Nahm transformation on the lattice

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The Nahm transformation is a duality mapping between self-dual Yang-Mills configurations on the torus, which exchanges the number of colours with the topological charge. We show how lattice gauge theory techniques can be used to implement it numerically. The method is presented and its precision illustrated with some applications.

1. Introduction

The Nahm transformation \cite{1,2} is a powerful mathematical tool to study self-dual Yang-Mills fields living on a 4-torus. It can furthermore be extended to the more general case of fields on $T^{4-n} \times \mathbb{R}^n$, $n = 0, \ldots, 4$ \cite{3}. In particular, when $n = 4$ the ADHM construction is recovered within this context. Some of the problems in which the Nahm transformation has succeeded are the construction of instanton solutions (calorons) with non-trivial holonomy at finite temperature ($n = 3$) \cite{3,4}, together with their associated fermion zero modes (ZM) \cite{5}, or a nonexistence proof for $Q = 1$ fields with trivial twist on $T^4$ \cite{2}.

A limitation to the usefulness of the Nahm transformation comes from the lack of analytic knowledge on self-dual fields on $T^4$; these can be, on the other hand, accurately approximated and studied by means of lattice techniques \cite{6}. It is therefore desirable to develop a numerical version of the Nahm transformation such that it can be applied directly on lattice configurations. Here we report on the lattice implementation of the Nahm transformation we have set up, together with some examples of its abilities and accuracy; some of the results obtained from its application are presented in \cite{7}.

2. The Nahm transformation

Here we remind briefly some basics of the Nahm transformation. Let us consider a $SU(N)$ self-dual gauge field $A_\mu$ of topological charge $Q$ on a 4-torus, and construct a Weyl operator $\bar{D}_z(x) = \bar{\sigma}_\mu(D^\mu(x) - 2\pi iz^\mu)$, with $\bar{\sigma}_\mu = (1, -i\vec{\tau})$ ($\tau_i$ being the Pauli matrices), and $z^\mu \in \mathbb{R}$. Now, from the index theorem it follows that, provided $A_\mu$ contains no trivial flat factors, the operator $\bar{D}_z$ has $Q$ ZM satisfying the Weyl equation $\bar{D}_z(x)\Psi_\alpha^z(x) = 0$, where $\alpha$ runs in the $Q$-dimensional space of ZM. The Nahm transform of $A_\mu$ is defined as:

$$\hat{A}_\mu^{\alpha\beta}(z) = i \int d^4x \, \Psi_\alpha^z(x) \frac{\partial}{\partial z^\mu} \Psi_\beta^z(x),$$ \hspace{1cm} (1)

where a trace is taken over all other indices in $\Psi$.

The basic properties of the transformation are: (1) $\hat{A}_\mu$ is a self-dual $SU(Q)$ potential with topological charge $N$, living on the dual torus; (2) applying it on $A_\mu$ gives back $A_\mu$; and (3) it induces an isometry between the original and dual moduli spaces.

The above construction works only in the case of trivial twist, i.e. when the twist vectors $\vec{k}$, $\vec{m}$ vanish modulo $N$, due to the fact that fermion fields transforming in the fundamental representation of $SU(N)$ do not support consistently nontrivial twisted boundary conditions. This can be circumvented, however, in two equivalent ways \cite{8}: either by replicating the twisted configuration along certain directions until an untwisted field is obtained, or by introducing a new flavour index on which a twist is imposed that compensates for the one acting over colour. The transfor-
mation can thus be applied on any self-dual field on the torus.\footnote{Both constructions have been implemented within our numerical framework, and prove to be perfectly equivalent. The results in Fig. 1 have been obtained with replicas, the ones in Fig. 2 with flavoured fermions.} Properties (1)-(3) still hold, with $Q$ replaced by $N_0Q$ and $N$ by $N/N_0$, $N_0$ being a twist-dependent integer. In this general case, the torus where $A_\mu$ lives is not the dual one (see \(2\) for details).

3. Numerical method and sample results

Our implementation of a lattice Nahm transformation involves two main stages: first, we construct a lattice gauge configuration whose transform interests us; for this we use improved cooling techniques, which provide us with a tight control over the structure of the field. Then, the relevant quantities entering the transformation itself, such as fermion ZM, must be computed. Here we describe how to construct $\hat{F}_{\mu\nu}$, as well as “dual” link variables $\hat{U}_\mu$, the latter allowing to calculate such quantities as Wilson loops.

For the computation of $\hat{F}_{\mu\nu}$ we use the identity:

$$\hat{F}_{\mu\nu}(z) = 8\pi^2 \eta_{\mu\nu} \int d^4x \Psi^\dagger_z(x) \tau_\alpha \chi_z(x) \ , \quad (2)$$

where $\eta_{\mu\nu} \sigma_\alpha = \bar{\sigma}_{\mu \nu} \sigma_\alpha$, and $\chi_z$ fulfills:

$$D^\mu_z(x)D^\nu_z(x)\chi_z(x) = \Psi^\dagger_z(x) \ . \quad (3)$$

We substitute directly all these continuum expressions by their lattice versions. Through the whole computation we use naive fermions; doublers will therefore appear, but it is always possible to isolate the physical modes, as explained below. Alternatively, Wilson fermions can be introduced from the beginning to spoil doublers; this technique was in fact implemented first, and is described in detail in [11].

The first step is the computation of lattice ZM in the background of a given self-dual field and for a given $z$. As no exact ZM will appear, what we do is to search for the lowest eigenvectors of the hermitian positive operator $\hat{D}^\dagger_z(x)\hat{D}_z(x)$ by using a standard conjugate gradient algorithm, which supplies both high stability and accurate solutions. Thus we end with a lowest 8Q-dimensional (for SU(2)) eigenspace of quasi-ZM, while higher modes carry $O(1)$ eigenvalues. The subspace of physical ZM is selected by diagonalising the Wilson-Dirac operator within this 8Q-dimensional lowest space. In this way one gets $Q$ exactly chiral lowest modes; this is the multiplicity given by the index theorem.

Once the ZM have been computed, the operator inversion involved in Eq. (3) must be carried out. For this we use a stabilised biconjugate gradient algorithm [11], which proves highly efficient. Finally, we compute $\hat{F}_{\mu\nu}$ in Eq. (2).

![Figure 1. Action density of a SU(2) Q = 1/2 instanton-like configuration with twist $\vec{k} = \vec{m} = (0, 1, 0)$ living on a $8^3 \times 32$ lattice vs. the Nahm transform of a $16^3 \times 4$ SU(2) Q = 1/2 caloron-like field with the same twist at coincident dual points.](image-url)
It is possible to construct a lattice Nahm transform, which is useful, for instance, in computing Wilson loops for the Nahm-dual field. The Nahm-dual lattice link variable $\hat{U}_{z,z+\Delta}$ is built up by first computing the lattice $ZM \, \Psi(\ell)$ at $z$ and $z + \Delta$; then the matrix $U(z) = \sum_{x \in \text{latt.}} \Psi^{(\ell)}(x)^\dagger \Psi^{(\ell)}(x+\Delta) (x)$ (which approximates a Wilson line, cf. Eq. (1)) is formed; and, finally, it is decomposed as $U(z) = H(z) \hat{U}_{z,z+\Delta}$, with $H(z)$ a hermitian positive matrix. The link is thus unitary and gauge-covariant by construction. We exemplify the procedure by applying it to a vortex-like $SU(2)$ configuration as those discussed in Ref. [13], which is Nahm-self-dual.

![Figure 2](image-url).

**4. Conclusions**

A lattice implementation of the Nahm transformation, useful to study the properties of self-dual Yang-Mills fields, has been set up in a consistent and perfectly satisfactory way. Several checks have been performed, and a remarkable accuracy is featured by the results.

The method has already been successfully applied to discover new properties of non-abelian, twisted gauge fields living on a torus. However, its possibilities are far from being exhausted, and a number of interesting applications are in sight.

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