Nonlinear Connections and Clifford Structures

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Abstract

We present an introduction to the geometry of higher order vector and co–vector bundles (including higher order generalizations of the Finsler geometry and Kaluza–Klein gravity) and review the basic results on Clifford and spinor structures on spaces with generic local anisotropy modeled by higher order nonlinear connections. Geometric applications in locally anisotropic gravity and matter field interactions are considered. This article contains the results outlined by authors and P. Stavrinos in theirs lectures.

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Contents

0.1 Introduction ................................................. 4

1 (Co-) Vector Bundles and Nonlinear Connections .............. 7
  1.1 Vector and Covector Bundles .................................. 8
    1.1.1 Vector and tangent bundles ................................ 8
    1.1.2 Covector and cotangent bundles .......................... 9
    1.1.3 Higher order vector/covector bundles .................... 9
  1.2 Nonlinear Connections ..................................... 13
    1.2.1 N–connections in vector bundles ....................... 13
    1.2.2 N–connections in covector bundles ..................... 14
    1.2.3 N–connections in higher order bundles ................. 15
    1.2.4 Anholonomic frames and N–connections ................. 16
  1.3 Distinguished connections and metrics ...................... 21
    1.3.1 D–connections ........................................ 21
    1.3.2 Metric structure ....................................... 24
    1.3.3 Some remarkable d–connections ......................... 27
    1.3.4 Almost Hermitian anisotropic spaces ................... 29
  1.4 Torsions and Curvatures .................................... 30
    1.4.1 N–connection curvature ................................ 30
    1.4.2 d–Torsions in v– and cv–bundles ...................... 31
    1.4.3 d–Curvatures in v– and cv–bundles .................... 32
  1.5 Generalizations of Finsler Spaces .......................... 33
    1.5.1 Finsler Spaces ....................................... 33
    1.5.2 Lagrange and Generalized Lagrange Spaces ............ 34
    1.5.3 Cartan Spaces ........................................ 36
    1.5.4 Generalized Hamilton and Hamilton Spaces ............ 37
  1.6 Gravity on Vector Bundles .................................. 38

2 Clifford Ha–Structures ........................................ 41
  2.1 Distinguished Clifford Algebras .............................. 41
  2.2 Clifford Ha–Bundles ......................................... 45
    2.2.1 Clifford d–module structure in dv–bundles ............ 46
    2.2.2 Clifford fibration ..................................... 47
  2.3 Almost Complex Spinor Structures .......................... 48
The spinors studied by mathematicians and physicists are connected with the general theory of Clifford spaces introduced in 1876 [14]. The theory of spinors and Clifford algebras play a major role in contemporary physics and mathematics. The spinors were discovered by Élie Cartan in 1913 in mathematical form in his researches on representation group theory [12]; he showed that spinors furnish a linear representation of the groups of rotations of a space of arbitrary dimensions. Physicists Pauli [45] and Dirac [17] (in 1927, respectively, for the three-dimensional and four-dimensional space–time) introduced spinors for the representation of the wave functions.

In general relativity theory spinors and the Dirac equations on (pseudo) Riemannian spaces were defined in 1929 by H. Weyl [24], V. Fock [18] and E. Schrödinger [50]. The books [46, 47, 48] by R. Penrose and W. Rindler monograph summarize the spinor and twistor methods in space–time geometry (see additional references [19, 1, 10, 23, 57, 11] on Clifford structures and spinor theory).

Spinor variables were introduced in Finsler geometries by Y. Takano in 1983 [56] where he dismissed anisotropic dependencies not only on vectors on the tangent bundle but on some spinor variables in a spinor bundle on a space–time manifold. Then generalized Finsler geometries, with spinor variables, were developed by T. Ono and Y. Takano in a series of publications during 1990–1993 [11, 12, 43, 14]. The next steps were investigations of anisotropic and deformed geometries with spinor and vector variables.
and applications in gauge and gravity theories elaborated by P. Stavrinos and his students, S. Koutroubis, P. Manouselis, and V. Balan beginning 1994 [53, 54, 55, 51, 52]. In those works the authors assumed that some spinor variables may be introduced in a Finsler-like way but they did not relate the Finsler metric to a Clifford structure and restricted the spinor–gauge Finsler constructions only for antisymmetric spinor metrics on two–spinor fibers with possible generalizations to four dimensional Dirac spinors.

Isotopic spinors, related with $SU(2)$ internal structural groups, were considered in generalized Finsler gravity and gauge theories also by G. Asanov and S. Ponomarenko [6] in 1988. In that book, and in other papers on Finsler geometry with spinor variables, the authors did not investigate the possibility of introducing a rigorous mathematical definition of spinors on spaces with generic local anisotropy.

An alternative approach to spinor differential geometry and generalized Finsler spaces was elaborated, beginning 1994, in a series of papers and communications by S. Vacaru with participation of S. Ostaf [54, 57, 53, 53]. This direction originates from Clifford algebras and Clifford bundles [22, 57] and Penrose’s spinor and twistor space–time geometry [46, 47, 48], which were re–considered for the case of nearly autoparallel maps (generalized conformal transforms) in Refs. [58, 53, 60]. In the works [53, 77, 85], a rigorous definition of spinors for Finsler spaces, and their generalizations, was given. It was proven that a Finsler, or Lagrange, metric (in a tangent, or, more generally, in a vector bundle) induces naturally a distinguished Clifford (spinor) structure which is locally adapted to the nonlinear connection structure. Such spinor spaces could be defined for arbitrary dimensions of base and fiber subspaces, their spinor metrics are symmetric, antisymmetric or nonsymmetric, depending on the corresponding base and fiber dimensions. This work resulted in the formation of spinor differential geometry of generalized Finsler spaces and developed a number of geometric applications to the theory of gravitational and matter field interactions with generic local anisotropy.

The geometry of anisotropic spinors and (distinguished by nonlinear connections) Clifford structures was elaborated for higher order anisotropic spaces [51, 71, 70] and, more recently, for Hamilton and Lagrange spaces [52].

We emphasize that the theory of anisotropic spinors may be related not only to generalized Finsler, Lagrange, Cartan and Hamilton spaces or their higher order generalizations, but also to anholonomic frames with associated nonlinear connections which appear naturally even in (pseudo) Riemannian geometry if off–diagonal metrics are considered [73, 74, 75, 76, 77]. In order to construct exact solutions of the Einstein equations in general relativity and extra dimension gravity (for lower dimensions see [72, 90, 91]), it is more convenient to diagonalize space–time metrics by using some anholonomic transforms. As a result one induces locally anisotropic structures on space–time which are related to anholonomic (anisotropic) spinor structures.

The main purpose of the present review is to present a detailed summary and new results on spinor differential geometry for generalized Finsler spaces and (pseudo) Riemannian space–times provided with anholonomic frame and associated nonlinear connection structure, to discuss and compare the existing approaches and to consider applications to modern gravity and gauge theories.
This article is organized in four Chapters:

In Chapter 1, we give the basic definitions from the theory of generalized Finsler, Lagrange, Cartan and Hamilton spaces on vector and co–vector (tangent and co–tangent spaces) and their generalizations for higher order vector–covector bundles following the monographs [37, 39, 70].

Chapter 2 is a generalization of the results on Clifford structures for higher order vector bundles.

Chapter 3 is devoted to the differential geometry of Spinors in Higher Order Anisotropic Spaces.

Chapter 4 contains geometric applications to the theory of locally anisotropic interactions [81, 71].

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Chapter 1

(Co-) Vector Bundles and Nonlinear Connections

In this Chapter the space–time geometry is modeled not only on a (pseudo) Riemannian manifold $V^{[n+m]}$ of dimension $n + m$ but it is considered on a vector bundle (or its dual, covector bundle) being, for simplicity, locally trivial with a base space $M$ of dimension $n$ and a typical fiber $F$ (cofiber $F^*$) of dimension $m$, or as a higher order extended vector/covector bundle (we follow the geometric constructions and definitions of monographs [37, 36, 39, 34, 35] which were generalized for vector superbundles in Refs. [69, 70]). Such fibered space–times (in general, with extra dimensions and duality relations) are supposed to be provided with compatible structures of nonlinear and linear connections and (pseudo) Riemannian metric. For the particular cases when: a) the total space of the vector bundle is substituted by a pseudo–Riemannian manifold of necessary signature we can model the usual pseudo–Riemannian space–time from the Einstein gravity theory with field equations and geometric objects defined with respect to some classes of moving anholonomic frames with associated nonlinear connection structure; b) if the dimensions of the base and fiber spaces are identical, $n = m$, for the first order anisotropy, we obtain the tangent bundle $TM$.

Such both (pseudo) Riemannian spaces and vector/covector (in particular cases, tangent/cotangent) bundles of metric signature $(\cdot,+,...,+)\,$ enabled with compatible fibered and/or anholonomic structures, the metric in the total space being a solution of the Einstein equations, will be called \textbf{anisotropic space–times}. If the anholonomic structure with associated nonlinear connection is modeled on higher order vector/covector bundles we shall use the term of \textbf{higher order anisotropic space–time}.

The geometric constructions are outlined as to present the main concepts and formulas in a unique way for both type of vector and covector structures. In this part of the book we usually shall omit proofs which can be found in the mentioned monographs [36, 37, 34, 33, 39, 70].
1.1 Vector and Covector Bundles

In this Section we introduce the basic definitions and denotations for vector and tangent (and theirs dual spaces) bundles and higher order vector/covector bundle geometry.

1.1.1 Vector and tangent bundles

A locally trivial vector bundle, in brief, v–bundle, $\mathcal{E} = (E, \pi, M, Gr, F)$ is introduced as a set of spaces and surjective map with the properties that a real vector space $F = \mathbb{R}^m$ of dimension $m$ ($\dim F = m$, $\mathbb{R}$ denotes the real number field) defines the typical fiber, the structural group is chosen to be the group of automorphisms of $\mathbb{R}^m$, i. e. $Gr = GL(m, \mathbb{R})$, and $\pi : E \rightarrow M$ is a differentiable surjection of a differentiable manifold $E$ (total space, $\dim E = n + m$) to a differentiable manifold $M$ (base space, $\dim M = n$). Local coordinates on $\mathcal{E}$ are denoted $u^\alpha = (x^i, y^a)$, or in brief $u = (x, y)$ (the Latin indices $i, j, k, ... = 1, 2, ..., n$ define coordinates of geometrical objects with respect to a local frame on base space $M$; the Latin indices $a, b, c, ... = 1, 2, ..., m$ define fiber coordinates of geometrical objects and the Greek indices $\alpha, \beta, \gamma, ...$ are considered as cumulative ones for coordinates of objects defined on the total space of a v-bundle).

Coordinate transforms $u^{\alpha'} = u^{\alpha'}(u^\alpha)$ on a v–bundle $\mathcal{E}$ are defined as

$$(x^i, y^a) \rightarrow (x'^i, y'^a),$$

where

$$x'^i = x^{i'}(x^i), \quad y'^a = K^a_{a'}(x^i)y^a$$

and matrix $K^a_{a'}(x^i) \in GL(m, \mathbb{R})$ are functions of necessary smoothness class.

A local coordinate parametrization of v–bundle $\mathcal{E}$ naturally defines a coordinate basis

$$\partial_\alpha = \frac{\partial}{\partial u^\alpha} = \left( \partial_i = \frac{\partial}{\partial x^i}, \partial_a = \frac{\partial}{\partial y^a} \right),$$

and the reciprocal to (1.2) coordinate basis

$$d^\alpha = du^\alpha = (d^i = dx^i, \; d^a = dy^a)$$

which is uniquely defined from the equations

$$d^\alpha \circ \partial_\beta = \delta^\alpha_\beta,$$

where $\delta^\alpha_\beta$ is the Kronecher symbol and by "$\circ$" we denote the inner (scalar) product in the tangent bundle $T\mathcal{E}$.

A tangent bundle (in brief, t–bundle) $(TM, \pi, M)$ to a manifold $M$ can be defined as a particular case of a v–bundle when the dimension of the base and fiber spaces (the last one considered as the tangent subspace) are identic, $n = m$. In this case both type of indices $i, k, ...$ and $a, b, ...$ take the same values $1, 2, ..., n$. For t–bundles the matrices of fiber coordinates transforms from (1.1) can be written $K^i_{i'} = \partial x^{i'} / \partial x^i$.

We shall distinguish the base and fiber indices and values which is necessary for our further geometric and physical applications.
1.1.2 Covector and cotangent bundles

We shall also use the concept of **covector bundle**, (in brief, **cv–bundles**)
\[ \tilde{E} = \left( \tilde{E}, \pi^*, M, Gr, F^* \right), \]
which is introduced as a dual vector bundle for which the typical fiber \( F^* \) (cofiber) is considered to be the dual vector space (covector space) to the vector space \( F \). The fiber coordinates \( p_a \) of \( \tilde{E} \) are dual to \( y^a \) in \( E \). The local coordinates on total space \( \tilde{E} \) are denoted \( \tilde{u} = (x, p) = (x^i, p_a) \). The coordinate transform on \( \tilde{E} \),
\[ \tilde{u} = (x^i, p_a) \rightarrow \tilde{u}' = (x'^i, p_{a'}), \]
are written
\[ x'^i = x^i(x^i), \quad p_{a'} = K_{a'}^a(x^i)p_a. \] (1.4)

The coordinate bases on \( E^* \) are denoted
\[ \tilde{\partial}_a = \frac{\partial}{\partial u^a} = \left( \partial_i = \frac{\partial}{\partial x^i}, \tilde{\partial}^a = \frac{\partial}{\partial p_a} \right) \] (1.5)
and
\[ \tilde{d}^a = d u^a = \left( d^i = dx^i, \tilde{d}_a = dp_a \right). \] (1.6)

We shall use ”breve” symbols in order to distinguish the geometrical objects on a cv–bundle \( \mathcal{E}^* \) from those on a v–bundle \( \mathcal{E} \).

As a particular case with the same dimension of base space and cofiber one obtains the **cotangent bundle** \( (T^*M, \pi^*, M) \), in brief, **ct–bundle**, being dual to \( TM \). The fibre coordinates \( p_i \) of \( T^*M \) are dual to \( y^i \) in \( TM \). The coordinate transforms (1.4) on \( T^*M \) are stated by some matrices \( K_{k'}^k(x^i) = \partial x^k/\partial x^{k'} \).

In our further considerations we shall distinguish the base and cofiber indices.

1.1.3 Higher order vector/covector bundles

The geometry of higher order tangent and cotangent bundles provided with nonlinear connection structure was elaborated in Refs. [34, 35, 38, 39] following the aim of geometrization of higher order Lagrange and Hamilton mechanics. In this case we have base spaces and fibers of the same dimension. In order to develop the approach to modern high energy physics (in superstring and Kaluza–Klein theories) one had to introduce (in Refs. [80, 71, 70, 69]) the concept of higher order vector bundle with the fibers consisting from finite ’shells” of vector, or covector, spaces of different dimensions not obligatory coinciding with the base space dimension.

**Definition 1.1.** A distinguished vector/covector space, in brief dvc–space, of type
\[ \tilde{F} = F[v(1), v(2), cv(3), ..., cv(z - 1), v(z)] \] (1.7)
is a vector space decomposed into an invariant oriented direct sum

\[ \tilde{F} = F_{(1)} \oplus F_{(2)} \oplus F_{(3)}^* \oplus \ldots \oplus F_{(z-1)}^* \oplus F_{(z)} \]

of vector spaces \( F_{(1)}, F_{(2)}, \ldots, F_{(z)} \) of respective dimensions

\[ \dim F_{(1)} = m_1, \dim F_{(2)} = m_2, \ldots, \dim F_{(z)} = m_z \]

and of covector spaces \( F_{(3)}^*, \ldots, F_{(z-1)}^* \) of respective dimensions

\[ \dim F_{(3)}^* = m_3^*, \ldots, \dim F_{(z-1)}^* = m_{(z-1)}^*. \]

As a particular case we obtain a distinguished vector space, in brief dv–space (a distinguished covector space, in brief dcv–space), if all components of the sum are vector (covector) subspaces. We note that we have fixed for simplicity an orientation of vector/covector subspaces like in (1.7); in general there are possible various types of orientations, number of subspaces and dimensions of subspaces.

Coordinates on \( \tilde{F} \) are denoted

\[ \tilde{y} = (y_{(1)}, y_{(2)}, p_{(3)}, \ldots, y_{(z)}) = \{y^{<a_2>}\} = (y^{a_1}, y^{a_2}, p_{a_3}, \ldots, p_{a_{z-1}}, y^{a_z}), \]

where indices run corresponding values:

\[ a_1 = 1, 2, \ldots, m_1; \ a_2 = 1, 2, \ldots, m_2, \ldots, z = 1, 2, \ldots, m_z. \]

**Definition 1.2.** A higher order vector/covector bundle (in brief, hv–bundle) of type \( \tilde{E} = \tilde{E}[v(1), v(2), cv(3), \ldots, cv(z-1), v(z)] \) is a vector bundle \( E = (\tilde{E}, p^{<d>}, \tilde{F}, M) \) with corresponding total, \( \tilde{E} \), and base, \( M \), spaces, surjective projection \( p^{<d>} : \tilde{E} \to M \) and typical fiber \( \tilde{F} \).

We define higher order vector (covector) bundles, in brief, hv–bundles (in brief, hcv–bundles), if the typical fibre is a dv–space (dcv–space) as particular cases of hv–bundles.

A hv–bundle is constructed as an oriented set of enveloping ‘shell by shell’ v–bundles and/or cv–bundles,

\[ p^{<s>} : \tilde{E}^{<s>} \to \tilde{E}^{<s-1>}, \]

where we use the index \( < s > = 0, 1, 2, \ldots, z \) in order to enumerate the shells, when \( \tilde{E}^{<0>} = M \). Local coordinates on \( \tilde{E}^{<s>} \) are denoted

\[ \tilde{u}^{(s)} = (x, \tilde{y}^{<s>}) = (x, y_{(1)}, y_{(2)}, p_{(3)}, \ldots, y_{(s)}) = (x^i, y^{a_1}, y^{a_2}, p_{a_3}, \ldots, y^{a_s}). \]

If \( < s >= < z > \) we obtain a complete coordinate system on \( \tilde{E} \) denoted in brief

\[ \tilde{u} = (x, \tilde{y}) = \tilde{u}^{(z)} = (x^i = y^{a_0}, y^{a_1}, y^{a_2}, p_{a_3}, \ldots, y^{a_z}). \]
We shall use the general commutative indices $\alpha, \beta, \ldots$ for objects on hvc—bundles which are marked by tilde, like $\tilde{u}, \tilde{u}^\alpha, \ldots, \tilde{E}, \ldots$.

The coordinate transforms for a hvc—bundle $\tilde{E}$,

$$\tilde{u} = (x, \tilde{y}) \rightarrow \tilde{u}' = (x', \tilde{y}')$$

are given by recurrent formulas

$$x'^i = x^i (x^i), \text{ rank} \left( \frac{\partial x'^i}{\partial x^i} \right) = n;$$

$$y'^a_i = K^{a_i}_{a_1} (x) y^{a_1}, K^{a_i}_{a_1} \in GL(m_1, \mathcal{R});$$

$$y'^a_i = K^{a_i}_{a_1} (x, y(1)) y^{a_2}, K^{a_i}_{a_2} \in GL(m_2, \mathcal{R});$$

$$p'_a = K^{a_3}_{a_1} (x, y(1), y(2)) p_a, K^{a_3}_{a_1} \in GL(m_3, \mathcal{R});$$

$$y'^a_i = K^{a_i}_{a_1} (x, y(1), y(2), p(3)) y^{a_4}, K^{a_i}_{a_4} \in GL(m_4, \mathcal{R});$$

$$\ldots \ldots \ldots \ldots$$

$$p'_{a_{z-1}} = K^{a_{z-1}}_{a_1} (x, y(1), y(2), p(3), \ldots, y(z-2)) p_{a_{z-1}}, K^{a_{z-1}}_{a_1} \in GL(m_{z-1}, \mathcal{R});$$

$$y'^a_i = K^{a_i}_{a_1} (x, y(1), y(2), p(3), \ldots, y(z-2), p_{a_{z-1}}) y^{a_z}, K^{a_i}_{a_z} \in GL(m_z, \mathcal{R}),$$

where, for instance, by $GL(m_2, \mathcal{R})$ we denoted the group of linear transforms of a real vector space of dimension $m_2$.

The coordinate bases on $\tilde{E}$ are denoted

$$\tilde{\partial}_\alpha = \frac{\partial}{\partial u^\alpha} = \left( \partial_i = \frac{\partial}{\partial x^i}, \partial_{a_1} = \frac{\partial}{\partial y^{a_1}}, \partial_{a_2} = \frac{\partial}{\partial y^{a_2}}, \partial_{a_3} = \frac{\partial}{\partial p_{a_3}}, \ldots, \partial_{a_z} = \frac{\partial}{\partial y^{a_z}} \right) \quad (1.8)$$

and

$$\tilde{d}^\alpha = \tilde{d} u^\alpha = \left( d^i = dx^i, d^{a_1} = dy^{a_1}, d^{a_2} = dy^{a_2}, d_{a_3} = dp_{a_3}, \ldots, d^{a_z} = dy^{a_z} \right). \quad (1.9)$$

We end this subsection with two examples of higher order tangent / cotangent bundles (when the dimensions of fibers/cofibers coincide with the dimension of bundle space, see Refs. [34, 35, 38, 39]).

**Osculator bundle**

The $k$—osculator bundle is identified with the $k$—tangent bundle $(T^k M, p^{(k)}, M)$ of a $n$—dimensional manifold $M$. We denote the local coordinates

$$\tilde{u}^\alpha = (x^i, y^{(1)}, \ldots, y^{(k)}) ,$$
where we have identified $y_i^{(1)} \simeq y^{a_1}, \ldots, y_i^{(k)} \simeq y^{a_k}, k = z$, in order to have similarity with denotations from [39]. The coordinate transforms 

$$\tilde{u}^\alpha \rightarrow \tilde{u}^\alpha' (\tilde{u}^\alpha)$$

preserving the structure of such higher order vector bundles are parametrized

$$x^i' = x^i' (x^i), \det \left( \frac{\partial x'^i}{\partial x^i} \right) \neq 0,$$

$$y_{(1)}' = \frac{\partial x'^i}{\partial x^i} y_i^{(1)},$$

$$2y_{(2)}' = \frac{\partial y'^i_{(1)}}{\partial x^i} y_i^{(1)} + 2 \frac{\partial y'^i_{(1)}}{\partial y_i^{(1)}} y_i^{(2)},$$

$$\ldots \ldots$$

$$ky_{(k)}' = \frac{\partial y'^i_{(1)}}{\partial x^i} y_i^{(1)} + \ldots + k \frac{\partial y'^i_{(k-1)}}{\partial y_i^{(k-1)}} y_i^{(k)},$$

where the equalities

$$\frac{\partial y'^i_{(s)}}{\partial x^i} = \frac{\partial y'^i_{(s+1)}}{\partial y_i^{(1)}} = \ldots = \frac{\partial y'^i_{(k)}}{\partial y_i^{(k-s)}}$$

hold for $s = 0, \ldots, k - 1$ and $y_{(0)}^i = x^i$.

The natural coordinate frame on $(T^k M, p^{(k)}, M)$ is defined

$$\tilde{\partial}_\alpha = \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y_i^{(1)}}, \ldots, \frac{\partial}{\partial y_i^{(k)}} \right)$$

and the coframe is

$$\tilde{d}_\alpha = (dx^i, dy_i^{(1)}, \ldots, dy_i^{(k)}).$$

These formulas are respectively some particular cases of (1.8) and (1.9).

**The dual bundle of k–osculator bundle**

This higher order vector/covector bundle, denoted as $(T^{*k} M, p^{*k}, M)$, is defined as the dual bundle to the k–tangent bundle $(T^k M, p^{k}, M)$. The local coordinates (parametrized as in the previous paragraph) are

$$\tilde{u} = (x, y_{(1)}, \ldots, y_{(k-1)}, p) = (x^i, y_{(1)}^i, \ldots, y_{(k-1)}^i, p_i) \in T^{*k} M.$$
The coordinate transforms on \((T^*kM, p^*(k))\) are

\[
x^i' = x^i'(x^i), \quad \det \left( \frac{\partial x^i'}{\partial x^i} \right) \neq 0,
\]

\[
y_{(1)}' = \frac{\partial x^i'}{\partial x^i} y_{(1)}^i,
\]

\[
2y_{(2)}' = \frac{\partial y_{(1)}'}{\partial x^i} y_{(1)}^i + 2 \frac{\partial y_{(1)}'}{\partial y^i} y_{(2)}^i,
\]

\[
(k - 1)y_{(k-1)}' = \frac{\partial y_{(k-2)}'}{\partial x^i} y_{(1)}^i + \ldots + k \frac{\partial y_{(k-1)}'}{\partial y_{(k-2)}^i} y_{(k-1)}^i,
\]

\[
p_{i}' = \frac{\partial x^i}{\partial p_i}.
\]

where the equalities

\[
\frac{\partial y_{(s)}'}{\partial x^i} = \frac{\partial y_{(s+1)}'}{\partial y_{(1)}^i} = \ldots = \frac{\partial y_{(k-1)}'}{\partial y_{(k-1-s)}^i}
\]

hold for \(s = 0, \ldots, k - 2\) and \(y_{(0)}^i = x^i\).

The natural coordinate frame on \((T^*kM, p^*(k))\) is defined

\[
\tilde{\partial}_\alpha = \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y_{(1)}^i}, \ldots, \frac{\partial}{\partial y_{(k-1)}^i}, \frac{\partial}{\partial p_i} \right)
\]

and the coframe is

\[
\tilde{\Lambda}_\alpha = (dx^i, dy_{(1)}^i, \ldots, dy_{(k-1)}^i, dp_i).
\]

These formulas are respectively another particular cases of (1.8) and (1.9).

### 1.2 Nonlinear Connections

The concept of **nonlinear connection**, in brief, N-connection, is fundamental in the geometry of vector bundles and anisotropic spaces (see a detailed study and basic references in \([36, 37]\)). A rigorous mathematical definition is possible by using the formalism of exact sequences of vector bundles.

#### 1.2.1 N–connections in vector bundles

Let \(\mathcal{E} = (E, p, M)\) be a v–bundle with typical fiber \(R^m\) and \(\pi^T: TE \to TM\) being the differential of the map \(P\) which is a fibre–preserving morphism of the tangent bundle \(TE, \tau_E, E) \to E\) and of tangent bundle \((TM, \tau, M) \to M\). The kernel of the vector
bundle morphism, denoted as \((VE, \tau_V, E)\), is called the **vertical subbundle** over \(E\), which is a vector subbundle of the vector bundle \((TE, \tau_E, E)\).

A vector \(X_u\) tangent to a point \(u \in E\) is locally written as
\[(x, y, X, Y) = (x^i, y^a, X^i, Y^a),\]
where the coordinates \((X^i, Y^a)\) are defined by the equality
\[X_u = X^i \partial_i + Y^a \partial_a.\]
We have \(\pi^T(x, y, X, Y) = (x, X)\). Thus the submanifold \(VE\) contains the elements which are locally represented as \((x, y, 0, Y)\).

**Definition 1.3.** A nonlinear connection \(N\) in a vector bundle \(\mathcal{E} = (E, \pi, M)\) is the splitting on the left of the exact sequence
\[0 \to VE \to TE \to TE/VE \to 0\]
where \(TE/VE\) is the factor bundle.

By definition (1.3) it is defined a morphism of vector bundles \(C : TE \to VE\) such the superposition of maps \(C \circ i\) is the identity on \(VE\), where \(i : VE \to VE\). The kernel of the morphism \(C\) is a vector subbundle of \((TE, \tau_E, E)\) which is the horizontal subbundle, denoted by \((HE, \tau_H, E)\). Consequently, we can prove that in a \(v\)-bundle \(\mathcal{E}\) a \(N\)-connection can be introduced as a distribution
\[\{N : E_u \to H_u E, T_u E = H_u E \oplus V_u E\}\]
for every point \(u \in E\) defining a global decomposition, as a Whitney sum, into horizontal, \(H \mathcal{E}\), and vertical, \(V \mathcal{E}\), subbundles of the tangent bundle \(T \mathcal{E}\)
\[T \mathcal{E} = H \mathcal{E} \oplus V \mathcal{E}.\] (1.10)

Locally a \(N\)-connection in a \(v\)-bundle \(\mathcal{E}\) is given by its coefficients
\[N^a_i(u) = N^a_i(x, y)\]
with respect to bases (1.2) and (1.3)
\[N = N^a_i(u)dx^i \otimes \partial_a.\]

We note that a linear connection in a \(v\)-bundle \(\mathcal{E}\) can be considered as a particular case of a \(N\)-connection when \(N^a_i(x, y) = K^a_{bi}(x) y^b\), where functions \(K^b_{ai}(x)\) on the base \(M\) are called the Christoffel coefficients.

### 1.2.2 \(N\)-connections in covector bundles:

A nonlinear connection in a \(cv\)-bundle \(\check{\mathcal{E}}\) (in brief a \(\check{N}\)-connection) can be introduced in a similar fashion as for \(v\)-bundles by reconsidering the corresponding definitions for \(cv\)-bundles. For instance, it is stated by a Whitney sum, into horizontal, \(H \check{\mathcal{E}}\), and vertical, \(V \check{\mathcal{E}}\), subbundles of the tangent bundle \(T \check{\mathcal{E}}\):
\[T \check{\mathcal{E}} = H \check{\mathcal{E}} \oplus V \check{\mathcal{E}}.\] (1.11)
Hereafter, for the sake of brevity we shall omit details on definition of geometrical objects on cv–bundles if they are very similar to those for v–bundles: we shall present only the basic formulas by emphasizing the most important particularities and differences.

Definition 1.4. A Ņ–connection on $\breve{E}$ is a differentiable distribution

$$\breve{\mathcal{N}} : \breve{E} \to \breve{\mathcal{N}}_u \in T^*_u \breve{E}$$

which is suplimentary to the vertical distribution $V$, i. e.

$$T_u \breve{E} = \breve{\mathcal{N}}_u \oplus \dot{V}_u, \forall \breve{\mathcal{E}}.$$ 

The same definition is true for Ņ–connections in ct–bundles, we have to change in the definition (1.4) the symbol $\breve{E}$ into $T^* M$.

A Ņ–connection in a cv–bundle $\breve{E}$ is given locally by its coefficients

$$\breve{\mathcal{N}}_{ia}(u) = \breve{\mathcal{N}}_{ia}(x,p)$$

with respect to bases (1.2) and (1.3)

$$\breve{\mathcal{N}} = \breve{\mathcal{N}}_{ia}(u)d^i \otimes \partial^a.$$

We emphasize that if a N–connection is introduced in a v–bundle (cv–bundle) we have to adapt the geometric constructions to the N–connection structure.

1.2.3 N–connections in higher order bundles

The concept of N–connection can be defined for higher order vector / covector bundle in a standard manner like in the usual vector bundles:

Definition 1.5. A nonlinear connection $\check{\mathcal{N}}$ in hvc–bundle $\check{\mathcal{E}}$ is a splitting of the left of the exact sequence

$$0 \to V\check{\mathcal{E}} \to T\check{\mathcal{E}} \to T\check{\mathcal{E}}/V\check{\mathcal{E}} \to 0$$

We can associate sequences of type (1.12) to every mappings of intermediary sub-bundles. For simplicity, we present here the Whitney decomposition

$$T\check{\mathcal{E}} = H\check{\mathcal{E}} \oplus V_{v(1)}\check{\mathcal{E}} \oplus V_{v(2)}\check{\mathcal{E}} \oplus V_{cv(3)}^{*}\check{\mathcal{E}} \oplus \ldots \oplus V_{cv(z-1)}^{*}\check{\mathcal{E}} \oplus V_{v(z)}\check{\mathcal{E}}.$$ 

Locally a N–connection $\check{\mathcal{N}}$ in $\check{\mathcal{E}}$ is given by its coefficients

$$\check{\mathcal{N}}_{i, a_1}, \check{\mathcal{N}}_{i, a_2}, \check{\mathcal{N}}_{a_1, a_3}, \ldots, \check{\mathcal{N}}_{a_1, a_{z-1}}, \check{\mathcal{N}}_{i, a_2},$$

$$0, \check{\mathcal{N}}_{a_1, a_2}, \check{\mathcal{N}}_{a_2, a_3}, \ldots, \check{\mathcal{N}}_{a_1, a_{z-1}}, \check{\mathcal{N}}_{a_2, a_2},$$

$$\ldots, \ldots, \ldots, \ldots, \ldots,$$

$$0, 0, 0, \ldots, \check{\mathcal{N}}_{a_2, a_{z-2}}, \check{\mathcal{N}}_{a_2, a_{z-2}},$$

$$0, 0, 0, \ldots, 0, \check{\mathcal{N}}_{a_1, a_{z-1}}.$$

which are given with respect to the components of bases (1.8) and (1.9).

We end this subsection with two examples of N–connections in higher order vector/covector bundles:
N–connection in osculator bundle

Let us consider the second order of osculator bundle (see subsection (1.1.3)) \( T^2 M = \text{Osc}^2 M \). A N–connection \( \tilde{N} \) in \( \text{Osc}^2 M \) is associated to a Whitney sum

\[
T^2 M = NT^2 M \oplus VT^2 M
\]

which defines in every point \( \tilde{u} \in T^2 M \) a distribution

\[
T_u T^2 M = N_0 (\tilde{u}) \oplus N_1 (\tilde{u}) \oplus VT^2 M.
\]

We can parametrize \( \tilde{N} \) with respect to natural coordinate bases as

\[
N_a^a, \quad N_a^{a_2},
\]

\[
N_0, \quad N_a^{a_1}.
\]

As a particular case we can consider \( N_a^{a_2} = 0 \).

N–connection in dual osculator bundle

In a similar fashion we can take the bundle \( (T^* M, p^* M) \) being dual bundle to the \( \text{Osc}^2 M \) (see subsection (1.1.3)). We have

\[
T^*^2 M = TM \otimes T^* M.
\]

The local coefficients of a N–connection in \( (T^*^2 M, p^*^2 M) \) are parametrized

\[
N_i^{a_1}, \quad N_i^{a_2},
\]

\[
0, \quad N_{a_1a_2}.
\]

As a particular case we can consider \( N_{a_1a_2} = 0 \).

1.2.4 Anholonomic frames and N–connections

Having defined a N–connection structure in a (vector, covector, or higher order vector / covector) bundle we can adapt to this structure, (by 'N–elongation’, the operators of partial derivatives and differentials and to consider decompositions of geometrical objects with respect to adapted bases and cobases.

Anholonomic frames in v–bundles

In a v–bundle \( \mathcal{E} \) provided with a N–connection we can adapt to this structure the geometric constructions by introducing locally adapted basis (N–frame, or N–basis):

\[
\delta_a = \frac{\delta}{\delta y^a} = \left( \delta_i = \frac{\delta}{\delta x^i} = \partial_i - N_i^a (u) \partial_a, \partial_a = \frac{\partial}{\partial y^a} \right),
\]

16
and its dual $N$–basis, $(N$–coframe, or $N$–cobasis),

$$\delta^\alpha = \delta u^\alpha = \left( d^i = \delta x^i = dx^i, \delta^a = \delta y^a + N_i^a(u) dx^i \right).$$  \hspace{1cm} (1.17)

The \textbf{anholonomic coefficients}, $w = \{w^\alpha_{\beta \gamma}(u)\}$, of $N$–frames are defined to satisfy the relations

$$[\delta^\alpha, \delta^\beta] = \delta^\alpha \delta^\beta - \delta^\beta \delta^\alpha = w^\alpha_{\beta \gamma}(u) \delta^\gamma.$$  \hspace{1cm} (1.18)

A frame bases is holonomic if all anholonomy coefficients vanish (like for usual coordinate bases), or anholonomic if there are nonzero values of $w^\alpha_{\beta \gamma}$.

So, we conclude that a $N$–connection structure splitting conventionally a $v$–bundle $E$ into some horizontal $HE$ and vertical $VE$ subbundles can be modeled by an anholonomic frame structure with mixed holonomic $\{x^i\}$ and anholonomic $\{y^a\}$ variables. This case differs from usual, for instance, tetradic approach in general relativity when tetradic (frame) fields are stated to have only for holonomic or only for anholonomic variables. By using the $N$–connection formalism we can investigate geometrical and physical systems when some degrees of freedoms (variables) are subjected to anholonomic constraints, the rest of variables being holonomic.

The operators (1.16) and (1.17) on a $v$–bundle $E$ enabled with a $N$–connection can be considered as respective equivalents of the operators of partial derivations and differentials: the existence of a $N$–connection structure results in 'elongation' of partial derivations on $x$–variables and in 'elongation' of differentials on $y$–variables.

The \textbf{algebra of tensorial distinguished fields} $DT(E)$ (d–fields, d–tensors, d–objects) on $E$ is introduced as the tensor algebra $T = \{T_{pq}^r\}$ of the $v$–bundle $E_{(d)} = (HE \oplus VE, p_d, E)$, where $p_d : HE \oplus VE \to E$.

An element $t \in T_{pq}^r$, d–tensor field of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, can be written in local form as

$$t = t^{i_1 \ldots i_r a_1 \ldots a_r}_{j_1 \ldots j_q b_1 \ldots b_r}(u) \delta^i_1 \times \ldots \delta^i_p \times \partial_{a_1} \times \ldots \times \partial_{a_r} \times d^j_1 \times \ldots \times d^j_q \times \delta^b_1 \times \ldots \times \delta^b_s.$$

We shall respectively use the denotations $X(E)$ (or $X(M)$), $\Lambda^p(E)$ or $(\Lambda_p(M))$ and $F(E)$ (or $F(M)$) for the module of d–vector fields on $E$ (or $M$), the exterior algebra of $p$–forms on $E$ (or $M$) and the set of real functions on $E$ (or $M$).

\textbf{Anholonomic frames in cv–bundles}

The anholonomic frames adapted to the $N$–connection structure are introduced similarly to (1.16) and (1.17):

the locally adapted basis ($\bar{N}$–basis, or $\bar{N}$–frame):

$$\bar{\delta}_\alpha = \frac{\bar{\delta}}{\delta u^\alpha} = \left( \delta_i = \frac{\delta}{\delta x^i} = \partial_i + \bar{N}_{i a}(\bar{u}) \partial^a, \delta^a = \frac{\partial}{\partial p_a} \right),$$ \hspace{1cm} (1.19)
and its dual (N–cobasis, or N–coframe):
\[ \tilde{\delta}^\alpha = \tilde{\delta} u^\alpha = (d^i = \delta x^i = dx^i, \tilde{\delta}_a = \tilde{\delta} p_a = dp_a - \tilde{N}_{ia} (\tilde{u}) dx^i). \] (1.20)

We note that for the singles of N–elongations are inverse to those for N–elongations.

The anholonomic coefficients, \( \tilde{\dot{\omega}} = \{ \tilde{\dot{\omega}}^\alpha_{\beta \gamma}(\tilde{u}) \} \), of N–frames are defined by the relations
\[ \left[ \tilde{\delta}_\alpha, \tilde{\delta}_\beta \right] = \tilde{\delta}_\alpha \tilde{\delta}_\beta - \tilde{\delta}_\beta \tilde{\delta}_\alpha = \tilde{\dot{\omega}}^\alpha_{\beta \gamma}(\tilde{u}) \tilde{\delta}_\alpha. \] (1.21)

The algebra of tensorial distinguished fields \( DT \left( \tilde{\mathcal{E}} \right) \) (d–fields, d–tensors, d–objects) on \( \tilde{\mathcal{E}} \) is introduced as the tensor algebra \( \tilde{T} = \{ \tilde{T}^{pr}_{qs} \} \) of the cv–bundle
\[ \tilde{\mathcal{E}}_{(d)} = \left( H\tilde{\mathcal{E}} \oplus V\tilde{\mathcal{E}}, \tilde{p}_d, \tilde{\mathcal{E}} \right), \]
where \( \tilde{p}_d : H\tilde{\mathcal{E}} \oplus V\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}} \).

An element \( \tilde{t} \in \tilde{T}^{pr}_{qs} \), d–tensor field of type \( \left( \begin{array}{ccc} p & q & r \\ \end{array} \right) \), can be written in local form as
\[ \tilde{t} = \tilde{t}^1_{j_1 \ldots j_q} \tilde{t}^{a_1 \ldots a_r} (\tilde{u}) \tilde{\delta}_{j_1} \otimes \ldots \otimes \tilde{\delta}_{j_q} \otimes \tilde{\delta}_{a_1} \otimes \ldots \otimes \tilde{\delta}_{a_r} \otimes \tilde{\partial}^{j_1} \otimes \ldots \otimes \tilde{\partial}^{j_q} \otimes \tilde{\partial}^{a_1} \otimes \ldots \otimes \tilde{\partial}^{a_r}. \]

We shall respectively use the denotations \( \mathcal{X} \left( \tilde{\mathcal{E}} \right) \) (or \( \mathcal{X} (M) \)), \( \Lambda^p \left( \tilde{\mathcal{E}} \right) \) or \( (\Lambda^p (M)) \) and \( \mathcal{F} \left( \tilde{\mathcal{E}} \right) \) (or \( \mathcal{F} (M) \)) for the module of d–vector fields on \( \tilde{\mathcal{E}} \) (or \( M \)), the exterior algebra of p–forms on \( \tilde{\mathcal{E}} \) (or \( M \)) and the set of real functions on \( \tilde{\mathcal{E}} \) (or \( M \)).

Anholonomic frames in hvc–bundles

The anholonomic frames adapted to a N–connection in hvc–bundle \( \tilde{\mathcal{E}} \) are defined by the set of coefficients \( \{1.13\} \); having restricted the constructions to a vector (covector) shell we obtain some generalizations of the formulas for corresponding N(or N)–connection elongation of partial derivatives defined by \( \{1.16\} \) (or \( \{1.19\} \)) and \( \{1.17\} \) (or \( \{1.20\} \)).

We introduce the adapted partial derivatives (anholonomic N–frames, or N–bases) in \( \tilde{\mathcal{E}} \) by applying the coefficients \( \{1.13\} \)
\[ \tilde{\delta}_\alpha = \frac{\tilde{\delta}}{\delta \tilde{u}^\alpha} = \left( \delta_i, \delta_{a_1}, \delta_{a_2}, \tilde{\delta}^{a_3}, \ldots, \tilde{\delta}^{a_{z-1}}, \partial_{a_z} \right), \]
where
\[
\delta_i = \partial_i - N_i^a \partial_{a1} - N_i^{a2} \partial_{a2} + N_{ia3} \partial_{a3} - \ldots + N_{ia_{a-1}} \partial_{a_{a-1}} - N_i^{a_z} \partial_{a_z},
\]
\[
\delta_{a1} = \partial_{a1} - N_a^{a2} \partial_{a2} + N_a^{a3} \partial_{a3} - \ldots + N_a^{a_{a-1}} \partial_{a_{a-1}} - N_a^{a_z} \partial_{a_z},
\]
\[
\delta_{a2} = \partial_{a2} + N_{a2}^{a3} \partial_{a3} - \ldots + N_{a2}^{a_{a-1}} \partial_{a_{a-1}} - N_{a2}^{a_z} \partial_{a_z},
\]
\[
\delta_{a3} = \partial_{a3} - N_a^{a4} \partial_{a4} - \ldots + N_a^{a_{a-1}} \partial_{a_{a-1}} - N_a^{a_z} \partial_{a_z},
\]
\[
\delta_{a_z-1} = \partial_{a_z-1} - N_a^{a_{a_z}} \partial_{a_z},
\]
\[
\partial_z = \partial/\partial y^a_z.
\]

These formulas can be written in the matrix form:
\[
\tilde{\delta} = \tilde{N}(u) \times \dot{\delta},
\]
(1.22)

where
\[
\tilde{\delta} = \begin{pmatrix}
\delta_i \\
\delta_{a1} \\
\delta_{a2} \\
\delta_{a3} \\
\vdots \\
\delta_{a_{a-1}} \\
\delta_{a_z}
\end{pmatrix},
\tilde{\delta} = \begin{pmatrix}
\dot{\partial}_i \\
\dot{\partial}_{a1} \\
\dot{\partial}_{a2} \\
\dot{\partial}_{a3} \\
\vdots \\
\dot{\partial}_{a_{a-1}} \\
\dot{\partial}_{a_z}
\end{pmatrix},
\]
(1.23)

and
\[
\tilde{N} = \begin{pmatrix}
1 & -N_i^a & -N_i^{a2} & N_{ia3} & -N_i^{a4} & \ldots & N_i^{a_{a-1}} & -N_i^{a_z} \\
0 & 1 & -N_a^{a2} & N_a^{a3} & -N_a^{a4} & \ldots & N_a^{a_{a-1}} & -N_a^{a_z} \\
0 & 0 & 1 & N_{a2}^{a3} & -N_{a2}^{a4} & \ldots & N_{a2}^{a_{a-1}} & -N_{a2}^{a_z} \\
0 & 0 & 0 & 1 & -N_{a3}^{a4} & \ldots & N_{a3}^{a_{a-1}} & -N_{a3}^{a_z} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & -N_{a_{a-1}a_z} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}.
\]

The adapted differentials (anholonomic N-coframes, or N-cobases) in \( \tilde{E} \) are introduced in the simplest form by using matrix formalism: The respective dual matrices to (1.23)
\[
\tilde{\delta}^* = \{ \tilde{\delta}^a \} = \begin{pmatrix}
d^a & \delta^a_1 & \delta^a_2 & \delta^a_3 & \ldots & \delta^a_{a_{a-1}} & \delta^a_{a_z}
\end{pmatrix},
\]
\[
\tilde{d}^* = \{ \tilde{d}^a \} = \begin{pmatrix}
d^a & d^a_1 & d^a_2 & d^a_3 & \ldots & d^a_{a_{a-1}} & d^a_{a_z}
\end{pmatrix},
\]
are related via a matrix relation
\[
\tilde{\delta}^* = \tilde{d}^* \tilde{M}
\]
(1.24)
which defines the formulas for anholonomic N–coframes. The matrix $\hat{M}$ from (1.24) is the inverse to $\hat{N}$, i.e. satisfies the condition
\[ \hat{M} \times \hat{N} = I. \] (1.25)

The anholonomic coefficients, $\tilde{w} = \{ \tilde{w}_{\beta\gamma}(\tilde{u}) \}$, on hcv–bundle $\tilde{E}$ are expressed via coefficients of the matrix $\hat{N}$ and their partial derivatives following the relations
\[ \left[ \tilde{\delta}_\alpha, \tilde{\delta}_\beta \right] = \tilde{\delta}_\alpha \tilde{\delta}_\beta - \tilde{\delta}_\beta \tilde{\delta}_\alpha = \tilde{w}_{\beta\gamma}(\tilde{u}) \tilde{\delta}_\alpha. \] (1.26)

We omit the explicit formulas on shells.

A d–tensor formalism can also be developed on the space $\tilde{E}$. In this case the indices have to be stipulated for every shell separately, like for v–bundles or cv–bundles.

Let us consider some examples for particular cases of hcv–bundles:

**Anholonomic frames in osculator bundle**

For the osculator bundle $T^2M = Osc^2M$ from subsection (1.2.3) the formulas (1.22) and (1.24) are written respectively in the form

\[ \tilde{\delta}_\alpha = \left( \delta, \frac{\partial}{\partial y^i(1)}, \frac{\partial}{\partial y^i(2)} \right), \]

where

\[ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(1)i}^j \frac{\partial}{\partial y^j(1)} - N_{(2)i}^j \frac{\partial}{\partial y^j(2)}, \]

\[ \frac{\delta}{\delta y^i(1)} = \frac{\partial}{\partial y^i(1)} - N_{(2)i}^j \frac{\partial}{\partial y^j(2)}, \]

and

\[ \tilde{\delta}^\alpha = (dx^i, \delta y^i(1), \delta y^i(2)), \] (1.27)

where

\[ \delta y^i(1) = dy^i(1) + M_{(1)j}^i dx^j, \]

\[ \delta y^i(2) = dy^i(2) + M_{(2)j}^i dy^j(1) + M_{(2)j}^i dx^j, \]

with the dual coefficients $M_{(1)j}^i$ and $M_{(2)j}^i$ (see (1.23)) expressed via primary coefficients $N_{(1)j}^i$ and $N_{(2)j}^i$ as

\[ M_{(1)j}^i = N_{(1)j}^i, M_{(2)j}^i = N_{(2)j}^i + N_{(1)m}^i N_{(1)j}^m. \]
Anholonomic frames in dual osculator bundle

Following the definitions for dual osculator bundle \((T^*M, p^*M, \pi^*M, \rho^*M)\) in subsection (1.2.3) the formulas (1.22) and (1.24) are written respectively in the form

\[
\tilde{\delta}_\alpha = \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i(1)}, \frac{\partial}{\partial p(2)i} \right),
\]

where

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(1)i}^j \frac{\partial}{\partial y^j(1)} + N_{(2)i}^j \frac{\partial}{\partial p(2)j},
\]

\[
\frac{\delta}{\delta y^i(1)} = \frac{\partial}{\partial y^i(1)} + N_{(2)ij} \frac{\partial}{\partial p(2)j},
\]

and

\[
\tilde{\delta}^\alpha = \left( dx^i, \delta y^i(1), \delta p(2)i \right),
\]

where

\[
\delta y^i(1) = dy^i(1) + N_{(1)i}^j dx^j,
\]

\[
\delta p(2)i = dp(2)i - N_{(2)ij} dx^j,
\]

with the dual coefficients \(M_{(1)i}^j\) and \(M_{(2)i}^j\) (see (1.22)) were expressed via \(N_{(1)i}^j\) and \(N_{(2)ij}\) like in Ref. [39].

1.3 Distinguished connections and metrics

In general, distinguished objects (d–objects) on a v–bundle \(\mathcal{E}\) (or cv–bundle \(\tilde{\mathcal{E}}\)) are introduced as geometric objects with various group and coordinate transforms coordinated with the N–connection structure on \(\mathcal{E}\) (or \(\tilde{\mathcal{E}}\)). For example, a distinguished connection (in brief, d–connection) \(D\) on \(\mathcal{E}\) (or \(\tilde{\mathcal{E}}\)) is defined as a linear connection \(D\) on \(\mathcal{E}\) (or \(\tilde{\mathcal{E}}\)) conserving under a parallelism the global decomposition (1.10) (or (1.11)) into horizontal and vertical subbundles of \(TE\) (or \(T\tilde{E}\)). A covariant derivation associated to a d–connection becomes d–covariant. We shall give necessary formulas for cv–bundles in round brackets.

1.3.1 D–connections

D–connections in v–bundles (cv–bundles)

A N–connection in a v–bundle \(\mathcal{E}\) (cv–bundle \(\tilde{\mathcal{E}}\)) induces a corresponding decomposition of d–tensors into sums of horizontal and vertical parts, for example, for every d–vector
\( X \in \mathcal{X}(\mathcal{E}) \) \((\tilde{X} \in \mathcal{X}(\tilde{\mathcal{E}}))\) and 1–form \( A \in \Lambda^1(\mathcal{E}) \) \((\tilde{A} \in \Lambda^1(\tilde{\mathcal{E}}))\) we have respectively

\[
\begin{align*}
X &= hX + vX & A &= hA + vA, \\
(\tilde{X}) &= h\tilde{X} + v\tilde{X} & (\tilde{A}) &= h\tilde{A} + v\tilde{A})
\end{align*}
\]

where

\[
hX = X^i \delta_i, \quad vX = X^a \partial_a \quad \text{and} \quad h\tilde{X} = \tilde{X}^i \delta_i, \quad v\tilde{X} = \tilde{X}^a \partial_a
\]

and

\[
hA = A_i \delta^i, \quad vA = A_a d^a \quad \text{and} \quad h\tilde{A} = \tilde{A}_i \delta^i, \quad v\tilde{A} = \tilde{A}_a d^a.
\]

In consequence, we can associate to every \(d\)-covariant derivation along the \(d\)-vector \((1.29), \quad D_X = X \circ D \quad (D_{\tilde{X}} = \tilde{X} \circ D)\) two new operators of \(h\)- and \(v\)-covariant derivations

\[
D^{(h)} X \quad \text{and} \quad D^{(v)} X, \quad \forall Y \in \mathcal{X}(\mathcal{E})
\]

for which the following conditions hold:

\[
D_X Y = D^{(h)} X Y + D^{(v)} X Y
\]

\[
(1.30)
\]

where

\[
D^{(h)} X Y = (hX)f \quad \text{and} \quad D^{(v)} X Y = (vX)f, \quad X, Y \in \mathcal{X}(\mathcal{E}), \quad f \in \mathcal{F}(M)
\]

\[
(1.31)
\]

The components \(\Gamma_{\alpha\beta\gamma}^\gamma \quad (\tilde{\Gamma}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}^\gamma)\) of a \(d\)-connection \(\tilde{D}_\alpha = (\tilde{\delta}_\alpha \circ D)\), locally adapted to the \(N\)—connection structure with respect to the frames \((1.16) \quad \text{and} \quad (1.17) \quad (1.19) \quad \text{and} \quad (1.20))\), are defined by the equations

\[
D_\alpha \delta_\beta = \Gamma_{\alpha\beta\gamma}^\gamma \quad (\tilde{D}_{\tilde{\alpha}} \tilde{\delta}_{\tilde{\beta}} = \tilde{\Gamma}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}^\gamma),
\]

from which one immediately follows

\[
\Gamma_{\alpha\beta\gamma}(u) = (D_\alpha \delta_\beta) \circ \delta_\gamma \quad (\tilde{\Gamma}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}(\tilde{u}) = (\tilde{D}_{\tilde{\alpha}} \tilde{\delta}_{\tilde{\beta}}) \circ \tilde{\delta}_\gamma).
\]

The coefficients of operators of \(h\)- and \(v\)-covariant derivations,

\[
D^{(h)}_k = \{\lambda^i_{jk}, \lambda^a_{ab}\} \quad \text{and} \quad D^{(v)}_c = \{\lambda^i_{jk}, \lambda^a_{bc}\}
\]

\[
(\tilde{D}^{(h)}_k = \{\tilde{\lambda}_i^{jk}, \tilde{\lambda}_a^{ab}\} \quad \text{and} \quad \tilde{D}^{(v)}_c = \{\tilde{\lambda}_i^{jk}, \tilde{\lambda}_a^{bc}\})
\]
(see (1.30)), are introduced as corresponding h- and v-parametrizations of (1.31)
\[
L^i_{jk} = (D_k \delta_j) \circ d^i, \quad L^a_{bk} = (D_k \partial_b) \circ \delta^a
\]
\[
(\tilde{L}^i_{jk}) = (\tilde{D}_k \delta_j) \circ d^i, \quad L^b_{ak} = (\tilde{D}_k \tilde{\partial}^b) \circ \tilde{\delta}_a
\]
and
\[
C_i^j_{jc} = (D_c \delta_j) \circ d^i, \quad C_a^b_{bc} = (D_c \partial_b) \circ \delta^a
\]
\[
(\tilde{C}^i_{jc}) = (\tilde{D}^c \delta_j) \circ d^i, \quad \tilde{C}_a^b_{bc} = (\tilde{D}^c \tilde{\partial}^b) \circ \tilde{\delta}_a.
\]
A set of components (1.32) and (1.33)
\[
\Gamma^\gamma_{\alpha\beta} = [L^i_{jk}, L^a_{bk}, C^i_{jc}, C_a^b_{bc}] \quad (\tilde{\Gamma}^\gamma_{\alpha\beta} = [\tilde{L}^i_{jk}, \tilde{L}^b_{ak}, \tilde{C}^i_{jc}, \tilde{C}_a^b_{bc}])
\]
completely defines the local action of a d-connection D in \(E\) (\(\tilde{D}\) in \(\tilde{E}\)).

For instance, having taken on \(E\) (\(\tilde{E}\)) a d-tensor field of type \(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\),
\[
t = t^i_{ja} \delta_i \otimes \partial_a \otimes d^j \otimes \delta^b,
\]
\[
\tilde{t} = \tilde{t}^i_{ja} \tilde{\delta}_i \otimes \tilde{\partial}_a \otimes d^j \otimes \tilde{\delta}_b,
\]
and a d-vector \(X\) (\(\tilde{X}\)) we obtain
\[
D_X t = D_X^{(h)} t + D_X^{(v)} t = \left( X^k t^i_{ja} + X^c t^i_{jc} \right) \delta_i \otimes \partial_a \otimes d^j \otimes \delta^b,
\]
\[
(\tilde{D}_X \tilde{t}) = (\tilde{D}_X)^{(h)} \tilde{t} + (\tilde{D}_X)^{(v)} \tilde{t} = \left( \tilde{X}^k \tilde{t}^i_{ja} + \tilde{X}^c t^i_{jc} \right) \tilde{\delta}_i \otimes \tilde{\partial}_a \otimes d^j \otimes \tilde{\delta}_b,
\]
where the h-covariant derivative is written
\[
t^i_{ja} \quad = \quad \delta_k t^i_{ja} + t^{i}_{aj}L^k_{jb} + L^a_{ck}t^i_{jc} - t^h_{jk} t^i_{ja} - L^c_{bk} t^{ia}
\]
\[
(\tilde{t}^i_{ja}) \quad = \quad \tilde{\delta}_k \tilde{t}^i_{ja} + \tilde{t}^i_{aj} \tilde{L}^k_{jb} + \tilde{L}^a_{ck} \tilde{t}^i_{jc} - \tilde{t}^h_{jk} \tilde{t}^i_{ja} - \tilde{L}^c_{bk} \tilde{t}^{ia}
\]
and the v-covariant derivative is written
\[
t^i_{ja} \quad = \quad \partial_i t^i_{ja} + C_{ck} t^i_{ja} + C^{c} d^{i} j^{a} - C^{h} d^{i} j^{a} - C^{c} d^{i} j^{a}
\]
\[
(\tilde{t}^i_{ja}) \quad = \quad \tilde{\partial}_i \tilde{t}^i_{ja} + \tilde{C}_{ck} \tilde{t}^i_{ja} + \tilde{C}^{c} d^{i} j^{a} - \tilde{C}^{h} d^{i} j^{a} - \tilde{C}^{c} d^{i} j^{a}.
\]
For a scalar function \(f \in \mathcal{F}(E)\) (\(f \in \mathcal{F}(\tilde{E})\)) we have
\[
D_k^{(h)} = \frac{\delta f}{\delta x^k} = \frac{\partial f}{\partial x^k} - N_a^k \frac{\partial f}{\partial y^a} \quad \text{and} \quad D_c^{(v)} f = \frac{\partial f}{\partial y^c},
\]
\[
(\tilde{D}_k)^{(h)} = \frac{\delta f}{\delta x^k} = \frac{\partial f}{\partial x^k} + N_a^k \frac{\partial f}{\partial p^a} \quad \text{and} \quad (\tilde{D})^{(v)} f = \frac{\partial f}{\partial p^c}.
\]
D–connections in hvc–bundles

The theory of connections in higher order anisotropic vector superbundles and vector bundles was elaborated in Refs. [1, 4, 7]. Here we re–formulate that formalism for the case when some shells of higher order anisotropy could be covector spaces by stating the general rules of covariant derivation compatible with the N–connection structure in hvc–bundle \( \tilde{\mathcal{E}} \) and omit details and cumbersome formulas.

For a hvc–bundle of type \( \tilde{\mathcal{E}} = \tilde{\mathcal{E}}[v(1), v(2), cv(3), \ldots, cv(z − 1), v(z)] \) a d–connection \( \tilde{\Gamma}^\gamma_{\alpha\beta} \) has the next shell decomposition of components (on induction being on the \( p \)-th shell, considered as the base space, which in this case a hvc–bundle, we introduce in a usual manner, like a vector or covector fiber, the \((p + 1)\)-th shell)

\[
\tilde{\Gamma}^\gamma_{\alpha\beta} = \{ \Gamma^\gamma_{\alpha_1\beta_1} = [L^a_{i_1 j_1 k_1}, L^a_{b_1 k_1}, C^a_{i_1 c_1}, C^a_{b_1 c_1}]; \\
\Gamma^\gamma_{\alpha_2\beta_2} = [L^a_{i_2 j_2 k_2}, L^a_{b_2 k_2}, C^a_{i_2 c_2}, C^a_{b_2 c_2}]; \\
\Gamma^\gamma_{\alpha_3\beta_3} = [L^a_{i_3 j_3 k_3}, L^a_{b_3 k_3}, C^a_{i_3 c_3}, C^a_{b_3 c_3}]; \\
\vdots \}
\]

These coefficients determine the rules of a covariant derivation \( \tilde{D} \) on \( \tilde{\mathcal{E}} \).

For example, let us consider a d–tensor \( \tilde{\mathbf{t}} \) of type

\[
\begin{pmatrix}
1 & l_1 & l_2 & \tilde{l}_3 & \ldots & l_z \\
1 & l_1 & l_2 & \tilde{l}_3 & \ldots & l_z
\end{pmatrix}
\]

with corresponding tensor product of components of anholonomic N–frames (1.22) and (1.24)

\[
\tilde{\mathbf{t}} = \tilde{t}^{i_1 a_1 b_1 \ldots b_{z−1} a_z} \delta_{\cdot} \otimes \partial_{a_1} \otimes d^j \otimes \delta^b_1 \otimes \partial_{a_2} \otimes \delta^b_2 \otimes \tilde{\delta} a_3 \otimes \delta b_3 \otimes \ldots \otimes \tilde{\delta} a_{z−1} \otimes \delta b_{z−1} \otimes \partial_{a_z} \otimes \delta b_z.
\]

The d–covariant derivation \( \tilde{D} \) of \( \tilde{\mathbf{t}} \) is to be performed separately for every shall according the rule (1.34) if a shell is defined by a vector subspace, or according the rule (1.35) if the shell is defined by a covector subspace.

1.3.2 Metric structure

D–metrics in v–bundles

We define a metric structure \( \mathbf{G} \) in the total space \( E \) of a v–bundle \( \mathcal{E} = (E, p, M) \) over a connected and paracompact base \( M \) as a symmetric covariant tensor field of type \((0, 2)\),

\[
\mathbf{G} = G^\alpha_\beta du^\alpha \otimes du^\beta
\]
being non degenerate and of constant signature on E.

Nonlinear connection $N$ and metric $G$ structures on $E$ are mutually compatible if there are satisfied the conditions:

$$G(\delta_i, \partial_a) = 0, \text{ or equivalently, } G_{ia}(u) - N^b_i(u) h_{ab}(u) = 0, \quad (1.36)$$

where $h_{ab} = G(\partial_a, \partial_b)$ and $G_{ia} = G(\partial_i, \partial_a)$, which gives

$$N^b_i(u) = h^{ab}(u) G_{ia}(u) \quad (1.37)$$

( the matrix $h^{ab}$ is inverse to $h_{ab}$). In consequence one obtains the following decomposition of metric:

$$G(X,Y) = hG(X,Y) + vG(X,Y), \quad (1.38)$$

where the d–tensor $hG(X,Y) = G(hX, hY)$ is of type $(0\ 0\ 2\ 0)$ and the d–tensor $vG(X,Y) = G(vX, vY)$ is of type $(0\ 0\ 0\ 2)$. With respect to anholonomic basis $\{1.16\}$ the d–metric $\{1.38\}$ is written

$$G = g_{\alpha\beta}(u) \delta^\alpha \otimes \delta^\beta = g_{ij}(u) d^i \otimes d^j + h_{ab}(u) \delta^a \otimes \delta^b, \quad (1.39)$$

where $g_{ij} = G(\delta_i, \delta_j)$.

A metric structure of type $\{1.38\}$ (equivalently, of type $\{1.39\}$) or a metric on $E$ with components satisfying the constraints $\{1.36\}$, (equivalently $\{1.37\}$) defines an adapted to the given $N$–connection inner (d–scalar) product on the tangent bundle $TE$.

We shall say that a d–connection $\tilde{D}_X$ is compatible with the d-scalar product on $TE$ (i. e. it is a standard d–connection) if

$$\tilde{D}_X (X \cdot Y) = \left(\tilde{D}_X Y\right) \cdot Z + Y \cdot \left(\tilde{D}_X Z\right), \ \forall X, Y, Z \in \mathcal{X}(E).$$

An arbitrary d–connection $D_X$ differs from the standard one $\tilde{D}_X$ by an operator $\hat{P}_X(u) = \{X^a \hat{\alpha}^\alpha_{\beta}(u)\}$, called the deformation d-tensor with respect to $\tilde{D}_X$, which is just a d-linear transform of $\mathcal{E}_u, \ \forall \ u \in \mathcal{E}$. The explicit form of $\hat{P}_X$ can be found by using the corresponding axiom defining linear connections $\{25\}$

$$\left(D_X - \tilde{D}_X\right) f Z = f \left(D_X - \tilde{D}_X\right) Z,$$

written with respect to $N$–elongated bases $\{1.16\}$ and $\{1.17\}$. From the last expression we obtain

$$\hat{P}_X(u) = \left[(D_X - \tilde{D}_X)\delta_\alpha(u)\right] \delta^\alpha(u),$$

25
therefore

\[ D_X Z = \tilde{D}_X Z + \hat{P}_X Z. \] (1.40)

A d–connection \( D_X \) is **metric** (or **compatible**) with metric \( G \) on \( E \) if

\[ D_X G = 0, \forall X \in \mathcal{X}(E). \]

With respect to anholonomic frames these conditions are written

\[ D_\alpha g_{\beta\gamma} = 0, \] (1.41)

where by \( g_{\beta\gamma} \) we denote the coefficients in the block form (1.39).

**D–metrics in cv– and hvc–bundles**

The presented considerations on self–consistent definition of N–connection, d–connection and metric structures in v–bundles can re–formulated in a similar fashion for another types of anisotropic space–times, on cv–bundles and on shells of hvc–bundles. For simplicity, we give here only the analogous formulas for the metric d–tensor (1.39):

- On cv–bundle \( \mathcal{E} \) we write

\[ \tilde{G} = \tilde{g}_{\alpha\beta}(\tilde{u}) \tilde{\delta}^\alpha \otimes \tilde{\delta}^\beta = \tilde{g}_{ij}(\tilde{u}) d^i \otimes d^j + \tilde{h}^{ab}(\tilde{u}) \tilde{\delta}_a \otimes \tilde{\delta}_b, \] (1.42)

where \( \tilde{g}_{ij} = \tilde{G}(\tilde{\delta}_i, \tilde{\delta}_j) \) and \( \tilde{h}^{ab} = \tilde{G}(\tilde{\delta}^a, \tilde{\delta}^b) \) and the N–coframes are given by formulas (1.20).

For simplicity, we shall consider that the metricity conditions are satisfied, \( \tilde{D}_\gamma \tilde{g}_{\alpha\beta} = 0 \).

- On hvc–bundle \( \mathcal{E} \) we write

\[ \tilde{G} = \tilde{g}_{\alpha\beta}(\tilde{u}) \tilde{\delta}^\alpha \otimes \tilde{\delta}^\beta = \tilde{g}_{ij}(\tilde{u}) d^i \otimes d^j + \tilde{h}_{a_{1}b_{1}}(\tilde{u}) \tilde{\delta}^{a_{1}} \otimes \tilde{\delta}^{b_{1}} + \tilde{h}_{a_{2}b_{2}}(\tilde{u}) \tilde{\delta}^{a_{2}} \otimes \tilde{\delta}^{b_{2}} + \tilde{h}^{a_{3}b_{3}}(\tilde{u}) \tilde{\delta}_{a_{3}} \otimes \tilde{\delta}_{b_{3}} + \ldots + \tilde{h}^{a_{z-1}b_{z-1}}(\tilde{u}) \tilde{\delta}_{a_{z-1}} \otimes \tilde{\delta}_{b_{z-1}} + \tilde{h}_{a_{z}b_{z}}(\tilde{u}) \tilde{\delta}^{a_{z}} \otimes \tilde{\delta}^{b_{z}}, \] (1.43)

where \( \tilde{g}_{ij} = \tilde{G}(\tilde{\delta}_i, \tilde{\delta}_j) \) and \( \tilde{h}_{a_{1}b_{1}} = \tilde{G}(\partial_{a_{1}}, \partial_{b_{1}}) \), \( \tilde{h}_{a_{2}b_{2}} = \tilde{G}(\partial_{a_{2}}, \partial_{b_{2}}) \), \( \tilde{h}^{a_{3}b_{3}} = \tilde{G}(\tilde{\delta}^{a_{3}}, \tilde{\delta}^{b_{3}}) \), .... and the N–coframes are given by formulas (1.24).

The metricity conditions are \( \tilde{D}_\gamma \tilde{g}_{\alpha\beta} = 0 \).

- On osculator bundle \( T^2 M = Osc^2 M \) we have a particular case of (1.43) when

\[ \tilde{G} = \tilde{g}_{\alpha\beta}(\tilde{u}) \tilde{\delta}^\alpha \otimes \tilde{\delta}^\beta = \tilde{g}_{ij}(\tilde{u}) d^i \otimes d^j + \tilde{h}_{i}^{\prime}(\tilde{u}) \delta y_{(1)}^i \otimes \delta y_{(1)}^i + \tilde{h}_{ij}(\tilde{u}) \delta y_{(2)}^i \otimes \delta y_{(2)}^i \] (1.44)

where the N–coframes are given by (1.27).
• On dual osculator bundle \((T^*M, p^*M, M)\) we have another particular case of (1.43) when

\[
\bar{G} = \bar{g}_{\alpha\beta}(\bar{u}) \bar{\delta}^\alpha \otimes \bar{\delta}^\beta
\]

\[
= \bar{g}_{ij}(\bar{u}) \, \delta^i \otimes \delta^j + \bar{h}_{ij}(\bar{u}) \, \delta y^i_{(1)} \otimes \delta y^j_{(1)} + \bar{h}^{ij}(\bar{u}) \, \delta p^i_{(2)} \otimes \delta p^j_{(2)}
\]  

(1.45)

where the N–coframes are given by (1.28).

**1.3.3 Some remarkable d–connections**

We emphasize that the geometry of connections in a v–bundle \(E\) is very reach. If a triple of fundamental geometric objects \((N_i^a(u), \Gamma^\alpha_{\beta\gamma}(u), g_{\alpha\beta}(u))\) is fixed on \(E\), a multi–connection structure (with corresponding different rules of covariant derivation, which are, or not, mutually compatible and with the same, or not, induced d–scalar products in \(\mathcal{T}E\)) is defined on this v–bundle. We can give a priority to a connection structure following some physical arguments, like the reduction to the Christoffel symbols in the holonomic case, mutual compatibility between metric and N–connection and d–connection structures and so on.

In this subsection we enumerate some of the connections and covariant derivations in v–bundle \(E\), cv–bundle \(\tilde{E}\) and in some hvc–bundles which can present interest in investigation of locally anisotropic gravitational and matter field interactions :

1. Every N–connection in \(E\) with coefficients \(N_i^a(x, y)\) being differentiable on \(y\)–variables, induces a structure of linear connection \(N^\alpha_{\beta\gamma}\), where

\[
N^a_{bi} = \frac{\partial N_i^a}{\partial y^b} \text{ and } N^a_{bc}(x, y) = 0.
\]

(1.46)

For some \(Y(u) = Y^i(u) \partial_i + Y^a(u) \partial_a\) and \(B(u) = B^a(u) \partial_a\) one introduces a covariant derivation as

\[
D_Y^{(N)} B = \left[ Y^i \left( \frac{\partial B^a}{\partial x^i} + N^a_{bi} B^i \right) + Y^a \frac{\partial B^a}{\partial y^b} \right] \frac{\partial}{\partial y^a}.
\]

2. The d–connection of Berwald type \([10]\) on v–bundle \(E\) (cv–bundle \(\tilde{E}\))

\[
\Gamma^{(B)\alpha}_{\beta\gamma} = \left( L^i_{jk}, \frac{\partial N_i^a}{\partial y^b}, 0, C^a_{bc} \right), \quad \tilde{\Gamma}^{(B)\alpha}_{\beta\gamma} = \left( \tilde{L}^i_{jk}, -\frac{\partial \tilde{N}_{ka}}{\partial p_b}, 0, \tilde{C}^a_{bc} \right)
\]

(1.47)
where

\[
L_{i,j,k}^i(x,y) = \frac{1}{2}g^i_{ir} \left( \frac{\delta g_{jk}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r} \right),
\]

(1.48)

\[
C^a_{bc}(x,y) = \frac{1}{2}h^{ab} \left( \frac{\partial h_{cd}}{\partial y^c} + \frac{\partial h_{ed}}{\partial y^d} - \frac{\partial h_{bd}}{\partial y^c} \right),
\]

(\tilde{L}_{i,j,k}(x,p) = \frac{1}{2}g^i_{ir} \left( \frac{\delta \tilde{g}_{jk}}{\delta x^k} + \frac{\delta \tilde{g}_{kr}}{\delta x^j} - \frac{\delta \tilde{g}_{jk}}{\delta x^r} \right),
\]

\[
\tilde{C}^a_{bc}(x,p) = \frac{1}{2}h^{ab} \left( \frac{\partial \tilde{h}_{cd}}{\partial p_c} + \frac{\partial \tilde{h}_{ed}}{\partial p_d} - \frac{\partial \tilde{h}_{bd}}{\partial p_c} \right),
\]

which is hv—metric, i.e. there are satisfied the conditions \(D_k^{(B)}g_{ij} = 0\) and \(D_c^{(B)}h_{ab} = 0\) (\(\tilde{D}_k^{(B)}g_{ij} = 0\) and \(\tilde{D}_c^{(B)}h_{ab} = 0\)).

3. The canonical d–connection \(\Gamma^{(c)}(\text{or } \tilde{\Gamma}^{(c)})\) on a v–bundle (or cv–bundle) is associated to a metric \(G\) (or \(\tilde{G}\)) of type \([1.39]\) (or \([1.42]\)),

\[
\Gamma^{(c)\alpha}_{\beta\gamma} = [L_{j,k}^{(c)i}, L_{bc}^{(c)a}, C_{jc}^{(c)i}, C_{bc}^{(c)a} ] (\tilde{\Gamma}^{(c)\alpha}_{\beta\gamma} = [\tilde{L}_{j,k}^{(c)i}, L_{ac}^{(c)b}, \tilde{C}_{jc}^{(c)i}, \tilde{C}_{bc}^{(c)a} ])
\]

with coefficients

\[
L_{j,k}^{(c)i} = \frac{\partial N_{i}^{a}}{\partial y^{b}} + \frac{1}{2}h^{ac} \left( \frac{\partial N_{i}^{d}}{\partial y^{b}}h_{dc} - \frac{\partial N_{i}^{d}}{\partial y^{c}}h_{db} \right),
\]

(1.49)

\[
L_{bi}^{(c)a} = \frac{\partial N_{i}^{a}}{\partial y^{b}} + \frac{1}{2}h^{ac} \left( \frac{\partial N_{i}^{d}}{\partial y^{b}}h_{dc} - \frac{\partial N_{i}^{d}}{\partial y^{c}}h_{db} \right),
\]

\[
(\tilde{L})_{a,b}^{(c)} = -\frac{\partial N_{i}^{a}}{\partial p_b} + \frac{1}{2}h^{ac} \left( \frac{\partial N_{i}^{d}}{\partial p_b}h_{dc} + \frac{\partial N_{i}^{d}}{\partial p_d}h_{db} \right),
\]

\[
C_{jc}^{(c)i} = \frac{1}{2}g^{jk} \frac{\partial g_{jk}}{\partial y^{c}} (\tilde{C}_{jc}^{(c)i} = \frac{1}{2}g^{jk} \frac{\partial \tilde{g}_{jk}}{\partial p_c}).
\]

This is a metric d–connection which satisfies conditions

\[
D_k^{(c)}g_{ij} = 0, D_c^{(c)}h_{ab} = 0, D_c^{(c)}h_{ab} = 0
\]

\[
(\tilde{D}_k^{(c)}g_{ij} = 0, \tilde{D}_k^{(c)}h_{ab} = 0, \tilde{D}_c^{(c)}h_{ab} = 0).
\]

In physical applications we shall use the canonical connection and for simplicity we shall omit the index \((c)\). The coefficients \([1.49]\) are to be extended to higher order if we are dealing with derivations of geometrical objects with “shell” indices. In this case the fiber indices are to be stipulated for every type of shell into consideration.

4. We can consider the N–adapted Christoffel d–symbols

\[
\tilde{\Gamma}^{a}_{\beta\gamma} = \frac{1}{2}g^{a\tau} (\delta_{\gamma}g_{\tau\beta} + \delta_{\beta}g_{\tau\gamma} - \delta_{\gamma}g_{\beta\gamma}),
\]

(1.50)

which have the components of d–connection \(\tilde{\Gamma}^{a}_{\beta\gamma} = (L_{j,k}^{i}, 0, 0, C_{bc}^{a})\), with \(L_{j,k}^{i}\) and \(C_{bc}^{a}\) as in \([1.48]\) if \(g_{\alpha\beta}\) is taken in the form \([1.39]\).
Arbitrary linear connections on a v-bundle \( \mathcal{E} \) can be also characterized by theirs deformation tensors (see (1.40)) with respect, for instance, to the d–connection (1.50):

\[
\Gamma_{\beta\gamma}^{(B)} = \tilde{\Gamma}_{\beta\gamma} + P_{\beta\gamma}^{(B)} \quad \Gamma_{\beta\gamma}^{(c)} = \tilde{\Gamma}_{\beta\gamma} + P_{\beta\gamma}^{(c)}
\]

or, in general,

\[
\Gamma_{\beta\gamma} = \tilde{\Gamma}_{\beta\gamma} + P_{\beta\gamma}
\]

where \( P_{\beta\gamma}^{(B)}, P_{\beta\gamma}^{(c)} \) and \( P_{\beta\gamma} \) are respectively the deformation d-tensors of d–connections (1.47), (1.49) or of a general one. Similar deformation d–tensors can be introduced for d–connections on cv–bundles and hvc–bundles. We omit explicit formulas.

### 1.3.4 Almost Hermitian anisotropic spaces

The are possible very interesting particular constructions [36, 37, 39] on t–bundle \( TM \) provided with N–connection which defines a N–adapted frame structure \( \delta_{\alpha} = (\delta_i, \dot{\delta}_i) \) (for the same formulas (1.16) and (1.17) but with identified fiber and base indices). We are using the 'dot' symbol in order to distinguish the horizontal and vertical operators because on t–bundles the indices could take the same values both for the base and fiber objects. This allow us to define an almost complex structure \( J = \{J_{\alpha}^{\beta}\} \) on \( TM \) as follows

\[
J(\delta_i) = -\dot{\delta}_i, \quad J(\dot{\delta}_i) = \delta_i. \tag{1.51}
\]

It is obvious that \( J \) is well–defined and \( J^2 = -I \).

For d–metrics of type (1.39), on \( TM \), we can consider the case when

\[
g_{ij}(x,y) = h_{ab}(x,y), \text{ i.e.}
\]

\[
G_{(t)} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)dy^i \otimes dy^j, \tag{1.52}
\]

where the index \( (t) \) denotes that we have geometrical object defined on tangent space.

An almost complex structure \( J_{\alpha}^{\beta} \) is compatible with a d–metric of type (1.52) and a d–connection \( D \) on tangent bundle \( TM \) if the conditions

\[
J_{\alpha}^{\beta}J_{\gamma}^{\delta}g_{\beta\delta} = g_{\alpha\gamma} \text{ and } D_{\alpha}J_{\beta}^{\gamma} = 0
\]

are satisfied.

The pair \((G_{(t)}, J)\) is an almost Hermitian structure on \( TM \).

One can introduce an almost sympletic 2–form associated to the almost Hermitian structure \((G_{(t)}, J)\),

\[
\theta = g_{ij}(x,y)dy^i \wedge dx^j. \tag{1.53}
\]

If the 2–form (1.53), defined by the coefficients \( g_{ij} \), is closed, we obtain an almost Kählerian structure in \( TM \).
Definition 1.6. An almost Kähler metric connection is a linear connection $D^{(H)}$ on $T\tilde{M} = TM \setminus \{0\}$ with the properties:

1. $D^{(H)}$ preserve by parallelism the vertical distribution defined by the $N$–connection structure;

2. $D^{(H)}$ is compatible with the almost Kähler structure $(G_{(t)}, J)$, i. e.

$$D^{(H)}_X g = 0, \ D^{(H)}_X J = 0, \ \forall X \in \mathcal{X}(TM).$$

By straightforward calculation we can prove that a d–connection $D\Gamma = (L^i_{jk}, L^i_{jk}, C^i_{jc}, C^i_{jc})$ with the coefficients defined by

$$D^{(H)}_t \delta_i \delta_j = L^i_{jk} \delta_i, \ D^{(H)}_t \dot{\partial}_j = L^i_{jk} \dot{\partial}_i;$$

$$D^{(H)}_t \delta_i \dot{\partial}_j = C^i_{jk} \delta_i, \ D^{(H)}_t \dot{\partial}_j = C^i_{jk} \dot{\partial}_i;$$

where $L^i_{jk}$ and $C^a_{bc} \rightarrow C^i_{jk}$, on $TM$ are defined by the formulas (1.48), define a torsion-less (see the next section on torsion structures) metric d–connection which satisfy the compatibility conditions (1.41).

Almost complex structures and almost Kähler models of Finsler, Lagrange, Hamilton and Cartan geometries (of first an higher orders) are investigated in details in Refs. [34, 35, 39, 70].

1.4 Torsions and Curvatures

In this section we outline the basic definitions and formulas for the torsion and curvature structures in v–bundles and cv–bundles provided with $N$–connection structure.

1.4.1 N–connection curvature

1. The curvature $\Omega$ of a nonlinear connection $N$ in a v–bundle $E$ can be defined in local form as [36, 37]:

$$\Omega = \frac{1}{2} \Omega^a_{ij} d^i \wedge d^j \otimes \partial_a,$$

where

$$\Omega^a_{ij} = \delta_j N^a_i - \delta_i N^a_j$$

$$= \partial_j N^a_i - \partial_i N^a_j + N^b_i N^a_{bj} - N^b_j N^a_{bi};$$

$N^a_{bi}$ being that from (1.40).
2. For the curvature \( \tilde{\Omega} \), of a nonlinear connection \( \tilde{\mathbf{N}} \) in a cv–bundle \( \tilde{\mathbf{E}} \) we introduce
\[
\tilde{\Omega} = \frac{1}{2} \tilde{\Omega}_{ija} d^i \wedge \tilde{\partial}^a,
\]
where
\[
\tilde{\Omega}_{ija} = -\delta_j \tilde{N}_{ia} + \delta_i \tilde{N}_{ja} = -\partial_j \tilde{N}_{ia} + \partial_i \tilde{N}_{ja} + \tilde{N}_{ib} \tilde{N}_{ja}^b - \tilde{N}_{jb} \tilde{N}_{ja}^b,
\]
\( \tilde{N}_{ja}^b = \tilde{\partial}^b \tilde{N}_{ja} = \partial \tilde{N}_{ja}/\partial p_b. \)

3. Curvatures \( \tilde{\Omega} \) of different type of nonlinear connections \( \tilde{\mathbf{N}} \) in higher order anisotropic bundles were analyzed for different type of higher order tangent/dual tangent bundles and higher order prolongations of generalized Finsler, Lagrange and Hamilton spaces in Refs. [34, 35, 39] and for higher order anisotropic superspaces and spinor bundles in Refs. [70, 80, 71, 69]: For every higher order anisotropy shell we shall define the coefficients (1.55) or (1.56) in dependence of the fact with type of subfiber we are considering (a vector or covector fiber).

1.4.2 d–Torsions in v- and cv–bundles
The torsion \( T \) of a d–connection \( D \) in v–bundle \( \mathbf{E} \) (cv–bundle \( \tilde{\mathbf{E}} \)) is defined by the equation
\[
T(X, Y) = XY^\circ T \hat{=} D_X Y - D_Y X - [X, Y].
\]
One holds the following h- and v–decompositions
\[
T(X, Y) = T(hX, hY) + T(hX, vY) + T(vX, hY) + T(vX, vY).
\]
We consider the projections:
\[
hT(X, Y), vT(hX, hY), hT(hX, hY), ... \]
and say that, for instance, \( hT(hX, hY) \) is the h(hh)–torsion of \( D \), \( vT(hX, hY) \) is the v(hh)–torsion of \( D \) and so on.

The torsion (1.57) in v-bundle is locally determined by five d–tensor fields, torsions, defined as
\[
T_{jk}^i = hT(\delta_k, \delta_j) \cdot d^i, \quad T_{ja}^a = vT(\delta_k, \delta_j) \cdot \delta^a,
\]
\[
P_{ji}^a = hT(\partial_i, \partial_j) \cdot d^i, \quad P_{ja}^a = vT(\partial_i, \partial_j) \cdot \delta^a, S_{bc}^a = vT(\partial_i, \partial_b) \cdot \delta^a.
\]
Using formulas (1.16), (1.17), (1.55) and (1.57) we can compute [36, 37] in explicit form the components of torsions (1.58) for a d–connection of type (1.32) and (1.33):
\[
T_{jk}^i = T_{jk}^i = L_{jk}^i - L_{kj}^i, \quad T_{ja}^i = C_{ja}^i, T_{ja}^i = -C_{ja}^i, T_{ja}^i = 0, \quad T_{ja}^i = -P_{ja}^i.
\]
\[
T_{bc}^a = C_{bc}^a - C_{cb}^a = \delta_j N_i^a - \delta_j N_i^a, \quad T_{ba}^a = P_{ba}^a = \partial_b N_i^a - L_{ba}^a.
\]
By a direct computation, using (1.16), (1.17), (1.32), (1.33) and (1.63) we get:
\[ \begin{align*}
\hat{T}_{jk}^i &= h T(\delta_k, \delta_j) \cdot d^i, \quad \hat{T}_{jka}^i = v T(\delta_k, \delta_j) \cdot \delta_a, \\
\hat{P}_{ja}^i &= h T(\delta^j, \delta_k) \cdot d^i, \quad \hat{P}_{aj}^i = v T(\delta^j, \delta_k) \cdot \delta_a, \quad \hat{S}_{a}^{bc} = v T(\delta^c, \delta^b) \cdot \delta_a.
\end{align*} \] (1.60)

and
\[ \begin{align*}
\check{T}_{jk}^i &= \hat{T}_{jk}^i = L_{jk}^i - L_{kj}^i, \quad \check{T}_{jk}^i = \check{C}^i_{jk}^a, \quad \check{T}_{jk}^a = -\check{C}^a_{jk}^i, \quad \check{T}_{jk}^j = 0, \quad \check{T}_{jk}^j = -\check{P}_{jk}^a, \\
\check{T}_{a}^{bc} &= \check{S}_{a}^{bc} = \check{C}_{a}^{bc} - \check{C}_{a}^{cb}, \quad \check{T}_{ija} = -\delta_j \check{N}_{ia} + \delta_i \check{N}_{ja}, \quad \check{T}_{a}^{bi} = \check{P}_{a}^{bi} = -\check{P}_{a}^{ib}.
\end{align*} \] (1.61)

The formulas for torsion can be generalized for hv–bundles (on every shell we must write (1.59) or (1.61) in dependence of the type of shell, vector or co-vector one, we are dealing).

### 1.4.3 d–Curvatures in v- and cv–bundles

The curvature \( R \) of a d–connection in v–bundle \( \mathcal{E} \) is defined by the equation
\[ R(X, Y) Z = XY^* R \bullet Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z. \]
One holds the next properties for the h– and v–decompositions of curvature:
\[ \begin{align*}
\text{v} R(X, Y) h Z &= 0, \quad \text{h} R(X, Y) v Z = 0, \\
\text{R} (X, Y) Z &= \text{h} R(X, Y) h Z + \text{v} R(X, Y) v Z.
\end{align*} \] (1.62)

From (1.62) and the equation \( R(X, Y) = -R(Y, X) \) we get that the curvature of a d–connection \( D \) in \( \mathcal{E} \) is completely determined by the following six d–tensor fields:
\[ \begin{align*}
R_{h,jk}^i &= d^i \cdot R(\delta_k, \delta_j) \delta_h, \quad R_{b,jk}^a = \delta^a \cdot R(\delta_k, \delta_j) \partial_b, \quad R_{a,jk}^a = \delta^a \cdot R(\delta_k, \delta_j) \partial_a, \\
P_{j,ck}^i &= d^i \cdot R(\partial_c, \delta_k) \delta_j, \quad P_{b,ck}^a = \delta^a \cdot R(\partial_c, \delta_k) \partial_b, \quad P_{c,ck}^a = \delta^a \cdot R(\partial_c, \partial_c) \partial_a.
\end{align*} \] (1.63)

By a direct computation, using (1.16), (1.17), (1.32), (1.33) and (1.63) we get:
\[ \begin{align*}
R_{h,jk}^i &= \delta_h L_{h,jk}^i - \delta_j L_{h,kj}^i - L_{h,jk}^i L_{m,kj}^i - L_{h,kj}^i L_{m,jk}^i + C_{h}^{i}_{a} R_{j,k}^a, \\
R_{b,jk}^a &= \delta_b L_{b,jk}^a - \delta_j L_{b,kj}^a + L_{b,jk}^a L_{c,kj}^a - L_{b,kj}^a L_{c,jk}^a + C_{b}^{a}_{c} R_{j,k}^c, \\
P_{j,ka}^i &= \partial_a L_{j,ka}^i - \partial_k L_{j,ka}^i + L_{j,ka}^i L_{l,ka}^i - L_{j,ka}^i L_{l,ka}^i + C_{j}^{i}_{a} P_{k,a}^b, \\
P_{b,ka}^c &= \partial_a L_{b,ka}^c - \partial_k L_{b,ka}^c + L_{b,ka}^c L_{l,ka}^c - L_{b,ka}^c L_{l,ka}^c + C_{b}^{c}_{a} P_{k,a}^d, \\
S_{j,bc}^{a} &= \partial_a C_{j}^{i}_{j} + \partial_b C_{j}^{i}_{j} + C_{j}^{i}_{b} C_{j}^{i}_{c} + C_{j}^{i}_{c} C_{j}^{i}_{b}, \\
S_{b,cd}^{a} &= \partial_d C_{b}^{i}_{c} + \partial_c C_{b}^{i}_{c} + C_{b}^{i}_{c} C_{b}^{i}_{d} + C_{b}^{i}_{d} C_{b}^{i}_{c}.
\end{align*} \] (1.64)

We note that d–torsions (1.59) and d–curvatures (1.64) are computed in explicit form by particular cases of d–connections (1.47), (1.49) and (1.50).

32
For cv–bundles we have
\begin{align}
\hat{R}_{h,jk}^i &= d^i \cdot R(\delta_k, \delta_j) \delta_h, \quad \hat{R}_{a,jk}^b = \delta_a \cdot R(\delta_k, \delta_j) \delta^b,
\hat{P}_{j,k}^i c &= d^i \cdot R(\delta^c, \delta_k) \delta_j, \quad \hat{P}_{a,k}^b c = \delta_a \cdot R(\delta^c, \delta_k) \delta^b,
\hat{S}_{j,a}^{abc} &= d^i \cdot R(\delta^c, \delta^b) \delta_j, \quad \hat{S}_{a}^{b,cd} = \delta_a \cdot R(\delta^d, \delta^c) \delta^b.
\end{align}

and
\begin{align}
\hat{R}_{h,jk}^i &= \hat{\delta}_h L_{h,j}^i - \hat{\delta}_j L_{h,k}^i + L_{h,j}^m L_{m,k}^i - L_{h,k}^m L_{j,m}^i + \check{C}_h^i a \hat{R}_{a,jk},
\hat{R}_{a,jk}^b &= \hat{\delta}_h L_{a,j}^b - \hat{\delta}_j L_{a,k}^b + L_{a,j}^c L_{c,k}^b - L_{a,k}^c L_{j,c}^b + \check{C}_a^b c \hat{R}_{c,jk},
\hat{P}_{j,k}^i a &= \hat{\delta}_a L_{j,k}^i - (\hat{\delta}_k \check{C}_j^a + L_{j,k}^c \check{C}_c^a - L_{j,k}^a \check{C}_c^d) + \check{C}_j^a b \hat{P}_{b,k}^a,
\hat{P}_{c,k}^a &= \hat{\delta}_a L_{c,k}^b - (\hat{\delta}_k \check{C}_c^b + L_{c,k}^d \check{C}_d^b - L_{d,k}^b \check{C}_c^d) + \check{C}_c^b d \hat{P}_{d,k}^a,
\hat{S}_{j,a}^{abc} &= \hat{\delta}_j L_{a,c}^b - \hat{\delta}_a L_{j,c}^b + \check{C}_a^b c \check{C}_j^c - \check{C}_a^b c \check{C}_j^b,
\hat{S}_{a}^{b,cd} &= \hat{\delta}_a L_{b,c}^d - \hat{\delta}_b L_{a,c}^d + \check{C}_a^c d \check{C}_b^c - \check{C}_a^c d \check{C}_b^a.
\end{align}

The formulas for curvature can be also generalized for hvc–bundles (on every shell we must write (1.59) or (1.60) in dependence of the type of shell, vector or co-vector one, we are dealing).

### 1.5 Generalizations of Finsler Spaces

We outline the basic definitions and formulas for Finsler, Lagrange and generalized Lagrange spaces (constructed on tangent bundle) and for Cartan, Hamilton and generalized Hamilton spaces (constructed on cotangent bundle). The original results are given in details in monographs [36, 37, 39].

#### 1.5.1 Finsler Spaces

The Finsler geometry is modeled on tangent bundle \( TM \).

**Definition 1.7.** A Finsler space (manifold) is a pair \( F^n = (M, F(x,y)) \) where \( M \) is a real \( n \)–dimensional differentiable manifold and \( F : TM \to \mathcal{R} \) is a scalar function which satisfy the following conditions:

1. \( F \) is a differentiable function on the manifold \( \tilde{TM} = TM \setminus \{0\} \) and \( F \) is continuous on the null section of the projection \( \pi : TM \to M \);

2. \( F \) is a positive function, homogeneous on the fibers of the \( TM \), i.e. \( F(x,\lambda y) = \lambda F(x,y) \), \( \lambda \in \mathcal{R} \);

3. The Hessian of \( F^2 \) with elements
\[
g_{ij}^{(F)}(x,y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}
\]
is positively defined on \( \tilde{TM} \).
The function $F(x, y)$ and $g_{ij}(x, y)$ are called respectively the fundamental function and the fundamental (or metric) tensor of the Finsler space $F$.

One considers "anisotropic" (depending on directions $y'$) Christoffel symbols, for simplicity we write $g_{ij}^{(F)} = g_{ij}$,

$$
\gamma^i_{jk}(x, y) = \frac{1}{2} g^{ir} \left( \frac{\partial g_{rk}}{\partial x^j} + \frac{\partial g_{jr}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^r} \right),
$$

which are used for definition of the Cartan N–connection,

$$
N^i_{(c) j} = \frac{1}{2} \frac{\partial}{\partial y^j} \left[ \gamma^i_{nk}(x, y) y^n y^k \right].
$$

This N–connection can be used for definition of an almost complex structure like in (1.51) and to define on $TM$ a d–metric

$$
G_{(F)} = g_{ij}(x, y) dx^i \otimes dx^j + g_{ij}(x, y) \delta y^i \otimes \delta y^j,
$$

with $g_{ij}(x, y)$ taken as (1.67).

Using the Cartan N–connection (1.68) and Finsler metric tensor (1.67) (or, equivalently, the d–metric (1.69)) we can introduce the canonical d–connection

$$
D \Gamma (N_{(c)}) = \Gamma^\alpha_{(c) \beta \gamma} = (L^i_{(c) jk}, C^i_{(c) jk})
$$

with the coefficients computed like in (1.54) and (1.48) with $h_{ab} \rightarrow g_{ij}$. The d–connection $D \Gamma (N_{(c)})$ has the unique property that it is torsionless and satisfies the metricity conditions both for the horizontal and vertical components, i. e. $D_\alpha g_{\beta \gamma} = 0$.

The d–curvatures

$$
\check{R}^i_{h, jk} = \{\check{R}^i_{h, jk}, \check{P}^i_{jk l}, S^i_{(c) jk l}\}
$$

on a Finsler space provided with Cartan N–connection and Finsler metric structures are computed following the formulas (1.64) when the $a, b, c...$ indices are identified with $i, j, k, ...$ indices. It should be emphasized that in this case all values $g_{ij}, \Gamma^\alpha_{(c) \beta \gamma}, R^\alpha_{(c) \beta \gamma \delta}$ are defined by a fundamental function $F(x, y)$.

In general, we can consider that a Finsler space is provided with a metric $g_{ij} = \partial^2 F^2 / 2 \partial y^i \partial y^j$, but the N–connection and d–connection are be defined in a different manner, even not be determined by $F$.

### 1.5.2 Lagrange and Generalized Lagrange Spaces

The notion of Finsler spaces was extended by J. Kern [24] and R. Miron [27]. It is widely developed in monographs [36, 37] and extended to superspaces by S. Vacaru [64, 69, 70].

The idea of extension was to consider instead of the homogeneous fundamental function $F(x, y)$ in a Finsler space a more general one, a Lagrangian $L(x, y)$, defined as a
differentiable mapping \( L : (x, y) \in TM \rightarrow L(x, y) \in \mathcal{R} \), of class \( C^\infty \) on manifold \( \tilde{TM} \) and continuous on the null section \( 0 : M \rightarrow TM \) of the projection \( \pi : TM \rightarrow M \). A Lagrangian is regular if it is differentiable and the Hessian

\[
g^{(L)}_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}
\]

is of rank \( n \) on \( M \).

**Definition 1.8.** A Lagrange space is a pair \( L^n = (M, L(x, y)) \) where \( M \) is a smooth real \( n \)-dimensional manifold provided with regular Lagrangian \( L(x, y) \) structure \( L : TM \rightarrow \mathcal{R} \) for which \( g_{ij}(x, y) \) from (1.70) has a constant signature over the manifold \( \tilde{TM} \).

The fundamental Lagrange function \( L(x, y) \) defines a canonical \( N \)-connection

\[
N_{(cL)}^i j = \frac{1}{2} \frac{\partial}{\partial y^j} \left[ g^{jk} \left( \frac{\partial^2 L^2}{\partial y^k \partial y^h} y^h - \frac{\partial L}{\partial x^k} \right) \right]
\]

as well a d-metric

\[
G_{(L)} = g_{ij}(x, y) dx^i \otimes dx^j + g_{ij}(x, y) \delta y^i \otimes \delta y^j,
\]

with \( g_{ij}(x, y) \) taken as (1.70). As well we can introduce an almost Kählerian structure and an almost Hermitian model of \( L^n \), denoted as \( H^{2n} \) as in the case of Finsler spaces but with a proper fundamental Lagrange function and metric tensor \( g_{ij} \). The canonical metric d-connection \( D_{\Gamma} (N_{(cL)}) = \Gamma^\alpha_{(cL)\beta\gamma} = (L^i_{(cL)} j k, C_i^{(cL)} j k) \) is to computed by the same formulas (1.54) and (1.48) with \( h_{ab} \rightarrow g_{ij}^{(L)} \), for \( N_{(cL)}^i j \). The d-torsions (1.59) and d-curvatures (1.64) are defined, in this case, by \( L^i_{(cL)} j k \) and \( C^i_{(cL)} j k \). We also note that instead of \( N_{(cL)}^i j \) and \( \Gamma^\alpha_{(cL)\beta\gamma} \) one can consider on a \( L^n \)-space arbitrary \( N \)-connections \( N^i j \), d-connections \( \Gamma^\alpha_{\beta\gamma} \) which are not defined only by \( L(x, y) \) and \( g_{ij}^{(L)} \) but can be metric, or non-metric with respect to the Lagrange metric.

The next step of generalization is to consider an arbitrary metric \( g_{ij} (x, y) \) on \( TM \) instead of (1.70) which is the second derivative of "anisotropic" coordinates \( y^i \) of a Lagrangian \( [27, 28] \).

**Definition 1.9.** A generalized Lagrange space is a pair \( GL^n = (M, g_{ij}(x, y)) \) where \( g_{ij}(x, y) \) is a covariant, symmetric d-tensor field, of rank \( n \) and of constant signature on \( \tilde{TM} \).

One can consider different classes of \( N \)- and d-connections on \( TM \), which are compatible (metric) or non compatible with (1.71) for arbitrary \( g_{ij}(x, y) \). We can apply all formulas for d-connections, N-curvatures, d-torsions and d-curvatures as in a v-bundle \( E \), but reconsidering them on \( TM \), by changing \( h_{ab} \rightarrow g_{ij}(x, y) \) and \( N_{i a} \rightarrow N^i_{a} \).
1.5.3 Cartan Spaces

The theory of Cartan spaces (see, for instance, [49, 23]) was formulated in a new fashion in R. Miron’s works [29, 30] by considering them as duals to the Finsler spaces (see details and references in [39]). Roughly, a Cartan space is constructed on a cotangent bundle $T^*M$ like a Finsler space on the corresponding tangent bundle $TM$.

Consider a real smooth manifold $M$, the cotangent bundle $(T^*M, \pi^*, M)$ and the manifold $\tilde{T}^*M = T^*M \{0\}$.

**Definition 1.10.** A Cartan space is a pair $C^n = (M, K(x, p))$ such that $K : T^*M \to \mathcal{R}$ is a scalar function which satisfy the following conditions:

1. $K$ is a differentiable function on the manifold $\tilde{T}^*M = T^*M \{0\}$ and continuous on the null section of the projection $\pi^* : T^*M \to M$;

2. $K$ is a positive function, homogeneous on the fibers of the $T^*M$, i. e. $K(x, \lambda p) = \lambda F(x, p), \lambda \in \mathcal{R}$;

3. The Hessian of $K^2$ with elements

$$\tilde{g}^{ij}_{(K)}(x, p) = \frac{1}{2} \frac{\partial^2 K^2}{\partial p_i \partial p_j}$$

(1.72)

is positively defined on $\tilde{T}^*M$.

The function $K(x, y)$ and $\tilde{g}^{ij}(x, p)$ are called respectively the fundamental function and the fundamental (or metric) tensor of the Cartan space $C^n$. We use symbols like ”$\tilde{g}$” as to emphasize that the geometrical objects are defined on a dual space.

One considers ”anisotropic” (depending on directions, momenta, $p_i$) Christoffel symbols, for simplicity, we write the inverse to (1.72) as $\tilde{g}^{ij}_{(K)} = \tilde{\gamma}_{ij}$,

$$\tilde{\gamma}_{jk}^i(x, p) = \frac{1}{2} \tilde{g}^{ir} \left( \frac{\partial \tilde{g}_{rk}}{\partial x^j} + \frac{\partial \tilde{g}_{jr}}{\partial x^k} - \frac{\partial \tilde{g}_{jk}}{\partial x^r} \right),$$

which are used for definition of the canonical $N$–connection,

$$\tilde{N}_{ij} = \tilde{\gamma}_{ji}^k p_k - \frac{1}{2} \tilde{\gamma}_{ni} p_k \tilde{\gamma}_{kj}^l \delta p_l, \ \tilde{\gamma}^n = \frac{\partial}{\partial p_n}.$$  

(1.73)

This $N$–connection can be used for definition of an almost complex structure like in (1.51) and to define on $T^*M$ a d–metric

$$\tilde{G}^{(k)} = \tilde{g}^{ij}(x, p) dx^i \otimes dx^j + \tilde{g}^{ij}(x, p) \delta p_i \otimes \delta p_j,$$

(1.74)

with $\tilde{g}^{ij}(x, p)$ taken as (1.72).
Using the canonical N–connection (1.73) and Finsler metric tensor (1.72) (or, equivalently, the d–metric (1.74) we can introduce the canonical d–connection

\[ D\tilde{\Gamma}(\tilde{\mathcal{N}}(k)) = \tilde{\Gamma}^\alpha_{(k)\beta\gamma} = \left( \tilde{H}^i_{(k)jk}, \tilde{C}^i_{(k)jk} \right) \]

with the coefficients are computed

\[ \tilde{H}^i_{(k)jk} = \frac{1}{2} \tilde{g}^{ir}(\delta_j \tilde{g}_{rk} + \delta_k \tilde{g}_{jr} - \delta_r \tilde{g}_{jk}), \quad \tilde{C}^i_{(k)jk} = \tilde{g}_{ia} \tilde{\partial}^a \tilde{g}^{jk}. \]

The d–connection \( D\tilde{\Gamma}(\tilde{N}(k)) \) has the unique property that it is torsionless and satisfies the metricity conditions both for the horizontal and vertical components, i. e. \( \tilde{D}_a \tilde{g}_{\beta\gamma} = 0 \).

The d–curvatures

\[ \tilde{R}^\alpha_{(k)\beta\gamma\delta} = \{ R^i_{(k)hjk}, P^i_{(k)jkm}, \tilde{S}^i_{jk} \} \]

on a Finsler space provided with Cartan N–connection and Finsler metric structures are computed following the formulas (1.66) when the \( a, b, c... \) indices are identified with \( i, j, k... \) indices. It should be emphasized that in this case all values \( \tilde{g}_{ij}, \tilde{\Gamma}^\alpha_{(k)\beta\gamma} \) and \( \tilde{R}^\alpha_{(k)\beta\gamma\delta} \) are defined by a fundamental function \( K(x,p) \).

In general, we can consider that a Cartan space is provided with a metric \( \tilde{g}^{ij} = \partial^2 K^2/2\partial p_i \partial p_j \), but the N–connection and d–connection could be defined in a different manner, even not determined by \( K \).

### 1.5.4 Generalized Hamilton and Hamilton Spaces

The geometry of Hamilton spaces was defined and investigated by R. Miron in the papers [33, 32, 31] (see details and references in [39]). It was developed on the cotangent bundle as a dual geometry to the geometry of Lagrange spaces. Here we start with the definition of generalized Hamilton spaces and then consider the particular case.

**Definition 1.11.** A generalized Hamilton space is a pair 
\( GH^n = (M, \tilde{g}^{ij}(x,p)) \) where \( M \) is a real \( n \)–dimensional manifold and \( \tilde{g}^{ij}(x,p) \) is a contravariant, symmetric, nondegenerate of rank \( n \) and of constant signature on \( T^*M \).

The value \( \tilde{g}^{ij}(x,p) \) is called the fundamental (or metric) tensor of the space \( GH^n \). One can define such values for every paracompact manifold \( M \). In general, a N–connection on \( GH^n \) is not determined by \( \tilde{g}^{ij} \). Therefore we can consider arbitrary coefficients \( \tilde{N}_{ij}(x,p) \) and define on \( T^*M \) a d–metric like (1.42)

\[ \tilde{G} = \tilde{g}_{\alpha\beta}(\tilde{u}) \tilde{\partial}^\alpha \otimes \tilde{\partial}^\beta = \tilde{g}_{ij}(\tilde{u}) d^i \otimes d^j + \tilde{g}^{ij}(\tilde{u}) \tilde{\partial}_i \otimes \tilde{\partial}_j, \quad (1.75) \]

This N–coefficients \( \tilde{N}_{ij}(x,p) \) and d–metric structure (1.73) allow to define an almost Kähler model of generalized Hamilton spaces and to define canonical d–connections, d–torsions and d-curvatures (see respectively the formulas (1.48), (1.49), (1.61) and (1.64) with the fiber coefficients redefined for the cotangent bundle \( T^*M \)).
A generalized Hamilton space $GH^n = (M, \tilde{g}^{ij}(x, p))$ is called reducible to a Hamilton one if there exists a Hamilton function $H(x, p)$ on $T^*M$ such that

$$\tilde{g}^{ij}(x, p) = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j}. \tag{1.76}$$

**Definition 1.12.** A Hamilton space is a pair $H^n = (M, H(x, p))$ such that $H : T^*M \to \mathbb{R}$ is a scalar function which satisfy the following conditions:

1. $H$ is a differentiable function on the manifold $\tilde{T}^*M = T^*M \setminus \{0\}$ and continuous on the null section of the projection $\pi^*: T^*M \to M$;

2. The Hessian of $H$ with elements (1.76) is positively defined on $\tilde{T}^*M$ and $\tilde{g}^{ij}(x, p)$ is nondegenerate matrix of rank $n$ and of constant signature.

For Hamilton spaces the canonical $N$–connection (defined by $H$ and its Hessian) exists,

$$\tilde{\nabla}_{ij} = \frac{1}{4} \{\tilde{g}_{ij}, H\} - \frac{1}{2} \left( \tilde{g}_{ik} \frac{\partial^2 H}{\partial p_k \partial x^j} + \tilde{g}_{jk} \frac{\partial^2 H}{\partial p_k \partial x^i} \right),$$

where the Poisson brackets, for arbitrary functions $f$ and $g$ on $T^*M$, act as

$$\{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial x^i}.$$

The canonical $d$–connection $D\tilde{\Gamma}(\tilde{\nabla}_{(c)}) = \tilde{\Gamma}^\alpha_{(c)\beta\gamma} = \left( \tilde{H}^i_{(c) jk}, \tilde{C}^i_{(c) jk} \right)$is defined by the coefficients

$$\tilde{H}^i_{(c) jk} = \frac{1}{2} \tilde{g}^{is} \left( \delta_j \tilde{g}_{sk} + \delta_k \tilde{g}_{js} - \delta_s \tilde{g}_{jk} \right), \tilde{C}^i_{(c) jk} = -\frac{1}{2} \tilde{g}_{is} \delta_j \tilde{g}^{sk}.$$

In result we can compute the $d$–torsions and $d$–curvatures like on cv–bundle or on Cartan spaces. On Hamilton spaces all such objects are defined by the Hamilton function $H(x, p)$ and indices have to be reconsidered for co–fibers of the co-tangent bundle.

### 1.6 Gravity on Vector Bundles

The components of the Ricci $d$–tensor

$$R^\tau_{\alpha\beta} = R^\tau_{\alpha,\beta\tau}$$

with respect to a locally adapted frame (1.17) are as follows:

$$R_{ij} = R^k_{i,jk}, \quad R_{ia} = -2P_{ia} = -P^k_{i,ka}, \quad R_{ai} = 1 \quad P_{ai} = P^b_{ai}, \quad R_{ab} = S^c_{a, bc}. \tag{1.77}$$
We point out that because, in general, \( P_{ai} \neq P_{ia} \) the Ricci d-tensor is non symmetric.

Having defined a d-metric of type in \( E \) we can introduce the scalar curvature of d–connection \( D \):

\[
\hat{R} = G^{\alpha\beta}R_{\alpha\beta} = R + S,
\]

where \( R = g^{ij} R_{ij} \) and \( S = h_{ab} S_{ab} \).

For our further considerations it will be also useful to use an alternative way of definition torsion (1.57) and curvature (1.62) by using the commutator

\[
\Delta_{\alpha\beta} \doteq \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha = 2 \nabla_{[\alpha} \nabla_{\beta]}.
\]

For components of d–torsion we have

\[
\Delta_{\alpha\beta} f = T^\gamma_{\alpha\beta} \nabla_\gamma f
\]

for every scalar function \( f \) on \( E \). Curvature can be introduced as an operator acting on arbitrary d-vector \( V^\delta \):

\[
(\Delta_{\alpha\beta} - T^\gamma_{\alpha\beta} \nabla_\gamma)V^\delta = R^\delta_{\gamma\alpha\beta} V^\gamma
\]

(1.79)

(we note that in this section we shall follow conventions of Miron and Anastasiei \[36, 37\] on d-tensors; we can obtain corresponding Penrose and Rindler abstract index formulas \[17, 18\] just for a trivial N-connection structure and by changing denotations for components of torsion and curvature in this manner: \( T^\gamma_{\alpha\beta} \rightarrow T^\gamma_{\alpha\beta} \) and \( R^\delta_{\gamma\alpha\beta} \rightarrow R^\delta_{\gamma\alpha\beta} \)).

Here we also note that torsion and curvature of a d-connection on \( E \) satisfy generalized for locally anisotropic spaces Ricci and Bianchi identities \[36, 37\] which in terms of components (1.79) are written respectively as

\[
R^\delta_{[\gamma\alpha\beta]} + \nabla_\alpha T^\delta_{\beta\gamma} + T^\nu_{[\alpha\beta} T^\delta_{\gamma]\nu] = 0
\]

(1.80)

and

\[
\nabla_\alpha R^\sigma_{[\nu\beta\gamma]} + T^\delta_{[\alpha\beta} R^\sigma_{\nu\gamma]\delta] = 0.
\]

(1.81)

Identities (1.80) and (1.81) can be proved similarly as in \[47\] by taking into account that indices play a distinguished character.

We can also consider a la-generalization of the so-called conformal Weyl tensor (see, for instance, \[47\]) which can be written as a d-tensor in this form:

\[
C^\gamma_{\alpha\beta} = R^\gamma_{\alpha\beta} - \frac{4}{n + m - 2} R^\gamma_{[\alpha \delta^\beta]} + \frac{2}{(n + m - 1)(n + m - 2)} \hat{R} \delta^\gamma_{[\alpha \delta^\beta]}.
\]

(1.82)

This object is conformally invariant on locally anisotropic spaces provided with d-connection generated by d-metric structures.
The Einstein equations and conservation laws on v-bundles provided with N-connection structures are studied in detail in [36, 37, 2, 3]. In Ref. [79] we proved that the locally anisotropic gravity can be formulated in a gauge like manner and analyzed the conditions when the Einstein locally anisotropic gravitational field equations are equivalent to a corresponding form of Yang-Mills equations. In this subsection we write the locally anisotropic gravitational field equations in a form more convenient for theirs equivalent reformulation in locally anisotropic spinor variables.

We define d-tensor $\Phi_{\alpha\beta}$ as to satisfy conditions

$$-2\Phi_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{n+m} \epsilon R g_{\alpha\beta}$$

which is the torsionless part of the Ricci tensor for locally isotropic spaces [47, 48], i.e. $\Phi_{\alpha^\alpha} = 0$. The Einstein equations on locally anisotropic spaces

$$\bar{\bar{G}}_{\alpha\beta} + \lambda g_{\alpha\beta} = \kappa E_{\alpha\beta},$$

where

$$\bar{\bar{G}}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} \epsilon R g_{\alpha\beta}$$

is the Einstein d-tensor, $\lambda$ and $\kappa$ are correspondingly the cosmological and gravitational constants and by $E_{\alpha\beta}$ is denoted the locally anisotropic energy-momentum d-tensor [36, 37], can be rewritten in equivalent form:

$$\Phi_{\alpha\beta} = -\frac{\kappa}{2} (E_{\alpha\beta} - \frac{1}{n+m} E_{\tau^\tau} g_{\alpha\beta}).$$

Because the locally anisotropic spaces generally have nonzero torsions we shall add to (1.80) (equivalently to (1.84)) a system of algebraic d-field equations with the source $S_{\alpha\beta\gamma}$ being the locally anisotropic spin density of matter (if we consider a variant of locally anisotropic Einstein–Cartan theory):

$$T^\gamma_{\alpha\beta} + 2 \delta_{[\alpha} T^\delta_{\beta]} = \kappa S^\gamma_{\alpha\beta}. $$

From (1.80) and (1.87) one follows the conservation law of locally anisotropic spin matter:

$$\nabla_\gamma S^\gamma_{\alpha\beta} - T^\delta_{\beta\gamma} S^\gamma_{\alpha\beta} = E_{\beta\alpha} - E_{\alpha\beta}. $$

Finally, in this section, we remark that all presented geometric constructions contain those elaborated for generalized Lagrange spaces [36, 37] (for which a tangent bundle $TM$ is considered instead of a v-bundle $E$). We also note that the Lagrange (Finsler) geometry is characterized by a metric with components parametized as $g_{ij} = \frac{1}{2} \frac{\partial^2 \ell}{\partial y^i \partial y^j} \left( g_{ij} = \frac{1}{2} \frac{\partial^2 \Lambda^2}{\partial y^i \partial y^j} \right)$ and $h_{ij} = g_{ij}$, where $\mathcal{L} = \mathcal{L} (x, y) (\Lambda = \Lambda (x, y))$ is a Lagrangian (Finsler metric) on $TM$ (see details in [36, 37, 24, 8]).
Chapter 2

Clifford Ha–Structures

The theory of anisotropic spinors formulated in the Part II is extended for higher order anisotropic (ha) spaces. In brief, such spinors will be called ha–spinors which are defined as some Clifford ha–structures defined with respect to a distinguished quadratic form (1.43) on a hvc–bundle. For simplicity, the bulk of formulas will be given with respect to higher order vector bundles. To rewrite such formulas for hvc–bundles is to consider for the "dual" shells of higher order anisotropy some dual vector spaces and associated dual spinors.

2.1 Distinguished Clifford Algebras

The typical fiber of dv–bundle \( \xi_d, \pi_d : HE \oplus V_1E \oplus ... \oplus V_mE \to E \) is a d-vector space, \( \mathcal{F} = h\mathcal{F} \oplus v_1\mathcal{F} \oplus ... \oplus v_z\mathcal{F} \), split into horizontal \( h\mathcal{F} \) and verticals \( v_p\mathcal{F}, p = 1, ..., z \) subspaces, with a bilinear quadratic form \( G(g, h) \) induced by a hvc–bundle metric (1.43). Clifford algebras (see, for example, Refs. [22, 57, 48]) formulated for d-vector spaces will be called Clifford d–algebras [67, 66, 84]. We shall consider the main properties of Clifford d–algebras. The proof of theorems will be based on the technique developed in Ref. [22] correspondingly adapted to the distinguished character of spaces in consideration.

Let \( k \) be a number field (for our purposes \( k = \mathbb{R} \) or \( k = \mathbb{C}, \mathbb{R} \) and \( \mathbb{C} \), are, respectively real and complex number fields) and define \( \mathcal{F} \), as a d-vector space on \( k \) provided with nondegenerate symmetric quadratic form (metric) \( G \). Let \( C \) be an algebra on \( k \) (not necessarily commutative) and \( j : \mathcal{F} \to C \) a homomorphism of underlying vector spaces such that \( j(u)^2 = G(u) \cdot 1 \) (1 is the unity in algebra \( C \) and d-vector \( u \in \mathcal{F} \)). We are interested in definition of the pair \( (C, j) \) satisfying the next universality conditions. For every \( k \)-algebra \( A \) and arbitrary homomorphism \( \varphi : \mathcal{F} \to A \) of the underlying d–vector spaces, such that \( (\varphi(u))^2 \to G(u) \cdot 1 \), there is a unique homomorphism of algebras \( \psi : C \to A \) transforming the diagram 1 into a commutative one.

The algebra solving this problem will be denoted as \( C(\mathcal{F}, A) \) [equivalently as \( C(G) \) or \( C(\mathcal{F}) \)] and called as Clifford d–algebra associated with pair \( (\mathcal{F}, G) \).

Theorem 2.1. The above-presented diagram has a unique solution \( (C, j) \) up to isomor-
The Clifford d-algebra

**Proof:** (We adapt for d-algebras that of Ref. [23], p. 127 and extend for higher order anisotropies a similar proof presented in the Part II). For a universal problem the uniqueness is obvious if we prove the existence of solution \( C(G) \). To do this we use tensor algebra \( \mathcal{L}(F) = \bigoplus_{i=1}^{\infty} T^i(F) \), where \( T^0(F) = k \) and \( T^i(F) = k \) and \( T^i(F) = F \otimes ... \otimes F \) for \( i > 0 \). Let \( I(G) \) be the bilateral ideal generated by elements of form \( \epsilon(u) = u \otimes u - G(u) \cdot 1 \) where \( u \in F \) and 1 is the unity element of algebra \( \mathcal{L}(F) \). Every element from \( I(G) \) can be written as \( \sum i \lambda_i \mu_i \), where \( \lambda_i, \mu_i \in \mathcal{L}(F) \) and \( u_i \in F \). Let \( C(G) = \mathcal{L}(F)/I(G) \) and define \( j : F \to C(G) \) as the composition of homomorphism \( i : \mathcal{L}(F) \to L^1(F) \subset \mathcal{L}(F) \) and projection \( p : \mathcal{L}(F) \to C(G) \). In this case pair \( (C(G), j) \) is the solution of our problem. From the general properties of tensor algebras the composition \( \varphi : F \to A \) can be extended to \( \mathcal{L}(F) \), i.e., the diagram 2 is commutative, where \( \rho \) is a monomorphism of algebras. Because \( \varphi(u)^2 = G(u) \cdot 1 \), then \( \rho \) vanishes on ideal \( I(G) \) and in this case the necessary homomorphism \( \tau \) is defined. As a consequence of uniqueness of \( \rho \), the homomorphism \( \tau \) is unique.

Tensor d-algebra \( \mathcal{L}(F) \) can be considered as a \( \mathbb{Z}/2 \) graded algebra. Really, let us introduce \( \mathcal{L}^{(0)}(F) = \sum T^{2i}(F) \) and \( \mathcal{L}^{(1)}(F) = \sum T^{2i+1}(F) \). Setting \( I^{(a)}(G) = I(G) \cap \mathcal{L}^{(a)}(F) \). Define \( C^{(a)}(G) \) as \( p(\mathcal{L}^{(a)}(F)) \), where \( p : \mathcal{L}(F) \to C(G) \) is the canonical projection. Then \( C(G) = C^{(0)}(G) \oplus C^{(1)}(G) \) and in consequence we obtain that the Clifford d-algebra is \( \mathbb{Z}/2 \) graded.

It is obvious that Clifford d-algebra functorially depends on pair \( (F, G) \). If \( f : F \to F' \) is a homomorphism of k-vector spaces, such that \( G'(f(u)) = G(u) \), where \( G \) and \( G' \) are, respectively, metrics on \( F \) and \( F' \), then \( f \) induces an isomorphism of d-algebras

\[
C(f) : C(G) \to C(G')
\]

with identities \( C(\varphi \cdot f) = C(\varphi) C(f) \) and \( C(Id_F) = Id_{C(F)} \).

If \( A^a \) and \( B^b \) are \( \mathbb{Z}/2 \)-graded d-algebras, then their graded tensorial product \( A^a \otimes B^b \) is defined as a d-algebra for k-vector d-space \( A^a \otimes B^b \) with the graded product induced as \( (a \otimes b)(c \otimes d) = (-1)^{ab} ac \otimes bd \), where \( b \in B^b \) and \( c \in A^a \) \( (a, b = 0, 1) \).

Now we re-formulate for d-algebras the Chevalley theorem [13]:

**Theorem 2.2.** The Clifford d-algebra

\[
C(hF \oplus v_1F \oplus ... \oplus v_2F, g + h_1 + ... + h_z)
\]

is naturally isomorphic to \( C(g) \otimes C(h_1) \otimes ... \otimes C(h_z) \).

**Proof.** Let \( n : hF \to C(g) \) and \( n'_{(p)} : v(p), F \to C(h_{(p)}) \) be canonical maps and map

\[
m : hF \oplus v_1F \oplus ... \oplus v_2F \to C(g) \otimes C(h_1) \otimes ... \otimes C(h_z)
\]

is defined as

\[
m(x, y_{(1)}, ..., y_{(z)}) = n(x) \otimes 1 \otimes ... \otimes 1 + 1 \otimes n'(y_{(1)}) \otimes ... \otimes 1 + 1 \otimes ... \otimes 1 \otimes n'(y_{(z)}),
\]
\[ x \in h\mathcal{F}, y(1) \in v(1)\mathcal{F}, \ldots, y(z) \in v(z)\mathcal{F}. \text{ We have} \]
\[
(m(x, y(1), \ldots, y(z)))^2 = \left[ (n(x))^2 + (n'(y(1)))^2 + \ldots + (n'(y(z)))^2 \right] \times 1
\]
\[
= [g(x) + h(y(1)) + \ldots + h(y(z))].
\]

Taking into account the universality property of Clifford d-algebras we conclude that \(m_1 + \ldots + m_z\) induces the homomorphism

\[
\zeta : C(h\mathcal{F} \oplus v_1\mathcal{F} \oplus \ldots \oplus v_z\mathcal{F}, g + h_1 + \ldots + h_z) \rightarrow C(h\mathcal{F}, g) \hat{\otimes} C(v_1\mathcal{F}, h_1) \hat{\otimes} \ldots C(v_z\mathcal{F}, h_z).
\]

We also can define a homomorphism

\[
v : C(h\mathcal{F}, g) \hat{\otimes} C(v_1\mathcal{F}, h(1)) \hat{\otimes} \ldots \hat{\otimes} C(v_z\mathcal{F}, h(z)) \rightarrow C(h\mathcal{F} \oplus v_1\mathcal{F} \oplus \ldots \oplus v_z\mathcal{F}, g + h_1 + \ldots + h_z)
\]

by using formula \(v(x \otimes y(1) \otimes \ldots \otimes y(z)) = \delta(x) \delta'_1(y(1)) \ldots \delta'_z(y(z))\), where homomorphisms \(\delta\) and \(\delta'_1, \ldots, \delta'_z\) are, respectively, induced by embedding of \(h\mathcal{F}\) and \(v_1\mathcal{F}\) into \(h\mathcal{F} \oplus v_1\mathcal{F} \oplus \ldots \oplus v_z\mathcal{F} : \]

\[
\delta : C(h\mathcal{F}, g) \rightarrow C(h\mathcal{F} \oplus v_1\mathcal{F} \oplus \ldots \oplus v_z\mathcal{F}, g + h_1 + \ldots + h_z),
\]

\[
\delta'_1 : C(v_1\mathcal{F}, h(1)) \rightarrow C(h\mathcal{F} \oplus v_1\mathcal{F} \oplus \ldots \oplus v_z\mathcal{F}, g + h_1 + \ldots + h_z),
\]

\[
\delta'_z : C(v_z\mathcal{F}, h(z)) \rightarrow C(h\mathcal{F} \oplus v_1\mathcal{F} \oplus \ldots \oplus v_z\mathcal{F}, g + h_1 + \ldots + h_z).
\]

Superposition of homomorphisms \(\zeta\) and \(v\) lead to identities

\[
v \zeta = Id_{C(h\mathcal{F}, g) \hat{\otimes} C(v_1\mathcal{F}, h(1)) \hat{\otimes} \ldots \hat{\otimes} C(v_z\mathcal{F}, h(z))}, \quad \zeta v = Id_{C(h\mathcal{F}, g) \hat{\otimes} C(v_1\mathcal{F}, h(1)) \hat{\otimes} \ldots \hat{\otimes} C(v_z\mathcal{F}, h(z))} (2.1)
\]

Really, d-algebra \(C(h\mathcal{F} \oplus v_1\mathcal{F} \oplus \ldots \oplus v_z\mathcal{F}, g + h_1 + \ldots + h_z)\) is generated by elements of type \(m(x, y(1), \ldots y(z))\). Calculating

\[
v \zeta (m(x, y(1), \ldots y(z))) = v(n(x) \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes n'_1(y(1)) \otimes \ldots \otimes 1
\]
\[
+ \ldots + 1 \otimes \ldots \otimes n'_z(y(z))) = \delta(n(x)) \delta'(n'_1(y(1))) \ldots \delta'(n'_z(y(z)))
\]
\[
= m(x, 0, \ldots, 0) + m(0, y(1), \ldots, 0) + \ldots + m(0, 0, \ldots, y(z))
\]
\[
= m(x, y(1), \ldots, y(z));
\]

we prove the first identity in \((2.1)\).

On the other hand, d-algebra

\[
C(h\mathcal{F}, g) \hat{\otimes} C(v_1\mathcal{F}, h(1)) \hat{\otimes} \ldots \hat{\otimes} C(v_z\mathcal{F}, h(z))
\]

43
is generated by elements of type
\[ n(x) \otimes 1 \otimes \ldots \otimes 1 \otimes n'(y(1)) \otimes \ldots \otimes 1, \ldots 1 \otimes \ldots \otimes n'(y(z)), \]
we prove the second identity in (2.1).

Following from the above-mentioned properties of homomorphisms \( \zeta \) and \( \nu \) we can assert that the natural isomorphism is explicitly constructed. □

In consequence of the presented in this section Theorems we conclude that all operations with Clifford d-algebras can be reduced to calculations for \( C(hF, g) \) and \( C(v(p)/F, h(p)) \) which are usual Clifford algebras of dimension \( 2^n \) and, respectively, \( 2^{m_p} \).

Of special interest is the case when \( k = R \) and \( F \) is isomorphic to vector space \( R^{p+q,a+b} \) provided with quadratic form
\[ -x_1^2 - \ldots - x_p^2 + x_{p+q}^2 - y_1^2 - \ldots - y_a^2 + y_{a+b}. \]

In this case, the Clifford algebra, denoted as \( (C^{p,q}, C^{a,b}) \), is generated by symbols \( e_{1}^{(x)}, e_{2}^{(x)}, \ldots, e_{p+q}^{(x)}, e_{1}^{(y)}, e_{2}^{(y)}, \ldots, e_{a+b}^{(y)} \) satisfying properties
\[
\begin{align*}
(e_i)^2 &= -1 \ (1 \leq i \leq p), \ (e_j)^2 = -1 \ (1 \leq j \leq a), \\
(e_k)^2 &= 1 \ (p + 1 \leq k \leq p + q), \\
(e_j)^2 &= 1 \ (n + 1 \leq s \leq a + b), \ e_i e_j = -e_j e_i, \ i \neq j.
\end{align*}
\]

Explicit calculations of \( C^{p,q} \) and \( C^{a,b} \) are possible by using isomorphisms \[\{22, 48\}\]
\[
C^{p+n,q+n} \simeq C^{p,q} \otimes M_{2} (R) \otimes \ldots \otimes M_{2} (R) \\
\simeq C^{p,q} \otimes M_{2^{n}} (R) \cong M_{2^{n}} (C^{p,q}),
\]
where \( M_s (A) \) denotes the ring of quadratic matrices of order \( s \) with coefficients in ring \( A \). Here we write the simplest isomorphisms \( C^{1,0} \simeq C, \ C^{0,1} \simeq R \oplus R \) and \( C^{2,0} = H \) where by \( H \) is denoted the body of quaternions.

Now, we emphasize that higher order Lagrange and Finsler spaces, denoted \( H^{2n} \)–spaces, admit locally a structure of Clifford algebra on complex vector spaces. Really, by using almost Hermitian structure \( J \alpha \beta \) and considering complex space \( C^n \) with nondegenerate quadratic form \( \sum_{a=1}^{n} |z_a|^2, \ z_a \in C^2 \) induced locally by metric (1.13) (rewritten in complex coordinates \( z_a = x_a + iy_a \)) we define Clifford algebra \( \overset{\sim}{C}^{n} = \overset{\sim}{C}^{1} \otimes \ldots \otimes \overset{\sim}{C}^{1}, \)
\[
\]
where \( \overset{\sim}{C}^{1} = C \otimes C = C \oplus C \) or in consequence, \( \overset{\sim}{C}^{n} \simeq C^{n,0} \otimes C \simeq C^{0,n} \otimes C \). Explicit calculations lead to isomorphisms
\[
\overset{\sim}{C}^{2} = C^{0,2} \otimes C \approx M_{2} (R) \otimes C \approx M_{2} (\overset{\sim}{C}^{n}), \ C^{2p} \approx M_{2p} (C)
\]
and
\[
\overset{\sim}{C}^{2p+1} \approx M_{2p} (C) \oplus M_{2p} (C),
\]

44
which show that complex Clifford algebras, defined locally for $H^{2n}$-spaces, have periodicity 2 on $p$.

Considerations presented in the proof of theorem 2.2 show that map $j : F \to C (F)$ is monomorphic, so we can identify space $F$ with its image in $C (F, G)$, denoted as $u \to \pi$, if $u \in C^{(0)} (F, G)$ \((u \in C^{(1)} (F, G))\); then $u = \pi$ (respectively, $\pi = - u$).

**Definition 2.1.** The set of elements $u \in C (G)^*$, where $C (G)^*$ denotes the multiplicative group of invertible elements of $C (F, G)$ satisfying $\pi F^{-1} \in F$, is called the twisted Clifford $d$-group, denoted as $\tilde{\Gamma} (F)$.

Let $\tilde{\rho} : \tilde{\Gamma} (F) \to GL (F)$ be the homomorphism given by $u \to \rho \tilde{u}$, where $\tilde{\rho}_u (u) = \bar{w} u u^{-1}$. We can verify that $\ker \tilde{\rho} = R^*$ is a subgroup in $\tilde{\Gamma} (F)$.

The canonical map $j : F \to C (F)$ can be interpreted as the linear map $F \to C (F)^0$ satisfying the universal property of Clifford $d$-algebras. This leads to a homomorphism of algebras, $C (F) \to C (F)^t$, considered by an anti-involution of $C (F)$ and denoted as $u \to ^t u$. More exactly, if $u_1 \ldots u_n \in F$, then $t_u = u_n \ldots u_1$ and $^t \pi = \pi (-1)^n u_n \ldots u_1$.

**Definition 2.2.** The spinor norm of arbitrary $u \in C (F)$ is defined as $S (u) = ^t \pi \cdot u \in C (F)$.

It is obvious that if $u, u', u'' \in \tilde{\Gamma} (F)$, then $S (u, u') = S (u) S (u')$ and $S (uu'u'') = S (u) S (u') S (u'')$. For $u, u' \in F S (u) = - G (u)$ and $S (u, u') = S (u) S (u') = S (uu')$.

Let us introduce the orthogonal group $O (G) \subset GL (G)$ defined by metric $G$ on $F$ and denote sets

\[ SO (G) = \{ u \in O (G), \det |u| = 1 \} , \quad Pin (G) = \{ u \in \tilde{\Gamma} (F), S (u) = 1 \} \]

and $Spin (G) = Pin (G) \cap C^0 (F)$. For $F \cong R^{n+m}$ we write $Spin (n_E)$, $Pin (n_E)$. By straightforward calculations (see similar considerations in Ref. [22]) we can verify the exactness of these sequences:

\[ 1 \to \mathbb{Z}/2 \to Pin (G) \to O (G) \to 1, \]
\[ 1 \to \mathbb{Z}/2 \to Spin (G) \to SO (G) \to 0, \]
\[ 1 \to \mathbb{Z}/2 \to Spin (n_E) \to SO (n_E) \to 1. \]

We conclude this subsection by emphasizing that the spinor norm was defined with respect to a quadratic form induced by a metric in $dv$–bundle $\mathcal{E}^{<z>}$. This approach differs from those presented in Refs. [3] and [14].

### 2.2 Clifford Ha– Bundles

We shall consider two variants of generalization of spinor constructions defined for $d$–vector spaces to the case of distinguished vector bundle spaces enabled with the structure of $N$-connection. The first is to use the extension to the category of vector bundles. The second is to define the Clifford fibration associated with compatible linear $d$-connection and metric $G$ on a $dv$–bundle. We shall analyze both variants.
2.2.1 Clifford d–module structure in dv–bundles

Because functor \( F \to C(\mathcal{F}) \) is smooth we can extend it to the category of vector bundles of type

\[
\xi^{<z>} = \{ \pi_d : HE^{<z>} \oplus V_1E^{<z>} \oplus ... \oplus V_tE^{<z>} \to E^{<z>} \}.
\]

Recall that by \( F \) we denote the typical fiber of such bundles. For \( \xi^{<z>} \) we obtain a bundle of algebras, denoted as \( C(\xi^{<z>}) \), such that \( C(\xi^{<z>})_u = C(F_u) \). Multiplication in every fiber defines a continuous map

\[
C(\xi^{<z>}) \times C(\xi^{<z>}) \to C(\xi^{<z>}).
\]

If \( \xi^{<z>} \) is a distinguished vector bundle on number field \( k \), the structure of the \( C(\xi^{<z>}) \)-module, the d-module, the d-module, on \( \xi^{<z>} \) is given by the continuous map \( C(\xi^{<z>}) \times \xi^{<z>} \to \xi^{<z>} \) with every fiber \( F_u \) provided with the structure of the \( C(F_u) \)-module, correlated with its k-module structure. Because \( F \subset C(\mathcal{F}) \), we have a fiber to fiber map \( F \times E\xi^{<z>} \to \xi^{<z>} \), inducing on every fiber the map \( F_u \times E\xi^{<z>} \to \xi^{<z>} \) \((R\text{-linear on the first factor and k-linear on the second one})\). Inversely, every such bilinear map defines on \( \xi^{<z>} \) the structure of the \( C(\xi^{<z>}) \)-module by virtue of universal properties of Clifford d–algebras. Equivalently, the above–mentioned bilinear map defines a morphism of v–bundles

\[
m : \xi^{<z>} \to HOM(\xi^{<z>}, \xi^{<z>}) \quad [HOM(\xi^{<z>}, \xi^{<z>})
\]

denotes the bundles of homomorphisms] when \( (m(u))^2 = G(u) \) on every point.

Vector bundles \( \xi^{<z>} \) provided with \( C(\xi^{<z>}) \)-structures are objects of the category with morphisms being morphisms of dv-bundles, which induce on every point \( u \in \xi^{<z>} \) morphisms of \( C(F_u) \)-modules. This is a Banach category contained in the category of finite-dimensional d-vector spaces on filed \( k \).

Let us denote by \( H^s(\mathcal{E}^{<z>}, GL_{n_E}(\mathcal{R})) \), where \( n_E = n + m_1 + ... + m_z \), the s-dimensional cohomology group of the algebraic sheaf of germs of continuous maps of dv-bundle \( \mathcal{E}^{<z>} \) with group \( GL_{n_E}(\mathcal{R}) \) the group of automorphisms of \( \mathcal{R}^{n_E} \) (for the language of algebraic topology see, for example, Refs. [22] and [21]). We shall also use the group \( SL_{n_E}(\mathcal{R}) = \{ A \subset GL_{n_E}(\mathcal{R}), \det A = 1 \} \). Here we point out that cohomologies \( H^s(M, Gr) \) characterize the class of a principal bundle \( \pi : P \to M \) on \( M \) with structural group \( Gr \). Taking into account that we deal with bundles distinguished by an N-connection we introduce into consideration cohomologies \( H^s(\mathcal{E}^{<z>}, GL_{n_E}(\mathcal{R})) \) as distinguished classes (d-classes) of bundles \( \mathcal{E}^{<z>} \) provided with a global N-connection structure.

For a real vector bundle \( \xi^{<z>} \) on compact base \( \mathcal{E}^{<z>} \) we can define the orientation on \( \xi^{<z>} \) as an element \( \alpha_d \in H^1(\mathcal{E}^{<z>}, GL_{n_E}(\mathcal{R})) \) whose image on map

\[
H^1(\mathcal{E}^{<z>}, SL_{n_E}(\mathcal{R})) \to H^1(\mathcal{E}^{<z>}, GL_{n_E}(\mathcal{R}))
\]
is the d-class of bundle \( \mathcal{E}^{<z>} \).
Definition 2.3. The spinor structure on \( \xi^{<z>} \) is defined as an element 
\[
\beta_d \in H^1(\mathcal{E}^{<z>}, \text{Spin}(n_E))
\]
whose image in the composition
\[
H^1(\mathcal{E}^{<z>}, \text{Spin}(n_E)) \to H^1(\mathcal{E}^{<z>}, \text{SO}(n_E)) \to H^1(\mathcal{E}^{<z>}, GL_{n_E}(R))
\]
is the \( d \)-class of \( \mathcal{E}^{<z>} \).

The above definition of spinor structures can be re-formulated in terms of principal bundles. Let \( \xi^{<z>} \) be a real vector bundle of rank \( n+m \) on a compact base \( E^{<z>} \). If there is a principal bundle \( P_d \) with structural group \( \text{SO}(n_E) \) or \( \text{Spin}(n_E) \), this bundle \( \xi^{<z>} \) can be provided with orientation (or spinor) structure. The bundle \( P_d \) is associated with element
\[
\alpha_d \in H^1(\mathcal{E}^{<z>}, \text{SO}(n_{<z>})) \quad \text{[or]} \quad \beta_d \in H^1(\mathcal{E}^{<z>}, \text{Spin}(n_E)).
\]

We remark that a real bundle is oriented if and only if its first Stiefel–Whitney \( d \)-class vanishes,
\[
w_1(\xi_d) \in H^1(\xi, \mathbb{Z}/2) = 0,
\]
where \( H^1(\mathcal{E}^{<z>}, \mathbb{Z}/2) \) is the first group of Chech cohomology with coefficients in \( \mathbb{Z}/2 \).

Considering the second Stiefel–Whitney class \( w_2(\xi^{<z>}) \in H^2(\mathcal{E}^{<z>}, \mathbb{Z}/2) \) it is well known that vector bundle \( \xi^{<z>} \) admits the spinor structure if and only if \( w_2(\xi^{<z>}) = 0 \).

Finally, we emphasize that taking into account that base space \( \mathcal{E}^{<z>} \) is also a \( v \)-bundle, \( p : E^{<z>} \to M \), we have to make explicit calculations in order to express cohomologies
\[
H^*(\mathcal{E}^{<z>}, GL_{n+m}) \quad \text{and} \quad H^*(\mathcal{E}^{<z>}, SO(n + m)) \quad \text{through cohomologies}
\]
\[
H^*(M, GL_n), H^*(M, SO(m_1)), \ldots H^*(M, SO(m_z)),
\]
which depends on global topological structures of spaces \( M \) and \( \mathcal{E}^{<z>} \). For general bundle and base spaces this requires a cumbersome cohomological calculus.

### 2.2.2 Clifford fibration

Another way of defining the spinor structure is to use Clifford fibrations. Consider the principal bundle with the structural group \( Gr \) being a subgroup of orthogonal group \( O(G) \), where \( G \) is a quadratic nondegenerate form defined on the base (also being a bundle space) space \( \mathcal{E}^{<z>} \). The fibration associated to principal fibration \( P(\mathcal{E}^{<z>}, Gr) \) with a typical fiber having Clifford algebra \( C(G) \) is, by definition, the Clifford fibration \( PC(\mathcal{E}^{<z>}, Gr) \). We can always define a metric on the Clifford fibration if every fiber is isometric to \( PC(\mathcal{E}^{<z>}, G) \) (this result is proved for arbitrary quadratic forms \( G \) on pseudo–Riemannian bases [57]). If, additionally, \( Gr \subset SO(G) \) a global section can be defined on \( PC(G) \).

Let \( \mathcal{P}(\mathcal{E}^{<z>}, Gr) \) be the set of principal bundles with differentiable base \( \mathcal{E}^{<z>} \) and structural group \( Gr \). If \( g : Gr \to Gr' \) is an homomorphism of Lie groups and \( P(\mathcal{E}^{<z>}, Gr) \subset \mathcal{P}(\mathcal{E}^{<z>}, Gr) \) (for simplicity in this subsection we shall denote mentioned bundles and sets of bundles as \( P, P' \) and respectively, \( \mathcal{P}, \mathcal{P}' \)), we can always construct a principal
bundle with the property that there is an homomorphism \( f : P' \to P \) of principal bundles which can be projected to the identity map of \( \mathcal{E}^{<z>} \) and corresponds to isomorphism \( g : Gr \to Gr' \). If the inverse statement also holds, the bundle \( P' \) is called as the extension of \( P \) associated to \( g \) and \( f \) is called the extension homomorphism denoted as \( \bar{g} \).

Now we can define distinguished spinor structures on bundle spaces.

**Definition 2.4.** Let \( P \in \mathcal{P}(\mathcal{E}^{<z>}, O(G)) \) be a principal bundle. A distinguished spinor structure of \( P \), equivalently a ds-structure of \( \mathcal{E}^{<z>} \) is an extension \( \tilde{P} \) of \( P \) associated to homomorphism \( h : \text{PinG} \to O(G) \) where \( O(G) \) is the group of orthogonal rotations, generated by metric \( G \), in bundle \( \mathcal{E}^{<z>} \).

So, if \( \tilde{P} \) is a spinor structure of the space \( \mathcal{E}^{<z>} \), then \( \tilde{P} \in \mathcal{P}(\mathcal{E}^{<z>}, \text{PinG}) \).

The definition of spinor structures on varieties was given in Ref.\[15\]. In Refs. \[16\] and \[16\] it is proved that a necessary and sufficient condition for a space time to be orientable is to admit a global field of orthonormalized frames. We mention that spinor structures can be also defined on varieties modeled on Banach spaces \[1\]. As we have shown similar constructions are possible for the cases when space time has the structure of a \( v \)-bundle with an \( N \)-connection.

**Definition 2.5.** A special distinguished spinor structure, ds-structure, of principal bundle \( P = \mathcal{P}(\mathcal{E}^{<z>}, \text{SO}(G)) \) is a principal bundle \( \tilde{P} = \mathcal{P}(\mathcal{E}^{<z>}, \text{SpinG}) \) for which a homomorphism of principal bundles \( \tilde{p} : \tilde{P} \to P \), projected on the identity map of \( \mathcal{E}^{<z>} \) and corresponding to representation \( R : \text{SpinG} \to \text{SO}(G) \), is defined.

In the case when the base space variety is oriented, there is a natural bijection between tangent spinor structures with a common base. For special ds-structures we can define, as for any spinor structure, the concepts of spin tensors, spinor connections, and spinor covariant derivations (see Refs. \[66, 84, 80\]).

### 2.3 Almost Complex Spinor Structures

Almost complex structures are an important characteristic of \( H^{2n} \)-spaces and of osculator bundles \( \text{Osc}^{k=2k_1}(M) \), where \( k_1 = 1, 2, \ldots \). For simplicity in this subsection we restrict our analysis to the case of \( H^{2n} \)-spaces. We can rewrite the almost Hermitian metric \[36, 37\], \( H^{2n} \)-metric in complex form \[67\]:

\[
G = H_{ab}(z, \xi) dz^a \otimes dz^b, \tag{2.2}
\]

where

\[
z^a = x^a + iy^a, \quad \overline{z}^a = x^a - iy^a, \quad H_{ab}(z, \overline{z}) = g_{ab}(x, y) \bigg|_{x=x(z, \overline{z}), y=y(z, \overline{z})},
\]
and define almost complex spinor structures. For given metric (2.2) on $H^{2n}$-space there is always a principal bundle $P^U$ with unitary structural group $U(n)$ which allows us to transform $H^{2n}$-space into v-bundle $\xi^U \approx P^U \times_{U(n)} R^{2n}$. This statement will be proved after we introduce complex spinor structures on oriented real vector bundles [22].

Let us consider momentarily $k = C$ and introduce into consideration [instead of the group $Spin(n)$] the group $Spin^c \times_{Z/2} U(1)$ being the factor group of the product $Spin(n) \times U(1)$ with the respect to equivalence

$$(y, z) \sim (−y, −a), \quad y \in Spin(m).$$

This way we define the short exact sequence

$$1 \to U(1) \to Spin^c(n) \xrightarrow{\pi} SO(n) \to 1,$$

(2.3)

where $\rho^\circ(y, a) = \rho^\circ(y)$. If $\lambda$ is oriented, real, and rank $n$, $\gamma$-bundle $\pi : E_\lambda \to M^n$, with base $M^n$, the complex spinor structure, spin structure, on $\lambda$ is given by the principal bundle $P$ with structural group $Spin^c(m)$ and isomorphism $\lambda \approx P \times_{Spin^c(n)} R^n$ (see (2.3)). For such bundles the categorial equivalence can be defined as

$$\epsilon^c : E^c_\xi (M^n) \to E^c_\xi (M^n),$$

(2.4)

where $\epsilon^c (E^c) = P \Delta_{Spin^c(n)} E^c$ is the category of trivial complex bundles on $M^n$, $E^c_\xi (M^n)$ is the category of complex v-bundles on $M^n$ with action of Clifford bundle $C(\lambda)$, $P \Delta_{Spin^c(n)}$ and $E^c$ is the factor space of the bundle product $P \times_M E^c$ with respect to the equivalence $(p, e) \sim (p \tilde{g}^{-1}, \tilde{g}e), p \in P, e \in E^c$, where $\tilde{g} \in Spin^c(n)$ acts on $E$ by via the imbedding $Spin(n) \subset C^{0,n}$ and the natural action $U(1) \subset C$ on complex v-bundle $\xi^c$, $E^c = tot\xi^c$, for bundle $\pi : E^c \to M^n$.

Now we return to the bundle $\xi = \xi_{<1>^c}$. A real v-bundle (not being a spinor bundle) admits a complex spinor structure if and only if there exist a homomorphism $\sigma : U(n) \to Spin^c(2n)$ making the diagram 3 commutative. The explicit construction of $\sigma$ for arbitrary $\gamma$-bundle is given in Refs. [22] and [7]. For $H^{2n}$-spaces it is obvious that a diagram similar to (2.4) can be defined for the tangent bundle $TM^n$, which directly points to the possibility of defining the $^cSpin$-structure on $H^{2n}$-spaces.

Let $\lambda$ be a complex, rank $n$, spinor bundle with

$$\tau : Spin^c(n) \times_{Z/2} U(1) \to U(1)$$

(2.5)

the homomorphism defined by formula $\tau(\lambda, \delta) = \delta^2$. For $P_s$ being the principal bundle with fiber $Spin^c(n)$ we introduce the complex linear bundle $L(\lambda^c) = P_s \times_{Spin^c(n)} C$ defined as the factor space of $P_s \times C$ on equivalence relation

$$(pt, z) \sim (p, t(t)^{-1}z),$$

where $t \in Spin^c(n)$. This linear bundle is associated to complex spinor structure on $\lambda^c$.49
If $\lambda^c$ and $\lambda^{c'}$ are complex spinor bundles, the Whitney sum $\lambda^c \oplus \lambda^{c'}$ is naturally provided with the structure of the complex spinor bundle. This follows from the homomorphism

$$\omega' : \text{Spin}^c(n) \times \text{Spin}^c(n') \to \text{Spin}^c(n+n'),$$

(2.6)
given by formula $[(\beta, z), (\beta', z')] \to [\omega(\beta, \beta'), zz']$, where $\omega$ is the homomorphism making the diagram 4 commutative. Here, $z, z' \in U(1)$. It is obvious that $L(\lambda^c \oplus \lambda^{c'})$ is isomorphic to $L(\lambda^c) \otimes L(\lambda^{c'}).$

We conclude this subsection by formulating our main result on complex spinor structures for $H^{2n}$-spaces:

**Theorem 2.3.** Let $\lambda^c$ be a complex spinor bundle of rank $n$ and $H^{2n}$-space considered as a real vector bundle $\lambda^c \oplus \lambda^{c'}$ provided with almost complex structure $J^c_\beta$; multiplication on $i$ is given by $\begin{pmatrix} 0 & -\delta_j^i \\ \delta_j^i & 0 \end{pmatrix}$. Then, the diagram 5 is commutative up to isomorphisms $e^c$ and $\bar{c}^c$ defined as in (2.4), $\mathcal{H}$ is functor $E^c \to E^c \otimes L(\lambda^c)$ and $\mathcal{E}^{0,2n}_c(M^n)$ is defined by functor $\mathcal{E}_c(M^n) \to \mathcal{E}^{0,2n}_c(M^n)$ given as correspondence $E^c \to \Lambda(\mathcal{C}^n) \otimes E^c$ (which is a categorial equivalence), $\Lambda(\mathcal{C}^n)$ is the exterior algebra on $\mathcal{C}^n$. $W$ is the real bundle $\lambda^c \oplus \lambda^{c'}$ provided with complex structure.

**Proof:** We use composition of homomorphisms

$$\mu : \text{Spin}^c(2n) \xrightarrow{\pi} \text{SO}(n) \xrightarrow{\varepsilon} U(n) \xrightarrow{\varphi} \text{Spin}^c(2n) \times_{\mathbb{Z}/2} U(1),$$

commutative diagram 6 and introduce composition of homomorphisms

$$\mu : \text{Spin}^c(n) \xrightarrow{\Delta} \text{Spin}^c(n) \times \text{Spin}^c(n) \xrightarrow{\omega^c} \text{Spin}^c(n),$$

where $\Delta$ is the diagonal homomorphism and $\omega^c$ is defined as in (2.6). Using homomorphisms (2.3) and (2.6) we obtain formula $\mu(t) = \mu(t) r(t)$.

Now consider bundle $P \times_{\text{Spin}^c(n)} \text{Spin}^c(2n)$ as the principal $\text{Spin}^c(2n)$-bundle, associated to $M \oplus M$ being the factor space of the product $P \times \text{Spin}^c(2n)$ on the equivalence relation $(p, t, h) \sim (p, \mu(t)^{-1} h)$. In this case the categorial equivalence (2.4) can be rewritten as

$$\epsilon^c(E^c) = P \times_{\text{Spin}^c(n)} \text{Spin}^c(2n) \Delta_{\text{Spin}^c(2n)} E^c$$

and seen as factor space of $P \times \text{Spin}^c(2n) \times_M E^c$ on equivalence relation

$$(pt, h, e) \sim (p, \mu(t)^{-1} h, e) \text{ and } (p, h_1, h_2, e) \sim (p, h_1, h_2^{-1} e)$$

(projections of elements $p$ and $e$ coincides on base $M$). Every element of $\epsilon^c(E^c)$ can be represented as $P \Delta_{\text{Spin}^c(n)} E^c$, i.e., as a factor space $P \Delta E^c$ on equivalence relation $(pt, e) \sim (p, \mu^c(t), e)$, when $t \in \text{Spin}^c(n)$. The complex line bundle $L(\lambda^c)$ can be
interpreted as the factor space of
\( P \times_{Spin^c(n)} C \) on equivalence relation \((pt, \delta) \sim (p, r(t)^{-1} \delta)\).

Putting \((p, e) \otimes (p, \delta)(p, \delta e)\) we introduce morphism

\[
e^c(E) \times L(\lambda^c) \rightarrow e^c(\lambda^c)
\]

with properties

\[
(pt, e) \otimes (pt, \delta) \rightarrow (pt, \delta e) = (p, \mu^c(t)^{-1} \delta e) ,

(p, \mu^c(t)^{-1} e) \otimes (p, l(t)^{-1} e) \rightarrow (p, \mu^c(t) r(t)^{-1} \delta e)
\]

pointing to the fact that we have defined the isomorphism correctly and that it is an isomorphism on every fiber. \(\Box\)
Chapter 3

Spinors and Ha–Spaces

3.1 D–Spinor Techniques

The purpose of this section is to show how a corresponding abstract spinor technique entailing notational and calculations advantages can be developed for arbitrary splits of dimensions of a d-vector space $\mathcal{F} = h\mathcal{F} \oplus v_1\mathcal{F} \oplus \ldots \oplus v_z\mathcal{F}$, where $\dim h\mathcal{F} = n$ and $\dim v_p\mathcal{F} = m_p$. For convenience we shall also present some necessary coordinate expressions.

The problem of a rigorous definition of spinors on locally anisotropic spaces (d–spinors) was posed and solved \[67, 66, 80\] in the framework of the formalism of Clifford and spinor structures on v-bundles provided with compatible nonlinear and distinguished connections and metric. We introduced d-spinors as corresponding objects of the Clifford d-algebra $\mathcal{C}(\mathcal{F}, G)$, defined for a d-vector space $\mathcal{F}$ in a standard manner (see, for instance, \[22\]) and proved that operations with $\mathcal{C}(\mathcal{F}, G)$ can be reduced to calculations for $\mathcal{C}(h\mathcal{F}, g), \mathcal{C}(v_1\mathcal{F}, h_1), \ldots$ and $\mathcal{C}(v_z\mathcal{F}, h_z)$, which are usual Clifford algebras of respective dimensions $2^n, 2^{m_1}, \ldots$ and $2^{m_z}$ (if it is necessary we can use quadratic forms $g$ and $h_p$ correspondingly induced on $h\mathcal{F}$ and $v_p\mathcal{F}$ by a metric $G$ (1.43)). Considering the orthogonal subgroup $O(G) \subset GL(G)$ defined by a metric $G$ we can define the d-spinor norm and parametrize d-spinors by ordered pairs of elements of Clifford algebras $\mathcal{C}(h\mathcal{F}, g)$ and $\mathcal{C}(v_p\mathcal{F}, h_p), p = 1, 2, \ldots z$. We emphasize that the splitting of a Clifford d-algebra associated to a dv-bundle $\mathcal{E}^{<z>}$ is a straightforward consequence of the global decomposition defining a N-connection structure in $\mathcal{E}^{<z>}$.

In this subsection we shall omit detailed proofs which in most cases are mechanical but rather tedious. We can apply the methods developed in \[46, 47, 48, 25\] in a straightforward manner on h- and v-subbundles in order to verify the correctness of affirmations.
3.1.1 Clifford d–algebra, d–spinors and d–twistors

In order to relate the succeeding constructions with Clifford d-algebras \([67, 66]\) we consider a la-frame decomposition of the metric (3.43):

\[
G_{\langle\alpha\rangle<\beta\rangle} (u) = l_{\langle\alpha\rangle} (u) l_{\langle\beta\rangle} (u) G_{\langle\alpha\rangle<\beta\rangle},
\]

where the frame d-vectors and constant metric matrices are distinguished as

\[
l_{\langle\alpha\rangle} (u) = \begin{pmatrix}
\hat{p}_{ij} (u) & 0 & \cdots & 0 \\
0 & \hat{p}_{a_1} (u) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{p}_{a_z} (u)
\end{pmatrix},
\]

\[
G_{\langle\alpha\rangle<\beta\rangle} = \begin{pmatrix}
g_{ij} & 0 & \cdots & 0 \\
0 & h_{a_1 b_1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & h_{a_z b_z}
\end{pmatrix},
\]

\(g_{ij}\) and \(h_{a_1 b_1}, \ldots, h_{a_z b_z}\) are diagonal matrices with \(g_{ii} = h_{a_1 a_1} = \ldots = h_{a_z a_z} = \pm 1\).

To generate Clifford d-algebras we start with matrix equations

\[
\sigma_{\langle\alpha\rangle} \sigma_{\langle\beta\rangle} + \sigma_{\langle\beta\rangle} \sigma_{\langle\alpha\rangle} = -G_{\langle\alpha\rangle<\beta\rangle} I,
\]

where \(I\) is the identity matrix, matrices \(\sigma_{\langle\alpha\rangle} (\sigma\text{-objects})\) act on a d-vector space \(\mathcal{F} = h\mathcal{F} \oplus v_1 \mathcal{F} \oplus \cdots \oplus v_z \mathcal{F}\) and theirs components are distinguished as

\[
\sigma_{\langle\alpha\rangle} = \left( \sigma_{\langle\alpha\rangle} \right)^{\gamma}_{\beta} = \begin{pmatrix}
\left( \sigma_{\langle\alpha\rangle} \right)^{k}_{l} & 0 & \cdots & 0 \\
0 & \left( \sigma_{\alpha_1} \right)^{z_1}_{\beta_1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \left( \sigma_{\alpha_z} \right)^{z_z}_{\beta_z}
\end{pmatrix},
\]

\(\beta, \gamma, \ldots\) refer to spin spaces of type \(\mathcal{S} = S_{(h)} \oplus S_{(m_1)} \oplus \cdots \oplus S_{(m_z)}\) and underlined Latin indices \(\underline{j}, \underline{k}, \ldots\) and \(\underline{b_1}, \underline{c_1}, \ldots, \underline{b_z}, \underline{c_z}\) refer respectively to h-spin space \(S_{(h)}\) and \(v_p\)-spin space \(S_{(v_p)}\), \((p = 1, 2, \ldots, z)\) which are correspondingly associated to a h- and \(v_p\)-decomposition of a dv-bundle \(\mathcal{E}_{\langle\zeta\rangle}\). The irreducible algebra of matrices \(\sigma_{\langle\alpha\rangle}\) of minimal dimension \(N \times N\), where \(N = N_{(n)} + N_{(m_1)} + \ldots + N_{(m_z)}\), \(\dim S_{(h)} = N_{(n)}\) and \(\dim S_{(v_p)} = N_{(m_p)}\), has these dimensions

\[
N_{(n)} = \begin{cases}
2^{(n-1)/2}, & n = 2k + 1 \\
2^{n/2}, & n = 2k;
\end{cases}
\]

\[
N_{(m_p)} = \begin{cases}
2^{(m_p-1)/2}, & m_p = 2k_p + 1 \\
2^{m_p}, & m_p = 2k_p;
\end{cases}
\]

where \(k = 1, 2, \ldots, k_p = 1, 2, \ldots\)
The Clifford d-algebra is generated by sums on \( n + 1 \) elements of form
\[
A_1 I + B^i \sigma_i + C^{ij} \sigma_{ij} + D^{ijk} \sigma_{ijk} + ... \tag{3.3}
\]
and sums of \( m_p + 1 \) elements of form
\[
A_{2(p)} I + B^{a_p} \sigma_{a_p} + C^{a_p b_p} \sigma_{a_p b_p} + D^{a_p b_p c_p} \sigma_{a_p b_p c_p} + ... \tag{3.4}
\]
with antisymmetric coefficients
\[
C^{ij} = C^{[i|j]}, C^{[a_p b_p]} = C^{[a_p b_p]}, D^{[ij]} = D^{[i|j]}, D^{[a_p b_p c_p]} = D^{[a_p b_p c_p]}, ...
\]
and matrices
\[
\sigma_{ij} = \sigma_{[i|j]}, \sigma_{a_p b_p} = \sigma_{[a_p b_p]}, \sigma_{ijk} = \sigma_{[i|j|k]}, ...
\]
Really, we have \( 2^{n+1} \) coefficients \((A_1, C^{ij}, D^{ijk}, ...)\) and \( 2^{m_p+1} \) coefficients \((A_{2(p)}, C^{a_p b_p}, D^{a_p b_p c_p}, ...)\) of the Clifford algebra on \( F \).

For simplicity, we shall present the necessary geometric constructions only for h-spin spaces \( S_{(h)} \) of dimension \( N_{(n)} \). Considerations for a v-spin space \( S_{(v)} \) are similar but with proper characteristics for a dimension \( N_{(m)} \).

In order to define the scalar (spinor) product on \( S_{(h)} \) we introduce into consideration this finite sum (because of a finite number of elements \( \sigma_{[i|...|k]} \)):
\[
(+)^k E_{kn} = \delta^k_i \delta^j_m + 2N_{(n)} \epsilon_{km} \epsilon_{ij} + \frac{2^2}{3!} (\sigma^{[i|j]} \delta^k_m + \frac{2^4}{3!} (\sigma^{[i|j]} \delta^k_m + \frac{2^4}{3!} (\sigma^{[i|j]} \delta^k_m + ... \tag{3.5}
\]
which can be factorized as
\[
(+)^k E_{kn} = N_{(n)} (+)^k \epsilon_{km} \epsilon_{ij} \] for \( n = 2k \)
\[
and
\[
(+)^k E_{kn} = 2N_{(n)} \epsilon_{km} \epsilon_{ij}, \quad (-)^k E_{kn} = 0 \] for \( n = 3(\text{mod} 4) \),\]
\[
(+)^k E_{kn} = 0, \quad (-)^k E_{kn} = 2N_{(n)} \epsilon_{km} \epsilon_{ij} \] for \( n = 1(\text{mod} 4) \).\]

Antisymmetry of \( \sigma_{i|...|k} \) and the construction of the objects \((3.3)-(3.6)\) define the properties of \( \epsilon \)-objects \((\pm) \epsilon_{km} \) and \( \epsilon_{km} \) which have an eight-fold periodicity on \( n \) (see details in [18] and, with respect to locally anisotropic spaces, [17]).

For even values of \( n \) it is possible the decomposition of every h-spin space \( S_{(h)} \) into irreducible h-spin spaces \( S_{(h)} \) and \( S'_{(h)} \) (one considers splitting of h-indices, for instance, \( l = L \oplus L', m = M \oplus M', ... \); for v-indices we shall write \( a_p = A_p \oplus A'_p, b_p = B_p \oplus B'_p, ... \) and defines new \( \epsilon \)-objects
\[
\epsilon_{lm} = \frac{1}{2} (+)^l \epsilon_{lm} + (-)^l \epsilon_{lm} \] and \( \epsilon_{lm} = \frac{1}{2} (+)^l \epsilon_{lm} - (-)^l \epsilon_{lm} \).\]
We shall omit similar formulas for ε-objects with lower indices.

In general, the spinor ε-objects should be defined for every shell of anisotropy where instead of dimension \( n \) we shall consider the dimensions \( m_p, 1 \leq p \leq z \), of shells.

We define a d–spinor space \( S_{(n,m_1)} \) as a direct sum of a horizontal and a vertical spinor spaces, for instance,

\[
S_{(8k,8k')} = S_0 \oplus S'_0 \oplus S_{|0} \oplus S'_{|0}, S_{(8k,8k'+1)} = S_0 \oplus S'_0 \oplus S_{(-)} \oplus ..., \]

\[
S_{(8k+4,8k'+5)} = S_{\Delta} \oplus S'_{\Delta} \oplus S_{(-)} \oplus ..., \]

The scalar product on a \( S_{(n,m_1)} \) is induced by (corresponding to fixed values of \( n \) and \( m_1 \)) ε-objects considered for h- and v₁-components. We present also an example for \( S_{(n,m_1 + ... + m_z)} \):

\[
S_{(8k+4,8k'+5)} = [S_{\Delta} \oplus S'_{\Delta} \oplus S_{(-)} \oplus ... \oplus S_{(p)\Delta} \oplus S'_{(p)\Delta} \oplus ... \oplus S_{(z)\circ} \oplus S'_{(z)\circ}].
\]

Having introduced d-spinors for dimensions \((n, m_1 + ... + m_z)\) we can write out the generalization for ha–spaces of twistor equations by using the distinguished σ-objects:

\[
(\sigma_{\{\hat{\alpha}\}})/|\beta| = \frac{\delta \omega_{\hat{\alpha}}}{\delta u_{\hat{\beta}}} = \frac{1}{n + m_1 + ... + m_z} \frac{G_{\{\hat{\alpha}\} \{\hat{\beta}\}}}{(\sigma_{\{\hat{\alpha}\}})/|\beta|} \frac{\delta \omega_{\hat{\alpha}}}{\delta u_{\hat{\beta}}}, \quad (3.8)
\]

where \( |\beta| \) denotes that we do not consider symmetrization on this index. The general solution of (3.8) on the d-vector space \( F \) looks like as

\[
\omega_{\hat{\beta}} = \Omega_{\hat{\beta}} + u^{\{\hat{\alpha}\}}(\sigma_{\{\hat{\alpha}\}})/|\beta| \Pi_{\hat{\beta}}, \quad (3.9)
\]

where \( \Omega_{\hat{\beta}} \) and \( \Pi_{\hat{\beta}} \) are constant d-spinors. For fixed values of dimensions \( n \) and \( m = m_1 + ...m_z \) we must analyze the reduced and irreducible components of h- and v₁-parts of equations (3.3) and their solutions (3.9) in order to find the symmetry properties of a d-twistor \( Z^\alpha \) defined as a pair of d-spinors

\[
Z^\alpha = (\omega_{\hat{\alpha}}, \pi'_{\hat{\beta}}),
\]

where \( \pi_{\hat{\beta}} = \pi'_{\hat{\beta}}(0) \in S_{(n,m_1,...,m_z)} \) is a constant dual d-spinor. The problem of definition of spinors and twistors on ha-spaces was firstly considered in [84] (see also [58]) in connection with the possibility to extend the equations (3.3) and their solutions (3.10), by using nearly autoparallel maps, on curved, locally isotropic or anisotropic, spaces. We note that the definition of twistors have been extended to higher order anisotropic spaces with trivial N– and d–connections.

### 3.1.2 Mutual transforms of d-tensors and d-spinors

The spinor algebra for spaces of higher dimensions can not be considered as a real alternative to the tensor algebra as for locally isotropic spaces of dimensions \( n = 3, 4 \).
The same holds true for ha–spaces and we emphasize that it is not quite convenient to perform a spinor calculus for dimensions \( n, m \gg 4 \). Nevertheless, the concept of spinors is important for every type of spaces, we can deeply understand the fundamental properties of geometrical objects on ha–spaces, and we shall consider in this subsection some questions concerning transforms of \( d \)-tensor objects into \( d \)-spinor ones.

### 3.1.3 Transformation of \( d \)-tensors into \( d \)-spinors

In order to pass from \( d \)-tensors to \( d \)-spinors we must use \( \sigma \)-objects (3.2) written in reduced or irreduced form (in dependence of fixed values of dimensions \( n \) and \( m \)):

\[
(\sigma_{\hat{\alpha}})^2, (\sigma_{\hat{\alpha}})^{\hat{\beta}}\gamma, (\sigma_{\hat{\alpha}})^{\hat{\beta}2}, \ldots, (\sigma_{\hat{\alpha}})^{\hat{k}4}, \ldots, (\sigma_{\hat{\alpha}})^{AA'}, \ldots, (\sigma^{\hat{i}J})_{II'}, \ldots.
\] (3.10)

It is obvious that contracting with corresponding \( \sigma \)-objects (3.10) we can introduce instead of \( d \)-tensors indices the \( d \)-spinor ones, for instance,

\[
\omega_{\hat{\beta}}\gamma = (\sigma_{\hat{\alpha}})^{\hat{\beta}}\gamma \omega_{\hat{\alpha}}, \quad \omega_{\hat{A}\hat{B}'} = (\sigma_{\hat{\alpha}})^{\hat{A}\hat{B}'} \omega_{\hat{\alpha}}, \quad \ldots, \zeta_{\hat{i}2} = (\sigma^{\hat{i}j})_{\hat{J}2} \zeta_{\hat{k}}, \ldots.
\]

For \( d \)-tensors containing groups of antisymmetric indices there is a more simple procedure of theirs transforming into \( d \)-spinors because the objects

\[
(\sigma_{\hat{\alpha}_{\ldots\hat{\gamma}}})^{\hat{d}w}, \quad (\sigma_{\hat{\alpha}_{\ldots\hat{X}}})^{\hat{d}w}, \quad \ldots, (\sigma^{\hat{j}_{\ldots\hat{k}}})_{II'}, \quad \ldots.
\] (3.11)

can be used for sets of such indices into pairs of \( d \)-spinor indices. Let us enumerate some properties of \( \sigma \)-objects of type (3.11) (for simplicity we consider only h-components having \( q \) indices \( \hat{i}, \hat{j}, \hat{k}, \ldots \) taking values from 1 to \( n \); the properties of \( v_p \)-components can be written in a similar manner with respect to indices \( \hat{a}_p, \hat{b}_p, \hat{c}_p, \ldots \) taking values from 1 to \( m \)):

\[
(\sigma_{\hat{i}_{\ldots\hat{j}}}^{ijkl}) \quad \text{is} \quad \begin{cases} \text{symmetric on} \ k, l \text{ for} \ n - 2q \equiv 1, 7 \ (\text{mod} \ 8); \\ \text{antisymmetric on} \ k, l \text{ for} \ n - 2q \equiv 3, 5 \ (\text{mod} \ 8) \end{cases}
\] (3.12)

for odd values of \( n \), and an object

\[
(\sigma_{\hat{i}_{\ldots\hat{j}}}^{IJ}) \left( (\sigma_{\hat{i}_{\ldots\hat{j}}}^{I'J'}) \right)
\]

is \( \begin{cases} \text{symmetric on} \ I, J (I', J') \text{ for} \ n - 2q \equiv 0 \ (\text{mod} \ 8); \\ \text{antisymmetric on} \ I, J (I', J') \text{ for} \ n - 2q \equiv 4 \ (\text{mod} \ 8) \end{cases} \) (3.13)

or

\[
(\sigma_{\hat{i}_{\ldots\hat{j}}}^{I'J'}) = \pm (\sigma_{\hat{i}_{\ldots\hat{j}}}^{IJ}) \begin{cases} n + 2q \equiv 6(\text{mod}8); \\ n + 2q \equiv 2(\text{mod}8), \end{cases}
\] (3.14)

with vanishing of the rest of reduced components of the \( d \)-tensor \( (\sigma_{\hat{i}_{\ldots\hat{j}}}^{ijkl}) \) with prime/unprime sets of indices.
3.1.4 Fundamental $d$–spinors

We can transform every $d$–spinor $\xi_{\alpha} = (\xi^i, \xi^{a_1}, ..., \xi^{a_z})$ into a corresponding $d$-tensor. For simplicity, we consider this construction only for a $h$-component $\xi^i$ on a $h$-space being of dimension $n$. The values

$$\xi_{\alpha} \xi_{\beta} (\hat{\sigma}^{i..\hat{j}})_{\alpha\beta} \quad (n \text{ is odd})$$  \hfill (3.15)

or

$$\xi^I \xi^J (\hat{\sigma}^{i..\hat{j}})_{IJ} \quad (\text{or } \xi^{I'} \xi^{J'} (\hat{\sigma}^{i..\hat{j}})_{I'J'}) \quad (n \text{ is even})$$  \hfill (3.16)

with a different number of indices $\hat{i}..\hat{j}$, taken together, defines the $h$-spinor $\xi^i$ to an accuracy to the sign. We emphasize that it is necessary to choose only those $h$-components of $d$-tensors (3.15) (or (3.16)) which are symmetric on pairs of indices $\alpha\beta$ (or $IJ$ (or $I'J'$)) and the number $q$ of indices $\hat{i}..\hat{j}$ satisfies the condition (as a respective consequence of the properties (3.12) and/or (3.13), (3.14))

$$n - 2q \equiv 0, 1, 7 \pmod{8}.$$  \hfill (3.17)

Of special interest is the case when

$$q = \frac{1}{2} (n \pm 1) \quad (n \text{ is odd})$$  \hfill (3.18)

or

$$q = \frac{1}{2} n \quad (n \text{ is even}).$$  \hfill (3.19)

If all expressions (3.15) and/or (3.16) are zero for all values of $q$ with the exception of one or two ones defined by the conditions (3.17), (3.18) (or (3.19)), the value $\xi^i$ (or $\xi^I$ ($\xi^{I'}$)) is called a fundamental $h$-spinor. Defining in a similar manner the fundamental $v$-spinors we can introduce fundamental $d$-spinors as pairs of fundamental $h$- and $v$-spinors. Here we remark that a $h(v_p)$-spinor $\xi^i$ ($\xi^{i_p}$) (we can also consider reduced components) is always a fundamental one for $n(m) < 7$, which is a consequence of (3.19).

3.2 Differential Geometry of $Ha$–Spinors

This subsection is devoted to the differential geometry of $d$–spinors in higher order anisotropic spaces. We shall use denotations of type

$$v^{<a>} = (v^i, v^{<a>}) \in \sigma^{<a>} = (\sigma^i, \sigma^{<a>})$$

and

$$\zeta^{\alpha p} = (\zeta^{i_p}, \zeta^{\alpha p}) \in \sigma^{\alpha p} = (\sigma^{i_p}, \sigma^{\alpha p})$$
for, respectively, elements of modules of d-vector and irreduced d-spinor fields (see details in [67]). D-tensors and d-spinor tensors (irreduced or reduced) will be interpreted as elements of corresponding $\sigma$–modules, for instance,

$$q^{<\alpha>_\beta}> \in \sigma^{<\alpha>}/ \sigma^{<\beta>}, \psi^{\alpha_\beta}_{\gamma_\delta} \in \sigma^{\alpha_\beta}_{\gamma_\delta}, \xi_{I_\mu}^{I_\nu} K_{J_\sigma} L_{N_\tau} \in \sigma^{I_\mu} J_{I_\nu} K_{J_\sigma} L_{N_\tau}, \ldots$$

We can establish a correspondence between the higher order anisotropic adapted to the N–connection metric $g_{\alpha\beta}$ (1.43) and d-spinor metric $\epsilon_{\alpha\beta}$ ($\epsilon$-objects for both $h$- and $v_p$-subspaces of $\mathcal{E}^{<z>}$) of a ha–space $\mathcal{E}^{<z>}$ by using the relation

$$g_{<\alpha><\beta>} = -\frac{1}{N(n) + N(m_1) + \ldots + N(m_z)} \times ((\sigma^{<\alpha>}(u))^{\alpha_\beta}_{\gamma_\delta}(\sigma^{<\beta>}(u))^{\gamma_\delta}_{\alpha_\beta}) \epsilon_{\alpha_\beta} \epsilon_{\gamma_\delta}, \quad (3.20)$$

where

$$(\sigma^{<\alpha>}(u))^{\alpha_\beta}_{\gamma_\delta} = I^{\alpha_\beta}_{<\alpha>} (u) (\sigma^{<\beta>}(u))^{\gamma_\delta}_{<\beta>}, \quad (3.21)$$

which is a consequence of formulas (3.1)–(3.7). In brief we can write (3.20) as

$$g_{<\alpha><\beta>} = \epsilon_{\alpha_\beta} \epsilon_{\gamma_\delta}, \quad (3.22)$$

if the $\sigma$-objects are considered as a fixed structure, whereas $\epsilon$-objects are treated as caring the metric “dynamics”, on higher order anisotropic space. This variant is used, for instance, in the so-called 2-spinor geometry [47, 48] and should be preferred if we have to make explicit the algebraic symmetry properties of d-spinor objects by using metric decomposition (3.22). An alternative way is to consider as fixed the algebraic structure of $\epsilon$-objects and to use variable components of $\sigma$-objects of type (3.21) for developing a variational d-spinor approach to gravitational and matter field interactions on ha-spaces (the spinor Ashtekar variables [3] are introduced in this manner).

We note that a d–spinor metric

$$\epsilon_{\mu\nu} = \begin{pmatrix}
\epsilon_{ij} & 0 & \ldots & 0 \\
0 & \epsilon_{a_{i} b_{i}} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \epsilon_{a_{i} b_{i}}
\end{pmatrix}$$

on the d-spinor space $\mathcal{S} = (\mathcal{S}_{(h)}, \mathcal{S}_{(v_1)}, \ldots, \mathcal{S}_{(v_z)})$ can have symmetric or antisymmetric $h$ $(v_p)$-components $\epsilon_{ij}$ ($\epsilon_{a_{i} b_{i}}$), see $\epsilon$-objects. For simplicity, in order to avoid cumbersome calculations connected with eight-fold periodicity on dimensions $n$ and $m_p$ of a ha-space $\mathcal{E}^{<z>}$, we shall develop a general d-spinor formalism only by using irreduced spinor spaces $\mathcal{S}_{(h)}$ and $\mathcal{S}_{(v_p)}$. 

58
3.2.1 D-covariant derivation on ha–spaces

Let $E^{<z>}$ be a ha-space. We define the action on a d-spinor of a d-covariant operator

$$\nabla^{<\alpha>} = (\nabla_i^{<\alpha>}, \nabla_a^{<\alpha>}) = (\sigma_i^{<\alpha>}, \sigma_a^{<\alpha>})$$

as maps

$$\nabla_{\alpha_1 \alpha_2} : \sigma^\beta \rightarrow \sigma_{<\alpha>}^\beta = \sigma_{\alpha_1 \alpha_2}^\beta =$$

$$(\sigma_i^\beta = \sigma_i^{<\alpha>}^\beta, \sigma_a^\beta = \sigma_a^{<\alpha>}^\beta, ...,$$,

$$...$$,

$$\sigma_{(z)}^\beta = \sigma_{(z)}^{<\alpha>}^\beta, ...,$$,

$$...)$$

satisfying conditions

$$\nabla^{<\alpha>}(\xi^\beta + \eta^\beta) = \nabla^{<\alpha>}\xi^\beta + \nabla^{<\alpha>}\eta^\beta,$$

and

$$\nabla^{<\alpha>}(f \xi^\beta) = f \nabla^{<\alpha>}\xi^\beta + \xi^\beta \nabla^{<\alpha>}f$$

for every $\xi^\beta, \eta^\beta \in \sigma^\beta$ and $f$ being a scalar field on $E^{<z>}$. It is also required that one holds the Leibnitz rule

$$\nabla^{<\alpha>}(\zeta^\beta \eta^\beta) = \nabla^{<\alpha>}((\zeta^\beta \eta^\beta) - \zeta^\beta \nabla^{<\alpha>}\eta^\beta$$

and that $\nabla^{<\alpha>}$ is a real operator, i.e. it commutes with the operation of complex conjugation:

$$\overline{\nabla^{<\alpha>} \psi_{\alpha \beta \gamma ...}} = \nabla^{<\alpha>}(\overline{\psi_{\alpha \beta \gamma ...}}).$$

Let now analyze the question on uniqueness of action on d–spinors of an operator $\nabla^{<\alpha>}$ satisfying necessary conditions. Denoting by $\nabla^{(1)}_{<\alpha>}$ and $\nabla^{<\alpha>}$ two such d-covariant operators we consider the map

$$(\nabla^{(1)}_{<\alpha>} - \nabla^{<\alpha>}): \sigma^\beta \rightarrow \sigma_{<\alpha>}^\beta = \sigma_{\alpha_1 \alpha_2}^\beta. \quad (3.23)$$

Because the action on a scalar $f$ of both operators $\nabla^{(1)}_{<\alpha>}$ and $\nabla^{<\alpha>}$ must be identical, i.e.

$$\nabla^{(1)}_{<\alpha>}f = \nabla^{<\alpha>}f,$$
the action (3.23) on $f = \omega_\beta \xi^\beta$ must be written as

$$(\nabla^{(1)}_{<\alpha>} - \nabla_{<\alpha>})(\omega_\beta \xi^\beta) = 0.$$ 

In consequence we conclude that there is an element $\Theta_{\alpha_1 \alpha_2 \beta} \in \sigma_{\alpha_1 \alpha_2 \beta}$ for which

$$\nabla^{(1)}_{\alpha_1 \alpha_2} \xi^\gamma = \nabla_{\alpha_1 \alpha_2} \xi^\gamma + \Theta_{\alpha_1 \alpha_2 \beta} \xi^\beta, \quad \nabla^{(1)}_{\alpha_1 \alpha_2} \omega_\gamma = \nabla_{\alpha_1 \alpha_2} \omega_\gamma - \Theta_{\alpha_1 \alpha_2 \beta} \omega_\gamma. \quad (3.24)$$

The action of the operator (3.23) on a d-vector $v^{<\beta>} = v^{\beta_1 \beta_2}$ can be written by using formula (3.24) for both indices $\beta_1$ and $\beta_2$:

$$(\nabla^{(1)}_{<\alpha>} - \nabla_{<\alpha>})v^{<\beta>} = \Theta_{<\alpha>\gamma_{12}}^{\beta_1 \beta_2} \xi_{12} + \Theta_{<\alpha>\gamma_{12}}^{\beta_2 \beta_1} \xi_{12}$$

$$= (\Theta_{<\alpha>\gamma_{12}}^{\beta_1 \beta_2} \delta_{12}^\beta_2 + \Theta_{<\alpha>\gamma_{12}}^{\beta_2 \beta_1} \delta_{12}^\beta_1) t \omega_\beta = Q^{<\beta>}_{<\alpha>\gamma} v^{<\gamma>},$$

where

$$Q^{<\beta>}_{<\alpha>\gamma} = Q^{\beta_1 \beta_2}_{\alpha_1 \alpha_2 \gamma_{12}} = \Theta_{<\alpha>\gamma_{12}}^{\beta_1 \beta_2} \delta_{12}^\beta_2 + \Theta_{<\alpha>\gamma_{12}}^{\beta_2 \beta_1} \delta_{12}^\beta_1. \quad (3.25)$$

The d-commutator $\nabla_{[<\alpha>, \nabla^{(1)}_{<\beta>}]$ defines the d-torsion. So, applying operators $\nabla^{(1)}_{[<\alpha>, \nabla^{(1)}_{<\beta>}]$ and $\nabla_{[<\alpha>, \nabla^{(1)}_{<\beta>}]$ on $f = \omega_\beta \xi^\beta$ we can write

$$T^{(1)}_{<\alpha>\gamma} = Q^{<\beta>}_{<\alpha>\gamma} - Q^{<\beta>}_{<\alpha>\gamma} = Q^{<\beta>}_{<\alpha>\gamma} - Q^{<\beta>}_{<\alpha>\gamma}$$

with $Q^{<\gamma>}_{<\alpha>\beta}$ from (3.25).

The action of operator $\nabla^{(1)}_{<\alpha>}$ on d-spinor tensors of type $\chi^{<\beta_1 \beta_2 \cdots}_{\alpha_1 \alpha_2 \cdots}$ must be constructed by using formula (3.24) for every upper index $\beta_1 \beta_2 \cdots$ and formula (3.23) for every lower index $\alpha_1 \alpha_2 \cdots$.

### 3.2.2 Infeld–van der Waerden coefficients

Let

$$\delta^\alpha_{\underline{\alpha}} = (\delta^1_{\underline{1}}, \delta^2_{\underline{1}}, \ldots, \delta^{N(m)}_{\underline{1}}, \delta^1_{\underline{N(n)}}, \delta^2_{\underline{N(n)}}, \ldots, \delta^{N(m)}_{\underline{N(m)}})$$

be a d-spinor basis. The dual to it basis is denoted as

$$\delta_{\underline{\alpha}}^\alpha = (\delta_{\underline{1}}^{\frac{1}{1}}, \delta_{\underline{1}}^{\frac{2}{1}}, \ldots, \delta_{\underline{N(n)}}^{\frac{N(n)}{1}}, \delta_{\underline{1}}^{\frac{1}{2}}, \ldots, \delta_{\underline{N(m)}}^{\frac{N(m)}{2}}).$$

A d-spinor $\kappa^{\underline{\alpha}} \in \sigma^{\underline{\alpha}}$ has components $\kappa^{\underline{\alpha}} = \kappa^\underline{\alpha} \delta^\alpha_{\underline{\alpha}}$. Taking into account that

$$\delta_{\underline{\alpha}}^\alpha \delta_{\underline{\beta}}^\beta \nabla_{\alpha \beta}^\underline{\alpha} = \nabla_{\alpha \beta}^\underline{\alpha},$$
we write out the components \( \nabla_{\alpha\beta} \kappa^\gamma \) as

\[
\delta_\alpha^\alpha \delta_\beta^\beta \delta_\gamma^\gamma \nabla_{\alpha\beta} \kappa^\gamma = \delta_\alpha^\alpha \nabla_{\alpha\beta} \kappa^\gamma + \kappa^\xi \delta_\xi^\gamma \nabla_{\alpha\beta} \delta_\gamma^\xi = \nabla_{\alpha\beta} \kappa^\gamma + \kappa^\xi \delta_\xi^\gamma \nabla_{\alpha\beta} \kappa^\gamma, \tag{3.26}
\]

where the coordinate components of the d–spinor connection \( \gamma_{\alpha\beta}^\gamma \) are defined as

\[
\gamma_{\alpha\beta}^\gamma = \delta_\alpha^\gamma \nabla_{\alpha\beta} \delta_\beta^\gamma. \tag{3.27}
\]

We call the Infeld - van der Waerden d-symbols a set of \( \sigma \)-objects \( (\sigma_\alpha)^{\alpha\beta} \) parametrized with respect to a coordinate d-spinor basis. Defining

\[
\nabla_{<\alpha>} = (\sigma_{<\alpha>})_{\alpha\beta} \nabla_{\alpha\beta},
\]

introducing denotations

\[
\gamma_{\alpha<\alpha>\tau}^\xi = (\sigma_{<\alpha>})_{\alpha\beta} \delta_\beta^\xi \tag{3.28}
\]

and using properties (3.26) we can write relations

\[
\nabla_{<\alpha>} \delta_\beta^\beta \nabla_{<\alpha>} \kappa_{\alpha\beta} = \nabla_{<\alpha>} \kappa_{\alpha\beta} + \kappa_{\xi\gamma} \gamma_{<\alpha>\xi},
\]

for d-covariant derivations \( \nabla_{\alpha} \kappa_{\beta} \) and \( \nabla_{\alpha} \mu_{\beta} \).

We can consider expressions similar to (3.28) for values having both types of d-spinor and d-tensor indices, for instance,

\[
\nabla_{<\alpha>} \delta_\beta^\beta \nabla_{<\alpha>} \mu_{\beta} = \nabla_{<\alpha>} \mu_{\beta} - \mu_{\xi\gamma} \delta_\xi^\gamma,
\]

we can consider some possible relations between components of d-connections \( \gamma_{\alpha<\alpha>}^\xi \) and \( \Gamma_{<\alpha><\beta><\gamma>} \) and derivations of \( (\sigma_{<\alpha>})^{\alpha\beta} \). We can write

\[
\Gamma_{<\alpha><\beta><\gamma>} = l_{<\alpha><\beta><\gamma>} \nabla_{<\gamma>} l_{<\alpha><\beta>} = l_{<\alpha><\beta>} \nabla_{<\gamma>} (\sigma_{<\beta>})^{\xi\gamma} l_{<\alpha><\beta>} \nabla_{<\gamma>} ((\sigma_{<\beta>})^{\xi\gamma} \delta_\delta^\xi \delta_\gamma^\tau)
\]

\[
= l_{<\alpha><\beta>}^{\alpha\gamma} \delta_\alpha^\alpha \delta_\beta^\beta \nabla_{<\gamma>} (\sigma_{<\beta>})^{\xi\gamma} \delta_\delta^\xi \nabla_{<\gamma>} \delta_\gamma^\tau + l_{<\alpha><\beta>}^{\alpha\gamma} \delta_\alpha^\alpha \delta_\beta^\beta \nabla_{<\gamma>} \delta_\gamma^\tau + l_{<\alpha><\beta>}^{\alpha\gamma} \delta_\alpha^\alpha \delta_\beta^\beta (\sigma_{<\beta>})^{\xi\gamma} \delta_\delta^\xi \nabla_{<\gamma>} \delta_\gamma^\tau
\]

\[
+ \delta_\alpha^\alpha \nabla_{<\gamma>} \delta_\gamma^\tau, \tag{3.29}
\]

where \( l_{<\alpha><\beta>}^{\alpha\beta} = (\sigma_{<\alpha>})^{\alpha\beta} \), from which one follows

\[
(\sigma_{<\alpha>})^{\mu\nu}(\sigma_{<\alpha>})^{\beta\gamma} \Gamma_{<\alpha><\beta><\gamma>} = (\sigma_{<\alpha>})^{\beta\gamma} \nabla_{<\gamma>} (\sigma_{<\alpha>})^{\mu\nu} + \delta_\beta^\mu \gamma_{<\alpha>\alpha} + \delta_\gamma^\mu \gamma_{<\beta>}.
\]
Connecting the last expression on \( \beta \) and \( \nu \) and using an orthonormalized d-spinor basis when \( \gamma_{\alpha \beta} <\gamma \beta> = 0 \) (a consequence from (3.27)) we have

\[
\gamma_{\alpha \beta} \mu = \frac{1}{N(n) + N(m_1) + \ldots + N(m_z)} (\Gamma^\mu_{\alpha \beta} <\gamma \nu> - (\sigma_{\alpha \beta}) <\beta> \nabla <\gamma> (\sigma_{\alpha \beta}) \mu),
\]

where

\[
\Gamma^\mu_{\alpha \beta} <\gamma \nu> = (\sigma_{\alpha \beta}) \mu \Gamma^\mu_{\alpha \beta} <\gamma \beta>.
\]

We also note here that, for instance, for the canonical and Berwald connections and Christoffel d-symbols we can express d-spinor connection (3.30) through corresponding locally adapted derivations of components of metric and N-connection by introducing corresponding coefficients instead of \( \Gamma^\mu_{\alpha \beta} <\gamma \beta> \) in (3.30) and than in (3.29).

### 3.2.3 D–spinors of ha–space curvature and torsion

The d-tensor indices of the commutator \( \Delta_{\alpha \beta} \) can be transformed into d-spinor ones:

\[
\Box_{\alpha \beta} = (\sigma_{\alpha \beta})_{\alpha \beta} \Delta_{\alpha \beta} = (\Box_{ij}, \Box_{ab}) = (\Box_{ij}, \Box_{ab}, \ldots, \Box_{ab}, \ldots, \Box_{ab}),
\]

with h- and v\( p \)-components,

\[
\Box_{ij} = (\sigma_{\alpha \beta})_{ij} \Delta_{\alpha \beta} \text{ and } \Box_{ab} = (\sigma_{\alpha \beta})_{ab} \Delta_{\alpha \beta},
\]

being symmetric or antisymmetric in dependence of corresponding values of dimensions \( n \) and \( m_p \) (see eight-fold parametizations. Considering the actions of operator (3.31) on d-spinors \( \pi^\gamma \) and \( \mu \), we introduce the d-spinor curvature \( X_{\alpha \beta} \) as to satisfy equations

\[
\Box_{\alpha \beta} \pi^\gamma = X_{\alpha \beta} \gamma \pi^\delta \text{ and } \Box_{\alpha \beta} \mu = X_{\alpha \beta} \gamma \mu, \quad (3.32)
\]

The gravitational d-spinor \( \Psi_{\alpha \beta \gamma \delta} \) is defined by a corresponding symmetrization of d-spinor indices:

\[
\Psi_{\alpha \beta \gamma \delta} = X_{\alpha \beta \gamma \delta}. \quad (3.33)
\]

We note that d-spinor tensors \( X_{\alpha \beta} \) and \( \Psi_{\alpha \beta \gamma \delta} \) are transformed into similar 2-spinor objects on locally isotropic spaces [47, 48] if we consider vanishing of the N-connection structure and a limit to a locally isotropic space.

Putting \( \delta_{\alpha \beta} \) instead of \( \mu \) in (3.32) and using (3.33) we can express respectively the curvature and gravitational d-spinors as

\[
X_{\gamma \delta \alpha \beta} = \delta_{\gamma \delta} \Box_{\alpha \beta} \delta_{\gamma \delta} \pi \text{ and } \Psi_{\delta \alpha \beta \gamma} = \delta_{\gamma \delta} \Box_{\alpha \beta} \delta_{\gamma \delta} \pi. \]

62
The d-spinor torsion $T_{\gamma\delta}^\alpha_{\alpha_1..\alpha_n}$ is defined similarly as for d-tensors by using the d-spinor commutator \((3.31)\) and equations

$$\square_{\alpha\beta} f = T_{\gamma\delta}^\alpha_{\alpha_1..\alpha_n} \nabla_{\gamma_1..\gamma_n} f.$$ 

The d-spinor components $R_{\gamma\delta}^\alpha_{\alpha_1..\alpha_n}$ of the curvature d-tensor $R_\gamma^\delta_{\alpha\beta}$ can be computed by using relations \((3.30)\), \((3.31)\) and \((3.33)\) as to satisfy the equations

$$(\square_{\alpha\beta} - T_{\gamma\delta}^\alpha_{\alpha_1..\alpha_n} \nabla_{\gamma_1..\gamma_n}) V^\gamma_{\alpha\beta} = R_{\gamma\delta}^\alpha_{\alpha_1..\alpha_n} \partial_{\alpha_{\alpha_1}} \partial_{\alpha_{\alpha_2}} V^\gamma_{\alpha_1..\alpha_n},$$

here d-vector $V^\gamma_{\alpha\beta}$ is considered as a product of d-spins, i.e. $V^\gamma_{\alpha\beta} = \nu^\gamma_{\alpha\beta}$. We find

$$R_{\gamma\delta}^\alpha_{\alpha_1..\alpha_n} = \left(X_{\gamma\delta}^\gamma_{\alpha_1..\alpha_n} + T_{\gamma\delta}^\gamma_{\alpha_1..\alpha_n} \gamma_{\alpha_{\alpha_1..\alpha_n}}\right) \delta_{\gamma\delta} + \left(X_{\gamma\delta}^\gamma_{\alpha_1..\alpha_n} + T_{\gamma\delta}^\gamma_{\alpha_1..\alpha_n} \gamma_{\alpha_{\alpha_1..\alpha_n}}\right) \delta_{\gamma\delta},$$

in order to get the d–spinor components of the Ricci d-tensor

$$R_{\gamma\delta}^\alpha_{\alpha_1..\alpha_n} = R_{\gamma\delta}^\alpha_{\alpha_1..\alpha_n} = \left(X_{\gamma\delta}^\gamma_{\alpha_1..\alpha_n} + T_{\gamma\delta}^\gamma_{\alpha_1..\alpha_n} \gamma_{\alpha_{\alpha_1..\alpha_n}}\right) \delta_{\gamma\delta} + \left(X_{\gamma\delta}^\gamma_{\alpha_1..\alpha_n} + T_{\gamma\delta}^\gamma_{\alpha_1..\alpha_n} \gamma_{\alpha_{\alpha_1..\alpha_n}}\right) \delta_{\gamma\delta},$$

and this d-spinor decomposition of the scalar curvature:

$$q^\gamma R = R_{\alpha\beta}^\gamma_{\alpha_1\alpha_2} X_{\alpha\beta}^\gamma_{\alpha_1..\alpha_n} \delta_{\alpha_{\alpha_1..\alpha_n}} + T_{\alpha\beta}^\gamma_{\alpha_1..\alpha_n} \partial_{\alpha_{\alpha_1..\alpha_n}} \gamma_{\alpha_{\alpha_1..\alpha_n}} + \left(X_{\alpha\beta}^\gamma_{\alpha_1..\alpha_n} + T_{\alpha\beta}^\gamma_{\alpha_1..\alpha_n} \gamma_{\alpha_{\alpha_1..\alpha_n}}\right) \delta_{\alpha_{\alpha_1..\alpha_n}}.$$ 

Putting \((3.34)\) and \((3.35)\) into \((1.78)\) and, correspondingly, \((4.14)\) we find the d–spinor components of the Einstein and $\Phi_{<\alpha<\beta>}$ d–tensors:

$$\nabla_{\gamma_{\alpha_{\alpha_1..\alpha_n}}} \Phi_{<\alpha<\beta>} = \nabla_{\gamma_{\alpha_{\alpha_1..\alpha_n}}} \Phi_{<\alpha<\beta>} = X_{\alpha_{\alpha_1..\alpha_n}} \delta_{\alpha_{\alpha_1..\alpha_n}} + T_{\alpha_{\alpha_1..\alpha_n}} \gamma_{\alpha_{\alpha_1..\alpha_n}} + \left(X_{\alpha_{\alpha_1..\alpha_n}} + T_{\alpha_{\alpha_1..\alpha_n}} \gamma_{\alpha_{\alpha_1..\alpha_n}}\right) \delta_{\alpha_{\alpha_1..\alpha_n}}.$$ 

63
and

$$\Phi_{<\gamma<\alpha>} = \Phi_{\gamma_1\alpha_1\gamma_2\alpha_2} = \frac{1}{2(n + m_1 + \ldots + m_z)} \varepsilon_{\gamma_1\alpha_1} \varepsilon_{\gamma_2\alpha_2} \left[ X_{\beta_1}^{\mu_1} \frac{\beta_2}{\beta_1} \mu_2, + \right.$$

$$T_{\varepsilon_{\beta_1}}^{\varepsilon_{\beta_2}} \frac{\beta_2}{\beta_1} \gamma_{\varepsilon_{\beta_1}}^{\mu_1} + X_{\beta_1}^{\mu_1 \mu_2} \beta_2^{\mu_2} \mu_1 + T_{\varepsilon_{\beta_1}}^{\varepsilon_{\beta_2}} \frac{\beta_2}{\beta_1} \delta_2 \gamma_{\varepsilon_{\beta_1}}^{\varepsilon_{\beta_2}}, -$$

$$\left. \frac{1}{2} \left[ X_{\varepsilon_{\beta_1}}^{\delta_1} \frac{\delta_1}{\alpha_1} \alpha_2 \delta_2 \gamma_{\varepsilon_{\beta_1}}^{\varepsilon_{\beta_2}} + T_{\varepsilon_{\beta_1}}^{\varepsilon_{\beta_2}} \alpha_1 \alpha_2 \delta_2 \gamma_{\varepsilon_{\beta_1}}^{\varepsilon_{\beta_2}}, + \right. \right.$$

$$\left. X_{\varepsilon_{\beta_1}}^{\delta_2} \frac{\delta_2}{\alpha_1} \alpha_2 \delta_1 \gamma_{\varepsilon_{\beta_1}}^{\varepsilon_{\beta_2}} + T_{\varepsilon_{\beta_1}}^{\varepsilon_{\beta_2}} \alpha_1 \alpha_2 \delta_1 \gamma_{\varepsilon_{\beta_1}}^{\varepsilon_{\beta_2}} \right]. \quad (3.38)$$

The components of the conformal Weyl d-spinor can be computed by putting d-spinor values of the curvature (3.34) and Ricci (3.35) d-tensors into corresponding expression for the d-tensor (1.77). We omit this calculus in this work.
Chapter 4

Ha-Spinors and Field Interactions

The problem of formulation gravitational and gauge field equations on different types of locally anisotropic spaces is considered, for instance, in [37, 8, 6] and [79]. In this Chapter we shall introduce the basic field equations for gravitational and matter field la-interactions in a generalized form for generic higher order anisotropic spaces.

4.1 Scalar field ha–interactions

Let \( \varphi (u) = (\varphi_1 (u), \varphi_2 (u), \ldots, \varphi_k (u)) \) be a complex k-component scalar field of mass \( \mu \) on ha-space \( \mathcal{E}^{<z>} \). The d-covariant generalization of the conformally invariant (in the massless case) scalar field equation [47, 48] can be defined by using the d’Alambert locally anisotropic operator [4, 63] \( \Box = D^{<\alpha>} D_{<\alpha>} \), where \( D_{<\alpha>} \) is a d-covariant derivation on \( \mathcal{E}^{<z>} \) and constructed, for simplicity, by using Christoffel d–symbols (all formulas for field equations and conservation values can be deformed by using corresponding deformations d–tensors \( P^{<\alpha><\beta><\gamma>} \) from the Cristoffel d–symbols, or the canonical d–connection to a general d-connection into consideration):

\[
(\Box + \frac{n_E - 2}{4(n_E - 1)} R + \mu^2) \varphi (u) = 0,
\]

where \( n_E = n + m_1 + \ldots + m_z \). We must change d-covariant derivation \( D_{<\alpha>} \) into \( \delta D_{<\alpha>} = D_{<\alpha>} + i e A_{<\alpha>} \) and take into account the d-vector current

\[
J_{<\alpha>}^{(0)} (u) = i((\varphi (u) D_{<\alpha>} \varphi (u) - D_{<\alpha>} \overline{\varphi} (u)) \varphi (u))
\]

if interactions between locally anisotropic electromagnetic field (d-vector potential \( A_{<\alpha>} \)), where \( e \) is the electromagnetic constant, and charged scalar field \( \varphi \) are considered. The equations (4.1) are (locally adapted to the N-connection structure) Euler equations for the Lagrangian

\[
\mathcal{L}^{(0)} (u) = \sqrt{|g|} \left[ g^{<\alpha><\beta>} \delta_{<\alpha>} \overline{\varphi} (u) \delta_{<\beta>} \varphi (u) - \left( \mu^2 + \frac{n_E - 2}{4(n_E - 1)} \right) \overline{\varphi} (u) \varphi (u) \right].
\]
where $|g| = \det g_{\alpha<\beta}$. 

The locally adapted variations of the action with Lagrangian (4.2) on variables $\varphi(u)$ and $\varphi'(u)$ leads to the locally anisotropic generalization of the energy-momentum tensor,

$$
E^{(0,\text{can})}_{\alpha<\beta}(u) = \delta_{\alpha>\beta}(u) \delta_{\alpha>\beta}(u) + \delta_{\alpha>\beta}(u) \delta_{\alpha>\beta}(u)
$$

$$
- \frac{1}{\sqrt{|g|}} g_{\alpha>\beta} \mathcal{L}^{(0)}(u),
$$

(4.3)

and a similar variation on the components of a d-metric (1.43) leads to a symmetric metric energy-momentum d-tensor,

$$
E^{(0)}_{\alpha<\beta}(u) = E^{(0,\text{can})}_{\alpha<\beta}(u)
$$

$$
- \frac{n}{n-2} \left[ R_{\alpha<\beta} + D_{\alpha} D_{\beta} - g_{\alpha<\beta} \Box \right] \varphi(u) \varphi(u).
$$

(4.4)

Here we note that we can obtain a nonsymmetric energy-momentum d-tensor if we firstly vary on $G_{\alpha<\beta}$ and than impose the constraint of compatibility with the N-connection structure. We also conclude that the existence of a N-connection in dv-bundle $\mathcal{E}<z>$ results in a nonequivalence of energy-momentum d-tensors (4.3) and (4.4), nonsymmetry of the Ricci tensor, nonvanishing of the d-covariant derivation of the Einstein d-tensor, $D_{\alpha}\Gamma_{\alpha<\beta} \neq 0$ and, in consequence, a corresponding breaking of conservation laws on higher order anisotropic spaces when $D_{\alpha}E^{\alpha<\beta} \neq 0$. The problem of formulation of conservation laws on locally anisotropic spaces is discussed in details and two variants of its solution (by using nearly autoparallel maps and tensor integral formalism on locally anisotropic and higher order multispaces) are proposed in [63].

In this Chapter we present only straightforward generalizations of field equations and necessary formulas for energy-momentum d-tensors of matter fields on $\mathcal{E}<z>$ considering that it is naturally that the conservation laws (usually being consequences of global, local and/or intrinsic symmetries of the fundamental space-time and of the type of field interactions) have to be broken on locally anisotropic spaces.

### 4.2 Proca equations on ha–spaces

Let consider a d-vector field $\varphi_{<\alpha>}(u)$ with mass $\mu^2$ (locally anisotropic Proca field) interacting with exterior la-gravitational field. From the Lagrangian

$$
\mathcal{L}^{(1)}(u) = \sqrt{|g|} \left[ -\frac{1}{2} f_{\alpha<\beta} (u) f^{\alpha<\beta} (u) + \mu^2 \varphi_{<\alpha>} (u) \varphi_{<\alpha>} (u) \right],
$$

(4.5)

where $f_{\alpha<\beta} = D_{\alpha} \varphi_{\beta} - D_{\beta} \varphi_{\alpha}$, one follows the Proca equations on higher order anisotropic spaces

$$
D_{\alpha} f^{\alpha<\beta} (u) + \mu^2 \varphi_{<\beta>} (u) = 0.
$$

(4.6)
Equations (4.6) are a first type constraints for $\beta = 0$. Acting with $D_{<\alpha>}$ on (4.6), for $\mu \neq 0$ we obtain second type constraints

$$D_{<\alpha>} \varphi^{<\alpha>} (u) = 0. \quad (4.7)$$

Putting (1.7) into (4.6) we obtain second order field equations with respect to $\varphi_{<\alpha>}$:

$$\Box \varphi_{<\alpha>} (u) + R_{<\alpha><\beta> \varphi^{<\beta>}} (u) + \mu^2 \varphi_{<\alpha>} (u) = 0. \quad (4.8)$$

The energy-momentum d-tensor and d-vector current following from the (4.8) can be written as

$$E^{(1)}_{<\alpha><\beta>} (u) = -g^{<\epsilon><\tau>} (\overline{f}_{<\beta><\tau>} f_{<\alpha><\epsilon>} + f_{<\alpha><\epsilon>} f_{<\beta><\tau>})$$

$$+ \mu^2 (\varphi_{<\alpha> \varphi_{<\beta>}} + \varphi_{<\beta> \varphi_{<\alpha>}}) - \frac{g_{<\alpha><\beta>}}{\sqrt{|g|}} L^{(1)} (u)$$

and

$$J^{(1)}_{<\alpha>} (u) = i (\overline{f}_{<\alpha><\beta>} (u) \varphi^{<\beta>} (u) - \varphi_{<\beta>} (u) f_{<\alpha><\beta>} (u)).$$

For $\mu = 0$ the d-tensor $f_{<\alpha><\beta>}$ and the Lagrangian (4.5) are invariant with respect to locally anisotropic gauge transforms of type

$$\varphi_{<\alpha>} (u) \rightarrow \varphi_{<\alpha>} (u) + \delta_{<\alpha>} \Lambda (u),$$

where $\Lambda (u)$ is a d-differentiable scalar function, and we obtain a locally anisotropic variant of Maxwell theory.

4.3 Higher order anisotropic Dirac equations

Let denote the Dirac d-spinor field on $E^{<zz>}$ as $\psi (u) = (\psi_{\alpha} (u))$ and consider as the generalized Lorentz transforms the group of automorphism of the metric $G_{<\alpha><\beta>}$ (see (4.43)). The d-covariant derivation of field $\psi (u)$ is written as

$$\nabla_{<\alpha>} \psi = \left[ \delta_{<\alpha>} + \frac{1}{4} C_{\alpha\beta\gamma} (u) \sqrt{\delta_{<\alpha>} (u) \sigma^{\beta} \sigma^{\gamma}} \right] \psi, \quad (4.9)$$

where coefficients $C_{\alpha\beta\gamma} = (D_{<\alpha}> l_{\alpha}^{\epsilon<\alpha>}) l_{\beta<\gamma}^{\epsilon<\gamma>}$ generalize for ha-spaces the corresponding Ricci coefficients on Riemannian spaces [18]. Using $\sigma$-objects $\sigma^{<\alpha>} (u)$ (see (3.2) and (3.12)–(3.14)) we define the Dirac equations on ha–spaces:

$$(i \sigma^{<\alpha>} (u) \nabla_{<\alpha>} - \mu) \psi = 0,$$

which are the Euler equations for the Lagrangian

$$L^{(1/2)} (u) = \sqrt{|g|} \{ [\psi^{+} (u) \sigma^{<\alpha>} (u) \nabla_{<\alpha>} \psi (u)$$

$$- (\nabla_{<\alpha>} \psi^{+} (u)) \sigma^{<\alpha>} (u) \psi (u)] - \mu \psi^{+} (u) \psi (u) \}, \quad (4.10)$$
where $\psi^+(u)$ is the complex conjugation and transposition of the column $\psi(u)$.

From (4.10) we obtain the d–metric energy-momentum d-tensor

$$ E^{(1/2)}_{\alpha\beta} = \frac{i}{4} \left[ \psi^+(u) \sigma_{\alpha} (u) \overrightarrow{\nabla_{\beta}} \psi(u) + \psi^+(u) \sigma_{\beta} (u) \overrightarrow{\nabla_{\alpha}} \psi(u) ight] $$

and the d-vector source

$$ J^{(1/2)}_{\alpha} (u) = \psi^+(u) \sigma_{\alpha} (u) \psi(u). $$

We emphasize that locally anisotropic interactions with exterior gauge fields can be introduced by changing the higher order anisotropic partial derivation from (4.9) in this manner:

$$ \delta_\alpha \rightarrow \delta_\alpha + ie^* B_\alpha, $$

where $e^*$ and $B_\alpha$ are respectively the constant d-vector potential of locally anisotropic gauge interactions on higher order anisotropic spaces (see [79] and the next section).

### 4.4 D–spinor Yang–Mills fields

We consider a dv–bundle $B_E$, $\pi_B : B \to \mathcal{E}^{<z>}$ on ha–space $\mathcal{E}^{<z>}$. Additionally to d-tensor and d-spinor indices we shall use capital Greek letters, $\Phi, \Upsilon, \Xi, \Psi, \ldots$ for fibre (of this bundle) indices (see details in [47, 48] for the case when the base space of the v-bundle $\pi_B$ is a locally isotropic space-time). Let $\overrightarrow{\nabla}_{\alpha}$ be, for simplicity, a torsionless, linear connection in $B_E$ satisfying conditions:

$$ \nabla_{\alpha} : \Upsilon^\Theta \rightarrow \nabla_{\alpha} \Upsilon^\Theta $$

$$ (\nabla_{\alpha} (\lambda^\Theta + \nu^\Theta) = \nabla_{\alpha} \lambda^\Theta + \nabla_{\alpha} \nu^\Theta, $$

$$ (f \lambda^\Theta) = \lambda^\Theta \nabla_{\alpha} f + f \nabla_{\alpha} \lambda^\Theta, \quad f \in \Upsilon^\Theta $$

where by $\Upsilon^\Theta$ (or $\Xi^\Theta$) we denote the module of sections of the real (complex) v–bundle $B_E$ provided with the abstract index $\Theta$. The curvature of connection $\nabla_{\alpha}$ is defined as

$$ K_{\alpha\beta}^\Theta \lambda^\Omega = \left( \nabla_{\alpha} \nabla_{\beta} - \nabla_{\beta} \nabla_{\alpha} \right) \lambda^\Theta. $$

For Yang-Mills fields as a rule one considers that $B_E$ is enabled with a unitary (complex) structure (complex conjugation changes mutually the upper and lower Greek indices). It is useful to introduce instead of $K_{\alpha\beta}^\Theta \lambda^\Omega$ a Hermitian matrix $F_{\alpha\beta}^\Theta = i K_{\alpha\beta}^\Theta$ connected with components of the Yang-Mills d-vector potential $B_{\alpha\beta}$ according the formula:
where the locally anisotropic space indices commute with capital Greek indices. The
gauge transforms are written in the form:

\[
\begin{align*}
B_{\langle \alpha \rangle \Theta} & \mapsto \hat{B}_{\langle \alpha \rangle \Theta} = B_{\langle \alpha \rangle \Theta} s_{\hat{\Phi}} \hat{\Phi} q_{\Theta} \Theta + i s_{\Theta} \hat{\Phi} \nabla_{\langle \alpha \rangle} q_{\Theta} \Theta, \\
F_{\langle \alpha \rangle \langle \beta \rangle \Xi} & \mapsto \hat{F}_{\langle \alpha \rangle \langle \beta \rangle \Xi} = F_{\langle \alpha \rangle \langle \beta \rangle \Xi} s_{\hat{\Phi}} \hat{\Phi} q_{\Xi} \Xi,
\end{align*}
\]

where matrices \( s_{\hat{\Phi}} \) and \( q_{\Xi} \) are mutually inverse (Hermitian conjugated in the unitary case). The Yang-Mills equations on torsionless locally anisotropic spaces [79] (see details in the next Section) are written in this form:

\[
\begin{align*}
\nabla_{\langle \alpha \rangle} F_{\langle \alpha \rangle \langle \beta \rangle \Theta} & = J_{\langle \beta \rangle \Theta}, \\
\nabla_{\langle \alpha \rangle} F_{\langle \beta \rangle \langle \gamma \rangle |\Theta} & = 0.
\end{align*}
\]

We must introduce deformations of connection of type \( \nabla^* \rightarrow \nabla + P_\alpha \) (the deformation d-tensor \( P_\alpha \) is induced by the torsion in dv-bundle \( B_E \)) into the definition of the curvature of gauge ha–fields (4.11) and motion equations (4.12) if interactions are modeled on a
generic higher order anisotropic space.

### 4.5 D–spinor Einstein–Cartan Theory

The Einstein equations in some models of higher order anisotropic supergravity have
been considered in [64, 70]. Here we note that the Einstein equations and conservation
laws on v–bundles provided with N-connection structures were studied in detail in [36, 37, 88, 86, 68]. In Ref. [79] we proved that the locally anisotropic gravity can be formulated in a gauge like manner and analyzed the conditions when the Einstein gravitational locally anisotropic field equations are equivalent to a corresponding form of Yang-Mills equations. Our aim here is to write the higher order anisotropic gravitational field equations in a form more convenient for theirs equivalent reformulation in higher
order anisotropic spinor variables.

#### 4.5.1 Einstein ha–equations

We define d-tensor \( \Phi_{\langle \alpha \rangle \langle \beta \rangle} \) as to satisfy conditions

\[
-2 \Phi_{\langle \alpha \rangle \langle \beta \rangle} \doteq R_{\langle \alpha \rangle \langle \beta \rangle} - \frac{1}{n + m_1 + \ldots + m_z} \langle R \rangle g_{\langle \alpha \rangle \langle \beta \rangle}
\]
which is the torsionless part of the Ricci tensor for locally isotropic spaces \[47, 48\], i.e. \(\Phi_{<\alpha><\alpha>}=0\). The Einstein equations on higher order anisotropic spaces

\[
\check{G}_{<\alpha><\beta>} + \lambda g_{<\alpha><\beta>} = \kappa E_{<\alpha><\beta>},
\]

where

\[
\check{G}_{<\alpha><\beta>} = R_{<\alpha><\beta>} - \frac{1}{2} \check{R} g_{<\alpha><\beta>}
\]
is the Einstein d–tensor, \(\lambda\) and \(\kappa\) are correspondingly the cosmological and gravitational constants and by \(E_{<\alpha><\beta>}\) is denoted the locally anisotropic energy–momentum d–tensor, can be rewritten in equivalent form:

\[
\Phi_{<\alpha><\beta>} = -\frac{\kappa}{2} (E_{<\alpha><\beta>} - \frac{1}{n + m_1 + \ldots + m_z} E_{<\tau><\tau>} g_{<\alpha><\beta>}).
\]

Because ha–spaces generally have nonzero torsions we shall add to (4.13) (equivalently to (4.14)) a system of algebraic d–field equations with the source \(S_{<\alpha><\beta><\gamma>}\) being the locally anisotropic spin density of matter (if we consider a variant of higher order anisotropic Einstein–Cartan theory):

\[
T_{<\alpha><\beta><\gamma>} + 2\delta_{<\alpha><\delta>} T_{<\beta><\gamma><\delta>} = \kappa S_{<\alpha><\beta><\gamma>}. \tag{4.15}
\]

From (4.15) one follows the conservation law of higher order anisotropic spin matter:

\[
\nabla_{<\gamma>} S_{<\alpha><\beta>} - T_{<\delta><\alpha><\beta><\gamma>} S_{<\alpha><\beta><\gamma>} = E_{<\beta><\alpha>} - E_{<\alpha><\beta>}.
\]

### 4.5.2 Einstein–Cartan d–equations

Now we can write out the field equations of the Einstein–Cartan theory in the d-spinor form. So, for the Einstein equations (1.78) we have

\[
\check{G}_{\gamma_{1}\gamma_{2}\alpha_{1}\alpha_{2}} + \lambda \varepsilon_{\gamma_{1}\alpha_{1}} \varepsilon_{\gamma_{2}\alpha_{2}} = \kappa E_{\gamma_{1}\gamma_{2}\alpha_{1}\alpha_{2}},
\]

with \(\check{G}_{\gamma_{1}\gamma_{2}\alpha_{1}\alpha_{2}}\) from (3.37), or, by using the d-tensor (3.38),

\[
\Phi_{\gamma_{1}\gamma_{2}\alpha_{1}\alpha_{2}} + \left(\frac{\check{R}}{4} - \frac{\lambda}{2}\right) \varepsilon_{\gamma_{1}\alpha_{1}} \varepsilon_{\gamma_{2}\alpha_{2}} = -\frac{\kappa}{2} E_{\gamma_{1}\gamma_{2}\alpha_{1}\alpha_{2}},
\]

which are the d-spinor equivalent of the equations (1.14). These equations must be completed by the algebraic equations (4.15) for the d-torsion and d-spin density with d-tensor indices changed into corresponding d–spinor ones.
4.5.3 Higher order anisotropic gravitons

Let a massless d-tensor field $h_{<\alpha><\beta>}(u)$ is interpreted as a small perturbation of the locally anisotropic background metric d-field $g_{<\alpha><\beta>}(u)$. Considering, for simplicity, a torsionless background we have locally anisotropic Fierz–Pauli equations

$$\Box h_{<\alpha><\beta>}(u) + 2R_{<\tau><\alpha><\beta><\nu>}(u) \ h^{<\tau><\nu>}(u) = 0$$

and d–gauge conditions

$$D_{<\alpha>} h_{<\alpha><\beta>}(u) = 0, \quad h(u) \equiv h_{<\alpha><\beta>}(u) = 0,$$

where $R_{<\tau><\alpha><\beta><\nu>}(u)$ is curvature d-tensor of the locally anisotropic background space (these formulae can be obtained by using a perturbation formalism with respect to $h_{<\alpha><\beta>}(u)$ developed in [20]; in our case we must take into account the distinguishing of geometrical objects and operators on higher order anisotropic spaces).

Finally, we remark that all presented geometric constructions contain those elaborated for generalized Lagrange spaces [36, 37] (for which a tangent bundle $TM$ is considered instead of a $v$-bundle $E^{<z>}$) and for constructions on the so called osculator bundles with different prolongations and extensions of Finsler and Lagrange metrics [38]. We also note that the higher order Lagrange (Finsler) geometry is characterized by a metric of type (dmetrichcv) with components parametrized as $g_{ij} = 1/2 \partial^2 L / \partial y^i \partial y^j \ (g_{ij} = 1/2 \partial^2 \Lambda^2 / \partial y^i \partial y^j)$ and $h_{\alpha\beta\rho\sigma} = g_{ij}$, where $L = L(x, y(1), y(2), \ldots, y(z)) \ (\Lambda = \Lambda(x, y(1), y(2), \ldots, y(z)))$ is a Lagrangian (Finsler metric) on $TM^{(z)}$ (see details in [36, 37, 20, 8]).
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