Eigenvalue Dynamics and the Matrix Chain

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Abstract

We introduce a general method for transforming the equations of motion following from a Das-Jevicki-Sakita Hamiltonian, with boundary conditions, into a boundary value problem in one-dimensional quantum mechanics. For the particular case of a one-dimensional chain of interacting $N \times N$ Hermitean matrices, the corresponding large $N$ boundary value problem is mapped into a linear Fredholm equation with Hilbert-Schmidt type kernel. The equivalence of this kernel, in special cases, to a second order differential operator allows us recover all previously known explicit solutions for the matrix eigenvalues. In the general case, the distribution of eigenvalues is formally derived through a series of saddle-point approximations. The critical behaviour of the system, including a previously observed Kosterlitz-Thouless transition, is interpreted in terms of the stationary points. In particular we show that a previously conjectured infinite series of sub-leading critical points are due to expansion about unstable stationary points and consequently not realized.


1 Introduction

The utility of studying the statistical mechanics of systems which can be encoded in terms of matrix variables has been evident over the past twenty years. From the generalization of Wigner and Dyson’s work by Brézin, Itzykson, Parisi and Zuber [1] to the use of matrix models in low dimensional gravity (see [2] for reviews), to modern implementations of matrix strings [3], matrix models have proven to be a convenient framework for calculation. Remarkably, throughout this period the dynamics of matrix models have been considered only in the ‘stationary’ case where solutions of the equations of motion are assumed to be time-independent. There arise circumstances though where the time evolution of a system governed by an action over matrix variables is relevant, especially if there are definite initial and final conditions on the state of the system. Examples include two-dimensional Yang-Mills theory on the cylinder and sphere [4, 5], matrix model approaches to the continuous Hirota equation [6] and lattice QCD [7] and, the case we will focus on here, linear chains of interacting Hermitean matrices. It is our goal to give a coherent framework in which in to study this class of matrix model problems.

When an action over matrix variables can effectively be reduced to the eigenvalues, the treatment of the eigenvalues as a single, collective coordinate [8] provides a convenient viewpoint for determining the Hamiltonian of the system. For the matrix actions we will consider here the particular form is known as the Das-Jevicki-Sakita Hamiltonian [9]. The associated equations of motion are the non-linear Euler equations of fluid dynamics in one dimension and with the addition of appropriate boundary conditions present a difficult mathematical problem. Here we will show how these boundary value problems can be interpreted in the familiar framework of the semi-classical limit of one-dimensional quantum mechanics. While in general solutions are still difficult to obtain, we will focus our attention on the case of a one-dimensional chain of interacting $N \times N$ Hermitean matrices which was shown to be a Das-Jevicki-Sakita-type system by Matytsin [10] in the limit of large $N$. In this situation a restatement of the equations of motion in terms of quantum mechanics allows one to consider a linear problem which is most conveniently stated as a Fredholm integral equation. Formal solution of this equation in terms of iterations of the integral kernel, which can be evaluated by saddle-point methods, will provide a framework for detailing the nature and position of critical points which arise in the system.

An interesting system from a statistical mechanical point of view, the one-dimensional matrix chain was discussed previously as a model for string theory with a discrete target space [12]. In addition to this direct application, the matrix chain is related by duality
transformation to string theory on a circle with radius $R = 1/a$ \[^{13,14}\] and to $O(2)$ sigma models coupled to two dimensional quantum gravity \[^{16,17}\]. The model itself consists of Hermitean matrices $\phi_x$ on the sites $\{x\}$ of a one-dimensional lattice interacting with nearest neighbours. The action of the system is

$$S_V = a \sum_x \text{Tr}V(\phi_x) - \sum_x \frac{\text{Tr}(\phi_x - \phi_{x+1})^2}{2a}$$  \hspace{1cm} (1.1)$$

where $a$ is the lattice spacing and $V$ is the potential at each site. Assuming the site potentials $V(\phi_x)$ to be analytic in $\phi_x$, the action can be reduced to the eigenvalues of the matrix variables by $SU(N)$ gauge rotations $\Lambda$ and the known result for the correlator \[^{11}\]

$$I[\phi_x, \phi_{x+1}] = \int d\Lambda e^{-N\text{Tr}\Lambda\phi_x\Lambda^\dagger\phi_{x+1}}$$  \hspace{1cm} (1.2)$$

Consequently, in the limit of large $N$, the calculation of the partition function reduces to determining solutions of equations of motion for the eigenvalues. Despite its relative simplicity and importance in low-dimensional string theory and quantum gravity, explicit solutions of the matrix chain are only known for a few particular choices of potential $V$. In Section Four we will show these cases arise when the equations of motion are equivalent to certain Sturm-Liouville eigenvalue problems.

This model exhibits a remarkable range of behaviour as one adjusts the lattice spacing, $a$. For instance, in the limit of vanishing $a$ one recovers the canonical example of a Das-Jevicki-Sakita system, matrix quantum mechanics (restricted to a singlet sector)

$$S_{MQM} = \int dx \text{Tr}\left[\frac{1}{2}\dot{\phi}^2(x) + V(\phi(x))\right]$$  \hspace{1cm} (1.3)$$

In fact, the methods we will introduce in the next Section and use to study the matrix chain find their most natural form when applied to such matrix mechanical systems on a finite interval $x \in [0,T]$.

More interestingly, fixing the potential $V$ to satisfy $V''(0) = -4$, at critical lattice spacing $a = 1$, the matrix chain was argued to undergo a Kosterlitz-Thouless phase transition \[^{13}\]. The physics of this phase transition are most transparent in the dual picture of string theory on the circle. Here strings wrapping around the non-contractible target space play the role of vortices which are liberated when the radius $R \sim 1/a$ becomes sufficiently small. In terms of a discrete lattice, at sufficiently small lattice spacing $a < 1$ the universal behaviour of the system is that of the continuous model (1.3) due to communication between lattice sites. When the spacing is larger than unity this communication is broken and the system can be expected to factorize into a product of single $D = 0$ matrix models, which in fact it does as $a \to \infty$. The main objective of studying the matrix chain is to understand precisely how this
change of character from $D = 1$ matrix quantum mechanics to isolated $D = 0$ matrix models takes place at $a = 1$.

In addition to $a = 0$ and $a = 1$, it was argued in [16] and [17] that at a discrete infinity of lattice spacings non-analytic behaviour arises in the distribution of eigenvalues of the matrix variables $\phi_x$. Considering iterative approximation of a functional solution to the equations of motion, it was argued these critical points arise when $a = \sin(p\pi/2q)$ where $p \leq q$ are each positive integers. In Section Five, where we develop the general solution for the distribution of eigenvalues as a series of saddle-point integrations, we will show that this behaviour is not realized in the matrix chain. In fact it will be shown that such critical behaviour arises when unstable stationary points are mistakenly considered relevant in saddle-point approximations. Moreover, in constructing the general solution it will become clear that a countable infinity of unstable stationary points appear in intermediate steps each of which can lead to misleading results.

In addition to the details just outlined, we recall the Kazakov-Migdal (lattice) model of induced QCD as an interesting application of the techniques developed here. Since the $D$-dimensional model is effectively reducible to the one-dimensional case, the non-trivial behaviour of the matrix chain demonstrated here may help to overcome some of the apparent deficiencies of that model of gauge interaction.

2 Collective field theory to quantum mechanics

To begin we will demonstrate a non-linear change of variable which allows the equations of motion derived from a Das-Jevicki-Sakita Hamiltonian [9]

\[
\mathcal{H}[\rho, \Pi] = \frac{1}{2} \int dx \rho(x) \left\{ \left( \frac{\partial \Pi(x)}{\partial x} \right)^2 - \frac{\pi^2}{3} \rho^2(x) \right\} + \int dx \rho(x) V(x, t)
\]  

(2.1)

to be determined as a temporal boundary problem in one-dimensional quantum mechanics. In (2.1), $\rho(\lambda)$ is a probability distribution function which returns the fraction of eigenvalues of the matrix variables $\phi_x$ with value $\lambda$. In the present context of collective field theory, $\rho$ plays the role of a coordinate and $\Pi$ is the canonically conjugate momentum. Here we have included a potential $V$ which is most generally time-dependent. Independent of time, (2.1) describes the evolution of the matrix quantum mechanics (1.3) with $V = V$.

The equations of motion following from this Hamiltonian are given by the functional
variations with respect to the canonical variables

\[ \frac{\partial}{\partial t} \rho = \frac{\delta H}{\delta \Pi} , \quad \frac{\partial}{\partial t} v = -\frac{\partial}{\partial x} \left( \frac{\delta H}{\delta \rho} \right) \quad (2.2) \]

where, for convenience, we have defined the ‘velocity’ field

\[ v(x, t) = \frac{\partial \Pi}{\partial x} \quad (2.3) \]

Explicitly we see that the equations of motion are of the form of the Euler equations for a one-dimensional inviscid fluid experiencing a time-dependent force derived from \( V \)

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0 \quad (2.4) \]

\[ \frac{\partial v}{\partial t} + \frac{1}{2} \left( \frac{\partial v^2}{\partial x} - \pi^2 \frac{\partial \rho^2}{\partial x} \right) = -\frac{\partial}{\partial x} V(x, t) \]

As in all evolution problems we need to specify boundary conditions to supplement the equations of motion. In the case of matrix quantum mechanics the classical treatment of \[1\] assumed free boundary conditions for the evolution of the system over infinite time. In this case one is led to a study of the ground state of the system and the equations of motion \( (2.4) \) are solved for the stationary case of \( v = 0 \). In general though we are interested in arbitrary boundary conditions for the evolution of the system through some time \( T \). To be definite we will specify the generic conditions on the fields \( v \) and \( \rho \) at time \( t = 0 \) and \( t = T \)

\[ \rho(x, 0) = \rho_0(x) , \quad v(x, 0) = v_0(x) \quad (2.5) \]

\[ \rho(x, T) = \rho_T(x) , \quad v(x, T) = v_T(x) \]

We would now like to rewrite the equations of motion in terms of a single quasi-linear first order differential equation. Introducing the complex function

\[ f(x, t) = v(x, t) + i\pi \rho(x, t) \quad (2.6) \]

it is easy to see that \( (2.4) \) leads to

\[ \frac{\partial f}{\partial t} + f \frac{\partial f}{\partial x} = -\frac{\partial}{\partial x} V(x, t) \quad (2.7) \]

For vanishing potential and real \( f \) this is the Hopf equation which is the prototypical one-dimensional model of an equation which admits wave solutions complete with shocks and other inherently non-linear behaviour \[18\]. We would like to solve this equation with the boundary conditions given in \( (2.5) \). While the solution of the (complex) Hopf equation is known explicitly \[10\] in terms of initial data, typically we are interested in cases where boundary data is
supplied. In these situations the explicit solution leads to an ill-posed, inverse-type problem for the boundary conditions. Consequently we would like to find a more natural presentation of the problem. To this end we will consider a generalized version of (2.7) and introduce a dispersive term \[18\] which will serve to smooth out the solutions and allow one to calculate the details of the solution \(f\) in terms of boundary conditions. Adding such a term leads to a complex version of the forced Burgers’ equation

\[
\frac{i\hbar}{\hbar} \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t} + f \frac{\partial f}{\partial x} + \frac{\partial}{\partial x} V(x,t) \tag{2.8}
\]

Obviously, in the limit as \(\hbar \to 0\) the forced Hopf equation (2.7) is recovered. The usefulness of this addition to the problem is that there exists a well-known (to mathematicians) change of variable which will reduce (2.8) to a second order, linear differential equation. This transformation is known as the Cole-Hopf transform and is given

\[
f(x,t) = -2i\hbar \frac{\partial}{\partial x} \log \psi(x,t) \tag{2.9}
\]

As is readily verified, this transformation leads (up to an irrelevant additive constant of integration) to the Schrödinger equation

\[
H\psi = \left[ -\hbar^2 \frac{\partial^2}{\partial x^2} + \frac{V(x,t)}{2} \right] \psi = i\hbar \frac{\partial}{\partial t} \psi \tag{2.10}
\]

Consequently we see that the time evolution of the collective field Hamiltonian (2.1) is given by the semi-classical (\(\hbar \to 0\)) limit of one-dimensional quantum mechanics with the same potential, up to a factor.

Restricting to time independent potentials \(V(x,t) = V(x)\), the boundary conditions (2.5) enter into the picture as the start and end points of evolution by \(H\) through time \(T\)

\[
\psi(x,T) = e^{-iHT/\hbar} \psi(x,0) \tag{2.11}
\]

where the wavefunction is defined in terms of the eigenvalue density and velocity

\[
\psi(x,t) = \exp \left[ -\frac{\pi}{2\hbar} \int^x d\eta \rho(\eta,t) + \frac{i}{2\hbar} \int^x d\eta v(\eta,t) \right] \tag{2.12}
\]

Given the Green’s function \(G(x,u|T)\) for the Schrödinger problem it may be convenient to cast (2.11) as an integral equation

\[
\int du \ G(x,u|T) \ \exp \left[ -\frac{\pi}{2\hbar} \int^u d\eta \rho_0(\eta) + \frac{i}{2\hbar} \int^u d\eta v_0(\eta) \right] = \exp \left[ -\frac{\pi}{2\hbar} \int^x d\eta \rho_T(\eta) + \frac{i}{2\hbar} \int^x d\eta v_T(\eta) \right] \tag{2.13}
\]

\(\overset{1}{\text{This restriction is not essential. In the general case the time evolution of the system is given by a Dyson series expansion.}}\)
In applications the implementation of boundary conditions and Green’s function varies depending on the situation. For example, in investigations of the continuous Hirota equation [6] one is interested in the case of time-dependent potentials \( V(x, t) \) leading to non-trivial Green’s functions. Here the initial and final velocities are specified and the initial and final eigenvalue distributions are identified \( (\rho_0 = \rho_T) \) so that (2.13) takes on the form of a non-linear integral equation for \( \rho_0 \).

Another, more familiar application is that of two-dimensional Yang-Mills theory [4, 5, 19]. In this case the coordinates \((x, u)\) in (2.13) are replaced by their periodic counter-parts since the matrix variables are unitary rather than Hermitean. The Green’s function is that of the heat equation on a circle and the boundary conditions on \( \rho \) and \( v \) specify the topology of the two-dimensional space-time. In general these cases again lead to non-linear integral equations for the unknown fields but as we will now see, there are situations where the general integral equation (2.13) leads to a tractable, linear equation.

### 3 An application: the matrix chain

An application of the preceding formalism, which we will be investigating for the remainder of this Paper is that of the one-dimensional matrix chain. Instead of (1.1), it will be convenient to consider a different form of the action

\[
S_U = \sum_x \text{Tr} U(\phi_x) - \sum_x \text{Tr} \phi_x \phi_{x+1}
\]

where we have rescaled the matrix field \( \phi_x \) and the new site potential \( U \) is related to \( V \) by

\[
U(\phi_x) = aV(\sqrt{a}\phi_x) + \phi_x^2
\]

The evolution of eigenvalues in this system, in the limit of large \( N \), was shown by Matytsin [10] to be given by the Das-Jevicki-Sakita Hamiltonian (2.1) with vanishing potential \( V \). Here the boundary conditions (2.5), which play a crucial role, are given by

\[
\rho_0(x) = \rho(x, 0) = \rho(x, 1)
\]

\[
v(x, 0) = -v(x, 1) = \frac{1}{2} U'(x) - x
\]

The objective is to solve for the initial and final eigenvalue density \( \rho_0 \) in terms of the potential \( U \).

With \( V = 0 \), and \( \epsilon = i\hbar \), the Schrödinger equation (2.10) reduces to the linear heat equation. Substituting the well known Green’s function in (2.13) and implementing the
boundary conditions (3.3), we find a linear integral equation of second Fredholm type for the initial eigenvalue density $\rho_0(x) = \rho(x, 0)$ in terms of the effective potential $U(x)$

$$\chi(x) = \frac{\lambda}{\sqrt{4\pi \epsilon}} \int du \, e^{-\frac{(x-u)^2}{4\epsilon} - \frac{x^2-U(x)+u^2-U(u)}{4\epsilon}} \chi(u)$$ (3.4)

where

$$\chi(x) = \exp \left[ -\frac{i\pi}{2\epsilon} \int^x d\eta \, \rho_0(\eta) \right]$$ (3.5)

and we have introduced the constant $\lambda$ which does not affect the asymptotic solution but is convenient for later discussion. In this form (3.4) has the natural interpretation of an eigen-problem with eigenvector $\chi$ and associated eigenvalue $\lambda$. It is interesting to note that the same integral equation arose in the context of matrix chains previously [12, 13]. In these instances the authors were seeking the lowest lying eigenvalues $\lambda$ in order to calculate the partition function. Here we are most concerned with the asymptotic ($\epsilon \to 0$) form of the eigenfunctions $\chi$ in order to recover $\rho_0$.

Hence we have re-stated the differential form of the boundary value problem (eqns. (2.4) and (2.5)) for the matrix chain as a well-posed, linear problem for the eigenvalue density $\rho_0$. Moreover, the integral form (3.4) allows one to write equations for the eigenvalue density immediately. Formally, in the limit of vanishing $\epsilon$, the integral in (3.4) can be calculated by saddle-point methods. This calculation is particularly straightforward when only a single stationary point, $u_s$ contributes to the integral (3.4). This stationary point is determined by the extrema condition

$$x = U'(u_s)/2 + i\pi \rho_0(u_s)$$ (3.6)

Likewise, at this stationary point $\rho_0$ can be formally evaluated using its relation to $\chi$ from equation (3.7)

$$u_s = U'(x)/2 - i\pi \rho_0(x)$$ (3.7)

Labeling $x$ by $G_+$ in equation (3.6) and $u_s$ by $G_-$ in equation (3.7), we see the saddle-point evaluation of the integral equation (3.4) leads to Matytsin’s solution [10] of the boundary value problem

$$x = G_+(G_-(x)) = G_-(G_+(x))$$ (3.8)

This form of the solution of the problem is well-suited to detailed local analysis as demonstrated in [16, 17]. However, for a global analysis which can accommodate the existence of multiple stationary points, we will take the (well-studied) integral form (3.4) over the functional equation (3.8). Before considering the particular characteristics of solutions though, we will utilize the interpretation of integral operators as self-adjoint operators to recover all previously known explicit solutions $\rho_0$ of the matrix chain.
4 Integral and differential operators

An analysis of an integral equation centers on the properties of the kernel $K$. In our case, the kernel of (3.4) is

$$K(x, u) = \frac{1}{\sqrt{4\pi\epsilon}} e^{\frac{u^2}{4\epsilon} - U(x) + U(u)}$$

This positive definite kernel is real, symmetric and hence Hermitean. In the following we will assume the kernel to have many convenient properties which follow from the boundedness of all $n$-fold iterations

$$\int dx_1 \cdots dx_{n-1} K(x_1, x_2) \cdots K(x_{n-1}, x_n) < \infty$$

Consequently we will only consider potentials $U(x)$ which increase at infinity faster than $x^2$. Under such conditions $K$ defines a compact operator and represents a self-adjoint operator on the space of square-integrable functions. From classical operator theory [20] there is a countable infinity of orthogonal eigenfunctions $\{\chi_n(x)\}$ which form a complete set on this space. It is our goal to obtain, as a function of the site potential $U$, the asymptotic form of these eigenfunctions which determine the large $N$ solution of the eigenvalue distribution $\rho_0$ through (3.5).

There are a number of cases in which the kernel $K$ is equivalent to familiar self-adjoint operators in physics. In fact only in these instances are explicit solutions of the large $N$ matrix chain known. In each case the eigenvalue distribution $\rho_0$ is the solution of a quadratic equation. Here we will show that such situations arise by considering the $\epsilon \to 0$ limit of second order self-adjoint differential operators $L$ for which $L$ and $K$ share a complete set of eigenfunctions and define the same action on the Hilbert space.

In order to determine the conditions that $L$ must satisfy to be equivalent to the integral kernel we act on $\chi_n$ as defined by the integral equation (3.4)

$$E_n\chi_n(x) = L_x\chi_n(x) = \int du \ L_xK(x, u)\chi_n(u)$$

Likewise, acting under the integral sign and using the self-adjoint property of $L$

$$E_n\chi_n(x) = \int du \ K(x, u)L_u\chi_n(u) = \int du \ \chi_n(u)L_uK(x, u)$$

Subtracting (4.4) from (4.3) we find that $L$ will be equivalent to the integral kernel $K$, if and only if we satisfy the natural commutativity condition

$$(L_x - L_u)K(x, u) = 0$$
It follows from the symmetry of the kernel that the action of $L$ generates symmetric functions of $x$ and $u$ which will cancel out in (4.5),

$$L_x K(x, u) = h(x + u, xu)K(x, u) = \sum_{mn} a_{mn}(x + u)^m(xu)^n K(x, u)$$  \hspace{1cm} (4.6)

where $h$ is an arbitrary function of two variables with Taylor expansion coefficients $\{a_{mn}\}$.

This construction is valid for any differential operator but for simplicity we will consider only second order differential operators of Sturm-Liouville form

$$L_x = -4\epsilon^2 \frac{d}{dx} \left( a_2(x) \frac{d}{dx} \right) + a_0(x)$$  \hspace{1cm} (4.7)

Now it is a simple matter to find the potentials $U$ and associated coefficients $a_2$ and $a_0$ which are consistent with the symmetry condition (4.6). Since a second order $L$ can only produce quadratic polynomials in $u$, (4.6) leads to the equation

$$-a_2(x)(u - U'(x)/2)^2 - 2\epsilon (a_2(x)(u - U'(x)/2))' + a_0(x) = \sum_{m+n\leq 2} a_{mn}(x + u)^m(xu)^n$$  \hspace{1cm} (4.8)

Expanding each side of the second equality in powers of $u$ and equating coefficients gives, to leading order in $\epsilon$, the result

$$U'(x) = -\frac{a_{11} x^2 + (2a_{20} + a_{01})x + a_{10}}{a_{02} x^2 + a_{11} x + a_{20}}$$  \hspace{1cm} (4.9)

$$a_2(x) = -(a_{02} x^2 + a_{11} x + a_{20})$$

$$a_0(x) = a_{00} + a_{10} x + a_{20} x^2 - \frac{(a_{11} x^2 + (a_{01} + 2a_{20})x + a_{10})^2}{4(a_{02} x^2 + a_{11} x + a_{20})}$$

Having determined the coefficients $a_2$ and $a_0$, we can solve for the explicit form of the associated eigenvalue distribution. The differential form of the integral equation (3.4) is

$$L_x \chi(x) = E \chi(x)$$  \hspace{1cm} (4.10)

where the eigenvalue $E$ is related to that of the integral form by $\lambda = e^{E/\epsilon}$. In the limit of vanishing $\epsilon$, the solution of this differential equation is given simply by the WKB approximation which implies the replacement of derivatives with ‘momenta’

$$i\epsilon \frac{d\chi}{dx} \rightarrow \frac{\pi}{2} a_0$$  \hspace{1cm} (4.11)

With this substitution, (4.10) reduces to a quadratic equation in the eigenvalue distribution which is solved by

$$\rho_0(x) = \frac{1}{\pi} \sqrt{\frac{E - a_0(x)}{a_2(x)}}$$  \hspace{1cm} (4.12)

$$= \sqrt{\frac{4(a_{02} x^2 + a_{11} x + a_{20})(a_{00} + a_{10} x + a_{20} x^2 - E) - (a_{11} x^2 + (a_{01} + 2a_{20})x + a_{10})^2}{2\pi(a_{02} x^2 + a_{11} x + a_{20})}}$$
Since \( \rho_0 \) is to be interpreted as a probability distribution describing the eigenvalues of the matrix variables, it must be properly normalized. This is carried out by integrating \( \rho_0 \) over the positive support of \( (a_0 - E)/a_2 \) and fixing the undetermined constant \( E \) to satisfy

\[
\int dx \rho_0(x) = 1 \quad (4.13)
\]

Integrating (4.9), we find the most general potential of the matrix chain which is consistent with a second order, self-adjoint differential operator

\[
U(x) = -\frac{a_{11}}{a_{02}} x - \frac{2a_{02}^2 a_{10} - a_{01} a_{02} a_{11} + a_{11}^2 - 2a_{02} a_{11} a_{20}}{a_{02}^2 \sqrt{4 a_{02} a_{20} - a_{11}^2}} \arctan \left( \frac{a_{11} + 2 a_{02} x}{\sqrt{4 a_{02} a_{20} - a_{11}^2}} \right) \\
- \frac{a_{01} a_{02} - a_{11}^2}{2a_{02}^2} \log \left[ a_{20} + a_{11} x + a_{02} x^2 \right] \quad (4.14)
\]

Contained in this general form are a number of previously examined examples of the matrix chain. First, setting \( a_{11} = 0 \) and \( a_{10} = 0 \) recovers the symmetric double Penner-type potential first considered by Matytsin \[10\]. Also the asymmetric double Penner considered in connection with the Kazakov-Migdal model of induced QCD \[21\] can be recovered (see \[22\] for recent work). In each of these cases the eigenfunctions \( \chi \) are related to prolate spheroidal wave functions. Additionally, with \( a_{20} = 1 \), the limit \( a_{02} \to 0 \) recovers the quadratic potential case solved by Gross \[23\] and Makeenko \[24\]. With \( U' = (a_{01} + 2) x = 2m^2 x \), \( \mathcal{L} \) is the Schrödinger operator for the simple harmonic oscillator and the general expression for the distribution of eigenvalues (4.12), properly normalized, reduces to

\[
\rho_0(x) = \frac{1}{\pi} \sqrt{2 \sqrt{m^4 - 1} - (m^4 - 1) x^2} \quad (4.15)
\]

In each of these cases the analytic structure of the eigenvalue distribution (4.12) is what one would expect of a \( D = 0 \) one-matrix model with a particular potential. This correspondence was previously suggested for arbitrary potential in the matrix chain, but such a simple solution of the integral equation (3.4) is not possible. In fact with further computation it can be argued that higher, finite, order self-adjoint differential operators are equivalent to the integral kernel \( K \) only if they are functions of the second order operator \( \mathcal{L}_x \) we have constructed above. Consequently, the large \( N \) eigenvalue distribution \( \rho_0 \) of the matrix chain is the solution of a polynomial equation only for potentials of the form (4.14). This suggests that for a generic potential the solution of the matrix chain problem follows from the semi-classical approximation to a pseudo-differential operator as suggested by naively extending the results for finite chains of matrices \[23\]. We will not pursue this line of reasoning here but return in the next Section to the integral form (3.4) and give a method for obtaining the general solution and describe its unique features.
5 General solution and classification of critical points

A phase transition in a system is characterized by a change in the analytic structure of observables as functions of external parameters. In standard matrix models this is commonly taken to include a change in analytic structure of the distribution of eigenvalues \(^2\). In the present case we have shown that the distribution of eigenvalues for a linear chain of Hermitian matrices is determined completely by a linear integral equation \((3.4)\) involving the kernel \(K\) \((4.1)\). In this Section we will go further and solve for the asymptotic eigenfunctions \(\chi\) as a series of iterations of \(K\). Given that iterations of \(K\) are calculable by saddle-point methods and the relation \((3.3)\), we can effectively solve for the distribution of eigenvalues \(\rho_0\). In particular we will be able to determine the critical structure of the matrix chain by analyzing changes in the analytic structure of the kernel \(K\) and its iterations.

We begin by slightly generalizing the problem of finding the eigenfunctions \(\chi\) in order to make use of well-known techniques \([20]\). Instead of the integral equation \((3.4)\) let us consider for a moment an inhomogeneous version with \(c\) some real constant

\[
\chi(x) = c + \lambda \int du \ K(x, u)\chi(u) \equiv c + \lambda K\chi
\]

(5.1)

Since there are no constant eigenfunctions of the kernel \(K\), the unique solution of \((5.1)\) is given by the Neumann series which builds the solution up through an iterative procedure

\[
\begin{align*}
\chi_0 &= c \\
\chi_1 &= c + \lambda K\chi_0 \\
&\quad \vdots \\
\chi_n &= c + \lambda K\chi_{n-1}
\end{align*}
\]

(5.2)

Taking the limit of this process generates the solution

\[
\chi(x) = \left(1 + \sum_{n=1}^{\infty} \lambda^n \int du \ K^n(x, u)\right) c \equiv R(x; \lambda)c
\]

(5.3)

where \(R\) is commonly referred to as the resolvent kernel and \(K^n\) is given by the convolution of \(n\) kernels

\[
K^n(x, u) = \int dz_1 \cdots dz_{n-1} K(x, z_1)K(z_1, z_2) \cdots K(z_{n-1}, u)
\]

(5.4)

\(^2\)This assumption of a physical phase transition following from a change in the analytic structure of the eigenvalue distribution does not always hold \([26]\). Nevertheless, in the absence of calculating physical observables, we will take a change in structure of the eigenvalue distribution to be a strong hint of a physical transition.
The convergence of the resolvent kernel is guaranteed by the boundedness of $K^n$ and the Hilbert-Schmidt theory of integral equations \[20\]. In order to recover the homogeneous solution we should set $c = 0$ which leads to a trivial solution for $\chi$. Fortunately for us only the $c$ independent logarithmic derivative of $\chi$ is required to recover the eigenvalue distribution $\rho_0$ of the matrix chain. From the definition of $\chi$, (3.5) and (5.3) we have the result

$$
\rho_0(x) = \lim_{\epsilon \to 0} \frac{2i \epsilon}{\pi} \frac{d}{dx} \log \chi(x) = \lim_{\epsilon \to 0} \frac{2i \epsilon}{\pi} \frac{d}{dx} \log R(x; \lambda) \quad (5.5)
$$

Hence we can extract the eigenvalue distribution from the resolvent kernel $R$, which is expressible solely in terms of iterations of the kernel $K$ as defined in (5.3). This is an important observation since the iterated kernels are calculable in the limit of vanishing $\epsilon$. Explicitly, the $n$th iterated kernel can be written for $n \geq 2$

$$
K^n(x, u) = \frac{1}{(4\pi\epsilon)^{n/2}} e^{-\frac{U(x)+U(u)}{4\epsilon}} \int dz_1 \cdots dz_{n-1} e^{-S/2\epsilon} \quad (5.6)
$$

where the ‘action’ $S$ is given by

$$
S = \sum_{i=1}^{n-1} U(z_i) - \sum_{i=1}^{n-2} z_i z_{i+1} - z_1 x - z_{n-1} u \quad (5.7)
$$

Evaluating the integral over $\tilde{z} = \{z_1, \ldots, z_{n-1}\}$ is a straightforward application of saddle-point methods in the limit of vanishing $\epsilon$. In this limit the dominant contributions to the integral come from the vicinity of the stationary points of the function $S$

$$
\frac{\partial S}{\partial z_i} \bigg|_{z^\alpha} = 0 \quad (5.8)
$$

In general there are many such stationary points, which we will label by $\alpha$. Expanding $S$ up to quadratic order about the stationary points, the resulting Gaussian integrations give the approximation to the iterated kernel

$$
K^{n+1}(x, u) = \frac{1}{\sqrt{4\pi\epsilon}} e^{-\frac{U(x)+U(u)}{4\epsilon}} \sum_{\alpha} \frac{(-1)^{\lambda_n(z^\alpha)}}{\sqrt{\det H_n(z^\alpha)}} e^{-S(z^\alpha)/2\epsilon} \quad (5.9)
$$

where $\lambda_n(z^\alpha)$ is the number of negative eigenvalues of the $n \times n$ Hessian $H_n$ of second derivatives of $S$ evaluated at the stationary point

$$
H_n(z^\alpha) = \frac{\partial^2 S}{\partial z_i \partial z_j} \bigg|_{z^\alpha} = \begin{pmatrix}
U''(z_1^\alpha) & -1 & 0 & 0 & \cdots \\
-1 & U''(z_2^\alpha) & -1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & U''(z_n^\alpha)
\end{pmatrix} \quad (5.10)
$$
While we have counted contributions from all stationary points of $S$ in (5.9), the only relevant ones are those with minimum ‘action’. The contributions of other, irrelevant, stationary points are exponentially suppressed in the limit of vanishing $\epsilon$.

Unfortunately it is difficult to calculate the iterated kernels and the resolvent in closed form to obtain explicit results for the eigenvalue distribution. It is possible though to extract enough qualitative information from (5.9) to outline a one-dimensional phase diagram of the matrix chain. To be definite we will consider a quartic potential

$$U(\phi) = \phi^4 + m^2 \phi^2$$

(5.11)

This simple form is convenient since it allows a clear view of the intrinsic behaviour of the matrix chain which depends on the strength of the quadratic term.

Starting with $m^2$ large and positive we expect that the eigenvalue distribution will be localized near the origin where the quadratic term of the potential is dominant. Consequently, the solution (4.13) for the pure quadratic potential is a good approximation to the true solution and is smooth deformation of it. Making contact with the saddle-point evaluation of the iterated kernels (5.9) it is easy to convince oneself that for large $m^2$ there is only one stationary point of the action (5.7) and it is located near the origin $\vec{z} = 0$. As $m^2$ is decreased, the solution (4.13) of the pure quadratic potential becomes unstable at $m^2 = 1$ and in fact this instability is present in the full solution. Information about the stability of stationary points is contained in the eigenvalues $\mu_r$ of the Hessian $H_s(z)$, which evaluated at the origin in $z$-space give

$$\mu_r = 2m^2 - 2 \cos \frac{r\pi}{s+1}, \quad r = 1, \ldots, s$$

(5.12)

Consequently we see that when evaluating arbitrarily high iterations of the kernel, $K^n$, the Hessian at the origin will begin to develop negative eigenvalues at $m^2 = 1$ and this stationary point becomes unstable. Since $S$ is bounded from below for our choice of potential, there are other stationary points where $S$ has a lower magnitude and it is these which will dominate the integral (5.9) for $K^n$.

This degeneration at $m^2 = 1$ signals a phase transition in the matrix chain and the physical meaning of it can be found by returning to the original form of the matrix chain action (1.1). The potential there, $V$ is related to the current $U$ by the one dimensional lattice spacing $a$ through

$$U(\phi) = aV(\sqrt{a}\phi) + \phi^2$$

(5.13)

Taking two derivatives with respect to $\phi$ and fixing (without loss of generality) $V''(0) = -4$ to agree with the conventions of [16], we find

$$m^2 = 1 - 2a^2$$

(5.14)
Hence, the instability at $m^2 = 1$ corresponds to the limit of vanishing lattice spacing and we are observing a phase transition to $D = 1$ matrix quantum mechanics (i.e. (1.3)) from the $m^2 > 1$ phase where the solutions are qualitatively what one would expect from a $D = 0$ single matrix model. In fact this observation can be made more concrete by returning to the integral equation (3.4) but in terms of the potential $V$ and lattice spacing $a$,

$$e^{-\frac{i\pi}{2} \int x d\eta \rho_0(\eta)} = \frac{\lambda}{\sqrt{4\pi \epsilon}} \int du e^{-\frac{(x-u)^2}{4\epsilon} - aV(x) + V(u)} e^{-\frac{i\pi}{2} \int u d\eta \rho_0(\eta)}$$

(5.15)

It is a straightforward calculation to show that for vanishing $\epsilon$, and to leading order in vanishing $a$, (5.13) is solved by

$$\rho_0(x) = \frac{1}{\pi} \sqrt{E - V(x)}$$

(5.16)

which is the well-known solution for the ground state of matrix quantum mechanics [1].

As $m^2$ is decreased further below this transition point, the stationary point of $S$ at the origin $\vec{z} = 0$ is seen to become progressively more unstable as the number of negative eigenvalues of the Hessian (5.12) grows. Here the stationary points relevant to the saddle-point integrations (5.9) are not in the neighbourhood of the origin and are stable under changes in $m^2$. The only instability that occurs with the relevant stationary points in this phase is due to a $\mathbb{Z}_2$ symmetry of the action $S$. Recalling its definition

$$S = \sum_{i=1}^{n-1} U(z_i) - \sum_{i=1}^{n-2} z_iz_{i+1} - z_1x - z_{n-1}u$$

(5.17)

we see that for vanishing $x$ and $u$ there is a symmetry under $z_i \rightarrow -z_i$. It follows that a relevant stationary point at $\vec{z}^*$ has a mirror image at $-\vec{z}^*$ and as $x$ and $u$ are tuned through the origin the relevant stationary point can change from one to the other. In this way non-analytic behaviour arises in the solution of the eigenvalue distribution $\rho_0$, presumably of the form

$$\rho_0 \sim |x|$$

(5.18)

which is consistent with the critical behaviour of $D = 1$ matrix quantum mechanics. Of course this detail is difficult to verify without explicit calculation.

In addition to this standard behaviour it was argued in [10] that there exist a countable infinity of additional points in this phase where the eigenvalue distribution $\rho_0$ exhibits sub-leading non-analyticities. In fact we can reproduce these results if we assume, incorrectly, that the origin $\vec{z} = 0$ is a relevant stationary point in the evaluation of $\rho_0$. Considering the vanishing of Hessian eigenvalues (5.12), we see that the naive saddle-point approximation to $K_s^{s+1}$ (5.9) will break down when

$$m^2 = \cos \frac{r\pi}{s+1}$$

(5.19)
It is well known that at degenerate stationary points there is a change in the analytic structure of a saddle-point approximated integral so it is natural to assume that when (5.19) is satisfied, \( K^{s+1} \) and the eigenvalue distribution \( \rho_0 \) will exhibit non-analyticities. In terms of lattice spacing the degeneracy condition (5.19) can be found by comparing with (5.14). The result

\[
a = \sqrt{1 - \cos \frac{r\pi}{2s+1}} = \sin \frac{r\pi}{2(s+1)}
\]

with \( r \) and \( s \) ranging over positive integers is exactly the condition for subleading critical behaviour as found in [16]. Again we stress that this behaviour is not realized in the matrix chain and only arises when one takes into account unstable stationary points in the evaluation of saddle-point integrations. This is the main shortcoming of the functional form (3.8): in general there are a countable infinity of solutions and it is a difficult task to discover the true solution with minimum action.

There is one more phase transition in the matrix chain as \( m^2 \) decreases further. From (5.12) we see that as \( m^2 \to -1 \) all eigenvalues of the Hessian at the origin become negative and this stationary point is degenerate in all directions. On the surface, besides this observation of an irrelevant stationary point, there is little change in the system. The relevant stationary points are found by solving polynomial equations and their stability is determined locally as a smooth function of \( m^2 \). This is misleading though since \( m^2 = -1 \) corresponds to a lattice spacing of unity which, in our scaling, is where a Kosterlitz-Thouless phase transition was shown to occur [13, 15, 16]. The only overt sign of the transition here is that the integral kernel \( K \) again changes analytic structure to

\[
K(x, u) = \frac{1}{\sqrt{4\pi \epsilon}} e^{\frac{(x+u)^2 - x^4 + u^4}{4\epsilon}}
\]

Near the origin it has the form of a translation kernel which is of a different character than the original kernel (4.1). If we consider the higher, quartic terms to only make the kernel well-defined, the behaviour of the eigenvalue distribution \( \rho_0 \) can be found by restricting the range of integration in the integral equation (3.4)

\[
e^{-\frac{i\pi}{2}} \int \rho_{KT}(\eta) d\eta = \frac{1}{\sqrt{4\pi \epsilon}} \int_{-b}^{b} du \ e^{\frac{(x+u)^2}{4\epsilon}} e^{-\frac{i\pi}{2}} \int \rho_{KT}(\eta) d\eta
\]

In this form it is clear that the solution will depend only on the constant \( b \) which sets a scale [10], otherwise it is a universal quantity. Unfortunately the asymptotic (\( \epsilon \to 0 \)) solutions of this equation are not known and so remains the eigenvalue distribution \( \rho_{KT} \) of the matrix chain at the Kosterlitz-Thouless transition.

Finally, for \( m^2 < -1 \) there persists a pair of \( \mathbb{Z}_2 \) related stationary points that at \( u \) and \( x \) vanishing trade off relevance. It is conjectured that this non-analytic behaviour is what one
would find in a $D = 0$ single matrix model, i.e.

$$\rho_0 \sim |x|^2$$

(5.23)

It would seem that the same mechanism which produced the critical exponents of matrix quantum mechanics for $-1 < m^2 < 1$ produces different exponents here. This may be because of the total degeneracy of the origin but explicit calculations of the iterated kernels (5.9) and the eigenvalue distribution $\rho_0$ are needed to make definite statements about the characteristics of the solutions of this phase of the model.

6 Conclusions and an application

To review, we have demonstrated a general technique for expressing the equations of motion for systems whose time evolution is governed by Das-Jevicki-Sakita Hamiltonians with boundary conditions as an evolution problem in one-dimensional quantum mechanics. Specifying to the particular example of the one-dimensional chain of interacting Hermitean matrices we showed that in this case the general formalism reduces to a linear integral equation for the large-$N$ eigenvalue density of the matrices. Analyzing the associated integral kernel we were able to recover all previous explicit solutions of the matrix chain from asymptotic solution of particular second order ordinary differential equations. In the general case, the solution of the integral equation was developed in terms of iterations of the kernel and from this solution the critical behavior of the system was found and a universal integral equation for the distribution of eigenvalues at the Kosterlitz-Thouless point was presented. In addition it was demonstrated that the sub-leading critical behaviour observed in [16] results from contributions from unstable stationary points in saddle-point approximations and hence is not observed.

What was not accomplished here is the explicit calculation of a non-trivial solution to the large-$N$ matrix chain using the formalism developed here. It would be nice to find a potential for which the series of iterated kernels could be explicitly calculated and summed and the eigenvalue distribution extracted. This is of particular interest in testing the conjectures involving the existence of a Kosterlitz-Thouless phase transition in the system. With a global solution for $\rho_{KT}$ one would be in a position to calculate explicitly the free energy of the matrix chain and check the critical behaviour. Of course only the leading non-analytic part of the free energy is required but in order to obtain this one should have a definite idea of the analytic structure of the distribution of eigenvalues.

The calculations we have performed here are not only relevant to string theory in one
dimension but also for other problems related to the matrix chain. An example is the Kazakov-Migdal lattice model of induced QCD in $D$-dimensions [7]. Devised as a lattice model of gauge interactions where the self-interaction of adjoint scalar fields $\phi_x$ are supposed to induce the standard Wilson term which is left out, it is defined by the following action

$$S_{KM} = \sum_x \text{Tr} V(\phi_x) - \sum_{\langle xy \rangle} \text{Tr} \Lambda_{xy} \phi_y \Lambda_{yx} \phi_x$$

(6.1)

Here the sub-scripts label sites on a hyper-cubic lattice on which the $N \times N$ adjoint scalar fields $\phi_x$ reside. Nearest neighbour interactions are mediated by $SU(N)$ gauge fields $\Lambda$ residing on the lattice links. From the large $N$ saddle-point equations it can be shown [10] that the one-dimensional matrix chain we have considered (3.1), with potential $U$, is equivalent to the Kazakov-Migdal model with a potential satisfying

$$V'(x) = DU'(x) - 2(D - 1) \int dz \frac{\rho_0(z)}{x - z}$$

(6.2)

where the eigenvalue density $\rho_0$ is common to both models.

The one drawback of this model of induced QCD is that the action (6.1) has a $\mathbb{Z}_N$ symmetry under the center of the gauge group which leads to super-confinement of gauge degrees of freedom, even on the lattice scale [27]. Obviously this not a feature that one would like to have in the continuum limit of the model and the easiest way to avoid it is for the system not to realize this symmetry faithfully in its solution. Unfortunately for the potentials $U$ of logarithmic form, for which we have $\rho_0$ explicitly (see Section Four), this symmetry persists [21, 28]. It would be interesting to see if solutions of the matrix chain for more general potentials might lead to non-trivial propagation of gauge fields in the continuum limit of the Kazakov-Migdal model. This can be checked in principle using the solution of the matrix chain for the eigenvalue density $\rho_0$ and the formalisms of [29] and [30] which express the correlation of gauge fields in the model in terms of $\rho_0$.

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