The existence of global weak solutions to the shallow water wave model with moderate amplitude

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Abstract: The existence of global weak solutions to the shallow water model with moderate amplitude, which is firstly introduced in Constantin and Lannes's work (2009), is investigated in the space $C([0, \infty) \times \mathbb{R}) \cap L^{\infty}((0, \infty); H^2(\mathbb{R}))$ without the sign condition on the initial value by employing the limit technique of viscous approximation. A new one-sided lower bound and the higher integrability estimate act a key role in our analysis.

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1. Introduction

In this paper, we consider the following model for shallow water wave with moderate amplitude

$$\begin{equation}
\begin{cases}
  u_t + u_x + \frac{3}{2} \sigma u u_x + \kappa \sigma^2 u^2 u_x + \kappa \sigma^3 u^3 u_x + \frac{\sigma}{12} (u_{xxx} - u_{txx}) \\
  = - \frac{7}{24} \sigma (u^2 u_{xxx} + 2 u_x u_{xx}), \\
  u(0, x) = u_0.
\end{cases}
\end{equation}
$$

system (1) is firstly found in Constantin and Lannes (2009) as a model for the evolution of the free surface $u$. Here the function $u = u(t, x)$, $\sigma$ and $\kappa$ are parameters, $\mu > 0$ is shallowness parameter, $\sigma > 0$ is amplitude parameter (see Alvarez-Samaniego & Lannes, 2008; Constantin & Lannes, 2009; Mi & Mu, 2013). It is shown in Constantin (2011) that unlike KdV and Camassa–Holm equation, system (1) does not have a bi-Hamiltonian integrable structure. However, the equation possesses solitary wave profiles that resemble those of C–H (Constantin & Escher, 2007). Recently, Constantin and Lannes (2009) established the local well-posedness of system (1) for any initial data $u_0 \in H^{s+1}(\mathbb{R})$.
with $s > \frac{3}{2}$ and also claimed that if the maximal existence time is finite, then blow-up occurs in form of wave breaking. In Duruk Mutlubas (2013), the local well-posedness of system (1) is proved for initial data in $H^s$ with $s > \frac{3}{2}$ using Kato’s semigroup method for quasi-linear equations. Orbital stability and existence of solitary waves for system (1) was obtained in Duruk Mutlubas and Geyer (2013), Geyer (2012). Mi and Mu (2013) investigated the local well-posedness of system (1) in Besov space using Littlewood–Paley decomposition and transport equation theory, and proposed that if initial data $u_0$ is analytic its solutions are analytic. Moreover, persistence properties on strong solutions were also presented (see Mi & Mu, 2013).

One of the close relatives of the first equation of problem (1) is the rod wave equation (Dai, 1998; Dai & Huo, 2000)

$$u_t - u_{txx} + 3uu_x = \gamma (2u_x u_{xx} + uu_{xxx}), \quad t > 0, \quad x \in \mathbb{R},$$

where $\gamma \in \mathbb{R}$ and $u = u(t, x)$ stands for the radial stretch relative to a prestressed state in non-dimensional variables. Equation (2) is a model for finite-length and small-amplitude axial-radial deformation waves in the cylindrical compressible hyperelastic rods. Since Equation (2) was derived by Dai (1998), Dai and Huo (2000), many works have been carried out to investigate its dynamic properties. In Constantin and Strauss (2000), Constantin and Strauss studied the Cauchy problem of the rod equation on the line (nonperiodic case), where the local well-posedness and blow-up solutions were discussed. Moreover, they also proved the stability of solitary waves for the equation (see Constantin & Strauss, 2000). Later, Yin (2003, 2004) and Hu and Yin (2010) discussed the smooth solitary waves and blow-up solutions. Zhou (2006), the precise blow-up scenario and several blow-up results of strong solutions to the rod equation on the circle (periodic case) were presented. For other techniques to study the problems relating to various dynamic properties of other shallow water wave equations, the reader is referred to Coclite, Holden, and Karlsen (2005), Yan, Li, and Zhang (2014), Fu, Liu, and Qu (2012), Guo and Wang (2014), Himonas, Misiolek, Ponce, and Zhou (2007), Holden and Raynaud (2009), Li and Olver (2000), Qu, Fu, and Liu (2014), Lai (2013) and the reference therein.

Xin and Zhang (2000) use the limit method of viscous approximations to analyze the existence of global weak solutions for Equation (2) with $\gamma = 1$ (Namely, Camassa–Holm equation). Motivated by the desire to extend the works (Xin & Zhang, 2000), the objective of this paper was to establish the existence of global weak solutions for the system (1) in the space $C([0, \infty) \times \mathbb{R}) \cap L^\infty([0, \infty); H^s(\mathbb{R}))$ under the assumption $u_0(x) \in H^s(\mathbb{R})$. Following the idea in Xin and Zhang (2000), the limit method of viscous approximations is employed to establish the existence of the global weak solution for system (1). In our analysis, a new one-sided lower bound (see Lemma 3.4) and the higher integrability estimate (see Lemma 3.3), which ensure that weak convergence of $q_\epsilon$ is equal to strong convergence, play a crucial role in establishing the existence of global weak solutions.

The rest of this paper is as follows. The main result is presented in Section 2. In Section 3, we state the viscous problem and give a corresponding result. Strong compactness of the derivative of viscous approximations is obtained in Section 4. Section 5 completes the proof of the main result.

2. The main results

Using the Green function $G(x) = \sqrt{\frac{\pi}{\mu}} e^{-\sqrt{\frac{\pi}{\mu}} |x|}$, we have $(1 - \mu \frac{d^2}{dx^2})^{-1} f = G(x) * f$ for all $f \in L^2$, and $G * (\frac{\mu}{48} u_{xx}^3) = u$, where we denote by * the convolution. Then we can rewrite system (1) as follows

$$\begin{cases}
    u_t - (1 + \frac{7}{4} \sigma u) u_x + \frac{\mu}{12} \frac{d^2 u}{dx^2} - \frac{7}{48} \sigma \mu u^3 = 0,
    \\
    u(0, x) = u_0,
\end{cases}$$

which is also equivalent to the elliptic–hyperbolic system

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\[
\begin{cases}
    u_t - \left( 1 + \frac{7}{2} \sigma u \right) u_x + \frac{\partial}{\partial x} \Lambda^{-2} \left[ 2u + \frac{\sigma}{2} u^2 + \frac{\sigma}{3} u^3 + \frac{\sigma}{4} u^4 - \frac{7}{48} \sigma \mu u_x^2 \right] = 0, \\
    \frac{\partial}{\partial x} = \partial_x \Lambda^{-2} \left[ 2u + \frac{\sigma}{2} u^2 + \frac{\sigma}{3} u^3 + \frac{\sigma}{4} u^4 - \frac{7}{48} \sigma \mu u_x^2 \right], \\
    u(0, x) = u_0(x),
\end{cases}
\]

(4)

where \( \Lambda = (1 - \frac{\sigma}{12} x^2)^{1/2} \).

Now we give the definition of a weak solution to the Cauchy problem (3) or (4).

**Definition 2.1** A continuous function \( u : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) is said to be a global weak solution to the Cauchy problem (4) if

(i) \( u \in C([0, \infty) \times \mathbb{R}) \cap L^\infty([0, \infty); H^4(\mathbb{R})) \);

(ii) \( \| u(t, \cdot) \|_{L^\infty(\mathbb{R})} \leq \| u_0 \|_{L^\infty(\mathbb{R})} \) for every \( t > 0 \);

(iii) \( u = u(t, x) \) satisfies (4) in the sense of distributions and takes on the initial value pointwise.

The existence of global weak solutions to the Cauchy problem (4) will be established by proving compactness of a sequence of smooth functions \( \{ u_j \}_{j=0}^\infty \) solving the following viscous problem

\[
\begin{cases}
    \frac{\partial u}{\partial t} - \left( 1 + \frac{7}{2} \sigma u \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \Lambda^{-2} \left[ 2u + \frac{\sigma}{2} u^2 + \frac{\sigma}{3} u^3 + \frac{\sigma}{4} u^4 - \frac{7}{48} \sigma \mu u_x^2 \right] = 0, \\
    \frac{\partial}{\partial x} = \partial_x \Lambda^{-2} \left[ 2u + \frac{\sigma}{2} u^2 + \frac{\sigma}{3} u^3 + \frac{\sigma}{4} u^4 - \frac{7}{48} \sigma \mu u_x^2 \right], \\
    u_j(0, x) = u_{j, 0}(x).
\end{cases}
\]

(5)

The main result of the present paper is collected in the following theorem.

**Theorem 2.2** Assume that \( u_0(x) \in H^4(\mathbb{R}) \). Then the Cauchy problem (4) has a global weak solution \( u(t, x) \) in the sense of Definition 2.1. In addition, there is a positive constant \( C = C(\| u_0 \|_{H^4(\mathbb{R})}) \) independent of \( \epsilon \), such that

\[
- \frac{4}{7 \sigma} - \sqrt{\frac{4C}{7 \sigma}} \leq \frac{\partial u(t, x)}{\partial x}, \quad \text{for} \quad (t, x) \in [0, T) \times \mathbb{R}.
\]

(6)

### 3. Viscous approximations

Defining

\[
\phi(x) = \begin{cases} 
    e^{-\frac{1}{x^2}}, & |x| < 1, \\
    0, & |x| \geq 1,
\end{cases}
\]

(7)

and setting the mollifier \( \phi_\epsilon(x) = \epsilon^{-\frac{1}{2}} \phi(\epsilon^{-\frac{1}{2}} x) \) with \( 0 < \epsilon < \frac{1}{4} \) and \( u_{\epsilon, 0} = \phi_\epsilon * u_0 \), we know that \( u_{\epsilon, 0} \in C^\infty \) for any \( u_0 \in H^4 \), \( s > 0 \) (see Lai & Wu, 2010).

In fact, suitably choosing the mollifier, we have

\[
\| u_{\epsilon, 0} \|_{H^4(\mathbb{R})} \leq \| u_0 \|_{H^4(\mathbb{R})}, \quad \text{and} \quad u_{\epsilon, 0} \to u_0 \quad \text{in} \quad H^4(\mathbb{R}).
\]

(8)

Differentiating the first equation of problem (5) with respect to variable \( x \) and letting \( q_\epsilon(t, x) = \frac{\partial u}{\partial x} \), we have
\[
\frac{\partial q_1}{\partial t} - \frac{\partial q_1}{\partial x} = \frac{12}{\mu} \left(2u_i + \frac{5}{2} \sigma u_i^2 + \frac{1}{3} \sigma^2 u_i^3 + \frac{k}{4} \sigma^3 u_i^4\right)
\]
\[
- \frac{12}{\mu} \left[2u_i + \frac{5}{2} \sigma u_i^2 + \frac{1}{3} \sigma^2 u_i^3 + \frac{k}{4} \sigma^3 u_i^4 - \frac{7}{4\sigma} \frac{\partial^2 q_1}{\partial x^2}\right]
\]
\[
= Q(t, x).
\]

The starting point of our analysis is the following well-posedness result for problem (5).

**Lemma 3.1** Assume \( u_0 \in H^1(\mathbb{R}) \). For any \( l \geq 2 \), there exists a unique solution \( u_l \in C([0, \infty); H^l(\mathbb{R})) \) to the Cauchy problem (5). Moreover, for any \( t > 0 \), it holds that

\[
\int_{\mathbb{R}} \left(u_l^2 + \frac{\mu}{12}(\frac{\partial u_l}{\partial x})^2\right) dx + 2\epsilon \int_{\mathbb{R}} \left((\frac{\partial u_l}{\partial x})^2 + \frac{\mu}{12} \frac{\partial^2 u_l}{\partial x^2}\right)(s, x) dx ds
\]
\[
= \int_{\mathbb{R}} \left(u_{l,0}^2 + \frac{\mu}{12}(\frac{\partial u_{l,0}}{\partial x})^2\right) dx < (1 + \frac{\mu}{12}) \left(u_{l,0}^2 + \frac{1}{\mu} \frac{\partial^2 u_{l,0}}{\partial x^2}\right) dx
\]
\[
= (1 + \frac{\mu}{12}) \lVert u_{l,0} \rVert^2_{H^1(\mathbb{R})}.
\]

**Proof** For any \( l \geq 2 \) and \( u_0 \in H^1(\mathbb{R}) \), we have \( u_{l,0} \in C([0, \infty); H^l(\mathbb{R})) \). From Theorem 2.1 in Coclite et al. (2005), we infer that problem (5) has a unique solution \( u_l \in C([0, \infty); H^l(\mathbb{R})) \).

The first equation of (5) is rewritten as

\[
\frac{\partial u_l}{\partial t} + \frac{\partial u_l}{\partial x} + 3 \sigma u_l \frac{\partial u_l}{\partial x} + \frac{1}{2} \sigma^2 u_l^2 \frac{\partial u_l}{\partial x} + \frac{k}{3} \sigma^3 u_l^3 \frac{\partial u_l}{\partial x} + \frac{\mu}{12} \left(\frac{\partial^3 u_l}{\partial x^3} - \frac{\partial^2 u_l}{\partial x^2}\right)
\]
\[
+ \frac{7}{24} \sigma u_l \frac{\partial^3 u_l}{\partial x^3} + 2 \frac{\partial u_l}{\partial x} \frac{\partial^2 u_l}{\partial x^2} = \epsilon \frac{\partial^2 u_l}{\partial x^2} - \frac{\mu}{12} \frac{\partial^4 u_l}{\partial x^4}.
\]

Multiplying (11) by \( u_l \), we derive that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left(u_l^2 + \frac{\mu}{12}(\frac{\partial u_l}{\partial x})^2\right) dx + \epsilon \int_{\mathbb{R}} \left((\frac{\partial u_l}{\partial x})^2 + \frac{\mu}{12} \frac{\partial^2 u_l}{\partial x^2}\right)(s, x) dx ds = 0,
\]

which finishes the proof.

For simplicity, in this paper, let \( c \) denote any positive constant which is independent of the parameter \( \epsilon \). From Lemma 3.1, we have

\[
\lVert u_l \rVert_{C^0(\mathbb{R})} \leq C \lVert u_{l,0} \rVert_{H^l(\mathbb{R})} \leq C \lVert u_{l,0} \rVert_{H^l(\mathbb{R})} \leq C \lVert u_0 \rVert_{H^l(\mathbb{R})}.
\]

**Lemma 3.2** For \( 0 < t < T \), there exists a positive constant \( C = C(\lVert u_0 \rVert_{H^l(\mathbb{R})}) \), independent of \( \epsilon \), such that

\[
\lVert P(t, \cdot) \rVert_{L^\infty(\mathbb{R})} \leq C,
\]
\[
\lVert P(t, \cdot) \rVert_{L^1(\mathbb{R})} \leq C,
\]
\[
\lVert P(t, \cdot) \rVert_{L^2(\mathbb{R})} \leq C
\]

and

\[
\lVert P(t, \cdot) \rVert_{L^1(\mathbb{R})} \leq C
\]
\[ \| \frac{\partial P_i(t, \cdot)}{\partial x} \|_{L^\infty(\mathbb{R})} \leq C, \quad \| \frac{\partial P_j(t, \cdot)}{\partial x} \|_{L^\infty(\mathbb{R})} \leq C, \quad \| \frac{\partial P_k(t, \cdot)}{\partial x} \|_{L^\infty(\mathbb{R})} \leq C, \quad \| Q_i(t, \cdot) \|_{L^\infty(\mathbb{R})} \leq C, \]

where \( u_i = u_i(t, x) \) is the unique solution of (5) and

\[ Q_i(t, \cdot) = \frac{12}{\mu} \left[ 2u_i + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 - \frac{7}{48} \sigma \mu q^2 \right], \]

Proof In the proof of this lemma, we will use the identity

\[ \Lambda^{-2} g(x) = \sqrt{\frac{3}{\mu}} \int_{\mathbb{R}} e^{-\sqrt{\frac{\nu}{\mu}} |x-y|} g(y) dy \quad g(x) \in L^2(\mathbb{R}). \]

For simplicity, setting \( u_i(t, x) = u(t, x) \), we have

\[ P_i(t, \cdot) = \frac{12}{\mu} \Lambda^{-2} \left[ 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 - \frac{7}{48} \sigma \mu q^2 \right], \]

and

\[ \frac{\partial P_i(t, \cdot)}{\partial x} = \frac{12}{\mu} \partial_x \Lambda^{-2} \left[ 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 - \frac{7}{48} \sigma \mu q^2 \right]. \]

Note that \( \int_{\mathbb{R}} e^{-\sqrt{\frac{\nu}{\mu}} |x|} dx = \sqrt{\frac{\nu}{3}} \) for \( x \in \mathbb{R} \). Using (22), one has

\[ |P_i| = \frac{12}{\mu} \left| \Lambda^{-2} \left[ 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 - \frac{7}{48} \sigma \mu q^2 \right] \right| \]

\[ \leq \frac{12}{\mu} \sqrt{\frac{3}{\mu}} \int_{\mathbb{R}} e^{-\sqrt{\frac{\nu}{\mu}} |x-y|} \left| 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 \right| dy \]

\[ + \frac{7\sigma}{4} \sqrt{\frac{3}{\mu}} \int_{\mathbb{R}} e^{-\sqrt{\frac{\nu}{\mu}} |x-y|} q^2 dy \]

\[ \leq C \left( \| u_0 \|_{L^\infty(-\infty, \infty)} + \| u_0 \|_{L^2(-\infty, \infty)}^2 + \| u_0 \|_{L^3(-\infty, \infty)}^3 + \| u_0 \|_{L^6(-\infty, \infty)}^6 \right), \]

which proves (14). \( \square \)

In view of Lemma 3.1 and Tonelli theorem, one has

\[ \left| \frac{7\sigma}{4} \Lambda^{-2}(q^2) \right| = \frac{7\sigma}{4} \sqrt{\frac{3}{\mu}} \int_{\mathbb{R}} e^{-\sqrt{\frac{\nu}{\mu}} |x-y|} q^2 dy \leq c \| u_0 \|_{L^6(-\infty, \infty)}^2, \]

and then, we get

\[ \int_{\mathbb{R}} \left| \frac{7\sigma}{4} \Lambda^{-2}(q^2) \right| dx = \frac{7\sigma}{4} \sqrt{\frac{3}{\mu}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\sqrt{\frac{\nu}{\mu}} |x-y|} q^2 dy dx \]

\[ \leq c \| u_0 \|_{L^6(-\infty, \infty)}^2, \]
Using the Tonelli theorem and the Hölder inequality, it holds

\[
\int_{\mathcal{R}} \left| \frac{12}{\mu} \Lambda^{-2} \left[ 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 \right] \right|^2 \, dx \\
\leq \frac{12}{\mu} \sqrt{\frac{3}{\mu}} \int_{\mathcal{R}} e^{-\sqrt{\frac{\mu}{2}}|x-y|} \left| 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 \right|^2 \, dy \\
\leq c \left( \|u_0\|_{H^1(R)}^2 + \|u_0\|_{H^1(R)}^3 + \|u_0\|_{H^1(R)}^4 + \|u_0\|_{H^1(R)}^6 \right).
\]

(28)

Making use of (27) and (28), we complete the proof of (15).

From (26)–(27) and the Hölder inequality, we have

\[
\int_{\mathcal{R}} |\frac{7σ}{4} \Lambda^{-2}(q^2)|^2 \, dx \leq \frac{7σ}{4} \Lambda^{-2}\|q^2\|_{L^2} \frac{7σ}{4} \Lambda^{-2}(q^2) \|_{L^2} \\
\leq c \|u_0\|_{H^1(R)}^2.
\]

(29)

Hence,

\[
\|\frac{7σ}{4} \Lambda^{-2}(q^2)\|_{L^2} \leq c \|u_0\|_{H^1(R)}^2.
\]

(30)

By (25) and (28), one has

\[
\int_{\mathcal{R}} \left| \frac{12}{\mu} \Lambda^{-2} \left[ 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 \right] \right|^2 \, dx \\
\leq c \left( \|u_0\|_{H^1(R)}^2 + \|u_0\|_{H^1(R)}^3 + \|u_0\|_{H^1(R)}^4 + \|u_0\|_{H^1(R)}^6 \right).
\]

(31)

From (30) and (31), we deduce (16).

On the other hand, from (24), we derive that

\[
\frac{\partial P}{\partial x} = \frac{12}{\mu} \Lambda^{-2} \left[ 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 - \frac{7}{48} \sigma \mu q^2 \right]
\]

\[
= \frac{12}{\mu} \sqrt{\frac{3}{\mu}} \int_{\mathcal{R}} e^{-\sqrt{\frac{\mu}{2}}|x-y|} \left[ 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 - \frac{7}{48} \sigma \mu q^2 \right] \, dy
\]

\[
= \frac{36}{\mu^2} \int_{\mathcal{R}} e^{-\sqrt{\frac{\mu}{2}}|x-y|} \left[ 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 \right]
\]

\[
- \frac{7}{48} \sigma \mu q^2 |\text{sign}(y-x)| \, dy.
\]

(32)

Inequalities (17), (18), and (19) are direct consequences of (25), (27), (28), (30), and (31).

Finally, note that

\[
\left| \frac{12}{\mu} \left[ 2u + \frac{5}{2} \sigma u^2 + \frac{1}{3} \sigma^2 u^3 + \frac{k}{4} \sigma^3 u^4 \right] \right|
\]

\[
\leq \frac{12}{\mu} \left( 2|u| + \frac{5}{2} \sigma |u|^2 + \frac{1}{3} \sigma^2 |u|^3 + \frac{k}{4} \sigma^3 |u|^4 \right)
\]

\[
\leq c \left( \|u_0\|_{H^1(R)}^2 + \|u_0\|_{H^1(R)}^3 + \|u_0\|_{H^1(R)}^4 + \|u_0\|_{H^1(R)}^6 \right).
\]

(33)
Using (25), we obtain (20).

**Lemma 3.3** Let $0 < \phi < 1$, $T > 0$, and $a, b \in \mathbb{R}$, $a < b$. Then there exists a positive constant $C_1$ depending only on $\|u_{0}\|_{L^p(\Omega)}$, $\phi$, $T$, $a$ and $b$, but independent of $\varepsilon$, such that

$$
\int_{0}^{T} \int_{a}^{b} \frac{\partial u}{\partial x}(t,x)^{2+q} dt dx \leq C_1,
$$

(34)

where $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of (5).

**Proof** The proof of Lemma 3.3 is similar to that of Lemma 4.1 in Xin and Zhang (2000). Here, we omit its proof. □

**Lemma 3.4** For an arbitrary $T > 0$, the following estimate on the first-order spatial derivative holds

$$
-\frac{4}{7\sigma} - \sqrt{\frac{4C}{7\sigma}} \leq \frac{\partial u(t,x)}{\partial x}, \quad \text{for} \quad (t,x) \in (0,T) \times \mathbb{R}.
$$

(35)

**Proof** Using (9), we get

$$
\frac{\partial (-q_{\varepsilon})}{\partial t} - \frac{\partial (-q_{\varepsilon})}{\partial x} - \varepsilon \frac{\partial^2 (-q_{\varepsilon})}{\partial x^2} + \frac{7}{4} (-q_{\varepsilon})^2 - \frac{7}{2} \sigma u \frac{\partial (-q_{\varepsilon})}{\partial x}
$$

$$
= -Q_{\varepsilon}(t,x) \leq C.
$$

(36)

Let $f = f(t)$ be the solution of

$$
\frac{df}{dt} + \frac{7}{4} \sigma f^2 = C, \quad t > 0, \quad f(0) = \|\frac{\partial u_{0}}{\partial x}\|_{L^\infty}.
$$

(37)

Since $f = f(t)$ is a supersolution of the parabolic equation (36) with initial value $u_{\varepsilon,0}$ due to the comparison principle for parabolic equations, we get

$$
-q_{\varepsilon}(t,x) \leq f(t,x).
$$

Consider the function $F(t) = \frac{a}{\sqrt{t}} + \sqrt{\frac{4C}{7\sigma}}$, observing that $\frac{df}{dt} + \frac{7}{4} \sigma f^2 - C = \frac{a}{\sqrt{t}} + \sqrt{\frac{4C}{7\sigma}} > 0$ for any $t > 0$ and using the comparison principle for ordinary differential equations, we have $f(t) \leq F(t)$ for all $t > 0$. It completes the proof. □

**Lemma 3.5** There exists a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ tending to zero and a function $u \in L^\infty([0,\infty); H^1(\mathbb{R})) \cap H^1([0,T] \times \mathbb{R})$, for each $T \geq 0$, such that

$$
u_{\varepsilon_j} \rightharpoonup u \quad \text{in} \quad H^1([0,T] \times \mathbb{R}), \quad \text{for each} \quad T \geq 0,
$$

(38)

$$
u_{\varepsilon_j} \rightarrow u \quad \text{in} \quad L^\infty([0,T] \times \mathbb{R}),
$$

(39)

where $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of (5).

**Proof** For fixed $T > 0$, using Lemmas 3.1 and 3.3, and

$$
\frac{\partial u}{\partial t} - \left(1 + \frac{7}{2} \sigma u \right) \frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2},
$$

we obtain
\[ \| \frac{\partial u}{\partial t} \|_{L^2([0,T]\times \mathbb{R})} \leq \sqrt{\frac{12T}{\mu}} \| u_0 \|_{L^2(\mathbb{R})} + \sqrt{\frac{42\sigma T}{\mu}} \| u_0 \|_{H^s(\mathbb{R})} + \sqrt{CT} + \sqrt{\frac{6\sigma}{\mu}} \| u_0 \|_{L^2(\mathbb{R})}. \]

Hence \( \{u_j\} \) is uniformly bounded in \( L^\infty([0,\infty); H^1(\mathbb{R})) \cup H^1([0,T] \times \mathbb{R}) \) and (38) follows.

Observe that, for each \( 0 \leq s, t \leq T \),
\[
\| u_j(t, \cdot) - u_j(s, \cdot) \|_{L^2(\mathbb{R})} \leq \sqrt{t - s} \int_s^t \left( \frac{\partial u}{\partial t}(r, x) \right)^2 dx \leq \sqrt{t - s} \int_0^T \left( \frac{\partial u}{\partial t}(r, x) \right)^2 dx.
\]

Moreover, \( \{u_j\} \) is uniformly bounded in \( L^\infty([0,\infty); H^1(\mathbb{R})) \) and \( H^1(\mathbb{R}) \subset L^\infty_{\text{loc}}(\mathbb{R}) \subset L_{\text{loc}}^2(\mathbb{R}) \). Using the results in Coclite et al. (2005), we know that (39) holds.

**Lemma 3.6** There exists a sequence \( \{e_j\}_{j=1}^\infty \) tending to zero and a function \( P \in L^\infty([0,\infty) \times \mathbb{R}) \) such that for each \( 0 < p < \infty \),
\[
P_j \to P \quad \text{strongly in} \quad L^p_{\text{loc}}([0,\infty) \times \mathbb{R}).
\]

**Proof** Using Lemma 3.2, we have the existence of pointwise convergence subsequence \( P_j \) which is uniformly bounded in \( L^\infty([0,\infty) \times \mathbb{R}) \). Inequalities (15) and (16) derive that (42) holds.

Throughout this paper we use overbars to denote weak limits.

**Lemma 3.7** There exists a sequence \( \{e_j\}_{j=1}^\infty \) tending to zero and two functions \( q \in L^\infty_{\text{loc}}([0,\infty) \cap \mathbb{R}) \), \( q^2 \in L^\infty_{\text{loc}}([0,\infty) \cap \mathbb{R}) \) such that
\[
q_j \to q \quad \text{in} \quad L^\infty_{\text{loc}}([0,\infty) \cap \mathbb{R}), \quad q_j^2 \to q^2 \quad \text{in} \quad L^\infty_{\text{loc}}([0,\infty) \cap L^2(\mathbb{R})),
\]

for each \( 1 < p < 3 \) and \( 1 < r < \frac{3}{2} \). Moreover,
\[
q^j(t, x) \leq q^2(t, x) \quad \text{for almost every} \quad (t, x) \in [0, T] \times \mathbb{R}
\]

and
\[
\frac{\partial u}{\partial x} = q \quad \text{in the sense of distributions on} \quad [0, T] \times \mathbb{R}.
\]

**Proof** Equations (43) and (44) are direct consequences of Lemmas 3.1. and 3.3. Inequality (45) is valid because of the weak convergence in (44). Finally, (46) is a consequence of definition of \( q_j \), Lemma 3.5. and (43).

In the following, for notational convenience, we replace the sequence \( \{u_j\}_{j=1}^\infty \) \( \{q_j\}_{j=1}^\infty \) and \( \{Q_j\}_{j=1}^\infty \) by \( \{u_j\}_{j>\alpha} \) \( \{q_j\}_{j>\alpha} \) and \( \{Q_j\}_{j>\alpha} \) separately.

Using (43), we conclude that for any convex function \( \eta \in C^1(\mathbb{R}) \) with \( \eta \) bounded, Lipschitz continuous on \( \mathbb{R} \) and any \( 1 < p < 3 \) we get
\[
\eta(q_j) \to \eta(q) \quad \text{in} \quad L^p_{\text{loc}}([0,\infty) \times \mathbb{R})
\]

and
\( \eta(q_s) \to \overline{\eta(q)} \) in \( L^\infty_{\text{loc}}([0, \infty); L^2(\mathbb{R})) \).

\[ \frac{\partial}{\partial t} \eta(q_s) - \frac{\partial}{\partial x} \eta(q_s) - \varepsilon \frac{\partial^2}{\partial x^2} \eta(q_s) + \frac{\partial^2}{\partial x^2} (\frac{7 \sigma}{2} u \eta(q_s)) \\
+ \frac{7 \sigma}{2} q_s \eta(q_s) = \frac{7 \sigma}{4} q_s^2 \eta'(q_s) + Q \eta'(q_s). \]

(48)

Multiplying Equation (9) by \( \eta'(q_s) \) yields

\[ \frac{\partial}{\partial t} \eta(q_s) - \frac{\partial}{\partial x} \eta(q_s) - \varepsilon \frac{\partial^2}{\partial x^2} \eta(q_s) + \frac{\partial^2}{\partial x^2} (\frac{7 \sigma}{2} u \eta(q_s)) \\
+ \frac{7 \sigma}{2} q_s \eta(q_s) = \frac{7 \sigma}{4} q_s^2 \eta'(q_s) + Q \eta'(q_s). \]

(49)

**Lemma 3.8** For any convex \( \eta \in C^1(\mathbb{R}) \) with \( \eta' \) bounded, Lipschitz continuous on \( \mathbb{R} \), it holds that

\[ \frac{\partial}{\partial t} \eta(q_s) - \frac{\partial}{\partial x} \eta(q_s) - \varepsilon \frac{\partial^2}{\partial x^2} \eta(q_s) + \frac{\partial^2}{\partial x^2} (\frac{7 \sigma}{2} u \eta(q_s)) \\
\leq \frac{7 \sigma}{4} q_s^2 \eta'(q_s) + O \eta'(q_s), \]

in the sense of distributions on \([0, T] \times \mathbb{R}\). Here \( \eta(q_s) \) and \( q_s^2 \eta'(q_s) \) denote the weak limits of \( q_s \eta(q_s) \) and \( q_s^2 \eta'(q_s) \) in \( L^\infty_{\text{loc}}([0, \infty) \times \mathbb{R}) \), for \( 1 < r < 2 \), respectively.

**Proof** In (49), by the convexity of \( \eta \), Lemmas 3.5–3.7, sending \( \varepsilon \to 0 \), gives rise to the desired result.

Remark 3.9 We know that

\[ q = q_+ + q_- = \overline{q}_+ + \overline{q}_-, \quad q^2 = (q_+)^2 + (q_-)^2, \quad \overline{q}^2 = (\overline{q}_+)^2 + (\overline{q}_-)^2 \]

almost everywhere in \((\lbrack 0, \infty \rbrack \times \mathbb{R})\), where \( \xi_+ = \xi_{\lbrack 0, +\infty \rbrack}(\xi), \xi_- = \xi_{\lbrack -\infty, 0 \rbrack}(\xi) \) for \( \xi \in \mathbb{R} \).

**Lemma 3.10** In the sense of distributions on \((\lbrack 0, \infty \rbrack \times \mathbb{R})\), it holds that

\[ \frac{\partial}{\partial t} q - \frac{\partial}{\partial x} q - \frac{7 \sigma}{2} \frac{\partial}{\partial x} (u q) = - \frac{7 \sigma}{4} q^2 + Q. \]

(50)

**Proof** Using Lemmas 3.5–3.8, (52) holds by sending \( \varepsilon \to 0 \) in (9).

**Lemma 3.11** For any \( \eta \in C^1(\mathbb{R}) \) with \( \eta' \in L^\infty(\mathbb{R}) \), it has

\[ \frac{\partial}{\partial t} \eta(q_s) - \frac{\partial}{\partial x} \eta(q_s) - \frac{7 \sigma}{2} \frac{\partial}{\partial x} (u \eta(q_s)) \\
= - \frac{7 \sigma}{2} q_s \eta(q_s) + \frac{7 \sigma}{2} \eta'(q_s)(q_s^2 - \frac{1}{2} \overline{q}^2) + \frac{\partial}{\partial x} \eta(q_s) + O \eta'(q_s) \]

in the sense of distributions on \((\lbrack 0, \infty \rbrack \times \mathbb{R})\).

**Proof** Let \( \{\omega_s\} \) be a family of mollifiers defined on \( \mathbb{R} \). Defined \( q_s(t, x) = (q(t, \cdot) * \omega_s)(x) \). The notation \( * \) is the convolution with respect to the \( x \) variable. 

Multiplying (52) by \( \eta'(q_s) \), it has

\[ \frac{\partial}{\partial t} \eta(q_s) - \frac{\partial}{\partial x} \eta(q_s) - \frac{7 \sigma}{2} \frac{\partial}{\partial x} (u \eta(q_s)) \\
= \eta'(q_s) \left( \frac{7 \sigma}{2} \frac{\partial}{\partial x} (u q) * \omega_s + \frac{\partial}{\partial x} q * \omega_s - \frac{7 \sigma}{4} \overline{q}^2 * \omega_s + Q * \omega_s \right) \\
= \eta'(q_s) \left( \frac{7 \sigma}{2} u q_s * \omega_s + \frac{7 \sigma}{2} q_s^2 * \omega_s \right) \]

(54)

\[ + \eta'(q_s) \left( q_s * \omega_s - \frac{7 \sigma}{4} \overline{q}^2 * \omega_s + Q * \omega_s \right) \]

and
\[-\frac{7\sigma}{2} \frac{\partial}{\partial x}(uw(q_x)) = -\frac{7\sigma}{2} q_x(q_x) - \frac{7\sigma}{2} w_x'(q_x)(q_x + \omega_x). \tag{55}\]

Using the boundedness of \(u, \eta'\) and letting \(\delta \to 0\) in the above two equations, we obtain (53).

Following the ideas in Xin and Zhang (2000), in next section we hope to improve the weak convergence of \(q\) in (43) to strong convergence, and then we have an existence result for problem (4). Since the measure \(\langle q^2 - q' \rangle \geq 0\), we will prove that if the measure is zero initially, then it will continue to be zero at all times \(t > 0\).

\[\square\]

4. Strong convergence of \(q_x\)

**Lemma 4.1** (see Coclite et al., 2005) Assume \(u_0 \in H^1(\mathbb{R})\). It holds that

\[
\lim_{t \to 0} \int_{\mathbb{R}} q^2(t, x) dx = \lim_{t \to 0} \int_{\mathbb{R}} \overline{q}(t, x) dx = \int_{\mathbb{R}} \left( \frac{du}{dx} \right)^2 dx. \tag{56}\]

**Lemma 4.2** (see Coclite et al., 2005) If \(u_0 \in H^1(\mathbb{R})\), for each \(M > 0\), it has

\[
\lim_{t \to 0} \int_{\mathbb{R}} \left( \eta'_0(q)(t, x) - \eta'_0(q(t, x)) \right) dx = 0, \tag{57}\]

where

\[
\eta_0(\xi) = \begin{cases} 
\frac{1}{2} \xi^2 - \frac{1}{2} (M - |\xi|)^2, & \text{if } |\xi| \leq M, \\
M|\xi| - \frac{1}{2} M^2, & \text{if } |\xi| > M,
\end{cases} \tag{58}\]

and \(\eta'_0(\xi) : = \eta_0(\xi) x_{(0, +\infty)} (\xi), \eta_0(\xi) : = \eta_0(\xi) x_{(-\infty, 0)} (\xi)\) for \(\xi \in \mathbb{R}\).

**Lemma 4.3** (see Coclite et al., 2005) Let \(M > 0\). Then for each \(\xi \in \mathbb{R}\)

\[
\begin{align*}
\eta_0(\xi) &= \frac{1}{2} \xi^2 - \frac{1}{2} (M - |\xi|)^2 x_{(0, +\infty)} (\xi), \\
\eta'_0(\xi) &= \eta_0(\xi) x_{(-\infty, 0)} (\xi), \\
\eta_0'(\xi) &= \frac{1}{2} \xi^2 - \frac{1}{2} (M + \xi)^2 x_{(-\infty, 0)} (\xi), \\
\eta'_0'(\xi) &= \eta_0'(\xi) x_{(0, +\infty)} (\xi).
\end{align*} \tag{59}\]

**Lemma 4.4** For almost all \(t > 0\), it holds that

\[
\begin{align*}
\int_{\mathbb{R}} [\eta_0'(q) - \eta_0'(q)] dx &= -\frac{7\sigma M}{4} \int_0^t \int_{\mathbb{R}} q(M - q) x_{(0, +\infty)} (q) dx dt \\
&\quad + \frac{7\sigma M}{4} \int_0^t \int_{\mathbb{R}} q(M - q) x_{(-\infty, 0)} (q) dx dt \\
&\quad + \int_0^t \left[ \frac{7\sigma M}{4} \left( q^2 - q' \right) dx ds \\
&\quad + \int_0^t \int_{\mathbb{R}} q(s, x) (\eta_0'(q) - \eta_0'(q)) dx ds. \tag{60}\end{align*}
\]

**Proof** For an arbitrary \(T > 0\), \(0 < t < T\). Using (50) minus (53), and the entropy \(\eta_0\) results in

\[
\begin{align*}
\frac{\partial}{\partial t} (\eta_0'(q) - \eta_0'(q)) &= -\frac{7\sigma}{2} (q \eta_0(q) - q \eta_0(q)) + \frac{7\sigma}{4} (q^2 - q'') x_{(0, +\infty)} (q) - q' x_{(-\infty, 0)} (q) \\
&\quad + \frac{\partial}{\partial x} (\eta_0'(q) - \eta_0'(q)) + Q((\eta_0')'(q) - (\eta_0')'(q)) \\
&\quad - \frac{7\sigma}{4} (q^2 - q') (\eta_0')'(q). \tag{61}\end{align*}
\]
Since $\eta'_u$ is increasing and $(\eta'_u)^2 \leq M$, from (45), we have
\[
0 \leq -\frac{7\sigma}{4}(q^2 - \overline{q}^2)(\eta'_u)'(q) \leq -\frac{7\sigma}{4}(q^2 - \overline{q}^2)M. \tag{62}
\]

It follows from Lemma 4.3 that
\[
q\eta'_u(q) - \frac{1}{2}q^2(\eta'_u)'(q) = -\frac{M}{2}q(M - q)\chi_{\{M,\infty\}}(q), \tag{63}
\]
From (61)–(63), we obtain the following result
\[
\frac{\partial}{\partial x}(\eta'_u(q) - \eta'_u(q)) - \frac{7\sigma}{2}\frac{\partial}{\partial x}((u(\eta'_u(q) - \eta'_u(q))) \\
\leq -\frac{7\sigma}{4}q(M - q)\chi_{\{M,\infty\}}(q) + \frac{7\sigma}{4}Mq(M - q)\chi_{\{M,\infty\}}(q) \\
+ \frac{\partial}{\partial x}(\eta'_u(q) - \eta'_u(q)) + Q((\eta'_u)'(q) - (\eta'_u)'(q)) \\
+ \frac{7\sigma}{4}(q^2 - \overline{q}^2). \tag{64}
\]
Integrating the resultant inequality over $(0, t) \times \mathbb{R}$ yields
\[
\int_{\mathbb{R}} (\eta'_u(q) - \eta'_u(q)))dx \leq \lim_{t \to 0} \int_{\mathbb{R}} (\eta'_u(q) - \eta'_u(q)))dx \\
- \frac{7\sigma}{4} \int_{0}^{t} \int_{\mathbb{R}} q(M - q)\chi_{\{M,\infty\}}(q)dxdt \\
+ \frac{7\sigma}{4} \int_{0}^{t} \int_{\mathbb{R}} q(M - q)\chi_{\{M,\infty\}}(q)dxdt \\
+ \int_{0}^{t} \int_{\mathbb{R}} \frac{7\sigma}{4}(q^2 - \overline{q}^2)dxds \\
+ \int_{0}^{t} \int_{\mathbb{R}} Q(s, x)(\eta'_u)'(q) - (\eta'_u)'(q))dxds. \tag{65}
\]
Using Lemma 4.2, we complete the proof. □

**Lemma 4.5** For almost all $t > 0$, it holds that
\[
\int_{\mathbb{R}} \frac{1}{2}(q^2 - \overline{q}^2)dx \leq \int_{0}^{t} \int_{\mathbb{R}} Q(q, q)dxds. \tag{66}
\]

**Proof** Let $M > 0$. Subtracting (53) from (50) and using entropy $\eta'_u$, we deduce
\[
\frac{\partial}{\partial x}(\eta'_u(q) - \eta'_u(q)) - \frac{7\sigma}{2}\frac{\partial}{\partial x}((u(\eta'_u(q) - \eta'_u(q))) \\
\leq -\frac{7\sigma}{2}(q\eta'_u(q) - q\eta'_u(q)) + \frac{7\sigma}{4}q^2(\eta'_u)'(q) - q^2(\eta'_u)'(q)) \\
+ \frac{\partial}{\partial x}(\eta'_u(q) - \eta'_u(q)) + Q((\eta'_u)'(q) - (\eta'_u)'(q)) \\
- \frac{7\sigma}{4}(q^2 - \overline{q}^2)(\eta'_u)'(q). \tag{67}
\]
Since $M \leq (\eta'_u)^2 \leq 0$, we get
\[
\frac{7\sigma M}{4}(q^2 - \overline{q}^2) \leq -\frac{7\sigma}{4}(q^2 - \overline{q}^2)(\eta'_u)'(q) \leq 0. \tag{68}
\]
It follows from Lemma 4.3 that
Integrating the above inequality over \( \int_{0}^{1} \), we have

\[
\begin{align*}
\frac{1}{2} q_\sigma^2(\eta_\sigma^\gamma(q)) &= -\frac{M}{2} q(M + q) \chi_{(-\infty, -M)}(q), \\
\frac{1}{2} q_\sigma^2(\eta_\sigma^\gamma(q)) &= -\frac{M}{2} q(M + q) \chi_{(-\infty, -M)}(q).
\end{align*}
\]

Using (35), we can find sufficiently large \( M > 0 \) such that \( q \geq -M \). Let \( \Omega_M = (\frac{4}{7 \sigma M^2}, \frac{1}{10}) \times \mathbb{R} \). Applying Lemma 4.2 gives rise to

\[
\begin{align*}
q_\sigma^2(\eta_\sigma^\gamma(q)) - \frac{1}{2} q_\sigma^2(\eta_\sigma^\gamma(q)) = -\frac{M}{2} q(M + q) \chi_{(-\infty, -M)}(q), \\
q_\sigma^2(\eta_\sigma^\gamma(q)) - \frac{1}{2} q_\sigma^2(\eta_\sigma^\gamma(q)) = -\frac{M}{2} q(M + q) \chi_{(-\infty, -M)}(q).
\end{align*}
\]

Substituting (68) and (69) into (67) gives

\[
\frac{d}{dt}(\eta_\sigma^\gamma(q) - \eta_\sigma^\gamma(q)) - \frac{7\sigma}{2} \frac{d}{dx}(u(\eta_\sigma^\gamma(q) - \eta_\sigma^\gamma(q))) \\
\leq \frac{d}{dt}(\eta_\sigma^\gamma(q) - \eta_\sigma^\gamma(q)) + Q((\eta_\sigma^\gamma(q) - \eta_\sigma^\gamma(q)).
\]

Integrating the above inequality over \( (0, t) \times \mathbb{R}, \) by (71), we obtain

\[
\int_{0}^{1} \frac{1}{2} (\eta_\sigma^\gamma(q) - q_\sigma^2)dx \\
\leq \int_{0}^{1} Q(\eta_\sigma^\gamma(q) - q_\sigma^2)dxds.
\]

**Lemma 4.6** It holds that

\[
\frac{1}{2} q_\sigma^2 = q_\sigma^2
\]

almost everywhere in \( [0, t) \times (-\infty, \infty). \)

**Proof** It follows from Lemma 4.3 that

\[
\begin{align*}
\eta_\sigma^\gamma(q) - \eta_\sigma^\gamma(q) &= \frac{1}{2} (\eta_\sigma^\gamma(q) - q_\sigma^2) + \frac{1}{2}(M - q)^2 \chi_{(-\infty, -M)}(q) \\
&\quad - \frac{1}{2}(M - q)^2 \chi_{(-\infty, -M)}(q).
\end{align*}
\]

From (60) and (74), we have

\[
\begin{align*}
\int_{0}^{1} \frac{1}{2} (\eta_\sigma^\gamma(q) - q_\sigma^2)dx &\leq -\frac{7\sigma M^2}{4} \int_{0}^{1} (M - q) \chi_{[M, \infty)}(q)dxds \\
&\quad + \frac{7\sigma M^2}{4} \int_{0}^{1} (M + q) \chi_{[M, \infty)}(q)dxds \\
&\quad + \frac{7\sigma M^2}{2} \int_{0}^{1} (\eta_\sigma^\gamma(q) - \eta_\sigma^\gamma(q))dxds \\
&\quad + \frac{7\sigma M^2}{2} \int_{0}^{1} \frac{1}{2} (\eta_\sigma^\gamma(q) - q_\sigma^2)dxds \\
&\quad + \int_{0}^{1} Q(\eta_\sigma^\gamma(q) - q_\sigma^2)dxds,
\end{align*}
\]

where we used the identity \( M(M - q)^2 + Mq(M - q) = M^2(M - q). \)
Combining (66) with (75) gets
\[
\int_{J_R} \left( \frac{1}{2} (q_+ - q_-)^2 \right) dx \\
\leq \frac{7\sigma M^2}{4} \int_{J_R} (M - q)_X(q) dx ds \\
- \frac{7\sigma M^2}{4} \int_{J_R} (M - q)_X(q) dx ds \\
+ \frac{7\sigma M}{2} \int_{J_R} \left| n_{q_0}^+(q) - n_{q_0}^-(q) \right| dx ds \\
+ \frac{7\sigma M}{2} \int_{J_R} \frac{1}{2} (q_+ - q_-)^2 dx ds \\
+ \int_{J_R} Q(s,x) (q_+ - q_-) + \left( n_{q_0}^+(q) - n_{q_0}^-(q) \right) dx ds.
\]

(76)

In fact, for \( 0 < t < T \), there exists a constant \( L > 0 \), depending only on \( \| u_0 \|_{\mathcal{H}^1} \), and \( T \) such that
\[ \| Q(t,x) \|_{L^\infty([0,T] \times \mathbb{R})} \leq L. \]

From Lemma 4.3, it has
\[
q_+ + (n_{q_0}^+)'(q) = q + (M - q)_X(q), \\
q_- + (n_{q_0}^-)'(q) = q + (M - q)_X(q).
\]

(77)

Since the map \( \xi \rightarrow \xi_+ + (n_{q_0}^+)'(\xi) \) is convex and concave, we get
\[
(q_+ - q_-) + \left( n_{q_0}^+(q) - n_{q_0}^-(q) \right) = - (M - q)_X(q) + (M - q)_X(q) \leq 0.
\]

(78)

Therefore,
\[
Q(s,x) (q_+ - q_-) + \left( n_{q_0}^+(q) - n_{q_0}^-(q) \right) \\
\leq - L \left( (M - q)_X(q) - (M - q)_X(q) \right).
\]

(79)

Choosing \( M \) large enough,
\[
\frac{7\sigma M^2}{4} (M - q)_X(q) - \frac{7\sigma M^2}{4} (M - q)_X(q) \\
+ Q(s,x) (q_+ - q_-) + \left( n_{q_0}^+(q) - n_{q_0}^-(q) \right) \\
\leq - L \left( (M - q)_X(q) - (M - q)_X(q) \right) \\
\leq 0.
\]

(80)

Hence, from (76) and (80), we obtain
\[
0 \leq \int_{J_R} \left( \frac{1}{2} (q_+ - q_-)^2 \right) dx \\
\leq \frac{7\sigma M}{2} \int_{J_R} \left( \frac{1}{2} (q_+ - q_-)^2 \right) dx ds.
\]

(81)

For \( t > 0 \), we conclude from Gronwall’s inequality and Lemma 4.1 and 4.2 that
\[
0 \leq \int_{J_R} \left( \frac{1}{2} (q_+ - q_-)^2 \right) dx = 0.
\]

(82)
By the Fatou lemma, sending $M \to \infty$, we obtain
\[
0 \leq \int_{\mathbb{R}} q^n - q^2 \, dx \leq 0,
\] (83)
which completes the proof.

5. Proof of main theorem

**Proof of Theorem 2.2.** From (8), (10), and Lemma 3.5, we know that the conditions (i) and (ii) in definition 2.1 are satisfied. We have to verify (iii). Due to Lemma 4.6, we have
\[
q_n \longrightarrow q \quad \text{in} \quad L^2_{\text{loc}}((0, \infty) \times \mathbb{R}).
\] (84)
Using (84) and Lemmas 3.5 and 3.6, we know that $u$ is a distributional solution to problem (1). In addition, Inequality (6) is deduced from Lemma 3.4. Then the proof of Theorem 2.2 is finished.

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