_0^\infty rdr \frac{d}{dk} \int_{-1}^1 dx P_l (x) \frac{\sin \left( \sqrt{2kr} \sqrt{1 - x} \right)}{\sqrt{2\sqrt{1 - x}}} \left[ V^2 (r) - \frac{1}{3} \nabla_r^2 V (r) \right]$$

$$- \frac{1}{240k} \int_0^\infty dr \frac{d^3}{dk^3} \int_{-1}^1 dx P_l (x) \frac{\sin \left( \sqrt{2kr} \sqrt{1 - x} \right)}{\sqrt{2\sqrt{1 - x}}} \left[ \frac{d}{dr} \nabla_r^2 V (r) - \frac{7}{2} V (r) r \frac{dV (r)}{dr} \right]. \quad (3.56)$$

Then, by using Eq. (3.25), we arrive at Eq. (3.2).

**4 Heat-kernel approach for phase shift: covariant perturbation theory**

In this section, based on the heat-kernel expansion given by the covariant perturbation theory [10–12], by the relation between partial-wave phase shifts and heat kernels given by Eq. (2.1), we establish an expansion for scattering phase shifts.

*The covariant perturbation theory type expansion for a partial-wave scattering phase shift is $\delta_l (k) = \delta^{(1)}_l (k) + \delta^{(2)}_l (k) + \cdots$ with*

$$\delta^{(1)}_l (k) = - \frac{\pi}{2} \int_0^\infty rdr V (r) J^2_{l+1/2} (kr), \quad (4.1)$$

$$\delta^{(2)}_l (k) = - \frac{\pi^2}{2} \int_0^\infty rdr J_{l+1/2} (kr) Y_{l+1/2} (kr) V (r) \int_0^r r' dr' J^2_{l+1/2} (kr') V (r'), \quad (4.2)$$
where $Y_\nu(z)$ is the Bessel function of the second kind [25].

A detailed calculation is as follows.

### 4.1 Covariant perturbation theory for heat-kernel expansion

The heat-kernel expansion is systematically studied in the covariant perturbation theory [10–12]. The heat-kernel expansion given by the covariant perturbation theory reads [4, 6]

$$K(t; r, r') = K^{(0)}(t; r, r') + K^{(1)}(t; r, r') + K^{(2)}(t; r, r') + \cdots$$

where

$$K^{(0)}(t; r, r') = \langle r | e^{-H_0 t} | r' \rangle = \frac{1}{(4\pi t)^{3/2}} e^{-|r-r'|^2/(4t)}$$

is the zero-order (free) heat kernel. Substituting the zero-order heat kernel (4.4) into Eq. (4.3), we obtain the first two orders of a heat kernel,

$$K^{(1)}(t; r, r') = \langle r | (-t) \int_0^\infty d\alpha_1 d\alpha_2 \delta(1 - \alpha_1 - \alpha_2) e^{-\alpha_1 H_0 t} V e^{-\alpha_2 H_0 t} | r' \rangle$$

and

$$K^{(2)}(t; r, r') = \langle r | (-t)^2 \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) e^{-\alpha_1 H_0 t} V e^{-\alpha_2 H_0 t} V e^{-\alpha_3 H_0 t} | r' \rangle$$

For the spherical potentials $V(r) = V(r)$, $K^{(1)}(t; r, r')$ and $K^{(2)}(t; r, r')$ given by Eqs. (4.5) and (4.6) become

$$K^{(1)}(t; r, r', \gamma) = -\int_0^\infty y^2 d\gamma d\Omega_y V(y) \int_0^t d\tau \exp \left(-\frac{1}{4\pi(t-\tau)} \left(r^2 + y^2 - 2ry \cos \gamma_{ry} \right) \right)$$

$$\times \exp \left(-\frac{1}{4\pi(t-\tau')} \left(r'^2 + y^2 - 2r'y \cos \gamma_{r'y} \right) \right)$$

(4.7)
Appendix A: 

where $\gamma$ is the angle between $r$ and $r'$, $\gamma_{ry}$ is the angle between $r$ and $y$, $\gamma_{r'y}$ is the angle between $r'$ and $y$, $\gamma_{yz}$ is the angle between $y$ and $z$, and $\gamma_{zar'}$ is the angle between $z$ and $r'$.

4.2 First-order phase shift $\delta_1^{(1)} (k)$

In this section, we calculate the first-order phase shift in the frame of the covariant perturbation theory.

The first-order phase shift $\delta_1^{(1)} (k)$ can be obtained by substituting the first-order heat kernel given by the covariant perturbation theory, Eq. (4.7), into the relation between partial-wave phase shifts and heat kernels, Eq. (2.1), and taking $r' = r$ (radial diagonal):

$$
\delta_1^{(1)} (k) = -2\pi^2 \int_0^\infty r^2 dr \int_0^{\pi \infty} d\gamma \frac{1}{2\pi i} \int_0^t dt \int_0^\infty y^2 dy \frac{1}{\tau} \exp \left( -\frac{r^2 + y^2}{4\pi (t - \tau)} \right) \exp \left( -\frac{r^2 + y^2}{4\pi \tau} \right) I_1,
$$

where $I_1$ is an integral with respect to the angle,

$$
I_1 = \int_{-1}^1 d\gamma \sin \gamma P_l (\cos \gamma) \int d\Omega_y \exp \left( \frac{ry}{2\tau (t - \tau)} \cos \gamma_{ry} \right) \exp \left( \frac{ry}{2\tau} \cos \gamma_{r'y} \right).
$$

To calculate $I_1$, we use the expansion [25]

$$
e^{iz \cos a} = \sum_{l=0}^\infty (2l + 1) i^l \sqrt{\frac{\pi}{2z}} J_{l + 1/2} (z) P_l (\cos a)
$$

(4.11)

to rewrite $I_1$ as

$$
I_1 = \sum_{l_1=0}^{\infty} (2l_1 + 1) i^{l_1} j_{l_1} \left( \frac{ry}{\tau (t - \tau)} \right) \sum_{l_2=0}^{\infty} (2l_2 + 1) i^{l_2} j_{l_2} \left( \frac{ry}{\tau \tau} \right)
$$

$$
\times \int_{-1}^1 d\cos \gamma P_l (\cos \gamma) \int d\Omega_y P_{l_1} (\cos \gamma_{ry}) P_{l_2} (\cos \gamma_{r'y}),
$$

(4.12)

where $j_{l'} (z) = \sqrt{\pi/ (2z)} J_{l' + 1/2} (z)$ is the spherical Bessel function of the first kind. Without loss of generality, we choose $r' = (r', 0, 0)$ and then $\gamma_{r'y} = \theta_y$ and $\gamma = \theta_r$. Now, the integral with respect to $\Omega_y$ can be worked out directly by using the integral formula (A.1) given in Appendix A:

$$
\int d\Omega_y P_{l_1} (\cos \gamma_{ry}) P_{l_2} (\cos \gamma_{r'y}) = \int d\Omega_y P_1 (\cos \gamma_{ry}) P_2 (\cos \theta_y) = P_1 (\cos \theta_r) \frac{4\pi}{2l_1 + 1} \delta_{l_1, l_2}.
$$

(4.13)
The integral with respect to $\gamma (= \theta_r)$, then, can also be worked out by using the orthogonality of the Legendre polynomials $\int_{-1}^{1} dx P_k (x) P_l (x) = 2 / (2^l + 1) \delta_{kl}$ [25]:

$$
\int_{-1}^{1} d \cos \theta_r P_l (\cos \theta_r) P_1 (\cos \theta_r) \frac{4 \pi}{2 l_1 + 1} \delta_{l_1, l_2} = \frac{8 \pi}{(2 l + 1)^2} \delta_{l, l_1} \delta_{l_1, l_2}.
$$  (4.14)

By Eqs. (4.13) and (4.14), we achieve

$$
\mathcal{I}_1 = \sum_{l_1=0}^{2l+1} (2 l_1 + 1) j_1(l_1) \left( \frac{r y}{i 2 (t - \tau)} \right) \sum_{l_2=0}^{2l+1} (2 l_2 + 1) j_1(l_2) \left( \frac{r y}{i 2 \tau} \right) \frac{8 \pi}{(2 l + 1)^2} \delta_{l, l_1} \delta_{l_1, l_2}
= 8 \pi r^2 \int_{i} j_1 \left( \frac{r y}{i 2 (t - \tau)} \right) j_1 \left( \frac{r y}{i 2 \tau} \right) \cdot
$$  (4.15)

Substituting Eq. (4.15) into Eq. (4.9) gives

$$
\delta_t^{(1)} (k) = -2 \pi^2 \frac{1}{2 \pi t} \int_{c-i \infty}^{c+i \infty} dt \frac{e^{k^2 t}}{t} \int_{0}^{t} d \tau \int_{0}^{\infty} y^2 dy V (y) \mathcal{I}_2,
$$  (4.16)

where

$$
\mathcal{I}_2 = 8 \pi r^2 \int_{0}^{\infty} r^2 dr \exp \left( \frac{r^2 + y^2}{4 (t - \tau)} \right) \exp \left( \frac{-r^2 + y^2}{4 \tau} \right) j_1 \left( \frac{r y}{i 2 (t - \tau)} \right) j_1 \left( \frac{r y}{i 2 \tau} \right). \quad (4.17)

To calculate the integral $\mathcal{I}_2$, we use the integral representation, Eq. (B.1), given in Appendix B to represent the factor $j_1 \left( \frac{r y}{i 2 (t - \tau)} \right) j_1 \left( \frac{r y}{i 2 \tau} \right)$ as

$$
\int_{-\infty}^{\infty} \sin \theta \left( \left( \frac{r y}{i 2 (t - \tau)} \right)^2 + \left( \frac{r y}{i 2 \tau} \right)^2 \right) \frac{d \theta}{\sqrt{\left[ \left( \frac{r y}{i 2 (t - \tau)} \right)^2 + \left( \frac{r y}{i 2 \tau} \right)^2 \right]^3}}.
$$  (4.18)

Substituting the integral representation (4.18) into Eq. (4.17) and working out the integral give

$$
\mathcal{I}_2 = 4 \pi r^2 \int_{-1}^{1} d \cos \theta P_1 (\cos \theta) \int_{0}^{\infty} r^2 dr \frac{\exp \left( \frac{-r^2 + y^2}{4 (t - \tau)} \right)}{\sqrt{\left[ \left( \frac{r y}{i 2 (t - \tau)} \right)^2 + \left( \frac{r y}{i 2 \tau} \right)^2 \right]^3}} \frac{\exp \left( \frac{r^2 + y^2}{4 \tau} \right)}{(4 \pi \tau)^{3/2}}
\times \sin \theta \left( \left( \frac{r y}{i 2 (t - \tau)} \right)^2 + \left( \frac{r y}{i 2 \tau} \right)^2 \right) \frac{d \theta}{\sqrt{\left[ \left( \frac{r y}{i 2 (t - \tau)} \right)^2 + \left( \frac{r y}{i 2 \tau} \right)^2 \right]^3}}
= \frac{4 \pi r^2}{(4 \pi \tau)^{3/2}} \int_{-1}^{1} d \cos \theta P_1 (\cos \theta) \left( \frac{-y^2}{2 \tau} \right) \exp \left( \frac{-y^2 \cos \theta}{2 \tau} \right). \quad (4.19)

Substituting Eq. (4.19) into Eq. (4.16) and performing the integral with respect to $\tau$, we have

$$
\delta_t^{(1)} (k) = -\frac{\sqrt{\pi}}{4} \int_{0}^{\infty} y^2 dy V (y) \left( \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} dt \frac{e^{k^2 t}}{\sqrt{t}} \int_{0}^{1} d \cos \theta P_1 (\cos \theta) \exp \left( \frac{-y^2 \cos \theta}{2 \tau} \right) \right). \quad (4.20)
$$
Using the expansion \( \exp\left(-y^2 \cos \theta / (2t)\right) = \sum_{l=0}^{\infty} (2l + 1) i^l j_l \left(i \frac{y^2}{2t}\right) P_l (\cos \theta) \) (see Eq. (4.11)) and the orthogonality of the Legendre polynomials, we can work out the integral:

\[
\int_{-1}^{1} d \cos \theta P_l (\cos \theta) \exp \left(-\frac{y^2 \cos \theta}{2t}\right) = \sum_{l'=0}^{\infty} (2l' + 1) i^{l'} j_{l'} \left(i \frac{y^2}{2t}\right) \int_{-1}^{1} d \cos \theta P_l (\cos \theta) P_{l'} (\cos \theta) = 2^{l' + 1} i^{l'} \frac{\sqrt{\pi} l'!}{y^{l'+1/2}} I_{l+1/2} \left(\frac{y^2}{2t}\right),
\]

where \( I_{l'} (z) \) is the modified Bessel function of the first kind and the relation \( j_{2l+1} (z) = \sqrt{\pi} (2l)! I_{l+1/2} (z) \) is used. Substituting Eq. (4.21) into Eq. (4.20), we have

\[
\delta^{(1)}_1 (k) = -\frac{\pi}{2} \int_0^\infty y dy V(y) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \frac{e^{kt}}{t} e^{-y^2/(2t)} I_{l+1/2} \left(\frac{y^2}{2t}\right).
\]

Finally, by performing the inverse Laplace transformation in Eq. (4.22),

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \frac{e^{kt}}{t} e^{-y^2/(2t)} I_{l+1/2} \left(\frac{y^2}{2t}\right) = j_{l+1/2} (kr),
\]

the first-order phase shift given by the covariant perturbation theory, Eq. (4.1), is obtained.

### 4.3 Second-order phase shift \( \delta^{(2)}_1 (k) \)

In this section, we calculate the second-order phase shift in the frame of the covariant perturbation theory.

The second-order phase shift \( \delta^{(2)}_1 (k) \) can be obtained by substituting the second-order heat kernel given by the covariant perturbation theory, Eq. (4.8), into the relation between partial-wave phases and heat kernels, Eq. (2.1), and taking \( r' = r \):

\[
\delta^{(2)}_1 (k) = 2\pi^2 \int_0^\infty r^2 dr \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \frac{e^{kt}}{t} \int_0^t d\tau \int_0^\tau d\tau' \times \int_0^\infty y^2 dy V(y) \int_0^\infty z^2 dz V(z) \frac{\exp \left(-\frac{r^2 + z^2}{4(t-\tau)}\right)}{[4\pi (t-\tau)]^{3/2}} \frac{\exp \left(-\frac{r^2}{4\tau}\right)}{4\pi \tau^{3/2}} \frac{\exp \left(-\frac{z^2}{4\tau'}\right)}{4\pi \tau'^{3/2}} \mathcal{I}_3,
\]

where

\[
\mathcal{I}_3 = \int_{-1}^{1} d \cos \gamma P_l (\cos \gamma) \int d\Omega_y \int d\Omega_z \exp \left(\frac{r y \cos \gamma \gamma y}{2 (t-\tau)}\right) \exp \left(\frac{y z \cos \gamma \gamma z}{2 (\tau-\tau')}\right) \exp \left(\frac{z r \cos \gamma \gamma z}{2\tau'}\right).
\]

Using Eq. (4.11), we rewrite the integral \( \mathcal{I}_3 \) as

\[
\mathcal{I}_3 = \sum_{l_1=0}^{\infty} (2l_1 + 1) i^{l_1} j_{l_1} \left(\frac{r y}{2 (t-\tau)}\right) \sum_{l_2=0}^{\infty} (2l_2 + 1) i^{l_2} j_{l_2} \left(\frac{y z}{2 (\tau-\tau')}\right) \sum_{l_3=0}^{\infty} (2l_3 + 1) i^{l_3} j_{l_3} \left(\frac{z r}{2\tau'}\right)
\times \int_{-1}^{1} d \cos \gamma P_l (\cos \gamma) \int d\Omega_y \int d\Omega_z P_{l_1} (\cos \gamma y) P_{l_2} (\cos \gamma z) P_{l_3} (\cos \gamma z) \cdot
\]

\[
\]

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Without loss of generality, we choose \( r' = (r', 0, 0) \) and then we have \( \gamma_{rr'} = \theta_z \). The integral with respect to \( \Omega_z \) can be then worked out by use of the integral formula, Eq. (A.1), given in Appendix A:

\[
\int d\Omega_z P_{l_2} (\cos \gamma_{yz}) P_{l_3} (\cos \gamma_{zr'}) = \int d\Omega_z P_{l_2} (\cos \gamma_{yz}) P_{l_3} (\cos \theta_z) = P_{l_2} (\cos \theta_y) \frac{4\pi}{2l_2 + 1} \delta_{l_2,l_3},
\]

The integral with respect to \( \Omega_y \) also can be integrated directly by Eq. (A.1), given in Appendix B, we rewrite (4.27)

\[
\int d\Omega_y P_{l_1} (\cos \gamma_{ry}) P_{l_2} (\cos \theta_y) = P_{l_1} (\cos \theta_r) \frac{4\pi}{2l_1 + 1} \delta_{l_1,l_2} \frac{4\pi}{2l_2 + 1} \delta_{l_2,l_3}.
\]

Then, performing the integral with respect to \( \gamma \) (\( \gamma = \theta_r \) when \( r' = (r', 0, 0) \)) in Eq. (4.26), we have

\[
\int_{-1}^{1} d\cos \theta_r P_{l} (\cos \theta_r) P_{l_1} (\cos \theta_r) = \frac{32\pi^2}{(2l+1)^3} \delta_{l,l_1} \delta_{l_1,l_2} \delta_{l_2,l_3},
\]

By Eqs. (4.27), (4.28), and (4.29), we have

\[
I_3 = 32\pi^2 i^3 j_l \left( \frac{ry}{i2(t-\tau)} \right) j_l \left( \frac{yz}{i2(t-\tau)} \right) j_l \left( \frac{zr}{i2(\tau')} \right).
\]

Substituting Eq. (4.30) into Eq. (4.24) gives

\[
\delta^{(2)}_l (k) = 64\pi^4 i^3 l \int_{-i\infty}^{i\infty} \frac{dt}{2\pi i} e^{klt} \int_{0}^{t} d\tau \int_{0}^{t} d\tau' \times \int_{0}^{\infty} y^2 dy V (y) \int_{0}^{\infty} z^2 dz V (z) \exp \left( -\frac{y^2 + z^2}{4(\tau - \tau')} \right) \frac{\exp \left( -\frac{r^2 + z^2}{4(\tau - \tau')} \right)}{[4\pi \sqrt{(t - \tau)}]^{3/2}} j_l \left( \frac{ry}{i2(t-\tau)} \right) \frac{\exp \left( -\frac{r^2 + z^2}{4(\tau')} \right)}{[4\pi \sqrt{(t - \tau')}^{3/2}} j_l \left( \frac{zr}{i2(\tau')} \right).
\]

To perform the integral with respect to \( r \), by using the integral representation (B.1) given in Appendix B, we rewrite

\[
\frac{1}{2} \int_{-1}^{1} d\cos \theta \frac{\sin \sqrt{\left( \frac{ry}{i2(t-\tau)} \right)^2 + \left( \frac{zr}{i2(\tau')} \right)^2 - 2 \frac{ry}{i2(t-\tau)} \frac{zr}{i2(\tau')} \cos \theta}}{\sqrt{\left( \frac{ry}{i2(t-\tau)} \right)^2 + \left( \frac{zr}{i2(\tau')} \right)^2 - 2 \frac{ry}{i2(t-\tau)} \frac{zr}{i2(\tau')} \cos \theta}} P_{l} (\cos \theta).
\]
Then, the integral with respect to $r$ can be worked out,

$$
\int_0^\infty r^2 dr \exp \left( -\frac{r^2 + y^2}{4(t-\tau)} \right) \exp \left( -\frac{z^2 + y^2}{4(t-\tau)} \right) j_t \left( \frac{ry}{\sqrt{2(t-\tau)}} \right) j_t \left( \frac{zy}{\sqrt{2(t-\tau)}} \right) = \frac{1}{2} \int_{-1}^1 \frac{d cos \theta P_t(\cos \theta)}{[4\pi (t-\tau)]^{3/2}} \int_0^\infty r^2 dr \exp \left( -\frac{r^2 + y^2}{4(t-\tau)} \right) \exp \left( -\frac{z^2 + y^2}{4(t-\tau)} \right) \sin \sqrt{\left[ \frac{ry}{\sqrt{2(t-\tau)}} \right]^2 + \left( \frac{zy}{\sqrt{2(t-\tau)}} \right)^2 - 2 \frac{ry}{\sqrt{2(t-\tau)}} \frac{zy}{\sqrt{2(t-\tau)}} \cos \theta} \times \sqrt{\left[ \frac{ry}{\sqrt{2(t-\tau)}} \right]^2 + \left( \frac{zy}{\sqrt{2(t-\tau)}} \right)^2 - 2 \frac{ry}{\sqrt{2(t-\tau)}} \frac{zy}{\sqrt{2(t-\tau)}} \cos \theta} = \frac{1}{8\pi} \int_{-1}^1 \frac{d cos \theta P_t(\cos \theta)}{[4\pi (t-\tau + \tau')]^{3/2}} \exp \left( -\frac{y^2 + z^2 + 2yz \cos \theta}{4(t-\tau + \tau')} \right). \tag{4.33}
$$

Substituting Eq. (4.33) into Eq. (4.31), we have

$$
\delta_l^{(2)}(k) = 8\pi^3 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \frac{e^{kt}}{t} \int_0^\tau d\tau' \int_0^\pi y^2 dy V(y) \int_0^\infty z^2 dz V(z) \exp \left( -\frac{y^2 + z^2}{4(t-\tau')} \right) \frac{\exp \left( -\frac{y^2 + z^2}{4(t-\tau)} \right)}{[4\pi (t-\tau)]^{3/2}} j_t \left( \frac{y}{\sqrt{2(t-\tau)}} \right) \times \frac{\exp \left( -\frac{y^2 + z^2}{4(t-\tau')} \right)}{[4\pi (t-\tau + \tau')]^{3/2}} \int_{-1}^1 d cos \theta P_t(\cos \theta) \exp \left( -\frac{y^2 + z^2 + 2yz \cos \theta}{2(t-\tau + \tau')} \right). \tag{4.34}
$$

Using the expansion (4.11) and the orthogonality of the Legendre polynomials, we have

$$
\int_{-1}^1 d cos \theta P_t(\cos \theta) \exp \left( -\frac{y^2 + z^2 + 2yz \cos \theta}{2(t-\tau + \tau')} \right) = \sum_{l'=0}^{2l+1} (2l'+1) l'^t j_l \left( -\frac{y}{\sqrt{2(t-\tau + \tau')}} \right) \times \int_{-1}^1 d cos \theta P_t(\cos \theta) P_{l'}(\cos \theta) = 2l^t j_l \left( -\frac{y}{\sqrt{2(t-\tau + \tau')}} \right). \tag{4.35}
$$

Substituting Eq. (4.35) into Eq. (4.34) and setting $T = \tau - \tau'$, we have

$$
\delta_l^{(2)}(k) = 16\pi^3 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \frac{e^{kt}}{t} \int_0^\tau d\tau' \int_0^\pi y^2 dy V(y) \int_0^\infty z^2 dz V(z) \exp \left( -\frac{y^2 + z^2}{4(t-\tau')} \right) \exp \left( -\frac{y^2 + z^2}{4(t-\tau)} \right) \frac{\exp \left( -\frac{y^2 + z^2}{4(t-\tau)} \right)}{[4\pi (t-\tau')]^{3/2}} j_t \left( \frac{y}{\sqrt{2T}} \right) \times \frac{\exp \left( -\frac{y^2 + z^2}{4(t-\tau)} \right)}{[4\pi (t-\tau)]^{3/2}} \int_{-1}^1 d cos \theta P_t(\cos \theta) \exp \left( -\frac{y^2 + z^2 + 2yz \cos \theta}{4(t-\tau)} \right). \tag{4.36}
$$

Exchanging the order of integrals $\int_0^t d\tau' \int_{\tau}^t d\tau \rightarrow \int_0^t d\tau' \int_{\tau'}^t d\tau$ and resetting $T = \tau'$ give

$$
\delta_l^{(2)}(k) = 16\pi^3 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \frac{e^{kt}}{t} \int_0^\tau d\tau' \int_0^\pi y^2 dy V(y) \int_0^\infty z^2 dz V(z) \exp \left( -\frac{y^2 + z^2}{4\tau'} \right) \exp \left( -\frac{y^2 + z^2}{4(t-\tau')} \right) \frac{\exp \left( -\frac{y^2 + z^2}{4(t-\tau')} \right)}{[4\pi (t-\tau')]^{3/2}} j_t \left( \frac{y}{\sqrt{2\tau'}} \right) \times \frac{\exp \left( -\frac{y^2 + z^2}{4(t-\tau')} \right)}{[4\pi (t-\tau)]^{3/2}} \int_{-1}^1 d cos \theta P_t(\cos \theta) \exp \left( -\frac{y^2 + z^2 + 2yz \cos \theta}{4(t-\tau)} \right). \tag{4.37}
$$
Integrating with respect to \( \tau \), we have

\[
\delta^{(2)}_l (k) = \frac{1}{4} \int_0^\infty y^2 dy V(y) \int_0^{2\pi} z^2 dz V(z) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dk e^{k^2 t I_4},
\]

where

\[
I_4 = \frac{1}{t} \int_0^t d\tau \frac{\exp\left(-\frac{y^2+z^2}{4(t-\tau)}\right) \exp\left(-\frac{y^2+z^2}{4\tau}\right)}{(t-\tau)^{1/2}} j_l \left(\frac{y z}{\sqrt{t-\tau}}\right) j_l \left(\frac{y z}{\sqrt{t}}\right).
\]

\[
= -8 \int_0^\infty dk e^{-k^2 t} \left\{ \begin{array}{ll}
   k^2 j_l (k y) n_l (k y) j_l^2 (k z), & y > z \\
   k^2 j_l^2 (k y) j_l (k z) n_l (k z), & y < z
\end{array} \right.
\]

where \( n_l (z) \) is the spherical Bessel function of the second kind. Thus, the inverse Laplace transformation of \( I_4 \) can be worked out:

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dk e^{k^2 t I_4} = -4 \left\{ \begin{array}{ll}
   k^2 j_l (k y) n_l (k y) j_l^2 (k z), & y > z \\
   k^2 j_l^2 (k y) j_l (k z) n_l (k z), & y < z
\end{array} \right.
\]

Substituting Eq. (4.40) into Eq. (4.38), we have

\[
\delta^{(2)}_l (k) = -k^2 \int_0^\infty y^2 dy j_l (k y) n_l (k y) V(y) \int_0^y z^2 dz j_l^2 (k z) V(z)

- k^2 \int_0^\infty y^2 dy j_l^2 (k y) V(y) \int_y^\infty z^2 dz j_l (k z) n_l (k z) V(z).
\]

By exchanging the order of integrals, \( \int_0^\infty dy \int_y^\infty dz \to \int_0^\infty dz \int_0^z dy \), we rewrite Eq. (4.41) as

\[
\delta^{(2)}_l (k) = -k^2 \int_0^\infty y^2 dy j_l (k y) n_l (k y) V(y) \int_0^y z^2 dz j_l^2 (k z) V(z)

- k^2 \int_0^\infty z^2 dz j_l (k z) n_l (k z) V(z) \int_0^z y^2 dy j_l^2 (k y) V(y).
\]

Obviously, the two parts in Eq. (4.42) are equal. Using \( j_l (z) = \sqrt{\pi/(2z)} J_{l+1/2} (z) \) and \( n_l (z) = \sqrt{\pi/(2z)} Y_{l+1/2} (z) \) gives Eq. (4.2).

5 Comparison with Born approximation

The approach for scattering problems established in the present paper is not only one single approach. Every method of calculating heat kernels can be converted into a method for scattering problems. As applications, in Secs. 3 and 4, we suggest two methods for scattering phase shifts, based on two heat kernel methods: the Seeley-DeWitt expansion and the covariant perturbation theory.

In scattering theory, there are many approximation methods, such as Born approximation, WKB method, eikonal approximation, and variational method. [27].

In this section, we compare our two methods with the Born approximation.
5.1 Comparison of first-order contribution

For clarity, we relist some results given by the above sections in the following.

The first-order phase shift given by the Seeley-DeWitt expansion given in Sec. 3 reads

\[ \delta^{(1)}_{l}(k)_{\text{Seeley-DeWitt}} = -\frac{\pi}{2} \int_{0}^{\infty} r dr J_{l+1/2}^{2}(kr) V(r) - \frac{\pi}{24} \int_{0}^{\infty} \frac{d^{2} J_{l+1/2}^{2}(kr) dV(r)}{dr} + \cdots . \]  

(5.1)

The first-order phase shift given by the covariant perturbation theory given in Sec. 4 reads

\[ \delta^{(1)}_{l}(k)_{\text{cpt}} = -\frac{\pi}{2} \int_{0}^{\infty} r dr J_{l+1/2}^{2}(kr) V(r) . \]  

(5.2)

As a comparison, the first-order phase shift given by the Born approximation reads [27]

\[ \delta^{(1)}_{l}(k)_{\text{Born}} = \arctan \left[ -\frac{\pi}{2} \int_{0}^{\infty} r dr V(r) J_{l+1/2}^{2}(kr) \right] \approx -\frac{\pi}{2} \int_{0}^{\infty} r dr J_{l+1/2}^{2}(kr) V(r) + \cdots . \]  

(5.3)

Obviously, the leading contributions of these three methods are the same (in the Born approximation, the first-order contribution is in fact \( -\arctan \left[ -\frac{\pi}{2} \int_{0}^{\infty} r dr V(r) J_{l+1/2}^{2}(kr) \right] \)), but the higher contribution can be safely ignored in the first-order contribution).

5.2 Comparison of second-order contribution

The second-order phase shift given by the Seeley-DeWitt expansion given in Sec. 3 reads

\[ \delta^{(2)}_{l}(k)_{\text{Seeley-DeWitt}} = \frac{\pi}{8k} \int_{0}^{\infty} r dr \frac{d J_{l+1/2}^{2}(kr)}{dk} \left[ V^{2}(r) - \frac{1}{3} \nabla^{2} V(r) \right] 
- \frac{\pi}{240k} \int_{0}^{\infty} dr \frac{d^{3} J_{l+1/2}^{2}(kr)}{dk^{3}} \left[ \frac{d}{dr} \nabla^{2} V(r) - \frac{7}{2} V(r) \frac{dV(r)}{dr} \right] + \cdots . \]  

(5.4)

The second-order phase shift given by the covariant perturbation theory given in Sec. 4 (Eqs. (4.2) and (4.42)) reads

\[ \delta^{(2)}_{l}(k)_{\text{cpt}} = -\frac{\pi^{2}}{2} \int_{0}^{\infty} r dr J_{l+1/2}(kr) Y_{l+1/2}(kr) V(r) \int_{0}^{r} r' dr' J_{l+1/2}^{2}(kr') V(r') 
- \frac{\pi^{2}}{4} \int_{0}^{\infty} r dr J_{l+1/2}(kr) Y_{l+1/2}(kr) V(r) \int_{0}^{r} r' dr' J_{l+1/2}^{2}(kr') V(r') 
- \frac{\pi^{2}}{4} \int_{0}^{\infty} r dr J_{l+1/2}^{2}(kr) V(r) \int_{r}^{\infty} r' dr' J_{l+1/2}(kr') Y_{l+1/2}(kr') V(r') . \]  

(5.5)
The second-order phase shift given by the Born approximation \[27\] reads

\[
\delta_l^{(2)}(k)_{\text{Born}} = \arctan \left[ -\frac{\pi^2}{4} \int_0^\infty rdr J_{l+1/2}(kr) Y_{l+1/2}(kr) V(r) \int_0^r r' dr' J_{l+1/2}^2(kr') V(r') \right] 
- \frac{\pi^2}{4} \int_0^\infty rdr J_{l+1/2}^2(kr) V(r) \int_r^\infty r' dr' J_{l+1/2}(kr') V(r') 
\approx -\frac{\pi^2}{4} \int_0^\infty rdr J_{l+1/2}(kr) Y_{l+1/2}(kr) V(r) \int_0^r r' dr' J_{l+1/2}^2(kr') V(r') 
- \frac{\pi^2}{4} \int_0^\infty rdr J_{l+1/2}^2(kr) V(r) \int_r^\infty r' dr' J_{l+1/2}(kr') Y_{l+1/2}(kr') V(r') + \cdots.
\]

(5.6)

It can be directly seen that the leading contribution of the second-order Born approximation and the leading contribution of the second-order covariant perturbation theory are the same. The second-order contribution given by the Seeley-DeWitt expansion is different from the other two methods. This is because the Born approximation and the covariant perturbation theory are both small \(V(r)\) expansions, but the Seeley-DeWitt expansion is a small \(t\) asymptotic expansion.

5.3 Comparison through an exactly solvable potential \(V(r) = \alpha/r^2\)

In this section, we compare the three methods, the Seeley-DeWitt expansion, the covariant perturbation theory method, and the Born approximation, through an exactly solvable potential:

\[
V(r) = \frac{\alpha}{r^2}.
\]

(5.7)

Using these three approximation methods to calculate an exactly solvable potential can help us to compare them intuitively.

The phase shift for the potential (5.7) can be solved exactly,

\[
\delta_l = -\frac{\pi}{2} \left[ \sqrt{\left( l + \frac{1}{2} \right)^2 + \alpha} - \left( l + \frac{1}{2} \right) \right].
\]

(5.8)

In order to compare the methods term by term, we expand the exact result (5.8) as \(\delta_l = \delta_l^{(1)} + \delta_l^{(2)} + \cdots\), where

\[
\delta_l^{(1)} = -\frac{\pi \alpha}{2(2l + 1)},
\]

(5.9)

\[
\delta_l^{(2)} = \frac{\pi \alpha^2}{2(2l + 1)^3}.
\]

(5.10)

First order: The first-order contribution given by the Seeley-DeWitt method, the covariant perturbation theory, and the Born approximation can be directly obtained by sub-
stituting the potential (5.7) into Eqs. (3.1), (4.1), and (5.3), respectively:

\[
\delta_l^{(1)}(k)_{\text{Seeley-DeWitt}} = -\frac{\pi\alpha}{2(2l + 1)} + \frac{4\pi\alpha}{24l^3 + 36l^2 - 78l - 45},
\]
(5.11)

\[
\delta_l^{(1)}(k)_{\text{cpt}} = -\frac{\pi\alpha}{2(2l + 1)},
\]
(5.12)

\[
\delta_l^{(1)}(k)_{\text{Born}} = \arctan\left[-\frac{\pi\alpha}{2(2l + 1)}\right] \simeq -\frac{\pi\alpha}{2(2l + 1)} - \frac{1}{3} \left[-\frac{\pi\alpha}{2(2l + 1)}\right]^3.
\]
(5.13)

Comparing with the direct expansion of the exact solution, Eqs. (5.9) and (5.10), we can see that all results are good approximations, and the result given by the covariant perturbation theory is the best result. It can be checked that the second term in the result given by the Seeley-DeWitt method, Eq. (5.11), is small; the maximum value of the second term appears at \(l = 0\), which is about 17% of the first term.

**Second order:** The second-order contribution given by the Seeley-DeWitt method, the covariant perturbation theory, and the Born approximation can be directly obtained by substituting the potential (5.7) into Eqs. (3.2), (4.2), and (5.6), respectively:

\[
\delta_l^{(2)}(k)_{\text{Seeley-DeWitt}} = \frac{2\pi\alpha(3\alpha - 2)}{24l^3 + 36l^2 - 78l - 45} - \frac{16\pi\alpha(7\alpha - 8)}{160l^5 + 400l^4 - 2800l^3 - 4600l^2 + 7890l + 4725},
\]
(5.14)

\[
\delta_l^{(2)}(k)_{\text{cpt}} = \frac{\pi\alpha^2}{2(2l + 1)^3},
\]
(5.15)

\[
\delta_l^{(2)}(k)_{\text{Born}} = \arctan\left[-\frac{\pi\alpha^2}{2(2l + 1)^3}\right] \simeq -\frac{\pi\alpha^2}{2(2l + 1)^3} - \frac{1}{3} \left[-\frac{\pi\alpha^2}{2(2l + 1)^3}\right]^3.
\]
(5.16)

Comparing with the second-order contribution, Eq. (5.10), we can see that, like that in the case of first-order contributions, the result given by the covariant perturbation theory is the best result. Numerical tests show that the result given by the above three methods are very close to each other.

Based on the above comparison, we can conclude that the method based on the covariant perturbation theory is better than other two methods.

### 6 Calculating global heat kernel from phase shift

The key result of this paper is a relation between partial-wave phase shifts and heat kernels. Besides solving a scattering problem from a known heat kernel, obviously, we can also calculate a heat kernel from a known phase shift. Here, we only give a simple example with the potential \(\alpha/r^2\). A systematic discussion on how to calculate heat kernels and other spectral functions, such as one-loop effective actions, vacuum energies, and spectral counting functions, from a solved scattering problem will be given elsewhere.

For the potential

\[
V(r) = \frac{\alpha}{r^2},
\]
(6.1)
the exact partial-wave phase shift is given by Eq. (5.8),
\[
\delta_l = -\frac{\pi}{2} \left[ \sqrt{l + \frac{1}{2}}^2 + \alpha - \left( l + \frac{1}{2} \right) \right].
\] (6.2)

By the relation between a global heat kernel and a scattering phase shift given by Ref. [14],
\[
K_f^s (t) = \frac{2}{\pi} t \int_0^\infty kdk\delta_l (k)e^{-k^2t} - \frac{\delta_l (0)}{\pi},
\] (6.3)
we can calculate the scattering part of the global heat kernel immediately,
\[
K_f^s (t) = -\frac{1}{2} \left[ \sqrt{\alpha + \left( l + \frac{1}{2} \right)^2} - \left(l + \frac{1}{2}\right) \right].
\] (6.4)

In this case, the bound part of heat kernel \(K^b (t) = 0\) and the free part of heat kernel \(K_f^f (t) = R/\sqrt{4\pi t} - \frac{1}{2} (l + \frac{1}{2})\), where \(R\) is the radius of the system. The global partial-wave heat kernel then reads
\[
K_l (t) = K_f^s (t) + K^b (t) + K_f^f (t)
= \frac{R}{\sqrt{4\pi t}} - \frac{1}{\alpha} \sqrt{\alpha + \left( l + \frac{1}{2} \right)^2}.
\] (6.5)

As a comparison, we calculate the partial-wave heat kernel for \(V(r) = \alpha/r^2\) by another approach.

The partial-wave heat kernel of a free particle, \(K_f^f\), which is the heat kernel of the radial operator \(D^{free} = -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2}\), can be calculated directly:
\[
K_f^f (t; r, r') = \frac{1}{2t\sqrt{rr'}} \exp \left( -\frac{r^2 + r'^2}{4t} \right) I_{l+1/2} \left( \frac{rr'}{2t} \right).
\] (6.6)

By setting \(s = \sqrt{\alpha + (l+1/2)^2} - 1/2\), we can obtain the partial-wave heat kernel of operator \(D_s = -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} + \frac{\alpha}{r^2}\),
\[
K_l (t; r, r') = \frac{1}{2t\sqrt{rr'}} \exp \left( -\frac{r^2 + r'^2}{4t} \right) I_{\sqrt{\alpha+(l+1/2)^2}} \left( \frac{rr'}{2t} \right).
\] (6.7)

Taking trace of \(K_l (t; r, r')\) gives the global partial-wave heat kernel \(K_l (t)\):
\[
K_D_l (t) = \int_0^R r^2 dr K_l (t; r, r')
= \frac{R^{2(1+\eta)}}{(4t)^{1+\eta} \Gamma(2+\eta)} 2F_2 \left( \eta + \frac{1}{2}, \eta + 1; \eta + 2, 2\eta + 1; -\frac{R^2}{t} \right),
\] (6.8)
where \(\eta = \sqrt{(l+1/2)^2 + \alpha}, \ pF_q (a_1, a_2 \cdots a_p; b_1, b_2 \cdots b_q; z)\) is the generalized Hypergeometric function [25]. Expanding \(K_D_l (t)\) at \(R \to \infty\) gives
\[
K_D_l (t) = \frac{R}{\sqrt{4\pi t}} - \frac{1}{2} \eta + \sqrt{t} \left( \alpha + l^2 + 1 \right) + \frac{i}{4\sqrt{\pi}} R e^{-R^2/4\pi t} \ldots.
\] (6.9)

When \(R \to \infty\), it recovers the heat kernel given by Eq. (6.5).
7 Conclusions and outlook

In this paper, based on two quantum field theory methods, the heat kernel method [1] and the scattering spectral method [2], we suggest a method for calculating the scattering phase shift. The method suggested in the present paper is indeed a series of different methods of calculating scattering phase shifts constructed from various heat kernel methods.

The key step is to find a relation between partial-wave phase shifts and heat kernels. This relation allows us to express a partial-wave phase shift by a heat kernel. Then, each method of the calculation of heat kernels can be converted to a method of the calculation of phase shifts.

As applications, we provide two methods for the calculation of phase shifts, corresponding to two heat kernel methods, the Seeley-DeWitt expansion and the covariant perturbation theory.

Furthermore, as emphasized above, by this approach, we can construct various methods for scattering problems with the help of various heat kernel methods. In subsequent works, we shall construct various scattering methods by using various heat-kernel expansions.

In this paper, as a coproduct, we also provide two kinds of off-diagonal heat-kernel expansions, based on the technique developed in the Seeley-DeWitt expansion and the covariant perturbation theory for diagonal heat kernels, since the heat kernel method for scatterings established in the present paper is based on the off-diagonal heat kernel rather than the diagonal heat kernel. It should be emphasized that many methods for calculating diagonal heat kernels can be directly applied to the calculation of off-diagonal heat kernels. That is to say, the method for calculating the diagonal heat kernel often can also be converted to a method for calculating off-diagonal heat kernels and scattering phase shifts, as we have done in the present paper. Therefore, we can construct scattering methods from many methods of diagonal heat kernels, e.g., [1, 9, 28].

The heat kernel theory is well-studied in both mathematics and physics. Here, as examples, we list some methods on the calculation of heat kernel. In Refs. [29–31], the author calculates the heat-kernel coefficient with different boundary conditions. In Ref. [32], using the background field method, the author calculates the fourth and fifth heat-kernel coefficients. In Refs. [33–35], the author calculates the third coefficient by the covariant technique. In Refs. [9, 36], by the string-inspired worldline path-integral method, the authors calculate the first seven heat-kernel coefficients. In Ref. [28], a direct, nonrecursive method for the calculation of heat kernels is presented. In Ref. [37], the first five heat-kernel coefficients for a general Laplace-type operator on a compact Riemannian space without boundary by the index-free notation are given. In Refs. [10–13, 38], a non-Seeley-DeWitt expansion of a heat kernel is established. In Refs. [39–41], a covariant pseudo-differential-operator method for calculating heat-kernel expansions in an arbitrary space dimension is given.

An important application of the method given by this paper is to solve various spectral functions by a scattering method. The problem of spectral functions is an important issue in quantum field theory [15, 42, 43]. A subsequent work on this subject is a systematic discussion on calculating heat kernels, effective actions, vacuum energies, etc., from a known...
phase shift. We will show that, based on scattering methods, we can obtain some new heat-kernel expansions. It is known that though there are many discussions on the high energy heat-kernel expansion, the low-energy expansion of heat kernels is relatively difficult to obtain. While, there are some successful low-energy scattering theories; by using the relation given in this paper, we can directly obtain some low-energy results for heat kernels.

Starting from the result given by the present paper, we can study many problems. The method presented in this paper can be applied to low-dimensional scatterings. One- and two-dimensional scatterings and their applications have been deeply studied, such as the transport property of low-dimensional materials [44–46]. We will also consider a systematic application of our method on relativistic scatterings. The relativistic scattering is an important problem, e.g., the collision of solitons in relativistic scalar field theories [47] and the Dirac scattering in the problem of the electron properties of graphene [48, 49]. We can also apply the method to low-temperature physics. There are many scattering problems in low-temperature physics, such as the scattering in the problem of the transition temperature of BEC [50, 51] and the transport property of spin-polarized fermions in low temperatures [52, 53].

The application of the method to inverse scattering problems is an important subject of our subsequent work. The inverse scattering problem has extremely important significance in physics [54, 55]. In practice, for example, the inverse scattering method can be applied to the problem of BEC [56] and the Aharonov–Bohm effect [57].

In Ref. [42], we provide a method for solving the spectral function, such as one-loop effective actions, vacuum energies, and spectral counting functions in quantum field theory. The key idea is to construct the equations obeyed by these quantities. We show that, for example, the equation of the one-loop effective action is a partial integro-differential equation. By the relation between partial-wave phase shifts and heat kernel, we can also construct an equation obeyed by phase shifts.

Moreover, in conventional scattering theory, an approximately large-distance asymptotics is used to seek an explicit result. In Ref. [58], we show that such an approximate treatment is not necessary; without the large-distance asymptotics, one can still rigorously obtain an explicit result. The result presented in this paper can be directly applied to the scattering theory without large-distance asymptotics.

\[ \int d\Omega' P_l (\cos \gamma) P_{l'} (\cos \theta') \]

In this appendix, we provide an integral formula:

\[ \int d\Omega' P_l (\cos \gamma) P_{l'} (\cos \theta') = \frac{4\pi}{2l + 1} P_l (\cos \theta) \delta_{ll'}, \quad (A.1) \]

where \( \gamma \) is the angle between \( r = (r, \theta, \phi) \) and \( r' = (r', \theta', \phi') \) and \( d\Omega' = \sin \theta' d\theta' d\phi' \).

**Proof.** Using the integral formula [59]

\[ \int d\Omega Y_{i\theta} (\gamma) Y_{i\theta'} (\theta') = \frac{4\pi}{2l + 1} Y_{i\theta} (\theta) \delta_{lw} \quad (A.2) \]
and the relation $Y_{lm}(\theta, \phi) = \sqrt{(2l+1)/(4\pi)} P_l(\cos \theta)$, we have

$$\int d\Omega' Y_{l0}(\gamma) Y_{l'0}(\theta') = \sqrt{\frac{2l+1}{4\pi}} \int d\Omega' P_l(\cos \gamma) P_{l'}(\cos \theta') = P_l(\cos \theta) \delta_{ll'}.$$  \hspace{1cm} (A.3)

This proves Eq. (A.1).

B Integral representation of $j_l(u) j_l(v)$

In this appendix, we provide an integral representation for the product of two spherical Bessel functions $j_l(u) j_l(v)$:

$$j_l(u) j_l(v) = \frac{1}{2} \int_{-1}^{1} d\cos \theta \frac{\sin w}{w} P_l(\cos \theta), \quad (u < v), \hspace{1cm} (B.1)$$

where $w = \sqrt{u^2 + v^2 - 2uv \cos \theta}$.

**Proof.** Using the expansion [60]

$$\frac{\sin w}{w} = \sum_{l=0}^{\infty} (2l + 1) j_l(u) j_l(v) P_l(\cos \theta), \hspace{1cm} (B.2)$$

where $u = |u|$ and $v = |v|$ with $\theta$ as the angle between $u$ and $v$. Multiplying the both sides of (B.2) by $P_{l'}(\cos \theta)$ and integrating from 0 to $\pi$ give

$$\int_{-1}^{1} d\cos \theta \frac{\sin w}{w} P_{l'}(\cos \theta) = \sum_{l=0}^{\infty} \int_{-1}^{1} d\cos \theta (2l + 1) j_l(u) j_l(v) P_l(\cos \theta) P_{l'}(\cos \theta) = 2j_{l'}(u) j_{l'}(v). \hspace{1cm} (B.3)$$

Here, the orthogonality, $\int_{-1}^{1} d\cos \theta P_l(\cos \theta) P_{l'}(\cos \theta) = 2/(2l+1) \delta_{ll'}$, is used. This proves Eq. (B.1).

C Integral representation of $j_l(kr) n_l(kr')$

In this appendix, we provide an integral representation for the product of the spherical Bessel function of the first kind and the spherical Bessel function of the second kind, $j_l(u) n_l(v)$:

$$j_l(u) n_l(v) = -\frac{1}{2} \int_{-1}^{1} d\cos \theta \frac{\cos w}{w} P_l(\cos \theta), \quad (u < v), \hspace{1cm} (C.1)$$

where $w = \sqrt{u^2 + v^2 - 2uv \cos \theta}$.

**Proof.** Using the expansion [60]

$$\frac{\cos w}{w} = -\sum_{\ell=0}^{\infty} (2\ell + 1) j_{l\ell}(u) n_{l\ell}(v) P_l(\cos \theta), \quad u < v, \hspace{1cm} (C.2)$$

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where \( u = |\mathbf{u}| \) and \( v = |\mathbf{v}| \) with \( \theta \) as the angle between \( \mathbf{u} \) and \( \mathbf{v} \). Multiplying on the both sides of (C.2) by \( P_\ell (\cos \theta) \) and integrating from 0 to \( \pi \) give

\[
\int_{-1}^{1} d \cos \theta \frac{\cos w}{w} P_\ell (\cos \theta) = - \sum_{\ell = 0}^{\infty} (2\ell + 1) j_\ell (u) n_\ell (v) \int_{-1}^{1} d \cos \theta P_\ell (\cos \theta) P_\ell (\cos \theta)
= - \sum_{\ell = 0}^{\infty} (2\ell + 1) j_\ell (u) n_\ell (v) \frac{2}{2\ell + 1} \delta_{\ell l}
= -2 j_l (u) n_l (v).
\]

This proves Eq. (C.1).

Acknowledgments

We are very indebted to Dr G. Zeitrauman for his encouragement. This work is supported in part by NSF of China under Grant No. 11075115.

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