A class of bicovariant differential calculi on Hopf algebras

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ABSTRACT We introduce a large class of bicovariant differential calculi on any quantum group $A$, associated to $Ad$-invariant elements. For example, the deformed trace element on $SL_q(2)$ recovers Woronowicz’ $4D_\pm$ calculus. More generally, we obtain a sequence of differential calculi on each quantum group $A(R)$, based on the theory of the corresponding braided groups $B(R)$. Here $R$ is any regular solution of the QYBE.

1 Introduction

The differential geometry of quantum groups was introduced by S.L. Woronowicz in [16] for $SU_q(2)$ and then formulated for any compact matrix quantum group in [17]. This theory is based on the idea that the differential and (co)algebraic structures of $A$ should be compatible in the sense that $A$ can coact covariantly on the algebra of its differential forms. This idea leads to the notions of right-, left- and bi-covariant differential calculi over $A$. The last of these fully respects the (co)algebraic structure of $A$ and therefore seems to be the most natural one. By now, the importance of this notion in physics has become clear and several examples are known, constructed by hand. In the present paper our goal is to show how to obtain such calculi more systematically.

First, let us note that the bicovariance condition, however restrictive, does not determine the differential structure of $A$ uniquely so that there is no preferred such structure. Some attempts have been undertaken [12] [11] [13] [14] in order to decrease the number of allowed differential calculi over $A$ by means of some additional requirements, e.g. that the differential structure of $A$ should be obtained from the differential structure on a quantum space on which $A$ acts covariantly. Such considerations can help us characterise uniquely and hence construct the desired differential calculus. So far, this kind of approach has proven successful in obtaining examples for $GL_q(N)$ and some other matrix quantum groups.

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There is another method of construction of bicovariant differential calculi due to Jurco\cite{Jurco}, which works for the standard quantum groups associated to simple Lie algebras. It is based on the observation that the space of left-invariant vector fields should correspond to the quantum universal enveloping algebra \( U_q(g) \) dual to \( A \). The correspondence is not quite direct. As noted in \cite{Jurco} left invariant vector fields are induced by certain linear functionals on \( A \) vanishing on an ideal (which we will denote by \( M \)) defined by the differential calculus. From these are obtained related linear functionals that can then be used to determine commutation relations between 1-forms and elements of \( A \). The latter functionals can be built up from the functionals \( l_\pm \) in the FRT description of \( U_q(g) \). Therefore, reversing these arguments one can use \( l_\pm \) to construct vector fields and to obtain the relevant differential calculus (cf. \cite{Jurco}). This method gives a special class of bicovariant differential calculi on the standard quantum groups. See also the preprint \cite{Jurco} for \( SL_q(N) \), received after the present work was completed.

By contrast to such specific methods, we introduce in the present paper a general construction for bicovariant differential calculi based in the abstract theory of Hopf algebras. In our construction we associate a bicovariant differential calculus to every element \( \alpha \) of the Hopf algebra that is invariant under the adjoint coaction. For example, the calculi described in \cite{Jurco} are related to the invariant quantum trace of a quantum matrix. It turns out that all previously-known examples of bicovariant differential calculi can be similarly obtained by our general construction. On the other hand, our construction is not limited either to matrix quantum groups or to the standard quantum groups associated to simple Lie algebras. Moreover, because the construction is not tied to specific applications, we obtain more novel and unexpected calculi even for the standard quantum groups. This is somewhat analogous to the commutative case, where the axioms of a differential structure admit exotic solutions in addition to the standard differential structures, as known even for \( \mathbb{R}^4 \). Several examples are collected at the end of Section 3 after proving our main theorem. A novel feature of our approach is that the differential calculi in our construction form a kind of algebra given by addition and multiplication of the underlying \( Ad \)-invariant elements.

We use the following notation: \( A \) is a Hopf algebra over the field \( k \) (such as \( \mathbb{R} \) or \( \mathbb{C} \)), with coproduct \( \Delta : A \to A \otimes A \), counit \( \epsilon : A \to k \) and antipode \( S : A \to A \).

2 Differential structures on quantum groups

We recall first some basic notions about differential calculus on quantum groups, according to the general theory formulated in \cite{Jurco} (cf. \cite{Jurco}). The section then introduces more technical facts needed for our construction. To begin with, one says that \( (\Gamma, d) \) is a first order differential calculus over a Hopf algebra \((A, \Delta, S, \epsilon)\) if \( d : A \to \Gamma \) is a linear map obeying the Leibniz rule, \( \Gamma \) is a bimodule over \( A \) and every
element of $\Gamma$ is of the form $\sum_k a_k db_k$, where $a_k, b_k \in A$. It is known that every differential calculus on an algebra $A$ can be obtained as a quotient of a universal one $(A^2, D)$, where $A^2 = \ker \mu (\mu : A \otimes A \rightarrow A$ is the multiplication map in $A)$ and $D : A \rightarrow A^2$ is defined by

$$Da = a \otimes 1 - 1 \otimes a.$$  \hspace{1cm} (1)

The map $D$ obeys Leibniz rule provided $A^2$ has an $A$ bimodule structure given by

$$c(\sum_k a_k \otimes b_k) = \sum_k ca_k \otimes b_k, \quad (\sum_k a_k \otimes b_k)c = \sum_k a_k \otimes b_k c$$ \hspace{1cm} (2)

for any $\sum_k a_k \otimes b_k \in A^2$, $c \in A$. Furthermore every element of $A^2$ can be represented in the form $\sum_k a_k Db_k$. Indeed, let $\rho = \sum_k a_k \otimes b_k \in A^2$. Then

$$\rho = -\sum_k a_k (b_k \otimes 1 - 1 \otimes b_k) + \sum_k a_k b_k \otimes 1 = -\sum_k a_k Db_k$$

Hence $(A^2, D)$ is a first order differential calculus over $A$ as stated. Any first order differential calculus on $A$ can be realized as $(\Gamma = A^2/N, d = \pi D)$ where $N \subset A^2$ is a sub-bimodule of $A^2$ and $\pi : A^2 \rightarrow A^2/N$ is a canonical epimorphism.

In the analysis of differential structures over quantum groups (Hopf algebras) a crucial rôle is played by the covariant differential calculi.

**Definition 2.1** Let $(\Gamma, d)$ be a first order differential calculus over a Hopf algebra $A$. We say that $(\Gamma, d)$ is:

1. left-covariant if for any $a_k, b_k \in A$,

   $$\left( \sum_k a_k db_k = 0 \right) \Rightarrow \left( \sum_k \Delta(a_k)(id \otimes d)\Delta(b_k) = 0 \right).$$

2. right-covariant if for any $a_k, b_k \in A$,

   $$\left( \sum_k a_k db_k = 0 \right) \Rightarrow \left( \sum_k \Delta(a_k)(d \otimes id)\Delta(b_k) = 0 \right).$$

3. bicovariant if it is left- and right-covariant.

The notion of a covariant differential calculus leads to:

**Definition 2.2** Let $A$ be a Hopf algebra and $\Gamma$ be an $A$-bimodule as well as a left $A$-comodule (right $A$-comodule) with coaction $\Delta_L$ ($\Delta_R$ resp.) Then we say that $\Gamma$ is a left-covariant (right-covariant) $A$-bimodule of if:

$$\Delta_L(R)(a\rho) = (\Delta a)\Delta_L(R)\rho, \quad \Delta_L(R)(\rho a) = (\Delta_L(R)\rho)\Delta a$$
for any \( a \in A \) and \( \rho \in \Gamma \). It is bi-covariant if both conditions hold.

Let \( \Gamma_1, \Gamma_2 \) be two bi-covariant \( A \)-bimodules with coactions \( \Delta_{1L(R)} \) and \( \Delta_{2L(R)} \) respectively. We say that the linear map \( \phi : \Gamma_1 \to \Gamma_2 \) is a bi-covariant bimodule map if:

\[
(id \otimes \phi) \Delta_{1L} = \Delta_{2L} \phi, \quad (\phi \otimes id) \Delta_{1R} = \Delta_{2R} \phi
\]

Left-covariance (resp. right-covariance) of a first order differential calculus \((\Gamma, d)\) implies that \( \Gamma \) is a left-covariant (resp. right-covariant) \( A \)-bimodule and that \( d \) is a bi-covariant bimodule map, i.e. if \( \Delta_L : \Gamma \to A \otimes \Gamma \), (resp. \( \Delta_R : \Gamma \to \Gamma \otimes A \)) is a left coaction (resp. right coaction) then

\[
\Delta_L d = (id \otimes d) \Delta, \quad (\Delta_R d = (d \otimes id) \Delta)
\]  

If \((\Gamma, d)\) is a bi-covariant differential calculus over \( A \) then \( \Gamma \) is a bi-covariant \( A \)-bimodule. Bicovariance of \((\Gamma, d)\) allows one to define a wedge product of forms, construct the exterior \((\mathbb{Z}_2\text{-graded})\) algebra \( \Omega(A) \) (with \( \Omega^1(A) = \Gamma \)) and extend the differential \( d \) uniquely to the whole of \( \Omega(A) \). Here \( d \) obeys the graded Leibniz rule, \( d^2 = 0 \) and equations \((\text{\textcolor{red}{3}})\) are obeyed also for the extensions of coactions to the whole of \( \Omega(A) \).

The exterior product in \( \Omega(A) \) is defined by the following construction. We say that an element \( \omega \in \Omega^1(A) \) is left-invariant (right-invariant) if \( \Delta_L(\omega) = 1 \otimes \omega \) (\( \Delta_R(\omega) = \omega \otimes 1 \)). A one-form \( \omega \) is bi-invariant if it is left- and right-invariant. Given a basis \( \{\omega_i\} \) of the space of all left-invariant 1-forms any element \( \rho \in \Omega^1(A) \) can be represented uniquely as \( \rho = \sum a_i \omega_i \), where \( a_i \in A \). Similarly any element \( \rho \) of \( \Omega^1(A) \otimes \Omega^1(A) \) can be represented as \( \rho = \sum a_{ij} \omega_i \otimes \eta_j \), where \( a_{ij} \in A \), \( \omega_i \) are left-invariant and \( \eta_j \) are right-invariant. Now define \( \Omega^2(A) = \Omega^1(A) \otimes \Omega^1(A)/\ker(id - \sigma) \), where \( \sigma : \Omega^1(A) \otimes \Omega^1(A) \to \Omega^1(A) \otimes \Omega^1(A) \), is such that \( \sigma : \omega_1 \otimes \omega_2 \mapsto \omega_2 \otimes \omega_1 \) for any left-invariant \( \omega_1 \) and right-invariant \( \omega_2 \). This definition extends to \( \Omega^n(A) \) for any positive \( n \). The exterior differentiation \( d \) is defined with the help of a bi-invariant one-form \( \theta \) in such a way that

\[
d \rho = \theta \wedge \rho - (-1)^{\partial \rho} \rho \wedge \theta
\]  

for any homogeneous \( \rho \in \Omega(A) \) of degree \( \partial \rho \).

Bicovariance of the differential calculus \((\Omega^1(A), d)\) means not only that \( \Omega^1(A) \) is a bi-covariant \( A \)-bimodule and that higher modules \( \Omega^n(A) \) can be defined but also that \( \Omega(A) \) is an exterior bialgebra (see \([\text{\textcolor{red}{3}}]\)), i.e. \( \Omega(A) \) is a \( \mathbb{Z}_2 \)-graded or super bialgebra and the graded-coproduct \( \Delta \) in \( \Omega(A) \) is compatible with \( d \). The tensor product \( \Omega(A) \otimes \Omega(A) \) is graded in a natural way according to

\[
\Omega(A) \otimes \Omega(A) = \bigoplus_{n=0}^{\infty} \bigoplus_{k=0}^{n} \Omega^k(A) \otimes \Omega^{n-k}(A).
\]
The action of $d$ on this tensor product is defined by means of the graded Leibniz rule. The graded-
coproduct $\Delta$ on $\Omega^1(A)$ is given by

$$\Delta = \Delta_R + \Delta_L$$

(5)

and then extended to the whole of exterior algebra by

$$\Delta(a_0 da_1 \cdots da_n) = \Delta(a_0)\Delta(da_1)\cdots \Delta(da_n).$$

The universal calculus $(A^2, D)$ is probably the most important example of a bicovariant differential
calculus over a general Hopf algebra $A$. The exterior algebra related to it will be denoted by $\Lambda A$
This $\Lambda A$ is a quotient of the unital differential envelope of $A$, $\Omega A$ (cf \[5\], \[8\]) by the kernel of the symmetrizer
$id - \sigma$. The comodule actions $\Delta^U_L$, $\Delta^U_R$ of $A$ on $A^2$ are given by

$$\Delta^U_L(\sum a_k \otimes b_k) = \sum a_k(1) b_k(1) \otimes a_k(2) \otimes b_k(2)$$

$$\Delta^U_R(\sum a_k \otimes b_k) = \sum a_k(1) \otimes b_k(1) \otimes a_k(2) b_k(2)$$

Here we have used standard notation \[15\].

Let $(\Gamma, d)$ be a bicovariant differential calculus over a Hopf algebra $A$ and let $N \subset A^2$ be such that
$\Gamma = A^2/N$. The maps $\Delta^U_L$, $\Delta^U_R$ defined above allow one to describe $N$ in terms of a right ideal $M \subset \ker \epsilon$.
The latter can be defined by the relation

$$A \otimes M = (id \otimes \epsilon \otimes id)\Delta^U_L N.$$  

This is equivalent to

$$M \otimes A = (\epsilon \otimes id \otimes id)\Delta^U_R N$$

because the differential calculus is bicovariant. The ideal $M$ has an additional property, namely it is an
$Ad_R$-invariant subspace of $A$. To be more precise let us recall the notion of the right adjoint coaction of
a Hopf algebra $A$ on itself. This is the linear map $Ad_R : A \rightarrow A \otimes A$ given by

$$Ad_R(a) = \sum a_{(2)} \otimes (S a_{(1)}) a_{(3)},$$

for any $a \in A$ (here $(id \otimes \Delta)\Delta a = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$). We say that an element $\alpha \in A$ is $Ad_R$-invariant if:

$$Ad_R(\alpha) = \alpha \otimes 1.$$  

Similarly we say that a subspace $B \subset A$ is $Ad_R$-invariant if $Ad_R(B) \subset B \otimes A$. In our case the ideal $M$
is such that

$$Ad_R(M) \subset M \otimes A.$$
We note that the map $(id \otimes \epsilon \otimes id)\Delta^U_R$ is an isomorphism of $A$-bimodules. This implies that $M$ defines $N$ uniquely, hence $M$ and $N$ can be treated equivalently with respect to the calculus they define.

In the case of the universal calculus we define a map $\omega_U : A \to A^2$ by

$$\omega_U(a) = \sum S a_{(1)} Da_{(2)} = \epsilon(a) 1 \otimes 1 - \sum S a_{(1)} \otimes a_{(2)}.$$  

This map, being the quantum group equivalent of the Maurer-Cartan form, will play an important role in the construction of bicovariant differential calculi on $A$. It is easy to check that $\omega_U(a)$ is a left-invariant 1-form for any $a \in A$ and that $\omega_U$ intertwines $\Delta^U_R$ and $Ad_R$.

We begin by showing how to use $\omega_U$ to reconstruct the universal differential $D$ on $A$. Let us recall (cf. [15]) that an element $\Lambda \in A$ is called a right-integral in $A$ if $\Lambda a = \epsilon(a) \Lambda$ for any $a \in A$. We have the following:

**Proposition 2.3** Let $A$ be a Hopf algebra and $\Lambda \in A$ a right integral in $A$ such that $\epsilon(\Lambda) \neq 0$. Put $\hat{\theta} = -\frac{1}{\epsilon(\Lambda)} \omega_U(\Lambda)$. Then

$$Da = \hat{\theta}a - a\hat{\theta}$$  

for any $a \in A$.

**Proof.** We compute the left hand side of (7) directly,

$$\hat{\theta}a - a\hat{\theta} = -\frac{1}{\epsilon(\Lambda)}(\epsilon(\Lambda)1 \otimes a - a \otimes 1 \epsilon(\Lambda) + (SL_{(1)} \otimes L_{(2)}a - a(SL_{(1)} \otimes L_{(2)}))$$

$$= Da + \frac{1}{\epsilon(\Lambda)}(a(SL_{(1)} \otimes L_{(2)} - a(0)(SL_{(1)})(SL_{(1)} \otimes L_{(2)}a_{(2)}))$$

$$= Da + \epsilon(\Lambda)^{-1}(a(SL_{(1)} \otimes L_{(2)} - a(0)S(L_{(1)}a_{(1)}) \otimes (L_{(2)}))$$

$$= Da + \epsilon(\Lambda)^{-1}(a(SL_{(1)} \otimes L_{(2)} - a(SL_{(1)} \otimes L_{(2)}))$$

$$= Da$$

as required. $\Box$

We notice here that the 1-form $\hat{\theta}$ of Proposition 2.3 is not necessarily bi-invariant. This implies further that $\hat{\theta}$ cannot necessarily be extended to a universal exterior derivative $d$ via equation (5) above. However, we see that a sufficient condition for $\hat{\theta}$ to be bi-invariant is that $\Lambda$ is $Ad_R$-invariant and that in this case we will obtain a bicovariant calculus and an exterior algebra. This observation is a leading idea behind our general construction in the next section.
3 Construction of bicovariant differential calculi on quantum groups

As explained in the previous section, bicovariance of the first order differential calculus $(\Omega^1(A), d)$ implies that $\Omega^1(A)$ is a bicovariant bimodule over $A$ and that higher modules $\Omega^n(A)$ can be defined. The problem remains how to construct examples of such $(\Omega^1(A), d)$. We do this now by extending our reconstruction of the universal calculus in Proposition 2.3 to the case of a calculus induced by an arbitrary $Ad_R$-invariant element.

The main result of the section is:

**Theorem 3.1** Let $A$ be a Hopf algebra and let $\alpha \in A$ ($\alpha \neq 0, 1$) be an $Ad_R$-invariant element of $A$. Then there exists bicovariant differential calculus $(\Gamma_\alpha, d)$ such that

$$da = \lambda^{-1}((\pi_\alpha \circ \omega_U(\alpha))a - a(\pi_\alpha \circ \omega_U(\alpha)))$$

(8)

for any $a \in A$, where $\pi_\alpha : A^2 \to \Gamma_\alpha$ is a natural projection and $\lambda \in k^*$. 

**Proof.** We consider the sub-bimodule $N_\alpha = \{bn_\alpha(a)c : a, b, c \in A\}$ of $A^2$ where

$$n_\alpha(a) = (\epsilon(\alpha) + \lambda)(a \otimes 1 - 1 \otimes a) + \sum S\alpha(1) \otimes \alpha(2)a - \sum aS\alpha(1) \otimes \alpha(2),$$

and define

$$\Gamma_\alpha \equiv A^2/N_\alpha.$$ 

We have to prove that $\Gamma_\alpha$ is a bicovariant bimodule of $A^2$ and that $d = \pi_\alpha \circ D$ is given by the equation (8). To prove the former statement we need to show that

$$\Delta_L^U N_\alpha \subset A \otimes N_\alpha, \quad \Delta_R^U N_\alpha \subset N_\alpha \otimes A$$

(9)

i.e. that $N_\alpha$ is both left- and right-invariant. It is enough to show this for elements $n_\alpha(a) \in N_\alpha$. We have

$$\Delta_L^U n_\alpha(a) = (\epsilon(\alpha) + \lambda) \sum (a(1) \otimes a(2) \otimes 1 - a(1) \otimes 1 \otimes a(2))$$

$$+ \sum ((S\alpha(2))\alpha(3)a(1) \otimes S\alpha(1) \otimes \alpha(4)a(2) - a(1) ((S\alpha(2))\alpha(3) \otimes a(2)S\alpha(1) \otimes \alpha(4)))$$

$$= \sum a(1) \otimes ((\epsilon(\alpha) + \lambda)(a(2) \otimes 1 - 1 \otimes a(2))$$

$$+ S\alpha(1) \otimes \alpha(2)a(2) - a(2)S\alpha(1) \otimes \alpha(2))$$

$$= \sum a(1) \otimes n_\alpha(a(2)) \in A \otimes N_\alpha.$$ 

To prove the second part of (8) we will need the following equality

$$S\alpha(2) \otimes \alpha(3) \otimes (S\alpha(1))\alpha(4) = S\alpha(1) \otimes \alpha(2) \otimes 1$$

(10)
which holds for any $Ad_R$-invariant element $\alpha \in A$. Indeed,
\[
(S \otimes id \otimes id)(\Delta \otimes id)Ad_R(\alpha) = \sum (S \otimes id \otimes id)(\Delta \otimes id)(\alpha(2) \otimes (S\alpha(1))\alpha(3))
\]
\[
= \sum (S \otimes id \otimes id)(\alpha(2) \otimes \alpha(3) \otimes (S\alpha(1))\alpha(4))
\]
\[
= S\alpha(2) \otimes \alpha(3) \otimes (S\alpha(1))\alpha(4).
\]
On the other hand, $\alpha$ is $Ad_R$-invariant, hence
\[
(S \otimes id \otimes id)(\Delta \otimes id)Ad_R(\alpha) = (S \otimes id \otimes id)(\Delta \otimes id)(\alpha \otimes 1) = \sum S\alpha(1) \otimes \alpha(2) \otimes 1.
\]
Now we have
\[
\Delta_R^U n_{\alpha}(a) = \sum ((\epsilon(\alpha) + \lambda)(a(1) \otimes 1 \otimes a(2)) - 1 \otimes a(1) \otimes a(2))
\]
\[
+ S\alpha(2) \otimes \alpha(3)a(1) \otimes (S\alpha(1))\alpha(4)a(2) - a(1)S\alpha(2) \otimes \alpha(3) \otimes a(2)(S\alpha(1))\alpha(4)
\]
\[
= \sum ((\epsilon(\alpha) + \lambda)(a(1) \otimes 1 - 1 \otimes a(1) \otimes a(2))
\]
\[
+ S\alpha(1) \otimes \alpha(2)a(1) \otimes a(2) - a(1)S\alpha(1) \otimes \alpha(2) \otimes a(2)
\]
\[
= \sum n_{\alpha}(a(1)) \otimes a(2) \in N_{\alpha} \otimes A.
\]
Hence we have proved that $(\Gamma_\alpha, d = \pi_\alpha \circ D)$ is a first order bicovariant differential calculus as stated.

To prove (8) it is enough to observe that
\[
n_{\alpha}(a) = \lambda Da - \omega_U(\alpha)a + a\omega_U(\alpha) \tag{11}
\]
Now applying $\pi_\alpha$ to the both sides of (11) we obtain (8). \qed

In this way we can assign a bicovariant differential calculus $(\Gamma_\alpha, d)$ on $A$ to any $Ad_R$-invariant $\alpha \in A$. The differential calculus $(\Gamma_\alpha, d)$ is universal in the following sense. Every bicovariant sub-bimodule $N \subset A^2$ such that $N_{\alpha} \subset N$ defines a bicovariant differential calculus $\Gamma = A^2/N$ such that $d = \pi D$ is given by (8) (with $\pi_\alpha$ replaced by $\pi$). In other words $N_{\alpha}$ is the smallest sub-bimodule of $A^2$ leading to the bicovariant differential calculus with a derivative $d$ given by (8).

Applying the map $(id \otimes \epsilon \otimes id)\Delta_L^U$ to $N_{\alpha}$ we see that the right ideal $M_{\alpha} \in \ker \epsilon$ corresponding to $N_{\alpha}$ is of the form
\[
M_{\alpha} = \{ r_{\alpha}(a)b; a, b \in A \}
\]
where
\[
r_{\alpha}(a) = (\lambda + \epsilon(\alpha) - \alpha)(\epsilon(a) - a).
\]
Using properties of the counit and the fact that $\alpha$ is an $Ad_R$-invariant element of $A$ one can easily show that $M_{\alpha}$ is $Ad_R$-invariant. Using that $(id \otimes \epsilon \otimes id)\Delta_L^U$ is an isomorphism of $A$-bimodules one can then
obtain an alternative proof of Theorem 3.1. From this point of view Theorem 3.1 can be thought of as a corollary of Theorem 1.8 of [17].

The universality of \((\Gamma_\alpha, d)\) stated above takes the following form in terms of \(M_\alpha\): If \(M \subset \ker \epsilon\) is an \(Ad_R\)-invariant ideal of \(A\) such that \(M_\alpha \subset M\), then \(M\) induces a bicovariant differential calculus with \(d\) given by (8).

The remainder of the section is devoted to developing several examples.

**Example 3.2** 4\(D_\pm\) calculi on \(SL_q(2)\). Let \(A = SL_q(2)\), generated by \(2 \times 2\) matrix \(t = (t_{ij})_{i,j=1}^2\). We use conventions such that \(t_{11}t_{12} = qt_{12}t_{11}\) etc. As is well-known the \(q\)-deformed trace of \(t\),

\[
\text{tr}_q t = t_{11}^1 + q^{-2}t_{22}^2
\]

is an \(Ad_R\)-invariant element of \(A\). Put \(\alpha = \text{tr}_q t\). The right ideal \(M_\alpha\) is generated by the four elements

\[
\begin{align*}
    r_{i,j} &= (\lambda + 1 + q^{-2} - \alpha)(\delta_{i,j} - t_{i,j}),
\end{align*}
\]

If we set \(\lambda = \lambda_\pm\) where

\[
\begin{align*}
    \lambda_+ &= (1 - q)(q^{-3} - 1), \\
    \lambda_- &= -(1 + q)(q^{-3} + 1)
\end{align*}
\]

then we can extend \(M_\alpha\) to the ideals \(M_\pm\) which are generated by \(r_{i,j}\) and the five additional elements

\[
\begin{align*}
    (t_{12}^2), & \quad t_{12}^i(t_{22}^2 - t_{11}^1), & \quad (t_{21}^2)^2, & \quad t_{21}^i(t_{12}^2 - t_{11}^1), \\
    q^2(t_{22}^2)^2 - (1 + q^2)(t_{11}^1 t_{22}^2 + q^{-1}t_{12}^1 t_{21}^2) + (t_{11}^1)^2.
\end{align*}
\]

The ideals \(M_\pm\) generate the 4\(D_\pm\) calculi introduced in [17]. Notice that the 4\(D_\pm\) calculi have a proper classical limit when \(q \to 1\) even though \(\lambda_\pm \to 0\) so that a singularity appears in the definition of \(d\) in (8). This is because the numerator also tends to zero in a suitable way. This example suggests that (8) can be formulated more generally with \(\lambda \in k\) provided that the effects of the singular point can be cancelled in a suitable way.

**Example 3.3** Bicovariant differential calculus on the quantum plane. Let \(A = \mathbb{C}_q\), where \(\mathbb{C}_q\) is the quantum plane generated by \(1\) and three elements \(x, x^{-1}, y\) modulo the relation \(xy = qyx\). The bialgebra structure on \(\mathbb{C}_q\) is given by \(\Delta x^{\pm 1} = x^{\pm 1} \otimes x^{\pm 1}\), \(\Delta y = y \otimes 1 + 1 \otimes y\), etc. The element \(x\) is \(Ad_R\)-invariant. Set \(\alpha = x\). The ideal \(M_\alpha\) is generated by two elements:

\[
\begin{align*}
    (\lambda + 1 - x)(1 - x), & \quad (\lambda + 1 - x)y.
\end{align*}
\]

Now if we put \(\lambda = q - 1\) and extend \(M_\alpha\) to the ideal \(M\) generated additionally by \(y^2\), then we obtain the bicovariant differential calculus \(III_{q,q^{-1}}\) described in [3].
Example 3.4 Bicovariant differential calculi on $A(R)$. This example is a generalisation of Example 3.2.

Let $A(R)$ be the bialgebra associated to a solution $R$ of the quantum Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, where $R_{12} = R \otimes I$ etc. and $R \in M_n(k) \otimes M_n(k)$. The algebra of $A(R)$ is generated by the matrix of generators $t = (t_{ij})_{i,j=1}^n$ modulo the relation $Rt_1t_2 = t_2t_1R$. It has a standard coalgebra structure induced by matrix multiplication. This $A(R)$ is usually made into a Hopf algebra by a taking suitable determinant-like quotient. $R$ need not be one of the standard $R$-matrices but we assume that it is regular in the sense that such a quotient can be made to give an honest (dual-quasitriangular) Hopf algebra $A$. In particular, we assume that $R^{-1}$ and $\tilde{R} = ((R^{t_2})^{-1})^{t_2}$ exist, where $t_2$ denotes transposition in the second matrix factor. Dual-quasitriangular means that $R$ extends to a functional $\mathcal{R} : A \otimes A \to k$ obeying axioms dual to those for a universal $R$-matrix. In particular, it obeys

$$\mathcal{R}(t_{i_1}^j, t_{k_1}^l) = R_{i_1}^j_{k_1}^l, \quad \mathcal{R}(t_{i_2}^j, St_{k_2}^l) = \tilde{R}_{i_2}^j_{k_2}^l \quad (12)$$

while its extension to products is as a bialgebra bicharacter (i.e., $\mathcal{R}(ab, c) = \sum \mathcal{R}(a, c_{(1)}) \mathcal{R}(b, c_{(2)})$ and $\mathcal{R}(a, bc) = \sum \mathcal{R}(a_{(1)}, c) \mathcal{R}(a_{(2)}, b)$).

In this context it has been shown in [9] that $A(R)$ can be transmuted to the bialgebra $B(R)$ living in the braided category of $A(R)$-modules. The algebra of $B(R)$ is generated by the matrix $u$ with relations

$$R_{21}u_1R_{12}u_2 = u_2R_{21}u_1R_{12} \quad (13)$$

These relations are known in various other contexts also. The braided coalgebra structure is the standard matrix one on the generators (extended with braid statistics). Likewise for their quotients: the Hopf algebra $A$ transmutes to a braided group $B$ living in the category of $A$-comodules. We need now the precise relation between the product in the braided $B(R)$ and the original $A(R)$ by which (13) were obtained (in an equivalent form) in [9]. This is given at the Hopf algebra level by the transmutation formula

$$a \cdot b = \sum a_{(2)}b_{(3)}\mathcal{R}(a_{(1)}, b_{(2)})\mathcal{R}(a_{(3)}, Sb_{(1)}) \quad (14)$$

where $\cdot$ denotes the product in $B$ while the right hand side is in $A$. Here $B = A$ as a linear space, as a coalgebra and as an $A$-comodule under the adjoint coaction $Ad_R$ [9].

We next define a matrix $\vartheta = (\vartheta_{ij})_{i,j=1}^n$, where $\vartheta_{ij} = \tilde{R}_{i}^{k}j^{k}$ (summation over repeated indices). It has been shown in [11] that

$$\alpha_k = \text{tr}(\vartheta u^k), \quad k = 1, 2, \ldots \quad (15)$$

are bosonic (i.e., invariant) central elements of $B(R)$ and $B$. The product in (15) is the braided one in $B(R)$. For the standard $R$-matrices this recovers the known Casimirs [8] of the enveloping algebras $U_q(g)$, as explained in [10]. However, we are not limited to this case.
We now proceed to use (14) to view the $\alpha_k$ as elements of $A(R)$ and $A$. The generators correspond, $u = t$, while the formula (14) is used to express the products of generators in $B$ in terms of products in $A$ and evaluated using (12) and the bicharacter property for $\mathcal{R}$, as explained in [10, Sec. 5]. Computing $u^n$ in this way, the first three $\alpha_k$ immediately come out as

\begin{align*}
\alpha_1 & = \text{tr}(\vartheta t) = \vartheta_{j}^{i} t_{i} \\
\alpha_2 & = \text{tr}(\vartheta R^{(1)} t \vartheta R^{(2)} t) = \vartheta_{j}^{i} R_{k}^{m} t_{n} t_{i} t_{j}^{m} t_{k}^{n} \\
\alpha_3 & = \text{tr}(\vartheta R^{(1)} R^{(1)} t \vartheta R^{(2)} R^{(2)} t \vartheta R^{(1)} R^{(2)} R^{(2)} t) \\
& = \vartheta_{j}^{i} R_{k}^{p} R_{l}^{w} t_{i}^{m} t_{n}^{p} t_{s}^{q} \vartheta_{r}^{u} t_{z}^{v} \vartheta_{u}^{v} \vartheta_{r}^{u} t_{i}^{j}
\end{align*}

where $R = R^{(1)} \otimes R^{(2)}$ and $R', R''$ are copies of $R$. One can compute all the $\alpha_k$ in a similar way from (14).

Because these $\alpha_k$ are invariant elements of $B$, they are also $Ad_R$-invariant elements of $A$. Hence we obtain a sequence of bicovariant differential calculi on the quantum group $A$ obtained from $A(R)$. The corresponding ideal $M_{\alpha_k}$ for fixed $k$ is generated by $n^2$ elements

$$(\lambda_k + \epsilon(\alpha_k) - \alpha_k)(\delta_i^j - t_{i}^{j}), \quad \epsilon(\alpha_k) = \text{tr} \vartheta.$$ 

Similarly, $\alpha$ given by any function of the $\alpha_k$ will also do to define a bicovariant differential calculus, which function can be chosen according to the needs of a specific application. Finally, for well-behaved $R$, these formulae can also be used at the level of $A(R)$ itself, after this is made into a Hopf algebra by formally inverting suitable elements.

This completes the set of basic examples of differential calculi obtained from $Ad_R$-invariant elements of $A$. We observe however that if $\alpha, \beta \in A$ are $Ad_R$ invariant elements of $A$ then $\alpha + \beta$ and $\alpha \beta$ are also $Ad_R$-invariant. So we have a kind of ‘algebra’ of differential calculi corresponding to the algebra of $Ad_R$-invariant elements.

For example, if we find a single non-trivial $Ad_R$-invariant element $\alpha$ of $A$ then we are able to define a hierarchy of bicovariant differential calculi on $A$, i.e. $\Gamma_{\alpha}, \Gamma_{\alpha^2}, \ldots$. We illustrate such a hierarchy by the following:

**Example 3.5** Classification of bicovariant differential calculi on the line. Let us consider the Hopf algebra $A = k[x^{-1}, x]$ with comultiplication $\Delta x^{\pm 1} = x^{\pm 1} \otimes x^{\pm 1}$, counit $\epsilon(x^{\pm 1}) = 1$ and antipode $S(x^{\pm 1}) = x^{\mp 1}$. Clearly, all powers of $x$ and hence all the elements of $A$ are $Ad_R$-invariant. We put $\alpha = x$, $\lambda = q - 1$ and consider the differential calculi corresponding to $\alpha, \alpha^2, \ldots$. For fixed $n > 0$ the
ideal $M_{\alpha^n}$ is generated by the element

$$(q - x^n)(1 - x).$$

We notice that this ideal determines all the commutation relations in the bimodule $\Gamma_{\alpha^n}$, which is now generated by $2n - 1$ elements $dx^{-n+1}, \ldots, dx^{-1}, dx, dx^2, \ldots, dx^n$. We have explicitly

$$xdx^m = dx^{m+1} - dxx^m, \quad -n < m < n$$

$$xdx^n = dx^n x + (q^{-1} - 1)dxx^n.$$

**Example 3.6** Let $A = C_q$ as in Example 3.3. We consider $M_{\alpha}$ with $\alpha = x$ and we build the hierarchy $M_{\alpha^n}$. Each $M_{\alpha^n}$ is generated by the two elements

$$(\lambda + 1 - x^n)(1 - x), \quad (\lambda + 1 - x^n)y.$$

Assuming that $\lambda = q - 1$ and adding one more generator $y^2$ we can extend $M_{\alpha^n}$ to an ideal $M_n$. One can easily see that the bimodule $\Gamma_n$ induced by $M_n$ is generated by $2n$ elements $dy, dx^{-n+1}, \ldots, dx^{-1}, dx, dx^2, \ldots, dx^n$. The commutation relations in $\Gamma_n$ read

$$xdx^n = dx^n x + (q^{-1} - 1)dxx^n$$

$$xdx^m = dx^{m+1} - dxx^m, \quad -n < m < n$$

$$xdy = dyx$$

$$ydx^m = q^{-m}dx^my + (q^{-1} - 1)dxx^m, \quad -n < m \leq n$$

$$ydy = q^{-1}dyy.$$

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