ON THE GENERALIZED HAMMING WEIGHTS OF CERTAIN
REED–MULLER-TYPE CODES

MANUEL GONZÁLEZ-SARABIA, DELIO JARAMILLO, AND RAFAEL H. VILLARREAL

Abstract. There is a nice combinatorial formula of P. Beelen and M. Datta for the $r$-th
generalized Hamming weight of an affine cartesian code. Using this combinatorial formula we
give an easy to evaluate formula to compute the $r$-th generalized Hamming weight for a family
of affine cartesian codes. If $X$ is a set of projective points over a finite field we determine th e
basic parameters and the generalized Hamming weights of the Veronese type codes on $X$ and
their dual codes in terms of the basic parameters and the generalized Hamming weights of the
corresponding projective Reed–Muller-type codes on $X$ and their dual codes.

1. Introduction

Let $K = \mathbb{F}_q$ be a finite field and let $C$ be an $[m, \kappa]$-linear code of length $m$ and dimension
$\kappa$, that is, $C$ is a linear subspace of $K^m$ with $\kappa = \dim_K(C)$. The multiplicative group of $K$ is
denoted by $K^*$. The dual code of $C$ is given by

$$C^\perp := \{b \in K^m : \langle b, c \rangle = 0 \ \forall \ c \in C\},$$

where $b = (b_1, \ldots, b_m)$, $c = (c_1, \ldots, c_m)$, and $\langle b, c \rangle = \sum_{i=1}^{m} b_i c_i$ is the inner product of $a$ and $b$.

Fix an integer $1 \leq r \leq \kappa$. Given a subcode $D$ of $C$ (that is, $D$ is a linear subspace of $C$), the
support $\chi(D)$ of $D$ is the set of non-zero positions of $D$, that is,

$$\chi(D) := \{i \mid \exists (a_1, \ldots, a_m) \in D, a_i \neq 0\}.$$

The $r$-th generalized Hamming weight of $C$, denoted $\delta_r(C)$, is the size of the smallest support
of an $r$-dimensional subcode $[14, 16, 29]$. Generalized Hamming weights have been extensively
studied; see $[2, 4, 9, 13, 15, 21, 25, 27, 30, 31]$ and the references therein. The study of these
weights is related to trellis coding, $t$–resilient functions, and was motivated by some applications
from cryptography $[29]$. If $r = 1$, $\delta_1(C)$ is the minimum distance of $C$ and is denoted $\delta(C)$.

In this note we give explicit formulas for the generalized Hamming weights of certain projective
Reed-Muller-type codes and study the basic parameters (length, dimension, minimum distance)
and the generalized Hamming weights of Veronese type codes and their dual codes.

These linear codes are constructed as follows. Let $\mathbb{P}^{s-1}$ be a projective space over $K$, let
$X = \{[P_1], \ldots, [P_m]\}$ be a subset of $\mathbb{P}^{s-1}$ where $m = |X|$ is the cardinality of the set $X$, $P_i \in K^s$
for all $i$, and let $S = K[t_1, \ldots, t_s] = \oplus_{d=0}^{\infty} S_d$ be a polynomial ring with the standard grading,
where $S_d$ is the $K$-vector space generated by the homogeneous polynomials in $S$ of degree $d$. Fix a
degree $d \geq 1$. For each $i$ there is $h_i \in S_d$ such that $h_i(P_i) \neq 0$. Indeed suppose $P_i = (a_1, \ldots, a_s)$,
there is at least one \( k \in \{1, \ldots, s\} \) such that \( a_k \neq 0 \). Setting \( h_i = t_k^d \) one has that \( h_i \in S_d \) and \( h_i(P_t) \neq 0 \). Consider the evaluation map

\[
ev_d : S_d \to K^m, \quad h \mapsto \left( \frac{h(P_1)}{h_1(P_1)}, \ldots, \frac{h(P_m)}{h_m(P_m)} \right).
\]

This is a linear map between the \( K \)-vector spaces \( S_d \) and \( K^m \). The Reed–Muller–type code of order \( d \) associated to \( X \) \([5, 11]\), denoted \( C_X(d) \), is the image of \( \ev_d \), that is

\[
C_X(d) = \left\{ \left( \frac{h(P_1)}{h_1(P_1)}, \ldots, \frac{h(P_m)}{h_m(P_m)} \right) : h \in S_d \right\}.
\]

The \( r \)-th generalized hamming weight \( \delta_r(C_X(d)) \) of \( C_X(d) \) is sometimes denoted by \( \delta_X(d, r) \). If \( r = 1 \), \( \delta_X(d, r) \) is the minimum distance of \( C_X(d) \) and is denoted by \( \delta_X(d) \). The map \( \ev_d \) is independent of the set of representatives \( P_1, \ldots, P_m \) that we choose for the points of \( X \), and the basic parameters of \( C_X(d) \) are independent of \( h_1, \ldots, h_m \) \([19]\) Lemma 2.13 and so are the generalized Hamming weights of \( C_X(d) \) \([8, \text{Remark 1}]\). The basic parameters of \( C_X(d) \) are related to the algebraic invariants of the quotient ring \( S/I(X) \), where \( I(X) \) is the vanishing ideal of \( X \) (see for example \([10, 20, 22]\)). Indeed, the dimension of \( C_X(d) \) is given by the Hilbert function \( H_X \) of \( S/I(X) \), that is,

\[
H_X(d) := \dim_K(S_d/I(X)_d) = \dim_K(C_X(d)),
\]

the length \( m = |X| \) of \( C_X(d) \) is the degree or the multiplicity of \( S/I(X) \). Moreover, the regularity index of \( H_X \) is the regularity of \( S/I(X) \) \([28, \text{pp. 226, 346}] \) and is denoted \( \text{reg}(S/I(X)) \). By the Singleton bound \([27]\) one has \( \delta_X(d) = 1 \) for \( d \geq \text{reg}(S/I(X)) \). Recall that the \( a \)-invariant of \( S/I(X) \), denoted \( a_X \), is the regularity index minus 1.

Let \( A_1, \ldots, A_{s-1} \) be subsets of \( K = \mathbb{F}_q \) and let \( X := [A_1 \times \cdots \times A_{s-1} \times \{1\}] \subset \mathbb{P}^{s-1} \) be a projective cartesian set, where \( d_i = |A_i| \) for all \( i = 1, \ldots, s-1 \) and \( 2 \leq d_1 \leq \cdots \leq d_{s-1} \). The Reed–Muller-type code \( C_X(d) \) is called an affine cartesian code \([17]\).

There is a recent expression for the \( r \)-th generalized Hamming weight of an affine cartesian code \([1, \text{Theorem 5.4}] \), which depends on the \( r \)-th monomial in ascending lexicographic order of a certain family of monomials (see \([1]\) and the proof of Theorem 2.4). Using this result in Section 2 we give an easy to evaluate formula to compute the \( r \)-th generalized Hamming weight for a family of affine cartesian codes (Theorem 2.11). Other formulas for the second generalized Hamming weight of an affine cartesian code are given in \([7, \text{Theorems 9.3 and 9.5}] \).

Let \( k \geq 1 \) be an integer and let \( M_1, \ldots, M_N \) be the set of all monomials in \( S \) of degree \( k \), where \( N = \binom{k+s-1}{s-1} \). The map

\[
\rho_k : \mathbb{P}^{s-1} \to \mathbb{P}^{N-1}, \quad [x] \mapsto [(M_1(x), \ldots, M_N(x))],
\]

is called the \( k \)-th Veronese embedding. Given \( X \subset \mathbb{P}^{s-1} \), the \( k \)-th Veronese type code of degree \( d \) is \( C_{\rho_k(X)}(d) \), the Reed–Muller-type code of degree \( d \) on \( \rho_k(X) \).

In Section 3 we are able to show that the Reed–Muller-type code \( C_X(kd) \) over the set \( X \) has the same basic parameters and the same generalized Hamming weights as the Veronese type code \( C_{\rho_k(X)}(d) \) over the set \( X \) for \( k \geq 1 \) and \( d \geq 1 \) (Theorem 3.2). As a consequence making \( X = \mathbb{P}^{s-1} \) we recover a result of Rentería and Tapia-Recillas \([23, \text{Proposition 1}] \). Also we show that the dual codes of \( C_X(kd) \) and \( C_{\rho_k(X)}(d) \) are equivalent (Theorem 3.5).

For all unexplained terminology and additional information we refer to \([3, 28] \) (for the theory of Gröbner bases), and \([18, 27] \) (for the theory of error-correcting codes and linear codes).
In this section we present our main result on Hamming weights of certain cartesian codes. To avoid repetitions, we continue to employ the notations and definitions used in Section 1.

Let \( \prec \) be a monomial order on \( S \) and let \( (0) \neq I \subset S \) be an ideal. If \( f \) is a non-zero polynomial in \( S \), the leading monomial of \( f \) is denoted by \( \text{in}_<(f) \). The initial ideal of \( I \), denoted by \( \text{in}_<(I) \), is the monomial ideal given by

\[
\text{in}_<(I) = \{ \{ \text{in}_<(f) \} \mid f \in I \}.
\]

A monomial \( t^a \) is called a standard monomial of \( S/I \), with respect to \( \prec \), if \( t^a \) is not in the ideal \( \text{in}_<(I) \). The set of standard monomials, denoted \( \Delta_<(I) \), is called the footprint of \( S/I \). The footprint of \( S/I \) is also called the Gröbner escalier of \( I \). The image of the standard polynomials of degree \( d \), under the canonical map \( S \mapsto S/I, \ x \mapsto \overline{x} \), is equal to \( S_d/I_d \), and the image of \( \Delta_<(I) \) is a basis of \( S/I \) as a \( K \)-vector space. This is a classical result of Macaulay [3] Chapter 5.

We come to our main result.

**Theorem 2.1.** Let \( X := [A_1 \times \cdots \times A_{s-1} \times \{1\}] \) be a subset of \( \mathbb{P}^{s-1} \), where \( A_i \subset \mathbb{F}_q \) and \( d_i = |A_i| \) for \( i = 1, \ldots, s-1 \). If \( 2 \leq d_1 \leq \cdots \leq d_{s-1} \) and \( d \geq 1 \), then

\[
\delta_r(C_X(d)) = \begin{cases} d_{k+r+1} \cdots d_{s-1} \lfloor (d_{k+1} - 1) \rfloor - 1 & \text{if } 1 \leq r < s-k-1, \\ (d_{k+1} - 1) d_{k+2} d_{s-1} - 1 & \text{if } 1 \leq r = s-k-1, \end{cases}
\]

where we set \( d_i \cdots d_j = 1 \) if \( i > j \) or \( i < 1 \), and \( k \geq 0 \), \( \ell \) are the unique integers such that \( d = \sum_{i=1}^k (d_i - 1) + \ell \) and \( 1 \leq \ell \leq d_{k+1} - 1 \).

**Proof.** Setting \( n = s-1 \), \( R = K[t_1, \ldots, t_n] \) a polynomial ring with coefficients in \( K = \mathbb{F}_q \), and \( L = (t_1^{d_1}, \ldots, t_n^{d_n}) \), we order the set \( M_{\leq d} := \Delta_<(L) \cap R_{\leq d} \) of all standard monomials of \( R/L \) of degree at most \( d \) with the lexicographic order (lex order for short), that is, \( t^a > t^b \) if and only if the first non-zero entry of \( a - b \) is positive. For \( r > 0 \), \( 0 \leq k \leq n-r \), the \( r \)-th monomial \( t_1^{a_1} \cdots t_n^{a_n} \) of \( M_{\leq d} \) in decreasing lex order is

\[
t_{k+1}^{d_{k+1} - \ell} t_{k+2}^{d_{k+2} - 1} \cdots t_{n-1}^{d_{n-1} - 1}.
\]

and the \( r \)-th monomial \( t_1^{a_1} \cdots t_n^{a_n} \) of \( M_{\geq c_0 - d} \) is given by

\[
t_{k+1}^{d_{k+1} - \ell} t_{k+2}^{d_{k+2} - 1} \cdots t_{n-1}^{d_{n-1} - 1}.
\]

Case (I): \( 0 \leq k < n-r \). The case \( r = 1 \) was proved in [17] Theorem 3.8. Thus we may also assume \( r \geq 2 \). Therefore, applying [1] Theorem 5.4, we obtain that \( \delta_r(C_X(d)) \) is given by

\[
1 + \sum_{i=1}^n a_{r,i} \prod_{j=i+1}^n d_j = 1 + (d_{k+1} - \ell) d_{k+2} \cdots d_n + \sum_{i=k+2, i \neq k+r}^n (d_i - 1) \prod_{j=i+1}^n d_j + (d_{k+r} - 2) d_{k+r+1} \cdots d_n
\]

\[
= (d_{k+1} - \ell) d_{k+2} \cdots d_n + \left( 1 + \sum_{i=k+2}^n (d_i - 1) \prod_{j=i+1}^n d_j \right) - d_{k+r+1} \cdots d_n
\]

\[
= (d_{k+1} - \ell) d_{k+2} \cdots d_n + (d_{k+2} \cdots d_n) - d_{k+r+1} \cdots d_n
\]

\[
= (d_{k+1} - \ell + 1) d_{k+2} \cdots d_n - d_{k+r+1} \cdots d_n = d_{k+r+1} \cdots d_n [(d_{k+1} - \ell + 1) d_{k+2} \cdots d_{k+r} - 1].
\]
Case (II): $k = n - r$. In this case the $r$-th monomial $\ell_{1}^{a_{r,1}} \cdots \ell_{n}^{a_{r,n}}$ of $M_{\geq 0} - d$ in ascending lex order is
\[
\ell_{k+1}^{d_{k+1}-\ell} \ell_{k+2}^{d_{k+2}-\ell} \cdots \ell_{k+r-1}^{d_{k+r-1}-\ell} \ell_{k+r}^{d_{k+r+2}-\ell} \cdots \ell_{n}^{d_{n}-\ell}.
\]

Therefore, applying Theorem 5.4, we obtain that $\delta_{r}(C_{X}(d))$ is given by
\[
1 + \sum_{i=1}^{n} a_{r,i} \prod_{j=i+1}^{n} d_{j} = 1 + (d_{k+1}-\ell)d_{k+2} \cdots d_{n} + \sum_{i=k+2}^{n} (d_{i} - 1) \prod_{j=i+1}^{n} d_{j}
\]
\[
= (d_{k+1}-\ell)d_{k+2} \cdots d_{n} + \left(1 + \sum_{i=k+2}^{n} (d_{i} - 1) \prod_{j=i+1}^{n} d_{j}\right) - 1
\]
\[
= (d_{k+1}-\ell)d_{k+2} \cdots d_{n} + (d_{k+2} \cdots d_{n}) - 1 = (d_{k+1}-\ell+1)d_{k+2} \cdots d_{n} - 1.
\]

Definition 2.2. The set $\mathbb{T} = \{[x_{1}, \ldots, x_{s}] \in \mathbb{P}^{s-1} | x_{i} \in K^{*} \forall i\}$ is called a projective torus.

Corollary 2.3. Let $\mathbb{T}$ be a projective torus in $\mathbb{P}^{s-1}$ and let $\delta_{r}(C_{X}(d))$ be the $r$-th generalized Hamming weight of $C_{\mathbb{T}}(d)$. Then
\[
\delta_{r}(C_{\mathbb{T}}(d)) = [(q-1)^{r-1}(q-\ell)} - 1] (q-1)^{s-k-r-1}
\]
for $1 \leq r \leq s - k - 1$, where $d = k(q-2) + \ell$, $k \geq 0$, $1 \leq \ell \leq q-2$.

Proof. It follows readily from Theorem 2.1 making $A_{i} = K^{*} = F_{q} \setminus \{0\}$ for $i = 1, \ldots, s - 1$.

This corollary generalizes the case when $X$ is a projective torus in $\mathbb{P}^{s-1}$ and $r = 1$:

Theorem 2.4. [24 Theorem 3.5] Let $\mathbb{T}$ be a projective torus in $\mathbb{P}^{s-1}$ and let $C_{\mathbb{T}}(d)$ be the Reed–Muller-type code on $\mathbb{T}$ of degree $d \geq 1$. Then its length is $(q-1)^{s-1}$, its minimum distance is given by
\[
\delta_{\mathbb{T}}(d) = \begin{cases} (q-1)^{s-k-2}(q-1-\ell) & \text{if } d \leq (q-2)(s-1) - 1, \\ 1 & \text{if } d \geq (q-2)(s-1), \end{cases}
\]
where $k$ and $\ell$ are the unique integers such that $k \geq 0$, $1 \leq \ell \leq q-2$ and $d = k(q-2) + \ell$, and the regularity of $S/I(\mathbb{T})$ is $(q-2)(s-1)$.

The case when $X$ is a projective torus in $\mathbb{P}^{s-1}$ and $r = 2$ is treated in [6 Theorem 18].

3. VERONESE TYPE CODES

Let $S = K[t_{1}, \ldots, t_{s}]$ be a polynomial ring over a field $K$ and let $\{M_{1}, \ldots, M_{N}\}$ be the set of all monomials of $S$ of degree $k \geq 1$, where $N = \binom{s+k-1}{s-1}$. The map
\[
\rho_{k}: \mathbb{P}^{s-1} \rightarrow \mathbb{P}^{N-1}, \quad [x] \mapsto ([M_{1}(x), \ldots, M_{N}(x)])
\]
is called the $k$-th Veronese embedding. Given $X \subset \mathbb{P}^{s-1}$, the $k$-th Veronese type code of degree $d$ is $C_{\rho_{k}(X)}(d)$, the Reed–Muller-type code of degree $d$ on $\rho_{k}(X)$. The next aim is to show that the Reed–Muller-type code $C_{X}(kd)$ has the same basic parameters and the same generalized Hamming weights as the Veronese type code $C_{\rho_{k}(X)}(d)$ for $k \geq 1$ and $d \geq 1$.

Lemma 3.1. $\rho_{k}$ is well-defined and injective.
Proof. If \([x] = [z], x, y \in \mathbb{P}^{s-1}, x = (x_1, \ldots, x_s), z = (z_1, \ldots, z_s),\) then \(x = \lambda z\) for some \(\lambda \in K^\ast\). Thus \(M_i(x) = \lambda^i M_i(z)\) for all \(i\), that is, \([M_i(x)] = [M_i(z)]\), here we are using \((M_i(x))\) as a short hand for \((M_i(x), \ldots, M_N(x))\). Thus \(p_k\) is well-defined. To show that \(p_k\) is injective assume that \(p_k([x]) = p_k([z])\). Then for some \(\mu \in K^\ast\) one has \(M_i(x) = \mu M_i(z)\) for all \(i\). Pick \(j\) such that \(z_j \neq 0\) and let \(\lambda = x_j/z_j\). Note that \(M_i = t_j^k\) for some \(i\). Then one has \(x_j^k = \mu z_j^k\), that is, \(\mu = \lambda^k\). For each \(1 \leq \ell \leq s\), using the monomial \(M_\ell = t_j^k\) one has
\[
x_j^{k-1} x_\ell = \mu z_j^{k-1} z_\ell = \lambda^k z_j^{k-1} z_\ell = \lambda (\lambda z_j)^{k-1} z_\ell.
\]
Thus \(x_\ell = \lambda z_\ell\) for all \(\ell\), that is, \([x] = [z]\). \(\square\)

We come to the main result of this section.

**Theorem 3.2.** If \(X \subset \mathbb{P}^{s-1}\), then the projective Reed–Muller-type codes \(C_{\rho_k(X)}(d)\) and \(C_X(kd)\) have the same basic parameters and the same generalized Hamming weights for \(k \geq 1\) and \(d \geq 1\).

Proof. Setting \(N = (k+s-1)\), let \(R = K[y_1, \ldots, y_N] = \oplus_{d=0}^{\infty} R_d\) be a polynomial ring over the field \(K\) with the standard grading. We can write \(X = \{[P_1], \ldots, [P_m]\}\), where \(m = |X|, P_1 \in K^\ast\), and the \([P_i]\)'s are in standard form, i.e., the first non-zero entry of \(P_i\) is 1 for all \(i\). By Lemma 3.1 the map \(p_k\) is injective. Thus \(C_X(kd)\) and \(C_{\rho_k(X)}(d)\) have the same length. As \([P_1], \ldots, [P_m]\) are in standard form, for each \(i\) there is \(g_i \in S_{kd}\) such that \(g_i(P_i) = 1\). Therefore, by [19] Lemma 2.13, we may assume that the Reed–Muller-type code \(C_X(kd)\) is the image of the evaluation map
\[
ev_{kd}: S_{kd} = K[t_1, \ldots, t_s]_{kd} \to K^m, \quad g \mapsto (g(P_1), \ldots, g(P_m)),
\]
and the Veronese type code \(C_{\rho_k(X)}(d)\) is the image of the evaluation map
\[
ev^1_d: R_d = K[y_1, \ldots, y_N]_d \to K^m, \quad f \mapsto \left(\begin{array}{c}
f(Q_1)
f(Q_2)
\vdots
f(Q_m)
\end{array}\right),
\]
where \(Q_i = (M_1(P_1), \ldots, M_N(P_i))\) for \(i = 1, \ldots, m\), and \(f_1, \ldots, f_m\) are polynomials in \(R_d\) such that \(f_i(Q_i) \neq 0\) for \(i = 1, \ldots, m\). For any polynomial \(f = f(y_1, \ldots, y_N) = \sum \lambda_a y^a\) in \(R_d\), \(\lambda_a \in K^\ast\), one has
\[
f(M_1, \ldots, M_N)(P_i) = \sum \lambda_a (M_1^{a_1} \cdots M_N^{a_N})(P_i)
\]
\[
= \sum \lambda_a M_1^{a_1}(P_i) \cdots M_N^{a_N}(P_i)
\]
\[
= f(M_1(P_i), \ldots, M_N(P_i)).
\]

As \(K[t_1, \ldots, t_s]_{kd}\) is equal to \(K[M_1, \ldots, M_N]_{kd}\), any \(g \in K[t_1, \ldots, t_s]_{kd}\) can be written as \(g = f(M_1, \ldots, M_N)\) for some \(f = f(y_1, \ldots, y_N)\) in \(R_d\). Therefore, using Eq. (3.3), we get
\[
C_X(kd) = \{(g(P_1), \ldots, g(P_m)) | g \in K[t_1, \ldots, t_s]_{kd}\}
\]
\[
= \{(f(Q_1), \ldots, f(Q_m)) | f \in K[y_1, \ldots, y_N]_d\}.
\]

As a consequence, setting \(\lambda_i = f_i(Q_i)\) and \(\lambda = (\lambda_1, \ldots, \lambda_m)\), one has
\[
C_X(kd) = \lambda \cdot C_{\rho_k(X)}(d) := \{\lambda \cdot a | a \in C_{\rho_k(X)}(d)\},
\]
where \(\lambda \cdot a := (\lambda_1 a_1, \ldots, \lambda_m a_m)\) for \(a = (a_1, \ldots, a_m)\) in \(C_{\rho_k(X)}(d)\). This means that the linear codes \(C_X(kd)\) and \(C_{\rho_k(X)}(d)\) are equivalent [8] Remark 1]. Thus the dimension and minimum distance of \(C_X(kd)\) and \(C_{\rho_k(X)}(d)\) are the same, and so are the generalized Hamming weights. \(\square\)

For convenience we recall the following classical result of Sørensen [26].
Corollary 3.6. If $((1, \ldots, 1), C_{p^{s-1}}((q - 1)(s - 1) - kd))$ is the subspace of $K^m$ generated by $(1, \ldots, 1)$ and $C_{p^{s-1}}((q - 1)(s - 1) - kd)$, then the linear code $C_{V_k}(d)$ is equivalent to

$$
\begin{cases}
C_{p^{s-1}}((q - 1)(s - 1) - kd) & \text{if } kd \not\equiv 0 \mod (q - 1), \\
((1, \ldots, 1), C_{p^{s-1}}((q - 1)(s - 1) - kd)) & \text{if } kd \equiv 0 \mod (q - 1),
\end{cases}
$$

where $(1, \ldots, 1), C_{p^{s-1}}((q - 1)(s - 1) - kd)$ is the subspace of $K^m$ generated by $(1, \ldots, 1)$ and $C_{p^{s-1}}((q - 1)(s - 1) - kd)$.

Veronese codes are a natural generalization of the classical projective Reed–Muller codes.

Corollary 3.4. [23 Proposition 1] If $\forall k = \rho_k([p^s - 1])$, then the projective Reed–Muller-type codes $C_{V_k}(d)$ and $C_{p^{s-1}}(kd)$ have the same basic parameters for $k \geq 1$ and $d \geq 1$.

Proof. This follows at once from Theorem 3.2 making $X = \mathbb{P}^{s-1}$.

As a byproduct we relate the dual codes of $C_{\rho_k(X)}(d)$ and $C_X(kd)$.

Theorem 3.5. If $X$ is a subset of $\mathbb{P}^{s-1}$, then $C_{\rho_k(X)}(d)$ and $C_X^{-1}(kd)$ are equivalent codes and

$$
C_{\rho_k(X)}^{-1}(d) = \lambda \cdot C_X^{-1}(kd),
$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$, with $\lambda_i = f_i(Q_i)$ for all $i = 1, \ldots, m$, is the vector that was given in the proof of Theorem 3.2.

Proof. Let $(u_1, \ldots, u_m) \in C_X^{-1}(kd)$. Then $\langle (u_1, \ldots, u_m), (v_1, \ldots, v_m) \rangle = \sum_{i=1}^{m} u_i v_i = 0$, for all $(v_1, \ldots, v_m) \in C_X(kd)$. By using Eq. (3.4) we conclude that

$$
\langle (u_1, \ldots, u_m), (\lambda_1 v'_1, \ldots, \lambda_m v'_m) \rangle = \sum_{i=1}^{m} u_i \lambda_i v'_i = 0,
$$

for all $(v'_1, \ldots, v'_m) \in C_{\rho_k(X)}(d)$. Therefore

$$
\langle (\lambda_1 u_1, \ldots, \lambda_m u_m), (v'_1, \ldots, v'_m) \rangle = \sum_{i=1}^{m} \lambda_i u_i v'_i = 0.
$$

for all $(v'_1, \ldots, v'_m) \in C_{\rho_k(X)}(d)$. Thus

$$
\lambda \cdot C_X^{-1}(kd) \subset C_{\rho_k(X)}^{-1}(d).
$$

Furthermore one has the equalities

$$
\dim_K \lambda \cdot C_X^{-1}(kd) = \dim_K C_X^{-1}(kd) = m - \dim_K C_X(kd)
$$

(3.6)

$$
= m - \dim_K C_{\rho_k(X)}(d) = \dim_K C_X^{-1}(kd),
$$

and the equality $C_{\rho_k(X)}^{-1}(d) = \lambda \cdot C_X^{-1}(kd)$ follows from Eqs. (3.5) and (3.6). Thus $C_{\rho_k(X)}^{-1}(d)$ and $C_X^{-1}(kd)$ are equivalent codes [8, Remark 1].

Corollary 3.6. If $X = \mathbb{P}^{s-1}$, $\forall k = \rho_k([p^s - 1])$, and $kd \leq (q - 1)(s - 1)$, then the linear code $C_{V_k}(d)$ is equivalent to

$$
\begin{cases}
C_{p^{s-1}}((q - 1)(s - 1) - kd) & \text{if } kd \not\equiv 0 \mod (q - 1), \\
((1, \ldots, 1), C_{p^{s-1}}((q - 1)(s - 1) - kd)) & \text{if } kd \equiv 0 \mod (q - 1),
\end{cases}
$$

where $((1, \ldots, 1), C_{p^{s-1}}((q - 1)(s - 1) - kd))$ is the subspace of $K^m$ generated by $(1, \ldots, 1)$ and $C_{p^{s-1}}((q - 1)(s - 1) - kd)$. 



Proof. This result follows at once from Theorem 3.3 and [26, Theorem 2].

The rest of this section is devoted to show some explicit examples.

Example 3.7. Let $K$ be the field $\mathbb{F}_8$. If $\mathbb{X} = \mathbb{P}^2$, then by Theorem 3.3 the basic parameters of the classical projective Reed–Muller-type code $C_{\mathbb{X}}(d)$ of degree $d$ are given by

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $|\mathbb{X}|$ | 73 | 73 | 73 | 73 | 73 | 73 | 73 | 73 | 73 | 73 | 73 | 73 | 73 | 73 |
| $H_{\mathbb{X}}(d)$ | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 52 | 58 | 63 | 67 | 70 | 72 | 73 |
| $\delta_{\mathbb{X}}(d)$ | 64 | 56 | 48 | 40 | 32 | 24 | 16 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

The dimension of $C_{\mathbb{X}}(d)$ is $H_{\mathbb{X}}(d)$. The regularity of $S/I(\mathbb{X})$ is 15 and the $a$-invariant is 14.

Example 3.8. Let $K$ be the field $\mathbb{F}_8$. If $k = 2$, $\mathbb{X} = \mathbb{P}^2$, and $\mathbb{V}_2 = \rho_2(\mathbb{X})$, then by Theorem 3.2 and Example 3.7 the parameters of the Veronese code $C_{\mathbb{V}_2}(d)$ of degree $d$ are given by

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $|\mathbb{V}_2|$ | 73 | 73 | 73 | 73 | 73 | 73 | 73 | 73 |
| $H_{\mathbb{V}_2}(d)$ | 6 | 15 | 28 | 45 | 58 | 67 | 72 | 73 |
| $\delta_{\mathbb{V}_2}(d)$ | 56 | 40 | 24 | 8 | 6 | 4 | 2 | 1 |

The regularity of $S/I(\mathbb{V}_2)$ is 8 and the $a$-invariant is 7.

Example 3.9. Let $K$ be the field $\mathbb{F}_5$. If $k = 2$, $\mathbb{T}$ is a projective torus in $\mathbb{P}^2$, and $\rho_2(\mathbb{T})$ is the corresponding Veronese type code, then by Corollary 2.3, Theorem 2.4, [6, Theorem 18], and Macaulay2 [12], we obtain the following information for $C_{\mathbb{T}}(d)$:

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|
| $|\mathbb{T}|$ | 16 | 16 | 16 | 16 | 16 | 16 |
| $H_{\mathbb{T}}(d)$ | 3 | 6 | 10 | 13 | 15 | 16 |
| $\delta_{\mathbb{T}}(d)$ | 12 | 8 | 4 | 3 | 2 | 1 |
| $\delta_2(C_{\mathbb{T}}(d))$ | 15 | 11 | 7 | 4 | 3 | 2 |
| $\delta_3(C_{\mathbb{T}}(d))$ | 16 | 12 | 8 | 6 | 4 | 3 |

and the regularity of $S/I(\mathbb{T})$ is 6. Therefore, by Theorem 3.2 we get the following information for the Veronese type code $C_{\rho_2(\mathbb{T})}(d)$:

| $d$ | 1 | 2 | 3 |
|-----|---|---|---|
| $|\rho_2(\mathbb{T})|$ | 16 | 16 | 16 |
| $H_{\rho_2(\mathbb{T})}(d)$ | 6 | 13 | 16 |
| $\delta_{\rho_2(\mathbb{T})}(d)$ | 8 | 3 | 1 |
| $\delta_2(C_{\rho_2(\mathbb{T})}(d))$ | 11 | 4 | 2 |
| $\delta_3(C_{\rho_2(\mathbb{T})}(d))$ | 12 | 6 | 3 |

and the regularity of $S/I(\rho_2(\mathbb{T}))$ is 3.

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M.G. Sarabia. Instituto Politécnico Nacional, UPIITA, Av. IPN No. 2580, Col. La Laguna Ticomán, Gustavo A. Madero C.P. 07340, Ciudad de México. Departamento de Ciencias Básicas

E-mail address: mgonzaleza@ipn.mx

Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14–740, 07000 Mexico City, D.F.

E-mail address: delio.jaramillo@cimat.mx

Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14–740, 07000 Mexico City, D.F.

E-mail address: vila@math.cinvestav.mx