Estimates of Amplitudes of Transient Regimes in Quasi–Controllable Discrete Systems

V. Kozyakin
Institute of Information Transmission Problems,
Russia, Moscow, Ermolovoj st.,19.
e-mail kozyakin@ippi.msk.su

A. Pokrovskii
Mathematics Department,
University of Queensland,
4072 Australia.
e-mail ap@maths.uq.oz.au

Abstract
Families of regimes for discrete control systems are studied possessing a special quasi–controllability property that is similar to the Kalman controllability property. A new approach is proposed to estimate the amplitudes of transient regimes in quasi–controllable systems. Its essence is in obtaining of constructive a priori bounds for degree of overshooting in terms of the quasi–controllability measure. The results are applicable for analysis of transients, classical absolute stability problem and, especially, for stability problem for desynchronized systems.

Key words. Controllability; convergence; mathematical system theory; stability; robustness.

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Introduction
Currently, there are a growing number of cases in which systems are described as operating permanently as if in the transient mode. Examples are flexible manufacturing systems, adaptive control systems with high level of external noises, so called desynchronized systems or asynchronous discrete event systems [1, 12, 13]. In connection...
with this, it is necessary to ensure that the state vector amplitude satisfies reasonable estimates within the whole time interval of the system functioning including the interval of transient regime and an infinite interval when the state vector is “close to equilibrium”. Emphasize, that this necessity often contradicts the usual desire to design a feedback which makes the system as stable as possible. The reason is that the stability property characterizes only the asymptotic behavior of a system and does not take into account system behavior during the transient interval. As a result, a stable system can have large overshooting or “peaks” in the transient process that can result in complete failure of a system. First mentions about systems with peak effects could be found in [3, 4] and [18]. In [11, 19, 20, 25, 30] this effect was studied for some classes of linear systems. As it was noted in [7, 8, 9, 11, 21, 23, 27] when the regulator in feedback links is chosen to guarantee as large degree of stability as possible then, simultaneously, overshooting of the system state during the transient process grows i.e., the peak effects are getting more dangerous. From the geometrical point of view peak effect means that when we are trying to design a feedback which improves the stability of the system we should “spoil” automatically a form of Lebesgue surfaces of respective Lyapunov functions.

The above papers were mainly concerned with continuous time control systems because, in completely controllable and observable discrete system it is possible to chose feedback which turns to zero the specter of the respective closed–loop system. Nevertheless, similar effects occurred when optimizing asymptotic behaviour of badly controllable or observable discrete time systems which arise in some applications, see further references in [28, 29]. Consider as the simplest, if trivial, example the linear system which is described by the relations

\[ x_{n+1} = Ax_n + bx^1_n, \quad n = 0, 1, 2, \ldots \]

(0.1)

Here \( x = (x^1, x^2)^T \) be a vector from \( \mathbb{R}^2 \), \( A \) be a matrix of the form

\[
\begin{pmatrix}
  a & \varepsilon \\
  \varepsilon & a
\end{pmatrix}
\]

with the small \( \varepsilon \) and \( b \in \mathbb{R}^2 \) defines a feedback to be constructed. From the asymptotical point of view the best vector \( b^* \) is \( \left(-2a, -\frac{a^2+\varepsilon^2}{\varepsilon}\right) \) which makes the eigenvalues of a closed system equal to zero. On the other hand, for small \( \varepsilon \) this vector \( b^* \) is the most dangerous at the first time step, because the system (0.1) can be written for this \( b^* \) as \( x_{n+1} = A_*x_n \) where

\[
\begin{pmatrix}
  -a & \varepsilon \\
  \varepsilon & a
\end{pmatrix}
\]

has a big element \( \frac{a^2}{\varepsilon} \) in the left bottom corner. There arises a general question if this kind of the peak effect is connected only with poor controllability or observability of the system? If an answer is positive, then the respective quantitative estimates are of interest. Especially urgent such estimates seems to be when a whole class of systems is examined just as in problems of absolute stability or in desynchronized systems. Another schemes of appearing peak effects in discrete systems see in [6, 10].
In this paper a new approach is developed presenting the means to solve for some classes of systems effectively the problem of estimation the state vector amplitude within the whole time interval. The key concept used is a quasi–controllability property of a system that is similar to the Kalman controllability property. The degree of quasi–controllability can be characterized by a numeric value. The main result of the paper is in proving the following: if a quasi–controllable system is stable then the amplitudes of all its state trajectories starting from the unit ball are bounded by the value reciprocal of the quasi–controllable measure. Due to the fact that the measure of quasi–controllability can be easily computed, this fact becomes an efficient tool for analysis of transients. It is shown also that for quasi–controllable systems the properties of stability or instability are robust with respect to small perturbation of system’s parameters. Some other results in this direction were announced in [14, 15].

1 Quasi–controllable families of matrices

The notion of quasi–controllability of the system will be introduced in this section. Degree of quasi–controllability will be estimated by some nonnegative value, the quasi–controllability measure. The basic property of quasi–controllability measure and some examples will be also discussed in this section.

1.1 Definition and the first properties

Let \( F = \{A_1, A_2, \ldots, A_M\} \) be a finite family of real \( N \times N \) matrices.

Definition 1.1 A family \( F \) is said to be quasi–controllable one if no nonzero proper subspace of \( \mathbb{R}^N \) is invariant for all matrices from \( F \).

Denote by \( F_k (k = 1, 2, \ldots) \) the set of finite products of matrices from \( F \cup \{I\} \) which contain no more that \( k \) factors. Define \( F_k(x), x \in \mathbb{R}^N \), as the set of vectors \( Lx \), with \( L \in F_k \). Denote by \( \text{co}(W) \) and \( \text{span}(W) \) respectively the convex and the linear hulls of the set \( W \subseteq \mathbb{R}^N \). Introduce also the set \( \text{abco}(W) = \text{co}(W \cup -W) \) which is called the absolute convex hull of \( W \). Let \( \| \cdot \| \) be a norm in \( \mathbb{R}^N \); a ball in this norm of the radius \( t \) centered at 0 denote by \( \mathbb{S}(t) \).

Theorem 1.2 Suppose that \( p \geq N - 1 \). Then a family \( F \) is quasi–controllable if and only if \( \text{span}\{F_p(x)\} = \mathbb{R}^N \) for each nonzero \( x \in \mathbb{R}^N \).

Proof. Let the family \( F \) be quasi–controllable and \( x \in \mathbb{R}^N \) be a given nonzero vector. Introduce the sets \( \mathcal{L}_0 = \text{span}\{x\} \) and \( \mathcal{L}_k = \text{span}\{F_k(x)\}, k \geq 1 \). Then

\[
\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \ldots \subseteq \mathcal{L}_p \subseteq \mathbb{R}^N.
\]

Therefore,

\[
1 \leq \dim \mathcal{L}_0 \leq \dim \mathcal{L}_1 \leq \ldots \leq \dim \mathcal{L}_p \leq N.
\]
On the other hand,
\[
A_i \mathcal{L}_j \subseteq \mathcal{L}_{j+1}, \quad A_i \in \mathcal{F}, \ 0 \leq j \leq p - 1.
\] (1.3)

If \( \dim \mathcal{L}_p = N \) then \( \mathcal{L}_p = \text{span}\{\mathcal{F}_p(x)\} = \mathbb{R}^N \). If \( \dim \mathcal{L}_p < N \) then by (1.2) and the condition \( p \geq N - 1 \), the equality \( \dim \mathcal{L}_j = \dim \mathcal{L}_{j+1} \) holds for some \( j \in [0, p - 1] \). The last equality and (1.1) imply \( \mathcal{L}_j = \mathcal{L}_{j+1} \). By the last equality and (1.3) the subspace \( \mathcal{L}_j \) should be invariant with respect to all matrices from \( \mathcal{F} \); due to quasi-controllability of the family \( \mathcal{F} \) this subspace coincides with \( \mathbb{R}^N \). Hence, \( \mathcal{L}_j = \mathcal{L}_{j+1} = \ldots = \mathcal{L}_p = \text{span}\{\mathcal{F}_p(x)\} = \mathbb{R}^N \).

Now suppose that \( \text{span}\{\mathcal{F}_p(x)\} = \mathbb{R}^N \), but the family \( \mathcal{F} \) is not quasi-controllable. Then there exists a nonzero proper subspace \( \mathcal{L} \subset \mathbb{R}^N \) which is invariant with respect to all matrices from \( \mathcal{F} \). In this case the inclusion \( \text{span}\{\mathcal{F}_p(x)\} \subseteq \mathcal{L} \) holds for each \( x \in \mathcal{L} \). Therefore, \( \text{span}\{\mathcal{F}_p(x)\} \neq \mathbb{R}^N \). This contradiction proves the quasi-controllability of the family \( \mathcal{F} \) class. The lemma is proved. \( \Box \)

**Definition 1.3** The value \( \sigma_p(\mathcal{F}) \) defined by
\[
\sigma_p(\mathcal{F}) = \inf_{x \in \mathbb{R}^N, \|x\| = 1} \sup \{ t : S(t) \subseteq \text{absco}[\mathcal{F}_p(x)] \}
\]
is called \( p \)-measure of quasi-controllability of the family \( \mathcal{F} \) (with respect to the norm \( \| \cdot \| \)).

**Theorem 1.4** Suppose that \( p \geq N - 1 \). The family \( \mathcal{F} \) is quasi-controllable if and only if \( \sigma_p(\mathcal{F}) \neq 0 \).

**Proof.** Suppose that \( \sigma_p(\mathcal{F}) \neq 0 \). Then \( S[\|x\|\sigma_p(\mathcal{F})] \subseteq \text{absco}[\mathcal{F}_p(x)] \) holds for each nonzero \( x \in \mathbb{R}^N \) and, further, \( \mathbb{R}^N = \text{span}\{\mathcal{F}_p(x)\} \). Therefore, by Theorem 1.2 the family \( \mathcal{F} \) is quasi-controllable.

Suppose now that the family \( \mathcal{F} \) is quasi-controllable but \( \sigma_p(\mathcal{F}) = 0 \). Then there exist \( x_n \in \mathbb{R}^N, \|x_n\| = 1 \), and \( y_n \in \text{absco}[\mathcal{F}_p(x_n)] \) such that \( y_n \to 0 \) and \( ty_n \not\in \text{absco}[\mathcal{F}_p(x_n)] \) for \( t > 1 \). Without loss of generality we can suppose that the sequences \( \{x_n\} \) and \( \{\frac{y_n}{\|y_n\|}\} \) are convergent: \( x_n \to x, \frac{y_n}{\|y_n\|} \to z \).

By Theorem 1.2 the linear hull of the set \( \{\mathcal{F}_p(x)\} \) coincides with \( \mathbb{R}^N \). Hence, there exist matrices \( L_1, L_2, \ldots, L_N \in \mathcal{F}_p \) such that the vectors \( L_1x_n, L_2x_n, \ldots, L_Nx_n \) are linearly independent. Then the vectors \( L_1x_n, L_2x_n, \ldots, L_Nx_n \) are also independent for all sufficiently large \( n \). It means that for any \( n \) there exist numbers
\[
\theta_1^{(n)}, \theta_2^{(n)}, \ldots, \theta_N^{(n)}, \quad \sum_{i=1}^N \theta_i^{(n)} = 1,
\]

such that the vector
\[
z_n = \sum_{i=1}^N \theta_i^{(n)} L_ix_n
\] (1.4)
is collinear to \( y_n \) i.e., \( z_n = \eta_n y_n \) (\( \eta_n > 0 \)).

By definition \( z_n \in \text{absco}\{L_1x_n, L_2x_n, \ldots, L_Nx_n\} \subseteq \text{absco}\{\mathcal{F}_p(x_n)\} \) and \( ty_n \) does not belong to the set \( \mathcal{F}_p(x_n) \) for \( t > 1 \). Therefore, \( \eta_n \leq 1 \). The last inequality and the condition \( y_n \to 0 \) imply \( z_n \to 0 \). Without loss of generality the sequences \( \{\theta_1^{(n)}\}, \{\theta_2^{(n)}\}, \ldots, \{\theta_N^{(n)}\} \) can be supposed to be convergent to some limits \( \theta_1, \theta_2, \ldots, \theta_N \).

Now, after transition to the limit in (1.4), we get

\[
\sum_{i=1}^{N} \theta_i L_i x = 0, \quad \sum_{i=1}^{N} \theta_i = 1.
\]

This contradicts the linear independence of the vectors \( L_1x, L_2x, \ldots, L_Nx \), and the theorem is proved. \( \blacksquare \)

The following theorem is useful when a family of matrices depends on a parameter.

**Theorem 1.5** Let \( p \geq N - 1 \) and the \( N \times N \) matrices

\[
A_1(\tau), A_2(\tau), \ldots, A_M(\tau)
\]

be continuous at the point 0 with respect to the real parameter \( \tau \). Suppose that the family \( \mathcal{F}(\tau) = \{A_1(\tau), A_2(\tau), \ldots, A_M(\tau)\} \) is quasi–controllable at \( \tau = 0 \). Then the family \( \mathcal{F}(\tau) \) is quasi–controllable for all sufficiently small \( \tau \) and the function \( \sigma_p[\mathcal{F}(\tau)] \) is continuous in \( \tau \) at the point \( \tau = 0 \).

The proof is relegated to the Appendix.

### 1.2 Examples

Let \( A \) be a matrix of the size \( N \) and \( b, c \in \mathbb{R}^N \). Consider the family \( \mathcal{F} = \mathcal{F}(A, b, c) \) which consists of the matrix \( A \) and the matrix \( Q = bc^T \) with elements \( q_{ij} = b_i c_j \), \( i, j = 1, \ldots, N \).

**Proposition 1.6** The family \( \mathcal{F}(A, b, c) \) is quasi–controllable if and only if the pair \( (A, b) \) is completely controllable and the pair \( (A, c) \) is completely observable.

**Proof.** Evidently, the subspace \( E \subseteq \mathbb{R}^N \) is invariant with respect to the matrix \( Q \) if and only if either \( b \in E \) or \( E \subseteq c^0 \) where

\[
c^0 = \left\{ x \in \mathbb{R}^N : \sum_{i=1}^{N} x^i c_i = 0 \right\}.
\]

Farther, the matrix \( A \) has a proper invariant subspace \( E_1 \) which contains the vector \( b \) if and only if

\[
\text{span}(\{b, Ab, \ldots, A^{N-1}b\}) = \mathbb{R}^N,
\]

that is if the pair \( (A, b) \) is completely controllable. At last, the matrix \( A \) has a proper invariant subspace \( E_2 \) which is contained in \( c^0 \) if and only if

\[
\text{span}(\{c, cA, \ldots, cA^{N-1}\}) = \mathbb{R}^N,
\]
that is if the pair \((A, c)\) is completely observable. Therefore the assertion is proved.

The following example is the most important for this paper. Let us consider a \(N \times N\) scalar matrix \(A = (a_{ij})\) of the size \(N\) and introduce the family \(\mathcal{F}_1(A) = \{A_1, A_2, \ldots, A_N\}\) by equalities

\[
A_i = \begin{pmatrix}
1 & 0 & \ldots & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{i1} & a_{i2} & \ldots & a_{ii} & \ldots & a_{iN} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & \ldots & 1
\end{pmatrix}.
\] (1.5)

The families \(\mathcal{F}_1(A)\) play a key role in the theory of desynchronized systems, see \([1, 12]\) and, also, Subsection 2.3.

The matrix \(A\) is said to be irreducible, if by any reordering of the basis elements in \(\mathbb{R}^N\) it cannot be represented in a block triangle form.

\[
A = \begin{pmatrix}
B & C \\
0 & D
\end{pmatrix}.
\]

Irreducibility of the matrix \(A\) means that this matrix has no nonempty proper invariant subspace which is the linear hull of a subset of the basic vectors

\[
e_i = (0, 0, \ldots, 1, \ldots, 0) \quad i = 1, 2, \ldots, N.
\]

Let the norm \(\| \cdot \|\) \(\mathbb{R}^N\) is defined by \(\|x\| = |x_1| + |x_2| + \ldots + |x_N|\). Let

\[
\alpha = \frac{1}{2N} \min\{\|(A - I)x\| : \|x\| = 1\} \quad \beta = \frac{1}{2} \min\{|a_{ij}| : i \neq j, a_{ij} \neq 0\},
\]

**Proposition 1.7** The family \(\mathcal{F}_1(A)\) is quasi–controllable, if and only if 1 is not an eigenvalue of \(A\) and the matrix \(A\) is irreducible. If \(\mathcal{F}_1(A)\) is quasi–controllable then

\[
\sigma_N[\mathcal{F}_1(A)] \geq \alpha\beta^{N-1}.
\]

**Proof.** Let 1 be an eigenvalue of \(A\) with an eigenvector \(x_*\). Then \(x_*\) is an eigenvector with the eigenvalue 1 for each matrix \(A_1, A_2, \ldots, A_N\). Hence, in this case the family \(\mathcal{F}_1(A)\) is not quasi–controllable.

Suppose that the matrix \(A\) is irreducible. Then we can assume without loss of generality that some subspace of the form \(E_p = \text{span}\{e_1, e_2, \ldots, e_p\}\) with \(p < N\) is invariant with respect to the matrix \(A\). Therefore, \(E_p\) should be also invariant with respect to each matrix \(A_1, A_2, \ldots, A_N\). That is, the family \(\mathcal{F}_1(A)\) is not quasi–controllable.

Let us now prove that the family \(\mathcal{F} = \mathcal{F}_1(A)\) is quasi–controllable, providing that 1 is not an eigenvalue of \(A\) and that \(A\) is irreducible. It will suffice to show that, for each nonzero vector \(x \in \mathbb{R}^N\),

\[
\text{span}\{\mathcal{F}_N(x)\} = \mathbb{R}^N.
\] (1.6)
Choose a vector $x \in \mathbb{R}^N, \|x\| = 1$, and consider the vectors $(A_1 - I)x, (A_2 - I)x, \ldots, (A_N - I)x \in \text{span}\{\mathcal{F}_1(x)\}$. By definition

$$(A - I)x = (A_1 - I)x + (A_2 - I)x + \ldots + (A_N - I)x,$$

and 1 is not an eigenvector of the matrix $A$. Therefore, at least one of the vectors $(A_1 - I)x, (A_2 - I)x, \ldots, (A_N - I)x$ is nonzero. Without loss of generality we can assume that $(A_1 - I)x \neq 0$ and $\|(A_1 - I)x\| \geq \frac{1}{N}\|(A - I)x\| \geq 2\alpha$. But

$$(A_i - I)x = \langle \tilde{a}_i, x \rangle e_i, \quad i = 1, 2, \ldots, N,$$  

(1.7)

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$ and the vectors $\tilde{a}_i$ are of the form

$$\tilde{a}_i = (a_{i1}, a_{i2}, \ldots, a_{ii} - 1, \ldots, a_{iN}), \quad i = 1, 2, \ldots, N.$$  

Hence, $\langle \tilde{a}_1, x \rangle e_1 \neq 0$, $\langle \tilde{a}_1, x \rangle e_1 \in \text{span}\{\mathcal{F}_1(x)\}$ and $\|\langle \tilde{a}_1, x \rangle e_1\| \geq 2\alpha$. This implies that

$$e_1 \in \text{span}\{\mathcal{F}_1(x)\}$$  

(1.8)

So the vector

$$\frac{1}{2}\langle \tilde{a}_1, x \rangle e_1 = \frac{1}{2}A_1 x - \frac{1}{2}x$$

belongs to $\text{absco}\{\mathcal{F}_1(x)\}$, and, further, $\text{absco}\{\mathcal{F}_N(x)\}$. Therefore,

$$\alpha e_1 \in \text{absco}\{\mathcal{F}_N(x)\}.$$  

Let $Ae_1 = (v_1, v_2, \ldots, v_N)$. By irreducibility of the matrix $A$, the subspace $\text{span}\{e_1\}$ is noninvariant with respect to $A$. So, at least one of the coordinates $v_1, v_2, \ldots, v_N$ of the vector $Ae_1$, with the index different from 1, is nonzero. Without loss of generality, assume that $v_2 \neq 0$. But the second coordinate of the vector $Ae_1$ coincides with the second coordinate of the vector $A_2e_1$ and, consequently, of the vector $(A_2 - I)e_1$. That is, $(A_2 - I)e_1 \neq 0$ and, by (1.8), $(A_2 - I)e_1 \in \text{span}\{\mathcal{F}_2(x)\}$. Therefore, by (1.7)

$$e_2 \in \text{span}\{\mathcal{F}_2(x)\}$$

and the vector $\frac{1}{2}\tilde{a}_2 e_2 = \frac{1}{2}\langle \tilde{a}_2, e_1 \rangle e_2 = \frac{1}{2}A_2 e_1 - \frac{1}{2}e_1$ belongs to $\text{absco}\{\mathcal{F}_2(e_1)\}$ and, further, belongs to $\text{absco}\{\mathcal{F}_N(e_1)\}$. Hence,

$$\alpha \beta e_2 \in \text{absco}\{\mathcal{F}_N(x)\}.$$  

Similarly, the irreducibility of the matrix $A$ implies the inclusions

$$e_i \in \text{span}\{\mathcal{F}_i(x)\}, \quad \alpha \beta^{i-1} e_i \in \text{absco}\{\mathcal{F}_N(x)\}, \quad i = 1, 2, \ldots, N$$  

(1.9)

for an appropriate reordering of the basic vectors $e_1, e_2, e_3, \ldots, e_N$. The equality (1.6) and the estimate

$$\sigma_N[\mathcal{F}_1(A)] \geq \alpha \beta^{N-1}$$

follow from the relations (1.9). The proof of the assertion is completed. \qed
2 Quasi–controllability and the peak effect

This section contains the main results of the paper. We investigate the influence of quasi–controllability on stability, instability and transient processes of dynamical systems generated by nonautonomous linear difference equations

\[ x(n + 1) = A(n)x(n). \]  

(2.1)

A conceptually simple and effective method to estimate norms of solutions of difference equations uniformly for all \( n = 0, 1, 2, \ldots \) will be described.

2.1 A priori estimate of oversooting measure

**Definition 2.1** Let \( \mathcal{F} \) be a family of \( N \times N \) matrices. The difference equation (2.1) is Lyapunov absolutely stable with respect to the family \( \mathcal{F} \), if there exists \( \mu < \infty \), such that for each sequence \( A(n) \in \mathcal{F} \) any solution \( x(n) \) of the corresponding equation satisfies the estimate

\[ \sup_{n \geq 0} \|x(n)\| \leq \mu \|x(0)\|. \]  

(2.2)

**Definition 2.2** The smallest \( \mu \) for which the estimate (2.2) holds is called the oversooting measure of the equation (2.1) with respect to the family \( \mathcal{F} \), and is denoted by \( \chi(\mathcal{F}) \).

From definition it follows that \( \chi(\mathcal{F}) \) coincides with the smallest \( \mu \), for which the estimate (2.2) holds with respect to all solutions (2.4).

**Theorem 2.3** Let the equation (2.1) be Lyapunov absolutely stable with respect to the quasi–controllable family \( \mathcal{F} \). Then the inequality

\[ \chi(\mathcal{F}) \leq \sigma_p^{-1}(\mathcal{F}) \]  

(2.3)

holds for each \( p \geq N - 1 \).

This assertion is the central result of the paper. The proof is relegated to the next subsection. Now we will discuss some applications of the inequality (2.3). Clearly, the Lyapunov absolute stability of the equation (2.1) is equivalent to the Lyapunov stability of the difference inclusion

\[ x(n + 1) \in F_{\mathcal{F}}x(n). \]  

(2.4)

where \( F_{\mathcal{F}} \) is defined by

\[ F_{\mathcal{F}}(x) = \text{co}\{Ax : A \in \mathcal{F}\}. \]

Inclusions of the form (2.4) embrace the usual systems of the discrete absolute stability theory [16, 17, 22]. On the other hand, the Lyapunov absolute stability follows from the absolute stability of the corresponding system. Consequently, when estimating oversooting measure of control systems, it is possible to combine the classical
methods of absolute stability theory with Theorem 2.3. A definitive example of using 
this approach will be presented in Subsection 2.3. Now let us give only some simple 
corollaries of Theorem 2.3.

Consider a difference equation 
\[ x_{n+1} = Ax_n + bu_n, \quad n = 0, 1, \ldots, \] (2.5)
with \( b \in \mathbb{R}^N \) and the scalars \( u_n \) satisfying for a fixed \( c \in \mathbb{R}^N \) the inequality 
\[ |u_n| \leq \gamma \langle c, x_n \rangle \]
where \( \gamma \) is a real parameter. Such equations are common in control theory [17].

**Corollary 2.4** Let the pair \((A, b)\) be completely controllable and the pair \((A, c)\) be 
completely observable and suppose that \( \max_{|\omega|=1} \|\omega I - A\|^{-1}b \| < 1 \). Then for each 
\( p \geq 1 \) any solution \( x_n, n = 0, 1, \ldots \) of the equation (2.5) satisfies the inequality 
\[ \|x_n\| \leq \sigma_p^{-1}(\mathcal{F}_*) \|x_0\| \] where \( \mathcal{F}_* = \{A - \gamma bc^T, A + \gamma bc^T\} \).

**Proof.** By virtue of the proposition 1.6 the class \( \mathcal{F}_* \) is quasi–controllable. So this 
corollary follows immediately from Theorem 2.3 and the circle criteria of absolute 
stability, [17]. \( \square \)

Consider again the general inclusion (2.4).

**Corollary 2.5** Let the family \( \mathcal{F} \) be quasi–controllable and suppose that each uni-
f ormly bounded solution \( \ldots, x_{-n}, \ldots, x_{-2}, x_{-1}, x_0 \) is the zero solution. Then for each 
\( p \geq 1 \) any solution \( x_n, n = 0, 1, \ldots \) of the inclusion (2.4) satisfies the inequality 
\[ \|x_n\| \leq \sigma_p(\mathcal{F}) \|x_0\| \].

**Proof.** This corollary follows from Theorem 2.3 and from the principle of absence 
of any bounded solution in the absolute stability problem [16]. \( \square \)

### 2.2 Proof of Theorem 2.3

Firstly, let us establish two auxiliary assertions. Let \( \mathcal{R} \) denote the set of all finite 
products of matrices from \( \mathcal{F} \). Define the length \( \ell(R) \) of a matrix \( R \in \mathcal{R} \) as the smallest 
number of factors \( A_1, A_2, \ldots, A_q \in \mathcal{F} \) in the representation \( R = A_1A_2 \ldots A_q \).

**Lemma 2.6** Let the family \( \mathcal{F} \) be quasi–controllable and suppose that the inequalities 
\[ \|Rx_*\| > \mu \frac{1}{\sigma_p(\mathcal{F})} \|x_*\|, \quad \mu > 1 \] (2.6)
hold for some \( x_* \in \mathbb{R}^N \) \( x_* \neq 0 \), \( p \geq N - 1 \), \( R \in \mathcal{R} \). Then for any \( x \in \mathbb{R}^N \), \( x \neq 0 \) 
there exists a matrix \( R_x \in \mathcal{R} \) such that \( \|R_x\| \geq \mu \|x\| \), and \( \ell(R_x) \leq \ell(R) + p \).
Proof. Let us fix an arbitrary \( x \in \mathbb{R}^N, x \neq 0 \). The vector \( \sigma_p(\mathcal{F})x \) belongs to the absolute convex hull of the set \( \mathcal{F}_p\left(\frac{\|x\|}{\|x\|}x\right) \) by the definition of the quasi-controllability measure. Therefore, there exist scalars \( \theta_1, \theta_2, \ldots, \theta_Q \) with

\[
\sum_{i=1}^{Q} |\theta_i| \leq 1, \tag{2.7}
\]

and matrices \( L_1, L_2, \ldots, L_Q \in \mathcal{F}_p \) such that

\[
\sum_{i=1}^{Q} \theta_i \frac{\|x\|}{\|x\|} L_i x = \sigma_p(\mathcal{F})x. \]

Hence,

\[
\sum_{i=1}^{Q} \theta_i R L_i x = \sigma_p(\mathcal{F}) \frac{\|x\|}{\|x\|} R x, \]

and, further, by (2.6),

\[
\sum_{i=1}^{Q} \|\theta_i L_i R x\| \geq \mu \|x\|. \tag{2.9}
\]

But then (see (2.7)) there exists an index \( i, 1 \leq i \leq Q \) such that the matrix \( R x = L_i R \) satisfies \( \|R x\| \geq \mu \|x\| \).

It remains to note that the length \( \ell(R x) \leq \ell(R) + p \), due to the inclusion \( L_i \in \mathcal{F}_p \), and the lemma is proved. \( \square \)

**Definition 2.7** The equation (2.1) is said to be absolutely exponentially unstable with degree \( \lambda > 1 \) in the family \( \mathcal{F} \), if for some \( \kappa > 0 \) and for each vector \( x \in \mathbb{R}^N, x \neq 0 \), there exists a sequence \( A(n) \in \mathcal{F} \), such that the solution \( x(n) \) of the equation (2.1) with the initial condition \( x(0) = x \) satisfies the estimate

\[
\|x(n)\| \geq \kappa \lambda^n \|x(0)\|, \quad n = 0, 1, 2, \ldots. \tag{2.8}
\]

**Lemma 2.8** Let the family \( \mathcal{F} \) be bounded and suppose that the conditions of Lemma 2.6 hold. Then the equation (2.1) is absolutely exponentially unstable in the family \( \mathcal{F} \).

**Proof.** Let us fix an arbitrary vector \( x \in \mathbb{R}^N, x \neq 0 \), and construct an auxiliary sequence of vectors \( \{z(m)\}, m = 0, 1, \ldots \), by relations \( z(0) = x \) and

\[
z(m) = R_{z(m-1)} z(m-1), \quad m = 1, 2, \ldots.
\]

Here \( R_{z(m)} \) are the matrices from Lemma 2.6. Then by Lemma 2.6

\[
\|z(m)\| \geq \mu^m \|z(0)\|, \quad m = 0, 1, 2, \ldots. \tag{2.9}
\]

By definition, matrices \( R_{z(m)}, m = 0, 1, \ldots \), can be represented in the form

\[
R_{z(m)} = A_{m,1}(m), \ldots, A_{m,2}, A_{m,1}, \quad A_{m,j} \in \mathcal{F},
\]

10
where \( l(m) \) is the length of \( R_z(m) \). Denote by \( \{A(n)\}, n = 0, 1, \ldots \), the sequence of matrices

\[
A_{0,1}, A_{0,2}, \ldots, A_{0,l(0)}, A_{1,1}, A_{1,2}, \ldots, A_{1,l(1)}, \ldots, A_{m,1}, A_{m,2}, \ldots, A_{m,l(m)}, \ldots,
\]

and consider the solution \( x(n) \) of the respective equation (2.1), with the initial condition \( x(0) = x \). Then the relations

\[
x(q_m) = z(m), \quad m = 0, 1, \ldots,
\]

hold with \( q_0 = 0 \) and

\[
q_m = \sum_{i=0}^{m-1} l(i), \quad m = 1, 2, \ldots.
\]

Estimates (2.9) imply

\[
\|x(n)\| \geq \mu^m \|x(0)\|, \quad n = q_m, \quad m = 0, 1, \ldots.
\]  \hspace{1cm} (2.10)

Norms of matrices from \( \mathcal{F} \) are uniformly bounded by the conditions of the lemma and also the estimates

\[
q_m - q_{m-1} = l(m - 1) \leq K, \quad m = 1, 2, \ldots,
\]  \hspace{1cm} (2.11)

hold by Lemma 2.6. Therefore, the inequality (2.10), in a slightly weaker form, can be extended on the positive integers \( n \) from the interval \( (q_{m-1}, q_m) \):

\[
\|x(n)\| \geq \nu \mu^m \|x(0)\|, \quad \nu > 0, \quad q_{m-1} < n \leq q_m, \quad m = 0, 1, \ldots.
\]  \hspace{1cm} (2.12)

Inequalities (2.12) for appropriate \( \kappa > 0, \lambda > 1 \) imply the estimate (2.8), taking into account that \( q_m \leq mK, \quad m = 0, 1, \ldots \), by virtue of (2.11). Therefore, the lemma is proved.

Let us return to and finish the proof of Theorem 2.3. Suppose that the theorem is false. Then there exists a sequence of matrices \( \{A(n) \in \mathcal{F}, \ n = 0, 1, \ldots \} \) and a solution \( x(n) \) of the corresponding equation (2.1), such that

\[
\|x(n_0)\| > \sigma_p^{-1}(\mathcal{F}) \|x(0)\|.
\]  \hspace{1cm} (2.13)

holds for some \( n_0 \geq 1, \ p \geq N - 1 \). The inequality (2.13) implies

\[
\|A(n_0 - 1) \ldots A(1)A(0)x(0)\| > \sigma_p^{-1}(\mathcal{F}) \|x(0)\|.
\]

Hence, by Lemma 2.8, the equation (2.1) is absolutely exponentially unstable with respect to the family \( \mathcal{F} \) and yet this equation is not even Lyapunov absolutely stable with respect to this family. This contradiction proves the theorem. \( \Box \)
2.3 Application to desynchronized systems

Recently much attention was paid to the development of methods for the analysis of dynamics of multicomponent systems with asynchronously interacting subsystems (see [12, 13] for further references). As examples we can mention the systems with faults in data transmission channels, multiprocessor computing and telecommunication systems, flexible manufacturing systems and so on. It turned out that under weak and natural assumptions systems of this kind possess strong properties like robustness. In applications the robustness is often treated as reliability of a system with respect to perturbations of various kinds such as drift of parameters, malfunctions or noises in data transmission channels, etc.

Let us introduce basic notions of the desynchronized systems theory. Consider a linear system $S$ consisting of $N$ subsystems $S_1, S_2, \ldots, S_N$ that interact at some discrete instants $\{T^n\}, -\infty < n < \infty$. The interaction times may be chosen according to some deterministic or stochastic law but generally they are not known in advance. Let the state of each subsystem $S_i$ be determined within the interval $[T^n, T^{n+1})$ by a numerical value $x_i(n), -\infty < n < \infty$.

Suppose that at each instant $T^n \in \{T^k : -\infty < k < \infty\}$ only one of the subsystems $S_i, i = i(n) \in \{1, 2, \ldots, N\}$, may change its state and the law of the state updating is linear:

$$x_i(n + 1) = \sum_{j=1}^{N} a_{ij} x_j(n), \quad i = i(n).$$

Consider the matrix $A = (a_{ij})$ and introduce for each $i = 1, 2, \ldots, N$ an auxiliary matrix $A_i$ (i-mixture of the matrix $A$) that is obtained from $A$ by replacing its rows with indexes $i \neq j$ with the corresponding rows of the identity matrix $I$ (see (1.5)). Then the dynamics equation for the system $S$ can be written in the following compact form:

$$x(n + 1) = A_{i(n)} x(n), \quad -\infty < n < \infty. \quad (2.14)$$

The system described above is referred to as the linear desynchronized or asynchronous system.

**Theorem 2.9** Suppose that 1 is not an eigenvalue of $A$ and that the matrix $A$ is irreducible. Suppose that the desynchronized system is Lyapunov absolutely stable. Then

$$\chi(\mathcal{F}) \leq \frac{1}{\alpha \beta^{N-1}} \quad (2.15)$$

3 Robustness of instability

In this short concluding section we will consider another application of the above methods to qualitative analysis of discrete systems.

Consider the difference equation (2.1), where matrices $A(n)$ belong to a family $\mathcal{F}(\tau) = \{A_1(\tau), A_2(\tau), \ldots, A_M(\tau)\}$, which depends on a real parameter $\tau$. 
Theorem 3.1 Let the family $\mathcal{F}(\tau)$ be quasi–controllable and be continuous at $\tau = 0$. Suppose that the equation (2.1) is not Lyapunov absolutely stable with respect to the family $\mathcal{F}(0)$. Then the equation (2.1) is not Lyapunov absolutely stable, and, in fact, is absolutely exponentially unstable, with respect to the family $\mathcal{F}(\tau)$, for all sufficiently small $\tau$.

Proof. Suppose that the equation (2.1) is not Lyapunov absolutely stable with respect to the family $\mathcal{F}(0)$. Then there exist matrices $A(n, \tau) \in \mathcal{F}(\tau)$, $n = 0, 1, \ldots$, such that at $\tau = 0$ the solution of the respective equation (2.1) satisfies for some $n_0 > 0$ the inequality
\[ \|x(n_0)\| > \sigma_{N-1}[\mathcal{F}(0)]\|x(0)\|. \]
Therefore
\[ \|A(n_0 - 1, 0) \ldots A(1, 0)A(0, 0)x(0)\| > \sigma_{N-1}[\mathcal{F}(0)]\|x(0)\|. \] (3.1)
On the other hand, the matrices $\{A(n, \tau)\}$ and, by Theorem 1.5, the functions $\sigma_{N-1}[\mathcal{F}(\tau)]$ are continuous at the point $\tau = 0$. Consequently, (3.1) implies
\[ \|A(n_0 - 1, \tau) \ldots A(1, \tau)A(0, \tau)x(0)\| > \sigma_{N-1}[\mathcal{F}(\tau)]\|x(0)\|. \]
and, by virtue of Lemma 2.8, the equation (2.1) is absolutely exponentially unstable with respect to the class $\mathcal{F}(\tau)$. Hence, the theorem is proved.

In some situations the following corollary from the theorems 1.5, 2.3 and 3.1 is useful.

Corollary 3.2 Let a quasi–controllable family of matrices $\mathcal{F} = \{A_1, A_2, \ldots, A_M\}$ be the limit of families $\mathcal{F}_m = \{A_{1,m}, A_{2,m}, \ldots, A_{M,m}\}$. Suppose that the equation (2.1) is Lyapunov absolutely stable with respect to families $\mathcal{F}_m$, $m = 1, 2, \ldots$. Then this equation is Lyapunov absolutely stable with respect to the family $\mathcal{F}$. More than that, the families $\mathcal{F}_m$ are quasi–controllable and the measures of overshooting $\chi(\mathcal{F}_m)$ are uniformly bounded.

The following two examples show that the previous corollary turns out to be false without the assumption about quasi–controllability of the family $\mathcal{F}$.

Example 3.3 Consider the sequence of families $\mathcal{E}_m = \{E_m\}$, each of which consists of the single matrix
\[ E_m = \begin{pmatrix} 1 - \frac{1}{m} & 1 \\ 0 & 1 - \frac{1}{m} \end{pmatrix}. \]
Then the limit family $\mathcal{E}$ consists of the matrix
\[ E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]
and is not quasi–controllable. Therefore the respective equation (2.1) is not exponentially stable with respect to the family $\mathcal{E}$, notwithstanding this equation is exponentially stable with respect to the families $\mathcal{E}_m$. 

13
Example 3.4 Consider the sequence of the families \( \mathcal{F}_m = \{F_m\} \), each of which consists of the single matrix
\[
F_m = \begin{pmatrix}
1 - \frac{1}{m^2} & \frac{1}{m} \\
0 & 1 - \frac{1}{m^2}
\end{pmatrix}.
\]
The respective limit family \( \mathcal{F} \) includes only the identity matrix \( I \) and, therefore, is not quasi–controllable. Evidently, the equation (2.1) is stable with respect to the family \( \mathcal{F} \), as well as with respect to the families \( \mathcal{F}_m, m = 1, 2, \ldots \). On the other hand, the measures of overshooting \( \chi(\mathcal{F}_m) \) are not uniformly bounded.

Appendix. Proof of Theorem 1.5

Proof. Establish first that, under conditions of the theorem, there exists \( \kappa > 0 \) satisfying
\[
\sigma_p[\mathcal{F}(\tau)] \geq \kappa \quad (3.2)
\]
for all sufficiently small \( \tau \). Suppose the contrary. Then there exist \( \tau_n \to 0, x_n \in \mathbb{R}^N \) \( (\|x_n\| = 1) \) and
\[
y_n \in \text{absco}[\mathcal{F}_p(\tau_n, x_n)]
\]
such that
\[
y_n \to 0, \quad ty_n \not\in \text{absco}[\mathcal{F}_p(\tau_n, x_n)] \quad \text{at} \quad t > 1.
\]
Without loss of generality we can suppose that the sequences \( \{x_n\} \) and \( \{\frac{y_n}{\|y_n\|}\} \) are convergent: \( x_n \to x, \frac{y_n}{\|y_n\|} \to z \).

By Theorem 1.2 the linear hull of the set \( \{\mathcal{F}_p(0, x)\} \) coincides with \( \mathbb{R}^N \). Therefore, there exist matrices \( L_1(0), L_2(0), \ldots, L_N(0) \in \mathcal{F}_p(0) \) such that the vectors \( L_1(0)x, L_2(0)x, \ldots, L_N(0)x \) are linearly independent. Then the vectors \( L_1(\tau_n)x_n, L_2(\tau_n)x_n, \ldots, L_N(\tau_n)x_n \) are also linearly independent for all sufficiently large \( n \). Hence, for any positive integer \( n \) there exist
\[
\theta_1^{(n)}, \theta_2^{(n)}, \ldots, \theta_N^{(n)},
\]
such that
\[
\sum_{i=1}^{N} \theta_i^{(n)} = 1, \quad (3.3)
\]
and the vectors \( y_n \) are collinear to the respective vectors
\[
z_n = \sum_{i=1}^{N} \theta_i^{(n)} L_i(\tau_n)x_n. \quad (3.4)
\]
That is,
\[
z_n = \eta_n y_n, \quad \text{with} \quad \eta_n > 0. \quad (3.5)
\]
By definition, \( z_n \in \text{absco}\{L_1(\tau_n)x_n, L_2(\tau_n)x_n, \ldots, L_N(\tau_n)x_n\} \subseteq \text{absco}\{\mathcal{F}_p(\tau_n, x_n)\} \)
where $ty_n$ does not belong to the set $\mathfrak{F}_p(\tau_n, x_n)$ at $t > 1$; hence (3.5) implies $\eta_n \leq 1$. The last inequality and the condition $y_n \to 0$ imply, in turn,

$$z_n \to 0.$$  

(3.6)

The sequences $\{\theta_1^{(n)}\}, \{\theta_2^{(n)}\}, \ldots, \{\theta_N^{(n)}\}$ we can suppose to be convergent to some limits $\theta_1, \theta_2, \ldots, \theta_N$. As the limit of (3.4) and (3.3) we have:

$$\sum_{i=1}^{N} \theta_i L_i(0)x = 0, \quad \text{and} \quad \sum_{i=1}^{N} \theta_i = 1. \quad (3.7)$$

The relations (3.7) contradict the linear independence of the vectors $L_1(0)x, L_2(0)x$, \ldots, $L_N(0)x$. This contradiction proves the estimate (3.2).

Let us return to the proof of the theorem. Denote

$$\varphi = \liminf_{\tau \to 0} [\mathfrak{F}(\tau)], \quad \psi = \limsup_{\tau \to 0} [\mathfrak{F}(\tau)]$$

and define $\mathfrak{F}_p(\tau) = \{L_1(\tau), L_2(\tau), \ldots, L_Q(\tau)\}$. Let us establish the inequality

$$\psi \leq \sigma_p(\mathfrak{F}(0)). \quad (3.8)$$

Chose arbitrary vectors $x \in S(1), y \in S(\psi)$. There exist a sequence $\tau_n \to 0$, a sequence $y_n \to y$ ($y_n \in S\{\sigma_p(\mathfrak{F}(\tau_n))\}$) and sequences of real values $\theta_1^{(n)}, \theta_2^{(n)}, \ldots, \theta_N^{(n)}$, such that the relations

$$y_n = \sum_{i=1}^{N} \theta_i^{(n)} L_i(\tau_n)x, \quad \sum_{i=1}^{N} \theta_i^{(n)} \leq 1 \quad (3.9)$$

hold. Without loss of generality the sequences $\{\theta_1^{(n)}\}, \{\theta_2^{(n)}\}, \ldots, \{\theta_N^{(n)}\}$ can be considered as convergent:

$$\theta_1^{(n)} \to \theta_1, \quad \theta_2^{(n)} \to \theta_2, \quad \ldots, \quad \theta_N^{(n)} \to \theta_N.$$

Then (3.9) imply:

$$y = \sum_{i=1}^{N} \theta_i L_i(0)x, \quad \sum_{i=1}^{N} \theta_i \leq 1. \quad (3.10)$$

Therefore, each vector $y \in S(\psi)$ can be written in the form (3.10) for any $x \in S(1)$. This proves (3.8).

Let us establish now the inequality

$$\varphi \geq \sigma_p(\mathfrak{F}(0)). \quad (3.11)$$

Because of the inequality $\varphi \leq \psi$, the assertion of the theorem will follow from (3.8) and (3.11).
If \( \sigma_p[\mathcal{F}(0)] = 0 \), we have nothing to prove. Suppose that \( \sigma_p[\mathcal{F}(0)] > 0 \) and choose some \( \gamma > 0 \) satisfying
\[
\sigma_p[\mathcal{F}(0)] - \gamma > 0. \tag{3.12}
\]
Let us fix a vector \( x \in S(1) \) and establish that the condition
\[
\|L_i(\tau) - L_i(0)\| \leq \frac{\gamma \kappa}{\sigma_p[\mathcal{F}(0)] - \gamma} \tag{3.13}
\]
with \( \kappa \) from (3.2) implies
\[
S(\sigma_p[\mathcal{F}(0)] - \gamma) \subseteq \text{absco} \mathcal{F}_p(\tau)x. \tag{3.14}
\]

Let \( y \) be an arbitrary vector from \( S(\sigma_p[\mathcal{F}(0)] - \gamma) \). There exist \( \theta_1, \theta_2, \ldots, \theta_Q \), such that
\[
y = \sum_{i=1}^{Q} \frac{\sigma_p[\mathcal{F}(0)] - \gamma}{\sigma_p[\mathcal{F}(0)]} \theta_i L_i(0)x
\]
and
\[
\sum_{i=1}^{Q} \theta_i \leq 1. \tag{3.15}
\]
Hence,
\[
y = \sum_{i=1}^{Q} \frac{\sigma_p[\mathcal{F}(0)] - \gamma}{\sigma_p[\mathcal{F}(0)]} \theta_i L_i(\tau)x + z \tag{3.16}
\]
where
\[
z = \sum_{i=1}^{Q} \frac{\sigma_p[\mathcal{F}(0)] - \gamma}{\sigma_p[\mathcal{F}(0)]} \theta_i (L_i(0) - L_i(\tau))x.
\]
By (3.13) and (3.15) the vector \( z \) satisfies the estimate
\[
\|z\| \leq \frac{\gamma \kappa}{\sigma_p[\mathcal{F}(0)] - \gamma}.
\]
Farther, by (3.2) there exist \( \eta_1(\tau), \eta_2(\tau), \ldots, \eta_Q(\tau) \) satisfying
\[
z = \sum_{i=1}^{Q} \frac{\sigma_p[\mathcal{F}(0)] - \gamma}{\sigma_p[\mathcal{F}(0)]} \eta_i(\tau)L_i(\tau)x, \quad \sum_{i=1}^{Q} \eta_i(\tau) \leq 1. \tag{3.17}
\]
Define now
\[
\theta_i(\tau) = \frac{\sigma_p[\mathcal{F}(0)] - \gamma}{\sigma_p[\mathcal{F}(0)]} \theta_i + \frac{\gamma}{\sigma_p[\mathcal{F}(0)] - \gamma} \eta_i(\tau). \tag{3.18}
\]
The relations (3.16) and (3.17) imply
\[
y = \sum_{i=1}^{Q} \theta_i(\tau)L_i(\tau)x,
\]
16
where

\[ \sum_{i=1}^{N} \theta_i(\tau) \leq 1 \]

by virtue of (3.15), (3.17), (3.18). We have just proven that the inclusion (3.14) holds for all \( \tau \) satisfying (3.13). Hence, for such \( \tau \)

\[ \sigma_p[\mathcal{F}(\tau)] \geq S(\sigma_p[\mathcal{F}(0)] - \gamma) . \]

Taking the lower limit of the last inequality at \( \tau \to 0 \), we obtain

\[ \theta \geq \sigma_p[\mathcal{F}(0)] - \gamma. \]

This and the arbitrariness of \( \gamma > 0 \) imply (3.12).

The inequalities (3.8) and (3.11) and, consequently, the theorem are proven.

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