The Menger and projective Menger properties of function spaces with the set-open topology

Alexander V. Osipov

Krasovskii Institute of Mathematics and Mechanics, Ural Federal University, Ural State University of Economics, 620219, Yekaterinburg, Russia

Abstract
For a Tychonoff space $X$ and a family $\lambda$ of subsets of $X$, we denote by $C_\lambda(X)$ the space of all real-valued continuous functions on $X$ with the set-open topology. In this paper, we study the Menger and projective Menger properties of a Hausdorff space $C_\lambda(X)$. Our main results state that if $\lambda$ is a $\pi$-network of $X$, then

1. $C_\lambda(X)$ is Menger space, if and only, if $C_\lambda(X)$ is $\sigma$-compact,
and, if $Y$ is a dense subset of $X$, then

2. $C_p(Y|X)$ is projective Menger space, if and only, if $C_p(Y|X)$ is $\sigma$-pseudocompact.

Keywords: Menger, projective Menger, set-open topology, $\sigma$-compact, $\sigma$-pseudocompact, $\sigma$-bounded, basically disconnected space, function space

2010 MSC: 54C25, 54C35, 54C40, 54D20

1. Introduction

Throughout this paper $X$ will be a Tychonoff space. Let $\lambda$ be a family non-empty subsets of $X$, $C(X)$ the set of all continuous real-valued function on $X$. Denote by $C_\lambda(X)$ the set $C(X)$ is endowed with the $\lambda$-open topology. The elements of the standard subbases of the $\lambda$-open topology will be denoted as follows: $[F, U] = \{f \in C(X) : f(F) \subseteq U\}$, where $F \in \lambda$, $U$ is an open subset of $\mathbb{R}$. Note that if $\lambda$ consists of all finite subsets of $X$ then the $\lambda$-open topology is equal to the topology of pointwise convergence, that...

Email address: OAB@list.ru (Alexander V. Osipov)
is $C_\lambda(X) = C_p(X)$. Denote be $C_p(Y|X) = \{ h \in C_p(Y) : h = f|_Y \text{ for } f \in C(X) \}$ for $Y \subset X$.

Recall that, if $X$ is a space and $\mathcal{P}$ a topological property, we say that $X$ is $\sigma$-$\mathcal{P}$ if $X$ is the countable union of subspaces with the property $\mathcal{P}$.

So a space $X$ is called $\sigma$-compact ($\sigma$-pseudocompact, $\sigma$-bounded), if $X = \bigcup_{i=1}^{\infty} X_i$, where $X_i$ is a compact (pseudocompact, bounded) for every $i \in \mathbb{N}$.

N.V. Velichko proved that $C_p(X)$ is $\sigma$-compact, if and only, if $X$ is finite. In [20], V.V. Tkachuk clarified when $C_p(X)$ is $\sigma$-pseudocompact and when $C_p(X)$ is $\sigma$-bounded, and considered similar questions for the space $C^*_p(X)$ of bounded continuous functions on $X$.

A space $X$ is said to be Menger [9] (or, [17]) if for every sequence $\{U_n : n \in \omega\}$ of open covers of $X$, there are finite subfamilies $V_n \subset U_n$ such that $\bigcup\{V_n : n \in \omega\}$ is a cover of $X$.

Every $\sigma$-compact space is Menger, and a Menger space is Lindelöf. The Menger property is closed hereditary, and it is preserved by continuous maps. It is well known that the Baire space $\mathbb{N}^\omega$ (hence, $\mathbb{R}^\omega$) is not Menger.

In [2], A.V. Arhangel’skii proved that $C_p(X)$ is Menger, if and only, if $X$ is finite.

Let $\mathcal{P}$ be a topological property. A.V. Arhangel’skii calls $X$ projectively $\mathcal{P}$ if every second countable image of $X$ is $\mathcal{P}$. Arhangel’skii consider projective $\mathcal{P}$ for $\mathcal{P} = \sigma$-compact, analytic [3], and other properties.

Lj.D.R. Kočinac characterized the classical covering properties of Menger, Rothberger, Hurewicz and Gerlits-Nagy in term of continuous images in $\mathbb{R}^\omega$. The projective selection principles were introduced and first time considered in [11].

Every Menger space is projectively Menger. It is known (Theorem 2.2 in [11]) that a space is Menger, if and only, if it is Lindelöf and projectively Menger.

Characterizations of projectively Menger spaces $X$ in terms a selection principle restricted to countable covers by cozero sets are given in [3].

In [16], M. Sakai proved that $C_p(X)$ is projectively Menger, if and only, if $X$ is pseudocompact and $b$-discrete.

In this paper we study the Menger property of Hausdorff space $C_\lambda(X)$, and the projective Menger property of $C_p(Y|X)$ where $Y$ is dense subset of $X$. 

2
2. Main definitions and notation

Recall that a family \( \lambda \) of non-empty subsets of a topological space \((X, \tau)\) is called a \( \pi \)-network for \( X \) if for any nonempty open set \( U \in \tau \) there exists \( A \in \lambda \) such that \( A \subset U \).

Throughout this paper, a family \( \lambda \) of nonempty subsets of the set \( X \) is a \( \pi \)-network. This condition is equivalent to the space \( C_\lambda(X) \) being a Hausdorff space \([12]\).

We will also need the following assertion \([1], [4]\).

**Proposition 2.1.** If \( \mathbb{I}_\alpha = \mathbb{I} = [0, 1] \) for \( \alpha \in A \) and \( Y \) is a subspace of the Tychonoff cube \( \mathbb{I}^A = \prod \{ \mathbb{I}_\alpha : \alpha \in A \} \) which, whatever the countable set \( B \subset A \), projects under the canonical projection \( \pi_B : \mathbb{I}^A \mapsto \mathbb{I}^B \) onto the whole cube \( \mathbb{I}^B = \prod \{ \mathbb{I}_\alpha : \alpha \in B \} \) of \( \mathbb{I}^A \), then \( Y \) is pseudocompact.

**Theorem 2.2.** (Nokhrin \([12]\)) For a Tychonoff space \( X \) the following statements are equivalent:

1. \( C_\lambda(X) \) is a \( \sigma \)-compact;
2. \( X \) is a pseudocompact, \( D(X) \) is a dense \( C^* \)-embedded set in \( X \) and family \( \lambda \) consists of all finite subsets of \( D(X) \), where \( D(X) \) is an isolated points of \( X \).

The closure of a set \( A \) will be denoted by \( \overline{A} \) (or \( cl(A) \)); the symbol \( \emptyset \) stands for the empty set. As usual, \( f(A) \) and \( f^{-1}(A) \) are the image and the complete preimage of the set \( A \) under the mapping \( f \), respectively.

A subset \( A \) of a space \( X \) is said to be bounded in \( X \) if for every continuous function \( f : X \mapsto \mathbb{R} \), \( f|A : A \mapsto \mathbb{R} \) is a bounded function. Every \( \sigma \)-bounded space is projectively Menger (Proposition 1.1 in \([3]\)).

3. Main results

In order to prove the main theorem we need to prove some statements that we call Lemmas, but note their self-importance.

Recall that a space \( X \) is called basically disconnected \([8]\), if every cozero-set has an open closure. Clearly, every basically disconnected (Tychonoff) space is zero-dimensional space.

**Lemma 3.1.** If \( C_\lambda(X) \) is Menger, then \( X \) is a basically disconnected space.
Proof. Let $U \subseteq X$ be a cozero set in $X$. Claim that $\overline{U} = \text{Int} \overline{U}$. Suppose that $\overline{U} \setminus \text{Int} \overline{U} \neq \emptyset$. Since $U$ is a cozero set, there are open sets $U_n$ of $X$ such that for each $n \in \mathbb{N}$, $\overline{U}_n \subseteq U_{n+1}$ and $\bigcup_{n=1}^{\infty} U_n = U$. For each $n,m \in \mathbb{N}$, we put 

$$Z_{n,m} = \{ f \in C(X, [0,1]) : f(\overline{U}) \equiv 0 \text{ and } f(U_n) \subset [\frac{1}{2^m}, 1] \}.$$ 

Note that $Z_{n,m}$ is closed subset of $C(X)$ for each $n, m \in \mathbb{N}$. Let $h \notin Z_{n,m}$.

If $x \in X \setminus \text{Int} \overline{U}$ such that $h(x) \neq 0$. Since $\lambda$ is $\pi$-network of $X$, there is $A \in \lambda$ such that $A \subset h^{-1}(h(x) - \frac{|h(x)|}{2}, h(x) + \frac{|h(x)|}{2}) \cap \text{Int} (X \setminus \text{Int} \overline{U})$. Then $h \in [A, (h(x) - \frac{|h(x)|}{2}, h(x) + \frac{|h(x)|}{2})]$ and $[A, (h(x) - \frac{|h(x)|}{2}, h(x) + \frac{|h(x)|}{2})] \cap Z_{n,m} = \emptyset$.

If $x \in U_n$ and $h(x) \notin [\frac{1}{2^m}, 1]$. Let $d = \frac{\text{diam}(h(x), [\frac{1}{2^m}, 1])}{2}$. Since $\lambda$ is a $\pi$-network of $X$, there is $A \in \lambda$ such that $A \subset h^{-1}((h(x) - d, h(x) + d) \cap U_n$. Then $h \in [A, (h(x) - d, h(x) + d)]$ and $[A, (h(x) - d, h(x) + d)] \cap Z_{n,m} = \emptyset$.

Assume that $\bigcap\{Z_{n,m} : n \in \mathbb{N}\} = \emptyset$ for all $m \in \mathbb{N}$. Using the Menger property of $C(X)$, we can take some $\varphi \in \mathbb{N}^\mathbb{N}$ such that $\bigcap\{Z_{\varphi(m),m} : m \in \mathbb{N}\} = \emptyset$. For each $m \in \mathbb{N}$, take any $g_m \in C(X)$ satisfying $g_m(X \setminus \text{Int} \overline{U}) \equiv 0$ and $g_m(U_{\varphi(m)}) = \{1\}$. Let $g = \sum_{j=1}^{\infty} 2^{-j} g_j$. Then, $g \in C(X)$ and $g(X \setminus \text{Int} \overline{U}) \equiv 0$. Fix any $m \in \mathbb{N}$, $1 \leq k \leq \varphi(m)$ and $x \in U_k$. Then we have $g(x) = \sum_{j=1}^{\infty} 2^{-j} g_j(x) \geq 2^{-m} g_m(x) = 2^{-m}$.

Hence, $g \in \bigcap\{Z_{\varphi(m),m} : m \in \mathbb{N}\}$. This is a contradiction. Thus, there is some $m \in \mathbb{N}$ such that $\bigcap\{Z_{n,m} : n \in \mathbb{N}\} \neq \emptyset$. Let $p \in \bigcap\{Z_{n,m} : n \in \mathbb{N}\}$. Then $p(U) \subset [\frac{1}{2^m}, 1]$ and $p((X \setminus \text{Int} \overline{U})) \equiv 0$. It follows that $\overline{U} \setminus \text{Int} \overline{U} = \emptyset$.

A subset $G \subset \omega^\omega$ is dominating if for every $f \in \omega^\omega$ there is a $g \in G$ such that $f(n) \leq g(n)$ for all but finitely many $n$.

**Theorem 3.2.** (Hurewicz [10]) A second countable space $X$ is Menger iff for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is not dominating.

"Second countable" can be extended to "Lindelöf":

**Theorem 3.3.** (Kočinac [11], Theorem 2.2) A Lindelöf space $X$ is Menger iff for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is not dominating.

**Lemma 3.4.** If $C(X)$ is Menger. Then $X$ is pseudocompact.

Proof. Assume that $X$ is not pseudocompact and $f \in C(X)$ is not bounded function. Without loss of generality we can assume that $\mathbb{N} \subset f(X)$. For each $n \in \mathbb{N}$ we choose $A_n \in \lambda$ such that $A_n \subset f^{-1}((n - \frac{1}{3}, n + \frac{1}{3}))$. By
Lemma 3.1. \( F_n = \overline{f^{-1}\left((n - \frac{1}{2}, n + \frac{1}{2})\right)} \) is clopen set for each \( n \in \mathbb{N} \). Let \( K = \{ f \in C(X) : f|F_n = s_{f,n} \text{ for each } n \in \mathbb{N} \text{ and } s_{f,n} \in \mathbb{R} \} \). Then \( K \) is closed subset of \( C_\lambda(X) \) and, hence, it is Menger. Fix \( a_n \in A_n \) for every \( n \in \mathbb{N} \). Note that \( D = \{a_n : n \in \mathbb{N}\} \) is a \( C \)-embedded copy of \( \mathbb{N} \) (3L (1) in [8]). So we have a continuous mapping \( F : K \mapsto \mathbb{R}^D \) the space \( K \) onto \( \mathbb{R}^D \). But \( F(K) = \mathbb{R}^D = \mathbb{R}^\omega \) is dominating, contrary to the Theorem 3.3. \[ \square \]

**Lemma 3.5.** If \( C_\lambda(X) \) is Menger, then \( \mu = \{ A \in \lambda : A \text{ is finite subset of } X \} \) is a \( \pi \)-network of \( X \).

**Proof.** Assume that there exist an open set \( U \) of \( X \) such that \( B \not\subset U \) for every \( B \in \mu \). Fix a family \( \{V_n : n \in \mathbb{N}\} \) of open subsets of \( X \) such that \( V_n \subset U \) for every \( n \in \mathbb{N} \) and \( V_n \cap V_{n'} = \emptyset \) for \( n \neq n' \). Fix \( x_n \in V_n \) and \( \epsilon > 0 \). For every \( f \in C_\lambda(X) \) and \( n \in \mathbb{N} \) consider \( B_{f,n} \subset \lambda \) such that \( B_{f,n} \subset f^{-1}\left((f(x_n) - \epsilon, f(x_n) + \epsilon)\right) \cap V_n \). Then \( U_n = \{[B_{f,n}, (f(x_n) - \epsilon, f(x_n) + \epsilon)] : f \in C_\lambda(X)\} \) is an open cover of \( C_\lambda(X) \) for every \( n \in \mathbb{N} \). Using the Menger property of \( C_\lambda(X) \), for sequence \( \{U_n : n \in \omega\} \) of open covers of \( C_\lambda(X) \), there are finite subfamilies \( S_n \subset U_n \) such that \( \bigcup S_n \in \omega \) is a cover of \( C_\lambda(X) \). Let \( S_n = \{ [B_{f_1,n}, (f_{1,n}(x_n) - \epsilon, f_{1,n}(x_n) + \epsilon)], ..., [B_{f_{k(n),n}}, (f_{k(n),n}(x_n) - \epsilon, f_{k(n),n}(x_n) + \epsilon)] \} \) for every \( n \in \mathbb{N} \). Since \( B_{f_{s,n}} \) is an infinite subset of \( X \), we fix \( z_{s,n} \in B_{f_{s,n}} \) for every \( s \in 1, k(n) \) and \( n \in \mathbb{N} \) such that \( z_{s',n} \neq z_{s'',n} \) for \( s' \neq s'' \). Let \( Z = \{ z_{s,n} : s \in 1, k(n) \text{ and } n \in \mathbb{N} \} \).

Define the function \( q : Z \mapsto \mathbb{R} \) such that \( q(z_{s,n}) = 0 \) if \( 0 \notin (f_{s,n}(x_n) - \epsilon, f_{s,n}(x_n) + \epsilon) \), else \( q(z_{s,n}) = 2\epsilon \) for \( s \in 1, k(n) \) and \( n \in \mathbb{N} \). By Lemma 3.1, \( X \) is a basically disconnected space.

Recall that (14N p.215 in [8]) every countable set in a basically disconnected space is \( C^* \)-embedded.

Hence, there is \( t \in C_\lambda(X) \) such that \( t|Z = q \). But \( t \notin \bigcup S_n : n \in \omega \}. \) This is a contradiction. \[ \square \]

Denote \( D(X) \) a set of isolated points of \( X \).

**Lemma 3.6.** If \( C_\lambda(X) \) is Menger, then \( D(X) \) is dense set in \( X \).

**Proof.** Assume that there exist an open set \( W \neq \emptyset \) such that \( W \cap D(X) = \emptyset \). By Lemma 3.5, \( \mu \) is \( \pi \)-network of \( X \), hence, there is \( A \in \mu \) such that \( A \subset W \). Note that \( X \setminus A \) is dense set in \( X \). The constant zero function defined on
Assume that $V$ is an open cover of $C$ that $\bigcup V = X \setminus A$. Let $\epsilon > 0$. Then $V = \{B_f, (f(x_f) - \frac{|f(x_f)|}{2}, f(x_f) + \frac{|f(x_f)|}{2}) : f \in C(X) \setminus \{f_0\} \cup A, (-\epsilon, \epsilon]\}$ is an open cover of $C(X)$. Since $C(X)$ is Menger and, hence, $C(X)$ is Lindelöf, there is a countable subcover $V' = \{B_{f_n}, (f_n(x_f) - \frac{|f_n(x_f)|}{2}, f_n(x_f) + \frac{|f_n(x_f)|}{2}) : n \in \mathbb{N}\} \cup A, (-\epsilon, \epsilon]\subset V$ of $C(X)$. Since $X$ is a basically disconnected space and every countable set in a basically disconnected space is $C^*$-embedded, there is $h \in C(X)$ such that $h|_{\bigcup_{n \in \mathbb{N}} B_{f_n} = 0}$ and $h(a) = \epsilon$ for some $a \in A$. Note that $h \notin \bigcup V'$, to contradiction.

**Lemma 3.7.** If $C(X)$ is Menger, then $D(X)$ is $C^*$-embedded.

**Proof.** Let $f$ be a bounded continuous function from $D(X)$ into $\mathbb{R}$, and $F_A = \{g \in C(X) : g|A = f|A\}$ for $A \in D(X)^\omega$. Note that $F_A$ is closed subset of $C(X)$ and, by Lemma 3.1, $F_A \neq \emptyset$. So $\xi = \{F_A : A \in D(X)^\omega\}$ is family of closed subspaces with the countable intersection property. Since $C(X)$ is Menger, hence, it is Lindelöf, and every family of closed subspaces of with the countable intersection property has non-empty intersection. It follows that $\bigcap \xi \neq \emptyset$. We thus get that $\tilde{f} \in \bigcap \xi$ such that $\tilde{f} \in C(X)$ and $\tilde{f}|D(X) = f$.

**Proposition 3.8.** Let $X = \mathbb{N}$ and let $\lambda = \{X\} \bigcup \{\{x\} : x \in X\}$. Then $C^*_\lambda(X)$ is not Menger.

**Proof.** Assume that $C^*_\lambda(X)$ is Menger. For every $i \in \mathbb{N}$ consider an open cover $\mathcal{V}_i = \{[N, (-2 + \frac{1}{i+1}, 2 - \frac{1}{i+1})] \cup \{(x, (-\infty, -2 + \frac{2i+1}{2(i+1)}, +\infty)) : x \in X\}$ of $C^*_\lambda(X)$. Using the Menger property of $C^*_\lambda(X)$, for sequence $\{\mathcal{V}_i : i \in \mathbb{N}\}$ of open covers of $C^*_\lambda(X)$, there are finite subfamilies $\mathcal{S}_i \subset \mathcal{V}_i$ such that $\bigcup \{\mathcal{S}_i : i \in \mathbb{N}\}$ is a cover of $C^*_\lambda(X)$.

Without loss of generality we can assume that $[N, (-2 + \frac{1}{i+1}, 2 - \frac{1}{i+1})] \in \mathcal{S}_i$ for each $i \in \mathbb{N}$.

By using induction, for each $i \in \mathbb{N}$, determine the values of the function $f$ at some points, depending on the $\mathcal{S}_i$, as follows:

For $i = 1$ and

$\mathcal{S}_1 = \{[N, (-2 + \frac{1}{2}, 2 - \frac{1}{2})], [x_1, (-\infty, -2 + \frac{3}{4}) \cup (2 - \frac{3}{4}, +\infty)], ..., [x_k, (-\infty, -2 + \frac{3}{4}) \cup (2 - \frac{3}{4}, +\infty)]\}$, define
Proof. Suppose that $A$ if $A$ is a countable set in a basically disconnected space is $C$ (note that either $z \in C$). Then a space $C$ with $A = \bigcup \{x_n : n \in \mathbb{N}\}$, where $P = \bigcup \{x_n : n \in \mathbb{N}\}$ for $i = m$.

$$S_m = \{[\mathbb{N}, (-2 + \frac{1}{m+1}, 2 - \frac{1}{m+1})], [x^m_1, (-\infty, -2 + \frac{2m+1}{2m(m+1)} \cup (2 - \frac{2m+1}{2m(m+1)}, +\infty)], \ldots, [x^m_{k(m)}, (-\infty, -2 + \frac{2m+1}{2m(m+1)} \cup (2 - \frac{2m+1}{2m(m+1)}, +\infty)]\},$$

$$f(x^m_n) = 0 \text{ where } x^m_n \notin P_{m-1} \text{ for } n \in 1, k(m) \text{ and }$$

$$f(s_m) = p_m \text{ where } p_m \in [-2 + \frac{2m+1}{2m(m+1)}, 2 - \frac{2m+1}{2m(m+1)+2}] \cup (2 + \frac{m+1}{1}, 2 - \frac{1}{m+1}) \text{ for some } s_m \in X \setminus P_{m-1}. \text{ Denote } P_m = \bigcup \{x^m_n \cup s_m \cup P_{m-1} \text{ and }$$

$$P = \bigcup_{m \in \mathbb{N}} P_m.$$ If $X \setminus P \neq \emptyset$, then let $f(x) = 1$ for $x \in X \setminus P$.

By construction of $f$, $f \notin S_i$ for every $i \in \mathbb{N}$, to contradiction. \hfill $\square$

**Lemma 3.9.** If $C_\lambda(X)$ is Menger, then each $A \in \lambda$ is a finite subset of $D(X)$.

**Proof.** Suppose that $C_\lambda(X)$ is Menger, $\tilde{X} = \{A\} \bigcup \{\{x\}, x \in D(X)\}$ and $A \in \lambda$ is an infinite subset of $X$. Then $C_\lambda(X)$ is Menger, too. Note that if $A$ is countable and $A \subset D(X)$, then we have a continuous mapping $g : C_\lambda(X) \rightarrow C^*_p \cup \{N\}$. Hence, $C^*_p \cup \{N\}$ is Menger, contrary to Proposition 3.8.

Let $V = (-1, 1) \cup [\mathbb{R} \setminus [-4, 4]]$. Consider $U = \{[A, V] \cup \{[x, \mathbb{R} \setminus [-\frac{2}{3}, \frac{2}{3}]] : x \in D(X)\}$. Since $D(X)$ is dense subset of $C_\lambda(X)$ (Lemma 3.6), $U$ is an open cover of $C_\lambda(X)$ and, hence, there is a countable subcover $U' \subset U$ of $C_\lambda(X)$. Let $U' = \{[A, V], [x_1, \mathbb{R} \setminus [-\frac{2}{3}, \frac{2}{3}]], \ldots, [x_n, \mathbb{R} \setminus [-\frac{2}{3}, \frac{2}{3}]], \ldots\}$. Let $z \in A \setminus \bigcup_n \{x_n\}$ (note that either $z \in A \setminus D(X)$ or $A \subset D(X)$ and $|A| > \aleph_0$). Since every countable set in a basically disconnected space is $C^*$-embedded, there is $h \in C_\lambda(X)$ such that $h \bigcup \{x_n\} = 0$ and $h(z) = 2$. It follows that $h \notin U'$, to contradiction. It follows that $A$ is finite subset of $D(X)$. \hfill $\square$

**Theorem 3.10.** Let $X$ be a Tychonoff space and let $\lambda$ be a $\pi$-network of $X$. Then a space $C_\lambda(X)$ is Menger, if and only if, $C_\lambda(X)$ is $\sigma$-compact.

**Proof.** By Lemma 3.4, $X$ is pseudocompact. By Lemmas 3.5 and 3.9 the family $\lambda$ consists of all finite subsets of $D(X)$, where $D(X)$ is an isolated
By Lemma 3.7, $D(X)$ is a dense $C^*$-embedded set in $X$. It follows that $C_\lambda(X)$ is $\sigma$-compact (Theorem 2.2).

Various properties between $\sigma$-compactness and Menger are investigated in the papers [19, 6]. We can summarize the relationships between considered notions in ([19], see Figure 1), Theorems 3.10 and 2.2. Then we have the next

**Theorem 3.11.** For a Tychonoff space $X$ and a $\pi$-network $\lambda$ of $X$, the following statements are equivalent:

1. $C_\lambda(X)$ is $\sigma$-compact;
2. $C_\lambda(X)$ is Alster;
3. (CH) $C_\lambda(X)$ is productively Lindelöf;
4. "TWO wins M-game" for $C_\lambda(X)$;
5. $C_\lambda(X)$ is projectively $\sigma$-compact and Lindelöf;
6. $C_\lambda(X)$ is Hurewicz;
7. $C_\lambda(X)$ is Menger;
8. $X$ is a pseudocompact, $D(X)$ is a dense $C^*$-embedded set in $X$ and family $\lambda$ consists of all finite subsets of $D(X)$, where $D(X)$ is an isolated points of $X$.

**4. Projectively Menger space**

According to Tkačuk [20], a space $X$ said to be $b$-discrete if every countable subset of $X$ is closed (equivalently, closed and discrete) and $C^*$-embedded in $X$.

**Lemma 4.1.** (Lemma 2.1 in [16]) The following are equivalent for a space $X$:

1. $X$ is $b$-discrete;
2. For any disjoint countable subsets $A$ and $B$ in $X$, there are disjoint zero-sets $Z_A$ and $Z_B$ in $X$ such that $A \subseteq Z_A$ and $B \subseteq Z_B$;
3. For any disjoint countably subsets $A$ and $B$ in $X$ such that $A$ is closed in $X$, there are disjoint zero-sets $Z_A$ and $Z_B$ in $X$ such that $A \subseteq Z_A$ and $B \subseteq Z_B$. 

8
Definition 4.2. For \( A \subset X \), a space \( X \) will be called \( b_A \)-discrete if every countable subset of \( A \) is closed in \( A \) and \( C^* \)-embedded in \( X \).

Lemma 4.3. The following are equivalent for a space \( X \) and \( A \subset X \):

1. \( X \) is \( b_A \)-discrete;
2. For any disjoint countable subsets \( D \) and \( B \) in \( A \), there are disjoint zero-sets \( Z_D \) and \( Z_B \) in \( X \) such that \( D \subset Z_D \) and \( B \subset Z_B \);
3. For any disjoint countably subsets \( D \) and \( B \) in \( A \) such that \( D \) is closed in \( A \), there are disjoint zero-sets \( Z_A \) and \( Z_B \) in \( X \) such that \( D \subset Z_D \) and \( B \subset Z_B \).

Similarly to the proof of implication \((C_p(X, \mathbb{I}) \) is projectively Menger \( \Rightarrow \) \( X \) is \( b \)-discrete) of Theorem 2.4 in [16], we claim the next

Lemma 4.4. Let \( C_\lambda(X) \) be a projectively Menger space, then \( X \) is \( b_A \)-discrete where \( A = \bigcup \lambda \).

Proof. Let \( C_\lambda(X) \) be a projectively Menger. We show the statement (3) in Lemma 4.3. Let \( D \) and \( B \) be a disjoint countable subsets in \( A \) such that \( D \) is closed in \( A \). Let \( B = \{b_n : n \in \mathbb{N}\} \), and let \( B_n = \{b_1, ..., b_n\} \).

For each \( n, m \in \mathbb{N} \), we put \( Z_{n,m} = \{f \in C_\lambda(X) : f(D) = \{0\} \text{ and } f(B_m) \subset [\frac{1}{2^m}, 1]\} \). Since \( D \) and \( B_m \) are countable and \( \lambda \) is a \( \pi \)-network of \( X \), each \( Z_{n,m} \) is a zero-set in \( C_\lambda(X) \). Assume that \( \bigcap\{Z_{n,m} : m \in \mathbb{N}\} = \emptyset \) for all \( n \in \mathbb{N} \). Using the projective Menger property of \( C_\lambda(X) \), Theorem 6 in [5], we can take some \( \varphi \in \mathbb{N}^\mathbb{N} \) such that \( \bigcap\{Z_{n,\varphi(n)} : n \in \mathbb{N}\} = \emptyset \). For each \( n \in \mathbb{N} \), take any \( g_n \in C_\lambda(X) \) satisfying \( g_n(D) = \{0\} \) and \( g_n(B_{\varphi(n)}) = \{1\} \).

Let \( g = \sum_{j=1}^\infty 2^{-j}g_j \). Then, \( g \in C_\lambda(X) \) and \( g(D) \equiv 0 \). Fix any \( n \in \mathbb{N} \), \( 1 \leq k \leq \varphi(m) \). Then we have

\[
g(b_k) = \sum_{j=1}^\infty 2^{-j}g_j(b_k) \geq 2^{-n}g_n(b_k) = 2^{-n}.
\]

Hence, \( g \in \bigcap\{Z_{n,\varphi(n)} : n \in \mathbb{N}\} \). This is a contradiction. Thus, there is some \( n \in \mathbb{N} \) such that \( \bigcap\{Z_{n,m} : m \in \mathbb{N}\} \neq \emptyset \). Let \( h \in \bigcap\{Z_{n,m} : m \in \mathbb{N}\} \). Then \( D \subset Z_A = h^{-1}(0) \) and \( B \subset Z_B = h^{-1}(\frac{1}{2m}, 1] \).

\[ \square \]

Theorem 4.5. Let \( X \) be a Tychonoff space and let \( Y \) be a dense subset of \( X \). Then the following statements are equivalent:

1. \( C_p(Y|X) \) is projectively Menger;

\[ \]
2. $C_p(Y|X)$ is $\sigma$-bounded;
3. $C_p(Y|X)$ is $\sigma$-pseudocompact;
4. $X$ is pseudocompact and $b_Y$-discrete.

Proof. Note that $C_p(Y|X)$ is homeomorphic to $C_\lambda(X)$ for $\lambda = [Y]^{<\omega}$.

(1) $\Rightarrow$ (4). By Lemma 4.4, $X$ is $b_Y$-discrete. Assume that $X$ is not pseudocompact and $f \in C(X)$ is not bounded function. Without loss of generality we can assume that $\mathbb{N} \subseteq f(X)$. For each $n \in \mathbb{N}$ we choose $a_n \in Y$ such that $a_n \in f^{-1}((n - \frac{1}{3}, n + \frac{1}{3}))$. Note that $D = \{a_n : n \in \mathbb{N}\}$ is a $C$-embedded copy of $\mathbb{N}$ (3L (1) in [8]). So we have a continuous mapping $F : C_p(Y|X) \mapsto \mathbb{R}^D$ the Menger space $C_p(Y|X)$ onto $\mathbb{R}^D$. But $F(C_p(Y|X)) = \mathbb{R}^D = \mathbb{R}^\omega$ is dominating, contrary to the Theorem 3.3.

(4) $\Rightarrow$ (3). Since $C_p(Y,X,\mathbb{I})$ is a dense subset of $\mathbb{I}^Y$ and $X$ is $b_Y$-discrete, by Proposition 2.1, $C_p(Y|X,\mathbb{I})$ is pseudocompact. Hence, $C_p(Y|X)$ is $\sigma$-pseudocompact.

Note that every $\sigma$-pseudocompact space is $\sigma$-bounded, and every $\sigma$-bounded space is projectively Menger (Proposition 1.1 in [3]).

5. Examples

Using Theorem 3.10 and Theorem 4.5 we can construct example of projective Menger topological group $C_\lambda(X)$ such that it is not Menger.

Note that if $\lambda = \bigcup\lambda^{<\omega}$, then $C_\lambda(X)$ is a topological group (locally convex topological vector space, topological algebra) ([14], [15]).

Example 5.1. (Example 1 in [13]) Let $T$ be a $P$-space without isolated points, $X = \beta(T)$ and let $\lambda$ be a family of all finite subsets of $T$. Then $C_\lambda(X)$ is $\sigma$-countably compact (Theorem 1.2 in [13]), hence, the topological group $C_\lambda(X)$ is projective Menger. But the space $X$ does not contain isolated points, hence, $C_\lambda(X)$ is not Menger.

Example 5.2. (Example 2 in [13]) Let $D$ be an uncountable discrete space and $\lambda = D^{<\omega}$. Consider $F = \beta(D) \setminus \bigcup\{S : S \subseteq D, \text{and} S \text{ countable}\}$. Denote by $b(D)$ a quotient space obtained from $\beta(D)$ by identifying the set $F$ with the point $\{F\}$. Then the topological group $C_\lambda(b(D))$ is projective Menger ($\sigma$-countably compact), but is not Menger.

Example 5.3. (14) D.B.Shahmatov has constructed for an arbitrary cardinal $\tau \geq 2^{\aleph_0}$ an everywhere dense pseudocompact space $X_\tau$ in $\mathbb{I}^\tau$ such that
$X_\tau$ is a $b$-discrete. Hence, the topological group $C_p(X_\tau)$ is projective Menger ($\sigma$-pseudocompact and is not $\sigma$-countably compact), but is not Menger for an arbitrary cardinal $\tau \geq 2^{\aleph_0}$.

**Remark 5.4.** By Theorems 2.2 and 3.10 if $X$ is compact, $\lambda$ is a $\pi$-network of $X$ and $C_\lambda(X)$ is Menger, then $X$ is homeomorphic to $\beta(D)$, where $\beta(D)$ is Stone-$\check{C}$ech compactification of a discrete space $D$, and $\lambda = [D]^{<\omega}$.

**References**

[1] A.V. Arhangel’skii, *Continuous maps, factorization theorems, and function spaces*, Trudy Moskovsk. Mat. Obshch., 47, (1984), 3–21.

[2] A.V. Arhangel’skii, *Hurewicz spaces, analytic sets and fan tightness of function spaces*, Sov. Math. Dokl., 33, (1986), 396–399.

[3] A.V. Arhangel’skii, *Projective $\sigma$-compactness, $\omega_1$-caliber, and $C_p$-spaces*, Topology and its Applications, 157, (2000), 874–893.

[4] A.V. Arhangel’skii, V.I. Ponomarev, *Fundamentals of general topology: problems and exercises*, Reidel, 1984. (Translated from the Russian.)

[5] M. Bonanzinga, F. Cammaroto, M. Matveev, *Projective versions of selection principles*, Topology and its Applications, 157, (2010), 874–893.

[6] H. Duanmu, F.D. Tall, L. Zdomskyy, *Productively Lindelöf and indestructibly Lindelöf spaces*, Topology and its Applications 160:18 (2013), 2443-2453.

[7] R. Engelking, *General Topology*, PWN, Warsaw, (1977); Mir, Moscow, (1986).

[8] L. Gillman, M. Jerison, *Rings of continuous functions*, The University Series in Higher Mathematics. Princeton, New Jersey: D. Van Nostrand Co., Inc., 1960. 300 p.

[9] W. Hurewicz, *Über eine verallgemeinerung des Borelschen Theorems*, Math. Z. 24 (1925) 401-421.

[10] W. Hurewicz, *Über folger stetiger funktionen*, Fund. Math. 9 (1927) 193-204.
[11] Lj.D.R. Kočinac, *Selection principles and continuous images*, Cubo Math. J. 8 (2) (2006) 23–31.

[12] S.E. Nokhrin, *Some properties of set-open topologies*, Jurnal of Mathematical Sciences, issue 144, n 3, (2007) 4123–4151.

[13] A.V. Osipov, E.G. Pytkeev, *On the σ-countable compactness of spaces of continuous functions with the set-open topology*, Proceedings of the Steklov Institute of Mathematics, issue 285, n. S1, (2014) 153–162.

[14] A.V. Osipov, *Topological-algebraic properties of function spaces with set-open topologies*, Topology and its Applications, issue 3, n. 159, (2012) 800–805.

[15] A.V. Osipov, *Group structures of a function spaces with the set-open topology*, Sib. Èlektron. Mat. Izv., 14, (2017) 1440-1446.

[16] M. Sakai, *The projective Menger property and an embedding of $S_\omega$ into function spaces*, Topology and its Applications, Vol. 220 (2017) 118–130.

[17] M. Sakai, M. Scheepers, *The combinatorics of open covers* in: K.P. Hart, J. van Mill, P.Simon (Eds.), Recent Progress in General Topology III, Atlantic Press, 2014, pp. 751–799.

[18] D.B. Shahmatov, *A pseudocompact Tychonoff space all countable subsets of which are closed and $C^*$-embedded*, Topology and its Applications, 22:2, (1986), 139–144.

[19] F.D. Tall, *Productively Lindelöf spaces may all be $D$*, Canadian Mathematical Bulletin 56:1 (2013), 203–212.

[20] V.V. Tkačuk, *The spaces $C_p(X)$: decomposition into a countable union of bounded subspaces and completeness properties*, Topology and its Applications, n 22, (1986), 241–253.