FUKAYA CATEGORY FOR LANDAU-GINZBURG ORBIFOLDS
AND BERGLUND-HÜBSCH CONJECTURE FOR INVERTIBLE CURVE SINGULARITIES

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ABSTRACT. For a weighted homogeneous polynomial and a choice of a diagonal symmetry group, we define
a new Fukaya category based on the wrapped Fukaya category of its Milnor fiber together with monodromy
information. It is analogous to the variation operator in singularity theory. As an application, we formulate
a full version of Berglund-Hübsch homological mirror symmetry and prove it for the case of two variables.
Namely, given one of the polynomials $W = x^p + y^q, x^p + x y^q, x^p y + x y^q$ and a symmetry group $G$, we use
Floer theoretic construction to obtain the transpose polynomial $W^T$ with the transpose symmetry group
$G^T$ as well as derived equivalence between the new Fukaya category of $(W, G)$ and the matrix factorization
category of $(W^T, G^T)$. In this case, monodromy corresponds to the restriction of LG model to a hypersurface
in the mirror. For ADE singularities, Auslander-Reiten quivers for indecomposable matrix factorizations were
known from 80’s, and we find the corresponding Lagrangians as well as surgery exact triangles.

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1. Introduction

Singularity theory is a fascinating branch of mathematics with a long history and has deep relations
to many branches of mathematics, such as algebraic, complex and symplectic geometry, Lie groups and
algebras, commutative algebra and mathematical physics. Classifications as well as the topology and
geometry of singularities have been well-established (see [AGZV85], [AGZV12]). In commutative algebras, Cohen-Macaulay rings as coordinate rings of singularities have been investigated in 80’s and its indecomposable maximal Cohen-Macaulay-modules has been classified for ADE singularities (see [Knö87], [Nos90]). Eisenbud has shown that maximal Cohen-Macaulay modules are equivalent to matrix factorizations of a singularity [Eis80]. In mathematical physics, singularities are often called Landau-Ginzburg (LG for short) models, and together with a finite group $G$ preserving the singularity, they are called LG orbifold.

More recently, there has been much attention to the mirror symmetry of singularities, which revealed quite unexpected connections to different branches of mathematics. Symplectic study of Picard-Lefschetz theory by Seidel is mirror to the corresponding algebraic geometry of coherent sheaves [Sei08], and quantum singularity theory developed by Fan, Jarvis and Ruan on Witten equation is mirror to the corresponding integrable hierarchies [FJR13], just to mention a few.

In this paper, we investigate three different but interrelated topics. (1) Fukaya category for Landau-Ginzburg orbifolds, (2) Homological mirror symmetry for invertible curve singularities, and (3) Auslander-Reiten theory for matrix factorizations.

The main motivation is a mirror symmetry between LG orbifolds. A large class of conjectural examples can be formulated following Berglund-Hübsch [BH93]. A polynomial

$$W(x_1, \ldots, x_n) = \sum_{i=1}^{n} \prod_{j=1}^{n} x_{i_j}^{a_{i_j}}$$

is called invertible if the matrix of exponents $A = (a_{i_j})$ is an $n \times n$ invertible matrix. Its Berglund-Hübsch dual

$$W^T(x_1, \ldots, x_n) = \sum_{i=1}^{n} \prod_{j=1}^{n} x_{j_i}^{a_{j_i}}$$

is an invertible polynomial whose exponent matrix is the transpose $A^T$. The group of diagonal symmetries

$$G_W = \{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n \mid W(\lambda_1 z_1, \ldots, \lambda_n z_n) = W(z_1, \ldots, z_n)\}.$$ 

also plays an important role. For a subgroup $G < G_W$, define its dual group $G^T$ following [BH95] as

$$G^T := \text{Hom}(G_W / G, \mathbb{C}^*) \subset G_{W^T}.$$ 

Berglund-Hübsch mirror symmetry is a duality between two Landau-Ginzburg models $(W, G)$ and $(W^T, G^T)$. For closed string mirror symmetry, Fan-Jarvis-Ruan defined quantum singularity theory (FJRW invariants) of $(W, G)$. It should be mirror to Saito-Givental theory of $(W^T, G^T)$ (see [FJR13], [Kra09] for example).

For open string mirror symmetry, Berglund-Hübsch combined with Kontsevich’s homological mirror symmetry conjecture [Kon95] predicts a derived equivalence between the following two categories:

$$\{\text{Fukaya category associated to } (W, G) \} \rightsquigarrow \{G^T\text{-equivariant matrix factorization category } \mathcal{MF}(W^T, G^T)\}.$$ 

This conjecture has been studied extensively for the case of the trivial subgroup $G$ of $G_W$. In this case, the LHS can be defined as $\text{Fukaya-Seidel category } \mathcal{F}S(W)$, a directed $Z$-graded $A_{\infty}$-category of vanishing cycles [Sei08]. For the trivial $G$, we have $G^T = G_{W^T}$. The RHS (with an addition of $Z$-grading) is known as the category of maximally graded matrix factorization of $W^T$. There have been many interesting works in this direction, which proves the conjecture for the cases of trivial $G$. See Seidel [Sei01], Kajiura-Saito-Takahashi [KST07], Auroux-Katzarkov-Orlov [AKO08], Futaki-Ueda [FU11], [FU13], Lekili-Ueda [LU18], [LU20], Harbermann-Smith [HS19] and the references therein. We also refer readers to a nice survey by Ebeling [Ebe].

Unfortunately, there is no known definition for the Fukaya category for $(W, G)$ when $G$ is nontrivial. Namely, a rigorous definition of Fukaya-Seidel category for a nontrivial $G$ is currently out of reach (See [FU09], Problem 3). The main difficulty is that one needs to perturb $W$ to Morse function $W_\epsilon$ to obtain a legitimate collection of Lagrangian vanishing cycles. This procedure destroys the original symmetry $G_W$. 


This situation motivates us to seek a new definition of Fukaya category for a pair \((W, G)\) which does not require Morsification of \(W\). We propose an approach to define a Fukaya category using wrapped Fukaya category of Milnor fiber, maximal symmetry group \(G_W\) and monodromy of the singularity. Our approach is orthogonal to that of Fukaya-Seidel category in the sense that we will mainly work with non-compact Lagrangians (\(K\) in Figure 1) whereas Fukaya-Seidel category uses vanishing cycles (\(L\) in Figure 1).

A topological precursor of our construction is a variation operator

\[ \text{var} : H_{n-1}(\overline{M}_W, \partial \overline{M}_W) \to H_{n-1}(\overline{M}_W), \]

where \(M_W = W^{-1}(1)\) is a Milnor fiber of \(W\). It is defined as a difference of a cycle itself and its monodromy image (fixing the boundary), and it provides an alternative way to describe vanishing cycles.

On the other hand, for weighted homogeneous polynomials monodromy homeomorphism (not fixing the boundary) is known to be an action given by its weights (Milnor [Mil68], see (4.1)). One of our key observation is that this is a part of the maximal symmetry group \(G_W\), hence becomes trivial for the quotient \([M_W/G_W]\). Furthermore, we regard the monodromy on the boundary of the Milnor fiber as a time one Reeb flow on the link, which is a contact manifold. In particular, this provides a distinguished Hamiltonian orbit \(\Gamma_W\) in the quotient space \([M_W/G_W]\).

We remark that monodromy on symplectic Lefschetz fibration has played important roles also in the study of Fukaya-Seidel category. For example, Seidel has shown that monodromy provides a natural transformation from identity to monodromy functor [Sei08]. From this point of view, triviality of the monodromy for \([M_W/G_W]\) should provide a natural transformation from identity functor to itself which is an element of Hochschild cohomology of the wrapped Fukaya category, or a symplectic cohomology class [Sei06]. This should be (conjecturally) the geometric orbit \(\Gamma_W\) that we described above.

Now, let us explain our construction of the new Fukaya category using the orbit \(\Gamma_W\). First, classical variation operator may be viewed as the coequalizer sequence for monodromy \(\phi\),

\[ H_{n-1}(\overline{M}_W, \partial \overline{M}_W) \xrightarrow{\phi_{id}} H_{n-1}(\overline{M}_W, \partial \overline{M}_W) \xrightarrow{\text{var}} H_{n-1}(\overline{M}_W) \to 0 \]

To construct a symplectic analogue, we consider the quantum cap action of \(\Gamma_W\) on the wrapped Fukaya category of \([M_W/G_W]\) given by a version of closed-open map. It gives an \(A_\infty\)-bimodule map \(\cap \Gamma_W : \mathcal{W} \mathcal{F}([M_W/G_W]) \to \mathcal{W} \mathcal{F}([M_W/G_W]).\) As an analogue of coequalizer, we consider the cone of \(\cap \Gamma_W\), which is again an \(A_\infty\)-bimodule over \(\mathcal{W} \mathcal{F}([M_W/G_W]).\) We show that it also carries an \(A_\infty\)-category structure by constructing higher \(A_\infty\)-operations using \(J\)-holomorphic maps from popsicles with \(\Gamma_W\)-insertions (see Section 4).
Theorem 1.1 (Theorem 4.14). For a non-degenerate weighted homogeneous polynomial $W$ and its group of maximal diagonal symmetry $G_W$, there is an $A_{\infty}$-category $\mathcal{F}(W,G_W)$ which fits to a distinguished diagram of bimodules:

$$
\mathcal{W} \mathcal{F}([M_W/G_W]) \xrightarrow{\cap \Gamma_W} \mathcal{W} \mathcal{F}([M_W/G_W]) \rightarrow \mathcal{F}(W,G_W) \rightarrow
$$

(For a precise definition of $\Gamma_W$ and its action $\cap \Gamma_W$, see section 4).

This is the new Fukaya category for the maximal symmetry group $G_W$. For any subgroup $G < G_W$, we can define the associated Fukaya category using semi-direct product $\mathcal{F}(W,G) := \mathcal{F}(W,G_W) \times G^T$ (following Seidel [Sei15]).

Surprisingly, the category $\mathcal{F}(W,G_W)$ in our application has finite dimensional Hom spaces whereas wrapped Fukaya category has infinite dimensional Hom spaces. This is because the cap action kills most of (but not all) wrapped generators for non-compact Lagrangians. Given a symplectic cohomology class $\Gamma$ in a Liouville manifold $M$, similar construction provides a new $A_{\infty}$-category $\mathcal{C}_\Gamma(M)$ (see Section 2). We use popsicle structures that are developed by Abouzaid-Seidel [AS10] and Seidel [Sei18] in a different geometric context.

With this construction in hand, we can formulate the Berglund-Hübsch homological mirror symmetry conjecture, and we prove it for invertible polynomials of two variables (see Section 5). These come in three families

(1) Ferram) $x^p + y^q$, (Chain) $x^p + xy^q$, (Loop) $x^p y + xy^q$.

First, we prove HMS for the quotient of Milnor fiber. A Milnor fiber $M_W$ may have a complicated geometry but its maximal quotient $[M_W/G_W]$ for these singularities turns out to be an orbifold sphere with three special points (orbifold points or punctures). Therefore, we may consider Seidel Lagrangian $L \subset [M_W/G_W]$ (following Seidel [Sei11]) and define its potential function $W_L(b)$ for Maurer-Cartan elements $b = xX + yY + zZ$ (following [CHL17]). We can compute $W_L$ explicitly by counting suitable polygons with $X, Y, Z$-corners in the Milnor fiber $M_W$, and by the localized mirror functor [CHL17], we obtain the homological mirror $A_{\infty}$-functor.

Theorem 1.2 (Theorem 8.1). We have an $A_{\infty}$-functor $\mathcal{F}^L$

$$
\mathcal{F}^L : \mathcal{W} \mathcal{F}([M_W/G_W]) \rightarrow \mathcal{M} \mathcal{F}(W_L)
$$

where $W_L$ for Fermat, chain and loop cases are given as

$$
W_L = x^p + y^q + x y z, \ y^q + x y z, \ x y z
$$

This functor is fully faithful and gives a derived equivalence between two categories.

The mirror polynomial $W^T$ is related to $W_L$ in a quite interesting way. If we define $g$ to be the following polynomial:

$$
g(x, y, z) = \begin{cases} 
  z & \text{for (Fermat)} \\
  z - x^{p-1} & \text{for (Chain)} \\
  z - x^{p-1} - y^{q-1} & \text{for (Loop)} 
\end{cases}
$$

we find that

$$
W_L = W^T + x y \cdot g(x, y, z).
$$

Namely, if we restrict to the hypersurface $g(x, y, z) = 0$, we obtain the transpose polynomial $W^T$ from $W_L$.

This ad-hoc operation of the restriction to the hypersurface $g(x, y, z) = 0$ turns out to be mirror to the monodromy of the singularity $W$. The monodromy orbit $\Gamma_W$ is related to the polynomial $g(x, y, z)$ by a version of closed-open map (deformed by bounding cochains), called Kodaira-Spencer map [FOOO16].
Reiten almost split exact sequences are realized as Lagrangian surgery exact sequences. Moreover, Auslander-Reiten quiver records indecomposable objects as well as irreducible morphisms between them, for ADE (simple) singularities, there are only finitely many indecomposable MCM modules [Knö87]. The localization is more difficult problem than a derived equivalence.

For ADE curve singularity, any indecomposable matrix factorization can be shown to be a split-generator for the multiplication by $g$ on $\mathcal{M}(W^T + xyg)$. The category $\mathcal{F}(W,G_W)$ corresponds to the restriction of $\mathcal{M}(W^T + xyg)$ to the hypersurface $g(x,y,z) = 0$, which is $\mathcal{M}(W^T)$.

**Theorem 1.3** (Theorem 5.3). There is an derived equivalence of $\mathbb{Z}/2$-graded $A_\infty$-categories

$H(\mathcal{F}^T) : D^\pi(\mathcal{F}(W,G_W)) = \mathcal{H. M}(W^T)$

which extends the diagram (commuting up to homotopy) of distinguished triangles of bimodules:

$\mathcal{W}(\{M_W/G_W\}) \xrightarrow{\Gamma_W} \mathcal{W}(\{M_W/G_W\}) \longrightarrow \mathcal{F}(W,G_W) \longrightarrow$

$\mathcal{M}(W^T + xyg) \xrightarrow{g} \mathcal{M}(W^T + xyg) \longrightarrow \mathcal{M}(W^T) \longrightarrow$

As a corollary, we obtain Berglund-Hübsch duality for any subgroup $G < G_W$,

$D^\pi(\mathcal{F}(W,G)) = \mathcal{H.M}(W^T,G^T)$.

We expect the scheme of our proof to work also for higher dimensional Berglund-Hübsch pairs using Sheridan’s immersed spheres and we leave it for the future investigation.

One immediate question would be a relation between the new Fukaya category $\mathcal{F}(W,G)$ for the trivial subgroup $G < G_W$ and the Fukaya-Seidel category $\mathcal{FS}(W)$, as well as the homological mirror symmetry to maximally graded matrix factorizations of $W^T$. First, we expect that $\mathcal{FS}(W)$ can be embedded in a $\mathbb{Z}$-graded version of $\mathcal{F}(W,G)$ for the trivial subgroup $G$, where $\mathcal{F}(W,G) = \mathcal{F}(W,G_W) \times G_W$. Second, in a sequel to this paper, we develop a (Koszul) dual version of localized mirror functor theory for invertible curve singularities. Namely, we use non-compact Lagrangians, and their Maurer-Cartan equations in the new Fukaya category $\mathcal{F}(W,G_W)$ to define the mirror potential function $W^T$. Then the localized mirror functor takes vanishing cycles to their mirror matrix factorizations, which should recover the homological mirror symmetry from the Fukaya-Seidel category.

The third topic of this paper is related to Auslander-Reiten theory [AR75] of matrix factorizations. The category $\mathcal{M}(W^T)$ is equivalent to a category of maximal Cohen-Macaulay modules of a Cohen-Macaulay ring $R := \mathbb{C}[x_1, \ldots, x_n]/(W^T)$ [Eis80]. It has been studied intensively in the 80’s with tremendous success, including classification problem of indecomposable MCM modules. Knörrer showed that for ADE (simple) singularities, there are only finitely many indecomposable MCM modules [Knö87]. The Auslander-Reiten quiver records indecomposable objects as well as irreducible morphisms between them, and for ADE singularity, it is given in Yoshino’s textbook [Yos90].

We investigate geometry behind such Auslander-Reiten quiver via Berglund-Hübsch HMS. We know that $\mathcal{F}(W,G_W)$ is derived equivalent to $\mathcal{M}(W^T)$ (which can be verified using a split-generator). In fact, for ADE singularity, any indecomposable matrix factorization can be shown to be a split-generator for $\mathcal{M}(W^T)$ by AR exact sequences. Finding an explicit Lagrangian that matches a specific matrix factorization is more difficult problem than a derived equivalence.

For this purpose, we will find an explicit model of equivariant geometry for Milnor fibers of invertible curve singularities, and in particular for the case of ADE singularities, we obtain the following theorem in the last section.

**Theorem 1.4** (Theorem 10.5). For ADE curve singularity $W^T$, we find explicit Lagrangians in the Milnor fiber of $W$ corresponding to indecomposable matrix factorizations in the AR quiver. Moreover, Auslander-Reiten almost split exact sequences are realized as Lagrangian surgery exact sequences.
Buchweitz-Greuel-Schreyer proved the converse statement of Knörrer by constructing infinitely many non-isomorphic MCM modules for non-simple cases [BGS87]. It would be very interesting to find a geometric reason for such phenomena.

1.1. Baby example: $x^2 + y^2$. To illustrate our construction, let us explain the case of $W = x^2 + y^2$. Milnor fiber $M_W = \{x^2 + y^2 = 1\}$ is $T^*S^1$ or a cylinder. Its zero section is a vanishing cycle of $W$ and its cotangent fiber $K$ generates its wrapped Fukaya category (See Figure 1).

We also consider its maximal diagonal symmetry group $G_W$, which is a $\mathbb{Z}/2 \times \mathbb{Z}/2$ acting as a $(-1)$ multiplication on each factor. The quotient of $M_W$ by $G_W$ is an orbifold sphere $\mathbb{P}^1_{2,2,\infty}$ with two $\mathbb{Z}/2$-orbifold points, say $A, B$ and a single puncture $C$ as in Figure 2.

A mirror for this orbifold is a polynomial $x^2 + y^2 + xyz$. To see this, take Seidel’s Lagrangian $L$ (see Figure 15), which is an immersed Lagrangian with three odd immersed generators $X, Y, Z$. It is weakly unobstructed with bounding cochain $b = xX + yY + zZ$ and its Lagrangian potential function is exactly $W_L = x^2 + y^2 + xyz$ (see [CHL17]). This can be seen by taking lifts of $L$ to the cylinder, which gives four circles as in Figure 3. Pick a generic point, and count all rigid polygons passing through it. The reader can find two bigons with corners labeled by $X, X$ and $Y, Y$ together with a minimal triangle $XYZ$.

If we set $z = 0$, $W_L$ becomes a desired dual polynomial $W^T = x^2 + y^2$. This is related to the monodromy of $W$ as follows. Monodromy of $W$ (fixing the boundary) is given by a Dehn twist $\phi$ as in Figure 4(A). We will consider a version of monodromy $\psi$ which does not fix the boundary as in Figure 4(B). It turns out that the monodromy $\psi$ on the Milnor fiber $M_W$ is the same as the action by the element $(-1, -1)$ of a maximal symmetry group $G_W$. Also, consider the Reeb flows on the boundary $\partial M_W$ describing the
monodromy $\psi$, which give a Reeb orbit $\Gamma_W$ in the quotient $\mathbb{P}^1_{2,2,\infty}$. In this case, $\Gamma_W$ is the orbit that winds the puncture of the quotient once. Closed-open map takes this orbit to $z \cdot 1_L$.

A category $\mathcal{F}(W, G_W)$ that we are going to construct is essentially the wrapped Fukaya category of $[M_W / G_W] = \mathbb{P}^1_{2,2,\infty}$ with additional relation "$\Gamma_W = 0". We realize it by considering quantum cap action of $\Gamma_W$ on wrapped Fukaya category of $\mathbb{P}^1_{2,2,\infty}$. We consider $K$ as an object of the $A_\infty$-category $\mathcal{F}([M_W / G_W]) = \mathcal{F}(W, G_W)$ is different from the category of cones (where cones are considered as twisted complexes), and hence this surgery interpretation should be taken only as an intuition.

The actual $A_\infty$-category is defined using popsicle maps with $\Gamma_W$-insertions.

\[ \phi(K) \quad \text{var} \quad \psi(K) \]

\[ \text{Cone} \]

\[ (\text{A}) \text{ Variation} \]

\[ (\text{B}) \text{ Lagrangian surgery} \]

\[ \text{Figure 4. Two ways of representing vanishing cycle} \]

We have $A_\infty$-functor from $\mathcal{F}(W, G_W) \rightarrow \mathcal{M}(W^T)$ which is induced by localized mirror functor\cite{CHL17} (and setting $z = 0$). It sends a noncompact Lagrangian $K$ to Floer complex $\begin{pmatrix} CW^*(K, L), -m_1^{0,b} \end{pmatrix} |_{z=0}$. This complex is a matrix factorization of $W^T$. We can calculate it in Figure 3 and it is given by

\[ -\begin{pmatrix} y & x \\ x & -y \end{pmatrix} - \begin{pmatrix} y & x \\ x & -y \end{pmatrix} \]

which is a compact generator of $\mathcal{M}(W^T)$\cite{Dyc11}. We can check that $K$ is also a generator of $\mathcal{F}(W, G_W)$ and they have isomorphic endomorphism ring. Hence we obtain Berglund-Hübsch HMS.

1.2. Structure of the paper. In Section 2 for a fixed symplectic cohomology class $\Gamma$ in a Liouville manifold $M$, we construct an $A_\infty$-category $\mathcal{C}_\Gamma$ from wrapped Fukaya category of $M$ on which the quantum cap action of $\Gamma$ vanishes. It uses pseudo-holomorphic popsicles with $\Gamma$-insertions. This construction will be generalized to orbifold quotients of Milnor fibers of weighted homogeneous polynomials in Section 4. A parallel construction in algebraic geometry is described in Section 3 which is a restriction to a hypersurface. In Section 4 we explain our choice of distinguished Hamiltonian orbit $\Gamma_W$ for a Milnor fiber of a given weighted homogeneous polynomial $W$ which encodes the monodromy information. Then we define a new $A_\infty$-category $\mathcal{F}(W, G)$ by generalizing the construction of $\mathcal{C}_\Gamma$ to this particular class of orbifolds.

In Section 5 we briefly review Berglund-Hübsch conjecture as a start of a new chapter. In Section 6, we investigate the equivariant topology of the Milnor fibers of invertible curve singularities, and give a combinatorial description of the equivariant tessellations of the Milnor fibers with respect to the maximal
diagonal symmetry group $G_W$-action. In particular, it is shown that the quotient of Milnor fiber $M_W$ by $G_W$-action is an orbifold sphere with three special points which are either orbifold points are punctures.

In Section 7, we recall the setup of relevant Floer theory and localized mirror functor. We prove homological mirror symmetry for Milnor fibers in Section 8. This is before we take monodromy into consideration. In Section 9, we first investigate how the HMS for Milnor fiber intertwines monodromy information. For this purpose, we show that the cap action by $\Gamma_W$ in the symplectic side is homotopic to the multiplication by a polynomial $g(x, y, z)$ in the matrix factorizations under localized mirror functor. Finally we prove Berglund-Hübsch homological mirror symmetry for invertible polynomials of two variables, by using the combination of popsicles and localized mirror functor. We explain how to obtain the full version of Berglund-Hübsch HMS for any subgroup $G$ of $G_W$ from that of $G_W$ using semi-direct product construction.

Last but not least, we find in Section 10 explicit Lagrangians and surgery exact triangles which correspond to indecomposable matrix factorizations and Auslander-Reiten almost split exact sequences under Buglund-Hübsch HMS.

In Appendix A, we briefly describe the moduli spaces and perturbation scheme we use throughout the paper. Appendix B explains a compactification of popsicle moduli spaces. In Appendix C provided by Osamu Iyama, it is proved that any short exact sequence between indecomposable matrix factorizations that involves same modules as one of the AR exact sequences should be isomorphic to it.

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2. **Quantum Cap Action and the New $A_\infty$-Category**

Let $(M, \omega)$ be a Liouville domain with cylindrical ends. We assume that the reader is familiar to the standard definition of symplectic cohomology $SH^\bullet(M)$ and the wrapped Fukaya category $WF(M)$ (see appendix A for a brief account of their constructions). It is well-known that $SH^\bullet(M)$ acts on $WF(M)$. One way of describing it is via quantum cap action. Namely, given a symplectic cohomology class $\Gamma \in SH^0(M)$, quantum cap action gives an $A_\infty$-bimodule map from diagonal $A_\infty$-bimodule $WF(M)_\Delta$ to itself:

$$\cap \Gamma : WF(M)_\Delta \to WF(M)_\Delta$$

The purpose of this section is to define a new $A_\infty$-category $\mathcal{C}_\Gamma$ on which the cap action of $\Gamma$ vanishes.

**Theorem 2.1.** Given an element $\Gamma \in SH^0(M)$, we define a new $A_\infty$-category $\mathcal{C}_\Gamma$ such that

1. $\mathcal{C}_\Gamma$ has the same set of objects as the wrapped Fukaya category $WF(M)$,
2. for two objects $L_1, L_2$, its morphisms are given by
   $$\text{Hom}_{\mathcal{C}_\Gamma}(L_1, L_2) = CW(L_1, L_2) \oplus CW(L_1, L_2) e$$
   where $CW(L_1, L_2)$ is the morphism space for $WF(M)$ and $\deg e = -1$,
3. a natural inclusion $\Psi : WF(M) \to \mathcal{C}_\Gamma$ is an $A_\infty$-functor,
4. regarding the $A_\infty$-category $\mathcal{C}_\Gamma$ as an $A_\infty$-bimodule over $WF(M)$ (using $\Psi$), we have a distinguished triangle of $A_\infty$-bimodules
   $$WF(M)_\Delta \xrightarrow{\cap \Gamma} WF(M)_\Delta \to \mathcal{C}_\Gamma \to$$

The construction of $\mathcal{C}_\Gamma$ is based on pseudo-holomorphic maps from popsicles with $\Gamma$-insertions at sprinkles.
Remark 2.2. With the standard assumption \( c_1(M) = 0 \), all the structures we consider are \( \mathbb{Z} \)-graded. This will be not the case for our main application, so we will only get a \( \mathbb{Z}/2 \)-graded version.

2.1. Quantum cap action. A quantum cap action in the context of quantum cohomology and Fukaya category of a compact symplectic manifold [Aur07], and it can be also defined for wrapped Fukaya category [Can13]. Let us briefly recall its definition.

Let \( P_{n,[i]} \) \((1 \leq i \leq n)\) be a moduli space of discs with one interior marked point \( z_1^+ \) and boundary marked points \( z_0, \ldots, z_n \) such that \( z_0, z_1^+, z_i \) lie on a geodesic of \( D^2 \). By applying an automorphism of the disc, we may say that \( z_1^+, z_0, z_i \) are precisely 0, 1, \(-1\) of the disc. Consider a moduli space

\[
\mathcal{P}_{n,[i]}(\Gamma; a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n, a_0)
\]

of pseudo-holomorphic maps from \( S \in P_{n,[i]} \) with an interior insertion \( \Gamma \in SH^*(M) \) at the puncture \( z_1^+ \) and with boundary insertions \((a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)\). Here, we use a different symbol \( b \) to emphasize that it is a bimodule input.

**Definition 2.3.** A cochain level quantum cap action of \( \Gamma \) is an \( A_\infty \)-bimodule map is defined by

\[
\cap \Gamma : T(W F) \otimes W F \otimes T(W F) \to W F
\]

\[(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \to (-1)^{\deg a_{n,[i]} - \deg \Gamma} \mathcal{P}_{n,[i]}(\Gamma; a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n),
\]

\[
\star \mathcal{P}_{n,[i]} = \sum_{j<i} (j-1) \cdot \deg a_i + i \cdot \deg b + \sum_{i<j} j \deg a_j
\]

where \( \mathcal{P}_{n,[i]}(\Gamma; a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \) is a sum of orientation operators associated to the moduli spaces for all possible \( a_0 \) (see Appendix A).

It is easy to check that \( \cap \Gamma \) is a bimodule homomorphism from \( W F(M) \) to itself. Also, this action is closely related to the closed-open map (see Appendix A) in the following way.

**Proposition 2.4.** Up to homotopy, we have

\[
\cap \Gamma(b) = m_2 \left( b, CO_{L_2}(\Gamma) \right)
\]

where \( CO_{L_2} : SH^*(M) \to CW^*(L_2, L_2) \) is a word-length zero component of the closed-open map.

**Proof.** Consider a 1-parameter family of moduli space of holomorphic discs with

- one outgoing boundary marking \( z_0 \) at 1
- one incoming boundary marking \( z_1 \) at \(-1\)
- one moving interior marking \( z_1^+ \) at \(-it\), \( t \in [0, 1] \)

At \( t = 0 \), we get \( P_{1,[1]} \). At \( t = 1 \), we get a moduli space of discs with disc bubble containing interior marked point. See Figure 5. It corresponds to a disc moduli space governing \( m_2 \left( b, CO_{L_2}(\Gamma) \right) \). \( \square \)

By algebraic nonsense, we have found a distinguished triangle of \( A_\infty \)-bimodules:

\[
W F(M) \xrightarrow{\cap \Gamma} W F(M) \xrightarrow{} \mathcal{C}_{\Gamma} \xrightarrow{\partial}
\]

**Corollary 2.5.** As a complex,

\[
\text{Hom}_{\mathcal{C}_{\Gamma}}^*(L_1, L_2) \cong \text{Cone} \left( CW^*(L_1, L_2) \xrightarrow{\cap \Gamma} CW^*(L_1, L_2) \right)
\]

The action \( \cap \Gamma \) vanishes on \( \mathcal{C}_{\Gamma} \) in a homotopic way. What we are trying to do is extending the \( A_\infty \)-bimodule structure of \( \mathcal{C}_{\Gamma} \) to an \( A_\infty \)-category structure. Intuitively,
objects of $\mathcal{C}_\Gamma$ are twisted complexes

$$L \xrightarrow{\text{CO}(\Gamma)} L, \quad L \in \mathcal{W}_F(M).$$

In many cases, they can be realized as a geometric surgery of $L$ with itself along \text{CO}$(\Gamma)$.

• the space of morphisms is a "half" of the morphisms between twist complexes. It consists of 

$$"a + \epsilon b" = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \in CW^* \left( L_1 \xrightarrow{\text{CO}(\Gamma)} L_1, \ L_2 \xrightarrow{\text{CO}(\Gamma)} L_2 \right)$$

This intuition works well in differential graded world (see Section 3), but fails in all possible ways in the $A_\infty$-world. For example, $A_\infty$-compositions do not preserve the above space of morphisms. We will consider a new type of $J$-holomorphic curves to construct a precise $A_\infty$-structure.

2.2. Popsicles. We consider a generalization of the cap action using popsicles.

Definition 2.6 ([AS10], [Sei18]). A popsicle is a disc $D^2$ with following decorations;

1. boundary marked points: denoted by $z_0, z_1, \ldots, z_n$.
2. popsicle sticks: regarding the interior of $D^2$ as a hyperbolic disc, the geodesic connecting $z_i$ and $z_0$ (at infinity), denoted as $Q_i$.
3. flavour: for $F = \{1, \ldots, l\}$ and a non-decreasing map

$$\phi : F \to \{1, \ldots, n\}.$$ 

4. sprinkles (interior marked points): $l$ distinct interior marked points $z^+_1, \ldots, z^+_l$, such that $z^+_j$ lies on the geodesic $Q_{\phi(j)}$ for $j = 1, \ldots, l$.

We say that the above popsicle has type $(n, \phi)$ (see Figure 6). We call it stable if $n + l \geq 2$. We denotes a moduli space of popsicles of type $(n, \phi)$ by $P_{n,F,\phi}$.

Note that $\phi(j)$ records the popsicle stick on which the interior marked point $z^+_j$ lies. When $\phi(j) = \phi(j+1) = \cdots = \phi(j+k)$, then $z^+_j, \ldots, z^+_{j+k}$ lie on the same popsicle stick $Q_{\phi(j)}$. For such $\phi$, the moduli space $P_{n,F,\phi}$ carries an extra permutation symmetry of these interior marked points. Denote by $\text{Sym}^\phi \subset S_F$ a subgroup of a symmetry group $S_F$ which stabilizes $\phi$. $\text{Sym}^\phi$ is trivial if and only if $\phi$ is injective.

Remark 2.7. Our popsicle moduli space is slightly different from [AS10], [Sei18] where sprinkles were allowed to coincide. Hence, we take a different compactification of popsicle moduli space such that when two interior marked points collide in the same popsicle stick, sphere bubble is created as in the standard Gromov-Witten theory. Also we do not need the notion of weighted popsicles.
One can compactify the moduli space of popsicles by introducing broken popsicles, which is denoted as $\overline{P}_{n,F,\phi}$. This is almost the same as that of the references except the issues related to the remark. We give a detailed explanation about a gluing of two popsicles and the compactification of the moduli spaces for completeness. See Appendix B.

2.3. **Popsicle maps with $\Gamma$-insertions.** Let us introduce pseudo-holomorphic maps from popsicles with $\Gamma$-insertions.

**Definition 2.8.** Let $a_i \in CF^*(L_{i-1}, L_i)$ and $\Gamma \in SH^*(M)$. Define

$$\overline{P}_{n,F,\phi}(\Gamma; a_1, \ldots, a_n, a_0)$$

be a compactified moduli space of pseudo-holomorphic maps

$$\left\{ u : S \to M \mid S \in \overline{P}_{n,F,\phi} \right\}$$

satisfies

- a boundary segment from $z_i$ to $z_{i+1}$ goes to $L_i$,
- a boundary marking $z_i$ goes to $a_i$,
- all interior markings are asymptotic to $\Gamma$.

It can be described as a submanifold with corners

$$\overline{P}_{n,F,\phi}(\Gamma; a_1, \ldots, a_n, a_0) \subset \mathcal{M}|F|\fin(\Gamma, \ldots, \Gamma; a_1, \ldots, a_n, a_0)$$

cut out by popsicle conditions on interior marked points.

A standard compactness and transversality argument can be applied to $\overline{P}_{n,F,\phi}$. We prove in Appendix B that $\overline{P}_{n,F,\phi}$ is a manifold with corners. Therefore we can choose a Floer data consistently by extending it from the lowest dimensional strata in an inductive manner.

**Lemma 2.9.** For a generic choice of universal and consistent Floer data,

1. The moduli spaces $\overline{P}_{n,F,\phi}(\Gamma; a_1, \ldots, a_n, a_0)$ are smooth and compact.
2. For a given input $\Gamma$ and $a_i$, ($i = 1, \ldots, n$), there are only finitely many $a_0$ for which $\overline{P}_{n,F,\phi}(\Gamma; a_1, \ldots, a_n, a_0)$ is non-empty.
3. It is a manifold of dimension

$$|F|(1 - \deg \Gamma) + n - 2 + \deg a_0 - \sum_{i=1}^{n} \deg a_i$$
Proof. Compactness and transversality arguments are mostly the same as in Appendix A. A standard index formula tells us that
\[
\dim \mathcal{T}_{n,F,\phi}(\Gamma; a_1, \ldots, a_n, a_0) = \dim \mathcal{T}_{[F]_1,n,1}(\Gamma, \ldots, \Gamma; a_1, \ldots, a_n, a_0) - |F|
\]
\[
= (2|F| + n - 2) + \deg a_0 - \sum_{i=1}^{n} \deg a_i - |F| \cdot \deg \Gamma - |F|
\]
\[
= |F|(1 - \deg \Gamma) + n - 2 + \deg a_0 - \sum_{i=1}^{n} \deg a_i.
\]
\[\square\]

Let \( P_{n,F,\phi}^\Gamma \) be a sum of orientation operators associated to the zero-dimensional component of
\[
\mathcal{T}_{n,F,\phi}(\Gamma; a_1, \ldots, a_n, a_0)
\]
for all possible \( a_0 \). A degree of this operator is \( 2 - n - |F|(1 - \deg \Gamma) \). Also, \( P_{n,F,\phi}^\Gamma \) describes \( A_\infty \)-structure \( \{m_n\} \) if \( F \) is empty, and describes quantum cap action \( \cap \Gamma \) if \( |F| = 1 \).

Remark 2.10. Popsicle structures were introduced by Abouzaid-Seidel [AS10] and Seidel [Sei18]. It is a good moment to point out differences and similarities.

In [AS10], popsicles were used to mark the places to put sub-closed one forms for continuation maps between linear Hamiltonians. They localize the continuation map and obtain a big chain complex involving countable family of Hamiltonians \( \{nH\}_{n \in \mathbb{N}} \). This is a definition of wrapped Fukaya category in a linear Hamiltonian setting.

Later, Seidel [Sei18] considered the continuation map for Lefschetz fibration, but he also considered its cone
\[
CF^*(L_0, L_1; -H) \to CF^*(L_0, L_1; 0) \to CF^*(L_0, L_1; \text{conti}),
\]
not only its localization. Popsicle maps were used to construct an \( A_\infty \)-category structure on \( CF^*(L_0, L_1; \text{conti}) \). This brings out the effect of removing contributions from compact part of Lefschetz fibration, and recovers the Floer cohomology of the fiber.

Our geometric setting is different from the references because we regard sprinkles as genuine inputs for the symplectic cohomology class \( \Gamma \). But the algebraic properties we desire are similar to those of [Sei18]. In our case, we want to remove the effect of the quantum cap action \( \cap \Gamma \) from the wrapped Fukaya category. Namely, wrapped generator in the image of quantum cap action will be killed.

2.4. \( A_\infty \)-category \( \mathcal{C}_\Gamma \). We construct a new \( \mathbb{Z} \)-graded \( A_\infty \)-category \( \mathcal{C}_\Gamma \) under the assumption \( \deg \Gamma = 0 \). We start with the following important observation.

Proposition 2.11. (AS10) If \( \phi : F \to \{1, \ldots, n\} \) is not injective, then \( P_{n,F,\phi}^\Gamma \) vanishes.

Proof. The assumption means that at least one popsicle stick carries more than two interior markings. Then \( \text{Sym}^\phi \) contains a nontrivial transposition. Since we put a same class \( \Gamma \) for all interior markings, the transposition extends to \( \mathcal{T}_{n,F,\phi}(\Gamma; a_1, \ldots, a_n) \) also. It induces an orientation-reversal automorphism on \( \mathcal{T}_{n,F,\phi} \). Therefore the contribution of this moduli space should vanish. \[\square\]

Now we can focus on the case when \( \phi : F \to \{1, \ldots, n\} \) is injective. Then \( F \) can be considered as a subset \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \). In this case, we omit a notation \( \phi \) and simply write \( \mathcal{T}_{n,F} \) and \( P_{n,F}^\Gamma \).

Definition 2.12. An admissible cut of \( F \) consists of

1. \( n_1, n_2 \geq 1 \) such that \( n_1 + n_2 = n + 1 \)
(2) a number \( i \in \{1, \ldots, n\} \)

(3) \( F_1 \subseteq \{1, \ldots, n_1\} \) and \( F_2 \subseteq \{1, \ldots, n_2\} \) such that \(|F_1| + |F_2| = |F|\)

satisfying the following properties:

- \( F \ni \{k \in F_1, k < i\} \) and \( F \ni \{k + n_2 - 1|k \in F_1, k > i\} \)

- \( F \ni \{k + i - 1|k \in F_2\} \)

If \( i \notin F_1 \), then this completely recovers \( F \). Otherwise, \( F \) has one more element among \([i, i + 1, \ldots, (i + n_2 - 1)]\).

An admissible cut describes a stratum \( P_{n_1,F_1} \times P_{n_2,F_2} \) of a moduli space \( \bar{P}_{n,F} \). They describe precisely codimension 1 strata whose associated flavour \( \phi_j \) is still injective. Combined with \([2,11]\), we get a quadratic relation

\[
\sum_{\text{admissible cuts}} (-1)^{\mathfrak{a}} P_{n_1,F_1}^\Gamma (a_1, \ldots, a_{i-1}, a_i, \ldots, a_{i+|F_2|-1}, a_{i+|F_2|}, \ldots, a_n) = 0,
\]

where \( \mathfrak{a} = n_2i + i + 1 + n_1|F_2| + (n_2 + |F_2|) \left( \sum_{j \leq i + |F_2|} \deg a_j \right) \)

\[
\quad + \left| \{(f_1, f_2) \in F_1 \times F_2 | f_1 < f_2 \text{ inside } F\} \right|
\]

Now we are ready to define a new \( A_{\infty} \)-category.

**Definition 2.13.** Let \( \Gamma \in SH^0(M) \) be a fixed symplectic cohomology class of degree zero. An \( A_{\infty} \)-category \( \mathcal{C}_\Gamma \) consists of

1. a set of objects \( \text{Ob}(\mathcal{C}_\Gamma) = \text{Ob}(W\mathcal{F}(M)) \).
2. morphisms between two objects

\[
\text{Hom}_{\mathcal{C}_\Gamma}(L_1, L_2) = CW(L_1, L_2) \oplus CW(L_1, L_2)e
\]

Here \( \deg e = -1 \), and we denotes the element of this complex by \( c := a + eb \).

3. An \( A_{\infty} \)-structure \( \{M_n\}_{n=1}^{\infty} \) is given as follows. We may write

\[
M_n(c_1, \ldots, c_n) = M_n^a(c_1, \ldots, c_n) + e M_n^b(c_1, \ldots, c_n)
\]

(a) Suppose \( c_i = a_i \) for all \( i \) (all the inputs do not have \( e \) components), then we set

\[
M_n(a_1, \ldots, a_n) = m_n(a_1, \ldots, a_n)
\]

where \( m_n \) is the \( A_{\infty} \)-operation for \( W\mathcal{F}(M) \).

(b) Suppose \( c_i = eb_i \) for \( i \in \{i_1, \ldots, i_k\} \), and \( c_i = a_i \) for \( i \notin \{i_1, \ldots, i_k\} \). Then we set

\[
F = \{i_1, \ldots, i_k\}, \quad \hat{F} = \{i_1, \ldots, \hat{i}_j, \ldots, i_k\},
\]

and define

\[
M_n^a(c_1, \ldots, c_n) = (-1)^{\ast_n^a} P_{n,F}^\Gamma (a_1, \ldots, b_{i_1}, \ldots, b_{i_j}, \ldots, a_n)
\]

\[
M_n^b(c_1, \ldots, c_n) = \sum_{j=1}^{k} (-1)^{\ast_n^a + \ast_n^b} P_{n,F_j}^\Gamma (a_1, \ldots, b_{i_1}, \ldots, b_{i_j}, \ldots, a_n)
\]

If we use the common notion \( x_i \) to denote \( a_i \) and \( b_i \). Then

\[
\ast_n^a = \sum_j j \deg x_j + \sum_{f \in F, l > f} (\deg x_l - 1)
\]

\[
\ast_n^b = \sum_j j \deg x_j + \sum_{f \in \hat{F}, l > j} (\deg x_l - 1)
\]

\[
\ast_j = \sum_{i=1}^{j} (\deg x_i - 1)
\]
As a sanity check, let us check the degree of $M_n$. Since $\deg \Gamma = 0$, the degree of $P^\Gamma_{n, F}$ is $2 - n - |F|$. Additional degree shift $-|F|$ comes from interior markings. We correspondingly shift our inputs $b_i$ for each interior markings by multiplying $\epsilon$. Therefore, a degree of $M_n$ becomes $2 - n$.

We may use the sign analysis of [AS10] in our situation. First, recall that we are using the sprinkle as a place for interior $\Gamma$-insertion whereas in [AS10] sprinkle is just a marker for some other data. Hence orientation for the latter sprinkle is given by the orientation of $R$ (the popsicle stick), but in our case, we need $R \otimes o^\Gamma$ where $o^\Gamma$ is the orientation operator of the Reeb orbit $\Gamma$. It is important that our symplectic cohomology insertion $\Gamma$ has an even degree so that it does not affect any sign for switching places. The result will be a $\mathbb{Z}/2$-graded $\mathcal{C}^\Gamma$ at the expense of giving up $\mathbb{Z}$-grading. We refer readers to [AS10] for detailed explanation for signs, and leave the adaptation as an exercise.

2.5. Proof of $A_\infty$-identity.

**Proposition 2.14.** $\mathcal{C}_\Gamma$ is an $A_\infty$-category. Namely, for any composable $(c_1, \ldots, c_n)$, we have

$$\sum_{n_1 + n_2 = n+1} (-1)^{\sum_{j=1}^{n} |c_j|} M_{n_1}(c_1, \ldots, c_{i-1}, M_{n_2}(c_i, \ldots, c_{i+n_2-1}), \ldots, c_n) = 0.$$ 

**Proof.** We check the identity on each component of the output. We first show that

$$\sum (M^a_{n_1}(\ldots, M^a_{n_2}(\ldots), \ldots) \pm M^a_{n_1}(\ldots, M^b_{n_2}(\ldots), \ldots)) = 0.$$ 

This identity follows from the compactification of popsicle moduli spaces. Namely, a codimension one stratum of the popsicle moduli space corresponds to a term in the above equation. In Figure 7, we illustrated corresponding broken popsicles for the case $|F| = 4$. Even for broken popsicles, we can consider a sequence of hyperbolic geodesics connecting $z_0$ and $z_{ij}$. In the figure, dotted lines are such geodesics that do not contain a sprinkle.

![Figure 7. $A_\infty$-identity with $a$-output](image)

Next we show that

$$\sum (M^b_{n_1}(\ldots, M^a_{n_2}(\ldots), \ldots) \pm M^b_{n_1}(\ldots, M^b_{n_2}(\ldots), \ldots)) = 0.$$ 

This identity follows from the compactification of popsicle moduli spaces for $F^j$ for all $j$. In Figure 8, we illustrated corresponding broken popsicles for the case $|F| = 4$ and $j = 1$. We also expressed the forgotten geodesic for $M^b$-operation as dotted lines.

![Figure 8. $A_\infty$-identity with $b$-output](image)

2.6. Example: $M_2$-operation. Let us examine the following Leibniz rule for the input $(a, c b)$.

$$(2.1) \quad M_1(M_2(a, c b)) + M_2(M_1(a), c b) + (-1)^{|a|} M_2(a, M_1(c b)) = 0.$$ 

For simplicity, we will omit the signs from the formulas. From the definition

$$M_2(a, c b) = P^\Gamma_{2, |2|}(a, b) + \epsilon m_2(a, b)$$
We have
\[ M_1(M_2(a,\epsilon b)) = M_1(\mathbf{P}_{2,1}^r(a,\epsilon b)) + \epsilon m_2(m_1(a,b)) + \epsilon m_1(\mathbf{P}_{1,1}^r(a,\epsilon b)) \]
\[ M_2(M_1(a),\epsilon b) = M_2(m_1(a),\epsilon b) = \mathbf{P}_{2,1}^r(m_1(a),\epsilon b) + \epsilon m_2(m_1(a,b)) \]
\[ M_2(a,M_1(\epsilon b)) = M_2(a,\mathbf{P}_{1,1}^r(\epsilon b)) + \epsilon m_1(\mathbf{P}_{1,1}^r(\epsilon b)) = m_2(a,\mathbf{P}_{1,1}^r(\epsilon b)) + \mathbf{P}_{2,1}^r(a,m_1(b)) + \epsilon m_2(a,m_1(b)) \]

If we collect the terms with \( \epsilon \) in (2.1), we obtain the original \( A_\infty \)-identity
\[ \epsilon \left( m_1(m_2(a,b)) + m_2(m_1(a,b)) + (-1)^{|a|} m_2(a,m_1(b)) \right) = 0. \]

Collecting the terms without \( \epsilon \) in (2.1), we get the following (up to sign)
\[ \mathbf{P}_{2,1}^r(m_1(a),\epsilon b) + \mathbf{P}_{2,1}^r(a,m_1(b)) + m_2(a,\mathbf{P}_{1,1}^r(\epsilon b)) + \mathbf{P}_{1,1}^r(m_2(a,b)) + m_1\mathbf{P}_{2,1}^r(a,b) = 0. \]

These terms correspond to the codimension one degenerations (given by disc bubblings) in Figure 9. Here dotted lines just indicate paths to the 0-th vertex, and do not give any restriction to the domain. Hence one may remove dotted lines to find the corresponding \( A_\infty \)-operations.

Let us examine Leibniz rule for the input \((\epsilon b_1,\epsilon b_2)\). Namely, we want to verify
\[
(2.2) \quad M_1(M_2(\epsilon b_1,\epsilon b_2)) + M_2(M_1(\epsilon b_1),\epsilon b_2) + (-1)^{|b_1|} M_2(\epsilon b_1,M_1(\epsilon b_2)) = 0.
\]
We have

\[ M_1(M_2(\epsilon b_1, \epsilon b_2)) = M_1(\mathbb{P}^\Gamma_{1,2}(b_1, b_2) + \epsilon \mathbb{P}^\Gamma_{2,2}(b_1, b_2) + \epsilon \mathbb{P}^\Gamma_{2,1}(b_1, b_2)) = m_1(b_1, b_2) + \epsilon m_1(b_1, b_2) \]

\[ M_2(M_1(\epsilon b_1), \epsilon b_2) = M_2(\mathbb{P}^\Gamma_{1,1}(b_1) + \epsilon m_1(b_1), \epsilon b_2) = \mathbb{P}^\Gamma_{2,2}(\mathbb{P}^\Gamma_{2,1}(b_1, b_2) + \epsilon m_1(b_1)) \]

The following figure 10 describes the terms without \( \epsilon \) in the above (in the same order). It is not hard to see that these arise from codimension one boundary strata of \( \mathbb{P}^\Gamma_{2,1,2} \). The terms with \( \epsilon \) are similar.

3. Mirror counterpart: Hypersurface restriction

A mirror construction of last section for algebraic geometry is just a categorical reformulation of the restriction to a hypersurface.

3.1. Restriction to a hypersurface in \( D^b\text{Coh} \). Let \( S \) be an algebra. Choose an element \( g \in Z(S) \cong HH^0(S, S) \)

The DG-bimodule \( S \xrightarrow{g} S \) is quasi-isomorphic to an ideal quotient \( S/(g) \) which carries a natural algebra structure. One can directly construct DG algebra structure on the bimodule itself:

**Definition 3.1.** Define a DG algebra

\[ \mathcal{B} := S[\epsilon]/\left( \epsilon^2 = 0, \frac{d\epsilon}{\epsilon} = g \right), \]

Here \( \deg \epsilon = -1 \) and the differential \( d \) on \( S \) is set to be zero.
Further assume that $S$ is commutative. Consider an affine variety $X = \text{Spec}(S)$ and a hypersurface $Y = V(g)$ with an inclusion $i : Y \hookrightarrow X$. We have the following elementary lemma whose proof is omitted.

**Lemma 3.2.** We have an isomorphism $S \cong i_* \mathcal{O}_Y$. Moreover, we have the following.

1. A sheaf $\mathcal{F}$ on a hypersurface $Y$ corresponds to an $B$-module object. It is a pair $(i_* \mathcal{F}, h_{\mathcal{F}})$ where $i_* \mathcal{F}$ is a pushforward of $\mathcal{F}$ equipped with a homotopy $h_{\mathcal{F}}$ between the zero map and a multiplication of $g$. It is an action of $\epsilon \in B$.

2. Moreover,

$$\text{Hom}_Y(\mathcal{F}_1, \mathcal{F}_2) \cong \text{Hom}_B((i_* \mathcal{F}_1, h_{\mathcal{F}_1}), (i_* \mathcal{F}_2, h_{\mathcal{F}_2})).$$

For the sheaf $\mathcal{O}_Y$ on $Y$, its pushforward $i_* \mathcal{O}_Y$ has a simple free resolution.

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{g} \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y \longrightarrow 0$$

An action of degree $-1$ element $\epsilon$, or a homotopy $h$, is given as follows.

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\epsilon g} \mathcal{O}_X \longrightarrow 0$$

A category of coherent sheaves on $Y$ is described as $B$-modules of $X$.

**Theorem 3.3.** Let $Y \subset X$ as before. Then

$$\text{DCoh}(Y) \cong B - \text{mod}(\text{DCoh}(X))$$

**Proof.** A concise categorical proof using Lurie’s Barr-Beck theorem can be found in Corollary 3.3.1 in [Pre11]. We present an elementary proof to illustrate the idea. Since everything is affine, it is enough to consider a structure sheaf $\mathcal{O}_Y \in \text{DCoh}(Y)$. Computation shows that the morphism complex is

$$\text{Hom}^{-1}_B((i_* \mathcal{O}_Y, h), (i_* \mathcal{O}_Y, h)) \cong \left\{ \begin{array}{ccc} \mathcal{O}_X & \xrightarrow{g} & \mathcal{O}_X \\ \mathcal{O}_X & \xleftarrow{a_{21}} & \mathcal{O}_X \\ \mathcal{O}_X & \xleftarrow{\epsilon g} & \mathcal{O}_X \end{array} \right\}$$

$$\text{Hom}^0_B((i_* \mathcal{O}_Y, h), (i_* \mathcal{O}_Y, h)) \cong \left\{ \begin{array}{ccc} \mathcal{O}_X & \xrightarrow{g} & \mathcal{O}_X \\ \mathcal{O}_X & \xleftarrow{a_{11}} & \mathcal{O}_X \\ \mathcal{O}_X & \xleftarrow{\epsilon g} & \mathcal{O}_X \end{array} \right\}$$

$$\text{Hom}^1_B((i_* \mathcal{O}_Y, h), (i_* \mathcal{O}_Y, h)) \cong \left\{ \begin{array}{ccc} \mathcal{O}_X & \xrightarrow{g} & \mathcal{O}_X \\ \mathcal{O}_X & \xleftarrow{a_{12}} & \mathcal{O}_X \\ \mathcal{O}_X & \xleftarrow{\epsilon g} & \mathcal{O}_X \end{array} \right\}$$

Therefore, $\text{Hom}^*_B((i_* \mathcal{O}_Y, h), (i_* \mathcal{O}_Y, h))$ is isomorphic to

$$H^*( \text{Hom}_X(\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{g} \text{Hom}_X(\mathcal{O}_X, \mathcal{O}_X) ) \cong \text{Hom}_Y^*(\mathcal{O}_Y, \mathcal{O}_Y).$$

Intuitively, objects of $B - \text{mod}(\text{DCoh}(X))$ are cones

$$\left( \mathcal{F}[1] \xrightarrow{g} \mathcal{F} \right), \quad \mathcal{F} \in \text{DCoh}(X).$$
It is quasi-isomorphic to a quotient $\mathcal{F}/(g)$ and the space of morphisms is a half of the original one. It consists of

$$\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \in \text{Hom}_X^*(\mathcal{F}_1[1] \xrightarrow{g} \mathcal{F}_1, \mathcal{F}_2[1] \xrightarrow{g} \mathcal{F}_2).$$

This is closed under DG operations whereas $A_\infty$-analogue is not (see the paragraph after Corollary 2.5).

3.2. **Restriction to a graph hypersurface in Matrix factorizations.** Let us explain an analogous construction for matrix factorizations. First, let us recall its definition for readers’ convenience.

**Definition 3.4.** For $W \in R$, a DG category of matrix factorization of $W$, denoted by $\text{MF}(W)$ consists of the following:

1. its objects are matrix factorization $(P, \delta_P)$ of $W$. $P$ is $\mathbb{Z}/2$-graded free $R$-module and $\delta_P$ is an odd degree endomorphism such that $\delta_P^2 = W \cdot \text{id}$.
2. $\text{Hom}^•_{\text{MF}(W)}((P, \delta_P), (P', \delta_{P'})) := \{\text{Hom}^•_R(P, P'), d\}$ with usual composition $\circ$. A differential $d$ on morphisms is defined as

$$d(\phi) = \delta_{P'} \circ \phi - (-1)^{\text{deg}(\phi)} \phi \circ \delta_P.$$

We are interested in the following situation. Consider a polynomial of the form $U = U_1(x_1, \ldots, x_{n-1}) + x_n \cdot U_2(x_1, \ldots, x_{n-1})$. We consider a graph of some polynomial $f : A_{n-1} \rightarrow A_n$ and a pull-back $V(x_1, \ldots, x_{n-1}) = U(x_1, \ldots, x_{n-1}, f) = U_1 + f \cdot U_2$ along the graph. We explain how to obtain a similar relation between $\text{MF}(U)$ and $\text{MF}(V)$. We start by collecting functorial properties between two matrix factorization categories, which we refer to [Orl09] and [EP15].

Let $X = \{U = 0\} \subset \mathbb{C}^n$ and $Y = \{V = 0\} \subset \mathbb{C}^{n-1}$. We view $Y$ as a hypersurface $\{x_n = f\} \subset X$. A closed embedding $Y \hookrightarrow X$ is proper and has a finite tor-dimension. A usual adjoint pair of functors $(i^*, i_*)$ extends to categories of singularities.

$$i^* : D^b_{sg}(X) \hookrightarrow D^b_{sg}(Y) : i_*$$

On the other hand, there are Orlov’s equivalences

$$\text{MF}(U) \cong D^b_{sg}(X), \quad \text{MF}(V) \cong D^b_{sg}(Y)$$

Here, $\overline{\mathcal{C}}$ denotes Karoubi completion of a category $\mathcal{C}$. This functor sends

$$M = (M^{\text{odd}} \xrightarrow{\phi_{10}} M^{\text{even}}) \hookrightarrow \text{coker}(\phi_{10})$$

We have an induced pair

$$i^* : \text{MF}(U) \hookrightarrow \text{MF}(V) : i_*$$

**Proposition 3.5.** Let

$$M = (M^{\text{odd}} \xrightarrow{\phi_{10}} M^{\text{even}}) \in \text{MF}(U),$$

$$N = (N^{\text{odd}} \xrightarrow{\psi_{10}} N^{\text{even}}) \in \text{MF}(V).$$

Then
(1) \((i^*, i_*)\) is an adjoint pair.
(2) \(i^* M \cong M|_{x_n = f} \in \text{MF}(V)\).

\[
\begin{array}{ccc}
\rotatebox{90}{\(x_n-f\)} & \rotatebox{90}{\(x_n-f\)} \\
\C[x_1, \ldots, x_n] & \C[x_1, \ldots, x_n] \\
\U_2 & \U_2
\end{array}
\]

(3) \(i_* N \cong N \otimes \begin{array}{ccc}
\C[x_1, \ldots, x_n] & \C[x_1, \ldots, x_n] \\
\U_2 & \U_2
\end{array} \in \text{MF}(U)

(4) \((i_* \circ i^*)M = \text{Cone}((x_n - f) : M[1] \to M) \in \text{MF}(U)\)

Proof. The first proposition is proven in more general setup. See [EP15] Section 2.1. Second proposition follows from the fact that cokernel commutes with tensor product.

\[
\text{coker}(\phi_{10}) \otimes \C[x_1, \ldots, x_n] \cong \text{coker}(\phi_{10}|_{x_n = f})
\]

To prove a third proposition, we should specify Fourier-Mukai kernel of a push-forward functor. Write

\[
V(x_1, \ldots, x_{n-1}) - V(y_1, \ldots, y_{n-1}) = \sum_{i=1}^{n-1} (x_i - y_i) \cdot V_i
\]

Define a Koszul-type matrix factorization \(\Gamma\) of \(V(\bar{x}) - U(\bar{y}) = V(\bar{x}) - (V(\bar{y}) + (y_n - f)U_2(\bar{y}))\) as

\[
\Gamma := \left( \Lambda^* \langle e_1, \ldots, e_n \rangle, \left( \sum_{i=1}^{n-1} (x_i - y_i) i_{e_i} + (y_n - f) i_{e_n} + \sum_{i=1}^{n-1} V_i (\cdot \wedge e_i) + U_2 (\cdot \wedge V) \right) \right).
\]

Under Orlov’s equivalence \(\Gamma\) corresponds to a stabilization of a graph \(\Gamma_{\gamma \rightarrow \chi}\). Therefore a Fourier-Mukai functor associated to \(\Gamma\) is a pushforward functor. Notice that

\[
- \otimes \Gamma \cong - \otimes \Delta_V \otimes \begin{array}{ccc}
\rotatebox{90}{\(x_n-f\)} & \rotatebox{90}{\(x_n-f\)} \\
\C[x_1, \ldots, x_n] & \C[x_1, \ldots, x_n] \\
\U_2 & \U_2
\end{array}
\]

where \(\Delta_V\) is a stabilized diagonal of \(V\). This proves the third proposition.

For the fourth proposition, observe that \(i_* \circ i^* M\) goes to

\[
\text{coker}(\phi_{10}|_{x_n = f} : M^{\text{odd}}|_{x_n = f} \to M^{\text{even}}|_{x_n = f})
\]

under Orlov’s equivalence. It is easy to see that the periodic tail of a following double complex realizes the matrix factorization associated to that module.

\[
\begin{array}{cccccc}
\cdots & \phi_{10} & M^{\text{even}} & \phi_{01} & M^{\text{odd}} & \phi_{10} & M^{\text{even}} \\
\phi_{10} & M^{\text{even}} & \phi_{01} & M^{\text{odd}} & \phi_{10} & M^{\text{even}} \\
\phi_{10} & M^{\text{even}} & \phi_{01} & M^{\text{odd}} & \phi_{10} & M^{\text{even}} \\
\phi_{10} & M^{\text{even}} & \phi_{01} & M^{\text{odd}} & \phi_{10} & M^{\text{even}}
\end{array}
\]

This is equal to \(\text{Cone}((x_n - f) : M[1] \to M)\).

The fourth of Proposition 3.5 means

\[
(i_* \circ i^*)M \cong \left( M[1] \xrightarrow{x_n-f} M \right) = M[e]/\left( e^2 = 0, \quad de = x_n - f \right).
\]

This is a perfect analogy of \(\mathcal{B}\)-module objects we considered in the last subsection.

Definition 3.6. Define a DG category \(\text{MF}(U)|_{x_n-f}\) as follows:

- its objects consist of the matrix factorizations \((i_* \circ i^*)M\) for \(M \in \text{MF}(U)\),
• its morphisms $\text{Hom}_{\text{MF}(U)}((i_* \circ i^{*})\text{M}_1, (i_* \circ i^{*})\text{M}_2)$ consist of
  
  $$a + eb \in \text{Hom}_{\text{MF}(U)}((i_* \circ i^{*})\text{M}_1, (i_* \circ i^{*})\text{M}_2) = \text{Hom}(\text{M}_1[e], \text{M}_2[e])$$

  Differentials and compositions are induced from $\text{MF}(U)$. 

The next proposition explains the relation between MF of the restriction $V = U|_{x_n - f = 0}$ and the DG category $\text{MF}(U)|_{x_n - f}$.

**Corollary 3.7.** Let

$$i_* \circ i^{*} : \text{MF}(U) \to \text{MF}(U)|_{x_n - f}$$

be a natural inclusion and 

$$c : \text{MF}(U)|_{x_n - f} \to \text{MF}(V)$$

be a DG functor sending $(i_* \circ i^{*})\text{M}$ to the matrix factorization $\text{M}|_{x_n - f = 0}$.

1. The following diagram commutes.

   $\begin{array}{ccc}
   \text{MF}(U) & \xrightarrow{i_* \circ i^{*}} & \text{MF}(U)|_{x_n - f} \\
   & \searrow & \downarrow c \\
   & & \text{MF}(V)
   \end{array}$

2. A category $\text{MF}(U)|_{x_n - f}$ fits into a diagram of distinguished triangle of DG bimodules:

   $\begin{array}{ccc}
   \text{MF}(U) & \xrightarrow{\text{MF}(U)|_{x_n - f}} & \text{MF}(U)|_{x_n - f} \\
   & \searrow & \downarrow \text{MF}(V)
   \end{array}$

3. $c$ is fully faithful. It is a quasi-equivalence whenever $i^{*} : \text{MF}(U) \to \text{MF}(V)$ is essentially surjective.

**Proof.** The first and the second proposition follows directly from the definition. For the third one, observe

$$i_* \circ i^{*} \text{M} \simeq \text{M}[e]/\left(\begin{array}{c}
  e^2 = 0 \\
  de = x_n - f
\end{array}\right).$$

Therefore, we have

$$\text{Hom}_{\text{MF}(U)|_{x_n - f}}((i_* \circ i^{*})\text{M}_1, (i_* \circ i^{*})\text{M}_2) \simeq \text{Hom}_{\text{C}[x_1, \ldots, x_n, e]/(de = x_n - f)}(\text{M}_1[e], \text{M}_2[e])$$

$$\simeq \text{Hom}_{\text{MF}(U)}(\text{M}_1, \text{M}_2[e])/\left(\begin{array}{c}
  e^2 = 0 \\
  de = x_n - f
\end{array}\right)$$

$$\simeq \text{Hom}_{\text{MF}(U)}(\text{M}_1, (i_* \circ i^{*})\text{M}_2) \simeq \text{Hom}_{\text{MF}(V)}(i^{*} \text{M}_1, i^{*} \text{M}_2).$$

If $i^{*}$ is essentially surjective, we have $i_*, N \simeq \left(\begin{array}{c}
  M[1] \\
  x_n - f
\end{array}\right) \to \text{M}$ for some $M \in \text{MF}(U)$. This implies $c$ is also essentially surjective. \(\square\)

In our main application, $\text{MF}(U)|_{x_n - f}$ serve as another DG-model representing $\text{MF}(V)$.

4. $A_{\infty}$-Category for a Weighted Homogeneous Polynomial with a Symmetry Group

In this section, we turn our attention to a symplectic geometry of weighted homogeneous polynomial $W$ paired with its group $G_W$ of diagonal symmetries. From the monodromy transformation of $W$ and its flow induced on the boundary of Milnor fiber $M_W$, we define a Reeb orbit $\Gamma_W$ in the quotient orbifold $[M_W/G_W]$. Generalizing the construction of Section 2 to $([M_W/G_W], \Gamma_W)$, we define a new $\mathbb{Z}/2$-graded $A_{\infty}$-category $\mathcal{F}(W, G_W)$ for a pair $(W, G_W)$. 
4.1. Preliminaries. We recall basic facts about weighted homogeneous polynomial and variation operator in classical singularity theory.

Definition 4.1. A polynomial $W$ is called weighted homogeneous if

$$W(\lambda^{w_1}z_1, \ldots, \lambda^{w_n}z_n) = \lambda^hW(z_1, \ldots, z_n)$$

for $w_1, \ldots, w_n, h \in \mathbb{N}$ with $\text{gcd}(w_1, \ldots, w_n, h) = 1$. We say $W$ has weight $(w_1, \ldots, w_n; h)$.

Assume that $W$ has an isolated singularity at the origin. Set $V_t = V_t(W) = \{z \in \mathbb{C}^n \mid W(z) = t\}$. $V_0$ is an hypersurface of isolated singularity at 0 and $V_t$ ($t \neq 0$) is non-singular. Milnor fiber $M_W$ is defined to be $V_1(W)$. For the well-known Milnor fibration

$$\frac{W}{|W|} : S^{2n-1}_C \setminus K \to S^1$$

with $K = (S^{2n-1}_C \cap V_0)$, its fiber is diffeomorphic to $M_W$. Geometric monodromy $h : C^n \to C^n$ is defined by

$$h(x_1, \ldots, x_m) = (e^{2\pi i \lambda_1/h}x_1, \ldots, e^{2\pi i \lambda_n/h}x_m)$$

which restricts to $h : M_W \to M_W$. It is known that $S^{2n-1}_C \setminus K$ is diffeomorphic to the manifold obtained by identifying two ends of $M_W \times [0,1]$ by $h$ (see [Mil68] Lemma 9.4).

It is another famous theorem of Milnor that the homotopy type of $M_W$ is a bouquet of $(n-1)$ spheres. One may define the notion of vanishing skeleton, but it is usually very singular (for Brieskorn-Pham singularities, it is called Pham’s spine). If we perturb $W$ to a complex Morse function $W'$, then we may obtain vanishing spheres in the Milnor fiber of $W'$, and Fukaya-Seidel directed $A_\infty$-category of $W$ is defined on these objects. But since $G_W$-symmetry is broken after the perturbation, it is not known how to work equivariantly within Fukaya-Seidel category framework.

On the other hand, it is well-known that non-compact Lagrangians are related to vanishing cycles via the variation operator. Consider monodromy homomorphism (from a parallel transport fixing the boundary)

$$h_* : H_*(\overline{M}_W) \to H_*(\overline{M}_W), \quad h_* : H_*(\overline{M}_W, \partial \overline{M}_W) \to H_*(\overline{M}_W, \partial \overline{M}_W).$$

**Definition 4.2.** A variation operator (around the origin in $\mathbb{C}$)

$$\text{var} : H_{n-1}(\overline{M}_W, \partial \overline{M}_W) \to H_*(\overline{M}_W).$$

is defined by sending $[c] \mapsto (h_* - id)([c])$.

This map is known to be an isomorphism. We want to find a symplectic categorical analogue of this variation operator for weighted homogenous polynomials. At first, we will define a distinguished Reeb orbit $\Gamma_W$ from the geometric monodromy (4.1). The analogue of monodromy homomorphism (4.2) will be the quantum cap action by $\Gamma_W$.

$$\cap \Gamma_W : \mathcal{W}(L, L) \to \mathcal{W}(L, L).$$

Then, the analogue of the variation operator (4.3) will be the new Fukaya category $\mathcal{C}_\Gamma$, with a distinguished triangle of $A_\infty$-bimodules

$$\mathcal{W}(M) \xrightarrow{\cap \Gamma_W} \mathcal{W}(M) \to \mathcal{C}_\Gamma$$

Let us explain what it means. Recall that in Floer theory, it is well-known that taking a Lagrangian surgery corresponds to taking a cone complex. Roughly speaking, we are taking a surgery of non-compact Lagrangians for the Reeb chords at infinity to turn it into a compact object, namely the corresponding vanishing cycle. But its morphism space is not the same as that of a cone, and is more subtle as explained in Section[2]
4.2. Diagonal symmetry group of weighted homogeneous polynomials. We collect some facts about weighted homogeneous polynomials and their diagonal symmetry group from [FJR13] and a reference therein.

Let $W$ be a weighted homogeneous polynomial of weight $(w_1, \ldots, w_n; h)$.

**Definition 4.3.** The maximal diagonal symmetry group $G_W$ of $W$ is defined as

$$G_W := \left\{ \lambda \in (\mathbb{C}^*)^n \mid W(\lambda_1 \cdot x_1, \ldots, \lambda_n \cdot x_n) = W(x_1, \ldots, x_n) \right\}.$$ 

Note that geometric monodromy (4.1) is an element of $G_W$. A class of polynomials that we are interested in are those with finite diagonal symmetries.

**Definition 4.4 ([FJR13] Definition 2.1.5).** A weighted homogeneous polynomial $W$ is called non-degenerate if

1. $W$ contains no monomial of the form $x_i x_j$ for some $i \neq j$;
2. $W$ is an isolated singularity at the origin.

If $W$ is non-degenerate, the number of monomials of $W$ must be greater than or equal to the number of variables. The first finiteness condition that we need is the following.

**Proposition 4.5 ([HK12], [FJR13]).** Let $W$ be a non-degenerate weighted homogeneous polynomial. Then

1. The weight $(w_1, \ldots, w_n; h)$ of $W$ is bounded by $\frac{w_i h}{n} \leq \frac{1}{2}$ and it is unique.
2. The maximal diagonal symmetry group $G_W$ is a finite abelian group.

For a subgroup $K \subset G_W$, let $C^{N_K} := (\mathbb{C}^n)^K$ be the set of fixed points of $K$. Notice that they are always a coordinate plane. Let $W_K$ be a restriction of $W$ on $C^{N_K}$. The second finiteness condition that we will use is the following.

**Proposition 4.6 ([FJR13] Lemma 2.10).** Let $W$ be a non-degenerate weighted homogeneous polynomial. Then

1. $W_K$ is a non-degenerate weighted homogeneous polynomial on $C^{N_K}$ for $\forall K < G_W$.
2. $G_{W_K}$ is canonically embedded inside $G_W$.

An orbifold strata $\{\text{Fix}(K)\}_{K < G_W}$ on $C^n$ is compatible with restrictions. This will enable us to do inductive arguments on the number of variables later on. A pair $(W, G)$ of non-degenerate homogeneous polynomial $W$ and a subgroup $G < G_W$ of diagonal symmetries is called an orbifold Landau-Ginzburg model.

4.3. Orbifold wrapped Fukaya category. The starting point of our construction is an orbifold wrapped Fukaya category of $[M_W / G_W]$. We follow [Sei13] closely.

We consider the collection $\mathcal{W}$ of Lagrangians $L \in [M_W / G_W]$ with the following properties.

- It is given by the family of embedded Lagrangians $\{L_i\}_{i \in I}$ and an action of $G_W$ on the index set $I$ such that $g(L_i) = L_g i$.
- $L$ lies away from the singular locus.
- It is either compact or conical at the end.
- It is equipped with an orientation and a spin structure (compatible with $G_W$-action).
- $G_W$-orbits intersect transversally to each other without triple intersections.

**Definition 4.7.** A $\mathbb{Z}/2$-graded orbifold wrapped Fukaya category $\mathcal{W} \mathcal{F} ([M_W / G_W])$ consists of:

1. a set of objects $\mathcal{W}$;
(2) For two such Lagrangians $L_0, L_1 \in \mathcal{W}$, its morphism space is defined by

$$CW^*(L_0, L_1) := \bigoplus_{g, h \in G_W} CW^* \left( g \cdot \tilde{L}_0, h \cdot \tilde{L}_1 \right)$$

where $\tilde{L}_i$ is a lift of $L_i$ and we take $G_W$-invariant part.

(3) $A_\infty$-operations are induced from the $m_k$-operation of $\mathcal{WF}(M_W)$.

Following Seidel [Sei15], we can make the $A_\infty$-structure $G_W$-equivariant. We omit the details and refer readers to [Sei15]. $\mathcal{WF}([M_W/G_W])$ is only $\mathbb{Z}/2$-graded because the action of $G_W$ may not be special linear. The orbifold canonical bundle is not trivial and we cannot put a $\mathbb{Z}$-grading on $\mathcal{WF}([M_W/G_W])$.

A holomorphic disc $u : S \to M_W$ defining $A_\infty$-structure can be considered as a smooth holomorphic disc $\tilde{u} : S \to [M_W/G_W]$ ramified at orbifold points accordingly. Conversely, because $S$ has a vanishing orbifold fundamental group, all smooth holomorphic discs lifts to $M_W$ in a $G_W$-equivariant manner. Therefore, Fukaya category of $[M_W/G_W]$ is the same as $G_W$-equivariant Fukaya category of $M_W$.

4.4. Monodromy flow, orbits and $\mathcal{F}(W, G_W)$. Fix a non-degenerate weighted homogeneous polynomial $W$ of weight $(w_1, \ldots, w_n; h)$ once and for all. Choose a slightly rescaled symplectic form $\omega$ and a Liouville form $\lambda$ on $\mathbb{C}^n$ (see [KvK16] for the relation to the standard one) as

$$\omega = \sum_k \frac{1}{2\pi i w_k} dz_k \wedge d\bar{z}_k, \quad \lambda = \sum_k \frac{i}{4\pi w_k} (z_k d\bar{z}_k - \bar{z}_k dz_k).$$

**Definition 4.8.** The monodromy flow is the Hamiltonian flow

$$\Phi_W(s)(x_1, \ldots, x_n) := (e^{\frac{2\pi i w_1}{h} x_1}, \ldots, e^{\frac{2\pi i w_n}{h} x_n})$$

of a quadratic Hamiltonian $H := \frac{1}{2} \sum_{i=1}^n |x_i|^2$. The monodromy transformation $\Phi_W = \Phi_W(1)$ is a time-1 flow

$$x_i \mapsto e^{\frac{2\pi i w_i}{h}} x_i.$$

Geometrically, a Hamiltonian action of $H$ is a lifting of a rotation action on the base of a fibration $W : \mathbb{C}^n \to \mathbb{C}$. More precisely, we have

$$W(e^{\frac{2\pi i w_1}{h} x_1}, \ldots, e^{\frac{2\pi i w_n}{h} x_n}) = e^{2\pi i s}$$

which means that the flow of $H$ acts as a circle action of an $S^1$ family of Milnor fiber $W = e^{2\pi i s}$. Set $s = 1$ then we get a desired automorphism.

The monodromy flow restricts to a singular fiber $W^{-1}(0)$ and a link of $W$

$$L_{W, \delta} := W^{-1}(0) \cap S^{2n-1}_\delta$$

for small $\delta > 0$. The monodromy flow $\Phi_W(s)$ becomes a Reeb flow $R$ on $L_{W, \delta}$, where the contact one form is given by a restriction of $\lambda$. Starting from any point $x \in L_{W, \delta}$, we get a Reeb chords

$$\gamma : [0, 1] \to L_{W, \delta}, \quad \gamma(0) = x, \quad \gamma(1) = \Phi_W(x)$$

Notice that $\Phi_W$ always gives an element $g_W$ of a maximal symmetry group $G_W$. Reeb chords become orbits of a quotient $(L_{W, \delta}/G_W)$. A space of time-1 Reeb orbits of the quotient is a total space $L_{W, \delta}/G_W$.

Although we have not defined full-fledged orbifold symplectic cochains in general, the ideas of [CFHW96] and [KvK16] still work. We start with the following lemma.

**Lemma 4.9.** There is a Morse function $h$ on $L_{W, \delta}/G_W$ such that

- if a critical point $p \in \text{crit}(h)$ lies in $\text{Fix}(K)$ for some $K < G_W$, then the unstable manifold $W^-(p)$ of $p$ is also contained in $\text{Fix}(K)$. 

its Morse-Witten complex is well-defined and computes the cohomology of \((L_{W,\delta}/G_W)\).

**Proof.** Since orbifold strata \(\{F(x)\}_{x < G_W}\) of \(\mathbb{C}^n\) intersect transversally to the link, it induces orbifold strata for \(L_{W,\delta}\). We construct a Morse function \(h'\) in an inductive way. The action of a diagonal symmetry group is effective on the link. Therefore the deepest strata does not have a nontrivial fixed locus. Choose a Morse function on it and extend it strata by strata in a way that it depends on the radial direction of the normal bundle of the strata inside the next one. We may choose our extension so that the negative Morse flow line of our extension flows into the lower strata. This is possible because our orbifold strata are induced from coordinate planes. When several stratum intersect along a lower dimensional one, their normal directions are compatible.

Morse flow of \(h'\) respects orbifold strata. If \(p \in \text{crit}(h') \cap \text{Fix}(K)\), then its unstable manifold \(W^-(p)\) is contained in \(\text{Fix}(K)\). If \(q \in \text{crit}(h') \cap \text{Fix}(K')\) for some \(K \leq K'\), then its stable manifold \(W^+(q)\) intersect \(W^-(p)\) at smooth points of \(\text{Fix}(K)\) only. Therefore, the moduli space of gradient flow trajectories can be defined as in the smooth case. They intersect transversally after a small perturbation of \(h'\) in the smooth part of \(\text{Fix}(K)\). We do it strata by strata to obtain \(h\).

Because of this first property, Morse-Witten complex of \(h\) makes sense. i.e, its differential does square to zero. Moreover, the unstable manifolds of \(h\) provides a cell decomposition of \((L_{W,\delta}/G_W)\). Hence Morse-Witten complex computes the singular cohomology of \((L_{W,\delta}/G_W)\) (for a similar argument for \(G = S^1\) in the context of contact homology, see [Bou03]).

The link \(L_{W,\delta}\) can be symplectically identified with \(W^{-1}(1) \cap S^2_{\delta} - 1\) in the following way (see [Sei00]). Choose a cutoff function \(\psi\) with \(\psi(t^2) = 1\) for \(t^2 \leq \frac{\delta}{3}\) and \(\psi(t^2) = 0\) for \(t^2 \geq \frac{2\delta}{3}\). Define

\[
F := \{ x \in B^{2n}_{\delta} | W(x) = \psi(|x|^2) \}.
\]

It was shown in [Sei00] that \(F\) is symplectic manifold with \(\partial F = L_{W,\delta}\). Moreover \(F\) is diffeomorphic to \(W^{-1}(1) \cap B^{2n}\) by a smooth cobordism

\[
G_{\delta} := \{ x \in B^{2n}_{\delta} | W(x) = s\psi(|x|^2) + (1 - s) \}.
\]

Shrinking \(\delta\) if it is necessary, we see that \(F\) and \(M_W\) contain \(W^{-1}(1) \cap B^{2n}_{\delta/3}\) as a Liouville submanifold and Liouville isotopy compresses both \(F\) and \(M_W\) to \(\text{Int}\left(W^{-1}(1) \cap B^{2n}_{\delta/3}\right)\). Since \(\psi\) only depends on \(|x|^2\), the whole construction is compatible with the action of \(G_W\). Therefore \(L_{W,\delta}/G_W\) serves as a model for the contact type boundary of \(M_W/G_W\).

In this setup, Reeb flow \(\mathcal{R}\) of \(L_{W,\delta}/G_W\) can be viewed as a Hamiltonian flow \(H \in \mathcal{H}(M_W/G_W)\). It is Morse-Bott in a sense that a critical set of action functional is a total space \(L_{W,\delta}/G_W\) rather than isolated points. We adapt the idea of [CFHW96] and [KvK10]. Choose a small normal neighborhood \(v(L_{W,\delta}/G_W) = L_{W,\delta}/G_W \times (-\epsilon, \epsilon)\) inside \(M_W/G_W\). Choose a sufficiently small, nonnegative bump function \(\rho\) supported inside \((-\epsilon, \epsilon)\). Use our Morse function \(h\) in [4.9] to define an equivariant time-dependent perturbation term by

\[
\overline{h} : v(L_{W,\delta}/G_W) \times S^1 \to \mathbb{R},
\]

\[
(p, v, t) \mapsto h(F^t_{\mathcal{R}}(p))\rho(v).
\]

**Definition 4.10.** A local Floer cochain complex

\[
CF_{\text{loc}}^*(L_{W,\delta}/G_W, H_S) = \bigoplus_{\gamma \in C^1(H_S)} \mathcal{O}_\gamma
\]

is defined to be a free module generated by Hamiltonian orbits of time-dependent Hamiltonian \(H_S := H + \overline{h}\). Its differential is defined similarly as a counting of pseudo-holomorphic cylinders, but only those inside \(v(L_{W,\delta}/G_W)\). Its cohomology, called local Floer cohomology, is denoted by \(HF_{\text{loc}}^*(L_{W,\delta}/G_W, H_S)\).
Theorem 4.11. As a $\mathbb{Z}/2$-graded vector space,
\[ HF_{\text{loc}}^r(L_{W,\delta}/G_W, H_{S!}) \cong H_{\text{Morse}}^r(L_{W,\delta}/G_W, h; \mathbb{Z}) \]

**Proof.** See [KyK16], Proposition 8.4. One can check that $L_{W,\delta}/G_W$ satisfies all the conditions in the proposition except the first Chern class condition, which makes the isomorphism only $\mathbb{Z}/2$-graded. We can also check that flows and Hamiltonians in the proof are equivariant. \hfill \Box

**Definition 4.12.** The Hamiltonian orbit $\Gamma_W \in CF_{\text{loc}}^r(L_{W,\delta}/G_W, H)$ is defined to be a cocycle which corresponds to a fundamental class
\[ \Gamma_W \leftrightarrow [L_{W,\delta}/G_W] \in H^r(L_{W,\delta}/G_W; \mathbb{Z}) . \]

It is still possible to check that $\Gamma_W$ is "closed" in a suitable sense.

**Lemma 4.13.** Suppose $H_{S!}$ is $G$-equivariant, $H_{S!} > 0$, and $C^2$-small Morse perturbation of $H$ inside a compact region. Then there is no pseudo-holomorphic cylinder of finite energy satisfying
\[ u : S^1 \times \mathbb{R} \to [M_W/G_W] \quad \lim_{s \to -\infty} u(t, s) = \Gamma_W(t) . \]

whose output
\[ \gamma_-(t) := \lim_{s \to -\infty} u(s, t) \]
does not lie in $[L_{W,\delta}/G_W] \times \{1\}$.

**Proof.** At first, we can rule out the case when the output is on outside of a compact region using the idea of [Sei06] using action values: for a non-trivial orbit $\gamma \in \mathcal{O}(H_{S!})$ at the end, its action is given by
\[ A_{H_{S!}}(\gamma) := -\int_{S^1} \gamma^* \lambda + \int_0^1 H_{S!}(\gamma(t)) dt \]
\[ = -2 \int_0^1 r^2 dt + \int_0^1 H(\gamma(t)) dt + \int_0^1 F(\gamma(t)) dt , \quad (H_{S!} = H(r) + F(r, t)) \]
\[ = -\int_0^1 r^2 + \varepsilon \quad (\varepsilon << 1) \]

Nontrivial Hamiltonian orbits are appears as a small perturbation of orbits of level $n$, which means that there is a perturbation of Hamiltonian orbits $\gamma' \subset (L_{W,\delta}/G_W) \times \{n\}$ of $H$. An action value of such orbit is dominated by $-n^2$. The orbit $\Gamma_W(t)$ is an orbit of level 1.

Since $H_{S!} > 0$, a topological energy of $u$
\[ E_{\text{top}}(u) := \int_{S^1 \times \mathbb{R}} \omega - d(u^* H_{S!} \cdot dt) = A_{H_{S!}}(\gamma_-) - A_{H_{S!}}(\Gamma_W) \]

must be positive. Therefore, the output $\gamma_-$ cannot be an orbit of level $n \geq 2$.

Suppose $\gamma_-$ is an orbit in a compact region, a Morse critical point of $H$. Since $u$ provides a homotopy between orbifold loops, we must have $\Phi(\gamma_-(0)) = \gamma_-(1)$. It means that $\gamma_-$ must be a fixed point of a monodromy $\Gamma$. It is impossible because the only fixed point of $\Gamma$ is the origin, which is not contained in $M_W$. \hfill \Box

Now, we are ready to define the new $A_\infty$-category for a Landau-Ginzburg orbifold $(W, G_W)$ applying Theorem 2.1

**Theorem 4.14.** For a weighted homogeneous polynomial $W$ with the Hamiltonian orbit $\Gamma_W$ of $[M_W/G_W]$, we define a new $A_\infty$-category $\mathcal{C}_{\Gamma_W}$ such that

1. $\mathcal{C}_{\Gamma_W}$ has the same set of objects as the wrapped Fukaya category $\mathcal{W} \mathcal{F}([M_W/G_W])$,
2. for two objects $L_1, L_2$, its morphisms are given by

$$\text{Hom}_{\mathcal{C}_W}(L_1, L_2) = CW(L_1, L_2) \oplus CW(L_1, L_2)\psi$$

where $CW(L_1, L_2)$ is the morphism space for $\mathcal{W}(\mathcal{M}_W/G_W)$ and $\text{deg} \psi = -1$,

3. a natural inclusion $\Psi: \mathcal{W}(\mathcal{M}_W/G_W) \to \mathcal{C}_W$ is an $A_\infty$-functor,

4. regarding the $A_\infty$-category $\mathcal{C}_W$ as an $A_\infty$-bimodule over $\mathcal{W}(\mathcal{M}_W/G_W)$ (using $\Psi$), we have a distinguished triangle of $A_\infty$-bimodules

$$\mathcal{W}(\mathcal{M}_W/G_W)_\Delta \xrightarrow{\cap \Gamma_W} \mathcal{W}(\mathcal{M}_W/G_W)_\Delta \longrightarrow \mathcal{C}_W \longrightarrow$$

Definition 4.15. A $Z/2$-graded Fukaya category of a Landau-Ginzburg orbifold $(W, G_W)$ is defined to be

$$\mathcal{F}(W, G_W) := \mathcal{C}_W.$$

There are few things left to check to apply the construction of $\mathcal{C}_W$ for $W = \mathcal{W}(\mathcal{M}_W/G_W)$ and $\Gamma = \Gamma_W$. At first, one can use $\Gamma_W$ as if it is a symplectic cohomology class. We prove that all possible cylinder breakings of pseudo-holomorphic curves involving $\Gamma_W$ are appear as cancelling pairs at the level of local Floer cohomology in Lemma 4.13. Also notice that an orbifold nodal degeneration does not affect a standard analysis of codimension 1 boundary strata of moduli spaces because they are of codimension two. It follows from the fact that a local model of such degeneration is given by a family $z_1z_2: [C^2/(Z/n)] \to C$. Here, a group $Z/n$ acts on $C^2$ by $(z_1, z_2) \to (\xi \cdot z_1, \xi^{-1} \cdot z_2)$ with $\xi$ an $n$-th root of unity. Finally, the degree of $\Gamma_W$ is always even.

Therefore operations such as a closed-open map $\text{CO}(\Gamma_W)$, quantum cap action $\cap \Gamma_W$ and popsicle operations $\mathcal{P}_{n, F_G}$ on $\mathcal{W}(\mathcal{M}_W/G_W)$ make sense. The $A_\infty$-operation $\{M_k\}$ we introduced earlier is still well-defined in a $Z/2$-graded sense.

4.5. Fukaya category for a subgroup $G < G_W$. Let us explain how to define a Fukaya category $\mathcal{F}(W, G)$ for a subgroup $G < G_W$ using the dual group $G_W^*$ action on $\mathcal{F}(W, G_W)$.

Let us first explain the algebraic setting. Let $\mathcal{C}$ be an $A_\infty$-category with an action by finite abelian group $H$. By taking a $H$-invariant part, we may obtain an $A_\infty$-category $\mathcal{C}^H$ whose object is a $H$-family of objects $\{h \cdot K\}_{h \in H}$, denoted as $K$, and its morphisms between $K$ and $K'$ are given by

$$\bigoplus_{h \in H} \text{Hom}_\mathcal{C}(K, h'K') \cong \left( \bigoplus_{h_1, h_2 \in H} \text{Hom}_\mathcal{C}(h_1K, h_2K') \right)^H.$$  

Then the character group $H^* = \text{Hom}(H, C^*)$ naturally acts on the $A_\infty$-category $\mathcal{C}^H$: we define $\chi \in H^*$ action for an element $X \in \text{Hom}_\mathcal{C}(K, h'K')$ (resp. its dual formal variable $x$) as

$$\chi \cdot X = \chi(h^{-1})X \quad (\text{resp. } \chi \cdot x = \chi(h)x).$$

In [Sei15], Seidel observed that one can recover $\mathcal{C}$ from $\mathcal{C}^H$ using semi-direct product. By taking a tensor product of $\text{Hom}_\mathcal{C}(K, K')$ with the group ring $C[H^*]$, and extending the $A_\infty$-operation suitably, we obtain a semi-direct product $A_\infty$-category $\mathcal{C}^H \rtimes H^*$, which can be canonically identified with $\mathcal{C}$. More precisely, the component $\text{Hom}_{\mathcal{C}^H}(K, K') \otimes \chi$ in the morphism space can be identified with the $\chi$-eigenspace of $H$-action on $\mathcal{C}$ (see [CL20] for an explicit identification). Geometrically, it means that $A_\infty$-operations in $\mathcal{C}^H$ actually come from $\mathcal{C}$. For the case of Fukaya category, this corresponds to the fact that a smooth holomorphic discs in the quotient can be always lifted.

Let us discuss the case of the $A_\infty$-category $\mathcal{F}(W, G_W)$. For a $J$-holomorphic disc without $\Gamma_W$-insertion,

$$m_k(\chi \cdot w_1, \ldots, \chi \cdot w_k) = \chi \cdot m_k(w_1, \ldots, w_k)$$

holds because disc has a lift to upstairs and still is a disc. When we walk along the boundary Lagrangians, we come back to the original branch that we start with.
We have additional operations of popsicles which have $\Gamma_W$ interior insertions. These $J$-holomorphic discs do not have lifts to upstairs, so we may proceed as follows. Denote by $g_W \in G_W$ be the group element for the monodromy of $W$ in (4.1). For each sprinkle with $\Gamma_W$ insertion, we can make a branch cut in the domain $D^2$ to the boundary arc between two marked points $z_0, z_1$. We can make the cuts disjoint from each other. Then, the resulting disc lifts to the cover, but if we restrict to the arc $z_0 z_1$, it has discontinuous lift. Namely, $z_0 z_1$ boundary arc has as many $g_W^{-1}$ jumps as the number of sprinkles. This can be used to show that

$$m_{k,F}(\chi \cdot w_1, \ldots, \chi \cdot w_k) = \chi(g_W^{[F]} m_{k,F}(w_1, \ldots, w_k)$$

To accommodate this new factor, we introduce $G_W^*$-action on the formal parameter $e$ in the Definition 2.13. It is not hard to check that this gives the desired $G_W^*$-action on $\mathcal{C}_{\Gamma_W}$.

**Proposition 4.16.** We set $\chi \cdot e = \chi(g_W e)$. Then $A_{\infty}$-category $\mathcal{C}_{\Gamma_W}$ admits $G_W^*$-action.

For any subgroup $G \subset G_W$, we define its dual group $G^T = \text{Hom}(G_W / G, \mathbb{C}^*)$ following Berglund-Henningson [BH95]. We are ready to define a Fukaya category of the Landau-Ginzburg orbifold $(W, G)$.

**Definition 4.17.** We define a Fukaya $A_{\infty}$-category for $(W, G)$ by

$$\mathcal{F}(W, G) := \mathcal{C}_{\Gamma_W} \rtimes G^T.$$  

Here, the quotient map $G_W \to G_W / G$ induces the map $G^T \to G_W^*$ and hence a $G^T$-action on $\mathcal{C}_{\Gamma_W}$.

5. **Invertible singularities and Berglund–Hübsch conjecture**

With the definition of the Fukaya category $\mathcal{F}(W, G)$, we can formulate full homological mirror symmetry conjecture for Berglund-Hübsch pairs. In this section, we state these conjectures and our main theorems for invertible curve singularities, and explain briefly the strategy of its proof.

Recall that invertible polynomial is given by

$$W(x_1, \ldots, x_n) = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ij}}$$

where the matrix of exponents $A = (a_{ij})$ is an $n \times n$ invertible matrix. Kreuzer-Sharke [KS92] classified invertible singularities and showed that they are given by a Thom-Sebastiani sum of following three atomic polynomials:

- **Fermat type** $W = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ ($a_i \geq 2$),
- **Chain type** $W = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_n^{a_n-1} x_n + x_n^{a_n}$ ($a_i \geq 2$),
- **Loop type** $W = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_n^{a_n-1} x_n + x_n^{a_n} x_1$ ($a_i \geq 2$).

Thus we may assume that an invertible polynomial $W$ is weighted homogeneous, and non-degenerate. The maximal symmetry group $G_W$ has a simple description. Let $A$ be the matrix of exponents and $A^{-1} = (w_{ij})$ be its inverse. Then $G_W$ contains

$$g_j := \left( e^{2\pi i w_{1j}}, e^{2\pi i w_{2j}}, \ldots, e^{2\pi i w_{nj}} \right)$$

whose weights are given by column vectors of $A^{-1}$. The set $\{g_j\}$ is possibly a non-minimal generators of $G_W$. Moreover, $|G_W|$ is equal to det($A$) (see [FU11] and reference therein).

For a given pair $(W, G)$ of invertible polynomial $W$ and a subgroup $G \subset G_W$, define its Berglund–Hübsch dual pair as $(W^T, G^T)$ where $W^T$ is an invertible polynomial whose exponent matrix is $A^T$ and $G^T$ is the dual group of $G$. Berglund-Hübsch HMS conjecture can be formulated using $\mathcal{F}(W, G)$ as follows.
Conjecture 5.1 (Berglund–Hübsch HMS conjecture). There is a derived equivalence of \( A_{\infty} \)-categories
\[
D^T \mathcal{F}(W, G) \cong \mathcal{H} \mathcal{M} \mathcal{F}(W^T, G^T),
\]
where the latter is a homotopy category of \( G^T \)-equivariant matrix factorizations of \( W^T \).

Here \( \mathcal{M} \mathcal{F}(W) \) is an \( A_{\infty} \)-category version of DG category \( \text{MF}(W) \).

Definition 5.2. An \( A_{\infty} \)-category \( \mathcal{M} \mathcal{F}(W) \) is defined as follows. An object of \( \mathcal{M} \mathcal{F}(W) \) is a matrix factorization of \( W \). For two matrix factorizations \( P \) and \( P' \), its morphism space is
\[
\text{Hom}_{\mathcal{M} \mathcal{F}(W)}(P, P') = \text{Hom}_{\text{MF}(W)}(P', P)
\]
with \( A_{\infty} \)-operations
\[
m_1 := d, \ m_2(\phi, \psi) := (-1)^{\text{deg}(\phi)} \phi \circ \psi, \ m_{\geq 3} = 0.
\]

From now on, we restrict to the cases of two variables, called invertible curve singularities. Note that invertible curve singularities can be classified into the following three different classes:

- (Fermat) \( F_{p,q} = x^p + y^q \)
- (Chain) \( C_{p,q} = x^p + xy^q \)
- (Loop) \( L_{p,q} = x^p y + xy^q \)

Therefore, we may assume that \( W \) is one of the above three types. Our first main theorem is a proof of Berglund–Hübsch HMS conjecture for invertible curve singularities for the maximal diagonal symmetry group \( G_W \).

Theorem 5.3. Let \( W \) be an invertible polynomial of two variables. Then there is an \( A_{\infty} \)-functor
\[
\mathcal{G}^L : \mathcal{F}(W, G_W) \to \mathcal{M} \mathcal{F}(W^T + xy g)
\]
whose cohomology functor induces a derived equivalence
\[
H(\mathcal{G}^L) : D^T \mathcal{F}(W, G_W) \to \mathcal{H} \mathcal{M} \mathcal{F}(W^T).
\]

From this theorem, we can deduce the full version of HMS between Berglund–Hübsch pairs.

Corollary 5.4 (Berglund–Hübsch HMS). For any subgroup \( G < G_W \), we have derived equivalences of \( \mathbb{Z}/2 \)-graded categories
\[
D^T \mathcal{F}(W, G) \cong \mathcal{H} \mathcal{M} \mathcal{F}(W^T, G^T)
\]
A proof of Theorem 5.3 occupies next four sections. We first investigate the equivariant topology of Milnor fibers \( M_W \) in Section 6 and find that the quotient orbifold \( [M_W / G_W] \) is an orbifold partial compactification of a pair of pants. More precisely, \( [M_W / G_W] \) has one, two, three punctures (and hence two, one, zero orbifold points) for Fermat, Chain, Loop cases respectively. We further investigate the monodromies of branch points to determine the equivariant tessellation of the Milnor fiber explicitly. In Section 7 we recall Floer theory for such orbifold spheres from [Sei11] and localized mirror functor from [CHL17]. In Section 8 we prove the following HMS for Milnor fibers by showing that split-generators of the wrapped Fukaya category are mapped to split-generators of matrix factorization categories under the localized mirror functor \( \mathcal{F}^L \) (Theorem 6.1).

\[
\mathcal{F}^L : \mathcal{W} \mathcal{F}([M_W / G_W]) \longrightarrow \mathcal{M} \mathcal{F}(W_L),
\]
where the disc potential \( W_L \) can be written as \( W_L = W^T + xy \cdot g \). Here the polynomial \( g \) is given by
\[
g(x, y, z) = \begin{cases} 
  z & \text{for Fermat type} \\
  z - x^{p-1} & \text{for Chain type} \\
  z - x^{p-1} - y^{q-1} & \text{for Loop type}
\end{cases}
\]
In Section 9 we compute the class \( \Gamma_W \) explicitly and show that its Kodaira-Spencer (closed-open) map image is exactly \( g(x, y, z) \). Moreover, we show that quantum cap action by \( \Gamma_W \) and multiplication by \( g(x, y, z) \) are homotopic under the mirror functor \( \mathcal{F}^L \).
Theorem 5.5. There is a homotopy commuting diagram of $A_\infty$-bimodules:

\[
\begin{array}{c}
\mathcal{W} \mathcal{F}([M_W/G_W]) \\
\downarrow \quad \mathcal{W} \mathcal{F}([M_W/G_W]) \\
\mathcal{M} \mathcal{F}(W^T + xyg) \\
\downarrow \quad \mathcal{M} \mathcal{F}(W^T + xyg) \\
\mathcal{M} \mathcal{F}(W^T + xyg)|_g
\end{array}
\]

All vertical lines are quasi-isomorphisms. Furthermore, \( \mathcal{M} \mathcal{F}(W^T + xyg)|_g \simeq \mathcal{M} \mathcal{F}(W^T) \)

As a final step, we obtain the derived equivalence of two $A_\infty$-categories \( \mathcal{F}(W, G_W) \) and \( \mathcal{M} \mathcal{F}(W^T) \) as follows. It turns out there is a complication due to a fact that the natural geometric $A_\infty$-functor from \( \mathcal{F}(W, G_W) \) does not land directly on \( \mathcal{M} \mathcal{F}(W^T) \) nor \( \mathcal{M} \mathcal{F}(W^T + xyg)|_g \) but only up to some homotopies.

We overcome this difficulty in the following way. We can regard \( (\mathcal{L}, \mathcal{b}) \) as an element of \( \mathcal{F}(W, G_W) \), and consider a localized mirror functor with respective to \( (\mathcal{L}, \mathcal{b}) \):

\[ \mathcal{G}^L : \mathcal{F}(W, G_W) \rightarrow \mathcal{M} \mathcal{F}(W^T + xyg). \]

We show that the images of the functor of objects and morphisms do lie on \( \mathcal{M} \mathcal{F}(W^T + xyg)|_g \) up to explicit homotopies. This provides the desired derived equivalence

\[ H(\mathcal{G}^L) : D^Z \mathcal{F}(W, G_W) \rightarrow \mathcal{H} \mathcal{M} \mathcal{F}(W^T + xyg)|_g \simeq \mathcal{H} \mathcal{M} \mathcal{F}(W^T). \]

As mentioned in the introduction, a version of Berglund-Hübsch conjecture for \( G = \{ e \} \subset G_W \) can be formulated using Fukaya-Seidel category (which have been extensively studied). We recall its precise form and explain conjectural relation to our approach, and we plan to discuss it in the sequel. Define

\[ K_W := \{ (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n \mid W(\lambda_1 \cdot x_1, \ldots, \lambda_n \cdot x_n) = c \cdot W \text{ for some } c \in \mathbb{C}^* \} \]

and \( L_W \) be its character group \( \text{Hom}(K_W, \mathbb{C}^*) \). There is an exact sequence

\[ 1 \rightarrow G_W \rightarrow K_W \rightarrow \mathbb{C}^* \rightarrow 1 \quad \text{and dually, } \ 1 \rightarrow Z \rightarrow L_W \rightarrow G_W^T \rightarrow 1. \]

It has been conjectured that

\[ \overline{FS}^Z(W) \simeq MF(W^T, L_W), \]

and confirmed in many cases. Therefore it is natural to make the following conjecture.

Conjecture 5.6. There is an appropriate $\mathbb{Z}$-grading on \( \mathcal{F}(W, \{ e \}) \) so that there is an $A_\infty$-equivalence

\[ D^Z \overline{FS}^Z(W) \simeq D^Z \mathcal{F}^Z(W, \{ e \}). \]

6. Equivariant topology of Milnor fibers for invertible curve singularities

In this section, we first describe the topology of its Milnor fiber \( M_W = W^{-1}(1) \) and their maximal symmetry group \( G_W \). We show in Proposition 5.4 that the quotient \( [M_W / G_W] \) is homeomorphic to an orbifold sphere with three special points, which are either orbifold points or (orbifold) punctures. For later calculations, we give an explicit description of the tessellation on \( M_W \) induced from the associated orbifold covering \( M_W \rightarrow [M_W / G_W] \).

6.1. Topology of Milnor fiber. Recall that Milnor fiber is homotopy equivalent to the bouquet of $\mu$-circles where $\mu$ is the Milnor number of the singularity.

Lemma 6.1. The weights (up to gcd) and Milnor numbers of curve singularities are as follows.

1. Weights of \( F_{p,q} \) are \( (q, p; pq) \). Its Milnor number is \( \mu_F = (p - 1)(q - 1) \).
2. Weights of \( C_{p,q} \) are \( (q, p - 1; pq) \). Its Milnor number is \( \mu_C = pq - p + 1 \).
3. Weights of \( L_{p,q} \) are \( (q - 1, p - 1; pq - 1) \). Its Milnor number is \( \mu_L = pq \).
Then, it has isolated singularity at the origin whose Milnor number is given by

$$
\mu = \left(\frac{h}{w_1} - 1\right) \cdots \left(\frac{h}{w_n} - 1\right)
$$

It is easy to check the following.

**Lemma 6.3.** \(G_{F_{p,q}} \approx \mathbb{Z}/p \oplus \mathbb{Z}/q\), \(G_{C_{p,q}} \approx \mathbb{Z}/pq\) and \(G_{L_{p,q}} \approx \mathbb{Z}/(pq - 1)\).

**Proof.** \(G_{F_{p,q}} = \{\left\lfloor \frac{2k\pi}{p}\right\rfloor, \left\lfloor \frac{2k\pi}{q}\right\rfloor \mid 0 \leq k \leq p-1, 0 \leq l \leq q-1\}\). The generators of \(G_{C_{p,q}}\) and \(G_{L_{p,q}}\) are \((\xi^q, \xi^{-1})\) and \((\eta^q, \eta^{-1})\) respectively for \(\xi = \exp\left(\frac{2\pi i}{p}\right), \eta = \exp\left(\frac{2\pi i}{pq-1}\right)\). □

For curve singularities, \(M_W\) is given by a (non-compact) Riemann surface. The boundary of a Milnor fiber is a union of \(k\) circles for curve singularities. In our case, we compactify \(M_W\) to \(\hat{M}_W\) by shrinking each circle of the link to a point. \(G_W\) acts on \(M_W\) as well as \(\hat{M}_W\).

**Proposition 6.4.** For invertible curve singularities, the genus \(g\), the number of boundary components \(k\) of the Milnor fiber \(M_W\) are given as follows. Also the quotient \([\hat{M}_W/G_W]\) is an orbifold projective line \(\mathbb{P}^1_{a,b,c}\):

- **(Fermat)** \(g = \frac{\mu_F + 1 - d}{2}\), \(k = d\), \((a, b, c) = \left(p, q, \frac{pq}{d}\right)\) for \(d = \gcd(p, q)\).
- **(Chain)** \(g = \frac{\mu_C - d}{2}\), \(k = d + 1\), \((a, b, c) = \left(pq, q, \frac{pq}{d}\right)\) for \(d = \gcd(p-1, q)\).
- **(Loop)** \(g = \frac{\mu_L - 1 - d}{2}\), \(k = d + 2\), \((a, b, c) = \left(pq - 1, pq - 1, \frac{pd}{d}\right)\) for \(d = \gcd(p-1, q-1)\).

Here, \(c\) vertex for Fermat, \(a, c\)-vertices for Chain, \(a, b, c\)-vertices for Loop type are punctures of \([M_W/G_W]\).

**Proof.** It is well-known that the number of boundary components are the same as the number of irreducible factors of \(W\). Recall that for sufficiently small \(r\) and \(0 < \epsilon < r\), the link \(W^{-1}(0) \cap S_{r}^{n-1}\) and \(W^{-1}(e) \cap S_{r}^{n-1}\) are diffeomorphic and note that each factor of \(W\) gives a boundary component for \(W^{-1}(0)\). For Fermat type, \(x^p + y^q\) factors into \(d\) factors for \(d = \gcd(p, q)\). For Chain type, since \(x^p + x^q = x(x^{p-1} + y^q)\), \(C_{p,q}\) has \(d + 1\) factors with \(d = \gcd(p-1, q)\). For loop type, since \(x^p + x^q = x^y(x^{p-1} + y^q)\), \(L_{p,q}\) has \(d + 2\) factors with \(d = \gcd(p-1, q-1)\).

To compute the genus, note that \(M_W\) is obtained by removing \(k\) punctures from \(\hat{M}_W\). Hence, Euler characteristic is \(\varepsilon(M_W) = \varepsilon(\hat{M}_W) - k\). But \(M_W\) has the homotopy type of bouquet of \(\mu\)-circles for the Milnor number \(\mu\), and its Euler characteristic \(\varepsilon(M_W) = 1 - \mu\). Therefore, the genus of \(\hat{M}_W\) (and hence \(M_W\)) is obtained from \(2 - 2g - k = 1 - \mu\) or \(g = (\mu + 1 - k)/2\).

Now, to get the quotient orbifold \([\hat{M}_W/G_W]\), we first find that there are exactly three orbits of \(G_W\) with non-trivial stabilizer in \(\hat{M}_W\) and show that the quotient has genus zero using orbifold Euler characteristic. We will use the fact that \(\varepsilon(\hat{M}_W)/|G_W|\) equal the orbifold Euler characteristic of \([\hat{M}_W/G_W]\).

Let us consider the Fermat case. Orbits of \([0,1]\) and \([1,0]\) gives two singular orbits of \(Z/p \oplus Z/q\)-action on \(M_W\). They have stabilizers \(Z/p, Z/q\) respectively. For \(d = \gcd(p, q)\), we have \(d\) punctures and \(Z/p \oplus Z/q\) acts transitively on them. So the quotient has three orbifold points \((a, b, c) = (Z/p, Z/q, Z/(pq/d))\).

To see that the quotient is \(\mathbb{P}^1_{a,b,c}\):

$$
\varepsilon(\hat{M}_W) = 2 - 2g = k + 1 - \mu = d + 1 - (p-1)(q-1)
$$
Note that it equals $|G|\cdot e_{orb}(\mathbb{P}^1_{a,b,c})$ which is
\[
(pq)\cdot \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1\right) = pq \cdot \left(\frac{1}{p} + \frac{1}{q} + \frac{d}{pq} - 1\right)
\]
This proves the claim for the Fermat case.

The other cases are similar. For the chain case, the orbit of $[(1,0)]$ has stabilizer $\mathbb{Z}/q$. The other two orbifold points come from punctures. Note that $C_{p,q}$ is a product of $x$ and $x^{p-1} + y^q$. It is easy to see that $G_W$ action preserves each branches $x = 0$ as well as $x^{p-1} + y^q = 0$. Hence the puncture corresponding to the branch $x = 0$ has the full group $G_f$ as a stabilizer and the other $d$ punctures (for the factors of $x^{p-1} + y^q = 0$ with $d = \gcd(p-1,q)$) are acted by $G_f$ in a transitive way. Therefore, the orbifold point has stabilizer $\mathbb{Z}/(pq/d)$. For the loop type, $M_W$ has no fixed point of $G_W$-action, and the punctures for factors $x, y, x^{p-1} + y^q$ form three orbits with stabilizer $\mathbb{Z}/(p q - 1), \mathbb{Z}/(pq - 1), \mathbb{Z}/((pq - 1)/d)$. This finishes the proof.

6.2. Orbifold covering. We observed that $G_W$ acts on the Milnor fiber $M_W$ to produce the regular orbifold covering $\tilde{M}_W \to \mathbb{P}^1_{a,b,c}$. Given a Riemann surface, there may be two non-equivalent group actions with isomorphic quotient space (see Broughton [Bro91] for example). Hence, to determine the $G_W$-action on $M_W$ explicitly, we find an explicit group homomorphism

\[
(6.1) \quad \phi : \pi^\text{orb}_1(\mathbb{P}^1_{a,b,c}) \to G_W.
\]

For the kernel $\Gamma = \text{Ker}(\phi)$, $\tilde{M}_W$ is an orbifold covering of $\mathbb{P}^1_{a,b,c}$ corresponding to the kernel $\Gamma$ with deck transformation group $G_W$.

We use the following presentation of the orbifold fundamental group of $\mathbb{P}^1_{a,b,c}$

\[
(6.2) \quad \pi^\text{orb}_1(\mathbb{P}^1_{a,b,c}) = \left\langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^a = \gamma_2^b = \gamma_3^c = \gamma_1 \gamma_2 \gamma_3 = 1 \right\rangle.
\]

Here, $\gamma_1$ is a small loop going counter-clockwise around $0 \in \mathbb{P}^1$, $\gamma_2$ is for $1 \in \mathbb{P}^1$ and $\gamma_3$ is for $\infty \in \mathbb{P}^1$. Later on, this presentation will serve as an additional grading on a Floer theory.

**Proposition 6.5.** The homomorphism (6.1) is given as follows.

1. **(Fermat)** For the covering $M_{F_{p,q}} \to \mathbb{P}^1_{p,a,pq,\gcd(p,\mathbb{Z})}$, we have
   \[
   \phi(\gamma_1) = (1,0), \phi(\gamma_2) = (0,1), \phi(\gamma_3) = (-1,-1) \in \mathbb{Z}/p \times \mathbb{Z}/q
   \]
   identified with $G_W$ by $(k,l) \mapsto \left( e^{\frac{2\pi ki}{p}}, e^{\frac{2\pi lij}{q}} \right)$.

2. **(Chain)** For the covering $M_{C_{p,q}} \to \mathbb{P}^1_{pq,q,\gcd(p\mathbb{Z},q)}$, we have
   \[
   \phi(\gamma_1) = 1, \phi(\gamma_2) = -p, \phi(\gamma_3) = p - 1 \in \mathbb{Z}/p q
   \]
   identified with $G_W$ by $k \mapsto \left( e^{\frac{2\pi ki}{p}}, e^{\frac{-2\pi kj}{q}} \right)$.

3. **(Loop)** For the covering $M_{L_{p,q}} \to \mathbb{P}^1_{pq-1,pq-1,\gcd(p\mathbb{Z},q-1)}$, we have
   \[
   \phi(\gamma_1) = 1, \phi(\gamma_2) = -p, \phi(\gamma_3) = p - 1 \in \mathbb{Z}/(pq - 1)
   \]
   identified with $G_W$ by $k \mapsto \left( e^{\frac{2\pi ki}{pq-1}}, e^{-\frac{2\pi kj}{pq-1}} \right)$.

Let us give the proof in each case separately.
6.2.1. Fermat type \( F_{p,q} \). \( M_{F_{p,q}} \) is a locus of an equation \( x^p + y^q = 1 \). We regard them as a Riemann surface of a multivalued function
\[
y = \left(1 - x^p\right)^\frac{1}{q},
\]
with \( q \) branch points \( x_k = e^{\frac{2\pi ki}{q}} \) (for \( k = 0, \ldots, q - 1 \)). We connect each branch points \( x_k \) with \( \infty \) by a ray \( \{re^{\frac{2\pi i}{q}} \mid r \geq 1 \} \). With these branch cuts, \( M_{F_{p,q}} \) is a \( q \) sheeted covering of a complex plane \( \mathbb{C} \).

A fundamental domain of the quotient is a following "pizza" shape domain.
\[
(6.3) \quad \left\{ x = re^\theta \mid 0 \leq r, \ 0 \leq \theta \leq \frac{2\pi}{p} \right\}
\]

There are three distinguished paths \( \gamma_i : [0,1] \to \mathbb{C} \).
- \( \gamma_1(t) = e \cdot e^{\frac{2\pi ti}{p}}, \ (0 < e \ll 1) \), a small path around the origin.
- \( \gamma_2(t) = 1 + e \cdot e^{2\pi it}, \ (0 < e \ll 1) \), a small circle around the branch points.
- \( \gamma_3(t) = Re^{\frac{-2\pi i - 2\pi i}{p q}}, \ (R \gg 1) \), a boundary circle with opposite orientation.

These are orbifold loops that correspond to generators of \( \pi_1^{\text{orb}}(\mathbb{R}^1_{p,q}, \frac{1}{q}, \mathbb{C}) \) in \( (6.2) \).

As we realize \( F_{p,q} \) as a \( q \) sheeted covering of \( \mathbb{C} \), we label those sheets from 1 to \( q \) so that the crossing branch cuts increases the label number by +1. Each sheets has \( p \) copies of the fundamental domain. We put the label \( i_j \) on the following copy;
\[
\left\{ x = re^\theta \mid 0 \leq r, \ \frac{2(j-1)\pi}{p} \leq \theta \leq \frac{2j\pi}{p} \right\} \subset i\text{-th sheet}.
\]

In this setup, we can write down the \( \phi \) from a representation of the group to the group of permutation of the set of labels \( \{i_j \mid 1 \leq i \leq q, \ 1 \leq j \leq p \} \).

\[
\phi : \pi_1^{\text{orb}}(\mathbb{R}^1_{p,q}, \frac{1}{q}, \mathbb{C}) \to S_{pq}
\]

\[
\gamma_1 \mapsto (1, 1, 2, \ldots, 1p)(2, 2, \ldots, 2p) \cdots (q, q, \ldots, q_p) \quad \gamma_2 \mapsto (1, 1, 2, \ldots, q_1)(2, 2, \ldots, q_2) \cdots (1, 1, 2, \ldots, q_p) \quad \gamma_3 \mapsto (\gamma_1 \circ \gamma_2)^{-1}
\]

The image of this representation is isomorphic to \( \mathbb{Z}/p \times \mathbb{Z}/q \), generated by \( \gamma_1 \) and \( \gamma_2 \). It is compatible to the diagonal symmetry group action in the following way:

- \( \gamma_1 \) is a rotation of each sheets by \( \frac{2\pi}{p} \). It corresponds to a diagonal action \( x \to e^{\frac{2\pi i}{q}} \cdot x, \ y \to y \).
- \( \gamma_2 \) is a rotation of each sheets by \( \frac{2\pi}{q} \) so it corresponds to a diagonal action \( x \to x, \ y \to e^{\frac{2\pi i}{p}} \cdot y \).
- \( \gamma_3 \) corresponds to a diagonal action \( x \to e^{-\frac{2\pi i}{p}} \cdot x, \ y \to e^{-\frac{2\pi i}{q}} \cdot y \).

6.2.2. Chain type \( C_{p,q} \). \( M_{C_{p,q}} \) is a locus of an equation \( x^p + x y^d = 1 \). We regard them as a Riemann surface of a multivalued meromorphic function
\[
y = \left(1 - x^p\right)^\frac{1}{q}.
\]

This function has \( q \) zero branch points \( x_k = e^{\frac{2\pi ki}{q}} \) and a single pole branch point \( x = 0 \).

We connect each branch points with \( \infty \) by rays as before. Also, we overlap a ray from a pole \( x = 0 \) and a ray from a zero \( x = 1 \). Because they are coming out of different sources, they cancel each other on the overlap. With these choice of branch cuts, \( M_{C_{p,q}} \) is a \( q \) sheeted covering of \( \mathbb{C}^* \).
A fundamental domain of the quotient $M_{p,q}/G_{p,q}$ can be taken as the same domain $6.3$ with the puncture at the origin, and orbifold loops $\gamma_1, \gamma_2, \gamma_3$ are the same as in the Fermat case.

Due to the branch cut along the line segment $[0, 1]$ on the real axis, monodromy representation is different from the Fermat cases. It is not hard to see that we get the following symmetric group representation

$$\phi : \pi_1^{\text{orb}} \left( \mathbb{P}^1_{pq,pq-1} \right) \to S_{pq-1} \times S_{pq-1}$$

$$\gamma_1 \to (q, -1) : (a, b) \mapsto (a + q, b - 1)$$

$$\gamma_2 \to (-1, p) : (a, b) \mapsto (a - 1, b + p)$$

$$\gamma_3 \to (\gamma_1 \circ \gamma_2)^{-1}$$

Unlike the Fermat case, $\phi(\gamma_1)$ generates $\phi(\gamma_2)$ by the relation $\phi(\gamma_2) = \phi(\gamma_1)^{-p}$. Hence the image of $\phi$ is generated by $\gamma_1$, and isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Notice that $\phi(\gamma_1)$ rotates each sheet by $\frac{2\pi}{p}$, and change the label of a sheet by $+1$ after you apply it by $-p$ times. Therefore,

- $\gamma_1$ corresponds to $x \to e^{\frac{2\pi i}{p}} \cdot x$, $y \to e^{\frac{-2\pi i}{p}} \cdot y$.

6.2.3. Loop type $L_{p,q}$: $M_{p,q}$ is a locus of an equation $x^p y + xy^q = 1$. As we cannot realize $L_{p,q}$ as a Riemann surface of a single function, we work with the following parametrization by $z \in \mathbb{C}$

$$x = \left( \frac{z^q}{1 - z} \right)^{\frac{1}{p-1}}, \quad y = \left( \frac{(1 - z)^p}{z} \right)^{\frac{1}{q-1}},$$

with two branch points $z = 0, 1$. We connect each two with $\infty$ by half lines and let the one from the origin overlaps the one from $z = 1$. Then $M_{p,q}$ is a $pq - 1$ sheeted covering of a $z$-plane $\mathbb{C} \setminus \{0, 1\}$ as follows.

A fundamental domain is the whole $z$-plane minus two points $z = 0, 1$. The three distinguished paths are

- $\gamma_1(t) = e \cdot e^{2\pi it}$, $(0 < e \ll 1)$, a small circle around $z = 0$.
- $\gamma_2(t) = 1 + e \cdot e^{2\pi it}$, $(0 < e \ll 1)$, a small circle around $z = 1$.
- $\gamma_3(t) = Re^{-2\pi it}$, $(R \gg 1)$, a boundary circle with opposite orientation.

Let us compute the monodromy representation. Whenever we cross a branch cut inside $z$-plane, we change a covering sheet for $x$ and $y$ both. Each of them has $pq - 1$ possibilities, so there are $(pq - 1) \times (pq - 1)$ different sheets. Let's label them by $(i, j)$, $i, j = 1, \ldots, pq - 1$. But we don't need all of them because $\pi_1$-orbit of $(1, 1)$ consists of only $pq - 1$ sheets among them. The monodromy representation of $x_1^{\text{orb}}$ is written as

$$\phi : \pi_1^{\text{orb}} \left( \mathbb{P}^1_{pq-1, pq-1} \right) \to S_{pq-1} \times S_{pq-1}$$

$$\gamma_1 \to (q, -1) : (a, b) \mapsto (a + q, b - 1)$$

$$\gamma_2 \to (-1, p) : (a, b) \mapsto (a - 1, b + p)$$

$$\gamma_3 \to (\gamma_1 \circ \gamma_2)^{-1}$$

Since $\phi(\gamma_2) = \phi(\gamma_1)^{-p}$, the image of this representation is generated by $\gamma_1$ and isomorphic to $\mathbb{Z}/pq - 1$. Furthermore, it is easy to see that

- $\gamma_1$ corresponds to a diagonal action $x \to e^{\frac{2\pi i}{pq}} \cdot x$, $y \to e^{\frac{-2\pi i}{pq}} \cdot y$.

6.3. Equivariant tessellation of Milnor fibers. The equator of $\mathbb{P}^1_{a,b,c}$ contains three orbifold points and divides the orbisphere into two cells. From the orbifold covering $\overline{M} \to \mathbb{P}^1_{a,b,c}$ and considering lifts of
these two cells, we obtain a tessellation of Milnor fibers of invertible curve singularities. Let us give a combinatorial description of the tessellation of $\hat{M}_W$ as well as $G_W$-action on it.

Consider a $2m$-gon whose boundary edges are labelled by $a_1, \ldots, a_{2m}$ ordered and oriented in a counterclockwise way. We say edges are identified as $\pm(2p-1)$ pattern if $a_{2k}$ and $(a_{2k+2p-1})^{op}$ are identified, and $a_{2k-1}$ and $(a_{2k-2p})^{op}$ are identified for any $k$. Here indices are modulo $2m$, and $a^{op}$ is the orientation reversal of the edge. Note that even and odd numbered edges play different roles. See Figure 13 (B) for 16-gon identified as $\pm7$ pattern.

**Theorem 6.6.** Compactified Milnor fiber $\hat{M}_W$ and $G_W$ on it are explicitly described as follows

1. (Fermat) $\hat{M}_{F, pq}$ is given by $(2pq - 2p)$-gon with edges identified as $\pm(2p-1)$ pattern. An odd numbered edge corresponds to an oriented path from a-vertex to c-vertex in the quotient.

2. (Chain) $\hat{M}_{C, pq}$ is given by $(2pq)$-gon with edges identified as $\pm(2p-1)$ pattern. An odd numbered edge corresponds to an oriented path from b-vertex to c-vertex in the quotient.

3. (Loop) $\hat{M}_{L, pq}$ is given by $(2pq - 1)$-gon with edges identified as $\pm(2p-1)$ pattern. An odd numbered edge corresponds to an oriented path from b-vertex to c-vertex in the quotient.

**Proof.** Recall that we have $\hat{M}_W / G_W = \mathbb{P}^1_{a,b,c}$ from Proposition 6.4. Let $H$ be the universal cover of $\hat{M}_f$ or equivalently that of $\mathbb{P}^1_{a,b,c}$. We have $\pi_1^{orb}(\mathbb{P}_{a,b,c})$ action on $H$. Let $F$ be a fundamental domain in $H$ for this action as in the Figure 11 where the angle is measured in $S^2$, $\mathbb{R}^2$ or $\mathbb{H}$ depending on the universal cover. Here $x_1, x_2, x_3$ project down to $a, b, c$ orbifold points and at two of them, we have the full cone angle but the cone angle for the other is divided into half. Also, we will use Proposition 6.5 which describes the relation between generators $\gamma_1, \gamma_2, \gamma_3$ of $\pi_1^{orb}(\mathbb{P}_{a,b,c})$ and $G_f$.

![Figure 11. Fundamental domain of $\mathbb{P}^1_{a,b,c}$ in $\mathbb{H}$](image)

For the Fermat case, consider $\gamma_1, \gamma_2 \in \pi_1^{orb}(\mathbb{P}_{a,b,c})$ and collect the following $p \times q$ copies of $F$ to define a polygon

$$P := \left\{ \gamma_2^i \gamma_1^j F \mid 0 \leq i \leq p-1, 0 \leq j \leq q-1 \right\}.$$  

First, $P$ is a fundamental domain of $\hat{M}_W$ since $G_W = \mathbb{Z}/p \times \mathbb{Z}/q$ and $\phi(\gamma_1) = (1,0), \phi(\gamma_2) = (0,1)$ are the generators of $G_W$ by Proposition 6.5.

Also, one can check that $P$ is a $(2pq - 2p)$-gon in the following way. First, $\{\gamma_2^j F\}$ for $j = 0, \ldots, q-1$ can be glued counter-clockwise way around the vertex $x_2$ of Figure 11 to form a $2q$-gon, say $Q$. Then, by applying $\{\gamma_1^i\}$ for $i = 0, 1, \ldots, p - 1$ to $Q$, we get $p$-copies of $Q$ glued around the vertex $x_1$ to form a $(2pq - 2p)$-gon and this is exactly $P$. Because 2 edges of $Q$ meeting at the vertex $x_1$ become interior edges, number of boundary edges decrease by $2p$ from $2pq$. See Figure 12 (A) for the case of $\hat{M}_{F,2}$, where $Q$ is given by the union of $F$ and $\gamma_2 F$ and $P$ is the 10-gon.
To find the boundary identification, we consider additional tiles next to $P$. To see how two boundary edges from $\gamma_2^{-1} F$ are identified to the remaining edges, we consider the additional rotation action around vertex for $x_1$. Consider two tiles $\gamma_1 \gamma_2^{-1} F$, $\gamma_1^{-1} \gamma_2^{-1} F$ which are not in $P$. Each tile shares one boundary edge with $\gamma_2^{-1} F$. Note that $\gamma_1 \gamma_2^{-1} F$ and $\gamma_1^{-1} \gamma_2^{-1} F$ are identified with $\gamma_2^{-1} F$ and $\gamma_1^{-1} F$ respectively because of $\phi(\gamma_1 \gamma_2^{-1}) = \phi(\gamma_2^{-1} F)$. This observation implies a boundary edge $e = (\gamma_2^{-1} F) \cap (\gamma_1^{-1} F)$ should be identified with corresponding edge of $\gamma_2^{-1} F$. Similarly, boundary edge $e' = (\gamma_2^{-1} F) \cap (\gamma_1^{-1} F)$ is identified with corresponding edge of $\gamma_1^{-1} F$. One can check that it gives $\pm (2q - 1)$ identification on $\partial P$. This proves the proposition for the Fermat case.

Next, let us discuss the chain type. Recall that we have $G_{pq} = \mathbb{Z}/pq$, and $\phi(1) = 1 \in \mathbb{Z}/pq$ is the generator. We take the following $pq$-copies of $F$ to define a $2pq$-gon:

$$P := \left\{ \gamma_i^i F \mid 0 \leq i \leq pq - 1 \right\},$$

which is a fundamental domain for the Milnor fiber. Consider $\gamma_2^{-1} \gamma_1^k F$ and $\gamma_2 \gamma_1^{k+1} F$. Since $\phi(\gamma_2) = -p$, we have $\phi(\gamma_2^{-1} \gamma_1^k) = \phi(\gamma_1^{p+k})$. Therefore, $\gamma_2^{-1} \gamma_1^k F$ can be identified with $\gamma_1^{p+k} F$ as a tile in the Milnor fiber.
Hence \( x_2x_3 \) edge of \( \gamma^k F \) should be identified with \( x_2x'_3 \) edge of \( \gamma^{k+p} F \). See Figure 13(A). In terms of edges of \( P \), this is \((2p-1)\) identification. From the same argument for \( \gamma_2 \gamma_1^p F \), we find that \( x_2x'_3 \) edge of \( \gamma^k F \) should be identified with \(-(2p-1)\) pattern. This proves the chain case.

For the loop type, we can proceed similarly as in the chain case. We take \( P := \{ \gamma^i F \mid 0 \leq i \leq pq-2 \} \) and we get the same identification as in the chain case from Proposition 6.5.

\[ \square \]

**Remark 6.7.** A description of \( M_W \) is not unique. For example, we will use another shape of domain in \( F_{3,4} \), \( F_{3,5} \) and \( C_{3,3} \) case (see Section 10). A gluing rule for \( C_{3,3} \) is given as in the Figure 14.

**Remark 6.8.** We remark that \( \tilde{M}_W \) is a sphere for \( F_{2,2} \), and a torus for \( F_{3,2}, F_{4,2}, C_{2,2}, C_{3,2}, L_{2,2} \) and a higher genus surface for the rest of the cases. For the last case, universal cover can be taken as the hyperbolic plane \( \mathbb{H} \) and there exists a Fuchsian group \( \Gamma \) such that \( \mathbb{H}/\Gamma \simeq \tilde{M}_W \). Furthermore, a finite group \( G \) acts on \( M_W \) if and only if there exist a Fuchsian group \( \Gamma' \) and a surjective homomorphism \( \phi : \Gamma' \to \Gamma_W \) with kernel \( \Gamma' \) such that \( M \simeq \mathbb{H}/\Gamma' \) and \( M/G \simeq \mathbb{H}/\Gamma' \). It is exactly a homomorphism given in Proposition 6.5. We refer readers to [Kat92] about Fuchsian group and related facts.

From the universal cover of \( \tilde{M}_W \), we replace compactified points by punctures and get a cover of \( M_W \). We will use this cover of \( M_W \) in the next section.

### 7. Floer theory for Milnor fiber quotients and localized mirror functor

**7.1. \( \Omega \)- and \( H_1 \)-grading.** We consider additional gradings on \( \mathcal{W}(|M_W/G_W|) \) following [Sei11] (with small variation) in addition to \( \mathbb{Z}/2 \)-grading. We use a holomorphic volume form \( \Omega \) on \( \mathbb{P}^1 \) with poles of order one at \( 0, 1, \infty \in \mathbb{P}^1 \). This choice provides a trivialization of a tangent bundle \( T_{|M_W/G|} \) away from \( 0, 1 \) and \( \infty \). Time-1 orbits of \( H \) are always disjoint from \( 0, 1, \infty \in \mathbb{P}^1 \) after adding a small time-dependent perturbation term. Therefore, each Hamiltonian orbit still carries an honest cohomological Conley-Zehnder index. We use this integer as a degree.

We can put a grading on a Lagrangians and Hamiltonian chords between them in a similar fashion. For a Lagrangian \( L \) which is oriented and away from \( 0, 1, \infty \in \mathbb{P}^1 \), we get a phase map \( \Phi_L : L \to S^1 \) defined by

\[
\Phi_L(x) = \frac{\Omega(X)}{||\Omega(X)||}
\]
where $X$ is a nonvanishing vector field on $TL$ pointing positive direction.

**Definition 7.1.** An $\Omega$-grading on $L$ is a choice of lift $\phi_L : L \to \mathbb{R}$ of a phase map $\Phi_L$. An $\Omega$-graded Lagrangian $L$ is a Lagrangian submanifold with a specific choice of $\Omega$-grading $\phi_L$.

For any time-1 Hamiltonian chord $a \in \chi(L_0, L_1)$ between graded Lagrangian submanifolds, there is the unique homotopy class of Lagrangian path from $T_{t_0}, a(0)$ to $T_{t_1}, a(1)$ compatible to the grading. The absolute Maslov index $\mu_M(a)$ is now well-defined, and we use it as a degree of $a$.

A discrepancy occurs when we consider a moduli space of discs. A standard index formula starts to read an intersection number of a holomorphic maps and pole divisor of $\Omega$. Let

$$\overline{\mathcal{M}}_{m; n, 1; [u]}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$$

be a sub-moduli space of $\overline{\mathcal{M}}_{m; n, 1; [u]}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ whose relative class is $[u]$. Then a standard index formula is given by

$$\dim_{\mathbb{R}} \overline{\mathcal{M}}_{m; n, 1; [u]}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0) = (2m + n - 2) + \deg a_0 - \sum_{i=1}^{n} \deg a_i - \sum_{j=1}^{n} \deg \gamma_j$$

$$+ 2(\deg(u, 0) + \deg(u, 1))$$

The dimension of our moduli space may differ in even numbers. It breaks a $\mathbb{Z}$-grading into $\mathbb{Z}/2$-grading.

Meanwhile, there is a topological grading coming from an orbifold cohomology. Recall we use a notation $\gamma_1, \gamma_2, \gamma_3$ to denote a homotopy class of loops winding orbifold point 0, 1 or $\infty$ respectively. We denote by $H_1^{\text{orb}}$ the abelianization of $\pi_1^{\text{orb}}$. We get

$$H_1^{\text{orb}}([M_W/G_W]) \simeq \left\{ \begin{array}{l}
\mathbb{Z}\langle \gamma_1, \gamma_2, \gamma_3 \rangle / \langle p\gamma_1 = q\gamma_2 = \gamma_1 + \gamma_2 + \gamma_3 = 0 \rangle \quad \text{(Fermat)} \\
\mathbb{Z}\langle \gamma_1, \gamma_2, \gamma_3 \rangle / \langle q\gamma_2 = \gamma_1 + \gamma_2 + \gamma_3 = 0 \rangle \quad \text{(Chain)} \\
\mathbb{Z}\langle \gamma_1, \gamma_2, \gamma_3 \rangle / \langle \gamma_1 + \gamma_2 + \gamma_3 = 0 \rangle \quad \text{(Loop)}
\end{array} \right.$$  

Notice that any symplectic cochains, including Morse critical points, can be considered as an element of $H_1^{\text{orb}}$. Moreover, if the Lagrangian submanifold $L$ is simply connected, elements of $CW^*(L, L)$ can also be labeled by $H_1^{\text{orb}}$. Let’s call it an $H_1$-grading. The Floer theoretic operation uses pseudo-holomorphic curves whose homological boundary is a difference of homology class of an output and inputs. Therefore, we have

**Lemma 7.2.** A pseudo-holomorphic curve operation is homogeneous with respect to an $H_1$-grading.

7.2. **Seidel’s Lagrangian $L$.** Since $[M_W/G_W]$ is an orbifold sphere with three special points, we consider an immersed circle, called Seidel Lagrangian $L$ and its $A_\infty$-algebra following Seidel (see Figure 15 and [Seil]). $L$ is oriented so that edges of the front triangle in Figure 15 is oriented counter-clockwise. We briefly recall the algebra structure of $CF^*(\mathbb{L}, \mathbb{L})$. It has three immersed generators $X, Y, Z$ of odd degree, $\bar{X} = Y \wedge Z$, $\bar{Y} = Z \wedge X$, $\bar{Z} = X \wedge Y$ of even degree.

**Proposition 7.3.** An $\Omega$ and $H_1$-grading of $CF^*(\mathbb{L}, \mathbb{L})$ is given by the following table (see [7.1] and [7.2]).

|       | 1 | X | Y | Z | $\bar{X}$ | $\bar{Y}$ | $\bar{Z}$ | $|pt| = X \wedge Y \wedge Z$ |
|-------|---|---|---|---|-----------|-----------|-----------|---------------------------|
| $\Omega$-grading | 0 | 1 | 1 | -1 | 0         | 0         | 2         | 1                         |
| $H_1$-grading    | 0 | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $-\gamma_1$ | $-\gamma_2$ | $-\gamma_3$ | 0                         |

**Proof.** (See [Seil], [She15]) $H_1$-grading is still well-defined even though $\mathbb{L}$ is not simply-connected. Because the class of $\mathbb{L}$ is null-homotopic, any path inside $\mathbb{L}$ starting from an immersed point to its another lift defines the unique homotopy class of loop in $H_1$. An $\Omega$-grading can be computed as a Maslov index of that class. Because $\Omega$ has order 1 poles at 0 and 1, Maslov index of $X$ and $Y$ is 1. The index of $Z$ is differ by 2 because our $\Omega$ has no poles nor zeros at $\infty$. 

In the surface case, we will use the following Seidel’s convention \cite{Sei11} of signs for $A_\infty$-operation $\{m_k\}$, which are defined as counting convex polygons. Each polygon contributes $\pm 1$ to the coefficient of output and the sign is determined as follow. Let $L_0, \ldots, L_k$ be Lagrangian submanifolds which intersect transversally and $x_i$ be an intersection point of $L_{i-1}$ and $L_i$ for $0 \leq i \leq k$ (modulo $k+1$). Suppose that there is a polygon $P$ bounded by $L_0, \ldots, L_k$ which has $k$ inputs $x_1, \ldots, x_k$ and one output $x_0$. For each $1 \leq i \leq k-1$, if the orientation of $L_i$ does not match to the orientation of $\partial P$, there is a sign $(-1)^{|x_i|}$. If the orientation of $L_k$ does not match to the orientation of $\partial P$, it gives $(-1)^{|x_k| + |x_0|}$. If $L_i$ is equipped with a non-trivial spin structure, we represent it by a red dot on $L_i$. Each time when an edge of a polygon contains the red dot, the sign is multiplied by $(-1)$. Multiplying all these sign contributions determines the sign of polygon.

Using the reflection symmetry (take $L$ to be invariant under the involution), we can follow \cite{CHL17} to prove that $L$ is weakly unobstructed and compute its potential function $W_L$.

**Lemma 7.4.** Equip $L$ with a nontrivial spin structure (marked as red dot in Figure 15). Then

1. $L$ is weakly unobstructed.
2. $b = xX + yY + zZ$ is a weak bounding cochain with potential $W_L$ where

$$W_L = \begin{cases} 
  x^p + y^q + xyz, & \text{for } F_{p,q} \\
  y^q + xyz, & \text{for } C_{p,q} \\
  xyz, & \text{for } L_{p,q}
\end{cases}$$

**Remark 7.5.** Note that $W_L$ is independent of $p$ for the chain, $p, q$ for the loop case. This is not a contradiction because the quotient space $[M_W/G_W]$ is also independent of those indices as well.

**Remark 7.6.** Since Milnor fibers are exact and $L$ is an exact Lagrangian (since it is homologically trivial), there exist a change of coordinate such that $W$ does not have any area $T$-coefficient. Therefore, we will omit them in the paper.

**Proof.** Weakly unobstructedness can be proved exactly the same way as Theorem 7.5 of \cite{CHL17}. To compute $W_L$, we fix a generic point and count all polygons whose corners are given by $X, Y, Z$’s. Because of punctures, there are finitely many polygons contributing to $W_L$. Also, we are only counting smooth discs, which have lifts to the Milnor fiber. So we can count them in the cover.

In the Fermat case, recall that the Milnor fiber can be obtained by first taking $2p$-gon and taking $q$-copies of these $2p$-gon’s by rotation around the $\mathbb{Z}/q$ fixed point. Then, we have one $p$-gon and one $q$-gon and $XYZ$-triangle passing through a generic point. See Figure 16(A). Therefore, we have

$$W_L = x^p + y^q + xyz$$
Figure 16. Holomorphic polygons for the potentials

In the chain case, its Milnor fiber is given by a $2pq$-gon with $A$-puncture at the center and $B$-vertex and $C$-puncture as vertices of $2pq$-gon (with $\mathbb{Z}/pq$-action around $A$). To see the discs, it is more convenient to describe the Milnor fiber with the orbifold point $B$ at the center. Note that there is only $\mathbb{Z}/q$-action around $B$. We can easily find a $q$-gon for the potential. Hence, we obtain

$$W_L = y^q + xyz$$

In the loop case, all vertices are punctures and the only nontrivial disc is $X Y Z$-triangle. Hence we have

$$W_L = xyz$$

7.3. **Localized mirror functor to Matrix factorization category.** For weakly unobstructed $L$, localized mirror functor formalism [CHL17] provides a canonical $A_\infty$-functor from Fukaya category of $[M_W/G_W]$ to the matrix factorization category of $W_L$. We use the version that appeared in [CHL].

**Definition 7.7.** Let $W_L^\bullet$ be the disc potential of $L$. The localized mirror functor $\mathcal{F}^\bullet : \text{Fuk}(X) \to \mathcal{MF}(W_L)$ is defined as follows.

- For given Lagrangian $L$, $\mathcal{F}^\bullet(L) := (\text{CF}(L, L), -m_1^0, b_1) =: M_L$.
- Higher component

$$\mathcal{F}^\bullet_k : \text{CF}(L_1, L_2) \otimes \cdots \otimes \text{CF}(L_k, L_{k+1}) \to \mathcal{MF}(M_{L_1}, M_{L_{k+1}})$$

is given by

$$\mathcal{F}^\bullet_k(a_1, \ldots, a_k) := \sum_{l=0}^{\infty} m_{k+1+l}(a_1, \ldots, a_k, \bullet, b_{l+1}, \ldots, b)$$

Here the input $\bullet$ is an element in $M_{L_{k+1}} = \text{CF}(L_{k+1}, L)$.

**Theorem 7.8 (CHL17).** $\mathcal{F}^\bullet$ defines an $A_\infty$-functor, which is cohomologically injective on $L$.

**Example 7.9.** Consider a polynomial $x^2 + xy^4$ and its Milnor fiber ($D_7^2$ singularity). Its fundamental domain and lifts of Seidel Lagrangian are as in Figure 17 with $W_L = y^4 + xyz$. We calculate $\mathcal{F}^\bullet(L) = (\text{CF}(L, L), -m_1^0)$ for the $L$ in the Figure. It is generated by two intersection points $o_1$ and $e_1$.

There is one triangle $o_1 Y_1 e_1$ which contributes to $m_1^{0,b}(o_1)$ and it gives $y$. There are two polygon $e_1 Z X o_1$ and $e_1 Y_2 Y_3 Y_4 o_1$. They contribute to $m_1^{0,b}(e_1)$ and give $y^3 + xz$. For all three polygons, since the orientations
Figure 17. Polygons contributing to $m_1^{0,b}$

of discs and those of Lagrangians are matched, all signs of discs are +. Therefore, $(CF(L,L), -m_1^{0,b})$ is a (matrix) factorization

$$-y \cdot (-y^3 - xz)$$

of the potential $y^4 + xyz$. Later, we will consider a restriction $z = x$, which turns $W_L$ into $D_5$ singularity $y^4 + yx^2$.

Another important example is when $L$ is $\mathbb{L}$ itself. $\mathcal{F}^\perp(\mathbb{L})$ can be computed following [CHL17].

**Theorem 7.10.** (c.f. Theorem 7.6 [CHL17])

$$M_L \simeq k_{W_L}^{\text{stab}}$$

where $k_{W_L}^{\text{stab}}$ is a matrix factorization of stabilized residue field at the origin.

**Remark 7.11.** The singularity of $W_L$ is not isolated, and in fact contains some of the coordinate axes. Hence $k_{W_L}^{\text{stab}}$ alone does not generate $\mathcal{M}(W_L)$. In the next section, we will find non-compact Lagrangians $L$ whose associated matrix factorization $M_L$ corresponds to the desired coordinate axes.

The following lemma tells us that homotopic Lagrangians correspond to quasi-isomorphic matrix factorizations, which will be useful in our computations.

**Lemma 7.12.** Suppose two non-compact Lagrangian $L_1$ and $L_2$ connect same punctures and represent a same homotopy class with respect to their boundary in the cover of Milnor fiber. Then they are isomorphic.

**Proof.** Assume that there is no intersection between them. We have two even wrapped generators $\alpha \in \text{Hom}(L_2, L_1)$ and $\beta \in \text{Hom}(L_1, L_2)$. They satisfy $m_1(\alpha) = m_1(\beta) = 0$, $m_2(\beta, \alpha) = e_1$ and $m_2(\alpha, \beta) = e_2$. Hence $\alpha$ and $\beta$ give an isomorphism between them.

Figure 18. Isomorphism between two Lagrangians

In general case when they intersect arbitrarily, we choose a open set containing two Lagrangians and another Lagrangian $L'$ in the complement. Then there is a sequence of isomorphisms $L_1 \simeq L' \simeq L_2$. $\square$
8. Homological mirror symmetry for Milnor fibers (without monodromy action)

In this section, we consider homological mirror symmetry for Milnor fiber as a symplectic manifold. We will find that $G_W$-equivariant mirror of $M_W$ is a Landau-Ginzburg model $W_L$. By applying Theorem 7.8 to the wrapped Fukaya category of $[M_W/G_W]$, we obtain an $A_\infty$-functor which gives a derived equivalence.

**Theorem 8.1.** We have an $A_\infty$-functor

$$F^L: \mathcal{W} \mathcal{F}([M_W/G_W]) \rightarrow \mathcal{M} \mathcal{F}(W_L)$$

where $W_L$ for Fermat $F_{p,q}$, chain $C_{p,q}$ and loop $L_{p,q}$ cases are given as

$$W_L = x^p + y^q + xyz, \quad y^q + xyz, \quad xyz$$

This functor is fully faithful and gives a derived equivalence between two categories.

**Remark 8.2.** $W_L$ is related to the transposed potential $W^T$ as follows. If we set

$$g(x, y, z) = \begin{cases} 
  z, & \text{for } F_{p,q} \\
  z - x^{p-1}, & \text{for } C_{p,q} \\
  z - x^{p-1} - y^{q-1}, & \text{for } L_{p,q}
\end{cases}$$

then we have

$$W_L = W^T(x, y) + xyg.$$  

As we will explain later, if we add monodromy information and take our newly defined $A_\infty$-category, the mirror will be obtained by setting $g = 0$, hence we obtain the matrix factorization of $W^T(x, y)$.

We prove the above theorem in the rest of the section. Although we treat each case separately, the underlying strategies are basically the same.

8.1. Fermat cases. Recall that $G_W = \mathbb{Z}/p \times \mathbb{Z}/q$ is the maximal diagonal symmetry of $W = x^p + y^q$ and the quotient space $[M_{F_{p,q}}/G_W]$ has a single puncture $C$ of orbifold order $\gcd(p, q)$. Then, for a preimage $\tilde{C}$ in $M_{F_{p,q}}$, we connect $\tilde{C}$ and $(1,0) \cdot \tilde{C}$ by the shortest path $\tilde{L}_1$ as in the Figure 19 which we take as a non-compact Lagrangian. We denote by $\tilde{L}$ the embedded Lagrangian in $[M_{F_{p,q}}/G_W]$ given by its projection. Denote by $\tilde{L}$ the set of lifts of $L$ in $M_{F_{p,q}}$, which is exactly $G_W \cdot \tilde{L}_1$.

![Figure 19. Milnor fiber of $F_{4,2}$ and a choice of Lagrangian $L$ and its lifts](image-url)
To prove the theorem in Fermat case, we show that $G_W$ copies of $L$ split-generate $\mathcal{W}\mathcal{F}(M_{F,W})$. Also, we compute the mirror matrix factorization $\mathcal{F}(L)$, and prove that the functor is fully faithful. Finally, we show that $\mathcal{H}(W_1)$ is split generated by $\mathcal{F}(L)$, and this proves the theorem. 8.1

Recall that $CW^*(L, L)$ is defined as $(CW^*(\tilde{L}, \tilde{L}))^{G_W}$.

**Lemma 8.3.** Wrapped Floer complex $CW^*(L, L)$ satisfies the following:

1. as a vector space,
   $$CW^*(L, L) \cong T(a, b)/\mathcal{R}_{p,q}$$
   Here, $T(a, b)$ is a tensor algebra generated by two alphabets $a, b$. The ideal $\mathcal{R}_{p,q}$ is defined as
   $$\mathcal{R}_{p,q} = \langle a \otimes a = \delta_{2p}, b \otimes b = \delta_{2q} \rangle$$
   $\mathbb{Z}/2$-gradings of $a, b$ are odd and this induces $\mathbb{Z}/2$-grading on $T(a, b)/\mathcal{R}_{p,q}$.

2. $m_1$ vanishes and $m_2$ coincides with the tensor product.

3. $m_k(a, \ldots, a)$ is zero for $1 \leq k < p$ and it is equal to 1 for $k = p$. Likewise, $m_k(b, \ldots, b)$ is zero for $1 \leq k < q$ and equal to 1 for $k = q$.

Its $\Omega$ and $H_1$-grading is given by the following table.

|        | $a$ | $b$ |
|--------|-----|-----|
| $\Omega$-grading | 0   | 1   |
| $H_1$-grading | $-\frac{\gamma_2}{2}$ | $-\frac{\gamma_1}{2}$ |

**Proof.** We can choose $\tilde{L}_1$ so that $G_W$-orbits of $\tilde{L}_1$ are disjoint. Therefore $CW^*(L, L)$ consists only of Hamiltonian chords at infinity. Among such chords we choose the following two generators.

- **a**, the shortest chord $\in CW^*(\tilde{L}_1, (1, 0) \cdot \tilde{L}_1)$, $(1, 0) \in \mathbb{Z}/p \times \mathbb{Z}/q$.
- **b**, the shortest chord $\in CW^*(\tilde{L}_1, (0, 1) \cdot \tilde{L}_1)$, $(0, 1) \in \mathbb{Z}/p \times \mathbb{Z}/q$.

For example, take a rotation of $\tilde{L}_1$ around the $\mathbb{Z}/p$ fixed point and there is the unique wrapped generator $a$ between these two branches. By abuse of notation, we also denote by $a, b$ the generators in $(CW^*(\tilde{L}, \tilde{L}))^{G_W}$ given by the sum of $G_W$-copies of the above generators.

We can also concatenate them to create new Hamiltonian chords ($m_2$-products near the puncture), denoted by $\{a, b, ab, ba, aba, bab, \ldots\}$. One can check that $m_1$ vanishes. Note that if we consider $m_2$-operations near the puncture, $m_2(a, a), m_2(b, b)$ vanish for $p, q \geq 3$ as they are not composable. If $p = 2$ or $q = 2$, we could have an $m_2$-product coming from a global holomorphic polygon which contributes $m_2(a, a)$ or $m_2(b, b)$ respectively. In general, there are two global $J$-holomorphic polygons with all of its corners are of word length 1. They are $p$-gon and $q$-gon and come from lifts of upper/lower hemisphere of $(M_{F,W}/G_W) \setminus L)$. Their corners are Hamiltonian chords $a$ or $b$ at infinity. They cannot contribute to $m_{p-1}$ or $m_{q-1}$, only contribute to $m_p$ or $m_q$ respectively. The boundaries of these polygons are whole $G_W$-orbits of $L$ so they represents the unit element of $CW^*(L, L)$. The computation of grading is entirely analogous to Proposition 7.3. We leave it as an exercise.

**Lemma 8.4.** $\tilde{L}$ split-generates the wrapped Fukaya category of $M_{F,W}$.

**Proof.** We proceed as in the work of Heather Lee [Lee16]. To avoid confusion, let us denote by $a$ the sum over $G_W$ orbit of $a$ in this proof. From Abouzaid’s generating criterion, it is enough to show that the following open-closed map hits the unit.

$$\mathcal{O}_\mathcal{C}: CC_*(CW^*(\tilde{L}, \tilde{L})) \to SH^*(M_{F,W}).$$
Lemma 8.9. Its cohomology is

\[
\frac{\partial_0^p}{p} - \frac{\partial_1^q}{q} \in \text{CC}_* (CW^* (\tilde{L}, \tilde{L}))
\]

It is not hard to see that \( \tilde{L} \) provides a tessellation of \( M_{F_{p,q}} \), which consists of \( q \) distinct \( p \)-gons and \( p \) distinct \( q \)-gons. We first check that it is \( G_W \)-equivariant Hochschild cycle. From Lemma 8.3 it is enough to check \( m_p, \bar{m}_q \) operations respectively.

\[
\partial_{\text{Hoch}} (\partial_0^p / p - \partial_1^q / q) = (m_p (\bar{\alpha} \ldots, \bar{\alpha}) - \bar{m}_q (\bar{b} \ldots, \bar{b})) = 1 - 1 = 0.
\]

On the other hand, the image of the open-closed map of this Hochschild cycle is a cocycle represented by the bounded area of \( M_{F_{p,q}} \) covering each region with weight one. Note that the orientation of the boundary Lagrangians of \( p \)-gon and \( q \)-gon are opposite to each other, and thus \( p \)-gons and \( q \)-gons in the image add up despite the negative sign in the expression \(-\bar{m}_q / q\). \( \square \)

Let us discuss the mirror matrix factorization. Using localized mirror functor, we can explicitly compute the mirror matrix factorization. Since \( W_L \) has non-isolated singularity (singularity along \( z \)-axis), we need to be a little bit careful in the discussion. By counting appropriate polygons from the picture with 3-gons. We first check that it is \( G \)-invariant. From Lemma 8.3, it is enough to check \( m_p, \bar{m}_q \) operations respectively.

\[
\text{Corollary 8.7.} \text{ The matrix factorization } M_L \text{ is of Koszul type. Namely, we have an isomorphism } M_L \cong \{ S \theta_x, \theta_y \}, \partial_K + \partial_K'.
\]

Here, \( \theta_x, \theta_y \) are odd degree generators (hence anti-commute) and

\[
\partial_K = x \cdot \theta_x + y \cdot \theta_y, \quad \partial_K' := W_x \theta_x \wedge \cdot + W_y \theta_y \wedge \cdot
\]

where \( W_x = (x^{p-1} + y z), \quad W_y = y^{q-1} \).

\[
\text{Remark 8.8.} \text{ The following Koszul complex has cohomology } \mathbb{C}[z] \text{ concentrated on the right end.}
\]

\[
K(x, y) := 0 \xrightarrow{S(\theta_1 \wedge \theta_2)} S(\theta_1 \wedge \theta_2) \xrightarrow{\partial_K} S(\theta_1 \wedge \theta_2) \xrightarrow{\partial_K} S \rightarrow 0
\]

Following Dyckerhoff [Dyc11], we compute its endomorphism algebra \( \text{End}(M_L) \).

\[
\text{Lemma 8.9.} \text{ \text{End}}_{MF}(M_L) \text{ is quasi-isomorphic to a DG algebra of polynomial differential operators}
\]

\[
\text{Hom}_{MF}(M_L, M_L) = \{ S(\theta_x), (\theta_y \wedge, (\theta_x \wedge)), (\theta_y \wedge)), D \}
\]

\[
D(\partial_{\theta_x}) = W_x, \quad D(\partial_{\theta_y}) = W_y, \quad D(\theta_x \wedge) = x, \quad D(\theta_y \wedge) = y.
\]

Its cohomology is

\[
H^* (\text{Hom}_{MF}(M_L, M_L)) = \mathbb{C}[z][\Gamma_x, \Gamma_y]
\]

\[
\Gamma_x = [\partial_{\theta_x} - x^{p-2} (\theta_x \wedge) - z (\theta_y \wedge)], \quad \Gamma_y = [\partial_{\theta_y} - y^{q-2} (\theta_y \wedge)].
\]
Proof. The first part of the lemma is obvious because morphisms of a matrix factorization of Koszul type are those of exterior algebras. It is easy to check that the differential satisfies given equations. For example,

\[ D(\partial_{0j}) = [\partial_K + \partial'_{K}, \partial_{0j}] = [\partial'_{K}, \partial_{0j}] = [(W_x \theta_x \wedge), u_{0j}] = W_x. \]

To each differential operator, we can assign an order of its symbols. It provides a decreasing filtration \( \{ F^i \} \) on the complex. The first page of the spectral sequence associated to the filtration is a dual Koszul complex associated to a regular sequence \( (x, y) \);

\[ E_1 = H^* \left( k^\wedge (x, y) \otimes \mathbb{C}[\partial_{0j}, \partial_{0j}], \partial_{0j} \wedge 1 \right) \cong \mathbb{C}[z][\partial_{0j}, \partial_{0j}] \]

In particular we know that the cohomology algebra is a \( \mathbb{C}[z] \)-modules of rank less or equal to 4. On the other hand, the cycles generated by \( \Gamma_x, \Gamma_y \) in the lemma have already provided four \( \mathbb{C}[z] \)-linear independent elements. Therefore the spectral sequence degenerates at \( E_1 \) page. This finishes the proof. \( \square \)

We can show that our mirror functor is fully-faithful.

Lemma 8.10. The first-order part of the mirror functor is

\[ \mathcal{F}_1 : CW^* (L, L) \to \text{Hom}_{\mathcal{H}\mathcal{F}} (M_L, M_L) \]

\[ a \to \Gamma_x, b \to \Gamma_y. \]

It is a quasi-isomorphism. Therefore \( \mathcal{F}_1 \) embeds \( \mathcal{W} \mathcal{F} (M_{F_{\rho,q}}) \) as a full subcategory of \( \mathcal{H}\mathcal{F} (W_L) \).

Proof. From the Figure 19 we see that \( \mathcal{F}_1 \) sends \( a \) to \( \Gamma_x \) and \( b \) to \( \Gamma_y \). Moreover,

\[ [\Gamma_x, \Gamma_y] = [-z(\partial_y \wedge), \partial_{0j}] = z. \]

Therefore \( ab + ba \) hits \( z \) and \( \mathcal{F}_1 \) is surjective.

Notice that \( CW^* (L, L) \) and \( H^* (\text{Hom}_{\mathcal{H}\mathcal{F}} (M_L, M_L)) \) are filtered by

\[ F^k := (ab + ba)^k \cdot CW^* (L, L), \quad G^l := z^l \cdot H^* (\text{Hom}_{\mathcal{H}\mathcal{F}} (M_L, M_L)) \]

It is easy to check that \( \mathcal{F}_1 \) is a filtered map with respect to \( F^* \) and \( G^* \).

The graded piece \( F^0/F^1 \) is a 4 dimensional vector space spanned by four words \( <1, a, b, ab> \). This is because

\[ aba = (ab + ba) \cdot a - \delta_{2,p} b, \quad bab = (ab + ba) \cdot b - \delta_{2,q} a. \]

An element \( ab + ba \) is in the center of the algebra. Therefore

\[ F^k/F^{k+1} = (ab + ba)^k \cdot F^0/F^1 = (ab + ba)^k \cdot <1, a, b, ab>. \]

By a similar reason, we have

\[ G^k/G^{k+1} = z^k \cdot G^0/G^1 = z^k \cdot <1, \Gamma_x, \Gamma_y, (\Gamma_x \circ \Gamma_y) > \]

The induced morphism of associated graded \( \text{Gr} \mathcal{F}_1 \) is an isomorphism of vector spaces at every level. By the comparison theorem, so is \( \mathcal{F}_1 \). \( \square \)

Corollary 8.11. \( \mathcal{F}_1 : \mathcal{W} \mathcal{F} ([M_{F_{\rho,q}}/G_{F_{\rho,q}}]) \to \mathcal{H}\mathcal{F} (W^T + xyz) \) is a quasi-equivalence.

Proof. It is enough to show that \( M_L \) and \( M_L \) generate \( \text{MF}(W^T + xyz) \). Orlov’s equivalence

\[ \text{MF}(W^T + xyz) \cong D_{sg}(W^T + xyz) \]

\[ \left( \begin{array}{c} M^1 \\ \phi \psi \end{array} \right) \to \text{coker}(\psi) \]
sends $M_1$ to a skyscraper sheaf $\mathcal{O}_0$ at the origin and $M_L$ to a structure sheaf $\mathcal{O}_z$ of $z$-axis. These are two irreducible components of a critical locus of $W^T + xyz$. Therefore they generates $\text{MF}(W^T + xyz)$ (see [Ste13]).

8.2. Chain cases. The polynomial $W = C_{p,q} = x^p + xy^q$ has maximal symmetry group $G_W = \mathbb{Z}/pq$. We proceed as in the Fermat case. Denote by $\xi$ the following generator of $G_W$:

$$x \rightarrow e^{\frac{2\pi i}{p}} \cdot x, \quad y \rightarrow e^{\frac{-2\pi i}{pq}} \cdot y.$$ 

Recall that the quotient space $[MC_{p,q}]/G_W$ has one orbifold point of order $q$ and two punctures of order $pq$ and $\frac{pq}{\gcd(p-1,q)}$, respectively. Let us call them as $B_1, B_2$ respectively. The orbifold action near $B_1$ is generated by $\xi$ while the action near $B_2$ is generated by $\xi^{p-1}$ by Proposition 6.5.

![Figure 20. Milnor fiber of $E_7 = C_{3,3}$ and a choice of Lagrangian $L$](image)

We take a Lagrangian $L$ connecting $B_1$ and $B_2$ in $\mathbb{P}^1_{p,q,q,\frac{pq}{\gcd(p-1,q)}}$ (we may take the part of the equator between $B_1$ and $B_2$), and denote by $\tilde{L}$ the sum of all lifts of $L$ in the Milnor fiber.

**Lemma 8.12.** The wrapped Floer complex $CW^*(L, L)$ satisfies the following:

1. as a vector space,

$$CW^*(L, L) = \mathbb{C}[a, b]/(ab = 0)$$

Here, $a, b$ are even variables.

2. $m_1$ vanishes and $m_2$ coincides with a polynomial multiplication.

3. $m_k(a, b, a, b, \ldots) = 0$ for $1 \leq k \leq 2q - 1$ and $m_{2q}(a, b, a, \ldots, a, b) = 1$. Likewise, $m_k(b, a, b, a \ldots) = 0$ for $1 \leq k \leq 2q - 1$ and $m_{2q}(b, a, \ldots, b, a) = 1$.

Its $\Omega$ and $H_1$-grading is given by the following table.

| 1_L | a   | b   |
|-----|-----|-----|
| $\Omega$-grading | 0   | 0   | 2   |
| $H_1$-grading     | 0   | $-\gamma_1$ | $-\gamma_3$ |

**Proof.** Branches of $\tilde{L}$ do not intersect with each other in the interior. Therefore $CW^*(L, L)$ consists of Hamiltonian chords at infinity near $B_1$ or $B_2$. Among them we choose two generators between the nearest orbits. Namely, choose one lift $\tilde{L}_1$ and take the wrapped generator

- $a$, the shortest chord $\in CW^*(\tilde{L}_1, \xi^{-1} \cdot \tilde{L}_1)$ near $B_1$
• \( b \), the shortest chord \( \in CW^*(\tilde{L}_1, \xi_1^{1-p} \cdot \tilde{L}_1) \) near \( B_2 \).

Here, \( a \) (resp. \( b \)) is nothing but the chord between \( \tilde{L}_1 \) and its clockwise rotation at \( B_1 \) (resp. \( B_2 \)). Namely, recall that \( \xi, \xi^{p-1} \) correspond to \( \gamma_1, \gamma_3 \) of the orbifold fundamental group in the Proposition 6.5. And \( \gamma_1^{-1}, \gamma_3^{-1} \) are the minimal clockwise rotations in the uniformizing neighborhood of orbifold points. Therefore \( \xi \cdot \tilde{L}_1 \) is obtained by clockwise rotation of \( \tilde{L}_1 \) centered at \( B_1 \) sending \( B_2 \)-vertex to the nearest \( B_2 \)-vertex. The same holds for \( \xi^{p-1} \cdot \tilde{L}_1 \) switching the role of \( B_1 \) and \( B_2 \).

We can also concatenate them to create new Hamiltonian chords, namely \( a^2, a^3, \ldots, b^2, b^3, \ldots \). We cannot concatenate different words as their heads and tails are different from each other. The rest of the argument is similar to the Fermat case. \( m_1 \) vanishes because there are no \( J \)-holomorphic strip between them. Concatenating two chords corresponds to \( m_2 \) operation concentrated near the punctures. The first global \( J \)-holomorphic polygon contributes to a non-trivial \( A_\infty \) operation is a \( 2q \)-gon. It is a lift of an orbifold bigon \( (M_{C_{p,q}} / G_W) \setminus L_0 \). Its corners consists of \( q \) many \( a \) and \( b \) alternating each other.

**Lemma 8.13.** \( \tilde{L} \) generates the wrapped Fukaya category of \( M_{C_{p,q}} \).

**Proof.** We proceed as in the Fermat case. Milnor fiber \( M_{C_{p,q}} \) is tessellated by \( p \) copies of \( 2q \)-gons that are considered in the previous lemma. In Figure 20, this is given by 3 copies of hexagons. To show that open-closed map hits the unit, we take the following Hochschild cycle;

\[
\frac{1}{q}(\tilde{a} \otimes \tilde{b})^{\otimes q} \in CC^*(CW^*(\tilde{L}, \tilde{L}))
\]

It is indeed a cycle because

\[
\partial_{Hoch}\left(\frac{1}{q}(\tilde{a} \otimes \tilde{b})^{\otimes q}\right) = m_{2q}(\tilde{a}, \tilde{b}, \ldots, \tilde{a}, \tilde{b}) - m_{2q}(\tilde{b}, \tilde{a}, \ldots, \tilde{b}, \tilde{a}) = 1_{\tilde{L}} - 1_{\tilde{L}} = 0.
\]

On the other hand, the open-closed image of this Hochschild cycle is a cocycle represented by the bounded area of \( M_{C_{p,q}} \) covering each region with weight one.

If we solve the weak Maurer-Cartan equation for \( L \), we get the potential \( W_L = y^q + xyz \), which can be also written as \( W^T + xyg \) with \( g(x, y, z) = z - x^{p-1} \).

**Lemma 8.14.** The localized mirror functor \( \mathcal{F}^1 : \mathcal{W}\mathcal{F}([M_{C_{p,q}} / G_{C_{p,q}}]) \to \mathcal{M}\mathcal{F}(W_L) \) sends \( L \) to the matrix factorization \( M_L = (S \xrightarrow{\delta_0} S \xleftarrow{\delta_1} ) \) where

\[
\delta_0 = y, \quad \delta_1 = y^{q-1} + xz.
\]

**Proof.** This follows from the Figure 20.

The matrix factorization \( M_L \) is again of Koszul type. One can check directly that

\[
M_L = \{ S[\theta_y], (y \cdot i_{\theta_y} + W_y \cdot \theta_y \wedge) \}, \quad W_y = y^{q-1} + xz
\]

Using the same technique,

**Lemma 8.15.** The self-hom space of \( M_L \) is quasi-isomorphic to a DG algebra of polynomial differential operators

\[
\text{Hom}_{MF}(M_L, M_L) \simeq \left[ S[\theta_y], (\theta_y \wedge), D \right]
\]

\[
D(\theta_y) = W_y, \quad D(\theta_y \wedge) = y.
\]

Its cohomology is concentrated to even degree and isomorphic to

\[
H^*(\text{Hom}_{MF}(M_L, M_L)) \simeq \mathbb{C}[x, z]/(xz = 0)
\]
Proof. The first part of the lemma is same as Fermat case. The cohomology computation can be done in a similar way, but we found that it is much easier to do it by hands. This complex is isomorphic to the 2-periodic complex

\[ S^{\#2, \text{even}} \xrightarrow{D_0} S^{\#2, \text{odd}} \]

where \( D_0 = \begin{pmatrix} y & -y \\ W_y & -W_y \end{pmatrix}, \ D_1 = \begin{pmatrix} y & -W_y \\ y & -W_y \end{pmatrix}. \)

Therefore, we have

\[
\ker(D_0)/\text{im}(D_1) = \left\{ \left( \frac{a}{b} \right) \in S^{\#2, \text{even}} \mid a = b \right\} / S \cdot \left( \frac{y}{y} \right) = \mathbb{C}[x,y,z]/(y = W_y = 0) = \mathbb{C}[x,z]/(xz = 0),
\]

\[
\ker(D_1)/\text{im}(D_0) = \left\{ \left( \frac{a}{b} \right) \in S^{\#2, \text{odd}} \mid W_y \cdot a + y \cdot b = 0 \right\} / S \cdot \left( \frac{y}{-W_y} \right) \simeq 0.
\]

The last equality holds because \((y, W_y)\) is a regular sequence of \(S\). \(\square\)

Now we can show that our mirror functor is an equivalence.

Lemma 8.16. The first-order part of the mirror functor is given by

\[
\mathscr{F}_1 : CW^*(L, L) \to \text{Hom}_{\mathcal{M}_F}(M_L, M_L)
\]

\[
a \to x, b \to z.
\]

It is a quasi-isomorphism. Moreover \(\mathscr{F}_1 : \mathcal{W} \mathcal{F}([M_{C_{pq}}/G_{C_{pq}}]) \to \mathcal{M}_F(y^q + xyz)\) is a quasi-equivalence.

Proof. Similar to Fermat case. \(\square\)

8.3 Loop cases. The loop type polynomial \(W = x_1^p x_2 + x_1 x_2^q\) has \(G_W = \mathbb{Z}/pq - 1\) as the maximal diagonal symmetry group. One notable difference of a loop type from the others is that the action of \(G_W\) is free. The quotient \(M_{L_{pq,q}}/G_W\) is an honest three punctured sphere. Its wrapped Fukaya category and its homological mirror symmetry was proved in \(\text{[AAE+13]}\). The result in this section can be essentially found therein, except that we use localized mirror functor to define the explicit correspondences.

Let us introduce more notation. For loop type, we use variables \(x_i (i = 1, 2, 3)\) instead of \(x, y, z\). Let \(\xi\) denote the following generators of this group.

\[
x_1 \to e^{\frac{2\pi i}{pq}}, x_1, \quad x_2 \to e^{\frac{2\pi i}{pq}}, x_2
\]

Also recall three punctures are of order \(pq - 1, pq - 1\) and \(\frac{pq-1}{\gcd(p - 1, q - 1)}\). Let’s denote them by \(B_1, B_2,\) and \(B_3\) respectively. A cyclic orbifold action is generated by \(\xi\) near \(B_1\), by \(\xi^{-p}\) near \(B_2\) and by \(\xi^{p-1}\) near \(B_3\) by Proposition \(\text{[6.5]}\). As there are three punctures, we choose three shortest Lagrangians \(L_i\) from \(B_{i+1}\) to \(B_{i+2}\) for \(i = 1, 2, 3\) \(\mod 3\) which are part of the equator sphere passing through 3 punctures. The following can be checked from \(\text{[AAE+13]}\).

Lemma 8.17. The wrapped Floer complexes \(CW^*(L_i, L_j)\) satisfies the following:

1. as a vector space,

\[
CW^*(L_i, L_j) \simeq \left\{ \frac{\mathbb{C}[a_{i+1}, b_{i+2}]/(a_{i+1} b_{i+2} = 0)}{i = j} \quad \mathbb{C} < a_i^m \cdot c_i, j \cdot b_j^m, n, m \in \mathbb{N} \quad i \neq j \right\}
\]

Here, \(a_i, b_i\) are even and \(c_{i,j}\) are odd.

2. \(m_1\) vanishes and \(m_2\) coincides with a polynomial multiplication and an obvious bimodule structure.

3. \(m_3(c_{12}, c_{23}, c_{31}) = 1\)

Its \(\Omega\) and \(H_1\)-grading is given by the following table.
Consider the direct sum of lifts $\tilde{L}_i$ of $L_i$ in $M_{L_{p,q}}$. Then the following is well-known.

Lemma 8.18. $\{\tilde{L}_i\}_{i=1,2,3}$ split-generates the wrapped Fukaya category of $M_{L_{p,q}}$.

Next, we move on to the mirror computation. For the Seidel Lagrangian $L$ in the quotient $[M_{L_{p,q}}/G_W]$, the potential function can be computed as

$$W_L = xyz.$$ 

Recall that this is related to $W_T$ as $xyz = x^p y + xy^q + xy(z - x^{p-1} - y^{q-1}) = W_T + xy \cdot g(x, y, z)$.

From the picture, it is easy to check the following.

Lemma 8.19. The localized mirror functor $F_L : \mathcal{W}F([M_{L_{p,q}}/G_{L_{p,q}}]) \to \mathcal{M}F(x_1x_2x_3)$ sends $L_i$ to the matrix factorization $M_{L_i} = (S \overset{\delta_0}{\longrightarrow} S \overset{\delta_1}{\rightarrow} S)$ where

$$\delta_0 = x_i, \quad \delta_1 = \frac{x_1x_2x_3}{x_i}.$$ 

As before, we can also write them as

$$M_{L_i} = (S[\theta_{x_i}], (x_i \cdot l_{\theta_{x_i}} + W_{x_i} \cdot \theta_{x_i} \land)), \quad W_{x_i} = \frac{x_1x_2x_3}{x_i}.$$ 

For later purpose, we calculate hom complex by hand (in [AAE+13] it was proved using Orlov’s equivalence).

Lemma 8.20. The self-hom space of $M_{L_i}$ is quasi-isomorphic to a DG algebra of polynomial differential operators

$$\text{Hom}_{MF}(M_{L_i}, M_{L_i}) = \left(S[\theta_{x_i}], (\theta_{x_i} \land), D\right)$$

$$D(\partial_{\theta_{x_i}}) = W_{x_i}, \quad D(\theta_{x_i} \land) = x_i.$$ 

The cohomology of Floer complexes are given as follows:

$$H^i(\text{Hom}_{MF}(M_{L_i}, M_{L_i})) \simeq \left\{ \begin{array}{ll}
\mathbb{C}[x_1, x_2, x_3]/(x_i = W_{x_i} = 0) & i = j \\
\mathbb{C}[x_1, x_2, x_3]\cdot (x_i = x_j = 0) & i \neq j
\end{array} \right.$$ 

Proof. A computation of self-Floer complex is almost identical to that of chain type. A complex of morphism $\text{Hom}_{MF}(M_{L_i}, M_{L_j})$ is isomorphic to

$$D_0 \xleftarrow{D_0} S^{\otimes 2, even} \xrightarrow{D_1} S^{\otimes 2, odd}$$

where $D_0 = \begin{pmatrix} x_i & -x_j \\ W_{x_i} & -W_{x_i} \end{pmatrix}$, $D_1 = \begin{pmatrix} W_{x_i} & -x_j \\ W_{x_j} & -x_i \end{pmatrix}$. 
Therefore, we have

\[
\ker(D_0) / \text{im}(D_1) = \left\{ \begin{array}{l}
\{ (a, b) \in \mathbb{S}^{2, \text{even}} \mid a = b \} / S \{ x_i \} + S \{ W_{x_i} \} & (i = j) \\
\{ (a, b) \in \mathbb{S}^{2, \text{odd}} \mid x_i \cdot a = x_j \cdot b \} / S \{ x_j \} + S \{ W_{x_j} \} & (i \neq j)
\end{array} \right.
\]

\[
\simeq \begin{cases}
\mathbb{C}[x_1, x_2, x_3] / (x_i = W_{x_i} = 0) & (i = j) \\
0 & (i \neq j)
\end{cases}
\]

\[
\ker(D_1) / \text{im}(D_0) = \left\{ \begin{array}{l}
\{ (c, d) \in \mathbb{S}^{2, \text{odd}} \mid W_{x_i} \cdot c = x_i \cdot d \} / S \{ x_i \} & (i = j) \\
\{ (c, d) \in \mathbb{S}^{2, \text{odd}} \mid W_{x_i} \cdot c = x_j \cdot d \} / S \{ x_j \} + S \{ W_{x_j} \} & (i \neq j)
\end{array} \right.
\]

\[
\simeq \begin{cases}
\mathbb{C}[x_1, x_2, x_3] \cdot (x_i x_j x_k) / (x_i = x_j = 0) & (i = j) \\
0 & (i \neq j)
\end{cases}
\]

Now we can show that our mirror functor is an equivalence as before.

**Lemma 8.21.** The first-order part of the mirror functor is

\[
\mathcal{F}^L : CW^*(L_i, L_j) \to \text{Hom}_{\mathcal{A}}(\mathcal{M}_{L_i}, \mathcal{M}_{L_j})
\]

\[
a_i \mapsto x_i, \ b_i \mapsto x_i, \ c_{i,j} \mapsto \frac{x_1 x_2 x_3}{x_i x_j}.
\]

It is a quasi-isomorphism. Moreover, \(\mathcal{F}^L : \mathcal{W}[\{MC_{p,q} / \Gamma_{p,q}\}] \to \mathcal{A}(x_1 x_2 x_3)\) is a quasi-equivalence.

9. **BERGLUND–HÜBSCH HOMOLOGICAL MIRROR SYMMETRY FOR INVERTIBLE CURVE SINGULARITIES**

In this section, we will prove Theorem 5.3. We compute \(\Gamma_W\) explicitly and show that the localized mirror functor \(\mathcal{F}^L\) intertwines \(\Gamma_W\) and \(g(x, y, z)\). We enhance \(\mathcal{F}^L\) to an \(A_{\infty}\) equivalence \(\mathcal{G}^L\) to finish the proof.

9.1. **Computation of \(\Gamma_W\).** Recall \(\Gamma_W\) comes from the Reeb flow on the link \(L_{W, \delta}\). In this particular case, \(\Gamma_W\) is represented by a union of loops around punctures.

**Proposition 9.1.** A class \(\Gamma_W\) is given by a sum of Hamiltonian orbits, geometrically represented by a following element of \(\pi_1^{\text{orb}}(\mathbb{P}^1, \mathbb{P}^1_{a,b,c})\):

1. **Fermat type** \(\simeq \mathbb{P}^1_{p,q, \frac{pq}{gcd(p, q)}}\) : \(\Gamma_W \mapsto \gamma_3^{-1}\).
2. **Chain type** \(\simeq \mathbb{P}^1_{pq, q, \frac{pq}{gcd(p, q)}}\) : \(\Gamma_W \mapsto \gamma_1^{-p} + \gamma_3^{-1}\).
3. **Loop type** \(\simeq \mathbb{P}^1_{pq-1, pq-1, \frac{pq}{gcd(p, q)-1}}\) : \(\Gamma_W \mapsto \gamma_1^{-p} + \gamma_2^{-q} + \gamma_3^{-1}\).

**Proof.** Look at the covering homomorphisms \(\phi : \pi_1^{\text{orb}}(\mathbb{P}^1, \mathbb{P}^1_{a,b,c}) \to G_W\) in Proposition 6.5. If a winding number of \(\Gamma_W\) around a puncture corresponds to \(\gamma_1\) is \(m\), then the image \(\phi(\gamma_1^m)\) is equal to the monodromy \(g_W\). It is enough to find a negative integer \(k\) (because of the orientation) with minimal absolute value among those who satisfying

\[
\phi(\gamma_1^k) = g_W \in G_W.
\]
(1) **Fermat type** \( x^p + y^q \)

We have

\[
g_W = \phi(\gamma_3^{-1}) = (1, 1) \in \mathbb{Z}/p \times \mathbb{Z}/q.
\]

(2) **Chain type** \( x^p + xy^q \)

We have

\[
g_W = \phi(\gamma_1^{-1}) = \phi(\gamma_3^{-1}) = 1 - p \in \mathbb{Z}/pq.
\]

(3) **Loop type** \( x^p y + xy^q \)

We have

\[
g_W = \phi(\gamma_1^{-1}) = \phi(\gamma_2^{-1}) = \phi(\gamma_3^{-1}) = 1 - p \in \mathbb{Z}/pq - 1.
\]

\[\square\]

9.2. **Explicit computation of \( \Gamma_W \)-action on wrapped Fukaya category.** Since we have a clear representation of \( \Gamma_W \), we can calculate a cohomology of \( \mathcal{F}(W, G_W) \) quite explicitly. Although the definition of \( \mathcal{F}(W, G_W) \) is very complicated, its cohomology has a simple description. Namely, the cohomology of \( \mathcal{F}(W, G_W) \) is a wrapped Floer cohomology of the quotient \( [M_W/G_W] \), but the image of cap actions of \( \Gamma_W \) is killed off.

In the cohomology level, \( \Gamma_W \) acts on \( HW^*(L, L) \), and is the same as multiplication by \( CO_L(\Gamma_W) \in HW^*(L, L) \) by Proposition 9.2. For a non-compact Lagrangian \( L \), there is an obvious candidate for \( CO_L(\Gamma_W) \). Our \( \Gamma_W \) corresponds to time-1 periodic Reeb orbits of \( [L_{W,\delta}/G_W] \). In a similar way, we can apply the time-1 Reeb flow to generate Hamiltonian chords on the intersection points \( L \cap [L_{W,\delta}/G_W] \). We call it a **monodromy chord of \( L \)** and denote it by \( \gamma_{W,L} \in CW^*(L, L) \).

**Proposition 9.2.** We have \( CO_L(\Gamma_W) = \gamma_{W,L} \). Hence,

\[
\text{Hom}_{\mathcal{F}(W, G_W)}(L, L) \cong \text{Cone} \left( m_2(-, \gamma_{W,L}) : HW^*(L, L) \to HW^*(L, L) \right).
\]

**Proof.** By Lemma 7.2, \( CW^*(L, L) \) carries an \( H_1 \)-grading since \( L \) is simply connected. Also closed-open map preserves \( H_1 \)-grading as well. Note that for each component of \( \partial L \), there is the unique wrapped generator which has the same \( H_1 \)-grading as \( \Gamma_W \) on that conical end. The sum of these chords is \( \gamma_{W,L} \). Since our Milnor fiber is a Riemann surface, it is not hard to see that \( CO_L(\Gamma_W) \) is exactly \( \gamma_{W,L} \), following the standard example of cylinder (see [Pas19] for more details). \[\square\]

**Example 9.3.** Let us look at the explicit Lagrangian \( L \) for Fermat type \( W \) as in Lemma 8.3. Recall that we have

\[
CW^*(L, L) \cong T(a, b) / \mathcal{R}_{F_{p,q}} = \langle a \otimes a = \delta_{2,p}, b \otimes b = \delta_{2,q} \rangle
\]

with \( A_\infty \)-relation \( m_p(a, \ldots, a) = m_q(b, \ldots, b) = 1 \).

Since \( L \) intersect the boundary twice, we have \( \gamma_{W,L} = ab + ba \). An \( m_2 \)-multiplication is injective, so we conclude that the cone of \( m_2(-, \gamma_{W,L}) \) is a cokernel of it. We have

\[
\text{Hom}_{\mathcal{F}(W, G_W)}(L, L) \cong \langle 1, a, b, a \otimes b \rangle, \quad M_p(a, \ldots, a) = M_q(b, \ldots, b) = 1.
\]

Observe that we get a finite dimensional endomorphism space even though we start with a non-compact Lagrangian. This is because the action of \( \Gamma_W \) kills most of the wrapped generators. Also notice that the result matches to the computation of minimal \( A_\infty \)-model for \( \text{End}_{\text{MF}(W)}(k_{W})^{\text{stab}} \) in [Dyc11].
9.3. $\Gamma_W$ and the polynomial $g(x, y, z)$. We show that closed-open map image of orbit $\Gamma_W$ (with boundary $b$-deformations) provides the polynomial $g(x, y, z)$. For more precise formulation, let us first recall the notion of Kodaira-Spencer map defined by Fukaya-Oh-Ohta-Ono [FOOO16]. For a Lagrangian torus $L$ inside a compact toric manifold $M$, they constructed a map $KS: QH^*(M) \rightarrow \text{Jac}(W)$ from a closed-open map with boundary $L$. $T^n$-equivariant perturbation guarantees that the output is a multiple of the fundamental class $[L]$, and its coefficient gives an element $\text{Jac}(W)$. In [ACHL20], the case of $\mathbb{P}^1_{a,b,c}$ with $QH^*_\text{orb}(\mathbb{P}^1_{a,b,c})$ has been constructed, which we will use in our case.

Since our Milnor fiber is non-compact, we need to consider symplectic cohomology instead of quantum cohomology (Sei06). Following the general definition proposed in [CL20], it is natural to define Kodaira-Spencer map in our case as

$$KS^b: SH^*([M_W/G_W]) \rightarrow H^*(CF(L, L) \otimes \mathbb{C}[x, y, z], \tau_1^{b, b})$$

where we consider closed-open map with boundary on $(L, b)$. Here, the target of the map is nothing by Koszul complex of $W_L$. We omit its definition since we would need a proper definition of orbifold symplectic cohomology. Instead we just consider the image of $\Gamma_W$.

**Definition 9.4.** Kodaira-Spencer invariant of $\Gamma_W$ is defined as

$$KS^b(\Gamma_W) := \sum_l CO_l(\Gamma_W)(b, \ldots, b)$$

From [ACHL20] and [CL20], $KS^b(\Gamma_W)$ is a multiple of $[L]$, and its coefficient can be regarded as an element of $\text{Jac}(W_L)$. Let us explain how to compute this invariant. To relate the insertion of orbifold quantum cohomology generators and symplectic cohomology generators, we will use the work of Tonkonog [Ton19]. We may assume that $\Gamma_W$ is defined by autonomous Hamiltonian (see BO09).

**Proposition 9.5.**

$$KS^b(\Gamma_W) = 1_L \otimes g(x, y, z)$$

**Proof.** First, note that closed-open map preserves $\Omega$- and $H_1$-grading of $SH^*$ and $CF^*(L, L)$. Next, if we consider a compactification of $[M_W/G_W]$ into $\mathbb{P}^1_{a,b,c}$ holomorphic discs that contribute to the closed-open map from $QH^*(\mathbb{P}^1_{a,b,c}) \rightarrow \text{Jac}(W_L)$ has been worked out in [ACHL20]. Proceeding similarly, we can show that the image is a multiple of $[L]$ (from $\mathbb{Z}/2$-symmetry), and can be regarded as an element of $\text{Jac}(W)$. From the grading consideration, only possible contribution of $KS^b(\gamma^k)$ is a $k$-th power of a variable associated to the immersed corner of $L$ opposite to the puncture (multiplied by the fundamental class $[L]$).

We can see such a polygon in a picture explicitly as in Figure21.

1. Fermat type $F_{p,q}$: $KS^b(\Gamma_W) = z \cdot 1$
2. Chain type $C_{p,q}$: $KS^b(\Gamma_W) = (z - x^{p-1}) \cdot 1$
3. Loop type $L_{p,q}$: $KS^b(\Gamma_W) = (z - x^{p-1} - y^{q-1}) \cdot 1$

These polynomials are exactly $g(x, y, z)$ we want. More precisely, let $C$ be one of the punctures of $[M_W/G_W]$, and we replace a puncture $C$ (of branching order $N_C$), by a $\mathbb{Z}/N_C$ orbifold point. Then $\gamma^k_C$ corresponds to the orbifold cohomology class $[k/N_C]$ (coming from the twisted sector). The above (signed) computation indeed follows from the counting of orbifold holomorphic polygons with $[k/N_C]$ insertion in the interior according to Proposition 9.1. The orbifold $\mathbb{P}^1_{a,b,c}$ has a Riemann sphere as an underlying space, and orbifold holomorphic disc near the $[k/N_C]$ insertion is nothing but a holomorphic map from a disc to the Riemann sphere with tangency order $k$, which is Fredholm regular. Then we use Tonkonog’s domain stretching technique to compare the orbifold and $\Gamma_W$ invariants. In our case, the hypersurface...
\[ \Sigma \text{ in } \text{[Ton19]} \text{ is just a point (or points) playing the role of Donaldson hypersurface. Technically speaking, Tonkonog use variant of linear Hamiltonian, whereas we use quadratic Hamiltonian but since we are only concerned about the orbit } \Gamma_W, \text{ we can use the arguments by Seidel } \text{[Sei06]} \text{ that continuation map is compatible with closed-open map to relate these two settings.} \]

Let us explain more details assuming the familiarity with \text{[Ton19]}. By clever choice of sequence of Hamiltonians (called } S\text{-shaped) for the domain stretching, the standard } J\text{-holomorphic discs break into parts which share Reeb orbits as same asymptotics. Reeb orbits for } S\text{-shaped Hamiltonian are divided into types } I, II, III, IV_a, IV_b, \text{ depending on their positions with respect to Liouville collar. Key part of the proof is to show that only type } II \text{ Reeb orbit appears in the breaking. In our case, if breaking occurs at type } I, IV_a, IV_b \text{ Reeb orbits (which are constant orbits), then collecting the parts from this constant orbit to } \Sigma \text{ we get a non-trivial sphere that maps to } X. \text{ The starting polygon in } X \text{ do not intersect other vertices of } p^1_{a,b,c}, \text{ and this intersection number with perturbed } J\text{-holomorphic curve is positive, so the sphere should not intersect other vertices. Therefore, such a sphere cannot exist. This excludes these types of Reeb orbits as breaking orbits. The argument about type } III \text{ orbit using no escape lemma still applies to our case. One can see that any disc bubble would increase the intersection with vertices of the orbisphere, hence do not occur. The rest of the argument is the same as the reference and we obtain the desired comparison results. We refer readers to } \text{[Ton19]} \text{ for more details.} \]

9.4. Restriction to hypersurface } g = 0 \text{ is mirror to cap action of } \Gamma_W. \text{ Next, we prove that the mirror functor } \mathfrak{F}^L \text{ interwines the quantum cap action } \cap \Gamma_W (2.3) \text{ and the multiplication } \times g(x, y, z). \]

Let us start with the simplest but essential part of the whole theorem. Observe that for any Lagrangian } L, \text{ its mirror matrix factorization } M_L = \mathfrak{F}^L(L) \text{ carries two canonical endomorphisms. One is just a multiplication } \times g(x, y, z) : M_L \rightarrow M_L, \text{ while the other one is given as follows:}

\textbf{Definition 9.6.} An endomorphism } (\cap \Gamma_W)^b \text{ of } M_L \text{ is defined by}

\[ (\cap \Gamma_W)^b : M_L \rightarrow M_L, \quad \alpha \rightarrow \sum_L (\cap \Gamma_W)(\alpha, b, \ldots, b). \]

\[ (\cap \Gamma_W)^b - \times g(x, y, z) = [m^0_{i}, H^b]. \]

\textbf{Proposition 9.7.} There is a homotopy } H^L : M_L \rightarrow M_L \text{ such that}

\[ (\cap \Gamma_W)^b - \times g(x, y, z) = [m^0_i, H^b]. \]

\textbf{Proof.} Consider a moduli space similar to that of Proposition 2.4 such that an interior marked point is allowed to be in the lower part of the disc along a vertical geodesic as in Figure 22. We allow arbitrary
insertions of $b$ in the lower part of the boundary of the disc. The figure illustrates two of the boundary components of $\mathcal{M}$, which correspond to $(\cap \Gamma)^b$ and $\times g(x, y, z)$. For the latter, the disc bubble gives $\text{KS}^b(\Gamma_W) = 1_b \otimes g(x, y, z)$, and hence the main disc component has to be a constant disc, and it corresponds to the multiplication by $g(x, y, z)$. There are two additional boundary components which contributes to $[m^0_{1, b}, H^L]$. This proves the proposition.

![Diagram](image)

**Figure 22.** Deformed cap action and Kodaira-Spencer invariant $g(x, y, z)$

Proposition 9.7 can be generalized to a bimodule homomorphism. First, notice that an $A_{\infty}$-functor $\mathcal{F}^L$ makes $\mathcal{M}(W^T + xyg)$ a $\mathcal{W} \mathcal{F}$ $- \mathcal{W} \mathcal{F}$ bimodule with respect to which $\mathcal{F}^L$ is a bimodule homomorphism. We view a multiplication $\times g$ as a bimodule homomorphism from $\mathcal{M} \mathcal{F}$ to itself.

**Proposition 9.8.** A following diagram of bimodule homomorphisms commutes up to homotopy.

$$
\begin{array}{ccc}
\mathcal{W} \mathcal{F}(M_W/G_W) & \xrightarrow{\cap \Gamma_W^b} & \mathcal{W} \mathcal{F}(M_W/G_W) \\
\downarrow_{\mathcal{F}^L} & & \downarrow_{\mathcal{F}^L} \\
\mathcal{M} \mathcal{F}(W^T + xyg) & \xrightarrow{\times g} & \mathcal{M} \mathcal{F}(W^T + xyg)
\end{array}
$$

**Proof.** In the course of the proof, we use the notation

$$m^\text{bimod}(a_1, \ldots, a_k, b_1, \ldots, a_{k_1 + k_2 + 1}) = (-1)^{\alpha_1} m_{k_1 + k_2 + 1}(a_1, \ldots, a_k, b_1, \ldots, a_{k_1 + k_2 + 1})$$

$$\mathcal{F}^L_{\text{bimod}}(a_1, \ldots, a_k, b_1, \ldots, a_{k_1 + k_2 + 1}) = (-1)^{\alpha_1} \mathcal{F}^L_{k_1 + k_2 + 1}(a_1, \ldots, a_k, b_1, \ldots, a_{k_1 + k_2 + 1})$$

to indicate that $m$ or $\mathcal{F}^L$ are viewed as a bimodule structures and homomorphism between them. Here, $\alpha_j = \sum_{i=1}^j (\deg a_i - 1)$. As an intermediate step, we introduce the following operations (See Figure 23):

1. generalize $(\cap \Gamma_W)^b$ as follows:

$$(\cap \Gamma_W)^b : \text{CW}^*(L_0, L_1) \otimes \cdots \otimes \text{CW}^*(L_{k_1 - 1}, L_k) \to \text{Hom}(M_{L_0}, M_{L_k}),
(\cap \Gamma_W)^b(a_1, \ldots, a_k)(a) := \sum_i (\cap \Gamma_W)^{k_1 + k_2 + 1}(a_1, \ldots, a_k, a, b, \ldots, b).
$$

2. also generalize $H^L$ by allowing $\mathcal{W} \mathcal{F}$ inputs in the upper-boundary of the popsicle maps as in the Figure 23

3. define a pre-homomorphism of bimodules $\tilde{H}^L$ as follows:

$$\tilde{H}^L : \mathcal{W} \mathcal{F} \otimes k_1 \otimes \mathcal{W} \mathcal{F} \otimes k_2 \to \mathcal{M} \mathcal{F}(W^T + xyg),
\tilde{H}^L(a_1, \ldots, a_k, b_1, \ldots, a_{k_1 + k_2 + 1})(a) := \sum_i (\cap \Gamma_W)^{k_1 + k_2 + 2 + i}(a_1, \ldots, b, \ldots, a_{k_1 + k_2 + 1}, a, b, \ldots, b)$$
The result follows from codimension one boundary configurations associated to (2) and (3), which are also illustrated in Figure 23. Notice that $K^b(\Gamma_W)$ and a disc potential is a multiple of unit. If degenerations involve such configurations, then it contributes to zero except a single important case; $k = 0$ for (2). That is Proposition 9.7 with Figure 22. Otherwise, we can skip them.

Analyzing the case $k > 0$ for (2), we get

\[
(\cap \Gamma_W)^b(a_1, \ldots, a_k) = \sum (-1)^j H^k_1(a_1, \ldots, a_j, m_1(a_j+1), \ldots, a_k) + \sum (H^1 l(a_1, \ldots, a_j), \mathcal{F}^l(a_j+1), \ldots, a_k) + \sum m_2(\mathcal{F}^l(a_1, \ldots, a_j), H^l(a_j+1), \ldots, a_k).
\]

This relation says that $H^k_1$ provides a null-homotopy for $(\cap \Gamma_W)^b$. Combined with Proposition 9.7, we conclude that $H^k_1$ provides a homotopy between $(\cap \Gamma_W)^b$ and $\times g(x, y, z)$. Similarly,

\[
\sum m_2((\cap \Gamma_W)^b(a_1, \ldots, a_j), \mathcal{F}^l_{bimod}(a_j+1, \ldots, a_{k_1+k_2+1})) + \sum (\cap \Gamma_W)^b(a_1, \ldots, (\cap \Gamma_W)^b(a_j+1, \ldots, a_{k_1+k_2+1})) + \sum (H^l_1(a_1, \ldots, a_j), \mathcal{F}^l(a_j+1, \ldots, a_{k_1+k_2+1})) + \sum m_2(\mathcal{F}^l(a_1, \ldots, a_j), H^l(a_j+1, \ldots, a_{k_1+k_2+1})) + \sum (\cap \Gamma_W)^b(a_1, \ldots, m_{bimod,k+1}(a_j+1, \ldots, a_{j+k}), \ldots, a_k).
\]

This relation tells us that $\tilde{H}^l_1$ provides a homotopy between $\mathcal{F}^l_{bimod}(\cap \Gamma_W)$ and $(\cap \Gamma_W)^b \circ \mathcal{F}^l_{bimod}$.

---

**Figure 23.** $H^l_1$, $\tilde{H}^l_1$ and their boundary configurations
Combine these two and define $K^L := \tilde{\mathcal{F}}^L + H^L \circ \mathcal{F}_{\text{bimod}}^L$. It provides a desired homotopy between $\mathcal{F}_{\text{bimod}} \circ (\cap \Gamma_W)$ and $(xg) \circ \mathcal{F}_{\text{bimod}}^L$, which proves the theorem. 

From the algebraic non-sense, we obtain Theorem 5.5 as a corollary of the Proposition 9.8 (Notice that our polynomial $g$ is of the form $z - f(x, y)$ regardless of a type of $W$. Therefore we can apply Corollary 3.7 to $\mathcal{MF}(W^T + xyg)$.)

**Corollary 9.9.** The following diagram commutes up to homotopy:

$$
\begin{array}{cccc}
\mathcal{MF}(\{M_W/G_W\}) & \xrightarrow{\cap \Gamma_W} & \mathcal{MF}(\{M_W/G_W\}) & \xrightarrow{\mathcal{F}} \mathcal{F}(W, G_W) \\
\mathcal{MF}(W^T + xyg) & \xrightarrow{g} & \mathcal{MF}(W^T + xyg) & \xrightarrow{\mathcal{F}} \mathcal{MF}(W^T + xyg) \\
\end{array}
$$

Here we define the bimodule map $\tilde{\mathcal{F}}^L$ as

$$\tilde{\mathcal{F}}^L(a, b, \ldots) := (\mathcal{F}^L(a, b, \ldots) + K^L(a, b, \ldots)) + e\mathcal{F}^L(a, b, \ldots).$$

Each row is a distinguished triangle of bimodules. All vertical lines induce quasi-isomorphisms.

Moreover, we also have

**Lemma 9.10.** The restriction $i^* : \text{MF}(W^T + xyg) \to \text{MF}(W^T)$ is essentially surjective.

**Proof.** $W^T$ is an isolated singularity. Therefore $\text{MF}(W^T)$ is generated by the matrix factorization of stabilized residue field $k^*_{\text{stab}}$ (see [Dyc11]). It is enough to find matrix factorization $M \in \text{MF}(W^T + xyg)$ such that $M|_{g=0} = k^*_{\text{stab}}$.

Fermat type $M = \left( \begin{array}{cc} x & y \\ -y^q & x^p + yz \end{array} \right) \times \left( \begin{array}{cc} x^{p-1} + yz & -y \\ y^{q-1} & x \end{array} \right)$

Chain type $M = \left( \begin{array}{cc} x & y \\ -y^q & yz \end{array} \right) \times \left( \begin{array}{cc} yz & -y \\ y^{q-1} & x \end{array} \right)$

Loop type $M = \left( \begin{array}{cc} x & y \\ 0 & yz \end{array} \right) \times \left( \begin{array}{cc} yz & -y \\ 0 & x \end{array} \right)$

In fact, these matrix factorizations are $M_L = \mathcal{F}^L(L)$ for explicit Lagrangians $L$ (for example, see Remark 8.6 for the Fermat case).

Lemma 9.10 implies $\mathcal{MF}(W^T + xyg)|_g$ is another $A_\infty$-model for $\mathcal{MF}(W^T)$. This establishes an equivalence between $\mathcal{F}(W, G_W)$ and $\mathcal{MF}(W^T)$ as $A_\infty$-bimodules over $\mathcal{MF}(\{M_W/G_W\})$.

9.5. **Proof of Theorem 5.3** Observe that $L$, as an object of $\mathcal{F}(W, G_W)$, is still a weakly unobstructed Lagrangian with weak bounding cochains $b = xX + yY + zZ$ (placed in the first component), with potential function $W_1$. Therefore, we can define the localized mirror functor with respect to $(L, b)$ from $\mathcal{F}(W, G_W)$.

**Definition 9.11.** Define an $A_\infty$-functor

$$\mathcal{G}^L : \mathcal{F}(W, G_W) \to \mathcal{MF}(W^T + xyg)$$

as follows: at the level of objects, it is defined as

$$\mathcal{G}^L(L) = \text{Cone} \left( M_L[H^L] \xrightarrow{(\cap \Gamma_W)^b} M_L \right) \simeq M_L[e]/\left( \begin{array}{c} e^2 = 0 \\ de = (\cap \Gamma_W)^b \end{array} \right)$$
Denote an element of $\mathcal{G}^L(L)$ by $\alpha + \epsilon \beta$. Then we define

$$[\mathcal{G}^L_k(a_1, \ldots, \epsilon b_i, \ldots, \epsilon b_j, \ldots, a_k)] (\alpha + \epsilon \beta) = \sum_l M_{l+k+2}(a_1, \ldots, \epsilon b_i, \ldots, \epsilon b_j, \ldots, a_k, \alpha + \epsilon \beta, b, \ldots, b)$$

A target of the localized mirror functor $\mathcal{G}^L$ is just $\mathcal{M}_\mathcal{F}(W^T + x y g)$. In the ideal case, the image of $\mathcal{G}^L$ would lie on $\mathcal{M}_\mathcal{F}(W^T + x y g)|_g = \mathcal{M}_\mathcal{F}(W^T)$ in a direct manner, but because $(\cap \Gamma_W)^b$ is only homotopic to the multiplication by $g$, this does not hold.

We will show that this is true at the level of homotopy categories. To be more precise, recall from Definition 9.6 that

- objects of $\mathcal{M}_\mathcal{F}(W^T + x y g)|_g$ should be of the form Cone$(g)$. While $\mathcal{G}^L(L)$ is not, it is isomorphic to such matrix factorization. See Lemma 9.12.
- morphisms are of the form $a + \epsilon b \in \text{Hom}(M_1[\epsilon], M_2[\epsilon])$. As a $2 \times 2$ matrix, it only consists of

$$\begin{pmatrix} S & 0 \\ T & S \end{pmatrix} \in \text{Hom}(M_1 \oplus M_1[1], M_2 \oplus M_2[1]), \quad S, T \in \text{Hom}(M_1, M_2).$$

Although a matrix associated to $\mathcal{G}^L_k(a + \epsilon b)$ does not have such a form, we show that it does modulo coboundaries. See Proposition 9.13.

**Lemma 9.12.** $\mathcal{G}^L(L)$ is isomorphic to $\text{Cone}(g : M_L[1] \to M_L)$.

**Proof.** The relation $(\cap \Gamma_W)^b - (\times g) = [m_1^0, H^L]$ of Proposition 9.7 implies

$$I := \begin{pmatrix} \text{Id} & H^L \\ 0 & \text{Id} \end{pmatrix} : \text{Cone}((\cap \Gamma_W)^b : M_L[1] \to M_L) \to \text{Cone}(g : M_L[1] \to M_L)$$

is an isomorphism of matrix factorizations with an inverse $I^{-1} = \begin{pmatrix} \text{Id} & -H^L \\ 0 & \text{Id} \end{pmatrix}$.

**Lemma 9.12** induces a chain isomorphism

$$\mathcal{M}_\mathcal{F}(\mathcal{G}^L(L_1), \mathcal{G}^L(L_2)) \cong \text{Hom}(\mathcal{M}_\mathcal{F}\left(\begin{array}{cc} M_1[1] & M_2[1] \\ \xrightarrow{g} & \xrightarrow{g} \\ M_1 & M_2 \end{array}\right), \phi \to I \circ \phi \circ I^{-1}).$$

Under this identification, we can show that the image of $\mathcal{G}^L_1$ has a desired shape up to homotopy.

**Proposition 9.13.** Suppose $a + \epsilon b \in \text{Hom}(\mathcal{F}(W, G_W)(L_1, L_2))$ is a cocycle. Then $\mathcal{G}^L_1(a + \epsilon b)$ is contained in $\text{Hom}(\mathcal{M}_\mathcal{F}(W^T + x y g)|_g (M_{L_1}, M_{L_2})$ up to coboundary.

**Proof.** For $a \in \text{Hom}(\mathcal{F}(W, G_W)(L_1, L_2)$, the morphism $\mathcal{G}^L_1(a)$ under the isomorphism (9.1) is given as;

$$a + \epsilon \beta \mapsto (A(a) + B(\beta)) + \epsilon A(\beta),$$

$$A(a) = \mathcal{F}^L(a)(a),$$

$$B(\beta) = \sum_l M^{a_1}_{l+2+l}(a, \epsilon \beta, b, \ldots, b) + [H^L, \mathcal{F}^L_1(a)](\beta).$$

Notice that $\sum_l M^{a_1}_{l+2+l}(a, \epsilon \beta, b, \ldots, b)$ is equal to $(\cap \Gamma_W)^b(a)(\beta)$ up to sign. As a $2 \times 2$ matrix, the morphism has a form $\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$, and we want a null homotopy for $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. 

For $eb \in \text{Hom}_{\mathcal{F}(W,G_{\text{pt}})}(L_1, L_2)$, the situation is more complicated. The morphism $\mathcal{H}^l_1(\alpha b)$ under the isomorphism $\mathcal{H}^l_1$ is given as:

$$\alpha + e\beta \rightarrow C(\alpha) + D(\beta) + e \{ E(\alpha) + F(\beta) \},$$

$$C(\alpha) = \sum_{l} M^a_{2+1}(eb, \alpha, b, \ldots, b) + m_2(H^k, \mathcal{F}^l_{1}(b))(\alpha),$$

$$D(\beta) = \sum_{l} M^a_{2+1}(eb, \beta, b, \ldots, b) + H^k, \sum_{l} M^a_{2+1}(eb, b, \ldots, b) \} (\beta) + m_2(H^k, \mathcal{F}^l_{1}(b), H^k) (\beta)$$

$$+ m_2 \left( H^k, \sum_{l} M^a_{2+1}(b, \epsilon, b, b, \ldots, b) \right),$$

$$E(\alpha) = \mathcal{F}^l_{1}(b)(\alpha)$$

$$F(\beta) = \sum_{l} M^a_{2+1}(eb, \beta, b, \ldots, b) + m_2(\mathcal{F}^l_{1}(b), H^k)(\beta) + \sum_{l} M^a_{2+1}(b, \epsilon, b, b, \ldots, b).$$

Notice that $\sum_{l} M^a_{2+1}(eb, \beta, b, \ldots, b)$ is equal to $\tilde{H}^k(b)(\beta)$ up to sign. As a $2 \times 2$ matrix, it has a form $\begin{pmatrix} C & D \\ E & F \end{pmatrix}$, so we should find a null homotopy for $\begin{pmatrix} 0 & D \\ 0 & C - F \end{pmatrix}$.

A null-homotopy for $B$ or $(C - F)$ is rather easy to find. We will write it down explicitly, but the reader would notice that they are a special case of Proposition 9.8. But a null-homotopy for $D$ requires something more, and we consider the following three auxiliary operations to find one (see Figure 24):

1. Define $\mathcal{K}^k: \text{Hom}_{\mathcal{F}(W,G_{\text{pt}})}(L_1, L_2) \rightarrow \text{Hom}_{\mathcal{H}, \mathcal{F}}(M_{L_1}, M_{L_2})$ as follows: for $a + e\beta \in \text{Hom}_{\mathcal{F}(W,G_{\text{pt}})}(L_1, L_2)$ and $\beta \in M_{L_1}$, $\mathcal{K}^k(a + e\beta) (\beta)$ is defined similarly to $\sum_{l} M^a_{2+1}(a + e\beta, b, b, b, b)$, but we allow an interior marked point on the geodesic connecting $\beta$ and the output to be in the lower part of the disc along a vertical lines. Notice that $\mathcal{K}^k(a)$ is equal to $H^k(a)$ up to sign, while $\mathcal{K}^k(e\beta)$ is new.

2. Define $\mathcal{L}^k \in \text{Hom}_{\mathcal{H}, \mathcal{F}}(M_{L_1}, M_{L_2})$ by the moduli space $\mathcal{P}_{1+1, \phi,[1,2]}^w(\bullet, b, \ldots, b)$ with $\phi(1) = \phi(2) = 1$, but we allow two interior insertions to move down along the geodesics perpendicular to the (unique) popsicle stick.

3. Similarly, define $L^k \in \text{Hom}_{\mathcal{H}, \mathcal{F}}(M_{L_1}, M_{L_2})$ is defined as same as $\mathcal{L}^k$, but we allow only one of the interior insertions move away from the popsicle stick.

![Figure 24](image_url)  

**Figure 24.** Homotopies $\mathcal{K}^l_1$, $\mathcal{L}^l_1$ and $L^l_1$. 
From codimension one boundary analysis, we get the following relations (see Figure 25):

\[
[m_{1}^{0,b}, \mathcal{K}^{L}(a)](\beta) + \mathcal{X}(m_{1}(a))(\beta) = [H^{\mathcal{L}}, \mathcal{L}^{L}(a)](\beta) + \sum_{l} M_{l+1}^{a}(a, \epsilon \beta, b, \ldots, b),
\]

\[
[m_{1}^{0,b}, \mathcal{X}^{L}(e b)](\beta) + \mathcal{X}(M_{1}(e b))(\beta) = \sum_{l} M_{l+1}^{a}(e b, \epsilon \beta, b, \ldots, b) + [H^{\mathcal{L}}, \sum_{l} M_{l+1}^{a}(e b, \bullet, b, \ldots, b)](\beta)
\]

\[
+ m_{2} \left( (\cap_{W})^{b} \sum_{l} M_{l+1}^{a}(b, \epsilon \beta, b, \ldots, b) \right)(\beta) + m_{2}(L^{L}, \mathcal{L}_{1}(b))(\beta),
\]

\[
[m_{1}^{0,b}, \mathcal{L}^{L}] = m_{2}(H^{L}, H^{L}) + L^{L}.
\]
Now suppose $M_1(a) = m_1(a) = 0$. The first relation alone implies (as a $2 \times 2$ matrix)
\[
\varphi_1^a(a) \rightarrow \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + d \begin{pmatrix} 0 & \mathcal{K}^L(a) \\ 0 & 0 \end{pmatrix},
\]
where $d$ is a differential of a matrix factorization categories. Next, suppose that $M_1(\epsilon b) = 0$. Then if we combine above three relations plus Proposition 9.7, we get
\[
\varphi_1^a(\epsilon b) \rightarrow \begin{pmatrix} C & D \\ E & F \end{pmatrix} = \begin{pmatrix} C & 0 \\ E & C \end{pmatrix} + d \begin{pmatrix} 0 & \mathcal{K}^L(\epsilon b) + H^L \circ \mathcal{K}^L(b) + \mathcal{L}^L \circ \mathcal{F}^L(b) \\ -\mathcal{K}^L(b) \end{pmatrix}.
\]

Corollary 9.14. (Theorem 5.3) An $A_\infty$-functor $\mathcal{G}^L : \mathcal{F}(W, G_W) \rightarrow \mathcal{M}(W^T + xyg)$ induces
\[
H(\mathcal{G}^L) : D^\mathcal{F}(W, G_W) \rightarrow \mathcal{HM}(W^T + xyg)|_g \simeq \mathcal{HM}(W^T)
\]
which is a quasi-equivalence between homotopy categories.

Proof. Lemma 9.12 said that $\mathcal{G}^L(L)$ is isomorphic to the object of $\mathcal{M}(W^T + xyg)|_g$. Proposition 9.13 implies that $\mathcal{G}^L$ induces
\[
H(\mathcal{G}^L) : H^* \Hom_{\mathcal{F}(W, G_W)(L_1, L_2)} = H^* \Hom_{\mathcal{HM}(W^T + xyg)|_g} \begin{pmatrix} M_{L_1}[1] & \rightarrow & M_{L_1} \\ \rightarrow & & \rightarrow \end{pmatrix}.
\]
We know $H(\mathcal{F}^L)$ is an isomorphism of cohomologies, and so is $H(\mathcal{G}^L)$ by a simple filtration argument. The proof is now complete.

9.6. Proof of Corollary 5.4 Corollary 5.4 follows form the compatibility between $\mathcal{G}^L$ and the group action, which we are now going to explain.

In the setting of subsection 4.5, let $L$ be an object of $\mathcal{E}^H$ with weak bounding cochains $b = \sum_i x_i X_i$ with a potential function $W_i$. Then localized mirror functor is an $A_\infty$-functor $\mathcal{F}^L : \mathcal{E}^H \rightarrow \mathcal{M}(W_i)$, which takes an object $K$ to the matrix factorization $\mathcal{F}^L(K) = (\text{hom}(K, L), -m_1^0, b)$. Notice that $R = \mathbb{C}[x_1, \ldots, x_n]$ is canonically equipped with an action of a character group $H^*$.

Definition 9.15. A category of $H^*$-equivariant matrix factorizations $\text{MF}(W, H^*)$ consists of the following:

- its objects are $\mathbb{Z}/2$-graded matrix factorizations $(P^*, \delta_P)$ where $P$ is equipped with an $H^*$-action, and $\delta_P$ is $H^*$-equivariant.
- $\Hom_{\text{MF}(W, H^*)}((P^*, \delta_P), (Q^*, \delta_Q)) := \{\text{Hom}_{R}(P^*, Q^*|_{H^*}, d)\}$ with an usual differential $d$.

We denote an $A_\infty$-version of this category by $\mathcal{M}(W, H^*)$.

Notice that we can identify $\mathcal{E}$ with $\mathcal{E}^H \times H^*$ and also identify $\mathcal{M}(W_0, H^*)$ with $\mathcal{M}(W_0 \times H^*)$. Therefore, a $H^*$-equivariant lift $\mathcal{F}^L \times H^* : \mathcal{E} \rightarrow \mathcal{M}(W_0, H^*)$ exists if $\mathcal{F}^L$ is $H^*$-equivariant. Weak bounding cochains $b = \sum_i x_i X_i$ are always invariant under $H^*$-action since the action on $x_i$ and $X_i$ are opposite to each other. This implies that $\mathcal{F}^L$ is $H^*$-equivariant. For example, since $\chi \cdot b = b$, we obtain that $m_1^0, b (\chi \cdot a) = \chi \cdot m_1^0, b (a)$.

Proposition 9.16. An $A_\infty$-functor $\mathcal{G}^L : \mathcal{F}(W, G_W) \rightarrow \mathcal{M}(W^T + xyg)$ can be lifted to
\[
\mathcal{G}^L, G^T : \mathcal{F}(W, G) \rightarrow \mathcal{M}(W^T + xyg, G^T)
\]
for any subgroup $G < G_W$ which induces a derived equivalence
\[
H(\mathcal{G}^L, G^T) : D^\mathcal{F}(W, G) \rightarrow \mathcal{HM}(W^T, G^T).
\]
Proof. Apply the above discussion to $H = G_W$, $\mathcal{G}^H = \mathcal{W F}([M_W/G_W])$ and $\mathcal{M F}(W_L) = \mathcal{M F}(W^T + xyg)$. Notice that then for any $G < G_W$, we have a lift

$$\mathcal{G}^{L,G^T} : \mathcal{W F}([M_W/G_W]) \times G^T \to \mathcal{M F}(W^T + xyg) \times G^T.$$ 

We also observed that $\Gamma_W$-insertions are compatible to $G_W$-action in subsection 4.5. So we also have a corresponding lift $\mathcal{G}^{L,G^T} : \mathcal{W F}(W, G) \to \mathcal{M F}(W^T + xyg, G^T)$ which induces

$$H(\mathcal{G}^{L,G^T}) : D^a \mathcal{W F}(W, G) \to \mathcal{H M F}(W^T, G^T).$$

This functor induces obviously fully-faithful, so we have to show that its image generates the target. From Proposition 2.3.1 of [PVT8], we conclude that $\text{MF}(W^T, G^T)$ is generated by a sum of matrix factorizations of stabilized residue fields twisted by characters of $G$:

$$\bigoplus_{\chi \in G^T} k_{W^T}^{\text{stab}} \otimes \chi$$

Since $\mathcal{G}^{L}$ hits $L_{W^T}^{\text{stab}}$, its equivariant lift also hits exactly this generator. \qed

10. Relation to Auslander-Reiten theory of Cohen-Macaulay modules

For ADE singularity $W^T$, its matrix factorization category is of finite type, which means that there are only finitely many indecomposable ones. Its Auslander-Reiten quiver has been described by Yoshino [Yos90], and we find a corresponding Lagrangian Floer theory under Berglund-Hübsch duality. More precisely, we find non-compact Lagrangians in the Milnor fiber of $W$ for each indecomposable matrix factorizations and realize all of Auslander-Reiten almost split exact sequences as Lagrangian surgery exact sequences.

10.1. Auslander-Reiten theory. Recall that a ring $R$ is Cohen-Macaulay (CM) if its Krull dimension equals its depth. For example, complete intersections give CM rings. CM rings and their modules play a central role in the theory of commutative algebra. $R$-module $M$ whose depth (minimal length of projective resolution) equals dimension of $R$ is called maximal CM (in short MCM) module.

Auslander-Reiten(AR) developed a classification theory of indecomposable objects and in particular defined an associated quiver, which is called AR quiver $Q$. Vertices of $Q$ are indecomposable MCM modules and its arrows are given by irreducible morphisms (roughly the minimal morphisms that do not factor nontrivially). Also, there are dotted arrows, called AR translation $\tau$. Given an indecomposable module $M$ and its AR translation $\tau(M)$, there is an associated AR exact sequence:

$$0 \to \tau(M) \to N \to M \to 0$$

where $N$ is the direct sum of MCM modules that are sources of arrows to $M$ (or targets of arrows from $\tau(M)$). Therefore, one can read off all AR exact sequences whenever an AR quiver is given.

On the other hand Eisenbud proved that MCM modules of $R$ correspond to $\mathbb{Z}/2$-graded matrix factorizations of the defining function via periodic resolution, and hence the above classification results can be translated into those of matrix factorizations [Eis80]. We refer readers to the excellent book by Yoshino [Yos90] for more details.

10.2. Localized mirror functor and AR exact sequence. In this section, we remind readers that we work with $\mathbb{Z}/2$-graded $(A_\infty$ or DG) categories. Given an AR sequence,

$$(10.1) \quad 0 \to \tau(M) \to N \to M \to 0$$

there exist a corresponding extension element $a \in \text{Ext}^1(M, \tau(M))$. Given a $\mathbb{Z}/2$-graded matrix factorization $(d_{01} : P^1 \to P^0, d_{01} : P^0 \to P^1)$ of $W$, we may write $(d_{01}, d_{01})$ as a pair of polynomial matrices $(\phi, \psi)$ satisfying $\phi \cdot \psi = \psi \cdot \phi = W \cdot id$. We recall the following fact.
Proposition 10.1 (Yos90 Proposition 3.11, Proposition 7.7). For a matrix factorization \( M = (\phi, \psi) \) of \( W \), its AR translation \( \tau(M) \) is given by \((\psi, \phi)\).

We observe that a Lagrangian with opposite orientation corresponds to the Auslander translation.

Lemma 10.2. If a Lagrangian \( L \) maps to \( M = (\phi, \psi) \) under localized mirror functor, then its orientation reversal \( L[1] \) maps to a matrix factorization \( \tau(M) \).

Proof. Localized mirror functor takes \( L \) to a matrix factorization \( P^0 = CF^0(L, \mathbb{L}), P^1 = CF^1(L, \mathbb{L}) \) with \( d = -m^0_{1, b} \). For the orientation reversal \( L[1], CF^*(L, \mathbb{L}) \) and \( CF^*(L[1], \mathbb{L}) \) has the same set of generators, but with opposite \( \mathbb{Z}/2 \)-grading, and \( m^0_{1, b} \) is modified accordingly (this is given by the same polygon). It remains to check the related signs. Since orientation of \( L \) does not change at \( X, Y, Z \)-corner of \( L \), there are only 4 possible orientation choices for such a polygon, and we can check each case. From the sign convention of Fukaya category of surface, the orientation of \( L \) does not contribute to sign. The only difference after orientation reversing is that the degrees of first input and output are interchanged. But the sum of those two degrees is still 1, so total sign of the polygon is not changed. This proves the claim.

Now, we want to relate AR exact sequences with Lagrangian surgery exact sequences. First, let us recall that a Lagrangian surgery can be related to a cone. The following lemma is a modification of Lemma 5.4 of [Abo08].

![Figure 26. Lagrangian surgery at c](image)

Lemma 10.3. Let \( L_1 \) and \( L_2 \) be unobstructed non-compact curves which intersect transversally and minimally at a single point \( c \) in the interior so that \( c \in CW^1(L_1, L_2) \), or \( L_1 \) and \( L_2 \) are disjoint but conical to the same puncture, with the wrapped generator \( c \in CW^1(L_1, L_2) \) as in Figure 26. Then, \( L_3 \) that is obtained after Lagrangian surgery at \( c \) is isomorphic to the twisted complex \( Cone(c) := L_1 \sim L_2 \).

Proof. We find an explicit isomorphism as follows. We define \( a \in CW^0(L_3, L_1), b \in CW^0(L_2, L_3) \) as in Figure 26. In the first case, \( a, b \) are taken to be the sum of two generators, \( a = a_1 - a_2 \) and \( b = b_1 + b_2 \). Then we may regard \( a, b \) as \( a \in Hom^0(L_3, Cone(c)), b \in Hom^0(Cone(c), L_3) \). Then, it is enough to show that \( m^1_{T, w}(a, b) = 1_{L_3} \) and \( m^1_{T, w}(b, a) = 1_{Cone(c)} = 1_{L_1} + 1_{L_2} \). By definition of \( m^1_{T, w} \) for twisted complexes, we have \( m^1_{T, w}(a, b) = m_3(a, c, b) = 1_{L_3} \), and \( m^1_{T, w}(b, a) = m_3(c, b, a) + m_3(b, a, c) = 1_{L_1} + 1_{L_2} \). We can prove similarly when the orientations of \( L_1 \) and \( L_2 \) are reversed.

Lemma 10.4. In the setting of previous lemma, we have an exact triangle of matrix factorizations for \( W_L \).

\[
(CW^*(L_1, \mathbb{L}), -m^0_{1, b}) \xrightarrow{\Phi^1_{\text{f}}(a)} (CW^*(L_3, \mathbb{L}), -m^0_{1, b}) \xrightarrow{\Phi^1_{\text{f}}(b)} (CW^*(L_2, \mathbb{L}), -m^0_{1, b}) \rightarrow
\]

We may set \( g(x, y, z) = 0 \) in the above to obtain the exact triangle for \( W_T \).
Proof. Those two triangles are just

\[ \mathcal{F}^L(L_1) \to \mathcal{F}^L(L_3) \to \mathcal{F}^L(L_2) \to, \quad \text{and} \quad (i^* \circ \mathcal{F}^L)(L_1) \to (i^* \circ \mathcal{F}^L)(L_3) \to (i^* \circ \mathcal{F}^L)(L_2) \to. \]

The proposition follows from the fact that a localized mirror functor \( \mathcal{F}^L \) and the restriction \( g(x, y, z) = 0 \) are both exact (see Proposition 3.5). Notice that the second exact triangle is in fact isomorphic to

\[ \mathcal{F}^L(L_1) \xrightarrow{\phi^L(a)} \mathcal{F}^L(L_3) \xrightarrow{\phi^L(b)} \mathcal{F}^L_1(L_2) \to. \]

\[ \square \]

If an AR sequence (10.1) is the same as (10.2), we will say that AR sequence is realized as Lagrangian surgery exact sequence. In this section, we prove the following.

**Theorem 10.5.** For ADE curve singularity \( W^T \), we find explicit Lagrangians in the Milnor fiber of \( W \) corresponding to indecomposable matrix factorizations in the AR quiver. Also, all AR sequences of matrix factorizations for ADE curve singularities \( W^T \) can be realized as Lagrangian surgery exact sequence for \( W \). Namely, for any AR exact sequence (10.1), we can find Lagrangians \( L_1, L_3, L_2 \) in \([M_W/G_W] \) (with a Floer generator \( c \in CW^1(L_1, L_2) \)) such that (10.2) can be identified with (10.1) in the cohomology category.

**Remark 10.6.** Due to \( A_\infty \) convention for matrix factorization category, the direction of a morphism \( c \) in Floer theory is the opposite of \( \alpha \in \text{Ext}^1(M, \tau(M)) \).

**Remark 10.7.** For an \( A_\infty \) functor \( \mathcal{F}^L|_{g=0} : \mathcal{W} \mathcal{F}([M_W/G_W]) \to \mathcal{M}_T(W^T) \), we will find several non-isomorphic Lagrangians mapping to the same matrix factorization. This may look strange but this is because \( \mathcal{F}^L|_{g=0} \) is not an equivalence. In \( \mathcal{F}(W, G_W) \), these Lagrangians become isomorphic objects. Algebraically speaking, if \( \alpha \beta + \beta \alpha \) is in image of quantum cap action and hence set to vanish in \( \mathcal{F}(W, G_W) \), then \( \text{Cone}(\alpha \beta) \) and \( \text{Cone}(\beta \alpha) \) become isomorphic even though they may not be isomorphic in the original Fukaya category.

In our definition of \( \mathcal{M}_T(W^T) \), trivial matrix factorization \( (W^T, 1) \) is homotopic to a zero object, but for AR theory in [Yos90], \( R = \mathbb{C}[x, y]/W^T \) (cokernel of the above factorization) is taken as an indecomposable MCM module. Hence \( \mathcal{M}_T(W^T) \) corresponds to stable matrix factorization category in [Yos90].

Before going further, we recall a construction of mapping cone in the matrix factorization (DG) category. Let \( P = (P_1, P_2) \), \( P' = (P'_1, P'_2) \) be two matrix factorizations and \( \phi = (\phi_1, \phi_2) : (P_1, P_2) \to (P'_1, P'_2) \). We define a mapping cone of \( \phi \), \( \text{Cone}(\phi) \), as a block matrix given by

\[
\begin{pmatrix}
P'_1 & \phi_1 \\
0 & P_2
\end{pmatrix}
\begin{pmatrix}
P'_2 \\
0
\end{pmatrix}
\begin{pmatrix}
-\phi_2 \\
P_1
\end{pmatrix}.
\]

10.3. \( A_n \)-cases. For \( A_n \)-singularity given by \( F_{n+1,2} = x^{n+1} + y^2 \) with \( R = \mathbb{C}[x, y]/F_{n+1,2} \), the AR quiver can be described as follows. AR quiver depends on the parity of \( n \): for even \( n \),

\[
R \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow \cdots \leftrightarrow M_n
\]

for odd \( n \),

\[
R \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow \cdots \leftrightarrow M_{n-1} \leftrightarrow N_-
\]
Here $M_k$ is the $2 \times 2$ matrix factorization given by
\[
\begin{pmatrix} y & x^k \\ x^{n+1-k} & -y \end{pmatrix} \left( \begin{pmatrix} y & x^k \\ x^{n+1-k} & -y \end{pmatrix} \right)
\]
and $N_-$ is the $(1 \times 1)$ matrix factorization
\[
(x^{n+1} - iy) \cdot (x^{n+1} + iy)
\]
and $N_+$ is obtained from $N_-$ by switching two factors. In fact, one can easily check that $2 \times 2$ matrix factorization $N_- \oplus N_+$ is isomorphic to $M_{n+1}$, and this is what we obtain from a Lagrangian.

We will find that as in Figure 27(A), $M_k$’s correspond to the non-compact Lagrangian connecting two punctures of the $2(n+1)$-gons. Namely, if we label the punctures as $v_1, \ldots, v_{n+1}$ counter-clockwise direction, then Lagrangian $L_{M_k}$ is given by the curve connecting $v_1$ and $v_{1+k}$ (either project it to the quotient Milnor fiber, or consider the $GW$-copies in the Milnor fiber).

**Lemma 10.8.** $L_{M_k}$ maps to $M_k$ under $\mathcal{F}_\mathbb{P}|_{z=0}$.

**Proof.** This is given by direct computation. We give an explicit calculation for $M_1$ in $A_4$-case. The Lagrangian $L_{M_1}$ intersects $\mathbb{P}$ at 4 points, which we label as $o_1, o_2, e_1, e_2$. $m^{1,b}_1$ counts holomorphic polygons connecting these intersections, while allowing $X, Y, Z$-corners in $\mathbb{P}$. In Figure 27(B), we illustrated some of the holomorphic polygons. Instead of drawing all of them, we write down the labels of the corners. For example, a grey polygon is denoted by $e_2 X_4 X_6 X_8 X_{10} o_1$. It contributes $x^4$ to the coefficient of $o_1$ in $m^{1,b}_1(e_2)$. Here are the list of all 8 polygons for $m^{0,b}_1$.

\[
\begin{align*}
o_1 Y_3 e_1, & \quad o_1 X_2 e_2, \quad o_2 X_3 X_5 X_7 X_9 e_1, \quad o_2 Y_1 e_2, \quad e_1 Y_2 o_1, \quad e_1 X_1 o_2, \quad e_2 X_4 X_6 X_8 X_{10} o_1, \quad e_2 Y_4 o_2.
\end{align*}
\]

Signs can be computed accordingly (see the paragraph after Theorem 7.8) and we obtain the following.

\[
\begin{align*}
m^{0,b}_1(o_1) = -y \cdot e_1 + x \cdot e_2 & \\
m^{0,b}_1(e_1) = -y \cdot o_1 + x \cdot o_2
\end{align*}
\]

This can be made into two $2 \times 2$ matrices $M_4$, which is isomorphic to $M_1$ by a change of basis. The rest are similar and omitted. \qed
In Figure 28, we take $\gamma$ as an odd morphism from $L_{M_1}$ to $L_{M_1}$, and $\text{Cone}(\gamma)$ becomes $L_{M_2}$. This realizes the AR sequence

$$0 \to \tau(M_1) \to M_2 \to M_1 \to 0$$

In general, the AR sequence of $A_n$ case is of the form

$$0 \to M_k \to M_{k-1} \oplus M_{k+1} \to M_k \to 0$$

In Chapter 9 of [Yos90], a (half of) explicit morphism whose cone gives this AR sequence is specified:

(10.3)$$
\begin{pmatrix}
0 & x^{k-1} \\
-x^{n-k} & 0
\end{pmatrix}
$$

For $L_{M_k}$ with $k \geq 2$, we take an odd immersed generator $\gamma_k$ as follows. This is at the intersection between $L_{M_k}$ and the counter-clockwise rotation by $2\pi/(n+1)$ of $L_{M_k}$. It is not hard to check that $\mathcal{S}_1^k(\gamma_k)$ equals the morphism (10.3) by counting suitable polygons. For example, in the case of $A_4$ and $\gamma$ as in Figure 28, we get $-\begin{pmatrix} 0 & x^3 \\ -1 & 0 \end{pmatrix}$. This is the morphism given in (10.3) for $k = 4, n = 4$.

Remark 10.9. When we find any AR sequence, we do not calculate all morphisms in the sequence. It is enough to check whether Lagrangians in surgery sequence map to the corresponding indecomposable matrix factorizations in AR sequence. See Appendix C.

10.4. $D_n$-cases. For $D_n$ singularity $x^{n-1} + xy^2$, AR quiver of indecomposable matrix factorizations are given as follows. For odd $n$, we have
Here, \( A \) (resp. \( B \)) is a \( 1 \times 1 \) factorization \((x, x^{n-1} + y^2)\) (resp. \((x^{n-1} + y^2, y)\)).

\[
\phi_j = \begin{pmatrix} y & x^j \\ x^{n-j-2} & -y \end{pmatrix}, \quad \psi_j = \begin{pmatrix} xy & x^{j+1} \\ x^{n-j-1} & -xy \end{pmatrix}
\]

\( M_j \) (resp. \( N_j \)) is a \( 2 \times 2 \) matrix factorization given by \((\phi_j, \psi_j)\) (resp. \((\psi_j, \phi_j)\)).

\[(10.4)\]

\[
\xi_j = \begin{pmatrix} y & x^j \\ x^{n-j-1} & -xy \end{pmatrix}, \quad \eta_j = \begin{pmatrix} xy & x^j \\ x^{n-j-1} & -xy \end{pmatrix}
\]

\( X_j \) (resp. \( Y_j \)) is a \( 2 \times 2 \) matrix factorization given by \((\xi_j, \eta_j)\) (resp. \((\eta_j, \xi_j)\)). For even \( n \),

\[
\begin{array}{c}
A \\
B \\
\vdots \\
R
\end{array}
\rightarrow
\begin{array}{c}
Y_1 \\
M_1 \\
\vdots \\
X_1
\end{array}
\rightarrow
\begin{array}{c}
M_1 \\
Y_2 \\
\vdots \\
X_2
\end{array}
\rightarrow
\cdots
\rightarrow
\begin{array}{c}
M_1 \\
Y_{n-2} \\
\vdots \\
X_{n-2}
\end{array}
\rightarrow
\begin{array}{c}
C_+
\\
D_+
\\
C_-
\\
D_-
\end{array}
\]

To find the corresponding Lagrangians, we consider the dual singularity \( D_T^2 \) given by

\[
C_{2,n-1} = x^2 + xy^{n-1}.
\]

First, Milnor fiber of \( C_{2,n-1} \) is given as LHS of Figure 29 which is a \( 4(n-1) \)-gon with edges identified as \( \pm 3 \) pattern. We label immersed generators of Seidel Lagrangian as in Figure 29. Then the potential of Seidel Lagrangian is \( W_L = y^{n-1} + xyz \), and if we restrict to the hypersurface \( z - x = 0 \), we get \( D_n \) singularity \( y^{n-1} + xy^2 \).

![Figure 29. Milnor fiber for \( D_T^2 = C_{2,4} \), Lagrangians for indecomposable MF’s](image)

We now describe Lagrangians in Milnor fiber of \( D_T^2 \) corresponding to indecomposable matrix factorizations of \( D_n \). Note that our potential is \( y^{n-1} + xy^2 \), hence we have to switch \( x \) and \( y \) after calculating matrix factorization to match them with above list.

**Lemma 10.10.** Lagrangian \( L_A, L_{Y_1}, L_M \) (defined as in Figure 29) correspond to matrix factorizations \( A, Y_1, M_1 \). By Lemma 10.2, orientation reversals of these Lagrangians correspond to \( B, X_1, N_1 \) respectively.
Figure 30. Holomorphic polygons for $m_{0,b}^1$ in the case of $L_Y$ in $D^7_T$ Milnor fiber

**Proof.** Potential $W_L$ comes from two polygons, minimal $XYZ$ triangle, and $(n-1)$-gon with all $Y$-corners. Note that Lagrangian $L_A$ cuts each of these two polygons into two parts. This gives the corresponding $1 \times 1$ matrix factorization. For the other cases, each of $L_Y, L_M$ intersect $\tilde{L}$ at two even and two odd points and counting polygons with signs provides the desired $2 \times 2$ matrix factorizations. Let us illustrate the case of $L_Y$ and leave the rest as an exercise. In this case, there are 8 polygons given by

These polygons contribute to the following $m_{0,b}^1$ computations.

$$m_{1}^{0,b}(o_1) = x y \cdot e_1 - y^3 \cdot e_2$$
$$m_{1}^{0,b}(e_1) = x \cdot o_1 + y^3 \cdot o_2$$
$$m_{1}^{0,b}(o_2) = y \cdot e_1 + x \cdot e_2$$
$$m_{1}^{0,b}(e_2) = -y \cdot o_1 + x y \cdot o_2$$

Note that some of the signs in $(CF(L_Y, \mathbb{L}), -m_{0,b}^1)$ are different from (10.4), but by a simple change of basis, we can identify it with $(\eta_1, \xi_1)$.

**Lemma 10.11.** All AR exact sequences for $D_n$ singularity can be realized as Lagrangian surgeries.

**Proof.** It is enough to check the following exact sequences (the other four will correspond to the orientation reversal of Lagrangians).

$$0 \rightarrow B \rightarrow Y_1 \rightarrow A \rightarrow 0 \quad (10.5a)$$
$$0 \rightarrow X_1 \rightarrow B \oplus N_1 \rightarrow Y_1 \rightarrow 0 \quad (10.5b)$$
$$0 \rightarrow N_i \rightarrow Y_i \oplus X_i \rightarrow M_i \rightarrow 0 \quad (10.5c)$$
$$0 \rightarrow X_i \rightarrow M_{i-1} \oplus N_i \rightarrow Y_i \rightarrow 0 \quad (10.5d)$$

Corresponding Lagrangian surgeries are illustrated in Figure 31 and we obtain the above exact sequence, by applying Lemma 10.4.

10.5. **$E_6$-case.** For $E_6$ singularity $x^3 + y^4$, its AR quiver of indecomposable MF’s is the following.
Here, $M_j$ (resp. $N_j$) is a $2 \times 2$ matrix factorization given by $(\phi_j, \psi_j)$ (resp. $(\psi_j, \phi_j)$) for $j = 1, 2$.

$$\phi_1 = \begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix}, \psi_1 = \begin{pmatrix} x^2 & y \\ y^3 & -x \end{pmatrix}$$

$$\phi_2 = \begin{pmatrix} x & y^2 \\ y^2 & -x^2 \end{pmatrix}, \psi_2 = \begin{pmatrix} x^2 & y^2 \\ y^2 & -x \end{pmatrix}$$

A (resp. $B$) is a $3 \times 3$ matrix factorization given by $(\alpha, \beta)$ (resp. $(\beta, \alpha)$).

(10.6)

$$\alpha = \begin{pmatrix} y^3 & x^2 & xy^2 \\ xy & -y^2 & x^2 \\ x^2 & -xy & -y^3 \end{pmatrix}, \beta = \begin{pmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & -y \end{pmatrix}$$

$X$ is a $4 \times 4$ matrix factorization given by $(\xi, \eta)$ where

$$\xi = \begin{pmatrix} \phi_2 & \epsilon_3 \\ 0 & \psi_2 \end{pmatrix}, \eta = \begin{pmatrix} \psi_2 & \epsilon_4 \\ 0 & \phi_2 \end{pmatrix} \text{ with } \epsilon_3 = \begin{pmatrix} 0 & y \\ -xy & 0 \end{pmatrix}, \epsilon_4 = \begin{pmatrix} 0 & xy \\ -y & 0 \end{pmatrix}.$$

Since $F^T_{3,4} = F_{3,4}$, we consider the Milnor fiber of $F_{3,4} = x^3 + y^4$. First, it is given by $\mathbb{Z}/4$-copies of a hexagon glued as in Figure 32 whose boundary is identified with $(\pm 5)$ pattern. $\mathbb{Z}/4$-action is the rotation at the center, and $\mathbb{Z}/3$-action is the simultaneous rotation in every hexagon.

Lifts of Seidel Lagrangian are drawn in dotted lines, and XXX and XYZ-triangles, and YYYY-quadrangle produce the potential $W_L = x^3 + y^4 + xyz$ and after the restriction $z = 0$, we get $E_6$ singularity $x^3 + y^4$. 

---

**Figure 31.** Auslander-Reiten exact sequences for $D_5$ via Lagrangian surgery
Lemma 10.12. Lagrangians $L_M, L_B, L_X$ in Figure 32 correspond to matrix factorizations $M_1, B, X$. By Lemma 10.2, orientation reversals of these Lagrangians correspond to $N_1, A, X$ respectively.

Proof. We explain the case of $L_B$ only. There are 14 polygons;

$$o_1 Y_3 e_1, o_1 X_1 e_2, o_2 Y_6 Y_8 e_2, o_2 X_7 e_3, o_3 X_{10} e_1, o_3 Y_9 e_3,$$

$$e_1 Y_5 Y_7 Y_1 o_1, e_1 X_4 X_6 o_2, e_1 X_6 X_8 o_3, e_2 Y_4 Y_2 e_2, e_2 Y_2 X_8 o_3, e_3 X_9 Y_7 e_1, e_3 X_9 X_{11} o_2, e_3 Y_4 Y_6 o_3.$$

Last polygon is a part of polygon for $y^4$, cut out by $L_B$ along the edge $e_3 o_3$. It lies outside of the domain in Figure 33 but can be obtained by a group action of the vertex without prime. The above polygons

Figure 32. Milnor fiber for $E_6$ and Lagrangians corresponding to indecomposable MF’s

Figure 33. Holomorphic polygons for $m_{1}^{0,b}$ in the case of $L_B$ in $E_6$ Milnor fiber
contribute to $m_{1}^{0,b}$ as follows.

\[
\begin{align*}
    m_{1}^{0,b}(o_1) &= -y \cdot e_1 + x \cdot e_2 \\
    m_{1}^{0,b}(o_2) &= y^2 \cdot e_2 - x \cdot e_3 \\
    m_{1}^{0,b}(o_3) &= x \cdot e_1 + y \cdot e_3 \\
    m_{1}^{0,b}(e_1) &= -y^3 \cdot o_1 + xy \cdot o_2 + x^2 \cdot o_3 \\
    m_{1}^{0,b}(e_2) &= x^2 \cdot o_1 + y^2 \cdot o_2 + xy \cdot o_3 \\
    m_{1}^{0,b}(e_3) &= xy^2 \cdot o_1 - x^2 \cdot o_2 + y^3 \cdot o_3
\end{align*}
\]

These give the matrices in (10.6) (up to change of a basis).

**Lemma 10.13.** All AR exact sequences for $E_6$ singularity can be realized as Lagrangian surgeries.

**Proof.** We need to check following exact sequences (the other four are given by orientation reversals).

(10.7) \quad 0 \rightarrow M_1 \rightarrow A \rightarrow N_1 \rightarrow 0

(10.8) \quad 0 \rightarrow B \rightarrow X \oplus M_1 \rightarrow A \rightarrow 0

(10.9) \quad 0 \rightarrow X \rightarrow M_2 \oplus A \oplus B \rightarrow X \rightarrow 0

(10.10) \quad 0 \rightarrow M_2 \rightarrow X \rightarrow M_2 \rightarrow 0

We illustrated the corresponding Lagrangian surgery in Figure 34. Note that in the case of (10.9), a connected Lagrangian submanifold, denoted as $L_{A \oplus B}$, correspond to the direct sum $A \oplus B$. This does
not directly follow from the picture: $L_{A\oplus B}$ can be considered as a mapping cone of a morphism $\gamma$ from $L_A$ to $L_B$, but $\gamma$ can be shown to be zero up to homotopy. More precisely, matrix factorization from the Lagrangian $L_{A\oplus B}$ can be computed as

$$
\begin{pmatrix}
-y^3 & x^2 & xy^2 & y^2 & 0 & 0 \\
xy & y^2 & -x^2 & 0 & y^2 & 0 \\
x^2 & xy & y^3 & 0 & 0 & y^2 \\
0 & 0 & 0 & -y & 0 & x \\
0 & 0 & 0 & x & y^2 & 0 \\
0 & 0 & 0 & 0 & -x & y
\end{pmatrix}
- \begin{pmatrix}
y & 0 & x & -y^2 & 0 & 0 \\
x & y^2 & 0 & 0 & -y^2 & 0 \\
0 & -x & y & 0 & 0 & -y^2 \\
0 & 0 & 0 & -y^3 & x^2 & y^2 \\
0 & 0 & 0 & x^2 & x y & y^3
\end{pmatrix}
$$

after basis change. It is a (minus of) mapping cone of $\gamma$:

$\begin{pmatrix}
\phi \\
\psi
\end{pmatrix}
\in \text{Hom}(A, A)$.

We can find explicit null-homotopy $h = (h_1, h_2)$ of $(y^2 \cdot id, y^2 \cdot id) \in \text{Hom}(A, A)$.

$$
\begin{pmatrix}
-y^3 & x^2 & xy^2 \\
xy & y^2 & -x^2 \\
x^2 & xy & y^3
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
-y & 0 & x \\
x & y^2 & 0 \\
0 & -x & y
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
y^2 & 0 & 0 \\
y^2 & 0 & 0 \\
y^2 & 0 & 0
\end{pmatrix}
$$

Hence, this mapping cone is isomorphic to the direct sum $A \oplus B$.

**Remark 10.14.** We can prove that if $L$ maps to a matrix factorization $P$, then its involution image maps to $P$ or $P[1]$ in ADE case.

10.6. **$E_7$-case.** For $E_7$ singularity $x^3 + xy^3$, its AR quiver of indecomposable MF’s is the following.
Here, $A$ (resp. $B$) is a $1 \times 1$ factorization $(x, x^2 + y^3)$ (resp. $(x^2 + y^3, x)$). $C$ (resp. $D$) is a $2 \times 2$ matrix factorization given by $(\gamma, \delta)$ (resp. $(\delta, y)$).

\[
\gamma = \begin{pmatrix} x^2 & xy \\ xy^2 & -x^2 \end{pmatrix}, \delta = \begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix}
\]

$M_j$ (resp. $N_j$) is a $2 \times 2$ matrix factorization given by $(\phi_j, \psi_j)$ (resp. $(\psi_j, \phi_j)$) for $j = 1, 2$.

\[
\phi_1 = \begin{pmatrix} x & y \\ xy^2 & -x^2 \end{pmatrix}, \psi_1 = \begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix}
\]

\[
\phi_2 = \begin{pmatrix} x & y^2 \\ xy & -x^2 \end{pmatrix}, \psi_2 = \begin{pmatrix} x^2 & y^2 \\ xy & -x \end{pmatrix}
\]

$X_j$ (resp. $Y_j$) is a $3 \times 3$ matrix factorization given by $(\xi_j, \eta_j)$ (resp. $(\eta_j, \xi_j)$) for $j = 1, 2$.

\[
\xi_1 = \begin{pmatrix} xy^2 & -x^2 & -x^2 y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{pmatrix}, \eta_1 = \begin{pmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{pmatrix}
\]

\[
\xi_2 = \begin{pmatrix} x^2 & -y^2 & -xy \\ xy & x & -y^2 \\ xy^2 & xy & x^2 \end{pmatrix}, \eta_2 = \begin{pmatrix} x & 0 & y \\ -xy & x^2 & 0 \\ 0 & -xy & x \end{pmatrix}
\]

Finally, $X_3$ (resp. $Y_3$) is a $4 \times 4$ matrix factorization given by $(\xi_3, \eta_3)$ (resp. $(\eta_3, \xi_3)$).

\[
\xi_3 = \begin{pmatrix} y & \epsilon \\ 0 & \delta \end{pmatrix}, \eta_3 = \begin{pmatrix} \delta & -\epsilon \\ 0 & \gamma \end{pmatrix}
\]

with $\epsilon = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$.

**Remark 10.15.** The matrix $\epsilon$ is defined as $\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$ in [Yos90] and it seems to be a typo.

To find the corresponding symplectic geometry, we consider the Milnor fiber of $C_{3,3} = x^3 + xy^3$.

Recall that the fundamental domain is given by 3 copies of hexagons as we have seen in Figure 20 (with $\mathbb{Z}/3$-fixed point at the center of each hexagon). The potential is $W_L = y^3 + xyz$ and after we set $z = x^2$, we get $E_7$ singularity $y^3 + yx^3$. We remark that two punctures behave differently. Namely, puncture $A$ (drawn as a rectangle), and puncture $C$ (drawn as a circle) have different monodromies in Proposition 6.5, and this results in the relation $z = x^2$ via Kodaira-Spencer map. Because of similar reason in $D_n$ cases, we need to switch $x$ and $y$ for complete correspondence.
**Lemma 10.16.** Lagrangians \( L_A, L_D, L_M, L_Y \) in Figure 35 correspond to indecomposable matrix factorizations \( A, D, M_i, Y_i \) respectively. By Lemma 10.2, orientation reversals of these Lagrangians correspond to \( B, C, N_i, X_i \) respectively.

**Proof.** We explain the case of \( Y_3 \) only. There are 20 polygons;

\[
o_1 Y_0 e_1, o_1 Z_5 e_2, o_2 X_4 e_1, o_2 Y_7 e_2, o_3 Z_3 e_1, o_3 Y_1 Y_3 e_3, o_4 Y_4 e_1, o_4 Y_4 X_3 e_3, o_4 Y_4 Y_6 e_4.
\]

\[
e_1 Y_6 Y_8 o_1, e_1 Y_6 Z_4 o_2, e_2 X_5 Y_8 o_1, e_2 Y_5 Y_1 o_2, e_3 Y_5 o_3, e_3 Z_2 o_4, e_4 Y_8 o_1, e_4 Z_4 o_2, e_4 X_1 o_3, e_4 Y_2 o_4.
\]

We have the following \( m_1^{0,b} \) calculations.

**Figure 36.** Holomorphic polygons for \( m_1^{0,b} \) calculation

\[
m_1^{0,b}(o_1) = y \cdot e_1 + x^2 \cdot e_2 \quad m_1^{0,b}(e_1) = y^2 \cdot o_1 + x^2 y \cdot o_2
\]

\[
m_1^{0,b}(o_2) = x \cdot e_1 - y \cdot e_2 \quad m_1^{0,b}(e_2) = x y \cdot o_1 - y^2 \cdot o_2
\]

\[
m_1^{0,b}(o_3) = x^2 \cdot e_1 + y^2 \cdot e_3 + x^2 y \cdot e_4 \quad m_1^{0,b}(e_3) = y \cdot o_3 + x^2 \cdot o_4
\]

\[
m_1^{0,b}(o_4) = -y \cdot e_1 + x y \cdot e_3 - y^2 \cdot e_4 \quad m_1^{0,b}(e_4) = -y \cdot o_1 - x^2 \cdot o_2 + x \cdot o_3 - y \cdot o_4
\]

**Lemma 10.17.** The following AR exact sequences (and their AR translation) for \( E_7 \) singularity can be realized as Lagrangian surgeries.

\[
(10.11a) \quad 0 \to A \to M_2 \to B \to 0 \quad (10.11b) \quad 0 \to D \to Y_3 \to C \to 0
\]

\[
(10.11c) \quad 0 \to M_1 \to X_1 \to N_1 \to 0 \quad (10.11d) \quad 0 \to M_2 \to B \oplus Y_2 \to N_2 \to 0
\]

\[
(10.11e) \quad 0 \to Y_1 \to M_1 \oplus X_3 \to X_1 \to 0 \quad (10.11f) \quad 0 \to Y_2 \to N_2 \oplus Y_3 \to X_2 \to 0
\]

\[
(10.11g) \quad 0 \to Y_3 \to X_2 \oplus C \oplus Y_1 \to X_3 \to 0
\]
Proof. In the case of (10.11g), we need to show that \(LY_1 \oplus X_2\) corresponds to the direct sum \(Y_1 \oplus X_2\). \(LY_1 \oplus X_2\) maps to matrix factorization given by

\[
\begin{pmatrix}
x & 0 & y & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & x^2 & 0 & 0 & 0 & 0 \\
0 & y & -x & -x^2 & -x & -y \\
0 & 0 & 0 & x^2 + y^2 & x^2 & xy \\
0 & 0 & 0 & 0 & -y & x^2 + y^2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

As a mapping cone, corresponding morphism is homotopic to zero;

\[
\begin{pmatrix}
x & 0 & y \\
-1 & 0 & 0 \\
0 & x & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
x+y & y & xy \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
= \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
-x^2 & -x & -y \\
\end{pmatrix}
\]

(Figure 37. Auslander-Reiten exact sequences for \(E_7\) via Lagrangian surgery)

(A) Lagrangian surgery for (10.11a)  
(B) Lagrangian surgery for (10.11b)  
(C) Lagrangian surgery for (10.11c)  
(D) Lagrangian surgery for (10.11d)
10.7. **E₈-case.** For $E₈$ singularity $x^3 + y^5$, its AR quiver of indecomposable MF’s is the following.

Here, $M_i$ (resp. $N_i$) is a $2 \times 2$ matrix factorization given by $(φ_i, ψ_i)$ (resp. $(ψ_i, φ_i)$) for $i = 1, 2$.

$φ_i = \begin{pmatrix} x & x^i \\ y^{5-i} & -x^2 \end{pmatrix}, ψ_i = \begin{pmatrix} x^2 & x^i \\ y^{5-i} & -x \end{pmatrix}$

$A_i$ (resp. $B_i$) is a $3 \times 3$ matrix factorization given by $(α_i, β_i)$ (resp. $(β_i, α_i)$) for $i = 1, 2$.

$α_i = \begin{pmatrix} y & -x & 0 \\ 0 & y^i & -x \\ x & 0 & y^{4-i} \end{pmatrix}, β_i = \begin{pmatrix} y^4 & x y^4-i & x^2 \\ -x^2 & y^{5-i} & xy \\ -x y^i & -x^2 & y^{i+1} \end{pmatrix}$
$C_i$ (resp. $D_i$) is a $4 \times 4$ matrix factorization given by $(\gamma_i, \delta_i)$ (resp. $(\gamma_i, \delta_i)$) for $i = 1, 2$.

\[
\gamma_1 = \begin{pmatrix} 1 & -x & 0 & y^3 \\ x & -y^2 & 0 & -x \\ -y^2 & 0 & -x & 0 \\ 0 & -y^2 & -x & -y^2 \end{pmatrix}, \delta_1 = \begin{pmatrix} x^2 & -y^3 & 0 \\ -x^2 & xy & 0 & -y^3 \\ 0 & -y^2 & -x & 0 \\ y^2 & 0 & y & -x \end{pmatrix}
\]
\[
\gamma_2 = \begin{pmatrix} \phi_2 & \epsilon_1 \\ 0 & \epsilon_2 \end{pmatrix}, \delta_2 = \begin{pmatrix} \psi_2 & \epsilon_2 \\ 0 & \phi_2 \end{pmatrix} \text{ with } \epsilon_1 = \begin{pmatrix} 0 & y \\ -x & 0 \end{pmatrix}, \epsilon_2 = \begin{pmatrix} 0 & xy \\ -y^2 & 0 \end{pmatrix}.
\]

$X_1$ (resp. $Y_1$) is a $6 \times 6$ matrix factorization given by $(\xi_1, \eta_1)$ (resp. $(\eta_1, \xi_1)$).

\[
\xi_1 = \begin{pmatrix} \beta_2 & \epsilon_3 \\ 0 & \alpha_2 \end{pmatrix}, \eta_1 = \begin{pmatrix} \alpha_2 & \epsilon_4 \\ 0 & \beta_2 \end{pmatrix} \text{ with } \epsilon_3 = \begin{pmatrix} 0 & 0 & xy \\ -x & 0 & 0 \\ 0 & -xy & 0 \end{pmatrix}, \epsilon_4 = \begin{pmatrix} 0 & 0 & -x \\ xy & 0 & 0 \\ 0 & xy & 0 \end{pmatrix}.
\]

$X_2$ (resp. $Y_2$) is a $5 \times 5$ matrix factorization given by $(\xi_2, \eta_2)$ (resp. $(\eta_2, \xi_2)$).

\[
\xi_2 = \begin{pmatrix} y^4 & x^2 & 0 & -xy^2 & 0 \\ -x^2 & xy & 0 & -y^3 & 0 \\ 0 & -y^2 & -x & 0 & y^3 \\ -xy^2 & y^3 & 0 & x^2 & 0 \\ -y^3 & 0 & -y^2 & xy & -x^2 \end{pmatrix}, \eta_2 = \begin{pmatrix} y & -x & 0 & 0 & 0 \\ x & 0 & 0 & y^2 & 0 \\ -y^2 & 0 & -x^2 & 0 & -y^3 \\ 0 & -y^2 & 0 & x & 0 \\ 0 & 0 & y^2 & y & -x \end{pmatrix}.
\]

Since $F^T_{3,5} = F_{3,5}$, we consider the Milnor fiber of $F_{3,5} = x^3 + y^5$. First, it is given by $\mathbb{Z}/5$-copies of a hexagon glued as in Figure [39] whose boundary is identified with $(\pm 5)$ pattern. $\mathbb{Z}/5$-action is the rotation at the center, and $\mathbb{Z}/3$-action is the simultaneous rotation in every hexagons. Lifts of Seidel Lagrangian are drawn as dotted lines, and $XXX$ and $XYZ$-triangles, and pentagon with $Y$-corners produces the potential $W_L = x^3 + y^5 + xyz$. With the restriction $z = 0$, we get $E_8$ singularity $x^3 + y^5$.

**Figure 39.** Milnor fiber for $E_8$ and Lagrangians for indecomposable MF's

**Lemma 10.18.** Lagrangians $L_B, L_C, L_N, L_X$ in Figure [39] correspond to their respective matrix factorizations. By Lemma [10.2], orientation reversals of these Lagrangians correspond to $A_i, D_i, M_i, Y_i$ respectively.

**Proof.** We explain the case of $X_2$. There are 28 polygons;

\[
o_1 Y_3 Y_5 Y_7 Y_9 e_1, o_1 X_5 X_1 e_2, o_1 X_5 Y_2 Y_4 e_4, o_2 X_2 Y_5 e_1, o_2 Y_10 Y_2 Y_4 e_4, o_3 Y_7 X_4 e_1, o_3 Y_7 Y_9 e_2, o_3 X_9 e_3, o_3 Y_2 Y_4 Y_6 e_5, o_4 Y_5 Y_7 Y_9 e_2, o_4 X_10 X_12 e_4, o_5 Y_17 Y_19 e_3, o_5 Y_7 X_12 e_4, o_5 X_1 X_6 e_5, e_1 Y_20 o_1, e_1 X_6 o_2, e_2 X_5 o_1, e_2 Y_3 o_4, e_3 X_11 Y_8 o_2, e_3 X_11 X_7 o_3, e_3 Y_11 Y_13 Y_15 o_5, e_4 Y_6 Y_8 o_2, e_4 X_9 o_4, e_5 Y_18 Y_5 o_3, e_5 Y_18 o_4, e_5 X_13 o_5.
\]
The vertices marked with prime in the above expression lies outside of Figure 40 (but can be obtained by a group-action of the vertex without prime). Then $m^{0,b}_1$ is given as follow.

![Figure 40. Holomorphic polygons for $m^{0,b}_1$ calculation](image)

\[m^{0,b}_1(o_1) = y^4 \cdot e_1 - x^2 \cdot e_2 + xy^2 \cdot e_4\]
\[m^{0,b}_1(o_2) = x^2 \cdot e_2 + xy \cdot e_1 - y^3 \cdot e_4\]
\[m^{0,b}_1(o_3) = xy \cdot e_1 + y^2 \cdot e_2 - x \cdot e_3 + y^3 \cdot e_5\]
\[m^{0,b}_1(o_4) = xy^2 \cdot e_1 + y^3 \cdot e_2 + x^2 \cdot e_4\]
\[m^{0,b}_1(o_5) = y^2 \cdot e_3 + xy \cdot e_4 + x^2 \cdot e_5\]

Putting them into two matrices, we obtain $X_2$.

\[m^{0,b}_1(e_1) = y \cdot o_1 + x \cdot o_2\]
\[m^{0,b}_1(e_2) = -x \cdot o_1 + y^2 \cdot o_4\]
\[m^{0,b}_1(e_3) = xy \cdot o_2 - x^2 \cdot o_3 + y^3 \cdot o_5\]
\[m^{0,b}_1(e_4) = -y^2 \cdot o_2 + x \cdot o_4\]
\[m^{0,b}_1(e_5) = y^2 \cdot o_3 - y \cdot o_4 + x \cdot o_5\]

**Lemma 10.19.** The following AR exact sequences (and their AR translation) for $E_8$ singularity can be realized as Lagrangian surgeries.

\[(10.12a) \quad 0 \to N_1 \to A_1 \oplus R \to M_1 \to 0\]
\[(10.12b) \quad 0 \to B_1 \to D_1 \oplus N_1 \to A_1 \to 0\]
\[(10.12c) \quad 0 \to N_2 \to D_2 \to M_2 \to 0\]
\[(10.12d) \quad 0 \to B_2 \to X_1 \to A_2 \to 0\]
\[(10.12e) \quad 0 \to C_1 \to B_1 \oplus Y_2 \to D_1 \to 0\]
\[(10.12f) \quad 0 \to D_2 \to M_2 \oplus X_1 \to C_2 \to 0\]
\[(10.12g) \quad 0 \to X_2 \to Y_1 \oplus C_1 \to Y_2 \to 0\]
\[(10.12h) \quad 0 \to X_1 \to A_2 \oplus C_2 \oplus X_2 \to Y_1 \to 0\]

**Proof.** We show that $L_{C_2 \oplus X_2}$ corresponds to the direct sum $C_2 \oplus X_2$ in (10.12h). $L_{C_2 \oplus X_2}$ maps to $9 \times 9$ matrix factorization. As a mapping cone, we give a null-homotopy of corresponding morphism.
\[
\begin{pmatrix}
  x & y^3 & 0 & 0 \\
y^2 & -x^2 & 0 & 0 \\
0 & xy^2 & x^2 & y^3 \\
-y & 0 & y^2 & -x
\end{pmatrix}
\begin{pmatrix}
  x & -y & 0 & 0 & 0 \\
y^2 & -x & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 \\
-y & 0 & 0 & 0 & -1
\end{pmatrix}
+ \begin{pmatrix}
  y^4 & x^2 & xy & x^2y & 0 \\
-x^2 & xy & y^2 & y^3 & 0 \\
0 & 0 & -x & 0 & y^2 \\
x^2y^2 & -y^3 & 0 & x^2y & xy \\
0 & 0 & y^3 & 0 & x^2
\end{pmatrix}
\begin{pmatrix}
  0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
1 & 0 & y & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  y^4 & x^2 & xy & x^2y & 0 \\
-x^2 & xy & y^2 & y^3 & 0 \\
0 & 0 & -x & 0 & y^2 \\
x^2y^2 & -y^3 & 0 & x^2y & xy \\
0 & 0 & y^3 & 0 & x^2
\end{pmatrix}
\begin{pmatrix}
  x & -y & 0 & 0 & 0 \\
y^2 & -x & 0 & 0 & -1 \\
1 & 0 & y & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
-y & 0 & 0 & 0 & -1
\end{pmatrix}
= \begin{pmatrix}
  0 & 0 & y^2 & 0 & 0 \\
0 & 0 & 0 & 0 & -xy \\
0 & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\textbf{Figure 41.} Auslander-Reiten exact sequences for $E_8$ via Lagrangian surgery
Figure 42. Auslander-Reiten exact sequences for $E_8$ via Lagrangian surgery

Appendix A. Moduli Space of Pseudo-Holomorphic Curves and Perturbations

We briefly describe a perturbation scheme for moduli spaces of pseudo-holomorphic curves that we use. We refer [Sei08], [AS10], [Abo12], and especially [Abo10] and [Sei18] from which most of the material has been borrowed.

A.1. Setup. Let $(M^{2n}, \omega = d\lambda)$ be a Liouville manifold with cylindrical ends. By definition, $M$ can be decomposed into a union of a compact part and cylindrical ends

$$M = M_{\text{cpt}} \bigcup_{\partial M_{\text{cpt}}} \partial M_{\text{cpt}} \times [1, \infty),$$

and Liouville flow $Z$ is of the form $Z = r \frac{\partial}{\partial r}$ at the cylindrical end where $r$ is a coordinate of $[1, \infty)$. We also assume $c_1(TM) = 0$ to put $Z$-grading everywhere.

- We will work with a function $H \in C^\infty(M, \mathbb{R})$ such that $H > 0$, $C^2$-small on $M_{\text{cpt}}$ and quadratic at infinity ($H(x, r) = r^2$), and denote the class of such functions by $\mathcal{H}(M)$.
- Whenever we consider a time dependent perturbation $H_{S^1} = H + F: S^1 \times M \to \mathbb{R}$, we assume $H_{S^1} > 0$, $C^2$-small on $M_{\text{cpt}}$ so that the time-1 periodic orbits of $H_{S^1}$ are non-degenerate. This is true for a generic perturbation.
- A $\omega$-compatible almost complex structure $J$ is called contact type if $\lambda \circ J = dr$ at the end. We denote a class of such almost complex structure by $\mathcal{J}(M)$. 
\( \mathcal{W} \) is a collection of exact properly embedded Lagrangian submanifolds in \( M \), such that if non-compact, \( L \cap \partial M_{cp} \) is a Legendrian submanifold, and \( L \) is conical at the end. Furthermore, all such \( L \) is required to have vanishing relative first Chern class \( 2c_1(M, L) \). We attach a spin structure and a grading function on each \( L \).

Fix a small, time dependent perturbation \( H_{S^1} : S^1 \times M \to \mathbb{R} \) of \( H \). Let \( \mathcal{O} := \mathcal{O}(M, H_{S^1}) \) be a set of time-1 orbits \( \gamma \) of \( H_{S^1} \), and \( \chi(L_0, L_1; H) \) to be the set of time-1 Hamiltonian chords \( a \) of \( H \) from \( L_0 \) to \( L_1 \) for two Lagrangians \( L_0, L_1 \in \mathcal{W} \). We use a notation \( o_\gamma \) (resp. \( o_a \)) to denote its orientation operator. Its degree \( \text{deg} o_\gamma \) (resp. \( \text{deg} o_a \)) is given by its cohomological Conley-Zehnder index (resp. Maslov index).

A.2. Moduli spaces. The moduli space of the Riemann sphere with \( m + 1 \) marked points (\( m \) for positive punctures and 1 for the negative puncture) is denoted as \( \text{Moduli spaces} \). The moduli space of a disc with \( m \) interior marked points \( \{z_1^+, \ldots, z_m^+\} \) (all for positive punctures) and with \( n + 1 \) cyclically ordered boundary marked points \( \{z_1, \ldots, z_n\} \) for positive punct and \( \{z_0\} \) for the negative puncture) is denoted as \( \text{Moduli spaces} \). An oriented real blow-up of \( \mathbb{S} \) at marked points is a surface \( S \) such that boundary puncture \( z_k \) (resp. interior puncture \( z_k^+ \)) is replaced by \([0, 1] \times \{z_k\} \) (resp. \( S^1 \times \{z_k^+\} \)). Let us denote (with coordinate \((s, t)\))

\[
Z^+ = [0, \infty) \times [0, 1], \quad Z^- = (-\infty, 0] \times [0, 1], \quad C^+ = [0, \infty) \times S^1, \quad C^- = (-\infty, 0] \times S^1.
\]

**Definition A.1.** A set of ends \( \mathcal{S} \) for \( S \in \text{Moduli spaces} \) is a choice of

- strip-like ends \( \epsilon_k^\pm : Z_k \to S \), \( \lim_{s \to \pm \infty} \epsilon_k^\pm(s, t) = z_k \), \( (\epsilon_k^\pm)^{-1}(\partial S) = \{t = 0, 1\} \)
- cylindrical ends \( \delta_l^\pm : C_l \to S \), \( \lim_{s \to \pm \infty} \delta_l^\pm(s, t) = z_l^\pm \)

Such collection is said to be weighted if each strip and cylinder is endowed with a positive real number

- \( w_{S,k}^\pm \) for each strip-like end \( \epsilon_k^\pm \)
- \( v_{S,l}^\pm \) for each cylindrical end \( \delta_l^\pm \)

such that \( \sum_{S,k} w_{S,k}^\pm + \sum_{S,l} v_{S,l}^\pm = \sum_{S,k} w_{S,k}^- + \sum_{S,l} v_{S,l}^- \).

The choice of ends (without weights) is a choice complex coordinate \( z \) of \( S \) near punctures. It also provides an analytic coordinate \((\sigma, t)\) of \( S \) near puncture in the following way.

\[
\sigma = e^{\pi z}, \quad t = \arg(z)/\pi
\]

We denote Deligne-Mumford compactification of a family of surfaces \( |S| \) by \( |S|_{m,n,1} \).

**Definition A.2.** Let \((S, \mathcal{S})\) denote a holomorphic disc \( S \) with ends \( \{v_S\} \) with weight \( \{v_S\} \).

1. A basic, asymptotically compatible 1-form \( \alpha_S \) is a closed 1-form on \( S \) whose restriction on \( \partial S \) vanishes, extends smoothly to \( |S| \) and \( \alpha_S = v_S \) at the interval / circle at infinity. It implies

\[
\kappa^* \alpha_S = v_S dt + d(e^{\pi z} g_k(e^{\pi z}, t)), \quad \forall \kappa
\]

for sufficiently large \(|s|\) and for some smooth function \( g_k \) that vanishes for \( t = 0, 1 \).

2. A secondary, asymptotically compatible 1-form \( \beta_S \) is a sub-closed 1-form whose restriction on \( \partial S \) vanishes, extends smoothly to \( |S| \), \( \beta_S = v_S dt \) at the circle at infinity, and vanishes at the intervals at infinity. It implies

\[
\kappa^* \beta_S = \begin{cases} v_S dt + d(e^{\pi z} h_k(e^{\pi z}, t)) & \forall \text{cylindrical end } \kappa \\
 d(e^{\pi z} h_k(e^{\pi z}, t)) & \forall \text{strip-like end } \kappa
\end{cases}
\]

for sufficiently large \(|s|\) and for some smooth function \( h_k \) that vanishes for \( t = 0, 1 \) whenever \( \kappa \) is a strip-like ends.
space. Since the space of Floer data is contractible, we can extend it up to constant ambiguity in the Hamiltonian terms. Also, we say an \( \chi \)-choice of Floer data is a choice of Floer data \( \text{Floer} \) \( \chi \end{align*}
end with weights 1 for all ends. A form \( dt \) is a compatible sub-closed one form. A universal and consistent Hamiltonian \( F \), a Floer data \( \text{Floer} \) \( F \) for a fixed surface \( S \), Hamiltonian \( H \) \( S \), not on the choice of the set of ends \( \{ \kappa \} \). It allows us to choose a special kind of ends to deal with popsicle structures later on.

It implies
\[
\kappa^* J_S = (\phi^V)\gamma F_t + e^{\pi s}O_{\kappa,(s,t)}, \quad \forall \kappa
\]
for some error term \( O_{\kappa,(s,t)} \).

**Remark A.3.** In [Abo10], (1), (2), and (6) is required to be strictly compatible to \( \mathcal{S} \) which means
\[
\kappa^* \alpha_S = v_\kappa d\tau
\]
and similar for \( \beta_S \) and \( J_S \). In particular, it depends on the choice \( \mathcal{S} \). An idea of asymptotic compatibility is due to [Sei18]. It is more flexible because it only depends smoothly on \( |S| \), not on the choice of the set of ends \( \{ \kappa \} \). It allows us to choose a special kind of ends to deal with popsicle structures later on.

Finally, we define

**Definition A.4.** For a fixed surface \( S \), Hamiltonian \( H \in \mathcal{H}(M) \) and absolutely bounded time-dependent Hamiltonian \( F \), a Floer data \( \text{Floer}_S \) consists of

\( \kappa^* a_S = v_\kappa d\tau \)

1. A collection of weighted strip and cylinder data \( \mathcal{S} \);
2. A basic 1-form \( \alpha_S \) and secondary 1-form \( \beta_S \) asymptotically compatible to \( (S, \mathcal{S}) \);
3. An \( (S, \mathcal{S}) \)-adapted time-shifting map \( a_S \);
4. An \( S \)-dependent, \( (S, \mathcal{S}, H, F) \)-compatible Hamiltonian \( H_S \) and \( F_S \);
5. An \( S \)-dependent, asymptotically compatible almost complex structure \( J_S \).

Also, we say \( \text{Floer}_S^1 \) and \( \text{Floer}_S^2 \) are conformally equivalent if \( \text{Floer}_S^2 \) is a rescaling by Liouville flow of \( \text{Floer}_S^1 \), up to constant ambiguity in the Hamiltonian terms.

In the simplest case of a strip \( S \in S_{0;1,1} \) or a cylinder \( S \in S_{1,1;0} \), we choose a canonical strip-like/cylindrical end with weights 1 for all ends. A form \( dt \) is a compatible sub-closed one form. A universal and consistent choice of Floer data is a choice of Floer data \( \text{Floer}_S \) for all \( S \in S_{m;n,1} \) which varies smoothly over the moduli space. Since the space of Floer data is contractible, we can extend it to \( S_{m;n,1} \).

**Definition A.5.** Let \( \gamma_i \in \mathcal{S} \) be a time-1 Hamiltonian orbits and \( a_j \in \chi(L_{j-1}, L_j) \), \( j = 1, \ldots, n \) and \( a_0 \in \chi(L_n, L_0) \) be Hamiltonian chords. Define the moduli space \( \mathcal{M}_{m,n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0) \) of maps
\[
\left\{ u: S \to M \mid S \in S_{m;n,1} \right\}
\]
satisfying the inhomogeneous Cauchy-Riemann equation with respect to $J_S$
\[
(du - X_{H_S} \otimes \alpha_S - X_{F_S} \otimes \beta_S)^{0,1} = 0
\]
and the following asymptotic boundary conditions;
\[
\lim_{s \to \infty} u \circ \varepsilon^k_+(s, \cdot) = a_k \\
\lim_{s \to -\infty} u \circ \varepsilon^0_-(s, \cdot) = a_0 \\
\lim_{s \to \infty} u \circ \delta^+_i(s, \cdot) = \gamma_i
\]
\[u(z) \in \psi^{a_z}(L_i, z \in \partial_i S, \text{ an } i\text{-th boundary component of } S.\]

Additional perturbation terms $\alpha_S, \beta_S, H_S, F_S$ come from the universal and consistent choice of Floer data $\text{Floer}_S$.

The following compactness and transversality result is standard.

**Lemma A.6.** For a generic choice of universal and consistent Floer data, 

1. The moduli spaces $\mathcal{M}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ are compact.
2. For a given input $\gamma_i$, $i = 1, \ldots, m$ and $a_j$, $j = 1, \ldots, n$, there are only finitely many $a_0$ for which $\mathcal{M}_{m,n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ is non-empty.
3. It is a manifold of dimension
\[
\dim_{\mathbb{R}} \mathcal{M}_{m,n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0) = (2m + n - 2) + \deg a_0 - \sum_{i=1}^{m} \deg a_i - \sum_{j=1}^{n} \deg a_j
\]

**Proof.** See [Abo10]. For a compactness result, one need to assure that the energy of pseudo-holomorphic curves are a priori bounded in $M$. This estimate is carefully done therein. Transversality result is a standard application of Sard-Smale argument. The dimension formula is also a standard application of Atiyah-Singer index theorem on a linearized Fredholm operator. $\square$

When $\mathcal{M}_{m,n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ has dimension zero so that it is rigid, then a map $u : S \to M$ in that moduli space is isolated. An orientation of the moduli space provides a canonical isomorphism
\[
Q_u : \bigotimes_{i=1}^{m} o_{\gamma_i} \otimes \bigotimes_{j=1}^{n} o_{a_j} \to o_{a_0}.
\]
We sum up $Q_u$ for all $u \in \mathcal{M}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)$ and all $a_0$ and define
\[
F_{m,n,1}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n) = \sum_{\dim_{\mathbb{R}} \mathcal{M}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0) = 0} \sum_{u \in \mathcal{M}(\gamma_1, \ldots, \gamma_m; a_1, \ldots, a_n, a_0)} Q_u \left( \bigotimes_{i=1}^{m} o_{\gamma_i} \otimes \bigotimes_{j=1}^{n} o_{a_j} \right)
\]
We define $\mathcal{M}(\gamma_1, \ldots, \gamma_m, \gamma_0)$ and $F_{m,1,0}$ in a similar way.

Let us first recall the our setup and explain the detailed construction.

**A.3. Wrapped Fukaya category and symplectic cohomology.** Symplectic cohomology $SH^*(M)$ is a version of Hamiltonian Floer cohomology for Liouville domain introduced by Cieliebak, Floer and Hofer [CFH05] and Viterbo [Vit99]. For Lagrangian submanifolds in $M$ that are either compact or cylindrical at infinity, a wrapped Fukaya $A_\infty$-category $\mathcal{WF}(M)$ was defined by Abouzaid-Seidel [AS10]. We use the version with quadratic Hamiltonian of Abouzaid [Abo12] which we recall briefly here.

For two Lagrangian submanifolds $L_0, L_1 \in \mathcal{W}$, a wrapped Floer cochain complex is a vector space
\[
CW^*(L_0, L_1; H) = \bigoplus_{a \in \chi(L_0, L_1; H)} o_a
\]
It is graded by the degree \( \deg o_a \). We will use the notation \( a \) instead of \( o_a \) for generators, and \( CW^*(L_0, L_1) \) instead of \( CW^*(L_0, L_1; H) \) if it cause no confusion.

**Definition A.7.** A wrapped Fukaya category \( \mathcal{W} \mathcal{F}(M) \) consists of a set of objects \( \mathcal{W} \) with the space of morphisms \( CW^*(L_0, L_1) \) for \( L_i \in \mathcal{W} \) equipped with an \( A_\infty \) structure

\[
m_k : CW^*(L_0, L_1) \otimes \cdots \otimes CW^*(L_{k-1}, L_k) \to CW^*(L_0, L_k)
\]

\[
m_k(a_1, \ldots, a_k) = (-1)^{s_k} F_{k;1,1}(a_1, \ldots, a_k);
\]

\[
\alpha_k = \sum_{i=1}^{k} i \cdot \deg a_i;
\]

The Liouville flow \( \psi^t \) for time \( \log(\rho) \) defines a canonical isomorphism

\[
CW^*(L_0, L_1; H, J_t) \cong CW^* \left( \psi^t L_0, \psi^t L_1; \frac{H \circ \psi^t}{\rho}, (\psi^t)^* J_t \right)
\]

Also, \( H \circ \psi^t = \rho^2 t^2 \) at the cylindrical end. Therefore \( H \circ \psi^t \in \mathcal{H}(M) \). The proof of \( A_\infty \) relation follows from the degeneration patterns of pseudo-holomorphic discs which correspond to a codimension 1 boundary strata of Gromov bordification \( \mathcal{M}_{0,k,1}(a_1, \ldots, a_k, a_0) \). In particular, we have \( m_1^2 = 0 \). Its \( m_1 \)-cohomology \( HW^*(L_0, L_1) \) is called the wrapped Floer cohomology.

We are also interested in the symplectic cohomology of Liouville manifold.

**Definition A.8.** A symplectic cochain complex is a \( \mathbb{Z} \)-graded cochain complex

\[
CH^*(M; H_S) = \bigoplus_{\gamma \in \mathcal{O}(M; H_S)} o_\gamma
\]

graded by the degree \( \deg o_\gamma \). We will use the notation \( \gamma \) instead of \( o_\gamma \) for generators if it cause no confusion. A differential of this complex is

\[
d_{CH}(o_\gamma) = (-1)^{\deg o_\gamma} F_{1,1;0}(\gamma_1).
\]

Recall that \( F_{1,1;0}(\gamma_1) \) is given by a counting of a zero-dimensional component of a moduli space of pseudo-holomorphic cylinders \( \mathcal{M}_{1,1;0}(\gamma_1, \gamma_0) \). Its cohomology is denoted as \( SH^*(M) \).

A ring structure on its cohomology is induced from

\[
CH^*(M)^{\otimes 2} \to CH^*(M) : (\gamma_1, \gamma_2) \mapsto (-1)^{\deg \gamma_1} F_{2,1;0}(\gamma_1, \gamma_2)
\]

Let us recall the definition of a closed-open map to Hochschild cochain complex of wrapped Fukaya category.

**Definition A.9.** A closed-open map is a map

\[
CO = \{ CO_{i} \}_{i \geq 0} : CH^*(M) \to CC^*( \mathcal{W} \mathcal{F}(M), \mathcal{W} \mathcal{F}(M))
\]

\[
CO_{i}(\gamma)(a_1, \ldots, a_i) := (-1)^{\deg \gamma} F_{i,i;1,1}(\gamma; a_1, \ldots, a_i, a_0)
\]

A degeneration pattern of a moduli space \( \mathcal{M}_{1,n,1}(\gamma; a_1, \ldots, a_n, a_0) \) proves that \( CO \) is a cochain map.

**APPENDIX B. COMPACTIFICATIONS OF POPSICLE MODULI SPACES**

The Gromov bordification \( \overline{P}_{n,d} \) is larger than the original reference. Its boundary strata contains sphere bubbles as depicted in Figure 43. We remark that this extra component does not contribute to \( A_\infty \)-operations that we define.
Let us start by recalling a gluing process of two popsicles from \cite{Sei18}. Start with two surfaces $S_1$ and $S_2$, choice of $\zeta_1^+ \in \Sigma_{S_1}^{+,bd}$ and $\zeta_2^- \in \Sigma_{S_2}^{-,bd}$ and a gluing parameter $\gamma \in (0,1)$. Denote the coordinates on those ends by $(s_1,t)$. A parameter $\gamma$ determines a gluing length $l = -\frac{\log(\gamma)}{\pi} \in (0,\infty)$. We construct $S_\gamma$ as
\[
S_\gamma = (S_1 \setminus \epsilon_{\zeta_1}(\{|s_1| > l\}) \cup (S_2 \setminus \epsilon_{\zeta_2}(\{|s_2| < -l\})),
\]
\[
\epsilon_{\zeta_1}(s_1,t) \sim \epsilon_{\zeta_2}(s_2,t) \quad s_2 = s_1 - l.
\]

**Definition B.1.** (\cite{Sei18}) Let $S \in S_{m,n,1}$. A choice of strip-like ends $(\epsilon^-_0,\epsilon^+_1,\ldots,\epsilon^+_n)$ is called rational if the following condition holds.

- $\epsilon^-_0$ extends to an isomorphism $Z \to \tilde{S}$ such that $\epsilon^-_0(-\infty) = z_0$
- $\epsilon^+_k$ extends to an isomorphism $Z \to \tilde{S}$ such that $\epsilon^+_k(-\infty) = z_0$ and $\epsilon^+_k(+\infty) = z_k$

where $Z$ is an infinite strip.

$\epsilon_0$ determines an isomorphism $\tilde{S} \simeq \mathbb{H}$ such that $x_0$ corresponds to $i\infty$ and the image of $\epsilon_0^-$ is $\{|z| \geq 1\} \subset \mathbb{H}$. The rest $\{z_k\}_{k \geq 1}$ can be identified by the real numbers denoted by $x_k \in \mathbb{R} = \partial \mathbb{H}$ and corresponding end $\epsilon^+_k$ is determined by the radius the radius $\rho_k$ of the semicircle centered at $x_k$. Finally, the sprinkle $z^+_k \in Q_{\phi(l)}$ is determined by a positive real number $y_l$ and identified with a point $x_{\phi(l)} + iy_l \in \mathbb{H}$.

Rational ends help us to put a canonical popsicle structure on a gluing of two. Let $S_\gamma$ be a gluing of $S_1 \in P_{n_1,F_1}$ and $S_2 \in P_{n_2,F_2}$ along $z_{0,1} \in S_1$ and $z_{1,2} \in S_2$. Then positive boundary markings of $S_\gamma$ are $(n_1 + n_2 - 1)$ points on $\partial \mathbb{H}$:
\[
z_k \leftrightarrow x_{k,\gamma} = \begin{cases} 
  x_{k,1} & k < i, \\
  x_{i,1} + (\rho_{1,i} \times \gamma)x_{k-i+1,2} & i \leq k \leq i + n_2 - 1, \\
  x_{k-n_2+1,1} & i + n_2 \leq k.
\end{cases}
\]
Moreover, sprinkles are given as:
\[
z^+_k \leftrightarrow x_{\phi(f),\gamma} + iy_{f,\gamma} = \begin{cases} 
  x_{\phi_1(f),1} + iy_f & f \in F_1, \phi_1(f) \neq i, \\
  x_{i,1} + (\rho_{1,i} \times \gamma)x_{\phi_1(f)-i+1,2} + iy_f & f \in F_1, \phi_1(f) = i, \\
  x_{i,1} + (\rho_{1,i} \times \gamma)x_{\phi_2(f),2} + iy_{\phi_2(f)} & f \in F_2.
\end{cases}
\]

Now we move on to a description of a compactified moduli space.

**Definition B.2.** A rooted ribbon tree is a tree $T$ with

- a root and leaves: $d+1$ semi-infinite edges with a preferred choice of one among them. The preferred one is called the root, and the rest is called leaves.
- ribbon structure: a cyclic order on adjacent edges for each vertex $v$ of $T$.
The root and leaves determine a direction on edges. Each vertex \( v \) has a single adjacent edge \( e_0 \) emanating from the root, and the rest are cyclically ordered as \([e_1, \ldots, e_{\text{val}(v)−1}]\).

At first, let us describe a model for sphere bubbles.

**Definition B.3.** Let \( T \) be a rooted tree with no leaves. An \( F \)-flavoured icecream modelled on \( T \) consists of spheres \( \mathbb{P}^1_w \) for each vertex \( w \) with a following decorations.

- an anti-holomorphic involution \( \tau_w : \mathbb{P}^1_w \to \mathbb{P}^1_w \)
- \( \text{val}(w) \)-special points which is invariant under \( \tau_w \) and respects a cyclic order at \( w \).
- decomposition of a set of flavour \( F = \bigsqcup_w F_w \);
- a sprinkle function \( x_w : F_w \to \mathbb{P}^1_w \) whose image is also \( \tau_w \)-invariant and disjoint from special points.

We call \( F \)-flavoured icecream is stable if there are more than three special points on each \( \mathbb{P}^1_w \).

The reader would immediately notice that a \( \phi \)-flavoured icecream is just a model for a sphere bubble when two or more sprinkles on a same popsicle stick collide. A tree \( T \) only determines a configuration of a sphere bubble so it has no leaves. Extra markings other than nodal points are determined from a sprinkle function \( x_w \). An involution \( \tau \) comes from the following reason; a popsicle stick can be considered as a fixed locus of an anti-holomorphic involution on a disc. Whenever several sprinkles collide, a sphere bubble also carries an involution \( \tau \). All nodal points and sprinkles should be \( \tau \)-invariant.

**Definition B.4.** A \( \phi \)-flavoured broken popsicle with icecreams modeled on a rooted tree \( T \) and a set of rooted trees with no leaves \( \{T^\prime_{v,i}\}_{v,i} \) consists of

- decomposition of \( F \): decomposition
  \[ F = \bigcup_v F_v, \quad \text{where } F_v = F_v^\prime \cup \bigsqcup_i F^\prime_{v,i} \]
- decomposition of \( \phi \): a map \( \phi_v : F_v \to \{1, \ldots, \text{val}(v)−1\} \) satisfying the following two conditions.
  1. for each \( f \in F_v^\prime \), the vertex \( v \) must lie on the unique path from the root to \( e_{\text{val}(v)} \) at \( v \);
  2. an image \( \phi_v(F^\prime_{v,i}) \) is a single point.
- popsicles: an assignment of \( \phi_v \)-flavoured popsicle on each \( v \) such that the sprinkle map \( x_v \) is injective on \( F_v^\prime \) and constant on \( x_v(F^\prime_{v,i}) \). Images \( x_v(F^\prime_{v,i}) \) and \( x_v(F^\prime_{v,i}) \) are different.
- icecreams: a stable \( F^\prime_{v,i} \)-flavoured icecream structure modeled on \( T^\prime_{v,i} \) with no leaves for each \( (v,i) \).

A \( \phi \)-flavoured broken popsicle with icecreams is called stable if all popsicles and icecreams are stable.

Although the definition looks complicated, the geometric intuition should be clear. A decomposition of \( F_v \) consists of two parts; \( F_v^\prime \) is a part on which \( \phi \) is injective, and we assign an ordinary popsicle structure according to its image. On the other hand, \( F^\prime_{v,i} \) is a set of sprinkles that collides at the point \( \phi(F^\prime_{v,i}) \). We attach a sphere bubble, or icecream on that point. Notice that \(|F^\prime_{v,i}| \geq 2 \) when it is stable.

A moduli space of \( \phi \)-flavoured broken popsicle with icecream modeled on \( T \) is a product
\[
P^T_{n,F,\phi} = \prod_v P^{|\text{val}(v)−1,F_v,\phi_v|_\text{f}} \times \prod_{T^\prime_{v,i}} \mathbb{P}^{\vert F^\prime_{v,i} \vert + \vert \text{edge}(T^\prime_{v,i}) \vert − 3\vert \text{vert}(T^\prime_{v,i}) \vert}
\]
Take the disjoint union of those spaces, and denote it by
\[
\overline{P}^T_{n,F,\phi} := \bigsqcup_{T,F} P^T_{n,F,\phi^*}
\]

**Proposition B.5.** \( \overline{P}^T_{n,F,\phi} \) is a compact smooth manifold with corners.
Proof. The boundary strata is a mixture of two disjoint degenerations; one is when an underlying disc component breaks into several pieces, and the other is when several sprinkles collide.

The first part can be covered by the result of [AS10]. If we forget about icecream structure and simply allow a sprinkle function \( x_v \) may not be injective, then the corresponding moduli is the same as their moduli spaces of \( \phi \)-flavoured popsicles. They construct an algebro-geometric model (called holomorphic lollipops) for such moduli spaces and prove that a standard gluing procedure along strip-like end gives a structure of a smooth manifold with corner on the moduli space.

Then, a second kind of degeneration can be covered easily. This is essentially a compactification of a configuration space of points on \( S^1 \) (see Fulton–Macpherson [FM94]). Consider a fiber of a forgetful map \( \pi_v : P_{\text{val}(v) - 1, F_v, \phi_v} \to S_{\text{val}(v) - 1, 1} \).

It is an open complement of \( \mathbb{R}^{\vert F_v \vert} \) given by

\[
\left\{ \bar{x} \in \mathbb{R}^{\vert F_v \vert} : x_v(f_1) \neq x_v(f_2), \ \forall f_i \in F'_v \right\}
\]

A value \( x_v(F'_v, i) \) determines a limit point on a naive compactified fiber \( \mathbb{R}^{\vert F \vert} \). We perform a consecutive oriented real blow-up on the locus where two or more coordinate coincides until all coordinates are finally distinguished to each other. A rooted tree \( T'_{v, i} \) corresponds exactly to a possible boundary strata of this blow-ups. The number of vertices of \( T'_{v, i} \) determines the number of blow-ups you perform to reach that strata. A value of sprinkles \( x_w \) determines coordinates of a moduli.

A real oriented blow-ups of a smooth compact manifold with corners is again a smooth compact manifold with corners. We finish the proof. □

The structure of a manifold with corners are compatible to a canonical inclusion

\[ P_{n, F, \phi} \subset S_{\vert F \vert, n, 1}. \]

**APPENDIX C. DETERMINATION OF AUSLANDER-REITEN SEQUENCE**

This appendix is due to Osamu Iyama. The result in this appendix guarantees that the exact sequences from Lagrangian Floer theory indeed coincides with those in Auslander-Reiten theory.

**Definition C.1.** Let \( \mathcal{C} \) be an exact category. We say that an exact sequence \( 0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0 \) is determined by its terms if, for each exact sequence \( 0 \to A' \xrightarrow{a'} B' \xrightarrow{b'} C \to 0 \), there exists a commutative diagram whose vertical maps are isomorphism:

\[
\begin{array}{cccccc}
0 & \to & A & \xrightarrow{a} & B & \xrightarrow{b} & C & \to & 0 \\
& \downarrow{\cong} & \downarrow{\cong} & \downarrow{\cong} & \downarrow{\cong} & \\
0 & \to & A & \xrightarrow{a'} & B' & \xrightarrow{b'} & C & \to & 0
\end{array}
\]

For a Cohen-Macaulay local ring \( R \), we denote by \( \text{CM}_R \) the category of maximal Cohen-Macaulay \( R \)-modules.

**Theorem C.2.** Let \( R \) be a complete local Cohen-Macaulay isolated singularity.

1. Each split exact sequence in \( \text{CM}_R \) is determined by its terms.
2. Each almost split sequence in \( \text{CM}_R \) is determined by its terms.
Let $\mathcal{E}$ be an exact category with enough projectives. We denote by $\underline{\mathcal{E}}$ the stable category. It has the same objects as $\mathcal{E}$, and the morphisms between $X, Y \in \mathcal{E}$ are given by
\[ \underline{\text{Hom}}_{\mathcal{E}}(X, Y) := \text{Hom}_{\mathcal{E}}(X, Y)/P(X, Y), \]
where $P(X, Y)$ is the subgroup of $\text{Hom}_{\mathcal{E}}(X, Y)$ consisting of morphisms factoring through projective objects in $\mathcal{E}$. For example, we denote by $\underline{\text{CMR}}$ the stable category of CMR.

**Proposition C.3.** Let $R$ be a complete local Cohen-Macaulay isolated singularity. Then the stable category $\underline{\text{CMR}}$ is Hom-finite, that is, the $R$-module $\underline{\text{Hom}}_{\mathcal{R}}(X, Y)$ has finite length for each $X, Y \in \text{CMR}$.

We are ready to prove Theorem C.2.

**Proof of Theorem C.2**

(a) We need to show that each exact sequence $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ in CMR with $B \cong A \oplus C$ splits.

It is well-known that we have an induced exact sequence
\[ \underline{\text{Hom}}(\cdot, A) \xrightarrow{a} \underline{\text{Hom}}(\cdot, B) \xrightarrow{b} \underline{\text{Hom}}(\cdot, C) \]
of functors on $\text{mod } R$. Evaluating $C$, we obtain an exact sequence
\[ \underline{\text{Hom}}(C, A) \xrightarrow{a} \underline{\text{Hom}}(C, B) \xrightarrow{b} \underline{\text{Hom}}(C, C). \]
On the other hand, since $B \cong A \oplus C$,
\[ \text{length}_{\mathcal{R}}\underline{\text{Hom}}(C, A) + \text{length}_{\mathcal{R}}\underline{\text{Hom}}(C, C) = \text{length}_{\mathcal{R}}\underline{\text{Hom}}(C, B) \]
holds, where each term is finite by Proposition C.3. In particular, the right map in (C.1) has to be surjective. Thus
\[ \text{End}_{\mathcal{R}}(C) = b \cdot \text{Hom}_{\mathcal{R}}(C, B) + P(C, C) \]
holds. Since $b : B \to C$ is surjective, each morphism in $P(C, C)$ factors through $b$. Thus $P(C, C) \subset b \cdot \text{Hom}(C, B)$ and hence $\text{End}_{\mathcal{R}}(C) = b \cdot \text{Hom}_{\mathcal{R}}(C, B)$ holds. In particular, $b$ is a split epimorphism.

(b) Let $\alpha : 0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ be an almost split sequence, and $\beta : 0 \to A \xrightarrow{a'} B \xrightarrow{b'} C \to 0$ an arbitrary exact sequence.

If $\beta$ splits, then (a) implies that $\alpha$ also splits, a contradiction. Thus $\beta$ does not split. Since $b : B \to A$ is right almost split and $b' : B \to A$ is not a split epimorphism, there exists $f : B \to B$ such that $b' = bf$. Thus we obtain a commutative diagram:
\[
\begin{array}{c}
0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b'} C \longrightarrow 0 \\
\downarrow g & \downarrow f & \downarrow \text{id} \\
0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0
\end{array}
\]
The left commutative square gives rise to an exact sequence
\[ 0 \to A \xrightarrow{\cdot g} A \oplus B \xrightarrow{[a \ f]} B \to 0. \]
This sequence splits by (a). In particular, $[a \ f] : A \oplus B \to B$ is a split epimorphism. Since $a : A \to B$ belongs to the radical of the Krull-Schmidt category CMR, the morphism $f : B \to B$ has to be a split epimorphism. Thus $f$ is an isomorphism, and so is $g$. \qed

Note that above proof works for morphisms in $\underline{\text{CMR}}$ since only difference is that we consider morphisms up to homotopy in $\underline{\text{CMR}}$.

**Corollary C.4.** Theorem C.3 holds in $\underline{\text{CMR}}$ also.
Remark C.5. Not all exact sequences are determined by their terms. A simple example is given by a simple singularity $R = k[[x, y, z]]/(x^3 + xy^2 + z^2)$ of type $D_4$. For a unique indecomposable object $X \in \text{CMR}$ with rank 2, there exists a one-parameter family of non-isomorphic exact sequences of the form $0 \to X \to R^\oplus 2 \to X \to 0$.

The stable matrix factorization category is equivalent to $\text{CMR}$. Hence we get a following corollary for Lagrangian exact sequences.

Corollary C.6. Let $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ be some AR sequence in $\text{CMR}$. If Lagrangian surgery exact sequence is given by $0 \to A \xrightarrow{a'} B \xrightarrow{b'} C \to 0$, this exact sequence is isomorphic to AR sequence.

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