On the algebraic structures of the space of interval-valued intuitionistic fuzzy numbers

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Abstract

This study is inspired by those of Huang et al. (Soft Comput. 25, 2513–2520, 2021) and Wang et al. (Inf. Sci. 179, 3026–3040, 2009) in which some ranking techniques for interval-valued intuitionistic fuzzy numbers (IVIFNs) were introduced. In this study, we prove that the space of all IVIFNs with the relation in the method for comparing any two IVIFNs based on a score function and three types of entropy functions is a complete chain and obtain that this relation is an admissible order. Moreover, we demonstrate that IVIFNs are complete chains to the relation in the comparison method for IVIFNs on the basis of score, accuracy, membership uncertainty index, and hesitation uncertainty index functions.

Keywords: Interval-valued intuitionistic fuzzy numbers (IVIFNs); Complete lattice; Entropy function; Score function.

1. Introduction

Zadeh [26] proposed the notion of fuzzy sets (FSs) characterized by a membership function, which appoints to each object a grade of membership ranging between zero and one. Since Zadeh’s efficient study, some authors presented the generalization of the FSs, such as the concepts of type-1 fuzzy sets (T1FSs), interval type-2 fuzzy sets (IT2FSs), generalized type-2 fuzzy sets (GT2FSs) (see [4, 9, 10, 12, 18]). This generalization has induced substantial theoretical improvements and several applications [14, 27], and ensures a significant alternative other than probability theory to characterize uncertainty, imprecision, and vagueness in numerous areas [28].

Atanassov [1] introduced the intuitionistic fuzzy sets (IFSs) as a natural extension of Zadeh’s FSs, also called the Atanassov’s intuitionistic fuzzy sets (AIFSs). In the theory of IFSs, both the degree of membership and the degree of non-membership are given to each element of the universe, and their sum is smaller than or equal to 1. Hence, IFSs are more effective than FSs in handling with the uncertainty and fuzziness of systems. Atanasov [4] discussed type-1 fuzzy sets (T1FSs) that form the base of FSs extensions. Furthermore, a comparison between T1FSs and IFSs was presented in [4], and their some applications were studied in [7, 20]. Atanassov and Gargov [6] let the membership and nonmembership functions be interval values rather than single values. Thereby, they further generalized IFSs to the situation of interval-valued intuitionistic fuzzy sets (IVIFSs), which have been efficaciously used in the fields of supply and investment decision-making [21, 29]. Atanassov [2, 5] also defined some different functional laws of IVIFSs.

In recent times, Xu [23] applied the score function and accuracy function to compare interval-valued intuitionistic fuzzy numbers (IVIFNs). Subsequently, Ye [25], Nayagam et al. [17], Sahin [19], Zhang and Xu [30] and Nayagam et al. [16] provided some additional accuracy functions to compare IVIFNs and developed multicriteria fuzzy decision-making techniques. It is well known that these approaches cannot identify the difference between two arbitrary IVIFNs in some situations due to the particular properties of intervals. For this reason, to explore the difference between two IVIFNs, Wang et al. [22] proposed two new functions: the membership uncertainty index and the hesitation uncertainty index. Furthermore, a

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complete ranking method for two arbitrary IVIFNs using these functions was developed. Later, Huang et al. [13] introduced a novel comparison technique for IVIFNs, which tells the difference between any two IVIFNs. Their method exploits a score function and three types of entropy functions, where the entropies are crucial concepts for measuring the fuzziness of uncertain information.

This work is motivated by those of Huang et al. [13] and Wang et al. [22] in which more functional and more reasonable techniques for comparing two IVIFNs were proposed. The main of this paper is to enhance some results presented in the mentioned above works by providing new theorems related to the methodologies to compare two IVIFNs. In this paper, considering a score function and three types of entropy functions, we demonstrate that IVIFNs with the order in the procedure to compare any two IVIFNs proposed by [13], denoted by “$\leq_{\text{HZX}}$”, are complete totally order sets. Furthermore, we observe that the order $\leq_{\text{HZX}}$ is an admissible order on IVIFNs. Moreover, taking into account a prioritized sequence of score, accuracy, membership uncertainty index, and hesitation uncertainty index functions, we show that IVIFNs are complete totally order sets to the order in the ranking principle for IVIFNs given by [22], denoted by $\leq_{\text{WLW}}$.

The rest of this study is organized as follows. Section 2 reviews some basic concepts related to IVIFSs and some classical score functions and accuracy functions on them. In Section 3, we reveal that IVIFNs are complete totally order sets, regarding the order $\leq_{\text{HZX}}$ based on a score function and three types of entropy functions and prove that $\leq_{\text{HZX}}$ is an admissible order on IVIFNs. In Section 4, we propose that IVIFNs with the order $\leq_{\text{WLW}}$ on the basis of score, accuracy, membership uncertainty index, and hesitation uncertainty index functions are complete totally order sets. Section 5 provides concluding remarks of this research.

2. Preliminaries

2.1. Lattices

In the following, some basic notions and results about lattices are recalled. These terms will be used in the sequel.

Definition 2.1 ([8]). A binary relation $\preceq$ defined on a non-empty set $L$ is called a partial order on the set $L$ if, for all $a, b, c \in L$, it satisfies the following properties:

1. (Reflexivity) $a \preceq a$.
2. (Antisymmetry) If $a \preceq b$ and $b \preceq a$, then $a = b$.
3. (Transitivity) If $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

If, for all $a, b \in L$, either $a \preceq b$ or $b \preceq a$, then we say $\preceq$ is a total order on $L$. A non-empty set with a partial order on it is called a partially ordered set, or more briefly a poset. And if the relation is a total order, then we say it a totally ordered set or simply a chain.

A poset $(L, \preceq, 0, 1)$ is bounded if it has top and bottom elements, which are denoted as 1 and 0, respectively; that is, two elements 1, 0 $\in L$ exist such that $0 \preceq a \preceq 1$ for all $a \in L$.

Let $(L, \preceq)$ be a poset and $X \subset L$. An element $u \in L$ is said to be an upper bound of $X$ if $x \preceq u$ for all $x \in X$. An upper bound $u$ of $X$ is said to be its smallest upper bound or supremum, written as $\bigvee X$ or $\text{sup} X$, if $u \preceq y$ for each upper bound $y$ of $X$. Dually, we can define the greatest lower bound or infimum of $X$, written as $\bigwedge X$ or $\text{inf} X$. In the case of pairs of elements, it is customary to write

\[ x \lor y = \text{sup}\{x, y\} \text{ and } x \land y = \text{inf}\{x, y\}. \tag{2.1} \]

Definition 2.2 ([8]). A lattice $L$ is a poset in which for all $a, b \in L$ the set $\{a, b\}$ has a supremum and an infimum. If there are elements 0 and 1 in $L$ such that $0 \preceq a \preceq 1$ for all $a \in L$, then $L$ is called a bounded lattice.

Definition 2.3 ([11, Definition O-2.1]). A lattice $L$ is called complete if, for any subset $A$ of $L$, the greatest lower bound and the smallest upper bound of $A$ exist. A totally ordered complete lattice is called a complete chain.

Lemma 2.1 ([8]). Let $L$ be a bounded lattice. Then the following statements are equivalent:

(i) $L$ is a complete lattice;
(ii) Every nonempty subset of $L$ has an infimum;
(iii) Every nonempty subset of $L$ has a supremum.
2.2. Interval-valued intuitionistic fuzzy sets

In the following, we present classical concepts of intuitionistic fuzzy sets (IFSs) and interval-valued intuitionistic fuzzy sets (IVIFSs) as well as classical score functions and accuracy functions on IVIFSs to be helpful future discussions.

Definition 2.4 ([3, Definition 2.3]). Let X be the universe of discourse. An interval-valued intuitionistic fuzzy set (IVIFS) A in X is defined as an object having the following form

\[ A = \{ (x, \mu_A(x), \nu_A(x)) \mid x \in X \} , \]

where \( \mu_A(x) = [\mu_A^L(x), \mu_A^R(x)] \) and \( \nu_A(x) = [\nu_A^L(x), \nu_A^R(x)] \) are subintervals of \([0,1]\) denoting the membership degree and non-membership degree of the element \( x \) to the set \( A \), respectively, which meet the condition of \( \mu_A^R(x) + \nu_A^L(x) \leq 1 \).

Clearly, if \( \mu_A^L(x) = \mu_A^R(x) \) and \( \nu_A^L(x) = \nu_A^R(x) \), then an IVIFS reduces to a traditional IFS.

For an IVIFS \( A \), the indeterminacy degree \( \pi_A(x) \) of element \( x \) belonging to the IFS \( A \) is defined by

\[ \pi_A(x) = [1 - \mu_A^R(x) - \nu_A^R(x), 1 - \mu_A^L(x) - \nu_A^L(x)] \].

In [24], the pair \( (\mu_A(x), \nu_A(x)) \) is called an interval-valued intuitionistic fuzzy number (IVIFN), which was also called an interval valued intuitionistic fuzzy pair (IVIFP) in [5]. For convenience, we use \( \alpha = [\mu_A^L(x), \mu_A^R(x)] \) to represent an IVIFN, which satisfies

\[ [\mu_A^L(x), \mu_A^R(x)] \subset [0,1], [\nu_A^L(x), \nu_A^R(x)] \subset [0,1], \text{ and } \mu_A^R + \nu_A^L \leq 1, \]

and use \( \hat{\Theta} \) to denote the set of all IVIFNs.

Xu [23] introduced the score and accuracy functions for IVIFNs and used them to compare two IVIFNs. However, there is no general method which can rank any two arbitrary IVIFNs.

Definition 2.5 ([23]). Let \( \alpha = [\mu_A^L(x), [\nu_A^L(x), \nu_A^R(x)] \) be an IVIFN. A score function \( S \) is defined as follows

\[ S(\alpha) = \frac{\mu_A^L + \mu_A^R}{2} - \frac{\nu_A^L + \nu_A^R}{2}, \quad S(\alpha) \in [-1,1]. \]

Moreover, the accuracy function is defined as follows

\[ H(\alpha) = \frac{\mu_A^L + \mu_A^R}{2} + \frac{\nu_A^L + \nu_A^R}{2} . \]

Definition 2.6 ([23]). Let \( \alpha_1 \) and \( \alpha_2 \) be two IVIFNs. Then, the ranking principle is defined as follows:

1. If \( S(\alpha_1) < S(\alpha_2) \), then \( \alpha_1 < \alpha_2 \);
2. If \( S(\alpha_1) = S(\alpha_2) \), then
   - \( H(\alpha_1) < H(\alpha_2) \), then \( \alpha_1 < \alpha_2 \);
   - \( H(\alpha_1) = H(\alpha_2) \), then \( \alpha_1 = \alpha_2 \).

3. Algebraic structures of \((\hat{\Theta}, \leq_{\text{max}})\)

In Definition 2.6, a ranking procedure for IVIFNs by using a score function and an accuracy function was proposed. Unfortunately, because of the various properties of intervals, the score and accuracy functions together may not determine the difference between two arbitrary IVIFNs. Hence, Huang et al. [13] introduced a complete ranking method for IVIFNs via a score function and three kinds of entropy functions, which are essential notions for measuring the fuzziness of uncertain information. To be more precise, they proposed the following order \( \leq_{\text{max}} \) on IVIFNs, which can rank any two arbitrary IVIFNs. They also proved that it is a total order on IVIFNs.

Definition 3.1 ([13, Definition 3.1]). Let \( \alpha = [\mu_A^L(x), [\nu_A^L(x), \nu_A^R(x)] \) be an IVIFN. Define the score function \( S \) of \( \alpha \)

\[ S(\alpha) = \frac{\mu_A^L + \mu_A^R}{2} - \frac{\nu_A^L + \nu_A^R}{2} . \]

Moreover, three entropy functions of \( \alpha \) are defined as follows:

- \( E_1(\alpha) = \frac{1 - \mu_A^L - \mu_A^R}{2} + \frac{1 - \nu_A^L - \nu_A^R}{2} ; \)
- \( E_2(\alpha) = \frac{\mu_A^R - \mu_A^L + \nu_A^R - \nu_A^L}{2} . \)
\[ E_3(\alpha) = \mu^L_\alpha - \mu^R_\alpha. \]

Remark 1. Notice that \( E_1(\alpha) \) can be replaced by the accuracy function \( H(\alpha) \) defined by

\[ H(\alpha) = \frac{\mu^L_\alpha + \mu^R_\alpha}{2} + \frac{\nu^L_\alpha + \nu^R_\alpha}{2}. \]

**Definition 3.2** ([13, Definition 3.2]). Let \( \alpha_1 \) and \( \alpha_2 \) be two IVIFNs. Then, it gets the following ranking principle:

1. If \( S(\alpha_1) < S(\alpha_2) \), then \( \alpha_1 \) is smaller than \( \alpha_2 \), denoted by \( \alpha_1 \prec_{HZX} \alpha_2 \);
2. If \( S(\alpha_1) = S(\alpha_2) \), then
   - \( H(\alpha_1) < H(\alpha_2) \), then \( \alpha_1 \prec_{HZX} \alpha_2 \);
   - \( H(\alpha_1) = H(\alpha_2) \), then
     - \( E_2(\alpha_1) < E_2(\alpha_2) \), then \( \alpha_1 \prec_{HZX} \alpha_2 \);
     - \( E_2(\alpha_1) = E_2(\alpha_2) \), then
       * \( E_3(\alpha_1) < E_3(\alpha_2) \), then \( \alpha_1 \prec_{HZX} \alpha_2 \);
       * \( E_3(\alpha_1) = E_3(\alpha_2) \), then \( \alpha_1 = \alpha_2 \).

If \( \alpha_1 \prec_{HZX} \alpha_2 \) or \( \alpha_1 = \alpha_2 \), we will denote it by \( \alpha_1 \preceq_{HZX} \alpha_2 \).

**Theorem 3.1** ([13, Theorem 3.1]). *Definition 3.2 defines a total order on IVIFNs, i.e., the ranking principle given by Definition 3.2 gives a complete ranking in any class of IVIFNs.*

**Remark 2.** It is easy to see that the bottom and the top elements of \( \tilde{\Theta} \) are, respectively, \( \langle 0, 0 \rangle \) and \( \langle 1, 1 \rangle \) with respect to the order \( \preceq_{HZX} \).

**Definition 3.3.** Let \( \alpha_1 \) and \( \alpha_2 \) be two IVIFNs. A relation \( \preceq \) on \( \tilde{\Theta} \) is defined as: \( \alpha \preceq \beta \) if and only if \( \mu^L_\alpha \leq \mu^L_\beta, \mu^R_\alpha \leq \mu^R_\beta, \nu^L_\alpha \geq \nu^L_\beta \), and \( \nu^R_\alpha \geq \nu^R_\beta \).

**Definition 3.4** ([15, Definition 4.1]). A partial order \( \preceq \) on \( \tilde{\Theta} \) is said to be an *admissible order* if it is a total order and refines the order \( \preceq \) introduced in Definition 3.3, i.e., it is a total order satisfying that for any \( \alpha, \beta \in \tilde{\Theta}, \alpha \preceq \beta \) implies \( \alpha \preceq \beta \).

In the following Theorem 3.2, by applying a score function and three types of entropy functions, we demonstrate that the space of all IVIFSs with the order \( \preceq_{HZX} \) is a complete chain.

**Theorem 3.2.** \( \langle \tilde{\Theta}, \preceq_{HZX} \rangle \) is a complete chain.

**Proof.** By Theorem 3.1, \( \langle \tilde{\Theta}, \preceq_{HZX} \rangle \) is a chain. Thus, it suffices to check that it is complete. Given a nonempty subset \( \Omega \subset \tilde{\Theta} \), we claim that the smallest upper bound of \( \Omega \) exists.

For convenience, let \( \mathcal{S}(\Omega) = \{ S(\alpha) \mid \alpha \in \Omega \} \) and \( \xi_1 = \sup \mathcal{S}(\Omega) \). Then, we consider the following four cases:

1. \( \xi_1 \notin \mathcal{S}(\Omega) \) and \( \xi_1 \leq 0 \). Take \( \beta_1 = \langle [0, 0], [-\xi_1, -\xi_1] \rangle \). Clearly, \( \beta_1 \in \tilde{\Theta} \). Meanwhile, for any \( \alpha \in \Omega \), it is clear that \( S(\alpha) < \sup \mathcal{S}(\Omega) = \xi_1 \) by \( \xi_1 \notin \mathcal{S}(\Omega) \). This, together with Definition 3.2, implies that \( \alpha \prec_{HZX} \beta_1 \), i.e., \( \beta_1 \) is an upper bound of \( \Omega \). Given an upper bound \( \beta = \langle [\mu^L_\beta, \mu^R_\beta], [\nu^L_\beta, \nu^R_\beta] \rangle \in \tilde{\Theta} \) of \( \Omega \), by Definition 3.2, we have that \( S(\beta) \geq S(\alpha) \) holds for all \( \alpha \in \Omega \), implying that \( S(\beta) \geq \sup \{ S(\alpha) \mid \alpha \in \Omega \} = \xi_1 \).
2. \( S(\beta) > \xi_1 = S(\beta_1) \), by Definition 3.2, it is clear that \( \beta \succ_{HZX} \beta_1 \).
3. \( S(\beta) = \xi_1 = S(\beta_1) \), by \( S(\beta) = \frac{\mu^L_\beta + \mu^R_\beta}{2} - \frac{\nu^L_\beta + \nu^R_\beta}{2} \geq -\frac{\nu^L_\beta + \nu^R_\beta}{2} \), we have \( H(\beta) = \frac{\mu^L_\beta + \nu^R_\beta}{2} + \frac{\nu^L_\beta + \nu^R_\beta}{2} \geq \frac{\nu^L_\beta + \nu^R_\beta}{2} \geq -S(\beta) = -\xi_1 \).
4. If \( H(\beta) > -\xi_1 \), by Definition 3.2 and \( H(\beta_1) = -\xi_1 \), we have \( \beta \succ_{HZX} \beta_1 \).
5. If \( H(\beta) = -\xi_1 \), noting that \( H(\beta_1) = -\xi_1 \), \( E_2(\beta_1) = E_3(\beta_1) = 0 \), and \( E_2(\beta) \geq 0 \), \( E_3(\beta) \geq 0 \), by Definition 3.2, we have \( \beta \succeq_{HZX} \beta_1 \).
Therefore, $\beta_1$ is the smallest upper bound of $\Omega$.

(2) $\xi_1 \notin \mathcal{F}(\Omega)$ and $\xi_1 \geq 0$. Similarly to the proof of (1), it can be verified that $\beta_2 = \langle [\xi_1, \xi_1], [0, 0] \rangle$ is the smallest upper bound of $\Omega$.

(3) $\xi_1 \in \mathcal{F}(\Omega)$ and $\xi_1 \leq 0$. This implies that $\Omega = \{ \alpha \in \Omega \mid S(\alpha) = \xi_1 \} \neq \emptyset$. Then, let us take $\xi_2 = \sup \{ H(\alpha) \mid \alpha \in \Omega \}$.

3.1) If $\xi_2 \notin \{ H(\alpha) \mid \alpha \in \Omega \}$, then, for any $n \in \mathbb{N}$, there exists $\alpha_n \in \bar{\Omega}$ such that $\xi_2 - \frac{1}{n} < H(\alpha_n) < \xi_2$, i.e.,

\[
\begin{aligned}
S(\alpha_n) &= \frac{\mu_{\alpha_n}^L + \mu_{\alpha_n}^R}{2} - \frac{\nu_{\alpha_n}^L + \nu_{\alpha_n}^R}{2} = \xi_1, \\
\xi_2 - \frac{1}{n} < H(\alpha_n) &= \frac{\mu_{\alpha_n}^L + \mu_{\alpha_n}^R}{2} + \frac{\nu_{\alpha_n}^L + \nu_{\alpha_n}^R}{2} < \xi_2.
\end{aligned}
\]

This, together with $\alpha_n \in \bar{\Theta}$, implies that

\[
\xi_2 = \lim_{n \to +\infty} \left( \frac{\mu_{\alpha_n}^L + \mu_{\alpha_n}^R}{2} + \frac{\nu_{\alpha_n}^L + \nu_{\alpha_n}^R}{2} \right) \leq \lim_{n \to +\infty} \left( \frac{\mu_{\alpha_n}^L + \nu_{\alpha_n}^R}{2} \right) \leq 1,
\]

\[
\frac{\xi_1 + \xi_2}{2} = \lim_{n \to +\infty} \frac{S(\alpha_n) + H(\alpha_n)}{2} = \lim_{n \to +\infty} \frac{\mu_{\alpha_n}^L + \mu_{\alpha_n}^R}{2} \in [0, 1],
\]

and

\[
\frac{\xi_2 - \xi_1}{2} = \lim_{n \to +\infty} \frac{H(\alpha_n) - S(\alpha_n)}{2} = \lim_{n \to +\infty} \frac{\nu_{\alpha_n}^L + \nu_{\alpha_n}^R}{2} \in [0, 1],
\]

and thus $\beta_2 = \left( \left[ \frac{\xi_1 + \xi_2}{2}, \xi_1 + \xi_2 \right], \left[ \frac{\xi_2 - \xi_1}{2}, \xi_2 - \xi_1 \right] \right) \in \bar{\Theta}$. By direct calculation, we have $S(\beta_2) = \xi_1$ and $H(\beta_2) = \xi_2$. For any $\alpha \in \Omega$, from the choice of $\xi_1$, it follows that $S(\alpha) \leq \xi_1 = S(\beta_2)$.

- If $S(\alpha) < S(\beta_2)$, by Definition 3.2, it is clear that $\alpha <_{\text{hzw}} \beta_2$.
- If $S(\alpha) = \xi_1$, i.e., $\alpha \in \bar{\Theta}$, by the choice of $\xi_2$ and $\xi_2 \notin \{ H(\alpha) \mid \alpha \in \bar{\Omega} \}$, we have $H(\alpha) < \xi_2 = H(\beta_2)$, and thus $\alpha <_{\text{hzw}} \beta_2$ by Definition 3.2.

These imply that $\beta_2$ is an upper bound of $\Omega$. Given an upper bound $\beta = \langle [\mu_{\beta}^L, \mu_{\beta}^R], [\nu_{\beta}^L, \nu_{\beta}^R] \rangle \in \bar{\Theta} \setminus \Omega$, by Definition 3.2, it is clear that $S(\beta) \geq \xi_1$.

- If $S(\beta) > \xi_1$, by Definition 3.2 and $S(\beta_2) = \xi_1$, it is clear that $\beta >_{\text{hzw}} \beta_2$.
- If $S(\beta) = \xi_1$, for any $\alpha \in \bar{\Omega}$, by $\beta >_{\text{hzw}} \alpha$ and $S(\beta) = S(\alpha)$, then $H(\beta) \geq H(\alpha)$, and thus $H(\beta) \geq \sup \{ H(\alpha) \mid \alpha \in \bar{\Omega} \} = \xi_2 = H(\beta_2)$. This, together with $E_2(\beta_2) = 0 \leq E_2(\beta)$ and $E_3(\beta_2) = 0 \leq E_3(\beta)$, implies that $\beta >_{\text{hzw}} \beta_2$.

Therefore, $\beta_2$ is the smallest upper bound of $\Omega$.

3.2) If $\xi_2 \in \{ H(\alpha) \mid \alpha \in \bar{\Omega} \}$, i.e., $\bar{\Omega}_1 = \{ \alpha \in \bar{\Omega} \mid H(\alpha) = \xi_2 \} \neq \emptyset$, then let us take $\xi_3 = \sup \{ E_2(\alpha) \mid \alpha \in \bar{\Omega}_1 \}$.

3.2.1) If $\xi_3 \not= \sup \{ E_2(\alpha) \mid \alpha \in \bar{\Omega}_1 \}$, i.e., $\{ \alpha \in \bar{\Omega}_1 \mid E_2(\alpha) = \xi_3 \} \neq \emptyset$, then we take $\bar{\Omega}_2 = \{ \alpha \in \bar{\Omega}_1 \mid E_2(\alpha) = \xi_3 \}$ and $\xi_4 = \sup \{ E_3(\alpha) \mid \alpha \in \bar{\Omega}_2 \}$, and consider the following two subcases:

i) If $\xi_4 \not= \sup \{ E_3(\alpha) \mid \alpha \in \bar{\Omega}_2 \}$, then we choose

\[
\beta_3 = \left( \left[ \frac{\xi_1 + \xi_2 - \xi_4}{2}, \xi_1 + \xi_2 + \xi_4 \right], \left[ \frac{\xi_2 - \xi_1}{2}, \frac{2\xi_4 - \xi_2 + \xi_1}{2} \right] \right).
\]

Claim 1. $\beta_3 \in \bar{\Theta}$.

Proof of Claim 1: By $\xi_4 = \sup \{ E_3(\alpha) \mid \alpha \in \bar{\Omega}_2 \}$, we have that, for any $n \in \mathbb{N}$, there exists $\alpha_n = \langle [\mu_{\alpha_n}^L, \mu_{\alpha_n}^R], [\nu_{\alpha_n}^L, \nu_{\alpha_n}^R] \rangle \in \bar{\Omega}_2$ (i.e., $\alpha_n \in \bar{\Omega}_1$ and $E_2(\alpha_n) = \xi_3$) such that

\[
\xi_4 - \frac{1}{n} < E_3(\alpha_n) = \mu_{\alpha_n}^R - \mu_{\alpha_n}^L < \xi_4, \quad \text{i.e.,} \quad \lim_{n \to +\infty} E_3(\alpha_n) = \xi_4.
\]

From $\langle [\mu_{\alpha_n}^L, \mu_{\alpha_n}^R], [\nu_{\alpha_n}^L, \nu_{\alpha_n}^R] \rangle \in \bar{\Omega}_1$, it follows that

\[
\begin{aligned}
\frac{\mu_{\alpha_n}^L + \mu_{\alpha_n}^R}{2} - \frac{\nu_{\alpha_n}^L + \nu_{\alpha_n}^R}{2} &= \xi_1, \\
\frac{\mu_{\alpha_n}^L + \nu_{\alpha_n}^R}{2} + \frac{\nu_{\alpha_n}^L + \mu_{\alpha_n}^R}{2} &= \xi_2.
\end{aligned}
\]
and thus
\[ \frac{\mu^L_{\alpha_n} + \mu^R_{\alpha_n}}{2} = \frac{\xi_1 + \xi_2}{2} \quad \text{and} \quad \frac{\nu^L_{\alpha_n} + \nu^R_{\alpha_n}}{2} = \frac{\xi_2 - \xi_1}{2}. \]

This, together with \( E_2(\alpha_n) = \frac{\mu^R_{\alpha_n} - \mu^L_{\alpha_n} + \nu^R_{\alpha_n} - \nu^L_{\alpha_n}}{2} = \xi_3 \) and \( E_3(\alpha_n) = \mu^R_{\alpha_n} - \mu^L_{\alpha_n} \), implies that
\[ [\mu^L_{\alpha_n}, \mu^R_{\alpha_n}] = \left[ \frac{\xi_1 + \xi_2}{2} - \frac{E_3(\alpha_n)}{2}, \frac{\xi_1 + \xi_2}{2} + \frac{E_3(\alpha_n)}{2} \right] \subset [0, 1], \]
and
\[ [\nu^L_{\alpha_n}, \nu^R_{\alpha_n}] = \left[ \frac{\xi_2 - \xi_1}{2} - \frac{2\xi_3 - E_3(\alpha_n)}{2}, \frac{\xi_2 - \xi_1}{2} + \frac{2\xi_3 - E_3(\alpha_n)}{2} \right] \subset [0, 1]. \]

By \( \alpha_n = ([\mu^L_{\alpha_n}, \mu^R_{\alpha_n}]; [\nu^L_{\alpha_n}, \nu^R_{\alpha_n}]) \in \bar{\Theta} \), we have
\[ \frac{\xi_1 + \xi_2}{2} - \frac{E_3(\alpha_n)}{2} \geq 0, \]
\[ \frac{\xi_2 - \xi_1}{2} - \frac{2\xi_3 - E_3(\alpha_n)}{2} \geq 0, \]
and
\[ \left[ \frac{\xi_1 + \xi_2}{2} + \frac{E_3(\alpha_n)}{2} \right] + \left[ \frac{\xi_2 - \xi_1}{2} + \frac{2\xi_3 - E_3(\alpha_n)}{2} \right] = \xi_2 + \xi_3 \leq 1. \]
These, together with formula (3.1), imply that
\[ \frac{\xi_1 + \xi_2}{2} = \lim_{n \to +\infty} \left( \frac{\xi_1 + \xi_2}{2} - \frac{E_3(\alpha_n)}{2} \right) \geq 0, \quad \text{(3.2)} \]
\[ \frac{\xi_2 - \xi_1}{2} = \lim_{n \to +\infty} \left( \frac{\xi_2 - \xi_1}{2} - \frac{2\xi_3 - E_3(\alpha_n)}{2} \right) \geq 0, \quad \text{(3.3)} \]
and
\[ \frac{\xi_1 + \xi_2 + \xi_3}{2} \left( \frac{\xi_2 - \xi_1}{2} + \frac{2\xi_3 - E_3(\alpha_n)}{2} \right) = \xi_2 + \xi_3 \leq 1, \quad \text{(3.4)} \]
i.e., \( \beta_3 \in \bar{\Theta} \).

By direct calculation, we have

**Claim 2.** \( S(\beta_3) = \xi_1, H(\beta_3) = \xi_2, E_2(\beta_3) = \xi_3, \) and \( E_3(\beta_3) = \xi_4. \)

**Claim 3.** \( \beta_3 \) is an upper bound of \( \Omega. \)

**Proof of Claim 3:** For any \( \alpha \in \Omega, \)

- If \( \alpha \in \Omega \setminus \bar{\Omega}, \) by the choice of \( \bar{\Omega}, \) we have \( S(\alpha) < \xi_1 = S(\beta_3) \) (by Claim 2), and thus \( \alpha <_{\text{h.z}} \beta_3 \) by Definition 3.2.
- If \( \alpha \in \bar{\Omega} \setminus \bar{\Omega}_1, \) by the choices of \( \bar{\Omega} \) and \( \bar{\Omega}_1, \) we have \( S(\alpha) = \xi_1 = S(\beta_3) \) and \( H(\alpha) < \xi_2 = H(\beta_3) \) (by Claim 2), and thus \( \alpha <_{\text{h.z}} \beta_3 \) by Definition 3.2.
- If \( \alpha \in \bar{\Omega}_1 \setminus \bar{\Omega}_2, \) by the choices of \( \bar{\Omega}_1 \) and \( \bar{\Omega}_2, \) we have \( S(\alpha) = \xi_1 = S(\beta_3), H(\alpha) = \xi_2 = H(\beta_3), \) and \( E_2(\alpha) < \xi_3 = E_2(\beta_3) \) (by Claim 2), and thus \( \alpha <_{\text{h.z}} \beta_3 \) by Definition 3.2.
- If \( \alpha \in \bar{\Omega}_2, \) by the choices of \( \bar{\Omega}_2 \) and \( \xi_4, \) we have \( S(\alpha) = \xi_1 = S(\beta_3), H(\alpha) = \xi_2 = H(\beta_3), \) \( E_2(\alpha) = \xi_3 = E_2(\beta_3), \) and \( E_3(\alpha) < \xi_4 = E_3(\beta_3) \) (by Claim 2), and thus \( \alpha <_{\text{h.z}} \beta_3 \) by Definition 3.2.

**Claim 4.** \( \beta_3 \) is the smallest upper bound of \( \Omega. \)

**Proof of Claim 4:** Given an upper bound \( \beta = ([\mu^L_{\beta_3}, \mu^R_{\beta_3}]; [\nu^L_{\beta_3}, \nu^R_{\beta_3}]) \in \bar{\Theta} \) of \( \Omega, \) by Definition 3.2, it is clear that \( S(\beta) \geq \xi_1. \)
- If \( \beta > \xi_1, \) by Definition 3.2 and \( S(\beta_3) = \xi_1, \) it is clear that \( \beta >_{\text{h.z}} \beta_3. \)
- If \( \beta = \xi_1, \) for \( \alpha \in \Omega, \) by \( \beta \geq_{\text{h.z}} \alpha, \) then \( H(\beta) \geq H(\alpha), \) and thus \( H(\beta) \geq \sup \{H(\alpha) \mid \alpha \in \Omega\} = \xi_2 = H(\beta_3). \)
  - If \( H(\beta) > H(\beta_3), \) it is clear that \( \beta >_{\text{h.z}} \beta_3 \) by Definition 3.2.
If \( H(\beta) = H(\beta_3) \), by \( \beta \geq_{h\infty} \alpha \) for all \( \alpha \in \Omega_1 \), we have \( E_2(\beta) \geq \sup \{ E_2(\alpha) \mid \alpha \in \Omega_1 \} = \xi_3 = E_2(\beta_3) \).

* If \( E_2(\beta) > E_2(\beta_3) \), it is clear that \( \beta >_{h\infty} \beta_3 \) by Definition 3.2.

* If \( E_2(\beta) = E_2(\beta_3) \), by \( \alpha \geq_{h\infty} \alpha \) for all \( \alpha \in \Omega_2 \), we have \( E_3(\beta) \geq \sup \{ E_3(\alpha) \mid \alpha \in \Omega_2 \} = \xi_4 = E_3(\beta_3) \), and thus \( \beta >_{h\infty} \beta_3 \) by Definition 3.2.

Therefore, \( \beta_3 \) is the smallest upper bound of \( \Omega \).

ii) If \( \xi_4 \in \{ E_3(\alpha) \mid \alpha \in \bar{\Omega}_2 \} \), i.e., there exists \( \alpha \in \bar{\Omega}_2 \) such that \( E_3(\alpha) = \xi_4 \), then

\[
\begin{align*}
S(\alpha) &= \frac{\mu^L_{\alpha} + \mu^R_{\alpha}}{2} - \frac{\nu^L_{\alpha} + \nu^R_{\alpha}}{2} = \xi_1, \\
H(\alpha) &= \frac{\mu^L_{\alpha} + \mu^R_{\alpha}}{2} + \frac{\nu^L_{\alpha} + \nu^R_{\alpha}}{2} = \xi_2, \\
E_2(\alpha) &= \frac{\mu^L_{\alpha} - \nu^L_{\alpha} + \mu^R_{\alpha} - \nu^R_{\alpha}}{2} = \xi_3, \\
E_3(\alpha) &= \mu^R_{\alpha} - \mu^L_{\alpha} = \xi_4,
\end{align*}
\]

i.e.,

\[
\begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
\mu^L_{\alpha} \\
\mu^R_{\alpha} \\
\nu^L_{\alpha} \\
\nu^R_{\alpha}
\end{bmatrix}
= \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4
\end{bmatrix}.
\]

Since

\[
\begin{vmatrix}
\frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2}
\end{vmatrix}
= 1 \neq 0,
\]

we obtain that such \( \alpha (\alpha \in \bar{\Omega}_2 \) and \( E_3(\alpha) = \xi_4 \) is unique,

which is denoted by \( \hat{\alpha} \). Under this condition, it can be verified that \( \alpha \) is the maximum of \( \Omega \), and thus it is the smallest upper bound of \( \Omega \).

3.2.2) If \( \xi_3 \notin \{ E_2(\alpha) \mid \alpha \in \Omega_1 \} \), by \( \bar{\Omega}_1 \neq \varnothing \), there exists \( \alpha \in \bar{\Omega} \) such that \( H(\alpha) = \xi_2 \), i.e.,

\[
\begin{align*}
S(\alpha) &= \frac{\mu^L_{\alpha} + \mu^R_{\alpha}}{2} - \frac{\nu^L_{\alpha} + \nu^R_{\alpha}}{2} = \xi_1, \\
H(\alpha) &= \frac{\mu^L_{\alpha} + \mu^R_{\alpha}}{2} + \frac{\nu^L_{\alpha} + \nu^R_{\alpha}}{2} = \xi_2,
\end{align*}
\]

implying that

\[
\zeta_1 := \frac{\xi_1 + \xi_2}{2} = \frac{\mu^L_{\alpha} + \mu^R_{\alpha}}{2} \in [0, 1] \text{ and } \zeta_2 := \frac{\xi_2 - \xi_1}{2} = \frac{\nu^L_{\alpha} + \nu^R_{\alpha}}{2} \in [0, 1].
\]

Clearly, \( \zeta_1 \leq \zeta_2 \) and \( \zeta_1 + \zeta_2 = \frac{\mu^L_{\alpha} + \mu^R_{\alpha}}{2} + \frac{\nu^L_{\alpha} + \nu^R_{\alpha}}{2} \leq \mu^L_{\alpha} + \nu^R_{\alpha} \leq 1 \).

**Claim 5.** \( \zeta_1, \zeta_2, \zeta_3 \geq 0, \zeta_1 + \zeta_2 + \zeta_3 \leq 1, \) and \( \zeta_1 + \zeta_2 - \zeta_3 \geq 0 \).

**Proof of Claim 5:** Clearly, \( \zeta_1, \zeta_2, \zeta_3 \geq 0 \). By \( \xi_3 = \sup \{ E_2(\alpha) \mid \alpha \in \bar{\Omega}_1 \} \notin \{ E_2(\alpha) \mid \alpha \in \bar{\Omega}_1 \} \), we have that, for any \( n \in \mathbb{N} \), there exists \( \alpha_n \in \bar{\Omega}_1 \) such that \( \zeta_3 - \frac{1}{n} < E_2(\alpha_n) < \xi_3 \), implying that

\[
\frac{(\mu^L_{\alpha_n} + \mu^R_{\alpha_n}) + (\nu^L_{\alpha_n} + \nu^R_{\alpha_n})}{2} = \xi_2 = \zeta_1 + \zeta_2 \text{ (by } \alpha_n \in \bar{\Omega}_1),
\]

and

\[
\xi_3 - \frac{1}{n} < E_2(\alpha_n) = \frac{(\mu^R_{\alpha_n} - \mu^L_{\alpha_n}) + (\nu^R_{\alpha_n} - \nu^L_{\alpha_n})}{2} < \xi_3,
\]

and thus

\[
\xi_3 - \frac{1}{n} + (\zeta_1 + \zeta_2) < \left[ \frac{(\mu^L_{\alpha_n} + \mu^R_{\alpha_n}) + (\nu^L_{\alpha_n} + \nu^R_{\alpha_n})}{2} + \frac{(\mu^R_{\alpha_n} - \mu^L_{\alpha_n}) + (\nu^R_{\alpha_n} - \nu^L_{\alpha_n})}{2} \right] = \mu^R_{\alpha_n} + \nu^R_{\alpha_n} < \xi_3 + (\zeta_1 + \zeta_2),
\]

and

\[
0 \leq \mu^L_{\alpha_n} + \nu^L_{\alpha_n} = \left[ \frac{(\mu^L_{\alpha_n} + \mu^R_{\alpha_n}) + (\nu^L_{\alpha_n} + \nu^R_{\alpha_n})}{2} - \frac{(\mu^R_{\alpha_n} - \mu^L_{\alpha_n}) + (\nu^R_{\alpha_n} - \nu^L_{\alpha_n})}{2} \right] < (\zeta_1 + \zeta_2) - \left( \xi_3 - \frac{1}{n} \right).
\]
By $\mu^R_{\alpha_n} + \nu^R_{\alpha_n} \leq 1$ ($n \in \mathbb{N}$), letting $n \to +\infty$, we have

$$1 \geq \lim_{n \to +\infty} (\mu^R_{\alpha_n} + \nu^R_{\alpha_n}) = \xi_3 + (\zeta_1 + \zeta_2),$$

and

$$\zeta_1 + \zeta_2 - \xi_3 \geq 0.$$

Let us choose

$$\beta_4 = \begin{cases} [\{\zeta_1, \zeta_1\}, [\zeta_2 - \xi_3, \zeta_2 + \xi_3]], & \zeta_2 - \xi_3 \geq 0, \\
[\{\zeta_1 - (\xi_3 - \zeta_2), \zeta_1 + (\xi_3 - \zeta_2)], [0, 2\xi_2]], & \zeta_2 - \xi_3 < 0. \end{cases}$$

Claim 6. $\beta_4 \in \Theta$.

Proof of Claim 6: If $\zeta_2 - \xi_3 \geq 0$, by Claim 5, we have $[\zeta_1, \zeta_1] \subset [0, 1]$, $[\zeta_2 - \xi_3, \zeta_2 + \xi_3] \subset [0, 1]$, and $\zeta_1 + (\xi_3 - \zeta_2) \leq 1$, and thus $\beta_4 \in \Theta$.

If $\zeta_2 - \xi_3 < 0$, by Claim 5, we have $0 \leq \zeta_1 - (\xi_3 - \zeta_2) \leq \zeta_1 + (\xi_3 - \zeta_2) \leq \zeta_1 + (\xi_3 + \zeta_2) \leq 1$, $2\zeta_2 \leq \zeta_2 + \xi_3 \leq \zeta_1 + \zeta_2 + \xi_3 \leq 1$, i.e., $[\zeta_1 - (\xi_3 - \zeta_2), \zeta_1 + (\xi_3 - \zeta_2)] \subset [0, 1]$, $[0, 2\xi_2] \subset [0, 1]$; and $\zeta_1 + (\xi_3 - \zeta_2) + 2\zeta_2 = \zeta_1 + \zeta_2 + \xi_3 \leq 1$, and thus $\beta_4 \in \Theta$.

By direct calculation, we have $S(\beta_4) = \xi_1$, $H(\beta_4) = \xi_2$, and $E_2(\beta_4) = \xi_3$. Similarly to the proof of Claim 3, it can be verified that

Claim 7. $\beta_4$ is an upper bound of $\Omega$.

Claim 8. $\beta_4$ is the smallest upper bound of $\Omega$.

Proof of Claim 8: Given an upper bound $\beta = ([\mu^R_\beta, \mu^R_\beta], [\nu^R_\beta, \nu^R_\beta]) \in \Theta$ of $\Omega$, by Definition 3.2, it is clear that $S(\beta) \geq \xi_1$.

- If $S(\beta) = \xi_1$, by Definition 3.2 and $S(\beta_4) = \xi_1$, it is clear that $\beta >_{\max} \beta_4$.

- If $S(\beta) = \xi_1$, for any $\alpha \in \Omega$, by $\beta >_{\max} \alpha$, then $H(\beta) \geq H(\alpha)$, and thus $H(\beta) \geq \sup\{H(\alpha) \mid \alpha \in \Omega\} = \xi_2 = H(\beta_4)$.

* If $H(\beta) > H(\beta_4)$, it is clear that $\beta >_{\max} \beta_4$ by Definition 3.2.

* If $H(\beta) = H(\beta_4)$, by $\beta >_{\max} \alpha$ for all $\alpha \in \Omega_1$, we have $E_2(\beta) \geq \sup\{E_2(\alpha) \mid \alpha \in \Omega_1\} = \xi_3 = E_2(\beta_4)$. Consider the following two subcases:

  - If $\xi_2 - \xi_3 \geq 0$, by the choice of $\beta_4$, we have $E_4(\beta_4) = 0 \leq E_2(\beta)$. This, together with $S(\beta_4) = S(\beta)$, $H(\beta_4) = H(\beta)$, and $E_2(\beta_4) = E_2(\beta)$, implies that $\beta_4 <_{\max} \beta$.

  - If $\xi_2 - \xi_3 < 0$, by $\xi_2 - \xi_3 \geq 0$, we have $l_1 \leq \xi_2 < \xi_3$, implying that $\xi_3 - \xi_2 < \xi_3 - l_1$. This, together with $\beta_4 = ([\xi_1 - (\xi_3 - \xi_2), \zeta_1 + (\xi_3 - \xi_2)], [0, 2\xi_2])$, implies that $E_3(\beta_4) = 2(\xi_3 - \xi_2) < 2l_1 = E_3(\beta)$, and thus $\beta_4 <_{\max} \beta$ since $S(\beta_4) = S(\beta)$, $H(\beta_4) = H(\beta)$, and $E_2(\beta_4) = E_2(\beta)$.

Therefore, $\beta_4$ is the smallest upper bound of $\Omega$.

In the following Theorem 3.3, we show that the order $\leq_{\max}$ fulfills the order $\subseteq$ introduced in Definition 3.3; namely, it is an admissible order on IVIFNs.

Theorem 3.3. The order $\leq_{\max}$ in Definition 3.2 is an admissible order on $\Theta$.  \[\square\]
Proof. Based on Theorem 3.1, \( \leq_{\text{hxx}} \) is a total order on \( \tilde{\Theta} \). For two IVIFNs \( \alpha = (\langle \mu^L_{\alpha}, \mu^R_{\alpha}; [\nu^L_{\alpha}, \nu^R_{\alpha}] \rangle) \) and \( \beta = (\langle \mu^L_{\beta}, \mu^R_{\beta}; [\nu^L_{\beta}, \nu^R_{\beta}] \rangle) \), let \( \alpha \leq \beta \). By Definition 3.3, there holds \( \mu^L_{\alpha} \leq \mu^L_{\beta}, \mu^R_{\alpha} \leq \mu^R_{\beta}, \nu^L_{\alpha} \geq \nu^L_{\beta}, \) and \( \nu^R_{\alpha} \geq \nu^R_{\beta} \). Then, we have that
\[
S(\alpha) - S(\beta) = \left( \frac{\mu^L_{\alpha} + \mu^R_{\alpha} - \nu^L_{\alpha} - \nu^R_{\alpha}}{2} \right) - \left( \frac{\mu^L_{\beta} + \mu^R_{\beta} - \nu^L_{\beta} - \nu^R_{\beta}}{2} \right) \\
\geq \frac{\mu^L_{\alpha} - \mu^L_{\beta} + \mu^R_{\alpha} - \mu^R_{\beta} + \nu^L_{\alpha} - \nu^L_{\beta} + \nu^R_{\alpha} - \nu^R_{\beta}}{2} \leq 0.
\]

Case 1. Let \( S(\alpha) - S(\beta) < 0 \). Then, there holds \( S(\alpha) < S(\beta) \). By Definition 3.2, we have that \( \alpha \leq_{\text{hxx}} \beta \).

Case 2. Let \( S(\alpha) - S(\beta) = 0 \). Then, there holds \( S(\alpha) = S(\beta) \). In this case,
\[
S(\alpha) = \frac{\mu^L_{\alpha} + \mu^R_{\alpha}}{2} - \frac{\nu^L_{\alpha} + \nu^R_{\alpha}}{2} = \frac{\mu^L_{\beta} + \mu^R_{\beta}}{2} - \frac{\nu^L_{\beta} + \nu^R_{\beta}}{2} = S(\beta).
\]
This implies that
\[
\nu^L_{\alpha} + \nu^R_{\alpha} - \nu^L_{\beta} - \nu^R_{\beta} = \mu^L_{\alpha} + \mu^R_{\alpha} - \mu^L_{\beta} - \mu^R_{\beta}.
\]
Moreover, we have that
\[
H(\alpha) - H(\beta) = \left( \frac{\mu^L_{\alpha} + \mu^R_{\alpha} + \nu^L_{\alpha} + \nu^R_{\alpha}}{2} \right) - \left( \frac{\mu^L_{\beta} + \mu^R_{\beta} + \nu^L_{\beta} + \nu^R_{\beta}}{2} \right) \\
= \frac{\mu^L_{\alpha} + \mu^R_{\alpha} - \mu^L_{\beta} - \mu^R_{\beta} + \nu^L_{\beta} + \nu^R_{\beta} - \nu^L_{\alpha} - \nu^R_{\alpha}}{2}.
\]
By the formulas (3.5) and (3.6),
\[
H(\alpha) - H(\beta) = \mu^L_{\alpha} + \mu^R_{\alpha} - \mu^L_{\beta} - \mu^R_{\beta},
\]
which implies that \( H(\alpha) - H(\beta) \leq 0 \).

Case 2.1. Let \( H(\alpha) - H(\beta) < 0 \). Then, there holds \( H(\alpha) < H(\beta) \). By Definition 3.2, we have that \( \alpha \leq_{\text{hxx}} \beta \).

Case 2.2. Let \( H(\alpha) - H(\beta) = 0 \). By the formula (3.7),
\[
\nu^L_{\alpha} + \nu^R_{\alpha} = \nu^L_{\beta} + \nu^R_{\beta}.
\]
This, together with the formula (3.5), implies that
\[
\nu^L_{\alpha} + \nu^R_{\alpha} = \nu^L_{\beta} + \nu^R_{\beta}.
\]
By the formulas (3.8) and (3.9), respectively, we obtain that \( \mu^L_{\alpha} - \mu^L_{\beta} = \mu^R_{\alpha} - \mu^R_{\beta} \) and \( \nu^L_{\alpha} - \nu^L_{\beta} = \nu^R_{\alpha} - \nu^R_{\beta} \).

Since \( \mu^L_{\alpha} - \mu^L_{\beta} \leq 0 \) and \( \mu^R_{\alpha} - \mu^R_{\beta} \geq 0 \), then \( \mu^L_{\alpha} - \mu^L_{\beta} = \mu^R_{\alpha} - \mu^R_{\beta} = 0 \). That is, \( \mu^L_{\alpha} = \mu^L_{\beta} \) and \( \mu^R_{\alpha} = \mu^R_{\beta} \).

Similarly, we have that \( \nu^L_{\alpha} = \nu^L_{\beta} \) and \( \nu^R_{\alpha} = \nu^R_{\beta} \). Hence, \( \alpha = \beta \).

Therefore, we conclude that \( \alpha \leq_{\text{hxx}} \beta \). \( \square \)

Remark 3. Since \((\tilde{\Theta}, \leq_{\text{hxx}})\) is a complete lattice, we can establish the decomposition theorem and Zadeh’s extension principle for IVIFSs as follows: for an IVIFS \( A \) defined on the universe of discourse \( X \),

(Decomposition Theorem) For every \( x \in X \),
\[
A(x) = \vee \{\alpha \in \tilde{\Theta} \mid x \in A_{\alpha}\},
\]
where \( A_{\alpha} = \{z \in X \mid A(z) \geq_{\text{hxx}} \alpha\} \), \( \vee \) is the supremum under the linear order \( \leq_{\text{hxx}} \).

(Zadeh’s Extension Principle) Let \( X \) and \( Y \) be two nonempty sets and \( f : X \to Y \) be a mapping from \( X \) to \( Y \). Define a mapping \( \tilde{f} : \tilde{\Theta}^X \to \tilde{\Theta}^Y \) by
\[
\tilde{f} : \tilde{\Theta}^X \to \tilde{\Theta}^Y \\
\tilde{f}(A)(y) = \begin{cases} 
\langle [0, 0]; [1, 1] \rangle, & f^{-1}(\{y\}) = \emptyset, \\
\bigvee_{x \in f^{-1}(\{y\})} A(x), & f^{-1}(\{y\}) \neq \emptyset,
\end{cases}
\]
which is called the Zadeh’s extension mapping of \( f \) in the sense of IFSs.
4. Algebraic structures of \((\tilde{\Theta}, \leq_{WLW})\)

Wang et al. [22] introduced two additional functions to investigate the difference between two IVIFNs. In particular, they presented the membership uncertainty index and the hesitation uncertainty index as detailed below.

**Definition 4.1** ([22, Definition 3.3 and 3.4]). Let \(\alpha = ([\mu_\alpha^R, \mu_\alpha^L], [\nu_\alpha^R, \nu_\alpha^L])\) be an IVIFN. Define the membership uncertainty index \(T(\alpha)\) and the hesitation uncertainty index \(G(\alpha)\) of \(\alpha\) as

\[T(\alpha) = (\mu_\alpha^R - \mu_\alpha^L) - (\nu_\alpha^R - \nu_\alpha^L),\]

and

\[G(\alpha) = (\mu_\alpha^R - \mu_\alpha^L) + (\nu_\alpha^R - \nu_\alpha^L),\]

respectively.

By taking a prioritized sequence of score, accuracy, membership uncertainty index, and hesitation uncertainty index functions, the following procedure to compare any two IVIFNs was introduced by Wang et al. [22]. This prioritized sequence of the comparison method serves several application fields in reality. For example, many Canadian research-intensive institutions recruit their tenure-track faculty members following a priority order of research first, teaching second, and service last.

**Definition 4.2** ([22, Definition 3.5]). Let \(\alpha_1\) and \(\alpha_2\) be two IVIFNs. Then, it gets the following ranking principle:

1. If \(S(\alpha_1) < S(\alpha_2)\), then \(\alpha_1\) is smaller than \(\alpha_2\), denoted by \(\alpha_1 <_{WLW} \alpha_2\);
2. If \(S(\alpha_1) = S(\alpha_2)\), then
   - \(H(\alpha_1) < H(\alpha_2)\), then \(\alpha_1 <_{WLW} \alpha_2\);
   - \(H(\alpha_1) = H(\alpha_2)\), then
     - \(T(\alpha_1) < T(\alpha_2)\), then \(\alpha_1 <_{WLW} \alpha_2\);
     - \(T(\alpha_1) = T(\alpha_2)\), then
       - \(G(\alpha_1) < G(\alpha_2)\), then \(\alpha_1 <_{WLW} \alpha_2\);
       - \(G(\alpha_1) = G(\alpha_2)\), then \(\alpha_1 = \alpha_2\).

If \(\alpha_1 <_{WLW} \alpha_2\) or \(\alpha_1 = \alpha_2\), we will denote it by \(\alpha_1 \leq_{WLW} \alpha_2\).

**Remark 4.** It is easy to see that the relation \(\leq_{WLW}\) introduced in Definition 4.2 is transitive on IVIFNs. Then, by [22, Remark 3.1 and Theorem 3.1], we get that it is a total order on IVIFNs.

In the following Theorem 4.1, by applying score, accuracy, membership uncertainty index, and hesitation uncertainty index functions, we observe that IVIFNs in conjunction with the order \(\leq_{WLW}\) are complete chains.

**Theorem 4.1.** \((\tilde{\Theta}, \leq_{WLW})\) is a complete chain.

**Proof.** It is similar to that of Theorem 3.2. 

5. Concluding remarks

Methodologies that rank any two IVIFNs have been studied by many researchers, such as Xu [23], Ye [25], Nayagam et al. [17], Sahin [19], Zhang and Xu [30] and Nayagam et al. [16]. Following this purpose, they have introduced various novel accuracy functions. Nevertheless, their methodologies sometimes cannot assert the difference between two IVIFNs. Then, Wang et al. [22] and Huang et al. [13] have investigated the difference between two IVIFNs by introducing particular additional functions. Furthermore, they have proposed complete ranking methods for IVIFNs. The main contributions of this study are as follows: having regard to a score function and three kinds of entropy functions, we have shown that IVIFNs with the order in the comparison approach for IVIFNs introduced by [13] are complete chains. Furthermore, we have observed that IVIFNs with the order in the method for comparing IVIFNs introduced by [22] are complete chains by applying score, accuracy, membership uncertainty index, and hesitation uncertainty index functions.
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