DIFFEOMORPHISM GROUPS OF CIRCLE BUNDLES OVER INTEGRAL SYMPLECTIC MANIFOLDS

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ABSTRACT. We study the diffeomorphism and isometry groups of manifolds $\mathcal{M}_p$, $p \in \mathbb{Z}$, which are circle bundles over a closed $4n$-dimensional integral symplectic manifold. Equivalently, $\mathcal{M}_p$ is a compact $(4n + 1)$-dimensional contact manifold with closed Reeb orbits. We use Wodzicki-Chern-Simons forms to prove that $\pi_1(\text{Diff}(\mathcal{M}_p))$ and $\pi_1(\text{Isom}(\mathcal{M}_p))$ are infinite for $|p| \gg 0$. We also give the first high dimensional examples of nonvanishing Wodzicki-Pontryagin forms.

1. Introduction

In this paper, we study the diffeomorphism and isometry groups of manifolds $\mathcal{M}_p$, $p \in \mathbb{Z}$, which are circle bundles over closed $4n$-dimensional integral symplectic manifolds $\mathcal{M}$. Equivalently, $\mathcal{M}_p$ is a compact $(4n + 1)$-dimensional contact manifold with closed Reeb orbits. We use Wodzicki-Chern-Simons (WCS) forms to determine that $\pi_1(\text{Diff}(\mathcal{M}_p))$ and $\pi_1(\text{Isom}(\mathcal{M}_p))$ are infinite for $|p| \gg 0$. This extends results from for circle bundles over Kähler surfaces in [16] to symplectic manifolds in arbitrarily high dimensions.

For a closed manifold $\mathcal{M}$, current knowledge of the homotopy type of $\text{Diff}(\mathcal{M})$ is mostly limited to low dimensions [11]. In all known cases, the Smale conjecture that $\text{Diff}(\mathcal{M})$ has the same homotopy type as $\text{Isom}(\mathcal{M})$ holds. It is elementary that $\text{Diff}(S^1)$ has the homotopy type of $O(2)$. The theorems that $\text{Diff}(S^2) \sim O(3)$ and $\text{Diff}(S^3) \sim O(4)$, due to Smale [20] and Hatcher [10] respectively, are much more difficult. The homotopy type of $\text{Diff}_0(\Sigma^2)$ (the identity component of $\text{Diff}(\Sigma^2)$) for a closed Riemann surface is known [4]. For most classes of closed geometric 3-manifolds $\mathcal{M}$, the Smale conjecture is valid [9, 17], and so e.g. the components of $\text{Diff}(\mathcal{M})$ are contractible for $\mathcal{M}$ hyperbolic. Recently, Ricci flow has been successfully applied in two and three dimensions to the Smale conjecture [1]. In higher dimensions, Farrell-Hsiang [7] used algebraic K-theory to prove that some rational higher homotopy groups $\pi_k(\text{Diff}(S^n)) \otimes \mathbb{Q}$ are finite extensions of $\pi_k(SO(n + 1))$ in the stable range.

However, in general even the question of whether $\pi_1(\text{Diff}(\mathcal{M}))$ is finite or infinite seems difficult. As a trivial example, if $\mathcal{M} = M \times S^1$, then the subgroup of $\pi_1(\text{Diff}(\mathcal{M}))$ given by diffeomorphisms independent of $m \in M$ ($\theta \mapsto f(\theta)(m, \psi) = (m, f'((\theta, \psi))))$ is isomorphic to $\text{Diff}(S^1)$ by $f \mapsto f'$ for fixed $m \in M$. Thus $\pi_1(\text{Diff}(\mathcal{M}))$ is infinite.

In more generality, suppose $\mathcal{M}$ admits a nontrivial circle action $a : S^1 \times \mathcal{M} \to \mathcal{M}$. This gives a loop $a^D : S^1 \to \text{Diff}(\mathcal{M})$ of diffeomorphisms and hence an element of
π₁(Diff(M)). If the circle action is free, \( \overline{M} \) is the total space of a circle bundle over the orbit space \( M \), with the action given by rotation of the circle fibers. It is natural to conjecture that this loop of diffeomorphisms has infinite order in \( \pi_1(Diff(\overline{M})) \). This is not always true: for the canonical bundle \( \overline{M} = S^{2n+1} \) over \( M = \mathbb{C}P^n \), the fiber rotation is an isometry of the standard metric \( S^{2n+1} \). In fact, \( a^D \) is the generator of the isometry group \( SO(2n + 2) \cong \mathbb{Z}_2 \).

In this example, the first Chern number of the canonical bundle is 1. The main result is that for sufficiently high Chern number, rotation in the circle fiber gives an element of infinite order in \( \pi_1(Diff(\overline{M})) \). More precisely:

**Theorem 3.4:** Let \((M, \omega)\) be a closed integral symplectic manifold of dimension \( 4n \). For \( p \in \mathbb{Z} \), let \( \overline{M}_p \) be the circle bundle over \( M \) with first Chern class \( p[\omega] \). Then for \( |p| \gg 0 \), \( \pi_1(Diff(\overline{M}_p)) \) and \( \pi_1(\text{Isom}(\overline{M}_p)) \) are infinite. Equivalently, let \( \overline{M} \) be a closed \((4n + 1)\)-dimensional contact manifold with closed Reeb orbits. Then \( \overline{M} \) covers infinitely many such contact manifolds \( \overline{M}_p \) with \( \pi_1(Diff(\overline{M}_p)) \) and \( \pi_1(\text{Isom}(\overline{M}_p)) \) infinite.

In the concrete example of \( \mathbb{C}P^2 \), we proved in [16] that \( \pi_1(Diff(\overline{M}_p)) \) is infinite for \( p \neq \pm 1 \). In fact, the only example we know where \( a^D \) does not have infinite order in \( \pi_1(\text{Isom}(\overline{M}_p)) \) is for \( \mathbb{C}P^n \).

For \( \dim(M) = 4 \), \( \overline{M}_p \) is a Riemannian version of Kaluza-Klein theory, which is a classical unification of gravity and electromagnetism. In this setting, the nontriviality of \( \pi_1(Diff(\overline{M}_p)) \) becomes the nontriviality of the fundamental group of the gauge group \( G \) of the circle bundle. As a result, there could be a global gauge anomaly in the quantized theory, such as the appearance of a phase factor in correlators computed around a nontrivial element in \( \pi_1(G) \).

In §2 we give background material on pseudodifferential operators and WCS classes on loop spaces. In §3 we prove the main result, both by direct calculation and computer verification. In §4, we apply our theory to an example, perhaps the first, of a symplectic, non-Kähler manifold due to Thurston. Through explicit calculations, we get the results in Theorem 3.4 for all \( p \). In §5, we relate Pontryagin forms on \( M \) to WCS forms on \( \overline{M}_p \) in the Kähler case (Proposition 5.2). Using this Proposition, we prove that the Wodzicki-Pontryagin forms on the free loop space \( LS^{4n+1} \) are nonvanishing (Theorem 5.3). These forms were predicted to vanish in [14], and aside from a low dimensional example in [13] are the first examples of nonvanishing Wodzicki characteristic forms. In Appendix A, we discuss why symplectic manifolds of dimension \( 4k + 2 \) are more difficult to treat. The online files [5, 6] include a particularly long calculation for the Thurston example and computer codes verifying the main results.

2. Background material

2.1. Finite dimensional background material. The complexified tangent bundle of a Riemannian manifold \((M^{4n}, g)\) has Chern character \( ch(M) \in H^{4n}(M, \mathbb{Q}) \) with
2k-component
\[ ch_{2k}(M) = \frac{1}{k!(2\pi)^k} \mathrm{Tr}(\Omega^k) \in H^{2k}(M, \mathbb{R}), \]  
where \( \Omega = \Omega_M \) is the curvature form of \( g \). There are associated Pontryagin-type forms \( \tilde{p}_k(\Omega) = (-1)^k/[(2k)!(2\pi)^{2k}] \mathrm{Tr}(\Omega^{2k}) \) and classes
\[ \tilde{p}_k(M) = [\tilde{p}_k(\Omega)] = (-1)^k ch_{2k}(M) \in H^{4k}(M, \mathbb{Z}). \]

The usual Pontryagin classes \( p_k(M) := (-1)^k c_{2k}(M) \) are built from the even Chern classes \( c_{2k}(M) \). By invariant theory for \( SO(n) \), the rings generated by \( \{\tilde{p}_{2k}\} \) and \( \{p_{2k}\} \) are the same; this reduces to Newton’s identities relating the elementary symmetric functions in \( \lambda_1, \ldots, \lambda_n \) to \( \sum \lambda_1, \ldots, \sum \lambda^n \) [18, §16]. non-zero Pontryagin number \( p_f(M) \) iff it has a nonzero
As part of Chern-Weil theory, for connections \( \nabla^0, \nabla^1 \) on \( TM \) with curvature forms \( \Omega^0, \Omega^1 \), the Chern-Simons form \( \tilde{CS}_{4k-1}(\nabla^0, \nabla^1) \in \Lambda^{4k-1}(M) \),
\[ \tilde{CS}_{4k-1}(\nabla^0, \nabla^1) = 2k \int_0^1 \mathrm{Tr}((\omega_1 - \omega_0) \wedge \Omega_t \wedge \ldots \wedge \Omega_t) \, dt, \]
satisfies
\[ d\tilde{CS}_{4k-1}(\nabla^0, \nabla^1) = \tilde{p}_k(\Omega^0) - \tilde{p}_k(\Omega^1). \]  
Here \( \omega_t = t\omega_0 + (1-t)\omega_1 \), \( \Omega_t = d\omega_t + \omega_t \wedge \omega_t \). Our sign convention is \( \Omega(\partial_k, \partial_j)^0 = g(R(\partial_k, \partial_j)\partial_b, \partial_a) \) where \( R \) is the curvature tensor of \( g \).

2.2. Infinite dimensional background material. This material is taken from [15][16]. Let \( (M, g) \) be a Riemannian manifold. For fixed \( s_0 \gg 0 \), the loop space \( LM \) of \( s_0 \)-differentiable loops is a Banach manifold with tangent space at a loop \( \gamma : S^1 \rightarrow M \) given by \( T_s LM = \Gamma(\gamma^* TM \rightarrow S^1) \), where the sections of the pullback bundle are \( s_0 - 1 \) differentiable. \( LM \) has two preferred connections, the \( L^2 \) or \( s = 0 \) Levi-Civita connection \( \nabla^0 \) associated to the \( L^2 \) inner product \( \langle , \rangle_0 \), and the \( s = 1 \) Levi-Civita connection \( \nabla^1 \) associated to the inner products \( \langle , \rangle_1 \):
\[ \langle X, Y \rangle_0 = \int_{S^1} g(X_t, Y_t)_{\gamma(\theta)} dt, \quad \langle X, Y \rangle_1 = \int_{S^1} g((1 + \Delta)X_t, Y_t)_{\gamma(\theta)} dt. \]
Here \( \Delta = \nabla^* \nabla \) is the Laplacian associated to the pullback connection \( \nabla = \gamma^* \nabla^M \) of the Levi-Civita connection \( \nabla^M \) on \( M \). While the connection and curvature forms for \( \nabla^0 \) at \( \gamma \) take values in \( \mathrm{End}(\gamma^* TM) \), the corresponding forms for \( \nabla^1 \) take values in \( \Psi DO_{s=0} \), the Lie algebra of zeroth order pseudodifferential operators (\( \Psi DOs \)) on \( \Gamma(\gamma^* TM \otimes \mathbb{C}) \), with the understanding that zeroth order means order at most zero. Since endomorphisms of a bundle are zeroth order \( \Psi DOs \), we can consider \( \nabla^0, \nabla^1 \) to be \( \Psi DO_{s=0} \)-connections, where the Lie group \( \Psi DO_{s=0} \) of zeroth order invertible \( \Psi DOs \) with bounded inverse has Lie algebra \( \Psi DO_{s=0} \). In particular, the curvature forms for these connections take values in \( \Psi DO_{s=0} \).
In contrast to finite dimensions, there are two natural traces on \( \Psi DO_{<0} \). Recall that a zeroth order \( \Psi DO \) on \( \Gamma(\gamma^*TM \otimes \mathbb{C}) \) has a symbol sequence \( P \sim \sum_{k=0}^{\infty} \sigma^P_k(x, \xi) \), where \( x \in S^1, \xi \in T^*_x S^1 \); for \( \pi: T^* S^1 \to S^1 \) the projection, \( \sigma^P_k(x, \xi) \in \text{End}(\pi^* \gamma^*TM|_{(x, \xi)}) \) is homogeneous of degree \(-k\) in \( \xi \). The first trace is the leading order trace

\[
\text{Tr}^{lo}(P) = \frac{1}{4\pi} \int_{S^* S^1} \text{tr}(\sigma_0(x, \xi)) \, d\xi \, dx,
\]

where \( S^* S^1 \) is the unit cotangent bundle of \( S^1 \). For example, if \( P \in \text{End}(\gamma^*TM \otimes \mathbb{C}) \), then \( \text{Tr}^{lo}(P) = (1/2\pi) \int_{S^1} \text{tr}(P(x)) \, dx \). The second is the Wodzicki residue (see [8])

\[
\text{res}^W(P) = \frac{1}{4\pi} \int_{S^* S^1} \text{tr}(\sigma_{-1}(x, \xi)) \, d\xi \, dx.
\]

For \( P \in \text{End}(\gamma^*TM \otimes \mathbb{C}) \), \( \text{res}^W(P) = 0 \). The trace in (2.1) can be replaced by either trace to give a theory of characteristic classes on \( TLM \):

\[
\chi^{lo}_{[2k]}(LM) := \frac{1}{k!} [\text{Tr}^{lo}(\Omega^k)] \in H^{2k}(LM, \mathbb{R}), \quad \chi^W_{[2k]}(LM) := \frac{1}{k!} [\text{res}^W(\Omega^k)] \in H^{2k}(LM, \mathbb{R}).
\]

In fact, the \( \chi^W_{[2k]}(LM) \) always vanish, while there are many examples of nonvanishing \( \chi^{lo}_{[2k]}(LM) \) [14]. In this paper, we only consider the Wodzicki residue trace. There are corresponding Wodzicki-Pontryagin classes

\[
p_k^W(LM), \tilde{p}_k^W(LM) \in H^{4k}(LM, \mathbb{R}).
\]

Since these classes vanish, we focus on the associated Wodzicki-Chern-Simons (WCS) forms

\[
\tilde{C}S^W_{2k-1} = k \int_0^1 \text{res}^W((\omega_1 - \omega_0) \wedge \underbrace{\Omega_t \wedge \ldots \wedge \Omega_t}_{k-1}) \, dt \in \Lambda^{2k-1}(LM).
\]

Fix a loop \( \gamma(\theta) \in LM \) and complexified tangent vectors \( X_1, \ldots, X_{4k-1} \in \Gamma(\gamma^*TM \otimes \mathbb{C}) \) at \( \gamma \). By [16] Prop. 2.5, for the \( L^2 \) and \( s = 1 \) Sobolev connections, we have

\[
\tilde{C}S^W_{2k-1}(g)(X_1, \ldots, X_{2k-1}) = \frac{k}{2k-1} \sum_{\sigma} \text{sgn}(\sigma) \int_{S^1} \text{tr}([(-2R(X_{\sigma(1)}, \gamma) - R(\cdot, \gamma) X_{\sigma(1)} + R(X_{\sigma(1)}, \cdot) \gamma)]((\Omega^M)^{k-1}(X_{\sigma(2)}, \ldots, X_{\sigma(2k-1)})) \, d\theta,
\]

where \( R, \Omega \) are the curvature tensor and curvature two-form of \( g \), \( \mathcal{G}_{2k-1} \) is the permutation group of \( \{1, \ldots, 2k-1\} \), and we have omitted the \( \theta \) dependence of \( \gamma, X_i \).

The analog of (2.2) in this context is \( d\tilde{C}S^W_{4k-1}(\nabla^0, \nabla^1) = \tilde{p}_k(\Omega^0) - \tilde{p}_k(\Omega^1) \), where \( \Omega_0, \Omega_1 \) are the curvature of the \( L^2 \), resp. Sobolev \( s = 1 \), metrics on \( LM \). Since \( \Omega_0 \) takes values in endomorphisms of \( TLM \), its Wodzicki residue vanishes. Thus

\[
d\tilde{C}S^W_{4k-1}(\nabla^0, \nabla^1) = -\tilde{p}_k^W(\Omega^1) \in \Lambda^{4k}(LM).
\]
3. The higher dimensional symplectic case

In §3.1, we prove the main result Theorem 3.4. We first discuss the Riemannian geometry of circle bundles $\overline{M}_p$, $p \in \mathbb{Z}$, over symplectic manifolds $(M, \omega)$, where $c_1(\overline{M}_p) = (2\pi)^{-1}[p\omega]$. We compute the curvature as a function of $p$. Using the curvature calculations, we prove that the WCS class on $L\overline{M}_p$ is a polynomial in $p^2$ with nonzero top coefficient. As we explain, this proves the Theorem. In §3.2, we discuss computer calculations that verify our calculations.

3.1. Geometry of line bundles over integral symplectic manifolds. Let $(M, \omega)$ be a compact integral symplectic manifold of real dimension $4n$; equivalently, $M$ is projective algebraic. The symplectic form $\omega \in H^2(M, \mathbb{Z})$ determines a Riemannian metric $g(X, Y) = -\omega(JX, Y)$, where $J$ is a compatible almost complex structure.

We compute the symplectic volume form of $M$ in local coordinates, where $J^b_j = g_{jb}^b$. The symplectic volume form of $M$ is

$$\frac{1}{(2n)!} \omega^{2n} = \frac{1}{(2n)!2^{2n}} \sum_{\sigma \in S_{4n}} \text{sgn}(\sigma) J_{\sigma(1)} J_{\sigma(2)} \cdot \ldots \cdot J_{\sigma(4k-1)} J_{\sigma(4k)} e_1 \wedge \ldots \wedge e_{4n},$$

where $S_{4n}$ is the permutation group on $\{1, \ldots, 4n\}$.

Our convention throughout the paper is that $g_{ab}^b J^b_j = J^b_j$, so $g_{ab} J^b_j = J_{aj}$, not $J_{aj}$.

Proof. (i) Since $\omega(X, Y) = g(JX, Y)$, we get

$$\omega_{ij} = g(J \partial_i, \partial_j) = g(J^b_i \partial_j, \partial_j) = J^b_i g_{bj} = J_{ij}.$$

(ii) This follows from (i), since

$$\omega^{2n} = \frac{1}{2^{2n}} \sum_{\sigma \in S_{4n}} \text{sgn}(\sigma) \omega_{\sigma(1)} \omega_{\sigma(2)} \cdot \ldots \cdot \omega_{\sigma(4k-1)} \omega_{\sigma(4k)} e_1 \wedge \ldots \wedge e_{4n}.$$ 

Because $\omega$ is integral, it has an associated line bundle $L = L_1$ over $M$. Let $\overline{M}_p$ be the total space of the circle bundle $L_p \xrightarrow{\pi} M$ associated to $p\omega$ for $p \in \mathbb{Z}$. $L_p$ comes with a connection $\bar{\eta}$ with $d\bar{\eta} = p\pi^* \omega$, the curvature of $\bar{\eta}$. The metric $g$ induces a metric $\bar{g} = \bar{g}_p$ on $\overline{M}_p$ by

$$\bar{g}(\vec{X}, \vec{Y}) = g(\pi_1^* \vec{X}, \pi_1^* \vec{Y}) + \bar{\eta}(\vec{X}) \bar{\eta}(\vec{Y}).$$

We also denote $\bar{g}(\vec{X}, \vec{Y})$ by $(\vec{X}, \vec{Y})$.

Let $\xi$ be a vector tangent to the circle fiber with $\bar{\eta}(\xi) = 1$, and let $X^L$ denote the horizontal lift to $\overline{M}_p$ of a tangent vector $X$ to $M$. We have $\bar{\eta}(X^L) = 0$.

We compute the Levi-Civita connection $\nabla$ for $\bar{g}$.
Lemma 3.2. (i) $\nabla_x \xi = \mathcal{L}_x \xi = 0$;
(ii) $\nabla_{X^L} Y^L = (\nabla_X Y)^L - pg(JX, Y)\xi$;
(iii) $\nabla_{X^L} \xi = \nabla_X X^L = p(JX)^L$.

Here $\mathcal{L}$ is the Lie derivative.

Proof. (i) As in [16] §3.2, each circle fiber is the orbit of an isometric $S^1$ action on $M_p$, so each circle is a geodesic ($\nabla_x \xi = 0$), with $\xi$ preserved by the action ($\mathcal{L}_x \xi = 0$).
Alternatively, for the first part, since $d\bar{\eta}(\cdot, \xi) = 0$, we get $\mathcal{L}_\xi \bar{\eta} = d\bar{\eta} + i_{\xi} d\bar{\eta} = d1 + d\bar{\eta}(\xi, \cdot) = 0$. Thus $\mathcal{L}_{\nabla_x \xi} = \mathcal{L}_x (g^L + \bar{\eta} \otimes \bar{\eta}) = 0$, so $\xi$ is a Killing vector field. This implies $\bar{g}(\nabla_x \xi, Z) + \bar{g}(\xi, \nabla_x Z) = 0$. Setting $Z = \xi$ and then $Z \perp \xi$, we get $\nabla_x \xi = 0$.
(ii) We define $H(X, Y) \in \mathbb{R}, FX = F(X) \in TM$ by
$$\nabla_{X^L} Y^L = (\nabla_X Y)^L + H(X, Y)\xi, \quad (3.1)$$
$$\nabla_{X^L} \xi = (FX)^L, \quad (3.2)$$

These definitions are valid, since for $\xi L$, it follows from [19] Lemma 1] that $\pi_* (\nabla_x Y^L) = \nabla X Y$, so $\nabla_{X^L} Y^L = (\nabla_X Y)^L + H(X, Y)\xi$ for some $H(X, Y)$. For $\xi L$, $\xi, \xi = 0$ implies $\langle \nabla_{X^L} \xi, \xi \rangle = 0$, so $\nabla_{X^L} \xi = (FX)^L$ for some $FX$.

We note that $H(X, Y) = -H(Y, X)$: using $\bar{\eta}(X) = \bar{g}(\xi, X)$, we get
$$0 = (\mathcal{L}_\bar{g})(X, Y) = (\nabla_{\bar{\eta}} \bar{\eta})(Y) + (\nabla_{\bar{\eta}} \bar{\eta})(X) = \bar{g}(\xi, \nabla X Y) + \bar{g}(\xi, \nabla Y X)$$
$$= H(X, Y) + H(Y, X).$$

Thus
$$p\omega(X, Y) = d\bar{\eta}(X^L, Y^L) = \frac{1}{2} \left( \nabla_{X^L} \bar{\eta}(Y^L) - \bar{\eta}(\nabla_{X^L} Y^L) - \nabla_{Y^L} \bar{\eta}(X^L) - \bar{\eta}(\nabla_{Y^L} X^L) \right)$$
$$= \frac{1}{2} \left( -\bar{\eta}(\nabla_{X^L} Y^L) + \bar{\eta}(\nabla_{Y^L} X^L) \right) = -\bar{g}(\xi, \nabla_{X^L} Y^L) + \bar{g}(\xi, \nabla_{Y^L} X^L)$$
$$= \frac{1}{2} (\bar{\eta}(\nabla_Y X) + \bar{\eta}(\nabla_X Y)) = -\bar{g}(\xi, \nabla_{X^L} Y^L) + \bar{g}(\xi, \nabla_{Y^L} X^L)$$
$$= \frac{1}{2} (\bar{\eta}(\nabla_Y X) + \bar{\eta}(\nabla_X Y)) = -\bar{g}(\xi, \nabla_{X^L} Y^L) + \bar{g}(\xi, \nabla_{Y^L} X^L)$$
$$= \frac{1}{2} (\bar{\eta}(\nabla_Y X) + \bar{\eta}(\nabla_X Y)) = -\bar{g}(\xi, \nabla_{X^L} Y^L) + \bar{g}(\xi, \nabla_{Y^L} X^L)$$

This implies
$$H(X, Y) = -p\omega(X, Y) = p\omega(J^2 X, Y) = -pg(JX, Y).$$

(iii) From $\mathcal{L}_\xi X^L = 0$, we get the first equality in (iii):
$$\nabla_x X^L - \nabla_{X^L} \xi = \langle \xi, X \rangle = \mathcal{L}_x X^L = 0.$$

(This also gives an alternative proof of (i): since $\langle X^L, \xi \rangle = 0$, we have
$$\langle \nabla_x \xi, X \rangle = \langle \xi, \nabla_x X \rangle = \langle \xi, \nabla_{X^L} \xi \rangle = 0.$$
Since $\langle \xi, \xi \rangle = 1$ implies $\langle \nabla_x \xi, \xi \rangle = 0$, we get $\nabla_x \xi = 0$. Another proof that the circle fibers are geodesics is in [12] Thm. 5.2.13.)
Lemma 3.3.

\[ \text{It follows from } \langle Y^L, \xi \rangle = 0 \text{ that} \]
\[ \langle \nabla_{X^L} Y^L, \xi \rangle + \langle Y^L, \nabla_{X^L} \xi \rangle = 0, \text{ or } H(X, Y) + \langle Y^L, (FX)^L \rangle = 0. \]

Since \( \bar{g}(X^L, Y^L) = g(X, Y) \), we have \(-pg(JX, Y) = H(X, Y) = -g(FX, Y)\), so \(FX = pJX\).

\[ \square \]

For curvature conventions for \( \bar{M}_p \), we set
\[ \bar{R}(\partial_k, \partial_j)_a^b = \bar{R}_{kjb}, \quad \bar{R}(\partial_k, \partial_j, \partial_b, \partial_a) = \langle \bar{R}(\partial_k, \partial_j)\partial_b, \partial_a \rangle = \bar{R}_{kjab}, \]
with
\[ \bar{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \]

The curvature tensor \( \bar{R} \) of \( \bar{g} \) is related to the curvature tensor \( R \) of \( g \) as follows:

**Lemma 3.3.**

(i) \[ \bar{g}(\bar{R}(X^L, Y^L)Z^L, W^L) = g(R(X, Y)Z, W) + p^2[-g(JY, Z)g(JX, W) + \]
\[ + g(JX, Z)g(JY, W) + 2g(JX, Y)g(JZ, W)], \]

(ii) \[ \bar{g}((\nabla_X J)Y, Z) = \]

(iii) \[ \bar{g}((\nabla_J J)X, Z) = \]

(iv) \[ \bar{g}((\nabla_J J)Y, Z) = \]

**Proof.** (i) and (ii). We have
\[ \nabla_{X^L} \nabla_{Y^L} Z^L = \nabla_{X^L} ((\nabla_Y Z)^L - pg(JY, Z)\xi) \]
\[ = \nabla_{X^L} (\nabla_Y Z)^L - p\nabla_{X^L} (g(JY, Z)\xi) \]
\[ = \nabla_X (\nabla_Y Z)^L - pg(JX, \nabla_Y Z)\xi - p[X^L(g(JY, Z)\xi + g(JY, Z)\nabla_{X^L} \xi] \]
\[ = (\nabla_X \nabla_Y Z)^L - pg(JX, \nabla_Y Z)\xi \]
\[ - g((\nabla_X J)Y, Z)\xi + g(J\nabla_X Y, Z)\xi \]
\[ + g(JY, \nabla_X Z)\xi + g(JY, Z)(p(JX)^L), \]
\[ [X^L, Y^L] = \nabla_{X^L} Y^L - \nabla_{Y^L} X^L \]
\[ = (\nabla_X Y)^L - pg(JX, Y)\xi - (\nabla_Y X)^L + pg(JY, X)\xi \]
\[ = -2pg(JX, Y)\xi + [X, Y]^L, \]
so
\[ \bar{R}(X^L, Y^L)Z^L = (\nabla_X \nabla_Y Z)^L - pg(JX, \nabla_Y Z)\xi \]
\[ - p(g((\nabla_X J)Y, Z)\xi + g(J\nabla_X Y, Z)\xi) + g(JY, \nabla_X Z)\xi + pg(JY, Z)(JX)^L) \]
\[ - [(\nabla_Y \nabla_X Z)^L - pg(JY, \nabla_X Z)\xi \]

\[ \square \]
mers infinitely many such contact manifolds (a closed covariant derivative map

\[ p g((\nabla Y J) X, Z) \xi + g(J \nabla Y X, Z) \xi + g(J X, \nabla Y Z) \xi + pg(J X, Z)(J Y)^L] \]

\[ - \left( [\nabla_{(X,Y)}^L Z^L] - 2 pg(J X, Y) \nabla \xi Z^L \right) \]

\[ = (R(X, Y) Z)^L - p^2 g(J Y, Z)(J X)^L + p^2 g(J X, Z)(J Y)^L + 2 p^2 g(J X, Y)(J Z)^L \]

\[ - pg((\nabla X J) Y, Z) \xi + pg((\nabla Y J) X, Z) \xi. \]

Thus,

\[ \overline{p}(R(X^L, Y^L) Z^L, W^L) = (R(X, Y) Z, W) - p^2 g(J Y, Z)g(J X, W) \]

\[ + p^2 g(J X, Z)g(J Y, W) + 2 p^2 g(J X, Y)g(J Z, W), \]

\[ \overline{g}(R(X^L, Y^L) Z^L, \xi) = -pg((\nabla X J) Y, Z) + pg((\nabla Y J) X, Z). \]

(iii) and (iv). Using \([X^L, \xi] = 0\) and Lemma 3.2, we have

\[ \overline{R}(X^L, \xi) Y^L) = \overline{\nabla}_{X^L} \nabla_{\xi} Y^L - \nabla_{\xi} \overline{\nabla}_{X^L} Y^L \]

\[ = \nabla_{X^L}(p(J Y)^L) - \nabla_{\xi}(p(J X Y)^L) - pg(J X, Y) \xi) \]

\[ = p \nabla_{X^L}(J Y)^L - p(J \nabla X Y)^L \]

\[ = p((\nabla X J) Y)^L - pg(J X, J Y) \xi) - p(J \nabla X Y)^L \]

\[ = p((\nabla X J) Y)^L - p^2 g(X, Y) \xi. \]

In other words,

\[ \overline{g}(R(X^L, \xi) Y^L, Z^L) = pg((\nabla X J) Y, Z), \]

\[ \overline{g}(R(X^L, \xi) Y^L, \xi) = -p^2 g(X, Y). \]

In fact, (ii) and (iv) are equivalent; this uses the symmetry of \(\overline{R}\) and

\[ 0 = -d \omega(X, Y, Z) = d(g(J \cdot, \cdot))(X, Y, Z) \]

\[ \Rightarrow g((\nabla X J) Y, Z) + g((\nabla Y J) X, Z) - g((\nabla Z J) Y, X) = 0. \]

Here is the main result.

**Theorem 3.4.** Let \((M, \omega)\) be a closed integral symplectic manifold of dimension \(4n\). Then for \(|p| \gg 0\), \(\pi_1(\text{Diff}(M_p))\) and \(\pi_1(\text{Isom}(M_p))\) are infinite. Equivalently, let \(\overline{M}\) be a closed \((4n + 1)\)-dimensional contact manifold with closed Reeb orbits. Then \(\overline{M}\) covers infinitely many such contact manifolds \(M_p\) with \(\pi_1(\text{Diff}(M_p))\) and \(\pi_1(\text{Isom}(M_p))\) infinite.

In fact, \(\overline{M}_p\) is diffeomorphic to \(\overline{M}_{-p}\), since \(L_p\) is diffeomorphic to \(L_{-p} = L_p^*\) via the fiberwise map \(v \mapsto <\cdot, v>\).

**Proof.** For the equivalence, we note that the line bundle \(L_1\) covers \(L_p\) by the map \(z \mapsto z^p\) in each fiber, as can be seen by the Čech construction of \(c_1(L_p)\). The equivalence of
line bundles over symplectic manifolds and contact manifolds with closed Reeb orbits is given by the Boothby-Wang fibration theorem [2 Thm. 3.9].

We recall the approach of [16]. For any set $X$, the following sets are in bijection:

\[
\text{Maps}(S^1 \times X, X) \leftrightarrow \text{Maps}(S^1, \text{Maps}(X, X)) \leftrightarrow \text{Maps}(X, \text{Maps}(S^1, X)).
\]

In particular, let $a : S^1 \times \overline{M}_p \rightarrow \overline{M}_p$ be the isometric $S^1$ action of rotation in the fibers of $\overline{M}_p$. This gives equivalent maps $a^D \in \pi_1(\text{Diff}(\overline{M}_p))$ defined by $a^D(\theta)(\overline{m}) = a(\theta, \overline{m})$, and $a^L : \overline{M}_p \rightarrow L\overline{M}_p$ defined by $a^L(\overline{m})(\theta) = a(\theta, \overline{m})$. This induces $a^L_* : H_{4n+1}(\overline{M}_p) \rightarrow H_{4n+1}(L\overline{M}_p)$. If $a^D$ is trivial, then the trivializing homotopy $F : [0, 1] \times S^1 \rightarrow \text{Diff}(\overline{M}_p)$, $F(0, \cdot) = a^D$, $F(1, \cdot) = \text{Id} : \overline{M}_p \rightarrow \overline{M}_p$, induces $F' : [0, 1] \times \overline{M}_p \rightarrow L\overline{M}_p$ with $F'(0, \cdot) = a^L$, $F'(1, m) = (\theta \mapsto m)$ for all $\theta$, i.e., $F'(1, m)$ is the constant loop at $m$. Thus if $a^D$ is trivial, $a^L_* = F'(1, \cdot)_*$ takes the fundamental homology class $[\overline{M}_p]$ in $\overline{M}_p$ to the class $[\overline{M}_p, c]$ of constant loops in $L\overline{M}_p$.

If $\beta \in \Lambda^{4n+1}(L\overline{M}_p)$ is closed and has

\[
\int_{a^L_*[\overline{M}_p]} \beta = \int_{[\overline{M}_p]} (a^L)^* \beta \neq 0,
\]

then $a^L$ is not homotopic to the constant loop map. It follows that $a^D$ must be nontrivial in $\pi_1(\text{Diff}(\overline{M}_p))$ and in fact has infinite order [16 Prop. 3.4].

For our purposes, $\beta = \tilde{C}S^W_{4n+1}$; this is closed by [16 Thm. 2.6]. By (2.3), $\tilde{C}S^W_{4n+1}$ is the zero form on constant loops, since $\dot{\gamma} = 0$. Thus the second condition $\int_{[\overline{M}_p, c]} \beta = 0$ in (3.3) is satisfied.

For the first condition in (3.3), we want to compute $a^L_* \tilde{C}S^W_{4n+1}$ in a local frame. In our setting, $a^L(m)$ is the loop $\gamma = \gamma_m$ given by the fiber $\overline{M}_{p, m}$. As in [16 (3.5)], we may assume that $\dot{\gamma} = \tilde{x} = e_0$, the first element of an orthonormal frame $\{e_0, \ldots, e_{4n}\}$. Then (2.3) becomes

\[
a^L_* \tilde{C}S^W_{4n+1, \gamma} = \frac{2n + 1}{2^{2n-1}} \sum_{\sigma \in \mathfrak{S}_{4n+1}} \text{sgn}(\sigma) \overline{R}_{\sigma_0 \ell_1} \overline{R}_{\sigma_1 \sigma_2 \ell_2} \overline{R}_{\sigma_3 \sigma_4 \ell_3} \cdots \overline{R}_{\sigma_{4n-1} \sigma_{4n} \ell_{2k}}
\]

\[
\cdot e_0 \land \ldots \land e_{4n}
\]

\[
= \frac{2n + 1}{2^{2n-1}} \sum_{q=1}^{4n+2} S_{4n+1, q} p^q e_0 \land \ldots \land e_{4n},
\]

(3.4)

where $\mathfrak{S}_{4n+1}$ is the permutation group of $\{0, 1, \ldots, 4n\}$, and $\ell_i, r \in \{0, 1, \ldots, 4n\}$. We have used Lemma 3.3 to write $a^L_* \tilde{C}S^W_{4n+1}$ on $\overline{M}_p$ as a polynomial in $p$. In particular, it is easy to see that the top power of $p$ in $S_{4n+1}$ is $p^{4n+2}$ and that there is no term with power $p^0$.

We focus on the top term.
Claim:

\[ S_{4n+1,4n+2} = (-1)^{n+1}2^{n+1}(2n+1) \sum_{\sigma' \in S_{4n}} \text{sgn}(\sigma')J_{\sigma'_{\text{I}}\sigma'_{\text{II}}\ldots J_{\sigma'_{4n-1}\sigma'_{4n}}} \]

where \( S_{4n} \) is the permutation group of \( \{1, \ldots, 4n\} \).

By Lemma 3.1, the right hand side of (3.5) is a nonzero multiple of the symplectic volume form. As a result,

\[ 0 < \int_{M_p} a^{L^*c} S_{4n+1} \]

for \( |p| \gg 0 \), which is the first condition in (3.3). Thus the Claim implies that \([a^D]\) has infinite order in \( \pi_1(\text{Diff}(M_p)) \). The action is by isometries of the fibers, so if \([a^D]\) had finite order in \( \pi_1(\text{Isom}(M_p)) \), it would have finite order in \( \pi_1(\text{Diff}(M_p)) \).

Thus the Theorem follows from the Claim, which we now prove. By Lemma 3.3, the only way to get \( p^{4n+2} \) in a term in (3.4) is if \( \sigma_0 = 0 \). Therefore,

\[ S_{4n+1,4n+2}p^{4n+2} \]

\[ \equiv \sum_{\sigma_0=0} \text{sgn}(\sigma)T_{0,0}T_{\sigma_1\sigma_2\ell_1}T_{\sigma_3\sigma_4\ell_2} \ldots \]

\[ \equiv \sum_{\sigma_0=0} \text{sgn}(\sigma)T_{0,0}T_{\sigma_1\sigma_2\sigma_3\sigma_4\ell_2} \ldots \text{mod } p^{4n+2} \]

\[ \equiv \sum_{\sigma_0=0} \text{sgn}(\sigma)(\text{sgn}(\sigma)T_{0,0}T_{\sigma_1\sigma_2\sigma_3\sigma_4\ell_2} \ldots \text{mod } p^{4n+2} \]

\[ \equiv -p^2 \sum_{\sigma_0=0} \text{sgn}(\sigma)(T_{\sigma_1\sigma_2\sigma_3\sigma_4\ell_2} \ldots \text{mod } p^{4n+2} \]

where \((mod \ p^{4n+2})\) denotes all terms with power \( p^{4n+2} \). Therefore,

\[ S_{4n+1,4n+2}p^{4n+2} \]

\[ = -p^{4n+2} \sum_{\sigma_0=0} \text{sgn}(\sigma)(A_{1}^{\prime})_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}^{a_1}(A_{2}^{\prime})_{\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9}^{a_3} \]

\[ \ldots (A_{n}^{\prime})_{\sigma_{4n-3}\sigma_{4n-4}\ldots\sigma_{4n-1}\sigma_{4n}a_1}^{a_2n-1} \]

where

\[ (A_{i}^{\prime})_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}^{a_1} \]
formally of the form $\text{sgn}(\sigma S)$

Since, terms in (3.6) containing $g_{\sigma a}$ do not contribute to $S_{4n+1,4n+2}$. Indeed, by the symmetry of $g$, for fixed $\sigma$ the term in $S_{4n+1,4n+2}$ formally of the form $\text{sgn}(\sigma) J \cdot J \cdot \ldots \cdot J \cdot g_{\sigma a}$ is cancelled by the term with $(ij)\sigma$ in cycle notation.

As a result, we have

$$S_{4n+1,4n+2}p^{4n+2} = p^{4n+2} \sum_{\sigma=0}^{\infty} \text{sgn}(\sigma)(A_1)_{\sigma_{\sigma_{\sigma}}\sigma_{\sigma_{\sigma}}}$$

\hspace{1.5cm} \ldots (A'_n)_{\sigma_{\sigma_{\sigma}}\sigma_{\sigma_{\sigma}} \sigma_{\sigma_{\sigma}} \sigma_{\sigma_{\sigma}}} a_{2n-1}, \quad (3.7)

with

\begin{align*}
(A_1)_{\sigma_{\sigma_{\sigma}}\sigma_{\sigma_{\sigma}}} & = -2J_{\sigma_{\sigma}} a_{1} J_{\sigma_{\sigma_{\sigma}}} g_{\sigma_{\sigma_{\sigma}}} + 2J_{\sigma_{\sigma_{\sigma}}} a_{1} J_{\sigma_{\sigma_{\sigma}}} g_{\sigma_{\sigma_{\sigma}}} + 2J_{\sigma_{\sigma_{\sigma}}} a_{1} J_{\sigma_{\sigma_{\sigma}}} g_{\sigma_{\sigma_{\sigma}}} \\
& \quad - 2J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma_{\sigma}}} d_{\sigma_{\sigma_{\sigma}}} a_{1} - 4J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma_{\sigma}}} d_{\sigma_{\sigma_{\sigma}}} a_{1} \\
& = -4J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma}} a_{1} g_{\sigma_{\sigma_{\sigma}}} - 4J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma_{\sigma}}} d_{\sigma_{\sigma_{\sigma}}} a_{1} - 4J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma_{\sigma}}} d_{\sigma_{\sigma_{\sigma}}} a_{1} \\
& = -22J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma_{\sigma}}} d_{\sigma_{\sigma_{\sigma}}} a_{1} + J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma}} a_{1} g_{\sigma_{\sigma_{\sigma}}} + J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma_{\sigma}}} d_{\sigma_{\sigma_{\sigma}}} a_{1} \\
& = -22J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma_{\sigma}}} d_{\sigma_{\sigma_{\sigma}}} a_{1} + J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma}} a_{1} g_{\sigma_{\sigma_{\sigma}}} + J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma_{\sigma}}} d_{\sigma_{\sigma_{\sigma}}} a_{1}.
\end{align*}

(In the last line, we replaced $J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma}} a_{1} g_{\sigma_{\sigma_{\sigma}}}$ with $J_{\sigma_{\sigma_{\sigma}}} J_{\sigma_{\sigma_{\sigma}}} a_{1} g_{\sigma_{\sigma_{\sigma}}}$ using the sign preserving “change of variables” $\sigma \mapsto (13)(24)\sigma$; strictly speaking, this is valid only after we plug $(A_1)$ back into (3.7).)
Doing the same computations for \((A'_2), \ldots, (A'_n)\), we get
\[
S_{4n+1, 4n+2} = \sum_{s_0 = 0} \text{sgn}(\sigma) (A_1)_{\sigma_1 \sigma_2 \sigma_3 \sigma_4 a_3} \cdot (A_2)_{\sigma_5 \sigma_6 \sigma_7 \sigma_8 a_5} \cdot \ldots \cdot (A_n)_{\sigma_{4n} \sigma_{4n-1} \sigma_{4n-2} \sigma_{4n-3} a_1^2 a_n^2},
\]
where
\[
(A_2)_{\sigma_5 \sigma_6 \sigma_7 \sigma_8 a_5} = (-2^2)[J_{\sigma_5 \sigma_6} (J_{\sigma_7 \sigma_8} \delta_{a_5} + J_{\sigma_7} a_3 g_{\sigma a_5} + J_{\sigma_7 \sigma_8} \delta_{a_5})]
\]
\[
(A_n)_{\sigma_{4n-3} \sigma_{4n-2} \sigma_{4n-1} \sigma_{4n} a_1^2 a_n^2} = (-2^2)[J_{\sigma_{4n-3} \sigma_{4n-2}} (J_{\sigma_{4n-1}} \delta_{a_4} \delta_{a_5} + J_{\sigma_{4n-1}} a_2 g_{\sigma a_1} + J_{\sigma_{4n-2} \sigma_{4n-1}} \delta_{a_5} a_1^2 a_n^2)].
\]

We now begin to simplify (3.8).
\[
S_{4n+1, 4n+2} = (1) \sum_{s_0 = 0} \text{sgn}(\sigma) (A_1)_{\sigma_1 \sigma_2 \sigma_3 \sigma_4 a_3} \cdot (A_2)_{\sigma_5 \sigma_6 \sigma_7 \sigma_8 a_5} \cdot \ldots \cdot (A_n)_{\sigma_{4n} \sigma_{4n-1} \sigma_{4n-2} \sigma_{4n-3} a_1^2 a_n^2},
\]
where
\[
(A_2)_{\sigma_1 \sigma_2 \sigma_3 \sigma_4 a_3} = (A_1)_{\sigma_1 \sigma_2 \sigma_3 \sigma_4 a_3} \cdot (A_2)_{\sigma_5 \sigma_6 \sigma_7 \sigma_8 a_5} a_1
\]
\[
= (-2^2)^2 [J_{\sigma_1 \sigma_2} (J_{\sigma_3 \sigma_4} \delta_{a_1}, a_1 + J_{\sigma_3 \sigma_4} g_{\sigma a_3} + J_{\sigma_3 \sigma_4} \delta_{a_3} a_1)]
\]
\[
\cdot [J_{\sigma_5 \sigma_6} (J_{\sigma_7 \sigma_8} \delta_{a_5} a_2 + J_{\sigma_7 \sigma_8} a_3 g_{\sigma a_5} + J_{\sigma_7 \sigma_8} \delta_{a_5} a_3)]
\]
\[
= (-2^2)^2 \cdot J_{\sigma_1 \sigma_2} J_{\sigma_5 \sigma_6} [J_{\sigma_3 \sigma_4} \delta_{a_5} a_1 + J_{\sigma_5 \sigma_6} \delta_{a_3} a_2] [J_{\sigma_5 \sigma_6} \delta_{a_5} a_3 + J_{\sigma_7 \sigma_8} a_3 g_{\sigma a_5} + J_{\sigma_7 \sigma_8} \delta_{a_5} a_3]
\]
\[
= (-2^2)^2 \cdot J_{\sigma_1 \sigma_2} J_{\sigma_3 \sigma_4} \delta_{a_1} \delta_{a_3} a_1 + J_{\sigma_3 \sigma_4} g_{\sigma a_3} \delta_{a_5} a_3 + J_{\sigma_3 \sigma_4} \delta_{a_5} a_3 J_{\sigma_7 \sigma_8} a_3 \delta_{a_5} a_3
\]
\[
+ J_{\sigma_3} a_1 g_{\sigma a_3} J_{\sigma_7 \sigma_8} \delta_{a_5} a_3 + J_{\sigma_3} g_{\sigma a_3} J_{\sigma_7 \sigma_8} a_3 \delta_{a_5} a_3 + J_{\sigma_3} \delta_{a_5} a_3 J_{\sigma_7 \sigma_8} a_3 \delta_{a_5} a_3
\]
\[
= (-2^2)^2 \cdot J_{\sigma_1 \sigma_2} J_{\sigma_3 \sigma_4} \delta_{a_1} \delta_{a_3} a_1 + J_{\sigma_3 \sigma_4} g_{\sigma a_3} \delta_{a_5} a_3 + J_{\sigma_3 \sigma_4} \delta_{a_5} a_3 J_{\sigma_7 \sigma_8} a_3 \delta_{a_5} a_3
\]
\[
+ J_{\sigma_3} a_1 g_{\sigma a_3} J_{\sigma_7 \sigma_8} \delta_{a_5} a_3 + J_{\sigma_3} g_{\sigma a_3} J_{\sigma_7 \sigma_8} a_3 \delta_{a_5} a_3 + J_{\sigma_3} \delta_{a_5} a_3 J_{\sigma_7 \sigma_8} a_3 \delta_{a_5} a_3
\]
\[
= (-2^2)^2 \cdot J_{\sigma_1 \sigma_2} J_{\sigma_3 \sigma_4} \delta_{a_1} \delta_{a_3} a_1 + J_{\sigma_3 \sigma_4} g_{\sigma a_3} \delta_{a_5} a_3 + J_{\sigma_3 \sigma_4} \delta_{a_5} a_3 J_{\sigma_7 \sigma_8} a_3 \delta_{a_5} a_3
\]
\[
+ J_{\sigma_3} a_1 g_{\sigma a_3} J_{\sigma_7 \sigma_8} \delta_{a_5} a_3 + J_{\sigma_3} g_{\sigma a_3} J_{\sigma_7 \sigma_8} a_3 \delta_{a_5} a_3 + J_{\sigma_3} \delta_{a_5} a_3 J_{\sigma_7 \sigma_8} a_3 \delta_{a_5} a_3
\]
Continuing to simplify (3.8), we have
\[(A_{123})_{\sigma_1 \ldots \sigma_7} \sigma_1 \ldots \sigma_7^a \sigma_1 = (A_{12})_{\sigma_1 \sigma_8} a_1 (A_3)_{\sigma_9 a_{10} \sigma_{11} a_{12}}^a_5
\]
\[= (-2^2)^2 \cdot J_{\sigma_1 a_1} J_{\sigma_3 a_4} J_{\sigma_5 a_6} [J_{\sigma_7 a_5} a_{11} g_{\sigma_7 a_5} + J_{\sigma_7 a_5} a_{12} a_7]
\]
\[\cdot (-2) \cdot J_{\sigma_9 a_{10}} [J_{\sigma_{11} a_5} a_{12} a_7 + J_{\sigma_{11} a_5} g_{\sigma_{12} a_7} + J_{\sigma_{11} a_5} a_{12} a_7]
\]
\[= (-2^2)^1 \cdot J_{\sigma_1 a_1} J_{\sigma_3 a_4} J_{\sigma_5 a_6} J_{\sigma_7 a_8} J_{\sigma_9 a_{10}} [J_{\sigma_{11} a_5} a_{12} a_7 + J_{\sigma_{11} a_5} g_{\sigma_{12} a_7} + J_{\sigma_{11} a_5} a_{12} a_7],
\]
where the last line follows from computations as in (3.8).
In the end, we obtain
\[S_{4n+1, 4n+2}
\]
\[= (-1)^{n+1} 2^{2n} \sum_{\sigma_0 = 0}^{\sigma_5} \operatorname{sgn}(\sigma) J_{\sigma_1 a_1} J_{\sigma_3 a_1} J_{\sigma_5 a_1} [J_{\sigma_{4n-3} a_{11} a_{12}} a_{11} g_{\sigma_{4n-1} a_1} + J_{\sigma_{4n-3} a_{12} a_7} a_{12} a_7]
\]

This proves the Claim and finishes the proof of Theorem 3.4. □

3.2. A computer verification. Using the code at egison.org, we obtain the following results for $S_{4n+1, 4n+2}$ in Theorem 3.4 [5].

| dim(M) | 4 | 6 | 8 |
|--------|---|---|---|
| $S_{4n+1, 4n+2}$ | -192 | 0 | 61440 |

In this pointwise calculation, we have put the almost complex structure into the normal form
\[J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
This agrees with (3.12): for dim($M$) = 4 (i.e. $n = 1$), $\sum_{\sigma_0 = 0}^{\sigma_5} J_{\sigma_1 a_1} J_{\sigma_3 a_1} = -2!^2$, so $S_{4n+1, 4n+2} = (-1)^{2^3} (3)(-8) = -192$; for dim($M$) = 8 (i.e. $n = 2$), the corresponding sum over permutations gives $4!^2$, so $S_{4n+1, 4n+2} = (-1)^{2^5} (384) = 61440$.

The fact that $c_n = 0$ for dim($M$) = 6 is proven in Appendix A and applies to all manifolds of dimension $4n + 2$.

4. The Thurston example

We calculate explicitly the WCS class for the example given by Thurston [21] of a non-Kähler symplectic manifold $M^4$. By putting an explicit Riemannian metric on $M$, we can compute that $\pi_1(\text{Diff}(M))$ and $\pi_1(\text{Isom}(M))$ are infinite for all $p \in \mathbb{Z}$.
4.1. The metric. $M$ is a $T^2$ fibration over $T^2$. To construct $M$, we take coordinates $\theta_1, \theta_2, \theta_3, \theta_4 \in [0,1]$. The base $T^2$ has coordinates $\theta_1, \theta_2$, where we glue $\theta_1, \theta_2$ as usual to get a torus. For the fiber $T^2$, we take the linear transformation
\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]
on $\mathbb{R}^2 = \{(\theta_3, \theta_4)\}$ (so now these coordinates are real numbers), which glues the unit $(\theta_3, \theta_4)$-square to the parallelogram with sides given by the vectors $\vec{\theta}_3, \vec{\theta}_3 + \vec{\theta}_4$. We do this gluing in the $\theta_2$ direction, so that $M$ is well-defined on $[0,1]^4$ with the relations/gluings
\[
(0, \theta_2, \theta_3, \theta_4) \sim (1, \theta_2, \theta_3, \theta_4), \quad (\theta_1, 0, \theta_3, \theta_4) \sim (\theta_1, 1, \theta_3, \theta_3 + \theta_4).
\]
We claim that the metric
\[
d\theta_1^2 + d\theta_2^2 + d\theta_3^2 - \theta_2 d\theta_3 d\theta_4 + (1 + \theta_2) d\theta_4^2
\]is well-defined on $M$. Since $\partial_{\theta_4}$ at $\theta_2 = 0$ is glued to $\partial_{\theta_3} + \partial_{\theta_4}$ at $\theta_2 = 1$, this means we must have
\[
\langle \partial_{\theta_i}, \partial_{\theta_j} \rangle_{(\theta_2, \theta_3, \theta_4)} = \langle \partial_{\theta_i}, \partial_{\theta_j} \rangle_{(1, \theta_3, \theta_4)}, \quad i, j = 1, 2, 3, 4,
\]
\[
\langle \partial_{\theta_i}, \partial_{\theta_j} \rangle_{(\theta_1, 0, \theta_3, \theta_4)} = \langle \partial_{\theta_i}, \partial_{\theta_j} \rangle_{(\theta_1, 1, \theta_3, \theta_4)}, \quad i, j = 1, 2, 3,
\]
\[
\langle \partial_{\theta_i}, \partial_{\theta_4} \rangle_{(\theta_1, 0, \theta_3, \theta_4)} = \langle \partial_{\theta_i}, \partial_{\theta_3} + \partial_{\theta_4} \rangle_{(\theta_1, 1, \theta_3, \theta_4)}, \quad i = 1, 2, 3, j = 4,
\]
\[
\langle \partial_{\theta_4}, \partial_{\theta_4} \rangle_{(\theta_1, 0, \theta_3, \theta_4)} = \langle \partial_{\theta_3} + \partial_{\theta_4}, \partial_{\theta_3} + \partial_{\theta_4} \rangle_{(\theta_1, 1, \theta_3, \theta_4)}.
\]
Since the metric is independent of $\theta_1 \in [0,1]$, the first equation holds; since the metric is independent of $\theta_2$ for $i, j = 1, 2, 3$, the second equation holds. For the third equation, the left hand side is 0; the right hand side is also 0 for $i = 1, 2$, and for $i = 3$ we get
\[
\langle \partial_{\theta_i}, \partial_{\theta_3} + \partial_{\theta_4} \rangle_{(\theta_1, 1, \theta_3, \theta_4)} = 1 - (\theta_2 = 1) = 0.
\]
For the last equation, the left hand side is 1, and the right hand side is
\[
\langle \partial_{\theta_3}, \partial_{\theta_4} \rangle_{\theta_2=1} + 2 \langle \partial_{\theta_3}, \partial_{\theta_4} \rangle_{\theta_2=1} + \langle \partial_{\theta_4}, \partial_{\theta_4} \rangle_{\theta_2=1} = 1 + 2(-1) + 2 = 1.
\]
(Since $g_{33} = 1$ is independent of $\theta_2$, from the gluing $\partial_{\theta_3}|_{\theta_2=0} = \partial_{\theta_3}|_{\theta_2=1}, \partial_{\theta_4}|_{\theta_2=0} = (\partial_{\theta_3} + \partial_{\theta_4})|_{\theta_2=1}$, $g_{34}(\theta_2)$ must satisfy $g_{34}(0) = 0, g_{34}(1) = -1$ and $g_{44}(\theta_2)$ must satisfy $g_{44}(0) = 1, g_{44}(1) = 2$, so our choice of metric is the simplest one possible.)

As a check, we note that the volume form is
\[
(1 + \theta_2 - \theta_2^2) d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \wedge d\theta_4,
\]
which is equal at $\theta_2 = 0$ and $\theta_2 = 1$. It is also positive definite, since $1 + \theta_2 - \theta_2^2$ has no roots in $[0,1]$. 


4.2. The compatible AC structure and the new metric. Given a symplectic form $\omega$ and a Riemannian metric $g$, we want to find an AC structure $J$ and a new metric $\tilde{g}$ with the compatibility condition $\omega(u, v) = \tilde{g}(Ju, v)$. The usual procedure is to write
\[ \omega(u, v) = g(Au, v) \] (4.2)
for some skew-adjoint transformation $A$. (The matrix of $A$ is not necessarily skew-symmetric in the basis $\{\partial_\theta_i\}$, since this basis is only orthogonal at $\theta_2 = 0$.) For $A^*$ the adjoint of $A$ with respect to $g$, we set
\[ J = \sqrt{AA^*}^{-1} A = \sqrt{-A^2}^{-1} A, \quad \tilde{g}(u, v) = g(\sqrt{AA^*}u, v). \] (4.3)
It is easy to check that $J^2 = -1$ and that (4.2) holds. Note that $\tilde{g}(u, v) = g((AA^*)^{1/4}u, (AA^*)^{1/4}v)$ is positive definite.

We take the symplectic form $\omega = d\theta_1 \wedge d\theta_2 + \kappa d\theta_3 \wedge d\theta_4$, $\kappa \in \mathbb{Z} \setminus \{0\}$, so $(M, \omega)$ is integral. (For $\kappa < 0$, $\omega^2$ is the volume form for the reverse of the standard orientation.) For the metric $g$, we first have to compute $A$. (4.2) is equivalent to
\[ \omega_{ij} = A^k_i g_{kj}. \] (4.4)
A straightforward calculation gives
\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & \frac{\theta_2 \kappa}{1+\theta_2^2-\theta_2^2} & \frac{\kappa}{1+\theta_2^2-\theta_2^2} \\
0 & 0 & \frac{(-1-\theta_2) \kappa}{1+\theta_2^2-\theta_2^2} & \frac{-\theta_2 \kappa}{1+\theta_2^2-\theta_2^2}
\end{pmatrix}
\]

We now have to compute $\sqrt{AA^*}$. From (4.2) and
\[ \omega(u, v) = -\omega(v, u) = -g(Av, u) = g(-A^*u, v), \]
we get $A^* = -A$. Thus

\[ AA^* = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{\kappa^2}{1+\theta_2^2-\theta_2^2} & 0 \\
0 & 0 & 0 & \frac{\kappa^2}{1+\theta_2^2-\theta_2^2}
\end{pmatrix} \implies \sqrt{AA^*} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{\kappa}{(1+\theta_2^2-\theta_2^2)^{1/2}} & 0 \\
0 & 0 & 0 & \frac{\kappa}{(1+\theta_2^2-\theta_2^2)^{1/2}}
\end{pmatrix}, \]

and
\[
J = \sqrt{AA^*}^{-1} A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & \frac{\theta_2}{(1+\theta_2^2-\theta_2^2)^{1/2}} & \frac{1}{(1+\theta_2^2-\theta_2^2)^{1/2}} \\
0 & 0 & \frac{-\theta_2}{(1+\theta_2^2-\theta_2^2)^{1/2}} & \frac{-\theta_2}{(1+\theta_2^2-\theta_2^2)^{1/2}}
\end{pmatrix}
\] (4.5)

Note that $J$ is independent of $\kappa$. 
To compute $\bar{g}$, we have 

$$
\sqrt{AA^*} \begin{pmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    u_4
\end{pmatrix} = \begin{pmatrix}
    u_1 \\
    u_2 \\
    \frac{\kappa}{(1+\theta_2-\theta_2^2)^{1/2}} u_3 \\
    \frac{\kappa}{(1+\theta_2-\theta_2^2)^{1/2}} u_4
\end{pmatrix} \implies \bar{g} = g(\sqrt{AA^*}, \cdot) = \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & (1+\theta_2-\theta_2^2)^{1/2} & -\theta_2 \kappa \\
    0 & 0 & (1+\theta_2-\theta_2^2)^{1/2} & (1+\theta_2-\theta_2^2)^{1/2}
\end{pmatrix}
$$

(4.6)

4.3. The top WCS form. Let $(e_0, \ldots, e_4)$ be a local orthonormal frame of $\overline{M}_p$, with $e_0 = \xi$. By (2.3) with $k = 3$, 

$$
a^L \cdot C S_5^W (e_0, \ldots, e_4) = C S_5^W (a^L e_0, \ldots, a^L e_4)
$$

(4.7)

$$
= 3 \int_{S^1} \sum_{\sigma \in S_5} \text{sgn}(\sigma) \overline{R}_{\sigma_0 \ell_1} \overline{R}_{\sigma_1 \sigma_2 \ell_2} \overline{R}_{\sigma_3 \sigma_4 \ell_3} d \theta_0,
$$

where $S_5$ is the permutation group on $\{0, 1, 2, 3, 4\}$, $\ell_1, \ell_2, r \in \{0, 1, 2, 3, 4\}$, $\sigma = (\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$, and $\theta_0$ is the fiber coordinate with $\partial \theta_0 = \xi$. We have used that $a^L(m)$ is the circle fiber of $\tilde{m} \in \overline{M}_p$, so $\gamma$ in (2.3) equals $\xi$. Thus the integral over $S^1$ is the integral over the circle fiber in $\overline{M}_p$.

Set $\beta = \beta(\theta_2) = 1 + \theta_2 - \theta_2^2$

Proposition 4.1. We have

$$
\int_{\overline{M}_p} a^L \cdot C S_5^W = \frac{3 \pi^2 \kappa^2 p^{3/2}}{8} \int_0^1 (3072 p^4 - 640 p^2 \beta^{-2} - 25 \beta^{-4}) d \theta_2.
$$

(4.8)

Proof. We explain the constants on the right hand side of (4.8). By the construction of $\overline{g}$, $a$ acts via isometries on $\overline{M}_p$. This makes the integrand in (4.7) independent of $\theta_0$, so the integral is replaced with a factor of $2\pi$. Thus

$$
\int_{\overline{M}_p} a^L \cdot C S_5^W = \int_{\overline{M}_p} a^L \cdot C S_5^W (e_0, \ldots, e_4) e^0 \wedge \ldots \wedge e^4
$$

$$
= 2 \pi \cdot 3 \int_{\overline{M}_p} \sum_{\sigma \in S_5} \text{sgn}(\sigma) \overline{R}_{\sigma_0 \ell_1} \overline{R}_{\sigma_1 \sigma_2 \ell_2} \overline{R}_{\sigma_3 \sigma_4 \ell_3} d \text{dvol}.
$$

We now switch to the coordinates $\{\theta_0, \ldots, \theta_4\}$, so $\overline{R}$ is now computed in these coordinates, and $d \text{dvol} = \kappa d \theta_0 \wedge \ldots \wedge d \theta_4$. The integrand is again independent of the point in the fiber, so the integral over the fiber just detects the length of the fiber. By the construction of $\overline{g}$, the fiber in $\overline{M}_1$ has length $2\pi = \int_0^{2\pi} |\xi|$. For $\overline{M}_p$, $\overline{g}$ has the fiber term $p d \eta \otimes d \eta$, so $|\xi| = p^{1/2}$, and we only integrate from $0$ to $2\pi/p$. Thus the length of the fiber is $\int_0^{2\pi/p} p^{1/2} = 2\pi/p^{1/2}$. So

$$
\int_{\overline{M}_p} a^L \cdot C S_5^W = \left( \frac{2 \pi \cdot 3}{2} \right) \left( \frac{2 \pi \kappa}{p^{1/2}} \right) \int_{\overline{M}_p} \sum_{\sigma \in S_5} \text{sgn}(\sigma) \overline{R}_{\sigma_0 \ell_1} \overline{R}_{\sigma_1 \sigma_2 \ell_2} \overline{R}_{\sigma_3 \sigma_4 \ell_3} d \theta_1 \wedge \ldots \wedge d \theta_4.
$$
Thus the Proposition follows if

$$\int_M \sum_{\sigma \in S_5} \text{sgn}(\sigma) R_{\sigma_0 \ell_0} R_{\sigma_1 \sigma_2 \ell_2} R_{\sigma_3 \sigma_4 \ell_4} d\theta_1 \wedge \ldots \wedge d\theta_4$$

$$= \frac{p^2}{16} \int_0^1 (3072p^4 - 640p^2 \beta^{-2} - 25\beta^{-4}) d\theta_2. \quad (4.9)$$

The long calculation of (4.9) is in [6]. This result is verified by the computer calculations in a file at [5]. \[\square\] Since the top coefficient of $p$ is nonzero, we conclude from Thm. 3.4 that $\pi_1(\text{Diff}(\mathcal{M}_p))$ and $\pi_1(\text{Isom}(\mathcal{M}_p))$ are infinite for $|p| \gg 0$. We will improve this to all $p$ as follows:

**Theorem 4.2.** $\pi_1(\text{Diff}(\mathcal{M}_p))$ and $\pi_1(\text{Isom}(\mathcal{M}_p))$ are infinite for all $p$.

**Proof.** For $p = 0$, this follows as in the Introduction (cf. [16, Rmk. 3.2]). For $p \neq 0$, by (4.8), (4.9), it suffices to show that

$$\int_0^1 (3072p^4 - 640p^2 \beta^{-2} - 25\beta^{-4}) d\theta_2 \neq 0, \quad (4.10)$$

for $p \in \mathbb{Z}$. Either by a direct calculation or by Wolfram Alpha, we get (for $\theta = \theta_2$)

$$\int \beta^{-2} d\theta = \frac{2\theta - 1}{5(1 + \theta - \theta^2)} - \frac{2\ln(-2\theta + \sqrt{5} + 1)}{5\sqrt{5}} + \frac{2\ln(2\theta + \sqrt{5} - 1)}{5\sqrt{5}} + C,$$

$$\int \beta^{-4} d\theta = -\frac{1}{375} \left( \frac{1}{1 + \theta - \theta^2} \left( -60\theta^5 + 150\theta^4 + 50x^3 - 225x^2 - 75\theta + 80 \right) \ight.$$

$$+ 12\sqrt{5} \left( \ln(2\theta + \sqrt{5} - 1) - \ln(-2\theta + \sqrt{5} + 1) \right) + C.$$

The definite integrals are

$$\int_0^1 \beta^{-2} d\theta = \frac{2}{25} \left( 5 + 4\sqrt{5} \coth^{-1}(\sqrt{5}) \right),$$

$$\int_0^1 \beta^{-4} d\theta = \frac{16}{375} \left( 10 + 3\sqrt{5} \coth^{-1}(\sqrt{5}) \right).$$

Plugging this into (4.10), we must show that

$$10(-1 - 24p^2 + 288p^4) - 3\sqrt{5}(1 + 64p^2) \coth^{-1}(\sqrt{5}) \neq 0.$$ 

This quadratic equation in $p^2$ has solutions $p \approx \pm 0.159514i$, $\pm 0.424868$. Since there are no integral solutions, the theorem follows. \[\square\]

A second computer program verifying these calculations is in [5].
5. The higher dimensional Kähler case

In this section, we prove that the lowest order term in the WCS form has a geometric/topological interpretation on Kähler manifolds (Prop. 5.2); this appears to fail for general symplectic manifolds. We use this result to give non-vanishing results for Wodzicki-Pontryagin forms (Thm. 5.3). In contrast, the Wodzicki-Pontryagin classes vanish in $H^{4k}(LM)$; this non-vanishing of the representative forms gives the first known examples in arbitrarily high dimensions.

We start with some general remarks about the real homology/cohomology of loop spaces. For a manifold $N$, define the ring homomorphism $L : \Lambda^*(N) \rightarrow \Lambda^*(LN), \delta \mapsto \delta_L$ by

$$\delta_L(X_1, \ldots, X_k) = \delta(X_1(0), \ldots, X_k(0)).$$

Then $a^{L,*} \circ L = \text{Id}$. (5.1)

To see this, take $v \in T_pN$ and a curve $\gamma(s)$ tangent to $v$ at $p$. Then $a_L(v) = (d/ds)|_{s=0}a_L(\gamma(s))$, a vector field along the loop $a_L(\gamma(t))$. Since $a_L(\gamma(s)(0)) = \gamma(s), we get $a_L^*(\delta_L)(X_1, \ldots, X_k) = \delta([a_L^*(X_1)](0), \ldots, [a_L^*(X_r)](0))$.

Lemma 5.1. $L$ induces an injection $L^* : H^k(N, \mathbb{R}) \hookrightarrow H^k(LN, \mathbb{R})$ for all $k$.

Proof. We have

$$[(dL \circ L)\delta](X_1, \ldots, X_{k+1})$$

$$= \sum_i (-1)^{i-1} X_i(\delta_L(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}))$$

$$+ \sum_{i \leq j} (-1)^{i+j} \delta([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})$$

$$= \sum_i (-1)^{i-1} X_i(\delta(X_1(0), \ldots, \hat{X}_i(0), \ldots, X_{k+1}(0)))$$

$$+ \sum_{i \leq j} (-1)^{i+j} \delta([X_i, X_j](0), X_1(0), \ldots, \hat{X}_i(0), \ldots, \hat{X}_j(0), \ldots, X_{k+1}(0)),$$

$$[(L \circ dN)\delta](X_1, \ldots, X_{k+1}) = dN\delta(X_1(0), \ldots, X_{k+1}(0))$$

$$= \sum_i (-1)^{i-1} X_i(0)(\delta(X_1(0), \ldots, \hat{X}_i(0), \ldots, X_{k+1}(0)))$$

$$+ \sum_{i \leq j} (-1)^{i+j} \delta([X_i, X_j](0), X_1(0), \ldots, \hat{X}_i(0), \ldots, \hat{X}_j(0), \ldots, X_{k+1}(0)).$$
Let $\gamma_s(t)$ be a family of loops with tangent vector $X_i \in T_{\gamma_s}LN$. Extend the $X_j$ to vector fields near $\gamma = \gamma_0$. Then

$$X_i(\delta(X_1(0), \ldots, \dot{X}_i(0), \ldots, X_{k+1}(0)) = (d/ds|_{s=0})\delta_{\gamma(s)}(0)(X_1(0), \ldots, \dot{X}_i(0), \ldots, X_{k+1}(0))$$

$$= X_i(0)(\delta(X_1(0), \ldots, \dot{X}_i(0), \ldots, X_{k+1}(0))).$$

Similar computations imply $d_{LN} \circ L = L \circ d_N$, so $L : \Lambda^*(N) \to \Lambda^*(LN)$ induces $L^* : H^*(N) \to H^*(LN)$. Then $a^{L^*}L^* = Id$ implies $L^* : H^*(N) \to H^*(LN)$ is injective.

In contrast to this general cohomological result, our goal is to obtain information on the Wodizicki-Pontryagin forms from the characteristic cohomology ring of $TM$.

Let $(M, \omega)$ be an integral Kähler manifold of real dimension $4n$; equivalently, $M$ is projective algebraic. The Kähler form $\omega \in H_2(M, \mathbb{Z})$ determines the Riemannian metric $g(X, Y) = -\omega(JX, Y)$, where $J$ is the complex structure. The key feature of the Kähler case for us is that $\nabla J = 0$. Thus in Lemma 3.3 the terms (ii) and (iv) vanish.

By (2.3) and Lemma 3.3, the WCS forms $\tilde{C}S_{2k-1}^W$ on $LM_p$ and their pullbacks $a^{L^*}\tilde{C}S_{2k-1}^W$ to $M_p$ are polynomials in $p^2$:

$$\tilde{C}S_{2k-1}^W = \sum_{i=1}^{k} \tilde{C}S_{2k-1,2i}^W p^{2i} \in \Lambda^{2k-1}(LM_p),$$

$$a^{L^*}\tilde{C}S_{2k-1}^W = \sum_{i=1}^{k} a^{L^*}\tilde{C}S_{2k-1,2i}^W p^{2i} \in \Lambda^{2k-1}(M_p),$$

where the forms $\tilde{C}S_{2k-1,2i}^W$ are curvature expressions independent of $p$.

We can explicitly compute the relevant part of $\tilde{C}S_{2k-1,2}^W$ used in the computations below. Let $\Omega = \Omega_M$ be the curvature of the Kähler metric.

**Proposition 5.2.** Let $\pi : M_p \to M$ be the fibration. For $\xi$ the unit tangent vector to the fibers of $\pi$,

$$\varepsilon_\xi a^{L^*}\tilde{C}S_{4k+1,2}^W = (2k + 1)2 \cdot \pi^* \operatorname{tr}(\Omega_M^{2k}) = (-1)^k(4k + 2)(2\pi)^{2k}(2k)! \cdot \pi^*\tilde{p}_k(\Omega_M).$$

Thus the Pontryagin-type form $\tilde{p}_k(\Omega_M)$ is related to $\tilde{C}S_{4k+1,2}^W$.

**Proof.** Let $\xi = \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{4n+1}$ be an orthonormal frame of $M_p$ at $\tilde{m}$ with $\xi$ tangent to the fiber of the $S^1$ action and $\{\varepsilon_i\}_{i=2}^{4n+1}$ a horizontal lift of an orthonormal frame $\{e_i\}$ at $\pi(\tilde{m})$. We must show that $a^{L^*}\tilde{C}S_{4k+1,2}^W(\xi, \varepsilon_2, \ldots, \varepsilon_{4k+1})$ at $\tilde{m}$ is a specific multiple of $\operatorname{tr}(\Omega_M^{2k})(\pi_*e_2, \ldots, \pi_*e_{4n+1})$ at $\pi(\tilde{m})$. 


We denote e.g. \( \overline{R}(\bar{\epsilon}_{\sigma(2)}, \bar{\epsilon}_{\sigma(3)}, \bar{\epsilon}_{\ell_2}, \bar{\epsilon}_{\ell_1}) \) by \( \overline{R}(\sigma(2), \sigma(3), \ell_2, \ell_1) \), and denote \( \bar{\epsilon}_r \) by \( r \).

As in (2.3), we have
\[
a^{L+CSW}_{4k+1}(g)(\tau_1, \ldots, \tau_{4k+1}) = \frac{2k+1}{2^{2k-1}} \sum_{\sigma} \sgn(\sigma) \int_{S^1} \text{tr}[(R(e_{\sigma(1)}, \cdots, \bar{\epsilon}) (\Omega^M)^2 k (e_{\sigma(2)}, \cdots, e_{\sigma(4k+1)})]
\]
\[
= \frac{2k+1}{2^{2k-1}} \sum_{\sigma} \sgn(\sigma) \int_{S^1} \overline{R}(\sigma(1), \ell_1, \bar{\xi}, \ell_2) \overline{R}(\sigma(2), \sigma(3), \ell_2, \ell_1) \overline{R}(\sigma(4), \sigma(5), \ell_3, \ell_2) \ldots
\]
\[
\cdot \overline{R}(\sigma(4k-2), \sigma(4k-1), \ell_{4k}, \ell_{4k-1}) \overline{R}(4k, \sigma(4k+1), r, \ell_{4k}).
\]

We want to compute the terms in (5.3) of order \( p^2 \). These terms come from (\( \alpha \)) permutations with \( \sigma(1) = 1 \), and (\( \beta \)) permutations with \( \sigma(1) \neq 1 \).

We claim the (\( \beta \)) terms contribute zero (for all powers of \( p^2 \)). The term \( \overline{R}(\sigma(1), \ell_1, \bar{\xi}, n) \) with \( \sigma(1) \neq 1 \) is zero unless \( \ell_1 = 1 \) and \( \sigma(1) = n \). Note that \( \overline{R}(n, 1, \bar{\xi}, n) = -1 \). Thus
\[
(\beta) = \frac{2k+1}{2^{2k-1}} \cdot p^2
\]
\[
\cdot \int_{S^1} \sum_{\sigma(1)=n} \sum_{\ell_1=1} -\sgn(\sigma) \overline{R}(\sigma(2), \sigma(3), \ell_2, 1 = \bar{\xi}) \cdot \ldots \overline{R}(4k, \sigma(4k+1), r = \sigma(1), \ell_{4k})
\]
In the term \( \overline{R}(\sigma(2), \sigma(3), \ell_2, 1 = \bar{\xi}) \), we get zero unless either \( \sigma(2) = 1 \) and \( \ell_2 = \sigma(3) \) or \( \sigma(3) = 1 \) and \( \ell_2 = \sigma(2) \). Therefore
\[
(\beta) = \frac{2k+1}{2^{2k-1}} \cdot p^2
\]
\[
\cdot \int_{S^1} \sum_{\sigma(1)=r} \sum_{\ell_1=1} \sum_{\ell_2=1} \sgn(\sigma) \overline{R}(\sigma(4), \sigma(5), \ell_3, \ell_2) \ldots \overline{R}(4k, \sigma(4k+1), r = \sigma(1), \ell_{4k})
\]
\[
- \frac{2k+1}{2^{2k-1}} \cdot p^2
\]
\[
\cdot \int_{S^1} \sum_{\sigma(1)=r} \sum_{\ell_1=1} \sum_{\ell_2=1} \sgn(\sigma) \overline{R}(\sigma(4), \sigma(5), \ell_3, \ell_2) \ldots \overline{R}(4k, \sigma(4k+1), r = \sigma(1), \ell_{4k}).
\]

For fixed \( \ell_2 \), there is a bijection between \( \{ \sigma : \sigma(2) = 1, \sigma(3) = \ell_2 \} \) and \( \{ \tau : \tau(3) = 1, \tau(2) = \ell_2 \} \) given by \( \sigma \mapsto \sigma(1) \ell_2 \) in cycle notation. Since \( \sgn(\sigma) = -\sgn(\sigma(1) \ell_2) \), we get
\[
(\beta) = \frac{2k+1}{2^{2k-1}} \cdot p^2
\]
\[
\cdot \int_{S^1} \sum_{\sigma(1)=r} \sum_{\ell_1=1} \sum_{\ell_2=1} \sgn(\sigma) \overline{R}(\sigma(4), \sigma(5), \ell_3, \ell_2) \ldots \overline{R}(4k, \sigma(4k+1), r = \sigma(1), \ell_{4k}).
\]
The last term in (5.7) is
\[ \overline{R}(\sigma(4k), \sigma(4k + 1), r = \sigma(1), \ell_{4k}) = \overline{R}(\sigma(4k), \sigma(4k + 1), \xi, \ell_{4k}). \] (5.8)
This term vanishes if \( \ell_{4k} = \xi \). If \( \ell_{4k} \neq \xi \), then since \( \sigma(2) = 1 \), we have \( \sigma(4k) \neq \xi, \sigma(4k + 1) \neq \xi \). Thus (5.8) vanishes in all cases. Therefore (\( \beta \)) = 0.

The (\( \alpha \)) term is
\[
(\alpha) = \frac{2k + 1}{2^{2k-1}} \cdot p^2 \cdot \int_{S^1} \sum_{\sigma(1) = 1}^{\ell_1 = r \neq 1} \text{sgn}(\sigma) \overline{R}(\sigma(2), \sigma(3), \ell_2, \ell_1 = n) \cdots \overline{R}(\sigma(4k), \sigma(4k + 1), \ell_1 = r, \ell_{4k})
\] (5.9)
In Lemma 3.3 a nonzero product of terms of types (i) and (iii) with one term having \( \xi \) and having a power \( p^2 \) must include exactly one term from (iii) and only the first term on the right hand side of (i). Therefore
\[
(\alpha) = \frac{2k + 1}{2^{2k-1}} \cdot p^2 = \cdot \int_{S^1} \sum_{\sigma(1) = 1}^{\ell_1 = r \neq 1} \text{sgn}(\sigma) R(\sigma(2), \sigma(3), \ell_2, \ell_1 = r) \cdots R(\sigma(4k), \sigma(4k + 1), \ell_1 = r, \ell_{4k})
\] (5.10)
\[
= \frac{2k + 1}{2^{2k-1}} \cdot p^2 \cdot \sum_{\sigma(1) = 1}^{\ell_1 = r \neq 1} \text{sgn}(\sigma) R(\sigma(2), \sigma(3), \ell_2, \ell_1 = r) \cdots R(\sigma(4k), \sigma(4k + 1), \ell_1 = r, \ell_{4k})
\] (5.11)
\[
= \frac{2k + 1}{2^{2k-1}} \cdot p^2 \cdot 2^{2k} \cdot \text{Tr}(\Omega^{2k})(e_2, \ldots, e_{4k+1}),
\]
\[
= (-1)^k (4k + 2)(2\pi)^2(2k)! \cdot p^2 \cdot \tilde{p}_k(\Omega)(e_2, \ldots, e_{4k+1})
\]
\[
= (-1)^k (4k + 2)(2\pi)^2(2k)! \cdot p^2 \cdot \pi^* \tilde{p}_k(\Omega)(\overline{e}_2, \ldots, \overline{e}_{4k+1}).
\]

While the Wodzicki-Pontryagin classes \( \tilde{p}_k^W(\mathbb{L}S^{4n+1}) \) vanish, we now prove that the representative Wodzicki-Pontryagin forms, also denoted \( \tilde{p}_k^W(\mathbb{L}S^{4n+1}) \), are nonzero for \( k \leq n \). This answers a conjecture in \[11\] in the negative.

**Theorem 5.3.** For \( k \leq n \), the Wodzicki-Pontryagin forms \( \tilde{p}_k^W(\mathbb{L}S^{4n+1}) \) are not identically zero.

**Proof.** The integral Pontryagin classes \( \tilde{p}_k^{\text{top}} \) in the torsion-free ring \( H^*(\mathbb{C}P^{2n}, \mathbb{Z}) \) are nonzero, so there is an integral cycle \( z = \sum_i n_i \sigma_i, \sigma_i : \Delta^{4k} \rightarrow \mathbb{C}P^{2n}, [z] \in H_{4k}(\mathbb{C}P^{2n}, \mathbb{Z}) \), with \( \langle \tilde{p}_k^{\text{top}}, [z] \rangle \neq 0 \). By results of Thom, some nonzero integer multiple \( s[z] \) of the homology class of \( z \) is represented by a smooth, closed 4k-submanifold \( Z \). For simplicity of notation, we assume \( s = 1 \). We then have \( \int_Z \tilde{p}_k(\Omega) \neq 0 \).
Set $M = \mathbb{CP}^{2n}$. For the projection $\pi : \overline{M_1} = S^{4n+1} \longrightarrow \mathbb{CP}^{2n}$, $\pi^{-1}(Z)$ is a closed $(4k+1)$-submanifold on $S^{4n+1}$ and determines the homology class $[\pi^{-1}(Z)] \in H_{4k+1}^{+1}(S^{4n+1}, \mathbb{R})$. Of course, this class is zero for $k \neq n$. Also, $Z' := a^k(\pi^{-1}(Z))$ is a closed $(4k+1)$-submanifold of $LS^{4n+1}$.

By Proposition 5.2 for an explicit constant $c_k$,

$$a^{L^*} \widetilde{CS}_{4k+1,2}^W = c_k \xi^* \wedge \pi^* \tilde{p}_k + \beta_2,$$

where $\beta_2$ has no $\xi^*$ term. Then

$$\int_{Z'} \widetilde{CS}_{4k+1,2}^W = \int_{\pi^{-1}(Z)} a^{L^*} \widetilde{CS}_{4k+1,2}^W = \int_{\pi^{-1}(Z)} c_k \xi^* \wedge \pi^* \tilde{p}_k,$$

since the integral vanishes on terms not containing $\xi^*$. Integrating over the fiber gives

$$\int_{Z'} \widetilde{CS}_{4k+1,2}^W = c \cdot c_k \int_Z \tilde{p}_k \neq 0,$$

where $c$ is the length of the fiber of $\pi$.

If $d\widetilde{CS}_{4k+1,2}^W \equiv 0$, then $[\widetilde{CS}_{4k+1,2}^W] \in H_{4k+1}^{4n+1}(LS^{4n+1}, \mathbb{R})$ defines a nonzero cohomology class. However, by e.g., [3], $LS^{4n+1}$ has no cohomology in the range $1, \ldots, 4n$. Thus

$$0 \neq d\widetilde{CS}_{4k+1,2}^W \quad (5.12)$$
on $\overline{M_1}$.

We want to prove that (5.12) implies $\tilde{p}_k^W \neq 0$. We want to use (5.2), but because these forms live on different $\overline{M}_p$, we have to pull them back to the common space $\overline{M_1} = LS^{4n+1}$. The $p : 1$ covering map $\pi_p : S^{4n+1} \longrightarrow \overline{M}_p$ induces the map $L\pi_p : LS^{4n+1} \longrightarrow L\overline{M}_p$, $L\pi(\gamma)(\theta) = (\pi_p \circ \gamma)(\theta)$. By (5.2), as forms on $LS^{4n+1}$, we have

$$\begin{align*}
(L\pi_p)^* \tilde{p}_k^W(\overline{M}_p) &= (L\pi_p)^* d\widetilde{CS}_{4k+1,1}(\overline{M}_p) = (L\pi_p)^* d \sum_{i=1}^{2k} \widetilde{CS}_{4k+1,2i}^W \tilde{p}_{2i}^2 \\
&= \sum_{i=1}^{2k} d(L\pi_p)^* \widetilde{CS}_{4k+1,2i}^W \tilde{p}_{2i}^2.
\end{align*} \quad (5.13)$$

$\pi_p$ is a covering map, so for $X \in T_\gamma LS^{4n+1}$, $[(L\pi_p)_* X(\theta)] \in T_{\pi(\gamma)(\theta)} \overline{M}_p$ is canonically identified with $X(\theta) \in T_{\gamma(\theta)} S^{4n+1}$. $\widetilde{CS}_{4k+1,1}(\overline{M}_p)$ is the integral of expressions in Lemma 3.3 which are invariant in the circle fibers of $S^{4n+1} \longrightarrow \mathbb{CP}^{2n}$, and $\pi_p$ preserves these fibers, so for $X_i \in T_\gamma LS^{4n+1}$,

$$(L\pi_p)^* \widetilde{CS}_{4k+1,2i}^W(X_1, \ldots, X_{4k+1}) = \widetilde{CS}_{4k+1,2i}^W((L\pi_p)_*(X_1), \ldots, (L\pi_p)_*(X_{4k+1}))$$
is independent of $p$. In particular,

$$(L\pi_p)^* \widetilde{CS}_{4k+1,2i}^W = (L\pi_1)^* \widetilde{CS}_{4k+1,2i}^W = \widetilde{CS}_{4k+1,2i}^W \in \Lambda^{4k+1}(LS^{4n+1}).$$
Similarly, \((L\pi_p)^*\overline{p}_k^W(M_\overline{p}) = \overline{p}_k^W(M_1) = \overline{p}_k^W(LS^{4n+1})\).

Thus (5.13) becomes

\[
\overline{p}_k^W = \sum_{i=1}^{2k} d\overline{C}_S^{W,4k+1,2i} \overline{p}^2i \in \Lambda^{4k+1}(LS^{4n+1})
\]  

(5.14)

for all \(p\). If we assume \(\overline{p}_k^W = 0\), then (5.14) implies \(d\overline{C}_S^{W,4k+1,2i} = 0\) for all \(i\). Indeed, if a polynomial vanishes for all values of \(p\), then all the coefficients vanish. For \(i = 1\), this contradicts (5.12). \(\square\)

**Corollary 5.4.** On \(\overline{M}_p = S^{4k+1}/\mathbb{Z}_p\), for \(k \leq n\), the Wodzicki-Pontryagin forms \(\overline{p}_k^W(L\overline{M}_p)\) are not identically zero.

**Proof.** Because \(\overline{p}_k^W(LS^{4n+1})\) is computed by integrating a curvature expression on \(S^{4n+1}\), if \(\overline{p}_k^W \neq 0\) at a loop \(\gamma \in LS^{4n+1}\), then \(\overline{p}_k^W(L\overline{M}_p) \neq 0\) at the loop \((L\pi_p)(\gamma) \in L\overline{M}_p\). \(\square\)

**Appendix A. Manifolds of dimension 4n + 2**

We show that the coefficient of the highest power \(p^{4n+4}\) of \(\overline{C}_S^{W,4n+3}(M_p)\) vanishes if \(\dim M = 4n + 2\). We note that \(\overline{C}_S^{W,3} = 0\) for any 3-manifold by [16] Prop. 2.7.

**Proposition A.1.** For \(\dim M = 4n + 2\), \(\overline{C}_S^{W,4n+3}(M_p) = 0 \mod p^{4n+3}\).

We only do the case where \(\dim M = 6\) to keep the notation manageable.

**Proof.** In the notation of (3.4), we have

\[
S_{7,8} = \sum_{\sigma_0 = 0}^{\text{sgn}(\sigma) R_{0\ell_0} c_0 R_{\sigma_1 \sigma_2 \ell_2} c_1 R_{\sigma_3 \sigma_4 \ell_3} c_2 R_{\sigma_5 \sigma_6 \ell_3} c_3}
\]

\[
= \sum_{\sigma_0 = 0}^{\text{sgn}(\sigma)(p^2 \delta^b a_1)} \cdot p^2 \left[ -J_{a_1}^a J_{a_2}^a + J_{a_1 a_2} J_{a_2}^a + 2 J_{a_1 a_2} J_{a_2}^a \right]
\]

\[
\cdot p^2 \left[ -J_{a_3}^a J_{a_4 a_3}^a + J_{a_3 a_4} J_{a_4}^a + 2 J_{a_3 a_4} J_{a_4}^a \right] \cdot p^2 \left[ -J_{a_5}^a J_{a_6 b}^a + J_{a_5 b} J_{a_6}^a + 2 J_{a_5 a_6} J_{a_6}^a \right]
\]

\[
= p^8 \sum_{\sigma_0} \text{sgn}(\sigma) \left[ -J_{a_1}^a J_{a_2}^a + J_{a_1 a_2} J_{a_2}^a + 2 J_{a_1 a_2} J_{a_2}^a \right]
\]

\[
\cdot \left[ -J_{a_3}^a J_{a_4 a_3}^a + J_{a_3 a_4} J_{a_4}^a + 2 J_{a_3 a_4} J_{a_4}^a \right] \cdot \left[ -J_{a_5}^a J_{a_6 a_1}^a + J_{a_5 a_6} J_{a_6}^a + 2 J_{a_5 a_6} J_{a_6}^a \right].
\]

Because \(J_{a b} J_{a c} = J_{b c} J_{a d} J_{a b} = -\delta_{b}^d J_{a d} = -g_{a d}\), the product of the first two expressions in square brackets simplifies to

\[
S_{7,8} = p^8 \sum_{\sigma_0 = 0} \text{sgn}(\sigma) \left[ -J_{a_1}^a J_{a_2}^a J_{a_3}^a g_{a_2 a_3} + 2 g_{a_1 a_3} J_{a_2}^a J_{a_3}^a J_{a_4}^a - 2 J_{a_1 a_2} \left( -\delta_{a_1}^a \right) J_{a_4 a_3} J_{a_4}^a + 2 J_{a_1 a_2} \left( -\delta_{a_1}^a \right) J_{a_4 a_3} J_{a_4}^a \right]
\]

\[
+ J_{a_4 a_3} \left( \delta_{a_1}^a \right) J_{a_1 a_2} + 4 J_{a_1 a_2} J_{a_4}^a J_{a_3}^a \left( -\delta_{a_1}^a \right) J_{a_4 a_3} J_{a_4}^a \right]
\]

\[
- J_{a_1 a_2} J_{a_3}^a + J_{a_5 a_6} J_{a_6}^a + 2 J_{a_5 a_6} J_{a_6}^a J_{a_4}^a.
\]
Taking the product of the terms inside the first square brackets with the terms inside the second square brackets, we get 15 terms, all of which simplify. For example, the product of the first terms in each square brackets gives

\[2J^a_1J^a_3g_{a_a}g_{a_3}J^a_5J^a_{a_1} = -2g_{a_a}J^a_3J^a_5J^a_{a_1}.\]

For a term with a Kronecker delta, we have

\[-2J_{a_1}J_{a_2}(\delta^a_{a_3})J_{a_4}g_{a_a}g_{a_3}J^a_5J^a_{a_1} = -2J_{a_1}J_{a_2}J_{a_3}J^a_5J^a_{a_1} = -2J_{a_1}J_{a_2}J^a_5g_{a_3}J^a_{a_1} = -2J_{a_1}J_{a_2}g_{a_3}J_{a_a}J^a_5J^a_{a_1}.\]

Similarly, every product is of the form \(g_{a_1}J_{a_k}J_{a_3}J_{a_m}J_{a_n}\), except for the product of the two last terms, which is

\[8J_{a_1}J_{a_2}J_{a_3}(-\delta^a_{a_3})J_{a_5}J^a_{a_1} = 0,\]

since \(\delta^a_{a_3}J^a_{a_1} = J^a_{a_1} = 0\).

In summary, every nonzero term in \(S_{7,8}\) is of the form \(\text{sgn}(\sigma)g_{a_a}J_{a_1}J_{a_2}J_{a_3}J_{a_4}\), where \(\tau = (i, j, k, l, m, n) \in \mathcal{S}_6\). Under the change of variables \(\tau \mapsto \tau(12)\), the sign of \(\tau\) changes sign, but the term \(g_{a_a}J_{a_1}J_{a_2}J_{a_3}J_{a_4}\) does not change sign. Thus the terms corresponding to \(\tau\) and \(\tau(12)\) in \(S_{7,8}\) cancel, so \(S_{7,8} = 0\).

**Remark A.1.** In this proof, it was crucial that \(S_{7,8}\) contains an odd number of terms in \((\Delta.1)\). This is where the hypothesis \(M = 4n + 2\) is used.

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