Asymptotic behaviour of a solution for Kadomtsev-Petviashvili-2 equation *

O.M. Kiselev
Institute of Mathematics
112 Chernyshevsky str., Ufa, 450000, Russia
E-Mail: ok@imat.rb.ru

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Abstract

An asymptotic behaviour of solution of Kadomtsev-Petviashvili-2 equation is obtained as \( t \to \infty \) uniformly with respect to spatial variables.

1 Introduction

Properties of solutions and process of integration for the Kadomtsev-Petviashvili equation (KP) \[\text{(1)}\]:

\[ \partial_x(\partial_t u + 6u \partial_x u + \partial_x^3 u) = -3\sigma^2 \partial_y^2 u \]

depend on sign of \( \sigma^2 \). This equation is called KP-2 if \( \sigma^2 = 1 \).

In this work the asymptotic behaviour of decreasing solution of equation KP-2 is obtained as \( t \to \infty \). The main term of the asymptotics has an order by \( O(t^{-1}) \) and fast oscillates. An envelope of these oscillations depends on \( \xi = x/t \) and \( \eta = y/t \). Results of the work are formulated in terms of scattering data for auxiliary linear problem, which is associated with equation KP-2 in the inverse scattering transform (IST) \[\text{2} \  \text{3}\].

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The equation KP plays an important role in modern mathematical physics. Many applications of these equations are known in plasma physics, water waves and other fields of waves process [1]. Therefore questions about solvability of this equation in different functional classes were studied in detail. In particular, the existence of global solution of Cauchy problem in class of distribution functions was proved in [5]. A norm of solution in Sobolev space was estimated and an asymptotics of a solution were obtained as \( t \to \infty \) for KP-like equation but with more high nonlinearity, which called generalized equation KP, in [6].

The IST formalism allows to reduce a constructing of the solution of nonlinear integrable equation into solving of linear problems. One of most important achievements of IST is the constructing of asymptotic behaviours of solutions as \( t \to \infty \). These results are well-known for 1+1-dimensional equations (one spatial and one temporal variable) [7]-[13]. Asymptotic behaviours of solutions for (2+1)-dimensional equations were studied not so in detail. Rigorous results about temporal asymptotics of solutions, which is nonuniform with respect to spatial variables, for a special class of nondecreasing solutions were obtained in [14], [15]. In work [16] an formal asymptotics of decreasing solution of equation KP-1, which is nonuniform with respect to spatial variables, was constructed. Results about asymptotic behaviour of decreasing solutions of equation KP-2 as \( t \to \infty \) and about a remainder of this asymptotics, which are uniform with respect to spatial variables, are obtained in present work.

The formalism IST for solving of Cauchy problem for equation KP-2 was presented in [17]. This formalism will be used in this work. Below we remind basic steps of solving of Cauchy problem for equation KP-2 by IST.

Let us denote an initial condition for equation KP-2 as:

\[ u|_{t=0} = u_0(x,y). \] (2)

The process of solving of the Cauchy problem (1), (2) consists of several steps.

First step is solving direct scattering problem. On this step a boundary problem is solved for function \( \varphi \):

\[ -\partial_y \varphi + \partial_x^2 \varphi + 2ik\partial_x \varphi + u\varphi = 0, \quad \varphi|_{|k|\to\infty} = 1, \] (3)

and scattering data are constructed by a formula:

\[ F(k) = \frac{\text{sgn}(-\text{Re}(k))}{2\pi} \int \int_{\mathbb{R}^2} dx \, dy \, u_0(x,y) \varphi(x,y,k,0) \times \]

\[ \cdots \]
\[ \times \exp(-i(k + \bar{k})x - (k^2 - \bar{k}^2)y). \]  

(4)

It is useful to note, if \( u \) is real, then the function \( F(k) \) has property \( F(-\bar{k}) = -\bar{F}(k) \). This follows from formulas (3) and (4).

On the next step an evolution of scattering data is determined. This evolution is very simple:

\[ \mathcal{F}(k; t) = F(k) \exp(4it(k^3 + \bar{k}^3)). \]

Third step is solving of inverse scattering problem. This is reduced to so-called \( \bar{D} \)-problem:

\[ \begin{align*}
\partial_k \varphi &= \psi F(-\bar{k}) \exp(itS), \\
\partial_k \psi &= -\varphi F(k) \exp(-itS); \\
\end{align*} \]

\begin{align*}
(\varphi \psi) \big|_{|k| \to \infty} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{align*}

(5)

where \( S = 4(k^3 + \bar{k}^3) + (k + \bar{k})\xi - i(k^2 - \bar{k}^2)\eta, \quad \xi = x/t, \quad \eta = y/t \), the function \( F(k) \) is nonanalytic with respect to complex variable \( k \in \mathbb{C} \). Solving of this problem allows to obtain the functions \( \varphi \) and \( \psi \) at any time, if we know the evolution of scattering data.

At last we can obtain the solution of the Cauchy problem by using the formula [17]:

\[ u(x, y, t) = \partial_x \int \int_{\mathbb{C}} dk \wedge d\bar{k} F(k) \psi(k, x, y, t) \exp(itS). \]

(6)

However, this successive method of solving for Cauchy problems for general initial conditions allows to obtain very implicit answer with respect to initial data. Therefore the problem (5) and integral (6) are used usually for solving of integrable nonlinear equations and the scattering data is taking from some suitable functional class (see, for example, [14]-[16]). On one-hand side this gives for studying a functional class of solutions, but on the other hand-side this leads to implicit restrictions on class of initial conditions.

The auxiliary linear problem (3) and some related problems were studied in [18]-[23]. In the work [23] it was proved that if the function \( u_0(x, y) \) decreases exponentially, then the direct scattering problem (3) is solvable for \( \forall k \in \mathbb{C} \).

To construct the asymptotics of \( u \) as \( t \to \infty \) we use the asymptotics of \( \psi \) and evaluate an asymptotics of the integral (6) by stationary phase method [24]. As it turned out, the main term of asymptotics of the function \( u \) may be defined by using only the main term of an asymptotic expansion of \( \psi \) as \( t \to \infty \). Taking into account the
fast oscillating coefficients of the system from (5) one can guess, that
\(\psi = 1 + o(1)\) as \(t \to \infty\). But we cannot say something definitely about
the asymptotic behaviour of the function \(u\), until we do not know
analytic properties and an order of the remainder of asymptotics of \(\psi\)
more precisely. Thus we come to studying an asymptotic behaviour
of a solution of the \(\bar{D}\)-problem (5).

The asymptotic behaviour of the solution of the \(\bar{D}\)-problem as
\(t \to \infty\) with continuous and fast oscillated coefficients was obtained in
[25]. Here we study the \(\bar{D}\)-problem with discontinuous coefficients on
imaginary axis of complex parameter \(k\). Constructing of the asymptotics of solution for such problem is more complicated. Not only
stationary points of phase function of an oscillated exponent define a
structure of the asymptotic solution for (5) as in [25] (see also [26]),
but the location of these stationary points with respect to line of dis-
continuity of coefficients of the equations (5) as well. It contributes
additional difficulties into evaluating and leads to changes in results.
The uniform asymptotics of the solution of the problem (5) is con-
structed by matching method [27].

2 Main result

Theorem 1 Let \((1 + |k|)F \in L_1 \cap C, \partial^\alpha F \in L_1 \cap C\) as \(Re(k) \neq 0, \)
\(|a| \leq 2\) and:
\[
\sup_{z \in C} \int \int_{\mathbb{R}^2} d\kappa d\lambda \left| \frac{F(\kappa + i\lambda)}{\kappa + i\lambda - z} \right| < 2\pi,
\]
then the solution of the Cauchy problem of equation KP-2 for cor-
responding initial condition exists as \(\forall t > 0\). The asymptotic behaviour
of the solution as \(t \to \infty\) differs in different domains of variables
\((x,y,t)\):
as \(-(12\xi + \eta^2)t^{1/3} \gg 1\):
\[
\begin{align*}
\psi(x,y,t) &= -4t^{-1} \pi \frac{1}{12i \sqrt{-\eta^2 - 12\xi}} f \left( \frac{1}{2} \sqrt{-\eta^2 - 12\xi + \frac{i\eta}{12}} \right) \times \\
&\quad \times \exp \left( -11it \sqrt{-\frac{\eta^2}{t^2} - 12\frac{\xi}{t}} \right) + c.c. + o(1).
\end{align*}
\]
as \((12\xi + \eta^2)t^{1/3} \gg 1\):
\[
\psi = o(t^{-1});
\]
as $|12\xi + 12\eta^2| \ll 1$:

$$u(x, y, t) = 8it^{-1}\sqrt{\pi}f(i\eta/12)\left(\int_0^\infty dp_1\sqrt{p_1}\cos\left(8p_1^3 - zp_1\right) + \right.$$

$$\left. + \int_0^\infty dp_1\sqrt{p_1}\sin\left(8p_1^3 - zp_1\right)\right) + o(t^{-1}).$$

Here $\xi = x/t$, $\eta = y/t$,

$$z = 8\left(\frac{y^2}{12t^{4/3}} + \frac{x}{t^{1/3}}\right);$$

$$f(k) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} dx\, dy\, u_0(x, y)\varphi(x, y, k, 0) \times$$

$$\times \exp(-i(k + \bar{k})x - (k^2 - \bar{k}^2)y).$$

The domains of validity for the asymptotics of the solution of equation KP-2 are intersected and therefore they give combined asymptotics of the solution uniformly on plane of $x, y$.

### 3 An analytic behaviours of the scattering data

In this section we demonstrate analytic behaviour of the scattering data which corresponds to sufficiently smooth and decreasing initial condition (2).

First of all we show, that the solution of (5) exists.

**Theorem 2** Let $F(k)$ be such, that the condition (7) is fulfilled, then the solution of (5) exists in a space of continuous vector-functions bounded when $k \in \mathbb{C}$.

**Proof.** Let us consider a system of integral equations which is equivalent to (5):

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + G[F] \begin{pmatrix} \varphi \\ \psi \end{pmatrix},$$

where

$$G[F]V = \int \int_{m \in \mathbb{C}} dm \wedge d\bar{m} \times$$

$$\begin{pmatrix} 0 & \frac{F(-m)}{k-m} \exp(itS) \\ \frac{F(m)}{k-m} \exp(-itS) & 0 \end{pmatrix} V(m, \xi, \eta, t).$$
Using well-known results about integral operators (see, for example \cite{29}) one can show that the operator $G[F]$ transforms the space of continuous vector-functions into itself.

The operator $G[F]$ is contracting operator. It is follows from inequality (8). Hence we obtain the theorem statement.

To construct the asymptotics we must study smoothness of scattering data $F(k)$ in neighborhood of line of discontinuity $\text{Re}(k) = 0$.

**Lemma 1** Let $u_0(x, y)$ be a finite function, then in neighborhood of some point $k' \in \mathbb{C}$ the scattering data have to be represented in the form:

$$F(k) = \text{sgn}[\text{Re}(-k)] \sum_{|\alpha|=0}^2 f_{\alpha_1 \alpha_2}(k') (k - k')^{\alpha_1} (\overline{k - k'})^{\alpha_2} + O(|k - k'|^3),$$

where $f_{\alpha_1 \alpha_2}(k')$ are continuous functions, when $\text{Re}(k') \neq 0$ and

$$f_{\alpha_1 \alpha_2}(k') = f^{(1)}_{\alpha_1 \alpha_2}(k') + \text{sgn}[\text{Re}(-k)] f^{(2)}_{\alpha_1 \alpha_2}(k'),$$

when $\text{Re}(k') = 0$.

**Proof** of this lemma consists of successive evaluating of partial derivatives of scattering data with respect to $k$ and $\overline{k}$. For example let us evaluate $f^{(j)}_{10}(k')$ and $f^{(j)}_{10}(k')$ as $\text{Re}(k') = 0$.

Denote

$$f(k) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} dx dy u_0(x, y) \varphi(x, y, k, 0) \exp(-i(k + \overline{k})x - (k^2 - \overline{k}^2)y).$$

Then $f^{(1)}_{00}(k') = f(k')$, $f^{(0)}_{10}(k') \equiv 0$.

Evaluate $f_{10}(k') = f^{(1)}_{10}(k') + \text{sgn}[\text{Re}(-k)] f^{(2)}_{10}(k')$.

$$\partial_k f(k) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} dx dy u_0(x, y) \exp(-i(k + \overline{k})x - (k^2 - \overline{k}^2)y) \times$$

$$[(ix - 2ky) \varphi(x, y, k, 0) + \partial_k \varphi(x, y, k, 0)].$$

To evaluate the derivative $\partial_k \varphi$ we use the integral equation for the function $\varphi$:

$$\partial_k \varphi = \frac{-1}{2i\pi} \text{V.P.} \int \int_{\mathbb{C}} \frac{dm \wedge d\overline{m}}{(k - m)^2} F(-\overline{m}) \times$$

$$\exp(i(m + \overline{m}) - (m^2 - \overline{m}^2)y) \psi(x, y, m).$$
The functions $F(-\bar{m})$ and $\psi(x, y, m)$ are smooth on the left and right-hand sides of complex plane of $m$ and one can show that the integral exists. Let us represent it into more convenient form. For this we rewrite the integral into sum of two integrals over left-hand side and right-hand side of complex plane. Let us integrate by parts these integrals. As a result we obtain

$$\partial_{k} \varphi = \frac{-1}{2i\pi} \sum_{\pm} \int_{\partial \Omega^{\pm}} \frac{(\pm) d\bar{m}}{k - m} [f(-\bar{m}) \times \exp((m + \bar{m})x - (m^2 - \bar{m}^2)y)] -$$

$$\frac{-1}{2i\pi} \sum_{\pm} \int \int_{\Omega^{\pm}} \frac{d\bar{m} \wedge d\tilde{m}}{k - m} \left[ (ix - 2my)(\pm)f(-\bar{m}) \psi(x, y, m) + (\pm) \partial_{\bar{m}} f(-\bar{m}) \psi(x, y, m) - \varphi(x, y, m)f(-\bar{m})f(m) \right] \times \exp((m + \bar{m})x - (m^2 - \bar{m}^2)y).$$

Here $\Omega^{\pm}$ is right-hand side (+) and left-hand side (-) of complex plane. We mean the integral over $\partial \Omega^{+}$ as sum of integral over right-hand side and left-hand side of the circle with center at origin of coordinates and with large radius (which tends into infinity) and of improper integral over imaginary line and of an integral over the circle with small radius $\varepsilon$ with center at $k = m$. The direction of the integration over $\partial \Omega^{+}$ is determined by standard way.

Consider the sum of the integrals over $\partial \Omega^{\pm}$. Each of the integrals over large half-circles is equal to zero as $R \to \infty$ because the integrand decreases. The sum of the integrals over the imaginary axis gives doubled integral over the imaginary axis in positive direction. The integral over small circle equals zero as $\varepsilon \to 0$.

Let us evaluate the derivative $\partial_{\bar{m}} f(-\bar{m})$ of the integrand of the double integrals. For this we note that $\varphi(x, y, k, 0) = \psi(x, y, -\bar{k}, 0)$, hence $f(-\bar{m})$ has to be represented by the function $\psi(x, y, m, 0)$ and then:

$$\partial_{\bar{m}} f(-\bar{m}) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} dx \ dy \ u_0(x, y) [ - f(m) \varphi(x, y, m, 0) +$$

$$+ (ix - 2my) \psi(x, y, m, 0)] \exp(-i(m + \bar{m})x - (m^2 - \bar{m}^2)y).$$

So we obtain a formula $f_{10}(k')$:

$$f_{10}(k') = f^{(1)}_{10}(k') + \text{sgn}[\text{Re}(-k)] f^{(2)}_{10}(k'),$$

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where
\[
f^{(1)}_{10}(k') = -f(-\bar{k}')\psi(x, y, k', 0) \frac{1}{2\pi i} \int_{\mathbb{R}^2} dx dy u_0(x, y) \times \left[ (ix - 2ky)\phi(x, y, k, 0) + \left( -\frac{1}{2i\pi} \sum_{\pm} V.P. \int_{-i\infty}^{i\infty} \frac{dm}{k' - m} \right) (\pm f(-\bar{m})\psi(x, y, m, 0) - \frac{1}{2i\pi} \sum_{\pm} \int_{\Omega_{\pm}} \frac{dm}{k' - m} \left[ (ix - 2my)(\pm f(-\bar{m})\psi(x, y, m)) + (\pm)\partial\bar{m}f(-\bar{m})\psi(x, y, m) - \varphi(x, y, m)f(-\bar{m})f(m) \right] \times \exp(i(m + \bar{m})x - (m^2 - \bar{m}^2)y) \right] \times \exp(-i(k' + \bar{k}')x - (k'^2 - \bar{k}'^2)y) \right] \times \exp(2i(k' + \bar{k}')x - 2(k'^2 - \bar{k}'^2)y).
\]

\[f^{(2)}_{10} = -\frac{1}{\pi} \int_{\mathbb{R}^2} dx dy u_0(x, y) f(-\bar{k}')\psi(x, y, k', 0).\]

An expression of \( f_{01}(k') = f^{(1)}_{01}(k') + \text{sgn}[\text{Re}(-k)]f^{(2)}_{01}(k') \) has the form:

\[f^{(1)}_{01}(k') = \frac{1}{2\pi i} \int_{\mathbb{R}^2} dx dy u_0(x, y)(-ix + 2\bar{k}'y)\varphi(x, y, k', 0) \times \exp(i(k' + \bar{k}')x - (k'^2 - \bar{k}'^2)y),\]
\[f^{(2)}_{01}(k') = \frac{1}{2\pi i} \int_{\mathbb{R}^2} dx dy u_0(x, y)\psi(x, y, k', 0) \times \exp(2i(k' + \bar{k}')x - 2(k'^2 - \bar{k}'^2)y).\]

The expressions of other coefficients of the expansion have to be evaluated by the same way. The lemma is proved.

### 4 Asymptotic solution of \( \bar{D} \)-problem

In this section we construct an asymptotic solution as \( t \to \infty \) of the \( \bar{D} \)-problem:

\[\partial_{\bar{k}}\mu = \nu F(k) \exp(itS), \quad \partial_{\bar{k}}\nu = -\mu F(-\bar{k}) \exp(-itS); \quad \left( \begin{array}{c} \mu \\ \nu \end{array} \right) \bigg|_{|k|\to\infty} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right). \]
The solution of the problem (5) has to be obtained using the solution of (8) and the formula:

\[
\begin{pmatrix}
\varphi \\
\psi
\end{pmatrix} =
\begin{pmatrix}
\mu(k, \xi, \eta, t) \\
\nu(k, \xi, \eta, t)
\end{pmatrix} - \begin{pmatrix}
-\nu(-\bar{k}, \xi, \eta, t) \\
\mu(-\bar{k}, \xi, \eta, t)
\end{pmatrix}.
\tag{9}
\]

The asymptotic solution of the problem (8) is combined. Here we construct the asymptotics as \( t \to \infty \) uniformly with respect to all parameters. The stationary points of the functions \( S(k, \bar{k}, \xi, \eta) \) with respect to parameters \( k, \bar{k} \) plays the important role in these constructions.

The asymptotic expansion of the solutions is constructed using the inverse powers of large parameter \( t \) as a asymptotic sequence of asymptotic expansion outside of small neighborhoods of the stationary points of the function. Near the stationary points the asymptotic sequence has the form \( t^{-n/2} \), where \( n = 0, 1, 2, \ldots \). Following the terminology of matching method \[27\], we call the asymptotics outside of small neighborhoods of stationary points by outer expansion and we call by interior expansion the asymptotics near the stationary points. The right-hand side in the system of equations (8) is discontinuous on the line \( \text{Re}(k) = 0 \), therefore the asymptotic solution is differentiable out of the line \( \text{Re}(k) = 0 \).

The domains of validity for interior and exterior asymptotic expansions are intersected. This fact is used by matching method in order to construct unique combined asymptotic expansions.

The phase function \( S \) depends on two parameters \((\xi, \eta) \in \mathbb{R}^2\). On the curve \( 12\xi + \eta^2 = 0 \) the confluence of two stationary points of the function \( S \) occurs. In this case we have one confluent stationary point. The structure of the asymptotic expansion of solution of (8) is changed here. As \( |12\xi + \eta^2| \ll 1 \) the expansion is constructed on the powers of \( t^{-n/3}, n = 0, 1, 2, \ldots \) as a asymptotic sequence.

The formulas for the uniform asymptotic expansions of the solution of (8) are large and it seems convenient to formulate the results about general case in section 4.1 and about confluence case in section 4.2 for convenience.

4.1 Asymptotics in a general case

In this section the combined asymptotic solution of (8) is constructed when the phase function \( S \) has nondegenerate stationary points \( k = k_1 \) and \( k = k_2 \). Here we suppose that \( \forall k \in \mathbb{C}, \, t^{1/3}|\partial_k^3 S|_{k = k_1, 2} \gg 1 \) as
\( t \to \infty \). This leads to restriction on values of the parameters \( \xi \) and \( \eta \), namely, \( t^{1/3}|12\xi + \eta^2| \gg 1 \). The asymptotic expansion which is uniform with respect to \( k \in \mathbb{C} \) is formulated in the end of this section.

To construct the combined asymptotic solution which is valuable as \( k \in \mathbb{C} \) we obtain exterior and interior asymptotic expansions. These expansions are valid outside of small neighborhoods of \( k_j, j = 1, 2 \) and in the small neighborhoods of \( k_j \) respectively. Denote: \( \theta = \sqrt{-12\xi - \eta^2} \). Then one can obtain an expressions for the stationary points of \( S \) by using the parameter \( \theta \): \( k_1 = \frac{1}{\sqrt{2}}(i\eta + \theta), \quad k_2 = \frac{1}{\sqrt{2}}(i\eta - \theta) \).

### 4.1.1 The stationary points outside of the break line

Consider the case when \( \text{Re}(\theta) \neq 0 \), i.e. when the stationary points \( k_{1,2} \) are outside of the discontinuity line \( \text{Re}(k) = 0 \). Let us formulate a result of this section about the combined asymptotic solution:

**Lemma 2** Let the system of the equations (8) have not the homogeneous solutions, \( F(k) \in C^2 \cap L_1 \) as \( \text{Re}(k) \neq 0 \) and the parameters \( \xi \) and \( \eta \) satisfy the inequality \( t^{1/3}|\theta|^2 \gg 1 \), then:

1. When \( \sqrt{t|\theta||k - k_{1,2}|} \gg 1 \) the formal asymptotic solution of the system (8) with respect to \( \text{mod}(O(t^{-2}|\partial_k S|^{-3})) \) has the form:

   \[
   \tilde{\mu} = 1 + t^{-1} \tilde{\mu}_1(k, \xi, \eta),
   \]

   \[
   \tilde{\nu} = (t^{-1} \tilde{\nu}_1(k, \xi, \eta) + t^{-2} \tilde{\nu}_2(k, \xi, \eta)) \exp(-itS) + t^{-1} \tilde{\nu}_0(k, \xi, \eta),
   \]

   the functions \( \tilde{\mu}_1, \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_0 \) are defined by (18), (12), (16), (29);

2. When \( |\theta|^{-1}|k - k_j| \ll 1 \) the formal asymptotic solution of the system (8) with respect to \( \text{mod}(O(t^{-1}|\theta|^{-1})) \) has the form:

   \[
   \tilde{\mu} = 1 + t^{-1} \tilde{M}(l_j, \xi, \eta),
   \]

   \[
   \tilde{\nu} = (t^{-1/2} \tilde{\nu}_0(l_j, \xi, \eta) + t^{-1} \tilde{\nu}_1(l_j, \xi, \eta)) \exp(-itS),
   \]

   where \( l_j, j = 1, 2 \), are defined by formula:

   \[
   l_j = \sqrt{t(k - k_j)} \sqrt{\frac{\partial^2 S_j}{2}} + 4(k - k_j),
   \]

   the functions \( \tilde{\mu}_1(l_j, \xi, \eta), \tilde{\nu}_1(l_j, \xi, \eta) \) are defined by (31), (26), (30).
Proof. Let us construct the external asymptotic solution in the form:

\[
\mu^{ex} = 1 + t^{-1} \mu (k, \xi, \eta) + t^{-2} \mu_1 (k, \xi, \eta) \exp(itS) + \ldots, \tag{10}
\]

\[
\nu^{ex} = (t^{-1} \nu_1 (k, \xi, \eta) + t^{-2} \nu (k, \xi, \eta) + \ldots) \exp(-itS) +
+ t^{-1} \nu_0 (k, \xi, \eta) + \ldots. \tag{11}
\]

Let us substitute the formulas (10) and (11) into (8), equate coefficients with equal power of \( t \). As a result we obtain:

\[
-i \partial_k S \nu_1 (k, \xi, \eta) = -\text{sgn}(\text{Re}(k)) f(-\bar{k}),
\]

\[
\partial_k \mu (k, \xi, \eta) + i \partial_k S \mu_1 (k, \xi, \eta) \exp(itS) =
-\text{sgn}(\text{Re}(k)) f(k) \nu_1 (k, \xi, \eta) +
\text{sgn}(\text{Re}(k)) f(k) \nu_0 (k, \xi, \eta) \exp(itS),
\]

\[
-i \partial_k S \exp(-itS) \nu_1 (k, \xi, \eta) + \partial_k \nu_0 (k, \xi, \eta) =
( - \text{sgn}(\text{Re}(k)) f(-\bar{k}) \mu (k, \xi, \eta) \partial_k \nu_1 (k, \xi, \eta) \exp(-itS)).
\]

If we equate to zero the coefficients of oscillated terms and nonoscillated terms of the expansions (10) and (11) respectively, then we obtain formulas:

\[
\frac{1}{\nu} = \frac{\text{sgn}(\text{Re}(k)) f(-\bar{k})}{i \partial_k S}; \tag{12}
\]

\[
\partial_k \frac{1}{\nu_0} (k, \xi, \eta) = 0; \tag{13}
\]

\[
\partial_k \frac{1}{\nu_1} = -\frac{f(-\bar{k}) f(k)}{i \partial_k S}; \tag{14}
\]

\[
\frac{2}{\mu_1} (k, \xi, \eta) = \frac{\text{sgn}(\text{Re}(k)) f(k) \nu_0 (k, \xi, \eta)}{i \partial_k S}; \tag{15}
\]

\[
\frac{2}{\nu} = -\frac{1}{i \partial_k S} \left( -\text{sgn}(\text{Re}(k)) f(-\bar{k}) \frac{1}{\mu} \partial_k \frac{1}{\nu_1} \right). \tag{16}
\]

The function \( \frac{1}{\nu_1} \) has a jump on the imaginary axis of \( k \). This jump is \( \frac{2f(k)}{i \partial_k S} \). In order for the coefficients of the asymptotics of function \( \nu \) as
to be continuous, we add an analytic function of \( \tilde{k} \), which has the same jump on the imaginary axis in inverse direction, to \( \frac{1}{\nu} \exp(itS) \):

\[
\frac{1}{\nu} \nu'_{0}(\tilde{k}) = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} \frac{dnf(n)}{(k-n)\partial_{n}S}.
\]  

(17)

The function \( \nu'_{1} \) define an analytic function \( \nu_{0} \) of \( \tilde{k} \) uncompletely. The rest terms will be defined when we will match the exterior and interior expansion.

A Cauchy-Green formula gives solution of (14) which is decreasing as \( |k| \to \infty \) and bounded when \( \forall k \in \mathbb{C} \):

\[
\frac{1}{\nu} = \frac{-1}{2i\pi} \int \int_{\mathbb{C}} \frac{dp \wedge d\bar{p} f(-\bar{p})f(p)}{k-p} i\partial_{p}S.
\]  

(18)

We can obtain the domain of values of \( k \), where the external expansion is valid, using the condition \( |t^{-1} \frac{1}{\nu} / (t^{-2} \bar{\tilde{p}})\| \gg 1 \). As a result of calculations we obtain:

\[
\sqrt{t|\theta||k - k_j|} \gg 1.
\]

Let us construct the interior asymptotic solution which is valid in the neighborhood of point \( k_j \) as \( j = 1, 2 \). Denote new scaling variable by \( l_j \):

\[
l_j^2 = t(k-k_j)^2 \frac{\partial^2 S_j}{2} + 4t(k-k_j)^3. \]  

(19)

When \( |l_j| \) is not so large \( (t^{1/2}|\partial^2 S_j|^{3/2} \gg |l_j|) \) an asymptotic formula is valid as \( t \to \infty \)

\[
(k - k_j) = \sqrt{\frac{2}{t\partial^2 S_j} l_j} - \frac{8}{t(\partial^2 S_j)^2} l_j^2 + \ldots.
\]  

(20)

Rewrite the system (8) into terms of new variables \( l_j \) and \( \bar{l}_j \). Substitute the asymptotic expansion

\[
\mu^{in} = 1 + t^{-1} \frac{1}{M_j} (l_j, \xi, \eta) + \ldots,
\]  

(21)

\[
\nu^{in} = (t^{-1/2} \frac{1}{N_j} (l_j, \xi, \eta) + t^{-1} \frac{2}{N_j} (l_j, \xi, \eta) + \ldots) \exp(-itS),
\]  

(22)

into system (8).
As a result one can obtain equations for the coefficients of expansions (21) and (22):

\[
\partial_{l_j} \frac{1}{2} N_j - 2i l_j \frac{1}{2} N_j = -\sqrt{\frac{2}{\partial_k^2 S_j}} \operatorname{sgn}(\Re(k_j)) f(-\bar{k}_j); \tag{23}
\]

\[
\partial_{l_j} \frac{2}{2} N_j - 2i l_j \frac{2}{2} N_j = \operatorname{sgn}(\Re(k_j)) \left[ \left( \frac{16}{(\partial_k^2 S_j)^2} f(-\bar{k}_j) - \frac{2}{\partial_k^2 S_j} f_{10}(-\bar{k}_j) \right) l_j - \frac{2 l_j}{|\partial_k^2 S_j|} f_{01}(-\bar{k}_j) \right]; \tag{24}
\]

\[
\partial_{l_j} \frac{1}{2} M_j = -\operatorname{sgn}(\Re(k_j)) f(k_j) \frac{1}{2} N_j \sqrt{\frac{2}{\partial_k^2 S_j}}. \tag{25}
\]

Construct the solutions of the equations (23)–(25). Since the external expansion has not the terms of order \( t^{-1/2} \), then the boundary condition for function \( \frac{1}{2} N_1 (l_j, \xi, \eta) \) has the form:

\[
\frac{1}{2} N_j (l_j, \xi, \eta)|_{l_j \to \infty} = 0.
\]

The solution of the boundary condition for \( \frac{1}{2} N_j (l_j, x, y) \) is evaluated by formula:

\[
\frac{1}{2} N_j (l_j, \xi, \eta) = \operatorname{sgn}(\Re(k_j)) \sqrt{2} f(-\bar{k}_j) \exp(i(l_j^2 + \bar{l}_j^2)) \times \frac{\sqrt{2} f(-\bar{k}_j) \exp(i(l_j^2 + \bar{l}_j^2))}{2i\pi} \times \int \int_C \frac{dn \wedge d\bar{n}}{l_j - n} \exp(-i(n^2 + \bar{n}^2)); \tag{26}
\]

The solution of nonhomogeneous equation for the function \( \frac{2}{2} N_j (l_j, \xi, \eta) \) has the form:

\[
\frac{2}{2} N_j^s (l_j, \xi, \eta) = \operatorname{sgn}(\Re(k_j)) \left[ \frac{8 f(-\bar{k}_j)}{(\partial_k^2 S_j)^2} - \frac{f_{10}(-\bar{k}_j)}{\partial_k^2 S_j} \right. \left. \frac{2 f_{01}(-\bar{k}_j)}{|\partial_k^2 S_j|} \exp(i(l_j^2 + \bar{l}_j^2)) \right.
\]

\[
- l_j \frac{2 f_{01}(-\bar{k}_j)}{|\partial_k^2 S_j|} \frac{\exp(i(l_j^2 + \bar{l}_j^2))}{2i\pi} \int \int_C \frac{dn \wedge d\bar{n}}{l_j - n} \exp(-i(n^2 + \bar{n}^2)) \right]; \tag{27}
\]
The partial solution of the equation for the function $\frac{1}{M_j}(l_j, \xi, \eta)$ has to be written as four-multiple integral:

$$\frac{1}{M_j^s} = -2f(k_j)f(-\bar{k}_j)J,$$

where

$$J = \frac{1}{2i\pi} \int \int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{l_j - n} \exp(2i\pi \frac{i(n^2 + \bar{n}^2)}{2i\pi}) \times \int \int_{\mathbb{C}} \frac{dn \wedge d\bar{m}}{n - \bar{m}} \exp(-i(m^2 + \bar{m}^2)).$$

It is possible to reduce this four-multiple integral into two-multiple integral (see Appendix):

$$J = \bar{l}_j \exp(2i\pi \frac{i(l_j^2 + \bar{l}_j^2)}{2i\pi}) \int \int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{l_j - n} \exp(-i(n^2 + \bar{n}^2)) - \exp(2i\pi \frac{i(l_j^2 + \bar{l}_j^2)}{2i\pi}).$$

There exists a domain of values of complex parameter $k$, in which the external and internal asymptotic expansions of solution for the problem (8) are valid. In this domain these asymptotic expansions are equal up to the terms of order $o(t^{-1})$. In the domain, where \( t^{1/2}|k - k_j| \gg 1 \) and \(|k - k_j| \leq t^{-1/4}\), the external and internal expansion are valid. We compute asymptotics of external expansion as $k \to k_j$ and internal expansion as $|l_j| \to \infty$.

Let us present the asymptotics of the functions $\frac{1}{N_j}$ and $\bar{N}_j^s$ as $|l_j| \to \infty$:

$$\frac{1}{N_j}|_{|l_j|\to\infty} = -\text{sgn}(\text{Re}(k_j)) \sqrt{\frac{2}{\partial_k^2 S_j}} f(-\bar{k}_j) \times \left( \frac{1}{2il_j} + \frac{\exp(2i\pi \frac{i(l_j^2 + \bar{l}_j^2)}{l_j})}{\bar{l}_j} + O(|l_j|^{-3}) \right).$$

$$\bar{N}_j(l_j, \xi, \eta)|_{|l_j|\to\infty} = \text{sgn}(\text{Re}(k_j)) \left[ \frac{8if(-\bar{k}_j)}{(\partial_k^2 S_j)^2} - i\frac{f_{10}(-\bar{k}_j)}{\partial_k^2 S_j} \right] \frac{\bar{l}_j}{2il_j} + \frac{\bar{l}_j}{2il_j} \frac{-2f_{01}(-\bar{k}_j)}{|\partial_k^2 S_j|} + O(|l_j|^{-1}).$$
The matching condition for $\tilde{\nu}$ means that in the domain $t^{1/3}|k-k_j| \gg 1$ and $|k-k_j|=o(1)$:

\[
\left(t^{-1/2} \tilde{N}_j (l_j, \xi, \eta) + t^{-1} \tilde{N}_j (l_j, \xi, \eta)\right) \exp(-itS) -
\]

\[
t^{-1} \left( \tilde{b}_1 (k, \xi, \eta) \exp(-itS) + \tilde{b}_0 (k, \xi, \eta) \right) = o(t^{-1}).
\]

Using the asymptotics of interior expansions of $\tilde{N}_j$, which are rewritten in the terms of external variable $k$ as $k \to k_{1,2}$, and external expansion of $\tilde{b}_0$, we can obtain:

\[
\tilde{\nu}_0 (k, x, y) = \tilde{\nu}_0'(\bar{k}) + \frac{2f(-k_1)}{\partial^2_k S_1(k-k_1)} \exp(itS) \frac{-2f(-k_2)}{\partial^2_k S_2(k-k_2)},
\]

where the function $\tilde{\nu}_0'(\bar{k})$ is defined by (17);

\[
\tilde{N}_j = \tilde{N}_j^s + \tilde{\nu}_0'(k_j) \exp(i(l_j^2 + \bar{l}_j^2)) \exp(itS_j) +
\]

\[
+ \frac{2\text{sgn}(\text{Re}(k_m)) f(-k_m)}{\partial^2_k S_m(k_j-k_m)} \exp(itS_m) \exp(i(l_j^2 + \bar{l}_j^2)),
\]

where $m \neq j$, the function $\tilde{N}_j^s$ is defined by (27).

Let us construct $\tilde{M}_j (l_j, \xi, \eta)$. The asymptotics of (28) as $|l_j| \to \infty$ has the form:

\[
\tilde{M}_j^s |l_j| \to \infty = \frac{l_j}{l_j} \sqrt[12]{\frac{\partial^2_k S_j}{\partial^2_k S_j}} i \frac{f(-k_j)f(k_j)}{12\pi \partial^2_k S}. \]

Evaluate the asymptotics of $\mu_0$ as $k \to k_j$:

\[
\mu_0 |k \to k_j| = \frac{(k-k_j) f(-\bar{k}_j)f(k_j)}{k-k_j} - \frac{f(k_j)f(-\bar{k}_j)}{12i(k_j-k_n)} +
\]

\[
+ \frac{1}{2i\pi} \int \int_C d\bar{p} \wedge d\bar{p} \frac{f(-p)f(p) - f(k_j)f(-\bar{k}_j)}{12i(p-k_j)^2(p-k_n)},
\]

here $n \in 1, 2$, $n \neq j$. 

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The matching condition for \( \tilde{\mu} \) when \( t^{-1/3}|k-k_j| \gg 1 \) and \( |k-k_j| = o(t^{-1/4}) \) has the form:

\[
(1 + t^{-1} \mu_0 (k, \xi, \eta))(1 + t^{-1} M_j (l_j, \xi, \eta)) = o(t^{-1}).
\]

These condition allows to define \( \frac{1}{M_j} (l_j, \xi, \eta) \):

\[
\frac{1}{M_j} (l_j, \xi, \eta) = M_j^*(l_j, \xi, \eta) + C_j(\xi, \eta).
\] (31)

Here the function \( M_j^*(l_j, \xi, \eta) \) is defined by (28), \( C_j(\xi, \eta) \) has the form:

\[
C_j(\xi, \eta) = - f(k_j) f(-\bar{k}_j) + \frac{1}{2\pi i} \int \int_{\mathbb{C}} dp \wedge dp \frac{f(-\bar{p}) f(p) - f(k_j) f(-\bar{k}_j)}{12i(p-k_j)^2(p-k_n)},
\] (32)

where \( n \in 1, 2, n \neq j \).

Thus we have constructed the interior expansion in neighborhood of nondegenerate stationary point of \( S \) as \( |12\xi + \eta^2| \geq t^{-1/4} \).

Lemma is proved.

Constructed asymptotic solutions are nonuniform with respect to \( k \). But one can obtain an uniform asymptotic solution by using their combination. This uniform solution has the form (see, for example, [27]):

\[
\begin{pmatrix}
\tilde{\mu} \\
\tilde{\nu}
\end{pmatrix}
= \begin{pmatrix}
\tilde{\mu}_1 \\
\tilde{\nu}_1
\end{pmatrix} + \begin{pmatrix}
\tilde{M}_1 \\
\tilde{N}_1
\end{pmatrix} + \begin{pmatrix}
\tilde{M}_2 \\
\tilde{N}_2
\end{pmatrix} - A_{1,k} \begin{pmatrix}
\tilde{M}_1 \\
\tilde{N}_1
\end{pmatrix} - A_{1,k} \begin{pmatrix}
\tilde{M}_2 \\
\tilde{N}_2
\end{pmatrix}.
\] (33)

The result of action of operator \( A_{n,k} \) on the function \( \tilde{M} \) is defined as follows ([27]). Take the formula for \( \tilde{M}(l_j, \xi, \eta, t) \), and change the dependence on variable \( l_j \) into the dependence on variable \( k \) using the formula (14). Rewrite the sum of all terms of asymptotic expansion up to \( t \) with powers equal to \( -m \), where \( 0 \leq m \leq n \). For example for the functions \( \tilde{M}(l_1, \xi, \eta, t) \) and \( \tilde{N}(l_1, \xi, \eta, t) \) this process leads to the formulas:

\[
A_{1,k}(\tilde{M}_1) = 1 + t^{-1} \frac{\sqrt{\theta(k-k_1)^2 + 4(k-k_1)^3} f(-\bar{k}_1) f(k_1)}{\sqrt{\theta(k-k_1)^2 + 4(k-k_1)^3} 2|\partial_k^2 S_1|};
\]

\[
A_{1,k}(\tilde{N}_1) = 1 + t^{-1} \frac{\sqrt{\theta(k-k_1)^2 + 4(k-k_1)^3} f(-\bar{k}_1) f(k_1)}{\sqrt{\theta(k-k_1)^2 + 4(k-k_1)^3} 2|\partial_k^2 S_1|};
\]

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\[ A_{1,k}(\tilde{N}_1) = t^{-1} \left[ \frac{f(-\tilde{k}_1)}{2i\theta\sqrt{\theta(k-k_1)^2 + 4(k-k_1)^4}} - \frac{f(-\tilde{k}_1)\exp(-it(S-S_1))}{|\theta|\sqrt{\theta(k-k_1)^2 + 4(k-k_1)^4}} + 1 \nu' - \frac{f_0(-\tilde{k}_1)}{|\theta|} \frac{\sqrt{\theta(k-k_1)^2 + 4(k-k_1)^4}}{\sqrt{\theta(k-k_1)^2 + 4(k-k_1)^4}} \right]. \]

If we substitute (33) into (8) and evaluate a remainder, then we obtain:

**Theorem 3** The formal asymptotic solution of the problem (8) with respect to \( \text{mod}(O((t|\theta|^{-1}))) \), which is uniformly valuable when \( k \in \mathbb{C} \) and \( \theta^2t \gg 1 \), has the form (33).

### 4.1.2 The stationary points on the imaginary axis

If \( \text{Re}(\theta) = 0 \), then the stationary point of the phase function \( S \) belongs to the line, where the coefficients of the equation (8) are discontinuous. In this case constructing of the formal asymptotic solution of problem (8) differs from the asymptotic solution which was constructed before.

**Lemma 3** Let the system of equations (8) have no homogeneous solutions, \( F(k) \in C^2 \cap L_1 \) as \( \text{Re}(k) \neq 0 \); the parameters \( \xi \) and \( \eta \) are \(-t^{2/3}(12\xi + \eta^2) \gg 1\), then:

- when \( \sqrt{|\theta|}|k-k_{1,2}| \gg 1 \) the formal asymptotic solution of (8) with respect to \( \text{mod}(O(t^{-2}|\partial_k S|^{-3})) \) has the form:
  \[ \tilde{\mu} = 1 + t^{-1} \alpha(k,\xi,\eta), \]
  \[ \tilde{\nu} = (t^{-1} - 1_1 (k,\xi,\eta) + t^{-2} 2_1 (k,\xi,\eta)) \exp(-itS) + t^{-1} 1_0 (k,\xi,\eta); \]

- the functions \( \alpha, 1_1, 2_1, 1_0 \) are defined by (36), (37), (40), (46);

- when \( |\theta| |k-k_j| \ll 1 \) the formal asymptotic solution of (8) with respect to \( \text{mod}(O(t^{-1}|\theta|^{-1})) \) has the form:
  \[ \tilde{Y} = 1 + t^{-1} \tilde{Y}(l_j,\xi,\eta), \]
  \[ \tilde{Z} = (t^{-1/2} \tilde{Z}(l_j,\xi,\eta) + t^{-1} 2 \tilde{Z}(l_j,\xi,\eta)) \exp(-itS), \]

where the functions \( \tilde{Y}(l_j,\xi,\eta), \tilde{Z}(l_j,\xi,\eta), \tilde{Z}(l_j,\xi,\eta) \) are defined by (38), (44).
The proof. Let us construct the asymptotics. The external expansion is constructed similarly as in 4.1.1. The main difference is that the stationary points of the function $S$ are on the line of discontinuity of the coefficients of system (8). It leads to sufficient modifications in the formulas for asymptotics. Let us find the external expansion in the form:

$$
\mu^{ex} = 1 + t^{-1} \frac{1}{\alpha} (k, \xi, \eta) + t^{-2} \frac{2}{\alpha_1} (t, \xi, \eta) \exp(itS) + \ldots,
$$

(34)

$$
\nu^{ex} = t^{-1} (\beta (k, \xi, \eta) \exp(-itS) + \frac{1}{\beta_0} (k, \xi, \eta)) + t^{-2} (\frac{2}{\beta} \exp(-itS) + \ldots)
$$

(35)

After substituting of (34) and (35) into the system (8) we equate the coefficients with identical powers of $t$ and with oscillated and nonoscillated terms correspondingly. As a result we obtain equations for the coefficients of asymptotic expansions (34) and (35). The obvious formulas are

$$
\frac{1}{\alpha} = -\frac{1}{2\pi} \int \int_{C} \frac{f(-\bar{n})f(n)}{k-n} \, dn \wedge d\bar{n},
$$

(36)

$$
\frac{1}{\beta_1} = \frac{\text{sgn}(\text{Re}(k))}{i\partial_{\bar{k}}S} f(-\bar{k}).
$$

(37)

The function $\frac{1}{\beta_1}$ is discontinuous on the imaginary axis. We add an analytic function with the same jump on the imaginary axis in back direction then the coefficient of the expansion (35) at $t^{-1}$ is continuous:

$$
\beta'_0 = \frac{1}{\pi i \partial_{\bar{k}}S} \int_{-i\infty}^{i\infty} \frac{dn f(n) \partial_{\bar{n}}S}{(k-n) \partial_{n}S} = \frac{1}{\pi i \partial_{\bar{k}}S} \int_{-i\infty}^{i\infty} \frac{dn f(n)}{k-n}.
$$

(38)

Unlike $\nu'_0$, this function have singularities on the discontinuous line.

The formulas for the other coefficients of the expansions (34) and (35) have the forms:

$$
\frac{2}{\alpha_1} = \frac{\text{sgn}(\text{Re}(k))}{i\partial_{\bar{k}}S} f(k) \frac{1}{\beta_0} (k, \xi, \eta),
$$

(39)

$$
\frac{2}{\beta} = \frac{-1}{i\partial_{\bar{k}}S} \left( - \frac{\text{sgn}(\text{Re}(k))}{\partial_k} \frac{1}{\alpha} - \partial_k \frac{1}{\beta_1} \right).
$$

(40)

Here the analytic function $\frac{1}{\beta_0}$ is still undefined. We will finally define this function after matching of external and internal expansions.
The interior expansion in the neighborhood of the point \( k_j \) depend on the scaling variable \( l_j \). The asymptotics of the functions \( \mu \) and \( \nu \) has the same asymptotic sequence, but the equations for \( \frac{1}{2} Z_j \), \( \frac{1}{2} Z_j \) and \( \frac{1}{2} Y_j \) have discontinuous right-hand sides.

\[
\partial_{l_j} \frac{1}{2} Z_j - 2i l_j \frac{1}{2} Z_j = \sqrt{2} \partial_{k_S} S_j f(\bar{k}_j) \text{sgn}(\text{Re}(\bar{l} \exp(i\pi/4))); \quad (41)
\]

\[
\partial_{l_j} \frac{1}{2} Z_j - 2i l_j \frac{1}{2} Z_j = \text{sgn}(\text{Re}(\bar{l} \exp(i\pi/4))) \times
\]

\[
\times \left( \frac{16}{(\partial_{k_S} S_j)^2} f_{00}(-\bar{k}_j) + \frac{2i l_j}{\partial_{k_S} S_j} f_{01}^{(1)} (-\bar{k}_j) + \frac{2i l_j}{\partial_{k_S} S_j} f_{10}^{(1)} (-\bar{k}_j) + \frac{2i l_j}{\partial_{k_S} S_j} f_{01}^{(2)} (-\bar{k}_j) + \frac{2i l_j}{\partial_{k_S} S_j} f_{10}^{(2)} (-\bar{k}_j) \right). \quad (42)
\]

\[
\partial_{l_j} \frac{1}{2} Y_j = \text{sgn}(\text{Re}(l \exp(-i\pi/4))) f(k_j) \frac{1}{2} Z_j \sqrt{2} \partial_{k_S} S_j. \quad (43)
\]

Continuous partial solutions of the equations (41), (42) and (43) are obtained by Cauchy-Green formula. The formulas for the partial solutions of the equations (41) and (42) are differing from the formulas obtained in section 4.1.1 by function \( \text{sgn}(\text{Re}(\bar{n} \exp(i\pi))) \) under the sign of double integral:

\[
\frac{1}{2} Z_j \xrightarrow{s} = \mathcal{J}[\bar{H}_j](l_j), \quad \frac{2}{2} Z_j \xrightarrow{s} = \mathcal{J}[\bar{H}_j](l_j),
\]

where \( \bar{H}_j \) and \( \bar{H}_j \) are the right-hand side of the equations (41) and (42). An operator \( \mathcal{J} \) has the form:

\[
\mathcal{J}[h](l_j) = \frac{\exp(i(l_j^2 + \bar{l}_j^2))}{2i\pi} \int \int C \frac{dn \wedge d\bar{n}}{l_j - n} h(n) \exp(-i(l_j^2 + \bar{l}_j^2)).
\]

The partial solution of (43) contains of the double integral the same as in (28) and the term, which is defined by integral over the discontinuous line. The function \( \frac{1}{2} Y_j \xrightarrow{s} \) has the form:

\[
\frac{1}{2} Y_j \xrightarrow{s} = \frac{2f(k_j) f(-\bar{k}_j)}{|\partial_{k_S} S_j|} (J - 2J_{++} - 2J_{++}), \quad (45)
\]

where:

\[
J_{++} = \int \int_{\Omega^\pm} \frac{dn \wedge d\bar{n}}{l - m} \text{exp}(i(m^2 + \bar{m}^2)) \times \int \int_{\Omega^\pm} \frac{dn \wedge d\bar{n}}{n - m} \text{exp}(-i(n^2 + \bar{n}^2)).
\]
Here \( \Omega^\pm = \{ \pm \text{Re}[\exp(-i\pi/4)] > 0 \} \). Note that \( J_{++}(l) = J_{--}(-l) \).

In the Appendix we show, that we can reduce the integral \( J_{++} \) into double integral. As a result we obtain:

\[
J_{++} = -\frac{3}{4}i\pi + il \int \int_{\Omega^+} \frac{dm \wedge d\bar{m}}{il - \bar{m}} \exp(-i(m^2 + \bar{m}^2)), \text{ when } l \in \Omega^+;
\]

\[
J_{--} = i \exp(i(l^2 + \bar{l}^2)) \int \int_{\Omega^+} \frac{dm \wedge d\bar{m}}{il - \bar{m}} \exp(-i(m^2 + \bar{m}^2))
- \frac{1}{4}i\pi - \frac{1}{2}i\pi \exp(i(l^2 + \bar{l}^2)), \text{ when } l \in \Omega^-;
\]

Thus, the function \( Y_j^s \) is represented by sum of double integrals.

Let us consider matching of the external and internal asymptotic expansions. For that we need the asymptotic behaviour of the partial solutions of (41) and (42) as \( |l| \to \infty \). The asymptotic behaviour of the double integrals is evaluated in the Appendix. By using it we obtain:

\[
\frac{1}{Z^s_j} |_{|l| \to \infty} = \text{sgn}[\text{Re}(-\bar{l}_j \exp(i\pi/4))] f(-\bar{k}_j) \sqrt{\frac{2}{\partial_{\bar{k}_j}^2 S_j}} \left( \frac{-1}{2i l_j} + \exp(-i(l_j^2 + \bar{l}_j^2)) \right) + \frac{1}{2i l_j} + O(|l_j|^{-2});
\]

\[
\frac{2}{Z^s_j} |_{|l_j| \to \infty} = \text{sgn}[\text{Re}(-\bar{l}_j \exp(i\pi/4))] \times \left[ \left( \frac{-\bar{l}_j}{2i l_j} - \frac{1}{2} \exp(i(l_j^2 + \bar{l}_j^2)) \right) \frac{2i f^{(1)}_{10}}{\partial_{\bar{k}_j}^2 S_j} + \frac{1}{4} - \frac{1}{4} \exp(i(l_j^2 + \bar{l}_j^2)) \right] + \frac{2i f^{(2)}_{10}}{\partial_{\bar{k}_j}^2 S_j} \left( \frac{-\bar{l}_j}{2i l_j} + \frac{1}{2} \exp(i(l_j^2 + \bar{l}_j^2)) \right) + O(|l_j|^{-1}).
\]

The asymptotic behaviour of the function \( \beta_0'' \) is:

\[
\beta_0'' |_{k \to k_j} = \frac{1}{12i(k - k_j)(k_j - k_m)} \left[ f(k_j) \text{sgn}[\text{Re}(-\bar{k})] + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(n) - f(k_j)}{n - ik_j^2} \right] + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(n) - f(k_j)}{n - ik_j^2} \right] + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(n) - f(k_j)}{n - ik_j^2} \right]
\]
\[ -\frac{1}{12i(k_j - k_m)^2} \left[ f(k_j)\text{sgn}[\text{Re}(\tilde{k})] + \frac{1}{\pi i} \int_{-\infty}^{\infty} dn \frac{f(in) - f(k_j)}{n - ik_j} \right] + \frac{1}{12i(k_j - k_m)} \left[ f_{01}^{(1)}(k_j) + f_{01}^{(2)}(k_j)\text{sgn}[\text{Im}(i\tilde{k})] + \int_{-\infty}^{\infty} dn \frac{f(in) - f(k_j)(in - ik_j)f_{01}^{(2)}(k_j) - (in - ik_j)f_{10}^{(2)}(k_j)}{(n - ik_j)^2} \right] + o(1). \]

Using the matching conditions for the asymptotics of the external \(\nu^{ex}\) and internal \(\nu^{in}\) expansions we obtain:

\[ \frac{1}{\beta_0} - \frac{1}{\beta_0'} = \frac{C_1}{12i(k - k_1)(k_1 - k_2)} - \frac{C_2}{12i(k - k_2)(k_2 - k_1)}, \]  

(46)

where \(\frac{1}{\beta_0'}\) is defined by formula (38),

\[ C_j = \frac{1}{\pi i} \int_{-\infty}^{\infty} dn \frac{f(-n) - f(-k_j)}{n - ik_j}, \quad j = 1, 2; \]

\[ Z_j = Z_j^s, \quad \bar{Z}_j = \bar{Z}_j^s = \exp(-i(l_j^2 + \bar{l}_j^2))C_j^2, \]  

(47)

where functions \(Z_j^s\) and \(\bar{Z}_j^s\) are defined by (44),

\[ C_j^1 = \frac{i}{\partial_k^2 S_j} f_{10}^{(2)}(-\tilde{k_j}) - \frac{1}{\pi i} \int_{-\infty}^{\infty} dn \frac{f(n) - f(-\tilde{k}_j) - f_{01}^{(2)}(-\tilde{k}_j) - f_{10}^{(2)}(\tilde{k}_j)(n - ik_j)(n - ik_j)f_{10}^{(2)}(k_j)}{(n - ik_j)^2}. \]

The asymptotic behaviour of the function \(Y_j^s\) is:

\[ Y_j^s = \left(-i\pi + \frac{i\tilde{l}_j}{12\pi l_j}\right) f(-\tilde{k}_j)f(k_j) \frac{\partial_k^2 S_j}{|\partial_k^2 S_j|}. \]

The matching condition for the external expansion of \(\mu^{ex}\) and for the internal expansion of \(\mu^{in}\) gives:

\[ Y_j = Y_j^s + C_j(\xi, \eta) + i\pi \frac{f(-\tilde{k}_j)f(k_j)}{|\partial_k^2 S_j|}, \]  

(48)
where $Y_j^s$ is defined by (45), The function $C_j(\xi, \eta)$ is defined by (32).

The lemma is proved.

Constructed external and internal expansions are irregular with respect to parameters $k, \xi, \eta$. The uniform expansion with respect to $k$ when $(12\xi + \eta^2)t^{1/3} \gg 1$ has to be constructed by the same way as in preceding section:

\[
\left( \begin{array}{c}
\hat{\mu} \\
\hat{\nu}
\end{array} \right) = \left( \begin{array}{c}
\tilde{\mu} \\
\tilde{\nu}
\end{array} \right) + \left( \begin{array}{c}
\tilde{Y}_1 \\
\tilde{Z}_1
\end{array} \right) + A_{1,k} \left( \begin{array}{c}
\tilde{Y}_1 \\
\tilde{Z}_1
\end{array} \right) - A_{1,k} \left( \begin{array}{c}
\tilde{Y}_2 \\
\tilde{Z}_2
\end{array} \right).
\]

The operator $A_{n,k}$ was defined above. Let us substitute (49) into (8) and evaluate a remainder. As a result we obtain

**Theorem 4** *The formal asymptotic solution of the problem (8) with respect to $\text{mod}(O((t|\theta|^{-1})), \text{ which is uniform when } k \in \mathbb{C} \text{ and } -\theta^2t^{2/3} \gg 1, \text{ has the form } (44).*

4.2 Asymptotics in neighborhood of confluent stationary point

The system of equations in the problem (8) depends on two control parameters $\xi$ and $\eta$. On the parabola $12\xi + \eta^2 = 0$ the degeneracy of stationary points occurs: $k_1 = k_2 = k_0 = i\eta$. For this reason the asymptotics constructed in section 4.1 is invalid when parameter $\theta = \sqrt{\eta^2 + 12\xi}$ is close to zero. For example, the asymptotics of $\hat{\mu}$ as $k \to k_0$ and $\theta \to 0$ is discontinuous:

\[
\left[ \frac{1}{k} \mid_{k \to k_0} \right]_{\theta \to 0} = \left( \frac{|k - k_0|}{12i((k - k_0)^2 - \theta^2)} - \frac{k - k_0}{\theta 12i((k - k_0)^2 - \theta^2)} \right) \times f(-k_0)f(k_0) + o(1).
\]

This shows that we need new scaling of the parameter $\theta$. The scaled control parameter is:

\[ v = t^{1/3} \frac{\theta}{\sqrt{12}}. \]

The external expansion constructed in sec. 4.1 becomes discontinuous at $\theta = 0$ and the internal expansions constructed in sec. 4.1 become singular at the point $\theta = 0$ and lose their asymptotic properties. Therefore here we must change the internal variable for the internal asymptotic expansion:

\[ p = t^{1/3}(k - k_0). \]
In this section we construct a formal asymptotic solution of the problem (8) with respect to mod$(O(t^{-1}))$ when $|\theta| \ll 1$ uniform with respect to $k \in \mathbb{C}$. The result is formulated in the end of this section.

To construct the uniform asymptotic solution we need the external and internal asymptotics outside and inside a small neighborhood of the point $k_0$ respectively.

**Lemma 4** Let the system of the equations (8) be have no the homogeneous nontrivial solutions, $F(k) \in C^2 \cap L_1$ and the parameters $\xi$ and $\eta$ satisfy the inequality $|12\xi + \eta^2| \ll 1$, then:

when $|k - k_0| t^{1/3} \gg 1$ the formal asymptotic solution of the system (8) with respect to mod$(O(t^{-2/3}|k - k_0|) + O(t^{-1}))$ has the form:

\[
\tilde{\mu} = 1 + t^{-1} \frac{1}{M}(k, \xi, \eta),
\]
\[
\tilde{\nu} = t^{-2/3} \frac{1}{N_0} + t^{-1} \left( \frac{1}{N_1} \exp(itS) + \frac{2}{N_2} \right),
\]

the function $\frac{1}{M}$ is defined by (70), the functions $\frac{1}{N_0}$ and $\frac{2}{N_2}$ are defined by (72) and (73) respectively, the function $\frac{1}{N_1}$ is defined by (62); when $|k - k_0| \ll 1$ the asymptotic solution of the system (8) with respect to mod$(O(t^{-2/3}|k - k_0|) + O(t^{-1}))$ has the form:

\[
\tilde{\mu} = 1 + t^{-2/3} \frac{1}{M} t^{-1} \frac{2}{M},
\]
\[
\tilde{\nu} = (t^{-1/3} \frac{1}{N} + t^{-2/3} \frac{2}{N} + t^{-1} \frac{3}{N}) \exp(-itS);
\]

the functions $\frac{1}{M}, j = 1, 2$ and $\frac{1}{N}, j = 1, 2, 3$ are defined by (73), and (74) respectively.

**Proof.** Let us find the internal formal asymptotic expansion for the solution of the system (8) in the form:

\[
M^in = 1 + t^{-2/3} \frac{1}{M} + t^{-1} \frac{2}{M} + \ldots, \tag{51}
\]
\[
N^in = (t^{-1/3} \frac{1}{N} + t^{-2/3} \frac{2}{N} + t^{-1} \frac{3}{N} + \ldots) \exp(-itS). \tag{52}
\]

Change the variable $k$ into the variable $p$ in the system (8). The phase function of the exponent is depend on the variable $p$ as:

\[
tS \equiv \omega(p) \equiv 4(p^3 + p\bar{p}) - v^2(p + \bar{p}),
\]
where \( v = t^{1/3}\theta/\sqrt{12} \). Substitute (51) and (52) into the system (8), equate the coefficients with equal power of \( t \). As a result we obtain a sequence of equation for the coefficients of the expansions (51) and (52).

\[
\begin{align*}
\partial_p \mathcal{N}^1 - i(12p^2 - v^2) \mathcal{N}^1 = \text{sgn}[\text{Re}(\bar{p})] f(-\bar{k}_0), \\
\partial_p \mathcal{N}^2 - i(12p^2 - v^2) \mathcal{N}^2 = -\text{sgn}[\text{Re}(\bar{p})] (f^{(1)}_{01}(-\bar{k}_0)\bar{p} + f^{(2)}_{01}(-\bar{k}_0)\bar{p} - f^{(2)}_{10}(-\bar{k}_0)p) + f^{(1)}_{10}(-\bar{k})p. \\
\partial_p \mathcal{M} = \text{sgn}[\text{Re}(\bar{p})] f(k_0) \mathcal{N}^1, \\
\partial_p \mathcal{N}^3 - i(12p^2 - v^2) \mathcal{N}^3 = \frac{1}{2} \left[ \text{sgn}[ - \text{Re}(\bar{p})] f^{(1)}_{20}(-\bar{k}_0) + f^{(2)}_{20}(-\bar{k}_0) \right] p^2 - \frac{1}{2} \left[ \text{sgn}[ - \text{Re}(\bar{p})] f^{(1)}_{01}(-\bar{k}_0) + f^{(2)}_{02}(-\bar{k}_0) \right] \bar{p}^2 - \left[ \text{sgn}[ - \text{Re}(\bar{p})] f^{(1)}_{11}(-\bar{k}_0) + f^{(2)}_{12}(-\bar{k}_0) \right] |p|^2 - \text{sgn}[ - \text{Re}(\bar{p})] f(-\bar{k}_0) \mathcal{M}. \\
\partial_\theta \mathcal{M}^2 = \text{sgn}[\text{Re}(p)] f(k_0) \mathcal{N}^2 + \left[ \text{sgn}[ - \text{Re}(\bar{p})] f^{(1)}_{10}(k_0) + f^{(2)}_{10}(k_0) \right] \mathcal{N}^1 p + \left[ \text{sgn}[ - \text{Re}(\bar{k})] f^{(1)}_{01}(k_0) + f^{(2)}_{02}(k_0) \right] \mathcal{N}^1 \bar{p}.
\end{align*}
\]

These equations are obtained in a supposition that \(|p| t^{-1/3} \ll 1\). The uniform bounded partial solutions of the equations for the coefficients of the expansion (52) are obtained by using an integral operator:

\[
\mathcal{P}[g] = \frac{\exp(i\omega(p))}{2i\pi} \int \int_\mathbb{C} \frac{dr \land d\bar{r}}{p - r} \exp(-i\omega(r)) g(r),
\]

where \( g(r) \) is the right-hand side of corresponding equation.

Thus, the bounded partial solutions of the equations (53) and (54) (the functions \( \tilde{\mathcal{N}}^1 \) and \( \tilde{\mathcal{N}}^2 \)) are represented by double integrals.

Bounded partial solutions of equations for the coefficients of the expansion (51) are constructed by using the Cauchy-Green formula.
The formula for the partial solution of the equation (55) has the form:

\[ M^s(p, \xi, \eta) = f(k_0)f(-\bar{k}_0)(J_1(p, v^2) - 2J_1^+ - 2J_1^-). \quad (59) \]

Here

\[ J_1 = \frac{1}{2i\pi} \int \int_{\mathbb{C}} \frac{dn \wedge d\bar{n} \exp(i\omega(n))}{p - n} \int \int_{\mathbb{C}} \frac{dr \wedge d\bar{r} \exp(-i\omega(r))}{n - r}. \]

\[ J_1^\pm = \frac{1}{2i\pi} \int \int_{\Omega^\pm} \frac{dn \wedge d\bar{n} \exp(i\omega(n))}{p - n} \int \int_{\Omega^\pm} \frac{dr \wedge d\bar{r} \exp(-i\omega(r))}{n - r}. \]

In the Appendix the integrals \( J_1 \) and \( J_1^+ \) are reduced into double integrals. By the similar way we may represent \( \frac{1}{2}M^s \) as the sum of double integrals. Therefore we may represent \( M^s \) as sum of double integrals also. Corresponded formula is very large and we don’t write it here. But we will use this formula to evaluate an asymptotic behaviour of \( M^s \) as \(|p| \to \infty\).

The bounded partial solution of the equation (57) when \( p \in \mathbb{C} \) has to be built similarly. We give following statement about \( \frac{1}{2}M^s \).

**Lemma 5** Continuous partial solution of the equation for \( \frac{1}{2}M \) exists. This solution is uniformly bounded when \( p \in \mathbb{C} \).

**Sketch of proof.** The partial solution of the equation for \( \frac{1}{2}M \) has to be obtained as a result of using of the operator:

\[ \partial_p^{-1}[g] = \hat{M}^s = \frac{1}{2i\pi} \int \int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{p - n} g(n) \quad (60) \]

to the right-hand side of the equation. The integrals, which are obtained, are continuous by virtue of continuity of the integral operator with respect to the parameter \( p \). The boundedness of these integrals with respect to \( p \) when \( p \in \mathbb{C} \) may be obtained by using the asymptotic behaviour of the right-hand side terms of the equation for \( \frac{1}{2}M \) as \(|p| \to \infty\).

Let us to construct the external expansion in the form:

\[ m = 1 + t^{-1}m_0 (k, v) + t^{-5/3}(\hat{m}_1 \exp(itS) + \hat{m}_0) + \ldots; \quad (61) \]
\[ n = (t^{-1} \hat{n}_1 (k, v) + t^{-5/3} \hat{n}_2 (k, v) + t^{-2} \hat{n}_3 \ldots) \exp (iS) + t^{-2/3} \hat{n}_0 (k, v) + t^{-1} \hat{n}_0 (k, v) + \ldots. \] (62)

Substitute (61) and (62) into (8). As a result we get the equations for coefficients of the expansions:

\[ \frac{1}{n_1} = \text{sgn} \Re (−\tilde{k}) \frac{f (−\tilde{k})}{12i(k − k_0)^2}; \quad n_1 = -\text{sgn} \Re (−\tilde{k}) \frac{v^2 f (−\tilde{k})}{144i(k − k_0)^4}; \] (63)

\[ \partial_{\tilde{k}} \frac{1}{m} = \frac{f(k)f(−\tilde{k})}{12i(k − k_0)^2}; \] (64)

\[ \partial_{\tilde{k}} n_0 (k, \xi, \eta) = 0; \quad \partial_{\tilde{k}} 2 n_0 (k, \xi, \eta) = 0; \] (65)

\[ \partial_{\tilde{k}} \frac{2}{m_0} = \frac{f(k)f(−\tilde{k})v^2}{144i(k − k_0)^4}; \] (66)

\[ \tilde{m}_1 (k, \xi, \eta) = \text{sgn} \Re (−\tilde{k}) f(k) \frac{1}{12i(k − k_0)^2}; \] (67)

\[ \frac{3}{n_1} (k, \xi, \eta) = \frac{1}{12i(k − k_0)^2} \left( \text{sgn} \Re (−\tilde{k}) f(−\tilde{k}) \frac{1}{m_0} + \frac{2f (−\tilde{k})}{12i(k − k_0)^3} \text{sgn} \Re (−\tilde{k}) \left[ f_{10}^{(1)} (−\tilde{k}) \text{sgn} \Re (−\tilde{k}) \right] + f_{10}^{(2)} (−\tilde{k}) \right). \] (68)

The formulas (63) and (68) define the functions \( \frac{1}{n_1}, \frac{2}{n_1} \) and \( \frac{3}{n_1} \). Using the formulas (63) we can see, that the functions \( \frac{1}{n_0}, \frac{2}{n_0} \) are analytic of variable \( \tilde{k} \). The obvious form of this dependence is defined by two conditions. First one is the continuity of the asymptotics and second one is matching condition for the external and internal asymptotic expansions.

One can see, that the sufficient condition for the continuity of the coefficient of the asymptotics (62) as \( t^{-1} \) with respect to \( k \) is an addition into \( \frac{2}{n_0} \) of the term:

\[ \frac{2}{n_0}' = \frac{1}{\pi i} \frac{1}{12i(k − k_0)^2} \int_{i\infty}^{i\infty} \frac{d\lambda}{k − \lambda} f (−\tilde{l}). \] (69)
It follows from the formulas (67), that the coefficient of the asymptotics (61) as $t^{-5/3}$ is defined after evaluating of $\frac{1}{n_0}$, i.e. after matching of the coefficients as $t^{-2/3}$ of the external and internal expansions.

Consider the problem for defining of $\frac{1}{m_0}$ in detail. This function satisfies the boundary condition:

$$\frac{1}{m_0} \lim_{|k| \to \infty} = 0.$$ 

The external solution is valid outside of small neighborhoods of the points $k_j$, $j = 0, 1, 2$. Therefore solutions of the equation (54), which are smooth, bounded and decreased as $|k| \to \infty$:

$$\frac{1}{m_0}(k, \theta) = \frac{1}{2i\pi} \int \int_{C} \frac{dr \wedge d\bar{r}}{k - r} f(-\bar{r})f(r) - f(-k_0)f(k_0) + \frac{(k - k_0)f(k_0)f(-\bar{k}_0)}{12i(k - k_0)^2}$$

are defined to within an analytic function with respect to variable $k$, which has poles when $t^{1/3}|k - k_0| \ll 1$. The full definition of $\frac{1}{m_0}$ will be done by matching of the external and internal expansions.

For matching process we need asymptotic behaviour of the partial solutions of equations (53)-(56) as $|p| \to \infty$. Constructing of these asymptotics is reduced to evaluating of an integrals with weak singularity in the integrand. Evaluating of these integrals is done in the Appendix. Here we write the asymptotic behaviour of partial solutions of equations (53)-(56) as $|p| \to \infty$.

$$\mathcal{N}^s(p, \xi, \eta) = \frac{1}{p} \exp(i\omega(p))f(-\bar{k}_0)[\phi_{00}^+(v^2) - \phi_{00}^-(v^2)] + \frac{1}{p^2} \exp(i\omega(p))f(-\bar{k}_0)[\phi_{10}^+(v^2) - \phi_{10}^-(v^2)] + \frac{1}{12i} f(-\bar{k}_0) \text{sgn}[\text{Re}(-\bar{p})]\left(\frac{1}{p^2} - \frac{\exp(i\omega(p))}{p^2}\right) + O(|v^2||p|^{-3}).$$

$$\tilde{\mathcal{N}}^s(p, \xi, \eta) = \left[\frac{\exp(i\omega(p))}{p}(\phi_{10}^+(v^2) - \phi_{10}^-(v^2)) + \left(\frac{1}{p} - \frac{\exp(i\omega(p))}{p}\right) \frac{\text{sgn}[\text{Re}(-\bar{p})]}{12i}\right] f_{10}^{(1)}(-\bar{k}_0) + \left[\frac{1}{p} + \frac{\exp(i\omega(p))\phi_{10}^+(v^2)}{12i}\right] f_{10}^{(2)}(-\bar{k}_0) +$$

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\[ \frac{\exp(i\omega(p))}{p} (\phi_0^+ (v^2) - \phi_0^- (v^2)) + \left( \frac{\bar{p}}{p^2} - \frac{\exp(\omega(p))}{p} \right) \times \]
\[ \frac{\text{sgn}[\text{Re}(\bar{p})]}{12i} f_0^{(1)} (-\bar{k}_0) + \left[ \frac{1}{p} \exp(i\omega(p)) \phi_0 (v^2) + \frac{\bar{p}}{12ip^2} \right] f_0^{(2)} (-\bar{k}_0) + O(|p|^{-2}) + O(|v^2|p^{-2}), \]

\[ \mathcal{M}^*(p, \xi, \eta) = -f(k_0) f(-\bar{k}_0) \left[ -\frac{\bar{p}}{12ip^2} + \frac{1}{p} \left( -12i\phi_0 (v^2) \phi_0^- (v^2) + i\nu^2 \phi_0 (v^2) \phi_0^- (v^2) \right) \right] - \frac{2}{p} \left[ i\nu^2 (\phi_0^+ (v^2) \phi_0^- (v^2) + \phi_0^+ (v^2) \phi_0^- (v^2)) + 12i (\phi_0^+ (v^2) \phi_0^- (v^2) + \phi_0^+ (v^2) \phi_0^- (v^2)) - \frac{1}{2} \psi_0 \right] + O(|v^2|p^{-2}); \]

\[ \mathcal{M}^*(p, \xi, \eta) = f(k_0) f_0^{(1)} (-\bar{k}_0) \frac{\bar{p}}{12ip} + \]
\[ f(k_0) f_0^{(2)} (-\bar{k}_0) \left( \frac{\bar{p}}{12ip} + \frac{1}{12i} \text{sgn}[\text{Re}(p)] \right) + \]
\[ f(k_0) f_0^{(2)} (-\bar{k}_0) \left( \frac{\bar{p}^2}{24i} - \frac{1}{24i} \text{sgn}[\text{Re}(p)] \right) + \]
\[ f(k_0) f_0^{(1)} (-\bar{k}_0) \frac{\bar{p}^2}{24i} + \Phi(v^2) + O(|v^2|p^{-1}), \]

\[ \mathcal{N}^*(p, \xi, \eta) = \frac{1}{24i} \left[ \text{sgn}[\text{Re}(\bar{p})] (1 - \exp(i\omega(p))) f_0^{(1)} (-\bar{k}_0) \right] + \]
\[ \left[ \text{sgn}[\text{Re}(\bar{p})] (\frac{\bar{p}^2}{12ip^2} - \frac{\exp(i\omega(p))}{24i}) f_0^{(1)} (-\bar{k}_0) + \frac{\bar{p}^2 f_0^{(2)} (-\bar{k}_0)}{24i} \right] + \]
\[ \left[ \text{sgn}[\text{Re}(\bar{p})] (\frac{\bar{p}}{12ip} + \frac{\exp(i\omega(p))}{12i}) f_0^{(1)} (-\bar{k}_0) + \frac{\bar{p}}{12ip} f_1^{(2)} (-\bar{k}_0) \right] + \]
\[ f_0^2 (-\bar{k}_0) f(k_0) \exp(i\omega(p)) \Psi(v^2) + O(|v||p|^{-1}). \]

The functions \( \Phi(v^2) \) and \( \Psi(v^2) \) are smooth and uniformly bounded when \( v^2 \in \mathbb{R} \). Here we use notations:

\[ \phi_{mn}^\pm = \int_\mathbb{R} \int_{\text{Re}(r) > 0} \text{dr} \land d\bar{r} r^m \bar{r}^n \exp(i\omega(r)), \]
\[\psi^\pm_{mn} = \int \int \nabla \exp(-i\omega(r)), \]
\[\phi_{mn} = \int \int d\bar{r}^m \phi^n \exp(i\omega(r)), \]
\[\psi_{mn} = \int \int d\bar{r}^m \phi^n \exp(-i\omega(r)). \tag{71}\]

Evaluate an asymptotic behaviours of the coefficients of the external expansion as \(k \to k_0\) and \(\partial_k^2 S = o(1)\).

\[
\frac{1}{n_1} |_{k \to k_0} = \frac{f(-\bar{k}_0)}{12\pi(k-k_0)^2} \text{sgn} [\text{Re}(-\bar{k})] + \\
\frac{1}{12i(k-k_0)} \left[ f_{10}^{(1)}(-\bar{k}) + f_{01}^{(1)}(-\bar{k}) \frac{k-k_0}{k-k_0} + (f_{10}^{(2)}(-\bar{k}) + f_{01}^{(2)}(-\bar{k}) \frac{k-k_0}{k-k_0}) \text{sgn} [\text{Re}(-\bar{k})] \right] + \\
+ \frac{1}{2} f_{20}^{(1)}(-\bar{k}) + f_{11}^{(1)}(-\bar{k}) \frac{k-k_0}{k-k_0} + \frac{1}{2} f_{02}^{(1)} \frac{(k-k_0)^2}{(k-k_0)^2} + \\
- \frac{1}{2} f_{20}^{(2)}(-\bar{k}) + f_{11}^{(2)}(-\bar{k}) \frac{k-k_0}{k-k_0} + \\
\frac{1}{2} f_{02}^{(2)}(-\bar{k}) \frac{(k-k_0)^2}{(k-k_0)^2} \text{sgn} [\text{Re}(-\bar{k})] + o(1). \]

\[
\frac{2}{n_0'} = \frac{-1}{24\pi(k-k_0)^2} \times \\
\left[ \pi i f(-\bar{k}_0) \text{sgn} [\text{Re}(-\bar{\nu})] + \text{V.P.} \int_{-\infty}^{\infty} \frac{d\lambda f(i\lambda)}{\lambda - ik_0} \right] + \\
- \frac{1}{12\pi(k-k_0)} \left[ \pi i f_{01}^{(1)}(-\bar{k}_0) + \pi i f_{10}^{(1)}(-\bar{k}_0) + (\pi i f_{10}^{(2)}(-\bar{k}_0) + \\
+ \pi i f_{01}^{(2)}(-\bar{k}_0)) \text{sgn} [\text{Re}(-\bar{k}_0)] + \\
+ \text{V.P.} \int_{-\infty}^{\infty} \frac{d\lambda f(i\lambda)}{(\lambda - ik_0)^2} \right] + \frac{1}{12i} \left[ \pi i f_{20}^{(1)}(-\bar{k}_0) + \\
+ 2\pi i f_{11}^{(1)}(-\bar{k}_0) + \pi i f_{02}^{(1)}(-\bar{k}_0) + (\pi i f_{20}^{(2)}(-\bar{k}_0) + 2\pi i f_{11}^{(2)}(-\bar{k}_0) + \\
+ \pi i f_{02}^{(2)}(-\bar{k}_0)) \text{sgn} [\text{Re}(-\bar{k})] + \text{V.P.} \int_{-\infty}^{\infty} \frac{d\lambda f(i\lambda)}{(\lambda - ik_0)^3} \right] + o(1). \]

\[
\frac{1}{m^*} = \frac{(k-k_0)}{12i(k-k_0)^2} f(-\bar{k}_0)f(k_0) \]
Let us equate the coefficients with equal powers of the large parameter \( v \) bounded with respect to \( \nu \).

\[
\frac{k - k_0}{12i(k - k_0)} \left[ (f_0^{(1)}(\bar{k}_0)\text{sgn}[\text{Re}(\bar{k})] + f_0^{(2)}(\bar{k}_0)\text{sgn}[\text{Re}(\bar{k})] + f_0^{(1)}(k_0)\text{sgn}[\text{Re}(k)]) 
- \frac{(k - k_0)^2}{12i(k - k_0)^2} \left[ f(k_0)(f_0^{(1)}(\bar{k}_0)\text{sgn}[\text{Re}(\bar{k})] + f_0^{(2)}(\bar{k}_0)) + f(\bar{k}_0)(f_0^{(1)}(k_0)\text{sgn}[\text{Re}(k)]) + f_0^{(2)}(k_0) \right] + o(1).
\]

As a result we obtain:

Let us do matching of the external and internal asymptotic expansions of the function \( \tilde{\nu} \). The matching condition for \( \tilde{\nu} \) in domain \( t^{-1/3} \ll |k - k_0| \ll 1 \) when \( |\theta| \ll 1 \) has the form:

\[
(t^{-2/3} \frac{1}{n_0} + t^{-1} \frac{2}{n_0} + t^{-1} \frac{1}{n_0} \exp(itS)) - (t^{-1/3} \frac{N}{+t^{-2/3} \frac{2}{N} + t^{-1} \frac{3}{N}}) = o(t^{-1}).
\]

Let us equate the coefficients with equal powers of the large parameter \( t \). As a result we obtain:

\[
\frac{1}{n_0} = \frac{1}{k - k_0} f(-\bar{k}_0)[\phi_{00}^+(v^2) - \phi_{00}^-(v^2)],
\]

\[
\frac{2}{n_0} = \frac{1}{k - k_0} f(-\bar{k}_0)[\phi_{01}^+(v^2) - \phi_{01}^-(v^2)] + \frac{1}{k - k_0} f_0^{(1)}(\bar{k}_0)[\phi_{10}^+(v^2)\phi_{10}^-(v^2)] + \frac{1}{k - k_0} f_0^{(1)}(\bar{k}_0)[\phi_{10}^+(v^2) - \phi_{10}^-(v^2)] + \frac{1}{k - k_0} f_0^{(2)}(\bar{k}_0)\phi_{01}(v^2),
\]

where \( \phi, \psi, \phi^\pm, \psi^\pm \) are defined by (71);

\[
\frac{1}{N} = \mathcal{P}[g_1], \quad \frac{2}{N} = \mathcal{P}[g_2], \quad \frac{3}{N} = \mathcal{P}[g_3] f_0^2(-\bar{k}_0) f(k_0) \exp(i\omega(p)) \Psi(v^2),
\]

where \( g_j \) are the right-hand sides of the equations for \( \tilde{\nu} \), the operator \( \mathcal{P}[g] \) is defined by (58), the function \( \Psi(v^2) \) is smooth and uniformly bounded with respect to \( v^2 \) when \( v^2 \in \mathbb{R} \).

Matching condition for the function \( \tilde{\mu} \) in the domain \( t^{-1/3} \ll |k - k_0| \ll 1 \) when \( |\theta| \ll 1 \) has the form:

\[
(1 + t^{-1} \bar{m}) - (1 + t^{-2/3} \frac{1}{M} + t^{-1} \frac{2}{M}) = o(t^{-1}).
\]
As a result we obtain:

\[ \frac{1}{M} = \frac{1}{M^s}; \; \frac{2}{M} = \frac{2}{M^s} - \Phi(v^2), \]  

(75)

where \( \frac{1}{M^s} \) is defined by (69), the function \( \frac{2}{M^s} \) is defined by (70) and the function \( \Phi(v^2) \) is smooth and uniformly bounded when \( v^2 \in \mathbb{R} \).

\[ \frac{1}{m} = \frac{1}{m^s} - \frac{1}{k - k_0} \left( -12i \phi_{01}(v^2) \phi_{10}(v^2) + iv^2 \phi_{00}(v^2) \phi_{00}(v^2) \right) \]

\[ - \frac{2}{k - k_0} \left[ iv^2 \left( \phi_{00}(v^2) \psi_{00}(v^2) + \phi_{00}(v^2) \psi_{00}^+(v^2) \right) \right. \]

\[ + 12i \left( \phi_{01}^+(v^2) \psi_{01}^-(v^2) + \phi_{01}(v^2) \psi_{01}^+(v^2) \right) - \frac{1}{2} \psi_{01} \] \[ (76) \]

Here \( \frac{1}{m^s} \) is defined by (70) and the functions \( \psi^\pm, \phi^\pm, \phi \) and \( \psi \) are defined by formulas (71).

Thus we have matched internal and external expansions of \( \tilde{\mu} \) and \( \tilde{\nu} \). The lemma is proved.

These expansions are ununiform with respect to \( k \). Now we can construct uniform asymptotic expansion with respect to \( k \in \mathbb{C} \). Following the matching method the uniform expansion is:

\[ \begin{pmatrix} \tilde{\mu} \\ \tilde{\nu} \end{pmatrix} = \begin{pmatrix} \frac{1}{m} \\ \frac{1}{n} \end{pmatrix} + \begin{pmatrix} \tilde{M}_1 \\ \tilde{N}_1 \end{pmatrix} - A_{1,k} \begin{pmatrix} \tilde{M}_1 \\ \tilde{N}_1 \end{pmatrix}. \]  

(77)

Here the operator \( A_{n,k} \) processes on the function \( \tilde{M}_1 \) in the formula for \( M_1 \) by followed manner. One must change the variable \( p \) into the variable \( k \) using the formula (50) and write all terms of the asymptotic expansion with respect to \( t \) with the powers are equal to \( -m \), where \( 0 \leq m \leq n \). For example, one can obtain for the function \( \tilde{\mathcal{M}}(p, \xi, \eta, t) \):

\[ A_{1,k}[\tilde{\mathcal{M}}(p, \xi, \eta, t)] = 1 + t^{\frac{1}{2}} \left( -f(k_0) f(-\bar{k}_0) \left[ -\frac{k - k_0}{12i(k - k_0)^2} \right. \right. \]

\[ \left. \left. + \frac{1}{k - k_0} \left( -12i \phi_{01}(v^2) \phi_{10}(v^2) + iv^2 \phi_{00}(v^2) \phi_{00}(v^2) \right) \right. \right. \]

\[ \left. \left. \left. - \frac{2}{k - k_0} \left[ iv^2 \left( \phi_{00}(v^2) \psi_{00}(v^2) + \phi_{00}(v^2) \psi_{00}^+(v^2) \right) \right. \right. \right. \]

\[ \left. \left. \left. + 12i \left( \phi_{01}^+(v^2) \psi_{01}^-(v^2) + \phi_{01}(v^2) \psi_{01}^+(v^2) \right) - \frac{1}{2} \psi_{01} \right) \right] \right]. \]

\[ f(k_0) f^{(1)}_0 (-\bar{k}_0) \left[ -\frac{k - k_0}{12i(k - k_0)} \right. \]

\[ + \]
\[ f(k_0) f_{10}^{(2)} (-\bar{k}_0) \left( \frac{k - k_0}{12i(k - k_0)} + \frac{1}{12i} \right) \text{sgn}[\text{Re}(k)] + f(k_0) f_{01}^{(1)} (-\bar{k}_0) \frac{k - k_0^2}{24i(k - k_0)^2} + f(k_0) f_{01}^{(2)} (-\bar{k}_0) \left( \frac{k - k_0^2}{24i(k - k_0)^2} - \frac{1}{24i} \right) \text{sgn}[\text{Re}(k)] \],

The obviously formula for \( A_{1,k}[\tilde{N}] \) is more large and doesn’t shown here.

**Theorem 5** The formula (77) gives the asymptotics solution of the problem (8) with respect to \( \text{mod}(O(t^{-1})) \) as \( t \to \infty \). This asymptotic solution is uniform with respect to \( k \in \mathbb{C} \) and \( |\theta| \gg 1 \).

### 5 Justification of the asymptotics of the solution of \( \bar{D} \)-problem

In this section we prove that the remainder of the asymptotics has order by \( t^{-4/3} \) uniformly with respect to \( k \in \mathbb{C} \) and this remainder has to be differentiable with respect to \( x \). We call by the remainder of the asymptotics the difference between solution of the problem (8) and constructed asymptotic solutions (33) when \( \theta^2 t^{-2/3} \gg 1 \), (19) when \( -\theta^2 t^{-2/3} \gg 1 \) and (77) when \( |\theta| \ll 1 \). The differentiability of the remainder will be important when we will construct an asymptotic behaviour of solution of the equation KP-2.

**Theorem 6** Let \( \partial^\alpha f(k, \bar{k}) \in L_1 \cap C^1 \) when \( |\alpha| \leq 2 \) when \( k \) is out of the imaginary axis and

\[
\sup_{z \in \mathbb{C}} \left| \int \int_{\mathbb{C}} \frac{dk \wedge d\bar{k}}{|k - z|} |F(k)| \right| < 2\pi,
\]

then the solution of the problem (8) is:

\[
\left( \begin{array}{c} \mu \\ \nu \end{array} \right) = \left( \begin{array}{c} \tilde{\mu} \\ \tilde{\nu} \end{array} \right) + O(t^{4/3}),
\]

(78)

when \( k \in \mathbb{C}, \xi, \eta \in \mathbb{R} \). The remainder of the asymptotics has to be differentiable with respect to \( x \).
The proof. Let us write the system of differential equation for the remainder. Denote the remainder in (82) by $V$. Substitute (82) into (8). As a result we obtain:

$$
\begin{pmatrix}
\partial_k & 0 \\
0 & \partial_k
\end{pmatrix} V = \begin{pmatrix}
0 & F(-\tilde{k}) \exp(it S) \\
F(k) \exp(-it S) & 0
\end{pmatrix} V + f, \quad (79)
$$

$$
V|_{|k|\to\infty} = 0. \quad (80)
$$

We denote by the vector $f$ the residual which originates in (8) when we substitute the column into this equation $(\tilde{\mu}, \tilde{\nu})^T$:

$$
f_1 = -\partial_k \tilde{\mu} + F(-\tilde{k}) \exp(it S) \tilde{\nu},
$$

$$
f_2 = -\partial_k \tilde{\nu} + F(k) \exp(-it S) \tilde{\mu}.
$$

Let us denote by $X$ the space of bounded and continuous with respect to $k$ vector-functions with the norm:

$$
||W|| = \sup_{k \in \mathbb{C}, (\xi, \eta) \in \mathbb{R}^2} |W_1| + \sup_{k \in \mathbb{C}, (\xi, \eta) \in \mathbb{R}^2} |W_2|.
$$

Consider a system of integral equations instead of the problem (79), (80):

$$
V = G[F] V + H, \quad (81)
$$

where $G[F]$ is the integral operator:

$$
G[F] V = \int \int_{m \in \mathbb{C}} dm \wedge d\tilde{m} \times
\begin{pmatrix}
0 & F(-\tilde{m}) \exp(it S) \\
F(m) \exp(-it S) & 0
\end{pmatrix} V(m, \xi, \eta, t);
$$

$$
H = \int \int_{m \in \mathbb{C}} dm \wedge d\tilde{m} \begin{pmatrix}
f_1(m, \xi, \eta, t) \\
f_2(m, \xi, \eta, t)
\end{pmatrix}.
$$

Using obvious forms of the functions $f_1, f_2$ one can prove, that

$$
||H|| = O(t^{-4/3})
$$

uniformly with respect to $\xi, \eta \in \mathbb{R}$ when $k \in \mathbb{C}$.

The operator $G[F]$ is contracting in the space $X$, therefore the solution of the integral equation (81) exists in $X$ and is evaluated by $O(t^{-4/3})$ uniformly with respect to $\xi, \eta \in \mathbb{R}$.
Show that the remainder of the asymptotics is differentiable on \( x = t\xi \). Differentiate with respect to \( x \) the system of the equation for the remainder. Denote the derivative of the vector \( V \) by \( \chi \). Then we obtain:

\[
\chi = G[F]\chi + \partial_x G[F]V + \partial_x H.
\]

The terms \( \partial_x G[F]V + \partial_x H \) may be evaluated by order \( O(t^{-4/3}) \).

The operator \( G[F] \) is contracting, therefore one can obtain: \( ||\chi|| = O(t^{-4/3}) \). The theorem is proved.

6 Solution of the equation KP-2

An asymptotics of the solution of the problem (5) as \( t \to \infty \) may be written as:

\[
\left( \begin{array}{c} \phi \\ \psi \end{array} \right) = \left( \begin{array}{c} \tilde{\mu} \\ \tilde{\nu} \end{array} \right) (k, \bar{k}, \xi, \eta, t) + \left( \begin{array}{c} -\tilde{\nu} \\ -\tilde{\mu} \end{array} \right) (-\bar{k}, -k, \xi, \eta, t) + O(t^{-4/3}).
\]

The second term in this formula is the solution of the problem (5) with the boundary condition:

\[
\left. \left( \begin{array}{c} \tilde{\mu} \\ \tilde{\nu} \end{array} \right) \right|_{|k| \to \infty} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right).
\]

The proof of the theorem 1. Let us substitute the function \( \psi(k, \bar{k}, \xi, \eta, t) \) into the formula for the solution of the equation KP-2(5). Differentiate the integrand with respect to \( x \). The main terms of the integrand are the terms which appear after differentiating of the exponent. The derivatives with respect to \( x \) of others factors of integrand are small because they depend on \( x \) slowly. The integrals of such terms over the plane are evaluated by the order \( O(t^{-4/3}) \) uniformly with respect to \( (x, y) \in \mathbb{R} \). Rewrite the main term of the integral as the integral over real plane: \( (\kappa, \lambda) \in \mathbb{R}^2 \), where \( \kappa = \text{Re}(k), \quad \lambda = \text{Im}(k) \).

As a result we obtain:

\[
u(x, y, t) = -4 \int \int_{\mathbb{R}^2} d\kappa \wedge d\lambda |\kappa| f(\kappa + i\lambda) \times \\
\times \exp(it(8\kappa^3 - 24\kappa\lambda^2 + 2\kappa\xi + 4\kappa\lambda\eta)) + O(t^{-4/3}).
\]

Thus the main term of the asymptotics of the solution of the Cauchy problem for the equation KP-2 is given by integral with fast...
oscillating exponent. Let us evaluate the asymptotic behaviour of this integral.

Let \( \eta^2 + 12\xi > 0 \) and \( t^{1/3}|\eta^2 + 12\xi| \gg 1 \), then the stationary points of the exponent are: \( \lambda_{1,2} = \frac{\eta}{12} \pm \sqrt{\eta^2 + 12\xi} \), \( \kappa_{1,2} = 0 \). The stationary phase method (see for example [24]) gives:

\[
\lambda_{1,2} \approx \frac{\eta}{12} \pm \sqrt{\eta^2 + 12\xi}, \quad \kappa_{1,2} = 0.
\]

The asymptotic behaviour of the integral is:

\[
u(x, y, t) = o(t^{-1}).
\]

Let \( \eta^2 + 12\xi < 0 \) and \( t^{1/3}|\eta^2 + 12\xi| \gg 1 \), then \( \lambda_{1,2} = \frac{\eta}{12} \), \( \kappa_{1,2} = \pm \frac{1}{2} \sqrt{-\eta^2 - 12\xi} \). The asymptotic behaviour of the integral is:

\[
u(x, y, t) = -4t^{-1} \frac{\pi}{12i\sqrt{-\eta^2 - 12\xi}} f_0 \left( \frac{1}{2} \sqrt{-\eta^2 + 12\xi + i\eta} \right) \times 
\]

\[	imes \exp \left( -11it \sqrt{-\frac{y^2}{t^2} - 12\frac{\pi}{t}} \right) + c.c. + o(t^{-1}).
\]

To evaluate the main term of the asymptotics of the integral when \( |12\xi + \eta^2| = o(1) \), we substitute the scaled variables \( p_1 = t^{1/3} \Re(k-k_0) \), \( p_2 = t^{1/3} \Im(k-k_0) \) and the parameter \( v^2 = t^{2/3}(\eta^2 + 12\xi)/\sqrt{12} \) into integral (83). As a result we obtain:

\[
u(\xi, \eta, t) = 4it^{-1} f(k_0) \int_{\mathbb{R}^2} dp_1 dp_2 \times 
\]

\[
p_1 \exp(i(8p_1^3 - 2v^2p_1 - 24p_1p_2^2)) + o(t^{-1}).
\]

Let us integrate the internal integral with respect to the parameter \( p_2 \), use the even property of the integrand with respect to \( p_1 \). As a result we obtain:

\[
u(x, y, t) = 8it^{-1} \sqrt{\pi} f(i\eta/12) \left( \int_0^{\infty} dp_1 \sqrt{p_1} \cos(8p_1(p_1^2 - 8v^2)) + 
\]

\[
+ \int_0^{\infty} dp_1 \sqrt{p_1} \sin(8p_1(p_1^2 - 8v^2)) \right) + o(t^{-1}).
\]

The theorem 1 is proved.

A Asymptotic behaviour of double integral with weak singularity of the integrand

Here we obtain the asymptotic behaviour of integrals which are appeared when the asymptotic solution of (8) was studied.
Evaluating of the asymptotic behaviour of one-dimensional integrals with weak singular integrand and fast oscillated exponent was done in [24] p.26 and [28] p.332. The asymptotic behaviour of many-dimensional integrals with fast oscillating exponent was studied in [24], [30]. The asymptotic behaviour of the Cauchy integrals with fast oscillating exponent in one-dimensional case was studied in [24] and in many-dimensional case was studied in [31]. The asymptotic behaviour of the two-dimensional for some integrals over all complex plane with weak singularity was studied in [25].

A.1 Integrals over half-plane with general stationary point of fast oscillated exponent

Here we study an asymptotic behaviour of an integral:

\[
I = \int \int_{\Omega^+} \frac{dn \wedge d\bar{n}}{l - n} \exp(-i(n^2 + \bar{n}^2)),
\]

where \(|l| \to \infty\) and the domain \(\Omega^+ = \{\text{Re}(l + \bar{l} - i(l - \bar{l})) > 0\}\).

**Theorem 7** The asymptotic behaviour of the integral \(I\) as \(|p| \to \infty\) has the form:

\[
I = -2i\pi \frac{\exp(-i(l^2 + \bar{l}^2))}{2il} - \frac{3i\pi}{2l} + O(|l|^{-2}) \quad l \in \Omega^+;
\]

\[
I = -\frac{i\pi}{2l} + O(|l|^{-2}) \quad \text{where} \quad l \notin \Omega^+.
\]

**The proof.** Let us suppose that \(l \in \Omega^+\). Divide the domain \(\Omega^+\) into three domains: first one is \(\Omega_1^+ = \{\Omega^+ \setminus \{|n| \leq |l|/2| \cup |n - l| < \varepsilon\}\}\). This domain hasn’t stationary point of phase function of the exponent and the singularity of the integrand. Second one is \(\Omega_2^+ = \{\Omega^+ \setminus |n| \geq |l|/2\}\). This domain contains the stationary point of the phase function. At last, third domain is \(\Omega_3^+ = \{|n - l| < \varepsilon\}\). This domain contains the singularity of the integrand.

Let us integrate by parts over \(\Omega_1^+\). As a result we obtain:

\[
\int \int_{\Omega_1^+} \frac{dn \wedge d\bar{n}}{l - n} \exp(-i(n^2 + \bar{n}^2)) = \int \int_{\partial\Omega_1^+} \frac{d\bar{n}}{l - n} \exp(-i(n^2 + \bar{n}^2)) - 2i\pi \frac{\exp(-i(l^2 + \bar{l}^2))}{2il} - \int \int_{\Omega_1^+} \frac{dn \wedge d\bar{n}}{l - n} \exp(-i(n^2 + \bar{n}^2)) \frac{2in}{2in^2}.
\]

(84)
The boundary of the domain Ω₁⁺ includes a large half-circle of radius R as R → ∞, a circle of radius ε over the point l = n, a half-circle of radius |l|/2 and two segments: [R exp(3iπ/4), |l| exp(3iπ/2)] and [|l| exp(−iπ/4)/2, R exp(3iπ)].

Consider the integrals over boundary of the domain Ω₁⁺. The integral over large half-circle is equals to zero as R → ∞. The integral over half-circle of radius |l|/2 has order |l|−3 (because of oscillations and small value of the integrand). The integral over the circle at n = l equals to a residue of the integrand multiplied by 2iπ as ε → 0. The integral over the segments [R exp(3iπ/4), |l| exp(3iπ/2)] and [|l| exp(−iπ/4)/2, R exp(3iπ)] (let us denote their union by L) is reduced to the form:

\[ \int_{n \in L} \frac{d\bar{n}}{-2il(l-n)} = \frac{\exp(3i\pi/4)}{-2il} \int_{|\lambda| \geq |l|/2} \frac{d\lambda}{l - \lambda \exp(i\pi/4)}. \]

The second term of (84) has to be evaluated as by O(|l|−3).

Let us consider the integral over Ω₂⁺. Represent:

\[ \frac{1}{l-n} = \frac{1}{l}(1 + \frac{\bar{n}}{l-n}). \]

Then the integral over Ω₂⁺ has to be written as:

\[ I_2 = \int \int_{\Omega_2^+} dn \wedge d\bar{n} \exp(-i(n^2 + \bar{n}^2)) = \]

\[ \frac{1}{l} \int \int_{\Omega_2^+} dn \wedge d\bar{n} \frac{1}{l-n} \exp(-i(n^2 + \bar{n}^2)) + \]

\[ \frac{1}{l} \int \int_{\Omega_2^+} dn \wedge d\bar{n} \frac{\bar{n}}{l-n} \exp(-i(n^2 + \bar{n}^2)). \]

Here we integrate by parts the second term. After evaluations we obtain:

\[ I_2 = \frac{1}{l} \int \int_{\Omega_2^+} dn \wedge d\bar{n} \exp(-i(n^2 + \bar{n}^2)) + \]

\[ \frac{1}{2il} \int_{|l| \exp(-i\pi/4)/2}^{\exp(3i\pi/4)/2} \frac{dn}{l-n} + O(|l|^{-2}). \]

The integral over Ω₃⁺ as ε → 0 equals to zero.

Let us sum the obtained asymptotics:

\[ I = -2i\pi \frac{\exp(-i(l^2 + \bar{l}^2))}{2il} + \frac{\exp(i\pi/2)}{2il} \int_{-\infty}^{\infty} \frac{d\lambda}{l \exp(-i\pi/4) - \lambda} + \]

37
Thus the first statement of the theorem is proved. The second statement has to be proved by the same way.

The theorem is proved.

A.2 Asymptotic behaviour of the integral with confluent phase function

In this section we obtain an asymptotic behaviour of an integral as \(|p| \to \infty\)

\[
W^+ = \int \int_{\Omega^+} \frac{dr \wedge d\bar{r}}{p - r} \exp(-i\omega(r)).
\]

Theorem 8 The asymptotic behaviour of the integral (85) where \(\omega(p) = 4(p^3 + \bar{p}^3) - v^2(p + \bar{p})\) as \(|p| \to \infty\) and \(|p \pm \frac{v}{\sqrt{12}}| \geq 0\) has the form:

\[
W^+ = \left[ \begin{array}{c}
\frac{1}{p}\phi_{00}^+(v^2) + \frac{1}{p^2}\phi_{01}^+(v^2) + \frac{\pi i}{12p^2} + 2\pi i \frac{\exp(-i\omega(p))}{12p^2} + O(|p|^{-3} + |v|^2|p|^{-2}), \quad \text{when} \quad p \in \Omega^+; \\
\frac{1}{p}\phi_{00}^+(v^2) + \frac{1}{p^2}\phi_{01}^+(v^2) \frac{\pi i}{12p^2} + O(|p|^{-3} + |v|^2|p|^{-2}), \quad \text{when} \quad p \not\in \Omega^+.
\end{array} \right]
\]

Let us prove the theorem. Represent the integral in the form:

\[
W^+ = \frac{1}{p} \int \int_{\Omega^+} dn \wedge d\bar{n} \exp(-i\omega(n)) +
\]

\[
+ \frac{1}{p^2} \int \int_{\Omega^+} dn \wedge d\bar{n} \exp(-i\omega(n))\bar{n} +
\]

\[
+ \frac{v^2}{12p^2} \int \int_{\Omega^+} \frac{dn \wedge d\bar{n}}{p - n} n^2 \exp(-i\omega(n)) +
\]

\[
+ \frac{1}{12p^2} \int \int_{\partial\Omega^+} \frac{dn \wedge d\bar{n}}{p - n} (12\bar{n}^2 - v^2) \exp(-i\omega(n)).
\]

Integrate by parts the last term. As a result we obtain:

\[
W^+ = \frac{1}{p}\phi_{00}^+(v^2) + \frac{1}{p^2}\phi_{01}^+(v^2) +
\]

\[
+ \frac{1}{12p^2} \int_{\partial\Omega^+} \frac{dn}{p - n} \left( \frac{12\bar{n}^2 - v^2}{-i(12\bar{n}^2 - v^2)} \right) +
\]

\[
O(|p|^{-3} + |v|^2|p|^{-2}).
\]
Consider the integral over the boundary of the domain $\partial \Omega^+$. This integral may be considered as a sum of integrals over imaginary axis $\text{Re}(r) = 0$, over half-circle of radius $R$ as $R \to \infty$, $\text{Re}(r) > 0$ and over the circle $|p - r| = \varepsilon$ as $\varepsilon \to 0$ (if $p \in \Omega^+).$ We can see that the integral over the large half-circle tends to zero. As a result we obtain the statement of the theorem.

One more double integral which we need is:

$$U^+ = \int \int_{\Omega^+} \frac{dr \wedge d\bar{r}}{p - r} \exp(-i\omega(r)).$$

Its asymptotics can be evaluated by the same way as the asymptotics of the integral (85) as $|p| \to \infty$ $|p \pm v| \geq 0$:

$$U = \begin{cases} \frac{1}{p} \phi_{10}^+(v^2) + 2\pi i \frac{1}{12p} \exp(-i\omega(p)) + \frac{\pi i}{12p} + 2\pi i \frac{\exp(-i\omega(p))}{12p^2} + O(|p|^{-2} + |v|^2|p|^{-2}), & \text{when } p \in \Omega^+; \\ \frac{1}{p} \phi_{10}^+(v^2) \frac{\pi i}{12p} + O(|p|^{-2} + |v|^2|p|^{-2}), & \text{when } p \notin \Omega^+. \end{cases}$$

### B Reducing of four-multiply integral into double integral

This section is pure technical. Here we show as the four-multiply integrals have to be reduced into double integrals over half-plane and all complex plane.

#### B.1 Four-multiply integral with nondegenerate phase of the exponent.

Let us show that the four-multiply integral may be written as a sum of double integrals. Change the variable: $n - m = r$, then the integral $J$ has the form:

$$J = \frac{1}{2i\pi} \int \int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{l_j - n} \times$$

$$\frac{1}{2i\pi} \int \int_{\mathbb{C}} \frac{dr \wedge d\bar{r}}{r} \exp(i(r^2 + \bar{r}^2)) \exp(-2i(rn + \bar{r}\bar{n})).$$

Integrate by part over $\bar{n}$. As a result we obtain:

$$J = \frac{-1}{4\pi^2} \lim_{R \to \infty} \int_{|n| = R} \frac{dn\bar{n}}{l_j - n} \times$$
\[
\int \int_C \frac{dr \wedge d\bar{r}}{r} \exp(i(r^2 + \bar{r}^2)) \exp(-2i(rn + \bar{m})) + \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{|l-n|=\epsilon} \frac{dn\bar{m}}{l_j - n} \times \\
\frac{1}{2\pi} \int \int_C \frac{dr \wedge d\bar{r}}{r} \exp(i(r^2 + \bar{r}^2)) \exp(-2i(rn + \bar{m})) - \frac{1}{2\pi} \int \int_C \frac{dn \wedge d\bar{m}}{l_j - n} \times \\
\frac{1}{2\pi} \int \int_C dr \wedge d\bar{r}(-2i) \exp(i(r^2 + \bar{r}^2)) \exp(-2i(rn + \bar{m})).
\]

First term is equal to zero because of fast oscillating of the integrand, second term is equal to residue with sign minus of the integrand at \(n = l\), the internal integral in the third term has to be evaluated. Finally we obtain:

\[
J = \bar{l}_j \frac{\exp(i(l_j^2 + \bar{l}_j^2))}{2\pi} \int \int_C \frac{dn \wedge d\bar{m}}{l_j - m} \exp(-i(n^2 + \bar{m}^2)) - \exp(i(l_j^2 + \bar{l}_j^2)).
\]

**B.2 Four-multiply integral over different half-planes with nondegenerate phase function**

Let us consider an integral:

\[
J_{-+} = \int \int_{\Omega^-} \frac{dn \wedge d\bar{n} \exp(i(n^2 + \bar{n}^2))}{l - n} 2\pi \times \\
\int \int_{\Omega^+} \frac{dm \wedge d\bar{m} \exp(-i(m^2 + \bar{m}^2))}{n - m}.
\]

A partially integrating over \(\bar{n}\) gives:

\[
J_{-+} = \int_{\partial\Omega^-} \frac{dn}{l - n} \exp(i(n^2 + \bar{n}^2)) 2\pi \times \\
\int \int_{\Omega^+} \frac{dm \wedge d\bar{m} \exp(-i(m^2 + \bar{m}^2))}{n - m} - \\
\int \int_{\Omega^-} \frac{dn \wedge d\bar{n} \exp(i(n^2 + \bar{n}^2))}{l - n} \frac{\partial}{\partial \bar{n}} 2\pi \times \\
\int \int_{\Omega^+} \frac{dm \wedge d\bar{m} \exp(-i(m^2 + \bar{m}^2))}{n - m}.
\]

The derivative is:

\[
J_{-+} = \int_{\partial\Omega^-} \frac{dn}{l - n} \bar{n} \exp(i(n^2 + \bar{n}^2)) 2\pi \times 
\]
\[ \int \int_{\Omega^+} \frac{dm \wedge d\bar{m}}{n-m} \exp(-i(m^2 + \bar{m}^2)) - \frac{3}{4} \int_{\partial\Omega^-} \frac{dn}{l-n} \exp(i(n^2 + \bar{n}^2)). \]

Let us consider integral over part of the boundary in first term:

\[ I = \lim_{R \to \infty} \int_{R \exp(3i\pi/4)}^{R \exp(-i\pi/4)} \frac{dn}{l-n} \frac{\tilde{n}}{2i\pi} \int_{\Omega^+} \frac{dm \wedge d\bar{m}}{n-m} \exp(-i(m^2 + \bar{m}^2)). \]

Change the order of integration over \( m, \bar{m} \) and \( n \). The internal integral over \( n \) has to be evaluated:

\[
\lim_{R \to \infty} \int_{R \exp(3i\pi/4)}^{R \exp(-i\pi/4)} \frac{dn}{l-n} \frac{\tilde{n}}{(l-n)(n-m)} = \begin{cases} 
\frac{1}{2} + \frac{m}{l-m}, & \text{when } l \in \Omega^+; \\
-\frac{1}{2} & \text{when } l \in \Omega^+.
\end{cases}
\]

In last expression of \( J^-_+ \) the integrals over the large half of circle \( |n| = R \) as \( R \to \infty \) are equal to zero.

Final formulas for \( J^-_+ \) have to be written as:

\[
J^-_+ = \begin{cases} 
-\frac{5}{4}i\pi + iR \int \int_{\Omega^+} dm \wedge d\bar{m} \frac{\exp(-i(m^2 + \bar{m}^2))}{l-m}, & \text{when } l \in \Omega^+; \\
-\frac{5}{4}i\pi + \frac{3}{2}i\pi \exp(i(l^2 + \bar{l}^2)) - i\exp(i(l^2 + \bar{l}^2)) \times \int \int_{\Omega^+} dm \wedge d\bar{m} \frac{\exp(-i(m^2 + \bar{m}^2))}{l-m}, & \text{when } l \in \Omega^-.
\end{cases}
\]

### B.3 Four-multiply integral with confluent phase function

Let us reduce an four-multiply integral

\[
J_1 = \int \int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{p-n} \exp(i\omega(n)) \int \int_{\mathbb{C}} \frac{dm \wedge d\bar{m}}{n-m} \exp(-i\omega(m))
\]

into sum of double integrals. Denote:

\[
V(n, v^2) = \frac{\exp(i\omega(n))}{2i\pi} \int \int_{\mathbb{C}} \frac{dr \wedge d\bar{r}}{n-r} \exp(-i\omega(r)).
\]

The function \( V(p, v^2) \) has to be written as:

\[
\frac{\partial V}{\partial n} = 12i\tilde{n} \exp(i\omega(n))\phi_{00}(v^2) + 12i \exp(i\omega(n))\phi_{01}(v^2).
\]
We can obtain this formula by changing variable in the integrand \( \rho = n - r \) and differentiating the obtained expression of \( V(n, v^2) \) with respect to \( \bar{n} \) and, finally changing the variable back: \( r = n - \rho \).

Partial differentiating of the double integral over \( \bar{n} \) and \( n \) gives:

\[
\int \int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{p - n} V(n, v^2) = \lim_{R \to \infty} \int_{|n| = R} \frac{dn \bar{n}}{p - n} V(n, v^2) + \\
+ \lim_{\epsilon \to 0} \int_{|\rho - |n|\epsilon = \epsilon} \frac{dn \bar{n}}{p - n} V(n, v^2) - \int \int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{p - n} \frac{\partial V(n, v^2)}{\partial \bar{n}}.
\]

(87)

Asymptotic behaviour of \( V(n, v^2) \) as \( |n| \to \infty \) was obtained above. Using this asymptotics for the first part of the formula (87) gives zero. Second term of the right-hand side of (87) gives the residue of the integrand at \( p \) multiplied by \( 2i\pi \). Then we obtain:

\[
J_1(p, v^2) = \bar{p} V(p, v^2) - \phi_{00}(v^2) \exp(i\omega(p)) - \\
-12i\phi_{01}(v^2) \frac{1}{2i\pi} \int \int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{p - r} \bar{n} \exp(i\omega(n)) + \\
+ i v^2 \phi_{00}(v^2) \frac{1}{2i\pi} \int \int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{p - r} \exp(i\omega(n)).
\]

On the same way one can evaluate an integral

\[
J_1^{-+} = \int \int_{\Omega^-} \frac{dn \wedge d\bar{n}}{p - n} \exp(i\omega(n)) \int \int_{\Omega^+} \frac{dm \wedge d\bar{m}}{\bar{n} - m} \exp(-i\omega(m)).
\]

The difference between the integral \( J_1 \) and the integral \( J_1^{-+} \) consists in the addition terms over boundaries of the domains \( \Omega^+ \) and \( \Omega^- \) in the result. The finally forms are:

when \( p \in \Omega^+ \):

\[
J_1^{-+} = -\phi_{00}^+(v^2) \exp(i\omega(p)) + \\
+\bar{p} \exp(i\omega(p)) \int \int_{\Omega^+} \frac{dn \wedge d\bar{n}}{p - m} \exp(-i\omega(m)) + \\
\frac{1}{2} \int \int_{\Omega^-} \frac{dn \wedge d\bar{n}}{p - m} \bar{n} \exp(i\omega(n)) + \\
+ i v^2 \phi_{00}^+(v^2) \frac{1}{2i\pi} \int \int_{\Omega^-} \frac{dm \wedge d\bar{m}}{\bar{n} - m} \exp(i\omega(n)) + \\
-12i\phi_{01}^+(v^2) \frac{1}{2i\pi} \int \int_{\Omega^-} \frac{dm \wedge d\bar{m}}{\bar{n} - m} \exp(i\omega(n));
\]

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when $p \not\in \Omega^+$:

$$J_1^{-+} = -\phi_0^+(v^2) + p\int \int_{\Omega^+} \frac{dm \wedge d\bar{m}}{p + \bar{m}} \exp(-i\omega(m))$$

$$-\frac{1}{2} \int \int_{\Omega^-} \frac{dn \wedge d\bar{n}}{p - n} \bar{n} \exp(i\omega(n))$$

$$i v^2 \phi_0^+(v^2) \frac{1}{2i\pi} \int \int_{\Omega^-} \frac{dn \wedge d\bar{n}}{p - n} \exp(i\omega(n))$$

$$-12i\phi_{01}^+(v^2) \frac{1}{2i\pi} \int \int_{\Omega^-} \frac{dn \wedge d\bar{n}}{p - n} \exp(i\omega(n)).$$

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