NEW CANONICAL DECOMPOSITION IN MATCHING THEORY

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ABSTRACT. In matching theory, one of the most fundamental and classical branches of combinatorics, canonical decompositions of graphs are powerful and versatile tools that form the basis of this theory. However, the abilities of the known canonical decompositions, that is, the Dulmage-Mendelsohn, Kotzig-Lovász, and Gallai-Edmonds decompositions, are limited because they are only applicable to particular classes of graphs, such as bipartite graphs, or they are too sparse to provide sufficient information. To overcome these limitations, we introduce a new canonical decomposition that is applicable to all graphs and provides much finer information. We focus on the notion of factor-components as the fundamental building blocks of a graph; through the factor-components, our new canonical decomposition states how a graph is organized and how it contains all the maximum matchings. The main results that constitute our new theory are the following: (i) a canonical partial order over the set of factor-components, which describes how a graph is constructed from its factor-components; (ii) a generalization of the Kotzig-Lovász decomposition, which shows the inner structure of each factor-component in the context of the entire graph; and (iii) a canonically described interrelationship between (i) and (ii), which integrates these two results into a unified theory of a canonical decomposition. These results are obtained in a self-contained way, and our proof of the generalized Kotzig-Lovász decomposition contains a shortened and self-contained proof of the classical counterpart.

1. INTRODUCTION

This paper introduces a new canonical decomposition in matching theory. In this section, we give a brief explanation of our results.

Matching theory [25] is one of the most classical and fundamental fields in combinatorics. Given a graph, a matching is a set of edges in which no two are adjacent. As small matchings such as a singleton exist trivially by definition, maximum matchings typically attract great interest. As can be seen from the definition, a matching is a basic way to express pairings of elements, and therefore has been intensively studied not only in graph theory [3] but also in algebra [11, 25, 27, 31].

The role of matching theory in combinatorial optimization is especially important. In the decades since 1965, the remarkable growth of combinatorial optimization has been driven by polyhedral combinatorics [12, 30], which explores a systematic and unified approach to numerous types of combinatorial problems through linear programming theory. The maximum matching problem serves as an archetypal prototype in polyhedral combinatorics [23, 30]. Therefore, progress in the theory of matchings leads to benefits for the entire field of combinatorial optimization.

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Canonical decompositions are highly versatile tools that form the foundation of matching theory [25]. A type of structure theorems exist that define a uniquely determined partition of a graph, and then use this partition to state the matching theoretic properties of the graph. A canonical decomposition is a way to understand graphs that is naturally derived from one of these structure theorems.

The known canonical decompositions are the following: the Dulmage-Mendelsohn decomposition [5–7], Kotzig-Lovász decomposition [20–22, 26], and Gallai-Edmonds decomposition [8, 10]. The power of each canonical decomposition originates partly from its uniqueness for a given graph. Therefore, in matching theory, the adjective “canonical” has come to mean being unique for a given graph, and being canonical itself has been considered important.

However, we sometimes encounter problems that cannot be solved successfully with these canonical decompositions because they are applicable to only particular classes of graphs or do not provide sufficient information. The Dulmage-Mendelsohn and Kotzig-Lovász decompositions target bipartite graphs and consistently factor-connected graphs, respectively. The Gallai-Edmonds decomposition, by definition, targets all graphs, but tends to be too sparse and, therefore, some classes of graphs, such as factorizable graphs, fall into trivially irreducible cases, which is a limitation that cannot be disregarded.

To address these limitations, in this paper, we establish the basilica decomposition, which is a new canonical decomposition that is applicable to all graphs and provides much finer information than the Gallai-Edmonds decomposition. We derive this new canonical decomposition using the notion of factor-components, which serve as the fundamental building blocks that constitute a graph when studying matchings. The properties of the maximum matchings are captured by describing both how an entire given graph is constructed by factor-components and the inner structure of each factor-component. More precisely, the main results that constitute the new canonical decomposition are the following:

(i) The organization of a given graph in terms of its factor-components can be understood as a partially ordered structure. The set of factor-components forms a poset with respect to a certain canonical binary relation, which is similar to the Dulmage-Mendelsohn decomposition.

(ii) A generalization of the Kotzig-Lovász decomposition is provided that targets general graphs, which describes the inner structure of each factor-component in the context of the entire given graph.

(iii) Although [i] and [ii] are established independently, they have a certain canonical relationship that enables us to understand a graph as an architectural building-like structure in which these ideas are unified naturally. The integrated notion obtained from this relationship is our new canonical decomposition.

Regarding our proofs, we obtain this new canonical decomposition without using any known results; thus, it is purely self-contained. Additionally, the proof that establishes the generalization of the Kotzig-Lovász decomposition contains a greatly shortened and purely self-contained proof for the classical Kotzig-Lovász decomposition.

Considering the important role of canonical decompositions, we believe that our results will contribute to further development in combinatorics. In fact, several consequential results have been already obtained [14, 17, 19].
The remainder of this paper is organized as follows. In Section 2 we present preliminary definitions and lemmas. In Section 3 we explain more about the technical background of the canonical decompositions and what we aim to establish in this paper. In Section 4 we list some elementary well-known lemmas used in later sections, with self-contained proofs. The new results of this paper appear in Section 5 onward. In Section 6 we provide a statement about consistently factor-connected graphs that is used in later sections. The main theorems that establish the basilica decomposition are then presented; we present (i), (ii), and (iii) in Sections 6, 7, and 8 respectively.

In Section 10 we present some properties of the basilica decomposition. In Section 11 we propose a polynomial time algorithm for computing the basilica decomposition. Finally, in Section 12 we conclude this paper.

2. Definitions

2.1. General Statements. For standard definitions and notation for sets, graphs, and algorithms, we mostly follow Schrijver [30]. In the following, we list those that may be non-standard or exceptional. We denote the vertex set of a graph $G$ by $V(G)$ and the edge set by $E(G)$. We treat paths and circuits as graphs; that is, a path is a connected graph in which every vertex is of degree two or less and at least one vertex is of degree less than two, whereas a circuit is a connected graph in which every vertex is of degree two. Given a path $P$ and vertices $x, y \in V(P)$, $xP'y$ denotes the subpath of $P$ whose ends are $x$ and $y$. We sometimes regard a graph as its vertex set. As usual, a singleton \( \{x\} \) is sometimes denoted simply by $x$. In the remainder of this section, unless otherwise stated, let $G$ be a graph and let $X \subseteq V(G)$.

2.2. Operations on Graphs. The subgraph of $G$ induced by $X$ is denoted by $G[X]$, and $G[V(G) \setminus X]$ is denoted by $G - X$.

We denote by $G/X$ the contraction of $G$ by $X$. That is, $V(G/X) = V(G) \setminus X \cup \{x\}$, where $x \notin V(G)$, and $E(G/X) = E(G) \setminus (E(G[X]) \setminus \delta_G(X) \cup S)$, where $S$ is obtained by replacing each edge $uv \in \delta_G(X)$ with $u \in X$ and $v \notin X$ by $xv$. Let $\hat{G}$ be a supergraph of $G$, and let $F \subseteq E(\hat{G})$. We denote by $G + F$ and $G - F$ the graphs obtained by adding $F$ to $G$ and deleting $F$ from $G$ without removing any vertices, respectively.

The union of two subgraphs $G_1$ and $G_2$ of $G$ is denoted by $G_1 + G_2$.

For simplicity, regarding these operations of creating a new graph from given graphs, we identify the vertices, edges, and subgraphs of the newly created graph with those of old graphs to which they naturally correspond.

2.3. Functions on Graphs. A neighbor of $X$ is a vertex in $V(G) \setminus X$ that is joined to a vertex in $X$. The set of neighbors of $X$ is denoted by $N_G(X)$. Given $Y, Z \subseteq V(G)$, $E_G[Y, Z]$ denotes the set of edges joining $Y$ and $Z$, and $\delta_G(X)$ denotes $E_G[X, V(G) \setminus X]$. We sometimes denote $E_G[X, Y]$, $\delta_G(X)$, $N_G(X)$ simply by $E[X, Y]$, $\delta(X)$, $N(X)$, respectively, if their subscripts are apparent from the context.

2.4. Matchings. A set of edges is a matching if any distinct two are disjoint. We say that a matching $M$ covers a vertex $v$ if $v$ is adjacent to an edge in $M$, otherwise we say that $M$ exposes $v$. 

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A maximum matching is a matching with the greatest cardinality. A perfect matching is a matching that covers all vertices. Note that a perfect matching is a maximum matching but the converse does not necessarily hold. A graph is factorizable if it has a perfect matching. A near-perfect matching is a matching that covers all vertices except for one. A graph is factor-critical if, for any vertex, there is a near-perfect matching that exposes it. Let $M$ be a matching of a graph $G$. We say that $X$ is closed with respect to $M$ if $\delta_G(X) \cap M = \emptyset$. We denote $M \cap E(G[X])$ by $M_X$.

We say that a path or circuit is $M$-alternating if edges in $M$ and not in $M$ appear alternately. More precisely, a circuit $C$ is $M$-alternating if $M \cap E(C)$ is a perfect matching of $C$. We define three types of $M$-alternating paths. Let $P$ be a path with ends $x$ and $y$. We say that $P$ is $M$-saturated or $M$-exposed between $x$ and $y$ if $M \cap E(P)$ or $E(P) \setminus M$, respectively, is a perfect matching of $P$. We say that $P$ is $M$-forwarding from $x$ to $y$ if $M \cap E(P)$ is a near-perfect matching of $P$ that exposes $y$. Accordingly, a path with one vertex is $M$-forwarding. That is, $M$-saturated and -exposed paths have an odd number of edges, whose ends are covered and exposed by $M$, respectively. In contrast, an $M$-forwarding path from $x$ to $y$ has an even number of edges, in which $x$ is covered by $M$ as long as this path has any edge whereas $y$ is always exposed.

An ear relative to $X$ is a path with two distinct ends in $X$ such that any other vertex is disjoint from $X$, or a circuit such that exactly one vertex is in $X$. Let $P$ be an ear relative to $X$. Even if $P$ is a circuit, the ends are the vertices in $V(P) \cap X$, and the internal vertices are those in $V(P) \setminus X$. Hence, for convenience, if $x$ is the only end and $y$ is an internal vertex of $P$, we denote by $xPy$ one of the paths on $P$ between $x$ and $y$. The set of internal vertices of $P$ is denoted by $\text{int}(P)$. If $\text{int}(P)$ intersects $Y \subseteq V(G)$, then we say that $P$ traverses $Y$. We say that $P$ is an M-ear if $P \setminus X$ is an M-saturated path.

2.5. Gallai-Edmonds Family. Let $G$ be a graph. The set of vertices that are exposed by some maximum matchings is denoted by $D(G)$. The set $N_G(D(G))$ is denoted by $A(G)$, and the set $V(G) \setminus D(G) \setminus A(G)$ is denoted by $C(G)$. We call $\{D(G), A(G), C(G)\}$ the Gallai-Edmonds family of $G$, because the Gallai-Edmonds decomposition is derived from a structure theorem regarding $D(G)$, $A(G)$, and $C(G)$.

2.6. Factor-Connected Components. Let $G$ be a graph. An edge $e \in E(G)$ is allowed if there is a maximum matching of $G$ containing $e$. Let $C_1, \ldots, C_k$ be the connected components of the subgraph of $G$ determined by the union of allowed edges. We call $G[C_i]$ a factor-connected component or a factor-component of $G$ for each $i \in \{1, \ldots, k\}$. We denote the set of factor-connected components of $G$ by $\mathcal{G}(G)$.

Thus, a graph is composed of its factor-connected components and the edges joining distinct factor-connected components. In addition, a set of edges is a maximum matching if and only if it is a disjoint union of maximum matchings taken from each factor-component. Hence, we can regard factor-components as the fundamental building blocks that determine the matching structure of a graph.

A factor-component is consistent if it is disjoint from $D(G)$, otherwise it is inconsistent. It is also easily observed that a factor-component is a factorizable graph.
if and only if it is consistent. Therefore, given a maximum matching \( M \), a factor-component \( C \) is consistent if and only if \( M_C \) is a perfect matching of \( C \). The sets of consistent and inconsistent factor-components of \( G \) are denoted by \( G^+ (G) \) and \( G^- (G) \), respectively. A graph is factor-connected if it consists of only one factor-component. In particular, this graph is consistently factor-connected if its only factor-component is consistent. Note that any consistent factor-component is a consistently factor-connected graph.

### 3. Canonical Decompositions and Aim of Our Study

#### 3.1. Known Canonical Decompositions

We now explain more technical details of the canonical decompositions that were omitted from Section 1. The Dulmage-Mendelsohn [5–7], Kotzig-Lovász [20–22, 26], and Gallai-Edmonds decompositions [8, 10] are the three known canonical decompositions and have been extensively applied. They are provided by their respective structure theorems, which follow a certain common pattern:

- First, define a partition of a given graph into substructures, which is described matching theoretically and is, by definition, unique to each graph, such as the Gallai-Edmonds family or the set of factor-components.
- Second, provide statements about how the entire graph is structured and the maximum matchings it contains, such as where in the graph there are allowed or non-allowed edges, or the matching theoretic properties of the substructures determined by the partition.

Because these partitions are determined uniquely for a given graph, canonical decompositions can provide us with information about all maximum matchings, not just those of them that are specified in some way. They therefore exhibit a powerful and versatile nature.

The traits of the three canonical decompositions are the following.

- The Dulmage-Mendelsohn decomposition states that, for bipartite graphs, the structure of factor-components can be described as a partially ordered set with respect to a certain binary relation. This decomposition provides an efficient solution of a system of linear equations by utilizing the sparsity of matrices [4]. Additionally, it is the origin of principal partition theory [29], which is a branch of submodular function theory [9].
- The Kotzig-Lovász decomposition captures the structure of consistently factor-connected graphs by defining a certain binary relation that is proved to be an equivalence relation. This decomposition is especially effective in the polyhedral study of matchings. From the Kotzig-Lovász decomposition, many important results regarding the perfect matching polytopes have been obtained; see Lovász and Plummer [24] or Schrijver [30] for surveys.
- Among them, the Gallai-Edmonds decomposition is probably the best known, because it is the essence of characterizing the size of a maximum matching and designing algorithms for computing maximum matchings. It has contributed to matching theory from many aspects. This decomposition provides properties of graphs based on the Gallai-Edmonds family. Some algorithms for computing the maximum matching algorithms are proposed using this decomposition [2, 25]. It also has applications in linear algebra [11, 27].
The exact statements of the three canonical decompositions are given in the following. The structures of graphs provided by Theorems 3.1, 3.2, and 3.3 are the Dulmage-Mendelsohn, Kotzig-Lovász, and Gallai-Edmonds decompositions, respectively.

**Theorem 3.1** (Dulmage and Mendelsohn [5–7]). Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $G(G)$ be denoted by $\{G_i : i \in I\}$, where $I = \{1, \ldots, |G(G)|\}$. Let $A_i = V(G_i) \cap A$ and $B_i := V(G_i) \cap B$ for each $i \in I$. Then, there exists a partial order $\prec_A$ satisfying the following for any $i, j \in I$:

1. If $E[A_i, B_j] \neq \emptyset$, then $G_i \prec_A G_j$; and,
2. if $G_i \prec_A H \prec_A G_j$, yields $G_i = H$ or $G_j = H$, then $E[A_i, B_i] \neq \emptyset$.

**Theorem 3.2** (Kotzig [20]). Let $G$ be a consistently factor-connected graph. Define a binary relation $\sim$ as follows: for $u, v \in V(G)$, $u \sim v$ holds if $G - u - v$ is not factorizable. Then, $\sim$ is an equivalence relation on $V(G)$, and accordingly, $\mathcal{P}(G)$ is a partition of $V(G)$, where $\mathcal{P}(G) := V(G)/\sim$.

**Theorem 3.3** (the Gallai-Edmonds structure theorem; Gallai [10], Edmonds [8]). For any graph $G$, the following hold:

1. The graph $G[D(G)]$ consists of $|A(G)| + |V(G)| - 2\nu(G)$ connected components, and each of them are factor-critical, whereas each connected component of $G[C(G)]$ is factorizable.
2. Let $M$ be an arbitrary maximum matching of $G$. Then, for each connected component $K$ of $G[D(G)]$, the set $M_K$ is a near-perfect matching of $G$; each vertex in $A(G)$ is matched to a vertex in $D(G)$, and furthermore, if $u$ and $v$ are distinct vertices from $A(G)$, then the vertices to which they are matched belong to distinct connected components of $G[D(G)]$; for each connected component $L$ of $G[C(G)]$, the set $M_L$ is a perfect matching of $L$.
3. All edges in $E[A(G), D(G)]$ are allowed, whereas no edge in $E(G[A(G)])$ or $E[A(G), C(G)]$ is allowed.

In addition to the statements in these three structure theorems that derive the canonical decompositions, additional fundamental properties are known for each canonical decomposition. These include properties that use the respective canonical decompositions to describe what happens after basic operations that frequently occur in graph theory, such as adding or deleting vertices and edges. These properties accordingly tell us how to make good use of each canonical decomposition. See, for example, Lovász and Plummer [25] for these properties.

### 3.2 Limitations of Classical Canonical Decompositions.

Although the above canonical decompositions are quite useful, we sometimes encounter problems that cannot be solved with any of them, because each of them only targets a particular class of graphs or they can be too sparse to provide sufficient information.

The Dulmage-Mendelsohn and Kotzig-Lovász decompositions only target bipartite graphs and consistently factor-connected graphs, respectively. The Gallai-Edmonds decomposition, by definition, targets any graph $G$, however it mainly focuses on the structure of $G[A(G) \cup D(G)]$ and thus provides little information about the remainder of the graph, that is, $G[C(G)]$, which can be a vast portion. In particular, if a given graph $G$ is factorizable, then $D(G) = A(G) = \emptyset$ and $C(G) = V(G)$ hold; thus, the Gallai-Edmonds decomposition claims nothing about $G$. This cannot be disregarded because perfect matchings are themselves a notion
that attracts intense attention. Of course, the classical Kotzig-Lovász decomposition is applicable to each factor-component in $G(G(C(G)))$, ignoring the other part; however, the information obtained by this operation is meaningless for the entire given graph in most contexts.

3.3. Our New Canonical Decomposition. In this paper, we present a new canonical decomposition, the basilica decomposition, that overcomes the limitations of the classical decompositions; that is, this targets all graphs and simultaneously provides further information that the Gallai-Edmonds decomposition cannot. The main concepts and theorems that constitute this new canonical decomposition are the following.

(i) How a graph is organized from its factor-component can be described by a partially ordered structure; we find a canonically defined partial order between factor-components, which is similar to that in the Dulmage-Mendelsohn decomposition (Theorem 6.1).

(ii) We obtain a generalization of the Kotzig-Lovász decomposition for general graphs (Theorem 7.5). This generalization considers the entire structure of a given graph and provides finer information than repeated applications of the classical Kotzig-Lovász decomposition.

(iii) There is a relationship between the above two concepts, even though they are defined independently (Theorem 8.5). This relationship unites the two notions into a canonical decomposition, in which we can view a graph as an architectural building-like structure. We name this new canonical decomposition the basilica decomposition.

Note how this new canonical decomposition is obtained. All the new statements are provided with self-contained proofs in this paper, except for the algorithmic result in Section 11 that computes the basilica decomposition in polynomial time. Additionally, our results contain a greatly shortened proof of the Kotzig-Lovász decomposition, which is also completely self-contained.

4. Basic Properties

4.1. On Matchings. We now present some basic properties of matchings. We will sometimes use these properties implicitly. These are easily observed by parity arguments or by taking symmetric differences of matchings, and readers familiar with matching theory may wish to skip this subsection.

Lemma 4.1. Let $G$ be a graph and $M$ be a matching of $G$. Let $X \subseteq V(G)$ be closed with respect to $M$, and let $x \in V(G) \setminus X$ and $y \in X$. Let $P$ be a path that is $M$-forwarding from $x$ to $y$ or $M$-saturated between $x$ and $y$. Let $z \in V(P)$ be the first vertex in $X$ that we encounter if we trace $P$ from $x$. Then, $xPz$ is an $M$-forwarding path from $x$ to $z$ with $V(xPz) \cap X = \{z\}$.

Lemma 4.2. Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$, and let $xy \in E(G) \setminus M$. The following three properties are equivalent:

(i) The edge $xy$ is allowed in $G$.

(ii) There is an $M$-alternating circuit $C$ with $xy \in E(C)$.

(iii) There is an $M$-saturated path between $x$ and $y$. 
4.2. On the Gallai-Edmonds Family and Factor-components. We now present some observations about factor-components. Lemmas 4.4 and 4.5 are known statements that can be found in Edmonds’ algorithm for maximum matchings or the Gallai-Edmonds structure theorem. These are easily confirmed.

Definition 4.3. Let $G$ be a graph. The set of vertices that are exposed by some maximum matchings are denoted by $D(G)$. The set $N_G(D(G))$ is denoted by $A(G)$, and the set $V(G) \setminus D(G) \setminus A(G)$ is denoted by $C(G)$.

Lemma 4.4. Let $G$ be a graph and $M$ be a maximum matching.

(i) A vertex $x$ is in $D(G)$ if and only if there exists an $M$-forwarding path from a vertex exposed by $M$ to $x$.

(ii) If a vertex $x$ is in $A(G)$, then there exists an $M$-exposed path between $x$ and a vertex exposed by $M$.

Lemma 4.5. Let $G$ be a graph. For any maximum matching of $G$, the vertex to which a vertex in $A(G)$ is matched is in $D(G)$. Accordingly, no edge in $\delta(C(G))$ is allowed.

The next proposition, which characterizes consistent and inconsistent factor-components, follows immediately from Lemma 4.5.

Proposition 4.6. Let $G$ be a graph. A factor-component of $G$ is inconsistent if and only if it is a factor-component of $G[A(G) \cup D(G)]$. A factor-component of $G$ is consistent if and only if it is a factor-component of $G[C(G)]$.

4.3. On Factor-critical Graphs. We now present some fundamental properties of factor-critical graphs. Some of these are well-known, but we again present their proofs. The next one can be easily obtained by considering symmetric differences of matchings:

Lemma 4.7. Let $M$ be a near-perfect matching of a graph $G$ that exposes $v \in V(G)$. Then, $G$ is factor-critical if and only if for any $u \in V(G)$ there exists an $M$-forwarding path from $u$ to $v$.

Lemma 4.7 leads to the following three statements:

Lemma 4.8. Let $G$ be a graph and $M$ be a matching of $G$. Let $H_1$ and $H_2$ be factor-critical subgraphs of $G$ such that there exists $v \in V(H_1) \cap V(H_2)$ and, for each $i \in \{1, 2\}$, $M_{H_i}$ is a near-perfect matching of $H_i$ exposing only $v$. Then, $H_1 + H_2$ is factor-critical.

Proof. Obviously, $M_1 \cup M_2$ is a near-perfect matching of $H_1 + H_2$, exposing only $v$. As $H_1$ and $H_2$ are both factor-critical, the claim follows from Lemma 4.7.

Proposition 4.9 (implicitly stated in Lovász [23]). Let $G$ be a factor-critical graph, let $v \in V(G)$, and let $M$ be a near-perfect matching that exposes $v$. Then, for any $e \in \delta(v)$, there is an $M$-ear relative to $v$ that contains $e$.

Proof. Let $u \in V(G)$ be the end of the edge $e$ other than $v$. From Lemma 4.7 there is an $M$-forwarding path $P$ from $u$ to $v$. Thus, $P + e$ is a desired $M$-ear.

Theorem 4.10 (implicitly stated in Lovász [23]). Let $G$ be a factor-critical graph. For any factor-critical subgraph $G'$ such that $G - V(G')$ is factorizable, the graph $G/G'$ is factor-critical.
**Claim 5.2.** Let \( u \in x + V' \). Proof. Let \( M \) be a perfect matching of \( G - V(G) \). Note that \( M \) is also a near-perfect matching of \( G/G' \) that exposes the vertex \( g' \) corresponding to \( G' \). Arbitrarily choose \( v \in V(G') \), and let \( M' \) be a near-perfect matching of \( G' \) that exposes \( v \). Then, \( M' \cup M \) is a near-perfect matching of \( G \) that exposes \( v \). Let \( x \) be an arbitrary vertex in \( V(G) \). From Lemma 5.4, there is an \( M' \cup M \)-forwarding path \( P \) from \( x \) to \( v \). Trace \( P \) from \( x \), and let \( y \) be the first encountered vertex in \( V(G') \). Then, in the graph \( G/G' \), the path \( xPy \) corresponds to an \( M \)-forwarding path from \( x \) to \( g' \). Hence, from Lemma 5.7, again, \( G/G' \) is factor-critical. 

5. Structure of Alternating Paths in Consistently Factor-connected Graphs

We now present our new results. In this section, we prove a proposition about consistently factor-connected graphs to be used in later sections.

**Proposition 5.1.** Let \( G \) be a consistently factor-connected graph and \( M \) be a perfect matching of \( G \). Then, for any two vertices \( u, v \in V(G) \), there is an \( M \)-saturated path between \( u \) and \( v \), or an \( M \)-forwarding path from \( u \) to \( v \).

Proof. Let \( u \in V(G) \) be an arbitrary vertex. Let \( U \subseteq V(G) \) be the set of vertices that can be reached from \( u \) by \( M \)-saturated or \( M \)-forwarding paths. We obtain this proposition by showing \( U = V(G) \). Suppose, to the contrary, \( U \not\subseteq V(G) \).

**Claim 5.2.** Let \( v \in U \), and let \( P \) be an \( M \)-saturated path between \( u \) and \( v \) or an \( M \)-forwarding path from \( u \) to \( v \). Then, \( V(P) \subseteq U \) holds.

Proof. Let \( w \in V(P) \). Then, \( uPw \) is an \( M \)-saturated path between \( u \) and \( w \) or an \( M \)-forwarding path from \( u \) to \( w \). Therefore, \( w \in U \) holds. Hence, we have \( V(P) \subseteq U \).

As \( G \) is connected, it has some edges that join \( U \) and \( V(G) \setminus U \).

**Claim 5.3.** Let \( v \in U \cap N(V(G) \setminus U) \). Then, there is no \( M \)-saturated path between \( u \) and \( v \).

Proof. Suppose this claim fails, and let \( P \) be an \( M \)-saturated path between \( u \) and \( v \in U \cap N(V(G) \setminus U) \). From Claim 5.2, \( V(P) \subseteq U \) holds. Therefore, the vertex \( v' \) is in \( U \), and by letting \( w \in V(G) \setminus U \) be a vertex with \( vw \in E(G) \) we have \( vw \notin M \). Hence, \( P + vw \) is an \( M \)-forwarding path from \( u \) to \( w \), which contradicts \( w \notin U \), and this claim is proved.

**Claim 5.4.** No edge joining \( U \) and \( V(G) \setminus U \) is in \( M \).

Proof. Let \( vw \) be an edge with \( v \in U \) and \( w \in V(G) \setminus U \). From Claims 5.2 and 5.3 there is an \( M \)-forwarding path \( P \) from \( u \) to \( v \) with \( V(P) \subseteq U \). Hence, if \( vw \notin M \) then \( P + vw \) is an \( M \)-saturated path between \( u \) and \( w \), and this contradicts \( w \notin U \). Therefore, \( vw \notin M \) follows, and this claim is proved.

As \( G \) is factor-connected, some edges in \( E[U, V(G) \setminus U] \) are allowed. Let \( e = vw \) be one of these edges. From Claim 5.4, \( e \notin M \) holds, and therefore, from Lemma 4.7 there is an \( M \)-saturated path \( Q \) between \( v \) and \( w \). Trace \( P \) from \( u \), and let \( x \) be the first vertex we encounter that is in \( Q \); such \( x \) certainly exists under the current hypotheses because \( v \in V(P) \cap V(Q) \) holds. Note that by this definition of \( x \), \( uPx + xQ \alpha \) forms a path for each \( \alpha \in \{v, w\} \).
Claim 5.5. The path $uP x$ is $M$-forwarding from $u$ to $x$.

Proof. Suppose this claim fails, that is, $uP x$ is an $M$-saturated path. Then, we have $x' \in V(uP x)$; however, at the same time, we have $x' \in V(Q)$, because $x \in V(Q)$ holds and $Q$ is an $M$-saturated path. This contradicts the definition of $x$, and this claim is proved.

Note also that, for $\alpha$, which is equal to either $v$ or $w$, $xQ\alpha$ is an $M$-saturated path. Hence, from Claim 5.5 for this $\alpha$, it follows that $uP x + xQ\alpha$ is an $M$-saturated path between $u$ and $\alpha$. Thus, $w \in U$ holds, which is a contradiction. This completes the proof of this proposition.

6. Partially Ordered Structure

In this section, we prove that the factor-components of a graph form a partially ordered set with respect to a certain canonical binary relation that we define here. As stated before, factor-components are the fundamental building blocks of a graph, in that a graph consists of its factor-components and the edges between them. However, how a graph is constructed from factor-components and edges is not arbitrary but follows a certain rule. That is, given some factor-connected graphs, construct a new graph by joining them with edges in an arbitrary manner. The factor-components of the resulting graph will not be in general equal to the original factor-connected graphs. We show that the rule of the factor-components is an ordered structure.

Definition 6.1. Given a graph $G$, a set $X \subseteq V(G)$ is separating if it is a disjoint union of the vertex sets of some factor-components, i.e., if there exist $H_1, \ldots, H_k \in \mathcal{G}(G)$, where $k \geq 1$, such that $X = V(H_1) \cup \cdots \cup V(H_k)$.

Note that a nonempty set $X$ is separating if and only if $\delta(X) \cap M = \emptyset$ holds for any maximum matching $M$.

Definition 6.2. Let $G$ be a graph, and let $G_1, G_2 \in \mathcal{G}(G)$. A separating set $X$ is a critical-inducing set for $G_1$ if $V(G_1) \subseteq X$ holds and $G[X]/G_1$ is a factor-critical graph. Moreover, we say that $X$ is a critical-inducing set for $G_1$ to $G_2$ if $V(G_1) \cup V(G_2) \subseteq X$ holds and $G[X]/G_1$ is a factor-critical graph.

We say $G_1 \triangleleft G_2$ if there is a critical-inducing set for $G_1$ to $G_2$.

We show that $\triangleleft$ is a partial order in Theorem 6.14. Reflexivity is obvious from the definition, hence the following lemmas are provided for transitivity and antisymmetry. First of all, observe the following:

Lemma 6.3. Let $G$ be a graph. If $X$ is a critical-inducing set for a factor-component $H \in \mathcal{G}(G)$ such that $X \neq V(H)$, then $X \setminus V(H) \subseteq C(G)$ holds. Consequently, for any maximum matching $M$ of $G$, $M_{X \setminus V(H)}$ is a perfect matching of $G[X \setminus V(H)]$. Accordingly, if $G_1 \triangleleft G_2$ holds for two distinct factor-components $G_1$ and $G_2$, then $G_2$ is consistent.

Proof. As $G[X]/H$ is factor-critical, $X \setminus V(H)$ is a separating set such that $G[X \setminus V(H)]$ has a perfect matching. Therefore, the factor-components that comprise $X \setminus V(H)$ are consistent, which implies from Proposition 4.16 that they are contained in $C(G)$. The remaining claims now follows immediately.
The following three lemmas can be easily confirmed by analogy between factor-critical graphs and critical-inducing sets. The next lemma follows from Lemmas 4.17 and 4.19.

**Lemma 6.4.** Let $G$ be a graph, $M$ be a maximum matching of $G$, and $X \subseteq V(G)$ be a separating set, and let $G_1 \in \mathcal{G}(G)$. The following three statements are equivalent.

(i) The set $X$ is a critical-inducing set for $G_1$.

(ii) For any $x \in X \setminus V(G_1)$, there exists $y \in V(G_1)$ such that there is an $M$-forwarding path from $x$ to $y$ whose vertices except $y$ are in $X \setminus V(G_1)$.

(iii) For any $x \in X \setminus V(G_1)$, there exists $y \in V(G_1)$ such that there is an $M$-forwarding path from $x$ to $y$.

The next lemma is immediate from Lemma 4.19.

**Lemma 6.5.** Let $G$ be a graph, and let $G_1 \in \mathcal{G}(G)$. If $X_1, X_2 \subseteq V(G)$ are critical-inducing sets for $G_1$, then $X_1 \cup X_2$ is also a critical-inducing set for $G_1$.

The next one is easily obtained from Proposition 4.19 and Theorem 4.10.

**Lemma 6.6.** Let $G$ be a graph and $M$ be a maximum matching of $G$, and let $G_1 \in \mathcal{G}(G)$. Let $X$ and $X'$ be critical-inducing sets for $G_1$ with $X' \subseteq X$. Then, $G[X]/X'$ is factor-critical, and $M_{X \setminus X'}$ is a near-perfect matching of it that exposes only the contracted vertex corresponding to $X'$. Moreover, if $X \not\subseteq X$, holds, then there exists an $M$-ear relative to $X'$ whose internal vertices are not empty and are contained in $X \setminus X'$.

Transitivity of $\triangleleft$ now follows rather easily:

**Lemma 6.7.** Let $G$ be a graph and $G_1, G_2, G_3$ be factor-components of $G$. If $G_1 \triangleleft G_2$ and $G_2 \triangleleft G_3$ hold, then $G_1 \triangleleft G_3$ holds.

**Proof.** Let $M$ be a maximum matching of $G$. Let $X_1$ and $X_2$ be critical-inducing sets for $G_1$ to $G_2$ and for $G_2$ to $G_3$, respectively. We prove that $X_1 \cup X_2$ is a critical-inducing set for $G_1$ to $G_3$. First, $X_1 \cup X_2$ is obviously a separating set that contains $G_1$ and $G_3$. Take $x \in X_1 \cup X_2$ arbitrarily. If $x \in X_1$ holds, then, from Lemma 6.4, there exists an $M$-forwarding path $P_x$ from $x$ to a vertex in $V(G_1)$ with $V(P_x) \subseteq X_1$. If $x \in X_2 \setminus X_1$ holds, then, from Lemma 6.4, there is an $M$-forwarding path $Q_x$ from $x$ to a vertex in $G_2$. From Lemma 4.19, there exists $y \in X_1$ such that $xQ_xy$ is an $M$-forwarding path with $V(xQ_xy) \cap X_1 = \{y\}$. We obtain an $M$-forwarding path from $x$ to a vertex in $V(G_1)$, that is, $xQ_xy + P_y$. Therefore, from Lemma 6.4, $X_1 \cup X_2$ is a critical-inducing set for $G_1$ to $G_3$, and the proof is complete.

In the following, we provide definitions and lemmas to prove antisymmetry of $\triangleleft$.

**Definition 6.8.** Let $G$ be a graph and $M$ be a maximum matching of $G$. Let $X_0$ be a nonempty proper subset of $V(G)$.

(i) Let $X \subseteq V(G)$ be a nonempty set of vertices that is disjoint from $X_0$ and is closed with respect to $M$.

(ii) Let $P$ be an $M$-ear relative to $X_0$ with $int(P) \neq \emptyset$ and $int(P) \subseteq X$.

For each $x \in X$, define a set of paths $\Lambda(x; X, P; X_0)$ as follows: A path $Q$ is an element of $\Lambda(x; X, P; X_0)$ if it is $M$-forwarding from $x$ to a vertex $y \in int(P)$ with $V(Q) \subseteq X$ and $V(Q) \cap V(P) = \{y\}$. Additionally, we define a property
Proof. First confirm that \( x \) for each \( G \Ψ(\cdot) \) therefore \( \Lambda(\cdot) \). Let Lemma 6.11.

Lemma 6.10. Let \( \Psi(\cdot) \) be a nonempty proper subset of \( V(G) \). If \( \Psi(X, P, X_0, M, G) \) holds for \( X \) and \( P \), then, for any \( x \in X, P \) has an end \( w \) such that there exists an \( M \)-forwarding path \( R \) from \( x \) to \( w \) with \( V(R) \setminus \{w\} \subseteq X \).

Proof. Let \( x \in X, P \), and let \( Q \in \Lambda(x; X, P; X_0) \) be an \( M \)-forwarding path from \( x \) to a vertex \( y \in \text{int}(P) \).

Let \( w \) be the end of \( P \) such that \( yPw \) is an \( M \)-forwarding path from \( y \) to \( w \). Then, \( Q + yPw \) is an \( M \)-forwarding path with the desired property.

The next lemma is an observation about the intersection of a consistent factor-component and a set of vertices closed with respect to a maximum matching.

Lemma 6.9. Let \( G \) be a graph, \( M \) be a maximum matching of \( G \), and \( X_0 \) be a nonempty proper subset of \( V(G) \). If \( \Psi(X, P, X_0, M, G) \) holds for \( X, P \), then, for any \( x \in X, P \) has an end \( w \) such that there exists an \( M \)-forwarding path \( R \) from \( x \) to \( w \) with \( V(R) \setminus \{w\} \subseteq X \).

Proof. Let \( x \in X, P \), and let \( Q \in \Lambda(x; X, P; X_0) \) be an \( M \)-forwarding path from \( x \) to a vertex \( y \in \text{int}(P) \).

Let \( w \) be the end of \( P \) such that \( yPw \) is an \( M \)-forwarding path from \( y \) to \( w \). Then, \( Q + yPw \) is an \( M \)-forwarding path with the desired property.

The next lemma is an observation about the intersection of a consistent factor-component and a set of vertices closed with respect to a maximum matching.

Lemma 6.10. Let \( G \) be a graph and \( M \) be a maximum matching of \( G \). Let \( X \subseteq V(G) \) be closed with respect to \( M \), and let \( H \in \mathcal{G}^+(G) \) be such that \( V(H) \cap X \neq \emptyset \). Then, for any \( x \in V(H) \), there exist a vertex \( y \in X \) and an \( M \)-forwarding path \( P \) from \( x \) to \( y \) with \( V(P) \setminus \{y\} \subseteq V(H) \setminus X \).

Proof. Take \( z \in X \cap V(H) \) arbitrarily. From Proposition 4.1 there is a path \( Q \) that is \( M \)-forwarding from \( x \) to \( z \) or \( M \)-saturated between \( x \) and \( z \). Trace \( Q \) from \( x \), and let \( y \) be the first vertex in \( X \) that we encounter. Then, from Lemma 4.1, \( xQy \) is a desired path.

The next lemma is derived from Lemma 6.10 and is used to prove Lemma 6.13.

Lemma 6.11. Let \( G \) be a graph and \( M \) be a maximum matching of \( G \). Let \( X_0 \) be a nonempty proper subset of \( V(G) \) that is separating. If a set of vertices \( X \subseteq C(G) \) and an \( M \)-ear \( P \) relative to \( X_0 \) satisfy \( \Psi(X, P, X_0, M, G) \), then \( X^* \) is a separating set that satisfies \( \Psi(X^*, P, X_0, M, G) \), where \( X^* := X \cup \bigcup \{V(H) : H \in \mathcal{G}(G), V(H) \cap X \neq \emptyset\} \). Accordingly, if \( \Lambda(X_0 = V(G_1)) \) for some \( G_1 \in \mathcal{G}(G) \), then \( X^* \cup V(G_1) \) is a critical-inducing set for \( G_1 \).

Proof. First confirm that \( X^* \) is disjoint from \( X_0 \) and is separating. Obviously, for each \( x \in X \), any path in \( \Lambda(x; X, P; X_0) \) is also a path in \( \Lambda(x; X^*, P; X_0) \), and therefore \( \Lambda(x; X^*, P; X_0) \neq \emptyset \). Hence, it suffices to prove \( \Lambda(x; X^*, P; X_0) \neq \emptyset \) for each \( x \in V(H) \setminus X \), where \( H \) is a factor-component with \( V(H) \cap X \neq \emptyset \). As \( X \subseteq C(G) \) holds, Proposition 4.6 implies that \( H \) is consistent. From Lemma 6.10 there is an \( M \)-forwarding path \( R \) from \( x \) to a vertex \( y \in X \) with \( V(R) \setminus \{y\} \subseteq V(H) \setminus X \). For \( Q \in \Lambda(y; X, P; X_0) \), the concatenation \( R + Q \) is a path in \( \Lambda(y; X^*, P; X_0) \). Thus, we obtain \( \Psi(X^*, P; X_0, M, G) \). Accordingly, the remaining claim of the lemma also follows from Lemmas 6.4 and 6.9.
Note also the following observation about \( \Psi \), which is used in the proof of Lemma \ref{lemma:6.13}.

**Lemma 6.12.** Let \( G \) be a graph, \( M \) be a maximum matching of \( G \), and \( X_0 \) be a nonempty proper subset of \( V(G) \).

(i) If \( P_0 \) is an \( M \)-ear relative to \( X_0 \) with \( \text{int}(P_0) \neq \emptyset \), then \( \Psi(\text{int}(P_0), P_0; X_0, M, G) \) holds.

(ii) Let \( X \) and \( P \) be such that \( \Psi(X, P; X_0, M, G) \) holds, and let \( Q \) be an \( M \)-ear relative to \( X \) with \( \text{int}(Q) \neq \emptyset \) and \( \text{int}(Q) \cap X_0 = \emptyset \). Then, \( \Psi(X \cup \text{int}(Q), P; X_0, M, G) \) also holds.

**Proof.** The property \( \Psi(\text{int}(P_0), P_0; X_0, M, G) \) holds because each vertex \( y \in \text{int}(P_0) \) forms a trivial \( M \)-forwarding path of \( \Lambda(y; \text{int}(P_0), P_0; X_0) \). For each \( x \in X \), obviously \( \Lambda(x; X \cup \text{int}(Q), P; X_0) \neq \emptyset \) holds. For each \( x \in \text{int}(Q) \), let \( w \) be the end of \( Q \) such that \( xQw \) is an \( M \)-forwarding path from \( x \) to \( w \). For \( R \in \Lambda(w; X, P; X_0) \), \( xQw + R \) is a path of \( \Lambda(x; X \cup \text{int}(Q), P; X_0) \). Thus, \( \Psi(X \cup \text{int}(Q), P; X_0, M, G) \) is proved. \( \Box \)

The next lemma is the key to Theorem \ref{theorem:6.14}.

**Lemma 6.13.** Let \( G \) be a graph and \( M \) be a maximum matching of \( G \). Let \( G_1, G_2 \in \mathcal{G}(G) \) be such that \( G_1 \neq G_2 \) and \( G_1 \preceq G_2 \) hold. Then there exists a set of vertices \( X \subseteq V(G) \) and an \( M \)-ear \( P \) relative to \( G_1 \) such that \( V(G_2) \subseteq X \) and \( \Psi(X, P; G_1, M, G) \) hold.

**Proof.** Let \( X \subseteq V(G) \) be a critical-inducing set for \( G_1 \) to \( G_2 \). Define a family \( \mathcal{Y} \subseteq 2^{X \setminus V(G_1)} \) as follows: A set of vertices \( W \) is a member of \( \mathcal{Y} \) if \( W \) is a (inclusion-wise) maximal subset of \( X \setminus V(G_1) \) that satisfies \( \Psi(W, P; G_1, M, G) \) for some \( M \)-ear \( P \) relative to \( G_1 \).

Let \( X' := V(G_1) \cup \bigcup_{W \in \mathcal{Y}} W = \bigcup_{W \in \mathcal{Y}} V(G_1) \cup W \). Lemma \ref{lemma:6.11} implies that, for each \( W \in \mathcal{Y} \), \( V(G_1) \cup W \) is a critical-inducing set for \( G_1 \). Accordingly, from Lemma \ref{lemma:6.5} \( X' \) is also a critical-inducing set for \( G_1 \).

We prove this lemma by showing \( V(G_2) \subseteq X' \). Suppose the contrary. Then, \( X' \subseteq X \) holds. From Lemma \ref{lemma:6.6} there exists an \( M \)-ear \( Q \) relative to \( X' \) such that \( \text{int}(Q) \neq \emptyset \) and \( \text{int}(Q) \subseteq X \setminus X' \) hold. In the following, note that if any \( W \subseteq X \setminus V(G_1) \) with \( W \cap \text{int}(Q) \neq \emptyset \) satisfies \( \Psi(W, P; G_1, M, G) \) for some \( M \)-ear \( P \), then it contradicts the definition of \( \mathcal{Y} \) under the current hypothesis. Let \( \mathcal{Y}^* := \mathcal{Y} \cup \{V(G_1)\} \). First consider the case where both ends of \( Q \) are contained in the same member of \( \mathcal{Y}^* \), say, \( W \). If \( W \in \mathcal{Y} \) holds, then, from Lemma \ref{lemma:6.11}, \( W \cup \text{int}(Q) \) is a set of vertices that satisfies \( \Psi(W \cup \text{int}(Q), P; G_1, M, G) \), where \( P \) is an \( M \)-ear with \( \Psi(W, P; G_1, M, G) \); otherwise, that is, if \( W = V(G_1) \), then Lemma \ref{lemma:6.11} implies \( \Psi(\text{int}(Q), Q; G_1, M, G) \). This is a contradiction.

Next consider the case where two ends \( u_1 \) and \( u_2 \) of \( Q \) are contained in distinct members of \( \mathcal{Y}^* \), say, \( W_1 \) and \( W_2 \), respectively. For each \( i \in \{1, 2\} \), if \( W_i \) is a member of \( \mathcal{Y} \), then let \( P_i \) be an \( M \)-ear relative to \( G_1 \) such that \( \Psi(W_i, P_i; G_1, M, G) \) holds; according to Lemma \ref{lemma:6.9} there exists an \( M \)-forwarding path \( R_i \) from \( u_i \) to an end of \( P_i \), say, \( r_i \). Otherwise, that is, if \( W_i = V(G_1) \), let \( R_i \) be the trivial \( M \)-forwarding path that consists solely of \( u_i \), and let \( r_i := u_i \).

If \( (V(R_1) \setminus \{r_1\}) \cap (V(R_2) \setminus \{r_2\}) = \emptyset \), then let \( \hat{Q} := R_1 + Q + R_2 \). From Lemma \ref{lemma:6.12} \( Q \) is an \( M \)-ear relative to \( G_1 \) such that \( \Psi(\text{int}(\hat{Q}), \hat{Q}; G_1, M, G) \) holds, which is a contradiction.
Otherwise, that is, if \((V(R_1) \setminus \{u_1\}) \cap (V(R_2) \setminus \{u_2\}) \neq \emptyset\), then we have \(W_1, W_2 \in \mathcal{Y}\). Trace \(R_2\) from \(u_2\), and let \(x\) be the first vertex we encounter that is in \(W_1\). Then, from Lemma \ref{lem:1.1}, \(u_2R_2x\) is an \(M\)-forwarding path from \(u_2\) to \(x\), and therefore \(Q + u_2R_2x\) is an \(M\)-ear relative to \(W_1\). Thus, this case is reduced to the first case, and we are again lead to a contradiction.

Hence, we obtain \(X = X'\), and therefore \(V(G_2)\) is contained in some member of \(\mathcal{Y}\). This completes the proof. \(\square\)

We can finally prove that \(\triangleleft\) is a partial order.

**Theorem 6.14.** For any graph \(G\), the binary relation \(\triangleleft\) is a partial order on \(G(G)\).

**Proof.** Reflexivity is obvious from the definition. Transitivity follows from Lemma \ref{lem:6.7}. Hence, we prove antisymmetry. Let \(G_1, G_2 \in G(G)\) be factor-components with \(G_1 \triangleleft G_2\) and \(G_2 \triangleleft G_1\). Suppose antisymmetry fails, that is, \(G_1 \neq G_2\) holds. Note that, from Lemma \ref{lem:6.3}, \(G_1\) is consistent. Let \(M\) be a maximum matching of \(G\). From Lemma \ref{lem:6.13}, there exist a set of vertices \(X \subseteq V(G)\) with \(V(G_2) \subseteq X\) and an \(M\)-ear \(P_1\) relative to \(G_1\) that satisfy \(\Psi(X, P_1; G_1, M, G)\). Let \(u_1\) and \(v_1\) be the ends of \(P_1\).

By Lemma \ref{lem:6.4}, there exists \(w \in V(G_2)\) such that there is an \(M\)-forwarding path \(Q\) from \(u_1\) to \(w\). Trace \(Q\) from \(u_1\), and let \(x\) be the first vertex in \((X \cup \{v_1\}) \setminus \{u_1\}\) that we encounter; such a vertex exists because \(V(G_2) \subseteq X\) holds.

**Claim 6.15.** Without loss of generality, we can assume that \(x \neq v_1\), that is, \(x \in X\) holds and \(u_1Qx\) is a path with \(v_1 \notin V(u_1Qx) \setminus \{u_1\}\), which is \(M\)-forwarding from \(u_1\) to \(x\).

**Proof.** Suppose the claim fails, that is, \(x = v_1\) holds. Then, \(u_1 \neq v_1\) holds by the definition of \(x\). If \(u_1Qv_1\) is an \(M\)-saturated path, then \(P_1 + u_1Qv_1\) forms an \(M\)-alternating circuit that contains the non-allowed edges in \(E(P_1) \cap \delta(G_1)\), which contradicts Lemma \ref{lem:1.2}. Otherwise, that is, if \(u_1Qv_1\) is an \(M\)-forwarding path from \(u_1\) to \(v_1\), then \(v_1Qw\) is an \(M\)-forwarding path from \(v_1\) to \(w\) that is disjoint from \(u_1\). Redefine \(x\) as the first vertex in \(X\) that we encounter if we trace \(v_1Qw\) from \(v_1\). Then, \(v_1Qx\) is a path that is disjoint from \(u_1\) and is \(M\)-forwarding from \(v_1\) to \(x\), according to Lemma \ref{lem:4.1}. Therefore, by swapping the roles of \(u_1\) and \(v_1\), without loss of generality, we obtain this claim. \(\square\)

Therefore, hereafter let \(x \in X\) and let \(u_1Qx\) be an \(M\)-forwarding path from \(u_1\) to \(x\) with \(v_1 \notin V(u_1Qx) \setminus \{u_1\}\).

As \(x \in X\) holds, \(\Psi(X, P; G_1, M, G)\) implies that there is an \(M\)-forwarding path \(R\) from \(x\) to an internal vertex of \(P_1\), say, \(y\), such that \(V(R) \subseteq X\) and \(V(R) \cap V(P_1) = \{y\}\).

If \(u_1P_1y\) has an even number of edges, then \(u_1Qx + xRy + yP_1u_1\) is an \(M\)-alternating circuit that contains some non-allowed edges, say, the edges in \(E(P_1) \cap \delta(u_1)\), which contradicts Lemma \ref{lem:1.2}.

Hence hereafter we assume that \(u_1P_1y\) has an odd number of edges. From Proposition \ref{prop:5.1} there is a path \(L\) of \(G_1\) that is \(M\)-saturated between \(v_1\) and \(u_1\) or \(M\)-forwarding from \(v_1\) to \(u_1\). Trace \(L\) from \(v_1\), and let \(z\) be the first vertex on \(u_1Qz\); note that Lemma \ref{lem:4.1} implies that \(v_1Lz\) is an \(M\)-forwarding path from \(v_1\) to \(z\). Additionally, note that \(L\) is disjoint from \(X\), because \(V(L) \subseteq V(G_1)\) holds and \(X\) is disjoint from \(V(G_1)\). If \(u_1Qz\) has an odd number of edges, then \(zQu_1 + P_1 + v_1Lz\) is an \(M\)-alternating circuit that contains non-allowed edges, say,
the edges in $E(P_1) \cap \delta(G_1)$, which contradicts Lemma 4.2. If $u_1Qz$ has an even number of edges, then $v_1Lz + zQx + xRy + yP_1u_1$ is an $M$-alternating circuit, which is again a contradiction. Thus we obtain $G_1 = G_2$, and the theorem is proved.

Remark 6.16. The partially ordered structure determined by $\triangleright$ is not a generalization of the Dulmage-Mendelsohn decomposition. We can confirm that $\triangleright$ is determined totally unique for a graph, whereas the partial order for the Dulmage-Mendelsohn decomposition depends on the choice of color classes. If a graph $G$ is bipartite, then the poset $(G(G), \triangleright)$ has a trivial structure. In our next work [13], we give a generalization of the Dulmage-Mendelsohn decomposition using the results in this paper.

Remark 6.17. From Lemma 6.3, any inconsistent factor-component is minimal with respect to $\triangleright$. We call the partial order $\triangleright$ the basilica order.

7. Generalization of the Kotzig-Lovász decomposition

In this section, we give a generalization of the Kotzig-Lovász decomposition for general graphs. Given a graph $G$, the deficiency of $G$ is the number $|V(G)| - 2\nu(G)$ and is denoted by $\text{def}(G)$, where $\nu(G)$ is the size of a maximum matching. That is, $\text{def}(G)$ is the number of vertices exposed by a maximum matching. Note that $\text{def}(G) = 0$ if and only if $G$ is factorizable.

Definition 7.1. Let $G$ be a graph. For $u, v \in V(G)$, we say $u \sim_G v$ if $u = v$ holds or if $u$ and $v$ are contained in the same factor-component and $\text{def}(G - u - v) > \text{def}(G)$ holds.

By definition, if a graph $G$ is consistently factor-connected then the binary relation $\sim_G$ coincides with $\sim$ given by Kotzig [20–22]. We prove in Theorem 7.5 that $\sim_G$ is an equivalence relation. Lemmas 7.2 to 7.4 in the following are used to prove Theorem 7.5. These lemmas relate the deficiency and the Gallai-Edmonds family, and can be observed more easily from the Gallai-Edmonds structure theorem. However, we prove them without using the Gallai-Edmonds structure theorem to keep our results self-contained.

The next lemma implies that each vertex in $D(G)$ forms an equivalence class that is a singleton.

Lemma 7.2. Let $G$ be a graph, and let $u \in D(G)$. Then, for any $v \in V(G) \setminus \{u\}$, $\text{def}(G - u - v) \leq \text{def}(G)$ holds.

Proof. As $u \in D(G)$ holds, $\text{def}(G - u) = \text{def}(G) - 1$. Obviously, $|\text{def}(G - u - v) - \text{def}(G - u)| = 1$. Hence, $\text{def}(G - u - v) \leq \text{def}(G - u) + 1 = \text{def}(G)$.

The next lemma will be used in both Lemma 7.4 and Theorem 7.5.

Lemma 7.3. Let $G$ be a graph and $M$ be a maximum matching of $G$, and let $u, v \in V(G) \setminus D(G)$ be two distinct vertices. Then $\text{def}(G - u - v) \leq \text{def}(G)$ holds if and only if there exists an $M$-saturated path between $u$ and $v$.

Proof. We first prove the necessity. Let $P$ be an $M$-saturated path between $u$ and $v$. Then, $M \triangle E(P)$ is a matching of $G - u - v$ that covers any vertex that $M$ covers other than $u$ and $v$. Hence, $\text{def}(G - u - v) \leq \text{def}(G)$ holds.
Next, we prove the sufficiency. If $uv$ is an edge in $M$, then the claim obviously holds. Hence, in the following, we assume $uu', vv' \in M$ for some $u', v' \in V(G) \setminus \{u, v\}$. Then, $M \setminus \{uu', vv'\}$ is a matching of $G - u - v$ but is not maximum, because it exposes the vertices in $S \cup \{u', v'\}$, where $S$ is the set of vertices that $M$ exposes. Hence, $G - u - v$ has an $M$-exposed path $P$ whose ends are in $S \cup \{u', v'\}$. If both ends are in $S$, then $M \triangle E(P)$ is a bigger matching of $G$ than $M$, which is a contradiction. If one end $x$ is in $S$ and the other is equal to either $u'$ or $v'$, say, $u'$, then, in $G$, $P + uu'$ is an $M$-forwarding path from $u$ to $x$. This implies $u \in D(G)$ from Lemma 4.4(ii) which is a contradiction. Therefore, the ends of $P$ are $u'$ and $v'$, and $P + uu' + vv'$ is an $M$-saturated path between $u$ and $v$.

The next lemma implies that $A(G) \cap V(H)$ forms an equivalence class for each $H \in \mathcal{G}^-(G)$.

**Lemma 7.4.** Let $G$ be a graph, and let $M$ be a maximum matching of $G$. For any $u, v \in A(G)$, $\text{def}(G - u - v) > \text{def}(G)$ holds.

**Proof.** If $u = v$, then the claim obviously holds. Hence, assume $u \neq v$ and suppose the claim fails, that is, suppose $\text{def}(G - u - v) \leq \text{def}(G)$. Then, by Lemma 7.3 there exists an $M$-saturated path $P$ between $u$ and $v$. By Lemma 4.4(ii) there is an $M$-exposed path $Q$ between $u$ and a vertex $x$ exposed by $M$. Trace $Q$ from $x$, and let $y$ be the first encountered vertex in $V(P)$. Then, $xQy$ is an $M$-forwarding path from $x$ to $y$, and, for a vertex $w$ that is equal to either $u$ or $v$, the path $xQy + yPw$ is an $M$-forwarding path from $w$ to $x$. This implies $w \in D(G)$ according to Lemma 4.4(i) which is a contradiction. Hence, we obtain $\text{def}(G - u - v) > \text{def}(G)$.

The next theorem presents our generalization of the Kotzig-Lovász decomposition.

**Theorem 7.5.** For any graph $G$, the binary relation $\sim_G$ is an equivalence relation on $V(G)$.

**Proof.** Reflexivity and symmetry obviously hold by definition. We prove transitivity in the following. Let $u, v, w \in V(H)$ be such that $u \sim_G v$ and $v \sim_G w$. If any two among $u, v, w$ are identical, clearly the claim follows. Therefore, it suffices to consider the case that they are mutually distinct. If $H$ is inconsistent, then, from Lemma 7.2, $u, v, w \in A(G)$ follows. Thus, from Lemma 7.4, $u \sim_G w$ is obtained. Therefore, in the remainder of this proof, we assume that $H$ is consistent. Suppose that the claim fails, that is, $u \not\sim_G w$. From Lemma 7.3 there is an $M$-saturated path $P$ between $u$ and $w$. By Proposition 5.1 there is an $M$-forwarding path $Q$ from $v$ to $u$. Trace $Q$ from $v$ and let $x$ be the first vertex we encounter in $V(Q) \cap V(P)$. If $uPx$ has an odd number of edges, then $vQx + xPu$ is an $M$-saturated path between $u$ and $v$, which is a contradiction. If $uPx$ has an even number of edges, then $xPw$ has an odd number of edges, and by the same argument we have a contradiction.

If a graph $G$ is consistently factor-connected, then the family of equivalence classes under $\sim_G$, that is, $V(G)/\sim_G$, coincides with the original Kotzig-Lovász decomposition [20][22]. Therefore, for a general graph $G$, we denote $V(G)/\sim_G$ by $\mathcal{G}(G)$, and call it the generalized Kotzig-Lovász decomposition or simply the Kotzig-Lovász decomposition. By the definition of $\sim_G$, each equivalence class is contained in some factor-component. Therefore, for each $H \in \mathcal{G}(G)$, the family
\{S \in \mathcal{P}(G) : S \subseteq V(H)\} is a partition of \(V(H)\); we denote this partition by \(\mathcal{P}_G(H)\).

The next statement shows that our generalization of the Kotzig-Lovász decomposition provides information that the classical Kotzig-Lovász decomposition does not.

**Observation 7.6.** For a factorizable graph \(G\) and a factor-component \(H \in \mathcal{G}(G)\), the partition \(\mathcal{P}_G(H)\) is a refinement of \(\mathcal{P}(H)\); that is, if two vertices \(u, v \in V(H)\) satisfy \(u \sim_G v\) in \(G\), then \(u \sim v\) holds in \(H\).

In general, \(\mathcal{P}_G(H)\) is a proper refinement of \(\mathcal{P}(H)\). Therefore, our generalization of the Kotzig-Lovász decomposition is not trivial; that is, \(\mathcal{P}(G)\) is not merely a disjoint union of the Kotzig-Lovász decomposition of each factor-component.

Our proof of Theorem 7.5 provides a shortened and self-contained proof of the classical Kotzig-Lovász decomposition. Kotzig’s proof consists of three papers, so proving that \(\sim\) is an equivalence relation from first principles has been considered challenging [25]. Lovász’s proof uses the Gallai-Edmonds structure theorem, and, accordingly, is not self-contained. However, in fact, it can be proved in a simple way even without the premise of the Gallai-Edmonds structure theorem or the notion of barriers. All the results used to obtain Theorem 7.5 are self-contained in this paper.

Lovász [24] reformulated the classical Kotzig-Lovász decomposition using the notion of barriers [25]. In our next paper [14, 17], we discuss the relationship between barriers and our generalized Kotzig-Lovász decomposition, and show that our decomposition also provides a generalization of Lovász’s formulation.

The next observation follows from Lemmas 7.2 and 7.4.

**Observation 7.7.** Let \(G\) be a graph, and let \(H \in \mathcal{G}^{-}(G)\). Then, \(\mathcal{P}_G(H) = \{A(G) \cap V(H)\} \cup \bigcup\{\{x\} : x \in D(G) \cap V(H)\}\).

**Observation 7.8.** Let \(G\) be a graph. For a factor-component \(H \in \mathcal{G}(G)\), \(\mathcal{P}_G(H)\) consists of only a single member if and only if \(|V(H)| = 1\), which implies that its only vertex is in \(D(G)\).

**Remark 7.9.** An alternative way to define \(\sim_G\) is the following. Given a graph \(G\), for any \(u, v \in V(G) \setminus D(G)\), we say \(u \sim_G v\) if \(u\) and \(v\) are contained in the same factor-component and \(\text{def}(G - u - v) > \text{def}(G)\) holds. Obviously, \(\sim_G\) is also an equivalence relation over \(V(G) \setminus D(G)\), and its equivalence classes coincide with those given in this section, except for the trivial classes over \(D(G)\). We prefer this formulation for the generalized Kotzig-Lovász decomposition given the nature of matchings shown in our next paper [14, 17]. However, in this paper, we employ the other formulation.

8. **Basilica Type Relationship and Definition of New Canonical Decomposition**

8.1. **Relationship between \(\triangleleft\) and \(\sim_G\).** There is a relationship between the partial order \(\triangleleft\) and the generalized Kotzig-Lovász decomposition, even though they are given independently. We state this relationship in Theorem 8.5 using the following definitions and lemmas.

**Definition 8.1.** Let \(G\) be a graph, and let \(H \in \mathcal{G}(G)\). We denote by \(\mathcal{U}_G(H)\) the set of upper bounds of \(H\) in the poset \((\mathcal{G}(G), \triangleleft)\); that is, \(\mathcal{U}_G(H) := \{H' \in\)
\( \mathcal{G}(G) : H \triangleleft H' \). We define \( \mathcal{U}_G(H) := \mathcal{U}_G^*(H) \setminus \{H\} \) and denote by \( \mathcal{U}_G^*(H) \) and \( \mathcal{U}_G(H) \) the sets of vertices that are contained in the factor-components in \( \mathcal{U}_G^*(H) \) and in \( \mathcal{U}_G(H) \), respectively; that is, \( \mathcal{U}_G^*(H) := \bigcup_{H' \in \mathcal{U}_G(H)} V(H') \) and \( \mathcal{U}_G(H) := \bigcup_{H' \in \mathcal{U}_G(H)} V(H') \). We often omit the subscripts “\( G \)” if they are apparent from the context.

**Lemma 8.2.** Let \( G \) be a graph and \( M \) be a maximum matching of \( G \), and let \( G_1, G_2 \in \mathcal{G}(G) \) be distinct factor-components. If there exists an \( M \)-ear \( P \) with \( \text{int}(P) \subseteq C(G) \) that is relative to \( G_1 \) and traverses \( G_2 \), then \( G_1 \triangleright G_2 \) holds. Accordingly, any factor-component traversed by \( P \) is an upper bound of \( G_1 \).

**Proof.** As stated in Lemma 6.6, \( \Psi(\text{int}(P), P; G_1, M, G) \) holds. Thus, from Lemma 6.11 using \( \text{int}(P) \), we can construct a critical-inducing set for \( G_1 \) to \( G_2 \). Therefore, \( G_1 \triangleright G_2 \) holds, and accordingly the remaining statement is also obtained.

**Lemma 8.3.** Let \( G \) be a graph and \( M \) be a matching of \( G \), and let \( H \in \mathcal{G}(G) \). Let \( P \) be an \( M \)-ear relative to \( H \) with end vertices \( u, v \in V(H) \) and with \( \text{int}(P) \subseteq C(G) \). Then \( u \sim_G v \) holds.

**Proof.** First, note \( \text{int}(P) \subseteq U(H) \) according to Lemma 8.2. Hence, if \( H \) is an inconsistent factor-component, then \( u, v \in A(G) \cap V(H) \) holds. Therefore, we obtain \( u \sim_G v \) from Lemma 7.4.

Hence, in the following, assume that \( H \) is consistent. Suppose the claim fails, that is, \( u \not\sim_G v \) holds. Then, from Lemma 7.3 there is an \( M \)-saturated path \( Q \) between \( u \) and \( v \). Trace \( Q \) from \( u \), and let \( x \) be the first vertex we encounter that is in \( V(P) \setminus \{u\} \). If \( x = v \), then \( Q + P \) is an \( M \)-alternating circuit that contains some non-allowed edges of \( \delta_G(H) \), which contradicts Lemma 4.2. Hence, we assume \( x \in \text{int}(P) \setminus \{u\} \) in the following. If \( uPx \) has an even number of edges, then \( uQx + xPu \) is an \( M \)-alternating circuit with some non-allowed edges of \( \delta_G(H) \), which is again a contradiction. Hence, we assume that \( uPx \) has an odd number of edges. Let \( I \in \mathcal{G}(G) \) be the factor-component that contains \( x \). The connected components of \( uQx + xPu - E(I) \) are \( M \)-ears relative to \( I \), and one of them traverses \( H \). This implies \( I \triangleright H \) under Lemma 8.2 which contradicts \( H \triangleright I \) under Theorem 6.11.

**Lemma 8.4.** Let \( G \) be a graph and \( M \) be a maximum matching of \( G \), and let \( G_0 \in \mathcal{G}(G) \). Let \( X \subseteq C(G) \) be a set of vertices such that \( \Psi(X, P; G_0, M, G) \) holds for some \( M \)-ear \( P \) relative to \( G_0 \). Then,

(i) there exists a connected component \( K \) of \( G[U(G_0)] \) with \( X \subseteq V(K) \); and,

(ii) there exists \( T \in \mathcal{P}_G(G_0) \) such that \( N_G(X) \cap V(G_0) \subseteq T \) holds.

**Proof.** As Lemma 6.11 states that \( X \) is contained in a critical-inducing set for \( G_0 \), we have \( X \subseteq U(G_0) \). Additionally, \( \Psi(X, P; G_0, M, G) \) implies that \( G[X] \) is connected. Therefore, (i) follows.

Let \( u \) and \( v \) be the ends of \( P \). From Lemma 8.3 there exists \( T \in \mathcal{P}_G(G_0) \) with \( \{u, v\} \subseteq T \). Let \( w \in N_G(X) \cap V(G_0) \), and let \( z \in X \) be a vertex with \( wz \in E(G) \). From Lemma 6.3 there exists an \( M \)-forwarding path \( Q \) from \( z \) to \( r \in \{u, v\} \) with \( V(Q) \setminus \{r\} \subseteq X \). Then, \( wz + Q \) forms an \( M \)-ear relative to \( G_0 \) whose ends are \( w \) and \( r \). Therefore, from Lemma 8.3 \( w \in T \) follows, and we have (ii).

The relationship between the basilica order and generalized Kotzig-Lovász decomposition is shown in the next theorem.
Theorem 8.5. Let $G$ be a graph, and let $G_0 \in \mathcal{G}(G)$. For each connected component $K$ of $G[U(G_0)]$, there exists $T_K \in \mathcal{P}_G(G_0)$ such that $N_G(K) \cap V(G_0) \subseteq T_K$.

Proof. Let $M$ be a maximum matching of $G$.

Define a family $\mathcal{X} \subseteq 2^{V(K)}$ as follows: $X \subseteq V(K)$ is a member of $\mathcal{X}$ if $\Psi(X, P; G_0, M, G)$ holds for some $M$-ear $P$ relative to $G_0$.

Claim 8.6. It holds that $\bigcup_{X \in \mathcal{X}} X = V(K)$.

Proof. From the definition of $\mathcal{X}$, clearly $\bigcup_{X \in \mathcal{X}} X \subseteq V(K)$. In contrast, $V(K)$ is obviously separating, and, from Lemma 6.13, each factor-component that composes $V(K)$ is contained in a set of vertices $X$ with $\Psi(X, P; G_0, M, G)$ for some $M$-ear $P$ relative to $G_0$; from Lemma 8.4, this $X$ satisfies $X \subseteq V(K)$. Therefore, we have $\bigcup_{X \in \mathcal{X}} X \supseteq V(K)$.

For each $T \in \mathcal{P}_G(G_0)$, we define $\mathcal{X}_T \subseteq \mathcal{X}$ as follows: $X \in \mathcal{X}_T$ is a member of $\mathcal{X}_T$ if $N_G(X) \cap V(G_0) \subseteq T$ holds. From Lemma 8.4, if $S \neq T$, then $\mathcal{X}_S \cap \mathcal{X}_T = \emptyset$; additionally, $\bigcup_{T \in \mathcal{P}_G(G_0)} \mathcal{X}_T = \mathcal{X}$ holds.

Claim 8.7. Let $S, T \in \mathcal{P}_G(G_0)$. Let $X \in \mathcal{X}_S$ and $Y \in \mathcal{X}_T$. If $X \cap Y \neq \emptyset$, then $S = T$. If $X \cap Y = \emptyset$ and $E[X, Y] \neq \emptyset$, then $S = T$.

Proof. First assume $X \cap Y \neq \emptyset$. As both $X$ and $Y$ are closed with respect to $M$, so is $X \cap Y$. Take $x \in X \cap Y$ arbitrarily; from Lemma 6.9, we have an $M$-forwarding path $Q$ from $x$ to a vertex $r \in V(G_0)$ with $V(Q) \setminus \{r\} \subseteq X$; from Lemma 8.4, we have $r \in S$. Trace $Q$ from $r$, and let $y$ be the first vertex we encounter that is in $N_G(X \cap Y)$; let $z \in X \cap Y$ be such that $yz \in E[X \setminus Y, X \cap Y]$. Here, $rQy$ is an $M$-forwarding path with $V(rQy) \setminus \{r\} \subseteq X \setminus Y$. By contrast, we also have an $M$-forwarding path $R$ from $z$ to a vertex $s \in T$ with $V(R) \setminus \{s\} \subseteq Y$. Here, $rQy + yz + R$ is an $M$-ear relative to $G_0$ with ends $r$ and $s$. From Lemma 8.3, $S = T$ follows.

Next, assume $X \cap Y = \emptyset$ and $E[X, Y] \neq \emptyset$. Let $t_1 \in X$ and $t_2 \in Y$ be vertices with $t_1t_2 \in E[X, Y]$. From Lemma 6.9, for each $i \in \{1, 2\}$, we have an $M$-forwarding path $L_i$ from $t_i$ to a vertex $r_i \in V(G_0)$ with $V(L_i) \setminus \{r_i\} \subseteq X$ and $V(L_i) \setminus \{r_i\} \subseteq Y$; from Lemma 8.4, we have $r_1 \in S$ and $r_2 \in T$. Therefore, $L_1 + t_1t_2 + L_2$ forms an $M$-ear relative to $G_0$ with ends $r_1$ and $r_2$. From Lemma 8.3, again $S = T$ follows.

As $K$ is connected, Claim 8.7 implies $|\{T \in \mathcal{P}_G(G_0) : \mathcal{X}_T \neq \emptyset\}| = 1$. This completes the proof.

8.2. Declaration of New Canonical Decomposition. We can now declare a new canonical decomposition in which the basilica order and the generalized Kotzig-Lovász decomposition are unified through Theorem 8.5. According to Theorem 8.5, the strict upper bounds on a factor-component are each “attached” or “assigned” to an equivalence class of the generalized Kotzig-Lovász decomposition. That is, let $H$ be a factor-component of a graph $G$, let $I \in \mathcal{G}(G) \setminus \{H\}$ be such that $H \triangleleft I$, and let $K$ be the connected component of $G[U(H)]$ with $V(I) \subseteq V(K)$. If $S \in \mathcal{P}_G(H)$ is such that $N_G(K) \cap V(H) \subseteq S$ as in Theorem 8.5, then we can view $I$ as being “attached” or “assigned” to $S$ as an upper bound on $H$. Hence, a graph can be regarded as being constructed by repeatedly assigning and attaching each factor-component to an equivalence class possessed by a lower bound.
Although the basilica order structure and the generalized Kotzig-Lovász decomposition themselves can be considered individually as canonical decompositions, they are integrated into a single theory of a canonical decomposition through the relationship given by Theorem 8.5. We call this integrated concept the basilica decomposition, because this evokes the idea of a graph being structured like an architectural building. The term “basilica” comes from the cathedral theorem by Lovász [24, 25], which is an inductive characterization of saturated graphs. In fact, the cathedral theorem can be derived from our new canonical decomposition [18].

9. INCONSISTENT FACTOR-COMPONENTS VIA GALLAI-EDMONDS STRUCUTRE THEOREM

In this section, we use the Gallai-Edmonds structure theorem to obtain further information about the inner structure of inconsistent factor-components.

**Lemma 9.1.** Let $G$ be a graph, $M$ be a maximum matching of $G$, and $r$ be a vertex exposed by $M$. If $P$ is an $M$-forwarding path from some vertex to $r$, then all edges of $P$ are allowed and therefore $P$ is contained in a factor-component.

**Proof.** If $P$ is such a path, then $M \triangle E(P)$ is also a maximum matching of $G$. Hence, the claim follows.

**Lemma 9.2.** Let $G$ be a graph. If $K$ is a connected component of $G[D(G)]$, then the vertices in $V(K) \cup N_G(K)$ are contained in the same factor-component.

**Proof.** According to Theorem 3.3, $K$ is factor-critical. Let $r \in V(K)$, and let $M$ be a maximum matching of $G$ exposing $r$. Arbitrarily choose $x \in V(K)$. From Lemma 4.7, there is an $M$-forwarding path $P$ from $x$ to $r$. From Lemma 9.1, $x$ and $r$ are contained in the same factor-component. Therefore, the vertices in $N_G(K)$ are also contained in the same factor-component as the vertices of $K$.

The next theorem follows from Lemma 9.2 and Theorem 3.3.

**Theorem 9.3.** Let $G$ be a graph. Any subgraph $H$ is an inconsistent factor-component of $G$ if and only if it is a connected component of $G[D(G) \cup A(G)] \setminus E(G[A(G)])$.

10. PERTINENT PROPERTIES

10.1. Non-triviality of $\triangle$. The following theorem shows that most factorizable graphs with more than one factor-components have non-trivial structures as posets.

**Theorem 10.1.** Let $G$ be a factorizable graph, $G_1, G_2 \in \mathcal{G}(G)$ be factor-components for which $G_1 \triangle G_2$ does not hold, and let $G_1$ be minimal in the poset $(\mathcal{G}(G), \triangle)$. Then there are possibly identical complement edges $e$ and $f$ of $G$ between $G_1$ and $G_2$ with $\mathcal{G}(G + e + f) = \mathcal{G}(G)$ and $G_1 \triangle G_2$ in $(\mathcal{G}(G + e + f), \triangle)$.

**Proof.** First, we prove the case where there is an edge $xy$ with $x \in V(G_1)$ and $y \in V(G_2)$. Let $M$ be a perfect matching of $G$. Choose a vertex $w \in V(G_2)$ with $w \neq y$ in $G_2$, and let $P$ be an $M$-saturated path of $G_2$ between $w$ and $y$. If $xw \in E(G)$ holds, then $xy + P + wx$ is an $M$-ear that is relative to $G_1$ and
traverses \( G_2 \). This implies \( G_1 \triangleleft G_2 \) under Lemma 8.2, which is a contradiction. Thus, \( xw \notin E(G) \) holds.

Suppose \( \mathcal{G}(G + xw) \neq \mathcal{G}(G) \). Then, Lemma 12.2 implies that \( G + xw \) has an \( M \)-alternating circuit that contains \( xw \), hence \( G \) has an \( M \)-saturated path \( C \) between \( x \) and \( w \). Trace \( C \) from \( x \), and let \( z \) be the first vertex in \( V(G_2) \) that we encounter. Then, \( xy + xCz \) is an \( M \)-ear of \( G \) that is relative to \( G_2 \) and traverses \( G_1 \), which implies \( G_2 \triangleleft G_1 \) under Lemma 8.2, this contradicts the minimality of \( G_1 \). Thus, \( \mathcal{G}(G + xw) = \mathcal{G}(G) \), and we have proved this case.

We now consider the other case, where no edge of \( G \) connects \( G_1 \) and \( G_2 \). Choose \( x \in V(G_1) \) and \( y \in V(G_2) \) arbitrarily. If \( \mathcal{G}(G + xy) = \mathcal{G}(G) \) holds, then we can reduce it to the first case and the claim follows.

Therefore, it suffices to consider the case with \( \mathcal{G}(G + xy) \neq \mathcal{G}(G) \). Then, from Lemma 12.2 for any perfect matching \( M \) of \( G, G + xy \) has an \( M \)-alternating circuit that contains \( xy \). Thus, we have an \( M \)-saturated path \( C \) between \( x \) and \( y \) in \( G \). Trace \( C \) from \( y \), and let \( u \) be the first vertex in \( G_1 \) that we encounter. Furthermore, trace \( uCy \) from \( u \), and let \( v \) be the first vertex we encounter that is in \( G_2 \).

If \( \mathcal{G}(G + uv) = \mathcal{G}(G) \), then the claim follows by the same argument.

Otherwise, that is, if \( \mathcal{G}(G + uv) \neq \mathcal{G}(G) \), then Lemma 12.2 implies that \( G \) has an \( M \)-alternating circuit that contains \( uv \). Thus, we have an \( M \)-saturated path \( D \) between \( u \) and \( v \) in \( G \).

Trace \( D \) from \( u \), and let \( w \) be the first vertex of \( vCu - u \) that we encounter.

If \( wCu \) has an even number of edges, then \( wCu + uDw \) is an \( M \)-alternating circuit of \( G \) that contains non-allowed edges, which is a contradiction according to Lemma 12.2. Therefore, we assume that \( wCu \) has an odd number of edges. Let \( H \in \mathcal{G}(G) \) be the factor-component with \( w \in V(H) \).

Then, \( wCu + uDw - E(H) \) is an \( M \)-ear that is relative to \( H \) and traverses \( G_1 \); this implies \( G_1 \triangleleft H \) from Lemma 8.2, which contradicts the minimality of \( G_1 \). Thus, this completes the proof.

### 10.2. Vertices in Upper Bounds

From Theorem 8.3, the following is derived rather easily.

**Theorem 10.2.** Let \( G \) be a graph, and let \( H \in \mathcal{G}(G) \). Then, \( U^*(H) \) is the maximum critical-inducing set for \( H \); that is, the union of all the critical-inducing sets for \( H \) is also a critical-inducing set for \( H \) and equals \( U^*(H) \).

**Proof.** By Theorem 6.14, any critical-inducing set for \( H \) is contained in \( U^*(H) \). Therefore, by Lemma 6.5, the union of all the critical-inducing set for \( H \) is also a critical-inducing set for \( H \), contained in \( U^*(H) \).

Conversely, by the definition of \( U^*(H) \), for each \( I \in U^*(H) \), there is a critical-inducing set for \( H \) to \( I \). Therefore, \( U^*(H) \) is contained in, and accordingly coincides with the union of all the critical-inducing sets.

Thus, we have the following as a corollary of Theorem 10.2.

**Corollary 10.3.** Let \( G \) be a graph, and let \( H \in \mathcal{G}(G) \) and \( S \subseteq P_G(H) \). Let \( K_1, \ldots, K_l \), where \( l \geq 1 \), be the connected components of \( G[U(H)] \) such that \( N(K_i) \cap V(H) \subseteq S \) for each \( i \in \{1, \ldots, l\} \). Then, \( G[V(K_1) \cup \cdots \cup V(K_l) \cup S]/S \) is factor-critical.

### 10.3. Immediate Compatible Pair of Factor-Components
Lemma 10.4. Let $G$ be a graph and $M$ be a maximum matching of $G$. Let $X \subseteq V(G)$ be an critical-inducing set for $G_1 \in \mathcal{G}(G)$ and let $P$ be an M-ear relative to $X$ with $\text{int}(P) \neq \emptyset$ and $\text{int}(P) \subseteq C(G)$. Let $Y := X \cup V(H_1) \cup \cdots \cup V(H_k)$, where $H_1, \ldots, H_k$ are the factor-components that $P$ traverses. Then, $Y$ is a critical-inducing set for $G_1$.

Proof. According to Lemma 6.4 for each $v \in X$, there is an $M$-forwarding path $R_v$ from $v$ to a vertex in $G_1$. From Lemma 6.12 $\Psi(\text{int}(P), P; X, M, G)$ holds. Furthermore, from Lemma 6.11 $\Psi(Y \setminus X; P; X, M, G)$ holds. Hence, for each $x \in Y \setminus X$, there is an $M$-forwarding path $L_x$ from $x$ to a vertex $w$ that is equal to one of the ends of $P$, according to Lemma 6.9. Therefore, $L_x + R_w$ is an $M$-forwarding path from $x$ to a vertex in $G_1$. Thus, the statement is proved by Lemma 6.4.

Lemma 10.5. Let $G$ be a graph and $M$ be a maximum matching of $G$. Let $X \subseteq V(G)$ be an critical-inducing set for $G_1 \in \mathcal{G}(G)$ and let $P$ be an M-ear relative to $X$ with ends $u_1$ and $u_2$. Then, there are a factor-component $H$ with $V(H) \subseteq X$ and an $M$-ear $Q$ relative to $H$ such that $E(Q) \setminus E(G[X]) = E(P)$.

Proof. Under Lemma 6.4 for each $i \in \{1, 2\}$, there is an $M$-forwarding path $Q_i$ from $u_i$ to some vertex in $V(G_1)$. Trace $Q_2$ from $u_2$, and let $x$ be the first encountered vertex in a factor-component $H$ that also has some vertices of $Q_1$; such $H$ certainly exists because $G_1$ has shares some vertices with both $Q_1$ and $Q_2$. Furthermore, trace $Q_1$ from $u$, and let $z$ be the first vertex in $H$. Then, $H$ and $zQ_1u + PvQ_{2}y$ are desired factor-component and M-ear.

Proposition 10.6. Let $G$ be a graph and $M$ be a maximum matching of $G$. Let $G_1$ and $G_2$ are distinct factor-components with $G_1 \triangleleft G_2$. If $G_1$ and $G_2$ are immediate, that is, for any $H \in \mathcal{G}(G)$, $G_1 \triangleleft H \triangleleft G_2$ implies $G_1 = H$ or $G_2 = H$, then there is an M-ear relative to $G_1$ that traverses $G_2$.

Proof. Let $X$ be a critical-inducing set $X$ for $G_1$ to $G_2$. Let $X'$ be a maximal (in fact, the maximum) subset of $X \setminus V(G_2)$ that is critical-inducing for $G_1$; such $X'$ certainly exists because $V(G_1) \subseteq X \setminus V(G_2)$ is critical-inducing for $G_1$. Under Lemma 6.6 there is an $M$-ear $P$ relative to $X'$ with $V(P) \subseteq X$ and $\text{int}(P) \neq \emptyset$. From Lemma 10.5 the minimum separating set that contains $X' \cup V(P)$ is a critical-inducing set for $G_1$. Therefore, $P$ traverses $G_2$. Furthermore, Lemma 10.5 implies that we can use this $P$ to obtain an $M$-ear that is relative to a factor-component $H$ with $V(H) \subseteq X'$ and traverses $G_2$. Therefore, from Lemma 8.2 $G_1 \triangleleft H$ and $H \triangleleft G_2$ hold. As $H \neq G_2$ holds, we have $H = G_1$. This completes the proof of this proposition.

11. Algorithmic Results

11.1. Algorithmic Preliminaries. In the remainder of this paper, we present algorithms for computing the basilica decomposition. Section 11.2 presents some preliminary facts that will be used in the remaining sections. We denote by $n$ and $m$ the numbers of vertices and edges of an input graph, respectively. Note that we can assume $m = \Omega(n)$ and, accordingly, $O(n + m) = O(m)$ if an input graph is connected or factorizable. Section 11.2 provides an algorithm for computing the factor-components, and then Sections 11.3 and 11.4 present how to compute the
generalized Kotzig-Lovász decomposition and the basilica order. Each costs $O(nm)$ time, using Edmonds’ maximum matching algorithm as a subroutine [8].

**Theorem 11.1** (Micali and Vazirani [28], Vazirani [32]). Given a graph, one of its maximum matchings can be computed in $O(\sqrt{nm})$ time.

The following two statements can be found implicitly in Edmonds’s algorithm [8]. See also Lovász and Plummer [25].

**Theorem 11.2** (implicitly stated in Edmonds [8]). Given a graph $G$ and a maximum matching $M$, the set $D(G)$, $A(G)$, and $C(G)$ can be computed in $O(n + m)$ time.

**Proposition 11.3** (implicitly stated in Edmonds [9]). Let $G$ be a graph and $M$ be a maximum matching of $G$, and let $r \in V(G)$ be a vertex exposed by $M$. Let $C$ be the connected component of $G[D(G)]$ that contains $r$.

(i) Then, for any maximum matching $M'$ of $G$ that exposes $r$, $M'_X$ is a near-perfect matching of $C$.

(ii) Define $X \subseteq 2^V(G)$ as follows: $X \subseteq V(G)$ is a member of $X$ if $r \in X$ holds, $G[X]$ is factor-critical, and $M_X$ is a near-perfect matching of $G[X]$, exposing $r$. Then, the maximum member of $X$ is equal to $V(C)$.

(iii) Given $G$, $M$, and $r$, $C$ can be computed in $O(m)$ time.

The next statement can be deduced from Edmonds’ algorithm. See also Carvalho and Cheriyan [1].

**Proposition 11.4.** Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$, and let $u \in V(G)$.

(i) The set of vertices that can be reached from $u$ by an $M$-saturated path can be computed in $O(m)$ time.

(ii) All the allowed edges adjacent to $u$ can be computed in $O(m)$ time.

(iii) All the factor-components of $G$ can be computed in $O(nm)$ time.

### 11.2. Computing Factor-components.

Propositions [1.6] and [11.4] show how to compute inconsistent factor-components, whereas Theorem [9.3] implies an algorithm for computing inconsistent factor-components. Hence, we now obtain the following:

**Theorem 11.5.** Given a graph $G$, one of its perfect matchings $M$, and the sets $D(G)$, $A(G)$, and $C(G)$, the factor-components of $G$ are computed in $O(nm)$ time.

**Proof.** Under Proposition [1.6] we can compute $G(G)$ by computing $G(G[D(G) \cup A(G)])$ and $G(G[C(G)])$ individually. From Theorem [9.3], we can compute $G(G[D(G) \cup A(G)])$ in $O(n + m)$ time. From Proposition [11.4], we can compute $G(G[C(G)])$ in $O(nm)$ time. Therefore, we can obtain $G(G)$ in $O(nm)$ time. \[\]

### 11.3. Computing the Generalized Kotzig-Lovász Decomposition.

From Observation [7.7] and Proposition [11.4], we can compute the generalized Kotzig-Lovász decomposition.

**Theorem 11.6.** Given a graph $G$, one of its maximum matchings $M$, the set of factor-components $G(G)$, and the sets $D(G)$, $A(G)$, and $C(G)$, the generalized Kotzig-Lovász decomposition of $G$ can be computed in $O(nm)$ time.
Proof. We compute \( \mathcal{P}_G(H) \) for each \( H \in \mathcal{G}(G) \). According to Observation 7.4, if \( H \) is inconsistent then \( \mathcal{P}_G(H) = \{V(H) \cap A(G)\} \cup \{\{x\} : x \in V(H) \setminus A(G)\} \). Therefore, \( \mathcal{P}_G(H) \) for all \( H \in \mathcal{G}^+(G) \) can be computed in \( O(nm) \) time in total.

If \( H \) is consistent, we can compute \( \mathcal{P}_G(H) \) in a similar way as the Kotzig-Lovász decomposition of a consistently factor-connected graph \( H \). That is, for each \( v \in V(H) \), compute the set of vertices \( U \) that can be reached from \( v \) by an \( M \)-saturated path, and recognize \( V(H) \setminus U \) as a member of \( \mathcal{P}_G(H) \). Each \( U \in \mathcal{P}_G(H) \) can be computed in \( O(m) \) time according to Proposition 11.4. Therefore, computing \( \mathcal{P}_G(H) \) for all \( H \in \mathcal{G}^+(G) \) costs \( O(nm) \) time. Thus, the proof is completed. \( \square \)

11.4. Computing the Basilica Order. In this section, we present an algorithm for computing the basilica order in \( \mathcal{G}(G) \) time. We determine the poset by computing \( U(H) \) for each factor-component \( H \). The following lemmas are provided to associate \( U(H) \) with Proposition 11.3. Lemmas 11.7 and 11.8 are used to prove Lemma 11.9.

Lemma 11.7. Let \( G \) be a graph and \( M \) be a maximum matching of \( G \). Let \( H \in \mathcal{G}(G) \). For no \( x \in V(G) \setminus V(H) \) exposed by \( M \) and for no \( y \in V(H) \), there exists an \( M \)-exposed path from \( x \) to \( y \).

Proof. Suppose this lemma fails, and let \( P \) be an \( M \)-exposed path from \( x \in V(G) \setminus V(H) \) to \( y \in V(H) \). Then, \( y \) is covered by \( M \), because, otherwise \( M \Delta E(P) \) would be a bigger matching of \( G \) than \( M \). Hence, there is a vertex \( y' \in V(H) \) to which \( y \) is matched by \( M \). Then, \( P + yy' \) is an \( M \)-forwarding path from \( y' \) to \( x \), and therefore, from Lemma 9.1, \( y \) and \( x \) are contained in the same factor-component, which is a contradiction. \( \square \)

Lemma 11.8. Let \( G \) be a graph and \( M \) be a maximum matching of \( G \). Let \( H \in \mathcal{G}(G) \). Then, \( M_{V(G) \setminus V(H)} \) is a maximum matching of \( G/H \).

Proof. Suppose this lemma fails, that is, \( M_{V(G) \setminus V(H)} \) is not a maximum matching of \( G/H \). Then, \( G/H \) has an \( M \)-exposed path \( P \) in which one end is the contracted vertex \( h \) that corresponds to \( H \) and the other end is a vertex exposed by \( M \). In \( G \), \( P \) forms an \( M \)-exposed path between a vertex not in \( V(H) \) to a vertex of \( H \). This contradicts Lemma 11.7. \( \square \)

The next lemma associates the set of strict upper bounds with the special subgraph \( C \) depicted in Proposition 11.3.

Lemma 11.9. Let \( G \) be a graph and \( M \) be a maximum matching of \( G \), and let \( G_0 \in \mathcal{G}(G) \). Let \( G' := G/G_0 \), and let \( g_0 \) be the contracted vertex that corresponds to \( G_0 \). Then, there exists a connected component \( C \) of \( G'[\Delta(G')] \) with \( g_0 \in V(C) \) such that \( V(C) \setminus \{g_0\} \) is equal to \( U_G(G_0) \).

Proof. Let \( M' := M \setminus E(G_0) \). Define \( X \subseteq 2^{V(G)} \) as follows: \( X \subseteq V(G) \) is a member of \( X \) if \( V(G_0) \subseteq X \) holds, \( G[X]/G_0 \) is factor-critical, and \( M_X \) is a perfect matching of \( G[X] \). Additionally, define \( X' \subseteq 2^{V(G')} \) as follows: \( X' \subseteq V(G') \) is a member of \( X' \) if \( g_0 \in X' \) holds, \( G'[X'] \) is factor-critical, and \( M_{X'} \) is a near-perfect matching of \( G'[X'] \), exposing \( g_0 \). It is easy to see that for \( X \subseteq V(G) \) and \( X' \subseteq V(G') \) with \( X \setminus V(G_0) = X' \setminus \{g_0\} \), \( X \in X \) holds if and only if \( X' \in X' \) holds.

According to Lemma 11.8, \( M' \) is a maximum matching of \( G' \), which exposes \( g_0 \). Hence, from Proposition 11.3 there exists a connected component \( C \) of \( G'[\Delta(G')] \)
with \( g_0 \in V(C) \), and \( V(C) \) is equal to the maximum member of \( \mathcal{X}' \). Accordingly, \( \mathcal{X} \) has the maximum member \( X_0 \), with \( X_0 \setminus V(G_0) = V(C) \setminus \{g_0\} \).

In the following, we prove \( X_0 = U^*_G(G_0) \). From Proposition 11.3, with respect to any maximum matching of \( G' \) that exposes \( g_0 \), \( V(C) \) is closed. This implies that \( X_0 \) is a separating set of \( G \). Accordingly, \( X_0 \) is a critical-inducing set for \( G_0 \), and therefore, from Theorem 10.2 \( X_0 \subseteq U^*_G(G_0) \) holds. By contrast, \( X_0 \supseteq U^*_G(G_0) \) holds, because \( U^*_G(G_0) \in \mathcal{X} \) holds. Hence, we obtain \( X_0 = U^*_G(G_0) \), and therefore, \( U_G(G_0) = V(C) \setminus \{g_0\} \).

The next statement immediately follows from Proposition 11.3 and Lemma 11.9.

Lemma 11.10. Given a graph \( G \), a maximum matching \( M \) of \( G \), and \( H \in \mathcal{G}(G) \), \( U(H) \) can be computed in \( O(m) \) time.

The next theorem shows how to compute the poset of the basilica order using Lemma 11.10.

Theorem 11.11. Given a graph \( G \), one of its maximum matchings \( M \), and \( \mathcal{G}(G) \), we can compute the poset \( (\mathcal{G}(G), \prec) \) in \( O(nm) \) time.

Proof. It is sufficient to list all the strict upper bounds for each factor-component of \( G \) by the following procedure.

1: Initialize \( f : \mathcal{G}(G) \to 2^{\mathcal{G}(G)} \) by \( f(H) := \emptyset \) for each \( H \in \mathcal{G}(G) \);
2: for all \( H \in \mathcal{G}(G) \) do
3: \hspace{1em} compute \( U(H) \) according to Lemma 11.10;
4: \hspace{1em} for all \( x \in U(H) \) do
5: \hspace{2em} let \( I \in \mathcal{G}(G) \) be such that \( x \in V(I) \);
6: \hspace{2em} \( f(H) := f(H) \cup \{I\} \).
7: \hspace{1em} end for
8: end for

The correctness of the algorithm is obvious. For each \( H \in \mathcal{G}(G) \), the above procedure costs \( O(m) \) time; therefore, the entire computation costs \( O(nm) \) time.

11.5. Concluding Algorithms. From Theorems 11.1, 11.2, and 11.3 a maximum matching, the Gallai-Edmonds family, and the set of factor-components can be computed in \( O(nm) \) time in total. Therefore, from Theorems 11.6 and 11.11 we obtain an \( O(nm) \) time algorithm for computing the basilica decomposition.

Theorem 11.12. Given a graph \( G \), the basilica order \( \prec \) over \( \mathcal{G}(G) \) and the generalized Kotzig-Lovász decomposition can be computed in \( O(nm) \) time.

12. Conclusion

We have introduced a new canonical decomposition, the basilica decomposition. The central results that support our new theory are the basilica order, a canonical partial order over the set of factor-components (Theorem 6.14), the generalized Kotzig-Lovász decomposition (Theorem 7.5), and the structure described by a relationship between these two, which unites them into a canonical decomposition (Theorem 8.5). We have also presented an \( O(nm) \) time algorithm for computing the basilica decomposition. As canonical decompositions have formed the theoretical foundation of matching theory, we believe that the results in this paper will be beneficial to this field, and, by extension, to the entire field of combinatorics. We have already obtained some important results using the ideas in this paper.
• The structure of barriers, which are a classically important notion that corresponds to the dual of maximum matchings, has been revealed [14,17].
• A new proof of Lovász’s cathedral theorem, which is an inductive characterization of saturated graphs, has been obtained [18].
• A purely graph theoretic proof of the celebrated tight cut lemma, which has contributed to almost all the results about the perfect matching polytope since 1982, has been obtained [19].

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