Fractional vortex lattice structures in spin-triplet superconductors

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Abstract. Motivated by recent interest in spin-triplet superconductors, we investigate the vortex lattice structures for this class of unconventional superconductors. We discuss how the order parameter symmetry can give rise to $U(1) \times U(1)$ symmetry in the same sense as in spinor condensates, making half-quantum vortices (HQVs) topologically stable. We then calculate the vortex lattice structure of HQVs, with particular attention on the roles of the crystalline lattice, the Zeeman coupling and Meissner screening, all absent in spinor condensates. Finally, we consider how spin–orbit coupling leads to a breakdown of the $U(1) \times U(1)$ symmetry in free energy and whether the HQV lattice survives this symmetry breaking. As examples, we examine simpler spin-triplet models proposed in the context of $\text{Na}_x\text{CoO}_2 \cdot y\text{H}_2\text{O}$ and Bechgaard salts, as well as the better known and more complex model for $\text{Sr}_2\text{RuO}_4$.

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1. Introduction

A half-quantum vortex (HQVs) with vorticity $h/4e$, which is half that of the usual Abrikosov vortex with vorticity $\Phi_0 \equiv h/2e$, presents an exciting example of fractionalized topological defects. Quantization of collective topological defects provides clear-cut access to the nature of the ground state. For instance, the vorticity $h/2e$ of the Abrikosov vortex in a type II superconductor clearly shows that the circulation associated with the vortex is that of charge $2e$ Cooper pairs. Since the vorticity in a condensate is determined by the requirement of single-valuedness of the order parameter describing the condensate, ‘fractionalization of vorticity’ is possible with a multi-component order parameter when different components are allowed to wind separately. For instance, in a triplet superconductor, the additional Cooper pair spin degree of freedom can be free to rotate in-plane $[1]–[3]$, giving rise to additional $U(1)$ symmetry and an associated spin winding number; the cases where vortex fractionalization is due to the $U(1) \times U(1)$ symmetry of different physical origin has also been studied $[4]$. Hence, observation of fractionalization of vortices can serve as an indicator of the structure of the order parameter in a given condensate. In addition, the recent proposals predicting non-Abelian fractional statistics for the composite of a HQV and the Majorana fermions bound at its core in the chiral triplet superconductors, brought about a rise in attention to and interest in the possibility of using HQVs in triplet superconductors $[5]–[8]$. This type of non-Abelian statistics was first studied for quasiholes in the spin-polarized $\nu = 5/2$ quantum Hall state.
Therefore, if we want to obtain the same statistics for vortices in a spinful superconductor, the vortices should be HQVs so that there would be phase winding only for a single component.

However, although there are a number of candidate triplet superconductors such as the single layer ruthenate Sr$_2$RuO$_4$ [11, 12], the cobaltate Na$_x$CoO$_2$·yH$_2$O [13] and organic superconductors [14], HQVs have never been observed in bulk systems, in line with the energetic stability issues raised by two of us in [3]. It was pointed out in [3] that due to the absence of screening for the spin supercurrent circulation required for HQV in triplet superconductors, HQVs can be energetically unstable in bulk samples toward combining into full Abrikosov vortices despite their advantage in magnetic energy. Related considerations have appeared, in the context of spin-triplet superconductivity in UPt$_3$, in a paper by Zhitomirsky [15].

The main aim of this work is to investigate the possibility of using high enough fields to generate a HQV lattice in triplet superconductors where the vortex lattice serves two purposes at once: (i) stabilizing HQVs at finite separation and (ii) providing an unambiguous signature of its formation (halving of the vortex lattice unit cell). Experiments have already determined the vortex lattice structure successfully at low fields in Sr$_2$RuO$_4$ [16, 17] and the observed square lattice geometry was consistent with the theoretical prediction by one of the present authors based on a chiral triplet order parameter in [18]. However, Agterberg [18] considered the limit of strong spin–orbit coupling, which leads to vortex lattices of full quantum Abrikosov vortices. Recently, measurements of the Knight shift for the field along the c-axis [19] as well as ARPES data [20, 21] have indicated that the spin–orbit coupling is perhaps not so strong. Therefore, in this paper, we extend the studies of [18] to allow for weak spin–orbit coupling, leading to the possibility of HQV lattices for fields along the c-axis. Additionally, the organic superconductor (TMTSF)$_2$ClO$_4$ naturally has weak spin–orbit coupling and Knight shift measurements provide evidence for a spin-triplet state at high magnetic fields [14], which is precisely the situation we consider here. We also provide an analysis of this case and the closely related case for cobaltate spin-triplet superconductors [13].

In this paper, we study the energetics of different vortex lattice configurations. The key additional physical ingredient is the $U(1)$ spin-rotational invariance of the Cooper pairs that arises in a magnetic field. This generically leads to two different species of fractional vortices whose fractional fluxes sum to $\Phi_0$. When stable, these fractional vortices form interlacing lattices analogous to vortex–antivortex lattice configurations proposed by [22, 23] in the context of a two-dimensional (2D) superfluid and the configuration in two-component Bose condensates proposed by [24].

The rest of the paper is organized as follows. In section 2, we give a pedagogical introduction to the symmetry properties of the triplet order parameter. In particular, we will show how $U(1) \times U(1)$ symmetry can arise in the order parameter of such systems. In section 3, we discuss the form of Gibbs free energy that is allowed by various symmetries in the case when spin–orbit coupling is not included. In section 4, we provide the general theoretical framework for the vortex lattice phases. In section 5, we show that the lowest Landau level solution often provides an adequate description, and we discuss this solution for the lattice of HQVs. In section 6, we consider the effect of $U(1) \times U(1)$ symmetry breaking driven by spin–orbit coupling. In section 7, we lay out predictions on how to detect the proposed HQV lattice structures and we conclude with a summary and outlook in section 8.
2. The triplet order parameter

The order parameter of a triplet superconductor takes a matrix form in the spin space [12, 25]:

$$\hat{\Delta}(\mathbf{k}) = \begin{bmatrix} \Delta_{1\uparrow\downarrow}(\mathbf{k}) & \Delta_{1\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{1\uparrow\downarrow}(\mathbf{k}) & \Delta_{1\uparrow\downarrow}(\mathbf{k}) \end{bmatrix} = \begin{bmatrix} -d_x + id_y & d_z \\ d_z & d_x + id_y \end{bmatrix},$$  \hspace{1cm} (2.1)

where the spin quantization axis is along the z-direction. The triplet pairing requires that $\Delta_{1\uparrow\downarrow} = \Delta_{1\downarrow\uparrow}$, and a set of three complex functions of $\mathbf{k}$, namely $(d_x(\mathbf{k}), d_y(\mathbf{k}), d_z(\mathbf{k}))$, was introduced to parameterize the gap matrix. When the three functions are collectively represented using a vector notation, the ‘unit vector’ $\hat{\mathbf{d}}(\mathbf{k})$ represents the symmetry direction (zero projection direction) with respect to the rotation of the Cooper pair spin. In the presence of the sufficiently high field along the c-axis, Zeeman splitting between electrons with opposite spins prohibits pairing, leading to $\Delta_{1\uparrow\downarrow} = \Delta_{1\downarrow\uparrow} = 0$. In the $\mathbf{d}$-vector notation, this implies that the $\mathbf{d}$-vector lies in-plane (perpendicular to the applied field). In the rest of the paper, we assume that the field is sufficiently large so that this is the case. In the context of strontium ruthenate, our results apply for the field along the $c$-axis (this is also true for the cobaltates when we are discussing spin–orbit coupling in hexagonal systems). For organic and cobaltate superconductors, our results apply for the field along any two-fold or higher symmetry axis of symmetry.

For non-chiral triplet order parameter symmetry, which is expected for the cobaltate Na$_x$CoO$_2$·yH$_2$O [13] and organic superconductors [14], the spin pairing gap matrix takes the form

$$\hat{\Delta}(\mathbf{k}) = f(\mathbf{k}) \begin{bmatrix} \Delta_{1\uparrow\downarrow} & 0 \\ 0 & \Delta_{1\downarrow\uparrow} \end{bmatrix},$$  \hspace{1cm} (2.2)

where the function $f(\mathbf{k})$ depends on the specific odd angular momentum channel. The key simplifying feature is that the orbital dependence is described by a 1D representation encoded by $f(\mathbf{k})$. It has been suggested that the cobaltate Na$_x$CoO$_2$·yH$_2$O has a spin-triplet pairing through an f-wave channel [13], although data from the Knight shift experiments remain controversial [26, 27]. In this case a common choice is $f(\mathbf{k}) = k_\perp(k_x^2 - 3k_y^2)$. However, the precise form of $f(\mathbf{k})$ is not needed for our results. For the organic superconductor (TMTSF)$_2$ClO$_4$, there is a strong case that the system becomes a triplet superconductor under sufficient field $H \gtrsim 20$ kOe. In this case, there are many proposals for $f(\mathbf{k})$. However, again, the specific form is not needed for our results.

For the chiral order parameter symmetry expected for Sr$_2$RuO$_4$, with $\mathbf{d}$ in the basal plane and no spin–orbit coupling, the order parameter has four complex degrees of freedom:

$$\hat{\Delta}(\mathbf{k}) = \sum_{\sigma = \pm} \bar{f}(k_\sigma, k_\perp) \begin{bmatrix} \Delta_{1\uparrow\downarrow,\sigma} & 0 \\ 0 & \Delta_{1\downarrow\uparrow,\sigma}\end{bmatrix},$$  \hspace{1cm} (2.3)

where the function $\bar{f}$, like the non-chiral case discussed above, depends on the specific odd angular momentum channel and $k_\sigma = k_x + i\sigma k_y$. This is equivalent to

$$\mathbf{d}(\mathbf{k}) = \Delta_+ \hat{\mathbf{d}}_+ \exp(in\varphi_\mathbf{k}) + \Delta_- \hat{\mathbf{d}}_- \exp(-in\varphi_\mathbf{k}),$$  \hspace{1cm} (2.4)

where $n$ is an integer ($n = 1$ for p-wave and $n = 3$ for f-wave) and $\varphi_\mathbf{k}$ is the azimuthal angle associated with a unit vector $\hat{\mathbf{k}}$ in the 2D plane (assuming a quasi-2D setting with the angular
momentum along the c-axis: \( \hat{c} = \hat{\mathbf{z}} \). Although \( \Delta_+ = 0 \) for a homogeneous chiral superconductor, we will show that \( \Delta_- \neq 0 \) often plays an important role in describing the vortex lattice structure of a chiral superconductor.

Equations (2.2) and (2.4) clearly show that under these circumstances, the order parameter symmetry takes the \( U(1) \times U(1) \) form, which can allow for HQVs with \( h/4e \) vorticity associated with \( \pi \) orbital phase winding and \( \pi \) d-vector winding.

3. The Gibbs free energy

In order to identify stable vortex types and the lattice structure itself, we start with the Gibbs free energy including all the terms allowed by symmetry up to quartic order. As usual, the quartic terms determine the vortex lattice structure. Due to additional spin degrees of freedom, the full expression for the Gibbs free energy involves a number of additional terms compared to the singlet superconductor case, and it is instructive to consider different contributions separately:

\[
f = f_{\text{mag}} + f_0^{(2)} + f_Z^{(2)} + f_{\text{SO}}^{(2)} + f_{\text{hom}} + f_{\text{inh}}^{(4)},
\]

where \( f_{\text{mag}} = h^2/8\pi - hH/4\pi \) is the magnetic energy (the field \( h \) is the sum of the external field \( H \) and the screening field), the superscript ‘(2)’ indicates terms quadratic in the order parameter and ‘(4)’ those quartic in the order parameter.

Other than the conventional quadratic term \( f_0^{(2)} \):

\[
f_0^{(2)} = -\alpha \sum_i |\Delta_i|^2,
\]

the remaining quadratic terms in equation (3.1) are the consequences of additional spin degrees of freedom for the triplet superconductors. The Zeeman coupling term

\[
f_Z^{(2)} = -\tilde{\kappa} h (|\Delta_{\uparrow\uparrow}|^2 - |\Delta_{\downarrow\downarrow}|^2)
\]

plays an important role for the HQV lattice by introducing a slight spin-polarization. This slight spin-polarization gives rise to two phase transitions as in the case of the \( A_1/A_2 \) phase of \( ^3\text{He} \) [28, 29]. The inhomogeneous part of the quadratic terms \( f_{\text{inh}}^{(2)} \) are of the form

\[
K_{ij:kl}(D_i \Delta_k)(D_j \Delta_l)^* + c.c.,
\]

where \( D_i = \nabla_i + (2\pi i/\Phi_0)A_i \). For these terms, we require rotational invariance up to the lattice symmetry with respect to orbital degrees of freedom only, which means that we require invariance with respect to rotating \( D_i \)s and the orbital component of \( \Delta_i \)s. The complex structure of \( f_{\text{inh}}^{(2)} \) can result in a condensate wavefunction of a different form than that of a conventional superconductor. \( f_{\text{SO}}^{(2)} \) is the quadratic spin–orbit coupling term assuming the spin–orbit coupling to be small and is discussed in section 6. In the presence of spin–orbit coupling, the free energy has to be invariant under the combined discrete rotation of the orbital and spin degrees of freedom specific for the given lattice symmetry. For lattices with orthogonal or tetragonal symmetry, spin–orbit coupling may reduce the symmetry of the Gibbs free energy and tends to suppress HQV formation by introducing a length scale beyond which the HQVs cannot exist (this length scale diverges as the spin–orbit coupling vanishes). This implies that the vortex
lattice spacing must be less than this length scale for the HQV lattice to appear. However, we show that for spin-triplet hexagonal materials (specifically the 2D $\Gamma_6^-$ and $\Gamma_5^-$ representations in the notation of Sigrist and Ueda [25]), even large spin–orbit coupling still allows for the existence of a fractional vortex lattice. This consideration may apply to Na$_x$CoO$_2$·yH$_2$O.

Among the quartic terms, $f^{(4)}_{\text{hom}}$ represents the usual set of homogeneous terms. As is shown in appendix A, certain quartic terms vanish in the weak-coupling theory. The terms that vanish are those that lift the energy degeneracy between the full quantum vortex (QV) and the HQV lattice. For this reason, we also include the inhomogeneous quartic term $f^{(4)}_{\text{in}}$. This term accounts for the difference between the spin phase stiffness $\rho_{\text{sp}}$ and the overall phase stiffness $\rho_{\text{s}}$. Not only does this difference play an important role in the stability of isolated HQVs as was shown in [3], it also plays the role of tuning parameter for the vortex lattice structure.

With multiple systems in mind, we consider contributions to the Gibbs free energy specifically for non-chiral and chiral superconductors, respectively.

### 3.1. Non-chiral triplet superconductor

As mentioned earlier, here we assume the orbital dependence of the gap function to be the same for all spin-triplet components. Formally, this means that the orbital degree of freedom belongs to a 1D irreducible representation of the point group. We apply our analysis to materials that have orthorhombic, tetragonal or hexagonal point groups. One relevant example is a non-chiral triplet f-wave superconductor with hexagonal symmetry, which has been proposed in the context of the cobaltates Na$_x$CoO$_2$·yH$_2$O; in this case $f(k) = k_x(k_x^2 - 3k_y^2)$ from equation (2.2). When the $d$-vector lies in the $xy$-plane, the inclusion of spin–orbit coupling implies that formally this order parameter belongs to the $\Gamma_6^-$ representation of the hexagonal point group (the consequences of spin–orbit coupling for this representation are discussed in more detail in section 6).

With the in-plane spin rotational invariance, the relevant free energy within the assumptions stated above is given by

$$
\begin{align}
    f^{(2)}_{\text{in}} &= \sum_{i=x,y,z} K_i (|D_i \Delta_{\uparrow\uparrow}|^2 + |D_i \Delta_{\downarrow\downarrow}|^2), \\
    f^{(4)}_{\text{hom}} &= \beta_1 \left( \sum_i |\Delta_i|^2 \right)^2 + \beta_2 |\Delta_{\uparrow\uparrow}|^2 |\Delta_{\downarrow\downarrow}|^2, \\
    f^{(4)}_{\text{in}} &= \gamma [\Delta_{\uparrow\uparrow}^* \Delta_{\downarrow\downarrow} (D_{\perp\perp} \Delta_{\uparrow\uparrow}) \cdot (D_{\perp\perp} \Delta_{\downarrow\downarrow})^* + \text{c.c.}].
\end{align}
$$

For tetragonal and hexagonal point groups, $K_x = K_y$, while for orthorhombic point groups, $K_x \neq K_y$. For the high field limit we are considering, it is possible to rescale lengths in two directions perpendicular to the applied field such that $\tilde{K}_i = \tilde{K}_j$ for $i \neq j$ (where $\tilde{K}_i$ refers to the new coefficient in the rescaled coordinates). We will therefore ignore the difference between the $K_i$ and assume that for orthorhombic point groups we are working in rescaled coordinates. The term $f^{(4)}_{\text{in}}$ is not the most general such term allowed by symmetry. However, it is this term that allows the Ginzburg–Landau (GL) theory to give the same physics as in [3]. Indeed, one can gain more insight into the vortex lattice solutions that minimize equation (3.2) and equations (3.6) and (3.7) by relating the coefficient of the inhomogeneous quartic term $\gamma$ to
the stiffness ratio $\rho_{sp} < \rho_s$ which controls the energetic stability of a pair of HQVs [3]. Within
the London approximation, the gradient terms in equations (3.5) and (3.7) amount to phase
bending energy, which will be proportional to $(\rho_s + \rho_{sp})$ and $(\rho_s - \rho_{sp})$, respectively. Combining
equations (3.5) and (3.7) with the homogeneous solution $|\Delta_1^\uparrow|^2 = |\Delta_1^\downarrow|^2 = \alpha/(\beta_1 - \beta^2)$, we
obtain the following relation between $\gamma$ and $\rho_{sp}/\rho_s$:

$$\gamma = \frac{K_1(\beta_1 - \beta_2)}{1 - \rho_{sp}/\rho_s} \frac{1 - \rho_{sp}/\rho_s}{1 + \rho_{sp}/\rho_s}.$$  (3.8)

Hence $\gamma > 0$ would imply stability of HQVs and double transitions into two possible vortex
phases: a lattice of ordinary Abrikosov vortices and a lattice of HQVs. This transition is
determined by the $\beta_2$ term of equation (3.6) and the $\gamma$ term of equation (3.7).

### 3.2. Chiral triplet superconductor

With the ruthenate $\text{Sr}_2\text{RuO}_4$ in mind, we consider a chiral triplet p-wave superconductor with
square symmetry for which $\tilde{f}(k_s) = k_s + i\sigma k_y$ in equation (2.3) with

$$\hat{\Delta}(\mathbf{k}) = \sum_{\sigma = \pm} (k_s + i\sigma k_y) \begin{bmatrix} \Delta_{\sigma^+} & 0 \\ 0 & \Delta_{\sigma^+} \end{bmatrix},$$  (3.9)

where $\Delta_{s,\sigma}$ ($s = \uparrow\uparrow, \uparrow\downarrow$ and $\sigma = \pm$) form expansion parameters for the Gibbs free energy. In
terms of the $\mathbf{d}$-vector notation $\mathbf{d} \equiv \hat{\mathbf{x}}(\eta_{xx} k_x + \eta_{xy} k_y) + \hat{\mathbf{y}}(\eta_{yx} k_x + \eta_{yy} k_y)$,

$$\Delta_{\sigma^+} = -(\eta_{xx} - i\eta_{xy} - i\eta_{yx} - \eta_{yy})/2,$$

$$\Delta_{\sigma^+} = -(\eta_{xx} + i\eta_{xy} - i\eta_{yx} + \eta_{yy})/2,$$

$$\Delta_{\sigma^+} = (\eta_{xx} - i\eta_{xy} + i\eta_{yx} + \eta_{yy})/2,$$

$$\Delta_{\sigma^+} = (\eta_{xx} + i\eta_{xy} + i\eta_{yx} - \eta_{yy})/2.$$  (3.10)

Formally, without spin–orbit coupling, this order parameter is a direct product of an $E_u$
orbital representation of the tetragonal point group and the in-plane vector representation for
spin rotations. When spin–orbit coupling is included the order parameter contains the four
different 1D representations of the tetragonal point group. In the case without spin–orbit
coupling, the relevant free energy for this representation can be constructed using the known
free energy for the $E_u$ representation [25], we list below $f_{\text{in}}^{(2)}$, $f_{\text{hom}}^{(4)}$, and $f_{\text{in}}^{(4)}$. Before listing $f_{\text{in}}^{(2)}$,
we note that this free energy term should respect the $C_4$ symmetry on the $xy$-plane only for the
orbital degrees of freedom:

$$(D_x, D_y, \Delta_{s,+}, \Delta_{s,-}) \rightarrow (D_y, -D_x, i\Delta_{s,+}, -i\Delta_{s,-}).$$  (3.11)

However, for simplicity, we consider cylindrical symmetry in the orbital degrees of freedom;
this does not significantly alter the arguments below. This symmetry gives us $f_{\text{in}}^{(2)} = \sum_s f_{\text{in}}^{(2,s)}$
where

\[ f^{(2,s)}_{\text{in}} = K_1([D\Delta_{s,+}|^2 + |D\Delta_{s,-}|^2] + K_2[[((D_x\Delta_{s,+})(D_x\Delta_{s,-})^* - (D_y\Delta_{s,+})(D_y\Delta_{s,-})^*)/2 + \{((D_x\Delta_{s,-})(D_x\Delta_{s,+})^* - (D_y\Delta_{s,-})(D_y\Delta_{s,+})^*)/2 + i\{(D_x\Delta_{s,-})(D_y\Delta_{s,+})^* + (D_y\Delta_{s,-})(D_y\Delta_{s,-})^*\}/2 + K_4(|D_{s,+}|^2 + |D_{s,-}|^2)\].

(3.12)

In addition, the following term is also allowed by symmetry

\[ \delta K \frac{2\pi}{\Phi_0} \hbar \sum_s (-|\Delta_{s,+}|^2 + |\Delta_{s,-}|^2), \]

(3.13)

which stabilizes this in-plane chiral phase for strong enough magnetic field (note the similarity to the Zeeman term for the condensate spin degrees of freedom). As for the homogeneous quartic terms,

\[ f^{(4)}_{\text{hom}} = \sum_s [\beta_1(|D_{s,+}|^4 + |D_{s,-}|^4)/2 + \beta_1'|D_{s,+}|^2|D_{s,-}|^2 - \sum_{\sigma = \pm} (\beta_2|\Delta_{\uparrow,\sigma}|^2|\Delta_{\downarrow,\sigma}|^2
\]

\[ + \beta_2'|\Delta_{\uparrow,\sigma}|^2|\Delta_{\downarrow,\sigma}|^2\] \[ - \beta_3[(\Delta_{\uparrow,\pm}\Delta_{\downarrow,\mp}) - (\Delta_{\uparrow,\mp}\Delta_{\downarrow,\pm})^* + c.c.]\].

(3.14)

The \( \beta_2, \beta_2' \) and \( \beta_3 \) terms originate from interaction between spin up–up pairs and down–down pairs. Again, for simplicity, we have written the free energy in the limit of a cylindrical Fermi surface. Lastly, we have

\[ f^{(4)}_{\text{in}} = \gamma \sum_{\sigma = \pm} [\Delta_{\uparrow,\sigma}^* \Delta_{\downarrow,\sigma} (D\Delta_{\uparrow,\sigma}) \cdot (D\Delta_{\downarrow,\sigma}^*) + c.c.] + \gamma' \sum_{\sigma = \pm} [\Delta_{\uparrow,\sigma}^* \Delta_{\downarrow,\sigma} (D\Delta_{\uparrow,\sigma}) \cdot (D\Delta_{\downarrow,\sigma}^*) + c.c.]\]

(3.15)

Note that the form of equation (3.15) is consistent with the form of the interaction terms in equation (3.14). Again, this is not the most general term allowed by symmetry, but it is the minimal term that captures the physics in the London limit described in [3].

4. Determining the vortex lattice structure—general considerations

We consider the vortex lattice phases near the upper critical field to map out the stability condition for HQV lattice phases. As usual, the first step toward determining the vortex lattice structure is to identify the eigenstates of the linearized GL equations. In order to obtain a linearized GL equation, we take a variation of the quadratic terms in the free energy, for example:

\[ f_0^{(2)} + f_{\text{in}}^{(2)} + f_Z^{(2)} = -\alpha \sum_j |\Delta_j|^2 - \tilde{\kappa} h(|\Delta_{\uparrow}|^2 - |\Delta_{\downarrow}|^2) + [K_{j,k,l,m}(D_j\Delta_l)(D_k\Delta_m)^* + c.c.]

(4.1)
with respect to one component of the order parameter $\Delta_s^*$. This gives an equation of the form

$$\alpha \Delta_s = K_{h,s} D_{l} D_{s} \Delta_s^* - \kappa H \Delta_s$$  \hspace{1cm} (4.2)

(note that we are ignoring the difference between $h$ and $H$ in this approximation). Since the gradient terms in equation (4.2) cannot, in general, be reduced into a $D_s^2 + D_s^2$ form, the lowest Landau level wave functions are not sufficient for calculating the condensate wave function in general. However, the solution of this equation can still be expressed in terms of Landau level wave functions:

$$\phi_n(r) = [2^n \pi^{1/2} (n!)]^{-1/2} \sum_{m} q_m e^{i m x} e^{-(y - k_m)^2/2} H_n(y - k_m),$$ \hspace{1cm} (4.3)

where $H_n$ is the Hermite polynomial of $n$th order, $x'$ and $y'$ are $x$, $y$ coordinates in units of the magnetic length $l = (\Phi_0/2\pi H)^{1/2}$. This is because $D_s \phi_n$ can be expressed as a linear combination of the raising and lowering operators of the Landau levels $\Pi_{\pm}$, since $\Pi_{\pm} = i (D_s \pm i D_s)/\sqrt{2}$.

This wavefunction describes a vortex lattice when $|\phi_n(r)|$ is periodic in the lattice vectors $a_1 = a l(1, 0)$ and $a_2 = b l (\cos \theta, \sin \theta)$, and $\phi_n(r)$ vanishes at $m_1 a_1 + m_2 a_2$ when $m_1$ and $m_2$ are integers. This requires

$$k_m = 2\pi (m - 1/2)/a = (m - 1/2) \sqrt{2\pi \sigma},$$ \hspace{1cm} (4.4)

where $\sigma = (b/a) \sin \theta$ and $\xi = (b/a) \cos \theta$. Note that we used the flux quantization condition $ab \sin \theta = 2\pi$. For a lattice of HQVs, we need to consider a second lattice that is translated by $l \tau = l(\tau_s, \tau_y)$ with respect to the first lattice. For the wavefunction of this lattice, we can use [30]

$$\bar{\phi}_n(r) = e^{i \tau_x} \phi_n(r - \tau),$$ \hspace{1cm} (4.5)

the phase factor being chosen so that $\Pi_- \phi_0(r) = 0$. This latter condition ensures that the translated eigenstates have the same gauge as the untranslated eigenstates.

The formalism considered here gives us not only the energy due to interaction between vortices but also the core energy of vortices as well. This is because our vortex lattice wavefunction gives a full description of the core regions. From the linearized GL equation we used here, a full QV is merely two HQVs of opposite spins coinciding at the same point. This means that the full QV core energy, if we ignore the cross term between two spin components in $f_{\text{hom}}^{(4)}$, is approximately twice the core energy of a HQV. If the HQV core energy is actually larger than this, that would make stabilization of the HQV more difficult, i.e. the largest value allowed for $\rho_{\text{op}}/\rho_s$ for the HQV lattice would be smaller than what we obtain through the formalism used here.

The lattice structure can be determined by finding $(\sigma, \xi, \tau)$ that minimize the free energy expectation value. For this, we first set the amplitude of the order parameter to minimize the energy for given $(\sigma, \xi, \tau)$ (the amplitude depends on $H_{s2} - H$) and then compare the energies for different values of $(\sigma, \xi, \tau)$. To determine these structures, we will need to take the spatial integral of the product of four Landau level wavefunctions. We have computed these integrals in appendix C.
5. The lowest Landau level solution

In the bulk of this section, we provide a detailed analysis of the lowest Landau level solution for the non-chiral triplet superconductors and briefly comment on the chiral case in section 5.3. In this case the relevant free energy (for orthorhombic, tetragonal and hexagonal materials) is

\[
f = \sum_{s=\uparrow,\downarrow,\downarrow} \left[ -\alpha |\Delta_s|^2 + \beta_1 |\Delta_s|^4/2 + \left( \sum_{i=x,y,z} K_i D_i \Delta_s \right)^2 \right] - \beta_2 |\Delta_{\uparrow\uparrow}|^2 |\Delta_{\downarrow\downarrow}|^2 - \tilde{k} \hbar (|\Delta_{\uparrow\uparrow}|^2 - |\Delta_{\downarrow\downarrow}|^2) + \gamma [\Delta_{\uparrow\uparrow}^* \Delta_{\downarrow\downarrow} (\mathbf{D} \Delta_{\uparrow\uparrow}) \cdot (\mathbf{D} \Delta_{\downarrow\downarrow})^* + \text{c.c.}] + \frac{\hbar^2}{8\pi} - \frac{HH}{4\pi}. \quad (5.1)
\]

As mentioned before, we assume that we have rescaled lengths so that we can take \( K_x = K_z = K_y = K \). First look at the upper critical field problem. The linearized GL equation is

\[
\frac{\alpha l^2}{K} \Delta_s = \left( 1 + 2\Pi_+ \Pi_- - s \frac{Hl^2}{K} \right) \Delta_s. \quad (5.2)
\]

The largest \( H_{c2} \) occurs when the \( \Delta_s \) are in the lowest Landau level. This leads to two possible values for the upper critical field,

\[
H_{c2}^\pm = \frac{\alpha \Phi_0}{2\pi (K \pm Hl^2 \tilde{k})}. \quad (5.3)
\]

We assume that \( \tilde{k} > 0 \) so that \( \Delta_{\uparrow\uparrow} \) has the larger \( H_{c2} \). We do not assume that the splitting between these two critical fields is large since it is given by the small Zeeman term \( \tilde{k} \).

Now consider the expectation value of \( f \) in terms of the Landau level wavefunctions. Applying equation (4.2) to the first two terms of equation (5.1) gives the lowest Landau level solutions \( \Delta_{\uparrow\uparrow} = C_{\uparrow\uparrow} \phi_0 \) and \( \Delta_{\downarrow\downarrow} = C_{\downarrow\downarrow} e^{i2\alpha} \tilde{\phi}_0 \). Inserting this solution as determined at \( H = H_{c2} \) gives

\[
\langle f \rangle = -\frac{1}{c} (\mathbf{j} \cdot \delta \mathbf{A}) + \beta_1 \langle |\phi_0|^4 \rangle (C_{\uparrow\uparrow}^4 + C_{\downarrow\downarrow}^4)/2 + \left[ \frac{2\gamma}{l^2} (\langle |\phi_0|^2 |\tilde{\phi}_0|^2 \rangle - \langle |\phi_0|^2 |\tilde{\phi}_0|^2 \rangle) \right] - \beta_2 \langle |\phi_0|^2 |\tilde{\phi}_0|^2 \rangle C_{\uparrow\uparrow}^2 C_{\downarrow\downarrow}^2 + \left( \frac{\hbar^2}{8\pi} - \frac{H^2}{8\pi} + \alpha \left[ 1 - \frac{K + H_{c2}^2 \tilde{k}}{K - H_{c2}^2 \tilde{k}} \right] \right) C_{\downarrow\downarrow}^2. \quad (5.4)
\]

where \( \mathbf{j} \) is the supercurrent, \( \delta \mathbf{A} \) is the deviation of the vector potential from what we would have for \( h = H_{c2} \), and \( \mathbf{h}_s \) is the screening field of the superconductor. Note that in this approximation, \( \mathbf{j} \) is calculated solely from quadratic terms, ignoring \( \gamma \) terms, and by the Maxwell equation \( \nabla \times \mathbf{h}_s = 4\pi \mathbf{j} / c \). More specifically,

\[
\mathbf{j} = 2eK [\Delta_{\uparrow\uparrow}^* (\mathbf{D} \Delta_{\uparrow\uparrow}) + \Delta_{\downarrow\downarrow}^* (\mathbf{D} \Delta_{\downarrow\downarrow}) + \text{c.c.}] - e\tilde{k} \nabla \times \tilde{\zeta} (|\Delta_{\uparrow\uparrow}|^2 - |\Delta_{\downarrow\downarrow}|^2). \quad (5.5)
\]
Since \( \nabla \times \delta A = \hat{z}(h_s + H - H_{c2}) \) and, from Maxwell’s equations, \( \nabla \times h_s = 4\pi j/c \), partial integration leads to [31]

\[
\frac{1}{c} (j \cdot \delta A) = \frac{1}{4\pi} \langle h_s \cdot (h_s + H - \hat{z}H_{c2}) \rangle
\]

\[
= \frac{\langle h_s^2 \rangle}{4\pi} + \frac{H_{c2} - H}{4\pi} \langle h_s \rangle. \tag{5.6}
\]

Meanwhile, when we calculate the expectation value of \( \gamma \), we set \( H = H_{c2} \). This leads to a free energy of

\[
\langle f \rangle = -\frac{H_{c2} - H}{4\pi} \langle h_s \rangle - \frac{\langle h_s^2 \rangle}{8\pi} - \frac{H^2}{8\pi} + \alpha \left[ 1 - \frac{K + H_{c2}^2}{K - H_{c2}^2} \right] C_{\uparrow\downarrow}^2 + \frac{\beta}{2} \langle |\phi_0|^4 \rangle (C_{\uparrow\uparrow}^4 + C_{\downarrow\downarrow}^4)
\]

\[
+ \left[ \frac{2\gamma}{I} \left( \langle |\phi_0|^2 |\tilde{\phi}_0|^2 \rangle - \langle |\phi_0|^2 |\bar{\phi}_1|^2 \rangle \right) - \beta_2 \langle |\phi_0|^2 |\bar{\phi}_0|^2 \rangle \right] C_{\uparrow\downarrow}^2 C_{\downarrow\uparrow}^2
\]

\[
= -\frac{H^2}{8\pi} + \langle \tilde{f} \rangle. \tag{5.7}
\]

The screening field \( h_s \) can be calculated by assuming \( |\tilde{A}| \propto (H_{c2} - H)^{1/2} \) near the second-order phase transition at \( H = H_{c2} \). Here, we will deal only with \( O(|\tilde{A}|^4) \) (or equivalently, \( O(1 - H/H_{c2}^2) \)) and ignore higher order terms. This allows us to calculate \( j \), and consequently \( h_s \), solely from quadratic terms. In the lowest Landau level, this yields the screening field:

\[
h_s = \frac{8\pi^2 K}{\Phi_0} \left( |\Delta_{\uparrow\uparrow}|^2 + |\Delta_{\downarrow\downarrow}|^2 \right) - 4\pi \tilde{k} (|\Delta_{\uparrow\uparrow}|^2 - |\Delta_{\downarrow\downarrow}|^2)
\]

\[
= \left( \frac{8\pi^2 K}{\Phi_0} - 4\pi \tilde{k} \right) C_{\uparrow\downarrow}^2 |\phi_0|^2 + \left( \frac{8\pi^2 K}{\Phi_0} + 4\pi \tilde{k} \right) C_{\downarrow\uparrow}^2 |\tilde{\phi}_0|^2. \tag{5.8}
\]

Inserting equation (5.8) into equation (5.7), the free energy takes the following form:

\[
\langle \tilde{f} \rangle = -\tilde{\alpha}_1 C_{\uparrow\downarrow}^2 - \tilde{\alpha}_2 C_{\downarrow\uparrow}^2 + \tilde{\beta}_1 C_{\uparrow\uparrow}^4 + \tilde{\beta}_2 C_{\downarrow\downarrow}^4 + \tilde{\beta}_3 C_{\uparrow\downarrow}^2 C_{\downarrow\uparrow}^2 \tag{5.9}
\]

with terms quadratic or quartic in \( C_{\uparrow\uparrow} \) and \( C_{\downarrow\downarrow} \) with coefficients that are independent in general (we will further specify these coefficients in the next two subsections). In the absence of screening fields, Zeeman fields and the term proportional to \( \gamma \), the form of the free energy in equation (5.7) is similar to that examined in [24] in the context of two-component Bose condensates (spin-half spinor condensates). In that case, the vortex lattice structure is solely determined by the competition between the \( \beta_1 \) and \( \beta_2 \) terms of equation (5.1); the \( \beta_1 \) term determines the interaction energy within each vortex lattice, and the \( \beta_1 \) term determines the interaction energy between two fractional vortex species each forming lattices. Specifically, the quartic term \(-\beta_2 \langle |\phi_0|^2 |\tilde{\phi}_0|^2 \rangle C_{\uparrow\downarrow}^2 C_{\downarrow\uparrow}^2 \) determines the stability of the HQV lattice. If \( \beta_2 < 0 \), then a HQV lattice is the ground state. If \( \beta_2 > 0 \), then the full QV lattice is the ground state. In appendix A, we show that \( \beta_2 = 0 \) in weak-coupling theories, so that the two lattice structures are degenerate. In the rest of the paper, we will focus on aspects that are unique to triplet superconductors; in sections 5.1 and 5.2: the effects of the screening fields \( f_{in}^{(4)} \), and the Zeeman field.
5.1. The effects of screening and $f_{in}^{(4)}$

Here we look into the effect of screening. Ignoring the Zeeman field (in weak-coupling theories, the Zeeman field is vanishing in the clean limit), we have symmetry between $C_{\uparrow \uparrow}$ and $C_{\downarrow \downarrow}$, giving us

$$\tilde{\alpha}_1 = \tilde{\alpha}_2 \equiv \tilde{\alpha} = \frac{2\pi K(H_{c2} - H)}{\Phi_0} \langle |\phi_0|^2 \rangle$$ (5.10)

and

$$\tilde{\beta}_1 = \tilde{\beta}_2 \equiv \tilde{\beta} = \left( \frac{\beta}{2} - \frac{8\pi^3 K^2}{\Phi_0^2} \right) \langle |\phi_0|^4 \rangle,$$ (5.11)

which means that the free energy in equation (5.9) takes a simpler form:

$$\langle f \rangle = -\tilde{\alpha}(C_{\uparrow \uparrow}^2 + C_{\downarrow \downarrow}^2) + (C_{\uparrow \downarrow}^4 + C_{\downarrow \uparrow}^4) + \tilde{\beta}_3 C_{\uparrow \uparrow}^2 C_{\downarrow \downarrow}^2,$$ (5.12)

with

$$\tilde{\beta}_3 = \frac{2\gamma}{\beta} \left( |\phi_0|^2 \langle |\phi_0|^2 \rangle^2 - \langle |\phi_1|^2 \rangle \right) - \left( \frac{16\pi^3 K^2}{\Phi_0^2} + \beta_2 \right) \langle |\phi_0|^2 |\phi_0|^2 \rangle.$$ (5.13)

Now the free energy of equation (5.7) can be minimized by choosing $C_{\uparrow \uparrow}^2 = C_{\downarrow \downarrow}^2 = \tilde{\alpha}/(2\tilde{\beta} + \tilde{\beta}_3)$ (which gives $|\hat{\Delta}| \propto (H_{c2} - H)^{1/2}$ as mentioned), giving us the free energy expectation value

$$\langle f \rangle = -\frac{H^2}{8\pi} - \frac{\tilde{\alpha}^2}{2\beta + \tilde{\beta}_3}.$$ (5.14)

Equations (5.13) and (5.14) allow for understanding the role of both screening and $f_{in}^{(4)}$. The screening affects the vortex lattice structure through the dependence of terms proportional to $K^2$ in equations (5.11) and (5.13) on the lattice structure parameters $\sigma$, $\sigma$ and $\tau$. Since all quartic expectation values depend on the lattice structure, the lattice structure will be determined through minimizing $(2\tilde{\beta} + \tilde{\beta}_3)$. Since the magnitude of the $K^2$ term in equation (5.13) is larger for full QVs (equation (5.11) is not affected), screening tends to disfavor HQV lattices, in line with the earlier observation for isolated HQVs [3]. However, since weak-coupling theories lie near the point $\tilde{\beta}_3 = 0$, we expect that the screening effect will put the physical system in a fine balance between interlacing lattices of HQVs and the ordinary Abrikosov vortex lattice. Hence it should be possible to observe the transition between the two phases upon small changes of field and temperature. Indeed, the interaction $f_{in}^{(4)}$ plays this role. In particular, we numerically find that for $\gamma > 0$, this term tends to favor HQV lattices. A positive sign of $\gamma$ occurs when $\rho_{sp} < \rho_0$ and this is to be expected in spin-triplet superconductors [3]. Note that unlike the other contributions in $\tilde{\beta}_3$, the contribution from $f_{in}^{(4)}$ vanishes as $T \rightarrow T_c$. Consequently, this term can drive a field- and temperature-dependent transition between a HQV lattice and a full quantum lattice. Figures 1 and 2 show the phase diagram for $\gamma = 0$. Note the similarity between the calculated phase diagram and that found in the context of two-component Bose condensates [24, 32].
5.2. The effects of the Zeeman term

For simplicity, we consider here the effect of Zeeman field alone, ignoring screening and setting $\gamma = 0$ (ignoring $f_{in}^{(4)}$). In the presence of Zeeman field, the free energy equation (5.9) would have different coefficients for two quartic terms:

$\langle \tilde{f} \rangle = -\tilde{\alpha}_1 C_{11}^2 - \tilde{\alpha}_2 C_{11}^2 + \tilde{\beta}(C_{11}^4 + C_{22}^4) + \tilde{\beta}_3 C_{11}^2 C_{22}^2,$  \hspace{1cm} (5.15)

where $\tilde{\alpha}_1 = \alpha + \frac{k}{\Phi_0} + H \tilde{k}$, $\tilde{\alpha}_2 = \alpha - \frac{k}{\Phi_0} - H \tilde{k}$, $\tilde{\beta} = \beta_1 \langle |\phi_0|^4 \rangle$ and $\tilde{\beta}_3 = -\beta_2 \langle |\phi_0|^2 |\tilde{\phi}_0|^2 \rangle$. The Zeeman field has two main consequences: (i) it typically leads to two phase transitions. In the first phase $C_{11} \neq 0$ and $C_{22} = 0$, and in the second phase both components are nonzero. The first phase is analogous to the $^3$He $A_1$ phase, with a non-unitary spin-triplet order parameter.
However, weak-coupling theories prefer unitary spin-triplet states and this drives the second transition. (ii) In the fractional vortex lattice phase where both components are nonzero, the magnetic flux contained by isolated fractional vortices is no longer a half-integral flux quanta. Instead, two types of vortices carry fractional flux values of

\[ \Phi_i = \Phi_0 \frac{|c_i|^2}{|c_1|^2 + |c_2|^2}. \]  

The double transition is possible if \( 2\tilde{\beta} + \tilde{\beta}_3 > 0 \) (note again that weak-coupling theories yield \( \tilde{\beta}_3 = 0 \) and \( \tilde{\beta} > 0 \)), there can be two transitions with a second transition appearing at a temperature

\[ T_{c2} - T_{c1} = \frac{4\tilde{\beta}}{2\tilde{\beta} + \tilde{\beta}_3} \frac{K + H_c l^2\tilde{k}}{K_1 - H_c l^2\tilde{k}} T_{c1}. \]  

(5.17)

In the high-temperature phase, the vortex lattice is hexagonal and, at the second transition, the lattice will remain hexagonal and the second component will either coincide with the first or be displaced half a hexagonal vortex lattice vector from the first. As the temperature is further reduced below the second transition, the lattice will continuously deform, asymptotically approaching the phases presented in section 5.1 (those shown in figure 2). The resulting phase diagram is qualitatively shown in figure 3.

Both consequences of the Zeeman field stem from breaking the additional \( \mathbb{Z}_2 \) symmetry that is present when \( \tilde{\alpha}_1 = \tilde{\alpha}_2 \). In general, the existence of fractional vortices is the result of the \( U(1) \times U(1) \) symmetry of the free energy. When there is additional \( \mathbb{Z}_2 \) symmetry due to \( \tilde{\alpha}_1 = \tilde{\alpha}_2 \), the flux contained in each fractional vortex is restricted to be half the flux quantum since the two components of the order parameter are no longer degenerate in a magnetic field. In section 7, we will see that this helps us distinguish a lattice of HQVs from a lattice of full QVs.
5.3. Chiral triplet superconductors: lowest Landau level solution

The chiral triplet superconductor with tetragonal symmetry, because of the inhomogeneous quadratic terms we have already seen in equation (3.12),

\[
f^{(2,s)}_\text{in} = K_1 (| \mathbf{D} \Delta_{s,+} |^2 + | \mathbf{D} \Delta_{s,-} |^2) + K_2 [(D_x \Delta_{s,+})^*(D_x \Delta_{s,-}) - (D_y \Delta_{s,+})^*(D_y \Delta_{s,-})]/2 \\
+ (D_y \Delta_{s,+})^*(D_y \Delta_{s,-}) - (D_x \Delta_{s,+})^* (D_x \Delta_{s,-}) - i(\Delta_{s,+}) (D_y \Delta_{s,-})^*]
\]

has a much more complicated quadratic free energy,

\[
f_\text{0}^{(2)} = \sum_{s=\uparrow, \downarrow} [-\alpha (| \Delta_{s,+} |^2 + | \Delta_{s,-} |^2) + f^{(2,s)}_\text{in}],
\]

even when we exclude the Zeeman field and any spin–orbit coupling.

Due to these inhomogeneous quadratic terms, we cannot put both chirality components in the lowest Landau level. This is due to the presence of \([(D_x \Delta_{s,+})^*(D_y \Delta_{s,-}) + c.c.]\) terms in \(f^{(2,s)}_\text{in}\). The above quadratic free energy of equation (5.19), together with equation (3.13) that gives the energy splitting between two chiralities, leads to the linearized GL equation

\[
\alpha^2 \left( \begin{array}{c} \Delta_{s,+} \\ \Delta_{s,-} \end{array} \right) = \left[ \begin{array}{cc} K_1 (1 + 2 \Pi_\uparrow \Pi_-) - \delta K & K_2 \Pi_\uparrow^2 \\ K_2 \Pi_-^2 & K_1 (1 + 2 \Pi_\uparrow \Pi_-) + \delta K \end{array} \right] \left( \begin{array}{c} \Delta_{s,+} \\ \Delta_{s,-} \end{array} \right).
\]

However, a lowest Landau level solution that satisfies equation (5.20) may still have the highest \(H_{c,2}\) and may therefore be possible [15]. This would lead to a nonzero order parameter for only one chirality — \((\Delta_{s,+}, \Delta_{s,-}) = C(0, \phi_0)\) — and requires \(\delta K < -\frac{K_2^2}{4 \delta_1^0}\). In this case, the vortex energetics of the chiral triplet superconductor is identical to that of the non-chiral triplet superconductor, for inserting this lowest Landau level solution into the full Gibbs free energy leads us back to equation (5.4).

However, equation (5.20) can also give us a solution with Landau level mixing; in this case, both \(\Delta_{s,+}\) and \(\Delta_{s,-}\) are nonzero. We have presented a discussion on this Landau level mixing in appendix B.

6. Role of spin–orbit coupling

6.1. Generic case

If the material has orthogonal or tetragonal symmetry (although not necessarily true for hexagonal symmetry, which is discussed in the next subsection), then there will exist spin–orbit coupling terms of the type

\[
\epsilon \Delta_{\uparrow\uparrow} \Delta_{\downarrow\downarrow}^*.
\]

Such terms break the \(U(1) \times U(1)\) symmetry and consequently isolated fractional flux vortices are no longer stable. Nevertheless, a fractional flux quantum vortex lattice can still exist, provided that the separation between vortices is less than \(\xi_{so}\), which is defined by \(\xi_{so}^2 = K/\epsilon\).
For completeness, we write here the spin–orbit coupling terms that appear in the context of a chiral spin-triplet superconductor. While we do not include these terms in calculations, they may be useful in other contexts. Due to the tetragonal $C_4$ symmetry, homogeneous spin–orbit coupling terms should be invariant under the transformation

$$(\Delta_{\uparrow\uparrow,+}, \Delta_{\uparrow\uparrow,-}, \Delta_{\downarrow\downarrow,+}, \Delta_{\downarrow\downarrow,-}) \rightarrow (-\Delta_{\uparrow\uparrow,+}, \Delta_{\uparrow\uparrow,-}, \Delta_{\downarrow\downarrow,+}, -\Delta_{\downarrow\downarrow,-}).$$

To quadratic order, this condition is satisfied by the lowest Landau level (which will minimize the free energy when the above equation appears). These states belong to the 2D representations labelled $\Gamma_1$.

For hexagonal materials, there exist spin-triplet pairing states for which no terms like that in $6.2$. Hexagonal materials

6.2. Hexagonal materials

For hexagonal materials, there exist spin-triplet pairing states for which no terms like that in the above equation appear. These states belong to the 2D representations labelled $\Gamma_5^-$ and $\Gamma_6^-$ in the review article by Sigrist and Ueda [25]. Consequently, these materials need to be considered more carefully.

We will now show that in hexagonal materials, a little away from $H_{c2}$, spin–orbit coupling does not break $U(1) \times U(1)$ symmetry. For hexagonal materials, the only term that exists in the GL free energy that is due to spin–orbit coupling is (note that the inclusion of this term gives rise to the complete free energy found that is found in Sigrist and Ueda for the $\Gamma_{5,6}$ representations):

$$f_{SO} = K_{so} \left\{ (D_1 \Delta_{\downarrow\downarrow})(D_1 \Delta_{\uparrow\uparrow})^* - (D_3 \Delta_{\downarrow\downarrow})(D_3 \Delta_{\uparrow\uparrow})^* + (D_3 \Delta_{\uparrow\uparrow})(D_3 \Delta_{\downarrow\downarrow})^* 
- (D_3 \Delta_{\downarrow\downarrow})(D_3 \Delta_{\uparrow\uparrow})^* - i[(D_1 \Delta_{\downarrow\downarrow})(D_3 \Delta_{\uparrow\uparrow})^* + (D_3 \Delta_{\downarrow\downarrow})(D_1 \Delta_{\uparrow\uparrow})^*]
+ i[(D_1 \Delta_{\uparrow\uparrow})(D_3 \Delta_{\downarrow\downarrow})^* + (D_3 \Delta_{\uparrow\uparrow})(D_1 \Delta_{\downarrow\downarrow})^*] \right\}/2. \tag{6.3}$$

With the field along the $c$-axis, the solution to the quadratic problem satisfies

$$\frac{\alpha l^2}{K} \left( \Delta_{\uparrow\uparrow} \right) = \left( 1 + 2N - K_{\zeta} \right) \left( \tilde{K}_{so} \Pi_{\zeta} \frac{2}{1 + 2N + K_{\zeta}} \right) \left( \Delta_{\downarrow\downarrow} \right),$$

where $K_{\zeta} = \tilde{K} H l^2 / K$ and $\tilde{K}_{so} = K_{so} / K$. All the eigenstates for this problem can be found analytically [30]. Typically, $|\tilde{K}_{so}| \ll 1$, so we will be interested in the eigenstates that contain the lowest Landau level (which will minimize the free energy when $\tilde{K}_{so} = 0$). The two relevant eigenstates that we wish to keep are: $(\Delta_{\uparrow\uparrow}, \Delta_{\downarrow\downarrow}) = (\phi_0, \epsilon \phi_2)$ and $(\Delta_{\uparrow\uparrow}, \Delta_{\downarrow\downarrow}) = (0, \phi_0)$, where $\epsilon$ is proportional to $\tilde{K}_{so}$. Note that unlike in section 5.3, we can keep both solutions because while we examined $H \sim H_{c2}$ in that subsection, we are a little away from $H_{c2}$ in this subsection.

In the absence of out-of-plane magnetic field, such spin–orbit interactions support a superconductor with order parameter symmetry analogous to that of the $^3$He-B phase. This class of phases has been recently discussed as a time-reversal invariant topological superconductor [33].
We therefore write $\Delta_{+} = \gamma_{1}\phi_{0}$ and $\Delta_{\downarrow} = \gamma_{1}\epsilon\phi_{2} + \gamma_{2}\tilde{\phi}_{0}$ to include these two eigenstates. For simplicity, we ignore screening and the Zeeman field to find the following free energy:

$$\langle f \rangle = -(1 - H/H_{c,1})|\gamma_{1}|^{2} - (1 - H/H_{c,2})|\gamma_{2}|^{2} + \beta_{1}[|\gamma_{1}|^{4}(|\phi_{0}|^{4}) + (|\gamma_{1}\epsilon\phi_{2} + \gamma_{2}\tilde{\phi}_{0}|^{4})]
- \beta_{2}(|\gamma_{1}\phi_{0}|^{2}|\gamma_{1}\epsilon\phi_{2} + \gamma_{2}\tilde{\phi}_{0}|^{2}),$$ \hfill (6.4)

where $H_{c,i}$ ($i = 1, 2$) is the upper critical field for eigenstate $i$. Since spin–orbit coupling is expected to be small, this implies that $\epsilon \ll 1$, so keeping to linear order in $\epsilon$ yields:

$$\langle f \rangle = -(1 - H/H_{c,1})|\gamma_{1}|^{2} - (1 - H/H_{c,2})|\gamma_{2}|^{2} + \beta_{1}[|\gamma_{1}|^{4}(|\phi_{0}|^{4}) + (|\gamma_{1}\epsilon\phi_{2} + \gamma_{2}\tilde{\phi}_{0}|^{4})]
+ \beta_{2}(|\phi_{0}|^{2}|\tilde{\phi}_{0}|^{2})|\gamma_{1}|^{2}|\gamma_{2}|^{2} + \epsilon[\beta_{1}|\gamma_{2}|(\gamma_{1}\phi_{0})^{*}(|\phi_{0}|^{2}\tilde{\phi}_{0}^{*} + c.c.)]
- \epsilon[\beta_{2}|\gamma_{1}|^{2}|\gamma_{2}|^{2}(|\phi_{0}|^{2}\phi_{2}\tilde{\phi}_{0}^{*} + c.c.).]$$ \hfill (6.5)

Without the last two terms, this theory is the same as that found for non-chiral spin-triplet superconductors with a Zeeman field but without any spin–orbit coupling. At the upper critical field, one of the two components $\gamma_{1}$ or $\gamma_{2}$ order and the vortex lattice will be hexagonal (this conclusion is correct even when including terms that are second order in $\epsilon$). As the temperature or magnetic field is reduced, the last two terms in equation (6.5) can play an important role. These two terms break the $U(1) \times U(1)$ symmetry of the theory and therefore will tend to remove any HQV lattice phases. However, the spatial averages $\langle |\tilde{\phi}_{0}|^{2}\phi_{0}\phi_{2}^{*} \rangle$ and $\langle |\phi_{0}|^{2}\phi_{2}\tilde{\phi}_{0}^{*} \rangle$ vanish for a hexagonal vortex lattice (loosely speaking, this follows from noting that $\phi_{n}$ picks up a factor $e^{in\phi}$ under a rotation about $\hat{z}$ and that a hexagonal vortex lattice is symmetric under rotations of $\pi/3$). The hexagonal symmetry of the materials conspires to remove this form of $U(1) \times U(1)$ symmetry breaking and the HQV lattice structures are still possible (indeed the theory is the same as that given for the non-chiral spin-triplet superconductors with a Zeeman field, but without spin–orbit coupling). Note that if $\tau \neq 0$ (signaling the existence of the fractional vortex lattice), then $\langle |\tilde{\phi}_{0}|^{2}\phi_{0}\phi_{2}^{*} \rangle = \langle |\phi_{0}|^{2}\phi_{2}\tilde{\phi}_{0}^{*} \rangle = 0$ for any lattice geometry. Consequently, the last two terms of equation (6.5) do not play any role in the theory of the fractional vortex lattices.

It is reasonable to ask if there are any other $U(1) \times U(1)$ symmetry breaking terms that we have neglected in the above analysis. Indeed there is one that appears at order $\epsilon^{2}$: $\epsilon^{2}\beta_{1}|\gamma_{1}|^{4}(\gamma_{2}^{*}2|\phi_{0}^{*}\tilde{\phi}_{0}^{*}|)^{*}$. This term allows for the existence of a fractional vortex lattice phase subject to the constraint that $\tau$ is half a vortex lattice translation vector [30]. There are also $U(1) \times U(1)$ that appear at order $\epsilon^{3}$, but these vanishes for the same reason as the order $\epsilon$ term. Consequently, the spin–orbit coupling for the hexagonal 2D representations plays essentially the same role as the Zeeman field.

### 7. Observation of the vortex lattice

The best way to determine both the vortex lattice structure and the vortex type is to observe the magnetic field distribution through the small angle neutron scattering. What we will see in this experiment is the Fourier transform $f(G)$ of the screening field of equation (5.8),

$$h_{s}(r) = \left(\frac{8\pi^{2}}{K_{0}} - 4\pi\tilde{\kappa}\right)C_{\uparrow\uparrow}^{2}|\phi_{0}(r)|^{2} + \left(\frac{8\pi^{2}}{K_{0}} + 4\pi\tilde{\kappa}\right)C_{\downarrow\downarrow}^{2}|\tilde{\phi}_{0}(r)|^{2},$$ \hfill (7.1)
Figure 4. Contour plots of the screening field in the real (that is, position) space for different types of vortex lattices: (a) a lattice of ordinary Abrokosov (full quantum) vortices, (b) a HQV lattice and (c) a lattice of fractional vortices in the presence of a Zeeman field. The position is measured in units of magnetic length. Note the halving of the unit cell in going from the HQV lattice to the full QV lattice. When the Zeeman field is added in, the periodicity is that of the full quantum lattice, though the vortex lattice unit cell now has additional structure due to the appearance of fractional flux at both the corners and center of the unit cell.

that is, \( h_s(r) = \sum_G f(G) \exp(iG \cdot r) \), where \( G \) is the reciprocal lattice vector of the vortex lattice in units of the inverse magnetic length.

The characteristic feature of the vortex lattice with half-vortices in the small angle neutron scattering experiment is the modulation of the Bragg peaks. The form factor of the Bragg peaks would be the \( f(G) \) of the last paragraph. Using

\[
|\phi_0(r)|^2 = \sum_G (-1)^{m_1 + m_2 + m_3} e^{-G^2/2},
\]

where \( G = m_1 G_1 + m_2 G_2 \), and \( G_i \)s are the basis vectors of the reciprocal lattice, we obtain the form factor

\[
f(G) = (-1)^{m_1 + m_2 + m_1 m_2} e^{-G^2/2} \left[ \left( \frac{8\pi^2 K}{\Phi_0} - 4\pi \tilde{\kappa} \right) |C_{11}|^2 + \left( \frac{8\pi^2 K}{\Phi_0} + 4\pi \tilde{\kappa} \right) |C_{11}|^2 e^{iG \cdot \tau} \right].
\]

This equation implies that the intensity \( |f(G)|^2 \) for our Bragg peaks does not come out the same for all \( G \)s. This is because for almost all vortex lattice structures (the single exception being the not very robust honeycomb lattice) \( \tau \) is half a vortex translation vector so that we have \( e^{iG \cdot \tau} = -1 \) for one half of \( G \)s and \( e^{iG \cdot \tau} = 1 \) for the other half. When there is no Zeeman field, \( e^{iG \cdot \tau} = -1 \) peaks disappear completely; this is natural given that the magnetic field cannot distinguish the spin up and the spin down HQVs at all and thus sees the unit lattice vector halved. However, when the Zeeman field breaks down the \( Z_2 \) symmetry between the spin up–up pairs and down–down pairs, we now see a secondary peak for \( e^{iG \cdot \tau} = -1 \) as shown in figure 4.

Another promising direction for detecting fractional vortex lattices would be to use spin-polarized scanning tunneling microscopy to probe the vortex cores. The key point is that the low-energy quasi-particle spins have opposite polarization in the two different HQVs. This is
because for half of HQV cores, we have $\Delta_{\uparrow\uparrow} = 0$ and $\Delta_{\downarrow\downarrow} \neq 0$, so that only spin-down quasi-particles are gapped. On the other hand, for the other half of HQV cores, only spin-up quasi-particles are gapped. This spin-polarization of the subgap core modes should be readily detected through spin-polarized scanning tunneling microscopy.

8. Conclusion

In this paper, we explored various possibilities for fractional vortex lattice structures in spin-triplet superconductors starting from the most general form of Gibbs free energy that is allowed by the symmetry of the order parameter and that of the lattice symmetries relevant for three candidate spin-triplet superconductors, namely the single layer ruthenate Sr$_2$RuO$_4$ [11, 12], the cobaltate Na$_x$CoO$_2$·$y$H$_2$O [13] and the organic (TMTSF)$_2$ClO$_4$. The focus of our analysis was on the role of aspects unique to triplet superconductors, such as (i) Cooper pair Zeeman field, (ii) spin–orbit coupling, (iii) screening and (iv) interaction effects in the energetics of the vortex lattice structure. (i) The Cooper pair Zeeman field breaks a $Z_2$ symmetry of the free energy whose presence constrains the fractional vortices to contain half integral flux quanta. The resulting structure is that of two interlacing lattices of vortices containing arbitrary fractions of flux quanta that add up to one flux quanta. Such fractional vortex lattices will have interesting field distributions in the vortex lattice unit cell due to internal structures within the unit cell. (ii) The effect of spin–orbit coupling is lattice symmetry specific. In hexagonal lattice systems such as cobaltates Na$_x$CoO$_2$·$y$H$_2$O, spin–orbit coupling has the same effect as the Cooper pair Zeeman field, supporting fractional vortex lattices. However, for tetragonal or orthorhombic lattices, sufficiently strong spin–orbit coupling generally favors the ordinary Abrikosov vortex lattice over HQVs. However, such an effect is relatively mild in a dense vortex lattice, provided that the separation between the HQVs is less than a length set by the spin–orbit coupling. (iii) The Meissner screening effectively generates attraction between two HQVs with opposite winding of the spin phase and weakly destabilizes the HQVs. (iv) The interaction effects clearly support energetic stability of HQVs within the GL theory. The interaction effects represented by inhomogeneous (unique to triplet superconductors) quartic terms can drive the difference in effective superfluid stiffness $\rho_s > \rho_p$ which stabilizes HQVs in the London limit. When the above effects are put together, all weak coupling theories we examined appear to lie at the point of fine balance between the ordinary Abrikosov vortex lattice and lattices of HQVs. Hence, it should be possible to observe transitions between these structures with small changes of parameters. This further motivates experimental searches for these fractional vortex lattices. We have sketched possible routes for such searches using neutron scattering or spin polarized scanning tunneling microscopy.

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Appendix A. GL energy: fourth order terms from weak-coupling theory

The GL free energy can be determined in the weak-coupling limit. In the context of the existence of HQV lattice structures, the result for the fourth order terms in the free energy turns out to be highly relevant. As shown here, this reveals that weak-coupling theories sit at a point in which the HQV and the full QV lattices are degenerate. This indicates that interactions beyond the weak-coupling limit are essential for determining which lattice structure actually appears (screening plays a role here as well, as shown earlier).

The portion of the free energy we calculate here is given in equation (3.6)

\[ f^{(4)}_{\text{hom}} = \beta_1 \left( \sum_i |\Delta_i|^2 \right)^2 + \beta_2 |\Delta_{\uparrow\uparrow}|^2 |\Delta_{\downarrow\downarrow}|^2. \]  

(A.1)

The weak-coupling limit (without spin–orbit coupling) yields (this follows from [25])

\[ f^{(4)}_{\text{hom}} \propto \langle |d(k)|^4 \rangle + \langle q^2(k) \rangle, \]  

(A.2)

where \( q(k) = i d(k) \times d^*(k) \), \( \langle h(k) \rangle \) means average \( h(k) \) over all \( k \) on the Fermi surface, and the proportionality constant can be found but it is not important for our considerations. When \( q \) is nonzero, then the superconducting state is called non-unitary. In weak-coupling theories, non-unitary states cost energy and typically do not appear. Using the gap structure of equation (2.2), we find

\[ f^{(4)}_{\text{hom}} \propto \langle |f(k)|^4 \rangle [(|\Delta_{\uparrow\uparrow}|^2 + |\Delta_{\downarrow\downarrow}|^2)^2 + (|\Delta_{\uparrow\downarrow}|^2 - |\Delta_{\downarrow\uparrow}|^2)] = 2 \langle |f(k)|^4 \rangle (|\Delta_{\uparrow\uparrow}|^4 + |\Delta_{\downarrow\downarrow}|^4). \]  

(A.3)

This implies that \( \beta_2 = -2\beta_1 \), independent of the shape of the Fermi surface. The lack of interaction between the two components of the gap function leads to degeneracy between the HQV and the full QV lattice structures.

Appendix B. Ruthenate—the Landau level mixing

We show here how we can have the Landau level mixing in a chiral triplet superconductor. The case we are considering here is in the weak pairing regime and has tetragonal crystalline symmetry and a cylindrical Fermi surface. Let us consider again the linearized GL equation:

\[ i^2 \left( \frac{\Delta_{s+}}{\Delta_{s-}} \right) = \frac{K}{\alpha} \left( 1 + \frac{2\Pi_+ \Pi_-}{\Pi_+^2} \right) \left( \frac{\Delta_{s+}}{\Delta_{s-}} \right), \]  

(B.1)

where \( s = \uparrow\uparrow, \downarrow\downarrow \). (Note that, although otherwise the same as equation (5.20), we now ignore the energy splitting between the \( \pm \) chiralities and set \( K_1 = K_2 = K \).) This matrix equation has a solution in the form

\[ \begin{pmatrix} \Delta_{s+} \\ \Delta_{s-} \end{pmatrix} = C \begin{pmatrix} \phi_0 \\ -\delta \phi_2 \end{pmatrix}, \]  

(B.2)

where \( \delta = \sqrt{3} - \sqrt{2} \).
When we ignore the Zeeman field, much of the vortex lattice energetics of the lowest Landau level case remains valid with the Landau level mixing. For instance, the two main formulae of section 5.1, equation (5.12),

\[
\langle \bar{f} \rangle = -\bar{\alpha}(C_{\uparrow\uparrow}^2 + C_{\downarrow\downarrow}^2) + \bar{\beta}(C_{\uparrow\downarrow}^4 + C_{\downarrow\uparrow}^4) + \bar{\beta}_3 C_{\uparrow\uparrow}^2 C_{\downarrow\downarrow}^2,
\]

and equation (5.14)

\[
\langle f \rangle = -\frac{H^2}{8\pi} - \frac{\bar{\alpha}^2}{2\bar{\beta} + \bar{\beta}_3},
\]

remain valid, mainly due to \(|\hat{\Delta}| \propto (H_{c2} - H)^{1/2}\). This means we can still calculate \(h_s\), solely from quadratic terms. For quadratic terms, we simply have two copies (for \(s = \uparrow \uparrow\) and \(\downarrow \downarrow\)) of what was obtained for the case of \(d = (k_x + i k_y)\hat{z}\) by one of us [18]; we can use the formula for \(h_s\) for that case:

\[
h_s = \frac{8\pi^2 K}{\Phi_0} [C_{\uparrow\uparrow}^2 ((1 - 3\delta/\sqrt{2} + 2\delta^2)|\phi_0|^2 + (2\delta^2 - \delta/\sqrt{2})|\phi_1|^2 + \delta^2|\phi_2|^2)\]

\[+ C_{\downarrow\downarrow}^2 ((1 - 3\delta/\sqrt{2} + 2\delta^2)|\tilde{\phi}_0|^2 + (2\delta^2 - \delta/\sqrt{2})|\tilde{\phi}_1|^2 + \delta^2|\tilde{\phi}_2|^2)].\]

(B.5)

The spatial average of this equation is still proportional to \((C_{\uparrow\uparrow}^2 + C_{\downarrow\downarrow}^2)\) just like equation (5.8). Also, \(\tilde{\alpha} \propto (H_{c2} - H)\) still stands:

\[\tilde{\alpha} = \frac{2\pi K (H_{c2} - H)}{\Phi_0} [(1 - 3\delta/\sqrt{2} + 2\delta^2)|\phi_0|^2 + (2\delta^2 - \delta/\sqrt{2})|\phi_1|^2 + \delta^2|\phi_2|^2)].\]

(B.6)

However, the formula for \(2\tilde{\beta} + \tilde{\beta}_3\) is much more complicated here, especially when we include all terms of equations (3.14) and (3.15) for these coefficients. For the sake of convenience, instead of directly writing down \(\tilde{\beta}\) and \(\tilde{\beta}_3\), we will list \(\overline{h_z^2}\) (terms that are proportional to \(K^2\)), \(f_{\text{hom}}^{(4)}\) (terms involving coefficients of equation (3.14)) and \(f_{\text{in}}^{(4)}\) (terms involving coefficients of equation (3.15)); to obtain \(2\tilde{\beta} + \tilde{\beta}_3\), we can use the relation

\[
2\tilde{\beta} + \tilde{\beta}_3 = f_{\text{hom}}^{(4)} + f_{\text{in}}^{(4)} - \frac{\overline{h_z^2}}{8\pi}.
\]

(B.7)

The following is the full listing of \(\overline{h_z^2}/8\pi\), \(f_{\text{hom}}^{(4)}\) and \(f_{\text{in}}^{(4)}\) (note that we have set \(\beta_3\) of equation (3.14 to be zero):

\[
\frac{\overline{h_z^2}}{8\pi} = \frac{8\pi^3 K^2}{\Phi_0^2} \left[ (1 - 3\delta/\sqrt{2} + 2\delta^2)|\phi_0|^4 + 2(2\delta^2 - \delta/\sqrt{2})(1 - 3\delta/\sqrt{2} + 2\delta^2)|\phi_0|^2|\phi_1|^2 \right]
\]

\[+ 2\delta^2(1 - 3\delta/\sqrt{2} + 2\delta^2)|\phi_0|^2|\phi_2|^2 + (2\delta^2 - \delta/\sqrt{2})^2|\phi_1|^4 \]

\[+ 2\delta^4(2\delta^2 - \delta/\sqrt{2})(|\phi_1|^2|\phi_2|^2) + \delta^4|\phi_2|^4 + (1 - 3\delta/\sqrt{2} + 2\delta^2)^2|\phi_0|^2|\phi_0|^2 \]

\[+ 2(2\delta^2 - \delta/\sqrt{2})(1 - 3\delta/\sqrt{2} + 2\delta^2)|\phi_0|^2|\tilde{\phi}_1|^2 + 2\delta^2(1 - 3\delta/\sqrt{2} + 2\delta^2)|\phi_0|^2|\tilde{\phi}_2|^2 \]

\[+ (2\delta^2 - \delta/\sqrt{2})(|\phi_1|^2|\phi_2|^2) + 2\delta^2(2\delta^2 - \delta/\sqrt{2})(|\phi_1|^2|\tilde{\phi}_2|^2) + \delta^4(|\phi_2|^2|\tilde{\phi}_2|^2)\].

(B.8)
\[ f^{(4)}_{\text{hom}} = \beta_1 (|\phi_0|^4 + \delta^4 (|\phi_2|^4)) + 2\beta'_1 \delta^2 (|\phi_0|^2 |\phi_2|^2) \]
\[ - \beta_2 (|\phi_0|^2 |\phi_0|^2 + \delta^4 (|\phi_2|^2 |\phi_2|^2)) - 2\delta^2 \beta'_2 (|\phi_0|^2 |\phi_2|^2) \]  \hspace{1cm} \text{(B.9)}

and
\[ f^{(4)}_{\text{in}} = \frac{2\gamma'}{f} ((|\phi_0|^2 |\phi_0|^2) - (|\phi_0|^2 |\phi_1|^2)) + 3\delta^2 (|\phi_0|^2 |\phi_2|^2) - 3\delta^2 (|\phi_0|^2 |\phi_3|^2) \]
\[ + \frac{2\gamma'}{f^2} (3\delta^2 (|\phi_0|^2 |\phi_2|^2) + 2\delta^4 (|\phi_0|^2 |\phi_1|^2) + \delta^4 (|\phi_0|^2 |\phi_2|^2) \]
\[ - 3\delta^4 (|\phi_0|^2 |\phi_3|^2) + 2\delta^4 (|\phi_1|^2 |\phi_2|^2) + \delta^4 (|\phi_1|^2 |\phi_3|^2) - 3\delta^4 (|\phi_2|^2 |\phi_2|^2) \]
\[ + 3\delta^4 (|\phi_2|^2 |\phi_3|^2) - 3\delta^4 (|\phi_2|^2 |\phi_3|^2) \] \hspace{1cm} \text{(B.10)}

**Appendix C. Correlation functions**

In calculating \( f^{(4)}_{\text{in}} \), note that
\[ (Df) \cdot (Dg)^* = \frac{1}{f^2} [(\Pi_+ f)(\Pi_+ g)^* + (\Pi_- f)(\Pi_- g)^*] \]  \hspace{1cm} \text{(C.1)}

and
\[ \Pi_+ \phi_n = \sqrt{n + 1} \phi_{n+1}, \quad \Pi_- \phi_n = \sqrt{n} \phi_{n-1}. \]  \hspace{1cm} \text{(C.2)}

Together with partial integration,
\[ \langle (\Pi_+ \phi_n) (\phi_m^*) (\phi_p^*) \rangle = \langle \phi_n (\Pi_- \phi_m^*) (\phi_p^*) \rangle - \langle \phi_n \phi_m^* (\Pi_+ \phi_p^*) \rangle + \langle \phi_n \phi_m^* \phi_p^* (\Pi_- \phi_q^*) \rangle, \]  \hspace{1cm} \text{(C.3)}

these equations give
\[ \frac{1}{f^2} (\phi_m^* \phi_0 (D\phi_0) \cdot (D\phi_0)^* = \langle |\phi_0|^2 |\phi_0|^2 \rangle - \langle |\phi_0|^2 |\phi_1|^2 \rangle, \]
\[ \frac{1}{f^2} (\phi_m^* \phi_2 (D\phi_0) \cdot (D\phi_2)^* = \langle |\phi_0|^2 |\phi_2|^2 \rangle - \langle |\phi_0|^2 |\phi_3|^2 \rangle. \]  \hspace{1cm} \text{(C.4)}

\[ \frac{1}{f^2} (\phi_m^* \phi_2 (D\phi_2) \cdot (D\phi_2)^* = \langle |\phi_0|^2 |\phi_2|^2 \rangle + \langle |\phi_0|^2 |\phi_3|^2 \rangle + \langle |\phi_1|^2 |\phi_2|^2 \rangle + \langle |\phi_1|^2 |\phi_3|^2 \rangle \]
\[ - 3\langle |\phi_0|^2 |\phi_3|^2 \rangle - 3\langle |\phi_1|^2 |\phi_3|^2 \rangle + 3\langle |\phi_2|^2 |\phi_3|^2 \rangle - 3\langle |\phi_2|^2 |\phi_3|^2 \rangle. \]

These can be evaluated using
\[ \frac{\langle |\phi_p|^2 |\phi_q|^2 \rangle}{\langle |\phi_0|^2 \rangle} = \sum_{r,s} L_n^0 (k_{rs}^2/2) L_q^0 (k_{rs}^2/2) e^{-k_{rs}^2/2}, \]
\[ \frac{\langle |\phi_p|^2 |\phi_2|^2 \rangle}{\langle |\phi_0|^2 \rangle} = \sum_{r,s} L_n^0 (k_{rs}^2/2) L_q^0 (k_{rs}^2/2) e^{-k_{rs}^2/2} \cos (k_{rs} \cdot \tau), \]  \hspace{1cm} \text{(C.5)}

where \( L_n^0 \) is a Laguerre polynomial of \( n \)th order and \( k_{rs} = (\sqrt{2\pi} \sigma r, \sqrt{2\pi} \sigma (s - \varsigma r)) \).
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