Automaton Research Project

Matrix Graph Automata

Author: Joshua Herman
Collaborator: Keith Pedersen

January 24, 2010
1 Abstract

The Graph Automata have been the paradigm in the expression of utilizing Graphs as a language. Matrix Graph grammars \([\text{?}]\) are an algebratization of graph rewriting systems. Here we present the dual of this formalizm which some extensions which we term Graph Field Automata. The advantage to this approach is a framework for expressing machines that can use Matrix Graph Grammars.

Keywords : Graph Theory, Turing machines, Automaton, Galois Theory, Graph Machine, Abstract Graph Galois Machine , Category Theory

2 Introduction

This paper describes an automata based upon an analogous extension of adjacency matrices to finite fields from \([\text{?}]\), which map the edges of a graph as members of a matrix. In a matrix describing graph A, the matrix position A\([3,4]\) will store a 1 iff vertex 3 is connected to vertex 4, otherwise it will store a 0 (the reverse position A\([4,3]\) will be identical to A\([3,4]\)). But one can see that a simple adjacency matrix is a binary device, incapable of representing more than a Turing machine. Thus an Graph Field Automata starts only with a graph and the simple adjacency matrix that describes the graph. Then it allows the adjacency matrix to hold real number values and performs real number operations upon the adjacency matrix. This creates an algebraic data structure that we call an abstract adjacency matrix. A Euclidean space is extended using two automorphisms on this abstract adjacency matrix, creating a Galois Field. Abstract Graph Galois Machines are more favorable than some current alternatives to Turing machines because, due to the ease of algebratization, this allows for the tools of algebra and finite fields to be used. Possible further extensions would involve the use of representation theory to reason about groups. But the greatest virtue of the Graph Field Automata is that, since its architecture is defined by graphs in Euclidean space, it is relatively straightforward to build a Graph Field Automata that expresses languages in BQP and other elements of the polynomial heirarchy. And if it is possible to emulate a quantum computer, this functionality would allow one to design algorithms that would run on both machines but would also take advantage of the exponential speedup on quantum computers. But regardless of its application to quantum computers, the Graph Field Automata shows great promise in its ability to allow for new and novel mathematical tools to be applied to graph rewriting systems.

3 Definitions

One of the difficulties when dealing with graph theory is the plethora of labels that are used interchangeably to describe the same thing. We define a graph as the interrelation between sets of vertices. These vertices possess no location in space (Cartesian or otherwise) because it is only the interrelation between vertices that is important. When two vertices are related or connected, they share an edge. Therefore, a graph is defined by a set of vertices and a set of edges connecting the vertices. A linear graph is one in which there is only one path connecting two distant vertices (i.e., there is only one set of edges one
can use to travel from vertex A to vertex Q). A non-linear graph is one in which there are multiple paths between at least one pair of vertices (or, stated differently, the edges form at least one loop). A planar graph can be drawn in two dimensions with no intersecting/overlapping edges and a non-planar graph can be drawn in two dimensions only when at least two edges intersect. A polar graph is one where at least one vertex has a different number of edges than the rest of the vertices. If all the vertices have the same number of edges, then there is some form of symmetry present and no edge is automatically given a unique status; this graph would be non-polar. Finally, any vertex with only one edge is considered an endpoint because it will represent a terminus to the graph. If one were travelling along the edges of the graph, upon reaching an endpoint one would have no choice but to turn back as the endpoint vertex has only one edge to choose from.

4 The Graph Field Grammar

While a graph may be simple to draw, describing the structure of any graph in graph theory is vastly more difficult because there is no unique way of labeling the vertices. With the multitude of different definition of graphs we seek to create an abstract representation which will work for any possible graph in graph theory. The construction that is presented allows for the expression of any graph in a concise language by abstracting the concept of an incidence matrix. The abstract graph representation was created to alleviate some of these woes by describing graphs in a concise string form. By defining some labeling rules, the number of possible descriptions for each unique graph is greatly reduced (though these rules provide the greatest benefit for linear graphs and very little benefit for non-planar graphs). The meta graph language is quite simple to describe, especially since its architecture is based upon a stack. The informal grammatical rules are as follows.

- a graph is always labeled with preference to endpoints. In any graph possessing endpoints, only the endpoints are candidates to be the initial vertex.
- However, if a graph has no endpoints, then only its polar vertices are candidates for the initial vertex.
- obviously this provides the most benefit for linear graphs, because most non-linear graphs have no endpoints
- since most non-planar graphs are also non-polar, they are they hardest to label. One can only follow the first rule To define the language we must define a decision problem for the language so that we can recognize it. Given the input of a string written in the following manner we can generate a given graph. The generation of the graph and the string convey the same information and therefore have the same structure.

4.1 Formal Definition

Let the grammar be an alphabet $\sigma$ where the strings over $\sigma^*$ are as follows. The null string is denoted as $\epsilon$ which is a string over $\sigma$. It also follows that each
element of the set is either a member of $Z_2$ or $Z_2$ where $n$ is a member of the
vertex set. If there is other information such as directionality or color, this is
encoded with the string. If more than one connection is allowed then utilize
members of $Z_n$.

4.2 Definition of incidence field

The abstract incidence field is a field representation of a graph. The graph has
the added property of an abstract labeling. The field is defined as follows. Let
$V = v_1...v_n$ be the set of vertices, $G(V) = e_1...e_n$ be the function mapping
the set of vertices which defines $E = e_1...e_m$ the set of edges.

Note both $V$ and $E$ are parallel to each other and they can’t interact. Also
the field $G$ defines a function which maps elements of $V$ to $E$.

\[
\{ \text{AIM}(G) = \begin{pmatrix} Z_a & Z_b & \ldots & Z_e \\ Z_c & Z_d & \ldots & 1, 2, 3, 4 \end{pmatrix} \}
\]

What is added is a new set representing the labeling of the incidence field which
is an infinite dimensional subset of $G_n$; this labeling gives information about
which vertex has connectivity to another vertex. Elementary row and column
operations can be generalized to changing the order of the labels and therefore
the columns. This generates a new field representation of the graph. Therefore an
abstract graph machine is all possible labeling of the graph which is a symmetry
group of $C_n$. Furthermore, let the field be defined as As a vector space $C_{e,n}^+$, *
and a Vector sort also sorted by lexicographic order. The two axioms are as
follows

Connectivity is denoted by subsets of the set $Z_n$. Let the information of con-
nectivity be a unique number for each graph. The number is then partitioned
based upon the connectivity of the graph. Depending on the amount of points
connected, each point has an associated number which is less than the amount
of vertices. Any other descriptions of graphs which denote some type of connec-
tivity are numbered as a binary sequence of $Z_n$. The union of this is as follows:
$\bigcup_{i=1}^n Z_i$ where $i$ is the sequence of descriptions of graphs. This numbering sys-
tem allows for the expression of all possible graphs in this abstract incidence
field.

The formalization of the machine is as follows.

5 Formal Definition of the Field Graph Machine

5.1 Definition

A deterministic finite graph machine or acceptor deterministic finite state ma-
chine is a quintuple $(\varepsilon, S, s_0, \rho, F)$, where:

1. $1-\varepsilon, S$ is the input digraphs (a finite, non-empty set of digraphs).
2. $2-\varepsilon, S$ is a finite, non-empty set of states consisting of a nilhation $F_2$
3. $3-\varepsilon, s_0$ is an initial state, an element of $S$. 

3
4. $\rho$ is the state-transition function: $\rho S \times \sigma \rightarrow S$

5. $F$ is the set of final states, a (possibly empty) subset of $S$.

6. Each set of states is denoted by an adjacency field with a vertex field.

### 5.2 Outline of the Machine

The input to the machine is any given abstract graph representation and a set of operations on that specific graph. The output is another processed abstract algebraic graph representation. The central data structure that is used is an infinite dimensional Euclidean space over the real numbers. The actual automaton is a computer over the real numbers. It achieves this by performing a mapping function over a subset of the reals and then iterating over the necessary values.

### 5.3 Rationale

The two automorphisms on the field create a Galois Field. This field is then extended using algorithms to create a Galois Machine. The cycling automorphism algorithm allows for all of the possible states of the computer to easily be cycled through. This allows for Monte Carlo algorithms or Las Vegas algorithms. It is to be noted that both algorithms still need a pseudo- or true random source to operate. If a Turing complete language is needed, SKI combinators are provided. If gate like operations are needed, one can use the combinators or gate functions without running the cycling algorithm.

### 6 Examples of the Machine

#### 6.0.1 Outline of Machine

Using the definitions from [?] we define the Galois machine with a Graph alphabet as follows. turn the new labeling.

#### 6.1 Galois Machines

Galois machines are an algebraic model of computation. They have the following structure.

The universe of the model is defined as the labeling. The functions of the model are the binary functions of a Real field. The relation is the automorphisms that have been previously described. The category that we will study is automorphisms between specific graph representations and the actual category of all possible graphs.

We define a functor which maps a actual graph to its representation. This is a bijective homomorphism between each individual graph and each individual representation. This functor has the ability to create a set of homomorphisms between a type of graph and the representation.

We define an abstract algebraic data structure as the abstract graph representation and its associated encoding algorithms. The example implementation that is used in this paper is a set of matrices but given the properties any other

Once the graph is transformed into its representation.
To create branching in the machine a copy of the machine is created in another space. The input language is the abstract adjacency field. The formal definition of the input is as follows:

The field can also be generalized into any amount of dimensions. Given the above cyclic algorithm to operate the machine, you need to place both the vector, which determines the lexicographic order of the matrices, and the data themselves. Both store the information contained within a graph.

6.1.1 Operation list

6.2 Example Run

State 1

\[
(D, L, T, O) \text{ where } D_{a1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}
\]

\[D_{a2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad L = (1, 2, 3, 4)T = (+, -, /) \cup \text{CurrentOp}(GCP(a1))\]

State 2

\[
(D, L, T, O) \text{ where } D_{a1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_{a2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad L = (1, 2, 3, 4)T = (-, /)O = (+)\]

Both structures are embedded in a dimension of a Euclidean space.

Algorithm 1 Graph Cyclic Procedure

Let the input be a set \(C_n\) listing the vertex labels of a graph with elements of the set \(Z_n\). Steps of algorithm

0 Store the value of \(C_0\) in a temporary field \(C_0\).

1 Starting at \(C_0\), replace each member of the set \(C_n\) with its successor (\(C_i = C_{i+1}\)).

2 If you have reached the end of the set \(C_n\), replace the terminating value with \(C_0\).

3 Return the new labeling.

7 Constructive Proof of the Permutation group of abstract incidence matrices

Algorithm 2 Permutation Automorphism Algorithm Steps of algorithm

Step 0 Given the Abstract Graph Machine, prove that all elements obey the conjecture that it is a symmetry group of order \(C_n\).
Step 1 Create an $N \times N$ Abstract Graph Machine the label vector is of size $N$.

Step 2 Starting at $C_0$, replace a member of the set $C_n$ with another member $C_m$.

Step 3 Return the new permuted abstract graph machine.

**Algorithm 3** Mirror Automorphism Algorithm

Steps of algorithm

Step 0 Given the Abstract Graph Machine prove that all elements obey the conjecture that it is a symmetry group of order $C_n$.

Step 1 Create a copy of the selected $N \times N$ Abstract Graph Machine the label vector is of size $N$.

Step 2 If a row mirror is required

Step 3 Return the new permuted abstract graph machine.

When the labeling is returned this can be applied to the Abstract Adjacency Field and this creates a mapping of the information represented. The graph is then relabeled in this new order to return a new state.

### 8 Any Algorithm You Like

#### 8.1 New Methods of Operation of the Machine

To branch the computation, create a new copy abstract graph machine in the Galois space in another dimension. Then apply the new labeling to reorder the field to get a new adjacency field and relabel the graph with a new label vector. For each possible branch, a Galois machine is implemented. On the other hand, if you want a totally deterministic operation follow the NOR gate operation or the SKI combinatory logic, which are outlined below.

**Algorithm 4** Proof of Functional Completeness

Let us investigate a NOR gate expressed in an abstract graph machine. A 1 stands for a value of 1 in the complex plane for the vector. A NOR gate is defined as an adjacency field as follows: 

\[
N = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \cup (1, 2, 3, 4)
\]

This gate is a binary field which is the field representation of a graph. By changing how the graph is connected utilizing elementary row operations, the set of zeros can be changed into the resulting graph using two elementary row operations. These operations can be done by changing the labels of the graph. Change the labeling of row 2 to row 4 and then reverse the column labeling. Since the NOR gate is functionally complete, we have a proof of the functional completeness of the computer. Depending on the amount of NOR gates, you select a row and put a branch representing a continuation of the next gate. Q.E.D.
8.1.1 Random Operation Methods

Given a verification function and a random number generator, one can operate the computer randomly quite easily. All transitions instead of being of the set \(0, 1\) we express them as a probability density of the rationals from \(0, 1\). What would occur is one needs a verification function and randomly permutes the information until a correct result is obtained. To operate the machine randomly without gambling resources a generalized Las Vegas operation mode can be done if a verification function is written which verifies that a correct result has occurred.

\[\text{Note: for this operation to work the solution space must be a part of the input. Therefore it is recommended that while operating randomly a verification function must be input.}\]

9 Notices and Acknowledgments

Without the paper by Blum Shub Smale I could not have implemented the hypercomputer. Special thanks to Jim Otto Jr. for helping me. Thanks to Keith Pedersen. Without his help I could not come to this conclusion.

References

[1] Perez Velasco, P. P., de Lara, J. 2006. Towards a New Algebraic Approach to Graph Transformation. Basic Concepts, Sequentialization and Parallelism: Long Version. Tech. Rep. of the School of Comp. Sci., Univ. Autonoma Madrid. Available at: http://www.ii.uam.es/jlara/investigacion/techrep0306.pdf.

[2] Abstract Algebra By David S. Dummit, Richard M. Foote Published by Wiley, 2003 ISBN 0471433349, 9780471433347 944 pages

[3] Crocker, D.; Overell, P. (January 2008). "Augmented BNF for Syntax Specifications: ABNF" (plain text) 16. RFC Editor. Retrieved on 2008-05-26.

[4] Herbert Wiklicky Quantitative Computation by Hilbert Machines 1998

[5] Ilwoo Cho and Palle E. T. Jorgensen Applications of Automata and Graphs: Labeling-Operators in Hilbert Space I http://www.citebase.org/abstract?id=oai:arXiv.org:0803.2034 2008

[6] Curry, Haskell B., Feys, R & Craig, W. Combinatory Logic Vol 1 North-Holland

[7] Algorithms and Theory of Computation Handbook, CRC Press LLC, 1999. "Las Vegas algorithm", in Dictionary of Algorithms and Data Structures [online], Paul E. Black, ed., U.S. National Institute of Standards and Technology, 17 July 2006.

[8] Reinhard Diestel Graph Theory Electronic Edition 2005 Springer-Verlag Heidelberg, New York 2005

[9] Richard P. Feynman, Anthony Hey, Tony Hey, Robin W. Allen \LaTeX: Feynman Lectures on Computation. Westview Press; 1 edition, Massachusetts, 1st Edition, (July 2000).
[10] Reinhard Diestel \LaTeX: Graph Theory. Springer-Verlag Heidelberg, New York 3rd Edition, (2005).

[11] Felix Klein, Robert Hermann Development of Mathematics in the 19th Century Published by Math Sci Press, 1979 ISBN 0915692287, 9780915692286 630 pages