Coefficient Estimates for Certain Subclasses of Bi-Univalent Functions

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Abstract

Let $\Sigma$ denote the class of bi-univalent functions in $D = \{z \in \mathbb{C} : |z| < 1\}$. In this paper, we consider two subclasses of $\Sigma$ defined in the open unit disk $D$ which are denoted by $S_{s,\Sigma}^*(\phi)$ and $C_{s,\Sigma}(\phi)$. Besides, we find upper bounds for the second and third coefficients for functions in these subclasses.

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1 Introduction

Let \( A \) denote the class of functions \( f(z) \) normalized by the following Taylor-Maclaurin series:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in D
\]

which are analytic in the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Further, let \( S \) denote the subclass of functions in \( A \) which are univalent in \( D \). Some of the important and well-investigated subclasses of \( S \) include the class of starlike functions and the class of convex functions which are denoted by \( S^* \) and \( C \) respectively. By definition, we have

\[
S^* = \left\{ f : f \in A \text{ and } \Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \ z \in D \right\}
\]

and

\[
C = \left\{ f : f \in A \text{ and } \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \ z \in D \right\}
\]

It readily follows from definitions (2) and (3) that

\[
f(z) \in C \iff zf'(z) \in S^*.
\]

The Koebe one-quarter theorem [4] states that the image of \( D \) under every function \( f(z) \) from \( S \) contains a disk of radius \( \frac{1}{4} \). Thus every function \( f(z) \in S \) has an inverse \( f^{-1}(f(z)) \) defined by \( f^{-1}(f(z)) = z \ (z \in D) \) and

\[
f \left( f^{-1}(w) \right) = w \left( |w| < r_0(f); r_0(f) \geq \frac{1}{4} \right).
\]

In fact, the inverse function \( f^{-1}(w) \) is given by

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.
\]

A function \( f(z) \in A \) is said to be bi-univalent in \( D \) if both \( f(z) \) and \( f^{-1}(w) \) are univalent in \( D \). Let \( \Sigma \) denote the class of bi-univalent functions given by the Taylor-Maclaurin series expansion (1). Some examples of function in the class \( \Sigma \) are \( z, \log(1-z) \) and \( \left( \frac{1+z}{1-z} \right) \). However, the familiar Koebe function is not a member of \( \Sigma \). Other examples of function in \( S \) such as \( z - z^2 \) and \( \frac{1}{1-z^2} \) are also not members of \( \Sigma \).

Lewin [5] investigated the class \( \Sigma \) and showed that \( |a_2| < 1.51 \). Subsequently, Brannan and Clunie [1] conjectured that \( |a_2| \leq \sqrt{2} \) for \( f \in \Sigma \). Netanyahu [7], on the other hand, showed that \( \max_{f \in \Sigma} |a_2| = \frac{4}{3} \). Brannan and Taha [2] introduced certain subclasses of \( \Sigma \) similar to the familiar subclasses...
of $S$ consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and obtained estimates on the initial coefficients. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \ (n \in N \setminus \{1, 2\}; N := 1, 2, 3, \ldots)$$

is still an open problem.

If the functions $f(z)$ and $g(z)$ are analytic in $D$ then $f(z)$ is said to be subordinate to $g(z)$ written as $f(z) \prec g(z), (z \in D)$ if there exists a Schwarz function $w(z)$, analytic in $D$, with $w(0) = 0, |w(z)| < 1, (z \in D)$ such that $f(z) = g(w(z)), (z \in D)$.

In [6], the authors introduced the class $S^*(\phi)$ of Ma-Minda starlike functions and the class $C(\phi)$ of Ma-Minda convex functions, unifying previously studied classes related to starlike and convex functions. The class $S^*(\phi)$ consists of all the functions $f \in A$ satisfying the subordination $zf'(z)f(z) \prec \phi(z)$ whereas $C(\phi)$ is formed with functions $f \in A$ for which the subordination $1 + zf''(z)f'(z) \prec \phi(z)$ holds. The function $\phi$ is analytic and univalent function with positive real part in $D$ with $\phi(0) = 0, \phi'(0) > 0$ and $\phi$ maps the unit disk $D$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Taylor’s series expansion of such function is of the form

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots \quad (6)$$

where all coefficients are real and $B_1 > 0$.

In [10], Sakaguchi introduced the class $S^*_s$ of starlike functions with respect to symmetric points in $D$, consisting of functions $f \in A$ that satisfy the condition

$$Re\left(\frac{zf''(z)}{f(z) - f(-z)}\right) > 0, \ z \in D$$

and in [3], Das and Singh introduced the class $C_s$ of convex functions with respect to symmetric points in $D$, consisting of functions $f \in A$ that satisfy the condition

$$Re\left(\frac{(zf'(z))'}{(f(z) - f(-z))'}\right) > 0, \ z \in D.$$

Motivated by the earlier works of [10], [3] and [6] and considering functions $f \in \Sigma$, this paper introduce two subclasses of $\Sigma$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses.
2 Preliminary Result and Definitions

In order to derive our main results, we need the following lemma.

**Lemma 2.1.** ([9]) If \( p(z) \in P \) then \( |p_k| \leq 2 \) for each \( k \), where \( P \) is the family of all functions \( p(z) \) analytic in \( D \), for which \( \text{Re}(p(z)) > 0 \), \( p(z) = 1 + p_1z + p_2z^2 + \ldots \) for \( z \in D \).

**Definition 2.1.** A function \( f(z) \in \Sigma \) is said to be in class \( S^*_s,\Sigma(\phi) \) if the following subordinations hold:

\[
\frac{zf'(z)}{f(z) - f(-z)} < \phi(z)
\]

and

\[
\frac{wg'(w)}{g(w) - g(-w)} < \phi(w)
\]

where \( g(w) = f^{-1}(w) \) is given by (5).

**Definition 2.2.** A function \( f(z) \in \Sigma \) is said to be in class \( C^*_s,\Sigma(\phi) \) if the following subordinations hold:

\[
\frac{(zf'(z))'}{(f(z) - f(-z))'} < \phi(z)
\]

and

\[
\frac{(wg'(w))'}{(g(w) - g(-w))'} < \phi(w)
\]

where \( g(w) = f^{-1}(w) \) is given by (5).

3 Main Results

For functions in the class \( S^*_s,\Sigma(\phi) \), the following result is obtained.

**Theorem 3.1.** If \( f \in S^*_s,\Sigma(\phi) \) is given by (1) then

\[
|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{2|B_1^2 + 2(B_1 - B_2)|}}
\]

and

\[
|a_3| \leq \frac{1}{2}B_1 \left( 1 + \frac{1}{2}B_1 \right).
\]
Proof. Let \( f \in S_{\phi}^*(\phi) \) and \( g = f^{-1} \). Then there are analytic functions \( u, v : D \to D \), with \( u(0) = v(0) = 0 \), satisfying

\[
\frac{zf'(z)}{f(z) - f(-z)} = \phi(u(z)) \tag{13}
\]

and

\[
\frac{wg'(w)}{g(w) - g(-w)} = \phi(v(w)). \tag{14}
\]

Define the functions \( r_1 \) and \( r_2 \) by

\[
r_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + ...
\]

and

\[
r_2(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + b_1 z + b_2 z^2 + ...
\]

or equivalently

\[
u(z) = \frac{r_1(z) - 1}{r_1(z) + 1} = \frac{1}{2} \left( c_1 z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + ... \right) \tag{15}
\]

and

\[
v(z) = \frac{r_2(z) - 1}{r_2(z) + 1} = \frac{1}{2} \left( b_1 z + \left( b_2 - \frac{b_1^2}{2} \right) z^2 + ... \right). \tag{16}
\]

Then \( r_1 \) and \( r_2 \) are analytic in \( D \) with \( r_1(0) = r_2(0) = 1 \). Since \( u, v : D \to D \), the functions \( r_1 \) and \( r_2 \) have a positive real part in \( D \) and \( |b_i| \leq 2 \) and \( |c_i| \leq 2 \). In view of (13)-(16), clearly

\[
\frac{zf'(z)}{f(z) - f(-z)} = \phi \left( \frac{r_1(z) - 1}{r_1(z) + 1} \right) \tag{17}
\]

and

\[
\frac{wg'(w)}{g(w) - g(-w)} = \phi \left( \frac{r_2(w) - 1}{r_2(w) + 1} \right). \tag{18}
\]

Using (15) and (16) together with (6), it is evident that

\[
\phi \left( \frac{r_1(z) - 1}{r_1(z) + 1} \right) = 1 + \frac{1}{2} B_1 c_1 z + \left( \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right) z^2 + \ldots \tag{19}
\]

and

\[
\phi \left( \frac{r_2(w) - 1}{r_2(w) + 1} \right) = 1 + \frac{1}{2} B_1 b_1 w + \left( \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2 \right) w^2 + \ldots \tag{20}
\]
Since $f \in \Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion
\[ g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \]

Since
\[ \frac{zf'(z)}{f(z) - f(-z)} = 1 + 2a_2z + 2a_3z^2 + \ldots \]
and
\[ \frac{wg'(w)}{g(w) - g(-w)} = 1 - 2a_2w + 2(2a_2^2 - a_3)w^2 + \ldots \]
it follows from (17)-(20) that
\begin{align*}
2a_2 &= \frac{1}{2} B_1 c_1 \quad (21) \\
2a_3 &= \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \quad (22) \\
-2a_2 &= \frac{1}{2} B_1 b_1 \quad (23)
\end{align*}
and
\[ 2(2a_2^2 - a_3) = \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2 \quad (24) \]
From (21) and (23), it follows that
\[ c_1 = -b_1. \quad (25) \]
Now (21)-(25) yield
\[ a_2^2 = \frac{B_1^3 (b_2 + c_2)}{8 \left( B_1^2 + 2 (B_1 - B_2) \right)} \]
which, in view of the inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives us the estimate on $|a_2|$ as asserted in (11).

By subtracting (24) from (22), further computation using (21) and (25) lead to
\[ a_3 = \frac{B_2^2 (c_1^2 + b_1^2)}{32} + \frac{B_1 (c_2 - b_2)}{8} \]
and this yields the estimate given in (12). The proof of Theorem 3.1 is completed.

The result in Theorem 3.1 is similar to Theorem 2.3 in [8] if $\alpha = 0$.

By using the similar approach as Theorem 3.1, we obtain the following result for functions $f \in C_{s,\Sigma}(\phi)$. 
Theorem 3.2. If \( f \in C_{\Sigma}(\phi) \) is given by (1) then

\[
|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{2} |3B_1^2 + 8(B_1 - B_2)|} \tag{26}
\]

and

\[
|a_3| \leq \frac{1}{2} B_1 \left( \frac{1}{3} + \frac{1}{8} B_1 \right). \tag{27}
\]

Proof. Let \( f \in C_{\Sigma}(\phi) \) and \( g = f^{-1} \). Then there are analytic functions \( u, v : D \), with \( u(0) = v(0) = 0 \), satisfying

\[
\frac{(zf'(z))'}{(f(z) - f(-z))'} = \phi(u(z)) \tag{28}
\]

and

\[
\frac{(wg'(w))'}{(g(w) - g(-w))'} = \phi(v(w)). \tag{29}
\]

Since

\[
\frac{(zf'(z))'}{(f(z) - f(-z))'} = 1 + 4a_2z + 6a_3z^2 + ...
\]

and

\[
\frac{(wg'(w))'}{(g(w) - g(-w))'} = 1 - 4a_2w + 6(2a_2^2 - a_3)w^2 + ...
\]

it follows from (19), (20), (28) and (29) that

\[
4a_2 = \frac{1}{2} B_1 c_1 \tag{30}
\]

\[
6a_3 = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \tag{31}
\]

\[
-4a_2 = \frac{1}{2} B_1 b_1 \tag{32}
\]

and

\[
6(2a_2^2 - a_3) = \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2 \tag{33}
\]

From (30) and (32), it follows that

\[
c_1 = -b_1. \tag{34}
\]

Equations (30)-(34) yield

\[
a_2^2 = \frac{B_1^3 (b_2 + c_2)}{8 (3B_1^2 + 8(B_1 - B_2))}
\]
which, in view of the inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives the estimate on $|a_2|$ as asserted in (26).

Further computation using (30)-(34) lead to

$$a_3 = \frac{B_1^2 (b_1^2 + c_1^2)}{128} + \frac{B_1 (c_2 - b_2)}{24}$$

and this yields the estimate given in (27). The proof of Theorem 3.2 is completed.

The result in Theorem 3.2 is similar to Theorem 2.3 in [8] if $\alpha = 1$.

For functions in the class $S^*_s, \Sigma(\phi)$, we obtained the result on Fekete-Szegö inequalities as follows.

**Theorem 3.3.** Let $f$ given by (1) be in the class $S^*_s, \Sigma(\phi)$ and $\mu \in \mathbb{R}$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{2}, & |\mu - 1| \leq 1 + 2 \left( \frac{B_1 - B_2}{B_1^2} \right) \\ \frac{|1-\mu| B_1^2}{2|B_1^2 + 2(B_1 - B_2)|}, & |\mu - 1| \geq 1 + 2 \left( \frac{B_1 - B_2}{B_1^2} \right) \end{cases}$$

Finally, we give the result on Fekete-Szegö inequalities for functions in the class $C_s, \Sigma(\phi)$.

**Theorem 3.4.** Let $f$ given by (1) be in the class $C_s, \Sigma(\phi)$ and $\mu \in \mathbb{R}$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{6}, & |\mu - 1| \leq \frac{1}{3} \left[ 3 + 8 \left( \frac{B_1 - B_2}{B_1^2} \right) \right] \\ \frac{|1-\mu| B_1^2}{2|3B_1^2 + 8(B_1 - B_2)|}, & |\mu - 1| \geq \frac{1}{3} \left[ 3 + 8 \left( \frac{B_1 - B_2}{B_1^2} \right) \right] \end{cases}$$

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