An Optimal $\chi$-Bound for $(P_6, \text{diamond})$-Free Graphs

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Abstract

Given two graphs $H_1$ and $H_2$, a graph $G$ is $(H_1, H_2)$-free if it contains no induced subgraph isomorphic to $H_1$ or $H_2$. Let $P_t$ be the path on $t$ vertices and $K_t$ be the complete graph on $t$ vertices. The diamond is the graph obtained from $K_4$ by removing an edge. In this paper we show that every $(P_6, \text{diamond})$-free graph $G$ satisfies $\chi(G) \leq \omega(G) + 3$, where $\chi(G)$ and $\omega(G)$ are the chromatic number and clique number of $G$, respectively. Our bound is attained by the complement of the famous 27-vertex Schl"afli graph. Our result unifies previously known results on the existence of linear $\chi$-binding functions for several graph classes. Our proof is based on a reduction via the Strong Perfect Graph Theorem to imperfect $(P_6, \text{diamond})$-free graphs, a careful analysis of the structure of those graphs, and a computer search that relies on a well-known characterization of 3-colourable $(P_6, K_3)$-free graphs.

1 Introduction

All graphs in this paper are finite and simple. We say that a graph $G$ contains a graph $H$ if $H$ is isomorphic to an induced subgraph of $G$. A graph $G$ is $H$-free if it does not contain $H$. For a family $\mathcal{H}$ of graphs, $G$ is $\mathcal{H}$-free if $G$ is $H$-free for every $H \in \mathcal{H}$. When $\mathcal{H}$ consists of two graphs, we write $(H_1, H_2)$-free instead of $\{H_1, H_2\}$-free. As usual, $P_t$ and $C_s$ denote the path on $t$ vertices and the cycle on $s$ vertices, respectively. The complete graph on $n$ vertices is denoted by $K_n$. The graph $K_3$ is also referred to as the triangle. Let the diamond be the graph obtained from $K_4$ by removing an edge. For two graphs $G$ and $H$, we use $G + H$ to denote the disjoint union of $G$ and $H$. For a positive integer $r$, we use $rG$ to denote the disjoint union of $r$ copies of $G$. The complement of $G$ is denoted by $\overline{G}$. A clique (resp. stable set) in a graph is a set of pairwise adjacent (resp. non-adjacent) vertices. A $q$-colouring of a graph $G$ is a function $\phi : V(G) \rightarrow \{1, \ldots, q\}$ such that $\phi(u) \neq \phi(v)$ whenever $u$ and $v$ are adjacent in $G$. Equivalently, a $q$-colouring of $G$ is a partition of $V(G)$ into $q$ stable sets. A graph is $q$-colourable if it admits a $q$-colouring. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimum number $q$ for which $G$ is $q$-colourable. The clique number of $G$, denoted by $\omega(G)$, is the size of a largest clique in $G$. Obviously, $\chi(G) \geq \omega(G)$ for any graph $G$.

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A family $G$ of graphs is said to be $\chi$-bounded if there exists a function $f$ such that for every graph $G \in G$, every induced subgraph $H$ of $G$ satisfies $\chi(H) \leq f(\omega(H))$. The function $f$ is called a $\chi$-binding function for $G$. The notion of $\chi$-bounded families was introduced by Gyárfás [11] in 1987. Since then it has received considerable attention for $H$-free graphs.

We briefly review some results in this area. A hole in a graph is an induced cycle of length at least 4. An antihole is the complement of a hole. A hole or antihole is odd or even if it is of odd or even length, respectively. The famous Strong Perfect Graph Theorem [6] says that the class of graphs without odd holes or odd antiholes is $\chi$-bounded and the $\chi$-binding function is the identity function $f(x) = x$. If we only forbid odd holes, then the resulting class remains $\chi$-bounded but the best known $\chi$-binding function is double exponential [17]. On the other hand, if even holes are forbidden, then a linear $\chi$-binding function exists [1]: every even-hole-free graph $G$ satisfies $\chi(G) \leq 2\omega(G) - 1$. In recent years, there has been an ongoing project led by Scott and Seymour that aims to determine the existence of $\chi$-binding functions for classes of graphs without holes of various lengths. We refer the reader to the recent survey by Scott and Seymour [18] for various nice results. One thing to note is that most $\chi$-binding functions in this setting are exponential.

Another line of research is the study of $H$-free graphs for a fixed graph $H$. A classical result of Erdős [8] shows that the class of $H$-free graphs is not $\chi$-bounded if $H$ contains a cycle. Gyárfás [10] conjectured that the converse is also true (known as the Gyárfás Conjecture), and proved the conjecture when $H = P_4$ [11]: every $P_4$-free graph $G$ has $\chi(G) \leq (t - 1)^{\omega(G)} - 1$. Similar to results in [18], this $\chi$-binding function is exponential in $\omega(G)$. It is natural to ask the following question.

- Is it possible to improve the exponential bound for $P_t$-free graphs to a polynomial bound?

This turns out to be a very difficult question, and not much progress has been made over the past 30 years. It remains open whenever $t \geq 5$. (For $t \leq 4$, $P_t$-free graphs are perfect and hence $f(x) = x$ is the $\chi$-binding function.) Therefore, researchers have started to investigate subclasses of $P_t$-free graphs, hoping to discover techniques and methods that would be useful for tackling the problem. A natural type of subclass is to forbid a second graph in addition to forbidding $P_t$. For example, it was shown by Gaspers and Huang [9] that every $(P_6, C_4)$-free graph $G$ has $\chi(G) \leq \frac{3}{2}\omega(G)$. This $3/2$ bound was improved recently by Karthick and Maffray [13] to the optimal bound $5/4$: every $(P_6, C_4)$-free graph $G$ has $\chi(G) \leq \frac{5}{4}\omega(G)$. In another work, Karthick and Maffray [14] showed that every $(P_5, \text{diamond})$-free graph $G$ satisfies $\chi(G) \leq \omega(G) + 1$. Bharathi and Choudum [2] gave a cubic $\chi$-binding function for the class of $(P_2 + P_3, \text{diamond})$-free graphs. For the class of $(P_6, \text{diamond})$-free graphs, a common superclass of $(P_3, \text{diamond})$-free graphs and $(P_2 + P_3, \text{diamond})$-free graphs, Karthick and Mishra [15] proved that $f(x) = 2x + 5$ is a $\chi$-binding function, greatly improving the result for $(P_2 + P_3, \text{diamond})$-free graphs. In the same paper, they also obtained an optimal $\chi$-bound for $(P_6, \text{diamond})$-free graphs when the clique number is 3: every $(P_6, \text{diamond}, K_4)$-free graph is 6-colourable. For more results of this flavor, see [4, 5, 12, 19].

Our Contributions

In this paper, we give an optimal $\chi$-bound for the class of $(P_6, \text{diamond})$-free graphs. We prove that each $(P_6, \text{diamond})$-free graph $G$ satisfies $\chi(G) \leq \omega(G) + 3$ (Theorem 3 in Section 4). The bound is tight since it is attained by the complement of the famous 27-vertex Schläfli graph [15]. Our result unifies the results on the existence of linear $\chi$-binding functions for the class of $(P_3, \text{diamond})$-free graphs [14], $(P_2 + P_3, \text{diamond})$-free graphs [2], and $(P_6, \text{diamond})$-free graphs [15], and answers an open question in [15].

The remainder of the paper is organized as follows. We present some preliminaries in
Section 2 and prove some structural properties of imperfect \((P_6, \text{diamond})\)-free graphs in Section 3. We prove our main result in Section 4 and give some open problems in Section 5.

## 2 Preliminaries

For general graph theory notation we follow [3]. Let \(G = (V, E)\) be a graph. The *neighbourhood* of a vertex \(v\), denoted by \(N_G(v)\), is the set of vertices adjacent to \(v\). For a set \(X \subseteq V(G)\), let \(N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X\) and \(N_G[X] = N_G(X) \cup X\). The *degree* of \(v\), denoted by \(d_G(v)\), is equal to \(|N_G(v)|\). For \(x \in V\) and \(S \subseteq V\), we denote by \(N_S(x)\) the set of neighbours of \(x\) that are in \(S\), i.e., \(N_S(x) = N_G(x) \cap S\). For \(X, Y \subseteq V\), we say that \(X\) is *complete* (resp. *anti-complete*) to \(Y\) if every vertex in \(X\) is adjacent (resp. non-adjacent) to every vertex in \(Y\). For \(x \in V\) and \(Y \subseteq V\), we say \(x\) is *complete* (resp. *anti-complete*) to \(Y\) if \(x\) is adjacent (resp. non-adjacent) to every vertex in \(Y\). A vertex subset \(K \subseteq V\) is a *clique cutset* if \(G - K\) has more components than \(G\) and \(K\) induces a clique. For \(S \subseteq V\), the subgraph *induced* by \(S\) is denoted by \(G[S]\). We shall often write \(S\) for \(G[S]\) if the context is clear. We say that a vertex \(v\) *distinguishes* \(u\) and \(w\) if \(v\) is adjacent to exactly one of \(u\) and \(w\). A component of a graph is *trivial* if it has only one vertex, and *non-trivial* otherwise.

A graph \(G\) is *perfect* if \(\chi(H) = \omega(H)\) for each induced subgraph \(H\) of \(G\). An *imperfect* graph is a graph that is not perfect. One of the most celebrated theorems in graph theory is the Strong Perfect Graph Theorem [6].

**Theorem 1** ([6]). A graph is perfect if and only if it does not contain an odd hole or an odd antihole as an induced subgraph.

Another useful result is a characterization of 3-colourable \((P_6, K_3)\)-free graphs.

**Theorem 2** ([16]). A \((P_6, K_3)\)-free graph is 3-colourable if and only if it does not contain the Grötzsch graph (see Figure 1) as an induced subgraph.

![Figure 1: The Grötzsch graph.](image-url)

## 3 Structure of Imperfect \((P_6, \text{diamond})\)-Free Graphs

In this section we study the structure of imperfect \((P_6, \text{diamond})\)-free graphs. It follows from Theorem 1 that every imperfect \((P_6, \text{diamond})\)-free graph contains an induced \(C_5\). Let \(G = (V, E)\) be an imperfect \((P_6, \text{diamond})\)-free graph and let \(Q = \{v_1, v_2, v_3, v_4, v_5\}\) induce a \(C_5\) in
$G$ with edges $v_i v_{i+1}$ for $i = 1, \ldots, 5$. Note that all indices are modulo 5. We partition $V(G) \setminus Q$ into the following subsets:

\[
A_i = \{ v \in V \setminus Q : N_Q(v) = \{v_i\} \},
\]

\[
B_{i,i+1} = \{ v \in V \setminus Q : N_Q(v) = \{v_i, v_{i+1}\} \},
\]

\[
C_{i,i+2} = \{ v \in V \setminus Q : N_Q(v) = \{v_i, v_{i+2}\} \},
\]

\[
F_i = \{ v \in V \setminus Q : N_Q(v) = \{v_i, v_{i-2}, v_{i+2}\} \},
\]

\[
Z = \{ v \in V \setminus Q : N_Q(v) = \emptyset \}.
\]

Let $A = \bigcup_{i=1}^5 A_i$, $B = \bigcup_{i=1}^5 B_{i,i+1}$, $C = \bigcup_{i=1}^5 C_{i,i+2}$, and $F = \bigcup_{i=1}^5 F_i$. Since $G$ is diamond-free, it follows that $N(Q) = A \cup B \cup C \cup F$ and thus $V(G) = Q \cup A \cup B \cup C \cup F \cup Z$. We now prove a number of useful properties about those subsets.

1. Each component of $A_i$ is a clique. This follows directly from the fact that $G$ is diamond-free.

2. The sets $A_i$ and $A_{i+1}$ are anti-complete.

   If $a_1 \in A_i$ and $a_2 \in A_{i+1}$ are adjacent, then $\{a_1, a_2, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$ induces a $P_6$.

3. The sets $A_i$ and $A_{i+2}$ are complete.

   If $a_1 \in A_i$ and $a_2 \in A_{i+2}$ are not adjacent, then $\{a_2, v_{i+2}, v_{i+3}, v_{i+4}, v_i, a_1\}$ induces a $P_6$, a contradiction.

4. Each $B_{i,i+1}$ is a clique.

   If $b_1, b_2 \in B_{i,i+1}$ are not adjacent, then $\{b_1, b_2, v_i, v_{i+1}\}$ induces a diamond.

5. The set $B = B_{i,i+1} \cup B_{i+2,i+3}$ for some $i$.

   We show that for each $i$ either $B_{i,i+1}$ or $B_{i-1,i}$ is empty. Suppose not. Let $b_1 \in B_{i,i+1}$ and $b_2 \in B_{i-1,i}$. Then either $\{b_1, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, b_2\}$ induces a $P_6$ or $\{b_1, b_2, v_i, v_{i+1}\}$ induces a diamond, depending on whether $b_1$ and $b_2$ are adjacent. Therefore, the property holds.

6. The set $B_{i,i+1}$ is anti-complete to $A_i \cup A_{i+1}$.

   By symmetry, it suffices to show that $B_{i,i+1}$ is anti-complete to $A_i$. If $a \in A_i$ and $b \in B_{i,i+1}$ are adjacent, then $\{a, b, v_i, v_{i+1}\}$ induces a diamond.

7. The set $B_{i,i+1}$ is complete to $A_i \cup A_{i+2}$.

   By symmetry, it suffices to show that $B_{i,i+1}$ is complete to $A_{i+2}$. If $a \in A_{i+2}$ and $b \in B_{i,i+1}$ are not adjacent, then $\{a, v_{i+2}, v_{i+3}, v_{i+4}, v_i, b\}$ induces a $P_6$.

8. Each $C_{i,i+2}$ is a stable set.

   If $c_1, c_2 \in C_{i,i+2}$ are adjacent, then $\{c_1, c_2, v_i, v_{i+2}\}$ induces a diamond.

9. Each vertex in $C_{i,i+2}$ is either complete or anti-complete to each component of $A_i$ and $A_{i+2}$.

   If $c \in C_{i,i+2}$ is adjacent to $a_1 \in A_i$ ($A_{i+2}$) but not adjacent to $a_2 \in A_i$ ($A_{i+2}$) with $a_1 a_2 \in E$, then $\{a_1, a_2, c, v_i\}$ ($\{a_1, a_2, c, v_{i+2}\}$) induces a diamond.

10. Each vertex in $C_{i,i+2}$ has at most one neighbour in each component of $A_{i+1}$, $A_{i+3}$ and $A_{i+4}$.

   Suppose that a vertex $c \in C_{i,i+2}$ has two neighbours $a_1$ and $a_2$ in the same component of $A_j$ where $j \neq i$ and $j \neq i+1$. Since each component of $A_j$ is a clique by (1), $a_1 a_2 \in E$. Then $\{c, a_1, a_2, v_j\}$ induces a diamond in $G$.\[\]
(11) Each vertex in $C_{i,i+2}$ is anti-complete to each non-trivial component of $A_{i+1}$.

Suppose that $c \in C_{i,i+2}$ has a neighbour $a_1$ in a non-trivial component of $A_{i+1}$. Let $a_2$ be a vertex in that component other than $a_1$. By (10), we have that $ca_2 \notin E$. Then \{a_2, a_1, c, v_{i+2}, v_{i+3}, v_{i+4}\} induces a $P_6$.

(12) The set $C_{i,i+2}$ is anti-complete to $B_{j,j+1}$ if $j \neq i + 3$. Moreover each vertex in $C_{i,i+2}$ has at most one neighbour in $B_{i+3,i+4}$.

Suppose that $c \in C_{i,i+2}$ is adjacent to some $b \in B_{j,j+1}$ for some $j \neq i + 3$. Since $b$ and $c$ have exactly one common neighbour in $Q$, it follows that \{b, c, v_j, v_{j+1}\} induces a diamond. This proves the first part of the claim. Suppose that $c$ is adjacent to two vertices $b_1, b_2 \in B_{i+3,i+4}$. By (4), $b_1b_2 \in E$. Then \{c, b_1, b_2, v_{i+3}\} induces a diamond.

(13) Each $F_i$ has at most one vertex. Moreover, $F$ is a stable set.

If $F_i$ contains two vertices $f_1$ and $f_2$, then either \{v_i, v_{i+2}, f_1, f_2\} or \{v_{i-2}, v_{i+2}, f_1, f_2\} induces a diamond, depending on whether $f_1f_2 \in E$. This proves the first part of the claim. Let $f_1 \in F_i$ and $f_2 \in F_j$ with $i \neq j$. Note that there exists an index $k$ such that $v_k$ is a common neighbour of $f_1$ and $f_2$ and $v_{k+1}$ is adjacent to exactly one of $f_1$ and $f_2$. If $f_1f_2 \in E$, then \{f_1, f_2, v_k, v_{k+1}\} induces a diamond.

(14) The set $F_i$ is anti-complete to $A_{i+2} \cup A_{i+3}$.

By symmetry, it suffices to show that $F_i$ is anti-complete to $A_{i+2}$. If $f \in F_i$ is adjacent to $a \in A_{i+2}$, then \{a, f, v_{i+2}, v_{i+3}\} induces a diamond.

(15) Each vertex in $F_i$ is either complete or anti-complete to each component of $A_i$.

If $f \in F_i$ is adjacent to $a_1 \in A_i$ but not to $a_2 \in A_i$ with $a_1a_2 \in E$, then \{a_1, a_2, f, v_i\} induces a diamond.

(16) Each vertex in $F_i$ has at most one neighbour in each component of $A_{i+1}$ and $A_{i+2}$.

If $f \in F_i$ has two neighbours $a_1$ and $a_2$ in the same component of $A_{i+1}$ ($A_{i+2}$), then \{f, a_1, a_2, v_{i+1}\} induces a diamond.

(17) The set $F_i$ is anti-complete to $B_{j,j+1}$ if $j \neq i + 2$ and complete to $B_{j,j+1}$ if $j = i + 2$.

If $f \in F_i$ is not adjacent to $b \in B_{i+2,i+3}$, then \{f, b, v_{i+2}, v_{i+3}\} induces a diamond. This proves the second part of the claim. Note that if $f \in F_i$ and $b \in B \setminus B_{i+2,i+3}$ have exactly one common neighbour in $Q$, say $v_k$. If $fb \in E$, then \{f, b, v_k, v_{k+1}\} or \{f, b, v_k, v_{k-1}\} induces a diamond.

(18) The set $F_i$ is anti-complete to $C_{j,j+2}$ if $j \neq i - 1$.

Let $f \in F_i$. Note that if $j \neq i - 1$, then each vertex $c \in C_{j,j+2}$ is adjacent to exactly one of $v_{i+2}$ and $v_{i+3}$. If $fc \in E$, then \{v_{i+2}, v_{i+3}, f, c\} induces a diamond.

(19) If $A_i$ is not stable, then $A_{i+2} = A_{i+3} = B_{i+1,i+2} = B_{i-1,i-2} = \emptyset$.

Suppose that $A_i$ contains an edge $a_1a_2$. If there is a vertex $x$ in $A_{i+2} \cup A_{i+3} \cup B_{i+1,i+2} \cup B_{i-1,i-2}$, then $x$ is adjacent to $a_1$ and $a_2$ by (3) and (7). Then \{x, a_1, a_2, v_i\} induces a diamond.

(20) If $A_i$ is not empty, then each of $B_{i+1,i+2}$ and $B_{i-1,i-2}$ contains at most one vertex.

Let $a \in A_i$. If $B_{i+1,i+2}$ (resp. $B_{i-1,i-2}$) contains two vertices $b_1$ and $b_2$, then \{a, b_1, b_2, v_{i+1}\} (resp. \{a, b_1, b_2, v_{i-1}\}) induces a diamond by (4) and (7).
4 The Optimal χ-Bound

In this section, we derive an optimal χ-bound for $(P_6, \text{diamond})$-free graphs. An atom is a graph without clique cutsets. Two non-adjacent vertices $u$ and $v$ in a graph $G$ are comparable if $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$. The following is the main result of this paper.

**Theorem 3.** Let $G$ be a $(P_6, \text{diamond})$-free graph. Then $\chi(G) \leq \omega(G) + 3$.

**Proof.** We prove this by induction on $|V(G)|$. If $G$ is disconnected, then we are done by applying the inductive hypothesis to each connected component of $G$. If $G$ contains a clique cutset $S$ such that $G - S$ is the disjoint union of two subgraphs $H_1$ and $H_2$, then it follows from the inductive hypothesis that $\chi(G) = \max\{\chi(G[V(H_1) \cup S]), \chi(G[V(H_2) \cup S])\} \leq \omega(G) + 3$. If $G$ contains two non-adjacent vertices $u$ and $v$ such that $N(v) \subseteq N(u)$, then $\chi(G) = \chi(G - v)$ and $\omega(G) = \omega(G - v)$, and we are done by applying the inductive hypothesis to $G - v$. Therefore, we can assume that $G$ is a connected atom with no pair of comparable vertices. If $G$ is perfect, then $\chi(G) = \omega(G)$ by Theorem 1. Otherwise the theorem follows from Theorem 4 below.

**Theorem 4.** Let $G$ be a connected atom with no pair of comparable vertices. If $G$ is $(P_6, \text{diamond})$-free and imperfect, then $\chi(G) \leq \omega(G) + 3$.

The remainder of the section is devoted to the proof of Theorem 4. We begin with a simple lemma that will be useful later. A matching in a graph is a set of edges such that no two edges in the set meet a common vertex.

**Lemma 1.** Let $G$ be a graph that can be partitioned into two cliques $X$ and $Y$ such that the edges between $X$ and $Y$ form a matching. If $\max\{|X|, |Y|\} \leq k$ for some integer $k \geq 2$, then $G$ is $k$-colourable.

**Proof.** Note that either $X$ or $Y$ is a maximum clique of $G$ unless $X$ and $Y$ are singletons in which case the maximum size of a clique of $G$ is at most 2. Moreover, $G$ is perfect by Theorem 1. Since $\max\{|X|, |Y|\} \leq k$ and $2 \leq k$, it follows that $G$ is $k$-colourable.

**Proof of Theorem 4.** Let $G = (V, E)$ be a graph satisfying the assumptions of the theorem. Since $G$ is imperfect, it contains an induced $C_5$, say $Q = \{v_1, v_2, v_3, v_4, v_5\}$ (in order). We partition $V(G) \setminus Q$ as in Section 3. Let $\omega := \omega(G)$. If $\omega \leq 3$, then the theorem follows from a known result that every $(P_6, \text{diamond})$-free graph without a $K_4$ is 6-colourable [15]. Therefore, we can assume that $\omega \geq 4$. The idea is to colour $Q \cup A \cup B$, $C \cup F$, and $Z$ independently using as few colours as possible. However, to obtain the optimal bound, we need to reuse colours in some smart way. In particular, we show that we can reuse one colour from $Q \cup A \cup B$ on some $C_{i+2}$, so that the remaining of $C \cup F$ can be coloured with only 3 colours (Claim 2 and Claim 4). The proof of Claim 4 relies on a computer search combined with a well-known result characterizing 3-colourable $(P_6, K_3)$-free graphs (Theorem 2). See Figure 2 for a diagram illustrating our colouring of $G$ with $\omega + 3$ colours.

We first deal with the components of $Z$.

**Claim 1.** Each component of $Z$ is $(\omega - 1)$-colourable.
Proof. Let $K$ be an arbitrary component of $Z$. Suppose first that $K$ has a neighbour $c$ in $C_{i,i+2}$ for some $i$. If $c$ is adjacent to $k_1 \in K$ but not to $k_2 \in K$ with $k_1 k_2 \in E$, then \{$k_2, k_1, c, v_{i+2}, v_{i+3}, v_{i+4}$\} induces a $P_6$. So, $c$ is complete to $K$. Since $G$ is diamond-free, $K$ is a clique of size at most $\omega - 1$. Thus, $K$ is $(\omega - 1)$-colourable.

Suppose now that $K$ has no neighbour in $C$. It follows from (21) that $K$ is anti-complete to $A \cup B$. Since $G$ is connected, $K$ has a neighbour $f \in F$. Let $L_1 := N(f) \cap K$ and $L_{i+1} := N(L_i) \cap K$ for $i \geq 1$. Since $K$ is connected, $K = \bigcup_{i=1}^{\ell} L_i$. Suppose that $L_i$ contains a vertex $s_i$ for some $i \geq 3$. By definition, there is an induced path $f, s_1, \ldots, s_i$ such that $s_j \in L_j$ for each $1 \leq j \leq i$. Then for some $1 \leq k \leq 5$, we have that $v_{k+1}, v_k, f, s_1, \ldots, s_i$ contains an induced $P_6$, where $v_k f \in E$ and $v_{k+1} f \notin E$. This shows that $K = L_1 \cup L_2$. Since $G$ is diamond-free, each component of $L_1$ is $P_3$-free and thus is a clique. Since $G$ is $P_6$-free, each vertex in $L_1$ is either complete or anti-complete to each component of $L_2$. Moreover, if a component $X$ of $L_2$ has two neighbours in a component of $L_1$, then two such neighbours, a vertex in $X$ and $f$ induce a diamond.

First, we claim that $L_2$ is stable. Suppose not. Let $X$ be a component of $L_2$ with at least two vertices $x$ and $x'$. By definition, $X$ has a neighbour $y$ in some component $Y$ of $L_1$. If $X$ has a neighbour $y' \in L_1$ with $y' \neq y$, then $y'$ is in a component other than $Y$, and so $\{y, y', x, x'\}$ induces a diamond. So, $y$ is the only neighbour of $X$ in $L_1$. Since $G$ has no clique cutset, $X$ has a neighbour $f' \in F$. Note that there is an induced path $f, v_i, v_j, f'$ where $v_i, v_j \in Q$. If $f'$ distinguishes an edge $xx'$ in $X$, then $f, v_i, v_j, f', x', x$ induces a $P_6$. So, $f'$ is complete to $X$. If $f'y \notin E$, then $\{f', y, x, x'\}$ induces a diamond. So, $f'y \in E$. Note that the above argument works for each neighbour of $X$ in $F$. Hence, if there is another neighbour $f'' \in F$ of $X$, then $f'f'' \notin E$ by (13) and so $\{f', f'', x, x'\}$ induces a diamond. This shows that $\{f', y\}$ is a clique cutset that separates $X$ from $G$, a contradiction. This proves that $L_2$ is stable.

Secondly, we claim that if $L_2 \neq \emptyset$, then each component of $L_1$ has size at most 2. Let $x \in L_2$. If $x$ has no neighbour in $F$, then $N(x) \subseteq L_1 \subseteq N(f)$. This contradicts the assumption that $G$ has no pair of comparable vertices. So $x$ has a neighbour $f' \in F$. Note that there is an induced path $f, v_i, v_j, f'$ where $v_i, v_j \in Q$.

Figure 2: A $(\omega + 3)$-colouring of $G$. A solid line means that the edges between the two sets are arbitrary and a dashed line means that the two sets are anti-complete.
We now complete the proof using the above two claims. If \( L_2 = \emptyset \), then \( K = L_1 \) is a clique of size at most \( \omega - 1 \), and so is \((\omega - 1)\)-colourable. If \( L_2 \neq \emptyset \), then \( K \) is 3-colourable by the two claims. Since \( \omega \geq 4 \), it follows that \( K \) is \((\omega - 1)\)-colourable. \( \Box \)

Next we deal with \( Q \cup A \cup B \cup C_{i,i+2} \) for some \( i \).

**Claim 2.** There exists an index \( 1 \leq i \leq 5 \) such that \( Q \cup A \cup B \cup C_{i,i+2} \) can be coloured with \( \omega \) colours such that \( C_{i,i+2} \) is monochromatic.

**Proof.** By (5), we may assume that \( B = B_{2,3} \cup B_{4,5} \). We consider several cases. In each case we give a desired colouring explicitly. In the following, when we say that we colour a set with a certain colour, we mean that we colour each vertex in the set with that colour. For convenience, we always colour \( C_{i,i+2} \) with colour \( \omega \) below.

**Case 1.** \( A_1 \) is not stable.

By (19), we have that \( A_3 = A_4 = B_{2,3} = B_{4,5} = \emptyset \). Moreover, \( A_1 \) is anti-complete to \( A_2 \cup A_5 \) by (2), and \( A_2 \) and \( A_5 \) are complete to each other by (3). By (19), if \( A_2 \) is not stable, then \( A_5 \) is empty. This implies that one of \( A_2 \) and \( A_5 \) is stable. By symmetry, we may assume that \( A_5 \) is stable. We now colour \( Q \cup A \cup B \cup C_{1,3} \) as follows.

- Colour \( Q = v_1, v_2, v_3, v_4, v_5 \) with colours 1, 2, 1, 2, 3 in order.
- Colour \( A_5 \) with colour 2.
- Colour each component of \( A_1 \) with colours in \( \{2, 3, \ldots, \omega\} \) using the smallest colour available.
- Colour each component of \( A_2 \) with colours in \( \{1, 3, \ldots, \omega\} \) using the smallest colour available.
- Colour \( C_{1,3} \) with colour \( \omega \).

We now show that this is a \( \omega \)-colouring of \( Q \cup A \cup B \cup C_{1,3} \). Observe first that each trivial component of \( A_1 \) is coloured with 2 and each trivial component of \( A_2 \) is coloured with 1. By (1), the colouring is proper on \( Q \cup A \cup B \). It remains to show that the colour of \( C_{1,3} \) does not conflict with those colours of \( A \). By (11), no vertex in \( C_{1,3} \) can have a neighbour in a non-trivial component of \( A_2 \). So, \( C_{1,3} \) does not conflict with \( A_2 \). Suppose that there exists a vertex \( c \in A_1 \) with colour \( \omega \) who has a neighbour \( c \in C_{1,3} \). Let \( K \) be the component of \( A_1 \) containing \( a \). Then \( c \) is complete to \( K \) by (9). This implies that \( K \cup \{v_1, c\} \) is a clique and so \( |K| \leq \omega - 2 \). This contradicts that \( a \) is coloured with colour \( \omega \). So, \( C_{1,3} \) does not conflict with \( A_1 \). This proves that the colouring is a proper colouring.

**Case 2.** \( A_1 \) is stable but not empty.

By (19), we have that \( A_3 \) and \( A_4 \) are stable. By (20), each of \( B_{2,3} \) and \( B_{4,5} \) contains at most one vertex. Let \( b_{2,3} \) and \( b_{4,5} \) be the possible vertex in \( B_{2,3} \) and \( B_{4,5} \), respectively. By (19), we have that one of \( A_2 \) and \( A_5 \) is stable. By symmetry, we may assume that \( A_5 \) is stable. If \( A_2 \) is stable, then it is easy to verify that the following is a 3-colouring of \( Q \cup A \cup B \cup C_{1,i+2} \) for any \( i \). We now assume that \( A_2 \) is not stable. By (19), we have that \( A_4 = A_5 = \emptyset \). We can colour \( Q \cup A \cup B \cup C_{1,3} \) as follows.

- Colour \( Q = v_1, v_2, v_3, v_4, v_5 \) with colours 1, 2, 3, 1, 3 in order.
- Colour \( A_1 \) and \( A_3 \) with colours 3 and 1, respectively.
- Colour each component of \( A_2 \) with colours in \( \{1, 3, \ldots, \omega\} \) using the smallest colour available.
Case 3.1. A vertex in B available.

By Lemma 1, there exists a (ω−1, . . . , ω)−colouring of Q ∪ A ∪ B ∪ C_{1,3}.

An argument similar to that in Case 1 shows that this is indeed an ω-colouring of Q ∪ A ∪ B ∪ C_{1,3}.

Case 3. A 1 is empty. We further consider two subcases.

Case 3.1. A 2 is not stable.

By (19), we have that A_4 = A_5 = ∅. By (20), either A_3 is empty or B_{4,5} has at most one vertex.

Suppose first that A_3 not empty. Then B_{4,5} has at most one vertex. Let b_{4,5} be the possible vertex in B_{4,5}. Consider the following colouring of Q ∪ A ∪ B ∪ C_{1,3}.

• Colour Q = v_1, v_2, v_3, v_4, v_5 with colours 1, 2, 3, 1, 3 in order.

• Colour each component of A_2 with colours in {1, 3, . . . , ω} using the smallest colour available.

• Colour each component of A_3 with colours in {1, 2, 4, . . . , ω} using the smallest colour available.

• Colour b_{4,5} with 2 if b_{4,5} exists, and colour vertices in B_{2,3} with colours in {1, 4, . . . , ω}.

• Colour C_{1,3} with colour ω.

By (4), we have that |B_{2,3}| ≤ ω − 2. By (12), we have that C_{1,3} and B_{2,3} are anti-complete. By (11), we have that C_{1,3} is anti-complete to each non-trivial component of A_2. If b_{4,5} exists, then A_3 is stable by (20). It then follows from the definition that each vertex in A_3 is coloured with 1. One can easily verify that the above is a proper ω-colouring of Q ∪ A ∪ B ∪ C_{1,3}. If b_{4,5} does not exist, then an argument similar to that in Case 1 shows that this is indeed an ω-colouring of Q ∪ A ∪ B ∪ C_{1,3}.

Suppose now that A_3 is empty. Since G is diamond-free, the edges between B_{4,5} and B_{2,3} form a matching. For the same reason, the edges between B_{4,5} and each component of A_2 form a matching. Consider the following colouring of Q ∪ A ∪ B ∪ C_{1,3}.

• Colour Q = v_1, v_2, v_3, v_4, v_5 with colours 3, ω, 1, ω, 1 in order.

• For each component K of A_2, pick an arbitrary vertex a_K in the component and colour it with 1. By Lemma 1, there exists a (ω − 2)-colouring of B_{4,5} ∪ (K \ a_K) using colours 2, 3, . . . , ω − 1.

• Colour C_{1,3} with colour ω.

Since B_{2,3} and A_2 are anti-complete, the above colouring (by permuting colours in A_2) gives an ω-colouring of Q ∪ A ∪ B ∪ C_{1,3}.

Case 3.2. A_2 is stable. By symmetry, A_5 is stable.

Suppose first that A_3 is not stable. By (19), we have that A_5 = B_{4,5} = ∅. If A_4 is stable, one can easily verify that the following is an ω-colouring of Q ∪ A ∪ B ∪ C_{2,4}.

• Colour Q = v_1, v_2, v_3, v_4, v_5 with colours 1, 2, ω, 3, 2 in order.

• Colour A_2 and A_4 with 1 and 2, respectively, and colour each component of A_3 with colours in {1, 2, . . . , ω − 1}.

• Colour vertices in B_{2,3} with colours in {1, 3, . . . , ω − 1}.
• Colour $C_{2,4}$ with colour $\omega$.

If $A_4$ is not stable, then $B_{2,3} = \emptyset$ by (19). One can obtain a desired colouring as in Case 1.

Now suppose that $A_3$ is stable. By symmetry, $A_4$ is stable. So, each $A_i$ is stable for $2 \leq i \leq 5$. We first claim that if both $A_2$ and $A_5$ are not empty, then each of $B_{2,3}$ and $B_{4,5}$ contains at most one vertex. Let $a_2 \in A_2$ and $a_5 \in A_5$. By (3), it follows that $a_5a_2 \in E$. If $b \in B_{2,3}$ is not adjacent to $a_5$, then $\{b, v_3, v_4, v_5, a_5, a_2\}$ induces a $P_6$. So, $B_{2,3}$ is complete to $a_5$. Then $B_{2,3}$ contains at most one vertex, for otherwise two vertices in $B_{2,3}$, $a_5$ and $v_2$ induce a diamond. Similarly, $B_{4,5}$ contains at most one vertex. This proves the claim. Now if $A_2$ and $A_5$ are not empty, then since $\omega \geq 4$, the following is an $\omega$-colouring of $Q \cup A \cup B \cup C_{2,4}$: \{v_1, v_3, b_{4,5}\} \cup A_4 \cup A_5, \{v_2, v_4\}, \{v_5, b_{2,3}\} \cup A_2 \cup A_3$, and $C_{2,4}$. So, we can assume by symmetry that $A_2 = \emptyset$. By (20), either $A_3 = \emptyset$ or $B_{4,5}$ has at most one vertex. One can easily verify that the following is an $\omega$-colouring of $Q \cup A \cup B \cup C_{2,4}$:

• Colour $Q = v_1, v_2, v_3, v_4, v_5$ with colours 3, 1, 2, 1, 2 in order.

• Colour $A_4$ and $A_5$ with 2 and 1, respectively, and colour $A_3$ with 3 if $A_3 \neq \emptyset$.

• By Lemma 1, there exists a $(\omega - 2)$-colouring of $B_{4,5} \cup B_{2,3}$ using colours in $\{3, \ldots, \omega\}$. If $B_{4,5}$ contains at most one vertex $b_{4,5}$, we may assume that $b_{4,5}$ is coloured with colour $\omega$.

• Colour $C_{2,4}$ with colour $\omega$.

Finally, we deal with $C \cup F$.

**Claim 3.** The subgraph $C \cup F$ is triangle-free.

**Proof.** Suppose, by contradiction, that $C \cup F$ contains a triangle $T$ with vertices $h_i$ for $i = 1, 2, 3$. Since $F$ is stable by (13), it follows that $T$ contains at least two vertices from $C$. Moreover, vertices in $T \cap C$ are in different $C_{i,i+2}$, since each $C_{i,i+2}$ is stable by (8). If $T$ contains a vertex of $F$, then the other two vertices of $T$ are from $C_{i-1,i+1}$ by (18). But this contradicts the fact that $C_{i-1,i+1}$ is stable. So, all vertices of $T$ are in $C$. If the three vertices of $T$ are from $C_{i,i+2}$, $C_{i+1,i+3}$, and $C_{i+2,i+4}$ for some $i$, then \{h_1, h_2, h_3, v_{i+2}\} induces a diamond in $G$. If the three vertices of $T$ are from $C_{i,i+2}$, $C_{i,i-2}$, and $C_{i+2,i+4}$ for some $i$, then \{h_1, h_2, h_3, v_{i+2}\} induces a diamond in $G$.

**Claim 4.** For each $1 \leq i \leq 5$, the subgraph $(C \cup F) \setminus C_{i,i+2}$ is 3-colourable.

**Proof.** We show via a computer program that $(C \cup F) \setminus C_{i,i+2}$ does not contain the Grötzsch graph as an induced subgraph. Since $C \cup F$ is triangle-free by Claim 3, it follows from Theorem 2 that $(C \cup F) \setminus C_{i,i+2}$ is 3-colourable.

We now explain the algorithm and give the pseudocode. Let $H$ be an induced copy of the Grötzsch graph (see Figure 1). For each vertex $v \in V(H)$, a *label* of $v$ is an element in the set $S = \{1, 2, 3, 4, 5, 13, 14, 24, 25, 35\}$. The meaning of the label of $v$ is to indicate where $v$ comes from. For example, if the label of $v$ is 1, it indicates that $v \in F_1$, and if the label of $v$ is 13, it indicates that $v \in C_{1,3}$. A *labelling* of $H$ is a function $\mathcal{L} : V(H) \to S$. We denote by $H\mathcal{L}$ the copy of $H$ with labelling $\mathcal{L}$. For a labelling $\mathcal{L}$ of $H$, we say that $\mathcal{L}$ is *valid* if the graph obtained by taking the union of $H$ and $Q$, where the edges between $H$ and $Q$ are connected according to $\mathcal{L}$, is $(P_6, \text{diamond})$-free.

We use a simple recursive algorithm that uses certain reduction rules to find all valid labellings of $H$. The algorithm MAIN (see Algorithm 1) takes two parameters $\mathcal{L}$ and $\mathcal{F}$ as inputs, where $\mathcal{L}$ is a function from $V(H)$ to the power set $2^S$ of $S$ and $\mathcal{F}$ is a set to store valid labellings of $H$, and returns a set $\mathcal{F}$ of valid labellings of $H$ where the label of each $v \in V(H)$ is in $\mathcal{L}(v)$. 

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The algorithm recursively checks if a vertex \( v \in V(H) \) can be labelled with label \( \ell \) for each label \( \ell \in L(v) \). Once a label \( \ell \) is assigned to \( v \), the algorithm calls the subroutine \textsc{UpdateLabels} (see Algorithm 2) to update possible labels for other vertices using certain reduction rules (see Rule 1-Rule 3 below). If at some point \( L(v) \) becomes empty for some \( v \in V(H) \), we discard the search, since the current labelling is not valid. If at some point \( L(v) \) becomes a singleton for each \( v \in V(H) \), then \( L \) is a labelling of \( H \). The algorithm then checks whether it is valid. If so, the labelling is added to the list \( \mathcal{F} \) of valid labelling, and is discarded otherwise.

**Rule 1** If \( v \in V(H) \) has label \( i(i+2) \), then the label of each neighbour of \( v \) is in \( \{13, 14, 24, 25, 35\} \setminus \{i(i+2)\} \cup \{i+1\} \).

This follows from (8) and (18).

**Rule 2** If \( v \in V(H) \) has label \( i \), then the only possible label for each neighbour of \( v \) is \( (i-1)(i+1) \).

This follows from (8), (13) and (18).

**Rule 3** If \( v \in V(H) \) has two neighbours \( u_1 \) and \( u_2 \) whose labels are \( i(i+2) \), then the label of \( v \) cannot contain the numbers \( i \) and \( i+2 \) in its label.

Suppose not. Then \( v \) is adjacent to \( v_i \) or \( v_{i+2} \). But now \( \{v, u_1, u_2, v_1\} \) or \( \{v, u_1, u_2, v_{i+2}\} \) induces a diamond.
Algorithm 1: A recursive algorithm $\text{MAIN}(\mathcal{L}, \mathcal{F})$.

**Input:** A function $\mathcal{L}: V(H) \to 2^S$.

**Output:** All valid labellings of $H$ such that the label of each $v \in V(H)$ is in $\mathcal{L}(v)$.

// Base cases
1. if there exists a vertex $v \in V(H)$ such that $\mathcal{L}(v) = \emptyset$ then
   2. return $\mathcal{F}$;
end
3. else if $|\mathcal{L}(v)| = 1$ for each $v \in V(H)$ then
   4. Let $H' = H \cup Q$ where the edges between $H$ and $Q$ are constructed according to $\mathcal{L}$.
   5. if $H'$ is not $(P_6,\text{diamond})$-free then
      6. return $\mathcal{F}$;
   7. end
   8. else
      9. return $\mathcal{F} \cup \{H_L\}$;
   10. end
end

// Recursive call
11. else
12. for each $v \in V(H)$ with $|\mathcal{L}(v)| \geq 2$ do
   13. for each label $\ell \in \mathcal{L}(v)$ do
      14. $\mathcal{L}' := \mathcal{L}$;
      15. $\mathcal{L}'(v) := \ell$;
      16. $\text{UPDATELABELS}(\mathcal{L}')$;
      17. return $\text{MAIN}(\mathcal{L}', \mathcal{F})$;
   18. end
19. end
20. end

Algorithm 2: The subroutine $\text{UPDATELABELS}(\mathcal{L})$.

1. for each $v \in V(H)$ do
   // Rule 1
2. if $\mathcal{L}(v) = i(i + 2)$ then
   3. for each $u \in N_H(v)$ do
      4. $\mathcal{L}(u) := \mathcal{L}(u) \cap ((\{13, 14, 24, 25, 35\} \setminus \{i(i + 2)\}) \cup \{i + 1\})$;
   5. end
   6. end
   // Rule 2
7. if $\mathcal{L}(v) = i$ then
8. for each $u \in N_H(v)$ do
9. $\mathcal{L}(u) := \mathcal{L}(u) \cap \{(i - 1)(i + 1)\}$;
10. end
11. end
   // Rule 3
12. if there exist $u_1, u_2 \in N_H(v)$ such that $\mathcal{L}(u_1) = \mathcal{L}(u_2) = i(i + 2)$ then
13. $\mathcal{L}(v) := \mathcal{L}(v) \cap (S \setminus \{i(i + 2), (i - 2)(i), (i + 2)(i + 4), i, i + 2\})$;
14. end
15. end
By symmetry, we can assume the label of vertex 0 in \( H \) is \( \{1\} \) or \( \{25\} \). To find all valid labellings of \( H \), we make a single call to \( \text{MAIN}(L^*, \emptyset) \) where \( L^* : V(H) \to 2^S \) such that \( L^*(0) = \{1,25\} \) and \( L^*(v) = S \) for each \( v \in V(H) \setminus \{0\} \). The algorithm returns 20 valid labellings of \( H \). Each of these labellings needs a vertex from \( C_{i,i+2} \) for each \( 1 \leq i \leq 5 \). This shows that \( (C \cup F) \setminus C_{i,i+2} \) does not contain the Grötzsch graph. The output of the algorithm is given in the Appendix.

We now give a \((\omega + 3)\)-colouring of \( G \). By Claim 2, there exists an index \( i \) such that \( Q \cup A \cup B \cup C_{i,i+2} \) can be coloured with colours from \( \{1,2,\ldots,\omega\} \), and that \( C_{i,i+2} \) is coloured with colour \( \omega \). By Claim 1, we can colour each component of \( Z \) with colours from \( \{1,2,\ldots,\omega-1\} \). By Claim 4, we can colour \( (C \cup F) \setminus C_{i,i+2} \) with colours \( \omega+1, \omega+2, \omega+3 \). This is a \((\omega+3)\)-colouring of \( G \) (See Figure 2).

5 Conclusion

In this paper, we proved that each \((P_6, \text{diamond})\)-free graph \( G \) satisfies \( \chi(G) \leq \omega(G) + 3 \). This answers an open question in [15] and gives an optimal \( \chi \)-bound for the class. It is not difficult to see that one can turn our proof into a polynomial-time algorithm for colouring a \((P_6, \text{diamond})\)-free graph \( G \) using \( \omega(G) + 3 \) colours. A natural question is whether one can decide the chromatic number of these graphs in polynomial time. To answer this question, it may be useful to consider whether there exists a structure theorem for the class of \((P_6, \text{diamond})\)-free graphs. We point out that Chudnovsky, Seymour, Spirkl and Zhong [7] give such a structure theorem for a subclass, namely \((P_6, K_3)\)-free graphs.

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Appendix

We use an array of size 11 to represent a valid labelling of $H$. The array is indexed by the vertices of $H$ (see Figure 1), which means that the first element of the array is the label for vertex 0 of $H$, and the second element of the array is the label for vertex 1 of $H$, etc. The algorithm returns the following 20 valid labellings of $H$.

| [25], | [13], | [13], | [14], | [13], | [14], | [25], | [35], | [24], | [35], | [24] |
| [25], | [13], | [13], | [14], | [13], | [14], | [35], | [25], | [24], | [35], | [24] |
| [25], | [13], | [14], | [13], | [13], | [14], | [35], | [24], | [25], | [35], | [24] |
| [25], | [13], | [14], | [13], | [13], | [14], | [35], | [24], | [25], | [35], | [24] |
| [25], | [13], | [14], | [13], | [14], | [13], | [35], | [24], | [35], | [24], | [35] |
| [25], | [13], | [14], | [13], | [14], | [13], | [35], | [24], | [35], | [24], | [35] |
| [25], | [13], | [14], | [13], | [13], | [14], | [35], | [25], | [24], | [35], | [24], | [35] |
| [25], | [14], | [13], | [13], | [14], | [13], | [24], | [25], | [35], | [24], | [35], | [24] |
| [25], | [14], | [13], | [13], | [14], | [13], | [24], | [35], | [25], | [24], | [35], | [24] |
| [25], | [14], | [13], | [13], | [14], | [13], | [24], | [35], | [25], | [24], | [35], | [24] |
| [25], | [14], | [13], | [13], | [14], | [13], | [24], | [35], | [25], | [24], | [35], | [24] |
| [25], | [14], | [13], | [13], | [14], | [13], | [24], | [35], | [25], | [24], | [35], | [24] |
