CATEGORICAL CREPANT RESOLUTIONS
OF SINGULARITIES AND THE
TITS-FREUDENTHAL MAGIC SQUARE

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Abstract

We prove that the tangent developables of the varieties appearing in the third row of the Tits-Freudenthal magic square admit categorical crepant resolutions of singularities.
1 Introduction

We work over \( \mathbb{C} \) the field of complex numbers. If \( X \) is an algebraic scheme of finite type over \( \mathbb{C} \), we denote by \( \mathcal{D}^b(X) \) (resp. \( \mathcal{D}^-(X), \mathcal{D}^{\text{perf}}(X) \)), the derived category of bounded complexes of coherent sheaves on \( X \) (resp. derived category of unbounded complexes from below of coherent sheaves on \( X \), the full subcategory of \( \mathcal{D}^b(X) \) consisting of complexes of vector bundles).

1.1 Categorical crepant resolution of singularities

Let \( X \) be an algebraic variety with Gorenstein singularities. A crepant resolution of singularities of \( X \) (that is a resolution \( \pi : \tilde{X} \to X \) such that \( \pi^*\omega_X = \omega_{\tilde{X}} \)) is often considered to be a "minimal" resolution of \( X \). The following conjecture (see [BO02]) gives a precise meaning to that notion of minimality:

**Conjecture 1.1.1 (Bondal-Orlov)** Let \( X \) be a variety with Gorenstein and rational singularities and let \( \tilde{X} \to X \) be a crepant resolution of \( X \). Then, for any other resolution \( \tilde{X}' \to X \), there exists a fully faithful embedding:

\[
\mathcal{D}^b(\tilde{X}) \hookrightarrow \mathcal{D}^b(\tilde{X}').
\]

Unfortunately, crepant resolution of singularities are quite rare. For instance, a cone over \( v_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+1)}{2}} \) never admits a crepant resolution of singularities when \( n \) is odd (it is \( \mathbb{Q} \)-factorial with terminal singularities). Thus it seems interesting to look for "categorical crepant resolution of singularities".

The notion of categorical crepant resolution of singularities has been formalized by Kuznetsov (see [Kuz08]) in the case of Gorenstein varieties with rational singularities.

**Definition 1.1.2** Let \( X \) be an algebraic variety with Gorenstein and rational singularities. A categorical resolution of singularities of \( X \) is a triangulated category \( \mathcal{F} \) with a functor \( R\pi_{\mathcal{F}*} : \mathcal{F} \to \mathcal{D}^b(X) \) such that:

- there exists a resolution of singularities \( \pi : \tilde{X} \to X \) with a fully faithful admissible functor \( \delta : \mathcal{F} \to \mathcal{D}^b(\tilde{X}) \) such that \( R\pi_{\mathcal{F}*} = R\pi_* \circ \delta \),

- for all \( \mathcal{F} \in \mathcal{D}^{\text{perf}}(X) \), we have:

\[
R\pi_{\mathcal{F}*}L\pi^*_{\mathcal{F}} \mathcal{F} \simeq \mathcal{F},
\]

where \( L\pi^*_{\mathcal{F}} \) is the left adjoint to \( R\pi_{\mathcal{F}*} \).
Moreover, if for all $T \in \mathcal{T}$ we have:

$$\delta(S_{\mathcal{T}}(F)) = \delta T \otimes \pi^* \omega_X [\dim X],$$

where $S_{\mathcal{T}}$ is the Serre functor of $\mathcal{T}$, we say that $\mathcal{T}$ is strongly crepant.

If for all $F \in D^{\text{perf}}(X)$, there is a quasi-isomorphism:

$$L\pi^* F \simeq L\pi_! F,$$

where $L\pi_!$ is the right adjoint of $R\pi_*$, we say that $\mathcal{T}$ is weakly crepant.

Obviously, if $\mathcal{T} \to D^b(X)$ is a strongly crepant resolution, then it is weakly crepant. The converse is false as shown in section 7 and 8 of [Kuz08]. If $\pi : \tilde{X} \to X$ is a crepant resolution of singularities then $R\pi_* : D^b(\tilde{X}) \to D^b(X)$ is a strongly crepant categorical resolution of singularities. The converse is partially true:

**Proposition 1.1.3** Let $X$ be a projective irreducible Gorenstein variety with rational singularities. Let $\pi : \tilde{X} \to X$ be a proper morphism with $\tilde{X}$ irreducible, such that $R\pi_* : D^b(\tilde{X}) \to D^b(X)$ is a weakly crepant categorical resolution of singularities. Then $\pi : \tilde{X} \to X$ is a crepant resolution of singularities.

**Proof:**

As $D^b(\tilde{X})$ is a weakly crepant categorical resolution of $X$, we have the equality:

$$R\pi_* L\pi^* C(x) \simeq C(x),$$

for all $x \in X_{\text{smooth}}$, which implies that $\pi$ is dominant. As it is proper, it is surjective. By hypothesis, $D^b(\tilde{X})$ is an admissible subcategory of the derived category of a smooth projective variety. It implies that $D^b(\tilde{X})$ is Ext-bounded, so that $\tilde{X}$ is smooth. Moreover, we deduce that the right adjoint to $R\pi_*$ satisfies the formula (see [Nee96]):

$$L\pi_! \mathcal{F} \simeq L\pi^* \mathcal{F} \otimes \omega_{\tilde{X}/X} [\dim \tilde{X} - \dim X].$$

Since $D^b(\tilde{X})$ is a weakly crepant categorical resolution of singularities, we have $\dim \tilde{X} = \dim X$ and $\omega_{\tilde{X}} = \pi^* \omega_X$. But the morphism $\pi$ is surjective, so that the equality $\dim \tilde{X} = \dim X$ implies that $\pi$ is generically finite.

Using again the fact that $D^b(\tilde{X})$ is a weakly crepant categorical resolution, we have $R\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$. As $\pi$ is proper, generically finite and $X$ is normal, Zariski’s Main Theorem implies that $\pi$ is birational.
1.2 Main result and connections with other works

Notations 1.2.1 From now on, we will exclusively focus on weakly crepant categorical resolution of singularities. We will simply call them categorical crepant resolution of singularities.

The main result of this chapter is the following:

**Theorem 1.2.2** The tangent developables of the following embedded varieties admit categorical crepant resolutions of singularities:

- The symplectic Grassmannian $G_{\omega}(3, 6) \subset \mathbb{P}^{13}$;
- The Grassmannian $G(3, 6) \subset \mathbb{P}^{19}$;
- The spinor variety $S_{12} \subset \mathbb{P}^{31}$;
- The octonionic Grassmannian: $G_{\omega}(O^3, O^6) \subset \mathbb{P}^{55}$.

These four varieties have a uniform description in terms of complex composition algebras (this will be discussed in section 2). They are the "symplectic Grassmannians" of $A^3 \subset A^6$ for $A$ the complexification of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ and they appear as the varieties in the third row of the Freudenthal’s magic square (see [LM01]).

In [Abu12], we define the notion of wonderful resolution of singularities (see definition 2.1.2) and we prove the following (see theorem 2.3.2):

**Theorem 1.2.3** Let $X$ be a Gorenstein variety with rational singularities. Assume that $X$ admits a wonderful resolution of singularities, then $X$ admits a categorical crepant resolution of singularities.

As a corollary of this result, we obtain (see example 2.1.3 in [Abu12]):

**Corollary 1.2.4** All Gorenstein determinantal varieties (square, symmetric or skew-symmetric) admit categorical crepant resolutions of singularities.

In Example 2.1.6 of [Abu12], we observed that the tangent developable of $G(3, 6)$ does not admit a wonderful resolution of singularities. So the construction of a categorical crepant resolution of singularities for the tangent variety of $G(3, 6)$ was still an open question. We solve this problem in the present chapter. Note that the construction of such a categorical resolution of singularities could also be useful for the (still conjectural) determination of a homological projective dual to $G(3, 6)$ (see [Del11]).
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2 Resolution of singularities and the Tits-Freudenthal magic square

2.1 Basic description of the magic square

One incarnation of the Tits-Freudenthal magic square is a table of 16 varieties which are linked to each other by very interesting geometric and representation-theoretic properties (see [LM01] for a detailed study of the magic square):

\[
\begin{array}{cccc}
\nu_2(Q_1) & \mathbb{P}(T_{\mathbb{P}^2}) & G_\omega(2, 6) & \mathbb{O}\mathbb{P}^2_6 \\
\nu_2(\mathbb{P}^2) & \mathbb{P}^2 \times \mathbb{P}^2 & G(2, 6) & \mathbb{O}\mathbb{P}^2 \\
G_\omega(3, 6) & G(3, 6) & S_{12} & E_7/\mathbb{P}_7 \\
F^\text{ad}_4 & E^\text{ad}_6 & E^\text{ad}_7 & E^\text{ad}_8
\end{array}
\]

The second row of this table enumerates the Severi varieties. Recall that a Severi variety is a smooth variety \( X \subset \mathbb{P}^N \) such that \( \frac{3}{2} \dim X + 2 = N \) and the secant variety of \( X \) does not fill \( \mathbb{P}^N \) (see [Zak93] for the classification of the Severi varieties). The varieties in the first row are hyperplane sections of the Severi varieties. The ones in the last row are the closed orbits of the adjoint representations of the exceptional groups \( F_4, E_6, E_7 \) and \( E_8 \), while the third row gives the varieties of lines through a point of the corresponding adjoint varieties.

One can also describe these varieties in terms of complex composition algebras. Let \( \mathcal{A} \) denotes the complexification of one of the four real division algebra (\( \mathbb{R}, \mathbb{C}, \mathbb{H} \) and \( \mathbb{O} \)). Let \( W_\mathcal{A} \) be the space of 3 × 3 Hermitian matrices over \( \mathcal{A} \). The varieties of the second row can be seen as the varieties of matrices of rank 1 in \( \mathbb{P}(W_\mathcal{A}) \), thus they are Veronese embeddings of the projective planes over \( \mathcal{A} \) (we will denote them by \( \mathcal{A}\mathbb{P}^2 \)). The varieties in the first row are the traceless matrices of rank 1 in \( \mathbb{P}(W_\mathcal{A}) \): they are hyperplane sections of the previous ones. The varieties in the third row can be described as \( G_\omega(\mathcal{A}^3, \mathcal{A}^6) \), the isotropic Grassmannians of \( \mathcal{A}^3 \) in \( \mathcal{A}^6 \). The varieties in the last row are the so-called \( F \)-symplecta, which we denote by \( E(\mathcal{A})^\text{ad} \). We refer to [LM01].
for more details on the description of the magic square in terms of complex composition algebras. In the following, we let $m_A = \dim_C A$ and $\text{Sp}_6(A)$ denotes the groups: $\text{Sp}_6, \text{SL}_6, \text{Spin}_{12}$ and $E_7$. We summarize our notations:

| $A$ | $R \otimes R C$ | $C \otimes R C$ | $H \otimes R C$ | $O \otimes R C$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| $\mathbb{P}^2_0$ | $v_2(Q_1)$ | $\mathbb{P}(T_{\mathbb{P}^2})$ | $G_\omega(2, 6)$ | $\text{OP}_0^2$ |
| $\mathbb{P}^2$ | $v_2(\mathbb{P}^2)$ | $\mathbb{P}^2 \times \mathbb{P}^2$ | $G(2, 6)$ | $\text{OP}_0^2$ |
| $G_\omega(A^3, A^6)$ | $G_\omega(3, 6)$ | $G(3, 6)$ | $S_{12}$ | $E_7/\text{P}_7$ |
| $E(\tilde{\omega})^{ad}$ | $F_4^{ad}$ | $E_6^{ad}$ | $E_7^{ad}$ | $E_8^{ad}$ |

We are especially interested in the varieties in the third row. Let us give another description of these varieties which is more concrete and which will be useful for further computations. The space $W_A$ is naturally endowed with a cubic form: the determinant. We will denote it by $C$. Thus, $C$ is a linear form $S^3 W_A \to \mathbb{C}$ and can also be considered as a linear map $S^2 W_A \to W_A^*$. We denote by $V_A$ the space $\mathbb{C} \oplus W_A \oplus W_A^* \oplus \mathbb{C}$, which coordinates are $(\alpha, A, B, \beta)$. Denote by $\phi$ the rational map:

$$\phi : \mathbb{P}(\mathbb{C} \oplus W_A) \dasharrow \mathbb{P}(V_A)$$

and denote by $\mathcal{Q}$ the quartic defined on $V_A$ by:

$$\mathcal{Q}(\alpha, A, B, \beta) = (3\alpha \beta - \frac{1}{2} (A, B))^2 + \frac{1}{3} (\beta C(A^{\otimes 3}) + C^*(B^{\otimes 3}))$$

$$- \frac{1}{6} (C^*(B^{\otimes 2}), C(A^{\otimes 2})),$$

where $\langle , \rangle$ is the natural pairing between $W_A$ and $W_A^*$ and $C^*$ denotes the determinant on $W_A^*$. The equation of the secant variety to $A \mathbb{P}^2 \subset \mathbb{P}(W_A)$ is $\{C(A^{\otimes 3}) = 0\}$. The following result is proved in [LM01]:

**Theorem 2.1.1** The variety $G_\omega(A^3, A^6) \subset \mathbb{P}(V_A)$ is the image of the rational map $\phi$. The quartic $\mathcal{Q}$ is an $\text{Sp}_6(A)$-invariant form on $V_A$ and the hypersurface $\mathcal{Q} = 0$ is the tangent variety of $G_\omega(A^3, A^6)$ in $\mathbb{P}(V_A)$.

### 2.2 Desingularization of the tangent variety of $G_\omega(A^3, A^6)$

The orbit stratification of the action of $\text{Sp}_6(A)$ on $\mathbb{P}(V_A)$ is given as follows (in the upper parentheses, we let the dimension of the corresponding orbit):
Proposition 2.2.1 There is a natural diagram:

\[
\begin{array}{ccc}
\mathbb{P}(\mathcal{Y}) & \longrightarrow & \sigma_{+}(G_{\omega}(A^{3}, A^{6})) \\
\downarrow & & \downarrow \\
\text{Sp}_{6}^{ad}(A) & \longrightarrow & \sigma_{+}(G_{\omega}(A^{3}, A^{6}))
\end{array}
\]

where the map \( p : \mathbb{P}(\mathcal{Y}) \to \sigma_{+}(G_{\omega}(A^{3}, A^{6})) \) is a resolution of singularities.

Here \( \text{Sp}_{6}^{ad}(A) \) is the closed orbit of the adjoint representation of \( \text{Sp}_{6}(A) \). The bundle \( \mathcal{Y} \) is a homogeneous vector bundle on \( \text{Sp}_{6}^{ad}(A) \). The map \( \theta \) makes the exceptional divisor \( F \) of \( p \) a fibration into smooth quadrics of dimension \( \text{dim} \, A + 2 \) over \( \text{Sp}_{6}^{ad}(A) \), while the map \( q \) makes it a fibration into \( \mathbb{A} \mathbb{P}^{2} \) over \( G_{\omega}(A^{3}, A^{6}) \). Note that the variety \( \mathbb{P}(\mathcal{Y}) \) is the blow-up of \( \sigma_{+}(G_{\omega}(A^{3}, A^{6})) \) along \( G_{\omega}(A^{3}, A^{6}) \).

The orbit closure \( \tau(G_{\omega}(A^{3}, A^{6})) \) can also be desingularized in a similar way:

Proposition 2.2.2 Let \( \check{T}G_{\omega}(A^{3}, A^{6}) \) be the projective bundle of embedded tangent spaces to \( G_{\omega}(A^{3}, A^{6}) \subset \mathbb{P}(V_{6}) \). There is a natural diagram:
\[
\begin{align*}
E &= \sigma(\mathbb{A}\mathbb{P}^2) \\
\xrightarrow{\mu} &\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \\
\xrightarrow{T} &\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \\
\xrightarrow{\pi} &\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \\
\xrightarrow{\rho} &G_\omega(\mathbb{A}^3, \mathbb{A}^6)
\end{align*}
\]

where \( \pi : \tilde{T}G_\omega(\mathbb{A}^3, \mathbb{A}^6) \to \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \) is a resolution of singularities.

The map \( \rho : \tilde{T}G_\omega(\mathbb{A}^3, \mathbb{A}^6) \to G_\omega(\mathbb{A}^3, \mathbb{A}^6) \) is the projective bundle whose fiber over \( x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6) \) is the embedded projective tangent space to \( G_\omega(\mathbb{A}^3, \mathbb{A}^6) \) at \( x \). This map makes the exceptional divisor \( E \) a fibration over \( G_\omega(\mathbb{A}^3, \mathbb{A}^6) \) whose fibers are secant varieties of \( \mathbb{A}\mathbb{P}^2 \subset \mathbb{P}(W_\lambda) \). We denote it by \( E = \sigma(\mathbb{A}\mathbb{P}^2) \).

The map \( \mu : E \to \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \) is not flat. Its fiber over \( p \in G_\omega(\mathbb{A}^3, \mathbb{A}^6) \) is a cone over \( \mathbb{A}\mathbb{P}^2 \), while its fiber over \( p \in \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \) is a smooth quadric of dimension \( \dim \mathbb{A} + 1 \). The map \( \tilde{T}G_\omega(\mathbb{A}^3, \mathbb{A}^6) \to \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \) is the blow-up of \( \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \) along \( \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \).

The restriction of \( \rho \) to the singular locus of the exceptional divisor \( E \) makes it a fibration in \( \mathbb{A}\mathbb{P}^2 \) over \( G_\omega(\mathbb{A}^3, \mathbb{A}^6) \). We denote it by \( E_{\text{sing}} = \mathbb{A}\mathbb{P}^2 \subset \sigma(\mathbb{A}\mathbb{P}^2) \). The divisor \( E \) can be desingularized by blowing up its singular locus and the desingularization is also the projectivization of a homogeneous bundle. One also notices that \( \tilde{T}G_\omega(\mathbb{A}^3, \mathbb{A}^6) \) is the blow-up of \( \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \) along \( \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \). Since \( \mu \) is smooth outside \( G_\omega(\mathbb{A}^3, \mathbb{A}^6) \), we have \( E_{\text{sing}} \subset \mu^{-1}(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \). But a simple count of dimension shows that this inclusion is an equality. We refer to [LM01], section 7 for more details on this desingularization.

**Remark 2.2.3** Though this resolution of singularities of \( \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \) is quite simple and very explicit, it will not be useful in order to find a categorical crepant resolution of singularities of \( \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \). Indeed, one of the key points in order to construct such a categorical resolution would be to find a semi-orthogonal decomposition:

\[
\begin{align*}
\text{D}^b(E) &= (\mu^*\text{D}^b(\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)))) \otimes \mathcal{O}_E(r_\lambda E), \ldots, \mu^*\text{D}^b(\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)))) \otimes \mathcal{O}_E(E), \mathcal{D},
\end{align*}
\]

where \( E \) is the exceptional divisor of the resolution:

\[
\pi : \tilde{T}G_\omega(\mathbb{A}^3, \mathbb{A}^6) \to \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)),
\]
$r_\lambda$ is the unique integer (well-defined since $E$ is integral) such that:

$$K_{\mathcal{F}_G(A^3,A^6)} = \pi^* K_{\tau(G_w(A^3,A^6))} \otimes \mathcal{O}_{\mathcal{F}_G(A^3,A^6)}(r_\lambda E)$$

and $\mathcal{F}$ is the left orthogonal to the subcategory generated by the:

$$\mu^* D^b(\sigma_+(G_w(A^3,A^6))) \otimes \mathcal{O}_E(kE),$$

for $1 \leq k \leq r_\lambda$. Unfortunately the map:

$$\mu : E \rightarrow \sigma_+(G_w(A^3,A^6))$$

is not flat and $\sigma_+(G_w(A^3,A^6))$ is singular, thus $\mu^* D^b(\sigma_+(G_w(A^3,A^6)))$ lies a priori in $D^-(E)$ and not in $D^b(E)$ (we prove in the Appendix A that $\mu$ has infinite Tor-dimension, so that $\mu^* D^b(\sigma_+(G_w(A^3,A^6)))$ really lies in $D^-(E)$ and not in $D^b(E)$). Though $\mu^* D^-(\sigma_+(G_w(A^3,A^6)))$ is an admissible subcategory of $D^-(E)$, it is very unlikely (at least I am not able to prove it) that it is the negative completion of an admissible subcategory of $D^b(E)$.

At this point, one could argue that the definition of a categorical crepant resolution should be somehow modified and everything should be considered over $D^-(\tau(G_w(A^3,A^6)))$. Thus, a categorical resolution of $\tau(G_w(A^3,A^6))$ would be a triangulated category $\mathcal{T}$, with a natural functor:

$$\pi_{\mathcal{T}*} : \mathcal{T} \rightarrow D^-(\tau(G_w(A^3,A^6))),$$

such that $\mathcal{T}$ is an admissible subcategory of $D^-(Y)$, for some "geometric" resolution of singularities $\pi : Y \rightarrow \tau(G_w(A^3,A^6))$. We should again have:

$$\pi^* D^{\text{perf}}(\tau(G_w(A^3,A^6))) \subset \mathcal{T}$$

and crepancy would be described as before:

$$\pi_{\mathcal{T}}^*(\mathcal{F}) = \pi_{\mathcal{T}}^1(\mathcal{F}),$$

for all $\mathcal{F} \in D^{\text{perf}}(\tau(G_w(A^3,A^6)))$, where $\pi_{\mathcal{T}}$ and $\pi_{\mathcal{T}}^1$ are the left and right adjoint of $\pi_{\mathcal{T}*}$. However, this definition is not meaningful if one does not require that $\mathcal{T}$ comes from an admissible subcategory of $D^b(Y)$. Otherwise, the theorem of Grauert-Riemenschneider would show that for any resolution of singularities $\pi : Y \rightarrow \tau(G_w(A^3,A^6))$, the category $\pi^* D^- (\tau(G_w(A^3,A^6)))$ is always a categorical crepant resolution of $\tau(G_w(A^3,A^6))$. This is something we want to avoid, since we cannot consider $\pi^* D^- (\tau(G_w(A^3,A^6)))$ as a "smooth" triangulated category.

Hence, we see that we have to find another resolution of singularities of $\tau(G_w(A^3,A^6))$, which would allow us to work over $D^b(\tau(G_w(A^3,A^6)))$. 
Theorem 2.2.4  Let $\pi_1 : X_1 \to \tau(G_\omega(A^3, A^6))$ be the blow-up of $\tau(G_\omega(A^3, A^6))$ along $G_\omega(A^3, A^6)$ and let $\pi_2 : X_2 \to X_1$ be the blow-up of $X_1$ along the strict transform of $\sigma_+(G_\omega(A^3, A^6))$ through $\pi_1$. The variety $X_2$ is a resolution of singularities of $\tau(G_\omega(A^3, A^6))$.

Note that the strict transform of $\sigma_+(G_\omega(A^3, A^6))$ through $\pi_1$ (which we denote by $\pi_1^* \sigma_+(G_\omega(A^3, A^6))$) is the blow-up of $\sigma_+(G_\omega(A^3, A^6))$ along $G_\omega(A^3, A^6)$ and it is smooth by proposition 2.2.1. As a consequence, the sequence of blow-ups $\pi_1 : X_1 \to \tau(G_\omega(A^3, A^6))$ and $\pi_2 : X_2 \to X_1$ only consists of blow-ups along smooth centers (which we will later prove to be normally flat). In such a case, the projection of any exceptional divisor to the corresponding center of blow-up has finite Tor-dimension, which will be very convenient for us.

Unfortunately, we are not able to describe this resolution as the total space of a projective bundle over a flag variety. In fact, I believe that there is no projective bundle over a flag variety whose total space coincide with $X_2$. Thus, we have to check locally that this sequence of blow-ups really produces a resolution of singularities. We recall the equation of the tangent variety of $G_\omega(A^3, A^6) \subset \mathbb{P}(V_\lambda)$:

$$
\mathcal{Q}(\alpha, A, B, \beta) = (3\alpha \beta - \frac{1}{2}(A, B))^2 + \frac{1}{3}(\beta C(A^{\otimes 3}) + \alpha C^*(B^{\otimes 3})) - \frac{1}{6} \langle C^*(B^{\otimes 2}), C(A^{\otimes 2}) \rangle,
$$

where $(\alpha, A, B, \beta)$ is a system of coordinates for $V_\lambda = \mathbb{C} \oplus W_\lambda \oplus W_\lambda^* \oplus \mathbb{C}$. In the following we denote by $E_1$ the exceptional divisor of $\pi_1$, $E_2$ the exceptional divisor for $\pi_2$ and $E_1^{(2)}$ the strict transform of $E_1$ through $\pi_2$. Before diving into the proof of theorem 2.2.4 we introduce some more notations in the diagrams below:
The proof of this result will be divided into several steps.

Step 1: Tangent cones to $\tau (G_{\omega}(A^3, A^6))$ along its different orbits.

We are going to compute the tangent cones to $\tau (G_{\omega}(A^3, A^6))$ at points of its different strata. By $\text{Sp}_6(A)$-equivariance, the hypersurface $\tau (G_{\omega}(A^3, A^6))$ is normally flat along the orbit $\sigma_+(G_{\omega}(A^3, A^6)) \setminus G_{\omega}(A^3, A^6)$. So, proposition 2.2.2 shows that the tangent cone to $\tau (G_{\omega}(A^3, A^6))$ at any point $x \in \sigma_+(G_{\omega}(A^3, A^6)) \setminus G_{\omega}(A^3, A^6)$ is a cone over a smooth quadric of dimension $m_A + 1$ with vertex $T_{\sigma_+(G_{\omega}(A^3, A^6)), x}$ (where $T_{\sigma_+(G_{\omega}(A^3, A^6)), x}$ is the embedded tangent space to $\sigma_+(G_{\omega}(A^3, A^6))$ at $x$).

We also compute the tangent cone to $\tau (G_{\omega}(A^3, A^6))$ at $x \in G_{\omega}(A^3, A^6)$. Since $\tau (G_{\omega}(A^3, A^6))$ is invariant under the action of $\text{Sp}_6(A)$ and $G_{\omega}(A^3, A^6)$ is a closed orbit in $\tau (G_{\omega}(A^3, A^6))$, we only need to compute the tangent cone at any given point in $G_{\omega}(A^3, A^6)$, say $x_0 = (1, 0, 0, 0)$. The first partial derivatives of $\mathcal{Q}$ all vanish at $x_0$ (because we know that $\sigma_+(G_{\omega}(A^3, A^6))$ is the singular locus of $\tau (G_{\omega}(A^3, A^6))$). Furthermore, the polynomials $C(A^{\otimes 3})$ and $C^*(B^{\otimes 3})$ are homogeneous cubic polynomials in the variables $A$ and $B$, thus we have (with a slight abuse of notations):

$$
\frac{\partial^2 C(A^{\otimes 3})}{\partial A^2}(1, 0, 0, 0) = \frac{\partial^2 C^*(B^{\otimes 3})}{\partial B^2}(1, 0, 0, 0) = 0.
$$
The polynomial $C(A^\otimes 2)$ and $C^*(B^\otimes 2)$ are homogeneous of degree 2, thus we have:

$$\frac{\partial^2 \langle C^*(B^\otimes 2), C(A^\otimes 2) \rangle}{\partial A^2}(1, 0, 0, 0) = \frac{\partial^2 \langle C^*(B^\otimes 2), C(A^\otimes 2) \rangle}{\partial B^2}(1, 0, 0, 0) = 0.$$ 

The same type of arguments show that the only second partial derivative of $Q$ which does not vanish at $(1, 0, 0, 0)$ is $\frac{\partial^2 Q}{\partial \beta^2}(1, 0, 0, 0) = 18$. Thus the tangent cone to $\tau(G_x(A, A^6))$ at $x_0$ is given by the equation $18\beta^2 = 0$, this is a double hyperplane. This means that $E_1$, the exceptional divisor of $\pi_1 : X_1 \to \tau(G_x(A, A^6))$, is a fibration into doubled $\mathbb{P}^3_{\mathbb{R}} + 2$ over $G_x(A, A^6)$. Suppose that $|E_1|_{\text{red}}$ is a Cartier divisor on $X_1$. Then $X_1$ is smooth along $|E_1|_{\text{red}}$ because $|E_1|_{\text{red}}$ is smooth. But $\tau(G_x(A, A^6))$ is singular along $\sigma_+(G_x(A, A^6))$, so that $X_1$ is singular along $\pi_1^{-1}(\sigma_+(G_x(A, A^6)))|G_x(A, A^6))$ (because $\pi_1$ is an isomorphism outside $G_x(A, A^6)$). By semi-continuity of the multiplicity, $X_1$ is singular along the Zariski closure:

$$\pi_1^{-1}(\sigma_+(G_x(A, A^6)))|G_x(A, A^6)) = \pi_1^*(\sigma_+(G_x(A, A^6))).$$

But $E_1 \cap \pi_1^*(\sigma_+(G_x(A, A^6)))$ is not empty since it is the exceptional divisor of the blow-up of $\sigma_+(G_x(A, A^6))$ along $G_x(A, A^6)$. This is a contradiction and shows that $|E_1|_{\text{red}}$ is not Cartier on $X_1$.

The fact that $|E_1|_{\text{red}}$ is not Cartier on $X_1$ is a source of troubles. Indeed, we cannot discuss the smoothness of $X_1$ along $E_1 \cap E_1 \cap \pi_1^*(\sigma_+(G_x(A, A^6)))$. So we have to introduce an intermediate device which enables us to prove the smoothness of $X_2$.

Note that we proved that all tangent cones to $\tau(G_x(A, A^6))$ are at most quadratic, so that there is no point of multiplicity strictly bigger than two in $\tau(G_x(A, A^6))$.

**Step 2 : Resolution and polar divisors.**

**Step 2.1 : Strategy of the proof.**

Let $p = (p_0, P_1, P_2, p_3) \in \mathbb{P}(V_h)$ be a general point and let $P(\mathcal{O}, p)$ be the polar to $\tau(G_x(A, A^6))$ with respect to $p$, that is:

$$P(\mathcal{O}, p) = \tau(G_x(A, A^6)) \cap \{\mathcal{H}_p = 0\},$$

where $\mathcal{H}_p = p_0 \frac{\partial}{\partial a} + P_1 \frac{\partial}{\partial A} + P_2 \frac{\partial}{\partial B} + p_3 \frac{\partial}{\partial \beta} = 0$. It is clear that $\sigma_+(G_x(A, A^6)) = \tau(G_x(A, A^6))_{\text{sing}} \subset P(\mathcal{O}, p)$. Before going any further, we summarize the situation in the following diagram:
Our goal is to show that the strict transform of $P(\mathcal{Q}, p)$ through $\pi = \pi_1 \circ \pi_2$ (which we denote by $P(2)(\mathcal{Q}, p)$) is smooth. Indeed, if we do so, we get that $X_2$ is smooth along $P(2)(\mathcal{Q}, p)$ (because $P(2)(\mathcal{Q}, p)$ is a Cartier divisor on $X_2$). Moreover, if we can prove that $X_2$ is smooth along $E_2$ and along 

$$E_1^{(2)} \setminus \left( (P(2)(\mathcal{Q}, p) \cup E_2) \cap E_1^{(2)} \right) = E_1^{(2)} \setminus \left( E_1^{(2)} \cup E_{1,2} \right),$$

then we have won. Indeed, we already know that $\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ is smooth outside $P(\mathcal{Q}, p)$ so that $X_2$ is also smooth outside

$$P(2)(\mathcal{Q}, p) \cup E_2 \cup \left( E_1^{(2)} \setminus \left( E_1^{(2)} \cup E_{1,2} \right) \right)$$

because $\pi$ is an isomorphism outside this locus and we have:

$$\pi \left( P(2)(\mathcal{Q}, p) \cup E_2 \cup E_1^{(2)} \right) \subset P(\mathcal{Q}, p).$$

**Step 2.1 : Smoothness along $P(\mathcal{Q}, p)^{(2)}$.**

**Step 2.2.a : Tangent cones to the polar divisors.**

First, we show that $\{ \mathcal{H}_p = 0 \}$ is a smooth cubic hypersurface. Indeed, let $y \in \{ \mathcal{H}_p = 0 \}$ such that:

$$\frac{\partial \mathcal{H}_p}{\partial \alpha}(y) = \frac{\partial \mathcal{H}_p}{\partial A}(y) = \frac{\partial \mathcal{H}_p}{\partial B}(y) = \frac{\partial \mathcal{H}_p}{\partial \beta}(y) = 0.$$ 

Since $p$ is a general point, the above equalities imply that all second partial derivatives of $\mathcal{Q}$ vanish at $y$. But $\mathcal{Q}$ is a homogeneous polynomial, so that $\mathcal{Q}$
and all its first partial derivatives also vanish at $y$. As a consequence, $y$ is a point of multiplicity $3$ in $\tau(G_\omega(\mathbf{A}^3, \mathbf{A}^6))$, which is impossible by hypothesis.

Let us also prove that the tangent cone to $P(\mathcal{Q}, p)$ at any point $x \in \sigma_+(G_\omega(\mathbf{A}^3, \mathbf{A}^6)) \setminus G_\omega(\mathbf{A}^3, \mathbf{A}^6)$ is a cone over a smooth quadric of dimension $m_A$ with vertex $T_{\sigma_+(G_\omega(\mathbf{A}^3, \mathbf{A}^6))} x$.

Let $x \in \sigma_+(G_\omega(\mathbf{A}^3, \mathbf{A}^6)) \setminus G_\omega(\mathbf{A}^3, \mathbf{A}^6)$, the tangent space to $\{ H_p = 0 \}$ at $x$ is given by the equation:

\[
t_t(\alpha, A, B, \beta) \left( \frac{\partial^2 g}{\partial \alpha A}(x) \frac{\partial^2 g}{\partial \alpha B}(x) \frac{\partial^2 g}{\partial \alpha A}(x) \frac{\partial^2 g}{\partial \alpha B}(x) \right) (p_0, p_1, p_2, p_3) = 0
\]

and the tangent cone to $\tau(G_\omega(\mathbf{A}^3, \mathbf{A}^6))$ at $x$ is given by:

\[
t_t(\alpha, A, B, \beta) \left( \frac{\partial^2 g}{\partial \alpha A}(x) \frac{\partial^2 g}{\partial \alpha B}(x) \frac{\partial^2 g}{\partial \alpha A}(x) \frac{\partial^2 g}{\partial \alpha B}(x) \right) (\alpha, A, B, \beta) = 0.
\]

But we already showed that the tangent cone to $\tau(G_\omega(\mathbf{A}^3, \mathbf{A}^6))$ at any $x \in \sigma_+(G_\omega(\mathbf{A}^3, \mathbf{A}^6)) \setminus G_\omega(\mathbf{A}^3, \mathbf{A}^6)$ is a cone with vertex $T_{\sigma_+(G_\omega(\mathbf{A}^3, \mathbf{A}^6))} x$ over a smooth quadric of dimension $m_A + 1$. From this we deduce two facts:

- the projective dual of this tangent cone is a smooth quadric in $T_{\sigma_+(G_\omega(\mathbf{A}^3, \mathbf{A}^6))}^\perp x$,
- the image of the Hessian matrix of $\mathcal{Q}$ (seen as a map $\mathbb{P}(V_A) \to \mathbb{P}(V_A)^*$) is the whole $T_{\sigma_+(G_\omega(\mathbf{A}^3, \mathbf{A}^6))}^\perp x$.

As a consequence, since $p$ is general in $\mathbb{P}(V_A)$, the point:

\[
\left( \frac{\partial^2 g}{\partial \alpha A}(x) \frac{\partial^2 g}{\partial \alpha B}(x) \frac{\partial^2 g}{\partial \alpha A}(x) \frac{\partial^2 g}{\partial \alpha B}(x) \right) (p_0, p_1, p_2, p_3)
\]

does not lie in the projective dual to the tangent cone to $\tau(G_\omega(\mathbf{A}^3, \mathbf{A}^6))$ at $x$. This amounts to say that the intersection of $T_{\{ H_p = 0 \}} x$ with the tangent cone to $\tau(G_\omega(\mathbf{A}^3, \mathbf{A}^6))$ at $x$ is transverse. Hence, the tangent cone to $P(\mathcal{Q}, p)$ at $x$ is a cone over a smooth quadric of dimension $m_A$ with vertex $T_{\{ H_p = 0 \}, x}$. 

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Now, we are interested in the tangent cone to $P(\mathcal{Q}, p)$ at $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$. We will compute it at $x_0 = (1, 0, 0, 0)$ for simplicity. The Taylor expansion of $\mathcal{Q}$ at $x_0$ is:

$$
\mathcal{Q}(\alpha, A, B, \beta) = 9\beta^2 + \frac{1}{3}C^*(B^\otimes 3) - 3\beta(A, B) + \text{terms of order 4},
$$

and the expansion of $\mathcal{K}_p$ at $x_0$ is:

$$
\mathcal{K}_p(\alpha, A, B, \beta) = 18p_3\beta + \text{terms of order 2}.
$$

The tangent cone to $P(\mathcal{Q}, p)$ is defined by the ideal generated by all the leading forms of the equations in the ideal generated by $\mathcal{Q}$ and $\mathcal{K}_p$. Let $f = 2p_3\mathcal{Q} - (\beta + \frac{(A, B)}{3p_3})\mathcal{K}_p$. Then one checks that the Taylor expansion of $f$ at $x_0$ is:

$$
f(\alpha, A, B, \beta) = \frac{2p_3}{3}C^*(B^\otimes 3) + \beta.(\text{terms of order 2}) + \text{terms of order 4}.
$$

As a consequence, the tangent cone to $P(\mathcal{Q}, p)$ at $x_0$ (which we denote by $\mathcal{C}_{P(\mathcal{Q}, p), x_0}$) is given by the equation $\{\beta = 0\}$ and $\{C^*(B^\otimes 3) = 0\}$. This is the cone over the secant variety to $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}(V_A) = |\hat{\pi}_1^{-1}(x_0)|_{\text{red}}$ with vertex $T_{G_\omega(\mathbb{A}^3, \mathbb{A}^6), x_0}$. Notice that this tangent cone does not depend on the general point $p$ chosen to define the polar $P(\mathcal{Q}, p)$. Hence, by $\text{Sp}_6(\mathbb{A})$-equivariance, this is true for all $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$. Thus, for all $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$, the tangent cone $\mathcal{C}_{P(\mathcal{Q}, p), x}$ is the cone over the secant variety $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}(V_A) = |\hat{\pi}_1^{-1}(x)|_{\text{red}}$ with vertex $T_{G_\omega(\mathbb{A}^3, \mathbb{A}^6), x}$.

**Step 2.2.b : Explicit resolution of the polar divisors.**

Let $q_1 : P(\mathcal{Q}, p)^{(1)} \to P(\mathcal{Q}, p)$ be the blow-up of $P(\mathcal{Q}, p)$ along $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ ($P(\mathcal{Q}, p)^{(1)}$ is also the strict transform of $P(\mathcal{Q}, p)$ along $\pi_1$) and denote by $E'_1$ the exceptional divisor of that blow-up. The above description of the tangent cones of $P(\mathcal{Q}, p)$ at any $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ shows that the map $q_1 : E'_1 \to G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ is a fibration into secant varieties to $\mathbb{A}\mathbb{P}^2 \subset |\hat{\pi}_1^{-1}(x)|_{\text{red}}$, for $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$. Since $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}(V_A)$ is exactly the singular locus of its secant variety, the singular locus of $E'_1$ is a fibration into $\mathbb{A}\mathbb{P}^2$ over $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$. Moreover, by Proposition 2.2.1, the fiber over $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ of the exceptional divisor of the blow-up of $\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ along $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ is the secant variety to $\mathbb{A}\mathbb{P}^2 \subset \mathbb{P}(V_A) = |\hat{\pi}_1^{-1}(x)|_{\text{red}}$. Therefore, we have:

$$E'_{1, \text{sing}} = E'_1 \cap \pi_1^*\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) = E_1 \cap \pi_1^*\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)).$$
Note that $P(\mathcal{D}, p)^{(1)}$ is smooth along $E'_1 \setminus E'_{1 \text{sing}}$, because $E'_1$ is a Cartier divisor on $P(\mathcal{D}, p)^{(1)}$. We discussed the tangent cones to $P(\mathcal{D}, p)$ at points in $\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \setminus G_\omega(\mathbb{A}^3, \mathbb{A}^6)$; these are cones over smooth quadrics of dimension $m_A$ with vertex $T_{\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))}$. Thus for any $x$ in

$$\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \setminus \left( E'_1 \cap (\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \right),$$

the tangent cone to $P(\mathcal{D}, p)^{(1)}$ at $x$ is again a cone over a smooth quadric of dimension $m_A$ with vertex $T_{\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)))}$.

Let us compute the tangent cone to $P(\mathcal{D}, p)^{(1)}$ at any point $x \in E'_{1 \text{sing}}$. We know that $E'_1$ is a fibration into secant varieties of $\mathbb{A}P^2$ over $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$. But the tangent cone to this secant variety at any point $x \in \mathbb{A}P^2$ is a cone over a smooth quadric of dimension $m_A$ with vertex $T_{\mathbb{A}P^2}$, hence the tangent cone to $E'_1$ at $x \in E'_{1 \text{sing}}$ is a cone over a smooth quadric of dimension $m_A$ with vertex $T_{E'_{1 \text{sing}}}$, since $E'_1$ is a Cartier divisor in $P(\mathcal{D}, p)^{(1)}$, we have:

$$\text{mult } \partial_{P(\mathcal{D}, p)^{(1)}, x} \leq \text{mult } \partial_{E'_1, x} = 2,$$

for any $x \in E'_{1 \text{sing}}$. Moreover, we know that $\text{mult } \partial_{P(\mathcal{D}, p)^{(1)}, y} = 2$ for all $y \in \pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \setminus E'_{1 \text{sing}}$. Thus, by semi-continuity of the multiplicity, we have:

$$\text{mult } \partial_{P(\mathcal{D}, p)^{(1)}, x} = 2,$$

for all $x \in E'_{1 \text{sing}}$. Since the tangent cone to $E'_1$ at $x \in E'_{1 \text{sing}}$ is a cone over a smooth quadric of dimension $m_A$, with vertex $T_{E'_{1 \text{sing}}}$, we deduce that the tangent cone to $P(\mathcal{D}, p)^{(1)}$ at $x$ is a cone over the same smooth quadric of dimension $m_A$, but with vertex $T_{\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)))}$ (recall that $\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ is smooth by proposition 2.2.1).

Let $q_2 : P(\mathcal{D}, p)^{(2)} \to P(\mathcal{D}, p)^{(1)}$ be the blow-up of $P(\mathcal{D}, p)^{(1)}$ along $\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$; $(P(\mathcal{D}, p)^{(2)})$ is the strict transform of $P(\mathcal{D}, p)^{(1)}$ along $\pi_2$) and denote by $E'_2$ be the exceptional divisor of that blow-up. The above description of the tangent cones to $P(\mathcal{D}, p)^{(1)}$ at any $x \in \pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ shows that the map $q_2 : E'_2 \to \pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ is a fibration into smooth quadrics of dimension $m_A$. This implies that $E'_2$ is smooth, from which we deduce that $P(\mathcal{D}, p)^{(2)}$ is smooth along $E'_2$. Moreover, we proved that $P(\mathcal{D}, p)^{(1)}$ is smooth along $E'_1 \setminus (E'_1 \cap \pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)))$. As a consequence, $P(\mathcal{D}, p)^{(2)}$ is also smooth along $E'_1^{(2)}$, the total transform of $E'_1$ through $q_2$. Since $P(\mathcal{D}, p)$ is smooth outside $\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$, we get that $P(\mathcal{D}, p)^{(2)}$ is also smooth outside $E'_2 \cup E'_1^{(2)}$ and this completes the proof of the smoothness of $P(\mathcal{D}, p)^{(2)}$. 16
Finally $\mathcal{O}_{X_2}(P(\mathcal{O}, p)^{(2)}) = \pi^* \mathcal{O}_{X_2}(P(\mathcal{O}, p)) \otimes \mathcal{O}_{X_2}(k_1 E_1^{(2)} + k_2 E_2)$, where $k_1$ and $k_2$ are some integers. We deduce that $P(\mathcal{O}, p)^{(2)}$ is a Cartier divisor in $X_2$. Hence the smoothness of $P(\mathcal{O}, p)^{(2)}$ implies the smoothness of $X_2$ along $P(\mathcal{O}, p)^{(2)}$.

**Step 2.3 : Smoothness along $E_2$**

The Cartier divisor $E_2 \subset X_2$ is a fibration into smooth quadrics of dimension $m_\mathbb{A} + 1$ over $\pi_1 \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$, from which we deduce that it is smooth. As a consequence, the variety $X_2$ is also smooth along $E_2$.

**Step 2.4 : “Final step“: smoothness along $E_1^{(2)} \setminus \left( (P(\mathcal{O}, p)^{(2)} \cup E_2) \cap E_1^{(2)} \right)$**

In the following, we denote by $\text{SL}_3(\mathbb{A})$, the groups : $\text{SL}_3$, $\text{SL}_3 \times \text{SL}_3$, $\text{SL}_6$ and $E_6$.

We finally show that $X_2$ is smooth. The only fact left to demonstrate is that $X_2$ is smooth along:

$$E_1^{(2)} \setminus \left( (P(\mathcal{O}, p)^{(2)} \cup E_2) \cap E_1^{(2)} \right) = E_1^{(2)} \setminus \left( E_1^{(2)} \cup E_{1,2} \right).$$

To do so, we need to exploit the action of $\text{Sp}_6(\mathbb{A})$ on $\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$. The universal property of the blow-up implies that the stabilizer of $x$ in $\text{Sp}_6(\mathbb{A})$ acts on $\pi_1^{-1}(x)$. The reductive part of this stabilizer is $\text{SL}_3(\mathbb{A})$ (see [LM01]). Any non-trivial orbit closure of the action of this stabilizer on $|\pi_1^{-1}(x)|_{\text{red}}$ is an orbit closure for the action of $\text{SL}_3(\mathbb{A})$ on $\mathbb{P}(W_\mathbb{A})$. Hence, the orbit diagram of the action on $|\pi_1^{-1}(x)|_{\text{red}}$ of the stabilizer of $x$ in $\text{Sp}_6(\mathbb{A})$ is:

$$\mathbb{A} \mathbb{P}^2 \subset \sigma(\mathbb{A} \mathbb{P}^2) \subset \mathbb{P}(W_\mathbb{A}) = |\pi_1^{-1}(x)|_{\text{red}}.$$ The group $\text{Sp}_6(\mathbb{A})$ acts on $X_1$ and $E_1$ is stable under this action. The above description of the action on $|\pi_1^{-1}(x)|_{\text{red}}$ of the stabilizer of $x$ in $\text{Sp}_6(\mathbb{A})$ shows that the dense orbit in $|E_1|_{\text{red}}$ is the complement of $P(\mathcal{O}, p)^{(1)} \cap E_1 = E_1$. The group $\text{Sp}_6(\mathbb{A})$ also acts on $X_2$ and $E_1^{(2)}$ is the stable for this action. The dense orbit inside $|E_1^{(2)}|_{\text{red}}$ is the complement in $|E_1^{(2)}|_{\text{red}}$ of $E_1^{(2)} \cup E_{1,2}$. As a consequence, the multiplicity of $X_2$ along $|E_1^{(2)}|_{\text{red}} \setminus \left( E_1^{(2)} \cup E_{1,2} \right)$ is less than the multiplicity of $X_2$ along $E_{1,2}$. But we know that $X_2$ is smooth along $E_2$, so that $X_2$ is smooth along $E_1^{(2)} \setminus \left( E_1^{(2)} \cup E_{1,2} \right)$ and we are done! ▪

In fact, we believe that a much more general statement than Theorem 2.2.4 holds. To state our conjecture, we need some recollections on prehomogeneous vector spaces (we refer to [Kim03] for a detailed treatment of prehomogeneous spaces).
Definition 2.2.5  A strongly prehomogeneous vector space is the data \((G, V)\) of an algebraic group \(G\) acting linearly on a finite dimensional vector space \(V\) with a finite number of orbits.

Let us denote by \(V_G^0, \ldots, V_G^m\) the orbits of \(G\) on \(V\). We say that the orbit diagram of \((G, V)\) is linear if \(V_G^0 = \{0\}\) and up to a reordering, we have:

\[ V_G^i \subset \overline{V_G^{i+1}}, \]

for all \(i \geq 0\), where \(\overline{V_G^{i+1}}\) denotes the Zariski closure of \(V_G^{i+1}\).

Example 2.2.6  
- The square determinantal varieties of size \(n\) are the orbits of the action of \(\text{GL}_n \times \text{GL}_n\) on \(\mathbb{C}^n \otimes \mathbb{C}^n\). Their orbit diagram is linear.
- The symmetric (resp. skew-symmetric) determinantal varieties of size \(n\) are the orbits of the action of \(\text{GL}_n\) (resp. \(\text{GL}_n\)) on \(S^2 \mathbb{C}^n\) (resp. \(\wedge^2 \mathbb{C}^n\)). Their orbit diagram is also linear.
- The pair \((\mathbb{C}^* \times \text{Sp}_6(\mathbb{A}), V_A)\) is a strongly prehomogeneous vector space whose orbit diagram is again linear.
- The pair \((\text{GL}_8, \wedge^3 \mathbb{C}^8)\) is a strongly prehomogeneous space whose orbit diagram is not linear (see [Hol11]).
- The pair \((\text{GL}_9, \wedge^3 \mathbb{C}^9)\) is not a prehomogeneous space (see [Hol11]).

We can now state our conjecture:

**Conjecture 2.2.7**  Let \((G, V)\) be a strongly prehomogenous vector space whose orbit diagram \(\{V_G^0, \ldots, V_G^m\}\) is linear and let \(X = \mathbb{P}(\overline{V_G^1})\) be the projectivization of the closure of any orbit. Consider the sequence:

\[ X_i \xrightarrow{\pi_i} X_{i-1} \rightarrow \cdots \rightarrow X_2 \xrightarrow{\pi_2} X_1 = X, \]

where \(\pi_k : X_k \rightarrow X_{k-1}\) is the blow-up of the strict transform of \(\mathbb{P}(\overline{V_G^{k-1}})\) through \(\pi_1 \circ \ldots \circ \pi_{k-1}\). Then \(X_i\) is smooth.

This conjecture is well-known for all square, symmetric and skew-symmetric determinantal varieties (see Example 2.1.3 of [Abu12]). Theorem 2.2.4 and Proposition 2.2.1 show that the conjecture holds for the pair \((\mathbb{C}^* \times \text{Sp}_6(\mathbb{A}), V_A)\).
2.3 Some vanishing lemmas

In this section, we state the vanishing lemmas we will need for the proof of our main theorem. Recall that $E_1$, the exceptional divisor of the map $\pi_1 : X_1 \to \tau(G_\omega(A^3, A^6))$, is a fibration into doubled $\mathbb{P}^{3m_\lambda + 2}$ over $G_\omega(A^3, A^6)$. As for $E_2$, the exceptional divisor of the map $\pi_2 : X_2 \to X_1$, it is a fibration in smooth quadrics of dimension $m_\lambda + 1$ over the strict transform of $\sigma_+(G_\omega(A^3, A^6))$ through $\pi_1$.

We also recall some notations we used in the proof of Theorem 2.2.4. $E^{(2)}_1$ denotes the total transform of $E_1$ through $\pi_2$. Since the intersection of the proper transform of $\sigma_+(G_\omega(A^3, A^6))$ through $\pi_1$ (which we denote by $\pi^*_1\sigma_+(G_\omega(A^3, A^6))$) with $E_1$ is proper, the divisor $E^{(2)}_1$ is also the blow-up of $E_1$ along $\pi^*_1\sigma_+(G_\omega(A^3, A^6)) \cap E_1$. The divisor $E_{1,2}$ the intersection $E^{(2)}_1 \cap E_2$, which is also the exceptional divisor of the blow-up of $E_1$ along $\pi^*_1\sigma_+(G_\omega(A^3, A^6)) \cap E_1$. The morphism $\pi$ is the composition $\pi_1 \circ \pi_2$ and $\pi_{1,2}$ is the restriction of $\pi_2$ to $E^{(2)}_1$. Finally, we denote by $\tilde{\pi}_1$ (resp. $\tilde{\pi}_2$, $\tilde{\pi}$ and $\tilde{\pi}_{1,2}$) the restriction of $\pi_1$ (resp. $\pi_2$, $\pi$ and $\pi_{1,2}$) to $E_1$ (resp. $E_2$, $E^{(2)}_1$ and $E_{1,2}$). We summarize these notations in the following diagrams (which already appeared in the proof of 2.2.4).

![Diagram](image_url)
We start with transformation formulas for the canonical bundle through the maps $\pi_1$ and $\pi_2$.

**Lemma 2.3.1** We have the formulas:

$$\omega_{X_1} = \pi_1^*\omega_{\tau(G_\omega(A^3, A^6))} \otimes \mathcal{O}_{X_1}((-3m_A + 1)E_1),$$

and

$$\omega_{X_2} = \pi_2^*\omega_{\tau(G_\omega(A^3, A^6))} \otimes \mathcal{O}_{X_2}(m_A E_2).$$

The existence of an integer $p$ such that $\omega_{X_2} = \pi_2^*\omega_{\tau(G_\omega(A^3, A^6))} \otimes \mathcal{O}_{X_2}(pE_2)$ is trivial as $E_2$, the scheme-theoretic exceptional locus of $\pi_2$, is an integral divisor on $X_2$. The existence of such a formula for $\omega_{X_1}$ is less obvious. Indeed, since $E_1$ is not reduced, one could imagine an equality $\omega_{X_1} = \pi_1^*\omega_{\tau(G_\omega(A^3, A^6))} \otimes \mathcal{O}_{X_1}(qE'_1)$, where $E'_1$ is a Cartier divisor on $X_1$ with $|E_1|_{\text{red}} = |E'_1|_{\text{red}}$, but such that $qE'_1$ is not a multiple of $E_1$.

**Proof:**

We start with the formula for $\omega_{X_1}$. We divide the proof of this formula into two steps:

- we prove that the blow-up $\pi_1 : X_1 \to \tau(G_\omega(A^3, A^6))$ is the contraction of a negative extremal ray (see [KM98], section 3),
- we prove that the bundle $\omega_{X_1} \otimes \pi_1^*\omega_{\tau(G_\omega(A^3, A^6))}^{-1} \otimes \mathcal{O}_{X_1}((-3m_A - 1)E_1)$ is trivial.

**Step 1 : The blow-up $X_1 \to \tau(G_\omega(A^3, A^6))$ is a Mori contraction.**

Let $\widetilde{\mathbb{P}(V_A)}$ be the blow-up of $\mathbb{P}(V_A)$ along $G_\omega(A^3, A^6)$ and denote by $H_1$ the exceptional divisor of that blow-up. We have $E_1 = H_1|_{X_1}$. The map
$q_1 : H_1 \to G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ is a projective bundle of relative dimension $3m_H + 3$ over $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ and the restriction of $H_1$ to any fiber $q_1^{-1}(x) = \mathbb{P}^{3m_H+3}$ is $\mathcal{O}_{\mathbb{P}^{3m_H+3}}(1)$. As a consequence, we have the equality:

$$E_1|_{\pi^{-1}_1(x)} = \mathcal{O}_{\mathbb{P}^{3m_H+3}}(1)|_{\pi^{-1}_1(x)},$$

for all $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$. We will denote this last bundle by $\mathcal{O}_{\tilde{\pi}^{-1}_1(x)}(1)$. Recall that the proof of Theorem 2.2.4 shows that $\tilde{\pi}^{-1}_1(x)$ is a doubled $\mathbb{P}^{3m_H+2}$ in $q_1^{-1}(x)$. Thus we have:

$$\omega_{\tilde{\pi}^{-1}_1(x)} = \mathcal{O}_{\tilde{\pi}^{-1}_1(-3m_H - 2)}.$$

By the adjunction formula we have $\omega_{E_1} = \omega_{X_1} \otimes \mathcal{O}_{E_1}(E_1)$. The morphism $\tilde{\pi}_1 : E_1 \to G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ is flat and $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ is smooth so that the normal bundle $N_{\tilde{\pi}^{-1}_1(x)}/E_1$ is trivial. By the adjunction formula, we get:

$$\omega_{E_1}|_{\tilde{\pi}^{-1}_1(x)} = \omega_{\tilde{\pi}^{-1}_1(x)} = \mathcal{O}_{\tilde{\pi}^{-1}_1}(-3m_H - 2)$$

Let $NE_{\pi_1}(X)$ be the cone of effective 1-cycles in $X$ contracted by $\pi_1$ and let $R \in NE_{\pi_1}(X)$ be the numerical class of a line in $\pi_1^{-1}(x)$. The above formula shows that:

$$\omega_{X_1}.R < 0,$$

so that $R$ is a negative ray for $X_1$ with respect to $\pi_1$. Let us prove that $NE_{\pi_1}(X) = \langle R \rangle$. We have an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^{3m_H+2}}(-1) \to \mathcal{O}_{\mathbb{P}^{3m_H+2}} \to \mathcal{O}_{\mathbb{P}^{3m_H+2}} \to 0,$$

where $\mathcal{O}_{\mathbb{P}^{3m_H+2}}$ is the structure sheaf of a doubled $\mathbb{P}^{3m_H+2}$ in $\mathbb{P}^{3m_H+3}$. Note that $\mathcal{O}_{\mathbb{P}^{3m_H+2}}(-1)$ consists of nilpotent elements of $\mathcal{O}_{\mathbb{P}^{3m_H+2}}$, so we can lift the above exact sequence to an exact sequence of groups sheaves (see [Gro], ExposA© XI, section 1):

$$0 \to \mathcal{O}_{\mathbb{P}^{3m_H+2}}(-1) \to \mathcal{O}_{\mathbb{P}^{3m_H+2}}^\times \to \mathcal{O}_{\mathbb{P}^{3m_H+2}}^\times \to 1,$$

where $\mathcal{O}_X^\times$ is the sheaf of units of the scheme $X$. Taking the long exact sequence of cohomology, we find that:

$$H^1(2\mathbb{P}^{3m_H+2}, \mathcal{O}_{\mathbb{P}^{3m_H+2}}^\times) = H^1(\mathbb{P}^{3m_H+2}, \mathcal{O}_{\mathbb{P}^{3m_H+2}}^\times),$$

that is:

$$\text{Pic}(2\mathbb{P}^{3m_H+2}) = \text{Pic}(\mathbb{P}^{3m_H+2}) = \mathbb{Z}.$$
Thus, we see that the cone of effective 1-cycles (modulo numerical equivalence) on $\pi_1^{-1}(x)$ is of dimension 1. Since the morphism $\tilde{\pi}_1 : E_1 \to \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ is flat, the cone $NE_{\pi_1}(X)$ is also of dimension 1. Hence, we have:

$$NE_{\pi_1}(X) = \langle R \rangle,$$

so that $R$ is a negative extremal ray for $NE_{\pi_1}(X)$. As $X_1$ is Gorenstein with rational singularities (hence canonical singularities, by [Ko97, proposition 11.13]), we can apply the relative Cone theorem to $X_1$ and $R$ (see [KM98, Theorem 3.25]) and we find a commutative diagram:

$$\begin{array}{ccc}
X_1 & \xrightarrow{p} & Y \\
\downarrow{\pi_1} & & \downarrow{q} \\
\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) & & \\
\end{array}$$

where $p$ is the contraction of the extremal ray $R$. We know that $NE_{\pi_1}(X) = \langle R \rangle$. Therefore, for all $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$, all the curves lying in $\pi_1^{-1}(x)$ are contracted by $p$.

We want to demonstrate that $q$ is an isomorphism. Let $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ and assume that $\dim p(\pi_1^{-1}(x)) > 0$. We can find two curves $C \in p(\pi_1^{-1}(x))$ and $C' \subset \pi_1^{-1}(x)$ such that $p(C') = C$. But this is a contradiction since all curves lying in $\pi_1^{-1}(x)$ are contracted by $p$. As a consequence, for all $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$, the scheme $p(\pi_1^{-1}(x))$ is a point. We deduce that $q : Y \to \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ is a birational finite morphism such that $Rq_* \mathcal{O}_Y = \mathcal{O}_{\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))}$. The variety $\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ is normal, so by the Main Theorem of Zariski, the morphism $q$ is an isomorphism. We deduce that $\pi_1 : X_1 \to \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ is the contraction of the extremal ray generated by $R$.

Step 2 : The bundle $\omega_{X_1} \otimes \pi_1^*\omega_{\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))} \otimes \mathcal{O}_{X_1}((-3m_A - 1)E_1)$ is trivial.

Let $L = \omega_{X_1} \otimes \mathcal{O}_{X_1}((-3m_A - 1)E_1)$. The formulae for the restrictions of $\omega_{X_1}$ to $E_1$ and $\omega_{E_1}$ to $\pi_1^{-1}(x)$ show that $L_{\pi_1^{-1}(x)} = \mathcal{O}_{\pi_1^{-1}(x)}$, for any $x \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$. Thus, we can apply again the relative Cone Theorem and we get $L = \pi_1^*L'$ for some line bundle $L'$ on $\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$.

Let us prove that $L' = \omega_{\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))}$. Since $\mathbf{R}\pi_1_* \mathcal{O}_{X_1} = \mathcal{O}_{\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))}$ and $\dim X_1 = \dim \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$, Grothendieck duality shows that the complex $\mathbf{R}\pi_1_* \omega_{X_1} [\dim X_1]$ is a dualizing complex for $\tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$. We apply the Grauert-Riemenschneider theorem to $\pi_1$ and we get $\mathbf{R}^i\pi_1_* \omega_{X_1} = 0$ for $i > 0$. 

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As a consequence, we have \( \pi_1 \omega_{X_1} = \omega_{\tau(\mathbb{G}_m^3, \mathbb{A}^6)} \). Moreover, the divisor \( E_1 \) is effective and contracted by \( \pi_1 \). The variety \( \tau(\mathbb{G}_m^3, \mathbb{A}^6) \) being normal and \( \pi_1 \) being birational, the sheaf \( \pi_1 \mathcal{O}_{X_1}(3m_A + 2E_1) \) is trivial. Finally, we apply \( \pi_1 \ast \) on both sides of the equality:

\[
\omega_{X_1} = \pi_1^\ast (L') \otimes \mathcal{O}_{X_1}(3m_A + 1),
\]

and the projection formula gives:

\[
\pi_1 \ast \omega_{X_1} = L'.
\]

As we showed that \( \pi_1 \ast \omega_{X_1} = \omega_{\tau(\mathbb{G}_m^3, \mathbb{A}^6)} \), this concludes the proof that:

\[
\omega_{X_1} = \pi_1^\ast \omega_{\tau(\mathbb{G}_m^3, \mathbb{A}^6)} \otimes \mathcal{O}_{X_1}(3m_A + 1E_1).
\]

The formula for \( \omega_{X_2} \) is proved in a similar fashion, but is much easier. Indeed, as \( E_2 \) is the scheme-theoretic exceptional locus of \( \pi_2 \) and is an integral divisor on \( X_2 \), there exists an integer \( p \) such that \( \omega_{X_2} = \pi_2^\ast \omega_{X_1} \otimes \mathcal{O}_{X_2}(pE_2) \). We determine the integer \( p \) by restricting this equality to the fibers of \( \pi_2|_{E_2} : E_2 \to \pi_2^\ast \sigma_+ (\mathbb{G}_m^3, \mathbb{A}^6) \). The adjunction formula shows again that

\[
\omega_{X_2} \otimes \mathcal{O}_{E_2}(E_2) \otimes \mathcal{O}_{\pi_2^{-1}(x)} = \omega_{\pi_2^{-1}(x)}
\]

and we conclude using the fact that \( \pi_2^{-1}(x) \) is a smooth quadric of dimension \( m_A + 1 \).

\[\square\]

**Proposition 2.3.2** We have the vanishings:

- \( \text{R}^i \pi_2^\ast \mathcal{O}_{E_2}(kE_2) = 0 \), for all \( i \geq 0 \) and for all \( 1 \leq k \leq m_A \),
- \( \text{R}^i \pi_2^\ast \mathcal{O}_{E_1^{(2)}}(kE_1^{(2)}) = 0 \), for all \( i \geq 0 \) and for all \( 1 \leq k \leq 3m_A + 1 \),
- \( \text{R}^i \pi_1^\ast \mathcal{O}_{E_{1,2}}(kE_{1,2}) \), for all \( i \geq 0 \) and for all \( 1 \leq k \leq m_A \).

**Proof:**

The three points are more or less direct consequences of the Kawamata-Viehweg relative vanishing theorem and of the Grauert-Riemenschneider vanishing theorem.

For the first point, we have \( \omega_{X_2} = \pi_2^\ast \omega_{X_1} \otimes \mathcal{O}_{X_2}(m_AE_2) \) and \( -E_2 \) is relatively ample with respect to \( \pi_2 \). Since \( X_2 \) is Gorenstein with rational singularities (in fact it is smooth), we apply the Kawamata-Viehweg relative vanishing theorem and we get:

\[
\text{R}^1 \pi_2^\ast \mathcal{O}_{X_2}(kE_2) = 0,
\]
for all $i > 0$ and for all $k < m_A$. The vanishing:

$$R^1\pi_2*\mathcal{O}_{X_2}(m_AE_2) = 0,$$

for all $i > 0$, is a consequence of the theorem of Grauert-Riemenschneider.

Now, for all $k \in \mathbb{Z}$, we have an exact sequence:

$$0 \to \mathcal{O}_{X_2}((k - 1)E_2) \to \mathcal{O}_{X_2}(kE_2) \to \mathcal{O}_{E_2}(kE_2) \to 0.$$

We take the long exact sequence associated to the functor $R\pi_2*$ and taking into account the above vanishing, we find:

$$R^1\tilde{\pi}_2*\mathcal{O}_{E_2}(kE_2) = 0,$$

for all $i > 0$ and for all $k \leq m_A$.

Finally, we want to prove some vanishing for $\tilde{\pi}_2*\mathcal{O}_{E_2}(kE_2)$. Notice first that $E_2$ is an effective divisor contracted by the birational morphism $\pi_2$. The variety $X_1$ being normal, we have:

$$\pi_2*\mathcal{O}_{X_2}(kE_2) = \mathcal{O}_{X_1},$$

for all $k \geq 0$. Thus, the long exact sequence associated to the above short exact sequence and the vanishing results already proved imply:

$$\tilde{\pi}_2*\mathcal{O}_{E_2}(kE_2) = 0,$$

for all $k \geq 1$. This concludes the first point.

The second point is proved in the same manner with the following observation. We have $R\pi_2*,\mathcal{O}_{X_2}(E_1^{(2)}) = \mathcal{O}_{X_1}(E_1)$ by the projection formula. Thus, to prove the vanishing result for $R^1\tilde{\pi}_1*\mathcal{O}_{E_1^{(2)}}(kE_1^{(2)})$, it is sufficient to prove it for $R^1\tilde{\pi}_1*\mathcal{O}_{E_1}(kE_1)$. This is done exactly in the same way as for the first point of the proposition.

The third point needs a slightly more involved argument. The intersection $E_1^{(2)} \cap E_2 = E_{1,2}$ is proper, so we have a resolution:

$$0 \to \mathcal{O}_{X_2}(-E_1^{(2)} - E_2) \to \mathcal{O}_{X_2}(-E_1^{(2)}) \oplus \mathcal{O}_{X_2}(-E_2) \to \mathcal{O}_{X_2} \to \mathcal{O}_{E_{1,2}} \to 0.$$

We tensor this resolution by $\mathcal{O}_{X_2}(kE_2)$, for any integer $k$, and we get:

$$0 \to \mathcal{O}_{X_2}(-E_1^{(2)} + (k - 1)E_2) \to \mathcal{O}_{X_2}(-E_1^{(2)}) \oplus \mathcal{O}_{X_2}((k - 1)E_2) \to \mathcal{O}_{X_2}(kE_2) \to \mathcal{O}_{E_{1,2}}(kE_{1,2}) \to 0.$$
Recall that the Kawamata-Viehweg relative vanishing theorem and the Grauert-Riemenschneider vanishing theorem imply that
\[ R^i \pi_2_\ast \mathcal{O}_{X_2}(kE_2) = 0, \]
for all \( i > 0 \) and all \( k \leq m_A \). Finally, we chop the above resolution into two short exact sequences. We take the long exact sequences associated to the functor \( R\pi_2_\ast \) for these two short exact sequences and we find:
\[ R^i \tilde{\pi}_{1,2}\ast \mathcal{O}_{E_{1,2}}(kE_{1,2}) = 0, \]
for all \( i > 0 \) and all \( k \leq m_A \).

The vanishing:
\[ \tilde{\pi}_{1,2}\ast \mathcal{O}_{E_{1,2}}(kE_{1,2}) = 0, \]
for all \( k \geq 1 \) is proved as for the first point of the proposition. Indeed, we have:
\[ \pi_2\ast \mathcal{O}_{X_2}(kE_2) = \mathcal{O}_{X_1}, \]
for all \( k \geq 0 \). We again chop the above long exact sequence into two short exact sequences and we go on as in the proof of the first point of the proposition.

3 Proof of the main theorem

In this section we are going to prove our main result:

**Theorem 3.0.3** The variety \( \tau(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \) admits a categorical crepant resolution of singularities.

From now on, for any proper morphism \( f : X \to Y \) of schemes of finite type, we denote by \( f_\ast \) the total derived functor \( Rf_\ast : D^b(X) \to D^b(Y) \), by \( f^\ast \) the total derived functor \( Lf^\ast : D^-(Y) \to D^-(X) \) and by \( f! \) the right adjoint to \( Rf_\ast : D^b(X) \to D^b(Y) \). In case we need to use specific homology sheaves of these functors, we will denote them by \( R^i f_\ast, L^i f^\ast \) and \( L^i f! \). If \( F, G \) are two objects of \( D^-(X) \), we denote by \( F \otimes^L G \) the derived tensor product \( F \otimes^L G \).
### 3.1 Standard reductions

Denote by $i_1 : E_1^{(2)} \hookrightarrow X_2$ and $i_2 : E_2 \hookrightarrow X_2$ the embeddings of the exceptional divisors. We define the following subcategories of $D^b(X_2)$:

$$\mathcal{B}_k = i_{2*} \left( \pi_2^* D^b(\pi_1^* \sigma_+ (G_\omega(A^3, A^6)) \otimes \mathcal{O}_{E_2}(kE_2) \right),$$

for all $1 \leq k \leq m_\delta$ and:

$$\mathcal{A}_l = i_{1*} \left( \pi_1^* D^b(G_\omega(A^3, A^6)) \otimes \mathcal{O}_{E_1^{(2)}}(lE_1^{(2)} + m_\delta E_2) \right),$$

for all $1 \leq l \leq 3m_\delta + 1$. Our key proposition is the following:

**Proposition 3.1.1** We have a semi-orthogonal decomposition:

$$D^b(X_2) = \langle \mathcal{A}_{3m_\delta + 1}, \ldots, \mathcal{A}_1, \mathcal{B}_{m_\delta}, \ldots, \mathcal{B}_1, \mathcal{D}_{X_2} \rangle,$$

where $\mathcal{D}_{X_2}$ is the left orthogonal to the full admissible subcategory generated by the $\mathcal{A}_i$ and $\mathcal{B}_k$. Moreover we have the property:

$$\pi^* D^{\per}(\tau(G_\omega(A^3, A^6))) \subset \mathcal{D}_{X_2}.$$

Before diving into the proof of Proposition 3.1.1, we explain how it implies our main result. We will prove that $\mathcal{D}_{X_2}$ is a categorical crepant resolution of singularities of $\tau(G_\omega(A^3, A^6))$.

**Proof:**

First note that $\mathcal{D}_{X_2}$ is an admissible subcategory of $D^b(X_2)$ and that $\pi^* D^{\per}(\tau(G_\omega(A^3, A^6))) \subset \mathcal{D}_{X_2}$. Thus, we only have to prove that for all $\mathcal{F} \in D^{\per}(\tau(G_\omega(A^3, A^6)))$, we have:

$$\pi_\mathcal{D}^* (\mathcal{F}) \simeq \pi_\mathcal{\mathcal{D}}^! (\mathcal{F}),$$

where $\pi^\mathcal{D}$ and $\pi_\mathcal{\mathcal{D}}^!$ are the left and right adjoints to the functor

$$\pi_\mathcal{\mathcal{D}} : \mathcal{D}_{X_2} \to D^b(\tau(G_\omega(A^3, A^6))).$$

Let $\delta : \mathcal{D}_{X_2} \hookrightarrow D^b(X_2)$ be the fully faithful admissible embedding. We must prove that $\delta^* \pi^* (\mathcal{F}) = \delta^! \pi^! (\mathcal{F})$, for all $\mathcal{F} \in D^{\per}(\tau(G_\omega(A^3, A^6)))$. Recall that

$$\pi^! (\mathcal{F}) = \pi^* (\mathcal{F}) \otimes \pi^* \left( \omega_{\tau(G_\omega(A^3, A^6))}^{-1} \right) \otimes \omega_{X_2} = \pi^* (\mathcal{F}) \otimes \mathcal{O}_{X_2}(m_\delta E_2 + (3m_\delta + 1) E_1^{(2)}).$$

Now, since the functor $\delta$ is fully faithful, the equality $\delta^* \pi^* (\mathcal{F}) = \delta^! \pi^! (\mathcal{F})$ is equivalent to $\delta(\delta^* \pi^* (\mathcal{F})) = \delta(\delta^! \pi^! (\mathcal{F}))$. As $\pi^* D^{\per}(\tau(G_\omega(A^3, A^6))) \subset \mathcal{D}_{X_2}$,
we have $\delta(\delta^*\pi^*(\mathcal{F})) = \pi^*(\mathcal{F})$. We are going to show that $\delta(\delta^!\pi^!(\mathcal{F})) = \pi^!(\mathcal{F})$.

For $1 \leq k \leq m_A$ and for $1 \leq l \leq 3m_A + 1$, we have exact sequences:

$$0 \to \mathcal{O}_X((k - 1)E_2) \to \mathcal{O}_X(kE_2) \to i_2*\mathcal{O}_{E_2}(kE_2) \to 0,$$

$$0 \to \mathcal{O}_X((l - 1)E_1(2) + m_AE_2) \to \mathcal{O}_X(lE_1(2) + m_AE_2) \to i_1*\mathcal{O}_{E_1}(lE_1(2) + m_AE_2) \to 0.$$

Tensoring the above exact sequences with $\pi^*\mathcal{F}$, we get exact triangles:

$$\mathcal{O}_X((k - 1)E_2) \otimes \pi^*\mathcal{F} \to \mathcal{O}_X(kE_2) \otimes \pi^*\mathcal{F} \to i_2*\mathcal{O}_{E_2}(kE_2) \otimes \pi^*\mathcal{F},$$

$$\mathcal{O}_X((l - 1)E_1(2) + m_AE_2) \otimes \pi^*\mathcal{F} \to \mathcal{O}_X(lE_1(2) + m_AE_2) \otimes \pi^*\mathcal{F} \to i_1*\mathcal{O}_{E_1}(lE_1(2) + m_AE_2) \otimes \pi^*\mathcal{F}.$$ 

We deduce a long sequence of triangles:

$$\pi^*(\mathcal{F}) \to F_1^{(2)} \to \ldots \to F_{m_A}^{(2)} \to F_1^{(1)} \to \ldots \to F_{3m_A}^{(1)} \to F_{3m_A+1}^{(1)}$$

where $F_k^{(2)} = \pi^*\mathcal{F} \otimes \mathcal{O}_X(kE_2)$, $\mathcal{F}_k^{(2)} = i_2*\left(\mathcal{O}_{E_2}(kE_2) \otimes i_2^*\pi^*\mathcal{F}\right)$, $F_l^{(1)} = \pi^*\mathcal{F} \otimes \mathcal{O}_X(lE_1(2) + m_AE_2)$ and $\mathcal{F}_l^{(1)} = i_1*\left(\mathcal{O}_{E_1}(lE_1(2) + m_AE_2) \otimes i_1^*\pi^*F\right)$.

But we have commutative diagrams:
so that

\[ i_2^* \tilde{\pi}^* \mathcal{F} = \tilde{\pi}_2^* j_2^* \pi_1^* \mathcal{F} \subset \tilde{\pi}_2^* \tau \left( G_\omega(\mathbb{A}^3, \mathbb{A}^6) \right) \]

and

\[ i_1^* \pi^* \mathcal{F} = \tilde{\pi}^* j_1^* \mathcal{F} \subset \tilde{\pi}^* \tau \left( G_\omega(\mathbb{A}^3, \mathbb{A}^6) \right). \]

Thus, \( \pi^* \mathcal{F} \) is the \( \mathcal{D}_{X_2} \)-component of \( F^{(1)}_{3m+1} = \pi_1^*(\mathcal{F}) \) in the semi-orthogonal decomposition of proposition \[3.1.1]\. As a consequence, we have \( \pi^* \mathcal{F} = \delta \delta^!(\pi_1^!(\mathcal{F})) \), which is what we wanted to prove.

\[ \blacksquare \]

### 3.2 The key proposition

In this section, we prove Proposition \[3.1.1\]. We first recall the statement of Proposition 4.1 of \[Kuz08\]:

**Proposition 3.2.1 (Kuznetsov's Lefschetz decomposition)** Let \( E \) be a Cartier divisor on a variety \( X \). Assume that there is a semi-orthogonal decomposition:

\[ D^b(E) = \langle \mathcal{A}_m \otimes \mathcal{O}_E(mE), \ldots, \mathcal{A}_1 \otimes \mathcal{O}_E(E), \mathcal{A}_0 \rangle, \]

with \( \mathcal{A}_m \subset \ldots \subset \mathcal{A}_0 \) admissible subcategories of \( D^b(E) \). Then there is a semi-orthogonal decomposition:

\[ D^b(X) = \langle i_*(\mathcal{A}_m \otimes \mathcal{O}_E(mE)), \ldots, i_*(\mathcal{A}_1 \otimes \mathcal{O}_E(E)), \mathcal{D} \rangle, \]

where \( i : E \hookrightarrow X \) is the natural inclusion and \( \mathcal{D} = \{ \mathcal{F} \in D^b(X), i^* \mathcal{F} \in \mathcal{A}_0 \} \).

This result will be very useful to deduce semi-orthogonal decompositions on \( X_2 \), starting from semi-orthogonal decompositions on \( E_2 \) and \( E_1^{(2)} \). To prove \[3.1.1\] we need the following lemma:

**Lemma 3.2.2** We have the following semi-orthogonal decomposition:

\[ D^b(E_2) = \langle (T_{E_2} \otimes \mathcal{O}_{E_2}(mE_2)), \ldots, (T_{E_2} \otimes \mathcal{O}_{E_2}(E_2)), \mathcal{E}_2 \rangle, \]
where \( T_{E_2} = \tilde{\pi}_2^* \mathbb{D}(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \) and \( \mathcal{E}_2 \) is the left orthogonal to the subcategory generated by the \( T_{E_2} \otimes \mathcal{O}_{E_2}(kE_2) \), for \( 1 \leq k \leq m_A \). Moreover, we have the inclusion:

\[
\tilde{\pi}_2^* \mathbb{D}(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \subset \mathcal{E}_2.
\]

We also have the semi-orthogonal decomposition:

\[
\mathbb{D}^b(E_1^{(2)}) = (T_{E_1^{(2)}} \otimes \mathcal{O}_{E_1^{(2)}})((3m_A + 1)E_1^{(2)} + m_A E_2), \ldots, T_{E_1^{(2)}} \otimes \mathcal{O}_{E_1^{(2)}}(E_1^{(2)} + m_A E_2), i_{1,2*} \left( T_{E_1,2} \otimes \mathcal{O}_{E_1,2}(m_A E_{1,2}) \right), \ldots, i_{1,2*} \left( T_{E_1,2} \otimes \mathcal{O}_{E_1,2}(E_{1,2}) \right), \mathcal{E}_1^{(2)},
\]

with \( T_{E_1^{(2)}} = \tilde{\pi}_1^* \mathbb{D}(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \), \( T_{E_1,2} = \tilde{\pi}_1^* \mathbb{D}(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \cap E_1) \) and \( \mathcal{E}_1 \) is the left orthogonal to the subcategory generated by the \( T_{E_1^{(2)}} \otimes \mathcal{O}_{E_1^{(2)}}(kE_1^{(2)} + m_A E_2) \) and the \( i_{1,2*} \left( T_{E_1,2} \otimes \mathcal{O}_{E_1,2}(lE_{1,2}) \right), \) for \( 1 \leq k \leq 3m_A + 1 \) and \( 1 \leq l \leq m_A \). Moreover we have the inclusion:

\[
T_{E_1^{(2)}} \subset \mathcal{E}_1^{(2)}.
\]

**Proof:** We start with the proof of the first point. By Proposition 2.3.2, we have:

\[
\tilde{\pi}_2^* \mathcal{H}om(\mathcal{O}_{E_2}(kE_2), \mathcal{O}_{E_2}(kE_2)) = \mathcal{O}_{\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))},
\]

for all \( k \in \mathbb{Z} \). This implies that the subcategories \( \tilde{\pi}_2^* \mathbb{D}(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \otimes \mathcal{O}_E(kE) \) are full admissible subcategories of \( \mathbb{D}^b(E_2) \), for all \( k \in \mathbb{Z} \). Let \( 1 \leq k < l \leq m_A + 1 \) be integers. We have:

\[
\text{Hom} \left( \mathcal{O}_E(kE) \otimes \tilde{\pi}_2^* \mathbb{D}(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))), \mathcal{O}_E(lE) \otimes \tilde{\pi}_2^* \mathbb{D}(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \right) = \text{Hom} \left( \tilde{\pi}_2^* \mathbb{D}(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))), \mathcal{O}_E((l - k)E) \otimes \tilde{\pi}_2^* \mathbb{D}(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \right) = 0,
\]

where the last equality holds by Proposition 2.3.2 because \( 1 \leq l - k \leq m_A \). As a consequence, we have a semi-orthogonal decomposition:

\[
\mathbb{D}^b(E_2) = \langle (T_{E_2} \otimes \mathcal{O}_{E_2}(m_A E_2)), \ldots, (T_{E_2} \otimes \mathcal{O}_{E_2}(E_2)) \rangle,
\]

with \( T_{E_2} = \tilde{\pi}_2^* \mathbb{D}(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \) and \( \mathcal{E}_2 \) is the left orthogonal to the admissible subcategory generated by the \( \mathcal{O}_E(kE) \otimes \mathbb{D}(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \) for \( 1 \leq k \leq m_A \). It only remains to show that:
\[
\tilde{\pi}_2^* D^b(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \subset \mathcal{E}_1^2.
\]
Equivalently, we need to prove that the subcategory \( \tilde{\pi}_2^* D^b(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \) is left orthogonal to the admissible subcategory generated by the \( \mathcal{O}_E(kE) \otimes \tilde{\pi}_2^* D^b(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))) \) for \( 1 \leq k \leq m_{\mathbb{A}} \). As before, this is a consequence of Proposition 2.3.2.

For the second point, we first note that the same proof as for the first point yields the following semi-orthogonal decomposition:

\[
D^b(E_{1, 2}) = \langle T_{1, 2} \otimes \mathcal{O}_{E_{1, 2}}(m_{\mathbb{A}} E_{1, 2}), \ldots, T_{1, 2} \otimes \mathcal{O}_{E_{1, 2}}(E_{1, 2}), \mathcal{E}_{1, 2} \rangle,
\]
with \( T_{1, 2} = \tilde{\pi}_{1, 2}^* D^b(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \cap E_1) \) and \( T_{1, 2} \subset \mathcal{E}_{1, 2} \). So by Proposition 3.2.1, the categories

\[
i_{1, 2*} \left( \mathcal{O}_{E_{1, 2}}(kE_{1, 2}) \otimes \tilde{\pi}_{1, 2}^* D^b(\pi_1^* \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \cap E_1) \right),
\]
are full admissible subcategories of \( D^b(E_{1}^{(2)}) \) which are left orthogonal to each other, for \( 1 \leq k \leq m_{\mathbb{A}} \).

Using again Proposition 2.3.2, we prove that the subcategories:

\[
\tilde{\pi}_1^* D^b(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \otimes \mathcal{O}_{E_{1}^{(2)}}(lE_1^{(2)} + m_{\mathbb{A}} E_2)
\]
are full admissible subcategories of \( D^b(E_{1}^{(2)}) \) which are left orthogonal to each other, for \( 1 \leq l \leq 3m_{\mathbb{A}} + 1 \). The adjunction formula shows that \( \omega_{E_{1}^{(2)}} = \mathcal{O}_{E_{1}^{(2)}}(m_{\mathbb{A}} E_2) \otimes \mathcal{O}_{E_{1}^{(2)}}(lE_1^{(2)} + m_{\mathbb{A}} E_2) \) for some \( \in T_{E_{1}^{(2)}} \). Then, by Serre duality, we have:

\[
\text{Hom} \left( i_{1, 2*} \left( T_{E_{1}^{(2)}} \otimes \mathcal{O}_{E_{1}^{(2)}}(kE_{1, 2}) \right), T_{E_{1}^{(2)}} \otimes \mathcal{O}_{E_{1}^{(2)}}(lE_1^{(2)} + m_{\mathbb{A}} E_2) \right)
\]

\[
= \text{Hom} \left( T_{E_{1}^{(2)}} \otimes \mathcal{O}_{E_{1}^{(2)}}(lE_1^{(2)}), i_{1, 2*} \left( T_{E_{1, 2}} \otimes \mathcal{O}_{E_{1, 2}}(kE_{1, 2}) \right) \right)^*,
\]
for \( 1 \leq k \leq m_{\mathbb{A}} \) and \( 1 \leq l \leq 3m_{\mathbb{A}} + 1 \). Recall that \( T_{E_{1}^{(2)}} = \pi_{1, 2}^* T_{E_1} \), with \( T_{E_1} = \tilde{\pi}_1^* D^b(G_\omega(\mathbb{A}^3, \mathbb{A}^6)) \) and that \( \mathcal{O}_{E_{1}^{(2)}}(E_1^{(2)}) = \pi_{1, 2}^* \mathcal{O}_{E_1}(E_1) \). Thus, the adjunction formula for \( \pi_{1, 2} \) gives:

\[
\text{Hom} \left( T_{E_{1}^{(2)}} \otimes \mathcal{O}_{E_{1}^{(2)}}(lE_1^{(2)}), i_{1, 2*} \left( T_{E_{1, 2}} \otimes \mathcal{O}_{E_{1, 2}}(kE_{1, 2}) \right) \right)
\]

\[
= \text{Hom} \left( T_{E_1} \otimes \mathcal{O}_{E_1}(lE_1), \pi_{1, 2*} \left( T_{E_{1, 2}} \otimes \mathcal{O}_{E_{1, 2}}(kE_{1, 2}) \right) \right).
\]

But we have a commutative diagram:

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so that:

\[ \pi_{1,2} \left( i_{1,2} \left( T_{E_{1,2}} \otimes \mathcal{O}_{E_{1,2}}(kE_{1,2}) \right) \right) = \kappa_{1*} \left( \tilde{\pi}_{1,2} \left( T_{E_{1,2}} \otimes \mathcal{O}_{E_{1,2}}(kE_{1,2}) \right) \right) \]

and by Proposition 2.3.2 we have \( \tilde{\pi}_{1,2*} \left( \mathcal{O}_{E_{1,2}}(kE_{1,2}) \right) = 0 \) for all \( 1 \leq k \leq m_\mathcal{A} \).

As a consequence, we have proved that we have a semi-orthogonal decomposition:

\[ \mathcal{D}^b(E_1^{(2)}) = \langle T_{E_{1}^{(2)}} \otimes \mathcal{O}_{E_{1}^{(2)}}((3m_\mathcal{A} + 1)E_1^{(2)} + m_\mathcal{A}E_2), \ldots, T_{E_{1}^{(2)}} \otimes \mathcal{O}_{E_{1}^{(2)}}(E_1^{(2)} + m_\mathcal{A}E_2), \ldots, T_{E_{1}^{(2)}} \otimes \mathcal{O}_{E_{1}^{(2)}}(m_\mathcal{A}E_2), \pi_{1,2*} \left( T_{E_{1,2}} \otimes \mathcal{O}_{E_{1,2}}(m_\mathcal{A}E_1, E_2) \right), \ldots, \rangle, \]

with \( T_{E_{1}^{(2)}} = \tilde{\pi}^* \mathcal{D}^b(G_\omega(A^3, A^6)) \), \( T_{E_{1,2}} = \tilde{\pi}_{1,2}^* \mathcal{D}^b(\pi_+^*(G_\omega(A^3, A^6)) \cap E_1) \) and \( \mathcal{E}_1^{(2)} \) is the left orthogonal to the admissible subcategory generated by the \( T_{E_{1}^{(2)}} \otimes \mathcal{O}_{E_{1}^{(2)}}(lE_1^{(2)} + m_\mathcal{A}E_2) \) and \( \pi_{1,2*} \left( T_{E_{1,2}} \otimes \mathcal{O}_{E_{1,2}}(kE_{1,2}) \right) \) for \( 1 \leq l \leq 3m_\mathcal{A} + 1 \) and \( 1 \leq k \leq m_\mathcal{A} \).

It remains to prove that \( T_{E_{1}^{(2)}} \subset \mathcal{E}_1^{(2)} \). This is done as before using Proposition 2.3.2. We leave the proof to the reader.

\[ \nabla \]

Using this lemma, we can finish the proof of proposition 3.1.1.

\[ \text{Proof:} \]

\[ \nabla \text{Proof of Proposition 3.1.1} \]

By Proposition 3.2.1 we know that the categories

\[ \mathcal{A}_l = \pi_{1*} \left( \tilde{\pi}^* \mathcal{D}^b(G_\omega(A^3, A^6)) \otimes \mathcal{O}_{E_{1}^{(2)}}(lE_1^{(2)} + m_\mathcal{A}E_2) \right) \]

and

\[ \mathcal{B}_k = \pi_{2*} \left( \tilde{\pi}_2^* \mathcal{D}^b(\pi_+^*(G_\omega(A^3, A^6))) \otimes \mathcal{O}_{E_2}(kE_2) \right) \]

are full admissible subcategories of \( \mathcal{D}^b(X_2) \) for \( 1 \leq l \leq 3m_\mathcal{A} + 1 \) and \( 1 \leq k \leq m_\mathcal{A} \). Moreover, again by Proposition 3.2.1, the \( \mathcal{A}_l \) are left orthogonal to each other for \( 1 \leq l \leq 3m_\mathcal{A} + 1 \), while the \( \mathcal{B}_k \) are left orthogonal to each other for
\[1 \leq k \leq m_k. \text{ We start by proving that the } B_k \text{ are left orthogonal to the } A_l. \]

We have:

\[
\text{Hom}(B_k, A_l) = \text{Hom}(i_2^* (\tilde{\pi}_2^* D^b(\pi_1^* \sigma_+(G_\omega(A^3, A^6))) \otimes \mathcal{O}_{E_2}(kE_2)),
\]

\[
i_1^* (\tilde{\pi}_1^* D^b(G_\omega(A^3, A^6)) \otimes \mathcal{O}_{E_1^{(2)}}(lE_1^{(2)} + m_k E_2))
\]

\[
= \text{Hom}(i_1^* [i_2^* (\tilde{\pi}_2^* D^b(\pi_1^* \sigma_+(G_\omega(A^3, A^6))) \otimes \mathcal{O}_{E_2}(kE_2))],
\]

\[
\tilde{\pi}_1^* D^b(G_\omega(A^3, A^6)) \otimes \mathcal{O}_{E_1^{(2)}}(lE_1^{(2)} + m_k E_2)).
\]

But we have a cartesian square:

\[
\begin{array}{ccc}
E_{1,2} & \xrightarrow{i_{1,2}} & E_1^{(2)}, \\
\downarrow i_{2,1} & & \downarrow i_1 \\
E_2 & \xrightarrow{i_2} & X_2
\end{array}
\]

with \(\dim E_{1,2} = \dim E_2 + \dim E_1^{(2)} - \dim X_2\) and \(i_1, i_2\) are locally complete intersection embeddings. Thus, this diagram is Tor-neutral (see [Kuz06], Corollary 2.27) and we have:

\[
i_1^* i_2^* \mathcal{F} = i_{1,2}^* i_{2,1}^* \mathcal{F},
\]

for all \(\mathcal{F} \in D^-(E_2)\). So we have:

\[
i_1^* [i_2^* (\tilde{\pi}_2^* D^b(\pi_1^* \sigma_+(G_\omega(A^3, A^6))) \otimes \mathcal{O}_{E_2}(kE_2))]
\]

\[
= i_{1,2}^* [i_{2,1}^* (\tilde{\pi}_2^* D^b(\pi_1^* \sigma_+(G_\omega(A^3, A^6))) \otimes \mathcal{O}_{E_2}(kE_2))].
\]

The commutative diagram:

\[
\begin{array}{ccc}
E_{1,2} & \xrightarrow{i_{2,1}} & E_2 \\
\downarrow \pi_{2,1} & & \downarrow \pi_2 \\
E_1 \cap \pi_1^* \sigma_+(G_\omega(A^3, A^6)) & \xrightarrow{j_2} & \pi_1^* \sigma_+(G_\omega(A^3, A^6))
\end{array}
\]

shows that:

\[
i_{2,1}^* (\tilde{\pi}_2^* D^b(\pi_1^* \sigma_+(G_\omega(A^3, A^6)))) = \tilde{\pi}_{2,1}^* (j_{2,1}^* D^b(\pi_1^* \sigma_+(G_\omega(A^3, A^6)))).
\]
As $\pi_+^* \sigma_+ (G_\omega(A^3, A^6))$ is smooth, we have the inclusion:

$$\tilde{\pi}_{1,2}^* (j_{2*} \mathbb{D}^b (\pi_+^* \sigma_+ (G_\omega(A^3, A^6)))) \subset T_{E_{1,2}}.$$

Hence, to prove that $\text{Hom}(\mathcal{B}_k, \mathcal{A}_l^c) = 0$, it is sufficient to prove that:

$$\text{Hom}(\tilde{i}_{1,2*} (T_{E_{1,2}} \otimes \mathcal{O}_{E_{1,2}^c} (kE_2)), T_{E_{1,2}} \otimes \mathcal{O}_{E_{1,2}^c} (lE_1^2 + m_A E_2)) = 0.$$

Since $\mathcal{O}_{E_{1,2}^c} (E_2) = \mathcal{O}_{E_{1,2}^c} (E_{1,2})$, this last vanishing comes from the semi-orthogonal decomposition of $\mathbb{D}^b (E_1)$ found in Lemma 3.2.2. As a consequence, we have proved that we have a semi-orthogonal decomposition:

$$\mathbb{D}^b (X_2) = \langle \mathcal{A}_{3m_A+1}, \ldots, \mathcal{A}_1, \mathcal{B}_{m_A}, \ldots, \mathcal{B}_1, \mathcal{D}_{X_2} \rangle,$$

where $\mathcal{D}_{X_2}$ is the left orthogonal to the admissible subcategory generated by the $\mathcal{A}_l$’s and the $\mathcal{B}_k$’s. The only fact left to complete the proof of Proposition 3.1.1 is the inclusion $\pi^* \mathbb{D}^\text{perf} (\tau (G_\omega(A^3, A^6))) \subset \mathcal{D}_{X_2}$. Equivalently, we have to prove that $\pi^* \mathbb{D}^\text{perf} (\tau (G_\omega(A^3, A^6)))$ is left orthogonal to the $\mathcal{A}_l$’s and the $\mathcal{B}_k$’s. By Proposition 3.2.1, the left orthogonal to the $\mathcal{B}_k$’s is:

$$\{ \mathcal{F} \in \mathbb{D}^b (X_2), i_{2*} \mathcal{F} \in \mathcal{E}_2 \}$$

and we know (Proposition 3.2.2) that $\tilde{\pi}_{2*} \mathbb{D}^b (\pi_+^* \sigma_+ (G_\omega(A^3, A^6))) \subset \mathcal{E}_2$. Moreover the commutative diagram:

shows that $i_{2*} \pi^* \mathbb{D}^\text{perf} (\tau (G_\omega(A^3, A^6))) \subset \tilde{\pi}_{2*} \mathbb{D}^b (\pi_+^* \sigma_+ (G_\omega(A^3, A^6)))$, which implies that $\pi^* \mathbb{D}^\text{perf} (\tau (G_\omega(A^3, A^6)))$ is left orthogonal to the $\mathcal{B}_k$’s, for $1 \leq k \leq m_A$. We prove in the same fashion that $\pi^* \mathbb{D}^\text{perf} (\tau (G_\omega(A^3, A^6)))$ is left orthogonal to the $\mathcal{A}_l$’s for $1 \leq l \leq 3m_A + 1$. This concludes the proof of Proposition 3.1.1.
4 Conclusion

In this paper, we developed methods in order to build categorical crepant resolutions of singularities for some varieties which do not admit any wonderful resolution of singularities. In [Abu12], we noticed that strongly crepant categorical resolutions have very interesting minimality properties. Unfortunately, they seem to be much more difficult to construct. As a corollary of the main result of [Abu12], we know that all determinantal varieties admit categorical crepant resolutions of singularities. Which determinantal varieties have a strongly crepant resolution is yet a widely open problem:

**Question 4.0.3** Which determinantal varieties admit strongly crepant categorical resolution of singularities?

From [ACGHS5], section 2.2, we know that all square determinantal varieties have a small resolution of singularities, hence a (geometric) crepant resolution. Thus, the above question is only interesting for symmetric and Pfaffian determinantal varieties. In the Appendix B, we will prove that all Pfaffians are $\mathbb{Q}$-factorial with terminal singularities, so that they do not admit any geometric crepant resolution of singularities.

Let us mention some obstructions to the construction of strongly crepant categorical resolution of singularities. Let $X$ be a projective variety with Gorenstein rational singularities and let:

$$\pi_\mathcal{F}_*: \mathcal{F} \to D^b(X),$$

a categorical crepant resolution of $X$. If $S_\mathcal{F}$ is a Serre functor for $\mathcal{F}$, then, for all $\mathcal{F} \in D^{\text{perf}}(X)$, we have:

$$S_\mathcal{F}(\pi_\mathcal{F}^* \mathcal{F}) = \pi_\mathcal{F}^* \mathcal{F} \otimes \pi_\mathcal{F}^* \omega_X[\dim X],$$

where $\pi_\mathcal{F}^*$ is the left adjoint to $\pi_\mathcal{F}^{**}$. In order for $\mathcal{F}$ to be a strongly crepant resolution, we need:

$$S_\mathcal{F}(T) = T \otimes \pi_\mathcal{F}^* \omega_X[\dim X],$$

for all $T \in \mathcal{F}$. If $\mathcal{F} \simeq D^b(Y)$ for some variety $Y$, then the Serre functor of $\mathcal{F}$ is the tensor product with the dualizing complex of $Y$. As $\mathcal{F}$ is a crepant resolution of $X$, we deduce that the dualizing complex of $Y$ is $\pi_\mathcal{F}^* \omega_X[\dim X]$. Nevertheless, if $\mathcal{F}$ is not geometric, then we can not predict how the Serre functor of $\mathcal{F}$ acts on objects which are not in $\pi_\mathcal{F}^* D^{\text{perf}}(X)$. Kuznetsov gives examples of categorical crepant resolutions which are not strongly crepant
In particular, the Serre functor does not act the same on all objects of these categories. In order to construct strongly crepant resolutions of singularities, it seems necessary to understand in details the objects $T \in \mathcal{T}$ such that $S_{\mathcal{T}}(T) \neq T \otimes \pi_* \omega_X[\dim X]$. This more or less reduces to understand more precisely categorical crepant resolutions of singularities. A complete answer to the following question could prove to be very helpful:

**Question 4.0.4** Let $X$ be a projective variety with Gorenstein rational singularities. Let $\pi : \tilde{X} \rightarrow X$ be a resolution of singularities and:

\[ \begin{array}{ccc}
\mathcal{T} & \xrightarrow{\delta} & \text{D}^b(\tilde{X}) \\
\downarrow \pi_* & & \downarrow \pi_* \\
\text{D}^b(X) & \xrightarrow{\pi_*} & \text{D}^b(\tau(G(3,6)))
\end{array} \]

be a categorical crepant resolution of $X$. When is $\mathcal{T}$ the derived category of a (non-commutative) moduli space of objects in $\text{D}^b(\tilde{X})$?

Note that the idea of linking a non trivial component of a Lefschetz decomposition of $\text{D}^b(Y)$ (for smooth $Y$) to a moduli space of objects in $\text{D}^b(Y)$ is not new (see [Kuz04], [Kuz09]). It had been fruitfully exploited in [BMMS12].

Let us come back to the case of $G(3, 6) \subset \mathbb{P}^{19}$. We prove that the tangent variety of $G(3, 6)$ admit a categorical crepant resolution of singularities. One would like to know if this resolution is non-commutative, more precisely:

**Question 4.0.5** Let

\[ \begin{array}{ccc}
\mathcal{D}_{X_2} & \xrightarrow{\delta} & \text{D}^b(X_2) \\
\downarrow \pi_* & & \downarrow \pi_* \\
\text{D}^b(\tau(G(3,6))) & \xrightarrow{\pi_*} & \text{D}^b(\tau(G(3,6)))
\end{array} \]

be the categorical crepant resolution of the tangent variety to $G(3, 6)$ built in the theorem 3.0.3. Is there a sheaf of algebras $\mathcal{A}_{\tau(G(3,6))}$ on $\tau(G(3,6))$ such that:

$\mathcal{D}_{X_2} \simeq \text{D}^b(\tau(G(3,6)), \mathcal{A}_{\tau(G(3,6))})$?

\footnote{A daring mind would not restrict to the sole algebras, but would also consider DG-algebras and perhaps $A_{\infty}$-algebras...}

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As a consequence of theorem 5.2 in [Kuz08], it is sufficient to find "very good" semi-orthogonal decompositions of the exceptional divisors of $X_2$. In our situation, it would be sufficient to find a pair of vector bundles $\{\mathcal{V}, \mathcal{W}\}$ on $X_2$, which is exceptional with respect to $\pi$, such that:

$$E_2 = \langle \mathcal{V}|_{E_2} \otimes \tilde{\pi}_2^*D^b(\pi_1^*\sigma_+(G(3, 6))) \rangle$$

and

$$E_1^2 = \langle \mathcal{W}|_{E_1^{(2)}} \otimes \tilde{\pi}^*D^b(G(3, 6)) \rangle,$$

where $E_2$ and $E_1^2$ are defined in the lemma 3.2.2.

The natural projection $\tilde{\pi}_2 : E_2 \rightarrow \pi_1^*\sigma_+(G(3, 6))$ is a fibration into smooth quadrics. It seems plausible to find two vector bundles on $X_2$ which would specialize into the relative spinor bundles once restricted to $E_2$. This would rule out the case of $E_2$. The case of $E_1^{(2)}$ is much more subtle. Indeed, the projection $\tilde{\pi}_1 : E_1 \rightarrow G(3, 6)$ is a fibration into doubled $\mathbb{P}^8$. As a consequence, the divisor $E_1$ is globally non reduced and so is the divisor $E_1^{(2)}$. The existence of a vector bundle $W$ on $X_2$ such that:

$$E_1^{(2)} = \langle W|_{E_2} \otimes \tilde{\pi}_2^*D^b(\pi_1^*\sigma_+(G(3, 6))) \rangle$$

would imply that $D^b(E_1^{(2)})$ has finite homological dimension: it is impossible! One can however hope that the theorem 5.2 of [Kuz08] could be extended in the following way: it is sufficient to find a sheaf of algebras $\mathcal{A}_{X_2}$ on $X_2$ with finite homological dimension, such that $D^b(E_1^{(2)}, \mathcal{A}_{X_2} \otimes \mathcal{O}_{E_1^{(2)}})$ still has a "very good" semi-orthogonal decomposition and the natural projection:

$$D^b(E_1^{(2)}, \mathcal{A}_{X_2} \otimes \mathcal{O}_{E_1^{(2)}}) \rightarrow D^b(E_1^{(2)}),$$

is a categorical resolution of singularities (in some extended sense for non-reduced schemes). Fortunately enough, part of this program has been already carried out in [KL12]. Indeed, theorem 5.23 of this paper enables us to construct a sheaf of algebras $\mathcal{A}_{E_1}$ on $E_1$ such that:

- the natural projection $D^b(E_1, \mathcal{A}_{E_1}) \rightarrow D^b(E_1)$ is a categorical resolution of singularities,

- there is a semi-orthogonal decomposition:

$$D^b(E_1, \mathcal{A}_{E_1}) = \langle \langle \mathcal{O}_{E_1|\text{red}}^{\alpha}(\mathcal{O}_{E_1|\text{red}}(-8) \otimes \tilde{\pi}_1^*D^b(G(3, 6))) \rangle, \ldots \rangle \mathcal{O}_{E_1|\text{red}}^{\alpha}(\mathcal{O}_{E_1|\text{red}}(-8) \otimes \tilde{\pi}_1^*D^b(G(3, 6))), \ldots \rangle \mathcal{O}_{E_1|\text{red}}^{\beta}(\mathcal{O}_{E_1|\text{red}}(-8) \otimes \tilde{\pi}_1^*D^b(G(3, 6))), \ldots \rangle \mathcal{O}_{E_1|\text{red}}^{\beta}(\mathcal{O}_{E_1|\text{red}}(-8) \otimes \tilde{\pi}_1^*D^b(G(3, 6))), \ldots \rangle.$$
where $\mathcal{O}_{E_1 \mid \text{red}}^\alpha(1)$ and $\mathcal{O}_{E_1 \mid \text{red}}^\beta(1)$ are sheaves of $\mathcal{A}_{E_1}$-modules which identify to the relatively very ample generator of the relative Picard group of the projective bundle:

$$|E_1|_{\text{red}} \rightarrow G(3, 6),$$

when they are restricted to $|E_1|_{\text{red}}$, the reduced scheme underlying $E_1$.

Finally, there are many points left to check in order to demonstrate the existence of a non-commutative crepant resolution of $\tau(G(3, 6))$:

- find a sheaf of algebras $\mathcal{A}_{E_1^{(2)}}$ on $E_1^{(2)}$ such that $D^b(E_1^{(2)}, \mathcal{A}_{E_1^{(2)}})$ admits a semi-orthogonal decomposition compatible with the one of $D^b(E_1, \mathcal{A}_{E_1})$,
- show that $\mathcal{A}_{E_1^{(2)}}$ is the restriction to $E_1^{(2)}$ of a sheaf of algebras $\mathcal{A}_{X_2}$ on $X_2$ (this should be the trickiest part!),
- show that the natural projection:

$$r_* : D^b(X_2, \mathcal{A}_{X_2}) \rightarrow D^b(X_2),$$

satisfy $r_* r^* = \text{id}$,
- prove that the category $D^b(E_2, \mathcal{A}_{X_2} \otimes \mathcal{O}_{E_2})$ still has a "very good" semi-orthogonal decomposition which is compatible with the decomposition of $D^b(E_1^{(2)}, \mathcal{A}_{E_1^{(2)}})$.

The first point of this program is the certainly the easiest to complete. Indeed, the divisor $E_1^{(2)}$ is the bow-up of $E_1$ along a smooth subscheme which meet transversally all fibers of $\tilde{\pi}_1 : E_1 \rightarrow G(3, 6)$. 

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A The map $\mu : E \to \sigma_+(G_\omega(A^3, A^6))$ has infinite Tor-dimension

A.1 Basic facts on finite Tor-dimension

We recall the following definition:

**Definition A.1.1** Let $f : X \to Y$ be a morphism of schemes of finite type over an algebraically closed field $k$. We say that $f$ has finite Tor-dimension if $\mathcal{O}_X$ has a finite projective resolution as a $\mathcal{O}_Y$-module.

The following are the best known examples of morphism with finite Tor-dimension:

- morphisms $f : X \to Y$, with $Y$ smooth,
- flat morphisms,
- locally complete intersection morphisms,
- any composition of the three above examples.

The result below implies that any resolution of singularities has infinite Tor-dimension:

**Proposition A.1.2** Let $f : X \to Y$ a proper morphism of varieties over an algebraically closed field $k$. Let $y \in Y_{\text{sing}}$ and assume that $f^{-1}(y)$ is not included in the singular locus of $X$. Then $f$ has infinite Tor-dimension.

As I was not able to find any proper reference for this standard fact, I provide a proof of it.

**Proof:**

Since $f$ takes closed point to closed points, this question can be localized at the neighborhood of any point in $X$. Thus, we have to prove the following: Let $f : A \to B$ be a morphism of local Noetherian rings whose residue fields are $k$ and with $B$ regular. Assume that $f$ has finite Tor-dimension. Then $A$ is also regular.

We first consider a finite free resolution of $k$ as a $B$-module:

$$0 \to M_r \to \cdots \to M_p \to \cdots \to M_0 \to k \to 0.$$  

Then let:

$$\cdots \to N_{q,p} \to \cdots \to N_{0,p} \to M_p \to 0$$
be a (possibly infinite) resolution of $M_p$ by free $A$-modules. Since all $N_{q,p}$ are free $A$-modules, the map $M_p \to M_{p-1}$ lifts to a map $N_{q,p} \to N_{q,p-1}$ for all $0 \leq p \leq r$ and $q \geq 0$. Thus, we get an infinite double complex of free $A$-modules whose terms are the $N_{q,p}$ for $0 \leq p \leq r$ and $q \geq 0$. Hence $k$ admits a finite resolution by flat $A$-modules, so that $A$ is regular (see [Mat86], Theorem 19.2).

**A.2 Growth of infinite free resolutions**

In this section, we come back to the case of the morphism $\mu : E \to \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ and we prove the following:

**Proposition A.2.1** The morphism $\mu : E \to \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ has infinite Tor-dimension.

Note that this result is not completely obvious. Indeed, the singular locus of $E$ is precisely the inverse image by $\mu$ of the singular locus of $\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$, so that we cannot apply Proposition A.1.2 to prove the statement. However we will stick to the following principle:

*If $f : X \to Y$ has finite Tor-dimension, then the singularities of $Y$ can’t be much worse than the singularities of $X$."

We will make this idea precise using the theory of growth of Betti numbers for infinite free resolutions. We refer to [Avr10] for a nice exposition of this theory.

**Definition A.2.2** Let $B$ be a local noetherian ring with residue field $k$, an algebraically closed field of char 0. Let $\mathcal{F}$ be a module of finite type on $B$ and let:

$$\cdots \to M_n \to M_{n-1} \to \cdots \to M_1 \to \mathcal{F} \to 0,$$

be a (possibly infinite) minimal resolution of $\mathcal{F}$ by free $B$-modules. The $n$-th Betti number of $\mathcal{F}$, which we denote by $\beta^n(\mathcal{F})$, is the rank of $M_n$. 
Note that $\beta^n(\mathcal{F})$ is also equal to the dimension of $\text{Tor}_B^n(\mathcal{F}, k)$.

**Definition A.2.3** With the same hypothesis as above, we define the complexity of $\mathcal{F}$ to be:

$$\text{cp}(\mathcal{F}) = \min\{d, \exists \alpha \in \mathbb{R} \text{ such that } \beta^n(\mathcal{F}) \leq \alpha.n^{d-1} \text{ for all } n >> 0\}.$$  

The following result characterizes locally complete intersection in terms of complexity (see [Avr10] remark 8.1.3).

**Theorem A.2.4** Let $B$ be a local notherian ring whose residue field is $k$. Assume that $\text{cp}(k) < +\infty$. Then $B$ is a complete intersection in a regular local ring. Moreover, assume that $B$ is Cohen-Macaulay and that $\text{cp}(k) \leq 1$, then $B$ is a hypersurface ring in a regular local ring.

The converse of the above theorem holds and is much easier. In the case of hypersurfaces, there is even a more precise result. We start with a definition:

**Definition A.2.5** Let $B$ be a notherian local ring with residue field $k$. Let $\mathcal{F}_\bullet$ be an unbounded from below complex of modules over $B$. We say that $\mathcal{F}_\bullet$ is periodic at infinity of period $p > 0$, if there exists an integer $m$ such that for all $i < m$, we have:

$$\mathcal{F}_{i-p} = \mathcal{F}_i$$

and a commutative diagram:

$$\begin{array}{ccc}
\mathcal{F}_{i-p+1} & \xrightarrow{\partial_{i-p+1}} & \mathcal{F}_{i-p} \\
\downarrow & & \downarrow \\
\mathcal{F}_{i+1} & \xrightarrow{\partial_{i+1}} & \mathcal{F}_i
\end{array}$$

The following is one of the fundamental results in the theory of matrix factorizations (see [Avr10], construction 5.1.2):

**Theorem A.2.6** Let $B$ be a notherian local ring which is a hypersurface ring in some local regular ring. Then, any module of finite type $\mathcal{F}$ over $B$ admits a resolution by a complex $\tilde{\mathcal{F}}^\bullet$ of finite free $B$-modules, periodic at infinity of period 2.

We can now prove the main result of this section:
**Theorem A.2.7** Let $f : A \to B$ be a local morphism of noetherian local rings with residue field $k$ and with $A$ Cohen-Macaulay. Assume that $B$ is a hypersurface ring in a regular local ring and that $f$ has finite Tor-dimension. Then $A$ is also a hypersurface ring in a regular local ring.

This result (and its proof) is somehow similar to its analogue A.1.2.

**Proof:**

We start with a periodic resolution of $k$ by finite free $B$-modules:

\[
\cdots M_p \xrightarrow{\partial_p^B} M_{p-1} \xrightarrow{\partial_{p-1}^B} \cdots \xrightarrow{\partial_0^B} M_0 \to k \to 0,
\]

with

\[ M_{i-2} = M_i \]

and a commutative diagram:

\[
\begin{array}{ccc}
M_{i-1} & \xrightarrow{\partial_{i-1}^N} & M_{i-2} \\
\downarrow & & \downarrow \\
M_{i+1} & \xrightarrow{\partial_{i+1}^N} & M_i
\end{array}
\]

for all $i \ll 0$.

Since $f$ has finite Tor-dimension, the same argument as in the proof of proposition A.1.2 shows that we can find a double complex $N_{\bullet, \bullet}$ of flat $A$-modules, such that $N_{\bullet, p}$ is a finite resolution of $M_p$ by flat $A$-modules. Since the complex $M_{\bullet}$ is periodic at infinity of period 2, we get:

\[ N_{\bullet, i-2} = N_{\bullet, i} \]

and

\[
\begin{array}{ccc}
N_{\bullet, i-1} & \xrightarrow{\partial_{i-1}^{N_{\bullet}}} & N_{\bullet, i-2} \\
\downarrow & & \downarrow \\
N_{\bullet, i+1} & \xrightarrow{\partial_{i+1}^{N_{\bullet}}} & N_{\bullet, i}
\end{array}
\]

for $i \ll 0$. Let $G_{\bullet}$ be the Cartan-Eilenberg resolution of $N_{\bullet, \bullet}$. This is an unbounded from below, periodic at infinity, complex of flat $A$-modules which is
quasi-isomorphic to $k$. Since all the $G_q$ are flat $A$-modules, the $\text{Tor}^n_A(k, k)$ are the homology groups of the complex $G_\bullet \otimes_A k$. But the very definition of periodicity at infinity implies that the sequence of homology groups $\mathcal{H}_i(G_\bullet \otimes_A M)$ is periodic for all $A$-modules $M$ and $i \ll 0$. As a consequence, the $\text{Tor}^n_A(k, k)$ are periodic for $n >> 0$. But the ring $A$ is Cohen-Macaulay, so by Proposition [A.2.3] the ring $A$ is a hypersurface in a regular local ring. \hfill ▪

Now we can prove that the map $\mu : E \to G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ has infinite Tor-dimension. We proceed by contradiction. Assume that $\mu$ has finite Tor-dimension. Since $E$ is a Cartier divisor in a smooth variety and $\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ is Cohen-Macaulay, we can apply Theorem [A.2.7] and we find that for any $x \in \sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$, there exists an open subset $U_x$ of $\sigma_+(G_\omega(\mathbb{A}^3, \mathbb{A}^6))$ containing $x$ such that $U_x$ is a hypersurface in a smooth scheme, say $V_x$. Let $\kappa : \tilde{V}_x \to V_x$ be the blow-up of $V_x$ along $U_x \cap G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ and denote by $E_V$ the exceptional divisor. The strict transform of $U_x$ by $\kappa$ is the blow-up of $U_x$ along $U_x \cap G_\omega(\mathbb{A}^3, \mathbb{A}^6)$, whose exceptional divisor $E_U$ is a fibration into $\mathbb{A}\mathbb{P}^2$ over $U_x$ (see Proposition [2.2.1]). Since $U_x$ is a hypersurface in $V_x$, the fibers of $E_U$ over $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ are hypersurfaces in the fibers of $E_V$ over $G_\omega(\mathbb{A}^3, \mathbb{A}^6)$. As a consequence $\mathbb{A}\mathbb{P}^2$ is a hypersurface in some projective space. We will show that it is impossible.

Indeed, let us first consider the case $\mathbb{A} = \mathbb{R}$. Then $\mathbb{A}\mathbb{P}^2 = \mathbb{P}^2$. All embeddings of $\mathbb{P}^2$ in projective spaces are given by powers of $\mathcal{O}_{\mathbb{P}^2}(1)$ followed by linear projections. The only embedding of $\mathbb{P}^2$ as a hypersurface is thus the embedding in $\mathbb{P}^3$ as a hyperplane. But if the tangent cone of $U_x$ at $y \in G_\omega(\mathbb{A}^3, \mathbb{A}^6)$ is a hyperplane in the tangent space to $V_x$ at $y$, then $U_x$ is smooth at $y$ which is a contradiction.

For $\mathbb{A} = \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, we use a topological argument to get a contradiction. We recall that in the cases $\mathbb{A} = \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, the Severi varieties are $\mathbb{P}^2 \times \mathbb{P}^2$, $\text{Gr}(2, 6)$ and $\mathbb{O}\mathbb{P}^2$. The first integer cohomology groups of these varieties are described in the following table:

| $\mathbb{A}\mathbb{P}^2$ | dim $\mathbb{A}\mathbb{P}^2$ | $H^0(\mathbb{A}\mathbb{P}^2, \mathbb{Z})$ | $H^2(\mathbb{A}\mathbb{P}^2, \mathbb{Z})$ | $H^4(\mathbb{A}\mathbb{P}^2, \mathbb{Z})$ |
|-------------------------|-----------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathbb{P}^2 \times \mathbb{P}^2$ | 4 | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ |
| Gr(2, 6) | 8 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
| $\mathbb{O}\mathbb{P}^2$ | 16 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |

| $H^6(\mathbb{A}\mathbb{P}^2, \mathbb{Z})$ | $H^8(\mathbb{A}\mathbb{P}^2, \mathbb{Z})$ | $H^{10}(\mathbb{A}\mathbb{P}^2, \mathbb{Z})$ | $H^{12}(\mathbb{A}\mathbb{P}^2, \mathbb{Z})$ | $H^{14}(\mathbb{A}\mathbb{P}^2, \mathbb{Z})$ |
|-------------------------|-----------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | 0 | 0 |
| $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ |

To fill this table we need:

- the Künneth formula for $\mathbb{P}^2 \times \mathbb{P}^2$,
• the fact that the Schubert classes form a basis of the integral cohomology of the Grassmannian (see [Man98]) for $\text{Gr}(2, 6)$,

• the beginning of section 3 of [IM05] for $\mathbb{O}\mathbb{P}^2$.

Assume that $\mathbb{A}\mathbb{P}^2$ is embedded in $\mathbb{P}^{2m_A+1}$ as a hypersurface. Then, by Lefschetz hyperplane theorem, we have:

$$H^{2k}(\mathbb{A}\mathbb{P}^2, \mathbb{Z}) = H^{2k}(\mathbb{P}^{2m_A+1}, \mathbb{Z}) = \mathbb{Z},$$

for all $0 \leq k \leq m_A - 1$. The above array shows that it is impossible.
B  \(\mathbb{Q}\)-factoriality and resolution of singularities

B.1 Statement of the result and proof

Let \(X\) be a singular variety. Experience tells us that it is often possible to decide if \(X\) is normal, Cohen-Macaulay, Gorenstein, with rational singularities (see [KW12] and [KW13] for some lists about prehomogeneous vector spaces). However \(\mathbb{Q}\)-factoriality seems to be much harder to prove. In [KW12] and [KW13], \(\mathbb{Q}\)-factoriality is never discussed. In this appendix, we prove a criterion for \(\mathbb{Q}\)-factoriality and we apply it to concrete situations. This result was implicitly used in the proof of lemma 2.3.1 Notice that our result is similar to lemma 1.1.1 in [Nam09].

**Proposition B.1.1** Let \(\pi : Y \to X\) be a birational morphism such that \(X\) and \(Y\) have Gorenstein rational singularities. Assume that the scheme theoretic exceptional locus of \(\pi\) (denoted by \(E\)) is a Cartier divisor in \(Y\) such that:

* (i) all the fibers of \(\pi\) have Picard rank equal to 1,
* (ii) \(\omega_Y\) is relatively anti-ample with respect to \(\pi\),
* (iii) \(\pi(E)\) is irreducible.

Then we have:

\[Y \text{ is } \mathbb{Q}\text{-factorial} \implies X \text{ is } \mathbb{Q}\text{-factorial}.

Notice that we do not impose the fibers of \(\pi\) to be reduced. One also easily checks that if \(X\) has Gorenstein terminal singularities and:

\[X_n \to X_{n-1} \to \cdots \to X_0 = X\]

is a resolution of singularities where all \(\pi_i : X_i \to X_{i-1}\) are blow-ups along smooth normally flat centers, then all the \(\pi_i\)'s satisfy the hypotheses (ii) and (iii) of proposition B.1.1. Hence, to apply this proposition to such a resolution of singularities, the only non-trivial hypothesis to check is the condition (i). Moreover, this condition on the Picard rank is sharp, as shown by the following example.

**Example B.1.2** Let \(V\) be a vector space of dimension \(n \geq 2\) and let

\[X = \{A \in \text{End}(V) \text{ such that } \text{rk}(A) \leq 1\}.

This is a rational singularities Gorenstein variety which is only singular in \(0_{\text{End}(V)}\) (see [Wey03], corollary 6.1.5). Consider the incidence:

\[\tilde{X} = \{(A, L, M) \in X \times \mathbb{P}(V) \times \mathbb{P}(V^*) \text{ such that } \text{Im}(A) \subset L \text{ and } M \subset \text{Ker}(A)\}.

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The natural projection $\pi : \tilde{X} \to X$ is a resolution of singularities and the exceptional locus of $\pi$ (denoted by $E$) is isomorphic to $\mathbb{P}(V) \times \mathbb{P}(V^*)$: this is a Cartier divisor in $\tilde{X}$. One easily shows that $\omega_{\tilde{X}} = \pi^* \omega_X \otimes \mathcal{O}_{\tilde{X}}((n-1)E)$, so that the condition (ii) and (iii) of proposition [B.1.1] are satisfied. However $X$ is not $\mathbb{Q}$-factorial. Indeed, let

$$X' = \{(A, L) \in X \times \mathbb{P}(V) \text{ such that } \text{Im}(A) \subset L\}.$$ 

The projection $p : X' \to X$ is a resolution of singularities whose exceptional locus is isomorphic to $\mathbb{P}(V)$: it has codimension bigger than 2 in $X'$. As a consequence of [Deb01], 1.40, the variety $X$ is not $\mathbb{Q}$-factorial.

**Proof:**

of proposition [B.1.1] We first demonstrate that $\dim NE(\pi) = 1$.

Let $x_0 \in \pi(E)$ be a general point. We will show that for all $x \in \pi(E)$, there exists two curves $C_0 \subset \pi^{-1}(x_0)$ and $C \subset \pi^{-1}(x)$ such that $C_0$ and $C$ are numerically equivalent.

Let $S$ be a curve in $X$ passing through $x_0$ and $x$ with $S \subset \pi(E)$. Let $S' \to S$ be the normalization of $S$ and $p : S' \to X$ the induced morphism. Let us consider the fiber product:

$$Y' = Y \times_S S' \overset{f_{S'}}{\longrightarrow} S'$$

$$Y \overset{p'}{\longrightarrow} Y' \overset{f}{\longrightarrow} X$$

Let $d$ be the dimension of $f^{-1}(x_0)$. Let $Z \subset Y'$, the vanishing locus of $d - 1$ general sections of $\mathcal{O}_{Y'/S}(m)$ for $m >> 0$ (where $\mathcal{O}_{Y'/S}(1)$ is a relatively ample bundle for $f_{S'}$). Denote by $f_Z : Z \to S'$ the morphism obtained by restriction of $f_{S'}$. Then $f_Z^{-1}(x_0)$ is of dimension 1. Let $Z_0$ be the reduced space underlying an irreducible component of $Z$ which dominates $S'$. As $S'$ is smooth and $Z_0$ integral, we conclude that the restriction $f_{Z_0} : Z_0 \to S'$ is flat. The fiber $f_{Z_0}^{-1}(x)$ is then of dimension 1. Finally, the morphism $p' : Y' \to Y$ is finite over its image, so that $p'(f_{Z_0}^{-1}(x))$ and $p'(f_{Z_0}^{-1}(x_0))$ are two curves numerically equivalent (in $Y$) which are respectively included in $\pi^{-1}(x)$ and $\pi^{-1}(x_0)$.

We can now prove that $\dim NE(\pi) = 1$. It is sufficient to prove that if $C$ and $C_0$ are two curves in $\pi^{-1}(x)$ and $\pi^{-1}(x_0)$, then $C$ and $C_0$ are numerically
proportional in $Y$. But the Picard rank of $\pi^{-1}(x)$ is 1, so that all curves in $\pi^{-1}(x)$ are numerically proportional to each other. The same holds for $\pi^{-1}(x_0)$. Since we know that there exists two curves included in $\pi^{-1}(x)$ and $\pi^{-1}(x_0)$ which are numerically equivalent, we deduce that all curves in $\pi^{-1}(x)$ are numerically proportional to any curve in $\pi^{-1}(x_0)$. Hence, the curves $C$ and $C_0$ are numerically proportional.

Let $x \in \pi(E)$ and $C$ a curve in $\pi^{-1}(x)$. The condition $(ii)$ of proposition B.1.1 implies:

$$\omega_X|_{\pi^{-1}(x)} C < 0.$$ 

Thus, the class $C$ generates a negative ray in $NE(\pi)$. Since $\dim NE(\pi) = 1$, this is an extremal negative ray. As a consequence, we apply the relative cone theorem (see theorem 7.51 of [Deb01]) and we find a diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{c_R} & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi} & \downarrow q \\
\end{array}
\]

where $c_R$ is the contraction for the negative extremal ray $R = \mathbb{R}^+.[C]$.

Let $x \in \pi(E)$. Assume that $\dim c_R(\pi^{-1}(x)) > 0$. Then, we can find two curves $C \subset c_R(E)$ and $C' \subset \pi^{-1}(x)$ such that $c_R(C') = C$. But $\dim NE(\pi) = 1$, so that all curves included in $\pi^{-1}(x)$ are contracted by $c_R$ (see theorem 7.51 of [Deb01]), this is a contradiction. We conclude that $q$ is a finite morphism such that $Rq_*\mathcal{O}_Z = \mathcal{O}_X$. But $X$ is normal, so that by Zariski’s main theorem, the morphism $q$ is an isomorphism. As a consequence, the morphism $\pi$ is a divisorial contraction of a negative extremal ray. As $Y$ is $\mathbb{Q}$-factorial, proposition 7.44 of [Deb01] ensures that $X$ is also $\mathbb{Q}$-factorial.

\[\Box\]

### B.2 Applications

We apply proposition [B.1.1] to some examples.

**Corollary B.2.1** *All Pfaffians varieties are $\mathbb{Q}$-factorial.*

This result is certainly well-known to experts, but the only (implicit) reference I have been able to find is lemma 1.1.1 in [Nam09].
Proof:

Let $V$ be a vector space of dimension $n \geq 2$ and $p$ an integer such that $2p < n$. We denote by:

$$Z^{(p)} = \mathbb{P}\{ A \in \bigwedge^2 V, \text{rg}A \leq 2p \}$$

the Pfaffian variety of rank $2p$ in $\mathbb{P}(\bigwedge^2 V)$. Let us consider the resolution of singularities $\pi : \tilde{Z}^{(p)} \rightarrow Z^{(p)}$ where:

$$\tilde{Z}^{(p)} = \{(A, M) \in Z^{(p)} \times \text{Gr}(2p, V), \text{such that } \text{Im}(A) \subset M \}.$$  

The variety $\tilde{Z}^{(p)}$ is the total space of a projective bundle over $\text{Gr}(2p, V)$: it is irreducible. Hence $Z^{(p)}$ is also irreducible for any $p$. Let $E$ be the exceptional locus of $\pi : \tilde{Z}^{(p)} \rightarrow Z^{(p)}$. This is an integral Cartier divisor, and we have $\pi(E) = Z^{(p-1)}$. Hence, the condition (iii) of [B.1.1] is satisfied.

We show that $Z^{(p)}$ has terminal singularities. As $Z^{(p)}$ is Gorenstein (see [Wey03], proposition 6.4.3), there is an integer $m \in \mathbb{Z}$ such that $\omega_{Z^{(p)}} = \pi^*\omega_{Z^{(p)}} \otimes \mathcal{O}_{\tilde{Z}^{(p)}}(mE)$ (see definition 2.22 of [KM98]). By the adjunction formula we have:

$$\omega_E = \pi^*\omega_{Z^{(p)}} \otimes \mathcal{O}_E((m+1)E).$$

But the map $\pi : E \rightarrow \pi(E)$ is generically flat, so that the adjunction formula implies:

$$\omega_{\pi^{-1}(x)} = \mathcal{O}_E((m+1)E)|_{\pi^{-1}(x)},$$

for generic $x \in \pi(E)$. Moreover, we know that $\mathcal{O}_E(E)|_{\pi^{-1}(x)} = \mathcal{O}_{\pi^{-1}(x)}(-1)$ and that the generic fiber of $\pi : E \rightarrow \pi(E)$ is isomorphic to $\text{Gr}(2, \mathbb{C}^{n-2p+2})$. Thus, for generic $x \in \pi(E)$, we have $\omega_{\pi^{-1}(x)} = \mathcal{O}_E((n-2p+2)E)|_{\pi^{-1}(x)}$. As $2p < n$, we deduce that $m > 1$. The conditions (ii) [B.1.1] is satisfied.

Finally, for all $A \in Z^{(p)}$, the fiber of $\pi$ over $A$ is isomorphic to $\text{Gr}(2p - \text{rg}(A), V/\text{Im}(A))$ : it has Picard rank 1. As a consequence, the condition (i) of [B.1.1] is also satisfied for the morphism $\pi : \tilde{Z}^{(p)} \rightarrow Z^{(p)}$ and we get that $Z^{(p)}$ is $\mathbb{Q}$-factorial.

Another corollary of proposition [B.1.1] is the following:

Corollary B.2.2 The hypersurface $\tau(G_\omega(A^3, A^6))$ is $\mathbb{Q}$-factorial.

when $\mathbb{A} = \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$, the hypersurface $\tau(G_\omega(A^3, A^6))$ is smooth in codimension 3. Grothendieck’s factoriality theorem (see [Gro], Exposé XI, corollary 3.14) shows that $\tau(G_\omega(A^3, A^6))$ is factorial. When $\mathbb{A} = \mathbb{R}$, the singular locus of $\tau(G_\omega(A^3, A^6))$ has precisely codimension 3 in $\tau(G_\omega(A^3, A^6))$. So we can not apply Grothendieck’s result.
**Proof:**

By theorem 2.2.4, we have a sequence of blow-ups:

\[ X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X = \tau(G_\omega(A^3, A^6)) \]

such that \(X_2\) is smooth and the morphisms \(\pi_1\) and \(\pi_2\) satisfy the items (i) and (iii) of proposition B.1.1. Moreover, by lemma 2.3.1, these morphisms also satisfy the condition (ii) of B.1.1. As a consequence, we can apply proposition B.1.1 to \(\pi_1\) and \(\pi_2\). We deduce that \(\tau(G_\omega(A^3, A^6))\) is \(\mathbb{Q}\)-factorial.

\[ \square \]
References

[Abu12] Roland Abuaf. Lefschetz decompositions and categorical resolution of singularities II: The wonderful resolutions. arXiv:1209.1564, 2012.

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. Geometry of algebraic curves. Vol. I, volume 267 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1985.

[Avr10] Luchezar L. Avramov. Infinite free resolutions. In Six lectures on commutative algebra, Mod. Birkhäuser Class., pages 1–118. Birkhäuser Verlag, Basel, 2010.

[BMMS12] Marcello Bernardara, Emanuele Macrì, Sukhendu Mehrotra, and Paolo Stellari. A categorical invariant for cubic threefolds. Adv. Math., 229(2):770–803, 2012.

[BO02] A. Bondal and D. Orlov. Derived categories of coherent sheaves. Li, Ta Tsien (ed.) et al., Proceedings of the International Congress of Mathematicians, ICM 2002, Beijing, China, August 20-28, 2002. Vol. II: Invited lectures. Beijing: Higher Education Press. 47-56 (2002), 2002.

[Deb01] Olivier Debarre. Higher-dimensional algebraic geometry. Universitext. Springer-Verlag, New York, 2001.

[Del11] Dragos Deliu. Homological Projective Duality for Gr(3,6). Dissertation for the degree of Doctor in Philosophy at the University of Pennsylvania. Available at http://repository.upenn.edu/dissertations/AAI3463052, 2011.

[Gro] Alexander Grothendieck. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2). Documents Mathématiques (Paris), 4. Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d’un exposé de Michèle Raynaud.

[Hol11] Frédéric Holweck. Singularities of duals of Grassmannians. J. Algebra, 337:369–384, 2011.

[IM05] Atanas Iliev and Laurent Manivel. The Chow ring of the Cayley plane. Compos. Math., 141(1):146–160, 2005.

[Kim03] Tatsuo Kimura. Introduction to prehomogeneous vector spaces, volume 215 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 2003. Translated from
the 1998 Japanese original by Makoto Nagura and Tsuyoshi Nishitani and revised by the author.

[KL12] Alexander Kuznetsov and Valery Lunts. Categorical resolutions of irrational singularities. 2012. arXiv:1212.6170.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol97] János Kollár. Singularities of pairs. In János (ed.) et al., *Algebraic geometry*. Proceedings of the Summer Research Institute, Santa Cruz, CA, USA, July 9–29, 1995. Providence, RI: American Mathematical Society. Proc. Symp. Pure Math. 62(pt.1), 221-287, 1997.

[Kuz04] Alexander Kuznetsov. Derived category of a cubic threefold and the variety $V_{14}$. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):183–207, 2004.

[Kuz06] Alexander Kuznetsov. Hyperplane sections and derived categories. *Izv. Math.*, 70(3):447–547, 2006.

[Kuz08] Alexander Kuznetsov. Lefschetz decompositions and categorical resolutions of singularities. *Selecta Math. (N.S.*), 13(4):661–696, 2008.

[Kuz09] Alexander Kuznetsov. Derived categories of Fano threefolds. *Tr. Mat. Inst. Steklova*, 264(Mnogomernaya Algebraicheskaya Geometriya):116–128, 2009.

[KW12] Witold Kraskiewicz and Jerzy Weyman. Geometry of orbit closures for the representations associated to gradings of Lie algebras of types $E_6$, $F_4$ and $G_2$. 2012. arXiv:1201.1102.

[KW13] Witold Kraskiewicz and Jerzy Weyman. Geometry of orbit closures for the representations associated to gradings of Lie algebras of types $E_7$. 2013. arXiv:1301.0720.

[LM01] J. M. Landsberg and L. Manivel. The projective geometry of Freudenthal’s magic square. *J. Algebra*, 239(2):477–512, 2001.

[Man98] Laurent Manivel. *Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence*, volume 3 of *Cours Spécialisés*. Société Mathématique de France, Paris, 1998.
[Mat86] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.

[Nam09] Yoshinori Namikawa. Induced nilpotent orbits and birational geometry. *Adv. Math.*, 222(2):547–564, 2009.

[Nee96] Amnon Neeman. The Grothendieck duality theorem via Bousfield’s techniques and Brown representability. *J. Amer. Math. Soc.*, 9(1):205–236, 1996.

[Wey03] Jerzy M. Weyman. *Cohomology of vector bundles and syzygies*. Cambridge Tracts in Mathematics 149. Cambridge: Cambridge University Press. xiv, 371 p., 2003.

[Zak93] F.L. Zak. *Tangents and secants of algebraic varieties*. Translations of Mathematical Monographs. 127. Providence, RI: American Mathematical Society (AMS). vii, 164 p., 1993.