Controllability and lack of controllability with smooth controls in viscoelasticity via moment methods

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Abstract: In this paper we study controllability of a linear equation with persistent memory when the control belongs to $H^k_0(0,T;L^2(\Omega))$. In the case the memory is zero, our equation is reduced to the wave equation and a result due to Everdoza and Zuazua informally states that smoother targets can be reached by using smoother controls. In this paper we prove that this result can be partially extended to systems with memory, but that the memory is an obstruction to a complete extensions.

Key words: viscoelasticity, controllability, smooth controls

AMS classification: 45K05, 93B03, 93B05, 93C22

1 Introduction

Stimulated by the applications to the quadratic regulator problem, controllability for distributed parameter systems is usually studied with square integrable controls. Such general controls are hardly realizable in practice and only smooth or piecewise smooth controls, like bang-bang controls, can be implemented. Moreover, when a control is implemented numerically, via discretization, convergence estimates depend on the smoothness of the control (see [4]). This fact revived interest on controllability with smooth controls.

*This papers fits into the research program of the GNAMPA-INDAM and has been written in the framework of the “Groupement de Recherche en Contrôle des EDP entre la France et l’Italie (CONEDP-CNRS)”.
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and the crucial results are in [3] (see also [15, Theorem 2.1] where reachability of the wave equation under controls of class $H^s$ acting on the entire boundary is studied). A natural guess (which is true for the wave equation but which we in part disprove for systems with memory) is that if the control is smooth then the reachable targets are “smooth” too, and the problem is to identify the targets which can be reached by using controls in certain smoothness classes.

In this paper we are going to examine the following equation, which is encountered in viscoelasticity and in diffusion processes when the material has a complex molecular structure:

$$w'' = (\Delta w + bw) + \int_0^t K(t - s)w(s) \, ds.$$ (1.1)

Here $w = w(x,t)$, the apex denotes time derivative, $w''(x,t) = w_{tt}(x,t)$, $x \in \Omega \subseteq \mathbb{R}^d$ is a bounded region with $C^2$ boundary, $K(t)$ is a real continuous function and $\Delta = \Delta_x$ is the laplacian in the variable $x$.

Dependence on the time and especially space variable is not explicitly indicated unless needed for clarity so that we shall write $w = w(t) = w(x,t)$ according to convenience.

We associate the following initial/boundary conditions to system (1.1):

$$w(0) = w_0, \quad w'(0) = w_1,$$ $$w(x,t) = \begin{cases} f(x,t) & x \in \Gamma \\ 0 & x \in \partial \Omega \setminus \Gamma \end{cases}$$ (1.2)

($\Gamma$ is a relatively open subset of $\partial \Omega$).

The function $f$ is a control, which is used to steer the pair $(w(t), w'(t))$ to hit a prescribed target $(\xi, \eta)$ at a certain time $T$.

The spaces of the initial data and final targets and of the control $f$ will be specified below.

There is no assumption on the sign of $b$ whose presence is explained in Remark 5. Furthermore we note:

- It is known (and recalled in Sect. 1.1) that when $f \in L^2(0,T; L^2(\Gamma))$ and $(w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ then problem (1.1)-(1.2) admits a unique mild solution $(w(t), w'(t)) \in C([0,T]; L^2(\Omega) \times H^{-1}(\Omega))$ for every $T > 0$;

- when $K = 0$ (i.e. when we consider the wave equation) the solution of (1.1) is denoted $u$;
when we want to stress the dependence on \( f \) of the solution of (1.1) we use the notation \( w_f \) (the notation \( u_f \) when \( K = 0 \)).

In order to describe the result proved in [3] it is convenient to introduce the following operators \( A \) and \( \mathcal{A} \) in \( L^2(\Omega) \):

\[
\text{dom } A = H^2(\Omega) \cap H_0^1(\Omega), \quad A\phi = \Delta \phi + b\phi, \quad \mathcal{A} = (-A)^{1/2}
\] (1.3)

(note that \( A \) is a positive operator if \( b \geq 0 \) and in this case \( \mathcal{A} \) is defined in a standard way; if \( b < 0 \) the definition of \( \mathcal{A} \) is discussed in Sect. 1.1).

It turns out that \( \text{dom } A = H_0^1(\Omega) \).

Definition 1: Let \( T > 0 \) and let \( \mathcal{F} \) be a closed subspace of \( L^2(0, T; L^2(\Gamma)) \). We say that Eq. (1.1) is \( \mathcal{A}^{k+1} \times \mathcal{A}^k \)-controllable at time \( T \) with controls \( f \in \mathcal{F} \) when the following properties hold:

1. If \((w_0, w_1) \in \text{dom } \mathcal{A}^{k+1} \times \text{dom } \mathcal{A}^k \) and \( f \in \mathcal{F} \) then \((w_f(T), w'_f(T)) \in \text{dom } \mathcal{A}^{k+1} \times \text{dom } \mathcal{A}^k \);

2. for every \( w_0, \xi_0 \) in \( \text{dom } \mathcal{A}^{k+1} \) and every \( w_1, \eta \) in \( \text{dom } \mathcal{A}^k \) there exists \( f \in \mathcal{F} \) such that \((w_f(T), w'_f(T)) = (\xi, \eta)\).

The following result is proved in [10] (see also [11, 12, 13]).

Theorem 2: There exists a time \( T_0 \) and a relatively open set \( \Gamma \subseteq \partial \Omega \) which have the following properties: Let \( T > T_0 \). For every \( w_0 \) and \( \xi \) in \( L^2(\Omega) \) and for every \( w_1 \) and \( \eta \) in \( H^{-1}(\Omega) \) there exists \( f \in L^2(0, T; L^2(\partial \Omega)) \) such that \((w_f(T), w'_f(T)) = (\xi, \eta)\). The set \( \Gamma \) and the number \( T_0 \) do not depend on the continuous memory kernel \( K(t) \).

Note that Theorem 2 holds in particular for the wave equation (i.e. when \( K = 0 \)) and the proofs in the references above do depend on the known controllability result of the wave equation.

The result in [3] can be adapted to the case of the wave equation (without memory) as described in [3] Sect. 5.2 (see item [1] in Remark 5 to understand the exponents):

Theorem 3: Let \( T, T_0 \) and \( \Gamma \) be as in Theorem 2. System (1.1) with \( K = 0 \) is \( \mathcal{A}^k \times \mathcal{A}^{k-1} \)-controllable at time \( T \) with controls \( f \in H_0^k(0, T; L^2(\Gamma)) \).
In the light of Theorem 2 (which extends the well known controllability result of the wave equation) it is natural to guess that Theorem 3 can be extended too. Instead we have the following result:

**Theorem 4:** Let $T_0$, $T$ and $\Gamma$ have the properties in Theorem 2. Then we have:

1. System \((1.1)\) is $\text{dom}\, A \times L^2(\Omega)$-controllable at time $T$ with controls $f \in H^1_0(0, T; L^2(\Gamma))$ (note that $L^2(\Omega) = \text{dom}\, A^0$).

2. System \((1.1)\) $\text{dom}\, A^2 \times \text{dom}\, A$-controllable at time $T$ with controls $f \in H^2_0(0, T; L^2(\Gamma))$.

3. Let $k \geq 3$. For every $T > 0$ there exist controls $f \in H^k_0(0, T; L^2(\Gamma))$ such that $(w(T), w'(T)) \notin \text{dom}\, A^k \times \text{dom}\, A^{k-1}$.

**Remark 5:**

1. The operator $A$ in [3, Sect. 5.2] is defined as the laplacian with domain $H^1_0(\Omega)$ (and image $H^{-1}(\Omega)$) while we used $\text{dom}\, A = H^2(\Omega) \cap H^1_0(\Omega)$.

2. For the sake of brevity, the properties in Theorems 2 will be called “controllability in $L^2(\Omega) \times H^{-1}(\Omega)$”.

3. In the case $\Omega = (a, b)$, item 1 of Theorem 4 has been proved in [4].

4. it is clear that when studying controllability we can reduce ourselves to study the system with zero initial conditions, $w_0 = 0$, $w_1 = 0$.

5. The usual form of the system with persistent memory which is encountered in viscoelasticity is

$$w'' = \Delta w + \int_0^t M(t - s)\Delta w(s) \, ds.$$  

We formally solve this equation as a Volterra integral equation in the “unknown” $\Delta w$. Two integrations by parts in time (followed by an exponential transformation) lead to Eq. (1.1), with $b \neq 0$ (if the initial conditions are different from zero also an affine term, which contains the initial conditions appear, but when studying controllability we can assume $w_0 = 0$, $w_1 = 0$). For this reason we kept the addendum $bw$ in Eq. (1.1). This transformation is known as *MacCamy trick* and it is detailed in [11].
1.1 Preliminaries

The operator $A$ in $L^2(\Omega)$ was already defined: $A\phi = \Delta \phi + b\phi$, $\text{dom} A = H^2(\Omega) \cap H^1_0(\Omega)$ (we recall that $\Omega$ is a region with $C^2$ boundary).

The operator $A$ is selfadjoint, possibly non positive since there is no assumption on the sign of $b$. Its resolvent is compact so that the Hilbert space $L^2(\Omega)$ has an orthonormal basis $\{\phi_n\}$ of eigenvectors of the operator $A$. We denote $-\lambda_n^2$ the eigenvalue of $\phi_n$ since $\lambda_n^2 > 0$ for large $n$ (it might be $\lambda_n^2 \leq 0$ if $n$ is small). The eigenvalues are repeated according to their multiplicity (which is finite).

We shall use the following known asymptotic estimate for the eigenvalues (see [1]):

$$\lambda_n^2 \sim n^{2/d}.$$ (1.4)

In particular, $\lambda_n^2$ is positive for large $n$.

Let $\gamma \in (0,1)$. If $n$ is large and $\lambda_n^2 \geq 0$ then $\lambda_n^{2\gamma}$ is the nonegative determination; otherwise we fix one of the determinations. When $\gamma = 1/2$, for example the one with nonnegative imaginary part. So, $\lambda_n$ denotes the chosen determination of the square root of $\lambda_n^2$, $\lambda_n > 0$ is $n$ is large.

By definition,

$$\xi = \sum_{n=1}^{+\infty} \xi_n \phi_n(x) \in \text{dom} (-A)^{\gamma} \iff \{\lambda_n^{2\gamma} \xi_n\} \in l^2$$

and $(-A)^{\gamma} \xi = \sum_{n=1}^{+\infty} \lambda_n^{2\gamma} \xi_n \phi_n$ so that in particular

$$A\xi = i (-A)^{1/2} \xi = i \left( \sum_{n=1}^{+\infty} \lambda_n \xi_n \phi_n(x) \right).$$

Furthermore we define (we recall the that $\lambda_n$ is real when $n$ is large)

$$R_+(t) \left( \sum_{n=1}^{+\infty} \xi_n \phi_n(x) \right) = \sum_{n=1}^{+\infty} (\cos \lambda_n t) \xi_n \phi_n(x),$$

$$R_-(t) \left( \sum_{n=1}^{+\infty} \xi_n \phi_n(x) \right) = i \left( \sum_{n=1}^{+\infty} (\sin \lambda_n t) \xi_n \phi_n(x) \right).$$

Finally, we introduce the operator $D$: $L^2(\Gamma) \mapsto L^2(\Omega)$:

$$u = Df \iff \begin{cases} \Delta u + bu = 0 \text{ in } \Omega, \\ u = f \text{ on } \Gamma, \\ u = 0 \text{ on } \partial \Omega \setminus \Gamma. \end{cases}$$
It is known that $\text{im } D \subseteq H^{1/2}(\Omega) \subseteq \text{dom } (-A)^{1/4-\epsilon}$ for every $\epsilon > 0$.

It is known (see [6]) that the mild solution of the wave equation

$$u'' = \Delta u + F,$$

with initial and boundary conditions (1.2) is

$$u(t) = R_+(t)w_0 + A^{-1}R_-(t)w_1 - A \int_0^t R_-(t-s)Df(s) \, ds$$

$$+ A^{-1} \int_0^t R_-(t-s)F(s) \, ds. \quad (1.5)$$

By definition, the mild solution of problem (1.1)-(1.2) is the solution of the following Volterra integral equation

$$w(t) = u(t) + A^{-1} \int_0^t \left[ \int_0^{t-s} K(r)R_-(t-s-r)w(s) \, dr \right] \, ds \quad (1.6)$$

where $u(t)$ is given by (1.5) with $F = 0$.

We note that $w'(t)$ is given by

$$w'(t) = u'(t) + \int_0^t \left[ \int_0^{t-s} K(s)R_+(t-s-r)w(s) \, dr \right] \, ds \quad (1.7)$$

where

$$u'(t) = AR_-(t)w_0 + R_+(t)w_1 - A \int_0^t R_+(t-s)Df(s) \, ds. \quad (1.8)$$

The following result is known (see [11, Ch. 2]):

**Theorem 6:** If $(\xi, \eta, f) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(0, T; L^2(\partial \Omega))$ then $(w_f(t), w'_f(t)) \in C([0, T]; L^2(\Omega)) \times C([0, T]; H^{-1}(\Omega))$ for every $T > 0$ and the linear transformation $(\xi, \eta, f) \mapsto (w, w')$ is continuous in the indicated spaces.

Finally, let $w_0 = 0$, $w_1 = 0$. We introduce the following operators which are continuous from $L^2(0, T; L^2(\Gamma))$ to $L^2(\Omega) \times H^{-1}(\Omega)$:

$$f \mapsto \Lambda_E(T)f = \left( \begin{array}{c} \Lambda_{E,1}(T)f \\ \Lambda_{E,2}(T)f \end{array} \right) = \left( \begin{array}{c} u_f(T) \\ u'_f(T) \end{array} \right)$$

$$f \mapsto \Lambda_V(T)f = \left( \begin{array}{c} \Lambda_{V,1}(T)f \\ \Lambda_{V,2}(T)f \end{array} \right) = \left( \begin{array}{c} w_f(T) \\ w'_f(T) \end{array} \right).$$

Our assumption is that the set $\Gamma$ and the time $T$ have been chosen so that both these operators are surjective.
2 Controllability with square integrable controls and moment problem

From now on we study controllability and so we assume \( w_0 = 0, \) \( w_1 = 0. \)

We expand the solutions of Eq. (1.1) in series of \( \phi_n, \)

\[
 w(x, t) = \sum_{n=1}^{+\infty} \phi_n(x)w_n(t), \quad w_t(x, t) = \sum_{n=1}^{+\infty} \phi_n(x)w'_n(t).
\]

(2.1)

It is easily seen that \( w_n(t) \) solves

\[
 w''_n(t) = -\lambda_n^2 w_n(t) + \int_0^t K(t-s)w_n(s) \, ds - \int_\Gamma (\gamma_1 \phi_n) f(x, t) \, d\Gamma
\]

where \( \gamma_1 \) is the exterior normal derivative and \( d\Gamma \) is the surface measure.

The initial conditions are zero since \( w(0) = 0, \) \( w'(0) = 0. \) In order to represent the solution of the previous equation, we introduce \( \zeta_n(t), \) the solution of

\[
 \zeta''_n(t) = -\lambda_n^2 \zeta_n(t) + \int_0^t K(t-s)\zeta_n(s) \, ds, \quad \begin{cases} \zeta_n(0) = 0, \\ \zeta'_n(0) = 1. \end{cases}
\]

(2.2)

Then we have

\[
 \begin{cases}
 w_n(t) = \int_\Gamma \int_0^t [\zeta_n(t-s)\gamma_1 \phi_n(x)] f(x, s) \, ds, \\
 w'_n(t) = \int_\Gamma \int_0^t [\zeta'_n(t-s)\gamma_1 \phi_n(x)] f(x, s) \, ds.
\end{cases}
\]

(2.3)

Let the target \( (\xi, \eta) \in L^2(\Omega) \times H^{-1}(\Omega) \) have the expansion

\[
 \xi(x) = \sum_{n=1}^{+\infty} \xi_n \phi_n(x), \quad \eta(x) = \sum_{n=1}^{+\infty} (\eta_n \lambda_n) \phi_n(x).
\]

Then, this target is reachable at time \( T \) if and only if there exists \( f \in L^2(0, T; L^2(\Gamma)) \) such that

\[
 M_0 f = c_n = \eta_n + i\xi_n
\]
where \( M_0 \) is the moment operator

\[
M_0 f = \int_{\Gamma} \int_0^T E_n^{(0)}(s) \Psi_n f(x, T - s) \, ds \, d\Gamma,
\]

(2.4)

\[
\Psi_n = \frac{\gamma_n \Phi_n}{\lambda_n}, \quad E_n^{(0)}(s) = [\zeta_n'(s) + i\lambda_n \zeta_n(s)].
\]

Note that the operator \( M_0 \) takes values in \( l^2 \). So, we should write \( M_0 f = \{c_n\} \). The brace here is usually omitted.

The sequence \( \{\Psi_n\} \) is bounded in \( L^2(\partial\Omega) \) and it is almost normalized if \( \Gamma \) has been chosen as in Theorem 2 (see [11, Theorem 4.4]).

It is known from [7, 11] that the operator \( M \) is continuous. Our assumption is that \( T \) and \( \Gamma \) have been so chosen that this operator, defined on \( L^2(0, T; L^2(\Gamma)) \), is surjective in \( l^2(\mathbb{C}) \). This implies (see [11, Sect. 3.3]) that:

**Lemma 7:**
1. The sequence \( \{e_n\} \)

\[
e_n = [\zeta_n'(s) + i\lambda_n \zeta_n(s)] \Psi_n
\]

is a Riesz sequence in \( L^2(0, T; L^2(\Gamma)) \), i.e. it can be transformed to an orthonormal sequence using a linear, bounded and boundedly invertible transformation.

2. The series

\[
\sum \alpha_n \Psi_n(x) [\zeta_n'(s) + i\lambda_n \zeta_n(s)]
\]

converges in \( L^2(0, T; L^2(\Gamma)) \) if and only if \( \{\alpha_n\} \in l^2 \).

3. If \( K(t) = 0 \) then \( e_n(x, t) = \Psi_n(x)e^{i\lambda_n t} \).

### 3 The system with controls of class \( H^1_0([0, T]; L^2(\Gamma)) \)

The property \( f \in H^1_0([0, T]; L^2(\Gamma)) \) can be written as follows:

\[
f(x, t) = \int_0^t g(x, s) \, ds \quad g \in \mathcal{N}_1,
\]

\[
\mathcal{N}_1 = \left\{ g \in L^2(0, T; L^2(\Gamma)) : \int_0^T g(x, s) \, ds = 0 \right\}.
\]

(3.1)
So, when \( f \in H^{1,0}_{0}([0, T]; L^2(\Omega)) \) we can integrate by parts the integral in (1.3) (with \( F = 0 \), see [9] for the rigorous justification) and we get (using that the initial conditions are zero):

\[
\begin{align*}
    u_f(t) &= -A \int_0^t R_-(t-s)D \int_0^s g(r) \, dr \, ds = \\
    &\quad = D \int_0^t g(r) \, dr - \int_0^t R_+(t-s)Dg(s) \, ds = \hat{u}_g(t), \quad \text{(3.2)} \\
    u'_f(t) &= -A \int_0^t R_-(t-s)Dg(s) \, ds = \tilde{u}_g(t). \quad \text{(3.3)}
\end{align*}
\]

**Remark 8:** These expressions show an interesting fact (compare also [3, Corollary 1.5]). We see from Theorem 6 that the integrals take values respectively in \( H^{1,0}_{0}(\Omega) \) and \( L^2(\Omega) \) and \( g \mapsto \tilde{u}_g(t) \) is a linear continuous map from \( N_1 \) to \( C([0, T]; L^2(\Omega)) \). Instead, \( g \mapsto \hat{u}_g(t) \) is a linear continuous map from \( N_1 \) to \( C([0, T]; H^{1/2}(\Omega)) \) since \( \text{im} \, D \subseteq H^{1/2}(\Omega) \). We have \( \hat{u}_g(T) \in H^{1}_{0}(\Omega) \) only when \( t = T \).

Of course these maps are continuous among the specified spaces but in every numerical computation we expect that the value of \( T \) is affected by some error, and the fact that \( u_f(t) \notin H^{1}_{0}(\Omega) \) if \( t \neq T \) might raise some robustness issues in the numerical approximation of the steering control. This issue seemingly is still to be studied.

Let us introduce

\[
N_1 \ni g \mapsto \Lambda_E^{(1)}(T)g = (-\hat{u}_g(T), -\tilde{u}_g(T))
\]

\[
= \left( \int_0^T R_+(T-s)Dg(s) \, ds, \quad A \int_0^T R_-(T-s)Dg(s) \, ds \right). \quad \text{(3.4)}
\]

Controllability with \( H^1_0 \)-controls \( f \) of the wave equation (the case \( K = 0 \), proved in [3]) is equivalent to surjectivity of the map \( \Lambda_E(T) \) from \( N_1 \) to \( H^1_0(\Omega) \times L^2(\Omega) \).

We introduce formulas (3.2)-(3.3) in (1.6) and (1.7). We get:
\[ w_f(t) = \hat{u}_g(t) + A^{-1} \int_0^t \left[ \int_0^{t-s} K(r)R_-(t-s-r)w(s) \, dr \right] \, ds = \hat{w}_g(t) \]  

(3.5)

\[ w'_f(t) = \tilde{u}_g + \int_0^t \left[ \int_0^{t-s} K(r)R_+(t-s-r)w(s) \, dr \right] \, ds = \tilde{w}_g(t). \]  

(3.6)

We introduce the operator \( \Lambda_V^{(1)}(T) \), to be compared with the operator \( \Lambda_E^{(1)}(T) \) in (3.4):

\[ \Lambda_V^{(1)}(T)g = (-\hat{w}_g(T), -\tilde{w}_g(T)) , \quad g \in \mathcal{N}_1. \]

Controllability in \( H_0^1(\Omega) \times L^2(\Omega) \) is equivalent to surjectivity of the map \( \Lambda_V^{(1)}(T) \) from \( \mathcal{N}_1 \) to \( H_0^1(\Omega) \times L^2(\Omega) \).

We see from here that the functions \( \hat{w}_g(t) \) and \( \tilde{w}_g(t) \) have the same properties as stated above for \( \hat{u}_g(t) \) and \( \tilde{u}_g(t) \), in particular \( \hat{w}_g(t) \in H^{1/2}(\Omega) \) and \( \tilde{w}_g(T) \in H_0^1(\Omega) \) but there is an additional difficulty: if we want to consider \( \hat{u}_g(T) \) we can simply ignore the contribution of \( Dg(T) \). Instead, due to the Volterra structure of Eq. (3.5), the term \( Dg(t) \) which comes from \( \hat{u}_g(t) \) cannot be simply ignore when looking at the function \( \hat{w}_g \) for \( t = T \).

In spite of this, using \( \text{im} D \subseteq H^{1/2}(\Omega) = \text{dom} (-A)^{1/4} - \epsilon/2 \) (for every \( \epsilon > 0 \)) and solving (3.5)-(3.6) via Picard iteration, it is simple to prove\(^1\)

Lemma 9: We have:

- \( w_f(T) \in H_0^1(\Omega), w'_f(T) \in L^2(\Omega); \)

- The operator \( \Lambda_V^{(1)}(T) - \Lambda_E^{(1)}(T) \) is compact in \( H_0^1(\Omega) \times L^2(\Omega) \) and so \( \text{Im} \Lambda_V^{(1)}(T) \) is closed in \( H_0^1(\Omega) \times L^2(\Omega) \) and \( \left[ \text{Im} \Lambda_V^{(1)}(T) \right]^\perp \) is finite dimensional.

The goal is the proof of Theorem\(^2\) i.e. the proof that every element in \( \left[ \text{Im} \Lambda_V^{(1)}(T) \right]^\perp \) is equal zero. We prove this result by using the properties of the moment operator.

\(^1\) we use \([\cdot]^\perp\) to denote the subspace of the annihilators in the dual space.
3.1 Controllability with $H^1_0([0, T]; L^2(\Gamma))$ controls

Our point of departure is the expansion (2.1) and the representation (2.3) of $w_n(t)$, $w'_n(t)$. Let

$$K_1(t) = \int_0^t K(s) \, ds$$

When $f$ has the representation (3.1) we can manipulate (2.3) as follows:

$$w_n(t) = \left(-\frac{1}{\lambda^2_n}\right) \int_\Gamma \gamma_1 \phi_n \int_0^t \left(-\lambda^2_n \psi_\nu(n(s))\right) \int_0^{t-s} g(r) \, dr \, ds \, d\Gamma =$$

$$= \left(-\frac{1}{\lambda^2_n}\right) \int_\Gamma \gamma_1 \phi_n \int_0^t \left[\int_0^{t-s} K(s-\nu) \psi_\nu(n) \, d\nu\right] \int_0^{t-s} g(r) \, dr \, ds \, d\Gamma =$$

$$= \frac{1}{\lambda^2_n} \int_\Gamma \gamma_1 \phi_n \int_0^t g(x, t-r) \, d\Gamma \, dr -$$

$$-\frac{1}{\lambda^2_n} \int_\Gamma \gamma_1 \phi_n \int_0^t g(x, t-r) \left[\int_0^r K_1(r-\nu) \psi_\nu(n) \, d\nu\right] \, dr \, d\Gamma$$

$$w'_n(t) = \int_\Gamma \gamma_1 \phi_n \int_0^t g(x, t-r) \psi_\nu(n) \, dr \, d\Gamma.$$ (3.7)

Let now

$$H^1_0(\Omega) \ni \xi = \sum_{n=1}^{+\infty} \frac{\xi_n}{\lambda_n} \phi_n(x), \quad L^2(\Omega) \ni \eta = \sum_{n=1}^{+\infty} \eta_n \phi_n(x), \quad \{\xi_n\}, \{\eta_n\} \in l^2$$

and let

$$c_n = -\xi_n + i\eta_n$$

Of course, $\{c_n\}$ is an arbitrary element of $l^2 = l^2(\mathbb{C})$ and our goal is the proof that the following moment problem is solvable for every $\{c_n\} \in l^2$:

$$\int_\Gamma \int_0^T g(T-r) E^{(1)}_n(r) \Psi_n \, dr \, d\Gamma = c_n,$$

$$E^{(1)}_n(r) = \left(\psi'_n(r) - \int_0^r K_1(r-\nu) \psi_\nu(n) \, d\nu\right) + i\lambda_n \psi_\nu(n).$$

Here $g$ is not an arbitrary $L^2$ function, i.e. this is not a moment problem in the space $L^2(0, T; L^2(\Gamma))$; it is a moment problem in the Hilbert space $N_1$. 
So, it is not really $E_n^{(1)}(r)\Psi_n$ which enters this moment problem but any projection of $E_n^{(1)}(r)\Psi_n$ on the Hilbert space $N_1$: the moment problem to be studied is

$$M_1g = c_n = \int_T \int_0^T g(T-r)\mathcal{P}_{N_1} \left(E_n^{(1)}(\cdot)\Psi_n\right) \, dr \, d\Gamma$$

where $\mathcal{P}_{N_1}$ is any fixed projection on $N_1$. The operator $M_1$ is the moment operator of our control problem.

So, the controllability problem can be reformulated as follows: to prove the existence of a suitable projection $\mathcal{P}_{N_1}$ such that the moment problem (3.8) is solvable for every $\{c_n\} \in l^2$. In fact, surely there exist projections for which the moment problem is not solvable: the projection $P_h = 0$ for every $h$ is an example.

We are going to prove that the following special projection does the job:

$$\langle \mathcal{P}_{N_1}f \rangle (t) = f(t) - \frac{1}{T} \int_0^T f(s) \, ds.$$  \hspace{1cm} (3.9)

**Remark 10:** The projection $\mathcal{P}_{N_1}$ is the orthogonal projection of $L^2(0, T; L^2(\Gamma))$ onto $N_1$. In fact, for every $f \in L^2(0, T; L^2(\Gamma))$ and every $g \in N_1$ we have

$$\int_T \int_0^T \overline{g}(x, t) [f - \mathcal{P}_{N_1}f] (x, t) \, dt \, d\Gamma = \frac{1}{T} \int_T \left[ \int_0^T f(x, s) \, ds \right] \left[ \int_0^T \overline{g}(x, t) \, dt \right] \, d\Gamma = 0. \quad \blacksquare$$

Let us note that the results reported in Section 3 in particular show that the operator $M_1$ is continuous and the image of $M_1$ is closed with finite codimension (this is Lemma 9).

So, it is sufficient that we prove that if $\{\bar{\alpha}_n\} \perp \text{im} \, M_1$ then $\{\alpha_n\} = 0$ i.e. we must prove that

$$\langle \alpha_n, M_1g \rangle = 0 \quad \forall g \in N_1 \quad \implies \quad \{\alpha_n\} = 0$$

i.e. we prove that if the following equality holds then $\{\alpha_n\} = 0$:

$$\sum_{n=1}^{+\infty} \alpha_n \mathcal{P}_{N_1} \left( \Psi_n \left[ \overline{\zeta_n}(\cdot) - \int_0^{(\cdot)} K(\cdot - \nu)\zeta_n(\nu) \, d\nu + i\lambda_n \overline{\zeta_n}(\cdot) \right]\right) = 0.$$  \hspace{1cm} (3.10)
We introduce explicitly the projection (3.9) and we see that we must prove \( \{\alpha_n\} = 0 \) when the following equality holds:

\[
\sum_{n=1}^{+\infty} \alpha_n \Psi_n \left[ \zeta_n'(t) - \int_0^t K_1(t - \nu) \zeta_n(\nu) \, d\nu + i\lambda_n \zeta_n(t) \right] = 0
\]

\[
= \frac{1}{T} \sum_{n=1}^{+\infty} \alpha_n \Psi_n \left[ \zeta_n(T) - \int_0^T \int_0^t K_1(t - \nu) \zeta_n(\nu) \, d\nu \, dt + i\lambda_n \int_0^T \zeta_n(\nu) \, d\nu \right]
\]

(3.11)

Note that the numerical series on the right side of (3.11) converges since both the series (3.10) and the series on the left of (3.11) converge, thanks to Lemma 7. The series on the right side of (3.11) is constant. This implies that also the sum of the series on the left is constant and so its derivative is equal zero.

We shall prove that the derivative can be computed termwise. Accepting this fact, the proof that \( \{\alpha_n\} = 0 \) is simple: the termwise derivative is

\[
0 = \sum_{n=1}^{+\infty} \alpha_n \Psi_n \left[ \zeta_n''(t) - \int_0^t K(t - \nu) \zeta_n(\nu) \, d\nu + i\lambda_n \zeta_n'(t) \right] = 0
\]

(3.12)

We noted (in Lemma 7 item 1) that \( L^2(\Omega) \times H^{-1}(\Omega) \)-controllability with square integrable controls of the viscoelastic system is equivalent to the fact that \( \{\Psi_n [\zeta_n'(t) + i\lambda_n \zeta_n(t)]\} \) is a Riesz sequence in \( L^2(0, T; L^2(\Gamma)) \) and so \( \{\lambda_n \alpha_n\} = 0 \) i.e. \( \{\alpha_n\} = 0 \). Of course we implicitly used \( \{\lambda_n \alpha_n\} \in l^2 \), a fact we shall prove now.

In order to complete the proof we must see that it is legitimate to distribute the derivative on the series (3.11) and that this implies in particular \( \{\lambda_n \alpha_n\} \in l^2 \).

The fact that \( \{\Psi_n [\zeta_n' + i\lambda_n \zeta_n]\} \) is a Riesz sequence in \( L^2(0, T; L^2(\Gamma)) \) shows that we can distribute the series on the left hand side of (3.11), which
can be written as
\[
\sum_{n=1}^{+\infty} \alpha_n \Psi_n \left[ \zeta'_n(t) + i \lambda_n \zeta_n(t) \right] = \int_0^t K_1(s - \nu) \left[ \sum_{n=1}^{+\infty} \alpha_n \Psi_n(\nu) \right] d\nu + \text{const}.
\] (3.13)

So, it is sufficient that we study the differentiability of the series
\[
\sum_{n=1}^{+\infty} \alpha_n \Psi_n Z_n(t), \quad Z_n(t) = [\zeta'_n(t) + i \lambda_n \zeta_n(t)].
\]

Using the definition (2.2) we see that
\[
\zeta_n(t) = \frac{1}{\lambda_n} \sin \lambda_n(t) + \frac{1}{\lambda_n} \int_0^t \int_0^s K(s - \tau) \sin \lambda_n \tau \, d\tau \, \zeta_n(t - s) \, ds,
\]
\[
\zeta'_n(t) = \cos \lambda_n(t) + \frac{1}{\lambda_n} \int_0^t \int_0^s K(s - \tau) \sin \lambda_n \tau \, d\tau \, \zeta'_n(s) \, ds
\]
and so we get the following formula for \(Z_n(t)\):
\[
\left\{ \begin{array}{l}
Z_n = E_n + \frac{1}{\lambda_n} K * S_n * Z_n \\
\text{where} * \text{ denotes the convolution and where we define} \\
S_n(t) = \sin \lambda_n t, \quad C_n(t) = \cos \lambda_n t, \quad E_n(t) = e^{i\lambda_n t}.
\end{array} \right.
\] (3.14)

Gronwall inequality shows that \(\{Z_n(t)\}\) is bounded on bounded intervals. We introduce the notations
\[
F^{(s)}(t) = F(t), \quad F^{(s)} = F * F^{(s-1)}.
\]

With these notation the formula for \(Z_n(t)\) shows also that
\[
Z_n = E_n + \frac{1}{\lambda_n} K * S_n * E_n + \frac{1}{\lambda_n^2} K^{(s2)} * S^{(s2)} * Z_n,
\]
\[
= E_n + \sum_{r=1}^{K} \frac{1}{\lambda_n^r} K^{(sr)} * S^{(sr)} * E_n + \frac{1}{\lambda_n^{k+1}} P_{K,n}(t).
\] (3.15)
The functions $P_{n,K}(t)$ and $P'_{n,K}(t)$ are bounded on bounded intervals,

$$|P_{n,K}(t)| < M_K, \quad |P_{n,K}(t)| < M_K$$

where $M_K$ does not depend on $n$ and $t \in [0, T]$.

So, using the fact that $\{\Psi_n\}$ is bounded in $L^2(\Gamma)$ and (1.4) (see [11, Lemma 4.4]) we see that

$$\sum_{n=1}^{+\infty} \alpha_n \Psi_n \frac{1}{\lambda_n} M_{n,K}(t)$$

is of class $C^1$ (and termwise differentiable) when $K$ is large enough. We fix an index $K$ with this property and we consider the series of each one of the terms in (3.15) for which $r \geq 1$:

$$\sum_{n=1}^{+\infty} \alpha_n \Psi_n \frac{1}{\lambda_n} K \ast S_n \ast E_n, \quad \sum_{n=1}^{+\infty} \alpha_n \Psi_n \frac{1}{\lambda_n} K^{(sr)} \ast S_n^{(sr)} \ast E_n. \quad (3.16)$$

$L^2$-convergence of the series is clear. We prove that they converge to $H^1$-functions.

We consider the first series. We compute the convolution $(S_n \ast E_n)$ and we see that this series is equal to

$$\frac{1}{2i} \sum_{n=1}^{+\infty} \alpha_n \Psi_n \frac{1}{\lambda_n} \int_0^t K(s) \left[ (t-s) e^{i\lambda_n(t-s)} - \frac{1}{\lambda_n} \sin \lambda_n(t-s) \right] ds.$$

The series inside the integral converges in $L^2(0, T)$ thanks to item 3 in Lemma 7. Its termwise derivative is:

$$\frac{1}{2i} \int_0^t K(s) \left[ \sum_{n=1}^{+\infty} \alpha_n \Psi_n \left( \frac{1}{\lambda_n} e^{i\lambda_n(t-s)} + i(t-s) e^{i\lambda_n(t-s)} - \cos \lambda_n(t-s) \right) \right] ds =$$

$$\frac{1}{2i} \int_0^t K(s) \left[ \sum_{n=1}^{+\infty} \alpha_n \Psi_n \left( \frac{1}{\lambda_n} e^{i\lambda_n(t-s)} + i(t-s) e^{i\lambda_n(t-s)} - \cos \lambda_n(t-s) \right) \right] ds.$$

This series is $L^2$-convergent thanks to Lemma 7.

A similar argument holds for every $r \leq K$. So, $\sum_{n=1}^{+\infty} \alpha_n \Psi_n E_n$ converges in $H^1(0, T)$ for every $T$, in particular $T > T_0$.

From [11, Lemma 3.4] and the appendix, we see that $\alpha_n = \delta_n / \lambda_n$, $\{\delta_n\} \in l^2$, and that the derivative of the series can be computed termwise.

This ends the proof of item [4] in Theorem 4.
Remark 11: This proof holds in particular if $K = 0$, and gives an alternative proof to the result in [3]. An important additional property in [3] is that a smooth steering control solves an optimization problem (and that its essential support is relatively compact in $[0, T] \times \Gamma$).

4 When the control is smoother

In this section we prove item 2 and 3 in Theorem 4.

We note that $f \in H^2_0(0, T; L^2(\Gamma))$ if and only if

$$f(t) = \int_0^t (t - s)g(s) \, ds,$$

$g \in \mathcal{N}_2 = \left\{ g \in L^2(0, T; L^2(\Gamma)), \int_0^T g(s) \, ds = 0, \int_0^T (T - r)g(r) \, dr = 0 \right\}$.

An analogous representation holds if $f \in H^k_0(0, T; L^2(\Gamma))$.

Using these characterizations, we integrate by parts formulas (1.5) (with $F = 0$) and (1.8) (with zero initial conditions) and when $f \in H^2_0$ we find

$$\begin{cases} u(t) = D \int_0^t (t - r)g(r) \, dr - A^{-1} \int_0^t R_-(t - s)Dg(s) \, ds = \hat{u}_g(t), \\ u'(t) = D \int_0^t g(r) \, dr - \int_0^t R_+(t - s)Dg(r) \, dr = \tilde{u}_g(t). \end{cases} \tag{4.1}$$

This is similar to (3.2) and (3.3) (and now Remark 8 applies both to $u$ and $u'$). Let

$$L_-(t)w = \int_0^t K(r)R_-(t - r)w \, dr, \quad L_+(t)w = \int_0^t K(r)R_+(t - r)w \, dr.$$

We have, from (1.6) and (1.7),

$$\begin{cases} w_f(t) = \hat{u}_g(t) + A^{-1} \int_0^t L_-(t - s)w(s) \, ds, \\ w'_f(t) = \tilde{u}_g(t) + \int_0^t L_+(t - s)w(s) \, ds. \end{cases} \tag{4.2}$$

The proof of item 2 in Theorem 4 consist in two parts: first we prove the regularity property $(w_f(T), w'_f(T)) \in \text{dom} \mathcal{A}^2 \times \text{dom} \mathcal{A}$ and then we prove that $f \mapsto (w_f(T), w'_f(T)) \in \text{dom} \mathcal{A}^2 \times \text{dom} \mathcal{A}$ is surjective in this space.

The proof of item 3 in Theorem 4 is the proof that the corresponding regularity does not hold, i.e. that due to the memory, there exist functions $f \in H^k(0, T]; L^2(\Gamma))$ such that $(w_f(T), w'_f(T)) \notin \text{dom} \mathcal{A}^k \times \text{dom} \mathcal{A}^{k-1}$. 


4 When the control is smoother

We proceed as follows: we first examine the regularity issue (i.e. the positive result, when \( f \in H^2_0(0, T; L^2(\Gamma)) \) and the lack of regularity if \( f \) is smoother) in subsection 4.1. In the subsection 4.2 we prove \((\text{dom } \mathcal{A}^2 \times \text{dom } \mathcal{A})\)-controllability when \( f \in H^2_0(0, T; L^2(\Gamma)) \).

4.1 Regularity and lack of regularity

We consider \( w(t) \) in (4.2). A step of Picard iteration gives

\[
 w(t) = \hat{u}_g(t) + A^{-1} \int_0^t L_-(t-s) \hat{u}_g(s) \, ds + A^{-1} \left( L^{(s_2)} \ast w \right)(t). \tag{4.3}
\]

We know from [3] that \( \hat{u}_g(T) \in H^2_0(\Omega) = \text{dom } A = \text{dom } \mathcal{A}^2 \) (also seen from (4.1)). As we noted, \( \hat{u}_g(t) \) has this regularity for \( t = T \) but not for \( t \in (0, T) \). Instead, we prove that \( w(t) - \hat{u}_g(t) \in \text{dom } \mathcal{A}^2 \) for every \( t \in [0, T] \).

This is clear for the third addendum on the right side of (4.3). We examine the second term, which is

\[
 A^{-1} \int_0^t L_-(t-s) D \int_0^s (s-r) g(r) \, dr \, ds - A^{-1} (L_\ast R \ast g)(t).
\]

The fact that \( w(T) \in \text{dom } \mathcal{A} \) follows from the representation of \( w(T) \) in terms of \( w(t) \) in the second line of (4.2). This fact shows that the integral even takes values in \( \text{dom } \mathcal{A}^2 = \text{dom } A \) for every \( t \in [0, T] \) while the first addendum \( \hat{u}_g(T) \in \text{dom } \mathcal{A} \) from [3] (also seen from (4.1)).

In conclusion, we proved that

\[
 f \in H^2_0(0, T; L^2(\Gamma)) \implies (w(T), w'(T)) \in \text{dom } \mathcal{A}^2 \times \text{dom } \mathcal{A}.
\]
Now we prove that this result cannot be improved when \( f \) is smoother. It is sufficient that we show

\[
f \in H^3(0, T; L^2(\Gamma)) \implies w_f(T) \in \text{dom} \mathcal{A}^3.
\]

Note that \( f \in H^3(0, T; L^2(\Gamma)) \) when

\[
f(t) = \int_0^t (t - s)^2 g(s) \, ds, \quad \int_0^T (T - s)^k g(s) \, ds = 0 \quad k = 0, 1, 2.
\]

We use this representation of \( f \) and we integrate by parts the integral in (1.5) with \( F = 0 \). We get

\[
u_f(t) = D \int_0^t (t - r)^2 g(r) \, dr + 2A^{-1}D \int_0^t g(r) \, dr + A^{-3} \left[ -2A \int_0^t R_+(t - s)Dg(s) \, ds \right].
\]

The last addendum belongs to \( \text{dom} \mathcal{A}^3 \). In the following computation we write \( G(t) \) for any term which takes values in \( \text{dom} \mathcal{A}^3 \), not the same at every occurrence. So,

\[
u_f(t) = \int_0^t P(t - r)g(r) \, dr + G(t), \quad P(t) = t^2D + 2A^{-1}D.
\]

We replace in the expression of \( w_f(t) \) in (1.6). Two steps of Picard iteration gives the following representation for \( w(t) \):

\[
w(t) = \int_0^t P(t - \nu)Dg(\nu) \, d\nu + A^{-1} \int_0^t L_+(t - r) \int_0^r (r - \nu)^2 Dg(\nu) \, d\nu \, dr
\]

\[+ A^{-2} \int_0^t L_-(t - r) \int_0^r L_-(r - s) \int_0^s (s - \nu)^2 Dg(\nu) \, d\nu \, ds \, dr + G(t).
\]

The integral in the second line can be integrated by parts so that the second line takes values in \( \text{dom} \mathcal{A}^3 \). The first addendum is zero for \( t = T \). So, we
must study the regularity of
\[
\mathcal{A}^{-1} \int_0^t L_-(t - r) \int_0^r (r - \nu)^2 Dg(\nu) \, d\nu \, dr
\]
\[
= -\mathcal{A}^{-2} \int_0^t K(t - s) \int_0^s \frac{d}{dr} R_+(s - r) \int_0^r (r - \nu)^2 Dg(\nu) \, d\nu \, dr \, ds
\]
\[
= -\mathcal{A}^{-2} \int_0^t K(t - s) \int_0^s (s - \nu)^2 Dg(\nu) \, d\nu \, ds
\]
\[
+ 2\mathcal{A}^{-2} \int_0^t K(t - s) \int_0^s R_+(s - r) \int_0^r (r - \nu) Dg(\nu) \, d\nu \, dr \, ds.
\]

The last integral can be integrated by parts again and subsumed in the term \(G(t)\). Instead,
\[
\mathcal{A}^{-2} D \int_0^T (T - \nu)^2 \int_0^\nu K(s) g(\nu - s) \, ds \, d\nu \notin \text{dom} \mathcal{A}^3
\]
since
\[
D \int_0^T (T - \nu)^2 \int_0^\nu K(s) g(\nu - s) \, ds \, d\nu \notin \text{dom} \mathcal{A}
\]
as it is seen for example when \(g(x, \nu) = g_0(x)g_1(\nu)\) and \(g_1\) such that
\[
\int_0^T (T - \nu)^2 \int_0^\nu K(s) g_1(\nu - s) \, ds \, d\nu \neq 0.
\]

As a specific example, when \(\Omega = (0, 1)\) (and \(\Gamma = \{0\}\)) then \(Df = (1 - x)f \notin H_0^1(0, 1)\) unless \(f = 0\).

Similar arguments hold if \(f \in H^k(0, T; L^2(\Gamma))\) and \(k > 3\).

**4.2 \(\text{dom } \mathcal{A}^2 \times \text{dom } \mathcal{A}\)-controllability when \(f \in H_0^2(0, T; L^2(\Gamma))\)**

This part of the proof is similar to that in the case \(f \in H_0^1(0, T; L^2(\Gamma))\) and it is only sketched.

A simple examination of formulas (4.2) and (4.3) shows that the map \(f \mapsto (w_f(T) - u_f(T), w'_f(T) - u'_f(T))\) from \(H_0^2(0, T; L^2(\Gamma))\) to \(\text{dom } \mathcal{A}^2 \times \text{dom } \mathcal{A}\) is compact. Hence we must prove
\[
\left\{ (w_f(T), w'_f(T)) : f \in L^2(0, T; L^2(\Gamma)) \right\}^\perp = \{0\}
\]
(the orthogonal is respect to $\text{dom} \mathcal{A}^2 × \text{dom} \mathcal{A} = (H^2(Ω) \cap H_0^1(Ω)) × H_0^1(Ω)$).

We use again the formulas for $w_n(t)$ and $w'_n(t)$ in (3.7), where $g(t) = f'(t)$ has now to be replaced by $\int_0^t g(s) \, ds$. We see that

$$ w_n(T) = \frac{1}{\lambda_n} \int_\Gamma \int_0^T g(T - r) \left[ \Psi_n \left( \zeta_n(r) + \int_0^r K_2(r - s) \zeta_n(s) \, ds \right) \right] \, dr \, d\Gamma, $$

$$ w'_n(T) = \frac{1}{\lambda_n} \int_\Gamma \int_0^T g(x, T - r) \left[ \Psi_n \left( -\zeta'_n(r) + \int_0^r K_1(r - s) \zeta_n(s) \, ds \right) \right] \, dr \, d\Gamma $$

where

$$ K_2(t) = \int_0^t K_1(s) \, ds = \int_0^t (t - s)K(s) \, ds. $$

We want to reach

$$ \xi(x) = \sum_{n=1}^{+\infty} \frac{\xi_n}{\lambda_n^2} \phi_n(x), \quad \eta(x) = \sum_{n=1}^{+\infty} \frac{\eta_n}{\lambda_n} \phi_n(x), \quad \{\xi_n\} \in l^2, \quad \{\eta_n\} \in l^2. $$

So, we must solve the moment problem in $\mathcal{N}_2$

$$ \int_\Gamma \int_0^T g(T - r) P_2(\Psi_n E_n^{(2)}) \, dr \, d\Gamma = c_n, \quad \{c_n\} = \{-\eta_n + i\xi_n\} \in l^2 $$

where $P_2$ is the orthogonal projection on $\mathcal{N}_2$:

$$ (P_2 f)(x, t) = f(x, t) - sA_f - B_f, \quad A_f = \frac{1}{T} \int_0^T (s - \frac{T}{2}) f(x, s) \, ds, \quad B_f = \frac{1}{T} \int_0^T f(s) \, ds - \frac{1}{2} TA_f \quad (4.4) $$

and

$$ E_n^{(2)}(r) = \left( \zeta'_n(r) - \int_0^r K_1(r - s) \zeta_n(s) \, ds \right) + i \left( \lambda_n \zeta_n(r) + \lambda_n \int_0^r K_2(r - s) \zeta_n(s) \, ds \right). $$

Proceeding as in the case $f \in H_0^1(0, T; L^2(\Gamma))$ we see that we must prove $\{\alpha_n\} = 0$ when the following equality holds:

$$ \sum_{n=1}^{+\infty} \alpha_n \Psi_n \left( \zeta'_n(r) + i\lambda_n \zeta_n(r) \right) = \sum_{n=1}^{+\infty} \alpha_n \Psi_n \left[ \int_0^r K_1(r - s) \zeta_n(s) \, ds + i \int_0^r K_2(r - s) \lambda_n \zeta_n(s) \, ds \right] $$

$$ + \sum_{n=1}^{+\infty} \alpha_n \Psi_n [sA_n + B_n] \quad (4.5) $$
where $A_n$ and $B_n$ are as in (14.4) with $f$ replaced by

$$(\zeta'_{n} - K_1 * \zeta_{n}) + i\lambda_n (\zeta_{n} + K_2 * \zeta_{n}) .$$

In fact, it is legitimate to distribute the series on the sum since each one of these series converge because $\{\Psi_n (\zeta'_n(r) + i\lambda_n \zeta_n(r))\}$ is a Riesz sequence in $L^2(0, T; L^2(\Gamma))$ (see Lemma 7 item 4).

Equality (4.5) is similar to (3.11), with the right hand side of class $C^2$.

So we get

$$\alpha_n = \frac{\beta_n}{\lambda_n}; \{\beta_n\} \in l^2$$

and we can compute the termwise derivative of the series. Computing the derivative and noting that $K'_1 = K_1$, $K'_2 = K_2$ we get

$$\sum_{n=1}^{+\infty} \beta_n \Psi_n [-\lambda_n \zeta_n(r) + i\zeta'_n(r)] = \sum_{n=1}^{+\infty} \beta_n \Psi_n \int_0^r K_1(r-s) \zeta_n(s) \, ds - i \sum_{n=1}^{+\infty} \alpha_n \Psi_n A_n .$$

This is the same as (3.13) and so we get $\{\beta_n\} = 0$, i.e. $\{\alpha_n\} = 0$ as wanted.

**Appendix**

In this appendix we prove the following simple result, which however is crucial in the proof of our theorem (see also [5, p. 323]).

**Lemma 12:** Let $\mathbb{J}$ be a denumerable set and let the sequence $\{e_n\}_{n \in \mathbb{J}}$ in a Hilbert space $H$ have the following properties:

1. if $\{\alpha_n\} \in l^2$ then $\sum \alpha_n e_n$ converges in $H$;
2. $\{\langle f, e_n \rangle\} \in l^2$ for every $f \in H$;
3. the subspace $M = \{\{\langle f, e_n \rangle\} , f \in H\}$ is closed and its codimension is finite, equal to $k$.

Under these conditions, there exists a set $K \subseteq \mathbb{J}$ of precisely $k$ indices such that $\{e_n\}_{n \notin K}$ is a Riesz sequence.

**Proof.** Let $\mathbb{M}$ be the operator

$$\mathbb{M}f = \{\langle f, e_n \rangle\} \quad H \mapsto l^2 .$$
It is known that $\mathbb{M}$ is a closed operator (see [2] and [11, Theorem 3.1]) and Assumption 2 shows that its domain is closed, equal to $H$. Hence it is a continuous operator.

$M = \text{im} \mathbb{M}$ and Assumption 3 shows that there exist $k$ (and not more) linearly independent sequences $\{\alpha^i_n\}_{n \in J}$, $i = 1, \ldots, k$ such that

$$\sum_{n \in J} \alpha^i_n \langle f, e_n \rangle = 0 \quad \forall f \in H. \quad (4.6)$$

Note that the assumptions of this lemma does not depend on the order of the elements $e_n$. If we exchange the order of two elements $e_{n_1}$ and $e_{n_2}$ then the corresponding elements $\alpha^i_{n_1}$ and $\alpha^i_{n_2}$ are exchanged for every $i = 1, \ldots, k$. So we can assume $\alpha^i_1 \neq 0$ and without restriction $\alpha^i_1 = 1$. Hence, every $e_i$, $i = 1, \ldots, k$ is a linear combination of the elements $e_n$, $n \in J \setminus \{1, \ldots, k\}$.

The operator

$$\mathbb{M}'f = \{\langle f, e_n \rangle_{n \in J \setminus K} \} \quad H \mapsto l^2(J \setminus K)$$

has dense image in $l^2(J \setminus K)$ otherwise we can find $\gamma$ orthogonal to its image, and adding $k$ entries equal to zero in front, we have an element orthogonal to $\text{im} \mathbb{M}$, linearly independent of $\{\alpha^i_1\}$ and this is not possible.

Let $\mathbb{M}_0$ be the operator on $H$

$$\mathbb{M}_0f = \{\langle f, e_n \rangle \quad n \in K\}.$$ 

Then, $\mathbb{M}$ is the direct sum $\mathbb{M} = \mathbb{M}_0 \oplus \mathbb{M}'$ and Lemma 9 shows that $\text{im} \mathbb{M}$ is closed. Hence, the image of $\mathbb{M}'$ is closed too, since $\text{im} \mathbb{M}$ is closed. Consequently, $\mathbb{M}'$ is surjective from $H$ to $l^2(J \setminus \{1, \ldots, k\})$ and boundedly invertible. It follows that $\{e_n\}_{n \in J \setminus K}$ is a Riesz sequence. $\blacksquare$

References

[1] Agmon, S., Lectures on elliptic boundary value problems. D. Van Nostrand Co., Princeton (1965).

[2] Avdonin, S.A., Ivanov, S.A.: Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems. Cambridge University Press, New York (1995).
[3] Ervedoza, S., Zuazua E., A systematic method for building smooth controls from smooth data, *Discr. Cont. Dyn. Systems Discrete Contin. Dyn. Syst. Ser. B*, **14** 1375-1401 (2010).

[4] Ervedoza S., Zuazua E., *Numerical approximation of exact controls for waves*. Springer, New York (2013).

[5] Gohberg, I.C., Krejn, M.G.: *Opérateurs Linéaires non Auto-adjoints dans un Espace Hilbertien*. Dunod, Paris (1971).

[6] Lasiecka, I., Triggiani, R.: A cosine operator approach to $L_2(0,T;L_2(\Gamma))$–boundary input hyperbolic equations. *Appl. Math. Optim.* **7**, 35-93 (1981).

[7] Lions, J-L.: *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*. Vol. 1, Recherches en Mathématiques Appliquées 9, Masson, Paris (1988).

[8] Komornik, V., *Exact controllability and stabilization. The multiplier method*. RAM: Research in Applied Mathematics. John Wiley & Sons, Ltd., Chichester (1994).

[9] Pandolfi, L., The controllability of the Gurtin-Pipkin equation: a cosine operator approach. *Appl. Math. Optim.* **52**, 143-165 (2005) (a correction in *Appl. Math. Optim.* **64**, 467-468 (2011)).

[10] Pandolfi, L., Sharp control time for viscoelastic bodies. *J. Integral Equations Appl.* **27** 103-136 (2015).

[11] Pandolfi, L., *Distributed systems with persistent memory. Control and moment problems*. Springer Briefs in Electrical and Computer Engineering. Control, Automation and Robotics. Springer, Cham (2014).

[12] Pandolfi, L., Controllability for the heat equation with memory: a recent approach, *Riv. Matem. Univ. Parma*, **7** 259-277 (2016).

[13] L. Pandolfi: Controllability of isotropic viscoelastic bodies of Maxwell-Boltzmann type, in print, *ESAIM control. Calc. Var.*, https://doi.org/10.1051/cocv/2016068

[14] Pandolfi, L., Triulzi, D., Regularity of the steering control for systems with persistent memory, *Appl. Math. Lett.*, **51** 34-40 (2016).
[15] Russel, D. A unified boundary controllability theory for hyperbolic and parabolic partial differential equations, in *Studies in Applied Mathematics* **52** 189-211 (1973).