Ricci-cubic holographic dark energy

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ABSTRACT: In this work, we propose the Ricci-cubic holographic dark energy model. The model is inspired by the cubic curvature invariant formed by the contraction of three Riemann tensors. A combination of Ricci scalar and the cubic invariant is used to describe the infrared cutoff of the holographic dark energy. Such a construction is extremely useful since the evolution does not depend on the past or future features of the universe, but completely on its present features. Moreover, the use of invariants makes the theory more fundamental in nature. We have constructed the model and studied its cosmological features. The analytical solutions of various cosmological parameters such as the density parameter, equation of state parameter, and deceleration parameter are extracted and their behaviour is studied. It is seen that the holographic dark energy model can exhibit all the cosmological epoch, sequentially starting from radiation in the early universe, followed by matter, and finally the dark energy dominated epoch at late times. The equation of state parameter shows that the model can exhibit quintessence nature, phantom-divide crossing, and even phantom nature depending on the choice of parameter spaces.

KEYWORDS: Ricci scalar, cubic invariant, holographic dark energy, cosmology, equation of state.
1 Introduction

Observations from the SN Ia supernovae have confirmed that of late, the universe has entered in an accelerated expansion phase \[1, 2\]. It is obvious that the long standing matter dominated epoch has come to an end. There are various theoretical framework which can explain this unusual phenomenon, of which, the most usual one is the cosmological constant \(\Lambda\) \[3\]. But in order to find a solution of the dynamical nature one needs to introduce extra degrees of freedom beyond the standard framework of general relativity (GR) \[4\] and standard model of particle physics. Moreover explaining the entire thermal history of the universe including the early time inflation is always an issue. Modification of GR can be done via two different avenues. The first one is by introducing modifications in the geometrical sector giving rise to modified gravity theories \[5–7\]. The other way is to modify the matter sector thus introducing exotic components with negative pressure known as dark energy (DE) \[8\]. Both these concepts employ extra degrees of freedom as desirable.

There are various candidates of DE available in the literature. Chaplygin gas models \[9–13\] and scalar field models are notable examples. The holographic principle, which has its origin in the black hole thermodynamics, states that the entropy of a system is characterized by its area and not by its volume \[14, 15\]. Holographic dark energy (HDE) \[16–18\] has been developed in connection with this holographic principle, which also has connections with the string theory \[14, 19\]. It is known that a quantum field theory has connections with an ultraviolet cutoff which is the largest distance possible under the framework \[20\]. This ultraviolet cutoff in turn has direct connection with the vacuum energy, which will be a form of dark energy of the holographic origin. For an extensive review on HDE the reader may refer to \[21\]. There have been extensive research on HDE both in its basic and extended forms and with time the model has been quite successful \[22–28\]. One of the major success of HDE models have been its compatibility with the observational data \[29–31\].

It is an accepted fact that the HDE density is proportional to the inverse square of the infrared cutoff \(L\) given by,

\[
\rho_{DE} = \frac{3c}{\kappa^2 L^2}
\]  \hspace{1cm} (1.1)

where \(\kappa^2\) is the gravitational constant and \(c\) is a parameter. However in connection with the cosmological application of the holographic principle there is no accepted idea about what the infrared cutoff should be. The most common choices are the Hubble radius and the particle horizon which are incapable of driving the cosmic acceleration \[32\]. Finally it is the future event horizon
that suits the scenario and can suitably act as the infrared cutoff [18]. Although this choice suits the scenario well, there are some logical problems associated with it. The present value of the dark energy density is actually determined by the future evolution of the DE, which is quite an uncomfortable concept to deal with. So further attempts have been made in the quest of finding modified holographic dark energy models, where the DE density does not depend on the future evolution, but on the past and present evolution. Models where the infrared cutoff can be given by the quantities depending on the past features of the universe are called the agegraphic dark energy [33–35]. The model which depend upon the present evolution of the universe involves the use of the inverse square root of Ricci scalar as the infrared cutoff [36]. This is called the Ricci holographic dark energy. Using Ricci scalar as the infrared cutoff has the added advantage that the evolution of the dark energy is governed by a gravitational invariant which has fundamental importance in gravitational theories. Moreover its evolution depends only on the present features of the universe, which is theoretically much more sound concept. Apart from this, Ricci holographic dark energy has very interesting cosmological applications [36–39]. Some phenomenological problems of minimal holographic dark energy are given in [40, 41]. Since the use of invariants is fundamental in physics, naturally a we can use other invariants to represent the infrared cutoff and thus give rise to new forms of holographic dark energy. The simplest such extension would be using the Gauss-Bonnet invariant. Saridakis in [47] addressed this problem by using a combination of the Ricci scalar and Gauss-Bonnet invariant to describe the infrared cutoff as given below,

\[ \frac{1}{L^2} = -\alpha R + \beta \sqrt{|G|} \]  

(1.2)

This model of HDE was called the Ricci-Gauss Bonnet HDE, and it exhibited very interesting cosmological features including the entire thermal history of the universe, from the radiation followed by matter, and finally dark energy dominated epoch.

Since we are talking about invariants, there can be many such possibilities from the mathematical point of view. The Gauss-Bonnet invariant was formed from the contractions of two Riemann tensors. So it is a quadratic theory whose spectrum coincides with that of the Einstein gravity. We can always play around with the Riemann tensor \( R_{\mu\nu\rho\sigma} \), Ricci tensor \( R_{\mu\nu} \), energy momentum tensor \( T_{\mu\nu} \), etc. and explore their possible contractions and produce different scalar invariants from such exercises. It is understandable that their physical significance and importance to cosmology is a different issue and needs thorough study, their mathematical importance is irrefutable. For the cubic case there exists a six parameter family of cubic theories whose spectrum is identical to that of the Einstein gravity [48]. In 4-dimensional spacetime a general non-topological cubic term would be given by [48, 49]

\[ P = \beta_1 R_{\mu \nu} R_{\rho \sigma} R_{\gamma \delta} + \beta_2 R_{\mu \nu} R_{\rho \sigma} R_{\gamma \delta} + \beta_3 R_{\mu \nu} R_{\rho \sigma} R_{\gamma \delta} + \beta_4 R_{\mu \nu} R_{\rho \sigma} R_{\gamma \delta} + \beta_5 R_{\mu \nu} R_{\rho \sigma} R_{\gamma \delta} + \beta_6 R_{\mu \nu} R_{\rho \sigma} R_{\gamma \delta} + \beta_7 R_{\mu \nu} R_{\rho \sigma} R_{\gamma \delta} + \beta_8 R_{\mu \nu} R_{\rho \sigma} R_{\gamma \delta} \]  

(1.3)

where \( \beta_i \) are parameters. If the theory possesses a spectrum identical to that of general relativity then the following parameter conditions are satisfied,

\[ \beta_7 = \frac{1}{12} (3\beta_1 - 24\beta_2 - 16\beta_3 - 48\beta_4 - 5\beta_5 - 9\beta_6) \]  

(1.4)

\[ \beta_8 = \frac{1}{12} (-6\beta_1 + 36\beta_2 + 22\beta_3 + 64\beta_4 + 3\beta_5 + 9\beta_6) \]  

(1.5)
The cubic invariant has been used to develop the cubic gravity theory [48] and even extended to generalized $f(P)$ gravity theories [49–51]. Motivated from [47], here we are interested in constructing a holographic dark energy model where the infrared cutoff is given by a combination of Ricci scalar and the cubic invariant given in eqn.(1.3). This will be an extension of the Ricci holographic dark energy and the Ricci Gauss-Bonnet holographic dark energy. We term it as Ricci-cubic holographic dark energy and here we are interested in constructing the model and exploring its cosmological features. Such a construction is more generalized and theoretically more tangible since higher order invariants have their contributions in the set up. This will allow for a far richer cosmological structure which is the basic motivation behind this study. The paper is organized as follows: In section II we construct the Ricci-cubic holographic dark energy and present the basic equations. Section III is dedicated to the cosmological evolution of the universe filled with the HDE model. Finally the paper ends with a discussion and conclusion in section IV.

2 Ricci-cubic holographic dark energy

Here we will construct a model of holographic dark energy where the infrared cutoff is given by a combination of the Ricci and cubic scales. We will consider a homogeneous and isotropic universe modelled by the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric given by,

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)$$

(2.1)

where $a(t)$ is the cosmological scale factor and $k$ is the spatial curvature, such that $k = -1, 0, +1$ corresponds to open, flat and closed spatial geometry respectively. $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ represents the 2-sphere. Here we will concentrate on the flat geometry which may be easily extended to the open and closed universes.

It is known that Ricci holographic dark energy uses the Ricci scalar $R$ in the FLRW metric as the infrared cutoff. Since the dimensions of the Ricci scalar is $1/(\text{Length})^2$, the energy density of the dark energy is obtained proportional to $R$. Whenever we use a modification using any curvature invariants, it is imperative to use the new invariants in the same order in order to preserve the consistency. We know that in the FLRW geometry the cubic invariant $P$ occurs in the order of $R^3$. Therefore it can be argued that in any model of HDE where the IR cutoff is given by $R$, there should be contributions from $P^{1/3}$ also. Following this motivation here we will consider an HDE model where the inverse square of the IR cutoff is given by,

$$\frac{1}{L^2} = -\alpha R + \lambda P^{1/3}$$

(2.2)

where the constants $\alpha$ and $\lambda$ are model parameters. It can be clearly seen that for $\lambda = 0$, we retrieve the usual Ricci HDE, while for $\alpha = 0$, we get a pure cubic HDE. Using eqn.(2.2) in eqn.(1.1) we get the energy density of the Ricci-cubic HDE as,

$$\rho_{DE} = \frac{3}{\kappa^2} \left( -\alpha R + \lambda P^{1/3} \right)$$

(2.3)

where the constant $c$ has been absorbed in the model parameters $\alpha$ and $\lambda$ for convenience. Now for the flat FLRW geometry the Ricci scalar and the cubic invariant are respectively given by,

$$R = -6 \left( 2H^2 + \dot{H} \right)$$

(2.4)

and

$$P = 6\beta H^4 \left( 2H^2 + 3\dot{H} \right)$$

(2.5)
where \( H = \dot{a}/a \) is the Hubble function and the dots denote derivatives with respect to time. Moreover in the above expression we have defined \( \tilde{\beta} \) as,
\[
\tilde{\beta} \equiv -\beta_1 + 4\beta_2 + 2\beta_3 + 8\beta_4
\]

(2.6)

It should be mentioned here that for the cubic invariant we have considered derivatives only upto first order, so that the FLRW equations are of the second order. The condition for the second order field equations is satisfied if we consider,
\[
\beta_6 = 4\beta_2 + 2\beta_3 + 8\beta_4 + \beta_5
\]

(2.7)

Using these, the energy density of Ricci-cubic HDE becomes,
\[
\rho_{DE} = \frac{3}{\kappa^2} \left[ 6\alpha \left( 2H^2 + \dot{H} \right) + \lambda \left\{ 6\tilde{\beta}H^4 \left( 2H^2 + 3\dot{H} \right) \right\}^{1/3} \right]
\]

(2.8)

Now the first FLRW equation is given by,
\[
3H^2 = \kappa^2 \left( \rho_m + \rho_{DE} \right)
\]

(2.9)

where \( \rho_m \) is the energy density of matter. The equation of state (EoS) parameter of matter is given by \( w_m = p_m/\rho_m \), where \( p_m \) is the pressure of matter. Finally the matter sector follows the conservation relation given by,
\[
\dot{\rho}_m + 3H (\rho_m + p_m) = 0
\]

(2.10)

Using eqns.(2.9) and (2.10) one can fully determine the evolution of the universe, provided the matter equation of state is known. Generally we consider a pressureless matter sector, i.e. \( p_m = 0 \) leading to \( w_m = 0 \). Using this in the conservation equation (2.10) we get the matter energy density as \( \rho_m = \rho_{m0}a^{-3} \), where \( \rho_{m0} \) is the density of matter in the present time. Thus we have the matter energy density and also from eqn.(2.8) we have the dark energy density. Using them in the FLRW equation (2.9) we have a differential equation in terms of the scale factor \( a \), which may be solved to get the evolution of the universe filled with Ricci-cubic holographic dark energy. In the following section we will study the cosmological evolution of such a universe.

3 Cosmological Evolution

In this section we will study the cosmological evolution of a universe filled with Ricci-cubic HDE and pressureless matter in the form of dust. Let us introduce the density parameters,
\[
\Omega_m \equiv \frac{\kappa^2 \rho_m}{3H^2}, \quad \Omega_{DE} \equiv \frac{\kappa^2 \rho_{DE}}{3H^2}
\]

(3.1)

In terms of the above density parameters the FLRW equation (2.9) becomes \( \Omega_m + \Omega_{DE} = 1 \). Using the energy density for dust we can write \( \Omega_m \) as \( \Omega_m = \Omega_{m0}H_0^2/H^2a^3 \), where \( \Omega_{m0} \) is the present value of \( \Omega_m \) given by \( \Omega_{m0} = \kappa^2 \rho_{m0}/3H_0^2 \). Similarly \( H_0 \) is the present value of the Hubble function. Using this result we can easily get the Hubble function as below,
\[
H = \frac{H_0\sqrt{\Omega_{m0}}}{\sqrt{a^3(1-\Omega_{DE})}} \]

(3.2)

Here we will use \( x = \ln a \) as the independent variable. Differentiating eqn.(3.2) we have,
\[
\dot{H} = -\frac{H^2}{2(1-\Omega_{DE})}\left[3(1-\Omega_{DE})-\Omega'_{DE}\right]
\]

(3.3)
where prime represents derivative with respect to $x$. Using the above equation in eqns. (2.4) and (2.5) we get,

$$R = -3H^2 \left( 1 + \frac{\Omega_{DE}}{1 - \Omega_{DE}} \right)$$ (3.4)

$$P = 3\tilde{\beta}H^6 \left( \frac{3\Omega_{DE}' - 5}{1 - \Omega_{DE}} \right)$$ (3.5)

Using the above expressions for $R$ and $P$ in eqn.(2.3) we get,

$$\rho_{DE} = \frac{3H^2}{\kappa^2} \left[ 3^{1/3} \lambda \left( \tilde{\beta} \left( \frac{5 - 5\Omega_{DE} - 3\Omega_{DE}'}{\Omega_{DE} - 1} \right) \right)^{1/3} + \frac{3\alpha (-1 + \Omega_{DE} - \Omega_{DE}')}{\Omega_{DE} - 1} \right]$$ (3.6)

Putting the above expression for energy density in (3.1) we get the dimensionless energy density as,

$$\Omega_{DE} - 3^{1/3} \lambda \left( \frac{\tilde{\beta} (5 - 5\Omega_{DE} - 3\Omega_{DE}')}{\Omega_{DE} - 1} \right)^{1/3} - \frac{3\alpha (-1 + \Omega_{DE} - \Omega_{DE}')}{\Omega_{DE} - 1} = 0$$ (3.7)

The above differential equation governs the evolution of the Ricci-cubic HDE for a flat universe with matter in the form of dust. Unfortunately this equation does not have any general analytic solution. So we try to find out a solution under some assumptions. The first term on the right hand side being a cube root over $\Omega_{DE}$ and its derivatives is the most complicated term. So we expand the term binomially and consider only the linear powers of $\Omega_{DE}$ and its derivative. We are quite justified in doing this since it can be considered that the energy density of any component of the universe in the late universe should be low enough, so that higher powers may be neglected. A solution for the simplified equation is obtained as,

$$\Omega_{DE} = \left( \frac{3(2\lambda + 1)}{15\lambda + 3} \right)^{1/3}$$ (3.8)

where $\xi = (15\tilde{\beta})^{1/3}$ and $C_1$ is the constant of integration. The integration constant can easily determined by considering the scale factor of the present universe as $a = a_0 = 1$ and the corresponding energy density as $\Omega_{DE} = \Omega_{DE0}$. So eqn.(3.8) gives the evolution of the Ricci-cubic HDE in terms of the logarithm of the scale factor. This can easily be evaluated in terms of the redshift by using the relation $z = \frac{a_0 - a}{a_0}$, since $x \equiv \ln a = -\ln (1 + z)$. Finally using eqn.(3.8) in eqn.(3.2) we can easily obtain the Hubble function, which can further be integrated to obtain the scale factor $a(t)$.

Another important parameter regarding the dark energy model is the equation of state parameter given by $w_{DE} = p_{DE}/\rho_{DE}$, where $p_{DE}$ and $\rho_{DE}$ are the pressure and density of holographic dark energy respectively. Since matter is conserved according to eqn.(2.10), Ricci-cubic HDE will be conserved according to the equation,

$$\dot{\rho}_{DE} + 3H (1 + w_{DE}) \rho_{DE} = 0$$ (3.9)

Using eqn.(2.3) in the above equation we get

$$w_{DE} = -1 + \Omega_{DE}^{-1} \left( \frac{\alpha R'}{3H^2} - \frac{\lambda P'}{9H^2 \rho^{2/3}} \right)$$ (3.10)

Now differentiating eqns.(2.4) and (2.5) and using eqn.(3.3) we have,

$$\frac{R'}{3H^2} = 3 + \frac{2\Omega_{DE}'}{1 - \Omega_{DE}} - \frac{2(\Omega_{DE}')^2}{(1 - \Omega_{DE})^2} - \frac{\Omega_{DE}''}{1 - \Omega_{DE}}$$ (3.11)
and

\[
\frac{P'}{9H^2P^{2/3}} = \frac{\tilde{b}^{1/3}}{3^{2/3} (1 - \Omega_{DE})^2} \left[ \Omega_{DE}' (1 - \Omega_{DE}) + 15 (1 - \Omega_{DE})^2 + 2\Omega_{DE}' (2\Omega_{DE}' + 7\Omega_{DE} - 7) \right] \left( \frac{3\Omega_{DE}'}{(1 - \Omega_{DE})} - 5 \right)^{2/3}
\]

This expression gives the EoS parameter of the Ricci-cubic HDE in terms of \(\ln a\), i.e. as a function of the redshift. Finally we can introduce the deceleration parameter which is crucial for any model of dark energy, and is given by,

\[
w_{DE} = -1 - \Omega_{DE}^{-1} \left[ \alpha \left( 3 + \frac{2\Omega_{DE}'}{1 - \Omega_{DE}} - \frac{2(\Omega_{DE}')^2}{(1 - \Omega_{DE})^2} - \frac{\Omega_{DE}''}{1 - \Omega_{DE}} \right) \right]
- \frac{\lambda\tilde{b}^{1/3}}{3^{2/3} (1 - \Omega_{DE})^2} \left\{ \Omega_{DE}' (1 - \Omega_{DE}) + 15 (1 - \Omega_{DE})^2 + 2\Omega_{DE}' (2\Omega_{DE}' + 7\Omega_{DE} - 7) \right\} \left( \frac{3\Omega_{DE}'}{(1 - \Omega_{DE})} - 5 \right)^{2/3}
\]

Using eqns.(3.11) and (3.12) in eqn.(3.10) we get the EoS parameter as,

\[
w_{DE} = -1 - \Omega_{DE}^{-1} \left[ \alpha \left( 3 + \frac{2\Omega_{DE}'}{1 - \Omega_{DE}} - \frac{2(\Omega_{DE}')^2}{(1 - \Omega_{DE})^2} - \frac{\Omega_{DE}''}{1 - \Omega_{DE}} \right) \right]
- \frac{\lambda\tilde{b}^{1/3}}{3^{2/3} (1 - \Omega_{DE})^2} \left\{ \Omega_{DE}' (1 - \Omega_{DE}) + 15 (1 - \Omega_{DE})^2 + 2\Omega_{DE}' (2\Omega_{DE}' + 7\Omega_{DE} - 7) \right\} \left( \frac{3\Omega_{DE}'}{(1 - \Omega_{DE})} - 5 \right)^{2/3}
\]

It is clearly seen that the usual evolution of the universe can be obtained from the Ricci-cubic holographic dark energy with the transition from deceleration to acceleration occurring at \(z \approx 0.45\) as the observations suggest.

In Fig.(1) we present the dimensionless density parameters \(\Omega_{DE}(z)\) and \(\Omega_m(z) = 1 - \Omega_{DE}(z)\) as functions of the redshift \(z\). We see that as the universe evolves the dark energy dominates over the matter sector, which complies with known results. In Fig.(2) we have plotted the equation of state (EoS) parameter against the redshift as it arises from eqn.(3.13). From the plot we see that presently \((z = 0)\) the universe has entered a dark energy dominated phase \((w_{DE} < -1/3)\). In fact it has the plunged deep into the DE dominated region with a fair possibility of phantom-divide crossing \((w_{DE} < -1)\). In Fig.(3) the deceleration parameter \(q\) have been plotted against the redshift \(z\). It is clearly seen that the usual evolution of the universe can be obtained from the Ricci-cubic holographic dark energy with the transition from deceleration to acceleration occurring at \(z \approx 0.45\) as the observations suggest.

From eqn.(3.10) it is seen that even if \(\Omega_{DE} \to 1\) in the future universe \((z < 0)\), the asymptotic value of \(w_{DE}\) still depends on the model parameters \(\alpha\) and \(\lambda\). So the effects of these parameters on the EoS parameter is significant, and it must be studied in detail. So we have plotted \(w_{DE}\) for various choices of \(\lambda\) and \(\alpha\) in figs.(4) and (5) respectively. In Fig.(4) we have presented the EoS parameter against the redshift for different values of the parameter \(\lambda\), which scales the cubic term \(P\) in the energy density of the Ricci-cubic HDE. From the plot we see that with the increase in the value of \(\lambda\) there is a decrease in both the current \((z = 0)\) and future \((z < 0)\) values of the EoS parameter. Moreover it is seen that the dependence of the EoS parameter on \(\lambda\) is highly pronounced in the future epoch and declines both in the present and past times. It is evident from the figure that in all the trajectories there is a transition from \(w_{DE} > -1\) to \(w_{DE} < -1\) around \(z \approx 0.45\). This means that the universe has evolved from the quintessence regime to the phantom regime of late (around the present time). We can also comprehend that for proper parameter spaces, we can have trajectories lying completely in the quintessence regime going asymptotically to \(w_{DE} \to -1\), resulting in a de-Sitter universe. This is because from the eqns.(2.8) and (2.9) we see that a late de-Sitter like evolution is possible if \(12\alpha + \left(12\beta\right)^{1/3} = 1\). Similarly for suitable parameter spaces, it is also possible to obtain trajectories lying completely in the phantom regime \((w_{DE} < -1)\), where the universe is driven towards a Big Rip. In Fig.(5) we have plotted the EoS parameter against the redshift for different values of the parameter \(\alpha\), which scales the Ricci scalar component of the
Fig. 1 shows the evolution of the Ricci-cubic holographic dark energy density parameters $\Omega_{DE}$ and $\Omega_m$ as a function of the redshift $z$ for $\alpha = -0.45$, $\lambda = 1.09$ and $\tilde{\beta} = 100$. The constant of integration $C_1$ has been fixed in order to obtain $\Omega_{DE}(z = 0) \equiv \Omega_{DE0} \approx 0.68$ at present time.

Fig. 2 shows the evolution of the equation of state (EoS) parameter $w_{DE}$ of Ricci-cubic holographic dark energy as a function of the redshift $z$ for $\alpha = 1.5$, $\lambda = 0.09$, $\tilde{\beta} = 10^6$. The constant of integration $C_1$ has been fixed in order to obtain $\Omega_{DE}(z = 0) = \Omega_{DE0} \approx 0.68$.

HDE. It is see that with an increase in the value of $\alpha$ there is a corresponding decrease in the EoS parameter. We also see that the evolution of the EoS parameter depends significantly on $\alpha$ in the future epoch ($z < 0$). It should be mentioned that in all these plots we have fixed the constant $C_1$ in order to obtain $\Omega_{DE}(z = 0) = \Omega_{DE0} \approx 0.68$ consistent with the observations [52]. To gain more insights into the future evolution of the universe described by Ricci-cubic HDE and to understand the stability of the various asymptotic solutions one must employ the dynamical system analysis. Such a study is beyond the scope of the present work and can be a potential future project.

If we consider $\lambda = 0$, we get the usual Ricci dark energy and the corresponding equations
Fig. 3 shows the evolution of the deceleration parameter $q$ as a function of the redshift $z$ for $\alpha = 0.1$, $\lambda = 0.5$ and $\beta = 100$. The constant of integration $C_1$ has been fixed in order to obtain $q = 0$ at around $z \approx 0.45$ when the transition occurs.

obtained above are considerably simplified. From eqn.(3.7) we get $\Omega_{DE}$ in the form,

$$\Omega_{DE|\text{Ric}} = \frac{e^{\frac{3\alpha}{3\alpha - 1}C_2} - 3\alpha e^{x + 3\alpha C_2}}{e^{\frac{3\alpha}{3\alpha - 1}C_2} - e^{x + 3\alpha C_2}}$$

which is exactly the form obtained in [47]. In the above expression $C_2$ is the constant of integration. Using this the Hubble parameter in eqn.(3.2) becomes,

$$H_{\text{Ric}} = H_0 \sqrt{\Omega_{m0}} e^{-\frac{2x}{3\alpha - 1}} \left[ \frac{(3\alpha - 1)e^{x + 3\alpha C_2}}{e^{\frac{3\alpha}{3\alpha - 1}C_2} - e^{x + 3\alpha C_2}} \right]^{-1/2}$$

Using the above expressions in eqn.(3.1) we get,

$$\rho_{DE|\text{Ric}} = \frac{3H_0^2\Omega_{m0}}{\kappa^2(3\alpha - 1)} \left[ e^{\left(\frac{15\beta}{3\alpha - 1}\right)x + C_2(3\alpha - 1)} - 3\alpha e^{-3x} \right]$$

This result exactly coincides with the expression given in [36] with the constants being redefined. Finally the using eqn.(3.13) the EoS parameter for Ricci dark energy can be given by,

$$w_{DE|\text{Ric}} = \frac{1}{3\Omega_{DE|\text{Ric}}} - \frac{1}{9\alpha} = \frac{3\alpha - 1}{9\alpha \left( 1 - 3\alpha e^{\left(\frac{15\beta}{3\alpha - 1}\right)x + 3\alpha C_2} \right)}$$

In the same way we can obtain pure cubic HDE by putting $\alpha = 0$. From eqn.(3.7) we get $\Omega_{DE}$ for pure cubic HDE as,

$$\Omega_{DE|\text{cub}} = \frac{e^{\frac{5\alpha + \zeta}{(3\alpha - 1)^3}x + (15\beta)^{1/3}\lambda} e^{C_3 + (15\beta)^{1/3}\lambda C_3}}{e^{\frac{5\alpha + \zeta}{(3\alpha - 1)^3}x} - e^{C_3 + (15\beta)^{1/3}\lambda C_3}}$$

where $\zeta = \frac{5^5/3^3x}{(3\alpha - 1)^3}$ and $C_3$ is the constant of integration. The Hubble parameter in eqn.(3.2) can be written as,

$$H_{\text{cub}} = H_0 \sqrt{\Omega_{m0}} \left[ \frac{e^{3x}e^{C_3 + (15\beta)^{1/3}\lambda C_3} \left( 1 + (15\beta)^{1/3}\lambda \right)}{e^{C_3 + (15\beta)^{1/3}\lambda C_3} - e^{\frac{5\alpha + \zeta}{(3\alpha - 1)^3}x + (15\beta)^{1/3}\lambda C_3}} \right]^{-1/2}$$
Fig. 4 shows the evolution of the equation of state (EoS) parameter \( w_{DE} \) of Ricci-cubic holographic dark energy as a function of the redshift \( z \) for different values of \( \lambda \) keeping \( \alpha \) unaltered. The constant of integration \( C_1 \) has been fixed in order to obtain \( \Omega_{DE}(z = 0) = \Omega_{DE0} \approx 0.68 \). We have taken \( \tilde{\beta} = 10^6 \).

Using eqn. (3.1) we get the energy density of cubic HDE as,

\[
\rho_{DE|cub} = 3H_0^2\Omega_m e^{-3x} \left[ \alpha \left( \frac{15 + \frac{152}{3^3/3\lambda}}{3^3/3\lambda} \right) - C_3 \left( 1 + (15\tilde{\beta})^{1/3} \lambda \right) \right] + \left( 15\tilde{\beta} \right)^{1/3} \lambda \]

\( (3.21) \)

The EoS parameter of pure cubic HDE can be obtained using eqns. (3.8) and (3.13) as follows,

\[
w_{DE|cub} = \frac{\Omega_{DE|cub}^4 + 15\Omega_{DE|cub} \tilde{\beta} \lambda^3 - 27\Omega_{DE|cub}^2 \tilde{\beta} \lambda^3 - 27\tilde{\beta}^{5/3} \lambda^5 \left( \frac{\Omega_{DE|cub}}{\beta \lambda^3} \right)^{2/3}}{27\tilde{\beta}^{5/3} \lambda^5 \left( \frac{\Omega_{DE|cub}}{\beta \lambda^3} \right)^{2/3}}
\]

\( (3.22) \)

Finally we will study two different cosmological set up using the Ricci-cubic holographic dark energy. In the first case we will explore an interacting scenario between the Ricci-cubic holographic dark energy and dark matter. In the second case we will include radiation sector in our set-up to describe the entire thermal history of the universe.

### 3.1 Interacting scenario between Ricci-cubic HDE and dark matter

A generalized scenario should allow for an interaction between dark energy and dark matter sectors since it cannot be eliminated by any logical arguments. Moreover inclusion of interaction in cosmological models helps to alleviate the cosmic coincidence problem, which raises question on the equality of the density parameters of the dark energy and dark matter sectors although these components have evolved following completely different scales \([53, 54]\). Including the interacting term the conservation equations for matter and HDE become respectively,

\[
\dot{\rho}_m + 3H (\rho_m + p_m) = -Q
\]

\( (3.23) \)

and

\[
\dot{\rho}_{DE} + 3H (1 + w_{DE}) \rho_{DE} = Q
\]

\( (3.24) \)
Fig. 5 shows the evolution of the equation of state (EoS) parameter $w_{DE}$ of Ricci-cubic holographic dark energy as a function of the redshift $z$ for different values of $\alpha$ keeping $\lambda$ unaltered. The constant of integration $C_1$ has been fixed in order to obtain $\Omega_{DE}(z = 0) = \Omega_{DE0} \approx 0.68$. We have taken $\beta = 10^9$.

where $Q$ represents the interaction between dark energy and dark matter sectors. Here $Q > 0$ indicates that there is a transfer from the dark matter sector to the dark energy sector and $Q < 0$ indicates the flow in the reverse direction. So a positive interaction term basically means a growth the dark energy sector, whereas a negative interaction indicates a decay of dark energy. A satisfactory form of the interaction term is theorized in various ways in the literature. Almost in all such forms it is evident that the interaction should be proportional to the energy density of the constituents of the universe. A well motivated form of interaction widely studied in literature is $Q = \eta H \rho_m$, where $\eta$ represents the rate of interaction between the sectors [53, 54]. Now we proceed to explore the cosmology of Ricci-cubic holographic dark energy in the presence of interaction with the dark matter sector. Using eqn.(3.23) we get the matter energy density as,

$$\rho_m = \frac{\rho_{m0}}{a^{3+\eta}} \tag{3.25}$$

Using this the Hubble parameter gets modified as below,

$$H = \frac{H_0 \sqrt{\Omega_{m0}}}{\sqrt{a^{3+\eta}(1 - \Omega_{DE})}} \tag{3.26}$$

The time gradient of the Hubble parameter is consequently modified as,

$$\dot{H} = -\frac{H^2}{2(1 - \Omega_{DE})} [(3 + \eta)(1 - \Omega_{DE}) - \Omega'_{DE}] \tag{3.27}$$

Further the Ricci and the cubic invariants get respectively modified as,

$$R = -3H^2 \left(1 - \eta + \frac{\Omega'_{DE}}{1 - \Omega_{DE}}\right) \tag{3.28}$$

$$P = 3\beta H^6 \left(\frac{3\Omega'_{DE}}{1 - \Omega_{DE}} - (5 + 3\eta)\right) \tag{3.29}$$
Using the above invariants the modified form of the differential equation (3.7) is obtained as,

\[
\Omega_{DE} + \frac{3\alpha(\eta - 1)(\Omega_{DE} - 1) + \Omega'_{DE}}{\Omega_{DE} - 1} + 3^{1/3} \lambda \left[ \frac{3(5 + 3\eta)(\Omega_{DE} - 1) + 3\Omega'_{DE}}{\Omega_{DE} - 1} \right]^{1/3} = 0 \quad (3.30)
\]

This equation does not admit any analytic solution and so we search for approximations that will allow for a solution. We expand binomially the third term on the left hand side of the above equation and consider only the linear terms. The solution is obtained as,

\[
\Omega_{DE} = \frac{e^{3\alpha(\eta + C_4)} + 3^{5/3} \alpha e^{3\alpha C_4} + 3 \alpha \eta (\eta + C_4) + 3^{1/3} \beta^{1/3}(5 + 3\eta)^{1/3} \lambda (\eta + C_4)}{e^{3\alpha(\eta + C_4)} - 3^{2/3} \alpha e^{3\alpha C_4} + 3 \alpha \eta (\eta + C_4) + 3^{1/3} \beta^{1/3}(5 + 3\eta)^{1/3} \lambda (\eta + C_4)}
\]

where \( q = -\frac{(5 + 3\eta)^{1/3}}{15 \alpha + 9 \alpha \eta + 3^{1/3} \beta^{1/3}(5 + 3\eta)^{1/3} \lambda} \) and \( C_4 \) is the constant of integration. For \( \eta > 0 \), we have a dark energy dominated universe soon enough, but for \( \eta < 0 \) there will be a delayed dark energy dominated epoch.

### 3.2 Evolution in the presence of radiation

In order to properly explain the entire thermal history of the universe we have to include the radiation component in our set up and then properly explore the scenario to obtain the radiation dominated early universe as indicated by the observations. In this section we will study the evolution equations of the Ricci-cubic holographic dark energy in presence of radiation. We begin by considering the density parameter for radiation as,

\[
\Omega_r \equiv \frac{k^2}{3H^2 \mu_r}
\]

(3.32)

Including the radiation component in the FLRW eqn. (2.9) we get \( \Omega_m + \Omega_{DE} + \Omega_r = 1 \). The Hubble parameter gets consequently modified as,

\[
H = \frac{H_0 \sqrt{\Omega_{m0}}}{\sqrt{a^3(1 - \Omega_{DE} - \Omega_r)}}
\]

(3.33)

The time gradient of the Hubble parameter is then given by,

\[
\dot{H} = -\frac{H^2}{2(1 - \Omega_{DE} - \Omega_r)} \left[ 3(1 - \Omega_{DE} - \Omega_r) - \Omega'_{DE} - \Omega'_r \right]
\]

(3.34)

The scalar invariants are modified as follows,

\[
R = -3H^2 \left( 1 + \frac{\Omega'_{DE} + \Omega'_r}{1 - \Omega_{DE} - \Omega_r} \right)
\]

(3.35)

\[
P = 3\tilde{\beta}H^6 \left( \frac{3(\Omega'_{DE} + \Omega'_r)}{1 - \Omega_{DE} - \Omega_r} - 5 \right)
\]

(3.36)

Using these results with eqns. (2.3) and (3.1) we get a differential equation in terms of \( \Omega_{DE} \) as given below,

\[
\Omega_{DE} + 3^{1/3} \lambda \left( \frac{\tilde{\beta}(-5 + 3\Omega_{DE} + 3\Omega'_{DE} + 5\Omega_r + 3\Omega'_r)}{\Omega_{DE} + \Omega_r - 1} \right)^{1/3} - 3\alpha \left( -1 + \Omega_{DE} - \Omega'_{DE} + \Omega_r - \Omega'_r \right) = 0
\]

(3.37)

The equations (3.11) and (3.12) are modified as follows,

\[
\frac{R'}{3H^2} = 3 + 2 \left( \frac{\Omega'_{DE} + \Omega'_r}{1 - \Omega_{DE} - \Omega_r} \right) - \frac{2(\Omega'_{DE} + \Omega'_r)^2}{(1 - \Omega_{DE} - \Omega_r)^2} - \frac{\Omega''_{DE} + \Omega''_r}{1 - \Omega_{DE} - \Omega_r}
\]

(3.38)
and

\[
\frac{P'}{9H^2P^{2/3}} = \frac{\beta^{1/3}}{3^{2/3}(1 - \Omega_{DE} - \Omega_r)^{4/3}[3(\Omega''_{DE} + \Omega_E) - 5(1 - \Omega_{DE} - \Omega_r)]^{2/3} \times [(\Omega''_{DE} + \Omega_E)(1 - \Omega_{DE} - \Omega_r)}
\]

\[
+ 15(1 - \Omega_{DE} - \Omega_r)^{2} + 2(\Omega''_{DE} + \Omega_E)\{2(\Omega''_{DE} + \Omega_E) + 7(\Omega_{DE} + \Omega_r) - 7\} \right]
\]

(3.39)

Using the above expressions the EoS parameter given in eqn.(3.13) gets modified to,

\[
w_{DE} = -1 - \Omega_{DE}^{-1}\left[3 + 2 \left(\frac{\Omega''_{DE} + \Omega_E}{1 - \Omega_{DE} - \Omega_r}\right) - \frac{2(\Omega''_{DE} + \Omega_E)^2}{(1 - \Omega_{DE} - \Omega_r)^2} - \frac{\Omega''_{DE} + \Omega_E}{1 - \Omega_{DE} - \Omega_r}\right] - \frac{\lambda\beta^{1/3}}{3^{2/3}(1 - \Omega_{DE} - \Omega_r)^{4/3}[3(\Omega''_{DE} + \Omega_E) - 5(1 - \Omega_{DE} - \Omega_r)]^{2/3} \times [(\Omega''_{DE} + \Omega_E)(1 - \Omega_{DE} - \Omega_r)}
\]

\[
+ 15(1 - \Omega_{DE} - \Omega_r)^{2} + 2(\Omega''_{DE} + \Omega_E)\{2(\Omega''_{DE} + \Omega_E) + 7(\Omega_{DE} + \Omega_r) - 7\} \right]
\]

(3.40)

In this case eqn.(3.37) does not admit any analytical solution and one should elaborate it numerically. In Fig.(6) we have presented the behaviour of the Ricci-cubic HDE density parameter \(\Omega_{DE}\), the matter density parameter \(\Omega_m\) and the radiation density parameter \(\Omega_r\) against the redshift.

In order to comply with the observations we have imposed the constraints \(\Omega_{DE}(z = 0) \equiv \Omega_{DE0} \approx 0.68\), \(\Omega_m(z = 0) \equiv \Omega_{m0} \approx 0.32\) and \(\Omega_r(z = 0) \equiv \Omega_{r0} \approx 0.0001\) at the present time. It is seen from the figure that with the evolution of the universe \((z \rightarrow 0)\) filled with the HDE fluid there is a transition of the universe from a matter dominated to a dark energy dominated epoch, with the radiation slowly decaying out. The universe sequentially evolves into radiation dominated, matter-dominated and finally, a dark energy-dominated regime which are in agreement with known results.

4 Discussion & Conclusion

In this work, we have proposed the Ricci-cubic holographic dark energy, inspired by the cubic curvature invariant. Here the infrared cutoff is determined by a combination of the Ricci scalar and the cubic curvature invariant. The advantage of such a formulation over the standard ones (like the Ricci holographic dark energy or the agegraphic dark energy model) is that, here the infrared cutoff and consequently the holographic dark energy density is not dependent on the future or the past evolution of the universe. The complete evolution is dependent on the current features of the universe, which is big advantage over the standard formulations. One more advantage is that here the infrared cutoff is given by invariants, which are of fundamental theoretical importance in gravity theories. Adopting a generalized approach we included contributions from both the Ricci scalar and cubic curvature invariant of the same order in the infrared cutoff and consequently in the energy density of the holographic dark energy.

First of all we formed the model and set up the necessary equations of the Ricci-cubic holographic dark energy. Then considering the Friedmann metric we proceeded to study the cosmological evolution of the universe filled by the Ricci-cubic holographic dark energy. Suitable dimensionless density parameters for matter and dark energy were constructed. A differential equation involving the dimensionless density parameter of the holographic dark energy was formed in terms of the logarithm of the scale factor (directly related to the logarithm of redshift). Under some assumptions
Fig. 6 shows the evolution of the dimensionless density parameters $\Omega_{DE}$, $\Omega_m$ and $\Omega_r$ of the various components of the universe as a function of the redshift $z$. Here we have imposed the conditions $\Omega_{DE}(z = 0) \equiv \Omega_{DE0} \approx 0.68$, $\Omega_m(z = 0) \equiv \Omega_{m0} \approx 0.32$ and $\Omega_r(z = 0) \equiv \Omega_{r0} \approx 0.0001$ at the present time. We have taken $\alpha = -0.2$, $\lambda = 0.02$, $\beta = 0.01$.

this equation admitted an analytical solution. We also determined the equation of state parameter and the deceleration parameter in terms of the dimensionless density parameters.

The Cosmological features of the Ricci-cubic holographic dark energy model are quite interesting and presents some increased capabilities in its dynamics. This is due to the presence of two model parameters controlling the two curvature invariants. Plots were generated for the cosmological parameters and it was seen that they comply with the cosmological evolution of the universe. It was also seen that the onset of the accelerated expansion can be fine tuned at $z \approx 0.45$ which is in agreement with the observations. Various plots for the equation of state parameter were generated to explore the effects of the model parameters on the evolution of the universe. It was seen that for suitable parameter spaces, it is possible to have complete quintessence like behaviour, complete phantom like behaviour, and phantom-divide crossing during the cosmological evolution. The equation of state parameter can also take the value exactly $-1$ mimicking the $\Lambda$CDM universe. An asymptotic de-Sitter like evolution is also well supported by this model under suitable conditions. When the contribution from the cubic invariant is set to zero, we get the usual Ricci holographic dark energy. Moreover when the contribution from the Ricci scalar is set to zero, we get the "pure" cubic holographic dark energy. Finally we included radiation in our set-up, and it was seen that, sequentially radiation, matter and dark energy epoch are possible. In the end the universe entered into a dark energy dominated regime, resulting in the late cosmic acceleration. It should be mentioned here that each plot is generated by using a different set of parameter values to comply with the known results. In future this can be well-adjusted by constraining the parameters using observational data.

Eventually we conclude by stating that there are some future work that needs to be done to deeply understand the nature of Ricci-cubic holographic dark energy, and consequently consider it as a successful candidate of dark energy. To explore the global behaviour of the scenario at late times, we need to employ the dynamical system analysis. In order to constrain the model parameters we need to use observational data from Type Ia supernovae (Sn Ia), baryon acoustic oscillations (BAO), cosmic microwave background (CMB) shift parameter and Hubble parameter observations. Apart from this, in order to check the model stability we need to perform a perturbation analysis.
These necessary studies are promising future projects related to Ricci-cubic holographic dark energy model.

Data Availability Statement

No new data were used or generated during the preparation of this manuscript.

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