**Normalized ground state solutions for the fractional Sobolev critical NLSE with an extra mass supercritical nonlinearity**

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**Abstract**

This paper is concerned with existence of normalized ground state solutions for the mass supercritical fractional nonlinear Schrödinger equation involving a critical growth in the fractional Sobolev sense. The compactness of Palais-Smale sequences is obtained by a special technique, which borrows from the ideas of Soave (J. Funct. Anal. 279 (6) (2020), art. 1086102020). This paper represents an extension of previously known results, both in the local and the nonlocal cases.

**Keywords:** Normalized solutions; fractional Schrödinger equation; mass supercritical; Sobolev critical.

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**1 Introduction**

This paper is devoted to the following fractional Sobolev critical nonlinear Schrödinger equation (NLSE) in $\mathbb{R}^N$ ($N \geq 2$)

$$
\begin{cases}
(-\Delta)^s u = \mu u + |u|^{2^*_s - 2} u + \eta |u|^{p-2} u, \\
\|u\|_{L^2}^2 = m^2,
\end{cases}
$$

where $s \in (0, 1)$, $\mu \in \mathbb{R}$ is an unknown real number which will appear as a Lagrange multiplier, $2^*_s$ is the fractional Sobolev critical exponent, $\eta > 0$, $p \in (2 + \frac{4}{N}, 2^*_s)$, $m > 0$ is a finite parameter, and $(-\Delta)^s$ is the fractional Laplace operator defined by

$$
(-\Delta)^s u(x) = C(N, s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,
$$

where $C(N, s)$ is the dimensional constant which depends on $N$ and $s$ (for more details we refer the interested reader to Di Nezza et al. [6]).

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The fractional Schrödinger equation originated from Laskin’s work [10], and in recent years, the interest in its study has grown considerably. It is worthwhile and very interesting to look for normalized solutions to such equations which have a prescribed $L^2$-norm, since they represent the physical view of the conservation of mass.

Existence and properties of normalized solutions for certain problems strongly depend on the behavior of combined nonlinearity $g(u) = |u|^{q-2}u + \eta|u|^{p-2}u$, where $2 < p < q < 2^*_s$. Zhang et al. [13] investigated a class of Sobolev subcritical fractional NLSEs where the parameters $p$ and $q$ are in different order, and they obtained several interesting results concerning existence of normalized solutions. In particular, when the degree of nonlinearity $g(u)$ exceeds the mass critical index $2 + 4s/N$, the functional turns out to be unbounded from below, which makes it impossible to adopt the direct variational method.

Secchi and Appolloni [2] studied existence and multiplicity of ground state normalized solutions for the fractional mass supercritical NLSEs by using the min-max theory under more general assumptions, but in the Sobolev subcritical sense. Under certain conditions on the potential, Peng et al. [14] showed that NLSE has at least one normalized solution, with the help of a new min-max argument and the splitting lemma for nonlocal version also in the case when the mass is supercritical and Sobolev subcritical.

However, to the best of our knowledge, there are very few papers on normalized solutions of fractional NLSEs. Moreover, they only consider the Sobolev subcritical case. Therefore it is natural to inquire what difficulties appear when a fractional Sobolev critical nonlinearity is considered. For example, Zhen and Zhang [18] investigated a critical fractional NLSE with the $L^2$-supercritical perturbation, but their coefficient of perturbation was not allowed to be large enough, while in the present paper, $\eta$ can be large enough since $\eta \in [\eta^*, +\infty)$. Very recently, almost at the same time as our study, Li and Zou [11] considered the same problem using the concentration compactness principle for overcoming the lack of compactness. With the above methods and techniques, Zuo and Rădulescu [20] investigated the existence and nonexistence of normalized solutions for a class of fractional mass supercritical nonlinear Schrödinger coupled systems with Sobolev critical nonlinearities. In response to this difficulty, we consider a technical analysis method combined with the Brézis-Lieb lemma, which comes from the ideas of Soave [16].

Of course, in the case when $s \to 1$, the fractional Laplacian $(-\Delta)^s$ reduces to the classical Laplace operator $-\Delta$, the literature on the relevant problem (1.1) is very large. Here we shall only mention some key papers which are relevant to our study. Brézis and Nirenberg [4] presented a pioneering work. Later, many made important progress in this field. For $L^2$-supercritical perturbation $\eta|u|^{p-2}u$, Soave [16] made the first contribution concerning existence of normalized solutions for NLSEs in the Sobolev critical case. Next, Alves et al. [1] obtained a similar result for this kind of NLSEs when dimension $N \geq 5$, and $\eta$ large enough. In particular, under weaker, more general conditions, Jeanjean and Lu [9] proved existence of ground states and asymptotic behavior of ground state energy with mass change. They also obtained infinitely many radial solutions when $N \geq 2$, and established existence and multiplicity of nonradial sign-changing solutions for every $N \geq 4$.

Inspired by the work mentioned above, we consider in this paper the problem of existence of ground state normalized solutions of fractional Sobolev critical NLSEs with a mass supercritical nonlinearity. The compactness can be restored combining some of the main ideas of Brézis and Nirenberg [4] and Jeanjean [8]. In order to introduce the main result of this paper, we first define fractional Sobolev space:

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \mid \|u\|_{H^s}^2 = \int_\mathbb{R}^{2N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < \infty \right\},$$

where $s > 0$ and $N$ is the dimension of the underlying space.
in which the norm is defined by

\[ \|u\| = (\|u\|_{L^2}^2 + [u]_{H^s}^2)^{1/2}. \]

For convenience, we shall simply denote the norm of Lebesgue space \( L^p(\mathbb{R}^N) \) by \( \|u\|_p \) for \( p \in [1, \infty) \).

A standard method for investigating problem (1.1) is to find critical points of the following energy functional

\[
I_\eta(u) = \frac{1}{2} [u]_{H^s}^2 - \frac{1}{2s} \|u\|_{2s}^{2s} - \frac{\eta}{p} \|u\|_p^p,
\]

constrained to the set

\[
S(m) = \{ u \in H^s(\mathbb{R}^N) | \|u\|_2^2 = m^2 \}.
\]

Obviously, \( I_\eta \) is of class \( C^1 \) in \( H^s(\mathbb{R}^N) \).

We can now state our main result.

**Theorem 1.1.** Assume that \( N \geq 2 \) and \( p \in (2 + \frac{4s}{N}, 2^*_s) \). Then for every \( m > 0 \), there exists \( \eta^* = \eta^*(m) > 0 \) such that for every \( \eta \geq \eta^* \), problem (1.1) admits a radial normalized solution \( \tilde{u} \), whose associated Lagrange multiplier \( \mu \) is negative.

**Remark 1.1.** Our conclusion can be regarded as an extension of Alves et al. [1, Theorem 1.1].

**Remark 1.2.** According to Theorem 1.1 and Zhen and Zhang [18, Theorem 1.3(1)-(2)], we know that existence of radial normalized ground state is possible when \( \eta \) is small enough or large enough, however it remains an open problem for the rest of the range of \( \eta \).

**Remark 1.3.** If \( N = 4s \) or \( \frac{p}{p-1}2s < N < 4s \) or \( 2s < N < \frac{p}{p-1}2s \), we can get our conclusion without any restriction on \( \eta \), see Zhen and Zhang [18, Theorem 1.3(3)]. If we further assume that \( N^2 > 8s^2 \), then we can also arrive at a similar conclusion, but in this case we consider the mass \( m \) as the parameter instead of \( \eta \), see Zhang and Han [17, Theorem 1.3].

The paper is organized as follows. Section 2 contains the proofs of some important lemmas, which play a key role in the proof of the compactness condition. In Section 3, we prove the strong convergence of the Palais-Smale sequence at some level set, using a special technique. In Section 4, we prove Theorem 1.1.

### 2 Main lemmas

Although the study of normalized solutions is convenient for applications, it also presents some difficulties. For example, the Nehari manifold method cannot be used because the constant \( \mu \) is unknown. This also makes it difficult to verify the boundedness of Palais-Smale sequences when employing some common methods.

To this end, following Soave [16], let

\[
\zeta_p = (Np - 2N)/2ps, \quad \text{for every } p \in (2, 2^*_s],
\]

it is easy to see that \( \zeta_p \in (0, 1] \) and define the Pohozaev manifold

\[
\mathcal{P}_{\eta,m} = \{ u \in S(m) | P_\eta(u) = 0 \}, \quad \text{where } P_\eta(u) = [u]_{H^s}^2 - \|u\|_{2s}^{2s} - \eta \zeta_p \|u\|_p^p,
\]

where the definition of \( \zeta_p \) is related to (2.4). It is well-known that any critical point of \( I_\eta|_{S(m)} \) stays in \( \mathcal{P}_{\eta,m} \), as a consequence of Zhen and Zhang [18, Proposition 2.1 and Remark 2.1].
In order to get the mountain pass geometry, we are going to make the scaling transformation. For \( u \in S(m) \) and \( \xi \in \mathbb{R} \), we let
\[
(\xi \ast u)(x) = e^{\frac{2N}{s}} u(e^\xi x) = v(x), \quad \text{for a.e. } x \in \mathbb{R}^N,
\]
which is based on a very interesting idea from Jeanjean [8]. A careful analysis shows that the transformed functional \( I_\eta = I_\eta(\xi \ast u) \) has the same mountain pass geometry and mountain pass level as the original functional \( I_\eta(u) \).

For reader’s convenience, we give the proof of the following lemma, which can also be found in Li and Zou [11].

**Lemma 2.1.** Assume that \( u \in S(m) \) is arbitrary but fixed. Then we have:

1. \( [\xi \ast u]_H^2 \rightarrow 0 \) and \( I_\eta(\xi \ast u) \rightarrow 0 \), as \( \xi \rightarrow -\infty \);
2. \( [\xi \ast u]_H^2 \rightarrow +\infty \) and \( I_\eta(\xi \ast u) \rightarrow -\infty \), as \( \xi \rightarrow +\infty \).

**Proof.** By a direct calculation, we get
\[
[E \ast u]_H^2 = e^{2N} \int \int \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx 
\]
\[
\|E \ast u\|^\beta_B = e^{\frac{(\beta-2)N\xi}{2}} \|u\|^\beta_B \quad \text{for every } \beta \geq 2.
\]

On the basis of (2.2), we have
\[
[E \ast u]_H^2 \rightarrow 0, \quad \text{as } \xi \rightarrow -\infty,
\]
\[
I_\eta(\xi \ast u) = \frac{1}{2} [E \ast u]_H^2 - \frac{1}{2s} \|E \ast u\|^2_{2s} - \frac{\eta}{p} \|E \ast u\|^p_p \rightarrow 0, \quad \text{as } \xi \rightarrow -\infty,
\]
thereby demonstrating (1).

On the other hand, it follows from (2.2) that \( [\xi \ast u]_H^2 \rightarrow +\infty \), as \( \xi \rightarrow +\infty \). Besides,
\[
I_\eta(\xi \ast u) = \frac{1}{2} e^{2N} [u]_H^2 - \frac{1}{2s} e^{\frac{(2s-2)N\xi}{2}} \|u\|^2_{2s} - \frac{\eta}{p} e^{\frac{(p-2)N\xi}{2}} \|u\|^p_p \rightarrow -\infty, \quad \text{as } \xi \rightarrow +\infty,
\]
since \( p \in (2 + \frac{4s}{N}, 2s) \), which in turn, demonstrates (2). This completes the proof of Lemma 2.1.

The following two inequalities (fractional Sobolev inequality (2.3) and fractional Gagliardo-Nirenberg inequality (2.4)) play an important role in our proof of the main result in Section 4.

Thanks to Servadei and Valdinoci [15], there exists a optimal fractional critical Sobolev constant \( S > 0 \) such that
\[
S \|u\|^2_2 \leq [u]_H^2, \quad \text{for every } u \in H^s(\mathbb{R}^N).
\]

Also, according to Frank et al. [7], there exists an optimal constant \( C(N, p, s) \) such that for every \( p \in (2, 2s) \),
\[
\|u\|^p_p \leq C^p(N, p, s) [u]_H^{p\zeta_p} \|u\|_{2s}^{p(1-\zeta_p)}, \quad \text{for every } u \in H^s(\mathbb{R}^N),
\]
where \( \zeta_p \) is given by (2.1).

In the following lemma, we give a specific value of \( \rho(m, \eta) \) and analyze the asymptotic behavior of \( \rho(m, \eta) \) when \( \eta \) is large enough, which is more detailed than in Luo and Zhang [13, Lemma 5.2].

Let
\[
S_r(m) = S(m) \cap H^s_{rad}(\mathbb{R}^N) = \{ u \in S(m) : u(x) = u(|x|) \}.
\]
Lemma 2.2. There exists a small enough $\rho(m, \eta) > 0$ such that

$$0 < \inf_{u \in X} I_\eta(u) \leq \sup_{u \in X} I_\eta(u) < \inf_{u \in Y} I_\eta(u),$$

with

$$X = \{ u \in S_r(m), [u]_{H^s}^2 \leq \rho(m, \eta) \}, \ Y = \{ u \in S_r(m), [u]_{H^s}^2 = 2\rho(m, \eta) \}.$$ 

Moreover, $\rho(m, \eta) \to 0$, as $\eta \to \infty$.

Proof. In view of (2.3) and (2.4), we get

$$\frac{1}{2^s} \|v\|_{2^s}^2 + \frac{\eta}{p} \|v\|_p^p \leq \frac{1}{2^s} \left( [v]_{H^s}^2 \right)^{2^s} + \left( \frac{\eta C_p(N, p, s)}{p} \right) \left( [v]_{H^s}^2 \right)^{\frac{2^s}{2^s}} \frac{N_p - 2N}{4s} m^{\frac{2p - N_p + 2N}{2s}}. \quad (2.5)$$

Then for every $u \in S_r(m)$, fixing $[u]_{H^s}^2 \leq \rho(m, \eta)$ and $[v]_{H^s}^2 = 2\rho(m, \eta)$, where $\rho(m, \eta)$ is a positive number which depends on $m$ and $\eta$, one has

$$I_\eta(v) - I_\eta(u) \geq \frac{1}{2} \left( [v]_{H^s}^2 - \frac{1}{2^s} \|v\|_{2^s}^2 - \frac{\eta}{p} \|v\|_p^p \right) \geq \frac{1}{2} \rho(m, \eta) - \frac{2^s}{2^s} \rho(m, \eta)^{2^s} \left( \frac{\eta C_p(N, p, s)}{p} \right) \frac{N_p - 2N}{4s} m^{\frac{2p - N_p + 2N}{2s}}.$$

Thus, choosing

$$\rho(m, \eta) = \min \left\{ \left( \frac{2^s}{8} \frac{2^s}{8} \frac{2^s}{8} \right) \left( \frac{2^s}{8} \frac{2^s}{8} \frac{2^s}{8} \right) \right\}, \quad (2.6)$$

we can infer that

$$\frac{1}{2} \rho(m, \eta) - \frac{2^s}{2^s} \rho(m, \eta)^2 - \frac{\eta C_p(N, p, s)}{p} \frac{N_p - 2N}{4s} m^{\frac{2p - N_p + 2N}{2s}} > 0.$$

Now, by (2.5) and the definition of $\rho(m, \eta)$ in (2.6), we get

$$I_\eta(u) \geq \frac{1}{2} [u]_{H^s}^2 - \frac{2^s}{2^s} [u]_{H^s}^2 - \frac{\eta C_p(N, p, s)}{p} \frac{N_p - 2N}{4s} m^{\frac{2p - N_p + 2N}{2s}} [u]_{H^s}^2 > 0,$$

which means that the inequality in Lemma 2.2 holds. Finally, $\lim_{\eta \to \infty} \rho(m, \eta) = 0$ also follows from (2.6). This completes the proof of Lemma 2.2.

Next, fix $u_0 \in S_r(m)$. It follows from Lemma 2.1 and Lemma 2.2 that there exist numbers $\xi_1 = \xi_1(m, \eta, u_0) < 0$ and $\xi_2 = \xi_2(m, \eta, u_0) > 0$ such that the functions $u_{1, \eta} = \xi_1 \ast u_0$, $u_{2, \eta} = \xi_2 \ast u_0$ satisfy

$$[u_{1, \eta}]_{H^s}^2 < \frac{\rho(m, \eta)}{2}, \ [u_{2, \eta}]_{H^s}^2 > 2\rho(m, \eta), \ I_\eta(u_{1, \eta}) > 0, \ \text{and} \ I_\eta(u_{2, \eta}) < 0.$$
Now, similarly to the discussion in Jeanjean [8] or Luo and Zhang [13, Proposition 5.3], we fix the minimax

\[ E_\eta(m) = \inf_{\psi \in \Gamma} \max_{t \in [0,1]} I_\eta(\psi(t)), \text{ where } \Gamma = \{ \psi \in C([0,1], S_r(m)) : [\psi(0)]^2_{H^s} < \rho(m, \eta)/2, I_\eta(\psi(1)) < 0 \}. \]

By virtue of Lemma 2.2, we know that

\[ [\psi(1)]^2_{H^s} > \rho(m, \eta), \text{ for every } \psi \in \Gamma. \]

Therefore there exists \( t_0 \in (0,1) \) such that

\[ [\psi(t_0)]^2_{H^s} = \rho(m, \eta)/2 \text{ and } \max_{t \in [0,1]} I_\eta(\psi(t)) \geq I_\eta(\psi(t_0)) \geq \inf_{u \in X} I_\eta(u) > 0, \]

therefore \( E_\eta(m) > 0 \).

The following lemma is a key step to analyze the level value of mountain pass, so we present a more detailed calculation process (in comparison with Li and Zou [11]).

**Lemma 2.3.** \( \lim_{\eta \to \infty} E_\eta(m) = 0. \)

**Proof.** Fix \( u_0 \in S_r(m) \), and consider the path \( \psi_0(t) = [(1-t)\xi_1 + t\xi_2] \ast u_0 \in \Gamma \). We have

\[
E_\eta(m) \leq \max_{t \in [0,1]} I_\eta(\psi_0(t)) \leq \max_{r \geq 0} \left\{ \frac{1}{2} r^2 [u_0]^2_{H^s} - \frac{\eta}{p} r^{\frac{Np-2N}{2s}} \|u_0\|_p^p \right\}.
\]

Thus, setting \( C_1 = [u_0]^2_{H^s} \) and \( C_2 = \|u_0\|_p^p \), we consider the maximum value of the following function

\[
f(r) = \frac{1}{2} C_1 r^2 - \frac{\eta}{p} C_2 r^{\frac{Np-2N-2s}{2s}}, \text{ for any } r \geq 0.
\]

Letting

\[
f'(r) = C_1 r - \left( \frac{Np-2N}{2s} \right) \frac{\eta}{p} C_2 r^{\frac{Np-2N-2s}{2s}} = 0,
\]

we get the maximum of \( f(r) \) at

\[
r_{\text{max}} = \left( \frac{2spC_1}{(Np-2N)\eta C_2} \right)^{\frac{2s}{Np-2N-4s}}.
\]

Hence,

\[
\max_{r \geq 0} \left\{ \frac{1}{2} r^2 [u_0]^2_{H^s} - \frac{\eta}{p} r^{\frac{Np-2N}{2s}} \|u_0\|_p^p \right\} = \frac{1}{2} \left( \frac{2spC_1}{(Np-2N)\eta C_2} \right)^{\frac{4s}{Np-2N-4s}} C_1 - \frac{\eta}{p} \left( \frac{2spC_1}{(Np-2N)\eta C_2} \right)^{\frac{Np-2N}{Np-2N-4s}} C_2
\]

\[
\leq \frac{1}{2} \left( \frac{2spC_1}{(Np-2N)\eta C_2} \right)^{\frac{4s}{Np-2N-4s}} C_1,
\]

so there exists \( C > 0 \), which is not dependent on \( \eta > 0 \), such that

\[ E_\eta(m) \leq C \left( \frac{1}{\eta} \right)^{\frac{4s}{Np-2N-4s}} \to 0, \text{ as } \eta \to \infty, \]

since \( p > 2 + 4s/N \). This completes the proof of Lemma 2.3. \( \square \)
Similarly to Luo and Zhang [13, Propositions 5.3-5.4], for \( \{ \xi_n \} \subset \mathbb{R} \) we know that \( I_\eta(u_n) \) and \( I_\eta(\xi_n \ast u_n) \) have the same mountain pass level value. Moreover, there is a certain relationship between their Palais-Smale sequences.

**Lemma 2.4.** Let \( \{ \xi_n \ast u_n \} \subset S_r(m) \) be a Palais-Smale sequence for \( I_\eta \) at level \( E_\eta(m) \), i.e.,

\[
I_\eta(\xi_n \ast u_n) \to E_\eta(m) > 0 \quad \text{and} \quad I_\eta'(\xi_n \ast u_n) \to 0, \quad \text{as} \quad n \to \infty.
\]

Then \( \lim_{n \to \infty} P_\eta(\xi_n \ast u_n) = 0. \)

**Proof.** We first have

\[
I_\eta(\xi_n \ast u_n) = \frac{1}{2} \| \xi_n \ast u_n \|^2_{H^s} - \frac{1}{2s} \| \xi_n \ast u_n \|^2_{2^*_s} - \frac{\eta}{p} \| \xi_n \ast u_n \|^p
\]

\[
= \frac{1}{2} e^{2\xi_n s}[u_n]^2_{H^s} - \frac{1}{2s} e^{\frac{(2^*_s - 2)N \xi_n}{2}} \| u_n \|^2_{2^*_s} - \frac{\eta}{p} e^{\frac{(p-2)N \xi_n}{2}} \| u_n \|^p,
\]

and \( I_\eta(\xi_n \ast u_n) \) is \( C^1 \) with respect to \( \xi_n \). Now, by taking the derivative

\[
\frac{\partial}{\partial \xi_n} I_\eta(\xi_n \ast u_n) = 2se^{2\xi_n s}[u_n]^2_{H^s} - se^{\frac{2^*_s - 2)N \xi_n}{2}} \| u_n \|^2_{2^*_s} - s\eta \xi p e^{\frac{(p-2)N \xi_n}{2}} \| u_n \|^p,
\]

noticing that

\[
P_\eta(\xi_n \ast u_n) = 2e^{2\xi_n s}[u_n]^2_{H^s} - e^{\frac{(2^*_s - 2)N \xi_n}{2}} \| u_n \|^2_{2^*_s} - \eta \xi p e^{\frac{(p-2)N \xi_n}{2}} \| u_n \|^p,
\]

we can infer that

\[
\frac{\partial}{\partial \xi_n} I_\eta(\xi_n \ast u_n) = sP_\eta(\xi_n \ast u_n).
\]

Thus, the conclusion of Lemma 2.4 is a consequence of the following limit

\[
\lim_{n \to \infty} \frac{\partial}{\partial \xi_n} I_\eta(\xi_n \ast u_n) = 0,
\]

since \( I_\eta'(\xi_n \ast u_n) \to 0 \), as \( n \to \infty \). This completes the proof of Lemma 2.4.

**Lemma 2.5.** Let \( \{ u_n \} \subset S_r(m) \) be a Palais-Smale sequence for \( I_\eta \) with the level \( E_\eta(m) \). If \( \lim_{n \to \infty} P_\eta(u_n) = 0 \), then \( \{ u_n \} \) is bounded in \( S_r(m) \).

**Proof.** We note that \( \xi_p p > 2 \), since \( p > 2 + 4s/N \). It follows from \( \lim_{n \to \infty} P_\eta(u_n) = 0 \) that

\[
I_\eta(u_n) = \frac{\eta}{2p} (\xi_p p - 2) \| u_n \|^p + \frac{s}{N} \| u_n \|^2_{2^*_s} + o(1),
\]

and from the boundedness of \( I_\eta(u_n) \), that \( \| u_n \|^p \) and \( \| u_n \|^2_{2^*_s} \) are both bounded, therefore \( \{ u_n \}^2_{H^s} \) is bounded. This completes the proof of Lemma 2.5.
3 Compactness condition

In this section we give a very important proof of the compactness conditions, inspired by the ideas of Soave [16].

Proposition 3.1. Let \( \{u_n\} \subset S_r(m) \) be a Palais-Smale sequence for \( I_\eta \) with the level

\[
0 < E_\eta(m) < \frac{sS^N}{N},
\]

where \( S \) is the best fractional Sobolev constant defined in (2.3). If \( \lim_{n \to \infty} P_\eta(u_n) = 0 \), then one of the following properties holds:

1. either up to subsequence, \( u_n \to \tilde{u} \) converges weakly in \( H^s(\mathbb{R}^N) \) but not strongly, where \( \tilde{u} \neq 0 \) is a solution of the first equation of (1.1) for some \( \mu < 0 \), and \( I_\eta(\tilde{u}) < E_\eta(m) - \frac{sS^N}{N} \);

2. or up to subsequence, \( u_n \to \tilde{u} \) converges strongly in \( H^s(\mathbb{R}^N) \), \( I_\eta(\tilde{u}) = E_\eta(m) \), and \( \tilde{u} \) is a solution of (1.1) for some \( \mu < 0 \).

Proof. In general, the embedding \( H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \) is not compact for any \( p \in (2, 2^*_s) \), so we need to restore compactness in the radial function space. According to Lemma 2.5, we know that sequence \( \{u_n\} \) is bounded and the embedding \( H^s_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \) is compact for every \( p \in (2, 2^*_s) \) (see Lions [12, Proposition I.1]). Therefore there exists \( \tilde{u} \in H^s_{rad}(\mathbb{R}^N) \) such that up to a subsequence, \( u_n \to \tilde{u} \) converges weakly in \( H^s(\mathbb{R}^N) \), \( u_n \to \tilde{u} \) converges strongly in \( L^p(\mathbb{R}^N) \), and a.e. in \( \mathbb{R}^N \). Since \( \{u_n\} \) is a Palais-Smale sequence for \( I_\eta |_{S(m)} \), by the Lagrange multipliers rule there exists \( \{\mu_n\} \subset \mathbb{R} \) such that for every \( \phi \in H^s(\mathbb{R}^N) \),

\[
\int \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))((\phi(x) - \phi(y)))}{|x - y|^{N+2s}}dxdy - \int_{\mathbb{R}^N} (\mu_n u_n \phi + |u_n|^{2^*_s - 2} u_n \phi + \eta |u_n|^{p-2} u_n \phi)dx = o(1)||\phi||,
\]

as \( n \to \infty \). Setting \( \phi = u_n \), we can infer that \( \{\mu_n\} \) is also bounded, and therefore up to a subsequence, \( \mu_n \to \mu \in \mathbb{R}^N \). By invoking \( \lim_{n \to \infty} P_\eta(u_n) = 0 \), the compactness of the embedding \( H^s_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \), and \( \zeta_p < 1 \), we get

\[
\mu m^2 = \lim_{n \to \infty} \mu_n ||u_n||^2_{H^s} = \lim_{n \to \infty} \left( [u_n]_{H^s}^2 - ||u_n||^2_{2^*_s} + \eta ||u_n||_p^p \right) = \lim_{\eta \to \infty} \eta (\zeta_p - 1) ||u_n||_p^p = \eta (\zeta_p - 1) \tilde{u}^p \leq 0,
\]

where \( \mu = 0 \) if and only if \( \tilde{u} \equiv 0 \). We shall now show that

\[
\tilde{u} \not\equiv 0. \tag{3.3}
\]

Suppose to the contrary, that \( \tilde{u} \equiv 0 \). Since \( \{u_n\} \) is bounded in \( H^s(\mathbb{R}^N) \), it follows that up to a subsequence, \( [u_n]_{H^s}^2 \to \gamma \in \mathbb{R} \). Since \( P_\eta(u_n) \to 0 \) and \( u_n \to 0 \) converges strongly in \( L^p(\mathbb{R}^N) \), it follows that

\[
||u_n||^2_{2^*_s} = [u_n]_{H^s}^2 - \eta \zeta_p ||u_n||_p^p \to \gamma,
\]

\[
\int \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))((\phi(x) - \phi(y)))}{|x - y|^{N+2s}}dxdy - \int_{\mathbb{R}^N} (\mu_n u_n \phi + |u_n|^{2^*_s - 2} u_n \phi + \eta |u_n|^{p-2} u_n \phi)dx = o(1)||\phi||,
\]
therefore by (2.3), $\gamma \geq S\frac{\gamma}{\bar{p}}$. Furthermore, we can infer that

either $\gamma = 0$ or $\gamma > S\frac{\gamma}{\bar{p}}$.

If $\gamma > S\frac{\gamma}{\bar{p}}$, then due to $I_\eta(u_n) \to E_\eta(m)$ and $\lim_{n \to \infty} P_\eta(u_n) = 0$, we get

$$E_\eta(m) + o(1) = I_\eta(u_n) = \frac{s}{N}[u_n]^2_{H^s} - \frac{\eta}{p} \left(1 - \frac{\bar{p}p}{2s}\right)\|u_n\|^p_p + o(1)$$

$$= \frac{s}{N}[u_n]^2_{H^s} + \frac{\gamma s}{N} + o(1),$$

so $E_\eta(m) = \frac{\gamma s}{N}$, thereby $E_\eta(m) \geq \frac{s\gamma}{N\bar{p}}$, which contradicts our conditions.

If instead, we have $\gamma = 0$, we note that $[u_n]^2_{H^s} \to 0$, $\|u_n\|_{2s}^{2s} \to 0$ and $\|u_n\|_p^p \to 0$. Therefore $I_\eta(u_n) \to 0$, which is a contradiction as well. So (3.3) is proved. Furthermore, it follows from (3.2) and (3.3) that $\mu < 0$. Invoking the limit weak convergence in (3.1), we get

$$(-\Delta)^s\bar{u} = \mu\bar{u} + |\bar{u}|^{2s-2}\bar{u} + \eta|\bar{u}|^{p-2}\bar{u} \quad \text{in } \mathbb{R}^N,$$

and thus by the Pohozaev identity (see Chang and Wang [5, Proposition 4.1]) and related explanations in Zhen and Zhang [18, Proposition 2.1 and Remark 2.1], we have $P_\eta(\bar{u}) = 0$. We know that $w_n = u_n - \bar{u} \to 0$ in $H^s(\mathbb{R}^N)$, and according to Zuo et al. [19, Lemma 2.4] and the Brézis-Lieb lemma [3], we have

$$\|u_n\|_p^p = [u_n]^2_{H^s} + |u_n|^{2s}_p + \eta|u_n|^{p-2}_p \quad \text{in } \mathbb{R}^N,$$

and thus by the Pohozaev identity (see Chang and Wang [5, Proposition 4.1]) and related explanations in Zhen and Zhang [18, Proposition 2.1 and Remark 2.1], we have $P_\eta(\bar{u}) = 0$. We know that $w_n = u_n - \bar{u} \to 0$ in $H^s(\mathbb{R}^N)$, and according to Zuo et al. [19, Lemma 2.4] and the Brézis-Lieb lemma [3], we have

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and thus by the Pohozaev identity (see Chang and Wang [5, Proposition 4.1]) and related explanations in Zhen and Zhang [18, Proposition 2.1 and Remark 2.1], we have $P_\eta(\bar{u}) = 0$. We know that $w_n = u_n - \bar{u} \to 0$ in $H^s(\mathbb{R}^N)$, and according to Zuo et al. [19, Lemma 2.4] and the Brézis-Lieb lemma [3], we have

$$\|u_n\|_p^p = [u_n]^2_{H^s} + |u_n|^{2s}_p + \eta|u_n|^{p-2}_p \quad \text{in } \mathbb{R}^N,$$

and thus by the Pohozaev identity (see Chang and Wang [5, Proposition 4.1]) and related explanations in Zhen and Zhang [18, Proposition 2.1 and Remark 2.1], we have $P_\eta(\bar{u}) = 0$. We know that $w_n = u_n - \bar{u} \to 0$ in $H^s(\mathbb{R}^N)$, and according to Zuo et al. [19, Lemma 2.4] and the Brézis-Lieb lemma [3], we have

$$\|u_n\|_p^p = [u_n]^2_{H^s} + |u_n|^{2s}_p + \eta|u_n|^{p-2}_p \quad \text{in } \mathbb{R}^N.$$
If instead, we have \( \gamma = 0 \), then we claim that \( u_n \to \bar{u} \) in \( H^s(\mathbb{R}^N) \). Indeed, we have \( \lim_{n \to \infty} [w_n]_{H^s} = 0 \), so it follows from \( w_n = u_n - \bar{u} \) that \( [u_n - \bar{u}]^2_{H^s} \to 0 \).

Next, it suffices to verify that \( u_n \to \bar{u} \) in \( L^2 \). Choosing \( \phi = u_n - \bar{u} \) in (3.1), invoking (3.4) with \( u_n \to \bar{u} \) and subtracting, we get
\[
[u_n - \bar{u}]^2_{H^s} - \int_{\mathbb{R}^N} (\mu_n u_n - \mu \bar{u}) (u_n - \bar{u}) \, dx = \\
\int_{\mathbb{R}^N} |u_n|^{2s-2} u_n - |\bar{u}|^{2s-2} \bar{u} (u_n - \bar{u}) \, dx + \int_{\mathbb{R}^N} (|u_n|^p - 2u_n - |\bar{u}|^{p-2} \bar{u}) (u_n - \bar{u}) \, dx + o(1).
\]

We note that \( \lim_{n \to \infty} \|w_n\|_{2^*_s}^2 = 0 \). It follows from (3.5) that \( \|w_n\|_{2^*_s}^2 \to \|\bar{u}\|_{2^*_s}^2 \), therefore, in the formula above, the first term, the third term, and the fourth term converge to 0. As a result,
\[
0 = \lim_{n \to \infty} \int_{\mathbb{R}^N} (\mu_n u_n - \mu \bar{u}) (u_n - \bar{u}) \, dx = \lim_{n \to \infty} \mu \int_{\mathbb{R}^N} (u_n - \bar{u})^2 \, dx.
\]
Thus also assertion (2) of Proposition 3.1 has been established, i.e., up to subsequence, \( u_n \to \bar{u} \) converges strongly in \( H^s(\mathbb{R}^N) \), \( I_\eta(\bar{u}) = E_\eta(m) \), and \( \bar{u} \) is a solution of (1.1) for some \( \mu < 0 \). The proof of Proposition 3.1 is complete. \( \square \)

4 Proof of Theorem 1.1

Lemma 2.4 and [13, Propositions 5.3-5.4] imply that for a given Palais-Smale sequence \( \{u_n\} \subset S_r(m) \) for \( I_\eta \) with the level \( E_\eta(m) \), if \( \lim_{n \to \infty} P_\eta(\xi_n \ast u_n) = 0 \), then the sequence \( \{\xi_n \ast u_n\} \subset S_r(m) \) is also a Palais-Smale sequence for \( I_\eta \) on the same level, thus we can apply Lemma 2.5. In order to prove our main result, it remains to verify the condition \( E_\eta(m) < \frac{s \mathcal{S}^N}{N} \) of Proposition 3.1, which is a consequence of Lemma 2.3.

Therefore, we know that one of the two conclusions of Proposition 3.1 must hold. We show that conclusion (1) does not hold. Indeed, if it did, then \( \bar{u} \) would be a nontrivial solution of (1.1), i.e., up to a subsequence \( u_n \to \bar{u} \) converges weakly in \( H^s(\mathbb{R}^N) \) but not strongly, where \( \bar{u} \neq 0 \) is a solution of the first equation of (1.1) for some \( \mu < 0 \), and
\[
I_\eta(\bar{u}) < E_\eta(m) - \frac{s \mathcal{S}^N}{N} < 0.
\]
However, since \( P_\eta(\bar{u}) = 0 \) by the Pohozaev identity and \( \zeta_p p > 2 \), we also get
\[
I_\eta(\bar{u}) = \frac{\eta}{2p} (\zeta_p p - 2) \|u_n\|_p^p + \frac{s}{N} \|u_n\|_{2^*_s}^2 > 0,
\]
which is a contradiction.

Therefore, conclusion (2) must hold and \( \bar{u} \) is a radial normalized solution of (1.1) for some \( \mu < 0 \). This completes the proof of Theorem 1.1.
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