Derived equivalences of self-injective 2-Calabi–Yau tilted algebras

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Abstract
Consider a $k$-linear Frobenius category $ℰ$ such that the corresponding stable category $ℂ$ is 2-Calabi–Yau, Hom-finite with split idempotents. Let $l, m ∈ ℂ$ be maximal rigid objects with self-injective endomorphism algebras. We will show that their endomorphism algebras $ℂ(l, l)$ and $ℂ(m, m)$ are derived equivalent. Furthermore, we will give a description of the two-sided tilting complex that induces this derived equivalence.

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1 | INTRODUCTION

In [3], August showed that given two objects $M, N ∈ CM(R)$ with Spec $R$ being a complete local isolated cDV singularity, such that $M$ and $N$ are maximal rigid objects connected through a number of mutations, the contraction algebras $\text{End}(M)$ and $\text{End}(N)$ are derived equivalent. Note that in this setting, $\text{End}(M)$ and $\text{End}(N)$ are symmetric algebras.

In this paper, we generalize the result mentioned above to the setting of a more general Frobenius category than $CM(R)$. Our general course of action and a number of the proofs will be based on those in [3]. We shall use a result from [23] that will give a conflations that is able to replace the exchange sequences you would get from mutations. This will allow us to prove that $\text{End}(M)$ and $\text{End}(N)$ are derived equivalent if they are self-injective and $M$ and $N$ are maximal rigid objects. This will therefore lead to significantly more general results, allowing for categories without the condition $Σ^2 ≅ \text{id}$ and categories that may have infinitely many maximal-rigid objects.

Two-sided tilting complex. The definition of a tilting complex was introduced by Rickard in [20]. He showed that given two derived equivalent algebras, there exists a tilting complex inducing such an equivalence.

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Let $k$ be an algebraically closed field, $A$ a $k$-algebra. Let $\text{proj} A$ denote the category of f.g. projective left $A$-modules, $K^b(\text{proj} A)$ its bounded homotopy category.

**Definition.** A complex $T \in K^b(\text{proj} A)$ is called a tilting complex if the following are satisfied.

- $\text{Hom}(T, T[i]) = 0$ for all $i \neq 0$.
- $\text{thick}(T) = K^b(\text{proj} A)$, where $\text{thick}(T)$ indicates the thick closure of $\text{add}(T)$.

The notion of a two-sided tilting complex is also due to Rickard [19]. We use the following form of the definition due to Keller [15, 8.1.4]. Let unadorned tensor products be over $k$ and let $D$ denote the derived category.

**Definition 1.1.** Let $B_T A \in D(B \otimes A^{\text{op}})$ be a complex of $(B, A)$-bimodules. Let $B_T$ (resp. $T_A$) be $B_T A$ seen as a complex in $D(B)$ (resp. $D(A^{\text{op}})$). $B_T A$ is a two-sided tilting complex if the following are satisfied.

1. The canonical map $B \to \text{Hom}_{D(A^{\text{op}})}(T_A, T_A)$ is bijective and $\text{Hom}_{D(A^{\text{op}})}(T_A, T_A[i]) = 0$ for $i \neq 0$.
2. $T_A$ is quasi-isomorphic to a complex in $K^b(\text{proj} A^{\text{op}})$.
3. $\text{thick}(T_A) = K^b(\text{proj} A^{\text{op}})$.

Let $\mathcal{E}$ be a $k$-linear Frobenius category. Then, given two maximal rigid objects $l, m \in \mathcal{E}$ with suitable projective summands, the complex $T = \mathcal{E}(l, m)$ is a two-sided tilting complex in $D(\mathcal{E}(m, m) \otimes \mathcal{E}(l, l)^{\text{op}})$ (see [14, Prop 5.1]), making $A = \mathcal{E}(l, l)$ and $B = \mathcal{E}(m, m)$ derived equivalent. Looking at the stable category $\mathcal{C} = \mathcal{E}$, a similar choice of the module $\mathcal{E}(l, m)$ does not necessarily give a tilting module, and it is not necessarily true that $A = \mathcal{E}(l, l)$ and $B = \mathcal{E}(m, m)$ are derived equivalent. However, we are able to prove the following main result.

**Theorem A** (Corollary 3.11). Let $\mathcal{E}$ be $k$-linear Frobenius category such that the associated stable category $\mathcal{C} = \mathcal{E}$ is 2-CY and Hom-finite with split idempotents.

Let $l, m \in \mathcal{E}$ be maximal rigid objects. Denote $A = \mathcal{E}(l, l)$ and $B = \mathcal{E}(m, m)$. Assume that $A$ and $B$ are self-injective. Then, they are derived equivalent. Furthermore, there exist objects $l', m' \in \mathcal{E}$ with $l \cong l'$ and $m \cong m'$, such that by denoting $A = \mathcal{E}(l', l')$, $B = \mathcal{E}(m', m')$, and $T = \mathcal{E}(l', m')$, there is a two-sided tilting complex of $B \otimes A^{\text{op}}$-modules

$$\mathcal{T}_A = \left( \begin{array}{ccc} \mathcal{E}(l') & \mathcal{E}(l') \\ \mathcal{E}(m') & \mathcal{E}(m') \\ \end{array} \right) \subseteq 1,$$

inducing this derived equivalence. The subscript $\subseteq 1$ denotes the soft truncation to homological degrees $\leq 1$.

In the last section, we will focus on two examples. The first example is on cluster-tilting objects in the cluster category $C(D_{2n})$. In [21], Ringel has listed the cluster-tilting objects of $C(D_{2n})$ with self-injective endomorphism algebras. It has already been shown in [4, Lem. 4.5] that these endomorphism algebras are derived equivalent. This result was achieved by supplying a tilting complex ad hoc. We will use our result to recover this tilting complex by manually calculating the tilting complex $\mathcal{T}_A$. 
The second example will be on a class of examples based on Postnikov diagrams. We will use the work of Pasquali [18], which describes how reduced and symmetric \((k, n)\)-Postnikov diagrams give rise to cluster tilting objects with self-injective endomorphism algebras. This will lead to the following result.

**Corollary B** (Corollary 4.4). Let \(k, n \in \mathbb{N}\), with \(k < n\). Let \(\hat{B}\) be the completion of the so-called boundary algebra (see Section 4). Let \(D, D'\) be two symmetric and reduced \((k, n)\)-Postnikov diagrams, with associated cluster tilting objects \(T, T'\) (resp.) in \(\text{CM}(\hat{B})\), the stable category of Cohen–Macaulay modules. Then, the self-injective algebras \(\text{End}(T)\) and \(\text{End}(T')\) are derived equivalent.

In the setting of Frobenius categories, derived equivalences between endomorphism algebras have previously been studied by several authors. Assume that \(\mathcal{E}\) is a sufficiently nice Frobenius category with stable category \(\mathcal{C}\) and choose two cluster tilting objects in \(\mathcal{E}\). In a 2007 article, Iyama showed that if you look at these objects as objects in \(\mathcal{E}\), then they have derived equivalent endomorphism algebras (see [11, Cor. 5.3.3]). Furthermore, Iyama constructs a two-sided tilting complex that induces the derived equivalence. This result was extended by Palu to a more general Frobenius category (see [17, Prop. 4]). Both of these results concern derived equivalences of endomorphism algebras in the Frobenius category. A natural question to ask is whether this extends to derived equivalences of endomorphism algebras in the stable category. Although this is not always true, under certain conditions it will be. Dugas showed that if we look at the Frobenius category \(\text{CM}(R)\) for some odd-dimensional Gorenstein hypersurface \(R\) that is an isolated singularity, the endomorphism algebras of cluster-tilting objects in the stable category are derived equivalent (see [6, Cor. 5.5]). This was done by providing a one-sided tilting complex. If we instead have a complete local isolated cDV singularity \(R\) and two cluster-tilting objects linked by a path of mutations, a result by August provides us with a two-term tilting complex between the two stable endomorphism algebras (see [3, Thm. 3.2, Cor 3.3]). This result was shown to hold for maximal rigid objects, a generalization of cluster-tilting objects. The two latter articles consider Frobenius categories whose stable categories satisfy \(\Sigma^2 \cong \text{id}\). This article aims to generalize these results to a more general Frobenius category and to replace the global assumption of \(\Sigma^2 \cong \text{id}\) with local assumptions on the maximal rigid objects considered. Further, we do not assume that there exists a path of mutations between the objects.

# Preliminaries

In this section, the following setup will be assumed.

**Setup 2.1.** Let \(k\) be an algebraically closed field. Let \(\mathcal{E}\) be a \(k\)-linear Frobenius category. Let \(\mathcal{C} := \mathcal{E}\) be the associated stable category. We will assume that \(\mathcal{C}\) is a 2-Calabi–Yau, Hom-finite category with split idempotents. Observe that \(\mathcal{C}\) has the same objects as \(\mathcal{E}\) but different morphisms.

It is well known that \(\mathcal{C}\) is a triangulated category, whose suspension functor will be denoted as \(\Sigma\).

**Definition 2.2.**
- \(x \in \mathcal{E}\) is called **rigid** if \(\mathcal{C}(x, \Sigma x) = 0\).
- \(x \in \mathcal{E}\) is called **maximal rigid** if it is rigid, and \(\mathcal{C}(x \oplus y, \Sigma (x \oplus y)) = 0\) implies \(y \in \text{add}_e(x)\).
The following result is due to Zhou and Zhu [23]. It generalizes a similar result from [10], which then can be applied to our setup. We will use it to construct conflations that will then connect maximal rigid objects in a way that can “replace” exchange sequences of mutations.

**Theorem 2.3** [23, Cor. 2.5]. Let \( x \in \mathcal{C} \) be maximal rigid, and let \( y \in \mathcal{C} \) be rigid, then \( y \in \text{add}_\mathcal{C}(x) \ast \text{add}_\mathcal{C}(\Sigma x) \), that is, there exists a triangle

\[
x_1 \rightarrow x_0 \rightarrow y \rightarrow \Sigma x_1,
\]

with \( x_i \in \text{add}_\mathcal{C}(x) \).

The following lemma is a collection of useful results when working in the context of Setup 2.1. See also [14, Lem. A.1].

**Lemma 2.4.** Let \( x, y, z \in \mathcal{E} \). Define \( A := \mathcal{E}(x, x) \) and \( A_\ast := \mathcal{E}(x, x) \).

(a) If \( x \cong 0 \) in \( \mathcal{C} \), then \( x \) is a projective object in \( \mathcal{E} \).
(b) For each triangle \( x \rightarrow y \rightarrow z \rightarrow \Sigma x \) in \( \mathcal{C} \), there is a conflation \( 0 \rightarrow x \rightarrow y' \rightarrow z' \rightarrow 0 \) in \( \mathcal{C} \) such that \( y \cong y' \) and \( z \cong z' \) in \( \mathcal{C} \).
(c) \( x \cong y \) in \( \mathcal{C} \) if and only if there exist projective objects \( p, p' \in \mathcal{E} \) such that \( x \oplus p \cong y \oplus p' \) in \( \mathcal{E} \).
(d) If \( x \in \text{add}_\mathcal{C}(y) \) or \( z \in \text{add}_\mathcal{C}(y) \), then composition of morphisms induces a \( k \)-linear bijection \( \mathcal{C}(y, z) \otimes \mathcal{E}(x, y) \rightarrow \mathcal{C}(x, z) \) that is natural in \( x, z \), where \( B = \mathcal{C}(y, y) \).
(e) If \( \bar{x} \in \text{add}_\mathcal{C}(x) \), then composition of morphisms induces a \( k \)-linear bijection \( \mathcal{C}(x, y) \otimes A \rightarrow \mathcal{C}(\bar{x}, y) \) that is natural in \( \bar{x}, y \).
(f) If \( y \in \text{add}_\mathcal{C}(x) \), then the canonical map \( \mathcal{C}(x, -) : \mathcal{C}(y, z) \rightarrow \text{Hom}_A(\mathcal{C}(x, y), \mathcal{C}(x, z)) \) is a bijection.
(g) If \( x \) is maximal rigid and \( y, z \) are rigid, then the map \( \mathcal{C}(y, z) \rightarrow \text{Hom}_A(\mathcal{C}(x, y), \mathcal{C}(x, z)) \) is surjective.

**Proof.** (a)–(c) are standard results, and are straightforward to prove.

(d)–(f) are also standard results, but they are a bit less straightforward. (d) can be proved by proving it for \( x = y^m \) and then using that any \( x \in \text{add}_\mathcal{C}(y) \) is a direct summand of \( y^n \) for some \( n \in \mathbb{N} \). Similar methods can be used for (e) and (f).

(g) Since \( x \) is maximal rigid, and \( y, z \) are rigid, Theorem 2.3 says that there exist triangles

\[
x_1^y \rightarrow x_0^y \rightarrow y \rightarrow \Sigma x_1^y \quad \text{and} \quad x_1^z \rightarrow x_0^z \rightarrow z \rightarrow \Sigma x_1^z,
\]

with \( x_1^y, x_1^z \in \text{add}_\mathcal{C}(x) \). Since \( x \) is rigid, this induces two exact sequences

\[
\mathcal{C}(x, x_1^y) \rightarrow \mathcal{C}(x, x_0^y) \rightarrow \mathcal{C}(x, y) \rightarrow 0 \quad \text{and} \quad \mathcal{C}(x, x_1^z) \rightarrow \mathcal{C}(x, x_0^z) \rightarrow \mathcal{C}(x, z) \rightarrow 0.
\]

Now let \( f \in \text{Hom}_A(\mathcal{C}(x, y), \mathcal{C}(x, z)) \). Since \( x_1^y, x_1^z \in \text{add}_\mathcal{C}(x) \), (f) says that there exist morphisms \( g \in \mathcal{E}(x_1^y, x_1^z) \) making the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{C}(x, x_1^y) & \rightarrow & \mathcal{C}(x, x_0^y) \\
\downarrow g_{x_1^y} & & \downarrow g_{x_0^y} \\
\mathcal{C}(x, x_1^z) & \rightarrow & \mathcal{C}(x, x_0^z) \\
\end{array}
\begin{array}{ccc}
\mathcal{C}(x, y) & \rightarrow & 0 \\
\downarrow f & & \\
\mathcal{C}(x, z) & \rightarrow & 0.
\end{array}
\]
By the axioms of triangulated categories, there exists a morphism \( g \in \mathcal{C}(y, z) \), making \((g_1, g_0, g)\) a morphism of triangles

\[
\begin{align*}
x_1^y & \longrightarrow x_0^y \quad \alpha \quad y \quad \longrightarrow \quad \Sigma x_1^y \\
\downarrow g_1 & \quad \downarrow g_0 \quad \downarrow g \quad \downarrow \Sigma g_1 \\
x_1^z & \longrightarrow x_0^z \quad \beta \quad z \quad \longrightarrow \quad \Sigma x_1^z.
\end{align*}
\]

Thus, we get a diagram

\[
\begin{align*}
\mathcal{C}(x, x_1^y) & \longrightarrow \mathcal{C}(x, x_0^y) \quad \alpha_\ast \longrightarrow \quad \mathcal{C}(x, y) \quad \longrightarrow \quad 0 \\
\downarrow g_1_\ast & \quad \downarrow g_0_\ast \quad \downarrow f \quad \downarrow g_\ast \\
\mathcal{C}(x, x_1^z) & \longrightarrow \mathcal{C}(x, x_0^z) \quad \beta_\ast \longrightarrow \quad \mathcal{C}(x, z) \quad \longrightarrow \quad 0,
\end{align*}
\]

with \( f \alpha_\ast = \beta_\ast g_0_\ast = g_\ast \alpha_\ast \). Since \( \alpha_\ast \) is an epimorphism, this implies that \( f = g_\ast \). \(\square\)

Let \( D(\cdot) = \text{Hom}_k(\cdot, k) \) denote \( k \)-duality.

**Lemma 2.5.** Let \( x \in \mathcal{C} \) be a maximal rigid object, then \( \Sigma^2 x \cong x \) if and only if \( \mathcal{C}(x, x) \) is a self-injective algebra.

**Proof.** Let \( x' \in \mathcal{C} \), and assume that \( \Sigma^2 x \cong x \). Since \( \mathcal{C} \) is 2-Calabi–Yau, there are isomorphisms

\[
D\mathcal{C}(x', x) \cong D\mathcal{C}(x', \Sigma^2 x) \cong \mathcal{C}(x, x'),
\]

which are functorial in \( x' \). This gives an isomorphism \( D\mathcal{C}(x, x) \cong \mathcal{C}(x, x) \) of left \( \mathcal{C}(x, x) \)-modules making \( \mathcal{C}(x, x) \) self-injective. For the opposite implication, one can do an argument similar to that of [12, Prop. 3.6]. \(\square\)

Let \( \Omega \) denote the syzygy, \( \Omega^{-1} \) the cosyzygy in \( \mathcal{B} \). With this, the following is a standard result.

**Lemma 2.6.** Let \( x, y \in \mathcal{B} \), then \( \text{Ext}^1_\mathcal{C}(x, \Omega y) \cong \mathcal{C}(x, y) \cong \text{Ext}^1_\mathcal{C}(\Omega^{-1} x, y) \).

**Condition 2.7.** Let \( x \in \mathcal{C} \) be rigid such that \( \Sigma^2 x \cong x \). Then, \( x \) is said to satisfy **Condition 2.7** with projective objects \( q_1, q_2 \) if there are conflations in \( \mathcal{C} \):

\[
\begin{align*}
0 & \longrightarrow \Omega x \quad g \quad q_1 \quad f \quad x \quad \longrightarrow \quad 0, \\
0 & \longrightarrow x \quad g' \quad q_2 \quad f' \quad \Omega x \quad \longrightarrow \quad 0
\end{align*}
\]

with \( q_i \in \text{add}_\mathcal{C}(x) \) being projective objects.

**Lemma 2.8.** Let \( x \in \mathcal{C} \) be rigid such that \( \Sigma^2 x \cong x \), and denote \( A = \mathcal{C}(x, x) \). Assume that \( x \) satisfies **Condition 2.7** with projective objects \( q_1, q_2 \in \text{add}_\mathcal{C}(x) \), then the corresponding conflations induce augmented projective resolutions:
Here, the first sequence is a projective resolution of right $A$-modules, and the second sequence is a projective resolution of left $A$-modules.

**Proof.** Since $x$ satisfies Condition 2.7 with projective objects $q_1, q_2$, there are conflations

$$0 \to \Omega x \xrightarrow{g} q_1 \xrightarrow{f} x \to 0,$$

$$0 \to x \xrightarrow{f'} q_2 \xrightarrow{f'} \Omega x \to 0.$$

Hence, there is an exact sequence spliced from these two conflations.

$$0 \to x \xrightarrow{f'} q_2 \xrightarrow{f} q_1 \xrightarrow{f} x \to 0.$$

Applying the functor $\mathcal{E}(x, -)$ gives the exact sequence.

$$0 \to \mathcal{E}(x, x) \xrightarrow{g'} \mathcal{E}(x, q_2) \xrightarrow{f} \mathcal{E}(x, q_1) \xrightarrow{f} \mathcal{E}(x, x).$$

That $f'_s$ is surjective follows directly from $x$ being rigid and therefore $\text{Ext}^1(x, x) = 0$, see Lemma 2.6. Since $\text{Ext}^1(x, q_1) = 0$, it follows directly from Lemma 2.6 that this is a projective resolution of $\mathcal{E}(x, x)$ over right $A$-modules. This shows that we have the resolution from (2). The method for finding the resolution from (3) is similar. \qed

**Lemma 2.9.** Let $x \in \mathcal{C}$ be rigid such that $\Sigma^2 x \cong x$ and assume that it satisfies Condition 2.7 with projective objects $q_1, q_2 \in \text{add}_{\mathcal{E}}(x)$. Let $p \in \mathcal{E}$ be a projective object. Then, $x \oplus p$ satisfies Condition 2.7 with projective objects $q_1 \oplus p, q_2 \oplus p$.

**Proof.** Consider the conflations.

$$0 \to 0 \to p \xrightarrow{=} p \to 0,$$

$$0 \to p \xrightarrow{=} p \to 0 \to 0.$$

By adding these to those in (1), one obtains the needed criteria to satisfy Condition 2.7. \qed

**Lemma 2.10.** Let $x \in \mathcal{C}$ be rigid, such that $\Sigma^2 x \cong x$. Then there exists a projective object $p \in \mathcal{E}$ such that $x' = x \oplus p$ satisfies Condition 2.7 with projective objects $q_1, q_2 \in \text{add}_{\mathcal{E}}(p)$.

**Proof.** Let $0 \to \Omega x \to p_1 \to x \to 0$ and $0 \to \Omega^2 x \to p_2' \to \Omega x \to 0$ be conflations in $\mathcal{E}$ where $p_1, p_2'$ are projective objects. Since $\Omega^2 x \cong \Sigma^{-2} x \cong x$ in $\mathcal{C}$, Lemma 2.4(c) says that there are
projective objects $p_3, q \in \mathcal{E}$ such that $\Omega^2 x \oplus q \cong x \oplus p_3$. Hence, there are two exact sequences

$$0 \longrightarrow \Omega x \overset{\varepsilon}{\longrightarrow} p_1 \overset{f}{\longrightarrow} x \longrightarrow 0,$$

$$0 \longrightarrow x \oplus p_3 \overset{\varepsilon'}{\longrightarrow} p_2 \overset{f'}{\longrightarrow} \Omega x \longrightarrow 0,$$

where $p_2 = p'_2 \oplus q$. By adding some ‘trivial’ conflations, we obtain two different conflations.

$$0 \longrightarrow \Omega x \overset{(\varepsilon, 0)}{\longrightarrow} p_1 \oplus (p_1 \oplus p_2 \oplus p_3) \overset{(f, 0)}{\longrightarrow} x \oplus (p_1 \oplus p_2 \oplus p_3) \longrightarrow 0,$$

$$0 \longrightarrow x \oplus p_3 \oplus (p_1 \oplus p_2) \overset{(\varepsilon', \text{id})}{\longrightarrow} p_2 \oplus (p_1 \oplus p_2) \overset{(f', \text{id})}{\longrightarrow} \Omega x \longrightarrow 0.$$

Let $x' = x \oplus p_1 \oplus p_2 \oplus p_3$, $q_1 = p_1 \oplus (p_1 \oplus p_2 \oplus p_3)$, $q_2 = p_2 \oplus (p_1 \oplus p_2)$. Using this, the two conflations above can be written as follows:

$$0 \longrightarrow \Omega x' \longrightarrow q_1 \longrightarrow x' \longrightarrow 0,$$

$$0 \longrightarrow x' \longrightarrow q_2 \longrightarrow \Omega x' \longrightarrow 0. \quad (4)$$

This implies that $x'$ satisfies Condition 2.7 with projective objects $q_1, q_2 \in \text{add}_\mathcal{E}(p)$ for $p = p_1 \oplus p_2 \oplus p_3$.

**Lemma 2.11.** Let $x \in \mathcal{C}$ be rigid such that $\Sigma^2 x \cong x$ and assume that it satisfies Condition 2.7. Let $y \in \mathcal{C}$. Define $A := \mathcal{C}(x, x)$, and $A := \mathcal{C}(x, x)$. Then $\text{Tor}_2^A(\mathcal{C}(x, y), A_A) \cong 0$.

**Proof.** By Lemma 2.8, there is an augmented projective resolution

$$0 \longrightarrow \mathcal{E}(x, x) \longrightarrow \mathcal{E}(p_1, x) \longrightarrow \mathcal{E}(p_2, x) \longrightarrow \mathcal{E}(x, x) \longrightarrow 0,$$

of $A_A$ over left $A$-modules, with $p_1 \in \text{add}_\mathcal{E}(x)$ being projective objects. Using Lemma 2.4(c) gives that

$$\mathcal{C}(x, y) \otimes_\mathcal{A} A \cong \mathcal{C}(x, y) \otimes_\mathcal{A} \left( 0 \longrightarrow \mathcal{E}(x, x) \longrightarrow \mathcal{E}(p_1, x) \longrightarrow \mathcal{E}(p_2, x) \longrightarrow \mathcal{E}(x, x) \longrightarrow 0 \right)$$

$$\cong 0 \longrightarrow \mathcal{E}(x, y) \longrightarrow \mathcal{E}(p_1, y) \longrightarrow \mathcal{E}(p_2, y) \longrightarrow \mathcal{E}(x, y) \longrightarrow 0$$

$$\cong 0 \longrightarrow \mathcal{E}(x, y) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{E}(x, y) \longrightarrow 0.$$

Hence, $\text{Tor}_2^A(\mathcal{C}(x, y), A_A) \cong 0$. \hfill \Box

**Lemma 2.12.** Let $x \in \mathcal{C}$ be rigid such that $\Sigma^2 x \cong x$ and such that it satisfies Condition 2.7. Define $A := \mathcal{C}(x, x)$, and $A := \mathcal{C}(x, x)$. Let $a$ be the ideal in $A$ of morphisms factoring through a projective object. Let $h : M \to Q$ be a morphism of right $A$-modules with $Q$ being a projective object, such that $\text{Coker} h \cong \mathcal{C}(x, y)$ for some $y \in \mathcal{C}$. Then, $Ma \cap \text{Ker} h = (\text{Ker} h)a$.

**Proof.** Using the Tor vanishing of Lemma 2.11, this can be proved by the same argument as [3, Lem. 3.7]. \hfill \Box
**Definition 2.13.** Let $x, y \in \mathcal{C}$ be maximal rigid such that $\Sigma^2 x \cong x, \Sigma^2 y \cong y$ and assume that $x, y$ satisfy Condition 2.7 with projective objects $p_1^x, p_2^x \in \mathcal{E}$ and $p_1^y, p_2^y \in \mathcal{E}$, respectively, then we say that $(x, y)$ is a compatible pair if

1. $p_1^x, p_2^y \in \text{add}_\mathcal{E}(x) \cap \text{add}_\mathcal{E}(y)$.
2. There are conflations

$$0 \rightarrow x_1 \rightarrow x_0 \rightarrow y \rightarrow 0 \quad \text{and} \quad 0 \rightarrow x \rightarrow y_0 \rightarrow y_1 \rightarrow 0,$$

where $x_i \in \text{add}_\mathcal{E}(x)$ and $y_i \in \text{add}_\mathcal{E}(y)$.

**Remark 2.14.** Given $x, y \in \mathcal{C}$ such that $(x, y)$ is a compatible pair, notice that this does not necessarily mean that $(y, x)$ is a compatible pair.

**Lemma 2.15.** Let $x', y' \in \mathcal{C}$ be maximal rigid objects such that $\Sigma^2 x' \cong x'$ and $\Sigma^2 y' \cong y'$. Then, there exist objects $x, y \in \mathcal{C}$ with $x \cong x'$ and $y \cong y'$ such that $(x, y)$ is a compatible pair.

**Proof.** By Lemma 2.10, there exist projective objects $P_{x'}, P_{y'} \in \mathcal{E}$ such that $x' \oplus P_{x'}$ and $y' \oplus P_{y'}$ satisfy Condition 2.7 with projective objects in $\text{add}_\mathcal{E}(P_{x'})$ and $\text{add}_\mathcal{E}(P_{y'})$, respectively. Let $\tilde{x} = x' \oplus P_{x'} \oplus P_{y'}$ and $\tilde{y} = y' \oplus P_{x'} \oplus P_{y'}$. By Lemma 2.9, $\tilde{x}$ and $\tilde{y}$ satisfy Condition 2.7 with projective objects in $\text{add}_\mathcal{E}(P_{x'} \oplus P_{y'})$, and thereby satisfy condition (1) of Definition 2.13.

By Theorem 2.3, there exist triangles

$$\tilde{x}_1 \rightarrow \tilde{x}_0 \rightarrow \tilde{y} \rightarrow \Sigma \tilde{x}_1$$

$$\tilde{y}_0 \rightarrow \tilde{y}_1 \rightarrow \Sigma \tilde{x} \rightarrow \Sigma \tilde{y}_0,$$

with $\tilde{x}_1 \in \text{add}_\mathcal{E}(\tilde{x})$ and $\tilde{y}_1 \in \text{add}_\mathcal{E}(\tilde{y})$. By rotating the latter triangle, we get that there also is a triangle

$$\tilde{x} \rightarrow \tilde{y}_0 \rightarrow \tilde{y}_1 \rightarrow \Sigma \tilde{x}.$$ 

By Lemma 2.4(b), there exist conflations

$$0 \rightarrow \tilde{x}_1' \rightarrow \tilde{x}_0' \rightarrow \tilde{y} \rightarrow 0$$

$$0 \rightarrow \tilde{x} \rightarrow \tilde{y}_0' \rightarrow \tilde{y}_1' \rightarrow 0,$$

with $\tilde{x}_1', \tilde{y}_1' \in \mathcal{E}$ such that $\tilde{x}_1 \cong \tilde{x}_1'$ and $\tilde{y}_1 \cong \tilde{y}_1'$ in $\mathcal{E}$. Thus, there exist projective objects $p_1^{\tilde{x}}, p_1^{\tilde{y}}, p_1^{\tilde{y}} \in \mathcal{E}$ such that

$$\tilde{x}_1 \oplus p_1^{\tilde{x}} \cong \tilde{x}_1' \oplus p_1^{\tilde{x}}' \quad \text{and} \quad \tilde{y}_1 \oplus p_1^{\tilde{y}} \cong \tilde{y}_1' \oplus p_1^{\tilde{y}}' \quad (5)$$

in $\mathcal{E}$, see Lemma 2.4(c).

Since $\tilde{x}_1 \in \text{add}_\mathcal{E}(\tilde{x})$, there exist $n \in \mathbb{N}$ and $t \in \mathcal{C}$ such that $\tilde{x}_1 \oplus t \cong \tilde{x}_1^n$. Therefore, by Lemma 2.4(c), there exist projective objects $q_i, r_i \in \mathcal{E}$ such that $\tilde{x}_1 \oplus t \oplus q_i \cong \tilde{x}_1^n \oplus r_i$ in $\mathcal{E}$. This means that $\tilde{x}_1 \in \text{add}_\mathcal{E}(\tilde{x} \oplus r_i)$, and by (5), $\tilde{x}_1' \in \text{add}_\mathcal{E}(\tilde{x} \oplus r_i \oplus p_1^{\tilde{x}})$. Let $Q_i^{\tilde{x}} = r_i \oplus p_1^{\tilde{x}}$, and similarly, we can find projective objects $Q_i^{\tilde{y}} \in \mathcal{E}$ such that $\tilde{y}_1' \in \text{add}_\mathcal{E}(\tilde{y} \oplus Q_i^{\tilde{y}})$. 
Let \( P = Q^x_0 \oplus Q^x_1 \oplus Q^y_0 \oplus Q^y_1 \), \( x = \tilde{x} \oplus P \), and \( y = \tilde{y} \oplus P \). By Lemma 2.9, \( x, y \) satisfy Condition 2.7 with projective objects in \( \text{add}_\mathcal{E}(P^{x'}_0 \oplus P^{y'}_0 \oplus P) \), and therefore, satisfy condition (1) of Definition 2.13. By adding “trivial” conflations to the ones above, we get following conflations:

\[
0 \longrightarrow \tilde{x}'_1 \longrightarrow \tilde{x}'_0 \oplus P \longrightarrow y \longrightarrow 0,
\]

\[
0 \longrightarrow x \longrightarrow \tilde{y}'_0 \oplus P \longrightarrow \tilde{y}'_1 \longrightarrow 0.
\]

Letting \( x_1 = \tilde{x}'_1 \), \( x_0 = \tilde{x}'_0 \oplus P \), \( y_1 = \tilde{y}'_1 \), \( y_0 = \tilde{y}'_0 \oplus P \), we get that \((x, y)\) is a compatible pair. \(\square\)

### 3. Derived Equivalences

In this section, Setup 2.1 together with the following setup will be assumed.

**Setup 3.1.** Let \( l, m \in \mathcal{C} \) be maximal rigid objects, such that \( \Sigma^2 l \cong l \) and \( \Sigma^2 m \cong m \) in \( \mathcal{C} \). Without loss of generality, by Lemma 2.15, we may assume that \((l, m)\) is a compatible pair. Let \( A = \mathcal{E}(l, l) \), \( B = \mathcal{E}(m, m) \), and \( B \otimes_A \text{T}_A = \mathcal{E}(l, m) \). The following construction of a two-sided tilting complex is inspired by [16, p. 5123] and [3, Thm. 1.1].

\[
B \otimes_A \text{T}_A = \left( \begin{array}{c} L \\ B \otimes_T \otimes A \\ \subseteq \mathcal{C} \end{array} \right)_{\leq 1},
\]

where \( \subseteq \mathcal{C} \) refers to taking a soft truncation, keeping the homological degrees \( \leq 1 \).

The main goal of this section is to show that \( B \otimes_A \text{T}_A \) is a 2-sided tilting complex, making \( A \) and \( B \) derived equivalent (Corollary 3.11).

**Lemma 3.2** [14, Prop 5.1]. \( B \otimes_A \text{T}_A \) is a two-sided tilting complex viewed as a \( B \otimes \text{A}^{\text{op}} \)-complex.

**Proof.** Given that \((l, m)\) is a compatible pair, the proof is similar to that of [14, Prop 5.1]. \(\square\)

Since \((l, m)\) is a compatible pair, there are conflations in \( \mathcal{E} \):

\[
0 \longrightarrow \Omega m \overset{g}{\longrightarrow} p_1 \overset{f}{\longrightarrow} m \longrightarrow 0,
\]

\[
0 \longrightarrow m \overset{g'}{\longrightarrow} p_2 \overset{f'}{\longrightarrow} \Omega m \longrightarrow 0
\]

where \( p_i \in \text{add}_{\mathcal{E}(m)} \cap \text{add}_{\mathcal{E}(l)} \) are projective objects, giving a projective resolution \( Q_B : 0 \longrightarrow \mathcal{E}(m, m) \longrightarrow \mathcal{E}(m, p_2) \longrightarrow \mathcal{E}(m, p_1) \longrightarrow \mathcal{E}(m, m) \) (7) of \( B_B \) (see Lemma 2.8).

**Lemma 3.3.** In \( D(A^{\text{op}}) \), the object \( B \otimes_B \text{T}_A \) is quasi-isomorphic to the complex

\[
0 \longrightarrow \mathcal{E}(l, m) \overset{g'}{\longrightarrow} \mathcal{E}(l, p_2) \overset{(gf')}{\longrightarrow} \mathcal{E}(l, p_1) \overset{f'}{\longrightarrow} \mathcal{E}(l, m),
\]

(8)
with homology

\[ H_1(B \otimes T_A) = \begin{cases} 
    \mathcal{E}(l, m) & i = 0, \\
    \mathcal{E}(l, \Omega m) & i = 1, \\
    0 & \text{otherwise.}
\end{cases} \]

**Proof.** Using the projective resolution \( Q_B \) of \( B_B \) from (7), calculate

\[
B_B \otimes T_A \cong Q_B \otimes T_A \cong 0 \to \mathcal{E}(m, m) \otimes T_A \to \mathcal{E}(m, p_2) \otimes T_A \to \mathcal{E}(m, p_1) \otimes T_A \to \mathcal{E}(m, m) \otimes T_A
\]

\[
\cong 0 \to \mathcal{E}(l, m) \to \mathcal{E}(l, p_2) \to \mathcal{E}(l, p_1) \to \mathcal{E}(l, m),
\]

where the last isomorphism follows from Lemma 2.4(d).

This complex can also be seen as the result of applying the functor \( \mathcal{E}(l, -) \) to the concatenation of the conflations from (6):

\[
0 \longrightarrow \mathcal{E}(l, m) \xrightarrow{\ell'} \mathcal{E}(l, p_2) \xrightarrow{(gf')} \mathcal{E}(l, p_1) \xrightarrow{\ell} \mathcal{E}(l, m).
\]

Thus, \( H_i(B \otimes L_B T_A) = 0 \) for \( i \neq 0, 1 \). Since \( \text{Ext}^1_\mathcal{E}(l, p_1) \cong 0 \), it follows from Lemma 2.6 together with the long exact Ext sequence of the conflation

\[
0 \longrightarrow \Omega m \longrightarrow p_1 \longrightarrow m \longrightarrow 0,
\]

that \( H_0(B \otimes L_B T_A) \cong \mathcal{E}(l, m) \).

To calculate \( H_1 \), notice that \( \text{Ker}(f_*) \cong \mathcal{E}(l, \Omega m) \), and \( \text{Im}((gf')_*) \cong \text{Im}(f'_*) \), due to \( g_* \) being injective. Therefore, \( H_1(B \otimes L_B T_A) \cong \mathcal{E}(l, \Omega m) / \text{Im}(f'_*) \). Using that \( \text{Ext}^1_\mathcal{E}(l, p_2) \cong 0 \), the long exact Ext sequence of the conflation

\[
0 \longrightarrow m \longrightarrow p_2 \longrightarrow \Omega m \longrightarrow 0
\]

gives that \( H_1(B \otimes L_B T_A) \cong \text{Ext}^1_\mathcal{E}(l, m) \cong \mathcal{E}(l, \Omega^{-1} m) \), with the last isomorphism coming from Lemma 2.6. However, \( \Sigma^2 m \cong m \) means \( \Omega^{-1} m \cong m \), so \( \Omega^{-1} m \cong \Omega m \).

**Corollary 3.4.** There is a quasi-isomorphism in \( D(A^{op}) \) from \( B \otimes L_B T_A \) to the complex

\[
\mathcal{E}(l, p_1) / \text{Im}((gf')_*) \xrightarrow{f_*} \mathcal{E}(l, m). \tag{9}
\]

**Proof.** This follows directly from Lemma 3.3 by using soft truncation. □

The next goal is to obtain an alternative description of \( B \otimes L_B T_A \). Since \( l, m \) are maximal rigid and \( (l, m) \) is a compatible pair, there exists a conflation

\[
0 \longrightarrow l_1 \xrightarrow{\delta_1} l_0 \xrightarrow{\delta_2} m \longrightarrow 0,
\]
where $l_i \in \text{add}_E(l)$. For the rest of this section, denote by $P_A$ the complex of $A^{op}$-modules

$$P_A: \mathcal{C}(l, l_1) \xrightarrow{\phi_{l_1}} \mathcal{C}(l, l_0),$$

concentrated in degrees 0, 1. The claim is that $B \otimes_A T_A \cong P_A$ in $D(A^{op})$. To show this is the case, we will find a complex of projective objects which is quasi-isomorphic to the complex from Corollary 3.4, and show that it is also quasi-isomorphic to $P_A$.

First, notice that since $p_1$ is projective, there is a push-out diagram

$$\begin{array}{ccc}
0 & \rightarrow & \Omega m \\
\downarrow \gamma_1 & & \downarrow \gamma_0 \\
0 & \rightarrow & l_1 \\
\phi_1 & \rightarrow & l_0 \\
\phi_0 & \rightarrow & m \\
& & \rightarrow 0,
\end{array}$$

which gives a conflation (see [5, Lem. 2.12])

$$0 \rightarrow \Omega m \xrightarrow{(g, \gamma_1)} p_1 \oplus l_1 \xrightarrow{(\gamma_0, \phi_1)} l_0 \rightarrow 0. \hspace{1cm} (10)$$

**Lemma 3.5.** There is a complex $\mathcal{S}_A$ in $D(A^{op})$ of projective objects, given by

$$\mathcal{C}(l, l_1) \xrightarrow{\phi_{l_1}} \mathcal{C}(l, l_0) \xrightarrow{(g, \phi_0)} \mathcal{C}(l, p_2) \xrightarrow{(g, f', \gamma_1)} \mathcal{C}(l, l_0), \hspace{1cm} (11)$$

which is quasi-isomorphic to the complex from (9).

**Proof.** Notice that the complex from (9) is isomorphic to cone($f_\ast$). We will show that cone($f_\ast$) is quasi-isomorphic to the complex in (11).

By Lemma 2.6, there is an exact sequence

$$0 \rightarrow \mathcal{C}(l, l_1) \xrightarrow{\phi_{l_1}} \mathcal{C}(l, l_0) \xrightarrow{\phi_0} \mathcal{C}(l, m) \rightarrow 0. \hspace{1cm} (12)$$

This is an augmented projective resolution of $\mathcal{C}(l, m)$, and we write

$$Q: \mathcal{C}(l, l_1) \xrightarrow{\phi_{l_1}} \mathcal{C}(l, l_0).$$

By concatenating the sequences from (8) and (12), followed by a truncation and adding the cokernel, the following exact sequence is obtained:

$$0 \rightarrow \mathcal{C}(l, l_1) \xrightarrow{\phi_{l_1}} \mathcal{C}(l, l_0) \xrightarrow{(g, \phi_0)} \mathcal{C}(l, p_2) \xrightarrow{(g, f', \gamma_1)} \mathcal{C}(l, p_1) \xrightarrow{\text{pt}} \mathcal{C}(l, p_1)/\text{Im}(gf'), \rightarrow 0.$$
Now lift $f_*$ to a morphism of chain complexes $\tilde{f}_* : Q' \to Q$. This lift is illustrated in the following diagram.

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \mathcal{C}(l, l_1) & \xrightarrow{ϕ_1} & \mathcal{C}(l, l_0) & \xrightarrow{(g'ϕ_1)} & \mathcal{C}(l, p_2) & \xrightarrow{(g'f')} & \mathcal{C}(l, p_1) & \xrightarrow{pr} & \mathcal{C}(l, p_1) / \text{Im}(gf')_* & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \xrightarrow{ϕ_1} & \mathcal{C}(l, l_1) & \xrightarrow{ϕ_*} & \mathcal{C}(l, l_0) & \xrightarrow{ϕ_*} & \mathcal{C}(l, m) & \longrightarrow & 0,
\end{array}
\]

where each row is an augmented projective resolution of $A^{op}$-modules. Thus, there is a quasi-isomorphism

\[
\text{cone}(\tilde{f}_*) \xrightarrow{\sim} \text{cone}(f_*).
\]

Using [22, Ex. 1.2.8], $\text{cone}(\tilde{f}_*)$ is seen to be the total complex $\text{Tot}^{\oplus}(C)$, of the double complex $C$ induced by $\tilde{f}_*$, which exactly is the complex from (11).

For the rest of this section, we will use the complex $\mathcal{P}_A$ from the previous lemma.

**Lemma 3.6.** The degree-wise projection $Φ : \mathcal{P}_A \to P_A$

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \mathcal{C}(l, l_1) & \xrightarrow{ϕ_*} & \mathcal{C}(l, l_0) & \xrightarrow{(g'ϕ_1)} & \mathcal{C}(l, p_2) & \xrightarrow{(g'f')} & \mathcal{C}(l, p_1) & \xrightarrow{pr} & \mathcal{C}(l, p_1) / \text{Im}(gf')_* & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \xrightarrow{ϕ_1} & \mathcal{C}(l, l_1) & \xrightarrow{ϕ_*} & \mathcal{C}(l, l_0) & \xrightarrow{ϕ_*} & \mathcal{C}(l, m) & \longrightarrow & 0,
\end{array}
\]

is a quasi-isomorphism.

**Proof.** By Lemma 3.5, the top row $\mathcal{P}_A$ is quasi-isomorphic to $B \otimes_T A$, and hence is exact in all degrees other than 0,1 by Lemma 3.3. Thus, to prove the lemma, it is enough to show that $H_i(Φ)$ is an isomorphism for $i = 0, 1$. First, we check that the homology of the two complexes agrees, after which it is enough to check that $H_i(Φ)$ is surjective because the homology spaces are finite dimensional over $k$.

Since there is a triangle $Ωm \to l_1 \to l_0 \to m$, and since $l$ is rigid with $Ωl \cong Σ^{-1}l \cong Σl$, there is an exact sequence

\[
0 \longrightarrow \mathcal{C}(l, Ωm) \longrightarrow \mathcal{C}(l, l_1) \longrightarrow \mathcal{C}(l, l_0) \longrightarrow \mathcal{C}(l, m) \longrightarrow 0.
\]

In combination with Corollary 3.4 and Lemmas 3.3 and 3.5, this gives that $H_i(P_A) \cong H_i(\mathcal{P}_A)$, for every $i \in \mathbb{Z}$.

Next, we show that $H_i(Φ)$ is surjective. For $i = 0$, this is straightforward. For $i = 1$, consider the diagram

\[
\begin{array}{cccccccc}
\mathcal{C}(l, p_1 \oplus l_1) & \xrightarrow{(-γ_0 \phi_1)} & \mathcal{C}(l, l_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}(l, l_1) & \xrightarrow{ϕ_*} & \mathcal{C}(l, l_0).
\end{array}
\]
Let $\alpha \in \mathcal{C}(l, l_1)$ such that the projection $\alpha \in \mathcal{C}(l, l_1)$ lies in $\text{Ker}\, \phi_1$. Thus, $\phi_1 \alpha = 0$, which means that $\phi_1 \alpha$ factors through a projective object $q \in \mathcal{C}$, say by $\phi_1 \alpha = \rho' \rho$, where $\rho : l \to q$ and $\rho' : q \to l_0$. By using (10), there is a commutative diagram

$$
\begin{array}{ccc}
q & \xrightarrow{\rho} & p_1 \oplus l_1 \\
\alpha \downarrow & & \downarrow \phi_1 \\
l & \xrightarrow{\alpha} & l_1 \\
\end{array}
$$

where $a_1, a_2$ exist by using the fact that $q$ is projective. Now calculate

$$\phi_1 \alpha = \left(-\gamma_0 \phi_1\right) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

Therefore,

$$0 = \phi_1(a_2 \rho - \alpha) - \gamma_0 a_1 \rho = (-\gamma_0 \phi_1) \begin{pmatrix} a_1 \rho \\ a_2 \rho - \alpha \end{pmatrix} = (-\gamma_0 \phi_1)_* \begin{pmatrix} a_1 \rho \\ a_2 \rho - \alpha \end{pmatrix}.$$ 

This means that $\begin{pmatrix} a_1 \rho \\ a_2 \rho - \alpha \end{pmatrix} \in \text{Ker}\, (-\gamma_0 \phi_1)_*$, making $\begin{pmatrix} a_1 \rho \\ a_2 \rho - \alpha \end{pmatrix} + \text{Im}\, \left(g_1 f'_1\right)_*$ an element of $H_1(\mathcal{P}_A)$. Now

$$H_1(\Phi) \left(-\begin{pmatrix} a_1 \rho \\ a_2 \rho - \alpha \end{pmatrix} + \text{Im}\, \left(g_1 f'_1\right)_* \right) = \alpha - a_2 \rho = \alpha,$$

which means that $H_1(\Phi)$ is surjective, and therefore, $\Phi$ is surjective on homology. \qed

**Corollary 3.7.** There are two isomorphisms

(a) $\mathcal{T}_A = (B \otimes_B L_A) \cong P_A$ in $D(A^{\text{op}})$.

(b) $\mathcal{T} \otimes A \cong B \otimes_B L_A$ in $D(A^{\text{op}})$.

**Proof.** (a) Corollary 3.4 and Lemma 3.5 imply $\mathcal{P}_A \cong B \otimes_B L_A$. Lemma 2.4(e) implies that $\mathcal{P} \otimes A \cong P_A \oplus P_A[3]$. Thus,

$$\frac{B \otimes T \otimes A}{A} \cong \mathcal{P} \otimes A \cong P_A \oplus P_A[3],$$

giving that

$$\mathcal{T}_A \cong (P_A \oplus P_A[3]) \cong P_A.$$

(b) It follows from (a) combined with Corollary 3.4 and Lemmas 3.5 and 3.6 that

$$\mathcal{T} \otimes A \cong P \otimes A \cong P_A \cong \mathcal{P}_A \cong B \otimes_B T_A.$$
We now have a one-sided isomorphism $\mathcal{F} \otimes^L_A A \cong B \otimes^L_B T_A$ in $D(A^{\text{op}})$. Next, we see that this isomorphism can be lifted to a two-sided isomorphism, that is, an isomorphism in $D(B \otimes A^{\text{op}})$.

**Theorem 3.8.** In the derived category $D(B \otimes A^{\text{op}})$ (resp. $D(B \otimes_A A)$), there is an isomorphism

$$B \mathcal{F} \otimes^L_A A \cong B \otimes^L_B T_A \quad \left(\text{resp. } B \mathcal{F} \otimes^L_A A \cong B \otimes^L_B T_A \right).$$

**Proof.** We will prove the first isomorphism, the second one can be done in a symmetric fashion. Let $B_Q \Rightarrow B \otimes^L_B T_A$ be a projective resolution over $B \otimes A^{\text{op}}$. Denote by $\alpha : B_Q \rightarrow B \otimes_A A$ the morphism defined by $q \mapsto q \otimes 1$. Consider the composition of morphisms

$$\text{pr} \circ \alpha : B_Q \rightarrow B \otimes_A A \rightarrow \left( B \otimes T \otimes_A A \right) \subseteq 1.$$  

By construction, this is isomorphic to

$$B \otimes^L_B T_A \rightarrow B \otimes B \otimes^L_A A \rightarrow \left( B \otimes T \otimes^L_A A \right) \subseteq 1 \cong B \mathcal{F} \otimes^L_A A.$$  

Writing out the composition gives the following, which we will show to be a quasi-isomorphism:

$$\ldots \rightarrow Q_2 \rightarrow Q_1 \rightarrow Q_0 \rightarrow 0 \rightarrow \ldots$$  

The projection $\text{pr} : B \otimes_A A \rightarrow (B \otimes_A A) \subseteq 1$ of soft truncation is an isomorphism on homology in degrees $\leq 1$. By Lemma 3.3, we have $H_i(B_Q) = 0$ for $i \neq 0, 1$. Therefore, it is enough to check that $\alpha$ is an isomorphism on homology in degrees 0 and 1. Moreover, since we are dealing with finite-dimensional vector spaces, Corollary 3.7(b) gives that for $i = 0, 1$, the dimensions of $H_i(B \otimes^L_B T_A) \cong H_i(B_Q)$ and $H_i(B \mathcal{F} \otimes^L_A A) \cong H_i(B \otimes A^{\text{op}} A)$ are the same, making it enough to check that $H_i(\alpha)$ is injective, for $i = 0, 1$.

Let $i \in \{0, 1\}$, and let $h \in \text{Ker}(d_i)$ be such that $h + \text{Im}(d_{i+1}) \subseteq \text{Ker}(H_i(\alpha))$. Then there exist morphisms $h_j \in Q_{i+1}$ and $a_j \in A$ such that

$$\text{Im}(d_{i+1} \otimes \text{id}) \ni \alpha_i(h) = h \otimes 1$$  

$$= (d_{i+1} \otimes \text{id}) \left( \sum_j h_j \otimes a_j \right)$$  

$$= \sum_j (d_{i+1} h_j \otimes a_j).$$
= \sum_j (d_{i+1}(h_j a_j) \otimes 1) \\
= \left( \sum_j d_{i+1}(h_j a_j) \right) \otimes 1.

Thus, \((h - \sum_j d_{i+1}(h_j a_j)) \otimes A = 0\). Letting \(a\) be the ideal of morphisms in \(A\) factoring through a projective object gives that \(h - \sum_j d_{i+1}(h_j a_j) \in Q_a \cap \ker d_i\). However, this intersection equals \((\ker d_i) a\), by Lemma 2.12. Lemma 2.12 applies for \(i = 0\) because the cokernel of \(Q_0 \to 0\) is \(H_0(Q_A) \cong H_0(B \otimes_B^L T_A)\) that has the form \(C(l, y)\) by Lemma 3.3. Since \(h - \sum_j d_{i+1}(h_j a_j) \in (\ker d_i) a\), we have

\[
\sum_j \tilde{h}_j \tilde{a}_j,
\]

for some \(\tilde{h}_j \in \ker d_i\) and \(\tilde{a}_j \in a\). By Lemma 3.3, there are isomorphisms over \(A^{\text{op}}\):

\[
\xi_i : H_i(Q_A) \to \begin{cases} 
C(l, m) & i = 0, \\
C(l, \Omega m) & i = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Notice that \(a\) annihilates \(\text{Im}(\xi_i)\), meaning that

\[
\xi_i \left( \sum_j \tilde{h}_j \tilde{a}_j + \text{Im}(d_{i+1}) \right) = \sum_j \xi_i(\tilde{h}_j + \text{Im}(d_{i+1}))\tilde{a}_j = 0.
\]

Since \(\xi_i\) is injective \(h - \sum_j d_{i+1}(h_j a_j) = \sum_j \tilde{h}_j \tilde{a}_j \in \text{Im}(d_{i+1})\). Thus, \(h \in \text{Im} d_{i+1}\). \(\square\)

**Theorem 3.9.** The canonical morphisms

\[
B \to \text{End}_{D(A^{\text{op}})}(B^T_A) \quad \text{and} \quad A \to \text{End}_{D(B)}(B^T_A),
\]

induced by \(- \otimes_B^L T_A\) and \(B^T \otimes_A^L -\) are isomorphisms.

**Proof.** We will show the first isomorphism. The second one is done similarly in a symmetric fashion. From the isomorphism in Theorem 3.8, the following commutative diagram is produced:

\[
\begin{array}{ccc}
\text{End}_{D(B^{\text{op}})}(B^T_A) & \to & \text{End}_{D(B^{\text{op}})}(B_B) \\
\downarrow \Phi_B^T \circ & & \downarrow \Phi_B^T \\
\text{End}_{D(A^{\text{op}})}(T_A) & \to & \text{End}_{D(A^{\text{op}})}(T_A)
\end{array}
\]

The morphism induced by \(- \otimes_B^L B_B\) is an isomorphism, and since \(B^T_A\) is a tilting complex (by Lemma 3.2), the morphism induced by \(- \otimes_B^L T_A\) is also an isomorphism. Hence, the map induced
by $- \otimes^L_{\Delta} A_A$ is surjective. Thus, to show $- \otimes^L_{\Delta} \mathcal{T}_A$ is an isomorphism, it is enough to show that

$$\text{End}_{D(A^{\text{op}})}(\mathcal{T}_A) \cong \text{End}_{D(A^{\text{op}})}(\mathcal{T}_A),$$

making $- \otimes^L_{\Delta} A_A$ an isomorphism. This is seen using the following calculation:

$$\text{RHom}_{D(A^{\text{op}})}(\mathcal{T}_A, \mathcal{T}_A) \cong \text{RHom}_{D(A^{\text{op}})}\left(\mathcal{T} \otimes^L A_A, \text{RHom}_{D(A^{\text{op}})}(A_A, \mathcal{T}_A)\right)$$

$$\cong \text{RHom}_{D(A^{\text{op}})}(L_A \otimes^L A_A, \mathcal{T}_A)$$

$$\cong \text{RHom}_{D(A^{\text{op}})}(L_B \otimes T \otimes A_A, \mathcal{T}_A)$$

(by 3.7(b))

$$\cong \text{RHom}_{D(A^{\text{op}})}(P \otimes A_A, \mathcal{T}_A)$$

(by Corollary 3.4 and Lemma 3.5)

$$\cong \text{RHom}_{D(A^{\text{op}})}(P_A \oplus P_A[3], \mathcal{T}_A).$$

(by Lemma 2.4(e))

By taking the 0th homology, one obtains that

$$\text{End}_{D(A^{\text{op}})}(\mathcal{T}_A) \cong \text{Hom}_{D(A^{\text{op}})}\left(P_A \oplus P_A[3], \mathcal{T}_A\right) \cong \text{Hom}_{D(A^{\text{op}})}\left(P_A, \mathcal{T}_A\right) \cong \text{End}_{D(A^{\text{op}})}(\mathcal{T}_A),$$

where the second isomorphism comes from $\mathcal{T}_A$ being isomorphic in $D(A^{\text{op}})$ to a two term complex of projective objects, see Corollary 3.7(a), and the last isomorphism is also by Corollary 3.7(a). \(\square\)

**Lemma 3.10.** The complex $P_A$.

$$(\mathcal{C}(l, l_1) \xrightarrow{\phi_1} \mathcal{C}(l, l_0),$$

is a two-term tilting complex in $D(A^{\text{op}})$.

**Proof.** $P_A$ is a silting complex by [2, Thm 2.18 and Rmk 2.19(2)]. Thus, to show that $P_A$ is a tilting complex, it is enough to show that $\text{Hom}_{\mathcal{S}^b(\text{proj} A^{\text{op}})}(P_A, P_A[-1]) = 0$.

Recall that $P_A$ is defined using the triangle

$$l_1 \xrightarrow{\phi_1} l_0 \xrightarrow{\phi_0} m \xrightarrow{\Sigma \phi_2} \Sigma l_1.$$`

By applying $\mathcal{C}(l, -)$ to this triangle, we obtain an exact sequence

$$0 \longrightarrow \mathcal{C}(l, \Sigma^{-1}m) \xrightarrow{\phi_2} \mathcal{C}(l, l_1) \xrightarrow{\phi_1} \mathcal{C}(l, l_0) \xrightarrow{\phi_0} \mathcal{C}(l, m) \longrightarrow 0.$$
This shows that $\text{Ker}(\phi_1^*) = \mathcal{C}(l, \Sigma^{-1}m)$, and $\text{Coker}(\phi_1^*) = \mathcal{C}(l, m)$. Let a chain map $f \in \text{Hom}_{\mathcal{X}^{\text{proj}}}(P_\Lambda, P_\Lambda[-1])$ be given:

$$
\mathcal{C}(l, l_1) \xrightarrow{\phi_1^*} \mathcal{C}(l, l_0) \xrightarrow{f_0} 0 \\
0 \xrightarrow{f_0} \mathcal{C}(l, l_1) \xrightarrow{\phi_1^*} \mathcal{C}(l, l_0).
$$

$f$ being a chain map implies that $f_0\phi_1^* = 0$. Thus, $f_0$ factors through $\text{Coker}(\phi_1^*)$, say $f_0 = f_0'\phi_0^*$. If we use that $f$ is a chain map, again we get that $\phi_1^*f_0' = 0$, giving that $f_0'$ factors through $\text{Ker}(\phi_1^*)$, say $f_0' = \phi_2^*f_0''$, with $f_0'' \in \text{Hom}_\Lambda(\mathcal{C}(l, m), \mathcal{C}(l, \Sigma^{-1}m))$. Therefore, $f_0 = \phi_2^*f_0'' \phi_0^*$.

Since $m$ is rigid and $\Sigma^2m \cong m$, we have that $\mathcal{C}(m, \Sigma^{-1}m) = 0$. Thus, Lemma 2.4(g) gives that $f_0'' = 0$, and thereby $f_0 = 0$. Hence, $f = 0$, making $P_\Lambda$ a tilting complex.

**Corollary 3.11.** $\mathcal{B} \mathcal{T}_\Lambda$ is a two-sided tilting complex. In particular,

$$
- \bigotimes_{\mathcal{B}} \mathcal{T}_\Lambda : D(\mathcal{B}^{\text{op}}) \longrightarrow D(\mathcal{A}^{\text{op}})
$$

is a triangulated equivalence, making $\mathcal{A}$ and $\mathcal{B}$ derived equivalent.

**Proof.** Recall that $\mathcal{T}_\Lambda \cong P_\Lambda$ in $D(\mathcal{A}^{\text{op}})$ (by Corollary 3.7(a)), and that $P_\Lambda$ is a tilting complex (by Lemma 3.10). This establishes the second half of part (1) as well as parts (2) and (3) of Definition 1.1. The first half of part (1) holds by Theorem 3.9. The triangulated equivalence follows from \cite[Prop. 8.1.4]{15}.

**4 | EXAMPLES**

**4.1 | Cluster-tilting objects from $C(D_{2n})$**

It was shown in \cite[Lem. 4.5]{4} that the self-injective cluster tilted algebras of the cluster category $\mathcal{C} = C(D_{2n})$ are derived equivalent. This was done by finding a tilting complex ad hoc. In this example, we will see how our results can be used to find such a tilting complex, and thereby recover the tilting complex from \cite{4}.

To apply the results from the previous section on this example, we need to ensure that $\mathcal{C}$ has a Frobenius model $\mathcal{E}$ that satisfies Setup 2.1.

**Theorem 4.1.** There exists a Frobenius category $\mathcal{E}$, such that $\mathcal{C} = \mathcal{E}$. Furthermore, the pair of $\mathcal{E}$ and $\mathcal{C}$ satisfies Setup 2.1.
Figure 1 Auslander–Reiten quiver of mod(kD₅).

Proof. Let $I$ be the direct sum of all injective indecomposable objects in mod($kD_{2n}$), and $M = I \oplus \tau I$. This $M$ has the needed properties to apply [8, Thm. 2.1], giving a Frobenius category $\mathcal{E}$ such that $\mathcal{E} = \mathcal{E}$, which also satisfies Setup 2.1.

Let $n \in \mathbb{N}$, with $n \geq 4$. Consider the quiver $D_{2n}$.

There are indecomposable projective representations $P(a), P(b), P(c)$ corresponding to the vertices $a, b, c$, and these can be viewed as objects of $\mathcal{E}$. In [21], Ringel described two cluster-tilting objects in $\mathcal{E}$, see Figure 1:

$T_1 = \bigoplus_{i=0}^{n-1} \tau^{-2i}P(a) \oplus \bigoplus_{i=0}^{n-1} \tau^{-2i-1}P(b)$,

$T_2 = \bigoplus_{i=0}^{n-1} \tau^{-2i}P(a) \oplus \bigoplus_{i=0}^{n-1} \tau^{-2i-1}P(c)$.

Denote their endomorphism algebras by $A_i = \text{End}_{\mathcal{E}}(T_i) = \mathcal{E}(T_i, T_i)$. To describe the endomorphism algebras, we define the following quivers:

$Q_1: \alpha \quad \cdots \quad \alpha$

$Q_2: \begin{array}{c}
\overset{\beta}{\overset{n-1}{\circ}} \overset{3}{\circ} \overset{\beta}{\cdots} \\
\overset{\alpha}{\overset{\beta}{\overset{n}{\circ}}} \overset{\alpha}{\overset{\beta}{\overset{2}{\circ}}} \overset{\alpha}{\overset{\beta}{\overset{1}{\circ}}}
\end{array}$
Then, $A_i = kQ_i/I_i$, with $I_1 = \langle \alpha^{2n-1} \rangle$, and $I_2 = \langle \alpha \beta, \beta \alpha, \beta^2 - \alpha^{n-1} \rangle$. Both of these algebras are self-injective by Lemma 2.5, but neither of them is symmetric. Now Corollary 3.11 says that $A_1$ and $A_2$ are derived equivalent. Furthermore, there is a direct way to calculate the associated one-sided tilting complexes.

Using the Auslander–Reiten quiver of mod($kD_{2n}$) (see Figure 1 for an example in the case $n = 4$), one can use the dimension vectors to see that there is an exact sequence (see [1, Prop. IX.3.1, Lem. IX.1.1(a)])

$$0 \longrightarrow P(b) \xrightarrow{\phi} \tau^{-1}P(a) \xrightarrow{\psi} \tau^{-2}P(c) \longrightarrow 0$$

inducing a triangle

$$P(b) \xrightarrow{\phi} \tau^{-1}P(a) \xrightarrow{\psi} \tau^{-2}P(c) \longrightarrow \Sigma P(b)$$

in the cluster category $\mathcal{C}$. This implies that for each $i$, there is a triangle

$$\tau^i P(b) \xrightarrow{\phi_i} \tau^{i-1}P(a) \xrightarrow{\psi_i} \tau^{i-2}P(c) \longrightarrow \Sigma \tau^i P(b)$$

in $\mathcal{C}$. With this, we construct the following triangle. Let $\Phi' = \bigoplus_{i=0}^{n-1} \phi_{-2i-1}$ and let $\Psi' = \bigoplus_{i=1}^{n} \psi_{-2i}$, then there exists a triangle

$$\begin{array}{c}
\bigoplus_{i=0}^{n-1} \tau^{-2i-1}P(b) \\
\bigoplus_{i=1}^{n} \tau^{-2i}P(a) \\
\bigoplus_{i=1}^{n} \tau^{-2i}P(c)
\end{array} \xrightarrow{\Phi} \\
\bigoplus_{i=1}^{n} \tau^{-2i}P(a) \xrightarrow{\Psi} \\
\bigoplus_{i=1}^{n} \tau^{-2i-1}P(c)$$

with $T_1^j \in \text{add}(T_1)$. Notice that we have used that $\tau^{2n} \cong \text{id}$ when describing $T_2$. It follows from $T_1$ being rigid that $\Psi$ is an add($T_1$) precovers, namely, if there is a morphism $\Psi : S \rightarrow T_2$, with $S \in \text{add}(T_1)$, then $\mathcal{C}(S, \Sigma T_1^2) = 0$, and $\Psi$ will therefore factor through $\Psi$. To see that $\Psi$ is a cover, it is enough to check that $\Phi \in \text{rad}_{\mathcal{C}}$ (see [7, Lem 3.12]). All the components $\phi_{-2i-1}$ of $\Phi$ are morphisms between two different indecomposable objects, and are therefore all in the radical. Thus, $\Phi$ is in the radical.

Define the following complex concentrated in degrees 0,1:

$$P_1 : \mathcal{C}(T_1, T_1^2) \xrightarrow{\Phi} \mathcal{C}(T_1, T_1^1).$$

This complex is a tilting complex over $A_1$ by Lemma 3.10. By Theorem 3.9, there is an isomorphism $A_2 \cong \text{End}_{D(A_1^{op})}(P_1)$. Similarly, a tilting complex $P_2$ could be found such that $A_1 \cong \text{End}_{D(A_2^{op})}(P_2)$.

To verify that this is indeed the case we check the isomorphism $A_2 \cong \text{End}_{D(A_1^{op})}(P_1)$. For each $T' \in \text{add}(T_1)$, the right $A_1$-module $\mathcal{C}(T_1, T')$ is projective. We can therefore identify the Hom-space from $T_1$ to indecomposable summands of $T_1$ with indecomposable projective modules over
We will do that as follows:
\[ P_{A_1}(2i) \cong C(T_1, \tau^{-2i}P(a)) \quad \text{and} \quad P_{A_1}(2j + 1) \cong C(T_1, \tau^{-2j+1}P(b)) \]
for \(0 < i \leq n\), and \(0 \leq j < n\). The morphisms between projective objects can be described as follows:
\[
\dim \text{Hom}(P_{A_1}(i), P_{A_1}(j)) = \begin{cases} 
0 & \text{if } j = i - 1 \\
0 & \text{if } i = 1, j = 2n \\
1 & \text{otherwise.}
\end{cases}
\]
Now \(\text{End}_{DA_1^{op}}(P_1)\) can be calculated. There are \(2n\) indecomposable components in \(P_1\):
\[
B_i : P_{A_1}(2i + 1) \xrightarrow{\phi_{-2i-1}} P_{A_1}(2i + 2), \quad C_j : 0 \xrightarrow{\alpha} P_{A_1}(2j),
\]
with \(0 \leq i < n\), and \(0 < j \leq n\). To describe the morphisms, we split it into cases.

(i) \(\gamma \in \text{Hom}(C_i, B_{i-1}) \neq 0\), for \(0 < i \leq n\). Here, \(\gamma\) is the inclusion.
(ii) \(\beta \in \text{Hom}(B_i, C_i) \neq 0\), for \(0 < i \leq n-1\) and \(\beta \in \text{Hom}(B_0, C_n) \neq 0\). Here, \(\beta\) is induced by the morphism \(P(a) \to \tau^2 P(a)\).
(iii) \(\alpha \in \text{Hom}(C_i, C_{i+1}) \neq 0\), for all \(0 < i \leq n - 1\), and \(\alpha \in \text{Hom}(C_n, C_1) \neq 0\). Here, \(\alpha\) is induced by the morphism \(P(a) \to \tau^{-2} P(a)\).
(iv) \(\text{Hom}(B_i, B_{i+1}) = 0\), for all \(0 \leq i < n - 1\), and \(\text{Hom}(B_{n-1}, B_0) = 0\). This is due to the “potential” morphisms being null-homotopic.
(v) \(\delta \in \text{Hom}(C_i, C_{i-1}) \neq 0\), for all \(1 < i \leq n\), and \(\delta \in \text{Hom}(C_1, C_n) \neq 0\). But notice that \(\delta\) factors through \(B_{i-1}\):

\[
\begin{align*}
C_i & \quad 0 \quad P_{A_1}(2i) \\
\downarrow \gamma & \quad \downarrow & \quad \downarrow \\
B_{i-1} & \quad P_{A_1}(2i-1) \quad P_{A_1}(2i) \\
\downarrow \beta & \quad \downarrow & \quad \downarrow \\
C_{i-1} & \quad 0 \quad P_{A_1}(2i - 2).
\end{align*}
\]
From this, we can determine the quiver of the endomorphism algebra \(\text{End}_{DA_1^{op}}(P_1)\):
It is straightforward to check that $\beta \gamma \cong \alpha^{n-1}$ using (v). It follows from the form of $I_1$ that $\alpha \beta = 0$. Lastly, there is the relation $\gamma \alpha = 0$, which is due to $\gamma \alpha$ being null-homotopic. It follows from $\alpha^{n-1} \neq 0$ and $\beta \gamma \neq 0$, that there are no more relations. This now means that

$$\text{End}_{D(A_1^{\text{op}})}(P_1) \cong Q/\langle \alpha^{n-1} - \beta \gamma, \alpha \beta, \gamma \alpha \rangle \cong A_2.$$ 

### 4.2 Example from symmetric $(k, n)$-Postnikov diagrams

It turns out that a good source of examples for finding derived equivalent algebras using Corollary 3.11 are Postnikov diagrams, as shown in Figures 2 and 3. A $(k, n)$-Postnikov diagram $D$ is a Postnikov diagram with $n$ vertices, and strands going from vertices $i$ to vertices $i + k$. Such a diagram is called symmetric if it is invariant under rotation by $k$ vertices, see Figures 2 and 3. Furthermore, $D$ is called reduced if no “untwisting” moves can be applied. For a detailed description of these properties, see [18, Sec. 4].

To each Postnikov diagram $D$, one can associate an ice quiver with potential $(Q, W, F)$, where

- $Q$ is the quiver associated to $D$. The vertices of $Q$ correspond to the alternating regions of $D$. There is an arrow between two vertices if their corresponding regions meet at an intersection of strands, the arrow will point with the “flow” of those intersecting stands. See Figures 2 and 3.
- $W$ is the potential given by the sum of clockwise cycles in $Q$ minus anti-clockwise cycles in $Q$, and
- $F$ is the frozen vertices, which is the set of vertices on the boundary of $D$.

For further details, see [18, Sec. 4]. Denote the associated frozen Jacobian algebra $P(Q, W, F)$.

Next, we construct the boundary algebra as described in [18, Sec. 6]. Given $k, n \in \mathbb{N}$, with $k < n$, consider a $\mathbb{Z}/n\mathbb{Z}$-grading on $\mathbb{C}[x, y]$ given by $\deg x = 1$, and $\deg y = -1$. Now let $R = \ldots$
Theorem 4.2. Let $k, n \in \mathbb{N}$ with $k < n$. We then have the following results:

1. $\text{CM}(\hat{B})$ is a Frobenius category ([13, Cor. 3.7]).
2. $\text{CM}(\hat{B})$ is 2-Calabi–Yau ([13, Cor. 4.6] and [9, Prop. 3.4]).
3. $\text{CM}(\hat{B})$ is Hom-finite ([13, Cor. 4.6] and [9, Sec. 3.1, 3.2]).
4. $\text{CM}(\hat{B})$ has split idempotents.

Proof. That $\text{CM}(\hat{B})$ has split idempotents comes from the fact that $\text{Sub} Q_k$ from [13, Cor. 4.6] and [9, Sec. 3] has split idempotents. This is due to it being the full subcategory of submodules of sums of $Q_k$, in a module category that has split idempotents. Since $\text{Sub} Q_k$ has split idempotents, $\text{Sub} Q_k$ has split idempotents, and by [13, Cor. 4.6], there is a triangle equivalence $\text{Sub} Q_k \cong \text{CM}(\hat{B})$. □

The next collection of result describes the objects we want to work with.

Theorem 4.3. We have the following results.

1. For every reduced $(k, n)$-Postnikov diagram $D$, there is an associated cluster-tilting object $T(D)$ of $\text{CM}(\hat{B})$ ([18, Thm. 7.2]).
Given a reduced \((k,n)\)-Postnikov diagram \(D\), then \(D\) is symmetric if and only if the endomorphism ring \(\text{End}(T(D))\) is self-injective ([18, Thm. 8.2, Lem 7.7]).

If \(D\) is a \((k,n)\)-Postnikov diagram, \(T\) the associated cluster tilting object, and \((Q,W,F)\) the associated ice quiver with potential, then \(\widehat{\text{End}}(T) \cong \mathcal{P}(Q, W, F)/(F)\), where \((F)\) is the ideal generated by the frozen vertices ([18, Lem. 7.5, Prop 7.6, Sec. 3]).

**Corollary 4.4.** Let \(k, n \in \mathbb{N}\), with \(k < n\). Let \(D, D'\) be two symmetric and reduced \((k,n)\)-Postnikov diagrams, with associated cluster tilting objects \(T = T(D)\) and \(T' = T(D')\) in \(\text{CM}(\widehat{B})\). Then, \(\text{End}(T)\) and \(\text{End}(T')\) are derived equivalent.

**Proof.** Since \(D\) and \(D'\) are symmetric Postnikov diagrams, Theorem 4.3(2) gives that \(\text{End} T\) and \(\text{End} T'\) are self-injective. Therefore, Corollary 3.11 gives that \(\text{End}(T)\) and \(\text{End}(T')\) are derived equivalent. \(\square\)

As an example of the use of Corollary 4.4, see Figures 2 and 3 for two symmetric and reduced \((3,9)\)-Postnikov diagrams and their associated ice quivers. The frozen vertices are exactly the ones on the boundary. Denote these ice quivers with potential by \((Q, W, F)\) and \((Q', W', F')\), respectively. Using Theorems 4.3(2) and 4.3(3) together with Corollary 4.4, we get that the associated algebras \(\mathcal{P}(Q, W, F)/(F)\) and \(\mathcal{P}(Q', W', F')/(F')\) are self-injective and derived equivalent. Note that in contrast to [3], the 2-CY-tilted algebras considered here are not symmetric.

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