Singularity Problem in Teleparallel Dark Energy Models

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Abstract

We study future singularity in teleparallel dark energy models, particularly its behavior and its (non)occurrence in the observationally viable models. For the models with a general self-potential of the scalar field, we point out that both at early times and in the future near the singularity the behavior of dark energy can be described by the analytic solutions of the scalar field we obtained for the model with no self-potential. As to the (non)occurrence in the viable models, we consider a natural binding-type self-potential, the quadratic potential, when fitting observational data, and illustrate the constraining region up to the 3σ confidence level as well as the region where a singularity will occur. As a result, the singularity region is outside the 3σ constraint. Thus, although the future singularity problem potentially exists in teleparallel dark energy models, the observationally viable models may not suffer this problem.
I. INTRODUCTION

The late-time acceleration of the cosmic expansion has been confirmed by a variety of observational data, such as those from type-Ia supernovae (SNIa) [1,2], cosmic microwave background radiation (CMB) [3–5] and baryon acoustic oscillation (BAO) [6]. This salient phenomenon may be explained simply by a cosmological constant, or suggest the existence of a new dynamical degree of freedom, either in the energy contents or in gravity, that provides anti-gravity. The origin and the nature of such anti-gravity is one of the most important problems in cosmology and astrophysics.

The new degree of freedom in the energy contents is generally called “dark energy” [7], while that in gravity gives a modification of gravity. The simplest degree of freedom is a scalar field, which is called quintessence when minimally coupled to gravity. In the present paper we will consider teleparallel dark energy, a dynamical scalar field non-minimally coupled to teleparallel gravity. It can be regarded as a new degree of freedom both in the energy contents and in the modification of gravity. The teleparallel dark energy model as an extension of teleparallel gravity is analogous to the minimal extension of the quintessence model in general relativity, i.e. the scalar-tensor theory, but has a richer structure [8–18].

We have studied in Ref. [12] the teleparallel dark energy model with no potential but simply the non-minimal coupling. We derived the analytic solutions of the scalar field in the radiation dominated (RD), matter dominated (MD) and scalar field/dark energy dominated (SD) eras, respectively. In SD we found a finite-time singularity that the Hubble expansion rate will go to infinity at a finite scale factor $a_s$ and a finite value of the scalar field $\phi_s$. This is caused by the non-minimal coupling that effectively changes the gravitational coupling strength and can even make it diverge when $\phi$ is driven to some specific value $\phi_s$.

In the present paper we study the future singularity problem in teleparallel dark energy models with a self-potential of the scalar field. In particular, we investigate (1) the behavior of the future singularity in the models with a general self-potential and (2) the (non)occurrence of the future singularity in the observationally viable models with a binding-type potential that provides an opportunity to avoid the singularity by confining the scalar field and keeping it away from $\phi_s$. We will fit observational data, obtain the observational constraints of the model, and then examine whether the constraining region overlaps with the singularity region. If no overlap, the singularity problem does not appear in the data-favored
region of the model, although it potentially exists in teleparallel dark energy models.

This paper is organized as follows. In Sec. II we introduce the teleparallel dark energy model with a general potential and study the singularity problem analytically. In Sec. III we numerically analyze a quadratic potential (binding-type) and an exponential potential (unbinding-type), and fit the former to observational data. A summary is given in Sec. IV.

II. TELEPARALLEL DARK ENERGY MODEL

In the theory of teleparallel gravity, gravity is described by torsion instead of curvature, and the dynamical variable is the vierbein field $e_a(x^\mu)$ (also called tetrad) that forms an orthonormal coordinate at each space-time point $x^\mu$,

$$e_a \cdot e_b = \eta_{ab} = \text{diag}(1, -1, -1, -1).$$ (1)

The metric is given by the vierbein field as $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. The torsion tensor is defined as the anti-symmetric part of the connection,

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} = e^\lambda_a \partial_\mu e^a_\nu - e^\lambda_a \partial_\nu e^a_\mu,$$ (2)

where the Weitzenböck connection $\Gamma^\lambda_{\nu\mu} \equiv e^\lambda_a \partial_\mu e^a_\nu$. The gravity Lagrangian with teleparallelism is given by the torsion scalar,

$$T = \frac{1}{4} T^\rho_{\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^\rho_{\mu\nu} T_{\nu\mu\rho} - T_{\rho\mu} T^\nu_{\nu\rho}.$$ (3)

The teleparallel dark energy model invokes a scalar field which is non-minimally coupled to teleparallel gravity. Its action reads

$$S = \int d^4 x \left[ \frac{T}{2\kappa^2} + \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi + \xi T \phi^2 \right) - V(\phi) + L_m \right],$$ (4)

where $\kappa \equiv \det(e^a_\mu) = \sqrt{-g}$, $\kappa$ and $\xi$ are coupling constants, $V(\phi)$ is the self-potential of the scalar field and $L_m$ the matter Lagrangian. For a flat, homogeneous and isotropic universe where $\phi = \phi(t)$ and $g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2)$, the vierbein field $e^b_\mu = \text{diag}(1, a, a, a)$, and the scalar and gravitational field equations derived from the above action read

$$\ddot{\phi} + 3H \dot{\phi} + 6\xi H^2 \phi + V,\phi = 0,$$ (5)

$$H^2 = \frac{\kappa^2}{3} \left( \rho_\phi + \rho_m + \rho_r \right),$$ (6)

$$\dot{H} = -\frac{\kappa^2}{2} \left( \rho_\phi + p_\phi + \rho_m + 4\rho_r/3 \right).$$ (7)
Here the Hubble expansion rate $H \equiv \dot{a}/a$, the energy density of non-relativistic matter $\rho_m \propto a^{-3}$, that of radiation $\rho_r \propto a^{-4}$, and the energy density and pressure of the scalar field

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) - 3\xi H^2 \phi^2,$$

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) + 3\xi H^2 \phi^2 + 2\xi \frac{d}{dt}(H\phi^2).$$

The equation of state (EoS) of the scalar field is defined as $w_\phi = p_\phi/\rho_\phi$. The gravitational field equations can be rewritten as

$$H^2 = \frac{1}{3} \left( \frac{\kappa^2}{1 + \xi \kappa^2 \phi^2} \right) \left( \frac{1}{2} \dot{\phi}^2 + V + \rho_m + \rho_r \right),$$

$$-\dot{H} = \frac{1}{2} \left( \frac{\kappa^2}{1 + \xi \kappa^2 \phi^2} \right) \left( \dot{\phi}^2 + 4\xi H \dot{\phi} \phi + \rho_m + 4\rho_r/3 \right),$$

where $\kappa^2/(1 + \xi \kappa^2 \phi^2)$ may be regarded as the effective gravitational coupling strength.

For the singularity problem we will consider the case with negative $\xi$ as in Ref. [12]. In this case the non-minimal coupling term in the scalar field equation (5) tends to drive the scalar field to infinity. The above equations show that a singularity occurs when $\kappa \phi$ is driven to $\kappa \phi_s \equiv \pm 1/\sqrt{-\xi}$ where the effective gravitational coupling strength goes to infinity. In Eqs. (6)–(9) both a positive potential and the term $3\xi H^2 \phi^2$ contribute negative pressure (while the term $2\xi H(\phi^2)/dt$ is undetermined). Accordingly, the non-minimal coupling with a negative coupling constant $\xi$ may provide repulsive gravitation onto the universe as well as a “repulsive force” onto the scalar field.

III. SINGULARITY IN TELEPARALLEL DARK ENERGY MODELS

In this section we investigate the behavior of the future singularity in the teleparallel dark energy model. According to the behavior of the scale factor, the effective dark energy density $\rho_{\text{eff}}$ and pressure $p_{\text{eff}}$ at the singularity, the future singularities have been classified [23] as follows. When $t \to t_s$,

- **Type I (Big Rip):** $a(t) \to \infty$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$,
- **Type II (Sudden):** $a(t) \to a_s$, $\rho_{\text{eff}} \to \rho_s$ and $|p_{\text{eff}}| \to \infty$,
- **Type III:** $a(t) \to a_s$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$,
- **Type IV:** $a(t) \to a_s$, $\rho_{\text{eff}} \to 0$, $|p_{\text{eff}}| \to 0$ and higher derivatives of $H$ diverge,
where $t_s$ and $a_s$ are finite. Recent studies on future singularity for other models related to the torsion scalar can be found in Refs. [24, 25].

In the following we will first analytically study the models with a general potential, then numerically analyze two specific types of potentials, and perform data fitting for the binding-type potential that has the opportunity to avoid the singularity.

A. Teleparallel dark energy with a general potential

In the viable models consistent with data, the potential is generally negligible at early times. Accordingly, in RD and MD the scalar field is mainly driven by the non-minimal coupling, and can be approximately described by the analytic solutions we obtained in Ref. [12] for the model with no potential. The solutions are presented as follows.

In RD and MD the Hubble expansion rate $H = \alpha/t$, where the constant $\alpha = (2/3)(1 + w_D)^{-1}$ and $w_D$ is the constant EoS of the dominant energy source. When $V(\phi)$ is negligible, the solution of the scalar field in these two eras is a linear combination of two modes:

$$\phi(t) = C_1 t^{l_1} + C_2 t^{l_2},$$

(12)

where $C_{1,2}$ are integration constants and

$$l_{1,2} = \frac{1}{2} \left[ \pm \sqrt{(3\alpha - 1)^2 - 24\xi \alpha^2 - (3\alpha - 1)} \right].$$

(13)

With negative $\xi$, $l_1$ and $l_2$ are positive and negative, respectively. Therefore, the $l_1$ mode, as an increasing mode, will soon dominates over the other decreasing ($l_2$) mode.

Henceforth we simply use the $l_1$ mode to describe the scalar field. The EoS of this mode

$$w_\phi = -1 + \frac{2(1 - l_1)}{3\alpha},$$

(14)

which is independent of the initial condition, a tracker behavior pointed out in Ref. [12]. We note that $w_\phi$ is always smaller than that of the dominant energy source, $w_D = -1 + 2/(3\alpha)$, if $w_D > -1$. This guarantees the existence of the late-time SD following MD, a generic feature of the teleparallel dark energy model.

At the late times when dark energy becomes important, the self-potential $V(\phi)$ competes with the non-minimal coupling to teleparallel gravity for the evolution of the scalar field. If $V(\phi)$ can prevent $\phi$ from reaching the singularity point $\phi_s$, the singularity can be avoided. On the contrary, if $\phi$ is still driven to $\phi_s$ even under $V(\phi)$, the singularity will occur.
Around the singularity, because of the extremely rapid expansion, $V(\phi)$, $\rho_m$ and $\rho_r$ can be ignored in the Friedmann equation when compared with the term proportional to $H^2\phi^2$. Equation (10) then gives
\[-(\kappa\phi')^2/6 + \xi(\kappa\phi)^2 + 1 \approx 0,\]
where the prime denotes the derivative with respect to the number of e-folding, $N \equiv \ln a$. In this case the behavior of the scalar field can be approximately described by the analytic solution $\phi(N)$ in SD we obtained in Ref. \cite{12} with no potential:
\[\phi(N) = \pm \sin \theta(N)/\sqrt{-\xi},\]
\[\theta(N) \equiv \sqrt{-6\xi N + C},\]
where $C$ is an integration constant and $\theta$ linearly increases with the e-folding number of the cosmic expansion. The dark energy EoS
\[w_\phi = -1 - \sqrt{-32\xi/3} \tan \theta.\]
As a result, when the cosmic scale factor increases to the value $a_s$ at which $\sin \theta = \pm 1$, $\kappa\phi = \kappa\phi_s \equiv \pm 1/\sqrt{-\xi}$ and $\phi' = 0$, the universe meets a type-III singularity:
\[H, \rho_\phi \to \infty, \quad p_\phi, w_\phi \to -\infty.\]

B. Numerical analysis

To demonstrate the features of teleparallel dark energy pointed out in the above analytical study, in this section we consider two specific potentials, a quadratic and an exponential potential, that respectively represent the binding-type and the unbinding-type potentials. We will show how the former may avoid the singularity while the latter cannot. We will then fit the former to observational data and thereby examine whether the singularity can be avoided in the observationally viable region of this model.

We write the quadratic potential as $V(\phi) = pV_0(\kappa\phi)^2$, where $V_0 = \rho_m^0/3$, i.e. one third of the present matter energy density, and $p$ is a dimensionless constant and the only free parameter of the potential. This self-interaction potential tends to confine the scalar field around zero, while the non-minimal coupling tends to drive the scalar field to infinity. The competition between them determines whether $\phi$ can reach the singularity point $\phi_s$, i.e., whether the singularity will occur.
FIG. 1. The evolution of the dark energy EoS for $V(\phi) = pV_0(\kappa\phi)^2$ with $(p, \xi) = (2, -0.4)$ and with the initial conditions: $\kappa\phi(i) \in [10^{-14}, 2 \times 10^{-9}]$ and $\phi'(i) = 0$ at $N = -20$. The black area is formed by the curves with respect to this wide range of initial conditions.

FIG. 2. Five initial conditions are picked to present the more detailed evolution of the dark energy EoS in the same quadratic-potential model as Fig. 1: $\kappa\phi(i) = 10^{-14}, 5 \times 10^{-10}, 10^{-9}, 1.5 \times 10^{-9}$ and $2 \times 10^{-9}$ (from top to bottom) at $N = -20$. The dashed line denotes the present time.

As an example for demonstration, here we set $p = 2$ and $\xi = -0.4$, and numerically obtain the evolution of the scalar field for a wide range of initial conditions: $\kappa\phi(i) \in [10^{-14}, 2 \times 10^{-9}]$ and $\phi'(i) = 0$ at $N = -20$. This setting is made according to the phenomenological requirement that $\Omega_{\phi 0} \sim \mathcal{O}(1)$ and $w_{\phi 0} \sim \mathcal{O}(-1)$. The evolution of the dark energy EoS along with the e-folding number of the cosmic expansion, $w_{\phi}(N)$, is presented in Fig. 1. Among these initial conditions, we pick five of them and present the more detailed evolution of $w_{\phi}$.
FIG. 3. The evolution of $\phi$ and $\phi'$ in the same quadratic-potential model as Fig. 1 with the same five initial conditions picked in Fig. 2. (The initial value $\phi(i)$ decreases from top to bottom.)

in Fig. 2 and the evolution of $\phi$ and $\phi'$ in Fig. 3. The occurrence of a future singularity is indicated when $w_\phi$ goes down rapidly, e.g., the lower three curves in Fig. 2.

These three figures manifest the features we have pointed out in the above analytical study: the tracker behavior of $w_\phi$, the late-time dominance of dark energy, and the possibility of avoiding the singularity. This possibility depends on the initial condition: For a larger initial value $\phi(i)$, i.e., closer to the singularity point $\phi_s$, the universe enters SD earlier and will meet the singularity even under a binding-type potential, as shown by the lower three curves in Fig. 2 and the upper three in Fig. 3. The singularity occurs when $\sqrt{-\xi \kappa \phi} = 1$ and $\phi' = 0$ at $\ln a_s \simeq 0.04, 0.44$ and 1.61, respectively. In contrast, for a smaller initial value $\phi(i)$ the scalar field will be pulled back by the potential before reaching $\phi_s$; thereafter, both the scalar field and its EoS oscillate around zero, i.e. behaving as a massive field, as shown by the upper two curves in Fig. 2 and the lower two in Fig. 3.

In addition to the initial condition, the value of $p$ is another key to the singularity. A larger $p$ corresponds to a steeper potential and therefore a larger “binding force” on $\phi$ that gives a better chance to avoid the singularity. This feature is illustrated in Fig. 4 where we consider a larger value, $p = 5$, and present $w_\phi(N)$ for the same five initial conditions as Fig. 2. This figure shows (i) the curve in the middle is previously singular but now becomes non-singular, (ii) the upper two non-singular curves enter the oscillating phase earlier, and  

1 The horizontal line in Fig. 3 shows the evolution for the smallest initial value $\phi(i)$. It oscillates at late times, but the amplitude is too small to be seen.
(iii) the lower two singular curves enter SD and then meet the singularity later than the previous case with \( p = 2 \).

For comparison, here we consider an unbinding-type potential, \( V(\phi) = pV_0e^{-\kappa \phi} \). We choose \( p = 2 \) and the initial conditions: \( \kappa \phi(i) \in [10^{-10}, 2 \times 10^{-9}] \) and \( \phi'(i) = 0 \) at \( N = -20 \). This unbinding potential apparently does not provide a binding force against the non-minimal coupling when the scalar field is driven towards \(+\infty\). Therefore the singularity will always occur, as shown by Fig. 5 where \( w_\phi \) always goes down rapidly after MD.

To examine the cosmological viability of the teleparallel dark energy models with no
FIG. 6. The 1σ–3σ constraints on the quadratic-potential model, where $\xi = -0.58$ and $p = 9.0$ are respectively set in the left and the right panel. The black area denotes the region where a future singularity will occur.

singularity, we fit the quadratic-potential model to the SNIa, BAO and CMB data, following the procedure in Ref. [9]. The result is presented in Fig. 6 where three contours show the 1σ–3σ constraints on the present matter energy density fraction $\Omega_{m0}$ together with the model parameters $p$ and $\xi$; the black area denotes the region where a singularity will occur. This figure shows that the observationally viable region up to the 3σ confidence level in this model is free of singularity. We note that the $\chi^2$ value of the best-fit of this model is 566.0 that is smaller than that of the $\Lambda$CDM model, $\chi^2_{\Lambda\text{CDM}} = 567.5$. In addition, the reduced $\chi^2$ values of these two models are both around 1.01, although $\Lambda$CDM invokes fewer parameters to give a better chance of having a smaller reduced $\chi^2$ value. This is because the current data favor the crossing of the phantom divide line ($w_\phi = -1$) from the phantom phase ($w_\phi < -1$) to quintessence one ($w_\phi > -1$) as the redshift increases, a characteristic of the teleparallel dark energy [8, 9] as well as other $f(T)$ models [26]. As a result, this model can fit the observational data well and the model in the fitting region suffers no future singularity problem.

IV. SUMMARY

In this paper we study the future singularity in the teleparallel dark energy models with a self-potential of the scalar field. A future singularity may appear due to the non-minimal
coupling of the scalar field to teleparallel gravity that tends to drive the scalar field to infinity. This singularity may be avoided by a binding-type potential that tends to confine the scalar field around a finite value. The destiny is determined by the competition between the self-interaction and the non-minimal coupling.

For the model with a general potential that looks difficult to analyze, we point out that the potential may be negligible at early times in the observationally viable models, as well as in the future near the singularity (if it exists). Therefore, in these epochs the scalar field can be approximately described by the analytic solutions in RD, MD and SD we obtained in Ref. [12] for the model with no potential. These analytic solutions show the tracker behavior of the dark energy EoS at early times (RD and MD), the inevitability of the late-time SD, and the possibility of meeting a type-III singularity in the future.

To demonstrate the possibility of avoiding the future singularity under a binding-type potential, we numerically analyze the model with a quadratic potential. With the numerical results we illustrate the above features read from the analytic solutions, and show how the (non)occurrence of the future singularity depends on the initial conditions and the steepness of the potential, both of which affect the competition between the self-interaction and the non-minimal coupling.

To examine whether the (non)occurrence of the singularity is favored by observational data, we fit the quadratic-potential model to the SNIa, CMB and BAO data. We present the 1σ–3σ constraints on the cosmological and the teleparallel dark energy model parameters, including the present matter energy density fraction, the non-minimal coupling constant and the mass scale of the quadratic potential (corresponding to the steepness of the potential). In addition, we illustrate in this parameter space the region where a future singularity will occur. As a result, the singularity region is outside the 3σ constraint, i.e., the current data favor the model region with no singularity. Thus, although the teleparallel dark energy models potentially have the future singularity problem, the observationally viable models may suffer no such problem.

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