On the equation $\nabla \phi + \phi X = 0$ and its relation to Schrödinger ground states

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To the memory of Joseph Hersch 1925-2012

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Abstract

We present some simple relations between the absolute minimizers of the functional $||\nabla \phi + \phi X||$, where $X$ is a vector field on $\mathbb{R}^n$, and ground state solutions to the (non-relativistic) Schrödinger equation. This article is a byproduct of the study of the more general functional $||A Du + F'(u)X||$.

1 Ground states

In the following let $X, Y, \ldots$ denote vector fields on $\mathbb{R}^n$, and $\langle f, g \rangle$ and $||f||$ the $L^2(\mathbb{R}^n)$ scalar product and norm respectively. We will use the same notation for vector fields as well, that is $\langle X, Y \rangle = \sum \langle X_i, Y_i \rangle$ and $||X|| = \sqrt{\langle X, X \rangle}$. All vector fields and functions shall be real valued. For any $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define the Schrödinger ground state energy as

$$E_0(V) = \inf \left\{ ||\nabla \phi||^2 + \int_{\mathbb{R}^n} V(x) |\phi(x)|^2 dx : \phi \in C^\infty_0(\mathbb{R}^n), ||\phi|| = 1 \right\}, \quad (1)$$

where $C^\infty_0(\mathbb{R}^n)$ means, as usual, the space of smooth functions having compact support. Note that we use units such that the (non-relativistic, time independent) Schrödinger equation has the form

$$-\Delta \psi + V \psi = E \psi \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n). \quad (2)$$

The leading actor in this article is the non-coercive functional

$$\mathcal{J}_X(\phi) = \int_{\mathbb{R}^n} |\nabla \phi + \phi X|^2 dx = ||\nabla \phi + \phi X||^2, \quad (3)$$

which is well defined on $C^\infty_0(\mathbb{R}^n)$ for any locally square integrable vector field $X$. 
Definition 1 A function $\phi_0$ is called a ground state to $\mathcal{J}_X(\phi)$, if the following conditions are satisfied:

i. $\phi_0 \not\equiv 0$,

ii. $\mathcal{J}_X(\phi_0) = 0$.

iii. $\|\nabla \phi_0\| < \infty$.

Thus, when we set
\[
\Lambda(X) = \inf \left\{ \mathcal{J}_X(\phi) : \phi \in C_0^\infty(\mathbb{R}^n), \|\phi\| = 1 \right\},
\]
a necessary condition for a ground state to exist is $\Lambda(X) = 0$. Formally, the Euler equation to the minimum problem above is easily calculated to be
\[
- \Delta u + (|X|^2 - \text{div } X)u = \Lambda(X)u,
\]
revealing the connection with the Schrödinger equation (2). On the other hand, if there is a ground state to $\mathcal{J}_X$, say $\phi_0$, it has to satisfy the first order equation (a.e.)
\[
\nabla \phi_0(x) + \phi_0(x)X(x) = 0
\]
since $\mathcal{J}_X(\phi_0) = \|\nabla \phi_0 + \phi_0 X\| = 0$. Indeed, applying the divergence operator to (6) and using the latter again in the resulting expression, gives (5) with $\Lambda(X) = 0$, as expected. It is clear that the equation (6) is generally easier to solve than (5).

Now let us state a simple proposition:

Proposition 1 Let $X \in L^2_{\text{loc}}$ such that $\text{div}(X) \in L^1_{\text{loc}}(\mathbb{R}^n)$, then for all $\phi \in C_0^\infty(\mathbb{R}^n)$ the following assertions are true:

\[
\|\nabla \phi + X\phi\|^2 = \|\nabla \phi\|^2 + \|X\phi\|^2 - \int_{\mathbb{R}^n} \text{div}(X)|\phi(x)|^2dx
\]
and
\[
\|\nabla \phi\|^2\|X\phi\|^2 \geq \frac{1}{4} \left( \int_{\mathbb{R}^n} \text{div}(X)|\phi(x)|^2dx \right)^2
\]
with equality if and only if
\[
\|\nabla \phi_0 + X\phi_0\| = 0
\]
for some functions $\phi_0$ such that $\|\nabla \phi_0\| < \infty$. For any such function $\phi_0$ then
\[
\|\nabla \phi_0\|^2 = \|X\phi_0\|^2 = \frac{1}{2} \int_{\mathbb{R}^n} \text{div}(X)|\phi_0(x)|^2dx
\]
holds.

Proof When expanding $\|\nabla \phi + X\phi\|^2$ one gets $\|\nabla \phi\|^2 + \|X\phi\|^2 + 2\langle \nabla \phi, X\phi \rangle$, where the scalar product $2\langle \nabla \phi, X\phi \rangle$ may be rewritten to $\langle \nabla |\phi|^2, X \rangle$. Since $\text{div}(X) \in L^1_{\text{loc}}(\mathbb{R}^n)$ by definition, the identity (7) follows by application of the divergence theorem. Inequality (8) and (9) are immediate consequences of the discriminant condition applied to $\|\nabla \phi + \lambda X\phi\|^2 = \|\nabla \phi\|^2 + \lambda^2\|X\phi\|^2 - 2\lambda\langle \phi, \phi \text{div}(X) \rangle \geq 0$ for all $\lambda \in \mathbb{R}$. Finally, (10) follows from (9) and (7). □
When we define
\[ V_{X,\lambda} = |X|^2 - \text{div}(X) + \lambda, \]
for \( X \in L^2_{\text{loc}}, \text{div}(X) \in L^1_{\text{loc}}, \lambda \in \mathbb{R}, \) then (7) reads
\[
\|\nabla \phi + X\phi\|^2 = \int_{\mathbb{R}^n} |\nabla \phi|^2 dx + \int_{\mathbb{R}^n} (V_{X,\lambda}(x) - \lambda)|\phi(x)|^2 dx,
\]
and recalling (1), we get
\[
\inf_{\|\phi\|=1} \|\nabla \phi + X\phi\|^2 = E_0(V_{X,\lambda}) - \lambda \geq 0,
\]
which motivates the next proposition:

**Proposition 2** For \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( E_0(V) \) as defined in (1) it holds
\[
E_0(V) \geq \sup_X \left\{ \Lambda(X) + \inf_{\|\phi\|=1} \int_{\mathbb{R}^n} (V(x) - |X|^2 + \text{div} X)|\phi(x)|^2 dx \right\}. \tag{11}
\]
and
\[
E_0(V) \leq \inf_X \left\{ \Lambda(X) + \sup_{\|\phi\|=1} \int_{\mathbb{R}^n} (V(x) - |X|^2 + \text{div} X)|\phi(x)|^2 dx \right\}. \tag{12}
\]

If \( X \) and \( V \) are pointwise defined (e.g. continuous) then there is a simple lower bound to the Schrödinger ground state energy:
\[
E_0(V) \geq \inf_{\mathbb{R}^n} \{V(x) - |X|^2 + \text{div} X\},
\]
valid for any reasonable vector field. The lower bound above was was orally communicated to us by the late Joseph Hersch many years ago and, actually, we found a reference to it in one of his numerous papers [1]. Note that (11) may be simplified by taking the supremum over ground states only, because then \( \Lambda(X) = 0 \).

**Proof** Setting \( B_{V,X}(\phi) = \int_{\mathbb{R}^n} (V(x) - |X|^2 + \text{div} X)|\phi(x)|^2 dx \), the upper bound
\[
E_0(V) \leq \|\nabla \phi + X\phi\|^2 + B_{V,X}(\phi)
\]
follows easily from (11) and (7). Adding \( \int V|\phi|^2 dx \) to (7) and rearranging, yields
\[
\|\nabla \phi + X\phi\|^2 + B_{V,X}(\phi) = \|\nabla \phi\|^2 + \int_{\mathbb{R}^n} V(x)|\phi(x)|^2 dx.
\]
Taking the infimum over \( \{\|\phi\| = 1\} \) on both sides gives
\[
\Lambda(X) + \inf_{\|\phi\|=1} B_{V,X}(\phi) \leq E_0(V).
\]
Since the right hand side is independent of \( X \), taking the supremum over \( X \) proves (11). Now (12) follows from the upper bound: \( \inf_{\|\phi\|=1} \{E_0(V) - B_{V,X}(\phi)\} \leq \Lambda(X) \).
Indeed, \( E_0(V) + \inf_{\|\phi\|=1} (-B_{V,X}(\phi)) = E_0(V) - \sup_{\|\phi\|=1} B_{V,X}(\phi) \leq \Lambda(X). \)
Remark 1 It is intuitively obvious that not every $X$ gives rise to a ground state, or in other terms, the equation (6) may have no (nontrivial) solutions at all. Indeed, as will be seen shortly, $X$ must be a gradient (one would say exact in terms of differential forms). It is well known that Schrödinger ground states may be chosen positive, so that $X = -\nabla \log \psi_0$ is an admissible vector field satisfying $J_X(\psi_0) = 0$.

Noting that the linear functional

$$T_X(\phi) = \int_{\mathbb{R}^n} (\nabla \phi + \phi X, X) dx$$

satisfies

$$|T_X(\phi)| \leq \|X\|_{2,K} \|\nabla \phi\|_{2,K} + \|X\|_{2,K}^2 \|\phi\|_{\infty} \leq C_K \sup_{|\alpha| \leq 1} \|\partial^\alpha \phi\|_{\infty},$$

for any $X \in L^2_{\text{loc}}$, thus $T \in \mathcal{D}'(\mathbb{R}^n)$, so that Propostion 4 may be extended to more general potentials (e.g. measures) along the same lines.

The ground states to $J_X$ have some nice properties.

Proposition 3 Let $\phi_0, \phi_1$ ground states to $J_X, J_Y$ respectively, then $\phi_0 \cdot \phi_1$ is a ground state to $J_{X+Y}$.

Proof $\|\nabla(\phi_0 \phi_1) + (X + Y)(\phi_0 \phi_1)\| = \|\phi_0(\nabla \phi_1 + Y \phi_1) + \phi_1((\nabla \phi_0 + X \phi_0))\| = 0$ because both terms $\nabla \phi_1 + Y \phi_1 = \nabla \phi_0 + X \phi_0 = 0$ in $L^2(\mathbb{R}^n)$ by supposition. □

Proposition 4 Suppose $\phi_0$ is a ground state to $J_X$, and let $P$ be a harmonic, homogeneous polynomial, satisfying $2 \nabla P(x) \cdot X(x) + W(x)P(x) = 0$, then

$$\phi_P(x) = P(x)\phi_0(x)$$

is a solution of $-\Delta \phi_P + (W + |X|^2 - \text{div} X)\phi_P = 0$, in $\mathcal{D}'(\mathbb{R}^n)$.

Proof Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, then

$$\langle \Delta \varphi, P\phi_0 \rangle = -\langle \nabla \varphi, \phi_0 \nabla P + P \nabla \phi_0 \rangle$$

Now, since

$$\nabla \phi_0 + \phi_0 X = 0$$

by supposition, it follows $\langle \Delta \varphi, P\phi_0 \rangle = -\langle \nabla \varphi, \phi_0 \nabla P - P \phi_0 X \rangle$, furthermore (using $\Delta P = 0$), $\langle \Delta \varphi, P\phi_0 \rangle = -\langle \varphi, \nabla \phi_0 \nabla P - \phi_0 \nabla P X - P \nabla \phi_0 X - \phi_0 P \text{div} X \rangle$, using $\nabla \phi_0 + \phi_0 X = 0$ again (two times)

$$\langle \Delta \varphi, P\phi_0 \rangle = \langle \varphi, -\phi_0 X \nabla P - \phi_0 P X - \phi_0 |X|^2 - \phi_0 P \text{div} X \rangle.$$ Then, writing $\phi_P = P\phi_0$ and using the supposition $2 \nabla P X = WP$

$$\int_{\mathbb{R}^n} \Delta \varphi \phi_P dx = \int_{\mathbb{R}^n} \varphi(W \phi_P + \phi_P |X|^2 - \phi_P \text{div} X) dx.$$ □

Now, let us look at some examples.
**Example 1** Let $X(x) = \alpha \frac{x}{|x|^p}$, $\alpha, p \in \mathbb{R}$. Noticing that $X$ is a gradient,

$$X = \nabla u = \frac{\alpha}{2-p} \nabla |x|^{2-p}$$

the equation

$$\nabla \phi(x) + \phi(x) \nabla u(x) = 0$$

is easily solved:

$$\nabla (\log \phi + u) = 0 \Rightarrow \phi(x) = Ce^{-u(x)},$$

thus we get

$$\phi_p(x) = C \exp \left[ \frac{\alpha}{p-2} |x|^{2-p} \right]$$

where $C = C(n, p, \alpha)$ is a normalization constant. A straightforward computation gives

$$|X|^2 = \alpha^2 |x|^{2-2p} \in L^1_{\text{loc}}(\mathbb{R}^n) \ldots \text{if } p < 1 + \frac{n}{2},$$

$$\text{div } X = \alpha \frac{n}{|x|^p} - \alpha p \frac{1}{|x|^p} = \alpha \frac{n-p}{|x|^p} \in L^1_{\text{loc}}(\mathbb{R}^n) \ldots \text{if } p < 1 + n,$$

hence

$$V_{X, \lambda}(x) - \lambda = \frac{\alpha^2}{|x|^{2(p-1)}} - \frac{\alpha}{|x|^p} \in L^1_{\text{loc}}(\mathbb{R}^n) \ldots \text{if } p < 1 + \frac{n}{2}.$$ 

Inserting into equations (7), (8) gives

$$\|\nabla \phi + X \phi\|^2 = \int_{\mathbb{R}^n} |\nabla \phi|^2 dx + \int_{\mathbb{R}^n} \left( \frac{\alpha^2}{|x|^{2(p-1)}} - \frac{n-p}{|x|^p} \right) |\phi(x)|^2 dx \geq 0$$

and

$$\int_{\mathbb{R}^n} |\nabla \phi|^2 dx \int_{\mathbb{R}^n} |x|^{2-2p} |\phi(x)|^2 dx \geq \frac{(n-p)^2}{4} \left( \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^p} dx \right)^2,$$

where the equality sign holds if $\phi = \phi_p$ (provided that, of course, $\phi_p$ satisfies the conditions of a ground state. For instance $n \geq 2$, $\alpha > 0$, $p < 1 + n/2$.

Let us have a closer look to the cases $p = 0, 1, 2, \ldots$, revealing some old friends:

**Example 2** Uncertainty, harmonic oscillator $X = \alpha x$. Setting $p = 0$ in (13) and (15) yields

$$\|\nabla \phi + X \phi\|^2 = \int_{\mathbb{R}^n} |\nabla \phi|^2 dx + \alpha^2 \int_{\mathbb{R}^n} |x|^2 |\phi(x)|^2 dx - \alpha n \int |\phi(x)|^2 dx,$$

which gives the Schrödinger ground state eigenvalue $E_0 = \alpha n$ with eigenfunction (13)

$$\phi_0(x) = C e^{-\frac{\alpha}{n} |x|^2},$$

and the well known Heisenberg uncertainty relation:

$$\int_{\mathbb{R}^n} |\nabla \phi(x)|^2 dx \int_{\mathbb{R}^n} |x|^2 |\phi(x)|^2 dx \geq \frac{n^2}{4} \left( \int_{\mathbb{R}^n} |\phi(x)|^2 dx \right)^2$$
Therefore, \( \phi_0 \) is a nice function, it is in \( S(\mathbb{R}^n) \) and real analytic (like \( X \)). Equation (5) goes to
\[
- \Delta \phi_0(x) + \alpha^2 |x|^2 \phi_0(x) = n \alpha \phi_0(x)
\]
which is Schrödinger’s equation for the harmonic oscillator.

Proposition 6 shows moreover that \( E_k = \alpha(n + 2k) \), \( k = 0, 1, 2, \ldots \) are the higher eigenvalues with eigenfunctions
\[
P_k(x) \exp \left( -\frac{\alpha}{2} |x|^2 \right),
\]
where \( P_k \) is a harmonic, homogeneous polynomial of degree \( k \).

**Example 3** The one electron atom: \( X = \alpha \frac{x}{|x|} \). Setting \( p = 1 \), we get analogously
\[
\| \nabla \phi + X \phi \|^2 = \| \nabla \phi \|^2 + \int_{\mathbb{R}^n} \left( \alpha^2 - \frac{\alpha(n-1)}{|x|} \right) |\phi(x)|^2 dx,
\]
and
\[
\int_{\mathbb{R}^n} |\nabla \phi(x)|^2 dx \int_{\mathbb{R}^n} |\phi(x)|^2 dx \geq \frac{(n-1)^2}{4} \left( \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|} dx \right)^2.
\]
Now, (5) goes to
\[
-\Delta \phi_0 - \frac{\alpha(n-1)}{|x|} \phi_0 = -\alpha^2 \phi_0
\]
which is, when setting \( n = 3 \), \( 2\alpha = Z \), the Schrödinger equation of an electron in the field of a nucleus of charge \( Z \). The eigenfunction (13) is
\[
\phi_0 = C \exp \left( -\frac{Z}{2} |x| \right)
\]
and the corresponding eigenvalue
\[
E_0 = -\frac{Z^2}{4}
\]
so that
\[
\int_{\mathbb{R}^1} |\nabla \phi(x)|^2 dx - Z \int_{\mathbb{R}^1} \frac{|\phi(x)|^2}{|x|} \geq -\frac{Z^2}{4} \int \frac{|\phi(x)|^2}{|x|} dx
\]
with equality for the \( \phi_0 \) above.

This \( \phi_0 \) is still a nice function, but it is not in \( S(\mathbb{R}^n) \) and fails to be continuously differentiable at the origin, i.e. \( \phi_0 \in C^\omega(\mathbb{R}^n \setminus \{0\}) \) only. Again, Proposition 6 provides higher eigenvalues and eigenfunctions:
\[
E_k = - \left( \frac{2}{n-1+2k} \right)^2 \alpha^2, \ k = 0, 1, 2, \ldots
\]
\[
\phi_k(x) = P_k(x) \exp \left( -\frac{\alpha |x|}{(n-1+2k)} \right)
\]
for any harmonic, homogeneous polynomial \( P_k \) of degree \( k \).
Example 4 Hardy’s inequality: \( X = \alpha \frac{x}{|x|^p} \): this is the case \( p = 2 \), so that \(|X|^2\) and \( \text{div} \, X \) have the same exponent. The singularity at 0 causes again no problems (assuming \( n \geq 3 \)) and we get

\[
\|\nabla \phi + X \phi\|^2 = \|\nabla \phi\|^2 + \alpha (\alpha - n + 2) \int_\mathbb{R}^n \frac{|\phi(x)|^2}{|x|^2} \, dx
\]

and

\[
\int_\mathbb{R}^n |\nabla \phi(x)|^2 \, dx \geq \frac{(n - 2)^2}{4} \int_\mathbb{R}^n \frac{|\phi(x)|^2}{|x|^2} \, dx.
\]

But this time we cannot use (13). Instead, one has to solve

\[
\nabla \phi + \alpha \frac{x}{|x|^2} \phi = 0 \Rightarrow \nabla (\log \phi + \alpha \log |x|) = 0,
\]

yielding

\[
\log(\phi|x|^{\alpha}) = C \Rightarrow \phi_0(x) \sim |x|^{-\alpha}.
\]

Indeed, there is no reasonable minimizer, though formally \( J_X(\phi_0) = 0 \). Clearly, neither \( \|\nabla \phi_0\| \) nor \( \|\phi_0 \cdot X\| \) is finite.

Remark 2 Scaling behavior. Let \( \phi_\lambda = \lambda^{n/2} \phi(\lambda x) \), \( X_\lambda(x) = \lambda^{-1} X \left( \frac{x}{\lambda} \right) \) then

\[
J_X(\phi_\lambda) = \int_\mathbb{R}^n \left| \lambda \nabla \phi(y) + \phi(y) X \left( \frac{y}{\lambda} \right) \right|^2 \, dy
\]

thus

\[
J_X(\phi_\lambda) = \lambda^2 J_{X_\lambda}(\phi).
\]

The example \( X_\lambda = \lambda^{p-2} x |x|^{-p} \) shows clearly that for \( p \geq 2 \) things are going odd. The examples also show that the singularities of \( X \) decrease the regularity of the ground state. It is possible, of course, to extend Proposition (2) to punctured domains, so that the results may be extended beyond \( p = 2 \).

The reader may be puzzled by the form of the eigenfunctions (17) and (18), but this is easily resolved when recollecting the fact that a homogeneous polynomial may look quite differently if restricted to a sphere (the factor \( x_1^2 + x_2^2 + x_3^2 \ldots = 1 \) drops out). There is, by the way, an interesting connection to the fact that the Fourier transform leaves functions of the form

\[
f(x) = P_k(x) \psi(|x|)
\]

invariant, in the sense that

\[
\hat{f}(\xi) = P_k(\xi) \hat{\psi}(|\xi|).
\]

This is obvious in the case \( p = 0 \), \( \alpha = 1 \), as we then have the eigenfunctions of the Fourier transform, but not so trivial in the case \( p = 1 \).

2 Regularity

We will give here only some elementary facts and assume some smoothness of the solutions. It is well known that if \( u \) and \( F \) are continuous and

\[
\nabla u(x) = F(x) \quad \text{in} \ B_r(x_0)
\]

in the distributional sense, then it also holds in the classical sense [2].
**Proposition 5** Let $\Omega \subset \mathbb{R}^n$ be an open set and $\phi \in C^1(\Omega)$ a solution of
\[
\nabla \phi(x) + \phi(x)X(x) = 0 \quad \forall x \in \Omega
\]
where $X$ is a vector field on $\Omega$. Then $X$ is locally the gradient of a $C^1$ function (and therefore continuous) on the open set $\{x \in \Omega : \phi(x) \neq 0\}$. Moreover, if $X \in C^k(\Omega)$, $k \in \mathbb{N}$, then $\phi \in C^{k+1}(\Omega)$.

**Proof** Suppose $\phi(x_0) > 0$ for some point $x_0 \in \Omega$, then $\phi(x) > 0$ in the ball $B_\varepsilon(x_0)$ for some $\varepsilon = \varepsilon(x_0) > 0$ by continuity. Thus, $X(x) = -\nabla \log \phi(x)$ in $B_\varepsilon(x_0)$. If $\phi(x_0) < 0$ then apply the same argument to the function $\phi'(x) := -\phi(x)$, giving $X(x) = -\nabla \log \phi'(x)$. Since $\log \phi$ is $C^1$ wherever $\phi > 0$ is, the assertion follows. The regularity claim can be proved by applying Leibniz’s rule. □

**Lemma 1** Let $\phi$ be a function defined in the ball $\{|x| < R_0\}$ satisfying
\[
|\phi(x)| \leq C|x|^\alpha \sup_{|y| \leq |x|} |\phi(y)|
\]
for some constants $C \geq 0$, $\alpha > 0$. Then $\phi \equiv 0$ in the ball $\{|x| < \min(R_0, C^{-\alpha})\}$.

**Proof** Let $M(r) := \sup_{|y| \leq r} |\phi(y)|$ and note that it is a non-decreasing function of $r$. Then we have by assumption $|\phi(x)| \leq C|x|^\alpha M(|x|)$, therefore, taking the supremum:
\[
\sup_{|x| \leq r} |\phi(x)| = M(r) \leq Cr^\alpha \sup_{|x| \leq r} M(|x|) = Cr^\alpha M(r)
\]
thus, $M(r) = 0$ for $Cr^\alpha < 1$. □

**Proposition 6** Let $\phi \in C^1(\mathbb{R}^n)$ be a solution of
\[
\nabla \phi(x) + \phi(x)X(x) = 0 \quad \forall x \in \mathbb{R}^n
\]
where $X$ is a vector field on $\mathbb{R}^n$. Suppose $\phi(x_0) = 0$ for a $x_0 \in \mathbb{R}^n$. If $X$ is locally bounded then $\phi \equiv 0$ in $\mathbb{R}^n$.

**Proof** Without loss of generality let $x_0 = 0$. Since $\phi \in C^1(\mathbb{R}^n)$ it follows that
\[
\frac{d}{dt}\phi(tx) = (\nabla \phi(tx), x) = -\langle \phi(tx)X(tx), x \rangle \Rightarrow \phi(x) - \phi(0) = -\int_0^1 \langle \phi(tx)X(tx), x \rangle dt.
\]
Thus
\[
|\phi(x)| = \left|\int_0^1 \langle \phi(tx)X(tx), x \rangle dt\right| \leq |x| \sup_{|y| \leq |x|} |\phi(y)X(y)|.
\]
□

**Remark 3** Local boundedness is necessary as the standard $C_0^\infty$ function $\chi_{B_1} \frac{1}{e^{\|x\|^2} - 1}$ shows.
3 Concluding remarks

We have seen that the Schrödinger ground states and those of the function $J_X$ are essentially the same. Moreover, ground states cannot change sign and the vector field $X$ has (therefore) to be a gradient. Thus, they have to satisfy the simple first order equation

$$\nabla \phi(x) + \phi(x) \nabla u(x) = 0$$

with solution

$$\phi(x) = C e^{-u(x)}.$$ 

The qualitative properties of $\phi$ are determined in an essential way by the function $u$. For instance, the critical points of $\phi$ are those of $u$, and $\phi$ is log-concave if $u$ is convex.

In a certain way the same is true for the “inhomogeneous” equation

$$\nabla \phi(x) + \phi(x) X(x) = Y(x),$$

which we have not touched here.

Now, what is the physical meaning of $X$? When we multiply \((19)\) by $-i$, we get

$$p\phi = -i\nabla \phi = iX\phi,$$

where $p$ is the momentum operator. So, indeed, $X$ indicates the momentum of the ground state. In the same sense it holds for the angular momentum,

$$L\phi = (x \wedge p) \phi = i(x \wedge X)\phi.$$ 

For any radial function $u(x) = u(|x|)$, for example:

$$X = \nabla u = u'(|x|) \frac{x}{|x|}$$

so that the angular momentum of the corresponding ground state has to be zero. In other words, one can prescribe the momentum ‘field’ $X = \nabla u$ (necessarily a potential field), then the (Schrödinger) ground state is completely determined. Clearly, the function $u(x)$ is closely related to the classical action function $S(x)$ and the parallels to Hamilton-Jacobi theory are quite obvious, but we have found that the deeper reason for \((19)\) stems from the quantum mechanical phase-space measure

$$d\mu_\phi = |\phi(x)|^2 |\hat{\phi}(k)|^2 dx dk,$$

which is a Radon measure generated by any normalized $L^2(\mathbb{R}^n)$ function $\phi$, so that one can write for the quantum mechanical energy:

$$\mathcal{E}(\phi) = \int_\Gamma \mathcal{H}(x, \hbar k)d\mu_\phi,$$

where $\mathcal{H}$ is the classical Hamilton function. We cannot go into details, but minimizing $\mathcal{E}$ with respect to $\phi$, gives not the usual Schrödinger equation, however, a kind of “double one” on $L^2(\Gamma)$, where $\Gamma = \mathbb{R}^n \times \mathbb{R}^n$ is the phase space. When we write \((8)\) in the form

$$\int_{\mathbb{R}^n} |k|^2 |\hat{\phi}(k)|^2 dk \int_{\mathbb{R}^n} |X|^2 |\phi(x)|^2 dx \geq \frac{1}{4} \left( \int_{\mathbb{R}^n} \text{div} X |\phi(x)|^2 dx \right)^2,$$
and rewrite it to
\[
\int_{\Gamma} |k|^2 |X|^2 d\mu_\phi \geq |\langle \hat{X}\phi, k\phi \rangle|^2
\]
the symmetry between \(k\) and \(X\) may be apparent. It might give a clue why it is the Fourier transform that connects configuration and momentum space in quantum mechanics (the way the Heisenberg group acts on \(\Gamma \times \mathbb{R}\) and the behaviour of \(d\mu_\phi\) under canonical transformations also support this).

Another way to gain some physical insight is to look at the associated energy-momentum tensors to the equations \(\text{(2)}\) and \(\text{(5)}\):
\[
T_S = \nabla \phi \otimes \nabla \phi - \mathbb{I}(|\nabla \phi|^2 + (V(x) - E)|\phi|^2)
\]
and
\[
T_X = \nabla \phi \otimes \nabla \phi - \mathbb{I}(|\nabla \phi|^2 + (|X|^2 - \text{div}(X))|\phi|^2).
\]
Now,
\[
\text{Div}(T_S) = -\nabla V(x)|\phi|^2
\]
and (think of a force \(F(x) = -\nabla V(x)\))
\[
\text{Tr}(T_S) = (2 - n)|\nabla \phi|^2 + n(V(x) - E)|\phi|^2.
\]
When we insert \(\nabla \phi = -\phi X\) into \(T_X\), it follows
\[
T_X = |\phi|^2 (X \otimes X - \mathbb{I}(2|X|^2 - \text{div}(X)),
\]
and therefore
\[
\text{Tr}(T_X) = |\phi|^2 ((2 - n)|X|^2 + n \text{div} X).
\]
The meaning of \(X\) and \(\text{div} X\) is now quite obvious as \(T_X\) and \(T_S\) are factually the same. For instance, if the potential \(V\) is homogeneous of degree \(k\), then (formally)
\[
\text{div}(T_S x) = \text{Div}(T_S) \cdot x + \text{Tr}(T_S) = -x \cdot \nabla V(x)|\phi|^2 + (2 - n)|\nabla \phi|^2 + n(V(x) - E)|\phi|^2
\]
\[
= (2 - n)|\nabla \phi|^2 + (n - k)V(x)|\phi|^2 - nE|\phi|^2,
\]
which is nothing more than the virial theorem, so that we must have
\[
\text{div}(T_X x) = \{(2 - k)|X|^2 + (k - n) \text{div} X - kE\}|\phi|^2.
\]
If, as we have seen in the examples, the ground states fall off sufficiently rapid, then
\[
\int_{\mathbb{R}^n} \text{div}(T_S x) dx = 0,
\]
hence
\[
2 \int_{\mathbb{R}^n} |\nabla \phi|^2 dx = k \int_{\mathbb{R}^n} V(x)|\phi|^2 dx.
\]

References

[1] Joseph Hersch, On the Methods of One-Dimensional Auxiliary Problems and of Domain Partitioning: Their Application to Lower Bounds for the Eigenvalues of Schrödinger’s Equation.. Journal of Mathematics and Physics, Vol. XLIII, No. 1, March 1964.

[2] Lars Hörmander, Linear Partial Differential Operators. Springer Verlag, Berlin, Fourth Printing, (Theorem 1.4.2) 1976.