ON CACTI AND CRYSTALS
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To Anthony Joseph, with admiration

Abstract. In the present work we study actions of various groups generated by involutions on the category $\mathcal{O}_q^{int}(g)$ of integrable highest weight $U_q(g)$-modules and their crystal bases for any symmetrizable Kac-Moody algebra $g$. The most notable of them are the cactus group and (yet conjectural) Weyl group action on any highest weight integrable module and its lower and upper crystal bases. Surprisingly, some generators of cactus groups are anti-involutions of the Gelfand-Kirillov model for $\mathcal{O}_q^{int}(g)$ closely related to the remarkable quantum twists discovered by Kimura and Oya in [28].

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1. Introduction

In the present work we study the action of various groups generated by involutions on the category $\mathcal{O}^\text{int}_q(\mathfrak{g})$ of integrable highest weight $U_q(\mathfrak{g})$-modules for any symmetrizable Kac-Moody algebra $\mathfrak{g}$ (the necessary notation is introduced in Section 2).

Let $V \in \mathcal{O}^\text{int}_q(\mathfrak{g})$. We claim that for every node $i$ of the Dynkin diagram $I$ of $\mathfrak{g}$ there exists a unique linear operator $\sigma^i_V$ on $V$ such that

$$\sigma^i_V(E^{(k)}_i(u)) = E^{(l-k)}_i(u)$$

for all $l \geq k \geq 0$ and for all $u \in \ker F_i \cap \ker (K_i - q_i^{-1})$. Clearly, $(\sigma^i_V)^2 = \text{id}_V$. Denote by $W(V)$ be the subgroup of $\text{GL}_k(V)$ generated by the $\sigma^i_V$, $i \in I$.

**Theorem 1.1.** For any non-zero module $V \in \mathcal{O}^\text{int}_q(\mathfrak{g})$, the assignments

$$\sigma^i_V \mapsto \begin{cases} 1, & i \in J(V) \\ s_i, & \text{otherwise} \end{cases}$$

where $J(V) = \{ i \in I : F_i(V) = \{0\} \}$, define a homomorphism $\psi_V$ from $W(V)$ to the Weyl group $W$ of $\mathfrak{g}$.

We prove Theorem 1.1 in §3.3 by showing that the image of $\psi_V$ can be described in terms of a natural action of $W$ on a certain set of extremal vectors in $V$. In particular, $\psi_V$ is surjective if and only if $J(V) = \emptyset$. Moreover, we show that $\sigma^i_V = \text{id}_V$ if and only if $i \in J(V)$. This suggests the following

**Conjecture 1.2.** The homomorphism $\psi_V$ is injective for any $V \in \mathcal{O}^\text{int}_q(\mathfrak{g})$.

Clearly, it is equivalent to $(\sigma^i \sigma^j)^{m_{ij}} = \text{id}_V$, $i \neq j \in I$ for appropriate choices of $m_{ij}$. We proved it for $m_{ij} = 2$ and we have ample evidence that this conjecture holds for $m_{ij} = 3$. 

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We also verified it for all modules in which weight spaces of non-zero weight are one dimensional (see Theorem 7.2). This class of modules includes all miniscule and quasi-miniscule ones. Conjecture 1.2 combined with Theorem 1.1 implies that \( W \) acts naturally and faithfully on objects in \( \mathcal{O}^{\text{int}}_q(\mathfrak{g}) \), which is quite surprising. Informally speaking, this conjecture asserts that Kashiwara’s action of the Weyl group on crystal bases lifts to an action on the corresponding module (see Remark 5.7).

**Remark 1.3.** The definition (1.1) of \( \sigma^i \) makes sense for any integrable \( U(\mathfrak{g}) \)-module where \( \mathfrak{g} \) is a semisimple or a (not necessarily symmetrizable) Kac-Moody Lie algebra. The “classical” Theorem 1.1 holds verbatim. Moreover, Conjecture 1.2 implies its classical version for all (even not symmetrizable) Kac-Moody algebras.

It turns out that we can extend the group \( \mathcal{W}(V) \) by adding involutions \( \sigma^J \) for any non-empty \( J \subseteq I \) such that the subgroup \( W_J = \langle s_i : i \in J \rangle \) is finite; we denote the set of all such \( J \) by \( \mathcal{J} \). Note that \( \{i\} \in \mathcal{J} \) for all \( i \in I \) and in particular \( \mathcal{J} \) is non-empty.

**Proposition 1.4** (Proposition 4.14). For any \( V \in \mathcal{O}^{\text{int}}_q(\mathfrak{g}) \), \( J \in \mathcal{J} \) there exists a unique \( \mathbb{k} \)-linear map \( \sigma^J = \sigma^J_V : V \to V \) such that

(a) \( \sigma^J(v) = v^J \) for any \( v \in \bigcap_{i \in J} \ker E_i \) where \( v^J \) is a distinguished element in \( \bigcap_{i \in J} \ker F_i \cap U_q(\mathfrak{g}^J) \)

(b) \( \sigma^J(F_j(v)) = E_j(\sigma^J(v)) \), \( \sigma^J(E_j(v)) = F_j(\sigma^J(v)) \) for all \( j \in J \), \( v \in V \) where \( \ast : J \to J \) is the involution on \( J \) induced by the longest element \( w_0^J \) of \( W_J \) via \( s_j \ast = w_0^J s_j w_0^J \), \( j \in J \) (see \( \S 2.1 \)).

Moreover, for any morphism \( f : V \to V' \) in \( \mathcal{O}^{\text{int}}_q(\mathfrak{g}) \) the following diagram commutes

\[
\begin{array}{ccc}
V & \xrightarrow{\sigma^J_V} & V \\
\downarrow{f} & & \downarrow{f} \\
V' & \xrightarrow{\sigma^J_{V'}} & V'
\end{array}
\]  

(1.2)

By definition, \( \sigma^J = \sigma^i \) if \( J = \{i\} \). The following is the main result of this paper.

**Theorem 1.5.** Let \( V \in \mathcal{O}^{\text{int}}_q(\mathfrak{g}) \). Then for any \( J \in \mathcal{J} \) we have in \( \text{GL}_k(V) \)

(a) \( \sigma^J \circ \sigma^J = 1 \);

(b) If \( J = J' \cup J'' \) where \( J' \) and \( J'' \) are orthogonal (that is, \( J' \cap J'' = \emptyset \) and \( s_{j'} s_{j''} = s_{j''} s_{j'} \) for all \( j' \in J' \), \( j'' \in J'' \) then \( \sigma^J = \sigma^{J'} \circ \sigma^{J''} \); in particular, \( \sigma^J \circ \sigma^{J''} = \sigma^{J''} \circ \sigma^J \) if \( J', J'' \in \mathcal{J} \) are orthogonal.

(c) \( \sigma^J \circ \sigma^K = \sigma^K \ast \circ \sigma^J \) for any \( K \subset J \), where \( \ast : J \to J \) is as in Proposition 1.4(b).

We prove Theorem 1.5 in \( \S 4.3 \) using appropriate modifications of Lusztig’s symmetries (which we introduce in \( \S 4.1 \)).
Following (and slightly generalizing) [30] (see also [11]), we denote $\text{Cact}_W$ the group generated by the $\tau_J$, $J \in \mathcal{J}$ subject to all relations of Theorem 1.5. Indeed, this definition coincides with that in [30, (1.1)] if $W$ is finite because $\tau_J = \tau_{J'}\tau_{J''}$ for any $J$ as in Theorem 1.5(b). By definition, the assignments $\tau_J \mapsto \sigma_J^I$, $J \in \mathcal{J}$ define a representation of $\text{Cact}_W$ on $V$. In view of (1.2) we obtain the following immediate corollary of Theorem 1.5 (see Section 4 for the notation).

**Corollary 1.6.** The group $\text{Cact}_W$ acts on the category $\mathcal{O}_q^{\text{int}}(g)$ via $\tau_J \mapsto \sigma_J^I$, $J \in \mathcal{J}$.

The study of cactus groups began with $\text{Cact}_n := \text{Cact}_{S_n}$ which appeared, to name but a few, in [14,15,17,34,36] in connection with the study of moduli spaces of rational curves with $n+1$ marked points and their applications in mathematical physics. It is easy to see that $\text{Cact}_n$ is generated by involutions $\tau_{i,j} = \tau_{\{i,...,j-1\}}$, $1 \leq i < j \leq n$ subject to the relations

\[
\tau_{i,j}\tau_{k,l} = \tau_{k,l}\tau_{i,j}, \quad i < j < k < l
\]
\[
\tau_{i,j}\tau_{j,k} = \tau_{i+l-k,j+l-j}\tau_{i,j}, \quad i \leq j < k \leq l.
\]

Categorical actions of $\text{Cact}_n$ on $n$-fold tensor products in symmetric coboundary categories (first introduced in [16]) were studied in [21, 35] and also implicitly in [10] where the braided structure on the category $\mathcal{O}_q^{\text{int}}(g)$ was converted into a symmetric coboundary structure for any complex reductive Lie algebra $g$ (for non-abelian examples of coboundary categories see the discussion after Theorem 1.8). It would be interesting to compare these actions of $\text{Cact}_n$ with the one given by Corollary 1.6. We expect that they are connected in some cases via the celebrated Howe duality (see e.g. the forthcoming paper [4]). In view of Corollary 1.6 it is natural to seek other categorical representations of $\text{Cact}_W$ for all Coxeter groups $W$.

Conjecture 1.2 suggests that our representation of $\text{Cact}_W$ on $\mathcal{O}_q^{\text{int}}(g)$ is not faithful. Therefore, we can pose the following

**Problem 1.7.** Find the kernel $K_g$ of this representation of $\text{Cact}_W$.

For example, if $g = \mathfrak{sl}_3$ then $\tau_{1,2}\tau_{1,3} = \tau_{1,3}\tau_{2,3}$ and so $\text{Cact}_W$ is freely generated by involutions $\tau_{1,2}$ and $\tau_{1,3}$. It is easy to see that $\tau_{1,2} \notin K_{\mathfrak{sl}_3}$, while $\tau_{1,3} \notin K_{\mathfrak{sl}_3}$ by Remark 7.14. Thus, if Conjecture 1.2 holds then $K_{\mathfrak{sl}_3} = \{(\tau_{1,2}\tau_{1,3})^{6n} : n \in \mathbb{Z}\}$ which would solve the problem for $W = S_3$.

To outline an approach to Problem 1.7 in general, denote $\Phi_V$, $V \in \mathcal{O}_q^{\text{int}}(g)$ the subgroup of $\text{GL}_k(V)$ generated by the $\sigma_J^I$, $J \in \mathcal{J}$. Then clearly $K_g$ is the intersection of kernels of canonical homomorphisms $\text{Cact}_W \to \Phi_V$ over all $V \in \mathcal{O}_q^{\text{int}}(g)$. We show (Proposition 4.18) that $\Phi_V \cong \Phi_{\mathcal{L}}$ where $\mathcal{L} = \bigoplus_{\lambda \in P^+: \text{Hom}_{\mathcal{O}_q}(V_{\lambda}, V) \neq 0} V_{\lambda}$. In particular, $\text{Cact}_W / K_g$ is isomorphic to $\Phi_{\mathcal{C}_g}$ where $\mathcal{C}_g = \bigoplus_{\lambda \in P^+} V_{\lambda}$ is the Gelfand-Kirillov model for $\mathcal{O}_q^{\text{int}}(g)$; in fact, it has a structure of an associative algebra (see Section 6). Thus, in view of the above we
expect that \( \mathbb{C} / \mathbb{K} \) is isomorphic to the dihedral group of order 12. However, it is likely that \( \Phi \mathbb{C} \) is infinite for simple \( \mathfrak{g} \) different from \( \mathfrak{sl}_2 \) and \( \mathfrak{sl}_3 \).

It turns out that the action of \( \mathbb{C} \) on \( \mathcal{G} \) descends to a permutation representation on any crystal basis of any object \( V \) (see §2.5 for definitions and notation). Thus, we obtain the following refinement of [20, Theorem 5.9].

**Theorem 1.8.** Let \( V \in \mathcal{G} \). Then for any lower or upper crystal basis \((L,B)\) of \( V \) at \( q = 0 \) the group \( \Phi V \) preserves \( L \) and acts on \( B \) by permutations.

We prove Theorem 1.8 in Section 5 by means of what we call \( c \)-crystal bases, which allow one to treat lower and upper crystal bases uniformly. Taking into account that \( B \) is graded by the weight lattice of \( \mathfrak{g} \), all weights occur in a crystal basis of \( \mathcal{C} \) and that \( W \) acts faithfully on the weight lattice, we obtain an immediate

**Corollary 1.9.** The assignments \( \sigma^J : \mathcal{G} \to W \) define a surjective homomorphism \( \mathbb{C} \to W \) which refines the natural epimorphism \( \mathbb{C} \to W \) from [30].

Analogously to the notion of the pure braid group, one calls the kernel of the natural homomorphism \( \mathbb{C} \to W \) the pure cactus group (this term was used for \( \mathbb{C} \) in e.g. [17, 34, 36]). Thus, Corollary 1.9 asserts that \( K \) is pure.

The involution \( \sigma^J \) was first defined in [8] for \( \mathfrak{g} = \mathfrak{gl}_n \) and simple polynomial representations \( V_\lambda \) and explicitly computed on the corresponding crystal in [29]. In fact, it coincides with the famous Schützenberger involution (see Remark 4.11). Following a suggestion of the first author and [29], an action of \( \mathbb{C} \) on the category of crystal bases was constructed in [21] thus turning it into a symmetric coboundary category.

We expect that to solve Problem 1.7 it suffices to find kernels of permutation representations of \( \mathbb{C} \) on all \( B \).

Since \( W(V) \) is naturally a subgroup of \( \Phi V \), its action on \( V \) induces an action on \( B \) by permutations which coincides with Kashiwara’s crystal Weyl group (see Remark 5.7).

In case when \( \mathfrak{g} \) is reductive we can refine Theorem 1.8 as follows.

**Theorem 1.10.** Let \( \mathfrak{g} \) be a reductive Lie algebra and let \( V \in \mathcal{G} \). Then for any crystal basis \((L,B)\) of \( V \) the involution \( \sigma^J \) preserves the corresponding upper global crystal basis \( B \) of \( V \).

An analogous result for \( J \subseteq I \) is weaker. We prove (Proposition 5.8) that the image of any element of \( B \) under \( \sigma^J \) is a \( \bar{\cdot} \)-invariant element of \( V \) where \( \bar{\cdot} \) is the anti-linear involution fixing \( B \). However, as explained in Remark 7.18, \( \sigma^J \) does not need to preserve \( B \) if \( J \subseteq I \). For example, if \( V \) is the 27-dimensional simple module \( V_{27} \) for \( \mathfrak{g} = \mathfrak{sl}_3 \) then the \( \sigma^i \), \( i = 1, 2 \) do not preserve the canonical basis of \( V \).

An analogue of Theorem 1.10 for a simple \( V \) and its lower global crystal basis was deduced from [31, Proposition 21.1.2] in [21, Theorem 5].
We prove Theorem 1.10 in §6.5. A central role in our argument is played by the following surprising property of $\sigma^I$ on the aforementioned quantum Gelfand-Kirillov model $C_q(\mathfrak{g})$ of $\mathcal{O}^{\text{int}}_q(\mathfrak{g})$.

**Theorem 1.11** (Theorem 6.21). For $\mathfrak{g}$ reductive finite dimensional, $\sigma^I_{C_q(\mathfrak{g})}$ is an anti-involution on $C_q(\mathfrak{g})$.

We do not expect an analogous result for $J \subset I$; for example, for $\mathfrak{g} = \mathfrak{sl}_3$, the $\sigma^i$, $i \in \{1, 2\}$ are not anti-automorphisms of $C_q(\mathfrak{g})$. Our proof of Theorems 1.10 and 1.11 rely in a crucial way on properties of a remarkable quantum twist defined in [28].

In view of Theorem 1.8 we can refine Conjecture 1.2 for every $V \in \mathcal{O}^{\text{int}}_q(\mathfrak{g})$ with $\mathbf{V} = C_q(\mathfrak{g})$ as follows. We expect that in the notation of Theorem 1.8 the group $\Phi_V$ acts on $B$ faithfully. Morally, this means that each element of $\Phi_V$ is semisimple in $\text{GL}_k(V)$. Similarly to Remark 1.3, our constructions, results and conjectures make sense if one replaces $U_q(\mathfrak{g})$ by $U(\mathfrak{g})$ for any (symmetrizable or not) Kac-Moody algebra $\mathfrak{g}$. Some results (for example, Theorem 1.8) should be possible to rescue even when $W$ is not crystallographic (and so $\mathfrak{g}$ does not exists) with the aid of theory of continuous crystals initiated by A. Joseph in [22].

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2. Preliminaries

2.1. **Coxeter groups.** Let $I$ be a finite set. Let $W$ be a Coxeter group with Coxeter generators $s_i$, $i \in I$ subject to the relations $(s_i s_j)^{m_{ij}} = 1$ where $m_{ii} = 1$, $m_{ij} = m_{ji}$ and $m_{ij} \in \{0\} \cup \mathbb{Z}_{\geq 2}$ for $i \neq j \in I$. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the Coxeter length function, that is, $\ell(w)$ is the minimal length of a presentation of $w$ as a product of the $s_i$, $i \in I$. We say that $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ is reduced if $\ell(s_{i_1} \cdots s_{i_r}) = r$ and denote by $R(w)$ the set of reduced words for $w$, that is, $R(w) = \{(i_1, \ldots, i_{\ell(w)}) \in I^{\ell(w)} : w = s_{i_1} \cdots s_{i_{\ell(w)}}\}$.

Given $J \subset I$ we denote by $W_J$ the subgroup of $W$ generated by the $s_i$, $i \in J$. We will need the following standard fact (see [12, IV.1.8, Théorème 2]).

**Lemma 2.1.** For any $J, J' \subset I$

(a) $W_J \cap W_{J'} = W_{J \cap J'}$;

(b) $W_J \subset W_{J'}$ if and only if $J \subset J'$.

Let $\mathcal{J} = \{J \subset I, : |W_J| < \infty\}$. If $J \in \mathcal{J}$ we denote by $w_J^\ell$ the unique longest element of $W_J$; thus, $\ell(s_j w_J^\ell) < \ell(w_J^\ell)$ for all $j \in J$. If $I \in \mathcal{J}$ we abbreviate $w_\infty = w_I^\ell$. Given
Given \( J \subseteq I \), we set \( J^\perp = \{ i \in I \setminus J : m_{ij} = 2, \forall j \in J \} = \{ i \in I \setminus J : s_is_j = s_ks_i, \forall j \in J \} \). We say that \( J, J' \subseteq I \) are orthogonal if \( J \cap J' = \emptyset \) and \( J' \subseteq J^\perp \) (whence \( J \subseteq J'^\perp \)).

Define a relation \( \sim \) on \( I \) by \( i \sim j \) if \( i = j \) or \( m_{ij} > 2 \). Then the transitive closure of this relation is an equivalence on \( I \) which we still denote by \( \sim \). In particular, if \( i \sim i' \) then there exists a sequence (called admissible) \( (i_0, \ldots, i_d) \in I^{d+1} \) with \( i_0 = i, i_d = i' \) and \( m_{i_{r-1}i_r} > 2, 1 \leq r \leq d \). Define \( \text{dist}(i,i') \) to be the minimal length of an admissible sequence beginning with \( i \) and ending with \( i' \). Clearly, this defines a metric on \( I \).

Define a topology on \( I \) by declaring that the fundamental neighborhood of each \( i \in I \) is its equivalence class with respect to \( \sim \). In particular, each open set is closed and vice versa and is a union of equivalence classes. For \( J \subseteq I \) we denote by \( \text{cl}(J) \) its closure in that topology, that is, the union of equivalence classes of elements of \( J \). Denote \( \partial(J) \) the boundary of \( J \), that is, the complement of \( J \) in \( \text{cl}(J) \). The following is immediate.

**Lemma 2.2.** Let \( J \subseteq I \). Then \( I = J \cup J^\perp \) if and only if \( J \) is closed in the above topology.

The following is a reformulation of a well-known fact ([12, IV.1.9, Proposition 2])

**Lemma 2.3.** Let \( J \subseteq I \) be a closed subset. Then \( W \) is the internal direct product of \( W_J \) and \( W_J' \).

Given a group \( G \) acting on a set \( X \), denote by \( K_X(G) \) the kernel of the natural homomorphism of groups \( G \to \text{Bij}(X) \) induced by the action. By definition, the action of \( G \) on \( X \) is faithful if and only if \( K_X(G) = \{1\} \).

The following is the main result of §2.1 (which is probably known although we could not find it in the literature).

**Theorem 2.4.** We have \( K_J : = K_{W/W_J}(W) = W_{I \setminus \text{cl}(I \setminus J)} \) for any \( J \subseteq I \). In particular, if \( J \) is connected and \( J \subseteq I \) then \( W \) acts faithfully on \( W/W_J \).

**Proof.** The following is immediate.

**Lemma 2.5.** Let \( G \) be a group and \( H \) be a subgroup of \( G \). Then \( K_{G/H}(G) = \{ k \in H : g^{-1}kg \in H, \forall g \in G \} \) is a subgroup of \( H \).

The following Lemmata are apparently well-known. We provide their proof for the reader’s convenience.

**Lemma 2.6.** Let \( w \in W \) and let \( J \subseteq I \) be such that \( \ell(s_jw) = \ell(w) - 1 \) for all \( j \in J \). Then \( W_J \) is finite and \( w = w_J'w' \) for some \( w' \in W \) with \( \ell(w) = \ell(w') + \ell(w_J') \).

**Proof.** By [12, Ch. IV, Ex. 3], every \( u \in W \) can be written uniquely as \( [u]_J \cdot J[u] \) where \( [u]_J \in W_J, J[u] = \{ x \in W : \ell(s_jx) > \ell(x), \forall j \in J \} \) and \( \ell(u) = \ell([u]_J) + \ell(J[u]) \).
The uniqueness of such a presentation implies that \([s_j w]_j = s_j[w]_j\) and \(J[s_j w] = J[w]\) for all \(j \in J\) and so that \(\ell(s_j[w]_j) < \ell([w]_j)\) for all \(j \in J\). This implies that \(W_J\) is finite and \([w]_j\) is its longest element \(w_j'\). The assertion follows with \(w' = J[w]\).

**Lemma 2.7.** For \(i \in I\) and \(u \in W_{I \setminus \{i\}}\) the following are equivalent.

(a) \(u \in W_{\{i\}^\perp}\) (in particular, \(s_i u = u s_i\));

(b) \(s_i u s_i \in W_{I \setminus \{i\}}\).

**Proof.** The implication (a) \(\implies\) (b) is obvious. To prove the opposite implication, note that the assumption in (b) implies that \(s_i u = u' s_i\) for some \(u' \in W_{I \setminus \{i\}}\). Then \(\ell(s_i u) = \ell(u) + 1\) and \(\ell(u' s_i) = \ell(u') + 1\) where \(\ell(u) = \ell(u')\). We prove the assertion

\[
s_i u = u' s_i \implies u = u' \in W_{\{i\}^\perp}
\]

by induction on \(\ell(u) = \ell(u')\), the case \(\ell(u) = \ell(u') = 0\) being obvious. If \(\ell(u) = \ell(u') > 0\) then there exists \(j \neq i \in I\) such that \(\ell(s_j u') < \ell(u')\). Let \(w = s_i u\). Then \(\ell(s_j w) < \ell(w)\) and \(\ell(s_j w) < \ell(w)\). Applying Lemma 2.6 to \(w\) and \(J = \{i, j\}\) we conclude that \(w = (s_is_j \cdots) u'\) with \(\ell(w) = m_{ij} + \ell(u')\) and so \(u = (s_is_j \cdots) u'\) with \(\ell(u) = m_{ij} - 1 + \ell(u')\). Since \(u \in W_{I \setminus \{i\}}\), a reduced word for \(u\) cannot contain \(i\), yet for any \((i_1, \ldots, i_r) \in R(u')\), \((j, i, \ldots, i_1, \ldots, i_r) \in R(u)\). Thus, \(m_{ij} = 2\) and so \(j \in \{i\}^\perp\). Then \(s_i(s_j u) = s_j s_i u = (s_j u') s_i\). Thus, \(s_j u, s_j u'\) satisfy (2.1) and \(\ell(s_j u) < \ell(u)\). Then the induction hypothesis implies that \(s_j u = s_j u' \in W_{\{i\}^\perp}\) and hence \(u = u' \in W_{\{i\}^\perp}\). \(\square\)

**Lemma 2.8.** Let \(i, i'\) be connected in \(I\) and let \((i = i_0, i_1, \ldots, i_d = i') \in I^{d+1}\) be an admissible sequence with \(d = \text{dist}(i, i')\). Suppose that \(w \in W_{I \setminus \{i\}}\) and \(s_{i_0} \cdots s_{i_d} \cdot w s_{i_d} \cdots s_{i_0} \in W_{I \setminus \{i\}}\). Then \(w \in W_{\{i_0, \ldots, i_d\}^\perp}\).

**Proof.** The argument is by induction on \(d\). The case \(d = 0\) (that is, \(i = i'\)) is established in Lemma 2.7. Suppose that \(d > 0\). Let \(u = s_{i_1} \cdots s_{i_d} \cdot w s_{i_d} \cdots s_{i_0}\). By Lemma 2.7, \(u \in W_{\{i\}^\perp}\). Since \(m_{i_1 i_0} > 2\), \(i_1 \not\in \{i\}^\perp\). Thus, \(u \in W_{I \setminus \{i\}}\) where \(I' = I \setminus \{i\}\) and \(\text{dist}(i_1, i') = d - 1\). By the induction hypothesis, \(u \in W_{\{i_1, \ldots, i_d\}^\perp}\) and in particular \(u = w\). But then \(w \in W_{\{i\}^\perp} \cap W_{\{i_1, \ldots, i_d\}^\perp} = W_{\{i\}^\perp} \cap (I \setminus \{i\})^\perp = W_{\{i_0, \ldots, i_d\}^\perp}\) where we used Lemma 2.1(a) and the observation that \(J^\perp \cap J' = (J \cup J')^\perp\). \(\square\)

By Lemma 2.5, \(K_J = \{w \in W : w u w u^{-1} \in W_J, \forall u \in W\}\) and is a subgroup of \(W_J\). Suppose that \(w \in K_J\); in particular, \(w \in W_J\). Furthermore, using Lemma 2.7 with \(u = w\) and \(i \in I \setminus J\), we conclude that \(w \in \bigcap_{i \in I \setminus J} W_{\{i\}^\perp} = W_{(I \setminus J)^\perp}\). Let \(i' \in \partial(I \setminus J)\). By definition, there exists \(i \in I \setminus J\) and an admissible sequence \((i_0, \ldots, i_d)\) with \(d = \text{dist}(i, i')\), \(i_0 = i\) and \(i_d = i'\). Since \(w w u^{-1} \in W_J\) with \(u = s_{i_0} \cdots s_{i_d}\), it follows from Lemma 2.8
that $w \in W_{\{i_0, \ldots, i_d\}} \subset W_{\{i\}}$. Thus, $w \in W_{(I \setminus J)^+ \cap (J \setminus I)^+} = W_{\{\{\emptyset \}, \{J\}\}} = W_{I \setminus cl(I \setminus J)}$. We proved that $K_J \subset W_{J_0}$ where $J_0 = I \setminus cl(I \setminus J)$.

To complete the proof of Theorem 2.4 we need the following.

Lemma 2.9. Let $J' \subset J$ which is closed in $I$. Then $W_{J'} \subset K_J$.

Proof. Since $J'$ is closed, $W_J = W_{J'} \times W_{J \setminus J'}$ and $W = W_{J'} \times W_{I \setminus J'}$ by Lemma 2.3. Then $W/W_J = W_{P \setminus J'}/W_{J \setminus J'}$. Since $W_{J'}$ acts by left multiplication in the first factor, this implies that $W_{J'}$ acts trivially on $W/W_J$. □

Applying Lemma 2.9 with $J' = J_0 = I \setminus cl(I \setminus J)$ we conclude that $W_{J_0} \subset K_J$. Thus, $K_J = W_{J_0}$. This completes the proof of Theorem 2.4. □

2.2. Cartan data and Weyl group. In this section we mostly follow [24]. Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix, that is $a_{ii} = 2$, $i \in I$, $-a_{ij} \in \mathbb{Z}_{\geq 0}$ and $a_{ij} = 0 \implies a_{ji} = 0$, $i \neq j$ and $d_i a_{ij} = d_j a_{ji}$ for some $d = (d_i)_{i \in I} \in \mathbb{Z}_+^I$. We fix the following data:

- a finite dimensional complex vector space $\mathfrak{h}$;
- linearly independent subsets $\{\alpha_i\}_{i \in I}$ of $\mathfrak{h}^*$ and $\{\alpha_i^\vee\}_{i \in I}$ of $\mathfrak{h}$;
- a symmetric non-degenerate bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$, and
- a lattice $P \subset \mathfrak{h}^*$ of rank $\dim \mathfrak{h}^*$

such that

1° $\alpha_j(\alpha_i^\vee) = a_{ij}$, $i, j \in I$;
2° $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{\geq 0}$;
3° $\lambda(\alpha_i^\vee) = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ for all $\lambda \in \mathfrak{h}^*$;
4° $\alpha_i \in P$ for all $i \in I$;
5° $\lambda(\alpha_i^\vee) \in \mathbb{Z}$ for all $\lambda \in P$;
6° $(P, P) \subset \frac{1}{d} \mathbb{Z}$ for some $d \in \mathbb{Z}_{>0}$.

These assumptions imply, in particular, that $\dim \mathfrak{h} \geq 2|I| - \operatorname{rank} A$.

Denote by $Q$ (respectively, $Q^+$) the subgroup (respectively, the submonoid) of $P$ generated by the $\alpha_i$. Let $P^+ = \{\lambda \in P : \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0}, \forall i \in I\}$.

Define $\omega_i \in \mathfrak{h}^*$, $i \in I$, by $\omega_i(\alpha_j^\vee) = \delta_{i,j}$, $j \in J$ and $\omega_i(h) = 0$ for all $h \in \bigcap_{i \in I} \ker \alpha_i$. We will assume that $\omega_i \in P$, $i \in I$ and denote by $P_{\text{int}}$ (respectively, $P_{\text{int}}^+$) the subgroup (respectively, the submonoid) of $P$ generated by the $\omega_i$, $i \in I$. Given any $J \subset I$, denote $\rho_J = \sum_{j \in J} \omega_j \in P$; we abbreviate $\rho_J = \rho$.

Let $W$ be the Weyl group associated with the matrix $A$, that is, the Coxeter group with $m_{ij} = 2$ if $a_{ij} = 0$, $m_{ij} = 3$ if $a_{ij}a_{ji} = 1$, $m_{ij} = 4$ if $a_{ij}a_{ji} = 2$, $m_{ij} = 6$ if $a_{ij}a_{ji} = 3$ and $m_{ij} = 0$ if $a_{ij}a_{ji} > 3$. It is well-known that $W$ is finite if and only if $A$ is positive definite. It should be noted that in that case $\alpha_i \in P_{\text{int}}$ for all $i \in I$. The group $W$ acts on $\mathfrak{h}$ (respectively, on $\mathfrak{h}^*$) by $s_i h = h - \alpha_i(h)\alpha_i^\vee$ (respectively, $s_i \lambda = \lambda - \lambda(\alpha_i^\vee)\alpha_i$), $h \in \mathfrak{h}$, $\lambda \in \mathfrak{h}^*$ and $i \in I$. Then we have $(w\lambda)(h) = \lambda(w^{-1}h)$ for all $w \in W$, $h \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$. 
Clearly, \( W(P) = P \) and \( P = P_{int} \oplus P_W \) where \( P_W = \{ \lambda \in P : w\lambda = \lambda, \forall w \in W \} = \{ \lambda \in P : \lambda(\alpha_i^\vee) = 0, \forall i \in I \} \).

Given \( J \subset I \) we define a linear map \( \rho_I^v : h^* \rightarrow \mathbb{C} \) by \( \rho_I^v(\alpha_i) = 1 \), \( i \in J \) and \( \rho_I^v(\lambda) = 0 \) if \((\lambda, \alpha_i) = 0 \) for all \( i \in J \). As before, we abbreviate \( \rho_I^v = \rho^v \). If \( J \in \mathcal{J} \) then it can be shown that \( \rho_I^v(\lambda) \) is equal to \( \frac{1}{2}\lambda(\sum_{h \in R_I^J} h) \) where \( R_I^J = \{ h \in h : h \in (\bigcup_{i \in J} W_J\alpha_i^\vee) \cap \sum_{i \in J} \mathbb{Z}_{\geq 0}\alpha_i^\vee \} \) is the set of positive co-roots of \( W_J \). In particular, this implies that \( \rho_I^v(P) \subset \frac{1}{2}\mathbb{Z} \).

If \( J \in \mathcal{J} \) then for each \( j \in J \) we have \( w^{v_J}(\alpha_j) = -\alpha_j^\vee \).

Given \( \lambda \in P^+ \), denote \( J_\lambda = \{ i \in I : (\lambda(\alpha_i^\vee)) = 0 \} = \{ i \in I : s_i\lambda = \lambda \} \). It is well-known that \( \text{Stab}_W \lambda = W_J_\lambda \) for \( \lambda \in P^+ \).

2.3. Quantum groups. We associate with the datum \((\Delta, h, \{ \alpha_i \}_{i \in J}, \{ \alpha_i^\vee \}_{i \in I})\) a complex Lie algebra \( g \) generated by the \( e_i, f_i, i \in I \) and \( h \) subject to the relations

\[
[h, h'] = 0, \quad [h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i, \quad [e_i, f_j] = \delta_{i,j}\alpha_i^\vee, \quad h, h' \in h, i, j \in I
\]

\[
(\text{ad } e_i)^{1-\alpha_i}(e_j) = 0 = (\text{ad } f_i)^{1-\alpha_i}(f_j), \quad i \neq j.
\]

If \( A \) is positive definite then \( g \) is a reductive finite dimensional Lie algebra. For \( J \subset I \) we denote by \( g_J^I \) the subalgebra of \( g \) generated by the \( e_i, f_i, i \in J \) and \( h \). It can also be regarded as the Lie algebra corresponding to the datum \((\Delta|_{J \times J}, h, \{ \alpha_i \}_{i \in J}, \{ \alpha_i^\vee \}_{i \in I})\). In particular, if \( J \in \mathcal{J} \) then \( g_J^I \) is a reductive finite dimensional Lie algebra.

Let \( k \) be any field of characteristic zero containing \( q^{\frac{1}{2}} \) which is purely transcendental over \( \mathbb{Q} \). Given any \( v \in k^* \) with \( v^2 \neq 1 \) define \( (n)_v = \frac{v^n - v^{-n}}{v - v^{-1}} \), \( (n)_v! = \prod_{s=1}^{n}(s)_v \), \( (n)_v\frac{k}{k} = \prod_{s=1}^{k}(n-s+1)_{v} \).

Let \( q_i = q^{\frac{1}{2}(\alpha_i, \alpha_i)} \). Henceforth, given any associative algebra \( A \) over \( k \) and \( X_i \in A, i \in I \) denote \( X_i^{(n)} := X_i^n/(n)_v! \). We will always use the convention that \( X_i^{(n)} = 0 \) if \( n < 0 \).

Define the Drinfeld-Jimbo quantum group \( U_q(g) \) corresponding to \( g \) as the associative algebra over \( k \) with generators \( K_\lambda, \lambda \in \frac{1}{2}P \) and \( E_i, F_i, i \in I \) subject to the relations

\[
K_\lambda E_i = q^{(\lambda, \alpha_i)}E_i K_\lambda, \quad K_\lambda F_i = q^{-(\lambda, \alpha_i)}F_i K_\lambda, \quad [E_i, F_j] = \delta_{ij} \frac{K_{\alpha_i} - K^{-\alpha_i}}{q_i - q^{-1}_i}, \quad \lambda \in \frac{1}{2}P, i, j \in I
\]

\[
\sum_{r+s=1-a_{ij}} (-1)^r E_i^{(r)} F_j E_i^{(s)} = 0 = \sum_{r+s=1-a_{ij}} (-1)^r F_j^{(r)} F_j F_i^{(s)}.
\]

This is a Hopf algebra with with the “balanced” comultiplication

\[
\Delta(E_i) = E_i \otimes K_{\frac{1}{2} \alpha_i} + K_{-\frac{1}{2} \alpha_i} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_{\frac{1}{2} \alpha_i} + K_{-\frac{1}{2} \alpha_i} \otimes F_i, \quad i \in I, \quad (2.2)
\]

while \( \Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \lambda \in \frac{1}{2}P \). Denote \( U_q^+(g) \) (respectively, \( U_q^-(g) \)) the subalgebra of \( U_q(g) \) generated by the \( E_i \) (respectively, the \( F_i \), \( i \in I \). Then \( U_q^\pm(g) \) is graded by
where the sum is direct and each summand is simple and isomorphic to \( v \), any dimensional. Furthermore, every \( V \) object in \( O \) (a direct summand) of objects are simple highest weight modules \( V \).

Given \( J \subset I \) we denote by \( U_q(\mathfrak{g}^J) \) the subalgebra of \( U_q(\mathfrak{g}) \) generated by the \( E_j, F_j, j \in I \) and \( K_\lambda, \lambda \in \frac{1}{2}P \) and set \( U_q^\pm(\mathfrak{g}^J) = U_q^\pm(\mathfrak{g}) \cap U_q(\mathfrak{g}^J) \).

If \( J \in \mathcal{J} \) then the algebra \( U_q(\mathfrak{g}^J) \) admits an automorphism \( \theta_J \) defined by \( \theta_J(E_i) = F_i, \theta_J(F_i) = E_i, \) and \( \theta_J(K_\lambda) = K_{w_\lambda}^{-1}, \lambda \in \frac{1}{2}P \). If \( I \in \mathcal{J} \) we abbreviate \( \theta_I = \theta \).

2.4. Integrable modules. We say that a \( U_q(\mathfrak{g}) \)-module \( M \) is integrable if \( M = \bigoplus_{\beta \in P} M(\beta) \) where \( M(\beta) = \{ m \in M : K_\lambda(m) = q^{(\lambda, \beta)}m, \forall \lambda \in \frac{1}{2}P \} \) and the \( E_i, F_i, i \in I \) act locally nilpotently on \( M \). Given any \( m \in M \) we can write uniquely

\[
m = \bigoplus_{\beta \in P} m(\beta)
\]  

where \( m(\beta) \in M(\beta) \) and \( m(\beta) = 0 \) for all but finitely many \( \beta \in P \). Denote \( \text{supp} \ m = \{ \beta \in P : m(\beta) \neq 0 \} \). By definition, if \( m \in M(\beta) \) and \( u_\pm \in U_q^\pm(\mathfrak{g})(\pm \nu), \nu \in \pm Q^+ \) then \( u_\pm(m) \in M(\beta \pm \nu) \). We say that \( m \in M(\beta), \beta \in P \) is homogeneous of weight \( \beta \) and call \( M(\beta) \) a weight subspace of \( M \).

**Definition 2.10.** The category \( \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \) is the full subcategory of the category of \( U_q(\mathfrak{g}) \)-modules whose objects are integrable \( U_q(\mathfrak{g}) \)-modules \( M \) with the following property: given \( m \in M \), there exists \( N(m) \geq 0 \) such that \( U_q^+(\mathfrak{g})(\nu)(m) = 0 \) for all \( \nu \in Q^+ \) with \( q^\nu(\nu) \geq N(m) \).

Given \( V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \), let \( V_+ = \bigcap_{i \in I} \ker E_i \) where the \( E_i \) are regarded as linear endomorphisms of \( V \). For any subset \( S \) of an object \( V \) in \( \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \) we denote \( S(\beta) = S \cap V(\beta) \) and \( S_+ = S \cap V_+ \).

It is well-known (see e.g. [31, Theorem 6.2.2]) that \( \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \) is semisimple and its simple objects are simple highest weight modules \( V_\lambda, \lambda \in P^+ \) with \( (V_\lambda)_+ = (V_\lambda)_+(\lambda) \) one-dimensional. Furthermore, every \( V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \) is generated by \( V_+ \) as a \( U_q(\mathfrak{g}) \)-module and \( V_+(\lambda) \neq 0 \) implies that \( \lambda \in P^+ \). Given \( \lambda \in P^+ \) and \( V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \) denote \( \mathcal{I}_\lambda(V) \) the \( \lambda \)-isotypical component of \( V \) as a \( U_q(\mathfrak{g}) \)-module. Thus, every simple submodule (and hence a direct summand) of \( \mathcal{I}_\lambda(V) \) is isomorphic to \( V_\lambda \) and \( \mathcal{I}_\lambda(V)_+ = V_+(\lambda) \). Furthermore, for any \( v \in V_+ \) we have the following equality of \( U_q(\mathfrak{g}) \)-submodules of \( V \)

\[
U_q(\mathfrak{g})(v) = \sum_{\lambda \in \text{supp} \ v} U_q(\mathfrak{g})(v(\lambda)),
\]  

where the sum is direct and each summand is simple and isomorphic to \( V_\lambda \).

It is immediate from the definition that every object \( \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \) can be regarded as an object in \( \mathcal{O}_q^{\text{int}}(\mathfrak{g}^J), J \subset I \). Denote \( P^+_J = \{ \mu \in P : \mu(\alpha^*_j) \in \mathbb{Z}_{\geq 0}, \forall j \in J \} \). Given \( V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \) and \( \lambda_j \in P^+_J \) denote by \( \mathcal{I}^J_{\lambda_j}(V) \) the \( \lambda_j \)-isotypical component of \( V \) as a
2.5. **Crystal operators, lattices and bases.** Here we recall some necessary facts from Kashiwara’s theory of crystal bases. To treat lower and upper crystal operators and lattices uniformly, we find it convenient to interpolate between them using c-crystal operators and lattices (for other generalizations see e.g. [13, 19]).

The following fact is standard (for example, see [31, Lemma 16.1.4]).

**Lemma 2.11.** Let $V \in \mathcal{O}^{\text{int}}_q(g)$ and fix $i \in I$. Then

$$V = \bigoplus_{0 \leq n \leq l} F_i^n (\ker E_i \cap \ker (K_{\alpha_i} - q_i^l)) = \bigoplus_{0 \leq n \leq l} E_i^n (\ker F_i \cap \ker (K_{\alpha_i} - q_i^{-l}))$$

Let $D = \{(l, k, s) \in D : k - l \leq s \leq k \leq l \}$. Fix a map $c : D \to \mathbb{Q}(z)^\times$ and denote its value at $(l, k, s)$ by $c_{l, k, s}$. We use the convention that $c_{l, k, s} = 0$ whenever $(l, k, s) \in \mathbb{Z}^3 \setminus D$. Using Lemma 2.11 we can define **generalized Kashiwara operators** $\bar{c}_{i,s}^c \in \text{End}_k V$, $s \in \mathbb{Z}$ by

$$\bar{c}_{i,s}^c (F_i^k(u)) = c_{l, k, s}(q_i) F_i^{k-s}(u),$$

for every $u \in \ker E_i \cap \ker (K_{\alpha_i} - q_i^l)$, $0 \leq k \leq l$. Note that under these assumptions on $u$, $\bar{c}_{i,s}^c (F_i^k(u)) \neq 0$ if and only if $(l, k, s) \in D$. Clearly, like lower or upper Kashiwara operators, the generalized ones commute with morphisms in $\mathcal{O}^{\text{int}}_q(g)$.

**Lemma 2.12.** Let $u \in \ker E_i \cap \ker (K_{\alpha_i} - q_i^l)$ and $u' \in \ker F_i \cap \ker (K_{\alpha_i} - q_i^{-l})$, $0 \leq k \leq l$. Then

$$\bar{c}_{i,s}^c (F_i^k(u)) = \mathcal{U}_{(l, k, s)}(q_i) F_i^{k-s}(u), \quad \bar{c}_{i,s}^c (E_i^k(u')) = \mathcal{U}_{(-l, k, s)}(q_i) E_i^{k+s}(u'),$$

for all $(l, k, s) \in D$, where $\mathcal{U}_{(l, k, s)}(q_i) = c_{l, k, s}(q_i) / (q_i)^s$. 

**Proof.** The first identity in (2.6) is immediate from (2.5). To prove the second, note that $E_i^{l+k}(u') = 0$ and so $u = E_i^{l+k}(u') \in \ker E_i \cap \ker (K_{\alpha_i} - q_i^l)$. It follows from [31, §3.4.2] that $E_i^{l+k}(u') = F_i^{l+k}(u)$. Using the first identity in (2.6) we obtain

$$\bar{c}_{i,s}^c (E_i^{l+k}(u')) = \bar{c}_{i,s}^c (F_i^{l-k}(u)) = \mathcal{U}_{(-l, k, s)}(q_i) F_i^{l-k}(u) = \mathcal{U}_{(-l, k, s)}(q_i) E_i^{k+s}(u').$$

The following is immediate.

**Lemma 2.13.** Given $c : D \to \mathbb{Q}(z)^\times$ we have

(a) $\bar{c}_{i,0}^c = \text{id}_V$ for all $V \in \mathcal{O}^{\text{int}}_q(g)$ if and only if $c_{l, k, 0} = 1$ for all $0 \leq k \leq l$;

(b) $\bar{c}_{i,t}^c \circ \bar{c}_{i,s}^c = \bar{c}_{i,s+t}^c$ for all $s, t \in \mathbb{Z}$ with $st \geq 0$ and for all $V \in \mathcal{O}^{\text{int}}_q(g)$ if and only if $c_{l, k, s+t} = c_{l, k, s} c_{l, k, s+t}$ for all $0 \leq k \leq l$, $s, t \in \mathbb{Z}$, $st \geq 0$.

The following is easy to deduce from [26, §3.1]
Lemma 2.14. Define $c^{\text{low}}, c^{\text{up}} : \mathbb{D} \to \mathbb{Q}(z)^\times$ by
\[
 c^{\text{low}}_{i,k,s} = \frac{(k)_z!}{(k-s)_z!}, \quad c^{\text{up}}_{i,k,s} = \frac{(l-k+s)_z!}{(l-k)_z!}, \quad (l, k, s) \in \mathbb{D}.
\]
(2.7)
We have
\[
 (\tilde{e}_i^{\text{low}})^s = \tilde{e}_i^{\text{low}}, \quad (\tilde{e}_i^{\text{up}})^s = \tilde{e}_i^{\text{up}}, \quad i \in I, s \in \mathbb{Z},
\]
where $\tilde{e}_i^{\text{low}}$ (respectively, $\tilde{e}_i^{\text{up}}$) are lower (respectively, upper) Kashiwara’s operators as defined in [26, §3.1].

Fix $c : \mathbb{D} \to \mathbb{Q}(z)^\times$ and let $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$. Let $\mathbb{A}$ be the local subring of $\mathbb{Q}(q) \subset k$ consisting of rational functions regular at 0. Generalizing well-known definitions of Kashiwara, we say that an $\mathbb{A}$-submodule $L$ of $V$ is a $c$-crystal lattice if $V = k \otimes_\mathbb{A} L$, $L = \bigoplus_{\beta \in B}(L \cap V(\beta))$, and $\tilde{e}_i^{c}(L) \subset L$ for all $i \in I, s \in \mathbb{Z}$.

We will be mostly interested in a special class of crystal lattices which we refer to as monomial. We need the following notation. Given $v \in V$ set
\[
 M^c_I(v) = \{v\} \cup \bigcup_{k \in \mathbb{Z}_{>0}} \{\tilde{e}^{c}_{i_1,m_1} \cdots \tilde{e}^{c}_{i_k,m_k}(v) : (i_1, \ldots, i_k) \in J^k, (m_1, \ldots, m_k) \in \mathbb{Z}^k\}.
\]
We abbreviate $M^c(v) = M^c_I(v)$. We call an $\mathbb{A}$-submodule $L$ of $V$ a $(c, J)$-monomial lattice if
\[
 L = \sum_{v_+} M^c_J(v_+)
\]
where the sum is over all $v_+ \in L \cap V_+^J(\lambda_J)$, $\lambda_J \in P^+$. Clearly, $L$ inherits a weight decomposition from $V$ and $\tilde{e}_i^{c}(L) \subset L$ for all $j \in J, a \in \mathbb{Z}$. In particular, if $L$ is a $(c, I)$-monomial and $k \otimes_\mathbb{A} L = V$ then $L$ is a $c$-crystal lattice.

Denote $\bar{L}$ the $\mathbb{Q}$-vector space $L/qL$. Given $\bar{v} \in \bar{L}$, denote
\[
 \bar{M}^c_J(\bar{v}) = \{\bar{v}\} \cup \bigcup_{k \in \mathbb{Z}_{>0}} \{\tilde{e}^{c}_{i_1,m_1} \cdots \tilde{e}^{c}_{i_k,m_k}(\bar{v}) : (i_1, \ldots, i_k) \in J^k, (m_1, \ldots, m_k) \in \mathbb{Z}^k\} \subset \bar{L}.
\]
As before, we abbreviate $\bar{M}^c(v) = \bar{M}^c_I(v)$.

By [25, Theorem 3] and [26, Theorem 3.3.1], if $c = c^{\text{low}}$ or $c = c^{\text{up}}$ then every object in $\mathcal{O}_q^{\text{int}}(\mathfrak{g})$ admits a $c$-crystal lattice. Moreover, in that case for any $\lambda \in P^+$, and any $v_\lambda \in V_\lambda(\lambda)$ the smallest $\mathbb{A}$-submodule of $V_\lambda$ containing $v_\lambda$ and invariant with respect to the $\tilde{e}_i^{c}, i \in I, s \in \mathbb{Z}_{<0}$ is a $c$-crystal lattice.

Let $L$ be a $c$-crystal lattice of $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$. Clearly, operators $\tilde{e}_i^{c}$ commute with the action of $q$ on $L$ and thus factor through to $\mathbb{Q}$-linear operators on $\bar{L} = L/qL$ denoted by the same symbols. Similarly to [25, 26], we say that $(L, B)$, where $B$ is a weight basis of $\bar{L}$, is a $c$-crystal basis of $V$ at $q = 0$ if $\tilde{e}_i^{c}(B) \subset B \cup \{0\}$, $i \in I, s \in \mathbb{Z}$. By [25, 26], every object in $\mathcal{O}_q^{\text{int}}(\mathfrak{g})$ admits a crystal basis provided that $c \in \{c^{\text{up}}, c^{\text{low}}\}$.

The following is well-known (cf. [25, 26]).
Lemma 2.15. Let $V \in \mathcal{O}_q^{\text{int}}(g)$, $c \in \{c^{\text{low}}, c^{\text{up}}\}$ and let $(L, B)$ be a $c$-crystal basis at $q = 0$. Then for any $J \subset I$

(a) $L$ is a $(c, J)$-monomial lattice;
(b) $M_{j}^c(b) \subset B \cup \{0\}$ for any $b \in B$;
(c) $B = \bigcup_{b_i \in B} M_{j}^c(b_i) \setminus \{0\}$ where $B_I = \cap_{j \in J} \ker e_{j,1}^c \subset B$.

Remark 2.16. It is not hard to see that if for given $c : \mathbb{D} \to \mathbb{Q}(z)^\times$ and $J \subset I$ Lemma 2.15(a)–(c) hold then $(L, B)$ is a $c$-basis at $q = 0$ of $V$ regarded as a $U_q(\mathfrak{g})$-module.

3. Properties of $\sigma^i$ and proof of Theorem 1.1

3.1. Special monomials in $U_q^{\pm}(g)$. Given any reduced sequence $i = (i_1, \ldots, i_m) \in I^m$ and $\lambda \in P^+$ we define $F_{i,\lambda} \in U_q^-(g)$ and $E_{i,\lambda} \in U_q^+(g)$, $\lambda \in P^+$ by

$$F_{i,\lambda} = F_{i_1}^{(a_1)} \cdots F_{i_m}^{(a_m)}, \quad E_{i,\lambda} = E_{i_1}^{(a_1)} \cdots E_{i_m}^{(a_m)},$$

where $a_k = a_k(i, \lambda) = s_{i_{k+1}} \cdots s_{i_m} \lambda(\alpha_k^\vee) = \lambda(s_{i_m} \cdots s_{i_{k+1}} \alpha_k^\vee) \in \mathbb{Z}_{\geq 0}$.

Lemma 3.1. Let $w \in W$, $\lambda \in P^+$, $i \in I$. Then

(a) $F_{w,\lambda} = F_{V,\lambda}$ and $E_{w,\lambda} = E_{V,\lambda}$ for any $i, i' \in R(w)$ and $\lambda \in P^+$. Thus, we can define $F_{w,\lambda} := F_{i,\lambda}$ and $E_{w,\lambda} := E_{i,\lambda}$ for some $i \in R(w)$;
(b) If $\ell(s_i w) = \ell(w) + 1$ then $F_{s_i w,\lambda} = F_{i}^{(w\lambda(\alpha_i^\vee))} F_{w,\lambda}$ and $E_{s_i w,\lambda} = E_{i}^{(w\lambda(\alpha_i^\vee))} E_{w,\lambda}$;
(c) If $\ell(s_i w) = \ell(w) - 1$ then $F_{w,\lambda} = F_{i}^{(-w\lambda(\alpha_i^\vee))} F_{s_i w,\lambda}$ and $E_{w,\lambda} = E_{i}^{(-w\lambda(\alpha_i^\vee))} E_{s_i w,\lambda}$;
(d) If $s_i \lambda = \lambda$ then $F_{w s_i,\lambda} = F_{w,\lambda}$;
(e) $\deg F_{w,\lambda} = - \deg E_{w,\lambda} = w \lambda - \lambda$;
(f) Suppose that $W$ is finite. Then $\theta(F_{w,\lambda}) = E_{w_0 w w_0, -w_0 \lambda}$.

Proof. It is well-known that $i$ can be obtained from $i'$ by a finite sequence of rank 2 braid moves of the form $s_i s_j \cdots = s_j s_i \cdots$ with $m_{ij}$ finite. Thus, it suffices to prove part (a) in case when $w$ is the longest element in the subgroup of $W$ generated by $s_i, s_j, i \neq j \in I$. But in that case it was established in [31, Proposition 39.3.7].

Parts (b), (c) and (d) are obtained from part (a) by choosing appropriate reduced decompositions. To prove parts (e) and (f) we use induction on $\ell(w)$, the case $\ell(w) = 0$ being obvious. For the inductive step, suppose that $\ell(s_i w) = \ell(w) + 1$. Since $\mu(\alpha_i^\vee) \alpha_i = \mu - s_i \mu$, $\mu \in P$, by part (b) and the induction hypothesis we have $\deg F_{s_i w,\lambda} = - w \lambda(\alpha_i^\vee) \alpha_i + w \lambda - \lambda = s_i w \lambda - \lambda$. This proves the inductive step in part (e). Since $s_i \ast = w_0 s_i w_0$ we have by Lemma 3.1(b)

$$\theta(F_{s_i w,\lambda}) = \theta(F_{i}^{(w\lambda(\alpha_i^\vee))} F_{w,\lambda}) = F_{i}^{(w\lambda(\alpha_i^\vee))} E_{w_0 w w_0, -w_0 \lambda} = F_{i}^{(-w\lambda(\alpha_i^\vee))} E_{w_0 w w_0, -w_0 \lambda}$$
3.1(d). By an obvious induction on $2.4$ we have (b).

**Remark 3.3.** It is not hard to show that $E_i(V) = \{0\}$ for every $i \in J(V)$.

We will need the following basic properties of these sets.

**Proposition 3.4.** Let $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$. Then

(a) $J(v) = J_{\alpha}$ for any $v \in V_+(\lambda) \setminus \{0\}$, $\lambda \in P^+$;

(b) There exists $v \in V_+$ such that $J(V) = J(U_q(\mathfrak{g}))(v))$.

**Proof.** It is well-known (see e.g. [31, Chap. 6]) that the annihilating ideal of $v$ in $U_q^{-} (\mathfrak{g})$ is generated by the $F_i(\lambda(\alpha_i') + 1)$, $i \in I$. Thus, $F_i(v) = 0$ if and only if $\lambda(\alpha_i') = 0$. This proves part (a). To prove part (b), note the following obvious fact.

**Lemma 3.5.** Let $R$ be a ring and let $M = \bigoplus_{\alpha \in A} M_\alpha$ as $R$-modules. Let $S$ be a subset of $R$. Then $\text{Ann}_S M = \bigcap_{\alpha \in A'} \text{Ann}_S M_\alpha = \text{Ann}_S M'$ where $A'$ is any subset of $A$ such that for each $\alpha \in A$ there exists $\alpha' \in A'$ such that $M_\alpha \cong M_{\alpha'}$ and $M' = \bigoplus_{\alpha \in A'} M_\alpha$. In particular, if $S$ is finite then $\text{Ann}_S M = \bigcap_{\alpha \in A_0} \text{Ann}_S M_\alpha = \text{Ann}_S M_0$ where $A_0$ is a finite subset of $A'$ and $M_0 = \bigoplus_{\alpha \in A_0} M_\alpha$. 

The inductive step in part (f) is proven. \qed
Apply this Lemma to \( R = U_q(\mathfrak{g}) \) and \( S = \{ F_i : i \in I \} \), which identifies with \( I \) and \( M = V \). Clearly, \( J(V) = \{ i \in I : F_i \in \text{Ann}_S V \} \). Since \( S \) is finite and \( V \) is a direct sum of simple modules, it follows from Lemma 3.5 that \( \text{Ann}_S V = \text{Ann}_S V' \) where \( V' = \bigoplus_{\lambda \in \Omega} U_q(\mathfrak{g})(v_\lambda) \) for some finite \( \Omega \subset \{ \lambda \in P^+ : \mathcal{I}_\lambda(V) \neq 0 \} \) and \( v_\lambda \in V_+(\lambda) \setminus \{ 0 \} \), \( \lambda \in \Omega \). Since \( V' = U_q(\mathfrak{g})(v) \) with \( v = \sum_{\lambda \in \Omega} v_\lambda \), part (b) follows.

\[ \square \]

**Proposition 3.6.** Let \( V \in \mathcal{O}^{\text{int}}_q(\mathfrak{g}) \). For each \( i \in I \), \( v \in V_+ \setminus \{ 0 \} \) the following are equivalent.

(a) \( (\lambda, w_\alpha) = 0 \) for all \( \lambda \in \text{supp} v \), \( w \in W \);

(b) \( \text{cl}(\{ i \}) \in J(U_q(\mathfrak{g})(v)) \).

**Proof.** Let \( J \) be the neighborhood of \( i \). In particular, \( J \cup J^\perp = I \).

(a) \( \implies \) (b) We need the following Lemma.

**Lemma 3.7.** For every \( i \sim j \in I \), \( i \neq j \) there exists \( w = w_{i,j} \in W \) such that \( w_{i,j} \alpha_i \in \mathbb{Z}_{>0} \alpha_j + \sum_{k \in I \setminus \{ j \}} \mathbb{Z}_{\geq 0} \alpha_k \).

**Proof.** Let \( i = (i = i_0, i_1, \ldots, i_d = j) \in I^{d+1} \) be an admissible sequence with \( d = \text{dist}(i, j) \). In particular, this sequence is repetition free. Denote \( \beta_k = s_{i_k} \cdots s_{i_1} (\alpha_i) \). We claim that \( \beta_k \in \sum_{0 \leq r \leq k} \mathbb{Z}_{>0} \alpha_i \). We argue by induction on \( k \), the case \( k = 0 \) being obvious. For the inductive step, note that

\[
\beta_k = s_{i_k}(\beta_{k-1}) = \sum_{0 \leq r \leq k-1} \mathbb{Z}_{>0} s_{i_k} \alpha_i = \sum_{0 \leq r \leq k-1} \mathbb{Z}_{>0} (\alpha_i - \alpha_i (\alpha_i^\vee) \alpha_k).
\]

Since \( \alpha_i \) is admissible, \( (\alpha_i)_{\alpha_i^\vee} < 0 \) while \( (\alpha_i \alpha_i^\vee) \leq 0 \) for all \( 0 \leq r \leq k - 2 \). Therefore, \( \beta_k \in \sum_{0 \leq r \leq k} \mathbb{Z}_{>0} \alpha_i \). In particular, \( w = s_{i_d} \cdots s_{i_1} \) is the desired \( w_{i,j} \in W \). \( \square \)

Write \( \lambda \in \text{supp} v \) as \( \lambda = \lambda' + \sum_{k \in I} l_k \omega_k \) where \( \lambda' \in P^W \) and \( l_k \in \mathbb{Z}_{\geq 0}, k \in I \). Let \( j \in J \). In the notation of Lemma 3.7, we have \( w_{i,j} \alpha_i = \sum_{k \in I} n_k \alpha_k \) with \( n_j \in \mathbb{Z}_{>0} \) and \( n_k \in \mathbb{Z}_{\geq 0} \), \( k \in I \setminus \{ j \} \), for some \( w_{i,j} \in W \). Then \( 0 = 2(\lambda, w_{i,j} \alpha_i) = (\alpha_j, \alpha_j) l_j n_j + \sum_{k \in I \setminus \{ j \}} (\alpha_k, \alpha_k) l_k n_k \).

Since \( n_j > 0 \) and \( n_k \geq 0 \) for all \( k \neq j \) this forces \( l_j = 0 \).

In particular, \( J \subset J(v) \). Since \( J \) is closed in \( I \), \( [U_q(\mathfrak{g})^{I^\perp}, F_j] = 0 \). Let \( V' = U_q(\mathfrak{g})(v) \).

Then \( V' = U_q^{-}(\mathfrak{g})(v) = U_q^{-}(\mathfrak{g})_{I^\perp}(v) \) and so \( J \subset J(V') \).

(b) \( \implies \) (a) Since \( J \subset J(U_q(\mathfrak{g})(v)) \) it follows that \( (\lambda, \alpha_j) = 0 \) for all \( j \in J \). Since \( W \alpha_j \in \mathbb{Z}_{\geq 0} \mathfrak{a}_j \) for any \( j \in J \), the assertion follows. \( \square \)

**Lemma 3.8.** Let \( V \in \mathcal{O}^{\text{int}}_q(\mathfrak{g}) \) such that \( V = U_q(\mathfrak{g})(v) \) for some \( v \in V_+ \). The following are equivalent for \( i \in I \).

(a) \( i \in J(V) \).

(b) \( [v(\lambda)]_{s_{i,w}} = [v(\lambda)]_w \) for all \( \lambda \in \text{supp} v \) and for all \( w \in W \);
Proof. The condition in part (b) implies that $s_i w \lambda = w \lambda$ for all $\lambda \in \text{supp} v$ and $w \in W$. Since $s_i w \lambda = w \lambda - (w \lambda)(\alpha_i^\vee) \alpha_i$, it follows that $(\lambda, w \alpha_i) = 0$ for all $\lambda \in \text{supp} v$ and $w \in W$ and so $i \in J(V)$ by Proposition 3.6.

Conversely, if $i \in J(V)$ then $F_i([v]_w) = 0$ for all $w \in W$. In particular, if $\ell(s_i w) = \ell(w) + 1$ then, since $[v]_{s_i w} = F_i(\sigma^i([\lambda]_w)) [v]_w \neq 0$ it follows that $(w \lambda, \alpha_i) = 0$ and thus $[v]_{s_i w} = [v]_w$. Similarly, if $\ell(s_i w) = \ell(w) - 1$ applying the previous argument to $w' = s_i w$ we obtain the same equality. \qed

3.3. Proof of Theorem 1.1. We will now express the action of the $\sigma^i$, $i \in I$ on extremal vectors in terms of the natural action of $W$ on $W/W_J$.

Proposition 3.9. Let $V \in \mathcal{O}_q^{\text{int}}(g)$ and $v \in V_+ \setminus \{0\}$. Then
(a) The set $[v]_W$ is $W(V)$-invariant. More precisely, $\sigma^i([v(\lambda)]_w) = [v(\lambda)]_{s_i w}$ for all $i \in I$, $w \in W$ and $\lambda \in \text{supp} v$;
(b) The canonical image of $W(V)$ in $\text{Bij}([v]_W)$ is isomorphic to $W_{J_0}$ where $J_0 = I \setminus \text{cl}(U_q(g)(v))$.

Proof. To prove part (a), let $\lambda \in P^+$, $w \in W$, $i \in I$ and suppose first that $\ell(s_i w) = \ell(w) + 1$. Then $w \lambda(\alpha_i^\vee) > 0$ and $[v(\lambda)]_w \in \ker F_i$, whence $\sigma^i([v(\lambda)]_w) = F_i(\sigma^i([\lambda]_w)) [v(\lambda)]_w = F_i(s_i w \lambda)(v(\lambda)) = [v(\lambda)]_{s_i w}$ where we used Lemma 3.1(b). If $\ell(s_i w) = \ell(w) - 1$ then $w = s_i w'$. By the above, $[v(\lambda)]_w = [v(\lambda)]_{s_i w'} = \sigma^i([v(\lambda)]_{s_i w})$. Since $\sigma^i$ is an involution, it follows that $\sigma^i([v(\lambda)]_w) = [v(\lambda)]_{s_i w}$.

To prove (b), the bijections from Proposition 3.2(a) allow one to identify $[v]_W$ with $\bigcup_{\lambda \in \text{supp} v} W/W_{J_\lambda}$. In particular, this induces an action of $W$ on $[v]_W$ via $w \cdot [v(\lambda)]_{w'} = [v(\lambda)]_{w w'}$, $\lambda \in \text{supp} v$, $w, w' \in W$. By part (a) the canonical images of $W(V)$ and $W$ in $\text{Bij}([v]_W)$ coincide. We need the following general fact.

Lemma 3.10. Let $G$ be a group acting on $X = \bigsqcup_{\alpha \in A} X_\alpha$. Then the canonical image of $G$ in $\text{Bij}(X)$ is isomorphic to $G/K$ where $K = \bigcap_{\alpha \in A} K_\alpha$ and $K_\alpha = \{g \in G : gx = x, \forall x \in X_\alpha\}$ is the kernel of the action of $G$ on $X_\alpha$.

Applying this Lemma to $X = [v]_W$, $X_\lambda = [v(\lambda)]_W$ and $G = W$ and using the fact that $K_\lambda = W_{J_\lambda}$ by Theorem 2.4 we conclude that $K = \bigcap_{\lambda \in \text{supp} v} W_{J_\lambda}$. By Lemma 2.1, $K = W_{J'}$ where $J'$ is the set of $i \in I$ such that $s_i$ fixes $[v]_W$ elementwise. By Lemma 3.8, $J' = J(V')$ where $V' = U_q(g)(v)$. Since $J'$ is closed, being an intersection of closed sets, $W/W_{J'} \cong W_{I \setminus J'}$. \qed

Proof of Theorem 1.1. It follows from Proposition 3.2(a) that the assignments $[v]_w \mapsto wW_J$ define a bijection $J : [v]_W \to W/W_J$. This induces a group homomorphism $\xi_V : W(V) \to \text{Bij}(W/W_J)$ via $\xi_V(\sigma^i([v]_w)) = J(\sigma^i([v]_w)) = J([v]_{s_i w}) = s_i w W_J$. It follows that $\xi_V(W(V))$ coincides with the image of $W$ in $\text{Bij}(W/W_J)$ given by the natural action. By Lemma 2.3, the latter is canonically isomorphic to $W_{I \setminus J_0}$ where $J_0 = \text{cl}(I \setminus J)$. \qed
4. Modified Lusztig symmetries and involutions $\sigma'$

Let $\mathcal{C}$ be a $k$-linear category whose objects are $k$-vector spaces and let $G$ be a group. An action of $G$ on $\mathcal{C}$ is an assignment $g \mapsto g_* = \{g_V : V \in \mathcal{C}\}$, $g \in G$, where $g_V \in \text{GL}_k(V)$ such that $(gg')_V = g_V \circ g'_V$ for all $g, g' \in G$ and $V \in \mathcal{C}$ and $g_V \circ f = f \circ g_V$ for any $g \in G$ and any morphism $f : V \to V'$ in $\mathcal{C}$.

Recall that the braid group $\text{Br}_W$ associated with a Coxeter group $W$ is generated by the $T_i$, $i \in I$ subject to the relations $T_i T_j \cdots = T_j T_i \cdots$ for all $i \neq j \in I$ in the notation of §2.1. In this section we discuss modified Lusztig symmetries which provide an action of $\text{Br}_W$ associated with the Weyl group $W$ of $\mathfrak{g}$ on the category $\mathcal{O}^\text{int}_q(\mathfrak{g})$ and use them to construct an action of $\text{Cact}_W$ on the same category.

4.1. Modified Lusztig’s symmetries. Given $i \in I$ and $V \in \mathcal{O}^\text{int}_q(\mathfrak{g})$ define $T_i^\pm \in \text{End}_k V$ by

$$T_i^+ = T_i K_{\frac{1}{2} \alpha_i}, \quad T_i^- = T_i K_{-\frac{1}{2} \alpha_i}$$

where $T_i, T_i' \in \text{End}_k V$ are Lusztig symmetries (see [31, §5.2]). We refer to these operators as modified Lusztig symmetries. By definition, $T_i^+(V(\beta)) = V(s_i \beta)$ and so $T_i^{\pm} K_{\lambda} = K_{s_i \lambda} T_i^{\pm}$, $\lambda \in \frac{1}{2} P$.

**Lemma 4.1.** The assignments $T_i \mapsto T_i^+$ (respectively, $T_i \mapsto T_i^-$) define an action of $\text{Br}_W$ on $\mathcal{O}^\text{int}_q(\mathfrak{g})$.

**Proof.** It can be deduced from [31, Proposition 39.4.3] along the lines of [2, Lemma 5.2] that the $T_i^+$ (and the $T_i^-$), $i \in I$, satisfy the defining relations of $\text{Br}_W$ as endomorphisms of $V$ for each $V \in \mathcal{O}^\text{int}_q(\mathfrak{g})$. To prove that this action of $\text{Br}_W$ commutes with morphisms, write, using [33, §3.1], for $V \in \mathcal{O}^\text{int}_q(\mathfrak{g})$ and $v \in V$

$$T_i^+(v) = \sum_{(a, b, c) \in \mathbb{Z}^3} (-1)^b q_i^{b-a-c} K_{\frac{1}{2}(a-c-1) \alpha_i} F_i^{(a)} E_i^{(b)} F_i^{(c)} K_{\frac{1}{2}(a-c) \alpha_i}(v),$$

$$T_i^-(v) = \sum_{(a, b, c) \in \mathbb{Z}^3} (-1)^b q_i^{b-a-c} K_{\frac{1}{2}(c-a+1) \alpha_i} E_i^{(a)} F_i^{(b)} E_i^{(c)} K_{\frac{1}{2}(c-a) \alpha_i}(v)$$

where the sum is finite since $V$ is integrable. It is now obvious that the $T_i^\pm$, $i \in I$ commute with homomorphisms of $U_q(\mathfrak{g})$-modules. \qed

It is well-known (see e.g. [31, §39.4.7]) that the element $T_{i_1} \cdots T_{i_r}$ with $i = (i_1, \ldots, i_r) \in I^r$ reduced depends only on $w = s_{i_1} \cdots s_{i_r}$ and not on $i$. This allows to define the canonical section of the natural group homomorphism $\text{Br}_W \to W$, $T_i \mapsto s_i$, $i \in I$, by $w \mapsto T_w := T_{i_1} \cdots T_{i_r}$ where $(i_1, \ldots, i_r) \in R(w)$. Denote $T_w^\pm$ the linear endomorphisms of any $V \in \mathcal{O}^\text{int}_q(\mathfrak{g})$ arising from Lemma 4.1 which correspond to the canonical element $T_w$ of $\text{Br}_W$. The elements $T_w, T_w^\pm$ are characterized by the following well-known property.
Lemma 4.2 ([31, §39.4.7]). Let \( w, w' \in W \) be such that \( \ell(w) + \ell(w') = \ell(ww') \). Then
\[ T_{ww'} = T_w T_{w'} \quad \text{and} \quad T^\pm_{ww'} = T^\pm_w \circ T^\pm_{w'}, \]
as linear endomorphisms of \( V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \).

Proposition 4.3. Let \( V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \). Then for any \( w, w' \in W, \lambda \in P^+ \) and \( v \in V_+(\lambda) \) we have:

(a) if \( \ell(ww') = \ell(w) + \ell(w') \) then
\[ T^+_w(F_{w',\lambda}(v)) = q^{\frac{1}{2}(w'\lambda,\rho - w^{-1}\rho)} F_{ww',\lambda}(v), \]
\[ T^-_w(F_{w',\lambda}(v)) = (-1)^{\rho^\vee (w'\lambda - w\lambda)} q^{\frac{1}{2}(w'\lambda,\rho - w^{-1}\rho)} F_{ww',\lambda}(v). \]

(b) if \( \ell(ww') = \ell(w) - \ell(w') \) then
\[ T^+_w(F_{w',\lambda}(v)) = (-1)^{\rho^\vee (w'\lambda - w\lambda)} q^{-\frac{1}{2}(w'\lambda,\rho - w^{-1}\rho)} F_{ww',\lambda}(v), \]
\[ T^-_w(F_{w',\lambda}(v)) = q^{-\frac{1}{2}(w'\lambda,\rho - w^{-1}\rho)} F_{ww',\lambda}(v). \]

(c) if \( \ell(ww') = \ell(w) - \ell(w') \) then
\[ T^+_w(F_{w',\lambda}(v)) = (-1)^{\rho^\vee (w'\lambda - \lambda)} q^{\frac{1}{2}(\lambda,2\rho - w^{-1}(\rho + w^{-1}\rho))} F_{ww',\lambda}(v), \]
\[ T^-_w(F_{w',\lambda}(v)) = (-1)^{\rho^\vee (\lambda - w'\lambda)} q^{\frac{1}{2}(\lambda,2\rho - w^{-1}(\rho + w^{-1}\rho))} F_{ww',\lambda}(v). \]

Proof. To prove (a) we argue by induction on \( \ell(w) \), the case \( \ell(w) = 0 \) being vacuously true. The following Lemma is the main ingredient in the proof of inductive steps in Proposition 4.3.

Lemma 4.4. In the notation of Proposition 4.3 we have, for all \( i \in I \)
\[ T^+_i(F_{w',\lambda}(v)) = \begin{cases} q^{\frac{1}{2}(w'\lambda,\alpha_i)} F_{s_iw',\lambda}(v), & (w'\lambda, \alpha_i) \geq 0, \\ (-1)^{w'\lambda(\alpha_i^\vee)} q^{-\frac{1}{2}(w'\lambda,\alpha_i)} F_{s_iw',\lambda}(v), & (w'\lambda, \alpha_i) \leq 0, \end{cases} \]
\[ T^-_i(F_{w',\lambda}(v)) = \begin{cases} (-1)^{w'\lambda(\alpha_i^\vee)} q^{\frac{1}{2}(w'\lambda,\alpha_i)} F_{s_iw',\lambda}(v), & (w'\lambda, \alpha_i) \geq 0, \\ q^{-\frac{1}{2}(w'\lambda,\alpha_i)} F_{s_iw',\lambda}(v), & (w'\lambda, \alpha_i) \leq 0. \end{cases} \]  

(4.2)

Proof. Clearly, \( F_{w',\lambda}(v) \) is either a highest or a lowest weight vector in the \( i \)-th simple quantum \( \mathfrak{sl}_2 \)-submodule \( V_m \) it generates where \( m = |(w'\lambda, \alpha_i^\vee)|. \) Then (4.2) follows from [31, Propositions 5.2.2, 5.2.3]. Namely, let \( V_m \) be the standard simple \( U_q(\mathfrak{sl}_2) \)-module with the standard basis \( \{ z_k \}_{0 \leq k \leq m} \) such that \( K(z_k) = v^{m-2k} z_k \) and \( z_k = E^{m-k}(z_m), \) \( 0 \leq k \leq m. \) Recall that \( T^+ = T_1^K K^{1/2} \) and \( T^- = T_1^n K^{-1/2}. \) Then by [31, Propositions 5.2.2, 5.2.3] we have
\[ T^+(z_k) = (-1)^{m-k} v^{k(m-k)+\frac{1}{2}m} z_{m-k}, \quad T^-(z_k) = (-1)^{m-k} v^{k(m-k)+\frac{1}{2}m} z_{m-k}, \] \( 0 \leq k \leq m. \) (4.3)
Thus, (4.2) is obtained by applying (4.3) with \( k = 0 \) if \( w'\lambda(\alpha_i^\vee) \geq 0 \) and \( k = m \) if \( w'\lambda(\alpha_i^\vee) \leq 0. \) \( \square \)
To prove inductive steps in part (a) of Proposition 4.3, suppose that \( w, w' \in W \) and \( i \in I \) satisfy \( \ell(s_i w w') = \ell(w) + \ell(w') + 1 \). In particular, \( \ell(s_i w) = \ell(w) + 1 \) and \( \ell(w w') = \ell(w) + \ell(w') \). Then we have, by the induction hypothesis and (4.2)

\[
T_{s_i w}^+(F_{w', \lambda}(v)) = T_{s_i}^+ T_{w}^+(F_{w', \lambda}(v)) = q^{\frac{1}{2}(w', \lambda, \rho - w', \rho)} T_{s_i}^+(F_{w w', \lambda}(v)) = q^{\frac{1}{2}(w', w - w', \rho)} F_{s_i w, w', \lambda}(v) = q^{\frac{1}{2}(w', w - w', \rho)} F_{s_i w, w', \lambda}(v),
\]

where we used the \( W \)-invariance of \((\cdot, \cdot)\) and the obvious observation that \( \alpha_i = \rho - s_i \rho \).

The second identity in part (a) is proved similarly using the observation that \( \mu(\alpha_i') = \rho(\mu - s_i \mu) \).

The proof of part (b) is identical, the only difference being that we assume \( \ell(s_i w w') = \ell(w') - \ell(w) - 1 \) which implies that \( \ell(s_i w) = \ell(w) + 1 \) and \( \ell(w w') = \ell(w) - \ell(w') \).

To prove part (c), denote \( w_1 = w w' \). Then \( \ell(w) = \ell(w_1) + \ell(w_1 w') - 1 \), \( w = w_1 w' - 1 \) and so \( T_{w}^+ = T_{w_1}^+ \circ T_{w'}^+ \), by Lemma 4.2. Applying part (b) with \( w = w' - 1 \) and then part (a) with \( w' = 1 \) and \( w = w_1 \) we obtain

\[
T_{w_1}^+(F_{w', \lambda}(v)) = T_{w_1}^+(T_{w'}^+(F_{w', \lambda}(v))) = (-1)^{\rho(\lambda, \lambda)} q^{-\frac{1}{2}(w', \lambda, \rho - w', \rho)} T_{w_1}^+(F_{w, \lambda}(v)) = (-1)^{\rho(\lambda, \lambda)} q^{\frac{1}{2}(\lambda, w - w', \rho)} F_{w_1, \lambda}(v) = (-1)^{\rho(\lambda, \lambda)} q^{\frac{1}{2}(\lambda, 2 \rho - w' - 1, \rho)} F_{w w', \lambda}(v).
\]

The identity for \( T_{w}^- \) is proved similarly.

Recall from [31, Chapter 37] that \( Br_W \) also acts on \( U_q(\mathfrak{g}) \) via Lusztig symmetries and let \( T_i^\pm \) be the automorphisms of \( U_q(\mathfrak{g}) \) defined as \( T_i^+ = T_{i, 1}^+ \) ad \( K_{\frac{1}{2} \alpha_i} \), and \( T_i^- = T_{i, 1}^- \) ad \( K_{-\frac{1}{2} \alpha_i} \), \( i \in I \) where ad \( K_{\lambda}(u) = K_{\lambda} K_{-\lambda}, \lambda \in \frac{1}{2} P, u \in U_q(\mathfrak{g}) \).

**Remark 4.5.** The operators \( T_i^\pm \), viewed as automorphisms of \( U_q(\mathfrak{g}) \), were already used in [2] for studying double canonical bases of \( U_q(\mathfrak{g}) \).

**Lemma 4.6.** On \( U_q(\mathfrak{g}) \) we have

(a) \( T_i^+ \circ \text{ad} \ K_{\lambda} = \text{ad} \ K_{s_i \lambda} \circ T_i^+ \) for all \( \lambda \in \frac{1}{2} P, i \in I \);

(b) \( T_w^+ = T_{w, 1}^+ \circ \text{ad} \ K_{\frac{1}{2}(\rho - w, -1, \rho)} \) and \( T_w^- = T_{w, 1}^- \circ \text{ad} \ K_{\frac{1}{2}(w - 1, \rho - \rho)} \) for all \( w \in W \).

**Proof.** Part (a) is immediate, while part (b) follows from (a) by induction similar to that in Proposition 4.3. \( \square \)

**Lemma 4.7.** Suppose that \( W \) is finite. Then for all \( i \in I \) we have

\[
T_{w_0}^\pm(E_i) = -q^{-\frac{1}{2}(\alpha_i, \alpha_i)} T_i^\pm K_{\alpha_i}, \quad T_{w_0}^\pm(F_i) = -q^{-\frac{1}{2}(\alpha_i, \alpha_i)} T_i^\pm K_{\alpha_i}.
\]
Proof. By [1, Lemma 2.8] and Lemma 4.6(b) we have $T^\pm_w(E_i) = q^\mp \frac{1}{2}(\rho - w^{-1} \rho, \alpha_i) E_i$ provided that $w \alpha_i = \alpha_j$. Since $(w^{-1} \rho, \alpha_i) = (\rho, w \alpha_i) = (\rho, \alpha_j)$ and $\langle \alpha_i, \alpha_i \rangle = \langle \alpha_j, \alpha_j \rangle$ it follows that $T^\pm_w(E_i) = E_j$. In particular, since $w_0 s_i \alpha_i = \alpha_i$, we have $T^\pm_w(s_i)(E_i) = E_i$. On the other hand, $T^\pm_{w_0 s_i}(E_i) = T^\pm_{s_i w_0}(E_i)$ and $\ell(w_0) = \ell(s_i) + 1$ whence $T^\pm_{w_0}(E_i) = T^\pm_i(E_i) = -q^{-\frac{1}{2} \langle \alpha_i, \alpha_i \rangle} F_i K_{s_i, i}$. The argument for $T^\pm_{w_0}(F_i)$ is similar.

The following Lemma is immediate from [31, Proposition 37.1.2].

Lemma 4.8. Let $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$. Then $T^\pm_i(x(v)) = T^\pm_i(x)(T^\pm_i(v))$ for all $i \in I$, $v \in V$, $x \in U_q(\mathfrak{g})$.

Lemma 4.9. We have $T^\pm_w(\mathcal{I}_\lambda(V)) = \mathcal{I}_\lambda(V)$ and $T^\pm_w(V(\beta)) = V(w \beta)$ for any $w \in W$, $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$, $\lambda \in P^+$ and $\beta \in P$.

4.2. $\sigma$ via modified Lusztig’s symmetries. Let $\mathfrak{g}$ be finite dimensional reductive and define

$$
\sigma^\pm(v) = (-1)^{\rho'(\lambda \pm \beta)} q^\pm \frac{1}{2} \langle (\beta, \beta) - \langle \lambda, \lambda \rangle \rangle - \langle \lambda, \rho \rangle T^\pm_{w_0}(v), \quad v \in V(\beta) \cap \mathcal{I}_\lambda(V).
$$

The main ingredient in our proof of Theorem 1.5 is the following result which generalizes [32, Proposition 5.5] to all reductive algebras including those whose semisimple part is not necessarily simply laced.

Theorem 4.10. Let $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$. Then

(a) $\sigma^+ = \sigma^-$ and is an involution which thus will be denoted by $\sigma$;

(b) $\sigma(x(v)) = \theta(x)(\sigma(v))$ for any $x \in U_q(\mathfrak{g})$, $v \in V$;

(c) $\sigma$ commutes with morphisms in $\mathcal{O}_q^{\text{int}}(\mathfrak{g})$.

Remark 4.11. For $\mathfrak{g} = \mathfrak{sl}_n$ the involution $\sigma$ coincides with the famous Schützenberger involution on Young tableau which was established for the first time in [8]. Thus, we can regard $\sigma$ as the generalized Schützenberger involution.

Proof. We need the following properties of $\sigma^\pm$.

Lemma 4.12. For any $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$ we have

(i) $\sigma^\pm(F_{w, \lambda}(v_\lambda)) = F_{w_0 w, \lambda}(v_\lambda)$ for any $v_\lambda \in \mathcal{I}_\lambda(V) \lambda$, $\lambda \in P^+$;

(ii) $\sigma^\pm(x(v)) = \theta(x)(\sigma^\pm(v))$ for any $x \in U_q(\mathfrak{g})$, $v \in V$;

(iii) $\sigma^+ = \sigma^-$ and $(\sigma^\pm)^2 = \text{id}_V$.

Proof. Let $v_\lambda \in \mathcal{I}_\lambda(V) \lambda$, $\lambda \in P^+$. Since $\rho + w_0 \rho = 0$, $\rho'(w_0 \mu) = -\rho'(\mu)$ for any $\mu \in P$ and $\ell(w_0 u) = \ell(w_0) - \ell(u)$ for any $u \in W$, Proposition 4.3(c) with $w = w_0$ and $w' = u$ yields

$$
T^\pm_{w_0}(F_{u, \lambda}(v_\lambda)) = (-1)^{\rho'(\pm u \lambda - \lambda)} q^\langle \lambda, \rho \rangle F_{w_0 u, \lambda}(v_\lambda),
$$

Since $F_{w, \lambda}(v_\lambda) \in V(u \lambda)$, (4.4) yields

$$
\sigma^\pm(F_{u, \lambda}(v_\lambda)) = (-1)^{\rho'(\lambda \mp u \lambda)} q^{-\langle \lambda, \rho \rangle} T^\pm_{w_0}(F_{u, \lambda}(v_\lambda)) = F_{w u, \lambda}(v_\lambda).
$$
This proves part (i). To prove part (ii), let \( v \in V(\beta) \). Then \( E_i(v) \in V(\beta + \alpha_i) \), and we obtain, by (4.4) and Lemmata 4.6, 4.7
\[
\sigma^\pm(E_i(v)) = (-1)^{\rho'(\lambda \mp (\beta + \alpha_i))} q^{\mp((\beta + \alpha_i, \beta + \alpha_i)-(\lambda, \lambda))-(\lambda, \rho)} T^\pm_{\omega_0}(E_i(v)) \\
= (-1)^{\rho'(\lambda \mp \beta)} q^{(\beta, \alpha_i)+\frac{1}{2}(\alpha_i, \alpha_i)-(\lambda, \lambda))-(\lambda, \rho)} T^\pm_{\omega_0}(E_i)(T^\pm_{\omega_0}(v)) \\
= (-1)^{\rho'(\lambda \mp \beta)} q^{(\beta, \alpha_i)+\frac{1}{2}(\beta, \beta)-(\lambda, \lambda))-(\lambda, \rho)} F_\epsilon K_i T^\pm_{\omega_0}(v) \\
= \theta(E_i)(q^{(\beta, \alpha_i+\omega_2 \alpha_i)} \sigma^\pm(v)) = \theta(E_i) \sigma^\pm(v).
\]
The identity \( \sigma^\pm(F_1(v)) = \theta(F_1)(\sigma^\pm(v)) \) is proved similarly. Finally, for any \( \mu \in \frac{1}{2}P \) we have
\[
\sigma^\pm(K_\mu(v)) = q^{(\beta, \mu)} \sigma^\pm(v) = q^{(\omega_2 \beta, \omega_2 \mu)} \sigma^\pm(v) = \theta(K_\mu)(\sigma^\pm(v)).
\]
Let \( \epsilon, \epsilon' \in \{+,-\} \). It follows from part (ii) that
\[
\sigma^\epsilon \circ \sigma^{\epsilon'}(x(v)) = \sigma^\epsilon(\theta(x(v))) = \theta^2(x(v)) = x(v)
\]
for any \( x \in U_q(\mathfrak{g}) \) and \( v \in V \) since \( \theta \) is an involution. Thus, \( \sigma^\epsilon \circ \sigma^{\epsilon'} \) is an endomorphism of \( V \) as a \( U_q(\mathfrak{g}) \)-module. By part (i) we have
\[
\sigma^\epsilon \circ \sigma^{\epsilon'}(F_{w,\lambda}(v\lambda)) = \sigma^\epsilon(F_{w_0 w,\lambda}(v\lambda)) = F_{w,\lambda}(v\lambda)
\]
for any \( w \in W, \lambda \in P^+ \) and \( v\lambda \in \mathcal{I}_\lambda(V)(\lambda) \). In particular, \( \sigma^\epsilon \circ \sigma^{\epsilon'}(v\lambda) = v\lambda \) and so \( \sigma^\epsilon \circ \sigma^{\epsilon'} \) is the identity map on the (simple) \( U_q(\mathfrak{g}) \)-submodule of \( V \) generated by \( v\lambda \). Since \( V \) is generated by \( \bigoplus_{\lambda \in P^+} \mathcal{I}_\lambda(V)(\lambda) \), \( \sigma^\epsilon \circ \sigma^{\epsilon'} = \text{id}_V \). This proves part (iii). \( \Box \)

Parts (a) and (b) of Theorem 4.10 were established in Lemma 4.12. To prove part (b), note the following obvious fact.

**Lemma 4.13.** Let \( \xi_* = \{\xi_V \in \text{End}_{U_q(\mathfrak{g})} V : V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})\} \) and suppose that \( \xi_* \) commute with morphisms in \( \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \), that is \( \xi_V \circ f = f \circ \xi_V \) for any morphism \( f : V \to V' \) in \( \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \). Let \( \chi : P^+ \times P \to \mathbb{K} \) and define \( \xi_*^\chi \) by by \( \xi_*^\chi(v) = \sum_{\beta \in \text{supp} \chi} \chi(\lambda, \beta) \xi_V(v(\beta)) \), \( v \in \mathcal{I}_\lambda(V), V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \). Then \( \xi_*^\chi \) also commutes with morphisms in \( \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \).

It is immediate from the definition (4.4) of \( \sigma \) that \( \sigma_* = (T_*^\pm)^\chi_{\pm} \) where \( \chi_{\pm}(\lambda, \beta) = (-1)^{\rho'(\lambda \mp \beta)} q^{\mp((\beta, \beta)-(\lambda, \lambda))-(\lambda, \rho)} \), \( \lambda, \beta \in P^+ \). Then part (c) follows from Lemma 4.13. \( \Box \)

### 4.3. Parabolic involutions and proof of Theorem 1.5

In view of Theorem 4.10, given \( V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \) and \( J \in \mathcal{J} \), let \( \sigma^J : V \to V \) be \( \sigma \) defined by (4.4) with \( U_q(\mathfrak{g}) \) replaced by \( U_q(\mathfrak{g}^J) \). Thus,
\[
\sigma^J(v) = (-1)^{\rho'(\lambda J \mp \beta)} q^{\mp((\lambda J, \lambda J)-(\beta, \beta))-(\lambda J, \rho_J)} T^\pm_{w_J}(v), \tag{4.5}
\]
for any \( \lambda J \in P^+_J, \beta \in P \) and \( v \in \mathcal{I}_\lambda^J(V)(\beta) \). Note that \( \mathcal{I}_\lambda^J(V)(\beta) = 0 \) unless \( \lambda J - \beta \in \sum_{j \in J} \mathbb{Z}_{\geq 0} \alpha_j \). We need the following properties of \( \sigma^J \).
Proposition 4.14. Let $V \in \mathcal{O}^\text{int}_q(\mathfrak{g})$. Then

(a) $\sigma^J(F_{w,\lambda}(v)) = F_{w^J,w,\lambda}(v)$ for any $v \in \mathcal{I}^J_{\lambda}(V)(\lambda_J)$, $\lambda_J \in P^+$. In particular, $\sigma^J(v) = v^J := F_{w^J,w,\lambda}(v)$.

(b) $\sigma^J$ is an involution;

(c) $\sigma^J$ commutes with morphisms in $\mathcal{O}^\text{int}_q(\mathfrak{g}')$ and satisfies $\sigma^J(x(v)) = \theta_J(x)(\sigma^J(v))$, $x \in U_q(\mathfrak{g}')$, $v \in V$;

(d) $\sigma^J(V(\beta)) = V(w_0^J\beta)$, $\beta \in P$;

(e) for any $\lambda_J \in P^+_J$, $\beta \in P$ and $v \in \mathcal{I}^J_{\lambda}(V)(\beta)$ we have

$$\sigma^J(v) = (-1)^{\rho^J_\beta(\lambda_J,\lambda)} q^{\frac{1}{2}((\lambda_J,\lambda_J) - (\lambda_J,\beta)) + (\lambda_J,\rho_J)(T^\pm_{w_0})^{-1}(v)}.$$

Proof. Replacing $\mathfrak{g}$ by $\mathfrak{g}'$ we obtain part (a) from Lemma 4.12(i) and parts (b), (c) from Theorem 4.10. Part (d) is immediate from (4.5) and Lemma 4.9. To prove part (e), let $v' = \sigma^J(v)$. Then $v = \sigma^J(v')$ and $v' \in \mathcal{I}^J_{\lambda}(V)(\beta')$ where $\beta' = w_0^J\beta$. Applying $(T^\pm_{w_0})^{-1}$ to (4.5) with $v$ replaced by $v'$ we obtain

$$(T^\pm_{w_0})^{-1}(\sigma^J(v')) = (-1)^{\rho^J_\beta(\lambda_J,\lambda)} q^{-\frac{1}{2}((\lambda_J,\lambda_J) - (\lambda_J,\beta')) - (\lambda_J,\rho_J)v'}.$$

Since $\rho^J_\beta(\beta') = \rho^J_\beta(w_0^J\beta) = -\rho^J_\beta(\beta)$ and $(\cdot, \cdot)$ is $W$-invariant, it follows that

$$(T^\pm_{w_0})^{-1}(v) = (-1)^{\rho^J_\beta(\lambda_J,\lambda)} q^{-\frac{1}{2}((\lambda_J,\lambda_J) - (\lambda_J,\beta)) - (\lambda_J,\rho_J)\sigma^J(v)}.$$

The assertion is now immediate. \qed

We need the following results.

Proposition 4.15. For any $J \subset J' \in \mathcal{J}$, $\sigma^{J'} \circ \sigma^J = \sigma^{J'} \circ \sigma^J$ where $*: J' \to J'$ is the unique involution satisfying $\alpha_{j*} = -w_0^{J'}\alpha_j$, $j \in J'$.

Proof. We may assume, without loss of generality, that $J' = J$ (and so $\mathfrak{g}$ is reductive finite dimensional). Let $w_J = w_J w_J^J$. Note that $w_0 = w_J^J w_0^J = w_0^J w_J$ and $\ell(w_0) = \ell(w_J) + \ell(w_J^J) = \ell(w_J) + \ell(w_J^J)$. Then by Lemma 4.2

$$T^\pm_{w_J} = T^\pm_{w_0} \circ (T^\pm_{w_0^J})^{-1} = (T^\pm_{w_0^J})^{-1} \circ T^\pm_{w_0}. \quad (4.6)$$

Let $v \in V(\beta) \cap \mathcal{I}_\lambda(V) \cap \mathcal{I}^J_{\lambda}(V)$, $\lambda \in P^+$, $\lambda_J \in P^+_J$, $\beta \in P$. Using Lemma 4.14(e), (4.4), Lemma 4.9 and (4.6) we obtain

$$\sigma \circ \sigma^J(v) = (-1)^{\rho^J_\beta(\lambda_J,\lambda)} q^{\frac{1}{2}((\lambda_J,\lambda_J) - (\lambda_J,\beta)) + (\lambda_J,\rho_J)}\sigma((T^\pm_{w_0})^{-1}(v))$$

$$= (-1)^{\rho^J_\beta(\lambda_J,\lambda)} q^{\frac{1}{2}((\lambda_J,\lambda_J) - (\lambda_J,\beta)) + (\lambda_J,\rho_J)T^\pm_{w_0}}(T^\pm_{w_0^J})^{-1}(v)$$

$$= (-1)^{\rho^J_\beta(\lambda_J,\lambda)} q^{\frac{1}{2}((\lambda_J,\lambda_J) - (\lambda_J,\beta)) + (\lambda_J,\rho_J)T^\pm_{w_J}(v)}.$$

Similarly,
\[\sigma^J \circ \sigma(v) = (-1)^{\rho^J(\lambda \pm \beta)} q^{-\frac{1}{2}((\lambda,\lambda) - \langle \beta, \beta \rangle)} T_{w_0}^\pm(v) \]

\[= (-1)^{\rho^J(\lambda \pm \beta) + \rho^J_\omega(w_0 \lambda \pm w_0 \beta)} q^{\frac{1}{2}((\lambda,\lambda) - \langle \beta, \beta \rangle) - (\lambda,\rho)} T_{w_0}^\pm(v) \]

\[= (-1)^{\rho^J(\lambda \pm \beta) + \rho^J_\omega(-\lambda \pm 2\beta)} q^{\frac{1}{2}((\lambda,\lambda) - \langle \beta, \beta \rangle) + (\lambda,\rho) - (\lambda,\rho)} T_{w_0}^\pm(v) \]

since \(\rho^J_\omega(w_0 \mu) = -\rho^J_\omega(\mu), \mu \in P\) and \(w_0 \rho_{J'} = -\rho_J\). Since \(-\rho^J_\omega(\beta) = \rho^J_\omega(w_0 \beta)\) and \(\rho^J(\beta - w\beta) = \rho^J_\omega(\beta - w\beta)\) for any \(w \in W_J\), it follows that \(\sigma^J \circ \sigma = \sigma \circ \sigma^J\).

**Proposition 4.16.** Let \(J, J' \in \mathcal{J}\) be orthogonal. Then \(\sigma^{J \cup J'} = \sigma^J \circ \sigma^{J'}\).

**Proof.** As before we may assume, without loss of generality, that \(I = J \cup J'\). Then \(w_0 = w_0^J w_0^{J'} = w_0^{J'} w_0^J\) and hence \(T_{w_0}^\pm = T_{w_0^J}^\pm \circ T_{w_0^{J'}}^\pm = T_{w_0^{J'}}^\pm \circ T_{w_0^J}^\pm\) by Lemma 4.2. Let \(\lambda \in P^+, \lambda_J, \lambda_{J'} \in P^+_J, \beta \in P, v \in \mathcal{I}_J(V) \cap \mathcal{I}_{J'}(V) \cap \mathcal{I}_{J'}(V').\) Then \(\gamma_J, \lambda_J, - \beta \in \sum_{j \in J} \mathbb{Z}_{\geq 0} \alpha_j, \gamma_J = \lambda_{J'} - \beta \in \sum_{j \in J'} \mathbb{Z}_{\geq 0} \alpha_{j'}, \gamma = \gamma_J + \gamma_{J'}\). Then we can rewrite (4.5) and (4.4) as

\[\sigma^J(v) = (-1)^{\rho^J_\omega(\gamma_J)} q^{-\frac{1}{2}(\gamma_J,\gamma_J) + (\lambda_J,\gamma_J) - (\lambda_J,\rho)} T_{w_0^J}^\pm(v) \]

\[\sigma^{J'}(v) = (-1)^{\rho^J_\omega(\gamma_{J'})} q^{-\frac{1}{2}(\gamma_{J'},\gamma_{J'}) + (\lambda_{J'},\gamma_{J'}) - (\lambda_{J'},\rho)} T_{w_0^{J'}}^\pm(v) \]

\[\sigma(v) = (-1)^{\rho^J_\omega(\gamma)} q^{-\frac{1}{2}(\gamma,\gamma) + (\lambda,\gamma) - (\lambda,\rho)} T_{w_0}^\pm(v) \]

Since \(w_0^J(\gamma_J) = \gamma_J\), we have

\[\sigma^J(\sigma^{J'}(v)) = (-1)^{\rho^J_\omega(\gamma_J) + \rho^J_\omega(\gamma_{J'})} q^{-\frac{1}{2}(\gamma_J,\gamma_J) + (\lambda_J,\gamma_J) - (\lambda_J,\rho) + (\lambda_{J'},\gamma_{J'}) - (\lambda_{J'},\rho)} T_{w_0}^\pm(v) \]

\[= (-1)^{\rho^J_\omega(\gamma_J) + \rho^J_\omega(\gamma_{J'})} q^{-\frac{1}{2}(\gamma,\gamma) + (\lambda,\gamma) + (\lambda,\rho) + (\lambda,\rho)} T_{w_0}^\pm(v) = \sigma(v) \]

since \(\rho^J_\omega(\gamma) = \rho^J_\omega(\gamma_J) + \rho^J_\omega(\gamma_{J'})\), \(\gamma, \gamma_J = \gamma_J, \gamma_J + (\gamma_{J'}, \gamma_{J'})\), \(\lambda_J, \gamma_J, \gamma_{J'}, \zeta + \zeta' = (\lambda_J, \gamma_J, \gamma_{J'}, \zeta, \zeta')\) for any \(\zeta \in \sum_{j \in J} \mathbb{Q}\alpha_j, \zeta' \in \sum_{j \in J'} \mathbb{Q}\alpha_{j'}\). \(\square\)

**Proof of Theorem 1.5.** Parts (a) (respectively, (b), (c)) of Theorem 1.5 were established in Lemma 4.14(b) (respectively, Proposition 4.16, Proposition 4.15).

**4.4. Kernels of actions of cactus groups.** For any \(V \in \mathcal{O}_q^{\text{int}}(g)\), denote \(\Phi_V\) be the subgroup of \(\text{GL}_k(V)\) generated by the \(\sigma^J_V, J \in \mathcal{J}\). We need the following basic properties of \(\Phi_V\).

**Lemma 4.17.** For any injective morphism \(f : V' \rightarrow V \in \mathcal{O}_q^{\text{int}}(g)\) the assignments \(\sigma^J_V \mapsto \sigma^J_V, J \in \mathcal{J}\) define a surjective homomorphism \(f^* : \Phi_V \rightarrow \Phi_{V'}\). In particular, if \(f\) is an isomorphism then so is \(f^*\).

**Proof.** Let \(V'' = f(V')\). By Theorem 4.10(c), we have \(\sigma^J_V \circ f = f \circ \sigma^J_V\) for all \(J \in \mathcal{J}\) and so the group \(\Phi_V\) acts on \(V''\), that is, there is a canonical homomorphism of groups \(\rho : \Phi_V \rightarrow \text{GL}_k(V'')\). Clearly, the assignments \(g \mapsto f \circ g \circ f^{-1}, g \in \text{GL}_k(V'')\) define an
isomorphism $\rho_f : \text{GL}_k(V'') \to \text{GL}_k(V')$. Let $f^* = \rho_f \circ \rho : \Phi_V \to \text{GL}_k(V')$. We claim that $f^*(\Phi_V) = \Phi_{V'}$. Indeed, $f^*(\sigma_J^\ell) = \sigma_J^\ell$, for all $J \in \mathcal{J}$. Since $\Phi_{V'}$ is generated by the $\sigma_J$, $\Phi_V$ is generated by the $\sigma_J^\ell$, $J \in \mathcal{J}$, and $f^*$ is a homomorphism of groups, the assertion follows. \hfill $\square$

**Proposition 4.18.** Let $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$. Then $\Phi_V \cong \Phi_{V'}$ where $V = \bigoplus_{\lambda \in P^+ : \text{Hom}_{U_q(\mathfrak{g})}(V, V) \neq 0} V^\lambda$. In particular, for any $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$ the group $\Phi_V$ is a quotient of $\Phi_{C_q(\mathfrak{g})}$ where $C_q(\mathfrak{g}) = \bigoplus_{\lambda \in P^+} V^\lambda$.

**Proof.** Fix $f_\lambda \in \text{Hom}_{U_q(\mathfrak{g})}(V, V) \setminus \{0\}$ for all $\lambda \in P^+$ with $\text{Hom}_{U_q(\mathfrak{g})}(V, V) \neq 0$ and let $f : V \to V$ be the direct sum of these $f_\lambda$. Then $f$ is injective. Applying Lemma 4.17 with $V' = V$ we obtain a surjective group homomorphism $f^* : \Phi_V \to \Phi_{V'}$. It remains to prove that its kernel is trivial. We apply Lemma 3.5 with $R = \mathbb{k}[\Phi_V]$ and $S = \{g - 1 : g \in \Phi_V\} \subset R$. Since $\Phi_V$ is a subgroup of $\text{GL}_k(V)$, $\text{Ann}_S V = \{0\}$. By our choice of $V$, $M = V$ and $M' = f(V)$ satisfy the assumptions of Lemma 3.5 and so $\text{Ann}_S f(V) = \{0\}$. Since $\ker f^* = \{g \in \Phi_V : g \circ f = \text{id}_V\}$, it follows that $\ker f^*$ is trivial. The second assertion is immediate from the first one and Lemma 4.17. \hfill $\square$

5. An action of $\text{Cact}_W$ on $\mathfrak{c}$-crystal bases and proof of Theorem 1.8

Retain the notation of §2.5 and observe that the assignment $(l, k, s) \mapsto (l, l - k, -s)$, $(l, k, s) \in \mathbb{D}$, defines an involution on $\mathbb{D}$. The following is the main result of this section.

**Theorem 5.1.** Let $\mathfrak{g}$ be reductive. Let $\mathfrak{c} : \mathbb{D} \to \mathbb{Q}(z)^\times$ satisfying

$$c_{l, k, s} = c_{l, l - k, -s}, \quad c_{l, 0, -t} = 1, \quad (l, k, s) \in \mathbb{D}$$

in the notation of Lemma 2.12. Then for any $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$

(a) $\sigma^I(L) = L$ for any $(\mathfrak{c}, I)$-monomial lattice $L$ in $V$;

(b) If $(L, B)$ is a $\mathfrak{c}$-crystal basis such that $B_+ = \{b \in B : \tilde{e}^\mathfrak{c}_{i, 1}(b) = 0, i \in I\}$ is a basis of $L_+ / qL_+$ where $L_+ = L \cap V_+$, then the induced $\mathbb{Q}$-linear map $\tilde{\sigma}^I : L / qL \to L / qL$ preserves $B$.

**Proof.** We abbreviate $\sigma = \sigma^I$, $V_+^I = V_+$ and $\mathcal{M}^\mathfrak{c}(v_+) = \mathcal{M}^\mathfrak{c}_I(v_+)$ for any homogeneous $v_+ \in V_+$. The key ingredient of our argument is the following

**Proposition 5.2.** Let $\mathfrak{g}$ be reductive, let $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$ and let $\mathfrak{c} : \mathbb{D} \to \mathbb{Q}(z)^\times$. Then

(a) If $\mathfrak{c}$ satisfies the first condition in (5.1) then $\sigma \circ \tilde{e}^\mathfrak{c} = \tilde{e}^\mathfrak{c} \circ \mathfrak{c}$ in $\text{End}_k V$ for any $i \in I$, $s \in \mathbb{Z}$.

(b) If $\mathfrak{c}$ satisfies (5.1) then $\sigma(\mathcal{M}^\mathfrak{c}(v_+)) = \mathcal{M}^\mathfrak{c}(v_+)$ for any homogeneous $v_+ \in V_+$.
Proof. In view of Lemma 2.11 and (2.6), to prove (a) it suffices to verify the identity for all \( v \in V \) of the form \( v = F_i^{(k)}(u) \), \( u \in \ker E_i \cap \ker (K_{\alpha_i} - q_i^l) \), \( 0 \leq k \leq l \). We have
\[
\sigma \circ \tilde{e}_{i,s}^c(v) = \mathfrak{c}_{j,k,s}(q_i)\sigma(F_i^{(k-s)}(u)) = \mathfrak{c}_{j,k,s}(q_i)E_i^{(k-s)}(\sigma(u)).
\]
(5.2)

We need the following

**Lemma 5.3.** \( \sigma(u) \in \ker F_i^{*} \cap \ker (K_{\alpha_i}^* - q_i^l) \).

**Proof.** Indeed, \( F_i^{*}(\sigma(u)) = \sigma(E_i(u)) = 0 \) and \( K_{\alpha_i}^*(\sigma(u)) = \sigma(K_{-\alpha_i}(u)) = q_i^l\sigma(u) = q_i^{-l}\sigma(u). \)

Using Lemmata 5.3 and 2.12, we obtain
\[
\tilde{e}_{i,-s}^c(\sigma(v)) = \tilde{e}_{i,-s}^c(E_i^{(k)}(\sigma(u))) = \mathfrak{c}_{l-k,s}(q_i^*)E_i^{(k-s)}(\sigma(u)).
\]
(5.3)

Since \( q_i^* = q_i \), by assumptions of Proposition 5.2 we have \( \mathfrak{c}_{l-k,s}(q_i^*) = \mathfrak{c}_{l,k,s}(q_i) \). Then (5.2) and (5.3) imply that \( \tilde{e}_{i,-s}^c(\sigma(v)) = \sigma(\tilde{e}_{i,s}^c(v)) \).

To prove part (b), we need the following

**Lemma 5.4.** Suppose that \( c_{i,0,-1} = 1/(l)! \) for all \( l \in \mathbb{Z}_{\geq 0} \) (that is, \( c_{(0,-1)} = 1 \) in the notation of Lemma 2.12). Then for any \( v_+ \in V_+(\lambda) \) and \( i = (i_1, \ldots, i_m) \in I^m \) reduced, \( \lambda \in P^+ \) we have in the notation of (3.1), \( F_{i,\lambda}(v_+) = \tilde{e}_{i_1,-a_1(1,\lambda)} \cdots \tilde{e}_{i_m,-a_m(1,\lambda)}(v_+) \). In particular,
\[
\sigma(v_+) = \tilde{e}_{i_1,-a_1(1,\lambda)} \cdots \tilde{e}_{i_N,-a_N(1,\lambda)}(v_+)
\]
where \( i = (i_1, \ldots, i_N) \in \mathcal{R}(w_0) \).

**Proof.** We use induction on \( m \), the case \( m = 0 \) being trivial. For \( i \) and \( \lambda \in P^+ \) fixed we abbreviate \( a_k = a_k(i, \lambda) \). For the inductive step, note that \( F_{i,\lambda} = F_{i_1,a_1(1)}F_{i_2,a_2} \) where \( i' = (i_2, \ldots, i_m) \) and so \( F_{i,\lambda}(v_+) = F_{i_1,a_1(1)}(v') \) where \( v' = F_{i_2,a_2} \cdots F_{i_m,a_m}(v_+) \) by the induction hypothesis. Since \( v' \in \ker E_i \), it follows from assumptions of the lemma and the first identity in (2.6) with \( i = i_1 \), \( k = 0 \) and \( l = a_1 = -s \) that \( F_{i_1,a_1(1)}(v') = \tilde{e}_{i_1,-a_1(1)}(v') = \tilde{e}_{i_1,-a_1} \cdots \tilde{e}_{i_m,-a_m}(v_+) \). Since \( \sigma(v_+) = F_{w_0,\lambda}(v_+) \) by Lemma 4.12(i) with \( w = 1 \), the second assertion follows from the first and Lemma 3.1(a).

Suppose now that \( v \in M^c(v_+) \) that is \( v = e_{j_{1,1},m_1} \cdots e_{j_{r,m_r}}(v_+) \in M^c(v_+) \), for some \( (j_1, \ldots, j_r) \in I^r \) and \( (m_1, \ldots, m_r) \in \mathbb{Z} \). Using Lemma 5.4 and Proposition 5.2(a), we obtain
\[
\sigma(v) = e_{j_{1,1},m_1} \cdots e_{j_{r,m_r}}(\sigma(v_+)) = e_{j_{1,1},m_1} \cdots e_{j_{r,m_r}} e_{i_1,-a_1} \cdots e_{i_N,-a_N}(v_+) \in M^c(v_+)
\]
where \( i = (i_1, \ldots, i_N) \in R(w_0) \) and \( a_k = a_k(i, \lambda) \), \( 1 \leq k \leq N \). Thus, \( \sigma(M^c(v_+)) \subseteq M^c(v_+) \). Since \( \sigma \) is an involution, it follows that \( \sigma(M^c(v_+)) = M^c(v_+) \). \( \square \)
Part (a) is immediate from Proposition 5.2(b). In particular, for each \((c,I)\)-monomial lattice \(L\) in \(V\) the involution \(\sigma_V\) induces an involution \(\tilde{\sigma}\) on the \(\mathbb{Q}\)-vector space \(\tilde{L} = L/qL\) satisfying
\[
\tilde{\sigma} \circ \tilde{e}_{i,s} = \tilde{e}_{i,-s} \circ \tilde{\sigma}.
\] (5.4)
The following is immediate from Proposition 5.2(b).

**Corollary 5.5.** Let \(L\) be a \((c,I)\)-monomial lattice in \(V\). Then \(\tilde{\sigma}(\tilde{M}_c(\tilde{v}_+)) = \tilde{M}_c(\tilde{v}_+)\) for any \(\tilde{v}_+ \in L_+/qL_+\).

Using the assumptions of part (b) of Theorem 5.1 we conclude that \(\bigcup_{b_+ \in B^+} \tilde{M}_c(b_+) = B \cup \{0\}\). Then it follows from Corollary 5.5 that \(\tilde{\sigma}\) preserves \(B \cup \{0\}\). Since \(\tilde{\sigma}\) is an involution, it follows that \(\tilde{\sigma}(B) = B\). This completes the proof of Theorem 5.1(b). \(\square\)

Note that (5.4) implies that for any upper crystal basis \((L,B)\) of \(V \in \mathcal{O}^\text{int}_q(g)\) the operator \(\tilde{\sigma}_V^I\) satisfies
\[
\tilde{\sigma} \circ (\tilde{e}_i^{up})^s = (\tilde{e}_i^{up})^{-s} \circ \tilde{\sigma}_V^I.
\] (5.5)
In particular, we obtain the following

**Corollary 5.6.** Let \(\lambda \in P^+\) and \((L_\lambda,B_\lambda)\) be the upper crystal basis of \(V_\lambda\). If \(f\) is any non-zero map \(B_\lambda \cup \{0\} \to B_\lambda \cup \{0\}\) satisfying (5.5) then \(f = \tilde{\sigma}_\lambda^I|_{B_\lambda \cup \{0\}}\).

**Proof of Theorem 1.8.** Note that
\[
\underline{c}_{l,k,s}^{\text{low}} = 1, \quad \underline{c}_{l,k,s}^{\text{up}} = \frac{(l - k + s)_! (k - s)_!}{(l)_! (k)_!}, \quad (l,k,s) \in \mathbb{D}.
\] (5.6)
It is now immediate that (5.1) holds for \(c \in \{c^{up}, c^{low}\}\).

Furthermore, by Lemma 2.15 and Remark 2.16, Theorem 5.1 applies to every \(c\)-crystal basis at \(q = 0\) for any \(J \in \mathcal{J}\) with \(g\) replaced by \(g^J\) and \(c \in \{c^{up}, c^{low}\}\). Thus, \(\sigma^J\) preserves a lower or upper crystal lattice \(\tilde{L}\) and \(\tilde{\sigma}^J\) preserves \(B\).

In particular, \(\Phi_V\) acts on \(L\) and this action factors through to an action on \(L/qL\) and induces an action on \(B\) by permutations. \(\square\)

**Remark 5.7.** Let \(L\) be an upper crystal lattice for \(V \in \mathcal{O}^\text{int}_q(g)\). It follows from the definition of \(\sigma^I_V\) that for any \(v \in L(\beta), \beta \in P\) we have \(\sigma^I_V(v) = \tilde{e}_{i}^{-\beta(\alpha_i^\vee)}(v)\). In particular, the action of \(\tilde{\sigma}^I_V\) on an upper crystal basis \((L,B)\) of \(V\) coincides with Kashiwara’s crystal Weyl group action (see [27]).

We conclude this section with a discussion of the action of \(\text{Cact}_W\) on upper global crystal bases. Let \(\tilde{\gamma}\) be any field involution of \(k\) such that \(\tilde{\gamma}(q^m) = q^{-\frac{m}{2}}\).

**Proposition 5.8.** Let \((L,B)\) be an upper crystal basis of \(V \in \mathcal{O}^\text{int}_q(g)\) and let \(G^{up}(B)\) be the corresponding upper global crystal basis. Denote by \(\tilde{\gamma}\) the (unique) additive map
$V \to V$ satisfying $\overline{f \cdot b} = \overline{f} \cdot \overline{b}$ for all $f \in \mathbb{k}$, $b \in \mathcal{G}_{\text{up}}(B)$. Then $\overline{\sigma^J(b)} = \sigma^J(b)$ for any $J \in \mathcal{J}$.

Proof. Denote by $\bar{\cdot}$ the ring automorphism of $U_q(g)$ satisfying $\overline{E_i} = E_i$, $\overline{F_i} = F_i$, $i \in I$, $\overline{K_\lambda} = K_{-\lambda}$, $\lambda \in \frac{1}{2}P$ and $\overline{fu} = \overline{f} \cdot \overline{u}$ for all $f \in \mathbb{k}$, $u \in U_q(g)$. The following is immediate from the properties of the upper global crystal basis ([26]).

Lemma 5.9. The map $\bar{\cdot} : V \to V$ defined in Proposition 5.8 satisfies
\[ \overline{u(v)} = \overline{u(\overline{v})}, \quad v \in V, \ u \in U_q(g). \] (5.7)

The following is immediate.

Lemma 5.10. Let $\eta : U_q(g) \to U_q(g)$ be any algebra automorphism commuting with $\bar{\cdot} : U_q(g) \to U_q(g)$ and let $V \in \mathcal{G}_{\text{int}}^\text{f}(g)$ with a fixed set $V_0 \subset V$ generating $V$ as a $U_q(g)$-module. Let $\bar{\cdot} : V \to V$ be any map satisfying (5.7) and let $\sigma \in \text{End}_\mathbb{k} V$ be such that:

(i) $\sigma(u(v)) = \eta(u)(\sigma(v))$, $u \in U_q(g)$, $v \in V$;

(ii) $\sigma(\overline{v}) = \overline{\sigma(v)}$ for any $v \in V_0$.

Then $\sigma(\overline{v}) = \overline{\sigma(v)}$ for all $v \in V$.

The set $V_0 = \mathcal{G}_{\text{up}}(B) \cap V_\perp$ generates $V$ as a $U_q(g')$-module. Clearly, $\theta_J$ commutes with $\bar{\cdot}$-involution on $U_q(g')$. The condition (i) of Lemma 5.10 holds with $\eta = \theta_J$ by Proposition 4.14(d). By Proposition 4.14(a), $\sigma^J(b) = F_{w_{\perp,\lambda}}(b)$ for any $b \in V_0(\lambda)$, $\lambda \in P_\perp$. Since $\overline{F_{w_{\perp,\lambda}}} = F_{w_{\perp,\lambda}}$ and $\overline{b} = b$, the condition (ii) of Lemma 5.10 is also satisfied. The assertion follows by Lemma 5.10.

6. $\sigma^J$ and the Canonical Basis

6.1. Automorphisms and skew derivations of localizations. Let $R$ be a unital $\mathbb{k}$-algebra. Given a monoid $\Gamma$ written multiplicatively and acting on $R$ by algebra automorphisms, define the semidirect product of $R$ with the monoidal algebra $\mathbb{k}[\Gamma]$ of $\Gamma$ as $R \rtimes \mathbb{k}[\Gamma]$ with the multiplication defined by
\[ (r \otimes \gamma) \cdot (r' \otimes \gamma') = r(\gamma \triangleright r') \otimes \gamma \gamma', \quad r, r' \in R, \ \gamma, \gamma' \in \Gamma, \]
where $\triangleright$ denotes the action of $\Gamma$ on $R$. Since $(r \otimes 1)(1 \otimes \gamma) = r \otimes \gamma$, we will henceforth omit the symbol $\otimes$ when writing elements of $R \rtimes \mathbb{k}[\Gamma]$. In other words, $R \rtimes \mathbb{k}[\Gamma]$ is generated by $R$ as a subalgebra and $\Gamma$ subject to the relations
\[ \gamma \cdot r = (\gamma \triangleright r) \cdot \gamma, \quad r \in R, \ \gamma \in \Gamma. \]

The following characterization of cross products is immediate.

Lemma 6.1. Let $f : R \to \widehat{R}$ be a homomorphism of $\mathbb{k}$-algebras and let $g : \Gamma \to \widehat{R}$ be a homomorphism of multiplicative monoids. Suppose that $R$ is a $\mathbb{k}[\Gamma]$-module algebra. Then
assignments $r \cdot \gamma \mapsto f(r) \cdot g(\gamma)$, $r \in R$, $\gamma \in \Gamma$ define a homomorphism of $\kappa$-algebras if and only if
\[ f(\gamma \triangleright r)g(\gamma) = g(\gamma)f(r), \quad r \in R, \gamma \in \Gamma. \tag{6.1} \]

Let $S$ be a submonoid of $R \setminus \{0\}$. Denote $S^{\text{op}}$ the opposite monoid of $S$ and denote its elements by $[s]$, $s \in S$. Suppose that $R$ is a $\kappa[S^{\text{op}}]$-module algebra with $[s] \triangleright r = \Sigma_s(r)$, $s \in S$ where $\Sigma_s$ is an algebra automorphism of $R$ and assume that
\[ \Sigma_s(s) = s, \quad s \in S. \tag{6.2} \]
Denote $R[S^{-1}] := (R \rtimes \kappa[S^{\text{op}}])/(s[s] - 1 : s \in S)$. We say that $S$ as above is an Ore submonoid if
\[ rs = s\Sigma_s(r), \quad r \in R, s \in S. \tag{6.3} \]
We use the convention that $\Sigma_\lambda s = \Sigma_s$ for all $\lambda \in \kappa^\times$. This notation is justified by the following

**Lemma 6.2.** Suppose that (6.2) holds. Then the following are equivalent:

(i) the natural homomorphism $1_{R,S} : R \to R[S^{-1}]$ is injective;

(ii) $S$ is an Ore submonoid of $R$ and the assignments $r \cdot [s] \mapsto rs^{-1}$, $r \in R$, $s \in S$ define an isomorphism $R[S^{-1}] \to R[S^{-1}]$ where $R[S^{-1}]$ is the Ore localization of $R$ by $S$.

**Proof.** In $R \rtimes \kappa[S^{\text{op}}]$ we have
\[ [s] \cdot r = \Sigma_s(r) \cdot [s], \quad s \in S, r \in R. \tag{6.4} \]
In particular, $[s] \cdot s = s \cdot [s]$, $s \in S$. Multiplying both sides of (6.4) by $s$ on the left and on the right we conclude that $rs = s\Sigma_s(r)$ in $R[S^{-1}]$ for all $r \in R$, $s \in S$. This identity clearly holds in $1_{R,S}(R)$. Since $1_{R,S}$ is injective, this implies that the corresponding identity holds in $R$ and so $S$ satisfies the two-sided Ore condition and so $R$ admits the Ore localization $R[S^{-1}]$. The assignments $r[s] \mapsto rs^{-1}$, $r \in R$, $s \in S$ define a surjective homomorphism from $R[S^{-1}] \to R[S^{-1}]$ which is easily seen to be injective. Thus, (i) implies (ii).

Conversely, the natural homomorphism $R \to R[S^{-1}]$, $r \mapsto r \cdot 1$, $r \in R$ is injective. Since it equals the composition of $1_{R,S}$ and the isomorphism $R[S^{-1}] \to R[S^{-1}]$, it follows that $1_{R,S}$ is injective. \qed

The following Lemma is immediate.

**Lemma 6.3.** Let $R$ be a $\kappa$-algebra and $S \subset R \setminus \{0\}$ be an Ore submonoid. Let $R'$ be a $\kappa$-subalgebra of $R$ and suppose that $S' \subset R' \cap S$ is an Ore submonoid of $R'$. Then $R'[S'^{-1}]$ is isomorphic to the subalgebra of $R[S^{-1}]$ generated by $R'$ and $\{s'^{-1} : s' \in S'\}$.

**Lemma 6.4.** Suppose that (6.2) and the assumptions of Lemma 6.2(ii) hold. Let $\varphi : R \to R'$ be any $\kappa$-algebra homomorphism, $S$ an Ore submonoid of $R$ and $S'$ an Ore submonoid of $R'$ such that $\varphi(S) \subset S'$. Suppose that $\Sigma_{\varphi(s)} \circ \varphi = \varphi \circ \Sigma_s$ for all $s \in S$. Then there exists a unique homomorphism $\hat{\varphi} : R[S^{-1}] \to R'[S'^{-1}]$ such that $\hat{\varphi}|_R = \varphi$. 

Proof. We apply Lemma 6.1 with $\Gamma = S^{op}$, $\hat{R} = R' \rtimes \mathbb{k}[S^{op}]$ and $g : S^{op} \to \hat{R}$ defined by $g([s]) = [\varphi(s)]$. Then

$$[\varphi(s)]\varphi(r) = \Sigma_{\varphi(s)}(\varphi(r))[\varphi(s)] = \varphi(\Sigma_{s}(r))[\varphi(s)] = \varphi([s \triangleright r])[\varphi(s)],$$

and so (6.1) holds. Therefore, the assignments $r[s] \mapsto \varphi(r)[\varphi(s)]$, $r \in R$, $s \in S$, define a homomorphism $\hat{\varphi} : R \rtimes \mathbb{k}[S^{op}] \to R' \rtimes \mathbb{k}[S^{op}]$. Since $\hat{\varphi}(s[s]) = \varphi(s)[\varphi(s)]$ it follows that the image of the defining ideal of $R[S^{-1}]$ under $\hat{\varphi}$ is contained in the defining ideal of $R'[S'^{-1}]$. Thus, $\hat{\varphi}$ factors through to the desired homomorphism $\hat{\varphi} : R[S^{-1}] \to R'[S'^{-1}]$. □

Let $L_{\pm} : R \to R'$ be $\mathbb{k}$-algebra homomorphisms and $E : R \to R'$ be a $\mathbb{k}$-linear map. We say that $E$ is an $(L_{+},L_{-})$-derivation from $R$ to $R'$ if $E(rr') = E(r)L_{+}(r') + L_{-}(r)E(r')$ for all $r, r' \in R$. Denote $\text{Der}_{L_{+},L_{-}}(R, R')$ the $\mathbb{k}$-subspace of $\text{Hom}_{\mathbb{k}}(R, R')$ of $(L_{+},L_{-})$-derivations from $R$ to $R'$. We refer to an $(L, L^{-1})$-derivation as an $L$-derivation and abbreviate $\text{Der}_{L_{+},L_{-}} R = \text{Der}_{L_{+},L_{-}}(R, R)$. The following is immediate.

**Lemma 6.5.** Let $R_0$ be a generating subset of $R$ and let $D, D' \in \text{Der}_{L_{+},L_{-}}(R, R')$. Then $D|_{R_0} = D'|_{R_0}$ implies that $D = D'$.

Given $r' \in R'$, denote by $D_{r'}^{\pm}$ the linear maps $R \to R'$

$$D_{r'}^{+}(x) = r'L_{+}(x) - L_{-}(x)r', \quad D_{r'}^{-}(x) = L_{-}(x)r' - r'L_{+}(x), \quad x \in R. \quad (6.5)$$

**Lemma 6.6.** Let $L_{\pm} : R \to R'$ be $\mathbb{k}$-algebra homomorphisms. The assignments $r' \mapsto D_{r'}^{+}$ (respectively, $r' \mapsto D_{r'}^{-}$), $r' \in R'$ define $\mathbb{k}$-linear maps $R' \to \text{Der}_{L_{+},L_{-}}(R, R')$.

Proof. For any $x, x' \in R$ we have

$$D_{r'}^{-}(xx') = r'L_{+}(xx') - L_{-}(xx')r'$$

$$= (r'L_{+}(x) - L_{-}(x)r')L_{+}(x') + L_{-}(x)(r'L_{+}(x') - L_{-}(x)r')$$

$$= D_{r'}^{-}(x)L_{+}(x') + L_{-}(x)D_{r'}^{-}(x').$$

Thus, $D_{r'}^{-} \in \text{Der}_{L_{+},L_{-}}(R, R')$. The computation for $D_{r'}^{+}$ is similar and is omitted. The linearity of both maps in $r'$ is obvious. □

6.2. Gelfand-Kirillov model for the category $G_{q}^{\text{int}}(\mathfrak{g})$. Throughout this section we mostly follow the notation from [6, Section 6]. Let $\Gamma$ be the monoid $P^{+}$ written multiplicatively, with its elements denoted by $\lambda, \lambda \in P^{+}$. Let $\mathcal{A}_{q}(\mathfrak{g})$ be an isomorphic copy of $U_{q}^{-}(\mathfrak{g})$ whose generators are denoted by $x_i, i \in I$. We denote the degree of a homogeneous element $x \in \mathcal{A}_{q}(\mathfrak{g})$ with respect to its $Q$-grading by $|x| \in -Q^{+}$. Define an action of $\Gamma$ on $\mathcal{A}_{q}(\mathfrak{g})$ by $v_{\lambda} \triangleright x = q^{\langle \lambda, |x| \rangle}x$ for $x \in \mathcal{A}_{q}(\mathfrak{g})$ homogeneous. Let $\mathcal{B}_{q}(\mathfrak{g}) = \mathcal{A}_{q}(\mathfrak{g}) \rtimes \mathbb{k}[\Gamma]$. In particular, we have

$$v_{\lambda}x = q^{\langle \lambda, |x| \rangle}xv_{\lambda} \quad (6.6)$$
for all \( \lambda \in P^+ \) and for all \( x \in A_q(\mathfrak{g}) \) homogeneous. We extend the \( Q \)-grading on \( A_q(\mathfrak{g}) \) to a \( P \)-grading on \( B_q(\mathfrak{g}) \) via \( |v_\lambda| = \lambda \) for \( \lambda \in P^+ \). Let \( \mathcal{O}_q(\mathfrak{g}) \) be the category of \( U_q(\mathfrak{g}) \)-modules whose objects satisfy all assumptions on objects of \( \mathcal{O}_q^{int}(\mathfrak{g}) \) except that we do not assume that the \( E_i, F_i, i \in I \), act locally nilpotently while assuming that all weight subspaces are finite dimensional. The following essentially coincides with \cite[Lemma 6.1]{6}.

**Lemma 6.7.** The algebra \( B_q(\mathfrak{g}) \) is a module algebra in the category \( \mathcal{O}_q(\mathfrak{g}) \) with respect to the action given by the following formulae for all \( \lambda \in \frac{1}{2} P, i \in I \)

- \( K_\lambda(y) = q^{\lambda|y|} y \) for all homogeneous elements \( y \in B_q(\mathfrak{g}) \) and \( \lambda \in \frac{1}{2} P \);
- \( F_i(y) = \frac{x_i K_{\frac{1}{2} \alpha_i}(y) - K_{\frac{1}{2} \alpha_i}(y) x_i}{q_i - q_i^{-1}} \) for all \( y \in B_q(\mathfrak{g}) \) and thus is a \( K_{\frac{1}{2} \alpha_i} \)-derivation of \( B_q(\mathfrak{g}) \);
- \( E_i \) is the unique \( K_{\frac{1}{2} \alpha_i} \)-derivation of \( B_q(\mathfrak{g}) \) such that \( E_i(x_j) = \delta_{ij} \) and \( E_i(v_\mu) = 0 \) for all \( i, j \in I, \mu \in P^+ \).

Thus for all \( x, y \in B_q(\mathfrak{g}) \) we have

\[
X_i(xy) = X_i(x)K_{\frac{1}{2} \alpha_i}(y) + K_{-\frac{1}{2} \alpha_i}(x)X_i(y)
\]

and more generally, for all \( n \geq 0 \)

\[
X_i^{(n)}(xy) = \sum_{r+s=n} X_i^{(r)}K_{-\frac{1}{2} \alpha_i}(x)X_i^{(s)}K_{\frac{1}{2} \alpha_i}(y)
\]

where \( X_i \) is either \( E_i \) or \( F_i \). The following is immediate from the definition of \( B_q(\mathfrak{g}) \) and its \( U_q(\mathfrak{g}) \)-module structure.

**Corollary 6.8.** \( B_q(\mathfrak{g}) = \sum_{\lambda \in P^+} A_q(\mathfrak{g})v_\lambda \) where \( A_q(\mathfrak{g})v_\lambda \) is a \( U_q(\mathfrak{g}) \)-submodule of \( B_q(\mathfrak{g}) \) for each \( \lambda \in P^+ \) and the sum is direct.

In the sequel we will also use \( E_i^* \) which is defined as the unique \( K_{-\frac{1}{2} \alpha_i} \)-derivation of \( B_q(\mathfrak{g}) \) satisfying \( E_i^*(x_j) = \delta_{ij}, E_i^*(v_\lambda) = 0 \) for all \( \lambda \in P^+, j \in I \). It is easy to check that \( E_i^*(x) = (E_i(x^*))^*, x \in A_q(\mathfrak{g}) \), where \( ^* : A_q(\mathfrak{g}) \to A_q(\mathfrak{g}) \) is the unique anti-involution preserving the \( x_i, i \in I \).

In the spirit of \cite{23}, using the decomposition from Corollary 6.8 we can define a linear map \( j : B_q(\mathfrak{g}) \to A_q(\mathfrak{g}) \) by

\[
j(x \cdot v_\lambda) = q^{-\frac{1}{2} \lambda|x|})x
\]

for all \( \lambda \in P^+ \) and \( x \in A_q(\mathfrak{g}) \) homogeneous. Clearly, \( j|_{A_q(\mathfrak{g})v_\lambda} \) is a bijection onto \( A_q(\mathfrak{g}) \).

**Lemma 6.9.** For any symmetrizable Kac-Moody \( \mathfrak{g} \) we have:

(a) \( j \) is a surjective homomorphism of \( U_q^+(\mathfrak{g}) \)-modules, with respect to the action defined in Lemma 6.7.

(b) \( j(x \cdot y) = q^{\frac{1}{2} (\lambda|j(y)|) - \frac{1}{2} (\mu|j(x)|) j(x) \cdot j(y) \) for all \( x \in V_\lambda, y \in V_\mu \) homogeneous.
Proof. Part (a) is easily checked using Corollary 6.8. To prove part (b) note that
\[ x = q^{\frac{1}{2}((\lambda,\lambda))}j(x)v_\lambda \] for all \( x \in V_\lambda \) homogeneous and so we can write
\[ x \cdot y = q^{\frac{1}{2}((\lambda,\lambda)) + \frac{1}{2}((\mu,\mu))} j(x \cdot y)v_{\lambda + \mu} = q^{\frac{1}{2}((\lambda,\lambda)) + \frac{1}{2}((\mu,\mu)) + ((\lambda,\lambda))} j(x)v_j(y)v_\mu = q^{\frac{1}{2}((\lambda,\lambda)) + \frac{1}{2}((\mu,\mu)) + ((\lambda,\lambda))} j(x) \cdot j(y)v_{\lambda + \mu}. \]

The assertion is now immediate. \( \square \)

Given a \( U_q(\mathfrak{g}) \)-module \( M \), denote by \( M^{\text{int}} \) the set of all \( m \in M \) such that \( U_q(\mathfrak{g})(m) \in \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \). The following is well-known and in fact is easy to check.

**Lemma 6.10.** The assignment \( M \mapsto M^{\text{int}} \) for every \( U_q(\mathfrak{g}) \)-module \( M \) and \( f \mapsto f \) for any morphism of \( U_q(\mathfrak{g}) \)-modules defines an additive submonoidal functor from the tensor category of \( U_q(\mathfrak{g}) \)-modules to \( \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \), that is \( M^{\text{int}} \otimes N^{\text{int}} \subset (M \otimes N)^{\text{int}} \) for any \( U_q(\mathfrak{g}) \)-modules \( M, N \). In particular, if \( M \) is an algebra object in the category of \( U_q(\mathfrak{g}) \)-modules then \( M^{\text{int}} \) is its \( U_q(\mathfrak{g}) \)-module subalgebra and an algebra object in the category \( \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \).

**Proposition 6.11.** For any \( \lambda \in P^+ \) the \( U_q(\mathfrak{g}) \)-submodule of \( \mathcal{A}_q(\mathfrak{g})v_\lambda \) generated by \( v_\lambda \) is naturally isomorphic to \( V_\lambda \) and coincides with \( (\mathcal{A}_q(\mathfrak{g})v_\lambda)^{\text{int}} \).

**Proof.** Given \( M \in \mathcal{O}_q(\mathfrak{g}) \), define \( M^\vee = \bigoplus_{\beta \in P^+} M^\vee(\beta) \) where \( M^\vee(\beta) = \text{Hom}_k(M(\beta), k) \). Endow \( M^\vee \) with a \( U_q(\mathfrak{g}) \)-module structure via \( (u \cdot f)(m) = f(u^T(m)) \), \( u \in U_q(\mathfrak{g}) \), \( f \in M^\vee \), \( m \in M \), where \( u \mapsto u^T \), \( u \in U_q(\mathfrak{g}) \) is the unique anti-involution of \( U_q(\mathfrak{g}) \) such that \( E_i^T = F_i \) and \( K_{\mu}^T = K_{\mu} \). The following is well-known:

**Lemma 6.12.** For any symmetrizable Kac-Moody \( \mathfrak{g} \), we have:

(a) The assignments \( M \mapsto M^\vee, M \in \mathcal{O}_q(\mathfrak{g}) \) define an involutive contravariant functor on \( \mathcal{O}_q(\mathfrak{g}) \).

(b) For any \( M \in \mathcal{O}_q(\mathfrak{g}) \), \( (M^{\text{int}})^\vee = (M^\vee)^{\text{int}} \).

(c) For any \( V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g}) \), \( V^\vee \) is naturally isomorphic to \( V \).

We need the following well-known fact which essentially coincides with [3, Lemma 2.10].

**Lemma 6.13.** There exists a unique non-degenerate pairing \( \langle \cdot, \cdot \rangle : U_q^-(\mathfrak{g}) \otimes \mathcal{A}_q(\mathfrak{g}) \to k \) such that \( \langle F_i, x_j \rangle = \delta_{i,j}, i, j \in I \), \( \langle uF_i, x \rangle = \langle u, E_i^*(x) \rangle \), \( \langle F_i u, x \rangle = \langle u, E_i(x) \rangle \), \( u \in U_q^-(\mathfrak{g}) \), \( x \in \mathcal{A}_q(\mathfrak{g}) \) and \( \langle u, x \rangle = 0 \) for \( u \in U_q^-(\mathfrak{g}) \), \( x \in \mathcal{A}_q(\mathfrak{g}) \) homogeneous unless \( deg u = |x| \).

Denote \( M_\lambda, \lambda \in P \), the Verma module with highest weight \( \lambda \) (see e.g. [31, §3.4.5]). For every \( \lambda \in P^+ \) we fix \( m_\lambda \in M_\lambda(\lambda) \setminus \{0\} \). Since \( M_\lambda \) is free as a \( U_q^-(\mathfrak{g}) \)-module, every element of \( M_\lambda \) can be written, uniquely, as \( um_\lambda \) for some \( u \in U_q^-(\mathfrak{g}) \). Let \( \mathcal{M}_q(\mathfrak{g}) = \bigoplus_{\lambda \in P^+} M_\lambda \). Define \( \langle \cdot, \cdot \rangle : \mathcal{M}_q(\mathfrak{g}) \otimes \mathcal{B}_q(\mathfrak{g}) \to k \) by \( \langle u(m_\lambda), xv_\mu \rangle = \delta_{\lambda, \mu} \langle u(x) \rangle \), \( u \in U_q^-(\mathfrak{g}), x \in \mathcal{A}_q(\mathfrak{g}), \lambda, \mu \in P^+ \). It is immediate from the definition that \( \langle \mathcal{M}_q(\mathfrak{g})(\beta), \mathcal{B}_q(\mathfrak{g})(\beta') \rangle = 0, \beta, \beta' \in P \), unless \( \beta = \beta' \).
The following Lemma seems to be well-known. We provide a proof for reader’s convenience.

**Lemma 6.14.** The pairing \( \langle \cdot, \cdot \rangle \) is non-degenerate and contragredient, that is
\[
\langle u'(m), b \rangle = \langle m, u'^T(b) \rangle, \quad u' \in U_q(\mathfrak{g}), \ m \in M_q(\mathfrak{g}), \ b \in B_q(\mathfrak{g}).
\]
(6.10)
In particular, \( A_q(\mathfrak{g})v_\lambda, \ \lambda \in P^+ \) naturally identifies with \( M_\lambda^\vee \).

**Proof.** The pairing \( \langle \cdot, \cdot \rangle \) is non-degenerate as a direct sum of non-degenerate (in view of Lemma 6.13) pairings \( M_\lambda \otimes A_q(\mathfrak{g})v_\lambda \to \mathfrak{k} \). To prove that it is contragredient, it suffices to prove (6.10) for \( u' \in \{ K_\mu, E_i, F_i \}, \ \mu \in \frac{1}{2}P, \ i \in I \). For all \( i \in I \). Moreover, we may assume without loss of generality that \( m = u(m_\lambda) \) and \( b = xv_\lambda \) with \( u \in U_q^-(\mathfrak{g}), \ x \in A_q(\mathfrak{g}) \) homogeneous. We have
\[
\langle K_\mu(u(m_\lambda)), xv_\lambda \rangle = q^{(\mu, \lambda + \deg u)} \delta_{\deg u, |x|} \langle u(m_\lambda), xv_\lambda \rangle = \langle u(m_\lambda), K_\mu(xv_\lambda) \rangle = \langle u(m_\lambda), K_\mu^T(xv_\lambda) \rangle.
\]
Furthermore, by Lemmata 6.9(a) and 6.13 we obtain
\[
\langle F_i(u(m_\lambda)), xv_\lambda \rangle = \langle (F_iu)(m_\lambda), xv_\lambda \rangle = \langle F_iu, j(x) \rangle = \langle u, E_i(j(x)) \rangle = \langle u(m_\lambda), E_i(xv_\lambda) \rangle = \langle u(m_\lambda), F_i^T(xv_\lambda) \rangle.
\]
In particular, we proved (6.10) for \( u' \in \{ K_\mu, F_i \}, \ \mu \in \frac{1}{2}P, \ i \in I \) for all \( m \in M_q(\mathfrak{g}) \) and \( b \in B_q(\mathfrak{g}) \).

It remains to prove that
\[
\langle E_i(m), b \rangle = \langle m, F_i(b) \rangle,
\]
for all \( m \in M_\lambda \) and for all \( b \in A_q(\mathfrak{g})v_\lambda \) homogeneous. We argue by induction on \( \rho^\vee(\lambda - \beta) \) where \( m \in M_\lambda(\beta) \). If \( \beta = \lambda \) then \( E_i(m) = 0 \) while \( F_i(b) = b' \) with \( |b'| = |b| - \alpha_i \). Since \( |b|, \lambda - Q^+ \), \( |b'| \neq \beta \) and so \( \langle m, b' \rangle = 0 \). For the inductive step, it suffices to assume that \( m = F_j(m') \) where \( m' \) is homogeneous. We have
\[
\langle E_i(F_j(m')), b \rangle = \langle [E_i, F_j](m'), b \rangle + \langle (F_jE_i)(m'), b \rangle = \langle m', [E_j, F_i](b) \rangle + \langle m', F_iE_j(b) \rangle = \langle m', E_j(F_i(b)) \rangle = \langle F_j(m'), F_i(b) \rangle,
\]
where we used (6.10) for \( u' = F_j \) and \( u' = [E_i, F_j] = \delta_{i,j}(q_i - q_i^{-1})^{-1}(K_{\alpha_i} - K_{-\alpha_i}) = [E_j, F_i] \).

The second assertion of the Lemma is immediate from the first. \( \square \)

By [31, Proposition 3.5.6], \( M_\lambda \) has a unique integrable quotient isomorphic to \( V_\lambda \). Applying \( int^\vee \) to the surjection \( M_\lambda \to V_\lambda \) we obtain, by Lemmata 6.12 and Lemma 6.14, the desired isomorphism \( V_\lambda \cong (M_\lambda^\vee)^{int} = (A_q(\mathfrak{g})v_\lambda)^{int} \) such that \( v_\lambda \mapsto v_\lambda \). \( \square \)

In view of Proposition 6.11, from now on we identify \( V_\lambda, \ \lambda \in P^+ \), with the \( U_q(\mathfrak{g}) \)-submodule of \( A_q(\mathfrak{g})v_\lambda \) generated by \( v_\lambda \).
Lemma 6.15. For any $\lambda, \mu \in P^+$ we have $V_\lambda \cdot V_\mu = V_{\lambda+\mu}$ in $\mathcal{B}_q(\mathfrak{g})$.

Proof. It is immediate from the definition of $\mathcal{B}_q(\mathfrak{g})$ and Corollary 6.8 that $V_\lambda \cdot V_\mu \subset \mathcal{A}_q(\mathfrak{g})v_{\lambda+\mu}$. Furthermore, since $V_\lambda \cdot V_\mu$ is the image of $V_\lambda \otimes V_\mu$ which is integrable, by Proposition 6.11 we have $V_\lambda \cdot V_\mu \subset (\mathcal{A}_q(\mathfrak{g})v_{\lambda+\mu})^{int} = V_{\lambda+\mu}$ and the latter is a simple $U_q(\mathfrak{g})$-module, $V_\lambda \cdot V_\mu = V_{\lambda+\mu}$.

Denote $C_q(\mathfrak{g}) = \mathcal{B}_q(\mathfrak{g})^{int}$. The following is an immediate corollary of Proposition 6.11 and Lemma 6.15.

Corollary 6.16. $C_q(\mathfrak{g})$ decomposes as $C_q(\mathfrak{g}) = \sum_{\lambda \in P^+} V_\lambda$ as a $U_q(\mathfrak{g})$-module algebra.

Proposition 6.17. For any symmetrizable Kac-Moody $\mathfrak{g}$ and $\lambda \in P^+$ we have:
\[ \mathfrak{j}(V_\lambda) = \bigcap_{i \in I} \ker E_i^{\lambda(\alpha_i^\vee)+1}. \] (6.11)

Proof. Note that Lemma 6.13 yields an isomorphism of $k$-vector spaces $\xi : \mathcal{A}_q(\mathfrak{g}) \to U_q^-(\mathfrak{g})^\vee := \bigoplus_{\gamma \in Q^+} \text{Hom}_k(U_q^-(\mathfrak{g})(-\gamma), k)$ defined by $\xi(x)(u) = (u, x)$, $x \in \mathcal{A}_q(\mathfrak{g})$, $u \in U_q^-(\mathfrak{g})$. Define $\phi_\lambda^\vee : M_\lambda^\vee \to U_q^-(\mathfrak{g})^\vee$ by $\phi_\lambda^\vee(f)(u) = f(u(m_\lambda))$ for all $f \in M_\lambda^\vee$, $u \in U_q^-(\mathfrak{g})$.

Define an action of $U_q^+(\mathfrak{g})$ on $U_q^-(\mathfrak{g})^\vee$ by $(u_+ \cdot f)(u_-) := f(u_+ u_-)$, $u_+ \in U_q^+(\mathfrak{g})$.

Lemma 6.18. For any $\lambda \in P^+$, $\phi_\lambda^\vee$ is an isomorphism of $U_q^+(\mathfrak{g})$-modules. Moreover, the following diagram in the category of $U_q^+(\mathfrak{g})$-modules commutes
\[ \begin{array}{ccc}
\mathcal{B}_q(\mathfrak{g}) & \xrightarrow{j} & \mathcal{A}_q(\mathfrak{g}) \\
\downarrow & & \downarrow \xi \\
M_\lambda^\vee & \xrightarrow{\phi_\lambda^\vee} & U_q^-(\mathfrak{g})^\vee
\end{array} \] (6.12)

where the left vertical arrow is obtained by the identification $M_\lambda^\vee \cong \mathcal{A}_q(\mathfrak{g})v_\lambda$ from Proposition 6.11.

Let $J_\lambda, \lambda \in P^+$, be the kernel of the canonical projection of $M_\lambda$ on $V_\lambda$. It is well-known (see e.g. [31, Proposition 3.5.6]) that $J_\lambda = \sum_{i \in I} U_q^-(\mathfrak{g})F_i^{\lambda(\alpha_i^\vee)+1}(m_\lambda)$. Applying $\vee$ to the projection $M_\lambda \to V_\lambda$ and using that $V_\lambda \cong V_\lambda^\vee$ we obtain an embedding $V_\lambda \to M_\lambda^\vee$. Note that
\[ \phi_\lambda^\vee(V_\lambda) = \{ f \in U_q^-(\mathfrak{g})^\vee : f\left(\sum_{i \in I} U_q^-(\mathfrak{g})F_i^{\lambda(\alpha_i^\vee)+1}\right) = \{0\} \}.
\]

Therefore, $\phi_\lambda^\vee(V_\lambda) = \bigcap_{i \in I} K_i$ where $K_i = \{ f \in U_q^-(\mathfrak{g})^\vee : f(U_q^-(\mathfrak{g})F_i^{\lambda(\alpha_i^\vee)+1}) = 0 \}$. By Lemma 6.13, $\xi^{-1}(K_i) = \ker E_i^{\lambda(\alpha_i^\vee)+1}$. Using (6.12) we obtain $\mathfrak{j}(V_\lambda) = \bigcap_{i \in I} \xi^{-1}(K_i) = \bigcap_{i \in I} \ker E_i^{\lambda(\alpha_i^\vee)+1}$. \qed
6.3. Realization of $\sigma^l$ via quantum twist. Let $v_{w,\lambda} = F_{w,\lambda}(v_{\lambda}), \lambda \in P^+$ where we use the notation from §3.1 (see also §3.2). This notation agrees with that in [6, (6.3)]. We need the following

Lemma 6.19. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. Then for any $w, w' \in W$ and $\lambda, \mu \in P^+$ we have

(a) $v_{w,\lambda} \cdot v_{w,\mu} = v_{w,(\lambda + \mu)}$. In particular, for any $w \in W$, the assignments $v_{\lambda} \mapsto v_{w,\lambda}, \lambda \in P^+$, define a homomorphism of monoids $g_w : \Gamma \to B_q(\mathfrak{g})$;

(b) if $\ell(w') = \ell(w) + \ell(w')$ then we have

$$v_{w,\lambda} \cdot v_{w,\lambda} = q^{(w\lambda - \lambda,\mu)} v_{w,\lambda} \cdot v_{w,\lambda};$$

(c) if $\ell(s_i w) = \ell(w) - 1, i \in I$ then $v_{w,\lambda} x_i = q^{(w,\alpha_i)} x_i v_{w,\lambda}$ for all $\lambda \in P^+$.

(d) If $\mathfrak{g}$ is finite dimensional then $v_{w,\lambda} x = q^{-|w,\lambda|} x v_{w,\lambda}$ for all $x \in A_q(\mathfrak{g})$ homogeneous.

Proof. To prove (a) we use induction on $\ell(w)$, the induction base being trivial. For the inductive step, suppose that $\ell(s_i w) = \ell(w) + 1$. Then by Lemma 3.1(b) and the induction hypothesis

$$v_{s_i w,(\lambda + \mu)} = F_{s_i w,\lambda + \mu}(v_{\lambda + \mu}) = F_i^{(w,\lambda + \mu)(\alpha_i')}(v_{w,(\lambda + \mu)}) = F_i^{(w\lambda + \mu)(\alpha_i')}(v_{w,\lambda} \cdot v_{w,\mu}).$$

Using (6.8) and observing that $F_i^{(r)}(v_{w,\lambda}) F_i^{(s)}(v_{w,\mu}) = 0$ if $r > w \lambda(\alpha_i')$ or $s > \mu(\alpha_i')$ we obtain by Lemma 3.1(b)

$$F_{s_i w,(\lambda + \mu)} = \sum_{r + t = w,\lambda + \mu(\alpha_i')} q^{|w \lambda - t w \lambda,\alpha_i|} F_i^{(r)}(v_{\lambda}) \cdot F_i^{(t)}(v_{\lambda}).$$

Part (b) was established in [6, Lemma 6.4]. To prove part (c), note that if $\ell(s_i w) = \ell(w) - 1$ then $F_i(v_{w,\lambda}) = 0$. Then $x_i K_{\alpha_i}(v_{w,\lambda}) - K_{-\alpha_i}(v_{w,\lambda}) x_i = 0$, whence $x_i v_{w,\lambda} = q^{-|w,\lambda|} v_{w,\lambda} x_i = q^{(w,\lambda,\alpha_i)} v_{w,\lambda} x_i$. In particular, applying part (c) with $w = w_o$ we obtain, using an obvious induction on $-\rho''(x)$,

$$v_{w_o,\lambda} x = q^{-|w_o,\lambda|} x v_{w_o,\lambda},$$

which yields part (d).

Following [6, §6.1], define generalized quantum minors $\Delta_{w,\lambda} \in A_q(\mathfrak{g}), w \in W, \lambda \in P^+$ by $\Delta_{w,\lambda} := \chi(v_{w,\lambda})$. In particular,

$$v_{w,\lambda} = q^{|w\lambda - \lambda|} \Delta_{w,\lambda} v_{\lambda}. \quad (6.13)$$

We list some properties of generalized quantum minors which will be used in the sequel.

Lemma 6.20. Let $w, w' \in W, \lambda, \mu \in P^+$. Then

(a) $\Delta_{w,\lambda} \cdot \Delta_{w,\mu} = q^{(w \mu - w^{-1} \mu,\lambda)} \Delta_{w,(\lambda + \mu)}$;
(b) $\Delta_{w\mu} \cdot \Delta_{w\nu} = q^{(w_{\mu} - w_{\nu} + \lambda)} \Delta_{w\nu} \cdot \Delta_{w\mu}$;

(c) If $\mathfrak{g}$ is finite-dimensional reductive then $\Delta_{w_\nu} \cdot \Delta_{w_\mu} = \Delta_{w_\nu(\lambda + \mu)}$ and $\Delta_{w_\nu} x = q^{-(w_\nu, \lambda | x)} x \Delta_{w_\nu}$ for any $x \in A_q(\mathfrak{g})$ homogeneous.

Proof. Parts (a) and (b) follow immediately from Lemma 6.19(a) and (b), respectively, by applying Lemma 6.9(b). The first assertion of part (c) is a special case of (a). Finally, using (6.6), (6.10) and Lemma 6.19(d) we can write

$$q^{(\lambda,|x|)} \Delta_{w_\nu} x v_\lambda = \Delta_{w_\nu} v_\lambda x = q^{-(w_\nu, \lambda | x)} x \Delta_{w_\nu} v_\lambda.$$ 

It remains to apply $j$ and use the fact that $j|_{A_q(\mathfrak{g})v_\lambda}$ is injective.

Let $S_w = \{\Delta_{w_\nu} : \lambda \in P^+\}$. It follows from Lemma 6.20(c) that $S_{w_\nu}$ is an abelian submonoid of $A_q(\mathfrak{g})$ and in fact is an Ore submonoid with $\Sigma_{w_\nu}(x_i) = q^{(\lambda, \alpha_i - \alpha_i)} x_i$ for $\lambda \in P^+, i \in I$.

Define $\tilde{B}_q(\mathfrak{g}) := B_q(\mathfrak{g})[S_{w_\nu}^{-1}]$ and let $\tilde{A}_q(\mathfrak{g})$ be the subalgebra of $\tilde{B}_q(\mathfrak{g})$ generated by $A_q(\mathfrak{g})$, as a subalgebra, and the $\Delta_{w_\nu}, \lambda \in P^+$. Clearly, $\tilde{A}_q(\mathfrak{g})$ is isomorphic to $A_q(\mathfrak{g})[S_{w_\nu}^{-1}]$. The following is the main result of Section 6.

**Theorem 6.21.** Let $\mathfrak{g}$ be finite dimensional. Then

(a) the assignments $x_i \mapsto q_i^{\frac{1}{2}(\delta_i, \epsilon_i)} E_i(\Delta_{w_\nu}) \Delta_{w_\nu}^{-1}$, $v_\lambda \mapsto v_{w_\nu}, \lambda \in P^+$, define an injective algebra homomorphism $\sigma : B_q(\mathfrak{g}) \to B_q(\mathfrak{g})^\text{op}$;

(b) $\tilde{\sigma}(V_\lambda) = V_\lambda$ and $\tilde{\sigma}|_{V_\lambda} = \sigma_{V_\lambda}^{T}$. In particular, the restriction of $\tilde{\sigma}$ to $C_q(\mathfrak{g})$ is an anti-involution on $C_q(\mathfrak{g})$.

Proof. The first step is to construct a homomorphism of algebras $\sigma_0 : A_q(\mathfrak{g}) \to \tilde{A}_q(\mathfrak{g})^\text{op}$.

**Proposition 6.22.** The assignments

$$x_i \mapsto q_i^{\frac{1}{2}(1-\delta_i, \epsilon_i)} E_i(\Delta_{w_\nu}) \Delta_{w_\nu}^{-1} = q_i^{\frac{1}{2}(1-\delta_i, \epsilon_i)} \Delta_{w_\nu}^{-1} E_i(\Delta_{w_\nu}), \quad i \in I,$$

define a homomorphism $\sigma_0 : A_q(\mathfrak{g}) \to \tilde{A}_q(\mathfrak{g})^\text{op}$ such that $\sigma_0(A_q(\mathfrak{g})(-\gamma)) \subset \tilde{A}_q(\mathfrak{g})(-w_\nu \gamma)$, $\gamma \in Q^+$.

Proof. Let $\delta$ be the unique automorphism of $A_q(\mathfrak{g})$ defined by $\delta(x_i) = x_i^{*}, i \in I$. Then $\kappa : A_q(\mathfrak{g}) \to A_q(\mathfrak{g})$ is the anti-automorphism defined by $\kappa(x) = (\delta(x))^*$. We need the following

**Lemma 6.23.** For any $\lambda \in P^+$, $\kappa(\Delta_{w_\nu}) = \epsilon_\lambda \Delta_{w_\nu}$ where $\epsilon_\lambda \in \{\pm 1\}$.

**Remark 6.24.** Later we will show that $\epsilon_\lambda = 1$. However, for the purposes of proving Proposition 6.22 this is irrelevant.

Proof. Define

$$A_q(\mathfrak{g})^\lambda := \{x \in A_q(\mathfrak{g})_{w_\nu} : (E_i^*)^{\lambda(\alpha_i) + 1}(x) = 0, \forall i \in I\}.$$
It follows from the definition and Lemma 6.9 that
\[ \mathcal{A}_q(x) = j(V_{\lambda}(w_\lambda\lambda)) = k\Delta_{w_\lambda\lambda}. \]
In particular, \( E_i^{1-w_{\lambda}(\alpha_i^\vee)}(\mathcal{A}_q(x)) = 0 \) for all \( i \in I \). Since \( \kappa(E_i(x)) = E_i^*(\kappa(x)) \), it follows that \( \kappa(\Delta_{w_\lambda\lambda}) \in \mathcal{A}_q(x) \) and so is a multiple of \( \Delta_{w_\lambda\lambda} \). Since \( \kappa \) is an involution, the assertion follows. \( \Box \)

It follows from Lemmata 6.4 and 6.23 that \( \kappa \) lifts to an anti-involution \( \hat{\kappa} \) on \( \hat{\mathcal{A}}_q(x) \).

By [28, Theorem 5.4], for any \( c = (c_i)_{i \in I} \in (k^*)^I \) the assignments
\[ x_i \mapsto c_i E_i^*(\Delta_{w_\lambda\omega_i})\Delta_{w_\lambda\omega_i}^{-1} = c_i q_i^{-\delta_i^{i,i^*}-1}\Delta_{w_\lambda\omega_i}^{-1}, \]
define a homomorphism of algebras \( \zeta_c : \mathcal{A}_q(x) \to \hat{\mathcal{A}}_q(x) \). Let \( c_0 = (q_i^{(1-\delta_i^{i,i^*})})_{i \in I} \) and set \( \sigma_0 := \hat{\kappa} \circ \zeta_c \). Since \( \hat{\kappa} \) is an anti-involution, we have
\[ \sigma_0(x_i) = q_i^{(1-\delta_i^{i,i^*})}(\kappa(\Delta_{w_\lambda\omega_i}))^{-1}\kappa(E_i^*(\Delta_{w_\lambda\omega_i})) = q_i^{(1-\delta_i^{i,i^*})} \Delta_{w_\lambda\omega_i}^{-1} \Delta_{w_\lambda\omega_i} \kappa(E_i^*(\Delta_{w_\lambda\omega_i})). \]
Thus, \( \sigma_0 \) is the desired homomorphism \( \mathcal{A}_q(x) \to \hat{\mathcal{A}}_q(x) \). Since \( |\sigma_0(x_i)| = \alpha_i^\ast = -w_\alpha \alpha_i \), it follows that \( |\sigma_0(x)| = w_\alpha |x| \) for all \( x \in \mathcal{A}_q(x) \) homogeneous. \( \Box \)

Now we have all necessary ingredients to prove Theorem 6.21(a). We apply Lemma 6.1 with \( \hat{R} = \hat{\mathcal{A}}_q(x) \), \( \hat{R} = \hat{\mathcal{B}}_q(x)^{op} \), \( f = \sigma_0 \) and \( g = g_{w_\lambda} \) viewed as a homomorphism \( \Gamma \to \hat{R} \) since \( \Gamma \) is abelian. Take \( x \in \mathcal{A}_q(x) \) homogeneous. Then the following holds in \( \mathcal{B}_q(x) \)
\[ g_{w_\lambda}(v_{\lambda})\sigma_0(v_{\lambda} \lambda) = q^{\lambda(\lambda,x)(w_\lambda\lambda)}v_{w_\lambda\lambda} \sigma_0(x) = q^{\lambda(\lambda,x)(w_\lambda\lambda)}v_{w_\lambda\lambda} \sigma_0(x) v_{w_\lambda\lambda} = \sigma_0(x) \]
which is (6.1) in \( \hat{R} \). Then by Lemma 6.1, \( \hat{\sigma} : \mathcal{B}_q(x) \to \hat{\mathcal{B}}_q(x)^{op}, v_{\lambda} x \mapsto \sigma_0(x) v_{w_\lambda\lambda}, x \in \mathcal{A}_q(x), \lambda \in P^+ \), is a well-defined homomorphism of algebras. Part (a) of Theorem 6.21 is proven.

Note that the \( K_\lambda, \lambda \in \frac{1}{2}P \), satisfy the assumptions of Lemma 6.4 and so can be lifted to automorphisms \( \hat{K}_\lambda \) of \( \mathcal{B}_q(x) \). Define
\[ \hat{F}_i(x) = \frac{x_i \hat{K}_{\frac{1}{2}\alpha_i}(x) - \hat{K}_{\frac{1}{2}\alpha_i}(x)x_i}{q_i - q_i^{-1}}, \quad \hat{E}_i(x) = \frac{\hat{K}_{\frac{1}{2}\alpha_i}(x) - \hat{K}_{\frac{1}{2}\alpha_i}(x)z_i - z_i \hat{K}_{\frac{1}{2}\alpha_i}(x)}{q_i - q_i^{-1}}, \]
where \( z_i = \hat{\sigma}(x_i) = q_i^{(1-\delta_i^{i,i^*})}E_i(\Delta_{w_\lambda\omega_i})\Delta_{w_\lambda\omega_i}^{-1}. \)

**Proposition 6.25.** We have for all \( \lambda \in \frac{1}{2}P, i \in I \):
(a) \( \hat{K}_\lambda|_{\mathcal{B}_q(x)} = K_\lambda, \hat{F}_i|_{\mathcal{B}_q(x)} = E_i \) and \( \hat{E}_i|_{\mathcal{B}_q(x)} = E_i \);
(b) \( \hat{K}_\lambda \circ \hat{\sigma} = \hat{\sigma} \circ \hat{K}_w_\lambda, \hat{F}_i \circ \hat{\sigma} = \hat{\sigma} \circ \hat{E}_i, \) and \( \hat{E}_i \circ \hat{\sigma} = \hat{\sigma} \circ \hat{E}_i \).

**Proof.** The first and the second assertions in part (a) are obvious. Furthermore, since \( \hat{F}_i = D_\pm^{(q_i - q_i^{-1})^{-1}x_i} \) and \( \hat{E}_i = D_\pm^{(q_i - q_i^{-1})^{-1}z_i} \) with \( L_\pm = \hat{K}_{\frac{1}{2}\alpha_i} \) in the notation of (6.5), we immediately obtain the following
Lemma 6.26. \( \hat{F}_i \) and \( \hat{E}_i \), \( i \in I \) are \( K_{\frac{1}{2} \alpha_i} \)-derivations of \( \hat{B}_q(g) \).

Thus, by Lemma 6.5 the last assertion in part (a) is equivalent to
\[
\hat{E}_i(v_\lambda) = 0, \quad \hat{E}_i(x_j) = \delta_{i,j}, \quad \lambda \in P^+, \, i, \, j \in I.
\]
Since \( |z_i| = \alpha_i \) and \( z_i \in \hat{A}_q(g) \), we have
\[
K_{\frac{1}{2} \alpha_i}(v_\lambda) z_i - z_i K_{\frac{1}{2} \alpha_i}(v_\lambda) = q^{-\frac{1}{2}(\alpha_i, \lambda)} v_\lambda z_i - q^\frac{1}{2} (\lambda, \alpha_i) z_i v_\lambda = 0.
\]
Thus, \( \hat{E}_i(v_\lambda) = 0 \) for all \( \lambda \in P^+ \). We need the following

Lemma 6.27. The following identity holds in \( A_q(g) \) for all \( i, j \in I \)
\[
q_i \hat{E}_i(x_j) \Delta_{w_0 i^*} = q_i \Delta_{w_0 i^*} x_j - q_i \Delta_{w_0 i^*} x_j = q_i \delta_{i,j} x_j.
\]

Proof. This follows by a straightforward computation by applying \( E_i \) to the identity
\[
x_j \Delta_{w_0 i^*} = q_j \delta_{i,j} \Delta_{w_0 i^*} x_j
\]
which is a special case of Lemma 6.20(c).

Since \( \Delta_{w_0 i^*} x_j \Delta_{w_0 i^*} = q^{- (w_0 i^* + i^*)} x_j = q_j \delta_{i,j} x_j \), we can write
\[
(q_i - q_i^{-1}) \hat{E}_i(x_j) \Delta_{w_0 i^*} = q_i \delta_{i,j} x_j.
\]
Using Lemma 6.27 we conclude that \( (q_i - q_i^{-1}) \hat{E}_i(x_j) \Delta_{w_0 i^*} = \delta_{i,j} \), \( (q_i - q_i^{-1}) \Delta_{w_0 i^*} \), and so \( \hat{E}_i(x_j) = \delta_{i,j} \). Part (a) of Proposition 6.25 is proven.

The first assertion in Proposition 6.25(b) is immediate since \( |\hat{\sigma}(x)| = w_0 |x| \) for \( x \in B_q(g) \) homogeneous. Furthermore, by (6.14) we obtain for all \( x \in B_q(g) \),
\[
\hat{\sigma}(F_i(x)) = \frac{\hat{\sigma}(x_i \hat{K}_{-\frac{1}{2} \alpha_i}(x_i)) - \hat{\sigma}(\hat{K}_{-\frac{1}{2} \alpha_i}(x_i))}{q_i - q_i^{-1}} = \frac{\hat{K}_{-\frac{1}{2} \alpha_i^*}(\hat{\sigma}(x)) z_i - z_i \hat{K}_{-\frac{1}{2} \alpha_i^*}(\hat{\sigma}(x))}{q_i - q_i^{-1}} = \hat{E}_i(\hat{\sigma}(x)).
\]

It remains to prove that \( \hat{\sigma}(E_i(x)) = \hat{F}_i(\hat{\sigma}(x)) \) for all \( x \in B_q(g) \). Let \( D_i = \hat{\sigma} \circ E_i - \hat{F}_i \circ \hat{\sigma} \). Since \( \hat{\sigma} \circ K_{\pm \frac{1}{2} \alpha_i} = \hat{K}_{\pm \frac{1}{2} \alpha_i} \circ \hat{\sigma} \) it follows that \( D_i \) is a \( \hat{K}_{\frac{1}{2} \alpha_i} \circ \hat{\sigma} \)-derivation from \( B_q(g) \) to \( \hat{B}_q(g) \). We have
\[
D_i(v_\lambda) = \hat{\sigma}(E_i(v_\lambda)) - \hat{F}_i(v_{w_0 \lambda}) = 0,
\]
By Proposition 6.25(a) we have
\[
\delta_{i,j} = \hat{E}_i(x_j) = \frac{q_j^{\frac{1}{2}(\alpha_i, \alpha_j)} x_j z_i - q_j^{-\frac{1}{2}(\alpha_i, \alpha_j)} z_i x_j}{q_i - q_i^{-1}} = \frac{q_j - q_j^{-1}}{q_i - q_i^{-1}} \hat{F}_j(z_i) \quad (6.15)
\]
and so
\[ D_i(x_j) = \tilde{\sigma}(E_j(x_i)) - \hat{F}_i^*(z_j) = \delta_{i,j} - \delta_{i,j}^* = 0, \]
Thus \( D_i = 0 \) on generators of \( B_q(\mathfrak{g}) \). Then \( D_i = 0 \) by Lemma 6.5. This completes the proof of Proposition 6.25(b). \( \square \)

To prove part (b), we need to show that \( \tilde{\sigma}(V_{\lambda}) \subset V_{\lambda} \). The following Lemma follows from Proposition 6.25(a) by an obvious induction.

**Lemma 6.28.** For any \( b \in B_q(\mathfrak{g}) \), \( r \geq 1, (i_1, \ldots, i_r) \in I^r \), \( \hat{E}_{i_1} \cdots \hat{E}_{i_r}(b) = E_{i_1} \cdots E_{i_r}(b) \in B_q(\mathfrak{g}) \). In particular, for any \( v \in V_{\lambda}, \lambda \in P^+ \) we have \( \hat{E}_{i_1} \cdots \hat{E}_{i_r}(v) \in V_{\lambda} \).

Since \( V_{\lambda} \) is spanned the \( F_i \cdots F_i(v_{\lambda}), r \geq 0, (i_1, \ldots, i_r) \in I^r \), it suffices to show that \( \tilde{\sigma}(F_i \cdots F_i(v_{\lambda})) \in V_{\lambda} \). We have by Proposition 6.25(b)
\[ \tilde{\sigma}(F_i \cdots F_i(v_{\lambda})) = \hat{E}_{i_1} \cdots \hat{E}_{i_r}(v_{\lambda}) \in V_{\lambda} \]
by Lemma 6.28 applied with \( v = v_{\lambda} \).

Consider the operator \( \sigma^I_{\lambda} \circ \tilde{\sigma} \). Clearly, it maps \( v_{\lambda} \) to itself and commutes with the \( U_q(\mathfrak{g}) \)-action by Proposition 6.25(b). Since \( V_{\lambda} \) is a simple \( U_q(\mathfrak{g}) \)-module generated by \( v_{\lambda} \), it follows that \( \tilde{\sigma}_{|V_{\lambda}} = (\sigma^I_{\lambda})^{-1} = \sigma^I_{\lambda} \). In particular, \( \tilde{\sigma} \) is an involution on each \( V_{\lambda} \) and hence an anti-involution on \( C_q(\lambda) \). \( \square \)

**Corollary 6.29.** Let \( \mathfrak{g} \) be finite-dimensional reductive. Then \( \sigma^I \) is an anti-involution on the algebra \( C_q(\mathfrak{g}) \).

### 6.4. \( \sigma^I \) on upper global crystal basis

Denote \( B^{up} \) the dual canonical basis in \( A_q(\mathfrak{g}) \) and denote \( B_\lambda \) the upper global crystal basis of \( V_{\lambda} \). Let \( B = \bigcup_{\lambda \in P^+} B_\lambda \) be the upper global crystal basis of \( C_q(\mathfrak{g}) \).

**Theorem 6.30.** For any finite dimensional reductive \( \mathfrak{g} \) we have \( \tilde{\sigma}(B) = B \). In particular, \( \sigma^I_{\lambda}(B_\lambda) = B_\lambda \).

**Proof.** We need the following

**Lemma 6.31** (see e.g. [28, Proposition 2.33]). For any \( \lambda \in P^+, j(B_\lambda) \subset B^{up} \).

In particular, since \( v_{\lambda} \in B_\lambda \), it follows that \( \Delta_{\lambda} \in B^{up} \).

Denote \( B^{up} = \{ q^{\frac{1}{2}(\lambda, \lambda)} b \Delta_{\lambda}^{-1} : b \in B^{up}, \lambda \in P^+ \} = \{ q^{-\frac{1}{2}(\lambda, \lambda)} \Delta_{\lambda}^{-1} b : b \in B^{up}, \lambda \in P^+ \} \).

We need the following

**Lemma 6.32.** In the notation of Proposition 6.22, \( \sigma_0(B^{up}) \subset \hat{B}^{up} \).
Proof. Note that \( \kappa(B^{up}) = B^{up} \) since it is a composition of two involutions preserving \( B^{up} \).
In particular, \( \kappa(\Delta_{w_\lambda}) = \Delta_{w_\lambda} \) for all \( \lambda \in P^+ \).

Let \( b \in B^{up} \), \( \lambda \in P^+ \). We have
\[
\hat{\kappa}(q^{(w_\lambda+\lambda,|b|)}b\Delta_{w_\lambda}^{-1}) = q^{(w_\lambda+\lambda,|b|)}(\kappa(\Delta_{w_\lambda}))^{-1}\kappa(b) = q^{-\frac{1}{2}(w_\lambda+\lambda,|b|)}\Delta_{w_\lambda}^{-1}\kappa(b) \in \hat{B}^{up}.
\]
By [28, Theorem 5.4] we have \( \zeta_{c_0}(B^{up}) \subset \hat{B}^{up} \) where \( \zeta_{c_0} : A_q(g) \to \hat{A}_q(g) \) is as in the proof of Proposition 6.22. Since \( \sigma_0 = \hat{\kappa} \circ \zeta_{c_0} \), the assertion follows. \( \square \)

Define
\[
\hat{B} = \{ q^{\frac{1}{2}(\lambda,|b|)}bv_\lambda : b \in B^{up}, \lambda \in P^+ \}.
\]
It is immediate from the definition that \( j(\hat{B}) = B^{up} \) and that \( \hat{B} \) is a basis in \( B_q(g) \).
Moreover, it follows from Lemma 6.31 that \( B \subset \hat{B} \) and
\[
B = C_q(g) \cap \hat{B}. \tag{6.16}
\]

Finally, define
\[
\hat{B} = \{ q^{-\frac{1}{2}(w_\lambda,|b|)}bv_{w_\lambda} : b \in \hat{B}^{up}, \lambda \in P^+ \} \subset \hat{B}_q(g).
\]

Proposition 6.33. \( \hat{B} = \{ q^{\frac{1}{2}(\lambda,|b|)}bv_\lambda : b \in \hat{B}^{up}, \lambda \in P^+ \} \). In particular, \( \hat{B} \) is a basis of \( \hat{B}_q(g) \). Finally, \( \hat{B} \subset \hat{B} \).

Proof. We need the following

Lemma 6.34. Let \( R \) be a \( k \)-algebra and let \( S \subset R \setminus \{ 0 \} \) be a commutative Ore submonoid.
Let \( B \) be a basis of \( R \) and suppose that \( \hat{B} = \{ \tau_s(b)s^{-1} : b \in B, s \in S \} \) is a basis of \( R[S^{-1}] \)
where \( \tau_s : R \to R \) is some family of automorphisms satisfying \( \tau_{ss'} = \tau_s \circ \tau_{s'} \), \( \tau_s|S = \text{id}_S \).
Then \( \hat{\tau}_s(\hat{B})s^{-1} = \hat{B} \) for any \( s \in S \), where \( \hat{\tau}_s \) is the unique lifting of \( \tau_s \) to \( R[S^{-1}] \) provided by Lemma 6.4.

Proof. Define \( f_s : R[S^{-1}] \to R[S^{-1}] \) by \( f_s(x) = \hat{\tau}_s(x)s^{-1} \), \( x \in R[S^{-1}] \). We claim that
\[
f_s \circ f_{s'} = f_{ss'}, \quad s, s' \in S \tag{6.17}
\]
and \( f_s \) is invertible with \( f_s^{-1}(x) = \hat{\tau}_s^{-1}(x)s \). Indeed, for all \( x \in R[S^{-1}] \) we have
\[
f_s(f_{s'}(x)) = \hat{\tau}_s(\hat{\tau}_{s'}(x)s^{-1}) = \hat{\tau}_{ss'}(x)s^{-1}s^{-1} = f_{ss'}(x).
\]
and also \( f_s(\hat{\tau}_s^{-1}(x)s) = x \) and \( \hat{\tau}_s^{-1}(f_s(x)) = x = f_s(\hat{\tau}_s^{-1}(x)) \).

We have \( \hat{B} = \{ f_s(b) : b \in B, s \in S \} \) and the assertion is equivalent to \( f_s(\hat{B}) = \hat{B} \)
for all \( s \in S \). Clearly, (6.17) implies that \( f_s(\hat{B}) \subset \hat{B} \) for all \( s \in S \). To prove the opposite inclusion, let \( \hat{b} \in \hat{B} \). Write
\[
f_s^{-1}(\hat{b}) = \sum_{\hat{b} \in \hat{B}} \lambda_{\hat{b}}. \hat{b}'.
\]
Then
\[ \hat{b} = \sum_{\hat{b}' \in \hat{B}} \lambda_{f_s'(\hat{b}')} = \sum_{\hat{b}' \in f_s(\hat{B}) \subset \hat{B}} \lambda_{f_s^{-1}(\hat{b}')} \hat{b}', \]

where we used that \( f_s(\hat{B}) \subset \hat{B} \). Since \( \hat{B} \) is a basis, this implies that \( \lambda_{f_s^{-1}(\hat{b}')} = \delta_{\hat{b}, \hat{b}'} \) and so \( \hat{b} \in f_s(\hat{B}) \). Therefore, \( \hat{B} \subset f_s(\hat{B}) \). \( \square \)

Apply this lemma to \( R = \mathcal{A}_q(\mathfrak{g}) \), \( S = \mathcal{S}_{w_0} \), \( B = \mathbf{B}^{up} \) and \( \hat{B} = \mathbf{B}^\perp \). By [28, Proposition 3.9], \( \mathbf{B}^{up} \) is a basis of \( \hat{\mathcal{A}}_q(\mathfrak{g}) \). We have \( \tau_{\Delta_{w_0 \lambda}}(b) = q^{\frac{1}{2}(w_0 \lambda + \lambda, b)} b \). Then all assumptions of Lemma 6.34 are satisfied. Indeed, \( \tau_{\Delta_{w_0 \lambda}}(b) = q^{\frac{1}{2}(w_0 \lambda + \lambda, b)} b = \tau_{\Delta_{w_0}(\lambda, b)}(b) \) and \( \tau_{\Delta_{w_0 \lambda}}(\Delta_{w_0 \mu}) = q^{\frac{1}{2}(w_0 \lambda + \lambda, \mu)} \Delta_{w_0 \mu} = \Delta_{w_0 \mu} \). Thus by Lemma 6.34 we have, for any \( \lambda \in P^+ \),
\[ \mathbf{B}^\perp = \tau_{\Delta_{w_0 \lambda}}(\mathbf{B}^\perp) \Delta_{w_0 \lambda} = \{ q^{\frac{1}{2}(w_0 \lambda + \lambda, b)} b \Delta_{w_0 \lambda} : b \in \mathbf{B}^\perp \}. \]

We have \( v_{w_0 \lambda} = q^{\frac{1}{2}(w_0 \lambda - \lambda, \lambda)} \Delta_{w_0 \lambda} v_\lambda \),
\[ \hat{\mathbf{B}} = \{ q^{\frac{1}{2}(w_0 \lambda, b)} v_{w_0 \lambda} : b \in \mathbf{B}^\perp \} = \{ q^{\frac{1}{2}(\lambda, b)} \tau_{\Delta_{w_0 \lambda}}^{-1}(b) v_{w_0 \lambda} : b \in \mathbf{B}^\perp \} \]
\[ = \{ q^{\frac{1}{2}(\lambda, b)} \tau_{\Delta_{w_0 \lambda}}^{-1}(b) \Delta_{w_0 \lambda} v_\lambda : b \in \mathbf{B}^\perp \} \]
\[ = \{ q^{\frac{1}{2}(\lambda, b')} b' v_\lambda : b' \in \mathbf{B}^\perp \} \]
where we denoted \( b' = \tau_{\Delta_{w_0 \lambda}}^{-1}(b) \Delta_{w_0 \lambda} \) and observed that \( |b'| = |b| + w_0 \lambda - \lambda \). This proves the first assertion of Proposition 6.33. The second and third assertions are now immediate. \( \square \)

Now we can complete the proof of Theorem 6.30. It follows from Proposition 6.33 and Lemma 6.32 that for any \( b \in \mathbf{B}_{up} \), \( \lambda \in P^+ \) we have
\[ \hat{\sigma}(q^{\frac{1}{2}(\lambda, b)} v_\lambda) = q^{\frac{1}{2}(\lambda, b)} v_{w_0 \lambda} \sigma_0(b) = q^{\frac{1}{2}(w_0 \lambda, \sigma_0(b))} \sigma_0(b) v_{w_0 \lambda} \in \hat{\mathbf{B}}. \]
Thus, \( \hat{\sigma}(\hat{\mathbf{B}}) \subset \hat{\mathbf{B}} \). Since \( \mathbf{B} \subset \hat{\mathbf{B}} \), it follows that \( \hat{\sigma}(\mathbf{B}) \subset \hat{\mathbf{B}} \). Then by Theorem 6.21(b) we conclude that \( \hat{\sigma}(\mathbf{B}) \subset \hat{\mathbf{B}} \cap \mathcal{C}_q(\mathfrak{g}) \). On the other hand, since \( \hat{\mathbf{B}} \) is linearly independent by Proposition 6.33, its intersection with \( \mathcal{C}_q(\mathfrak{g}) \) is also linearly independent. Since \( \hat{\mathbf{B}} \subset \hat{\mathbf{B}} \) by Proposition 6.33, it follows from (6.16) that \( \mathbf{B} = \hat{\mathbf{B}} \cap \mathcal{C}_q(\mathfrak{g}) \subset \hat{\mathbf{B}} \cap \mathcal{C}_q(\mathfrak{g}) \). But \( \mathbf{B} \) is a basis of \( \mathcal{C}_q(\mathfrak{g}) \) and so \( \mathbf{B} = \hat{\mathbf{B}} \cap \mathcal{C}_q(\mathfrak{g}) \). Thus, \( \hat{\sigma}(\mathbf{B}) \subset \mathbf{B} \). Since by Theorem 6.21(b) \( \hat{\sigma} \) is an involution on \( \mathcal{C}_q(\mathfrak{g}) \), \( \hat{\sigma}(\mathbf{B}) = \mathbf{B} \) which completes the proof of the first assertion of Theorem 6.30. The second assertion is immediate from the first and Theorem 6.21(b). \( \square \)

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[1] The difference between our notation and that of [7, 28] is in the linear automorphism of \( \mathcal{A}_q(\mathfrak{g}) \) defined on homogeneous elements \( x \) by \( x \mapsto q^{\frac{1}{2}(|x|, |x|) - (|x|, \rho)} x \).
6.5. **Proof of Theorem 1.10.** Let $V$ be any object in $\mathcal{O}_q^{\text{int}}(\mathfrak{g})$. Let $(L^{\text{up}}, B^{\text{up}})$ be an upper crystal basis of $V$ and let $G(B^{\text{up}})$ be the corresponding upper global crystal basis (see [26]). By [26, Theorem 3.3.1] there exists a direct sum decomposition $V = \sum_j V^j$ such that $B^j := G(B^{\text{up}}) \cap V^j$ is a basis of $V^j$ and each $V^j \cong V_{\lambda_j}$, $\lambda_j \in P^+$. The latter isomorphism identifies $B^j$ with $B_{\lambda_j}$. Since by Theorem 4.10, $\sigma_V^I$ is compatible with direct sum decompositions, the restriction of $\sigma_V^I$ to $V^j$ coincides with $\sigma_{V^j}^I$ and under the above isomorphism it identifies with $\sigma_{V_{\lambda_j}}^I$ and thus preserves $B_{\lambda_j}$ by Theorem 6.30.

7. **Examples**

7.1. **Thin modules.** Let $\lambda \in P^+$. We say that $V_\lambda$ is *quasi-miniscule* if $V_\lambda(\beta) \neq 0$ implies that $\beta \in W\lambda \cup \{0\}$. For example, $V_{\omega_i}$, $i \in I$ for $\mathfrak{g} = \mathfrak{sl}_n$ are (quasi)-miniscule, as well as the quantum analogue of the adjoint representation of $\mathfrak{g}$.

**Lemma 7.1.** Conjecture 1.2 holds for any quasi-miniscule $V = V_\lambda$.

**Proof.** Let $v = v(\lambda) \in V_\lambda(\lambda)$. Then in the notation of (3.2) we have $V_\lambda = k \cdot [v]_W \oplus V_\lambda(0)$. As shown in Proposition 3.9, the action of $W(V)$ on the basis $[v]_W$ of $k \cdot [v]_W$ is given by the Weyl group action on $W/W_\lambda$. It remains to observe that $\sigma_V^I|_{V_\lambda(0)} = \text{id}_{V_\lambda(0)}$, $i \in I$. \qed

This result can be extended to a larger class of modules. We say that $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$ is *thin* if $\dim V(\beta) \leq 1$ for all $\beta \in P \setminus \{0\}$. By definition, every quasi-miniscule module is thin. Furthermore, all modules $V_{m\omega_1}$, $m \in \mathbb{Z}_{\geq 0}$ are thin for $\mathfrak{g} = \mathfrak{sl}_{n+1}$.

**Theorem 7.2.** Conjecture 1.2 holds for thin modules.

**Proof.** Let $(L, B)$ be an upper crystal basis of $V \in \mathcal{O}_q^{\text{int}}(\mathfrak{g})$ and let $G^{\text{up}}(B) \subset V$ be the corresponding global crystal basis. We say that $b \in G^{\text{up}}(B)$ of weight $\beta \in P$ is thin if either $\beta = 0$ or $V(\beta) = kb$. Denote $G^{\text{up}}_0(B)$ the set of thin elements in $G^{\text{up}}(B)$. Clearly, $V$ is thin if and only if $G^{\text{up}}_0(B) = G^{\text{up}}(B)$. We need the following

**Proposition 7.3.** For any $b \in G^{\text{up}}(B) \cap V(\beta)$, $\beta \in P$ with $\dim V(\beta) = 1$ we have $\Phi_V(b) \subset G^{\text{up}}(B)$.

**Proof.** It suffices to prove that $\sigma^J(b) \in G^{\text{up}}(B)$ for all $J \in \mathcal{J}$. Since $\sigma^J(V(\beta)) = V(w^J_\beta)$ and $\dim V(w^J_\beta) = \dim V(\beta) = 1$, it follows that $\sigma^J(b) = cb'$ for some $b' \in G^{\text{up}}(B) \cap V(w^J_\beta)$. Let $b$ be the image of $b$ in $B$ under the quotient map $L \mapsto L/qL$. By Theorem 1.8, $\tilde{\sigma}^J(b) = b'$ and so $c \in 1 + q\mathbb{A}$. It follows from Proposition 5.8 that $\varpi = c$ and so $c = 1$. \qed

Let $g : B \rightarrow G^{\text{up}}(B)$ be Kashiwara’s bijection (cf. [26]) and let $B_0 = g^{-1}(G^{\text{up}}_0(B))$. Then by Theorem 1.8 and Proposition 7.3 we have $g(\tilde{\sigma}^i(b)) = \sigma^i(g(b))$ for all $b \in B_0$. Since the action of $\tilde{\sigma}^i$ on $B$ coincides with the action of $W$ defined in [27], it follows from [27, Theorem 7.2.2] that $W(V)$ is a homomorphic image of $W$. 
We may assume, without loss of generality, that $V = V_\lambda$ and $J(V) = \emptyset$. In view of Proposition 3.9(b), the action of $W(V)$ on the set $[v_\lambda]_W$, $v_\lambda \in V_\lambda(\lambda)$ is faithful and coincides with that of $W$. This implies that $\psi_V$ from Theorem 1.1 is an isomorphism. \hfill \Box

7.2. **Crystallizing cactus group action for $\mathfrak{g} = \mathfrak{sl}_3$.** We now describe combinatorial consequences of Theorem 1.8 for $\mathfrak{g} = \mathfrak{sl}_3$. It turns out that the corresponding action of $\text{Cact}_{\mathfrak{s}_3}$ lifts to the ambient set

$$\hat{M} = \{(m_1, m_2, m_{12}, m_{21}, m_{01}, m_{02}) \in \mathbb{Z}_+^2 \times \mathbb{Z}^4 : m_1m_2 = 0\}$$

where the crystal basis for $C_q(\mathfrak{sl}_3)$ identifies with $\hat{M} = \hat{M} \cap \mathbb{Z}_{\geq 0}^6$.

We need some notation. Define $\text{wt}_i : \hat{M} \to \mathbb{Z}$ by

$$\text{wt}_i(m) = m_{0i} - m_i - m_j - m_{ij}, \quad \{i, j\} = \{1, 2\}.$$ 

Furthermore, define $e_i' : \hat{M} \to \hat{M}$, $i \in \{1, 2\}$, $r \in \mathbb{Z}$ by

$$e_i'(m_1, m_2, m_{12}, m_{21}, m_{01}, m_{02}) = (m'_1, m'_2, m'_{12}, m'_ {21}, m'_{01}, m'_{02})$$

where, for $m = (m_1, m_2, m_{12}, m_{21}, m_{01}, m_{02}) \in \hat{M}$ we set

$$m'_i = [m_i - m_j - r]_+, \quad m'_j = [m_j - m_i + r]_+, \quad m'_{ij} = m_{ji}, \quad m'_{0j} = m_{0j},$$

$$m'_{ij} = m_{ij} + \min(m_i - r, m_j), \quad m'_{0i} = m_{0i} + r + \min(m_i - r, m_j), \quad \{i, j\} = \{1, 2\},$$

and $[x]_+ := \max(x, 0)$, $x \in \mathbb{Z}$. The following is well-known (cf. [5, Example 6.26]).

**Lemma 7.4.** The $e_i'$, $i \in \{1, 2\}$, $r \in \mathbb{Z}$ satisfy

$$e_i'e_i = e_i^{r+s}, \quad e_2'e_2 = e_2^{r+s}, \quad e_1'e_2e_1 = e_2'e_1e_2 = e_1^{r+s}e_2 = e_2e_1e_2, \quad r, s \in \mathbb{Z}. \quad (7.1)$$

In particular, $e_i^0 = \text{id}$ and $(e_i')^{-1} = e_i^{-r}$, $r \in \mathbb{Z}$, $i \in \{1, 2\}$.

**Proof.** Define a map $\hat{k} : \hat{M} \to \mathbb{Z}^5$ by $\hat{k}(m_1, m_2, m_{12}, m_{21}, m_{01}, m_{02}) \mapsto (a_1, a_2, a_3, l_1, l_2)$ where

$$a_1 = m_1 + m_{21}, \quad a_2 = m_2 + m_{12} + m_{21}, \quad a_3 = m_{12}, \quad l_i = m_i + m_{3-i}, \quad m_{0i}$$

with $i \in \{1, 2\}$. It is easy to see that $\hat{k}$ is a bijection with its inverse given by

$$(a_1, a_2, a_3, l_1, l_2) \mapsto (m_1, m_2, m_{12}, m_{21}, m_{01}, m_{02}),$$

where $m_1 = [a_1 + a_3 - a_2]_+$, $m_2 = [a_2 - a_1 - a_3]_+$, $m_{12} = a_3$, $m_{21} = \min(a_1, a_2 - a_3)$, $m_{01} = l_1 - a_1$, $m_{02} = l_2 - a_2 + \min(a_1, a_2 - a_3)$.

The action of operators $e_i'$, $i \in \{1, 2\}$, $r \in \mathbb{Z}$ on $\mathbb{Z}^5$ induced by this bijection coincides with the action constructed in [5, Example 6.26]

$$e_1'(a_1, a_2, a_3, l_1, l_2) = (a_1 + [\delta - r]_+ - [\delta]_+, a_2, a_3 + [\delta]_+ - \max(\delta, r), l_1, l_2)$$

where $\delta = a_1 + a_3 - a_2$, and

$$e_2'(a_1, a_2, a_3, l_1, l_2) = (a_1, a_2 - r, a_3, l_1, l_2).$$
The identities from the Lemma are now easy to obtain by using tropicalized relations for the \( e_i \) given after Definition 2.20 in [5] in the context of [5, Example 6.26].

Define \( \sigma = \sigma^{(1,2)} : \widehat{M} \to \widehat{M} \) by

\[
(m_1, m_2, m_{12}, m_{21}, m_{01}, m_{02}) \mapsto (m_1, m_2, m_{02}, m_{01}, m_{21}, m_{12}).
\]

Clearly, \( \sigma \) is an involution and \( \sigma(M) = M \). Furthermore, define \( \sigma^i : \widehat{M} \to \widehat{M}, i \in \{1, 2\} \) by

\[
\sigma^i(m) = e_i^{-\text{wt}_i(m)}(m), \quad m \in \widehat{M}.
\]

**Proposition 7.5.** The following identities hold in \( \text{Bij}(\widehat{M}) \)

\[
\sigma^i \circ \sigma^i = \text{id}, \quad \sigma^i \circ e_i = e_i^{-r} \circ \sigma^i, \quad \sigma \circ e_i = e_i^{-r} \circ \sigma.
\]

\[
\sigma^i \circ \sigma^j = \sigma^j \circ \sigma^i \circ \sigma^j, \quad \sigma^i \circ \sigma = \sigma \circ \sigma^i.
\]

where \( \{i, j\} = \{1, 2\} \). In particular, the assignments \( \tau_{i,i+1} \mapsto \sigma^i \), \( i \in \{1, 2\} \), \( \tau_{13} \mapsto \sigma \) define an action of \( \text{Cact}_S \) on \( \widehat{M} \).

**Proof.** Since \( \text{wt}_i(e_i^r(m)) = \text{wt}_i(m) + 2r \) for any \( m \in \widehat{M} \), we have

\[
\sigma^i \circ \sigma^i(m) = e_i^{-(\text{wt}_i(m) - 2 \cdot \text{wt}_i(m) - \text{wt}_i(m))}(m) = m,
\]

while

\[
\sigma^i \circ e_i^r(m) = e_i^{-(\text{wt}_i(m) - \text{wt}_i(m))}(m) = e_i^{-r} \circ \sigma^i(m).
\]

To prove the third identity, note that \( e_i^r(\sigma(m)) = (\tilde{m}_1, \tilde{m}_2, \tilde{m}_{12}, \tilde{m}_{21}, \tilde{m}_{01}, \tilde{m}_{02}) \), where

\[
\tilde{m}_i = [m_i - m_j - r]_+, \quad \tilde{m}_j = [m_j - m_i + r]_+, \quad \tilde{m}_{ji} = m_{0i}, \quad \tilde{m}_{0j} = m_{ij},
\]

\[
\tilde{m}_{ij} = m_{0j} + \min(m_i - r, m_j), \quad \tilde{m}_{0i} = m_{ji} + \min(m_i, m_j + r), \quad \{i, j\} = \{1, 2\}.
\]

which is easily seen to coincide with \( \sigma(e_j^{-r}(m)) \). The braid identity follows from the last relation in (7.1) (known as Verma relations) and the identity \( \text{wt}_j(e_j^r(m)) = \text{wt}_j(m) - r, \{i, j\} = \{1, 2\} \). Finally,

\[
\sigma \circ \sigma^j(m) = \sigma \circ e_j^{-\text{wt}_j(m)}(m) = e_j^{\text{wt}_j(m)}(\sigma(m)) = e_i^{-\text{wt}_i(\sigma(m))}(\sigma(m)) = \sigma^i \circ \sigma(m),
\]

where we used the identity \( \text{wt}_i(\sigma(m)) = m_{ji} - m_i + m_j - m_{0j} = -\text{wt}_j(m) \). \( \square \)

**Remark 7.6.** It would be interesting to define analogues of \( \widehat{M} \) for other \( \mathfrak{g} \) and study the action of the corresponding cactus groups on \( \widehat{M} \). We plan to study this in a subsequent publication via the approach of [5].

Given \( l_1, l_2 \in \mathbb{Z} \) define

\[
\widehat{M}_{l_1,l_2} = \{(m_1, m_2, m_{12}, m_{21}, m_{01}, m_{02}) \in \widehat{M} : m_{01} + m_1 + m_{21} = l_1, m_{02} + m_2 + m_{12} = l_2 \}.
\]
Clearly, $\vartheta$, $\vartheta^i$, $e_i^r$, $i \in \{1, 2\}$, $r \in \mathbb{Z}$ preserve $\widehat{M}_{l_1,l_2}$ for any $l_1, l_2 \in \mathbb{Z}$. Set $M_{l_1,l_2} = \widehat{M}_{l_1,l_2} \cap M$.

In view of [5, Example 6.26], $\widehat{k}(M_{l_1,l_2})$, where $\widehat{k}$ is defined in the proof of Lemma 7.4, identifies with the upper crystal basis $B^{up}(V_{l_1\omega_1+l_2\omega_2})$ of $V_{l_1\omega_1+l_2\omega_2}$. In particular, $\widehat{k}(M)$ identifies with the upper crystal basis $B^{up}(C_2) = \bigsqcup_{\lambda \in P^+} B^{up}(\mathcal{V}_\lambda)$ of $C_2 = C_q(\mathfrak{sl}_3)$. We use this identification throughout the rest of this chapter.

**Proposition 7.7.** Under the above identification, the restrictions of $\vartheta$, $\vartheta^i$, $i \in \{1, 2\}$ to $\mathcal{M}$ coincide with the action of $\text{Cact}_3$ on $B^{up}(C_2)$ provided by Theorem 1.8 with $\mathfrak{g} = \mathfrak{sl}_3$ and $V = C_2$.

*Proof.* It follows from Corollary 5.6 applied to $f = \vartheta$ extended to $B_\lambda \cup \{0\}$, and Proposition 7.5 that $\vartheta$ coincides with $\tilde{\vartheta}^{(1,2)}_\lambda$ for all $\lambda \in P^+$. On the other hand, by Remark 5.7 we have $\vartheta^i = \tilde{\vartheta}^{(i)}_\lambda$, $i \in \{1, 2\}$, $\lambda \in P^+$. \hfill \Box

### 7.3. Gelfand-Kirillov model for $\mathfrak{g} = \mathfrak{sl}_3$

Our goal here is to illustrate results and constructions from Section 6 for $\mathfrak{g} = \mathfrak{sl}_3$ and provide some evidence for Conjecture 1.2. We freely use the notation from Section 6 and §7.2. In this case the algebra $A_2 = A_q(\mathfrak{g})$ is generated by the $x_i$, $i \in \{1, 2\}$ subject to the relations

\[ x_i^2 x_j - (q + q^{-1}) x_i x_j x_i + x_j x_i^2 = 0, \quad \{i, j\} = \{1, 2\}. \]  

(7.2)

Define

\[ x_{ij} = \frac{q^\frac{1}{2} x_i x_j - q^{-\frac{1}{2}} x_j x_i}{q - q^{-1}}, \quad \{i, j\} = \{1, 2\}. \]

Then $x_i x_j = q^{\frac{1}{2}} x_{ij} + q^{-\frac{1}{2}} x_{ji}$, $\{i, j\} = \{1, 2\}$ and (7.2) is equivalent to $x_i x_{ij} = q x_{ij} x_i$ or $x_i x_{ji} = q^{-1} x_{ji} x_i$, $\{i, j\} = \{1, 2\}$. The following is well-known \(^1\) (see e.g. [7]).

**Lemma 7.8.** The dual canonical basis $B^{up}$ in the algebra $A_2$ is

\[ B^{up} = \{q^{\frac{1}{2}(m_1-m_2)(m_{21}-m_{12})} x_1^{m_1} x_2^{m_2} x_{12}^{m_{12}} x_{21}^{m_{21}} : (m_1, m_2, m_{12}, m_{21}) \in \mathbb{Z}^4_{\geq 0} \}. \]

We have $\widehat{A}_2 = A_2[S_{w_0}^{-1}] = A_2[x_{12}^{-1}, x_{21}^{-1}]$. It follows from Lemma 7.8 that

\[ \widehat{B}^{up} = \{q^{\frac{1}{2}(m_1-m_2)(m_{21}-m_{12})} x_1^{m_1} x_2^{m_2} x_{12}^{m_{12}} x_{21}^{m_{21}} : (m_1, m_2, m_{12}, m_{21}) \in \mathbb{Z}^2_{\geq 0} \times \mathbb{Z}^2, m_1 m_2 = 0 \}. \]

The following is immediate

**Lemma 7.9.** (a) The algebra $B_2 := B_q(\mathfrak{g})$ is generated by $A_2$ and $\mathbb{K}[v_1, v_2]$, where $v_i = v_{\omega_i}$, as subalgebras subject to the relations

\[ v_i x_j = q^{-\delta_{ij}} x_j v_i, \quad i, j \in \{1, 2\}. \]

(b) $B_2$ is a $U_q(\mathfrak{g})$-module algebra with the $U_q(\mathfrak{g})$-action defined in Lemma 6.7.
Abbreviate \( z_i = F_i(v_i) = v_{s_i\omega_i} \) and \( z_{ij} = F_i F_j(v_j) = v_{s_i s_j \omega_j} \). Clearly,

\[
    z_i = q^{-\frac{1}{2}} x_i v_i, \quad z_{ij} = q^{-\frac{1}{2}} x_{ij} v_j, \quad \{i, j\} = \{1, 2\}.
\]  

(7.3)

The following Lemma is an immediate consequence of Lemma 7.9

**Lemma 7.10.** (a) The algebra \( C_2 = C_4(\mathfrak{sl}_3) \) is generated by \( v_1, v_2, z_1, z_2, z_{12} \) and \( z_{21} \) subject to the relations

\[
v_1 v_2 = v_2 v_1, \quad v_i z_j = q^{-\delta_{i,j}} z_j v_i, \quad v_i z_{12} = q^{-1} z_{12} v_i, \quad v_i z_{21} = q^{-1} z_{21} v_i, \quad i, j \in \{1, 2\}.
\]

\[
z_i z_j = q v_i z_{ij} + q^{-1} z_{ij} v_j, \quad z_k z_{ij} = q^{-\delta_{i,j,k}} z_{ij} z_k, \quad \{i, j\} = \{1, 2\}, k \in \{1, 2\}
\]

and \( z_{12} z_{21} = z_{21} z_{12} \).

(b) \( C_2 \) is a \( U_q(\mathfrak{g}) \)-module algebra via

\[
E_k(v_i) = 0, \quad E_k(z_i) = \delta_{k,i} v_i, \quad E_k(z_{ij}) = \delta_{k,j} z_i, \quad F_k(v_i) = \delta_{i,k} z_i, \quad F_k(z_i) = \delta_{i,j} z_{ij}, \quad F_k(z_{ij}) = 0, \quad \{i, j\} = \{1, 2\}, k \in \{1, 2\}.
\]

(c) The \( P \)-grading on \( C_4(\mathfrak{g}) \) is given by \( |v_i| = \omega_i, |z_i| = s_i \omega_i = \omega_i - \alpha_i, |z_{ij}| = s_j s_i \omega_j = \omega_j - \alpha_i - \alpha_j, \{i, j\} = \{1, 2\} \).

The following is an immediate corollary of Theorem 6.21.

**Corollary 7.11.** The assignments

\[
v_i \mapsto z_{ji}, \quad z_i \mapsto z_i, \quad z_{ij} \mapsto v_j, \quad \{i, j\} = \{1, 2\}.
\]

define an anti-involution of \( C_2 \) which coincides with \( \sigma = \sigma_{C_2}^{\{1,2\}} \).

Given \( m \in \hat{M} \), define \( b_m \in C_2 \) as

\[
b_m = q^{\frac{1}{2}(m_1(m_2-m_0)+m_2(m_1-m_0)-(m_1+m_2)(m_0+m_2))} z_1^{m_1} z_2^{m_2} z_{12}^{m_{12}} z_{21}^{m_{21}} v_1^{m_0} v_2^{m_0}
\]

(7.4)

if \( m \in M \) and \( b_m = 0 \) if \( m \in \hat{M} \setminus M \). Thus, \( |b_m| = (m_0 + m_1 + m_{21}) \omega_1 + (m_0 + m_2 + m_{12}) \omega_2 - (m_1 + m_{12} + m_{21}) \alpha_1 - (m_2 + m_{12} + m_{21}) \alpha_2 = wt_1(m) \omega_1 + wt_2(m) \omega_2, m \in M \).

The following is a consequence of Lemma 7.10, (6.16) and Proposition 6.33

**Lemma 7.12.** The upper global crystal basis \( B \) of \( C_2 \) is \( B = \{ b_m : m \in M \} \). Furthermore, for each \( \lambda = l_1 \omega_1 + l_2 \omega_2 \in P^+ \) we have \( B_\lambda = \{ b_m : m \in M_{l_1, l_2} \} \). Moreover, under the identification of \( M \) with the upper crystal basis of \( C_4(\mathfrak{g}) \) (cf. §7.2), the map \( M \rightarrow B \) defined by \( m \mapsto b_m, m \in M \) is Kashiwara’s bijection \( G \) ([26]) between an upper crystal basis of \( C_2 \) and its upper global crystal basis.

The following is an explicit form of Theorem 6.30 for \( g = \mathfrak{sl}_3 \) and follows from (7.4) and Corollary 7.11.

**Lemma 7.13.** We have \( \sigma(b_m) = b_{\sigma(m)} \) for all \( m \in M \).
Remark 7.14. It is easy to check that
\[ |B_\lambda(0)| = |\{ m \in M_{l_1,t_2} : w_1(m) = w_2(m) = 0 \}| = \begin{cases} \min(l_1, l_2) + 1, & l_1 \equiv l_2 \pmod{3}, \\ 0, & \text{otherwise}. \end{cases} \]

It follows from the definition of \( \sigma \) that \( \sigma \) is trivial on \( B_\lambda(0) \) if and only if \( \dim V_\lambda(0) = 1 \) (that is, if and only if \( \min(l_1, l_2) = 0 \) and \( \max(l_1, l_2) \in 3\mathbb{Z}_{>0} \)). Thus, \( \tau_{1,3} \notin K_{sl_3} \) in the notation introduced after Problem 1.7. On the other hand, it is immediate from the definitions that the \( \sigma^i, i \in \{1, 2\} \) act trivially on \( V_\lambda(0) \) for any \( V \in \mathcal{O}_{\text{int}}(g) \). In particular, \( \sigma \) is not contained in \( W(V_\lambda) \) if \( \dim V_\lambda(0) > 1 \).

In order to calculate \( \sigma^i, i \in \{1, 2\} \) be need the following result.

Lemma 7.15. For any \( m \in M, r \geq 0 \) and \( i \in \{1, 2\} \) we have
\[
E_i^{(r)}(b_m) = \left( \begin{array}{c} m_i + m_{ij} \\ r \\
\end{array} \right)_q b_{e_i^r(m)} + \sum_{1 \leq t \leq r} C_i^{(r)}(m_j + m_{ij}, m_i + m_{ij}) b_{e_i^{-r}(m) + t a_i^+}
\]
\[
F_i^{(r)}(b_m) = \left( \begin{array}{c} m_j + m_{0i} \\ r \\
\end{array} \right)_q b_{e_i^{-r}(m)} + \sum_{1 \leq t \leq r} C_i^{(r)}(m_i + m_{0i}, m_j + m_{0i}) b_{e_i^{-r}(m) - t a_i^+}
\]
where \( a_1^+ = (0, 0, -1, 1, 0), a_2^+ = -a_1^+ \) and
\[
C_i^{(r)}(c, d) = \begin{cases} \left( \begin{array}{c} c \\ t \end{array} \right)_q \left( \begin{array}{c} d - t \\ r - t \end{array} \right)_q, & d - c \geq r, \\
\left( \begin{array}{c} d - c \\ t \end{array} \right)_q \left( \begin{array}{c} d - t \\ r \end{array} \right)_q, & d - c < r, 
\end{cases}
\]
with the convention that \( \left( \begin{array}{c} k \\ t \end{array} \right)_q = 0 \) if \( k < l \).

Proof. Both identities can be proven by induction on \( r \) using Lemma 7.10 and the fact that the \( E_i, F_i \) act on \( \mathcal{C}_2 \) by \( K_{\frac{1}{2}a_i} \)-derivations. \( \square \)

Denote
\[ b_{m}^{(i)} = E_i^{(r_i)}(b_{e_i^{-r_i}(m)}), \quad r_i = m_j + m_{0i}, i \in \{1, 2\}, m \in M. \]

The following is immediate from Lemma 7.15.

Lemma 7.16. For each \( i \in \{1, 2\}, B^{(i)} = \{ b_{m}^{(i)} : m \in M \} \) is a basis of \( \mathcal{C}_2 \). Moreover, for each \( i \in \{1, 2\}, \lambda = l_1 \omega_1 + l_2 \omega_2 \in P^+, B_{\lambda}^{(i)} := \{ b_{m}^{(i)} : m \in M_{l_1,t_2} \} \) is a basis of \( V_\lambda \).

Finally,
\[ \sigma^i(b_{m}^{(i)}) = b_{e^i(m)}^{(i)}, \quad i \in \{1, 2\}, m \in M. \]

In particular, \( \sigma^i(B_{\lambda}^{(i)}) = B_{\lambda}^{(i)} \).
Thus, $\sigma_i^i$, $i \in \{1,2\}$ are easy to calculate in respective bases $B^{(i)}$. To attack Conjecture 1.2 we need to find the matrix of both of them in a same basis. Note the following consequence of Lemma 7.15.

**Corollary 7.17.** For each $\lambda = l_1\omega_1 + l_2\omega_2 \in P^+$ we have

$$b^{(i)}_m = \sum_{m' \in M_{l_1,l_2}} C^{i,\lambda}_{m',m} b_{m'}$$

where $C^{i,\lambda}$ is an $M_{l_1,l_2} \times M_{l_1,l_2}$-matrix given by

$$C^{i,\lambda}_{m',m} = \begin{cases} 
\binom{m_i + m_0i + m_j + m_{ij}}{m_i + m_{ij}}, & m' = m, \\
C_t^{(m_j + m_0)}(m_j + m_{ij}, m_i + m_0i + m_j + m_{ij}), & m' - m = ta_1^+, t \in \mathbb{Z}_{>0}, \\
0, & \text{otherwise}.
\end{cases}$$

**Remark 7.18.** The bases $B^{(i)}_\lambda$, $i \in \{1,2\}$, $\lambda \in P^+$ are in fact Gelfand-Tsetlin bases. The matrices $C^{i,\lambda}$ appeared first in the classical limit ($q = 1$) in [18]. According to [18, Theorem 10] their entries are closely related to Clebsch-Gordan coefficients, and so one should expect that our matrices are related to quantum Clebsch-Gordan coefficients. It is easy to see that $\sigma^i(B_\lambda) = B_\lambda$ if and only if $V_\lambda$ is thin.

**Theorem 7.19.** For each $\lambda = l_1\omega_1 + l_2\omega_2 \in P^+$ and $i \in \{1,2\}$ the matrix $N^{i,\lambda}$ of $\sigma^i$ with respect to the basis $B_\lambda$ of $V_\lambda$ is given by

$$N^{i,\lambda} = C^{i,\lambda} P^{i,\lambda}(C^{i,\lambda})^{-1}$$

where $P^{i,\lambda} = (P^{i,\lambda}_{m',m})_{m,m' \in M_{l_1,l_2}}$ with

$$P^{i,\lambda}_{m',m} = \delta_{m',\lambda(m)}, \quad m, m' \in M_{l_1,l_2}.$$

**Conjecture 7.20** (Conjecture 1.2 for $g = sl_3$). For each $\lambda = l_1\omega_1 + l_2\omega_2 \in P^+$ we have

$$(N^{1,\lambda} A^{2,\lambda})^3 = 1.$$  

This was verified using Mathematica® for all $l_1, l_2 \in \mathbb{Z}_{\geq 0}$ such that $l_1 + l_2 \leq 14$.

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