Nambu, A Foreteller of Modern Physics III

Professor Nambu, string theory, and the moonshine phenomenon

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1. Farewell to Prof. Nambu

It is very sad that we no longer see and talk to Professor Nambu these days. As everybody knows, Prof. Nambu was one of the greatest theoretical physicists of our time. He was an idol of my generation. He contributed so much to transform and bring particle physics to the present-day standard model. He introduced string theory, which will be the main ingredient of particle physics for future generations. We no longer have the special privilege of having a great man among us.

The last time I saw him was in early June 2015, in a hospital in Osaka. It was a Sunday and I wanted to come to Osaka and see Prof. Nambu; I used to see him about once a month in those days. It turned out that on the previous day he had had a serious pain in his heart, which looked like an after-effect of the heart attack he had suffered before, and he was kept in an ICU.

I was not sure if I could see him, but it turned out I was allowed to enter the ICU room. Prof. Nambu seemed to have recovered from the pain of the day before and was in a good condition and spirit. He was smiling with his family. When first he saw me, he said “zero point life,” pointing to himself. He was making a joke of his health condition.

I think this was one of a few happy moments during his last fight against illness. I stayed for a few hours and went back to Tokyo. I want to keep the memory of this scene in ICU forever.

Now let me turn to a discussion of physics. About five years ago, together with my collaborators, I found a curious phenomenon in string theory: the appearance of an exotic discrete symmetry in the theory [1]. This phenomenon is now called Mathieu moonshine, and is under intensive study. Prof. Nambu was curious to hear about the story; however, there was no chance to tell him the details. Today I would like to give you a brief introduction to the moonshine phenomenon, which will possibly play an interesting role in string theory in the future.
Before going into the moonshine phenomenon in string theory, let me briefly recall the story of monstrous moonshine, which is very well known. The modular $J$ function has a $q$-series expansion

$$J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3$$

$$+ 20245856256q^4 + 333202640600q^5 + \cdots$$

$$q = e^{2\pi i \tau}, \quad \text{Im}(\tau) > 0, \quad J(\tau) = J\left(\frac{a\tau + b}{c\tau + d}\right), \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z}).$$

It turns out that the $q$-expansion coefficients of the $J$-function are decomposed into a sum of dimensions of irreducible representations of the monster group $M$ as

$$196884 = 1 + 196883,$$
$$21493760 = 1 + 196883 + 2129676,$$
$$864299970 = 2 \times 1 + 2 \times 196883 + 2129676 + 842609326,$$
$$20245856256 = 1 \times 1 + 3 \times 196883 + 2 \times 2129676$$

$$+ 842609326 + 19360062527,$$

$$\cdots$$

The dimensions of some irreducible representations of the monster group are in fact given by

$$\{1, 196883, 2129676, 842609326, 18538750076, 19360062527, \ldots\}$$

The monster group is the largest sporadic discrete group, of order $\approx 10^{53}$, and the strange relationship between the modular form and the largest discrete group was first noted by McKay.

To be precise, we may write

$$J_1(\tau) = J(q) - 744 = \sum_{n=-1} c(n) q^n, \quad c(0) = 0$$

$$= \sum_{n=-1} \text{Tr}_{V(n)} 1 \times q^n, \quad \dim V(n) = c(n)$$

The McKay–Thompson series is given by

$$J_g(\tau) = \sum_{n=-1} \text{Tr}_{V(n)} g \times q^n, \quad g \in M,$$

where $\text{Tr}_{V(n)} g$ denotes the character of a group element $g$ in the representation $V(n)$. This depends on the conjugacy class $g$ of $M$. If the McKay–Thompson series is known for all conjugacy classes, the decomposition of $V(n)$ into irreducible representations becomes uniquely determined. The series $J_g$ are modular forms with respect to subgroups of $SL(2, \mathbb{Z})$ and possess similar properties to the modular $J$-function, such as the genus = 0 (Hauptmodul) property.

The phenomenon of monstrous moonshine was understood mathematically in the early 1990s using the technology of vertex operator algebra. However, we still do not have a “simple” physical explanation of this phenomenon. (A possible connection to heterotic string theory is discussed in a recent article [2].)
2. Mathieu moonshine

We consider string theory compactified on the $K_3$ surface, which is a complex two-dimensional hyperKähler manifold and is ubiquitous in string theory. It possesses $SU(2)$ holonomy and a holomorphic two-form. Thus, string theory on $K_3$ has an $\mathcal{N} = 4$ superconformal symmetry with the central charge $c = 6$ that contains $SU(2)_{k=1}$ affine symmetry.

Now, instead of the modular $J$-function we consider the elliptic genus of the $K_3$ surface. The elliptic genus describes the topological invariants of the target manifold and counts the number of Bogomol’nyi–Prasad–Sommerfield (BPS) states in the theory. Using world-sheet variables it is written as

$$Z_{\text{elliptic}}(z; \tau) = \text{Tr}_{\mathcal{H}_L \times \mathcal{H}_R} (-1)^{F_L + F_R} e^{4\pi i z L_0} q^{L_0} \bar{q}^{\bar{L}_0}.$$

Here, $L_0$ denotes the zero mode of the Virasoro operators and $F_L$ and $F_R$ are left- and right-moving fermion numbers. $J^3_0$ denotes the Cartan generator of affine $SU(2)_1$. In the elliptic genus the right-moving sector is frozen to the supersymmetric ground states (BPS states), while in the left-moving sector all the states in the left-moving Hilbert space $\mathcal{H}_L$ contribute.

In general it is difficult to compute elliptic genera; however, we were able to evaluate it by making use of Gepner models [3]. The elliptic genus is given by

$$Z_{K_3}(\tau, z) = 8 \left[ \left( \frac{\theta_2(\tau, z)}{\theta_2(\tau, 0)} \right)^2 + \left( \frac{\theta_3(\tau, z)}{\theta_3(\tau, 0)} \right)^2 + \left( \frac{\theta_4(\tau, z)}{\theta_4(\tau, 0)} \right)^2 \right].$$

Here, $\theta_i(\tau, z)$ are Jacobi theta functions.

We want to see how the Hilbert space $\mathcal{H}_L$ in the elliptic genus decomposes into irreducible representations of $\mathcal{N} = 4$ superconformal algebra (SCA).

The highest weight states of $\mathcal{N} = 4$ SCA are parametrized by the eigenvalues of $L_0$ and $J^3_0$:

$$L_0 | h, \ell \rangle = h | h, \ell \rangle, \quad J^3_0 | h, \ell \rangle = \ell | h, \ell \rangle.$$

There are two different types of representation in $c = 6$ SCA. In the Ramond sector,

- BPS (massless) representation $h = \frac{1}{4}$; $\ell = 0, \frac{1}{2}$;
- non-BPS (massive) representation $h > \frac{1}{4}$; $\ell = \frac{1}{2}$.

The character of a representation is defined as

$$\text{Tr}_\mathcal{R} (-1)^F q^{L_0} e^{4\pi i z J^3_0},$$

where $\mathcal{R}$ denotes the representation space.

The index is given by the value of the character at $z = 0$,

$$\text{Index}(\mathcal{R}) = \text{Tr}_\mathcal{R} (-1)^F q^{L_0}.$$

BPS representations have a non-vanishing index,

$$\text{index (BPS, } \ell = 0) = 1, \quad \text{index (BPS, } \ell = \frac{1}{2}) = -2.$$
while non-BPS representations have vanishing indices,

\[ \text{index (non-BPS, } \ell = \frac{1}{2} \text{)} = 0. \]

Characters are given explicitly as [4]

\[ \chi_{BPS}^{h, \ell = 0} (\tau, z) = \frac{\theta_1(z; \tau)^2}{\eta(\tau)^3} \mu(z; \tau), \]

where

\[ \mu(z; \tau) = -ie^{\pi iz} \sum_n (-1)^n q^{\frac{1}{2} n(n+1)} e^{2\pi n z}, \]

while non-BPS characters are given by

\[ \chi_{\text{non-BPS}}^{h, \ell = \frac{1}{2}} = q^{h - \frac{1}{8}} \frac{\theta_1(z; \tau)^2}{\eta(\tau)^3}, \quad h > \frac{1}{4}. \]

The function \( \mu(\tau, z) \) is a typical example of a mock theta function (Lerch sum or Appel function). Mock theta functions look like theta functions but they have anomalous modular transformation laws and are difficult to handle. Recently there have been developments in understanding the nature of mock theta functions due to Zwegers [5]. He has introduced a method of regularization which is similar to those used in physics, and improves the modular property of mock theta functions so that they transform as analytic Jacobi forms.

Now let us make a decomposition of the elliptic genus into a sum of characters of \( \mathcal{N} = 4 \) representations:

\[ Z_{K3}(\tau, z) = 24 \chi_{BPS}^{h, \ell = 0} (\tau, z) + 2 \sum_{n \geq 0} A(n) \chi_{\text{non-BPS}}^{h, \ell = \frac{1}{2}} (\tau, z). \]

At smaller values of \( n \), the expansion coefficients \( A(n) \) may be found by direct series expansion of \( Z_{K3} \). We find that \( A(0) = -1 \), and

\[
\begin{array}{ccccccccccc}
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\
  A(n) & 45 & 231 & 770 & 2277 & 5796 & 13915 & 30843 & 65550 & \ldots \\
\end{array}
\]

The dimensions of some irreducible representations of the Mathieu group \( M_{24} \) appear in the above [1]:

\[
\text{dimensions : } \{ \ 45 \ 231 \ 770 \ 990 \ 1771 \ 2024 \ 2277 \ 3312 \ 3520 \ 5313 \ 5544 \ 5796 \ 10395 \ \ldots \} 
\]

\[
A(6) = 13915 = 3520 + 10395, \\
A(7) = 30843 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771. 
\]

\( M_{24} \) is a subgroup of \( S_{24} \) (the permutation group of 24 objects) and its order is given by \( \approx 10^9 \). \( M_{24} \) is known for its many interesting arithmetic properties, and in particular it is intimately tied to the Golay code of efficient error corrections:

Monster ⊃ Conway ⊃ Mathieu ⊃ \ldots
3. Mathieu moonshine conjecture

The expansion coefficients of the $K_3$ elliptic genus into $\mathcal{N} = 4$ characters are given by the sum of the dimensions of representations of the Mathieu group $M_{24}$.

We were able to derive analogues of the McKay–Thompson series [6,7]. Then, the multiplicities $C_R(n)$ of the decomposition of $A(n)$ into representations $R$,

$$A(n) = \sum_R C_R(n) \dim R,$$

are unambiguously determined. It turned out that $C_R(n)$ are all the positive integers up to $n \approx 1000$, and this gives very strong evidence of the Mathieu moonshine conjecture. The conjecture is now proved mathematically using the method of mathematical induction [8].

Unfortunately, the proof so far has not provided much insight into the nature of Mathieu moonshine. The situation looks a bit like the case of monstrous moonshine. The 24 of $M_{24}$ will certainly be the Euler number of $K_3$, and $M_{24}$ permutes homology classes. There are, however, various indications that string theory on $K_3$ cannot have such a high symmetry as $M_{24}$. Instead of the total Hilbert space the BPS subsector of the theory may possibly possess an enhanced symmetry. It will be interesting to look into the algebraic structures of BPS states to explain Mathieu moonshine.

4. More moonshine phenomena

Mathieu moonshine exists at the intersection of string theory, $K_3$ surface (geometry), (mock) modular forms, and sporadic discrete symmetry, and appears to possess an interesting mixture of physics and mathematics. Recently there has intense interest in exploring new types of moonshine phenomena other than Mathieu moonshine. Several types of new moonshine phenomena have already been discovered: umbral moonshine [9]; fermions on a 24-dimensional lattice [10]; and the spin-7 manifold [11].

Due to time limitations we discuss only umbral moonshine here. Umbral moonshine has a mysterious relationship with the Niemeier lattice. It is known there are 23 (24, if we include the Leech lattice) types of self-dual lattices in 24 dimensions. This is given by the combination of root lattices of A-D-E type together with appropriate weight vectors so that the lattice becomes self-dual. The simplest examples are

$$\begin{align*}
(A_1)^{24} & \quad (k = 1) \\
(A_2)^{12} & \quad (k = 2) \\
(A_3)^8 & \quad (k = 3) \\
\cdots & \quad \cdots
\end{align*}$$

If one divides the automorphism groups of the Niemeier lattice by the automorphism group of the A-D-E lattice, one obtains isolated discrete groups

$$G_k = \frac{[\text{automorphism group of lattice}]_k}{[\text{Weyl group of root lattice}]_k}.$$ 

It turns out that the $G_k$s become the symmetry groups of the umbral moonshine. In fact the first one agrees with the Mathieu group $G_1 = M_{24}$ and reproduces the Mathieu moonshine. The second one $G_2$ agrees with the Mathieu group $M_{12}$ and is assumed to be related to the four-dimensional hyperKähler manifold with $c = 12$ ($k = 2$).
An analogue of the $K_3$ elliptic genus is given by

$$Z(k = 2) = 4 \left[ \left( \frac{\theta_2(z)\theta_3(z)}{\theta_2(0)\theta_3(0)} \right)^2 + \left( \frac{\theta_2(z)\theta_4(z)}{\theta_2(0)\theta_4(0)} \right)^2 + \left( \frac{\theta_3(z)\theta_4(z)}{\theta_3(0)\theta_4(0)} \right)^2 \right].$$

By expanding $Z(k = 2)$ in terms of characters of representations of $c = 12$, $N = 4$ algebra one finds that the expansion coefficients decompose into the symmetry group $M_{12}$.

Here, however, there is something awkward: $Z(k = 2)$ does not contain the contribution of the vacuum operator ($\hbar = 0$ in the NS sector) and thus the theory appears to describe the geometry of a (singular) non-compact four-fold. The rest of the umbral moonshine series has the same property (absence of identity operator) and their geometrical interpretation is somewhat obscure.

Recently, we have used $N = 4$ Liouville theory [12], which is known to possess some special duality properties [13]. It is possible to embed the umbral series into $N = 4$ Liouville theory, and by using duality we can map the umbral theory at $c = 6k$ to its dual theory at $c = 6$. Thus an umbral moonshine at $c = 6k$ can be mapped to a dual moonshine at $c = 6$. We hope this is going to help the geometrical interpretation of umbral moonshine.

The moonshine symmetries recently discovered in string theory are still very mysterious, and we may encounter many more surprises in the near future.

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