SCATTERING FOR THE NON-RADIAL FOCUSING INHOMOGENEOUS NONLINEAR SCHRÖDINGER-CHOQUARD EQUATION

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Abstract. In this paper, we study the long-time behavior of global solutions to the Schrödinger-Choquard equation

\[ i\partial_t u + \Delta u = -(I_\alpha * |\cdot|^b|u|^p)|\cdot|^b|u|^{p-2}u. \]

Inspired by Murphy, who gave a simple proof of scattering for the non-radial inhomogeneous NLS, we prove scattering theory below the ground state for the intercritical case in energy space without radial assumption.

Key Words: Schrödinger-Choquard equation; Scattering theory.

1. Introduction

Considering the initial value problem (IVP), also called the Cauchy problem for the inhomogeneous nonlinear Schrödinger-Choquard equation

\[
\begin{aligned}
i\partial_t u + \Delta u &= -(I_\alpha * |\cdot|^b|u|^p)|\cdot|^b|u|^{p-2}u, \\
u(0, x) &= u_0(x) \in H^1(\mathbb{R}^N)
\end{aligned}
\]  

(1.1)

where \( u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C} \) and \( N \geq 3 \). The inhomogeneous term is \( |\cdot|^b \) for some \( b < 0 \). The Riesz-potential is defined on \( \mathbb{R}^N \) by

\[
I_\alpha := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\alpha)\pi^{\frac{N}{2}}2^{\alpha}} |\cdot|^{N-\alpha} = \frac{\mathcal{K}}{|\cdot|^{N-\alpha}}, \quad 0 < \alpha < N.
\]

And

\[
1 + 2 + \alpha + 2b \frac{N}{N-2} < p < 1 + 2 + \alpha + 2b \frac{N}{N-2}.
\]

We also assume that

\[
\min\{2 + \alpha + 2b, N + b, N + 4b + 2\alpha, 4 + \alpha + 2b - N\} > 0.
\]  

(1.2)

The class of solutions to (1.1) is left invariant by the scaling

\[
u_\lambda(t, x) = \lambda^{\frac{2+2\alpha+\alpha}{2(p-1)}} u(\lambda^2 t, \lambda x),
\]

(1.3)

which is also invariant in \( \dot{H}^{s_c}(\mathbb{R}^N) \) norm with \( s_c = \frac{N}{2} - \frac{2+2b+\alpha}{2(p-1)} \). Thus, we call that the equation (1.1) is \( \dot{H}^{s_c}(\mathbb{R}^N) \) critical. Moreover, equation (1.1) conserves the mass, defined by

\[
M(u) := \int_{\mathbb{R}^N} |u|^2 \, dx,
\]
and the energy, defined as the sum of the kinetic and potential energies:

$$E(u) := \int_{\mathbb{R}^N} \left[ \frac{1}{2} |\nabla u|^2 - \frac{1}{p} (I_\alpha * | \cdot |^b |u|^b |x|^b |u(x)|^p) \right] dx.$$ 

The particular case $b = 0$, Eq. (1.1) becomes the nonlinear generalized Hartree equation. In [5], Feng-Yuan have studied the Cauchy problem for the generalized Hartree equation in the energy subcritical. Miao-Xu-Zhao [10, 11] have studied the well-posedness issues of the Hartree equation which corresponded to the particular case $p = 2$. Saanouni [13] proved the scattering theory for the radial case via concentration-compactness roadmap.

For the Eq. (1.1), Alharbi-Saanouni [1] studied the local theory and finite time blow-up. In [14], Saanouni proved the scattering theory with the radial setting. In this work, we extend Saanouni’s work to non-radial case via a new approach established by Murphy [12].

As we known, the equation (1.1) admits a global but non-scattering solution

$$u(t, x) = e^{it} Q(x),$$

where $Q$ is the ground state, i.e., the solution to elliptic equation

$$-\Delta Q + Q - (I_\alpha * | \cdot |^b |Q|^p) |x|^b |Q|^p Q = 0.$$ 

We refer to Alharbi-Saanouni [1] about the existence of the ground state for the above elliptic equation. The uniqueness is still open.

Now, we state our main result.

**Theorem 1.1.** Let $p \geq 2$ and $b, \alpha$ satisfy (1.2). Suppose that $u_0 \in H^1(\mathbb{R}^N)$ satisfies

$$M(u_0)^{1-s_c} E(u_0)^{s_c} < M(Q)^{1-s_c} E(Q)^{s_c} \quad \text{and} \quad \|u_0\|_{L^2}^{1-s_c} \|\nabla u_0\|_{L^2}^{s_c} < \|Q\|_{L^2}^{1-s_c} \|\nabla Q\|_{L^2}^{s_c}.$$ 

Then, there exists a unique global solution $u$ to (1.1). Moreover, the global solution $u$ scatters in the sense that there exists $u_\pm \in H^1(\mathbb{R}^N)$ such that

$$\lim_{t \to \pm \infty} \|u(t, x) - e^{it\Delta} u_\pm\|_{H^1(\mathbb{R}^N)} = 0. \quad (1.4)$$

Theorem 1.1 was established first in the radial case by Saanouni [14]. For the non-radial case, the non-radial approach by Dodson-Murphy [5] fails since the lack of Galilean invariant. However, we have a key observation that the nonlinearity has a decay factor $|x|^b$ at infinity. Hence, this allow us treat the non-radial setting as the radial case (for more details, see [12]).

This paper is organized as follows: In Section 2, we recall some basic estimates to prove some main tools and establish a new scattering criterion. In Section 3, we prove a Morawetz estimate, which in turn implies ‘energy evacuation’ as $t \to \infty$. Thus, these two ingredients quickly complete Theorem 1.1. Finally, a Morawetz identity is proved in Appendix.

We conclude the introduction by giving some notations which will be used throughout this paper. We always use $X \lesssim Y$ to denote $X \leq CY$ for some constant $C > 0$. Similarly, $X \lesssim_u Y$ indicates there exists a constant $C(u)$ depending on $u$ such that $X \leq C(u)Y$. We also use the notation $\mathcal{O}$: e.g. $A = \mathcal{O}(B)$ indicates $A \leq CB$ for constant $C > 0$. The derivative operator $\nabla$ refers to the spatial variable only. We use $L^r(\mathbb{R}^N)$ to denote the Banach space of functions $f : \mathbb{R}^N \to \mathbb{C}$ whose norm

$$\|f\|_{L^r} := \|f\|_{L^r} = \left( \int_{\mathbb{R}^N} |f(x)|^r dx \right)^{\frac{1}{r}}.$$
is finite, with the usual modifications when \( r = \infty \). For any non-negative integer \( k \), we denote by \( H^{k,r}(\mathbb{R}^N) \) the Sobolev space defined as the closure of smooth compactly supported functions in the norm \( ||f||_{H^{k,r}} = \sum_{|\alpha| \leq k} \| \partial^\alpha_x f \|_r \), and we denote it by \( H^k \) when \( r = 2 \). For a time slab \( I \), we use \( L^q_t(I; L^r_x(\mathbb{R}^N)) \) to denote the space-time norm with the usual modifications when \( q \) or \( r \) is infinite, sometimes we use \( \| f \|_{L^q_t(I; L^r_x)} \) or \( \| f \|_{L^q_t L^r_x(I \times \mathbb{R}^N)} \) for short.

2. Preliminaries

Let us start this section by introducing the notation used throughout the paper. We recall some Strichartz estimates associated to the linear Schrödinger propagator.

We say the pair \((q, r)\) is \( \dot{H}^s\)-admissible, if it satisfies the condition

\[
\frac{2}{q} = \frac{N}{2} - \frac{N}{r} - s, \quad 2 \leq q, r \leq \infty, \quad N \geq 3.
\]

For \( s \in (0,1) \), We define the sets \( \Lambda_s \):

\[
\Lambda_s := \left\{ (q, r) \text{ is } \dot{H}^s\text{-admissible;} \left( \frac{2N}{N-2s} \right)^+ \leq r \leq \left( \frac{2N}{N-2} \right)^- \right\},
\]

and

\[
\Lambda_{-s} := \left\{ (q, r) \text{ is } \dot{H}^{-s}\text{-admissible;} \left( \frac{2N}{N-2s} \right)^+ \leq r \leq \left( \frac{2N}{N-2} \right)^- \right\},
\]

and let \( \Lambda_0 \) denote the \( L^2\)-admissible. Here, \( a^- \) is a fixed number slightly smaller than \( a \) \((a^- = a - \epsilon, \text{ where } \epsilon \text{ is small enough})\). \( a^+ \) can be defined by the same way.

Next, we define the following Strichartz norm

\[
\| u \|_{S(\dot{H}^s, I)} = \sup_{(q, r) \in \Lambda_s} \| u \|_{L^q_t L^r_x(I)}
\]

and dual Strichartz norm

\[
\| u \|_{S'(\dot{H}^{-s}, I)} = \inf_{(q, r) \in \Lambda_{-s}} \| u \|_{L^{q'}_t L^{r'}_x(I)},
\]

where \( q' \) denotes the dual exponent to \( q \), i.e. the solution to \( \frac{1}{q} + \frac{1}{q'} = 1 \). If \( I = \mathbb{R} \), \( I \) is omitted usually.

Now, let us recall some results about Strichartz estimates [3, 7, 8] and Hardy-Littlewood-Sobolev’s inequality [9].

**Lemma 2.1.** Let \( 0 < t \subset \mathbb{R} \), the following statement hold

(i) (linear estimate)

\[
\| e^{it\Delta} f \|_{S(\dot{H}^s)} \leq C \| f \|_{\dot{H}^s};
\]

(ii) (nonlinear estimate I)

\[
\left\| \int_0^t e^{i(t-s)\Delta} g(\cdot, s) ds \right\|_{S(L^2, I)} \leq C \| g \|_{S'(L^2, I)};
\]
(iii) (nonlinear estimate II)
\[ \| \int_0^t e^{i(t-s)\Delta} g(s, s) \|_{S(I^{\theta}, I)} \leq C \| g \|_{S(I^{\theta}, I)}. \]

**Lemma 2.2.** Let \( N \geq 3, 0 < \lambda < N \) and \( 1 < r, s < \infty < \infty \) and \( f \in L^r, g \in L^s \). If \( \frac{1}{r} + \frac{1}{s} + \frac{1}{N} = 2 \), then
\[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^N} \, dx \, dy \leq C(N, s, \lambda) \| f \|_{L^r} \| g \|_{L^s}. \]

Next, we prove some interpolation estimates for nonlinearities, which plays an important role in proving scattering theory.

**Lemma 2.3** (Nonlinear estimate). Let \( N \geq 3 \), \( b, \alpha \) satisfy \([1, 2]\) and \( p \geq 2 \). Then there exists \( \theta \in (0, 2(p-1)) \) sufficiently small such that
\[
\begin{align*}
(i) & \quad \| (I_\alpha \ast \cdot |^b |u|^p) \cdot |^b |u|^{p-2} u \|_{S'_{(H^\infty)}} \leq \| u \|_{L^p}^{\theta} \| u \|_{S'(H^\infty)}^{2p-1-\theta}; \\
(ii) & \quad \| (I_\alpha \ast \cdot |^b |u|^p) \cdot |^b |u|^{p-2} u \|_{S_{(H^\infty)}} \leq \| u \|_{L^p}^{\theta} \| u \|_{S(H^\infty)}^{2p-1-\theta} \| u \|_{S(L^2)}; \\
(iii) & \quad \| \nabla (I_\alpha \ast \cdot |^b |u|^p) \cdot |^b |u|^{p-2} u \|_{S'_{(H^\infty)}} \leq \| u \|_{L^p}^{2(p-1)-\theta} \| u \|_{S(H^\infty)}^{\theta} \| \nabla u \|_{S(L^2)} + \| u \|_{L^{1+\theta}}. 
\end{align*}
\]

**Proof.** In view of the singular factor \(|x|^b\) in the nonlinearity, we frequently divide our analysis in two region. Let \( B_1(0) \) denote the unit ball of radius 1 and center in origin, and \( B_1^\gamma(0) \) be \( \mathbb{R}^N \setminus B_1(0) \).

We introduce the parameters
\[ \hat{r} = \frac{2 + 1 - \theta}{2(p-1)} \]
Choosing \( \hat{q}, \hat{a}, \hat{\alpha} \) and \( \theta \) such that \( (\hat{q}, \hat{r}) \in \Lambda_0, (\hat{a}, \hat{\alpha}) \in \Lambda_{s_c} \) and \( (\hat{a}, \hat{\alpha}) \in \Lambda_{-s_c} \). These exponents obey the relations
\[ \frac{1}{\hat{a}} = \frac{2p - 1 - \theta}{\hat{\alpha}} \text{ and } \frac{1}{\hat{q}} = \frac{2(p-1) - \theta}{\hat{\alpha}} + \frac{1}{\hat{q}}. \]

We first prove (i), using the pair \( L^\hat{t}_L^\hat{r} \). Let \( r_1 \) be chosen later and define \( \mu \) so that
\[ 1 + \frac{\alpha}{N} = \frac{2}{\mu} + \frac{2 - \theta}{\hat{r}} + \frac{1}{r_1}. \]

Then, \( A \in \{ B_1(0), B_1^\gamma(0) \} \), using the Hardy-Littlewood-Paley inequality to estimate
\[
\| (I_\alpha \ast \cdot |^b |u|^p) \cdot |^b |u|^{p-2} u \|_{L^r} \lesssim \| x |^b \|_{L^p}^2 \| u \|_{L^p}^{\theta} \| u \|_{L^{r_{1}}}^{2p-1-\theta}. \]

Using the scaling relation above, we derive
\[ \frac{2N}{\mu} + 2b = \frac{2(p-1) + 2b}{2(p-1)} - \frac{N}{r_1}. \]

Thus, if \( A = B_1(0) \) we choose \( r_1 \) so that \( \theta r_1 = \frac{2N}{2(p-1)} \). On the other hand, if \( A = B_1^\gamma(0) \), we choose \( \theta r_1 = 2 \). In both case, we have Sobolev embedding that \( H^1 \subset L^{b+r_1} \). Thus, we take the \( L^\hat{r} \)-norm, apply Hölder’s inequality, and use the above scaling relation \([1, 2]\) to obtain
\[
\| (I_\alpha \ast \cdot |^b |u|^p) \cdot |^b |u|^{p-2} u \|_{L^{\hat{t}}_L^\hat{r}} \lesssim \| u \|_{L^{\hat{t}}_L^\hat{r}}^{\theta} \| u \|_{L^{\hat{r}}_L^\hat{t}}^{2p-1-\theta}. \]
The estimates of (ii) is similar. In this case, we use the second relation of (2.6) to get
\[
\| (I_\alpha \ast | \cdot |^b |u|^p) \cdot | \cdot |^b |u|^{p-2} u \|_{L^q_t L_x^\nu} \lesssim \| u \|_{L^p_t L^\nu_x}^6 \| u \|_{L^2_t \dot{H}^{1+\theta}^\nu} \| u \|_{L^q_t L^\nu_x}^{2(p-1)-\theta} \| \nabla u \|_{L^r_t L^\xi_x},
\]
Consider now (iii). We choose the exponents \( \bar{q}, \bar{r}, \bar{a} \)
\[
\bar{r} = \frac{4(p-1)N(2p-1-\theta)}{2(p-1)(N+2+4b+2\alpha)-\theta(2+\alpha+2b)},
\]
such that \( (\bar{q}, \bar{r}) \in \Lambda_0, (\bar{a}, \bar{r})_\Lambda \). Since
\[
\| \nabla (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^{p-2} u \|_{L^\nu_x} \lesssim \| (I_\alpha \ast | \cdot |^b-1 |u|^p) |x|^b |u|^{p-1}
\]
\[
+ (I_\alpha \ast | \cdot |^b |u|^{p-1} \nabla u) |x|^b |u|^{p-1}
\]
\[
+ (I_\alpha \ast | \cdot |^b |u|^p) |x|^{b-1} |u|^{p-1}
\]
\[
+ (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^{p-2} |\nabla u|,
\]
by using Hardy-Littlewood-Sobolev’s inequality, we get
\[
\| \nabla (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^{p-2} u \|_{L^\nu_x} \lesssim \| |x|^b \|_{L^{\nu_1}_x(A)} \| |x|^{-1} |u|^p \|_{L^{\nu_2}_x} + \| |u|^{p-1} \nabla u \|_{L^{\nu_3}_x} \| |x|^{-1} |u|^{p-1} \|_{L^{\nu_4}_x}
\]
\[
+ \| |x|^b \|_{L^{\nu_5}_x(A)} \| |u|^p \|_{L^{\nu_6}_x} \| |x|^{-1} |u|^{p-1} \|_{L^{\nu_7}_x} + \| |u|^{p-2} \nabla u \|_{L^{\nu_8}_x},
\]
with the following scaling relation:
\[
1 + \frac{\alpha}{N} = \frac{N-2}{2N} + \frac{2}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.
\]
And, we choose \( l \) such that
\[
\frac{2}{r_1} + \frac{2b}{N} = l \quad \text{and} \quad l := \begin{cases} \frac{\theta(1-s_c)}{N}, & \text{if } A = B, \\ -\frac{\theta s_c}{N}, & \text{if } A = B^c. \end{cases}
\]
Since \( 1 < \frac{2N}{N+2+4b+2\alpha} < N \), if we choose \( \theta \) small enough, we conclude that \( \| |x|^b \|_{L^{\nu_1}_x(A)} < \infty \) and that \( 1 < r_2, r_3 < N \). Hence, by Hardy’s inequality, we have
\[
\| |x|^{-1} f \|_{L^{\nu_2}_x} \leq \| \nabla f \|_{L^{\nu_2}_x}.
\]
Now, by splitting
\[
\frac{1}{r_2} + \frac{1}{r_3} = \left( \theta \left( \frac{1}{2} - \frac{s_c}{N} \right) - l \right) \left( \frac{1}{r_4} \right) + \frac{2(p-1)-\theta}{\bar{r}} + \frac{1}{\bar{r}},
\]
\[
\frac{1}{r_3} = \frac{p-1}{\bar{r}},
\]
we can get \( 2 \leq \theta r_4 \leq 2N/(N-2) \). Thus, using Hölder’s inequality and Sobolev inequality
\[
\| \nabla (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^{p-2} u \|_{L^\nu_x} \lesssim \| u \|_{L^\nu_x \dot{H}^{1+\theta}^\nu} \| u \|_{L^\nu_x L^\nu_x}^{2(p-1)-\theta} \| \nabla u \|_{L^\nu_x L^\nu_x}^{2(p-1)-\theta},
\]
which completes lemma.
\[\Box\]
2.1. **Scattering criterion.** To show Theorem 1.1, we first establish a scattering criterion by following that in [12].

**Lemma 2.4 (Scattering Criterion).** Let $b, \alpha, p$ satisfy the condition of Theorem 1.1. Suppose $u$ is the global solution to (1.1) satisfying

\[ \|u\|_{L_t^\infty H_x^1} \leq E. \]

There exists $\epsilon = \epsilon(E) > 0$ and $R = R(E) > 0$ such that if

\[ \liminf_{t \to \infty} \int_{|x| \leq R} |u(t, x)|^2 dx \leq \epsilon^2, \]

then $u$ scatters forward in time.

**Proof.** By Duhamel’s formula, we have

\[ u = e^{i(t-T_0)\Delta} u(T_0) + i \int_{T_0}^t e^{i(t-s)\Delta} F(u(s)) ds \]

where $F(u) = (I_\alpha \ast |b|^p |u|^p x |u|^{p-2} u$.

By Lemma 2.3 and continuity argument, we need to show

\[ \| e^{i(t-T_0)\Delta} u(T_0) \|_{L_t^\hat{a} L_x^\hat{r}([T_0, \infty))} \ll 1. \]

where the exponent $(\hat{a}, \hat{r})$ is as in Lemma 2.3.

Next, we use the Duhamel formula to write

\[ e^{i(t-T_0)\Delta} u(T_0) = e^{it\Delta} u_0 - iG_1(t) - iG_2(t), \]

where

\[ G_j(t) := \int_{I_j} e^{i(t-s)\Delta} F(u(s)) ds, \quad j = 1, 2, \]

here $I_1 = [0, T_0 - \epsilon^{-\theta}], I_2 = [T_0 - \epsilon^{-\theta}, T_0]$.

Choosing $T_0$ large enough, we have

\[ \|e^{it\Delta} u_0\|_{L_t^\hat{a} L_x^\hat{r}([T_0, \infty))} \ll 1. \]

It remains to show

\[ \|G_j(t)\|_{L_t^\hat{a} L_x^\hat{r}([T_0, \infty))} \ll 1, \quad \text{for } j = 1, 2. \]

**Estimation of $G_1(t)$:** We may use the dispersive estimate, Hardy-Littlewood-Sobolev’s inequality and Hölder’s inequality, we have

\[ \|G_1(t)\|_{L_t^\hat{a} L_x^\hat{r}([T_0, \infty))} \ll \left( \int_{T_0}^\infty \left( \int_{I_1} |t-s|^{-\frac{\hat{a}}{\hat{r}}-1} \|u\|_{H_x^2}^{2p-1} ds \right)^{\frac{q_0}{\hat{r}}} dt \right)^{\frac{1}{q_0}} \]

\[ \ll E \left( \int_{T_0}^\infty |t-T_0 + \epsilon^{-\theta}|^{-2} ds dt \right)^{\frac{1}{q_0}} \]

\[ \ll E \epsilon^{\frac{n}{\hat{r}}}, \]

where let $q_0$ satisfy $(q_0, \hat{r}) \in \Lambda_1$.

On the other hand, we may rewrite $G_1$ as

\[ G_1(t) = e^{i(t-T_0+\epsilon^{-\theta})\Delta} u(T_0 - \epsilon^{-\theta}) - e^{it\Delta} u_0. \]
By \((\hat{q}, \hat{r}) \in \Lambda_0\), then
\[
\frac{1}{\hat{a}} = \frac{1 - s_\epsilon}{\hat{q}} + \frac{s_\epsilon}{q_0}.
\]
Using Strichartz estimates and \((2.11)\), we have
\[
\|G_1(t)\|_{L^p_t L^r_x} \lesssim 1.
\]
Thus, by interpolation, we get
\[
\|G_1(t)\|_{L^t_t \chi^2_{(T_0, \infty)}} \lesssim \epsilon^{\frac{2}{\beta}}.
\]

**Estimation of \(G_2(t)\):** By Strichartz’s estimates, we have
\[
\|G_2(t)\|_{L^t_t \chi^2_{(T_0, \infty)}} \lesssim \|F(u)\|_{L^t_t \chi^2_{(I_2)}} \lesssim \|\chi_R F(u)\|_{L^t_t \chi^2_{(I_2)}} + \|(1 - \chi_R) F(u)\|_{L^t_t \chi^2_{(I_2)}},
\]
where \(\chi_R\) is a smooth cutoff to \(\{x : |x| \leq R\}\). Using Hardy-Littlewood-Sobolev’s inequality and Hölder’s inequality, we get
\[
\|
\chi_R F(u)\|_{L^{r'}} \lesssim \|u\|_{H^{\frac{2(p-1)}{p}}} \|
\chi_R u\|_{L^r} \lesssim \|\chi_R u\|^\beta_{L^2},
\]
where \(\beta\) satisfies \(1/r = \beta/2 + (1 - \beta)/2^*\) with \(2^* = \frac{2N}{N-2}\). On the other hand, we have
\[
\|(1 - \chi_R) F(u)\|_{L^{r'}} \lesssim \|x|^b\|_{L^{\mu_1}(A)} \|x|^b\|_{L^{\mu_2}(\{|x| \geq R\})} \|u\|_{L^{\theta r_1}} \|u\|^{2 - \theta}_{L^p},
\]
where the exponents have the scaling relation
\[
\frac{N}{\mu_1} + \frac{N}{\mu_2} + 2b = \frac{\theta(2 + \alpha + 2b)}{2(p-1)} - \frac{N}{r_1}.
\]
Since \(\frac{\theta(N-2)}{2} < \frac{\theta(2 + \alpha + 2b)}{2(p-1)} < \frac{\theta N}{2}\), there exists \(\mu_2 > \frac{N}{b}\), if \(A = B_1(0)\) we can choose \(\theta r_1 = \frac{2N}{N-2}\), or else \(A = B_1^\epsilon(0)\) we can choose \(\theta r_1 = 2\). In both case, we have \(H^1 \subset L^{\theta r_1}\) and \(|x|^b \in L^{\mu_1}(A)\).

Thus, we have
\[
\|(1 - \chi_R) F(u)\|_{L^{r'}} \lesssim \epsilon^{\frac{2}{\beta}} R^\frac{\theta - b + N}{2} \|u\|^{2 - \theta}_{H^{\frac{2}{p}}}.
\]

Let \(T_0\) be large enough. By the assumption \((2.14)\) and identity \(\partial_t |u|^2 = -2\nabla \cdot Im(\bar{u} \nabla u)\), together with integration by parts and Cauchy-Schwarz, we deduce
\[
\left| \partial_t \int_{I_2} \chi_R |u|^2 ds \right| \lesssim \frac{1}{R}.
\]
Choosing \(R \geq \max\{\epsilon^{-(2 + \theta)}, \epsilon^{\frac{2N}{\theta(N-2)}}\}\), we find
\[
\|\chi_R u\|_{L^t_t \chi^2_{(I_2 \times \R^N)}} \lesssim \epsilon,
\]
and \(\|F(u)\|_{L^{r'}} \lesssim \epsilon^{\beta}\).

Then, let \(\eta = \frac{\hat{a}' \beta}{2}\), we bound
\[
\|G_2(t)\|_{L^t_t \chi^2_{(T_0, \infty)}} \lesssim \|F(u)\|_{L^t_t \chi^2_{(I_2)}} \lesssim |I_2|^{\frac{1}{\hat{a}'}} \epsilon^\beta \lesssim \epsilon^\frac{\beta}{2},
\]
which completes the proof of lemma. □
2.2. Local theory and Variational analysis. The local theory for (1.1) is standard via Strichartz estimates and the fixed point argument. For any \(u_0 \in H^1(\mathbb{R}^N)\), there exists a unique maximal-lifespan solution \(u\). This solution belongs to \(C_t H^1(I_{\max} \times \mathbb{R}^N)\) and conserves the mass and energy. Because the nonlinear term is \(H^1\)-subcritical, we have an \(H^1\) blow-up criterion. In particular, if \(u\) is uniformly bounded in \(H^1\), then it is global. For more details, we refer the reader to [1, 13].

We briefly review some of the variational analysis related to the ground state \(Q\). For more details, see [1, 13].

The ground state \(Q\) optimizes the sharp Gagliardo-Nirenberg inequality:

\[
\int_{\mathbb{R}^N} (I_\alpha * | \cdot |^{b}|u|^p)|x|^{b}|u|^p \ dx \leq C_0 \|u\|^{A}_{L^2} \|\nabla u\|^{B}_{L^2},
\]

where \(B := Np - N - \alpha - 2b = 2(p - 1)s_c + 2, A := 2p - B = 2(p - 1)s_c\). And \(Q\) satisfies

\[
\|\nabla Q\|^2 = \frac{B}{2}\|Q\|^2,
\]

Thus

\[
\frac{2p}{B} \leq C_0 \|u\|^{2(p-1)(1-s_c)}_{L^2} \|\nabla u\|^{2(p-1)s_c}_{L^2}.
\]

In the spirit of Dodson-Murphy [4, 5], we can obtain the following coercivity, which can be founded in [14].

**Lemma 2.5** (Coercivity I). Suppose that \(u_0 \in H^1(\mathbb{R}^N)\) satisfies

\[
M(u_0)^{1-s_c}E(u_0)^{s_c} < (1-\delta)M(Q)^{1-s_c}E(Q)^{s_c}
\]

and \(\|u_0\|^{1-s_c}_{L^2} \|\nabla u_0\|^{s_c}_{L^2} < \|Q\|^{1-s_c}_{L^2} \|\nabla Q\|^{s_c}_{L^2}\). Then, there exists \(\delta' = \delta'(\delta) > 0\) so that

\[
\|u(t)\|^{1-s_c}_{L^2} \|\nabla u(t)\|^{s_c}_{L^2} < (1-\delta')\|Q\|^{1-s_c}_{L^2} \|\nabla Q\|^{s_c}_{L^2}
\]

for all \(t \in I, \) where \(u : I \times \mathbb{R}^N \to \mathbb{C}\) is the maximal-lifespan solution to (1.1). In particular, \(I = \mathbb{R}\) and \(u\) is uniformly bounded in \(H^1\).

**Lemma 2.6** (Coercivity II). Suppose \(\|f\|^{1-s_c}_{L^2} \|\nabla f\|^{s_c}_{L^2} \leq (1-\delta)\|Q\|^{1-s_c}_{L^2} \|\nabla Q\|^{s_c}_{L^2}\). Then there exists \(\delta' > 0\) so that

\[
\int_{\mathbb{R}^N} |\nabla f|^2 \ dx - \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^{b}|u|^p)|x|^{b}|u|^p \ dx \geq \delta' \int_{\mathbb{R}^N} |\nabla f|^2 \ dx.
\]

**Proof.** Using the identity

\[
\int_{\mathbb{R}^N} |\nabla f|^2 \ dx - \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^{b}|u|^p)|x|^{b}|u|^p \ dx = \frac{B}{2} E(f) - \frac{B - 2}{2} \|\nabla f\|_{L^2}^2.
\]

By the sharp Gagliardo-Nirenberg inequality

\[
E(f) \geq \|\nabla f\|_{L^2} \left[1 - \frac{C_0}{p} \|f\|^{A}_{L^2} \|\nabla f\|^{B-2}_{L^2}\right]
\geq \|\nabla f\|_{L^2} \left[1 - \frac{(1-\delta)C_0}{p} \|Q\|^{A}_{L^2} \|\nabla Q\|^{B-2}_{L^2}\right]
\geq \frac{B - 2(1-\delta)^{2(p-1)}}{B} \|\nabla f\|_{L^2}^2.
\]
Thus
\[\int_{\mathbb{R}^N} |\nabla f|^2 \, dx - \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b |u|^p) |x|^b |u|^p \, dx \geq \delta' \|\nabla f\|_{L^2}^2.\]

\[\square\]

### 3. Proof of Theorem 1.1

In this section, we turn to prove Theorem 1.1. Assume that \(u\) is a solution to (1.1) satisfying the hypothesis of Theorem 1.1. It follows from Lemma 2.5 that \(u\) is global, and
\[
\sup_{t \in \mathbb{R}} \|u(t)\|_{L^2_x}^{1-s_c} \|\nabla u(t)\|_{L^2_x}^{1-s_c} < (1 - \delta') \|Q\|_{L^2_x}^{1-s_c} \|\nabla Q\|_{L^2_x}^{1-s_c}.
\]

First, we need a lemma that gives Lemma 2.6 on large balls, so that we can exhibit the necessary coercivity. Let \(\chi(x)\) be a radial smooth function such that
\[
\chi(x) = \begin{cases} 
1, & |x| \leq \frac{1}{2} \\
0, & |x| > 1.
\end{cases}
\]

Set \(\chi_R(x) := \chi\left(\frac{x}{R}\right)\) for \(R > 0\).

**Lemma 3.1** (Coercivity on ball, [14]). There exists \(R = R(\delta, M(u), Q) > 0\) sufficiently large that
\[
\sup_{t \in \mathbb{R}} \|\chi_R u\|_{L^2_x}^{1-s_c} \|\nabla (\chi_R u)\|_{L^2_x}^{1-s_c} \leq (1 - \delta') \|Q\|_{L^2_x}^{1-s_c} \|\nabla Q\|_{L^2_x}^{1-s_c}.
\]

In particular, by Lemma 2.6, there exists \(\delta' > 0\) such that
\[
\int_{\mathbb{R}^N} |\nabla (\chi_R f)|^2 \, dx - \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\cdot|^b |\chi_R u|^p) |x|^b |\chi_R u|^p \, dx \geq \delta' \int_{\mathbb{R}^N} |\nabla (\chi_R f)|^2 \, dx.
\]

Let \(R \gg 1\) to be chosen later. We take \(a(x)\) to be a radial function satisfying
\[
a(x) = \begin{cases} 
|x|^2; & |x| \leq R \\
3R|x|; & |x| > 2R,
\end{cases}
\]
and when \(R < |x| \leq 2R\), there holds
\[
\partial_r a \geq 0, \quad \partial_r r a \geq 0 \quad \text{and} \quad |\partial^\alpha a| \lesssim R |x|^{-|\alpha|+1}.
\]

Here \(\partial_r\) denotes the radial derivative. Under these conditions, the matrix \((a_{jk})\) is non-negative.

It is easy to verify that
\[
\begin{cases} 
a_{jk} = 2\delta_{jk}, & \Delta a = 2N, & \Delta\Delta a = 0, \\
a_{jk} = \frac{2R}{|x|^2} \delta_{jk} - \frac{2R}{|x|^2} |x|^2, & \Delta a = \frac{3N(N-1)R}{|x|^3}, & \Delta\Delta a = -\frac{3(N-1)(N-3)R}{|x|^4}, \quad |x| \leq R, \\
a_{jk} = \frac{2R}{|x|^2} \delta_{jk} - \frac{2R}{|x|^2} |x|^2, & \Delta a = \frac{3N(N-1)R}{|x|^3}, & \Delta\Delta a = -\frac{3(N-1)(N-3)R}{|x|^4}, \quad |x| > 2R.
\end{cases}
\]

**Lemma 3.2** (Morawetz identity). Let \(a : \mathbb{R}^N \to \mathbb{R}\) be a smooth weight. Define
\[
M(t) = 2\text{Im} \int_{\mathbb{R}^N} \bar{u} \nabla u \cdot \nabla u \, dx.
\]
Then
\[
\frac{d}{dt} M_a(t) = \int_{\mathbb{R}^N} (-\Delta \Delta a)|u|^2 \, dx + 4 \int_{\mathbb{R}^N} a_j k \text{Re}(\partial_j \bar{u} \partial_k u) \, dx
\]
\[
- \left( 2 - \frac{4}{p} \right) \int_{\mathbb{R}^N} \Delta a |x|^p |u|^p (I_\alpha * |^p |u|^p) \, dx
\]
\[
+ \frac{4b}{p} \int_{\mathbb{R}^N} \nabla a \cdot \frac{x}{|x|^2} (I_\alpha * |^p |u|^p) |u|^p \, dx
\]
\[
- \frac{2K(N - \alpha)}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\nabla a(x) - \nabla a(y)) \cdot \frac{x - y}{|x - y|^{N - \alpha + 2}} |^p |u|^p |^p |u|^p \, dy \, dx.
\]
where subscripts denote partial derivatives and repeated indices are summed.

**Proposition 3.3** (Morawetz estimates). Let \( T > 0 \) and choosing \( R = R(\delta, M(u_0), Q) \) sufficiently large, then
\[
\frac{1}{T} \int_0^T \int_{|x| < R} |u(t,x)|^\frac{2N}{N - 2} \, dx \, dt \lesssim_{u, \delta} \frac{R}{T} + \frac{1}{R^3}.
\]
In particular, there exists a sequence of times \( t_n, R_n \to \infty \) so that
\[
\lim_{n \to \infty} \int_{|x| \leq R_n} |u(t,x)|^\frac{2N}{N - 2} \, dx = 0.
\]

**Proof.** Note that by Cauchy-Schwartz, the uniform \( H^1 \) bounds for \( u \), and the choice of weight function, we have
\[
\sup_{t \in \mathbb{R}} |M(t)| \lesssim_u R.
\]

We compute
\[
\frac{d}{dt} M(t) = 8 \int_{|x| < R} |\nabla u|^2 - \frac{B}{2p} (I_\alpha * |^b |u|^p)|^b |u|^p \, dx
\]
\[
+ \int_{|x| > 2R} \frac{3R(N - 1)(N - 3)}{|x|^3} |u|^2 + \frac{9R}{|x|} |\nabla u|^2 \, dx
\]
\[
+ \int_{R < |x| < 2R} 4 \text{Re} a_j k \bar{u} \partial_j u_k + \mathcal{O} \left( \frac{R}{|x|^3} |u|^2 \right) \, dx
\]
\[
+ \mathcal{O} \left( \int_{|x| > R} (I_\alpha * |^p |u|^p)|^b |u|^p \, dx \right),
\]
where \( \nabla = \nabla - \frac{1}{|x|^2} x \cdot \nabla \) denotes the angular part of the derivative. In (3.14), the angular derivation term is nonnegative, while the mass term is estimated by \( R^{-2} \). Similarly, in (3.15) the first term is nonnegative and the second term is estimated by \( R^{-2} \). Using Hardy-Littlewood-Sobolev’s inequality, we get
\[
\text{(3.16)} \lesssim_E R^p |||x|^b||_{L^r(A)} ||u||_{L^r}^{2p},
\]
the exponents satisfy the scaling relation,
\[
1 + \frac{\alpha}{N} = \frac{1}{r_1} + \frac{1}{r}.
\]
Since \( p \geq 2 \), for every \( A \in \{ B_1(0), B_1^*(0) \} \), there exists \( r \in [2, \frac{2N}{N - 2}] \) such that \( |x|^b \in L^r(A) \).
Thus, (3.16) can be estimated by \( R^p \).
We will make use the following identity, which can be checked by direct computation:
\[ \int_{\mathbb{R}^N} \chi_R^2 |\nabla u|^2 = \int_{\mathbb{R}^N} \left( |\nabla (\chi_R u)|^2 + \chi_R \Delta (\chi_R) |u|^2 \right) dx. \] (3.17)

In (3.13) we may insert \( \chi_R^2 \), we can write
\[ (3.13) = 8 \left( \int_{\mathbb{R}^N} |\nabla (\chi_R u)|^2 dx - \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha * | \cdot |^b |\chi_R u|^p) |\chi_R u|^p dx \right) \]
\[ + \int_{\mathbb{R}^N} O \left( \frac{|u|^2}{R^2} \right) dx + O \left( \int_{|x| > R} (I_\alpha * | \cdot |^b |u|^p) |u|^p dx \right). \] (3.18)

Continuing from above, we deduce
\[ \frac{1}{T} \int_0^T \int_{|x| < R} \frac{|u(t, x)|^{2N}}{R^N} dx dt \lesssim u, R \frac{R}{T} + \frac{1}{R^{1+b}}. \]

Choosing \( T \) sufficiently large and \( T = R^{1-b} \) implies
\[ \frac{1}{T} \int_0^T \int_{|x| < T} |u(t, x)|^{2N} dx dt < T^{-b}, \]
which suffices to give the desired result. \( \square \)

**Appendix: Morawetz Type Estimate**

In this appendix, we consider the Morawetz estimate for NLS in \( \mathbb{R}^N \). Supposing the function \( u(t, x) \) solves
\[ i \partial_t u + \Delta u = F(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \]

Define Morawetz action
\[ M_a(t) := 2 \text{Im} \int_{\mathbb{R}^N} \nabla a \cdot \nabla u \bar{u} dt. \]

By a simple computation shows

**Lemma 3.4 (Morawetz Identity [15]).** There holds
\[ \frac{d}{dt} M_a(t) = \int_{\mathbb{R}^N} (-\Delta a) |u|^2 dx + 4 \int_{\mathbb{R}^N} a_{jk} \text{Re}(\partial_j \bar{u} \partial_k u) dx + 2 \int_{\mathbb{R}^N} \nabla a(x) \cdot \{F, u\}_p dx. \]

where \( \{f, g\}_p = \text{Re}(\bar{f} \nabla g - \bar{g} \nabla f) \).

Next, we give a direction examples, i.e., \( F(t, x) = \lambda (I_\alpha * | \cdot |^b |u|^p) |x|^b |u|^{p-2} u, \) then
\[ 2 \int_{\mathbb{R}^N} \nabla a(x) \cdot \{F, u\}_p dx = \lambda \left( 2 - \frac{4}{p} \right) \int_{\mathbb{R}^N} \Delta a |u|^p (I_\alpha * | \cdot |^b |u|^p) |x|^b dx \]
\[ - \frac{4 \lambda}{p} \int_{\mathbb{R}^N} \nabla a(x) \cdot \nabla [(I_\alpha * | \cdot |^b |u|^p) |x|^b] |u|^p dx \]
\[ = \lambda \left( 2 - \frac{4}{p} \right) \int_{\mathbb{R}^N} \Delta a |u|^p (I_\alpha * | \cdot |^b |u|^p) |x|^b dx \]
\[ - \frac{4 \lambda}{p} \int_{\mathbb{R}^N} \nabla a \nabla (|x|^b) (I_\alpha * | \cdot |^b |u|^p) |u|^p dx \] (3.20)
\[ - \frac{4 \lambda}{p} \int_{\mathbb{R}^N} \nabla a ((\nabla I_\alpha) * | \cdot |^b |u|^p) |x|^b |u|^p dx. \] (3.21)
The estimate of (3.20): By a simple computation, we have

\[ (3.20) = -\frac{4b\lambda}{p} \int_{\mathbb{R}^N} \nabla a \cdot \frac{x}{|x|^2} (I_\alpha * \cdot |b|^p |u|^p)|u|^p dx. \]

The estimate of (3.21): Using the definition of \( I_\alpha * \cdot |b|^p |u|^p \), we obtain

\[ (3.21) = 4K(N - \alpha) \lambda \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \nabla a(x) \cdot \frac{x - y}{|x - y|^{N - \alpha + 2}} |y|^b |u|^p |x|^b |u|^p dy dx \]

\[ = - \frac{4K(N - \alpha) \lambda}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \nabla a(y) \cdot \frac{x - y}{|x - y|^{N - \alpha + 2}} |y|^b |u|^p |x|^b |u|^p dy dx. \]

Thus, we get

\[ (3.21) = 2K(N - \alpha) \lambda \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\nabla a(x) - \nabla a(y)) \cdot \frac{x - y}{|x - y|^{N - \alpha + 2}} |y|^b |u|^p |x|^b |u|^p dy dx. \]

Hence, we get the Morawetz estimate

\[
\frac{d}{dt} M_a(t) = \int_{\mathbb{R}^N} (-\Delta a) |u|^2 dx + 4 \int_{\mathbb{R}^N} a_{jk} \text{Re}(\partial_j \bar{u} \partial_k u) dx \\
+ \gamma \left( \frac{2 - 4}{p} \right) \int_{\mathbb{R}^N} \Delta a |u|^p (I_\alpha * \cdot |b|^p |u|^p) |x|^b dx \\
- \frac{4b\lambda}{p} \int_{\mathbb{R}^N} \nabla a \cdot \frac{x}{|x|^2} (I_\alpha * \cdot |b|^p |u|^p)|u|^p dx \\
+ \frac{2K(N - \alpha) \lambda}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\nabla a(x) - \nabla a(y)) \cdot \frac{x - y}{|x - y|^{N - \alpha + 2}} |y|^b |u|^p |x|^b |u|^p dy dx.
\]

Next, we choose \( a(x) \) as in (3.12). Let \( D := \{ x \in \mathbb{R}^N \| x \| \leq R \} \times \{ y \in \mathbb{R}^N \| y \| \leq R \} \subset \mathbb{R}^{2N} \) and \( D^c = \mathbb{R}^{2N} - D \). Since \( \nabla a(x) - \nabla a(y) = 2(x - y) \) on \( D \), then

\[
\frac{p}{2K\lambda(N - \alpha)} (3.21) = 2 \int_{D^c} \int_D \frac{|y|^b |u(y)|^p |x|^b |u(x)|^p}{|x - y|^{N - \alpha}} dy dx \\
+ \int_{D^c} \int_{D} (\nabla a(x) - \nabla a(y)) \cdot \frac{x - y}{|x - y|^{N - \alpha + 2}} |y|^b |u(y)|^p |x|^b |u(x)|^p dy dx.
\]

For the last integral, according to the definition of \( a(x) \), we have \( |\partial^\beta a(x)| \lesssim R^{2 - |\beta|} (|\beta| \geq 1) \) when \( |x - y| < R \). At the same time, we can obtain

\[
\left| (\nabla a(x) - \nabla a(y)) \cdot \frac{x - y}{|x - y|^2} \right| \lesssim \| D^2 a(x) \|_{L^\infty}.
\]

Else, when \( |x - y| \geq R \), then

\[
\left| (\nabla a(x) - \nabla a(y)) \cdot \frac{x - y}{|x - y|^2} \right| \lesssim \frac{\| D^2 a(x) \|_{L^\infty}}{R}.
\]

Moreover, by symmetry

\[
\left| \int_{D^c} (\nabla a(x) - \nabla a(y)) \cdot \frac{x - y}{|x - y|^{N - \alpha + 2}} |y|^b |u(y)|^p |x|^b |u(x)|^p dy dx \right| \leq \frac{2}{K} \int_{|x| > R} (I_\alpha * \cdot |b|^p |u|^p) |x|^b |u(x)|^p dx.
\]
Thus, We have the Morawetz estimate,
\[
\frac{d}{dt}M_a(t) = \int_{\mathbb{R}^N} (-\Delta \Delta a)|u|^2\,dx + 4\text{Re} \int_{\mathbb{R}^N} a_{jk} \partial_j \bar{u} \partial_k u\,dx \\
+ \lambda \left(2 - \frac{4}{p}\right) \int_{\mathbb{R}^N} \Delta a|x|^b|u|^p(I_\alpha * |\cdot|^b|u|^p)\,dx \\
+ \frac{4K\lambda(N - \alpha)}{p} \int \int_D \frac{|y|^b|u(y)|^p|x|^b|u(x)|^p}{|x - y|^{N - \alpha}}\,dy\,dx \\
+ O\left(\int_{|x| > R} (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^p\,dx\right).
\]

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