Limits of noise squeezing in Kerr effect

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It is well known that the optical Kerr effect can be a source of highly squeezed light, however the analytical limit of the noise suppression has not been found yet. The process is reconsidered and an analytical estimation of the optimal quadrature noise level is presented. The validity of the new scaling law is checked numerically and analytically.

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1. INTRODUCTION

The cubic non-linearity of isotropic media, like liquid or fiber, can be responsible for the self-phase modulation, self-induced rotation of polarization, self-modulation of intensity profile or self-focusing of strong optical beam. These phenomena can simply be explained by a linear dependence of the refraction index on the intensity of the propagating light, which is usually referred to as the optical Kerr effect. This effect plays also an important role in, e.g., the non-diffracting beam technics, soliton propagation or non-demolition measurements. From the viewpoint of quantum optics, the Kerr effect can be a source of the squeezed light [1–11] (for a review see Ref. [12]), also referred to as the self-squeezed light. Unfortunately, the Kerr non-linearity is usually very small, thus the effective application of the Kerr effect requires the long-interaction times (or lengths) and high-intensity lasers. The simplest strategy for the effective Kerr process is to use a long optical fiber and strong laser pulses. In recent experiments with the Kerr fibers, the quantum noise was successfully reduced by 0.7 dB [13] or even by 3.5 dB [14] (for a discussion of some experimental results see [3,15]).

Quantum description of the Kerr-type evolution is among a few non-trivial quantum dynamical models, which are fully solvable. This is one of the reasons for its long-term popularity and fundamental importance. Mathematically, the Kerr process can simply be described by the well-known interaction Hamiltonian

$$\hat{H} = \frac{1}{2} \hbar g \hat{a}^{\dagger 2} \hat{a}^2 = \frac{1}{2} \hbar g \hat{n} (\hat{n} - 1),$$

where $\hat{a}$ and $\hat{a}^{\dagger}$ are, respectively, the annihilation and creation operators satisfying the boson commutation relation; $\hbar$ is the Planck constant and $g$ is the Kerr non-linearity. Since the photon-number operator $\hat{n} = \hat{a}^{\dagger} \hat{a}$ commutes with the Hamiltonian, the photocount statistics of the field propagating through the Kerr medium is conserved. Nevertheless, the photon number fluctuations can be reduced via the Kerr effect, e.g., using the nonlinear Mach-Zehnder interferometer with the Kerr medium in one of the arms [2].

Assuming the input field to be in coherent state $|\alpha\rangle$, the solution of the Schrödinger equation is the Kerr state [12]

$$|\psi_K\rangle = \exp \left[ -\frac{i\tau}{2} \hat{n} (\hat{n} - 1) \right] |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp \left[ -\frac{i\tau}{2} n (n - 1) \right] |n\rangle,$$

where $\tau = gL/v_g$ is the dimensionless rescaled interaction length of the Kerr medium or the interaction time; $v_g$ is the group velocity of light in the medium. The only and trivial difference between the spatial and temporal evolution is the opposite sign in front of $\tau$. Thus, we use the terms interaction time or length interchangeably.

II. NOISE SQUEEZING ANALYSIS

The Kerr dynamics has been thoroughly studied from different points of view. Many results predicted by these theoretical studies are well known. For example, it can be proved that the time evolution is exactly periodic with the period $2\pi$. At the time moments $\tau_N = 2\pi M/N$, being rational fractions of the period, the input coherent state is transformed into a superposition of two [16] or more [17] well-localized coherent states, referred to as the Schrödinger cats and kittens, respectively. For the present study, the most important result is the fact that the Kerr states evolve periodically into the quadrature noise squeezed states [1,4–10]. By defining the general quadrature component as $X_\theta = \hat{a} \exp (i\theta) + \hat{a}^{\dagger} \exp (-i\theta)$, where $\theta$ represents the controlled phase of a homodyne detector, the principal squeezing can be defined as the minimum of all the quadrature variances with respect to all the possible phases $\theta$ and the following relation holds [6,18]

$$S = \min_\theta \text{Var} \ X_\theta$$

$$= 1 + 2 \left( \langle \hat{a} \hat{a}^{\dagger} \rangle - |\langle \hat{a} \rangle|^2 \right) - 2 \left| \langle \hat{a}^{\dagger} \rangle - \langle \hat{a} \rangle \right|^2.$$

(a) (b)
The time dependence of $S$ for some amplitudes is presented in figure 1. As for the Kerr state, the evolution of the principal squeezing is periodic with the period of $2\pi$. Moreover, $S$ is an even function, therefore it is sufficient to study the evolution of $S$ in the interval of $(0, \pi)$ representing a half of the whole period. From figure 1 we can see that the principal squeezing (4) starts from the vacuum noise level $S(0) = 1$ and then evolves into the squeezed states with $S < 1$ for some limited time interval. At the moment $\tau_{\text{min}}$ the quantum noise is maximally squeezed up to the value $S_{\text{min}} = S(\tau_{\text{min}})$. The phase portrait of the state has a crescent shape. The noise level can be very highly squeezed, especially for the large input amplitude $|\alpha|$, but it never reaches the absolute zero. In the following, we will give an analytical estimation of the maximum possible noise squeezing. For larger interaction times, the quantum noise of the Kerr state goes over from the vacuum noise level $S = 1$ up to the value of $S \approx 1 + 2|\alpha|^2$. That maximum is very flat (see figure 1(a)) and not particularly interesting for our purpose. The corresponding phase portrait rapidly changes from the crescent shape to the ring or to a discrete superposition of coherent states. At the time $\tau = \pi/2$, the Kerr state is an equally-weighted superposition of four coherent states. At last, close to $\tau \approx \pi$, the quantum noise drops down to the vacuum noise level $S \approx 1$. The phase portrait at this moment corresponds to an equally-weighted superposition of two coherent states.

The Kerr non-linearity is usually very small and, practically, the interaction times of the order of $\tau \approx 10^{-6}$ [20,12] can be reached in the optical domain. Fortunately, as follows from the above analysis and presented in figures 1(b) and 2, the highest noise suppression can be expected for short interaction times. It is worth noting that much higher Kerr-type non-linearities are observed in atom optics in, e.g., schemes based on Raman laser excitations of a trapped atom [21].

The noise power (4) depends on the interaction time $\tau$ and the input coherent amplitude $|\alpha|$. Instead of the amplitude, the Kerr parameter $r = |\alpha|^2\tau$ is often used as a good measure of the Kerr interaction. For intense laser pulses, the Kerr parameter can reach the values of the order of unity [15]. In real experiments, the interaction time is fixed by the optical fiber length, but the optical power, and thus the Kerr parameter, can simply be controlled. In the following, we will study the principal squeezing $S(r)$ in its dependence on the Kerr parameter for the fixed interaction times.

From the point of view of potential applications, it is interesting to know the level of the minimum noise, which can be suppressed in the Kerr process. Equation (4) is tricky to find the minimum of quantum noise level analytically. In the former studies, it was solved only numerically and analyzed graphically. We have plotted the minima of the noise level for some values of the scaled interaction times in figure 2. The regular dependence of the noise minimum on the interaction time is evident. From the linear dependence of the minimum points on the logarithmic scale we can guess the scaling law of the form $S_{\text{min}} \sim r_{\text{min}}^7$, which will be derived in next section.

It is even hard to analyze numerically equation (4). The trouble comes from computations that require extra computer precision. For example, the curves in figure 2 for parameters $\tau < 10^{-6}$ cannot be computed directly from (4) using the standard 16-digit arithmetics. The correct computation requires a special computing system, like MAPLE or MATHEMATICA, that enables at least the 100-digit arithmetics. On the other hand, the same numerical problems can sometimes be overcome by applying the saddle-point technique [8].

III. NOISE SQUEEZING APPROXIMATION

Here, as the main result of this Letter, we will derive an approximation of (4) for large amplitudes and short interaction times. This approximation can be used for simple calculation of the noise power bypassing the computer rounding error and will be used further for finding an analytic estimation of the quadrature noise minimum.

First, we exclude the amplitude from (4) by the substituting $|\alpha|^2 = r/\tau$. Assuming that the interaction time $\tau$ is very short, we can expand the terms $\cos \tau - 1$, $\exp(-2i\tau) - 1$ and $\exp(-i\tau) - 1$ in Taylor series. To derive a useful approximation, we assume the Kerr parameter
noise level. Equation (6) depends on the Kerr parameter which is indeed not sufficient to determine the minimum
circles represent apparently lie on the straight line on the log-log scale. The
used to estimate the minimum quadrature noise with a good
figure shows that, for (8). Note a good accuracy of our estimation of the minimum
The accuracy of the approxima-
tions given by (4), with
leading to $S'_{\text{min}} = 5/(12r^2_{\text{min}})$. This estimation of the
minimun noise level is an analytical confirmation of the
law as observed in figure 2. We have shown
that the minimum level of quadrature noise is proportional
to the 2/5th power of the interaction time or the
fiber length. The shorter interaction time $\tau \pmod{2\tau}$
results in the deeper reduction of noise, but the higher
optical power is required to reach the minimum since it
holds $|\alpha|^2 = r_{\text{min}}/\tau \sim \tau^{-6/5}$. For example, to obtain
$S'_{\text{min}} \approx 4.5 \times 10^{-3}$ at the interaction time $\tau \approx 10^{-6}$, the
optimum values of the Kerr parameter and the intensity are $r_{\text{min}} \approx 9.6$ and $|\alpha|^2 \approx 10^3$, respectively. A compar-
son of the exact values $S$ and $\tau$, calculated from (4), with
their estimations $S'$ and $\tau'$, given by the scaling law (8),
is presented in table 1 and figure 2, where the estimations
are marked by small circles.

$r$ only and thus gives physical grounds for the introduction of $r$. As shown in figure 3, the $S_0$ approximation
is monotonic in $r$ and has the only minimum of $S_0 = 0$
in the limit of $r \rightarrow \infty$. For sufficiently large $r$, $S_0$
can be estimated as $S_0 \approx 1/(4r^2)$. On the other hand, for
the relatively small Kerr parameters, (6) can be approxi-
imated linearly as $S_0 \approx 1 - 2r$. The vast majority of the
up-to-date measurements have been performed in the domain
where the approximation (6) is valid.

By assuming that the Kerr parameter of the minimum searched is large (see the values in Table 1), we can even more simplify (5). We expand (5) in the asymptotic se-
ries for large $r$ and keep the largest two terms only. Thus,
we obtain

$$S_1 \approx S' = \frac{1}{4r^2} + 2r^3 \tau. \quad (7)$$

From this elementary approximation we obtain directly
a formula for the minimum of the principal squeezing

$$S'_{\text{min}} = \frac{5}{12} (12r)^{2/5} \quad \text{at } r'_{\text{min}} = (12r)^{-1/5} \quad (8)$$

which is linear in $\tau$. The accuracy of the approxima-
tion (5) in comparison to the exact values of squeezing
for two different values of $\tau$ can be seen in figure 3. In
the less accurate approximation, the linear correction in
(5) can be ignored. In that case we get the well-known
approximation [8,15]

$$S_0 = 1 + 2r^2 - 2r \sqrt{1 + r^2}, \quad (6)$$

which is indeed not sufficient to determine the minimum
noise level. Equation (6) depends on the Kerr parameter

$\tau$ & $S_{\text{min}}$ & $S'_{\text{min}}$ & $r_{\text{min}}$ & $r'_{\text{min}}$ \\
--- & --- & --- & --- & --- \\
$10^{-1}$ & 0.476 & 0.448 & 0.627 & 0.964 \\
$10^{-2}$ & 0.196 & 0.178 & 1.312 & 1.528 \\
$10^{-3}$ & $7.503 \times 10^{-2}$ & $7.103 \times 10^{-2}$ & 2.288 & 2.421 \\
$10^{-4}$ & $2.899 \times 10^{-2}$ & $2.828 \times 10^{-2}$ & 3.755 & 3.839 \\
$10^{-6}$ & $4.501 \times 10^{-3}$ & $4.482 \times 10^{-3}$ & 9.609 & 9.642 \\
$10^{-9}$ & $2.829 \times 10^{-4}$ & $2.828 \times 10^{-4}$ & 38.38 & 38.38 \\
$10^{-12}$ & $1.784 \times 10^{-5}$ & $1.784 \times 10^{-5}$ & 152.8 & 152.8 \\

TABLE I. Several numerical values of the principal squeezing minimum, $S_{\text{min}}$, and the corresponding Kerr parameter $r_{\text{min}}$ obtained at the scaled interaction times $\tau$. Their estimations $S'_{\text{min}}$ and $r'_{\text{min}}$ are based on the scaling law approximation given by (8). It can be concluded from the table that
(i) the level of the minimum quadrature noise regularly drops down with the shorter interaction time $\tau$ which, however,
requires higher intensity such that the Kerr parameter $r = |\alpha|^2\tau$ increases, and (ii) the estimation is valid for $\tau < 10^{-3}$. 

![FIG. 2. Comparison of the principal squeezing minima as a function of $r = |\alpha|^2\tau$ for several values of $\tau$. The minima apparently lie on the straight line on the log-log scale. The circles represent $S'_{\text{min}}$ calculated from the scaling law formula (8). Note a good accuracy of our estimation of the minimum noise by $S'_{\text{min}}$ for $\tau < 10^{-3}$.](image1)

![FIG. 3. Comparison of the approximations $S_0$ and $S_1$, and the exact values $S$ of the principal squeezing as a function of the Kerr parameter $r$ for (a) $\tau = 10^{-1}$ and (b) $\tau = 10^{-2}$. The figure shows that, for $\tau \leq 10^{-2}$, the approximation $S_1$ can be used to estimate the minimum quadrature noise with a good precision.](image2)
IV. CONCLUSION

We have analyzed ordinary coherent light interacting with a non-absorbing non-linear Kerr medium, modelled as an anharmonic oscillator. Former numerical studies have shown that the model leads to almost complete quadrature squeezing. We have found, as we believe for the first time, an approximate analytical formula for the optimal noise level for high intensity coherent light assuming realistic values (see, e.g., [20]) of the Kerr non-linearity.

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