An order result for the exponential divisor function

LÁSZLÓ TÓTH
University of Pécs, Institute of Mathematics and Informatics,
Ifjúság u. 6, 7624 Pécs, Hungary,
ltoth@ttk.pte.hu

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Abstract: The integer \( d = \prod_{i=1}^{s} p_i^{b_i} \) is called an exponential divisor of \( n = \prod_{i=1}^{s} p_i^{a_i} > 1 \) if \( b_i \mid a_i \) for every \( i \in \{1, 2, ..., s\} \). Let \( \tau^{(c)}(n) \) denote the number of exponential divisors of \( n \), where \( \tau^{(c)}(1) = 1 \) by convention. The aim of the present paper is to establish an asymptotic formula with remainder term for the \( r \)-th power of the function \( \tau^{(c)} \), where \( r \geq 1 \) is an integer. This improves an earlier result of M. V. SUBBARAO [5].

1. Introduction

Let \( n > 1 \) be an integer of canonical form \( n = \prod_{i=1}^{s} p_i^{a_i} \). The integer \( d \) is called an exponential divisor of \( n \) if \( d = \prod_{i=1}^{s} p_i^{b_i} \), where \( b_i \mid a_i \) for every \( i \in \{1, 2, ..., s\} \), notation: \( d \mid_{e} n \). By convention \( 1 \mid_{e} 1 \).

Let \( \tau^{(c)}(n) \) denote the number of exponential divisors of \( n \). The function \( \tau^{(c)} \) is called the exponential divisor function. J. Wu [7] showed, improving an earlier result of M. V. SUBBARAO [5], that

\[
\sum_{n \leq x} \tau^{(c)}(n) = Ax + Bx^{1/2} + O(x^{2/9} \log x),
\]

where

\[
A := \prod_{p} \left( 1 + \sum_{a=2}^{\infty} \frac{\tau(a) - \tau(a-1)}{p^a} \right), \quad B := \prod_{p} \left( 1 + \sum_{a=5}^{\infty} \frac{\tau(a) - \tau(a-1) - \tau(a-2) + \tau(a-3)}{p^{a/2}} \right),
\]

\( \tau \) denoting the usual divisor function. The \( O \)-term can further be improved.

Other properties of the function \( \tau^{(c)} \), compared with those of the divisor function \( \tau \) were investigated in papers [1], [2], [4], [5].

M. V. SUBBARAO [5] remarked that for every positive integer \( r \),

\[
\sum_{n \leq x} (\tau^{(c)}(n))^r \sim A_r x,
\]

where

\[
A_r := \prod_{p} \left( 1 + \sum_{a=2}^{\infty} \frac{(\tau(a))^r - (\tau(a-1))^r}{p^a} \right).
\]

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It is the aim of the present paper to establish the following more precise asymptotic formula for the \( r \)-th power of the function \( \tau^{(e)} \), where \( r \geq 1 \) is an integer:

\[
\sum_{n \leq x} (\tau^{(e)}(n))^r = A_r x + x^{1/2} P_{2r-2}(\log x) + O(x^{u_r+\varepsilon}),
\]

for every \( \varepsilon > 0 \), where \( A_r \) is given by (3), \( P_{2r-2} \) is a polynomial of degree \( 2^r - 2 \) and \( u_r := \frac{2^{r+1} - 1}{2^{r+1} + 1} \).

Note that a similar formula is known for the divisor function \( \tau \), namely for any integer \( r \geq 2 \),

\[
\sum_{n \leq x} (\tau(n))^r = x Q_{2r-1}(\log x) + O(x^{v_r+\varepsilon}),
\]

valid for every \( \varepsilon > 0 \), where \( v_r := \frac{2^{r+1} - 1}{2^{r+1} + 1} \) and \( Q_{2r-1} \) is a polynomial of degree \( 2^r - 1 \), this goes back to the work of S. Ramanujan, cf. [8].

Formula (4) is a direct consequence of a simple general result, given in Section 2 as Theorem, regarding certain multiplicative functions \( f \) such that \( f(n) \) depends only on the \( \ell \)-full kernel of \( n \), where \( \ell \geq 2 \) is a fixed integer.

We also consider a generalization of the exponential divisor function, see Section 4.

Let \( \phi^{(e)}(n) \) denote the number of divisors \( d \) of \( n \) such that \( d \) and \( n \) have no common exponential divisors. The function \( \phi^{(e)} \) is multiplicative and for every prime power \( p^a \ (a \geq 1) \), \( \phi^{(e)}(p^a) = \phi(a) \), where \( \phi \) is the Euler function.

As another consequence of our Theorem we obtain for every integer \( r \geq 1 \) that

\[
\sum_{n \leq x} (\phi^{(e)}(n))^r = B_r x + x^{1/3} R_{2r-2}(\log x) + O(x^{t_r+\varepsilon}),
\]

for every \( \varepsilon > 0 \), where \( t_r := \frac{2^{r+1} - 1}{3^{r+1} + 1} \), \( R_{2r-2} \) is a polynomial of degree \( 2^r - 2 \) and \( \phi^{(e)}(p^a) = \phi(a) \),

\[
B_r := \prod_{p} \left( 1 + \sum_{a=1}^{\infty} \frac{(\phi(a))^r - (\phi(a-1))^r}{p^a} \right).
\]

In the case \( r = 1 \) formula (6) was proved in [6] with a better error term. Our error terms depend on estimates for

\[
D(1, \ell, \ell, \ldots, \ell; x) := \sum_{ab_1 b_2 \cdots b_{k-1} \leq x} 1,
\]

where \( k, \ell \geq 2 \) are fixed and \( a, b_1, b_2, \ldots, b_{k-1} \geq 1 \) are integers.

2. A general result

We prove the following general result.

**Theorem.** Let \( f \) be a complex valued multiplicative arithmetic function such that

a) \( f(p) = f(p^2) = \cdots = f(p^{\ell-1}) = 1, f(p^\ell) = f(p^{\ell+1}) = k \) for every prime \( p \), where \( \ell, k \geq 2 \) are fixed integers and

b) there exist constants \( C, m > 0 \) such that \( |f(p^n)| \leq C a^m \) for every prime \( p \) and every \( a \geq \ell + 2 \).

Then for \( s \in \mathbb{C} \)

i) \( F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \zeta^{k-1}(\ell s) V(s), \quad \text{Re } s > 1, \)

where the Dirichlet series \( V(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s} \) is absolutely convergent for \( \text{Re } s > \frac{1}{4\ell^2} \),

ii) \[
\sum_{n \leq x} f(n) = C_f x + x^{1/\ell} P_{\ell,k-2}(\log x) + O(x^{u_{k, \ell}+ \varepsilon}),
\]
for every \( \varepsilon > 0 \), where \( P_{\ell, k-2} \) is a polynomial of degree \( k - 2 \), \( u_{k, \ell} := \frac{2k-1}{3^j(2k-1)^j} \) and

\[
C_{\ell} := \prod_p \left( 1 + \sum_{a=\ell}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a} \right).
\]

iii) The error term can be improved for certain values of \( k \) and \( \ell \). For example in the case \( k = 3 \), \( \ell = 2 \) it is \( O(x^{8/25} \log^3 x) \).

3. Proofs

The proof of the Theorem is based on the following Lemma. For an integer \( \ell \geq 1 \) let \( \mu_\ell(n) = \mu(m) \) or 0, according as \( n = m^\ell \) or not, where \( \mu \) is the Möbius function. Note that function \( \mu_\ell \) is multiplicative and for any prime power \( p^\alpha \) (\( a \geq 1 \)),

\[
\mu_\ell(p^\alpha) = \begin{cases} -1, & \text{if } a = \ell, \\ 0, & \text{otherwise}. \end{cases}
\]

Furthermore, for an integer \( h \geq 1 \) let the function \( \mu_\ell^{(h)} \) be defined in terms of the Dirichlet convolution by

\[
\mu_\ell^{(h)} = \mu_\ell \ast \mu_\ell \ast \cdots \ast \mu_\ell.
\]

The function \( \mu_\ell^{(h)} \) is also multiplicative.

**Lemma.** For any integers \( h, \ell \geq 1 \) and any prime power \( p^\alpha \) (\( a \geq 1 \)),

\[
\mu_\ell^{(h)}(p^\alpha) = \begin{cases} (-1)^j \binom{h}{j}, & \text{if } a = j\ell, \quad 1 \leq j \leq h, \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof of the Lemma.** By induction on \( h \). For \( h = 1 \) this follows from (8). We suppose that formula (9) is valid for \( h \) and prove it for \( h + 1 \). Using the relation \( \mu_\ell^{(h+1)} = \mu_\ell^{(h)} \ast \mu_\ell \) and (8) we obtain for \( a < \ell \),

\[
\mu_\ell^{(h+1)}(p^\alpha) = \mu_\ell^{(h)}(p^\alpha) = 0
\]

and for \( a \geq \ell \),

\[
\mu_\ell^{(h+1)}(p^\alpha) = \mu_\ell^{(h)}(p^\alpha) - \mu_\ell^{(h-\ell)}(p^{\alpha-\ell}) = \begin{cases} \mu_\ell^{(h)}(p^\ell) - 1 = (-1)^1 \binom{h}{1} - 1 = -(-1)^1 \binom{h+1}{1}, & \text{if } a = \ell, \\ (-1)^j \binom{h}{j} - (-1)^{j-1} \binom{h}{j-1} = (-1)^j \binom{h+1}{j}, & \text{if } a = j\ell, \quad 2 \leq j \leq h, \\ -\mu_\ell(p^{h\ell}) = -(-1)^h \binom{h}{h} = (-1)^{h+1} \binom{h+1}{h+1}, & \text{if } a = (h+1)\ell, \\ 0, & \text{otherwise}, \end{cases}
\]

which proves the Lemma.

**Proof of the Theorem.** i) We can formally obtain the desired expression by taking \( v = f \ast \mu \ast \mu^{(k-1)} \). Here \( v \) is multiplicative and easy computations show that \( v(p^\alpha) = 0 \) for any \( 1 \leq a \leq \ell + 1 \) and for \( a \geq \ell + 2 \),

\[
v(p^\alpha) = \sum_{j \geq 0} (-1)^j \binom{k-1}{j} (f(p^{\alpha-j\ell}) - f(p^{\alpha-j\ell-1})),
\]

where, according to the Lemma, the number of nonzero terms is at most \( k \).

Let \( M_k = \max_{0 \leq j \leq k-1} \binom{k-1}{j} \). We obtain that for every prime \( p \) and every \( a \geq \ell + 2 \),

\[
|v(p^\alpha)| \leq 2kM_kCa^m.
\]
For every $\varepsilon > 0$, $a^n \leq 2^{a^2}$ for sufficiently large $a$, $a \geq a_0$ say, where $a_0 \geq \ell + 2$. For $Re\ s > 1/(\ell + 2)$ choose $\varepsilon > 0$ such that $Re\ s - \varepsilon > 1/(\ell + 2)$. Then

\[
\sum_{p} \sum_{a \geq a_0} \frac{|v(p^a)|}{p^{as}} \leq 2kM_kC \sum_{p} \sum_{a \geq a_0} \frac{2^{a\varepsilon}}{p^{as}} \leq 2kM_kC \sum_{p} \sum_{a \geq a_0} \frac{1}{p^{a(s - \varepsilon)}} = \]

\[
= 2kM_kC \sum_{p} \frac{1}{p^{a(s - \varepsilon)}} \left(1 - \frac{1}{p^{s - \varepsilon}}\right)^{-1} \leq 2kM_kC \left(1 - \frac{1}{2^{1/(\ell + 2)}}\right)^{-1} \sum_{p} \frac{1}{p^{a(s - \varepsilon)}},
\]

and obtain that $V(s)$ is absolutely convergent for $Re\ s > 1/(\ell + 2)$.

Note that $v(p^{\ell + 2}) = f(p^{\ell + 2}) - k$ for every $\ell \geq 3$, $k \geq 2$ and for $\ell = 2, k \geq 2$ it is $v(p^2) = f(p^2) - (k+1)/2$.

ii) Consider the $k$-dimensional generalized divisor function

\[
d(1, \ell, 1; n) = \sum_{ab(n)} 1.
\]

According to i),

\[
f(n) = \sum_{ab(n)} d(1, \ell, \ell, \ell; a) v(b).
\]

One has, see [3], Ch. 6,

\[
(10) \quad \sum_{n \leq x} d(1, \ell, \ell, \ell; n) = K_1 x + x^{1/\ell} \left(K_2 \log^{k-2} x + K_3 \log^{k-3} x + \ldots + K_{k-1} \log x + K_k\right) + O(x^{u_{k,\ell} + \varepsilon}),
\]

for every $\varepsilon > 0$, where $u_{k,\ell} = \frac{2k - 1}{s\ell(2k - 1)}$ (see [3], Theorem 6.10). $K_1, K_2, \ldots, K_{k-1}, K_k$ are absolute constants depending on $k$ and $\ell$ and $K_2 = \zeta^{-1}(\ell)$. For example for $k = 2$ one has $K_2 = \zeta(1/2)$, and for $k = 3$: $K_2 = \frac{1}{2} \zeta(1/3), K_3 = (2\gamma - 1) \zeta(1/3) + \frac{1}{2} \zeta'(1/3)$, where $\gamma$ is Euler’s constant.

We obtain

\[
\sum_{n \leq x} f(n) = \sum_{ab \leq x} d(1, \ell, \ell, \ell; a) v(b) = \sum_{b \leq x} v(b) \sum_{a \leq x/b} d(1, \ell, \ell, \ell; a) = \sum_{b \leq x} v(b) \left(K_1(x/b) + (x/b)^{1/\ell} \left(K_2 \log^{k-2}(x/b) + K_3 \log^{k-3}(x/b) + \ldots + K_{k-1} \log(x/b) + K_k\right) + O((x/b)^{u + \varepsilon})\right),
\]

and obtain the desired result by partial summation and by noting that $u_{k,\ell} > 1/(\ell + 2)$.

iii) For $k = 3, \ell = 2$ the error term of (10) is $O(x^{8/25} \log^3 x)$, cf. [3], Theorem 6.4.

4. Applications. 1. In case $f(n) = (\tau(r(n))^r$, where $r \geq 1$ is an integer, we obtain formula (4) applying the Theorem for $\ell = 2, k = 2^r$.

2. For $k \geq 2$ consider the multiplicative function $f(n) = \tau_2^{(r)}(n)$, where for every prime power $p^a$ ($a \geq 1$), $\tau_2^{(r)}(p^a) := \tau_2(a)$ representing the number of ordered $k$-tuples of positive integers $(x_1, \ldots, x_k)$ such that $a = x_1 \cdot \ldots \cdot x_k$. Here $\tau_2(p^b) = \binom{b+k-1}{k-1}$ for every prime power $p^b$ ($b \geq 1$). In case $k = 2$, $\tau_2^{(r)}(n) = \tau^{(r)(n)}$.

Taking $\ell = 2$ and $k := k$ we obtain that $v(p^2) = \tau_2(4) - k(1 + 2)/2 = 0$ and $V(s)$ is absolutely convergent for $Re\ s > 1/5$ (and not only for $Re\ s > 1/4$ given by the Theorem),

\[
(11) \quad \sum_{n \leq x} \tau_2^{(r)}(n) = C_2 x + x^{1/2} S_{k-2}(\log x) + O(x^{u_{k,\ell} + \varepsilon}),
\]

4
for every $\varepsilon > 0$, where $S_{k-2}$ is a polynomial of degree $k - 2$, $w_k := \frac{k-1}{2k+1}$ and

$$C_k = \prod_p \left( 1 + \sum_{a=2}^{\infty} \frac{\tau_k(a) - \tau_k(a-1)}{p^a} \right).$$

For $k = 3$ the error term of (11) can be improved into $O(x^{8/25} \log^3 x)$.

A similar formula can be obtained for $\sum_{n \leq x} (\tau_k^{(e)}(n))^{\ell}$.  

3. For the function $\phi^{(e)}(n)$ defined in the Introduction we obtain formula (6) by choosing $\ell = 3$, $k = 2^\ell$.

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