ON NEW APPROACH HADAMARD-TYPE INEQUALITIES FOR 
S-GEOMETRICALLY CONVEX FUNCTIONS

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ABSTRACT. In this paper we achieve some new Hadamard type inequalities using elementary well known inequalities for functions whose first derivatives absolute values are s-geometrically and geometrically convex. And also we get some applications for special means for positive numbers.

1. INTRODUCTION

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex mapping defined on the interval \( I \) of real numbers and \( a, b \in I, \) with \( a < b. \) The following double inequalities:

\[
f \left( \frac{a + b}{2} \right) \leq \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

hold. This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions.

In recent years many authors established several inequalities connected to this fact. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard-type inequalities see [1]-[9].

In this section we will present definitions and some results used in this paper.

Definition 1. Let \( I \) be an interval in \( \mathbb{R}. \) Then \( f : I \to \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R} \) is said to be convex if

\[
f (tx + (1 - t)y) \leq tf(x) + (1 - t)f(y),
\]

for all \( x, y \in I \) and \( t \in [0, 1]. \)

Definition 2. [1] Let \( s \in (0, 1]. \) A function \( f : I \subset \mathbb{R}_0 = [0, \infty) \to \mathbb{R}_0 \) is said to be s-convex in the second sense if

\[
f (tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y)
\]

for all \( x, y \in I \) and \( t \in [0, 1]. \)

It can be easily checked for \( s = 1, \) s-convexity reduces to the ordinary convexity of functions defined on \( [0, \infty). \)

Recently, in [3], the concept of geometrically and s-geometrically convex functions was introduced as follows.

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Definition 3. A function \( f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}_+ \) is said to be a geometrically convex function if
\[
(1.3) \quad f(x^t y^{1-t}) \leq [f(x)]^t [f(y)]^{1-t}
\]
for all \( x, y \in I \) and \( t \in [0, 1] \).

Definition 4. A function \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+ \) is said to be a \( s \)-geometrically convex function if
\[
(1.4) \quad f(x^t y^{1-t}) \leq [f(x)]^t [f(y)]^{(1-t)s}
\]
for some \( s \in (0, 1] \), where \( x, y \in I \) and \( t \in [0, 1] \).

If \( s = 1 \), the \( s \)-geometrically convex function becomes a geometrically convex function on \( \mathbb{R}_+ \).

Example 1. Let \( f(x) = x^s/s, x \in (0, 1], 0 < s < 1, q \geq 1 \), and then the function
\[
(1.5) \quad |f'(x)|^q = x^{(s-1)q}
\]
is monotonically decreasing on \((0, 1]\). For \( t \in [0, 1] \), we have
\[
(1.6) \quad (s - 1)q (t^s - t) \leq 0, \quad (s - 1)q ((1-t)^s - (1-t)) \leq 0.
\]
Hence, \( |f'(x)|^q \) is \( s \)-geometrically convex on \((0, 1]\) for \( 0 < s < 1 \).

2. Hadamard's type inequalities

In order to prove our main theorems, we need the following lemma [2].

Lemma 1. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) where \( a, b \in I \) with \( a < b \). If \( f' \in L[a, b] \), then the following equality holds:
\[
(2.1) \quad \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx = \frac{b - a}{4} \left[ \int_0^1 (-t) f' \left( \frac{1 + t}{2} a + \frac{1 - t}{2} b \right) \, dt + \int_0^1 t f' \left( \frac{1 + t}{2} b + \frac{1 - t}{2} a \right) \, dt \right].
\]
A simple proof of this equality can be also done integrating by parts in the right hand side. The details are left to the interested reader.

The next theorems gives a new result of the upper Hermite-Hadamard inequality for \( s \)-geometrically convex functions.

In the following part of the paper;
\[
(2.2) \quad \alpha(u, v) = |f'(a)|^u |f'(b)|^{-v}, \quad u, v \geq 0,
\]
\[
(2.3) \quad g_1(\alpha) = \begin{cases} \frac{1}{\alpha \ln \alpha - \alpha + 1} & \alpha = 1 \\ \alpha & \alpha \neq 1 \end{cases}
\]
and
\[
(2.4) \quad g_2(\alpha) = \begin{cases} \frac{1}{\alpha - 1} & \alpha = 1 \\ \frac{\alpha - 1}{\ln \alpha} & \alpha \neq 1 \end{cases}
\]
Theorem 1. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable mapping on $I^\circ$, $a, b \in I$ with $a < b$ and $f'$ is integrable on $[a, b]$. If $|f'|$ is $s$-geometrically convex and monotonically decreasing on $[a, b]$, and $s \in (0, 1]$ then the following inequality holds:

$$
(2.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b - a}{4} \left[ \int_0^1 |t| \left| f' \left( \alpha \left( \frac{s}{2}, s \right) \right) \right| \, dt + \int_0^1 |t| \left| f' \left( \beta \left( \frac{b - s}{4}, \frac{b + s}{4} \right) \right) \right| \, dt \right]
$$

Proof. Since $|f'|$ is a $s$-geometrically convex and monotonically decreasing on $[a, b]$, from Lemma [1] we get

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b - a}{4} \left[ \int_0^1 |t| \left| f' \left( \alpha \left( \frac{s}{2}, s \right) \right) \right| \, dt + \int_0^1 |t| \left| f' \left( \beta \left( \frac{b - s}{4}, \frac{b + s}{4} \right) \right) \right| \, dt \right]
$$

If $0 < k \leq 1, 0 < m, n \leq 1$,

$$
(2.6) \quad k^{m^n} \leq k^{mn}
$$

When $|f'(a)| = |f'(b)| = 1$, by (2.6), we get

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4}
$$

When $0 < |f'(a)|, |f'(b)| < 1$, by (2.6), we get

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b - a}{4} \left[ \int_0^1 |t| \left| f'(a)^{\frac{s}{2}} \right| \left| f'(b)^{\frac{s}{2}} \right| \, dt + \int_0^1 |t| \left| f'(b)^{\frac{s}{2}} \right| \left| f'(a)^{\frac{s}{2}} \right| \, dt \right]
$$

$$
= \frac{b - a}{4} \left| f'(a) f'(b) \right|^{\frac{s}{2}} \left( \int_0^1 |t| \left| f'(a)^{\frac{s}{2}} \right| \, dt + \int_0^1 |t| \left| f'(b)^{\frac{s}{2}} \right| \, dt \right)
$$

which completes the proof. \qed

Theorem 2. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable on $I^\circ$, $a, b \in I$, with $a < b$ and $f' \in L ([a, b])$. If $|f'|^s$ is $s$-geometrically convex and monotonically decreasing
from Lemma 1 and the well known Hölder inequality, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)}{4 (p+1)^\frac{p}{q}} \| f'(a) f'(b) \| \left\{ \left[ g_2 \left( \alpha \left( \frac{sq}{2} \right) \right) \right] \right\}^\frac{1}{q} + \left[ g_2 \left( \alpha \left( \frac{-sq}{2} \right) \right) \right] \right\}^\frac{1}{q},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1. \)

**Proof.** Since \( |f'|^q \) is a \( s \)-geometrically convex and monotonically decreasing on \([a, b]\), from Lemma 11 and the well known Hölder inequality, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left\{ \int_0^1 \left| f' \left( a + \frac{t}{2} \frac{b-a}{2} \right) \right|^q \, dt \right\}^\frac{1}{q} + \left[ \int_0^1 \left| f' \left( b + \frac{t}{2} \frac{b-a}{2} \right) \right|^q \, dt \right\}^\frac{1}{q}
\]

\[
= \frac{b-a}{4 (p+1)^\frac{p}{q}} \left\{ \int_0^1 \left| f' \left( a + \frac{t}{2} \frac{b-a}{2} \right) \right|^q \, dt \right\}^\frac{1}{q} + \left[ \int_0^1 \left| f' \left( b + \frac{t}{2} \frac{b-a}{2} \right) \right|^q \, dt \right\}^\frac{1}{q}
\]

\[
\leq \frac{b-a}{4 (p+1)^\frac{p}{q}} \left\{ \int_0^1 \left| f'(a) \right|^q \left| f'(b) \right|^q \, dt \right\}^\frac{1}{q} + \left[ \int_0^1 \left| f'(b) \right|^q \left| f'(a) \right|^q \, dt \right\}^\frac{1}{q}
\]

When \( |f'(a)| = |f'(b)| = 1 \), by (2.6), we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2 (p+1)^\frac{p}{q}}
\]

When \( 0 < |f'(a)|, |f'(b)| < 1 \), by (2.6), we get
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\[ \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left\{ \left[ \int_0^1 |f' (a)|^{\frac{p}{q}} |f' (b)|^{\frac{q}{q}} \right]^{\frac{1}{q}} + \left[ \int_0^1 |f' (b)|^{\frac{p}{q}} |f' (a)|^{\frac{q}{q}} \right]^{\frac{1}{q}} \right\} \]

\[ = \frac{(b-a)}{4(p+1)^{\frac{1}{p}}} |f' (a) f' (b)|^{\frac{1}{p}} \left\{ \left( \int_0^1 \left| \frac{f' (a)}{f' (b)} \right| \frac{dt}{f' (b)} \right)^{\frac{1}{p}} + \left( \int_0^1 \left| \frac{f' (b)}{f' (a)} \right| \frac{dt}{f' (a)} \right)^{\frac{1}{p}} \right\} \]

which completes the proof.

\[ \square \]

**Corollary 1.** Let \( f : I \subseteq (0, \infty) \rightarrow (0, \infty) \) be differentiable on \( I^o \), \( a, b \in I \) with \( a < b \), and \( f' \in L([a,b]). \) If \( |f'|^q \) is \( s \)-geometrically convex and monotonically decreasing on \([a,b]\) for \( p, q > 1 \) and \( s \in (0,1] \), then

\( i) \) When \( p = q = 2 \), one has

\[ \left| \frac{f (a) + f (b)}{2} - \frac{1}{b-a} \int_a^b f (x) dx \right| \leq \frac{(b-a)}{4\sqrt{3}} |f' (a) f' (b)|^{\frac{1}{2}} \left\{ \sqrt{g_2 (\alpha, s, s)} + \sqrt{g_2 (\alpha, -s, -s)} \right\} \]

\( ii) \) If we take \( s = 1 \) in (2.7), we have for geometrically convex, one has

\[ \left| \frac{f (a) + f (b)}{2} - \frac{1}{b-a} \int_a^b f (x) dx \right| \leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}} |f' (a) f' (b)|^{\frac{1}{p}} \left\{ [g_2 (\alpha, q, q)]^{\frac{1}{p}} + [g_2 (\alpha, -q, -q)]^{\frac{1}{p}} \right\} \]

**Theorem 3.** Let \( f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be differentiable on \( I^o \), \( a, b \in I \) with \( a < b \) and \( f' \in L([a,b]). \) If \( |f'|^q \) is \( s \)-geometrically convex and monotonically decreasing on \([a,b]\), for \( q \geq 1 \) and \( s \in (0,1] \), then

\[ \left( \text{2.8} \right) \]

\[ \left| \frac{f (a) + f (b)}{2} - \frac{1}{b-a} \int_a^b f (x) dx \right| \leq \frac{b-a}{4} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left\{ \left( \frac{f' (a)}{f' (b)} \right)^{\frac{1}{q}} \left[ g_1 (\alpha, \frac{sq}{2}, \frac{sq}{2}) \right]^{\frac{1}{q}} + \left( \frac{f' (b)}{f' (a)} \right)^{\frac{1}{q}} \left[ g_1 (\alpha, \frac{-sq}{2}, \frac{-sq}{2}) \right]^{\frac{1}{q}} \right\} \]
Proof: Since \(|f'|^q\) is a \(s\)-geometrically convex and monotonically decreasing on \([a, b]\), from Lemma 1 and the well known power mean integral inequality, we have

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{b - a}{4} \left\{ \left[ \int_0^1 t \left| f' \left( \frac{1 + t}{2} a + \frac{1 - t}{2} b \right) \right| \, dt \right]^{\frac{1}{q}} \right\}^{\frac{q}{2}}
\]

\[
+ \left[ \int_0^1 \left| f' \left( \frac{1 + t}{2} b + \frac{1 - t}{2} a \right) \right| \, dt \right]^{\frac{1}{q}}
\]

\[
\leq \frac{b - a}{4} \left\{ \left[ \int_0^1 t \left| f' \left( a^{1^q} + t^{1^q} b \right) \right| \, dt \right]^{\frac{1}{q}} \right\}^{\frac{q}{2}} + \left[ \int_0^1 \left| f' \left( b^{1^q} + t^{1^q} a \right) \right| \, dt \right]^{\frac{1}{q}}
\]

\[
\leq \frac{b - a}{4} \left\{ \left[ \int_0^1 t \left| f' \left( a^{1^q} + t^{1^q} b \right) \right| \, dt \right]^{\frac{1}{q}} \right\}^{\frac{q}{2}} + \left[ \int_0^1 \left| f' \left( b^{1^q} + t^{1^q} a \right) \right| \, dt \right]^{\frac{1}{q}}
\]

When \(|f'(a)| = |f'(b)| = 1\), by (2.6), we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4}
\]

When \(0 < |f'(a)|, |f'(b)| < 1\), by (2.6), we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4}
\]

\[
\left\{ \left[ \int_0^1 t \left| f' \left( a^{1^q} + t^{1^q} b \right) \right| \, dt \right]^{\frac{1}{q}} \right\}^{\frac{q}{2}} + \left[ \int_0^1 \left| f' \left( b^{1^q} + t^{1^q} a \right) \right| \, dt \right]^{\frac{1}{q}}
\]

\[
\leq \frac{b - a}{4} \left\{ \left[ \int_0^1 t \left| f' \left( a^{1^q} + t^{1^q} b \right) \right| \, dt \right]^{\frac{1}{q}} \right\}^{\frac{q}{2}} + \left[ \int_0^1 \left| f' \left( b^{1^q} + t^{1^q} a \right) \right| \, dt \right]^{\frac{1}{q}}
\]

\[
= \frac{b - a}{4} \left\{ \left[ \int_0^1 t \left| f' \left( a^{1^q} + t^{1^q} b \right) \right| \, dt \right]^{\frac{1}{q}} \right\}^{\frac{q}{2}} \left[ g_1 \left( \frac{1}{a^2} \right) \right]^{\frac{1}{2}} + \frac{1}{\left| f'(a) \right|} \left[ g_1 \left( \frac{1}{a^2} \right) \right]^{\frac{1}{2}}
\]

which completes the proof. 

\(\square\)
Theorem 4. Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+ \) be differentiable on \( I^o \), \( a, b \in I \), with \( a < b \) and \( f' \in L ([a, b]) \). If \( |f'| \) is \( s \)-geometrically convex and monotonically decreasing on \([a, b]\) for \( \mu_1, \mu_2, \eta_1, \eta_2 > 0 \) with \( \mu_1 + \eta_1 = 1 \) and \( \mu_2 + \eta_2 = 1 \) and \( s \in (0, 1) \), then

\[
(2.9) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq b - a \left( \int_0^1 |t| |f'(a)|^{\frac{s+1}{s}} dt + \int_0^1 t |f'(b)|^{\frac{s+1}{s}} dt \right) + \eta_1 \eta_2 \left( \alpha \left( \frac{s}{2\eta_1^2}, \frac{s}{2\eta_2} \right) \right).
\]

Proof. Since \( |f'| \) is a \( s \)-geometrically convex and monotonically decreasing on \([a, b]\), from Lemma \( \text{II} \) we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq b - a \left[ \int_0^1 |t| |f'(a)|^{\frac{s+1}{s}} dt + \int_0^1 t |f'(b)|^{\frac{s+1}{s}} dt \right] + \eta_1 \eta_2 \left( \alpha \left( \frac{s}{2\eta_1^2}, \frac{s}{2\eta_2} \right) \right).
\]

When \( 0 < |f'(a)|, |f'(b)| \leq 1 \), by (2.6), we get

\[
(2.10) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq b - a \left[ \int_0^1 |t| |f'(a)|^{\frac{s+1}{s}} |f'(b)|^{\frac{s+1}{s}} dt + \int_0^1 t |f'(b)|^{\frac{s+1}{s}} |f'(a)|^{\frac{s+1}{s}} dt \right] = \frac{b - a}{4} |f'(a) f'(b)|^{\frac{s}{2}} \left[ \int_0^1 \left| \frac{f'(a)}{f'(b)} \right|^{\frac{s}{2}} dt + \int_0^1 \left| \frac{f'(b)}{f'(a)} \right|^{\frac{s}{2}} dt \right].
\]
for all \( t \in [0, 1] \). Using the well known inequality \( mn \leq \mu m^\frac{1}{\mu} + \eta n^\frac{1}{\eta} \), on the right side of (2.11), we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left| f'(a) f'(b) \right| \frac{1}{\mu_1} \left\{ \frac{\mu_1}{1 + \mu_1} + \eta_1 g_2 \left( \alpha \left( \frac{s}{2\eta_1}, \frac{s}{2\eta_1} \right) \right) \right\}
\]

and we get, in here, if \( |f'(a)| = |f'(b)| = 1 \), we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left[ \frac{(1 + \mu_2) \mu_1^2 + (1 + \mu_1) \mu_2^2}{(1 + \mu_1) (1 + \mu_2)} + \eta_1 + \eta_2 \right]
\]

which the proof is completed. \( \square \)

3. Applications to special means for positive numbers

Let

\[
A(a, b) = \frac{a + b}{2}, \quad L(a, b) = \frac{b - a}{\ln b - \ln a} \quad (a \neq b),
\]

\[
L_p(a, b) = \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{1/p}, \quad a \neq b, \ p \in \mathbb{R}, \ p \neq -1, 0
\]

be the arithmetic, logarithmic, generalized logarithmic means for \( a, b > 0 \) respectively.

In the following propositions, \( \alpha(u, v) = \left| \frac{f'(a)u}{f'(b)v} \right| = \left| \frac{u^{v-1}}{v^{u-1}} \right| \).
Proposition 1. Let $0 < a < b \leq 1$, with $a \neq b$, and $0 < s < 1$. Then, we have

$$\frac{1}{s} |A(a^s, b^s) - L_s(a, b)^s| \leq \frac{(b-a)}{4(ab)^{\frac{s}{2}}(s-1)} \left\{ \left| \frac{\alpha}{a} \right| \ln \left| \frac{b}{a} \right|^{(s-1)\frac{p}{2}} - \frac{\left| \frac{b}{a} \right|^{(s-1)\frac{p}{2}} - 1}{\left( \ln \left| \frac{b}{a} \right|^{(s-1)\frac{p}{2}} \right)^2} \right\}$$

Proof. Let $f(x) = \frac{x^s}{s}$, $x \in (0, 1]$, $0 < s < 1$, then $|f'(x)| = x^{s-1}$, $x \in (0, 1]$ is a $s$-geometrically convex mapping. The assertion follows from Theorem 1 applied to $s$-geometrically convex mapping $|f'(x)| = x^{s-1}$, $x \in (0, 1]$. \qed

Example 2. Let $f(x) = \frac{x^s}{s}$, $x \in (0, 1]$, $0 < s < 1$, then $|f'(x)| = x^{s-1}$, $x \in (0, 1]$ is a $s$-geometrically convex mapping. If we apply in Theorem 1 for $s = 0.5, a = 0.89, b = 0.9$, we get

$$\frac{1}{s} \left| \frac{a^s + b^s}{2} - \left( \frac{b^{s+1} - a^{s+1}}{(s+1)(b-a)} \right) \right| \leq \frac{(b-a)}{4(ab)^{\frac{s}{2}}(s-1)} \times \left\{ \left| \frac{\alpha}{a} \right| \ln \left| \frac{b}{a} \right|^{(s-1)\frac{p}{2}} - \frac{\left| \frac{b}{a} \right|^{(s-1)\frac{p}{2}} - 1}{\left( \ln \left| \frac{b}{a} \right|^{(s-1)\frac{p}{2}} \right)^2} \right\}$$

And similarly, if we apply for $s = 0.2, a = 0.15, b = 0.6$, we obtain

$$4.921 067 116 \times 10^{-6} \leq 0.136 819 309 576 863 680 170 486$$

for $s = 0.75, a = 0.45, b = 0.86$, we obtain

$$6.115 413 651 \times 10^{-2} \leq 0.112 144 032 368 736 206 184 243$$

etc.

Proposition 2. Let $0 < a < b \leq 1$, with $a \neq b$, and $0 < s < 1$, and $p, q > 1$. Then, we have

$$\frac{1}{s} |A(a^s, b^s) - L_s(a, b)^s| \leq \frac{(b-a)}{4(p+1)^{\frac{s}{2}}} \left| ab \right|^{\frac{s}{2}} \left( b^{(1-s)/2} + a^{s(1-s)/2} \right) \left[ \left( L \left( a^{(s-1)\frac{p}{2}}, b^{(s-1)\frac{p}{2}} \right) \right) \right]^{\frac{1}{q}}.$$
Proposition 3. Let $0 < a < b < 1$, with $a \neq b$, and $0 < s < 1$, and $q \geq 1$. Then, we have
\[
\frac{1}{s} |A(a^s, b^s) - L_s(a, b)^s| 
\leq \frac{b - a}{4} \left( \frac{1}{2} \right)^{1 - \frac{s}{q}} \left\{ \left| a \right|^{\frac{s}{q}} - \left| b \right|^{\frac{s}{q}} \right\} \left[ \left| \frac{a}{b} \right|^{\frac{s}{q}} \frac{(s-1)\frac{s}{q}}{\ln \left| \frac{a}{b} \right|^{s-1} - \left| \frac{a}{b} \right|^{\frac{s}{q}-(s-1)\frac{s}{q}}} \right]^{\frac{s}{q}} 
+ \left| b \right|^{\frac{s}{q}} \left[ \left| \frac{a}{b} \right|^{\frac{s}{q}} \frac{(s-1)\frac{s}{q}}{\ln \left| \frac{a}{b} \right|^{s-1} - \left| \frac{a}{b} \right|^{\frac{s}{q}-(s-1)\frac{s}{q}}} - 1 \right]^{\frac{s}{q}} \right\}. 
\]

Proof. The assertion follows from Theorem 3 applied to $s$-geometrically convex mapping $|f'(x)| = \frac{x^s}{s}$, $x \in (0, 1)$. \qed

Proposition 4. Let $0 < a < b < 1$, with $a \neq b$, and $0 < s < 1$, and $q \geq 1$. Then, we have
\[
\frac{1}{s} |A(a^s, b^s) - L_s(a, b)^s| 
\leq \frac{b - a}{4} \left( \frac{1}{2} \right)^{1 - \frac{s}{q}} \left(1 + \mu_2\right) \frac{\mu_2^2}{(1 + \mu_1)(1 + \mu_2)} \left[ \eta_1 \left[ \frac{\left| \frac{a}{b} \right|^{(s-1)\frac{s}{q}} - 1}{\ln \left| \frac{a}{b} \right|^{s-1} - \left| \frac{a}{b} \right|^{\frac{s}{q}-(s-1)\frac{s}{q}}} \right] + \eta_2 \left[ \frac{\left( \frac{a}{b} \right)^{\frac{s}{q}} - 1}{\ln \left| \frac{a}{b} \right|^{\frac{s}{q}-(s-1)\frac{s}{q}}} \right] \right]. 
\]

Proof. The assertion follows from Theorem 4 applied to $s$-geometrically convex mapping $|f'(x)| = \frac{x^s}{s}$, $x \in (0, 1)$. \qed

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