ORTHOGLONAL POLYNOMIALS OF DISCRETE VARIABLE AND LIE ALGEBRAS OF COMPLEX SIZE MATRICES

DIMITRY LEITES, ALEXANDER SERGEEV

Abstract. We give a uniform interpretation of the classical continuous Chebyshev's and Hahn's orthogonal polynomials of discrete variable in terms of Feigin's Lie algebra $\mathfrak{gl}(\lambda)$ for $\lambda \in \mathbb{C}$. One can similarly interpret Chebyshev's and Hahn's $q$-polynomials and introduce orthogonal polynomials corresponding to Lie superalgebras.

We also describe the real forms of $\mathfrak{gl}(\lambda)$, quasi-finite modules over $\mathfrak{gl}(\lambda)$, and conditions for unitarity of the quasi-finite modules. Analogs of tensors over $\mathfrak{gl}(\lambda)$ are also introduced.

This is a transcript of the talk at MPI, Bonn in the memory of Misha Saveliev (preprint MPI-1999-44 at www.mpim-bonn.mpg.de), see also Theor. Math. Phys., v. 123, no. 2, 2000, 205–236 (Russian), 582–609 (English). It consists of three parts: the description of orthogonal polynomials proper ($\S$1, 8, 9) and some auxiliary results: the description of the generating function of the trace and its analogs ($\S$2) and a description of quasi-finite modules over $\mathfrak{gl}(\lambda)$ for $\lambda \in \mathbb{C}$.

For a continuation, see [S].

§1. Introduction

For recapitulations, see [NU], [NSU].

1.0. Definitions. The equation of the form

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0, \quad (1.0.1)$$

where $\sigma(x)$ is a polynomial of degree 2, $\tau(x)$ is a polynomial of degree 1 and $\lambda$ is a constant, is called a hypergeometric type equation and its solutions are called functions of hypergeometric type. By multiplying (0.1) by an appropriate $\rho(x)$ one reduces (1.0.1) to the self-adjoint form

$$(\sigma(x)\rho(x)y')' + \lambda \rho(x)y = 0, \text{ where } (\sigma(x)\rho(x))' = \tau(x)\rho(x). \quad (1.0.2)$$

Let $y_m$ and $y_n$ be solutions of (1.0.2) with distinct eigenvalues $\lambda_m$ and $\lambda_n$, respectively. If, for some $a$ and $b$, not necessarily finite, $\rho(x)$ satisfies the conditions

$$\sigma(x)\rho(x)x^k |_{x=a,b} = 0 \text{ for } k = 0, 1, \ldots,$$

then

$$\int_{a}^{b} y_m(x)y_n(x)\rho(x)dx = 0. \quad (1.0.3)$$

(Clearly, if $a$ and $b$ are finite, it suffices to require that $\sigma(x)\rho(x)|_{x=a,b} = 0$.)

Example: the Jacobi polynomials $P^{(\alpha,\beta)}_n(x)$ defined for $\alpha, \beta > -1$ as polynomial solutions of (0.2) for $\sigma(x) = 1-x^2$, $\rho = (1-x)^{\alpha}(1+x)^{\beta}$, $(a,b) = (-1,1)$ and $\tau = \beta - \alpha - (\alpha + \beta + 2)x$. For $\alpha = \beta = 0$, the Jacobi polynomials are called Chebyshev polynomials.

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The difference equation which approximates (1.0.1) on the uniform lattice is, clearly, of the form
\[ \sigma(x) \frac{1}{h} \left[ \frac{y(x+h) - y(x)}{h} - \frac{y(x) - y(x-h)}{h} \right] + \frac{\tau(x)}{2} \left[ \frac{y(x+h) - y(x)}{h} + \frac{y(x) - y(x-h)}{h} \right] + \lambda y = 0. \] (1.0.4)

Set \( h = 1 \) and \( \Delta f(x) = f(x+1) - f(x), \nabla f(x) = f(x) - f(x-1) \). Then the difference equation (1.0.4) takes the form
\[ \Delta(\sigma(x)\rho(x)\nabla y) + \lambda \rho(x)y = 0, \text{ where } \Delta(\sigma(x)\rho(x)) = \tau(x)\rho(x) \] (1.0.5)
while the orthogonality relations take the form
\[ \sum_{x_i=a}^{b-1} y_m(x_i)y_n(x_i)\rho(x_i) = 0 \] (1.0.6)
provided \( \rho(x) \) satisfies the conditions
\[ \sigma(x)\rho(x)x^k|_{x=a,b} = 0 \text{ for } k = 0, 1, \ldots \]

**Examples:** The Hahn polynomial \( h_{n}^{(\alpha,\beta)}(x, N) \) is a polynomial solution of (1.0.5) defined for \( \alpha, \beta > -1 \) for \( \sigma(x) = x(N + \alpha - x), \rho = \frac{\Gamma(N+\alpha-x)\Gamma(\beta+1+x)}{\Gamma(x+1)\Gamma(N-x)}, (a, b) = (0, N) \) and \( \tau = (\beta + 1)(N - 1) - (\alpha + \beta + 2)x \). (As noted in [R], actually, Hahn’s polynomials were known and studied by Chebyshev; Hahn rediscovered them together with their \( q \)-analogs.)

The Chebyshev polynomial \( t_n(x, N) \) is a polynomial solution of (1.0.5) defined for \( \alpha = \beta = 0, \sigma(x) = x(N - x), \rho = 1, (a, b) = (0, N) \) and \( \tau = N - 1 - 2x \).

A usual way to obtain \( q \)-analogs of these polynomials is to consider nonuniform partitions of the segment \([a, b]\), see [NSU]. Our scheme applied to \( U_q(\mathfrak{sl}(2)) \) instead of \( U(\mathfrak{sl}(2)) \) considered here gives another approach.

Until recently, the conventional inner products (1.0.3) and (1.0.6) were supposed to be positive definite. Though \( N \), the number of nodes of the uniform partition of segment \((a, b)\) is integer, it is possible to replace it with any complex number in expressions for \( h_n^{(\alpha,\beta)}(x, N) \) and \( t_n(x, N) \), since the latter analytically depend on \( N \). It was observed (see [NSU]) that, for \( N \) purely imaginary, there is a measure leading to a positive definite inner product of type (1.0.3). The discovery of this fact reflects the luck and serendipity of the researchers since no explanation of the phenomenon was seen.

It was also due to ingenious calculations that various identities orthogonal polynomials satisfy were discovered; the completeness of the list of such identities was never discussed.

1.1. **Our result: a summary.** We observed that the definition and all identities the continuous Chebyshev and Hahn orthogonal polynomials of discrete variable satisfy (and their versions on inhomogeneous lattice, or, in modern terms, \( q \)-analogs) follow from the properties of the Casimir elements for the Lie algebra \( \mathfrak{gl}(\lambda) \) of matrices of complex size (and its \( q \)-analog). Our starting point was an attempt to explicitly calculate the value of the quadratic Casimir operator on quasi-finite \( \mathfrak{gl}(\lambda) \)-modules — a source of orthogonality relations and various identities the said polynomials satisfy. (For a more detailed than ours description of quasi-finite modules of level 1, see [SI].)

Let, first, \( \lambda = n > 0 \) be an integer. The invariant functional, trace, on \( \mathfrak{g} = \mathfrak{gl}(n) \) gives rise to the non-degenerate bilinear form \((A, B) = \text{tr} AB\). The homogeneous components \( \mathfrak{g}_i \), where \( \mathfrak{g}_i \) is the space of matrices with support on the \( i \)th over-(for \( i > 0 \)) and under-diagonals (for \( i < 0 \)), are orthogonal with respect to this form and the form is non-degenerate on \( \mathfrak{g}_{-i} \oplus \mathfrak{g}_i \) and on \( \mathfrak{g}_0 \). The reduction of the form to the canonical form on these spaces leads to the classical Hahn and Chebyshev polynomials of discrete variable, respectively. By
expressing the Casimir elements — the generators of the center of \(U(\mathfrak{g})\) — in terms of these polynomials in the standard basis one gets all the identities the polynomials satisfy. More exactly, we express elements of \(\mathfrak{g}_0\), as well as those of \(\mathfrak{g}_{-i} \oplus \mathfrak{g}_i\), in terms of polynomials in one variable, \(H\). The details are given in the main text; here is a gist of the idea.

In \(\mathfrak{sl}(2)\), consider the basis \(Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) with the commutation relations
\[
[X, Y] = H, \quad [H, Y] = -2Y, \quad [H, X] = 2X.
\] (1.1.1)

Consider the principal embedding of \(\mathfrak{sl}(2)\) into \(\mathfrak{gl}(n)\) (this embedding corresponds to the \(n\)-dimensional irreducible \(\mathfrak{sl}(2)\)-module):
\[
Y \mapsto \sum_{i=1}^{n-1} E_{i+1,i}, \quad H \mapsto \sum_{i=1}^{n} (n - 2i + 1)E_{ii}, \quad X \mapsto \sum_{i=1}^{n-1} i(n - i)E_{i,i+1}.
\] (1.1.6)

Then \(\mathfrak{g}\) is of the form \(\mathfrak{g} = \bigoplus_{|i|\leq n-1} \mathfrak{g}_i\), where \(\mathfrak{g}_i = \{X^i\mathfrak{g}_0\}\) and \(\mathfrak{g}_{-i} = \{\mathfrak{g}_0Y^i\}\) for \(i > 0\) and \(\mathfrak{g}_0 = \mathbb{C}[H]/(P_n(H))\) is the Cartan subalgebra; here \(P_n(H) = \prod_{1 \leq i \leq n} (H - n + 2i - 1)\).

By applying this approach to \(\mathfrak{gl}(\lambda)\), \(\lambda \in \mathbb{C}\), we clarify the known results on the so-called continuous versions of the classical Hahn and Chebyshev polynomials of discrete variable, cf. \([\text{NU}], [\text{NSU}]\), concerning purely imaginary values of \(N = \lambda\) and get some new ones. So let us explain, first, what is \(\mathfrak{gl}(\lambda)\), the Lie algebra B. Feigin introduced in [F].

The quadratic Casimir operator of \(\mathfrak{sl}(2)\)
\[
\Omega = 2XY + \frac{1}{2} H^2 + H
\] (1.1.2)
lies in the center of \(U(\mathfrak{sl}(2))\). Let \(I_\lambda\) be the two-sided ideal in the associative algebra \(U(\mathfrak{sl}(2))\) generated by \(\Omega - \frac{1}{2} (\lambda^2 - 1)\). It turns out that the associative algebra \(\mathfrak{A}_\lambda = U(\mathfrak{sl}(2))/I_\lambda\) is simple for \(\lambda \notin \mathbb{Z} \setminus \{0\}\), otherwise \(\mathfrak{A}_\lambda\) contains an ideal such that the quotient is isomorphic to the matrix algebra \(\text{Mat}(|\lambda|)\). Set (1.1.1)
\[
\mathfrak{A}_\lambda = \begin{cases} 
\mathfrak{A}_\lambda & \text{if } \lambda \notin \mathbb{Z} \setminus \{0\} \\
\text{Mat}(|\lambda|) & \text{otherwise}.
\end{cases}
\] (1.1.3)

As associative algebra with unit, \(\mathfrak{A}_\lambda\) is generated by \(X, Y\) and \(H\) subject to the relations (1.1.1) and
\[
XY = \frac{1}{4} (\lambda^2 - (H - 1)^2)
\] (1.1.4)
and one more relation for integer values of \(\lambda\):
\[
X|\lambda| = 0 \text{ if } \lambda \in \mathbb{Z} \setminus \{0\}.
\] (1.1.5)

In what follows we set \(\mathfrak{gl}(\lambda) := L(\mathfrak{A}_\lambda)\), the Lie algebra associated with the associative algebra \(\mathfrak{A}_\lambda\) when we replace the dot product with the bracket.

If in (1.3) we replace \(U(\mathfrak{sl}(2))\) with its \(q\)-deformation, \(U_q(\mathfrak{sl}(2))\), we get \(q\)-versions of the classical polynomials, identities, etc. The details will be given elsewhere.

The classics considered orthogonal polynomials with respect to a sign definite scalar product determined by a measure (or the corresponding sum for the difference equations). The existence of such a measure for complex values of certain parameters looks like a miracle in the traditional approaches, cf. \([\text{NU}], [\text{NSU}], [\text{VK}]\). Contrariwise, our approach makes it manifest that there exists a measure for any \(\lambda \in \mathbb{Z} \setminus \{0\}\) and for \(\lambda\) purely imaginary, see formula (9.2).

Moreover, we immediately see that (9.2) is sign definite for \(\lambda\) real and such that \(0 < |\lambda| < 1\).
We also see that since at generic point the scalar product is INDEFINITE, it is natural to consider such scalar products as well. Moreover, if we replace in the above construction $\mathfrak{sl}(2)$ with a Lie superalgebra $\mathfrak{g}$ we never have a sign definite scalar product on its Cartan subalgebra. Indefinite (but non-degenerate and symmetric) scalar products were considered in the literature, but the corresponding orthogonal polynomials satisfy differential equations of order $> 2$, cf. [LK], [MK], [D]. (We are thankful to T. Ya. Azizov for these references.) Such equations describe, perhaps, some reality; this is plausible due to the natural way they appear; M. Vasiliev’s ideas [V] show a possible scope of their applicability. Still, modern physicists prefer 2nd order equations, as describing processes more readily at hand. Though this attitude to solutions of higher order equations might slacken with progress of science, we are glad that polynomials orthogonal with respect to the restriction of $\text{tr}$ on $\mathfrak{gl}(\lambda)$ (and their superanalogs) satisfy a 2nd order difference relation.

1.2. Several variables. In the known to us attempts to generalize the classical continuous polynomials of discrete variable to several variables one actually considers the representation of the direct sum of several copies of $\mathfrak{A}_\lambda$, see (1.3) in the tensor products of Verma modules over $\mathfrak{sl}(2)$. This leads to the sums of products of polynomials in one variable; more exactly, to slightly more involved polynomials, but still, compositions of polynomials in one variable.

Our approach leads to intrinsically much more involved polynomials, namely, we suggest: in the above scheme, replace $\mathfrak{sl}(2)$ with ANY Lie algebra or Lie superalgebra $\mathfrak{g}$. This is possible provided $\mathfrak{g}$ possesses (a) Verma modules with finitely many generators and (b) several finite dimensional representations. The theory seems to be richest if $\mathfrak{g}$ is simple or “close” to simple (nontrivial central extension, the derivation algebra, etc.). Such generalizations of $\mathfrak{gl}(\lambda)$ appear in spin $> 2$ models and Calogero-Sutherland model, see [V].

The results obtained in this way will be considered in separate papers. Finally, recently one of us (A. S.) observed how to embrace ALL classical polynomials of discrete variable, not just Chebyshev and Hahn ones) in a scheme slightly generalizing the one described here. This generalization and a superization of the above scheme (with $\mathfrak{sl}(2)$ replaced with $\mathfrak{osp}(1|2)$ and $\mathfrak{sl}(1|2)$ was delivered at the Special Invited Lecture at Advanced Study Institute, Special Functions 2000: Current Perspective and Future Directions Arizona State University, Tempe, Arizona, U.S.A., May 29 to June 9, 2000.

§2. The trace formula

2.1. Lemma. Let $A$ be an associative algebra generated by the Lie algebra $\mathfrak{g}$, consisdered as a subspace, i.e., $A$ is a quotient of $U(\mathfrak{g})$. Then $[A,A] = [\mathfrak{g},A]$.

Proof. It suffices to show that $[x_1 \ldots x_n, a] \in [\mathfrak{g},A]$ for any $a \in A$ and $x_1, \ldots, x_n \in \mathfrak{g}$. The induction on $n$: for $n = 1$, the statement is obvious. Let $n > 1$. Let us make use of the identity $[ab,c] = [a,bc] + [b,ca]$ true in any associative algebra, cf. [M]. Then

$$[x_1(x_2 \ldots x_n), a] = [x_1, x_2 \ldots x_n a] \pm [x_2 \ldots x_n, ax_1].$$

By inductive hypothesis, the second summand in the right hand side lies in $[\mathfrak{g},A]$. \qed

2.1.1. Corollary. Let $\mathfrak{g}$ be a simple Lie algebra or $\mathfrak{osp}(1|2n)$. Then $U(\mathfrak{g}) = Z(U(\mathfrak{g})) \oplus [U(\mathfrak{g}), U(\mathfrak{g})]$.

Proof. Let $U(\mathfrak{g}) = Z(U(\mathfrak{g})) \oplus \left( \bigoplus_{\lambda \neq 0} n_\lambda L^\lambda \right)$ be the decomposition into irreducible $\mathfrak{g}$-modules with respect to the adjoint representation. Since the modules $L^\lambda$ are irreducible, $[\mathfrak{g}, U(\mathfrak{g})] =$
\[ \left( \bigoplus_{\lambda \neq 0} n_{\lambda} L^{\lambda} \right). \] (This argument does not work for Lie superalgebras distinct from \( \mathfrak{osp}(1|2n) \) due to the lack of complete reducibility.) \(\square\)

Hereafter \( \rho \) is a half-sum of positive roots.

2.1.2. Corollary. Let \( \mathfrak{h} \) be a Cartan subalgebra in \( \mathfrak{g} \), which is either a simple finite dimensional Lie algebra; let \( \lambda \in \mathfrak{h}^* \), and \( M_{\lambda}^{\rho} \) the Verma module with highest weight \( \lambda \) (NOT \( \lambda - \rho \)); let \( J_{\lambda} \) be the (left) ideal of \( U(\mathfrak{g}) \) equal to the kernel of the representation of \( U(\mathfrak{g}) \) in \( M_{\lambda}^{\rho} \). Set \( \mathfrak{A}_\lambda = U(\mathfrak{g})/J_{\lambda} \). Then \( \mathfrak{A}_\lambda = \mathbb{C} \oplus [\mathfrak{A}_\lambda, \mathfrak{A}_\lambda] \).

Proof. By Lemma 2.1 \([\mathfrak{A}_\lambda, \mathfrak{A}_\lambda] = [\mathfrak{g}, \mathfrak{A}_\lambda] \); hence, as in Corollary 2.1.1, \( \mathfrak{A}_\lambda = Z(\mathfrak{A}_\lambda) \oplus [\mathfrak{A}_\lambda, \mathfrak{A}_\lambda] \). But \( Z(\mathfrak{A}_\lambda) \) is a homomorphic image of \( Z(U(\mathfrak{g})) \); hence, is equal to \( \mathbb{C} \). \(\square\)

2.2. If \( \mathfrak{g} \) is a simple Lie algebra and \( \lambda \) is a highest weight of a finite dimensional module, then \( P(\lambda) = \dim L^\lambda \) is a polynomial in \( \lambda = (\lambda_1, \ldots, \lambda_n) \). Consider \( P(\lambda) \) for any \( \lambda \in \mathfrak{h}^* \) such that \( \prod_{i \neq j} (\lambda_i - \lambda_j) \neq 0 \). By Corollary 2.1.2, there is a unique functional \( \text{tr} \) on \( \mathfrak{A}_\lambda \) such that \( \text{tr}(ab) = \text{tr}(ba) \) and \( \text{tr}(1) = P(\lambda) \).

Lemma. Let \( u \in U(\mathfrak{g}) \), \( \tilde{u} \in \mathfrak{A}_\lambda \) its image. Then \( \text{tr}(\tilde{u}) \) is also a polynomial in \( \lambda \).

Proof. Let \( \sharp : Z(U(\mathfrak{g})) \oplus [U(\mathfrak{g}), U(\mathfrak{g})] \longrightarrow Z(U(\mathfrak{g})) \) be the natural projection. By definition of \( \sharp \) we have:
\[
\text{tr}(\tilde{u}) = \chi_{\lambda}(u^\sharp)P(\lambda) = \varphi(u^\sharp)(\lambda)P(\lambda),
\]
where \( \varphi : Z(U(\mathfrak{g})) \longrightarrow \mathbb{C}[\mathfrak{h}] \) is the Harish-Chandra homomorphism, see [Di]. \(\square\)

2.3. Theorem. Let \( \mathfrak{g} \) be a simple Lie algebra, \( \mathfrak{h} \) its Cartan subalgebra. Let us identify the space \( (\mathbb{C}[\mathfrak{h}])^* \) with the algebra of formal power series on indeterminates \( t = (t_1, \ldots, t_n) \) by the following formula:
\[
f \mapsto \sum_{\nu=(\nu_1,\ldots,\nu_n)} f \left( \frac{h_1^{\nu_1}}{\nu_1!} \cdots \frac{h_n^{\nu_n}}{\nu_n!} \right) t_1^{\nu_1} \cdots t_n^{\nu_n},
\]
where \( h_1, \ldots, h_n \) is a basis of \( \mathfrak{h} \) and \( \nu_i \in \mathbb{Z}_+ \) for all \( i \). Let \( e^\lambda(t) = e^{\lambda_1 t_1 + \cdots + \lambda_n t_n} \). (Recall that the \( \lambda_i \) are supposed to be distinct.)

Then to the functional \( \text{tr} \) there corresponds the series
\[
\psi(\lambda, t) = \bigoplus_{w \in W} \varepsilon(w)e^{w(\lambda)}(t).
\]

Proof. By Lemma 1.2 \( \text{tr}(h_1^{\nu_1} / \nu_1! \cdots h_n^{\nu_n} / \nu_n!) \) is a polynomial in \( \lambda \). If \( \lambda \in P_{++} \) (here \( P_{++} \) is the set of highest weights of finite dimensional modules), then
\[
\hat{\psi}(\lambda, t) = \sum_{\nu=(\nu_1,\ldots,\nu_n)} \text{tr} \left( \frac{h_1^{\nu_1}}{\nu_1!} \cdots \frac{h_n^{\nu_n}}{\nu_n!} \right) t_1^{\nu_1} \cdots t_n^{\nu_n}
\]
coincides with (2.3.2) thanks to the Weyl character formula. But \( \psi(\lambda, t) = \sum_{\nu} P_{\nu} t^\nu \), where \( P_{\nu} \) are some polynomials. Indeed, since \( P_{\nu}(\lambda) = \text{tr} \left( \frac{h_1^{\nu_1}}{\nu_1!} \cdots \frac{h_n^{\nu_n}}{\nu_n!} \right) \) for \( \lambda \in P_{++} \), this is true for any \( \lambda \in \mathfrak{h}^* \). Hence, \( \hat{\psi}(\lambda, t) = \psi(\lambda, t) \). \(\square\)
§3. Highest weights of quasi-finite modules over $\mathfrak{gl}(\lambda)$

Let $L^i$ be the irreducible $\mathfrak{sl}(2)$-module with highest weight $i$. The Lie algebra $\mathfrak{gl}(\lambda)$ considered as $\mathfrak{sl}(2)$-module with respect to the adjoint action of the latter is of the form

$$\mathfrak{gl}(\lambda) = \begin{cases} L^0 \oplus L^2 \oplus \ldots L^{2|\lambda|-2} & \text{for } \lambda \notin \mathbb{Z} \setminus \{0\} \\ L^0 \oplus L^2 \oplus \ldots & \text{otherwise.} \end{cases} \quad (3.0.1)$$

The Lie algebra $\mathfrak{gl}(\lambda)$ possesses a $\mathbb{Z}$-grading $\mathfrak{gl}(\lambda) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{gl}(\lambda)_i$, where

$$\mathfrak{gl}(\lambda)_i = \{ z \in \mathfrak{gl}(\lambda) : [H, z] = 2iz \} \quad (3.0.2)$$

and the increasing filtration: $\mathfrak{gl}(\lambda)_i = \bigoplus_{k \leq i} \mathfrak{gl}(\lambda)_k$. Therefore, we can speak about the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ of $\mathfrak{g} = \mathfrak{gl}(\lambda)$, where $\mathfrak{g}_\pm = \bigoplus_{i \geq 0} \mathfrak{gl}(\lambda)_{\pm i}$, as well as about parabolic subalgebras, highest weights, etc., see [KR].

Following [KR] we say that the $\mathfrak{g}$-module $V$ is quasi-finite if $V = \bigoplus_{j \in \mathbb{Z}} V_j$ and $\dim V_j < \infty$ for every $j$. (We only consider graded $\mathfrak{g}$-modules, i.e., such that $\mathfrak{g}_j V_j \subset V_{j+j}$.)

The well-known fact that $\mathfrak{gl}(n)$ has a nontrivial one-dimensional representation ($A \mapsto \text{tr} A$) has its counterpart for $\mathfrak{gl}(\lambda)$.

3.1. A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is called a parabolic one if it contains $\mathfrak{g}_0 \oplus \mathfrak{g}_+$ as a proper subalgebra. For example, for any $P \subseteq \mathbb{C}[H]$, set $\mathfrak{p}_-1(P) = PC[H]Y$ and let $\mathfrak{p}(P)$ be generated by $\mathfrak{p}_-1(P)$ and $\mathfrak{g}_0 \oplus \mathfrak{g}_+$. This is the minimal parabolic subalgebra corresponding to $P$.

**Lemma.** The minimal parabolic subalgebra $\mathfrak{p}(P)$ is of the form $\mathfrak{p}(P) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{p}(P)_k$, where $\mathfrak{p}(P)_k = \mathfrak{g}_k$ for $k \geq 0$ and $\mathfrak{p}(P)_{-k} = I_k \cdot Y^k$, where $I_k$ is the ideal of $\mathbb{C}[H]$ generated by $P(H)P(H+2) \ldots P(H+2k-2)$.

**Proof.** See [KR], mutatis mutandis. \hfill \Box

3.2. Let $\lambda \in \mathfrak{g}_0^*$, $\mathfrak{g}_+ v_\lambda = 0$ and $M^\lambda = \text{ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^\mathfrak{g} (\mathbb{C}v_\lambda)$ the corresponding Verma module; let $L^\lambda$ be the irreducible $\mathfrak{g}$-module with highest weight $\lambda$.

**Lemma.** ([KR]) The following conditions are equivalent:

i) $M^\lambda$ contains a vacuum vector that lies in $M^\lambda_{(-1)}$, the space of filtration $-1$;

ii) The module $L^\lambda$ is quasi-finite;

iii) $L^\lambda$ is a quotient of the generalized Verma module $M^{\lambda,P} = \text{ind}_{\mathfrak{p}(P)}^\mathfrak{g} (\mathbb{C}v_\lambda)$ for some $P \in \mathbb{C}[H]$.

3.3. Shoikhet calls $\text{deg} P$ the level of the module $M^{\lambda,P}$. In [Sh] he described modules of level 1 in more details than we give here.

We have already established that $\mathfrak{g}_0^* \simeq (\mathbb{C}[H])^* \simeq \mathbb{C}[t]$, the algebra isomorphism being given by the map $F$:

$$F : (\mathbb{C}[H])^* \rightarrow \mathbb{C}[t], \quad \theta \mapsto F_\theta (t) = \sum_{k=0}^\infty \frac{\theta (H^k)}{k!} t^k.$$

The following theorem describes the set of formal power series corresponding to the highest weights of quasi-finite modules. (Recall that a quasi-polynomial is an expression of the form $\sum R_i(t) e^{\alpha_i t}$, where the $R_i(t)$ are polynomials.)
The formal power series corresponding to the highest weight of any quasi-finite module over $\mathfrak{gl}(\lambda)$ is of the form $\frac{R(t)}{1 - e^{-2t}}$, where $R(t)$ is a quasi-polynomial such that $R(0) = 0$.

Proof. First, observe the following easy to verify statements: (a) if $\theta \in (\mathbb{C}[H])^*$ and $F_\theta(t) = \sum_{k=0}^\infty \frac{\theta(a^k)}{k!} t^k$ the corresponding series, then

$$\sum_{k=0}^\infty \frac{\theta((H + a)^k)}{k!} t^k = e^{at} F_\theta(t);$$

and (b) for any $R(H) \in \mathbb{C}[H]$ we have

$$\sum_{k=0}^\infty \frac{\theta(R(H) H^k)}{k!} t^k = R\left(\frac{d}{dt}\right) F_\theta(t) \quad \text{and} \quad \sum_{k=0}^\infty \frac{\theta(R(a)(H + a)^k)}{k!} t^k = R\left(\frac{d}{dt}\right) (e^{at} F_\theta(t)).$$

If $\theta$ is the highest weight of a quasi-finite module over $\mathfrak{gl}(\lambda)$, then by Lemma 3.2 there exists a polynomial $P(H) \in \mathbb{C}[H]$ such that $\theta$ can be extended to a one-dimensional representation of the minimal parabolic subalgebra corresponding to $P$. It is not difficult to verify that $[p,p] \cap \mathfrak{g}_0 = [g_1, p_-]$. Hence, $\theta([g_1, p_-]) = 0$.

Denote:

$$T(H) = XY = \frac{1}{4}(\lambda^2 - (H + 1)^2). \quad (3.2)$$

Then the condition $\theta([g_1, p_-]) = 0$ can be expressed as $\theta([X, P(H)H^k Y]) = 0$, or, equivalently, as $\theta(X P(H)H^k Y - P(H)H^k Y X) = 0$, or, as

$$\theta(T(H - 2) P(H - 2)(H - 2)^k - P(H)H^k T(H)) = 0.$$ 

Therefore,

$$\sum_{k=0}^\infty \frac{\theta(T(H - 2) P(H - 2)(H - 2)^k)}{k!} t^k - \sum_{k=0}^\infty \frac{\theta(T(H) P(H)H^k)}{k!} t^k = T\left(\frac{d}{dt}\right) P\left(\frac{d}{dt}\right)(e^{-2t} F_\theta(t)) - T\left(\frac{d}{dt}\right) P\left(\frac{d}{dt}\right)(F_\theta(t)) =$$

$$T\left(\frac{d}{dt}\right) P\left(\frac{d}{dt}\right)((e^{-2t} - 1) F_\theta(t)) = 0.$$ 

Thus, the function $(e^{-2t} - 1) F_\theta(t)$ is a solution of an ordinary differential equation with constant coefficients, hence, is a quasi-polynomial. Obviously, $R(0) = 0$.

Conversely, if $R(t)$ is a quasi-polynomial, $R(0) = 0$ and $P\left(\frac{d}{dt}\right) R(t) = 0$, then $T\left(\frac{d}{dt}\right) P\left(\frac{d}{dt}\right) R(t) = 0$ and, by setting $F_\theta(t) = \frac{R(t)}{e^{-2t} - 1}$ (which is well defined since $R(0) = 0$) we satisfy the condition $\theta([g_1, p_-]) = 0$ for $p_- = P(H)\mathfrak{g}_0 Y$. Hence, $\theta$ is the highest weight of a quasi-finite module. \qed

3.4. The trace on $\mathfrak{gl}(0)$. If $P = 1$, then the parabolic subalgebra of $\mathfrak{g} = \mathfrak{gl}(0)$ coincides with the whole algebra and, therefore, $[g, \mathfrak{g}] \cap \mathfrak{g}_0 = [\mathfrak{g}_1, g_{-1}] \cap \mathfrak{g}_0 \neq 1$. This proves that $\mathfrak{g} \neq [g, \mathfrak{g}]$ and, therefore, proves also existence of an invariant functional $\theta$ on $\mathfrak{g} = \mathfrak{gl}(\lambda)$. Therefore, $R = (1 - e^{-2t}) F_\theta(t)$ satisfies the equation $T\left(\frac{d}{dt}\right) R(t) = 0$; explicitly:

$$(\lambda^2 - (\frac{d}{dt} + 1)^2) R(t) = 0.$$
If $\lambda \neq 0$, then solutions are of the form $R(t) = c_1 e^{(\lambda - 1)t} + c_2 e^{-(\lambda + 1)t}$ and the initial condition $R(0) = 0$ implies that

$$F_\theta(t) = \frac{c e^{(\lambda - 1)t} - e^{-(\lambda + 1)t}}{1 - e^{-2t}}.$$  

Let us identify the scalars with the corresponding scalar matrices and norm the functional $\theta$ (trace) naturally, i.e., by setting $\theta(1) = \lambda$. This fixes $c$, namely, $c = 1$.

Clearly, if $\lambda = 0$, then the characteristic equation has multiple roots and

$$F_\theta(t) = c \cdot \frac{e^{-t}}{1 - e^{-2t}}.$$  

Thus, $\mathfrak{sl}(0)$ is an infinite dimensional Lie algebra with an infinite-dimensional identity module.

\section*{4. The Lie Algebras $\mathfrak{gl}(\lambda)$ and $\mathfrak{gl}^-(\infty)$, $\mathfrak{gl}^+(\infty)$, $\mathfrak{gl}^*(\infty)$}

Let $V$ be a vector space with a fixed basis $v_i$ for $i \in \mathbb{Z}$, let $V^+$ be its subspace spanned by $v_i$ for $i \geq 0$ and $V^-$ its subspace spanned by $v_i$ for $i < 0$. Let $\mathfrak{gl}^-(\infty)$, $\mathfrak{gl}^+(\infty)$, and $\mathfrak{gl}^*(\infty)$ be the Lie algebras of linear transformations of $V^-$, $V^+$ and $V$, respectively, whose matrices in the fixed bases are supported on a finite number (depending on the matrix) of diagonals parallel to the main one.

On $\mathfrak{gl}^*(\infty)$, there is a (unique) nontrivial 2-cocycle

$$c(A, B) = \text{tr} ([J, A]B), \quad \text{where } J = \sum_{i \leq 0} E_{ii} - \sum_{i > 0} E_{ii}. \quad (4.0)$$  

Denote the corresponding central extension by $\widehat{\mathfrak{gl}}(\infty)$; the bracket in $\widehat{\mathfrak{gl}}(\infty)$ is of the form

$$[A, B] = AB - BA + \text{tr} ([J, A]B) \cdot z, \quad \text{where } z \text{ is the new central element.}$$  

In particular, for matrix units, we have

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj} + \delta_{il}\delta_{jk}(\kappa(i) - \kappa(j)) \cdot z,$$

where $\kappa(i) = 1$ if $i \leq 0$ and 0 otherwise.

\subsection*{4.0. Harish-Chandra modules over $\mathfrak{sl}(2)$}

Harish-Chandra modules $M^{\lambda,s} = \text{Span}(v_i : i \in \mathbb{Z})$ for any $\lambda, s \in \mathbb{C}$ are defined over $\mathfrak{sl}(2)$ as the span of the $v_i$ that satisfy the following relations

\begin{align*}
Hv_i &= (s + 2i)v_i, \\
Xv_i &= \frac{1}{2} \sqrt{(\lambda - s - 2i - 1)(\lambda + s + 2i + 1)}v_{i+1}, \\
Yv_i &= \frac{1}{2} \sqrt{(\lambda - s + 2i + 1)(\lambda + s - 2i - 1)}v_{i-1}. \quad (4.0.1)
\end{align*}

It is not difficult to observe that the quadratic Casimir operator $\Omega$ acts on $M^{\lambda,s}$ as a scalar operator of multiplication by $\frac{1}{2}(\lambda^2 - 1)$ and $M^{\lambda,s} \cong M^{\lambda,s'}$ if $s - s' \in 2\mathbb{Z}$.

Suppose $\lambda \in \mathbb{Z} \setminus \{0\}$. Then by $\text{[Di]}$ if $\lambda - s \not\in 2\mathbb{Z} + 1$, then $M^{\lambda,s}$ is irreducible, whereas if $\lambda - s \in 2\mathbb{Z} + 1$, then $M^{\lambda,s} = M^{\lambda,s}_+ \oplus M^{\lambda,s}_-$, where $M^{\lambda,s}_+$ is an irreducible $\mathfrak{sl}(2)$-module with the lowest weight $\lambda + 1$ and $M^{\lambda,s}_-$ is an irreducible $\mathfrak{sl}(2)$-module with the highest weight $\lambda - 1$ and these module exhaust all irreducible $\mathfrak{sl}(2)$-modules with a diagonal $H$-action on which $\Omega$ acts as a scalar operator of multiplication by $\frac{1}{2}(\lambda^2 - 1)$.

There is a more transparent realization of Harish-Chandra modules, namely, set

$$X^+ = x \frac{\partial}{\partial y} - 2(\lambda - 1) x \frac{y}{y}, \quad X^- = y \frac{\partial}{\partial x}; \quad \text{hence, } H = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - (\lambda - 1)$$

and set $w_i = x^{a+i} y^{b-i}$. Then, $\lambda = a + b + 1$, $s = a - b$ and $w_i = v_i$ up to a scalar factor.
4.1. The module $M^{\lambda,s}$ over $\mathfrak{gl}(\lambda)$ determines an embedding $\phi: \mathfrak{gl}(\lambda) \rightarrow \mathfrak{gl}(\infty)$. Since $H^2(\mathfrak{gl}(\lambda)) = 0$ (see [2]), there exists a lift of the embedding $\phi$ to a homomorphism $\hat{\phi}: \mathfrak{gl}(\lambda) \rightarrow \hat{\mathfrak{gl}}(\infty)$ which is of the form $\hat{\phi}(u) = \phi(u) + \theta(u) \cdot z$, where $\theta$ is a linear functional on $\mathfrak{gl}(\lambda)$ such that $\theta([u, v]) = c(\phi(u), \phi(v))$ for the cocycle on $\mathfrak{gl}(\infty)$ which makes it into $\hat{\mathfrak{gl}}(\infty)$. Observe that $\theta$ is determined uniquely up to a functional proportional to the trace on $\mathfrak{gl}(\lambda)$.

The following theorem describes $\hat{\phi}$ more explicitly.

**Theorem.** Let $g = \mathfrak{gl}(\lambda)$. Then $\hat{\phi}(g_i) = \phi(g_i)$ for $i \neq 0$ and

$$\hat{\phi}(e^{tH}) = \phi(e^{tH}) - \frac{e^{st} - e^{-(\lambda+1)t}}{1 - e^{-2t}}z,$$  \hspace{1cm} (4.1.1)

**Proof.**

\begin{align*}
\phi(X) &= \sum \alpha_i E_{i+1,i}, \text{ where } \alpha_i = \frac{1}{2} \sqrt{(\lambda - s - 2i - 1)(\lambda + s + 2i + 1)}, \\
\phi(H) &= \sum \gamma_i E_{i,i}, \text{ where } \gamma_i = s + 2i, \\
\phi(Y) &= \sum \beta_i E_{i-1,i}, \text{ where } \beta_i = \frac{1}{2} \sqrt{(\lambda - s - 2i + 1)(\lambda + s + 2i - 1)}.
\end{align*}

Therefore,

$$[J, \phi(X)] = \sum (\kappa(i) - \kappa(i + 1)) \alpha_i E_{i+1,i}$$

and $\phi(H^k)\phi(Y) = \sum \gamma_i^k \beta_{i+1} E_{i,i+1}$. Hence,

$$[J, \phi(X)]\phi(H^k)\phi(Y) = \sum \gamma_i^k \beta_{i+1} \alpha_i (\kappa(i) - \kappa(i + 1)) E_{i+1,i+1}$$

and

$$c(\phi(X), \phi(H^kY)) = \text{tr}([J, \phi(X)]\phi(H^kY)) = \sum \gamma_i^k \beta_{i+1} \alpha_i (\kappa(i) - \kappa(i + 1)) = \gamma_0^k \beta_1 \alpha_0 = \frac{1}{4} s^k (\lambda^2 - (s + 1)^2) = T(s)s^k.$$

Therefore,

$$T(\frac{d}{dt})(e^{st}) = T(s)e^{st} = \sum_{k \geq 0} \frac{T(s)s^k}{k!} t^k =$$

$$\sum_{k \geq 0} \frac{c(\phi(X), \phi(H^kY))}{k!} t^k =$$

$$\sum_{k \geq 0} \frac{\theta([X, H^kY])}{k!} t^k = \sum_{k \geq 0} \frac{\theta(XH^kY - H^kYX)}{k!} t^k =$$

$$\sum_{k \geq 0} \frac{\theta(T(H - 2)(H - 2)^k - H^kT(H))}{k!} t^k =$$

$$T(\frac{d}{dt})((e^{-2t} - 1)\theta(e^{tH})).$$

Therefore, $T(\frac{d}{dt})((e^{-2t} - 1)\theta(e^{tH} - e^{st})) = 0$ and

$$(e^{-2t} - 1)\theta(e^{tH}) = \begin{cases} 
  e^{st} + c_1 e^{-(\lambda+1)t} + c_2 e^{(\lambda-1)t} & \text{if } \lambda \neq 0 \\
  e^{st} + c_1 e^{-t} + c_2 te^{-t} & \text{if } \lambda = 0.
\end{cases}$$

In both cases, subtracting the summand proportional to the trace we get

$$(e^{-2t} - 1)\theta(e^{tH}) = e^{st} - e^{-(\lambda+1)t}.$$
4.2. $\mathfrak{gl}(\infty)$ over the algebra of truncated polynomials. Let $R_m = \mathbb{C}[\varepsilon]$, where $\varepsilon^{m+1} = 0$. By extending the module $M^\lambda$, a module $M^\lambda_{R_m}$ over $R_m$, i.e., by assuming that $s \in R_m$, we may define the action of $\mathfrak{gl}(2)$ in $M^\lambda_{R_m}$ by the same formulas as in $M^\lambda_R$. If the $v_i$ is the initial basis, then for the fixed basis of the extended module $M^\lambda_{R_m}$ we take $\varepsilon^j v_i$ for all $i \in \mathbb{Z}$ and $j = 1, \ldots, m$.

Observe that in this basis of $M^\lambda_{R_m}$ the action of $H$ is not a diagonal one.

The algebra $\mathfrak{gl}(\infty; R_m)$ and its subalgebras $\mathfrak{gl}^\pm(\infty; R_m)$ are naturally defined and the central extension $\hat{\mathfrak{gl}}(\infty; R_m)$ is determined by the same formula.

By considering $s+\varepsilon$ instead of $s \in \mathbb{C}$, we get the following corollary of Theorem 4.1:

**Corollary.** The homomorphism $\hat{\phi} : \mathfrak{gl}(\lambda) \rightarrow \hat{\mathfrak{gl}}(\infty; R_m)$ induced by the embedding $\phi : \mathfrak{gl}(\lambda) \rightarrow \mathfrak{gl}(\infty; R_m)$ satisfies $\hat{\phi}(\mathfrak{gl}(\lambda)_i) = \phi(\mathfrak{gl}(\lambda)_i)$ for $i \neq 0$ and

$$\hat{\phi}(e^{tH}) = \phi(e^{tH}) - \left(\frac{e^{st} - e^{-(\lambda+1)t}}{1 - e^{-2t}} + \frac{e^{st}}{1 - e^{-2t}} \sum_{j=1}^m \varepsilon^j j! \right) z. \quad (4.2)$$

4.3. **Remark.** In order to describe quasi-finite modules over $\mathfrak{gl}(\lambda)$ we are “forced” to consider the central extension $\hat{\mathfrak{gl}}(\infty)$ because $\hat{\mathfrak{gl}}(\infty)$ has more quasi-finite representations than $\mathfrak{gl}(\infty)$ does and it turns out to be insufficient to consider $\mathfrak{gl}(\infty)$-modules only.

4.4. The highest weights of quasi-finite modules. The following statements describe the restrictions onto the highest weight of an irreducible module necessary and sufficient for the module to be quasi-finite.

Denote by $\mathfrak{gl}_f(\infty)$ and $\mathfrak{gl}^*_f(\infty)$ the subalgebras of $\mathfrak{gl}(\infty)$ and $\mathfrak{gl}^*(\infty)$, respectively, consisting of matrices with finite support. Let $\mathfrak{h}$ denote the Cartan subalgebra of any of these Lie algebras; consider it spanned by the diagonal elements $E_{ii}$. Observe that the restriction of the embedding $\hat{\phi} : \mathfrak{gl}_f(\infty) \rightarrow \hat{\mathfrak{gl}}(\infty)$ onto $\mathfrak{h}$ is of the form $E_{ii} \mapsto \hat{\phi}(E_{ii}) + \kappa(i)z$, where $\phi$ is the natural embedding of $\mathfrak{gl}_f(\infty)$ into $\mathfrak{gl}(\infty)$. If $\theta$ is a weight of a (highest weight) module over one of the above Lie algebras, both $\mathfrak{gl}(\infty)$ and its subalgebras and $\mathfrak{gl}_f(\infty)$ and its subalgebras, we may, considering $\mathfrak{h}$ in the above way, determine the coordinates $\theta_i = \theta(E_{ii})$ of $\theta$.

4.4.1. **Proposition.** The $\mathfrak{gl}_f(\infty)$-module with highest weight $\Lambda$ is quasi-finite if and only if there are only finitely many distinct coordinates $\lambda_i$, where $\lambda_i = \Lambda(E_{ii})$.

**Proof.** If $V$ is quasi-finite and generated by the highest weight vector $v$. Since dim $V < \infty$, it follows that $V_1 = \text{Span}(E_{i+1,i}v)_{i \in I}$ for a finite set $I$. Let $k \notin I$ and $k+1 \notin I$. Then $E_{k+1,k}v = \sum_{i \in I} \alpha_i E_{i+1,i}v$, so $E_{k,k+1}(E_{k+1,k}v) = 0$ or, equivalently, $(E_{k,k} - E_{k+1,k+1})v = 0$, i.e., $\lambda_k = \lambda_{k+1}$.

The opposite implication is obvious. \hfill $\square$

4.4.2. **Proposition.** The $\mathfrak{gl}(\infty)$-module $V$ with highest weight $\Lambda$, where $\lambda_i = \lambda(E_{ii})$, is quasi-finite if and only if there are only finitely many nonzero coordinates $\lambda_i$, where $\lambda_i = \Lambda(E_{ii})$.

**Proof.** By considering $V$ as $\mathfrak{gl}_f(\infty)$-module we deduce that there are only finitely many distinct coordinates $\lambda_i$, so we can set $\lambda_k = \lambda_+$ and $\lambda_{-k} = \lambda_-$ for sufficiently large $k$. Hence, $E_{k,k+1}(E_{k+1,k}v) = 0$ and, thanks to irreducibility of $V$, $E_{k+1,k}v = 0$ for all $k$ but finitely many.
Let \( A = \sum_{i \in \mathbb{Z}} E_{i,i+1} \) and \( B_k = \sum_{i \geq k} E_{i+1,i} \). Then

\[
[A, B_k] = \sum_{i \geq k} E_{i,i} - \sum_{i \geq k} E_{i+1,i+1} = E_{k,k}.
\]

Therefore,

\[
[A, B_k]v = E_{k,k}v = \lambda_k v = AB_k v - B_k Av = AB_k v =
A(\sum_{i \geq k, i \in I} E_{i+1,i} v) = \sum_{i \geq k, i \in I} [A, E_{i+1,i}] v = \sum_{i \geq k, i \in I} (E_{i,i} - E_{i+1,i+1}) v =
\left( \sum_{i \geq k, i \in I} (\lambda_i - \lambda_{i+1}) \right) v = (\lambda_k - \lambda_j) v,
\]

where \( j \) is the largest index in the finite set \( I \). Therefore, \( \lambda_k = \lambda_k - \lambda_j \) and \( \lambda_j = 0 \).

Similarly, if \( C_k = \sum_{i \leq k} E_{i+1,i} \), then

\[
[A, C_k] = \sum_{i \leq k} E_{i,i} - \sum_{i \leq k} E_{i+1,i+1} = E_{k+1,k+1}.
\]

Hence,

\[
[A, C_k]v = \lambda_{k+1} v = AC_k v = A(\sum_{i \leq k} E_{i+1,i}) v = \sum_{i \leq k} [A, E_{i+1,i}] v
\sum_{i \leq k} (E_{i,i} - E_{i+1,i+1}) v = \sum_{i \leq k} (\lambda_i - \lambda_{i+1}) v = (\lambda_j - \lambda_{k+1}) v.
\]

So, \( \lambda_j = 0 \) for sufficiently large \( j \). The converse statement follows from Proposition 4.4.1.

**4.4.3. Proposition.** The \( \widehat{gl}(\infty) \)-module \( V \) with highest weight \((\Lambda, c)\), where \( c \) is the value of the central charge on \( z \), is quasi-finite if and only if there are only finitely many nonzero coordinates \( \lambda_i \), where \( \lambda_i = \Lambda(E_{ii}) \).

**Proof.** As in the proof of Proposition 4.4.2 we show that there are only finitely many distinct coordinates \( \lambda_i \), and \( E_{k+1,k} v \neq 0 \) for finitely many values of \( k \).

Let \( A = \sum_{i \in \mathbb{Z}} E_{i,i+1} \) and \( J = \sum_{i \leq 0} E_{i,i} \). Then \( [A, J] = E_{01} \), so \( c(A, B) = \text{tr} (E_{01} B) \) for any \( B \in \mathfrak{gl}(\infty) \), so \( c(A, B_k) = \kappa(k) \). We have

\[
[A, B_k]v = E_{k,k}v = (E_{k,k} + \kappa(k) z) v = AB_k v =
A(\sum_{i \geq k, i \in I} E_{i+1,i} v) = \sum_{i \geq k, i \in I} [A, E_{i+1,i}] v = \sum_{i \geq k, i \in I} (E_{i,i} - E_{i+1,i+1} + \delta_{i0} z) v =
\kappa(k) z + \left( \sum_{i \geq k, i \in I} (\lambda_i - \lambda_{i+1}) \right) v = \kappa(k) cv + (\lambda_k - \lambda_+) v = (\kappa(k)c + (\lambda_k) v,
\]

so \( \lambda_+ = 0 \).

For \( C_k = \sum_{i \leq k} E_{i+1,i} \), we have \( c(A, C_k) = \kappa(-k) \), so

\[
[A, C_k]v = (-E_{k+1,k+1} - \kappa(-k)) v =
(-\lambda_{k+1} - \kappa(-k)) v = AC_k v = A \sum_{i \leq k} E_{i+1,i} v = \sum_{i \leq k} [A, E_{i+1,i}] v =
\sum_{i \leq k} (E_{i,i} - E_{i+1,i+1} + \delta_{i0} z) v = (\kappa(-k) c + \lambda_- - \lambda_{k+1}) v.
\]

So, \( \lambda_- = 0 \).
4.4.4. Theorem. Let $\mathfrak{g}$ be one of the Lie algebras $\mathfrak{gl}(\infty; R_m)$ or $\mathfrak{gl}^\pm(\infty; R_m)$ or their hatted versions. The $\mathfrak{g}$-module $V$ with highest weight $\theta$ given by its coordinates $\theta_{ij} = (\epsilon^j E_{ii})$ is quasi-finite if and only if there are only finitely many nonzero coordinates $\theta_{ij}$.

Proof. The proof is similar to that of 4.4.1–4.4.3.

Let $V$ be one of the modules $M_{R_m}^{\lambda,s}$ or, if $M_{R_m}^{\lambda,s}$ is reducible, one of its irreducible submodules. We obtain a homomorphism of $\mathfrak{gl}(\lambda)$ into $\mathfrak{g} = \mathfrak{gl}(\infty; R_m)$ or $\mathfrak{gl}^\pm(\infty; R_m)$. Let $\theta$ be a linear functional satisfying the conditions of Theorem 4.4.4. We may consider $\theta$ as a linear functional on the Cartan subalgebra of $\mathfrak{gl}(\lambda)$. Let us calculate the corresponding generating function.

4.5. Theorem. The generating function $F_\theta(t)$ for the functional $\theta$ is as follows:

i) For $M_{R_m}^{\lambda,s}$:

$$F_\theta(t) = \frac{\sum_{i \in \mathbb{Z}} e^{(s-2i)t} \sum_{j=0}^{m} (\theta_{ij} - \theta_{i-1,j}) t^j}{1 - e^{-2t}} - \left( \frac{e^{st} - e^{-(\lambda+1)t}}{1 - e^{-2t}} c + \frac{e^{st} \sum_{j=1}^{m} c_j}{1 - e^{-2t}} \right)$$  \hspace{1cm} (4.5.1)

ii) For the module with highest weight $\lambda - 1$:

$$F_\theta(t) = \frac{\sum_{i \in \mathbb{Z}} e^{(\lambda-2i-1)t} \sum_{j=0}^{m} (\theta_{ij} - \theta_{i-1,j}) t^j}{1 - e^{-2t}}$$  \hspace{1cm} (4.5.2)

iii) For the module with highest weight $-\lambda - 1$:

$$F_\theta(t) = \frac{\sum_{i \in \mathbb{Z}} e^{(-\lambda-2i-1)t} \sum_{j=0}^{m} (\theta_{ij} - \theta_{i-1,j}) t^j}{1 - e^{-2t}}$$  \hspace{1cm} (4.5.3)

iv) For the module with lowest weight $\lambda + 1$:

$$F_\theta(t) = \frac{\sum_{i \in \mathbb{Z}} e^{(\lambda+2i+1)t} \sum_{j=0}^{m} (\theta_{ij} - \theta_{i-1,j}) t^j}{1 - e^{-2t}}$$  \hspace{1cm} (4.5.4)

v) For the module with lowest weight $1 - \lambda$:

$$F_\theta(t) = \frac{\sum_{i \in \mathbb{Z}} e^{(1-\lambda-2i)t} \sum_{j=0}^{m} (\theta_{ij} - \theta_{i-1,j}) t^j}{1 - e^{-2t}}$$  \hspace{1cm} (4.5.5)

Proof. We will only prove i): the other cases are similar.

Let $\phi$ be the homomorphism that determines the $\mathfrak{gl}(\lambda)$-action on $M_{R_m}^{\lambda,s}$. Then $\phi(H) = \sum_{i \in \mathbb{Z}} (s + \epsilon - 2i) E_{ii}$, so $\phi(e^{tH}) = \sum_{i \in \mathbb{Z}} e^{(s+\epsilon-2i)t} E_{ii}$ and

$$\phi((1 - e^{-2t}) e^{tH}) = \phi(e^{tH}) - \phi(e^{t(H-2)}) = \sum_{i \in \mathbb{Z}} e^{(s+\epsilon-2i)t} (E_{ii} - E_{i-1,i-1}) =$$

$$\sum_{i \in \mathbb{Z}} e^{(s-2i)t} \sum_{j=0}^{m} \epsilon^j (E_{ii} - E_{i-1,i-1}) \frac{t^j}{j!}.$$  \hspace{1cm} 

Therefore,

$$(1 - e^{-2t}) F_\theta(t) = \theta (\phi(e^{tH}) - \phi(e^{t(H-2)})) =$$

$$\sum_{i \in \mathbb{Z}} e^{(s-2i)t} \sum_{j=0}^{m} \frac{\theta_{ij} - \theta_{i-1,j}}{j!} t^j.$$
§5. Quasi-finite modules over \( \mathfrak{gl}(\lambda) \)

In what follows we will show that not all quasi-finite modules over \( \mathfrak{gl}(\lambda) \) can be represented in a canonical form similar to that of \( W_{1+\infty} \) from \( \text{KR} \).

Let \( \mathfrak{gl}^{\text{hol}}(\lambda) \) be the holomorphic completion of \( \mathfrak{gl}(\lambda) \), i.e., \( \mathfrak{gl}^{\text{hol}}(\lambda) \) is the algebra of functions in \( H \) holomorphic on the whole complex line \( \mathbb{C} \) and \( \mathfrak{gl}^{\text{hol}}(\lambda) = \{ f(H)X^i \} \) for \( i > 0 \) whereas \( \mathfrak{gl}^{\text{hol}}(\lambda) = \{ f(H)Y^{-i} \} \) for \( i < 0 \) and \( f \in \mathfrak{gl}_0^{\text{hol}}(\lambda) \). The relations in the completed algebra follow from (1.1) and (1.4), namely, they are

\[
X f(H) = f(H - 2)X, \quad Y f(H) = f(H + 2)Y \quad \text{and} \quad XY = \frac{1}{4}(\lambda^2 - (H - 1)^2).
\]

The following Proposition is proved via the same lines as its counterpart in \( \text{KR} \).

5.1. Proposition. Let \( V \) be quasi-finite \( \mathfrak{gl}(\lambda) \)-module. Then the \( \mathfrak{gl}(\lambda) \)-action can be naturally extended to a \( \mathfrak{gl}_i^{\text{hol}}(\lambda) \)-action for \( i \neq 0 \).

5.2. On \( \mathbb{C} \), introduce an equivalence relation by setting

\[
[s] = \begin{cases} 
  s + 2\mathbb{Z} & \text{if } s + \lambda \notin 2\mathbb{Z} + 1 \\
  s - 2\mathbb{Z}_+ & \text{if } s = \pm \lambda + 1 \\
  s + 2\mathbb{Z}_+ & \text{if } s = \pm \lambda - 1 
\end{cases}
\]

To each class \( [s] \) assign an irreducible \( \mathfrak{sl}(2) \)-module \( M^s \) with a diagonal action of \( H \) whose set of eigenvalues coincides with \( [s] \); denote by \( \mathfrak{gl}_s(\infty) \) the Lie algebra of linear transformations of \( M^s \) with finitely many nonzero diagonals in the \( H \)-diagonal basis.

The following Theorem is similar and is proved via the same lines as its counterpart in \( \text{KR} \).

Theorem. Consider \( \bigoplus_{i=1}^{k} M^s_{R_{m_i}} \), where the \( [s_i] \) are distinct and the \( m_i \) are non-negative integers. Let \( \phi : \mathfrak{gl}(\lambda) \longrightarrow \mathfrak{g} = \bigoplus_{i=1}^{k} \mathfrak{gl}_s(\infty; R_{m_i}) \) determined by this module. Then, for any quasi-finite \( \mathfrak{g} \)-module \( V \), any its \( \mathfrak{gl}(\lambda) \)-submodule is a \( \mathfrak{g} \)-submodule. In particular, if \( V \) is irreducible as \( \mathfrak{g} \)-module, it is irreducible as a \( \mathfrak{gl}(\lambda) \)-module.

Following \( \text{KR} \), let us describe now the structure of quasi-finite \( \mathfrak{gl}(\lambda) \)-modules. Let \( \theta \) be the highest weight of a quasi-finite \( \mathfrak{gl}(\lambda) \)-module and \( F_\theta(t) = \frac{R(t)}{1 - e^{-2t}} \), where \( R(t) = \sum r_i(t)e^{s_it} \) the corresponding formal power series, see 3.2. The polynomials \( r_i(t) \) will be referred to as multiplicities.

5.3. For every \( s \), denote by \( R_s(t) \) the sum of all quasi-polynomials with the exponents from class \( [s] \). Then \( R(t) = \sum R_s(t) \), where the sum runs over representatives of different equivalence classes. Let \( R_s(t) = \sum r_{i,s}(t)e^{(s-2i)t} \). The following properties of quasi-polynomials follow easily from definitions:

i) \( R(0) = 0 \);

ii) the sum of all multiplicities of \( R_{\lambda+1}(t) \) as well as that of \( R_{-\lambda+1}(t) \) are equal to 0;

iii) the sum of all multiplicities of \( R_{\lambda-1}(t) \) as well as that of \( R_{-\lambda-1}(t) \) are equal to a constant.

Denote: \( \Lambda = [\lambda - 1] \cap [\lambda + 1] \cap [-\lambda - 1] \cap [-\lambda + 1] \). To a quasi-polynomial \( R \) satisfying i)–iii) assign a \( \mathfrak{gl}(\lambda) \)-module \( V(s) \) as follows:
1) if \( s \not\in \Lambda \), let \( V(s) \) be the irreducible \( \widehat{\mathfrak{gl}}(\infty) \)-module with central charges \( c_j = -\sum_{i \leq l} r_i^{(j)}(0) \), where \( j = 0, \ldots, \max(\deg r_i) \), and the other coordinates of the highest weight are \( \theta_{ij} = r_i^{(j)}(0) + \delta_{i0} c_j \);

2) if \( s \not\in [\lambda - 1] \cap [-\lambda - 1] \) and \( R_s = \sum r_i(y)e^{s-2t} \), set \( c = -\sum r_i(0) \) and \( \theta_{ij} = r_i^{(j)}(0) + \delta_{i0} c_j \); let \( V(s) \) be the corresponding \( \mathfrak{gl}^-(\infty) \)-module.

3) if \( s \not\in [-\lambda + 1] \cap [\lambda + 1] \), set \( \theta_{ij} = r_i^{(j)}(0) \); let \( V(s) \) be the corresponding \( \mathfrak{gl}^+(\infty) \)-module.

**Theorem.** Let the quasi-polynomial \( R = \sum_{i=1}^{k} R_i(t) \) be decomposed with respect to the distinct equivalence classes of the exponents; let \( R \) satisfy the conditions i)–iii) above. Then the irreducible quasi-finite module with highest weight \( R(t) \) is isomorphic to

\[
V = V(s_1) \otimes \ldots \otimes V(s_k) \otimes (\alpha \text{ tr}),
\]

where \( \alpha \text{ tr} \) is the irreducible 1-dimensional module corresponding to the trace on \( \mathfrak{gl}(\lambda) \) and the \( V(s_i) \) are constructed according to 1)–3) above.

**Proof.** By Theorem 4.5 \( V \) is irreducible as \( \mathfrak{gl}(\lambda) \)-module. Therefore, it suffices to show that the restriction of the highest weight of this module onto \( \mathfrak{gl}(\lambda) \) is equal to \( \frac{R(t)}{1 - e^{-2t}} \). Let \( s = s_i \) be one of the exponents such that \( s \not\in \Lambda \). Then Proposition 5.1 implies that there exists a highest weight \( \theta_s \) for \( \widehat{\mathfrak{gl}}(\infty; R_m) \) whose generating function is equal to \( \frac{R_s(t) + c_s e^{-(\lambda+1)t}}{1 - e^{-2t}} \), where \( c_s = -R_s(0) \).

If \( s \in \Lambda \), then the same Proposition implies that the generating function is equal to \( \frac{R_s(t)}{1 - e^{-2t}} \), where the sum of all the exponents of \( R_s \) vanishes.

Finally, the generating function of the trace is equal to \( \alpha \frac{e^{(\lambda-1)t} - e^{-(\lambda+1)t}}{1 - e^{-2t}} \). Therefore, for our \( V \), the generating function is of the form \( \frac{R(t)}{1 - e^{-2t}} \), where

\[
R(t) = \sum_{s \not\in \Lambda} (R_s(t) + c_s e^{-(\lambda+1)t}) + \sum_{s \in \Lambda} R_s(t) + R_{\lambda+1}(t) + R'_{-\lambda-1}(t) + R'_{\lambda-1}(t) + \alpha (e^{(\lambda-1)t} - e^{-(\lambda+1)t}) = (R'_{-\lambda-1}(t) - \alpha e^{-(\lambda+1)t}) + (R'_{\lambda-1}(t) + \alpha e^{(\lambda-1)t}) = \sum_{s \not\in \Lambda} R_s(t).
\]

\( \square \)

**§6. Unitary modules over \( \mathfrak{gl}(\lambda) \)**

Recall that an *anti-involution* of the Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \) is an \( \mathbb{R} \)-linear map \( \omega : \mathfrak{g} \rightarrow \mathfrak{g} \) such that

\[
\omega(\alpha x) = \bar{\alpha} \omega(x), \quad \omega([x, y]) = [\omega(x), \omega(y)] \quad \text{and} \quad \omega^2 = \text{id}
\]

for any \( \alpha \in \mathbb{C} \) and \( x, y \in \mathfrak{g} \).
In presence of an anti-involution $\omega$ we can endow $V^*$, the dual of the $g$-module $V$, with another $g$-module structure, namely, set
\[(xl)(v) = l(\omega(x)v)\] for any $l \in V^*$, $v \in V$ and $x \in g$. \hfill (6.0.1)

In particular, if $g = g_- \oplus g_0 \oplus g_+$ and $\omega$ interchanges $g_-$ with $g_+$, then a $g$-homomorphism $M \rightarrow M^*$ is well-defined, where $M$ is a Verma module with highest weight vector $v$ and the $g$-module structure on $M^*$ is given by (6.0.1).

Indeed, let $v^* \in M^*$ be such that $v^*(v) = 1$ and $v^*(u) = 0$ for all $u$ of lesser weight than that of $v$. Hence, if $x \in g_+$, then $(xv^*)(u) = v^*(\omega(x)u) = 0$, so $v^*$ is also a highest vector and if $\theta(\omega(H)) = \theta(H)$, where $\theta$ is the weight of $v$, then the weight of $v^*$ is also equal to $\theta$. Hence, there exists a $g$-isomorphism $M \rightarrow M^*$ or an Hermitian $g$-invariant form $\langle \cdot, \cdot \rangle$ on $M$. If $\langle \cdot, \cdot \rangle$ is positive definite, $M$ is called unitary.

In this section we indicate the conditions on the highest weight of the quasi-finite $gl(\lambda)$-module to be unitary. First, let us describe automorphisms of $gl(\lambda)$.

As is well-known, the description of automorphisms of the algebra of functions, $\mathcal{F}$, especially polynomial ones, is a wild problem. Hence, so is the problem of description of automorphisms of the Lie algebra $\mathfrak{der}(\mathcal{F})$ of the differentiations of $\mathcal{F}$. To diminish the amount of automorphisms to a reasonable number, it is natural to consider the outer automorphisms only, i.e., the classes of automorphisms modulo the group of inner ones. Which automorphisms should be considered as inner ones? In the case of $\mathfrak{der}(\mathcal{F})$ the automorphisms induced by the automorphisms of $\mathcal{F}$ naturally qualify. Similarly, we say that the automorphism $\phi$ of $gl(\lambda)$ is an inner one if it also serves as an automorphism of the associative algebra $A_\lambda$.

We extend this definition to $LU_g(\lambda)$ and call its automorphism an inner one if it also serves as an automorphism of the associative algebra $U_g(\lambda)$.

**Problem.** How to define inner automorphisms for Lie subalgebras of $LU_g(\lambda)$, such as $\mathfrak{o}/\mathfrak{sp}(\lambda)$, to start with?

Recall that a linear map $\phi : g \rightarrow g$ is an anti-automorphism of the Lie algebra $g$ if $\phi([x, y]) = [\phi(y), \phi(x)]$. For the associative algebras, definition of an anti-automorphism is similar. For example, the map $t$ such that $t|_g = -\text{id}$ is called the principal anti-automorphism of the Lie algebra $g$. The principal anti-automorphism of $g$ can be extended to an anti-automorphism of the associative algebra $U(g)$: $t(x \otimes y) = t(y) \otimes t(x)$. Clearly, $t$ preserves Casimir elements.

**6.1. Theorem.** The group of outer automorphisms of $gl(\lambda)$ is isomorphic to $\mathbb{Z}/2$ and is generated by the class of $-t$, where $-t : x \mapsto -t(x)$.

**Proof.** Let $\psi \in Aut(gl(\lambda))$. Then $\psi(Y)$, $\psi(H)$, and $\psi(X)$ span a Lie algebra isomorphic to $sl(2)$. Moreover, they generate $A_\lambda$ because the weight of $\psi(X)^n$ is equal to $2n$ with respect to $\psi(H)$. Hence, there exists a homomorphism
\[\tau : U(sl(2)) \rightarrow A_\lambda,\quad X \mapsto \psi(X), \quad H \mapsto \psi(H), \quad Y \mapsto \psi(Y).\]

Since $\tau(\Omega) \in \mathbb{C}$, there exists a $\mu \in \mathbb{C}$ such that $\tau(\Omega) = \frac{1}{2}(\mu^2 - 1)$. Therefore, $\tau$ is surjective, hence, an isomorphism $A_\mu \rightarrow A_\lambda$. In a very difficult technical paper [Dix], Dixmier proved that this can only happen if $\lambda^2 = \mu^2$. Making use of this, we may assume that $\lambda = \mu$ and $\tau$ is an automorphism. Let $\psi_1 = \tau^{-1}\psi$. Then $\psi_1|_{sl(2)} = \text{id}$.

Therefore, $\psi_1$ is an automorphism of $gl(\lambda)$, as $sl(2)$-module. Therefore, $\psi_1|_{L^2i} = c_i \in \mathbb{C}$. Since the Lie algebra $gl(\lambda)$ is generated by $L^2$ and $L^4$, we deduce that $c_i = c_{2i}^{-1}$ for every $i \geq 1$. Moreover, since $[L^4, L^4] \supset L^2$, it follows that $c_2^2 = 1$. Hence, $c_2 = \pm 1$; if $c_2 = 1$, then $\psi_1 = \text{id}$, and if $c_2 = -1$, then $\psi_1 = \phi$. \hfill $\Box$
6.2. **Theorem.** If \( \lambda^2 \not\in \mathbb{R} \), then \( \mathfrak{gl}(\lambda) \) has no real forms, i.e., no involutive anti-linear automorphisms.

**Proof.** Let \( \omega \) be an involutive anti-linear automorphism of \( \mathfrak{gl}(\lambda) \). Then \( \omega(X), \omega(H) \) and \( \omega(Y) \) generate \( \mathfrak{a}_\lambda \). Hence, there exists a surjective homomorphism

\[
\tau : U(\mathfrak{sl}(2)) \longrightarrow \mathfrak{a}_\lambda, \quad \tau = \text{id on } X, H, Y;
\]

hence, we obtain an isomorphism \( \tau : \mathfrak{a}_\mu \longrightarrow \mathfrak{a}_\lambda \). By [Di], this may only happen if \( \lambda^2 = \mu^2 \). Hence, we may assume that \( \lambda = \mu \) and \( \tau \) is an automorphism. Then \( \omega_1 = \tau^{-1}\omega \) is an antilinear automorphism of \( \mathfrak{gl}(\lambda) \) such that

\[
\omega_1(X) = X, \quad \omega_1(H) = H, \quad \omega_1(Y) = Y.
\]

Set \( z_i = (\text{ad } X)^i(Y^2) \); in particular, \( z_0 = z \). Then \( \text{Span}(z_i : i = 1, \ldots, 4) \) form a basis of \( L^4 \) and if \( \omega_1(Y^2) = cY^2 \), then, clearly, \( \omega_1(z_i) = cz_i \) for \( i = 1, \ldots, 4 \). According to [GL], in \( \mathfrak{gl}(\lambda) \) the following relations hold:

\[
3[z_1, z_2] - 2[z, z_3] = 24(\lambda^2 - 4)Y \\
4[z_3, [z, z_1]] - 3[z_2, [z, z_2]] = 576(\lambda^2 - 9)z
\]

Having applied \( \omega_1 \) to both sides of these relations we get

\[
c^2(\lambda^2 - 4) = \bar{\lambda}^2 - 4 \\
c^3(\lambda^2 - 9) = c(\bar{\lambda}^2 - 9)
\]

Therefore, \( c^2 = 1 \) and \( \lambda^2 = \bar{\lambda}^2 \). \( \square \)

**Corollary.** If \( \lambda^2 \neq \bar{\lambda}^2 \), then \( \mathfrak{gl}(\lambda) \) has no involutive anti-linear automorphisms.

**Proof.** Let \( \omega \) be an involutive anti-linear automorphism; clearly \( \tau : X \leftrightarrow Y \) and \( \tau(H) = H \) determines an anti-automorphism of \( \mathfrak{gl}(\lambda) \). Then \( \tau \circ \omega \) is an involutive anti-linear automorphism. \( \square \)

6.3. If \( \lambda^2 = \bar{\lambda}^2 \), then \( \mathfrak{a}_\lambda \) possesses an involutive anti-linear automorphism \( \omega \) given on generators by the formulas:

\[
\omega(X) = Y, \quad \omega(Y) = X, \quad \omega(H) = H
\]

(6.3)

In what follows, the unitary modules are considered with respect to this automorphism.

**Theorem.** Let \( \lambda^2 \) be real and let \( F_\theta(t) = \frac{R(t)}{1 - e^{-2\pi t}} \), where \( R(t) \) is a quasi-polynomial none of whose exponents \( s_i \) belong to \( \Lambda = [-\lambda - 1] \cap [-\lambda + 1] \cap [-\lambda - 1] \cap [-\lambda + 1] \). The \( \mathfrak{gl}(\lambda) \)-module with character \( F_\theta(t) \) is unitary if and only if

\[
F_\theta(t) = \sum n_i e^{-(\lambda + 1)t} - e^{s_it} \quad \text{where } n_i \in \mathbb{Z}_+.
\]

**Proof.** Let \( V \) be the irreducible module with character \( F_\theta(t) \) and \( P(H) \) the annihilator of \( Yv \), where \( v \) is the highest weight vector. Then \( P(H) \) is the characteristic polynomial of the operator \( H \) in \( V_{-1} \). But since the unitary form is invariant and \( \omega(H) = H \), it follows that \( H \) is self-adjoint; hence, all the roots of \( P \) are real. Further, if \( \alpha \) is a multiple root of multiplicity \( m > 1 \), the polynomial \( P \) is of the form \( P = (H - \alpha)^{m-1}Q(H) \). For \( u = (H - \alpha)^{m-1}Q(H)v \), we have

\[
\langle u, u \rangle = \langle (H - \alpha)^{m-1}Q(H)v, (H - \alpha)^{m-1}Q(H)v \rangle = \langle Q(H)v, (H - \alpha)^{2m-2}Q(H)v \rangle = 0.
\]

Thanks to Hermitian property, \( u = 0 \). Thus, \( m = 1 \).
By Theorem 5.3 $V$ is of the form $V(s_1) \otimes \cdots \otimes V(s_k)$, where the $V(s_i)$ are irreducible $\mathfrak{gl}(\infty)$-modules. Clearly, $V$ is unitary if and only if each $V(s_i)$ is unitary. Now, thanks to [KR] we know that the $\mathfrak{gl}(\infty)$-module $V(s)$ is unitary if and only if $\theta_i - \theta_{i+1} + \delta_{i0} c \in \mathbb{Z}_+$ for all $i$, where the $\theta_i$ are the coordinates of the highest weight and $c$ the value of the central charge. \hfill \Box

§7. $\mathfrak{gl}(\lambda)$ and the symmetric group

In this section we deduce an explicit realization of certain irreducible $\mathfrak{gl}(\lambda)$-modules. Namely, we decompose the tensor powers of the Verma module over $\mathfrak{sl}(2)$, indicate the corresponding characteristic polynomials and $q$-characters.

7.1. Lemma. Let $A$ be an associative algebra; let $\mathfrak{S}_n$ naturally act on $A^{\otimes n}$. Then the algebra $(A^{\otimes n})_{\mathfrak{S}_n}$ of $\mathfrak{S}_n$-invariants is generated by the elements of the form

$$a \otimes 1 \otimes 1 \otimes \cdots \otimes 1 + \cdots 1 \otimes 1 \otimes 1 \otimes \cdots \otimes a.$$  

Proof. Denote: $s(a_1, \ldots, a_n) = \sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$. Let $B$ be the algebra generated by the $s(a, 1, \ldots, 1)$ for all $a \in A$. Let $|s(a_1, \ldots, a_n)|$ be the number of the $a_i$ distinct from 1. Let us prove that $B \cong (A^{\otimes n})_{\mathfrak{S}_n}$.

Let us carry on an induction on $|s(a_1, \ldots, a_n)|$ in order to prove that $s(a_1, \ldots, a_n) \in B$. Indeed, if $|s(a_1, \ldots, a_n)| = 1$, by definition $s(a_1, \ldots, a_n) = s(a, 1, \ldots, 1) \in B$.

Let $|s(a_1, \ldots, a_n)| = l > 1$. Consider

$$s(a_1, \ldots, a_{l-1}, 1, \ldots, 1)s(a_l, 1, \ldots, 1) = \alpha s(a_1, \ldots, a_l, 1, \ldots, 1) + \ldots,$$

where $\alpha$ is a nonzero constant and the dots stand for the linear combination of the terms $s(b_1, \ldots, a_m, 1, \ldots, 1)$ with $m < l$. \hfill \Box

7.2. Theorem. Let $V = M^{\lambda-1}$ be the Verma module with highest weight $\lambda - 1$ over $\mathfrak{sl}(2)$ (hence, a $\mathfrak{gl}(\lambda)$-module). Then

$$V^{\otimes n} = \bigoplus_{\nu} V' \otimes S'^{\nu},$$

where $V'$ is an irreducible $\mathfrak{gl}(\lambda)$-module and $S'^{\nu}$ is an irreducible $\mathfrak{S}_n$-module and $\nu$ runs over the partitions of $n$.

Proof. Clearly, $V$ is irreducible not only as $\mathfrak{gl}(\lambda)$-module but also as $\mathfrak{A}_\lambda$-module. Then, obviously, $W = V^{\otimes n}$ is irreducible as $\mathfrak{A}_\lambda^{\otimes n}$-module.

The image of $U(\mathfrak{gl}(\lambda))$ in $\operatorname{End}(W)$ coincides with the subalgebra generated by $s(a, 1, \ldots, 1)$ for $a \in \mathfrak{gl}(\lambda)$, so by Lemma 7.1 it is isomorphic to $(\mathfrak{A}_\lambda^{\otimes n})_{\mathfrak{S}_n}$.

Let us decompose $W$ into isotypical $\mathfrak{S}_n$-modules: $W = \bigoplus_{\nu} W' \otimes S'^{\nu}$, where $V'$ is an irreducible $\mathfrak{gl}(\lambda)$-module. There is no unique way to do this, for example, set $V' = e_\nu(W)$ for any minimal idempotent in $\mathbb{C}[\mathfrak{S}_n]$ corresponding to the partition $\nu$.

To show that $V'$ is irreducible as a $\mathfrak{gl}(\lambda)$-module, consider $V_1' = \operatorname{Hom}_{\mathfrak{S}_n}(S', W)$. As a $\mathfrak{gl}(\lambda)$-module, $V_1'$ is isomorphic to $V'$. Let $\phi, \psi \in V_1'$ and $\phi \neq 0$. Let us show that there exists a $u \in U(\mathfrak{gl}(\lambda))$ such that $u\phi = \psi$. Indeed, since $W$ is irreducible as $\mathfrak{A}_\lambda^{\otimes n}$-module, then the density theorem ([L], Ch. XVII, §3, Th.1) states that there is a $w \in \mathfrak{A}_\lambda^{\otimes n}$ such that $w\phi(v_i) = \psi(v_i)$, where the $v_i$ form a basis of $S'$. (Since $S'$ is irreducible and $\phi \neq 0$, then the vectors $\psi(v_i)$ are linearly independent for $i = 1, \ldots, \dim S'$.)

Let us average the element $w\phi$ with respect to $\mathfrak{S}_n$:

$$(w\phi)^2 = \frac{1}{|\mathfrak{S}_n|} \sum \sigma(w\phi)\sigma^{-1} = \frac{1}{|\mathfrak{S}_n|} \sum \sigma(w)\sigma^{-1}(\phi) = w^2\phi.$$
But, on the other hand, since $w\phi = \psi$, we deduce that $(w\phi)^2 = (\psi)^2 = \psi$, i.e., $w^2 \phi = \psi$ and we may assume that $w \in (\mathfrak{gl}(\lambda))^\otimes \otimes \mathfrak{sl}(\lambda)$.

**7.2.1. Corollary.** 1) Let $\nu = (\nu_1, \ldots, \nu_n)$. The generating function corresponding to the highest weight $\nu$ is of the form

$$\sum_{i=1}^{n} \nu_i e^{(\lambda - 2i + 1)t}, \tag{7.2.1.1}$$

2) Represent $\nu$ in the form $\nu = (\theta_1^{\alpha_1} \ldots \theta_m^{\alpha_m})$, where $\theta_1 > \cdots > \theta_m > 0$ and $\alpha_i \neq 0$ for all $i$ and where $\theta^\alpha$ denotes the product of $\alpha$ copies: $\theta \ldots \theta$. Then the characteristic polynomial of $V^\nu$ is equal to

$$P(H) = \prod_{i=1}^{m} (H - \lambda - 2\alpha_1 - 2\alpha_2 - \cdots - 2\alpha_i - 1).$$

**Proof.** Heading 1) follows from Theorem 4.5. Multiply (7.2.1.1) by $1 - e^{-2t}$; we see that the product is of the form

$$\nu_1 e^{(\lambda - 1)t} + \sum_{i=1}^{n-1} \left( \nu_{i+1} - \nu_i \right) e^{(\lambda - 2i + 1)t} - \nu_n e^{(\lambda - 2n - 1)t}. \tag{7.2.1.2}$$

Therefore, sum (7.2.1.2) without the first summand is a solution of an ordinary differential equation with constant coefficients whose characteristic equation is precisely of the form indicated. \hfill \Box

**7.2.2. Corollary.** Set $a = e^\lambda$, $q = e^{-\alpha}$, where $\alpha$ is the positive root of $\mathfrak{sl}(2)$. Then the $q$-character of $V^\nu$ is equal to

$$\chi^\nu = \frac{a^{|\nu|} q^{n(\nu)}}{\prod_{x \in \nu} (1 - q^{h(x)})}, \tag{7.2.2}$$

where $|\nu|$ is the number of cells in the Young tableau corresponding to $\nu$, $n(\nu) = \sum_{i \geq 1} \nu_i$ and $h(x)$ is the length of the hook corresponding to the cell $x$.

**Proof.** By Mac, Example 2 in §3, for the $S$-function $s^\nu$, we have

$$\chi^\nu = s^\nu(a, aq, aq^2, \ldots) = a^{|\nu|} s^\nu(1, q, q^2, \ldots) = \text{rhs of (7.2.2)}.$$

\hfill \Box

§8. ORTHOGONAL POLYNOMIALS FOR $\mathfrak{gl}(n)$

For an overview, see [NU], [NSU]. (Setting $T_l(\alpha_i) \ldots T_l(\alpha_i) = 1$ for $l = 0$ we make formulas (8.1.1) and the like look uniform.)

**8.1. Theorem.** In $\mathfrak{gl}(n)$:

i) Consider the basis $e_{kl} = (\text{ad } Y)^{k-l}(X^k)$ for $0 \leq k \leq n-1$ and $-k \leq l \leq k$. We have

$$\langle e_{kl}, e_{k'l'} \rangle = \delta_{k,k'} \delta_{l+l',0}. \tag{8.1.1}$$

ii) Determine the elements $f_{kl}$ from the equations ($0 \leq l \leq k$)

$$(\text{ad } Y)^{k-1}(X^k) = X^l f_{kl} \quad \text{and} \quad (\text{ad } Y)^{k+l}(X^k) = f_{k,-l} Y^l.$$
Set \( T_i(H) = \frac{1}{4}(n^2 - (H + 2i - 1)^2) \) and \( \alpha_i = n - 2i + 1 \) for \( i = 1, \ldots, n \). For a fixed \( l \geq 0 \) and any \( k \geq l \), the polynomials \( f_{kl} \) form an orthogonal basis with respect to the form

\[
\langle f, g \rangle = \begin{cases} 
\sum_{i=1}^{n} f(\alpha_i)g(\alpha_i)T_1(\alpha_i) \ldots T_l(\alpha_i) & \text{for } l > 0 \\
\sum_{i=1}^{n} f(\alpha_i)g(\alpha_i) & \text{for } l = 0.
\end{cases}
\tag{8.1.1}
\]

iii) Up to a constant factor the polynomials \( f_{kl} \) coincide with the Hahn polynomial of one discrete variable:

\[
f_{kl}(H) = \sum_{k=0}^{\infty} \binom{\alpha_1}{\alpha_2} \binom{\alpha_2}{\beta_1, \beta_2} z^i i!
\tag{8.1.3}
\]

is a generalized hypergeometric function, \( (\alpha)_0 = 1 \) and \( \alpha_i = (\alpha + i - 1) \) for \( i > 0 \).

**Proof.** i) First, observe that the subspaces \( L^{2k} \) in the decomposition of \( \mathfrak{gl}(n) = L^0 \oplus L^2 \oplus \cdots \oplus L^{2n-2} \) are pairwise orthogonal. Indeed, the form \( A, B \mapsto \text{tr} AB \) defines an invariant pairing of \( L^k \) and \( L^l \), hence, an \( \mathfrak{sl}(2) \)-homomorphism \( L^k \rightarrow L^l \) which is only possible if \( k = l \). Moreover, it is clear that \( \langle \mathfrak{gl}(n)_k, \mathfrak{gl}(n)_l \rangle \neq 0 \) if and only if \( k + l = 0 \). Now observe that \( \mathfrak{gl}(n)_l \cap L^{2k} = \text{Span}((\text{ad}Y)^{k-l}(X)) \). This proves i).

ii) Let \( X^lf(H) \in \mathfrak{gl}(n)_k \) and \( g(H)Y^l \in \mathfrak{gl}(n)_{-l} \). Then

\[
\text{tr} (g(H)Y^lX^lf(H)) = \text{tr} (f(H)g(H)T_1(H) \ldots T_l(H)).
\]

As is easy to verify

\[
\text{tr} f(H) = \sum_{f(1 \leq i \leq n)} f(\alpha_i) \text{ for any } f(H) \in \mathfrak{gl}(n)_0.
\]

This implies (8.1.1).

Moreover, i) implies that for any fixed \( l \) lying between \(-k \) and \( k \) the polynomials \( f_{kl} \) and \( f_{k, -l} \) form two mutually dual bases, i.e., \( \langle f_{kl}, f_{k', l'} \rangle = \delta_{l+l', 0} \). But the degrees of polynomials \( f_{kl} \) and \( f_{k, -l} \) are equal, which means that they coincide up to a constant factor. This proves ii).

iii) follows from the fact that orthogonal polynomials are uniquely determined by the weight function and the interval over which we consider the scalar product. Eq. (8.1.2) follows from comparison of the coefficients of the leading terms.

**Remark.** 1) Hahn polynomials are determined for three parameters \( \alpha, \beta \) and \( N \) as

\[
h_p^{(\alpha, \beta)}(z, N) = \frac{(-1)^p \Gamma(N)(\beta)_p}{p! \Gamma(N - p)} \sum_{i=0}^{\infty} \binom{\alpha}{\beta} \binom{\beta}{\beta_1, \beta_2} z^i \tag{8.1.4}
\]

Our \( f_{kl} \) coincides, up to a constant factor, with \( h_p^{(\alpha, \beta)}(z, N) \) at \( \alpha = \beta = l, p = k - l, z = \frac{1}{2}(H + n + 1) \) and \( N = n - 1 \).

2) Having finished this paper we have realized that sometimes it is more convenient to consider the form \( A, B \mapsto \text{tr} AB^t \), i.e., an invariant pairing on each \( \mathfrak{gl}(n)_k \), cf. [S].
8.2. Let us show now how the main properties of the Hahn polynomials follow from the representation theory of $\mathfrak{sl}(2)$. For any $f(H) \in \mathbb{C}[H]$, set
\[
\nabla f(H) = f(H + 2) - f(H), \quad \nabla f(H) = f(H) - f(H - 2). \tag{8.1.5}
\]

**Theorem.** Consider $\mathfrak{gl}(n)$ as $\mathfrak{sl}(2)$-module with respect to the image of the principal embedding, let $\Omega$ be the quadratic Casimir operator (1.2).

i) $\Omega$ is self-adjoint with respect to the form $\langle \cdot, \cdot \rangle$ and the polynomials $X^l f_{kl}$ are eigenfunctions of $\Omega$ corresponding to eigenvalue $2k(k + 1)$. The polynomials $f_{kl}$ satisfy the difference equation
\[
T_0(H) \nabla \Delta(f) - (l + 1)(H + l)\Delta(f) + (k - l)(k + l + 1)f = 0. \tag{8.2.i}
\]

ii) $f_{kl} = \begin{cases} \frac{\nabla^{k-l}(T_1 \cdots T_k)}{T_1 \cdots T_l} & \text{if } l > 0 \\
\frac{\nabla^k(T_1 \cdots T_k)}{T_1 \cdots T_l} & \text{if } l = 0. \end{cases}$

iii) $\langle f_{kl}, f_{kl} \rangle = \frac{(k - l)!(k!^2)}{(k + l)!(2k + 1)^2} n(n^2 - 1^2) \cdots (n^2 - k^2)$.

**Proof.** i) Since the form $\langle \cdot, \cdot \rangle$ is $\mathfrak{sl}(2)$-invariant, $\langle [w, u], v \rangle = -\langle u, [w, v] \rangle$ for any $u, v \in \mathfrak{gl}(n)$ and $w \in \mathfrak{sl}(2)$. Denote the $U(\mathfrak{sl}(2))$-action in $\mathfrak{gl}(n)$ by $*$; by induction we easily deduce that $\langle w * u, v \rangle = -\langle u, w^* v \rangle$ where now $w \in U(\mathfrak{sl}(2))$ and $t$ is the principal anti-involution. Since, as is easy to verify, $\Omega^t = \Omega$, we have $\langle \Omega * u, v \rangle = \langle u, \Omega * v \rangle$, i.e., $\Omega$ is self-adjoint. Since $\mathfrak{gl}(n) = \oplus L^{2^n}$ and $X^l f_{kl} = (\text{ad}Y)^{k-l}(X^k) \in L^{2k}$, it follows that $X^l f_{kl}$ is an eigenfunction of $\Omega$ with eigenvalue $2k(k + 1)$. Applying $\Omega$ to $X^l f_{kl}$ we obtain eq. (8.2.i).

ii) Recall the identity
\[
(\text{ad}y)^p(a) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} y^{p-j} ay^j.
\]
Set $y = Y$, $a = X^k$, $p = k - l$ and multiply by $Y^l$ from the left. We get
\[
Y^l X^l f_{kl} = Y^l (\text{ad}Y)^{k-l}(X^k) = Y^l \left( \sum_{j=0}^{k-l} (-1)^j \binom{k-l}{j} Y^{k-l-j} X^k Y^j \right) = \sum_{j=0}^{k-l} (-1)^j \binom{k-l}{j} y^{k-j} X^k Y^j = \sum_{j=0}^{k} (-1)^j \binom{k}{j} T_{j-l} \cdots T_0 T_1 \cdots T_{k-j}.
\]

But $\nabla^{k-l}(f)(H) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(H - 2j)$ for any function $f$, so
\[
Y^l X^l f_{kl} = \nabla^{k-l}(T_1 \cdots T_k).
\]
Since $Y^l X^l = T_1 \cdots T_1$, we are done.

iii) The module $L^{2k}$ is the linear span of the vectors $v_l = (\text{ad}Y)^{k-l}(X^k)$ for $-k \leq l \leq k$; hence,
\[
\langle v_l, v_m \rangle = -\langle v_{l-1}, v_{m+1} \rangle = \cdots = (-1)^l \langle v_0, v_{l+m} \rangle
\]
y by invariance of the form. Hence, $\langle v_l, v_{-l} \rangle = (-1)^l \langle v_0, v_0 \rangle$. But $v_l = X^l f_{kl} = (\text{ad} Y)^{k-l}(X^k)$ and $v_{-l} = f_{k,-l} Y^l = (\text{ad} Y)^{k+l}(X^k)$.

Now recall that $f_{kl}$ and $f_{k,-l}$ are identical up to a constant factor and compare the leading coefficients. We see that
\[
f_{k,-l} = (-1)^l \frac{(k + l)!}{(k - l)!} f_{k,l}.
\]
Hence,
\[ \langle f_{k,l}, f_{k,i} \rangle = \text{tr} \left( f_{k,l} Y^l \cdot X^i f_{k,l} \right) = (-1)^l \frac{(k+l)!}{(k-l)!} \langle v_l, v_{-l} \rangle = \frac{(k+l)!}{(k-l)!} \langle v_0, v_0 \rangle. \]

It remains to demonstrate that
\[ \langle v_0, v_0 \rangle = \frac{n(n^2 - 1^2) \ldots (n^2 - k^2)}{2k+1} (k!)^2. \]

Obviously, \( f_k = ((\text{ad} Y)^k(X^k))^2 \in U(\mathfrak{sl}(2)) \). So \( \langle v_0, v_0 \rangle = \text{tr}_n \phi_n(f_k) \), where \( \text{tr}_n \) is the trace on \( \mathfrak{gl}(n) \) and \( \phi_n \) is the homomorphism of \( U(\mathfrak{sl}(2)) \) into \( \mathfrak{gl}(n) \) induced by the principal embedding.

It is clear now that \( P_k(n) = \text{tr}_n(\phi_n(f_k)) \) is a polynomial of degree \( 2k + 1 \); moreover, this polynomial is an odd one. But \( \phi_n(X^k) = 0 \) for \( n \leq k \) and \( \phi_n(f_k) = 0 \); hence, \( P_k(n) = 0 \) if \( n \leq k \) and \( P_k(-n) = 0 \) because \( P_k \) is odd. Therefore,
\[ P_k(n) = c_k n(n^2 - 1^2) \ldots (n^2 - k^2). \]

To calculate the constant \( c_k \), it suffices to compute \( P_k(k+1) = \text{tr}_{k+1}(\phi_{k+1}(f_k)) \). But in this case \( X^k = (k!)^2 e_{1,k+1} \) and
\[ (\text{ad} Y)^k(X^k) = (k!)^2 (\text{ad} Y)^k(e_{1,k+1}) = (k!)^2 \sum_{j=0}^{k} (-1)^j \binom{k}{j} e_{k+1-j,k+1-j}. \]

Thus,
\[ \text{tr}_{k+1}(\phi_{k+1}(f_k)) = (k!)^4 \sum_{j=0}^{k} \binom{k}{j}^2 = (k!)^4 \binom{2k}{k} \]

implying \( c_k = \frac{(k!)^2}{2k+1} \) \( \square \)

8.3.1. Proposition. The following orthogonality relations (i) and (ii) hold:
\[ (i) \quad \sum_i f_{kl}(\alpha_i)f_{k,l}(\alpha_i)T_1(\alpha_i) \ldots T_l(\alpha_i) = \delta_{k,k_1} c_{k,l}, \]
where \( c_{k,l} = \frac{(k-l)!(k!)^2}{(k+l)!(2k+1)} n(n^2 - 1^2) \ldots (n^2 - k^2). \)
\[ (ii) \quad \sum_{k=l}^{n-1} \frac{1}{c_{kl}} f_{kl}(\alpha_i)f_{kl}(\alpha_j)T_1(\alpha_i) \ldots T_l(\alpha_i) = \delta_{ij} \text{ for any } i, j > l. \]

Proof. (i) follows from Theorem 8.2, iii). To prove ii), express \( e_{ij} \) via \( X^i f_{kl} \) and \( f_{k,-l} Y^l \). We have, in particular, \( e_{i,i+l} = \sum_{k=l}^{n-1} \alpha_{ik} X^i f_{kl} \) for any \( k \) between \( l \) and \( n-1 \). But, as is easy to verify,
\[ f_{kl} Y^l = (\sum_{i=1}^{n} f_{kl}(\alpha_i)e_{ii})(\sum_{i=1}^{n-1} e_{i+1,i})^l = \]
\[ (\sum_{i=1}^{n} f_{kl}(\alpha_i)e_{ii})(\sum_{i=1}^{n-l} e_{i+l,i}) = \sum_{i=1}^{n-l} f_{kl}(\alpha_{i+l})e_{i+l,i}. \]

Hence, by Theorem 8.2 we have
\[ f_{kl}(\alpha_{i+l}) = \langle e_{i,i+l}, f_{kl} Y^l \rangle = \alpha_{ik} \langle f_{kl} Y^l, X^i f_{kl} \rangle = \alpha_{ik} c_{kl}. \]
In other words,
\[ e_{i,i+l} = \sum_{k=l}^{n-1} \frac{f_{kl}(\alpha_{i+l})}{c_{kl}} X^l f_{kl}. \]

Similarly,
\[ e_{i+l,i} = \sum_{k=l}^{n-1} \frac{f_{kl}(\alpha_{i+l}) T_1(\alpha_{i+1}) \ldots T_i(\alpha_{i+l-1})}{c_{kl}} f_{kl} Y^l. \]

Therefore, by setting \( i' = i + l, j' = j + l \) we deduce that
\[
\langle e_{i+l,i}, e_{j,j+l} \rangle = \sum_{i=l}^{n-1} \frac{1}{c_{kl}} f_{kl}(\alpha_{i+l}) f_{kl}(\alpha_{j+i}) T_1(\alpha_{i+1}) \ldots T_i(\alpha_{i+l-1}) =
\sum_{i=l}^{n-1} \frac{1}{c_{kl}} f_{kl}(\alpha_{i'}) f_{kl}(\alpha_{j'}) T_1(\alpha_{i' - 1}) \ldots T_i(\alpha_{i' - l + 1}) =
\sum_{i=l}^{n-1} \frac{1}{c_{kl}} f_{kl}(\alpha_{i'}) f_{kl}(\alpha_{j'}) T_1(\alpha_{i'}) \ldots T_i(\alpha_{i'}).
\]

\[ \square \]

8.3.2. Expressing Casimir elements \( \Omega_2, \Omega_3, \ldots, \Omega_n \) for \( \mathfrak{gl}(n) \) in terms of the orthogonal polynomials rather than matrix units, we can derive various identities relating the polynomials. Some of these identities might be even new, at least, for non-integer values of \( n \), cf. 9.3. For example, for \( \Omega_2 \), we have:

**Proposition.** In \( \mathfrak{gl}(n) \), we have
\[
\sum_{k=0}^{n-1} \left( \frac{1}{c_{k0}} f_{k0}^2 + 2 \sum_{l=1}^{k} \frac{1}{c_{kl}} f_{kl}^2 T_1 \ldots T_l - \frac{2k+1}{n} \right) = -H.
\]

**Proof.** Let us rewrite the Casimir operator for \( \mathfrak{gl}(n) \) expressed in terms of matrix units
\[
\Omega_2 = \sum_{k=1}^{n} \epsilon_{kk} \otimes \epsilon_{kk} + \sum_{i<j} (\epsilon_{ij} \otimes \epsilon_{ji} + \epsilon_{ji} \otimes \epsilon_{ij}) =
\sum_{k=1}^{n} \epsilon_{kk} \otimes \epsilon_{kk} + \sum_{k=1}^{n} (n - 2k + 1) \epsilon_{kk} + 2 \sum_{i<j} \epsilon_{ji} \otimes \epsilon_{ij}
\]
via orthogonal polynomials:
\[
\Omega_2 = \sum_{k=0}^{n-1} \frac{1}{c_{k0}} f_{k0} \otimes f_{k0} + \sum_{k=1}^{n-1} \sum_{l=1}^{n-1} \frac{1}{c_{kl}} (X^l f_{kl} \otimes f_{kl} Y^l + f_{kl} Y^l \otimes X^l f_{kl}) =
\sum_{k=0}^{n-1} \frac{1}{c_{k0}} f_{k0} \otimes f_{k0} + 2 \sum_{l=1}^{k} \sum_{k=0}^{n-1} \frac{1}{c_{kl}} f_{kl} Y^l \otimes X^l f_{kl} + \sum_{k>l} \frac{1}{c_{kl}} [X^l f_{kl}, f_{kl} Y^l].
\]

Let us now transform the last sum. For this, consider a homomorphism \( \varphi : U(\mathfrak{gl}(n)) \rightarrow \mathfrak{gl}(n) \) induced by the identity representation. Since
\[
\frac{1}{c_{k0}} f_{k0} \otimes f_{k0} + \sum_{l=1}^{k} \frac{1}{c_{kl}} (X^l f_{kl} \otimes f_{kl} Y^l + f_{kl} Y^l \otimes X^l f_{kl})
\]
is \( \mathfrak{sl}(2) \)-invariant, its image in \( \mathfrak{gl}(n) \) is also an \( \mathfrak{sl}(2) \)-invariant, i.e., is a constant, or better say, scalar matrix, \( D(k) \):
\[
\frac{1}{c_{k0}} f_{k0} \cdot f_{k0} + 2 \sum_{l=1}^{k} \frac{1}{c_{kl}} f_{kl} Y^l \cdot X^l f_{kl} + \sum_{l=1}^{k} \frac{1}{c_{kl}} [X^l f_{kl}, f_{kl} Y^l] = D(k).
\]
Let us calculate the trace of both sides. We see that
\[ nD(k) = \frac{1}{c_{k0}} \text{tr}(f_{k0}^2) + 2 \sum_{l=1}^{k} \frac{1}{c_{kl}} \text{tr}(f_{kl}^2T_1 \ldots T_l) = 2k + 1 \]

implying \( D(k) = \frac{2k + 1}{n} \). Therefore,
\[ \sum_{l=1}^{k} \frac{1}{c_{kl}}[X^lf_{kl}, f_{kl}Y^t] = 2k + 1 \frac{1}{n} - \left( \frac{1}{c_{k0}} f_{k0} \cdot f_{k0} + 2 \sum_{l=1}^{k} \frac{1}{c_{kl}} f_{kl}Y^t \cdot X^lf_{kl} \right). \]

Summing over \( k \) we obtain
\[ \sum_{k=0}^{n-1} \left( \frac{1}{c_{k0}} f_{k0} \cdot f_{k0} + 2 \sum_{l=1}^{k} \frac{1}{c_{kl}} f_{kl}Y^t \cdot X^lf_{kl} - \frac{2k + 1}{n} \right) = -\sum_{k=0}^{n-1} \sum_{l=1}^{k} \frac{1}{c_{kl}}[X^lf_{kl}, f_{kl}Y^t]. \]

The linear parts of the same Casimir operator but expressed in different bases coincide, so the last sum is equal to \( -\sum_{i=1}^{n} (n - 2i + 1) c_{ii} = -H. \)  

\section{9. Orthogonal Polynomials and \( \mathfrak{gl}(\lambda) \)}

The above results for \( \mathfrak{gl}(n), n = 1, 2, \text{etc.} \) also hold \textit{mutatis mutandis} for \( \mathfrak{gl}(\lambda) \) with any complex \( \lambda \notin \mathbb{Z} \setminus \{0\} \). In this section we only consider such values of \( \lambda \).

Observe that \( \mathfrak{gl}(\lambda) \), unlike \( \mathfrak{gl}(\infty) \) or \( \mathfrak{gl}^{L}(\infty) \), has no basis consisting of matrix units. But we can represent it in the form
\[ \mathfrak{gl}(\lambda) = \oplus_{k=0}^{\infty} L^{2k} \text{ and } \mathfrak{gl}(\lambda) = \oplus_{t=-\infty}^{\infty} \mathfrak{gl}(\lambda)_t, \]

where \( \mathfrak{gl}(\lambda)_0 = \mathbb{C}[H] \) while \( \mathfrak{gl}(\lambda)_i = X^i \mathfrak{gl}(\lambda)_0 \) and \( \mathfrak{gl}(\lambda)_{-i} = \mathfrak{gl}(\lambda)_0 Y^i \) for \( i > 0 \).

As was shown above, there exists a trace on \( \mathfrak{gl}(\lambda) \) whose restriction onto \( \mathfrak{gl}(\lambda)_0 \) has for generating function
\[ \frac{e^{\lambda t} - e^{-\lambda t}}{e^t - e^{-t}} \text{ if } \lambda \neq 0 \text{ and } \frac{2t}{e^t - e^{-t}} \text{ if } \lambda = 0. \]

In the first case we have normalized the trace so that \( \text{tr}(1) = \lambda \) by analogy with the finite dimensional case when the scalars are naturally represented by scalar matrices and \( \text{tr}(1_n) = n \); in the second case, \( \lambda = 0 \), we assume that \( \text{tr}(1) = 1. \)

Observe that, for any integer \( \lambda = n \), we have
\[ \frac{e^{nt} - e^{-nt}}{e^t - e^{-t}} = e^{(1-n)t} + e^{(3-n)t} + \ldots + e^{(n-3)t} + e^{(n-1)t} \]

and the respective functional is of the form
\[ \text{tr}(f(H)) = \sum_{i=1}^{n} f(\alpha_i), \quad \alpha_i = n - 2i + 1. \]

The functional \( \text{tr} \) on \( \mathfrak{gl}(\lambda) \) gives rise to a non-degenerate symmetric invariant bilinear form \( \langle u, v \rangle = \text{tr} uv. \)

\textbf{9.1. Theorem .} \textit{In notations of §8: i) } \( \langle e_{k,l}, e_{k',l'} \rangle = \delta_{k,k'} \delta_{l,l'} \delta_{0,0}. \)

\textit{ii) The polynomials } \( f_{k,l} \) \textit{for a fixed } \( l \geq 0 \text{ and } k \geq l \text{ form an orthogonal basis with respect to the scalar product (8.1.1).} \)

\textit{iii) The polynomials } \( f_{k,l} \) \textit{coincide, up to a constant factor, with continuous Hahn polynomials of discrete variable given by (8.1.2) for } \( n = \lambda. \)
9.2. Theorem. Let $\Omega$ be the quadratic Casimir operator for $\mathfrak{sl}(2) \subset \mathfrak{gl}(\lambda)$.

i) $\Omega$ is self-adjoint with respect to the form (8.1.1) and the polynomials $X^lf_{kl}$ for $k \in \mathbb{Z}_+$ and $l \in [-k,k] \cap \mathbb{Z}$ are eigenfunctions of $\Omega$ corresponding to eigenvalue $2k(k+1)$. The polynomials $f_{kl}$ satisfy the difference equation (8.2.1) and are of the form (8.2.ii)

$$\langle f_{kl}, f_{kl} \rangle = \begin{cases} 
\frac{(k-l)!}{(k+l)!} \frac{(k!)^2}{2k+1} \lambda(\lambda^2 - 1^2) \ldots (\lambda^2 - k^2) & \text{if } \lambda \neq 0 \\
(-1)^k \frac{(k-l)!}{(k+l)!} \frac{(k!)^4}{2k+1} & \text{if } \lambda = 0.
\end{cases}$$

Proof. Proof of heading i) is the same as that of the corresponding statements of Theorem 8.2. If $\lambda \neq 0$, then both sides of eq. ii) are polynomials in $\lambda$ which coincide at integer values of $\lambda$ and, therefore, are equal by continuity in Zariski topology. To embrace $\lambda = 0$, consider $\lim_{\lambda \to 0} \lambda^{-1}$; as before, both sides become polynomials in $\lambda$ implying ii).

It is somewhat unexpected that the dual orthogonality relations hold for sufficiently large $|\lambda|$. Indeed, set

$$c_{kl} = \begin{cases} 
\frac{(k-l)!}{(k+l)!} \frac{(k!)^2}{2k+1} \lambda(\lambda^2 - 1^2) \ldots (\lambda^2 - k^2) & \text{if } \lambda \neq 0 \\
(-1)^k \frac{(k-l)!}{(k+l)!} \frac{(k!)^4}{2k+1} & \text{if } \lambda = 0.
\end{cases}$$

9.3. Conjecture. For sufficiently large $|\lambda|$, we have (for $\alpha_i = \lambda - 2i + 1$):

$$\sum_{k=1}^{\infty} \frac{1}{c_{kl}} f_{kl}(\alpha_i) f_{kl}(\alpha_j) T_1(\alpha_i) \ldots T_l(\alpha_i) = \delta_{ij} \text{ for } i, j > 1.$$ (9.3.1)

$$\sum_{k=0}^{\infty} \left( \frac{1}{c_{k0}} f_{k0}(\alpha_i)^2 + 2 \sum_{l=1}^{\infty} \frac{1}{c_{kl}} f_{kl}(\alpha_i)^2 T_1(\alpha_i) \ldots T_l(\alpha_i) - \frac{2k+1}{\lambda} \right) = -\alpha_i \text{ for } i \in \mathbb{N}. \quad (9.3.2)$$

A priori, $|\lambda|$ depends on $l$. We do not know a uniform proof and only checked Conjecture in certain particular cases. We are thankful to V. Gerdt and P. Grozman who helped us with numerical experiments on Maple and Mathematica, respectively.

9.4. On positive definiteness of the form $\langle \cdot, \cdot \rangle$. Formulas above make it clear that if we divide the form $\langle \cdot, \cdot \rangle$ by $\lambda$ and consider polynomials of even degree only we get a sign-definite form not only for integer values of $\lambda$ but also for real values such that $0 < |\lambda| < 1$ and for purely imaginary $\lambda$. The last observation suggests to divide the trace by $\lambda$ from the very beginning.

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Correspondence: Department of Mathematics, University of Stockholm, Roslagsvägen 101,Kräftiket hus 6, S-104 05, Stockholm, Sweden, mleites@math.su.se (On leave of absence from Balakovo University of Technique, Technology and Control, Balakovo, Saratov region, Russia)