Recovery-Based Error Estimators for Diffusion Problems: Explicit Formulas

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Abstract. We introduced and analyzed robust recovery-based a posteriori error estimators for various lower order finite element approximations to interface problems in [9, 10], where the recoveries of the flux and/or gradient are implicit (i.e., requiring solutions of global problems with mass matrices). In this paper, we develop fully explicit recovery-based error estimators for lower order conforming, mixed, and non-conforming finite element approximations to diffusion problems with full coefficient tensor. When the diffusion coefficient is piecewise constant scalar and its distribution is local quasi-monotone, it is shown theoretically that the estimators developed in this paper are robust with respect to the size of jumps. Numerical experiments are also performed to support the theoretical results.

1 Introduction

A posteriori error estimation for finite element methods has been extensively studied for the past three decades (see, e.g., books by Verfürth [24, 25], Ainsworth and Oden [3], Babuška and Strouboulis [4], and references therein). The widely adapted estimator is probably the Zienkiewicz-Zhu (ZZ) recovery-based error estimator [26, 27] due to its easy implementation, generality, and ability to produce quite accurate estimations. By first recovering a gradient in the conforming $C^0$ linear vector finite element space from the numerical gradient, the ZZ estimator is defined as the $L^2$ norm of the difference between the recovered and the numerical gradients.

Despite popularity of the ZZ estimator, it is also well known that the ZZ estimator over-refines regions where there are no error, and hence, they fail to reduce the global error. This is shown by Ovall in [22] through some interesting and realistic examples. Such a failure is simply caused by using continuous functions (recovered gradient/flux) to approximate discontinuous functions (true gradient/flux) in the recovery procedure. By recovering flux and/or gradient in the respective $H(\text{div}; \Omega)$ and $H(\text{curl}; \Omega)$ conforming finite element spaces, in [9, 10], we developed and studied robust recovery-based implicit and explicit error estimators for various lowest order finite element approximations to

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the interface problems. The implicit error estimator requires solution of a global $L^2$ minimization problem, and the explicit error estimator uses a simple edge average.

The explicit recovery introduced in [9, 10] is limited to the Raviart-Thomas ($RT$) [6] and the first type of Nédélec ($NE$) [20] elements of the lowest order for the respective flux and gradient recoveries. This simple averaging approach may not be extended to the Brezzi-Douglas-Marini ($BDM$) [6] and the second type of Nédélec [21] ($ND$) elements of the lowest order and to the diffusion problem with full coefficient tensor. The purpose of this paper is first to introduce a general approach for constructing explicit recovery of the flux/gradient for various lower order finite element approximations to the diffusion problem with the full coefficient tensor. The approach, similar to [13], is to localize the implicit recovery through a partition of the unity. For various lower order elements, we are able to reduce the local patch problem to the edge/face patch which contains at most two elements. Hence, by solving a local minimization problem on this two-element patch, we explicitly recover the flux/gradient. We then define the corresponding estimators and establish their reliability and efficiency. When the diffusion coefficient is piecewise constant and its distribution is local quasi-monotone, we are able to show theoretically that these estimators are robust with respect to the size of jumps. For a benchmark test problem, whose coefficient is not local quasi-monotone, numerical results also show the robustness of the estimators.

For the conforming finite element approximation to the interface problem, robust error estimators have been studied by Bernardi and Verfürth [3] and Petzoldt [23] for the residual-based estimator, Luce and Wohlmuth [19] for an equilibrated estimator on a dual mesh, and by us [9] for the recovery-based error estimator. Ainsworth in [12] studied robust error estimators for nonconforming and mixed methods, respectively. Robust error estimators for locally conserved methods were studied by Kim [18]. Recently, we studied robust recovery-based estimators for lowest order nonconforming, mixed, and discontinuous Galerkin methods (see [10, 8]) via the $L^2$ recovery and for higher-order conforming elements in [11] via a weighted $H(div)$ recovery. Robust equilibrated residual error estimator are constructed by us in [13]. For interface problems with flux jumps, we studied robust residual- and recovery-based error estimators in [12].

The paper is organized as follows. Section 2 describes the diffusion problem and its variational forms. Conforming, mixed, and nonconforming finite element methods are presented in Section 3. Section 4 introduces the explicit recoveries of the flux/gradient for those finite element approximations. The corresponding a posteriori error estimators are introduced in Section 5 and their reliability and efficiency bounds are established in Section 6. Finally, Section 7 provides numerical results for a benchmark test problem.

2 Diffusion Problem and Variational Form

Let $\Omega$ be a bounded polygonal domain in $\mathbb{R}^2$, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, and measure $(\Gamma_D) \neq 0$, and let $n$ be the outward unit vector normal to the boundary. Consider diffusion equation

$$-\nabla \cdot (A(x)\nabla u) = f \quad \text{in} \quad \Omega$$  \hspace{1cm} (2.1)
with boundary conditions
\[-A\nabla u \cdot n = g_N \text{ on } \Gamma_N \quad \text{and} \quad u = g_D \text{ on } \Gamma_D. \quad (2.2)\]

For simplicity of presentation, assume that \( f \in L^2(\Omega) \), that \( g_D \) and \( g_N \) are piecewise affine functions and constants, respectively, and that \( A \) is a symmetric, positive definite piecewise constant matrix.

Here and thereafter, we use standard notations and definitions for the Sobolev spaces. Let
\[ H_{g,D}^1(\Omega) = \{ v \in H^1(\Omega) \mid v = g_D \text{ on } \Gamma_D \} \quad \text{and} \quad H_D^1(\Omega) = H_{0,D}^1(\Omega). \]

Then the corresponding variational problem is to find \( u \in H_{g,D}^1(\Omega) \) such that
\[ a(u, v) = (A \nabla u, \nabla v) = (f, v) - (g_N, v)_{\Gamma_N} = f(v) \quad \forall \ v \in H_D^1(\Omega), \quad (2.3) \]
where \((\cdot, \cdot)\omega\) is the \( L^2 \) inner product on the domain \( \omega \). The subscript \( \omega \) is omitted when \( \omega = \Omega \).

In two dimensions, for \( \tau = (\tau_1, \tau_2)^t \), define the divergence and curl operators by
\[ \nabla \cdot \tau := \frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2} \quad \text{and} \quad \nabla \times \tau := \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}, \]
respectively. For a scalar-valued function \( v \), define the operator \( \nabla \perp \) by
\[ \nabla \perp v := (\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1})^t. \]

We shall use the following Hilbert spaces
\[ H(\text{div}; \Omega) = \{ \tau \in L^2(\Omega)^2 \mid \nabla \cdot \tau \in L^2(\Omega) \} \quad \text{and} \quad H(\text{curl}; \Omega) = \{ \tau \in L^2(\Omega)^2 \mid \nabla \times \tau \in L^2(\Omega) \} \]
equipped with the norms
\[ \| \tau \|_{H(\text{div}; \Omega)} = \left( \| \tau \|_{0, \Omega}^2 + \| \nabla \cdot \tau \|_{0, \Omega}^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \| \tau \|_{H(\text{curl}; \Omega)} = \left( \| \tau \|_{0, \Omega}^2 + \| \nabla \times \tau \|_{0, \Omega}^2 \right)^{\frac{1}{2}}, \]
respectively. Let
\[ H_{g,N}(\text{div}; \Omega) = \{ \tau \in H(\text{div}; \Omega) \mid \tau \cdot n|_{\Gamma_N} = g_N \}, \quad H_N(\text{div}; \Omega) = H_{0,N}(\text{div}; \Omega), \quad H_D(\text{curl}; \Omega) = \{ \tau \in H(\text{curl}; \Omega) \mid \tau \cdot t|_{\Gamma_D} = 0 \}, \]
where \( n = (n_1, n_2)^t \) and \( t = (t_1, t_2)^t = (-n_2, n_1)^t \) are the unit vectors outward normal to and tangent to the boundary \( \partial \Omega \), respectively.

Define the flux by
\[ \sigma = -A(x) \nabla u \quad \text{in} \quad \Omega, \]
then the mixed variational formulation is to find \( (\sigma, u) \in H_{g,N}(\text{div}; \Omega) \times L^2(\Omega) \) such that
\[ \begin{cases} \quad (A^{-1} \sigma, \tau) - (\nabla \cdot \tau, u) = -(\tau \cdot n, g_D)_{\Gamma_D} & \forall \tau \in H_N(\text{div}; \Omega), \\ (\nabla \cdot \sigma, v) = (f, v) & \forall v \in L^2(\Omega). \end{cases} \quad (2.4) \]
3 Finite Element Approximation

3.1 Finite Element Spaces

For simplicity, consider only triangular elements. Let $\mathcal{T} = \{K\}$ be a regular triangulation of the domain $\Omega$, and denote by $h_K$ the diameter of the element $K$. We assume that $A$ is piecewise constant matrix on the mesh $\mathcal{T}$. Denote the set of all nodes of the triangulation by $\mathcal{N} := \mathcal{N}_I \cup \mathcal{N}_D \cup \mathcal{N}_N$, where $\mathcal{N}_I$ is the set of all interior nodes and $\mathcal{N}_D$ and $\mathcal{N}_N$ are the sets of all boundary nodes belonging to the respective $\Gamma_D$ and $\Gamma_N$. Denote the set of all edges of the triangulation by $\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N$, where $\mathcal{E}_I$ is the set of all interior element edges and $\mathcal{E}_D$ and $\mathcal{E}_N$ are the sets of all boundary edges belonging to the respective $\Gamma_D$ and $\Gamma_N$.

For each $F \in \mathcal{E}$, denote by $\mathbf{n}_F = (n_{1,F}, n_{2,F})^t$ a unit vector normal to $F$; then $\mathbf{t}_F = -(n_{2,F}, n_{1,F})^t$ is a unit vector tangent to $F$. Let $K^-_F$ and $K^+_F$ be two elements sharing the common edge $F$ such that the unit outward normal vector of $K^-_F$ coincide with $\mathbf{n}_F$. When $F \in \mathcal{E}_D \cup \mathcal{E}_N$, $\mathbf{n}_F$ is the unit outward vector normal to $\partial \Omega$ and denote by $K^-_F$ the element having the edge $F$. For interior edges $F \in \mathcal{E}_I$, the selection of $\mathbf{n}_F$ is arbitrary but globally fixed. For a function $v$ defined on $K^-_F \cup K^+_F$, denote its traces on $F$ by $v|^{\pm}_F$ respectively. The jump over the edge $F$ is denoted by

$$\llbracket v \rrbracket_F := \begin{cases} v|^{-}_F - v|^{+}_F & \text{if } F \in \mathcal{E}_I, \\ v|^{-}_F & \text{if } F \in \mathcal{E}_D \cup \mathcal{E}_N. \end{cases}$$

(When there is no ambiguity, the subscript or superscript $F$ in the designation of jump and other places will be dropped.)

For each $K \in \mathcal{T}$, let $P_k(K)$ be the space of polynomials of degree $k$. Denote the linear conforming and nonconforming (Crouzeix-Raviart) finite element spaces [15, 17] associated with the triangulation $\mathcal{T}$ by

$$S = \{v \in H^1(\Omega) \mid v|_K \in P_1(K) \quad \forall \, K \in \mathcal{T}\}$$

and

$$S^{nc} = \{v \in L^2(\Omega) \mid v|_K \in P_1(K) \quad \forall \, K \in \mathcal{T}, \text{ and } v \text{ is continuous at } m_F \quad \forall \, F \in \mathcal{E}_I\},$$

respectively. Let

$$S_{g,\mathcal{D}} = \{v \in S \mid v = g_F \text{ on } \Gamma_D\}, \quad S^{nc}_{g,\mathcal{D}} = \{v \in S^{nc} \mid v(m_F) = g_F(m_F) \quad \forall \, F \in \mathcal{E}_D\},$$

$$S_D = \{v \in S \mid v = 0 \text{ on } \Gamma_D\}, \quad S^{nc}_D = \{v \in S^{nc} \mid v(m_F) = 0 \quad \forall \, F \in \mathcal{E}_D\}.$$ 

The $H(\text{div}; \Omega)$ conforming Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) spaces [6] of the lowest order are defined by

$$\text{RT} = \{\tau \in H(\text{div}; \Omega) \mid \tau|_K \in \text{RT}(K) \quad \forall \, K \in \mathcal{T}\}$$

and

$$\text{BDM} = \{\tau \in H(\text{div}; \Omega) \mid \tau|_K \in \text{BDM}(K) \quad \forall \, K \in \mathcal{T}\},$$

respectively, where $\text{RT}(K) = P_0(K)^2 + (x_1, x_2)^t P_0(K)$ and $\text{BDM}(K) = P_1(K)^2$. The $H(\text{curl}; \Omega)$-conforming first [20] and second [21] types of Nédélec spaces of the lowest
order are defined by

\[ NE = \{ \tau \in H(\text{curl}; \Omega) \mid \tau|_K \in NE(K) \forall K \in \mathcal{T} \} \]

and

\[ ND = \{ \tau \in H(\text{curl}; \Omega) \mid \tau|_K \in ND(K) \forall K \in \mathcal{T} \}, \]

respectively, where \( NE(K) = P_0(K)^2 + (x_2, -x_1)^tP_0(K) \) and \( ND(K) = P_1(K)^2 \). For convenience, denote \( RT(K) \) and \( BDM(K) \) by \( V(K) \), \( RT \) and \( BDM \) by \( V \), \( NE(K) \) and \( ND(K) \) by \( W(K) \), and \( NE \) and \( ND \) by \( W \). Also, let

\[ P_0 = \{ v \in L^2(\Omega) \mid v|_K \in P_0(K) \forall K \in \mathcal{T} \}. \]

Definitions and properties of bases for the \( RT, BDM, NE, \) and \( ND \) spaces on an element \( K \) are presented in Appendix A.

Finally, we define the discrete gradient, divergence, and curl operators by

\[ (\nabla_h v)|_K := \nabla(v|_K), \quad (\nabla_h \cdot \tau)|_K := \nabla \cdot (\tau|_K), \quad \text{and} \quad (\nabla_h \times \tau)|_K := \nabla \times (\tau|_K) \]

for all \( K \in \mathcal{T} \), respectively.

### 3.2 Finite Element Approximation

The conforming finite element method is to seek \( u_c \in S_{g,D} \) such that

\[ (A\nabla u_c, \nabla v) = (f, v) \quad \forall v \in S_D, \quad (3.1) \]

the mixed finite element method is to seek \( (\sigma_m, u_m) \in (\mathcal{V} \cap H_{g,n}(\text{div}; \Omega)) \times P_0 \) such that

\[
\begin{align*}
- (A^{-1}\sigma_m, \nabla \cdot \tau, u_m) &= - (\tau \cdot n|_D)_{\Gamma_D} \quad \forall \tau \in \mathcal{V} \cap H_N(\text{div}; \Omega), \\
(A\nabla \cdot \sigma_m, v) &= (f, v) \quad \forall v \in P_0,
\end{align*}
\]

and the nonconforming finite element method is to find \( u_{nc} \in S_{g,D}^{nc} \) such that

\[ (A\nabla_h u_{nc}, \nabla_h v) = (f, v) \quad \forall v \in S_D^{nc}. \quad (3.3) \]

### 4 Explicit Flux and Gradient Recoveries

In [9, 10, 8], we studied flux and/or gradient recoveries for various lower order finite element approximations to the diffusion problem. A unique feature of those recoveries is that the recovered quantities are in proper finite element spaces. However, those recoveries require solutions of global problems with mass matrices. In this section, we introduce explicit recovery procedures. This is done by first decomposing the error of the flux/gradient through a partition of the unity and then approximating the flux/gradient error by local patch problems. The partition of the unity is based on nodal basis functions of the non-conforming linear element, and hence the local patch problems contain at most two elements.
4.1 Explicit Flux Recovery for Conforming Method

Let \( u_c \) be the conforming linear finite element approximation defined in (3.1). Denote by
\[
\hat{\sigma}_c = -A \nabla u_c, \quad e_c = u - u_c, \quad \text{and} \quad E_c = \sigma - \hat{\sigma}_c = -A \nabla e_c,
\]
the numerical flux, the solution error, and the flux error, respectively. In this section, we introduce an explicit flux recovery procedure. This will be done through approximating the error flux \( E_c \) by local patch problems.

To this end, let \( \phi_{nc}^F(x) \in S_{nc}^F \) be the nodal basis function of the linear nonconforming element associated with the edge \( F \in \mathcal{E} \). Denote by \( \omega_F = \text{supp}(\phi_{nc}^F(x)) \) the support of \( \phi_{nc}^F(x) \), which contains either two or one triangles for the respective interior or boundary edges. Denote the collection of triangles in \( \omega_F \) by
\[
\mathcal{T}_F = \{ K \in \mathcal{T} : \omega_F \cap K \neq \emptyset \}.
\]
Let \( \mathcal{E}_{b,F} \) be the collection of the boundary edges of \( \omega_F \) that does not contain the edge \( F \). Then the collection of edges of triangles in \( \mathcal{T}_F \) is given by
\[
\mathcal{E}_F = \{ E \in \mathcal{E} : E \cap \omega_F \neq \emptyset \} = \{ F \} \cup \mathcal{E}_{b,F} \quad \forall \ F \in \mathcal{E}.
\]
It is also easy to check that
\[
\phi_{nc}^F(x) \equiv 1 \quad \text{on} \quad F \quad \text{and} \quad \int_E \phi_{nc}^F \, ds = 0 \quad \forall \ E \in \mathcal{E}_{b,F}. \tag{4.1}
\]
The set of functions \( \{ \phi_{nc}^F \}_{F \in \mathcal{E}} \) forms a partition of the unity in \( \Omega \):
\[
\sum_{F \in \mathcal{E}} \phi_{nc}^F(x) \equiv 1 \quad \forall \ x \in \Omega,
\]
which leads to the following decomposition of the error flux:
\[
E_c = \sum_{F \in \mathcal{E}} (\phi_{nc}^F E_c) = \sum_{F \in \mathcal{E}} (-\phi_{nc}^F A \nabla e_c).
\]

On edge \( F \in \mathcal{E}_I \cup \mathcal{E}_N \), denote the normal components of the numerical flux by
\[
\hat{\sigma}_{c,F}^+ = (\sigma_{c|K_F^+} \cdot n_F)|_F \quad \text{and} \quad \hat{\sigma}_{c,F}^- = (\sigma_{c|K_F^-} \cdot n_F)|_F \tag{4.2}
\]
and the jump of the numerical flux by
\[
\hat{j}_{f,F}^c = [\sigma_c \cdot n_F]|_F = \begin{cases} 
\hat{\sigma}_{c,F}^- - \hat{\sigma}_{c,F}^+, & \forall \ F \in \mathcal{E}_I, \\
\hat{\sigma}_{c,F}^- - g_N, & \forall \ F \in \mathcal{E}_N.
\end{cases}
\]
By the first equality in (4.1) and the continuity of the normal component of the true flux, it is easy to see that the jump of the normal component of the local error flux \( \phi_{nc} E_c \) on edge \( F \in \mathcal{E}_i \cup \mathcal{E}_N \) satisfies
\[
[\phi_{nc} E_c \cdot n_F] = -\hat{j}_{fc} F. \tag{4.3}
\]
Note that
\[
\phi_{nc} E_c \cdot n_F = (-\phi_{nc} A \nabla e_c) \cdot n_E \approx 0 \quad \text{on} \quad E \in \mathcal{E}_{b,F}.
\]
Therefore, we introduce the following approximation to the local error flux \( \phi_{nc} E_c \) on the local patch \( \omega_F \):

1. for every \( F \in \mathcal{E}_D \), set \( \sigma_{\Delta,c,F} = 0 \) on \( K_F^- \); \( \tag{4.4} \)
2. for every edge \( F \in \mathcal{E}_i \cup \mathcal{E}_N \), find \( \sigma_{\Delta,c,F} \in \mathcal{V}^c_{-1,F} \) such that
\[
\| A^{-1/2} \sigma_{\Delta,c,F} \|_{0,\omega_F} = \min_{\tau \in \mathcal{V}^c_{-1,F}} \| A^{-1/2} \tau \|_{0,\omega_F}, \tag{4.5}
\]
where \( \mathcal{V}^c_{-1,F} \) with \( \mathcal{V} = RT \) or BDM is a local finite element space defined by
\[
\mathcal{V}^c_{-1,F} = \{ \tau \in L^2(\omega_F) | \tau|_K \in \mathcal{V}(K) \forall K \in \mathcal{T}_F, [\tau \cdot n_F]_F = -j_{fc}^c, \tau|_K \cdot n_K = 0 \forall E \in \mathcal{E}_{b,F} \}.
\]

With the approximations defined in (4.4) and (4.5), the global approximation to the error flux is then defined by
\[
\sigma_{\Delta,c} = \sum_{F \in \mathcal{E}_I} \sigma_{\Delta,c,F} = \sum_{F \in \mathcal{E}_I} \sigma_{\Delta,c,F} + \sum_{F \in \mathcal{E}_N} \sigma_{\Delta,c,F}. \tag{4.6}
\]
This yields the following recovered flux for the conforming linear element:
\[
\sigma_c = \sigma_{\Delta,c} + \hat{\sigma}_c \in H(\text{div}; \Omega). \tag{4.7}
\]
The fact that \( \sigma_c \in H(\text{div}; \Omega) \) follows from (4.3) that
\[
[\sigma_c \cdot n_F] = [(\sigma_{\Delta,c} + \hat{\sigma}_c) \cdot n_F] = -j_{fc}^c + [\hat{\sigma}_c \cdot n_F] = 0 \quad \text{on} \quad F \in \mathcal{E}_I.
\]

### 4.1.1 Solution of (4.5)

The recovered flux defined in (4.7) requires solutions of the local problems defined in (4.5), which are constrained minimization problems. This section studies solutions of (4.5).

To this end, let \( \phi_{rt}^F \) be the local RT basis function given in (A.1) in Appendix A, define the global RT basis function associated with the edge \( F \) by
\[
\psi_{rt}^F = \begin{cases} 
\phi_{rt}^F |_{K_F^-}, & \text{x} \in K_F^-; \\
-\phi_{rt}^F |_{K_F^+}, & \text{x} \in K_F^+, \quad \forall F \in \mathcal{E}_i \quad \text{and} \quad \psi_{rt}^F = \begin{cases} 
\phi_{rt}^F |_{K_F^-}, & \text{x} \in K_F^-; \\
0, & \text{x} \notin \omega_F, \quad \forall F \in \mathcal{E}_D \cup \mathcal{E}_N.
\end{cases}
\end{cases}
\]
for any \( F \in \mathcal{E}_i \) and by
\[
\psi_{rt}^F = \begin{cases} 
\phi_{rt}^F |_{K_F^-} & x \in K_F^- , \\
0 & x \notin \omega_F 
\end{cases}
\]
for any \( F \in \mathcal{E}_D \cup \mathcal{E}_N \). Denote by \( \psi_{rt}^{F,-} \) and \( \psi_{rt}^{F,+} \) the restriction of \( \psi_{rt}^F \) on \( K_F^- \) and \( K_F^+ \), respectively. To solve (4.5), set
\[
\sigma_{j,F}^c = -j_{F}^c \psi_{rt}^{F,-} \quad \text{on} \quad K_F^-
\]
(4.8)
for any \( F \in \mathcal{E}_N \) and set
\[
\sigma_{j,F}^c = \begin{cases} 
-j_{F}^c \psi_{rt}^{F,-}, & \text{on} \quad K_F^- , \\
0, & \text{on} \quad K_F^+
\end{cases}
\]
(4.9)
for any \( F \in \mathcal{E}_i \). By (A.5), it is easy to check that for \( F \in \mathcal{E}_i \cup \mathcal{E}_N \)
\[
\left[ \sigma_{j,F}^c \cdot n_F \right]_F = -j_{F}^c \quad \text{and} \quad \sigma_{j,F}^c \cdot n_F = 0 \quad \text{on} \quad E \in \mathcal{E}_{b,F}.
\]
Hence, for any Neumann boundary edge \( F \in \mathcal{E}_N \), we have
\[
\sigma_{c,F}^\Delta = \sigma_{j,F}^c \quad \text{on} \quad K_F^- ,
\]
(4.10)
and for any interior edge \( F \in \mathcal{E}_i \), we have
\[
\sigma_{c,F}^\Delta - \sigma_{j,F}^c \in H(\text{div}; \omega_F) \quad \text{and} \quad \left( \sigma_{c,F}^\Delta - \sigma_{j,F}^c \right) |_E \cdot n_E = 0 \quad \forall \ E \in \mathcal{E}_{b,F}.
\]
Let
\[
\tilde{\sigma}_{c,F} = \sigma_{c,F}^\Delta - \sigma_{j,F}^c, \quad H_0(\text{div}; \omega_F) = \{ \tau \in H(\text{div}; \omega_F) \mid \tau \cdot n|_{\partial \omega_F} = 0 \},
\]
and
\[
\mathcal{V}_F^c = \{ \tau \in H_0(\text{div}; \omega_F) \mid \tau |_K \in \mathcal{V}(K) \forall K \in \mathcal{T}_F \} ,
\]
then the minimization problem in (4.5) for \( F \in \mathcal{E}_i \) is equivalent to finding \( \tilde{\sigma}_{c,F} \in \mathcal{V}_F^c \) such that
\[
\| A^{-1/2} (\tilde{\sigma}_{c,F} + \sigma_{j,F}^c) \|_{0,\omega_F} = \min_{\tau \in \mathcal{V}_F^c} \| A^{-1/2} (\tau + \sigma_{j,F}^c) \|_{0,\omega_F} .
\]
The corresponding variational formulation is to find \( \tilde{\sigma}_{c,F} \in \mathcal{V}_F^c \) such that
\[
(A^{-1} \tilde{\sigma}_{c,F}, \tau)_{0,\omega_F} = - (A^{-1} \sigma_{j,F}^c, \tau)_{0,\omega_F} \quad \forall \ \tau \in \mathcal{V}_F^c .
\]
(4.11)
Note that for any interior edge (4.11) has either one (RT) or two (BDM) unknowns. Their explicit formulas will be introduced in the subsequent section.
4.1.2 Explicit Formula for Flux Recovery

This section derives explicit formulas for the solution of (4.11) and, hence, for the RT and BDM recoveries.

First, we consider the RT recovery. Since $\tilde{\sigma}_{c,rt,F} \in RT^c_F \subset H_0(\text{div}; \omega_F)$, we have

$$\tilde{\sigma}_{c,rt,F} = \sigma^F_{c,rt,F} - \sigma^c_{j,F} = a_{rt,F} j^c_{f,F} \psi_{rt,F}$$

on $\omega_F^c$, for all $F \in \mathcal{E}_I$, which, together with (4.11), yields

$$a_{rt,F} = \beta_{rt,F}^{-} - \beta_{rt,F}^{+} \quad \text{with} \quad \beta_{rt,F}^{\pm} = (A^{-1} \psi_{rt,F}^r, \psi_{rt,F}^r)_{K_F^\pm}.$$

Hence, for any interior edge $F \in \mathcal{E}_I$, we have

$$\sigma^c_{c,rt,F} = \sum_{F \in \mathcal{E}_I \cap K^-} j_{f,F}^c \psi_{rt,F}^r - \sum_{F \in \mathcal{E}_N \cap K^+} j_{f,F}^c \psi_{rt,F}^r,$$  (4.13)

with $\sigma_{c,rt,F}^\pm$ defined in (4.12).

Since the numerical flux is a piecewise constant vector, it has the following local representation on each element $K \in \mathcal{T}$ (see Lemma 4.4 of [9]):

$$\hat{\sigma}_c|_K = \sum_{F \in \partial K} (\hat{\sigma}_c|_F \cdot n_K) \phi_{rt,F}^r,$$

where $n_K$ is the unit outward vector normal to $\partial K$. Globally, for any interior edge $F \in \mathcal{E}_I$, we have

$$\hat{\sigma}_c^F = \left\{ \begin{array}{ll} \hat{\sigma}_{c,F}^r \psi_{rt,F}^r, & \text{on } K_F^-, \\ \hat{\sigma}_{c,F}^+ \psi_{rt,F}^r, & \text{on } K_F^+. \end{array} \right.$$

(4.14)

Now, by (4.7) and (4.13), the explicit formula for the recovered flux using the RT element is then

$$\sigma^r_{c,F} = \sigma^\Delta_{c,rt,F} + \sigma_c = \sum_{F \in \mathcal{E}} \sigma^r_{c,F} \psi_{rt,F},$$

(4.15)

where the nodal value (i.e., the normal component of $\sigma^r_{c,F}$ on the edge $F$), $\sigma^r_{c,F}$, is given by

$$\sigma^r_{c,F} = \left\{ \begin{array}{ll} a_{rt,F} \hat{\sigma}_{c,F}^- + (1 - a_{rt,F}) \hat{\sigma}_{c,F}^+, & F \in \mathcal{E}_I, \\ g_N, & F \in \mathcal{E}_N, \\ \hat{\sigma}_{c,F}^-, & F \in \mathcal{E}_D \end{array} \right.$$

(4.16)
with the nodal values of the numerical fluxes, \( \hat{\sigma}_{c,F}^+ \) and \( \hat{\sigma}_{c,F}^- \), defined in (4.2). Note that for any interior edge \( F \in \mathcal{E}_I \), the nodal value of the recovered flux is an average of the numerical fluxes.

For interface problems, the recovered flux in (4.15) and the resulting estimator are similar to those introduced and analyzed in [9]. To this end, let \( A|_K = \alpha_K I \) for any \( K \in \mathcal{T} \), where \( \alpha_K \) and \( I \) are constant and the identity matrix, respectively. Let

\[
\alpha_F^- = \alpha_K^- \quad \text{and} \quad \alpha_F^+ = \alpha_K^+,
\]

then

\[
\beta_F^- = \frac{1}{\alpha_F^-} \left( \psi_F^r, \psi_F^{rt} \right)_K^- \quad \text{and} \quad \beta_F^+ = \frac{1}{\alpha_F^+} \left( \psi_F^r, \psi_F^{rt} \right)_K^+.
\]

For a regular triangulation, the ratio of \((\psi_F^r, \psi_F^{rt})_K^-\) and \((\psi_F^r, \psi_F^{rt})_K^+\) are bounded above and below. Thus

\[
a_{rt,F} = \frac{\beta_F^-}{\beta_F^- + \beta_F^+} \approx \frac{\alpha_F^+}{\alpha_F^- + \alpha_F^+} \quad \text{and} \quad 1 - a_{rt,F} = \frac{\beta_F^+}{\beta_F^- + \beta_F^+} \approx \frac{\alpha_F^-}{\alpha_F^- + \alpha_F^+}. \tag{4.17}
\]

(Here and thereafter, we will use \( x \approx y \) to mean that there exist two positive constants \( C_1 \) and \( C_2 \) independent of the mesh size such that \( C_1 x \leq y \leq C_2 x \).) (4.17) indicates that the weights in the nodal values of the recovered flux may be replaced by \( \frac{\alpha_F^+}{\alpha_F^- + \alpha_F^+} \) and \( \frac{\alpha_F^-}{\alpha_F^- + \alpha_F^+} \), respectively.

Next, we consider the BDM recovery. For edge \( F \in \mathcal{E} \), let \( s_F \) and \( e_F \) be endpoints of \( F \) such that \( e_F - s_F = h_F t_F \). Let \( \phi_{bdm}^c \) and \( \phi_{bdm}^e \) be the two local BDM basis functions associated with the vertices \( s_F \) and \( e_F \), respectively. For \( i = \{s, e\} \), define the global BDM basis functions associated with the edge \( F \) by

\[
\psi_{i,F}^{bdm} = \begin{cases} 
\phi_{bdm}^{|K_F|} \quad & x \in K_F^- , \\
-\phi_{bdm}^{|K_F|} \quad & x \in K_F^+ , \\
0 \quad & x \notin \omega_F , 
\end{cases}
\]

Denote by \( \psi_{i,F}^{bdm,-} \) and \( \psi_{i,F}^{bdm,+} \) the restriction of \( \psi_{i,F}^{bdm} \) on \( K_F^- \) and \( K_F^+ \), respectively.

Again, since \( \hat{\sigma}_{e,bdm,F} = \sigma_{e,bdm,F}^c - \sigma_{j,F}^c \in BDM_c^\omega \subset H_0(\text{div}; \omega_F) \), we have

\[
\hat{\sigma}_{e,bdm,F} = a_{bdm,F} J_{j,F} \psi_{s,F}^{bdm} + b_{bdm,F} J_{j,F} \psi_{e,F}^{bdm} \quad \forall \ F \in \mathcal{E}_I.
\]

For \( i, j \in \{s, e\} \), let

\[
\beta_{ij,F}^+ = \left( A^{-1} \psi_{i,F}^{bdm}, \psi_{j,F}^{bdm} \right)_{K_F^+} \quad \text{and} \quad \beta_{ij,F}^- = \beta_{ij,F}^+ + \beta_{ij,F}^c.
\]

Solving (4.11) yields

\[
a_{bdm,F} = \frac{(\beta_{ss,F}^- + \beta_{se,F}^-) \beta_{ee,F} - (\beta_{se,F}^- + \beta_{ee,F}^-) \beta_{se,F}}{\beta_{ss,F} \beta_{ee,F} - \beta_{se,F}^2}. \tag{4.18}
\]
and

$$b_{bdm,F} = \frac{(\beta_{ss,F}^+ + \beta_{se,F}^-)(\beta_{ss,F}^- - (\beta_{se,F}^- + \beta_{ee,F}^-))\beta_{se,F}}{\beta_{ss,F}^- \beta_{ee,F}^- - \beta_{se,F}^-}$$  \hfill (4.19)

Since $\psi^{rt,F}_{F} = \psi_{s,F}^{bdm,F} + \psi_{e,F}^{bdm,F}$, we have

$$\sigma^\Delta_{c,bdm,F} = \sigma_{c,bdm,F} + \sigma_{j,F}^c = \hat{a}_{bdm,F} j^c_F \psi_{s,F}^{bdm} + \hat{b}_{bdm,F} j^c_F \psi_{e,F}^{bdm} \quad \forall \, F \in E_i$$  \hfill (4.20)

with

$$\hat{a}_{bdm,F} = \begin{cases} 1 - a_{bdm,F}, & \text{on } K_F^-, \\ a_{bdm,F}, & \text{on } K_F^+, \end{cases} \quad \text{and} \quad \hat{b}_{bdm,F} = \begin{cases} 1 - b_{bdm,F}, & \text{on } K_F^-, \\ b_{bdm,F}, & \text{on } K_F^+. \end{cases}$$

Thus, by (4.6) and (4.10), the error flux is given by

$$\sigma_{c,bdm}^\Delta \equiv \sum_{F \in E_i} \sigma^\Delta_{c,bdm,F} - \sum_{F \in E_N} j^c_F (\psi_{s,F}^{bdm,-} + \psi_{e,F}^{bdm,-}).$$  \hfill (4.21)

with $\sigma^\Delta_{c,bdm,F}$ defined in (4.20). Now, by (4.7) and (4.14), the explicit formula for the recovered flux using the BDM element is then

$$\sigma_{c}^{bdm} = \sigma_{c,bdm}^\Delta + \hat{\sigma}_{c} = \sum_{F \in E} \sigma_{c,s,F}^{bdm,\psi_{s,F}^{bdm}} + \sum_{F \in E} \sigma_{c,e,F}^{bdm,\psi_{e,F}^{bdm}},$$  \hfill (4.22)

where

$$\sigma_{c,s,F}^{bdm} = \begin{cases} a_{bdm,F} \hat{\sigma}_{c,F}^-(1 - a_{bdm,F}) \hat{\sigma}_{c,F}^+, & F \in E_I, \\ g_N, & F \in E_N, \\ \hat{\sigma}_{c,F}^-, & F \in E_D, \end{cases}$$

and

$$\sigma_{c,e,F}^{bdm} = \begin{cases} b_{bdm,F} \hat{\sigma}_{c,F}^-(1 - b_{bdm,F}) \hat{\sigma}_{c,F}^+, & F \in E_I, \\ g_N, & F \in E_N, \\ \hat{\sigma}_{c,F}^-, & F \in E_D. \end{cases}$$

Note that, for any interior edge $F \in E_i$, the coefficients of the recovered flux are again weighted averages of the numerical fluxes.

For interface problems $A|_K = \alpha_K I$ with a regular triangulation, by a careful calculation, we can show that

$$a_{bdm,F} \approx \frac{\alpha_F^+}{\alpha_F^- + \alpha_F^+} \approx b_{bdm,F} \quad \text{and} \quad 1 - a_{bdm,F} \approx \frac{\alpha_F^-}{\alpha_F^- + \alpha_F^+} \approx 1 - b_{bdm,F}.$$  \hfill (4.23)

### 4.2 Explicit Gradient Recovery for Mixed Method

This section introduces an explicit gradient recovery based on the mixed finite element approximation in (3.2). Since derivation is similar to that in the previous section, we briefly describe the recovery procedure and present an explicit formula of the recovered gradient.
Let \((\sigma_m, u_m)\) be the solution of (3.2). Denote by
\[
\hat{\rho}_m = -A^{-1}\sigma_m \quad \text{and} \quad \mathbf{E}_m = \sigma - \sigma_m = -A(\nabla u - \hat{\rho}_m)
\]
the numerical gradient and the flux error, respectively. Then the gradient error is given by
\[
\nabla u - \hat{\rho}_m = -A^{-1}\mathbf{E}_m. \tag{4.24}
\]
Denote the tangential components of the numerical gradient on edge \(F \in \mathcal{E}\) by
\[
\hat{\rho}_{m,F}^+ = \left(\hat{\rho}_m |_{K_F^+} \cdot \mathbf{t}_F\right) |_F \quad \text{and} \quad \hat{\rho}_{m,F}^- = \left(\hat{\rho}_m |_{K_F^-} \cdot \mathbf{t}_F\right) |_F \tag{4.25}
\]
and the edge jump of the numerical gradient on edge \(F \in \mathcal{E}_I \cup \mathcal{E}_D\) by
\[
\hat{j}_{g,F}^m = [\hat{\rho}_m \cdot \mathbf{t}_F]_F = \begin{cases} 
\hat{\rho}_{m,F}^+ - \hat{\rho}_{m,F}^-, & \forall F \in \mathcal{E}_I, \\
\hat{\rho}_{m,F}^- - \nabla g_D \cdot \mathbf{t}_F, & \forall F \in \mathcal{E}_D.
\end{cases}
\]
By the continuity of the tangential components of the true gradient, the edge jump of the tangential component of the local error gradient is given by
\[
\left[ -\phi^{ac} A^{-1}\mathbf{E}_m \cdot \mathbf{t}_F \right]_F = -\hat{j}_{g,F}^m. \tag{4.26}
\]

Since \(\hat{\rho}_{m,F}^+\) and \(\hat{\rho}_{m,F}^-\) are affine functions defined on \(F \in \mathcal{E}\), the ND element is needed for their approximations. To this end, as in the BDM case, for edge \(F \in \mathcal{E}\), let \(s_F\) and \(e_F\) be endpoints of \(F\) such that \(e_F - s_F = h_F \cdot \mathbf{t}_F\), let \(\phi_{i,F}^{nd}\) \((i = s, e)\) be the local ND basis functions given in (A.4) in Appendix A, and define the global ND basis functions associated with edge \(F\) by
\[
\psi_{i,F}^{nd} = \begin{cases} 
\phi_{i,F}^{nd} |_{K_F^+}, & \mathbf{x} \in K_F^-, \\
-\phi_{i,F}^{nd} |_{K_F^-}, & \mathbf{x} \in K_F^+, \forall F \in \mathcal{E}_I \\
0, & \mathbf{x} \notin \omega_F, \forall F \in \mathcal{E}_D \cup \mathcal{E}_N.
\end{cases}
\]

Denote by \(\psi_{i,F}^{nd,-}\) and \(\psi_{i,F}^{nd,+}\) the restriction of \(\psi_{i,F}^{nd}\) on \(K_F^-\) and \(K_F^+\), respectively.

Denote the tangential components of the numerical gradient, \(\hat{\rho}_{m,F}^+\) and \(\hat{\rho}_{m,F}^-\), at the endpoints by
\[
d_{s,F}^\pm = \hat{\rho}_{m,F}^\pm (s_F) \quad \text{and} \quad d_{e,F}^\pm = \hat{\rho}_{m,F}^\pm (e_F),
\]
respectively. Then the numerical gradient has the following representation in local ND bases
\[
\hat{\rho}_m = d_{s,F}^- \psi_{s,F}^{nd,-} - d_{e,F}^- \psi_{e,F}^{nd,-}, \text{ on } K_F^-, \quad \text{for } F \in \mathcal{E}_D
\]
and
\[
\hat{\rho}_m = \begin{cases} 
d_{s,F}^- \psi_{s,F}^{nd,-} - d_{e,F}^- \psi_{e,F}^{nd,-}, & \text{on } K_F^-, \\
d_{s,F}^+ \psi_{s,F}^{nd,+} - d_{e,F}^+ \psi_{e,F}^{nd,+}, & \text{on } K_F^+, \quad \text{for } F \in \mathcal{E}_I.
\end{cases}
\]
Let
\[ c_{s,F} = \begin{cases} d_{s,F}^- - d_{s,F}^+, & \forall F \in \mathcal{E}_I, \\ d_{s,F}^-, & \forall F \in \mathcal{E}_D \end{cases} \]
and \[ c_{e,F} = \begin{cases} d_{e,F}^- - d_{e,F}^+, & \forall F \in \mathcal{E}_I, \\ d_{e,F}^-, & \forall F \in \mathcal{E}_D. \]

A simple calculation leads to
\[ j^m_{b,F} = \begin{cases} c_{s,F}(\psi_{s,F}^{nd} \cdot t_F) - c_{e,F}(\psi_{e,F}^{nd} \cdot t_F), & \forall F \in \mathcal{E}_I, \\ c_{s,F}(\psi_{s,F}^{nd} \cdot t_F) - c_{e,F}(\psi_{e,F}^{nd} \cdot t_F) - \nabla g_D \cdot t_F, & \forall F \in \mathcal{E}_D. \end{cases} \] (4.27)

Let \[ \rho^m_{j,F} = -c_{s,F} \psi_{s,F}^{nd} + c_{e,F} \psi_{e,F}^{nd}, \text{ on } K_F^- \] (4.28) for \( F \in \mathcal{E}_D \), and let
\[ \rho^m_{j,F} = \begin{cases} -c_{s,F} \psi_{s,F}^{nd} + c_{e,F} \psi_{e,F}^{nd}, & \text{on } K_F^-, \\ 0, & \text{on } K_F^+ \end{cases} \] (4.29) for \( F \in \mathcal{E}_I \). By the properties of the ND basis functions in (A.9) and (A.10), it is easy to check that \[ [\rho^m_{j,F} \cdot t_F]_F = -j^m_{b,F} \text{ and } \rho^m_{j,F} \cdot t_E = 0 \text{ on } E \in \mathcal{E}_{b,F} \] for \( F \in \mathcal{E}_I \cup \mathcal{E}_D \).

Let \[ H_0(\text{curl}; \omega_F) = \{ \tau \in H(\text{curl}; \omega_F) \mid \tau \cdot t|_{\partial \omega_F} = 0 \}, \] and let \[ ND^m_F = \{ \tau \in H_0(\text{curl}; \omega_F) \mid |\tau|_K \in ND(K) \forall K \in \mathcal{T}_F \}. \]

In a similar fashion as that of the previous section, by (4.26), we introduce the following approximation to the error gradient:
\[ \rho^\Delta_{m,nd} = \sum_{F \in \mathcal{E}_D} \rho^\Delta_{m,nd,F} + \sum_{F \in \mathcal{E}_I} \rho^\Delta_{m,nd,F}, \] (4.30)

where
\[ \rho^\Delta_{m,nd,F} = \begin{cases} \rho^m_{j,F}, & F \in \mathcal{E}_D, \\ \tilde{\rho}_{m,F} + \rho^m_{j,F}, & F \in \mathcal{E}_I. \end{cases} \] (4.31)

Here, \( \tilde{\rho}_{m,F} \in ND^m_F \) is the solution of the following minimization problem:
\[ \|A^{1/2}([\tilde{\rho}_{m,F} + \rho^m_{j,F}] \cdot t_F)\|_{0,\omega_F} = \min_{\tau \in ND^m_F} \|A^{1/2}(\tau + \rho^m_{j,F})\|_{0,\omega_F}. \] (4.32)

Let
\[ \gamma^-_{ij,F} = (A\psi_{i,F}^{nd}, \psi_{j,F}^{nd})_{K_F^-}, \quad \gamma^+_{ij,F} = (A\psi_{i,F}^{nd}, \psi_{j,F}^{nd})_{K_F^+}, \quad \text{and} \quad \gamma_{ij,F} = \gamma^-_{ij,F} + \gamma^+_{ij,F}. \]
for $i, j \in \{s, e\}$. Solving (4.32) leads to
\[
\tilde{p}_{m,F} = \rho_{s,F} \psi_{s,F}^{nd} + \rho_{e,F} \psi_{e,F}^{nd}
\]
with coefficients given by
\[
\rho_{s,F} = \frac{(c_{s,F} \gamma_{ss,F} - c_{e,F} \gamma_{se,F}) \gamma_{ee,F} - (c_{s,F} \gamma_{se,F} - c_{e,F} \gamma_{ee,F}) \gamma_{se,F}}{\gamma_{ss,F} \gamma_{ee,F} - \gamma_{se,F}^2}
\]
\[
= c_{s,F} \left( \gamma_{ss,F} \gamma_{ee,F} - \gamma_{se,F} \gamma_{se,F} - c_{e,F} \left( \gamma_{se,F} \gamma_{ee,F} - \gamma_{ee,F} \gamma_{se,F} \right) \right)
\]
and
\[
\rho_{e,F} = \frac{(c_{s,F} \gamma_{se,F} - c_{e,F} \gamma_{ee,F}) \gamma_{ss,F} - (c_{s,F} \gamma_{ss,F} - c_{e,F} \gamma_{se,F}) \gamma_{ee,F}}{\gamma_{ss,F} \gamma_{ee,F} - \gamma_{se,F}^2}
\]
\[
= c_{s,F} \left( \gamma_{se,F} \gamma_{ss,F} - \gamma_{ss,F} \gamma_{se,F} - c_{e,F} \left( \gamma_{ee,F} \gamma_{ss,F} - \gamma_{ss,F} \gamma_{ee,F} \right) \right).
\]

Hence, we have
\[
\rho_{m,nd,F}^{\Delta} = \tilde{p}_{m,F} + \rho_{m,F}^{nd} = \begin{cases} 
(\rho_{s,F} - c_{e,F}) \psi_{s,F}^{nd} + (\rho_{e,F} + c_{e,F}) \psi_{e,F}^{nd}, & \text{on } K_F^-, \\
\rho_{s,F} \psi_{s,F}^{nd} + \rho_{e,F} \psi_{e,F}^{nd}, & \text{on } K_F^+
\end{cases}
\]
(4.33)
for interior edge $F \in \mathcal{E}_I$. Now, the explicit formula for the recovered gradient using ND element is then
\[
\rho_{m,F}^{nd} = \rho_{m,nd}^{\Delta} + \tilde{p}_{m,F} = \sum_{F \in \mathcal{E}} a_{m,F}^{nd} \psi_{s,F}^{nd} + \sum_{F \in \mathcal{E}} b_{m,F}^{nd} \psi_{e,F}^{nd},
\]
(4.34)
where the coefficients are given by
\[
a_{m,F}^{nd} = \begin{cases} 
\rho_{s,F} + d_{s,F}^+, & F \in \mathcal{E}_I, \\
d_{s,F}^-, & F \in \mathcal{E}_N, \\
\nabla g_D \cdot t_F, & F \in \mathcal{E}_D
\end{cases}
\]
and
\[
b_{m,F}^{nd} = \begin{cases} 
\rho_{e,F} - d_{e,F}^+, & F \in \mathcal{E}_I, \\
d_{e,F}^-, & F \in \mathcal{E}_N, \\
-\nabla g_D \cdot t_F, & F \in \mathcal{E}_D.
\end{cases}
\]

Notice that
\[
\rho_{s,F} + d_{s,F}^+ = \ell_{s,F} d_{s,F}^- + (1 - \ell_{s,F}) d_{s,F}^+ - \frac{c_{e,F} \left( \gamma_{se,F} \gamma_{ee,F} - \gamma_{ee,F} \gamma_{se,F}^+ \right)}{\gamma_{ss,F} \gamma_{ee,F} - \gamma_{se,F}^2}
\]
with
\[
\ell_{s,F} = \frac{\gamma_{ss,F} \gamma_{ee,F} - \gamma_{se,F} \gamma_{se,F}}{\gamma_{ss,F} \gamma_{ee,F} - \gamma_{se,F}^2},
\]
and
\[
\rho_{e,F} - d_{e,F}^+ = -\ell_{e,F} d_{e,F}^- - (1 - \ell_{e,F}) d_{e,F}^+ + \frac{c_{s,F} \left( \gamma_{se,F} \gamma_{ss,F} - \gamma_{ss,F} \gamma_{se,F}^+ \right)}{\gamma_{ss,F} \gamma_{ee,F} - \gamma_{se,F}^2}.
\]
with
\[
\ell_{e,F} = \left( \frac{\gamma_{ee,F} \gamma_{ss,F} - \gamma_{se,F} \gamma_{se,F}}{\gamma_{ss,F} \gamma_{ee,F} - \gamma_{se,F}^2} \right).
\]

Note that, for any interior edge \( F \in \mathcal{E}_I \), the coefficients of the recovered gradient are weighted averages of the numerical gradients plus some high order terms.

For interface problems \( A|_K = \alpha_K I \) with a regular triangulation, by a careful calculation, we can show that
\[
\ell_{s,F} \approx \frac{\alpha_F^-}{\alpha_F^- + \alpha_F^+} \approx \ell_{e,F} \quad \text{and} \quad 1 - \ell_{s,F} \approx \frac{\alpha_F^+}{\alpha_F^- + \alpha_F^+} \approx 1 - \ell_{e,F}.
\]

### 4.3 Explicit Flux and Gradient Recoveries for Nonconforming Method

Let \( u_{nc} \) be the solution of (3.3). Denote by
\[
\hat{\rho}_{nc} = \nabla_h u_{nc} \quad \text{and} \quad \hat{\sigma}_{nc} = -A^{-1} \nabla_h u_{nc} = -A^{-1} \hat{\rho}_{nc}
\]
the numerical gradient and the numerical flux, respectively. This section introduces explicit formulas of the recovered flux \( \sigma_{nc} \in H(\text{div}, \Omega) \) and the recovered gradient \( \rho_{nc} \in H(\text{curl}, \Omega) \) based on \( \hat{\sigma}_{nc} \) and \( \hat{\rho}_{nc} \). Again, derivations are similar to those in the previous sections and, hence, descriptions in this section are brief.

Denote the solution error, the flux error, and the gradient error by
\[
e_{nc} = u - u_{nc}, \quad E_{nc} = \sigma - \hat{\sigma}_{nc} = -A \nabla_h e_{nc}, \quad \text{and} \quad \nabla u - \nabla_h u_{nc} = \nabla_h e_{nc} = -A^{-1} E_{nc},
\]
respectively. Denote the normal components of the numerical flux on edge \( F \in \mathcal{E} \) by
\[
\hat{\sigma}_{nc,F}^+ = \left( \sigma_{nc}|_{K_F^+} \cdot \mathbf{n}_F \right) |_F \quad \text{and} \quad \hat{\sigma}_{nc,F}^- = \left( \sigma_{nc}|_{K_F^-} \cdot \mathbf{n}_F \right) |_F
\]
and the edge jump of the numerical flux by
\[
\mathcal{J}_{f,F}^{nc} \equiv [\hat{\sigma}_{nc} \cdot \mathbf{n}_F]_F = \begin{cases} \hat{\sigma}_{nc,F}^- - \hat{\sigma}_{nc,F}^+, & \forall F \in \mathcal{E}_I, \\ \hat{\sigma}_{nc,F}^- - g_N, & \forall F \in \mathcal{E}_N. \end{cases}
\]
Denote the tangential components of the numerical gradient on edge \( F \in \mathcal{E} \) by
\[
\hat{\rho}_{nc,F}^+ = \left( \rho_{nc}|_{K_F^+} \cdot \mathbf{t}_F \right) |_F \quad \text{and} \quad \hat{\rho}_{nc,F}^- = \left( \rho_{nc}|_{K_F^-} \cdot \mathbf{t}_F \right) |_F
\]
and the edge jump of the numerical gradient by
\[
\mathcal{J}_{g,F}^{nc} \equiv [\hat{\rho}_{nc} \cdot \mathbf{t}_F]_F = \begin{cases} \hat{\rho}_{nc,F}^- - \hat{\rho}_{nc,F}^+, & \forall F \in \mathcal{E}_I, \\ \hat{\rho}_{nc,F}^- - \nabla g_D \cdot \mathbf{t}_F, & \forall F \in \mathcal{E}_D. \end{cases}
\]
By the continuity of the true flux and true gradient, we have
\[
[\phi_F^{nc} \mathbf{E}_{nc} \cdot \mathbf{n}_F]_F = -\mathcal{J}_{f,F}^{nc} \quad \text{and} \quad [-\phi_F^{nc} A^{-1} \mathbf{E}_{nc} \cdot \mathbf{t}_F]_F = -\mathcal{J}_{g,F}^{nc}.
\]
4.3.1 Explicit Formula for Flux Recovery

In a similar fashion as in Section 4.1, the approximation to the error flux using the RT element is given by

\[ \sigma_{nc,rt}^{\Delta} = \sum_{F \in E_I} \sigma_{nc,rt,F}^{\Delta} + \sum_{F \in E_N} \sigma_{nc,rt,F}^{\Delta} = \sum_{F \in E_I} \sigma_{nc,rt,F}^{\Delta} - \sum_{F \in E_N} J_{f,F}^{nc} \psi_{F}^{rt,-} \] (4.39)

with

\[ \sigma_{nc,rt,F}^{\Delta} = \begin{cases} 
- (1 - a_{rt,F}) j_{f,F}^{nc} \psi_{F}^{rt,-}, & \text{on } K_F^-, \\
 a_{rt,F} j_{f,F}^{nc} \psi_{F}^{rt,+}, & \text{on } K_F^+. 
\end{cases} \] (4.40)

where \( a_{rt,F} \) is defined in Section 4.1.2. Now, the explicit flux recovery using the RT element is given by

\[ \sigma_{nc}^{rt} = \sigma_{nc,rt}^{\Delta} + \hat{\sigma}_e = \sum_{F \in E} \sigma_{nc,F}^{rt} \psi_{F}^{rt} \in H(\text{div}, \Omega), \] (4.41)

where the nodal value \( \sigma_{nc,F}^{rt} \) is given by

\[ \sigma_{nc,F}^{rt} = \begin{cases} 
 a_{rt,F} \hat{\sigma}_{nc,F}^- + (1 - a_{rt,F}) \hat{\sigma}_{nc,F}^+, & F \in E_I, \\
g_N, & F \in E_N, \\
 \hat{\sigma}_{nc,F}, & F \in E_D. 
\end{cases} \] (4.42)

Using the BDM element, the approximation to the error flux is given by

\[ \sigma_{nc,bdm}^{\Delta} = \sum_{F \in E_I} \sigma_{nc,bdm,F}^{\Delta} + \sum_{F \in E_N} \sigma_{nc,bdm,F}^{\Delta} = \sum_{F \in E_I} j_{f,F}^{nc} \left( a_{bdm,F} \psi_{s,F}^{bdm} + b_{bdm,F} \psi_{e,F}^{bdm} \right) - \sum_{F \in E_N} J_{f,F}^{nc} \left( \psi_{s,F}^{bdm} - \psi_{e,F}^{bdm} \right), \] (4.43)

where \( a_{bdm,F} \) and \( b_{bdm,F} \) are defined in Section 4.1.2. Now, the explicit flux recovery using the BDM element is given by

\[ \sigma_{nc}^{bdm} = \sum_{F \in E} \sigma_{nc,s,F}^{bdm} \psi_{s,F}^{bdm} + \sum_{F \in E} \sigma_{nc,e,F}^{bdm} \psi_{e,F}^{bdm} \in H(\text{div}, \Omega) \] (4.44)

where \( \sigma_{nc,s,F}^{bdm} \) and \( \sigma_{nc,e,F}^{bdm} \) is similar to \( \sigma_{c,s,F}^{bdm} \) and \( \sigma_{c,e,F}^{bdm} \) defined in Section 4.1.2, i.e.,

\[ \sigma_{nc,s,F}^{bdm} = \begin{cases} 
 a_{bdm,F} \hat{\sigma}_{nc,F}^- + (1 - a_{bdm,F}) \hat{\sigma}_{nc,F}^+, & F \in E_I, \\
g_N, & F \in E_N, \\
 \hat{\sigma}_{nc,F}, & F \in E_D. 
\end{cases} \]

and

\[ \sigma_{nc,e,F}^{bdm} = \begin{cases} 
 b_{bdm,F} \hat{\sigma}_{nc,F}^- + (1 - b_{bdm,F}) \hat{\sigma}_{nc,F}^+, & F \in E_I, \\
g_N, & F \in E_N, \\
 \hat{\sigma}_{nc,F}, & F \in E_D. 
\end{cases} \]
4.3.2 Explicit Formula for Gradient Recovery

Let $\phi_F^{ne}$ be the local NE basis function given in Appendix A, define the global NE basis function associated with the edge $F$ by

$$\psi_F^{ne} = \begin{cases} \phi_F^{ne}|_{K_F^-}, & \mathbf{x} \in K_F^-, \\
-\phi_F^{ne}|_{K_F^+}, & \mathbf{x} \in K_F^+, \ \forall F \in \mathcal{E}_I \text{ and } \psi_F^{ne} = \begin{cases} \phi_F^{ne}|_{K_F^-}, & \mathbf{x} \in K_F^-, \\
0, & \mathbf{x} \notin \omega_F, \ \forall F \in \mathcal{E}_D \cup \mathcal{E}_N. \end{cases} \end{cases}$$

Denote by $\psi_F^{ne,-}$ and $\psi_F^{ne,+}$ the restriction of $\psi_F^{ne}$ on $K_F^-$ and $K_F^+$, respectively. Let

$$a_{ne,F} = \frac{\beta_{ne,F}^{+} - \beta_{ne,F}^{-}}{\beta_{ne,F}^{0}} \text{ with } \beta_{ne,F}^{\pm} = (A^{-1} \psi_F^{ne}, \psi_F^{ne})_{K_F^\pm}.$$

Then the approximation to the gradient error is

$$\rho_{nc,ne} = \sum_{F \in \mathcal{E}_I} \rho_{nc,ne,F}^\Delta + \sum_{F \in \mathcal{E}_D} \rho_{nc,ne,F}^\Delta = \sum_{F \in \mathcal{E}_I} \rho_{nc,ne,F}^\Delta + \sum_{F \in \mathcal{E}_D} j_{g_F}^{nc} \psi_F^{ne,-} \quad (4.45)$$

with

$$\rho_{nc,ne,F}^\Delta = \begin{cases} -(1 - a_{ne,F}) j_{g_F}^{nc} \psi_F^{ne,-}, & \mathbf{x} \in K_F^-, \\
a_{ne,F} j_{g_F}^{nc} \psi_F^{ne,+}, & \mathbf{x} \in K_F^+. \end{cases} \quad (4.46)$$

Now, the explicit gradient recovery using the NE element is given by

$$\rho_{nc,F}^{ne} = \rho_{nc,ne} + \rho_{nc} = \sum_{F \in \mathcal{E}} \rho_{nc,F}^{ne} \psi_F^{ne} \in H(\text{curl}, \Omega), \quad (4.47)$$

where the nodal value $\rho_{nc,F}^{ne}$ is given by

$$\rho_{nc,F}^{ne} = \begin{cases} a_{ne,F} \hat{\rho}_{nc,F} + (1 - a_{ne,F}) \hat{\rho}_{nc,F}^+, & F \in \mathcal{E}_I, \\
\nabla g_F \cdot t_F, & F \in \mathcal{E}_D, \\
\hat{\rho}_{nc,F}, & F \in \mathcal{E}_N. \end{cases} \quad (4.48)$$

Next, we describe the recovered gradient using the ND element. Let

$$a_F^{nc} = \frac{(\gamma_{ss,F} - \gamma_{se,F}) \gamma_{ee,F} - (\gamma_{se,F} - \gamma_{ee,F}) \gamma_{se,F}}{\gamma_{ss,F} \gamma_{ee,F} - \gamma_{se,F}^2} > 0, \ \text{ and}$$

$$b_F^{nc} = \frac{(\gamma_{se,F} - \gamma_{ee,F}) \gamma_{ss,F} - (\gamma_{ss,F} - \gamma_{se,F}) \gamma_{se,F}}{\gamma_{ss,F} \gamma_{ee,F} - \gamma_{se,F}^2} < 0$$

with $\gamma_{ij,F}^{\pm}$ and $\gamma_{ij,F}$, $(i, j \in \{s, e\})$ defined in Section 4.2. Similar to the gradient recovery using the ND element for the mixed method, the approximation to the error gradient is

$$\rho_{nc,nd}^\Delta = \sum_{F \in \mathcal{E}_I} (a_F^{nc} j_{g_F} \psi_{s,F}^{nd} + b_F^{nc} j_{g_F} \psi_{e,F}^{nd}) - \sum_{F \in \mathcal{E}_D} j_{g_F}^{nc} (\psi_{s,F}^{nd,-} - \psi_{e,F}^{nd,-}). \quad (4.49)$$
where the coefficients $\hat{a}_{F}$ and $\hat{b}_{F}$ are given by

$$
\hat{a}_{F} = \begin{cases} 
1 - a_{F}, & \text{on } K_{F}^{-}, \\
-a_{F}, & \text{on } K_{F}^{+}, 
\end{cases}
$$

and

$$
\hat{b}_{F} = \begin{cases} 
1 + b_{F}, & \text{on } K_{F}^{-}, \\
b_{F}, & \text{on } K_{F}^{+}, 
\end{cases}
$$

Now, the recovered gradient using the $ND$ element is given by

$$
\rho_{nd} = \rho_{nc} + \hat{\rho}_{nc} = \sum_{F \in E} \rho_{nc,s,F} \psi_{s,F} + \sum_{F \in E} \rho_{nc,c,F} \psi_{c,F},
$$

where the coefficients of $\psi_{nd}$ and $\psi_{nc}$ are given by

$$
\hat{\rho}_{nc,s,F} = \begin{cases} 
a_{F} \hat{\rho}_{nc} - \rho_{nc} + \left(1 - a_{F}\right) \hat{\rho}_{nc}, & F \in E_{I}, \\
\nabla g_{D} \cdot t, & F \in E_{D}, \\
\frac{\partial \rho_{nc}}{\partial n}, & F \in E_{N}, 
\end{cases},
$$

and

$$
\hat{\rho}_{nc,c,F} = \begin{cases} 
b_{F} \hat{\rho}_{nc} + b_{F} \hat{\rho}_{nc} + \left(1 + b_{F}\right) \hat{\rho}_{nc}, & F \in E_{I}, \\
-\nabla g_{D} \cdot t, & F \in E_{D}, \\
-\frac{\partial \rho_{nc}}{\partial n}, & F \in E_{N}.
\end{cases}
$$

### 5 Explicit A Posteriori Error Estimators

With the explicit recoveries of the flux and gradient introduced in Section 4 for various finite element approximations, this section describes the corresponding recovery-based a posteriori error estimators.

For the conforming linear element, we study two estimators using the respective $RT$ and $BDM$ recoveries. The global $RT$ a posteriori error estimator is given by

$$
\eta_{rt} = \|A^{-1/2} (\sigma_{c} + A \nabla u_{c})\|_{0, \Omega} = \|A^{-1/2} \sigma_{c,rt}\|_{0, \Omega},
$$

and the $RT$ local error indicators on element $K \in T$ and on edge $F \in E$ are given by

$$
\eta_{rt} = \|A^{-1/2} \sigma_{c,rt}\|_{0, K} \quad \text{and} \quad \eta_{rt} = \|A^{-1/2} \sigma_{c,rt,f}\|_{0, \omega_{F}},
$$

respectively, where $\sigma_{c,rt}$ is given in (4.13) and $\sigma_{c,rt,f}$ in (4.10) and (4.12). The global $BDM$ a posteriori error estimator is given by

$$
\eta_{bdt} = \|A^{-1/2} \sigma_{c,bdm}\|_{0, \Omega} = \|A^{-1/2} \sigma_{c,bdm}\|_{0, \Omega},
$$

and the $BDM$ local error indicators on element $K \in T$ and on edge $F \in E$ are given by

$$
\eta_{bdt} = \|A^{-1/2} \sigma_{c,bdm}\|_{0, K} \quad \text{and} \quad \eta_{bdt} = \|A^{-1/2} \sigma_{c,bdm,f}\|_{0, \omega_{F}},
$$

respectively, where $\sigma_{c,bdm}$ is defined in (4.21) and $\sigma_{c,bdm,f}$ in (4.10) and (4.20).
For the lowest-order mixed element, we study one estimator based on the explicit $ND$ recovery. The local error indicators on element $K \in \mathcal{T}$ and on edge $F \in \mathcal{E}$ are defined by

\[ \eta_{nc,K}^{nd} = \| A^{1/2} \rho_{nc,nd,K} \|_{0,K} \quad \text{and} \quad \eta_{nc,F}^{nd} = \| A^{1/2} \rho_{nc,nd,F} \|_{0,\Omega} , \]

respectively, where $\rho_{nc,nd}$ is defined in (4.30) and $\rho_{nc,nd,F}$ in (4.31) and (4.33). The global error estimator is then defined by

\[ \eta_{nc}^{nd} = \| A^{-1/2} (\sigma_m + A \rho_{nc,m}^{nd}) \|_{0,\Omega} = \| A^{1/2} \rho_{nc,nd} \|_{0,\Omega} . \]

For the nonconforming linear element, again we introduce two estimators based on the $RT-NE$ and $BDM-ND$ recoveries. Let $c_1, c_2 \in (0,1)$ be parameters to be determined such that $c_1 + c_2 = 1$ (e.g., $c_1 = c_2 = 1/2$). The global $RT-NE$ error estimator is defined by

\[ \eta_{nc}^{rh} = \left( c_1 \| A^{-1/2} \sigma_{nc,rt}^{\Delta} \|_{0,\Omega}^2 + c_2 \| A^{1/2} \rho_{nc,ne}^{\Delta} \|_{0,\Omega}^2 \right)^{1/2} , \]

and the $RT-WH$ local error indicators on element $K \in \mathcal{T}$ and on edge $F \in \mathcal{E}$ are defined respectively by

\[ \eta_{nc,K}^{rh} = \left( c_1 \| A^{-1/2} \sigma_{nc,rt,K}^{\Delta} \|_{0,K}^2 + c_2 \| A^{1/2} \rho_{nc,ne,K}^{\Delta} \|_{0,K}^2 \right)^{1/2} , \]

and \[ \eta_{nc,F}^{rh} = \left( c_1 \| A^{-1/2} \sigma_{nc,rt,F}^{\Delta} \|_{0,\omega}^2 + c_2 \| A^{1/2} \rho_{nc,ne,F}^{\Delta} \|_{0,\omega}^2 \right)^{1/2} , \]

where $\sigma_{nc,rt}, \sigma_{nc,rt,F}, \rho_{nc,ne}$ and $\rho_{nc,ne,F}$ are defined in (4.39), (4.40), (4.45), and (4.46), respectively.

Similarly, The global $BDM-ND$ error estimator is defined by

\[ \eta_{nc}^{rh} = \left( c_1 \| A^{-1/2} \sigma_{nc,bdm}^{\Delta} \|_{0,\Omega}^2 + c_2 \| A^{1/2} \rho_{nc,nd}^{\Delta} \|_{0,\Omega}^2 \right)^{1/2} , \]

and the local $BDM-ND$ error indicators on element $K \in \mathcal{T}$ and on edge $F \in \mathcal{E}$ are defined respectively by

\[ \eta_{nc,K}^{rh} = \left( c_1 \| A^{-1/2} \sigma_{nc,bdm,K}^{\Delta} \|_{0,K}^2 + c_2 \| A^{1/2} \rho_{nc,nd,K}^{\Delta} \|_{0,K}^2 \right)^{1/2} , \]

and \[ \eta_{nc,F}^{rh} = \left( c_1 \| A^{-1/2} \sigma_{nc,bdm,F}^{\Delta} \|_{0,\omega}^2 + c_2 \| A^{1/2} \rho_{nc,nd,F}^{\Delta} \|_{0,\omega}^2 \right)^{1/2} , \]

where $\sigma_{nc,bdm}$ and $\rho_{nc,nd}$ are defined in (4.43) and (4.49), respectively.

6 Efficiency and Reliability

This section establishes efficiency and reliability bounds of the estimators defined in Section 5 for interface problems (i.e., $A = \alpha I$ and $\alpha(x)$ is a piecewise constant with respect to the triangulation $\mathcal{T}$). In order to show that the reliability constant are independent of the jump of $\alpha$, as usual, we assume that the distribution of the coefficients $\alpha_K$ for all
$K \in \mathcal{T}$ is locally quasi-monotone \cite{23}, which is slightly weaker than Hypothesis 2.7 in \cite{5}. For convenience of readers, we restate it here.

Let $\omega_z$ be the union of all elements having $z$ as a vertex. For any $z \in \mathcal{N}$, let

$$
\hat{\omega}_z = \{ K \in \omega_z : \alpha_K = \max_{K' \in \omega_z} \alpha_{K'} \}.
$$

**Definition 6.1.** Given a vertex $z \in \mathcal{N}$, the distribution of the coefficients $\alpha_K$, $K \in \omega_z$, is said to be quasi-monotone with respect to the vertex $z$ if there exists a subset $\tilde{\omega}_{K,z,qm}$ of $\omega_z$ such that the union of elements in $\tilde{\omega}_{K,z,qm}$ is a Lipschitz domain and that

- if $z \in \mathcal{N} \setminus \mathcal{N}_D$, then $\{ K \} \cup \hat{\omega}_z \subset \tilde{\omega}_{K,z,qm}$ and $\alpha_K \leq \alpha_{K'} \forall K' \in \tilde{\omega}_{K,z,qm}$;
- if $z \in \mathcal{N}_D$, then $K \in \tilde{\omega}_{K,z,qm}$, $\partial \tilde{\omega}_{K,z,qm} \cap \Gamma_D \neq \emptyset$, and $\alpha_K \leq \alpha_{K'} \forall K' \in \tilde{\omega}_{K,z,qm}$.

The distribution of the coefficients $\alpha_K$, $K \in \mathcal{T}$, is said to be locally quasi-monotone if it is quasi-monotone with respect to every vertex $z \in \mathcal{N}$.

Let $f_T$ be the $L^2$ projection of $f$ onto the space of piecewise constant defined on elements of $\mathcal{T}$, let

$$
H_f = \left( \sum_{K \in \mathcal{T}} H^2_{f,K} \right)^{1/2} \quad \text{with} \quad H_{f,K} = \frac{h_K}{\sqrt{\alpha_K}} \| f - f_T \|_{l_0,K} \quad \forall K \in \mathcal{T},
$$

and let

$$
\hat{H}_f = \left( \sum_{z \in \mathcal{N} \setminus (\mathcal{S} \cup \Gamma_D)} \sum_{K \subset \omega_z} \frac{h^2_K}{\alpha_K} \| f \|_{l_0,K}^2 + \sum_{z \in \mathcal{N} \setminus (\mathcal{S} \cup \Gamma_D)} \sum_{K \subset \omega_z} \frac{h^2_K}{\alpha_K} \| f - \int_{\omega_z} f \psi_z dx \|_{l_0,K}^2 \right)^{1/2},
$$

where $\int_{\omega_z} f \psi_z dx = \int_{\hat{\omega}_z} f \psi_z dx / \int_{\hat{\omega}_z} \psi_z dx$ is a weighted average of $f$ over $\hat{\omega}_z$ and $\psi_z$ is a linear nodal basis function at $z \in \mathcal{N}$.

**Remark 6.2.** For various lower order finite element approximations, the second term in $\hat{H}_f$ is of higher order than $\eta_E$ (defined below in (6.2)) for $f \in L^2(\Omega)$ and so is the first term for $f \in L^p(\Omega)$ with $p > 2$ (see \cite{14}).

### 6.1 Conforming Elements

**Theorem 6.3.** Assume that the distribution of the coefficients are quasi-monotone. Then the error estimators $\eta^{rt}_c$ and $\eta^{bdm}_c$ satisfy the global reliability bound:

$$
\| \alpha^{1/2} \nabla e_c \|_{l_0,\Omega} \leq C(\eta^{rt}_c + \hat{H}_f) \quad \text{and} \quad \| \alpha^{1/2} \nabla e_c \|_{l_0,\Omega} \leq C(\eta^{bdm}_c + \hat{H}_f),
$$

(6.1)

where the constants above are independent of $\alpha$ and the mesh size.

**Proof.** Inequalities (6.1) may be established in a similar fashion as those in \cite{9,12}. \qed
To prove the efficiency bound, consider the edge error estimator and indicator of the residual type:

\[
\eta_{c,E} := \left( \sum_{F \in E \cup E_N} \eta_{c,F}^2 \right)^{1/2} \quad \text{with} \quad \eta_{c,F} = \begin{cases} 
\frac{h_F j_F^c}{\sqrt{\alpha_F^+ + \alpha_F^-}} & F \in E, \\
\frac{h_F j_F^c}{\sqrt{\alpha_F^-}} & F \in E_N.
\end{cases}
\] (6.2)

Without assumptions on the distribution of the coefficient \(\alpha\), it was proved by Petzoldt (see equation (5.7) in [23]) that there exists a constant \(C > 0\) independent of \(\alpha\) and the mesh size such that

\[
\eta_{c,F}^2 \leq C \left( \|\alpha^{-1/2} \nabla e_c\|_{0,\omega_F}^2 + \sum_{K \in T_F} H_{f,K}^2 \right).
\] (6.3)

Let \(T_K = \{T \in T : T \text{ and } K \text{ share at least one edge}\}\).

**Theorem 6.4.** The local indicators \(\eta_{c,F}^{rt}, \eta_{c,K}^{rt,}, \eta_{c,F}^{bdm}\), and \(\eta_{c,K}^{bdm}\) defined in Section 5 are efficient, i.e., there exists a constant \(C > 0\) independent of \(\alpha\) and the mesh size such that

\[
\eta_{c,F}^{bdm} \leq \eta_{c,F}^{rt} \leq C \|\alpha^{-1/2} \nabla e_c\|_{0,\omega_F} + \sum_{K \in T_F} H_{f,K}^2)^{1/2}
\] (6.4)

and

\[
\eta_{c,K}^{bdm} \leq \eta_{c,K}^{rt} \leq C \|\alpha^{-1/2} \nabla e_c\|_{0,\omega_K} + \sum_{T \in T_K} H_{f,T}^2)^{1/2}.
\] (6.5)

**Proof.** Without loss of generality, we establish the efficiency bounds only for interior edges. The first inequality of (6.4) is a direct consequence of the minimization problem in (4.5) and the fact that \(RT_{c_{1,F}} \subset BDM_{c_{1,F}}\). To prove the second inequality of (6.4), we assume that the triangulation is regular. By the equivalence in (4.17) and the fact that \(\|\eta_{F}^{rt}\|_{0,\omega_F} \leq C h_{F}^2\), we have

\[
\left(\eta_{c,F}^{rt}\right)^2 = \|\alpha_F^{-1/2} \sigma_{e,rt,F}\|_{0,K_F^-}^2 + \|\alpha_F^{-1/2} \sigma_{e,rt,F}\|_{0,K_F^+}^2 \\
\leq C \left( \frac{1}{\alpha_F^-} \left( \frac{\alpha_F^-}{\alpha_F^- + \alpha_F^+} \right)^2 + \frac{1}{\alpha_F^+} \left( \frac{\alpha_F^+}{\alpha_F^- + \alpha_F^+} \right)^2 \right) (j_F^c, h_F) = C \eta_{c,F}^2,
\]

which, combining with (6.3), implies the second inequality of (6.4). It is easy to see that

\[
\left(\eta_{c,K}^{rt}\right)^2 \leq \sum_{F \in E_K} \left(\eta_{c,F}^{rt}\right)^2 \quad \text{and} \quad \left(\eta_{c,K}^{bdm}\right)^2 \leq \sum_{F \in E_K} \left(\eta_{c,F}^{bdm}\right)^2.
\]

Now, (6.5) follows from (6.4). This completes the proof of the theorem. \(\square\)
6.2 Mixed Elements

**Theorem 6.5.** Assume that the distribution of the coefficient $\alpha$ is quasi-monotone. Then the error estimator $\eta_{m}^{nd}$ satisfies the following global reliability bound:

$$\|\alpha^{-1/2}E_{m}\|_{0,\Omega} \leq C(\eta_{m}^{nd} + H_{f} + G_{\nabla h \times (\alpha^{-1}\sigma_{m})}),$$  \hfill (6.6)

where $G_{\nabla h \times (\alpha^{-1}\sigma_{m})}$ is a higher order term if $\nabla h \times (\alpha^{-1}\sigma_{m}) \in L^{p}(\Omega)$ with $p > 2$. Moreover, if $\mathcal{V} = RT$, then

$$\|\alpha^{-1/2}E_{m}\|_{0,\Omega} \leq C(\eta_{m}^{nd} + H_{f}).$$  \hfill (6.7)

**Proof.** Let $\hat{\eta}_{m}$ be the implicit recovery-based estimator introduced in [10], i.e.,

$$\hat{\eta}_{m} = \min_{\tau \in \mathcal{ND}} \|\alpha^{1/2}\tau + \alpha^{-1/2}\sigma_{m}\|_{0,\Omega}. $$

It is obvious that $\hat{\eta}_{m} \leq \eta_{m}^{nd}$. Now, the theorem is a direct consequence of Theorem 6.2 of [10].

The efficiency of the $\eta_{m}^{nd}$ may be established by a direct calculation similar to the proof of Theorem 6.4. However, the calculation is quite complicated in this case. We will prove it through the following Helmholtz decomposition (see, e.g., [17]) of the error flux $E_{m}$: there exist $\xi_{m} \in H^{1}_{D}(\Omega)$ and $\zeta_{m} \in H^{1}_{N}(\Omega) \equiv \{v \in H^{1}(\Omega) \mid v = 0 \text{ on } \Gamma_{N}\}$ such that

$$E_{m} = \alpha\nabla\xi_{m} + \nabla^{\perp}\zeta_{m}$$  \hfill (6.8)

and

$$\|\alpha^{-1/2}E_{m}\|_{0,\Omega}^{2} = \|\alpha^{1/2}\nabla\xi_{m}\|_{0,\Omega}^{2} + \|\alpha^{-1/2}\nabla^{\perp}\zeta_{m}\|_{0,\Omega}^{2}. $$

**Theorem 6.6.** The local indicators $\eta_{m,F}^{nd}$ and $\eta_{m,K}^{nd}$ and the global error estimator $\eta_{m}^{nd}$ are efficient, i.e., there exists a constant $C > 0$ independent of $\alpha$ and the mesh size such that

$$\eta_{m,F}^{nd} \leq C\|\alpha^{-1/2}\nabla^{\perp}\zeta_{m}\|_{0,\omega_{F}}, \quad \eta_{m,K}^{nd} \leq C\|\alpha^{-1/2}\nabla^{\perp}\zeta_{m}\|_{0,\omega_{K}},$$  \hfill (6.9)

and

$$\eta_{m,F}^{nd} \leq C\|\alpha^{-1/2}\nabla^{\perp}\zeta_{m}\|_{0,\Omega} \leq C\|\alpha^{1/2}E_{m}\|_{0,\Omega}.$$  \hfill (6.10)

**Proof.** Without loss of generality, we establish the efficiency bounds only for interior edges. Let $\eta_{m,F}$ and $\eta_{m}$ be the respective edge indicator and estimator defined in [10], where

$$\eta_{m,F}^{2} = \frac{\alpha_{F}^{-} + \alpha_{F}^{+}}{2} \int_{F} |j_{m}^{g,F}|^{2} ds. $$  \hfill (6.11)

It is proved in Proposition 6.6 of [10] that

$$\eta_{m,F} \leq \|\alpha^{-1/2}\nabla^{\perp}\zeta_{m}\|_{0,\omega_{F}} \quad \text{and} \quad \eta_{m} \leq C\|\alpha^{-1/2}\nabla^{\perp}\zeta_{m}\|_{0,\Omega} \leq C\|\alpha^{1/2}E_{m}\|_{0,\Omega}. $$  \hfill (6.12)

Since $\|\psi_{i,F}^{nd}\|_{K} \approx C h_{F}$ for $i = s, e$, it follows from (4.32) with $\tau = 0$, (4.29), and the triangle inequality that

$$\eta_{m,F}^{nd} = \|\alpha^{1/2}\rho_{m,F}^{nd}\|_{0,\omega_{F}} \leq \|\alpha^{1/2}\rho_{m,F}^{nd}\|_{0,\omega_{F}} + \|\rho_{m,F}^{nd}\|_{0,\omega_{F}} \leq \sqrt{\alpha_{F}^{-}} \left( |c_{s,F}| + |c_{e,F}| \right) \leq C h_{F} \sqrt{\alpha_{F}^{-}} (|c_{s,F}| + |c_{e,F}|).$$

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Note that
\[ c_{s,F} = j_{g,F}^m(s_F) \quad \text{and} \quad c_{e,F} = j_{g,F}^m(e_F) \]
and that \( j_{g,F}^m \) is an affine function on \( F \), it is then easy to check that there exists a constant \( C > 0 \) independent of \( \alpha \) and \( h_F \) such that
\[ |c_{s,F}| + |c_{e,F}| \leq C h_F^{-1/2} \left( \int_F |j_{g,F}^m|^2 \, ds \right)^{1/2}. \]

By using the above two inequalities, we have
\[ \eta_{nd}^m \leq C h_F^{1/2} \sqrt{\alpha_F \left( \int_F |j_{g,F}^m|^2 \, ds \right)^{1/2}} \leq C \eta_{m,F}, \]
which, together with (6.12), implies the validity of the first inequality in (6.9). Now, the second inequality in (6.9) and (6.10) are straightforward from the definitions and (6.12).

### 6.3 Nonconforming Elements

**Theorem 6.7.** Assume that the distribution of the coefficient \( \alpha \) is quasi-monotone. Then the error estimators \( \eta_{nc}^r \) and \( \eta_{nc}^d \) satisfy the global reliability bounds:
\[ \| \alpha^{1/2} \nabla_h e_{nc} \|_{0,\Omega} \leq C \left( \eta_{nc}^d + H_f \right) \]  \hspace{1cm} (6.13)
and
\[ \| \alpha^{1/2} \nabla_h e_{nc} \|_{0,\Omega} \leq C \left( \eta_{nc}^r + H_f \right). \]  \hspace{1cm} (6.14)

**Proof.** Let \( \eta_{nc} \) be the implicit recovery-based estimator introduced in [10]:
\[ \eta_{nc}^2 = c \eta_{nc,1}^2 + (1 - c) \eta_{nc,2}^2 \]
with \( c \in (0, 1) \) being a parameter to be determined, where
\[ \eta_{nc,1} = \min_{\tau \in BDM} \| \alpha^{-1/2} \tau + \alpha^{1/2} \nabla_h u_{nc} \|_{0,\Omega} \quad \text{and} \quad \eta_{nc,2} = \min_{\tau \in ND} \| \alpha^{1/2} (\tau - \nabla_h u_{nc}) \|_{0,\Omega}. \]

It is obvious that \( \eta_{nc,1} \leq \eta_{nc}^b \leq \eta_{nc}^r \) and that \( \eta_{nc,2} \leq \eta_{nc}^d \leq \eta_{nc}^d \). Now, (6.13) and (6.14) follow from Theorem 6.4 of [10].

To prove the efficiency of the explicit error estimators, consider the weighted edge error estimator introduced in [10]:
\[ \eta_{nc,E} := \left( \sum_{F \in \mathcal{E}} \eta_{nc,F}^2 \right)^{1/2} \quad \text{with} \quad \eta_{nc,F}^2 = \begin{cases} \frac{2h_F^2}{\alpha_F + \alpha_F^-} (j_{f,F}^r)^2 + \frac{h_F^2 \alpha_F^r \alpha_F^-}{\alpha_F + \alpha_F^-} (j_{g,F}^r)^2, & F \in \mathcal{E}_I, \\ \frac{h_F^2}{\alpha_F} (j_{f,F}^r)^2, & F \in \mathcal{E}_N, \\ h_F^2 \alpha_F^- (j_{g,F}^r)^2, & F \in \mathcal{E}_D. \end{cases} \]
Lemma 6.8. There exist a positive constant $C$ independent of $\alpha$ and the mesh size such that
\[
\eta_{nc,F}^{rt} \leq C\eta_{nc,F} \quad \text{and} \quad \eta_{nc,F}^{ne} \leq C\eta_{nc,F}
\] (6.15)

Proof. Without loss of generality, we prove the validity of the lemma only for interior edges. Assume that the triangulation is regular, then $\|\phi_F^{rt}\|_{0,K} \leq C h_F$. It follows from the definition of $\eta_{nc,F}^{rt}$, (4.39), and the equivalence (4.17) that
\[
\eta_{nc,1,F}^{rt} = \|\alpha^{-1/2}\sigma_{nc,rt,F}^{\Delta}\|_{0,\omega_F} = \left(\|\alpha^{-1/2}\sigma_{nc,rt,F}^{\Delta}\|_{0,K_F^-}^2 + \|\alpha^{-1/2}\sigma_{nc,rt,F}^{\Delta}\|_{0,K_F^+}^2\right)^{1/2}
\]
\[
\leq C h_F j_{nc,F}^{\Delta} \left(\frac{1}{\alpha^-} \left(\frac{\alpha^-}{\alpha^- + \alpha^+}\right)^2 + \frac{1}{\alpha^+} \left(\frac{\alpha^+}{\alpha^- + \alpha^+}\right)^2\right)^{1/2}
\]
\[
= C \frac{h_F j_{nc,F}^{\Delta}}{\sqrt{\alpha^- + \alpha^+}} \leq C\eta_{nc,F},
\]
which implies the first inequality in (6.15).

To prove the second inequality in (6.15), for any $F \in \mathcal{E}_1$, introduce
\[
\rho_{j,F}^{nc} = \begin{cases} j_{g,F}^{nc} h_F \psi_{F}^{nc}, & \text{on } K_F^- , \\ 0, & \text{on } K_F^+ . \end{cases}
\]
Without loss of generality, we assume that $\alpha^- \leq \alpha^+$. (Otherwise, $\rho_{j,F}^{nc}$ may be redefined by exchanging $K_F^-$ and $K_F^+$.) Since $j_{g,F}^{nc}$ is a constant on $F$ and $\|\psi_{F}^{nc}\|_{K} \approx C h_F$, by the definitions of $\rho_{nc,ne,F}^{\Delta}$ in (4.46), we have
\[
\eta_{nc,F}^{ne} = \|\alpha^{1/2}\rho_{nc,ne,F}^{\Delta}\|_{0,\omega_F} \leq \|\alpha^{1/2}\rho_{j,F}^{nc}\|_{0,\omega_F} = \sqrt{\alpha^-} \|\rho_{j,F}^{nc}\|_{0,K_F^-}
\]
\[
= \sqrt{\alpha^-} J_{nc,F} h_F \|\psi_{F}^{ne}\|_{0,K_F^-} \leq C \sqrt{\alpha^-} J_{nc,F} h_F
\]
\[
\leq C \left(\frac{\alpha^+}{\alpha^+ + \alpha^-}\right)^{1/2} J_{nc,F} h_F \leq C\eta_{nc,F}.
\]
This completes the proof of the second inequality in (6.15) and, hence, the lemma. \qed

Theorem 6.9. The local indicators $\eta_{nc,F}^{rh}$, $\eta_{nc,K}^{rh}$, $\eta_{nc,F}^{bd}$, and $\eta_{nc,K}^{bd}$ are efficient, i.e., there exists a constant $C > 0$ independent of $\alpha$ and the mesh size such that
\[
\eta_{nc,F}^{bd} \leq \eta_{nc,F}^{rh} \leq C \|\alpha^{1/2} \nabla h e_{nc}\|_{0,\omega_F} + C \left(\sum_{K \in \mathcal{T}_F} H_{f,K}^2\right)^{1/2}
\] (6.16)

and that
\[
\eta_{nc,K}^{rh}, \eta_{nc,K}^{bd} \leq C \|\alpha^{1/2} \nabla h e_{nc}\|_{0,\omega_K} + C \left(\sum_{T \in \mathcal{T}_K} H_{f,T}^2\right)^{1/2}.
\] (6.17)
Proof. Let
\[ V_{nc}^{-1,F} = \{ \tau \in L^2(\omega_F) \mid \tau|_K \in V(K) \forall K \in T_F, \left[ \tau \cdot n_F \right]_F = -j_{nc,F}^{\tau}, \ \tau \cdot n_E = 0 \text{ on } E \in E_{b,F} \} \]
with \( V = RT \) or BDM and let
\[ W_{nc}^{-1,F} = \{ \tau \in L^2(\omega_F) \mid \tau|_K \in W(K) \forall K \in T_F, \left[ \tau \cdot t_F \right]_F = -j_{nc,F}^{\tau}, \ \tau \cdot t_E = 0 \text{ on } E \in E_{b,F} \}. \]
with \( W = NE \) or ND. Similar to Section 4.1, the approximation error fluxes \( \sigma_{nc,v,F}^{\Delta} \) with \( v = rt \) or bdm and the approximation error gradients \( \rho_{nc,w,F}^{\Delta} \) with \( w = ne \) or nd are then the solutions of the minimization problems:
\[ \| A^{-1/2} \sigma_{nc,v,F}^{\Delta} \|_{0,\omega_F} = \min_{\tau \in V_{nc}^{-1,F}} \| A^{-1/2} \tau \|_{0,\omega_F} \] (6.18)
and
\[ \| A^{1/2} \rho_{nc,w,F}^{\Delta} \|_{0,\omega_F} = \min_{\tau \in W_{nc}^{-1,F}} \| A^{1/2} \tau \|_{0,\omega_F}, \] (6.19)
respectively. Since \( RT_{nc}^{-1,F} \subset BDM_{nc}^{-1,F} \) and \( NE_{nc}^{-1,F} \subset ND_{nc}^{-1,F} \), the first inequality in (6.16) follows from their definitions. The second inequality in (6.16) is from the minimization problems in (6.18) and (6.19), Lemma 6.8 and Theorem 6.8 of [10]. The bounds in (6.17) are straightforward from their definitions and inequality (6.16).

7 Numerical Experiments

In this section, we report some numerical results for an interface problem with intersecting interfaces used by many authors, e.g., [18, 9, 10, 11], which is considered as a benchmark test problem. For simplicity, we only test the conforming element with explicit RT recovery. Other cases behave similarly.

Let \( \Omega = (-1,1)^2 \) and \( u(r, \theta) = r^\gamma \mu(\theta) \) in the polar coordinates at the origin with \( \mu(\theta) \) being a smooth function of \( \theta \) [9]. The function \( u(r, \theta) \) satisfies the interface equation with \( A = \alpha I, \Gamma_N = \emptyset, f = 0 \), and
\[ \alpha(x) = \begin{cases} R & \text{in } (0,1)^2 \cup (-1,0)^2, \\ 1 & \text{in } \Omega \setminus ([0,1]^2 \cup [-1,0]^2). \end{cases} \]
The \( \gamma \) depends on the size of the jump. In our test problem, \( \gamma = 0.1 \) is chosen and is corresponding to \( R \approx 161.4476387975881 \). Note that the solution \( u(r, \theta) \) is only in \( H^{1+\gamma-\epsilon}(\Omega) \) for any \( \epsilon > 0 \) and, hence, it is very singular for small \( \gamma \) at the origin. This suggests that refinement is centered around the origin.

Mesh generated by \( \eta_{rt}^c \) is shown in Figure 1. The refinement is centered at origin. Similar meshes for this test problem generated by other error estimators can be found in [9, 10, 13]. The comparison of the error and the \( \eta_{rt}^c \) is shown in Figure 2. The effectivity index is close to 1. Moreover, the slope of the log(dof)-log(relative error) for \( \eta_{rt}^c \) is \(-1/2\), which indicates the optimal decay of the error with respect to the number of unknowns.
A  Basis Functions of the Lowest Order  $RT$, $BDM$, $NE$, and $ND$ Spaces

This appendix describes basis functions for the $RT$, $BDM$, $NE$, and $ND$ finite element spaces of the lowest order. The definition of these basis functions can also be found in Section 2.6 of [7].

For a triangle $K$, denote by $x_i$, $x_j$, and $x_k$ its three vertices sorted counterclockwise and denote by $F_i$, $F_j$, and $F_k$ the edges opposite to the vertices $x_i$, $x_j$, and $x_k$, respectively. The lengths, the unit tangent vectors, and the heights of the edges are denoted by

$$h_l = |e_l|, \quad t_l = \frac{e_l}{h_l}, \quad \text{and} \quad H_l$$

for $l = i, j, k$, respectively. Let $\lambda_i$, $\lambda_j$, and $\lambda_k$ denote the barycentric coordinates of the triangle $K$ associated with vertices $x_i$, $x_j$, and $x_k$, respectively. Then the unit outward vectors normal to the edges are

$$n_l = -\frac{\nabla \lambda_l}{|\nabla \lambda_l|} \quad \text{for} \quad l = i, j, k.$$

Finally, denote by $|K|$ the area of the triangle $K$. Now, we state basis functions associated with the edge $F_k$ as follows:

- for RT
  \[
  \phi^{rt}_{r_k}|_K := \frac{1}{H_k}(x - x_k), \quad (A.1)
  \]

- for BDM (two basis functions associated with vertices $x_i$ and $x_j$)
  \[
  \phi^{bdm}_{i,F_k}|_K := \frac{1}{H_k}(x_i - x_k)\lambda_i, \quad \text{and} \quad \phi^{bdm}_{j,F_k}|_K := \frac{1}{H_k}(x_j - x_k)\lambda_j, \quad (A.2)
  \]

- for NE
  \[
  \phi^{ne}_{r_k}|_K := h_k(\lambda_j \nabla \lambda_i - \lambda_i \nabla \lambda_j), \quad (A.3)
  \]
for $ND$ (two basis functions associated with vertices $x_i$ and $x_j$)

$$\phi^{nd}_{i,F_k}|_K := h_k \lambda_i \nabla \lambda_j, \quad \text{and} \quad \phi^{nd}_{j,F_k}|_K := h_k \lambda_j \nabla \lambda_i. \quad (A.4)$$

It is easy to check that these basis functions satisfy the following properties:

1. for $RT$

$$\phi^{rt}_{F_k} = \phi^{bdm}_{i,F_k} + \phi^{bdm}_{j,F_k} \quad \text{and} \quad (\phi^{rt}_{F_k} \cdot n_\ell)|_{F_\ell} = \delta_{\ell k}, \quad \ell = i, j, k; \quad (A.5)$$

2. for $BDM$

$$\left(\phi^{bdm}_{i,F_k} \cdot n_k\right)|_{F_k} = \lambda_i \delta_{\ell k} \quad \text{and} \quad \left(\phi^{bdm}_{j,F_k} \cdot n_k\right)|_{F_k} = \lambda_j \delta_{\ell k}, \quad \ell = i, j, k; \quad (A.6)$$

Then it is clear that a linear function on $F_k$ can be represented by $\left(\phi^{bdm}_{i,F_k} \cdot n_k\right)|_{F_k}$ and $\left(\phi^{bdm}_{j,F_k} \cdot n_k\right)|_{F_k}$. Let $p$ be an affine function on $F_k$, then

$$p = p(x_i) \left(\phi^{bdm}_{i,F_k} \cdot n_k\right)|_{F_k} + p(x_j) \left(\phi^{bdm}_{j,F_k} \cdot n_k\right)|_{F_k}; \quad (A.7)$$

3. for $NE$

$$\phi^{ne}_{F_k} = \phi^{nd}_{i,F_k} - \phi^{nd}_{j,F_k} \quad \text{and} \quad (\phi^{ne}_{F_k} \cdot t_\ell)|_{F_\ell} = \delta_{\ell k}, \quad \ell = i, j, k; \quad (A.8)$$

4. for $ND$

$$\left(\phi^{nd}_{i,F_k} \cdot t_k\right)|_{F_k} = \lambda_i \delta_{\ell k}, \quad \text{and} \quad \left(\phi^{nd}_{j,F_k} \cdot t_k\right)|_{F_k} = -\lambda_j \delta_{\ell k}. \quad (A.9)$$

Let $p$ be an affine function on $F_k$, then

$$p = p(x_i) \left(\phi^{nd}_{i,F_k} \cdot t_k\right)|_{F_k} - p(x_j) \left(\phi^{nd}_{j,F_k} \cdot t_k\right)|_{F_k}. \quad (A.10)$$

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