Asymptotic Autonomy of Attractors for Stochastic Fractional Nonclassical Diffusion Equations Driven by a Wong–Zakai Approximation Process on $\mathbb{R}^n$

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Abstract: In this paper, we consider the backward asymptotically autonomous dynamical behavior for fractional non-autonomous nonclassical diffusion equations driven by a Wong–Zakai approximations process in $H^s(\mathbb{R}^n)$ with $s \in (0, 1)$. We first prove the existence and backward time-dependent uniform compactness of tempered pullback random attractors when the growth rate of nonlinearities have a subcritical range. We then show that, under the Wong–Zakai approximations process, the components of the random attractors of a non-autonomous dynamical system in time can converge to those of the random attractor of the limiting autonomous dynamical system in $H^s(\mathbb{R}^n)$.

Keywords: fractional nonclassical diffusion equation; Wong–Zakai approximations; random attractor; asymptotically autonomous

1. Introduction

In this paper, we study the asymptotically autonomous dynamics of the following fractional nonclassical diffusion equation driven by white noise on $\mathbb{R}^n$:

$$u_t + (-\Delta)\delta u_t + (-\Delta)^s u + \lambda u = f(t, x, u) + g(t, x) + h(t, x, u) \circ \frac{dW}{dt}, \quad t > \tau,$$

where $\lambda$ is positive constant, $s \in (0, 1)$, $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, $f$ and $h$ are nonlinear functions that satisfy certain dissipative conditions. $W = W(t, \omega)$ is a one-dimensional two-sided Brownian motion over a Wiener Probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{ \omega \in C(\mathbb{R}, R) : \omega(0) = 0 \}$ equipped with the compact-open topology, $\mathcal{F} = \mathcal{B}(\Omega)$ is the Borel $\sigma$-algebra of $\Omega$, $\mathbb{P}$ is the Wiener measure. The classical transformation $\{ \theta_t \}_{t \in \mathbb{R}}$ on $\Omega$ is given by $\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for all $\omega, t \in \Omega \times \mathbb{R}$, and thus $(\Omega, \mathcal{F}, \mathbb{P}, \{ \theta_t \}_{t \in \mathbb{R}})$ forms a metric dynamical system, see [1]. For each $0 \neq \delta \in \mathbb{R}$, we define a random variable

$$G_\delta : \Omega \rightarrow R \quad G_\delta(\omega) = \frac{1}{\delta} \omega(\delta).$$

Then the process $G_\delta(\theta_t \omega)$ is called a stationary stochastic process with a normal distribution, which satisfies

$$G_\delta(\theta_t \omega) = \frac{1}{\delta} (\omega(t + \delta) - \omega(t)), \quad \int_0^t G_\delta(\theta_s \omega) \, ds = \int_t^{t+\delta} \frac{\omega(s)}{\delta} \, ds + \int_0^\delta \frac{\omega(s)}{\delta} \, ds,$$

$$s \in [0, \infty).$$

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see Refs. [2,3]. As a consequence, we study the following Wong–Zakai approximations of Equation (1) driven by a multiplicative noise of $G_{\mathbb{C}}(\theta_{t}\omega)$:

$$
\begin{align*}
&u_{t} + (-\Delta)^{s}u_{t} + (-\Delta)^{s}u + \lambda u = f(t, x, u) + g(t, x, u) + h(t, x, u)G_{\mathbb{C}}(\theta_{t}\omega), \\
&u(\tau, x) = u_{\tau}(x), \quad x \in \mathbb{R}^{n}, \quad \tau > 0, \quad \tau \in \mathbb{R}.
\end{align*}
$$

(3)

where $\lambda > 0$, $s \in (0, 1)$, $g \in L^{2}_{\text{loc}}(\mathbb{R}, L^{2}(\mathbb{R}^{n}))$. The nonlinear functions $f, h : \mathbb{R} \times \mathbb{R}^{n} \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions: for all $(t, x, u) \in \mathbb{R} \times \mathbb{R}^{n} \times \mathbb{R}$,

$$
\begin{align*}
&f(t, x, u) \leq -\alpha_{1}|u|^{p} + \psi_{1}(t, x), \\
&|f(t, x, u)| \leq \alpha_{2}|u|^{p-1} + \psi_{2}(t, x), \\
&\frac{\partial}{\partial u}f(t, x, u) \leq \psi_{3}(t, x), \\
|h(t, x, u)| \leq \beta_{1}(t, x)|u|^{q-1} + \beta_{2}(t, x), \\
&|\frac{\partial}{\partial u}h(t, x, u)| \leq \beta_{3}(t, x),
\end{align*}
$$

(4) (5) (6) (7) (8)

where $\alpha_{1}, \alpha_{2} > 0$ are constants, and $\psi_{1} \in L^{1}_{\text{loc}}(\mathbb{R}, L^{1}(\mathbb{R}^{n}))$, $\psi_{2} \in L^{p}_{\text{loc}}(\mathbb{R}, L^{p}(\mathbb{R}^{n}))$, $\psi_{3} \in L^{\infty}(\mathbb{R}, L^{p}(\mathbb{R}^{n}))$, $\beta_{1} \in L^{\infty}(\mathbb{R}, L^{\frac{2p}{p-2}}(\mathbb{R}^{n}) \cap L^{\frac{q}{q-2}}(\mathbb{R}^{n}) \cap L^{\frac{2q}{q-2}}(\mathbb{R}^{n}))$, $\beta_{2} \in L^{\infty}(\mathbb{R}, L^{2}(\mathbb{R}^{n}))$, $\beta_{3} \in L^{\infty}(\mathbb{R}, L^{\infty}(\mathbb{R}^{n}))$ and

$$
2 \leq 2q < p < \infty \text{ if } n = 1 \text{ and } s \in \left[\frac{1}{2}, 1\right]; \text{ otherwise, } 2 \leq 2q < p < \frac{2q}{n-2}.
$$

(9)

where number $\frac{2q}{n-2}$ is called the fractional critical Sobolev embedding exponent. $\psi \in L^{p}_{\text{loc}}(\mathbb{R}, X)$ means that there exists a finite interval $I$ such that $\int_{I} \|\psi(t)\|_{X}^{p} dt < \infty$.

When $s = 1$, the fractional operator $(-\Delta)^{s}$ becomes the standard Laplacian $-\Delta$. In this special case, the pullback attractor both in the deterministic and stochastic case for the nonclassical diffusion equation has been studied in Refs. [4–10]. For the fractional stochastic equations driven by colored noise with $s \in (0, 1)$, the existence and the upper semicontinuity when $\delta \to 0$ of random attractors were also studied by Refs. [11–13]. As far as we know, there is no result available in the literature on the time-dependent uniform compactness and asymptotically autonomous upper semicontinuity of random attractors for the Wong–Zakai approximations of fractional stochastic nonclassical diffusion equation driven by a multiplicative noise of $G_{\mathbb{C}}(\theta_{t}\omega)$.

Wong and Zakai first proposed the Wong–Zakai approximations for employing deterministic differential equations to approximate stochastic differential equations [14,15]. Their approach was then extended to multidimensional stochastic differential equations [16–18] and stochastic differential equations driven by martingales and semimartingales [19,20], etc. Wong–Zakai approximations and random attractors for stochastic differential equations and lattice systems have lately been investigated in Refs. [2,3,21–26] and [27–30], respectively.

A random pullback attractor is a bi-parametric random set in the form $A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$. The time-dependence character reflects the non-autonomous feature of the system, which should be the most significant characteristic distinguished from autonomous cases, so both the existence and some time-dependent properties of the pullback random attractor $A(\tau, \omega)$ for the cocycle generated by system (3) in $H^{1}(\mathbb{R}^{n})$ with $s \in (0, 1)$ are concerned. We are more specifically interested in the backward compactness of random attractor $A(\tau, \omega)$, i.e., the compactness of $\cup_{\varsigma \leq \tau} A(\varsigma, \omega)$ for each $\tau \in \mathbb{R}$. It is worth noting that the absorbing set is a union of some random sets over an uncountable index set $(-\infty, \tau]$, hence the measurability of attractor $A$ is uncertain. To solve this difficulty, we shall prove a crucial fact that the attractor does not vary between the two attracted universes, see Theorem 1. The time-dependence of a pullback attractor, such as the backward compactness of attractor, has recently been studied in the literature, see Refs. [31–35].
In order to establish the existence result of a backward compact random attractor in $H^s(\mathbb{R}^n)$, we shall prove the backward asymptotic compactness of solutions, which indicates that the usual asymptotic compactness is uniform in the past. The main difficulties for proving such compactness comes from the fact that the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow H^r(\mathbb{R}^n)$ with $r > s$ is not compact, and the Wiener process is nowhere differentiable with respect to time. We cannot utilize the technique of differentiating the equation with respect to $t$. To get around these problems, we will use the methods of a cut-off technique [36] and flattening properties [37] of solutions to establish the desired backward asymptotic compactness in $H^s(\mathbb{R}^n)$.

Finally, we will show that the random attractor $\mathcal{A}(\tau, \omega)$ is backward asymptotically autonomous to random attractor $\mathcal{A}_\infty(\omega)$ in the sense of Hausdorff semi-distance. More precisely, for $P$-a.s. $\omega \in \Omega$,

$$\lim_{\tau \to -\infty} \text{dist}_{H^s(\mathbb{R}^n)}(\mathcal{A}(\tau, \omega), \mathcal{A}_\infty(\omega)) = 0. \tag{10}$$

where $\mathcal{A}_\infty(\omega)$ is a random attractor of the following limiting equation:

$$\begin{cases}
    \dot{u}_t + (-\Delta)^s \dot{u}_t + (-\Delta)^s \dot{u} + \lambda \ddot{u} = \tilde{f}(x, \ddot{u}) + \tilde{g}(x) + \tilde{h}(x, \dot{u}) \ddot{G}_\delta(\theta_t \omega), \\
    \dot{u}(0, \dot{u}_0) = u_0, \ x \in \mathbb{R}^n, \ t > 0,
\end{cases} \tag{11}$$

where $\tilde{g} \in L^2(\mathbb{R}^n)$ and $\tilde{f}, \tilde{h}$ will be specified later, see Theorem 2.

Since system (3) contains time-dependent nonlinear terms whereas system (11) does not, it is more complicated to determine the limit process (10), see Lemma 7. As a result, we will make certain assumptions about nonlinear terms and forcing terms, as shown in (83)–(85), which differs from the earlier works on an asymptotically autonomous problem in Refs. [31,34,35,38,39].

The outline of the paper is as follows. In the next section, we give some necessary assumptions and define a non-autonomous co-cycle for system (3). In Section 3, we derive backward uniform estimates on the solutions in $H^s(\mathbb{R}^n)$ including the backward uniform estimates on the tails and the bounded domains. In Section 4, we prove the existence, uniqueness, and backward compactness of random attractors for problem (3) in $H^s(\mathbb{R}^n)$. In the Section 5, we establish the asymptotic upper semicontinuity of these attractors in $H^s(\mathbb{R}^n)$ as $\tau \to -\infty$. Finally, we illustrate with one example the results of the paper for $n = 2$ briefly.

2. Non-Autonomous Co-Cycles for Fractional Equations

In this section, we first briefly review the concepts of fractional derivatives and fractional Sobolev spaces. Let $(-\Delta)^s$ with $s \in (0, 1)$ be the non-local, fractional Laplace operator defined by

$$(-\Delta)^s u(x) = C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}}dy, \ x \in \mathbb{R}^n,$$

where P.V. means the principal value of the integral and $C(n, s)$ is a constant given by

$$C(n, s) = \frac{s4^n \Gamma(\frac{n+2s}{2})}{\pi^n \Gamma(1-s)} > 0.$$

Let $H^s(\mathbb{R}^n)$ with $s \in (0, 1)$ be the fractional Sobolev space given by

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}dy < \infty \right\},$$
which is a Hilbert space equipped with the inner product and the norm:

\[
(u,v)_{H^p(\mathbb{R}^n)} = (u,v)_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy,
\]

\[
\|u\|_{H^p(\mathbb{R}^n)} = \left( \|u\|_{L^2(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{\frac{1}{2}}, \quad u, v \in H^p(\mathbb{R}^n).
\]

By Ref. [40], we also have

\[
H^p(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^n) \right\},
\]

and thus

\[
\|u\|_{H^p(\mathbb{R}^n)}^2 = \left\| u \right\|_{L^2(\mathbb{R}^n)}^2 + \frac{2}{C(n,s)} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}^2, \quad u \in H^p(\mathbb{R}^n).
\]

The reader is referred to Ref. [40] for more details on fractional operators.

It follows from Refs. [40,41] and (9) that

\[
\|u\|_{L^r(\mathbb{R}^n)} \leq c \|u\|_{H^p(\mathbb{R}^n)}, \quad \forall \ 2 \leq r \leq p \ \text{and} \ 0 < s < \min(1, \frac{n}{2}), \quad (12)
\]

\[
\|u\|_{L^2(\mathbb{R}^n)} \leq c \|u\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}, \quad \forall \ 0 < s < 1 \ \text{and} \ n \geq 1, \quad (13)
\]

\[
\|u\|_{L^{2p}(\mathbb{R}^n)} \leq c \|u\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}, \quad \forall \ 0 < s < 1, n \geq 1. \quad (14)
\]

where \(p\) is the number in (4) and \(\mu = \frac{np-2n}{2p} \in (0,1)\).

Under conditions (4)–(8), for each \(\tau \in \mathbb{R}, \omega \in \Omega\), then as in [42], the system (3) has a unique solution \(u(\cdot, \tau, \omega, u_\tau) \in C([\tau, \infty), H^p(\mathbb{R}^n))\), where \(u_\tau \in H^p(\mathbb{R}^n)\). This allow us to define a continuous cocycle (see [43]) \(\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n)\) given by

\[
\Phi(t, \tau, \omega, u_\tau) = u(t+\tau, \tau, \omega, u_\tau), \quad (t, \tau, \omega, u_\tau) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H^p(\mathbb{R}^n).
\]

We take a universe \(\bar{\mathcal{D}}\) on \(H^p(\mathbb{R}^n)\), which consists of all backward tempered set-valued mappings \(\bar{\mathcal{D}} : \mathbb{R} \times \Omega \to 2^{H^p(\mathbb{R}^n)} \setminus \{\emptyset\}\) satisfying

\[
\lim_{t \to +\infty} e^{-\alpha t} \sup_{\zeta \leq \tau} \|\bar{\mathcal{D}}(\zeta - t, \theta_{-t} \omega)\|_{H^p(\mathbb{R}^n)}^2 = 0, \quad \forall \alpha > 0, \ \tau \in \mathbb{R}, \ \omega \in \Omega.
\]

where \(\|\bar{\mathcal{D}}\|\) denotes the supremum of norms of all elements. Meanwhile, let \(\mathcal{D}\) be the usual universe of all tempered sets on \(H^p(\mathbb{R}^n)\), which satisfies

\[
\lim_{t \to +\infty} e^{-\alpha t} \|\mathcal{D}(\tau - t, \theta_{-t} \omega)\|_{H^p(\mathbb{R}^n)}^2 = 0, \quad \forall \alpha > 0, \ \tau \in \mathbb{R}, \ \omega \in \Omega.
\]

(17)

Obviously, \(\bar{\mathcal{D}}\) is a subset of \(\mathcal{D}\).

In order to obtain the \(\bar{\mathcal{D}}\)-pullback attractor \(A_{\bar{\mathcal{D}}}\), we make the following assumption for some \(a_0 > 0\),

\[
\sup_{\zeta \leq \tau} \int_{-\infty}^0 e^{a_0 r} \left( \|g(r + \zeta, \cdot)\|^2 + \|\psi_1(r + \zeta, \cdot)\|_1 + \|\psi_2(r + \zeta, \cdot)\|^2 \right) \, dr < \infty, \quad \forall \tau \in \mathbb{R}.
\]

(18)
where we use $\| \cdot \|_r$ (resp. $\| \cdot \|_\infty$) to denote the norm of $L'(\mathbb{R}^n)$ (resp. $L^2(\mathbb{R}^n)$) here and after. Indeed, by ([35], Lemma 4.2), one can show that the growth rate $a_0$ can be arbitrary, which means for $\forall \alpha > 0$, we have

$$\sup_{\xi \leq r} \int_{-\infty}^{0} e^{\alpha r} \left( \|g(r + \xi, \cdot)\|^2 + \|\psi_1(r + \xi, \cdot)\|_1 + \|\psi_2(r + \xi, \cdot)\|^2 \right)dr < \infty, \ \forall \tau \in \mathbb{R}. \ (19)$$

Based on this, (19) can imply the following tempered condition with arbitrary growth rates:

$$\int_{-\infty}^{0} e^{\alpha r} \left( \|g(r + \tau, \cdot)\|^2 + \|\psi_1(r + \tau, \cdot)\|_1 + \|\psi_2(r + \tau, \cdot)\|^2 \right)dr < \infty, \ \forall \tau \in \mathbb{R}. \ (20)$$

Since the systems defined on the entire space $\mathbb{R}^n$, we also need the following assumptions

$$\lim_{k \to \infty} \sup_{\xi \leq r} \int_{-\infty}^{0} e^{\alpha r} \int_{|x| \geq k} (|g(r + \xi, \cdot)|^2 + |\psi_1(r + \xi, \cdot)_{1} + |\psi_2(r + \xi, \cdot)|^2)dxd\tau = 0, \ \forall \tau \in \mathbb{R}. \ (21)$$

$$\lim_{k \to \infty} \sup_{\xi \leq r} \int_{-\infty}^{0} e^{\alpha r} \int_{|x| \geq k} (|\beta_1(r + \xi, \cdot)|^2 + |\beta_2(r + \xi, \cdot)|^2)dxd\tau = 0, \ \forall \tau \in \mathbb{R}. \ (22)$$

3. Backward Uniform Estimates of Solutions

In this section, we derive uniform estimates on the solutions of (3) when $t \to +\infty$, with the purpose of proving the existence of a bounded $\mathcal{D}$-pullback absorbing set that is uniform in the past and the backward asymptotic compactness of the random dynamical system associated with the equation.

**Lemma 1.** Let (4)-(9) and (19) be satisfied. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} = \{ \mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \subset \mathcal{D}$, there is a $T := T(\tau, \omega, \mathcal{D}) > 0$ such that for all $t \geq T$, the solution of (3) with $u_{\xi-t} \in \mathcal{D}(\xi-t, \theta_{-t} \omega)$ satisfies

$$\sup_{\xi \leq r} \|u(\xi, \xi-t, \theta_{-t} \omega, u_{\xi-t})\|^2_{H^r(\mathbb{R}^n)} \leq M + M \sup_{\xi \leq r} R(\xi, \omega), \ (23)$$

$$\sup_{\xi \leq r} \int_{\xi-t}^{\xi} e^{\lambda(r-\xi)} \|u(r, \xi-t, \theta_{-t} \omega, u_{\xi-t})\|^2_{H^r(\mathbb{R}^n)}dr \leq M + M \sup_{\xi \leq r} R^{p-1}(\xi, \omega), \ (24)$$

where $M > 0$ is a constant independent of $\tau$, $\omega$ and $\mathcal{D}$, $R(\tau, \omega)$ is given by

$$R(\tau, \omega) = \int_{-\infty}^{0} e^{1/2r} \left( 1 + \|g(r + \tau)\|^2 + \|\psi_1(r + \tau)\|_1 + |\mathcal{G}(\theta_{\tau} \omega)|^2 \right)dr, \ (25)$$

which is well-defined due to (19).

**Proof.** Let $\tau \in \mathbb{R}$ be fixed, and $\xi \leq \tau$. Multiplying Equation (3) by $u$ and integrate over $\mathbb{R}^n$, we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \|u\|^2 + \|(-\Delta)^{\frac{1}{2}} u\|^2 \right) + \lambda \|u\|^2 + \|(-\Delta)^{\frac{1}{2}} u\|^2 = (f(t, u), u) + \mathcal{G}(\theta_{\tau} \omega)(h(t, u), u) + (g(t), u) \ (26)$$

We now consider the right hand side of (26) one by one. For the nonlinear term, we have
\[(f(t, u), u) + G_\delta(\theta_t \omega)(h(t, u), u)\]
\[\leq -\alpha_1 \|u\|^p + \|\psi_1(t)\|_1 + |G_\delta(\theta_t \omega)| \int_{\mathbb{R}^n} |\beta_1(t, x)||u|^q dx + |G_\delta(\theta_t \omega)| \int_{\mathbb{R}^n} |\beta_2(t, x)||u|dx\]
\[\leq \|\psi_1(t)\|_1 + \frac{\lambda}{4} \|u\|^2 + c|G_\delta(\theta_t \omega)|^2 \|\beta_1(t)\|_{L^p} + c|G_\delta(\theta_t \omega)|^2 \|\beta_2(t)\|^2\]
\[\leq \frac{\lambda}{4} \|u\|^2 + \|\psi_1(t)\|_1 + c|G_\delta(\theta_t \omega)|^2 + c.\]

For the last term, by Young’s inequality, we have
\[(g(t), u) \leq \frac{\lambda}{4} \|u\|^2 + c\|g(t)\|^2.\] (28)

Substituting (27), (28) into (26), we see that
\[
\frac{d}{dt}(\|u\|^2 + \|(-\Delta)^{\frac{1}{2}} u\|^2) + \kappa (\|v\|^2 + \|(-\Delta)^{\frac{1}{2}} u\|^2)
\leq c\|g(t)\|^2 + c\|\psi_1(t)\|_1 + c|G_\delta(\theta_t \omega)|^2 + c,
\] (29)

where \(\kappa = \min(\lambda, 1)\).

Multiplying (29) by \(e^{st}\), then integrating over \((\zeta - t, \zeta)\) with \(t \geq 0\), and replace \(\omega\) by \(\theta^\omega\), we find that
\[
\|u(\zeta, \xi - t, \theta^\omega, u_{\zeta - t})\|^2 + \|(-\Delta)^{\frac{1}{2}} u(\zeta, \xi - t, \theta^\omega, u_{\zeta - t})\|^2
\leq e^{-st}(\|u_{\zeta - t}\|^2 + \|(-\Delta)^{\frac{1}{2}} u_{\zeta - t}\|^2)
\[\quad + c \int_{t-1}^{0} e^{sx} (1 + \|g(r + \zeta)\|^2 + \|\psi_1(r + \zeta)\|_1 + |G_\delta(\theta_r \omega)|^2) dr\]
\[\leq e^{-st}(\|u_{\zeta - t}\|^2 + \|(-\Delta)^{\frac{1}{2}} u_{\zeta - t}\|^2)
\[\quad + c \int_{t-1}^{0} e^{sx} (1 + \|g(r + \zeta)\|^2 + \|\psi_1(r + \zeta)\|_1 + |G_\delta(\theta_r \omega)|^2) dr,
\] (30)

Taking the supremum with respect to the time over \(\zeta \leq \tau\) in (30) and using (2) and (19), we see that
\[
c \sup_{\zeta \leq \tau} e^{-st} \|u_{\zeta - t}\|_{L^p(\mathbb{R}^n)}^2\]
\[\leq c \sup_{\zeta \leq \tau} e^{-st} \|\tilde{D}(\xi - t, \theta^\omega)\|_{L^p(\mathbb{R}^n)}^2 \to 0\text{ as } t \to +\infty,
\] (31)

and
\[
\sup_{\zeta \leq \tau} \int_{-t}^{0} e^{sx} (1 + \|g(r + \zeta)\|^2 + \|\psi_1(r + \zeta)\|_1 + |G_\delta(\theta_r \omega)|^2) dr
\leq \sup_{\zeta \leq \tau} \int_{-\infty}^{0} e^{sx} (1 + \|g(r + \zeta)\|^2 + \|\psi_1(r + \zeta)\|_1 + |G_\delta(\theta_r \omega)|^2) dr.
\] (32)

Therefore, it follows from (30) to (32), and we have
\[
\sup_{\zeta \leq \tau} \|u(\zeta, \xi - t, \theta^\omega, u_{\zeta - t})\|_{L^p(\mathbb{R}^n)}^2
\leq c \sup_{\zeta \leq \tau} \int_{-\infty}^{0} e^{sx} (1 + \|g(r + \zeta)\|^2 + \|\psi_1(r + \zeta)\|_1 + |G_\delta(\theta_r \omega)|^2) dr,
\] (33)

which implies (23).
Furthermore, taking \((p - 1)\)-th power of \((30)\) and multiplying by \(e^{\kappa(t - \xi)}\), we integrate the result over \((\xi - t, \xi)\) with \(t \geq 0\) to get

\[
\int_{\xi-t}^{\xi} e^{\kappa(t - \xi)} \|u(r, \xi - t, \theta - \xi \omega, u_{\xi - t})\|^{2p-2} H^p(\mathbb{R}^n) \, dr \\
\leq c \int_{\xi-t}^{\xi} e^{\kappa(t - \xi) - (p-1)\kappa(t - \xi) + \tau} \|u_{\xi - t}\|^{2p-2} H^p(\mathbb{R}^n) \\
+ c \left[ \int_{-\infty}^{0} e^{t \kappa t} (1 + \|g(r + \xi)\|^2 + \|\psi_1(r + \xi)\|_1 + |G_{\xi}(\theta_t \omega)|^2) \, dr \right]^{p-1}
\]

which, together with \((34)\), implies the desired result \((24)\). This completes the whole proof. \(\square\)

**Lemma 2.** Let \((4)-(9)\) hold. Then for every \(\tau \in \mathbb{R}, \omega \in \Omega \) and \(u_{\tau} \in H^p(\mathbb{R}^n)\), the derivative of the solution \(u(t, \tau, \omega, u_{\tau})\) of \((3)\) satisfies, for all \(t \geq \tau\),

\[
\|u_t(t, \tau, \omega, u_{\tau})\|_{H^p(\mathbb{R}^n)} \leq c\left(1 + \|u\|_{H^p(\mathbb{R}^n)}^{2p-2} + \|g(t)\|^2 + \|\psi_2(t)\|^2 + |G_{\xi}(\theta_t \omega)|^{\frac{2p}{p-2}}\right).
\]

**Proof.** Multiplying \((3)\) by \(u_t\) and integrating over \(\mathbb{R}^n\), we have

\[
\|u_t\|^2 + \|(-\Delta)^{\frac{1}{2}} u_t\|^2 + \lambda(u, u_t) + ((-\Delta)^{\frac{1}{2}} u, (-\Delta)^{\frac{1}{2}} u_t) \\
= (f(t, u), u_t) + G_{\xi}(\theta_t \omega)(h(t, u), u_t) + (g(t), u_t).
\]

Using the assumption \((5)\) and the Sobolev embedding inequality given in \((12)\), we obtain

\[
|f(t, u)| \leq \int_{\mathbb{R}^n} (a_2 |u|^{p-1} + |\psi_2(t)|) |u_t| \, dx \\
\leq a_2 \|u\|_p^{p-1} \|u_t\| + c |\psi_2(t)| \|u_t\| \\
\leq \frac{1}{4} \|u_t\|^2 + \frac{1}{4} \|(-\Delta)^{\frac{1}{2}} u_t\|^2 + c \|u\|_{H^p(\mathbb{R}^n)}^{2p-2} + c |\psi_2(t)|^2
\]

By \((12)\) and \((7)\), we obtain

\[
G_{\xi}(\theta_t \omega)(h(t, u), u_t) \leq |G_{\xi}(\theta_t \omega)| \int_{\mathbb{R}^n} \|\beta_1(t, x)\| |u|^{q-1} + |\beta_2(t, x)| |u_t| \, dx \\
\leq \frac{1}{4} \|u_t\|^2 + c \int_{\mathbb{R}^n} |G_{\xi}(\theta_t \omega)|^2 \|\beta_1(t, x)\|^2 |u|^{2q-2} \, dx + c |G_{\xi}(\theta_t \omega)|^2 \|\beta_2(t)\|^2
\]

which, together with \((34)\), implies the desired result \((24)\). This completes the whole proof. \(\square\)
Applying Young’s inequality, we have
\[
\lambda |(u, u_t)| + |((-\Delta)\frac{1}{2} u, (-\Delta)\frac{1}{2} u_t)| + |(g(t), u_t)|
\leq \frac{1}{4} ||u_t||^2 + \frac{1}{2} ||(-\Delta)\frac{1}{2} u_t||^2 + c ||u||_{H^1(\mathbb{R}^n)}^2 + c ||g(t)||^2 + c.
\] (39)

It follows from (36) to (39) that the desired result (35) follows immediately. □

We now derive the uniform estimates on the tails of the solutions. To this end, we introduce a smooth function \( \rho(s) \) defined for \( 0 \leq s < \infty \) such that \( 0 \leq \rho(s) \leq 1 \) for \( s \geq 0 \) and satisfies \( \rho(s) \equiv 0 \) for all \( 0 \leq s \leq \frac{1}{2} \) and \( \rho(s) \equiv 1 \) for all \( s \geq 1 \). Letting \( \rho_k(x) = \rho(\frac{|x|}{k}) \) for \( x \in \mathbb{R}^n \) and \( k \in \mathbb{N} \), by ([44], Lemma 3.4), we have
\[
\sup_{y \in \mathbb{R}^n} \left| \rho_k(x) - \rho_k(y) \right|^2 \leq \frac{c}{k^{25}} \quad \text{and} \quad \sup_{y \in \mathbb{R}^n} \left| (-\Delta)^{\frac{3}{2}} \rho_k(y) \right| \leq \frac{c}{k^{25}}.
\] (40)

**Lemma 3.** Let (4)–(9) and (19) be satisfied. Then for every \( \epsilon > 0, \tau \in \mathbb{R}, \omega \in \Omega \) and \( \tilde{D} = \{ \tilde{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \), there exists \( T = T(\tau, \omega, \tilde{D}, \epsilon) > 1, K = K(\tau, \omega, \tilde{D}, \epsilon) > 1 \), such that for all \( t \geq T \) and \( k \geq K \), the solution of (3) with \( u_{\xi-\tau} \in \tilde{D}(\xi - t, \theta_{-\epsilon} \omega) \) satisfies
\[
\sup_{\xi \leq \tau} \int_{O_k^c} \left| u(\xi, \xi - t, \theta_{-\epsilon} \omega, u_{\xi-\tau})(x) \right|^2 dx \leq \epsilon,
\] (41)
\[
\sup_{\xi \leq \tau} \int_{(O_k^c)^c} \frac{\left| u(\xi, \xi - t, \theta_{-\epsilon} \omega, u_{\xi-\tau})(x) - u(\xi, \xi - t, \theta_{-\epsilon} \omega, u_{\xi-\tau})(y) \right|^2}{|x - y|^{n+25}} dx dy \leq \epsilon,
\] (42)
where \( O_k = \{ x \in \mathbb{R}^n : |x| < k \}, O_k^c = \mathbb{R}^n \setminus O_k \) and \( (O_k^c)^c = \mathbb{R}^n \times \mathbb{R}^n \setminus O_k \times O_k \).

**Proof.** By Ref. [12], Lemma 2 and a slight modification, we have the following energy equation:
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^n} \rho_k(x) |u|^2 dx + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x) |u(x) - u(y)|^2 dx dy \right)
+ \kappa \left( \int_{\mathbb{R}^n} \rho_k(x) |u|^2 dx + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x) |u(x) - u(y)|^2 dx dy \right)
\leq c k^{-s} (1 + ||u||_{H^1(\mathbb{R}^n)}^2 + ||g(t)||^2 + ||\psi_2(t)||^2 + ||G_{\xi}(\theta_t \omega)||^{\frac{2p}{4 - s}})
+ c \int_{|x| \geq \frac{k}{2}} \left( \left| g(t, x) \right|^2 + \left| \psi_1(t, x) \right| \right) dx
+ c \left( ||G_{\xi}(\theta_t \omega)||^{\frac{p}{4 - s}} + ||G_{\xi}(\theta_t \omega)||^2 \right) \int_{|x| \geq \frac{k}{2}} \left( \left| \beta_1(t, x) \right|^{\frac{p}{4 - s}} + \left| \beta_2(t, x) \right|^2 \right) dx.
\] (43)
We now multiply the above by $e^{xt}$, then integrate the result over $(\zeta - t, \zeta)$ with $t \geq 0$ and replace $\omega$ by $\theta_\omega$ in the resulting inequality to obtain

$$
\int_{\mathbb{R}^n} \rho_k(x)|u(\zeta - t, \theta_\omega u_{\zeta - t})(x)|^2 dx + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)|(u(\zeta - t, \theta_\omega u_{\zeta - t})(x) - u(\zeta - t, \omega t)(y))|^2 dxdy \\
\leq e^{-xt} \left( \int_{\mathbb{R}^n} \rho_k(x)|u_{\zeta - t}|^2 dx + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)|u_{\zeta - t}(x) - u_{\zeta - t}(y)|^2 dxdy \right) \\
+ c k^{-s} \int_{\zeta - t}^\zeta e^{(r-\zeta)} \|u(r, \zeta - t, \theta_\omega u_{\zeta - t})\|_{L^p(\mathbb{R}^n)}^{2p-2} dr
$$

(44)

By (2) and (19), we have

$$
k^{-s} \sup_{\zeta \leq T} \int_{\zeta - t}^\zeta e^{(r-\zeta)} \|u(r, \zeta - t, \theta_\omega u_{\zeta - t})\|_{L^p(\mathbb{R}^n)}^{2p-2} dr
$$

(45)

In particular, for the second term on the right-hand side of (44), by Lemma 1, there exists $T := T(\zeta, \omega, \bar{D}) > 0$ such that for all $t \geq T$, as $k \to \infty$,

$$
k^{-s} \sup_{\zeta \leq T} \int_{\zeta - t}^\zeta e^{(r-\zeta)} \|u(r, \zeta - t, \theta_\omega u_{\zeta - t})\|_{L^p(\mathbb{R}^n)}^{2p-2} dr
$$

(46)

tends to 0 as $k \to \infty$. In addition, it follows from assumption (21), we get that

$$
\sup_{\zeta \leq T} \int_{-\infty}^0 e^{xr} \left( |g(r + \zeta)|^2 + |\psi_2(r + \zeta)|^2 + |G_\theta(\theta \omega)| \right) dr \to 0 \quad \text{as} \quad k \to \infty.
$$

(47)

From the last term on the right-hand side of (44), apply (2) and the assumption (22), we have

$$
\sup_{\zeta \leq T} \int_{-\infty}^0 e^{xr} \left( |G_\theta(\theta \omega)| \right) dxr \to 0 \quad \text{as} \quad k \to \infty.
$$

(49)

which tends to 0 as $k \to \infty$. In addition, it follows from assumption (21), we get that

$$
\sup_{\zeta \leq T} \int_{-\infty}^0 e^{xr} \left( |G_\theta(\theta \omega)| \right) dxr \to 0 \quad \text{as} \quad k \to \infty.
$$

(48)
Finally, it follows from (45) to (49) that there exists \( T = T(\tau, \omega, \tilde{D}, \varepsilon) \geq 1, K = K(\tau, \omega, \tilde{D}, \varepsilon) \geq 1 \), such that for all \( t \geq T \) and \( k \geq K \), we take the supremum over \( \zeta \in (-\infty, \tau] \) in (44), and we have

\[
\sup_{\zeta \leq \tau} \int_{\mathbb{R}^n} \rho_k(x) |u(\zeta, \zeta - t, \theta, \omega, \epsilon_{\zeta-1})(x)|^2 \, dx \leq \varepsilon, \quad (50)
\]

\[
\sup_{\zeta \leq \tau} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x) \left( \frac{|u(\zeta, \zeta - t, \theta, \omega, \epsilon_{\zeta-1})(x) - u(\zeta, \zeta - t)(y)|^2}{|x - y|^{n+2s}} \right) \, dx \, dy \leq \varepsilon, \quad (51)
\]

which implies the first result (41). Furthermore, by (51), it is easy to see that

\[
\sup_{\zeta \leq \tau} \int_{\mathbb{R}^n} \int_{|x| \geq k} \frac{|u(\zeta, \zeta - t, \theta, \omega, \epsilon_{\zeta-1})(x) - u(\zeta, \zeta - t)(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \leq \varepsilon. \quad (52)
\]

Interchanging \( x \) and \( y \) in (52) yields

\[
\sup_{\zeta \leq \tau} \int_{\mathbb{R}^n} \int_{|y| \geq k} \frac{|u(\zeta, \zeta - t, \theta, \omega, \epsilon_{\zeta-1})(x) - u(\zeta, \zeta - t)(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \leq \varepsilon. \quad (53)
\]

Combining (52) and (53), we obtain the second result (42). This completes the whole proof. \( \square \)

In order to establish the backward pullback asymptotic compactness of solutions in \( H^s(\mathbb{R}^n) \), we need to approximate \( \mathbb{R}^n \) be a family of bounded domains. For every \( x \in \mathbb{R}^n \) and \( k \in \mathbb{N} \), let

\[
\tilde{u}(t, \tau, \omega, \tilde{u}_x)(x) = \tilde{\zeta}_k(x) u(t, \tau, \omega, \tilde{u}_x)(x), \quad \text{where} \quad \tilde{\zeta}_k(x) = 1 - \rho(\frac{|x|}{\tilde{R}}).
\]

Then \( \tilde{u}(t, \tau, \omega, \tilde{u}_x)(x) = 0 \) for \( k \in \mathbb{N} \) and \( x \in O^c_k \) and \( ||\tilde{u}||_{H^s(\mathbb{R}^n)} \leq c ||u||_{H^s(\mathbb{R}^n)} \) for some constant \( c > 0 \) independent of \( k \in \mathbb{N} \).

Multiplying (3) by \( \tilde{\zeta}_k(x) \) and by [12], we have

\[
\tilde{u}_t + (-\Delta)^s \tilde{u}_t + (-\Delta)^s \tilde{u} + \lambda \tilde{u} = \tilde{\zeta}_k(x) f(t, x, u) + G(\theta, \omega) \tilde{\zeta}_k(x) h(t, x, u) + \tilde{\zeta}_k(x) g(t, x)
\]

\[
+ u(t)(-\Delta)^s \tilde{\zeta}_k(x) + C(n, s) \text{P.V.} \int_{\mathbb{R}^n} \frac{(\tilde{\zeta}_k(x) - \tilde{\zeta}_k(y))(u(x) - u(y))}{|x - y|^{n+2s}} \, dy
\]

\[
+ u(x)(-\Delta)^s \tilde{\zeta}_k(x) + C(n, s) \text{P.V.} \int_{\mathbb{R}^n} \frac{(\tilde{\zeta}_k(x) - \tilde{\zeta}_k(y))(u(x) - u(y))}{|x - y|^{n+2s}} \, dy,
\]

with initial-boundary conditions:

\[
\tilde{u}(\tau, x) = \tilde{\zeta}_k(x) u(\tau, x), \quad \forall x \in \mathbb{R}^n \quad \text{and thus} \quad \tilde{u}(\tau, x) = 0, \quad \forall x \in O^c_k.
\]

Let \( H = \{ u \in L^2(\mathbb{R}^n) : u = 0 \text{ on } O^c_k \} \) and \( V = \{ u \in H^s(\mathbb{R}^n) : u = 0 \text{ on } O^c_k \} \) which are considered as subspaces of \( L^2(\mathbb{R}^n) \) and \( H^s(\mathbb{R}^n) \), respectively. Considering the following eigenvalue problem in \( V \):

\[
(-\Delta)^s u = \lambda u, \quad \forall x \in O^c_k \quad \text{and} \quad u = 0, \quad \forall x \in O^c_k
\]

It is easy to see that the above has a family of eigenvalues \( \{ \lambda_i \}_{i=1}^\infty \) such that \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_i \to \infty \) as \( i \to \infty \) and the corresponding eigenfunctions \( \{ e_i \}_{i=1}^\infty \) in \( V \) form an orthonormal basis of \( H \). Write \( H_m = \text{span} \{ e_1, e_2, \cdots, e_m \} \), \( H^\perp_m = \text{span} \{ e_{m+1}, e_{m+2}, \cdots \} \), \( m \in \mathbb{N} \) and let \( P_m : H \to H_m, Q_m = I - P_m : H \to H^\perp_m \) be the orthonormal projector defined by

\[
P_m \tilde{u} = \tilde{u}_m, \quad \text{for } \tilde{u} \in H \quad \text{with} \quad \tilde{u} = \tilde{u}_{m+1} + \tilde{u}_{m+2}, \quad \tilde{u}_{m+1} \in H_m, \quad \tilde{u}_{m+2} \in H^\perp_m,
\]

where \( m \in \mathbb{N} \).
Lemma 4. Let (4)–(9) and (19) be satisfied. Then for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( \tilde{D} = \{ \tilde{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) in \( \tilde{D} \), there exists \( T = T(\tau, \omega, \tilde{D}, e) \geq 1 \), \( N = N(\tau, \omega, \tilde{D}, e) \geq 1 \), such that for all \( t \geq T \) and \( m \geq N \), the solution of (3) with \( u_{c-1} \in \tilde{D}(\xi, t, \theta-\omega) \) satisfies

\[
\sup_{\xi \leq \tau} \| (I - P_m) \xi_k u(\xi, \xi - t, \theta - \omega, u_{\xi - t}) \|_{H^p(\mathbb{R}^n)} \leq e \quad \text{for each } k \in \mathbb{N}.
\] (57)

Proof. Multiplying (55) by \( \tilde{u}_{m,2} \), we get

\[
\frac{1}{2} \frac{d}{dt} \| \tilde{u}_{m,2} \|^2 + \| (-\Delta)\hat{\xi}_k \tilde{u}_{m,2} \|^2 + \lambda \| \tilde{u}_{m,2} \|^2 + \| (-\Delta)\hat{\xi}_k \tilde{u}_{m,2} \|^2 = \int_{\mathbb{R}^n} \xi_k f(t, x, u) \tilde{u}_{m,2} dx \\
+ G_\delta(\theta_1 \omega) \int_{\mathbb{R}^n} \xi_k h(t, x, u) \tilde{u}_{m,2} dx + \int_{\mathbb{R}^n} \xi_k (g(t) + u_t (-\Delta)^s + u (-\Delta)^s) \tilde{u}_{m,2} dx \\
+ C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y)) (u(x) - u(y))}{|x - y|^{n+2s}} dy \tilde{u}_{m,2} dx \\
+ C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y)) (u(x) - u(y))}{|x - y|^{n+2s}} dy \tilde{u}_{m,2} dx.
\] (58)

By (5) and Gagliardo–Nirenberg inequality (14), the first term on the right-hand side of (58) is bounded by

\[
\int_{\mathbb{R}^n} \xi_k f(t, x, u) \tilde{u}_{m,2} dx \leq c \| \tilde{u}_{m,2} \|_{L^p(\mathbb{R}^n)} \| u \|_{L^p(\mathbb{R}^n)} + c \| \tilde{u}_{m,2} \|_{H^2(\mathbb{R}^n)}
\]

\[
\leq c \| u \|_{L^p(\mathbb{R}^n)} + c \| \tilde{u}_{m,2} \|_{H^2(\mathbb{R}^n)}
\]

\[
\leq c \lambda \| u \|_{H^2(\mathbb{R}^n)}
\]

\[
\leq \frac{1}{16} \| (-\Delta)^s \tilde{u}_{m,2} \|^2 + c \lambda \| u \|_{H^2(\mathbb{R}^n)}^2 + \lambda \| \tilde{u}_{m,2} \|_{H^2(\mathbb{R}^n)}^2.
\] (59)

where \( \mu = \frac{np - 2n}{2p} \in (0, 1) \) due to \( p \leq \frac{2n}{n - 2s} \). On the other hand, by assumption (7) and embedding inequality (12) and (13) as well as Hölder’s inequality, we have

\[
G_\delta(\theta_1 \omega) \int_{\mathbb{R}^n} \xi_k h(t, x, u) \tilde{u}_{m,2} dx
\]

\[
\leq |G_\delta(\theta_1 \omega)| \int_{\mathbb{R}^n} (|\beta_1(t, x)| |u|^{q - 1} + |\beta_2(t, x)|) |\tilde{u}_{m,2}| dx
\]

\[
\leq |G_\delta(\theta_1 \omega)| \| \tilde{u}_{m,2} \|_{L^p(\mathbb{R}^n)} \| \beta_1(t) \|_{L^{2p} \mathbb{R}^n} |u|_{L^{2q} \mathbb{R}^n} + |G_\delta(\theta_1 \omega)| \| \tilde{u}_{m,2} \|_{H^2(\mathbb{R}^n)}
\]

\[
\leq c |G_\delta(\theta_1 \omega)| \| \tilde{u}_{m,2} \|_{H^2(\mathbb{R}^n)} + |G_\delta(\theta_1 \omega)| \| \tilde{u}_{m,2} \|_{H^2(\mathbb{R}^n)}
\]

\[
\leq c \lambda \| \tilde{u}_{m,2} \|_{H^2(\mathbb{R}^n)}
\]

\[
\leq \frac{1}{16} \| (-\Delta)^s \tilde{u}_{m,2} \|^2 + c \lambda \| \tilde{u}_{m,2} \|_{H^2(\mathbb{R}^n)}^2.
\] (60)

Since \( (-\Delta)^s \xi_k = (-\Delta)^s \rho_k \) and by (40), we obtain in the second line of (58)

\[
\int_{\mathbb{R}^n} \xi_k (g(t) + u_t (-\Delta)^s + u (-\Delta)^s) \tilde{u}_{m,2} dx
\]

\[
\leq c \lambda \| \tilde{u}_{m,2} \|_{H^2(\mathbb{R}^n)} + \| u \|_{H^2(\mathbb{R}^n)} + \| g(t) \|
\]

\[
\leq \frac{1}{8} \| (-\Delta)^s \tilde{u}_{m,2} \|^2 + c \lambda \| u \|_{H^2(\mathbb{R}^n)}^2 + \| u \|_{H^2(\mathbb{R}^n)}^2.
\] (61)
By [44], the remaining terms on the right-hand side of (58) is bounded by
\[
C(n,s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y))(u_t(x) - u_t(y))}{|x - y|^n + 2s} dy \, dx \\
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y))(u(x) - u(y))}{|x - y|^n + 2s} dy \, dx
\]
(62)
where \(\hat{K}\) have then we take the supremum over inequality over \(t\). Substituting (59)–(62) into (58), we see from Lemma 2 that
\[
\frac{d}{dt} \left( \|\hat{u}_{m,2}\|^2 + \|(-\Delta)^{\frac{k}{2}} \hat{u}_{m,2}\|^2 \right) + \kappa \left( \|\hat{u}_{m,2}\|^2 + \|(-\Delta)^{\frac{k}{2}} \hat{u}_{m,2}\|^2 \right)
\leq c\lambda_m \left( 1 + \|u\|_{H^2(\mathbb{R}^n)}^2 + \|g(t)\|^2 + \|\psi(t)\|^2 + \|G_\epsilon(\theta_t \omega)\|_{H^2(\mathbb{R}^n)}^{\frac{2p}{2p-2}} \right),
\]
(63)
where \(\lambda_m := \lambda_{m+1}^{-1} \lambda_{m+1}^{-1} \lambda_{m+1}^{-1} \rightarrow 0 \) as \(m \rightarrow +\infty\). Applying the uniform Gronwall inequality over \((\xi - t, \xi)\) with \(t \geq 0\) and replacing \(\omega\) by \(\theta_{-\epsilon} \omega\) in the resulting inequality, then we take the supremum over \(\epsilon \in (-\infty, \tau]\), and we conclude that
\[
\sup_{\xi \leq \tau} \|\hat{u}_{m,2}(\xi, \xi - t, \theta_{-\epsilon} \omega, u_{\xi - l})\|^2 + \sup_{\xi \leq \tau} \|(-\Delta)^{\frac{k}{2}} \hat{u}_{m,2}(\xi, \xi - t, \theta_{-\epsilon} \omega, u_{\xi - l})\|^2
\leq e^{-\kappa t} \sup_{\xi \leq \tau} \left( \|(-\Delta)^{\frac{k}{2}} u_{m,2}(\xi, \xi - t, \theta_{-\epsilon} \omega, u_{\xi - l})\|^2 \right)
+ c\lambda_m \sup_{\xi \leq \tau} \int_{\xi - l}^{\xi} e^{\kappa(t-s)} \|u(s, \xi - t, \theta_{-\epsilon} \omega, u_{\xi - l})\|_{H^2(\mathbb{R}^n)}^{2p-2} ds
+ c\lambda_m \sup_{\xi \leq \tau} \int_{\xi - l}^{\xi} e^{\kappa(t-s)} \|g(s)\|^2 + \|\psi(s)\|^2 + \|G_\epsilon(\theta_t \omega)\|_{H^2(\mathbb{R}^n)}^{\frac{2p}{2p-2}} ds,
\]
(64)
Due to \(\|I - P_m\| \leq 1\), \(\|\xi\|_\infty \leq 1\) and \(u_{\xi - l} \in \bar{D}(\xi - t, \theta_{-\epsilon} \omega)\) with \(\bar{D} \subseteq \tilde{D}\) for all \(\xi \leq \tau\), we have
\[
e^{-\kappa t} \sup_{\xi \leq \tau} \left( \|(-\Delta)^{\frac{k}{2}} u_{m,2}(\xi, \xi - t, \theta_{-\epsilon} \omega, u_{\xi - l})\|^2 \right)
\leq c e^{-\kappa t} \sup_{\xi \leq \tau} \left( \|\bar{D}(\xi - t, \theta_{-\epsilon} \omega)\|_{H^2(\mathbb{R}^n)}^{2p-2} \right) \rightarrow 0 \) as \(t \rightarrow +\infty\).

Given \(\epsilon > 0\), let \(T = T(\tau, \omega, \tilde{D}, \epsilon) \geq 1\), \(N = N(\tau, \omega, \bar{D}, \epsilon) \geq 1\), such that for all \(t \geq T\) and \(m \geq N\). Then by (19) and Lemma 1, we find that the remaining terms on the right-hand side of (64) satisfies
\[
\sup_{\xi \leq \tau} \|u_{m,2}(\xi, \xi - t, \theta_{-\epsilon} \omega, (I - P_m)\xi_k u_{\xi - l})\|_{H^2(\mathbb{R}^n)} \leq \epsilon,
\]
which implies (57) as desired. \(\square\)

4. Existence of Backward Pullback Random Attractors

In this section, we show that the cocycle \(\Phi\), generated by the Wong–Zakai approximations for fractional nonclassical diffusion equations, has unique \(\bar{D}\)-pullback random attractors in \(H^p(\mathbb{R}^n)\). We establish the existence of \(\bar{D}\)-pullback absorbing sets in \(H^p(\mathbb{R}^n)\) firstly.

Lemma 5. Assume that (4)–(9) and (19) hold. Then the co-cycle \(\Phi\) for problem (3) has a \(\bar{D}\)-pullback random absorbing set \(K_{\bar{D}} = \{K_{\bar{D}}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \bar{D}\) given by
\[
K_{\bar{D}}(\tau, \omega) := \{u \in H^p(\mathbb{R}^n) : \|u\|_{H^p(\mathbb{R}^n)}^2 \leq M + MR(\tau, \omega)\}, \quad (\tau, \omega) \in \mathbb{R} \times \Omega,
\]
and a $\tilde{\mathcal{D}}$-pullback absorbing set $\mathcal{K}_D = \{K_D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \tilde{\mathcal{D}}$ given by

$$K_D(\tau, \omega) := \{u \in H^s(\mathbb{R}^n) : \|u\|_{H^s(\mathbb{R}^n)}^2 \leq M + \sup_{\xi \leq \tau} R(\xi, \omega)\}, \ (\tau, \omega) \in \mathbb{R} \times \Omega,$$

where $M$ and $R(\tau, \omega)$ are the same as given in Lemma 1.

**Proof.** By Lemma 1, $\mathcal{K}_D$ is an absorbing set and $\omega \to R(\tau, \omega)$ is measurable since it is the integral of some random variables. Hence, $\mathcal{K}_D$ is a random set. Next, we show $K_D \in \mathcal{D}$. For this end, we first prove that $\sup_{\xi \leq \tau} R(\xi, \omega)$ is tempered with any growth rate $\alpha > 0$. Indeed, letting $\bar{\alpha} = \min\{\frac{\alpha}{2}, \frac{\alpha}{4}\}$, by (2) and (19), we have,

$$e^{-at} \sup_{\xi \leq \tau} R(\xi, \omega)$$

$$= e^{-at} \sup_{\xi \leq \tau} \int_{-\infty}^{0} e^{\bar{\alpha} r} (\|g(r + s - t)\|^2 + \|\psi_1(r + s - t)\|_1 + |G_\xi(\theta_{\tau - \omega})|^{2\rho} + |G_\xi(\theta_{\tau - \omega})|^2) dr$$

$$\leq e^{-at} \sup_{\xi \leq \tau} \int_{-\infty}^{0} e^{\bar{\alpha} r} (\|g(r + s - t)\|^2 + \|\psi_1(r + s - t)\|_1 + |G_\xi(\theta_{\tau - \omega})|^{2\rho} + |G_\xi(\theta_{\tau - \omega})|^2) dr$$

$$= e^{-(\alpha - \bar{\alpha}) t} \sup_{\xi \leq \tau} \int_{-\infty}^{0} e^{\bar{\alpha} r} (\|g(r + s)\|^2 + \|\psi_1(r + s)\|_1 + |G_\xi(\theta_{\omega})|^{2\rho} + |G_\xi(\theta_{\omega})|^2) dr,

which tends to 0 as $t \to +\infty$. So, $\mathcal{K}_D \in \mathcal{D}$ and thus $\mathcal{K}_D \in \mathcal{D}$ due to $\mathcal{K}_D \subset \mathcal{K}_D$. 

On the other hand, it is easy to see that $\mathcal{K}_D$ is increasing, which means $\mathcal{K}_D(\tau_1, \omega) \subset \mathcal{K}_D(\tau_2, \omega)$ if $\tau_1 \leq \tau_2$. Given $\alpha > 0$, we have

$$e^{-at} \sup_{\xi \leq \tau} \|K_D(\xi - t, \theta_{\tau - \omega})\|_{H^s(\mathbb{R}^n)}^2 = e^{-at} \|K_D(\tau - t, \theta_{\tau - \omega})\|_{H^s(\mathbb{R}^n)}^2 \to 0 \text{ as } t \to +\infty, \quad (66)$$

where we have used $\mathcal{K}_D \in \mathcal{D}$. Then we have $\mathcal{K}_D \in \tilde{\mathcal{D}}$ as required. $\square$

**Remark 1.** In this lemma, the measurability of $\mathcal{K}_D$ is unknown since it is the union of some random sets $\mathcal{K}_D(\xi, \cdot)$ over an uncountable index set $\xi \leq \tau$. However, we will prove the measurability of $\mathcal{K}_D$ in Theorem 1.

We then show the $\tilde{\mathcal{D}}$-pullback backward asymptotic compactness of $\Phi$.

**Lemma 6.** Under conditions (4)–(9) and (19), the continuous co-cycle $\Phi$ associated with problem (3) is $\tilde{\mathcal{D}}$-pullback backward asymptotically compact in $H^s(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $\tilde{\mathcal{D}} = \{\tilde{D}(\tau, \omega) : \xi \in \mathbb{R}, \omega \in \Omega\} \in \tilde{\mathcal{D}}$, the sequence $\{\Phi(t_n, \xi_n - t_n, \theta_{\tau - \omega}, u_{0n})\}_{n=1}^{\infty}$ has a convergent sub-sequence in $H^s(\mathbb{R}^n)$ whenever $t_n \to +\infty, u_{0n} \in \tilde{D}(\xi_n - t_n, \theta_{\tau - \omega})$ and $\xi_n \leq \tau.$

**Proof.** By Lemmas 1, 3 and 4, the proof is standard, so we omit it. $\square$

Finally, we will establish the existence, uniqueness, and backward compactness of pullback random attractors of $\Phi$ in $H^s(\mathbb{R}^n)$.

**Theorem 1.** Suppose that conditions (4)–(9) and (19) hold. Let $\Phi$ be the continuous co-cycle for problem (3) and $\tilde{\mathcal{D}}$ (resp. $\mathcal{D}$) be the universe of backward tempered sets (resp. tempered sets). Then

(i) $\Phi$ has a $\tilde{\mathcal{D}}$-pullback bi-parametric attractor $A_{\tilde{\mathcal{D}}} \in \tilde{\mathcal{D}}$ such that $A_{\tilde{\mathcal{D}}}$ is backward compact in $H^s(\mathbb{R}^n)$;

(ii) $\Phi$ has a $\mathcal{D}$-pullback random attractor $A_{\mathcal{D}} \in \mathcal{D}$;

(iii) $A_{\tilde{\mathcal{D}}}(\tau, \omega) = A_{\mathcal{D}}(\tau, \omega)$ for all $(\tau, \omega) \in \mathbb{R} \times \Omega$, that is, $A_{\tilde{\mathcal{D}}} \in \tilde{\mathcal{D}}$ is a $\mathcal{D}$-pullback random attractor with the backward compactness.
Proof. (i) By Lemma 5, we find that \( K_{\bar{D}} = \{ K_{\bar{D}}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is a \( \bar{D} \)-pullback absorbing set for \( \Phi \) in \( H^s(\mathbb{R}^n) \). On the other hand, by Lemma 6 we know that \( \Phi \) is \( \bar{D} \)-pullback backward asymptotically compact in \( H^s(\mathbb{R}^n) \). Then it follows from the abstract result as given by [43] we show that \( \Phi \) has a \( \bar{D} \)-pullback bi-parametric attractor \( A_{\bar{D}} \in \bar{D} \) given by

\[
A_{\bar{D}}(\tau, \omega) = \bigcap_{t_0 > 0} \bigcup_{t \geq t_0} \Phi(t, \tau - t, \theta_{-t} \omega) K_{\bar{D}}(\tau - t, \theta_{-t} \omega)^{Hs(\mathbb{R}^n)},
\]

(67)

where the measurability of \( K_{\bar{D}} \) is unknown. However, we will prove that \( A_{\bar{D}} \) is a random attractor in (ii) and (iii).

Next, we prove \( A_{\bar{D}} \) is backward compact. We define

\[
W(\tau, \omega, K_{\bar{D}}) = \bigcap_{T > 0} \bigcup_{t \geq T} \bigcup_{r \leq T} \Phi(t, r - t, \theta_{-t} \omega) K_{\bar{D}}(r - t, \theta_{-t} \omega)^{Hs(\mathbb{R}^n)}, \forall \tau \in \mathbb{R}, \omega \in \Omega.
\]

(68)

In this case, the \( W(\tau, \omega, K_{\bar{D}}) \) is compact. Indeed, if there is a \( y \in W(\tau, \omega, K_{\bar{D}}) \), then

\[
y \in \bigcup_{t \geq n} \bigcup_{r \leq T} \Phi(t, r - t, \theta_{-t} \omega) K_{\bar{D}}(r - t, \theta_{-t} \omega)^{Hs(\mathbb{R}^n)}, \forall n \in \mathbb{N}.
\]

We choose three sequences by \( r_n \leq t_n \geq n \) and \( x_n \in K_{\bar{D}}(r_n - t_n, \theta_{-t_n} \omega) \) such that

\[
|\Phi(t_n, r_n - t_n, \theta_{-t_n} \omega)x_n - y| \leq \frac{1}{n}, \forall n \in \mathbb{N},
\]

which implies that

\[
y = \lim_{n \to +\infty} \Phi(t_n, r_n - t_n, \theta_{-t_n} \omega)x_n.
\]

Let \( \{ y_n \} \subset W(\tau, \omega, K_{\bar{D}}) \), then there are \( r_n \leq \tau, t_n \geq \max\{ t_{n-1}, n \} \) and \( x_n \in K_{\bar{D}}(r_n - t_n, \theta_{-t_n} \omega) \) such that

\[
d(\Phi(t_n, r_n - t_n, \theta_{-t_n} \omega)x_n, y_n) \leq \frac{1}{n}, \forall n \in \mathbb{N}.
\]

(69)

By the \( \bar{D} \)-backward asymptotic compactness of \( \Phi \) given by Lemma 6, there is a \( x \in X \) and a index subsequence \( \{ n_k \} \), such that

\[
\Phi(t_{n_k}, r_{n_k} - t_{n_k}, \theta_{-t_{n_k}} \omega)x_{n_k} \to x.
\]

(70)

Both (69) and (70) imply \( y_{n_k} \to x \in W(\tau, \omega, K_{\bar{D}}) \) as \( k \to \infty \) and thus \( W(\tau, \omega, K_{\bar{D}}) \) is compact as desired.

On the other hand, we have

\[
\bigcup_{r \leq \tau} A_{\bar{D}}(r, \omega) = \bigcup_{r \leq \tau} \bigcap_{T > 0} \bigcup_{t \geq T} \Phi(t, r - t, \theta_{-t} \omega) K_{\bar{D}}(r - t, \theta_{-t} \omega)^{Hs(\mathbb{R}^n)}
\]

\[
\subset \bigcup_{r \leq \tau} W(r, \omega, K_{\bar{D}}) = W(\tau, \omega, K_{\bar{D}})
\]

Hence, \( A_{\bar{D}} \) is backward compact.

(ii) By Lemma 5, we know that co-cycle \( \Phi \) has a \( \bar{D} \)-pullback absorbing set \( K_{\bar{D}} \in \bar{D} \) as given in Lemma 5 and \( K_{\bar{D}} \) is a random set. Using a similar process in Section 3, one can
show that \( \Phi \) is \( \mathcal{D} \)-pullback asymptotically compact. Then, by the abstract result in [43], \( \Phi \) has a \( \mathcal{D} \)-pullback random attractor \( \mathcal{A}_\mathcal{D} \in \mathcal{D} \) given by
\[
\mathcal{A}_\mathcal{D}(\tau, \omega) = \bigcap_{t_0 > 0} \bigcup_{t \geq t_0} \Phi(t, \tau - t, \theta_{-t}\omega)\mathcal{K}_\mathcal{D}(\tau - t, \theta_{-t}\omega)^{HF(\mathbb{R}^n)}.
\] (71)

(iii) By the definition of \( \mathcal{K}_\mathcal{D}(\tau, \omega) \) and \( \mathcal{K}_\mathcal{D}(\tau, \omega) \) in Lemma 5, we have the inclusion \( \mathcal{K}_\mathcal{D}(\tau, \omega) \subseteq \mathcal{K}_\mathcal{D}(\tau, \omega) \). Therefore, by (67) and (71), we see that
\[
\mathcal{A}_\mathcal{D}(\tau, \omega) \subseteq \mathcal{A}_\mathcal{D}(\tau, \omega), \quad \forall \tau \in \mathbb{R}, \tau \in \Omega.
\] (72)

On the other hand, since \( \mathcal{A}_\mathcal{D} \) is a subset of \( \mathcal{D} \) and \( \mathcal{A}_\mathcal{D} \in \mathcal{D} \), we have \( \mathcal{A}_\mathcal{D} \in \mathcal{D} \) and thus the \( \mathcal{D} \)-pullback attractor \( \mathcal{A}_\mathcal{D} \) attracts \( \mathcal{A}_\mathcal{D} \). Since \( \mathcal{A}_\mathcal{D} \) is invariant, we have
\[
\text{dist}_{HF(\mathbb{R}^n)}(\mathcal{A}_\mathcal{D}(\tau, \omega), \mathcal{A}_\mathcal{D}(\tau, \omega)) = \text{dist}_{HF(\mathbb{R}^n)}(\Phi(t, \tau - t, \theta_{-t}\omega)\mathcal{A}_\mathcal{D}(\tau - t, \theta_{-t}\omega), \mathcal{A}_\mathcal{D}(\tau, \omega)) \to 0 \text{ as } t \to +\infty.
\] (73)

By the compactness of \( \mathcal{A}_\mathcal{D} \), we have \( \mathcal{A}_\mathcal{D}(\tau, \omega) \subseteq \mathcal{A}_\mathcal{D}(\tau, \omega) \), which together with (72) implies that \( \mathcal{A}_\mathcal{D}(\tau, \omega) = \mathcal{A}_\mathcal{D}(\tau, \omega) \) for all \( \tau \in \mathbb{R}, \omega \in \Omega \). These finish the proof. \( \square \)

5. Asymptotic Upper Semicontinuity

In this section, we will prove the pullback random attractor \( \mathcal{A}_\mathcal{D}(\tau, \omega) \) of original non-autonomous stochastic Equation (3) backward converges to the autonomous random attractor \( \mathcal{A}_\infty \) of the following problem:
\[
\dot{u} + (-\Delta)^{\frac{\beta}{2}}u + (-\Delta)^{\frac{\alpha}{2}}u + \lambda u = \tilde{f}(x, u) + \tilde{g}(x) + \tilde{h}(x, \bar{u}) G_{\epsilon}(\theta_{t}\omega)
\] (74)
with the initial value
\[
\dot{u}(0, \omega, \bar{u}) = 0
\] (75)
The nonlinear functions \( \tilde{f}, \tilde{h} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) satisfy the following conditions: for all \( (x, u) \in \mathbb{R}^n \times \mathbb{R} \),
\[
\tilde{f}(x, u)u \leq -a_1 |u|^p + \tilde{\psi}_1(x),
\] (76)
\[
|\tilde{f}(x, u)| \leq a_2 |u|^{p-1} + \tilde{\psi}_2(x),
\] (77)
\[
\frac{\partial}{\partial u} \tilde{f}(x, u) \leq \tilde{\psi}_3(x),
\] (78)
\[
|\tilde{h}(x, u)| \leq \tilde{\beta}_1(x) |u|^{\theta-1} + \tilde{\beta}_2(x),
\] (79)
\[
|\frac{\partial}{\partial u} \tilde{h}(x, u)| \leq \tilde{\beta}_3(x),
\] (80)
where \( a_1, a_2 > 0 \) are constants, and \( \tilde{\psi}_1 \in L^1(\mathbb{R}^n), \tilde{\psi}_2 \in L^{p-1}(\mathbb{R}^n), \tilde{\psi}_3 \in L^{\infty}(\mathbb{R}^n), \tilde{\beta}_1 \in L^{\frac{p}{p-1}(\mathbb{R}^n)} \cap L^{\infty}(\mathbb{R}^n), \tilde{\beta}_2 \in L^{2\theta}(\mathbb{R}^n), \tilde{\beta}_3 \in L^{\infty}(\mathbb{R}^n). \)

Let \( \mathcal{D}_\infty \) be a universe of all tempered parametric sets \( \mathcal{D}_\infty \) in \( H^{s}(\mathbb{R}^n) \), satisfying that
\[
\lim_{t \to +\infty} e^{-\alpha t} \|\mathcal{D}_\infty(\theta_{-t}\omega)\|^2_{HF(\mathbb{R}^n)} = 0, \quad \forall \omega \in \Omega.
\] (81)
In fact, similar to Sections 3 and 4, we can also prove that problem (74) and (75) under (76)–(80) and \( \tilde{g} \in L^2(\mathbb{R}^n) \) generates an autonomous co-cycle \( \Phi_\infty : \mathbb{R}^+ \times \Omega \times H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \) given by
\[
\Phi_\infty(t, \omega) \bar{u}_0 = \bar{u}(t, \omega, \bar{u}_0), \quad (t, \omega) \in \mathbb{R}^+ \times \Omega.
\] (82)
Hence, we can prove that $\Phi_\infty$ has a $D_\infty$-pullback random attractor $A_\infty = \{A_\infty(\omega)\}$ in $H^1(\mathbb{R}^n)$.

In order to consider the backward asymptotically autonomous problem, we further assume the functions $f$ and $h$ satisfy

\[
\begin{align*}
|f(t, x, u) - \bar{f}(x, u)| &\leq |\psi_1(t, x) - \bar{\psi}_1(x)|; \\
h(t, x, u) - \bar{h}(x, u) &\leq |\beta_1(t, x) - \bar{\beta}_1(x)||u|^{q-1} + |\beta_2(t, x) - \bar{\beta}_2(x)|
\end{align*}
\]

and

\[
\lim_{\tau \to -\infty} \int_{\tau-1}^{\tau} \left( ||\psi_1(r) - \bar{\psi}_1||^2 + ||\psi_1(r) - \bar{\psi}_1|| + \sum_{i=1,2} ||\beta_i(r) - \bar{\beta}_i||^2 + ||g(r) - \bar{g}||^2 \right) dr = 0,
\]

where the assumption (85) is weaker than the convergence condition used in [34,45] that $\lim_{\tau \to -\infty} \int_{\tau}^{\infty} ||g(r) - \bar{g}||^2 dr = 0$.

**Lemma 7.** Suppose that assumption (4)–(9), (19), (76)–(80), and (83)–(85) hold. Then the solution $u$ of (3) is backward asymptotically autonomous to the solution $\bar{u}$ of (74). More precisely,

\[
\lim_{\tau \to -\infty} \|u(t + \tau, \tau, \theta_{-\tau} \omega, u_{\tau}) - \bar{u}(t, \omega, \bar{u}_0)\|_{H^1(\mathbb{R}^n)} = 0, \text{ for each } (t, \omega) \in \mathbb{R}^+ \times \Omega,
\]

whenever $\|u_{\tau} - \bar{u}_0\|_{H^1(\mathbb{R}^n)} \to 0$ as $\tau \to -\infty$.

**Proof.** Given $T > 0$ for $t \in (0, T)$. Let $\bar{u}(t, \tau) = u(t + \tau, \tau, \theta_{-\tau} \omega, u_{\tau}) - \bar{u}(t, \omega, \bar{u}_0)$, then minus (3) by (74), and take the inner product of result with $\bar{u}$, we obtain that

\[
\frac{1}{2} \frac{d}{dt} \left( ||\bar{u}||^2 + ||(-\Delta)^{\frac{1}{2}} \bar{u}||^2 \right) + \lambda ||\bar{u}||^2 + ||(-\Delta)^{\frac{1}{2}} \bar{u}||^2 = \int_{\mathbb{R}^n} \left( f(t + \tau, x, u) - \bar{f}(x, \bar{u}) \right) \bar{u} dx
\]

\[
+ \int_{\mathbb{R}^n} (g(t + \tau) - \bar{g}) \bar{u} dx + \int_{\mathbb{R}^n} G_2(\theta_{t} \omega) \left( h(t + \tau, x, u) - \bar{h}(x, \bar{u}) \right) \bar{u} dx.
\]

For the nonlinear term, let $\bar{f}(x, s) = \frac{d}{ds} \bar{f}(x, s)$, there exists $\xi_1$ between $u$ and $\bar{u}$, by (78), (83), and (85), we have

\[
\int_{\mathbb{R}^n} \left( f(t + \tau, x, u(t + \tau)) - \bar{f}(x, \bar{u}(t)) \right) \bar{u} dx
\]

\[
= \int_{\mathbb{R}^n} \left( f(t + \tau, x, u(t + \tau)) - \bar{f}(x, u(t + \tau)) + \bar{f}(x, u(t + \tau)) - \bar{f}(x, \bar{u}(t)) \right) \bar{u} dx
\]

\[
\leq \int_{\mathbb{R}^n} |\psi_1(t + \tau, x) - \bar{\psi}_1(x)||\bar{u}| dx + \int_{\mathbb{R}^n} \bar{f}(x, \xi_1) \bar{u}^2 dx
\]

\[
\leq ||\psi_1(t + \tau) - \bar{\psi}_1||^2 + ||u||^2_{L^\infty} + ||\bar{\psi}_1||_{L^\infty} ||\bar{u}||^2_{L^2}.
\]

For the external force term, we use the Young inequality, and have

\[
\int_{\mathbb{R}^n} (g(t + \tau) - \bar{g}) \bar{u} dx \leq ||g(t + \tau) - \bar{g}||^2 + ||\bar{u}||^2_{L^2}.
\]

For the last term on the right hand side of (87), let $\bar{h}(x, s) = \frac{d}{ds} \bar{h}(x, s)$, there exists $\xi_2$ between $u$ and $\bar{u}$, by (80) and (84), we obtain

\[
\int_{\mathbb{R}^n} (h(t + \tau, x, u(t + \tau)) - \bar{h}(x, \bar{u}(t))) \bar{u} dx
\]

\[
\leq ||\psi_2(t + \tau) - \bar{\psi}_2||^2 + ||u||^2_{L^\infty} + ||\bar{\psi}_2||_{L^\infty} ||\bar{u}||^2_{L^2}.
\]
\[
\int_{\mathbb{R}^n} G_\delta(\theta_1) (h(t, x, u) - \hat{h}(x, \hat{u})) \, dx
\]
\[
= \int_{\mathbb{R}^n} G_\delta(\theta_1) (h(t + \tau, x, u(t + \tau)) - \hat{h}(x, u(t + \tau)) + \hat{h}(x, u(t + \tau)) - \hat{h}(x, \hat{u}(t))) \, dx
\]
\[
\leq \int_{\mathbb{R}^n} |G_\delta(\theta_1)| \beta_1(t, \tau, x) - \hat{\beta}_1(x) \, |u(t)|^q \, dx
\]
\[
+ \int_{\mathbb{R}^n} |G_\delta(\theta_1)| \beta_2(t, \tau, x) - \hat{\beta}_2(x) \, |\hat{u}(t)|^q \, dx
\]
\[
\leq \|\beta_1(t + \tau) - \hat{\beta}_1(x)\| + \|G_\delta(\theta_1)\|^2 \|u\|_{L^2}^2 \|\hat{\beta}_2\|^2
\]
\[
+ \|G_\delta(\theta_1)\| \|u\|_{L^2}^2 \|\beta_2(t + \tau) - \hat{\beta}_2\|^2,
\]
where
\[
\|\beta_1(t + \tau) - \hat{\beta}_1(x)\| + \|G_\delta(\theta_1)\|^2 \|u\|_{L^2}^2 \|\hat{\beta}_2\|^2
\]
\[
+ \|G_\delta(\theta_1)\| \|u\|_{L^2}^2 \|\beta_2(t + \tau) - \hat{\beta}_2\|^2.
\]

Thus by (87)–(90), we have
\[
\frac{d}{dt} \|u\|^2_{L^2} \leq c(1 + \|G_\delta(\theta_1)\| + \|G_\delta(\theta_1)\|^2 \|u\|_{L^2}^2 + \|\hat{\beta}_1\|^2)
\]

Applying the Gronwall inequality to (91) over \([0, T]\), we get
\[
\int_{0}^{T} \|\phi(r + \tau)\|^2 \, dr \leq c e^{c \int_{0}^{T} \|\phi(r + \tau)\|^2 \, dr} \left( \|\phi(0)\|^2_{L^2} + \int_{0}^{T} \|\phi(r + \tau)\|^2 \, dr \right),
\]
where we have used the fact that the continuity of \|G_\delta(\theta_1)\| on \([0, T]\). By assumption (85), for a positive integer positive \(N \geq T\)
\[
\int_{0}^{T} \|\phi(r + \tau)\|^2 \, dr \leq \sum_{n=0}^{N} \int_{T+n}^{T+n+1} \|\phi(r + \tau)\|^2 \, dr \to 0, \text{ as } \tau \to -\infty.
\]

On the other hand, by using the energy inequality (29) on \(u(t + \tau)\) for \(t \in [0, T]\), we obtain
\[
\frac{d}{dt} \|u(t + \tau)\|^2_{L^2} \leq c \|g(t + \tau)\|^2_{L^2} + c \|\psi_1(t + \tau)\| + C(T, \omega)
\]
where \(C(T, \omega)\) are positive constants independent of \(\tau\) and \(t\). The Gronwall inequality implies that for all \(t \in [0, T]\),
\[
\|u(t + \tau)\|^2_{L^2} \leq c \left(1 + C(T, \omega) + \|u_\tau\|^2_{L^2} + \|g\|^2 + \|\psi_1\| \right)
\]
\[
+ \int_{0}^{T} \|g(r + \tau) - g\|^2 + \|\psi_1(r + \tau) - \psi_1\|^2 \, dr
\]
which is bounded as \(\tau \to -\infty\) due to (85). Hence, \(\int_{0}^{T} (1 + \|u(t + \tau)\|_{L^2}^2) \, dt\) is bounded as desired. Since \(\|u(0)\|_{L^2(\mathbb{R}^n)} = \|u_\tau - \hat{u}_0\|_{L^2(\mathbb{R}^n)} \to 0\) as \(\tau \to -\infty\), the desired result follows immediately.

Finally, we summarize the main results as follows.

**Theorem 2.** Suppose that assumption (4)–(9), (19), (76)–(80), and (83)–(85) hold. Then the stochastic non-autonomous system (3) has a \(\hat{\mathcal{D}}\)-pullback backward compact random attractor.
\[ A_\Delta = \{ A_\Delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \tilde{\mathcal{D}} \] such that it is backward asymptotically autonomous to the random attractor \( A_{\infty} = \{ A_{\infty}(\omega) : \omega \in \Omega \} \in \mathcal{D}_{\infty} \) in \( H^p(\mathbb{R}^n) \). More precisely,

\[
\lim_{\tau \to -\infty} \text{dist}_{H^p(\mathbb{R}^n)}(A_\Delta(\tau, \omega), A_{\infty}(\omega)) = 0, \quad \mathbb{P} \text{-a.s. } \omega \in \Omega. \tag{93}
\]

**Proof.** Denote by

\[
\Omega_1 = \{ \omega \in \Omega : \lim_{\tau \to -\infty} \text{dist}_{H^p(\mathbb{R}^n)}(A_\Delta(\tau, \omega), A_{\infty}(\omega)) = 0 \}.
\]

Then it suffices to prove \( \mathbb{P}(\Omega_1) = 1 \). Let \( \Omega_2 = \Omega \setminus \Omega_1 \). If \( \mathbb{P}(\Omega_1) < 1 \), then \( \Omega_2 \neq \emptyset \), and hence there exists \( \omega \in \Omega_2 \). Suppose that the backward upper semi-continuity (93) is not true, then there is a \( \varepsilon > 0 \) and \( 0 > \tau_n \downarrow -\infty \) such that

\[
\text{dist}_{H^p(\mathbb{R}^n)}(A_\Delta(\tau_n, \omega), A_{\infty}(\omega)) \geq 4\varepsilon \text{ for all } n \in \mathbb{N}.
\]

We can take a \( x_n \in A(\tau_n, \omega) \) such that

\[
d_{H^p(\mathbb{R}^n)}(x_n, A_{\infty}(\omega)) \geq \text{dist}_{H^p(\mathbb{R}^n)}(A_\Delta(\tau_n, \omega), A_{\infty}(\omega)) - \varepsilon \geq 3\varepsilon. \tag{94}
\]

Similar to the Lemma 5, we also find that \( A_{\infty} \in \mathcal{D}_{\infty} \) and the attraction of \( A_{\infty} \), then there exists a \( n_0 \in \mathbb{N} \) such that \( \tau_{n_0} \leq \tau_0 \) and

\[
\text{dist}_{H^p(\mathbb{R}^n)}(\Phi_{\infty}(|\tau_{n_0}|, \theta_{\tau_{n_0}} \omega), \mathcal{A}_{\Delta_{\tau_{n_0}}} (\theta_{\tau_{n_0}} \omega), A_{\infty}(\omega)) < \varepsilon.
\]

Furthermore, by the continuity of \( \Phi_{\infty} : H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n) \), we see that

\[
\text{dist}_{H^p(\mathbb{R}^n)}(\Phi_{\infty}(|\tau_0|, \theta_{\tau_0} \omega)\mathcal{A}_{\Delta_{\tau_0}} (\theta_{\tau_0} \omega), A_{\infty}(\omega)) < \varepsilon. \tag{95}
\]

By the invariance of random attractor \( A_{\Delta} \), we have

\[
\mathcal{A}_{\Delta}(\tau_n, \omega) = \Phi(|\tau_{n_0}|, \tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}} \omega)A_{\Delta_{\tau_{n_0}}}(\tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}} \omega).
\]

Hence, there exists a \( y_n \in A_{\Delta}(\tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}} \omega) \) such that

\[
x_n = \Phi(|\tau_{n_0}|, \tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}} \omega)y_n, \text{ for any } x_n \in A_{\Delta}(\tau_n, \omega).
\]

If \( n \geq n_0 \), then \( \tau_n - |\tau_{n_0}| \leq \tau_n \leq \tau_0 \) and thus

\[
\{ y_n : n \geq n_0 \} \subset \bigcup_{\tau \leq \tau_0} A_{\Delta}(\tau, \theta_{\tau_0} \omega) = A_{\Delta_{\tau_0}}(\theta_{\tau_0} \omega).
\]

By Theorem 1, \( A_{\Delta} \) is backward compact, then \( A_{\Delta_{\tau_0}}(\theta_{\tau_0} \omega) \) is a pre-compact set, which means that there is a subsequence \( \{ y_{n_k} \} \) of \( \{ y_n \} \) such that

\[
y_{n_k} \to y \text{ in } H^p(\mathbb{R}^n) \text{ as } k \to \infty \text{ with } y \in A_{\Delta_{\tau_0}}(\theta_{\tau_0} \omega).
\]

Note that \( \theta_{\tau_{n_k}} \omega \in \theta_{\tau_{n_0}} \Omega \subseteq \Omega \) due to the \( \{ \theta_t \} \) \( t \in \mathbb{R} \)-invariance of \( \Omega \). Applying the backward asymptotically autonomous (86) at sample \( \theta_{\tau_{n_0}} \omega \) for \( t = |\tau_{n_0}| \) and \( \tau = \tau_{n_k} - |\tau_{n_0}| \to -\infty \) as \( k \to \infty \), we have

\[
\| x_{n_k} - \Phi_{\infty}(|\tau_{n_0}|, \theta_{\tau_{n_0}} \omega)y \|_{H^p(\mathbb{R}^n)}
\]

\[
= \| \Phi(|\tau_{n_0}|, \tau_{n_k} - |\tau_{n_0}|, \theta_{\tau_{n_0}} \omega)y_{n_k} - \Phi_{\infty}(|\tau_{n_0}|, \theta_{\tau_{n_0}} \omega)y \|_{H^p(\mathbb{R}^n)} \leq \varepsilon, \tag{96}
\]
if \( k \) is large enough. This together with (95), we obtain that

\[
d_{H^s(\mathbb{R}^n)}(x_{n_k}, \mathcal{A}_{\omega}(\omega))) \\
\leq ||x_{n_k} - \Phi_{\omega}(|\tau_{n_k}|, \theta_{n_k} \omega)y||_{H^s(\mathbb{R}^n)} + d_{H^s(\mathbb{R}^n)}(\Phi_{\omega}(|\tau_{n_k}|, \theta_{n_k} \omega)y, \mathcal{A}_{\omega}(\omega))) \\
\leq \varepsilon + \text{dist}_{H^s(\mathbb{R}^n)}(\Phi_{\omega}(|\tau_{n_k}|, \theta_{n_k} \omega), \mathcal{A}_{\mathcal{B}_k}(\theta_{n_k} \omega), \mathcal{A}_{\omega}(\omega))) \leq 2\varepsilon,
\]

which contradicts with (94). Hence, the backward upper semicontinuity (93) holds true. \( \square \)

6. Conclusions

In view of Theorem 1, we can see that fractional nonclassical diffusion equations driven by Wong–Zakai approximations in \( H^s(\mathbb{R}^n) \) with \( s \in (0, 1) \) exists in backward time-dependent uniform compact random attractors when the growth rate of nonlinearities have a subcritical range. Theorem 2 implies that the components of the random attractors of a non-autonomous dynamical system in time can converge to those of the random attractor of the limiting autonomous dynamical system (time-independent) in \( H^s(\mathbb{R}^n) \). In this paper, system (3) contains time-dependent nonlinear terms, so it is more complicated to determine the limit process (10), which differs from the earlier works.

We give an example for \( n = 2 \) and the nonlinearity \( f \) and forcing term \( g \), which satisfy all the assumptions in the paper.

Given \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^2 \), let

\[
f(t, x, u) = -a_1|u|^{p-2}u + \frac{\phi(t, x)}{1 + u^2} + \frac{\theta_0}{t_0},
\]

where \( a_1 > 0, r_0 > 0 \) and \( p > 2, \phi(t, x) = |t|^{-\theta_0} \phi_0(x) \) for some \( r > 0 \) and \( \phi_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \). Then, we can verify that

\[
f(t, x, u)u = -a_1|u|^p + \frac{\phi(t, x)}{1 + u^2} \leq -a_1|u|^p + \phi(t, x),
\]

\[
|f(t, x, u)| \leq a_1|u|^{p-1} + \phi(t, x),
\]

\[
\frac{\partial f}{\partial u}(t, x, u) \leq \phi(t, x).
\]

So we find that \( f \) and \( g \) satisfy all the assumptions with

\[
\psi_1(t, x) = \psi_2(t, x) = \psi_3(t, x) = \phi(t, x), \text{ and } a_1 = a_2.
\]

In this case, Theorems 1 and 2 are clearly true.

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