LETTER

Discrete changes of current statistics in periodically driven stochastic systems

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Received 28 March 2010
Accepted 14 June 2010
Published 23 July 2010

Abstract. We demonstrate that the counting statistics of currents in periodically driven ergodic stochastic systems can show sharp changes of some of its properties in response to continuous changes of the driving protocol. To describe this effect, we introduce a new topological phase factor in the evolution of the moment generating function which is akin to the topological geometric phase in the evolution of a periodically driven quantum mechanical system with time-reversal symmetry. This phase leads to the prediction of a sign change for the difference of the probabilities to find even and odd numbers of particles transferred in a stochastic system in response to cyclic evolution of control parameters. The driving protocols that lead to this sign change should enclose specific degeneracy points in the space of control parameters. The relation between the topology of the paths in the control parameter space and the sign changes can be described in terms of the first Stiefel–Whitney class of topological invariants.

Keywords: driven diffusive systems (theory), current fluctuations, large deviations in non-equilibrium systems

ArXiv ePrint: 1003.3905
1. Introduction

When a system is driven using a periodic protocol, dimensionless currents can be introduced that describe the fluxes per driving period, rather than per unit time. The appearance of stochastic currents in response to periodic changes of parameters is generally referred to as the stochastic pump effect [1]. Statistical properties of such currents usually change continuously with changes of the shape of the parameter path. This is, to some degree, expected because usually small changes of kinetic rates in an ergodic Markovian stochastic system, which consists of only a finite number of discrete states, do not lead to sharp changes of system behavior. Discrete changes of current characteristics usually can be achieved in stochastic systems either with a formally infinite number of states or in the limit of low temperature [2]–[4] when fluctuations are suppressed and noise cannot smear the behavior near points of parameter degeneracy. In this letter, we confront this view and demonstrate that some of the statistical characteristics of currents can depend discontinuously on the choice of the driving protocol; moreover, this property can be found in finite-size ergodic Markov chains.

2. Moment generating function of currents in a two-state model

Consider a simple model in figure 1: a particle can randomly jump between two sites along two paths. Each path can be passed in both directions. We assume that the system is coupled to a heat bath at constant temperature so that kinetic rates, \( k^\alpha_j \), of transitions from state \( j \) into the other state via the path \( \alpha (j, \alpha = 1, 2) \) satisfy the detailed balance condition and can be parameterized as \( k^\alpha_j = \kappa_j g_\alpha \), where \( \kappa_j = e^{\beta E_j} \), \( g_\alpha = e^{-\beta W_\alpha} \) and \( \beta = 1/(k_B T) \) is the inverse temperature; \( E_j \) is called the energy of the potential well \( j \) and \( W_\alpha \) is the height of the potential barrier of the path \( \alpha \).
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Figure 1. A model of stochastic transitions between two states along two possible paths. Different paths are characterized by sizes of corresponding potential barriers $W_1$ and $W_2$. The two states are characterized by sizes of their well depths, $E_1$ and $E_2$. Periodic modulation of these parameters leads to the appearance of nonzero, on average, circulating current in the preferred (clockwise or counterclockwise) direction.

We will investigate the currents that circulate in a counterclockwise direction in figure 1 under the action of a cyclic periodic perturbation. Those currents are stochastic and, generally, can be characterized by two sets of probabilities $\pi_j(n; t)$ so that at time $t$ the system is at state $j$ having performed $n$ transitions on the time segment $[0, t]$ through link 1 (with the barrier $W_1$), counting a counterclock/clockwise transition with the ± sign, respectively. In what follows, we will notationally suppress the time parameter $t$ in $\pi_j(n; t)$ and other variables. The master equation for $\pi_j(n)$ is given by

$$
\dot{\pi}_1(n) = -(k_1^1 + k_1^2)\pi_1(n) + k_2^1\pi_2(n-1) + k_2^2\pi_2(n),
\dot{\pi}_2(n) = -(k_1^2 + k_2^2)\pi_2(n) + k_1^1\pi_1(n+1) + k_1^2\pi_1(n).
$$

Upon a Fourier transform, (1) translates into the following equation:

$$
\dot{Z} = \hat{H}(\chi; t)Z, \quad Z = (Z_1, Z_2), \quad Z_j(\chi; t) = \sum_{n=-\infty}^{\infty} \pi_j(n; t)e^{i\chi n},
$$

with the ‘twisted’ master operator:

$$
\hat{H}(\chi) = \begin{pmatrix}
-k_1(g_1 + g_2) & \kappa_2(g_1e^{i\chi} + g_2) \\
\kappa_1(g_1e^{-i\chi} + g_2) & -\kappa_2(g_1 + g_2)
\end{pmatrix}.
$$

The average over the final state MGF $Z(\chi; t) = Z_1(\chi; t) + Z_2(\chi; t)$ which contains complete information on the current distribution function, including the statistics of rare events, is obtained by solving equation (2) and can be expressed in terms of a time-ordered exponential $[5]$:

$$
Z(\chi; t) = \langle 1| T \exp \left( \int_0^t \hat{H}(\chi, t') dt' \right) |\pi\rangle,
$$

where $\langle 1| = (1, 1)$ and $|\pi\rangle = (\pi_1, \pi_2)$ is the vector of the initial system populations.

doi:10.1088/1742-5468/2010/07/L07001
3. Degeneracy point of $\hat{H}(\chi)$ eigenvalues

Lowest current cumulants are described by the properties of $Z(\chi)$ near the $\chi = 0$ point, and their behavior in the limit of adiabatically slow evolution of parameters is always defined by the eigenvalue of $\hat{H}(\chi)$ with the largest real part and the corresponding eigenvector [6]. In the vicinity of $\chi = 0$, the eigenvalues of $\hat{H}(\chi)$ are nondegenerate if all kinetic rates in (3) are nonzero (finite temperature). This follows from the observation that $\hat{H}(0)$ describes the evolution of probabilities in a two-state Markov chain [6]. Such an evolution matrix has a unique zero-mode solution corresponding to the unique steady state of an ergodic Markov chain. This uniqueness guarantees that the eigenvalue of $\hat{H}(\chi)$ with the largest real part is nondegenerate for sufficiently small $\chi$.

We will investigate the effect of geometric phases on the full counting statistics of currents that do not apparently appear when only lowest cumulants of the current distribution are measured, namely the matrix (3) has the unique point in the space of parameters where its eigenvalues are degenerate. This happens when simultaneously three conditions $\chi = \pi$, $W_1 = W_2$ and $E_1 = E_2$ are satisfied, which can be seen by explicitly inspecting the eigenvalue problem for the $2 \times 2$ matrix given by equation (3).

Interestingly, the degeneracy conditions do not depend on the inverse temperature $\beta$. The fact that the degeneracy point is encountered at a finite value of the counting parameter, $\chi = \pi$, means that it influences the properties of the full counting statistics rather than the lowest cumulants of a current distribution. We note also that at $\chi = \pi$ the matrix $\hat{H}(\pi)$ has all real entries.

The MGF at $\chi = \pi$:

$$Z(\pi) = P_0 - P_1 \equiv \sum_{j=1; k=-\infty}^{2; \infty} \pi_j(2k) - \sum_{j=1; k=-\infty}^{2; \infty} \pi_j(2k + 1),$$

has a simple physical meaning, namely it is the difference of the probabilities $P_0$ and $P_1$ to make an even and odd number of transitions through a given link, respectively.

The time evolution of $P_0 - P_1$ must be influenced by the presence of the degeneracy point. The evolution operator $\hat{H}(\pi)$ has two nonpositive eigenvalues:

$$\varepsilon_{\pm} = -\kappa_+ g_+ \pm \sqrt{\kappa_2^2 g_2^2 + \kappa_-^2 (g_-^2 - g_2^2)},$$

with $\kappa_{\pm} = (\kappa_1 \pm \kappa_2)/2$ and $g_{\pm} = g_1 \pm g_2$. Near their degeneracy point, we parameterize $g_- \kappa_+ = r \sin \phi$ and $\kappa_- g_+ = r \cos \phi$ which yields $\varepsilon_{\pm} \approx -\kappa_+ g_+ \pm r$ for the eigenvalues in the limit $g_- \ll g_+, \kappa_- \ll \kappa_+$. The eigenvalues as functions of $(r, \phi)$ lie on the two-cone surface. Taking the higher eigenvalue $\varepsilon_+$, the eigenstate $|u_+(\pi)\rangle \equiv (u_1, u_2)$, determined by solving a $2 \times 2$ linear problem, has a form $(u_1, u_2) = a(\phi)(\sin(\phi/2), -\cos(\phi/2))$, with $a(\phi)$ being an arbitrary periodic function of $\phi$. The requirement on the eigenstate not to be zero imposes that $a(\phi)$ is either a strictly positive or strictly negative function. However, the functions $\sin(\phi/2)$ and $\cos(\phi/2)$ change sign when changing $\phi$ around a full cycle. If one chooses the smooth field of eigenstates of $\hat{H}(\pi)$ over the space of all possible values of parameters $E_i$ and $W_j$, except the degeneracy point, such a vector field has to be double-valued and, along a nonintersecting path around the degeneracy point, the eigenvectors of $\hat{H}(\pi)$ change sign. To understand, how this effect can influence the stochastic currents, we should look at the evolution of MGF in response to externally induced periodic changes of the kinetic rates in the model.

doi:10.1088/1742-5468/2010/07/L07001
4. Evolution of MGF

4.1. The case of constant parameters

When all parameters in the evolution operator are time-independent the time-ordered exponential in (4) becomes just a matrix exponent, which can be simplified by introducing the left and right eigenstates $\langle u_\pm |$ and $| u_\pm \rangle$ of the matrix $\hat{H}_\pi$, corresponding to eigenvalues in (6). Then (4) gives us

$$P_0 - P_1 = C_+ e^{\epsilon_+ t} + C_- e^{\epsilon_- t}, \quad C_\pm = \langle 1 | u_\pm (t = 0) \rangle \langle u_\pm (t = 0) | \pi \rangle. \quad (7)$$

Since both eigenvalues $\epsilon_\pm$ are negative, the absolute value of $P_0 - P_1$ decays exponentially with time. Moreover, since $\epsilon_+ > \epsilon_-$, the term $C_- e^{\epsilon_- t}$ quickly becomes exponentially suppressed in comparison with $C_+ e^{\epsilon_+ t}$. The coefficients $C_\pm$ are totally determined by the initial probability distribution and kinetic rates. Their values describe initial relaxation processes to the steady-state distribution and remain the same even if kinetic rates are changing with time adiabatically [7]. At timescales larger than $1/(\epsilon_+ - \epsilon_-)$ the sign of $P_0 - P_1$ is determined by the sign of the coefficient $C_+$, which can be found and analyzed explicitly. Depending on the initial choice of parameters, including initial state probabilities, this coefficient can be either negative or positive. However, the sign of $P_0 - P_1$ is fixed after the term with coefficient $C_-$ becomes exponentially suppressed. In other words, the sign of $P_0 - P_1$ is the invariant of the evolution which can take values in a discrete two-state set.

4.2. Geometric phase

Here we ask a question whether the sign of $P_0 - P_1$ can be switched by adiabatically slow evolution of the parameters $g_-$ and $\kappa_-$ around a cycle. In the adiabatic limit, a naive strict quasi-steady-state approximation predicts that a slow perturbation merely modulates the decay exponents in (7) and the vector $Z(\pi)$ is given by the instantaneous eigenvector $| u_+ \rangle$. This means that, in principle, the sign of $P_0 - P_1$ is determined by the sign of the coefficient $C_+$, which can be found and analyzed explicitly. Depending on the initial choice of parameters, including initial state probabilities, this coefficient can be either negative or positive. However, the sign of $P_0 - P_1$ is fixed after the term with coefficient $C_-$ becomes exponentially suppressed. In other words, the sign of $P_0 - P_1$ is the invariant of the evolution which can take values in a discrete two-state set.

First, consider the case when a cyclic path in the space of control parameters does not enclose the point $g_- = \kappa_- = 0$. We can always choose a smooth parametrization of the eigenvectors of $\hat{H}(\pi)$ in the region $S_C$ inside the contour. According to [1,6], [9]–[11], $P_0 - P_1$, after a periodic and adiabatically slow evolution of parameters around the contour $C$ during time $T$, is given by

$$P_0 - P_1 = C_+ \exp \left( \int_0^T \epsilon_+(t) \, dt + \varphi_{\text{geom}}(C) \right), \quad (8)$$

where the geometric contribution to the exponent in (8) is given by

$$\varphi_{\text{geom}}(C) = \int_{S_C} dg_- d\kappa_- F_{g_-,\kappa_-}. \quad (9)$$

doi:10.1088/1742-5468/2010/07/L07001
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Figure 2. Evolution of the parity probability difference in a driven two-state stochastic system. The choice of parameters is $\kappa_+ = 2$, $g_+ = 1$, $\kappa_- = 0.8 + 2\cos(2\pi t/T)/3$, $g_- = 0.4 + \sin(2\pi t/T)/3$ and $T = 30$. The contour in the space of control parameters does not enclose the degeneracy point.

where $F_{g_-,\kappa_-} = \langle \partial_{g_-} u_+ | \partial_{\kappa_-} u_+ \rangle - \langle \partial_{\kappa_-} u_+ | \partial_{g_-} u_+ \rangle$ or explicitly

$$F_{g_-,\kappa_-} = -\frac{\kappa_- g_- \kappa_+ g_+}{2(\kappa_+^2 g_-^2 + \kappa_-^2 (g_+^2 - g_-^2))^{3/2}}.$$

The eigenvalues $\varepsilon_+(t)$ and the geometric phase $\varphi_{\text{geom}}(C)$ in (8) are both real, and therefore the exponential term does not affect the sign of $P_0 - P_1$. Figure 2 shows the results of our numerical solution of evolution equation (2) at $\chi = \pi$. The fact that $P_0 - P_1$ has the same sign during the whole evolution is not generic. Generally, it can change but the fact that it does not change after each period of driving protocol is always confirmed if the contour does not enclose the degeneracy point. In other words, the sign of $P_0 - P_1$ is invariant to the adiabatic change of parameters along a closed contour that does not enclose the degeneracy point.

4.3. Robust switching of the sign of $P_0 - P_1$

We now consider the case when the contour is cyclic and encloses the degeneracy point $g_- = \kappa_- = 0$. The previous arguments cannot be applied directly since at the degeneracy point the adiabatic approximation breaks down and the parametrization of eigenstates inside the contour cannot be chosen smoothly. They do, however, imply that $\text{sgn}(P_0 - P_1)$ does not change upon deformations of $C$ that do not encounter the degeneracy point, reducing our analysis to the contours, restricted to a small neighborhood of the degeneracy point. Our analysis, in section 3, of the vicinity of the crossing point showed that the smooth choice of eigenstates along a contour requires that, upon completion of the cycle, the eigenvector changes sign. Physically, this change of sign means that the probability difference $P_0 - P_1$ changes sign. Our numerical solution of equations (2) for evolution along a contour that encloses the degeneracy point is shown in figure 3. It confirms the sign change prediction. Note that this result is non-perturbative. It cannot appear as an effect of small non-adiabatic corrections to the decaying exponent in (8). The
Figure 3. Evolution of the probability difference \( P_0 - P_1 \) in a driven two-state stochastic system when a contour in the space of control parameters encloses the degeneracy point. The choice of parameters is \( \kappa_+ = 1, g_+ = 1, \kappa_- = 2 \cos(2\pi t/T)/3, g_- = \sin(2\pi t/T)/3 \) and \( T = 30 \).

The sign of the probability difference in (5) is, in fact, a topological invariant of the driving protocol, the latter being viewed as a cycle in the space \( M \cong \mathbb{R}^2 \setminus \{0\} \) of parameters \( x = (g_-, \kappa_-) \) where degeneracy of \( \tilde{H}(\pi; x) \) never occurs. By associating with \( x \in M \) the ground state of \( \tilde{H}(\pi; x) \), the former defined up to a real multiplicative factor, we build a linear real fiber bundle \( E \) over \( M \), or equivalently a map \( g : M \to \mathbb{RP}^1 \). Here the projective space \( \mathbb{RP}^1 \) is interpreted as a circle \( S^1 \) of ‘normalized states’ \( (u_1)^2 + (u_2)^2 = 1 \), with the opposite points on the circle identified, since there is no preferred way to define the sign of the ground state. The map \( g \) associates with a driving protocol \( C \) a cycle \( g(C) \) in \( \mathbb{RP}^1 \cong S^1 \), and each time \( C \) winds around the degeneracy point, \( g(C) \) produces a single rotation in \( \mathbb{RP}^1 \), so that \( n(C) = n(g(C)) \). The last statement can be reformulated in terms of the first Stiefel–Whitney class \( c_1 = c_1(E) \) [12] of \( E \) which can be viewed as...
a $\mathbb{Z}_2$-valued function on cycles in $M$, so that $c_1(C) = n(g(C)) \mod 2$ and the invariant $(-1)^{n(C)} = (-1)^{c_1(C)}$ of the counting statistics is obtained by evaluating the Stiefel–Whitney class on the driving protocol.

In this letter we studied in some detail a $\mathbb{Z}_2$-invariant in a simple two-node system. In a general network represented by a finite connected graph there is a set of invariants we are going to describe briefly here, with the analysis of the general case postponed to a further publication. With any ‘independent’ set $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ of links we can associate a $\mathbb{Z}_2$-invariant $I_\alpha(C) = \text{sgn} \sum_a (-1)^{\sum_{\alpha_j} a_j} P_\alpha(\alpha; C)$, where $a = \{a_1, \ldots, a_k\}$ is a set of binary variables and $P_\alpha(\alpha; C)$ is the joint probability that, for protocol $C$, all the number of jumps over link $\alpha_j$ was even/odd for $a_j = 0, 1$, respectively. The protocol is assumed to go through the space $M$ of parameters that avoids the ground-state degeneracy of the twisted operator $\hat{H}(\chi)$, with $\chi_\gamma = \pi$ for chosen links $\gamma \in \alpha$ and $\chi_\gamma = 0$ otherwise. Similar to the simple two-node case the ground state defines a linear bundle $E_\alpha$ over $M_\alpha$ with the first Stiefel–Whitney class $c_1 = c_1(E_\alpha)$, or equivalently a map $g : M_\alpha \to \mathbb{RP}^N$ with $N$ being the number of nodes in the network. Viewing $\mathbb{RP}^{N-1}$ as an $(N - 1)$-sphere with opposite points identified we have a twofold cover $S^{N-1} \to \mathbb{RP}^{N-1}$. Since all cycles in $S^{N-1}$ are contractible, there are two equivalence classes of cycles $s$ in the projective space: the ones produced by cycles in $S^{N-1}$ and the ones produced by contours that connect opposite points in $S^{N-1}$; this is referred to as $[s] = 0$ and $s = 1$, respectively. The topological factor associated with a driving protocol is given by $(-1)^{[g(C)]} = (-1)^{c_1(C)}$ so that the counting statistics invariant $I_\alpha(C) = (-1)^{c_1(C)} \text{sgn} C_+$, with $C_+$ depending on the initial distribution only, being independent of the driving protocol.

The topological factor is very similar to the topological Berry phase in quantum mechanics of systems with time-reversal symmetry (no magnetic fields), where $\hat{H}(x)$ is a family of Hamiltonians, represented by real Hermitian operators [13,8]. In our case the operators are real but, however, apparently non-Hermitian. They are, in fact, Hermitian with respect to a scalar product that depends on the kinetic rates. The latter results in the geometric contribution $\varphi_{\text{geom}}(C)$ in equation (8) that does not affect $\text{sgn}(P_0 - P_1)$.

6. Discussion

We proved that some of the properties of the full counting statistics of currents in finite-size stochastic models can be controlled by periodic changes of control parameters without continuous dependence of the result of the operation on the choice of the path in the parameter space. This finding reveals the importance of the degeneracy points of eigenvalues of the matrix $\hat{H}(\chi)$, governing the evolution of the current MGF. Although the effect is the property of the full counting statistics [5], [14]–[18], it is measurable experimentally because the probabilities of rare events in the model that we considered are suppressed, at least exponentially. As a result, only a finite number of terms in the definition of the parity difference (5) should be determined, which is achievable by repeating the measurement of the number of transitions through a link after sufficiently many cycles. The simplicity of our model suggests that the topological geometric factors can be a generic property of the current statistics. The role of other topological invariants, such as Chern numbers, in the evolution of the current counting statistics remains to be understood.

doi:10.1088/1742-5468/2010/07/L07001
Acknowledgments

We are grateful to John R Klein for useful discussions and comments. This material is based upon work supported by NSF under grant nos. CHE-0808910 and ECCS-0925365, and in part by DOE under contract no. DE-AC52-06NA25396.

References

[1] For a review, see Sinitsyn N A, 2009 J. Phys. A: Math. Theor. 42 193001
[2] Chernyak V Y and Sinitsyn N A, 2009 J. Chem. Phys. 131 181101
[3] Astumian R D and Dernyi I, 2001 Phys. Rev. Lett. 86 3859
[4] Shi Y and Niu Q, 2002 Europhys. Lett. 59 324
[5] Bagrets D A and Nazarov Y V, 2003 Phys. Rev. B 67 085316
[6] Sinitsyn N A and Nemenman I, 2007 Eur. Phys. Lett. 77 58001
[7] Sinitsyn N A and Nemenman I, 2010 arXiv:1001.4212
[8] Mehri-Dehnavi H and Mostafazadeh A, 2008 J. Math. Phys. 49 082105
[9] Ohkubo J, 2008 J. Stat. Mech. P02011
[10] Sinitsyn N A and Nemenman I, 2007 Phys. Rev. Lett. 99 220408
[11] Ren J, Hänggi P and Li B, 2010 Phys. Rev. Lett. 104 170601
[12] Milnor J W and Stasheff J D, 1974 Characteristic Classes (Princeton, NJ: Princeton University Press)
[13] Herzberg By G and Longuet-Higgins H C, 1963 Discuss Faraday Soc. 35 77
[14] Jarzynski C, 1997 Phys. Rev. Lett. 78 2690
[15] Bochkov G N and Kuzovlev Yu E, 1977 Zh. Eksp. Teor. Fiz. 72 238
[16] Crooks G, 1998 J. Stat. Phys. 90 1481
[17] Buttiker M and Moskalets M, 2006 Lect. Notes Phys. 690 33
[18] Elgart V and Kamenev A, 2004 Phys. Rev. E 70 051205

doi:10.1088/1742-5468/2010/07/L07001