Gluon self-energy in the color-flavor-locked phase

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We calculate the self-energies and the spectral densities of longitudinal and transverse gluons at zero temperature in color-superconducting quark matter in the color-flavor-locked (CFL) phase. We find a collective excitation, a plasmon, at energies smaller than two times the gap parameter and momenta smaller than about eight times the gap. The dispersion relation of this mode exhibits a minimum at some nonzero value of momentum, indicating a van Hove singularity.

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At asymptotically large quark chemical potentials $\mu$ and sufficiently small temperatures $T$, quark matter is a color superconductor \cite{1}. While there are, in principle, many different color-superconducting phases, corresponding to the different possibilities to form quark Cooper pairs with definite color, flavor, and spin quantum numbers, for three quark flavors and quark chemical potentials much larger than the strange quark mass, the ground state of color-superconducting quark matter is the so-called color-flavor-locked (CFL) phase \cite{2}.

In the CFL phase, the $SU(3)_c \times SU(3)_f$ color and (vector) flavor symmetry of QCD is broken to the diagonal subgroup $SU(3)_{c+f}$. Consequently, all eight gluons become massive due to the Anderson-Higgs mechanism. The situation is quite similar to that in the 2SC case, although there $SU(3)_c$ is broken to $SU(2)_c$, and only five gluons become massive, while the three gluons of the residual $SU(2)_f$ color symmetry remain massless. These general symmetry considerations can be confirmed by an explicit calculation of the gluon Meissner masses in a given color-superconducting phase. The Meissner mass (squared) of a gluon with adjoint color $a$ is the zero-momentum limit of the transverse component of the gluon self-energy at zero energy, $\lim_{p_\perp \to 0} \Pi_{ab}^{\perp} (p_{\perp} = 0, \mathbf{p})$.

At asymptotically large $\mu$, the QCD coupling constant $g \ll 1$, and the gluon self-energy is dominated by the contributions from one quark and one gluon loop. The quark loop is $\sim g^2 \mu^2$, while the gluon loops are $\sim g^2 T^2$. Since the color-superconducting gap parameter is $\phi \sim \mu \exp(-1/g) \ll \mu$ \cite{3}, and since the transition temperature to the normal conducting phase is $T_c \sim \phi$, for temperatures where quark matter is in the color-superconducting phase, $T \ll T_c \ll \mu$, the gluon loop contribution can be neglected. Following this line of arguments, the gluon Meissner masses have been computed for the 2SC phase in Ref. \cite{4} and for the CFL phase in Ref. \cite{5, 6}. The full energy-momentum dependence of the one-loop gluon self-energy has also been computed, but so far only for the 2SC phase \cite{7, 8}. The corresponding calculation for the CFL phase is the goal of the present paper.

For the 2SC phase, the derivation of the gluon self-energies and propagators was explicitly discussed in Sec. II of Ref. \cite{8}, and we briefly remind the reader of the main steps. Just like in other systems (ordinary superconductors, the standard model of electroweak interactions) where gauge symmetries are broken, there are Nambu-Goldstone fluctuations which mix with the longitudinal components of the gauge field. For a particular choice of gauge (‘t Hooft gauge), the Nambu-Goldstone modes decouple from the longitudinal components of the gauge field. Furthermore, for a particular choice of the ‘t Hooft gauge parameter, the Nambu-Goldstone modes can be eliminated and the gauge field propagator is explicitly 4-transverse.

In our case, i.e., for the CFL phase, the computational steps leading to the gluon self-energies $\Pi_{ab}^{\mu\nu}$ and the gluon propagators $\Delta_{ab}^{\mu\nu}$ are quite similar to those in the 2SC phase \cite{8}, if not even simpler because all eight gluons are affected similarly by the breaking of the gauge symmetry,

$$\Pi_{ab}^{\mu\nu}(P) = \delta_{ab} \Pi^{\mu\nu}(P) \ , \ \Delta_{ab}^{\mu\nu}(P) = \delta_{ab} \Delta^{\mu\nu}(P) \ , \ a, b = 1, \ldots, 8 \ . \tag{1}$$

Therefore, we do not give the details of this lengthy but straightforward calculation and just quote the final result. The final answer for the propagator of transverse gluons reads [cf. Eq. (56) of Ref. \cite{8}]

$$\Delta^t(P) = \frac{1}{P^2 - \Pi^t(P)} \ . \tag{2}$$
FIG. 1: The imaginary parts of (a) $\Pi_{00}$, (b) $-\Pi_{0i}^\mu\hat{p}_i$, (c) $\Pi^t$, (d) $\Pi^\ell$, and (e) $\hat{\Pi}_{00}$ as a function of energy $p_0$ for a gluon momentum $p = 4\phi$. The solid lines are for the CFL phase, the dotted lines correspond to the HDL self-energy.

while for longitudinal gluons we have [cf. Eq. (57) of Ref. [8]]

$$\hat{\Delta}_{00}(P) = -\frac{1}{\hat{p}^2 - \hat{\Pi}_{00}(P)}.$$  (3)

Here, the transverse and longitudinal components of the gluon self-energy are computed from projections of $\Pi^{\mu\nu}$,

$$\Pi^t(P) = \frac{1}{2} (\delta^{ij} - \hat{p}^j\hat{p}^j) \Pi^{ij}(P),$$  (4a)

$$\hat{\Pi}_{00}(P) = \hat{p}^2 \frac{\Pi_{00}(P) \Pi^t(P) - [\Pi_{0i}(P) \hat{p}_i]^2}{\hat{p}_0^2 \Pi_{00}(P) + 2 \hat{p}_0 \hat{p} \Pi_{0i}(P) \hat{p}_i + \hat{p}^2 \Pi^t(P)},$$  (4b)

where

$$\Pi^t(P) = \hat{p}^j \Pi^{ij}(P) \hat{p}^j.$$  (4c)

The computation of the individual components and projections $\Pi_{00}$, $\Pi_{0i}$, $\Pi^t$, and $\Pi^\ell$ is lengthy and thus deferred to the appendix. Note that the particular form of the longitudinal self-energy $\hat{\Pi}_{00}$ arises from decoupling spatially longitudinal and time-like gluon degrees of freedom, see Ref. [8].

In Fig. 1 we show the imaginary part of several components of the gluon self-energy for a gluon momentum $p = 4\phi$ as a function of the gluon energy $p_0$. For comparison, we also show the corresponding results for the gluon self-energy in the “hard-dense loop” (HDL) limit, $\Pi_{0\mu}^{\nu}$, cf. Eqs. (63a), (65a), (69a) and (69b) of Ref. [8]. The imaginary parts are quite similar to those of the 2SC case, cf. Fig. 1 of Ref. [8]. There are subtle differences, though, due to appearance
of two kinds of gapped quark excitations, one so-called singlet excitation with a gap $\phi_1$, and eight so-called octet excitations with a gap $\phi_8 \equiv \phi \[2\]$. In weak coupling, the singlet gap is approximately twice as large as the octet gap, $\phi_1 \simeq 2\phi_8 \equiv 2\phi \[3\]$. The one-loop gluon self-energy in the CFL phase has two types of contributions, depending on whether the quarks in the loop correspond to singlet or octet excitations, cf. Eq. (23b) of Ref. \[9\]. For the first type, both quarks in the loop are octet excitations, and for the second, one is an octet and the other a singlet excitation. There is no contribution from singlet-singlet excitations.

Nonvanishing octet-octet excitations require gluon energies to be larger than $2\phi_8 \equiv 2\phi_8$, while octet-singlet excitations require a larger gluon energy, $p_0 \geq \phi_1 + \phi_8 \equiv 3\phi$. This introduces some additional structure in the imaginary parts at $p_0 = 3\phi$, which can be seen particularly well in Figs. 1 (d) and (e).

In Fig. 2 we show the real parts of the gluon self-energy corresponding to the imaginary parts shown in Fig. 1. These are quite similar to the ones in the 2SC phase, cf. Fig. 2 of Ref. \[8\]. An explanation of the various structures can be given following arguments similar to those of Refs. \[7, 8\]. In essence, when computing the real part from a dispersion integral over the imaginary part, cf. Eq. (A15), a change of gradient in the imaginary part leads to a cusp-like structure in the real part. The only function that does not fit this general rule is $\text{Re} \hat{\Pi}^{00}$. The reason is that

\[ p = 4\phi \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The real parts of (a) $\Pi^{00}$, (b) $-\Pi^{00}\hat{p}_i$, (c) $\Pi^I$, (d) $\Pi^I$, and (e) $\hat{\Pi}^{00}$ as a function of energy $p_0$ for a gluon momentum $p = 4\phi$. The solid lines are for the CFL phase, the dotted lines correspond to the HDL self-energy.}
\end{figure}
this function is computed from the real part of the right-hand side of Eq. (4b). Note the singularity at \( p_0 \approx 1.6 \phi \). This singularity is caused by a root of the denominator in Eq. (4b), \( P_\mu \Pi^{\mu\nu}(P) \). As shown in Ref. \[11\] (see also Ref. \[8\] for the 2SC case), this condition determines the dispersion relation of the Goldstone excitations. As one expects, for large energies \( p_0 \gg \phi \) the real parts of the self-energies approach the corresponding HDL limit. Deviations from the HDL limit occur only for gluon energies \( p_0 \approx \phi \).

We now compute the spectral densities from the real and imaginary parts of the gluon self-energies. When \( \text{Im} \hat{\Pi}^{00}(p_0, p) \), \( \text{Im} \Pi^t(p_0, p) \neq 0 \), the spectral densities are regular and given by

\[
\hat{\rho}^{00}(p_0, p) = \frac{1}{\pi} \frac{\text{Im} \hat{\Pi}^{00}(p_0, p)}{[p^2 - \text{Re} \hat{\Pi}^{00}(p_0, p)]^2 + [\text{Im} \hat{\Pi}^{00}(p_0, p)]^2},
\]

\[
\hat{\rho}^t(p_0, p) = \frac{1}{\pi} \frac{\text{Im} \Pi^t(p_0, p)}{[p_0^2 - p^2 - \text{Re} \Pi^t(p_0, p)]^2 + [\text{Im} \Pi^t(p_0, p)]^2}.
\]

If \( \text{Im} \hat{\Pi}^{00}(p_0, p) \) or \( \text{Im} \Pi^t(p_0, p) \) vanish, the corresponding spectral density has a simple pole given by

\[
[p^2 - \text{Re} \hat{\Pi}^{00}(p_0, p)]_{p_0=\omega^{00}(p)} = 0
\]

for longitudinal gluons and

\[
[p_0^2 - p^2 - \text{Re} \Pi^t(p_0, p)]_{p_0=\omega^t(p)} = 0
\]

for transverse gluons.

FIG. 3: The spectral densities for (a) longitudinal and (b) transverse gluons for a gluon momentum \( p = m_g/2, \) with \( m_g = 8 \phi \). The dashed lines correspond to the HDL limit.

In Fig. 3 we show the spectral densities for longitudinal and transverse gluons in the CFL phase in comparison to the HDL limit. Note that there is a delta function-like peak in the transverse spectral density at an energy \( p_0 \approx 0.21 m_g \). This peak corresponds to a collective excitation, the so-called "light plasmon" predicted in Ref. \[12\] (see also \[13\]).
We show the dispersion relation of this collective mode in Fig. 4 (b). The mass \( m_{\text{coll}} \simeq 1.35 \phi \) is roughly in agreement with the value \( m_{\text{coll}} \simeq 1.362 \phi \) of Ref. [12]. As the momentum increases, the energy of the light plasmon excitation approaches \( 2 \phi \) from below. For momenta larger than \( \sim 8 \phi \), the location of this excitation branch becomes numerically indistinguishable from the continuum in the spectral density above \( p_0 = 2 \phi \), cf. Fig. 3. Close inspection reveals that the dispersion relation of the light plasmon has a minimum at a nonzero value of \( p \simeq 1.33 \phi \), indicating a van Hove singularity.

In Fig. 4 we also show the dispersion relations for the “regular” longitudinal and transverse excitations, as well as for the Nambu-Goldstone excitation defined by the root of \( P^\mu \Pi_{\mu\nu} (P) P^\nu = 0 \) [8, 11]. For our choice of gauge the gluon propagator is 4-transverse and this mode does not mix with the longitudinal component of the gauge field [8]. Therefore, the Nambu-Goldstone mode does not appear as a peak in the longitudinal spectral density, cf. Fig. 3. We finally note that other collective excitations have been investigated in Ref. [14].

In conclusion, we have computed the gluon self-energy in the CFL phase as a function of energy and momentum. While the imaginary parts of the gluon self-energy could be expressed analytically in terms of elliptic functions (see appendix), the real parts had to be computed numerically with the help of dispersion integrals. From the real and imaginary parts we constructed the spectral densities. We confirmed the existence of a low-energy collective excitation, the so-called “light plasmon” predicted in Ref. [12].

![dispersion relations](image-url)
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APPENDIX A: CALCULATION OF THE GLUON SELF-ENERGY

In this appendix, we compute the individual components of the gluon self-energy. According to Eq. (31a) of Ref. [13],

\[
\Pi^{00}(P) = -\frac{g^2}{12} \int \frac{d^3k}{(2\pi)^3} \sum_{\epsilon_1, \epsilon_2 = \pm} (1 + \epsilon_1 \epsilon_2 \hat{k}_1 \cdot \hat{k}_2) \times \left[ \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (1 - \hat{N}_1 - N_2) \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 - \hat{\phi}_1 \phi_2}{2 \epsilon_1 \epsilon_2} \right. \\
+ \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (1 - N_1 - \hat{N}_2) \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 - \phi_1 \hat{\phi}_2}{2 \epsilon_1 \epsilon_2} \\
+ 7 \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (1 - N_1 - N_2) \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 - 2 \phi_1 \phi_2/7}{2 \epsilon_1 \epsilon_2} \\
+ \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (\hat{N}_1 - N_2) \frac{\epsilon_1 \epsilon_2 + \xi_1 \xi_2 + \hat{\phi}_1 \phi_2}{2 \epsilon_1 \epsilon_2} \\
+ \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (N_1 - \hat{N}_2) \frac{\epsilon_1 \epsilon_2 + \xi_1 \xi_2 + \phi_1 \hat{\phi}_2}{2 \epsilon_1 \epsilon_2} \\
+ 7 \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (N_1 - N_2) \frac{\epsilon_1 \epsilon_2 + \xi_1 \xi_2 + 2 \phi_1 \phi_2/7}{2 \epsilon_1 \epsilon_2} \right]. \quad (A1a)
\]

Furthermore, from Eq. (31b) of Ref. [13] we obtain

\[
\Pi^\prime(P) = -\frac{g^2}{12} \int \frac{d^3k}{(2\pi)^3} \sum_{\epsilon_1, \epsilon_2 = \pm} (1 - \epsilon_1 \epsilon_2 \hat{k}_1 \cdot \hat{p} \hat{k}_2 \cdot \hat{p}) \times \left[ \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (1 - \hat{N}_1 - N_2) \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 + \hat{\phi}_1 \phi_2}{2 \epsilon_1 \epsilon_2} \right. \\
+ \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (1 - N_1 - \hat{N}_2) \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 + \phi_1 \hat{\phi}_2}{2 \epsilon_1 \epsilon_2} \\
+ 7 \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (1 - N_1 - N_2) \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 + 2 \phi_1 \phi_2/7}{2 \epsilon_1 \epsilon_2} \\
+ \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (\hat{N}_1 - N_2) \frac{\epsilon_1 \epsilon_2 + \xi_1 \xi_2 - \hat{\phi}_1 \phi_2}{2 \epsilon_1 \epsilon_2} \\
+ \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (N_1 - \hat{N}_2) \frac{\epsilon_1 \epsilon_2 + \xi_1 \xi_2 - \phi_1 \hat{\phi}_2}{2 \epsilon_1 \epsilon_2} \\
+ 7 \left( \frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (N_1 - N_2) \frac{\epsilon_1 \epsilon_2 + \xi_1 \xi_2 - 2 \phi_1 \phi_2/7}{2 \epsilon_1 \epsilon_2} \right]. \quad (A1b)
\]
and

\[
\Pi^\ell(P) = -\frac{g^2}{12} \int \frac{d^3k}{(2\pi)^3} \sum_{e_1, e_2 = \pm} (1 - e_1 e_2 \hat{k}_1 \cdot \hat{k}_2 + 2 e_1 e_2 \hat{k}_1 \cdot \hat{p} \hat{k}_2 \cdot \hat{p}) \\
\times \left[ \left( \frac{1}{p_0 + \hat{e}_1 + \hat{e}_2} - \frac{1}{p_0 - \hat{e}_1 - \hat{e}_2} \right) (1 - \hat{N}_1 - \hat{N}_2) \frac{\hat{\xi}_1 e_2 - \hat{\xi}_1 \hat{\xi}_2 + \hat{\phi}_1 \hat{\phi}_2}{2 \hat{e}_1 e_2} \right] \\
+ 7 \left( \frac{1}{p_0 + \hat{e}_1 + \hat{e}_2} - \frac{1}{p_0 - \hat{e}_1 - \hat{e}_2} \right) (1 - \hat{N}_1 - \hat{N}_2) \frac{\hat{\xi}_1 e_2 - \hat{\xi}_1 \hat{\xi}_2 + \hat{\phi}_1 \hat{\phi}_2}{2 \hat{e}_1 e_2} \\
+ 7 \left( \frac{1}{p_0 + \hat{e}_1 + \hat{e}_2} - \frac{1}{p_0 - \hat{e}_1 - \hat{e}_2} \right) (1 - \hat{N}_1 - \hat{N}_2) \frac{\hat{\xi}_1 e_2 - \hat{\xi}_1 \hat{\xi}_2 + \hat{\phi}_1 \hat{\phi}_2}{2 \hat{e}_1 e_2} \right]. \tag{A1c}
\]

The projection \(\Pi^0(P) \hat{p}_i\) was not explicitly given in Ref. \cite{6}. Starting with Eq. (24) of Ref. \cite{6} and following similar steps as for the other components, we find

\[
\Pi^0(P) \hat{p}_i = \frac{g^2}{12} \int \frac{d^3k}{(2\pi)^3} \sum_{e_1, e_2 = \pm} (e_1 \hat{k}_1 \cdot \hat{p} + e_2 \hat{k}_2 \cdot \hat{p}) \\
\times \left[ \left( \frac{1}{p_0 + \hat{e}_1 + \hat{e}_2} + \frac{1}{p_0 - \hat{e}_1 - \hat{e}_2} \right) (1 - \hat{N}_1 - \hat{N}_2) \frac{\hat{\xi}_1 e_2 - \hat{\xi}_1 \hat{\xi}_2}{2 \hat{e}_1 e_2} \right] \\
+ 7 \left( \frac{1}{p_0 + \hat{e}_1 + \hat{e}_2} + \frac{1}{p_0 - \hat{e}_1 - \hat{e}_2} \right) (1 - \hat{N}_1 - \hat{N}_2) \frac{\hat{\xi}_1 e_2 - \hat{\xi}_1 \hat{\xi}_2}{2 \hat{e}_1 e_2} \\
+ 7 \left( \frac{1}{p_0 + \hat{e}_1 + \hat{e}_2} + \frac{1}{p_0 - \hat{e}_1 - \hat{e}_2} \right) (1 - \hat{N}_1 - \hat{N}_2) \frac{\hat{\xi}_1 e_2 - \hat{\xi}_1 \hat{\xi}_2}{2 \hat{e}_1 e_2} \right]. \tag{A1d}
\]

Here \(\xi_i \equiv e_i k_i - \mu, k_1 \equiv \hat{k} \cdot \hat{p}\), where \(\hat{k}\) is the quark three-momentum and \(\hat{p}\) is the gluon three-momentum. The octet and singlet gap functions for quasiparticles \((e_i = +1)\) and quasiparticles \((e_i = -1)\) are \(\phi_i \equiv \phi_i^k\) and \(\hat{\phi}_i \equiv \hat{\phi}_i^k\), respectively, and the corresponding excitation energies are \(\epsilon_i \equiv \sqrt{\xi_i^2 + \phi_i^2}\) and \(\hat{\epsilon}_i \equiv \sqrt{\hat{\xi}_i^2 + \hat{\phi}_i^2}\). The thermal distribution functions are \(N_i \equiv [\exp(\epsilon_i/T) + 1]^{-1}\), and \(\hat{N}_i \equiv [\exp(\hat{\epsilon}_i/T) + 1]^{-1}\), respectively. In the limit \(T \to 0\), the latter vanish, since \(\epsilon_i, \hat{\epsilon}_i > 0\). Then the equations simplify to

\[
\Pi^{00}(P) = -\frac{g^2}{12} \int \frac{d^3k}{(2\pi)^3} \sum_{e_1, e_2 = \pm} (1 + e_1 e_2 \hat{k}_1 \cdot \hat{k}_2) \\
\times \left[ \left( \frac{1}{p_0 + \hat{e}_1 + \hat{e}_2} - \frac{1}{p_0 - \hat{e}_1 - \hat{e}_2} \right) \frac{\hat{\xi}_1 e_2 - \hat{\xi}_1 \hat{\xi}_2 - \hat{\phi}_1 \hat{\phi}_2}{2 \hat{e}_1 e_2} \right] \\
+ 7 \left( \frac{1}{p_0 + \hat{e}_1 + \hat{e}_2} - \frac{1}{p_0 - \hat{e}_1 - \hat{e}_2} \right) \frac{\hat{\xi}_1 e_2 - \hat{\xi}_1 \hat{\xi}_2 - \hat{\phi}_1 \hat{\phi}_2}{2 \hat{e}_1 e_2} \\
+ 7 \left( \frac{1}{p_0 + \hat{e}_1 + \hat{e}_2} - \frac{1}{p_0 - \hat{e}_1 - \hat{e}_2} \right) \frac{\hat{\xi}_1 e_2 - \hat{\xi}_1 \hat{\xi}_2 - 2 \hat{\phi}_1 \hat{\phi}_2/7}{2 \hat{e}_1 e_2} \right]. \tag{A2a}
\]
\[ \Pi'(P) = -\frac{g^2}{12} \int \frac{d^3k}{(2\pi)^3} \sum_{\epsilon_1, \epsilon_2 = \pm} (1 - \epsilon_1 \epsilon_2 \hat{k}_1 \cdot \hat{p} \hat{k}_2 \cdot \hat{p}) \]
\[
\times \left[ \frac{1}{p_0 + \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right] \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 + \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \\
+ \left[ \frac{1}{p_0 + \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right] \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 + \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \\
+ \frac{7}{2 \epsilon_1 \epsilon_2} \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 + 2 \phi_1 \phi_2/7}{2 \epsilon_1 \epsilon_2} \right] , \quad (A2b) \]

\[ \Pi''(P) = -\frac{g^2}{12} \int \frac{d^3k}{(2\pi)^3} \sum_{\epsilon_1, \epsilon_2 = \pm} (1 - \epsilon_1 \epsilon_2 \hat{k}_1 \cdot \hat{k}_2 + 2 \epsilon_1 \epsilon_2 \hat{k}_1 \cdot \hat{p} \hat{k}_2 \cdot \hat{p}) \]
\[
\times \left[ \frac{1}{p_0 + \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right] \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 + \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \\
+ \left[ \frac{1}{p_0 + \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right] \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 + \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \\
+ \frac{7}{2 \epsilon_1 \epsilon_2} \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 + 2 \phi_1 \phi_2/7}{2 \epsilon_1 \epsilon_2} \right] , \quad (A2c) \]

\[ \Pi^{\phi}(\hat{p}_i) = \frac{g^2}{12} \int \frac{d^3k}{(2\pi)^3} \sum_{\epsilon_1, \epsilon_2 = \pm} (\epsilon_1 \hat{k}_1 \cdot \hat{p} + \epsilon_2 \hat{k}_2 \cdot \hat{p}) \]
\[
\times \left[ \frac{1}{p_0 + \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} + \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right] \frac{\epsilon_1 \epsilon_2 - \xi_1 \epsilon_2}{2 \epsilon_1 \epsilon_2} \\
+ \left[ \frac{1}{p_0 + \epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} + \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right] \frac{\epsilon_1 \epsilon_2 - \xi_1 \epsilon_2}{2 \epsilon_1 \epsilon_2} \\
+ \frac{7}{2 \epsilon_1 \epsilon_2} \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2}{2 \epsilon_1 \epsilon_2} \right] . \quad (A2d) \]

In the following, we evaluate the imaginary part of \( \Pi^{\phi} \) explicitly; the calculation of the other components is similar.

1. Imaginary parts

We are interested in the retarded self-energy, so we analytically continue \( p_0 \to p_0 + i\eta \) in Eqs. \( A2 \). Then, using the Dirac identity
\[
\frac{1}{x + i\eta} = \mathcal{P} \frac{1}{x} - i\pi \delta(x) , \quad (A3) \]
where \( \mathcal{P} \) stands for the principal-value prescription, we find
\[
\text{Im} \Pi^{\phi}(P) = \pi \frac{g^2}{12} \int \frac{d^3k}{(2\pi)^3} \sum_{\epsilon_1, \epsilon_2 = \pm} (1 + \epsilon_1 \epsilon_2 \hat{k}_1 \cdot \hat{k}_2) \]
\[
\times \left\{ \delta(p_0 + \epsilon_1 + \epsilon_2) - \delta(p_0 - \epsilon_1 - \epsilon_2) \right\} \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 - \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \\
+ \left\{ \delta(p_0 + \epsilon_1 + \epsilon_2) - \delta(p_0 - \epsilon_1 - \epsilon_2) \right\} \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 - \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \\
+ \left\{ \delta(p_0 + \epsilon_1 + \epsilon_2) - \delta(p_0 - \epsilon_1 - \epsilon_2) \right\} \frac{7(\epsilon_1 \epsilon_2 - \xi_1 \xi_2) - 2\phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \right\} \quad (A4) \]
Since Eq. (A4) is an odd function of \( p_0 \), \( \text{Im} \, \Pi(-p_0, \mathbf{p}) = -\text{Im} \, \Pi(p_0, \mathbf{p}) \), the calculation can be restricted to positive values of the energy \( p_0 \geq 0 \). Hence, the first delta-function in each square bracket in Eq. (A4) can be dropped.

Moreover, we are interested in gluon energies and momenta \( p_0, \mathbf{p} \ll \mu \). Consequently, in order to have a vanishing argument of the remaining delta-functions, the quasiparticle energies should not be too large either, \( \epsilon_i, \hat{\epsilon}_i \ll \mu \). Then, only the terms with \( \epsilon_1 = \epsilon_2 = +1 \) will contribute; for either \( \epsilon_1 = -1, \epsilon_2 = +1 \), \( \epsilon_1 \simeq \hat{\epsilon}_1 \simeq |k_i + \mu| \sim \mu \) is too far from the Fermi surface to make a contribution for \( p_0 \ll \mu \). Shifting the integration variable so that \( \mathbf{k}_1 = \mathbf{k} + \mathbf{p}/2 \) and \( \mathbf{k}_2 = \mathbf{k} - \mathbf{p}/2 \), we find using \( p \ll \mu \)

\[
\mathbf{k}_1 \cdot \mathbf{k}_2 \simeq 1, \quad \xi_{1,2} \simeq \xi \pm \frac{\mathbf{p} \cdot \mathbf{k}}{2} \equiv \xi_{\pm}, \tag{A5}
\]

where we have defined \( \xi \equiv k - \mu \). Furthermore, we denote

\[
\epsilon_{\pm} = \sqrt{\xi_{\pm}^2 + \phi^2}, \quad \hat{\epsilon}_{\pm} = \sqrt{\xi_{\pm}^2 + \hat{\phi}^2}. \tag{A6}
\]

Setting \( \phi_1 \simeq \phi_2 \equiv \phi, \hat{\phi}_1 \simeq \hat{\phi}_2 \equiv \hat{\phi} \simeq 2\phi \) in weak coupling, we obtain

\[
\text{Im} \, \Pi^{00}(P) = -\frac{g^2 \pi}{6} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left[ \frac{\delta(p_0 - \epsilon_+ - \epsilon_-) \hat{\epsilon}_+ \epsilon_- - \xi_+ \xi_- - 2\phi^2}{2\epsilon_+ \epsilon_-} \right. \\
+ \frac{\delta(p_0 - \epsilon_+ - \hat{\epsilon}_-) \epsilon_+ \hat{\epsilon}_- - \xi_+ \xi_- - 2\phi^2}{2\epsilon_+ \hat{\epsilon}_-} \\
\left. + \delta(p_0 - \epsilon_- + \hat{\epsilon}_-) (\epsilon_- + \hat{\epsilon}_-) - 2\phi^2 \right] \tag{A7}
\]

Choosing \( \mathbf{p} = (0, 0, p) \) the integration over the polar angle \( \varphi \) becomes trivial. The integral over \( \xi \) can be performed using the delta-functions. Denoting \( x = \cos \theta \), the roots of the arguments of the delta-functions in Eq. (A7) are (in the order of appearance)

\[
\xi_{1,2}^*(x) = -\frac{3px\phi^2 \pm p_0 \sqrt{(p_x^2 x^2 - p_0^2 + 9\phi^2)(p_x^2 x^2 - p_0^2 + \phi^2)}}{2(p_x^2 x^2 - p_0^2)}, \tag{A8a}
\]

\[
\xi_{3,4}^*(x) = \frac{3px\phi^2 \pm p_0 \sqrt{(p_x^2 x^2 - p_0^2 + 9\phi^2)(p_x^2 x^2 - p_0^2 + \phi^2)}}{2(p_x^2 x^2 - p_0^2)}, \tag{A8b}
\]

\[
\xi_{5,6}^*(x) = \pm \frac{p_0}{2} \sqrt{-1 - \frac{4\phi^2}{p_0^2 - p_x^2 x^2}}. \tag{A8c}
\]

Then,

\[
\text{Im} \, \Pi^{00}(P) = -\frac{\pi m_g^2 c^2}{3} \frac{\phi^2}{p_0} \Theta(p_0 - 3\phi) \int_0^{u_1} dy \left\{ \frac{9}{\sqrt{(y^2 - 1 + 9\Phi^2)(y^2 - 1 + \Phi^2)}} - \frac{10 + 9\Phi^2}{(1 - y^2)^2} \right. \\
\left. \frac{18\Phi^2}{(1 - y^2)^2} + \frac{5 + 9y^2}{(y^2 - 1 + 9\Phi^2)(y^2 - 1 + \Phi^2)} \right\}, \tag{A9}
\]

where \( y = p x/p_0, \Phi \equiv \phi/p_0, u_1 = \min(p/p_0, \sqrt{1 - 9\phi^2/p_0^2}), u_2 = \min(p/p_0, \sqrt{1 - 4\phi^2/p_0^2}) \) and \( m_g \) is the gluon mass parameter (squared), \( m_g^2 = N_f g_s^2 \mu^2 / (6\pi^2) \), for \( N_f = 3 \).

Using the elliptic integrals of first, second, and third kind,

\[
F(\varphi, k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}, \tag{A10a}
\]

\[
E(\varphi, k) = \int_0^{\varphi} d\alpha \sqrt{1 - k^2 \sin^2 \alpha}, \tag{A10b}
\]

\[
\Pi(\varphi, l, k) = \int_0^{\varphi} \frac{d\alpha}{(1 + l \sin^2 \alpha) \sqrt{1 - k^2 \sin^2 \alpha}}, \tag{A10c}
\]
and the complete elliptic integrals of the first, $K(k) = F(\pi/2, k)$, the second kind, $E(k) = E(\pi/2, k)$ and the third kind $\Pi(l, r) = \Pi(\pi/2, l, r)$ in Eq. (A10), we obtain the final result

\[
\begin{align*}
\text{Im} \Pi^{00}(P) &= \frac{\pi m^2 p_0}{p} \left( \Theta(p_0 - 3\phi) \left( \frac{s^2}{\sqrt{4 - s^2}} \left( \Theta(E^{18}_p - p_0) \left[ 9K(t') - \left( 10 + \frac{9}{4} s^2 \right) \Pi(l, t') \right] \right. \right. \\
&- \frac{9}{2} t' \left. \sqrt{\frac{1}{dn}} \Pi(l, t') \right|_{n=1} \left. \right) + \Theta(p_0 - E^{18}_p) \left[ 9F(\alpha', t') - \left( 10 + \frac{9}{4} s^2 \right) \Pi(\alpha', l, t') \right] \\
&- \frac{9}{2} t' \sqrt{\frac{1}{dn}} \Pi(\alpha', l, t') \left. \right|_{n=1} \right) - \Theta(p_0 - 2\phi) \left( \Theta(E^{88}_p - p_0) \left[ 7E(t) - \frac{9}{2} s^2 K(t) \right] \\
&+ \Theta(p_0 - E^{88}_p) \left[ 7E(\alpha, t) - \frac{9}{2} s^2 F(\alpha, t) - \frac{p}{p_0} \sqrt{1 - \frac{4\phi^2}{p_0^2 - p^2}} \right] \right) \right)
\end{align*}
\]

where $t' = \sqrt{(p_0^2 - 9\phi^2)/(p_0^2 - \phi^2)}$, $t = \sqrt{1 - 4\phi^2/p_0^2}$, $\alpha' = \arcsin[p/\sqrt{p_0^2 - 9\phi^2}]$, $\alpha = \arcsin[p/(tp_0)]$, $l = -1 + 9\phi^2/p_0^2$ and $s = 2\phi/p_0$.

The imaginary parts of the other components of the gluon self-energy can be obtained analogously from Eqs. (A2a), (A2b), and (A2c). In addition to Eq. (A5) we employ

\[
\begin{align*}
\mathbf{k}_1 \cdot \mathbf{p} \mathbf{k}_2 \cdot \mathbf{p} &\simeq (\mathbf{k} \cdot \mathbf{p})^2, \quad \mathbf{k}_1 \cdot \mathbf{p} + \mathbf{k}_2 \cdot \mathbf{p} \simeq 2 \mathbf{k} \cdot \mathbf{p}.
\end{align*}
\]

and find

\[
\begin{align*}
\text{Im} \Pi^t(P) &= \frac{\pi m^2 p_0}{12p} \left[ \frac{s^2}{\sqrt{4 - s^2}} \Theta(p_0 - 3\phi) \left( \Theta(E^{18}_p - p_0) \left[ \frac{p_0^2}{p^2} \left( 1 - \frac{s^2}{4} \right) E(t') + \left( 1 - \frac{p_0^2}{p^2} \right) (11 + 2s^2) \right] \\
&\times K(t') - \left( 10\left( 1 - \frac{p_0^2}{p^2} \right) + \frac{9}{2} \left( 1 - \frac{p_0^2}{p^2} \right) \right) \Pi(l, t') - \frac{9}{2} \left( 1 - \frac{p_0^2}{p^2} \right) \frac{d}{dn} \Pi(l', t') \left. \right|_{n=1} \right) \\
&+ \Theta(p_0 - E^{18}_p) \left( \Theta(E^{88}_p - p_0) \left[ \left( 5p_0^2/p^2 \right) s^2 - 7 \left( 1 - \frac{p_0^2}{p^2} \right) E(t) + \frac{s^2}{2} \left( 5 - 19\frac{p_0^2}{p^2} \right) K(t) \right] + \Theta(p_0 - E^{88}_p) \\
&\times \left( \left[ \frac{5p_0^2}{2p^2} \right] s^2 - 7 \left( 1 - \frac{p_0^2}{p^2} \right) E(\alpha, t) + \frac{s^2}{2} \left( 5 - 19\frac{p_0^2}{p^2} \right) F(\alpha, t) + \frac{7p}{p_0} \left( 1 - \frac{p_0^2}{p^2} \right) \sqrt{1 - \frac{4\phi^2}{p_0^2 - p^2}} \right) \right) \right) \right)
\end{align*}
\]

\[
\begin{align*}
\text{Im} \Pi^t(P) &= -\frac{\pi m^2 p_0}{6p^3} \left( \frac{s^2}{\sqrt{4 - s^2}} \Theta(p_0 - 3\phi) \left( \Theta(E^{18}_p - p_0) \left[ \left( 1 - \frac{s^2}{4} \right) E(t') + \left( 10 + \frac{27}{4} s^2 \right) \Pi(l, t') \right] \\
&- \left( 11 + 2s^2 \right) K(t') + \frac{9}{2} s^2 \frac{d}{dn} \Pi(l, t') \left. \right|_{n=1} \right) + \Theta(p_0 - E^{18}_p) \left( \left( 1 - \frac{s^2}{4} \right) E(\alpha', t') \\
&+ \left( 10 + \frac{27}{4} s^2 \right) \Pi(\alpha', l, t') - \left( 11 + 2s^2 \right) F(\alpha', t') + \frac{9}{2} s^2 \frac{d}{dn} \Pi(\alpha', l, t') \left. \right|_{n=1} \right) \right) \\
&+ \Theta(p_0 - 2\phi) \left( \Theta(E^{88}_p - p_0) \left[ \left( \frac{5}{2} s^2 \right) E(t) - \frac{19}{2} s^2 K(t) \left. \right) \right) + \Theta(p_0 - E^{88}_p) \left[ \left( 7 + \frac{5}{2} s^2 \right) E(\alpha, t) - \frac{19}{2} s^2 F(\alpha, t) - \frac{7p}{p_0} \sqrt{1 - \frac{4\phi^2}{p_0^2 - p^2}} \right) \right) \right) \right),
\end{align*}
\]
\[ \text{Im} \Pi^{0i}(P) \hat{\phi}_i = -\frac{\pi m_g^2}{6} \frac{1}{p_0^2} \left( \frac{2s^2}{\sqrt{4-s^2}} \Theta(p_0 - 3\phi) \left\{ \Theta(E_p^{18} - p_0) \left[ 5K(t') - (5 + \frac{9}{4}s^2)\Pi(l, t') \right] \right\} - \frac{9}{4}s^2 \frac{d}{dn} \left\{ \Pi(l, t') \right\}_{n=1} \right) + \Theta(p_0 - E_p^{18}) \left[ 5F(\alpha', t') - (5 + \frac{9}{4}s^2)\Pi(\alpha', l, t') \right] \] 
\[ \quad - \frac{9}{4}s^2 \frac{d}{dn} \left\{ \Pi(\alpha', l, t') \right\}_{n=1} \right) - 7\Theta(p_0 - 2\phi) \left\{ \Theta(E_p^{88} - p_0) \left[ E(t) - s^2K(t) \right] \right\} \]. \quad (A13c)

In the limit \( \phi \to 0 \), we reproduce the HDL limit,

\[ \lim_{\phi \to 0} \text{Im} \Pi^{0i}(P) \equiv \text{Im} \Pi_{0i}^{0i}(P) , \quad \text{(A14a)} \]
\[ \lim_{\phi \to 0} \text{Im} \Pi^t(P) \equiv \text{Im} \Pi_{0i}^0(P) , \quad \text{(A14b)} \]
\[ \lim_{\phi \to 0} \text{Im} \Pi^t(P) \equiv \text{Im} \Pi_{0i}^0(P) , \quad \text{(A14c)} \]
\[ \lim_{\phi \to 0} \text{Im} \Pi^{0i}(P) \hat{\phi}_i \equiv \text{Im} \Pi_{0i}^{0i}(P) \hat{\phi}_i . \quad \text{(A14d)} \]

2. Real parts

There are two possibilities to compute the real parts of the gluon self-energy. Either, one evaluates a principal-value integral, cf. Eq. (A3), or one employs the dispersion integral

\[ \text{Re} \Pi(p_0, p) = \frac{\pi}{P} \int_{-\infty}^{\infty} d\omega \text{Im} \Pi(\omega, p) \frac{1}{\omega - p_0} + C , \quad (A15) \]

where \( C \) is a (subtraction) constant. If \( \text{Im} \Pi(\omega, p) \) is an odd function of \( \omega \), \( \text{Im} \Pi(-\omega, p) \equiv -\text{Im} \Pi(\omega, p) \) the dispersion integral becomes

\[ \text{Re} \Pi(p_0, p) = \frac{\pi}{P} \int_0^{\infty} d\omega \text{Im} \Pi_{\text{odd}}(\omega, p) \left( \frac{1}{\omega + p_0} - \frac{1}{\omega - p_0} \right) + C , \quad (A16) \]

and if it is an even function of \( \omega \), \( \text{Im} \Pi(-\omega, p) \equiv +\text{Im} \Pi(\omega, p) \) we have

\[ \text{Re} \Pi(p_0, p) = \frac{\pi}{P} \int_0^{\infty} d\omega \text{Im} \Pi_{\text{even}}(\omega, p) \left( \frac{1}{\omega - p_0} - \frac{1}{\omega + p_0} \right) + C , \quad (A17) \]

where in both cases \( \Pi(p_0, p) \) is assumed to be analytic in the upper complex \( p_0 \) plane.

The values of the constants \( C^{00}, C^t, C^{0t} \), and \( C^{0i} \) are determined by the large-\( p_0 \) dependence of the self-energy. Thus, it does not matter which color-superconducting phase we consider, and the constants assume the same values as for the 2SC phase, \( C^{00} = C^{0i} = 0, C^t = C^{0t} = m_g^2 \), cf. Ref. [5].

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