Topological Poincaré type inequalities and lower bounds on the infimum of the spectrum for graphs

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We study topological Poincaré type inequalities on general graphs. We characterize graphs satisfying such inequalities and then turn to the best constants in these inequalities. Invoking suitable metrics we can interpret these constants geometrically as diameters and inradii. Moreover, we can relate them to spectral theory of Laplacians once a probability measure on the graph is chosen. More specifically, we obtain a variational characterization of these constants as infimum over spectral gaps of all Laplacians on the graphs associated to probability measures.

Introduction

The Poincaré inequality is a fundamental tool in analysis and geometry. For graphs in the ℓ²-setting it has received much attention not least as it is equivalent to a spectral gap condition and plays a crucial role in investigation of heat kernel estimates, see e.g. [2, 11, 4, 7, 18] and references therein. Here, with notation to be explained later in detail we study Poincaré type inequalities on graphs \((X, b)\) of the form

\[ |\sup f - \inf f|^2 \leq c \mathcal{E}(f) \]  

(TPI)

for all \(f\) belonging to the set \(\mathcal{D}\) of finite energy functions as well as

\[ \sup |f|^2 \leq c \mathcal{E}(f) \]

for all \(f\) of finite energy vanishing somewhere. Note that there is no measure involved in these quantities so we think of these inequalities as topological Poincaré inequalities, explaining the (TPI) in the formula above.

We study validity and spectral consequences of these inequalities in the three most relevant instances viz for all \(f \in \mathcal{D}\), for all \(f \in \mathcal{F}_c(X)\) (the finitely supported functions) and for all \(f \in \mathcal{D}\) vanishing outside a fixed proper subset \(\Omega\) of \(X\). In all these cases we

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• give geometric characterizations for validity of these inequalities and determine the value of the best constant $c$ in (TPI) in terms of suitable metrics;

• show pure point spectrum for the Laplacians (when an additional ingredient in form of a probability measure is given);

• prove a variational principle characterizing the best constant in (TPI) via taking an infimum over all probability measures of the lowest eigenvalues of the Laplacians.

Specifically, the paper is structured as follows:

In Section 1 we present the basic setup use in the paper and in Section 2 we discuss the necessary background on the metrics.

Our first result characterizes those graphs which admit (TPI) for all functions $f$ of finite energy (Theorem 3.2). These turn out to be the canonically compactifiable graphs. Such graphs have recently been brought forward [9] as graphs with strong intrinsic compactness features and many claims to serve as discrete analogues to open relatively compact subsets of Euclidean spaces, see [16, 12, 14] as well for subsequent studies supporting this point of view. Our second result connects the first result with spectral theory. It shows that in this case the best constant $c$ in (TPI) is given via a variational formula by taking the infimum over the second Neumann eigenvalues for all probability measures of full support on $X$ (Theorem 3.6). As a corollary we obtain a lower bound for the infimum of the spectrum in terms of the diameter of the graph, which is - even for finite combinatorial graphs - better (by the factor 4) than the usual textbook bounds (Corollary 3.7). Along the way we also give a geometric characterization of the best constant in terms of a diameter of the graph with respect to the resistance metric (Proposition 3.4). All these results can be found in Section 3.

Section 4 then deals with (TPI) for functions with finite support. We can characterize graphs admitting such an inequality (Theorem 4.3). These turn out to be the uniformly transient graphs. Such graphs have already featured in various places in the literature, see e.g. [1, 22]. A systematic study of certain geometric and spectral theoretic properties has recently been given in [15]. We then go on to present a variational formula for the best constant in terms of spectral theory which in this case is given by taking the infimum over all lowest eigenvalues for all probability measures of full support on $X$ (Theorem 4.5). Along the way we also obtain a metric characterization of this constant (Proposition 4.4).

In Section 5 we then study the restriction of (TPI) to functions vanishing outside a prescribed proper subset $\Omega$ of $X$. Here again we can characterize geometrically those subsets admitting such an inequality (Theorem 5.2) and present a variational formula for the best constant in this context (Theorem 5.1). To a certain extent this can be seen as a generalization of the earlier two sections. We provide a discussion of this point of view in the finishing remark of Section 5.

As a byproduct of our considerations and a method of [15] we also obtain estimates for higher eigenvalues of the Laplacian (with Neumann boundary conditions) on graphs satisfying a Poincaré inequality. This is discussed in Section 6.

As far as methods go, we note that a crucial part of our considerations concerns geometric interpretation of the best constants in terms of diameters and inradii. More specifically our basic approach relies on considering metrics $\delta$ on $X$ with

$$|f(x) - f(y)|^2 \leq \delta(x, y)\mathcal{E}(f)$$

for all $x, y \in X$ for suitable $f$. Validity of the topological Poincaré inequalities then follows once the diameter of the graph in this pseudometric is bounded and - with the right pseudometric
- this diameter turns out to be the best constant. The corresponding metrics are discussed in Section 2. The sharpness of our results relies on exhibiting for each situation the correct metric. Indeed, from the structural point of view it can be seen as a main achievement of the present article to find the right metric for each of these situations. As a result there are three (slightly) different metrics appearing in our considerations. They can all be seen as variants of the well-known resistance metric. All of them are dominated by a pseudometric $d$ which is the generalization of the combinatorial pseudometric to our case. Thus, in all cases we obtain lower bounds on the spectral theory in terms of the pseudometric $d$. To make this transparent we have included some of these results in the corresponding sections.

Our methods are not confined to the setting of discrete graphs. They can be applied equally well to fractals and quantum graphs. For convenience of the reader and to ease the presentation we have decided to present them in the graph setting only. Details for fractals and quantum graphs will be provided elsewhere. We close this introduction by highlighting the following two points:

Of course, it would equally be possible to take square roots in the Poincaré inequalities we presented above and this case can be treated along very similar lines (compare comments in subsequent sections). Here, we stick to the above form of the inequalities for two reasons: One reason is that the metrics appearing in this context are close - and in some cases even equal - to the (natural generalization of the) combinatorial metric. So, we obtain particularly convincing geometric interpretations of the best constants. The other reason is that in our dealing with spectral theory we avoid taking square roots of eigenvalues.

Finally, let us note that we allow for rather general graphs in our considerations. More specifically, all the literature we quote is concerned with graphs satisfying some form of local bounds on the vertex degree (either in form of local finiteness of the graph or, more generally, in form of a local summability condition, see (S) in Remark 3(a) below). We do not need such restrictions for most of our results.

1 The set-up

In this section we present the set-up we deal with.

A weighted graph $(X, b)$ is given by

- a countable set $X$, finite or infinite;
- a symmetric $b : X \times X \to [0, \infty)$ called the weight function satisfying $b(x, x) = 0$ for all $x \in X$.

Note that our setting is substantially more general than the one usually assumed in the literature dealing with Laplacians on graphs, see e.g. [3, 13, 20, 23] and references therein, in that we do not require any form of local restrictions on the weights $b$. In particular, we do not assume any summability condition on the weights. The vector space of all real valued functions on $X$ is denoted as $\mathcal{F}(X)$. The set of all functions in $\mathcal{F}(X)$ with finite support is denoted by $\mathcal{F}_c(X)$.

Here, the support $\text{supp}(f)$ of a function $f$ is defined as

$$\text{supp}(f) := \{x \in X \mid f(x) \neq 0\}.$$  

We define the set of functions of finite energy associated to the graph by

$$\mathcal{D} := \{f \in \mathcal{F}(X) \mid \sum_{x,y \in X} b(x, y) |f(x) - f(y)|^2 < \infty\}.$$
and define the energy form $\mathcal{E}$ associated to the graph on $\mathcal{D} \times \mathcal{D}$ via

$$\mathcal{E}(f,g) := \frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x) - f(y))(g(x) - g(y)).$$

For $f \in \mathcal{D}$ we let $\mathcal{E}(f) := \mathcal{E}(f,f)$. On the space $\mathcal{F}(X)$ we define the variational semi-norm

$$\| \cdot \|_V : \mathcal{F}(X) \longrightarrow [0, \infty], \|f\|_V := \sup f - \inf f.$$  

The set of all bounded functions on $X$ is denoted by $\ell^\infty(X)$ and equipped with the supremum norm $\| \cdot \|_\infty$ given by $\| f \|_\infty := \sup_{x \in X} |f(x)|$. Clearly, $\| \cdot \|_V$ is finite if and only if $f$ is bounded and in this case we have

$$\|f\|_V = \sup f - \inf f$$

$$= \sup_{s \in [\inf f, \sup f]} \left\{ \|f - s1\|_\infty \right\}$$

$$= 2 \inf_{c \in \mathbb{R}} \{ \|f - c1\|_\infty \}.$$  

Therefore, $\| \cdot \|_V$ is (up to the constant 2) the quotient norm of $\ell^\infty$ modulo the constant functions.

We now turn to operator theory. We will need an additional ingredient viz a measure $m$ on $X$. We will assume that the measure has full support (i.e. any element of $X$ positive measure) and is finite on finite sets. Our basic Hilbert space is

$$\ell^2(X,m) := \{ f \in \mathcal{F}(X) | \|f\|_2^2 = \sum_{x \in X} |f(x)|^2 m(x) < \infty \}.$$  

We will rely on the theory of forms to provide selfadjoint operators. As we do not assume any local finiteness condition the following discussion may be in order: Our forms will in general not be densely defined in $\ell^2(X,m)$. Now, the theory of forms is usually discussed under the assumption that the forms are densely defined. However, as is well-known this is not necessary as one can just develop the theory in the - potentially smaller - Hilbert space arising from closing the form domain in the original Hilbert space. So, the operators associated to forms below do not necessarily have dense domain in $\ell^2(X,m)$ but rather are selfadjoint operators in a suitable closed subspace of $\ell^2(X,m)$.

The form $\mathcal{E}$ induces the closed form $\mathcal{E}^{(N)}$ with domain given by $\mathcal{D} \cap \ell^2(X,m)$, see e.g. Proposition 1.12 in [19], whose proof carries over to our setting. The associated operator will be denoted by $H_m^{(N)}$. We will often be interested in the situation that $m$ is a probability measure and $\mathcal{D}$ consists of bounded functions. In this case we clearly have $\mathcal{D} \cap \ell^2(X,m) = \mathcal{D}$. Whenever $\Omega$ is a subset of $X$ we can restrict the form $\mathcal{E}^{(N)}$ to functions in $\mathcal{D} \cap \ell^2(X,m)$ that vanish outside of $\Omega$. This restriction is a closed form on $\ell^2(\Omega,m)$ and will be denoted by $\mathcal{E}^{\Omega}$ and the associated operator will be called $H^{\Omega}$. If $\mathcal{F}_c(X)$ is contained in $\mathcal{D}$ then there is another natural closed form coming from $\mathcal{E}$ viz. $\mathcal{E}^{(D)}$ whose domain is the closure of $\mathcal{F}_c(X)$ with respect to the form norm of $\mathcal{E}^{(N)}$. The corresponding operator will be denoted by $H^{(D)} = H_m^{(D)}$.

Remark. We could also consider restrictions of $\mathcal{E}^{(D)}$ to $\Omega \subset X$. However, in order to ease the presentation we refrain from giving details for this case as well.

A basic ingredient for the spectral theory we need is the following simple lemma.
Lemma 1.1. If \( m \) is a finite measure on \( X \) and \( Q \) is a closed form and there exists a \( C > 0 \) with
\[
\|f\|_\infty \leq CQ(f)
\]
for all \( f \in D(Q) \), then the associated operator \( A \) has pure point spectrum, \( e^{-A} \) is trace class and we have \( \lambda_0 \geq \frac{1}{Cm(X)} \) for the lowest eigenvalue \( \lambda_0 \) of \( A \).

Proof. Clearly \( e^{-\frac{1}{2}A} \) maps into the form domain and hence into \( \ell^\infty(X) \). As \( m \) is a finite measure, we have an embedding \( \ell^\infty(X) \to \ell^2(X,m) \). Hence, \( e^{-\frac{1}{2}A} \) factors over \( \ell^\infty(X) \). From a factorisation principle we infer that \( e^{-\frac{1}{2}A} \) is Hilbert-Schmidt: see 11.2 and 11.6 in [6] as well as the discussion on p. 318 in [21]. Hence, \( e^{-A} \) is trace class and \( A \) has pure point spectrum.

As for the last part of the statement we note the obvious inequality
\[
\|f\|_2^2 \leq m(X)\|f\|_\infty.
\]
This finishes the proof. \( \Box \)

2 Metrics and Poincaré inequality

Our approach to Poincaré inequalities relies on using suitable pseudometrics. The background is discussed in this section.

We consider a graph \((X,b)\) together with a pseudometric \( \delta \) on \( X \), by which we mean that \( \delta : X \times X \to [0,\infty) \) is symmetric and satisfies the triangle inequality. We denote by \( U_s(x) := \{ y \in X \mid \delta(x,y) < s \} \) and \( B_s(x) := \{ y \in X \mid \delta(x,y) \leq s \} \) the open and closed balls of radius \( s \), respectively. We define the inradius of \( \Omega \subset X \) with \( \Omega \neq X \) by
\[
\text{Inr}(\Omega) := \sup \{ s > 0 \mid \exists x \in \Omega : U_s(x) \subset \Omega \}.
\]
Similarly, we define the diameter of \( X \) by
\[
\text{diam}(X) := \sup_{x,y} \delta(x,y).
\]
If the pseudometric is not clear from the context, we will write \( \text{Inr}_\delta \) and \( \text{diam}_\delta \).

We say that the pseudometric \( \delta \) satisfies a topological Poincaré inequality if
\[
|f(x) - f(y)|^2 \leq \delta(x,y)\mathcal{E}(f)
\]
holds for all \( f \in D \) and all \( x,y \in X \). We will be particularly concerned with two specific metrics satisfying this condition. These will be discussed next.

We start with a pseudometric which can be seen as a direct generalization of the combinatorial metric. An edge of the weighted graph \((X,b)\) is a set \( \{x,y\} \) with positive weight \( b(x,y) > 0 \). Denote by \( E \) the set of all edges. Clearly, that induces the structure of a combinatorial graph \((X,E)\). A path is a finite sequence of edges with nonempty intersections that can most easily be written as \( \gamma = (x_0,x_1,\ldots,x_k) \) where \( b(x_j,x_{j+1}) > 0 \) for all \( j = 0,\ldots,k-1 \); if we want to specify the endpoints we say that \( \gamma \) is a path from \( x_0 \) to \( x_k \). The length of such a path \( \gamma \) is given by
\[
L(\gamma) := \sum_{j=0}^{k-1} \frac{1}{b(x_j,x_{j+1})}.
\]
In particular the length of an edge \( \{x, y\} \) is given by \( \frac{1}{b(x,y)} \). To include trivial cases we also allow trivial paths \((x, x)\) from \(x\) to \(x\) whose length is 0. We will throughout assume that our graph is connected in the sense that every pair of points is connected by a path. The distance between \(x\) and \(y\) is given by

\[
d(x, y) := \inf \{ L(\gamma) \mid \gamma \text{ a path from } x \text{ to } y \}.
\]

Clearly, \(d\) is symmetric and satisfies the triangle inequality. If \(\sup_y b(x,y) < \infty\) for each \(x \in X\), then \(d\) is a metric. This pseudometric \(d\) satisfies a topological Poincaré inequality. For graphs satisfying \(\mathcal{F}_c(X) \subset \mathcal{D}\) this has has been noted in various places, including the recent [9]. The proof carries over to our setting. For the sake of completeness we include it next.

**Proposition 2.1** (\(d\) satisfies a topological Poincaré inequality). Let \(x, y \in X\) be arbitrary. Then for any path \(\gamma = (x_0, \ldots, x_k)\) from \(x\) to \(y\) and \(f \in \mathcal{D}\) the inequality

\[
|f(x) - f(y)|^2 \leq L(\gamma) \sum_{j=0}^{k-1} b(x_j, x_{j+1})(f(x_j) - f(x_{j+1}))^2
\]

holds. In particular

\[
|f(x) - f(y)|^2 \leq d(x, y) \mathcal{E}(f)
\]

is valid.

**Proof.** It suffices to show the first inequality. Take a path \(\gamma = (x_0, \ldots, x_k)\) from \(x\) to \(y\). Using the triangle inequality and the Cauchy-Schwarz inequality we can estimate

\[
|f(x) - f(y)| \leq \sum_{j=0}^{k-1} b(x_j, x_{j+1})^{1/2} |f(x_j) - f(x_{j+1})| \frac{1}{b(x_j, x_{j+1})^{1/2}}
\]

\[
\leq L(\gamma)^{1/2} \left( \sum_{j=0}^{k-1} b(x_j, x_{j+1})(f(x_j) - f(x_{j+1}))^2 \right)^{1/2}.
\]

This finishes the proof. \(\square\)

We now turn to the resistance metric \(r\) defined in the following way: Define \(r : X \times X \rightarrow [0, \infty]\) such that for any \(x, y \in X\) the number \(r(x, y)\) is minimal with

\[
|f(x) - f(y)|^2 \leq r(x, y) \mathcal{E}(f)
\]

for all \(f \in \mathcal{D}\).

**Proposition 2.2.** The function \(r\) is a pseudometric with \(r \leq d\).

**Remark.** For finite graphs it is well known that \(r\) is a pseudometric and this can be extended to locally finite graphs (see e.g. discussion in [9]). However, even for finite graphs the proof is involved and it is far from clear that it carries over graphs which are not locally finite, let alone to our more general setting. Thus, we provide an alternative argument. This has the advantage that variants of it can be applied to other cases that will be needed later on.
Proof. The inequality \( r \leq d \) is clear from the previous proposition. In particular, \( r \) is finite. Obviously, \( r \) is symmetric. It remains to show the triangle inequality. So, let \( x, y, z \in X \) be given. Let \( f \in D \) be arbitrary. We have to show

\[
|f(x) - f(z)|^2 \leq (r(x, y) + r(y, z))\mathcal{E}(f).
\]

The case \( f(x) = f(z) \) is obvious. Thus, we can assume without loss of generality \( f(x) > f(z) \). We distinguish three cases.

Case 1: \( f(y) \geq f(x) \). In this case we have

\[
|f(x) - f(z)|^2 \leq |f(y) - f(z)|^2 \leq r(y, z)\mathcal{E}(f) \leq (r(x, y) + r(y, z))\mathcal{E}(f).
\]

Case 2: \( f(y) \leq f(z) \). In this case we have

\[
|f(x) - f(z)|^2 \leq |f(x) - f(y)|^2 \leq r(x, y)\mathcal{E}(f) \leq (r(x, y) + r(y, z))\mathcal{E}(f).
\]

Case 3: \( f(x) > f(y) > f(z) \): Without loss of generality we can assume \( f(y) = 0 \) (as otherwise we could shift everything by \(-f(y)\)). Decomposing \( f \) into positive and negative part we obtain

\[
f_+(x)^2 = (f_+(x) - f_-(y))^2 \leq r(x, y)\mathcal{E}(f_+)
\]

and hence

\[
f_+(x) \leq \sqrt{r(x, y)\mathcal{E}(f_+)} =: T_x
\]

and similarly

\[
f_-(z) \leq \sqrt{r(y, z)\mathcal{E}(f_-)} =: T_z.
\]

By the inequality between geometric and arithmetic mean we have

\[
T_x T_z = \sqrt{(r(x, y)\mathcal{E}(f_-))(r(y, z)\mathcal{E}(f_+))} \leq \frac{r(x, y)\mathcal{E}(f_-) + r(y, z)\mathcal{E}(f_+)}{2}.
\]

Put together this gives

\[
|f(x) - f(z)|^2 = |f_+(x) + f_-(z)|^2 \\
\leq T_x^2 + 2T_x T_y + T_y^2 \\
\leq r(x, y)\mathcal{E}(f_+) + r(x, y)\mathcal{E}(f_-) + r(y, z)\mathcal{E}(f_+) + r(y, z)\mathcal{E}(f_-) \\
= (r(x, y) + r(y, z))(\mathcal{E}(f_+) + \mathcal{E}(f_-)) \\
\leq (r(x, y) + r(y, z))(\mathcal{E}(f_+) - 2\mathcal{E}(f_+, f_-) + \mathcal{E}(f_-)) \\
= (r(x, y) + r(y, z))\mathcal{E}(f).
\]

Here the previous to the last inequality follows as \( \mathcal{E}(f_+, f_-) \leq 0 \) (which can be seen immediately). By the definition of \( r \) and as \( f \) was arbitrary this shows

\[
r(x, z) \leq r(x, y) + r(y, z)
\]

and the proof is finished.

\[ \Box \]

Corollary 2.3. The pseudometric \( r \) satisfies a topological Poincaré inequality and for any other pseudometric \( \delta \) satisfying such an inequality \( r \leq \delta \) holds.
Proof. As shown in the previous proposition \( r \) is a pseudo metric. The remaining claim is clear from the definition.

As \( r \) is the smallest pseudometric admitting a topological Poincaré inequality, we will in the sequel mostly work with \( r \) (as this gives the sharpest estimates). However, we emphasize that all estimates will also hold for any other pseudometric \( \delta \) admitting a Poincaré inequality. In particular, they also hold for \( d \) (which often is much easier to calculate than \( r \)). In fact, for locally finite trees the metrics \( d \) and \( r \) agree (see e.g. [9]).

As mentioned already in the introduction we will also be concerned with the companion inequality

\[
\| f \|_\infty^2 \leq c \mathcal{E}(f)
\]

for \( f \in \mathcal{D} \) vanishing on a prescribed set. In this context, we say that a pseudometric \( \delta \) satisfies a topological Poincaré inequality on \( \Omega \subset X \) if

\[
| f(x) - f(y) |^2 \leq \delta(x,y) \mathcal{E}(f)
\]

(TPI\(_\Omega\))

for \( f \in \mathcal{D} \) with \( f \geq 0 \) vanishing outside \( \Omega \). To deal with this situation we need one more metric. For \( \Omega \subset X \) with \( \Omega \neq X \) we define \( r_\Omega : X \times X \rightarrow [0, \infty] \) such that for any \( x, y \in X \) the number \( r_\Omega(x,y) \) is minimal with

\[
| f(x) - f(y) |^2 \leq r_\Omega(x,y) \mathcal{E}(f)
\]

for all \( f \in \mathcal{D} \) with \( f \geq 0 \) and \( \text{supp}(f) \subset \Omega \). Thus, we have

\[
r_\Omega(x,y) = \sup\{| f(x) - f(y) |^2 | f \in \mathcal{D} \text{ with } f \geq 0, \mathcal{E}(f) \leq 1 \text{ and } f = 0 \text{ outside } \Omega}\}
\]

Proposition 2.4. For any \( \Omega \subset X \) with \( \Omega \neq X \) the function \( r_\Omega \) is a pseudometric with \( r_\Omega \leq r \).

Remark. We are not aware of an appearance of this pseudometric earlier in the literature. On a conceptual level bringing up this pseudometric can be seen as a key step in the present work.

Proof. The inequality \( r_\Omega \leq r \) is clear and directly gives that \( r_\Omega \) take finite values. That \( r_\Omega \) is a pseudometric follows by a slight variant of the argument given for \( r \). In fact, Case 1 and Case 2 go through without any changes. In the discussion of Case 3 the decomposition \( f = f_+ - f_- \) has to be replaced by the decomposition

\[
f = f_{c,+} + f_{c,-}
\]

with \( f_{c,+} := (f - c)_{+} \) and \( f_{c,-} := \min \{ f, c \} = c - (f - c)_{-} \) for \( c = f(y) \). It is then easy to see that \( f_{c,+} \) and \( f_{c,-} \) both are supported in \( \Omega \) and

\[
\mathcal{E}(f_{c,+}, f_{c,-}) = \mathcal{E}((f - c)_{+}, c_{1} - (f - c)_{-}) = - \mathcal{E}((f - c)_{+}, (f - c)_{-}) \geq 0
\]

holds. Given this inequality the argument can be carried through as in the case of \( r \). We omit the details.

As in the case of \( r \) the following is immediate from the construction.

Corollary 2.5. The pseudometric \( r_\Omega \) satisfies (TPI\(_\Omega\)) and for any other pseudometric \( \delta \) satisfying such an inequality \( r_\Omega \leq \delta \) holds.
Remark. As already mentioned in the introduction it would equally be possible to take square roots in the Poincaré inequalities we presented above. It turns out that the square roots of the metrics $r$ and $r_\Omega$ are metrics again. Indeed, we can consider the pseudometric $\varrho$ defined via

$$\varrho(x, y) := \sup\{f(x) - f(y) \mid f \in \mathcal{D}, E(f) \leq 1\}.$$ 

This pseudometric was introduced by Davies [5] in his study of non-commutative Dirichlet forms and then used in a similar spirit in [10] (see [15] as well). From its very definition we have for any $x, y \in X$

$$|f(x) - f(y)|^2 \leq \varrho(x, y)^2 E(f)$$

for all $f \in \mathcal{D}$ and $\varrho(x, y)^2$ is the smallest number with this property. So, the pseudometric $\varrho$ is the square root of $r$, see e.g. [15]. Similar considerations apply to $r_\Omega$ giving rise to the pseudometric $\varrho_\Omega$. This means that all results presented below could also be framed in terms of the corresponding metrics $\varrho$ and $\varrho_\Omega$.

3 Global Poincaré inequality on $\mathcal{D}$

In this section we investigate graphs satisfying a global topological Poincaré inequality in that there exists a $c > 0$ with

$$|f(x) - f(y)|^2 \leq cE(f)$$

for all $f \in \mathcal{D}$ and all $x, y \in X$.

The relevance of $r$ in our context comes from the following proposition.

Proposition 3.1. Let $(X, b)$ be a graph. Then,

$$\text{diam}_r(X) = \sup\{\|f\|^2 \mid f \in \mathcal{D} \text{ with } E(f) \leq 1\},$$

where the value $\infty$ is possible.

Proof. We calculate

$$\text{diam}_r(X) = \sup_{x,y} r(x, y)$$

$$= \sup\{|f(x) - f(y)|^2 \mid x, y \in X, f \in \mathcal{D} \text{ with } E(f) \leq 1\}$$

$$= \sup\{\|f\|^2 \mid f \in \mathcal{D} \text{ with } E(f) \leq 1\}.$$ 

This finishes the proof. 

Theorem 3.2 (Characterizing validity of a global Poincaré inequality). Let $(X, b)$ be a graph. Then, the following statements are equivalent:

(i) A global Poincaré type inequality holds, i.e. there exists a $c > 0$ with

$$\|f\|^2 \leq cE(f)$$

for all $f \in \mathcal{D}$ and $x, y \in X$.

(ii) The graph $(X, b)$ satisfies $\mathcal{D} \subset \ell^\infty$. 

(iii) \(\text{diam}_r(X) < \infty\).

**Proof.** (i) \(\iff\) (iii) is clear from the previous proposition. (i) \(\iff\) (ii): This is clear as (i) implies 
\[
\|f\|_V \leq cE(f) < \infty
\]
for all \(f \in \mathcal{D}\).

(ii) \(\iff\) (i) follows from a closed graph argument. Consider the closed subspace \(\text{lin}\{1\} = \{\lambda 1 \mid \lambda \in \mathbb{R}\}\) of \(\ell^\infty(X)\). Then \(\ell^\infty(X)/\text{lin}\{1\}\) is a Banach space with respect to 
\[
\|f + \text{lin}\{1\}\|_V := \sup f - \inf_{c \in \mathbb{R}} \{\|f - c1\|_\infty\},
\]
since the latter norm agrees with the quotient norm \(\|f + \text{lin}\{1\}\|_{\ell^\infty(X)/\text{lin}\{1\}}\).

Moreover it is not hard to see that \(\mathcal{D}/\text{lin}\{1\}\) is a Banach space with the norm \(\|\cdot\|_E := E(f)^\frac{1}{2}\). This follows e.g. from Proposition 1.6 in [19], whose proof carries over to our setting; see as well the outline of proof below.

By assumption the identity maps 
\[
\text{id}: \mathcal{D}/\text{lin}\{1\} \to \ell^\infty(X)/\text{lin}\{1\}.
\]

Thus, the desired estimate follows from the closed graph theorem once we can show that \(\text{id}\) is closed, so that it remains to verify that 
\[
[f_n]^{\|\cdot\|_V} \to [f] \text{ and } [f_n]^{\|\cdot\|_E} \to [g] \Rightarrow [f] = [g], \quad (\star)
\]
where we abbreviate \([f] := f + \text{lin}\{1\}\). To this end we fix \(o \in X\) and consider 
\[
\lambda: \ell^\infty(X)/\text{lin}\{1\} \to \ell^\infty(X), \lambda[f](x) := f(x) - f(o).
\]

Note that \(\lambda[f]\) is well-defined and \(\lambda\) is a lifting of the quotient map, i.e., \([\lambda[f]] = [f]\). The remaining claim (\(\star\)) will be settled once we see that for \(\bullet \in \{V, E\}\),
\[
[f_n]^{\|\cdot\|_\bullet} \to [f] \Rightarrow \lambda[f_n] \to \lambda[f] \text{ pointwise}, \quad (\star\star)
\]
and this is what we see next. Clearly, (\(\star\star\)) holds for \(\bullet = V\) since \(\lambda\) is continuous. For \(\bullet = E\) we can use Proposition 2.1. In fact, let \(x \in X\) and choose \(y = o\). Then the inequality in the latter Proposition gives 
\[
|\lambda[f_n](x) - \lambda[f](x)|^2 \leq d(x, o)E(f_n - f),
\]
and this finishes the proof.

The lifting \(\lambda\) can be used to prove the completeness of \(\mathcal{D}/\text{lin}\{1\}\) with respect to \(\|\cdot\|_E\) as follows. By what we just proved, for any Cauchy sequence \(([f_n])\) in the latter space, the functions \(\lambda[f_n]\) converge pointwise to some \(f\). A standard completeness proof settles that \([f]\) is the desired limit. \(\square\)
Remarks. (a) The notion of canonically compactifiable graph is put forward in [9] to denote graphs satisfying (ii) and the additional summability condition
\[ \sum_{y \in X} b(x, y) < \infty \] (S)
for all \( x \in X \). For such graphs the equivalence between (ii) and (iii) is already contained in [9].

(b) Whenever a probability measure \( m \) is given, a global Poincaré inequality clearly implies an \( \ell^2 \)-Poincaré inequality of the form
\[ \| f - m(f)1 \|_2^2 \leq c \mathcal{E}(f) \]
with \( m(f) = \sum x f(x)m(x) \) and \( \| \cdot \|_2 \) the norm on \( \ell^2(X, m) \) (compare the discussion in the introduction). The validity of such an \( \ell^2 \)-Poincaré inequality for canonically compactifiable graphs has recently been observed in [16].

Corollary 3.3. If \((X, b)\) satisfies a global Poincaré inequality and \(m\) is a probability measure on \(X\), then \(H^{(N)}_m\) has pure point spectrum.

Proof. Pure point spectrum is well known for canonically compactifiable graphs with finite measures [9]. Hence, the corollary follows immediately from the previous theorem in the case of locally summable \(b\). For the general case we note that the constant function 1 is clearly an eigenfunction to the eigenvalue 0. We can then conclude the statement from Lemma [14] applied to the restriction of \(\mathcal{E}^{(N)}\) to the orthogonal complement of 1.

We aim at studying the best constant \(c\) in the Poincaré inequality. More specifically, whenever \((X, b)\) satisfies
\[ \| f \|_V^2 \leq c \mathcal{E}(f) \]
for all \( f \in \mathcal{E} \), we can consider the infimum, which is actually a minimum, over all possible such \(c\). This minimum will be denoted by \(c_P\). Here come simple geometric and functional analytic characterisations of this constant.

Proposition 3.4 (Geometric and functional analytic interpretation of \(c_P\)). The equalities
\[ c_P = \text{diam}_r(X) \]
\[ = \left( \text{Norm of the embedding } j : (D/\text{lin}\{1\}, \| \cdot \|_E) \rightarrow (\ell^\infty(X)/\text{lin}\{1\}, \| \cdot \|_V) \right)^2 \]
hold.

Proof. This is immediate from the definitions and the proof of the previous theorem. \(\square\)

Remark. Clearly, any finite graph has finite diameter with respect to \(r\). However, it is not hard to construct infinite graphs with finite diameter. In fact, it is easy to see that any infinite graph with
\[ \kappa := \sum_{x, y : b(x, y) > 0} \frac{1}{b(x, y)} < \infty \]
has finite diameter (bounded by \(\kappa\) with respect to \(d\) see [9] for further discussion as well). Due to \(r \leq d\) it has then finite diameter with respect to \(r\) as well.
Our second main result gives a formula for $c_P$. We need the following proposition as preparation.

**Proposition 3.5.** Let $X$ be a countable set and $m$ a probability measure on $X$.

(a) Then,
\[ \|f\|_2^2 \leq \frac{1}{4} \|f\|_V^2 \]

for any bounded $f$ with zero mean i.e. with $\sum x f(x)m(x) = 0$.

(b) If $f$ is bounded and satisfies furthermore $\sup f = -\inf f$ we even have
\[ \sup_m \|f\|_2^2 = \frac{1}{4} \|f\|_V^2, \]

where the supremum is taken over all probability measures with full support such that $f$ has zero mean.

**Proof.** (a) We decompose $f$ into positive and negative part $f = f_+ - f_-$ and set
\[ S := \sup_{x \in X^+} f_+ \text{ and } I := \sup_{x \in X^-} f_. \]

Hence, $S$ is the supremum of $f$ and $I$ is the negative Infimum. So, we clearly have
\[ \|f\|_V^2 = (S + I)^2. \]

Set
\[ X^+ := \{ x \in X \mid f(x) > 0 \} \text{ and } X^- := \{ x \in X \mid f(x) < 0 \}. \]

Then, the condition on the mean of $f$ can be written as
\[ \sum_{x \in X^+} f_+(x)m(x) = \sum_{x \in X^-} f_-(x)m(x) =: T \]

Then, with $p := \sum_{x \in X^+} m(x)$ and $q := \sum_{x \in X^-} m(x)$ we can estimate $T$ via
\[ 0 \leq T \leq Sp \text{ as well as } 0 \leq T \leq Iq \]
with
\[ p + q \leq 1 \]
(as $m$ is a probability measure and $f$ may well take the value zero). From these estimates we obtain
\[ T^2 \leq SIpq \]
and after taking square roots
\[ T \leq \sqrt{SI}\sqrt{pq}. \]

Now, due to $p + q \leq 1$ we can easily infer $pq \leq \frac{1}{4}$. Combined with the inequality between arithmetic and geometric mean we infer
\[ T \leq \frac{1}{4}(S + I). \quad (*) \]
After this preparation we now easily finish the proof:

\[ \| f \|_2^2 = \sum_{x \in X_+} f_+(x)^2 m(x) + \sum_{x \in X_-} f_-(x)^2 m(x) \]
\[ \leq S \sum_{x \in X_+} f_+(x) m(x) + I \sum_{x \in X_-} f_-(x) m(x) \]

(definition of \( T \)) \[ = (S + I)T \]
\[ \leq \frac{1}{4} (S + I)^2 \]
\[ = \frac{1}{4} \| f \|_V^2. \]

This shows (a).

(b) The inequality \( \leq \) is clear from (a). Thus, it remains to show \( \geq \). Assume first that \( f \) attains both its maximum and its minimum in, say, \( x_M \) and \( x_m \). Consider now the measure \( m \) which gives the mass \( \frac{1}{2} \) to \( x_m \) and \( x_M \). Then, a short calculation shows that \( \| f \|_2^2 = \frac{1}{4} \| f \|_V \).

The problem with this argument is that in general we do not know that \( f \) attains maximum and minimum and that \( m \) will not be a supported on the whole of \( X \). So, we will have to modify the argument slightly: We chose \( x_m \) and \( x_M \) such that the values are very close to the supremum and infimum of \( f \) and we then distribute a very small mass on the remaining points, calculate the mean on these remaining points (which will be close to zero due to the smallness of the mass on the remaining points) and now give almost the mass \( \frac{1}{2} \) and \( \frac{1}{2} \) to \( x_M \) and \( x_m \). We omit more details.

Theorem 3.6 (Computing \( c_P \) via \( \ell^2 \)-theory). Let \( (X, b) \) satisfy a global Poincaré inequality. Then, the best possible constant \( c_P \) satisfies

\[ \frac{4}{c_P} = \inf_m \{ \text{second lowest eigenvalue of } H^{(N)}_m \}, \]

where the infimum runs over all probability measures of full support.

Remark. By Corollary 3.3 above the operator \( H^{(N)}_m \) has pure point spectrum. Clearly, the infimum of the spectrum is zero with the constant functions being eigenfunctions.

Proof. We denote the second lowest eigenvalue of \( H^{(N)}_m \) by \( \lambda_1 \) (and thereby suppress the dependence on \( m \)).

We first show “\( \leq \)”: Let \( m \) be an arbitrary probability measure on \( X \) with full support. Then, the first eigenvalue of \( H^{(N)}_m \) is 0 with the constant functions as eigenfunctions. Consider now an arbitrary \( f \in D \) with \( f \perp 1 \). Hence, \( f \) has zero mean. So, from (a) of the previous proposition we infer

\[ \mathcal{E}(f) \geq \frac{1}{c_P} \| f \|_V^2 \geq \frac{4}{c_P} \| f \|_2^2. \]

As \( f \in D \) with \( f \perp 1 \) was arbitrary we obtain \( \lambda_1 \geq \frac{4}{c_P} \).

We now turn to showing “\( \geq \)”: Let \( \varepsilon > 0 \) be arbitrary. Choose an \( f \in D \) with

\[ \mathcal{E}(f) < \frac{1 + \varepsilon}{c_P} \| f \|_V^2. \]
Clearly we can replace \( f \) by \( f - s1 \) for any \( s \in \mathbb{R} \) without changing the last inequality. In particular, we can assume without loss of generality \( \sup f = - \inf f \). Now, by (b) of the previous proposition we can chose a probability measure \( m \) of full support on \( X \) such that \( f \) has zero mean with respect to this measure and

\[
(1 + \varepsilon)\|f\|_V^2 \geq \frac{1}{4}\|f\|_V^2
\]

holds. Combining these estimates we infer

\[
\mathcal{E}(f) < \frac{4}{c_P}(1 + \varepsilon)^2\|f\|_V^2.
\]

As \( f \) has zero mean we have \( f \perp 1 \) and the last inequality gives

\[
\lambda_1 \leq \frac{4}{c_P}(1 + \varepsilon)^2.
\]

As \( \varepsilon > 0 \) was arbitrary the desired statement follows.

**Remark.** The constant \( c_P \) appears in the denominator in the preceding theorem. This is due to the fact that we have chosen to write the original inequality in the form \( \|f\|_V^2 \leq c\mathcal{E}(f) \) with the constant appearing on the right hand side. Of course, one could do differently. However, it seems that it is usual to have the constant on the right hand side and, moreover, as shown above, it is possible to give a direct geometric meaning of it in form of a diameter.

**Corollary 3.7.** Assume \( \text{diam}_d(X) < \infty \). Then, \( (X, b) \) satisfies the equivalent conditions of Theorem 3.2 and for any probability measure \( m \) on \( X \) of full support, the second lowest eigenvalue \( \lambda_1 \) of \( H^{(N)} \) is bounded below by

\[
\lambda_1 \geq \frac{4}{\text{diam}_d(X)}.
\]

**Proof.** As \( r \leq d \) the assumption implies \( \text{diam}_r(X) \leq \text{diam}_d(X) < \infty \) and the equivalent assumptions of Theorem 3.2 are satisfied. Moreover, from Proposition 3.4 we have

\[
c_P = \text{diam}_r(X) \leq \text{diam}_d(X).
\]

Given this the last statement follows from Theorem 3.6.

**Remark.** That a graph with \( \text{diam}_d(X) < \infty \) (satisfying the additional summability condition (S) given in Remark 3 (a)) is canonically compactifiable is already contained in [9]. The main statement of the corollary is the bound for the lowest eigenvalues. Even in the case of finite combinatorial graphs with combinatorial distances \( d_{\text{comb}} \) this bound is stronger (by the factor 4) than what can be found in the textbooks, e.g. [3].

**Remark** (Poincaré via \( \rho \)). It is instructive to interpret the Poincaré inequality via the pseudometric \( \rho \). As noted already we have \( \rho^2 = r \). So, the preceding considerations imply \( c_P = (\text{diam}_d(X))^2 \). It is now natural to define the **radius** of \( X \) with respect to \( \rho \) as \( R(\rho) := \frac{\text{diam}_d(X)}{2} \). Given this we can interpret the quantity \( \frac{4}{c_P} \) appearing in the second theorem of the last section as

\[
\lambda_1 \geq \frac{4}{c_P} = \frac{1}{R(\rho)^2}.
\]
4 Global Poincaré inequality for $\mathcal{F}_c(X)$

In this section we assume that $X$ is infinite and $\mathcal{F}_c(X) \subset \mathcal{D}$ holds. In this case it is easy to see that the aforementioned summability condition (S) from Remark 3 (a) is true. The following elementary observation provides further characterizations for this inclusion.

For $x \in X$ we write $\delta_x$ for the function that equals 1 at $x$ and is 0 otherwise.

**Proposition 4.1.** Let $(X, b)$ be a weighted graph. Then the following are equivalent:

(i) $(X, b)$ satisfies the summability condition (S):

$$\sum_{y \in X} b(x, y) < \infty$$
for all $x \in X$.

(ii) $\delta_x \in \mathcal{D}$ for all $x \in X$.

(iii) $\mathcal{F}_c(X) \subset \mathcal{D}$.

(iv) $\mathcal{F}_c(X) \cap \mathcal{D}$ is dense in $\mathcal{F}_c(X)$ w.r.t. $\| \cdot \|_{\infty}$, i.e., for one [any] measure $m$ on $X$ with full support, $\mathcal{E}(\mathcal{D})$ is a regular Dirichlet form on $\ell^2(X, m)$ (regular w.r.t. the discrete topology on $X$.)

**Proof.** (i) $\iff$ (ii) is clear from the fact that

$$\sum_{y \in X} b(x, y) = \mathcal{E}(\delta_x).$$

(ii) $\implies$ (iii) $\implies$ (iv) is clear.

(iv) $\implies$ (ii): Fix $x \in X$. By assumption, for any $\varepsilon > 0$ there is $\varphi_\varepsilon \in \mathcal{D}$ such that $\| \varphi_\varepsilon - \delta_x \|_{\infty} < \varepsilon$. For $\varepsilon \leq \frac{1}{12}$ we have that $\varphi_\varepsilon \leq \frac{1}{4}$ for $y \neq x$ and $\varphi_\varepsilon(x) \geq \frac{1}{4}$. Clearly, $\psi := (\varphi_\varepsilon - \frac{1}{4})_+ \in \mathcal{D}$ and $\psi(x) > 0$, so that $\delta_x = c\psi \in \mathcal{D}$. \hfill $\Box$

**Remark.** The equivalence of (iii) and (iv) is a slight extension of Lemma 2.1 in [13].

Under the assumption $\mathcal{F}_c(X) \subset \mathcal{D}$ it is natural to study whether there is $c$ such that

$$\|f\|_V^2 \leq c\mathcal{E}(f)$$

for all $f \in \mathcal{F}_c(X)$. We will see in the sequel that analogues of the considerations of the results in previous section hold in this case as well. For $f \in \mathcal{F}_c(X)$ we clearly have that $\| \cdot \|_V$ and $\| \cdot \|_\infty$ are equivalent in the sense that

$$\|f\|_\infty \leq \|f\|_V \leq 2\|f\|_\infty$$

for all $f \in \mathcal{F}_c(X)$. For this reason we will now restrict attention to the inequality

$$\|f\|_\infty^2 \leq c\mathcal{E}(f)$$

(TPI$_c$)

and say that a *global Poincaré inequality holds for $\mathcal{F}_c(X)$* provided it is satisfied for a suitable constant. Graphs satisfying this inequality are termed *uniformly transient* in [15] and have been thoroughly studied in this latter reference. Again, we can characterize graphs satisfying
this property in various ways. We will need one more pseudometric to do so. Specifically, we define \( r_0 \) as

\[
    r_0 := \sup_{\Omega \subset X: \text{finite}} r_{\Omega}.
\]

Hence,

\[
    r_0(x, y) = \sup\{ |f(x) - f(y)|^2 \mid f \in F_c(X), f \geq 0, \mathcal{E}(f) \leq 1 \}.
\]

The relevance of \( r_0 \) comes from the following proposition.

**Proposition 4.2.** The equality

\[
    \text{diam}_{r_0}(X) = \sup\{ \|f\|_{\infty}^2 \mid f \in F_c(X) \text{ with } \mathcal{E}(f) \leq 1 \}
\]

holds.

**Proof.** We note first that for \( f \in F_c(X) \) with \( f \geq 0 \) the equality \( \sup_{x,y}|f(x) - f(y)|^2 = \|f\|_{\infty}^2 \) holds and that \( \mathcal{E}(|f|) \leq \mathcal{E}(f) \) for any \( f \in D \) (as a direct calculation shows). Now, we can argue as follows

\[
    \text{diam}_{r_0}(X) = \sup_{x,y}\{|f(x) - f(y)|^2 : f \in F_c^+(X), \mathcal{E}(f) \leq 1\}
    = \sup\{\|f\|_{\infty}^2 \mid f \in F_c^+(X), \mathcal{E}(f) \leq 1\}
    = \sup\{\|f\|_{\infty}^2 \mid f \in F_c(X), \mathcal{E}(f) \leq 1\}
    = \sup\{\|f\|_{\infty}^2 \mid f \in F_c(X), \mathcal{E}(f) \leq 1\}.
\]

This finishes the proof. \( \square \)

We denote the closure of \( F_c(X) \) in \( D \) with respect to the norm \( \mathcal{E}(f)^{1/2} \) by \( D_0 \).

**Theorem 4.3 (Characterization validity Poincaré for \( F_c(X) \)).** The following assertions are equivalent:

(i) A global Poincaré inequality \([\text{TPI}_c]\) holds for \( F_c(X) \).

(ii) The set \( D_0 \) is contained in \( F_0(X) := \overline{F_c(X)}\|\cdot\|_{\infty} \).

(iii) \( \text{diam}_{r_0}(X) < \infty \).

**Proof.** Since \((X, b)\) satisfies \([S]\), the equivalence between (i) and (ii) is already contained in \([9]\). The equivalence of (i) and (iii) follows immediately from the previous proposition. \( \square \)

Again, we may ask for the best constant \( c_0^P \) in the Poincaré inequality. Here, the geometric and the functional analytic description is as follows.

**Proposition 4.4 (Geometric and functional analytic interpretation of \( c_0^P \)).** The equalities

\[
    c_0^P = \text{diam}_{r_0}(X)
    = (\text{norm of the embedding }(D_0, \| \cdot \|_\infty) \rightarrow (C_0(X), \| \cdot \|_\infty))^2
\]

hold.

**Proof.** This is clear from the proof of the previous theorem. \( \square \)
As in the previous section we may also give a description of the best constant in terms of eigenvalues.

**Theorem 4.5** (Computing $c_p^0$ via $\ell^2$-theory).

\[
\frac{1}{c_p^0} = \inf \{ \text{lowest eigenvalue of } H_m^{(\mathcal{D})} \},
\]

where the infimum is taken over all probability measures on $X$ of full support.

**Proof.** We show two inequalities:

"$\leq$": Let $f \in F_c(X)$ be arbitrary. Then,

\[
\|f\|_2^2 \leq \|f\|_\infty^2 \leq c_p^0 \mathcal{E}(f)
\]

and the desired inequality follows.

"$\geq$": Let $\varepsilon > 0$ be arbitrary. Then, there exists an $f \in F_c(X)$ with

\[
\|f\|_\infty^2 \geq \frac{c_p^0}{1 + \varepsilon} \mathcal{E}(f).
\]

We can now choose a probability measure on $X$ of full support that gives almost all of its mass to the point, where $f(x)^2$ attains its maximum. Then, $\|f\|_2^2 \geq \frac{1}{1 + \varepsilon} \|f\|_\infty^2$. Put together we see

\[
\|f\|_2^2 \geq \frac{c_p^0}{(1 + \varepsilon)^2} \mathcal{E}(f)
\]

for all $f \in F_c(X)$. This implies

\[
\frac{(1 + \varepsilon)^2}{c_p^0} \geq \lambda_0
\]

for the lowest eigenvalue $\lambda_0$ of $H_m^{(\mathcal{D})}$ and the desired inequality follows. 

Whenever $(X, b)$ is a graph we define the *Dirichlet inradius* with respect to $d$ by

\[
\text{Inr}_d^D(X) := \sup_{A \subset X: \text{finite}} \text{Inr}_d(A).
\]

Clearly, $\text{Inr}_d^D(X) \leq \text{diam}_d(X)$. Hence, finiteness of the diameter implies finiteness of the Dirichlet inradius. However, as shown by an example below we may have finiteness of the Dirichlet inradius for graphs with unbounded diameter.

**Corollary 4.6.** Let $(X, b)$ be a graph with $\text{Inr}_d^D(X) < \infty$. Then, $(X, b)$ is uniformly transient and for any probability measure $m$ of full support we have for the lowest eigenvalue $\lambda_0$ of $H_m^{(\mathcal{D})}$ the estimate

\[
\lambda_0 \geq \frac{1}{\text{Inr}_d^D(X)}.
\]

**Example** ($\text{Inr}_d^D(X) < \infty$ and $\text{diam}_d(X) = \infty$). The example is given by a copy of $\mathbb{Z}$ in which at each point a weighted copy of $\mathbb{N}$ is attached such that the weights make the total diameter of each of these copies of $\mathbb{N}$ not exceed 1. Specifically, the vertex set $X$ is given as $\mathbb{Z} \times \mathbb{N}$ and there is an edge with weight 1 between $(n, 0)$ and $(n + 1, 0)$ for any $n \in \mathbb{Z}$ and an edge with
weight $1/2^{k+1}$ between $(n, k)$ and $(n, k + 1)$ for any $n \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$. Then, any ball of size, say, 2, contains a set of the form $\{(n, k) : k \in \mathbb{N}\}$ for some $n \in \mathbb{Z}$ and hence has infinitely many points. Thus, the Dirichlet inradius is bounded by 2. At the same time, for any natural number $n$ the ball around $(0, 0)$ of size $n$ does not cover the graph (as clearly it does not contain e.g. $(n + 1, 0)$). Hence the diameter of the graph is infinite. Let us emphasize that the example is a locally finite tree and hence the metrics $d$ and $r$ agree (see [9]). Thus, the above inradius and diameter could also be taken with respect to $r$.

5 Subgraphs of finite inradius and finite measure

The preceding sections were concerned with full graphs satisfying global Poincaré type inequalities. Here, we show that any subgraph satisfies a global Poincaré type inequality in terms of its inradius.

We consider a graph $(X, b)$. A subset $\Omega$ of $X$ is said to satisfy a global topological Poincaré inequality if there exists a $c > 0$ with

$$\|f\|_2^2 \leq cE(f)$$

for all $f \in D$ with $\text{supp}(f) \subset \Omega$. (As in the previous section validity of this inequality is equivalent to validity of the corresponding inequality with $\| \cdot \|_V$ instead of $\| \cdot \|_\infty$.) In this case, we denote the best constant $c$ in this inequality by $c^\Omega_P$. For the sake of definiteness we set $c^\Omega_P := \infty$ in all other cases.

**Theorem 5.1** (Computing $c^\Omega_P$ via $\ell^2$-theory). Let $(X, b)$ be a graph and $\Omega$ as subset of $X$ satisfying a global topological Poincaré inequality. The $n, H^\Omega_m$ has pure point spectrum for any probability measure $m$ of full support in $\Omega$. Moreover,

$$\frac{1}{c^\Omega_P} = \inf \{\lambda_0 \mid \lambda_0 \text{ lowest eigenvalue of } H^\Omega_m\}$$

holds.

**Proof.** The first statement follows from Lemma [1.1] The second statement follows along the same lines as the proof of Theorem [4.5]. \qed

Our next aim is to give a geometric characterization of validity of a topological Poincaré inequality. To do so, we will invoke the pseudometric $r_\Omega$. Recall that it is defined via

$$r_\Omega(x, y) := \sup \{|f(x) - f(y)|^2 \mid f \in D \text{ with } f \geq 0, \text{supp}(f) \subset \Omega, E(f) \leq 1\}.$$

**Theorem 5.2** (Geometric characterisation of $c^\Omega_P$). Let $(X, b)$ be an arbitrary graph and $\Omega$ be a proper subset of $X$. The equality

$$c^\Omega_P = \text{Inr}_r(\Omega) = \text{diam}_r(\Omega)$$

holds. In particular, $\Omega$ allows for a topological Poincaré inequality if and only if its inradius with respect to $r_\Omega$ is finite.

**Remark.** may seem surprising that inradius and diameter of $\Omega$ agree (with respect to $r_\Omega$). The reason is that the pseudometric $r_\Omega$ treats the whole complement of $\Omega$ as one point and this makes any two points close to $X \setminus \Omega$ close to each other (even if they are rather far apart in terms of, say, the pseudometric $r$ on the set $X$).
Proof. The equality \( c_P^0 = \text{diam}_{r_0}(X) \) can be shown as in the proof of Proposition 4.2 (with \( \mathcal{F}_c(X) \) replaced by the set of functions in \( D \) which vanish outside of \( \Omega \)). It remains to show \( c_P^0 = \text{Inr}_{r_0}(\Omega) \). To do so, we establish two inequalities:

The inequality \( \text{Inr}_{r_0}(\Omega) \leq c_P^0 \) holds: Choose \( x \in X \) and \( p \in X \setminus \Omega \). Then, for any non-trivial \( f \in D \) with \( f \geq 0 \) and \( f = 0 \) on \( X \setminus \Omega \) we find

\[
|f(x) - f(p)|^2 = |f(x)|^2 \leq \|f\|_\infty^2 \leq c_P^0 \mathcal{E}(f).
\]

This implies \( r_0(x, p) \leq c_P^0 \) and, therefore, \( \text{Inr}_{r_0}(\Omega) \leq c_P^0 \).

The inequality \( c_P^0 \leq \text{Inr}_{r_0} \) holds: Choose \( \text{Inr}_{r_0}(\Omega) < s \). Then for \( x \in X \) we can find an \( p \in X \setminus \Omega \) with

\[
r_0(x, p) < s.
\]

For \( f \in D \) with \( f \geq 0 \) supported in \( \Omega \) and \( \mathcal{E}(f) \leq 1 \) this gives

\[
|f(x)|^2 = |f(x) - f(p)|^2 \leq s
\]

and we conclude \( \|f\|_\infty^2 \leq s \). As \( f \) was arbitrary, we obtain \( c_P^0 \leq s \). This finishes the proof. \( \square \)

As an application of the abstract results above we now obtain the following corollary.

**Corollary 5.3** (The finite measure case). Let a non-empty \( \Omega \subset X \) with \( \Omega \neq X \) be given and assume \( m(\Omega) < \infty \) and \( \text{Inr}_d(\Omega) < \infty \). Then, \( H^\Omega \) has pure point spectrum and for its lowest eigenvalue \( \lambda_\Omega \) we have

\[
\lambda_\Omega \geq \frac{1}{\text{Inr}_d(\Omega)m(\Omega)}.
\]

**Remark.** This generalizes a result of [17], which required a uniform version of [8].

**Proof.** We clearly have \( r_\Omega \leq r \leq d \). Now, the desired statements follow from the previous two theorems. \( \square \)

**Remark** (Recovering best constants). The previous considerations deal with proper subsets \( \Omega \) of \( X \). So, it may come as a surprise that they can actually be used to say something on Poincaré type inequalities in the case \( \Omega = X \) as well. In fact, we can (partially) recover results of the previous sections from it. As this is instructive we briefly discuss this:

Recovering the best constant in \( \|f\|_\infty^2 \leq c \mathcal{E}(f) \) for \( f \in \mathcal{F}_c(X) \): We have already noticed that the pseudometric \( r_0 \) appearing in the dealing with this inequality is the supremum over all \( r_\Omega \) for finite \( \Omega \subset X \). Indeed, validity of a Poincaré inequality for all \( f \in \mathcal{F}_c(X) \) is certainly equivalent to validity of a Poincaré inequality on \( \Omega \) for any finite \( \Omega \subset X \). Hence, the considerations of this section give another way to obtain the best constant \( c_P^0 = \text{diam}_{r_0}(X) \). In fact, this constant can be seen as a form of 'intrinsic Dirichlet inradius' as due to the definition of \( r_0 \) and Theorem 5.2 we have

\[
c_P^0 = \text{diam}_{r_0}(X) \quad \text{(definition of } r_0 \text{)} \quad \text{sup} \{ \text{diam}_{r_0}(X) \mid \Omega \subset X \text{ finite} \} \quad \text{sup} \{ \text{Inr}_{r_0}(\Omega) \mid \Omega \subset X \text{ finite} \}.
\]

Recovering \( \|f\|_\infty^2 \leq c \mathcal{E}(f) \) for all \( f \in D \): We may define the pseudometric \( r' \) via

\[
r' := \sup_{\Omega \subset X} r_\Omega.
\]
Then, the considerations of this section show that validity of a Poincaré inequality for all \( f \) vanishing somewhere is governed by \( r' \). In fact, this is already sufficient to obtain the Poincaré inequality for all \( f \in D \): Consider an arbitrary \( f \in D \). If \( f \) is constant there is nothing to show. If \( f \) is not constant we can assume without loss of generality that \( f_+ \) is not trivial. Then, by replacing \( f \) by \( g = f + \lambda 1 \) with a suitable \( \lambda \in \mathbb{R} \) we can assume that \( \|g_+\|_\infty \) is as close to \( \|f\|_V \) as we wish and at the same time \( g_- \) is not trivial. Then, \( g_+ \) must be zero somewhere and from the Poincaré inequality for \( r' \) we find

\[
\|g_+\|_\infty^2 \leq \text{diam}_{r'}(X)\mathcal{E}(g_+).
\]

As \( \|g_+\|_\infty \) is as close to \( \|f\|_V \) as we wish and \( \mathcal{E}(g_+) \leq \mathcal{E}(g) \) holds we can then infer

\[
\|f\|_V^2 \leq \text{diam}_{r'}(X)\mathcal{E}(f).
\]

As clearly, \( r' \leq r \) we see that the best constant is given by

\[
c_P = \text{diam}_{r'}(X).
\]

Indeed, with a similar argument it is not hard to see that \( r' \) equals \( r \). Here, again this best constant can be seen as a form of ‘intrinsic Neumann inradius’ as due to the definition of \( r' \) and Theorem 5.2 and we have

\[
c_P = \text{diam}_{r'}(X) = \sup \{ \text{diam}_{r_0}(X) \mid \Omega \subset X \} = \sup \{ \text{Inr}_{r_0}(\Omega) \mid \Omega \subset X \}.
\]

### 6 Bounds on higher eigenvalues

When \( m \) is finite a Poincaré inequality of the form \( \|f\|_V^2 \leq c\mathcal{E}(f) \) can also be employed to obtain bounds for large eigenvalues of the operator \( H^{(N)}_m \). This is discussed in the present section. We suppress the dependence on \( m \) and denote by

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots
\]

the Eigenvalues of \( H^{(N)}_m \), counted with multiplicity. The possibility of controlling higher eigenvalues by Poincaré inequalities comes from the following lemma and the considerations in Section 5.

**Lemma 6.1.** Let \((X,b)\) a graph and let \( m \) a measure of full support on \( X \) and suppose that \( H^{(N)}_m \) has pure point spectrum. For all \( n \in \mathbb{N} \) and all sets \( F \subset X \) of cardinality \( n \) such that \( \delta_x \in D \) for all \( x \in F \) we have

\[
\lambda_{n+1} \geq \lambda_{X\setminus F},
\]

where \( \lambda_{X\setminus F} \) is the smallest eigenvalue of \( H^{(N)}_m \setminus F \).

**Proof.** Using the min-max principle and the fact that 1 is an eigenfunction to the eigenvalue \( \lambda_0 = 0 \), for \( F = \{ x_1, \ldots, x_n \} \) we obtain

\[
\lambda_{n+1} \geq \inf \left\{ \frac{\mathcal{E}(g)}{\|g\|_2^2} \mid 0 \neq g \in D(\mathcal{E}^{(N)}) \cap \{ \delta_{x_1}, \ldots, \delta_{x_n}, 1 \}^\perp \right\}
\]

\[
= \inf \left\{ \frac{\mathcal{E}(g)}{\|g\|_2^2} \mid 0 \neq g \in D(\mathcal{E}^{(N)}) \text{ with } g|_F = 0 \text{ and } g \perp 1 \right\}
\]

\[
\geq \lambda_{X\setminus F}.
\]

This finishes the proof. 

\[\square\]
Recall that $c_P$ is the best constant $c$ for which the inequality $\|f\|_2^2 \leq c\mathcal{E}(f)$ holds for all $f \in D$ and that $c_P^{\Omega}$ is the best constant $c'$ for which the inequality $\|f\|_\infty^2 \leq c'\mathcal{E}(f)$ holds for all $f \in D$ with $\text{supp} f \subset \Omega$. We also use the convention $c_p = \infty$ respectively $c_P^{\Omega} = \infty$ if there exists no such constant.

**Theorem 6.2.** Let $(X,b)$ be a graph and let $m$ be a finite measure of full support on $X$. If $c_P < \infty$, then $H_m^{(N)}$ has pure point spectrum and for all $n \in \mathbb{N}$ and all sets $F \subset X$ of cardinality $n$ such that $\delta_x \in D$ for all $x \in F$ the following holds.

(a) \[ \lambda_{n+1} \geq \frac{4}{c_P m(X \setminus F)}. \]

(b) If, additionally, $n \geq 1$, then $c_P^{X\setminus F} \leq c_P < \infty$ and \[ \lambda_{n+1} \geq \frac{1}{c_P^{X\setminus F} m(X \setminus F)}. \]

**Proof.** That the spectrum of $H_m^{(N)}$ is pure point was already observed in Corollary 3.3.

(a): As seen in the proof of Lemma 6.1 we have \[ \lambda_{n+1} \geq \inf \left\{ \frac{\mathcal{E}(g)}{\|g\|_2^2} \mid 0 \neq g \in D(\mathcal{E}(N)) \text{ with } g|_F = 0 \text{ and } g \perp 1 \right\}. \]

For $g \in \ell^2(X,m)$ with $g|_F = 0$ and $g \perp 1$ Proposition 3.5 yields \[ \|g\|_V \leq \frac{4}{c_P m(X \setminus F)} \|g\|_2. \] This inequality combined with the Poincaré inequality for $\mathcal{E}$ shows the claim.

(b): A function which vanishes in (more than) one point of $X$ satisfies $\|f\|_\infty \leq \|f\|_V$. Therefore, $c_P^{X\setminus F} \leq c_P < \infty$. With this at hand the claim follows from Lemma 6.1 and Theorem 5.1. \qed

**Remarks.** (a) The method of proof is taken from [15], where a similar statement as in (a) is proven for the operator $H_m^{(D)}$ for uniformly transient graphs. Note that the graphs there always satisfy the additional summability condition (S).

(b) We chose to present both estimates because (a) gives a better constant (which is sharp for $\lambda_1$, cf. Theorem 3.6), while (b) may provide better asymptotics for large eigenvalues.

(c) Note that we have defined the Poincaré inequality on proper subsets of $X$ via the supremum norm. If we had chosen to define it via the variational norm the estimate in (b) would contain the factor 4 (as the estimate in (a)) does.

(d) We note that independent of our work related estimates on higher eigenvalues for canonically compactifiable graphs are also contained in [12].

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