Dynamics analysis of multi-field coupled piezoelectric energy harvester under random excitation

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Abstract. Automatic fully mechanized mining face requires highly equipment condition monitoring. Complex environment of coalmine means the difficulty of wiring and power supply of wired monitoring. The multi-field coupled piezoelectric energy harvester (MPEH) could power wireless monitoring nodes by capture vibration energy from mining equipment. This paper makes a complete investigation of the distance effect of magnet, noise spectrum density and damping on the dynamic response characteristics of the MPEH under random excitation. Taking white Gaussian noise as input excitation, a dynamic model of the MPEH with random excitation in multiple fields is established. The probability density function of dynamic response under different system parameters is obtained by FPK equation, and the stationary mean square value and power spectral density function of system response are given by the ignored high order cumulant truncation method. In order to verify the accuracy of analytical results, the Monte Carlo numerical analysis is presented. Results show that the decrease of the magnetic distance will cause the moving of peak power spectrum frequency to low frequency. Reducing system damping ratio, narrowing magnetic distance or increasing excitation spectral density can enhance the probability of large period vibration and increase the average power of the system.

1. Introduction

Vibration energy harvesting is a new technology for energy recovery and reuse, which converts vibrational energy in the environment into electrical energy. Due to the great application potential for powering wireless sensors and small portable devices, the vibration energy harvesting have been widely discussed [1-5]. Complex environment of coalmine means the difficulties of wiring and power supply of wired monitoring. Using vibration energy harvesting, which captures vibration energy to power the wireless monitoring node, will solve the trouble [6]. The piezoelectric vibration energy harvesting, due to its superiority in structure and application, has been receiving more attention [7-12]. Erturk and Inman [13-15] derived the distributed parameter model of the cantilever beam-based energy harvesters, and the high-energy interwell oscillation under harmonic excitations enhances the energy harvesting performance was experimentally verified. A flexible longitudinal zigzag energy harvester designed by Zhou et al. [16, 17], the exact theoretical model was established and checked by finite element method and experiments under harmonic excitations. Palagummi et al. [18] proposed a new form of bi-stable system based on the passive friction-free horizontal diamagnetic levitation mechanism and the characteristics of the bi-stable potential well were discussed theoretically and experimentally. Sun et al.
[19] introduced a horizontal asymmetric U-shaped vibration-based piezoelectric energy harvester, which exhibits a good performance on the energy transfer. Zhang et al. [20] designed an arc-shaped piezoelectric bistable vibration energy harvester and the energy harvesting performance enhancement of the system under harmonic excitations has been verified.

The environmental vibration of nature is usually expressed as a broad-band random signal [21, 22]. However, most of the studies on the performance of piezoelectric energy harvester are based on simple harmonic excitation [23-26], while the research under random excitation, which is closer to the actual working conditions, is relatively less.

At present, the performance of piezoelectric energy harvester under random excitation is mainly studied by numerical simulation and experiment. Zhao et al. [27] have carried out numerical simulation and experimental study of bimorph piezoelectric energy harvester under random fundamental excitation. The average piezoelectric power output and mean square vibration response under a wide range of resistance loads from short circuit to open circuit are studied. He et al. [28] established a bistable piezoelectric energy harvester model with nonlinear stiffness, damping and electromechanical coupling coefficient under random excitation. The dynamic response characteristics of bistable piezoelectric energy harvester with different frequency, amplitude and varying magnetic spacing are obtained by numerical simulation analysis. Andò B et al. [29] established the model of double resonance structure, and the output response of system is obtained under the random excitation threshold frequency of 100 Hz by numerical calculation with matlab/simulink. The dynamic characteristics of bistable piezoelectric energy harvester under band-limited noise are studied by limiting the noise excitation signal to 15 Hz, and the advantages of the response under random excitation are determined in this paper. Leng et al. [30] simulated the low frequency random excitation condition in environment by using the filtered Gaussian noise with bandwidth of 0-120 Hz. The Fokker-Planck-Kolmogorov (FPK) equation of the system is solved by finite element analysis. The output displacement and voltage of the elastic bistable piezoelectric energy harvester are observed by using Monte Carlo method and Runge-Kutta algorithm. Qin et al. [31, 32] compared with the response analysis of bistable piezoelectric energy capture after adding auxiliary magnetism under random excitation. The coherent resonance phenomenon under random vibration of the system is revealed. Meanwhile, Zhou et al. [33,34] proposed the tristable energy harvester, which was theoretically and experimentally to have a good performance under harmonic and random excitations [35, 36].

The research on piezoelectric energy harvester under random excitation is still little, relatively. Especially, the precise analytical response of the system is still in its infancy. The response of piezoelectric energy harvester under random excitation is in line with the actual situation and will guide the industrial design of piezoelectric energy harvester device more effectively. In this paper, white Gaussian noise excitation is selected as the input of the multi-field coupled piezoelectric energy harvester (MPEH), and the theoretical model of the MPEH under random excitation is established. Using FPK equation of the model with random vibration, the response probability density functions under different system parameters are obtained. The stationary mean square value and the power spectral density function of the MPEH are obtained by ignoring high order cumulant truncation method. The analytical results are verified by Monte Carlo numerical analysis and random experiments.

2 Theoretical modelling

2.1 System structure

The MPEH consists of four linear-arch composite beams, a mass block with permanent magnets and a pair of adjustable magnets, as shown in Figure 1. The linear-arch composite beam is based on metal beam, to which the flexible piezoelectric material Polyvinylidene fluoride (PVDF) adheres. Four composite beams connected to the same mass block, which has two permanent magnets on it. The distance between two adjustable magnets can be changed by adjusting the screw thread of the casing. The force between permanent magnets set as attractive force. The MPEH is installed in a base structure which transfers the random excitation z(t) to the harvester.
Theoretical model

The schematic diagram of linear-arch composite beam is shown in Figure 2. In order to obtain the distributed parameter model of the MPEH, Generalized Hamilton principle, Rayleigh-Ritz method, piezoelectric theory and Euler-Bernoulli beam theory are used [12,13,37]. The function can be expressed as:

\[ M \ddot{r}(t) + C \dot{r}(t) + K r(t) - \theta v(t) - K_1 r(t) - K_2 r(t)^3 = -H_z \ddot{z}(t) \]  

(1)

\[ \theta r(t) + C_p v(t) + q = 0 \]  

(2)

where \( r(t) \) and \( v(t) \) are the displacement mode function and voltage mode function, respectively. \( \ddot{z}(t) \) denotes the acceleration of the base excitation.

\( M \) and \( K \) are the modal mass and the modal stiffness of the MPEH. They are represented as:

\[ M = \int_{\Omega_b} \rho_b \psi_{1r}^2(X) \, d\Omega_b + \int_{\Omega_p} \rho_p \psi_{1r}^2(X) \, d\Omega_p + \frac{1}{4} m_0 \psi_{1r}^2(L) \]  

(3)

\[ K = \int_{\Omega_b} c_{11}^S z^2 \psi_{1r}''^2(X) \, d\Omega_b + \int_{\Omega_p} c_{11}^E z^2 \psi_{1r}''^2(X) \, d\Omega_p \]  

(4)

where \( \rho_b \) and \( \rho_p \) is the density of the substrate and PVDF, respectively. \( m_0 \) denotes the mass of block with permanent magnets. \( \Omega_b \) and \( \Omega_p \) are the path of integration of the substrate and PVDF, respectively. \( \psi_{1r}(X) \) is the first mode shape of the beam. \( c_{11}^S \) is stiffness coefficient of the substrate and \( c_{11}^E \) is the elasticity coefficient of PVDF. \( z \) is the distance from the surface of the piezoelectric beam to the neutral layer.

\( C \) is the modal damping coefficient of the MPEH, which is a constant determined by the geometric structure and environment of the harvester. It can be given by:

\[ C = \int_{\Omega_b} c \psi_{1r}^2(X) \, d\Omega_b \]  

(5)

where \( c \) is the damping coefficient.

\( \theta \) is the dimensionless electromechanical coupling coefficient, and \( C_p \) is internal capacitance of PVDF. They are represented respectively as:

\[ \theta = \int_{\Omega_p} e_{31} z \psi_{1p}'(Y) \psi_{1r}''(X) \, d\Omega_p \]  

(6)
\[ C_p = \int_{\Omega_p} (e_{33}^v \psi_{1v}^2(Y)) \, d\Omega_p \]  \hspace{1cm} (7)

where \( \psi_{1v}(Y) \) denotes potential distribution function, and \( e_{33} \) is piezoelectric stress constant.

The external excitation factor \( H_s \), and the electricity generated by the system \( q \) are defined as:

\[ H_s = \int m(X) \psi_{1r}(X) \, d\Omega_p + m_0 \psi_{1r}(L) \]  \hspace{1cm} (8)

\[ q = Q + \int \sigma \, dS \]  \hspace{1cm} (9)

where \( Q \) is the effective current and \( \sigma \) is the surface electron density.

In order to nondimensionalize equations (1) and (2), the variables are standardized as:

\[ \begin{align*}
    x &= \frac{r}{L}, \quad u = \frac{v}{v_0}, \quad \tau = \omega_1 t, \quad v^* = \frac{L \theta}{C_p} 
\end{align*} \]  \hspace{1cm} (10)

Assuming that the external load is a pure resistance \( R_l \) and the first-order mode frequency is \( \omega_1 \), the voltage is \( v = R_l \cdot \frac{dq}{dt} \) and damping ratio is \( \zeta = c/(2M \omega_1) \). Thus, the dynamic model excited by Gaussian white noise is:

\[ \ddot{x} + 2\zeta \dot{x} + (1 - \kappa_1)x - \kappa_2 x^3 - \vartheta u = -\rho W(\tau) \]  \hspace{1cm} (11)

\[ \dot{x} + \dot{u} + \omega u = 0 \]  \hspace{1cm} (12)

where

\[ \theta = \frac{\theta^2}{KC_p}, \quad \kappa_1 = \frac{K_1}{K}, \quad \kappa_2 = \frac{K_2 L^2}{K}, \quad \rho = \frac{H_s}{KL} z(0), \quad \omega = \frac{1}{R_l C_p \omega_1}, \quad L = L_1 + L_2 \]  \hspace{1cm} (13)

\( W(\tau) \) is Gaussian white noise, its spectral density is \( K_n \). The related function can be defined, as follows:

\[ E[W(\tau)W(\tau + \tau')] = 2\pi K_n \delta(\tau') \]  \hspace{1cm} (14)

Set \( x_1 = x, x_2 = \dot{x}, x_3 = u \), the equations (11) and (12) can be converted to

\[ \begin{align*}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= -2\zeta x_2 - (1 - \kappa_1)x_1 + \kappa_2 x_1^3 + \vartheta x_3 - \rho W(\tau) \\
    \dot{x}_3 &= -x_2 - \omega x_3
\end{align*} \]  \hspace{1cm} (15)

It is assumed that the random response of the MPEH is a vector Markov process. The simplest form of vector Markov process is Wiener process, also called Brownian motion, which is expressed as \( B(t) \). A stochastic process \( B(t) \), called a Wiener process, can be used to construct other Markov diffusion processes through stochastic differential equations. Therefore, Gaussian white noise can be replaced by the derivative of Wiener process, and the system is expressed as:

\[ \begin{align*}
    \frac{dx_1}{d\tau} &= x_2 \\
    \frac{dx_2}{d\tau} &= (-2\zeta x_2 - (1 - \kappa_1)x_1 + \kappa_2 x_1^3 + \vartheta x_3 - \rho W(\tau)) \, d\tau - \rho \sqrt{2\pi K_n} \, dB(\tau) \\
    \frac{dx_3}{d\tau} &= (-x_2 - \omega x_3) \, d\tau
\end{align*} \]  \hspace{1cm} (16)

where \( B(\tau) \) is the independent unit Wiener process.
2.3 Stationary probability density

The response of the MPEH under random excitation is uncertain, which can be described only by probability measure. The probability density function is expressed as the probability that the amplitude of a random signal falls within a certain range.

Assuming the environmental random excitation belongs to a stationary stochastic process, the system response is governed by a simplified FPK equation:

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} G_j = 0$$  \hspace{1cm} (17)

where $G_j$ is the probability flow in the $j$ direction and is given by:

$$G_j = a_j(x)p - \frac{1}{2} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} [b_{jk}(x)p]$$  \hspace{1cm} (18)

where $a_j$ and $b_{jk}$ are the first-order and second-order derivative moments of the system, respectively.

Based on the equation (16), we can deduce the following equations:

$$a_1 = x_2, \quad a_2 = -2\zeta x_2 - (1 - \kappa_1)x_1 + \kappa_2 x_1^3 + \vartheta x_3, \quad a_3 = -x_2 - \sigma x_3$$  \hspace{1cm} (19)

$$b_{11} = b_{33} = b_{12} = b_{13} = b_{23} = 0, \quad b_{22} = 2\pi K$$  \hspace{1cm} (20)

The stability condition of the system is unknown. In order to satisfy the FPK equation, the reversible and irreversible parts of the first derivative moment can be defined, as follows:

$$a_j(x) = a_j^R(x) + a_j^I(x)$$  \hspace{1cm} (21)

Thus,

$$a_1^R = x_2, \quad a_1^I = 0, \quad a_2^R = -(1 - \kappa_1)x_1 + \kappa_2 x_1^3 + \vartheta x_3, \quad a_2^I = -2\zeta x_2, \quad a_3^R = -x_2, \quad a_3^I = -\sigma x_3$$  \hspace{1cm} (22)

Based on equations (17), (18) and (21), the FPK equation can be expressed as:

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} G_j = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_j^R(x)p + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (a_j^I(x)p) - \frac{1}{2} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} [b_{jk}(x)p] = 0$$  \hspace{1cm} (23)

A sufficient set of conditions for equation (23) is

$$\begin{cases}
    a_j^I(x)p - \frac{1}{2} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} [b_{jk}(x)p] = 0 \\
    \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_j^R(x)p = 0
\end{cases}$$  \hspace{1cm} (24)

When the probability density function satisfies equation (24), the system belongs to the detailed balance class. A special case of detailed balance is the stationary potential. The stationary potential condition of the system is $a_j^I(x) = a_j(x)$ and $a_j^R(x)=0$. Thus, equation (24) can be expressed as:
\[
\begin{align*}
\frac{\partial (x_1 p)}{\partial x_k} & = \frac{1}{2} \sum_{k=1}^{n} \left[ \frac{\partial h_{jk}(x)}{\partial x_k} - b_{jk}(x) \frac{\partial \phi}{\partial x_k} \right] \\
\sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_r^p(x) & = \sum_{j=1}^{n} a_r^p(x) \frac{\partial \phi}{\partial x_j}
\end{align*}
\]  
(25)

Combined equation (22), equation (25) can be given by:

\[
\begin{align*}
\pi K_n \frac{\partial \phi}{\partial x_2} & = 2 \zeta x_2 \\
x_2 \frac{\partial \phi}{\partial x_1} - [(1 - \kappa_1) x_1 - \kappa_2 x_1^3 - \vartheta x_3] \frac{\partial \phi}{\partial x_2} - x_2 \frac{\partial \phi}{\partial x_3} & = 0
\end{align*}
\]  
(26)  
(27)

The general solution of equation (27) is

\[
\phi(x_1, x_2, x_3) = \phi(\lambda), \quad \lambda = \frac{1 - \kappa_1}{2} x_1^2 + \frac{\kappa_2}{4} x_1^4 + \frac{1}{2} x_2^2 + \frac{\vartheta}{2} x_3^2
\]  
(28)

Based on equations (26) and (28), we find out that:

\[
\frac{d \phi}{d \lambda} = \frac{2 \zeta}{\pi K_n} = C
\]  
(29)

Therefore, the system has an accurate and stable solution and the random process \( x_1(t), x_2(t) \) and \( x_3(t) \) have a stationary probability density.

\[
p(x_1, x_2, x_3) = C \cdot \exp\{-\frac{2 \zeta}{\pi K_n} \left[ \frac{1 - \kappa_1}{2} x_1^2 + \frac{\kappa_2}{4} x_1^4 + \frac{1}{2} x_2^2 + \frac{\vartheta}{2} x_3^2 \right] \}
\]  
(30)

where \( C \) is the normalization constant.

Contributes to the method of solving continuous edge density function, the probability of the system displacement density is obtained.

\[
p(x_1) = C_1 \cdot \exp\{-\frac{2 \zeta}{\pi K_n} \left[ \frac{1 - \kappa_1}{2} x_1^2 + \frac{\kappa_2}{4} x_1^4 \right] \}
\]  
(31)

where \( C_1 \) is the normalization constant. The probability density function and the joint probability density function of the response can be obtained by substituting the parameters of the multi-field coupled piezoelectric energy harvester system.

### 2.4 Stationary mean square value

A stationary mean square value is an expression of the average energy (power) of a signal. It can be known from equation (16) that the Ito equations for \( dx_1^2, dx_2^2, dx_3^2, dx_1 x_2, dx_1 x_3, dx_2 x_3 \) are obtained by using the Ito differential rule.

\[
\begin{align*}
\frac{dx_1^2}{dt} & = 2 x_1 x_2 dr \\
\frac{dx_2^2}{dt} & = [(2 \zeta x_2 - (1 - \kappa_1) x_1 + \kappa_2 x_1^3 + \vartheta x_3) \cdot 2 x_2 + 2 \pi K_1 \rho^2] d\tau + \rho \sqrt{2 \pi K_n} \cdot 2 x_2 dB(\tau) \\
\frac{dx_3^2}{dt} & = (-x_2 - \varrho x_3) \cdot 2 x_3 dr \\
dx_1 x_2 & = [x_2 \cdot x_2 + (-2 \zeta x_2 - (1 - \kappa_1) x_1 + \kappa_2 x_1^3 + \vartheta x_3) \cdot x_1] d\tau + \rho \sqrt{2 \pi K_n} \cdot x_1 dB(\tau) \\
dx_1 x_3 & = [x_2 \cdot x_3 + (-x_2 - \varrho x_3) \cdot x_1] d\tau \\
dx_2 x_3 & = [(-2 \zeta x_2 - (1 - \kappa_1) x_1 + \kappa_2 x_1^3 + \vartheta x_3) \cdot x_3 + (-x_2 - \varrho x_3) \cdot x_2] d\tau + \rho \sqrt{2 \pi K_n} \cdot x_3 dB(\tau)
\end{align*}
\]  
(32)

Set \( M = X_1^2 X_2^2 X_3^2, m_{ijk} = [X_1 X_2 X_3]^T \), the first-order moment and second-order moment equation of the system can be obtained by:
\[
\begin{align*}
\frac{dm_{100}}{d\tau} &= m_{010} \\
\frac{dm_{010}}{d\tau} &= -2\zeta m_{010} - (1 - \kappa_1)m_{100} + \kappa_2 m_{300} + \vartheta m_{001} \\
\frac{dm_{001}}{d\tau} &= -m_{010} - \omega m_{001} \\
\frac{dm_{200}}{d\tau} &= 2m_{110} \\
\frac{dm_{020}}{d\tau} &= -4\zeta m_{020} - 2(1 - \kappa_1)m_{110} + 2\kappa_2 m_{310} + 2\vartheta m_{011} + 2\pi K_n \rho^2 \\
\frac{dm_{002}}{d\tau} &= -2m_{011} - 2\omega m_{002} \\
\frac{dm_{110}}{d\tau} &= m_{020} - 2\zeta m_{110} - (1 - \kappa_1)m_{200} + \kappa_2 m_{400} + \vartheta m_{101} \\
\frac{dm_{101}}{d\tau} &= m_{011} - m_{110} - \omega m_{101} \\
\frac{dm_{011}}{d\tau} &= -(2\zeta + \omega)m_{011} - (1 - \kappa_1)m_{101} + \kappa_2 m_{301} + \vartheta m_{002} - m_{020}
\end{align*}
\]

Since the equation contains a moment higher than the second order, the equation cannot be solved. By using the truncation method of Gaussian cumulant, the accumulative quantity above the second order can be reduced to zero.

\[m_{300} = 0, \ m_{310} = 3m_{200} \cdot m_{110}, \ m_{400} = 3m_{200}^2, \ m_{301} = 3m_{200} \cdot m_{101}\]

The equation is converted into

\[
\begin{align*}
\frac{dm_{100}}{d\tau} &= m_{010} \\
\frac{dm_{010}}{d\tau} &= -2\zeta m_{010} - (1 - \kappa_1)m_{100} + \vartheta m_{001} \\
\frac{dm_{001}}{d\tau} &= -m_{010} - \omega m_{001} \\
\frac{dm_{200}}{d\tau} &= 2m_{110} \\
\frac{dm_{020}}{d\tau} &= -4\zeta m_{020} - 2(1 - \kappa_1)m_{110} + 6\kappa_2 m_{200} \cdot m_{110} + 2\vartheta m_{011} + 2\pi K_n \rho^2 \\
\frac{dm_{002}}{d\tau} &= -2m_{011} - 2\omega m_{002} \\
\frac{dm_{110}}{d\tau} &= m_{020} - 2\zeta m_{110} - (1 - \kappa_1)m_{200} + 3\kappa_2 m_{200}^2 + \vartheta m_{101} \\
\frac{dm_{101}}{d\tau} &= m_{011} - m_{110} - \omega m_{101} \\
\frac{dm_{011}}{d\tau} &= -(2\zeta + \omega)m_{011} - (1 - \kappa_1)m_{101} + 3\kappa_2 m_{200} \cdot m_{101} + \vartheta m_{002} - m_{020}
\end{align*}
\]

Substituting the multi-field coupled piezoelectric energy harvester system parameters into equation (35), the stationary mean square value of the system response can be obtained.

2.5 Power Spectral Density
The power spectral density is the distribution of the power with frequency of random signal, which describes the distribution of the mean square value over the entire frequency domain. Multiply each equation in equation (16) by $X_1(t_0)$ and averaged.

\[
\begin{align*}
\frac{dQ_{100}}{dt'} &= Q_{010} \\
\frac{dQ_{010}}{dt'} &= -2\zeta Q_{010} - (1 - \kappa_1)Q_{100} + \kappa_2 Q_{300} + \vartheta Q_{001} \quad (36) \\
\frac{dQ_{001}}{dt'} &= -Q_{010} - \omega Q_{001}
\end{align*}
\]

where

\[
\tau' = \tau - t_0, \quad Q_{ijk} = E[X_1^i(t)X_2^j(t)X_3^k(t)X_1(t') - \tau]]
\quad (37)
\]

Set all cumulants above the second order be zero by using the truncation method of Gaussian cumulant. It can be known that

\[
Q_{300} = E[X_1^3(t)X_1(t') - \tau] = 3E[X_1^2(t)]E[X_1(t)X_1(t') - \tau] = 3m_{200}Q_{100} \quad (38)
\]

Combined equation (38), the equation (36) can be expressed by:

\[
\begin{align*}
\frac{dQ_{100}}{dt'} &= Q_{010} \\
\frac{dQ_{010}}{dt'} &= -2\zeta Q_{010} - (1 - \kappa_1 - 3\kappa_2m_{200})Q_{100} + \vartheta Q_{001} \quad (39) \\
\frac{dQ_{001}}{dt'} &= -Q_{010} - \omega Q_{001}
\end{align*}
\]

Define the integral transformation to

\[
\bar{Q}_{ijk}(\omega) = \Im[Q_{ijk}(\tau')] = \frac{1}{\pi} \int_0^\infty Q_{ijk}(\tau') e^{-i\omega t'} dt' 
\quad (40)
\]

\[
\Im \left[ \frac{dQ_{ijk}(\tau')}{dt'} \right] = i\omega \Im[Q_{ijk}(\tau')] - \frac{1}{\pi} Q_{ijk}(0) = i\omega \bar{Q}_{ijk}(\omega) - \frac{1}{\pi} m_{i+j+k} 
\quad (41)
\]

The equation (39) can be converted into

\[
\begin{align*}
i\omega \bar{Q}_{100} - \frac{1}{\pi} m_{200} &= \bar{Q}_{010} \\
i\omega \bar{Q}_{010} - \frac{1}{\pi} m_{110} &= -2\zeta \bar{Q}_{010} - (1 - \kappa_1 - 3\kappa_2m_{200})\bar{Q}_{100} + \vartheta \bar{Q}_{001} \\
i\omega \bar{Q}_{001} - \frac{1}{\pi} m_{101} &= -\bar{Q}_{010} - \omega \bar{Q}_{001}
\end{align*}
\quad (42)
\]

Thus, the spectral density of the displacement $X_1(\tau)$ is

\[
\phi_{X_1X_1}(\omega) = \Re[\bar{Q}_{100}(\omega)] 
\quad (43)
\]

Substituting the parameters of the MPEH into equation (35), the system non-zero stationary second-order moment will be obtained. We substitute the result into equation (42) to obtain the correlation function expression and the actual part is the system response power spectral density, which is expressed as equation (43).

3. Stochastic dynamic response analysis
In order to study the effect of magnet distance, noise spectral density and system damping on the response of the MPEH, the dynamics models and theoretical analysis above are used. In this part, the Monte Carlo method is also used to simulate the response results to verify the correctness of the theoretical approximate solution.

3.1 Stationary probability density
Figure 3 shows the probability distribution of the system response under different parameters. The probability density function of the MPEH response is solved by the FPK equation method and compared with the Monte Carlo simulation results. It can be seen that the error between two methods is small. The results solved by the FPK equation method, which obey the Gaussian distribution, have good consistency under different parameter conditions. Figure 3(a) shows the response probability distribution of the system when the Gaussian white noise spectral density $K_n$ is 0.5 and set the magnet spacing $d$ is 15mm, 20mm and no magnetic force, respectively. As the magnetic distance decreases, the nonlinear parameter of the system increases and the mean square deviation of the probability density function distribution increases. It indicates that the probability of large period motion of the system response increases with the decrease of the magnetic distance, and the output power increases. Figure 3(b) shows the response probability distribution of the system when the Gaussian white noise spectral density $K_n$ is 0.5, 1, and 2 under no magnetic force, respectively. As the Gaussian white noise excitation spectral density increases, the mean square deviation of probability density function increases, which indicates that with the increase of the noise excitation spectral density, the large period motion probability of the system increases and the output power increases.

![Figure 3. The stationary probability distribution of the MPEH.](image)

3.2 Stationary mean square value
Figure 4 shows the stationary mean square value of the MPEH under different parameters. The Gaussian cumulant truncation method is used to solve the stationary mean square value of the system. Compared with the Monte Carlo simulation results, the method can obtain results that are more accurate. Figure 4(a) shows the variation of the stationary mean square value of the system response with different damping ratios when the Gaussian white noise spectral density $K_n$ is 0.5, and the magnet spacing $d$ is 15mm, 20mm and no magnetic force, respectively. As the damping of the system increases, the stationary mean square value and the average power output of the system decreases, especially the damping ratio is [0, 0.1]. Figure 4(b) shows the stationary mean value of the system under different Gaussian white noise spectral densities when the damping ratio $\zeta$ is 0.5, the magnet spacing $d$ is 15mm, 20mm and no magnetic force, respectively. As the noise spectral density increases, the stationary mean square value increases and the average power output increases. The addition of magnetic force to the system can increase the stationary mean square value and the output power of the system. However, when the spectral density is one, it will be less than the magnetic input. This is because the response
amplitude of the system has exceeded 15mm at this time, which hinders its motion over the distance between the magnets.

![Image](image_url)

**Figure 4.** The stationary mean square value of the MPEH.

### 3.3 Power Spectral Density

Figure 5 shows the power spectral density of the system under different parameters. Figure 5(a) shows the variation of the system response power when the Gaussian white noise spectral density $K_n$ is 0.5, the damping ratio $\zeta$ is 0.096, and the magnet spacing $d$ is 15 mm, 20 mm and no magnetic force, respectively. With the decrease of magnetic distance, the response peak of the system moves to low frequency and the peak value of response increases, which means the output power of the system increases. Set the threshold is 0.1; the system response band is widened. Figure 5(b) shows the variation of the system response power when the Gaussian white noise spectral density $K_n$ is 0.5, the magnet spacing $d$ is 15 mm, and the damping ratio $\zeta$ is 0.05, 0.1 and 0.2, respectively. As the system damping ratio increases, the peak frequency of the system response shifts to the low frequency and the peak decreases. It is obviously that the low frequency output power increases and the system response band broadens.

![Image](image_url)

**Figure 5.** The power spectral density of the MPEH.

### 4. Conclusion

This paper selects Gaussian white noise excitation as the input of the system, establishes the dynamic equation of the MPEH under random excitation. The FPK equation is used to obtain the response probability density function under different system parameters. By ignoring the high-order cumulant truncation method, the system response stationary mean square value and power spectral density function are obtained, and the analytical results are compared with Monte Carlo numerical simulation. The results show that when the system is added with magnetic attraction and the magnetic distance is
reduced, the variance of vibration probability distribution of the system increases and the probability of large period vibration increases. The average power of the system increases, and the peak frequency of the power spectrum shifts to the lower. Reducing the damping ratio or increasing the excitation spectral density will increase the probability of large period vibration and enhance the average power.

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