Parabolic induction and perverse sheaves on $W \backslash \mathfrak{h}$

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Alle Gestalten sind ähnlich und keine gleichet der andern; 
Und so deutet das Chor auf ein geheimes Gesetz...

Goethe, *Die Metamorphose der Pflanzen*

Abstract

For a complex reductive Lie group $G$ with Lie algebra $\mathfrak{g}$, Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and Weyl group $W$, we describe the category of perverse sheaves on $W \backslash \mathfrak{h}$ smooth w.r.t the natural stratification. The answer is given in terms of mixed Bruhat sheaves, which are certain mixed sheaf-cosheaf data on cells of a natural cell decomposition of $W \backslash \mathfrak{h}$. Using the parabolic Bruhat decomposition, we relate mixed Bruhat sheaves with properties of various procedures of parabolic induction and restriction that connect different Levi subgroups in $G$.

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0 Introduction

A. For a complex reductive Lie algebra $\mathfrak{g}$, the quotient $W\backslash \mathfrak{h}$ of the Cartan subalgebra by the Weyl group is isomorphic to an affine space and carries a natural stratification $S^{(0)}$. For example, for $\mathfrak{g} = \mathfrak{gl}_n$ we get the space of monic polynomials $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ and $S^{(0)}$ is given by the singularities of the discriminantal hypersurface $\Delta(f) = 0$.

The first goal of this paper is to give an elementary (quiver-type) description of $\text{Perv}(W\backslash \mathfrak{h})$, the category of perverse sheaves on $W\backslash \mathfrak{h}$ smooth with respect to $S^{(0)}$. Our main result, Theorem 2.6, identifies $\text{Perv}(W\backslash \mathfrak{h})$, with the category of objects of mixed functoriality related to a natural cell decomposition $\{U_m\}_{m \in \Xi}$ of $W\backslash \mathfrak{h}$ refining $S^{(0)}$, objects which we call mixed Bruhat sheaves. A mixed Bruhat sheaf $E$ consists of vector spaces $E(m)$, one for each cell $U_m$ and behaves like a cellular sheaf with respect to one part of cell inclusions $U_m \supset U_n$ and like a cellular cosheaf with respect to another part, see Definition 2.1.

This mixed nature harmonizes well with the self-dual, intermediate position of perverse sheaves themselves, half-way between the abelian categories of sheaves and of cosheaves (understood as objects of the derived category which are Verdier dual to sheaves). At the more geometric level, this corresponds to the position of intersection homology as half-way between cohomology and homology.

The indexing set $\Xi$ for our cell decomposition (and for mixed Bruhat sheaves) is the 2-sided Coxeter complex of Petersen [41]. For $\mathfrak{g} = \mathfrak{gl}_n$ this is the set of contingency matrices of content $n$ known in statistics [17, 35].

B. Our second and wider goal is to relate $\text{Perv}(W\backslash \mathfrak{h})$ to a classical subject of representation theory which well predates perverse sheaves: the “algebra of parabolic induction”. By this we mean the entire package of results related to principal series (parabolically induced) representations of reductive groups (in all contexts: finite field, real, $p$-adic, adelic, automorphic), their intertwiners, Eisenstein series, constant terms of automorphic forms and other procedures which pass from one Levi subgroup to another.

The rules of this “algebra” are familiar to all practitioners of representation theory (it underlies the philosophy of cusp forms of Gelfand and Harish-Chandra), but it was somehow considered $sui generis$, its interpretation in terms of something else being not clear or not looked for. For groups $GL_n$, one can interpret parts of the structure in terms of braided Hopf algebras (via the concept of the Hall algebra) [31, 37] and braided monoidal categories [30, 42]. For a more general reductive group $G$ with Lie algebra $\mathfrak{g}$, this is not possible.
Our observation is that perverse sheaves (and their categorical analogs, perverse schobers [32]) on $W \backslash \mathfrak{h}$ provide a conceptual encoding of this peculiar algebra, giving it a name, so to say. We illustrate this on two simplest examples in §7 and sketch some other examples in §8. A germ of this connection can be seen in the fact that the principal series intertwiners form a representation of the braid group $\text{Br}_g = \pi_1(W \backslash \mathfrak{h}^\text{reg})$ and so give a local system on the generic stratum $W \backslash \mathfrak{h}^\text{reg} \subset W \backslash \mathfrak{h}$. The examples we consider indicate that the correspondence between the two theories should hold in many different contexts.

Using $GL_n$ as a point of departure, our approach can be seen as importing, into the general theory of representations and automorphic forms, the new 2-dimensional point of view on Hopf algebras coming from their relation to $E_2$-algebras (J. Lurie).

C. The reason for the relation between $\text{Perv}(W \backslash \mathfrak{h})$ and parabolic induction comes from the elementary but remarkable matching between elements $\mathbf{m} \in \Xi$, i.e., cells of $U_\mathbf{m} \subset W \backslash \mathfrak{h}$ and Bruhat orbits, i.e., $G$-orbits $O_\mathbf{m} \subset F_I \times F_J$ in the pairwise products of all possible flag varieties for $G$. This matching is just the parabolic Bruhat decomposition; the remarkable fact is that some topological relations among the cells $U_\mathbf{m}$ have, as their counterparts, algebroid-geometric relations among the orbits $O_\mathbf{m}$. For example, the property for a cell inclusion $U_\mathbf{m} \supset U_\mathbf{n}$ to be anodyne (i.e., such that both cells lie in the same stratum of $\mathcal{S}^{(0)}$) corresponds to the property that the projection of orbits $p_{\mathbf{m},\mathbf{n}} : O_\mathbf{m} \to O_\mathbf{n}$ has fibers isomorphic to an affine space and so, for example, gives an isomorphism in the category of Voevodsky motives [5].

The data appearing in the theory of parabolic induction, are usually labelled by the standard Levis, i.e., by subsets $I \subset \Delta_\text{sim}$ of simple roots. This gives a bicube, i.e., a diagram of $2^{|\Delta_\text{sim}|}$ vector spaces or categories related by maps or functors back and forth for $I \subset J$ (e.g., induction/restriction, Eisenstein series/constant term), see §2C. In §7-8 we extend some of these bicube diagrams to depend on arbitrary $\mathbf{m} \in \Xi$, i.e., on an arbitrary Bruhat orbit $O_\mathbf{m}$. Informally, such a larger diagram has the parabolic intertwiners already “pre-installed”, since among the orbits we find the correspondences used to define the intertwiners.

D. The organization of the paper is as follows. In Section 1 we recall the 2-sided Coxeter complex $\Xi$ and introduce the cell decomposition of $W \backslash \mathfrak{h}$ into cells $U_\mathbf{m}$ labelled by $\Xi$. The definition (and thus the whole approach of the paper) involves separating the real and imaginary parts of a point of $\mathfrak{h} = \mathfrak{h}_R \oplus i\mathfrak{h}_R$. Thus, for $\mathfrak{g} = \mathfrak{gl}_n$, a given cell in $W \backslash \mathfrak{h} = \text{Sym}^n(\mathbb{C})$ consists of polynomials whose zeroes follow a given pattern of coincidences among their real and imaginary parts, given by a contingency matrix.

In Section 2 we define mixed Bruhat sheaves and formulate the main result, Theorem 2.6. We also explain the relation of mixed Bruhat sheaves with bicubes and work out the examples of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$.

Sections 3 and 4 are devoted to the proof of Theorem 2.6. The proof is based on the techniques of Cousin complexes, used in different forms in [33, 34, 36]. They are certain explicit complexes of sheaves whose terms are constructible with respect to an intermediate
real stratification (in fact, also a cell decomposition) $S^{(1)}$ of $W \backslash h$ whose strata we call Fox-Neuwirth-Fuchs cells. As in the classical cell decompositions of configuration spaces [22, 23], the definition of these cells involves making a preference of the real parts over the imaginary parts. At the same time, these complexes represent (i.e., are isomorphic to) perverse sheaves from $\text{Perv}(W \backslash h)$, so their cohomology sheaves are $S^{(0)}$-constructible. Our proof of this cohomological $S^{(0)}$-constructibility is based on the remarkable property

$$S^{(1)} \vee \tau S^{(1)} = S^{(0)}.$$ 

Here $\tau S^{(1)}$ is a stratification similar to $S^{(1)}$ but with the roles of the real and imaginary parts interchanged, and the statement means that $S^{(0)}$ is the smallest stratification of which both $S^{(1)}$ and $\tau S^{(1)}$ are refinements.

In Section 5 and 6, we discuss the geometry of the Bruhat orbits $O_m$ and its relation to the properties of the corresponding labels $m$ which can themselves be viewed as $W$-orbits in products of two quotients of $W$. In particular, we establish the "$\mathbb{A}^1$-equivalence" property (Proposition 6.1) of the orbit projection corresponding to an anodyne inclusion $m \geq n$.

In Section 7 we consider the simplest example of a motivic Bruhat sheaf coming from Bruhat orbits: the collection of appropriate spaces of functions on $\mathbb{F}_q$-points. We also work out an even easier "$\mathbb{F}_1$-version", when we consider functions on the sets $(W$-orbits) $m$ themselves. The resulting perverse sheaves are then identified in terms of representations of the Hecke algebras and symmetric groups.

The concluding Section 8 sketches some further constructions in the same spirit which we plan to develop in subsequent papers. In particular, we discuss a natural categorical generalization of mixed Bruhat sheaves.

Finally, the Appendix collects notations and conventions related to constructible sheaves and stratifications that are used in the main body of the paper.

E. It would be interesting to understand the relation between $\text{Perv}(W \backslash h)$ and parabolic induction in a more direct, intrinsic way.

Geometrically, parabolic subgroups in $G$ corresponds to various ways of approaching the infinity either in $G$ itself, or in the arithmetic quotients $G/\Gamma$. So one can think of realizing the cell decomposition $\{U_m\}$ of $W \backslash h$ as some combinatorial complex describing regions at infinity in $G$ or in a related space. The closest picture of this kind that we know, involves the wonderful compatification $\overline{G} \supset G$, see [16, 6]. This is a smooth projective $G \times G$-variety with $G \times G$-orbits $X_I$ labelled by $I \subset \Delta_\text{sim}$. For $G$ of adjoint type, $X_I$ fibers over $F_I \times F_I$ with fiber being the adjoint quotient of the corresponding Levi. So considering the action of the diagonal $G \subset G \times G$ on $\overline{G}$ (which extends the action of $G$ on itself by conjugation) does lead to the appearance of Bruhat orbits but only in the $F_I \times F_I$ instead of arbitrary $F_I \times F_J$.

In a somewhat different direction, it seems interesting to understand the relation of the cell decomposition $\{U_m\}$ with the characteristic map

$$\chi : g \longrightarrow W \backslash h.$$
In particular, the topology of the regions \( \chi^{-1}(U_m) \subset \mathfrak{g} \), for example (in the case of a semisimple \( \mathfrak{g} \)), the way they approach the nilpotent cone \( \mathcal{N} = \chi^{-1}(0) \), seems worth studying.

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1 The 2-sided Coxeter complex as a cell decomposition of \( W \setminus h \).

A. Notation. We consider the classical situation, denoting:

\( \mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h} \): a split reductive Lie algebra over \( \mathbb{C} \), its chosen Borel and Cartan subalgebras. It is standard that these data are in fact defined over \( \mathbb{Z} \). In particular, we have the real vector space \( \mathfrak{h}_R \), the real part of \( \mathfrak{h} \).

\( \mathfrak{h}_R^\mathbb{R} \supset \Delta \supset \Delta^+ \supset \Delta_{\text{sim}} \): the space of real weights, with the subsets of all roots (weights of \( \mathfrak{g} \)), positive roots (weights of \( \mathfrak{b} \)) and simple roots.

\( W \): the Weyl group acting on \( \mathfrak{h} \). For \( \alpha \in \Delta \) we denote by \( s_\alpha \in W \) the corresponding reflection.

\( \mathcal{H} = \{ h^\alpha = (\alpha^\perp)_R, \alpha \in \Delta^+ \} \): the arrangement of root hyperplanes in \( \mathfrak{h}_R \).

\( \mathcal{H}_C = \{ h^\alpha = \alpha^\perp \} \): the complexified arrangement in \( \mathfrak{h} \).

\( C \): the Coxeter complex, i.e., the decomposition of \( \mathfrak{h}_R \) into faces of \( \mathcal{H} \), see [33] §2A. We think of \( C \) as both a geometric decomposition of \( \mathfrak{h}_R \) and as a poset \((C, \leq)\) of faces ordered by inclusion of closures.

\( C^+ \in C \): the dominant Weyl chamber given by the conditions \( \alpha > 0 \) for all \( \alpha \in \Delta^+ \). It is an open simplicial cone with faces \( C_I^+ \), \( I \subset \Delta_{\text{sim}} \) given by the conditions \( \alpha = 0 \) for \( \alpha \in I \) and \( \alpha > 0 \) for \( \alpha \in \Delta_{\text{sim}} \setminus I \). By \( \overline{C_I^+} \) we denote the closure of \( C_I^+ \), i.e., the union of all the \( C_I^+ \).

The \( W \)-action on \( C \) induces an identification

\[
\bigsqcup_{I \in \Delta_{\text{sim}}} W/W_I \rightarrow C, \quad wW_I \mapsto w(C_I^+).
\]

We will use notations \( c, d \) etc. for cosets \( wW_I \) and \( A_c, A_d \) etc for the corresponding faces, i.e., elements of \( C \).

\( p : \mathfrak{h} \rightarrow W \setminus \mathfrak{h} \): the canonical projection of \( \mathfrak{h} \) to its quotient by \( W \). By Chevalley’s theorem, \( W \setminus \mathfrak{h} \) is an algebraic variety isomorphic to an affine space. Like \( \mathfrak{h} \) itself, \( W \setminus \mathfrak{h} \) is in fact defined over \( \mathbb{Z} \).

\( K = W \setminus \mathfrak{h}_R \). This is a closed curvilinear cone in the real affine space \( (W \setminus \mathfrak{h})(\mathbb{R}) \), so that \( p : \overline{C^+} \rightarrow K \) is a homeomorphism. We denote \( K_I = p(C_I^+) \subset K, I \subset \Delta_{\text{sim}} \), the faces of \( K \).
$S_h^{(0)}$: the stratification of $H$ into generic parts of the flats of $H$, see [33] §2D. This stratification is $W$-invariant and so induces a complex stratification of $W\setminus h$ which we denote $S^{(0)}$. Thus each stratum of $S^{(0)}$ is the image of the generic part of a flat of $H$.

B. The two-sided Coxeter complex. We further denote:

$\Xi = W\setminus(C \times C)$: the two-sided Coxeter complex of Petersen [41]. We consider it as a poset, with the order on $\Xi$ induced by the product order on $C \times C$. Thus

$$\Xi = \bigsqcup_{I,J \in \Delta_{\text{sim}}} \Xi(I, J), \quad \Xi(I, J) := W\setminus((W/W_I) \times W/W_J)).$$

The sets $\Xi(I, J)$ are connected by the horizontal and vertical contraction maps

$$\varphi^\prime_{(I_1,J_1)\rightarrow (I_2,J_2)}: \Xi(I_1, J_1) \rightarrow \Xi(I_2, J_2), \quad I_1 \subset I_2, \quad J_1 \subset J_2.$$

They are induced by the natural projections $W/W_I \rightarrow W/W_J$ and $W/W_J \rightarrow W/W_J$ respectively. The two types of maps commute with each other: for $I_1 \subset I_2$ and $J_1 \subset J_2$ the diagram below commutes:

$$\Xi(I_1, J_1) \xrightarrow{\varphi^\prime_{(I_1,J_1)\rightarrow (I_2,J_2)}} \Xi(I_2, J_1)$$

$$\Xi(I_1, J_2) \xrightarrow{\varphi^\prime_{(I_1,J_2)\rightarrow (I_2,J_2)}} \Xi(I_2, J_2).$$

Indeed, this diagram is obtained, by taking quotients by $W$, from the $W$-equivariant commutative square

$$\begin{array}{ccc}
(W/W_I) \times (W/W_J) & \xrightarrow{\pi^\prime} & (W/W_I) \times (W/W_J) \\
\pi^\prime & & \pi^\prime \\
(W/W_I) \times (W/W_J) & \xrightarrow{\pi^\prime} & (W/W_I) \times (W/W_J).
\end{array}$$

A typical element of $\Xi(I, J)$ will be denoted $m = W(c,d)$, where $c \in W/W_I$ and $d \in W/W_J$. We write $m \geq n$, if $n$ is obtained by a horizontal contraction of $m$ and $m \geq n$, if $n$ is obtained by a vertical contraction of $m$. The following is obvious from the definition of the order on $\Xi$:

**Proposition 1.4.** For $m,n \in \Xi$ the following are equivalent:

(i) $m \geq n$.

(ii) There exists a (unique) $m'$ such that $m \geq m' \geq n$. 

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(iii) There exists a (unique) \( n' \) such that \( m \geq'' n \geq' n \). \( \square \)

In particular, for \( m \in \Xi(I_1, J_2) \) and \( n \in \Xi(I_2, J_2) \) the inequality \( m \geq n \) implies that \( I_1 \subseteq I_2 \) and \( J_1 \subseteq J_2 \), and the arrows in (1.3) define a (necessarily surjective) \( W \)-invariant map

\[
\pi_{m,n} : m \to n,
\]

where we regard \( m \) and \( n \) as subsets (orbits) in the corresponding terms of the diagram (1.3). These maps are transitive, i.e., define a covariant functor from the poset \( (\Xi, \geq) \) (considered as a category) to the category of sets with \( W \)-action.

C. Mixed supremum in \( \Xi \). Let \( m', n \in \Xi \). Their mixed supremum is the subset

\[
\text{Sup}(m', n) = \{ m \in \Xi | m' \leq'' m \geq' n \}.
\]

For this set to be nonempty, it is necessary that there be \( I_1 \subseteq I_2 \) and \( J_2 \subseteq J_1 \) such that

\[
m' \in \Xi(I_1, J_2), \quad n \in \Xi(I_2, J_1),
\]

in which case

\[
\text{Sup}(m', n) = \{ m \in \Xi(I_1, J_1) | \varphi''_{(I_1|J_1,J_2)}(m) = m', \ \varphi'_{(I_2|I_1,J_1)}(n) \}.
\]

So for nonemptiness of \( \text{Sup}(m', n) \) it is further necessary that

\[
\varphi'_{(I_1,J_2|J_1)}(m') = \varphi''_{(I_2|I_1,J_1)}(n).
\]

Denoting this common value by \( n' \), we have \( m' \geq' n' \leq'' n \).

The following is then straightforward.

**Proposition 1.7.** \( \text{Sup}(m', n) \) is the set of \( W \)-orbits in the fiber product

\[
m' \times_{n'} n = m' \times_{(W/W_{I_2}) \times (W/W_{J_2})} n,
\]

where we consider \( m', n', n \) as subsets (\( W \)-orbits) in the corresponding terms of the diagram (1.3). \( \square \)

D. The cell decomposition of \( W \setminus \mathfrak{h} \) and anodyne inequalities. We further denote:

- \( S^{(2)}_{\mathcal{H}} \): the cell decomposition of \( \mathfrak{h} = i\mathfrak{h}_{R} \oplus \mathfrak{h}_{R} \) into the product cells \( iC + D \) where \( C, D \in C \) are faces of the arrangement \( \mathcal{H} \). Recall that each face of \( \mathcal{H} \) has the form \( C = A_c, c \in W/W_I, I \subseteq \Delta_{\text{sim}}. \)

- \( \mathcal{S}^{(2)} = p(S^{(2)}_{\mathcal{H}}) \): the decomposition of \( W \setminus \mathfrak{h} \) into the images \( U_m = p(iA_c + A_d), \ m = W(c,d) \in \Xi. \)

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Proposition 1.8. The restriction of $p$ to the closure of each $iA_c + A_d$ is a homeomorphism to its image. Therefore $S^{(2)}$ is a quasi-regular cell decomposition of $W \setminus \mathfrak{h}$ into the cells $U_m$, $m \in \Xi$, refining the complex stratification $S^{(0)}$.

Proof: We need to show the following: if $z, z' \in \overline{U}_m = i\overline{A}_c + \overline{A}_d$ and $p(z) = p(z')$ (i.e., $z' = wz$ for some $w \in W$), then $z = z'$. Now, a similar statement for the $W$-action on $\mathfrak{h}_R$ is standard: if $C \in \mathfrak{C}$ and $x, x' \in \overline{C}$ are such that $x' = wx$ for some $w \in W$, then $x = x'$. This is because $p$ maps the closure of any face bijectively to the closure of a face of $K$. Therefore writing $z = ix + y, z' = ix' + y'$ with $x, y, x', y' \in \mathfrak{h}_R$, the condition $z' = wz$ means $x' = wx$ and $y' = wy$ and so by the above $x' = x, y' = y$, i.e., $z' = z$. \hfill \Box

We note that $m \succeq n$ if and only if $U_m \supseteq U_n$.

Definition 1.9. An inequality $m \succeq n$, resp. $m \succ n$, resp. $m \succcurlyeq n$ is called anodyne, if $U_m$ and $U_n$ lie in the same stratum of $S^{(0)}$.

Proposition 1.10. Let $m \succeq n$. The following are equivalent:

(i) The inequality $m \succeq n$ is anodyne.

(ii) Considering $m, n$ as subsets in $\mathfrak{C} \times \mathfrak{C}$, we have $|m| \geq |n|$.

(iii) The map $\pi_{m,n} : m \rightarrow n$ is a bijection.

Proof: Let us prove (i)$\iff$(ii). For $c \in \mathfrak{C}$ let $W^c \subseteq W$ be the stabilizer of $c$ in $W$. Let $m = W(c, d)$. Then $|m| = |W|/|W^c \cap W^d|$. Now, $W^c \cap W^d$ is the stabilizer of the cell $iA_c + iA_d \subseteq \mathfrak{h}$ or, what is the same by Proposition 1.8, the stabilizer of any point in this cell.

Since $m \succeq n$, we can represent $n = W(a, b)$ with $c \succeq a$ and $d \succeq b$, i.e., $A_c \supseteq A_a$ and $A_d \supseteq A_b$. Further, let $S_m$ and $S_n$ be the strata of $S^{(0)}$ containing $U_m$ and $U_n$, so that $S_m \supseteq S_n$. That is, $S_m = p(L_{c,d})$ and $S_n = p(L_{a,b})$, where $L_{c,d}$ is the stratum of $S^{(0)}_H$ (the generic part of a flat of $\mathcal{H}_c$) containing $iA_c + A_d$, and similarly for $L_{a,b}$.

Note that $L_{c,d} \supseteq \overline{T}_{a,b}$ and, moreover, $L_{c,d} = L_{a,b}$ if and only if $S_m = S_n$. Note further, that the points in $L_{c,d}$ have the same stabilizer, namely $W^c \cap W^d$ while points in $L_{a,b}$ have the same stabilizer $W^a \cap W^b$.

Now, generic points of a (strictly) smaller complex flat of the root arrangement, have (strictly) bigger stabilizer in $W$. Therefore we have $|W^a \cap W^b| \geq |W^c \cap W^d|$ with equality meaning that $L_{c,d} = L_{a,b}$, i.e., $S_m = S_n$, i.e., that the inequality $m \succeq n$ is anodyne. This proves (i)$\iff$(ii). The equivalence (ii)$\iff$(iii) is clear since $p_{m,n}$ is a surjective map. \hfill \Box

Proposition 1.11. Suppose that $m' \succ n'$ is a and at least one of the inequalities is anodyne. Then $\text{Sup}(m', n)$ consists of exactly one element $m$. Further, if $m' \succ n'$ is anodyne, then $m \succeq n$ is anodyne. If $n \succ n'$ is anodyne, then $m \succeq m'$ is anodyne.

Proof: We apply Proposition 1.7. Note that the diagram (1.3) is Cartesian, being the external Cartesian product of two arrows

$$\{W/W_{I_1} \longrightarrow W/W_{J_1}\} \times \{W/W_{I_2} \longrightarrow W/W_{J_2}\}.$$
Suppose that \( \mathbf{m} \geq' \mathbf{n} \) is anodyne. Then the map \( \pi_{\mathbf{m}, \mathbf{n}} : \mathbf{m} \to \mathbf{n} \) (induced by \( \pi' \)) is a bijection and so in the Cartesian product diagram

\[
\begin{array}{ccc}
\mathbf{m} \times_{\mathbf{n'}} \mathbf{n} & \longrightarrow & \mathbf{n} \\
\downarrow & & \downarrow \\
\mathbf{m} & \longrightarrow & \mathbf{n'}
\end{array}
\]

the arrow \( \rho' \) is a bijection, i.e., \( \mathbf{m} \times_{\mathbf{n'}} \mathbf{n} \) consists of one \( W \)-orbit, so by Proposition 1.7 there is only one possible \( \mathbf{m} \) as claimed. Further, \( \mathbf{m} \geq' \mathbf{n} \) is anodyne by Proposition 1.10, since \( \rho' \) is a bijection. The case when \( \mathbf{n} \geq'' \mathbf{n'} \) is anodyne, is treated similarly.

\[\square\]

2 Main result: Perv\((W \backslash h)\) and mixed Bruhat sheaves

A. Mixed Bruhat sheaves. Let \( k \) be a field and \( \text{Vect}_k \) be the category of finite-dimensional \( k \)-vector spaces.

**Definition 2.1.** By a mixed Bruhat sheaf (of type \( g \)) we mean a datum of finite-dimensional \( k \)-vector spaces \( E(\mathbf{m}), \mathbf{m} \in \Xi \) and linear operators

\[
\begin{align*}
\partial'_{m,n} &= \partial'_{m,n,E} : E(\mathbf{m}) \to E(\mathbf{n}), \quad \mathbf{m} \geq' \mathbf{n}, \\
\partial''_{m,n} &= \partial''_{m,n,E} : E(\mathbf{n}) \to E(\mathbf{m}), \quad \mathbf{m} \geq'' \mathbf{n},
\end{align*}
\]

satisfying the conditions:

\(\text{(MBS1)}\) The \( \partial'_{m,n} \) are transitive, i.e., form a covariant functor from the poset \( (\Xi, \geq') \) (considered as a category) to \( \text{Vect}_k \). Similarly, the \( \partial''_{m,n} \) are transitive, i.e., form a contravariant functor \( (\Xi, \geq'') \to \text{Vect}_k \).

\(\text{(MBS2)}\) The \( \partial' \)- and \( \partial'' \)-maps commute with each other. That is, suppose \( \mathbf{m} \geq' \mathbf{n}' \leq'' \mathbf{n} \). Then

\[
\partial''_{m,n} \partial'_{m',n'} = \sum_{m \in \text{Sup}(m',n')} \partial'_{m,n} \partial''_{m,m'}.
\]

\(\text{(MBS3)}\) If \( \mathbf{m} \geq' \mathbf{n} \) is an anodyne inequality, then \( \partial'_{m,n} \) is an isomorphism. If \( \mathbf{m} \geq'' \mathbf{n} \) is an anodyne inequality, then \( \partial''_{m,n} \) is an isomorphism.

We denote by \( \text{MBS} = \text{MBS}_g \) the category of mixed Bruhat sheaves of type \( g \).

**Proposition 2.2.** Let \( X \) be a stratum of \( S^{(0)} \). The part of a mixed Bruhat sheaf \( E \) consisting of \( E(\mathbf{m}) \) with \( U_\mathbf{m} \subset X \) and the maps \( (\partial')^{-1}, \partial'' \) between such \( E(\mathbf{m}) \), gives a local system \( \mathcal{L}_{E,X} \) on \( X \).
In particular, taking $X = W \setminus \mathfrak{h}^{\text{reg}}$ to be the open stratum, we get, very directly, a representation of the braid group $\operatorname{Br}_g = \pi_1(W \setminus \mathfrak{h}^{\text{reg}})$.

**Proof:** To define a local system $\mathcal{G}$ on $S$, we need to give vector spaces $G_m$ for each $U_m \subset S$ and generalization maps $\gamma_{n,m} : G_n \to G_m$ for any inclusion $U_n \subset \overline{U}_m$ with $U_m \subset S$, which are isomorphisms and satisfy the transitivity conditions, see Proposition A.4. We put $G_m = E(m)$. When $U_n \subset \overline{U}_m$ with $U_m \subset S$, we have an anyonic inequality $n \leq m$. Now, if, in this situation, $n \leq m$, we define $\gamma_{n,m} = (\partial''_{m,n})^{-1}$, the inverse of an isomorphism $\partial'_{m,n}$, see (MBS3) If $n \leq'' m$, we define $\gamma_{n,m} = \partial''_{m,n}$, also an isomorphism by (MBS3). Proposition 1.11 together with (MBS2) implies that the two types of isomorphisms $\gamma_{n,m}$ thus defined, commute with each other. Together with the transitivity of the $\partial'$ and of the $\partial''$, this implies that these two types of $\gamma_{n,m}$ extend uniquely to any $n \leq m$ such that $U_m \subset S$ and are transitive. $\square$

Let $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the permutation of the factors. It induces an involution $\tau : \Xi \to \Xi$. For $m \in \Xi$ we write $m^{\tau} = \tau(m)$. Thus, if $m = W(c, d)$, the $m^{\tau} = W(d, c)$. The category MBS carries a perfect duality $E \mapsto E^{\tau}$, where the “dual” mixed Bruhat sheaf $E^{\tau}$ has

$$E^{\tau}(m) = E(m^{\tau})^*, \quad \partial'_{m,n,E^{\tau}} = (\partial''_{n', m', E})^*, \quad \partial''_{m,n,E^{\tau}} = (\partial''_{n', m', E})^*.$$

**B. Perverse sheaves and the main result.** Let

$$\operatorname{Perv}(W \setminus \mathfrak{h}) = \operatorname{Perv}(W \setminus \mathfrak{h}, S^{(0)}) \subset D^b_{S^{(0)}} \operatorname{Sh}(W \setminus \mathfrak{h})$$

be the category of perverse (middle perversity) sheaves of $k$-vector spaces on $W \setminus \mathfrak{h}$ which are (cohomologically) constructible with respect to $S^{(0)}$. In this paper we use the standard normalization of perversity conditions from [2] so that a local system on a smooth $d$-dimensional subvariety $Z$ is perverse, if put in degree $(- \dim_C Z)$. Thus, explicitly, $\mathcal{F} \in D^b_{S^{(0)}} \operatorname{Sh}(W \setminus \mathfrak{h})$ is perverse, if:

(Perv$^-$) For each $q$, the sheaf $H^q(\mathcal{F})$ is supported on a complex analytic subvariety of complex dimension $\leq -q$ (so $H^q(\mathcal{F}) = 0$ for $q > 0$).

(Perv$^+$) The condition (Perv$^-$) holds also for the Verdier dual complex $\mathbb{D}(\mathcal{F})$.

By definition, the Verdier duality $\mathbb{D}$ preserves $\operatorname{Perv}(W \setminus \mathfrak{h})$. In addition, consider the involution

$$\tau : \mathfrak{h} \longrightarrow \mathfrak{h}, \quad x + iy \mapsto y + ix, \quad x, y \in \mathfrak{h}_R.$$

This involution commutes with the $W$-action and preserves the strata of $S^{(0)}_H$. Therefore it descends to an involution of $W \setminus \mathfrak{h}$ which we denote by the same letter $\tau : W \setminus \mathfrak{h} \to W \setminus \mathfrak{h}$ and which preserves the strata of $S^{(0)}$. This means that the pullback $\tau^*$ preserves $\operatorname{Perv}(W \setminus \mathfrak{h})$, and we define the twisted dual of $\mathcal{F} \in \operatorname{Perv}(W \setminus \mathfrak{h})$ to be

$$\mathcal{F}^{\tau} = \tau^*(\mathbb{D}(\mathcal{F})) = \mathbb{D}(\tau^* \mathcal{F}).$$
Let \( \text{sgn} : W \to k^* \) be the sign character of \( W \), defined by \( \text{sgn}(s_\alpha) = -1 \). As we have the surjection
\[
\text{Br}_g = \pi_1(W \backslash \mathfrak{h}^\text{reg}) \to W,
\]
\( \text{sgn} \) is also a character of \( \text{Br}_g \). In particular, we have the 1-dimensional local system \( \mathcal{L}_{\text{sgn}} \) of \( k \)-vector spaces on \( W \backslash \mathfrak{h}^\text{reg} \).

The following is the main result of this paper.

**Theorem 2.6.** (a) We have an equivalence of categories \( \mathcal{E} : \text{MBS}_g \to \text{Perv}(W \backslash \mathfrak{h}, S^{(0)}) \) taking the duality (2.3) on \( \text{MBS}_g \) to the twisted Verdier duality (2.5).

(b) For mixed Bruhat sheaf \( E \), the restriction of the perverse sheaf \( \mathcal{E}(E) \) to the open stratum \( W \backslash \mathfrak{h}^\text{reg} \), is isomorphic to the shifted local system \( \mathcal{L}_E \otimes \mathcal{L}_{\text{sgn}}[r] \). Here \( r = \dim_C \mathfrak{h} \) and \( \mathcal{L}_E \) is the local system associated to \( E \) by Proposition 2.2.

The proof will be given in the next two sections.

C. **Examples. The bicube point of view.** Many examples of mixed Bruhat sheaves appear, most immediately, in the form of simpler diagrams which we call bicubes. More precisely, let \( S \) be a finite set. By an \( S \)-bicube we mean a diagram
\[
Q = (Q_I, u_{IJ}, v_{IJ}), \quad I, J \in S.
\]

- \( Q_I \in \text{Vect}_k \) is a vector space, given for any subset \( I \subseteq S \).

- \( v_{IJ} : Q_I \to Q_J \) and \( u_{IJ} : Q_J \to Q_I \) are linear maps given for any \( I \subseteq J \subseteq S \) and satisfying the transitivity properties:
\[
v_{II} = \text{Id}, \quad u_{II} = \text{Id}, \quad v_{IK} = v_{JK} v_{IJ}, \quad u_{IK} = u_{IJ} u_{JK}, \quad I \subseteq J \subseteq K.
\]

In other words, a bicube consists of two commutative cubes superimposed on the same set of vertices so that the arrows in the two cubes go in the opposite directions. We denote by \( \text{Bic}_S \) the category of \( S \)-bicubes.

Given a mixed Bruhat sheaf \( E \in \text{MBS}_g \), we associate to it a \( \Delta_{\text{sim}} \)-bicube \( Q = Q(E) \) as follows. For \( I \subseteq \Delta_{\text{sim}} \) put
\[
\mathbf{m}'_I = W(K_I, 0), \quad \mathbf{m}_I = W(K_I, K_I), \quad \mathbf{m}''_I = W(0, K_I) \in E.
\]
Then \( \mathbf{m}'_I \preceq \mathbf{m}_I \succeq \mathbf{m}''_I \), both inequalities being anodyne. Given \( E \), we put \( Q_I = E(\mathbf{m}'_I) \) and note the identification \( \varphi_I : Q_I \to E(\mathbf{m}''_I) \) defined as the composition
\[
Q_I = E(\mathbf{m}'_I) \xrightarrow{(\varphi_{\mathbf{m}'_I, \mathbf{m}_I'})^{-1}} E(\mathbf{m}_I) \xrightarrow{\varphi_{\mathbf{m}_I, \mathbf{m}''_I}} E(\mathbf{m}''_I).
\]
If \( I \subseteq J \), then \( K_I \supseteq K_J \), so \( \mathbf{m}'_I \succeq \mathbf{m}'_J \) and \( \mathbf{m}''_I \succeq \mathbf{m}''_J \). We define
\[
v_{IJ} = \varphi_{\mathbf{m}'_I, \mathbf{m}'_J} : Q_I \to Q_J, \quad u_{IJ} = \varphi_I^{-1} \circ \varphi_{\mathbf{m}_I, \mathbf{m}''_J} \circ \varphi_I : Q_J \to E_Q.
\]
Transitivity of the $\partial'$ and $\partial''$ implies that $Q(E)$ is indeed a bicube. This gives a functor

$$Q : \text{MBS}_g \longrightarrow \text{Bic}_{\Delta_{\text{sim}}}, \quad E \mapsto Q(E).$$

In general, we do not know if this functor is fully faithful, i.e., if all any mixed Bruhat sheaf can be recovered from the corresponding bicube. The following examples show that this is true for $g = \mathfrak{sl}_2$ or $g = \mathfrak{sl}_3$.

**Example 2.7.** Let $g = \mathfrak{sl}_2$. Then $\mathfrak{h} = \mathbb{C}$, $\mathfrak{h}_R = \mathbb{R}$, we have only one simple root $\alpha$ and the arrangement of hyperplanes $\mathcal{H}$ in $\mathfrak{h}_R = \mathbb{R}$ consists of one hyperplane $\{0\}$. The Coxeter complex $\mathcal{C}$ consists of $\mathbb{R}_{<0}, \{0\}$ and $\mathbb{R}_{>0}$. The Weyl group $W = \{1, s\}$, where $s : \mathfrak{h} \rightarrow \mathfrak{h}$ takes $z \mapsto -z$. The quotient $W \backslash \mathfrak{h}$ is identified with $\mathbb{C}$ by the function $z^2$. Thus $\text{Perv}(W \backslash \mathfrak{h}, S^{(0)}) = \text{Perv}(\mathbb{C}, 0)$ is the classical category of perverse sheaves on $\mathbb{C}$ with the only possible singularity at 0. The cell decompositions $S^{(2)}_R$ of $\mathfrak{h} = \mathbb{C}$ and $S^{(2)}$ of $W \backslash \mathfrak{h} = \mathbb{C}$ are depicted in Fig. 1.

![Figure 1: Cell decompositions $S^{(2)}_R$ and $S^{(2)}$ for $g = \mathfrak{sl}_2$.](image)

For simplicity we label the five cells of $S^{(2)}_R$ by their representative points $0, \pm 1, \pm i$. Then a mixed Bruhat sheaf is a diagram

(2.8)

$$
\begin{array}{c}
E_0 \xrightarrow{v} E_1 \\
\downarrow{b} \quad \updownarrow{a} \\
E_{-1} \xrightarrow{u} E_0 \xrightarrow{v} E_1 \\
\downarrow{d} \quad \updownarrow{c} \\
E_{-1} 
\end{array}
$$

with $a, b, c, d$ isomorphisms and $uv = ab + cd$. A description of $\text{Perv}(\mathbb{C}, 0)$ in terms of such diagrams is equivalent to the classical description in terms of diagrams

(2.9)

$$\Phi \xrightarrow{v}{u} \Psi, \quad T_\Psi := \text{Id}_\Psi - vu \text{ is an isomorphism}.$$ 

That is, $E_0$ is identified with $\Phi$ and the other 4 spaces are identified with $\Psi$. Note that the diagram (2.9) is an $\Delta_{\text{sim}}$-bicube, and passing from (2.8) to (2.9) is a particular case of the functor $Q$. So in this case $Q$ is fully faithful and its essential image is described by the condition of $T_\Psi$ being invertible.
Example 2.10. Let $\mathfrak{g} = \mathfrak{sl}_3$. In this case $W = S_3$, and the stratification $S^{(0)}$ of $W\setminus \mathfrak{h} \simeq \mathbb{C}^2$ is given by the semiicubic parabola

$$Y = \{(a, b) \in \mathbb{C}^2 \mid 4a^3 + 27b^2 = 0\}.$$ 

That is, the strata are $\mathbb{C}^2 \setminus Y$, $Y \setminus \{0\}$ and $\{0\}$. Further, in this case $\Delta_{\text{sim}}$ consists of two elements which we denote $\alpha_1$ and $\alpha_2$. Accordingly, the bicube associated to $E \in \text{MBS}_g$ is labelled by subsets of $\{1, 2\}$ and has the form (bisquare)

where we have omitted the indexing of the $v$- and $u$-maps. A description of $\text{Perv}(W\setminus \mathfrak{h})$ in this case was given in [26] (see also [34] §5.3 for a discussion) and proceeds in terms of bisquares as above satisfying certain conditions. This means that the functor $Q$ is fully faithful in this case as well.

3 The Cousin complex of a mixed Bruhat sheaf

Our proof of Theorem 2.6 is, similarly to [33, 36], based on associating to a mixed Bruhat sheaf $E$ a certain complex of sheaves $E^\bullet = E^\bullet(E)$ which we call the Cousin complex. A priori, $E^\bullet$ is only $\mathbb{R}$-constructible but it turns to be (cohomologically) constructible with respect to $S^{(0)}$ and, moreover, a perverse sheaf. Here we describe this construction.

A. Imaginary strata in $W\setminus \mathfrak{h}$. The “imaginary part” map $\text{Im} : \mathbb{C} \to \mathbb{R}$ induces the map

$$J : W\setminus \mathfrak{h} \longrightarrow W\setminus \mathfrak{h}_R = \bigsqcup_{I \in \Delta_{\text{sim}}} K_I.$$ 

We put

$$X^\text{Im}_I = J^{-1}(K_I) \hookrightarrow W\setminus \mathfrak{h}.$$ 

The $X^\text{Im}_I$ form a (real) stratification of $W\setminus \mathfrak{h}$ which we call the imaginary stratification and denote $S^\text{Im}$; the $X^\text{Im}_I$ will be refereed to as the imaginary strata. For the case $\mathfrak{g} = \mathfrak{gl}_n$, these strata were considered in [34].

Note that

$$X^\text{Im}_I = \bigsqcup_J \bigsqcup_{m \in \Xi(J,I)} U_m,$$

so $S^{(2)}$ refines $S^\text{Im}$. Note further that for any two cells $U_m, U_n \subset X^\text{Im}_I$ the inclusion $U_n \subset \overline{U}_m$ is equivalent to $n \leq m$.
Example 3.1. For the case $g = sl_2$, the two imaginary strata are depicted in Fig. 2, with $K_\emptyset$ being the union of three cells of $S^{(2)}$ and $K_{\{\alpha\}}$ being the union of two such cells.

![Figure 2: Imaginary strata for $g = sl_2$.](image)

B. The Cousin complex. Given a mixed Bruhat sheaf $E = (E(m), \partial', \partial'')$, we define a cellular sheaf $\tilde{E}_I$ on $X^m_I$ with stalk at $U_m$, $m \in \Xi(I, J)$ being $E(m)$ and the generalization map $E(n) \to E(n)$ for $U_n \subset \overline{U}_m$ being $\partial''_{m,n}$. Because of the transitivity of the $\partial''$-maps in (MBS1), this gives a well-defined sheaf $\tilde{E}_I$. We further put $E_I = j_{I*}\tilde{E}_I$, a sheaf on $W \setminus \emptyset$.

For $I \subset \Delta_{\text{sim}}$ let $k^I$ be the $k$-vector space spanned by $I$, i.e., the space with basis $e_\alpha, \alpha \in I$. Let $\det(I) = \Lambda'^{(I)}(k^I)$ be the top exterior power of $k^I$. For $I_1 \subset I_2$ such that $|I_2| = |I_1| + 1$, i.e., $I_2 = I_1 \cup \{\alpha\}$ for some $\alpha$, we have the map

$$\varepsilon_{I_1,I_2} : \det(I_1) \longrightarrow \det(I_2), \quad v \mapsto v \wedge e_\alpha.$$ 

We now define the complex of sheaves

$$\mathcal{E}^\bullet = \mathcal{E}^\bullet(E) = \left\{ \mathcal{E}_\emptyset \overset{d}{\longrightarrow} \bigoplus_{|I|=1} \mathcal{E}_I \otimes \det(I) \overset{d}{\longrightarrow} \bigoplus_{|I|=1} \mathcal{E}_I \otimes \det(I) \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \mathcal{E}_{\Delta_{\text{sim}}} \otimes \det(\Delta_{\text{sim}}) \right\}$$

graded so that $E_\emptyset$ in in degree $(-\dim_C(\emptyset))$. The differential $d$ is induced by the maps $\partial'$. More precisely, let $I_1 \subset I_2$ be such that $|I_2| = |I_1| + 1$, i.e., $I_2 = I_1 \cup \{\alpha\}$ for some $\alpha$. Let $m \in \Xi(I_2, J)$ so $U_m \subset X^m_I$. The definition $\mathcal{E}_{I_1} = (j_{I_1})_*\tilde{E}_{I_1}$ implies that $\mathcal{E}_{I_1}$ is (locally, hence globally) constant on $U_m$ and its stalk there is identified as

$$\mathcal{E}_{I_1}(U_m) := \Gamma(U_m, \mathcal{E}_{I_1}) = \bigoplus_{n \in \Xi(I_1, J) \cap \Xi(I_2, J)} E(n).$$

The stalk of $E_{I_2}$ at $U_m$ is, by definition, $E(m)$. Now, the matrix element

$$(3.3) \quad d_{I_1,I_2} : \mathcal{E}_{I_1} \otimes \det(I_1) \longrightarrow \mathcal{E}_{I_2} \otimes \det(I_2)$$

is defined, over $U_m$ to be given by the map

$$d_{I_1,I_2,m} : \sum_{\substack{n \in \Xi(I_1, J) \\ n \supseteq m}} \partial''_{n,m} \otimes \varepsilon_{I_1,I_2} : \bigoplus_{n \in \Xi(I_1, J) \cap \Xi(I_2, J) \cap \Xi(m,J)} E(n) \otimes \det(I_1) \longrightarrow E(m) \otimes \det(I_2).$$
Proposition 3.4. (a) The linear maps $d_{I_1,I_2,m}$ define a morphism of sheaves $d_{I_1,I_2}$ as in (3.3).

(b) The morphisms of sheaves $d$ with matrix elements $d_{I_1,I_2}$ define a complex of sheaves $\mathcal{E}^\bullet$ as in (3.2), i.e., satisfy $d^2 = 0$.

Proof: (a) follows at once from (MBS2) (commutativity of $\partial'$ with $\partial''$), while (b) follows from (MBS1) (transitivity of $\partial'$).

We call $\mathcal{E}^\bullet$ the Cousin complex associated to $E$. Theorem 2.6 will be a consequence of the following more precise result.

Theorem 3.5. (a) For $E \in \text{MBS}_g$ the complex $\mathcal{E}^\bullet(E)$ is an object of $\text{Perv}(W\backslash \mathfrak{h}, S^{(0)})$. In particular, it is (cohomologically) $S^{(0)}$-constructible.

(b) Further, $\mathcal{E}^\bullet(E')$, see (2.3), is naturally quasi-isomorphic to the twisted dual $(\mathcal{E}^\bullet(E))^\tau$, defined by (2.5).

(c) The functor $\mathcal{G}: E \mapsto \mathcal{E}^\bullet(E)$ is an equivalence of categories $\mathcal{G}: \text{MBS}_g \rightarrow \text{Perv}(W\backslash \mathfrak{h}, S^{(0)})$.

(d) For $E \in \text{MBS}_g$, the restriction of $\mathcal{E}(E)$ to the open stratum $W\backslash \mathfrak{h}^{\text{reg}}$, is isomorphic to the shifted local system $\mathcal{L}_E \otimes \mathcal{L}_{\text{sgn}}[\tau]$.

C. The Fox-Neuwirth-Fuchs cells. For any subset $S \subset \mathfrak{h}_R$ let $\text{Lin}_R(S)$ be the $\mathbb{R}$-linear subspace spanned by $L$.

Recall from [10] and [33] §2 the “intermediate”, or Björner-Ziegler stratification of $\mathfrak{h}$ induced by the root arrangement $\mathcal{H}$. This is a quasi-regular cell decomposition of $\mathfrak{h}$ into cells $[C,D]$ labelled by face intervals, i.e., pairs $(C,D) \in C$ such that $C \leq D$. By definition, $[C,D]$ consists of $x + iy \in \mathfrak{h}$ with $x,y \in \mathfrak{h}_R$ satisfying:

(a) $y \in C$.

(b) $x$ is congruent to an element of $D$ modulo the subspace $\text{Lin}_R(C) \subset \mathfrak{h}_R$.

Thus

$$S^{(2)}_{\mathcal{H}} < S^{(1)}_{\mathcal{H}} < S^{(0)}_{\mathcal{H}}.$$  

The action of $W$ on $\mathfrak{h}$ preserves the stratification $S^{(1)}_{\mathcal{H}}$ and so defines a stratification $S^{(1)} := p(S^{(1)}_{\mathcal{H}})$ of $W\backslash \mathfrak{h}$ such that

$$S^{(2)} < S^{(1)} < S^{(0)}.$$  

Proposition 3.6. Every stratum of $S^{(1)}$ is a topological cell, so $S^{(1)}$ is a (non-quasi-regular, in general) cell decomposition of $W\backslash \mathfrak{h}$.

Proof: Since each $[C,D]$ is known to be a cell, it suffices to prove that $p: [C,D] \rightarrow p([C,D])$ is a homeomorphism, i.e., to prove that if $z, z' \in [C,D]$ and $z' = wz$ for some $w \in W$, then $z' = z$.  

Let $z = x + iy, z' = x' + iy'$ with $x, y, x', y' \in \mathfrak{h}_R$. Then $z' = wz$ implies $y' = wy$. But since $y, y' \in C$ lie in the same face of $C$, we have $y' = y$. This also means that $w$ stabilizes $C$ (pointwise) and therefore stabilizes (also pointwise) the vector space $L := \text{Lin}_R(C)$, which is a flat of $\mathcal{H}$.

By conjugating with an appropriate element of $W$, we can assume that $L$ is a flat of which some open part lies in the closure of the dominant Weyl chamber $C^+$, i.e., $L = \cap_{\alpha \in \Delta_{\text{sim}}} \alpha^\perp$ for some $I \subset \Delta_{\text{sim}}$. Then the subgroup in $W$ fixing $L$ pointwise, is $W_L \subset W$. In this case the quotient arrangement $\mathcal{H}/L$ in $\mathfrak{h}/L$ (see [33] §2B) is the root arrangement associated to the semi-simplification of the Levi subalgebra associated to $I$. The group $W_L$ is the Weyl group of that semi-simplification. In particular, no two points on the same face of $\mathfrak{h}/L$ can be congruent under the action of $W_L$.

Now, the condition $z, z' \in [C, D]$ means $x = d + l, x' = d' + l'$ with $d, d' \in D$ and $l, l' \in L$. Moreover, $w(z) = z'$ implies $w(x) = x'$, so

$$w(d) + l = w(d + l) = w(x) = x' = d' + l', \quad \text{i.e.,} \quad w(d) = d' + l' - l,$$

which means that the images of $d$ and $d'$ in $\mathfrak{h}_R/L$, while lying in the same face of $\mathcal{H}/L$, are taken into each other under the action of $w \in W_L$. This implies that these images are equal, i.e., $d' = d + l''$ for some $l'' \in L$. Combining this with the above equality, we get

$$w(d) = d + \lambda, \quad \lambda := l'' + l' - l \in L.$$ 

Note that $\lambda$ is fixed by $w$. Now, for $\lambda \neq 0$ the last equality is impossible, since for any $n > 0$ we have $w^n(d) = d + n\lambda$ which contradicts the fact that, $W$ being finite, we must have $w^n = \text{Id}$ for some $n$.

So $\lambda = 0$ and $w(d) = d$, and therefore

$$w(x) = w(d + l) = w(d) + l = d + l = x.$$

Together with the equality $w(y) = y$ proved earlier, this implies $z' = w(z) = z$. \hfill $\square$

We will call the cells $p([C, D])$ of the cell decomposition $S^{(1)}$ the Fox-Neuwirth-Fuchs cells of $W \setminus \mathfrak{h}$.

**Examples 3.7.** (a) For $\mathfrak{g} = \mathfrak{sl}_2$, the Fox-Neuwirth-Fuchs cell decomposition $S^{(1)}$ of $W \setminus \mathfrak{h} = \mathbb{C}$ consists of $\mathbb{C} \setminus \mathbb{R}_{\geq 0}, \mathbb{R}_{> 0}$ and $\{0\}$, see Fig. 2.

(b) For $\mathfrak{g} = \mathfrak{gl}_n$ the Fox-Neuwirth-Fuchs cells of $W \setminus \mathfrak{h} = \text{Sym}^n(\mathbb{C})$ have been discussed in [35].

We refer to the Appendix for the meaning of the notations $\wedge, \vee$ for stratifications.

**Proposition 3.8.** (a) We have $S^{(1)} = S^{(1m)} \wedge S^{(0)}$. In particular, $S^{(1)}$ refines both $S^{(1m)}$ and $S^{(0)}$.

(b) We also have $S^{(0)} = S^{(1)} \vee \tau(S^{(1)})$, where $\tau : W \setminus \mathfrak{h} \to W \setminus \mathfrak{h}$ is the involution (2.4). In particular (Proposition A.2(b)), any sheaf which is $S^{(1)}$-constructible and $\tau(S^{(1)})$-constructible, is $S^{(0)}$-constructible.
Proof: As $S^{(2)}$ refines $S^{(1)}$, $S^{(1)}$ and $S^{(0)}$, we have equivalence relations $\equiv_{S^{(1)}}$, $\equiv_{S^{(1)}}$ and $\equiv_{S^{(0)}}$ on the set $\Xi$ labelling cells of $S^{(2)}$ describing when two cells lie in the same stratum of the corresponding coarser stratification.

Now, by definition, $S^{(1)}$ is the equivalence closure of the relation $\geq''$. At the same time, $S^{(1)}$ is the equivalence closure of the relation $R''$ defined by $mR''n$ if, first, $m \geq'' n$ and, second, the inequality is anodyne, i.e., $m \equiv_{S^{(0)}} n$. This implies (a).

Let us prove (b). Note that $S^{(2)}$ also refines $\tau(S^{(1)})$ and so we have the equivalence relation $\equiv_{\tau(S^{(1)})}$ on $\Xi$ describing how $S^{(2)}$-cells are arranged into $\tau(S^{(1)})$-cells. As before, $\equiv_{\tau(S^{(1)})}$ is the equivalence closure of the relation $R'$ defined by $mR'n$ if, first, $m \geq' n$ and, second, the inequality is anodyne. It follows that $R' \cup R''$, considered as a subset of $\Xi \times \Xi$, is contained in $\equiv_{S^{(0)}}$, and so $(R' \cup R'')^\sim$, the equivalence closure of $R' \cup R''$, is contained in $\equiv_{S^{(0)}}$. Now, $S^{(1)} \vee \tau(S^{(1)})$ is, by definition, the partition of $W\setminus h$ into unions of $S^{(2)}$-cells corresponding to the classes of $(R' \cup R'')^\sim$. So each part of $S^{(1)} \vee \tau(S^{(1)})$ is contained in a single $S^{(0)}$-stratum. Further, given an $S^{(0)}$-stratum $S$, we consider the corresponding $\equiv_{S^{(0)}}$-class $\Xi_S \subset \Xi$. Since the $U_m$, $m \in \Xi_S$, form a quasi-regular cell decomposition of $S$, the sub-poset $\Xi_S \subset \Xi$ is closed under taking intermediate points. That is, if $m, n, p \in \Xi$ are such that $m \leq n \leq p$ and $m, p \in \Xi_S$, then $n \in \Xi_S$. Now, $S$ being connected, any two $m, n \in \Xi_S$ are connected by a chain of anodyne inequalities $\geq, \leq$. But any anodyne inequality, say $m \geq n$, factors into two anodyne inequalities $m \geq' m' \geq'' n$. This means that $\Xi_S$ is a class for $(R' \cup R'')^\sim$, thus proving (b). \hfill \Box

D. Perversity of the Cousin complex. Here we prove parts (a) and (b) of Theorem 3.5. The argument is similar to that of [36] §5-6, so we give a more condensed presentation.

For $I \subset \Delta_{sim}$ let $X^R_I = \tau(X^I_{Im})$. Thus $X^R_I = \mathcal{R}^{-1}(K_I)$, where $\mathcal{R} : W\setminus h \rightarrow W\setminus h_R$ is induced by $Re : \mathcal{C} \rightarrow \mathbb{R}$.

For $m \in \Xi(I, J)$ be have a commutative diagram of embeddings

$$
\begin{array}{ccc}
U_m \xrightarrow{j_m} X^I \xrightarrow{k'^m} W\setminus h,
\end{array}
$$

Lemma 3.9. For any $V \in \text{Vect}_k$ we have canonical isomorphisms

$$(k'^m)_! (j'^m)_* \mathcal{V}_{I_{Im}} \xrightarrow{\sim} (k'_I)_! (j'_I)_* \mathcal{V}_{I_{Im}} ;
(k'_I)_! (j'_I)_* \mathcal{V}_{I_{Im}} \xrightarrow{\sim} (k'^m)_! (j'^m)_* \mathcal{V}_{I_{Im}} .$$

Proof: Let $m = W(c,d)$. By Proposition 1.8, $p : hW\setminus h$ induces a homeomorphism of closures $\overline{\mathcal{A}_c + \mathcal{A}_d} \rightarrow \overline{U_m}$. So our statement reduces to the similar statement about the
product diagram

\[
\begin{array}{c}
\begin{array}{c}
i A_c + A_d \\
i A_c + i h_R + A_d \\
i A_c + h_R \rightarrow \mathfrak{h},
\end{array}
\end{array}
\]

which is [36] Prop. 5.2.

For any \( J \subset \Delta_{\text{sim}} \) let \( \overline{J} = \Delta_{\text{sim}} \setminus J \) be the complement. Let also \( r = \dim_{\mathbb{C}}(\mathfrak{h}) \).

**Proposition 3.10.** For any \( E \in \text{MBS} \) we have a natural isomorphism in the derived category \( \mathbb{D}(\mathcal{E}(E)) \cong \tau^* \mathcal{E}(E^*) \).

**Proof:** The definition of \( \mathcal{E}_I = \mathcal{E}_I(E) \) as a cellular sheaf on \( X^\text{im}_I \) with stalk at \( U_m \) \( m \in \Xi(I, J) \) being \( E(m) \) and the generalization maps being \( \partial''_{m,n} \), realizes the sheaf \( \mathcal{E}_I = j_I^* \mathcal{E}_I \) as the total complex of the following complex in derived category:

\[
\bigoplus_{|I| = 0} \bigoplus_{m \in \Xi(I, J)} (k_I^*)_*(j_m^*)! E(m)_{U_m} \longrightarrow \bigoplus_{|J| = 1} \bigoplus_{m \in \Xi(I, J)} (k_I^*)_*(j_m^*)! E(m)_{U_m} [1] \longrightarrow \cdots
\]

see [33] (1.12). So the Cousin complex \( \mathcal{E}^*(E) \) is the total object of the complex in derived category with the \((p,q)\)th term being

\[
(3.11) \quad \bigoplus_{|I| = 0} \bigoplus_{m \in \Xi(I, J)} (k_I^*)_*(j_m^*)! E(m)_{U_m} [p],
\]

the horizontal differentials (corresponding to the generalization maps of the \( \mathcal{E}_I \)) given by the \( \partial'' \), and the vertical differentials given by the \( \partial' \).

Let us apply the Verdier duality \( \mathbb{D} \) to this double complex. Recall that \( \mathbb{D} \) interchanges any \( f_* \) with \( f^! \), and for a constant sheaf on a cell, we have

\[
\mathbb{D} \left( E(m)_{U_m} \right) = E(m)^*_{U_m} [\dim_{\mathbb{R}} U_m].
\]

If \( m \in \Xi(I, J) \), then \( \dim_{\mathbb{R}} U_m = 2r - |I| - |J| \). Therefore \( \mathbb{D}(\mathcal{E}^*(E)) \) is quasi-isomorphic to the total object of the double complex in the derived category whose \((p,q)\)th term is

\[
(3.12) \quad \bigoplus_{|I| = 0} \bigoplus_{m \in \Xi(I, J)} (k_I^*)_*(j_m^*)! E(m)^*_{U_m} [q],
\]

the horizontal differentials given by the duals to the \( \partial''_E \) and the vertical differentials given by the duals to the \( \partial'_{E^*} \). Taking into account Lemma 3.9, we recognize in (3.12) a version of the double complex (3.11) but for \( E^* \) instead of \( E \) and with the roles of the real and imaginary parts exchanged. In other words, we recognize \( \tau^* \mathcal{E}^*(E^*) \), whence the claim. \( \square \)

**Corollary 3.13.** \( \mathcal{E}^*(E) \) is (cohomologically) \( S^{(0)} \)-constructible.
Proof: Recall from the proof of Proposition 3.8 that the relation $\equiv_{S(1)}$ on the set $\Xi$ parametrizing the cells of $S(2)$, is the equivalence closure of the relation “anodyne $\geq''$. Since $\partial''_{m,n}$ for anodyne $m \geq'' n$ is an isomorphism, each $E_I(E)$ is $S(1)$-constructible and so $E^\bullet(E)$ is cohomologically $S(1)$-constructible.

On the other hand, Proposition 3.10 implies, in the same way, that $\mathbb{D}(E^\bullet(E))$ is quasi-isomorphic to a complex of which each term is $\tau(S(1))$-constructible, and so it is cohomologically $\tau(S(1))$-constructible. Since Verdier duality preserves cohomologically constructible complexes, $E^\bullet(E)$ is also cohomologically $\tau(S(1))$-constructible. Our statement now follows from Proposition 3.8(b). \hfill $\square$

Proposition 3.14. $E^\bullet(E)$ is perverse.

Proof: Let us prove $\text{Perv}^-$. Let $q \in \mathbb{Z}$. Since $E^\bullet(E)$ is $S(0)$ is $S(0)$-constructible, the set $Z = \text{Supp} H^q(E^\bullet(E))$ is a complex algebraic subvariety in $W \setminus \mathfrak{h}$. Clearly,

$$Z \subset \text{Supp} E^q(E) = \bigcup_{|I| \geq r+q} X_I^{\text{im}}.$$  

Let $\tilde{Z} = p^{-1}(Z) \subset \mathfrak{h}$. We conclude that for any $z = x + iy \in \tilde{Z}$, $x, y \in \mathfrak{h}_R$, the point $y$ lies in the union of (real) flats of $\mathcal{H}$ of codimension $\geq r + q$. But because $\tilde{Z}$ is a complex algebraic subvariety of $\mathfrak{h}$, this implies that $\tilde{Z}$ lies in the union of (complex) flats of $\mathcal{H}_C$ of complex codimension $\geq r + q$. Since $p : \mathfrak{h} \to W \setminus \mathfrak{h}$ is a finite map, $\dim_C(\tilde{Z}) = \dim_C(Z)$, and we obtain $\text{Perv}^-$ for $E^\bullet(E)$. The condition $\text{Perv}^+$ follows from this by Proposition 3.10. \hfill $\square$

Propositions 3.10, 3.14 and Corollary 3.13 now imply parts (a) and (b) of Theorem 3.5.

4 The Cousin complex of a perverse sheaf

Here, we prove part (c) of Theorem 3.5 by constructing a quasi-inverse to the functor

$\mathfrak{G} : \text{MBS}_q \to \text{Perv}(W \setminus \mathfrak{h}), \quad E \mapsto E^\bullet(E).$

For this, similarly to [33, 34, 36], we start from a perverse sheaf $\mathcal{F}$ and construct geometrically a Cousin-type resolution of $\mathcal{F}$.

A. Cousin complex II. Recall the embedding $j_I : X_I^{\text{im}} \hookrightarrow W \setminus \mathfrak{h}$ of the imaginary stratum. Let also $r = \dim_C \mathfrak{h}$.

Proposition 4.1. Let $\mathcal{F} \in \text{Perv}(W \setminus \mathfrak{h})$. Then:

(a) The complex $j_I^! \mathcal{F}$ is quasi-isomorphic to a single sheaf $\tilde{E}_I = \tilde{E}_I(\mathcal{F})$ in degree $|I| - r$.

(b) The complex $j_{Is}^! \tilde{E}_I$ is quasi-isomorphic to a single sheaf $E_I = E_I(\mathcal{F}) = R^0j_{Is}^! j_I^! \mathcal{F}$.

(c) $\mathcal{F}$ has an explicit representative (Cousin resolution) of the form

$$E^\bullet(\mathcal{F}) = \left\{ \mathcal{E}_{\mathfrak{G}}(\mathcal{F}) \xrightarrow{\delta} \bigoplus_{|I| = 1} E_I(\mathcal{F}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} E_{\Delta_{\text{sim}}}(\mathcal{F}) \right\},$$
graded so that \( E_\emptyset(\mathcal{F}) \) is in degree \((-r)\).

**Proof:** This is analogous to [34] which corresponds to the particular case \( g = \mathfrak{g}l_n \). To start, note that \( p^*\mathcal{F} \in \mathrm{Perv}(\mathfrak{h}, S^{(0)}_I) \) and \( \mathcal{F} = (p_*p^*\mathcal{F})^W \). Finding the preimage \( p^{-1}(X^\text{Im}_I) \), we get a Cartesian square

\[
\begin{array}{ccc}
\mathfrak{h} & \xrightarrow{p} & W \setminus \mathfrak{h} \\
\bigcup_{C \in \mathcal{C}} \mathfrak{h}_r + iC & \xleftarrow{p_I} & X^\text{Im}_I \\
\end{array}
\]

so

\[
(4.3) \quad j^*_I\mathcal{F} \simeq (p_{I *} l^*_I p^*\mathcal{F})^W.
\]

But \( l_I \) is the disjoint union of the embeddings \( l_C : \mathfrak{h}_r + iC \hookrightarrow \mathfrak{h} \). It remains to notice that each \( l^*_C p^*\mathcal{F} \) is quasi-isomorphic to a single sheaf in degree \( \text{codim}(C) - r = |I| - r \), by [33] Cor. 4.11 (we also need to take into account the difference in normalizations of the conditions of perversity).

(b) Let \( z \in W \setminus \mathfrak{h} \). The stalk at \( z \) of \( R^q j^*_I \hat{\mathcal{E}}_I \) is, by definition, \( H^q(U \cap X^\text{Im}_I, \hat{\mathcal{E}}_I) \) for a small ball \( U \) around \( z \). By (4.3) and the Cartesian square (4.2), this \( H^q \) is a subspace in the direct sum

\[
\bigcup_{p(C) = K_I} H^q(p^{-1}(U) \cap (\mathfrak{h}_r + iC), l^*_C p^*\mathcal{F}).
\]

Let us prove that for \( q > 0 \) each summand in this sum vanishes. Indeed, since \( p^{-1}(U) \) is a disjoint union of balls, this vanishing follows from a similar statement about perverse sheaves on arrangement, namely

\[
R^q l_{C *} (l^*_C p^*\mathcal{F}[\text{codim}(C) - r]) = 0, \quad q > 0,
\]

which is [33] Cor. 4.11(a).

(c) Given (a) and (b), this is a purely formal consequence of the Postnikov system associated to \( \mathcal{F} \) and the increasing filtration of \( W \setminus \mathfrak{h} \) by closed subspaces \( X^I_{\leq m} = \bigcup_{|I| \leq m} X^I_{\leq m} \), see [33] §1B.

**B. From a perverse sheaf to a mixed Bruhat sheaf.** Let \( \mathcal{F} \in \mathrm{Perv}(W \setminus \mathfrak{h}) \). Because of Proposition 3.8(a), each \( E_I(\mathcal{F}) \) is \( S^{(1)} \)-constructible, in particular, \( S^{(2)} \)-constructible. For \( \mathfrak{m} \in \Xi(I, J) \) let \( E(\mathfrak{m}) \) be the stalk of \( E_I(\mathcal{F}) \otimes \det(I) \otimes (-1) \) at \( U_{\mathfrak{m}} \). The generalization maps of the \( E_I(\mathcal{F}) \) and the differential \( \delta \) in the complex \( E^*(\mathcal{F}) \) translate directly into linear maps \( \partial_{\mathfrak{m}, n}^p : \mathfrak{c}_{\mathfrak{m}, n}^p \rightarrow \mathfrak{c}_{\mathfrak{m}, n} \) as in Definition 2.1, which satisfy (MBS1-2). More precisely, transitivity of the generalization maps gives transitivity of the \( \partial' \), the condition \( \delta^2 = 0 \) gives transitivity of the \( \partial'' \), and the fact that \( \delta \) is a morphism of cellular sheaves, gives (MBS2).
Proposition 4.4. The diagram $E = (E(m), \partial', \partial'')$ satisfies also (MBS3), so it is a mixed Bruhat sheaf/.

Proof: $\mathcal{S}^{(1)}$-constructibility of each $\mathcal{E}_I(F)$ gives one half of (MBS3): $\partial''_{m,n}$ is an isomorphism for anodyne $m \geq'' n$. Let us prove the other half. For this, we represent $\mathcal{E}^\bullet(F)$, similarly to the proof of Proposition 3.10, as the total object of the double complex in derived category consisting of shifted sheaves of the form $(k_i')^* j_i^* E(m)|_{U_m}$ and apply Verdier duality. This will give an explicit complex of sheaves $\mathcal{G}^\bullet$ which, on one hand, is quasi-isomorphic to $\mathcal{D}(F)$ and, on the other complex, has the form

$$\mathcal{G}^\bullet = \left\{ k_{\partial^*} \tilde{\mathcal{G}}_I \to \bigoplus_{|I|=1} k_{I_1} \tilde{\mathcal{G}}_I \to \cdots \right\}$$

(leftmost term in degree $(-r)$). Here $k_{I} : X_{I_1}^{Re} = \tau(X_{I_1}^{Im}) \hookrightarrow W \backslash \mathfrak{h}$ and $\tilde{\mathcal{G}}_I$ is an $\mathcal{S}^{(2)}$-cellular sheaf on $X_{I_1}^{Re}$. Explicitly, the stalks of $\tilde{\mathcal{G}}_I$ are the $E(m)^*$ and the generalization maps are the $(\partial'_{m,n})^*$. Note that for such $\mathcal{G}^\bullet$ we necessarily have

$$(4.5) \quad \tilde{\mathcal{G}}_I \simeq k_{I_1}^! \mathcal{G}^\bullet[r - |I|],$$

because for $I_1 \subsetneq I$ we have $k_{I_1}^! k_{I_1}^* \tilde{\mathcal{G}}_I = 0$. This means that $\mathcal{G}^\bullet$, as an explicit complex of sheaves, is isomorphic to the intrinsic Cousin complex of the perverse sheaf $\mathcal{D}(F) \simeq \mathcal{G}^\bullet$ but formed using the $X_{I_1}^{Re}$ instead of $X_{I_1}^{Im}$. In particular, each $\tilde{\mathcal{G}}_I$ is constructible with respect to $\mathcal{S}^{Re} \wedge \mathcal{S}^{(0)} = \tau(\mathcal{S}^{(1)})$. This means that its generalization maps $(\partial'_{m,n})^*$ associated to anodyne $m \geq' n$ are isomorphisms, and so the corresponding $\partial'_{m,n}$ themselves are isomorphisms, which gives (MBS3) for $E$. \qed

The proposition means that we have a functor

$$\mathbb{E} : \text{Perv}(W \backslash \mathfrak{h}) \to \text{MBS}_q.$$ 

It is clear that $\mathbb{G} \circ \mathbb{E} \simeq \text{Id}$, as $\mathcal{F}$ is quasi-isomorphic to $\mathcal{E}^\bullet(F) = \mathbb{G}(\mathcal{E}(F))$. Conversely, given $E \in \text{MBS}_q$ and denoting $\mathcal{F} = \mathbb{G}(E) = \mathcal{E}^\bullet(E)$, we see that $\mathcal{E}^\bullet(E)$, as an explicit complex, is isomorphic to the intrinsic Cousin complex of the perverse sheaf $\mathcal{F}$. This follows from the identification $\mathcal{E}_I(E) = j_I^! \mathcal{E}^\bullet(E)[r - |I|]$ obtained in the same way as $(4.5)$. This means that $\mathbb{E}(\mathbb{G}(E)) \simeq E$. This finishes the proof of parts (a)-(c) of Theorem 3.5.

C. Origin of the sign twist. Let us now prove part (d) of of Theorem 3.5. Denote $k : W \backslash \mathfrak{h}^{\text{reg}} \hookrightarrow W \backslash \mathfrak{h}$ the embedding. Let $E \in \text{MBS}_q$. As $\mathcal{E}(E) \in \text{Perv}(W \backslash \mathfrak{h})$, the restriction $k^* \mathcal{E}(E)$ has the form $\mathcal{L}[r]$, where $\mathcal{L}$ is a local system, found explicitly as

$$\mathcal{L} = \ker \left\{ k^* \mathcal{E}(E) \to \bigoplus_{|I|=1} k^* \mathcal{E}_I(E) \otimes \det(I) \right\},$$

see (3.2). Note that we have the embedding $l : X_{\partial}^{Im} \hookrightarrow W \backslash \mathfrak{h}^{\text{reg}}$ whose composition with $k$ is $j_{\partial} : X_{\partial}^{Im} \hookrightarrow W \backslash \mathfrak{h}$. Now, by definition $\mathcal{E}(E) = j_{\partial^*} \tilde{\mathcal{E}}_{\partial}(E)$, where $\tilde{\mathcal{E}}_{\partial}(E)$ is the local system
on the imaginary stratum $X^\mathrm{Im}_\langle \rangle$ with stalks $E(m)$ and generalization maps $\partial''_{m,n}$ for anodyne $m \geq n$ such that $U_n, U_m \subset X^\mathrm{Im}_\langle \rangle$. In other words, $\hat{\mathcal{E}}_{\langle \rangle}(E) = l^*\mathcal{L}_E$, and $\mathcal{E}_{\langle \rangle}(E) = j_{\langle \rangle} l^*\mathcal{L}_E$. So over the open part $X^\mathrm{Im}_\langle \rangle \subset W\setminus \mathfrak{h}^\text{reg}$, we have an identification of local systems

$$\text{Ker}(d) \longrightarrow \mathcal{L}_E$$

(no sign yet!). Indeed, $\mathcal{E}_I(E)$ is not present on $X^\mathrm{Im}_\langle \rangle$ for $|I| = 1$ so passing to $\text{Ker}(d)$ does not change the source.

For two faces $C, D \in \mathcal{C}$ we have the orbit $W(C, D) \in \Xi$ and we denote by $U_{C,D} = U_{W(C,D)} \in S^{(2)}$ the corresponding cell. To identify two local systems on the entire $W\setminus \mathfrak{h}^\text{reg}$ it suffices to do so outside of the union of the $U_{C,D}$ which have real codimension $\leq 2$, i.e., over the union of the $U_{C,D}$ with $\text{codim}(C) + \text{codim}(D) \leq 1$.

If $\text{codim}(C) = 0$ and $\text{codim}(D) \leq 1$, then $U_{C,D} \in X^\mathrm{Im}_\langle \rangle$, so by the above we have an identification $\text{Ker}(d) = \mathcal{L}_E$ over $U_{C,D}$.

Suppose that $\text{codim}(C) = 1$ and $\text{codim}(D) = 0$. Then $C$ lies in the closure of exactly two codimension 0 faces, say $C_1$ and $C_2$. This means that the open set $X^\mathrm{Im}_\langle \rangle$ approaches the cell $U_{C,D}$ from two sides, similarly to what is depicted in the left part of Fig. 2. So the local system structure on $\mathcal{L}_E$ gives an identification of $\mathcal{E}_{\langle \rangle}(E) = j_{\langle \rangle} l^*\mathcal{L}_E$ with the direct sum of two copies of $\mathcal{L}_E$ over $U_{C,D}$. The relevant part of the local system structure on $\mathcal{L}_E$ is given by the inverses of the anodyne $\partial'$ corresponding to the inequalities $W(C_i, D) \supset W(C, D)$, $i = 1, 2$. At the same time, the differential $d$ of which we take the kernel, is given by these same $\partial'$ (not inverses). This means that sections of $\text{Ker}(d)$ near $U_{C,D}$ will be pairs of sections of $\mathcal{L}_E$ on the two sides of the “cut” $U_{C,D}$ whose values on $U_{C,D}$ (with respect to the local system structure on $\mathcal{L}_E$) sum up to 0. Such pairs can be seen as sections of $\mathcal{L}_E \otimes \mathcal{L}_{\text{sgn}}$. This finishes the proof of part (d) of of Theorem 3.5, so the theorem is proved.

5 Geometry of Bruhat orbits

A. Parabolic Bruhat decomposition. Let $\mathbb{K}$ be a field. In this section we consider the split reductive Lie algebra $\mathfrak{g}$ as defined over $\mathbb{K}$, i.e., as a Lie $\mathbb{K}$-algebra, and similarly for $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. The root system $\Delta \supset \Delta^+ \supset \Delta_{\text{sim}}$ is then embedded into the $\mathbb{K}$-vector space $\mathfrak{h}^\ast$. For $\alpha \in \Delta$ we denote by $e_\alpha \in \mathfrak{g}$ the Chevalley root generator corresponding to $\alpha$.

At the same time we will still use the geometry of the complex Cartan subalgebra which we will denote $\mathfrak{h}_C$ and of its real part $\mathfrak{h}_R$. In particular, we will use the hyperplane arrangement $\mathcal{H}$ in $\mathfrak{h}_R$ and view the Coxeter complex $\mathcal{C}$ as the poset of real faces of this arrangement, the set $\Delta$ being also embedded into $\mathfrak{h}_R^\ast$. We will also use the geometry of the quotient $W\setminus \mathfrak{h}_C$ studied in the previous sections.

Let $G$ be a split reductive algebraic group over $\mathbb{K}$ with Lie algebra $\mathfrak{g}$ and $T \subset B \subset G$ be the maximal torus and the Borel subalgebra with Lie algebras $\mathfrak{h}$ and $\mathfrak{b}$ respectively. A parabolic subgroup $P \subset G$ (resp. parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$) is called standard, if $P \supset B$
As well known, standard parabolics correspond to subsets \( I \subset \Delta_{\text{sim}} \). We denote
\[
P_I = G_I U_I, \quad p_I = g_I \oplus u_I, \quad \mathfrak{p}_I = \text{Lie}(P_I), \quad \mathfrak{g}_I = \text{Lie}(G_I), \quad \mathfrak{u}_I = \text{Lie}(U_I)
\]
the standard parabolic subgroup corresponding to \( I \) with its standard Levi subgroup \( G_I \) and unipotent radical \( U_I \), as well as the corresponding standard parabolic subalgebra \( \mathfrak{p}_I \) with its standard Levi \( \mathfrak{g}_I \) and nilpotent radical \( \mathfrak{u}_I \). Thus \( \mathfrak{p}_I \) is generated by \( \mathfrak{b} \) and the root generators \( e_\alpha, \alpha \in \Delta \).

A parabolic subgroup \( P \subset G \), resp. subalgebra \( \mathfrak{p} \subset \mathfrak{g} \) will be called semi-standard, if \( P \supset T \) resp. \( \mathfrak{p} \supset \mathfrak{h} \). Again, the following is well known.

**Proposition 5.1.**  
(a) Semi-standard parabolics are in bijection with faces \( C \in \mathcal{C} \) of the Coxeter complex. Given \( C \in \mathcal{C} \), the corresponding semi-standard parabolic subgroup and subalgebra with its Levi and uni/nilpotent radical
\[
P_C = G_C U_C, \quad \mathfrak{p}_C = \mathfrak{g}_C \oplus \mathfrak{u}_C, \quad \mathfrak{p}_C = \text{Lie}(P_C), \quad \mathfrak{g}_C = \text{Lie}(G_C), \quad \mathfrak{u}_C = \text{Lie}(U_C)
\]
are characterized by the following conditions:

(SSP) The roots of \( \mathfrak{p}_C \) are those \( \alpha \in \Delta \) for which \( \alpha|_C > 0 \). Among these, the roots of \( \mathfrak{g}_C \) are the \( \alpha \) satisfying \( \alpha|_C = 0 \) and the roots of \( \mathfrak{u}_C \) are the \( \alpha \) satisfying \( \alpha|_C > 0 \).

(b) Two semi-standard parabolics are conjugate with respect to \( G \), if and only if they are conjugate with respect to the normalizer \( N(T) \subset G \), and such conjugation corresponds to the action of \( W = N(T)/T \) on \( \mathcal{C} \). \( \square \)

We denote \( F_I = G/P_I \) the flag space associated to \( I \subset \Delta_{\text{sim}} \). We consider it as an algebraic variety over \( K \). As well known \( G/P_I \) can be seen parametrizing parabolic subgroups \( P \subset G \) conjugate to \( P_I \) as well as parabolic subalgebras \( \mathfrak{p} \subset \mathfrak{g} \) conjugate to \( \mathfrak{p}_I \). We refer to such parabolics as parabolics of type \( I \). If \( I_1 \subset I_2 \), then \( P_{I_1} \subset P_{I_2} \) so we have the projection
\[
q_{I_1,I_2} : F_{I_1} \longrightarrow F_{I_2}. \tag{5.2}
\]

By a Bruhat orbit of type \((I, J)\) we will mean a \( G \)-orbit \( O \) on \( F_I \times F_J \). Such an \( O \) is a quasi-projective variety over \( K \) which we think of as consisting of pairs of parabolics \((P, P')\). The parabolic Bruhat decomposition can be formulated as follows.

**Proposition 5.3.** Let \( I, J \subset \Delta_{\text{sim}} \). We have a bijection
\[
G\backslash(F_I \times F_J) \cong W\backslash((W/W_I) \times (W/W_J)) = \Xi(I, J) \subset \Xi = W\backslash(\mathcal{C} \times \mathcal{C}).
\]
More precisely, each \( G \)-orbit on \( F_I \times F_J \) contains a pair of semi-standard parabolics \((P_C, P_D)\) for some pair of faces \((C, D) \in \mathcal{C} \times \mathcal{C} \) defined uniquely up to a simultaneous \( W \)-action. \( \square \)
Proof: The standard formulation, see, e.g., [12] §14.16 or [13] Ch. 4, §2.5, Rem. 2, is in terms of an identification of the sets of double cosets

\[ P_I \backslash G / P_J \simeq W_I \backslash W / W_J. \]

To get the statement in our form, recall that for any group \( H \) and subgroups \( K, L \) we have a bijection

\[ H \backslash ((H/K) \times (H/L)) \xrightarrow{\sim} K \backslash H / L, \quad H(h_1 K, h_2 L) \mapsto K(h_1^{-1} h_2) L. \]

The remaining details are left to the reader. \( \square \)

Thus the 2-sided Coxeter complex \( \Xi \) parametrizes Bruhat orbits in all the \( F_I \times F_J \). For \( m \in \Xi(I, J) \) we denote

\[ F_I \overset{\nu_m}{\leftarrow} O_m \overset{\nu_m}{\rightarrow} F_J \]

the corresponding Bruhat orbit with its projections to the factors. According to the above Proposition this diagram may be identified with

\[ G / P_C \overset{\nu_m}{\leftarrow} G / (P_C \cap P_D) \overset{\nu_m}{\rightarrow} G / P_D \]

B. Bruhat order on \( \Xi(I, J) \). The identification \( \Xi(I, J) \simeq G / (F_I \times F_J) \) makes manifest the Bruhat order on \( \Xi(I, J) \), which we denote \( \succeq \). It reflects the relation of inclusion of orbit closures. That is, \( m \succeq n \) iff \( O_m \subset \overline{O_n} \). With respect to the identification \( \Xi(I, J) \simeq W_I \backslash W / W_J \), this order is induced by the two-sided Bruhat order on \( W \). This latter identification implies the following.

Proposition 5.4. The contraction maps (1.1) are monotone with respect to the Bruhat orders \( \succeq \) in their source and target.

Proof: For example, \( \varphi_{(I_1, I_2)J} : \Xi(I_1, J) \to \Xi(I_2, J), I_1 \subset I_2 \), is the map

\[ W_{I_1} \backslash W / W_J \to W_{I_2} \backslash W / W_J \]

induced by the inclusion \( W_{I_1} \subset W_{I_2} \). Since the orders \( \succeq \) on the source and target of this map are induced by the same Bruhat order on \( W \), the map is monotone. \( \square \)

C. Structure of the orbits. As for any real hyperplane arrangement, the set \( C \) of faces of \( \mathcal{H} \) carries the composition, or Tits product operation \( \circ \), see [43] 2.30, [10] or [33] §2B. For two faces \( C, D \) the new face \( C \circ D \) can be described, geometrically, as follows. Choose any \( c \in C, d \in D \) and draw a straight line interval \( [c, d] \subset \mathfrak{h}_R \). Then \( C \circ D \) is the face containing the points \( c' \in [c, d] \) which are very close to \( c \) but not equal to \( c \). By construction, \( C \subseteq C \circ D \). The operation \( \circ \) is not commutative.
Let now \( m \in \Xi(I, J) \). Representing \( m \) as an orbit \( m = W(C, D) \), we have two \( W \)-orbits \( W(C \circ D) \) and \( W(D \circ C) \) associated to \( m \). Define two subsets \( \text{Hor}(m), \text{Ver}(m) \subset \Delta_{\text{sim}} \) called the \textit{horizontal} and \textit{vertical readings} of \( m \) by the conditions
\[
W(C \circ D) \ni K_{\text{Hor}(m)}, \quad W(D \circ C) \ni K_{\text{Ver}(m)}
\]
(see 1.A for the notation \( K_I \)). Note that \( C \leq C \circ D \) implies \( \text{Hor}(m) \subset I \) and \( D \leq D \circ C \) implies \( \text{Ver}(m) \subset J \).

**Proposition 5.5.** Let \( m \in \Xi(I, J) \) and \( O_m \subset F_I \times F_J \) be the corresponding Bruhat orbit.

(a) For any pair of parabolic subgroups \((P, P') \in O_m\) with unipotent radicals \( U, U' \), the subgroup
\[
P \circ P' := (P \cap P')U \subset P
\]
is a parabolic subgroup in \( G \) of type \( \text{Hor}(m) \), and \( P' \circ P = (P \cap P')U' \subset P' \) is a parabolic subgroup in \( G \) of type \( \text{Ver}(m) \).

(b) Associating to \((P, P')\) the subgroups \( P \circ P' \) and \( P' \circ P \) defines projections \( r'_m, r''_m \) in the commutative diagram
\[
\begin{array}{ccc}
F_{\text{Hor}(m)} & \xrightarrow{r'_m} & O_m & \xrightarrow{r''_m} & F_{\text{Ver}(m)} \\
q_{\text{Hor}(m), I} \downarrow & & \downarrow & & \downarrow q_{\text{Ver}(m), J} \\
F_I & & F_J & & F_J
\end{array}
\]

(c) The fibers of \( r'_m, r''_m \) are affine spaces.

**Proof:** (a) By Proposition 5.3, we can assume \( P \) and \( P' \) semi-standard: \( P = P_C, P' = P_D \) for some faces \( C, D \in C \). We claim that
\[
P_C \circ P_D = P_{C \circ D}.
\]
It suffices to prove the equality of the Lie algebras
\[
(p_C \cap p_D) \oplus u_C = p_{C \circ D}.
\]

For this, we recall the algebraic definition of \( C \circ D \), see [10] or [33] §2C. That is, consider the set \( \{0, +, -\} \) with the partial order \( 0 < + \) and \( 0 < - \) while \( + \) and \( - \) are non-comparable. For any \( \alpha \in \Delta \) and any \( C \in C \) we have the sign \( \text{sgn}(\alpha|C) \in \{0, +, -\} \). Then
\[
\text{sgn}(\alpha|C) = \begin{cases} 
\text{sgn}(\alpha|D), & \text{if } \text{sgn}(\alpha|C) < \text{sgn}(\alpha|D), \\
\text{sgn}(\alpha|C), & \text{otherwise}.
\end{cases}
\]
This means :
\[
\alpha|_{C\circ D} = 0 \iff \alpha|_C = \alpha|_D = 0, \\
\alpha|_{C\circ D} > 0 \iff \left( (\alpha|_D > 0, \alpha|_C = 0) \text{ or } \alpha|_C > 0 \right),
\]

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which, in view of Condition (SSP) of Proposition 5.1(a), gives precisely the roots of $(p_C \cap p_D) \oplus u_C$.

This proves part (a). Part (b) is now clear. To see (c), we again look at the semi-standard representatives above. In this case $O_m = G/(P_C \cap P_D)$, as $P_C \cap P_D$ is the stabilizer in $G$ of the point $(P_C, P_D) \in F_I \times F_J$. Now, the subgroup $P_C \cap P_D \subset G$ may not be parabolic but has the same Levi quotient as $(P_C \cap P_D)U_C$. Therefore the fibers of $r'_m$ are isomorphic to

$$(P_C \cap P_D)U_C / (P_C \cap P_D) = U_C / (U_C \cap P_C \cap P_D) = U_C / (U_C \cap P_D)$$

which is the factor of a unipotent group by a unipotent subgroup so it is isomorphic to an affine space. Similarly for $r''_m$.

The fundamental diagram 5.5 (b) may be rewritten as

$$G/P_{C \cap D} \xleftarrow{r'_m} G/(P_C \cap P_D) \xleftarrow{r''_m} G/P_{D \cap C}$$

$$\downarrow q_{Hor(m), J} \quad \downarrow p_m \quad \downarrow q'_{Ver(m), J}$$

$$G/P_C \quad \xrightarrow{p'_m} \quad G/P_{C \cap D} \quad \xrightarrow{p''_m} \quad G/P_D.$$

D. The diagram of Bruhat orbits. Let $m, n \in \Xi$ and $m \geq n$. In particular, if $m \in \Xi(I_1, J_1)$ and $n \in \Xi(I_2, J_2)$, then $I_1 \subset I_2$ and $J_1 \subset J_2$, so we have the projection

$$q_{I_1, I_2} \times q_{J_1, J_2} : F_{I_1} \times F_{J_1} \longrightarrow F_{I_2} \times F_{J_2}.$$  

**Proposition 5.6.** (a) If $m \geq n$, then $q_{I_1, I_2} \times q_{J_1, J_2}$ takes $O_m$ to $O_n$, so we have a projection $p_{m, n} : O_m \rightarrow O_n$.

(b) The projections $p_{m, n}$, $m \geq n$, are transitive, so they form a contravariant functor from $(\Xi, \leq)$ to the category of algebraic varieties over $\mathbb{K}$.

**Proof:** (a) If $m \geq n$, then we can represent $m = W(A, B)$, $n = W(C, D)$, where $A, B, C, D \in C$ are such that $A \geq C$ and $B \geq D$. This means that $P_A \subset P_C$ and $P_B \subset P_D$. But the first inclusion means that $P_C$, considered as a point of $F_{I_2}$, is the image of $P_A$, considered as a point of $F_{I_1}$, under $q_{I_1, I_2}$. Similarly for $P_B$ and $P_D$. This shows that one point of $O_m$, namely $(P_A, P_B) \in F_{I_1} \times F_{J_1}$, is mapped into a point of $O_n$, namely $(P_C, P_D) \in F_{I_2} \times F_{J_2}$. Since both $O_m$ and $O_n$ are $G$-orbits and the projection in question is $G$-equivariant, we conclude that $O_m$ is mapped onto $O_n$, in a surjective way, thus proving (a). Now part (b) is obvious because of the transitiveness of the projections $q$ in (5.2) for any three subsets $I_1 \subset I_2 \subset I_3$.  

E. Maps of orbits and maps of flag varieties.

**Proposition 5.7.** (a1) If $m \geq n$, then $\text{Ver}(m) \subset \text{Ver}(n)$, and we have a commutative diagram

$$\begin{align*}
O_m \xrightarrow{p_{m, n}} O_n \\
\downarrow r''_m \quad \downarrow r''_n \\
F_{\text{Ver}(m)} \xrightarrow{q_{\text{Ver}(m), \text{Ver}(n)}} F_{\text{Ver}(n)}
\end{align*}$$
(a2) If, moreover, \( m \gg n \) is anodyne, then \( \text{Ver}(m) = \text{Ver}(n) \).

(b1) If \( m \gg n \), then \( \text{Hor}(m) \subset \text{Hor}(n) \), and we have a commutative diagram

\[
\begin{array}{ccc}
O_m & \xrightarrow{p_{m,n}} & O_n \\
\downarrow r'_m & & \downarrow r'_n \\
F_{\text{Hor}(m)} & \xrightarrow{q_{\text{Hor}(m),\text{Hor}(n)}} & F_{\text{Hor}(n)}
\end{array}
\]

(b2) If, moreover, \( m \gg n \) is anodyne, then \( \text{Hor}(m) = \text{Hor}(n) \).

Proof: It suffices to prove (a1-2), the other two statements being similar. Suppose \( m \gg n \). Then we can represent \( m = W(A,C) \), \( n = W(B,C) \) with \( A, B, C \in \mathcal{C} \) such that \( A \geq B \). Now, the Tits product \( \circ \) is monotone in the second argument, see [33] Prop. 2.7(a). Therefore \( A \geq B \) implies \( C \circ A \geq C \circ B \), which means \( \text{Ver}(m) \subset \text{Ver}(m) \). The commutative diagram follows directly from the definitions of the maps, thus proving (a1).

To prove (a2), introduce the following notation. For any two flats \( M, N \subset \mathfrak{h}_R \) of \( \mathcal{H} \) let \( M \circ N \) be the minimal flat of \( \mathcal{H} \) containing them both. Suppose now that \( m \gg n \) is anodyne. This means that we can represent \( m = W(A,C) \), \( n = W(B,C) \) with \( A, B, C \in \mathcal{C} \) such that \( A \geq B \) and the product cells \( iA + C, iB + C \subset \mathfrak{h}_C \) lie in the same stratum of \( S_{\mathfrak{h}_C} \), i.e., in the generic part of the complexification \( L_C \) of the same real flat of \( \mathcal{H} \). This last condition means that

\[
\text{Lin}_R(A) \cup \text{Lin}_R(C) = L = \text{Lin}_R(B) \cup \text{Lin}_R(C).
\]

As before, we have \( C \circ A \geq C \circ B \). Recall now that \( \text{Lin}_R(C \circ A) = \text{Lin}_R(C) \cup \text{Lin}_R(A) \), see [33] Prop. 2.7(b), and similarly for \( C \circ B \). This means that \( C \circ A \) and \( C \circ B \) have the same linear envelope and thus \( C \circ A = C \circ B \). This implies that \( \text{Ver}(m) = \text{Ver}(n) \), proving (a2).

F. Maps of orbits and correspondences between flag varieties. For future use, we record a companion result to Proposition 5.7, dealing with the other type of projections. Namely, if \( m \gg n \), then, in general, \( \text{Hor}(m) \not\subset \text{Hor}(n) \), so \( F_{\text{Hor}(m)} \) and \( F_{\text{Hor}(n)} \) are not connected by a map. However, they are connected by a correspondence, as the following proposition shows.

Proposition 5.8. (a) If \( m \gg n \), then the image under \( p_{m,n} \) of any fiber of \( r'_m \) is a union of fibers of \( r'_n \). Therefore we have a commutative diagram with a Cartesian square

\[
\begin{array}{ccc}
O_m & \xrightarrow{p_{m,n}} & O_n \\
\downarrow r'_m & \downarrow s & \downarrow r'_n \\
F_{\text{Hor}(m)} & \xleftarrow{p_1} & Z & \xrightarrow{p_2} & F_{\text{Hor}(n)}
\end{array}
\]

where

\[
Z = \{(x_1, x_2) \in F_{\text{Hor}(m)} \times F_{\text{Hor}(n)} | (r'_m)^{-1}(x_1) \supset (r'_n)^{-1}(x_2)\},
\]

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\( \rho_1 \) and \( \rho_2 \) are the projections to the first and second factor, and

\[
s(o) = (r'_m(o), r'_n(p_{m,n}(o)), \quad o \in O_m.
\]

Further, \( Z \) is a single \( G \)-orbit. More precisely, if \( m = W(A, C) \), \( n = W(B, C) \) with \( A, B, C \in \mathcal{C} \) and \( A \supset B \), then \( Z = O_x \), where \( x = W(A \circ C, B \circ C) \).

(b) Likewise, if \( m \supseteq n \), then the image under \( p_{m,n} \) of any fiber of \( r'_m \) is a union of fibers of \( r'_n \), and we have a commutative diagram similar to (a).

Proof: We prove (a), since (b) is similar. Consider first the following general situation. Let \( K_1 \supset H_1 \subset H_2 \subset K_2 \) be subgroups of \( G \), so we have a diagram of projections of homogeneous spaces

\[
\begin{array}{ccc}
G/H_1 & \xrightarrow{p} & G/H_2 \\
r_1 \downarrow & & \downarrow r_2 \\
G/K_1 & \rightarrow & G/K_2.
\end{array}
\]

The condition that \( p(r_1^{-1}(x_1)) \), for any coset \( x_1 \in G/K_1 \), is a union of fibers \( r_2^{-1}(x_2) \), is, by homogeneity, equivalent to the condition that it is such a union for single \( x_1 \), e.g., for \( x_1 \) being the coset \( K_1 \). In this case \( p(r_1^{-1}(x_1)) \subset G/H_2 \) is the set of left cosets by \( H_2 \) contained in the right \( H_2 \)-invariant subset \( K_1 H_2 \subset G \). The condition that it is a union of some \( r_2^{-1}(x_2) \) is then that the set \( K_1 H_2 \) is a union of cosets by \( K_2 \), i.e., it is invariant under right multiplication with \( K_2 \). This can be expressed as \( K_1 H_2 = K_1 K_2 \).

We now apply this to our situation as follows. As \( m \supseteq n \), we can represent \( m = W(A, C), n = W(B, C) \) for \( A, B, C \in \mathcal{C} \) with \( A \supset B \). Let \( P_A, P_B, P_C \) be the corresponding semi-standard parabolics, with unipotent radicals \( U_A, U_B, U_C \). Since \( A \supset B \), we have \( P_A \subset P_B \), and the unipotent radicals are included in the opposite direction: \( U_B \subset U_A \). Our original situation is then a particular case of (5.9) corresponding to

\[
K_1 = (P_A \cap P_C)U_A = U_A(P_A \cap P_C),
\]

\[
H_1 = P_A \cap P_C, \quad H_2 = P_B \cap P_C,
\]

\[
K_2 = (P_B \cap P_C)U_B = U_B(P_B \cap P_C).
\]

So \( K_1 H_2 = U_A(P_B \cap P_C) \), while

\[
K_1 K_2 = (U_A(P_A \cap P_C))(U_B(P_B \cap P_C)) = U_A(P_B \cap P_C) U_B = U_A U_B (P_B \cap P_C) = U_A(P_B \cap P_C),
\]

which is the same. This shows the existence of the diagram, in particular, of the correspondence \( Z \).

Further, the morphism \( s : O_m \rightarrow Z \) is surjective, since \( p_{m,n} \) and \( r'_n \) are surjective. Therefore \( Z \) is a single \( G \)-orbit. To show that it is exactly the orbit \( O_x \) as claimed, it suffices to find the image of the point \( (P_A, P_C) \). Now, \( p_{m,n} \) takes \( P_A \) to \( P_B \), so \( s(P_A, P_C) = (P_A \circ C, P_B \circ C) \), whence the statement.
Example: Associated parabolics and intertwiner correspondences. For future reference we recall some elementary instances of the above constructions.

Two faces $C, D \in \mathcal{C}$ will be called associated, if $\text{Lin}_\mathbb{R}(C) = \text{Lin}_\mathbb{R}(D)$. Denote this latter space $L$ and put $m = \dim(L) = \dim(C) = \dim(D)$. We further call two associated faces $C$ and $D$ adjacent, if they are separated by an $(m-1)$-dimensional face $\Pi$, that is $C > \Pi < D$. For adjacent $C$ and $D$ we put

$$\Delta(C, D) = \{ \alpha \in \Delta : \alpha|_C > 0, \alpha|_D < 0 \}, \quad \delta(C, D) = |\Delta(C, D)|.$$  

A gallery joining two associated faces $C, D$ is a sequence of $m$-dimensional faces $(C_0 = C, C_1, \cdots, C_l = D)$ all lying in $L$ such that for each $i = 1, \cdots, l$, the faces $C_{i-1}$ and $C_i$ are adjacent: $C_{i-1} \supset \Pi_i \supset C_i$, $\dim(\Pi_i) = m - 1$. The number $l$ is called the length of the gallery. A minimal gallery is a gallery of minimal possible length. The length of a minimal gallery is called the face distance between associated faces.

Two semi-standard parabolics $P_C, P_D, C, D \in \mathcal{C}$, are called associated, resp. adjacent, if $C$ and $D$ are associated, resp. adjacent.

If $C$ and $D$ are associated. Note that in this case $C \circ D = C$ and $D \circ C = D$. Let $I(C), I(D) \subset \Delta_{\text{sim}}$ be the types of $P_C, P_D$. Putting $\mathfrak{m} = W(C, D) \in \Xi(I(C), I(D))$, we find that $\text{Hor}(\mathfrak{m}) = I(C)$ and $\text{Ver}(\mathfrak{m}) = I(D)$.

Let us denote for simplicity $F_C = F_{I(C)}$ (the space of parabolics conjugate to $P_C$) and similarly $F_D = F_{I(D)}$. Note that $\dim(F_C) = \dim(F_D)$, since $P_C$ and $P_D$ have the same Levi. Denote $O_{C,D} = O_{\mathfrak{m}} \subset F_C \times F_D$ the orbit corresponding to $\mathfrak{m}$. Proposition 5.5 shows that in the diagram

$$F_C \overset{\nu_{C,D}}{\longrightarrow} O_{C,D} \overset{\nu_{C,D}^\prime}{\longrightarrow} F_D$$

the fibers of both projections are affine spaces (of the same dimension). This diagram is the classical intertwiner correspondence used to define principal series intertwiners (and, more generally, intertwiners between parabolically induced representations).

**Proposition 5.12.** Let $C, D$ be associated faces. Then:

(a) The dimension of the fibers of $\nu_{C,D}$ and $\nu_{C,D}^\prime$ is equal to $\delta(C, D)$.

(b) If $(C_0 = C, C_1, \cdots, C_l = D)$ is a minimal gallery joining $C$ and $D$, then the correspondence $O_{C,D}$ is the fiber product of the correspondences

$$F_{C_{i-1}} \overset{\nu_{C_{i-1},C_i}}{\longrightarrow} O_{C_{i-1},C_i} \overset{\nu_{C_{i-1},C_i}^\prime}{\longrightarrow} F_{C_i}, \quad i = 1, \cdots, l.$$ 

In particular, $\delta(C, D) = \sum_{i=1}^l \delta(C_{i-1}, C_i)$.

(c) Consider chambers (faces of maximal dimension) $\tilde{C}, \tilde{D}$ of $\mathcal{H}$ such that $\tilde{C} \supset C$ and $\tilde{D} \supset D$. Then, the minimal face distance between such $\tilde{C}$ and $\tilde{D}$ is $\delta(C, D)$. If $\tilde{C}, \tilde{D}$ are
chambers with this face distance, then in the following diagram of projections the squares are Cartesian:

\[
\begin{array}{ccc}
F_C & \xleftarrow{\nu'_{C,D}} & O_{C,D} & \xrightarrow{\nu_{C,D}} & F_D \\
\downarrow & & \downarrow & & \downarrow \\
F_C & \xleftarrow{\nu'_{C,D}} & O_{C,D} & \xrightarrow{\nu_{C,D}} & F_D.
\end{array}
\]

Proof: This is classical material. Let us only comment on part (c). Note that \(F_C = F_D\) is the full flag space \(G/B\), so the orbit \(O_{C,D}\) corresponds to some element \(w\) of the Weyl group \(W\). The condition that, say, the left square in the diagram is Cartesian is equivalent to the property that the fibers of \(p'_{C,D}\) project isomorphically to the fibers of \(p_{C,D}\). Given that these fibers are affine spaces and given the surjectivity of the projections, such a property reduces to the equality of the dimensions of these affine spaces, i.e., to the equality \(l(w) = \delta(C, D)\). Here \(l(w)\) is the length of \(w\), i.e., the face distance between \(C\) and \(D\).

\[\square\]

6 Bruhat orbits as a motivic Bruhat cosheaf

The diagram \((O_m, p_{m,n})\) can be seen as a cellular cosheaf on \((W \setminus h, S^{(2)})\) with values in the category of algebraic varieties over \(K\). This diagram will be the source of several examples of mixed Bruhat sheaves, obtained by applying various natural constructions (such as, e.g., passing to the spaces of functions). In this section we highlight the geometric properties of \((O_m, p_{m,n})\) which will imply the axioms of a mixed Bruhat sheaf for such constructions.

A. An analog of (MBS3).

**Proposition 6.1.** If \(m \triangleright n\) is an anodyne inequality, then the fibers of \(p_{m,n}\) are affine spaces.

**Proof:** Any anodyne inequality \(\triangleright\) factors into a composition of an anodyne \(\triangleright'\) and an anodyne \(\triangleright''\). So it suffices to prove the statement under additional assumption that \(m \triangleright' n\) or \(m \triangleright'' n\). Suppose \(m \triangleright' n\) is anodyne. Then in the square of Proposition 5.7(a) the lower horizontal arrow is the identity, and the fibers of the vertical arrows are affine spaces by Proposition 5.5(c). This means that taking a point \(o \in O_m\), the Levi quotients of the \(G\)-stabilizers of \(o\) and \(p_{m,n}(o)\) will be the same, so the fibers of \(p_{m,n}\) are affine spaces as well. The case when \(m \triangleright'' n\) is anodyne is similar. \[\square\]

**Remark 6.2.** Proposition 6.1 implies that after passing to the category \(D_M\) of Voevodsky motives (where \(A^1\)-homotopy equivalences become isomorphisms, see [5]), we get an \(D_M\)-valued cosheaf on \(W \setminus h\) that is \(S^{(0)}\)-constructible.

\[30\]
B. Fiber products of orbits. An analog of (MBS2). Let \( m' \geq n' \leq n \) be elements of \( \Xi \), so we have the projections
\[
\begin{array}{c}
O_{n'} \\
\downarrow_{p_{n',n}} \\
o_{n'} \\
\end{array}
\rightarrow
\begin{array}{c}
O_{m'} \\
\downarrow_{p_{m',n}} \\
o_{m'} \\
\end{array}
\rightarrow
\begin{array}{c}
o_{n}.
\end{array}
\]

The following property is the geometric analog of the condition (MBS2) for mixed Bruhat sheaves. It is also analogous to Proposition 1.7.

**Proposition 6.3.** For any \( m' \geq n' \leq n \) we have the following decomposition into the union of orbits:
\[
O_{m'} \times_{O_n} O_{n'} \simeq \bigcup_{m \in \text{Sup}(m',n)} O_m.
\]

**Proof:** The assumption \( m' \geq n' \leq n \) implies that there are \( I_1 \subset I_2 \) and \( J_1 \subset J_2 \) such that
\[
m' \in \Xi(I_1, J_2), \quad n' \in \Xi(I_2, J_1), \quad n \in \Xi(I_2, J_2).
\]

Any \( m \in \text{Sup}(m', n) \) must then lie in \( \Xi(I_1, J_1) \). Note that
\[
O_{m'} \times_{O_n} O_{n'} = O_{m'} \times_{F_{I_2} \times F_{J_2}} O_{n'}.
\]

Note further that the square
\[
\begin{array}{ccc}
F_{I_1} \times F_{J_1} & \longrightarrow & F_{I_2} \times F_{J_1} \\
\downarrow & & \downarrow \\
F_{I_1} \times F_{J_2} & \longrightarrow & F_{I_2} \times F_{J_2}
\end{array}
\]
is Cartesian, being the external Cartesian product of two arrows
\[
\{ F_{I_1} \rightarrow F_{I_2} \} \times \{ F_{J_1} \rightarrow F_{J_2} \}.
\]
Therefore \( O_{m'} \times_{O_n} O_{n'} \) is contained in \( F_{I_1} \times F_{J_1} \) and is the union of those orbits \( O_m \) that project to \( O_{m'} \) and \( O_{n'} \), i.e., of the \( O_m \) with \( m \in \text{Sup}(m', n) \) as claimed. \( \square \)

7 Functions on \( \mathbb{F}_q \)-points.

A. Appearance as a bicube. In this section we present the simplest example of a mixed Bruhat sheaf encoding the algebra behind parabolic induction and restriction. Like other examples, it appears most immediately (and is well known) in the form of a bicube, see §2C.

We specialize the situation of §5 to the case when \( K = \mathbb{F}_q \) is a finite field. This field has to be distinguished from the “coefficient” field \( k \) as in §2A. In this section we assume that \( k \)
is algebraically closed of characteristic 0. For a variety $X/\mathbb{F}_q$ we denote $\text{Fun}(X)$ the $k$-vector space of functions $X(\mathbb{F}_q) \to k$. If $f : X \to Y$ is a morphism of varieties over $\mathbb{F}_q$, we denote $f^* : \text{Fun}(Y) \to \text{Fun}(X)$, $f_* : \text{Fun}(X) \to \text{Fun}(Y)$ the inverse image (pullback) and the direct image (sum over the fibers) of functions on $\mathbb{F}_q$-points.

For $I \subset \Delta_{\text{sim}}$ we have the flag space $F_I = G/P_I$. For $I \subset J \subset \Delta_{\text{sim}}$ we have the projection $q_{IJ} : F_I \to F_J$, cf. (5.2). We define a $\Delta_{\text{sim}}$-bicube $Q$ by

$$Q_I = \text{Fun}(F_I), \ v_{IJ} = (q_{IJ})_* : \text{Fun}(F_I) \to \text{Fun}(F_J), \ u_{IJ} = q_{IJ}^* : \text{Fun}(F_J) \to \text{Fun}(F_I).$$

Thus $Q_I = \text{Ind}^{G(\mathbb{F}_q)}_P k$ is the simplest parabolically induced representation. We now proceed to upgrade this bicube to a mixed Bruhat sheaf.

**B. Definition of the diagram $E_q$.** For $m \in \Xi$ we have the orbit $O_m$ and define the $k$-vector space $\tilde{E}(m) = \text{Fun}(O_m)$. If $m \gg n$, we define

$$\tilde{c}_{m,n}' = (p_{m,n})_* : \tilde{E}(m) = \text{Fun}(O_m) \longrightarrow \text{Fun}(O_n) = \tilde{E}(n).$$

If $m \gg n$, we define

$$\tilde{c}_{m,n}'' = p_{m,n}^* : \tilde{E}(n) = \text{Fun}(O_n) \longrightarrow \text{Fun}(O_m) = \tilde{E}(m).$$

**Proposition 7.1.** The diagram $\tilde{E} = (\tilde{E}(m), \tilde{c}_{m,n}', \tilde{c}_{m,n}'')$ satisfies the conditions (MBS1-2) of Definition 2.1.

**Proof:** (MBS1), i.e., the transitivity of the $\tilde{c}'$ and the $\tilde{c}''$, follows from the transitivity of the $p_{m,n}$ and from the compatibility with the direct and inverse images with composition. The condition (MBS2) follows directly from Proposition 6.3. More precisely, that proposition implies that for any $m' \gg n' \ll n$ we have a Cartesian square of finite sets

$$\begin{array}{ccc}
O_m(\mathbb{F}_q) & \longrightarrow & O_{n'}(\mathbb{F}_q) \\
\downarrow & & \downarrow \\
O_{n'}(\mathbb{F}_q) & \longrightarrow & O_n(\mathbb{F}_q).
\end{array}$$

So $\tilde{c}_{m',n'}' \circ \tilde{c}_{m',n'}'' = p_{n,n'}^* (p_{m',n})_*$ is equal, by the base change formula, to the result of first pulling back to the disjoint union of the $O_m(\mathbb{F}_q)$ and then pushing forward to $O_{n'}(\mathbb{F}_q)$, which is precisely the right hand side of (MBS2). \[\square\]

However, the diagram $\tilde{E}$ does not satisfy (MBS3). So for each $m$ we define the subspace

$$E_q(m) = (r_m')^* \text{Fun}(F_{\text{Hor}(m)}) \subset \tilde{E}(m), \quad r_m' : O_m \longrightarrow F_{\text{Hor}(m)},$$

to consist of functions pulled back from $F_{\text{Hor}(m)}$.

**Theorem 7.2.** (a) The maps $\tilde{c}'$ and $\tilde{c}''$ of $\tilde{E}$ preserve the subspaces $E_q(m)$.

(b) The diagram $E_q = (E_q(m), \tilde{c}'_{m,n}, \tilde{c}''_{m,n})$ is a mixed Bruhat sheaf, i.e., it satisfies all three conditions (MBS1-3).
C. Proof of Theorem 7.2. We start with part (a). Look first at the map \( \partial''_{m,n} = p''_{m,n} \). In this case we have a commutative square of Proposition 5.7(b1) so \( p''_{m,n} \) takes functions pulled back by \( r'_n \) to functions pulled back by \( r'_m \).

Look now at the map \( \partial'_{m,n} = (p_{m,n})_* \). We then have the diagram of Proposition 5.7(b2). Using the base change for the Cartesian square in that diagram in part (c) of Proposition 5.12, show that the following classical fact.

Let \( m \geq n \) be anodyne. Then, by Proposition 5.7(b2), \( \text{Hor}(m) = \text{Hor}(n) \), so the square in part (b2) of that proposition becomes a triangle. This shows that \( \partial''_{m,n} = (p_{m,n})^* \) takes \( E_q(n) = (r')^* \text{Fun}(F_{\text{Hor}(m)}) \) to \( E_q(m) = (r'_m)^* \text{Fun}(F_{\text{Hor}(m)}) \) isomorphically.

Further, the conditions (MBS1-2) for \( E \) follow from the validity of these conditions for \( \tilde{E} \). Let us prove (MBS3).

Let \( m \geq n \) be anodyne. Then, by Proposition 5.7(b2), \( \text{Hor}(m) = \text{Hor}(n) \), so the square in part (b2) of that proposition becomes a triangle. This shows that \( \partial''_{m,n} = (p_{m,n})^* \) takes \( E_q(n) = (r')^* \text{Fun}(F_{\text{Hor}(m)}) \) to \( E_q(m) = (r'_m)^* \text{Fun}(F_{\text{Hor}(m)}) \) isomorphically.

Let now \( m \geq n \) be anodyne. Proposition 5.8(a) shows that the map \( \partial''_{m,n} : E_q(m) \to E_q(n) \) is isomorphic to the map \( \rho_2 \circ \rho^*_1 : \text{Fun}(F_{\text{Hor}(m)}) \to \text{Fun}(F_{\text{Hor}(n)}) \). Now, if \( m = W(A,C) \) and \( n = W(B,C) \) with \( A \geq B \), then the condition that \( m \geq n \) is anodyne means that, similarly to the proof of Proposition 5.7, we have

\[
\text{Lin}_R(A \circ C) = \text{Lin}_R(A) \cup \text{Lin}_R(C) = \text{Lin}_R(B) \cup \text{Lin}_R(C) = \text{Lin}_R(B \circ C).
\]

In other words, \( A \circ C \) and \( B \circ C \) are associated faces, see §5G. The last claim in Proposition 5.8(a) implies then that \( Z \) is a particular case of the intertwiner correspondence (5.11) for two associated parabolics. So the isomorphicity of \( \partial''_{m,n} \) in this case is a particular case of the following classical fact.

**Proposition 7.3.** Let \( P_C, P_D \) be two associated semi-standard parabolics. Then the intertwiner

\[
(p''_{C,D})^* \circ (p''_{C,D})^* : \text{Fun}(F_C) \to \text{Fun}(F_D)
\]

is an isomorphism.

**Proof:** For convenience of the reader we recall the argument by reduction to the simplest case. First, the Cartesian squares in the diagram in part (c) of Proposition 5.12, show that the intertwiner for \( \tilde{C}, \tilde{D} \) takes functions pulled back from \( F_C \) to functions pulled back from \( F_D \), so it is enough to prove the isomorphicity of such a Borel intertwiner. Next, in this case \( O_{\tilde{C}, \tilde{D}} \) corresponds to some element \( w \in W \), and by 5.12(b) it is enough to consider the case when \( w = s_\alpha \) is a simple reflection. In this case we have the \( \mathbb{P}^1 \)-fibration \( q_\alpha : G/B \to G/P_\alpha \) and the corresponding orbit consists of \( (x,y) \in (G/B) \times (G/B) \) such that \( q_\alpha(x) = q_\alpha(y) \) but \( x \neq y \). The isomorphicity of the intertwiner in this case reduces to the case of a single fiber of \( q_\alpha \), i.e., to the case of the correspondence

\[
\mathbb{P}^1 \leftarrow (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \text{diag.} \to \mathbb{P}^1,
\]

in which case the isomorphicity is obvious. \( \square \)
D. The case of “$\mathbb{F}_1$-points”. Let us indicate an even simpler example corresponding to the formal limit $q = 1$ when, instead of groups of $\mathbb{F}_q$-points, we consider the Weyl groups.

For $m \in \Xi(I, J)$ put $E_1(m) = \text{Fun}(m)$, where we consider $m$ as a subset (orbit) in $(W/W_J) \times (W/W_I)$. If $m \geq n$, we have a $W$-equivariant surjection $\pi_{m,n} : m \to n$, see (1.5). If $m \geq' n$, we define

$$\tilde{\varphi}_{m,n} = (\pi_{m,n})_* : E_1(m) = \text{Fun}(m) \longrightarrow \text{Fun}(n) = E_1(n).$$

If $m \geq'' n$, we define

$$\tilde{\varphi}_{m,n} = \pi_{m,n}^* : E_1(n) = \text{Fun}(n) \longrightarrow \text{Fun}(m) = E_1(m).$$

**Proposition 7.4.** The diagram $E_1 = (E_1(m), \tilde{\varphi}_', \tilde{\varphi}_''$) is a mixed Bruhat sheaf of type $g$.

**Proof:** (MBS1) follows from the transitivity of the $\pi_{m,n}$. The condition (MBS2) follows from base change and Proposition 1.7. Finally, (MBS3) follows from Proposition 1.10.

The mixed Bruhat sheaf $E_1$ consists of $W$-modules, so it gives a perverse sheaf $\mathcal{F}_1 \in \text{Perv}(W \backslash h)$ with $W$-action. In particular, for any irreducible $W$-module $V$ we have the mixed Bruhat sheaf and perverse sheaf formed by the vector spaces of multiplicities of $V$:

$$E_1(V) = (E_1 \otimes_k V)^W = (E_1^V(m) = (E_1(m) \otimes_k V)^W), \quad \mathcal{F}_1^V = (\mathcal{F}_1 \otimes_k V)^W.$$ 

If we choose a representative $(C, D) \in m \subset C \times C$, then we have the “parabolic” subgroup $W_{C,D} \subset W$, the stabilizer of the pair $(C, D)$. It is conjugate to the “standard” parabolic subgroups $W_{\text{Hor}(m)}$ as well as $W_{\text{Ver}(m)}$. The choice of $(C, D)$ allows us to identify

$$E_1^V(m) \simeq V^{W_{C,D}}$$

Thus we can say that the mixed Bruhat sheaf $E_1^V$ is formed by the spaces of invariants in $V$ with respect to all the parabolic subgroups in $W$.

To identify $\mathcal{F}_1$ and $\mathcal{F}_1^V$, consider the diagram of projections and open embeddings

(7.7)

$$\begin{array}{ccc}
\mathfrak{h}\text{reg}^\text{reg} & \overset{p\text{reg}}{\longrightarrow} & W\backslash \mathfrak{h}\text{reg} \\
\downarrow & & \downarrow \\
\mathfrak{h} & \underset{p}{\longrightarrow} & W\backslash \mathfrak{h}.
\end{array}$$

For any $W$-module $V$ we denote by $\mathcal{L}_V$ the corresponding local system on $W\backslash \mathfrak{h}\text{reg}$. As before, let $r = \dim_C \mathfrak{h}$.

**Proposition 7.8.** (a) The perverse sheaf $\mathcal{F}_1$ is identified with $p_*k_\mathfrak{h} [r]$, with the $W$-action being the natural $W$-action on the direct image twisted by the sign character. In particular, $\mathcal{F}_1$ reduces to a single sheaf in degree $(-r)$.

(b) For any irreducible $W$-module $V$ we have an identification

$$\mathcal{F}_1^V \simeq \check{R}^0 j_* (\mathcal{L}_V \otimes_{\text{sgn}} [r]) = j_* (\mathcal{L}_V \otimes_{\text{sgn}} [r]).$$

In particular, $\mathcal{F}_1^V$ reduces to a single sheaf in degree $(-r)$. 

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Here $j_*$ is the perverse extension of a local system, i.e., the image of the natural map $j_! \to j_*$ in the abelian category $\text{Perv}(W\backslash \mathfrak{h})$. In our case it coincides with $R^0 j_*$. 

**Proof:** (a) Since $k_h[r]$ is perverse, $p_* k_h[r]$ is perverse and lies in $\text{Perv}(W\backslash \mathfrak{h})$. Let $E'$ be the mixed Bruhat sheaf associated to it. We identify $E' = E_1 \otimes \text{sgn}$. By definition, the $E'(m)$ are found as the stalks of the terms of the Cousin complex of $p_* k_h[r]$, i.e., of the sheaves $j_! j^* p_* k_h[[I] - r]$, where $j_I : X_{I}^{\text{Im}} \hookrightarrow W\backslash \mathfrak{h}$ is the embedding of the imaginary stratum. These sheaves can be found “upstairs” in $\mathfrak{h}$, as we are dealing with a direct image from $\mathfrak{h}$. The preimage $p^{-1}(X_{I}^{\text{Im}})$ is the union of the tube cells $\mathfrak{h}_{z} + iC \ni h$ over all faces $C \in C$ in the $W$-orbit of $K_I$. For any such, tube cell, $j^! h$ is found by the local Poincaré duality. It is the constant sheaf in degree $\text{codim}_{\mathfrak{h}}(C)$ with stalk being $\text{or}(C)$, the $1$-dimensional (co-)orientation $k$-vector space of $C$. To descend back to $W\backslash \mathfrak{h}$, let $m \in \Xi(I, J)$, which we think of as a subset (orbit) in $C \times C$. Then by the above, $E'(m)$, which is the stalk of $j_! j^! p_* k_h[[I] - r]$ at the cell $U_m$, is found as

$$E'(m) \simeq \bigoplus_{(C, D) \in m} \text{or}(C) = \text{Fun}(m) \otimes \text{sgn} = E_1(m) \otimes \text{sgn}.$$ 

The remaining details are left to the reader.

(b) Obviously,

$$k_h = R^0 j_* j^* k_h = \tilde{j}_* \tilde{j}^* k_h.$$ 

This implies that

$$p_* k_h = R^0 j_* j^* p_* k_h = j_! p_* k_h,$$

and so the same relation will hold after we take the space of multiplicities of any irreducible $W$-module $V$. Now, from (a) it follows that $j^* F_1$ is the local system corresponding to the regular representation of $W$ but with the “external” $W$-action twisted by sign. Therefore $j^* F_1 \simeq \mathcal{L}_{V \otimes \text{sgn}}$, whence the statement. 

□

**E. The perverse sheaf $F_q$. The Hecke algebra picture.** Let us use the notation $F = G/B$ for the full flag space of $G$ and $G_q \supset B_q$ for the finite groups $G(\mathbb{F}_q) \supset B(\mathbb{F}_q)$.

Return to the mixed Bruhat sheaf $E_q$ from §B. Let $F_q \in \text{Perv}(W\backslash \mathfrak{h})$ be the corresponding perverse sheaf. As $E_q$ and $F_q$ consist of $G_q$-modules, for any $G_q$-module $V$ we have the mixed Bruhat sheaf $E_q^V$ and the perverse sheaf $F_q^V$ formed by the multiplicities of $V$, as in (7.5). As in (7.6), we can say that $E_q^V$ “consists of” spaces of invariants in $V$ under various parabolic subgroups in $G_q$. These subgroups are, however, not necessarily the standard ones $P_I(\mathbb{F}_q)$ so certain conjugations are involved.

Call a $G_q$-module $V$ special, if it appears in $\text{Fun}(F) = \text{Ind}_{B_q}^{G_q} k$. It is clear that $E_q^V$ and $F_q^V$ are nonzero only if $V$ is special. Let $H_q = H(G_q, B_q) \subset k[G_q]$ be the Hecke algebra formed by $B_q$-bi-invariant functions on $G_q$. Let $\mathfrak{S}$ be the set of irreducible special representations of $G_q$. As in the case of any finite group and subgroup, it is classical that $\mathfrak{S}$ in bijection with the set of irreducible $H_q$-modules. More precisely, we the decomposition

$$\text{Fun}(F) = \bigoplus_{V \in \mathfrak{S}} V \otimes R_V,$$

(7.9)
where \( R_V \) is the irreducible \( H_q \)-module corresponding to \( V \). It is also classical [29] that \( H_q \) can be given by the generators \( \sigma_\alpha, \alpha \in \Delta_{\text{sim}} \) subject to the braid relations and the quadratic relations

\[
(\sigma_\alpha + 1)(\sigma_\alpha - q) = 0.
\]

In particular, we have a morphism of algebras

\[ k[\text{Br}_q] \longrightarrow H_q, \]

and so any \( H_q \)-module \( R \) gives a local system \( \mathcal{L}_R \) on \( W \setminus \mathfrak{h}_{\text{reg}} \).

**Proposition 7.10.** Let \( V \in \mathfrak{g} \) be an irreducible special representation of \( G_q \). The perverse sheaf \( \mathcal{F}^V \) is isomorphic to \( j_*(\mathcal{L}_{R(V)} \otimes \mathcal{L}_{\text{sgn}}[r]) \), with \( j \) as in (7.7).

**Proof:** Consider first the local system of \( G_q \)-modules \( \mathcal{L}_{E_q} \) on \( W \setminus \mathfrak{h}_{\text{reg}} \) associated to the mixed Bruhat sheaf \( E_q \) by Proposition 2.2. Its stalk at the cell \( U_{W(C^+, 0)} \) is \( \text{Fun}(F) \), because the Bruhat orbit associated to \( W(C^+, 0) \) is \( F \). Further, the action of the braid group generators \( \sigma_\alpha \) on that stalk of \( \mathcal{L}_{E_q} \) are found as the standard intertwiners \( q_\alpha^* q_\alpha - 1 \), see the proof of Proposition 7.3. These operators give precisely the action of \( H_q \) on \( \text{Fun}(F) \) giving the decomposition (7.9). This means that

\[ \mathcal{L}_{E_q} \cong \bigoplus_{V \in \mathfrak{g}} V \otimes \mathcal{L}_{R(V)} \]

as a local system of \( G_q \)-modules. By Theorem 2.6(b) this means that

\[ j^* \mathcal{F}^V \cong \bigoplus_{V \in \mathfrak{g}} V \otimes \mathcal{L}_{R(V)} \otimes \mathcal{L}_{\text{sgn}}[r]. \]

as a local system of \( G_q \)-modules. Therefore for \( V \in \mathfrak{g} \),

\[ j^* \mathcal{F}^V \cong \mathcal{L}_{R(V)} \otimes \mathcal{L}_{\text{sgn}}[r], \]

a (shifted) irreducible local system. Therefore \( j_*(\mathcal{L}_{R(V)} \otimes \mathcal{L}_{\text{sgn}}[r]) \), an irreducible perverse sheaf, is contained in \( \mathcal{F}^V_q \). It remains to show that \( \mathcal{F}^V_q \) is an irreducible perverse sheaf. For this it is enough to show that \( E^V_q \) is an irreducible mixed Bruhat sheaf, which we do now.

Each nonzero element of any \( E^V_q(m) \) can be brought, by a chain of isomorphisms (anodyne \((\overline{\partial}')\pm 1, (\overline{\partial}^\prime)\pm 1\) which are part of the structure, to a (nonzero) element of \( E^V_q(n) \), where \( n \) is of the form \( W(K_I, 0) \). So \( E^V_q(n) = Q^V_I \) is the \( V \)-multiplicity space of the \( I \)th component of the bicube \( Q \) from \( \S A \). So it is enough to show that the bicube \( Q^V \) of \( V \)-multiplicities of \( Q \) is irreducible in the category of bicubes.

Now, \( Q_{\emptyset} = \text{Fun}(F) \), and for each \( I \subset \Delta_{\text{sim}} \) the structure map \( v_{\emptyset, I} : Q_{\emptyset} \rightarrow Q_I \) (direct image of functions) is surjective, while \( u_{\emptyset, I} : Q_I \rightarrow Q_{\emptyset} \) (inverse image of functions) is injective. These properties will still be true for the cube \( Q^V \), since taking invariants under a finite group is an exact functor. So if we have a nonzero element \( f \) in some \( Q^V_I \), then \( u_{\emptyset, I}(f) \) is a nonzero vector in \( Q^V_{\emptyset} = R_V \), an irreducible \( H_q \)-module. As mentioned earlier in the proof, the action of the generators of \( H_q \) is expressed as \( q_\alpha^* q_\alpha - 1 \), i.e., in terms of the bicube structure. Therefore the minimal sub-bicube in \( Q^V \) containing \( f \), contains the entire \( Q^V_{\emptyset} \), and since \( v_{\emptyset, I} : Q^V_{\emptyset} \rightarrow Q^V_I \) is surjective, it contains each \( Q^V_I \), so it coincides with \( Q^V \).
Remarks 7.11. (a) The components of the bicube $Q^V$ can be defined directly in terms of the $H_q$-module $R_V$ as “invariants” with respect to the Hecke algebra of the standard Levi $G_I$. This suggests that the full $E_q^V$ can also be defined purely in Hecke algebra terms. Since Hecke algebras make sense for more general Coxeter groups, our construction may generalize to such cases as well.

(b) Let us extend the correspondences $V \mapsto R_V, E_q^V, \mathcal{F}_q^V$ to arbitrary $G_q$-modules $V$ by additivity with respect to direct sums. Then (it is a general property of Hecke algebras) $R_V \simeq V^{B_q}$, the space of $B_q$-invariants. In particular, $R_{\text{Fun}(F)} = H_q$ as a module over itself. Therefore $E_q^{H_q}(\mathfrak{m})$ consists of functions on $O_\mathfrak{m}(\mathbb{F}_q)$ pulled from $B_q$-invariant functions on $F_{\text{Hor}(\mathfrak{m})}(\mathbb{F}_q)$, cf [28] §9. This space has the same dimension as $E_1(\mathfrak{m}) = \text{Fun}(\mathfrak{m})$, and so $\mathcal{F}_q^{H_q} = (\mathcal{F}_q)^{B_q}$ has the same numerical invariants as $\mathcal{F}_1$ (“$q$-deformation”). Note also that $E_q^{H_q}(\mathfrak{m})$ can be seen as a decategorified version of the parabolic category $\mathcal{O}$, cf. Remark 8.1(a) below.

Remark 7.12. We can go “downstairs”, i.e., decategorify one more time and consider the dimensions of the spaces $\text{Fun}(O_\mathfrak{m})$. For an algebraic subgroup $H \subset G$ defined over $\mathbb{Z}$ consider the number $n_{G/H}(q) = |(G/H)(\mathbb{F}_q)|$. Considered as a function of $q$, $n_{G/H}(q)$ is a polynomial.

Let $O_\mathfrak{m} \cong G/H$ be a Bruhat orbit. The dimension of $\text{Fun}(O_\mathfrak{m})$ is equal to $n_{O_\mathfrak{m}}(q) = n_{G/H}(q)$. There are two possibilities:

(a) $H$ is a parabolic, in this case we call $O_\mathfrak{m}$ compact (the space $O_\mathfrak{m}(\mathbb{C})$ is compact).

(b) $H$ is proper intersection of two parabolics, i.e., $H = P \cap P'$ and $H \neq P, H \neq P'$. In this case we call $O_\mathfrak{m}$ noncompact.

If $H$ is parabolic, say $H = P_I$, then the polynomial $n_{G/H}(q) \in \mathbb{Z}[q]$ is prime to $(q - 1)$ and $q$, since

$$n_{G/H}(1) = \text{Card}(W/W_I), \quad n_{G/H}(0) = 1,$$

the second equality following from the Bruhat decomposition.

Notice that we have a $q$-analogue of Proposition 1.10 (iii): namely, $\mathfrak{m} \geq \mathfrak{n}$ is anodyne iff there exists $i \in \mathbb{Z}_{\geq 0}$ such that

$$n_{O_\mathfrak{m}}(q) = q^n n_{O_\mathfrak{n}}(q).$$

This is true since the fibers of the projection $O_\mathfrak{m} \to O_\mathfrak{n}$ are affine spaces.

Define a polynomial $\tilde{n}_{O_\mathfrak{m}}(q)$ by

$$n_{O_\mathfrak{m}}(q) = q^i \tilde{n}_{O_\mathfrak{m}}(q), \quad (\tilde{n}_{O_\mathfrak{m}}(q), q) = 1.$$

Applying Proposition 5.5, we see that:

- For any $\mathfrak{m} \in \Xi$ the polynomial $n_{O_\mathfrak{m}}(q)$ is not divisible by $q - 1$.
- $O_\mathfrak{m}$ is compact iff $n_{O_\mathfrak{m}}(q)$ is not divisible by $q$.
- $\tilde{n}_{O_\mathfrak{m}}(q) = n_{F_{\text{Hor}(\mathfrak{m})}}(q) = n_{F_{\text{Ver}(\mathfrak{m})}}(q)$.

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8 Further directions and applications

Our approach can be pursued further in several directions. In this final section we sketch several such possibilities, leaving the details for future work. For simplicity we assume that \( k \) is algebraically closed of characteristic 0.

A. Example: braided Hopf algebras. Let \( \mathfrak{g} = \mathfrak{gl}_n \). In this case:

\( \Delta \) is a root system of type \( A_{n-1} \), so elements of \( \Delta_{\text{sim}} = \{ \alpha_1, \cdots, \alpha_{n-1} \} \) correspond to the “intervals” between consecutive integers \( \{1, \cdots, n\} \). Subsets \( I \subset \Delta_{\text{sim}} \) correspond to ordered partitions of \( n \), i.e., vectors \( \alpha = (\alpha_1, \cdots, \alpha_p) \) of positive integers summing up to \( n \). The space \( F_I = F_\alpha \) consists of flags (filtrations) of \( L = k^n \)

\[
V_\ast = (V_1 \subset \cdots \subset V_p = L), \quad \dim \gr^V L = \alpha_i.
\]

Let \( \Xi = \Xi_n \) be the 2-sided Coxeter complex for \( \mathfrak{gl}_n \). If \( I, J \) correspond to ordered partitions \( \alpha, \beta \) as above, then \( \Xi_n(I, J) \) is identified with the set of contingency matrices with margins \( \alpha \) and \( \beta \), i.e., integer matrices \( M = [m_{ij}]_{i=1, \ldots, \alpha, j=1, \ldots, \beta} \), \( m_{ij} \geq 0 \) with row sums being \( \beta_j \) and column sums being \( \alpha_i \), see [41] §6 and [35].

The relation \( M \succeq N \), resp. \( M \succeq^p N \), means that \( N \) is obtained from \( M \) by summing some groups of adjacent columns, resp. rows.

The orbit \( O_M \subset F_\alpha \times F_\beta \) consists of pairs of filtrations \((V_\ast, V_\ast')\) such that \( \dim \gr^V \gr^{V'} L = m_{ij} \), cf.?[3]. Note that \( \gr^V \gr^{V'} \simeq \gr^{V'} \gr^V L \) (Zassenhaus lemma).

Our methods, specialized to the case \( \mathfrak{g} = \mathfrak{gl}_n \), lead to a very simple proof and a clear understanding of the main result of [34] (developing a part of [25]) on braided Hopf algebras. More precisely:

Note that Theorem 2.6 can be formulated and proved for perverse sheaves with values in any abelian category \( \mathcal{V} \). The concept of a “sheaf” can be understood as a sub-analytic sheaf, similarly for complexes, see [34]. The Verdier dual of a constructible sub-analytic complex is understood as taking values in the opposite category \( \mathcal{V}^{\text{op}} \).

Let \((\mathcal{V}, \otimes, R, 1)\) is a braided monoidal abelian category with bi-exact \( \otimes \) and \( A = \bigoplus_{i=0}^\infty A_i \), \( A_0 = 1 \), be a graded bialgebra in \( \mathcal{V} \), see [34], §2.4. For each \( n \) we associate to it a mixed Bruhat sheaf \( E = E_n \) on \( \Xi_n \) by

\[
E_n(M) = \bigotimes_{i,j} A_{m_{ij}}.
\]

Here the tensor product is understood in the “2-dimensional” sense, using the interpretation of braided monoidal structures as having \( N \)-fold tensor operations labelled by arrangements of \( N \) distinct points in the Euclidean plane \( \mathbb{R}^2 \). We read the matrix structure of \( M \) to position each factor \( A_{m_{ij}} \) at the point \((-i, j) \in \mathbb{Z}^2\) of a rectangular grid in \( \mathbb{R}^2 \). After this, each map \( \partial'_{M,N} \) is given by the multiplication in \( A \), while \( \partial''_{M,N} \) is given by the comultiplication.

The space \( W \setminus \emptyset \) for \( \mathfrak{gl}_n \) is \( \text{Sym}^n(\mathbb{C}) \), the symmetric product of \( \mathbb{C} \). Denoting \( \mathcal{F}_n \) the perverse sheaf on \( \text{Sym}^n(\mathbb{C}) \) corresponding to \( E_n \) by Theorem 2.6, we get a system \((\mathcal{F}_n)_{n \geq 0}\) of perverse
sheaves that is manifestly factorizable ([34], Def. 3.2.5), and the main result of [34] (Theorem 3.3.1) follows easily.

Note that the bicube associated to $E_n$ consists of “1-dimensional” (linearly ordered) tensor products

$$A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_p}, \quad \alpha_1 + \cdots + \alpha_p = n,$$

with the $u$-maps given by multiplication (bar-construction) and the $v$-maps given by the comultiplication (cobar-construction).

Passing from a bicube to a mixed Bruhat sheaf in this and other examples can be seen as “unfolding” of a naive 1-dimensional structure to a more fundamental 2-dimensional one.

B. Eisenstein series and constant terms. An example of a braided Hopf algebra is given by $\mathcal{H}(\mathcal{A})$, the Hall algebra of a hereditary abelian category $\mathcal{A}$ with appropriate finiteness conditions [27]. One can apply this approach to $\mathcal{A} = \text{Coh}(X)$, the category of coherent sheaves on a smooth projective curve $X/\mathbb{F}_q$, see [31, 37]. Considering functions supported on vector bundles, we get a $\mathbb{Z}$-graded, braided Hopf algebra $\mathcal{H}^\text{Bun} = \bigoplus_{n \geq 0} \mathcal{H}^\text{Bun}_n$ where $\mathcal{H}^\text{Bun}_n$ consists of unramified automorphic forms for the group $GL_n$ over the function field $\mathbb{F}_q(X)$. The multiplication is given by forming (pseudo) Eisenstein series and comultiplication by taking the constant term of an automorphic form. Because $\text{Coh}(X)$ does not fully satisfy the finiteness conditions (an object may have infinitely many subobjects, but only finitely many subobjects of any given degree), the comultiplication in $\mathcal{H}^\text{Bun}$ must be understood using generating functions or rational functions of a spectral parameter, see [37]. With this taken into account (i.e., after extending the field of scalars to allow the dependence on the extra parameter), $\mathcal{H}^\text{Bun}$ gives, for each $n$, a mixed Bruhat sheaf on $\Xi_n$ and so a perverse sheaf on $\text{Sym}^n(\mathbb{C})$, as explained in §A.

If now $G$ is a general split reductive group over $\mathbb{Z}$ with Lie algebra $\mathfrak{g}$, we still have the classical theory of unramified automorphic forms and Eisenstein series for $G$ over $\mathbb{F}_q(X)$, see [40]. It usually appears in the form of a bicube $Q$, where

$$Q_I = \text{Fun}(\text{Bun}_{G_I}(X)), \quad I \subset \Delta_{\text{sim}}$$

is the space of automorphic forms for the standard Levi $G_I$, i.e., of functions on the set of isomorphic classes of principal $G_I$-bundles on $X$. For $I \subset J$ the map $v_{IJ} : Q_I \to Q_J$ is given by taking the (pseudo) Eisenstein series and $u_{IJ} : Q_J \to Q_I$ is given by taking the constant term of an automorphic form.

For general $G$, this theory does not have a Hopf algebra interpretation. However, one can extend the above bicube $Q$ to a mixed Bruhat sheaf $E$ and so obtain a perverse sheaf on $W \setminus \mathfrak{h}$. For this, given $m \in \Xi(I, J)$, one should consider the moduli space $\text{Bun}_{G,m}(X)$ formed by principal $G$-bundles together with a $P_I$-structure and a $P_J$-structure (i.e., sections of the associated bundles with fibers $G/P_I$ and $G/P_J$), everywhere in relative position $m$. The corresponding $E(m)$ is then found inside the space of functions on $\text{Bun}_{G,m}(X)$. 39
C. Categorical upgrade: mixed Bruhat schobers. The concept of a mixed Bruhat sheaf is very convenient for a categorical upgrade, i.e., replacing vector spaces with $k$-linear dg-enhanced triangulated categories (simply “triangulated categories” below). The possibility of such upgrade of the theory of perverse sheaves was raised in [32], where such hypothetical objects were called perverse schobers, see also [11, 18].

Recall [1] that a diagram of triangulated categories

$$
\mathcal{C} \xrightarrow{f} \mathcal{D} \xleftarrow{g}
$$

consisting of a (dg-) functor $f$ and its right adjoint $g = f^*$ is called a spherical adjunction (and $f$ is called a spherical functor), if the cones of the unit and counit of the adjunction

$$
T_\mathcal{C} = \text{Cone}\{e : \text{Id}_\mathcal{C} \to gf[-1]\}, \quad T_\mathcal{D} = \text{Cone}\{\eta : fg \to \text{Id}_\mathcal{D}\}
$$

are equivalences (i.e., quasi-equivalences of dg-categories). As noticed in [32], such a diagram can be seen as a categorical upgrade of a perverse sheaf $F \in \text{Perv}(\mathcal{C}, 0)$ in its $(\Phi, \Psi)$-description, see Example 2.7.

Now, $\text{Perv}(\mathcal{C}, 0) = \text{Perv}(W/\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}_2$, so Theorem 2.6 suggests a generalization of the concept of a spherical functor to arbitrary $\mathfrak{g}$. Let us call such structures mixed Bruhat schobers and sketch the main features of the definition.

So a mixed Bruhat Schober $\mathcal{E}$ should consist of triangulated categories $\mathcal{E}(m), m \in \Xi$ and dg-functors

$$
\mathcal{g}'_{m,n} = \mathcal{g}'_{m,n,\Xi} : \mathcal{E}(m) \to \mathcal{E}(n), \quad m \geq' n,
$$

$$
\mathcal{g}''_{m,n} = \mathcal{g}''_{m,n,\Xi} : \mathcal{E}(n) \to \mathcal{E}(m), \quad m \geq'' n,
$$

satisfying the following analogs of (MBS1-3). First, so that the $\mathcal{g}'$, as well as the $\mathcal{g}''$ must be transitive (up to coherent homotopies). Second, (MBS2) is upgraded into the data of a “filtration” on the functor $\mathcal{g}'_{n,m} : \mathcal{E}(m) \to \mathcal{E}(n)$ for $m$ running in the poset $(\text{Sup}(m, n), \preceq)$. Such a filtration can be understood either as a Postnikov system (see [33] §1A) or as a Waldhausen diagram (see [19] §5 or [20] §7.3), adapted for the case of a partially ordered indexing set. The analog of the condition (MBS3) is that $\mathcal{g}'_{m,n}$ for any anodyne $m \geq' n$ and $\mathcal{g}''_{m,n}$ for any anodyne $m \geq'' n$ must be an equivalence (i.e., a quasi-equivalence of dg-categories). Further, we should impose natural adjointness conditions meaning that $\mathcal{g}'_{m,n}$ is identified with the right adjoint of $\mathcal{g}''_{m',n'}$ after composing with appropriate “homotopies” connecting $m$ with $m'$ and $n$ with $n'$ (note that the cells $U_m$ and $U_{m'}$ always lie in the same stratum of $\mathcal{S}^{(0)}$, and so $m$ and $m'$ can be connected by a chain of anodyne $\leq', \leq''$ or their inverses).

Precise details will be given in a subsequent paper. Let us list two natural sources of such structures.

D. Constructible sheaves on Bruhat orbits. We can upgrade the constructions of §7 by replacing the space of functions on $\mathbb{F}_q$-points of a variety with the category of constructible complexes.
We consider the simplest setting when \( k = \mathbb{C} \). For an algebraic variety \( X/\mathbb{C} \) let \( D(X) = D^b_{\text{constr}}(X) \) be the derived category of bounded complexes with cohomology sheaves constructible with respect to some \( \mathbb{C} \)-algebraic stratification, see Appendix A.

We have a \( \Delta_{\text{sim}} \)-bicube \( \Omega \) of triangulated categories similar to that \( \S 7A \). It consists of the categories \( D(F_I) \) and functors

\[
(q_{IJ})_* = (q_{IJ})! : D(F_I) \longrightarrow D(F_J), \quad (q_{IJ})^* : D(F_J) \rightarrow D(F_I), \quad I \subset J.
\]

Note that for \( g = \mathfrak{s} \mathfrak{l}_2 \) the bicube reduces to the diagram

\[
D(\mathbb{C}P^1) \xrightarrow{\pi_*} D(\text{pt}) \quad \pi : \mathbb{C}P^1 \rightarrow \text{pt},
\]

which is a proto-typical example of a spherical adjunction, \( \mathbb{C}P^1 \) being the sphere \( S^2 \), see [32] Ex. 1.10. To extend the bicube \( \Omega \), we proceed similarly to \( \S 7 \).

Given \( m \in \Xi \), we consider first the category \( D(O_m) \). For \( m \triangleright n \) we define \( \vartheta_m^n : D(O_m) \rightarrow D(O_n) \) to be the functor \( (p_{m,n})_! \), the (derived) direct image with proper supports. For \( m \triangleright n \) we define \( \vartheta_m^n : D(O_n) \rightarrow D(O_m) \) to be the functor \( (p_{m,n})^* \). We define the category \( \mathcal{E}(m) \) to be the essential image of the pullback functor \( (r_m^*)^* : D(F_{\text{Hor}(m)}) \rightarrow D(O_m) \). As in \( \S 7 \), we see that \( \vartheta', \vartheta'' \) preserve the \( \mathcal{E}(m) \), so we have a diagram of triangulated categories

\[
\mathcal{E} = (\mathcal{E}(m), \vartheta', \vartheta'')
\]

upgrading the mixed Bruhat sheaf \( E_q \) of Theorem 7.2.

**Remarks 8.1.** (a) We can also consider the diagram \( \mathcal{E}^B \) formed by \( B \)-equivariant objects in the \( \mathcal{E}(m) \). Then \( \mathcal{E}^B(m) \) is identified with the category of \( B \)-equivariant constructible complexes of \( F_{\text{Hor}(m)} \), so the diagram consists of various (graded derived versions of) parabolic categories \( O \).

(b) Instead of \( D(O_m) \), we can use other types of “categories of sheaves” on \( O_m \) which possess an appropriate formalism of pullbacks and pushforwards. For example, we can use the category of mixed motives over \( O_m \), see [15].

(c) We can also take the “quasi-classical” approach, i.e., consider, instead of constructible complexes (i.e., complexes of holonomic regular \( D \)-modules) on the \( O_m \), complexes of coherent sheaves on \( \text{T}^*(O_m) \), thus establishing a connection with the braid group actions on the coherent derived categories of such cotangent bundles via flops [9, 14, 39].

**E. Parabolic induction and restriction.** We consider the simplest case of finite Chevalley groups. That is, take \( k = \mathbb{F}_q \). For any \( I \subset \Delta_{\text{sim}} \) let \( \mathcal{E}_I \) be the derived category of finite-dimensional \( k \)-linear representations of the finite group \( G_I(\mathbb{F}_q) \). If \( I \subset J \), then \( G_I \subset G_J \) and we have the classical *parabolic induction* and *restriction* functors

\[
\text{Ind}_{I,J} : \mathcal{E}_I \longrightarrow \mathcal{E}_J, \quad M \mapsto \text{Ind}_{G_J(\mathbb{F}_q)}^{G_I(\mathbb{F}_q)}(M),
\]

\[
\text{Res}_{I,J} : \mathcal{E}_J \longrightarrow \mathcal{E}_I, \quad N \mapsto \text{Res}_{G_I(\mathbb{F}_q)}^{G_J(\mathbb{F}_q)}(N),
\]

41
These two functors are both left and right adjoint to each other. Further, the Ind and Res- functors are transitive, so we have a bicube \( \mathcal{C} \) of triangulated categories. The passing to the derived categories, seemingly unnecessary for this simple case (all the functors are exact at the level of abelian categorie) makes the following example more salient.

**Example 8.2.** Let \( g = \mathfrak{sl}_2 \). Then the bicube has the form

\[
\mathcal{C}_{\emptyset} = D^b \text{Rep}(\mathbb{F}_q^*) \xrightarrow{\text{Ind}} D^b \text{Rep}(\text{SL}_2(\mathbb{F}_q)) = \mathcal{C}_{\{1\}},
\]

where \( \text{Ind} \) is the functor of forming the principal series representation and \( \text{Res} \) is the functor of invariants with respect to the standard unipotent subgroup. It follows from elementary theory of representations of \( \text{SL}_2(\mathbb{F}_q) \), see, e.g., [24], that this is in fact a spherical adjunction.

To extend the bicube \( \mathcal{C} \) to more general Bruhat orbits, we associate to each \( m \in \Xi \), the derived category of \( G(\mathbb{F}_q) \)-equivariant vector bundles \( V \) on the discrete set \( O_m(\mathbb{F}_q) \) such that for any element \( x \in O_m(\mathbb{F}_q) \), the unipotent radical of the stabilizer of \( x \) acts in the fiber \( V_x \) trivially. Such category is equivalent to the derived category of representations of the standard Levi \( G_{\text{Hor}}(m)(\mathbb{F}_q) \).

### A Stratifications and constructible sheaves

**A 1. Stratifications.** We fix some terminology to be used in the rest of the paper. By a *space* we mean a real analytic space. For a space \( X \) we can speak about subanalytic subsets in \( X \), see, e.g., [38] §8.2 and references therein. Subanalytic subsets form a Boolean algebra.

**Definition A.1.** Let \( X \) be a space.

(a) A *partition* of \( X \) is a finite family \( \mathcal{S} = (X_a)_{a \in A} \) of subanalytic subsets in \( X \) such that we have a disjoint decomposition \( X = \bigsqcup_{a \in A} X_a \) of \( X \) as a disjoint union of subanalytic sets. The sets \( X_a \) are called the *strata* of the partition \( \mathcal{S} \).

(b) A *locally closed decomposition* (l.c.d.) of \( X \) is a partition \( \mathcal{S} = (X_a)_{a \in A} \) such that each \( X_a \) is locally closed and the closure of each \( X_a \) is a union of strata. In this case the set \( A \) becomes partially ordered by \( a \preceq b \) if \( X_a \subseteq X_b \).

(c) A *stratification* of \( X \) is an l.c.d. such that each \( X_a \) is an analytic submanifold and the Whitney conditions are satisfied. A *stratified space* is a real analytic space with a stratification.

Further a *cell decomposition* of \( X \) is a stratification \( \mathcal{S} \) such that each stratum is homeomorphic to an open \( d \)-ball \( B^d \) for some \( d \). A cell decomposition is called *regular*; if for each cell (stratum) \( X_a \) there exists a homeomorphism \( B^d \to X_a \) which extends to an embedding of the closed ball \( \overline{B}^d \to X \) whose image is a union of cells. We will say that \( (X, \mathcal{S}) \) is a *regular cellular space.*
A cell decomposition of $X$ if called quasi-regular, if $X$, as a stratified space, can be represented as $Y \setminus Z$ where $Y$ is a regular cellular space and $Z \subset Y$ is a closed cellular subspace.

Given two partitions $S$ and $T$ of $X$, we say that $S$ refines $T$ and write $S < T$, if each stratum of $T$ is a union of strata of $S$.

Given two partitions $S = (X_a)$ and $T = (Y_b)$ of $X$, their maximal common refinement $S \wedge T$ is the partition into subsets defined as connected components of the $X_a \cap Y_b$. If $S, T$ are l.c.d.’s or stratifications, then so is $S \wedge T < S, T$.

We also have the maximal common coarsening $S \vee T$. This is a partition consisting of equivalence classes of the equivalence relation $\equiv$ on $X$ defined as follows. We first form the relation $R$ defined by: $xRy$ if $x$ and $y$ lie in the same stratum $X_a$ of $S$ or in the same stratum $Y_b$ of $T$ and then define $\equiv$ as the equivalence closure of $R$. Thus the strata of $S \vee T$ are certain unions of the $X_a \cap Y_b$. If $S, T$ both have the property of being l.c.d.’s or stratifications, $S \vee T$ may not have such property. For example, the strata of $S \vee T$ may not be locally closed even if the strata of $S$ and $T$ are.

We will be particularly interested in the cases when $S, T$ and $S \vee T$ are all stratifications.

**Proposition A.2.** (a) Let $S, T$ be two stratifications of $X$ such that $S < T$. Then each $T$-constructible sheaf is $S$-constructible.

(b) Let $S, T, U$ be three stratifications of $X$ such that $U = S \vee T$. Suppose a sheaf $G$ is both $S$-constructible and $T$-constructible. Then $G$ is $U$-constructible. \qed

Let $S = (X_a)_{a \in A}$ be a quasi-regular cell decomposition of $X$, so $(A, \leq)$ is naturally a poset (inclusion of closures of cells). Recall, see, e.g., [33] §1D, that an $S$-constructible (cellular) sheaf $G$ on $X$ is uniquely determined by the data of its stalks $G_a = \Gamma(X_a, G)$ at the cells and generalization maps $\gamma_{a,b} : G_a \to G_b$, $a \leq b$, which satisfy the transitivity conditions

$$ \gamma_{a,a} = \text{Id}, \quad \gamma_{a,c} = \gamma_{b,c} \circ \gamma_{a,b}, \quad a \leq b \leq c. \quad (A.3) $$

A datum $R = (G_a, \gamma_{ab})$ formed by finite-dimensional vector spaces $G_a$ and linear maps $\gamma_{ab}$ satisfying (A.3), will be called a representation of $A$. It is simply a covariant functor
from $(A, \leq)$ (considered as a category) to $\text{Vect}_k$. Representations of $A$ form an abelian category $\text{Rep}(A)$; we denote $D^b\text{Rep}(A)$ the corresponding bounded derived category. Given $R = (G_a, \gamma_{ab}) \in \text{Rep}(A)$, one defines directly the cellular sheaf $\text{sh}(R)$ with stalks $G_a$ and generalization maps $\gamma_{ab}$, thus giving a functor $\text{sh}: \text{Rep}(A) \to \text{Sh}(X, S)$. The above discussion can be formulated more precisely as follows, see, e.g., [33] Prop. 1.8:

**Proposition A.4.** Let $S = (X_a)_{a \in A}$ be a quasi-regular cell decomposition of $X$. Then:

(a) The functor $\text{sh}: \text{Rep}(A) \to \text{Sh}(X, S)$ is an equivalence of abelian categories.

(b) The termwise extension of $\text{sh}$ to complexes defines an equivalence of triangulated categories $D\text{sh}: D^b\text{Rep}(A) \to D^b\text{Sh}(X)$. □

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