QCD trilemma

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We find that the chiral phase transition (chiral crossover) in QCD at physical point is triggered by big imbalance among three fundamental quantities essential for the QCD vacuum structure: susceptibility functions for the chiral symmetry, axial symmetry, and the topological charge. The balance, dubbed the QCD trilemma, is unavoidably violated when one of the magnitudes among them is highly dominated, or suppressed. Based on a three-flavor Nambu-Jona-Lasinio model, we explicitly evaluate the amount of violation of the QCD trilemma at physical point, and show that the violation takes place not only at vacuum, but even in a whole temperature regime including the chiral crossover epoch. This work confirms and extends the suggestion recently reported from lattice QCD with 2 flavors on dominance of the axial and topological susceptibilities left in the chiral susceptibility at high temperatures. It turns out that the imbalance is essentially due to the flavor symmetry violation of the lightest three flavors. The violation of QCD trilemma and its flavor dependence can be tested by lattice simulations with 2 + 1 flavors in the future, and would also give a new guiding principle to deeper understand the QCD phase structure, such as the Columbia plot, including possible extension with external fields.

The chiral phase transition is of importance to comprehend the QCD vacuum. The order parameter is given by the quark condensate, or equivalently difference of meson correlation functions (susceptibility functions) for the chiral partners. Though being so simple and well-defined, the chiral order parameter in real-life QCD is more involved due to presence of finite quark masses, which explicitly break the chiral symmetry.

Indeed, the chiral symmetry is restored at high temperature only in part, referred to as the chiral crossover, and gets more intricate because the chiral order parameter couples with the axial and topological features of the QCD vacuum via finite quark masses. This tagging is captured by a robust relation between order parameters for the chiral SU(2) × SU(2) and SU(1)A axial symmetry, which is constructed from a set of generic anomalous Ward identities for the three-flavor chiral SU(3) L × SU(3) R symmetry [1, 2]:

\[
\chi_{\eta-\delta} = \chi_{\pi-\delta} + \frac{4}{m_l^2} \chi_{\text{top}}, \tag{1}
\]

where \(m_l = m_u = m_d\) is the isospin-symmetric mass for the lightest up and down quarks; \(\chi_{\eta-\delta} \equiv \chi_{\eta} - \chi_{\delta}\) and \(\chi_{\pi-\delta} \equiv \chi_{\pi} - \chi_{\delta}\) are differences of meson susceptibilities related to the partners for the chiral \((\chi_{\eta} \text{ and } \chi_{\delta})\) and axial \((\chi_{\pi} \text{ and } \chi_{\pi})\) symmetries; \(\chi_{\text{top}}\) is the topological susceptibility. By the chiral SU(2) and axial rotations, the meson susceptibilities exchange their partners: \(\chi_{\eta} \leftrightarrow \chi_{\delta} \text{ (chiral)}\) and \(\chi_{\pi} \leftrightarrow \chi_{\delta} \text{ (axial)}\), hence \(\chi_{\eta-\delta} = 0\) and \(\chi_{\pi-\delta} = 0\) are signals of restorations of the associated symmetries. More details on the susceptibilities are presented in Appendix A. \((\chi_{\text{top}} < 0 \text{ and other susceptibilities are positive in our sign convention. See also Appendix A.})\) Thus Eq.(1) dictates coherence of the chiral SU(2) symmetry breaking and U(1)A breaking, linked with the transition rate of the topological charge, where all the breaking is controlled by nonzero quark masses.

One might simply think that the above three amplitudes should take the same order of magnitude, namely, no preference among them in magnitude. This case is thought to be “balanced”, and we dub this situation as “QCD trilemma”, and depict a triangle cartoon in Fig.1. The degree of formation of QCD trilemma can be evaluated via the following quantity:

\[
R = \frac{4}{m_l^2} \chi_{\text{top}} + \chi_{\pi-\delta} - \frac{4}{m_l^2} \chi_{\text{top}} = 1 + \frac{4}{m_l^2} \chi_{\pi-\delta}. \tag{2}
\]

By using this \(R\) the Ward identity in Eq.(1) is rewritten as

\[
\chi_{\eta-\delta} = R \cdot \chi_{\pi-\delta}, \tag{3}
\]

so that \(R\) measures the size of gap in magnitude between the chiral \((\chi_{\eta-\delta})\) and axial \((\chi_{\pi-\delta})\) order parameters. One may quantify the amount of balance to keep the QCD trilemma, by saying that the three susceptibilities are balanced when

\[
0.1 < R < 0.9, \tag{4}
\]

otherwise imbalanced due to an accidental big cancellation requiring a tuning by more than 10%. We shall dub \(R\) as the trilemma estimator.
To quantitatively clarify the imbalanced chiral phase transition in real-life QCD, we work on a Nambu-Jona-Lasinio (NJL) model with three flavors. In addition to the standard-scalar four-fermion interaction terms the model includes a determinant term (Kobayashi-Maskawa-'t Hooft [8][11]), which would be induced from the QCD instanton coupled to quarks and explicitly breaks the $U(1)_A$ axial symmetry [3]. We refer readers to a review paper [12] and Appendix B as the details on the Lagrangian, the formalism at finite temperature, and the formulae of the chiral and axial susceptibilities. We adopt the same inputs as in the literature, working in the isospin symmetric limit ($m_u = m_d = m_\pi$), to fix the model parameters at zero temperature ($T = 0$). QCD at the physical point that we call real-life QCD, presently modeled by the NJL description, predicts $m_\pi = 5.5$ MeV and $m_\eta = 138$ MeV [12], which well reproduces the hadronic observables (pion decay constant, pion, kaon and eta prime masses, and so forth).

We have also found that the value of $\chi_{\text{top}}$ at $T = 0$ estimated from the present model is in good agreement with the lattice result. Furthermore, the (subtracted and normalized) chiral condensate as well as the (normalized) $\chi_{\text{top}}$ exhibit perfectly consistent $T/T_c$-scalings in a qualitative sense, including below and above the psuedocritical temperature $T_c|\text{NJL} \simeq 188$ MeV, in comparison with the lattice data. For details, see Appendix C. This confirms that the present NJL model describes the chiral crossover phenomenon in real-life QCD quite well.

Figure 2 shows values of the trilemma estimator $R$ evolved with $T$, allowing $m_s$ off the physical point with $m_l$ kept physical. See the middle-solid curve with $m_s = 138$ MeV, which corresponds to real-life QCD. Comparison with the available $2 + 1$ flavor-lattice QCD data (with $m_\pi = 135$ MeV) on $R$ [11] — reconstructed from the data on $\chi_{\pi-\delta}$ and $\chi_{\text{top}}$ through the relation Eq. [8] — has also been displayed (in the zoomed-in window), which shows a good agreement including the error bars, for $140$ MeV \(\lesssim T \lesssim 200\) MeV.\(^2\)

Remarkably, in a whole temperature regime including the chiral crossover regime, real-life QCD stays outside the “balanced” region defined as in Eq. [3]. We have observed $R \simeq 0.05$ at around $T$ covering the crossover point ($T_c|\text{NJL} \simeq 188$ MeV: $140$ MeV \(\lesssim T \lesssim 200\) MeV), consistently with the lattice data, and $R \lesssim 0.01$ at $T \gtrsim 300$.

\(^1\) We will not consider intrinsic-temperature dependent couplings, instead, all $T$ dependence should be induced only from the thermal quark loop corrections to the couplings defined and introduced at vacuum. Actually, the present NJL shows good agreement with lattice QCD results on the temperature scaling for the chiral, axial, and topological susceptibilities, as shown in Appendix C. In this sense, we do not need to introduce such an intrinsic $T$ dependence for the model parameters in the regime up to temperatures around the chiral crossover.

\(^2\) The individual $\chi_{\pi-\delta}$ and $\chi_{\text{top}}$ involve some discrepancy between the NJL estimate and the lattice data, on both of which the NJL tends to give larger values. For details, see Appendix C.

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The lattice QCD simulations have revealed a faster drop of $\chi_{\pi-\delta}$, than $\chi_{\pi-\delta}$ around and above the pseudo-critical temperature of the chiral crossover [3][4], which indicates $R \ll 1$ in Eq. [3]. This is also supported from a rigorous argument based on QCD-inequality like relations [5] and its generalized evidence based on the lattice QCD setup [6]. Furthermore, a recent lattice study (with two lightest flavors) has shown significant contributions from the axial and topological susceptibilities ($\chi_{\pi-\delta}$ and $\chi_{\text{top}}$) left in the chiral susceptibility ($\chi_{\pi-\delta}$) in the chiral crossover domain [7]. This is rephrased in terms of the QCD trilemma as presence of the big imbalance, $R \ll 1$, in Eq. [3]. Thus, real-life QCD might be imbalanced in realizing the chiral crossover. Actually, though not explicitly addressed and restricted only around the crossover regime, this imbalance could be read off from the existing lattice QCD data [1] and also [7] with taking into account possible finite volume effects and statistical errors.

In this write-up, prior to the lattice simulations, we demonstrate that based on a chiral effective theory, real-life QCD having $2 + 1$ flavors at physical point indeed yields $R \ll 1$, i.e., exhibits the violation of QCD trilemma, in a whole temperature regime including the chiral crossover regime. We find that the violation of QCD trilemma, and the related dominance of $\chi_{\pi-\delta}$ and $\chi_{\text{top}}$ in the chiral order parameter in the crossover regime are due to the three-flavor symmetry violation.
MeV. Namely, the amount of imbalance is slightly amplified by thermal loop effects as $T$ develops from zero. 3

One might note that subtraction by $\chi_\delta$ in Eq.(1) is ambiguous, and can be replaced by another chiral susceptibility in the sigma meson channel ($\chi_\sigma$). We have checked that this replacement does not alter our main conclusion that real-life QCD involves big imbalance. We have also found that $\chi_{\pi-\sigma} \gg \chi_{\eta-\sigma}$ at $T = 0$, $\chi_{\pi-\sigma} \ll \chi_{\eta-\sigma}$ around the chiral crossover, and finally $\chi_{\pi-\sigma}$ and $\chi_{\pi-\delta}$ will get close to zero with keeping $\chi_{\pi-\sigma} \ll \chi_{\eta-\sigma}$ above the pseudo-critical temperature, which is consistent with the lattice data.

Although the present model parameters are fixed at the physical point, we may deduce some conjectures on the violation of QCD trilemma in a view of the quark mass difference. Extrapolating off real-life QCD, one can then observe that the “imbalance” domain still covers the two-flavor limit case with $m_s = 50$ GeV (bottom-dot-dashed curve), where strange quark is decoupled, and the amount of imbalance is greater than that in the real-life QCD case. Taking the three-flavor symmetric limit $m_s = m_l$ with $m_l$ fixed to the physical value, we find “balanced” QCD (top-dashed curve), which keeps almost constant $R$ at any finite $T$ within the “balanced” interval in Eq.(4). This implies that the three-flavor symmetry would be related to the relaxation of the QCD trilemma.

Since the order of magnitude for $R$ tends to be almost fixed at $T = 0$, we may focus only on $R$ at $T = 0$, and look into the flavor-symmetry dependence on $R$, by varying $m_s$ in a wide range, with fixed $m_l$ to the physical value. Figure 3 shows plots on $R$ as a function of $m_s/m_l$, together with the “balanced” interval in Eq.(4). As $m_s$ goes off the flavor symmetric limit in the “balanced” domain to be smaller, $R$ tends to get larger, to flow into the “imbalance” domain with gigantic suppressed $\chi_{\text{top}}$. The figure clearly shows that “balanced” QCD should have had some approximate three-flavor symmetry for up, down and strange quarks with $0.06 \lesssim m_s/m_l \lesssim 6$. Part 5 of Appendix C supplements detailed $m_l$-$m_s$ scaling laws on individual susceptibilities.

We have observed that an approximate three-flavor symmetry ($m_l \sim m_s$) would make the QCD vacuum balanced. This implies that $R$ can be kicked up into the “balanced” regime at higher $T$ in the thermal history of Universe, where the three light quarks act as almost massless, hence the flavor symmetry can approximately work. This possibility could not be covered by the framework of the present NJL model with $T \lesssim 300$ MeV, though.

As $T$ gets higher, the QCD instanton effects will be extremely diluted by an inverse Boltzman-like suppression $\sim e^{-T^2/\mu_{\text{QCD}}^2}$ (with a collective scale factor of QCD, $\mu_{\text{QCD}}$) as predicted from the dilute instanton gas description, so that the $U(1)_A$-breaking determinant term among quarks, $\mathcal{L}_{\text{det}} = K \det(q_L q_R) + \text{h.c.}$, will also

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3 Above $T \sim 300$ MeV, the NJL description as the effective theory of QCD will be somewhat unreliable because the deconfining color degrees of freedom would be significant.
get significantly suppressed [15]. Actually, $R$ becomes larger as $K$ gets smaller, because $\chi_{\text{top}} \propto K$ and gets smaller (see Eq. [2]). Thereby one would suspect that $R$ could jump in the “balanced” regime for $0.1 < R < 0.9$, above $T \gtrsim 300$ MeV [4].

To monitor $R$ in such a higher-$T$ QCD, we may introduce a model having only $\mathcal{L}_{\text{det}}$ plus the quark kinetic term, with the determinant coupling $K$ simply scaled by temperature as $K(T) = e^{-T^2/\mu_{\text{QCD}}} \cdot K$. This determinant-type interaction would be of the most minimal form relevant to $R$ dictating the chiral and axial breaking, which could mimic a “gluonic-interacting cloud” covering the quarks in quark-gluon plasma. More details on the model description and parameter setup for evaluation of $R$ are provided in Appendix D. See Fig. 4, which indicates a possibility that $R$ can be trapped in the “imbalanced” regime for $0.9 < R < 1$ at $T \sim 500 - 1000$ MeV. This takes place due to the highly suppressed $K(T)$, which promptly drives $R$ up to close to 1 as $T$ gets higher. A decisive conclusion on $R$ for $T \gtrsim 300$ MeV can be derived from lattice QCD simulations with chiral fermions.

In conclusion, real-life QCD is required to relax the trilemma ($R \ll 1$), meaning that the much smaller chiral order parameter is given by a big cancellation between the axial order parameter and the transition rate of the topological charge. This is schematically depicted in Fig. 1. This is “imbalance” of the QCD vacuum, present in a whole temperature regime of thermal QCD including the vacuum at $T = 0$. We conjectured that this imbalance or violation of QCD trilemma is triggered due to the three-flavor symmetry violation for up, down and strange quarks. The present work confirms and extends the suggestion recently reported from lattice QCD with 2 flavors on dominance of the axial and topological susceptibilities left in the chiral susceptibility at high temperatures over the chiral crossover. Our findings can directly be tested on lattice QCD with $2 + 1$ flavors at physical point and also off physical point, in the future. The notion of QCD trilemma and its violation would also be of great help to better understand the QCD phase structure in a various aspect, such as the Columbia plot, and also in application to thermomagnetic QCD, which are deserved in another publication.

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Appendix A: Scalar, pseudoscalar, and topological susceptibilities in QCD

The pseudoscalar susceptibilities $\chi_{P}^{uu,dd,ud}$, $\chi_{P}^{ss}$ and $\chi_{P}^{us,ds}$, and the pion susceptibility $\chi_{\pi}$ are defined as

$$
\chi_{P}^{f_{1}f_{2}} = \int_{T} d^{4}x \langle (\bar{q}_{f_{1}}(0)i\gamma_{5}q_{f_{1}}(0))(\bar{q}_{f_{2}}(x)i\gamma_{5}q_{f_{2}}(x)) \rangle, \quad \text{for} \quad q_{f_{1,2}} = u, d, s, \\
\chi_{\pi} = \int_{T} d^{4}x \left[ \langle (\bar{u}(0)i\gamma_{5}u(0))(\bar{u}(x)i\gamma_{5}u(x)) \rangle_{\text{conn}} + \langle (\bar{d}(0)i\gamma_{5}d(0))(\bar{d}(x)i\gamma_{5}d(x)) \rangle_{\text{conn}} \right], \quad (A1)
$$

with $\langle \cdots \rangle_{\text{conn}}$ being the connected part of the correlation function. Here $\int_{T} d^{4}x \equiv \int_{0}^{1/T} d\tau \int d^{3}x$ with the imaginary time $\tau = ix_{0}$.

The topological susceptibility $\chi_{\text{top}}$ is related to the $\theta$ vacuum configuration of QCD. It is defined as the curvature
of the $\theta$-dependent vacuum energy $V(\theta)$ in QCD at $\theta = 0$:

$$\chi_{\text{top}} = -\left. \int d^4x \frac{\delta^2 V(\theta)}{\delta \theta(x) \delta \theta(0)} \right|_{\theta=0}. \quad (A2)$$

Performing the $U(1)_A$ rotation for quark fields together with flavor singlet condition \[16\] \[17\], one can transfer the \( \theta \) dependence coupled to the topological gluon configurations, via the axial anomaly, to current quark mass terms. Thus $\chi_{\text{top}}$ goes like \[2\]

$$\chi_{\text{top}} = \bar{m}^2 \left[ \frac{(\bar{u}u)}{m_l} + \frac{(\bar{d}d)}{m_l} + \frac{(\bar{s}s)}{m_s} + \chi^{uu}_P + \chi^{dd}_P + \chi^{uu}_P + 2\chi^{ud}_P + 2\chi^{ds}_P + 2\chi^d_{d} \right] \approx \frac{1}{4} \left[ m_l \left( (\bar{u}u) + (\bar{d}d) \right) + m_l^2 \left( \chi^{uu}_P + \chi^{dd}_P + 2\chi^{ud}_P \right) \right] = m_s (\bar{s}s) + m_s^2 \chi^{ss}_s, \quad (A3)$$

where we have taken the isospin symmetric limit $m_u = m_d = m_l$; $\bar{m} \equiv \left( \frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right)^{-1}$. The signs of the quark masses and condensates are chosen to be positive and negative, respectively, such that $\chi_{\text{top}} > 0$. Note that $\chi_{\text{top}} \to 0$, when either of quarks becomes massless ($m_l$ or $m_s \to 0$), reflecting the flavor-singlet nature of the QCD vacuum.

A set of generic anomalous Ward identities for the three-flavor chiral $SU(3)_L \times SU(3)_R$ symmetry is derived directly by the chiral variations of the QCD action, in which only the current quark mass term breaks the chiral symmetry explicitly. It goes like \[1\] \[2\]

$$\langle \bar{u}u \rangle + \langle \bar{d}d \rangle = -m_l \chi_\pi, \quad \langle \bar{u}u \rangle + \langle \bar{d}d \rangle + 4 \langle \bar{s}s \rangle = - \left[ m_l \left( \chi^{uu}_P + \chi^{dd}_P + 2\chi^{ud}_P \right) - 2(m_s + m_l) \left( \chi^{us}_s + \chi^{ds}_d \right) + 4m_s \chi^{ss}_s \right], \quad (A4)$$

Combining Ward identities in Eq.(A4), we find

$$\chi_{\text{top}} = \frac{1}{2} m_l m_s \left( \chi^{uu}_P + \chi^{dd}_P \right) = \frac{1}{4} m_l^2 \chi_\eta, \quad (A5)$$

where $\chi_\eta$ is the eta meson susceptibility, defined as

$$\chi_\eta = \int d^4x \left[ \langle (\bar{u}(0)i\gamma_5 u(0))(\bar{u}(x)i\gamma_5 u(x)) \rangle + \langle (\bar{d}(0)i\gamma_5 d(0))(\bar{d}(x)i\gamma_5 d(x)) \rangle \right] + 2 \langle (\bar{u}(0)i\gamma_5 u(0))(\bar{d}(x)i\gamma_5 d(x)) \rangle \equiv \chi^{uu}_P + \chi^{dd}_P + 2\chi^{ud}_P. \quad (A6)$$

The last line of Eq.(A5) can be written as

$$(\chi_\eta - \chi_\delta) = (\chi_\pi - \chi_\delta) + \frac{4}{m_l^2} \chi_{\text{top}}, \quad (A7)$$

where $\chi_\delta$ is the susceptibility for the delta meson channel ($a_0$ meson in terms of the Particle Data Group identification), defined in the same way as $\chi_\pi$ in Eq.(A1) with the factors of $(i\gamma_5)$ replaced with identity 1. $\chi_{\eta-\delta} \equiv \chi_\eta - \chi_\delta$ and $\chi_{\pi-\delta} \equiv \chi_\pi - \chi_\delta$ play the roles of the chiral and axial order parameters, which signal the restorations when those (asymptotically) reach zero.

Appendix B: The NJL model and relevant formulas

1. Model description

The three-flavor NJL model Lagrangian that we work on is constructed as follows (for a review, see \[12\]):

$$\mathcal{L} = \bar{q} (i\gamma_\mu \partial^\mu - m) q + \mathcal{L}_{4f} + \mathcal{L}_{\text{KMT}},$$

$$\mathcal{L}_{4f} = \frac{g_8}{2} \sum_{a=0}^{8} \left[ \langle \bar{q} \lambda^a q \rangle^2 + \langle \bar{q} i\gamma_5 \lambda^a q \rangle^2 \right],$$

$$\mathcal{L}_{\text{KMT}} = g_D \left[ \det \bar{q}_i(1 + \gamma_5)q_j + \text{h.c.} \right], \quad (B1)$$
where $q$ is the $SU(3)$ triplet-quark field, $q = (u, d, s)^T$, and $\lambda^a$ ($a = 0, \cdots, 8$) are the Gell-Mann matrices in the flavor space with $\lambda^0 = \sqrt{2/3} \text{diag}(1, 1, 1)$. The current quark masses are embedded in the mass matrix $m$ of the form $m = \text{diag}(m_u, m_d, m_s)$.

The four-fermion interaction term $\mathcal{L}_{4f}$ is invariant under the chiral $U(3)_L \times U(3)_R$ transformation: $q \rightarrow U \cdot q$ with $U = \exp[-i\gamma_5 \sum_{a=0}^{8}(\lambda^a/2)\theta^a]$ and the chiral phases $\theta^a$. The mass term in $\mathcal{L}$ explicitly breaks $U(3)_L \times U(3)_R$ symmetry. The determinant term $\mathcal{L}_{KMT}$, called the Kobayashi-Maskawa-'t Hooft [8–11] term, induced from the QCD instanton configuration, preserves $SU(3)_L \times SU(3)_R$ invariance (associated with the chiral phases labeled as $a = 1, \cdots, 8$) but breaks the $U(1)_A$ (corresponding to $a = 0$) symmetry, measured by the effective coupling constant $g_D$.

2. Quark condensate

At finite temperatures, the expectation value of an operator $\Theta$ is given by the statistical thermal average:

$$
\langle \Theta \rangle = \frac{\text{Tr} \Theta e^{-H/T}}{\text{Tr} e^{-H/T}},
$$

where $H$ is the Hamiltonian operator.

We employ the mean-field approximation (MFA) and then obtain the gap equation and the thermodynamic potential. Then, the quark condensates act as the parameters in the MFA and are $T$-dependent, which we define as

$$
\langle \bar{u}u \rangle \equiv \alpha, \quad \langle \bar{d}d \rangle \equiv \beta, \quad \langle \bar{s}s \rangle \equiv \gamma.
$$

Searching for the minimum point of the thermodynamic potential with respect to $\alpha, \beta$ and $\gamma$ as variational parameters, the quark condensate formula are given as follows:

$$
\langle \bar{q}_i q_i \rangle = -2N_c \int \frac{d^3p}{(2\pi)^3} \frac{M_i}{E_i} \left[ 1 - 2(\exp(E_i/T) + 1)^{-1} \right],
$$

where $E_i = \sqrt{M_i^2 + p^2}$, $N_c$ denotes the number of colors to be fixed to three, and $M_i$ are the dynamical masses:

$$
M_u = m_u - 2g_s\alpha - 2g_D\beta\gamma
$$
$$
M_d = m_d - 2g_s\beta - 2g_D\alpha\gamma
$$
$$
M_s = m_s - 2g_s\gamma - 2g_D\alpha\beta.
$$

3. Chiral and axial susceptibilities

In this subsection, we introduce susceptibilities for pseudoscalar and scalar meson channels and give their explicit formulas.

a. Pseudoscalar meson channel

In the $\eta - \eta'$ coupled channel, the pseudoscalar meson susceptibility is defined on the generator basis as

$$
\chi_P^{ij} = \int_T d^4x (i\bar{q}(x)\gamma_5\lambda^i q(x))(i\bar{q}(0)\gamma_5\lambda^j q(0)),
$$

where $i, j = 0, 8$. This $\chi_P^{ij}$ takes a matrix form

$$
\chi_P = \frac{-1}{1 + G_P \Pi_P (0, 0)} \cdot \Pi_P (0, 0),
$$

where $G_P$ is the coupling strength matrix and $\Pi_P$ is the polarization tensor matrix:

$$
G_P = \begin{pmatrix}
G_P^{00} & G_P^{08} \\
G_P^{80} & G_P^{88}
\end{pmatrix} = \begin{pmatrix}
g_s - \frac{2}{3}(\alpha + \beta + \gamma)g_D & -\frac{\sqrt{2}}{6}(2\gamma - \alpha - \beta)g_D \\
-\frac{\sqrt{2}}{6}(2\gamma - \alpha - \beta)g_D & g_s - \frac{1}{3}(\gamma - 2\alpha - 2\beta)g_D
\end{pmatrix},
$$
By moving on to the flavor base via the base transformation, the scalar susceptibilities are cast into the form:

\[
\Pi_P = \begin{pmatrix} \Pi_P^{00} & \Pi_P^{08} \\ \Pi_P^{80} & \Pi_P^{88} \end{pmatrix} = \begin{pmatrix} \frac{2}{3}(2I_P^{uu} + I_P^{dd}) & 2\sqrt{2}(I_P^{uu} - I_P^{ss}) \\ 2\sqrt{2}(I_P^{uu} - I_P^{ss}) & \frac{2}{3}(I_P^{uu} + 2I_P^{ss}) \end{pmatrix},
\]

(B9)

with \(I_P^{ij}(\omega, p)\) being the pseudoscalar one-loop polarization functions [20],

\[
I_P^{ii}(0,0) = -\frac{N_c}{\pi^2} \int_0^\Lambda dp\,p^2 \frac{1}{E_i} \left[1 - 2\left(\exp(M_i/T) + 1\right)^{-1}\right], \quad \text{for } i = u, d, s.
\]

(B10)

By performing the basis transformation, the pseudoscalar susceptibilities defined in Eq. (A1) on the flavor basis are thus obtained as

\[
\left(\frac{1}{2}\chi_P^{uu} + \frac{1}{2}\chi_P^{dd} + \frac{1}{2}\chi_P^{ss} \right) = \left(\begin{array}{c} \frac{1}{6} \sqrt{2} \\ -\frac{\sqrt{2}}{3} \\ \frac{1}{6} \end{array} \right)
\]

\[
\left(\begin{array}{c} \chi_P^{00} \\ \chi_P^{08} \\ \chi_P^{80} \\ \chi_P^{88} \end{array} \right),
\]

(B11)

where we have taken the isospin symmetric limit into account, i.e., \(\chi_P^{uu} = \chi_P^{dd}\) and \(\chi_P^{ss} = \chi_P^{ds}\).

For \(\chi_\pi\) defined in Eq. (A1), the explicit formula in the NJL model reads

\[
\chi_\pi = \frac{-1}{1 + G_\pi \Pi_\pi(0,0)} \cdot \Pi_\pi(0,0),
\]

(B12)

where \(G_\pi = g_s + g_D\gamma\), which is the coupling strength in the pion channel, and \(\Pi_\pi\) is the quark-loop polarization function for \(\chi_\pi\), which is evaluated by using \(I_P^{ii}\) in Eq. (B10) as

\[
\Pi_\pi = I_P^{uu} + I_P^{dd} = 2I_P^{uu}.
\]

(B13)

b. Scalar meson channel

The definitions of scalar susceptibilities are similar to those for pseudoscalars’, which are given just by removing \(\gamma_5\) in the definition of pseudoscalar susceptibilities, and supplying the appropriate one-loop polarization functions and the corresponding coupling constants.

In the 0 - 8 coupled channel, the scalar susceptibility matrix \(\chi_S\) is evaluated in the present NJL on the generator basis as

\[
\chi_S = \frac{-1}{1 + G_S \Pi_S(0,0)} \cdot \Pi_S(0,0),
\]

(B14)

where \(G_S\) is the coupling strength matrix,

\[
G_S = \begin{pmatrix} G_{00}^S & G_{08}^S \\ G_{80}^S & G_{88}^S \end{pmatrix} = \begin{pmatrix} g_s + \frac{2}{3}(\alpha + \beta + \gamma)g_D & \frac{\sqrt{2}}{6}(2\gamma - \alpha - \beta)g_D \\ \frac{\sqrt{2}}{6}(2\gamma - \alpha - \beta)g_D & g_s + \frac{1}{3}(\gamma - 2\alpha - 2\beta)g_D \end{pmatrix}.
\]

(B15)

The scalar polarization tensor matrix \(\Pi_S\) in Eq. (B14) is given by

\[
\Pi_S = \begin{pmatrix} \Pi_{00}^S & \Pi_{08}^S \\ \Pi_{80}^S & \Pi_{88}^S \end{pmatrix} = \begin{pmatrix} \frac{2}{3}(2I_S^{uu} + I_S^{dd}) & 2\sqrt{2}(I_S^{uu} - I_S^{ss}) \\ 2\sqrt{2}(I_S^{uu} - I_S^{ss}) & \frac{2}{3}(I_S^{uu} + 2I_S^{ss}) \end{pmatrix},
\]

(B16)

\[
I_S^{ii}(0,0) = -\frac{N_c}{\pi^2} \int_0^\Lambda dp\,p^2 \frac{1}{E_i} \frac{E_i^2 - M_i^2}{E_i^4} \left\{1 - 2\left[\exp(M_i/T) + 1\right]\right\} \quad i = u, d, s.
\]

(B17)

By moving on to the flavor base via the base transformation, the scalar susceptibilities are cast into the form:

\[
\begin{pmatrix} \frac{1}{2}\chi_S^{uu} + \frac{1}{2}\chi_S^{dd} \\ \chi_S^{ud} \\ \chi_S^{su} \\ \chi_S^{ss} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \sqrt{2} \\ \frac{1}{6} - \frac{\sqrt{2}}{3} \\ \frac{1}{6} \\ -\frac{\sqrt{2}}{3} \end{pmatrix}
\]

\[
\begin{pmatrix} \chi_S^{00} \\ \chi_S^{08} \\ \chi_S^{80} \\ \chi_S^{88} \end{pmatrix},
\]

(B18)
in which we have read $\chi_{uu}^{S} = \chi_{dd}^{S}$ and $\chi_{us}^{S} = \chi_{ds}^{S}$.

The $\sigma$ meson susceptibility is defined as

$$\chi_{\sigma} = \int_{T} d^{4}x \left[ \langle \bar{u}(0)u(0)\rangle(\bar{u}(x)u(x)) + \langle \bar{d}(0)d(0)\rangle(\bar{d}(x)d(x)) \right] + 2\langle \bar{u}(0)u(0)\rangle(\bar{d}(x)d(x))$$

$$= 2\chi_{uu}^{S} + 2\chi_{ud}^{S}.$$  \hspace{1cm} (B19)

For the $\delta$ meson susceptibility, it is defined as

$$\chi_{\delta} = \int_{T} d^{4}x \left[ \langle \bar{u}(0)u(0)\rangle(\bar{u}(x)u(x)) \rangle_{\text{conn}} + \langle \bar{d}(0)d(0)\rangle(\bar{d}(x)d(x)) \rangle_{\text{conn}} \right].$$  \hspace{1cm} (B20)

Similar to $\chi_{\sigma}$ in Eq. (B12), the explicit formula for $\chi_{\delta}$ reads

$$\chi_{\delta} = \frac{-\Pi_{\delta}(0,0)}{1 + G_{\delta}\Pi_{\delta}(0,0)},$$  \hspace{1cm} (B21)

where $G_{\delta} = g_{s} - g_{D}\gamma$, which is the coupling strength in the $\delta$ channel, and $\Pi_{\delta} = I_{S}^{uu} + I_{S}^{dd} - 2I_{S}^{uu}$ is the corresponding quark-loop polarization function.

**Appendix C: NJL predictions**

In this subsection, we give the NJL predictions to the (subtracted) quark condensate, scalar and pseudoscalar susceptibilities, and topological susceptibility. We also check the consistency with the recent lattice QCD data on 2 + 1 flavors at physical point, and also with other effective models of QCD.

1. **Parameter setting**

In the present model, in Eq. (B1), we have five parameters that need to be fixed: the light quark mass $m_{l}$, the strange quark mass $m_{s}$, the coupling constants $g_{s}$ and $g_{D}$, and the three-momentum cutoff $\Lambda$. To fix the parameters, we take the following four hadronic observables at $T = 0$ as inputs:

$$m_{\pi} = 136 \text{ MeV}, \quad f_{\pi} = 93 \text{ MeV}, \quad m_{K} = 495.7 \text{ MeV}, \quad m_{\eta'} = 957.5 \text{ MeV}.$$  \hspace{1cm} (C1)

To fix the remaining one degree of freedom, we follow the literature [21] to take light quark mass $m_{l} = 5.5$ MeV (at the renormalization scale of 1 GeV). Thus all the model parameters are fixed, which are presented in Table I [12].

| parameters                  | values     |
|-----------------------------|------------|
| light quark mass $m_{l}$    | 5.5 MeV    |
| strange quark mass $m_{s}$  | 138 MeV    |
| four-fermion coupling constant $g_{s}$ | 0.358 fm$^{2}$ |
| six-fermion coupling constant $g_{D}$ | $-0.0275$ fm$^{2}$ |
| cutoff $\Lambda$           | 631.4 MeV |

The present NJL model predicts $\chi_{\text{top}} \approx 0.025$/fm$^{4}$. For this $\chi_{\text{top}}$, comparison with the results from the lattice QCD simulations is available, which are $\chi_{\text{top}} = 0.019(9)/$fm$^{4}$ [22], and $\chi_{\text{top}} = 0.0245(24)_{\text{stat}}(03)_{\text{flow}}(12)_{\text{cont}}/$fm$^{4}$ [23]. Here, for the latter the first error is statistical, the second one comes from the systematic error, and the third one arises due to changing the upper limit of the lattice spacing range in the fit. Although their central values do not agree each other, we may conservatively say that the difference between them is interpreted as a systematic error from the individual lattice QCD calculation.
2. Subtracted quark condensate

The quark condensate in the NJL model involves a ultraviolet divergence (which is dominated by a quadratic divergence) due to its vacuum part \((\langle -\bar{q}q \rangle \sim N_c m_q \Lambda^2 / (4\pi^2))\), and is needed to be renormalized when compared with lattice data. Since the quadratic divergences in the quark condensate come along with current quark masses (as above), we use a subtracted quark condensate as the chiral order parameter, which has been adopted in the lattice simulations: \(\Delta_{l,s}(T) \equiv \langle \bar{l}l \rangle - \frac{m_l}{m_s} \langle \bar{s}s \rangle\), where \(\langle \bar{u}u \rangle = \langle \bar{d}d \rangle\).

Figure 5 shows the subtracted quark condensate as a function of temperature predicted from the present NJL model, in comparison with the 2+1 flavor data from the lattice QCD at the physical point \[24\]. The pseudo-critical temperature \(T_{pc}\) is (for the NJL prediction) defined as \(\left.\frac{d^2 \langle \bar{l}l \rangle / dT^2 \right|_{T=T_{pc}} = 0\). We have found \(T_{pc}|_{NJL} \approx 188\text{ MeV}\), which is compared with the lattice result \(T_{pc}|_{lat} \approx 155\text{ MeV}\) \[25–29\]. In the figure, we have normalized \(T\) by their \(T_{pc}\).

![Figure 5: Temperature scaling of the subtracted quark condensate, in comparison with data from the lattice QCD with 2 + 1 flavors \[25\]. The normalization factor, the pseudo-critical temperature for the chiral crossover \((T_{pc})\) has been set to individual values estimated from the present NJL model \((T_{pc}|_{NJL} \approx 188\text{ MeV})\) and the lattice simulation \((T_{pc}|_{lat} \approx 155\text{ MeV})\).

From Fig. 5 we see that the present NJL prediction is perfectly consistent with the lattice data, confirming that the present model describes the chiral crossover phenomenon quite well.

3. Chiral and axial susceptibility partners

The scalar and pseudoscalar susceptibilities \(\chi_\eta, \chi_\pi, \chi_\sigma, \chi_\delta\) presented in Eqs. \[B11\], \[B12\], \[B19\], and \[B21\] are correlated with each other by the chiral \(SU(3)_L \times SU(3)_R\) and \(U(1)_A\) transformations \[30\]:

\[
\begin{align*}
\chi_\pi & \quad SU(2) \quad \chi_\sigma \\
U(1)_A & \quad \chi_\delta \quad U(1)_A \\
\chi_\eta & \quad SU(2)
\end{align*}
\]

The chiral and axial partners will be degenerate each other in the symmetric limits:

\[
\begin{align*}
\chi_\pi &= \chi_\sigma, \quad \chi_\delta = \chi_\eta \quad \text{(chiral SU(2) symmetric limit)} \\
\chi_\sigma &= \chi_\delta, \quad \chi_\sigma = \chi_\eta \quad \text{(U(1)_A axial symmetric limit)}
\end{align*}
\]
Then, observation of null difference between the above partners can effectively monitor the restoration of the related symmetry. We show the susceptibility differences as a function of temperature, in comparison to the lattice QCD result [4].

From Fig. 6 we see that the chiral-partner difference $\chi_{\eta-\delta}$ fits quite well with the lattice data, while the $U(1)_A$-partner difference $\chi_{\pi-\delta}$ shows some discrepancy. Note that $\chi_{\eta-\delta}$ and $\chi_{\pi-\delta}$ are both normalized by $T^2$ in Fig. 6. For the unnormalized ones, the present NJL model seems to yield larger values for both $\chi_{\eta-\delta}$ and $\chi_{\pi-\delta}$ than what the lattice simulation currently predicts.

4. Topological susceptibility

We numerically evaluate $\chi_{\text{top}}$ in Eq. (B11) as a function of temperature. In Fig. 7 we plot the temperature dependence of unnormalized topological susceptibility $\chi_{\text{top}}$, where we have taken the absolute value of $\chi_{\text{top}}$. Comparison with the dilute instanton gas approximation (DIGA) [13, 14], the linear sigma model result (denoted as CJT in the figure) [2] and the result from lattice simulation in the continuum limit [22, 23, 31] have also been displayed. The DIGA prediction has been quoted from the literature [31]. For the way of error bars associated with the DIGA, see the cited reference. The temperature is normalized by the pseudo-critical temperature in the figure, where we have taken $T_{pc}|_{\text{NJL}} = 188$ MeV for the NJL case, $T_{pc}|_{\text{CJT}} = 215$ MeV for the linear sigma model case, and $T_{pc}|_{\text{lat}} = 155$ MeV for the lattice and DIGA cases.

Figure 7 shows perfect consistency between the NJL analysis and lattice result. We see that they maintain good agreement with each other in the whole range of the available lattice data, $T/T_{pc} \sim 0 - 4$. At $T < T_{pc}$, all the results fit perfectly with each other, including the linear sigma model’s. In contrast, when $T > T_{pc}$, we see substantial deviation for the linear sigma model prediction from the NJL’s and lattice results. In the literature [2], the pseudoscalar susceptibility terms were not able to evaluate, because the authors did not include the higher order terms in the current quark masses, and therefore, performing the second order derivative on the mass parameter to obtain pseudoscalar susceptibility would not work out. Thus, their $\chi_{\text{top}}$ only includes the quark condensate terms. The present NJL model is able to give the pseudoscalar susceptibility contribution to $\chi_{\text{top}}$, to achieve an improved estimate on the quark condensate, and therefore, the figure shows better consistency with the lattice result.

5. Flavor symmetry dependence on the QCD trilemma

We may simply suppose that the scalar and pseudoscalar susceptibilities are scaled with the associated meson masses, like $\chi_{\delta} \propto 1/m_\delta^2$, $\chi_{\eta} \propto 1/m_\eta^2$ and $\chi_{\pi} \propto 1/m_\pi^2$. First, consider the light quark mass $m_l$ to generically differ from the strange quark mass $m_s$, including the real-life QCD case with the three-flavor symmetry broken. Among
the susceptibilities, $\chi_{\pi}$ is most sensitive to the current mass of the light quarks ($m_l$), because it is the pseudo Nambu-Goldstone boson of spontaneous breaking of $SU(2)_L \times SU(2)_R$ symmetry carried by the light quarks. The $\chi_{\pi}$ thus monotonically gets smaller (larger), as $m_l$ gets larger (smaller), by following $\chi_{\pi} \propto 1/m_{\pi}^2 \sim 1/m_l$. On the other hand, the other pseudoscalar susceptibility $\chi_{\eta}$ significantly involves the $U(1)$-axial anomaly contribution in $m_s$, so it almost keeps constant in $m_l$. The scalar susceptibility $\chi_{\delta}$, free from the Nambu-Goldstone boson nature, also keeps constant with the change of $m_l$. Besides, the topological susceptibility $\chi_{\text{top}}$ also simply scales with $m_l$, respecting the flavor-singlet condition: $\chi_{\text{top}} \to 0$ as $m_l \to 0$, and will be completely constant in $m_l$ for $m_l > m_s$ due to decoupling of the “light” quarks. Thus the difference in magnitude of susceptibilities are simply originated from the scaling properties with respect to the current mass of the light quarks. We plot those $m_l$ scaling behaviors (at $T = 0$) in Fig. 8. The light quark mass is allowed to vary from $10^{-2}\text{eV}$ to the cutoff scale of the presently employed NJL model (613.4 MeV), since the result from $m_l$ above the cutoff scale would be of poor reliability. From the figure, the $m_l$ dependence is read off and the susceptibilities are found to take simple power laws when $m_l \lesssim m_s$:

$$\begin{align*}
\chi_{\pi} &\sim m_l^{-1}, \\
\chi_{\eta} &\sim \text{constant for } m_l \lesssim m_s, \\
\chi_{\delta} &\sim \text{constant for } m_l \lesssim m_s, \\
\chi_{\text{top}} &\sim m_l \text{ for } m_l \lesssim m_s. 
\end{align*}$$

(C3)

The $m_l$ dependence on the trilemma estimator $R$ with fixed $m_s$ at physical value is shown in Fig. 9.

Next, consider the three-flavor symmetric limit, where $m_l = m_s$. In this case QCD is balanced as emphasized in the main text. It also turns out that the scaling law of $\chi_{\eta}$ in Eq. (C3) is broken: the Ward identity Eq. (A5) tells us that the difference between $\chi_{\pi}$ and $\chi_{\eta}$ is controlled by the $4m_{\pi}^2\chi_P^L$ term (where $\chi_P^L = \chi_P^u = \chi_P^d$). Since no preference among quark flavors is present in the flavor symmetric case, $\chi_P^L$ should be on the same order of magnitude as that of $\chi_{\pi}$, which we have indeed numerically confirmed. Since $m_l = m_s$, there is no extra power scaling of $1/m_l$ which is present in the flavor asymmetric case and leads to big enhancement of the $(4m_{\pi}^2\chi_P^L)$ part to destructively interfere with $\chi_{\pi}$, yielding a much suppressed $\chi_{\eta}$ compared to $\chi_{\pi}$ (See Fig. 8). Thus the scaling law of $\chi_{\eta}$ is the same as that of $\chi_{\pi}$, i.e., $\chi_{\eta} \sim m_l^{-1}$, as shown in Fig 10.

This scaling violation in the flavor symmetric case can also be understood as a big suppression of the $U(1)_A$ anomaly contribution, coupled to the flavor violation, to $m_{\pi}^2$, which dominates in $\chi_{\eta}$ in the flavor asymmetric case: in the flavor symmetric case we have $\chi_{\pi} = \chi_P^{88}$, and $\chi_{\eta} = \chi_{\pi} + 4\chi_P^L$ with $\chi_P^L = 1/6(\chi_P^{00} - \chi_P^{88})$. Straightforward numerical evaluation reveals that $\chi_P^{88} \gg \chi_P^{00}$ for small $m_l$. Then, we find $\chi_{\eta} \approx \chi_{\pi}/3 \sim 1/m_l$ for small $m_l$. In particular, note that $\chi_P^{88} = \chi_{\pi}$ does not include the $U(1)_A$ anomaly effect, and is now much larger than the $U(1)_A$-anomaly affected $\chi_P^{00}$ part, which implies the $U(1)_A$ anomaly contribution is much suppressed in $\chi_{\eta}$, hence in $m_{\eta}$ as well.
FIG. 8: (Without the three-flavor symmetry): the $m_l$ dependence on different susceptibilities [MeV$^2$] at $T = 0$ with $m_s$ fixed at physical value.

FIG. 9: QCD trilemma estimator $R$ at $T = 0$ as a function of $m_l$ with $m_s$ fixed at physical value. The light quark mass is virtually varied in a wide range, from $10^{-2}$ eV to the cutoff scale of the NJL model, 613.4 MeV.

This $m_l$ scaling violation for $\chi_\eta$ leads to the balanced QCD. When $m_l (= m_s)$ is smaller than 1 MeV, $R$ stays in the “balanced” region constantly at around 0.34. This is due to the modified scaling law for $\chi_\eta$ ($\chi_\eta \sim m_l^{-1}$).

Appendix D: A conjectured high T-QCD model

To evaluate the trilemma estimator $R$ for a higher $T$ like $T > 300$ MeV, we may consider a quarkonic-interacting model in a sense of quantum field theory, instead of dilute instanton gas by which it would be hard to compute thermally-averaged Green functions like $R$. We shall approximate such an interacting quark gas at high $T$ (say, quark-gluon plasma) by only the kinetic and determinant terms which would play a role of the gluonic counterpart acting as if it is like a dilute instanton effect among quarks. The size of the determinant coupling should be small, so that the chiral symmetry is broken only by the current quark masses: the coupling should not exceed the critical value. In addition, as the temperature gets higher, the QCD instanton effect should be suppressed by a Boltzmann-like factor as predicted from the dilute instanton gas description [13, 14], so that in terms of quarkonic interactions the determinant term would be suppressed as well.

Thus we write a conjectured model for high-$T$ QCD with three quark flavors as

$$\mathcal{L}_{\text{high-}T} = \bar{q}(i\gamma_\mu \partial^\mu - m)q + K(T)[\det(\bar{q}_L q_R) + \text{h.c.}], \quad (D1)$$

where $K(T)$ is the temperature-dependent coupling strength of the determinant term, which simply models the
Boltzmann-like suppression for the QCD instanton configuration: \( K(T) = e^{-T^2/\mu^2_{QCD}} K \), with a collective scale factor of QCD, \( \mu^2_{QCD} \), and the overall constant \( K \) being scaled by the model cutoff \( \Lambda \) along with a dimensionless parameter \( k \) like \( K = k/\Lambda^2 \).

For a benchmark evaluation of \( R \), we set the parameters as

\[
\mu_{QCD} = 200, 300, 400 \text{ MeV}, \quad \Lambda = 3 \text{ GeV}, \quad k = -0.5 \, .
\]

where the parameter \( k \) has been chosen so as not to exceed the critical coupling \( k_{cr} = -\pi^2/3 \). With the parameter setting above, this model can work for \( T \sim 300 \text{ MeV} \) up to the order of \( O(1 \text{ GeV}) \).

To calculate all the quantities relevant to \( R \), we only need to refer to the NJL model case, just by replacing \( g_D \) with \( K(T) \) and setting the four-fermion coupling strength \( g_s = 0 \). The plots of \( R \) are thus given in Fig.4 in the main text.
[22] C. Bonati, M. D’Elia, G. Martinelli, F. Negro, F. Sanfilippo and A. Todaro, JHEP 11 (2018), 170 
doi:10.1007/JHEP11(2018)170 [arXiv:1807.07954 [hep-lat]].

[23] S. Borsanyi, Z. Fodor, J. Guenther, K. H. Kampert, S. D. Katz, T. Kawanai, T. G. Kovacs, S. W. Mages, A. Pasztor and F. Pittler, et al. Nature 539 (2016) no.7627, 69-71 doi:10.1038/nature20115 [arXiv:1606.07494 [hep-lat]].

[24] Y. Aoki, S. Borsanyi, S. Durr, Z. Fodor, S. D. Katz, S. Krieg and K. K. Szabo, JHEP 06 (2009), 088 
doi:10.1088/1126-6708/2009/06/088 [arXiv:0903.4155 [hep-lat]].

[25] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz and K. K. Szabo, Nature 443 (2006), 675-678 
doi:10.1038/nature05120 [arXiv:hep-lat/0611014 [hep-lat]].

[26] S. Borsanyi et al. [Wuppertal-Budapest], J. Phys. Conf. Ser. 316 (2011), 012020 
doi:10.1088/1742-6596/316/1/012020 [arXiv:1109.5032 [hep-lat]].

[27] H. T. Ding, F. Karsch and S. Mukherjee. Int. J. Mod. Phys. E 24 (2015) no.10, 1530007 
doi:10.1142/S0218301315300076 [arXiv:1504.05274 [hep-lat]].

[28] A. Bazavov et al. [HotQCD], Phys. Lett. B 795 (2019), 15-21 
doi:10.1016/j.physletb.2019.05.013 [arXiv:1812.08239 [hep-lat]].

[29] H. T. Ding, Nucl. Phys. A 1005 (2021), 121940 
doi:10.1016/j.nuclphysa.2020.121940 [arXiv:2002.11957 [hep-lat]].

[30] A. Bazavov et al. [HotQCD], Phys. Rev. D 86 (2012), 094503 
doi:10.1103/PhysRevD.86.094503 [arXiv:1205.3535 [hep-lat]].

[31] P. Petreczky, H. P. Schadler and S. Sharma, Phys. Lett. B 762 (2016), 498-505 
doi:10.1016/j.physletb.2016.09.063 [arXiv:1606.03145 [hep-lat]].