CONSTRUCTION OF DISCONTINUOUS ENRICHMENT FUNCTIONS FOR ENRICHED FEM’S FOR INTERFACE ELLIPTIC PROBLEMS IN 1D

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Abstract. We introduce an enriched unfitted finite element method to solve 1D elliptic interface problems with discontinuous solutions, including those having implicit or Robin-type interface jump conditions. We present a novel approach to construct a one-parameter family of discontinuous enrichment functions by finding an optimal order interpolating function to the discontinuous solutions. In the literature, an enrichment function is usually given beforehand, not related to the construction step of an interpolation operator. Furthermore, we recover the well-known continuous enrichment function when the parameter is set to zero. To prove its efficiency, the enriched linear and quadratic elements are applied to a multi-layer wall model for drug-eluting stents in which zero-flux jump conditions and implicit concentration interface conditions are both present.

Key Words. enriched finite element, elliptic interface, implicit interface jump condition, Robin interface jump condition, linear and quadratic finite elements.

1. Introduction

Consider the interface two-point boundary value problem

\begin{align}
\begin{cases}
-(\beta(x)p'(x))' + w(x)p(x) = f(x), & x \in I = (a, b), \\
p(a) = p(b) = 0,
\end{cases}
\end{align}

where $w(x) \geq 0$, and $0 < \beta \in C[a, \alpha] \cup C[\alpha, b]$ is discontinuous across the interface $\alpha$ with the jump conditions on $p$ and its flux $q := \beta p'$:

\begin{align}
[p]_\alpha &= \lambda F(q^+, q^-, [p']_\alpha), & \lambda \in \mathbb{R}, & F : [c, d] \to \mathbb{R}, \\
[\beta p']_\alpha &= g, & g \in \mathbb{R}
\end{align}

where the jump quantity

\begin{align}
[s]_\alpha := s(\alpha^+) - s(\alpha^-), & & s^\pm := s(\alpha^\pm) := \lim_{\epsilon \to 0^\pm} s(\alpha \pm \epsilon).
\end{align}

The primary variable $p$ may stand for the pressure, temperature, or concentration in a medium with certain physical properties and the derived quantity $q := -\beta p'$ is the corresponding Darcy velocity, heat flux, or concentration flux, which is equally

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important. The piecewise continuous $\beta$ reflects a nonuniform material or medium property (we do not require $\beta$ to be piecewise constant). The function $w(x)$ reflects the surroundings of the medium. The case of $\lambda = 0$ is widely studied, while the case of $\lambda > 0$ gives rise to a more difficult situation. For example, the case of rightward concentration flow \cite{27, 28, 29} imposes

$$\begin{align*}
[p]_\alpha &= \lambda(\beta p)'(\alpha^-) \\
[\beta p']_\alpha &= 0,
\end{align*}$$

which generates an implicit condition since the left-sided derivative is unknown. Implicit interface conditions abound in higher dimensional applications \cite{1, 15, 19, 20}. For definiteness, we will study a class of efficient enriched methods for problem \cite{1} under the jump conditions \cite{4}, but our methods apply to problem \cite{1} subject to the general conditions \cite{2, 3} with a well-posed weak formulation. After a simple calculation, it is easy to see that (4) is equivalent to

$$\begin{align*}
[p]_\alpha &= \gamma [p']_\alpha, \quad \gamma = -\frac{\lambda \delta^- \delta^+}{[p]_\alpha} \\
[\beta p']_\alpha &= 0,
\end{align*}$$

which is indeed of the type \cite{2, 3}.

Numerical methods for the interface problem \cite{1} under \cite{4} generally use meshes that are either fitted or unfitted with the interface. A method allowing unfitted meshes would be very efficient when one has to follow a moving interface \cite{17} in a temporal problem. For the unfitted methods, there are available geometrically unfitted finite element methods typified in \cite{7} and the reference therein, the immersed finite and finite difference methods \cite{8, 13, 14, 18, 21, 22, 23}, the stable generalized finite element methods (SGFEM) \cite{5, 3, 4, 10, 31}, among others. In an unfitted method, the mesh is made up of interface elements where the interface intersects elements and non-interface elements where the interface is absent. On a non-interface element, one uses standard local shape functions, whereas on an interface element one uses specialized local shape functions reflecting the jump conditions. For an enriched method, the standard finite elements are enriched with some enrichment functions that reflect the presence of the interfaces. It was originally designed to handle crack problems \cite{6, 12, 24}, but for recent years efforts have been made to generalize it to fluid problems, see \cite{27} and the references therein.

The construction of the local shape basis of an immersed finite element or finite difference method uses information on discontinuous $\beta$ while an enriched method does not. Thus an enriched method does not require the discontinuous diffusion coefficient to be piecewise constant, which is an advantage. On the other hand, it makes the choice of the enrichment function less intuitive and the error analysis arguably harder. The purpose of this paper is to propose an approach to constructing the enrichment function from optimal order error analysis. The general idea is as follows. In the error analysis, we use the principle that says, roughly, the error in the finite element solution $p_h$ should be bounded by the approximation error in the finite element space $V_h$:

$$\|p - p_h\| \leq C \inf_{\chi \in V_h} \|p - \chi\| + \text{consistency error},$$

Suppose that the consistent error is of optimal order, then the optimal order analysis is completed if we can demonstrate an optimal order approximate piecewise
polynomial from \( V_h \). In an enriched finite element method, \( V_h \) takes the form of

\[
V_h := S_h \oplus \psi S_h = \{ p_h + q_h \psi : p_h, q_h \in S_h \},
\]

where \( S_h \) is a standard finite element space (e.g., \( P_k \)-conforming, \( k \geq 1 \)), and the function \( \psi \) is an enrichment function to reflect the jump conditions. In the literature, \( \psi \) is usually given beforehand and then one tries to find the optimal order interpolating polynomial to prove convergence. Our new approach is to connect the construction of \( \psi \) and the interpolating polynomial together and finds \( \psi \) through error analysis. In this way, we also have a unified theory for constructing enrichment functions for continuous and discontinuous finite element solutions.

Let’s mention how we were motivated to come up with the new approach. In view of (6), for standard continuous conforming finite element methods there are familiar interpolating polynomials that do the job [11]. For problem (1) with a continuous solution ([11]), Deng [10] proved the convergence of finite element solutions in all \( P_i \)-conforming spaces \((i \geq 1)\) enriched by the same well-known hat function [5, 3, 4, 10] (cf. Eq. (49) below). The crux of the proof was again the existence of a simple interpolating polynomial. However, for an enriched immersed or unfitted method approximating a discontinuous solution, it is impossible to find the same type of interpolation operator due to the finite jump of \( [p]_\alpha \) (cf. [9]). On the other hand, we in [2], unaware of [10], used a different interpolation operator to prove optimal order convergence. The approach was motivated by an additional presence of an interface deviation. In this paper we generalize the analysis of [2], modifying that operator (\( I_{ch} ^\alpha \) of (15) below) to find the desired interpolating polynomial. The new family of the enrichment functions is a result of this analysis, not given beforehand. However, the formula of the enrichment function (cf. (8) below) is simple and intuitive, and can be used without knowing the detail of the analysis.

The organization of this paper is as follows. In Section 2, we state the weak formulation for the implicit interface condition problem, define enrichment functions and spaces, and put their role in perspective in Remark 2.1. In Section 3, we carry out the error analysis and show how the construction of the enrichment function is related to it. Optimal order convergence in the broken \( H^1 \) and \( L^2 \)-norms is given in Theorem 3.4. In addition, the second-order accuracy of \( p_h \) at the nodes is proven in Theorem 3.5. In Section 4, we provide numerical examples of a porous wall model to demonstrate the effectiveness of the present enriched finite element method and confirm the convergence theory. Furthermore, following the viewpoint of the SGFEM [3, 4, 10], we compare the condition numbers of our (discontinuous solution) method with those in the continuous solution case [2], and numerically show that they are comparable for the same mesh sizes. Both linear and quadratic enriched elements are tested. Finally, in Section 5 we give some concluding remarks and discuss possible extensions of the present approach to multiple dimensions.

2. ENRICHMENT FUNCTIONS AND SPACES

2.1. We&omacul;#39;ll Formulation. Let \( I^- = (a, \alpha) \) and \( I^+ = (\alpha, b) \), and define

\[
H^1_{\alpha,0}(I) = \{ v \in L^2(I) : v \in H^1(I^-) \cap H^1(I^+), v(a) = v(b) = 0 \}.
\]

We use conventional Sobolev norm notation. For example, \( \|u\|_{1, J}^2 \) denotes the usual \( H^1 \)-seminorm for \( u \in H^1(J) \), and \( \|u\|^2_{1, I^- \cup I^+} = \|u\|^2_{1, I^-} + \|u\|^2_{1, I^+} \), \( i = 1, 2 \) for
$u \in H_0^2(I)$, where

$$H_0^2(I) = H^2(I^-) \cap H^2(I^+).$$

The space $H_{1,0}^2(I)$ is endowed with the norm $\| \cdot \|_{1, I^-} + \| \cdot \|_{1, I^+}$ norm, and $H_0^2(I)$ with the

Remark 2.1. In the paper, we assume that the functions $\beta$ are such that the solution $p$ exists.

The above weak formulation can be easily derived by integration-by-parts and by

Remark 2.1. By the Lax Milgram theorem, a unique solution of (4) is: Given $f \in L^2(I)$, find $p \in H_{1,0}^2(I)$ such that

(7)

$$a(p, q) = (f, q) \quad \forall q \in H_{1,0}^2(I),$$

where

$$a(p, q) = \int_0^b \beta(x)p'(x)q'(x)dx + \int_0^b w(x)p(x)q(x)dx + \frac{|p|_\alpha|q|_\alpha}{\lambda},$$

$$(f, q) = \int_0^b f(x)q(x)dx.$$ 

The above weak formulation can be easily derived by integration-by-parts and by

Remark 2.1. Since $\lambda > 0$, the bilinear form $a(\cdot, \cdot)$ is coercive and is bounded due to Poincaré inequality. By the Lax Milgram theorem, a unique solution $p$ exists. Throughout 

Remark 2.1. the paper, we assume that the functions $\beta$, $f$, and $w$ are such that the solution $p \in H_0^2(I)$.

2.2. Enrichment Functions. We now introduce an approximation space for the solution $p$. Let $a = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = b$ be a partition of $I$ and the interface point $\alpha \in (x_k, x_{k+1})$ for some $k$. As usual, the meshsize $h := \max_i h_i, h_i = x_{i+1} - x_i, i = 0, \ldots, n - 1$. Define the enrichment function

(8)

$$\psi(x) := \begin{cases} 0 & x \in [a, x_k] \\ m_1(x - x_k) & x \in [x_k, \alpha) \\ m_2(x - x_{k+1}) & x \in (\alpha, x_{k+1}] \\ 0 & x \in [x_{k+1}, b] \end{cases}$$

where

(9)

$$m_1 = \frac{\alpha - x_{k+1}}{x_{k+1} - x_k}, \quad m_2 = \frac{(\alpha - x_k - \gamma)(\alpha - x_{k+1})}{(x_{k+1} - x_k)(\alpha - x_{k+1} - \gamma)}, \quad \gamma = -\frac{\lambda \beta - \beta^+}{|\beta|_\alpha}$$

Remark 2.1.

- Note that $\psi$ satisfies the following conditions:

(10)

$$\psi(x_k) = \psi(x_{k+1}) = 0, \quad [\psi']_\alpha \neq 0, \quad [\psi]_\alpha = \begin{cases} 0 & \text{nonzero} \\ \gamma = 0 & \gamma \neq 0. \end{cases}$$

- We can view $\psi$ as parameterized by $\gamma$, and for different problems we would have to define $\gamma$. In the present application case, $\gamma = -\frac{\lambda \beta - \beta^+}{|\beta|_\alpha}$. Our theory depends on $\gamma$, not its specific definition.

- If we set $\gamma = 0 (|p|_\alpha = 0)$, we recover the familiar continuous enriched function for the continuous case in which $|\psi'|_\alpha = 1$ (cf. Eq. (11)).

In other words, the continuous case is the limiting case of the discontinuous ones.

- Notice that the slopes are uniformly bounded, i.e., there exists a constant $C > 0$ such that

(11)

$$|m_1| + |m_2| \leq C \quad \forall x_k, x_{k+1}, 0 \leq h \leq 1.$$
The main gist of this paper is to obtain \( \psi \) as a natural consequence of our error analysis. The definition of \( m_2 \) is a result of zeroing out of infinite coefficient of \([p']_\alpha \) in the error analysis (cf. Eq. (39)).

Let us describe the enriched space associated with \( \psi \). Let \( \bar{I} = \bigcup_{i=0}^{n-1} I_i, I_i = [x_i, x_{i+1}] \) and let \( S_h \) be the conforming linear finite element space

\[
S_h = \{ v_h \in C(\bar{I}) : v_h|_{I_i} \in \mathbb{P}_1, i = 0, \ldots, n-1, v_h(a) = v_h(b) = 0 \} = \text{span}\{ \phi_i, i = 1, 2, \ldots, n-1 \}
\]

where \( \phi_i \)'s are the Lagrange nodal basis hat functions. We denote the usual \( \mathbb{P}_1 \)-interpolation operator by \( \pi_h : C(\bar{I}) \rightarrow S_h \),

\[
\pi_h g = \sum_{i=1}^{n-1} g(x_i) \phi_i,
\]

and define the enriched finite element space

\[
\bar{S}_h = S_h \oplus \psi S_h = \{ p_h + q_h \psi : p_h, q_h \in S_h \} = \text{span}\{ \phi_1, \phi_2, \ldots, \phi_{n-1}, \phi_k \psi, \phi_{k+1} \psi \}.
\]

Consider the enriched finite element method for problem (1): Find \( p_h \in \bar{S}_h \subset H^1_{\alpha,0} \) such that

\[
a(p_h, q_h) = (f, q_h) \quad \forall q_h \in \bar{S}_h.
\]

2.3. Optimal Order Interpolating Polynomial \( I_h p \). It is essential for the enriched space to have good approximation properties for the functions in \( H^2_\alpha(I) \) that satisfy the jump conditions \([9]\). For \( p \in H^2_\alpha(I) \), let \( p_i, i = 1, 2 \) be the extensions of \( p \) restricted to \( I^- \) and \( I^+ \) to \( H^2(I) \), respectively \([10]\). Thus \( p_2 - p_1 \) is in \( H^1(I) \subset C(I) \) due to the Sobolev inequality, and as a result the usual \( \mathbb{P}_1 \)-interpolation operator \( \pi_h(p_2 - p_1) \in S_h \) is well defined. To exhibit approximation properties of \( \bar{S}_h \) for functions in \( H^2_\alpha(I) \) that satisfy \([9]\), we first define the interpolation operator

\[
I_h^\alpha : H^2_\alpha(I) \rightarrow \bar{S}_h
\]

\[
I_h^\alpha p = \pi_h p + \pi_h(p_2 - p_1)\psi.
\]

In particular

\[
I_h^\alpha p = \frac{p_1(x_k)(x_{k+1} - x) + p_2(x_{k+1})(x - x_k)}{x_{k+1} - x_k} + \pi_h(p_2 - p_1)\psi(x) \quad \forall x \in [x_k, x_{k+1}].
\]

This interpolation operator has been used successfully in \([2]\), but our experience showed that it is not capable of handling the discontinuous case \([p]_\alpha \neq 0 \). To emphasize we use a superscript \( c \) to indicate continuity. Now we modify it with an added correction term to accommodate the case of \([p]_\alpha \neq 0 \): Define the interpolation operator

\[
I_h : H^2_\alpha(I) \rightarrow \bar{S}_h
\]

\[
I_h p = I_h^\alpha p + \delta \psi,
\]

where

\[
\delta = -\frac{h_k^{-1}[p]_\alpha}{m_1} = -\frac{[p]_\alpha}{\alpha - x_{k+1}}.
\]

The \( \delta \)-term is motivated by the error analysis in Lemma \( 3.1 \) below. Its presence is to kill the jump term in \( p \) across \( \alpha \) that may go to infinity as \( h \) goes to zero (See Eq. \( (25) \)).
Let \( \chi_i, i = 1, 2 \) be the characteristic functions of \( I^- \) and \( I^+ \), respectively, and let

\[
V_h := \{ v = v_{h,1}\chi_1 + v_{h,2}\chi_2; \, v_{h,i} \in S_h, i = 1, 2 \}.
\]

Note that functions in the above space may be discontinuous at \( \alpha \). Define the auxiliary interpolations \( \bar{I}_h : H^2_{\alpha}(I) \to V_h \),

\[
\bar{I}_hp = \pi_h p_1 \chi_1 + \pi_h p_2 \chi_2.
\]

To derive a bound for the term \(|p - \bar{I}_hp|_{1, I^- \cup I^+}^2\) we split the error as follows:

\[
|p - \bar{I}_hp|_{1, I^- \cup I^+} \leq |p - \bar{I}_hp|_{1, I^- \cup I^+} + |\bar{I}_hp - \bar{I}_hp|_{1, I^- \cup I^+}.
\]

From the classical approximation theory

\[
|p - \bar{I}_hp|_{1, I^- \cup I^+} \leq Ch\|p\|_{2, I^- \cup I^+}.
\]

Thus it suffices to estimate the second term on the right side of (19), which is done in the following two lemmas. We mention in passing that all the constants in the estimates should be independent of the interface position as well. This fact is important if one wants to use the method for moving interface problems.

### 3. Construction of Enrichment Functions in relation to Error Analysis

**Lemma 3.1.** There exists a positive constant \( C \) independent of \( h \) and \( \alpha \) such that

\[
|\bar{I}_hp - \bar{I}_hp|_{1, I^-} \leq Ch\|p\|_{2, I^-} \quad \forall p \in H^2_{\alpha}(I).
\]

**Proof.** It suffices to show the detailed analysis on the interface element \([x_k, x_{k+1}]\).

For the interval \([x_k, \alpha]\), from the definition (15) of \( \bar{I}_hp \) and the addition and subtraktion of the same quantity yield

\[
(\bar{I}_hp - \bar{I}_hp)' = (\bar{I}_hp - \pi_hp)' - (\pi_h(p_2' - p_1')(x)\psi(x))'
\]

\[
\text {where}
\]

\[
J_1 := (\bar{I}_hp - \pi_hp)' + \frac{(x_{k+1} - \alpha)(p_2'(x) - p_1'(x))}{x_{k+1} - x_k},
\]

\[
J_2 := \frac{(x_{k+1} - \alpha)(p_2'(x) - p_1'(x))}{x_{k+1} - x_k} + (\pi_h(p_2' - p_1')(x)\psi(x))',
\]

Since \( \bar{I}_hp = \pi_h p_1 \chi_1 \) on \([x_k, \alpha]\), the first term in \( J_1 \)

\[
(\bar{I}_hp - \pi_hp)'(x) = \frac{p_1(x_{k+1}) - p_2(x_{k+1})}{x_{k+1} - x_k}
\]

\[
= \frac{p_1(x_{k+1}) - p_1(\alpha) + p_2(\alpha) - p_2(x_{k+1})}{x_{k+1} - x_k},
\]

and combining this with the second term in \( J_1 \) leads to

\[
J_1 = J_2 - \frac{|p|_\alpha}{x_{k+1} - x_k},
\]

\[
J_1 = J_3 - \frac{|p|_\alpha}{x_{k+1} - x_k},
\]
where using the Taylor’s expansion with integral remainder form

\[
|J_3| = \left| \frac{p_1(x_{k+1}) - p_1(\alpha) + p_2(\alpha) - p_2(x_{k+1})}{x_{k+1} - x_k} + \frac{(x_{k+1} - \alpha)(p_2'(\alpha) - p_1'(\alpha))}{x_{k+1} - x_k} \right|
\]

(24)

\[
= \left| \frac{p_1(x_{k+1}) - p_1(\alpha) - p_1'(\alpha)(x_{k+1} - \alpha)}{x_{k+1} - x_k} - \frac{p_2(x_{k+1}) - p_2(\alpha) - p_2'(\alpha)(x_{k+1} - \alpha)}{x_{k+1} - x_k} \right|
\]

\[
= \left| \frac{1}{x_{k+1} - x_k} \left( \int_{\alpha}^{x_{k+1}} p_1''(t)(x_{k+1} - t)dt - \int_{\alpha}^{x_{k+1}} p_2''(t)(x_{k+1} - t)dt \right) \right|
\]

\[
\leq \frac{x_{k+1} - \alpha}{x_{k+1} - x_k} \left( \int_{\alpha}^{x_{k+1}} |p_1''(t)|dt + \int_{\alpha}^{x_{k+1}} |p_2''(t)|dt \right)
\]

\[
\leq 2 \frac{x_{k+1} - \alpha}{x_{k+1} - x_k} h^{1/2} \|p''\|_{0,I-\cup I^+}
\]

\[
\leq C h^{1/2} \|p''\|_{0,I-\cup I^+},
\]

where the constant \( C = 2 \), independent of \( h \) and \( \alpha \). Note that \( J_1 \) is the difference between a small quantity \( J_3 \) and a large quantity \( |p_1\alpha/(x_{k+1} - x_k) \) as \( h \) goes to zero. The latter is controlled by the \( \psi' \) terms in (18) through the \( \delta \)-parameter in the following relation

(25)

\[
(\bar{I}_h p - I_h p)' = (\bar{I}_h p - I_h p)' - \delta \psi'
\]

\[
= J_1 - J_2 - \delta \psi'
\]

\[
= J_1 - J_2 - \frac{|p|_{1\alpha}}{x_{k+1} - x_k} - \delta m_1
\]

\[
= J_3 - J_2
\]

by the way we defined \( \delta \) in (18). Next we show that \( J_2 \) is the difference between a small quantity and a large term we can control.

To avoid clustering of expressions, let \( \Delta := p_2 - p_1 \) so that \( \Delta' = p_2' - p_1' \) and \( \Delta'' = p_2'' - p_1'' \). We also denote \( \Delta'(x_k) \) by \( \Delta_k' \), and \( \Delta'(x_{k+1}) \) by \( \Delta_{k+1}' \). Below, we use these notations when necessary. First note that with \( \psi = m_1(x - x_k) \) and \( \Delta'(\alpha) = |p|_{1\alpha} \)

\[
(\pi_h(p_2' - p_1')(x)\psi)'
\]

\[
= h^{-1}_k (\Delta'_{k+1} - \Delta'_{k})\psi + h^{-1}_k [(\Delta'_{k+1}(x - x_k) - \Delta'_{k}(x - x_{k+1})] \psi'
\]

\[
= m_1 h^{-1}_k [2 \Delta'_{k+1}(x - x_k) - \Delta'_{k}(x - x_k) - \Delta'_{k}(x - x_{k+1})]
\]
and hence
\[ J_2 := \frac{(x_{k+1} - x)(p_2'(\alpha) - p_1'(\alpha))}{x_{k+1} - x_k} + (\pi_h(\Delta')(x)p(x))' \]
\[ = h_k^{-1}(x_{k+1} - x)[p']_\alpha + m_1 h_k^{-1}\left[2\Delta'_{k+1}(x - x_k) - \Delta'_k(x - x_k) - \Delta'_k(x - x_{k+1})\right] \]
\[ = h_k^{-1}(x - \alpha)[p']_\alpha + m_1 h_k^{-1}\left[2\Delta'_{k+1}(x - x_k) - \Delta'_k(x - x_k) - \Delta'_k(x - x_{k+1})\right] \]
\[ = h_k^{-1}(x - \alpha)[p']_\alpha + m_1 h_k^{-1}\left[\sum_{i=1}^{4} I_i + [p']_\alpha h_k\right], \]
\[ = h_k^{-1}(x - \alpha)[p']_\alpha + (m_1 + 1)[p']_\alpha + m_1 h_k^{-1}\sum_{i=1}^{4} I_i \]
\[ := J_4 = m_1 h_k^{-1}\sum_{i=1}^{4} I_i \quad (\text{by } (29)) \]

where
\[ I_1 = (\Delta'(x_{k+1}) - \Delta'(x_k))(x - x_{k+1}), \]
\[ I_2 = (\Delta'(x_{k+1}) - \Delta'(x_k))(x - x_k), \]
\[ I_3 = (\Delta'(x_k) - \Delta'(\alpha))(x_{k+1} - x_k), \]
\[ I_4 = (\Delta'(x_k) - \Delta'(x_k))(x_{k+1} - x_k). \]

Each of the \(m_1 h_k^{-1} I_i\) terms in \(J_4\) can be estimated similarly using the Cauchy-Schwarz inequality, e.g.,
\[ |m_1 h_k^{-1} I_1| \leq |m_1| \frac{x_{k+1} - x}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} |(p_2' - p_1')'(y)| dy \]
\[ \leq |m_1| \frac{x_{k+1} - x}{x_{k+1} - x_k} \left( \int_{x_k}^{x_{k+1}} |(p_2' - p_1')'(y)|^2 dy \right)^{1/2} (x_{k+1} - x_k)^{1/2} \]
\[ \leq |m_1| h^{1/2} ||p''||_{0,T-\cup I^+} \]
\[ \leq C h^{1/2} ||p''||_{0,T-\cup I^+}, \quad (|m_1| \leq 1) \]

where \(C\) is a constant independent of \(h\) and \(\alpha\). Combining these estimates we see that
\[ J_2 = J_4 \]

with
\[ |J_4| = |m_1 h_k^{-1}\sum_{i=1}^{4} I_i| \leq C h^{1/2} ||p''||_{0,T-\cup I^+}. \]

From (25) and using (23), (26), and (28).
\[ (J_h p - I_h p)' = (J_h p - I_h p)' - \delta \psi' \]
\[ = J_3 - J_4. \]
Gathering all the local estimates and integrating, we have

\begin{equation}
|\bar{I}_h p - I_h p|_{1,1^-} \leq C h \|p\|_{2,1^- \cup \Gamma}^2 \quad \forall p \in H^2_0(I)
\end{equation}

where $C$ is independent of $h$ and $\alpha$. \hfill \Box

**Lemma 3.2.** There exist a positive constant $C$ independent of $h$ and $\alpha$ such that

\begin{equation}
|\bar{I}_h p - I_h p|_{1,1^+} \leq C h \|p\|_{2,1^- \cup \Gamma}^2 \quad \forall p \in H^2_0(I) \text{ satisfying (5).}
\end{equation}

**Proof.** For the interval $[\alpha, x_{k+1}]$, from the definition (15) of $I_h p$ and adding and subtracting of the same quantity, $h_k^{-1}(x_{k+1} - \alpha)[p']_\alpha$, yield

\begin{equation}
(\bar{I}_h p - I_h p)' = (\bar{I}_h p - \pi_h p)' - (\pi_h (p'_2 - p'_1)(x) \psi(x))' = (\bar{I}_h p - \pi_h p)' + \frac{(x_k - \alpha)(p'_2(\alpha) - p'_1(\alpha))}{x_{k+1} - x_k} - \frac{(x_k - \alpha)(p'_2(\alpha) - p'_1(\alpha))}{x_{k+1} - x_k} + (\pi_h (p'_2 - p'_1)(x) \psi(x))' \quad \forall x \in [\alpha, x_{k+1}]
\end{equation}

Noting that $\bar{I}_h p = \pi_h p_2 \chi_2$ for $x \in [\alpha, x_{k+1}]$, we see that

\begin{align*}
(\bar{I}_h p - \pi_h p)'(x) &= \frac{p_1(x_k) - p_2(x_k)}{x_{k+1} - x_k} \\
&= \frac{p_1(x_k) - p_1(\alpha) + p_2(\alpha) - p_2(x_k)}{x_{k+1} - x_k} - \frac{[p]_\alpha}{x_{k+1} - x_k}
\end{align*}

and hence

\begin{equation}
\bar{J}_1 = \bar{J}_3 - \frac{[p]_\alpha}{x_{k+1} - x_k}
\end{equation}

where

\begin{align*}
|\bar{J}_3| &= \left|\frac{p_1(x_k) - p_1(\alpha) + p_2(\alpha) - p_2(x_k)}{x_{k+1} - x_k} + \frac{(x_k - \alpha)(p'_2(\alpha) - p'_1(\alpha))}{x_{k+1} - x_k}\right| \\
&= \left|\frac{p_1(x_k) - p_1(\alpha) - p'_1(\alpha)(x_k - \alpha)}{x_{k+1} - x_k} - \frac{p_2(x_k) - p_2(\alpha) - p'_2(\alpha)(x_k - \alpha)}{x_{k+1} - x_k}\right| \\
&= \frac{1}{x_{k+1} - x_k} \left|\int_{\alpha}^{x_k} p''_1(t)(x_k - t) dt - \int_{\alpha}^{x_k} p''_2(t)(x_k - t) dt\right| \\
&\leq \frac{x_k - \alpha}{x_{k+1} - x_k} \left(\int_{\alpha}^{x_k} |p''_1(t)| dt + \int_{\alpha}^{x_k} |p''_2(t)| dt\right) \\
&\leq 2 \frac{x_k - \alpha}{x_{k+1} - x_k} h^{1/2} ||p''||_{0, I^- \cup \Gamma^+} \\
&\leq 2 h^{1/2} ||p''||_{0, I^- \cup \Gamma^+}.
\end{align*}

(32)
Having decomposed $\tilde{J}_1$ as the difference of a small term and a large term plus a finite term, we do the same for $\tilde{J}_2$. First, with $\psi = m_2(x - x_{k+1})$ we have

$$\left(\pi_h(p'_2 - p'_1)(x)\psi\right)' = h_k^{-1}(\Delta'_{k+1} - \Delta'_k)\psi + h_k^{-1}\left[(\Delta'_{k+1}(x - x_k) - \Delta'_k(x - x_{k+1})\right] \psi'$$

$$= m_2h_k^{-1}\left[(\Delta'_{k+1} - \Delta'_k)(x - x_{k+1}) + \Delta'_{k+1}(x - x_k) - \Delta'_k(x - x_{k+1})\right]$$

$$= m_2h_k^{-1}\left(\sum_{i=1}^{3} I_i + \Delta'(\alpha)(x_{k+1} - x_k)\right),$$

where

$$I_1 = (\Delta'(x_{k+1}) - \Delta'(x_k))(x - x_{k+1}),$$

$$I_2 = (\Delta'(x_{k+1}) - \Delta'(x_k))(x - x_k),$$

$$I_3 = (\Delta'(x_k) - \Delta'(\alpha))(x_{k+1} - x_k),$$

and hence

$$\tilde{J}_2 := \frac{(x_k - \alpha)(p'_2(\alpha) - p'_1(\alpha))}{x_{k+1} - x_k} + (\pi_h(p'_2 - p'_1)(x)\psi(x))'$$

$$= h_k^{-1}((x_k - \alpha)p'_2|\alpha + m_2h_k[p'_2]_\alpha) + m_2h_k^{-1}\left(\sum_{i=1}^{3} I_i\right),$$

where the last term can be estimated as before. Thus,

$$\tilde{J}_2 = \tilde{J}_4 + h_k^{-1}((x_k - \alpha)p'_2|\alpha + m_2h_k[p'_2]_\alpha)$$

with due to \((\ref{11})\)

$$|\tilde{J}_4| = |m_2h_k^{-1}\left(\sum_{i=1}^{3} I_i\right)| \leq Ch_1^{1/2}\|p''\|_{0, I- \cup I+}.$$

Now

$$\left(\tilde{I}_hp - I_hp\right)' = \left(\tilde{I}_hp - I_hp\right)' - \delta\psi'$$

will be estimated as follows. From \((\ref{30}), (\ref{31}),\) and \((\ref{5})\) we have

$$\left(\tilde{I}_hp - I_hp\right)' = \tilde{J}_3 + \tilde{J}_4 - h_k^{-1}(x_k - \alpha)[p'_2]_\alpha - h_k^{-1}[p]_\alpha - m_2[p'_2]_\alpha - \frac{h_k^{-1}[p]_\alpha}{m_1}m_2$$

$$= \tilde{J}_3 + \tilde{J}_4 - \tilde{J}_5$$

where

$$\tilde{J}_5 := (h_k^{-1}(x_k - \alpha) + h_k^{-1}\gamma + m_2 + m_2h_k^{-1}\gamma/m_1)[p'_2]_\alpha.$$
Gathering all the above local estimates and integrating, we conclude that there exists a constant \( C > 0 \) independent of \( h \) and \( \alpha \) such that
\[
|I_h p - I_h p|_{1,I^+} \leq Ch\|p\|_{2,I-\cup I^+} \quad \forall p \in H^2_I(I).
\]
\( \square \)

Using Lemmas 3.1 and 3.2, we obtain

**Theorem 3.3.** There exists a constant \( C > 0 \) independent of \( h \) such that
\[
\|p - I_h p\|_{0,I-\cup I^+} \leq Ch\|p\|_{2,I-\cup I^+} \quad \forall p \in H^2_I(I) \text{ satisfying (5).}
\]

Since our enriched finite element method is conforming, the convergence analysis is routine except for the step of checking the constant in the estimate to be independent of maximum meshsize \( h \) and the interface position \( \alpha \).

**Theorem 3.4.** Let \( p \) be the exact solution and \( p_h \) be the approximate solution of (7) and (14), respectively. Then there exists a constant \( \beta^* \) with \( \beta^* = \sup_{x \in [a,b]} \beta(x) \) and \( \beta_* = \inf_{x \in [a,b]} \beta(x) \).

**Proof.** Subtracting (7) from (14), we have
\[
|\beta| \independent \max h
\]

Using Lemmas 3.1 and 3.2, we obtain

Then the usual duality argument leads to
\[
\|p - p_h\|_{0,I-\cup I^+} \leq Ch\|p\|_{2,I-\cup I^+}.
\]

We note that the jump ratios \( \rho := \frac{\beta^*}{\beta_*} \) are of moderate size for the wall model in the next section.

**Theorem 3.5. Second order accuracy at nodes.** Suppose that \( \beta \in C^1(a,\alpha) \cap C^1(\alpha,b) \) and \( 0 \leq w \in C[a,b] \). Let \( p \) be the exact solution and \( p_h \) be the approximate solution of (7) and (14), respectively. Then there exists a constant \( C > 0 \) such that
\[
|p(\xi) - p_h(\xi)| \leq Ch^2\|p\|_{2,I-\cup I^+}, \quad \xi = x_i, 1 \leq i \leq n - 1.
\]
where \( C \) depends on certain norms of the Green’s function at \( \xi \).
Proof. Let $G(x, \xi), \xi \neq \alpha$ be the Green’s function satisfying

$$a(G(\cdot, \xi), v) = <\delta(x - \xi), v>, \quad v \in H^1_{\alpha, \Omega}(a, b)$$

whose existence is guaranteed by the Lax-Milgram theorem, since in 1D point evaluation is a bounded operator. Then from [2, 24], we know without loss of generality that for $\xi < \alpha, g = G(\cdot, \xi) \in H^2_\Omega$, for $\Omega = (\alpha, \xi), (\xi, x_k), (x_k, \alpha), (\alpha, x_{k+1}), (x_{k+1}, b)$. Similar regularity holds if $\xi$ lies elsewhere. Since $g$ satisfies (5) we can use the local estimates in Section 3 and conclude that there exists $I_h g \in \bar{S}_h$ such that

$$|g - I_h g|_{1, \Omega} \leq Ch||g''||_{0, \Omega}$$

for all the $\Omega$’s listed above. Now

$$e(x_i) = a(g, e) = a(g - I_h g, e)$$

implies that

$$|e(x_i)| \leq C h ||g||_{2, \ast} h ||p||_{2, I-\cup I+} \leq C h^2 ||g||_{2, \ast} ||p||_{2, 1-\cup I+}$$

where $||g||_{2, \ast}^2 := \sum ||g||_{2, \Omega}^2$, the summation being over all the $\Omega$’s listed above. $\square$

4. Numerical Examples

In this section, we test our method using the multi-layer porous wall model for the drug-eluting stents [25] that has been studied using the immersed finite element methods [27, 28, 29, 30]. In this one-dimensional wall model of layers, a drug is injected or released at an interface and gradually diffuses rightward. The concentration is thus discontinuous across the injection interface and continuous in the other layers. At all interface points, a zero-flux condition is imposed. We run tests on both enriched linear and quadratic finite element spaces.

4.1. Enriched Linear Elements. In this subsection we test the efficiency of our method on three problems. In Problem 1, we place only one interface point to model the layer where the drug is delivered. In Problem 2, we place two interfaces to model the layers where the concentration has continuously spread. Finally, in Problem 3 we combine the previous two cases and place three interface points to simulate the full wall model. In all three problems, we confirm in Table 1-Table 3 the optimal order convergence in the broken $H^1$ and the $L^2$ norms. In addition, the nodal errors are shown to be second order in all these tables as well. We are interested in the behavior of the condition numbers of the associated stiffness matrices. Following the viewpoint of the SGFEM [3, 4, 10], we compare the condition numbers in Problem 1 and Problem 3 (discontinuous solutions) with those in Problem 2 (continuous solution) [2] that are displayed in Table 4. We can see that the condition numbers in Table 4 and Table 5 are comparable in the order of magnitude with those in Table 2 for the same mesh sizes.

Problem 1. Discontinuous Solution. Consider the two-point boundary value problem with one interface point $\alpha_0 = 1/9$

$$\frac{\partial}{\partial x} \left( -D \frac{\partial u}{\partial x} + 2\delta u \right) + \gamma u = f \quad \text{in } (0, 1)$$
subject to the no-flux Neumann condition at \( x = 0 \) and the Dirichlet condition at \( x = 0 \):

\[
D_0 u'(0) = 0, \quad u(1) = \frac{1}{3}.
\]

Here the drug reaction coefficient \( \gamma = 0 \), and the drug diffusivity \( D \) and the characteristic convection parameter \( \delta \) are piecewise continuous with respect to \([0, 1/9]\) and \([1/9, 1]\):

\[
D(x) = \begin{cases} 
  D_0 = 1 & x \in [0, 1/9] \\
  D_1 = \frac{18(n-1)}{10n} & x \in [1/9, 1]; 
\end{cases}
\]

\[
\delta(x) = \begin{cases} 
  \delta_0 = 0 & x \in [0, 1/9] \\
  \delta = 0.5(9nD_1 - 8.1(n-1)) & x \in [1/9, 1]. 
\end{cases}
\]

Furthermore, at the interface point \( \alpha_0 \), one of the jump conditions is implicit

\[
[u]_{\alpha_0} = \lambda D_0 u'(\alpha_0),
\]

\[
-D_0 u'(\alpha_0) = -D_1 u'(\alpha_0^+) + 2\delta_1 u(\alpha_0^+)
\]

where \( \lambda = \frac{1}{81(n-1)} D_0 \). The exact solution

\[
u(x) = \begin{cases} 
  u_0 = x^{n-1}/30, & x \in [0, 1/9], \\
  u_1 = x^n/3, & x \in [1/9, 1]. 
\end{cases}
\]

We test the effectiveness of the method with \( n = 4 \) and with the enrichment function in \([8]\). The run results are displayed in Table 1.

| Problem 1 | \( L_2 \) error | \( H^1 \) error | nodal error | condition number |
|-----------|------------------|-----------------|-------------|-----------------|
| \( h = 1/8 \) | 1.43943e-03 | 6.59920e-02 | 4.85121e-03 | 0.137850e+06 |
| \( h = 1/16 \) | 3.40683e-04 | 3.24574e-02 | 1.01654e-03 | 0.143171e+06 |
| \( h = 1/32 \) | 8.39493e-05 | 1.61603e-02 | 2.46808e-04 | 0.271847e+06 |
| \( h = 1/64 \) | 2.09652e-05 | 8.07152e-03 | 6.12768e-04 | 0.346975e+06 |
| \( h = 1/128 \) | 5.22499e-06 | 4.03474e-03 | 1.53121e-04 | 0.125160e+07 |
| \( h = 1/256 \) | 1.30868e-06 | 2.01723e-03 | 3.82653e-06 | 0.271847e+07 |
| \( h = 1/512 \) | 3.26874e-07 | 1.00859e-03 | 9.56397e-07 | 0.835384e+08 |

| order | \( \approx 2 \) | \( \approx 1 \) | \( \approx 2 \) | - |

Table 1. \( L_2 \), broken \( H^1 \), nodal errors, and condition numbers with discontinuous jump condition

**Problem 2. Continuous Solution.** Consider the two-point boundary value problem with two interface points \( \alpha_1 = 1/3, \alpha_2 = 2/3 \)

\[
\frac{\partial}{\partial x} \left( -D \frac{\partial u}{\partial x} + 2\delta u \right) + \gamma u = f \quad x \in (0, 1)
\]

with the boundary conditions

\[
D_0 u'(0) = 0, \quad u(1) = 0.
\]

Here with \( n = 4 \)

\[
D(x) = \begin{cases} 
  D_1 = \frac{18(n-1)}{10n} & x \in [0, 1/3] \\
  D_2 = \frac{6nD_1 - 25}{3(n+1)} & x \in [1/3, 2/3] \\
  D_3 = \frac{8n - 3(n+1)D_2}{3(n+5)} & x \in [2/3, 1]; 
\end{cases}
\]
\[ \delta(x) = \begin{cases} 
\delta = 0.5(9nD_1 - 8.1(n - 1)) & x \in [0, 1/3] \\
\delta_2 = 0.5(3(n + 1)D_2 - 3nD_1 + 2\delta) & x \in [1/3, 2/3] \\
\delta_3 = 0.25(3(n - 1)D_3 - 3(n + 1)D_2 + 4\delta) & x \in [2/3, 1], 
\end{cases} \]

and \( \gamma = 10, 1, 0.1 \) in respective subintervals. At the interface points \( \alpha_i \) for \( i = 1, 2 \), the solution \( u \) is continuous and

\[ \begin{align*}
[u]_{\alpha_i} &= 0, \\
-D_iu'(\alpha^-_i) + 2\delta_iu(\alpha^-_i) &= -D_{i+1}u'(\alpha^+_i) + 2\delta_{i+1}u(\alpha^+_i).
\end{align*} \]

The exact solution is

\[ u(x) = \begin{cases} 
x^n/3 & x \in [0, 1/3] \\
x^{n+1} & x \in [1/3, 2/3] \\
3(1-x)x^{n+1} & x \in [2/3, 1].
\end{cases} \]

The enrichment function \( \psi \) is well-known \[3, 10, 2]:

\[ \psi(x) = \begin{cases} 
0 & x \in [0, x_k] \\
(x_{k+1} - \alpha)(x_k - x) & x \in [x_k, \alpha] \\
(x_{k+1} - x_k)(\alpha - x_k)(x - x_{k+1}) & x \in [\alpha, x_{k+1}] \\
0 & x \in [x_{k+1}, 1].
\end{cases} \]

The run results are displayed in Table 2.

| Problem 2 | \( L_2 \) error | \( H^1 \) error | nodal error | condition number |
|-----------|------------------|-----------------|-------------|-----------------|
| \( h = 1/8 \) | 8.58406e-03 | 2.91716e-01 | 2.07071e-02 | 0.127626e+05 |
| \( h = 1/16 \) | 2.11391e-03 | 1.46341e-01 | 4.56597e-03 | 0.109720e+06 |
| \( h = 1/32 \) | 5.30238e-04 | 7.35572e-02 | 1.11087e-03 | 0.304583e+06 |
| \( h = 1/64 \) | 1.32359e-04 | 3.67855e-02 | 2.76462e-04 | 0.175135e+07 |
| \( h = 1/128 \) | 3.31638e-05 | 1.84188e-02 | 6.91011e-05 | 0.511390e+07 |
| \( h = 1/256 \) | 8.29035e-06 | 9.21011e-03 | 1.72680e-05 | 0.277080e+08 |
| \( h = 1/512 \) | 2.07405e-06 | 4.60678e-03 | 4.31719e-06 | 0.825348e+08 |
| order & \( \approx 2 \) & \( \approx 1 \) & \( \approx 2 \) & - |

**Table 2.** \( L^2 \)-, broken \( H^1 \)-, nodal errors, and condition numbers with homogeneous jump conditions

**Problem 3. Implicit and Explicit Conditions Both Present.** In this problem, we combine the interfaces of the last two problems. The interface points are \( \alpha_0 = 1/9, \alpha_1 = 1/3 \) and \( \alpha_2 = 2/3 \). The two-point boundary value problem is

\[ \frac{\partial}{\partial x} \left( -D\frac{\partial u}{\partial x} + 2\delta u \right) + \gamma u = f \quad x \in (0, 1) \]

subject to the boundary conditions

\[ D_0u'(0) = 0 \quad u(1) = 0. \]

The coefficients are defined as follows:
\[
D(x) = \begin{cases} 
D_0 = 1 & x \in [0, 1/9] \\
D_1 = \frac{18(n-1)}{10n}, & x \in [1/9, 1/3] \\
D_2 = \frac{6nD_1-2\delta}{3(n+1)} & x \in [1/3, 2/3] \\
D_3 = \frac{8\delta - 3(n+1)D_2}{3(n+5)} & x \in [2/3, 1];
\end{cases}
\]

\[
\delta(x) = \begin{cases} 
\delta_0 = 0 & x \in [0, 1/9] \\
\delta = 0.5(9nD_1 - 8.1(n-1)) & x \in [1/9, 1/3] \\
\delta_2 = 0.5(3(n+1)D_2 - 3nD_1 + 2\delta) & x \in [1/3, 2/3] \\
\delta_3 = 0.25(3(n-1)D_3 - 3(n+1)D_2 + 4\delta_2) & x \in [2/3, 1];
\end{cases}
\]

\[n = 4 \text{ and } \gamma = 0, 10, 1, 0.1 \text{ in respective subintervals. The exact solution is}
\]

\[
u(x) = \begin{cases} 
u_0 = x^{n-1}/30 & x \in [0, 1/9] \\
u_1 = x^n/3 & x \in [1/9, 1/3] \\
u_2 = x^{n+1} & x \in [1/3, 2/3] \\
u_3 = 3(1-x)x^{n+1} & x \in [2/3, 1]
\end{cases}
\]

and satisfies the jump condition at 1/9 and \[48\] at the interface points 1/3 and 2/3. For the discontinuous interface point 1/9 we use the enrichment function defined in \[51\] and for the continuous interface points 1/3 and 2/3 we use the enrichment function in \[49\]. The run results are displayed in Table 3.

| Problem 3 | \(L_2\) error | \(H^1\) error | nodal error | condition number |
|-----------|----------------|----------------|-------------|-----------------|
| \(h = 1/8\) | 8.58383e-03 | 2.91715e-01 | 2.07048e-02 | 0.516955e+06 |
| \(h = 1/16\) | 2.11387e-03 | 1.46341e-01 | 4.56552e-03 | 0.207890e+06 |
| \(h = 1/32\) | 5.30234e-04 | 7.35577e-02 | 1.11075e-03 | 0.422880e+06 |
| \(h = 1/64\) | 1.32358e-04 | 3.67868e-02 | 2.76434e-04 | 0.175140e+07 |
| \(h = 1/128\) | 3.31640e-05 | 1.84202e-02 | 6.90932e-05 | 0.511405e+07 |
| \(h = 1/256\) | 8.29083e-06 | 9.21222e-03 | 1.72659e-05 | 0.277086e+08 |
| \(h = 1/512\) | 2.07413e-06 | 4.61030e-03 | 5.16955e-06 | 0.129948e+09 |

| order | \(\approx 2\) | \(\approx 1\) | \(\approx 2\) | - |

Table 3. \(L_2\)-, broken \(H^1\)-, nodal errors, and condition numbers with continuous and discontinuous jump conditions

4.2. Enriched Quadratic Elements. In Problems 4 and 6, we test our method on the conforming \(P_2\) elements enriched by the enrichment function in \(51\). All the conclusions in subsection 4.1 hold, including the statements of optimal order convergence and condition numbers. The purpose of this section is to see what to expect when going on to higher order elements. The theory will be developed in another paper.

**Problem 4. Discontinuous Solution.** The BVP setting is the same as in Problem 1, Eq. \(12\), of the previous section.

We test the effectiveness of the method with \(n = 4\) and with the enrichment function in \(51\). The run results are displayed in Table 4.
Table 4. $L_2$-, broken $H^1$-, nodal errors and condition numbers with discontinuous jump conditions

Problem 5. Continuous Solution. The BVP setting is exactly the same as in Problem 2, Eq. (47), of the previous section. The enrichment function $\psi$ is (49). The run results are displayed in Table 5.

Table 5. $L_2$-, broken $H^1$-, nodal errors and condition numbers with homogeneous jump conditions

Problem 6. Implicit and Explicit Conditions Both Present. The BVP setting is exactly the same as in Problem 3, Eq. (50), of the previous section. The run results are displayed in Table 6.

Table 6. $L_2$-, broken $H^1$-, nodal errors and condition numbers with continuous and discontinuous jump conditions
5. CONCLUDING REMARKS.

Based on optimal order analysis, we derived a family of enrichment functions for the conforming $P_1$ finite element, and the resulting enriched method can approximate discontinuous solutions in optimal order in the broken $H^1$ and $L^2$ norms. Encouraged by the preliminary numerical results for the quadratic element in subsection 4.2, we hope to extend the same approach to all $P_i, i \geq 2$.

Extension of our approach to higher dimensions is highly desirable. The tools we used in our approach in one dimension include Taylor’s expansion, extension operators in Sobolev spaces, and balancing the lower order large terms resulting from extension operators with the higher-order terms in the multipliers with the enrichment function. All these have their counterparts in higher dimensions. The new ingredient in higher dimensions will include some contamination from the added geometric complexity near the interface. An analogous 1D parameter called interface deviation $\epsilon$ was introduced in [2] to mimic the geometric complexity (The enrichment function breaks at $\alpha - \epsilon$ instead of the interface point $\alpha$). We wish to further investigate the effect of this parameter on our present method.

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