The pion-nucleon amplitude is considered in the vicinity of the elastic scattering threshold within a relativistic dynamical model dressing the $\pi NN$ and $\pi N\Delta$ vertices self-consistently with an infinite number of meson loops. The dressing is formulated as solution of a system of coupled integral equations incorporating unitarity, crossing symmetry and analyticity constraints. The calculated scattering lengths and the sigma-term agree with recent data analyses. The dressing is important in this model both below and at threshold. The contribution of the $\Delta$ resonance is discussed, including effects of the consistent dressing of the $\pi N\Delta$ vertex. A comparison with the approaches of chiral perturbation theory and the Bethe-Salpeter equation is outlined.
I. INTRODUCTION

The pion-nucleon scattering amplitude near the physical threshold is an interesting object to study for a number of reasons. At the threshold point itself, the amplitude is proportional to the s-wave scattering lengths, whose values are known to be strongly constrained by chiral symmetry [1]. For the amplitude below threshold, one can establish other chiral low-energy theorems [2,3] involving such quantities as the nucleon sigma-term [4] and thus related to the pattern of the explicit chiral symmetry breaking of QCD [5]. To extract the sigma-term from scattering data one usually analyses the amplitude at the Cheng-Dashen point. Although corresponding to unphysical kinematics, this point is of special importance because both pions are on-shell there and the difference between the amplitude and the sigma-term is minimal [3,6,7].

Chiral perturbation theory has been used to study loop corrections to the low-energy theorems and, in particular, to calculate the sigma-term [8]. However, the near-threshold region is significantly affected by the presence of singularities which may make a non-relativistic perturbative expansion unreliable [9]. In general, the pion-nucleon amplitude is ill-defined (and hence non-analytic) at the Cheng-Dashen point and threshold is a branch point dictated by unitarity. To obey unitarity for pion-nucleon scattering exactly, one can solve the relativistic Bethe-Salpeter equation with a tree-level potential; this yields a good description of the phase shifts and scattering lengths as well as allows one to calculate the sigma-term [10]. At the same time, the models based on the Bethe-Salpeter equation usually do not preserve crossing symmetry (however, see [11]) which plays an important role in the derivation of the low-energy theorems [3].

In this paper the pion-nucleon amplitude is studied in the near-threshold region using a relativistic dynamical model which incorporates essential constraints from unitarity, analyticity and crossing symmetry. The effective lagrangian of the model includes pions, nucleons, the Δ resonance, the ρ and σ mesons. The main distinguishing feature of this approach is a special method of calculating pion-nucleon and other meson-baryon loop corrections to free propagators and bare vertices. An infinite series of loops is summed up by solving a system of coupled integral equations for the dressed vertices and propagators. These equations are formulated so that constraints from unitarity, crossing symmetry and analyticity are fulfilled. This formalism was expounded in Refs. [12], with a simplified treatment of the Δ resonance. In the present paper, not only the πNN vertex and the nucleon self-energy, but also the πNΔ vertex and the Δ self-energy are dressed up to infinite order. The dressing procedure of [12] was extended in [13] to include photons while preserving gauge-invariance. Unitarity of the coupled-channel S-matrix above threshold was ensured since the dressing is consistent with the application of the dressed vertices and propagators in the K-matrix approach. In this way a good description of intermediate-energy pion-nucleon scattering, pion photoproduction and Compton scattering was obtained, and the nucleon electromagnetic polarisabilities were evaluated and found to be in agreement with experiment [14,15]. Since all parameters of the model (resonance coupling constants and a regularising cutoff) were fixed at the intermediate energies in Ref. [14], the present calculation of the pion-nucleon amplitude below and at threshold is determined solely by the loop dynamics.

The outline of the paper is as follows. The pion-nucleon amplitude at threshold and at the Cheng-Dashen point is defined in Section I, where also relevant low-energy theorems are
cited. The formalism of the model is described in Section III, including the integral equations for the dressed vertices and propagators and the method of their solution. In Section IV the invariant pion-nucleon amplitudes in the near-threshold region are explicitly expressed via the dressed functions. The main results of the paper will be presented in Section V. In particular, we will examine effects of multiple meson loops, of the Δ resonance and of the σ and ρ mesons, both below and at threshold. The role of the Δ will be discussed further, including the effects of the dressing of the πNΔ vertex. We will argue that the analyticity constraints incorporated in the dressing procedure are essential for the description of both the scattering lengths and the subthreshold coefficients in the same dynamical approach. Our formalism and results will be compared with the approaches of chiral perturbations theory and the Bethe-Salpeter equation in Section VI. Concluding remarks are made in Section VII.

II. PION-NUCLEON AMPLITUDE NEAR THRESHOLD

The standard isospin decomposition of the pion-nucleon amplitude is \[ M_{\alpha\beta} = \delta_{\alpha\beta} M^+ + \frac{1}{2}[\tau_\alpha, \tau_\beta] M^-, \]

where \( \tau_\alpha \) are Pauli matrices for the pion isospins. The spin structure of the amplitude is \[ M^\pm = \pi(p') \left\{ A^\pm + \frac{k' + k}{2} B^\pm \right\} u(p) = \pi(p') \left\{ D^\pm - \frac{1}{4m}[k', k] B^\pm \right\} u(p), \]

where \( k \) and \( k' \) (\( p \) and \( p' \)) are the four-momenta of the initial and final pions (nucleons), respectively, \( u(p) \) is the Dirac four-spinor. The invariant amplitudes \( A^\pm, B^\pm, D^\pm \) depend on the Mandelstam variables \( s = (p + k)^2, u = (p - k')^2 \) and \( t = (k - k')^2 \). We will also use the standard kinematic variables \( \nu = (s - u)/(4m) \) and \( \nu_B = (t - 2\mu^2)/(4m) \), where \( m = 0.939 \text{ GeV} \) and \( \mu = 0.138 \text{ GeV} \) are the nucleon and pion masses. The two sets of invariant functions in Eq. (2) are related as \( D = A + \nu B \).

In the following we shall calculate the value of the pion-nucleon amplitude at the subthreshold Cheng-Dashen point, i.e. at \( \nu = 0, t = 2\mu^2 \), with both pions on shell, \( k^2 = k'^2 = \mu^2 \). One quantity of special interest is the sigma-term, which is related to the pion-nucleon amplitude at the Cheng-Dashen point as

\[ \Sigma = F^2_\pi \lim_{\nu \to 0} \overline{B}^+(\nu, t = 2\mu^2), \]

where \( F_\pi = 92.4 \text{ MeV} \) is the pion decay constant. The bar indicates that the tree-level amplitude evaluated with the pseudovector πNN vertex (usually called the “pseudovector Born contribution”) is subtracted,

\[ \overline{B}^+ = B^+ - \frac{g^2}{m \nu_B^2 - \nu^2} \nu, \]
\[ \overline{B}^- = B^- - \frac{g^2}{m} \left( \frac{\nu_B}{\nu_B^2 - \nu^2} - \frac{1}{2m} \right), \]
\[ D^+ = D^+ - \frac{g^2}{m} \frac{\nu_B^2}{\nu_B^2 - \nu^2}, \]  
\[ D^- = D^- - \frac{g^2}{m} \left( \frac{\nu \nu_B}{\nu_B^2 - \nu^2} - \frac{\nu}{2m} \right). \]

We take the value \( g = 13.02 \) [17] for the \( \pi N N \) coupling constant. The physical masses and coupling constants of the particles used in this calculation are summarised in Table I.

According to a chiral low-energy theorem [3], the sigma-term Eq. (3) at the Cheng-Dashen point equals the scalar form factor of the nucleon up to corrections of order \( \mathcal{O}(\mu^4) \). The scalar form factor can be related to the explicit chiral symmetry breaking (see, e.g., Refs. [8,9]).

Another low-energy theorem [2,3] concerns the amplitude \( D^- \), requiring that the coefficient

\[ C = 2F_\pi^2 \lim_{\nu \to 0} \frac{D^-(\nu, t = 2\mu^2)}{\nu} \]

should approach unity in the chiral limit (i.e. for a vanishing pion mass), up to corrections of order \( \mathcal{O}(\mu^2) \).

In addition to the pion-nucleon amplitude in the subthreshold region, we shall also calculate the s-wave scattering lengths which characterise the amplitude at the threshold point \( T_h \equiv \{ s = (m + \mu)^2, u = (m - \mu)^2, t = 0 \} \):

\[ a^{1/2} = \left. \frac{D^+ + 2D^-}{4\pi(1 + \mu/m)} \right|_{T_h}, \]

\[ a^{3/2} = \left. \frac{D^+ - D^-}{4\pi(1 + \mu/m)} \right|_{T_h}, \]

corresponding to the total isospins 1/2 and 3/2, respectively. The low-energy theorem [3] asserts that at lowest order the numerators in Eqs. (9) and (10) equal \( \mu/(F_\pi^2) \) and \( -\mu/(2F_\pi^2) \), respectively.

### III. DESCRIPTION OF THE MODEL

Our calculation of the near-threshold pion-nucleon amplitude is based on the approach of Refs. [12,13]. However, the treatment of the \( \Delta \) resonance is significantly improved in the present version of the model, as will be explained in more detail below. In this section we describe our approach, focusing on the ingredients which are most relevant in the near-threshold region.

#### Structure of the amplitude

The \( \pi N \) amplitude below and at threshold is purely real. In this model it is constructed as the sum of the s-, u-channel nucleon and \( \Delta \) exchange graphs, plus the t-channel \( \rho \) and \( \sigma \) meson exchange graphs,
\[ M = M_s + M_u + M_s^A + M_u^A + M^{\rho \sigma}, \] (11)

as shown in Fig. 1. These are not simple tree diagrams, but rather skeleton diagrams as they comprise *dressed* vertices and propagators. Being a solution of a system of coupled integral equations, the nucleon and \( \Delta \) propagators and vertices are dressed with meson loops up to infinite order, while the \( \rho \) and \( \sigma \) propagators are calculated in a one \( \pi \pi \) loop approximation. Thus the central element of the approach is the calculation of the dressed vertices and propagators, which will be described in the following.

**Structure of the dressed vertices and propagators of the nucleon and \( \Delta \)**

The \( \pi N N \) vertex required throughout the dressing procedure has only one of the nucleons off the mass shell with the other nucleon and the pion being on-shell (the so-called half-off-shell vertex). For an incoming off-shell nucleon with the four-momentum squared \( p^2 \), the most general Lorentz- and CPT-covariant structure of such a vertex is [19]

\[ \tau_\alpha \Gamma(p) = \tau_\alpha \gamma^5 \left[ G_{PS}(p^2) + \frac{\not{p} + m}{2m} G_{PV}(p^2) \right], \] (12)

where \( G_{PS,PV}(p^2) \) are pseudovector and pseudoscalar form factors, to be computed below. The Lorentz-invariant expression for the nucleon self-energy is written in terms of two invariant functions \( A(p^2) \) and \( B(p^2) \):

\[ \Sigma(p) = \Sigma_L(p) - (Z_2 - 1)(\not{p} - m) - Z_2 \delta m, \quad \Sigma_L(p) = A(p^2)\not{p} + B(p^2)m, \] (13)

where \( \Sigma_L \) denotes the loop contributions to the self-energy. The complete self-energy \( \Sigma(p) \) contains also the counter-term contribution with renormalisation constants \( Z_2 \) and \( \delta m \) adjusted to provide the correct pole properties Eq. (32) of the dressed nucleon propagator [21]

\[ S(p) = \frac{1}{\not{p} - m - \Sigma(p)} = \frac{\not{p} + \xi(p^2)}{\alpha(p^2)[p^2 - \xi^2(p^2)]}, \] (14)

where for later use we have introduced the self-energy functions

\[ \alpha(p^2) = Z_2 - A(p^2), \quad \xi(p^2) = \frac{mB(p^2) + Z_2(m - \delta m)}{\alpha(p^2)}. \] (15)

The nucleon self-energy will be computed consistently with the \( \pi N N \) vertex.

We choose the following form of the \( \pi N \Delta \) vertex:

\[ T_\alpha V_\mu(k, p) = T_\alpha \frac{\not{k} - (p \cdot k)^\gamma_\mu}{m_\Delta^2} F_{\pi N \Delta}((p - k)^2) G_\Delta(p^2), \] (16)

\[ ^1 \text{We use the fully relativistic formalism, with the metric tensor, } \gamma \text{ matrices and other general conventions of [21].} \]
where \( p \) and \( k \) are the 4-momenta of an incoming \( \Delta \) and of an outgoing pion, respectively, and \( m_\Delta \) is the mass of the \( \Delta \). The part \( G_\Delta(p^2) \) of the form factor in Eq. (16) will be calculated in the dressing procedure. The real function \( F_{\pi N\Delta}( (p - k)^2) \) depending on the nucleon momentum is needed for convergence of the procedure. The isospin transition operators \( T_\alpha \) are defined by the relations [22]

\[
T_\alpha T_\beta^\dagger = \delta_{\alpha\beta} - \frac{\tau_\alpha \tau_\beta}{3}, \quad T_\alpha^\dagger T_\alpha = 1. \tag{17}
\]

To keep the calculations tractable, the \( \pi N \Delta \) vertex in Eq. (16) is not chosen in the most general Lorentz-covariant form (for comparison, throughout the calculations we maintain the most general structure of the \( \pi NN \) vertex). It is important however that the vertex Eq. (16) has the property of “gauge-invariance” [23],

\[
p \cdot V(k, p) = 0, \tag{18}
\]

which allows us to eliminate the background spin \( 1/2 \) component of the \( \Delta \) propagator [24] and to keep only the spin \( 3/2 \) component

\[
S^{\mu\nu}_\Delta(p) = \frac{1}{p - m_\Delta - \Sigma_\Delta(p)} P_{3/2}^{\mu\nu}(p) = \frac{\dot{p} + \omega(p^2)}{\eta(p^2)[p^2 - \omega^2(p^2)]} P_{3/2}^{\mu\nu}(p), \tag{19}
\]

where the spin \( 3/2 \) projection operator

\[
P_{3/2}^{\mu\nu}(p) = g^{\mu\nu} - \frac{\gamma^{\mu} \gamma^{\nu}}{3} - \frac{\dot{p} \gamma^{\mu} p^{\nu} + p^\mu \gamma^{\nu} \dot{p}}{3p^2}. \tag{20}
\]

Formulae completely analogous to Eqs. (13,15) hold for the \( \Delta \) self-energy. Although treating the \( \Delta \) as a pure spin \( 3/2 \) state does not improve the description of pion-nucleon scattering phase shifts as compared to the conventional treatment [10], it significantly simplifies the dynamical calculation of the \( \Delta \) self-energy: we need to compute only two self-energy functions \( A_\Delta(p^2) \) and \( B_\Delta(p^2) \), instead of 10 invariant functions [25] which would be required if the spin 1/2 background were not eliminated.

### Integral equations for the dressing and their solution

The \( \pi NN \) form factors \( G_{PV,PS}(p^2) \), the nucleon self-energy functions \( A(p^2), B(p^2) \), the \( \pi N \Delta \) form factor \( G_\Delta(p^2) \) and the \( \Delta \) self-energy functions \( A_\Delta(p^2), B_\Delta(p^2) \) are calculated by solving a system of coupled integral equations. This amounts to dressing these two- and three-point Green’s functions with meson loops up to infinite order. In the earlier version of the model [12,14] the \( \Delta \) resonance was not treated completely consistently with the nucleon: the \( \Delta \) self-energy was computed up to one \( \pi N \) loop only and the dressing of the \( \pi N \Delta \) vertex was not included. However, considering nucleon Compton scattering, we showed [15] that such simplified \( \Delta \) dressing, while being generally adequate, can lead to problems at low energies. Therefore, in the present work we refine the dressing procedure so that the nucleon and \( \Delta \) are now treated on the same footing.

The dressing equations will be formulated using the following notation. A generic Green’s function \( G(q) \) is a sum of independent Lorentz-structures (e.g. 1, \( \dot{q} \), \( \gamma_\mu \), etc.), each of which
is multiplied with a Lorentz-invariant function depending on $q^2$ (such as form factors or self-energy functions). If we use only imaginary or only real parts of the invariant functions, the result will be denoted as $\mathcal{G}_I(q)$ or $\mathcal{G}_R(q)$, respectively. If $\mathcal{G}(q)$ is calculated from a loop integral, then according to Cutkosky rules [24] $\mathcal{G}_I(q)$ is proportional to the discontinuity of the integral across the unitary cut in the complex $q^2$ plane (due to pinching poles of the propagators in the integrand) and $\mathcal{G}_R(q)$ is the principal-value part of the integral.$^2$

In our case, the principal-value and pole parts of the dressed $\pi NN$ vertex are denoted as $\Gamma_R(p)$ and $\Gamma_I(p)$, respectively. The expression for $\Gamma_R(p)$ or $\Gamma_I(p)$ is obtained by using, respectively, only the real or only the imaginary parts of the form factors $G_{PV,PS}(p^2)$ in the right-hand side of Eq. (12). The same applies to the $\pi N\Delta$ vertex Eq. (13): to obtain $(V_\mu)_R(k,p)$ or $(V_\mu)_I(k,p)$ we use only Re$G_\Delta(p^2)$ or Im$G_\Delta(p^2)$, respectively. Similarly, the pole part $\Sigma_I(p)$ of the nucleon self-energy Eq. (13) contains only Im$A(p^2)$ and Im$B(p^2)$, and the principal-value part $\Sigma_R(p)$ only Re$A(p^2)$ and Re$B(p^2)$. The pion propagator $D(k) = [k^2 - \mu^2 + i0]^{-1}$ does not get dressed, therefore its imaginary part comes from the on-shell pions: $D_I(k) = \delta(k^2 - \mu^2)\theta(k_0)$. In the same way, we retain only the dominant pole contribution to the discontinuity of the nucleon propagator: $S_I(p) = (p + m)\delta(p^2 - m^2)\theta(p_0)$. The resonance propagators do not have poles on the physical Riemann sheet, so their discontinuous parts come solely from their self-energies. For example, the discontinuity of the dressed $\Delta$ propagator Eq. (19) is obtained by keeping only the imaginary parts of its invariant functions:

$$\langle S^\mu_\nu\rangle_I(p) = \left\{ \frac{\Im}{\eta(p^2)} \left[ \frac{1}{p^2 - \omega^2(p)} \right] + \frac{\Im}{\eta(p^2)} \left[ \frac{\omega(p)}{p^2 - \omega^2(p)} \right] \right\} \theta(p_0),$$

and analogously for the $\rho$ and $\sigma$ mesons (unlike the $\Delta$, however, the propagators of the meson resonances are dressed in a one $\pi\pi$ loop approximation only, as will be discussed in more detail below and in Appendix A).

With the introduced notation, the system of dressing equations can be written

$$\Gamma_I(p) = \frac{1}{8\pi^2} \int d^4k \Gamma_R(p' - k) S(p' - k) \Gamma_R(p' - k) S_I(p - k) D_I(k) \Gamma_R(p)$$

$$+ \frac{1}{6\pi^2} \int d^4k (V_\mu)_R(-k,p' - k) S^\mu_\nu(p' - k) (V_\nu)_R(-q,p' - k)$$

$$\times S_I(p - k) D_I(k) \Gamma_R(p)$$

$$- \frac{1}{6\pi^2} \int d^4k \Gamma_R(p' - k) S(p' - k) (V_\mu)_R(q - p - k) (S^\mu_\nu)_I(p - k)$$

$$\times D_I(k) (V_\nu)_R(k,p - k) + \Gamma_I^{a\sigma}(p) + \Gamma_I^{a\sigma}(p),$$

$$\text{Re} \left\{ \begin{array}{l} G_{PV} \\ G_{PS} \end{array} \right\}(p^2) = \left\{ \begin{array}{l} G_{PV}^0 \\ G_{PS}^0 \end{array} \right\}(p^2) + \frac{\mathcal{P}}{\pi} \int_{(m + \mu)^2}^\infty dp^2 \frac{\Im \left\{ \begin{array}{l} G_{PV} \\ G_{PS} \end{array} \right\}(p^2)}{p^2 - p^2},$$

$^2$If the theory obeys analyticity (causality) constraints, the pole and principal-value parts of a loop must be related to each other through a dispersion integral [27].

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\[
\Sigma_I(p) = -\frac{3}{8\pi^2} \mathbf{r}(p) \int d^4k \, S_I(p-k) D_I(k) \Gamma_R(p), \tag{24}
\]

\[
\text{Re}\left\{ \frac{A}{B} \right\}(p^2) = \frac{\mathcal{P}}{\pi} \int_{(m+\mu)^2}^{\infty} dp^2 \frac{\text{Im}\left\{ \frac{A}{B} \right\}(p^2)}{p^2 - p'^2}, \tag{25}
\]

\[
(V_\mu)_I(q,p) = \frac{1}{4\pi^2} \int d^4k \, \Gamma_R(p' - k) S(p' - k) \Gamma_R(p - k) \times D_I(k)(V_\mu)_R(k,p) \\
+ \frac{1}{24\pi^2} \int d^4k \, (V_\nu)_R(-k,p' - k) S_\Delta^\nu\lambda(p' - k)(\overline{\nu}_\lambda)_R(-q,p' - k) \\
\times S_I(p-k) D_I(k)(V_\mu)_R(k,p) + (V_\mu^{\rho\sigma})_I(q,p), \tag{26}
\]

\[
\text{Re}G_\Delta(p^2) = G_\Delta^0(p^2) + \frac{\mathcal{P}}{\pi} \int_{(m+\mu)^2}^{\infty} dp^2 \frac{\text{Im}G_\Delta(p^2)}{p^2 - p'^2}, \tag{27}
\]

\[
\Sigma_I^\Delta(p) = \frac{\mathcal{P}_{3/2}(p)}{16\pi^2} \int d^4k \, (\overline{\nu}_\lambda)_R(k,p) S_I(p-k) D_I(k)(\nu_\lambda)_R(k,p), \tag{28}
\]

\[
\text{Re}\left\{ \frac{A_\Delta}{B_\Delta} \right\}(p^2) = \frac{\mathcal{P}}{\pi} \int_{(m+\mu)^2}^{\infty} dp^2 \frac{\text{Im}\left\{ \frac{A_\Delta}{B_\Delta} \right\}(p^2)}{p^2 - p'^2}, \tag{29}
\]

where \(\mathcal{P}\) denotes taking the principal-value of an integral, and isospin factors have been absorbed in the coefficients on the right-hand side. The inhomogeneities \(\Gamma^{\rho\sigma}_I(p)\) in Eq. (22) and \((V_\mu^{\rho\sigma})_I(q,p)\) in Eq. (26) contain \(\rho\) and \(\sigma\) mesons, as described in Appendix A. The real functions \(G_{PV,PS}^0(p^2)\) and \(G_\Delta^0\) are \(\pi NN\) and \(\pi N\Delta\) form factors, respectively. To see that the system of dressing equations is neither under- nor over-determined, note that Eqs. (22-24) and (28) have two independent spinor structures each. Thus Eqs. (22-24) are 14 scalar equations for 14 scalar functions \(\text{Im}G_{PV}^0(p^2), \text{Im}G_{PS}^0(p^2), \text{Im}A(p^2), \text{Im}B(p^2), \text{Im}G_\Delta(p^2), \text{Im}A_\Delta(p^2), \text{Im}B_\Delta(p^2), \text{Re}G_{PV}^0(p^2), \text{Re}G_{PS}^0(p^2), \text{Re}A(p^2), \text{Re}B(p^2), \text{Re}G_\Delta(p^2), \text{Re}A_\Delta(p^2), \text{Re}B_\Delta(p^2)\).

Formally, Eqs. (22-24) constitute a coupled system of nonlinear integral equations. Despite its quite complicated analytic form, this system of equations has a rather transparent meaning (see Fig. 3). The equations are solved by iteration, starting with input bare form factors \(G_{PV,PS}^0, G_\Delta^0\). In the course of iteration one effectively sums up an infinite series of meson-loop corrections to the bare vertices and free propagators. At each iteration step we first calculate the discontinuities of the loop integrals through the Cutkosky rules; these pole parts are then used in dispersion integrals to compute the corresponding principal-value parts. The details of the computation technique can be found in [12,13]. Here we will recapitulate only the most important points and discuss the new issues arising due to the consistent incorporation of the \(\Delta\) resonance in the dressing procedure.

The use of bare form factors \(G_{PV,PS}^0(p^2)\) and \(G_\Delta^0\) is necessary to regularise the equations.\(^3\)

\(^3\)An attempt to get rid of the bare form factors by using subtracted dispersion integrals in
We choose the purely pseudovector structure for the bare $\pi NN$ vertex, i.e. $G^0_{PV}(p^2) = 0$, since the derivative coupling of pions at low energies is dictated by chiral symmetry. The bare $\pi NN$ form factor is chosen in the form

$$G^0_{PV}(p^2) = f \exp \left[ -\ln \frac{(p^2 - m^2)^2}{\Lambda_N^4} \right], \quad (30)$$

where $\Lambda_N^4$ is a half-width. The bare coupling constant $f \equiv f_{\pi NN}$ is adjusted so that the dressed $\pi NN$ vertex is normalised on-shell to the physical $\pi NN$ coupling constant:

$$\lim_{\not{p} \to m} \Gamma(p) = g \gamma^5, \quad (31)$$

where the “sandwich” between the spinors of the initial and final nucleons is implicit. The usual renormalisation \cite{21} of the dressed nucleon propagator

$$\lim_{\not{p} \to m} S(p) = \frac{1}{\not{p} - m} \quad (32)$$

is imposed by adjusting the field renormalisation constant $Z_2 \equiv Z_2^N$ and the mass shift $\delta m \equiv \delta m_N$ (see Eqs. (13) and (14)). The renormalisation of the $\pi N\Delta$ vertex and of the $\Delta$ propagator is done similarly, except that now the pole properties are required of a propagator with only the real parts of the self-energy functions \cite{26,12}. There are three corresponding renormalisation constants: $f_{\pi N\Delta}$, $Z_2^\Delta$ and $\delta m_\Delta$. Note that due to the coupled nature of the dressing Eqs. (22–29) the six renormalisation conditions for the vertices and propagators can be obeyed only simultaneously. Thus $f_{\pi NN}$, $Z_2^N$, $\delta m_N$, and $f_{\pi N\Delta}$, $Z_2^\Delta$, $\delta m_\Delta$ are inter-dependent. The complete set of renormalisation constants obtained in the calculation are given in Table II.

We stress that the half-width $\Lambda_N^2$ is not a completely independent parameter: the iteration procedure converges only for $\Lambda_N^2 < (\Lambda_N^{max})^2$, and it is important that $(\Lambda_N^{max})^2$ is much larger than the energy scale due to the explicitly included particles. With the set of parameters used in this calculation, $(\Lambda_N^{max})^2 \approx 3 \text{ GeV}^2$. When a convergent solution does exist, it is reached in practice after about 30 iteration steps for $\Lambda_N^2 \approx (\Lambda_N^{max})^2$. The bare $\pi N\Delta$ form factor $G^0_{\pi\Delta}(p_\Delta^2)$ as well as the form factors $F_{\pi N\Delta}(p_N^2)$, $F_{\rho\pi\pi}(p_\rho^2)$, $F_{\sigma\pi\pi}(p_\sigma^2)$, $F_{\rho NN}(p_N^2)$ and $F_{\sigma NN}(p_N^2)$, appearing in the vertices with the resonances, have the same exponential form as Eq. (30), but peak at the masses of the corresponding particles (see Appendix A).

We assume the width $\Lambda_N^2$ of these bare form factors to be the same for all resonances, and set it close to the maximal value allowed by the convergence requirement. This is done in keeping with the general emphasis of our approach on the loop dynamics as determined by the dressing rather than on fitting additional parameters. Ideally, values of $\Lambda_N^2$ for different meson resonances should come from a dynamical dressing of these mesons on the same footing with the nucleon and the $\Delta$. Such an extension of the model is certainly feasible, although has not been done yet.

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Eqs. (22,25,27,29) fails since each new iteration step would require more subtractions than the previous.
The effects of the loop corrections on the $\pi NN$ vertex are similar to those discussed in detail in [12,14], where the $\Delta$ was not dressed consistently. We mention the main points here. The dressing generates an energy-dependent admixture of the pseudoscalar coupling, which at low energies remains much smaller than the pseudovector component and becomes more prominent only at intermediate energies. It is important that the dressing does not allow for a large pseudoscalar admixture to develop in the low-energy region. The pseudovector form factor in narrowed by the dressing. This softening persists independently of the functional form of the bare form factor (provided the latter falls sufficiently fast at infinity) and is stronger for wider bare form factors.

Like the half-widths $\Lambda^2_N$ and $\Lambda^2_R$, the coupling constants $g_{\rho NN}$, $\kappa_{\rho}$, $g_{\sigma NN}$, $g_{\sigma\pi\pi}$ and $f_{\sigma\pi\pi}$ of the $\rho$ and $\sigma$ mesons are mutually constrained by the requirement that a convergent solution of the dressing equations should exist. Fig. 3 shows the area of convergence in the space of the coupling constants $g_{\sigma NN}$ and $g_{\rho NN}$, the other parameters being fixed at their values given in Table II. The convergence area has a nontrivial shape for small $g_{\sigma NN}$ and large $g_{\rho NN}$ (or for small $g_{\rho NN}$ and large $g_{\sigma NN}$). For these values there is less cancellation between loops with $\sigma$'s and those with $\rho$'s, which precludes convergence. Generalising the illustration in Fig. 3, convergent solutions of Eqs. (22–29) can be found only in a certain subspace of the space spanned by $\Lambda^2_N$, $\Lambda^2_R$, $g_{\rho NN}$, $\kappa_{\rho}$, $g_{\sigma NN}$, $g_{\sigma\pi\pi}$ and $f_{\sigma\pi\pi}$. The remaining freedom in the parameters was removed by calculating the pion-nucleon phase shifts using the dressed K-matrix approach [14] and comparing them with data analyses [28] at intermediate energies. The dot in Fig. 3 corresponds to the thus optimised coupling constants $g_{\rho NN}$ and $g_{\sigma NN}$. The full set of relevant parameters from [14] are reproduced in Table II. We emphasise that in fixing these values we did not use any results of data analyses for the scattering lengths or for the subthreshold amplitudes. Therefore there are no free parameters in the present calculation of the near-threshold amplitudes.

IV. SCATTERING AMPLITUDE IN TERMS OF THE DRESSED VERTICES AND PROPAGATORS

Having solved the dressing equations, we proceed to evaluate the invariant amplitudes $A^\pm$ and $B^\pm$ defined in Eq. (2). First we write the s- and u-channel diagrams from Eq. (11) in terms of the dressed $\pi NN$, $\pi N\Delta$ vertices Eqs. (12,16) and dressed nucleon, $\Delta$ propagators Eqs. (13,19). Then we add the t-channel diagrams written in terms of the relevant meson vertices and propagators Eqs. (A3,A5,A10,A14,A15,A21). At the considered kinematics, only the real parts of the invariant functions of the two- and three-point Green's functions enter in the amplitude. After doing some straightforward algebra, the contribution of the nucleon, $\rho$ and $\sigma$ exchange diagrams to the invariant amplitudes can be written

$$A^+ = \frac{G^2_{PS}(s)[\xi(s) - m]}{\alpha(s)[s - \xi^2(s)]} + \frac{m}{4m^2} \left[ G^2_{PV}(s)[s - m^2] + \frac{G^2_{PS}(s)G_{PV}(s)}{m}[s - m^2] + \frac{G^2_{PV}(s)}{4m^2}[m + \xi(s)][s - m^2] \right]$$

$$B^+ = \frac{G^2_{PS}(u)[\xi(u) - m]}{\alpha(u)[u - \xi^2(s)]} + \frac{m}{4m^2} \left[ G^2_{PV}(u)[u - m^2] + \frac{G^2_{PS}(u)G_{PV}(u)}{m}[u - m^2] + \frac{G^2_{PV}(u)}{4m^2}[m + \xi(u)][u - m^2] \right]$$
\begin{align}
\frac{g_{\sigma NN} F_{\sigma\pi\pi}(t) \left[ g_{\sigma\pi\mu} - f_{\sigma\pi} \frac{k^2 + k^2 - t}{2\mu} \right]}{Z^s [t - \zeta^2(t)]},
\end{align}

\begin{align}
A^- = & \frac{G_{PS}(s)[\xi(s) - m]}{\alpha(s)[s - \xi^2(s)]} + \frac{G_{PS}(u)G_{PV}(u)}{\alpha(u)[u - \xi^2(u)]} \\
& - \frac{G_{PS}(u)[\xi(u) - m]}{\alpha(u)[u - \xi^2(u)]} + \frac{G_{PS}(u)G_{PV}(u)}{\alpha(u)[u - \xi^2(u)]} \\
& + \frac{g_{\rho NN} g_{\rho\pi\pi} \kappa_p F_{\rho\pi\pi}(t)(u - s)}{2mZ^s [t - \lambda^2(t)]},
\end{align}

\begin{align}
B^+ = & \frac{G_{PS}(s) + G_{PS}(s)G_{PV}(s)}{2m^2 s + 2m^2 + 2m \xi(s)} \\
& + \frac{G_{PS}(u)G_{PV}(u)}{\alpha(u)[u - \xi^2(u)]} \\
& - \frac{G_{PS}(u) + G_{PS}(u)G_{PV}(u)}{\alpha(u)[u - \xi^2(u)]} \\
& + \frac{g_{\rho NN} g_{\rho\pi\pi} (1 + \kappa_p) F_{\rho\pi\pi}(t)}{2m^2 s + 2m^2 + 2m \xi(s)}.
\end{align}

The \( \Delta \) exchange diagrams from Eq. (14) are given by (restoring the isospin indices)

\begin{align}
(M^\Delta_{a\beta})_{\alpha\beta} &= \left( \delta_{a\beta} - \frac{\tau_{a\beta}}{3} \right) \frac{G^2_{\Delta}(s)}{m^4 \eta(s)[s - \omega^2(s)]} \vec{t}(p') \left[ (p' + \vec{k}') k' - (p' + k') \cdot k' \gamma_{\mu} \right] \\
& \times [\vec{p} + \vec{k} + \omega(s)] P^{\mu\nu}_{3/2}(p + k) [(\vec{p} + \vec{k}) k_{\nu} - (p + k) \cdot k \gamma_{\nu}] u(p),
\end{align}

\begin{align}
(M^\Delta_{a\beta})_{\alpha\beta} &= \left( \delta_{a\beta} - \frac{\tau_{a\beta}}{3} \right) \frac{G^2_{\Delta}(u)}{m^4 \eta(u)[u - \omega^2(u)]} \vec{t}(p') \left[ -(p' - \vec{k}) k_{\mu} + (p' - k) \cdot k \gamma_{\mu} \right] \\
& \times [\vec{p} - \vec{k} + \omega(u)] P^{\mu\nu}_{3/2}(p - k) [-(\vec{p} - \vec{k}) k_{\nu} + (p - k) \cdot k \gamma_{\nu}] u(p).
\end{align}

For brevity we will not give the explicit decomposition of Eqs. (37,38) in terms of the invariant amplitudes.

**Properties of analyticity and crossing symmetry**

Due to the use of the cutting rules and dispersion integrals in the formulation of Eqs. (22–29), the two- and three-point Green’s functions obtained by solving these equations possess the correct analyticity structure associated with the nucleon and \( \Delta \) exchanges and obey the
two-body ($\pi N$) unitarity. Furthermore, the $\pi N$ amplitude obeys the crossing symmetry requirements that $A^+, A^-/(s-u), B^+/(s-u)$ and $B^-$ be invariant under the replacement $s \leftrightarrow u$. The crossing is respected due to our using the dressed two- and three-point Green’s functions in both s- and u-type diagrams. At the same time, the t-channel analyticity structure is not fully reproduced because the t-channel cuts are taken into account only through the $\pi\pi$ loops in the $\rho$ and $\sigma$ propagators but the loop corrections to the vertices with the $\rho$ and $\sigma$ mesons are not included. Also, the four-point one-particle irreducible diagrams—such as the box graph—are not included in the dressing.

The omitted dressed one-particle irreducible four-point diagrams can be thought of as being of order $O(a^2)$ in a certain formal expansion, where the parameter $a$ characterises the level of analyticity violation in the model. In this expansion, the lowest order $O(a^0)$ corresponds to an amplitude with no dressing, in which case the violation of analyticity is maximal. The next order $O(a^1)$ corresponds to an amplitude in which the one-particle reducible (with respect to the s-channel cuts) graphs contain the dressed propagators and vertices. Thus, at order $O(a^1)$ analyticity is restored at the level of two- and three-point Green’s functions, as is done in present dressing procedure. The higher orders in $a$ are described by induction in terms of reducibility of n-point Green’s functions constituting the amplitude.

The parameter $a$ can be defined as follows. If a scattering amplitude $T(\omega)$ of a process can be represented at small energies $\omega$ as a power series

$$ T(\omega) = c_0 + c_1 \omega + c_2 \omega^2 + \ldots, $$

then the coefficients $c_i$ could, in principle, be computed using the same model but in two different ways:

1. Low-energy (LE) evaluation: compute $c_i(LE)$ by evaluating $T(\omega)$ directly at low energies;

2. Sum-rule (SR) evaluation: compute $c_i(SR)$ by calculating appropriate total cross sections and integrating them in the sum rules corresponding to each particular coefficient.

These two ways of evaluation should give identical coefficients $c_i(LE) = c_i(SR) = c_i$ provided the analyticity of the model is exact. In practical calculations there will always be discrepancies between the two methods, which can be used to estimate the violation of analyticity in the model. Therefore one can quantify the parameter $a$ as

$$ a \sim |c_i(LE) - c_i(SR)|, $$

where the choice of a particular coefficient from the series Eq. (39) could be decided by additional considerations. This idea was tested for nucleon Compton scattering in Ref. [15], where the coefficients $c_1$ and $c_2, 3$ were related to the anomalous magnetic moment and to nucleon polarisabilities, respectively. The proposed formal expansion in the parameter $a$ can offer a systematic way of improving analyticity properties of dynamical approaches applicable at low and intermediate energies.
The properties of analyticity and crossing symmetry are crucial for the model to provide a good description of the amplitude both below and at the physical threshold. To study this in more detail, we shall focus below on the sigma-term, the Adler-Weisberger coefficient $C$, as defined by Eqs. (3,8), and on the scattering lengths, as defined by Eqs. (9,10). We will collectively call these quantities the “near-threshold coefficients”.

V. NEAR-THRESHOLD COEFFICIENTS

The sigma-term and coefficient $C$ at the Cheng-Dashen point

On expanding the amplitudes in Eqs. (33–38) around the Cheng-Dashen point $s = u = m^2, t = 2\mu^2$ (with $k^2 = k'^2 = \mu^2$) and using the definitions Eqs. (3,8), we obtain explicit formulae for the sigma-term and for the coefficient $C$ in terms of the dressed vertices and propagators:

$$\Sigma = -F^2 \sigma \left\{ \frac{G_{PS}(m^2)}{m \alpha(m^2)} + \frac{g_{\sigma NN} g_{\sigma \pi \pi} \mu F_{\sigma \pi \pi}(2\mu^2)}{Z_{\sigma}^2[2\mu^2 - \zeta^2(2\mu^2)]} \right\} + \Sigma_\Delta, \quad (41)$$

$$C = 2F^2 \left\{ \frac{G_{PS}(m^2)}{\alpha(m^2)} \left[ \frac{g}{m^2} - 4G'_{PS}(m^2) \right] - \frac{G_{PS}(m^2)}{2\alpha^2(m^2)} \left[ \frac{1}{m^2} - 4\alpha'(m^2) \right] \right\} + C_\Delta, \quad (42)$$

where $\Sigma_\Delta$ and $C_\Delta$ contain the effects of the $\Delta$ exchange in the s- and u-channel diagrams (see Appendix B). In Eqs. (41,42) we have made use of the relations

$$\xi(m^2) = m, \quad \xi'(m^2) = \frac{\alpha(m^2) - 1}{2m \alpha(m^2)}, \quad G_{PS}(m^2) + G_{PV}(m^2) = g, \quad (43)$$

which follow from the renormalisation conditions Eqs. (31,32).

Eqs. (41) and (42) show that the subthreshold parameters $\Sigma$ and $C$ are sensitive to the values and derivatives of the dressed $\pi NN$ form factor $G_{PS}(p^2)$ and of the nucleon self-energy function $\alpha(p^2)$ at the nucleon pole. Since the pseudoscalar form factor in the dressed $\pi NN$ vertex is very small in the vicinity of $m^2$, the nucleon contribution in Eqs. (11,12) is much smaller than that of the $\sigma$ and $\rho$ mesons. The $\sigma$ exchange in the t-channel gives a dominant numerical contribution to the sigma-term due to the presence of the non-derivative component $\sim g_{\sigma \pi \pi} \mu$ in the $\sigma \pi \pi$ vertex Eq. (A14). The $\rho$ meson plays a similar role for the coefficient $C$. It should be pointed out, however, that dissection of Eqs. (11,12) into a “nucleon contribution”, “$\sigma$ and $\rho$ meson contributions” and a “$\Delta$ contribution” can only be regarded as formal here: as already mentioned, Eqs. (22,29) not only determine the nucleon and $\Delta$ dressing, but also strongly constrain the other parameters of the model. For example,

4A similar conclusion was reached in the framework of the relativistic baryon chiral perturbation theory [3], where it was pointed out that the standard low-energy expansion does not reproduce the correct analyticity structure in the vicinity of singularities.
an arbitrary change of the values of the $\rho$ and $\sigma$ coupling constants would formally change
the dominant contributions in Eqs. (41,42), but with these altered coupling constants the
dressing procedure might not converge at all!

In what follows we will discuss various ingredients of the dressing by comparing the
“Dressed” and “Bare” calculations. The former contains the full dressing with the meson
loops whereas in the latter the bare vertices and free propagators have been used. Note that
since the $\rho\pi\pi$, $\rho NN$, $\sigma\pi\pi$ and $\sigma NN$ vertices do not get dressed in the model, in both calcu-
lations they are equipped with the bare form factors $F_{\pi\Delta}(p_N^2)$, $F_{\rho\pi\pi}(p_\rho^2)$, $F_{\sigma\pi\pi}(p_\sigma^2)$, $F_{\rho NN}(p_N^2)$
and $F_{\sigma NN}(p_N^2)$, as defined in Appendix A. The obtained values of the near-threshold coeffi-
cients are summarised in Table IV. Results of several data analyses are quoted in the last
row. Various ingredients of the fully dressed calculation are given in the other rows and will
be discussed below in more detail.

\textbf{Pion-nucleon scattering lengths}

The scattering lengths are evaluated by substituting the explicit expressions for the
amplitudes Eqs. (33–38) into Eqs. (9) and (10). The obtained values are listed in Table
IV for the different calculations considered. The large effect of the dressing on $a_1/2$
compared with the small effect on $a_3/2$

\textbf{Dependence on the bare form factor}

To see how our results depend on the choice of the bare form factor $G_{PV}^0$ in Eq. (30),
we did a calculation in which the width of the bare form factor was set to $\Lambda_N^2 = 2.8$
GeV$^2$ (i.e. near the upper limit $(\Lambda_N^2)^{\text{max}}$ dictated by the convergence) and all the other
parameters were kept as given in Table III. As was shown in [12], despite using a much
wider form factor, such a calculation leads to intermediate-energy phase shifts which are
similar to those obtained in the basic calculation with $\Lambda_N^2 = 1.8$ GeV$^2$. With this variation
of the bare width, the sigma-term and the coefficient $C$ change by less than 3%. The
sensitivity of the scattering lengths is similarly small. The important point here is that even
using quite different bare form factors the model yields results which are comparable with
data analyses both at the Cheng-Dashen point and at threshold. As long as the iteration
procedure converges, the loop dynamics depend only weakly on the details of the bare form
factor. The usage of the exponential bare form factor is also not essential and dipole-like
bare form factors lead to similar dressed vertices. By contrast, the role of the dressing is
quite significant. This can be seen by comparing the rows labelled “Dressed” and “Bare” in
Table IV.

\textbf{Effects of the dressing of the $\sigma$ and $\rho$ propagators}

In the fully dressed calculation, the self-energies of the $\sigma$ and $\rho$ mesons are dressed with
one $\pi\pi$ loop, as detailed in Appendix A. By using the free propagators instead, we obtain
the results listed in the rows “Free $\sigma$” and “Free $\rho$” in Table IV. The $\rho$ exchange contributes
solely to the isospin-odd amplitudes while the $\sigma$ exchange acts in the isospin-even channel, which is explicitly shown in Eqs. (33–36). Hence the sigma-term $\Sigma$ (being calculated from $D^+$ in Eq. (3)) is oblivious to the treatment of the $\rho$ meson while the coefficient $C$ (being calculated from $D^-$ in Eq. (8)) is not affected by the $\sigma$ meson.

**Role of the $\Delta$ resonance**

**$\Delta$ pole contribution**

By the $\Delta$ contribution to the pion-nucleon amplitudes one usually means the contribution of the s- and u-channel $\Delta$ exchange diagrams. This definition may be used for a comparison of our results with those of the chiral perturbation theory, where the $\Delta$ is usually not included as an explicit field in the lagrangian and thus does not appear in the loops $\Delta$. By comparing the rows “Dressed” and “No $\Delta$ poles” in Table IV, we see that the pole contribution of the $\Delta$ is small both below and at threshold (the explicit formulae are given in Appendix B). For example, it changes the sigma-term by 0.26 MeV, which is somewhat smaller than the values typically found in other calculations [7,9,35]. However, a quantitative comparison of our results with other approaches should be carried out carefully since typically one retains a spin 1/2 background in the $\Delta$ propagator (see, e.g., [32,9]). By contrast, in this model we deal with the pure spin 3/2 $\Delta$ due to the gauge-invariant structure of the $\pi N\Delta$ vertex Eq. (16).

**Effects of the dressing of the $\pi N\Delta$ vertex**

In addition to the s- and u-channel exchanges, the $\Delta$ resonance enters in the loop corrections to the $\pi NN$ and $\pi N\Delta$ vertices both of which are dressed up to infinite order. Such a contribution of the $\Delta$ through dressing has not been considered before in the context of the near-threshold $\pi N$ amplitude. Due to the coupled nature of Eqs. (22–29), the $\Delta$ dressing affects the $\pi N$ amplitude directly as well as through its effects on the $\pi NN$ vertex and nucleon propagator.

In the subthreshold region, the value of the sigma-term is slightly decreased by the $\Delta$ dressing, as can be seen by comparing the rows “Dressed” and “Bare $\Delta$” in Table IV (in the latter calculation the bare $\pi N\Delta$ vertex and the free $\Delta$ propagator were used). Comparison of these results with the column “No $\Delta$ poles” shows that the dressing moderates even further the small pole contribution of the $\Delta$ to the sigma-term. Notably, the effect of the consistent dressing of the $\pi N\Delta$ vertex is to decrease the sigma-term and to increase the coefficient $C$, whereas the $\Delta$ exchange has the opposite effect. This suggests that meson loops should be treated with special care in dynamical calculations of the $\Delta$ contribution to the subthreshold $\pi N$ amplitude.

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5There exist approaches [34] including the $\Delta$ explicitly in chiral lagrangians, in which case the $\Delta$ does enter in the loops. Such extensions of the standard chiral perturbation theory will not be discussed here.
Effects of multiple loops

The two- and three-point functions obtained as a solution of Eqs. (22–29) comprise an infinite series of meson-loop corrections. The effect of multiple meson loops on the near-threshold amplitude can be seen from the row labelled “One loop” in Table IV. This is a calculation in which the $\pi NN$ and $\pi N\Delta$ vertices, as well as the nucleon, $\Delta$, $\rho$ and $\sigma$ self-energies, include only one-loop corrections. These vertices and self-energies are obtained by iterating Eqs. (22–29) only once after discarding from them the effective two-loop contributions (the latter are the integrals containing one of the cut resonance propagators ($S_\Delta^{\mu\nu}I$, $(D_\rho^{\mu\nu})_I$ or $(D_\sigma)_I$). To focus on the genuine effects of the loops, no parameters were readjusted in this calculation, except the renormalisation constants $f_{\pi NN}$, $f_{\pi N\Delta}$, $Z_N^2$, $Z_\Delta^2$, $\delta m_N$, $\delta m_\Delta$. Note that in this case the $\pi N$ scattering amplitude is not exactly an amplitude including only one-loop corrections to the tree-level approximation. Nevertheless, taking into account that most of the effects of the dressing are due to the vertices, the difference between this “one loop” and the fully dressed calculations can serve as a reasonable estimate of the effects beyond few perturbative loop corrections in the amplitude. Table IV shows that the multiple-loop effects are small, being comparable in magnitude with the contribution of the $\Delta$ resonance (Ref. [35] gives an upper limit of $\approx 2$ MeV for the $\Delta$ contribution to the sigma-term). The value of the sigma-term comes mostly from the one-loop calculation, whereas the multiple loops and the $\Delta$ give contributions which are beyond the precision of the present data analyses.

VI. COMPARISON WITH THE APPROACHES OF CHIRAL PERTURBATION THEORY AND THE BETHE-SALPETER EQUATION

The near-threshold parameters obtained in this model are compared in Table V with calculations in chiral perturbation theory ($\chi$PT) and with those based on the Bethe-Salpeter equation (BSE). We quote the results of the heavy-baryon and relativistic (with the infrared regularisation) baryon formulations of chiral expansions, referred to as HB$\chi$PT and RB$\chi$PT, respectively. There is some similarity between the present model and the other approaches, as well as a number of differences. In principle, it would be desirable to compare the loops calculated in this model with the chiral loops and with the loops generated by the BSE at the most elementary level, i.e. by analysing the expansions of a simple Green’s function such as the nucleon self-energy. Such a comparison is hardly meaningful, however, since in general the Green’s functions are model- and representation-dependent in perturbative as well as nonperturbative approaches (see, e.g., [30] and references therein). Instead, the values of the near-threshold parameters assembled in Table V can be used to illustrate the most important similarities and differences between the four approaches: the present model, HB$\chi$PT, RB$\chi$PT and the BSE.

The basic dynamical degrees of freedom in all these approaches are the nucleon and the pion. However, unlike the two chiral formalisms, the present model and the BSE include the $\Delta$ resonance and the $\rho$ and $\sigma$ mesons explicitly. In general, these degrees of freedom give important contributions to the dressed Green’s functions. In chiral approaches effects of the resonances are typically encapsulated in low-energy constants using a resonance-saturation hypothesis [37,32] (as mentioned above, we do not touch upon extensions [34]...
of the chiral expansion with the \( \Delta \) as an explicit degree of freedom). In our model, as well as in \( \chi \)PT, the \( \pi NN \) vertex is predominantly pseudovector at low energies. A small pseudoscalar component in the vertex is associated with an explicit symmetry breaking. In all four approaches, the nontrivial dependence of the amplitudes on kinematical variables is determined by loop expansions. The methods of organising the expansions are different however, as are the methods of evaluating the loop diagrams.

The chiral expansions are series in small pion mass and momenta [38]. In the presence of a nucleon, a one-to-one correspondence is established in HB\( \chi \)PT between the power of each term in the expansion and the number of pion loops required for its calculation [39]. This nonrelativistic expansion is well-behaved in the part of the low-energy region which is not too close to the singularities of the amplitude dictated by unitarity and analyticity. However, it is necessary to rearrange the chiral expansion in order to calculate the amplitude near the singularities. This is most efficiently done in RB\( \chi \)PT by using an infrared regularisation in the relativistic formalism and representing the amplitude through dispersion integrals. (Within HB\( \chi \)PT, the correct singularity structure can be captured only by summing an infinite number of loop corrections.) The main emphasis of the approach of the BSE is on obeying unitarity in the relativistic formalism [10, 14]. The necessary loops are summed up effectively by solving the linear integral equation for the scattering amplitude. However, due to the use of a simplified kernel in practical calculations, the BSE does not yield a crossing symmetric amplitude.

The unitarity, crossing and analyticity constraints are used as the governing principles for organising the meson-loop expansion in this model. The (nonlinear) dressing equations represent a self-consistent procedure of using the cutting rules and dispersion relations. By solving them we effectively sum up an infinite series of loop diagrams in such a way that the essential singularity structure of the two- and three-point Green’s functions is correctly reproduced. However, in contrast to the chiral expansions or the BSE, one-particle irreducible four-point loop diagrams (such as the triangle or box graphs discussed in [9]) are not calculated explicitly in our model, which entails a violation of analyticity structure in the t-channel. Also, in this model we need to compute the dressed vertices only in the s- and u-type diagrams, i.e. for on-shell external pions only. Consequently, the dependence of the amplitude on the momenta of external pions is not obtained. For instance, to obey the Adler consistency condition [11] \( \mathcal{D}^\dagger (\nu = 0, t = \mu^2) = 0 \) (with either \( k^2 = \mu^2, k'^2 = 0 \) or \( k^2 = 0, k'^2 = \mu^2 \)), the \( \sigma NN \) vertex and the box graph will have to be dressed in this model, thus requiring an extension to order \( O(a^2) \) in the “analyticity violation expansion” outlined in Section [17].

In the calculation of RB\( \chi \)PT [9] the bulk of the sigma-term at the Cheng-Dashen point comes from a contact four-point vertex. This contact vertex is proportional to the pion mass squared and thus explicitly breaks chiral symmetry of the lagrangian. In the present approach, the largest contribution to the sigma-term is due to the t-channel \( \sigma \) exchange with the component \( \sim g_{\sigma \pi \pi} \mu \) in the \( \sigma \pi \pi \) vertex Eq. (A14). This non-derivative part of the \( \sigma \) exchange does not vanish in the chiral limit and is therefore analogous to the symmetry breaking contact term of RB\( \chi \)PT. It should not be concluded, however, that the loop contributions are of minor importance in the near-threshold region: it is through the loops that the essential unitarity, analyticity and crossing constraints are incorporated in this model as well as in RB\( \chi \)PT.
Although the means of regularisation of the loop integrals employed in this model and in the BSE (usage of bare form factors) are different to those utilised in the RBχPT (infrared and dimensional regularisation), they are similar in that they generate spurious singularities of the amplitude. These unwanted singularities should be removed from the region of physical interest. For example, the amplitude in the RBχPT has an unphysical pole at \( s = 0 \), which should be safely far from the relevant near-threshold region \([4]\). Similarly, the bare form factor in this model is rather wide, ensuring remoteness of its singularities. There is an important difference between the evaluation of the loop integrals in our approach and in the BSE. In the latter method the loops are computed using such techniques as, e.g., the Wick’s rotation and Feynman parametrisation, whereas in our framework the loops are calculated through the successive application of the Cutkosky rules and dispersion relations. While being equivalent in local field theories, these two methods of loop evaluation are likely to differ when the loops are regularised by form factors. Therefore the analytical properties of the amplitudes dressed in the BSE are probably different to those generated by our approach, which may have important consequences in the near-threshold region.

VII. CONCLUSIONS

A consistent dynamical calculation of the pion-nucleon amplitude in the near-threshold region should serve as a bridge between the physically accessible low-energy data and chiral low-energy theorems reflecting the QCD dynamics in the nonperturbative regime. In this paper we have shown that essential analyticity constraints can be incorporated in a self-consistent dressing procedure, resulting in a reliable description of the amplitude at the Cheng-Dashen point and at threshold. In particular, the pion-nucleon scattering lengths, the nucleon sigma-term and the Adler-Weisberger coefficient \( C \) evaluated in our approach are all consistent with the recent data analyses.

The values of the near-threshold coefficients depend crucially on the treatment of the \( \sigma \) and \( \rho \) meson exchanges. The approach also includes a consistent dressing of the \( \pi N \Delta \) vertex and of the \( \Delta \) propagator. This allows us to study the role of the \( \Delta \) for the nucleon sigma-term in more detail than has been done previously. In particular, we have found that the contribution of the \( \Delta \) in the near-threshold region should be considered on a par with the effects of multiple loops.

This dynamical model suggests that effective approaches incorporating the constraints of relativistic invariance, unitarity, crossing symmetry and analyticity can reveal important aspects of the low-energy strong interaction.

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APPENDIX A: CONTRIBUTIONS OF THE $\rho$ AND $\sigma$ TO THE DRESSING EQUATIONS

The term $\Gamma_1^{\rho\sigma}(p)$ on the right-hand side of Eq. (22) comprises 6 loop diagrams including the $\rho$ and $\sigma$ degrees of freedom,

$$\Gamma_1^{\rho\sigma}(p) = \sum_{n=1}^{3} \left\{ (\Gamma_1^\rho(n)) + (\Gamma_1^\sigma(n)) \right\}, \quad (A1)$$

where the different terms are described in the following.

$$(\Gamma_1^\rho(n))(p) = \frac{1}{8\pi^2} \int d^4k \Gamma_{\mu}^{\rho NN}(k-q, p-k) S_I(p-k) \Gamma^\rho_R(q-k) D_{\mu\nu}^{\rho}(q-k) D_I^{\rho}(p), \quad (A2)$$

with the $\rho\pi\pi$ vertex having the structure

$$\Gamma_\nu^{\rho\pi\pi}(q, q') = g_{\rho\pi\pi} \left[ q_\nu' - q_\nu + \frac{(q^2 - q'^2)(q_\nu + q_\nu')}{(q + q')^2} \right] F_{\rho\pi\pi}((q + q')^2). \quad (A3)$$

In order that a converging solution of the dressing should exist, the $\rho\pi\pi$ vertex is equipped with a form factor $F_{\rho\pi\pi}((q + q')^2)$ whose form is the same as that of the bare $\pi NN$ form factor in Eq. (30),

$$F_{\rho\pi\pi}(k^2) = \exp \left[ - \ln 2 \left( \frac{k^2 - m^2_\rho}{\Lambda^2_R} \right) \right], \quad (A4)$$

where the half-width $\Lambda^2_R$ is given in Table III and discussed in Section I. By analogy with the treatment of the $\Delta$ resonance (see Eq. (18)), we choose a gauge-invariant form of the $\rho\pi\pi$ vertex and therefore retain only the spin 1 part of the $\rho$ propagator

$$D^{\mu\nu}_\rho(k) = \frac{\mathcal{P}^{\mu\nu}_1(k)}{Z^\rho [k^2 - \lambda^2(k^2)]}, \quad (A5)$$

with the spin 1 projector

$$\mathcal{P}^{\mu\nu}_1(k) = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}. \quad (A6)$$

The $\rho$ self-energy function $\lambda^2(k^2)$ is calculated from one $\pi\pi$ loop as follows.

$$\lambda^2(p^2) = m^2_\rho - \delta m^2_\rho + \frac{\text{Re} \Pi^\rho_L(p^2)}{Z^\rho}, \quad (A7)$$

where $\Pi^\rho_L(p^2)$ is the $\pi\pi$ loop contribution to the self-energy and $Z^\rho, \delta m^2_\rho$ are renormalisation constants adjusted to ensure the correct pole properties of the dressed $\rho$ propagator Eq. (A5).

As all loop integrals in the model, this loop is evaluated using a dispersion relation:

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6 In this appendix we use the notation introduced in Section I.
\[ \text{Im}\Pi^\rho(k^2) = -\frac{P_1^{\mu\mu}(k)}{24\pi^2} \int d^4q \Gamma^{\rho\pi}(q, k - q) D_I(q) D_I(k - q) \Gamma^{\rho\pi}_\nu(q - k, -q), \quad (A8) \]

\[ \text{Re}\Pi_2^\rho(k^2) = \frac{P}{\pi} \int_{4\mu^2}^{\infty} dk^2 \frac{\text{Im}\Pi^\rho(k^2)}{k^2 - k^2}. \quad (A9) \]

The \(\rho NN\) vertex is chosen as

\[ \Gamma^{\rho NN}_\mu(k, p) = g_{\rho NN} \left[ \gamma_\mu + i\kappa_\rho \frac{\sigma_{\mu\nu}k^\nu}{2m} \right] F_{\rho NN}(p^2), \quad (A10) \]

where the regularising form factor \(F_{\rho NN}(p^2)\) depends on the four-momentum squared of an off-shell nucleon and has the form of Eq. (A4) in which \(m_N\) is substituted for \(m_\rho\). Describing Eq. (A1) further,

\[ (\Gamma_1^\rho)_{2}(p) = \frac{g_\gamma_\rho}{8\pi^3} \int d^4k S_I(p - k) \Gamma^{\rho NN}_\mu(-k, p) D(k - q) \Gamma^{\rho\pi}_\nu(-q, q - k)(D_\rho^{\mu\nu})_I(k), \quad (A11) \]

where Eq. (A1) has been used;

\[ (\Gamma_1^\rho)_{3}(p) = \frac{1}{32\pi^3} \int d^4k \Gamma^{\rho NN}_\mu(k, p' - k) S(p' - k) \Gamma_R(p' - k) S_I(p - k) \Gamma^{\rho NN}_\nu(-k, p)(D_\rho^{\mu\nu})_I(k). \quad (A12) \]

The loop integrals with the \(\sigma\) meson are similar in structure to those with the \(\rho\).

\[ (\Gamma_1^\sigma)(p) = -\frac{g_{\sigma NN}}{8\pi^2} \int d^4k S_I(p - k) \Gamma_R(p) D_\sigma(q - k) \Gamma^{\sigma\pi\pi}(q, q) D_I(k), \quad (A13) \]

where the \(\sigma\pi\pi\) vertex is chosen as

\[ \Gamma^{\sigma\pi\pi}(q, q') = \left[ g_{\sigma\pi\pi\mu} + f_{\sigma\pi\pi} \frac{q \cdot q'}{\mu} \right] F_{\sigma\pi\pi}((q + q')^2). \quad (A14) \]

The \(\sigma\) propagator \(D_\sigma(k)\) is obtained from one \(\pi\pi\) loop:

\[ D_\sigma(k) = \frac{1}{Z_\sigma \left[ k^2 - \zeta^2(k^2) \right]}, \quad (A15) \]

\[ \zeta^2(p^2) = m_\sigma^2 - \delta m_\sigma^2 + \frac{\text{Re}\Pi_2^\sigma(p^2)}{Z_\sigma}, \quad (A16) \]

\[ \text{Im}\Pi^\sigma(k^2) = -\frac{3}{16\pi^2} \int d^4q \Gamma^{\sigma\pi\pi}_\mu(q, k - q) D_I(q) D_I(k - q) \Gamma^{\sigma\pi\pi}_\nu(q - k, -q), \quad (A17) \]

\[ \text{Re}\Pi_2^\sigma(k^2) = \frac{P}{\pi} \int_{4\mu^2}^{\infty} dk^2 \frac{\text{Im}\Pi^\sigma(k^2)}{k^2 - k^2}. \quad (A18) \]

The two other loops with the \(\sigma\) read
\begin{align}
  (\Gamma^\sigma_1)_2(p) &= \frac{g_{\gamma S}}{8\pi^3} \int d^4k \, S_1(p-k) \Gamma^{\sigma NN}(-k, p) D(k-q) \Gamma^{\sigma \pi \pi}(-q, q-k)(D_\sigma)_1(k), \\
  (\Gamma^\sigma_1)_3(p) &= \frac{g_{\gamma NN}^2}{8\pi^3} \int d^4k \, \Gamma^{\sigma NN}(k, p' - k) S(p' - k) \Gamma_R(p' - k) S_I(p-k) \Gamma^{\sigma NN}(-k, p)(D_\sigma)_1(k),
\end{align}

where the \(\sigma NN\) vertex has the simple structure
\[
  \Gamma^{\sigma NN}(k, p) = g_{\sigma NN} F_{\sigma NN}(p^2),
\]

with the form factor \(F_{\sigma NN}(p^2) = F_{\rho NN}(p^2)\).

The term \((V_\mu^{\rho \sigma})_I(q, p)\) in Eq. (26) comprises two loop integrals whose form is given by Eq. (A2) and Eq. (A13) in which the \(\pi NN\) vertex \(\Gamma_R(p)\) is replaced with the \(\pi N\Delta\) vertex \((V_\mu)_R(k, p)\).

**APPENDIX B: CONTRIBUTION OF THE \(\Delta\) AT THE CHENG-DASHEN POINT**

The contributions of the \(\Delta\) resonance to the sigma-term and to the coefficient \(C\) in Eqs. (41,42) are obtained by decomposing Eqs. (37,38) into the invariant amplitudes \(D^\pm\) and evaluating the latter at the Cheng-Dashen point. The expressions in terms of the dressed \(\pi N\Delta\) form factor and the dressed \(\Delta\) self-energy functions (as defined in Eqs. (16,19)) are

\[
  \Sigma_\Delta = -\frac{2F^2_\pi G^2_\Delta(m^2) \omega(m^2)}{9m^4_\Delta \eta(m^2)[m^2 - \omega^2(m^2)]} \mu^4,
\]

\[
  C_\Delta = \frac{8F^2_\pi m \omega(m^2) G^2_\Delta(m^2)}{9m^4_\Delta \eta(m^2)[m^2 - \omega^2(m^2)]} \mu^2
  + 2F^2_\pi \left\{ \frac{G^2_\Delta(m^2) + 4m \omega(m^2) G_\Delta(m^2) G'_\Delta(m^2)}{9m^4_\Delta \eta(m^2)[m^2 - \omega^2(m^2)]} - \frac{2m G^2_\Delta(m^2) \eta(m^2) \omega^2(m^2)}{9m^4_\Delta \eta^2(m^2)[m^2 - \omega^2(m^2)]^2} \right\} \mu^4,
\]

explicitly showing the suppression by powers of the pion mass \(\mu\).
TABLES

TABLE I. Particle masses (in GeV) and coupling constants used in this calculation (the same values were used in Ref. [14]). The masses and the coupling constants of the Δ, ρ and σ correspond to the experimental widths and positions of the resonances as given in [18].

| m ≡ m_N | μ ≡ m_π | m_Δ | m_ρ | m_σ | g ≡ g_{πNN} | g_{πNΔ} | g_{ρππ} |
|----------|----------|------|------|------|-------------|---------|---------|
| 0.939    | 0.138    | 1.232| 0.77 | 0.76 | 13.02       | 19.76   | 6.07    |

TABLE II. The renormalisation parameters: bare coupling constants, mass shifts (in units of GeV) and field renormalisation factors.

| f ≡ f_{πNN} | f_{πNΔ} | δm ≡ δm_N | δm_Δ | δm_ρ | δm_σ | Z_2 ≡ Z_2^N | Z_2^Δ | Z_ρ | Z_σ |
|-------------|---------|------------|------|------|------|-------------|-------|-----|-----|
| 10.75       | 21.75   | -0.075     | -0.120| -0.089| -0.605| 0.80        | 1.16  | 1.17 | 1.14|

TABLE III. Values of the parameters of the model, as fixed by calculating the intermediate-energy pion-nucleon phase shifts in the dressed K-matrix approach of Ref. [14]. No parameters were readjusted in the present calculation.

| Λ_N^2 | Λ_R^2 | g_{ρNN} | κ_ρ | g_{σNN} | g_{σππ} | f_{σππ} |
|-------|-------|---------|-----|---------|---------|---------|
| 1.8   | 1.0   | 7.0     | 2.3 | 34      | 1.7     | 1.8     |
TABLE IV. The pion-nucleon sigma-term, the Adler-Weisberger coefficient $C$, evaluated at the Cheng-Dashen point from Eqs. (3,8), and the s-wave scattering lengths, evaluated from Eqs. (9,10). The rows represent the following calculations. “Dressed”: fully dressed calculation; “Bare”: bare calculation, i.e. using the free propagators and no loop corrections to the bare vertices; “Free $\sigma$”: full calculation, but using the free $\sigma$ propagator in the t-channel exchange; “Free $\rho$”: full calculation, but using the free $\rho$ propagator in the t-channel; “No $\Delta$ poles”: full calculation, but without the s- and u-channel $\Delta$ exchange pole diagrams; “Bare $\Delta$”: full calculation, but using the bare $\pi N\Delta$ vertex and the free $\Delta$ propagator; “One loop”: calculation in which the nucleon and $\Delta$ self-energies as well as the $\pi NN$ and $\pi N\Delta$ vertices are computed up to one-loop corrections only; “Data”: results of various data analyses.

|         | $\Sigma$ (MeV) | $C$     | $a^{1/2}(\mu^{-1})$ | $a^{3/2}(\mu^{-1})$ |
|---------|----------------|---------|----------------------|----------------------|
| Dressed | 73.99          | 1.16    | 0.175                | -0.087              |
| Bare    | 127.78         | 1.31    | 0.204                | -0.088              |
| Free $\sigma$ | 126.41        | 1.16    | 0.183                | -0.080              |
| Free $\rho$ | 73.99         | 1.40    | 0.210                | -0.105              |
| No $\Delta$ poles | 73.73     | 1.21    | 0.175                | -0.087              |
| Bare $\Delta$ | 74.05         | 1.14    | 0.175                | -0.087              |
| One loop | 71.98          | 1.21    | 0.181                | -0.093              |
| Data   | $64 \pm 8 [23]$ | $1.15 \pm 0.02 [20]$ | $0.173 \pm 0.003 [22]$ | $-0.101 \pm 0.004 [22]$ |
|        | $79 \pm 7 [30]$ | $0.175 \pm 0.004 [31]$ | $-0.085 \pm 0.027 [31]$ |                      |
|        | $71 \pm 9 [7]$  |         |                      |                      |

TABLE V. Comparison of the near-threshold parameters evaluated in the present model with results obtained in chiral perturbation theory and in the approach based on the Bethe-Salpeter equation. The third and fourth order HB$\chi$PT calculations [33] presented fits to three different phase-shifts, using Ref. [7] to relate the sigma-term and threshold parameters. The RB$\chi$PT calculation [9] used the data analyses of [29] as input. The BSE results are from Ref. [10].

|         | This model | HB$\chi$PT $\mathcal{O}(p^6)$ | HB$\chi$PT $\mathcal{O}(p^4)$ | RB$\chi$PT $\mathcal{O}(p^4)$ | BSE |
|---------|------------|-------------------------------|-------------------------------|-------------------------------|-----|
| $\Sigma$ (MeV) | 73.99 | 69                           | 73                           | 61                           | 23.6 |
| $C$     | 1.16       | 1.10                          | 1.13                          |                               |     |
|         | 0.82       | 1.09                          |                               |                               |     |
| $a^{1/2}(\mu^{-1})$ | 0.175 | 0.171                         | 0.171                         | 0.175                         | 0.177 |
|         | 0.159      | 0.159                         |                               |                               |     |
|         | 0.175      | 0.176                         |                               |                               |     |
| $a^{3/2}(\mu^{-1})$ | -0.087 | -0.101                        | -0.100                        | -0.100                        | -0.101 |
|         | -0.072     | -0.073                        |                               |                               |     |
|         | -0.086     | -0.084                        |                               |                               |     |
FIG. 1. Graphical representation of the pion-nucleon amplitude, corresponding to Eq. (11). The single solid lines are nucleons, the double lines are Δ’s, the dashed, zigzag and dotted lines are pions, ρ’s and σ’s, respectively. The empty and hatched circles denote the πNN and πNΔ vertices, respectively. The propagators and vertices are dressed with meson loops as described in Section 1.
FIG. 2. Graphical representation of the system of integral equations for the dressed two- and three-point Green’s functions. The notation for the propagators and vertices is as in Fig. 1. For each particle, the dressed and free propagators are denoted by thick and thin lines, respectively. The triangles denote the counterterms needed to fulfill the renormalisation conditions such as Eq. (32). The slashes through the loops and the integral signs indicate the use of the Cutkosky (cutting) rules and dispersion integrals in the iterative solution of the equations. The outgoing nucleons in the vertices (as well as the pions) are on-shell, which is denoted by the crossed lines. The correspondence with analytic equations is A ↔ (22, 23), B ↔ (24, 25), C ↔ (26, 27), D ↔ (28, 29), E ↔ (48, 49), F ↔ (417, 418).
FIG. 3. Illustration of the interdependence of the coupling constants $g_{\sigma NN}$ and $g_{\rho NN}$ (the other parameters being fixed as in Table III) due to the requirement of convergence of the dressing procedure. The area of convergence is sketched using 50 test solutions of Eqs. (22–29), but ignoring $\pi N$ phase shifts. The dot corresponds to the phase shifts fit obtained in Ref. [14].
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