Finite-size scaling and bulk critical behavior of a quantum spherical model with a long-range interaction: entropy, internal energy and specific heat

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Abstract. The entropy, the internal energy and the specific heat of a $d$-dimensional quantum spherical model with a long-range interaction, the decreasing of which is controlled by a parameter $\sigma (0 < \sigma \leq 2)$, are studied close to its quantum critical point in the context of the finite-size scaling (FSS) theory for $d = \sigma$. The obtained specific heat critical exponent $\alpha$ does not depend on $\sigma$ and satisfies the quantum hyperscaling relation. Based on the derived FSS forms and asymptotes of the universal scaling functions of the considered quantities, the leading temperature dependencies of the entropy, the internal energy and the specific heat are obtained in both the renormalized classical and the quantum disordered regions of the phase diagram. It has been shown that when the temperature $T \to +0$, the entropy and the specific heat in the renormalized classical region tend to zero as powers of $T$, while in the quantum disordered region they tend to zero exponentially.

1. Introduction

Over the past few decades the theory of the quantum phase transitions continues to be a subject of great interest (see e.g [1] and refs. therein). These phase transitions, caused by quantum fluctuations rather than thermal ones, appear at zero temperature ($T = 0$) as a function of some non-thermal control parameter $g$ (associated with pressure, doping concentration or magnetic fields) or as a competition between different parameters, describing the basic interaction of the system.

According to the dimensional crossover rule, the critical singularities with respect to $g$ of a $d$-dimensional quantum system at $T = 0$ close to its zero-temperature critical point $g_c$ are formally equivalent to those of a classical system with dimensionality $d + z$ ($z$ is the dynamical critical exponent) close to its critical temperature $T_c > 0$. This makes it possible to consider an infinite $d$-dimensional critical quantum system at low temperatures in the context of the finite-size scaling (FSS) theory [2] as an effective $(d + z)$-dimensional system with $d$ infinite spatial and $z$ finite temporal dimensions. A characteristic property of the quantum critical phenomena at $T \to 0$ is that not only the correlation length $\xi$ in the spatial dimension $d$ diverges as $\xi \propto |\delta g|^{-\nu}$, where $\delta g \propto (g - g_c)/g_c$ measures the distance from $g_c$, but the correlation length $\xi_t$, corresponding to the thermal fluctuations (along the direction of the imaginary time) also diverges as $\xi_t \propto \xi^z \propto |\delta g|^{-\nu z}$ [1].

In this paper scaling properties of the entropy, the internal energy and the specific heat of a $d$-dimensional quantum spherical model [3] with a long-range interaction, the decreasing of which is controlled by a parameter $\sigma (0 < \sigma \leq 2)$, are presented. This model, known as the "spherical quantum rotors" model (SQRM), does not describe quantum Heisenberg-Dirac spins, but rather quantum rotors [3].
Due to its exact solvability for each dimensionality, the SQRM is very useful for a rigorous study of finite-size effects [4–6]. The entropy, the internal energy and the specific heat of the model at $g = g_c$, i.e. in the quantum critical region, have been considered in [7].

Here, in the framework of the FSS theory, we study the entropy, the internal energy and the specific heat of the SQRM in both the renormalized classical and the quantum disordered regions of the phase diagram for the special case $d = \sigma$.

2. Scaling form of the entropy, the internal energy and the specific heat

Let us consider a $d$-dimensional infinite quantum system at low temperatures near to its quantum critical point ($T = 0, g = g_c$) in the absence of an ordering external field in space dimensions $d_l < d < d_u$, where $d_l$ and $d_u$ are the lower and the upper quantum critical dimensions, respectively.

Defining the finite size $L_\tau \propto \left( \frac{\hbar}{k_B T} \right)^{1/2}$ in the temporal (imaginary time) direction, we consider the infinite $d$-dimensional system as a system with a geometry of the form $\infty^d \times L_\tau^z$ (in the remainder we will set $k_B = \hbar = 1$).

Following [2], we can write the FSS form of the singular part of the free energy density $f_{\text{sing.}}(\delta g, T|d)$ as

$$f_{\text{sing.}}(\delta g, T|d) = L_\tau^{-(d+z)} (T L_\tau^z) X(x_\tau|d),$$

where $X(x_\tau|d)$ is a universal scaling function of the scaling variable $x_\tau = L_\tau^{1/\nu} \delta g$.

Based on fundamental thermodynamic relations, involving the free energy, by using its FSS form, given by equation (1), we derive the FSS forms of the entropy (per spin) $S$, the internal energy density $U$ and the specific heat (per spin) $C$, as follows:

$$S(\delta g, T|d) = -\frac{\partial f_{\text{sing.}}}{\partial T} = L_\tau^{-d} S_s(x_\tau|d),$$

$$U(\delta g, T|d) = f_{\text{sing.}} + TS = L_\tau^{-(d+z)} (T L_\tau^z) U_s(x_\tau|d)$$

and

$$c(\delta g, T|d) = -\frac{\partial^2 f_{\text{sing.}}}{\partial T^2} = L_\tau^{-d} C_s(x_\tau|d).$$

Here $S_s(x_\tau|d)$, $U_s(x_\tau|d)$ and $C_s(x_\tau|d)$ are the corresponding universal scaling functions:

$$S_s(x_\tau|d) = \frac{1}{\nu z} x_\tau X'(x_\tau|d) - \left( 1 + \frac{d}{z} \right) X(x_\tau|d),$$

$$U_s(x_\tau|d) = \frac{1}{\nu z} x_\tau X'(x_\tau|d) - \frac{d}{z} X(x_\tau|d)$$

and

$$C_s(x_\tau|d) = -\frac{1}{(\nu z)^2} x_\tau^2 X''(x_\tau|d) - \frac{1}{\nu z} \left( \frac{1}{\nu z} - 2 \frac{d}{z} - 1 \right) x_\tau X'(x_\tau|d) - \frac{d}{z} \left( \frac{d}{z} + 1 \right) X(x_\tau|d),$$

in which $X'(x_\tau|d)$ and $X''(x_\tau|d)$ are the first and the second derivatives of the function $X(x_\tau|d)$ with respect to $x_\tau$.

In a thermodynamic analogy with the classical case $T \to T_c \neq 0$, in which $c \approx \frac{\partial^2 f_{\text{sing.}}}{\partial (\delta g)^2} \propto |\delta g|^\alpha$, at $T \to 0$ near to the quantum phase transition, the specific heat critical exponent $\alpha$ may be defined by

$$\frac{c}{T} = \frac{\partial^2 f_{\text{sing.}}}{\partial T^2} \sim T^{-\alpha}.$$
From (4) for the FSS form of \( c/T \) we have

\[
\frac{c}{T} = L_{z}^{-d+z} \left( TL_{z}^{r} \right)^{-1} C_{s}(x_{r}|d).
\] (9)

For \( \nu = 1 \), in accordance with Ref. [2], we can write (9) in the form

\[
\frac{c}{T} = L_{z}^{p} \left( TL_{z}^{r} \right)^{-1} C_{s}(x_{r}|d)
\] (10)

with \( p = \alpha / \nu \), where the critical exponent \( \alpha \) satisfies the quantum hyperscaling relation [8],

\[
2 - \alpha = (d + z) \nu.
\] (11)

From (11) for the critical exponent \( \alpha \) one obtains \( \alpha = 1 - d/z \).

Now we proceed to study the scaling properties of the entropy, the internal energy and the specific heat, introduced above, on the example of one exactly solvable model.

3. The model

The Hamiltonian of the model is [3]

\[
H = \frac{1}{2g} \sum_{i} \mathbf{P}_{i}^{2} - \frac{1}{2} \sum_{i,j'} \mathbf{J}_{i,j'} \mathbf{S}_{i} \mathbf{S}_{j'} + \frac{1}{2} \mu \sum_{i} \mathbf{S}_{i}^{2} - h \sum_{i} \mathbf{S}_{i},
\] (12)

where \( \mathbf{S}_{i} \) are the spin operators at site \( i \), the operators \( \mathbf{P}_{i} \) play the role of "conjugated" momenta (i.e. \( [\mathbf{S}_{i}, \mathbf{P}_{i}] = 0 \), \( [\mathbf{P}_{i}, \mathbf{P}_{i'}] = 0 \), and \( [\mathbf{P}_{i}, \mathbf{S}_{i'}] = i \delta_{ii'} \)), \( \mathbf{J}_{i,j'} \) is the interaction matrix, the coupling constant \( g \) measures the strength of the quantum fluctuations, \( h \) is an ordering magnetic field, and the spherical field \( \mu \) is introduced so as to ensure the constraint

\[
\sum_{i} (\mathbf{S}_{i}^{s}) = N.
\] (13)

Here \( N \) is the total number of quantum spins, located at sites "1" of a hypercubical lattice \( L_{1} \times L_{2} \times \ldots \times L_{d} = N \), and \( \langle ... \rangle \) denotes the standard thermodynamic average, taken with the Hamiltonian, (12).

The normalized free energy density \( f(T,g)/J \) of a \( d \)-dimensional system with a long-range interaction (decreasing at long distances \( r \) as \( 1/r^{d+\sigma} \), \( 0 < \sigma \leq 2 \)) in the thermodynamic limit \( (N \rightarrow \infty) \) at \( h = 0 \) has the form [6]

\[
f(\frac{T,g}{J}) = \sup_{\phi} \left\{ k_{d} t \int_{0}^{x_{0}} x^{d-1} \ln \left[ 2 \sinh \left( \frac{\lambda}{2t} \sqrt{\phi + x^{\sigma}} \right) \right] dx - \frac{\phi}{2} \right\} - d,
\] (14)

where the supremum is attained at the solutions of the mean-spherical constraint, (13), that reads

\[
1 = \frac{\lambda}{2k_{d}} \int_{0}^{x_{0}} \frac{x^{d-1}}{\sqrt{\phi + x^{\sigma}}} \coth \left( \frac{\lambda}{2t} \sqrt{\phi + x^{\sigma}} \right) dx.
\] (15)

Here we have introduced the notations: \( \lambda = \sqrt{g/J} \) is the normalized quantum parameter, \( t = T/J \) - the normalized temperature, \( \phi = \mu/J \) - the scaled spherical field, \( x_{0} = 2\pi (d/S_{d})^{1/d} \) is the radius of the sphericalized Brillouin zone, \( S_{d} = 2\pi^{d/2}/\Gamma(d/2) \) is the surface of the \( d \)-dimensional unit sphere and \( k_{d}^{-1} = \frac{1}{2} (4\pi)^{d/2} \Gamma(d/2) \) (\( \Gamma \) is the Euler gamma function). From (15) after taking the limit \( t \rightarrow 0 \) and setting \( \phi = 0 \), one obtains the critical value \( \lambda_{c} \) of the quantum parameter \( \lambda \) at the quantum critical point, \( \lambda_{c} = (2 - \sigma/d)x_{0}^{\sigma/2} \).

Equations (14) and (15) provide the basis for studying the critical behavior of the model under consideration.
4. A universal scaling function of the free energy

Let us consider the model (12) in the low-temperature limit $(\lambda/t >> 1)$ close to its quantum critical point $(\phi << 1)$ in space dimensions $\sigma/2 < d < 3\sigma/2$, where $\sigma/2$ and $3\sigma/2$ are the lower and the upper critical dimensions, respectively, for the quantum critical point of the considered system.

By introducing the scaling variable $x_\tau = L_{\tau}^{-d} \delta \lambda$, where $L_{\tau} = (\lambda/t)^{1/2}$ is a temporal size, $\delta \lambda = 1/\lambda - 1/\lambda_c$ measures the deviation from $\lambda_c$, $z = \sigma/2$ is the dynamical critical exponent of the model and the critical exponent $y^{-1} = d - z$, after some algebra, the singular (the $\phi$ dependent) part of the free energy density (14), can be written in the form (1) with

$$X(x_\tau|d, \sigma) = -\frac{1}{2} x_\tau y_\tau^2 - \frac{k_d}{4\sigma \sqrt{\pi}} \Gamma\left(\frac{d}{\sigma}\right) \Gamma\left(\frac{d}{\sigma} - \frac{1}{2}\right) y_\tau^{d + 1} - \frac{k_d}{2\sigma \sqrt{\pi}} \Gamma\left(\frac{d}{\sigma}\right) y_\tau^{d + 1} \mathcal{K}\left(\frac{d}{\sigma} + \frac{1}{2}, \frac{1}{2}\right).$$

In (16) $y_\tau = L_{\tau}^{z} \phi^{1/2}$ is the solution of the corresponding spherical field equation

$$x_\tau = -\frac{k_d}{2\sigma \sqrt{\pi}} \Gamma\left(\frac{d}{\sigma}\right) \left[\Gamma\left(\frac{1}{2} - \frac{d}{\sigma}\right) y_\tau^{d - 1} + \frac{k_d}{\sigma \sqrt{\pi}} \Gamma\left(\frac{d}{\sigma}\right) y_\tau^{d - 1} \mathcal{K}\left(\frac{d}{\sigma} - \frac{1}{2}, \frac{1}{2}\right)\right].$$

In (16) and (17) $\mathcal{K}(v, y) \equiv \mathcal{K}_{1}(v, 1, y) = 2 \sum_{n=1}^{\infty} (m y)^{-n} K_{n}(2 m y)$, where $K_{n}(x)$ is the MacDonald function (the second modified Bessel function) [4, 5].

5. Universal scaling functions of the entropy, the internal energy and the specific heat in a particular case

At $d = \sigma$, taking into account that $K_{3/2}(x) = \sqrt{\pi/(2x)} \exp(-x)(1 + 1/x)$, the universal scaling function (16) considerably simplifies,

$$X(x_\tau|\sigma) = -\frac{1}{2} x_\tau y_\tau^2 - \frac{2k_d}{\sigma} \left[\frac{1}{6} y_\tau^3 + y_\tau L_{2}(\exp(-y_\tau)) + L_{3}(\exp(-y_\tau))\right],$$

where $y_\tau$ is the solution of the corresponding spherical field equation,

$$2 \sinh\left(\frac{y_\tau}{2}\right) = \exp\left(-\frac{\sigma}{2k_d} x_\tau\right)$$

and $L_{ip}(x)$ is the polylogarithm function.

At $x_\tau = 0$, taking into account that the solution $y_0$ of (19) is $y_0 = 2 \ln(\sqrt{5} + 1/2)$ and using the properties of the polylogarithm functions $L_{ip}(x)$ [9], from (18) one obtains the "temporal" Casimir amplitude [6], $\Delta_{\text{Cas}}(\sigma) = X(0|\sigma) = -16\zeta(3)/[5\sigma(4\pi)^{\sigma/2} \Gamma(\sigma/2)]$, where $\zeta(x)$ is the Riemann zeta function.

After substitution of (18) in equations (5), (6) and (7), taking into account that $z \nu = 1$, and $y_\tau' = -\left(\frac{\sigma}{k_d}\right) \left[4 \exp\left(\frac{\sigma}{k_d} x_\tau\right) + 1\right]^{-1/2}$ and $y_\tau'' = 2 \left(\frac{\sigma}{k_d}\right)^2 \exp\left(\frac{\sigma}{k_d} x_\tau\right) \left[4 \exp\left(\frac{\sigma}{k_d} x_\tau\right) + 1\right]^{-3/2}$ are the first and the second derivatives of $y_\tau$ with respect to $x_\tau$, we derive the universal scaling functions of the entropy (per spin), the internal energy density and the specific heat (per spin) for a $\sigma$-dimensional system,

$$S_{\phi}(x_\tau|\sigma) = x_\tau y_\tau^2 + 6 \frac{k_d}{\sigma} \left[\frac{1}{6} y_\tau^3 + y_\tau L_{2}(\exp(-y_\tau)) + L_{3}(\exp(-y_\tau))\right] + \frac{x_\tau y_\tau}{\sqrt{4 \exp\left(\frac{\sigma}{k_d} x_\tau\right) + 1}} \left[\frac{\sigma}{k_d} x_\tau + y_\tau + 2 \ln(1 - \exp(-y_\tau))\right],$$

$$U_{\phi}(x_\tau|\sigma) = \frac{1}{2} x_\tau y_\tau^2 + 4 \frac{k_d}{\sigma} \left[\frac{1}{6} y_\tau^3 + y_\tau L_{2}(\exp(-y_\tau)) + L_{3}(\exp(-y_\tau))\right] + \frac{x_\tau y_\tau}{\sqrt{4 \exp\left(\frac{\sigma}{k_d} x_\tau\right) + 1}} \left[\frac{\sigma}{k_d} x_\tau + y_\tau + 2 \ln(1 - \exp(-y_\tau))\right],$$

$$\nu(\sigma) = \frac{1}{2} x_\tau y_\tau^2 + 4 \frac{k_d}{\sigma} \left[\frac{1}{6} y_\tau^3 + y_\tau L_{2}(\exp(-y_\tau)) + L_{3}(\exp(-y_\tau))\right] + \frac{x_\tau y_\tau}{\sqrt{4 \exp\left(\frac{\sigma}{k_d} x_\tau\right) + 1}} \left[\frac{\sigma}{k_d} x_\tau + y_\tau + 2 \ln(1 - \exp(-y_\tau))\right].$$
and

\[ C_s(x_\tau|\sigma) = x_\tau y_\tau^2 + 12 \frac{k_\sigma}{\sigma} \left[ \frac{1}{6} y_\tau^3 + y_\tau Li_2(\exp(-y_\tau)) + Li_3(\exp(-y_\tau)) \right] 
\]

\[ + \frac{4x_\tau y_\tau}{\sqrt{4 \exp\left(\frac{\sigma}{k_\sigma} x_\tau\right) + 1}} \left[ \frac{\sigma}{2k_\sigma} x_\tau + y_\tau + \frac{\sigma}{2k_\sigma} x_\tau + y_\tau + 2 \ln(1 - \exp(-y_\tau)) \right] \]

\[ + 2\frac{2\sigma}{k_\sigma} \frac{x_\tau^2 y_\tau}{4 \exp\left(\frac{\sigma}{k_\sigma} x_\tau\right) + 1} \left[ \frac{\sigma}{2k_\sigma} x_\tau + y_\tau + \frac{\sigma}{2k_\sigma} x_\tau + y_\tau + \frac{\sigma}{2k_\sigma} x_\tau + y_\tau + 2 \ln(1 - \exp(-y_\tau)) \right] \cdot (22) \]

In (20), (21) and (22) \( y_\tau \) is the solution of (19).

At \( x_\tau = 0 \) (\( \lambda = \lambda_c \), i.e. in the quantum critical region) from (20), (21) and (22), taking into account that the solution of the corresponding spherical field equation is \( y_0 \) and using the properties of the polylogarithm functions \( Li_p(x) \) [9], we obtain the universal critical amplitudes of the entropy (per spin), the internal energy density and the specific heat (per spin):

\[ S_s(0|\sigma) = -3\Delta^t_{\text{Cas}}(\sigma), \quad U_s(0|\sigma) = -2\Delta^t_{\text{Cas}}(\sigma) \quad \text{and} \quad C_s(0|\sigma) = -6\Delta^t_{\text{Cas}}(\sigma), \]

(23)
in which \( \Delta^t_{\text{Cas}}(\sigma) \) is the "temporal" Casimir amplitude. The results (23) have been presented in [7].

5.1. Asymptotes of the scaling functions

For the considered case \( d = \sigma, z = 1 \) and the scaling variable \( x_\tau \) is \( x_\tau \propto (\lambda_c - \lambda)/T \). Therefore, the asymptotes of (20), (21) and (22) at \( x_\tau \to +\infty \) (\( \lambda_c > \lambda \)) refer to the renormalized classical region and these at \( x_\tau \to -\infty \) (\( \lambda_c < \lambda \)) - to the quantum disordered region.

5.1.1. Asymptotes of the scaling functions in the renormalized classical region

At \( x_\tau \to +\infty \), taking into account that the solution of (19) is \( y_\tau \approx \exp\left(-\frac{\sigma}{2k_\sigma} x_\tau\right) \), and \( Li_p(z) \approx \zeta(p) \) for \( z \to 1 \), we obtain the asymptotes of the scaling functions (20), (21) and (22):

\[ S_s(\infty|\sigma) = \frac{15}{4} \Delta^t_{\text{Cas}}(\sigma), \quad U_s(\infty|\sigma) = -\frac{5}{2} \Delta^t_{\text{Cas}}(\sigma) \quad \text{and} \quad C_s(\infty|\sigma) = -\frac{15}{2} \Delta^t_{\text{Cas}}(\sigma), \]

(24)

where \( \Delta^t_{\text{Cas}}(\sigma) \) is the "temporal" Casimir amplitude. It is easy to verify that the following general relations

\[ \frac{S_s(\infty|\sigma)}{S_s(\infty|2)} = \frac{U_s(\infty|\sigma)}{U_s(\infty|2)} = \frac{C_s(\infty|\sigma)}{C_s(\infty|2)} = \frac{\Delta^t_{\text{Cas}}(\sigma)}{\Delta^t_{\text{Cas}}(2)} \quad \text{and} \quad \frac{S_s(0|\sigma)}{S_s(\infty|\sigma)} = \frac{U_s(0|\sigma)}{U_s(\infty|\sigma)} = \frac{C_s(0|\sigma)}{C_s(\infty|\sigma)} = \frac{4}{5}, \]

(25)
hold. When \( \sigma \neq 2 \) the ratio \( \Delta^t_{\text{Cas}}(\sigma)/\Delta^t_{\text{Cas}}(2) \) is a decreasing function of \( \sigma \) [6]. The last result in (25) can be related with the value of the normalization factor \( \tilde{c} = 4/5 \), obtained for the model (12) at \( d = \sigma \) in [6].

5.1.2. Asymptotes of the scaling functions in the quantum disordered region

At \( x_\tau \to -\infty \), taking into account that the solution of (19) is \( y_\tau \approx -\frac{\sigma}{k_\sigma} x_\tau \), and \( Li_p(z) \approx z \) for \( z \to 0 \), we obtain that the scaling functions (20), (21) and (22) behave as:

\[ S_s(-\infty|\sigma) \approx \frac{2\sigma}{k_\sigma} x_\tau^2 \exp\left(\frac{\sigma}{k_\sigma} x_\tau\right), \]

(26)
\[ U_s(-\infty|\sigma) \approx -\frac{1}{6} \left( \frac{\sigma}{k_\sigma} \right)^2 \tau^3 \]  
\hfill (27)

and

\[ C_s(-\infty|\sigma) \approx -2 \left( \frac{\sigma}{k_\sigma} \right)^2 \tau^3 \exp \left( \frac{\sigma}{k_\sigma} \tau \right). \]  
\hfill (28)

6. Conclusion

The entropy, the internal energy and the specific heat of a \( d \)-dimensional quantum critical system with a long-range interaction, (12), are studied in the context of the FSS theory for \( d = \sigma \), where \( 0 < \sigma \leq 2 \) is a parameter, controlling the decay of the long-range interaction.

For a \( \sigma \)-dimensional system \( z \nu = 1 \) (\( z = \frac{d}{2} \) and \( \nu^{-1} = \frac{d}{2} \)) and the specific heat critical exponent \( \alpha \), defined in analogy with this one of a classical system by (8), satisfies the hyperscaling relation for quantum systems (11). The obtained from (11) critical exponent \( \alpha \) does not depend on \( \sigma \), \( \alpha = -1 \). This result is in accordance with [6], where for the model (12) it has been shown that the case of long-range interactions \( d = \sigma \) is a generalization of the case of short-range interactions \( d = 2 \).

Based on the derived here asymptotes (24), (26), (27) and (28) of the universal scaling functions (20), (21) and (22), the critical behavior of the considered quantities can be analyzed in the renormalized classical and the quantum disordered regions of the phase diagram. After replacing the asymptotes (24) in the corresponding scaling forms (2), (3) and (4), it can be seen that in the renormalized classical region the entropy \( S \), the internal energy \( U \) and the specific heat \( c \) at \( T \to +0 \) tend to zero as powers of the temperature \( T \): \( S \propto T^2 \), \( U \propto T^3 \) and \( c \propto T^2 \). Setting the asymptotes (26), (27) and (28) in the corresponding scaling forms (2), (3) and (4), one obtains that in the quantum disordered region, when \( T \to +0 \), the entropy and the specific heat tend to zero exponentially, \( S \propto \exp \left( -\frac{\sigma}{k_\sigma} \frac{|\delta\lambda|}{T} \right) \) and \( c \propto \exp \left( -\frac{\sigma}{k_\sigma} \frac{|\delta\lambda|}{T} \right) \), while the internal energy depends only on the deviation \( \delta\lambda \) from the quantum critical point \( \lambda_c \), \( U \propto |\delta\lambda|^3 \).

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References

[1] Sachdev S 2011 *Quantum Phase Transitions* (Cambridge: Cambridge University Press).
[2] Chamati H and Tonchev N S 2000 *J. Phys. A: Math. Gen.* 33 873.
[3] Vojta T 1996 *Phys. Rev. B* 53 710.
[4] Chamati H, Danchev D M, Pisanova E S and Tonchev N S 1997 Low-temperature regimes and finite-size scaling in a quantum spherical model. Preprint IC/97/82 (Trieste, Italy, July 1997); Preprint cond-mat/9707280.
[5] Chamati H, Pisanova E S and Tonchev N S 1998 Phys. Rev. B 57 5798.
[6] Chamati H, Danchev D M and Tonchev N S 2000 *Eur. Phys. J. B* 14 307.
[7] Pisanova E S and Ivanov I K 2019 *AIP Conf. Proc.* 2075 020010.
[8] Griffith M A and Continentino M A 2018 *Phys. Rev. E* 97 012107.
[9] Sachdev S 1993 *Phys. Lett. B* 309 285.