Symmetric Lie superalgebras and deformed quantum Calogero–Moser problems

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ABSTRACT

The representation theory of symmetric Lie superalgebras and corresponding spherical functions are studied in relation with the theory of the deformed quantum Calogero–Moser systems. In the special case of symmetric pair \( g = \mathfrak{gl}(n,2m), \mathfrak{t} = \mathfrak{osp}(n,2m) \) we establish a natural bijection between projective covers of spherically typical irreducible \( g \)-modules and the finite dimensional generalised eigenspaces of the algebra of Calogero–Moser integrals \( \mathcal{D}_{n,m} \) acting on the corresponding Laurent quasi-invariants \( \mathfrak{S}_{n,m} \).

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1. Introduction

In 1964 Berezin et al. [3] made an important remark that the radial part of the Laplace–Beltrami operator on the symmetric space \( X = SL(n)/SO(n) \)

\[
L = \Delta + \sum_{i<j}^{n} \coth(x_i - x_j)(\partial_i - \partial_j)
\]

is conjugated to the quantum Hamiltonian

\[
H = \Delta + \sum_{i<j}^{n} \frac{1}{2\sinh^2(x_i - x_j)}
\]

describing the pairwise interacting particle on the line.

This was probably the first recorded observation of the connection between the theory of symmetric spaces and the theory of what later became known as Calogero–Moser, or Calogero–Moser–Sutherland (CMS), integrable models [34]. Olshanetsky and Perelomov suggested a class of generalisations of CMS systems related to any root system and showed that the radial parts of all irreducible symmetric spaces are conjugated to some particular operators from this class [20]. The joint eigenfunctions of the corresponding commutative algebras of quantum integrals are zonal spherical functions. In the \( A_n \) case this leads to an important notion of the Jack polynomials introduced by H. Jack independently around the same time [14].

The discovery of the Dunkl operator technique led to an important link of the CMS systems with the representation theory of Cherednik algebras, see Etingof’s lectures [10].

It turned out that there are other integrable generalisations, which have only partial symmetry and called deformed CMS systems [8]. Their relation with symmetric superspaces was first discovered by one of the authors in [26] and led to a class of such operators related to the basic classical Lie superalgebras, which was introduced in [27].

In this paper we develop this link further to study the representation theory of symmetric Lie superalgebras and the related spherical functions. Such Lie superalgebra is a pair \((\mathfrak{g}, \theta)\), where \(\mathfrak{g}\) is a Lie superalgebra and \(\theta\) is an involutive automorphism of \(\mathfrak{g}\). It corresponds to the symmetric pair \(X = (\mathfrak{g}, \mathfrak{k})\), where \(\mathfrak{k}\) is \(\theta\)-invariant part of \(\mathfrak{g}\) and can be considered as an algebraic version of the symmetric superspace \(G/K\).
In the particular case of \( X = (\mathfrak{gl}(n, 2m), \mathfrak{osp}(n, 2m)) \) the radial part of the corresponding Laplace–Beltrami operator in the exponential coordinates is a particular case of the deformed CMS operator related to Lie superalgebra \( \mathfrak{gl}(n, m) \) \[27\]

\[
\mathcal{L} = \sum_{i=1}^{n} \left( x_i \frac{\partial}{\partial x_i} \right)^2 + k \sum_{j=1}^{m} \left( y_j \frac{\partial}{\partial y_j} \right)^2 - \sum_{i=1}^{n} \sum_{j=1}^{m} \left( x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right)
\]

\[
- \sum_{i<j}^{m} \frac{y_i + y_j}{y_i - y_j} \left( \frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_j} \right) - \sum_{i=1}^{n} \sum_{j=1}^{m} \left( x_i \frac{\partial}{\partial x_i} - k y_j \frac{\partial}{\partial y_j} \right)
\]

(1)

corresponding to the special value of parameter \( k = -\frac{1}{2} \). According to [27] it has infinitely many commuting differential operators generating the algebra of quantum deformed CMS integrals \( \mathfrak{D}_{n,m} \).

We study the action of \( \mathfrak{D}_{n,m} \) on the algebra \( \mathfrak{A}_{n,m} \) of \( S_n \times S_m \)-invariant Laurent polynomials \( f \in \mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}, y_1^{\pm1}, \ldots, y_m^{\pm1}]^{S_n \times S_m} \) satisfying the quasi-invariance condition

\[
x_i \frac{\partial f}{\partial x_i} - k y_j \frac{\partial f}{\partial y_j} \equiv 0
\]

(2)

on the hyperplane \( x_i = y_j \) for all \( i = 1, \ldots, n, \ j = 1, \ldots, m \) with \( k = -\frac{1}{2} \). It turns out that the generalised eigenspaces \( \mathfrak{A}_{n,m}(\chi) \) in the corresponding spectral decomposition

\[
\mathfrak{A}_{n,m} = \bigoplus_{\chi} \mathfrak{A}_{n,m}(\chi),
\]

where \( \chi \) are certain homomorphisms \( \chi : \mathfrak{D}_{n,m} \to \mathbb{C} \), are in general not one-dimensional, similarly to the case of Jack–Laurent symmetric functions considered in our recent paper \[32\]. We have shown there that the corresponding generalised eigenspaces have dimension \( 2^r \) and the image of the algebra of CMS integrals in the endomorphisms of such space is isomorphic to the tensor product of dual numbers

\[
\mathfrak{A}_r = \mathbb{C}[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r]/(\varepsilon_1^2, \varepsilon_2^2, \ldots, \varepsilon_r^2).
\]

It is known that the algebra \( \mathfrak{A}_r \) also appears as the algebra of the endomorphisms of the projective indecomposable modules over general linear supergroup (see Brundan–Stroppel [7]). It is natural therefore to think about possible links between our generalised eigenspaces and projective modules.

The main result of this paper is a one-to-one correspondence between the finite-dimensional generalised eigenspaces of \( \mathfrak{D}_{n,m} \) and projective covers of certain irreducible finite-dimensional modules of \( \mathfrak{gl}(n, 2m) \). More precisely, we prove the following main theorem.

Let \( Z(\mathfrak{g}) \) be the centre of the universal enveloping algebra of \( \mathfrak{g} = \mathfrak{gl}(n, 2m) \). For a \( \mathfrak{g} \)-module \( U \) we denote by \( U^\mathfrak{f} \) its subspace consisting of vectors invariant under \( \mathfrak{f} = \mathfrak{osp}(n, 2m) \).
Theorem 1.1. For any finite dimensional generalised eigenspace $\mathfrak{A}_{n,m}(\chi)$ there exists a unique projective indecomposable module $P$ over $\mathfrak{gl}(n,2m)$ and a natural map

$$\Psi : (P^*)^\mathbb{C} \longrightarrow \mathfrak{A}_{n,m}(\chi),$$

which is an isomorphism of $Z(\mathfrak{g})$-modules.

This establishes the bijection between the projective covers of spherically typical irreducible $\mathfrak{g}$-modules and the finite dimensional generalised eigenspaces of the algebra of deformed CMS integrals $\mathfrak{D}_{n,m}$ acting in $\mathfrak{A}_{n,m}$.

The corresponding projective modules can be described explicitly in terms of the highest weights of $\mathfrak{gl}(n,2m)$ under certain typicality conditions, which are natural generalisation of Kac’s typicality conditions [16].

As a corollary we have an algorithm for calculating the composition quotients in Kac flag of the corresponding projective covers in the spherically typical case (which may have any degree of atypicality in the sense of [5]). The number of the quotients is equal to the number of elements in the corresponding equivalence class, which can be described combinatorially, and equals $2^s$, where $s$ is the degree of atypicality (see sections 6 and 7 below). Our algorithm is equivalent to Brundan–Stroppel algorithm [7] in this particular case, but our technique is different and uses the theory of the deformed CMS systems.

The plan of the paper is following. In the next section we introduce the algebra $\mathfrak{D}_{n,m}$ of quantum integrals of the deformed CMS system (mainly following [31]) and study the corresponding spectral decomposition of its action on the algebra $\mathfrak{A}_{n,m}$.

In section 3 we introduce symmetric Lie superalgebras and derive the formula for the radial part of the corresponding Laplace–Beltrami operators. In particular, we show that for the four classical series of symmetric Lie superalgebras this radial part is conjugated to the deformed CMS operators introduced in [27].

The rest of the paper is dealing mainly with the particular case corresponding to the symmetric pairs $(\mathfrak{g},\mathfrak{k})$ with $\mathfrak{g} = \mathfrak{gl}(n,2m), \mathfrak{k} = \mathfrak{osp}(n,2m)$. We call a finite dimensional $\mathfrak{g}$-module $U$ spherical if the space of $\mathfrak{k}$-invariant vectors $U^\mathfrak{k}$ is non-zero. We describe the admissibility conditions on highest weight $\lambda$, for which the corresponding Kac module $K(\lambda)$ is spherical. Under certain assumptions of typicality we describe the conditions on admissible highest weights for irreducible modules to be spherical (see sections 5 and 6) and study the equivalence relation on the admissible weights defined by the equality of central characters.

These results are used in section 7 to prove the main theorem, which implies in particular that any finite-dimensional generalised eigenspace contains at least one zonal spherical function, corresponding to an irreducible spherically typical $\mathfrak{g}$-module. In the last section we illustrate all this, including explicit formulas for the zonal spherical functions, in the simplest example of symmetric pair $X = (\mathfrak{gl}(1,2),\mathfrak{osp}(1,2))$. 
2. Algebra of deformed CMS integrals and spectral decomposition

In this section we assume that the parameter $k$ is arbitrary nonzero, so everything is true for the special case $k = -\frac{1}{2}$ as well.

To define the algebra of the corresponding CMS integrals $\mathcal{D}_{n,m}$ it will be convenient to denote $x_{n+j} := y_j$, $j = 1, \ldots, m$ and to introduce parity function $p(i) = 0$, $i = 1, \ldots, n$, $p(i) = 1$, $i = n + 1, \ldots, n + m$. We also introduce the notation

$$\partial_j = x_j \frac{\partial}{\partial x_j}, \ j = 1, \ldots, n + m.$$ 

By definition the algebra $\mathcal{D}_{n,m}$ is generated by the deformed CMS integrals defined recursively in [27]. It will be convenient for us to use the following, slightly different choice of generators.

Define recursively the differential operators $\partial^{(p)}_i$, $1 \leq i \leq n + m$, $p \in \mathbb{N}$ as follows: for $p = 1$

$$\partial^{(1)}_i = k^{p(i)} \partial_i$$

and for $p > 1$

$$\partial^{(p)}_i = \partial^{(1)}_i \partial^{(p-1)}_i - \sum_{j \neq i} k^{1-p(j)} \frac{x_i}{x_i - x_j} \left( \partial^{(p-1)}_i - \partial^{(p-1)}_j \right).$$

Then the higher CMS integrals $L_p$ are defined as the sums

$$L_p = \sum_{i \in \mathcal{I}} k^{-p(i)} \partial^{(p)}_i.$$ 

In particular, for $p = 2$ we have

$$L_2 = \sum_{i=1}^{n+m} k^{-p(i)} \partial^2_i - \sum_{i<j}^{n+m} \frac{x_i + x_j}{x_i - x_j} \left( k^{1-p(j)} \partial_i - k^{1-p(i)} \partial_j \right),$$

which coincides with the deformed CMS operator (1).

**Theorem 2.1.** The operators $L_p$ are quantum integrals of the deformed CMS system:

$$[L_p, L_2] = 0.$$

**Proof.** Following the idea of our recent work [31] introduce a version of quantum Moser $(n + m) \times (n + m)$-matrices $L$, $M$ by

$$L_{ii} = k^{p(i)} \partial_i - \sum_{j \neq i} k^{1-p(j)} \frac{x_i}{x_i - x_j}, \quad L_{ij} = k^{1-p(j)} \frac{x_i}{x_i - x_j}, \quad i \neq j$$

and
\[ M_{ii} = -\sum_{j \neq i} \frac{2k^{1-p(j)}x_i x_j}{(x_i - x_j)^2}, \quad M_{ij} = \frac{2k^{1-p(j)}x_i x_j}{(x_i - x_j)^2}, \quad i \neq j. \]

Note that matrix \( M \) satisfies the relations

\[ Me = e^* M = 0, \quad e = (1, \ldots, 1)^t, \quad e^* = (1, \ldots, 1/k, \ldots, 1/k). \]

Define also matrix Hamiltonian \( H \) by

\[ H_{ii} = L_2, \quad H_{ij} = 0, \quad i \neq j. \]

Then it is easy to check that these matrices satisfy Lax relation

\[ [L, H] = [L, M]. \]

Indeed, matrix \( L \) is different from the Moser matrix (43) from [31] by rank one matrix \( e \otimes e^* \), which does not affect the commutator \([L, M]\) because of the relations \( Me = e^* M = 0 \). This implies as in [31,33] that the “deformed total trace”

\[ \mathcal{L}_p = \sum_{i,j} k^{-p(i)} (L^p)_{ij} \]

commute with \( L_2 \).

Define now the Harish-Chandra homomorphism

\[ \varphi : \mathfrak{D}_{n,m} \rightarrow \mathbb{C}[\xi_1, \ldots, \xi_{n+m}] \]

by the conditions (cf. [27]):

\[ \varphi(\partial_i) = \xi_i, \quad \varphi \left( \frac{x_i}{x_i - x_j} \right) = 1, \quad \text{if } i < j. \]

In particular, \( d_i^{(p)}(\xi) := \varphi(\partial_i^{(p)}) \) satisfy the following recurrence relations

\[ d_i^{(p)} = d_i^{(1)} d_i^{(p-1)} - \sum_{j > i} k^{-p(j)} (d_i^{(p-1)} - d_j^{(p-1)}), \quad (6) \]

which determine them uniquely with \( d_i^{(1)} = k^{p(i)} \xi_i, \quad i = 1, \ldots, n + m. \)

Let \( \rho(k) \in \mathbb{C}^{n+m} \) be the following deformed analogue of the Weyl vector

\[ \rho(k) = \frac{1}{2} \sum_{i=1}^n (k(2i - n - 1) - m)e_i + \frac{1}{2} \sum_{j=1}^m (k^{-1}(2j - m - 1) + n)e_{j+n} \quad (7) \]
and consider the bilinear form $(,)$ on $\C^{n+m}$ defined in the basis $e_1,\ldots,e_{n+m}$ by
\[(e_i,e_i) = 1, \ i = 1,\ldots,n, \ (e_j,e_j) = k, \ j = n+1,\ldots,n+m.\]

**Theorem 2.2.** [27] Harish-Chandra homomorphism is injective and its image is the subalgebra $\Lambda_{n,m}(k) \subset \C[\xi_1,\ldots,\xi_{n+m}]$ consisting of polynomials with the following properties:
\[f(w(\xi + \rho(k))) = f(\xi + \rho(k)), \ w \in S_n \times S_m\]
and for every $i \in \{1,\ldots,n\}, j \in \{n+1,\ldots,n+m\}$
\[f(\xi - e_i + e_j) = f(\xi) \tag{8}\]
on the hyperplane $(\xi + \rho(k), e_i - e_j) = \frac{1}{2}(1 + k)$.

**Corollary 2.3.** Operators $\mathcal{L}_p$ commute with each other.

From the results of [30] it follows that $\mathcal{L}_p$ generate the same algebra $\mathfrak{D}_{n,m}$ as commuting CMS integrals from [27], which gives another proof of their commutativity.

Let now $\mathfrak{A}_{n,m}$ be the algebra consisting of $S_n \times S_m$-invariant Laurent polynomials $f \in \C[x_1^{\pm 1},\ldots,x_n^{\pm 1},y_1^{\pm 1}\ldots,y_m^{\pm 1}]^{S_n \times S_m}$ satisfying the quasi-invariance condition
\[x_i \frac{\partial f}{\partial x_i} - k y_j \frac{\partial f}{\partial y_j} \equiv 0 \tag{9}\]
on the hyperplane $x_i = y_j$ for all $i = 1,\ldots,n, j = 1,\ldots,m$ with $k$ being arbitrary and the same as in the definition of $\mathfrak{D}_{n,m}$. We claim that the algebra $\mathfrak{D}_{n,m}$ preserves it.

For any Laurent polynomial
\[f = \sum_{\mu \in X_{n,m}} c_{\mu} x^{\mu}, \quad X_{n,m} = \Z^n \oplus \Z^m\]
consider the set $M(f)$ consisting of $\mu$ such that $c_{\mu} \neq 0$ and define the support $S(f)$ as the intersection of the convex hull of $M(f)$ with $X_{n,m}$.

**Theorem 2.4.** The operators $\mathcal{L}_p$ for all $p = 1,2,\ldots$ map the algebra $\mathfrak{A}_{n,m}$ to itself and preserve the support: for any $D \in \mathfrak{D}_{n,m}$ and $f \in \mathfrak{A}_{n,m}$
\[S(D f) \subseteq S(f).\]

**Proof.** The first part follows from the fact that if $f \in \mathfrak{A}_{n,m}$ then $\partial_i^{(p)} f$ is a polynomial. The proof is essentially repeating the arguments from [28] (see theorem 5 and lemmas 5 and 6), so we will omit it.
To prove the second part it is enough to show that \( S(\partial_i^{(p)} f) \subseteq S(f) \) for any \( f \in \mathfrak{A}_{n,m} \).
We use induction in \( p \). From the recursion (3) we see that it is enough to prove that

\[
g = \frac{x_i}{x_i - x_j} \left( \partial_i^{(p-1)} - \partial_j^{(p-1)} \right) (f)
\]
is a polynomial and \( S(g) \subseteq S(f) \).

The fact that \( g \) is a polynomial follows from lemma 6 of [28]. Denote the polynomial \( \left( \partial_i^{(p-1)} - \partial_j^{(p-1)} \right) (f) \) as \( h(x) \). By the induction assumption \( S(h) \subseteq S(f) \). Since

\[
h(x) = \left( 1 - \frac{x_j}{x_i} \right) g(x)
\]
and the support of a product of two Laurent polynomials is the Minkowski sum of the supports of the factors this implies that \( S(g) \subseteq S(h) \subseteq S(f) \). □

Now we are going to investigate the spectral decomposition of the action of the algebra of CMS integrals \( \mathfrak{D}_{n,m} \) on \( \mathfrak{A}_{n,m} \).

We will need the following partial order on the set of integral weights \( \lambda \in X_{n,m} = \mathbb{Z}^{n+m} \): we say that \( \mu \preceq \lambda \) if and only if

\[
\mu_1 \leq \lambda_1, \mu_1 + \mu_2 \leq \lambda_1 + \lambda_2, \ldots, \mu_1 + \cdots + \mu_{n+m} \leq \lambda_1 + \cdots + \lambda_{n+m}.
\]

Proposition 2.5. Let \( f \in \mathfrak{A}_{n,m} \) and \( \lambda \) be a maximal element of \( M(f) \) with respect to partial order. Then for any \( D \in \mathfrak{D}_{n,m} \) there is no \( \mu \) from \( M(D(f)) \), \( \mu \neq \lambda \) such that \( \lambda \preceq \mu \). The coefficient at \( x^\lambda \) in \( D(f) \) is \( \varphi(D)(\lambda)c_\lambda \), where \( c_\lambda \) is the coefficient at \( x^\lambda \) in \( f \).

If \( \lambda \) is the only maximal element of \( M(f) \) then \( \mu \preceq \lambda \) for any \( \mu \) from \( M(D(f)) \).

Proof. It is enough to prove this only for \( D = \partial_i^{(p)} \). We will do it by induction on \( p \). In the notations of the proof of Theorem 2.4 let us assume that there is \( \mu \in M(g) \) such that \( \lambda \preceq \mu \), \( \mu \neq \lambda \). Without loss of generality we can assume that \( \mu \) is maximal in \( M(g) \).

From \( h(x) = (1 - x_j/x_i)g(x) \) with \( i < j \) it follows that \( \mu \) is also maximal in \( M(h) \), which contradicts the inductive assumption. This implies that the coefficient at \( x^\lambda \) in \( \partial_i^{(p)}(f) \) satisfies the same recurrence relations (6) with the initial conditions multiplied by \( c_\lambda \). This proves the first part. The proof of the second part is similar. □

Let \( \chi : \mathfrak{D}_{n,m} \to \mathbb{C} \) be a homomorphism and define the corresponding generalised eigenspace \( \mathfrak{A}_{n,m}(\chi) \) as the set of all \( f \in \mathfrak{A}_{n,m} \) such that for every \( D \in \mathfrak{D}_{n,m} \) there exists \( N \in \mathbb{N} \) such that \( (D - \chi(D))^N(f) = 0 \). If the dimension of \( \mathfrak{A}_{n,m}(\chi) \) is finite then such \( N \) can be chosen independent on \( f \).

Proposition 2.6. Algebra \( \mathfrak{A}_{n,m} \) as a module over the algebra \( \mathfrak{D}_{n,m} \) can be decomposed in a direct sum of generalised eigenspaces.
where the sum is taken over the set of some homomorphisms χ (explicitly described below).

**Proof.** Let \( f \in \mathfrak{A}_{n,m} \) and define a vector space

\[
V(f) = \{ g \in \mathfrak{A}_{n,m} \mid S(g) \subseteq S(f) \}.
\]

By Theorem 2.4 \( V(f) \) is a finite dimensional module over \( \mathfrak{D}_{n,m} \). Since the proposition is true for every finite-dimensional module the claim now follows.

Now we describe all homomorphisms \( \chi \) such that \( \mathfrak{A}_{n,m}(\chi) \neq 0 \). We say that the integral weight \( \lambda \in X_{n,m} \subseteq \mathbb{Z}^{n+m} \) is dominant if

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \quad \lambda_{n+1} \geq \lambda_{n+2} \geq \cdots \geq \lambda_{n+m}.
\]

The set of dominant weights is denoted \( X^+_{n,m} \).

For every \( \lambda \in X^+_{n,m} \) we define the homomorphism \( \chi_\lambda : \mathfrak{D}_{n,m} \rightarrow \mathbb{C} \) by

\[
\chi_\lambda(D) = \varphi(D)(\lambda), \quad D \in \mathfrak{D}_{n,m}
\]

where \( \varphi \) is the Harish-Chandra homomorphism.

**Proposition 2.7.** 1) For any \( \lambda \in X^+_{n,m} \) there exists \( \chi \) and \( f \in \mathfrak{A}_{n,m}(\chi) \), which has the only maximal term \( x^\lambda \).

2) \( \mathfrak{A}_{n,m}(\chi) \neq 0 \) if and only if there exists \( \lambda \in X^+_{n,m} \) such that \( \chi = \chi_\lambda \).

3) If \( \mathfrak{A}_{n,m}(\chi) \) is finite dimensional then its dimension is equal to the number of \( \lambda \in X^+_{n,m} \) such that \( \chi_\lambda = \chi \).

**Proof.** Let \( \mu_1 = \lambda_1, \ldots, \mu_n = \lambda_n, \nu_1 = \lambda_{n+1}, \ldots, \nu_m = \lambda_{n+m} \). Consider the Laurent polynomial

\[
g(x, y) = s_\mu(x)s_\nu(y) \prod_{i,j}(1 - y_j/x_i)^2
\]

where \( s_\mu(x), s_\nu(y) \) are the Schur polynomials [18]. It is easy to check that \( g \) belongs to the algebra \( \mathfrak{A}_{n,m} \) and has the only maximal weight \( \lambda \). By Proposition 2.6 we can write \( g = g_1 + \cdots + g_N \), where \( g_i \) belong to different generalised eigenspaces. Therefore there exists \( i \) such that \( \lambda \in M(g_i) \). Since \( g_i \) can be obtained from \( g \) by some element from the algebra \( \mathfrak{D}_{n,m} \) (which is a projector to the corresponding generalised eigenspace in some finite-dimensional subspace containing \( g \)), then \( \lambda \) is the only maximal element of \( M(g_i) \) by Proposition 2.5. This proves the first part.
Let $\mathfrak{A}_{n,m}(\chi) \neq 0$. Pick up a nonzero element $f$ from this subspace and choose some maximal element $\lambda^{(1)}$ from $M(f)$ and an operator $D \in \mathfrak{D}_{n,m}$. Then according to Proposition 2.5 element $x^{\lambda^{(1)}}$ does not enter in $f_1 = (D - \chi^{(1)}(D))(f)$ and $S(f_1) \subset S(f)$. Repeating this procedure we get the sequence of nonzero elements $f_0 = f, f_1, \ldots, f_N$ and the numbers $a_1 = \chi^{(1)}(D), \ldots, a_N = \chi^{(N)}(D)$ such that

$$f_i = (D - a_i)f_{i-1}, \ i = 1, \ldots, N, \ (D - a_N)f_{N-1} = 0.$$ 

Therefore

$$P(t) = \prod_{i=1}^{N}(t - a_i)$$

is a minimal polynomial for $D$ in the subspace $<f_0, \ldots, f_{N-1}>$. But this subspace is in $\mathfrak{A}_{n,m}(\chi)$. Therefore this polynomial should be some power of $t - \chi(D)$ and hence $a_1 = a_2 = \cdots = a_N = \chi(D)$. In particular, this implies that $\chi(D) = a_1 = \chi^{(1)}(D)$ for some $\lambda^{(1)} \in X_{n,m}^+$ as required.

Conversely, let $\lambda \in X_{n,m}^+$. According to the first part there exists $\chi$ and $f \in \mathfrak{A}_{n,m}(\chi)$ such that $\lambda$ is its maximal weight. Therefore the previous considerations show that $\chi = \chi_\lambda$ and thus $\mathfrak{A}_{n,m}(\chi_\lambda) \neq 0$.

To prove the third part suppose that $\mathfrak{A}_{n,m}(\chi)$ is finite dimensional and that $\lambda^{(1)}, \ldots, \lambda^{(N)}$ are all different elements from $X_{n,m}^+$ such that $\chi^{(i)} = \chi, \ i = 1, \ldots, N$. According to the first two parts there exists $f_i \in \mathfrak{A}_{n,m}(\chi)$ with the only maximal weight $\lambda^{(i)}$. It is easy to see that $f_1, \ldots, f_N$ are linearly independent. To show that they form a basis consider any $f \in \mathfrak{A}_{n,m}(\chi)$ and take a maximal weight $\mu$ from $M(f)$. According to Proposition 2.5 $\chi_\mu = \chi$ and thus $\mu$ must coincide with one of $\lambda^{(i)}$. By subtracting from $f$ a suitable multiple of $f_i$ and using induction we get the result. $\square$

**Corollary 2.8.** The set of homomorphisms in Proposition 2.6 consists of $\chi = \chi_\lambda, \ \lambda \in X_{n,m}^+$.

### 3. Symmetric Lie superalgebras and Laplace–Beltrami operators

We will be using an algebraic approach to the theory of symmetric superspaces based on the notion of symmetric Lie superalgebras going back to Dixmier [9]. More geometric approach with relation to physics and random-matrix theory can be found in Zirnbauer [35]. For the classification of real simple Lie superalgebras and symmetric superspaces see Serganova [23].

**Symmetric Lie superalgebra** is a pair $(\mathfrak{g}, \theta)$, where $\mathfrak{g}$ is a complex Lie superalgebra, which will be assumed to be basic classical [15], and $\theta$ is an involutive automorphism

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1 Strictly speaking, the Lie superalgebra $\mathfrak{gl}(n, m)$ is not basic classical, but it is more convenient for us to consider than $\mathfrak{sl}(n, m)$. 
of $\mathfrak{g}$. We have the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and $-1$ eigenspaces of $\theta$:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$ 

Alternatively, one can talk about symmetric pair $X = (\mathfrak{g}, \mathfrak{k})$.

In this paper we restrict ourselves by the following 4 classical series of symmetric pairs (in Cartan’s notations [12,35]):

$$AI/AII = (\mathfrak{gl}(n, 2m), \mathfrak{osp}(n, 2m)), \quad DIII/CI = (\mathfrak{osp}(2l, 2m), \mathfrak{gl}(l, m)),$$

$$AIII = (\mathfrak{gl}(n_1 + n_2, m_1 + m_2), \mathfrak{gl}(n_1, m_1) \oplus \mathfrak{gl}(n_2, m_2)),$$

$$BDI/CII = (\mathfrak{osp}(n_1 + n_2, 2m_1 + 2m_2), \mathfrak{osp}(n_1, 2m_1) \oplus \mathfrak{osp}(n_2, 2m_2)).$$

In fact, we will give all the details only for the first series, which will be our main case (see next Section). For a more general approach we refer to the work by Alldridge et al. [2].

Commutative subalgebra $\mathfrak{a} \subset \mathfrak{p}$ is called Cartan subspace if it is reductive in $\mathfrak{g}$ and the centraliser of $\mathfrak{a}$ in $\mathfrak{p}$ coincides with $\mathfrak{a}$ [9]. We will consider only the cases when Cartan subspace can be chosen to be even (“even type” in the terminology of [2]).

The Lie superalgebra $\mathfrak{g}$ has an even invariant supersymmetric bilinear form with restriction on $\mathfrak{a}$ being non-degenerate. The corresponding quadratic form on $\mathfrak{a}$ we denote $Q$.

We have the decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ into nonzero eigenspaces

$$\mathfrak{g} = \mathfrak{g}_0^\alpha \oplus \bigoplus_{\alpha_0 \in R(X)} \mathfrak{g}_\alpha^\alpha .$$

The corresponding set $R(X) \subset \mathfrak{a}^*$ is called restricted root system of $X$ and $\mu_\alpha = \text{sdim} \ g_\alpha^\alpha$ are called multiplicities, where $\text{sdim} \ g_\alpha^\alpha$ is the super dimension: $\text{sdim} \ g_\alpha^\alpha = \dim \ g_\alpha^\alpha$ for even roots and $\text{sdim} \ g_\alpha^\alpha = - \dim \ g_\alpha^\alpha$ for odd roots.

For the symmetric pairs $X = (\mathfrak{gl}(n, 2m), \mathfrak{osp}(n, 2m))$ of type $AI/AII$ we have the following root system consisting of the even roots $\pm(x_i - x_j), 1 \leq i < j \leq n$ with multiplicity $\mu = 1$, $\pm(y_i - y_j), 1 \leq i < j \leq m$ with multiplicity $\mu = 4$ and odd roots $\pm(x_i - y_j), 1 \leq i \leq n, 1 \leq j \leq m$ with multiplicity $\mu = -2$. The corresponding invariant quadratic form is

$$Q = x_1^2 + \cdots + x_n^2 + k^{-1}(y_1^2 + \cdots + y_m^2)$$

with $k = -1/2$ (see the next section).

For the remaining 3 classical series we have the following restricted root systems of $BC(n, m)$ type, see [21,2].

For $X = (\mathfrak{osp}(2l, 2m), \mathfrak{gl}(l, m))$ of type $DIII/CI$ the restricted root system depends on the parity of $l$. For odd $l = 2n + 1$ the restricted even roots are $\pm x_i$ with $\mu = 4, \pm 2x_i$
with $\mu = 1$ for $i = 1, \ldots, n$, $\pm x_i \pm x_j$ with $\mu = 4$ for $1 \leq i < j \leq n$, $\pm 2y_i$ with $\mu = 1$ for $i = 1, \ldots, m$, $\pm y_i \pm y_j$ with $\mu = 1$ for $1 \leq i < j \leq m$ and odd roots $\pm x_i \pm y_j$ with $\mu = -2$ with for $1 \leq i \leq n$, $1 \leq j \leq m$. The quadratic form $Q$ is given by (13) with $k = -2$.

For even $l = 2n$ the restricted even roots are $\pm 2x_i$ with $\mu = 1$ for $i = 1, \ldots, n$, $\pm x_i \pm x_j$ with $\mu = 4$ for $1 \leq i < j \leq n$, $\pm 2y_i$ with $\mu = 1$ for $i = 1, \ldots, m$, $\pm y_i \pm y_j$ with $\mu = 1$ for $1 \leq i < j \leq m$ and odd roots $\pm x_i \pm y_j$ with $\mu = -2$ with for $1 \leq i \leq n$, $1 \leq j \leq m$. The quadratic form $Q$ is given by (13) with $k = -2$.

For the symmetric pairs $(\mathfrak{gl}(n_1 + n_2, m_1 + m_2), \mathfrak{gl}(n_1, m_1) \oplus \mathfrak{gl}(n_2, m_2))$ of type $AIII$ the even type means that $(n_1 - m_1)(n_2 - m_2) \geq 0$ (see [2]). We have then $n = \min(n_1, n_2)$, $m = \min(m_1, m_2)$ and the even roots $\pm x_i$ with $\mu = 2|n_1 - n_2|$, $\pm 2x_i$ with $\mu = 1$ for $i = 1, \ldots, n$, $\pm x_i \pm x_j$ with $\mu = 2$ for $1 \leq i < j \leq n$, $\pm y_i$ with $\mu = 2|n_1 - m_2| - 2y_i$ with $\mu = 1$ for $i = 1, \ldots, m$, $\pm y_i \pm y_j$ with $\mu = 2$ for $1 \leq i < j \leq m$ and odd roots $\pm x_i \pm y_j$ with $\mu = -2, \pm x_i$ with $\mu = -2|n_1 - m_2|$, $\pm y_j$ with $\mu = -2|n_1 - n_2|$ for $1 \leq i \leq n$, $1 \leq j \leq m$. The form $Q$ is given by (13) with $k = -1$.

For the even type $BDI/CII$ pairs $(\mathfrak{osp}(n_1 + n_2, 2m_1 + 2m_2), \mathfrak{osp}(n_1, 2m_1) \oplus \mathfrak{osp}(n_2, 2m_2))$ with $(n_1 - m_1)(n_2 - m_2) \geq 0$ we have again $n = \min(n_1, n_2)$, $m = \min(m_1, m_2)$ and the even roots $\pm x_i$ with $\mu = |n_1 - n_2|$ for $i = 1, \ldots, n$, $\pm x_i \pm x_j$ with $\mu = 1$ for $1 \leq i < j \leq n$, $\pm y_i$ with $\mu = 4|m_1 - m_2|$, $\pm 2y_i$ with $\mu = 3$ for $i = 1, \ldots, m$, $\pm y_i \pm y_j$ with $\mu = 4$ for $1 \leq i < j \leq m$ and odd roots $\pm x_i \pm y_j$ with $\mu = -2, \pm x_i$ with $\mu = -2|m_1 - m_2|$, $\pm y_j$ with $\mu = -2|n_1 - n_2|$ for $1 \leq i \leq n$, $1 \leq j \leq m$. The form $Q$ is given by (13) with $k = -1/2$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let $e_1, \ldots, e_N$ be a basis in $\mathfrak{g}$. The dual space $U(\mathfrak{g})^*$ is known to be the algebra isomorphic to the algebra of formal series $\mathbb{C}[X_1, \ldots, X_N]$, where $X_1, \ldots, X_N \in \mathfrak{g}^*$ is a dual basis (see Dixmier [9], Chapter 2).

By a zonal function for the symmetric pair $X = (\mathfrak{g}, \mathfrak{k})$ we mean a linear functional $f \in U(\mathfrak{g})^*$, which is two-sided $\mathfrak{k}$-invariant:

$$f(xu) = f(ux) = 0, \ x \in \mathfrak{k}, \ u \in U(\mathfrak{g}).$$

The space of such functions we denote $Z(X) \subset U(\mathfrak{g})^*$.

Let $Y = \mathfrak{t}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{t}$ be a subspace in $U(\mathfrak{g})$, on which the zonal functions vanish. Let also $U(\mathfrak{a}) = S(\mathfrak{a})$ be the symmetric algebra of $\mathfrak{a}$.

**Proposition 3.1.**

$$U(\mathfrak{g}) = S(\mathfrak{a}) + Y.$$

**Proof.** Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $R$ be the root system of $\mathfrak{g}$ and $X_\alpha$ are the corresponding root vectors with respect to $\mathfrak{h}$.

Let $\alpha \in \mathfrak{h}^*$ be a root of $\mathfrak{g}$ and $T_\alpha : U(\mathfrak{a}) \to U(\mathfrak{a})$ be the automorphism defined by $x \to x + \alpha(x)$, $x \in \mathfrak{a}$. Define $R_\alpha^\pm : U(\mathfrak{a}) \to U(\mathfrak{a})$ by
\[
R_\alpha^+ = \frac{1}{2} (T_\alpha + T_{-\alpha}), \quad R_\alpha^- = \frac{1}{2} (T_\alpha - T_{-\alpha}).
\]

Let \( X_\alpha \in \mathfrak{g}_\alpha, X_{-\alpha} \in \mathfrak{g}_{-\alpha} \) be corresponding root vectors and define

\[
h_\alpha = [X_\alpha, X_{-\alpha}] / (X_{\alpha}, X_{-\alpha}) \in \mathfrak{h}.
\]

Define also for any \( X \in \mathfrak{g} \)

\[
X^+ = \frac{1}{2} (X + \theta(X)), \quad X^- = \frac{1}{2} (X - \theta(X)).
\]

We will need the following lemma, which can be checked directly.

**Lemma 3.2.** For any \( u \in U(\mathfrak{a}) \) the following equalities hold true:

i) \( X_\alpha^+ u = R_\alpha^+ u X_\alpha^+ - R_\alpha^- u X_\alpha^- \)

ii) \( R_\alpha^- u X_\alpha X_{-\alpha} = T_\alpha u [X_\alpha^+, X_{-\alpha}] \in \mathcal{Y} \)

iii) \( [X_\alpha^+, X_{-\alpha}] = \frac{1}{2} h_\alpha (X_\alpha, X_{-\alpha}) \).

To prove the proposition it is enough to show that for \( q > 0 \)

\[
v = uX_{\alpha_1} \ldots X_{\alpha_q} \in \mathcal{Y}
\]

for any roots \( \alpha_1, \ldots, \alpha_q \), where \( u \in S(\mathfrak{a}) \). We prove this by induction in \( q \).

If \( q = 1 \) and \( w \in S(\mathfrak{a}) \), then by the first part of Lemma 3.2 we have \( R_\alpha^- w X_\alpha^- = R_\alpha^+ w X_\alpha^+ - X_\alpha^+ w \), which clearly belongs to \( \mathcal{Y} \). But any \( u \in S(\mathfrak{a}) \) can be represented in the form \( u = R_\alpha^- w \) for some \( w \in S(\mathfrak{a}) \), therefore \( uX_\alpha^- \in \mathcal{Y} \). Let now \( q > 1 \). Then modulo \( \mathcal{Y} \) we have using Lemma 3.2

\[
R_\alpha^- u X_{\alpha_1} \ldots X_{\alpha_q} = R_\alpha^+ u X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_q} - X_{\alpha_1}^+ u X_{\alpha_2} \ldots X_{\alpha_q}^-
\]

\[
\equiv R_\alpha^+ u X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_q} \equiv R_\alpha^+ u \cdot [X_{\alpha_1}^+, X_{\alpha_2} \ldots X_{\alpha_q}] -
\]

\[
= R_\alpha^+ u \cdot [X_{\alpha_1}^+, X_{\alpha_2}^-] X_{\alpha_3} \ldots X_{\alpha_q}^- + R_\alpha^+ u X_{\alpha_2} \cdot X_{\alpha_3} \ldots X_{\alpha_q}^- \ldots \in \mathcal{Y}
\]

by inductive assumption. \( \square \)

Let \( \alpha \in R \) be a root of \( \mathfrak{g} \) such that the restriction of \( \alpha \) on \( \mathfrak{a} \) is not zero. Let also \( f \in \mathcal{Z}(X) \) be a two sided \( \mathfrak{t} \)-invariant functional on \( U(\mathfrak{g}) \). By Proposition 3.1 \( f \) is uniquely determined by its restriction to \( U(\mathfrak{a}) = S(\mathfrak{a}) \), and thus we can consider \( \mathcal{Z}(X) \) as a subalgebra \( S(\mathfrak{a})^* \).

Identify \( S(\mathfrak{a})^* \) with the algebra of formal power series as follows (see [9]). Let \( e_1, \ldots, e_N \) be a basis in \( \mathfrak{a} \) and \( x_1, \ldots, x_N \in \mathfrak{a}^* \) be the dual basis. Then we can define for any \( f \in S(\mathfrak{a})^* \) the formal power series \( \hat{f} \in \mathbb{C}[[x_1, \ldots, x_N]] \) by

\[
\hat{f} = \sum_{M \in \mathbb{Z}_+^N} f(e_M) x^M,
\]
where
\[ e_M = \frac{1}{m_1! \ldots m_N!} e_1^{m_1} \ldots e_N^{m_N} \in U(\mathfrak{a}), \quad x^M = x_1^{m_1} \ldots x_N^{m_N}. \]

It is easy to see that the operator of multiplication by, say, \( e_1 \) corresponds to the partial derivative \( \frac{\partial}{\partial x_1} \) in this realisation:
\[ \hat{f}(ue_1) = \frac{\partial}{\partial x_1} \hat{f}(u). \]

Similarly, the shift operator \( T_\lambda, \lambda \in \mathfrak{a}^* \) corresponds to multiplication by \( e^\lambda \):
\[ \hat{f}(T_\lambda u) = e^\lambda \hat{f}(u), \quad u \in S(\mathfrak{a}). \]

Let \( S \subset S(\mathfrak{a})^* \) be the multiplicative set generated by \( e^{2\alpha} - 1, \alpha \in R(X) \) and \( S(\mathfrak{a})^*_{loc} = S^{-1}S(\mathfrak{a})^* \) be the corresponding localisation.

Choose an orthogonal basis \( h_i \in \mathfrak{h}, i = 1, \ldots, r \) and define the quadratic Casimir element \( C_2 \) from the centre \( Z(\mathfrak{g}) \) of the universal enveloping algebra \( U(\mathfrak{g}) \) by
\[ C_2 = \sum_{i=1}^{r} \frac{h_i^2}{(h_i, h_i)} + \sum_{\alpha \in R} \frac{X_\alpha X_{-\alpha}}{(X_{-\alpha}, X_\alpha)}, \quad (14) \]
where the brackets denote the invariant bilinear form on \( \mathfrak{g} \).

It can be defined invariantly as an image of the element of \( \mathfrak{g} \otimes \mathfrak{g} \) representing the invariant form itself and determines the corresponding Laplace–Beltrami operator \( \mathfrak{L} \) on \( X \) acting on left \( \mathfrak{g} \)-invariant functions \( f \in \mathfrak{F}(X) = U(\mathfrak{g})^\mathfrak{g} \) (which are algebraic analogues of the functions on the symmetric superspace \( X = G/K \)) by
\[ \mathfrak{L}f(x) = f(xC_2), \quad x \in U(\mathfrak{g}). \]

The restriction of the invariant bilinear form on \( \mathfrak{g} \) to \( \mathfrak{a} \) is a non-degenerate form, which we also denote by \( (, ) \). Let \( \Delta \) be the corresponding Laplace operator on \( \mathfrak{a} \) and \( \partial_\alpha, \alpha \in \mathfrak{a}^* \) be the differential operator on \( \mathfrak{a} \) defined by
\[ \partial_\alpha e^\lambda = (\alpha, \lambda)e^\lambda. \quad (15) \]

Consider the following operator \( \mathfrak{L}_{rad} : S(\mathfrak{a})^* \to S(\mathfrak{a})^* \) defined by
\[ \mathfrak{L}_{rad} = \Delta + \sum_{\alpha \in R_+(X)} \mu_\alpha \frac{e^{2\alpha} + 1}{e^{2\alpha} - 1} \partial_\alpha, \quad (16) \]
where the sum is taken over positive restricted roots considered with multiplicities \( \mu_\alpha \). This operator is the radial part of the Laplace–Beltrami operator \( \mathfrak{L} \) in the following sense.
Proposition 3.3. The following diagram is commutative

\[ \begin{array}{ccc} Z(X) & \xrightarrow{\ell} & Z(X) \\ \downarrow i^* & & \downarrow i^* \\ S(\mathfrak{a})_{loc}^* & \xrightarrow{\ell_{rad}} & S(\mathfrak{a})_{loc}^*. \end{array} \] (17)

Proof. For any root \( \alpha \) of \( \mathfrak{g} \) define the operators \( D_\alpha, \partial_\alpha : Z(X) \to S(\mathfrak{a})^* \) by

\[ D_\alpha(f)(u) = f\left( \frac{uX_\alpha X_\alpha}{(X_\alpha, X_\alpha)} \right), \quad \partial_\alpha(f)(u) = f(uh^-), \quad u \in S(\mathfrak{a}), \]

where we consider \( Z(X) \) as a subset of \( S(\mathfrak{a})^* \). One can check that the definition of the operator \( \partial_\alpha \) agrees with (15).

We claim that the operators \( D_\alpha, \partial_\alpha \) in the formal power series realisation satisfy the relation

\[ (e^\alpha - e^{-\alpha})D_\alpha = (-1)^{p(\alpha)}e^\alpha \partial_\alpha, \] (18)

where \( p(\alpha) \) is parity function: \( p(\alpha) = 0 \) for even roots and \( p(\alpha) = 1 \) for odd roots.

Indeed, since the restriction of \( f \in \mathfrak{g}(X) \) on \( Y \) vanishes, from parts \( ii) \) and \( iii) \) of Lemma 3.2 it follows that

\[ \hat{f} \left( \frac{R^-\alpha uX_\alpha X_\alpha}{(X_\alpha, X_\alpha)} \right) = \hat{f} \left( \frac{T_\alpha u[X_\alpha^+, X_\alpha^-]}{(X_\alpha, X_\alpha)} \right) = \frac{(-1)^{p(\alpha)}}{2} e^\alpha \hat{f}(uh^-), \]

since \( (X_\alpha, X_\alpha) = (-1)^{p(\alpha)}(X_\alpha, X_\alpha) \). Since \( R^-\alpha = \frac{1}{2}(T_\alpha - T_\alpha) \) we have

\[ \hat{f}(R^-\alpha uX_\alpha X_\alpha) = \frac{1}{2}(e^\alpha - e^{-\alpha})\hat{f}(uX_\alpha X_\alpha) \]

and thus the claim.

In the localisation \( S(\mathfrak{a})_{loc}^* \) we can write the operator \( D_\alpha \) as

\[ D_\alpha = \frac{(-1)^{p(\alpha)}e^\alpha}{e^\alpha - e^{-\alpha}} \partial_\alpha = \frac{(-1)^{p(\alpha)}e^{2\alpha}}{e^{2\alpha} - 1} \partial_\alpha \] (19)

and extend it to the whole \( S(\mathfrak{a})_{loc}^* \).

Summing over all \( \alpha \in R \) and taking into account that the multiplicities \( \mu_\alpha \) are defined with the sign \( (-1)^{p(\alpha)} \) after the restriction to \( \mathfrak{a} \) we have the second term in formula (16). One can check that the first part of the Casimir operator (14) gives the Laplace operator \( \Delta \). \( \square \)

Corollary 3.4. For 4 classical series of symmetric pairs (12) of even type the radial parts of Laplace–Beltrami operators are conjugated to the deformed CMS operators of classical type.

More precisely, for the classical series \( X = (\mathfrak{gl}(n, 2m), \mathfrak{osp}(n, 2m)) \) the corresponding radial part (16) is conjugated to the deformed CMS operator related to generalised root
system of type $A(n-1, m-1)$ from [27] with parameter $k = -1/2$ (as it was already pointed out in [26]).

For three other classical series the corresponding radial part is conjugated to the following deformed CMS operator of type $BC(n, m)$ introduced in [27]

$$L = -\Delta_n - k\Delta_m + \sum_{i<j}^n \left( \frac{2k(k+1)}{\sinh^2(x_i - x_j)} + \frac{2k(k+1)}{\sinh^2(x_i + x_j)} \right)$$

$$+ \sum_{i<j}^m \left( \frac{2(k-1) + 1}{\sinh^2(y_i - y_j)} + \frac{2(k-1) + 1}{\sinh^2(y_i + y_j)} \right)$$

$$+ \sum_{i=1}^n \sum_{j=1}^m \left( \frac{2(k+1)}{\sin^2(x_i - y_j)} + \frac{2(k+1)}{\sin^2(x_i + y_j)} \right) + \sum_{i=1}^n \frac{p(p+2q+1)}{\sinh^2 x_i}$$

$$+ \sum_{i=1}^n \frac{4q(q+1)}{\sinh^2 2x_i} + \sum_{j=1}^m \frac{kr(r+2s+1)}{\sinh^2 y_j} + \sum_{j=1}^m \frac{4ks(s+1)}{\sinh^2 2y_j},$$

(20)

where the parameters $k, p, q, r, s$ must satisfy the relation

$$p = kr, \quad 2q + 1 = k(2s + 1).$$

(21)

Indeed, using the description of the restricted roots given above and the definition of the deformed root system of $BC(n, m)$ type from [27], one can check that $n = \min(n_1, n_2), m = \min(m_1, m_2)$ and the parameters

$$k = -1, \quad p = |m_1 - m_2| - |n_1 - n_2| = -r, \quad q = s = -1/2$$

for the symmetric pairs $X = (\mathfrak{gl}(n_1 + n_2, m_1 + m_2), \mathfrak{gl}(n_1, m_1) \oplus \mathfrak{gl}(n_2, m_2))$,

$$k = -\frac{1}{2}, \quad p = |m_1 - m_2| - \frac{1}{2}|n_1 - n_2| = -\frac{1}{2}r, \quad q = 0, \quad s = -\frac{3}{2}$$

for the pairs $X = (\mathfrak{osp}(n_1 + n_2, 2m_1 + 2m_2), \mathfrak{osp}(n_1, 2m_1) \oplus \mathfrak{osp}(n_2, 2m_2))$. For the symmetric pairs $X = (\mathfrak{osp}(2l, 2m), \mathfrak{gl}(l, m))$ we have two different cases depending on the parity of $l$: when $l = 2n$ then

$$k = -2, \quad p = 0 = r, \quad q = s = -\frac{1}{2},$$

and when $l = 2n + 1$ then

$$k = -2, \quad p = -2, \quad r = 1, \quad q = s = -\frac{1}{2}.$$
maps the centre of the universal enveloping algebra of $\mathfrak{gl}(n,2m)$ to the algebra of the deformed CMS integrals $\mathcal{D}_{n,m}$.

4. Symmetric pairs $X = (\mathfrak{gl}(n,2m), \mathfrak{osp}(n,2m))$

Recall that the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(n,2m)$ is the sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{gl}(n) \oplus \mathfrak{gl}(2m)$ and $\mathfrak{g}_1 = V_1 \otimes V_2^* \oplus V_1^* \otimes V_2$, where $V_1$ and $V_2$ are the identical representations of $\mathfrak{gl}(n)$ and $\mathfrak{gl}(2m)$ respectively. As a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ we choose the diagonal matrices.

A bilinear form $(\ ,\ )$ on a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ is said to be even, if $V_0$ and $V_1$ are orthogonal with respect to this form and it is called to be supersymmetric if

$$(v, w) = (-1)^{p(v)p(w)}(w, v)$$

for all homogeneous elements $v, w$ in $V$. If $V$ is endowed with an even non-degenerate supersymmetric form $(\ ,\ )$, then the involution $\theta$ is defined by the relation

$$(\theta(x)v, w) + (-1)^{p(x)p(v)}(v, xw) = 0, \quad x \in \mathfrak{gl}(V). \quad (22)$$

When $\dim V_0 = n$, $\dim V_1 = 2m$ and the form $(\ ,\ )$ coincides with Euclidean structure on $V_0$ and symplectic structure on $V_1$ we can define the orthosymplectic Lie superalgebra $\mathfrak{osp}(n,2m) \subset \mathfrak{gl}(n,2m)$ as

$$\mathfrak{osp}(n,2m) = \{ x \in \mathfrak{gl}(n,2m) : \theta(x) = x \}.$$

Let $\varepsilon_1, \ldots, \varepsilon_{n+2m} \in \mathfrak{h}^*$ be the weights of the identical representation of $\mathfrak{gl}(n,2m)$. It will be convenient also to introduce $\delta_p := \varepsilon_{p+n}$, $1 \leq p \leq 2m$.

The root system of $\mathfrak{g}$ is $R = R_0 \cup R_1$, where

$$R_0 = \{ \varepsilon_i - \varepsilon_j, \delta_p - \delta_q : i \neq j : 1 \leq i, j \leq n, p \neq q, 1 \leq p, q \leq 2m \},$$

$$R_1 = \{ \pm (\varepsilon_i - \delta_p), \quad 1 \leq i \leq n, 1 \leq p \leq 2m \}$$

are even and odd (isotropic) roots respectively. We will use the following distinguished system of simple roots

$$B = \{ \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{2m-1} - \delta_{2m} \}.$$

The invariant bilinear form is determined by the relations

$$(\varepsilon_i, \varepsilon_i) = 1, \quad (\delta_p, \delta_p) = -1$$

with all other products to be zero. The integral weights are
\[ P_0 = \{ \lambda \in \mathfrak{h}^* \mid \lambda = \sum_{i=1}^{n+2m} \lambda_i \varepsilon_i = \sum_{i=1}^{n} \lambda_i \varepsilon_i + \sum_{p=1}^{2m} \mu_p \delta_p, \lambda_i, \mu_j \in \mathbb{Z} \}. \] (23)

The Weyl group \( W_0 = S_n \times S_{2m} \) acts on the weights by separately permuting \( \varepsilon_i, \ i = 1, \ldots, n \) and \( \delta_p, \ p = 1, \ldots, 2m \).

The involution \( \theta \) is acting on \( \mathfrak{h}^* \) by mapping \( \delta_{2j-1} \rightarrow -\delta_{2j}, \ \delta_{2j} \rightarrow -\delta_{2j-1}, \ j = 1, \ldots, m \) and \( \varepsilon_i \rightarrow -\varepsilon_i, \ i = 1, \ldots, n \). The dual \( \mathfrak{a}^* \) of Cartan subspace \( \mathfrak{a} \) can be described as the \( \theta \) anti-invariant subspace of \( \mathfrak{h}^* \)

\[ \mathfrak{a}^* = \{ \theta(x) - x, \ x \in \mathfrak{h}^* \} \]

and is generated by

\[ \tilde{\varepsilon}_i = \varepsilon_i, \ i = 1, \ldots, n, \quad \tilde{\delta}_j = \frac{1}{2}(\delta_{2j-1} + \delta_{2j}), \ j = 1, \ldots, m. \] (24)

The induced bilinear form in this basis is diagonal with

\[ (\tilde{\varepsilon}_i, \tilde{\varepsilon}_i) = 1, \quad (\tilde{\delta}_p, \tilde{\delta}_p) = -\frac{1}{2} \]

Let us introduce the following superanalogue of Gelfand invariants \[19\]

\[ Z_s = \sum_{i_1, \ldots, i_s} (-1)^{p(i_2)+\cdots+p(i_s)} E_{i_1, i_2} E_{i_2, i_3} \cdots E_{i_{s-1}, i_s} E_{i_s, i_1}, \ s \in \mathbb{N}, \] (25)

where \( E_{ij}, i, j = 1, \ldots, n + 2m \) is the standard basis in \( \mathfrak{gl}(n, 2m) \). One can define them also as \( Z_s = \sum_{i=1}^{n+2m} E_{ii}^{(s)} \), where elements \( E_{ij}^{(s)} \) are defined recursively by

\[ E_{ij}^{(s)} = \sum_{l=1}^{n+2m} (-1)^{p(l)} E_{il} E_{lj}^{(s-1)} \] (26)

with \( E_{ij}^{(1)} = E_{ij} \). One can check that these elements satisfy the following commutation relations

\[ [E_{ij}, E_{st}^{(l)}] = \delta_{js} E_{il}^{(l=1)} - (-1)^{(p(i)+p(j))(p(s)+p(t))} \delta_{it} E_{sj}^{(l)} \]

which imply that the elements \( Z_l \) are central.

Let as before \( \mathcal{Z}(X) \subset U(\mathfrak{g})^* \) be the subspace of zonal (two-sided \( \mathfrak{k} \)-invariant) functions and \( S(\mathfrak{a})^* \), \( S(\mathfrak{a})^*_{loc} \) be as in the previous section.

Let \( i^* : \mathcal{Z}(X) \rightarrow S(\mathfrak{a})^*_{loc} \) be the restriction homomorphism induced by the embedding \( i : \mathfrak{a} \rightarrow \mathfrak{g} \).
Theorem 4.1. The restriction homomorphism \( i^* \) is injective and there exists a unique homomorphism \( \psi : Z(\mathfrak{g}) \to \mathcal{D}_{n,m} \) such that the following diagram is commutative

\[
\begin{array}{cccc}
Z(X) & \xrightarrow{L_z} & Z(X) \\
\downarrow i^* & & \downarrow i^* \\
S(a)^*_{loc} & \xrightarrow{(\psi(z))} & S(a)^*_{loc},
\end{array}
\]

where \( L_z \) is the multiplication operator by \( z \in Z(\mathfrak{g}) \). The image of Gelfand invariants (25) are the deformed CMS integrals (4):

\[
\psi(Z_s) = 2^s L_s.
\]

We will call \( \psi \) the radial part homomorphism. For the Casimir element \( \mathcal{C} \) the operator \( \psi(\mathcal{C}) \) is the deformed CMS operator (1).

Proof. Let us prove first that for any \( z \in Z(\mathfrak{g}) \) there exists not more than one element \( \psi(z) \in \mathcal{D}_{n,m} \) which makes the diagram commutative. It is enough to prove this only when \( z = 0 \). Therefore we need to prove the following statement: if \( D \in \mathcal{D}_{n,m} \) and \( D(i^*(f)) = 0 \) for any \( f \in Z(X) \) then \( D = 0 \).

Let \( \mathfrak{a}_{n,m} \subset C(\mathfrak{a}) = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_m^{\pm 1}] \) be the subalgebra consisting of \( S_n \times S_m \)-invariant Laurent polynomials \( f \in C(\mathfrak{a}) \), satisfying the quasi-invariance conditions (2).

Let us take \( f = \phi_\lambda(x) \in Z(X) \) from Proposition 5.7, where \( \lambda \in P^+(X) \) satisfies Kac condition (31). By Proposition 5.7 (which proof is independent from the results of this section) \( i^*(f) \in \mathfrak{a}_{n,m} \) and by Proposition 2.5

\[
D(i^*(f)) = \varphi(D)(\lambda)e^\lambda + \ldots,
\]

where \( \ldots \) mean lower order terms in partial order (10) and \( \varphi \) is the Harish-Chandra homomorphism. If \( D(i^*(f)) = 0 \) then \( \varphi(D)(\lambda) = 0 \) for all \( \lambda \) which are admissible and \( K(\lambda) \) is irreducible. By Proposition 5.7 the set of such \( \lambda \) is dense in Zarisski topology in \( \mathfrak{a}^* \). Therefore \( \varphi(D) = 0 \) and since the Harish-Chandra homomorphism is injective we have \( D = 0 \).

Now let us prove that for every \( z \in Z(\mathfrak{g}) \) element \( \psi(z) \) indeed exists. It is enough to prove this only for the Gelfand generators \( Z_s \). Actually we prove now that \( \psi(Z_s) = 2^s L_s \), where \( L_s \) are the deformed CMS integrals defined by (4).

Let \( Y = \mathfrak{t} U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{t} \) and \( T_\alpha, R_\alpha : U(\mathfrak{a}) \to U(\mathfrak{a}) \) be defined as in the previous section for any root \( \alpha \in \mathfrak{h}^* \) of \( \mathfrak{g} \). For \( \alpha = \varepsilon_i - \varepsilon_j \) choose \( X_\alpha = E_{ij} \in \mathfrak{g}_\alpha \) and define

\[
X_\alpha^{(s)} = E_{ij}^{(s)} , \ h_\alpha^{(s)} = [X_\alpha, X_{-\alpha}^{(s)}] , \ s \in \mathbb{N},
\]

where \( E_{ij}^{(s)} \) are given by (26).
Lemma 4.2. For any \( u \in U(\mathfrak{a}) \) the following equalities hold true:

i) \( R^-_\alpha uX_\alpha X^{(s)}_{-\alpha} - T_\alpha u[X^+_\alpha, (X^{(s)}_{-\alpha})^-] \in Y \)

ii) \([X^+_\alpha, (X^{(s)}_{-\alpha})^-] = \frac{1}{2}(h^{(s)}_{\alpha})^-\), where

\[
h^{(s)}_{\alpha} = [X_{\alpha}, X^{(s)}_{-\alpha}] = E^{(s)}_{ii} - (1)^p(i) + p(j) E^{(s)}_{jj}.
\]

Proof. Since \( \partial_{\alpha} \), \( \partial^{(s)}_{\alpha}, \partial^{(s)}_{i} \) : \( \mathcal{Z}(X) \to S(\mathfrak{a})^* \) by the relations

\[
D^{(s)}_{\alpha}(i^*(f))(u) = f\left(uX_{\alpha}X^{(s)}_{-\alpha}\right), \quad \partial^{(s)}_{\alpha}(i^*(f))(u) = f\left(uh^{(s)}_{\alpha}\right),
\]

\[
\partial^{(s)}_{i}(i^*(f))(u) = f(E^{(s)}_{ii}u)
\]

for any \( f \in \mathcal{Z}(X) \), \( u \in S(\mathfrak{a}) \). Since \( i^* \) is injective these operators are well-defined.

Lemma 4.3. For any root \( \alpha = \varepsilon_i - \varepsilon_j \) of \( \mathfrak{g} \) the operators \( D^{(s)}_{\alpha}, \partial^{(s)}_{\alpha} \) in the formal power series realisation satisfy the relation

\[(e^{\alpha} - e^{-\alpha})D^{(s)}_{\alpha} = e^{\alpha}\partial^{(s)}_{\alpha}, \quad \partial^{(s)}_{\alpha} = \partial^{(s)}_{i} - (1)^p(i) + p(j) \partial^{(s)}_{j}.
\]

Proof. Since the restriction of \( f \in \mathcal{Z}(X) \) on \( Y \) vanishes, from Lemma 4.2 it follows that

\[
\hat{f}(R^-_\alpha uX_{\alpha}X^{(s)}_{-\alpha}) = \hat{f}(T_\alpha u[X^+_\alpha, (X^{(s)}_{-\alpha})^-] = \frac{1}{2}\hat{f}(T_\alpha u(h^{(s)}_{\alpha})^-) = \frac{1}{2}e^\alpha \hat{f}(u(h^{(s)}_{\alpha})^-) = \frac{1}{2}e^\alpha \partial^{(s)}_{\alpha} \hat{f}(u).
\]

Since \( R^-_\alpha = \frac{1}{2}(T_\alpha - T_{-\alpha}) \) we have

\[
\hat{f}(R^-_\alpha uX_{\alpha}X^{(s)}_{-\alpha}) = \frac{1}{2}(e^{\alpha} - e^{-\alpha})\hat{f}(uX_{\alpha}X^{(s)}_{-\alpha}) = \frac{1}{2}(e^{\alpha} - e^{-\alpha})D^{(s)}_{\alpha} \hat{f}(u),
\]

which implies the claim. \( \square \)

Now from the recurrence relation (26) and lemma we have

\[
\partial^{(s)}_{i} = (-1)^p(i)\partial^{(1)}_{i}\partial^{(s-1)}_{i} + \sum_{j \neq i} (-1)^p(j) \frac{e^{2\alpha}}{e^{2\alpha} - 1}(\partial^{(s-1)}_{i} - (1)^p(i) + p(j) \partial^{(s-1)}_{j}),
\]

\( i = 1, \ldots, n + 2m \), where \( \alpha = \varepsilon_i - \varepsilon_j \).

Define the operators \( \hat{\partial}^{(s)}_{i} = (-1)^p(i)\partial^{(s)}_{i} \), then the new operators satisfy the recurrence relation
\[
\hat{\delta}_i^{(s)} = \hat{\delta}_i^{(1)} \hat{\delta}_i^{(s-1)} - \sum_{j \neq i} (-1)^{1-p(j)} \frac{e^{2\alpha}}{e^{2\alpha} - 1} (\hat{\delta}_i^{(s-1)} - \hat{\delta}_j^{(s-1)}), \quad i = 1, \ldots, n + 2m.
\]

After the restriction to \( a \) we have

\[
\delta_{2j-1} = \delta_{2j} = \delta_j, \quad \varepsilon_i = \varepsilon_i, \quad \partial_{n+2j-1} = \partial_{n+2j}.
\]

Let us introduce

\[
x_i = e^{2\xi_i}, \quad i = 1, \ldots, n, \quad y_j = e^{2\delta_j}, \quad j = 1, \ldots, m.
\]

From the above recurrence relations we have \( \partial_{n+2j-1}^{(s)} = \partial_{n+2j}^{(s)} \). We also have

\[
\partial_i(x_i) = \partial_i(e^{2\xi}) = 2\xi_i (E_{ii} e^{2\xi}) = 2x_i, \quad i = 1, \ldots, n,
\]

\[
\partial_{n+2j}(y_j) = \partial_{n+2j}(e^{2\delta}) = 2\delta_j (E_{n+2j} e^{2\delta}) = y_j, \quad j = 1, \ldots, m.
\]

Therefore if we set \( x_{n+j} = y_j, \quad j = 1, \ldots, m \) we will have

\[
\hat{\delta}_i^{(s)} = \hat{\delta}_i^{(1)} \hat{\delta}_i^{(s-1)} - \sum_{j \neq i} (-1)^{1-p(j)} \frac{2^{p(j)} x_i}{x_i - x_j} (\hat{\delta}_i^{(s-1)} - \hat{\delta}_j^{(s-1)}),
\]

and \( \hat{\delta}_i^{(1)} = 2k^{p(i)} x_i \partial_i, \quad i = 1, \ldots, n + m \) with \( k = -1/2 \). So, if we define \( \tilde{\delta}_i^{(s)} = 2^{-s} \hat{\delta}_i^{(s)} \) we will have

\[
\tilde{\delta}_i^{(s)} = \tilde{\delta}_i^{(1)} \tilde{\delta}_i^{(s-1)} - \sum_{j \neq i} k^{1-p(j)} \frac{x_i}{x_i - x_j} (\tilde{\delta}_i^{(s-1)} - \tilde{\delta}_j^{(s-1)}), \quad \tilde{\delta}_i^{(1)} = k^{p(i)} x_i \partial_i, \partial_i x_i,
\]

where \( k = -1/2 \). This last recurrence relation and initial conditions coincide with (3) for this \( k \). So we have

\[
\psi(Z_s) = \sum_{i=1}^{n+2m} \tilde{\delta}_i^{(s)} = \sum_{i=1}^{n+2m} (-1)^{p(i)} \tilde{\delta}_i^{(s)} = 2^s \sum_{i=1}^{n+2m} (-\frac{1}{2})^{-p(i)} \tilde{\delta}_i^{(s)},
\]

which coincides with formula (4) with \( k = -1/2 \). Therefore we proved that \( \psi(Z_s) = 2^s \mathcal{L}_s \in D_{n,m} \).

So it is only left to prove that \( \psi \) is a homomorphism. Let \( \psi(z_1) = D_1, \psi(z_2) = D_2 \) then \( D_1 D_2 \) makes the diagram commutative for \( z_1 z_2 \). From the unicity it follows that \( \psi(z_1 z_2) = D_1 D_2 \). Theorem is proved. \( \square \)

### 5. Spherical modules and zonal spherical functions

Let \( X = (g, \mathfrak{t}) \) be symmetric pair and \( Z(X) \subset U(g)^* \) be the set of the corresponding zonal functions.
We will call such function \( f \in \mathcal{Z}(X) \) zonal spherical function if it is an eigenfunction of the action of the centre \( \mathcal{Z}(g) \) on \( U(g)^* \) (cf. [13]). Recall that the action of \( U(g) \) on \( U(g)^* \) is defined by the formula

\[
(g l)(x) = (-1)^{p(y)p(l)}l(y^tx), \quad x, y \in U(g),
\]

where \( y^t \) is the principal anti-automorphism of \( U(g) \) uniquely defined by the condition that \( y^t = -y, y \in g \).

If \( f \) is a generalised eigenfunction of \( \mathcal{Z}(X) \), we will call it generalised zonal spherical function on \( X \). The appearance of such functions is a crucial difference of the super case from the classical one.

Let now \( U \) be a finite dimensional \( g \)-module and \( U^\mathfrak{k} \) be the space of all \( \mathfrak{k} \)-invariant vectors \( u \in U \) such that \( xu = 0 \) for all \( x \in \mathfrak{k} \) and additionally that \( gu = u \) for all \( g \in O(n) \subset OSP(n, 2m) \). The last assumption is not essential but will be convenient. This allows us to exclude the possibility of tensor multiplication by one-dimensional representation given by Berezinian. Consider also similar space \( U^{*\mathfrak{k}} \) for the dual module \( U^* \).

For any \( u \in U^\mathfrak{k} \) and \( l \in U^{*\mathfrak{k}} \) we can consider the corresponding zonal function \( \phi_{u,l}(x) \in \mathcal{Z}(X) \subset U(g)^* \) defined by

\[
\phi_{u,l}(x) := l(xu), \quad x \in U(g).
\]

We denote the linear space of such functions for given \( U \) as \( \mathcal{Z}(U) \).

**Definition 5.1.** A finite dimensional \( g \)-module \( U \) is called spherical if the space \( U^\mathfrak{k} \) is non-zero.

The following result should be true in general, but we will prove it only in the special case for the symmetric pair \( X = (gl(n, 2m), osp(n, 2m)) \).

So from now on we assume that \( g = gl(n, 2m) \), \( \mathfrak{k} = osp(n, 2m) \).

**Theorem 5.2.** Let \( U \) be an irreducible spherical module. Then \( \dim \mathcal{Z}(U) = 1 \) and the corresponding function (29) is zonal spherical.

**Proof.** Let \( \theta \) be the automorphism of \( g = gl(n, 2m) \) defined in the previous section. It is easy to check that the correspondence

\[
F(U) = U^\theta, \quad x \circ u = \theta(x)u
\]

defines a functor on the category of finite dimensional \( g \)-modules. From the definition (22) it follows that for the standard representation we have \( U^\theta = U^* \). It turns out that this is true for any irreducible module.
Proposition 5.3. For any finite dimensional irreducible $\mathfrak{g}$-module $U$

$$U^\theta = U^*.$$

Proof. In the case of Lie algebras the proof uses the fact that the longest element of the

corresponding Weyl group $W$ maps the Borel subalgebra to the opposite one. In our case

we do not have proper Weyl group, so we have to choose a special Borel subalgebra to

prove this.

Let $\theta$ and $V$ be the same as in the previous section. Choose a basis in $V$:

$$V_0 = \langle f_1, \ldots, f_n \rangle, \quad V_1 = \langle f_{n+1}, \ldots, f_{n+2m} \rangle$$

such that

$$(f_i, f_{n-i+1}) = 1, i = 1, \ldots, n, \quad (f_{n+j}, f_{2m+n-j+1}) = 1 = -(f_{2m+n-j+1}, f_{n+j}),$$

where $j = 1, \ldots, m$ and other products are 0.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra consisting of the matrices, which are diagonal in

this basis, and the subalgebras $\mathfrak{n}_+, \mathfrak{n}_- \subset \mathfrak{g}$ be the set of upper triangular and low trian-
gular matrices respectively in the basis $(f_{n+1}, \ldots, f_{n+m}, f_1, \ldots, f_n, f_{n+m+1}, \ldots, f_{n+2m})$.

Consider the automorphism $\omega$ acting by conjugation by the matrix $C$ with

$$C_{i,n-i+1} = 1, \quad i = 1, \ldots, n, \quad C_{n+j,n+2m-j+1} = 1, \quad j = 1, \ldots, 2m$$

and all other entries being zero, corresponding to the product of two longest elements of

groups $S_n$ and $S_{2m}$.

Then one can check that $\theta(\mathfrak{h}) = \mathfrak{h}$, $\theta(\mathfrak{n}_+) = \mathfrak{n}_+$, $\omega(\mathfrak{h}) = \mathfrak{h}$, $\omega(\mathfrak{n}_+) = \mathfrak{n}_-$, and $\theta(h) + \omega(h) = 0$ for any $h \in \mathfrak{h}$.

Let $v \in U$ be highest weight vector with respect to Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}_+$ and let

$\lambda \in \mathfrak{h}^*$ be its weight. Then for module $U^\theta$ we also have $\mathfrak{n}_+ \circ v = \theta(\mathfrak{n}_+) v = \mathfrak{n}_+ v = 0$ and

$\theta(\lambda) \in \mathfrak{h}^*$ is the weight $v$ in $U^\theta$.

Let $u \in U$ be the highest vector with respect to Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}_-$, then the

corresponding weight is $\omega(\lambda)$ (cf. [4], Ch. 7, prop. 11).

Let $u^* \in U^*$ be the linear functional such that $u^*(u) = 1$ and $u^*(u') = 0$ for any other
eigenvector $u'$ of $\mathfrak{h}$ in $\in U$. Then it is easy to see that $\mathfrak{n}_+ u^* = 0$ and its weight with

respect to Cartan subalgebra $\mathfrak{h}$ is $-\omega(\lambda)$.

So we see that both irreducible modules $U^\theta$ and $U^*$ have the same highest weights

$\theta(\lambda) = -\omega(\lambda)$ with respect to the same Borel subalgebra. Therefore they are isomor-

phic. $\square$

Corollary 5.4. Every finite dimensional irreducible $\mathfrak{g}$-module $U$ has even non-degenerate

bilinear form $(,)$ such that
\[(\theta(x)u, v) + (-1)^{p(x)p(u)}(u, xv) = 0, \quad u, v \in U, \quad x \in \mathfrak{g}.\]

The \(\mathfrak{g}\)-modules \(U\) and \(U^*\) are isomorphic as \(\mathfrak{k}\)-modules.

Let us choose now the standard Borel subalgebra \(\mathfrak{b}\) consisting of upper triangular matrices in the basis \(e_1, \ldots, e_{n+2m}\) in \(V\) such that

\[V_0 = \langle e_1, \ldots, e_n \rangle, \quad V_1 = \langle e_{n+1}, \ldots, e_{n+2m} \rangle,\]

\[(e_i, e_i) = 1, \quad i = 1, \ldots, n, \quad (e_{n+2j-1}, e_{n+2j}) = -(e_{n+2j}, e_{n+2j-1}) = 1, \quad j = 1, \ldots, m \quad (\text{with all other products being zero}).\]

For every \(\mathfrak{g}\) module \(U\) we will denote by \(U^\mathfrak{k}\) the subspace of \(\mathfrak{k}\)-invariant vectors.

**Proposition 5.5.** For every irreducible finite dimensional \(\mathfrak{g}\)-module \(U\) we have

\[\dim U^\mathfrak{k} \leq 1.\]

**Proof.** Prove first that the map

\[\mathfrak{k} \times \mathfrak{b} \longrightarrow \mathfrak{g}, \quad (x, y) \rightarrow x + y\]

is surjective. The kernel of this map coincides with the set of the pairs \((x, -x), x \in \mathfrak{k} \cap \mathfrak{b}, which is the linear span of vectors

\[E_{n+2j-1,n+2j-1} - E_{n+2j,n+2j}, \quad E_{n+2j-1,n+2j}, \quad j = 1, \ldots, m\]

and has the dimension \(2m\). Since

\[\dim \mathfrak{k} + \dim \mathfrak{b} - 2m = \frac{1}{2}n(n - 1) + m(2m + 1) + 2nm\]

\[+ \frac{1}{2}(n + 2m)(n + 2m + 1) - 2m = (n + 2m)^2 = \dim \mathfrak{g},\]

which implies the claim.

Let \(v \in U\) be the highest weight vector with respect to Borel subalgebra \(\mathfrak{b}\) and \(w\) be vector invariant with respect to \(\mathfrak{k}\): \(yw = 0\) for all \(y \in \mathfrak{k}\), and such that \((w, v) = 0\). We claim that this implies that \(w = 0\).

To show this we prove by induction in \(N\) that \((w, x_1x_2 \ldots x_Nv) = 0\) for all \(x_1, \ldots, x_N \in \mathfrak{g}\). This is obviously true when \(N = 0\). Suppose that this is true for \(N\). Take any \(x \in \mathfrak{g}\) and represent it in the form \(x = y + z, \quad y \in \mathfrak{k}, \quad z \in \mathfrak{b}\), so that

\[(w, xx_1x_2 \ldots x_Nv) = (w, yx_1x_2 \ldots x_Nv) + (w, zx_1x_2 \ldots x_Nv).\]

\[\text{A different proof in the general even type case can be found in [1].}\]
By definition we have \( (w, yx_1x_2 \ldots x_Nv) = \pm (yw, x_1x_2 \ldots x_N) = 0 \). We have
\[
(w, zx_1x_2 \ldots x_Nv) = (w, [z, x_1x_2 \ldots x_N]v) \pm (w, x_1x_2 \ldots x_Nzv) = 0.
\]

By inductive assumption
\[
(w, [z, x_1x_2 \ldots x_N]v) = (w, [z, x_1]x_2 \ldots x_Nv) \cdots \pm (w, x_1x_2 \ldots [z, x_N]v) = 0
\]
and since \( zv = cv \) for any \( z \in \mathfrak{b} \) we also have \( (w, x_1x_2 \ldots x_Nzv) = 0 \). Thus we have \((w, u) = 0 \) for any \( u \in \mathcal{U} \), so \( \omega = 0 \) since the form is non-degenerate.

Let now \( w_1, w_2 \) be two \( \mathfrak{t} \)-invariant non-zero vectors. Then we have
\[
(w_1, v) = c_1 \neq 0, \quad (w_2, v) = c_2 \neq 0,
\]
so \( (c_1w_1 - c_2w_2, v) = 0 \) and thus \( c_1w_1 = c_2w_2 \). Thus the dimension of the space \( \mathcal{U}^t \) of \( \mathfrak{t} \)-invariant vectors in \( \mathcal{U} \) can not be greater than \( 1 \). □

Now let us deduce the Theorem. Let \( u \in \mathcal{U}^t \) be a non-zero vector, then by Corollary 5.4 there exists a non-zero \( l \in \mathcal{U}^{*t} \). By Proposition 5.5 they are unique up to a multiple, therefore \( \dim \mathcal{Z}(\mathcal{U}) \leq 1 \). So we only need to show that \( \mathcal{Z}(\mathcal{U}) \neq 0 \).

From the proof of Proposition 5.5 it follows that \( l(v) \neq 0 \) for a highest vector \( v \in \mathcal{U} \). Since module \( \mathcal{U} \) is irreducible \( v = xu \) for some \( x \in \mathcal{U}(\mathfrak{g}) \), so \( \phi_{u,l} \neq 0 \) and thus \( \mathcal{Z}(\mathcal{U}) \neq 0 \).

Since the space \( \mathcal{Z}(\mathcal{U}) \) is one-dimensional and centre \( \mathcal{Z}(\mathfrak{g}) \) preserves \( \mathcal{Z}(\mathcal{U}) \), it follows that the function \( \phi_{u,l} \) is zonal spherical. □

Now we would like to describe the conditions on the highest weights for irreducible modules to be spherical.

Let \( \varepsilon_i \) be the basis in \( \mathfrak{h}^* \) dual to the basis \( E_{ii}, i = 1, \ldots, n + 2m \). Let us call the weight
\[
\lambda = \sum_{i=1}^{n+2m} \lambda_i \varepsilon_i \in \mathfrak{h}^*
\]
admissible for symmetric pair \( X \) if
\[
\lambda_i \in 2\mathbb{Z}, \ i = 1, \ldots, n, \ \lambda_{n+2j-1} = \lambda_{n+2j} \in \mathbb{Z}, \ j = 1, \ldots, m.
\]
Denote the set of all such weights as \( P(X) \). Let also \( P^+(X) \subset P(X) \) be the subset of highest admissible weights:
\[
\lambda_1 \geq \cdots \geq \lambda_n, \ \lambda_{n+1} \geq \cdots \geq \lambda_{n+2m}.
\]

Let \( U = L(\lambda) \) be a finite-dimensional irreducible module with highest weight \( \lambda \) and \( U^* = L(\mu) \). Proposition 5.7 implies that both \( \lambda \) and \( \mu \) are admissible. We conjecture that this condition is also sufficient.
Conjecture. If the highest weights $\lambda$ and $\mu$ of both $U$ and $U^*$ are admissible then $U$ is spherical.

We will prove this only under additional assumption of typicality, which is a natural generalisation of Kac’s typicality conditions for Kac modules (see Corollary 6.10 below).

Let us remind the notion of Kac module [17]. Let $g_0$ be the even part of the Lie superalgebra $g$ and

$$p = p_0 \oplus p_1,$$

where $p_0 = g_0$ and $p_1 \subset g_1$ be the linear span of positive odd root subspaces.

Let $V^{(0)}$ be irreducible finite dimensional $g_0$-module. Define the structure of $p$-module on it by setting $p_1 V^{(0)} = 0$. The Kac module is defined as an induced module by

$$K(V^{(0)}) = U(g) \otimes_{U(p)} V^{(0)}$$

If $V^{(0)} = L^{(0)}(\lambda)$ is the highest weight $g_0$-module with weight $\lambda$, then the corresponding Kac module is denoted by $K(\lambda)$.

The following theorem describes the main properties of the Kac modules.

Recall that $\mathfrak{k} = \mathfrak{osp}(n, 2m) \subset g = \mathfrak{gl}(n, 2m)$.

**Theorem 5.6.** 1) We have the isomorphism of $\mathfrak{k}$-modules

$$K(\lambda) = U(\mathfrak{k}) \otimes_{U(\mathfrak{k}_0)} L^{(0)}(\lambda).$$

2) $K(\lambda)$ is projective as $\mathfrak{k}$ module.

3) As $g$ modules

$$K(\lambda)^* = K(2\rho_1 - w_0(\lambda)),$$

where $2\rho_1$ is the sum of odd positive roots and $w_0$ is the longest element of the Weyl group $S_n \times S_{2m}$.

4) $K(\lambda)$ is spherical if and only if $\lambda \in P^+(X)$, in which case

$$\dim K(\lambda)^{\mathfrak{k}} = 1.$$

**Proof.** We start with the following important fact, which can be easily checked:

$$(1 + \theta)p_1 = \mathfrak{k}_1,$$

where $\mathfrak{k}_1$ is the odd part of $\mathfrak{k}$. Let

$$\varphi : U(\mathfrak{k}) \otimes_{U(\mathfrak{k}_0)} L^{(0)}(\lambda) \to K(\lambda)$$
be the homomorphism of \(\mathfrak{g}\)-modules induced by natural inclusion \(L^{(0)}(\lambda) \subset K(\lambda)\). Let \(p_{-1}\) be the linear span of negative odd root subspaces and consider the filtration on \(K(\lambda)\) such that

\[
L^{(0)}(\lambda) = K_0 \subset K_1 \subset \cdots \subset K_{N-1} \subset K_N = K(L^{(0)}(\lambda)),
\]

where \(N = \dim p_{-1}\) and \(K_r\) is the linear span of \(x_1 \ldots x_s v, v \in L^{(0)}(\lambda)\), where \(x_1, \ldots, x_s \in p_{-1}, s \leq r\).

Now let us prove by induction in \(r\) that \(K_r \subset \text{Im } \varphi\). Case when \(r = 0\) is obvious. Let \(K_r \subset \text{Im } \varphi\), then \((x + \theta(x))K_r \subset \text{Im } \varphi\) for all \(x \in p_{-1}\) since \(x + \theta(x) \in \mathfrak{g}\). Since \(\theta(p_{-1}) = p_1\) we have \(\theta(x)K_r \subset K_{r-1}\).

Therefore \(xK_r \subset \text{Im } \varphi\) and thus \(K(\lambda) = K_N \subset \text{Im } \varphi\). This means that the homomorphism \(\varphi\) is surjective. Since both modules have the same dimension \(\varphi\) is an isomorphism. This proves the first part of the theorem.

Part 2) now follows since every induced module from \(p_0\) to \(p\) is projective, see [36]. Part 3) can be found in Brundan [6], see formula (7.7).

So we only need to prove part 4). From the first part we have the isomorphism of the vector spaces

\[
(K(\lambda)^*)^\mathfrak{g} = (L^{(0)}(\lambda)^*)^\mathfrak{p}_0.
\]

Thus we reduced the problem to the known case of Lie algebras. In particular, according to [11]

\[
\dim(L^{(0)}(\lambda)^*)^\mathfrak{p}_0 = 1
\]

if \(\lambda \in P^+(X)\) and 0 otherwise. From part 3) it follows that the same is true for the module \(K(\lambda)\). \(\square\)

We need some formula for the zonal spherical functions related to irreducible modules.

Let \(W(X) = S_n \times S_m\) be the restricted Weyl group. It acts naturally on \(\mathfrak{a}^*\) permuting \(\tilde{\xi}_i, \tilde{\delta}_j\) given by (24). Let

\[
C(\mathfrak{a}) = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_m^{\pm 1}]
\]

be the subalgebra of \(U(\mathfrak{a})^*\), where \(x_i = e^{2\tilde{\xi}_i}, y_j = e^{2\tilde{\delta}_j}\). The subalgebra \(\mathfrak{A}_{n,m} \subset C(\mathfrak{a})\) consists of \(S_n \times S_m\)-invariant Laurent polynomials \(f \in C(\mathfrak{a})\) satisfying the quasi-invariance conditions (2).

**Proposition 5.7.** Let \(L(\lambda)\) be an irreducible spherical finite-dimensional module and \(\phi_\lambda(x)\) be the corresponding zonal spherical function (29). Then its restriction to \(U(\mathfrak{a})\) belongs to \(\mathfrak{A}_{n,m}\) and has a form
\[ i^*(\phi_\lambda(x)) = \sum_{\mu \preceq \lambda} c_{\lambda,\mu} e^\mu(x), \quad x \in U(a), \quad (30) \]

where \( \lambda \in P^+(X) \), \( \mu \in P(X) \) and \( c_{\lambda,\lambda} = 1 \).

The restrictions to \( a \) of the weights \( \lambda \), for which \( L(\lambda) \) is an irreducible spherical finite-dimensional module, are dense in Zariski topology in \( a^* \).

**Proof.** The proof of the form (30) and of the invariance under \( W(X) \) can be reduced to the case of Lie algebras by considering \( L(\lambda) \) as a module over \( g_0 \) (see e.g. Goodman–Wallach [11]). Note that Goodman and Wallach consider the representations of Lie groups (rather than Lie algebras), so in order to use their result we need the invariance of vector under \( O(n) \), which is assumed in the definition of spherical modules.

To prove the quasi-invariance conditions we use the fact that \( i^*(\phi_\lambda) \) is an eigenfunction of the Laplace–Beltrami operator on \( X \). Since the radial part of this operator has the form (16) we see that the derivative \( \partial_\alpha \phi_\lambda \) must vanish when \( e^{2\alpha} - 1 = 0 \) for all roots \( \alpha \), which implies the conditions (2) in the variables \( x_i, y_j \).

According to [16] Kac module \( K(\lambda) \) is irreducible if and only if Kac’s typicality conditions

\[ \prod_{i=1}^{2m} \prod_{j=1}^{n} (\lambda + \rho, \varepsilon_i - \delta_j) \neq 0 \quad (31) \]

are satisfied, where \( \rho \) is given by

\[ \rho = \frac{1}{2} \sum_{i=1}^{n} (n - 2m - 2i + 1)\varepsilon_i + \frac{1}{2} \sum_{j=1}^{2m} (2m + n - 2j + 1)\varepsilon_{n+j}. \quad (32) \]

Let \( \lambda \in P^+(X) \) be an admissible weight, satisfying this condition. Then the corresponding Kac module \( K(\lambda) \) is irreducible and spherical by Theorem 5.6. Since such \( \lambda \) are dense in \( a^* \) the proposition follows. \( \square \)

6. Spherical typicality

Let \( g = \mathfrak{gl}(n,2m) \) and \( V \) be finite-dimensional irreducible \( g \)-module. By Schur lemma any element from the centre \( Z(g) \) of the universal enveloping algebra \( U(g) \) acts as a scalar in \( V \) and therefore we have the homomorphism

\[ \chi_V : Z(g) \longrightarrow \mathbb{C}, \quad (33) \]

which is called the central character of \( V \).

The following notion is an analogue of Kac’s typicality conditions [16].
Definition 6.1. Irreducible finite-dimensional spherical module \( L(\lambda) \) is called **spherically typical** if it is uniquely defined by its central character among the spherical irreducible \( \mathfrak{g} \)-modules.

In other words, if \( L(\mu) \) is another irreducible finite-dimensional spherical \( \mathfrak{g} \)-module and \( \chi_{\lambda} = \chi_{\mu} \), then \( \lambda = \mu \).

In order to formulate the conditions on the highest admissible weight to be spherically typical we need several results from the representation theory of Lie superalgebras [16, 22, 25].

Let \( \varphi : Z(\mathfrak{g}) \rightarrow S(\mathfrak{h}) \) be the Harish-Chandra homomorphism [16]. Let \( \varepsilon_i \in \mathfrak{h}^* \), \( i = 1, \ldots, n + 2m \) be the same as in the previous section and \( \rho \) is given by (32).

**Theorem 6.2.** [25] The image of the Harish-Chandra homomorphism \( \varphi \) is isomorphic to the algebra of polynomials from \( P(\mathfrak{h}^*) = S(\mathfrak{h}) \), which have the following properties:

1. \( f(w(\lambda + \rho)) = f(\lambda + \rho), w \in S_n \times S_{2m}, \lambda \in \mathfrak{h}^*, \)  
2. for all odd roots \( \alpha \in \mathfrak{h}^* \) and \( \lambda \in \mathfrak{h}^* \) such that \( (\lambda + \rho, \alpha) = 0 \)

\[ f(\lambda + \alpha) = f(\lambda). \]

It is generated by the following polynomials:

\[ P_r(\lambda) = \sum_{i=1}^{n} (\lambda + \rho, \varepsilon_i)^r - \sum_{j=1}^{2m} (\lambda + \rho, \varepsilon_{n+j})^r, \quad r \in \mathbb{N}. \tag{34} \]

We are going to apply this theorem in the case of admissible highest weights \( \lambda \in P^+(X) \). For any such weight define two sets

\[ A = \{a_1, \ldots, a_n\}, \quad B = \{b_1, \ldots, b_m\}, \]

where

\[ a_i = (\lambda + \rho, \varepsilon_i) - \frac{1}{2} (n - 2m - 1) = \lambda_i + 1 - i, \quad i = 1, \ldots, n, \]

\[ b_j = (\lambda + \rho, \varepsilon_{n+2j}) - \frac{1}{2} (n - 2m - 1) = -\lambda_{n+2j} - n + 2j, \quad j = 1, \ldots, m. \]

It is easy to check that this establishes a bijection between the set \( P^+(X) \) of the highest admissible weights and the set \( T \) of pairs \((A, B)\), where \( A, B \subset \mathbb{Z} \) are finite subsets of \( n \) and \( m \) elements respectively, which satisfy the following conditions:

\( i) \) If \( a, \tilde{a} \in A \) and there is no any other element in \( A \) between them (in the natural order in \( \mathbb{Z} \)), then \( a - \tilde{a} \) is odd integer.

\( ii) \) If we denote by \( B - 1 \) the shift of the set \( B \) by \(-1\) then \( B \cap (B - 1) = \emptyset \).

The dominance partial order on the set of admissible highest weights \( P^+(X) \) induces some partial order on \( T \), which we will be denote by the same symbol \( \prec \).
The Harish-Chandra homomorphism defines an equivalence relation on the set $P^+(X)$: 
$\lambda \sim \mu$ if and only if $\chi_\lambda = \chi_\mu$. The following lemma describes the corresponding equivalence relation on the set $\mathcal{T}$.

**Lemma 6.3.** Let $\lambda, \tilde{\lambda}$ be admissible highest weights and $(A, B), (\hat{A}, \hat{B})$ be the corresponding elements in $\mathcal{T}$. Then $\chi_\lambda = \chi_{\tilde{\lambda}}$ if and only if the following conditions are satisfied:

$$A \setminus (B \cup B - 1) = \hat{A} \setminus (\hat{B} \cup \hat{B} - 1),$$

$$(B \cup B - 1) \setminus A = (\hat{B} \cup \hat{B} - 1) \setminus \hat{A}.$$  

**Proof.** Let $C = B \cup (B - 1), \tilde{C} = \hat{B} \cup (\hat{B} - 1)$, then from (34) it follows that

$$\sum_{i=1}^{n} a_i^r - 2m \sum_{i=1}^{n} c_i^r = \sum_{i=1}^{n} \tilde{a}_i^r - 2m \sum_{i=1}^{n} \tilde{c}_i^r, \ r \in \mathbb{N}.$$  

Therefore

$$\sum_{i=1}^{n} a_i^r + 2m \sum_{i=1}^{n} c_i^r = \sum_{i=1}^{n} \tilde{a}_i^r + 2m \sum_{i=1}^{n} \tilde{c}_i^r, \ r \in \mathbb{N}.$$  

Hence the sequences $(a_1, \ldots, a_n, \tilde{c}_1, \ldots, \tilde{c}_{2m})$ and $(\tilde{a}_1, \ldots, \tilde{a}_n, c_1, \ldots, c_{2m})$ coincide up to a permutation. Hence $A \setminus C = \hat{A} \setminus \tilde{C}$ and $C \setminus A = \tilde{C} \setminus \hat{A}$.  

**Proposition 6.4.** If $A \cap B \neq \emptyset$, then there exists $(\hat{A}, \hat{B}) \in \mathcal{T}$, which is equivalent to $(A, B)$, such that $(\hat{A}, \hat{B}) \prec (A, B)$. In particular, every finite equivalence class contains a representative $(A, B)$ such that $A \cap B = \emptyset$.

If $A \cap B = \emptyset$, then the equivalence class of $(A, B)$ in $\mathcal{T}$ is finite and contains $2^s$ elements, where $s = |A \cap (B - 1)|$ and $(A, B) \preceq (\hat{A}, \hat{B})$ for any $(\hat{A}, \hat{B}) \in \mathcal{T}$.

**Proof.** Let $A \cap B \neq \emptyset$. Represent $B \cup (B - 1)$ as the disjoint union of the segments of integers

$$B \cup (B - 1) = \cup_i \Delta_i,$$

where a segment is a finite set of integers $\Delta$ such that $a, b \in \Delta$ and $a \leq c \leq b$ imply $c \in \Delta$. Since $B$ and $B - 1$ do not intersect then any $\Delta_i$ consists of even number of integers and if we set $C_i = B \cap \Delta_i$ then $C_i \cap (C_i - 1) = \emptyset$ and $\Delta_i = C_i \cup (C_i - 1)$.

Consider two cases. In the first case suppose that there exist $i$ and $a \in A \cap C_i$ such that if $a' \in \Delta_i$ and $a' < a$ then $a' \notin A$. Let $\Delta_i = [c, d]$ and set

$$\hat{A} = (A \setminus \{a\}) \cup \{c - 1\}, \quad \hat{B} = B \setminus \{a\} \cup \{c - 1\}.$$
We need to prove that \((\tilde{A}, \tilde{B})\) is equivalent to \((A, B)\) and that \((\tilde{A}, \tilde{B}) \in \mathcal{T}\).

Indeed, it is clear that \(c - 1 \notin B \cup (B - 1)\) and since \(a - (c - 1)\) is an even number we have \(c - 1 \notin A\). Therefore \((\tilde{A}, \tilde{B})\) is equivalent to \((A, B)\).

It is easy to see that if \(a' \in A\) is a neighbour of \(a\) (in the natural order on \(A\)) then the difference \(a' - (c - 1)\) is an odd integer. Further we have

\[
\tilde{B} \cup (\tilde{B} - 1) = \cup_{j \neq i} \Delta_j \cup [c - 1, a - 1] \cup [a + 1, d],
\]

which proves that \((\tilde{A}, \tilde{B}) \in \mathcal{T}\). Since \(c - 1 < a\) we have \((\tilde{A}, \tilde{B}) \prec (A, B)\) (see [7]).

Now suppose that the conditions of the first case are not fulfilled. From the assumption \(A \cap B \neq \emptyset\) we see that there exist \(i\) and \(a \in C_i \cap A\) such that there exists \(a' \in \Delta_i \cap A\) and \(a' < a\). Since \([a', a] \subset [c, d]\) we can assume that \(a', a\) are neighbours. Since the difference \(a - a'\) must be odd, we have \(a' \in (C_i - 1)\).

Let \(a_{\text{min}}\) be the minimal element from \(A\). Choose a set \(E = \{e - 1, e\}\) such that \(a_{\text{min}} - e\) is positive odd, \(E \cap (B \cup (B - 1)) = \emptyset\) and define

\[
\tilde{A} = (A \setminus \{a, a'\}) \cup E, \quad \tilde{B} = (B \setminus \{a\}) \cup \{e\}.
\]

It is easy to verify that the \((\tilde{A}, \tilde{B}) \sim (A, B), (\tilde{A}, \tilde{B}) \in \mathcal{T}\) and \((\tilde{A}, \tilde{B}) \prec (A, B)\). Note that in this case we have always an infinite equivalence class.

Now let us prove the second part.

Assume that \(A \cap B = \emptyset\). In that case every \(\Delta_i\) contains not more than one element of \(A\), which must belong to \((C_i - 1)\) since \(A \cap C_i = \emptyset\). Indeed, if \(\Delta_i \cap A\) contains two elements \(a\) and \(a'\) then \([a, a'] \subset \Delta_i\), so we can assume without loss of generality that \(a\) and \(a'\) are neighbours. Since both of them belong to \((C_i - 1)\) the difference \(a' - a\) is even, which is a contradiction.

Let \(a \in A \cap \Delta_i\) and \(\Delta_i = [c, d]\). Let \((\tilde{A}, \tilde{B}) \sim (A, B), (\tilde{A}, \tilde{B}) \in \mathcal{T}\). Then there are two possibilities: \(\tilde{B} \cup (\tilde{B} - 1)\) contains \(\Delta_i\) or not. In the last case the only possibility for \(\Delta_i\) is to be replaced by the union \([c, a - 1] \cup [a + 1, d + 1]\), which leads to \(2^i\) possibilities. \(\square\)

We now need the following Serganova’s lemma [24,22], which connects two highest weight vectors in an irreducible module with respect to an odd reflection.

Let \(g = \mathfrak{gl}(n, l)\) and \(b, b'\) be two Borel subalgebras such that

\[
b' = (b \setminus \{\gamma\}) \cup \{-\gamma\}
\]

where \(\gamma = \varepsilon_p - \varepsilon_{n+q}\) is a simple odd root, and \(\rho\) and \(\rho' = \rho - \gamma\) be the corresponding Weyl vectors.

Let \(V\) be a simple finite-dimensional \(g\)-module and \(v\) and \(v'\) be the highest weight vectors with respect to \(b\) and \(b'\) respectively. Let \(\lambda\) and \(\lambda'\) be the corresponding weights.

Define the sequences \(A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_l\}, A' = \{a'_1, \ldots, a'_n\}, B' = \{b'_1, \ldots, b'_l\}, \) where
\( a_i = (\lambda + \rho, \varepsilon_i), \ i = 1, \ldots, n, \ b_j = (\lambda + \rho, \varepsilon_{n+j}), \ j = 1, \ldots, l, \) (35)
\[ a'_i = (\lambda' + \rho', \varepsilon_i), \ i = 1, \ldots, n, \ b'_j = (\lambda' + \rho', \varepsilon_{n+j}), \ j = 1, \ldots, l. \]

**Lemma 6.5.** [24,22] If \( a_p \neq b_q \) then

\( A' = A, \ B' = B. \)

If \( a_p = b_q \) then

\[ a'_p = a_p + 1, \ b'_q = b_q + 1, \]
and \( a'_i = a_i, i \neq p, \ b'_j = b_j, j \neq q. \)

Define the following operation on the pairs of sequences \( F: (A, B) \rightarrow (\tilde{B}, \tilde{A}) \) recursively. If \( A \) and \( B \) consist of one element \( a \) and \( b \) respectively then

\[ F(a, b) = \begin{cases} (b, a), b \neq a \\ (b + 1, a + 1), b = a. \end{cases} \] (36)

If \( A = \{a_1, \ldots, a_n\}, \ B = \{b_1, \ldots, b_l\} \), then we repeat this procedure for all elements of \( A \) starting with \( a_n \) and moving them to the right of \( B \) using the rule (36).

**Example 6.6.** If \( A = (3, 2, 5), B = (3, 1, 2, 4) \) then

\[ F(A, B) = (\tilde{B}, \tilde{A}) = ((4, 1, 3, 5), (5, 3, 5)). \]

Let \( \mathfrak{b} \) be the standard Borel subalgebra of \( \mathfrak{gl}(n, l) \) and \( \tilde{\mathfrak{b}} \) be its “odd opposite” with the same even part and odd part replaced by the linear span of negative odd root vectors.

**Proposition 6.7.** [24,22] Let \((A, B)\) be the sequences (35) corresponding to the highest weight of \( \mathfrak{g} \)-module \( V \) with respect to standard Borel subalgebra \( \mathfrak{b} \), then the highest weight of \( V \) with respect to the odd opposite Borel subalgebra \( \tilde{\mathfrak{b}} \) is \((\tilde{A}, \tilde{B})\), where \((\tilde{B}, \tilde{A}) = F(A, B)\).

There is a natural bijection \( \sharp : P^+(X) \rightarrow X^+_{n,m} \) mapping the admissible weight \( \lambda = (2\lambda_1, \ldots, 2\lambda_n, \mu_1, \ldots, \mu_m) \) to

\[ \lambda^\sharp = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m). \] (37)

**Theorem 6.8.** A finite-dimensional irreducible \( \mathfrak{g} \)-module \( L(\lambda) \) is spherically typical if and only if \( \lambda \in P^+(X) \) and

\[ \prod_{1 \leq i \leq n, 1 \leq j \leq m} (\lambda + \rho, \varepsilon_i - \delta_{2j}) \neq 0, \] (38)
where $\rho$ is given by (32). In terms of the restricted roots $\alpha \in R(X) \subset \mathfrak{a}^*$ this condition can be written in the invariant form

$$\prod_{\alpha \in R_+(X)} [(\lambda^2 + \rho(k), \alpha) - \frac{1}{2}(\alpha, \alpha)] \neq 0,$$

(39)

where $\rho(k)$ is given by (7) with $k = -\frac{1}{2}$ and the form on $\mathfrak{a}^*$ is induced from the restricted form on $\mathfrak{a}$.

**Proof.** Let us prove first that the conditions are necessary.

Let $L(\lambda)$ be spherically typical irreducible finite dimensional $\mathfrak{g}$-module. Then the dual module $L(\lambda)^*$ is also spherical by Corollary 5.4. Since $L(\lambda)$ is the homomorphic image of $K(\lambda)$ the dual Kac module $K(\lambda)^* \supset L(\lambda)^*$ is spherical. But by part 3 of Theorem 5.6 we have $K(\lambda)^* = K(2\rho_1 - w_0(\lambda))$. Therefore by part 4 of the same theorem we have $2\rho_1 - w_0(\lambda) \in P^+(X)$, which implies that $\lambda \in P^+(X)$.

Now let us prove that $\lambda$ satisfies the condition (38). Suppose that this is not the case. This means that $A \cap B \neq \emptyset$ for the corresponding sets in $\mathcal{T}$. By Proposition 6.4 there exists $(\hat{A}, \hat{B}) \in \mathcal{T}$ such that $(\hat{A}, \hat{B}) \prec (A, B)$ and $(\hat{A}, \hat{B}) \sim (A, B)$. Therefore there exists $\mu \in P^+(X)$ such that $\mu \prec \lambda$ and $\chi_\mu = \chi_\lambda$. Then by Theorem 5.6, part 4 module $K(\mu)$ contains an invariant vector $\omega$. Consider the Jordan–Hölder series of $K(\mu)$:

$$K(\mu) = K_0 \supset K_1 \supset \cdots \supset K_N = 0.$$ 

There exists $0 \leq i \leq N - 1$ such that $\omega \in K_i, \omega \notin K_{i+1}$. Therefore sub-quotient $L(\nu) = K_i/K_{i+1}$ is an irreducible spherical module and $\chi_\nu = \chi_{\mu} = \chi_{\lambda}$. Since $L(\lambda)$ is spherically typical then $\lambda = \nu \preceq \mu$. Contradiction means that (38) is satisfied.

To prove that the conditions are sufficient assume that $\lambda \in P^+(X)$ and (38) is fulfilled. Then $A \cap B = \emptyset$. We claim that $L(\lambda)$ contains a $\mathfrak{t}$-invariant vector. Indeed, since $\lambda \in P^+(X)$ Kac module $K(\lambda)$ contains a $\mathfrak{t}$-invariant vector $\omega$. Let $L(\mu)$ be an irreducible spherical sub-quotient of $K(\lambda)$, such that the image of $\omega$ in $L(\mu)$ is non-zero. As before, $\mu \in P^+(X), \mu \preceq \lambda$ and $\chi_\mu = \chi_{\lambda}$. But since $A \cap B = \emptyset$ then by part 2 of Proposition 6.4 $\lambda \preceq \mu$. Therefore $\lambda = \mu$ and $L(\lambda)$ is spherical.

So we only need to prove that $L(\lambda)$ is spherically typical module.

Let $L(\nu)$ be an irreducible spherical module such that $\chi_{\lambda} = \chi_{\nu}$. As we have already shown $\nu \in P^+(X)$. Suppose that $\nu \neq \lambda$. By Corollary 5.4 the dual module $L(\nu)^*$ also spherical. It is known that $L(\nu)^* = L(-w_0(\mu))$, where $w_0$ is the longest element of the Weyl group $S_n \times S_{2m}$ and $\mu$ is the weight of the highest weight vector in $L(\nu)$ with respect to the odd opposite Borel subalgebra $\mathfrak{b}$. Therefore $\mu \in P^+(X)$.

Let $(\hat{A}, \hat{B}), (\tilde{A}, \tilde{B}) \in \mathcal{T}$ be the pairs corresponding to $\nu$ and $\mu$ respectively. By Proposition 6.7 we have $F(\hat{A}, \hat{B}) = (\tilde{B}, \tilde{A})$. Let us represent $\hat{B} \cup (\hat{B} - 1)$ as the disjoint union of integer segments

$$\hat{B} \cup (\hat{B} - 1) = \bigcup_i \Delta_i.$$
We assume that the labelling is done in the increasing order: if \( i' < i \) and \( x' \in \Delta_i, \ x \in \Delta_i \) then \( x' < x \). Note that since \((\hat{A}, \hat{B}) \in \mathcal{T}\) every segment consists of even number of points. From the description of the equivalence class containing \((A, B)\) (see Proposition 6.4) it follows that for any \( i \) there are the following possibilities:

1) \( \Delta_i \cap \hat{A} = \emptyset \),
2) \( \Delta_i \) contains one element from \( \hat{A} \cap (\hat{B} - 1) \),
3) \( \Delta_i = [c_i, d_i] \) contains one element \( a \in \hat{A} \cap \hat{B} \) and \( \hat{A} \cap [c_i, a - 1] = \emptyset \).

Let us choose \( i \) such that \( \Delta_i \) has property 3) and \( i \) is minimal. Since \( \hat{A} \cap \hat{B} \neq \emptyset \) such \( i \) indeed does exist. Then one can check using Proposition 6.7 that \( a - 1 \in \hat{B} \cup (\hat{B} - 1) \) and the segment containing this element consists of odd number of points. Therefore \((\hat{A}, \hat{B}) \notin \mathcal{T}\), which is a contradiction. Theorem is proved. \( \square \)

**Remark 6.9.** The usual typicality (31) for admissible weights implies the spherical typicality (38), but the converse is not true. As it follows from the proof of the theorem the degree of atypicality [5] of a spherically typical module \( L(\lambda) \) is equal to \( s \), where 2* is the number of elements in the equivalence class of \( \lambda \), and thus can be any number (see Proposition 6.4).

**Corollary 6.10.** If the highest weight \( \lambda \) of the irreducible module \( U = L(\lambda) \) is admissible and satisfies (38) then the highest weight \( \mu \) of \( U^* \) is also admissible and \( U \) is spherical.

We should mention that a different proof of the sphericity of \( U \) under the assumption that the weight \( \lambda \) is large enough was found in [1].

7. Proof of the main theorem

Let \( \mathfrak{D}_{n,m} \) be the algebra of quantum integrals of the deformed CMS system with parameter \( k = -\frac{1}{2} \). It acts naturally on the algebra \( \mathfrak{A}_{n,m} \) of \( S_n \times S_m \)-invariant Laurent polynomials \( f \in \mathbb{C}[x_1^\pm 1, \ldots, x_n^\pm 1, y_1^\pm 1, \ldots, y_m^\pm 1]^{S_n \times S_m} \) satisfying the quasi-invariance condition (2) with the parameter \( k = -\frac{1}{2} \). Let

\[
\mathfrak{A}_{n,m} = \bigoplus_{\chi} \mathfrak{A}_{n,m}(\chi)
\]

be the corresponding decomposition into the direct sum of the generalised eigenspaces (11).

On the set of highest admissible weights \( P^+(X) \) there is a natural equivalence relation defined by the equality of the corresponding central characters (33). Under bijection (37) it goes to the equivalence relation on \( X_{n,m}^+ \) when \( \lambda \sim \mu \) if \( \chi_\lambda = \chi_\mu \).

Consider \( \mathfrak{A}_{n,m} \) as \( Z(\mathfrak{g}) \)-module with respect to the radial part homomorphism \( \psi \).
Theorem 7.1. For any finite dimensional generalised eigenspace $\mathfrak{A}_{n,m}(\chi)$ there exists a unique projective indecomposable module $P$ over $\mathfrak{g}(n,2m)$ and a natural map from $\mathfrak{k}$-invariant part of $P^*$

$$\Psi : (P^*)^\mathfrak{k} \rightarrow \mathfrak{A}_{n,m}(\chi)$$

which is an isomorphism of $Z(\mathfrak{g})$-modules.

Proof. Let $\chi : \mathfrak{O}_{n,m} \rightarrow \mathbb{C}$ be a homomorphism such that $\mathfrak{A}_{n,m}(\chi)$ is a finite dimensional vector space. By Proposition 2.7 there exists $\nu \in X^+_{n,m}$ such that $\chi = \chi_\nu$.

Let $E$ be the equivalence class in $P^+(X)$ corresponding to the equivalence class of $\nu$ via $\equiv$ bijection. Since the corresponding equivalence class is finite by Proposition 6.4 and the definition of the sets $A$ and $B$ there exists $\lambda \in E$, which satisfies condition (38).

By Theorem 6.8 the corresponding irreducible module $L(\lambda)$ is spherically typical. Let $K(\lambda)$ is the corresponding Kac module. Consider the Jordan–Hölder series of $K(\lambda)$:

$$K(\lambda) = K_0 \supset K_1 \supset \cdots \supset K_N = 0.$$ 

Since $\lambda \in P^+(X)$ Kac module $K(\lambda)$ contains a non-zero $\mathfrak{k}$-invariant vector $v$. Let $L(\mu)$ be an irreducible spherical sub-quotient of $K(\lambda)$, such that the image of $v$ in $L(\mu)$ is non-zero. From the proof of Theorem 6.8 it follows that $\mu \in P^+(X)$, $\mu \preceq \lambda$ and $\chi_\lambda = \chi_\mu$. Since $L(\lambda)$ is spherically typical this implies that $\lambda = \mu$. Thus the image of $v$ under natural homomorphism $\varphi : K(\lambda) \rightarrow L(\lambda)$ is not zero: $\varphi(v) \neq 0$.

Let us consider the projective cover $P(\lambda)$ of $L(\lambda)$ (see e.g. [36]) and prove that it is generated by a $\mathfrak{k}$-invariant vector.

Since $K(\lambda)$ is projective as $\mathfrak{k}$-module (see Theorem 5.6), there exists a $\mathfrak{k}$-invariant vector $\omega \in P(\lambda)$ such that $\psi(\omega) \neq 0$ under natural homomorphism $\psi : P(\lambda) \rightarrow L(\lambda)$.

Let $N \subset P(\lambda)$ be the $\mathfrak{g}$-submodule generated by $\omega$. Since $\psi(\omega) \neq 0$ we have that $N$ is not contained in $\text{Ker}(\psi)$, which is known to be the only maximal submodule of $P(\lambda)$ [36]. Therefore $N = P(\lambda)$ and $\omega$ generates $P(\lambda)$ as $\mathfrak{g}$-module.

Now let us construct the map

$$\Psi : (P(\lambda)^*)^\mathfrak{k} \rightarrow \mathfrak{A}_{n,m}(\chi).$$

Let $l \in P(\lambda)^*$ be a $\mathfrak{k}$-invariant linear functional on $P(\lambda)$ and define $\Psi(l)$ as the restriction of $\phi_{\omega,l} = l(x\omega)$ on to $S(\mathfrak{a})$.

Let us show that $\Psi(l) \in \mathfrak{A}_{n,m}(\chi)$. Similarly to the proof of Proposition 5.7 we can claim that $\Psi(l)$ has a form

$$\Psi(l) = \sum_{\mu \in M} c_\mu e^\mu(x), \quad x \in U(\mathfrak{a}),$$

where the sum is taken over some finite subset $M \subset P(X)$. Let $\alpha$ be a root of $\mathfrak{g}$ and $X_\alpha, X_{-\alpha} \in \mathfrak{g}$ be the corresponding root vectors. The product $X_\alpha X_{-\alpha} \in U(\mathfrak{g})$ commutes
with \( a \), so the restriction of \( X_\alpha X_{-\alpha} \phi_{\omega,l} \) has a similar form with the same set \( M \). From (19) it follows that \( \partial_\alpha \Psi(l) \) is divisible by \( e^{2\alpha} - 1 \), which implies the claim.

From Theorem 4.1 it follows that \( \Psi : (P(\lambda))^\mathfrak{t} \to \mathfrak{A}_{n,m}(\chi) \) is a homomorphism of \( Z(\mathfrak{g}) \)-modules.

Now we are going to prove that \( \Psi \) is an isomorphism. Let us prove first that \( \Psi \) is injective. Suppose that \( \Psi(l) = 0 \). This means that the restriction of \( \phi_{\omega,l} \) on \( S(\mathfrak{a}) \) is zero. Therefore by Theorem 4.1 \( \phi_{\omega,l}(x) = l(x\omega) = 0 \) for all \( x \in U(\mathfrak{g}) \). Since \( \omega \) generates \( P(\lambda) \) as \( \mathfrak{g} \)-module we have that \( l = 0 \) and thus \( \Psi \) is injective.

In order to prove that \( \Psi \) is surjective it is enough to show that the dimension of \( (P(\lambda))^\mathfrak{t} \) is not less than the dimension of \( \mathfrak{A}_{n,m}(\chi) \). It is known that \( P(\lambda) \) has the Kac flag (see [36]). Let \( n_{\lambda,\mu} \) be the multiplicity of \( K(\mu) \) in the Kac flag of \( P(\lambda) \). Since Kac modules are projective as \( \mathfrak{t} \)-modules we have

\[
P(\lambda) = \bigoplus_{\mu \in Y} n_{\lambda,\mu} K(\mu),
\]

where \( Y \) is a finite subset of highest weights \( \mu \) of \( \mathfrak{g} \) such that \( n_{\lambda,\mu} > 0 \). Therefore by part 3 of Theorem 5.6

\[
P(\lambda)^* = \bigoplus_{\mu \in Y} n_{\lambda,\mu} K(2\rho_1 - w_0\mu)
\]

as \( \mathfrak{t} \)-modules. Hence by the same Theorem we have

\[
\dim(P(\lambda)^*)^{\mathfrak{t}} = \sum_{\mu \in Y \cap P^+(X)} n_{\lambda,\mu}^{\mathfrak{t}}.
\]

We claim that \( Y \subset P^+(X) \). Indeed by BGG duality for classical Lie superalgebras of type I (see [36]) we have \( n_{\lambda,\mu} = m_{\mu,\lambda} \), where \( m_{\mu,\lambda} \) is the multiplicity of \( L(\lambda) \) in the Jordan–Hölder series of \( K(\mu) \). So if \( n_{\lambda,\mu} > 0 \) then \( L(\lambda) \) is a subquotient of \( K(\mu) \). This means that there are submodules \( M \supset N \) in \( K(\mu) \) such that \( L(\lambda) = M/N \). Therefore we have natural homomorphism \( \phi : K(\lambda) \to M/N \). As we have just seen the image \( \phi(v) \) of \( \mathfrak{t} \)-invariant vector \( v \in K(\lambda) \) is not zero. Since \( K(\lambda) \) is projective as \( \mathfrak{t} \)-module we can lift previous homomorphism to a homomorphism \( K(\lambda) \to M \) and the image of vector \( v \) is a non-zero \( \mathfrak{t} \)-invariant vector in \( M \subset K(\mu) \). Therefore as before \( \mu \in P^+(X) \), so \( Y \subset P^+(X) \). Thus we have

\[
\dim(P(\lambda)^*)^{\mathfrak{t}} = \sum_{\mu \in Y} n_{\lambda,\mu} \geq |Y|.
\]

By Proposition 2.7 the dimension of \( \mathfrak{A}_{n,m}(\chi) \) is equal to the number of \( \tau \in X_{n,m}^+ \) such that \( \chi_\tau = \chi_\nu \), or equivalently, to the number \( |E| \) of the elements in the equivalence class \( E \). Let us show that \( E = Y \).
It is obvious that $Y \subseteq E$. To prove that $E \subseteq Y$ consider $\mu \in E$ and corresponding Kac module $K(\mu)$. By Theorem 5.6 it is spherical. Hence there exists its sub-quotient, which is spherical and irreducible. Since $\lambda$ is spherically typical this sub-quotient must be isomorphic to $L(\lambda)$. By BGG duality $\lambda \in Y$. Thus $\Psi$ is an isomorphism.

Now let us prove the uniqueness of $P$. If $P(\lambda)$ and $P(\mu)$ satisfy the conditions of the theorem then they have the same central characters and from spherical typicality it follows that $\lambda = \mu$. Theorem is proved.  

**Corollary 7.2.** Any finite-dimensional generalised eigenspace of $D_{n,m}$ in $A_{n,m}$ contains at least one zonal spherical function on $X$, corresponding to an irreducible spherically typical $g$-module.

**Remark 7.3.** Since $Y = E$ and all $n_{\lambda,\mu} = 1$ as another corollary we have an effective description of the Kac flag of the projective cover $P(\lambda)$ in the spherically typical case (which may have any degree of atypicality in the sense of [5], see above). Our description is equivalent to Brundan–Stroppel algorithm [5,7] in this particular case.

8. Zonal spherical functions for $X = (\mathfrak{gl}(1, 2), \mathfrak{osp}(1, 2))$

Let us illustrate this in the simplest example $m = 1, n = 1$, corresponding to the symmetric pair $X = (\mathfrak{gl}(1, 2), \mathfrak{osp}(1, 2))$.

The corresponding algebra $A_{1,1}$ consists of the Laurent polynomials $f \in \mathbb{C}[x, x^{-1}, y, y^{-1}]$, satisfying the quasi-invariance condition

$$\left(\partial_x + \frac{1}{2}\partial_y\right)f = 0$$

(40)
on the line $x = y$, where $\partial_x = x \frac{\partial}{\partial x}, \partial_y = y \frac{\partial}{\partial y}$. Writing $f = \sum_{i,j \in \mathbb{Z}} a_{i,j} x^i y^j$, where only finite number of coefficients are non-zero, we can write the quasi-invariance conditions as an infinite set of linear relations

$$\sum_{i+j = l} (2i+j)a_{i,j} = 0, \quad l \in \mathbb{Z}.$$  

Note that the algebra $A_{1,1}$ is naturally $\mathbb{Z}$-graded by the degree defined for the Laurent monomial $x^i y^j$ as $i+j$, so we have one linear relation in each degree.

The radial part of the Laplace–Beltrami operator (16) in these coordinates has the form

$$L_2 = \partial_x^2 - \frac{1}{2}\partial_y^2 - \frac{x+y}{x-y}(\partial_x + \frac{1}{2}\partial_y).$$

(41)

It commutes with the grading (momentum) operator $L_1 = \partial_x + \partial_y$, but in contrast with the usual symmetric spaces these two do not generate the whole algebra of the deformed CMS integrals $D_{1,1}$: one has to add the third order quantum integral
\[
\mathcal{L}_3 = \partial_x^2 + \frac{1}{4} \partial_y^2 - \frac{3}{2} \frac{x + y}{x - y} (\partial_x^2 - \frac{1}{4} \partial_y^2) + \frac{3}{4} \frac{x^2 + 4xy + y^2}{(x - y)^2} (\partial_x + \frac{1}{2} \partial_y). \quad (42)
\]

To describe the corresponding spectral decomposition let us introduce the functions

\[
\varphi_{ij} = x^i y^j - \frac{2i + j}{2i + j - 1} x^{i-1} y^{j+1}, \quad 2i + j \neq 1, \quad (43)
\]

\[
\psi_i = x^{i+1} y^{-1 - 2i} + x^{i-1} y^{1 - 2i}, \quad i \in \mathbb{Z}. \quad (44)
\]

One can easily check that they satisfy the quasi-invariance conditions and form a basis in the algebra \( \mathfrak{A}_{1,1} \). Denote also \( \varphi_{ij} \) with \( 2i + j = 0 \) as \( \varphi_i \):

\[
\varphi_i = x^i y^{-2i}, \quad i \in \mathbb{Z}.
\]

**Lemma 8.1.** The generators \( \mathcal{L}_i, i = 1, 2, 3 \) of algebra \( \mathfrak{D}_{1,1} \) act in the basis \((43, 44)\) as follows

\[
\mathcal{L}_1 \varphi_{ij} = (i + j) \varphi_{ij}, \quad \mathcal{L}_2 \varphi_{ij} = \lambda_{ij} \varphi_{ij}, \quad \lambda_{ij} = i(i - 1) - \frac{1}{2} j(j + 1),
\]

\[
\mathcal{L}_3 \varphi_{ij} = \mu_{ij} \varphi_{ij}, \quad \mu_{ij} = i^3 + \frac{1}{4} j^3 - \frac{3}{2} (i^2 - \frac{1}{4} j^2) + \frac{3}{4} (i + \frac{1}{2} j), \quad 2i + j \neq 1,
\]

\[
\mathcal{L}_1 \psi_i = -i \psi_i, \quad \mathcal{L}_2 \psi_i = -i^2 \psi_i - \varphi_i, \quad \mathcal{L}_2 \varphi_i = -i^2 \varphi_i,
\]

\[
\mathcal{L}_3 \psi_i = -i^3 \psi_i - 3 \varphi_i, \quad \mathcal{L}_3 \varphi_i = -i^3 \varphi_i, \quad i \in \mathbb{Z}.
\]

Let \( W_i = \langle \varphi_i, \psi_i \rangle \) be the linear span of \( \varphi_i \) and \( \psi_i \). We see that \( W_i \) is two-dimensional generalised eigenspace (Jordan block) for the whole algebra \( \mathfrak{D}_{1,1} \), while \( V_{ij} = \langle \varphi_{ij} \rangle \) with \( 2i + j \neq 0, 1 \) are its eigenspaces. In our terminology \( \varphi_{ij} \) with \( i + 2j \neq 1 \) are the zonal spherical functions, while \( \psi_i \) are the generalised zonal spherical functions of \( X = (\mathfrak{gl}(1,2), \mathfrak{osp}(1,2)) \).

**Theorem 8.2.** The spectral decomposition of \( \mathfrak{A}_{1,1} \) with respect to the action of \( \mathfrak{D}_{1,1} \) has the form

\[
\mathfrak{A}_{1,1} = \bigoplus_{2i + j \neq 0, 1} \langle \varphi_{ij} \rangle \oplus \bigoplus_{i \in \mathbb{Z}} \langle \varphi_i, \psi_i \rangle. \quad (45)
\]

This is in a good agreement with the equivalence relation \( \sim \) on

\[
\mathcal{T} = \{(a, b), \ a \in 2\mathbb{Z}, b \in \mathbb{Z}\},
\]

where \( a = 2i, \ b = -j + 1 \). Indeed, it is easy to check using Lemma 6.3 that the corresponding equivalence classes consist of one-element classes \( (a, b) \) with \( a \neq b, b - 1 \) and of two-element classes \( (a, a) \sim (a - 2, a - 1) \). In terms of \( i, j \) we have one element
classes \((i, j)\) with \(2i + j \neq 0, 1\), corresponding to \(V_{ij} = \langle \varphi_{ij} \rangle\) and two-element classes \((i, -2i + 1) \sim (i, -2i)\), corresponding to \(W_i = \langle \varphi_i, \psi_i \rangle\).

This also agrees with the representation theory. Let \(L(\lambda)\) and \(P(\lambda)\) be the irreducible spherical \(\mathfrak{g}\)-module with highest weight \(\lambda\) and its projective cover respectively. In our case \(\lambda = (2i, j, j), \ i, j \in \mathbb{Z}\). The spherical typicality condition (38) means that \(2i + j \neq 1\). In fact, one can show that if \(2i + j = 1\) the module \(L(\lambda)\) is not spherical, in agreement with the spectral decomposition (45). If \(2i + j \neq 0, 1\) the projective cover \(P(\lambda) = K(\lambda)\) coincides with the corresponding Kac module. One can check that it contains only one (up to a multiple) \(\mathfrak{osp}(1, 2)\)-invariant vector with the corresponding spherical function \(\varphi_{ij}\). A bit more involved calculations show that if \(2i + j = 0\) the projective cover \(P(\lambda)\) contains two dimensional \(\mathfrak{osp}(1, 2)\)-invariant subspace, corresponding to the generalised eigenspace \(W_i\).

9. Concluding remarks

Although we have considered only one series of the classical symmetric Lie superalgebras we believe that a similar relation of spectral decomposition of the algebra of the deformed CMS integrals and projective covers holds also at least for the remaining classical series (12). Note that the notion of spherically typical modules can be easily generalised to all these cases. Since the corresponding deformed root system is of \(BC(n, m)\) type, to describe the corresponding zonal spherical functions one can use the super Jacobi polynomials [29].

The type \(AI/AII\) we have considered in that sense is different since the corresponding deformed root system is of type \(A(n - 1, m - 1)\). To describe the zonal spherical functions in this case we can use the theory of Jack–Laurent symmetric functions developed in [30, 32].

Recall that such functions \(P^{(k,p_0)}_\alpha\) are certain elements of \(\Lambda^\pm\) labelled by bipartitions \(\alpha = (\lambda, \mu)\), where \(\Lambda^\pm\) is freely generated by \(p_a\) with \(a \in \mathbb{Z} \setminus \{0\}\) and variable \(p_0\) is considered as an additional parameter [30].

In [32] we considered the case of special parameters \(p_0 = n + k^{-1}m\) with natural \(m,n\). In that case the spectrum of the algebra of quantum CMS integrals acting on \(\Lambda^\pm\) is not simple. For generic \(k\) we showed that any generalised eigenspace has dimension \(2^n\), which coincides with the number of elements in the corresponding equivalence class of bipartitions. This equivalence can be described explicitly in terms of geometry of the corresponding Young diagrams \(\lambda, \mu\) (see [32]). In each equivalence class \(E\) there is only one bipartition \(\alpha\) such that the corresponding Jack–Laurent symmetric function \(P^{(k,p_0)}_\alpha\) is regular at \(p_0 = n + k^{-1}m\). At such \(p_0\) there is a natural homomorphism

\[\varphi_{n,m} : \Lambda^\pm \rightarrow \mathfrak{A}_{n,m}(k),\]

sending \(p_a\) to the deformed power sum

\[p_a(x, y, k) = x_1^a + \cdots + x_n^a + k^{-1}(y_1^a + \cdots + y_m^a), \ a \in \mathbb{Z}.\]
The image of the corresponding function \( \varphi_{n,m}(P_{\alpha}^{(k,n+k^{-1}m)}) \in \mathfrak{A}_{n,m}(k) \) is an eigenfunction of the algebra \( \mathfrak{D}_{n,m}(k) \) of the deformed CMS integrals, so its specialisation at \( k = -1/2 \) (provided it exists) determines a zonal spherical function for \( X = (\mathfrak{gl}(n,2m),\mathfrak{osp}(n,2m)) \). A natural question is whether this relation can be extended to an isomorphism of the corresponding generalised eigenspaces.

We would like to mention also that Brundan and Stroppel [7] showed that the algebra of the endomorphisms of a projective indecomposable module over general linear supergroup is isomorphic to

\[
\mathfrak{A}_r = \mathbb{C}[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r]/(\varepsilon_1^2, \varepsilon_2^2, \ldots, \varepsilon_r^2).
\]

We believe that using this and our main theorem it is possible to describe in a similar way the action of the algebra \( \mathfrak{D}_{n,m} \) in its generalised eigenspace, which in particular would imply that it contains only one zonal spherical function. This would be in a good agreement with the results of [32].

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