Effective actions on the squashed three-sphere

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Abstract

The effective actions of a scalar and massless spin-half field are determined as functions of the deformation of a symmetrically squashed three-sphere. The extreme oblate case is particularly examined as pertinent to a high temperature statistical mechanical interpretation that may be relevant for the holographic principle. Interpreting the squashing parameter as a temperature, we find that the effective ‘free energies’ on the three-sphere are mixtures of thermal two-sphere scalars and spinors which, in the case of the spinor on the three-sphere, have the ‘wrong’ thermal periodicities. However the free energies do have the same leading high temperature forms as the standard free energies on the two-sphere. The next few terms in the high-temperature series are also evaluated and briefly compared with the Taub-Bolt-AdS bulk result.

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1. Introduction

The holographic ‘principle’ says, in its barest form, that the information contained in the interior of a space-time domain is encoded in a field theory residing on the boundary. In accordance with this, the gravitational entropies of the Taub-Nut-AdS and Taub-Bolt-AdS space-times have been computed by Hawking, Hunter and Page [1] and by Chamblin et al [2] with the aim of comparing with boundary (conformal) field theories. The boundaries are symmetrically squashed three-spheres, possibly with identifications.

The object of this paper is to detail the calculation of the effective action, i.e. the functional determinants, of scalar and spinor fields as functions of the squashing on the three-sphere. The effective action will be associated with a free energy and thence with an entropy.

The fields will be free so this is only a prelude to a more realistic investigation and the present results may well have only a passing relevance to the holographic principle. Such a stopgap calculation has been suggested by Hawking et al in [1]. Nevertheless, the evaluation of the determinants is of interest in itself. Our results here will be confined to the determinants. Elaboration of the statistical mechanical interpretation is left for a later paper although some preliminary comparison with the bulk Taub-Bolt-AdS result is made.

2. The basic situation

The squashed 3-sphere appears as the spatial section of the frozen mixmaster universe and quantum field theory on this space-time was discussed by Hu, Fulling and Parker [3], Hu [4], Shen, Hu and O’Connor [5] and Critchley and Dowker [6]. It has been discussed in a Kaluza-Klein setting by Okada [7] and by Shen and Sobczyk [8].

A discussion of the vacuum energy of a massless spin-half field on the squashed 3-sphere has been given in [9] and for this reason we will reexamine this case before looking at the scalar field.

In reference [9], in order that rules of standard angular momentum theory should apply unmodified it was necessary to choose the radius of the unsquashed sphere, $S^3$, to be $a = 2$. and the (standard) metric in Euler angles is

$$ds^2 = (d\theta^2 + \sin^2 \theta d\phi^2) + l_3^2(d\psi + \cos \theta d\phi)^2$$

(1)
showing the (symmetrically) squashed $S^3$ as a twisted product, $S^2 \times S^1$. The circle has radius $2l_3$, and hence a circumference of $4\pi l_3$. If, illustratively, this periodicity is translated into a temperature, we find $\beta = 1/T = 4\pi l_3$. Another way of saying this is to note that the range of $\psi$ is $0 \rightarrow 4\pi$ and then to interpret $l_3 \psi$ as a Euclidean time. We will be particularly interested in the extreme oblate case, $l_3 \rightarrow 0$, when the metric reduces to that on the unit two-sphere. The relation between the $\zeta$–functions will constitute a type of twisted Selberg-Chowla formula.

In the notation of Hawking, Hunter and Page [1], $l_3^2 = E$. It is possible to identify points on the ‘$\psi$-circle’, and this is the more interesting situation. Nevertheless it will not be considered here. The work [2] also treats only this simplest case. In the notation of [2], $l_3^2 = 4n^2/l^2$. We denote the squashed sphere by $\tilde{S}^3$.

3. Spinor $\zeta$–functions on $\tilde{S}^3$.

We ignore all restrictions on the boundary spinor theory arising from its embedding in some bulk theory. For odd-dimensional spaces the relevant operator is thus the Pauli one which, on the squashed three-sphere, is $\Pi = -i\sigma^i \nabla_i$ in terms of (covariant) Pauli matrices and the spinor covariant derivatives, which we will not exhibit here. The dimension of spinor space on $\tilde{S}^3$ is 2, which is the same as that on $S^2$.

The eigenvalues of $\Pi$ are determined to be (Hitchin [10], Gibbons [11], and [9])

$$\omega_{\pm} = (2l_3)^{-1} \left( \frac{1}{2} l_3^2 \pm \left( \frac{n^2 + 4(l_3^2 - 1)q(n - q)}{2} \right)^{1/2} \right)$$

(2)

where the integers $n$ and $q$ emerge from the angular momentum quantum numbers labelling the unperturbed states (before the necessary secular diagonalisation). Thus $n = 2L + 1$ and $q = n/2 - M$ with $L$ an orbital label and $M$ the projection of the total angular momentum.

If $l_3 < 4$, $\omega_+$ is positive and $\omega_-$ negative, and for ease we will assume that this is so. The corresponding traced $\zeta$–functions are constructed separately,

$$\zeta_+(s) = \sum_{n=1}^{\infty} \sum_{q=0}^{n} \frac{n}{\omega_+^s}$$

and

$$\zeta_-(s) = \sum_{n=2}^{\infty} \sum_{q=1}^{n-1} \frac{n}{(-\omega_-)^s}$$
which exhibit the quantum number range restrictions with \( n \) the remaining degeneracy of the modes. We note that, as a function of \( l_3^2 \), nothing peculiar happens to the eigenvalues as \( l_3 \) passes through 1 and also that there is a square root branch point at \( l_3^2 = 0 \).

The \( \zeta \)-function for the squared operator \( \Pi^2 \) is

\[
\text{Tr}_3 \zeta_3(s) = \zeta_+(2s) + \zeta_-(2s).
\]

The awkwardness of the eigenvalues, (2), restricted the analysis in [9] to a power series expansion in the squashing parameter \( l_3 \). For the time being we shall continue with this expansion which, in any case, is adequate for the high temperature limit. The computation then reduces to that of the function, \( f(s) \), defined, for \( \text{Re } s > 3/2 \), by

\[
f(s) = \sum_{n=2}^{\infty} \sum_{q=1}^{n-1} \frac{n}{(n^2 + 4\delta^2 q(n-q))^s}, \quad \text{where } \delta^2 = l_3^2 - 1. \tag{3}
\]

In terms of \( f(s) \), \( \text{Tr}_3 \zeta_3(s) \) reads,

\[
\text{Tr}_3 \zeta_3(s) = 2(2l_3)^s \left( \zeta(2s-1, l_3^2) - w \zeta(2s, l_3^2) + f(s) + l_3^4 s(2s+1) f(s+1) + O(l_3^8) \right). \tag{4}
\]

This expansion is valid, numerically for \( l_3 < \sqrt{2} \). One can rejig the series to allow for larger values, but we will not bother. With this approach, it is not possible to discuss the \( l_3 \to \infty \) limit.

A preliminary aim is to calculate the (Euclidean) effective action, which we define to be \( \text{Tr}_3 \zeta_3(0)/2 \equiv W_{sp} \).

Noting that there is no conformal anomaly, \( \text{Tr}_3 \zeta_3(0) = 0 \), the complete formal series is

\[
W_{sp} = 2\zeta'(-1, l_3^2) - 2l_3^2 \zeta'(0, l_3^2) + f'(0) + l_3^4(2P + R) + \frac{1}{4} l_3^8 f(2) + O(l_3^{12}) \tag{5}
\]

where, since \( f(s) \) has a pole at \( s = 1 \), we have to write

\[
f(s) = \frac{P}{s-1} + R + O(s-1).
\]

It is expected that the interesting behaviour as \( l_3 \) becomes small is contained in \( f'(0) \) and we must determine the analytic continuation of this quantity. Initially we are looking for terms which diverge as \( l_3 \to 0 \).
In [9] we employed the standard Plana summation formula and, for convenience, will proceed in the same way although there are other approaches.

Extending again the $q$ sum to 0 to $n$ in (3), an application of the Plana summation formula yields, for deformations in the prolate direction, [9],

$$f(s) = -\zeta_R(2s - 1) + \zeta_R(2s - 2) \int_0^1 \frac{dy}{(1 + 4\delta^2 y(1 - y))^s}$$

$$- 2i \int_0^\infty \frac{dt}{\exp(2\pi t) - 1} \left\{ \sum_{n=1}^{\infty} \frac{n}{(n^2 + 4\delta^2(t^2 - itn))^s} - (t \to -t) \right\}$$

which is not a complete continuation as the $n$ summation has yet to be dealt with. The pole at $s = 1$ is apparent giving $P = -1/2$.

For deformations in the oblate direction ($0 \leq l_3 \leq 1$), which we need for the high temperature series, singularities of the summand encroach into the relevant part of the complex $q$ plane, i.e. $0 \leq \text{Re} \ q \leq n$. Concentrating now on this oblate case, ‘extra’ branch points occur at

$$q = \frac{n}{2} \pm i \frac{n}{2} \sqrt{1 - l_3^2} \equiv \frac{n}{2} \left(1 \pm i \tan \theta\right)$$

and we give the details of the resulting contributions. Following the usual procedure (as in Lindelöf [12]) we find (setting $b^2 = -\delta^2$ and omitting the $n$-summation for the moment) the additional pieces,

$$\int_U \frac{dz}{\exp(-2\pi i z)} - 1 \left(\frac{n}{n^2 - 4b^2 z(n - z)}\right)^s + \int_L \frac{dz}{\exp(2\pi i z)} - 1 \left(\frac{n}{n^2 - 4b^2 z(n - z)}\right)^s$$

where the infinite $U$ contour runs anticlockwise around the cut from $n (1+i \tan \theta)/2$ to $n/2 + i\infty$, while $L$ runs clockwise around the corresponding cut in the lower half plane. Symmetry means that the two contributions are equal and we get, after a change of variable, $z = (1 + i\zeta \tan \theta) n/2$, the extra contribution to $f(s)$,

$$\frac{i \tan \theta}{l_3^2} \sum_{n=1}^{\infty} \frac{1}{n^{2s-2}} \int_C \frac{d\zeta}{((-1)^n \exp(n\pi \zeta \tan \theta) - 1)(1 - \zeta^2)^s}$$

where $C$ is an anticlockwise contour running around the real $\zeta$-axis, cut from 1 to $\infty$. For $s$ a non-negative integer, this expression vanishes, while, if $s$ is a negative integer, it can be evaluated using residues, providing a useful numerical check.
Because of the exponential factor in the denominator, expression (7) converges for all $s$. It can be taken as the analytic continuation and should be added to (6), for deformations in the oblate direction, to give

\[
f(s) = -\zeta_R(2s - 1) + \zeta_R(2s - 2) \int_0^1 \frac{dy}{(1 - 4b^2y(1 - y))^{3/2}} - 2i \int_0^\infty \frac{dt}{\exp(2\pi t) - 1} \left\{ \sum_{n=1}^\infty \frac{n}{(n^2 - 4b^2(t^2 - itn))^{3/2}} - (t \to -t) \right\} + \left(8\right)
\]

\[
i \tan \theta \frac{l^2}{l_3^2} \sum_{n=1}^\infty \frac{1}{n^{2s-2}} \int_C \frac{d\zeta}{((-1)^n \exp(n\pi \zeta \tan \theta) - 1)} \frac{1}{(\zeta^2 - 1)^s},
\]

(9)

We shall denote the last two terms on the right-hand side of (8) by $f_2(s)$ and $f_3(s)$ respectively and consider $f_3$ first.

If $s$ is such that the integral converges at $\zeta = 1$, (7) can be converted to a real integral in the usual way by combining the upper and lower pieces of $C$. One gets, choosing appropriate phases,

\[
f_3(s) = 2 \sin(\pi s) \tan \theta \frac{l^2}{l_3^2} \sum_{n=1}^\infty \frac{1}{n^{2s-2}} \int_1^\infty \frac{d\zeta}{((-1)^n \exp(n\pi \zeta \tan \theta) - 1)} \frac{1}{(\zeta^2 - 1)^s},
\]

(9)

In particular we can differentiate with respect to $s$ to get the contribution to $f'(0)$,

\[
f_3'(0) = 2\pi \tan \theta \sum_{n=1}^\infty n^2 \int_1^\infty \frac{d\zeta}{((-1)^n \exp(n\pi \zeta \tan \theta) - 1)}. \]

(10)

This is a simple example of a Lerch function. Expansion of the integrand allows the integration to be done to yield, from (5), the contribution to $\frac{1}{2} \text{Tr}_3 \zeta_3'(0)$,

\[
W_{3p}^{(3)} = 2 \sum_{m=1}^\infty \frac{1}{m} \sum_{n=1}^\infty (-1)^{mn} ne^{-\pi mn \tan \theta}.
\]

(11)

The sum over $n$ produces

\[
W_{3p}^{(3)}(\beta') = f_3'(0) = \sum_{k=1}^\infty \frac{1}{4k \sinh^2(k\beta'/4)} - \sum_{k=0}^\infty \frac{1}{2(2k + 1) \cosh^2(2k + 1)\beta'/8}, \]

(12)

where, for convenience, we have denoted $4\pi \tan \theta$ by $\beta'$. This expression is useful numerically.

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For small $l_3$, the exponent in (11) can be written as $-mn\beta/4$ with $\beta$ defined by $\beta = 4\pi l_3$ and so, dividing by $\beta$ for normalisation purposes, we define the corresponding spinor ‘free energy’,

$$\Phi_{sp}^{(3)}(\beta) = -\frac{1}{\beta} W_{sp}^{(3)}(\beta')$$

We will postpone discussion of formula (13) until after the corresponding scalar case has been considered.

However before proceeding to this, it is necessary to consider the other terms in (6) and (8) which are needed for the complete determination of the effective action as a function of the squashing.

In [9], a further application of the Plana formula to the $n$-sum in (6) revealed a series of poles in $f(s)$ at $s = 3/2 - m$, $m = 1, 2, \ldots$, with residues

$$r_m = (-1)^{m+1} \frac{2^{2m-2}\Gamma(m-1/2)}{m!\Gamma(1/2)} (l_3^2 - 1)^m l_3^{2m-2} B_{2m},$$

where $B_{2m}$ is a Bernoulli number (using the definition in Bateman [13] e.g. In [9] we used Lindelöf’s signs [12]).

Here we will employ the Watson-Sommerfeld technique which is essentially equivalent to the Plana one. It has been used by Shen and Sobczyk [8] in the present context.

Completing the square, we have,

$$n^2 + 4\delta^2(t^2 - itn) = (n + iB)^2 + A^2$$

where

$$A^2 = 4l_3^2(l_3^2 - 1)t^2 = \overline{A}^2t^2, \quad B = 2(1 - l_3^2)t = \overline{B}t \quad \text{with} \quad \overline{A}^2 - \overline{B}^2 = 2\overline{B}.$$

The signs etc. are appropriate for the prolate case. For oblate $l_3 < 1$, we set $\overline{A}^2 = -\overline{C}^2$.

We leave the $t$-integration until last and just consider, for the oblate case first,

$$\sum_{n=1}^{\infty} \frac{n}{((n + iB)^2 - C^2)^s} = \frac{1}{2i} \int_L dz \frac{z}{((z + iB)^2 - C^2)^s} \cot \pi z$$

$$= \frac{1}{2i} \int_L d\zeta \frac{\zeta - iB}{((\zeta^2 - C^2)^s} \cot (\pi(\zeta - iB))$$

(16)
where \( L \) is the anticlockwise contour surrounding the poles of the cotangent at \( \zeta = n + iB, \ n = 1, 2, \ldots \). The \( \zeta \)-plane has branch points at \(-C\) and \(+C\). For symmetry’s sake the associated cuts are arranged to run down the imaginary axis to (almost) the origin, and thence along the real axis to \( \pm C \). To make a choice, the \(+C\)-cut comes in from \(+i\infty\) and the \(-C\) one from \(-i\infty\). The contour \( L \) is deformed to run down, just to the right of the imaginary axis, then around the right-hand side of the \( C \)-cut and finally to skirt the imaginary axis down to \(-i\infty\). We assume that \( s \) is such as to ensure convergence.

Taking the phases into account, the integral along the imaginary \( \zeta \)-axis is

\[
-\frac{1}{2} \int_0^\infty dy \frac{y - B}{(y^2 + C^2)^s} e^{-i\pi s} \coth \pi(y - B) - \frac{1}{2} \int_0^\infty dy \frac{y + B}{(y^2 + C^2)^s} e^{i\pi s} \coth \pi(y + B)
\]

and remembering that, according to (8), we need 4 times the imaginary part of this expression we find,

\[
2 \sin \pi s \left[ \int_0^\infty dy \frac{y - B}{(y^2 + C^2)^s} \coth \pi(y - B) - \int_0^\infty dy \frac{y + B}{(y^2 + C^2)^s} \coth \pi(y + B) \right]
\]

Now we make the split \( \coth \pi x = 1 + 2/(\exp 2\pi x - 1) \). The ‘1’ part of this allows the integral to be done,

\[
2 \sin \pi s \left[ -BC^{1-2s} \frac{\sqrt{\pi} \Gamma(s - 1/2)}{\Gamma(s)} - 2 \int_0^\infty dy \frac{y + B}{(y^2 + C^2)^s} \left( \exp(2\pi(y + B)) - 1 \right) \right. \\
\left. + 2 \int_0^\infty dy \frac{y - B}{(y^2 + C^2)^s} \left( \exp(2\pi(y - B)) - 1 \right) \right]
\]

and it is straightforward to check that the first term in (17) reproduces the poles in \( f(s) \) at \( s = 3/2 - m \), with residues (14), after putting back the \( t \)-integration.

The integral over the right-hand part of the \( C \)-cut can likewise be reduced to the form, if it converges,

\[
4 \sin \pi s \int_0^C dx \frac{1}{(C^2 - x^2)^s} \frac{x \sinh(2\pi B) - B \sin(2\pi x)}{\cosh(2\pi B) - \cos(2\pi x)}.
\]

From these expressions we can derive \( f'_2(0) \),

\[
f'_2(0) = \int_0^\infty \frac{4\pi t^2 dt}{\exp(2\pi t) - 1} \left[ \int_0^B dy \coth(\pi yt) - \int_0^C dx \frac{x \sinh(2\pi Bt) - B \sin(2\pi xt)}{\cosh(2\pi Bt) - \cos(2\pi xt)} \right]
\]
We note that, for \( s \) a positive integer, the integral in (18) does not converge so that this method is not convenient for these values.

The prolate case can be treated in the same way and gives

\[
f'_2(0) = \frac{AB}{\pi^2} \zeta(3) - 4\pi \int_0^\infty \frac{t^2 \, dt}{\exp(2\pi t) - 1} \int_{A+B}^{A-B} \frac{y \, dy}{\exp(2\pi yt) - 1}.
\]

In the near-round limit, \( l_3 \approx 1 \), \( f'_2(0) \) goes like \((1 - l^2_3)/3\) in both cases.

This approximation is most easily developed from the perturbation expansion of the original sum in (6). Since this provides a useful check, we give the first three terms obtained in this way. Defining \( l_3 = \cosh \gamma \) in the prolate and \( l_3 = \cos \phi \) in the oblate case, we find

\[
f'_2(0) \approx -\frac{1}{3} \gamma^2 - \left(\frac{1}{9} + \frac{2\pi^2}{45}\right) \gamma^4 - \left(\frac{2}{135} + \frac{4\pi^2}{45} - \frac{16\pi^4}{2835}\right) \gamma^6
\]

and the corresponding oblate form obtained by setting \( \gamma \rightarrow i\phi \). Numerically, for \( \gamma = 0.5 \), (21) gives \(-0.123\), in nice agreement with numerical integration applied to (20).

To order \( l^2_3 \), the complete effective action from (5) is

\[
W_{sp} = -\frac{1}{2\pi^2} \zeta_R(3) + f'_2(0) + f'_3(0) + O(l^4_3)
\]

where \( f'_2(0) \) is given by (20) in the prolate and by (19) in the oblate case. In the oblate case, \( f'_3(0) \) is given by (11), otherwise it is zero.

\( W_{sp} \), as given by (22), is the quantity we concentrate on. While the expressions in this section are numerically adequate, more suitable forms are developed in section 8.

For future reference, we note that when \( A \) and \( C \) are zero, but \( B \neq 0 \), both (19) and (20) give the same result, namely just the first part of (19). One must, however, be cautious with the \( C \rightarrow 0 \) limit in general, since the integrals in (17) may not converge. If one did wish to pursue this further, then a different contour deformation would be called for. Furthermore, it is possible to make a perturbation expansion in \( C \), or in \( A \), of the original summation in (6) or (8) the coefficients of which depend on the (imaginary part of) the \( \zeta \)-function

\[
\sum_{n=0}^\infty \frac{n}{(n+iB)^s}
\]

similar to the Hurwitz \( \zeta \)-function with imaginary parameter. The general nature of this expansion does not distinguish between prolate and oblate deformations.
4. The scalar field

We now turn to the massless scalar field. In the context of the squashed three-sphere, this has earlier been considered by Critchley and Dowker [6], Shen and Sobczyk [8], and more recently by Shyukov and Vassilevich [14] who were concerned with the heat-kernel expansion for the Laplacian. We will use the scalar operator, $-\Delta + 1/4$ for which the eigenvalues are

$$\lambda = \frac{1}{4l^2_3} \left( n^2 + 4\delta^2 \left( q + \frac{1}{2} \right) (n - q - \frac{1}{2}) \right)$$  \hspace{1cm} (23)

with degeneracy, $n$. The $n$ label runs from 1 to $\infty$, and $q$ from 0 to $n - 1$.

We recall that the scalar curvature on $\tilde{S}^3$ is

$$R = 2 - \frac{l^2_3}{2}$$

which correctly gives $3/2$ when $l_3 = 1$ for the round three-sphere (of radius 2) and 2 for the two-sphere at $l_3 = 0$. Our choice of operator restricts to $-\Delta + R/6$ in the round case, and corresponds to a conformally invariant wave equation in four dimensions. It also reduces to $-\Delta + R/8$ on the unit two-sphere when $l_3 = 0$.

Other possible operators would be the minimal one, $-\Delta$, and the one always conformally invariant in three dimensions $-\Delta + R/8$. It is not difficult to accommodate these by an expansion as in the spinor case, but since we are interested, for the moment, mostly in the leading high-temperature terms, the choice was dictated by the simplicity of the eigenvalues.

The $\zeta$-function, $\sum \lambda^{-s} \equiv \zeta_{sc}(s)$, is

$$\zeta_{sc}(s) = (2l^2_3)^{2s} \sum_{n=1}^{\infty} \sum_{q=0}^{n-1} \frac{n}{\left( n^2 + 4\delta^2 \left( q + \frac{1}{2} \right) (n - q - \frac{1}{2}) \right)^s}$$  \hspace{1cm} (24)

and the effective action is $W_{sc} = -\zeta'_{sc}(0)/2$.

The Plana summation formula is now applied, in the manner of [9] and discussed also in [8]. We will consider the oblate case. Again extra singularities appear in the relevant band of the complex $q$-plane. Replacing $q$ by $z$ and making the transformation of variable

$$z = \frac{n - 1}{2} + i \frac{n}{2} \zeta \tan \theta$$

the denominator becomes

$$(l^2_3n^2(1 - \zeta^2))^s$$
showing the branch points at $\zeta = \pm 1$, as before.

In this scalar case we use the alternative choice of summation contour, the band in the $z$ plane being defined by $-1/2 \leq \text{Re} \ z \leq (n - 1/2)$ (rather than 0 to $n - 1$). Then the full Plana formula, applied just to the $q$ summation, is

$$
\zeta_{sc}(s) = 2^{2s} \tan \theta \zeta_R(2s - 2) \int_0^{\cot \theta} \frac{dy}{(1 + y^2)^s} + 
2i(2l_3)^{2s} \int_0^\infty \frac{dt}{\exp(2\pi t) + 1} \sum_{n=1}^\infty \frac{n}{(n^2 + 4\delta^2(t^2 - itn))^s} (t \to -t)$$

$$
i2^{2s} \tan \theta \sum_{n=1}^\infty \frac{1}{n^{2s-2}} \int_C \frac{d\zeta}{((-1)^n \exp(n\pi \zeta \tan \theta) + 1)} \frac{1}{(1 - \zeta^2)^s},
$$

(25)

Look at the last term first and again combine the upper and lower parts of the clockwise loop $C$ to give

$$
-2^{2s+1} \sin(\pi s) \tan \theta \sum_{n=1}^\infty \frac{1}{n^{2s-2}} \int_1^\infty \frac{d\zeta}{((-1)^n \exp(n\pi \zeta \tan \theta) + 1)} \frac{1}{(1 - \zeta^2)^s},
$$

(26)

when the integration converges at $\zeta = 1$.

The contribution to $-\zeta_{sc}'(0)/2$ from this term is then

$$
W_{sc}^{(3)} = \pi \tan \theta \sum_{n=1}^\infty n^2 \int_1^\infty \frac{d\zeta}{(-1)^n \exp(n\pi \zeta \tan \theta) + 1}
$$

(27)

which can be written as

$$
W_{sc}^{(3)}(\beta') = \sum_{m=1}^\infty \frac{1}{m} \sum_{n=1}^\infty (-1)^{m(n+1)} n e^{-nm\beta'/4},
$$

(28)

and sums to

$$
W_{sc}^{(3)}(\beta') = \sum_{k=1}^\infty \frac{1}{8k \sinh^2(k\beta'/4)} + \sum_{k=0}^\infty \frac{1}{4(2k+1) \cosh^2((2k+1)\beta'/8)}.
$$

(29)

The corresponding contribution to the free energy $\Phi_{sc}^{(3)}(\beta) = \frac{1}{\beta} \zeta_{sc}'(0)$ is

$$
\Phi_{sc}^{(3)}(\beta) = -\frac{1}{\beta} W_{sc}^{(3)}(\beta')
$$

(30)

When $l_3$ is zero, the metric (1) reduces to that of the unit two-sphere. The scalar eigenvalues of $-\Delta + 1/4$ on the unit two-sphere are $n^2/4$ with odd $n$. The
degeneracy is $n$ and so the standard finite temperature correction to the bosonic free energy reads

$$F'_{sc}(\beta) = -\frac{1}{\beta} \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1,3,\ldots}^{\infty} n e^{-mn\beta/2}.$$  \hfill (31)

The eigenvalues of the massless Dirac operator the unit $S^2$ are $\pm n$, $n = 1, \ldots, \infty$ with degeneracy $2n$. So, (including positive and negative modes)

$$F'_{sp}(\beta) = \frac{2}{\beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{n=2,4,\ldots}^{\infty} n e^{-mn\beta/2}.$$  \hfill (32)

The effective action densities on the squashed three-sphere, in the highly oblate direction seem to possess both boson and fermion characteristics.

Split the sum in (28) into even and odd $n$,

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1,3,\ldots}^{\infty} n e^{-mn\beta'/4} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{n=2,4,\ldots}^{\infty} n e^{-mn\beta'/4}$$  \hfill (33)

so we can write, at least formally,

$$\Phi^{(3)}_{sc}(\beta) = \frac{\beta'}{\beta} \left( \frac{1}{2} F'_{sc}(\beta'/2) - \frac{1}{4} F'_{sp}(\beta'/2) \right).$$  \hfill (34)

Hence the small $\beta$ behaviour can be determined from known results on the two-sphere.

The standard expressions for the Weyl terms, for thermal behaviour on a two-manifold, here the two-sphere, are

$$F'_{sc}(\beta) \sim -\frac{\zeta_R(3)}{2\pi \beta^3} C^{sc}_0 = -\frac{2\zeta_R(3)}{\beta^3}$$  \hfill (35)

and

$$F'_{sp}(\beta) \sim -\frac{3\zeta_R(3)}{8\pi \beta^3} C^{sp}_0 = -\frac{3\zeta_R(3)}{\beta^3}$$  \hfill (36)

where $C^{sc}_0 = |\mathcal{M}|$ and $C^{sp}_0 = |\mathcal{M}| \text{Tr} 1$. The coefficients of the $C_0$’s and indeed all formal expressions, satisfy the functional relation

$$Q'_{sp}(\beta) = Q'_{sc}(\beta) - 2Q'_{sc}(2\beta).$$  \hfill (37)

From (34) the leading terms are

$$\Phi^{(3)}_{sc}(\beta) \sim -\frac{2\zeta_R(3)}{\beta^3} + \frac{1}{16\pi^2} \frac{\zeta_R(3)}{\beta} + \frac{1}{12\beta} \ln \beta + \frac{1}{\beta} \zeta_R(2)\zeta_R(1).$$  \hfill (38)
This result can be obtained more directly from the summed form (29), which differs from the spinor form, (12), just in the sign of the second term, and a factor of two. Moreover, this term has a logarithmic asymptotic dependence on $\beta$ as $\beta \to 0$, which is expected since the $C_1$ heat-kernel coefficient on the two-sphere is non-zero. The corresponding sub-leading terms can be seen in the last two terms in (38). The scaling of the $\ln \beta$ is provided by the size of the two-sphere (for which there is a conformal anomaly). The second term on the right-hand side of (38) is a simple consequence of the difference between ‘effective’ inverse temperature, $\beta'$, and the ‘true’ one, $\beta$.

For the record, we give the summed forms of the scalar and spinor free energies on the two-sphere,

\[ F'_{sc}(\beta) = -\frac{1}{2\beta} \sum_{m=1}^{\infty} \frac{\cosh(m\beta/2)}{m \sinh^2(m\beta/2)} \]  

\[ F'_{sp}(\beta) = \frac{1}{\beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m \sinh^2(m\beta/2)}. \]  

We must now turn to the $n$-summation in (25). The calculation is much the same as for the spinor field.

The prolate form obtained by the Watson-Sommerfeld method is,

\[ W^{(2)}_{sc} = \frac{3\mathcal{A} \mathcal{B}}{8\pi^2} \zeta_R(3) - 2\pi \int_0^\infty \frac{t^2 dt}{\exp(2\pi t) + 1} \int_{\mathcal{A} - \mathcal{B}}^{\mathcal{A} + \mathcal{B}} \frac{y\, dy}{\exp(2\pi y t) - 1}. \]  

This is handy if we want the extreme prolate (large $l_3$) limit. From the first term, the leading behaviour of the entire scalar effective action is

\[ W_{sc} \approx W^{(2)}_{sc} \sim -\frac{3l_3^4}{2\pi^2} \zeta_R(3), \quad l_3 \to \infty \]  

showing a sort of duality.

The oblate expression is found to be,

\[ W^{(2)}_{sc} = \int_0^\infty \frac{2\pi t^2 dt}{\exp(2\pi t) + 1} \left[ \int_0^\mathcal{B} dy \coth(\pi y t) - \int_0^\mathcal{C} dx \frac{x \sinh(2\pi B t) - B \sin(2\pi x t)}{\cosh(2\pi B t) - \cos(2\pi x t)} \right], \]

and the total scalar free energy, from (25), is

\[ W_{sc} = \frac{\zeta_R(3)}{4\pi^2} + W^{(2)}_{sc} + W^{(3)}_{sc}. \]
For prolate deformations, $W_{sc}^{(2)}$ is given by (41) and for oblate by (43). There is no $W_{sc}^{(3)}$ for the prolate case and it equals (29) in the oblate.

The corresponding ‘free energy’ in both cases is defined to be $\Phi_{sc}(\beta) = -W_{sc}/\beta$ and from (44) we see that $\Phi_{sc}(\beta) \sim \Phi_{sc}^{(3)}(\beta)$ so that from (38) this quantity has the same high temperature behaviour as on $S^2$. This statement is expanded in section 9.

Again we note that, in the $A = 0$ and $C = 0$ limits, both (41) and (43) reduce to just the first term of (43).

5. Spinors revisited.

We now return to (13) for the spin-half (Pauli) free energy on the squashed three-sphere and again decompose the sum into even and odd pieces,

$$\Phi_{sp}^{(3)}(\beta) = -\frac{2}{\beta} \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=2,4,\ldots}^{\infty} n e^{-m\beta' / 4} - \frac{2}{\beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{n=1,3,\ldots}^{\infty} n e^{-m\beta' / 4}$$

(45)

which presents a sort of twisted situation with thermal fermions being associated with scalar modes on $S^2$ and vice versa.

Let us rewrite this expression as

$$\Phi_{sp}^{(3)}(\beta) = \frac{\beta'}{\beta} \left( \frac{1}{2} \tilde{F}_{sp}(\beta'/2) - \tilde{F}_{sc}(\beta'/2) \right)$$

(46)

to be compared with (34).

To obtain the leading behaviour we can make use of the fact that the functional relation (37) takes the twisted form,

$$\tilde{Q}_{sc}^{'}(\beta) = \tilde{Q}_{sp}^{'}(\beta) - 2 \tilde{Q}_{sp}^{'}(2\beta).$$

(47)

The coefficients of the $C_0$’s in (35) and (36) will then be switched i.e.

$$\tilde{F}_{sp}^{'}(\beta) \sim -\frac{\zeta R(3)}{2\pi \beta^3} C_{sp}^{sc} = -\frac{4 \zeta R(3)}{\beta^3}$$

(48)

and

$$\tilde{F}_{sc}^{'}(\beta) \sim -\frac{3 \zeta R(3)}{8\pi \beta^3} C_{0}^{sc} = -\frac{3 \zeta R(3)}{2 \beta^3}$$

(49)

then from (46)

$$\Phi_{sp}^{(3)}(\beta) \sim -\frac{4 \zeta R(3)}{\beta^3} + \frac{1}{8\pi^2} \frac{\zeta R(3)}{\beta} - \frac{1}{6\beta} \ln \beta - \frac{2}{\beta} \zeta R(-1).$$

(50)

Again, this follows also from the summed form (12).

Once more we see that the leading behaviour on the extreme oblate three-sphere corresponds exactly to the high temperature form on the two-sphere, but not so the sub-leading terms.
6. Poles, residues and coefficients.

In [9] we determined the analytic structure of the spinor zeta function, and thence the coefficients in the heat-kernel expansion. The same procedure can be pursued of course in the scalar case and just needs the complete pole structure of $\zeta_{sc}(s)$. The intermediate Plana form (25) shows a pole at $s = 3/2$ coming from the Riemann $\zeta$–function. This is the ‘volume’ pole. It has a residue $4l_3$ which agrees with the general value,

$$C_m = \frac{(4\pi)^{d/2} \Gamma(d/2 - m)}{(4\pi)^{d/2} \Gamma(d/2 - m)}$$

for the residue at $s = d/2 - m$ ($m = 0, 1, \ldots, \neq d/2$) in a $d$-dimensional manifold, $\mathcal{M}$, if we recall the Weyl result, $C_0 = |\mathcal{M}|$. For our squashed three-sphere, $|\tilde{S}^3| = 16\pi^2 l_3$.

To obtain all the poles, we can proceed as described in section 2 the only differences being the change in sign and the bosonic factor. In the scalar case $\zeta_{sc}(s)$ has poles at $s = 3/2 - m$, $m = 1, 2, \ldots$ with residues

$$R_m = (2l_3)^{3-2m} (2^{1-2m} - 1)r_m$$

in terms of the spinor residues (14). Hence the heat-kernel coefficients are

$$C_m = |\tilde{S}^3|(2^{1-2m} - 1)(l_3^2 - 1)^m \frac{B_{2m}}{m!}.$$  \hfill (51)

Removing the volume factor, and taking the $l_3 \to 0$ limit yields Mullholland’s results for the unit two-sphere scalar coefficients,

$$\lim_{l_3 \to 0} \frac{C_m}{|\tilde{S}^3|} = (-1)^m (2^{1-2m} - 1) \frac{B_{2m}}{m!}.$$  

The spin-half case is similarly treated.

Shtykov and Vassilevich [14] have computed the coefficients on the deformed unit three-sphere for the minimal operator, $-\Delta$. To get these coefficients one can simply multiply the above expansion by the expansion of $\exp(t/4)$ in the usual way. For example

$$C^\text{minimal}_1 = |\tilde{S}^3| \frac{4 - l_3^2}{12} = \frac{|\tilde{S}^3|}{6} \frac{R}{6}$$

as required.
8. An alternative summation and a puzzling coincidence.

It is always a good idea to pursue different paths to any required quantity. Apart from inspiring confidence, or otherwise, in the final answer, it often reveals unexpected subtleties.

Instead of applying the Watson-Sommerfeld method to the \( n \)-summations in (6), (8) and (25), we can proceed as Okada does, [7], and employ the method first used by Minakshisundaram when discussing the \( \zeta \)-functions on spheres and later, more extensively, by Candelas and Weinberg [15] for the same purpose. The idea is to rewrite the summand using the Laplace transform

\[
\frac{1}{((n+iB)^2 + A^2)^s} = \frac{\sqrt{\pi} (2A)^{1/2-s}}{\Gamma(s)} \int_0^\infty dz \frac{e^{-(n+iB)z} z^{s-1}}{\sinh((z/2)^2)} J_{s-1/2}(Az) \quad (52)
\]

in order to effect the \( n \)-summation.

For oblate \( l_3 < 1 \), \( A \) becomes imaginary (\( \mathbf{A}^2 = -\mathbf{C}^2 \)) and the Bessel function becomes a modified one, \( I_{s-1/2} \). However there is an obstruction to the immediate application of a formula such as (52) in that, for \( t > 1/C \), \( C \) will be larger than at least one \( n \), violating the conditions of the identity. Our attitude in this section is to take the prolate expression as far as possible, and then to continue in \( l_3^2 \). The form of the eigenvalues shows that the singularities of the \( \zeta \)-function, as a function of \( l_2^3 \), all lie on the negative real axis and therefore should not prevent the continuation through \( l_3^2 = 1 \).

Inserting (52) into the middle term of (6) allows the \( n \)-sum to be done and we find

\[
-\frac{\sqrt{\pi} 2^{1/2-s}}{\Gamma(s)} \int_0^\infty dz \frac{z \sin Bz}{\sinh^2(z/2)} z^{2s-2} J_{s-1/2}(Az) \quad (53)
\]

omitting the overall \( t \)-integral for the time being. At this point, following Candelas and Weinberg, the \( z \) integral is replaced by a contour one in the cut, complex \( z \)-plane (this could have been done earlier) to get for (53)

\[
-\frac{\sqrt{\pi} 2^{1/2-s}}{(1 + \exp(2i\pi s))\Gamma(s)} \int_C dz \frac{z \sin Bz}{\sinh^2(z/2)} z^{2s-2} J_{s-1/2}(Az) \quad (54)
\]

where \( C \) can be taken as, say, \( z = x + iY \) (\(-\infty \leq x \leq \infty\)). The choice of the constant \( Y \) depends on any singularities possessed by the integrand. In this case, there are poles at \( z = 2p\pi \) (\( p = \pm 1, \pm 2, \ldots \)) and so \( Y \) should be less than \( 2\pi \).

At this point we can again check the form of the residues (14) at the poles of \( f(s) \), which, in the present representation, occur at the zeros of \( 1 + \exp(2i\pi s) \), i.e.
at half odd integer $s$. For $s > 1/2$, the original form, (53), is convergent so the only possible poles are at $s = 3/2 - m$, $m = 1, 2, \ldots$. For these values of $s$ there is no cut in the $z$-plane and, also, the integrand is an odd function of $z$. Hence, we can add the expression for the reversed contour, $-C \ (z = x - iY, \ x \ \text{running from} \ +\infty \ \text{to} \ -\infty)$ and divide by two. These two contours combine to give a clockwise contour around the pole at the origin and the integral is simply evaluated by residues.

For (54) we obtain the value, putting back the $t$-integration,

$$-\frac{2(-1)^{m+1}\sqrt{\pi}}{\Gamma(3/2-m)(m-1)!} A^{2m-2} B \int_0^\infty \frac{t^{2m-1}}{\exp(2\pi t) - 1} \, dt$$

and we regain (14) which expression can be continued without ambiguity into the oblate region. Of course, being geometrical, the final result must be valid in both the prolate and oblate cases.

Our main concern is with the derivative at zero, $f'(0)$. In (54) one must differentiate the $1/\Gamma(s)$ factor and set $s = 0$ in the rest to obtain a non-zero answer. We find for the corresponding contribution to $f'(0)$,

$$f'_2(0) = -\int_0^\infty \frac{dt}{\exp(2\pi t) - 1} \int_C \frac{dz}{z \sinh^2(z/2)} \sin Bt z \cos At z$$

and now the $t$-integral can be done (interchanging the limiting processes) using the standard formula

$$\int_0^\infty \frac{dt}{\exp(2\pi t) - 1} = \frac{1}{4} \coth(a/2) - \frac{1}{2a}, \quad |\text{Im} \ a| < 2\pi,$$

to produce

$$f'_2(0) = -\frac{1}{4} \int_C \frac{dz}{z \sinh^2(z/2)} \left[ \sum_{p=-\infty}^{\infty} \left( \frac{1}{((B+ \overline{A})z - 2\pi ip)} + \frac{1}{((B- \overline{A})z - 2\pi ip)} \right) \right]$$

$$= \frac{1}{4} \int_C \frac{dz}{z^2 \sinh^2(z/2)} \left( 1 - \frac{z \sinh Bz}{\cosh Bz - \cosh A z} \right)$$

$$= \frac{1}{2\pi^2} \zeta_R(3) - \frac{1}{4} \int_C \frac{dz}{z \sinh^2(z/2)} \left( \frac{\sinh Bz}{\cosh Bz - \cosh A z} \right)$$

The integrand is even in $z$ ensuring that the integral is real. Convergence at large $|z|$ is also assured. We also note that the $\zeta_R(3)$ term cancels against the contribution to $f'(0)$ from the first term in (22).
The price to be paid for performing the $t$-integration is a further constriction of the contour $C$, for which, in (54), $Y$ lies between 0 and $2\pi$. The condition on the validity of the $t$-integration means that now $0 < Y < 2\pi/|B \pm A|$, or, if we set $l_3 = \cosh \gamma$, $0 < Y < \pi(\coth \gamma \pm 1)$.

The term in brackets in (56) presents poles at

$$z_p^{(\pm)} = \frac{2i\pi p}{B \pm A} = ip\pi \left(1 \pm \frac{l_3}{\sqrt{l_3^2 - 1}}\right) \equiv ip\pi(1 \pm \coth \gamma), \quad p = \pm 1, \pm 2, \ldots \quad (57)$$

Starting from the prolate side, all the plus sign poles, $z_p^{(+)}$, $p > 0$, in (57) lie above the lowest existing pole at $2\pi i$ and cause no problems. However the minus sign, $p = -1$ pole, $z_{-1}^{(-)}$, falls below this pole when $l_3 > 3/(2\sqrt{2}) \approx 1.06066$ and approaches the origin as $l_3$ increases, as do all the other minus sign poles, symmetrically with the sign of $p$. The contour $C$ has to be adjusted to lie below this pole. This is what the above condition means.

Just as a check, we have numerically evaluated (55) and (56). Typically, for $\gamma = 0.5$, $(l_3 = 1.1276)$ we find that (55) gives $f_2'(0) \approx -0.123$, for values, $Y = 3$ and $Y = 4$, which straddle the pole at $z = i\pi(\coth \gamma - 1) \approx i3.6565$, the order of integration making no difference. In contrast, (56) gives $-0.123$ if $Y$ is below this pole, but $0.143$ if above. The difference is accounted for by the pole residue, as has been numerically checked.

As the oblate case is approached, $l_3 \downarrow 1$, the plus sign poles, $z_p^{(+)}$, with positive $p$ run away up the imaginary axis to $i\infty$, as do the minus sign poles, $z_p^{(-)}$, with negative $p$. Conversely, the plus sign poles with negative $p$, together with the minus sign poles with positive $p$, run off to $-i\infty$.

As $l_3$ passes into the oblate régime, the poles reappear at the complex positions

$$z_p^{(\pm)} = p\pi(i \pm \tan \theta) \quad (58)$$

We should follow them round carefully. It is apparent that some prolate poles with positive (negative) $p$, disappear with negative (positive) imaginary parts, only to reappear as oblate poles with imaginary parts of opposite sign. They have thus crossed the contour $C$ (at complex infinity) and, if we want to leave $C$ unchanged, it is necessary to include the corresponding residue contributions. These yield, choosing $\sqrt{l_3^2 - 1} = i\sqrt{1 - l_3^2}$,

$$\frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p \sinh^2 z_p^{(-)}}.$$
Symmetry under $A \to -A$ (or $C \to -C$) is maintained automatically because $\sinh^2 z_p(-) = \sinh^2 z_p(+) \sinh^2$, in tune with the fact that the branch point in (57) at $l_3^2 = 1$ is purely artificial.

The complete oblate contribution $f_2'(0)$ is then

$$f_2'(0) = \frac{1}{2\pi^2} \zeta_R(3) - \frac{1}{4} \int_C \frac{dz}{z \sinh^2(z/2)} \left( \frac{\sin Bz}{\cosh Bz - \cos Cz} \right) +$$

$$\frac{1}{2} \left( \sum_{m=1}^{\infty} \frac{1}{2m \sinh^2(m\pi \tan \theta)} - \sum_{m=0}^{\infty} \frac{1}{(2m + 1) \cosh^2 \left( (2m + 1)\pi \tan(\theta)/2 \right)} \right).$$

Other values of interest are $f(n)$, with $n$ a positive integer. These are needed in the expansion (5). Looking at (8) the first two terms are easily evaluated and the last one vanishes, as shown earlier. The remaining one, $f_2(s)$, is under present investigation and we find from (54),

$$f_2(n+1) = -\frac{4\sqrt{\pi}}{n!} \int_0^\infty dt \frac{\exp(2\pi t) - 1}{\exp(2\pi t)} \int_0^\infty dz \frac{\sin Btz \sinh^2(z/2)}{n}.$$ \[60\]

This time the $z$ integration can be done. As an example consider $f_2(1)$

$$f_2(1) = -4\sqrt{2} \int_0^\infty dt \frac{\exp(2\pi t) - 1}{\exp(2\pi t)} \int_0^\infty dz \frac{\sin Btz \sin \left( Atz \right)}{A \sinh^2 z/2}$$

and use the standard integral

$$\int_0^\infty dx \frac{\sin ax \sin bx}{\sinh^2 x/2} = 2\pi \frac{b \sinh(2a\pi) - a \sinh(2b\pi)}{\cosh(2a\pi) - \cosh(2b\pi)}$$ \[61\]

to leave a single numerical quadrature. In the oblate case ($A \to iC$), the integral in (61) always converges.

The other values can be reduced similarly to involve integrals obtained by repeated differentiation of (61).

For these values of $s > 3/2$, one can also calculate $f(s)$ by direct summation which, purely numerically, is probably more convenient.

For scalar fields, the evaluation of the $n$-summation term in (25) is precisely the same as described above for the spin-half case. The only changes are the overall sign of the term and the ‘bosonic’ factor $1/(\exp(2\pi t) + 1)$, which results in some algebraic differences. In place of (55) we have (remember $W = -\zeta'(0)/2$ here)

$$W_{sc}^{(2)} = -\frac{1}{2} \int_0^\infty dt \frac{\exp(2\pi t) + 1}{\exp(2\pi t) + 1} \int_C \frac{dz}{z \sinh^2(z/2)} \sin Btz \cos Atz$$ \[62\]
and this time we use the integral
\[ \int_0^\infty \frac{dt}{\exp(2\pi t) + 1} = \frac{1}{2a} - \frac{1}{4} \cosech \left( \frac{a}{2} \right), \quad |\text{Im} \ a| < 2\pi, \]
to give in the prolate case,

\[ W_{sc}^{(2)} = \frac{1}{8} \int C \frac{dz}{z \sinh^2(z/2)} \sum_{p=\infty}^\infty \left( \frac{(-1)^p}{((B + A)z - 2\pi ip)(B - A)z - 2\pi ip)} \right) \]
\[ = -\frac{1}{4\pi^2} \zeta_R(3) + \frac{1}{4} \int C \frac{dz}{z \sinh^2(z/2)} \frac{\sinh(Bz/2) \cosh(Az/2)}{\cosh Bz - \cosh Az} \]

(63)

with the same restriction on the contour, \( C \).

The oblate expression is found to be

\[ W_{sc}^{(2)} = -\frac{1}{4\pi^2} \zeta_R(3) + \frac{1}{4} \int C \frac{dz}{z \sinh^2(z/2)} \frac{\sinh(Bz/2) \cos(Cz/2)}{\cosh Bz - \cos Cz} - \frac{1}{4} \left( \sum_{m=1}^{\infty} \frac{1}{2m \sinh^2(m\pi \tan \theta)} + \sum_{m=0}^{\infty} \frac{1}{(2m + 1) \cosh^2 \left( \frac{(2m + 1)\pi \tan(\theta)/2} \right)} \right) \]

(64)

When this is substituted into the total expression (44) we again note the cancellation of the \( \zeta_R(3) \) terms.

There appears to be no ambiguity in the prolate calculation, and we have precise numerical agreement between (56) and (20) for spinors, and between (63) and (41) for scalars.

However, extra pole terms seem to arise when the Bessel technique is ‘continued’ from the prolate to the oblate régime, as exhibited in (59) and (64). These terms do not affect the perturbation expansion and indeed the first lines of (64) and (59) equal the whole of (43) and (19) respectively. Since they involve just a single quadrature, they at least provide better numerical alternatives.

The puzzle is, partly, that these extra terms precisely cancel the oblate contribution, \( W_{sc}^{(3)} \) (29), in the scalar case, and double this up in the spinor case. This is a curious coincidence that we cannot explain.

If one did believe the forms derived in this section, then there is no divergence in the total effective action as \( l_3 \to 0 \), for scalar fields. This would seem to run counter to a general expectation, derived from experience with the Selberg-Chowla formula, and also shows an unbelievable difference between scalars and spinors.

It is clear, however, that the results on the \( \overline{C} = 0 \) case based on a perturbation expansion of the original sum form about \( \overline{C} = 0 \) (or about \( \overline{A} = 0 \), discussed in the
earlier sections, show that there is no divergence in \(W^{(2)}\) as \(l_3 \to 0\) and hence that the results presented in the previous sections of this paper are the proper ones. We therefore discard the second lines of (64) and (59). Doing this allows one to find the \(l_3 \to 0\) limits of \(W^{(2)}\) in closed form,

\[
W^{(3)}_{sc} \bigg|_{l_3=0} + \frac{1}{4\pi^2} \zeta_R(3) = \frac{1}{16} \int_C \frac{dz}{z \sinh^3 z \cosh z} = \frac{1}{4} \ln 2 - \frac{1}{8\pi^2} \zeta_R(3) \tag{65}
\]

and

\[
W^{(3)}_{sp} \bigg|_{l_3=0} - \frac{1}{2\pi^2} \zeta_R(3) = -\frac{1}{8} \int_C \frac{\cosh 2z}{z \sinh^3 z \cosh z} = \frac{1}{2} \ln 2 + \frac{1}{4\pi^2} \zeta_R(3) \tag{66}
\]

where the contour has been pushed upwards through the poles of the integrand and the residues summed.

9. Preliminary comparison with the bulk results.

We are now in a position to gather our results together and give the total high temperature forms of the free energy, defined by \(\Phi = -W/\beta\).

For the scalar field, from (44), (65) and (38),

\[
\Phi_{sc}(\beta) \sim -\frac{2\zeta_R(3)}{\beta^3} + \frac{1}{12\beta} \ln \beta + \frac{1}{\beta} \left( \frac{3}{16\pi^2} \zeta_R(3) - \frac{1}{4} \ln 2 + \zeta'(-1) \right), \tag{67}
\]

while, for the spinor field from (22), (66) and (50),

\[
\Phi_{sp}(\beta) \sim -\frac{4\zeta_R(3)}{\beta^3} - \frac{1}{6\beta} \ln \beta - \frac{1}{\beta} \left( \frac{1}{8\pi^2} \zeta_R(3) + \frac{1}{2} \ln 2 + 2\zeta'(-1) \right). \tag{68}
\]

According to [1] and [2], the Taub-Bolt-AdS bulk, four-dimensional entropy goes like \(1/\beta^2\) at high temperatures. This behaviour agrees, of course, with our expressions for the free energy, or action, with which we prefer to work.

Extracting the leading terms from [1]

\[
\beta \Phi \sim -\alpha \left( \frac{1}{\beta^2} - \frac{9}{8\pi^2} + \frac{27(k + 2)^2}{1024\pi^4} \beta^2 \right) \tag{69}
\]

where \(\alpha\) is a constant which does not concern us here. The identification parameter \(k\) has been included and should be set equal to unity to conform to the assumptions of the present paper. In this case, (69) agrees with the calculations in [2].

We notice immediately the absence of any logarithmic or transcendental terms. These can be eliminated from our expressions by choosing a combination of two scalars plus one spinor, yielding the behaviour

\[
\beta \Phi \bigg|_{\text{two scalars + one spinor}} \sim -8\zeta_R(3) \left( \frac{1}{\beta^2} - \frac{1}{108\pi^2} \right) \tag{70}
\]

which is the best that can be done with the fields available.
10. Conclusion.

We have presented formulae for the determinants of spinor and scalar fields on the squashed three-sphere and have determined explicitly their leading \(1/\beta^2\) behaviours which exactly correspond to those on \(S^2 \times S^1\). The non-leading behaviours have also been found where the twisted nature of \(\tilde{S}^3\) shows up. The scalar expression is a mixture of thermal scalar and spinor on \(S^2\) while the spinor form combines twisted scalar and spinor, \textit{i.e.} fields with the ‘wrong’ thermal periodicity.

The next step in the calculation, which will be exposed at a later date, is to place \(k\) identifications on the \(\psi\)-circle. Since the resulting manifold is more or less locally unchanged, one would expect the highest terms to remain the same, depending, as they do, on the \(S^2\) geometry. This is born out in (69) and (70), and is a comment made also in [2].

The extreme \textit{prolate} limit presents certain technical difficulties which have still to be addressed. The action for the branch of the bulk action that leads to (69) tends to a constant as \(l_3 \to \infty\) while the other branch diverges like \(l_3^4\), in agreement with (42) for the scalar field. For the spinor field, one needs to go beyond the expansion in \(l_3\) and to allow for sign changes in the spectrum. For the scalar field we should also compute for the minimal Laplacian.

It should be noted that our \(W\) corresponds to \(-I\) of \([1,2]\). \(I\) is the difference between the actions of Taub-Bolt-AdS and Taub-Nut-AdS and is only defined for certain regions of \(l_3\). There appears to be no corresponding limitation in the present calculation since the round case, \(l_3 = 1\), is always accessible, whatever the operator.

Finally we remark that it is necessary to consider vector fields, for which the eigenvalues are also available, \([9,11]\).

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