A binary mixture of Bose-Einstein-condensates in a double-well potential: Berry phase and two-mode entanglement

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A binary mixtures of Bose-Einstein condensate (BEC) structures exhibit an incredible richness in terms of holding different kinds of phases. Depending on the ratio of the inter- and intra-atomic interactions, the transition from mixed to separated phase, which is also known as the miscibility-immiscibility transition, has been reported in different setups and by different groups. Here, we describe such type of quantum phase transition (QPT) in an effective Hamiltonian approach, by applying Holstein-Primakoff transformation in the limit of large number of particles. We demonstrate that non-trivial geometric phase near the critical coupling is present, which confirms the connection between Berry phase and QPT. We also show that, by using the spin form of Hillery & Zubairy criterion, a two mode entanglement accompanies this transition in the limit of large, but not infinite number of particles.

I. INTRODUCTION

In recent years, there has been growing interest in studying two-component quantum fluids. Phase mixing and separation of the two components due to relative strength of inter-species and intra-species interactions open gate to investigate variety of research topics including two-mode entanglement [1, 2], the dynamical phase transition [21], etc. has been done to performing a fully analytic derivation [18]. The characteristic of the self-trapping [19], entanglement [20], the dynamical phase transition [21] etc. has been be done within the BH model. The observation of controllable phase separation of a binary BEC was reported by using Feshbach resonance in the hyperfine levels [10–12] and in the isotopes [13] of the rubidium atoms. Additionally, the similar results were also obtained by using different kinds of atoms [14–17]. Theoretical investigations of these structures have shown that the relative interaction strength and number of particles play crucial role in characterizing the density profiles. A two species Bose-Hubbard (BH) model, in the limit of a weakly interacting gas, is widely used due to performing a fully analytic derivation [18]. The characteristic of the self-trapping [19], entanglement [20], the dynamical phase transition [21] etc. has been be done within the BH model.

In this work, we theoretically study collective dynamics of two-species BEC trapped in a double-well potential. First, we construct the Hamiltonian with two-mode approximation in the weak scattering limit. Then, we derive the effective Hamiltonians and the ground state wavefunctions analytically, by applying Holstein-Primakoff transformation [22,23]. We also perform numerical calculations in limit of the small number of particles, in which we obtain consistent results with analytical model through a phase transition. Next, we investigate the geometrical dynamics of the system in the presence of the miscibility-immiscibility transition. In many body systems, QPT can be observed when crossing between ground and excited states takes place. It is known that such level crossings generates singularities in the Hilbert space, and is therefore natural to expect the reflections of such behaviours in the wavefunction. The Berry phase is able to capture these points, and its connection with QPT has been studied in different models [24–26]. Here, we obtain non-trivial geometric phase by encircling the critical point and observe that with increasing number of particles, the transition in the value of the Berry phase becomes sharper around the critical point, and in the thermodynamic limit N→ ∞, there appears a step-like behaviour. Having the ability to adiabatically control the interatomic interaction strength between two species [12,13], makes this result valuable for the Berry phase related applications, and it also provides a tool to detect criticality in the presence of QPT.

Besides fundamental interest, the model with the use of Holstein-Primakoff representation also offers the possibility to detect bipartite entanglement from macroscopic observables. In large systems, it was shown that entanglement can be inferred from collective spin measurements [27,28], and experimental observations using this method have been reported between two spatially separated atomic ensembles [29,30], and between the spins of atoms in optical lattices [31,32]. Here we analyze such phenomena through the miscibility-immiscibility transition, by adopting the spin form of two-mode entanglement witness given in [30]. We observe that the criterion witnesses the entanglement onset in the separated phase. Unlike the Berry phase, entanglement decays faster with the increasing number of particles and/or interspecies interaction strength. We find that this is due to vanishing effective coupling term, which is responsible for two-mode squeezing.

The manuscript is organized as follows. In Sec. I we introduce the model of a two species BECs trapped in a double-well potential, and derive the effective Hamilto-
nians and associated ground state wavefunctions of the mixed and separated phases. The appearance of the non-trivial geometric phase around the critical coupling is observed in Sec. IV. In Sec. V we discuss the formation of bipartite entanglement by anticipating spin form of Hillery & Zubairy criterion. A summary appears in Sec. V

II. THE MODEL

We consider a two species $(a$ and $b$) condensate mixtures trapped in a double-well potential with large number of particles $N_{a(b)} \gg 1$. By assuming the interaction among the atoms is sufficiently weak $(t_a, t_b \gg g_a, g_b, g_{ab})$, we construct the Hamiltonian with two-mode approximation [19, 37], which is given by:

$$
\hat{H} = \frac{t_a}{2} (\hat{a}_L^{\dagger} \hat{a}_R + \hat{a}_R^{\dagger} \hat{a}_L) + \frac{t_b}{2} (\hat{b}_L^{\dagger} \hat{b}_R + \hat{b}_R^{\dagger} \hat{b}_L) + g_a (\hat{a}_L \hat{a}_R) + g_b (\hat{b}_L \hat{b}_R) + g_{ab} (\hat{a}_L \hat{b}_R + \hat{a}_R \hat{b}_L).
$$

Here, $\hat{a}_j^{\dagger}$ ($\hat{a}_j$) and $\hat{b}_j^{\dagger}$ ($\hat{b}_j$) are the creation (annihilation) operators of the species $a$ and $b$ respectively that residing in the $j$th well, $j = L, R$. The parameters $t_a$ and $t_b$ describe the coupling (tunneling) between two wells, $g_{ab}$ and $g_{ab}$ stand for intraspecies and interspecies interaction strength respectively, which is explicitly given by: $g_{ab} = 2\pi \hbar^2 A_{ab}/m_{a(b)} \int |\phi_a|^2 |\phi_b|^2 \, dr$ [38]. Here $A_{ab}$ is the $s$-wave scattering length between atoms, $m_{a(b)}$ is the reduced mass, where we denote $g_a = g_{aa}$ and $\alpha, \beta = a, b$.

The analysis of the Hamiltonian in Eq. (1) can be simplified by the introduction of the angular momentum operators for each species as [39]:

$$
\begin{align*}
\hat{J}_{ax} &= (\hat{a}_L^{\dagger} \hat{a}_L - \hat{a}_R^{\dagger} \hat{a}_R)/2, \\
\hat{J}_{ay} &= (\hat{a}_L^{\dagger} \hat{a}_R - \hat{a}_R^{\dagger} \hat{a}_L)/2i, \\
\hat{J}_{az} &= (\hat{a}_L^{\dagger} \hat{a}_L + \hat{a}_R^{\dagger} \hat{a}_L)/2,
\end{align*}
$$

where $\alpha = a, b$. These operators obey the usual angular momentum commutation relations: $[\hat{J}_{ax}^+, \hat{J}_{ay}^-] = 2\hat{J}_{az}$ and $[\hat{J}_{ax}^+, \hat{J}_{az}^-] = +\hat{J}_{ay}^+$, where $\hat{J}_{ax}^\pm = \hat{J}_{ax} \pm i\hat{J}_{ay}$. Inserting these definitions into the Eq. (1), the Hamiltonian can be rewritten as [1]

$$
\hat{H} = \sum_{\alpha = a, b} \{a_J (c_{a}^{\dagger} \hat{c}_a - N_{a}/2) + \frac{g_a}{2} (c_{a}^{\dagger} \hat{c}_a) \} + \frac{g_b}{2} (c_{b}^{\dagger} \hat{c}_b) + g_{ab} (\hat{c}_a^{\dagger} \hat{c}_b + \hat{c}_b^{\dagger} \hat{c}_a) + g_{ab} \sqrt{J_{ab}} (\hat{c}_a^{\dagger} \hat{c}_a + \hat{c}_b^{\dagger} \hat{c}_b).
$$

In the limit of large number of particles, one can make use of the Holstein-Primakoff (HP) representation of the angular momentum operators. In this representation, the operators defined in Eq. (2) can be written in terms of a bosonic mode in the following way [22, 23]

$$
\begin{align*}
\hat{J}_{ax} &= \hat{c}_a^{\dagger} \sqrt{N_{a} - \hat{c}_{a}^{\dagger} \hat{c}_{a}}, \\
\hat{J}_{ay} &= \sqrt{N_{a} - \hat{c}_{a}^{\dagger} \hat{c}_{a}} \hat{c}_{a}, \\
\hat{J}_{az} &= \hat{c}_{a}^{\dagger} \hat{c}_{a} - \hat{a}_{a},
\end{align*}
$$

which is analogous to that of two coupled oscillators. By defining the position and the momentum operators:

$$
\begin{align*}
\hat{x}_a &= \frac{1}{\sqrt{2}} (c_a^{\dagger} + c_a), \\
\hat{p}_a &= i \frac{1}{\sqrt{2}} (c_a^{\dagger} - c_a), \\
\hat{x}_b &= \frac{1}{\sqrt{2}} (c_b^{\dagger} + c_b), \\
\hat{p}_b &= i \frac{1}{\sqrt{2}} (c_b^{\dagger} - c_b),
\end{align*}
$$

we rewrite the effective Hamiltonian as

$$
\hat{H}^{(1)} = \sum_{\alpha = a, b} \{\frac{t_{\alpha}}{2} (\hat{p}_{\alpha}^2 + \hat{x}_{\alpha}^2) + \tilde{g}_{ab} \hat{x}_{\alpha} \hat{x}_{\beta} \} + 2 \tilde{g}_{ab} \hat{x}_{a} \hat{x}_{b},
$$

where $\tilde{g}_{ab} = g_{ab} \sqrt{J_{ab}}$. It is then straightforward to solve the resulting normal mode frequencies of the coupled oscillation, which is given by

$$
\epsilon_{\pm}^{(1)} = \sqrt{\omega_x^2 + \omega_y^2 + 4 \tilde{g}_{ab} \hat{x}_a \hat{x}_b}.
$$

where $\omega_{\pm} = \sqrt{g_{a} \hat{x}_a \pm \tilde{g}_{ab} \hat{x}_b + (t_a^2 \pm t_b^2)/2}$. Crucially, one can see that the normal mode frequencies can have complex values depending on the interaction strengths, when
In the following, we shall just consider the displacements these definitions into Eq. (5) and eliminate the first order critical interspecies coupling strength $g$. Here we use equal tunneling amplitudes; $t_a/j_a = 0.1 g$ and in the insets, we consider infinite number of particles case and take $t_a/j_a = 0$.

$g_{ab} > g^*_{ab}$ the system becomes unstable. Therefore, the critical interspecies coupling strength $g^*_{ab}$ can be given as

$$g^*_{ab} = \frac{1}{2} \sqrt{\left(\frac{t_a}{j_a} + 2g_a\right)\left(\frac{t_b}{j_b} + 2g_b\right)}.$$ (11)

In the thermodynamic limit, $j_a \to \infty$, it reduces to well-known criticality (e.g. $g^*_{ab} = \sqrt{g_ag_b}$) for the phase separation of two component BECs [42]. Depending on the ratio of the intraspecies ($g_{a(b)}$) and interspecies ($g_{ab}$) interaction strengths, different types of the phases has been extensively studied theoretically [4, 43, 44], and such phases also seen experimentally [12, 16].

The main motivation in this work is to find a solution above a critical coupling strength, where phase transition occurs. Such type of phase transition is well-known in the Dicke-type models, in which one can describe interaction of a single mode quantized field with an ensemble of N two-level atoms. It was shown that above a critical coupling strength such system can undergo quantum phase transition [15, 17], from normal to superradiant phase. Here we adopt this method [17, 48] to characterize the phase transition between mixed and separated phases.

To find a solution above the critical point ($g_{ab} > g^*_{ab}$), we displace the bosonic operators as

$$\hat{c}^\dagger_a = \hat{d}^\dagger_a \pm \sqrt{N_a} \alpha_a, \quad \hat{c}^\dagger_b = \hat{d}^\dagger_b \mp \sqrt{N_b} \beta_b.$$ (12)

In the following, we shall just consider the displacements as; $\hat{d}^\dagger_a = \hat{d}^\dagger_a + \sqrt{N_a} \beta_a$ and $\hat{d}^\dagger_b = \hat{d}^\dagger_b - \sqrt{N_b} \beta_b$. If we insert these definitions into Eq. (5) and eliminate the first order term in the boson operators, one can find the amounts of displacement of each mode by solving

$$\frac{t_a}{j_a} + 2\left(g_a - g_{ab}\sqrt{\frac{\beta_b(1-\beta_b)}{\beta_a(1-\beta_a)}}\right)(1 - 2\beta_a) = 0,$$ (13)

$$\frac{t_b}{j_b} + 2\left(g_b - g_{ab}\sqrt{\frac{\beta_a(1-\beta_a)}{\beta_b(1-\beta_b)}}\right)(1 - 2\beta_b) = 0.$$ (14)

The resulting effective Hamiltonian can be given by

$$\hat{H}^{(2)} = \sum_{\alpha=a,b} \{\omega_\alpha \hat{d}_\alpha^\dagger \hat{d}_\alpha + \kappa_\alpha (\hat{d}_\alpha^\dagger + \hat{d}_\alpha)^2\}$$

$$+ \lambda(\hat{d}_a^\dagger + \hat{d}_a)(\hat{d}_b^\dagger + \hat{d}_b),$$ (15)

where we consider the boson operators up to the second order, and the parameters can be found as

$$\omega_\alpha = t_\alpha + 2g_{ab}\beta_\alpha \sqrt{\beta_\alpha(1-\beta_\alpha)},$$ (16)

$$\kappa_\alpha = \omega_\alpha - t_\alpha + g_{a(b)} \frac{6\beta_\alpha (\beta_\alpha - 1) + 1}{2(1-\beta_\alpha)},$$ (17)

$$\lambda = g_{ab} \beta_\alpha \sqrt{(1-2\beta_\alpha)(1-2\beta_b)} \sqrt{(1-\beta_\alpha)(1-\beta_b)}.$$ (18)

where $\beta_\alpha = \beta_a$ and $\beta_b = a$. Before proceeding, let us check these parameters in the limits of $g_{ab} \leq g^*_{ab}$ and $g_{ab} \gg g^*_{ab}$. For the case; $g_{ab} \leq g^*_{ab}$ the displacement parameters take a single value, i.e. $\beta_\alpha = 0$, and the Hamiltonian in Eq. (15) reduces to the one given for the mixed phase in Eq. (9).

When $g_{ab} \gg g^*_{ab}$, the displacement parameters take a single value, i.e. $\beta_\alpha = 0.5$, as seen in FIG. 1. In this limit, two oscillators become uncoupled, $\lambda = 0$ and by increasing interaction strength of the interspecies, it only contributes to the effective strengths of the intraspecies, $\kappa_\alpha$, and tunneling, $\omega_\alpha, \beta$, coefficients, which makes the Hamiltonian $\hat{H}^{(2)}$ stable for all $g_{ab}$ values. To see this, we find the new eigen-frequencies, by moving to a position-momentum representation defined by

$$\hat{X}_a = \frac{1}{\sqrt{2}}(\hat{d}_a^\dagger + d_a), \quad \hat{P}_a = i\frac{1}{\sqrt{2}}(\hat{d}_a^\dagger - d_a),$$ (19)

$$\hat{X}_b = \frac{1}{\sqrt{2}}(\hat{d}_b^\dagger + d_b), \quad \hat{P}_b = i\frac{1}{\sqrt{2}}(\hat{d}_b^\dagger - d_b).$$ (20)

where one can see the relation between the coordinates: $\hat{X}_a = \hat{x}_\alpha - \chi_{a,b} \sqrt{2N_a} \alpha_a$, $\chi_{a,b} = \pm 1$. By following the same steps, as it is done for $\hat{H}^{(1)}$ in Eq. (9), one can find the corresponding oscillator energies for $\hat{H}^{(2)}$ as

$$\epsilon^{(2)}_{\pm} = \frac{1}{2} \left[2\omega_\alpha^2 + 2\omega_b^2 \pm \sqrt{(\omega_\alpha^2 - \omega_b^2)^2 + 16\lambda^2 \omega_\alpha \omega_b}\right]^{1/2}$$ (21)

where $\omega_b^2 = \omega_\alpha (\omega_\alpha + 4\kappa_\alpha)$. The new excitation energy, $\epsilon^{(2)}_{\pm}$, is real over all the parameter space and hence $\hat{H}^{(2)}$ describes the system in the phase separation region.

Next, we give the ground state wavefunctions of the mixed and separated phases as

$$\psi^{(j)}_{gs}(q_{ja}, q_{jb}) = \frac{m \Omega}{\pi} G_j(q_{ja}, q_{jb}) G_j(q_{ja}, q_{jb}),$$ (22)

where $j = 1, 2$ stands for the solutions of the mixed and separated phases respectively with $q_{ja} = x_\alpha$, $q_{ba} = X_a$, and $G_1(q_{ja}, q_{jb})$ and $G_2(q_{ja}, q_{jb})$ represent Gaussian functions defined by

$$G_1(q_{ja}, q_{jb}) = e^{\frac{m \Omega}{2} \left[(q_{ja}C - q_{jb}S)^2\right]}$$ (23)

$$G_2(q_{ja}, q_{jb}) = e^{\frac{m \Omega}{2} \left[(q_{ja}S + q_{jb}C)^2\right]}$$ (24)
where
\[ \xi = \frac{c_a + c_b + \sqrt{(c_a - c_b)^2 + 4\lambda^2}}{2K}, \]
\[ K = \sqrt{c_a c_b - \frac{\lambda}{2}}, \quad c_a = (\omega_a + 4\omega_a)\sqrt{\frac{\omega_a}{\omega_b}}, \]
\[ C = \cos(\phi), \quad S = \sin(\phi), \quad \tan(2\phi) = \frac{2\lambda}{c_a - c_b}, \]
\[ \Omega = \sqrt{K/m}, \quad m = 1/\sqrt{\omega_a \omega_b}, \]
where we denote \( \overline{a} = b \) and \( \overline{b} = a \). We define the parameters above for only \( \psi_{GA}^{(2)} \), by inserting \( \beta_a = 0 \) into the these parameters one can obtain desired solution for \( \psi_{GA}^{(3)} \).

In the following sections, since having derived the effective Hamiltonians and associated ground states that describe mixed and separated phases, we investigate the quantum features of this kind of phase transition in terms of geometric phase and bipartite entanglement.

### III. GEOMETRIC PHASE

In this section, we demonstrate that by encircling the critical point in parameter space, where the miscibility-immiscibility transition occurs, a non-trivial Berry phase can be obtained for the system considered in this work. Let us start with introducing the collective angular momentum operators after displacement operation is done. In the limit of large number of particles, it can be found as [see appendix]

\[ \hat{J}_{ax} \approx \chi_a \sqrt{N_a \beta_a (1 - \beta_a)} + \frac{1 - 2\beta_a}{\sqrt{2(1 - \beta_a)}} \hat{X}_a, \]
\[ \hat{J}_{ay} \approx -\frac{1 - \beta_a}{2} \hat{P}_a, \]
\[ \hat{J}_{az} \approx N_a (\beta_a - 1/2) + 2\chi_a \sqrt{N_a \beta_a} \hat{X}_a + \frac{1}{2} (\hat{P}_a^2 + \hat{X}_a^2 - 1), \]
where \( \chi_a = 1, \chi_b = -1 \) and \( X_{a,b} = X,Y \). Here we consider the terms up to \( 1/N \) order in the expansion. In the ground state, \( \langle \hat{J}_{az} \rangle = 0 \) and the main contribution to the expectation values of the \( \hat{J}_{ax} \) and \( \hat{J}_{az} \) comes from the first terms in Eqs. (29) and (31) respectively. Thus, we can safely neglect the other terms in the thermodynamic limit and obtain

\[ \frac{\langle \hat{J}_{az} \rangle}{N_a} = \begin{cases} 0, & g_{ab} \leq g_{ab}^*, \\ (\beta_a - 0.5), & g_{ab} > g_{ab}^* \end{cases} \]
and

\[ \frac{\langle \hat{J}_{az} \rangle}{\sqrt{N_a}} = \begin{cases} 0, & g_{ab} \leq g_{ab}^*, \\ \chi_a \sqrt{\beta_a (1 - \beta_a)}, & g_{ab} > g_{ab}^* \end{cases} \]
in which one can clearly observe that above \( g_{ab}^* \) there is a macroscopic excitation for each one. If we introduce a time-dependent unitary transformation \( U(\phi(t)) = e^{-i\phi(t)} \hat{J}_{az} \), where \( \hat{J}_{az} = \hat{J}_{az} + \hat{J}_{az} \) and \( \phi(t) \) is slowly time-varying parameter. When this phase, \( \phi(t) \), is varied adiabatically between 0 and 2\( \pi \), a state in phase space will encircle the origin. Then, Berry phase can be defined in the ground state as

\[ \gamma = i \int_0^{2\pi} d\phi \langle U^{\dagger}(\phi) \frac{d}{d\phi} U(\phi) \psi \rangle = 2\pi \langle \hat{J}_{az} \rangle, \]

where \( \langle \psi \rangle \) is the time-independent ground-state wavefunction. If we insert Eq. (32) into Eq. (34), the total scaled Berry phase of the system can be defined as

\[ \frac{\gamma}{2\pi} = \begin{cases} 0, & g_{ab} \leq g_{ab}^*, \\ \beta_a + \beta_b, & g_{ab} > g_{ab}^* \end{cases}, \]

where we use \( \gamma = 1 + \gamma/N \) with \( N_a = N_b = N \). In FIG. 2, we demonstrate the scaled Berry phase of the system as a function of coupling strength, \( g_{ab} \), for a finite and infinite number of particles. As it is shown, in the large particle limit, the scaled Berry phase has a zero value for \( g_{ab} \leq g_{ab}^* \) and above \( g_{ab}^* \) increases with increasing the coupling strength and its derivative becomes discontinuous at the critical value, \( g_{ab}^* \). Interestingly, in the thermodynamic limit, there is a step-like transition, which can be seen in the inset of the FIG. 2(a). This is due to solutions of the Eqs. (14) and (15). When \( t_a = 0 \), one can see that there is a single solution for each displacement, i.e. \( \beta_a = 0.5 \).

It is also possible to obtain non-trivial Berry phase for each species, if we define unitary transformation as \( U(\phi(t)) = e^{-i\phi(t)} \hat{J}_{ax} \) and one can obtain \( \gamma_a = 2\pi \langle \hat{J}_{ax} \rangle \). It can be read from Eq. (32) that above the critical coupling strength, \( g_{ab}^* \), there is a finite atomic inversion for each species. This illustrates the fact that each species has also non-trivial Berry phase.

As it is shown above, increasing the number of particles creates sharper transition. And/or increasing interspecies...
coupling ends up with higher value of the Berry phase. This scenario, however, is not the same for bipartite entanglement. In the following section, we discuss this in more detail.

IV. BIPARTITE ENTANGLEMENT

The precise control and characterization of large systems on a single particle level presents number of conundrums. To overcome such issues, atomic ensembles with large number of particles, in general, are controlled by global parameters, such as total spin. By having such observables in a system, it becomes possible to quantify entanglement [28, 49–52]. For example, in a recent experiments [31–33], bipartite entanglement has been reported for ultracold atomic BECs by measuring collective spins.

There are several practical criteria [27, 28, 36] for the detection of entanglement. These methods are, in general, sufficient, but not necessary. Depending on the structure in a given system, some criteria work better. For instance, in our recent work [41], we compare different types of the criteria when the system exhibits quantum phase transition.

In this section, we anticipate the spin form of the criterion derived by Hillery & Zubairy [36] to investigate bipartite entanglement. We first introduce the inequality for the detection of entanglement based upon the effective local spin operators [29, 31], which is given by [53]

\[ E_{\text{HZ}} = \langle \hat{J}_a^+ \hat{J}_a^- \hat{J}_b^+ \hat{J}_b^- \rangle - |\langle \hat{J}_a^+ \hat{J}_b^- \rangle|^2 < 0, \]

where \( E_{\text{HZ}} < 0 \) witnesses the presence of entanglement between the two collective spins. It is important to note that a complete analysis of the derivation of the criterion is beyond the scope of the present paper, and we address the reader to the Refs. [53, 55] for more details.

In FIG. 3, we demonstrate the results of Eq. (36) as a function of interspecies coupling strength \( g_{ab} \), and for various number of particles. When the interspecies coupling strength exceeds the critical value there appears a transition also in the entanglement. Unlike the Berry phase, entanglement decays at larger values of the \( g_{ab} \) and/or N. This is due to effective coupling strength, \( \lambda \), in Eq. (19), which can be thought as the source of bipartite entanglement. The explanation of this behaviour can be done as follows. When the interatomic coupling strength increases, \( g_{ab} \gg g_{ab}^* \), the value of the displacement parameters approaches to the single value, i.e., \( \beta_a \to 0.5 \), where effective coupling vanishes [see Eq. (18)]. The similar story is valid for the increasing number of particles. As, \( N \to \infty \), \( \beta_a \to 0.5 \) [see FIG. 1]. This can be observed in the inset of the FIG. 3.

V. SUMMARY

In summary, we have investigated theoretically the ground state properties of the two-component Bose-Einstein-condensate trapped in a double-well potential. We observe that the system can undergo a QPT at a critical coupling strength. We obtain the effective Hamiltonians and associated ground state wavefunctions to describe the system in each of its mixed and separated phases. The non-trivial geometric phase is found near the critical coupling in the limit of small and large number of particles, where we observe a step-like transition in the thermodynamic limit. We also anticipate the spin form of Hillery & Zubairy criterion to quantify entanglement across a QPT. It is observed that entanglement decays with increasing interspecies interaction strength and/or number of particles. The tunable interactions between the two species via Feshbach resonances, making the model a promising simulator for this kind of structures, and can find potential in the area of quantum communication [56] and quantum sensing [57].

ACKNOWLEDGMENTS

This research was supported by The Scientific and Technological Research Council of Turkey (TUBITAK) Grant No. 117F118.

APPENDIX

Here we show the derivations of Eqs. (29) and (30). By inserting displaced operators defined in Eq. (12) into the
Eq. (1), it becomes:

\[
\hat{J}_\alpha^+ = (\hat{d}_\alpha + \chi_\alpha \sqrt{N_\alpha \beta_\alpha}) \sqrt{1 - \beta_\alpha \sqrt{1 - \xi_\alpha}}, \quad \xi_\alpha \equiv \left(1 - \frac{\hat{d}_\alpha^+ \hat{d}_\alpha + \chi_\alpha \sqrt{N_\alpha \beta_\alpha}}{N_\alpha (1 - \beta_\alpha)}\right).
\]

After expanding the last term, \(\sqrt{1 - \xi_\alpha}\), in Eq. (37) up to \(1/N\) order, one can arrive:

\[
\hat{J}_\alpha^+ \approx \sqrt{1 - \beta_\alpha} \left[ (\hat{d}_\alpha^+ + \sqrt{N_\alpha \beta_\alpha}) - \frac{\beta_\alpha}{1 - \beta_\alpha} \hat{X}_\alpha \right].
\]

Similarly, one can derive lowering component, \(\hat{J}_\alpha^- = (\hat{J}_\alpha^+)^\dagger\). By using the definitions: \(\hat{J}_{ax} = (\hat{J}_\alpha^+ + \hat{J}_\alpha^-)/2\) and \(\hat{J}_{ay} = (\hat{J}_\alpha^+ - \hat{J}_\alpha^-)/2i\), we derive Eqs. (39) and (40) as

\[
\hat{J}_{ax} \approx \chi_\alpha \sqrt{N_\alpha \beta_\alpha (1 - \beta_\alpha)} + \frac{1 - 2\beta_\alpha}{\sqrt{2(1 - \beta_\alpha)}} \hat{X}_\alpha,
\]

\[
\hat{J}_{ay} \approx \sqrt{1 - \beta_\alpha} \frac{\hat{P}_\alpha}{2},
\]

It is straightforward to obtain \(z\)-component of the angular momentum operator, by inserting Eqs. (12), (19) and (20) into the definition of \(\hat{J}_{az}\) given in Eq. (1).

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