THE INTEGRATION THEORY OF CURVED ABSOLUTE $L\infty$-ALGEBRAS

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ABSTRACT. In this article, we introduce the notion of a curved absolute $L\infty$-algebra, a structure that behaves like a curved $L\infty$-algebra where all infinite sums of operations are well-defined by definition. We develop their integration theory by introducing two new methods in integration theory: the complete Bar construction and intrinsic model category structures. They allow us to generalize all essential results of this theory quickly and from a conceptual point of view. We provide applications of our theory to rational homotopy theory, and show that curved absolute $L\infty$-algebras provide us with rational models for finite type nilpotent spaces without any pointed or connected assumptions. Furthermore, we show that the homology of rational spaces can be recovered as the homology of the complete Bar construction. We also construct new smaller models for rational mapping spaces without any hypothesis on the source simplicial set. Another source of applications is deformation theory: on the algebraic side, we show that curved absolute $L\infty$-algebras are mandatory in order to encode the deformation complexes of $\infty$-morphisms of (co)-algebras. On the geometrical side, we construct a curved absolute $L\infty$-algebra from a derived affine stack and show that it encodes the formal geometry of any finite collection of points living in any finite field extension of the base field.

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INTRODUCTION

Global picture. In deformation theory, one studies how to "deform" structures (algebraic, geometric, etc) on an object. Here "deform" can have multiple meanings. Let $X$ be some type of object and $\mathcal{P}$ some type of structure. Ideally, one has a moduli space $\text{Def}_X(\mathcal{P})$ where the points are the $\mathcal{P}$-structures that we can endow $X$ with. If one considers these structures up to some equivalences, and remembers the equivalences between the equivalences and so on and so forth, one has an $\infty$-groupoid (or moduli stack). In this context, a first meaning of "deform" is, given a $\mathcal{P}$-structure on $X$, a point $x$ in $\text{Def}_X(\mathcal{P})$, to look at the formal neighborhood of $x$ inside $\text{Def}_X(\mathcal{P})$. This formal neighborhood encodes the infinitesimal deformations, structures "infinitely close" to $x$.

Over the years, it was noticed by D. Quillen, P. Deligne, V. Drinfeld and many others that in all practical examples of these deformations, the formal neighborhoods were "encoded"
by a differential graded (dg) Lie algebras. Examples include the work of K. Kodaira and D. Spencer in [KS58], encoding deformations of complex structures on manifolds or the work of M. Gerstenhaber in [Ger64], encoding deformations of associative algebra structures on a vector space. This point of view suggests that for every point \( x \) in \( \text{Def}_X(P) \), there should be a dg Lie algebra \( g_{x,X} \) such that infinitesimal deformations correspond to elements in \( g_{x,X} \) which satisfy the Maurer–Cartan equation. This principle was stated by Drinfel’d in one of his letter to Schechtman:

"Every infinitesimal deformation problem in characteristic zero is encoded by a dg Lie algebra."

This became recently a theorem by J. Lurie and J. P. Pridham, see [Lur11, Pri10]. The method was to formalize what an "infinitesimal deformation" is using the notion of K-pointed formal moduli problem and then show that their homotopy category is equivalent to the homotopy category of dg Lie algebras. But given a dg Lie algebra, how can one recover the \( \infty \)-groupoid of infinitesimal deformation it encodes?

Another point of view on the deformation theory of algebraic structures is given by operadic deformation complexes. In this context, given an operad \( P \) and a dg module \( M \), one can construct an explicit dg Lie algebra where Maurer–Cartan elements are exactly the \( P \)-algebra structures on \( M \) and where the equivalences (called \( \infty \)-isotopies) between these structures are given by the action of a group called the gauge group. Here "deforming" structures can also mean understanding when two \( P \)-algebra structures are gauge equivalent to each other or not. Again, this amounts to studying the \( \infty \)-groupoid that is encoded by the operadic deformation problem.

Integration theory amounts to constructing a way to recover these \( \infty \)-groupoids from their dg Lie algebras. Its most basic example can be traced back to Lie theory and the Baker–Campbell–Hausdorff formula which produces a group out of a nilpotent Lie algebra. This allows us to recover the gauge group in operadic deformation complexes. A first general approach to the integration of dg Lie algebras is given by the seminal work of V. Hinich in [Hin01], using methods from D. Sullivan [Sul77]. A refined version of the integration procedure was constructed by E. Getzler in [Get09], where he extended the integration procedure to nilpotent \( \mathcal{L}_\infty \)-algebras (homotopy Lie algebras). Inspired by the ideas of U. Buijs, Y. Félix, A. Murillo and D. Tanré in [BFMT20], generalized using operadic calculus, D. Robert-Nicoud and B. Vallette were able in [RNV20] to give a new characterizations of Getzler’s functor and obtained an adjunction

\[
\begin{array}{ccc}
\text{sSet} & \overset{L}{\longrightarrow} & \mathcal{L}_\infty\text{-alg}^{\text{comp}} \\
\downarrow & & \downarrow \\
\mathcal{R} & & 
\end{array}
\]

between simplicial sets and complete \( \mathcal{L}_\infty \)-algebras. This adjunction lies at the crossroad of three domains: Lie theory, deformation theory, and rational homotopy. Here the right adjoint functor \( \mathcal{R} \) produces an integration functor from complete \( \mathcal{L}_\infty \)-algebras to \( \infty \)-groupoids (Kan complexes). But the horn-fillers of this \( \infty \)-groupoid are a structure, not a property. They are given by explicit formulas which extend with the classical Baker–Campbell–Hausdorff formula in the specific case of Lie algebras. Finally, the functor \( L \) is shown to produce faithful rational models for pointed connected finite type nilpotent spaces, and greatly simplifies the original approach of D. Quillen [Qui69] to rational homotopy. Similar rational models using complete dg Lie algebras where also constructed in [BFMT20].

**Motivations.** The first goal of this article is to generalize the integration procedure to curved \( \mathcal{L}_\infty \)-algebras. Curved Lie or curved \( \mathcal{L}_\infty \)-algebras are a notion more general than their dg counterparts: these objects are endowed with a distinguished element, the curvature, which perturbs the relationships satisfied by the bracket and higher operations in a dg Lie or \( \mathcal{L}_\infty \)-algebra. In particular, the "differential" no longer squares to zero, thus the notion of quasi-isomorphism so useful in homotopical algebra disappears.
The main reason to consider curved $L_\infty$-algebras is that they are the "non-pointed analogue" of $L_\infty$-algebras. In a classical $L_\infty$-algebra, the element 0 is always a Maurer–Cartan element. This gives a canonical base point in the corresponding $\infty$-groupoid. It is no longer the case with curved $L_\infty$-algebras. From the point of view of Lie theory, these objects behave like Lie groupoids instead of Lie groups. From the point of view of rational homotopy theory, these objects provide us with rational models for non necessarily pointed and non necessarily connected spaces.

Developing the integration theory of curved $L_\infty$-algebras has also many applications to deformation theory. In his PhD. thesis [Nui19], J. Nuiten showed that, if $A$ is a dg unital commutative algebra, then the homotopy category of $A$-pointed formal moduli problems is equivalent to the homotopy category of dg Lie algebroids over $A$. Later, it was showed by D. Calaque, R. Campos and J. Nuiten in [CCN21] that the homotopy category of dg Lie algebroids over $A$ is equivalent to the homotopy category of certain curved $L_\infty$-algebras over the de Rham algebra of $A$, under some assumptions on $A$. Therefore deformation problems "parametrized by Spec$(A)$" can be encoded with curved $L_\infty$-algebras. "Parametrized by Spec$(A)$" means that instead of choosing a $K$-point in a moduli space $X$, one chooses a morphism $f : \text{Spec}(A) \to X$, and then looks at the formal neighborhood of Spec$(A)$ inside of $X$. Another approach to parametrized deformation problems is also given by $L_\infty$-spaces introduced by K. Costello in [Cos11]. These are families of curved $L_\infty$-algebras parametrized by smooth manifolds. They were constructed in order to treat parametrized deformation problems arising in fundamental physics. On the other side, operadic deformation complexes of unital algebraic structures form curved Lie algebras as shown in [RiL22b]. Furthermore, the space of $\infty$-morphisms between types of unital algebras or the space of $\infty$-morphisms between types of counital coalgebras are encoded, in both cases, by convolution curved $L_\infty$-algebras. In all the above cases, having a good integration theory for curved $L_\infty$-algebras is of primordial importance.

**Framework and methods.** The second main goal of this article is to introduce new methods in integration theory. This is first done by introducing a new kind of algebraic objects that we call curved absolute $L_\infty$-algebras. A curved absolute $L_\infty$-algebra can be thought as curved $L_\infty$-algebra where all infinite sums of operations have a well defined image without supposing any underlying topology. Infinite sums appear naturally in the theory of $L_\infty$-algebras with the Maurer–Cartan equation. So far, there has been two standard ways to deal with them either restrict to nilpotent $L_\infty$-algebras, or use the somewhat ad hoc approach of changing the base category in order to consider objects with an underlying complete topology, so that these infinite sums converge. Our approach solves these problems altogether. It also includes the classical examples of nilpotent Lie algebras and nilpotent $L_\infty$-algebras in the sense of [Get09].

Considering curved absolute $L_\infty$-algebras, which are encoded with a curved cooperad, allows us to use the operadic calculus developed by B. Le Grignou and D. Lejay in [GL22]. We introduce $u\mathbb{C}E_\infty$-coalgebras, a version of homotopy counital cocommutative coalgebras encoded by a dg operad as a Koszul dual notion of curved absolute $L_\infty$-algebras. We then construct a complete Bar–Cobar adjunction between these two types of objects. In particular, the complete Bar construction, given in terms of the non-nilpotent cofree $u\mathbb{C}E_\infty$-coalgebra, proves to be a new key ingredient in the theory. Using similar methods that in [RNV20], we can now construct a $u\mathbb{C}E_\infty$-coalgebra structure on the cellular chain functor and thus obtain the integration functor composing with the aforementioned adjunction. Notice that the methods of loc.cit. where definitely not generalizable on the nose, since in the curved case, one has to consider counital coalgebras, which are not conilpotent, and thus, which cannot be encoded by cooperads. This makes the usual operadic calculus fail in this context.

Another key ingredient to our results is the use of new model structures. In the curved setting, the notion of quasi-isomorphism no longer makes sense, and one needs to find another
notion of weak-equivalences. We solve this difficulty by first endowing the category of $\mathcal{uCC}_\infty$-coalgebras with a model category structure where weak-equivalences are given by quasi-isomorphisms and cofibrations by monomorphisms. Secondly, we transfer it onto the category of curved absolute $L_\infty$-algebras, thus obtaining a meaningful notion of weak-equivalence in the curved context. This intrinsic model category structure proves to be very natural, as it allows us to generalize directly all the main results of integration theory to our setting. From a conceptual point of view, this allows us to fully embed integration theory in the context of homotopy theory. The functor that associates to a nilpotent Lie algebra a group using the Baker–Campbell–Hausdorff formula is now naturally the restriction of a right Quillen functor.

The last main new method used in this article is the extensive use that we make of the duality squares that we introduced in [RiL22a, Section 2]. With them, one can understand homotopically the linear duality functors between types of coalgebras and types of algebras. This tool proves to be essential in order to obtain results in rational homotopy theory and in deformation theory.

**Main results.** We construct a cosimplicial $\mathcal{uCC}_\infty$-coalgebra $C_c^\bullet(\Delta^\bullet)$ by endowing the cellular chains on the simplicies with a $\mathcal{uCC}_\infty$-coalgebra. By Kan extension, this gives an adjunction

$$s\text{Set} \xrightarrow{\mathcal{C}_c^\bullet(\cdot)} \mathcal{uCC}_\infty\text{-coalg} \xleftarrow{\mathcal{R}} \text{curv abs } L_\infty\text{-alg},$$

where $\mathcal{C}_c^\bullet(\cdot)$ is the usual cellular chain functor with a $\mathcal{uCC}_\infty$-coalgebra structure. Since we endowed the category of $\mathcal{uCC}_\infty$-coalgebras with an intrinsic model category structure where weak-equivalences are given by quasi-isomorphisms, it is automatic to check that it forms a Quillen adjunction. By composing the above adjunction with the complete Bar-Cobar adjunction we get the following triangle.

Theorem A (Theorem 2.11). The following triangle of Quillen adjunctions commutes

Here $\hat{\Omega}_1$ is a completed Cobar construction, and $\hat{B}_1$ is a new complete Bar construction. Having this constructions allows to define the integration functor $\mathcal{R}$ as the composite of the functor $\mathcal{R}$ with the complete Bar construction. One of the main novelties of this is the fact that $\mathcal{R}$ defined in this way is automatically right Quillen functor with respect to transferred model category structure on curved absolute $L_\infty$-algebra. This simple fact directly implies the required properties that a well-behaved integration functor needs to satisfy.

Theorem B (Theorem 2.12).

1. For any curved absolute $L_\infty$-algebra $g$, the simplicial set $\mathcal{R}(g)$ is a Kan complex.
2. For any degree-wise epimorphism $f : g \to h$ of curved absolute $L_\infty$-algebras, the induced map

$$\mathcal{R}(f) : \mathcal{R}(g) \to \mathcal{R}(h)$$

is a fibrations of simplicial sets.
3. The functor $\mathcal{R}(\cdot)$ preserves weak equivalences. In particular, it sends any graded quasi-isomorphism between complete curved absolute $L_\infty$-algebras to a weak homotopy equivalence of simplicial sets.
The fact that $\mathcal{R}(g)$ is an $\infty$-groupoid is quintessential to integration theory as developed in [Hin01] and in [Get09]. The second statement is a generalization of one of the main theorems of [Get09]. The third point, the homotopy invariance of the integration functor, is the generalization of the celebrated Goldman–Milson Theorem of [GM88] and its extension by Dolgushev–Rogers to $\mathcal{L}_\infty$-algebras in [DR15]. Also, since $\mathcal{R}$ is a right Quillen functor, it automatically commutes with homotopy limits and therefore satisfies descent in the sense of [Hin97]. Moreover, the adjunction $\mathcal{L} \dashv \mathcal{R}$ is shown to be a non-abelian generalization of the Dold-Kan correspondence.

We then develop higher absolute Lie theory. We introduce and characterize gauge equivalences for Maurer–Cartan elements in this setting. We construct higher Baker–Campbell–Hausdorff products for curved absolute $\mathcal{L}_\infty$-algebras and we show that these are given by the same explicit formulae as in [RNV20]. We generalize A. Berglund’s Theorem of [Ber15] to the case of curved absolute $\mathcal{L}_\infty$-algebras with completely new proof based on a simple computation of the $\text{uCE}_\infty$-coalgebra structures on the cellular chains of the spheres. This gives a way to compute the homotopy groups of $\mathcal{R}(g)$ using the twisted homology of $g$. It is worth mentioning that Getzler’s and Hinich’s approach to integration theory would not have worked in this context, since the tensor product of a commutative algebra with a curved absolute $\mathcal{L}_\infty$-algebra has no natural curved absolute $\mathcal{L}_\infty$-algebra structure.

The third section is devoted to constructing rational homotopy models using the new functor $\mathcal{L}$. The fact that curved $\mathcal{L}_\infty$-algebras are the “non-pointed analogue” of $\mathcal{L}_\infty$-algebras allows to obtain ration Lie models for non necessarily pointed nor connected finite type nilpotent spaces using our functor $\mathcal{L}$.

**Theorem C** (Theorem 3.11). Let $X$ be a finite type nilpotent simplicial set. The unit of the adjunction

$$\eta_X : X \xrightarrow{\sim} \mathcal{R}(\mathcal{L}(X))$$

is a rational homotopy equivalence.

We show that if a curved absolute $\mathcal{L}_\infty$-algebra $g_X$ is a rational model for a space $X$, then we can recover the homology of $X$ by computing the homology of the complete Bar construction of $g_X$. This result can be viewed as Eckmann-Hilton dual to the A. Berglund’s theorem mentioned before. One can think of the complete Bar construction of a curved absolute $\mathcal{L}_\infty$-algebras as a higher Chevalley-Eilenberg complex.

We use the theory of B. Le Grignou developed in the forthcoming paper [Gri22b] to get a convolution curved absolute $\mathcal{L}_\infty$-algebra structure on the space of graded morphisms between a $\text{uCE}_\infty$-coalgebra and a curved absolute $\mathcal{L}_\infty$-algebra.

**Theorem D** (Theorem 3.26). Let $g$ be a curved absolute $\mathcal{L}_\infty$-algebra and let $X$ be a simplicial set. There is a weak equivalence of Kan complexes

$$\text{Map}(X, \mathcal{R}(g)) \xrightarrow{\sim} \mathcal{R}(\text{hom}(C^\infty_\ast(X), g)) ,$$

which is natural in $X$ and in $g$, where $\text{hom}(C^\infty_\ast(X), g)$ denotes the convolution curved absolute $\mathcal{L}_\infty$-algebra.

Notice that, for the first time, there are no assumptions on $X$ nor on $g$. If $Y$ is a finite type nilpotent simplicial set, the above theorem gives an explicit model for the mapping space of $X$ and the $Q$-localization of $Y$. Furthermore, this model is constructed using the cellular chains on $X$, hence it smaller than other rational models constructed before. See for instance [Ber15, BFM13, Laz13].

The last section is devoted to applications in deformation theory, where the new notion of curved absolute $\mathcal{L}_\infty$-algebras proves to be mandatory. We construct convolution curved algebra $\mathcal{L}_\infty$-algebras that encode $\infty$-morphisms between unital types of algebras as their Maurer–Cartan elements. This gives new notions of homotopies between $\infty$-morphisms and opens
the way to simplicial enrichments. The very nature of convolution algebras is "absolute"; they always admit infinite sums of operations without the need of an underlying filtration to make them converge. Therefore same methods could be applied to $\infty$-morphisms of coalgebras over dg operads or, more generally, $\infty$-morphisms of gebras over properads as defined by E. Hoffbeck, J. Leray and B. Vallette in [HLV21].

Finally, we explore the geometrical properties of curved absolute $L_\infty$-algebras. Starting from a derived affine stack $A$, that is, a dg unital commutative algebra, we construct an explicit curved absolute $L_\infty$-algebra model $g_A$. We then show that the formal neighborhood of any $L$-point of $A$ can be recovered from $g_A$, where $L$ is a finite field extension of our base field $K$. Furthermore, contrary to the $K$-pointed formal moduli approach, these points are not "specified in advance".

**Theorem E** (Theorem 4.20). Let $A$ be a dg $\mathrm{uCom}$-algebra. Let $B$ be a dg Artinian algebra. There is a weak equivalence of simplicial sets

$$\mathrm{Spec}(A)(B) \simeq R(\mathrm{hom}((\mathrm{Res}_\varepsilon B)^\circ, g_A)),$$

natural in $B$ and in $A$.

The trick behind the above theorem lies in the definition of a dg Artinian algebra. Since we are working with curved absolute $L_\infty$-algebras, dg Artinian algebras need not be augmented. In fact, we simply define them as dg unital commutative algebras in non-negative degrees with degree-wise finite dimensional homology. Conversely, any curved absolute $L_\infty$-algebra defines a deformation functor from dg Artinian algebras to $\infty$-groupoids. We view these results as a first step in the theory of "non-pointed" formal moduli problems.

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**Conventions.** We adopt the same conventions as in [RiL22a]. We work over a field $K$ of characteristic 0. The base category will be the symmetric monoidal category $(\mathrm{pdg-mod}, \otimes, K)$ of pre-differential graded $K$-modules with the tensor product. We denote $(\mathrm{pdg S-mod}, \circ, I)$ the monoidal category of graded $S$-modules endowed with the composition product $\circ$. This framework is explained with more details in [RiL22b].

1. **Curved absolute $L_\infty$-algebras**

In this section, we introduce the notion of curved absolute $L_\infty$-algebras. This is a new type of curved $L_\infty$-algebras, encoded by a conilpotent curved cooperad. They posses a much richer algebraic structure than usual curved $L_\infty$-algebras. Any infinite sum of operations has a well-defined image by definition. In particular, the Maurer–Cartan equation is always defined. The rest of this article is devoted to the study of this new notion and its applications. Notice that our degree conventions will correspond to shifted curved $L_\infty$-algebras. This shifted convention will be implicit from now on. Curved absolute algebras appear as the Koszul dual of a specific model for non-necessarily conilpotent $E_\infty$-coalgebras, which we call $\mathrm{uCE}_\infty$-coalgebras. We construct a complete Bar-Cobar adjunction that relates these two types of algebraic objects and show that it is a Quillen equivalence.
1.1. **Curved absolute \( \mathcal{L}_\infty \)-algebras.** Let \( \mathcal{uCom} \) be the operad encoding unital commutative algebras and let \( e : \mathcal{uCom} \to \mathcal{I} \) be the canonical morphism of \( S \)-modules given by the identity on \( \mathcal{uCom} (1) \cong \mathcal{I} \). We denote \( B^{s.a} \mathcal{uCom} \) its semi-augmented Bar construction with respect to this canonical semi-augmentation. We refer to Appendix 4.2 for more details on this particular construction.

**Definition 1.1 (Curved absolute \( \mathcal{L}_\infty \)-algebra).** A curved absolute \( \mathcal{L}_\infty \)-algebra \( g \) amounts to the data \( (g, \gamma_\theta, d_\theta) \) of a curved \( B^{s.a} \mathcal{uCom} \)-algebra.

**Remark 1.2.** Brief recollections on the notion of curved algebras over cooperads are given in [RiL22a, Section 1]. We also refer to [RiL22c] for a more thorough exposition of the theory developed in [GL22].

Let us unravel this definition. The data of a curved absolute \( \mathcal{L}_\infty \)-algebra structure on a pdg module \( (g, d_\theta) \) amounts to the data of a morphism of pdg modules

\[
\gamma_\theta : \prod_{n \geq 0} \text{Hom}_{n} (B^{s.a} \mathcal{uCom} (n), g^{\otimes n}) \to g ,
\]

which satisfies the conditions of [RiL22a, Definition 1.12 and 1.22]. This map admits a simpler description.

**Lemma 1.3.** There is an isomorphism of pdg modules

\[
\prod_{n \geq 0} \text{Hom}_{n} (B^{s.a} \mathcal{uCom} (n), g^{\otimes n}) \cong \prod_{n \geq 0} \hat{\Omega}^{s.a} \mathcal{uCom}^* (n) \hat{\otimes}_n g^{\otimes n} ,
\]

natural in \( g \), where \( \hat{\otimes} \) denotes the completed tensor product with respect to the canonical filtration on the complete Cobar construction.

**Proof.** By definition, \( B^{s.a} \mathcal{uCom} \) is a conilpotent curved cooperad. Let us denote by \( \mathcal{R}_\omega B^{s.a} \) the \( \omega \)-term of its coradical filtration. There is an isomorphism of conilpotent curved cooperads

\[
B^{s.a} \mathcal{uCom} \cong \operatorname{colim}_\omega \mathcal{R}_\omega B^{s.a} \mathcal{uCom} .
\]

Notice that for every \( n \geq 0 \) and every \( \omega \geq 0 \), \( \mathcal{R}_\omega B^{s.a} \mathcal{uCom} (n) \) is degree-wise finite dimensional. Thus we have

\[
\text{Hom}_{n} (B^{s.a} \mathcal{uCom} (n), g^{\otimes n}) \cong \text{Hom}_{n} \left( \operatorname{colim}_\omega \mathcal{R}_\omega B^{s.a} \mathcal{uCom} (n), g^{\otimes n} \right) ,
\]

\[
\cong \lim_{\omega} \text{Hom}_{n} \left( \mathcal{R}_\omega B^{s.a} \mathcal{uCom} (n), g^{\otimes n} \right) .
\]

for all \( n \geq 0 \). By Lemma 4.36, there is an isomorphism

\[
\mathcal{R}_\omega B^{s.a} \mathcal{uCom} \cong \hat{\Omega}^{s.c} \mathcal{uCom}^* / \mathcal{F}_\omega \hat{\Omega}^{s.c} \mathcal{uCom}^* ,
\]

where \( \mathcal{F}_\omega \hat{\Omega}^{s.c} \mathcal{uCom}^* \) denotes the \( \omega \)-term of its canonical filtration as an absolute partial operad. Therefore there are isomorphisms

\[
\lim_{\omega} \text{Hom}_{n} \left( \mathcal{R}_\omega B^{s.a} \mathcal{uCom} (n), g^{\otimes n} \right) \cong \lim_{\omega} \left( \hat{\Omega}^{s.c} \mathcal{uCom}^* / \mathcal{F}_\omega \hat{\Omega}^{s.c} \mathcal{uCom}^* (n) \otimes g^{\otimes n} \right)^{S_n}
\cong \left( \hat{\Omega}^{s.c} \mathcal{uCom}^* (n) \otimes g^{\otimes n} \right)^{S_n}
\cong \hat{\Omega}^{s.c} \mathcal{uCom}^* (n) \otimes g^{\otimes n} .
\]

Notice that the last isomorphism identifies invariants with coinvariants, which is possible because of the characteristic zero assumption. Nevertheless, this identification carries non-trivial coefficients, see Remark 1.6.

**Notation 1.4.** Let \( \text{CRT}_n^\omega \) denote the set of corked rooted trees of arity \( n \) and with \( \omega \) internal edges. A corked rooted tree is a rooted tree where vertices either have at least two incoming edges or zero incoming edges, which are called corks. The arity of a corked rooted tree is the
number of non-corked leaves. The unique rooted tree of arity \( n \) with one vertex is called the \( n \)-corolla, denoted by \( c_n \). Notice that for each \( n \), the set \( \text{CRT}_n \) of corked rooted trees of arity \( n \) is infinite. Likewise, for each \( \omega \), the set of corked rooted trees with \( \omega \) internal edges is also infinite. Nevertheless, the set \( \text{CRT}_n^\omega \) is finite. The only corked rooted tree of weight 0 is the trivial tree of arity one with zero vertices.

**Proposition 1.5.** Let \( (g, d_g) \) be a pdg module with a basis \( \{g_b \mid b \in B\} \). The pdg module

\[
\prod_{n \geq 0} \hat{\Omega}^{s,a,u}\text{Com}^*(n) \otimes S_n g^\otimes n
\]

admits a basis given by double series on \( n \) and \( \omega \) of corked rooted trees in \( \text{CRT}_n^\omega \) labeled by the basis elements of \( g^\otimes n \). These basis elements can be written as

\[
\left\{ \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \sum_{i \in I_\tau} \lambda_\tau^{(i)} \tau \left( g_{i_1}^{(1)}, \ldots, g_{i_n}^{(1)} \right) \right\},
\]

where \( I_\tau \) is a finite set, \( \lambda_\tau \) is a scalar in \( K \) and \((i_1, \ldots, i_n)\) is in \( B^n \). Here \( \tau(g_{i_1}, \ldots, g_{i_n}) \) is given by the rooted tree \( \tau \) with input leaves decorated by the elements \( \tau(g_{i_1}, \ldots, g_{i_n}) \). The degree of \( \tau(g_{i_1}, \ldots, g_{i_n}) \) is \( -\omega - 1 + |g_{i_1}| + \cdots + |g_{i_n}| \). An example of such decoration is given by

\[
\tau(g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}) =
\]

\[
\begin{array}{c}
g_{i_1} \\
g_{i_3} \\
g_{i_2} \\
g_{i_4} \\
1 \\
3 \\
2 \\
4
\end{array}
\]

for \( \tau =
\]

The pre differential of the pdg module is given by the sum of two terms: the first term applies the pre-differential \( d_g \) on each of the labels \( g_{i_1} \) of a corked rooted tree \( \tau \), the second splits each vertex into two vertices in all the possible ways including the splitting with a cork.

**Proof.** Corked rooted trees form a basis of the pdg \( S \)-module \( \hat{\Omega}^{s,a,u}\text{Com}^* \), hence the results follows by direct inspection. \( \square \)

**Remark 1.6 (Renormalization).** Let \( \tau \) be a corked rooted tree. It can be written as \( c_m \circ (\tau_1, \ldots, \tau_m) \). We define recursively the following coefficient for corked rooted trees:

\[
\mathcal{E}(c_n) := n! \quad \text{and} \quad \mathcal{E}(c_m \circ (\tau_1, \ldots, \tau_m)) := m! \prod_{t=1}^m \mathcal{E}(\tau_t).
\]

Then doubles series

\[
\left\{ \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \sum_{i \in I_\tau} \mathcal{E}(\tau) \lambda_\tau^{(i)} \tau \left( g_{i_1}^{(1)}, \ldots, g_{i_n}^{(1)} \right) \right\},
\]

form a basis of

\[
\prod_{n \geq 0} \text{Hom}_{S_n} \left( B^{s,a,u}\text{Com}(n), g^\otimes n \right) \cong \prod_{n \geq 0} \left( \hat{\Omega}^{s,a,u}\text{Com}^*(n) \otimes S_n g^\otimes n \right)^{S_n}.
\]

**Structural data.** Thus the data of a curved absolute \( \mathcal{L}_\infty \)-algebra structure on a pdg module \( (g, d_g) \) amounts to the data of a morphism of pdg modules

\[
\gamma_g : \prod_{n \geq 0} \hat{\Omega}^{s,a,u}\text{Com}^*(n) \otimes S_n g^\otimes n \to g,
\]
which satisfies the conditions of [RiL22a, Definition 1.12 and 1.22]. In particular, for any infinite sum
\[
\sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \sum_{l \in I_l} \lambda_{l}^{(i)} \tau \left( g_{l_1}^{(i)}, \ldots, g_{l_n}^{(i)} \right)
\]
there is a well-defined image
\[
\gamma_{\theta} \left( \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \sum_{l \in I_l} \lambda_{l}^{(i)} \tau \left( g_{l_1}^{(i)}, \ldots, g_{l_n}^{(i)} \right) \right),
\]
in the pdg module \( g \). This does not presuppose an underlying topology on the pdg module \( g \).

**Notation 1.7.** Since elements in \( \widehat{\mathcal{E}}_u \mathcal{C}^* / \mathcal{F}_u \widehat{\mathcal{E}}_u \mathcal{C}^* \otimes g^\otimes n \) might not be simple tensors, this makes the sums \( \sum_{i \in I_l} \) appear in the formulae. From now on, we will omit these sums and work only with simple tensors. Computations for infinite sums of non-simple tensors are, *mutatis mutandis*, the same.

**Pdg condition.** Let us make explicit each of the conditions satisfied by the structural morphism \( \gamma_{\theta} \). It is a morphism of pdg modules if only if, we have that
\[
\gamma_{\theta} \left( \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \lambda_{\tau} d_{\ell u \mathcal{C}^\theta} \left( \tau \right) (g_{l_1}, \ldots, g_{l_n}) \right) + \gamma_{\theta} \left( \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \sum_{j=0}^{n} (-1)^{j} \lambda_{\tau} (g_{l_1}, \ldots, d_{\theta} (g_{l_i}), \ldots, g_{l_n}) \right) =
\]
(1)
\[
d_{\theta} \left( \gamma_{\theta} \left( \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \lambda_{\tau} (g_{l_1}, \ldots, g_{l_n}) \right) \right).
\]

**Associativity condition.** The associativity condition on \( \gamma_{\theta} \) imposed by the diagram of [RiL22a, Definition 1.12] is equivalent to the following equality
\[
\gamma_{\theta} \left( \sum_{k \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_k^\omega} \lambda_{\tau} \gamma_{\theta} \left( \sum_{l_1 \geq 0} \sum_{\omega_1 \geq 0} \sum_{\tau_1 \in \text{CRT}_{l_1}^{\omega_1}} \lambda_{\tau_1} (g_{l_1}) \right) \ldots \gamma_{\theta} \left( \sum_{l_n \geq 0} \sum_{\omega_n \geq 0} \sum_{\tau_n \in \text{CRT}_{l_n}^{\omega_n}} \lambda_{\tau_n} (g_{l_n}) \right) \right) =
\]
(2)
\[
\gamma_{\theta} \left( \sum_{n \geq 0} \sum_{k \geq 0} \sum_{i_1 + \ldots + i_k = n} \sum_{\omega_{\tau} \geq 0} \sum_{\omega_{\tau} \geq 0} \sum_{\tau \in \text{CRT}_{\omega_{\tau}}^{\omega_{\tau}}} \lambda_{\tau} \left( \lambda_{\tau_1} \ldots \lambda_{\tau_k} \right) \tau \circ (\tau_{1}, \ldots, \tau_{k}) (g_{l_1}, \ldots, g_{l_k}) \right)
\]
where \( \bar{g}_{l_i} \) denotes an \( i_j \)-tuple of elements of \( g^{\otimes l} \) and where \( \tau \circ (\tau_{1}, \ldots, \tau_{k}) \) denotes the corked rooted tree obtained by grafting \( (\tau_{1}, \ldots, \tau_{k}) \) onto the leaves of \( \tau \). If \( \gamma_{\theta} \) satisfies the above conditions, it endows \( g \) with a pdg \( B^\mathcal{E}_u \mathcal{C} \)-algebra structure.

**Definition 1.8 (Elementary operations of a curved absolute \( L_\infty \)-algebra).** Let \( (g, \gamma_{\theta}, d_{\theta}) \) be a curved absolute \( L_\infty \)-algebra. The elementary operations of \( g \) are the family of symmetric operations of degree \(-1\) given by
\[
\{ \Gamma_{n} = \gamma_{\theta} (\ell_{n}(-, \ldots, -)) : g^{\otimes n} \to g \}
\]
for all \( n \neq 1 \).

**Curved condition.** The condition on \( \gamma_{\theta} \) imposed by the diagram of [RiL22a, Definition 1.22] amounts in this case to the following equation on the elementary operations
Conclusion. A curved absolute $\mathcal{L}_\infty$-algebra structure $\gamma_\theta$ on a pdg module $(g, d_g)$ amounts to the data of a degree 0 map

$$\gamma_\theta : \prod_{n \geq 0} \hat{\Omega}^{s.a}{\text{uCom}}^* (n) \otimes_S n \ g^\otimes_n \rightarrow g,$$

satisfying conditions 1, 2 and 3.

Remark 1.9 (Warning). In general, if $(g, \gamma_\theta, d_\theta)$ is a curved absolute $\mathcal{L}_\infty$-algebra, then

$$\gamma_\theta \left( \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \lambda_\tau \tau(g_{i_1}, \ldots, g_{i_n}) \right) \neq \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \lambda_\tau \gamma_\theta \left( \tau(g_{i_1}, \ldots, g_{i_n}) \right),$$

as the latter expression is not well-defined since infinite sums of elements in $g$ are not well-defined in general.

Morphisms. Let $(g, \gamma_\theta, d_\theta)$ and $(h, \gamma_\theta, d_\theta)$ be two curved absolute $\mathcal{L}_\infty$-algebras and let $f : g \rightarrow h$ be a morphism of pdg modules. The condition for $f$ to be a morphism of curved absolute $\mathcal{L}_\infty$-algebras can be written as

$$f \left( \gamma_\theta \left( \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \lambda_\tau \tau(g_{i_1}, \ldots, g_{i_n}) \right) \right) = \gamma_\theta \left( \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \lambda_\tau \tau(f(g_{i_1}), \ldots, f(g_{i_n})) \right).$$

Any curved absolute $\mathcal{L}_\infty$-algebra structure can be restricted along elementary operations to induce a curved $\mathcal{L}_\infty$-algebra structure in the standard sense.

Proposition 1.10. There is a restriction functor

$$\text{Res} : \text{curv abs } \mathcal{L}_\infty\text{-alg} \rightarrow \text{curv } \mathcal{L}_\infty\text{-alg},$$

from the category of curved absolute $\mathcal{L}_\infty$-algebras to the category of curved $\mathcal{L}_\infty$-algebras which is faithful.

Proof. Let $(g, \gamma_\theta, d_\theta)$ be a curved absolute $\mathcal{L}_\infty$-algebra, we can restrict the structural map

$$\text{Res}(\gamma_\theta) : \bigoplus_{n \geq 0} \hat{\Omega}^{s.c}{\text{uCom}}^* (n) \otimes_S n \ g^\otimes_n \xrightarrow{\iota_\theta} \prod_{n \geq 0} \hat{\Omega}^{s.c}{\text{uCom}}^* (n) \otimes_S n \ g^\otimes_n \xrightarrow{\gamma_\theta} g$$

along the natural inclusion $\iota_\theta$. It endows $g$ with a curved $\hat{\Omega}^{s.c}{\text{uCom}}^*$-algebra structure. By Proposition 4.38, this is equivalent to a classical curved $\mathcal{L}_\infty$-algebra structure in the sense of Definition 4.37. Any morphism of curved absolute $\mathcal{L}_\infty$-algebras is in particular a morphism of curved $\mathcal{L}_\infty$-algebras, and it faithful. □

Proposition 1.11. There restriction functor admits a left adjoint $\text{Abs}$, which called the absolute envelope of a curved $\mathcal{L}_\infty$-algebra. Therefore there is an adjunction

$$\text{curv } \mathcal{L}_\infty\text{-alg} \xleftarrow{\text{Res}} \xrightarrow{\text{Abs}} \text{curv abs } \mathcal{L}_\infty\text{-alg},$$

Proof. Both categories are presentable, by [AR94, Theorem 1.6.6] the functor Res admits as left-adjoint since it preserves all limits. □

Remark 1.12. For more comparison statements between the classical notion of curved $\mathcal{L}_\infty$-algebras as defined in Definition 4.37 and this new notion of curved absolute $\mathcal{L}_\infty$-algebra, we refer to [RiL22c, Chapter 3, Appendix A].
Completeness. The coradical filtration on the conilpotent curved cooperad $B^{s,a}u\mathcal{C}om$ induces a canonical filtration on any curved absolute $\mathcal{L}_\infty$-algebra. See [RiL22a, Definition 1.14]. In this case, this canonical filtration on $g$ is given by

$$W_{\omega}g := \text{Im} \left( \gamma_g |_{\mathcal{F}_g} : \prod_{n \geq 0} \mathcal{F}_g \hat{\mathcal{O}}^{s,a}u\mathcal{C}om^*[n] \otimes S_n g^\otimes n \rightarrow g \right),$$

for $\omega \geq 0$, where $\mathcal{F}_g \hat{\mathcal{O}}^{s,a}u\mathcal{C}om^*[n]$ is spanned by corked rooted trees with a number of internal edges greater or equal to $\omega$. Therefore an element $g$ is in $W_{\omega}g$ if and only if it can be written as

$$g = \gamma_g \left( \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \lambda_\tau \tau(g_{i_1}, \ldots, g_{i_n}) \right).$$

One can check that $g/W_{\omega}g$ has an unique curved absolute $\mathcal{L}_\infty$-algebra structure given by the quotient structural map.

**Remark 1.13.** Condition 1 implies that $d_g(W_{\omega}g) \subset W_{\omega}g$ and therefore the pre-differential $d_g$ is continuous with respect to the canonical filtration.

**Definition 1.14 (Complete curved absolute $\mathcal{L}_\infty$-algebra).** Let $(g, \gamma_g, d_g)$ be a curved absolute $\mathcal{L}_\infty$-algebra. It is complete if the canonical epimorphism

$$\varphi_g : g \twoheadrightarrow \lim_{\omega} g/W_{\omega}g$$

is an isomorphism of curved absolute $\mathcal{L}_\infty$-algebras.

**Remark 1.15.** We have that $\varphi_g$ is always an epimorphism by [RiL22a, Proposition 1.16], this phenomenon is explained in [RiL22a, Remark 1.17].

Let $(g, \gamma_g, d_g)$ be a complete curved absolute $\mathcal{L}_\infty$-algebra. One can write

$$\gamma_g \left( \sum_{n \geq 0} \sum_{\omega \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \lambda_\tau \tau(g_{i_1}, \ldots, g_{i_n}) \right) = \sum_{\omega \geq 0} \gamma_g \left( \sum_{n \geq 0} \sum_{\tau \in \text{CRT}_n^\omega} \lambda_\tau \tau(g_{i_1}, \ldots, g_{i_n}) \right),$$

using the fact that the canonical filtration on $g$ is complete.

**Definition 1.16 (Maurer–Cartan element of a curved absolute $\mathcal{L}_\infty$-algebra).** Let $(g, \gamma_g, d_g)$ be a curved absolute $\mathcal{L}_\infty$-algebra. A Maurer–Cartan element $\alpha$ is an element in $g$ of degree 0 which satisfies the following equation

$$\gamma_g \left( \sum_{n \geq 0, n \neq 0} \frac{c_n(\alpha, \ldots, \alpha)}{n!} \right) + d_g(\alpha) = 0.$$

The set of Maurer–Cartan elements in $g$ is denoted by $\mathcal{MC}(g)$.

**Remark 1.17.** The coefficients $1/n!$ appear because of the isomorphism between invariants and coinvariants in Lemma 1.3. When one considers the renormalization of this equation as in Remark 1.6, these coefficients disappear.

**Remark 1.18.** In a curved absolute $\mathcal{L}_\infty$-algebra, the element 0 is not in general a Maurer–Cartan element since:

$$\gamma_g \left( \sum_{n \geq 0, n \neq 0} \frac{c_n(0, \ldots, 0)}{n!} \right) + d_g(0) = \gamma_g(c_0) \neq 0.$$

Therefore the set $\mathcal{MC}(g)$ is not canonically pointed, neither non-necessarily non-empty.
REMARK 1.19. Notice that if one takes α to be a degree 0 element of g, the value of
\[
\gamma_\beta \left( \sum_{n \geq 0, n \neq 1} \frac{c_n(\alpha, \ldots, \alpha)}{n!} \right)
\]
is not determined in general by the values of the elementary operations \(l_n(\alpha, \ldots, \alpha)\). It can happen that \(l_n(\alpha, \ldots, \alpha) = 0\) for all \(n \geq 0\) but α is not a Maurer–Cartan element.

**Lemma 1.20.** Let \((g, \gamma_\beta, d_\beta)\) be a complete curved absolute \(L_\infty\)-algebra. Any degree 0 element \(\alpha\) in \(W_1 g\) satisfies
\[
\gamma_\beta \left( \sum_{n \geq 0, n \neq 1} \frac{c_n(\alpha, \ldots, \alpha)}{n!} \right) = \sum_{n \geq 0, n \neq 1} \frac{l_n(\alpha, \ldots, \alpha)}{n!}.
\]

**Proof.** Since \(\alpha\) is in \(W_1 g\), it can be written as
\[
\alpha = \gamma_\beta \left( \sum_{k \geq 0} \sum_{\omega \geq 1} \sum_{\tau \in \text{CRT}_\omega} \lambda_\tau \tau(g_\omega) \right).
\]
where \(g_\omega\) is a \(k\)-tuple in \(g\). A straightforward computation using the associativity condition 2 concludes the proof. \(\square\)

**Graded homology groups.** Let \((g, \gamma_\beta, d_\beta)\) be a curved absolute \(L_\infty\)-algebra. Recall that Condition 3 says that \(d_\beta^2(\cdot) = l_2(l_0, \cdot)\). Thus
\[
d_\beta^2(W_\omega g) \subseteq W_{\omega + 1} g.
\]
This implies that
\[
gr_\omega(g) := W_\omega g/W_{\omega + 1} g,
\]
forms a chain complex endowed with the differential induced by \(d_\beta\), for all \(\omega \geq 0\).

REMARK 1.21. Notice that any morphism \(f : g \longrightarrow h\) of curved absolute \(L_\infty\)-algebras is continuous with respect to the canonical filtration, i.e: \(f(W_\omega g) \subset W_\omega h\). Therefore the morphism of dg modules \(gr_\omega(f)\) is always well-defined.

**Definition 1.22** (Graded quasi-isomorphisms). Let \(f : g \longrightarrow h\) be a morphism between two curved absolute \(L_\infty\)-algebras. It is a graded quasi-isomorphism if
\[
gr_\omega(f) : gr_\omega(g) \longrightarrow gr_\omega(h)
\]
is a quasi-isomorphism of dg modules, for all \(\omega \geq 0\).

1.2. On \(u\mathcal{CE}_\infty\)-coalgebras. We study the Koszul dual notion of dg coalgebras over the dg operad \(\Omega B^{s-a}uCom\). We denote these coalgebras \(u\mathcal{CE}_\infty\)-coalgebras, since they correspond to counit coassociative coalgebras relaxed up to homotopy.

**Definition 1.23** (\(u\mathcal{CE}_\infty\)-coalgebra). A \(u\mathcal{CE}_\infty\)-coalgebra \(C\) is the data \((C, \Delta_C, d_C)\) of a dg \(\Omega B^{s-a}uCom\)-coalgebra.

**Lemma 1.24.** The data of a \(u\mathcal{CE}_\infty\)-coalgebra structure \(\Delta_C\) on a dg module \((C, d_C)\) is equivalent to the data of elementary decomposition maps
\[
\{ \Delta_\tau : C \longrightarrow C^{\otimes n} \}
\]
of degree \(\omega - 1\) for all corked rooted trees \(\tau\) in \(\text{CRT}^c_\omega\) for all \(n \geq 0\) and \(\omega \geq 0\). These operations are subject to the following condition: let \(E(\tau)\) denote the set of internal edges of \(\tau\), then
\[
\sum_{e \in E(\tau)} \Delta_{\tau(1)} \circ_{e} \Delta_{\tau(2)} - (-1)^{\omega-1} \sum_{e \in E(\tau)} \Delta_{\tau^c} + d_{C^{\otimes n}} \circ \Delta_\tau - (-1)^{\omega-1} \Delta_\tau \circ d_C = 0,
\]
where in the first term the corked rooted tree is split along \(e\) into \(\tau(1)\) and \(\tau(2)\) and where \(\tau^c\) denotes the corked rooted tree obtained by contracting the internal edge \(e\).
Proof. The data of a $uCC_{\infty}$-coalgebra structure $\Delta_C$ on a dg module $(C, d_C)$ is equivalent to the data of a morphism of dg operads

$$\Delta_C : \Omega B^{sa} u\mathbb{C}om \to \text{Coend}_C ,$$

which in turn is equivalent to the data of a curved twisting morphism

$$\delta_C : B^{sa} u\mathbb{C}om \to \text{Coend}_C .$$

Thus the operations $\Delta_\tau$ are given by $\delta_C(\tau)$ and their relationships are given by the Maurer-Cartan equation that $\delta_C$ satisfies. □

Notation 1.25. We denote by $\text{PCRT}^n_{\omega, \nu}$ the set of partitioned corked rooted trees with $\omega$ vertices, $\nu$ parenthesis, of arity $n$. Such a tree amounts to the data of a corked rooted tree $\tau$ together with the data of a partition of the set of vertices of $\tau$.

Remark 1.26. The dg operad $\Omega B^{sa} u\mathbb{C}om$ admits a basis given by partitioned corked rooted trees. The full structure of a $uCC_{\infty}$-coalgebra is given by decomposition maps

$$\{\Delta_\tau : C \to C^{\otimes n}\}$$

of degree $\omega - \nu$ if $\tau$ is in $\text{PCRT}^n_{\omega, \nu}$. Nevertheless, any such decomposition map is obtained as the composition of the elementary decomposition maps given by the sub-corked rooted trees contained inside each partition.

Let $(C, d_C)$ be a dg module endowed with a family of decomposition maps $\{\Delta_\tau\}$ for any partitioned corked rooted tree $\tau$. This data allows us to construct a morphism of dg modules

$$C \xrightarrow{\Delta_C} \prod_{n \geq 0} \text{Hom}_{S_n}(\Omega B^{sa} u\mathbb{C}om(n), C^{\otimes n})$$

and

$$c \mapsto \left[\text{ev}_c : \tau \mapsto \Delta_\tau(c)\right].$$

which satisfies the axioms of [RiL22a, Definition 1.17].

1.3. Model category structures. Since the dg operad encoding $uCC_{\infty}$-coalgebras is cofibrant, the category of $uCC_{\infty}$-coalgebras admits a canonical model category structure, left-transferred from the category of dg modules. See [RiL22a, Section 1] for an overview of the general results used in this subsection.

Proposition 1.27. There is a model category structure on the category of $uCC_{\infty}$-coalgebras left-transferred along the cofree-forgetful adjunction

$$\text{dg-mod} \xleftarrow{\mathcal{C}(uCC_{\infty})(-)} U \xrightarrow{\phi} uCC_{\infty}\text{-coalg} ,$$

where

(1) the class of weak equivalences is given by quasi-isomorphisms,
(2) the class of cofibrations is given by degree-wise monomorphisms,
(3) the class of fibrations is given by right lifting property with respect to acyclic cofibrations.

Proof. Consequence of the results of [GL22, Section 8] applied to this particular case, since $\Omega B^{sa} u\mathbb{C}om$ is cofibrant in the model category of dg operads. □

Furthermore, using the canonical curved twisting morphism

$$\iota : B^{sa} u\mathbb{C}om \to \Omega B^{sa} u\mathbb{C}om ,$$

we can construct a complete Bar-Cobar adjunction relating $uCC_{\infty}$-coalgebras and curved absolute $L_{\infty}$-algebras.
Proposition 1.28 (Complete Bar-Cobar adjunction). *The curved twisting morphism \( \iota \) induces a complete Bar-Cobar adjunction*

\[
\begin{align*}
\mathcal{U} \mathfrak{C} \mathfrak{C} \infty \text{-coalg} & \xrightarrow{\iota} \text{curv abs } \mathcal{L}_\infty \text{-alg}^\text{comp} \\
& \mathcal{L}_\infty \text{-alg}
\end{align*}
\]

between the category of \( \mathcal{U} \mathfrak{C} \mathfrak{C} \infty \)-coalgebras and the category of complete curved absolute \( \mathcal{L}_\infty \)-algebras.

*Proof.* Consequence of the results explained in [GL22, Section 9] applied to this particular case.

Using this adjunction, one can transfer the model category structure on \( \mathcal{U} \mathfrak{C} \mathfrak{C} \infty \)-coalgebras in order to endow curved absolute \( \mathcal{L}_\infty \)-algebras with a model category structure.

**Theorem 1.29.** There is a model category structure on the category of curved absolute \( \mathcal{L}_\infty \)-algebras right-transferred along the complete Bar-Cobar adjunction

\[
\begin{align*}
\mathcal{U} \mathfrak{C} \mathfrak{C} \infty \text{-coalg} & \xrightarrow{\iota} \text{curv abs } \mathcal{L}_\infty \text{-alg}^\text{comp} \\
& \mathcal{L}_\infty \text{-alg}
\end{align*}
\]

where

1. the class of weak equivalences is given by morphisms \( f \) such that \( \widehat{B}_i(f) \) is a quasi-isomorphism,
2. the class of fibrations is given by morphisms \( f \) such that \( \widehat{B}_i(f) \) is a fibration,
3. and the class of cofibrations is given by left lifting property with respect to acyclic fibrations.

Furthermore, this complete Bar-Cobar adjunction is a Quillen equivalence.

*Proof.* Consequence of the results explained in [GL22, Section 11] applied to this particular case.

One can give a complete characterization of the fibrations in this model category structure as follows.

**Proposition 1.30.** A morphism of curved absolute \( \mathcal{L}_\infty \)-algebras is a fibration if and only if it is a degree-wise epimorphism.

*Proof.* Follows from [GL22, Proposition 10.16].

**Remark 1.31.** In particular, all complete curved absolute \( \mathcal{L}_\infty \)-algebras are fibrant in this model structure.

There is a particular kind of weak-equivalences between curved absolute \( \mathcal{L}_\infty \)-algebras which admit an easy description.

**Proposition 1.32.** Let \( f : g \rightarrow h \) be a graded quasi-isomorphism between two complete curved absolute \( \mathcal{L}_\infty \)-algebras. Then

\[
\widehat{B}_i(f) : \widehat{B}_i(g) \rightarrow \widehat{B}_i(h)
\]
is a quasi-isomorphism of \( \mathcal{U} \mathfrak{C} \mathfrak{C} \infty \)-coalgebras, and \( f \) is a weak-equivalence of complete curved absolute \( \mathcal{L}_\infty \)-algebras.

*Proof.* Follows from [GL22, Theorem 10.25].

1.4. **Tensor product of \( \mathcal{U} \mathfrak{C} \mathfrak{C} \infty \)-coalgebras.** The tensor product of two \( \mathcal{U} \mathfrak{C} \mathfrak{C} \infty \)-coalgebras can naturally be endowed with a \( \mathcal{U} \mathfrak{C} \mathfrak{C} \infty \)-coalgebra structure. This gives a closed monoidal structure on the category of \( \mathcal{U} \mathfrak{C} \mathfrak{C} \infty \)-coalgebras which is compatible with its model category structure.

**Proposition 1.33.** The dg operad \( \Omega \mathfrak{B}^s \mathfrak{u} \mathfrak{C} \mathfrak{o} \mathfrak{m} \) is a dg Hopf operad, meaning there exists a diagonal morphism of operads

\[
\Delta_{\Omega \mathfrak{B} \mathfrak{u} \mathfrak{C} \mathfrak{o} \mathfrak{m}} : \Omega \mathfrak{B} \mathfrak{u} \mathfrak{C} \mathfrak{o} \mathfrak{m} \rightarrow \Omega \mathfrak{B} \mathfrak{u} \mathfrak{C} \mathfrak{o} \mathfrak{m} \otimes H \Omega \mathfrak{B} \mathfrak{u} \mathfrak{C} \mathfrak{o} \mathfrak{m}
\]
which is coassociative.
Proof. The operad \( u\mathcal{Com} \) can be obtained as the cellular chains on the operad \( u\mathcal{Com} \) in the category of sets given by \( \{ \} \) in non-negative arities. Any set-theoretical operad is a cocommutative Hopf operad on the nose. By Theorem 4.32, one has an isomorphism of dg operads

\[
\mathcal{C}_\xi^\ast (W(u\mathcal{Com})) \cong \Omega B^{s.a} (\mathcal{C}_\xi^\ast (u\mathcal{Com})) ,
\]

where \( W \) denotes the Boardmann-Vogt construction of \( u\mathcal{Com} \) seen as a discrete topological operad. Given a choice a cellular approximation of the diagonal on the topological interval \( I \), one has a coassociative Hopf operad structure on \( W(u\mathcal{Com}) \) which might not be cocommutative. The cellular chain functor \( \mathcal{C}_\xi^\ast (-) \) is both lax and colax (Eilenberg-Zilber and Alexander-Whitney maps) in a compatible way by [Gri22a, Proposition 26], therefore by [Gri22a, Proposition 25] it sends Hopf topological operads to dg Hopf operads.

\[ \square \]

**Corollary 1.34.** The category \( u\mathcal{CE}_\infty\)-coalgebras can be endowed with a monoidal structure given by the tensor product of the underlying dg modules.

**Proof.** Consider two \( u\mathcal{CE}_\infty\)-coalgebras \( (C_1, \Delta_{C_1}, d_{C_1}) \) and \( (C_2, \Delta_{C_2}, d_{C_2}) \). These \( u\mathcal{CE}_\infty\)-coalgebra structures amount to two morphisms of dg operads \( f_1 : \Omega B\mathcal{Com} \rightarrow \text{Coend}(C_1) \) and \( f_2 : \Omega B\mathcal{Com} \rightarrow \text{Coend}(C_2) \). Using the Hopf structure on gets

\[
\Omega B\mathcal{Com} \xrightarrow{\Delta_{\Omega B\mathcal{Com}}} \Omega B\mathcal{Com} \otimes \Omega B\mathcal{Com} \xrightarrow{f_1 \otimes f_2} \text{Coend}(C_1) \otimes \text{Coend}(C_2) \rightarrow \text{Coend}(C_1 \otimes C_2) ,
\]

which endows the dg module \( (C_1 \otimes C_2, d_{C_1 \otimes d_{C_2}}) \) with a \( u\mathcal{CE}_\infty\)-coalgebra structure. The counitality and coassociativity of \( \Delta_{\Omega B\mathcal{Com}} \) ensure that the category of \( u\mathcal{CE}_\infty\)-coalgebra together with the tensor product forms a monoidal category.

\[ \square \]

**Remark 1.35.** The dg Hopf operad structure on \( \Omega B^{s.a} u\mathcal{Com} \) constructed here extends the dg Hopf operad structure on \( u\mathcal{Com} \), therefore the tensor product of two counital cocommutative coalgebras seen as \( u\mathcal{CE}_\infty\)-coalgebras coincides with the usual structure on the tensor product of two counital cocommutative coalgebras.

**Proposition 1.36.** The category of \( u\mathcal{CE}_\infty\)-coalgebras together with their tensor product forms a monoidal model category.

**Proof.** It is straightforward to check the pushout-product axiom, the unit axiom is automatic since all objects are cofibrant, see [Hov99] for more details on this.

\[ \square \]

In fact, the tensor product of two coalgebras over a dg Hopf operad always forms a biclosed monoidal category. We specify the general constructions of the forthcoming paper [Gri22b] in the particular cases of interest for us. The construction of loc.cit holds for a wider variety of cases.

**Proposition 1.37 ([Gri22b]).** The category of dg \( u\mathcal{CE}_\infty\)-coalgebras is a biclosed monoidal category, meaning that there exists an internal hom bifunctor

\[
\langle -, - \rangle : (\mathcal{CE}_\infty\)-coalg\( )^{op} \times \mathcal{CE}_\infty\)-coalg \rightarrow \mathcal{CE}_\infty\)-coalg
\]

and, for any triple of \( u\mathcal{CE}_\infty\)-coalgebras \( C, D, E \), there exists isomorphisms

\[
\text{Hom}_{u\mathcal{CE}_\infty\text{-coalg}}(C \otimes D, E) \cong \text{Hom}_{u\mathcal{CE}_\infty\text{-coalg}}(C, \mathcal{D}_{(C, (D, E)}) \cong \text{Hom}_{u\mathcal{CE}_\infty\text{-coalg}}(D, (C, E)) ,
\]

which are natural in \( C, D \) and \( E \).

**Remark 1.38.** Let \( C_1 \) and \( C_2 \) be two \( u\mathcal{CE}_\infty\)-coalgebras. The \( u\mathcal{CE}_\infty\)-coalgebra \( \{C_1, C_2\} \) is given as the following equalizer

\[
\text{Eq}\left( \mathcal{C}(u\mathcal{CE}_\infty)(\text{hom}(C_1, C_2)) \xrightarrow{\rho} \mathcal{C}(u\mathcal{CE}_\infty)(\text{hom}(C_1, \mathcal{C}(u\mathcal{CE}_\infty)(C_2))) \right) ,
\]

\[ 15 \]
where \( \text{hom}(C_1, C_2) \) denotes the dg module of graded morphisms, where \( \rho \) is a map constructed using the comonad structure of \( \mathcal{C}(u\mathcal{C}_\infty) \) and the Hopf structure of the dg operad \( u\mathcal{C}_\infty \). This kind of construction works for coalgebras over any Hopf dg operad.

**Definition 1.39 (Convolution curved absolute \( \mathcal{L}_\infty \)-algebra).** Let \( C \) be a \( u\mathcal{C}_\infty \)-coalgebra and let \( g \) be a curved absolute \( \mathcal{L}_\infty \)-algebra. The pdg module of graded morphisms \( (\text{hom}(C, g), \partial) \) is endowed the following curved absolute \( \mathcal{L}_\infty \)-algebra structure.

The structural map \( \gamma_{\text{hom}(C,g)} \) is given by the following composition

\[
\begin{array}{ll}
\tilde{\mathcal{C}}(B^{s,a}\mathcal{uCom})(\text{hom}(C, g)) & \downarrow \text{coev}_C \\
\text{hom} \left( C, \tilde{\mathcal{C}}(B^{s,a}\mathcal{uCom})(\text{hom}(C, g)) \otimes C \right) & \downarrow (\Delta_C)_* \\
\text{hom} \left( C, \tilde{\mathcal{C}}(B^{s,a}\mathcal{uCom})(\text{hom}(C, g)) \otimes \tilde{\mathcal{C}}(\Omega B^{s,a}\mathcal{uCom})(C) \right) & \downarrow \xi \\
\text{hom} \left( C, \tilde{\mathcal{C}}(B^{s,a}\mathcal{uCom} \otimes \Omega B^{s,a}\mathcal{uCom})(\text{hom}(C, g) \otimes C) \right) & \downarrow \tilde{\mathcal{C}}(\delta_{B^{s,a}\mathcal{uCom}}) \\
\text{hom} \left( C, \tilde{\mathcal{C}}(B^{s,a}\mathcal{uCom})(g) \right) & \downarrow \text{ev}_C \\
\text{hom}(C, g), & \downarrow (\gamma g)_* \\
\end{array}
\]

where \( \text{coev}_C \) and \( \text{ev}_C \) are respectively the unit and the counit of the tensor-hom adjunction, and where \( \xi \) is the following natural inclusion

\[
\left( \prod_{n \geq 0} \text{Hom}_{\mathbb{S}}(M(n), V^\otimes n) \right) \otimes \left( \prod_{n \geq 0} \text{Hom}_{\mathbb{S}}(N(n), W^\otimes n) \right) \\
\downarrow \\
\prod_{n \geq 0} \text{Hom}_{\mathbb{S}}(M(n) \otimes N(n), (V \otimes W)^\otimes n).
\]

Finally

\[
\delta_{B^{s,a}\mathcal{uCom}} : B^{s,a}\mathcal{uCom} \to B^{s,a}\mathcal{uCom} \otimes \Omega B^{s,a}\mathcal{uCom}
\]

is a restriction of the diagonal \( \Delta_{\Omega B\mathcal{uCom}} \).

Convolution curved absolute \( \mathcal{L}_\infty \)-algebras and the internal hom-set of \( u\mathcal{C}_\infty \)-coalgebras are compatible in the following sense.
Theorem 1.40 ([Gri22b]). Let \( C \) be a \( u \mathcal{E}_\infty \)-coalgebra and let \( g \) be a curved absolute \( L_\infty \)-algebra. There is an isomorphism of \( u \mathcal{E}_\infty \)-coalgebras

\[
\left\{ C, \widehat{B}_i(g) \right\} \cong \widehat{B}_i \left( \text{hom}(C, g) \right),
\]

where \( \text{hom}(C, g) \) denotes the convolution curved absolute \( L_\infty \)-algebra of \( C \) and \( g \).

2. Higher absolute Lie theory

In this section, we follow an analogue approach to [RNV20] in order to integrate curved absolute \( L_\infty \)-algebras. For the first time, the integration functor and its left adjoint form a Quillen adjunction between curved absolute \( L_\infty \)-algebras and simplicial sets. The rest of this section is devoted to the study of the main properties of this adjunction.

2.1. Dupont’s contraction. In [Dup76], J-L Dupont proved that there is a homotopy contraction between the simplicial unital commutative algebra of polynomial differential forms on the geometrical simplicies and the simplicial sub-module of Whitney forms. Whitney forms on the simplicies are isomorphic to the cellular cochains on the simplicies and they are finite dimensional. This allows us to use the homotopy transfer theorem, and obtain a simplicial \( u \mathcal{E}_\infty \)-coalgebra structure on the cellular chains of simplicies. This coalgebra is not conilpotent, hence we encode it with an operad. This enables us to use the complete Bar-Cobar construction in order to construct a commuting triangle of Quillen adjunctions. This approach allows us to extend and to refine automatically many of the standard results in the theory of integration.

The simplicial dg unital commutative algebra of piece-wise polynomial differential forms on the standard simplex \( \Omega \), is given by

\[
\Omega_n = \frac{\mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]}{(t_0 + \cdots + t_n - 1, dt_0 + \cdots + dt_n)},
\]

with the obvious simplicial structure.

Theorem 2.1 (Dupont’s contraction). There exists a simplicial contraction

\[
\begin{array}{ccc}
\iota_* & : & \Omega_* \\
\hookrightarrow & & \rightarrow \\
\cong & & \rightarrow \\
p_* & : & C_*^\varepsilon(\Delta^*)
\end{array}
\]

where \( C_*^\varepsilon(\Delta^*) \) denotes the simplicial dg module given by the cellular cochains of the standard simplex.

A basis of \( C_*^\varepsilon(\Delta^n) \) is given by \( \{ \omega_I \} \) for \( I = \{ i_0, \ldots, i_k \} \subset [n] \), where \( |\omega_I| = -k \). Since we are working in the homological convention, both algebras are concentrated in degrees \( \leq 0 \), with \( dt_i \) being of degree \(-1\).

The data of a simplicial contraction amounts to the data of two morphisms of simplicial cochain complexes

\[
i_* : \Omega_* \rightarrow C_*^\varepsilon(\Delta^*), \quad \text{and} \quad p_* : C_*^\varepsilon(\Delta^*) \rightarrow \Omega_*;
\]

and a degree 1 linear map \( h_* : \Omega_* \rightarrow \Omega_* \), satisfying the following conditions:

\[
p_n i_n = \text{id}_{C_*^\varepsilon(\Delta^n)}, \quad i_n p_n - \text{id}_{\Omega_n} = d_n h_n + h_n d_n, \quad h_n^2 = 0, \quad p_n h_n = 0, \quad h_n i_n = 0,
\]

where \( d_n \) stands for the differential of \( \Omega_n \).

Remark 2.2. For \( n \geq 1 \), the dg unital commutative algebra \( \Omega_n \) is not augmented, as we have \( 1 = t_0 + \cdots + t_n \). In order to keep track of the full structure on \( \Omega^n \), one has to see it as a dg \( u \mathcal{E} \text{om} \)-algebra and not as a dg \( \mathcal{E} \text{om} \)-algebra.

Lemma 2.3. There is a simplicial \( u \mathcal{E}_\infty \)-algebra structure on \( C_*^\varepsilon(\Delta^*) \).
Proof. The dg operad $\Omega B^s u \mathcal{C}om$ is a cofibrant resolution of $u \mathcal{C}om$. Therefore we can apply the homotopy transfer theorem given in [HM12, Theorem 6.5.5] using the Dupont contraction of Theorem 2.1. Since this contraction is compatible with the simplicial structure, we obtain a simplicial dg $\Omega B^s u \mathcal{C}om$-algebra.

Let $C_c^*(\Delta^n)$ denote the cellular chains on the $n$-simplex.

**Lemma 2.4.** There is a cosimplicial $u \mathcal{E}_\infty$-coalgebra structure on $C_c^*(\Delta^*)$.

**Proof.** The linear dual of a degree-wise finite dimensional algebra over an dg operad is naturally a dg coalgebra over the same operad. Since $C_c^*(\Delta^n)$ is finite dimensional degree-wise for all $n$, its linear dual carries a canonical $u \mathcal{E}_\infty$-coalgebra structure. Applying a contravariant functor to a simplicial object gives a cosimplicial object. □

**Remark 2.5.** Notice that $C_c^*(\Delta^*)$ is a non-nilpotent $u \mathcal{E}_\infty$-coalgebra, therefore it cannot be encoded by a cooperad, like stated in [RNV20, Section 2].

We use this cosimplicial $u \mathcal{E}_\infty$-coalgebra to construct an adjunction between simplicial sets and $u \mathcal{E}_\infty$-coalgebras, using the following seminal result.

**Theorem 2.6 ([Kan58]).** Let $C$ be a locally small cocomplete category. The data of an adjunction

$$
\begin{array}{ccc}
\text{sSet} & \overset{\text{L}}{\to} & \mathcal{C} \\
\downarrow & & \downarrow \\
\text{R} & & \\
\end{array}
$$

is equivalent to the data of a cosimplicial object $F : \Delta \rightarrow \mathcal{C}$ in $\mathcal{C}$.

**Proof.** Given an adjunction $L \dashv R$, one can pullback the left adjoint $L$ along the Yoneda embedding $Y_0$

$$
\Delta \xrightarrow{Y_0} \text{sSet} \xrightarrow{L} \text{sSet},
$$

and obtain a cosimplicial object in $\mathcal{C}$. Given a cosimplicial object $F : \Delta \rightarrow \mathcal{C}$ in $\mathcal{C}$, one can consider its left Kan extension $\text{Lan}_{Y_0}(F)$ since $\mathcal{C}$ is cocomplete. Its right adjoint is given by

$$
R(-) := \text{Hom}_\mathcal{C}(F(\Delta^*), -).
$$

□

**Proposition 2.7.** There is an adjunction

$$
\begin{array}{ccc}
\text{sSet} & \overset{C_c^*(-)}{\to} & u \mathcal{E}_\infty\text{-coalg} \\
\downarrow & & \downarrow \\
\text{R} & & \\
\end{array}
$$

where $C_c^*(-)$ denotes the cellular chain functor endowed with a canonical $u \mathcal{E}_\infty$-coalgebra structure.

**Proof.** This is a straightforward application of Theorem 2.6 to the cosimplicial $u \mathcal{E}_\infty$-coalgebra constructed in Lemma 2.4. It is immediate to check that the left adjoint is given by the cellular chain functor $C_c^*(-)$ as a dg module, endowed with a $u \mathcal{E}_\infty$-coalgebra structure. This structure is given by the following colimit

$$
C_c^*(X) \cong \text{colim}_{E(X)} C_c^*(\Delta^*).
$$
on the category of elements $E(X)$ of $X$ in the category of $u \mathcal{E}_\infty$-coalgebras.

The right adjoint is therefore given, for a $u \mathcal{E}_\infty$-coalgebra $C$, by the functor

$$
\text{R}(C) := \text{Hom}_{u \mathcal{E}_\infty\text{-cog}}(C_c^*(\Delta^*), C).
$$

By pushing forward along the complete Cobar construction $\hat{\Omega}$, the cosimplicial $u \mathcal{E}_\infty$-coalgebra $C_c^*(\Delta^*)$, we also obtain a cosimplicial complete curved absolute $L_\infty$-algebra.
Definition 2.8 (Maurer–Cartan cosimplicial algebra). The Maurer–Cartan cosimplicial algebra is given by the cosimplicial complete curved absolute $L_\infty$-algebra
\[ \mathfrak{mc}^\bullet := \hat{\Omega}_i C^\ast_c (\Delta^\bullet) . \]

Proposition 2.9. There is an adjunction
\[ \text{sSet} \xleftarrow{\mathcal{L}} \text{curv abs } L_\infty\text{-alg} \xrightarrow{\mathcal{R}} \text{comp} \]

Proof. This follows directly from Theorem 2.6, applied to the cosimplicial object $\mathfrak{mc}^\bullet$. □

Remark 2.10. The right adjoint is therefore given by, for a curved absolute $L_\infty$-algebra $g$, by
\[ R(g)_\bullet := \text{Hom}_{\text{curv abs } L_\infty\text{-alg}} \left( \hat{\Omega}_i C^\ast_c (\Delta^\bullet), g \right) . \]

Theorem 2.11. The following triangle of adjunctions
\[ \text{uCE}_\infty\text{-coalg} \xrightarrow{\mathcal{R}} \text{curv abs } L_\infty\text{-alg} \xleftarrow{\hat{\mathcal{B}}}_i \ast \hat{\mathcal{B}}_i \]
commutes. Furthermore, all the adjunctions are Quillen adjunctions when we consider the model category structures on $\text{uCE}_\infty\text{-coalg}$ and on complete curved absolute $L_\infty$-algebras of Theorem 1.29; and the Kan-Quillen model category structure on simplicial sets.

Proof. Let $g$ be a curved absolute $L_\infty$-algebra. We have that
\[ R(g)_\bullet = \text{Hom}_{\text{curv abs } L_\infty\text{-alg}} \left( \hat{\Omega}_i C^\ast_c (\Delta^\bullet), g \right) \cong \text{Hom}_{\text{uCE}_\infty\text{-coalg}} \left( C^\ast_c (\Delta^\bullet), \hat{\mathcal{B}}_i g \right) \cong R \left( \hat{\mathcal{B}}_i g \right) . \]

The functor $C^\ast_c (\ast)$ sends monomorphisms of simplicial sets to degree-wise monomorphisms of $\text{uCE}_\infty\text{-coalg}$ and it sends weak homotopy equivalences to quasi-isomorphisms. Hence the triangle is made up of Quillen adjunctions. □

2.2. The integration functor. The functor $R$ in the above triangle is the integration functor we were looking for. Before giving an explicit combinatorial description of it, let us first state some of its fundamental properties.

Theorem 2.12.

1. For any curved absolute $L_\infty$-algebra $g$, the simplicial set $R(g)$ is a Kan complex.
2. Let $f : g \to h$ be a degree-wise epimorphism of curved absolute $L_\infty$-algebras. Then $R(f) : R(g) \to R(h)$ is a fibrations of simplicial sets.
3. The functor $R$ preserves weak equivalences. In particular, it sends any graded quasi-isomorphism $f : g \to h$ between complete curved absolute $L_\infty$-algebras to a weak homotopy equivalence of simplicial sets.

Proof. The functor $R$ is a right Quillen functor. □

Remark 2.13. Let us contextualize the results of Theorem 2.12.

1. The first result is quintessential to the classical integration theory of dg Lie algebras or nilpotent $L_\infty$-algebras as developed in [Hin01] and in [Get09].
(2) The second result generalizes one of the main theorems of [Get09], see Theorem 5.8 in \textit{loc.cit}.

(3) The third point, the homotopy invariance of the integration functor, is the generalization of Goldman–Milson’s invariance theorem proved in [GM88] and its generalization to $L_\infty$-algebras given in [DR15].

**Proposition 2.14.** The functor $\mathcal{R}$ commutes with homotopy limits.

**Proof.** The functor $\mathcal{R}$ is a right Quillen functor. \hfill $\square$

**Remark 2.15.** This means that it satisfies descent in the sense of [Hin97].

**Proposition 2.16.** Let $g$ be a complete curved absolute $L_\infty$-algebra. There is an isomorphism of simplicial sets

$$\mathcal{R}(g) \cong \lim_{W} \mathcal{R}(g/W) \, .$$

**Proof.** Since $g$ is complete, it can be written as

$$g \cong \lim_{W} g/W \, ,$$

where the limit is taken in the category of curved absolute $L_\infty$-algebras. Since $\mathcal{R}$ is right adjoint, it preserves all limits. \hfill $\square$

**Remark 2.17** (Comparison with Getzler integration functor). One way wonder how to compare the integration functor here defined with the integration functor defined by E. Getzler for nilpotent $L_\infty$-algebras in [Get09]. We refer the reader to [RiL22c, Chapter 3, Section 2.7]. There we show that our functor is isomorphic to Getzler’s original functor on the category of nilpotent $L_\infty$-algebra (which are particular examples of curved absolute $L_\infty$-algebras). We also show that it coincides with the integration functor of [RNV20] under some technical hypothesis on the filtrations considered.

**Dold-Kan correspondence.** The adjunction

$$\begin{array}{ccc}
\text{sSet} & \overset{\mathcal{L}}{\leftarrow} & \text{curv abs-}L_\infty\text{-alg}^{\text{comp}} \\
\mathcal{R} & \downarrow & \\
\text{curv abs-}L_\infty\text{-alg} & \overset{\mathcal{L}}{\rightarrow} & \text{sSet}
\end{array}$$

is a generalization of the Dold-Kan correspondence. Notice that a chain complex $(V, d_V)$ is a particular example of a complete curved absolute $L_\infty$-algebra where the structural morphism

$$\gamma_V : \prod_{n \geq 0} \hat{\Omega}^n \text{uCom}^\pi(n) \otimes S_n V \otimes n \rightarrow V$$

is the zero morphism. We call them abelian curved absolute $L_\infty$-algebras.

**Proposition 2.18.** Let $(V, d_V)$ be an abelian curved absolute $L_\infty$-algebra. Then

$$\mathcal{R}(V) \cong \Gamma(V) \, ,$$

where $\Gamma(-)$ is the Dold-Kan functor.

**Proof.** Recall that

$$\mathcal{R}(V) \cong \text{Hom}_{\text{uCom}_\infty\text{-cog}}(C^c(\Delta^\bullet), \hat{B}_i V) \, .$$

Since $\gamma_V = 0$, the complete Bar construction $\hat{B}_i V$ is simply given by the cofree $\text{uCom}_\infty\text{-coalgebra}$ generated by the dg module $(V, d_V)$. Thus

$$\mathcal{R}(V) \cong \text{Hom}_{\text{dg-mod}}(C^c_s(\Delta^\bullet), V) \cong \Gamma(V) \, .$$

\hfill $\square$
Remark 2.19. Abelian curved absolute $\mathcal{L}_\infty$-algebras concentrated in positive degrees are models for simplicial abelian groups. One can think of general curved absolute $\mathcal{L}_\infty$-algebras concentrated in positive degrees as models up to homotopy notion of a non-necessarily commutative group. This intuition can be made precise using higher Baker–Campbell–Hausdorff products; see [RiL22c, Chapter 3, Section 2.5] for a full treatment of this question.

**Derived adjunction.** The previous adjunction computes the derived functors of the adjunction $$\xymatrix{ \text{sSet} \ar[r]_{\mathbb{R}} & \text{uCE}_\infty \text{-coalg} \ar[l]_{\mathbb{L}}}.$$ Let $C$ be a $\text{uCE}_\infty$-coalgebra. It might not be fibrant, therefore one needs to take a fibrant resolution of $C$ in order to compute the right derived functor $\mathbb{R}C$ at $C$. The Theorem 1.29 provides us with $$\eta_C : C \to \mathbb{B}_t \Omega_\ast C,$$ a functorial fibrant resolution of $C$. Therefore we have that $$\mathbb{R}C(C)_\ast = \text{Hom}_{\text{uCE}_\infty \text{-cog}}(C^\ast, \mathbb{B}_t \Omega_\ast C) \cong \text{Hom}_{\text{curv abs}} \mathcal{L}_\infty \text{-alg} (\hat{\Omega}_t C^\ast, \hat{\Omega}_t C) \cong \mathbb{R}(\hat{\Omega}_t C).$$

Remark 2.20 (∞-morphisms). Let $C$ and $D$ be two $\text{uCE}_\infty$-coalgebras. The hom-set $$\text{Hom}_{\text{curv abs}} \mathcal{L}_\infty \text{-alg} (\hat{\Omega}_t C, \hat{\Omega}_t D)$$ is the set of ∞-morphisms of $\text{uCE}_\infty$-coalgebras between $C$ and $D$. See [GL22, Section 12].

### 2.3. Explicit version of the integration functor.

Let us now give a combinatorial description of the integration functor. Recall that the integration functor $\mathbb{R}$ is given by $$\mathbb{R}(-)_\ast := \text{Hom}_{\text{curv abs}} \mathcal{L}_\infty \text{-alg} (\hat{\Omega}_t (C^\ast \Delta^n)_\ast, -) \cong \text{Hom}_{\text{curv abs}} \mathcal{L}_\infty \text{-alg} (\text{mc}^\ast, -).$$

Thus it is primordial to describe the complete curved absolute $\mathcal{L}_\infty$-algebra structure on the cosimplicial Maurer–Cartan algebra $\text{mc}^\ast$ which represents this functor. Since it is given by the complete Cobar construction of the cosimplicial $\text{uCE}_\infty$-coalgebra $C^\ast(\Delta^n)$, let us describe this structure.

**Notation 2.21.** Consider $\Omega_n$ with its dg $u\text{Com}$-algebra structure. Let $$\mu_k : u\text{Com}(k) \otimes_{\mathbb{S}_k} \Omega_n \to \Omega_n$$ denote its dg $u\text{Com}$-algebra structural morphisms for $k \geq 1$, and $u : \mathbb{K} \to \Omega_n$ its unit.

**Proposition 2.22.** The transferred $\text{uCE}_\infty$-algebra structure on $C^\ast(\Delta^n)$ via the Dupont contraction of Theorem 2.1 is determined by morphisms $$\left\{ \mu_\tau : (C^\ast(\Delta^n))^{\otimes m} \to C^\ast(\Delta^n) \right\}$$ of degree $-\omega + 1$, where $\tau$ is a rooted tree in $\text{RT}_m$ of arity $m$ and weight $\omega$ with no corks.

For $\tau$ in $\text{RT}_m$, the operation $\mu_\tau$ is given by labeling all the vertices of $\tau$ with an operation $\mu_k$ of $\Omega_n$, where $k$ is the number of inputs of the vertex, by labeling all internal edges with the homotopy $h_n$, all the leaves with the map $i_n$ and the root with the map $p_n$, and composing the labeling operations along the rooted tree $\tau$. Pictorially, it is given by

For $\tau = $
The only corked rooted tree acting non-trivially is the single cork morphism given by \( p_n \circ u : \mathcal{K} \rightarrow C^\ast_\tau(\Delta^n) \).

**Proof.** The standard transfer formula for the homotopy transfer theorem given in [HM12, Theorem 6.5.5] involves operations \( \mu_\tau \) where \( \tau \) runs over all corked rooted trees in CRT. Two different types of corks can appear: the corks given by \( u \) and the corks given by \( u \circ h_n \). Pictorially, \( \tau \) can be as follows

\[
\begin{array}{c}
i_n \quad u \quad i_n \\
\mu_5 \quad h_n \quad \mu_2 \\
h_n \quad h_n \\
\mu_3
\end{array}
\]

where the first type of corks is represented on the left branch of the tree and the second type is represented on the right branch of the tree. Since we started with a strict \( u \mathcal{C}\mathcal{O}_\ast \mathcal{A}_\ast \mathcal{L}_\ast \) structure on \( \Omega_n \), the composition \( \mu_\tau \circ u \) simplifies into \( \mu_{\tau-1} \), hence the first type of corks disappears. Recall that \( \Omega_n \) is concentrated in non-positive degrees and that the unit of \( \Omega_n \) lies in degree 0. Hence \( h_n \circ u = 0 \) as \( h_n \) raises the degree by one and the second type of corks disappears as well. We are left with operations \( \mu_\tau \) where \( \tau \) runs over all rooted trees without corks \( \mathcal{R}_\mathcal{T}_\mathcal{M}^u \), with a single operation involving the unit \( u \) given by \( p_n \circ u \). \( \square \)

**Remark 2.23.** We recover exactly the same structure on \( C^\ast_\tau(\Delta^n) \) as the one given in [RVN20, Proposition 2.9] with the addition of the arity 0 operation \( p_n \circ u \).

Let us compute the \( u \mathcal{C}\mathcal{O}_\ast \mathcal{L}_\ast \mathcal{O}_\ast \mathcal{M}_\ast \) structure on its linear dual \( C^\ast_\tau(\Delta^n) \). The elementary decompositions

\[
\left\{ \Delta_\tau : C^\ast_\tau(\Delta^n) \rightarrow C^\ast_\tau(\Delta^n)^{\otimes m} \right\}
\]

which determine the \( u \mathcal{C}\mathcal{O}_\ast \mathcal{L}_\ast \mathcal{O}_\ast \mathcal{M}_\ast \) structure of \( C^\ast_\tau(\Delta^n) \) are given by \( \Delta_\tau := (\mu_\tau)^\ast \). Let \( \{a I\} = \{\omega_1^\ast\} \) be the dual basis of \( C^\ast_\tau(\Delta^n) \). Let \( \tau \) be a rooted tree in \( \mathcal{R}_\mathcal{T}_\mathcal{M}^u \). For any \( m \)-tuple \( (I_1, \cdots, I_m) \) of non-trivial subsets of \( [n] \), suppose we have that

\[
\mu_\tau(\omega_{I_1}, \cdots, \omega_{I_m}) = \sum_{I \subseteq [n]} \lambda_1^{\tau(I_1, \cdots, I_m)} \omega_{I_1}.
\]

where \( \{\omega_1\} \) is the basis of \( C^\ast_\tau(\Delta^n) \). Then

\[
\Delta_\tau(a I) = \sum_{I_1, \cdots, I_m \subseteq [n], I \neq 0} \lambda_1^{\tau(I_1, \cdots, I_m)} a_{I_1} \otimes \cdots \otimes a_{I_m}.
\]

Notice that \( a_{I_1} \otimes \cdots \otimes a_{I_m} \) appears in the decomposition of \( a_I \) under \( \Delta_\tau \) if and only if \( \omega_1 \) appears with a non-zero coefficient in \( \mu_\tau(\omega_{I_1}, \cdots, \omega_{I_m}) \).

**Remark 2.24.** Computing the coefficients \( \lambda_1^{\tau(I_1, \cdots, I_m)} \) is a hard task, see [RVN20, Remark 5.11].

**Complete Cobar construction.** Let \( (\mathcal{C}, \Delta_\mathcal{C}, d_\mathcal{C}) \) be a \( u \mathcal{C}\mathcal{O}_\ast \mathcal{L}_\ast \mathcal{O}_\ast \mathcal{M}_\ast \) coalgebra, where

\[
\Delta_\mathcal{C} : \mathcal{C} \rightarrow \prod_{n \geq 0} \text{Hom}_{\mathcal{S}_n}(\Omega \mathcal{B}^{s,a} \mathcal{U}\mathcal{C}\mathcal{O}(n), \mathcal{C}^{\otimes n})
\]
denotes its structural morphism. The complete Cobar construction $\Omega_1 C$ is given by the underlying graded module

$$\prod_{n \geq 0} \hat{\Omega}^S_C \mathbb{C} \mathbb{om}^*(n) \otimes_{S_n} C \otimes^n.$$

It is endowed with a pre-differential $d_{\text{cobar}}$ given by the difference of $d_1$ and $d_2$. The first term $d_1$ is given by

$$d_1 = -\hat{\mathcal{F}}^c (d_{\hat{\Omega}^S_C \mathbb{C} \mathbb{om}^*})(id) + \hat{\mathcal{F}}^c (id) (\Pi (id, d_C)).$$

The second term $d_2$ is induced by the $u C_{\infty}$-coalgebra structure of $C$, it is the unique derivation extending the map $\varphi$, given by the following composites

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\varphi} & \prod_{n \geq 0} \hat{\Omega}^S_C \mathbb{C} \mathbb{om}^*(n) \otimes_{S_n} C \otimes^n \\
\Lambda_c & \mapsto & \prod_{n \geq 0} \text{Hom}_{S_n} (\Omega B^a \mathbb{C} \mathbb{om}(n), C \otimes^n) \\
\hat{\mathcal{F}}^c (id) & \mapsto & \prod_{n \geq 0} \text{Hom}_{S_n} (B^a \mathbb{C} \mathbb{om}(n), C \otimes^n).
\end{array}$$

**Theorem 2.25.** Let $I = \{i_0, \ldots, i_k\} \subseteq [n]$ with $k > 0$, and $a_1$ be the corresponding basis element in $C^*_c (\Delta^n)$. When $\text{Card}(I) \geq 2$, the image of $a_1$ under the pre-differential $d_{\text{cobar}}$ is given by

$$d_{\text{cobar}}(a_1) = \sum_{1 \leq 0} (-1)^1 a_{i_0 \cdots i_1 \cdots i_k} - \sum_{m \geq 2} \sum_{\tau \in \text{RT}_{T_1, \ldots, T_m} \subseteq [n], I_1 \neq \emptyset} \frac{1}{\mathcal{E}(\tau) \lambda_{T_1}^{\tau_1, \cdots, \tau_m}} \tau(a_{I_1}, \cdots, a_{I_m}),$$

for $I = \{i_0, \ldots, i_k\}$, where $\mathcal{E}(\tau)$ is the renormalization coefficient of Remark 1.6. When $\text{Card}(I) = 1$, the image of $a_1$ under the pre-differential is given by

$$- \sum_{n \geq 0, n \neq 1} \frac{1}{n!} c_n (a_0, \cdots, a_0),$$

where $c_n$ denotes the n-corolla.

**Proof.** By Proposition 2.22, the only arity 0 operation on $C^*_c (\Delta^n)$ is given by $p_n \circ u$. In order to know in which decompositions the lonely cork is going to appear, it is enough to compute the image of the unit 1 in $\Omega_1$ by the morphism $p_n$. Since

$$p_n(1) = t_0 + \cdots + t_n \in C^*_c (\Delta^n)$$

for all $n \geq 0$, the cork only appears in the decomposition of the elements of the form $a_1$ where $\{i\} \subseteq [n]$. Consequently, all the formulas for $d_{\text{cobar}}(a_1)$ where $\text{Card}(I) \geq 2$ are only indexed by rooted trees without corks. The renormalization coefficient $\mathcal{E}(\tau)$ appears because an identification between invariants and coinvariants was done in the definition of $d_2$. \hfill \Box

**Corollary 2.26.** When $\text{Card}(I) \geq 2$, the formula for $d_{\text{cobar}}(a_1)$ coincides with the formula computed in [RNV20, Section 2.2] in the non-curved case.

**Proof.** Follows directly from the above theorem. \hfill \Box

**Corollary 2.27.** Let $(g, \gamma, d_g)$ be a curved absolute $L_{\infty}$-algebra. There is a bijection

$$R(g)_0 \cong MC(g),$$

between the 0-simplicies of $R(g)$ and the Maurer–Cartan elements of $g$. 

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Proof. At $n = 0$, the Dupont contraction is trivial since $h_0 = 0$ and $i_0$ and $p_0$ are isomorphism. Thus the resulting $u \in \mathfrak{C}_\infty$-coalgebra structure on $\mathbb{K} \mathfrak{a}_0$ is simply given by

$$
\mathbb{K} \mathfrak{a}_0 \longrightarrow \prod_{n \geq 0} \text{Hom}_{S_n} (\text{OB}^{a_u \text{Com}}_n (\mathbb{K} \mathfrak{a}_0) \otimes^n) \\
a_0 \longmapsto \sum_{n \geq 0, \, n \neq 1} [\text{ev}_{a_0} : c_n \mapsto a_0 \otimes \cdots \otimes a_0].
$$

Thus

$$
\mathfrak{m}^0 \cong \left( \prod_{n \geq 0} \hat{\text{O}}^{a_u \text{Com}}_n (\mathbb{K} \mathfrak{a}_0) \otimes^n, d_{\text{cobar}} (a_0) = - \sum_{n \geq 0, \, n \neq 1} \frac{1}{n!} c_n (a_0, \cdots, a_0) \right).
$$

The data of a morphism of curved absolute $\mathcal{L}_\infty$-algebras $f : \mathfrak{m}^0 \longrightarrow \mathfrak{g}$ is equivalent to the data of $\alpha$ in $\mathfrak{g}$ such that

$$
d_{\mathfrak{g}} (\alpha) = - \gamma_{\mathfrak{g}} \left( \sum_{n \geq 0, \, n \neq 1} \frac{1}{n!} c_n (\alpha, \cdots, \alpha) \right),
$$

since $\mathfrak{m}^0$ is freely generated by $a_0$ as a pdg $B^{a_u \text{Com}}$-algebra, and since the data of a morphism of pdg $B^{a_u \text{Com}}$-algebras is the same as the data of a morphism of curved $B^{a_u \text{Com}}$-algebras. \hfill \square

2.4. Gauge actions and higher BCH products. In this subsection, we briefly mention other results of interest in the theory of curved absolute $\mathcal{L}_\infty$-algebras which are developed in the author’s PhD. thesis [RiL22c, Chapter 3, Sections 2.4 and 2.5]. These are the characterization of paths in $\mathcal{R}(\mathfrak{g})$ as gauge actions by degree one elements in a curved absolute $\mathcal{L}_\infty$-algebra $\mathfrak{g}$ and the existence of higher Baker–Campbell–Hausdorff formulae for the horn-fillers in $\mathcal{R}(\mathfrak{g})$.

Gauge actions. Let $\lambda$ be a degree 1 element in a curved absolute $\mathcal{L}_\infty$-algebra $\mathfrak{g}$, there is a gauge action of $\lambda$ on the set of Maurer–Cartan elements of $\mathfrak{g}$. Given such a Maurer–Cartan element $\alpha$, there is another Maurer–Cartan element $\lambda \bullet \alpha$ which is given by an explicit formula in terms of rooted trees decorated by $\alpha$ and $\lambda$. Two Maurer–Cartan elements $\alpha$ and $\beta$ are gauge equivalent if there exists a degree one element $\lambda$ such that $\lambda \bullet \alpha = \beta$. One can show it is an equivalence relation on the set of Maurer–Cartan elements.

**Theorem 2.28.** Let $\mathfrak{g}$ be a complete curved absolute $\mathcal{L}_\infty$-algebra. There is a bijection

$$
\tau_0 (\mathcal{R}(\mathfrak{g})) \cong \mathfrak{M}(\mathfrak{g}) / \sim_{\text{gauge}},
$$

where the right-hand side denotes the set of Maurer–Cartan elements up to gauge equivalence.

**Remark 2.29.** See [RNV20, Section 4.2] for the case of complete $\mathcal{L}_\infty$-algebras.

Higher BCH formulae. We know from Theorem 2.12 that $\mathcal{R}(\mathfrak{g})$ is a Kan complex. It means that every horn $\Lambda^n_k \longrightarrow \mathcal{R}(\mathfrak{g})$ admits a horn-filler $\Delta^n \longrightarrow \mathcal{R}(\mathfrak{g})$. It the particular case of $\mathcal{R}(\mathfrak{g})$, horn-fillers can be completely characterized.

**Theorem 2.30.** Let $\mathfrak{g}$ be a complete curved absolute $\mathcal{L}_\infty$-algebra. There is a bijection

$$
\text{Hom}_{\text{Set}} (\Delta^n, \mathcal{R}(\mathfrak{g})) \cong \mathfrak{g}_n \times \text{Hom}_{\text{Set}} (\Lambda^n_k, \mathcal{R}(\mathfrak{g})),
$$

natural in $\mathfrak{g}$ with respect to morphisms of curved absolute $\mathcal{L}_\infty$-algebras.

Therefore, for every $\Lambda^n_k$-horn in $\mathcal{R}(\mathfrak{g})$, there is a canonical filler given by the element 0 in $\mathfrak{g}_n$ under the above bijection. That is, $\mathcal{R}(\mathfrak{g})$ has a canonical $\infty$-groupoid structure, which is no longer only a property. Moreover, the above theorem allows to define higher Baker–Campbell–Hausdorff products in the following way.
Definition 2.31 (Higher Baker–Campbell–Hausdorff products). Let \( g \) be a complete curved absolute \( \mathcal{L}_\infty \)-algebra and let \( y \) be an element in \( g \) of degree \( n \). The higher Baker–Campbell–Hausdorff product relative to \( y \) is given by

\[
\Gamma^y : \text{Hom}_{\text{sSet}}(\Lambda^n_k, \mathcal{R}(g)) \rightarrow g_{n-1}
\]

where \( x \) is sent to \( \Gamma^y(x) := \varphi^{(x,y)}(a_k) \) in \( g_{n-1} \), where \( \varphi^{(x,y)} \) is the curved twisting morphism associated to the element in \( \mathcal{R}(g)_n \) given by the bijection constructed in Theorem 2.30.

These higher Baker–Campbell–Hausdorff products are given by explicit formulae. In fact, we show that the formulae that give these products are exactly the same those found in [RNV20, Proposition 5.10]. We also show that, for a nilpotent Lie algebra concentrated in degree 1 (particular example of curved absolute \( \mathcal{L}_\infty \)-algebra), these higher Baker–Campbell–Hausdorff products recover the original Baker–Campbell–Hausdorff formula. Thus our integration procedure can be considered a generalization of the classical integration procedure that associates a unipotent group to a nilpotent Lie algebra via the Baker–Campbell–Hausdorff formula.

2.5. Higher homotopy groups. In this subsection, we compute the higher homotopy groups of the Kan complex \( \mathcal{R}(g) \). This is done by using the explicit \( u \mathcal{C}_\infty \)-coalgebra structures on the cellular chains of the \( n \)-spheres. This new method allows us to generalize the one of the main results of A. Berglund [Ber15] to curved absolute \( \mathcal{L}_\infty \)-algebras.

Let \( S^n \) denote the simplicial set given by one non-degenerate 0-simplex \([0]\) and one non-degenerate \( n \)-simplex \([n]\). This is a model for the simplicial set \( \Delta^n / \partial \Delta^n \), which does not form a Kan complex. Nevertheless, since for any complete curved absolute \( \mathcal{L}_\infty \)-algebra \( g, \mathcal{R}(g) \) is a Kan complex, one has

\[
\pi_n(\mathcal{R}(g), \alpha) \cong \text{Hom}_{\text{sSet}}(S^n, \mathcal{R}(g)) / \sim_{\text{hmt}},
\]

where the right-hand side is the set of morphisms of simplicial sets which send \([0]\) to \( \alpha \) in \( \mathcal{R}(g)_0 \) modulo the homotopy relation. Now notice that

\[
\text{Hom}_{\text{sSet}}(S^n, \mathcal{R}(g)) / \sim_{\text{hmt}} \cong \text{Hom}_{u \mathcal{C}_\infty \text{-coalg}}(C^\ast_\infty(S^n), \widehat{B}_1(g)) / \sim_{\text{hmt}}
\]

since \( C^\ast_\infty(-) \rightarrow \mathcal{R} \) is a Quillen adjunction and since both \( \mathcal{R}(g) \) and \( \widehat{B}_1g \) are fibrant objects. Hence by explicitly computing the \( u \mathcal{C}_\infty \)-coalgebra structure on \( C^\ast_\infty(S^n) \) one can compute the homotopy groups of \( \mathcal{R}(g) \) for any complete curved absolute \( \mathcal{L}_\infty \)-algebra.

Lemma 2.32. Let \( n \geq 1 \). The dg module \( C^\ast_\infty(S^n) \) is given by \( \mathbb{K}a_0 \) in degree 0 and \( \mathbb{K}a_{[n]} \) in degree \( n \) with zero differential. The \( u \mathcal{C}_\infty \)-coalgebra structure on \( C^\ast_\infty(S^n) \) is given by the elementary decomposition maps

\[
\begin{align*}
\xymatrix{ a_0 \ar[r] & \Delta_{c_m}(a_0) = a_0 \otimes \cdots \otimes a_0, } \\
\xymatrix{ a_{[n]} \ar[r] & \Delta_{c_m}(a_{[n]}) = a_0 \otimes \cdots \otimes a_0 \otimes a_{[n]}, }
\end{align*}
\]

where \( c_m \) denotes the \( m \)-corolla, for all \( m \geq 2 \), and where \( \Delta_\tau \) is the zero morphism for any other corked rooted tree \( \tau \) in CRT.
Proof. Consider the following pushout in the category of simplicial sets

\[
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & \{\ast\} \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & S^n.
\end{array}
\]

It induces a pullback in the category of $\vee C\mathcal{C}_\infty$-algebras

\[
\begin{array}{ccc}
C^\ast(S^n) & \rightarrow & C^\ast(\{\ast\}) \\
\downarrow & & \downarrow \\
C^\ast(\Delta^n) & \rightarrow & C^\ast(\partial \Delta^n).
\end{array}
\]

Therefore there is an epimorphism $C^\ast(S^n) \twoheadrightarrow C^\ast(\Delta^n)$ of $\vee C\mathcal{C}_\infty$-algebras. Using Proposition 2.22, one can see that, for degree reasons, the only operations that survive on $C^\ast(S^n)$ are the multiplications $\mu_{cm}$. Furthermore, it is clear that $\mu_{cm}(a_0, \ldots, a_0, a_{[n]}) = a_{[n]}$. Computing the $\vee C\mathcal{C}_\infty$-coalgebra structure of the linear dual $C^\ast_c(S^n)$ from this is straightforward. \qed

Let $\alpha$ be a Maurer–Cartan element, we define the twisted differential by $\alpha$ as

\[
d^\alpha_{\mathfrak{g}}(-) := d_{\mathfrak{g}}(-) + \gamma_{\mathfrak{g}} \left( \sum_{n \geq 2} \frac{1}{(n-1)!} c_n(\alpha, \ldots, \alpha, -) \right).
\]

**Lemma 2.33.** Let $\mathfrak{g}$ be a curved absolute $\mathcal{L}_\infty$-algebra and let $\alpha$ be a Maurer–Cartan element. Then $d^\alpha_{\mathfrak{g}}$ squares to zero.

**Proof.** The formal algebraic computations of [DSV18, Corollary 5.1] extend mutatis mutandis to this case using the associativity Condition 2. \qed

**Definition 2.34** ($\alpha$-homology groups). Let $\mathfrak{g}$ be a curved absolute $\mathcal{L}_\infty$-algebra and let $\alpha$ be a Maurer–Cartan element. We define the $\alpha$-homology groups of $\mathfrak{g}$ to be

\[
H^\alpha_n(\mathfrak{g}) := \frac{\text{Ker}(d^\alpha_{\mathfrak{g}})}{\text{Im}(d^\alpha_{\mathfrak{g}})}.
\]

**Theorem 2.35.** Let $\mathfrak{g}$ be a complete curved absolute $\mathcal{L}_\infty$-algebra and let $\alpha$ be a Maurer–Cartan element. There is a bijection

\[
\pi_n((\mathcal{R}(\mathfrak{g}), \alpha)) \cong H^\alpha_n(\mathfrak{g}),
\]

for all $n \geq 1$, which is natural in $\mathfrak{g}$.

**Proof.** Recall that the data of a morphism of $\vee C\mathcal{C}_\infty$-coalgebras $f_\nu : C^\ast_c(S^n) \rightarrow \hat{\mathcal{B}}_i(\mathfrak{g})$ is equivalent to the data of a curved twisting morphism $\nu : C^\ast_c(S^n) \rightarrow \mathfrak{g}$ relative to $i$. In this case, by Lemma 2.32, $\nu : C^\ast_c(S^n) \rightarrow \mathfrak{g}$ is a curved twisting morphism if and only if

\[
\gamma_{\mathfrak{g}} \left( \sum_{n \geq 0, \ n \neq 1} \frac{1}{n!} c_n(\nu(a_0), \ldots, \nu(a_0)) \right) + d_{\mathfrak{g}}(\nu(a_0)) = 0,
\]

that is, $\nu(a_0)$ is a Maurer–Cartan element, and if

\[
\gamma_{\mathfrak{g}} \left( \sum_{n \geq 0, \ n \neq 1} \frac{1}{n!} c_n(\nu(a_0), \ldots, \nu(a_{[n]})) \right) + d_{\mathfrak{g}}(\nu(a_{[n]})) = 0,
\]

that is, $d^\nu_{\mathfrak{g}}(\nu(a_{[n]})) = 0$, where $d^\nu_{\mathfrak{g}}(a_0)$ is the twisted differential by $\nu(a_0)$. Therefore we have a bijection

\[
\text{Hom}_{\text{Sets}}((S^n, a_0), (\mathcal{R}(\mathfrak{g}), \alpha)) \cong Z^\alpha_n(\mathfrak{g}).
\]
Let \( f_v : C^*_c(S^n) \rightarrow \hat{B}_1(g) \) and \( g_\rho : C^*_c(S^n) \rightarrow \hat{B}_1(g) \) be two morphisms of \( u\mathcal{C}_{\infty}\)-coalgebras which send \( a_0 \) to \( \alpha \). The data of an homotopy between them amounts to the data of a morphism

\[
h : C^*_c(S^n) \otimes C^*_c(\Delta^1) \rightarrow \hat{B}_1(g)
\]

such that \( h(- \otimes b_0) = f_v \) and \( h(- \otimes b_1) = g_\rho \), where \( C^*_c(\Delta^1) = \mathbb{K}.b_0 \oplus \mathbb{K}.b_1 \oplus \mathbb{K}.b_{01} \), since \( C^*_c(\Delta^1) \) is the interval object in the category of \( u\mathcal{C}_{\infty}\)-coalgebra. The \( u\mathcal{C}_{\infty}\)-coalgebra structure on \( C^*_c(S^n) \otimes C^*_c(\Delta^1) \) is given by the tensor product structure of Corollary 1.34. One can check that the data of such of a morphism \( h \) is equivalent to the data of an element \( \lambda \) of degree \( n + 1 \) such that

\[
d^\alpha(\lambda) = \nu(a_{[n]}) - \rho(a_{[n]}).
\]

Therefore there is a bijection

\[
\text{Hom}_{sSets}((S^n, a_0), (\mathcal{R}(g), \alpha)) / \sim_{hmt} \cong H^\alpha_n(g).
\]

\[\square\]

**Corollary 2.36.** Let \( \alpha \) and \( \beta \) be two gauge equivalent Maurer–Cartan elements of \( g \). There is an isomorphism

\[
H^\alpha_n(g) \cong H^\beta_n(g),
\]

for all \( n \geq 1 \).

**Proof.** It follows from Theorem 2.35 and Theorem 2.28. \[\square\]

**Remark 2.37 (Berglund map).** One can also prove this result by following the same ideas of [Ber15], that is, by constructing an explicit map

\[
\mathcal{R}^\alpha_n : H^\alpha_n(g) \rightarrow \pi_n(\mathcal{R}(g), \alpha)
\]

where \( f_{[u]} \) is the curved twisting morphism defined by \( f_{[u]}(a_{[n]}) = u \) and \( f_{[u]}(a_{[i]}) = \alpha \), for all \( 0 \leq i \leq n \), and which is zero on any other element \( a_1 \) of \( C^*_c(\Delta^n) \).

**Remark 2.38.** The group structure on \( \pi_n(\mathcal{R}(g), \alpha) \) is induced by the pinch map

\[
\text{pinch} : S^n \rightarrow S^n \vee S^n,
\]

which on cellular chains is given by \( C^*_c(\text{pinch})(a_{[n]}) = a_{[n]}^{(1)} + a_{[n]}^{(2)} \). Using this, it can be shown that for \( n \geq 2 \), there is an isomorphism of abelian groups

\[
\pi_n(\mathcal{R}(g), \alpha) \cong H^\alpha_n(g),
\]

where we consider the sum of homology classes on the right-hand side.

Let \( h : \Lambda^2_1(\lambda_1, \lambda_2) \rightarrow g \) be a \( \Lambda^2_2 \)-horn in \( \mathcal{R}(g) \). Let \( h(a_{[01]}) = \lambda_1 \) and \( h(a_{[12]}) = \lambda_2 \). These are two degree one elements in \( g \) which define a gauge equivalences \( \lambda_1 \circ \alpha = \alpha \) and \( \lambda_2 \circ \alpha = \alpha \). The Baker–Campbell–Hausdorff product relative to \( \alpha \) of \( \lambda_1 \) and \( \lambda_2 \) is given by

\[
\text{BCH}^\alpha(\lambda_1, \lambda_2) := \gamma_0(\Lambda^2_3(\lambda_1, \lambda_2)).
\]

**Remark 2.39.** The Baker–Campbell–Hausdorff product relative to \( \alpha \) is not well-defined in general for any two degree one elements of \( g \).

**Corollary 2.40.** Let \( \alpha \) be a Maurer–Cartan element of a curved absolute \( \mathcal{L}_{\infty} \)-algebra \( g \). The Baker–Campbell–Hausdorff product relative to \( \alpha \) defines a group structure with unit \( [\alpha] \) on \( H^\alpha_1(g) \), which is isomorphic to the group \( \pi_1(\mathcal{R}(g), \alpha) \).

**Proof.** This is straightforward from Theorem 2.35. \[\square\]
In this section, we show that curved absolute $L_\infty$-algebras are models for finite type nilpotent rational spaces without any pointed or connectivity assumptions. We do this by relating the adjunction $L \dashv R$ constructed in the previous section to the derived adjunction constructed by Bousfield and Gugenheim in [BG76] using Sullivan’s functor [Sul77]. We also construct small models for mapping spaces without any hypothesis on the sources using the machinery developed so far.

3.1. Comparing derived units of adjunctions. From now on, we assume the base field $K$ to be the field of rational numbers $Q$. Furthermore, we consider the category of simplicial sets endowed with the rational model structure constructed in [Bou75], where cofibrations are given by monomorphisms and where weak-equivalences are given by morphisms $f : X \rightarrow Y$ such that $H_*(f, Q) : H_*(X, Q) \rightarrow H_*(Y, Q)$ is an isomorphism.

Sullivan’s adjunction. One can use a dual version of Theorem 2.6 in order to induce the following contravariant adjunction

$$sSet \xrightarrow{\perp} \text{dg uCom-alg}^{\text{op}},$$

where $A_{\text{PL}}(-)$ is the piece-linear differential forms functor obtained by taking the left Kan extension of $\Omega^\bullet$. Its right adjoint functor, called the geometrical realization functor, is given by

$$(-)_\bullet := \text{Hom}_{\text{dg uCom-alg}}(-, \Omega^\bullet).$$

This adjunction is in fact a Quillen adjunction when one considers the standard model structure on dg unital commutative algebras, where weak-equivalences are given by quasi-isomorphisms and fibrations by degree-wise epimorphisms.

The quasi-isomorphism of dg operads $\varepsilon : \Omega B^a u\text{Com} \Rightarrow u\text{Com}$ induces the following Quillen equivalence

$$u\text{CE}_{\infty}-\text{alg} \xleftarrow{\text{Ind}_\varepsilon^{\text{op}}} \xrightarrow{\text{Res}_\varepsilon} \text{dg uCom-alg},$$

where $\text{Res}_\varepsilon$ is fully faithful. Thus one obtains the following commutative triangle of Quillen adjunctions

by defining $C_{\text{PL}}(-) := \text{Res}_\varepsilon^{\text{op}} \circ C_{\text{PL}}(-)$ and $(-)_\infty := (-) \circ \text{Ind}_\varepsilon^{\text{op}}$.

Lemma 3.1. Let $X$ be a simplicial set. There is a natural weak-equivalence of simplicial sets

$$R\langle C_{\text{PL}}(X) \rangle_\infty \simeq R\langle A_{\text{PL}}(X) \rangle.$$

Proof. The adjunction induced by $\varepsilon : \Omega B^a u\text{Com} \Rightarrow u\text{Com}$ is a Quillen equivalence. □

Cellular cochains functor. Consider again the simplicial $u\text{CE}_{\infty}$-algebra $\text{C}_C^* (\Delta^*)$ of Lemma 2.3 given by the cellular cochains on the standard simplicies together with their transferred $u\text{CE}_{\infty}$-algebra structure. It induces a Quillen adjunction
The above lemma implies that there is a natural weak-equivalence of curved $B$-algebras.

Our goal is to compare the derived unit of this adjunction with the derived unit of the adjunction obtained by extending Sullivan’s functor to the category of $u\mathcal{C}_\infty$-algebras. In order to do so, we need to be able to construct cofibrant resolutions. The canonical curved twisting morphism $\iota : B^{\s}u\mathcal{C}om \rightarrow \Omega B^{\s}u\mathcal{C}om$ induces a Quillen equivalence

$$\text{curv } B^{\s}u\mathcal{C}om\text{-coalg} \xrightarrow{\Omega_i} u\mathcal{C}_\infty\text{-alg},$$

where the model structure considered on the right hand side is the one obtained by transfer along this adjunction, see [LG19] for more details. An $\infty$-morphism of $u\mathcal{C}_\infty$-algebras $f : A \rightarrow B$ is the data of a morphism

$$f : B_iA \rightarrow B_iB$$

of curved $B^{\s}u\mathcal{C}om\text{-coalg}$s. It is an $\infty$-quasi-isomorphism if the term $f_{id} : A \rightarrow B$ is a quasi-isomorphism. This is equivalent to $f$ being a weak-equivalence in the transferred model structure.

**Lemma 3.2.** Let $X$ be a simplicial set. There is a pair of inverse $\infty$-quasi-isomorphism of $u\mathcal{C}_\infty$-algebras

$$(p_\infty)_X : C_{PL}(X) \rightarrow C^e_c(X), \quad (i_\infty)_X : C^e_c(X) \rightarrow C_{PL}(X),$$

which are natural in $X$.

**Proof.** Let $X$ be a simplicial set. Since Dupont’s contraction is compatible with the simplicial structures, it induces a homotopy retract

$$h_X \quad \xrightarrow{\text{coalg}} \quad C_{PL}(X) \xrightarrow{\text{alg}} \Omega_i C^e_c(X).$$

One can show, using analogue arguments to the proof of [RNV20, Proposition 7.12], that the transferred $u\mathcal{C}_\infty$-algebra structure from this contraction onto $C^e_c(X)$ is equal to the $u\mathcal{C}_\infty$-algebra structure obtained by considering the left Kan extension of $C^e_c(\Delta^\bullet)$. Therefore the morphisms of dg modules $p_X$ and $i_X$ can be extended to two inverse $\infty$-quasi-isomorphism $(p_\infty)_X$ and $(i_\infty)_X$, which are both natural in $X$, see [RNV20, Remark 7.14].

**Proposition 3.3.** Let $X$ be a simplicial set. There is a natural weak-equivalence of simplicial sets

$$\mathcal{R}\langle C_{PL}(X) \rangle_\infty \simeq \mathcal{R}\delta(C^e_c(X)).$$

**Proof.** The above lemma implies that there is a natural weak-equivalence of curved $B^{\s}u\mathcal{C}om\text{-coalg}$s

$$(p_\infty)_X : B_i C_{PL}(X) \rightarrow B_i C^e_c(X).$$

This in turn implies that there is a natural quasi-isomorphism

$$\Omega_i ((p_\infty)_X) : \Omega_i B_i C_{PL}(X) \rightarrow \Omega_i B_i C^e_c(X)$$

of $u\mathcal{C}_\infty$-algebras. Therefore we have

$$\mathcal{R}\langle C_{PL}(X) \rangle_\infty \simeq \text{Hom}_{u\mathcal{C}om\text{-alg}}(\text{Ind}_\epsilon \Omega_i B_i C_{PL}(X), \Omega_\ast) \simeq \text{Hom}_{u\mathcal{C}_\infty\text{-alg}}(\Omega_i B_i C_{PL}(X), \text{Res}_\epsilon \Omega_\ast) \simeq \text{Hom}_{u\mathcal{C}_\infty\text{-alg}}(\Omega_i B_i C^e_c(X), C^e_c(\Delta^\bullet))$$
where we used the existence of an $\infty$-quasi-isomorphism of $u\mathcal{E}_\infty$-algebras between $\text{Res}_\varepsilon \Omega \cdot$ and $C^*_c(\Delta^*)$, given again by the Dupont contraction. Notice that the intermediate equivalences are obtained using the fact that these are hom-spaces between a cofibrant and a fibrant object, thus stable by weak-equivalences.

**Finite type nilpotent spaces.** We recall what finite type nilpotent spaces are and we state the main theorem of this subsection.

**Definition 3.4 (Finite type simplicial set).** Let $X$ be a simplicial set. It is said to be of finite type if the homology groups $H_n(X, \mathbb{Q})$ are finite dimensional for all $n \geq 0$.

**Remark 3.5.** In particular, $X$ has a finitely many connected components since $H_0(X, \mathbb{Q})$ is finite dimensional.

**Definition 3.6 (Nilpotent simplicial set).** Let $X$ be a simplicial set. It is said to be nilpotent if for every 0-simplex $\alpha$ it satisfies the following conditions:

1. The group $\pi_1(X, \alpha)$ is nilpotent.
2. The $\pi_1(X, \alpha)$-module $\pi_n(X, \alpha)$-module is a nilpotent $\pi_1(X, \alpha)$-module.

**Proposition 3.7.** Let $X$ be a finite type simplicial set. There is a weak-equivalence of simplicial sets

$$\mathcal{RL}(X) \simeq \mathcal{RS}(C^*_c(X)),$$

which is natural on the subcategory of finite type simplicial sets.

**Proof.** Recall from the proof above that

$$\mathcal{RS}(C^*_c(X)) \simeq \text{Hom}_{u\mathcal{E}_\infty}(\Omega \cdot B \cdot C^*_c(X), C^*_c(\Delta^*)) .$$

The following square of Quillen adjunctions

$$
\begin{array}{ccc}
& & \text{curv } B^{s, a} u\text{Com-coalg}^{op} \\
\text{uC}^{\infty}-\text{alg}^{op} & \xleftarrow{\Omega^p} & \text{curv } B^{s, a} u\text{Com-coalg}^{comp} \\
\downarrow (-)^* & & \downarrow (-)^* \\
\text{uC}^{\infty}-\text{coalg} & \xleftarrow{\hat{\Omega}_1} & \text{curv } B^{s, a} u\text{Com-coalg}^{comp} \\
\end{array}
$$

commutes, by [RiL22a, Theorem 2.22] applied to this situation. Since $X$ is a finite type simplicial set, the homology of $\Omega \cdot B \cdot C^*_c(X)$ is degree-wise finite dimensional. Therefore the generalize Sweedler dual $(-)^0$ is homotopically fully-faithful by [RiL22a, Theorem 2.25], and we get

$$\text{Hom}_{u\mathcal{E}_\infty}(\Omega \cdot B \cdot C^*_c(X), C^*_c(\Delta^*)) \simeq \text{Hom}_{u\mathcal{E}_\infty}(C^*_c(\Delta^*), \hat{B}_1 \hat{\Omega}_1 C^*_c(X)) ,$$

which concludes the proof.

**Corollary 3.8.** Let $X$ be a finite type simplicial set. There is a weak-equivalence of simplicial sets

$$\mathcal{RL}(X) \simeq \mathcal{RL}(X) ,$$

which is natural in the subcategory of finite type simplicial sets.

**Proof.** This is a direct consequence of Lemma 3.1 and Propositions 3.3 and 3.7. □

Now we can transfer the known results about Sullivan’s rational models along the above equivalence.

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Theorem 3.9 ([LM15, Theorem C]). Let $X$ be a finite type nilpotent simplicial set. The unit

$$\eta_X : X \to \mathbb{R}\langle A_{PL}(X) \rangle$$

is a rational homotopy equivalence.

Remark 3.10. The original construction of [BG76] only works for connected finite type nilpotent simplicial set and was subsequently extended to disconnected finite type nilpotent simplicial sets by Markl–Lazarev in [LM15].

Theorem 3.11. Let $X$ be a finite type nilpotent simplicial set. The unit of adjunction

$$\eta_X : X \to RL(X)$$

is a rational homotopy equivalence.

Proof. Follows directly from Corollary 3.8 and Theorem 3.9. □

Corollary 3.12. Let $X$ be a pointed connected finite type simplicial set. The unit of adjunction

$$\eta_X : X \to RL(X)$$

is weakly equivalent to the $Q$-completion of Bousfield-Kan.

Proof. This is implied by [BG76, Theorem 12.2], using the equivalence $\mathbb{R}\langle A_{PL}(X) \rangle \simeq RL(X)$ of Corollary 3.8. □

3.2. Minimal models. In this subsection we show that curved absolute $\mathcal{L}_\infty$-algebras always admit a minimal resolutions, which are unique up to isomorphism. Furthermore, as a corollary of our constructions, we show that one can also recover the homology of a space $X$ via the homology of the complete Bar construction of its models.

Definition 3.13 (Minimal model). Let $g$ be a curved absolute $\mathcal{L}_\infty$-algebra. A minimal model $(V, \varphi_{d_V}, \psi_V)$ amounts to the data of

1. A graded module $V$ together with a map

$$\varphi_{d_V} : V \to \prod_{n \geq 0} \hat{\Omega}^{s,a}u\text{Com}^*(n)[\geq 1] \hat{\otimes}_{S_n} V^\otimes n$$

which lands on elements of weight greater or equal to 1, such that the induced derivation $d_V$ satisfies

$$d_V^2 = l_2 \circ_1 l_0 .$$

2. A weak equivalence of curved absolute $\mathcal{L}_\infty$-algebras

$$\psi_V : \left( \hat{\Omega}^{s,a}u\text{Com}^*(n) \hat{\otimes}_{S_n} V^\otimes n, d_V \right) \to g .$$

Remark 3.14. Given a graded module $V$, the data of the derivation $d_V$ amounts to a $u\mathcal{E}_\infty$-coalgebra structure on $V$. Thus the data $(V, \varphi_{d_V})$ above amounts to the data of a minimal $u\mathcal{E}_\infty$-coalgebra, that is, a $u\mathcal{E}_\infty$-coalgebra where the underlying differential is zero.

Proposition 3.15. Let $g$ be a curved absolute $\mathcal{L}_\infty$-algebra and let $(V, \varphi_{d_V}, \psi_V)$ and $(W, \varphi_{d_W}, \psi_W)$ be two minimal models of $g$. Then there is an isomorphism of graded modules $V \cong W$. 

Proof. If $(V, \varphi_{d_V}, \psi_V)$ and $(W, \varphi_{d_W}, \psi_W)$ are two minimal models of $g$, then, by definition, there exists a weak equivalence of curved absolute $\mathcal{L}_\infty$-algebras

$$\left( \hat{\Omega}^{s,a}u\text{Com}^*(n) \hat{\otimes}_{S_n} V^\otimes n, d_V \right) \Rightarrow \left( \hat{\Omega}^{s,a}u\text{Com}^*(n) \hat{\otimes}_{S_n} W^\otimes n, d_W \right) .$$

Therefore there is a quasi-isomorphism between the graded modules $V$ and $W$, which implies they are isomorphic. □
Proposition 3.16. Let \( \mathfrak{g} \) be a curved absolute \( \mathcal{L}_\infty \)-algebra. Then it admits a minimal model, where the graded module of generators is given by
\[
V \cong H_*(\hat{B}_i \mathfrak{g})
\]
that is, the homology of its complete Bar construction.

Proof. Since \( \mathbb{K} \) is a field of characteristic 0, one can always choose a contraction between \( \hat{B}_i \mathfrak{g} \) and its homology in order to apply the Homotopy Transfer Theorem. The transferred \( u\mathcal{C}_\infty \)-coalgebra structure provides us with the derivation by applying the complete Cobar construction. The image of the inclusion
\[
\iota : H_*(\hat{B}_i \mathfrak{g}) \to \hat{B}_i \mathfrak{g}
\]
under the complete Cobar construction allows us obtain the required weak-equivalence by post-compositing it with \( \eta : \mathfrak{g} \to \hat{\Omega}_* \hat{B}_i \mathfrak{g} \). \( \square \)

Remark 3.17. The same minimal model constructions hold for absolute \( \mathcal{L}_\infty \)-algebras using the complete Bar construction \( \hat{B}_i \) relative to \( \mathcal{C}_\infty \)-coalgebras.

Definition 3.18 (Rational curved absolute \( \mathcal{L}_\infty \)-algebras). Let \( \mathfrak{g} \) be a curved absolute \( \mathcal{L}_\infty \)-algebra. It is rational if there exists a simplicial set \( X \) and a zig-zag of weak equivalences of curved absolute \( \mathcal{L}_\infty \)-algebras
\[
\mathcal{L}(X) \leftarrow \ldots \to \mathfrak{g}
\]

Remark 3.19. If \( \mathfrak{g} \) is rational, then the homology of \( \hat{B}_i(\mathfrak{g}) \) is also concentrated in positive degrees. Indeed, \( \mathcal{L}(X) \) is weakly equivalent to \( \mathfrak{g} \) if and only if \( \mathcal{C}_\infty^*(X) \) is quasi-isomorphic to the complete Bar construction of \( \mathfrak{g} \).

Definition 3.20 (Rational models). Let \( \mathfrak{g} \) be a curved absolute \( \mathcal{L}_\infty \)-algebra. It is a rational model if it is rational for some simplicial set \( X \) and if furthermore \( \mathcal{R}(\mathfrak{g}) \) is weakly equivalent to \( X \).

Proposition 3.21. Let \( \mathfrak{g} \) be a curved absolute \( \mathcal{L}_\infty \)-algebra which is rational for a simplicial set \( X \). Then its minimal model is generated by \( H_*(X) \).

Proof. In this particular case, we have that
\[
H_*(\hat{B}_i \mathfrak{g}) \cong H_*(X),
\]
therefore by Proposition 3.16, the graded module \( H_*(X) \) is the generator of the minimal model. \( \square \)

Remark 3.22. Notice that the situation here is Koszul dual to Sullivan’s minimal models, where the minimal Sullivan model for \( A_{PL}(X) \) of a simply connected space \( X \) is generated by the linear dual of the homotopy groups \( \pi_*(X) \).

Recovering the homology groups. Let \( \mathfrak{g} \) be curved absolute \( \mathcal{L}_\infty \)-algebra, let us try to understand the homotopy type of \( \mathcal{R}(\mathfrak{g}) \). The homotopy groups of \( \mathcal{R}(\mathfrak{g}) \) can be fully described by the homology groups \( H_*(\mathfrak{g}^\alpha) \), where \( \alpha \) runs over the set of Maurer–Cartan elements of \( \mathfrak{g} \). In the case where \( \mathfrak{g} \) is a rational model, then one can also recover the homology groups of \( \mathcal{R}(\mathfrak{g}) \) using only the complete Bar construction with respect to \( \mathfrak{g} \).

Theorem 3.23. Let \( \mathfrak{g} \) be a curved absolute \( \mathcal{L}_\infty \)-algebra that is a rational model. The canonical morphism
\[
\mathcal{C}_\infty^*(\mathcal{R}(\mathfrak{g})) \to \hat{B}_i \mathfrak{g}
\]
is a quasi-isomorphism.

Proof. In this situation, \( \mathcal{L}(\mathcal{R}(\mathfrak{g})) \to \mathfrak{g} \) is a weak equivalence of curved absolute \( \mathcal{L}_\infty \)-algebras, which is equivalent to the canonical morphism \( \mathcal{C}_\infty^*(\mathcal{R}(\mathfrak{g})) \to \hat{B}_i \mathfrak{g} \) being a quasi-isomorphism of \( u\mathcal{C}_\infty \)-coalgebras. \( \square \)

Remark 3.24. One can think of \( \hat{B}_i \mathfrak{g} \) as a higher Chevalley-Eilenberg complex adapted to the setting of curved absolute \( \mathcal{L}_\infty \)-algebras.
3.3. Models for mapping spaces. In this section, we construct explicit rational models for mapping spaces, without any assumption on the source simplicial set. Furthermore, these models are relatively small, as they are constructed using the cellular chains on the source.

Recall that, if $X$ and $Y$ are simplicial sets, there is an explicit model for their mapping space given by

$$\text{Map}(X, Y)_\bullet := \text{Hom}_{\text{Set}}(X \times \Delta^\bullet, Y),$$

which forms a Kan complex when $Y$ is so.

**Lemma 3.25.** Let $X$ and $Y$ be two simplicial sets. There is an $\infty$-quasi-isomorphism

$$\psi_{X,Y}: C^\bullet_c(X \times Y) \cong C^\bullet_c(X) \otimes C^\bullet_c(Y),$$

of $\mathcal{C}_{\infty}$-coalgebras which is natural in $X$ and $Y$.

**Proof.** For any simplicial sets $X$, $Y$, recall (see [?]) that there is a natural quasi-isomorphism

$$\kappa: A_{PL}(X) \otimes A_{PL}(Y) \to A_{PL}(X \times Y)$$

of dg $u\mathcal{C}$-algebras. This gives a quasi-isomorphism

$$\text{Res}_\kappa(\kappa): C_{PL}(X) \otimes C_{PL}(Y) \to C_{PL}(X \times Y)$$

of $u\mathcal{C}_{\infty}$-algebras which is natural in $X$, $Y$, using the fact that the restriction functor $\text{Res}_\kappa$ is strong monoidal and preserves all quasi-isomorphisms.

Using the $\infty$-quasi-isomorphisms constructed in the proof of Proposition 3.3, we construct

$$C^\bullet_c(X) \otimes C^\bullet_c(Y) \xrightarrow{(\text{Res}_\kappa(\kappa))} C_{PL}(X) \otimes C_{PL}(Y) \xrightarrow{(\psi_{X,Y})} C^\bullet_c(X \times Y),$$

where the tensor product of two $\infty$-morphisms is still an $\infty$-morphism, since the 2-colored dg operad encoding $\infty$-morphisms of $\mathcal{C}_{\infty}$-algebras is cofibrant. This gives $\infty$-quasi-morphism of $u\mathcal{C}_{\infty}$-algebras

$$C^\bullet_c(X) \otimes C^\bullet_c(Y) \cong C^\bullet_c(X \times Y).$$

which is natural in $X$ and $Y$. Equivalently, a weak-equivalence $B_1(C^\bullet_c(X) \otimes C^\bullet_c(Y)) \to B_1(C^\bullet_c(X \times Y))$. Now we take $X$ and $Y$ to be finite simplicial sets (finite total number of non-degenerate simplicies). By applying the linear dual functor of [RiL22a, Theorem 2.22], this gives a weak-equivalence

$$\hat{\Omega}_1(C^\bullet_c(X) \otimes C^\bullet_c(Y)) \leftrightarrow \hat{\Omega}_1(C^\bullet_c(X \times Y))$$

of complete curved absolute $\mathcal{L}_\infty$-algebras. Now suppose $X$ and $Y$ are arbitrary simplicial sets, one can write them as the filtered colimit of finite simplicial sets

$$X \cong \colim_\alpha X_\alpha \quad \text{and} \quad Y \cong \colim_\beta Y_\beta.$$

Now we have that

$$\hat{\Omega}_1(C^\bullet_c(X) \otimes C^\bullet_c(Y)) \cong \colim_{\alpha, \beta} \hat{\Omega}_1(C^\bullet_c(X_\alpha) \otimes C^\bullet_c(Y_\beta)),$$

and

$$\hat{\Omega}_1(C^\bullet_c(X \times Y)) \cong \colim_{\alpha, \beta} \hat{\Omega}_1(C^\bullet_c(X_\alpha \times Y_\beta)),$$
since both bifunctors preserve filtered colimits in each variable. Both of this colimits are in fact homotopy colimits, hence there is a weak-equivalence
\[ \tilde{\Omega}_c(C^c_\ast(X) \otimes C^c_\ast(Y)) \to \tilde{\Omega}_c(C^c_\ast(X \times Y)), \]
which concludes the proof. \( \square \)

**Theorem 3.26.** Let \( g \) be a curved absolute \( \mathcal{L}_\infty \)-algebra and let \( X \) be a simplicial set. There is a weak equivalence of Kan complexes
\[ \text{Map}(X, \mathcal{R} (g)) \simeq \mathcal{R} \left( \text{hom}(C^c_\ast(X), g) \right), \]
which is natural in \( X \) and in \( g \), where \( \text{hom}(C^c_\ast(X), g) \) denotes the convolution curved absolute \( \mathcal{L}_\infty \)-algebra.

**Proof.** There is an isomorphism
\[ \text{Map}(X, \mathcal{R} (g))_\ast := \text{Hom}_{sSet}(X \times \Delta^\bullet, \mathcal{R} (g)) \cong \text{Hom}_{sSet}(C^c_\ast(X \times \Delta^\bullet), \hat{B}_1(g)). \]

We can pre-compose by the \( \infty \)-quasi-isomorphism of Lemma 3.25, giving a weak-equivalence of simplicial sets
\[ \text{Hom}_{u\epsilon e_{\infty}-cog}(C^c_\ast(X \times \Delta^\bullet), \hat{B}_1(g)) \to \text{Hom}_{u\epsilon e_{\infty}-cog}(C^c_\ast(X) \otimes C^c_\ast(\Delta^\bullet), \hat{B}_1(g)), \]
since both \( C^c_\ast(X \times \Delta^\bullet) \) and \( C^c_\ast(X) \otimes C^c_\ast(\Delta^\bullet) \) are Reedy cofibrant. Let’s compute this last simplicial set:
\[ \begin{align*} 
\text{Hom}_{u\epsilon e_{\infty}-cog}(C^c_\ast(X) \otimes C^c_\ast(\Delta^\bullet), \hat{B}_1(g)) & \cong \text{Hom}_{u\epsilon e_{\infty}-cog}\left( C^c_\ast(\Delta^\bullet), \left\{ C^c_\ast(X), \hat{B}_1(g) \right\} \right) \\
& \cong \text{Hom}_{u\epsilon e_{\infty}-cog}\left( C^c_\ast(\Delta^\bullet), \hat{B}_1\left( \text{hom}(C^c_\ast(X), g) \right) \right) \\
& \cong \text{Hom}_{\text{curv abs } \mathcal{L}_\infty-\text{alg}} \left( \tilde{\Omega}_c(C^c_\ast(\Delta^\bullet)), \text{hom}(C^c_\ast(X), g) \right) \\
& \cong \mathcal{R} \left( \text{hom}(C^c_\ast(X), g) \right). \end{align*} \]
\( \square \)

**Corollary 3.27.** Let \( X \) be a simplicial set and let \( Y \) be a finite type nilpotent simplicial set. There is a weak equivalence of simplicial sets
\[ \text{Map}(X, Y_Q) \simeq \mathcal{R} \left( \text{hom}(C^c_\ast(X), \mathcal{L}(Y)) \right), \]
where \( Y_Q \) denotes the \( \mathbb{Q} \)-localization of \( Y \). Therefore, given a map \( f : X \to Y_Q \), there is an isomorphism
\[ \pi_n \left( \text{Map}(X, Y_Q), f \right) \cong H_n^{C^c_\ast(f)} \left( \text{hom}(C^c_\ast(X), \mathcal{L}(Y)) \right), \]
of groups for \( n \geq 1 \).

**Proof.** There is a weak equivalence of simplicial sets
\[ Y_Q \simeq \mathcal{RL}(Y). \]
It induces equivalences
\[ \text{Map}(X, Y_Q) \simeq \text{Map}(X, \mathcal{RL}(Y)) \simeq \mathcal{R} \left( \text{hom}(C^c_\ast(X), \mathcal{L}(Y)) \right). \]
Finally, the second statement follows directly from Theorem 2.35. \( \square \)

**Remark 3.28.** Notice the following points about Corollary 3.27.
(1) There is no finiteness hypothesis on the simplicial set $X$.

(2) So far, the models for mapping spaces in the literature for mapping spaces are given in terms of complete tensor products

$$A_X \hat{\otimes} g_Y,$$

of a dg unital commutative algebra model $A_X$ of $X$ and a complete $L_\infty$-algebra model $g_Y$. Our model in comparison is much smaller as it only uses the cellular chains on $X$, see for instance [Ber15, BFM13, Laz13].

**Corollary 3.29.** Let $X$ be a finite type nilpotent simplicial set. The space of rational endomorphisms of $X$ is weak-equivalent to

$$\text{End}^h_Q(X) \xrightarrow{\sim} \mathbb{R}(\text{hom}(C_c^*(X), \mathcal{L}(X))).$$

**Proof.** This is immediate from the previous corollary. □

### 4. Deformation Theory

In this section, we consider different applications of the theory developed so far. First, we show that convolution (curved in our case) $L_\infty$-algebras constructed by D. Robert-Nicoud and F. Wierstra in [RNW20] posses naturally an "absolute" structure without the need of underlying filtrations. These convolution algebras encode $\infty$-morphisms between algebras over a (non-augmented) dg operad as their Maurer–Cartan elements and all the homotopies between those. Secondly, for any derived affine stack, we construct a curved absolute $L_\infty$-algebra which is "geometrical model". More concretely, we shows that this geometrical model recovers the formal neighborhood of any of the $L$-points of our derived affine stack, where $L$ is a finite algebraic extension of the base field $K$. These results should lead to a "non-pointed" version of formal moduli problems.

#### 4.1. Convolution curved absolute $L_\infty$-algebras and $\infty$-morphisms of algebras.

Let $P$ denote a dg operad and let $\mathcal{C}$ denote a conilpotent curved cooperad.

**Proposition 4.1.** There is a bijection

$$\text{Tw}(\mathcal{C}, P) \equiv \text{Hom}_{\text{curv ab pOp}}\left(\hat{\Omega}^{s,\mathcal{C}}\mathcal{U}\mathcal{C}om^*, \text{Hom}(\mathcal{C}, P)\right),$$

between morphisms of curved absolute partial operads $\phi_\alpha : \hat{\Omega}^{s,\mathcal{C}}\mathcal{U}\mathcal{C}om^* \rightarrow \text{Hom}(\mathcal{C}, P)$ and curved twisting morphisms $\alpha : \mathcal{C} \rightarrow P$ such that $\alpha(1)$ is the zero morphism.

**Proof.** First, notice that the convolution curved partial operad of [RiL22b, Subsection 6.2] is in fact a curved absolute partial operad in the sense of [RiL22b, Subsection 10.2]. Indeed, one can check that any infinite sum of compositions in it has a well-defined image in it since any such sum is locally finite because of the conilpotency of $\mathcal{C}$. There is an isomorphism

$$\text{Hom}(\mathcal{U}\mathcal{C}om^*, \text{Hom}(\mathcal{C}, P)) \cong \text{Hom}(\mathcal{C}, P)$$

of curved absolute partial operads since $\mathcal{U}\mathcal{C}om^*(n) = K$ for all $n \geq 0$. Therefore,

$$\text{Hom}_{\text{curv ab pOp}}\left(\hat{\Omega}\mathcal{U}\mathcal{C}om^*, \text{Hom}(\mathcal{C}, P)\right) \cong \text{Tw}(\mathcal{U}\mathcal{C}om^*, \text{Hom}(\mathcal{C}, P)) \cong \text{Tw}(\mathcal{C}, P),$$

using the curved operadic Bar-Cobar adjunction of [RiL22b, Section 6]. Now one can check that $\hat{\Omega}^{s,\mathcal{C}}\mathcal{U}\mathcal{C}om^*$ represents exactly those curved twisting morphisms that have a trivial arity one component. □

**Remark 4.2.** For an analogue statement concerning convolution $L_\infty$-algebra structures, see the constructions in [RNW20].
Let \((A, \gamma_A, d_A)\) be a dg \(\mathcal{P}\)-algebra and let \((C, \Delta_C, d_C)\) be a curved \(\mathcal{E}\)-coalgebra. The graded module of graded morphisms \(\text{hom}(C, A)\) is naturally a pdg module endowed with the pre-differential \(\partial(f) := d_A \circ f - f \circ d_C\).

**Proposition 4.3.** Let \((A, \gamma_A, d_A)\) be a dg \(\mathcal{P}\)-algebra and let \((C, \Delta_C, d_C)\) be a curved \(\mathcal{E}\)-coalgebra. The pdg module

\[
\text{hom}(C, A)
\]

can be endowed with a curved \(\mathcal{L}_\infty\)-algebra structure is given by

\[
\gamma_{\text{hom}(C, A)} : \bigoplus_{n \geq 0} \hat{\Omega} \text{uCom}^*(n) \otimes_{S_n} \text{hom}(C, A)^{\otimes n} \rightarrow \text{hom}(C, A)
\]

\[
c_n(f_1 \otimes \cdots \otimes f_n) \rightarrow \gamma_A \circ [\phi_\alpha(c_n) \otimes (f_1 \otimes \cdots \otimes f_n)] \circ \Delta_C,
\]

where \(c_n\) denotes the \(n\)-corolla.

**Proof.** This follows directly from Proposition 4.1 by considering the composition

\[
\hat{\Omega} \text{uCom}^* \xrightarrow{\phi_\alpha} \mathcal{H} \text{om}(\mathcal{E}, \mathcal{P}) \xrightarrow{\lambda_{\text{hom}(C,A)}} \text{End}_{\text{hom}(C,A)},
\]

of morphisms of curved operads, where \(\lambda_{\text{hom}(C,A)}\) is the natural curved algebra structure of

\[
\text{hom}(C, A)
\]

over \(\mathcal{H} \text{om}(\mathcal{E}, \mathcal{P})\).

**Remark 4.4 (Local nilpotency).** One can notice that the Maurer–Cartan equation is always well-defined without the need of imposing any filtration on the space \(\text{hom}(C, A)\). Indeed, let \(f : C \rightarrow A\) be a degree zero map, then the sum

\[
\gamma_A \circ \left[ \sum_{n \geq 0, n \neq 1} \phi_\alpha(c_n) \otimes f^{\otimes n} \right] \circ \Delta_C(c) + \partial(f)(c)
\]

is well defined for any element \(c\) in \(C\), since there are only a finite number of non-zero terms in \(\Delta_C(c)\). Filtrations introduced to make this type of sum converge like in [DHR15] are redundant.

**Proposition 4.5.** The extension of \(\gamma_{\text{hom}(C,A)}\) given by

\[
\prod_{n \geq 0} \hat{\Omega} \text{uCom}^*(n) \otimes_{S_n} \text{hom}(C, A)^{\otimes n} \rightarrow \text{hom}(C, A)
\]

\[
\sum_{n \geq 0} \sum_{\tau \in \text{CRT}_n^\alpha} \lambda_\tau(f_1 \otimes \cdots \otimes f_n) \rightarrow \gamma_A \circ \left[ \sum_{n \geq 0} \sum_{\tau \in \text{CRT}_n^\alpha} \phi_\alpha(\tau) \otimes (f_1 \otimes \cdots \otimes f_n) \right] \circ \Delta_C,
\]

defines a structure of curved absolute \(\mathcal{L}_\infty\)-algebra, denoted by \(\text{hom}(C, A)\).

**Proof.** One can check by hand that this formula satisfies conditions 1, 2, and 3.

**Remark 4.6.** This convolution curved absolute \(\mathcal{L}_\infty\)-algebra structure coincides with the one constructed in [Gri22b] between a dg \(\mathcal{O}\mathcal{E}\)-algebra and a curved \(\mathcal{E}\)-coalgebra, using the Hopf comodule structure of the cofibrant dg operad \(\mathcal{O}\mathcal{E}\).

**Proposition 4.7.** Let \(A\) and \(B\) be two dg \(\mathcal{P}\)-algebras. The simplicial set \(\mathcal{H}(\text{hom}(B_\alpha A, B))\) has as 0-simplicies the set of \(\infty_\alpha\)-morphisms between \(A\) and \(B\).

**Proof.** Maurer–Cartan elements correspond by definition to curved twisting morphisms \(B_\alpha A\) between and \(B\), which are in bijection with morphisms of curved \(\mathcal{E}\)-coalgebras between \(B_\alpha A\) and \(B_\alpha B\).
Therefore 1-simplicies in \( R(\text{hom}(B_\alpha A, B)) \) induce a notion of homotopies between these \( \infty \)-morphism, and similarly for \( n \)-simplicies which correspond to higher homotopies. Using these methods to simplicially enrich the category of dg \( P \)-algebras and thus obtain a model for the \( \infty \)-category of dg \( P \)-algebras will be the subject of a future work.

**Remark 4.8 (The dual case of \( \infty \)-morphism of coalgebras).** In [Gri22b], Brice Le Grignou constructs a convolution curved absolute \( \mathcal{L}_\infty \)-algebra from a dg \( \Omega \mathcal{C} \)-coalgebra and a curved \( \mathcal{C} \)-algebra. Integrating these convolution algebras provides us with the right set of \( \infty \)-morphisms between two coalgebras. This method also provides us with a way in which one can simplicially enrich the category of dg \( \Omega \mathcal{C} \)-coalgebras with \( \infty \)-morphisms.

**Remark 4.9 (Extension to dg properads).** Let \( \Omega \mathcal{C} \) be a properad. Consider the notion of \( \infty \)-morphism for \( \mathcal{C} \)-algebras over \( \Omega \mathcal{C} \) as defined in [HLV21]. It was privately communicated to us that the convolution algebra that governs these \( \infty \)-morphisms does not appear to carry any sort of filtration that would make the Maurer–Cartan equation converge. As a consequence, the authors plan to apply the present integration theory of absolute \( \mathcal{L}_\infty \)-algebras in order to settle a suitable simplicial enrichment for the category of \( \Omega \mathcal{C} \)-gebras with \( \infty \)-morphisms.

### 4.2. The formal geometry of Maurer–Cartan spaces.

Let \( A \) be a dg \( u \mathcal{C} \)-algebras, viewed equivalently as a derived affine stacks. Its functor of points is given by

\[
\text{Spec}(A)(−) : \text{dg } u\mathcal{C}\text{-}\text{alg}_{\geq 0} \longrightarrow \text{sSet}
\]

\[
B \longmapsto \text{Spec}(A)(B) := R\text{Hom}_{\text{dg } u\mathcal{C}\text{-}\text{alg}}(A, B \otimes \Omega_\ast),
\]

where \( \Omega_\ast \) is again the simplicial Sullivan algebra and where \( B \) is a dg \( u\mathcal{C} \)-algebra concentrated in non-negative homological degrees, that is, a derived affine scheme. The simplicial set \( \text{Spec}(A)(B)_\ast \) is called the \( B \)-points of \( A \).

The first goal of this subsection is to construct a functor from dg \( u\mathcal{C} \)-algebras to curved absolute \( \mathcal{L}_\infty \)-algebras that recovers the "formal geometry" of \( A \). Let \( x : A \longrightarrow K \) be a \( K \)-point of \( A \), it provides an \( K \)-augmentation of \( A \). The standard way of looking at infinitesimal thickenings of \( x \) inside \( A \) has been to test \( A \) against \( K \)-augmented dg Artinian algebras.

**Definition 4.10 (K-augmented dg Artinian algebra).** Let \( R \) be a dg \( u\mathcal{C} \)-algebra with homology concentrated in non-negative homological degrees. It is a \( K \)-augmented dg Artinian algebra if it satisfies the following conditions.

1. Its homology is degree-wise finite dimensional and concentrated in a finite number of degrees.
2. There is an unique augmentation morphism \( p : R \longrightarrow K \) (hence \( R \) is local).
3. Let \( \overline{R} \) be the non-unital dg commutative algebra given by the kernel of \( p \). Then \( \overline{R} \) is nilpotent.

This notion allows us to define the notion of a \( K \)-pointed formal moduli problem.

**Definition 4.11 (K-pointed formal moduli problem).** Let \( F : \text{dg } \mathcal{C}\text{-}\text{alg}_{\geq 0}^{K\text{-aug}} \longrightarrow \text{sSet} \) be a functor. It is a \( K \)-pointed formal moduli problem if it satisfies the following conditions.

1. We have that \( F(K) \simeq \{\ast\} \).
2. The functor \( F \) sends quasi-isomorphisms to weak-equivalences.
The functor $F$ preserves homotopy pullbacks of the dg Artinian algebras $X, Y$ and $Z$
\[
\begin{array}{ccc}
X & \times & Z \\
\downarrow & & \downarrow \\
X & \to & Z
\end{array}
\]
such that $\pi_1$ and $\pi_2$ are surjections on the zeroth homology groups $H_0$.

Example 4.12. Any $\mathbb{K}$-augmented derived affine stack $A$ defines such a pointed deformation problem by considering
\[
\text{Spec}^* (A)(-) = \mathcal{R}\text{Hom}_{\text{dg\,uCom-alg}_{K,\text{aug}}} (A, - \otimes \Omega_*),
\]
where morphisms of $\mathbb{K}$-augmented dg commutative algebras are required to preserve the augmentation. This functor preserves homotopy limits and sends quasi-isomorphisms to weak homotopy equivalences, hence it defines a pointed formal moduli problem.

The $\infty$-category of $\mathbb{K}$-pointed formal moduli problems is equivalent to the $\infty$-category of dg Lie algebras over $\mathbb{K}$, or for that matter, to the $\infty$-category of $\mathcal{L}\infty$-algebras, both of them constructed by consider quasi-isomorphisms as weak-equivalences. This was shown independently by J. Lurie in [Lur11] by working directly with $\infty$-categories and by J. P Pridham [Pri10] using model categories.

**Non-pointed version.** Heuristically speaking, curved absolute $\mathcal{L}\infty$-algebras should correspond to a non-pointed version of formal moduli problems. A first idea into what this notion might be is the following: in a pointed formal moduli problem, the point around which one considers the infinitesimal thickening is already specified in advance. In the non-pointed version, one should remove the $\mathbb{K}$-augmented hypothesis.

As a first approach to non-pointed deformation problems, we build a curved absolute $\mathcal{L}\infty$-algebra $g_A$ from a derived affine stack $A$. We consider the following commutative square given by [RiL22a, Theorem 2.22]:
\[
\begin{array}{ccc}
(dg\,u\text{Com-alg})^{\text{op}} & \xrightarrow{\Omega^?_{\mathcal{L}}} & (\text{curv}\,\mathcal{L}\infty\text{-coalg}^{\text{conil}})^{\text{op}} \\
\downarrow & & \downarrow \\
dg\,u\text{Com-coalg} & \xleftarrow{\widehat{\Omega}_{\mathcal{L}}} & \text{curv abs}\,\mathcal{L}\infty\text{-alg}^{\text{comp}},
\end{array}
\]
which relates the category of dg $u\text{Com}$-algebras to the category of curved absolute $\mathcal{L}\infty$-algebras.

Remark 4.13. The category of dg $u\text{Com}$-coalgebras does not admit a model category structure transferred from dg modules, where weak-equivalences are quasi-isomorphisms and where cofibrations are degree-wise monomorphisms. Therefore the left adjunction on this square is not a Quillen adjunction in this case, see [GL22, Section 9] for more details.

Definition 4.14 (Geometrical model). Let $A$ be a dg $u\text{Com}$-algebra. Its geometrical model $g_A$ is the complete curved absolute $\mathcal{L}\infty$-algebra given by $(B_{\mathcal{L}}A)^*$. It defines a functor
\[
g(-) := (B_{\mathcal{L}}(-))^*: (dg\,u\text{Com-alg})^{\text{op}} \to \text{curv abs}\,\mathcal{L}\infty\text{-alg}^{\text{comp}}.
\]
Remark 4.15. This approach is close to the one of [CCN19], where the authors consider the linear dual of the Bar construction relative to $k$ of a dg $P$-algebra. Nevertheless, they view it as a dg $P^!$-algebra instead of viewing this linear dual as dg algebra over the cooperad $P^!$. They use the linear dual of the Bar construction in order to show an equivalence between the $\infty$-category of pointed formal moduli problems defined using dg Artinian $P$-algebras and the $\infty$-category of dg $P^!$-algebras, under certain hypothesis on the operad $P$.

Proposition 4.16. The geometrical model functor $g(\_)$ sends quasi-isomorphisms to weak-equivalences.

Proof. Let $f : A \Rightarrow B$ be a quasi-isomorphism of dg $u\mathcal{C}om$-algebras, then $B_{\pi}(f) : B_{\pi}A \Rightarrow B_{\pi}B$ is a weak-equivalence of conilpotent curved $L_\infty$-coalgebras, since every dg $u\mathcal{C}om$-algebra is fibrant. The linear dual $(-)^*$ preserves all weak-equivalences since it is a right Quillen functor and since every conilpotent curved $L_\infty$-coalgebra is cofibrant (fibrant in the opposite category).

Definition 4.17 (dg Artinian algebra). Let $A$ be a dg $u\mathcal{C}om$-algebra in non-negative homological degrees. It is a dg Artinian algebra if its homology is degree-wise finite dimensional.

Example 4.18. Consider a dg Artinian algebra $A$ concentrated in degree zero. Since $A$ is a finite dimensional $K$-algebra, it can be written as a product

$$A \cong \prod_{i=0}^{n} A_i,$$

where $A_i$ are local Artinian algebras. Thus one can consider the following examples

1. the algebra $K^n$, which gives a collection of $n$ points,
2. any finite field extension $L$ of $K$,
3. classical thickenings like $K[t]/(t^n)$.

Lemma 4.19. There is an equivalence of $\infty$-categories between the opposite $\infty$-category of $u\mathcal{C}om_{\infty}$-coalgebras with finite dimensional homology concentrated in non-positive degrees and the $\infty$-category of dg Artinian algebras.

Proof. There are equivalences

$$\left(u\mathcal{C}om_{\infty}\text{-coalg}_{f_d \leq 0}\right)^{\infty} \xrightarrow{\sim} \left(u\mathcal{C}om_{\infty}\text{-coalg}_{\geq 0}\right)^{\infty} \xrightarrow{\sim} \text{dg Com-alg}_{f_d}^{\infty}$$

where the first equivalence is given by [RiL22a, Theorem 2.25] and the second equivalence is induced by the quasi-isomorphism of dg operads $\varepsilon : \Omega Bu\mathcal{C}om \Rightarrow u\mathcal{C}om$, which respects the degrees of the homology.

Theorem 4.20. Let $A$ be a dg $u\mathcal{C}om$-algebra. Let $B$ be a dg Artinian algebra. There is a weak homotopy equivalence of simplicial sets

$$\text{Spec}(A)(B) \simeq R(\text{hom}((\text{Res}_B B)^{\infty}, g_A)),$$

which is natural in $B$ and in $A$.

Proof. We start by computing the derived mapping space with the classical Bar-Cobar resolution

$$\text{Spec}(A)(B) \simeq \text{Hom}_{u\mathcal{C}om\text{-alg}}(\Omega_{\pi}B_{\pi}A, B \otimes \Omega_{\pi})$$

since $\Omega_{\pi}B_{\pi}A$ is a cofibrant resolution of $A$. Using the same methods as in the proof of Proposition 3.3, we show that

$$\text{Hom}_{u\mathcal{C}om\text{-alg}}(\Omega_{\pi}B_{\pi}A, B \otimes \Omega_{\pi}) \simeq \text{Hom}_{u\mathcal{C}om\text{-alg}}(\Omega_{\pi}B_{\pi} \text{Res}_{\pi} A, \text{Res}_{\pi}(B) \otimes C_{\pi}(\Delta^{\ast}))$$
We consider now the following quasi-isomorphism of $u\mathbb{C}_{\infty}$-algebras

$$
\text{Res}_\varepsilon(B) \otimes C^*_\varepsilon(\Delta^\bullet) \xrightarrow{\eta_{\text{Res}_\varepsilon(B)} \otimes \text{id}} ((\text{Res}_\varepsilon B)^\circ \otimes C^*_\varepsilon(\Delta^\bullet)) \xrightarrow{\Gamma} ((\text{Res}_\varepsilon B)^\circ \otimes C^*_\varepsilon(\Delta^\bullet))^*,
$$

where $\eta_{\text{Res}_\varepsilon(B)}$ is the unit of the $(-)^\circ \dashv (-)^\circ$ adjunction, which is a quasi-isomorphism since $B$ has degree-wise finite dimensional homology, and where $\Gamma$ is the lax monoidal structure of the linear dual functor $(-)^\circ$ with respect to the tensor product, which is a quasi-isomorphism since both objects have degree-wise finite dimensional homology. This quasi-isomorphism gives a direct natural weak-equivalence of simplicial sets

$$\Hom_{u\mathbb{C}_{\infty}-\text{alg}}(\Omega_i B_i \text{Res}_\varepsilon A, \text{Res}_\varepsilon(B) \otimes C^*_\varepsilon(\Delta^\bullet)) \cong \Hom_{u\mathbb{C}_{\infty}-\text{alg}}(\Omega_i B_i \text{Res}_\varepsilon A, ((\text{Res}_\varepsilon B)^\circ \otimes C^*_\varepsilon(\Delta^\bullet))^*)$$

There is an isomorphism of simplicial sets

$$\Hom_{u\mathbb{C}_{\infty}-\text{alg}}(\Omega_i B_i \text{Res}_\varepsilon A, ((\text{Res}_\varepsilon B)^\circ \otimes C^*_\varepsilon(\Delta^\bullet))^*) \cong \Hom_{u\mathbb{C}_{\infty}-\text{coalg}}((\text{Res}_\varepsilon B)^\circ \otimes C^*_\varepsilon(\Delta^\bullet), (\Omega_i B_i \text{Res}_\varepsilon A)^\circ)$$

induced by the adjunction $(-)^\circ \dashv (-)^\circ$. We compute that

$$(\Omega_i B_i \text{Res}_\varepsilon A)^\circ \cong \hat{B}_i(B_i \text{Res}_\varepsilon A)^* \cong \hat{B}_i(B_{\pi} A)^*.$$

Finally we get

$$\Hom_{u\mathbb{C}_{\infty}-\text{coalg}}((\text{Res}_\varepsilon B)^\circ \otimes C^*_\varepsilon(\Delta^\bullet), \hat{B}_i(B_{\pi} A)^*) \cong \Hom_{u\mathbb{C}_{\infty}-\text{coalg}}(C^*_\varepsilon(\Delta^\bullet), \hat{B}_i(\text{hom}((\text{Res}_\varepsilon B)^\circ, (B_{\pi} A)^*)))$$

$$\cong \Hom_{\text{curv abs } \mathbb{C}_{\infty}-\text{alg}}(\hat{\Omega}_i(C^*_\varepsilon(\Delta^\bullet)), \text{hom}((\text{Res}_\varepsilon B)^\circ, (B_{\pi} A)^*))$$

$$\cong \mathcal{R}(\text{hom}((\text{Res}_\varepsilon B)^\circ, g_A)),$$

which concludes the proof. \qed

**Example 4.21.** Let $A$ be a derived affine scheme, that is, a dg $u$Com-algebra concentrated in non-negative degrees. There is an isomorphism of constant simplicial sets

$$\text{Spec}(A)_{/(K)} \cong \mathcal{M}(g_A) \cong \mathcal{R}(g_A),$$

since $g_A$ is concentrated in non-positive degrees. For $A$ a derived affine stack, we have that

$$\text{Spec}(A)_{/(K)} \cong \mathcal{R}(g_A).$$

Geometrically speaking, these isomorphisms allow us to recover all the $K$-points of $A$ as well as the formal neighborhood of any of these points from its geometrical model $g_A$.

Furthermore, given a finite field extension $L$ of $K$, we have

$$\text{Spec}(A)_{/(L)} \cong \mathcal{M}(\text{hom}(L^\circ, g_A)) \cong \mathcal{M}(L \otimes g_A),$$

so we can base-change $g_A$ to any finite extension $L$, and recover the $L$-points of $A$ as well as their formal neighborhoods. Finally, any these combinations can be done for a finite number $n$ of points in $A$.
On the other hand, given a curved absolute $\mathcal{L}_\infty$-algebra $g$, one can construct a functor from dg Artinian algebras to simplicial sets.

**Definition 4.22** (Deformation functor). Let $g$ be a curved absolute $\mathcal{L}_\infty$-algebra. Its deformation functor

$$\text{Def}_g : \text{dg Art-alg}_{\geq 0} \longrightarrow \text{sSet}$$

is given by

$$\text{Def}_g(B) := \mathcal{R}(\text{hom}(\text{Res}_\epsilon(\cdot)^\circ, g)).$$

**Remark 4.23.** This construction is analogous to the following classical construction: given a dg Lie algebra $g$ and a dg Artinian algebra $A$, one can consider the simplicial set given by

$$\text{MC}(g \otimes \overline{A})_\bullet,$$

where $\overline{A}$ is the augmentation ideal of $A$ and $\text{MC}(\cdot)_\bullet$ is the integration functor constructed in [Hin01]. This construction has been the classical way to associate to a dg Lie/$\mathcal{L}_\infty$-algebra a "deformation problem".

**Proposition 4.24.** Let $f : h \rightarrow g$ be a weak-equivalence of complete curved absolute $\mathcal{L}_\infty$-algebras. It induces a natural weak-equivalence of simplicial sets $\text{Def}_f : \text{Def}_h \rightarrow \text{Def}_g$ between the associated deformation functors.

**Proof.** Let $f : h \rightarrow g$ be a weak-equivalence of complete curved absolute $\mathcal{L}_\infty$-algebras. It induces a natural weak-equivalence of curved absolute $\mathcal{L}_\infty$-algebras $f_* : \text{hom}(\cdot, h) \rightarrow \text{hom}(\cdot, g)$. Indeed, the map $f$ is a weak-equivalence if and only if

$$\hat{B}_i(f) : \hat{B}_i h \rightarrow \hat{B}_i g$$

is a quasi-isomorphism of $u\mathcal{C}_\infty$-coalgebras. The following composition is a quasi-isomorphism

$$\hat{B}_i(\text{hom}(\cdot, h)) \cong \{\cdot, \hat{B}_i h\} \rightarrow \{\cdot, \hat{B}_i g\} \cong \hat{B}_i(\text{hom}(\cdot, g)),$$

since $\{\cdot, -\}$ is a right Quillen functor by Proposition 1.36 and thus preserves weak-equivalences between fibrant objects, guaranteeing that $\{\cdot, \hat{B}_i(f)\}$ is a quasi-isomorphism. One then concludes using the fact that $\mathcal{R}$ sends weak-equivalences to weak homotopy equivalences of simplicial sets by Theorem 2.12.

**Remark 4.25.** The above proposition does not hold in general for the classical case of dg Lie/$\mathcal{L}_\infty$-algebras and their associated deformation functor when one considers weak-equivalences as quasi-isomorphisms. Indeed, the functor

$$\text{Def}_{(-)} : g \mapsto \left[\text{Def}_g := \text{MC}(g \otimes (-))_\bullet : \text{dg Art-alg}_{\geq 0}^{\text{coaug}} \longrightarrow \text{sSet}\right]$$

does not always send quasi-isomorphisms to natural weak-equivalence of simplicial sets. Some hypothesis are need for this, such as nilpotency. It just so happens that nilpotent dg Lie algebras are examples of dg absolute Lie algebras.

Let us explore other properties of the deformation functor.

**Lemma 4.26.** Let $g$ be a curved absolute $\mathcal{L}_\infty$-algebra. The functor

$$\mathcal{R}(\text{hom}(\cdot, g)) : (u\mathcal{C}_\infty\text{-coalg})^{\text{op}} \longrightarrow \text{sSet}$$

sends any homotopy colimit of $u\mathcal{C}_\infty$-coalgebras to a homotopy limit of simplicial sets. Furthermore, it sends quasi-isomorphisms to weak-equivalences.

**Proof.** Let $\text{hocolim} \, C_\alpha$ be a homotopy colimit of $u\mathcal{C}_\infty$-coalgebras, we consider the following equivalences

$$\mathcal{R}(\text{hom}(\text{hocolim} \, C_\alpha, g)) \cong \text{Hom}_{u\mathcal{C}_\infty\text{-coalg}}(C_\alpha^\bullet(\Delta^\bullet), (\text{hocolim} \, C_\alpha, \hat{B}_i g))$$

$$\cong \text{holim} \, \text{Hom}_{u\mathcal{C}_\infty\text{-coalg}}(C_\alpha^\bullet(\Delta^\bullet), (C_\alpha, \hat{B}_i g))$$
$\simeq \operatorname{holim} \mathcal{R}(\hom(C_\alpha, g))$. \hfill \Box

**Proposition 4.27.** Let $g$ be a curved absolute $L_\infty$-algebra.

1. The deformation functor $\operatorname{Def}_g$ sends quasi-isomorphisms to weak-equivalences of simplicial sets.
2. The deformation functor $\operatorname{Def}_g$ preserves homotopy limits.

**Proof.** This follows directly from Lemma 4.19 and Lemma 4.26. \hfill \Box

**Remark 4.28.** Developing the theory of non-pointed formal moduli problems will be the subject of a future work. For a brief heuristic exposition, see [RiL22c, Chapter 3, Section 4, Conclusion].

**Appendix**

In order to work with simpler algebraic structures, we recall the semi-augmented Bar construction introduced in [HM12, Section 3.3], which provides us with smaller but non-functorial cofibrant resolutions for non-augmented dg operads. We introduce its dual construction for dg counital partial cooperads in the sense of [RiL22b].

**Definition 4.29 (Semi-augmented dg operad).** A semi-augmented dg operad $(P, \gamma, \eta, d_P, \epsilon)$ is the data of a dg operad $(P, \gamma, \eta, d_P)$ together with a degree 0 morphism

$$\epsilon : P \rightarrow J,$$

such that $\epsilon \circ \eta = \text{id}_J$.

Let $\overline{P}$ be the graded $S$-module given by $\overline{P} := \text{Ker}(\epsilon)$.

**Definition 4.30 (Semi-augmented Bar construction).** Let $(P, \gamma, \eta, d_P, \epsilon)$ be a semi-augmented dg operad. Its semi-augmented Bar construction, denoted by $B_{sa}P$, is given by

$$B_{sa}P := (\mathcal{T}_c(s\overline{P}), d_{\text{bar}} := d_1 + d_2, \Theta_{\text{bar}}).$$

Here $\mathcal{T}_c(s\overline{P})$ denotes the cofree conilpotent coaugmented cooperad generated by the suspension of the graded $S$-module $\overline{P}$. The pre-differential $d_{\text{bar}}$ is given by the sum of two terms. The first term $d_1$ is given by the unique coderivation extending the map

$$\mathcal{T}_c(s\overline{P}) \xrightarrow{s d_{\overline{P}}} s\overline{P}.$$

The second term $d_2$ is given by the unique coderivation extending the map

$$\mathcal{T}_c(s\overline{P}) \xrightarrow{s^2(\overline{P}_{(1)} \circ_1 \overline{P})} s_{\overline{P}},$$

where $\overline{P}_{(1)}$ is given by the composition

$$\overline{P}_{(1)} \xrightarrow{\gamma_{(1)}} sP \xrightarrow{s^1 \gamma_{(1)}} s\overline{P}.$$

The curvature $\Theta_{\text{bar}}$ is given by

$$\Theta_{\text{bar}} : \mathcal{T}_c(\overline{P}) \xrightarrow{s^2(\overline{P}_{(1)} \circ_1 \overline{P})} s_{\overline{P}} \xrightarrow{s^{-2} \gamma_{(1)}} sP \xrightarrow{\epsilon} J.$$

The semi-augmented Bar construction $B_{sa}P$ forms a conilpotent curved cooperad.

It provides smaller but non-functorial resolutions. Indeed, it is only functorial with respect to morphisms of semi-augmented dg operads.

**Theorem 4.31 ([HM12, Theorem 3.4.4]).** Let $(P, \gamma, \eta, d_P, \epsilon)$ be a semi-augmented dg operad. There is a quasi-isomorphism of dg operads

$$\varepsilon_P : \Omega B_{sa}P \rightarrow P.$$
Furthermore, it coincides in certain cases with the Boardmann-Vogt construction.

**Theorem 4.32.** Let \( \mathcal{P} \) an operad in the category of cellular topological spaces. There is an isomorphism of dg operads

\[
\Omega \mathcal{B}^{s.a} C^*_c(\mathcal{P}, K) \cong C^*_c(\mathcal{W}\mathcal{P}, K),
\]

where \( C^*_c(\_, K) \) denotes the cellular chain functor and where \( \mathcal{W}(\_) \) denotes the Boardmann-Vogt construction.

**Proof.** This result is shown for reduced dg operads in [BM06]. See [Gri22b] for the extension to semi-augmented dg operads using this resolution. \( \square \)

**Example 4.33.** Let \( \text{uCom} \) be the operad in the category of sets defined by

\[
\text{uCom}(n) := \{ \ast \},
\]

for all \( n \geq 0 \), together with the obvious action of \( S_n \) and the obvious operad structure. Then

\[
\Omega \mathcal{B}^{s.a} \text{uCom} \cong \Omega \mathcal{B}^{s.a} C^*_c(\text{uCom}, K) \cong C^*_c(\text{WuCom}, K),
\]

where \( \text{uCom} \) is viewed as an operad in the category of cellular topological spaces by endowing it with the discrete topology.

We introduce the dual version for semi-coaugmented dg counital partial cooperads. For the definition of complete curved absolute (partial) operads, we refer to [RiL22b, Appendix].

**Definition 4.34 (Semi-coaugmented dg counital partial cooperad).** A semi-coaugmented dg counital partial cooperad \( (\mathcal{C}, \{ \Delta_i \}, \epsilon, d_C, \eta) \) amounts to the data of a dg counital partial cooperad \( (\mathcal{C}, \{ \Delta_i \}, \epsilon, d_C) \) together with a degree 0 morphism

\[
\eta : J \to \mathcal{C},
\]

such that \( \epsilon \circ \eta = \text{id}_J \).

Let \( \overline{\mathcal{C}} \) be the graded \( S \)-module given by \( \overline{\mathcal{C}} := \text{Ker}(\epsilon) \).

**Definition 4.35 (Semi-coaugmented complete Cobar construction).** Let \( (\mathcal{C}, \{ \Delta_i \}, \epsilon, d_C, \eta) \) be a semi-coaugmented dg counital partial cooperad. Its semi-coaugmented complete Cobar construction, denoted by \( \hat{\Omega}^{s.c} \mathcal{C} \), is given by

\[
\hat{\Omega}^{s.c} \mathcal{C} := \left( \mathcal{T}^{\wedge}(s^{-1}\overline{\mathcal{C}}), d_{\text{cobar}} := d_1 - d_2, \Theta_{\text{cobar}} \right).
\]

Here \( \mathcal{T}^{\wedge}(s^{-1}\overline{\mathcal{C}}) \) denotes the completed tree monad applied to the desuspension of the graded \( S \)-module \( \overline{\mathcal{C}} \). The pre-differential \( d_{\text{cobar}} \) is given by the difference of two terms. The first term \( d_1 \) is given by the unique derivation extending the map

\[
s^{-1}\overline{\mathcal{C}} \xrightarrow{s \text{d}_\overline{\mathcal{C}}} s^{-1}\overline{\mathcal{C}} \xrightarrow{} \mathcal{T}^{\wedge}(s^{-1}\overline{\mathcal{C}}).
\]

The second term \( d_2 \) is given by the unique derivation extending the map

\[
s^{-1}\overline{\mathcal{C}} \xrightarrow{s\overline{\Delta}_{(1)}} s^{-2}(\overline{\mathcal{C}} \circ_{(1)} \overline{\mathcal{C}}) \xrightarrow{} \mathcal{T}^{\wedge}(s^{-1}\overline{\mathcal{C}})
\]

where \( \overline{\Delta}_{(1)} \) is given by the composition

\[
\overline{\mathcal{C}} \xrightarrow{\Delta_{(1)}} \mathcal{C} \circ_{(1)} \mathcal{C} \xrightarrow{} \overline{\mathcal{C}} \circ_{(1)} \overline{\mathcal{C}}.
\]

The curvature \( \Theta_{\text{cobar}} \) is given by

\[
\Theta_{\text{cobar}} : J \to \mathcal{C} \xrightarrow{s^{-2}\overline{\Delta}_{(1)}} s^{-2}(\overline{\mathcal{C}} \circ_{(1)} \overline{\mathcal{C}}) \xrightarrow{} \mathcal{T}^{\wedge}(s^{-1}\overline{\mathcal{C}}).
\]
The semi-coaugmented complete Cobar construction $\hat{\Omega}^{s,c}\mathcal{O}$ forms a complete curved augmented absolute operad.

**Proposition 4.36.** Let $(\mathcal{P}, \gamma, \eta, d_\mathcal{P}, e)$ be a semi-augmented dg operad which is arity and degree-wise finite dimensional. There is an isomorphism of complete curved augmented absolute operads
\[
(\mathcal{B}^{s,a}_*\mathcal{P})^* \cong \hat{\Omega}^{s,c}_*\mathcal{P}^*.
\]

**Proof.** We first view $\mathcal{P}$ as a semi-augmented dg unital partial operad. Since it is arity and degree-wise finite dimensional, its linear dual is a semi-coaugmented dg counital partial operad. The result follows from a straightforward computation. □

Let us first make explicit what a curved algebra over $\hat{\Omega}^{s,c}\mathcal{O}$ is. Let us recall the definition of a **classical curved** $\mathcal{L}_\infty$-algebra.

**Definition 4.37 (Curved $\mathcal{L}_\infty$-algebra).** Let $g$ be a graded module. A **curved $\mathcal{L}_\infty$-algebra** structure $\{l_n\}_{n \geq 0}$ on $g$ is the data of a family of symmetric morphisms $l_n : g^{\otimes n} \to g$ of degree $-1$ such that, for all $n \geq 0$, the following equation holds:
\[
\sum_{p+q=n+1} \sum_{\sigma \in \text{Sh}^{-1}(p,q)} (l_p \circ_1 l_q)^\sigma = 0,
\]
where $\text{Sh}^{-1}(p,q)$ denotes the inverse of the $(p,q)$-shuffles. The morphism $l_0 : K \to g$ is equivalent to an element $\partial$ in $g_{-1}$ which is called the **curvature** of $g$.

Recall that any complete curved augmented absolute operad structure is in particular a curved operad in the sense of [RiL22b].

**Proposition 4.38.** Let $(V, d_V)$ be a pdg module. The data of a curved $\hat{\Omega}^{s,c}\mathcal{O}\mathcal{C}\mathcal{O}_m$-algebra structure on $V$ is equivalent to a family of symmetric operations
\[
\{l_n : V^{\otimes n} \to V\}
\]
of degree $-1$ for $n \neq 1$. When setting $l_1 := d_V$, the family $\{l_n\}_{n \geq 0}$ forms a curved $\mathcal{L}_\infty$-algebra structure in the classical sense.

**Proof.** This follows from a straightforward computation. □

**Remark 4.39 (Mixed curved $\mathcal{L}_\infty$-algebras).** One may wonder what type of algebraic structure would have appeared had we chosen to use the Bar-Cobar constructions that appear in [RiL22b] instead of their "semi-(co)augmented" counterparts. The type of algebraic structure that appears is not substantially different. A **mixed curved** $\mathcal{L}_\infty$-algebra $(g, \{l_n\}_{n \geq 0}, d_g)$ amounts to the data of a pdg module $(g, d_g)$ together with a family $\{l_n\}_{n \geq 0}$ of symmetric morphisms $l_n : g^{\otimes n} \to g$ of degree $-1$ such that
\[
\sum_{p+q=n+1} \sum_{\sigma \in \text{Sh}^{-1}(p,q)} (l_p \circ_1 l_q)^\sigma = -\partial(l_n),
\]
for all $n \geq 0$. In particular, for $n = 1$, the relation satisfied is
\[
d_g^2 - \partial(l_1) = l_1 \circ_1 l_1 + l_2 \circ l_0.
\]
It is immediate that $d_g - l_1$ together with the other structure maps does define a curved $\mathcal{L}_\infty$-algebra structure. Furthermore, this gives a forgetful functor
\[
\xymatrix{\text{mix curv-s}$\mathcal{L}_\infty$-alg} \ar[r]^-\text{blend} & \text{curv-$\mathcal{L}_\infty$-alg}.}
\]
This type of structures can be found in [CCN21]. Considering curved algebras over the curved cooperad $\mathcal{O}\mathcal{C}\mathcal{O}_m$ would result in "mixed curved absolute" $\mathcal{L}_\infty$-algebras.
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