HARDY INEQUALITIES FOR THE FRACTIONAL POWERS OF THE GRUSHIN OPERATOR

MANLI SONG
School of Mathematics and Statistics, Northwestern Polytechnical University
Xi'an 710129, China

JINGGANG TAN∗
Departamento de Matemática, Universidad Técnica Federico Santa María
Avda. España 1680, Valparaíso, Chile
Department of Mathematics, Jianghan University
Wuhan, Hubei, 430056, China

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Abstract. We establish uncertainty principles and Hardy inequalities for the fractional Grushin operator, which are reduced to those inequalities for the fractional generalized sublaplacian. The key ingredients to obtain them are an explicit integral representation and a ground state representation for the fractional powers of generalized sublaplacian.

1. Introduction. The uncertainty principle is a fundamental attribute of quantum mechanics. In the case of the Laplacian $\Delta = -\sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ on the Euclidean space $\mathbb{R}^d$ ($d \geq 3$), it states that

$$||f||_{L^2(\mathbb{R}^d)}^4 \leq C \int_{\mathbb{R}^d} |x|^2 |f(x)|^2 dx \int_{\mathbb{R}^d} |\Delta^{\frac{1}{2}} f(x)|^2 dx,$$

which can be deduced from Hardy inequality by Cauchy-Schwarz inequality. Here the fractional powers of Laplacian $\Delta^s = (-\Delta)^s$ is defined via spectral decomposition by

$$\Delta^s f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{2s} \hat{f}(\xi) e^{ix \xi} d\xi,$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix \xi}dx$ is the Fourier transform.

The well-known Hardy inequality of the Laplacian is given by

$$C_d \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^d} |\nabla f|^2 dx,$$  (1.1)

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∗ Corresponding author.
where $\nabla$ is the standard gradient operator on $\mathbb{R}^d$ and the sharp constant $C_d$ is given by
\[ C_d = \frac{(d - 2)^2}{4}. \]
The inequality (1.1) can be proved equivalent to
\[ C_d \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \, dx \leq \langle \Delta f, f \rangle, \]
where the symbol $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\mathbb{R}^d)$. Moreover, Hardy inequality has also been generalized to fractional powers of Laplacian $\Delta^s$. Now we state the Hardy type inequality for the fractional Laplacian $\Delta^s$. For $0 < s < d/2$, $f \in C_{0}^\infty(\mathbb{R}^d)$, we have
\[ C_{d,s} \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2s}} \, dx \leq \langle \Delta^s f, f \rangle, \tag{1.2} \]
where the sharp constant $C_{d,s}$ (see [2]) is given as
\[ C_{d,s} = 4^s \left( \frac{\Gamma \left( \frac{d+2s}{4} \right)}{\Gamma \left( \frac{d-2s}{4} \right)} \right)^2. \]
However, it is known the equality is never achieved. Later, Frank, Lieb and Seiringer [6] used a different approach called ground state representation to prove the inequality (1.2) when $0 < s < \min\{1, d/2\}$. Hardy’s inequality for the fractional powers of Laplacian has been extensively studied, referring the reader to [2, 7, 19, 14].

Recently, inspired by the ideas in Frank, Lieb and Seiringer [6], Hardy type inequalities concerning fractional powers have been developed to the Heisenberg group and the Grushin operator. Roncal and Thangavelu [10] proved analogues of Hardy type inequalities for fractional powers of the sublaplacian on the Heisenberg group and also obtained corresponding versions of Heisenberg uncertainty inequality.

Balhara [1] introduced the fractional powers of the Grushin operator and proved the non-homogeneous Hardy inequality for the fractional powers of the Grushin operator. They found an integral representation and a ground state representation for the fractional powers of generalized sublaplacian, which related to the Grushin operator by the Hecke-Bochner formula.

We are concerned with a Hardy inequality for the fractional powers of the Grushin operator involving a homogeneous weight. We consider the Grushin operator defined on $\mathbb{R}^{n+1}$ ($n \geq 2$) by
\[ \mathcal{G} = -\frac{1}{2} \left( \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + |x|^2 \frac{\partial^2}{\partial w^2} \right). \]
Instead of the fractional powers $\mathcal{G}^s$, we will study the related operators $\mathcal{G}_s$ and $\Lambda_s = \mathcal{G}_1^{-1-s} \mathcal{G}$ which behave like $\mathcal{G}^s$, see Section 2.1 for definitions, and prove the homogeneous Hardy inequality. Since the operators $\Lambda_s \mathcal{G}^{-s}$ and $\mathcal{G}_s \mathcal{G}^{-s}$ are bounded on $L^2(\mathbb{R}^{n+1})$, we can deduce the corresponding inequalities for $\mathcal{G}^s$ from the inequalities for $\Lambda_s$ and $\mathcal{G}_s$. Given an appropriate function in the Grushin space, it can be expressed by a form of solid harmonics. By spectral decomposition and Hecke-Bochner formula, the action of the fractional powers of the Grushin operator $\Lambda_s$ can be transformed to the fractional powers of generalized sublaplacian. Then it suffices to prove the homogeneous Hardy inequality involving the fractional powers of generalized sublaplacian. We obtain its fundamental solution and a positive modified kernel that derived from the heat kernel. Here the modified heat kernel
is adopted to establish an integral representation and the fundamental solution is used to compute a ground state representation of the fractional powers of generalized sublaplacian. Finally, as a consequence of the Hardy inequalities, we also obtain two versions of Heisenberg uncertainty for the fractional Grushin operator.

Now we state our Hardy inequality for $\Lambda_s$ with a homogeneous weight.

**Theorem 1.1.** For $0 < s < 1$, we have
\[
\frac{2^{2s} \Gamma\left(\frac{n+2s}{4}\right)^2}{\Gamma(1-s) \left(\Gamma\left(\frac{n}{4}\right)\right)^2} \int_{\mathbb{R}^{n+1}} \frac{|f(x,w)|^2}{(|x|^4 + 4w^2)^{\frac{s}{2}}} \, dxdw \leq \langle \Lambda_s f, f \rangle
\]
for all $f \in C_0^\infty(\mathbb{R}^{n+1})$.

Since we shall show that the operator $V_s = \Lambda_s \mathcal{G}^{-s}$ is bounded on $L^2(\mathbb{R}^{n+1})$ and its operator norm is given by the constant,
\[
\|V_s\| = \sup_{k \geq 0} \left(\frac{2k + n}{4}\right)^{1-s} \frac{\Gamma\left(\frac{2k+n}{4} + \frac{s}{2}\right)}{\Gamma\left(\frac{2k+n}{4} + \frac{n}{2}\right)} \leq \frac{n + 2(2-s)}{n + 2s},
\]
we can immediately deduce a Hardy inequality for the pure fractional power $\mathcal{G}^s$.

**Theorem 1.2.** For $0 < s < 1$, we have
\[
\frac{2^{2s} \Gamma\left(\frac{n+2s}{4}\right)^2}{\Gamma(1-s) \left(\Gamma\left(\frac{n}{4}\right)\right)^2} \int_{\mathbb{R}^{n+1}} \frac{|f(x,w)|^2}{(|x|^4 + 4w^2)^{\frac{s}{2}}} \, dxdw \leq \|V_s\| \langle \mathcal{G}^s f, f \rangle
\]
for all $f \in C_0^\infty(\mathbb{R}^{n+1})$.

We denote by $W^{s,2}(\mathbb{R}^{n+1})$ the Sobolev space consisting of all $L^2$ functions $f$ such that $\mathcal{G}^{s/2} f \in L^2$. Therefore, it is a Sobolev space naturally associated to $\mathcal{G}$. Note that an $L^2$ function $f$ belongs to $W^{s,2}(\mathbb{R}^{n+1})$ if and only if $\mathcal{G}_s f \in L^2$. We also give another version of Hardy inequality for $\mathcal{G}_s$ involving a non-homogeneous weight.

**Theorem 1.3.** For $0 < s < \frac{n+2}{4}$ and $\delta > 0$, we have
\[
(4\delta)^n \frac{\Gamma\left(\frac{n+2s+1}{2}\right)^2}{\Gamma\left(\frac{n-2s+1}{2}\right)^2} \int_{\mathbb{R}^{n+1}} \frac{|f(x,w)|^2}{\left(\delta + \frac{|x|^2}{2} + w^2\right)^{\frac{s+1}{2}}} \, dxdw \leq \langle \mathcal{G}_s f, f \rangle
\]
for all $f \in W^{s,2}(\mathbb{R}^{n+1})$. Moreover, the inequality is optimal and the equality is achieved by the function $f(x,w) = (\delta + |x|^2/2 + w^2)^{-\frac{n-2s}{2}}$.

It can be shown that the operator $U_s = \mathcal{G}_s \mathcal{G}^{-s}$ is bounded and its operator norm is given by
\[
\|U_s\| = \sup_{k \geq 0} \left(\frac{2k + n}{4}\right)^{-s} \frac{\Gamma\left(\frac{2k+n}{4} + \frac{1+s}{2}\right)}{\Gamma\left(\frac{2k+n}{4} + \frac{n}{2}\right)} \leq 1,
\]
From Theorem 1.3, we can immediately obtain a non-homogeneous Hardy inequality for the pure fractional $\mathcal{G}^s$.

**Theorem 1.4.** For $0 < s < \frac{n+2}{4}$ and $\delta > 0$, we have
\[
(4\delta)^n \frac{\Gamma\left(\frac{n+2s+1}{2}\right)^2}{\Gamma\left(\frac{n-2s+1}{2}\right)^2} \int_{\mathbb{R}^{n+1}} \frac{|f(x,w)|^2}{\left(\delta + \frac{|x|^2}{2} + w^2\right)^{\frac{s+1}{2}}} \, dxdw \leq \|U_s\| \langle \mathcal{G}^s f, f \rangle
\]
for all $f \in W^{s,2}(\mathbb{R}^{n+1})$. 
Finally, Heisenberg type uncertainty inequalities for $G_s$ and $\Lambda_s$ follow from our Hardy inequalities.

**Theorem 1.5.** For all functions $f \in W^{s,2}(\mathbb{R}^{n+1})$, we have

$$ (4\delta)^s \left( \frac{\Gamma(n/2+s+1)}{\Gamma(n/2+s+1)} \right)^2 \left( \int_{\mathbb{R}^{n+1}} |f(x,w)|^2 \, dx \, dw \right)^{2/s} \leq \left( \int_{\mathbb{R}^{n+1}} |f(x,w)|^2 \left( \delta + \frac{|x|^2}{2} + w^2 \right)^s \, dx \, dw \right) \langle G_s f, f \rangle $$

provided $0 < s < \frac{n+2}{4}$ and $\delta > 0$. In the smaller range $0 < s < 1$, we have

$$ \frac{2^{2s} \left( \frac{\Gamma(n+2s/2)}{\Gamma(n+2s)} \right)^2}{\Gamma(1-s) \left( \Gamma\left( \frac{n+2s}{2} \right) \right)^2} \left( \int_{\mathbb{R}^{n+1}} |f(x,w)|^2 \, dx \, dw \right)^{2/s} \leq \left( \int_{\mathbb{R}^{n+1}} |f(x,w)|^2 \left( |x|^4 + 4w^2 \right)^{s/2} \, dx \, dw \right) \langle \Lambda_s f, f \rangle. $$

For $\alpha \geq 0$. Let the space $X = \mathbb{R}^+ \times \mathbb{R}$ equipped with measure $d\mu_\alpha(x,w) = x^{2\alpha+1} \, dx \, dw$, where $dx$ and $dw$ are the standard Lebesgue measures. We denote by $L^p(X, d\mu_\alpha)$ ($1 \leq p \leq \infty$) the classical Lebesgue spaces with respect to the measure $d\mu_\alpha$ endowed with the norm $\| \cdot \|_p$. Let $\langle \cdot, \cdot \rangle_\alpha$ stand for the inner product derived by $L^2(X, d\mu_\alpha)$. The generalized sublaplacian $L_\alpha$ is defined on $X$ by

$$ L_\alpha = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x} \frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial w^2} \right). $$

To establish the Hardy inequality in Theorem 1.1 and 1.3, we turn to demonstrate the reduced corresponding inequalities for the fractional powers of generalized sublaplacian $L_\alpha$, which are stated as follows. For the definition of the following fractional operators $L_\alpha^s$, $L_{\alpha,s}$ and $\Lambda_{\alpha,s}$, we see Section 2.3.

**Theorem 1.6.** For $\alpha \geq 0$ and $0 < s < 1$, we have

$$ \frac{2^{2s} \left( \frac{\Gamma(n+2s/2)}{\Gamma(n+2s)} \right)^2}{\Gamma(1-s) \left( \Gamma\left( \frac{n+2s}{2} \right) \right)^2} \int_X \frac{|f(x,w)|^2}{x^{4s/2} + 4w^2} \, d\mu_\alpha(x,w) \leq \langle \Lambda_{\alpha,s} f, f \rangle $$

for all $f \in C_0^\infty(X)$.

The desired result comes from the combination of an integral representation, the fundamental solution and a ground state representation for the fractional generalized sublaplacian. Since the operator $V_{\alpha,s} = \Lambda_{\alpha,s} L_\alpha^{-s}$ is bounded on $L^2(X, d\mu_\alpha)$, we can immediately obtain a Hardy inequality for the pure fractional power $L_\alpha^s$.

**Theorem 1.7.** For $\alpha \geq 0$ and $0 < s < 1$, we have

$$ \frac{2^{2s} \left( \frac{\Gamma(n+2s/2)}{\Gamma(n+2s)} \right)^2}{\Gamma(1-s) \left( \Gamma\left( \frac{n+2s}{2} \right) \right)^2} \int_X \frac{|f(x,w)|^2}{x^{4s/2} + 4w^2} \, d\mu_\alpha(x,w) \leq \| V_{\alpha,s} \| \langle L_\alpha^s f, f \rangle $$

for all $f \in C_0^\infty(X)$.

In the same way, we denote by $W^{s,2}(X, d\mu_\alpha)$ the Sobolev space consisting of all $L^2(X, d\mu_\alpha)$ functions $f$ such that $L_\alpha^{s/2} f \in L^2(X, d\mu_\alpha)$, which is a Sobolev space naturally associated to $L_\alpha$. Again, we have that an $L^2(X, d\mu_\alpha)$ function $f$ belongs
Theorem 1.8. For $\alpha \geq 0$, $0 < s < \frac{\alpha}{2}$ and $\delta > 0$, we have
\[
(4\delta)^s \frac{\Gamma\left(\frac{\alpha+s+2}{2}\right)^2}{\Gamma\left(\frac{\alpha}{2}+s+2\right)} \int_X \frac{|f(x,w)|^2}{\left(\delta + \frac{x^2}{2}\right)^2 + w^2} d\mu_\alpha(x,w) \leq (L_{\alpha,s} f, f)_\alpha
\]
for all $f \in W^{s,2}(X, d\mu_\alpha)$. Moreover, the inequality is optimal and the equality is achieved by the function $f(x, w) = (\delta + x^2/2 + w^2)^{-\frac{\alpha+s+2}{2}}$.

Our proof employs the Cowling-Haagerup formula and Schur test, whose approach is different from the homogeneous case in Theorem 1.6. The Schur test lemma plays an important role in the proof and also in Ronal and Thangavelu [10].

Again, it can be shown that the operator $U_{\alpha,s} = L_{\alpha,s} \Lambda_{\alpha,s}^s$ is bounded. From Theorem 1.8, we can immediately obtain a non-homogeneous Hardy inequality for the pure fractional power $L_{\alpha,s}$.

Theorem 1.9. For $\alpha \geq 0$, $0 < s < \frac{\alpha}{2}$ and $\delta > 0$, we have
\[
(4\delta)^s \frac{\Gamma\left(\frac{\alpha+s+2}{2}\right)^2}{\Gamma\left(\frac{\alpha}{2}+s+2\right)} \int_X \frac{|f(x,w)|^2}{\left(\delta + \frac{x^2}{2}\right)^2 + w^2} d\mu_\alpha(x,w) \leq \|U_{\alpha,s}\| (L_{\alpha,s} f, f)_\alpha
\]
for all $f \in W^{s,2}(X, d\mu_\alpha)$.

We can also deduce Heisenberg type uncertainty inequalities for $L_{\alpha,s}$ and $\Lambda_{\alpha,s}$ from our Hardy inequalities.

Theorem 1.10. For $\alpha \geq 0$ and all functions $f \in W^{s,2}(X, d\mu_\alpha)$, we have
\[
(4\delta)^s \frac{\Gamma\left(\frac{\alpha+s+2}{2}\right)^2}{\Gamma\left(\frac{\alpha}{2}+s+2\right)} \left(\int_X |f(x,w)|^2 d\mu_\alpha(x,w)\right)^2 \\
\leq \left(\int_X |f(x,w)|^2 \left(\delta + \frac{x^2}{2}\right)^{2s} + w^2 \right)^2 d\mu_\alpha(x,w) (L_{\alpha,s} f, f)_\alpha
\]
provided $0 < s < \frac{\alpha}{2}$ and $\delta > 0$. In the smaller range $0 < s < 1$, we have and
\[
\frac{a^{2s} \left(\frac{\alpha+s+2}{2}\right)^2}{\Gamma(1-s)\left(\frac{\alpha+2}{2}\right)^2} \left(\int_X |f(x,w)|^2 d\mu_\alpha(x,w)\right)^2 \\
\leq \left(\int_X |f(x,w)|^2 (x^2 + 4w^2)^2 d\mu_\alpha(x,w)\right) (\Lambda_{\alpha,s} f, f)_\alpha.
\]

The paper is organized as follows. In Section 2 the fractional power of the Grushin operator $G_\alpha$ and modified fractional power of the Grushin operator are naturally introduced by spectral decomposition on the orthogonal projection responding to the Hermite operator on $\mathbb{R}^n$. The related fractional power of the generalized sublaplacian is presented by the Laguerre functions of type $\alpha$. It also includes the relation of the Laguerre translation operator and the modified heat kernel of generalized sublaplacian. In Section 3 we demonstrate Theorem 1.6 and 1.7 by combing the fundamental solution, the integral representation formula and the ground state representation. We also deduce Theorem 1.8 and 1.9 by employing the Cowling-Haagerup formula and Schur test. Finally, we obtain Theorem 1.10 from Theorem 1.6 and 1.8. In Section 4 by applying Spherical harmonics and Hecker–Bochner formula, we prove Theorem 1.1 and 1.3, then Theorem 1.2, 1.4 and 1.5 follow similarly.
2. Preliminaries.

2.1. Fractional powers of the Grushin operator on \( \mathbb{R}^{n+1} \) (\( n \geq 2 \)). The Grushin operator \( \mathcal{G} \) on \( \mathbb{R}^{n+1} \) (\( n \geq 2 \)) is defined by

\[
\mathcal{G} = -\frac{1}{2} \left( \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + |x|^2 \frac{\partial^2}{\partial w^2} \right),
\]

where \((x, w) \in \mathbb{R}^{n+1}\) equipped with Lebesgue measure and \(|x|\) is the Euclidean norm of \(x\). It is well known that \( \mathcal{G} \) is a self-adjoint positive operator on \( L^2(\mathbb{R}^{n+1}) \), which is shown to be closely related to the scaled Hermite operators on \( \mathbb{R}^n \) defined as

\[
\mathcal{H}(\lambda) = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \lambda^2 |x|^2, \quad \lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}.
\]

In fact, for any \( f \in L^2(\mathbb{R}^{n+1}) \), let \( f^\lambda \) stand for the Fourier transform of \( f \) in the variable \( w \)

\[
f^\lambda(x) = \int_{\mathbb{R}} f(x, w) e^{i\lambda w} dw.
\]

Applying the operator \( \mathcal{G} \) to the inverse Fourier transform of \( f^\lambda(x) \) in the variable \( \lambda \)

\[
f(x, w) = \frac{1}{2\pi} \int_{\mathbb{R}} f^\lambda(x) e^{-i\lambda w} d\lambda,
\]

we obtain

\[
\mathcal{G} f(x, w) = \frac{1}{4\pi} \int_{\mathbb{R}} \mathcal{H}(\lambda) f^\lambda(x) e^{-i\lambda w} d\lambda.
\]

On the other hand, we can write the spectral decomposition of \( \mathcal{H}(\lambda) \) as

\[
\mathcal{H}(\lambda) = \sum_{k=0}^{\infty} (2k+n)|\lambda| \mathcal{P}_k(\lambda),
\]

where \( \mathcal{P}_k(\lambda) \) stands for the orthogonal projection of \( L^2(\mathbb{R}^n) \) onto the \( k \)-th eigenspace corresponding to the eigenvalue \((2k+n)|\lambda|\) of \( \mathcal{H}(\lambda) \). More precisely, for any \( \phi \in L^2(\mathbb{R}^n) \),

\[
\mathcal{P}_k(\lambda) \phi = \sum_{|\beta|=k} \langle \phi, \Phi_\beta \rangle \Phi_\beta,
\]

where for \( \lambda \in \mathbb{R}^* \) and each \( \beta \in \mathbb{N}^n \),

\[
\Phi_\beta(x) = |\lambda|^\frac{n}{2} \Phi_\beta(\sqrt{|\lambda|} x), \quad x \in \mathbb{R}^n.
\]

Here, \( \Phi_\beta \) is the normalized Hermite function on \( \mathbb{R}^n \) (see [16, Chapter 1.4]).

Hence, the spectral decomposition of the Grushin operator \( \mathcal{G} \) is given by

\[
\mathcal{G} f(x, w) = \frac{1}{2\pi} \int_{\mathbb{R}} \left( \sum_{k=0}^{\infty} (2k+n)|\lambda| \mathcal{P}_k(\lambda) f^\lambda(x) \right) e^{-i\lambda w} d\lambda.
\]

Therefore, for any \( s > 0 \), a natural way to define fractional powers of the Grushin operator \( \mathcal{G} \) is by spectral decomposition

\[
\mathcal{G}^s f(x, w) = \frac{1}{2\pi} \int_{\mathbb{R}} \left( \sum_{k=0}^{\infty} (2k+n)|\lambda|^{s} \mathcal{P}_k(\lambda) f^\lambda(x) \right) e^{-i\lambda w} d\lambda.
\]

However, it is convenient to investigate the following modified fractional powers \( \mathcal{G}_s \) defined by

\[
\mathcal{G}_s f(x, w) = \frac{1}{2\pi} \int_{\mathbb{R}} \left( \sum_{k=0}^{\infty} \frac{\Gamma((2k+n+1)+s)}{\Gamma((2k+n+1)+\frac{s+1}{2})} (2|\lambda|)^{s} \mathcal{P}_k(\lambda) f^\lambda(x) \right) e^{-i\lambda w} d\lambda,
\]
Moreover, the inverse of the operator $G$ i.e., $G^{-1}$ and for $\lambda^2 = 0$.

\begin{equation}
G^{-1} f(x, w) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\Gamma(\frac{2k+n}{4} + \frac{1+\alpha}{2})}{\Gamma(\frac{2k+n}{4} + \frac{1+\alpha}{2})} (2|\lambda|)^{-s} \mathcal{P}_k(\lambda)f^{\lambda}(x)e^{-i\lambda w} d\lambda.
\end{equation}

Note that $G^{-1} = G$. As in [10], in order to prove a version of Hardy inequality for fractional powers of the Grushin operator with a homogeneous weight function, we do not deal directly with $G^s$ but the related operator

\begin{equation}
A_s = G_{-s}^{-1},
\end{equation}

which behaves like $G^s$.

2.2. 

**Laguerre transform on** $\mathbb{R}^+ \times \mathbb{R}$. For $\alpha \geq 0$. Let the space $X = \mathbb{R}^+ \times \mathbb{R}$ equipped with the measure $d\mu_\alpha(x, w) = x^{2\alpha+1}dx dw$, where $dx$ and $dw$ are the standard Lebesgue measures. We denote by $L^p(X, d\mu_\alpha) (1 \leq p \leq \infty)$ the classical Lebesgue spaces with respect to the measure $d\mu_\alpha$ endowed with the norm $\|\cdot\|_p$. Let $\langle \cdot, \cdot \rangle_\alpha$ stand for the inner product derived by $L^2(X, d\mu_\alpha)$. For convenience, we also denote the elements of $X$ by Greek letters $\xi = (x, w)$ and $\eta = (y, v)$.

For $\lambda \in \mathbb{R}^*$ and $k \in \mathbb{N}$, the scaled Laguerre functions of type $\alpha$ are defined by

\begin{equation}
\phi^{\alpha}_{k,\lambda}(x) = L^\alpha_\alpha(|\lambda|^2 x^2) e^{-\frac{1}{2}|\lambda|x^2},
\end{equation}

where $L^\alpha_\alpha$ is the Laguerre polynomial of type $\alpha$ (see [16, Chapter 1.4]). Then we can normalize $\phi^{\alpha}_{k,\lambda}(x)$ by defining

\begin{equation}
\tilde{\phi}^{\alpha}_{k,\lambda}(x) = \left( \frac{2\Gamma(k+1)|\lambda|^{\alpha+1}}{\Gamma(\alpha+k+1)} \right)^{\frac{1}{2}} \phi^{\alpha}_{k,\lambda}(x).
\end{equation}

The family $\{\tilde{\phi}^{\alpha}_{k,\lambda}(x) : k \in \mathbb{N}\}$ forms an orthonormal basis for $L^2(\mathbb{R}^+, x^{2\alpha+1}dx)$ (see [17, Proposition 2.4.2]).

We shall introduce a family of generalized Laguerre translation operators. For $f \in L^1(X, d\mu_\alpha)$, $(x, w), (y, v) \in X$, we define: for $\alpha > 0$

\begin{equation}
T^{(y,\nu)}_\alpha f(x, w) = \frac{1}{\alpha} \int_0^\pi \int_0^\pi f \left( (x^2 + y^2 - 2xy \cos \varphi \sin \theta)^{\frac{1}{2}}, w - v + xy \cos \varphi \sin \theta \right) \sin^{2\alpha-1} \varphi \sin^{2\alpha} \varphi d\varphi d\theta,
\end{equation}

and for $\alpha = 0$

\begin{equation}
T^{(y,\nu)}_0 f(x, w) = \frac{1}{\pi} \int_0^\pi f \left((x^2 + y^2 - 2xy \cos \varphi \sin \theta)^{\frac{1}{2}}, w - v - xy \sin \theta \right) d\theta.
\end{equation}

For $\lambda \in \mathbb{R}^*$ and $k \in \mathbb{N}$, define

\begin{equation}
\psi^{\alpha}_{k,\lambda}(x, w) = \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+1)} e^{i\lambda w} \tilde{\phi}^{\alpha}_{k,\lambda}(x).
\end{equation}

Though the definition of the generalized Leguerre translation operator is complicated, its action on $\psi^{\alpha}_{k,\lambda}$ is quite simple (see [12, Lemma 4.2]): setting $\eta^* = (y, -v)$,

\begin{equation}
T^{(y,\nu)}_\alpha \psi^{\alpha}_{k,\lambda}(\xi) = \psi^{\alpha}_{k,\lambda}(\xi) \psi^{\alpha}_{k,\lambda}(\eta).
\end{equation}
Using the generalized Laguerre translation operator, for \( f, g \in L^1(X, d\mu_\alpha) \), we define the convolution \( f *_\alpha g \) by

\[
f *_\alpha g(\xi) = \int_X T^\alpha f(\xi)g(\eta)d\mu_\alpha(\eta).
\]

From [12, Lemma 4.1], \( f *_\alpha g = g *_\alpha f \), i.e., the convolution is commutative.

For \( f \in L^1(X, d\mu_\alpha), \lambda \in \mathbb{R}^* \) and \( k \in \mathbb{N} \), its Laguerre transform \( \hat{f}(\alpha, \lambda, k) \) is defined by

\[
\hat{f}(\alpha, \lambda, k) = \int_X f(\xi)\psi^\alpha_{k,\lambda}(\xi)d\mu_\alpha(\xi).
\]

Then, we have \( \hat{f *_\alpha g}(\alpha, \lambda, k) = \hat{f}(\alpha, \lambda, k)\hat{g}(\alpha, \lambda, k) \) (see [12, Lemma 4.3]). For \( f \in L^1(X, d\mu_\alpha) \), if we define its Fourier transform in the second variable

\[
f^\lambda(x) = \int_{\mathbb{R}} f(x, w)e^{i\lambda w}dw,
\]

it can be easily checked that

\[
\hat{f}(\alpha, \lambda, k) = \frac{\Gamma(\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+k+1)}\int_0^\infty f^\lambda(x)\phi^\alpha_{k,\lambda}(x)|\lambda|^{\alpha+1}e^{-i\lambda w}d\lambda,
\]

(2.2)

Moreover, we have an representation for \( f \in L^2(X, d\mu_\alpha) \) (see [13, Lemma 3.1]).

**Lemma 2.1.** For \( f \in L^2(X, d\mu_\alpha) \), we have

\[
f(x, w) = \frac{1}{\pi\Gamma(\alpha+1)}\int_{\mathbb{R}} \sum_{k=0}^\infty \int_0^\infty \hat{f}(\alpha, \lambda, k)\phi^\alpha_{k,\lambda}(x)|\lambda|^{\alpha+1}e^{-i\lambda w}d\lambda.
\]

2.3. **Fractional powers of generalized sublaplacian.** For \( \alpha \geq 0 \), generalized sublaplacian \( \mathcal{L}_\alpha \) on \( X \) is defined by

\[
\mathcal{L}_\alpha = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{2\alpha + 1}{x} \frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial w^2} \right),
\]

which is self-adjoint and positive on \( L^2(X, d\mu_\alpha) \). By the following identity for Laguerre polynomials

\[
x \frac{d^2}{dx^2}L^\alpha_k(x) + (\alpha + 1 - x) \frac{d}{dx}L^\alpha_k(x) + kL^\alpha_k(x) = 0,
\]

we have

\[
\mathcal{L}_\alpha \psi^\alpha_{k,\lambda} = (2k + \alpha + 1)|\lambda|\psi^\alpha_{k,\lambda},
\]

i.e., \( \psi^\alpha_{k,\lambda} \) is the eigenfunction for \( \mathcal{L}_\alpha \) with the eigenvalue \((2k + \alpha + 1)|\lambda|\). Thus, for any \( s > 0 \), we can naturally define fractional powers of generalized sublaplacian \( \mathcal{L}_\alpha^s \) via spectral decomposition by

\[
\mathcal{L}_\alpha^sf(x, w) = \frac{1}{\pi\Gamma(\alpha+1)}\int_{\mathbb{R}} \sum_{k=0}^\infty ((2k+\alpha+1)|\lambda|^s\hat{f}(\alpha, \lambda, k)\phi^\alpha_{k,\lambda}(x))|\lambda|^{\alpha+1}e^{-i\lambda w}d\lambda.
\]

Note that \( \mathcal{L}_\alpha^sf(\alpha, \lambda, k) = ((2k + \alpha + 1)|\lambda|^s\hat{f}(\alpha, \lambda, k) \). However, it is convenient to work with the following modified fractional powers \( \mathcal{L}_{\alpha,s} \) defined by

\[
\mathcal{L}_{\alpha,s}f(x, w)
\]

\[
= \frac{1}{\pi\Gamma(\alpha+1)}\int_{\mathbb{R}} \sum_{k=0}^\infty \frac{\Gamma(2k+\alpha+1)}{\Gamma(2k+\alpha+1 + s)}\frac{1}{2^{1+\alpha}}(2|\lambda|^s\hat{f}(\alpha, \lambda, k)\phi^\alpha_{k,\lambda}(x))|\lambda|^{\alpha+1}e^{-i\lambda w}d\lambda,
\]
which implies that the spectral multiplier of $L_{\alpha,s}$ is 
\[
\Gamma\left(\frac{2k+\alpha+1}{2} + \frac{1+s}{2}\right)(2|\lambda|)^s.
\]
Also, the inverse of the operator $L_{\alpha,s}$ is given by
\[
L_{\alpha,s}^{-1} f(x, w) = \frac{1}{\pi \Gamma(\alpha+1)} \int_{\mathbb{R}} \left( \sum_{k=0}^{\infty} \Gamma\left(\frac{2k+\alpha+1}{2} + \frac{1-s}{2}\right)(2|\lambda|)^{-s} \hat{f}(\alpha, \lambda, k) \phi_{\alpha,k,\lambda}(x) \right) |\lambda|^{\alpha+1} e^{-i\lambda w} d\lambda.
\]

Note that $L_{\alpha,s}^{-1} = L_{\alpha,-s}$. We shall prove that $L_{\alpha,s}$ has an explicit fundamental solution, which makes it more suitable than $L_{\alpha}^s$, since the fundamental solution of $L_{\alpha}^s$ cannot be written down explicitly. Nevertheless, $L_{\alpha,s}$ is not very different from $L_{\alpha}^s$. We will see that $L_{\alpha,s} = U_{\alpha,s} L_{\alpha}^s$, where $U_{\alpha,s}$ is a bounded operator on $L^2(X, d\mu_\alpha)$. As [1], in order to prove a version of Hardy inequality with a non-homogeneous weight function, they do not deal directly with $L_{\alpha}^s$ but $L_{\alpha,s}$. We also need to study the related operator
\[
\Lambda_{\alpha,s} = L_{\alpha,1-s} L_{\alpha}.
\]
which behaves like $L_{\alpha}^s$.

2.4. Modified heat kernel for generalized sublaplacian. Generalized sublaplacian $L_{\alpha}$ is a self-adjoint positive operator on $L^2(X, d\mu_\alpha)$, which generates a heat semigroup $e^{-tL_{\alpha}}$ defined by
\[
e^{-tL_{\alpha}} f(\alpha, \lambda, k) = e^{-(2k+\alpha+1)|\lambda|t} \hat{f}(\alpha, \lambda, k).
\]
If we define $h_{\alpha,t}$ by the relation
\[
\hat{h}_{\alpha,t}(\alpha, \lambda, k) = e^{-(2k+\alpha+1)|\lambda|t}
\]
then we obtain
\[
e^{-tL_{\alpha}} f(x, w) = f *_{\alpha} h_{\alpha,t}(x, w).
\]
The function $h_{\alpha,t}$ is called the heat kernel associated with $L_{\alpha}$, which is positive and
\[
\int_X h_{\alpha,t}(x, w) d\mu_\alpha(x, w) = 1.
\]
In addition, we have the explicit expression for $h_{\alpha,t}^\lambda(x)$ (see[1, Proposition 2.7])
\[
h_{\alpha,t}^\lambda(x) = \frac{2}{\Gamma(\alpha+1)} \left( \frac{\lambda}{2 \sinh \lambda t} \right)^{\alpha+1} e^{-\frac{\lambda}{2} x^2 \coth \lambda t}. \quad (2.3)
\]
For $0 < s < 1$ and $t > 0$, we define the modified heat kernel $K_{\alpha,t,s}(x, w)$ by
\[
\int_{-\infty}^{\infty} K_{\alpha,t,s}(x, w) e^{i\lambda w} dw = h_{\alpha,t}^\lambda(x) \cosh \lambda t \left( \frac{\lambda t}{\sinh \lambda t} \right)^{2-s}. \quad (2.4)
\]
From (2.3), this kernel $K_{\alpha,t,s}(x, w)$ is an even function in the second variable.
We shall state the behavior of this kernel in the following lemma.

**Lemma 2.2.** For $\alpha \geq 0$, $0 < s < 1$ and $t > 0$, we have
\[
\int_X K_{\alpha,t,s}(\xi) d\mu_\alpha(\xi) = 1. \quad (2.5)
\]
\[
\int_X T_\alpha^\lambda K_{\alpha,t,s}(\xi) d\mu_\alpha(\xi) = \int_X T_\alpha^\lambda K_{\alpha,t,s}(\xi) d\mu_\alpha(\eta) = 1. \quad (2.6)
\]
Proof. Using (2.3) and (2.4), letting \( \lambda \) go to 0, we have
\[
\int_{-\infty}^{+\infty} K_{\alpha,t,s}(x,w)dw = \frac{2}{\Gamma(\alpha+1)} \left( \frac{\alpha}{2t} \right)^{\alpha+1} e^{-\frac{x^2}{4t}}.
\]
From this, it follows that
\[
\int_{X} K_{\alpha,t,s}(x,w)\,d\mu(x,w) = \frac{2}{\Gamma(\alpha+1)} \int_{0}^{+\infty} \left( \frac{\alpha}{2t} \right)^{\alpha+1} e^{-\frac{x^2}{4t}x^{\alpha+1}} dx = 1,
\]
which completes (2.5).

Then by [12, Lemma 3.1], we have
\[
\int_{X} T_{\alpha}^{\eta} f(\xi) g(\xi) d\mu_{\alpha}(\xi) = \int_{X} f(\xi) T_{\alpha}^{\eta^{*}} g(\xi) d\mu_{\alpha}(\xi),
\]
where \( \eta^{*} = (y, -v) \). Taking \( f = K_{\alpha,t,s} \) and \( g = 1 \), by (2.5), it implies that
\[
\int_{X} T_{\alpha}^{\eta} K_{\alpha,t,s}(\xi) d\mu_{\alpha}(\xi) = \int_{X} K_{\alpha,t,s}(\xi) T_{\alpha}^{\eta^{*}} 1(\xi) d\mu_{\alpha}(\xi).
\]
Using the identity (see [9, p. 411, 3.681.2])
\[
\int_{0}^{\pi} \sin^{z} \theta d\theta = B\left( z + 1, \frac{1}{2} \right) = \sqrt{\pi} \frac{\Gamma\left( \frac{z+1}{2} \right)}{\Gamma\left( \frac{z+2}{2} \right)}, \quad \forall z > -1,
\]
we obtain \( T_{\alpha}^{\eta^{*}} 1(\xi) = 1 \). Consequently, the left side of (2.6) immediately comes out.

In addition, since \( K_{\alpha,t,s} \) is an even function in the second variable, it can be easily checked that
\[
T_{\alpha}^{\eta} K_{\alpha,t,s}(\xi) = T_{\alpha}^{\xi} K_{\alpha,t,s}(\eta).
\]
Therefore,
\[
\int_{X} T_{\alpha}^{\xi} K_{\alpha,t,s}(\eta) d\mu_{\alpha}(\xi) = \int_{X} T_{\alpha}^{\eta} K_{\alpha,t,s}(\xi) d\mu_{\alpha}(\xi) = 1,
\]
which concludes the right side of (2.6). \( \square \)

3. Hardy’s inequalities for generalized sublaplacian.

3.1. The Cowling-Haagerup formula and a fundamental solution for \( \mathcal{L}_{\alpha,s} \).

For \( \alpha \geq 0 \) and \( \delta \geq 0 \), we set
\[
u_{\alpha,s,\delta}(x,w) = \left( \left( \delta + \frac{x^2}{2} \right)^{2} + w^2 \right)^{-\alpha+\frac{1}{2}},
\]
where \( (x, w) \in X \). Let us denote a distance function from the origin to \( \xi = (x, w) \) on \( X \) by
\[
\rho = \rho(\xi) = \left( x^4 + 4w^2 \right)^{\frac{1}{2}}.
\]
Then we can define the ball centered at the origin of radius \( R > 0 \), i.e., the set
\( B_R = \{ \xi \in X : \rho(\xi) < R \} \). Setting \( x = \rho \cos^{1/2} \theta, \ w = \frac{1}{2} \rho^2 \sin \theta \), we obtain the following explicit polar transform on \( X \)
\[
\int_{X} f(\xi) d\mu_{\alpha}(\xi) = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\frac{\rho}{2}} f\left( \rho \cos^{1/2} \theta, \frac{1}{2} \rho^2 \sin \theta \right) \rho^{2\alpha + 3} \cos^{\alpha} \theta d\rho d\theta.
\]
For $\delta > 0$ and $p \geq 1$, it is easy to observe that $u_{\alpha,s,\delta} \in L^p(X, d\mu_{\alpha}) \Leftrightarrow u_{\alpha,s,0} \cdot \chi_{B_{1}} \in L^p(X, d\mu_{\alpha}) \Leftrightarrow s > \left(\frac{p-1}{2}(\alpha+2)\right)$. Therefore, for $\delta > 0$, $u_{\alpha,s,\delta} \in L^1(X, d\mu_{\alpha})$ for any $s > 0$ and $u_{\alpha,s,\delta} \in L^2(X, d\mu_{\alpha})$ for any $s > -\frac{\alpha+2}{2}$.

We shall give the following generalized result by Cowling-Haagerup [4, Section 3], which is used in the proof of Hardy’s inequality.

We need to introduce an auxiliary function: for $a, b \in \mathbb{R}^+$ and $c \in \mathbb{R}$,

$$L(a, b, c) = \int_{0}^{+\infty} e^{-a(z+1)\frac{c}{z+1}}dz.$$ 

According to [4, Proposition 3.6], the function $L$ satisfies the following identity

$$\frac{(2\lambda)^{\alpha}}{\Gamma(a)}L(\lambda, a, b) = \frac{(2\lambda)^{b}}{\Gamma(b)}L(\lambda, b, a) \tag{3.1}$$

for all $a, b \in \mathbb{C}$ and $\lambda > 0$.

**Lemma 3.1.** For $\alpha \geq 0$, $\delta > 0$ and $-\frac{\alpha+2}{2} < s < \frac{\alpha+2}{2}$, we have

$$L_{\alpha, s}u_{\alpha, s, \delta}(\xi) = (4\delta)^{s} \left(\frac{\Gamma\left(\frac{\alpha+2}{2}\right)}{\Gamma\left(\frac{\alpha-s+2}{2}\right)}\right)^{2} u_{\alpha, s, \delta}(\xi).$$

**Proof.** Though it has been already proved in [1, Proposition 3.1]), we repeat it for the sake of completeness. We start with the generating function identity for the Laguerre functions of type $\alpha$, valid for $|r| < 1$ (see [16, 1.4. 24])

$$\sum_{k=0}^{\infty} r^{k}L_{k}(x^{2})e^{-\frac{1}{4}x^{2}} = (1 - r)^{-a-1}e^{-\frac{1}{4}(1+r)x^{2}}.$$

Therefore, taking $r = \frac{y}{y+|\lambda|}$ ($y > 0$) and $x^{2} \rightarrow |\lambda|x^{2}$, we get

$$\sum_{k=0}^{\infty} \left(\frac{y}{y+|\lambda|}\right)^{k} L_{k}(|\lambda|x^{2})e^{-\frac{1}{4}|\lambda|x^{2}} = |\lambda|^{-a-1}(1+y+|\lambda|)^{\alpha+1}e^{-\frac{1}{4}(2y+|\lambda|)x^{2}}. \tag{3.2}$$

For two appropriate functions $f, g$ on $\mathbb{R}^+$, recall the definition of their Laplace transforms by

$$F(a + ib) = \int_{0}^{+\infty} e^{-(a+ib)y}f(y)dy,$$

$$G(a + ib) = \int_{0}^{+\infty} e^{-(a+ib)y}G(y)dy, \quad a \in \mathbb{R}^+, b \in \mathbb{R}.$$

Letting $\beta = \frac{a+s+2}{2}$ and $f(x) = g(x) = \Gamma(\beta)^{-1}x^{\beta-1}e^{-\delta x}$, we have

$$F(a + ib) = G(a + ib) = (\delta + a + ib)^{-\beta}.$$

In addition, by the following formula (see [4, Lemma 3.4])

$$\int_{\mathbb{R}} F(a + ib)\overline{G(a + ib)}e^{-i|\lambda|b}db = 2\pi \int_{0}^{+\infty} f(y)g(y + |\lambda|)e^{-\alpha(2y+|\lambda|)}dy,$$

and taking $a = \frac{z^2}{2}$, it reduces that

$$\int_{\mathbb{R}} \left(\delta + \frac{x^2}{2}\right)^{-\frac{a+s+2}{2}}e^{-i|\lambda|b}db = 2\pi \int_{0}^{+\infty} f(y)g(y + |\lambda|)e^{-\frac{1}{2}(2y+|\lambda|)x^{2}}dy.$$
Since \( u_{\alpha,s,\delta}(x,w) \) is even in the second variable, the last left integral denotes \( u_{\alpha,s,\delta}^\lambda(x) \) and the last equality becomes

\[
u_{\alpha,s,\delta}^\lambda(x) = 2\pi \int_0^{+\infty} f(y)g(y + |\lambda|)e^{-\frac{1}{2}(2y + |\lambda|)x^2} dy. \tag{3.3}
\]

We can rewrite the expansion (3.2) by

\[
e^{-\frac{1}{2}(2y + |\lambda|)x^2} = |\lambda|^{\alpha+1} \sum_{k=0}^{\infty} y^k(y + |\lambda|)^{-(k+\alpha+1)} \phi^\lambda_{k,\lambda}(x).
\]

Plugging this to (3.3), we get

\[
u_{\alpha,s,\delta}^\lambda(x) = \frac{2|\lambda|^{\alpha+1}}{\Gamma(\alpha + 1)} \sum_{k=0}^{\infty} c_{\alpha,\delta}^\lambda(s) \phi^\lambda_{k,\lambda}(x), \tag{3.4}
\]

where the coefficients are given by

\[
c_{\alpha,\delta}^\lambda(s) = \frac{\pi\Gamma(\alpha + 1)}{(\Gamma(\frac{\alpha+s+2}{2}))^2} \int_0^{+\infty} e^{-\delta(2y+|\lambda|)} y^{\frac{2k+\alpha+s}{2}} (y + |\lambda|)^{-\frac{2k+\alpha+2-s}{2}} dy
\]

\[
= \frac{\pi\Gamma(\alpha + 1)|\lambda|^s}{(\Gamma(\frac{\alpha+s+2}{2}))^2} \int_0^{+\infty} e^{-\delta|\lambda|(2y+1)} y^{\frac{2k+\alpha+s}{2}} (y + 1)^{-\frac{2k+\alpha+2-s}{2}} dy
\]

\[
= \frac{\pi\Gamma(\alpha + 1)|\lambda|^s}{(\Gamma(\frac{\alpha+s+2}{2}))^2} L\left(\delta|\lambda|, \frac{2k + \alpha + 2 + s}{2}, \frac{2k + \alpha + 2 - s}{2}\right).
\]

In view of (3.1), it can be easily checked that

\[
c_{\alpha,\delta}^\lambda(-s) = (2\delta)^s|\lambda|^{-s} \left(\frac{\Gamma(\frac{\alpha+s+2}{2})}{\Gamma(\frac{\alpha-s+2}{2})}\right)^2 \frac{\Gamma(\frac{2k+\alpha+1}{2} + \frac{1+s}{2})}{\Gamma(\frac{2k+\alpha+1}{2} + \frac{1-s}{2})} c_{\alpha,\delta}^\lambda(s).
\]

On the other hand, by (2.2) and (3.4), we obtain

\[
u_{\alpha,s,\delta}(\alpha, \lambda, k) = \frac{\Gamma(\alpha + 1)\Gamma(k+1)}{\Gamma(\alpha + k + 1)} \int_0^{+\infty} u_{\alpha,s,\delta}^\lambda(x) \phi_{k,\lambda}^\alpha(x) x^{2\alpha+1} dx = c_{\alpha,\delta}^\lambda(s).
\]

So, combining with the spectral multiplier of \( \mathcal{L}_{\alpha,s} \), it follows that

\[
\mathcal{L}_{\alpha,s} u_{\alpha,-s,\delta}(\alpha, \lambda, k) = \frac{\Gamma(\frac{2k+\alpha+1}{2} + \frac{1+s}{2})}{\Gamma(\frac{2k+\alpha+1}{2} + \frac{1-s}{2})} (2|\lambda|)^s u_{\alpha,-s,\delta}(\alpha, \lambda, k)
\]

\[
= \frac{\Gamma(\frac{2k+\alpha+1}{2} + \frac{1+s}{2})}{\Gamma(\frac{2k+\alpha+1}{2} + \frac{1-s}{2})} (2|\lambda|)^s c_{\alpha,\delta}^\lambda(-s)
\]

\[
= (4\delta)^s \left(\frac{\Gamma(\alpha+s+2)}{\Gamma(\alpha-s+2)}\right)^2 c_{\alpha,\delta}^\lambda(s)
\]

\[
= (4\delta)^s \left(\frac{\Gamma(\alpha+s+2)}{\Gamma(\alpha-s+2)}\right)^2 u_{\alpha,s,\delta}(\alpha, \lambda, k).
\]

Our result immediately comes out. \(\square\)
Lemma 3.2. For \( \alpha \geq 0 \) and \(-\frac{\alpha+2}{2} < s < \frac{\alpha+2}{2}\) and \(\xi = (x,w) \in X\), the function

\[
g_{\alpha,s}(\xi) = \frac{2^{\alpha-2s+2} \Gamma^2 \left(\frac{\alpha-s+2}{2}\right)}{\pi \Gamma(\alpha+1) \Gamma(s)} \left(x^4 + 4w^2\right)^{-\frac{\alpha-s+2}{2}}
\]

(3.5)
is a fundamental solution of \(\mathcal{L}_{\alpha,s}\), i.e., it satisfies \(\mathcal{L}_{\alpha,s}g_{\alpha,s} = \delta_0\), where \(\delta_0\) is the Dirac delta function with support at 0.

Proof. From the proof of Lemma 3.1, we have

\[
u_{\alpha,-s,\delta}(\alpha,\lambda,k) = c_{\alpha,k,\delta}(-s) = \frac{\pi \Gamma(\alpha+1) |\lambda|^{-s}}{\Gamma^2(\frac{\alpha-s+2}{2})} L\left(\delta, \frac{2k+\alpha+2-s}{2}, \frac{2k+\alpha+2+s}{2}\right).
\]

Let \(\delta\) tend to 0 in the above formula and we get

\[
u_{\alpha,-s,0}(\alpha,\lambda,k) = \frac{\pi \Gamma(\alpha+1) |\lambda|^{-s}}{\Gamma^2(\frac{\alpha-s+2}{2})} L\left(0, \frac{2k+\alpha+2-s}{2}, \frac{2k+\alpha+2+s}{2}\right).
\]

One can easily check that

\[
L\left(0, \frac{2k+\alpha+2-s}{2}, \frac{2k+\alpha+2+s}{2}\right) = \frac{\Gamma(s)}{\Gamma(\frac{2k+\alpha+2+s}{2})}.
\]

Together with the spectral multiplier of \(\mathcal{L}_{\alpha,s}\), it gives that

\[
\mathcal{L}_{\alpha,s}\nu_{\alpha,-s,0}(\alpha,\lambda,k) = \frac{2|\lambda|^s}{\Gamma(\frac{2k+\alpha+2+s}{2})} \nu_{\alpha,-s,0}(\alpha,\lambda,k) = \frac{2s\pi \Gamma(\alpha+1) \Gamma(s)}{\Gamma^2(\frac{\alpha-s+2}{2})},
\]

which implies that \(g_{\alpha,s}\) is a fundamental solution for the operator \(\mathcal{L}_{\alpha,s}\). \(\square\)

3.2. Integral representations. In this subsection we find an integral representation for the operator \(\Lambda_{\alpha,s}\). For convenience, we work with \(\Lambda_{\alpha,1-s} = \mathcal{L}_{\alpha,s}^{-1} \mathcal{L}_{\alpha,I}\) and state the results for this operator. In terms of the kernel \(K_{\alpha,t,s}\), we define another kernel \(K_{\alpha,s}\) by

\[
K_{\alpha,s}(\xi) = \int_0^{+\infty} K_{\alpha,t,s}(\xi)t^{s-2}dt,
\]

which is a positive function and will be explicitly calculated. In order to compute the kernel \(K_{\alpha,s}\), we shall exploit several formulas and identities. We state them here together.

We have the formula (see [9, p. 498, 3.944.6])

\[
\int_0^{+\infty} z^{\mu-1} e^{-\beta z} \cos \delta zdz = \frac{\Gamma(\mu)}{(\delta^2 + \beta^2)^{\mu/2}} \cos \left(\mu \arctan \frac{\delta}{\beta}\right),
\]

(3.6)
valid for \(\text{Re}(\mu) > 0\) and \(\text{Re}(\beta) > |\text{Im}(\delta)|\). We also need the identity (see [9, p. 406, 3.663.1])

\[
\int_0^u (\cos z - \cos u)^{\nu-\frac{1}{2}} \cos azdz = \sqrt{\frac{\pi}{2}} \sin^{\nu} u \Gamma \left(\frac{\nu+1}{2}\right) P_{a-\frac{1}{2}}^{-\nu}(\cos u)
\]

(3.7)
valid for \(\text{Re}(\nu) > -\frac{1}{2}, a > 0\) and \(0 < u < \pi\), where \(P_{a-\frac{1}{2}}^{-\nu}\) is an associated Legendre function of the first kind (see [9, Section 8.7-8.8]). In particular, it is known that (see [9, p. 969, 8.755])

\[
P_{\nu}^{-\nu}(\cos u) = \frac{(\sin u)^{\nu}}{\Gamma(\nu+1)}.
\]

(3.8)
In addition, we have the equation
\[
\int_0^u (\cos z - \cos u)^{\nu-1} \cos((\nu+\beta)z) dz = \frac{\sqrt{\Gamma(\beta+1)} \Gamma(\nu) \Gamma(2\nu) \sin^{2\nu-1} u}{2^{\nu} \Gamma(\beta+2\nu) \Gamma(\nu+\frac{1}{2})} C_{\beta}^{\nu}(\cos u),
\] (3.9)
valid for \(Re(\nu) > 0, Re(\beta) > -1\) and \(0 < u < \pi\), where \(C_{\beta}^{\nu}\) is a Gegenbauer polynomial (see [9, Section 8.93]) and also has the following representation
\[
C_{\nu}^{\nu}(\cos u) = 2\nu \cos u.
\] (3.10)

**Proposition 3.3.** For \(\alpha \geq 0, 0 < s < 1\) and \((x, w) \in X\), we have
\[
K_{\alpha,s}(x, w) = c_{\alpha,s} \frac{x^2}{(x^4 + 4w^2)^{\frac{\alpha-s+3}{2}}} (x^4 + 4w^2)^{\frac{\alpha-s+3}{2}},
\]
where the constant \(c_{\alpha,s}\) is given by
\[
c_{\alpha,s} = \frac{2^{2-2s+4} \Gamma \left(\frac{\alpha-s+2}{2}\right) \Gamma \left(\frac{\alpha-s+4}{2}\right)}{\pi \Gamma(\alpha + 1)}.
\]

**Proof.** Calculations are borrowed from [10, Proposition 4.4]. Since tracking the constants is important, we still present a complete proof. Starting with the definition of \(K_{\alpha,s}\), we have
\[
\int_{-\infty}^{+\infty} K_{\alpha,s}(x, w) e^{i\lambda w} dw
= \int_{0}^{+\infty} K_{\alpha,t,s}^{\lambda}(x) t^{s-2} dt = \int_{0}^{+\infty} h_{\alpha,t}^{\lambda}(x) \cosh \lambda t \left(\frac{\lambda t}{\sinh \lambda t}\right)^{2-s} t^{s-2} dt
= \frac{1}{2^{s} \Gamma(\alpha + 1)} \int_{0}^{+\infty} \cosh \lambda t \left(\frac{\lambda t}{\sinh \lambda t}\right)^{\alpha-s+3} e^{-\frac{1}{2} \lambda x^2 \coth \lambda t} dt.
\]
Since the Fourier transform of \(K_{\alpha,s}\) in the \(w\) variable is an even function of \(\lambda\), take the inverse Fourier transform in the variable \(\lambda\) and we have
\[
K_{\alpha,s}(x, w) = \frac{1}{2^{s} \Gamma(\alpha + 1)} \int_{0}^{+\infty} \int_{0}^{+\infty} \cos(\lambda w) \cosh \lambda t \left(\frac{\lambda t}{\sinh \lambda t}\right)^{\alpha-s+3} e^{-\frac{1}{2} \lambda x^2 \coth \lambda t} d\lambda dt. \quad (3.11)
\]
With the change of variables \(\lambda \rightarrow \lambda x^{-2}\) and \(t \rightarrow tx^2\), we get
\[
K_{\alpha,s}(x, x^2 w) = x^{-2(\alpha-s+3)} K_{\alpha,s}(1, w), \quad (3.12)
\]
where
\[
K_{\alpha,s}(1, w)
= \frac{1}{2^{s} \Gamma(\alpha + 1)} \int_{0}^{+\infty} \int_{0}^{+\infty} \cos(\lambda w) \cosh \lambda t \left(\frac{\lambda t}{\sinh \lambda t}\right)^{\alpha-s+3} e^{-\frac{1}{2} \lambda x^2 \coth \lambda t} d\lambda dt
= \int_{0}^{+\infty} \left(\int_{0}^{+\infty} \cos(\lambda w) \lambda^{\alpha-s+2} e^{-\frac{1}{2} \lambda x^2 \coth t} d\lambda\right) \sinh t)^{-\alpha-s+3} \cosh t dt. \quad (3.13)
\]
The inner integral can be computed using (3.6), by taking \(\mu = \alpha-s+3, \beta = \frac{1}{2} \coth t\) and \(\delta = w\). Therefore, we obtain
\[
\int_{0}^{+\infty} \cos(\lambda w) \lambda^{\alpha-s+2} e^{-\frac{1}{2} \lambda x^2 \coth t} d\lambda
\]

\[
\int_{0}^{+\infty} \cos(\lambda w) \lambda^{\alpha-s+2} e^{-\frac{1}{2} \lambda x^2 \coth t} d\lambda
\]
By the change of variable $u = \frac{2w}{\coth t}$, the latter integral is equal to
\begin{equation}
2^\alpha w^{-1} \int_0^{2w} (1 - \frac{u^2}{4w^2})^{\frac{\alpha}{2}} (1 + u^2)^{-\frac{\alpha + 3}{2}} \cdot \cos \left((\alpha - s + 3) \arctan u\right) du.
\end{equation}
Therefore, this together with (3.14) gives that
\begin{equation}
K_{\alpha,s}(1, w) = \frac{2^{2s-\alpha} \Gamma(\alpha - s + 3)}{\pi \Gamma(\alpha + 1)} w^{-1} I,
\end{equation}
where
\begin{equation}
I = \int_0^{2w} \left(1 - \frac{u^2}{4w^2}\right)^{\frac{\alpha}{2}} (1 + u^2)^{-\frac{\alpha + 3}{2}} \cos \left((\alpha - s + 3) \arctan u\right) du. \tag{3.16}
\end{equation}
Next we shall show that the above integral can be explicitly calculated in terms of Legendre functions and Gegenbauer polynomials. Taking a second change of variable $\arctan u = z$, the integral $I$ becomes
\begin{equation}
I = \int_0^{\arctan 2w} \left(\cos^2 z - \frac{\sin^2 z}{4w^2}\right)^{\frac{\alpha}{2}} (\cos^2 z)^{\frac{1}{2}} \cos \left((\alpha - s + 3) z\right) dz,
\end{equation}
which can be rewritten as
\begin{equation}
I = \frac{x}{2} \int_0^{\arctan 2w} \left(\cos 2z - \frac{1 - \cos 2z}{2 \cdot 4w^2}\right)^{\frac{\alpha}{2}} (\cos 2z)^{\frac{1}{2}} \cos \left((\alpha - s + 3) z\right) dz
= 2^{-\frac{\alpha}{2}} \int_0^{\arctan 2w} \left(\cos 2z(1 + \frac{1}{4w^2}) - (\frac{1}{4w^2} - 1)\right)^{\frac{\alpha}{2}} \cdot \cos \left((\alpha - s + 3) z\right) dz.
\end{equation}
Thus,
\begin{equation}
\frac{I}{\left(\frac{1 + 4w^2}{4w^2}\right)^{\frac{\alpha}{2}}} = \int_0^{\arctan 2w} \left(\cos 2z - \frac{1 - 4w^2}{1 + 4w^2}\right)^{\frac{\alpha}{2}} (\cos z) \cos \left((\alpha - s + 3) z\right) dz
= \frac{1}{2} \int_0^{\arctan 2w} (\cos \beta - \frac{1 - 4w^2}{1 + 4w^2})^{\frac{\alpha}{2}} \cos \beta \cos \left(\frac{\alpha - s + 3}{2}\beta\right) d\beta
= \frac{1}{4} \int_0^{\arctan 2w} (\cos \beta - \cos \gamma)^{\frac{\alpha}{2}} \cos \beta \cos \left(\frac{\alpha - s + 3}{2}\beta\right) d\beta,
\end{equation}
where $\cos \gamma = \frac{1 - 4w^2}{1 + 4w^2}$. By applying the formula $2 \cos A \cos B = \cos(A + B) + \cos (A - B)$ with $A = \frac{\beta}{2}$ and $B = \frac{\alpha - s + 3}{2} \beta$, the latter integral is given by a sum of the following two integrals
\begin{equation}
J_1 = \int_0^{\arctan 2w} (\cos \beta - \cos \gamma)^{\frac{\alpha}{2}} \cos \left(\frac{\alpha - s + 2}{2}\beta\right) d\beta,
\end{equation}
and
\[ J_2 = \int_0^{2\arctan 2w} (\cos \beta \cos \gamma)^{\alpha-s} \cos \left( \frac{\alpha - s + 4}{2} \beta \right) d\beta. \]

One the one hand, the integral \( J_1 \) can be evaluated by using (3.7) with \( \nu = \frac{\alpha-s+1}{2} \) and \( a = \frac{\alpha-s+2}{2} \). Combined with the representation for the associated Legendre function (3.8), it becomes

\[
J_1 = \sqrt{\frac{\pi}{2}} (\sin \gamma)^{\alpha-s+1} \Gamma \left( \frac{\alpha-s+2}{2} \right) \frac{\Gamma \left( \frac{\alpha-s+1}{2} \right) \Gamma \left( \frac{\alpha-s+3}{2} \right)}{\Gamma \left( \frac{\alpha-s+1}{2} \right) \Gamma \left( \frac{\alpha-s+3}{2} \right)} (\cos \gamma)
\]

\[
= \sqrt{\frac{\pi}{2}} \left( \frac{\alpha-s+2}{2} \right) (\sin \gamma)^{\alpha-s+1} \Gamma \left( \frac{\alpha-s+3}{2} \right) (\cos \gamma)
\]

On the other hand, the integral \( J_2 \) can be evaluated using (3.9) and taking \( \nu = \frac{\alpha-s+2}{2} \) and \( \beta = 1 \). Together with the representation for the Gegenbauer polynomial (3.10), we have

\[
J_2 = \sqrt{\frac{\pi}{2}} \Gamma(2) \frac{\Gamma \left( \frac{\alpha-s+2}{2} \right) \Gamma(\alpha-s+2)}{\Gamma(\alpha-s+3)} (\sin \gamma)^{\alpha-s+1} \Gamma \left( \frac{\alpha-s+3}{2} \right) (1 + \cos \gamma)
\]

\[
= \sqrt{\frac{\pi}{2}} \Gamma(2) \frac{\Gamma \left( \frac{\alpha-s+2}{2} \right) \Gamma(\alpha-s+2)}{\Gamma(\alpha-s+3)} (\sin \gamma)^{\alpha-s+1} \Gamma \left( \frac{\alpha-s+3}{2} \right) (\cos \gamma)
\]

\[
= \sqrt{\frac{\pi}{2}} \frac{\Gamma \left( \frac{\alpha-s+2}{2} \right) \Gamma \left( \frac{\alpha-s+3}{2} \right)}{\Gamma(\alpha-s+3)} (\sin \gamma)^{\alpha-s+1} (1 + \cos \gamma)
\]

Besides, we obtain \( \sin \gamma = \frac{1}{1+4w^2} \) because \( \cos \gamma = \frac{1-4w^2}{1+4w^2} \). Thus, we get

\[
J_1 + J_2 = \sqrt{\frac{\pi}{2}} \frac{\Gamma \left( \frac{\alpha-s+2}{2} \right) \Gamma \left( \frac{\alpha-s+3}{2} \right)}{\Gamma(\alpha-s+3)} (\sin \gamma)^{\alpha-s+1} \left( 1 + \cos \gamma \right)
\]

\[
= \sqrt{\frac{\pi}{2}} \frac{\Gamma \left( \frac{\alpha-s+2}{2} \right) \Gamma \left( \frac{\alpha-s+3}{2} \right)}{\Gamma(\alpha-s+3)} \frac{4w}{1+4w^2} ^{\alpha-s+1} \left( 1 + 4w^2 \right) \left( 1 + 4w^2 \right)
\]

which gives

\[
I = \frac{1}{4} \sqrt{\frac{\pi}{2}} \frac{\Gamma \left( \frac{\alpha-s+2}{2} \right) \Gamma \left( \frac{\alpha-s+3}{2} \right)}{\Gamma(\alpha-s+3)} \left( 1 + 4w^2 \right) \frac{1}{8w^2} \left( 4w \right) ^{\alpha-s+1} \left( 1 + 4w^2 \right) \left( 1 + 4w^2 \right)
\]

\[
= \sqrt{\frac{\pi}{2}} \frac{\Gamma \left( \frac{\alpha-s+2}{2} \right) \Gamma \left( \frac{\alpha-s+3}{2} \right)}{\Gamma(\alpha-s+3)} \frac{4w}{1+4w^2} ^{\alpha-s+1} \left( 1 + 4w^2 \right) (1+4w^2) ^{-\frac{\alpha-s+4}{2}} \quad (3.17)
\]

Finally, plugging (3.17) into (3.15), it gives

\[
K_{\alpha,s}(1, w) = \frac{2^{\alpha-s} \Gamma(\alpha-s+3) \Gamma \left( \frac{\alpha-s+2}{2} \right) \Gamma \left( \frac{\alpha-s+3}{2} \right)}{\sqrt{\pi} \Gamma(\alpha+1)} \frac{1}{\Gamma \left( \frac{\alpha-s+2}{2} \right) (1+4w^2)^{\alpha-s+4}} \left( 1 + 4w^2 \right) ^{-\frac{\alpha-s+4}{2}}
\]

and by (3.12),

\[
K_{\alpha,s}(x, w) = x^{-2(\alpha-s+3)} K_{\alpha,s} \left( \frac{1}{x^2} \right) = c_{\alpha,s} \frac{x^2 (x^4 + 4w^2)^{\alpha-s}}{(x^4 + 4w^2)^{\alpha-s+3}}
\]
where the constant $c_{\alpha,s}$ is given by
\[
c_{\alpha,s} = \frac{2^{2-s} \Gamma(\alpha - s + 3)}{\sqrt{\pi} \Gamma(\alpha + 1)} \frac{\Gamma\left(\frac{\alpha-s+2}{2}\right)}{\Gamma\left(\frac{\alpha-s+3}{2}\right)}.
\]
By applying Legendre's duplication formula
\[
\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}),
\]
with $z = \frac{\alpha-s+3}{2}$, and after simplification, we obtain
\[
c_{\alpha,s} = \frac{2^{2-2s+4} \Gamma\left(\frac{\alpha-s+2}{2}\right)}{\pi \Gamma(\alpha + 1)} \Gamma\left(\frac{\alpha-s+2}{2}\right) \Gamma\left(\frac{\alpha-s+4}{2}\right).
\]
It completes the proof of the Proposition.

**Remark 3.4.** Observe that the kernel $K_{\alpha,s}$ is a positive function. Furthermore, since generalized translation is a positive operator (see [12, Proposition 3.2]), we have $T^{\eta}_{\alpha} K_{\alpha,s} \geq 0$.  

To give an integral representation of $\Lambda_{\alpha,1-s}$, we shall need this identity (see [9, p.382, 3.541.1])
\[
\int_{0}^{\infty} e^{-\mu t} \sinh^s \beta t dt = \frac{1}{\frac{\nu}{\beta^2} + 1} \frac{\Gamma\left(\frac{\nu}{\beta^2} + \frac{s}{2}\right)}{\Gamma\left(\frac{\nu}{\beta^2} + \frac{s}{2} + 1\right)}
\]
valid for $\Re \beta > 0$, $\Re \nu > -1$, and $\Re \mu > \Re (\beta \nu)$.

Now we can prove the following integral representation for $\Lambda_{\alpha,1-s}$ in an analogous way as that by Roncal and Thangavelu [10, Proposition 4.3].

**Theorem 3.5.** Let $\alpha \geq 0$ and $0 < s < 1$. Then for all $f \in W^{1-s,2}(X,d\mu_{\alpha})$, we have
\[
\Lambda_{\alpha,1-s} f(\xi) = \frac{1}{\Gamma(s-1)} \int_{0}^{\infty} \left(f(\xi) - f \ast_{\alpha} K_{\alpha,t,s}(\xi)\right) t^{s-2} dt.
\]
Moreover, the following pointwise representation is valid for all $f \in C_{0}^{\infty}(X)$,
\[
\Lambda_{\alpha,1-s} f(\xi) = \frac{1}{\Gamma(s-1)} \int_{X} \left(f(\xi) - f(\eta)\right) T^{\eta}_{\alpha} K_{\alpha,s}(\xi) d\mu_{\alpha}(\eta).
\]

**Proof.** Use the identity (3.18) and take $\nu = s - 1$, $\beta = 1$, which gives that
\[
2s \int_{0}^{\infty} e^{-\mu t} (\sinh t)^{s-1} dt = \frac{\Gamma(s) \Gamma\left(\frac{\mu}{2} + \frac{1-s}{2}\right)}{\Gamma\left(\frac{\mu}{2} + \frac{1-s}{2} + 1\right)}.
\]
It follows from the above identity that
\[
\frac{\mu}{\Gamma\left(\frac{\mu}{2} + \frac{1-s}{2}\right)} = 2s \int_{0}^{\infty} \frac{d}{dt} \left(1 - e^{-\mu t}\right) (\sinh t)^{s-1} dt
\]
\[
= 2s (1 - s) \int_{0}^{\infty} (1 - e^{-\mu t}) \cosh t (\sinh t)^{s-2} dt.
\]
So we have
\[
2^{-s} \frac{\mu}{\Gamma\left(\frac{\mu}{2} + \frac{1-s}{2}\right)} = (1 - s) \int_{0}^{\infty} (1 - e^{-\mu t}) \cosh t (\sinh t)^{s-2} dt.
\]
It is claimed that
\[ \int_0^{+\infty} \left( \cosh t \left( \sinh t \right)^{s-2} - t^{s-2} \right) dt = 0. \tag{3.20} \]

Indeed, for any \( \epsilon > 0 \), consider the integral
\[ \int_e^{+\infty} \cosh t \left( \sinh t \right)^{s-2} dt = \int_{\sinh \epsilon}^{+\infty} t^{s-2} dt = \int_{\sinh \epsilon}^{+\infty} t^{s-2} dt - \int_{\epsilon}^{\sinh \epsilon} t^{s-2} dt. \]

It implies that
\[ \int_{\epsilon}^{+\infty} \left( \cosh t \left( \sinh t \right)^{s-2} - t^{s-2} \right) dt = -\int_{\epsilon}^{\sinh \epsilon} t^{s-2} dt, \]
which converges to 0 as \( \epsilon \to 0 \). In view of (3.19) and (3.20), we obtain
\[ 2^{-s} \mu \frac{\Gamma \left( \frac{\mu}{2} + \frac{1-s}{2} \right)}{\Gamma \left( \frac{\mu}{2} + \frac{s}{2} \right)} = \frac{(1-s)}{\Gamma(s)} \int_0^{+\infty} \left( 1 - e^{-\mu t} \cosh \left( \frac{t}{\sinh t} \right)^{2-s} \right) t^{s-2} dt. \]

Hence, by taking \( \mu = 2k + \alpha + 1 \) and the change of variable \( t \to |\lambda| t \), we get
\[ 2^{-s} (2k + \alpha + 1) \Gamma \left( \frac{2k+\alpha+1}{2} + \frac{1-s}{2} \right) \Gamma \left( \frac{2k+\alpha+1}{2} + \frac{1+s}{2} \right) \]
\[ = \frac{(1-s)}{\Gamma(s)} \int_0^{+\infty} \left( 1 - e^{-2k+\alpha+1} t \cosh \left( \frac{|\lambda| t}{\sinh |\lambda| t} \right)^{2-s} \right) t^{s-2} dt. \]

Multiplying both sides by \( \frac{|\lambda|^{(s-1)/2}}{\pi \Gamma(\alpha+1)} f(\alpha, \lambda, k) \phi_{\alpha, \lambda}(x) \), and summing over \( k \), we have
\[ \frac{|\lambda|^{(s-1)/2}}{\pi \Gamma(\alpha+1)} \sum_{k=0}^{+\infty} \Gamma \left( \frac{2k+\alpha+1}{2} + \frac{1-s}{2} \right) \Gamma \left( \frac{2k+\alpha+1}{2} + \frac{1+s}{2} \right) \]
\[ = \frac{(1-s)}{\Gamma(s)} \frac{|\lambda|^{(s+1)/2}}{\pi \Gamma(\alpha+1)} \int_0^{+\infty} \left[ \sum_{k=0}^{+\infty} f(\alpha, \lambda, k) \phi_{\alpha, \lambda}(x) \right. \]
\[ - \left. \sum_{k=0}^{+\infty} e^{-2k+\alpha+1} |\lambda|^t \cosh |\lambda| t \left( \frac{|\lambda| t}{\sinh |\lambda| t} \right)^{2-s} \right] \]
\[ f(\alpha, \lambda, k) \phi_{\alpha, \lambda}(x) \right] t^{s-2} dt. \]

We multiply the above identity both sides by \( e^{-i\lambda w} \) and integral over \( \lambda \) variable, we obtain
\[ A_{\alpha,1-s} f(\xi) = \frac{(1-s)}{\Gamma(s)} \int_0^{+\infty} \left( f(\xi) - f * \kappa_{\alpha,t,s}(\xi) \right) t^{s-2} dt. \]

Because of \( \frac{(1-s)}{\Gamma(s)} = \frac{1}{\Gamma(s-1)} \), we have proved the representation
\[ A_{\alpha,1-s} f(\xi) = \frac{1}{\Gamma(s-1)} \int_0^{+\infty} \left( f(\xi) - f * \kappa_{\alpha,t,s}(\xi) \right) t^{s-2} dt. \]

By Lemma 2.2, we have
\[ f(\xi) - f * \kappa_{\alpha,t,s}(\xi) = f(\xi) - \int_X f(\xi) T_{\alpha}^s \kappa_{\alpha,t,s}(\xi) d\mu_{\alpha}(\eta) \]
\[ = \int_X \left( f(\xi) - f(\eta) \right) T_{\alpha}^s \kappa_{\alpha,t,s}(\xi) d\mu_{\alpha}(\eta). \]
Therefore we get
\[ \Lambda_{\alpha,1-s}f(\xi) = \frac{1}{|\Gamma(s-1)|} \int_0^{+\infty} \left( \int_X (f(\xi) - f(\eta)) T^s_\alpha K_{\alpha,t,s}(\xi) d\mu_\alpha(\eta) \right) t^{s-2} \, dt. \]

Under the assumption that \( f \in C_0^\infty(X) \), by using the stratified mean value theorem (see [5]), we can exchange the order of integration, which gives our desired representation.

The next proposition follows from the above Theorem.

**Proposition 3.6.** Let \( \alpha \geq 0 \) and \( 0 < s < 1 \). Then for all \( f, g \in W^{1-s,2}(X, d\mu_\alpha) \), we have
\[ \langle \Lambda_{\alpha,1-s}f, g \rangle_\alpha = \frac{1}{|\Gamma(s-1)|} \int_X \int_X \left( (f(\xi) - f(\eta)) \frac{g(\xi) - g(\eta)}{\vert g(\xi) - g(\eta) \vert} T^s_\alpha K_{\alpha,t,s}(\xi) d\mu_\alpha(\eta) \right) d\mu_\alpha(\xi). \]

**Proof.** Let \( f, g \in C_0^\infty(X) \). The integral representation obtained in last proposition gives that
\[ \langle \Lambda_{\alpha,1-s}f, g \rangle_\alpha = \frac{1}{|\Gamma(s-1)|} \int_X \int_X \left( (f(\xi) - f(\eta)) \frac{g(\xi) - g(\eta)}{\vert g(\xi) - g(\eta) \vert} T^s_\alpha K_{\alpha,t,s}(\xi) d\mu_\alpha(\eta) \right) d\mu_\alpha(\xi) = \frac{1}{|\Gamma(s-1)|} \int_X \int_X \left( (f(\eta) - f(\xi)) \frac{g(\eta) - g(\xi)}{\vert g(\eta) - g(\xi) \vert} T^s_\alpha K_{\alpha,t,s}(\eta) d\mu_\alpha(\xi) \right) d\mu_\alpha(\eta) = -\frac{1}{|\Gamma(s-1)|} \int_X \int_X \left( (f(\xi) - f(\eta)) \frac{g(\eta) - g(\xi)}{\vert g(\eta) - g(\xi) \vert} T^s_\alpha K_{\alpha,t,s}(\xi) d\mu_\alpha(\eta) \right) d\mu_\alpha(\xi). \]

Noting that (2.7), by Fubini’s theorem, we obtain
\[ \langle \Lambda_{\alpha,1-s}f, g \rangle_\alpha = -\frac{1}{|\Gamma(s-1)|} \int_X \int_X (f(\xi) - f(\eta)) \frac{g(\xi) - g(\eta)}{\vert g(\xi) - g(\eta) \vert} T^s_\alpha K_{\alpha,t,s}(\xi) d\mu_\alpha(\eta) d\mu_\alpha(\xi). \]

Therefore, we have
\[ \langle \Lambda_{\alpha,1-s}f, g \rangle_\alpha = \frac{1}{2|\Gamma(s-1)|} \int_X \int_X (f(\xi) - f(\eta)) \frac{g(\xi) - g(\eta)}{\vert g(\xi) - g(\eta) \vert} T^s_\alpha K_{\alpha,t,s}(\xi) d\mu_\alpha(\eta) d\mu_\alpha(\xi). \]

By using a density argument as [10, Theorem 5.4], we complete the proposition.

### 3.3. The ground state representation

Now we define the ground state representation for the operator \( \Lambda_{\alpha,1-s} \) by
\[ H_{\alpha,s}[f] = \langle \Lambda_{\alpha,1-s}f, f \rangle_\alpha - B_{\alpha,s} \int_X \frac{|f(x, w)|^2}{(x^4 + 4w^2)^{\frac{s+2}{2}}} \, d\mu_\alpha(x, w), \]
where \( B_{\alpha,s} \) is a positive constant defined by
\[ B_{\alpha,s} = 2^{2-2s} \left( \frac{\Gamma\left(\frac{a-s+2}{2}\right)}{\Gamma(s)\Gamma\left(\frac{a+1}{2}\right)} \right)^2. \]

**Theorem 3.7.** For \( \alpha \geq 0 \) and \( 0 < s < 1 \), let \( F \in C_0^\infty(X) \) be supported away from \( 0 \). If we define \( G_\alpha(\xi) = F(\xi)g_{\alpha,1}(\xi)^{-1} \), then
\[ H_{\alpha,s}[F] = \frac{1}{2|\Gamma(s-1)|} \int_X \int_X \left| G_\alpha(\xi) - G_\alpha(\eta) \right|^2 T^s_\alpha K_{\alpha,t,s}(\xi)g_{\alpha,1}(\xi)g_{\alpha,1}(\eta) d\mu_\alpha(\eta) d\mu_\alpha(\xi). \]
Proof. We apply the formula in the above proposition to \( g(\xi) = u_{\alpha,-1,\delta}(\xi) \) and \( f(\xi) = |F(\xi)|^2 g(\xi)^{-1} \), then

\[
\langle \Lambda_{\alpha,-1-s} f, g \rangle_\alpha = \frac{1}{2\Gamma(s-1)} \int_X \int_X \left( \frac{|F(\xi)|^2}{u_{\alpha,-1,\delta}(\xi)} - \frac{|F(\eta)|^2}{u_{\alpha,-1,\delta}(\eta)} \right) 
\cdot \left( u_{\alpha,-1,\delta}(\xi) - u_{\alpha,-1,\delta}(\eta) \right) T_\alpha^c K_{\alpha,s}(\xi) d\mu_\alpha(\eta) d\mu_\alpha(\xi).
\]

After simplification, the right hand side of (3.21) becomes

\[
\frac{1}{2\Gamma(s-1)} \int_X \int_X \left( |F(\xi) - F(\eta)|^2 - \frac{F(\xi)}{u_{\alpha,-1,\delta}(\xi)} \right) 
\cdot \left( u_{\alpha,-1,\delta}(\xi) - u_{\alpha,-1,\delta}(\eta) \right) T_\alpha^c K_{\alpha,s}(\xi) d\mu_\alpha(\eta) d\mu_\alpha(\xi).
\]

On the other hand, the left hand side of (3.21) can be simplified by the explicit formula for the Fourier transform of \( u_{\alpha,-1,\delta} \). Noting the fact that \( \Lambda_{\alpha,-1-s} \) is self-adjoint, we have

\[
\langle \Lambda_{\alpha,-1-s} f, g \rangle_\alpha = \langle f, \Lambda_{\alpha,-1-s} g \rangle_\alpha = \langle f, L_{\alpha,s}^{-1} L_\alpha u_{\alpha,-1,\delta} \rangle_\alpha = \langle f, \nu_{\alpha,s,\delta} \rangle_\alpha,
\]

where \( \nu_{\alpha,s,\delta}(\xi) = L_{\alpha,s}^{-1} L_\alpha u_{\alpha,-1,\delta} \). Consequently, we have the identity

\[
\langle f, \nu_{\alpha,s,\delta} \rangle_\alpha = \frac{1}{2\Gamma(s-1)} \int_X \int_X \left( |F(\xi) - F(\eta)|^2 - \frac{F(\xi)}{u_{\alpha,-1,\delta}(\xi)} \right) 
\cdot \left( u_{\alpha,-1,\delta}(\xi) - u_{\alpha,-1,\delta}(\eta) \right) T_\alpha^c K_{\alpha,s}(\xi) d\mu_\alpha(\eta) d\mu_\alpha(\xi).
\]

The Fourier transform of \( \nu_{\alpha,s,\delta} \) is given by

\[
\hat{\nu}_{\alpha,s,\delta}(\alpha, \lambda, k) = \frac{\Gamma\left(\frac{2k+\alpha+1}{2} + \frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2k+\alpha+1}{2} + 1\right)} (2|\lambda|)^{-(2k+\alpha+1)} |\lambda|^{2k+\alpha+1} \hat{u}_{\alpha,-1,\delta}(\lambda, k) 
= \frac{\Gamma\left(\frac{2k+\alpha+1}{2} + \frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2k+\alpha+1}{2} + \frac{1-\alpha}{2}\right)} (2|\lambda|)^{-(2k+\alpha+1)} |\lambda|^\lambda_{\alpha, k, \delta}(-1).
\]

Then

\[
\frac{(2|\lambda|)^{2\alpha} \hat{\nu}_{\alpha,s,\delta}(\alpha, \lambda, k)}{\Gamma\left(\frac{2k+\alpha+1}{2} + \frac{1-\alpha}{2}\right)} = (2k+\alpha+1) |\lambda| \frac{\pi \Gamma(\alpha+1)}{(\Gamma(\frac{\alpha+1}{2}))^2} \left(\frac{2k+\alpha+3}{2}\right)^{2k+\alpha+1} L(\delta |\lambda|, \frac{2k+\alpha+1}{2}, \frac{2k+\alpha+3}{2}) 
= 2\pi \Gamma(\alpha+1) \frac{2k+\alpha+1}{2} \frac{2k+\alpha+3}{2} L(\delta |\lambda|, \frac{2k+\alpha+1}{2}, \frac{2k+\alpha+3}{2}).
\]

Let \( \delta \) tend to 0, then

\[
L(0, \frac{2k+\alpha+1}{2}, \frac{2k+\alpha+3}{2}) = \frac{\Gamma\left(\frac{2k+\alpha+1}{2}\right)}{\Gamma\left(\frac{2k+\alpha+3}{2}\right)} = \frac{2}{2k+\alpha+1},
\]

which deduces that \( \nu_{\alpha,s,\delta} \) converges in the sense of distribution to

\[
\frac{2\pi \Gamma(\alpha+1)}{(\Gamma(\frac{\alpha+1}{2}))^2} g_{\alpha,s}(\xi) = \frac{2^{\alpha-2s+3} \Gamma(\frac{\alpha+s+2}{2})^2}{\Gamma(s) (\Gamma(\frac{\alpha+s+2}{2}))^2} (x^4 + 4w^2)^{-\frac{\alpha+s+2}{2}}.
\]

It follows that \( \langle f, \nu_{\alpha,s,\delta} \rangle \) converges to

\[
\frac{2^{\alpha-2s} \Gamma(\frac{\alpha+s+2}{2})^2}{\Gamma(s) (\Gamma(\frac{\alpha+s+2}{2}))^2} \int_X \frac{|F(x, w)|^2}{(x^4 + 4w^2)^{\frac{\alpha+s+2}{2}}} d\mu_\alpha(x, w).
\]
On the other hand, since $F$ is supported away from 0, the right hand side of (3.21) converges to
\[ \frac{1}{2|\Gamma(s-1)|} \int_X \int_X \left( |F(\xi) - F(\eta)|^2 - \left| \frac{F(\xi)}{g_{\alpha,1}(\xi)} - \frac{F(\eta)}{g_{\alpha,1}(\eta)} \right|^2 g_{\alpha,1}(\xi)g_{\alpha,1}(\eta) \right) T_{\alpha}^\alpha K_{\alpha,s}(\xi) d\mu(\eta) d\mu(\xi). \]

Using Proposition 3.6, we have
\[ \frac{1}{2|\Gamma(s-1)|} \int_X \int_X |F(\xi) - F(\eta)|^2 T_{\alpha}^\alpha K_{\alpha,s}(\xi) d\mu(\eta) d\mu(\xi) = \langle \Lambda_{\alpha,1-s} F, F \rangle_\alpha, \]
which completes the proof of the theorem.

3.4. The Hardy inequalities and uncertainty principle.

Proof of Theorem 1.6: By Remark 3.4, we conclude that $H_{\alpha,s}[F] \geq 0$. It immediately deduces Hardy’s inequality for $\Lambda_{\alpha,1-s}$ under the extra assumption that $F$ is supported away from 0. In fact, this condition can be removed. From (3.22), the inequality
\[ \int_X |F(\xi)|^2 u_{\alpha,-1,\delta}(\xi) d\mu_\alpha(\xi) \leq \langle \Lambda_{\alpha,1-s} F, F \rangle_\alpha \]
holds true for any $F \in C_0^\infty(X)$. As
\[ \int_X |F(\xi)|^2 \left( \left( \frac{x^2}{2} \right)^{\alpha+1} + w^2 \right)^{\alpha+1} \nu_{\alpha,s,\delta}(\xi) d\mu_\alpha(\xi) \leq \int_X |F(\xi)|^2 \nu_{\alpha,s,\delta}(\xi) d\mu_\alpha(\xi) \leq \langle \Lambda_{\alpha,1-s} F, F \rangle_\alpha, \]
we can pass to the limit as $\delta \to 0$. Since $\nu_{\alpha,s,\delta}$ converges in the sense of distribution to a constant multiple of $g_{\alpha,s}$, we obtain our desired inequality.

In order to prove Theorem 1.8, we need to introduce a version of Schur test.

Lemma 3.8 (Schur test). Let $(Y, d\nu_Y)$, $(Z, d\nu_Z)$ be two measurable spaces. $T$ is an integral operator given by
\[ Tf(z) = \int_Y K(y, z)f(y) \nu_Y(y) \]
with a non-negative and locally integrable kernel $K : Y \times Z \to \mathbb{R}^+$. If there exist real functions $p(y) > 0$ and $q(z) > 0$ and numbers $0 < A, B < +\infty$ such that
\[ \int_X K(y, z)p(y) d\nu_Y(y) \leq Aq(z) \text{ for almost every } y \in Y \quad (3.23) \]
and
\[ \int_Y K(y, z)q(z) d\nu_Z(z) \leq Bp(y) \text{ for almost every } y \in Y, \quad (3.24) \]
then the integral operator $T$ is well-defined for all $f \in L^2(Y, d\nu_Y)$, with
\[ \|T\|_{L^2(Y, d\nu_Y) \to L^2(Z, d\nu_Z)} \leq \sqrt{AB}. \]
Thus, we have whose adjoint operator is \( L \).

Moreover, we define another operator on \( L \).

By Schur test, it indicates that \( T \).

Observe that \( C \) we take \( f \) and \( f \).

Proof of Theorem 1.8. From Lemma 3.1, for \( 0 < s < \frac{\alpha+2}{2} \), we obtain

\[
\|Tf\|_{L^2(\mathbb{R}^n, d\mu)} \leq A \int_Y \left( \int_Z K(y, z) q(z) d\nu_Z(z) \right) \left| f(y) \right|^2 p(y) \, d\nu_Y(y)
\]

Integrating the above inequality in \( z \), by Fubini’s theorem and (3.24), it follows that

\[
\|Tf\|_{L^2(\mathbb{R}^n, d\mu)} \leq A \int_Y \left( \int_Z K(y, z) q(z) d\nu_Z(z) \right) \left| f(y) \right|^2 p(y) \, d\nu_Y(y)
\]

Proof of Theorem 1.8. From Lemma 3.1, for \( 0 < s < \frac{\alpha+2}{2} \), we obtain

\[
\mathcal{L}_{\alpha,-s} u_{\alpha,s,\delta} = C_{\alpha,-s,\delta} u_{\alpha,-s,\delta},
\]

where \( C_{\alpha,s,\delta} = (4\delta)^s \left( \frac{1}{\Gamma\left(\frac{\alpha+2}{2}\right)} \right)^2 \). Setting \( \varphi_\delta(x, w) = (\delta + x^2/2) + w^2 \), we get

\[
u_{\alpha,s,\delta}(x, w) = \varphi_\delta^{-s/2}(x, w) u_{\alpha,0,\delta}(x, w).
\]

In addition, we define an integral operator on \( L^2(X, d\mu_\alpha) \) by

\[
\mathcal{T}_{\alpha,s,\delta} f(\xi) = \varphi_\delta^{-s/2}(\xi) \mathcal{L}_{\alpha,-s} \left( \varphi_\delta^{-s/2} f \right)(\xi).
\]

Observe that \( \mathcal{T}_{\alpha,s,\delta} \) has a positive kernel and satisfies \( \mathcal{T}_{\alpha,s,\delta} u_{\alpha,0,\delta} = C_{\alpha,-s,\delta} u_{\alpha,0,\delta} \).

By Schur test, it indicates that \( \mathcal{T}_{\alpha,s,\delta} \) is bounded on \( L^2(X, d\mu_\alpha) \) and

\[
\|\mathcal{T}_{\alpha,s,\delta}\|_{L^2(X, d\mu_\alpha) \rightarrow L^2(X, d\mu_\alpha)} \leq C_{\alpha,-s,\delta}.
\]

Moreover, we define another operator on \( L^2(X, d\mu_\alpha) \) by

\[
N_{\alpha,s,\delta} f = \varphi_\delta^{-s/2} \mathcal{L}_{\alpha,-s}^{1/2} f,
\]

whose adjoint operator is \( N_{\alpha,s,\delta}^* f = \mathcal{L}_{\alpha,-s}^{1/2} (\varphi_\delta^{-s/2} f) \). It can be easily checked that \( \mathcal{T}_{\alpha,s,\delta} = N_{\alpha,s,\delta} N_{\alpha,s,\delta}^* \), which leads to

\[
\|N_{\alpha,s,\delta}\|_{L^2(X, d\mu_\alpha) \rightarrow L^2(X, d\mu_\alpha)} = \|\mathcal{T}_{\alpha,s,\delta}\|_{L^2(X, d\mu_\alpha) \rightarrow L^2(X, d\mu_\alpha)} \leq C_{\alpha,-s,\delta}.
\]

Thus, we have

\[
C_{\alpha,s,\delta} \|\varphi_\delta^{-s/2} \mathcal{L}_{\alpha,-s}^{1/2} f\|_{L^2(X, d\mu_\alpha)} = C_{\alpha,s,\delta} \|N_{\alpha,s,\delta} f\|_{L^2(X, d\mu_\alpha)} \leq \|f\|_{L^2(X, d\mu_\alpha)}.
\]

Noting that \( \mathcal{L}_{\alpha,-s}^{1/2} = \mathcal{L}_{\alpha,-s}^{-1/2} \) and applying the above inequality to \( \mathcal{L}_{\alpha,-s}^{1/2} f \), we immediately get the inequality in Theorem 1.8 on a dense subspace. On the other hand, if we take \( f = u_{\alpha,-s,\delta} \) in the theorem, both sides of the inequality equal

\[
C_{\alpha,s,\delta} \langle u_{\alpha,s,\delta}, u_{\alpha,-s,\delta} \rangle,
\]

which proves the optimality of the constant \( C_{\alpha,s,\delta} \).

Proof of Theorem 1.7 and 1.9. By estimating the norms of certain bounded operators \( V_{\alpha,s} \) and \( U_{\alpha,s} \), we can immediately deduce the Hardy inequalities for \( \mathcal{L}_{\alpha}^s \). We
Therefore, if \( \gamma \) valid for \( x \geq x \)

Indeed, from the formula (see [18, Section 7])

\[
\frac{\Gamma(x + \gamma)}{\Gamma(x + \beta + 1)} = \frac{1}{\Gamma(\beta - \gamma + 1)} \int_0^\infty e^{-(x+\gamma)z}(1 - e^{-z})^{\beta-\gamma}dz,
\]

valid for \( \beta - \gamma > 0, x + \gamma > 0 \), it follows that

\[
\frac{\Gamma(x + \gamma)}{\Gamma(x + \beta + 1)} \leq \frac{1}{\Gamma(\beta - \gamma + 1)} \int_0^\infty e^{-(x+\gamma)z}z^{\beta-\gamma}dz = (x + \gamma)^{-(\beta-\gamma)-1}.
\]

Therefore, if \( \gamma \geq 0 \) and \( x > 0 \), we obtain that

\[
x^{\beta-\gamma} \frac{\Gamma(x + \gamma)}{\Gamma(x + \beta)} \leq \frac{x + \beta}{x + \gamma}.
\] (3.25)

Taking \( x = \frac{2k + \alpha + 1}{2} \), \( \beta = \frac{2 - \alpha}{2} \) and \( \gamma = \frac{s}{2} \), we have

\[
\left( \frac{2k + \alpha + 1}{2} \right)^{1-s} \frac{\Gamma\left( \frac{2k + \alpha + 1}{2} + \frac{s}{2} \right)}{\Gamma\left( \frac{2k + \alpha + 1}{2} + \frac{2 - \alpha}{2} \right)} \leq \frac{2k + \alpha + 3 - s}{2k + \alpha + 1 + s} \leq \frac{\alpha + 3 - s}{\alpha + 1 + s}.
\]

Similarly, it can be shown that \( U_{\alpha,s} \) is bounded on \( L^2(X, d\mu_\alpha) \) and

\[
\|U_{\alpha,s}\| = \sup_{k \geq 0} \left( \frac{2k + \alpha + 1}{2} \right)^{-s} \frac{\Gamma\left( \frac{2k + \alpha + 1}{2} + \frac{1 + s}{2} \right)}{\Gamma\left( \frac{2k + \alpha + 1}{2} + \frac{1 + s}{2} \right)} \leq \max \left\{ \frac{\alpha + 1}{2} \right\}
\]

\[
\left( \frac{\alpha + 2 + s}{2} \right)^{1-s} \frac{\Gamma\left( \frac{\alpha + 2 + s}{2} + \frac{1 + s}{2} \right)}{\Gamma\left( \frac{\alpha + 2 + s}{2} + \frac{1 + s}{2} \right)} \leq \frac{2 + \alpha + s}{2 + \alpha + s} \left( \frac{2}{2 + \alpha + s} \right) \left( \frac{4 + \alpha - s}{2 + \alpha + s} \right),
\]

where \([\cdot]\) and \((\cdot)\) are the part and the fractional part of a real number respectively. Indeed, since \( \Gamma(x + 1) = x\Gamma(x), x > 0 \), we obtain

\[
x^{-s} \frac{\Gamma(x + \beta + s)}{\Gamma(x + \beta)} = x^{-(s-\gamma)} \frac{\Gamma(x + \beta + s - 1 + [s] + 1)}{\Gamma(x + \beta)} \leq \left( \frac{x + \beta + s - 1}{x} \right)^{[s] + 1} \frac{x^{-(s-\gamma)} \Gamma(x + \beta + s - 1)}{\Gamma(x + \beta)}.
\]

For \( k \geq 1 \), taking \( x = \frac{2k + \alpha - \gamma}{2}, \beta = 1, \gamma = \beta + (s) - 1 \), it is easily checked that \( x \geq \frac{2 + \alpha - \gamma}{2} > \frac{2 + \alpha}{2} \geq 0, \gamma = (s) \geq 0, x + \gamma > 0 \) and \( \beta - \gamma = 1 - (s) > 0 \). Hence, exploiting the estimate (3.25), we have

\[
\left( \frac{2k + \alpha - s}{2} \right)^{-s} \frac{\Gamma\left( \frac{2k + \alpha + 1}{2} + \frac{1 + s}{2} \right)}{\Gamma\left( \frac{2k + \alpha + 1}{2} + \frac{1 + s}{2} \right)} \leq \frac{2k + \alpha + s}{2k + \alpha - s} \left( \frac{2k + \alpha - s}{2k + \alpha - s + 2(s)} \right).
\]

So, for \( k \geq 1 \), we have

\[
\left( \frac{2k + \alpha + 1}{2} \right)^{-s} \frac{\Gamma\left( \frac{2k + \alpha + 1}{2} + \frac{1 + s}{2} \right)}{\Gamma\left( \frac{2k + \alpha + 1}{2} + \frac{1 + s}{2} \right)} \leq \frac{2k + \alpha + s}{2k + \alpha - s + 2(s)}.
\]
Hardy’s inequalities for the Grushin operator.

Proof of Theorem 1.10. Let $\mathcal{W}(x, w)$ denote either the homogeneous weight $(x^4 + 4w^2)^{\frac{s}{2}}$ or the non-homogeneous weight $((\delta + \hat{w}^2)^2 + w^2)^s$. By Cauchy-Schwarz inequality, we obtain

$$\left( \int_X |f(x, w)|^2 d\mu_\alpha(x, w) \right)^2 \leq \int_X |f(x, w)|^2 \mathcal{W}(x, w) d\mu_\alpha(x, w) \int_X |f(x, w)|^2 \mathcal{W}(x, w)^{-1} d\mu_\alpha(x, w).$$

By the Hardy inequality in Theorem 1.6 and 1.8, the last integral is bounded by $\Lambda_{\alpha, s}$ or $\mathcal{L}_{\alpha, s}$ times the corresponding constant. Finally, we obtain the uncertainty principle. 

4. **Spherical harmonics and Kecke-Bochner formula.** Above all, we recall some facts about spherical harmonic analysis and solid harmonics. For details, we refer to [11, Chapter 4]. For $m \in \mathbb{N}$, let $\mathcal{Y}_m$ denote the space of spherical harmonics of degree $m$. Let $a_m = \dim \mathcal{Y}_m$ and $\{Y_{m,j}\}_{j=1}^{a_m}$ be the orthonormal basis of $\mathcal{Y}_m$. We know that $L^2(S^{n-1}) = \oplus_{m=0}^{\infty} \mathcal{Y}_m$ and the collection $\{Y_{m,j} : m \in \mathbb{N}, j = 1, 2, \ldots, a_m\}$ forms an orthonormal basis for $L^2(S^{n-1})$. Corresponding to each spherical harmonic, we define solid harmonics on $\mathbb{R}^n$ by $P_{m,j}(x) = |x|^m Y_{m,j}(x/|x|)$. Let $\mathfrak{h}_m$ denote the space consisting of linear combination of functions of the form $g(|x|)P_{m,j}(x)$, where $g$ is radial such that $g(|x|)P_{m,j}(x) \in L^2(\mathbb{R}^n)$. With these definitions, we have $L^2(\mathbb{R}^n) = \oplus_{m=0}^{\infty} \mathfrak{h}_m$. Moreover, for any $f \in L^2(\mathbb{R}^{n+1})$, we have

$$f(x, w) = \sum_{m=0}^{\infty} \sum_{j=0}^{a_m} f_{m,j}(|x|, w) P_{m,j}(x), \quad (4.1)$$

where $f_{m,j}(|x|, w) = \int_{S^{n-1}} f(|x|\omega, w) P_{m,j}(|x|^{-1}\omega) d\sigma(\omega)$. By an easy calculation, we obtain

$$\int_{\mathbb{R}^n} |f(x, w)|^2 dx = \sum_{m=0}^{\infty} \sum_{j=0}^{a_m} \int_0^{\infty} |f_{m,j}(r, w)|^2 r^{n+2m-1} dr. \quad (4.2)$$

Moreover, we recall the Hecke-Bochner formula for the Hermite projection operators (see [15]).

**Lemma 4.1.** Suppose $f \in L^2(\mathbb{R}^n)$ is such that $f = gP$, where $g$ is radial and $P$ is a solid harmonic of degree $m$, then we have

$$\mathcal{P}_{2k+m}(\lambda) f(x) = R_{\lambda, k}^P(x) \phi_{\lambda, k}^P(|x|) P(x),$$

where $\phi_{\lambda, k}^P(|x|)$ is the spherical harmonic of degree $k$ and $P$ is the solid harmonic of degree $m$. 


where
\[ R_{m,k}^\alpha(g) = \frac{2|\alpha|^{\alpha+1} \Gamma(k+1)}{\Gamma(\alpha+k+1)} \int_0^{+\infty} g(r) \phi_{k,\lambda}(r) r^{2\alpha+1} dr \]
and \( \alpha = n/2 + m - 1 \). For other values \( j \in \mathbb{N} \), \( P_j f = 0 \).

4.2. Proof of the main theorem.

Proof of Theorem 1.1. Suppose that \( f \in C_\infty^\infty(\mathbb{R}^{n+1}) \) has a form of \( f(x, w) = g(|x|, w) P(x) \), where \( P \) is a solid harmonics of degree \( m \). Applying the spectral decomposition of \( \Lambda_s \) and Lemma 4.1, we obtain
\[ \Lambda_s f(x, w) = \frac{1}{2\pi} \int \sum_{k=0}^{+\infty} (2|\lambda|^{s-1}) \Gamma\left( \frac{k+n+2+\alpha}{2} \right) \Gamma\left( \frac{k+n+2}{2} \right) \lambda |\lambda| P_k(\lambda)f^\lambda(x)e^{-i\lambda w} d\lambda \]
\[ = \int \sum_{k=0}^{+\infty} \lambda |\lambda| P_k(\lambda)f^\lambda(x) \phi_{k,\lambda}(r) r^{n/2+m-1} e^{-i\lambda w} d\lambda, \]
where
\[ d_{m,k}(s) = \frac{2s^{-1}}{\pi} \Gamma\left( \frac{2k+m+n+2+s}{2} \right) \Gamma\left( \frac{k+n+1}{2} \right) \Gamma\left( \frac{k+n}{2} \right) (2k+m+n/2) \]
and
\[ A_{k,m}^\lambda(g) = \int_0^{+\infty} g^\lambda(r) \phi_{k,\lambda}(r) r^{n+2m-1} dr. \]
Thus, by the orthogonality of solid harmonics with respect to inner product inherited from \( L^2(\mathbb{S}^{n-1}) \), we obtain
\[ \langle \Lambda_s f, g \rangle = \left( \sum_{k=0}^{+\infty} d_{m,k}(s)|A_{m,k}^\lambda(g)|^2 \right) |\lambda|^{n/2+m+s} d\lambda. \]

On the other hand, we treat \( g \) as a function on \( X \), we get
\[ \langle \Lambda_s f, g \rangle = \langle \Lambda_n/2+m-1,s g, g \rangle_{n/2+m-1}. \]

Now for any \( f \in C_\infty^\infty(\mathbb{R}^{n+1}) \), Using (4.1) and (4.3), we obtain
\[ \langle \Lambda_s f, g \rangle = \sum_{m=0}^{\infty} \sum_{j=0}^{a_m} \langle \Lambda_s f, f \rangle_{n/2+m-1, s m, j} = \sum_{m=0}^{\infty} \sum_{j=0}^{a_m} \langle \Lambda_n/2+m-1, s f, f \rangle_{n/2+m-1}. \]

For any \( m \in \mathbb{N} \), by Theorem 1.6 and taking \( \alpha = n/2 + m - 1 \), we have
\[ \langle \Lambda_n/2+m-1, s f, f \rangle_{n/2+m-1} \geq \frac{2s^2 (\Gamma(m+n+2+s/2)^2)}{\Gamma(1-s)(\Gamma(m+n/2)^2)} \int \int_0^{+\infty} \left| f_{m,j}(x,w) \right|^2 \phi_{m,j}(x,w)^2 dx dw. \]
Thus, it deduces that

$$\langle A_s f, f \rangle \geq \sum_{m=0}^{\infty} \sum_{j=0}^{a_m} 2^{2s} \left( \frac{\Gamma(m+n/2+s)}{\Gamma(m/2)} \right)^2 \int_\mathbb{R} \int_0^\infty \frac{|f_{m,j}(x,w)|^2 x^{n+2m-1}}{(x^4 + 4w^2)^{s/2}} \, dx \, dw.$$ 

Since

$$\inf_{m \geq 0} \left\{ \frac{2^{2s} \left( \frac{\Gamma(m+n/2+s)}{\Gamma(m/2)} \right)^2}{\Gamma(1-s) \left( \frac{\Gamma(m/2)}{2} \right)^2} \right\} = \frac{2^{2s} \left( \frac{\Gamma(n/4)}{\Gamma(m/4)} \right)^2}{\Gamma(1-s) \left( \frac{\Gamma(m/4)}{2} \right)^2},$$

by (4.2), we have

$$\langle A_s f, f \rangle \geq \frac{2^{2s} \left( \frac{\Gamma(n/4)}{\Gamma(m/4)} \right)^2}{\Gamma(1-s) \left( \frac{\Gamma(m/4)}{2} \right)^2} \int_\mathbb{R} \int_0^\infty \frac{|f(x,w)|^2}{(x^4 + 4w^2)^{s/2}} \, dx \, dw.$$ 

As we obtained the inequality for $f \in C_c^\infty(\mathbb{R}^{n+1})$, by a density argument, we conclude that the inequality holds true for $f \in W^{s,2}(\mathbb{R}^{n+1})$. We observe that $0 < s < \frac{a + 2s}{2} = \frac{n + 2m + 2}{2}$ for all $m \in \mathbb{N}$ which requires the condition $0 < s < \frac{n+2}{4}$.

**Proof of Theorem 1.3.** We also suppose that $f \in C_c^\infty(\mathbb{R}^{n+1})$ has a form of $f(x,w) = g(|x|, w)P(x)$, where $P$ is a solid harmonics of degree $m$. By an argument analogous to the one used in the proof of Theorem 1.1, it can be checked that

$$\langle \mathcal{G}_s f, f \rangle = \langle \mathcal{L}_{n/2+m-1,s} g, g \rangle_{n/2+m-1}. \tag{4.4}$$

As in the proof of Theorem 1.1, it suffices to prove the inequality for any $f \in C_c^\infty(\mathbb{R}^{n+1})$. Using (4.1) and (4.4), we obtain

$$\langle \mathcal{G}_s f, f \rangle = \sum_{m=0}^{\infty} \sum_{j=0}^{a_m} \langle \mathcal{L}_s (f_{m,j} P_{m,j}), f_{m,j} P_{m,j} \rangle = \sum_{m=0}^{\infty} \sum_{j=0}^{a_m} \langle \mathcal{L}_{n/2+m-1,s} f_{m,j}, f_{m,j} \rangle_{n/2+m-1}.$$ 

For any $m \in \mathbb{N}$, by Theorem 1.8 and taking $\alpha = n/2 + m - 1$, we have

$$\langle \mathcal{L}_{n/2+m-1,s} f_{m,j}, f_{m,j} \rangle_{n/2+m-1} \geq (4\delta)^s \frac{\Gamma(\frac{m+n/2+s+1}{2})}{\Gamma(\frac{m+n/2+s-1}{2})} \int_\mathbb{R} \int_0^\infty \frac{|f_{m,j}(x,w)|^2 x^{n+2m-1}}{((\delta + x^2)^{2} + w^2)^s} \, dx \, dw.$$ 

Thus, it deduces that

$$\langle \mathcal{G}_s f, f \rangle \geq \sum_{m=0}^{\infty} \sum_{j=0}^{a_m} (4\delta)^s \frac{\Gamma(\frac{m+n/2+s+1}{2})}{\Gamma(\frac{m+n/2+s-1}{2})} \int_\mathbb{R} \int_0^\infty \frac{|f_{m,j}(x,w)|^2 x^{n+2m-1}}{((\delta + x^2)^{2} + w^2)^s} \, dx \, dw.$$
\[ \geq \inf_{m \geq 0} \left\{ (4\delta)^s \left( \frac{\Gamma\left(\frac{m+n/2+s+1}{2}\right)}{\Gamma\left(\frac{m+n/2-s+1}{2}\right)} \right)^2 \right\} \sum_{j=0}^{a_m} \int_{\mathbb{R}} |f_{m,j}(x,w)|^2 x^{n+2m-1} \left( \left( \delta + \frac{x^2}{2} \right)^s + w^2 \right)^{a_m} dx dw. \]

Since
\[ \inf_{m \geq 0} \left\{ (4\delta)^s \left( \frac{\Gamma\left(\frac{m+n/2+s+1}{2}\right)}{\Gamma\left(\frac{m+n/2-s+1}{2}\right)} \right)^2 \right\} = (4\delta)^s \left( \frac{\Gamma\left(\frac{n/2+s+1}{2}\right)}{\Gamma\left(\frac{n/2-s+1}{2}\right)} \right)^2, \]

by (4.2), we have
\[ \langle Gs f, f \rangle \geq (4\delta)^s \left( \frac{\Gamma\left(\frac{n/2+s+1}{2}\right)}{\Gamma\left(\frac{n/2-s+1}{2}\right)} \right)^2 \int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{|f(x,w)|^2}{\left( \left( \delta + \frac{|x|^2}{2} \right)^s + w^2 \right)^a_m} dx dw. \]

Finally, we shall prove the constant is sharp. If we take
\[ f(x, w) = \left( \left( \delta + |x|^2/2 \right)^s + w^2 \right)^{-\frac{n/2-s+1}{2}}, \]

then \( f(x, w) = u_{n/2-1-s, \delta}(|x|, w) \). By (4.4), it follows that
\[ \langle Ls f, f \rangle = \langle L_{n/2-1-s, \delta, u_{n/2-1-s, \delta}} \rangle_{n/2-1}. \]

Since \( u_{n/2-1-s, \delta} \) is the optimizer of the inequality in Theorem 1.8 for \( \alpha = n/2 - 1 \), we can see that
\[ \langle L_{n/2-1-s, \delta, u_{n/2-1-s, \delta}} \rangle_{n/2-1} = (4\delta)^s \left( \frac{\Gamma\left(\frac{n/2+s+1}{2}\right)}{\Gamma\left(\frac{n/2-s+1}{2}\right)} \right)^2 \int_{\mathbb{R}} \int_{\mathbb{R}^n} \left| u_{n/2-1-s, \delta, \delta}(x, w) \right|^2 x^{n-1} \left( \left( \delta + \frac{|x|^2}{2} \right)^s + w^2 \right)^a_m dx dw \]
\[ = (4\delta)^s \left( \frac{\Gamma\left(\frac{n/2+s+1}{2}\right)}{\Gamma\left(\frac{n/2-s+1}{2}\right)} \right)^2 \int_{\mathbb{R}} \int_{\mathbb{R}^n} \left| u_{n/2-1-s, \delta, \delta}(|x|, w) \right|^2 \left( \left( \delta + \frac{|x|^2}{2} \right)^s + w^2 \right) dx dw \]
\[ = (4\delta)^s \left( \frac{\Gamma\left(\frac{n/2+s+1}{2}\right)}{\Gamma\left(\frac{n/2-s+1}{2}\right)} \right)^2 \int_{\mathbb{R}^{n+1}} \frac{|f(x,w)|^2}{\left( \left( \delta + \frac{|x|^2}{2} \right)^s + w^2 \right)} dx dw. \]

Therefore, the constant involved in the inequality is sharp and the equality is achieved by \( f(x, w) = \left( \left( \delta + |x|^2/2 \right)^s + w^2 \right)^{-\frac{n/2-s+1}{2}} \). \( \square \)

In a similar way to generalized sublaplacian, we can also prove Theorem 1.2 and 1.4 and 1.5.

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**REFERENCES**

[1] R. Balhara, *Hardy’s inequality for the fractional powers of the Grushin operator*, Proc. Indian Acad. Sci. (Math. Sci.), 129 (2019), 33.

[2] W. Beckner, *Pitt’s inequality and the fractional Laplacian: sharp error estimates*, Forum Math., 24 (2012), 177–209.

[3] O. Ciaurri, L. Roncal and S. Thangavelu, *Hardy-type inequalities for fractional powers of the Dunkl-Hermite operator*, Proc. Edinb. Math. Soc., 61 (2018), 513–544.
[4] M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of simple Lie group of real rank one, Invent. Math., 96 (1989), 507–549.

[5] G. B. Folland and E. M. Stein, Hardy Spaces on Homogeneous Groups, Mathematical Notes, Vol. 28, Princeton University Press/University of Tokyo Press, Princeton, NJ/Tokyo, 1982.

[6] R. L. Frank, E. H. Lieb and R. Seiringer, Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators, J. Amer. Math. Soc., 21 (2008), 925–950.

[7] I. W. Herbst, Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$, Commun. Math. Phys., 53 (1977), 285–294.

[8] J. Huang, A heat kernel version of Cowling-Price theorem for the Laguerre hypergroup, Proc. Indian Acad. Sci., 120 (2010), 73–81.

[9] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, 7th edition, Elsevier Academic Press, Amsterdam, 2007.

[10] L. Roncal and S. Thangavelu, Hardy’s inequality for fractional powers of the sublaplacian on the Heisenberg group, Adv. Math., 302 (2016), 106–158.

[11] E. M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces (PMS-32), Vol. 32, Princeton University Press, 2016.

[12] K. Stempak, An algebra associated with the generalized sublaplacian, Studia Math., 88 (1988), 245–256.

[13] K. Stempak, Mean summability methods for Laguerre series, Trans. Amer. Math. Soc., 322 (1990), 671–690.

[14] J. Tan, X. Yu, Liouville type theorems for nonlinear elliptic equations on extended Grushin manifolds, J. Diff. Equa., 269 (2020), 523–541.

[15] S. Thangavelu, Lectures on Hermite and Laguerre Expansions, Math. Notes, Vol. 42, Princeton University Press, Princeton, NJ, 1993.

[16] S. Thangavelu, Harmonic Analysis on the Heisenberg Group, Progress in Mathematics, Vol. 159, Birkhäuser, Boston, MA, 1998.

[17] S. Thangavelu, An Introduction to the Uncertainty Principle. Hardy’s Theorem on Lie Groups, Progress in Mathematics, Vol. 217, Birkhäuser, Boston, MA, 2004.

[18] F. G. Tricomi and A. Erdélyi, The asymptotic expansion of a ratio of Gamma functions, Pacific J. Math., 1 (1951), 133–142.

[19] D. Yafaev, Sharp constants in the Hardy–Rellich inequalities, J. Funct. Anal., 168 (1999), 121–144.

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E-mail address: mlsong@nwpu.edu.cn
E-mail address: jinggang.tan@usm.cl