Distributed Compression of Correlated Classical-Quantum Sources or: The Price of Ignorance

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We resume the investigation of the problem of independent local compression of correlated quantum sources, the classical case of which is covered by the celebrated Slepian-Wolf theorem. We focus specifically on classical-quantum (cq) sources, for which one edge of the rate region, corresponding to the compression of the classical part, using the quantum part as side information at the decoder, was previously determined by Devetak and Winter [Phys. Rev. A 68, 042301 (2003)]. Whereas the Devetak-Winter protocol attains a rate-sum equal to the von Neumann entropy of the joint source, here we show that the full rate region is much more complex, due to the partially quantum nature of the source. In particular, in the opposite case of compressing the quantum part of the source, using the classical part as side information at the decoder, typically the rate sum is strictly larger than the von Neumann entropy of the total source.

We determine the full rate region in the generic case, showing that, apart from the Devetak-Winter point, all other points in the achievable region have a rate sum strictly larger than the joint entropy. We can interpret the difference as the price paid for the quantum encoder being ignorant of the classical side information. In the general case, we give an achievable rate region, via protocols that are built on the decoupling principle, and the principles of quantum state merging and quantum state redistribution. Our achievable region is matched almost by a single-letter converse, which however still involves asymptotic errors and an unbounded auxiliary system.

I. SOURCE AND COMPRESSION MODEL

Data compression can be regarded as the foundation of information theory in the treatment of Shannon [1], and it remains one of the most fruitful problems to be considered, especially when additional constraints on the source, the encoders or the decoder are imposed. In particular, the Slepian-Wolf problem of two sources correlated in a known way, but subject to separate, local compression [2] has proved to provide a unifying principle for much of Shannon theory, giving rise to natural information theoretic interpretations of entropy and conditional entropy, and exhibiting deep connections with error correction, channel capacities and mutual information (cf. [3]). The quantum case has been investigated for two decades, starting with the second author’s PhD thesis [4] and subsequently in [5], up to the systematic study [6], and while we still do not have a complete understanding of the rate region, it has become clear that the problem is of much higher complexity than the classical case. The quantum Slepian-Wolf problem, and specifically quantum data compression with side information at the decoder, has resulted in many fundamental advances in quantum information theory, including the protocols of quantum state merging [7, 8] and quantum state redistribution [9], which have given operational meaning to the conditional von Neumann entropy, the mutual information and the conditional quantum mutual information, respectively.

A variety of resource models and different tasks have been considered over the years: The source and its recovery was either modelled as an ensemble of pure states (following Schumacher [10]), or as a pure state between the encoders and a reference system; the communication resource required was either counted in qubits communicated, in addition either allowing or disallowing entanglement, or it was counted in ebits shared between the agents, but with free classical communication. While this latter model has lead to the
most complete picture of the general rate region, in the present paper we will go back to the original idea [4, 10] of quantifying the communication, counted in qubits, between the encoders and the decoder.

**Notation.** We use the following conventions throughout the paper. Quantum systems are associated with (finite dimensional) Hilbert spaces $A$, $B$, etc., whose dimensions are denoted $|A|$, $|B|$, respectively. We identify states with their density operators, and we use the notation $\phi = |\phi\rangle\langle\phi|$ as the density operator of the pure state vector $|\phi\rangle$. The von Neumann entropy is defined as $S(\rho) = -\text{Tr} \rho \log \rho$ (throughout this paper, log denotes by default the binary logarithm, and its inverse function exp, unless otherwise stated, is also to basis 2). Conditional entropy and conditional mutual information, $S(A|B)\rho$ and $I(A : B|C)\rho$, respectively, are defined in the same way as their classical counterparts:

$$S(A|B)\rho = S(AB)\rho - S(B)\rho,$$

and

$$I(A : B|C)\rho = S(A|BC)\rho - S(AC)\rho + S(BC)\rho - S(ABC)\rho - S(C)\rho.$$ 

The fidelity between two states $\rho$ and $\sigma$ is defined as

$$F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1 = \text{Tr} \sqrt{\rho^\dagger\sigma\rho^\frac{1}{2}}.$$ 

It relates to the trace distance in the following well-known way [11]:

$$1 - F(\rho, \sigma) \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)^2}.$$ 

The source model we shall consider is a hybrid classical-quantum one, with two agents, Alice and Bob, whose task is to compress the classical and quantum parts of the source, respectively. They then send their shares to a decoder, Debbie, who has to reconstruct the classical information with high probability and the quantum information with high (average) fidelity.

In detail, the source is characterised by a classical source, i.e. a probability distribution $p(x)$ on a discrete (in fact: finite) alphabet $X$ which is observed by Alice, and a family of quantum states $\rho_x$ on a quantum system $B_x$, given by a Hilbert space of finite dimension $|B|$. To define the problem of independent local compression (and decompression) of such a correlated classical-quantum source, we shall consider purifications $\psi^{BR}_{x}$ of the $\rho_x$, i.e. $\rho_x^B = \text{Tr}_R \psi^{BR}_{x}$. Thus the source can be described compactly by the cq-state

$$\omega^{XBR} = \sum_{x \in X} p(x) |x\rangle\langle x|^X \otimes |\psi^x\rangle\langle \psi^x|^B.$$ 

We will be interested in the information theoretic limit of many copies of $\omega$, i.e.

$$\omega^{X^nBR^nR^n} = (\omega^{XBR})^{\otimes n} = \sum_{x^n \in X^n} p(x^n) |x^n\rangle\langle x^n|^X \otimes |\psi^{x^n}\rangle\langle \psi^{x^n}|^{BR^n},$$ 

where we use the notation

$$x^n = x_1 x_2 \ldots x_n,$$

$$|x^n\rangle = |x_1\rangle |x_2\rangle \cdots |x_n\rangle,$$

$$p(x^n) = p(x_1)p(x_2)\ldots p(x_n),$$

and

$$|\psi^{x^n}\rangle = |\psi^{x_1}\rangle |\psi^{x_2}\rangle \cdots |\psi^{x_n}\rangle.$$ 

Alice and Bob, receiving their respective parts of the source, separately encode these using the most general allowed quantum operations; the compressed quantum information, living on a certain number of qubits, is passed to the decoder who has to output, again acting with a quantum operation, an element of $X$ and a state on $B^n$, in such a way as to attain a low error probability for $x^n$ and a high-fidelity approximation of the conditional quantum source state, $\psi^{BR^n}_{x^n}$. We consider two models: unassisted and entanglement-assisted, which we describe formally in the following (see Figs. 1 and 2).

**Unassisted model.** With probability $p(x^n)$, the source provides Alice and Bob respectively with states $|x^n\rangle^X$ and $|\psi^{x^n}\rangle^{BR^n}$. Alice and Bob then perform their respective encoding operations $E_X : X^n \rightarrow C_X$
and $\mathcal{E}_B : B^n \rightarrow C_B$, respectively, which are quantum operations, i.e., completely positive and trace preserving (CPTP) maps. Of course, as functions they act on the operators (density matrices) over the respective input and output Hilbert spaces. But as there is no risk of confusion, and not to encumber the notation, we will simply write the Hilbert spaces when denoting a CPTP map. Note that since $X$ is a classical random variable, $\mathcal{E}_X$ is entirely described by a cq-channel. We call $R_X = \frac{1}{n} \log |C_X|$ and $R_B = \frac{1}{n} \log |C_B|$ the quantum rates of the compression protocol. Since Alice and Bob are required to act independently, the joint encoding operation is $\mathcal{E}_X \otimes \mathcal{E}_B$. The systems $C_X$ and $C_B$ are then sent to Debbie who performs a decoding operation $\mathcal{D} : C_XC_B \rightarrow X^nB^n$. We define the extended source state

$$\omega^{X^nX'^nB^nR^n} = (\omega^{XX'R}) \otimes \sum_{x^n \in X^n} p(x^n) |x^n\rangle\langle x^n|^{X^n} \otimes |x^n\rangle\langle x^n|^{X'^n} \otimes |\psi_{x^n}\rangle\langle \psi_{x^n}|^{B^nR^n},$$

and say the encoding-decoding scheme has average fidelity $1 - \epsilon$ if

$$\mathcal{F} = F \left( \omega^{X^nX'^nB^nR^n}, (\mathcal{D} \circ (\mathcal{E}_X \otimes \mathcal{E}_B) \otimes \text{id}_{X'^nR^n}) \omega^{X^nX'^nB^nR^n} \right) \geq 1 - \epsilon,$$

(1)

where $\text{id}_{X'^nR^n}$ is the identity (ideal) channel acting on $X'^nR^n$. By the above fidelity definition and the linearity of CPTP maps, the average fidelity defined in (1) can be expressed equivalently as

$$\mathcal{F} = \sum_{x^n \in X^n} p(x^n) F \left( |x^n\rangle\langle x^n|^{X^n} \otimes |\psi_{x^n}\rangle\langle \psi_{x^n}|^{B^nR^n}, (\mathcal{D} \circ (\mathcal{E}_X \otimes \mathcal{E}_B) \otimes \text{id}_{R^n}) |x^n\rangle\langle x^n|^{X^n} \otimes |\psi_{x^n}\rangle\langle \psi_{x^n}|^{B^nR^n} \right).$$

We say that $(R_X, R_B)$ is an (asymptotically) achievable rate pair if there exist codes $(\mathcal{E}_X, \mathcal{E}_B, \mathcal{D})$ as above for every $n$, with fidelity $\mathcal{F}$ converging to 1, and classical and quantum rates converging to $R_X$ and $R_B$, respectively. The rate region is the set of all achievable rate pairs, as a subset of $\mathbb{R}^2_{\geq 0}$.

Figure 1. Circuits diagram of the unassisted model. Dotted lines are used to demarcate domains controlled by the different participants. The solid lines represent quantum information registers.

It is shown by Devetak and Winter [4, 5] that the rate pair

$$(R_X, R_B) = (S(X|B), S(B))$$

(2)

is achievable and optimal. The optimality is two-fold; first, the rate sum achieved, $R_X + R_B = S(X|B)$ is minimal, and secondly, even with unlimited $R_B$, $R_X \geq S(X|B)$. This shows that the Devetak-Winter point is an extreme point of the rate region. Interestingly, Alice can achieve the rate $S(X|B)$ using only classical communication. However, we will prove the converse theorems considering a quantum channel for Alice, which are obviously stronger statements. In Theorem 14, we show that our system model is equivalent to the
model considered in [4, 5], which implies the achievability and optimality of this rate pair in our system model. We remark that in [5], the rate $R_B = S(B)$ was not explicitly discussed, but it is clear that it can always be achieved by Schumacher’s quantum data compression [10], introducing an arbitrarily small additional error.

**Entanglement-assisted model.** This model generalizes the unassisted model, and it is basically the same, except that we let Bob and Debbie share entanglement and use it in encoding and decoding, respectively. In addition, we take care of any possible entanglement that is produced in the process. Consequently, while Alice’s encoding $\mathcal{E}_X : X^n \rightarrow C_X$ remains the same, the Bob’s encoding and the decoding map now act as $\mathcal{E}_B : B^n B'_0 \rightarrow C_B B'_0$ and $D : C_X C_B D_0 \rightarrow X^n B^n D'_0$, respectively, where $B_0$ and $D_0$ are $K$-dimensional quantum registers of Bob and Debbie, respectively, designated to hold the initially shared entangled state, and $B'_0$ and $D'_0$ are $L$-dimensional registers for the entanglement produced by the protocol. Ideally, both initial and final entanglement are given by maximally entangled states $\Phi_K$ and $\Phi_L$, respectively. Correspondingly, we say that the encoding-decoding scheme has average fidelity $1 - \epsilon$ if

$$F := F \left( \omega^{X^n X^n B^n R^n} \otimes \Phi_L^{B'_0 D'_0} (\mathcal{D} \circ (\mathcal{E}_X \otimes \mathcal{E}_B B_0 \otimes \text{id}_{D_0}) \otimes \text{id}_{X^n R^n}) \omega^{X^n X^n B^n R^n} \otimes \Phi_K^{B_0 D_0} \right) \geq 1 - \epsilon. \quad (3)$$

We call $E = \frac{1}{2}(\log K - \log L)$ the entanglement rate of the scheme. The CPTP map $\mathcal{E}_B$ takes the input systems $B^n B'_0$ to the compressed system $C_B$ plus Bob’s share of the output entanglement, $B'_0$. Debbie applies the decoding operation $\mathcal{D}$ on the received systems $C_X C_B$ and her part of the initial entanglement $D_0$, to produce an output state on systems $X^n B^n$ plus her share of the output entanglement, $D'_0$. We say $(R_X, R_B, E)$ is an (asymptotically) achievable rate triple if for all $n$ there exist entanglement-assisted codes as before, such that the fidelity $F$ converges to 1, and the classical, quantum and entanglement rates converge to $R_X$, $R_B$ and $E$, respectively. The rate region is the set of all achievable rate pairs, as a subset of $\mathbb{R}^2_0 \times \mathbb{R}$. In the following we will be mostly interested in the projection of this region onto the first two coordinates, $R_X$ and $R_B$, corresponding to unlimited entanglement assistance.

It is a simple consequence of the time sharing principle that the rate regions, both for the unassisted and the entanglement-assisted model, are closed convex regions. Furthermore, since one can always waste rate, the rate regions are open to the “upper right”. This means that the task of characterizing the rate regions boils down to describing the lower boundary, which can be achieved by convex inequalities. In the Slepian-Wolf problem, it is in fact linear inequalities, and we will find analogues of these in the present investigation.

Stinespring’s dilation theorem [12] states that any CPTP map can be built from the basic operations of isometry and reduction to a subsystem by tracing out the environment system [12]. Thus, the encoders and the decoder are without loss of generality isometries

$$U_X : X^n \rightarrow C_X W_X,$$

$$U_B : B^n B'_0 \rightarrow C_B B'_0 W_B,$$

$$V : C_X C_B D_0 \rightarrow X^n B^n D'_0 W_D,$$

where the new systems $W_X$, $W_B$ and $W_D$ are the environment systems of Alice, Bob and Debbie, respectively. They simply remain locally in possession of the respective party.

The following lemma states that for a code of block length $n$ and error $\epsilon$, the environment parts of the encoding and decoding isometries, i.e. $W_X$, $W_B$ and $W_D$, as well as the entanglement output registers $B'_0$ and $D'_0$, are decoupled from the reference $R^n$, conditioned on $X^n$. This lemma plays a crucial role in the proofs of converse theorems; it is proved in Appendix B.

**Lemma 1.** (Decoupling condition) For a code of block length $n$ and error $\epsilon$ in the entanglement-assisted model, let $W_X$, $W_B$ and $W_D$ be the environments of Alice’s and Bob’s encoding and of Debbie’s decoding isometries, respectively. Then,

$$I(W_X W_B W_D B'_0 D'_0 \mid X^n) \leq n \delta(n, \epsilon),$$

where $\delta(n, \epsilon) = 4\sqrt{\epsilon} \log(|X||B|) + \frac{\epsilon}{2} h(\sqrt{\epsilon})$, with the binary entropy $h(\epsilon) = -\epsilon \log \epsilon - (1 - \epsilon) \log(1 - \epsilon)$; the mutual information is with respect to the state

$$\xi^{X^n X^n B^n B'_0 D'_0 W_X W_B W_D R^n} = (\mathcal{D} \circ (\mathcal{E}_X \otimes \mathcal{E}_B \otimes \text{id}_{D_0}) \otimes \text{id}_{X^n R^n}) \omega^{X^n X^n B^n R^n} \otimes \Phi_K^{B_0 D_0}. $$
Figure 2. Circuits diagram of the entanglement-assisted model. Dotted lines are used to demarcate domains controlled by the different participants. The solid lines represent quantum information registers.

The structure of the rest of the paper is as follows. In the next section (Sec. II) we start looking at the important subproblem of compressing the quantum part of the source when the classical part is sent uncompressed, in other words we want to find the minimum achievable rate $R_B$ when $R_X$ is unbounded; this is the opposite edge of the rate region from the one determined in [4, 5]. We give a general lower (converse) bound and an upper (achievability) bound, which however do not match in general. Then, in Sec. II C we show that for a family of generic sources, the two bounds coincide, showing that for almost all sources in any open set of sources, the optimal quantum compression rate is $R_B = \frac{1}{2}(S(B) + S(B|X))$. These results hold in both models, entanglement-assisted and unassisted. In Sec. II C, we move to analysing the full rate region. We first extend the converse bound from Sec. II to a general outer bound on the rate region (Subsec. III A), which yields a tight, single-letter characterization of the rate region for generic sources, equally with or without entanglement-assistance (Subsec. III B); In general, however, can only give an outer bound on the rate region (Subsec. III C) Finally, in Sec. IV, we close with a discussion of what we have achieved and of the principal open questions left by our work.

II. QUANTUM DATA COMPRESSION WITH CLASSICAL SIDE INFORMATION

In this section, we assume that Alice sends her information to Debbie at rate $R_X = \log |X|$ such that Debbie can decode it perfectly, and we ask how much Bob can compress his system given that the decoder has access to classical side information $X^n$. This problem is a special case of the classical-quantum Slepian-Wolf problem, and we call it quantum data compression with classical side information at the decoder, in analogy to the problem of classical data compression with quantum side information at the decoder, in analogy to the classical-quantum Slepian-Wolf problem. We know from previous section that the Bob’s encoder is without loss of generality an isometry $U \equiv U_B : B^n B_0 \rightarrow CW B'_0$, taking $B^n$ and Bob’s part of the entanglement $B_0$ to systems $C \otimes W \otimes B'_0$, where $C \equiv C_B$ is the compressed information of rate $R_B = \frac{1}{n} \log |C|$; $W \equiv W_B$ is the environment of Bob’s encoding CPTP map, and $B'_0$ is the register carrying Bob’s share of the output entanglement (in this section, we drop subscript $B$ from $C_B$ and $W_B$). Having access to side information $X^n$, Debbie applies the decoding isometry $V : X^nCD_0 \rightarrow \hat{X}^n \hat{B}^nW_D D'_0$ to generate the output systems $\hat{X}^n \hat{B}^n$ and entanglement share $D'_0$, and where $W_D$ is the environment of the isometry. We call this encoding-decoding scheme a side information code of
block length $n$ and error $\epsilon$ if the average fidelity (3) is at least $1 - \epsilon$.

A. Converse bound

To state our lower bound on the necessary compression rate, we introduce the following quantity, which emerges naturally from the converse proof.

**Definition 2.** For the state $\omega^{XBR} = \sum_x p(x) |x\rangle\langle x|_X \otimes |\psi_x\rangle\langle\psi_x|_B^R$ and $\delta \geq 0$, define

$$I_\delta(\omega) := \sup_{\mathcal{T}} I(X : W)_\sigma \text{ s.t. } \mathcal{T} : B \rightarrow W \text{ ctp with } I(R : W | X)_\sigma \leq \delta,$$

where the mutual informations are understood with respect to the state $\sigma^{XWR} = (id_X \otimes \mathcal{T})\omega$ and $W$ ranges over arbitrary finite dimensional quantum systems.

The function $I_\delta = I_\delta(\omega)$ is non-decreasing and concave in $\delta$. Hence, it is also continuous for $\delta > 0$. Furthermore, let $\tilde{I}_0 := \lim_{\delta \rightarrow 0} I_\delta = \inf_{\delta > 0} I_\delta$.

Note that the system $W$ is not restricted in any way, which is the reason why in this definition we have a supremum and an infimum, rather than a maximum and a minimum. (It is a simple consequence of compactness of the domain of optimisation, together with the continuity of the mutual information, that if we were to impose a bound on the dimension of $W$ in the above definition, the supremum in $I_\delta$ would be attained, and for the infimum in $\tilde{I}_0$, it would hold that $\tilde{I}_0 = I_0$.)

**Proof of the properties of $I_\delta$.** The non-decrease with $\delta$ is evident from the definition, so we only have to prove concavity. For this consider $\delta_1, \delta_2 \geq 0$, $0 < p < 1$, and let $\delta = p\delta_1 + (1-p)\delta_2$. Let furthermore channels $\mathcal{T}_i : B \rightarrow W_i$ be given $(i = 1, 2)$ such that for the states $\sigma_i^{XWR} = (id_X \otimes \mathcal{T}_i)\omega$, $I(R : W_i | X)_{\sigma_i} \leq \delta_i$.

Now define $W := W_1 \otimes W_2$, so that $W_1$ and $W_2$ can be considered mutually orthogonal subspaces of $W$, and define the new channel $c\mathcal{T} := p\mathcal{T}_1 + (1-p)\mathcal{T}_2 : B \rightarrow W$. By the chain rule for the mutual information, one can check that w.r.t. $\sigma^{XWR} = (id_X \otimes \mathcal{T})\omega$,

$$I(R : W | X)_{\sigma} = pI(R : W_1 | X)_{\sigma_1} + (1-p)I(R : W_2 | X)_{\sigma_2} \leq p\delta_1 + (1-p)\delta_2 = \delta,$$

and likewise

$$I(X : W)_{\sigma} = pI(X : W_1)_{\sigma_1} + (1-p)I(X : W_2)_{\sigma_2}.$$

Hence, $I_\delta \geq pI(X : W_1)_{\sigma_1} + (1-p)I(X : W_2)_{\sigma_2}$; by maximizing over the channels, the concavity follows. ■

**Lemma 3.** The function $I_\delta(\omega)$ introduced in Definition 2, has the following additivity property. For any two states $\omega_1^{X_1B_1R_1}$ and $\omega_2^{X_2B_2R_2}$ and for $\delta \geq 0$,

$$I_\delta(\omega_1 \otimes \omega_2) = \max_{\delta_1 + \delta_2 = \delta} I_{\delta_1}(\omega_1) + I_{\delta_2}(\omega_2).$$

Consequently, $I_{n\delta}(\omega^\otimes n) = nI_\delta(\omega)$, and furthermore $I_0$ and $\tilde{I}_0$ are additive:

$$I_0(\omega_1 \otimes \omega_2) = I_0(\omega_1) + I_0(\omega_2), \quad \tilde{I}_0(\omega_1 \otimes \omega_2) = \tilde{I}_0(\omega_1) + \tilde{I}_0(\omega_2).$$

**Proof.** First, we prove that $I_\delta(\omega_1 \otimes \omega_2) \leq \max_{\delta_1 + \delta_2 = \delta} I_{\delta_1}(\omega_1) + I_{\delta_2}(\omega_2)$; the other direction of the inequality is trivial from the definition. Let $\mathcal{T} : B_1B_2 \rightarrow W$ be a CPTP map such that

$$\delta \geq I(W : R_1R_2X_1X_2) = I(W : R_1 | X_1X_2) + I(W : R_2 | X_1X_2)$$

$$= I(WX_2 : R_1 | X_1) + I(WX_1R_1 : R_2 | X_2),$$

where the second line is due to the independence of $\omega_1$ and $\omega_2$. We now define the new systems $W_1 := WX_2$ and $W_2 := WX_1R_1$. Then we have,

$$I(W : X_1X_2) = I(W : X_2) + I(W : X_1 | X_2) \leq I(WX_1R_1 : X_2) + I(WX_2 : X_1 | X_2).$$
where the second equality is due to the independence of $X_1$ and $X_2$. The inequality follows from data processing. From Eq. (4) we know that $I(W_1 : R_1 | X_1) \leq \delta_1$ and $I(W_2 : R_2 | X_2) \leq \delta_2$ for some $\delta_1 + \delta_2 = \delta$. Thereby, from Eq. (5) we obtain
\[
I_\delta(\omega_1 \otimes \omega_2) \leq I_{\delta_1}(\omega_1) + I_{\delta_2}(\omega_2) \\
\leq \max_{\delta_1 + \delta_2 = \delta} I_{\delta_1}(\omega_1) + I_{\delta_2}(\omega_2),
\]
Now, the multi-copy additivity follows easily: According to the first statement of the lemma, we have
\[
I_{n\delta}(\omega^{\otimes n}) = \max_{\delta_1 + \ldots + \delta_n = n \delta} I_{\delta_1}(\omega) + \ldots + I_{\delta_n}(\omega).
\]
Here, the right hand side is clearly $\geq nI_\delta(\omega)$ since we can choose all $\delta_i = \delta$. By the concavity of $I_\delta(\omega)$ in $\delta$, on the other hand, we have for any $\delta_1 + \ldots + \delta_n = n\delta$ that
\[
\frac{1}{n}(I_{\delta_1}(\omega) + \ldots + I_{\delta_n}(\omega)) \leq I_\delta(\omega),
\]
so the maximum is attained at $\delta_i = \delta$ for all $i = 1, \ldots, n$.
The first statement of the lemma also implies that $I_0$ and $I_\delta$ are additive.

We stop here briefly to remark on the curious resemblance of our function $I_\delta$ with the so-called information bottleneck function introduced by Tishby et al. [13], whose generalization to quantum information theory is recently being discussed [14, 15]. Indeed, the concavity and additivity properties of the two functions are proved by the same principles, although it is not evident to us, what—if any—, the information theoretic link between $I_\delta$ and the information bottleneck is.

**Theorem 4.** Consider any side information code of block length $n$ and error $\epsilon$, in the entanglement-assisted model. Then, the Bob’s quantum communication rate is lower bounded
\[
R_B \geq \frac{1}{2} \left( S(B) + S(B|X) - I_{\delta(n,\epsilon)} - \delta(n,\epsilon) \right),
\]
where $\delta(n,\epsilon) = 4\sqrt{6} \log(|X||B|) + \frac{2}{n} h(\sqrt{6}\epsilon)$. Any asymptotically achievable rate $R_B$ is consequently lower bounded
\[
R_B \geq \frac{1}{2} \left( S(B) + S(B|X) - I_0 \right).
\]

**Proof.** As already discussed in the introduction to this section, the encoder of Bob is without loss of generality an isometry $U : B^n B_0 \rightarrow CW B_0'$. The existence of a high-fidelity decoder using $X^n$ as side information is equivalent to decoupling of $WB_0'$ from $R^n$ conditional on $X^n$; indeed, by Lemma 1, $I(R^n : WB_0'|X^n) \leq n\delta(n,\epsilon)$. The first part of the converse reasoning is as follows:
\[
nR_B = \log |C| \geq S(C) \\
\geq S(CWB_0') - S(WB_0') \\
= S(B^n) + S(B_0) - S(WB_0'),
\]
where the second inequality is a version of subadditivity, and the equality in the last line holds because the encoding isometry $U$ does not change the entropy; furthermore, $B^n$ and $B_0$ are initially independent. Moreover, the decoder can be dilated to an isometry $V : X^n CD_0 \rightarrow X^n B^n D_0' WD$, where $W_D$ and $D_0'$ are the environment of Debbie’s decoding operation and the output of Debbie’s entanglement, respectively. Using the decoupling condition of Lemma 1 once more, we have
\[
nR_B + S(D_0) = \log |C| + S(D_0) \\
\geq S(C) + S(D_0) \\
\geq S(CD_0) \\
\geq S(X^n CD_0|X'^n) \\
= S(\hat{X}^n \hat{B}^n D_0' WD | X'^n) \\
= S(WB_0' R^n | X'^n) \\
\geq S(R^n | X'^n) + S(WB_0' | X'^n) - n\delta(n,\epsilon) \\
= S(B^n | X^n) + S(WB_0' | X'^n) - n\delta(n,\epsilon),
\]
where the third and fourth line are by subadditivity of the entropy; the fifth line follows because the decoding isometry $V$ does not change the entropy. The sixth line holds because for any given $x^n$ the overall state of the systems $X^nB^nB'_nD^n_D|WW_B^nR^n$ is pure. The penultimate line is due to the decoupling condition (Lemma 1), and the last line follows because for a given $x^n$ the overall state of the systems $B^nR^n$ is pure. Adding these two relations and dividing by $2n$, we obtain

$$R_B \geq \frac{1}{2}(S(B) + S(B|X)) - \frac{1}{2n}I(X^n : WB'_n) - \delta(n,\epsilon).$$

In the above inequality, the mutual information on the right hand side is bounded as

$$I(X^n : WB'_n) \leq I_{n\delta(n,\epsilon)}(\omega^{\otimes n}) = nI_{\delta(n,\epsilon)}(\omega),$$

To see this, define the CPTP map $T : B^n \rightarrow \tilde{W} := WB'_n$ as $T(\rho) := Tr_{CD^n}(U \otimes 1)(\rho \otimes \Phi_{B^nD^n}^R)(U \otimes 1)^\dagger$. Then we have $I(R^n : \tilde{W}|X^n) \leq n\delta(n,\epsilon)$, and hence the above inequality follows directly from Definition 2.

The second statement of the theorem follows because $\delta(n,\epsilon)$ tends to zero as $n \rightarrow \infty$ and $\epsilon \rightarrow 0$.

**Remark 5.** Notice that the term $\frac{1}{n}I(X^n : WB'_n)$ is not necessarily small. For example, suppose that the source is of the form $|\psi_x^{BR} = |\psi_x^{BR} \otimes |\psi_x^{B'}\rangle$ for all $x$; clearly it is possible to perform the coding task by coding only $B'$ and trashing $B''$ (i.e. putting it into $W$), because by having access to $x$ the decoder can reproduce $\psi_x^{B''}$ locally. In this setting, characteristically $\frac{1}{n}I(X^n : WB'_n)$ does not go to zero because $B''$ ends up in $W$.

### B. Achievable rates

In this subsection, we provide achievable rates both for the unassisted and entanglement-assisted model.

**Theorem 6.** In the unassisted model, there exists a sequence of side information codes that compress Bob’s system $B^n$ at the asymptotic qubit rate

$$R_B = \frac{1}{2}(S(B) + S(B|X)).$$

**Proof.** We can use the fully quantum Slepian-Wolf protocol (FQSW), also called coherent state merging [8], as a subprotocol since it considers the entanglement fidelity as the decodability criterion, which is more stringent than the average fidelity defined in (1). Namely, let

$$|\Omega_{X'X''}^{BR} = \sum_{x \in X} \sqrt{p(x)} |x\rangle^X |x\rangle^{X'} |\psi_x^{BR}$$

be the source in the FQSW problem, with the entanglement fidelity $F_\epsilon$ is the decodability criterion:

$$F_\epsilon = F\left(\Omega_{X'X''}^{BR R^n R^n}, (\mathcal{D} \circ (id_{X^n} \otimes \mathcal{E}_B) \otimes id_{X^n R^n}) \Omega_{X'X''}^{BR R^n R^n}\right) \leq F\left(\omega_{X'X''}^{BR R^n}, (\mathcal{D} \circ (id_{X^n} \otimes \mathcal{E}_B) \otimes id_{X^n R^n}) \omega_{X'X''}^{BR R^n}\right) = \overline{F},$$

where the inequality is due to the monotonicity of fidelity under CPTP maps, namely the projective measurement on system $X'$ in the computational basis $\{|x\rangle|x\rangle\}$. Therefore, if an encoding-decoding scheme attains an entanglement fidelity for the FQSW problem going to 1, then it will have average fidelity for the QC SW problem going to 1 as well. Hence, the FQSW rate

$$R_B = \frac{1}{2}I(B : X'R) = \frac{1}{2}(S(B) + S(B|X)_{\omega})$$

is achievable.
Remark 7. Notice that for the source considered at the end of the previous subsection in Remark 5, where  
\[ |\psi_x\rangle^{BR} = |\psi_x\rangle^{B'R} \otimes |\psi_x\rangle^{B''} \]  
for all \( x \), we can achieve a rate strictly smaller than the rate stated in the above theorem. The reason is that \( R \) is only entangled with \( B' \), so clearly it is possible to perform the coding task by coding only \( B' \) and trashing \( B'' \) because by having access to \( x \) the decoder can reproduce the state \( |\psi_x\rangle^{B''} \) locally. 

Thereby, the rate \( \frac{1}{2}(S(B') + S(B|X)) \) from Theorem 6 is not optimal. Looking for a systematic way of obtaining better rates, we have the following result in the entanglement-assisted model.

Theorem 8. In the entanglement-assisted model, there exists a sequence of side information codes with the following asymptotic entanglement and qubit rates:

\[ E = \frac{1}{2}(I(C:W)_\sigma - I(C:X)_\sigma) \quad \text{and} \quad R_B = \frac{1}{2}(S(B) + S(B|X)) - I(X:W)_\sigma, \]

where \( C \) and \( W \) are, respectively, the system and environment of an isometry \( V : B \rightarrow CW \) on \( \omega^{XBR} \) producing state \( \sigma^{XWBR} = (\text{id}_{XR} \otimes V)\omega \), such that \( I(W:R|X)_\sigma = 0 \).

Proof. First, Bob applies the isometry \( V \) to each copy of the \( n \) systems \( B_1, \ldots, B_n \):

\[ \sigma^{X'XWBR} = (V^{B \rightarrow CW} \otimes \mathbb{1}_{X'R})\omega^{X'XBR}(V^{B \rightarrow CW} \otimes \mathbb{1}_{X'R})^\dagger \]

\[ = \sum_x p(x) |x\rangle|x\rangle^X \otimes |x\rangle|x\rangle^X' \otimes |\phi_x\rangle|\phi_x\rangle^{WBR}. \]

Now suppose the following source state, where Bob and Debbie respectively hold the \( CW \) and \( X \) systems, and Bob wishes to send system \( C \) to Debbie while keeping \( W \) for himself:

\[ |\Sigma\rangle^{X'XWBR} = \sum_{x \in \mathcal{X}} \sqrt{p(x)} |x\rangle^{X'} |x\rangle^{X} |\phi_x\rangle^{WBR}. \]

For many copies of the above state, the parties can apply the quantum state redistribution (QSR) protocol [16, 17] for transmitting \( C \), having access to system \( W \) as side information at the encoder and to \( X \) as side information at the decoder. According to this protocol, Bob needs a rate of \( R_B = \frac{1}{2}I(C:W|X)_\Sigma = \frac{1}{2}I(C:W)_\sigma \) qubits of communication. The protocol requires a rate of \( \frac{1}{2}I(C:W)_\Sigma = \frac{1}{2}I(C:W)_\sigma \) qubits of entanglement shared between the encoder and decoder, and at the end of the protocol a rate of \( \frac{1}{2}I(C:W)_\Sigma = \frac{1}{2}I(C:X)_\sigma \) entanglement is distilled between the encoder and the decoder. This protocol attains high fidelity for the states \( \sigma^{X'XWBR} \) and, consequently for the state \( \sigma^{X'XWBR} \) due to the monotonicity of fidelity under CPTP maps:

\[ 1 - \epsilon \leq \mathcal{F}
\left(\Sigma^{X'XWBR} \otimes \Phi_{K}^{BR_D} \otimes \mathcal{D} \circ (\text{id}_{X} \otimes \mathcal{E}_{CWBR}) \otimes \text{id}_{X'BR'} \right), \]

\[ \leq \mathcal{F}
\left(\sigma^{X'XWBR} \otimes \Phi_{K}^{BR_D} \otimes \mathcal{D} \circ (\text{id}_{X} \otimes \mathcal{E}_{CWBR}) \otimes \text{id}_{X'BR'} \right), \quad (6) \]

where \( \mathcal{E}_{CWBR} \) and \( \mathcal{D} \) are respectively the encoding and decoding operations of the QSR protocol. The condition \( I(W:R|X)_\sigma = 0 \) implies that for every \( x \) the systems \( W \) and \( R \) are decoupled:

\[ \phi_x^{WR} = \phi_x^W \otimes \phi_x^R. \]

By Uhlmann’s theorem [18, 19], there exist isometries \( V_x : C \rightarrow VB \) for all \( x \in \mathcal{X} \), such that

\[ (\mathbb{1} \otimes V_x^{C \rightarrow VB})|\phi_x\rangle^{WBR} = |\psi_x\rangle^{VW} \otimes |\psi_x\rangle^{BR}. \]

After applying the decoding operation \( \mathcal{D} \) of QSR, Debbie applies the isometry \( V_x : C \rightarrow VB \) for each \( x \), which does not change the fidelity (6). By tracing out the unwanted systems \( V^nW^n \), due to the monotonicity of the fidelity under partial trace, the fidelity defined in (3) will go to 1 in this encoding-decoding scheme.
Remark 9. In Theorem 8, the smallest achievable rate, when unlimited entanglement is available, is equal to 
\( \frac{1}{2}(S(B) + S(B|X) - I_0) \). This rate resembles the converse bound \( R_B \geq \frac{1}{2}(S(B) + S(B|X) - I_0) \), except that \( I_0 \geq \tilde{I}_0 \). In the definition of \( \tilde{I}_0 \), it seems unlikely that we can take the limit of \( \delta \) going to 0 directly because there is no dimension bound on the systems \( C \) and \( W \), so compactness cannot be used directly to prove that \( \tilde{I}_0 \) and \( I_0 \) are equal.

Remark 10. Looking again at the entanglement rate in Theorem 8, \( E = \frac{1}{2} (I(C : W) - I(C : X)) \), we reflect that there may easily be situations where \( E \leq 0 \), meaning that no entanglement is consumed, and in fact no initial entanglement is necessary. In this case, the theorem improves the rate of Theorem 6 by the amount \( \frac{1}{2} I(X : W) \). This motivates the definition of the following variant of \( I_0 \),

\[ I_0(\omega) := \sup I(X : W) \text{ s.t. } I(R : W|X) = 0, \quad I(C : W) - I(C : X) \leq 0, \]

where the supremum is over all isometries \( V : B \rightarrow CW \).

As a corollary to these considerations, in the unassisted model the rate \( \frac{1}{2} (S(B) + S(B|X) - I_{0-}) \) is achievable.

C. Optimal compression rate for generic sources

In this subsection, we find the optimal compression rate for generic sources, by which we mean any source except for a submanifold of lower dimension within the set of all sources. Concretely, we will consider sources where there is at least one \( x \) for which the reduced state \( \psi_x = \text{Tr}_R [\psi_x]^{BR} \) has full support on \( B \). In this setting, coherent state merging as a subprotocol gives the optimal compression rate, so not only does the protocol not use any initial entanglement, but some entanglement is distilled at the end of the protocol.

Theorem 11. For any side information code of a generic source, with or without entanglement-assistance, the asymptotic compression rate \( R_B \) of Bob is lower bounded

\[ R_B \geq \frac{1}{2} (S(B) + S(B|X)), \]

so the protocol of Theorem 6 has optimal rate for a generic source. Moreover, in that protocol no prior entanglement is needed and a rate \( \frac{1}{2} I(X : B) \) ebits of entanglement is distilled between the encoder and decoder.

Proof. The converse bound of Theorem 4 states that the asymptotic quantum communication rate of Bob is lower bounded as

\[ R_B \geq \frac{1}{2} \left( S(B) + S(B|X) - \tilde{I}_0 \right), \]

where \( \tilde{I}_0 \) comes from Definition 2. We will show that for generic sources, \( \tilde{I}_0 = I_0 = 0 \). Moreover, Theorem 6 states that using coherent state merging, the asymptotic qubit rate of \( \frac{1}{2} (S(B) + S(B|X)) \) is achievable, that no prior entanglement is required and a rate of \( \frac{1}{2} I(X : B) \) ebits of entanglement is distilled between the encoder and the decoder.

We show that for any CPTP map \( T : B \rightarrow W \), which acts on a generic \( \omega^{XBR} \) and produces state \( \sigma^{XWR} = (\text{id}_X \otimes T) \omega^{XBR} \) such that \( I(R : W|X)_{\sigma} \leq \delta \) for \( \delta \geq 0 \), the quantum mutual information \( I(X : W)_{\sigma} \leq \delta' \log |X| + 2h(\frac{1}{2} \delta') \) where \( \delta' \) is defined in Eq. (8) below. Thus, we obtain

\[ \tilde{I}_0 = \lim_{\delta \rightarrow 0} I_{\delta} = 0. \]

To show this claim, we proceed as follows. From \( I(R : W|X)_{\sigma} \leq \delta \) we have

\[ I(R : W|X = x)_{\sigma} \leq \frac{\delta}{p(x)}, \]
so by Pinsker’s inequality [20] we obtain

\[ \| \phi_x^{WR} - \phi_x^W \otimes \phi_x^R \|_1 \leq \sqrt{\frac{2\delta \ln 2}{p(x)}}. \]

By Uhlmann’s theorem, there exists an isometry \( V_x : C \to BV \) such that

\[ \| (V_x \otimes \mathbb{1}_{WR}) \phi_x^{CWBR}(V_x \otimes \mathbb{1}_{WR})^\dagger - \theta_x^{WV} \otimes \psi_x^{BR} \|_1 \leq \sqrt{\frac{\delta \ln 2}{2p(x)}} \left( 2 - \sqrt{\frac{\delta \ln 2}{2p(x)}} \right), \]

where \( \theta_x^{WV} \) is a purification of \( \phi_x^W \). Since the source is generic by definition there is an \( x \), say \( x = 0 \), for which \( \psi_0^{BR} \) has full support on \( \mathcal{L}(H_B) \), i.e. \( \lambda_0 := \lambda_{\min}(\psi_0^{BR}) > 0 \). By Lemma 25 in Appendix A, for any \( \psi_x^{BR} \) there is an operator \( T_x \) acting on the reference system such that

\[ |\psi_x^{BR} \rangle = (\mathbb{1}_B \otimes T_x) |\psi_0^{BR} \rangle. \]

Using this fact, we show that the decoding isometry \( V_0 \) in Eq. (7) works for all states:

\[ \| (V_0 \otimes \mathbb{1}_{WR}) \phi_x^{CWBR}(V_0^\dagger \otimes \mathbb{1}_{WR}) - \theta_0^{WV} \otimes \psi_0^{BR} \|_1 \]

\[ = \| (V_0 \otimes \mathbb{1}_{WR})(\mathbb{1}_{CW} \otimes T_x) \phi_x^{CWBR}(\mathbb{1}_{CW} \otimes T_x)^\dagger (V_0^\dagger \otimes \mathbb{1}_{WR}) - \theta_0^{WV} \otimes (\mathbb{1}_B \otimes T_x) \psi_0^{BR}(\mathbb{1}_B \otimes T_x)^\dagger \|_1 \]

\[ = \| (\mathbb{1}_{BVW} \otimes T_x)(V_0 \otimes \mathbb{1}_{WR}) \phi_0^{CWBR}(V_0^\dagger \otimes \mathbb{1}_{WR}) (\mathbb{1}_{BVW} \otimes T_x)^\dagger - (\mathbb{1}_{BVW} \otimes T_x) \theta_0^{WV} \otimes \psi_0^{BR}(\mathbb{1}_{BVW} \otimes T_x)^\dagger \|_1 \]

\[ \leq \| \mathbb{1}_{BVW} \otimes T_x \|_2 \| (V_0 \otimes \mathbb{1}_{WR}) \phi_x^{CWBR}(V_0^\dagger \otimes \mathbb{1}_{WR}) - \theta_0^{WV} \otimes \psi_0^{BR} \|_1 \]

\[ \leq \frac{1}{\lambda_0} \sqrt{\frac{\delta \ln 2}{2p(0)}} \left( 2 - \sqrt{\frac{\delta \ln 2}{2p(0)}} \right), \]

where the last two inequalities follow from Lemma 18 and Lemma 25, respectively. By tracing out the systems \( VBR \) in the above chain of inequalities, we get

\[ \| \phi_x^W - \phi_0^W \|_1 \leq \frac{1}{\lambda_0} \sqrt{\frac{\delta \ln 2}{2p(0)}} \left( 2 - \sqrt{\frac{\delta \ln 2}{2p(0)}} \right) =: \delta'. \]

Thus, by triangle inequality we obtain

\[ \sum_{x} p(x) |x\rangle|x\rangle^X \otimes \phi_x^W - \sum_{x} p(x) |x\rangle|x\rangle^X \otimes \phi_0^W \|_1 \leq \sum_{x} p(x) \| \phi_x^W - \phi_0^W \|_1 \]

\[ \leq \frac{1}{\lambda_0} \sqrt{\frac{\delta \ln 2}{2p(0)}} \left( 2 - \sqrt{\frac{\delta \ln 2}{2p(0)}} \right) =: \delta'. \]

By applying the Alicki-Fannes inequality in the form of Lemma 24, to Eq. (9), we have

\[ I(X : W)_{\sigma} = S(X)_{\sigma} - S(X|W)_{\sigma} + S(X|W)_{\sigma_0} - S(X|W)_{\sigma_0} \]

\[ = S(X|W)_{\sigma_0} - S(X|W)_{\sigma} \]

\[ \leq \delta' \log |X| + 2h \left( \frac{1}{2} \delta' \right), \]

and the right hand side of the above inequality vanishes for \( \delta \to 0 \).
III. TOWARDS THE FULL RATE REGION

In this section, we consider the full rate region of the distributed compression of a classical-quantum source. The Devetak-Winter code, Eq. (2), and the code based on state merging, Theorem 6, we get two rate points in the unassisted (and hence also in the unlimited entanglement-assisted) rate region:

\[(R_X, R_B) = (S(X|B), S(B)), \quad (R_X, R_B) = \left( S(X), \frac{1}{2}(S(B) + S(B|X)) \right).\]

Their upper-right convex closure is hence an inner bound to the rate region, depicted schematically in Fig. 3 and described by the inequalities in the following theorem.

**Theorem 12.** For distributed compression of a classical-quantum source in unassisted model, the rate pairs satisfying the following inequalities are achievable:

\[
R_X \geq S(X|B), \\
R_B \geq \frac{1}{2}(S(B) + S(B|X)), \\
R_X + 2R_B \geq S(B) + S(XB).
\]

(10)

For generic sources we find that this is in fact the rate region. However, in general, we only present some outer bounds and inner bounds (achievable rates), which show the rate region to be much more complicated than the rate region of the classical Slepian-Wolf problem.

**A. General converse bounds**

For distributed compression of a classical-quantum source in general, we start with a general converse bound.
Theorem 13. The asymptotic rate pairs for distributed compression of a classical-quantum source in the entanglement-assisted model are lower bounded as

\[ R_X \geq S(X|B), \]
\[ R_B \geq \frac{1}{2} \left( S(B) + S(B|X) - I_0 \right), \]
\[ R_X + 2R_B \geq S(B) + S(BX) - I_0. \]

(11)

In the unassisted model, in addition to the above lower bounds, the asymptotic rate pairs are bounded as

\[ R_X + R_B \geq S(XB). \]

Proof. The individual lower bounds have been established already: \( R_X \geq S(X|B) \) is from [4, 5], in a slightly different source model. However, it also holds in our system model if Bob sends his information using unlimited communication such that Debbie can decode it perfectly. Namely, notice that the fidelity (1) is more stringent than the encoding criterion of [4, 5], so any converse bound considering the decoding criterion of [4, 5] is also a converse bound in our system model. The bound \( R_B \geq \frac{1}{2}(S(B) + S(B|X) - I_0) \) is from Theorem 11. These two bounds hold in the unassisted, as well as the entanglement-assisted model.

In the unassisted model, the rate sum lower bound \( R_X + R_B \geq S(XB) \) has been argued in [4, 5], too. As a matter of fact, for any distributed compression scheme for the source, \( E_X \otimes E_B \) jointly describes a Schumacher compression scheme with asymptotically high fidelity. Thus, its rate must be asymptotically lower bounded by the joint entropy of the source, \( S(XB) \) [4, 10, 21, 22].

This leaves the bound \( R_X + 2R_B \geq S(B) + S(BX) - I_0 \) to be proved in the entanglement-assisted model, which we tackle now. The encoders of Alice and Bob are isometries \( U_X : X^n \to C_XW_X \) and \( U_B : B^nB_0 \to C_BB_BB_0' \), respectively. They send their respective compressed systems \( C_X \) and \( C_W \) to Debbie and keep the environment parts \( W_X \) and \( W_B \) for themselves. Then, Debbie applies the decoding isometry \( V : C_XCBD_0 \to X^n \hat{B}'^nWD_0 \), where systems \( X^n \hat{B}'^nD_0 \) are the output states, and \( W_D \) and \( D_0' \) are the environment of Debbie’s decoding isometry and her output entanglement, respectively. We first bound the following sum rate:

\[ nR_X + nR_B + S(D_0) \geq S(C_X) + S(C_B) + S(D_0) \]
\[ \geq S(C_XC_BD_0) \]
\[ = S(\hat{X}^n\hat{B}^nWD_0') \]
\[ = S(\hat{X}^n\hat{B}^n) + S(W_DD_0'|\hat{X}^n\hat{B}^n) \]
\[ \geq S(\hat{X}^n\hat{B}^n) + S(W_DD_0'|\hat{X}^n\hat{B}^n) \]
\[ \geq S(X^nB^n) + S(W_DD_0'|\hat{X}^n\hat{B}^nX^n) - n\sqrt{2} \log(||X||B) - \sqrt{(2\epsilon)} \]
\[ \geq S(X^nB^n) + S(W_DD_0'|X^n) - 2n\delta(n, \epsilon) \]
\[ \geq S(X^nB^n) + S(W_DB_B'||X^n) - S(R^n\hat{B}^n\hat{X}^n|X^n) - 2n\delta(n, \epsilon) \]
\[ \geq S(X^nB^n) + S(W_DB_B'||X^n) - 3n\delta(n, \epsilon) \]
\[ = S(X^nB^n) + S(W_X|X^n) + S(W_DB_B'||X^n) - 3n\delta(n, \epsilon), \]

(12)

where the second line is by subadditivity, the equality in the third line follows because the decoding isometry \( V \) does not change the entropy. Then, in the fourth and fifth line we use the chain rule and strong subadditivity of entropy. The inequality in the sixth line follows from the decomposability of the systems \( X^nB^n \): the fidelity criterion (3) implies that the output state on systems \( X^n\hat{B}^n \) is \( 2\sqrt{2\epsilon} \)-close to the original state \( X^nB^n \) in trace norm; then apply the Fannes inequality (Lemma 23). The seventh line follows from the decoupling condition (Lemma 1), which implies that \( I(W_DD_0'|X^n\hat{B}^n|X^n) \leq n\delta(n, \epsilon) = 4n\sqrt{\epsilon} \log(||X||B) + 2\sqrt{(2\epsilon)} \).

In the eighth line, we use that for any given \( x^n \), the overall state of \( W_XW_DB_BD_0'R^n\hat{B}^n\hat{X}^n \) is pure, and invoking subadditivity; then, in line nine we use the decoding fidelity (3) once more, saying that the output state on systems \( X^n\hat{B}^nR^n\hat{X}^n \) is \( 2\sqrt{\epsilon} \)-close to the original state \( X^nB^nR^n\hat{X}^n \) in trace norm; then apply the Fannes inequality (Lemma 23). The equality in the eleventh line follows because for a given \( x^n \) the encoded states of Alice and Bob are independent.
Moreover, we bound $R_B$ as follows:

\[ nR_B \geq S(C_B) \]
\[ \geq S(C_B | W_B B'_0) \]
\[ = S(C_B W_B B'_0) - S(W_B B'_0) \]
\[ = S(B^n B_0) - S(W_B B'_0) \]
\[ = S(B^n) + S(B_0) - S(W_B B'_0). \]  \hspace{1cm} (13)

Adding Eqs. (12) and (13), and after cancellation of $S(B_0) = S(D_0)$ we get

\[ R_X + 2R_B \geq S(B) + S(XB) - \frac{1}{n} I(X^n : W_B B'_0) - 3n\delta(n, \epsilon) \]
\[ \geq S(B) + S(XB) - \frac{1}{n} I_{\delta(n, \epsilon)}(\omega \otimes n) - 3n\delta(n, \epsilon) \]
\[ = S(B) + S(XB) - I_{\delta(n, \epsilon)}(\omega) - 3n\delta(n, \epsilon), \]  \hspace{1cm} (14)

where given that $I(R^n : B'_0 W_B | X^n) \leq \delta(n, \epsilon)$, which we have from the decoupling condition (Lemma 1), the second equality follows directly from Definition 2, just as in the proof of Theorem 4. The equality in the last line follows from Lemma 3. In the limit of $n \to \infty$ and $\epsilon \to 0$, we have $\delta(n, \epsilon) \to 0$, and so $I_{\delta(n, \epsilon)}$ converges to $I_0$. \hfill $\blacksquare$

**B. Rate region for generic sources**

In this subsection, we find the complete rate region for generic sources, generalizing the insight of Theorem 11 for the subproblem of quantum compression with classical side information at the decoder.

**Theorem 14.** For a generic classical-quantum source, in particular one where there is an $x$ such that $\psi^B_x$ has full support, the optimal asymptotic rate region for distributed compression is the set of rate pairs satisfying

\[ R_X \geq S(X | B), \]
\[ R_B \geq \frac{1}{2} (S(B) + S(B | X)) , \]
\[ R_X + 2R_B \geq S(B) + S(XB). \]

Moreover, there are protocols achieving these bounds requiring no prior entanglement.

**Proof.** We have argued the achievability already at the start of this section (Theorem 12). As for the converse, we have shown in Theorem 11 that for a generic source, $\tilde{I}_0 = 0$, hence the claim follows from the outer bounds of Theorem 13. \hfill $\blacksquare$

This means that for generic sources, which we recall are the complement of a set of measure zero, the rate region has the shape of Fig. 3.

**C. General achievability bounds**

For general, non-generic sources, the achievability bounds of Theorem 12 and the outer bounds of Theorem 13 do not match. Here we present several more general achievability results that go somewhat towards filling in the unknown area in between, without, however, resolving the question completely.

**Theorem 15.** For distributed compression of a classical-quantum source in the entanglement-assisted model, any rate pairs satisfying the following inequalities are achievable: with $\alpha = \frac{2I(X;B)}{I(X;B) + I_0}$,

\[ R_X \geq S(X | B), \]
\[ R_B \geq \frac{1}{2} (S(B) + S(B | X) - I_0) , \]
\[ R_X + \alpha R_B \geq S(X | B) + \alpha S(B). \]  \hspace{1cm} (15)
More generally, for any auxiliary random variable $Y$ such that $Y - X - B$ is a Markov chain, all the following rate pairs (and hence also their upper-right convex closure) are achievable:

$$
R_X = I(X : Y) + S(X|BY) = S(X|B) + I(Y : B),
$$
$$
R_B = \frac{1}{2} (S(B) + S(B|Y) - I(Y : W)) = S(B) - \frac{1}{2} (I(Y : B) + I(Y : W)),
$$

where $C$ and $W$ are the system and environment of an isometry $V : B \rightarrow CW$ with $I(W : R|Y) = 0$.

**Proof.** The region described by Eq. (15) is precisely the upper-right convex closure of the two corner points $(S(X|B), S(B))$ and $(S(X), \frac{1}{2}(S(B) + S(B|X) - I_0))$. Their achievability follows from Theorems 14 and 8.

We use the two achievable points $(S(X|B), S(B))$ and $(S(X), \frac{1}{2}(S(B) + S(B|X) - I_0))$ to show the second statement. Namely, Alice and Debbie (the receiver) use the Reverse Shannon Theorem to simulate the channel taking $X$ to $Y$ in i.i.d. fashion, which costs $I(X : Y)$ bits of classical communication [23]. Now we are in a situation that we know, Bob has to encode $B_n$ with side information $Y_n$ at the decoder, which can be done at rate $\frac{1}{2}(S(B) + S(B|Y) - I(Y : W))$, by the quantum state redistribution protocol of Theorem 8. Then Alice has to send some more information to allow the receiver to decode $X_n$ which is an instance of classical compression of $X$ with quantum side information $BY$ that is already at the decoder, hence costing another $S(X|BY)$ in communication, by the Devetak-Winter protocol [4, 5]. For $Y = X$, we recover the rate point $(S(X), \frac{1}{2}(S(B) + S(B|X) - I_0))$, and for $Y = \emptyset$ we recover $(S(X|B), S(B))$. 

In Fig. 4, we show the situation for a general source, depicting the most important inner and outer bounds on the rate region in the entanglement-assisted model.

![Diagram](image-url)
IV. DISCUSSION AND OPEN PROBLEMS

After seeing no progress for over 15 years in the problem of distributed compression of quantum sources, we have decided to take a fresh look at the classical-quantum sources considered in [4, 5]. There, the problem of compressing the classical source using the quantum part as side information at the decoder was solved; here we were analyzing the full rate region, in particular we were interested in the other extreme of compressing the quantum source using the classical part as side information at the decoder. Like in the classical Slepian-Wolf coding, the former problem exhibits no rate loss, in that the quantum part of the source is compressed to the Schumacher rate, the local entropy, and the sum rate equals the joint entropy of the source. Interestingly, this is not the case for the latter problem: clearly, if the classical side information were available both at the encoder and the decoder, the optimal compression rate would be the conditional entropy \(S(B|X)\), which would again imply no sum rate loss. However, since the classical side information is supposed to be present only at the decoder, we have shown that in general the rate sum is strictly larger, in fact generically by \(\frac{1}{2} I(X:B)\), and with this additional rate there is always a coding scheme achieving asymptotically high fidelity. We term this additional rate “the price of ignorance”, as it corresponds to the absence of the side information at the encoder.

For the general case, we introduced information quantities \(I_0\) and \(\tilde{I}_0\) (Definition 2), to upper and lower bound the optimal quantum compression rate as

\[
\frac{1}{2} \left( S(B) + S(B|X) - \tilde{I}_0 \right) \leq R^*_B \leq \frac{1}{2} \left( S(B) + S(B|X) - I_0 \right),
\]

when unlimited entanglement is available. For generic sources, \(I_0 = \tilde{I}_0 = 0\), but in general we do not understand these quantities very well, and the first complex of open problems is about them: is \(I_0 = \tilde{I}_0\) in general, or are there examples of gaps? How to calculate either one of these quantities, given that a priori the auxiliary register \(W\) is unbounded? In fact, can one without loss of generality put a finite bound on the dimension of \(W\), for either optimization problem?

The second open problem is about the need for prior shared entanglement to achieve the optimal quantum compression rate \(R^*_B\). As a matter of fact, it would already be interesting to know whether the rate \(\frac{1}{2} (S(B) + S(B|X) - I_0)\) requires in general pre-shared entanglement.

Finally, the full rate region inherits these features: While it is simple, and in fact generated by the optimal codes for the two compression-with-side-information problems (quantum compression with classical side information, and classical compression with quantum side information), in the generic case, in general the picture is very complicated, and we have only been able to give several outer and inner bounds on the rate region, whose determination remains an open problem.

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Appendix A: Miscellaneous definitions and facts

For an operator $X$, the trace norm, the Hilbert-Schmidt norm and the operator norm are defined respectively in terms of $|X| = \sqrt{X^*X}$:

\[
\|X\|_1 = \text{Tr} |X|, \\
\|X\|_2 = \sqrt{\text{Tr}|X|^2}, \\
\|X\|_\infty = \lambda_{\text{max}}(|X|),
\]

where $\lambda_{\text{max}}(X)$ is the largest eigenvalue of $X$.

**Lemma 16** (Cf. [24]). For any operator $X$,

\[
\|X\|_1 \leq \sqrt{d} \|X\|_2 \leq d \|X\|_\infty,
\]

where $d$ equals the rank of $X$.

**Lemma 17** (Cf. [24]). For any self-adjoint operator $X$,

\[
\|X\|_1 = \max_{-I \leq Q \leq I} \text{Tr} (QX).
\]

**Lemma 18** (Cf. [24]). For any self-adjoint operator $X$ and any operator $T$,

\[
\|TXT^\dagger\|_1 \leq \|T\|_\infty \|X\|_1.
\]

The fidelity of two states is defined as

\[
F(\rho, \sigma) = \text{Tr} \sqrt{\sigma^{\frac{1}{2}} \rho \sigma^{\frac{1}{2}}}.
\]

When one of the arguments is pure, then

\[
F(\rho, |\psi\rangle\langle\psi|) = \sqrt{\text{Tr} (\rho |\psi\rangle\langle\psi|)} = |\langle\psi|\rho|\psi\rangle|.
\]

**Lemma 19.** The fidelity is related to the trace norm as follows [11]:

\[
1 - F(\rho, \sigma) \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)^2} = P(\rho, \sigma),
\]

where $P(\rho, \sigma)$ is the so-called purified distance, or Battacharya distance, between quantum states.

**Lemma 20** (Pinsker’s inequality, cf. [20]). The trace norm and relative entropy are related by

\[
\|\rho - \sigma\|_1 \leq 2\sqrt{2S(\rho||\sigma)}.
\]

**Lemma 21** (Uhlmann [18]). Let $\rho^A$ and $\sigma^A$ be two quantum states with fidelity $F(\rho^A, \sigma^A)$. Let $\rho^{AB}$ and $\sigma^{AC}$ be purifications of these two states, then there exists an isometry $V : B \rightarrow C$ such that

\[
F \left( (\mathbb{1}_A \otimes V^{B\rightarrow C})\rho^{AB}(\mathbb{1}_A \otimes V^{B\rightarrow C})^\dagger, \sigma^{AC} \right) = F(\rho^A, \sigma^A).
\]

A consequence of this, due to [25, Lemma 2.2], is as follows.

**Lemma 22.** Let $\rho^A$ and $\sigma^A$ be two quantum states with trace distance $\frac{1}{2} \|\rho^A - \sigma^A\|_1 \leq \epsilon$, and let $\rho^{AB}$ and $\sigma^{AC}$ be purifications of these two states. Then there exists an isometry $V : B \rightarrow C$ such that

\[
\| (\mathbb{1}_A \otimes V^{B\rightarrow C})\rho^{AB}(\mathbb{1}_A \otimes V^{B\rightarrow C})^\dagger - \sigma^{AC} \|_1 \leq \sqrt{\epsilon(2-\epsilon)}.
\]

**Lemma 23** (Fannes [26]; Audenaert [27]). Let $\rho$ and $\sigma$ be two states on $d$-dimensional space with trace distance $\frac{1}{2} \|\rho - \sigma\|_1 \leq \epsilon$, then

\[
|S(\rho) - S(\sigma)| \leq \epsilon \log d + h(\epsilon),
\]

where $h(\epsilon) = -\epsilon \log \epsilon - (1-\epsilon) \log (1-\epsilon)$ is the binary entropy.
There is also an extension of the Fannes inequality for the conditional entropy; this lemma is very useful especially when the dimension of the system conditioned on is unbounded.

Lemma 24 (Alicki-Fannes [28]; Winter [29]). Let \( \rho \) and \( \sigma \) be two states on a bipartite Hilbert space \( A \otimes B \) with trace distance \( \frac{1}{2} \| \rho - \sigma \|_1 \leq \epsilon \), then
\[
|S(A|B)_\rho - S(A|B)_\sigma| \leq 2 \epsilon \log |A| + 2h(\epsilon).
\]

Lemma 25. Let \( \rho \) be a state with full support on the Hilbert space \( A \), i.e. it has positive minimum eigenvalue \( \lambda_{\text{min}} \), and let \( |\psi\rangle^{AR} \) be a purification of \( \rho \) on the Hilbert space \( A \otimes R \). Then any purification of another state \( \sigma \) on \( A \) is of the form
\[
(\mathbb{1}_A \otimes T) |\psi\rangle^{AR},
\]
where \( T \) is an operator acting on system \( R \) with \( \|T\|_{\infty} \leq \frac{1}{\sqrt{\lambda_{\text{min}}}} \).

Proof. Let \( \rho = \sum_i \lambda_i |e_i\rangle\langle e_i| \) and \( \sigma = \sum_j \mu_j |f_j\rangle\langle f_j| \) be spectral decompositions of the states. The purification of \( \rho \) is \( |\psi\rangle^{AR} = \sum_i \sqrt{\lambda_i} |e_i\rangle |i\rangle \). Define \( |\phi\rangle^{AR} = \sum_j \sqrt{\mu_j} |f_j\rangle |j\rangle \). Any purification of the state \( \sigma \) is of the form \( \mathbb{1}_A \otimes V |\phi\rangle^{AR} \) where \( V \) is an isometry acting on system \( R \). Write the eigenbasis \( \{ |e_i\rangle \} \) as linear combination of eigenbasis \( \{ |f_j\rangle \} \). Then, we have \( |\phi\rangle^{AR} = \sum_{i,j} \sqrt{\mu_j \alpha_{ij}} |e_i\rangle |j\rangle \). Define the operator \( P = \sum_{j,k} p_{jk} |j\rangle\langle k| \) where \( p_{jk} = \alpha_{kj} \sqrt{\frac{\mu_j}{\lambda_k}} \). It is immediate to see that
\[
|\phi\rangle^{AR} = (\mathbb{1}_A \otimes P) |\psi\rangle^{AR}.
\]

Thus, we have \( (\mathbb{1}_A \otimes V) |\psi\rangle^{AR} = (\mathbb{1}_A \otimes VP) |\psi\rangle^{AR} \). Defining \( T = VP \), we then have
\[
\lambda_{\text{max}}(T^\dagger T) = \lambda_{\text{max}}(P^\dagger P) \leq \text{Tr}(P^\dagger P) = \sum_{j,k} |p_{jk}|^2 = \sum_{j,k} \frac{|\alpha_{kj}|^2 \mu_j}{\lambda_k} \leq \frac{1}{\lambda_{\text{min}}},
\]
where the last inequality follows from the orthonormality of the basis \( \{ |f_j\rangle \} \).

Appendix B: Proof of Lemma 1 (decoupling condition)

In this subsection, we show that the fidelity criterion (3) implies that given \( x^n \), the environments \( W_X, W_B \) and \( W_D \) of Alice’s, Bob’s and Debbie’s isometries are decoupled from the the rest of the output systems. For convenience, we restate the lemma we are aiming to prove.

Lemma 1. (Decoupling condition) For a code of block length \( n \) and error \( \epsilon \) in the entanglement-assisted model, let \( W_X, W_B \) and \( W_D \) be the environments of Alice’s and Bob’s encoding and of Debbie’s decoding isometries, respectively. Then,
\[
I(W_X W_B W_D \mathcal{B}_0 \mathcal{D}_0 : \hat{X}^n \hat{B}^n R^n | X'^n) \leq n \delta(n, \epsilon),
\]
where \( \delta(n, \epsilon) = 4\sqrt{6} \log(|X||B|) + \frac{1}{2} \epsilon h(\sqrt{6\epsilon}) \), with the binary entropy \( h(\epsilon) = -\epsilon \log \epsilon - (1-\epsilon) \log (1-\epsilon) \); the mutual information is with respect to the state
\[
\xi^{X^n X^n \hat{B}^n \hat{B}^n \mathcal{B}_0 \mathcal{D}_0 W_X W_B W_D R^n} = (\mathcal{D} \circ (\mathcal{E}_X \otimes \mathcal{E}_B \otimes \text{id}_{\mathcal{D}_0}) \otimes \text{id}_{X'^n R^n}) \omega^{X'^n X'^n \hat{B}^n \hat{B}^n \mathcal{B}_0 \mathcal{D}_0}.
\]

Proof. The parties share \( n \) copies of the state \( \omega^{X^n X^n B^n} \), where Alice and Bob have access to systems \( X^n \) and \( B^n \), respectively, and \( X'^n \) and \( R^n \) are the reference systems. Alice and Bob apply the following isometries to encode their systems, respectively:
\[
U_X : X^n \rightarrow C_X W_X, \quad U_B : B^n \mathcal{B}_0 \rightarrow C_B \mathcal{B}_0 W_B,
\]
where \( C_X, C_B \) are auxiliary systems with classical states and \( \xi^{X^n B^n \mathcal{B}_0 \mathcal{D}_0} = (\mathcal{D} \circ (\mathcal{E}_X \otimes \mathcal{E}_B \otimes \text{id}_{\mathcal{D}_0}) \otimes \text{id}_{X'^n R^n}) \omega^{X'^n X'^n \hat{B}^n \hat{B}^n \mathcal{B}_0 \mathcal{D}_0} \).
where Alice and Bob send respectively their compressed information $C_X$ and $C_B$ to Debbie and keep the environment parts $W_X$ and $W_B$ of their respective isometries for themselves. Debbie applies the decoding isometry $V : C_X C_B D_0 \rightarrow X^n B^n D'_0 W_D$ to the systems $C_X C_B$ and her part of the entanglement $D_0$, to generate the output systems $X^n B^n D'_0$, with $W_D$ the environment of her isometry. This leads to the following final state after decoding:

$$
\xi^{X^n X^n B^n B'_n D'_0 W_X W_B W_D R^n} = \sum_{x^n} p(x^n) |x^n \rangle \langle x^n|^{X^n} \otimes |\xi^{x^n} \rangle \langle \xi^{x^n}|^{X^n B^n B'_n D'_0 W_X W_B W_D R^n},
$$

where

$$
|\xi^{x^n} \rangle ^{X^n B^n B'_n D'_0 W_X W_B W_D R^n} = V^{C_X C_B D_0 \rightarrow X^n B^n D'_0 W_D} \left( U^{X^n \rightarrow C_X W_X} |x^n \rangle ^{X^n} \otimes U^{B^n B_0 \rightarrow C_B B'_0 W_B} (|\psi^{x^n} \rangle ^{B^n R^n} |\Phi^{K} \rangle ^{B_0 D_0}) \right).
$$

The fidelity defined in Eq. (3) is now bounded as follows:

$$
F = F \left( \omega^{X^n X^n B^n B'_n \otimes \Phi^{B'_n D'_0 \otimes \Phi^{B_0 D_0}}} / (D \otimes (id^{X^n D_0} \otimes \mathcal{E}_B) \otimes id^{X^n R^n}) \right) \omega^{X^n X^n B^n B'_n \otimes \Phi^{B_0 D_0}} \\
= F \left( \omega^{X^n X^n B^n B'_n \otimes \Phi^{B'_n D'_0 \otimes \xi^{X^n B^n B'_n D'_0 R^n}}} \right) \\
\leq \sum_{x^n \in X^n} p(x^n) F \left( |x^n \rangle \langle x^n|^{X^n} \otimes |\psi^{x^n} \rangle \langle \psi^{x^n}|^{B^n R^n} \right) \psi^{x^n} \\
\leq \sum_{x^n} p(x^n) \sqrt{\langle \xi^{x^n} \hat{B}^{R^n} \rangle} \parallel \xi^{x^n} \hat{B}^{R^n} \parallel, \tag{B1}
$$

where the inequality in the third line is due to the monotonicity of fidelity under partial trace, and $\parallel \xi^{x^n} \hat{B}^{R^n} \parallel$ denotes the operator norm, which in this case of a positive semidefinite operator is the maximum eigenvalue of $\xi^{x^n} \hat{B}^{R^n}$. Now, consider the Schmidt decomposition of the state $|\xi^{x^n} \rangle ^{X^n B^n B'_n D'_0 W_X W_B W_D}$ with respect to the partition $X^n B^n : B'_n D'_0 W_X W_B W_D$, i.e.

$$
|\xi^{x^n} \rangle ^{X^n B^n B'_n D'_0 W_X W_B W_D R^n} = \sum_{i} \sqrt{\lambda_{x^n}(i)} |u_{x^n}(i) \rangle ^{X^n B^n} \parallel |u_{x^n}(i) \rangle ^{B'_n D'_0 W_X W_B W_D}.
$$

High average fidelity $F \geq 1 - \epsilon$ implies that on average the above state has Schmidt rank approximately one. In other words, the two subsystems are nearly independent:

$$
\sum_{x^n} p(x^n) F \left( |\xi^{x^n} \rangle \langle \xi^{x^n}|^{X^n B^n B'_n D'_0 W_X W_B W_D R^n} \parallel \xi^{x^n} \rangle \langle \xi^{x^n}|^{X^n B^n B'_n D'_0 W_X W_B W_D R^n} \right) \\
= \sum_{x^n} p(x^n) \sqrt{\langle \xi^{x^n} \parallel \xi^{x^n} \rangle} \parallel \xi^{x^n} \parallel^2 \\
\geq \sum_{x^n} p(x^n) \left( \parallel \xi^{x^n} \parallel^2 \right)^{1/2} \\
\geq \left( \sum_{x^n} p(x^n) \right)^{1/3} \geq 1 - 3\epsilon, \tag{B2}
$$
where the inequality in the fifth line follows from the convexity of \( x^3 \) for \( x \geq 0 \), and in the sixth line we have used Eq. (B1). Based on the relation between fidelity and trace distance (Lemma 19), we thus obtain for the product ensemble

\[
\zeta \hat{X}^n \hat{Y}^n \hat{B}^n \hat{D}^n_0 \hat{W}_X \hat{W}_B \hat{W}_D R^n := \sum_{x^n} p(x^n) |x^n\rangle\langle x^n| ^n \hat{X}^n \hat{Y}^n \hat{B}^n \hat{D}^n_0 \hat{W}_X \hat{W}_B \hat{W}_D,
\]

that

\[
\|\xi - \zeta\|_1 = \sum_{x^n} p(x^n) \|\xi \hat{X}^n \hat{Y}^n \hat{B}^n \hat{D}^n_0 \hat{W}_X \hat{W}_B \hat{W}_D - \xi \hat{X}^n \hat{Y}^n \hat{B}^n \hat{D}^n_0 \hat{W}_X \hat{W}_B \hat{W}_D\|_1 \\
\leq 2\sqrt{6} \epsilon.
\]

By the Alicki-Fannes inequality (Lemma 24), this implies

\[
I(\hat{X}^n \hat{Y}^n \hat{B}^n \hat{D}^n_0 \hat{W}_X \hat{W}_B \hat{W}_D | X^n)_{\xi} = S(\hat{X}^n \hat{Y}^n \hat{B}^n \hat{D}^n_0 \hat{W}_X \hat{W}_B \hat{W}_D | X^n)_{\xi} - S(\hat{X}^n \hat{Y}^n \hat{B}^n \hat{D}^n_0 \hat{W}_X \hat{W}_B \hat{W}_D | X^n)_{\xi} \\
\leq 2\sqrt{6} \log(|X^n| |B^n| |R^n|) + 2h(\sqrt{6} \epsilon) \\
\leq 2\sqrt{6} \log(|X|^n |B|^n) + 2h(\sqrt{6} \epsilon) =: n \delta(n, \epsilon),
\]

where we note in the second line that \( S(\hat{X}^n \hat{Y}^n \hat{B}^n \hat{D}^n_0 \hat{W}_X \hat{W}_B \hat{W}_D | X^n)_{\xi} \geq S(\hat{X}^n \hat{B}^n \hat{R}^n | X^n)_{\xi} = S(\hat{X}^n \hat{B}^n \hat{R}^n | X^n)_{\xi} \), and in the third line that we can without loss of generality assume \(|R| \leq |X| |B|\), since that is the maximum possible dimension of the support of \( \omega^R \).

\[\blacksquare\]
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