Some questions on the class group of cyclotomic fields

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Abstract

This article deals with a study of the structure of the class group of the cyclotomic field $K = \mathbb{Q}(\zeta_p)$ for $p$ an odd prime number, starting from Stickelberger relation. The present state of this work leads me to set a question for all the prime numbers $h \neq p$ which divide the relative class number $h^-(K)$.

1 Some definitions

- Let $p$ be an odd prime.
  Let $\zeta$ be a primitive $p$-th root of unity.
  Let $K = \mathbb{Q}(\zeta)$ be the $p$-cyclotomic field.
  Let $C^-$ be the relative class group of $K$.
  Let $h$ be a prime, let us note $C(h)$ the $p$-Sylow subgroup of $C^-$.
  Let $h^-(K)$ be the relative class number of $K$.
  Let $K^+$ be the maximal totally real subfield of $K$.
  Let $h(K^+)$ be the class group of $K^+$.
  Let $G$ be the Galois group of the extension $K/\mathbb{Q}$.
  Let $v$ be a primitive root mod $p$.
  Let $\sigma$ be the generator of $G$ given by $\sigma : \zeta \rightarrow \zeta^v$.
  For $n \in \mathbb{Z}$, let us note $v_n$ for $v_n = v^n \mod p$ with $1 \leq v_n \leq p - 1$.

- Let, for the indeterminate $X$, $P(X) \in \mathbb{Z}[X]$ given by $P(X) = \sum_{i=0}^{p-2} v_{-i}X^i$.
  From Stickelberger relation, it can be shown that $P(\sigma)$ annihilates $C^-$. It can be also shown that for the indeterminate $X$,

$$P(X)(X - v) = p \times Q(X) + v(X^{p-1} - 1),$$
where $Q(X) \in \mathbb{Z}[X]$ is given by $Q(X) = \sum_{i=1}^{p-2} \frac{v_{i-1} - v_{i-1}}{p} \times X^i$ and $Q(\sigma)$ annihilates the relative $p$-class group $C_{p}^-$ of $K$ (the subgroup of exponent $p$ of $C^-$).

It can be shown that $\theta = \frac{P(\sigma)}{p} = \frac{Q(\sigma)}{\sigma - v}$ where $\theta$ is the Stickelberger element.

2 Numerical observations

- I have observed, with a little MAPLE program that:

  For all the prime numbers $p < 500$ and all the odd prime numbers $h < p^2$ then the $h$-rank of the $h$-Sylow subgroup $C(h)$ of $C^-$ is equal to the degree of the polynomial $D(X)$ given by the following relations:

  1. If $h \neq p$ then
     
     \begin{equation}
     D(X) = \gcd(P(X), X^{(p-1)/2} + 1) \mod h.
     \end{equation}

  2. if $h = p$ then
     
     \begin{equation}
     D(X) = \gcd(Q(X), X^{(p-1)/2} + 1) \mod h.
     \end{equation}

     where

     \begin{equation}
     \gcd(h, h^-(K)) = 1 \iff \deg(DX)) = 0.
     \end{equation}

   - These observations are derived from a comparison of my results in Queme [5] with R. Schoof [3] where the structure of minus-class group for $p < 500$ is displayed.

   - I think that, for $h \neq p$ the theorem 6.21 p. 103 in Washington [4] and for $h = p$ the relation, with Washington notations, $[R_p^- : I_p] = p$-part of $h^-(K)$ in the end of the proof of theorem 6.21 p. 106 bring a proof of the following assertion:

     \begin{equation}
     \deg(D(X)) \neq 0 \iff h \mid h^-(K),
     \end{equation}

   for all the odd prime numbers $h$, assertion which generalizes my numerical observations in relation [3]. Observe that this result uses only elementary arithmetical computations, hence is particularly suitable for the computer calculations of the prime divisors of the relative class number of the cyclotomic field $K$.

   - Let $D(X) = \prod_k D_k(X)^{n_k}$ be the factorization of the polynomial $D(X)$ in $\mathbb{F}_h[X]$. If $h = p$ then clearly $n_k = 1$ for all $k$. If $h \neq p$ there exist some primes $p$ with $n_k \neq 1$. As an example for $p = 331$, $h = 3$, then $D(X) = (X + 1)^2 \times (X^4 + 2X^3 + X^2 + 2X + 1)$ with the primitive root $v = 3, \sigma : \zeta \rightarrow \zeta^3$, and the rank of $C(3)$ is 6.
3 A question for the case $h \neq p$

Observe that, unfortunately, I am not able to compute the structure of the class group $C^-$ to suppress the doubt for large primes $p$ with my limited student MAPLE software.

Question: Does there exists known counter-examples to my observations in relation (1) p. 2 when $p > 500$ and $h$ is an odd prime with $h \neq p$?

Comment: In regards to the case $p = 4027$ and $h = 3$ candidate for a possible counter-example given in Washington p. 106, does $R^-/I^-$ not isomorphic to $C^-$ implies $\text{degree}(D(X)) \neq 3$-rank of $C(3)$, which should imply that $(p = 4027, h = 3)$ is a counter-example to my assertion (1) p. 2.

The case $(p = 4027, h = 3)$ described in Washington is not completely conclusive for our question: for instance we know from this description that $R^-/I^- \simeq C^-$ is false when the class group $C_2$ of the quadratic imaginary field $\mathbb{Q}(\sqrt{-p})$ is not cyclic:

1. Does it implies, looking at the full relative class group $C^-$, that the rank of the $h$-Sylow subgroups of $R^-/I^-$ and of $C^-$ are different. Note that $4026 = 2 \cdot 3 \cdot 11 \cdot 61$ and the difference of rank at quadratic field level could perhaps disappear at the level of one of the intermediate fields between $\mathbb{Q}(\sqrt{-p})$ and $K$ or at the level of the field $K$ itself.

2. Does it implies, in our elementary formulation, that the rank of $C(h) \neq \text{degree}(D(X))$?

In the other hand, in answer to my question, R. Schoof guess that the kind of behavior observed for quadratic fields also occurs for minus class groups of abelian fields of higher degree, more or less as the Cohen-Lenstra heuristics predict and thus that counter-examples are likely.

4 Context:

The motivations of this note appear more precisely in my preprint entitled On prime factors of class group of cyclotomic fields section 5 page 18. The URL of this preprint is at:

http://arxiv.org/PS_cache/math/pdf/0609/0609723.pdf

References

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