Dynamical Realizability for Quantum Measurement and Factorization of Evolution Operator

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Abstract

By building a general dynamical model for quantum measurement process, it is shown that the factorization of reduced evolution operator sufficiently results in the quantum mechanical realization of the wave packet collapse and the state correlation between the measured system and the measuring instrument-detector. This realizability is largely independent of the details of both the interaction and Hamiltonian of detector. The Coleman-Hepp model and all its generalizations are only the special cases of the more universal model given in this letter. An explicit example of this model is finally given in connection with coherent state.

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It is well-known that, though the theory of quantum mechanics and its applications are extremely successful, its interpretation in connection with the corresponding measurement is still a valued problem that the physicist must face [1-3]. Since an exactly-solvable model was presented twenty years ago [4] to describe von Neumann’s wave packet collapse (WPC) in measurement as a quantum dynamical process caused by the interaction between the measured system (S) and the measuring instrument-detector (D), the considerable studies have been focused this model [5-9] which is now called Coleman-Hepp (CH) model. More recently, it was respectively generalized to the case with energy exchanging between S and D [8] and to the case simultaneously with the classical limit—the large quantum number limit and the macroscopic limit—the large particle number limit [9]. Notice that another important problem in quantum measurement, the state correlation between S and D (SCBSD) can also be studied by making use of other solvable-models, e.g., the Cini model in ref. [10].

However, all the investigations mentioned above only proceeded with the concrete forms of interaction and thus the main conclusions seem to depend on the selection of concrete form of the interactions. It is undoubtful that the model-independent study for this problem is much appreciated. The present studies, which is briefly reported in this letter, are dedicated to seek of essence of the quantum mechanical realization of the WPC and SCBSD in the CH-model and its generalizations. To this end a more universal dynamical model of quantum measurement is proposed as interaction-independently as possible. Based on this model, we will show that the realizability of the WPC and SCBSD as a quantum dynamical process mainly depends on the factorizability of the reduced evaluation operator for the system. This crucial observation not only reveals the essence rooted in those well-established exactly-solvable models for quantum measurement, but also provides us with a guidance to find new exactly-solvable models. Finally, an exactly-solvable model associated coherent state will be analysed as an explicit example of them in detail.

1. The General Model. Our model can be regarded as an universal promotion of the original CH model. The measured system S is still represented by an ultrarelativistic particle with the free Hamiltonian $H_0 = c \hat{p}$, but the detector D is made of N particles with single-particle Hamiltonian $h_k(x_k), (k = 1, 2, ..., N)$ which is Hermitian. S is assumed to be independently subjected to the interaction $V_k(x, x_k)$ of each particle k. Here, $x$ and $x_k$ are the coordinates of S and the single particle k in D respectively and
the $k$'th interaction potential $V_k(x,x_k)$ only depends on $x$ and $x_k$ and $h_k(x_k)$ on the single particle coordinate $x_k$ and the corresponding momentum. Then, we can write down the total Hamiltonian for the ‘universe’ $\mathcal{S} + D$

$$H = H_0 + H' = H_I + H_D:\$$

$$H_I = \sum_{k=1}^N V_k(x,x_k), H_D = \sum_{k=1}^N h_k(x_k),$$

(1)

where

$$H' = H_I + H_D = \sum_{k=1}^N [h_k(x_k) + V_k(x,x_k)],$$

(2)

is a direct sum decomposition of single-particle forms. This fact, associated with the fact that the $H_0$ is of the first order of $\hat{p}$, will lead to the factorization of effective (reduced) evaluation operator and thereby produces the WPC in quantum measurement. This factorizability is also related to the SCBSD closely. Fortunately, to prove it we need not the further concrete forms of both $h_k(x_k)$ and $V_k(x,x_k)$. In this sense we say this model is more universal. It is worth notice that the original HC model and its generalizations are the special examples of this universal model.

2. The Evolution Operator. In order to interpretate the WPC and SCBSD as the consequence of the Schrödinger evolution of the universe $(S+D)$, we should consider the properties of the evolution operator defined by the general Hamiltonian (1). Following Hepp [4], we first transform into the ‘moving’ representation (also loosely called interaction representation) by assuming the evolution operator to be the following form

$$U(t) = e^{-ict\hat{p}/\hbar}U_e(t),$$

(3)

Obviously, the reduced evaluation operator $U_e(t)$ obeys an effective Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} U_e(t) = H_e(t)U_e(t),$$

(4)

with the effective Hamiltonian

$$H_e(t) = \sum_{k=1}^N h_{ek}(t) = \sum_{k=1}^N [h_k(x_k) + V_k(x + ct, x_k)],$$

(5)

depending on time. Since $H_e(t)$ is a direct sum of the time-dependent Hamiltonians $h_{ek}(t)$ ($k=1,2,\ldots,N$) parameterized by $x$, the $x$-dependent evolution operator, as the
formal solution to the eq.(4)

\[ U_e(t) = \prod_{k=1}^{N} \otimes U^{[k]}(t) = U^{[1]}(t) \otimes U^{[2]}(t) \otimes \ldots \otimes U^{[N]}(t), \]  

(6)

is factorizable, that is to say, \( U_e(t) \) is a direct product of the single-particle evolution operator

\[ U^{[k]}(t) = \mathfrak{S} \exp \left( \frac{1}{i\hbar} \int_0^t h_{ek}(t) dt \right), \]  

(7)

where \( \mathfrak{S} \) denotes the time-order operation. As proved in as follows, it is just the above factorizable property of the reduced evolution operator that results in the quantum dynamical realization of the WPC and closely related to the SCBSD in quantum measurement.

3. The Wave Packet Collapse as a Quantum Dynamical process. Let us use the ideal double-slit experiment of interference to show the WPC as the consequence of the quantum dynamical evolution of the universe \( S+D \) described by the above factorizable evolution operator. An incident wave is split by a divider into two branches \( | \psi_1 > \) and \( | \psi_2 > \) and the detector \( D \) is in the ground state

\[ | 0 > = | 0_1 > \otimes | 0_2 > \otimes \ldots | 0_N >, \]  

(8)

in the same time where \( | 0_k > \) is the ground state of \( h_k(x_k) \). Then, the initial state for the universe \( S+D \) is

\[ | \psi(0) > = (C_1 | \psi_1 > + C_2 | \psi_2 >) \otimes | 0 >, \]  

(9)

Notice that the ground state is required by a stable measuring instrument. Starting with this initial state the universe \( S+D \) will evolve according to the wave function

\[ | \psi(t) > = C_1 | \psi_1 > \otimes | 0 > + C_2 | \psi_2 > \otimes U_e(t) | 0 >, \]  

(10)

Here, like the authors in ref.[4-8], we have supposed that only the second branch wave \( | \psi_2 > \) interacts with \( D \) so that the double-slit experiment of interference can be realized. In fact, such a partiality of interaction for different states of \( S \) can be automatically given in the ‘momentum’ (p-) representation with the basis \( | p > \) if we use an improved Hamiltonian

\[ H = H_0 + H' = H_0 + H_I \delta_{p_0}(\hat{p}) + H_D, \]  

(11)

obtained by introducing an operator function \( \delta_{p_0}(\hat{p}) \):

\[ \delta_{p_0}(\hat{p}) | p > = \delta_{p_0,p} | p > \]
to the original Hamiltonian (1) and take

$$|\psi_1> = \sum_{p' \neq p_0} C_{p'} |p'>, |\psi_2> = |p_0>,$$  \hspace{1cm} (12)

From the final state (10) we explicitly write down the density matrix for the universe S+D

$$\rho(t) = |\psi(t)> <\psi(t)| = |C_1|^2 |\psi_1(t)> <\psi_1(t)| \otimes |0><0| +$$

$$|C_2|^2 |\psi_2(t)> <\psi_2(t)| \otimes U_e(t) |0><0| U_e(t)^+ +$$

$$+ C_1 C_2^* |\psi_1(t)> <\psi_2(t)| \otimes U_e(t) |0><0|$$

$$+ C_2 C_1^* |\psi_2(t)> <\psi_1(t)| \otimes |0><0| U_e(t)^+.$$  \hspace{1cm} (13)

In the problem of WPC, because we are only interest in the behaviors of the system S and the effect of the detector D on it, we only need the reduced density matrix for S

$$\rho(t)_S = Tr_D \rho(t) = |C_1|^2 |\psi_1(t)> <\psi_1(t)| + |C_2|^2 |\psi_2(t)> <\psi_2(t)| +$$

$$(C_1 C_2^* |\psi_1(t)> <\psi_2(t)| + C_2 C_1^* |\psi_2(t)> <\psi_1(t)|) <0| U_e(t) |0>,$$  \hspace{1cm} (14)

where $Tr_D$ represents the trace to the variables of the detector D. Let us recall that the WPC postulate means the reduction of pure state density matrix

$$\rho(t)_S \rightarrow \rho(t)_S = |C_1|^2 |\psi_1(t)> <\psi_1(t)| + |C_2|^2 |\psi_2(t)> <\psi_2(t)|,$$  \hspace{1cm} (15)

Obviously, under a certain condition to be determined, if $<0| U_e(t) |0> = 0$, then the coherent terms in eq.(14) vanish and the quantum dynamics automatically leads to this reduction, i.e., the WPC! Now, let us prove that this condition is just the macroscopic limit defined by very large particle number $N$ of D, i.e., $N \rightarrow \infty$. In fact, due to the factorization of the reduced evolution operator $U_e(t)$, the norm of $<0| U_e(t) |0>$ is

$$|<0| U_e(t) |0>| = \prod_{k=1}^N |<0_k| U_j^{[k]} |0_k>| = exp[-\sum_{k=1}^N \Delta_k(t)],$$  \hspace{1cm} (16)

where

$$e^{-\Delta_k(t)} = |<0_k| U_j^{[k]} |0_k>| = |1 - \sum_{n \neq 0} |<n| U_j^{[k]} |0_k>|^2|^{1/2} \leq 1,.$$  \hspace{1cm} (17)

Usually, $\Delta_k(t)$ is a non-zero and positive and thus the series $\sum_{k=1}^\infty \Delta_k(t)$ diverges to infinity, that is to say, $<0| U_e(t) |0>$ as well as its norm approach zero as $N \rightarrow \infty$. This just proves a central conclusion that the WPC can appear as a quantum dynamical
process for the universal model (1) in the macroscopic limit as long as the dynamical models are selected to have the factorizable evolution operators. However, in this case, there is not interactions among the particles in the detector. We understand it as an ideal case. Because the particles in a realistic measuring instrument must interact with each other, it is necessary to build the exactly-solvable dynamic model of quantum measurement with self-interacting detector. We believe the above-mentioned factorization property probably is also a clue to find such model.

In terms of the above model, we can also discuss the energy-exchanging process and the delicate behavior in $N \to \infty$ such as in ref. [8]. For the latter, we have

$$| < 0 | U_e(t) | 0 > | \sim e^{-N \bar{\Delta}_k(t)}$$

where $\bar{\Delta}_k(t)$ represents the average value of $\Delta_k(t)$. The above formula shows that gradual disappearance of the interference.

4. Correlation Between States of System and Detectors. Physically, the measurement is a scheme using the counting number of the measuring instrument $D$ to manifest the state of the measured system $S$. The state correlation between $S$ and $D$ (SCBSD) will enjoy this manifestation. Now, we show how this correlation occurs for the above dynamical model (1) in the certain limit. To simplify the problem, we also use the improved Hamiltonian (11). Let $c_{m_k}$ be a one with largest norm among the coefficients $c_{n_k} = < n_k | U^{[k]} | 0_k >$ ($k = 1, 2, \ldots$) and assume that the state $| m_k >$ with maximum probability amplitude is not degenerate. Then,

$$U^{[k]}(t) | 0 > = c_{m_k} \{ | m_k > + \sum_{n \neq m} [c_n/c_{m_k}] | n > \}.$$  \hspace{1cm} (18)

Except the coefficient of $| m_k >$, each of ones in $c_{m_k}^{-1} U^{[k]}(t) | 0 >$ has a norm less than 1. Because of the factorization of the reduced evolution operator, the wave function $U(t) | 0 > = \prod_{k=1}^{N} U^{[k]}$ will be strongly peaked around the state

$$| m > = | m_1 > \otimes | m_2 > \otimes | m_3 > \otimes \ldots \otimes | m_N > .$$

If the universe $S+D$ with the Hamiltonian (6) initially is in the state

$$| \psi(0) > = [ | p_0 > + | p > ] \otimes | 0 > , p \neq p_0 .$$

Then, it will evolve into a state around the state

$$| \psi(t) > = | p_0 > \otimes \prod_{k=1}^{N} c_{m_k} | m > + | p > \otimes | 0 > .$$  \hspace{1cm} (19)
This just manifests the correlation between the state $|p_0>$ of the system and $|m>$ of the detector. Notice that the SCBSD can exactly appears only for the 'classical' limit, in which some parameters or internal quantum quantum number take their limit values (e.g., in ref. [9,10], this is the limit with infinite spin). In fact, in the realistic problem, the correlation often occurs as a good approximation valid to quite high degree. A special example of such correlation problem was discussed in ref. [10]. According to the above general arguments, We will given a new example to deal with both the WPC and SCBSD.

4. New Exactly-Solvable Dynamical Model for Quantum Measurement and Coherent States. Up to now we have described the general form of the Hamiltonian as eq. (1) for the dynamics of quantum measurement. It is easy to see that the CH model and its various generalizations are only some special and explicit examples of the above more universal model. Now, let us apply above general results, as a guidance rule, to built new exactly-solvable model for quantum measurement. In this model, the WPC and SCBSD can be simultaneously described as a quantum dynamical process. The model Hamiltonian is

$$H = c\hat{p} + \sum_{k=1}^{N} f_k(x)[a_k^+ + a_k] + \sum_{k=1}^{N} \hbar \omega_k a_k^+ a_k,$$  \hspace{1cm} (20)$$

Here, the detector is made of $N$ harmonic oscillators linearly coupled to an ultrarelativistic particle as the measured system; $a_k^+$ and $a_k$ are the creation and annihilation operators for boson states respectively. The coupling function $f_k(x) = c_k x$ only depends on the coordinate of the ultrarelativistic particle. Because the effective Hamiltonian

$$H_e(t) = \sum_{k=1}^{N} (f_k(x + ct))[a_k^+ + a_k] + \hbar \omega_k a_k^+ a_k),$$  \hspace{1cm} (21)$$

is completely decomposable, the reduced evolution operator $U_e$ is factorizable, i.e., $U_e = \prod_k U^{[k]}(t)$, and its factors are$[11]$

$$U^{[k]}(t) = e^{-i\hbar \omega_k a_k^+ a_k e^{h_k(t)} e^{A_k(t)} a_k^+ e^{-B_k(t)} a_k},$$  \hspace{1cm} (22)$$

where the functions $h_k(t), A_k(t), B_k(t)$ are defined by

$$A_k(t) = B_k(t) = \left. \frac{c_k}{\hbar \omega_k^2} [(x + ct + ic/\omega_k)e^{i\hbar \omega_k t} - ic/\omega_k - x] \right|_{t=0},$$  \hspace{1cm} (23)$$

$$h_k(t) = \int_{0}^{t} A_k(s) \frac{\partial B_k(s)}{\partial s} ds$$
Notice that the real part of $h_k(t)$ is $-\eta_k(t)$:

$$
\eta_k(t) = \frac{c_k}{\hbar \omega_k^2} [(c^2 t^2 + xct + 2(x^2 + c^2/\omega_k^2)\sin^2 \omega_k t/2 - c \omega_k t (x \cos \omega_k t - (c/\omega_k) \sin \omega_k t)] .
$$

(24)

Since $\eta_k(t)$ is larger than zero after an interval of time, the norm

$$
| < 0 | U(t) | 0 > | = e^{-\sum_{k=1}^{N} \eta_k(t)}
$$

must approach zero as $N \to \infty$. This implies the dynamical realization of the WPC for quantum measurement.

Let us show how the correlation between the states of the system and the detector appears as a quantum dynamical process. If the detector is initially in its ground state $| 0 > = | 0_1 > \otimes | 0_2 > \otimes ... | 0_N >$, the state at $t$ is a direct product of the coherent states.

$$
| \psi_k(t) > = U[k](t) | 0 > = e^{h_k(t)} \sum_{n=0}^{\infty} e^{-i n \omega_k t} \frac{A_n^k(t)}{n!} a_k^{+n} | 0_k > ,
$$

(25)

Using the Stirling formula, we immediately determine the value $\bar{n}_k$ of quantum number $n$ for which the norm of the coefficient of Fock state $| n > f = 1/(n!)^{1/2} a_k^{+n} | 0_k >$ in the above expansion (25) is maximum, obtaining

$$
\bar{n}_k = | A_k(t) |^2 ,
$$

(26)

In this time the validity of Stirling formula require $\bar{n}_k = | A_k(t) |^2$ to be sufficiently large. This means that the counting number of the detector is macroscopically large. It is just what we expect for a measuring instrument. If we take the initial state of the system is $| \psi(0) > = [ v| p_0 > + w| p \neq p_0 > ] \otimes | 0 >$, then the correlation is enjoyed by the Schrodinger evolving state

$$
| \psi(t) > = v| p_0 > \otimes | \bar{n}_1 > \otimes | \bar{n}_2 > \otimes ... \otimes | \bar{n}_N > + w| p > \otimes | 0 > .
$$

(27)

6 Final Remarks. Finally, we should point out that in practical problems there must exist interactions among the particles constituting the detector D. They seem to break the factorization of the reduced evolution operator. How to realize the quantum measurement both for the WPC and SCBSD in this case is an open question we must face. It is expected, at least for some special case, that the certain canonical (or unitary) transformation possibly enable these particles to become the quasi-free
This is just similar to the system of harmonic oscillators with quartic coupling. In this case, we can imagine that the detector is made of free quasi-particles that do not interact with each other. If each quasi-particle interacts with the system independently, then the factorizability of the evolution operator can perseveres in the solvable models for quantum measurement.

We also remark on the realization of the double-slit type experiment where the interaction selects only one of the two branch wave functions. For the introduction of $\delta_{p_0}(\hat{p})$ in the Hamiltonian (11) (to realize the partiality of interaction), someone may not feel content. In fact, we can also enjoy this selection in a quite natural way. If the system is a spin-1/2 with the free Hamiltonian $H_0 = \hbar \omega \sigma_3$ and the detector is still defined by the general Hamiltonian in eq. (2), the following interaction

$$H_I = g(1 + \sigma_3) \sum_{k=1}^N \hbar_k(x_i),$$  \hspace{1cm} (28)

naturally results in the selection of interaction. Namely, the detector only acts on the spin-up state. The spin-down state is free of interaction. Thus, the eq. (28) defines a new dynamical model for quantum measurement, which is an extensive generalization of Cini’s model [10].

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