Numerical methods for determination the boundaries of unrecovered visco-plastic oil

I B Badiev¹, K S Golovanov¹, V L Gnedenkova¹ and O V Pankratova¹
¹Kazan Federal University, 18 Kremlyovskaya Street, 420008, Kazan, Russia

E-mail: ildar.badriev1@mail.ru

Abstract. In the present work, numerical methods are proposed for solving the problem of determining the boundaries of the unrecovered visco-plastic oil. This problem reduces to a stationary filtration problem for an incompressible fluid with an effective multi-valued law consisting in finding pressure fields and filtration velocity that satisfy the continuity equation and boundary conditions. A generalized formulation is formulated for pressure in the form of a mixed variational inequality with non-differentiable functional. To solve the variational inequality, an iterative splitting method is proposed that does not require inversion of the original operator. For the numerical implementation of this iterative method, finite element approximations of the variational inequality and the iteration method were constructed. A software package was developed in the Matlab environment. For model problems, numerical experiments were carried out for various initial data. The experimental results indicate the effectiveness of the iterative method.

1. Introduction
We consider the process of displacing of viscoplastic oil by water from the reservoir [1–3]. It is assumed that oil remains stationary in a porous medium, if the modulus of the pressure gradient does not exceed a certain limit value (limit gradient). In the process of repression, there may be areas of stationary oil (stagnant zones). As the oil displaces the water, the stagnant zone will turn out to be bypassed and eventually turn into the pillars of residual oil. The formation of the pillar due to the breakthrough of the displacing water inevitably occurs during any non-one-dimensional process of displacing. However, in the case of ordinary viscous fluids, following the Darcy law, due to a sufficiently long pumping of water, the pillars can be made arbitrarily small. The specificity of a viscoplastic fluid is that with sufficiently small sizes, the pillars can retain their shape in the stream of displacing water flowing around them indefinitely. Such pillars that do not change their form are called equilibrium. As the size of the pillar changes, a moment may come when the pressure gradient at its boundary will be equal to the limit. If this state is reached at all points of the pillar contour, that such pillar is called unrecovered. The unrecovered visco-plastic oil is the largest area that can be occupied by a residual viscoplastic fluid for an indefinitely long time, i.e., characterizes the contribution of the plastic properties of the fluid in the residual loss of oil. Thus, the calculation of the unrecovered visco-plastic oil is one of the possible approaches to the assessment of the influence of plastic properties of oil on the ultimate oil output, bypassing the direct consideration of the non-stationary process of displacement. Such a calculation is useful to assess the feasibility of drilling new wells.
In the present work, numerical methods are proposed for solving the problem of determining the boundaries of the unrecovered visco-plastic oil. In [4], it was shown that this problem reduces to a stationary filtration problem for an incompressible fluid with an effective multi-valued law consisting in finding pressure fields and filtration velocity that satisfy the continuity equation and boundary conditions.

Note that the study of filtration problems with nonlinear filtration laws is also of independent importance. Currently, the world’s hydrocarbon reserves are deteriorating and the share of hard-to-recover reserves is growing, including deposits in low-permeability reservoirs and high-viscosity oil deposits. In such deposits, deviations from the linear Darcy filtration law are observed, the magnitude of which becomes significant with decreasing permeability [5–14]. The urgency of the problem follows from the weak study of non-Newtonian fluids in porous media.

A generalized formulation is formulated for pressure in the form of a mixed variational inequality with an inversely strongly monotone operator [15] in Sobolev space [16]. The functional appearing in this variational inequality is lower semi-continuous, convex, proper, generally speaking, non-differentiable.

To solve the variational inequality, an iterative splitting method [17–23] is proposed that does not require inversion of the original operator. The main difficulty in its implementation is to solve the minimization problem that arises at each step of the method. For the considered filtration problem, the minimization problem was solved explicitly because it was possible to calculate the subdifferential of the functional conjugate to the minimized one. Moreover, each step of the method is actually reduced to the inverse of the Laplace operator. It should be noted that the method allows to find approximate values not only of the solution itself, but also of its characteristics, for filtration problems these are approximate values of the solution gradient, which is very useful from a practical point of view.

For the numerical implementation of this iterative method, finite element approximations of the variational inequality and the iteration method were constructed. A software package was developed in the Matlab environment. For model problems, numerical experiments were carried out for various initial data, and the optimal (by the number of iterations) iterative parameters were empirically determined. The experimental results indicate the effectiveness of the iterative method.

2. Problem statement

In [4], the problems of determining the boundaries of the unrecovered visco-plastic oil are considered. The calculation of residual oil pillars in the reservoir, in averaged formulation, is reduced to solving the stationary problem of filtration the displacing fluid with respect to the pressure fields \( p \) and the filtration velocity \( w \) with an effective multi-valued law

\[
-\nabla p \in \Phi(|w|w) |w|^{-1}, \quad 0 \leq \Phi(z) \leq \beta, \quad z = 0 \quad \Phi(z) = \beta, \quad 0 \leq z \leq \theta, \quad \Phi(z) = \alpha z, \quad z > \theta, \quad \alpha = \beta / \theta
\]  

(1)

Here the coefficients \( \beta, \ \theta \) are originally given values. The sets \( \Omega^+ = \{x \in \Omega : |\nabla p| > \beta\}, \ \Omega^0 = \{x \in \Omega : |\nabla p| = \beta\}, \ \Omega^- = \{x \in \Omega : |\nabla p| < \beta\} \) of the points of the filtration domain \( \Omega \) correspond to the zones of the oil reservoir in which the reservoir has been completely washed, partially or not washed.

In the following, we consider a more general filtration law than (1). The steady-state filtration process of incompressible fluid in a porous medium is studied. Filtration occurs in a domain \( \Omega \subset R^2 \) with a Lipschitz continuous boundary \( \Gamma = \Gamma_1 \bigcup \Gamma_2, \ \Gamma_1 \cap \Gamma_2 = \emptyset, \ mes\Gamma_1 > 0, \) (on \( \Gamma_1 \) a pressure is assumed to be zero, and on \( \Gamma_2 \) a no flow condition is given). The problem is considered in the presence of \( m \) point sources with constant intensities \( q_j, \ j = 1, 2, \ldots, m, \) in pairwise different interior points \( x^{(j)}), \ j = 1, 2, \ldots, m, \) of the domain \( \Omega. \) It is necessary to find the stationary fields of pressure \( u \) and fluid velocity \( v \) that satisfy the continuity equation and the boundary conditions

\[
\text{div} \ v(x) = \tilde{f}(x), \ x \in \Omega, \quad u(x) = 0, \ x \in \Gamma_1, \quad (v(x), n(x)) = 0, \ x \in \Gamma_2
\]  

(2)
(\tilde{f}) is the function characterizing the density of external sources, \(\mathbf{n}\) is the outward normal to \(\Gamma_2\) under the assumption that the filtrating fluid satisfies the multivalued filtration law
\[ -v(x) \in g(\nabla u(x)) \nabla u(x), x \in \Omega. \tag{3} \]

We assume that the function \(g\) that determines the filtration law can be represented as a sum \(g(\xi) = g_0(\xi) + \theta H(\xi - \beta)\), where a single-valued function and a multi-valued function are determined by the formulas \(g_0(\xi)\) and \(H(\xi)\) as follows:
\[ H(\xi) = \begin{cases} 0, & \xi \leq 0, \\ 1, & \xi > 0. \end{cases} \]

given constants, the function \(g : [0, +\infty) \to [0, +\infty)\) satisfies the conditions: \(g(0) = 0, g(\xi) > g(\zeta)\) for all \(\xi \geq \zeta \geq 0\) there are such constants \(k > 0, \xi^* > 0, L > 0\), that \(g(\xi) - g(\zeta) \geq k(\xi - \zeta)\) for all \(\xi > \zeta \geq \xi^*\), \(g(\xi) - g(\zeta) \leq L(\xi - \zeta)\) for all \(\xi > \zeta \geq 0\). We define an operator \(G : R^2 \to R^2\) by function \(g_0\) as follows:
\[ G(0) = 0, \quad G(0) = g_0(|y|y)|y|^{-1}, \quad y \neq 0. \]

Let’s denote \(V = \{\eta \in W^{1,2}_0(\Omega) : \eta(x) = 0, x \in \Gamma_1\}\). Following [5–7, 14, 24], by a generalized solution of problem (2), (3), we mean a function \(u \in V\) that is a solution of a variational inequality
\[ -v(x) \in g(\nabla u(x)) \nabla u(x), x \in \Omega. \tag{4} \]

where \((\cdot, \cdot)_{\nu}\) is the inner product in \(V\), operator \(A : V \to V\) and element \(f \in V\) are generated by the forms \((Aw, \eta)_{\nu} = \int_{\Omega} (G(\nabla w), \nabla \eta) dx, (f, \eta)_{\nu} = \int_{\Omega} f \eta dx\), the functional \(\Phi : Y = L^2(\Omega) \times L^2(\Omega) \to R\) is given by the formula
\[ \Phi(y) = \theta \int_{\Omega} \varphi(|y| - \beta) dx, \quad \varphi(\xi) = \begin{cases} 0, & \xi < 0, \\ \xi, & \xi \geq 0. \end{cases} \]

In [25], the existence of at least one solution \(u \in V\) of the variational inequality (4) was proved and the existence of a filtration velocity field constructed by \(u \in V\) and satisfying the continuity equation was established.

3. Iterative method

To solve the variational inequality (4), the following iterative process was proposed in [26–33]. Let \(\tau, \rho > 0\) be the iteration parameters, \(A^* : Y \to V\) be the conjugate operator to \(V\), \(w^{(0)} \in V, y^{(0)} \in Y, A^{(0)} \in Y\) be the arbitrary elements. For \(n = 0, 1, \ldots\) knowing \(w^{(n)}, y^{(n)}, A^{(n)}\) we find
\[ w^{(n+1)} = w^{(n)} - \tau A[w^{(n)} + A^* y^{(n)} + \rho(w^{(n)} - A^* y^{(n)})]. \tag{5} \]

Next, we find \(y^{(n+1)}\) by solving the minimization problem
\[ \rho(y^{(n+1)}, z - y^{(n+1)})_Y + \Phi(z) - \Phi(y^{(n+1)}) \geq (\rho A y^{(n+1)})_Y + \lambda^{(n)}, z_i - y^{(n+1)} \in Y, \quad \forall z \in Y. \tag{6} \]

Finally, we set
\[ A^{(n+1)} = A^{(n)} + \rho(A w^{(n+1)} - y^{(n+1)}). \tag{7} \]

It is not difficult to make sure that problem (6) is uniquely solvable.

Let us now dwell on the features of the application of the iterative method (5)–(7) for solving problems of established filtration with a multi-valued law.

The first step (5) is reduced, by virtue of the Riesz-Fisher theorem, to the inverse of the Laplace operator: \(-\Delta w^{(n+1)} = -(1 - \rho)\Delta w^{(n)} + \Delta w^{(n)} - \rho \Delta w^{(n)} + \tau f\) . The second step (relation (7)) is to computations by explicit formulas. The main difficulty in implementing the iterative method is the second step which consists in solving the minimization problem (6), that is, according to the definition of the proximal mapping in the form of a variational inequality (see [34]), the unconditional minimization problem of a strongly convex, lower semicontinuous functional \(\Phi_\rho : Y \to R\),
\[ \Phi_\rho(z) = 0.5 \rho \| z \|_F^2 + \Phi(z) + (\nabla w^{(n+1)} + \lambda^{(n)})_y . \]

Following [35], it can be verified that

\[ y^{(n+1)} = g_\rho^* (|t|) t / |t| , t = \nabla w^{(n+1)} + \lambda^{(n)} , \]

where \( g_\rho^* (\xi) = \begin{cases} \xi / \rho, & 0 \leq \xi \leq \rho \beta, \\ \beta, & \rho \beta \leq \xi \leq \rho \beta + \theta, \\ (\xi - \theta) / \rho, & \xi \geq \rho \beta + \theta. \end{cases} \)

By analogy with [36–39], finite-dimensional approximations of the problem and the iterative method were constructed. For the numerical implementation of approximate methods, a software complex was developed in the Matlab environment and numerical experiments were carried out to identify the optimal iterative parameters \( \tau, \rho \) for the model filtration problem. This model problem arises in the practice of developing non-Newtonian oil reservoirs when water flooding with a circular battery of wells. It is required to determine the boundaries of the of unrecovered visco-plastic oil when filtration in a domain with a circular contour of constant pressure. It is assumed that the water moves according to the multi-valued filtration law, and at the boundary of the pillar the modulus of the flow velocity is constant. In fig. 1 shows the boundaries of the pillars in the case of a battery of six symmetrically located wells, having a flow rate of \( q \), inside a circular power circuit with a radius \( R = 1 \). It shows due to the symmetry of the problem only a sector equal to the sixth part of the circular contour. The values \( \theta = 1, \beta = 1 \) were chosen and it was assumed that \( y(\xi) = \xi \). The pillars of a stationary fluid (oil) are located to the left of curves 1–3, corresponding to the values of the dimensionless flow parameter \( q / (\beta R) \) equal to 3.09, 1.85, 1.41, respectively. The distance from the center of the circle to the wells is 2/3 of the radius \( R \). The optimal (by the number of iterations) values of the iteration parameters \( \tau = 0.6, \rho = 1 \) are obtained empirically. The number of iterations is 67. Note that knowledge of domains of lost oil can be useful for assessing the feasibility of drilling new wells, which are potential sources of pollution of groundwater and the environment.

![Figure 1. Boundaries of unrecovered visco-plastic oil](image)

Acknowledgments

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University. The research was funded by the subsidy allocated to Kazan Federal University for the state assignment in the sphere of scientific activities, project № 1.12878.2018/12.1.

References

[1] Entov V M, Malakhova T A, Pankov V N and Pan'ko S V 1980 Calculation of the Limit-equilibrium of Retained Viscoplastic Oil Extracted from a Nonuniform Stratified Layer by Water Journal of Applied Mathematics and Mechanics 44 (1) 76-82
[2] Entov V M, Pankov V N and Pan'ko S V 1980 On the Analysis of Retained Residual Viscoplastic Petroleum Journal of Applied Mathematics and Mechanics 44 (5) 597-603
[3] Entov V M and Pan'ko S V 1984 Variational Formulation of the Problem of Retained Viscoplastic Oil Journal of Applied Mathematics and Mechanics 48 (6) 707–12
[4] Entov V M, Pankov V N and Pan'ko S V 1989 Mathematical Theory of Unrecovered Visco-Plastic Oil (Tomsk, Izdatel'ство Tomskogo universiteta). (in Russian)
[5] Lapin A V 1979 Investigation of Some Non-linear Problems of Filtering Theory USSR Computational Mathematics and Mathematical Physics 19 (3) 135-48
[6] Badriev I B 2013 On the solving of variational inequalities of stationary problems of two-phase
flow in porous media. *Applied Mechanics and Materials* **392** 183-7 DOI: 10.4028/www.scientific.net/AMM.392.183

[7] Lyashko A D and Karchevskii M M 1983 Difference Methods of Solving Nonlinear Problems in the Theory of Filtration *Soviet Mathematics* **27** (7) 34-56.

[8] Badriev I B and Nechaeva L A 2013 Mathematical simulation of steady filtration with multivalued law *PNRPU Mechanics Bulletin* **3** 37-65

[9] Badriev I B and Fanyuk B Y 2012 Iterative methods for solving seepage problems in multilayer beds in the presence of a point source *Lobachevskii Journal of Mathematics* **33** (4) 386-99. DOI: 10.1134/S1995080212040026

[10] Maksimov V M, Dmitriev N. M. and Dmitriev M N 2017 A new approach to determining symmetry groups of filtration properties of porous media in nonlinear filtration laws *Doklady Physics* **62**(4) 190-3 DOI: 10.1134/S1028335817040036

[11] Dmitriev N M, Maksimov V M and Mamedov M T 2010 Laws of flow with a limiting gradient in anisotropic porous media *Fluid Dynamics* **45** (2) 223-9 DOI: 10.1134/S0015462810020079

[12] Dmitriev N M, Maksimov V M and Ryabchukov E A 2006 Laws of viscoplastic fluid flow in anisotropic porous and fractured media *Fluid Dynamics* **41**(4) 585-92 DOI: 10.1007/s10697-006-0076-1

[13] Badriev I B, Banderov V V, Pankratova O V and Shangaraeva A I 2016 Mathematical simulation of a steady process of anisotropic filtration *IOP Conference Series: Materials Science and Engineering* **158** (1) 012013. DOI: 10.1088/1757-899X/158/1/012013

[14] Badriyev I B, Zadvornov O A, Ismagilov L N and Skvortsov E V 2009 Solution of plane seepage problems for a multivalued seepage law when there is a point source *Journal of Applied Mathematics and Mechanics* **73** (4) 434-42 DOI: 10.1016/j.jappmathmech.2009.08.007.

[15] Gol'shtein E G and Tret'yakov N V 1989 Modified Lagrangians (Moscow, Nauka). (in Russian)

[16] Adams R A 1975 *Sobolev Spaces* (New York, San Francisco, London, Academic Press)

[17] Badriev I B and Karchevskii M M 1989 Convergence of the iterative Uzawa method for the solution of the stationary problem of seepage theory with a limit gradient *Journal of Soviet Mathematics* **45**(4) 1302-9 DOI: 10.1007/BF01097083

[18] Badriev I B, Makarov M V and Paimushin V N 2018 Geometrically Nonlinear Problem of Longitudinal and Transverse Bending of a Sandwich Plate with Transversally Soft Core *Lobachevskii Journal of Mathematics* **39** (3) 448-57 DOI: 10.1134/S1995080218030046

[19] Badriev I B, Makarov M V and Paimushin V N 2017 Numerical investigation of a physically nonlinear problem of the longitudinal bending of the sandwich plate with a transversal-soft core *PNRPU Mechanics Bulletin* (1) 39-51 DOI: 10.15593/perm.mech/2017.1.03

[20] Badriev and Shagidullin R R 1992 Study of monomeric equations of static state of soft envelope and algorithm of their solution *Izvestiya vysshikh uchebnykh zavedenii. Matematika* (1) 8-16

[21] Badriev I B, Banderov V V and Makarov M V 2017 Mathematical Simulation of the Problem of the Pre-Critical Sandwich Plate Bending in Geometrically Nonlinear One Dimensional Formulation *IOP Conference Series: Materials Science and Engineering* **208** (1) 012002 DOI: 10.1088/1757-899X/208/1/012002

[22] Chebakova V J, Gerasimov A V and Kirpichnikov A P 2016 On the solving of one type of problems of mathematical physics *IOP Conference Series: Materials Science and Engineering* **158** (1) 012023 DOI: 10.1088/1757-899X/158/1/012023

[23] Solov'ev S I 1985 Fast methods for solving mesh schemes of the finite element method of second order accuracy for the Poisson equation in a rectangle *Izvestiya vysshikh uchebnykh zavedenii. Matematika* Mat. (10) 71-4

[24] Badriev I B, Kalacheva N V, Shangaraeva A I and Sudakov V A 2018 Numerical solving of highly viscous fluids filtration in porous media for nonlinear filtration laws with power growth *IOP Conference Series: Earth and Environmental Science* **155** (1) 012015
DOIs:

[25] Badriev I B, Zadvornov O A and Saddek A M 2001 Convergence Analysis of Iterative Methods for Some Variational Inequalities with Pseudomonotone Operators Differential Equations 37(7) 934-42 DOI: 10.1023/A:1011901503460

[26] Badriev I B and Zadvornov O A 2003 A decomposition method for variational inequalities of the second kind with strongly inverse-monotone operators Differential Equations 39 (7) 936-44 DOI: 10.1023/B:DIEQ.000009189.91279.93

[27] Badriev I B and Karchevskii M M 1994 Convergence of an iterative process in a Banach space Journal of Mathematical Sciences 124(6) 2727-35, DOI: 10.1007/BF02110578

[28] Badriev I B and Banderov V V 2014 Iterative methods for solving variational inequalities of the theory of soft shells Lobachevskii Journal of Mathematics 35 (4) 371-83 DOI: 10.1134/S1995080214040015

[29] Badriev I B and Banderov V V 2014 Iterative methods for solving variational inequalities of the theory of soft shells Lobachevskii Journal of Mathematics 35(4) 371-83 DOI: 10.1134/S1995080214040015

[30] Badriev I B, Banderov V V and Zadvornov O A 2013 On the solving of equilibrium problem for the soft network shell with a load concentrated at the point PNRPU Mechanics Bulletin (3) 17-35

[31] Badriev I B 1989 Application of duality methods to the analysis of stationary seepage problems with a discontinuous seepage law Journal of Soviet Mathematics 45 (4) 1310-4 DOI: 10.1007/BF01097084.

[32] Badriev I B, Makarov M V and Paimushin V N 2017 Contact statement of mechanical problems of reinforced on a contour sandwich plates with transversally-soft core Russian Mathematics 61(1) 69-75 DOI: 10.3103/S1066369X1701008X

[33] Badriev I B, Zheltukhin V S and Ju Chebakova V 2017 Numerical solution of the initial boundary value problems of radio-frequency capacitive coupled discharge Journal of Physics: Conference Series 927(1) 012008 DOI: 10.1088/1742-6596/927/1/012008

[34] Ekeland I and Temam R 1976 Convex Analysis and Variational Problems (Amsterdam: North Holland Publishing Company, New York: Oxford American Elsevier Publishing Company)

[35] Badriev I B, Zadvornov O A and Lyashko A D 2004 A study of variable step iterative methods for variational inequalities of the second kind Differential Equations 40 (7) 971-83 DOI: 10.1023/B:DIEQ.0000047028.07714.df

[36] Badriev I B 1983 Difference-schemes for linear-problems of the filtration theory with discontinuous law Izvestiya Vysshikh Uchebnykh Zavedenii Matematika 5 3-12

[37] Badriev I B, Banderov V V, Gnedenkova V L, Kalacheva N V, Korablev A I and Tagirov R R 2015 On the finite dimensional approximations of some mixed variational inequalities Applied Mathematical Science 9(113-6) 5697-705 DOI: 10.12988/ams.2015.57480

[38] Badriev I B, Banderov V V, Lavrentyeva E E and Pankratova O V 2016 On the Finite Element Approximations of Mixed Variational Inequalities of Filtration Theory IOP Conference Series: Materials Science and Engineering 158 (1) 012012 DOI: 10.1088/1757-899X/158/1/012012.

[39] Badriev I B and Pankratova O V 1992 Mixed finite-element method for nonlinear stationary problems of seepage theory Journal of Soviet Mathematics 61(6) 2405-16 DOI: 10.1007/BF01100574

[40] Badriev I B, Banderov V V and Singatullin M T 2015 Numerical Solution of Non-Linear Filtration Issues for High Viscous Fluids at the Presence of Wells Research Journal of Applied Sciences 10 (8) 343-6 DOI: 10.3923/ijasci.2015.343.346

[41] Badriev I B, Makarov M V and Paimushin V N 2016 Longitudinal and transverse bending by a cylindrical shape of the sandwich plate stiffened in the end sections by rigid bodies IOP Conference Series-Materials Science and Engineering 158 (1) 012011 DOI: 10.1088/1757-899X/158/1/012011

6