Lorentz Covariant Theory of Precise Doppler Measurements

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Received 15 November 1999, accepted dd.mm.yyyy by ue

Abstract. The Lorentz covariant theory of precise Doppler measurements (PDM) based on the retarded Liénard-Wiechert solution of the Einstein equations is described. An exact solution of equations of light propagation in the field of arbitrary moving bodies, which drastically extends the range of applicability of the new theory of PDM, is obtained. An explicit formula for the gravitational shift of frequency is given in analytic form. The limiting cases of the Doppler observations in gravitational lensing and of the spacecraft’s Doppler tracking are described in more detail. We also present the post-Newtonian theory of the PDM developed for searching relativistic effects in close optical binaries and massive planetary systems.

Keywords: General Relativity, the Doppler effect, Astrometry
PACS: 04.20.Cv, 04.25.-g, 04.80.-y

1 Background

Detection of low-frequency gravitational waves and verification of General Relativity in the Solar system Doppler tracking experiments, as well as searching for anisotropies in the cosmic background radiation (CMB) appeal to the development of the relativistic theory of precise Doppler measurements (PDM). Previous attempts to create such a theory in framework of General Relativity (GR) were based on making use of the post-Newtonian approximations (PNA) which operate with the instantaneous values of metric tensor taken on hypersurfaces of constant time. Hence, an adequate treatment of propagation of light rays in the gravitational field of an isolated astronomical system can be achieved using the PNA only for light-time intervals which do not exceed the characteristic Keplerian time of the system. If light propagates longer, the residual terms of the PNA theory of PDM become unmanageable. Moreover, the question about the instant of time at which positions and velocities of gravitating bodies deflecting light rays should be fixed to minimize calculational errors cannot be solved in the framework of the PNA unambiguously.

A natural way to overcome incompleteness in the PNA approach to the theory of PDM is to employ the post-Minkowskian approximation (PMA) scheme, which is Lorentz covariant and does not involve the slow-motion expansions with respect to
powers of $v/c$, where $v$ is the characteristic velocity of the light-ray-deflecting body, and $c$ is the speed of light. We have succeeded in developing of such a Lorentz covariant theory of PDM which is based on the retarded Liénard-Wiechert solution of the Einstein equations, linearized with respect to Newton’s gravitational constant $G$ \cite{1}. This makes it possible to obtain a unique exact solution of the equations of light propagation in the field of arbitrary moving spherically-symmetric bodies, which drastically extends the range of applicability of the new theory of PDM.

2 Doppler effect in Special and General Relativity

Special relativistic treatment of the frequency shift is well-known. It is based on two facts: 1) the proper time runs differently for identical clocks moving with different velocities; -2) electromagnetic waves propagate along straight lines in the flat space-time (see the paper \cite{2} for more details).

General relativistic formulation of the frequency shift in curved space-time is more involved. Two definitions of the Doppler shift are used \cite{3} - in terms of energy (A) and in terms of frequency (B)

\begin{align}
(A) \quad \frac{\nu}{\nu_0} &= \frac{u^\alpha K_{\alpha}}{u_0^\alpha K_{0\alpha}}, \\
(B) \quad \frac{\nu}{\nu_0} &= \frac{dT_0}{dT},
\end{align}

where $\nu_0$ and $\nu$ are emitted and observed electromagnetic frequencies of light; $T_0$, $u_0^\alpha$ and $T$, $u^\alpha$ are the proper time and 4-velocity of the source of light and observer; $K_{0\alpha}$ and $K_\alpha$ are null 4-vectors of the light particle at the points of emission and observation respectively. Despite the obvious difference in the two definitions they are identical as $u^\alpha = dx^\alpha/dT$, $K_\alpha = \partial \phi/\partial x^\alpha$, and the phase of electromagnetic wave $\phi$ is constant along the light rays. In GR there does not exist a parallel transport of 4-vectors in the sense of the flat space-time. Instead of that one has to integrate equations of light propagation to connect physical quantities at the points of emission and observation of light.

The integration of null geodesic equations in the case of space-times possessing symmetries was known for a long time and extensively used in astronomical practice \cite{4}. However, the most interesting astronomical phenomena in the propagation of light rays in curved space-time are caused by small, time-dependent perturbations of the background geometry. Usually, the 1st PNA in the relativistic N-body problem with fixed or uniformly moving bodies was applied to consider effects of such perturbations \cite{5}. Unfortunately, this approximation works properly if, and only if, time of propagation of light is much shorter than the characteristic Keplerian time of the N-body problem. An adequate treatment of the effects in propagation of light must account for the retardation effects in the propagation of gravitational field to the point of the field’s interaction with the electromagnetic wave.

3 Predictive Relativistic Mechanics of Photons

Physically consistent description of gravitational field in the problem of propagation of light rays is achieved in the framework of the post-Minkowskian approximations...
(PMA) of the Einstein gravitational field equations. For our purpose one PMA is completely enough. This approximation is linear with respect to the universal gravity constant $G$ and gives metric tensor as an algebraic sum of the Minkowski metric $\eta_{\alpha\beta}$ and its perturbation $h_{\alpha\beta}$ written in terms of the Liénard-Wiechert tensor potential. The perturbation $h_{\alpha\beta}[t, x, x_a(s), v_a(s)]$ is a function of a field point $(t, x)$, masses $m_a$ as well as coordinates $x_a(s)$ and velocities $v_a(s)$ of gravitating bodies taken at the retarded time $s = t - \frac{1}{c}|x - x_a(s)|$. Substitution of $h_{\alpha\beta}$ into the equations of the light ray geodesics bring them to the form of to the "retarded-functional differential system" (hereafter $G = c = 1$)

\[\ddot{x}(t) = \sum_{a=1}^{N} m_a F_a[x(t), \dot{x}(t), x_a(s), v_a(s)],\] (2)

because of the dependence of the gravitational light-deflecting force $F_a$ on the retarded time argument $s$. Studying of such equations belongs to the framework of "predictive relativistic mechanics" [6].

4 Solution of the light-ray geodesics

The system (2) is transformed to a simpler form by making use of a specific differential identity [1] applied to the force $F_a$ in order to change of the order of operations of taking partial derivatives and substitution for the unperturbed light-ray trajectory. Solution of the system (3) is, then, achieved by performing integrals from $F_a$ along the light-ray path by making use of transformation from the coordinate time $t$ to the retarded time $s$ which eliminates the time $t$ from all of the integrands of any integral. Hence, the integrals can be directly performed as soon as the motion of the light-ray-deflecting bodies is known. As a result, the time of light travelling from the point of emission $x_0$ to the point of observation $x$ is obtained in the following form [1]

\[t - t_0 = |x - x_0| + 2 \sum_{a=1}^{N} m_a \int_{s_0}^{s} \frac{[1 - \mathbf{k} \cdot \mathbf{v}_a(\zeta)]^2}{\sqrt{1 - \mathbf{v}_a^2(\zeta)}} \frac{d\zeta}{t^* + \mathbf{k} \cdot \mathbf{x}_a(\zeta) - \zeta},\] (3)

where $t^*$ is a pre-defined constant. General formula (3) is used for calculation of the Doppler shift in what follows.

5 Doppler Effect in Gravitational Lenses

In the case of a gravitational lens having a total mass $M$, moving with velocity $V^i$ with respect to the chosen coordinate system, and possesing a time-dependent quadrupole moment $I_{ij}$ the gravitational shift of frequency reads [1]

\[\left(\frac{\delta \nu}{\nu_0}\right)_{\text{obs}}^{\text{gr}} = 4 \left( -MV^i \dot{\theta}_i + \frac{1}{2} \dot{I}^{ij} \dot{\theta}_j \right) \ln |\xi| + \frac{r_0}{R} (\mathbf{v} \cdot \mathbf{\alpha}) + \frac{r}{R} (\mathbf{v}_0 \cdot \mathbf{\alpha}),\] (4)

where $r, r_0$ are distances from the lens to observer and source of light, $R \simeq r + r_0$, $\mathbf{\alpha}$ is the angle of the total deflection of light rays, $\mathbf{v}$ and $\mathbf{v}_0$ are velocities of observer
and the source of light respectively, $\mathbf{\xi} = \xi^i$ is the impact parameter of the light ray, $\hat{\partial}_i = \partial/\partial \xi^i$.

### 6 Spacecraft Doppler Tracking

Lorentz covariant approach allows to derive general relativistic corrections to the well-known special relativistic expression for the electromagnetic frequency shift in the spacecraft’s Doppler tracking. The corrections are summarized in the formula \[ \nu = \frac{1 - \mathbf{k} \cdot \mathbf{v}}{1 - c^2 \frac{\mathbf{k} \cdot \mathbf{v}}{c^2}} \left[ \frac{1 - \frac{v_0^2}{c^2}}{1 - \frac{v^2}{c^2}} \right]^{1/2} \left[ \frac{a(t_0)}{a(t)} \right]^{1/2} \frac{b(t)}{b(t_0)}, \] which consists of three factors. The first factor is the special relativistic Doppler shift while the second and the third factors are general relativistic corrections. They are rather complicated functions of positions and velocities of observer, spacecraft, and light-ray-deflecting masses. Exact expressions for these functions can be found in [1].

### 7 Post-Newtonian Theory for Doppler Measurements of Binary Stars

The determination of velocities of stars from precise Doppler measurements is described in [7]. We apply successive Lorentz transformations and the relativistic equation of light propagation to establish the exact treatment of the Doppler effect in binary systems both in Special and General Relativity theories. As a result, the Doppler shift is a sum of (1) linear in $c^{-1}$ terms, which include the ordinary Doppler effect and its variation due to the secular radial acceleration of the binary with respect to observer; (2) terms proportional to $c^{-2}$, which include the contributions from the quadratic Doppler effect due to the relative motion of binary star with respect to the Solar system, motion of the particle emitting light, orbital motion of the star around the binary’s barycenter, diurnal rotational motion of observer, and orbital motion of the Earth; and (3) terms proportional to $c^{-2}$, which include the contributions from redshifts due to gravitational fields of the star, star’s companion, Galaxy, Solar system, and the Earth. We briefly discuss in the paper [7] feasibility of practical implementation of these theoretical results, which crucially depends on further progress in the technique of precision Doppler measurements.

The author thanks G. Neugabauer for constant support and G. Schäfer for useful discussions.

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