Uncertainty relations based on Wigner–Yanase skew information

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Abstract

In this paper, we use certain norm inequalities to obtain new uncertain relations based on the Wigner–Yanase skew information. First for an arbitrary finite number of observables we derive an uncertainty relation outperforming previous lower bounds. We then propose new weighted uncertainty relations for two noncompatible observables. Two separable criteria via skew information are also obtained.

Keywords: uncertainty relations, skew information, entanglement

1. Introduction

Uncertainty relation is one of the fundamental building blocks of quantum theory, and plays an significant role in quantum information and quantum mechanics [1–4]. It reveals a fundamental limit with which certain pairs of physical properties of a particle, such as position and momentum, cannot be simultaneously known exactly.

The uncertainty relations dominated the developments of physics that ranges from foundations to quantum information, quantum communication and other areas as well, which give rise to wide applications in entanglement detection [5, 6], as well as security analysis of quantum key distribution in quantum cryptography [7], quantum metrology and quantum speed limit [8–10].

Generally the uncertainty relations are expressed in terms of the product of variances of the measurement results of two incompatible observables [11]. Besides variance-based uncertainty, there are also other ways to formulate the principle, such as in terms of entropies [12–16], majorization [17–20] and there are also fine-grained uncertainty relations [21–23].

The quantum uncertainty relation can be also described in terms of skew information [24]. In this work we will only focus on the skew information-based additive uncertainty relations. In 1963, Wigner and Yanase [25] introduced the skew information $I_\rho(H)$ of the observable $H$ as a measure of quantum information contained in a state $\rho$, namely,

$$I_\rho(H) = -\frac{1}{2} \text{Tr}([\sqrt{\rho}, H]^2) = \text{Tr}(\rho H^2) - \text{Tr}(\sqrt{\rho} H \sqrt{\rho} H).$$

In addition, the skew information can be cast as the norm form according to the Frobenius norm $\| \cdot \|$, that is

$$I_\rho(H) = \frac{1}{2} \| [\sqrt{\rho}, H] \|^2.$$

In this formulism $I_\rho(H)$ can be viewed as a kind of degree for non-commutativity between the quantum state $\rho$ and the observable $H$. It manifestly vanishes when $\rho$ commutes with $H$, and it is homogeneous in $\rho$. By means of the skew information and the decomposition of the variance, a stronger uncertainty relation was presented for mixed states [26, 27]. Since information is lost when separated systems are united such a measure should be decreasing under the mixing of state [28], that is, convex in $\rho$.

The Wigner–Yanase skew information has become a useful tool in quantum information theory, for instance, characterizing entanglement [29], begging a measure of the $H$ coherence of the state $\rho$, and quantifying the dynamics of some physical phenomena. In this paper, we present more...
tighter uncertainty relations based on the Wigner–Yanase skew information, and the newly given uncertainty principle is shown to be applicable to judge separability.

2. Uncertainty relation based on skew information for multi operators

In this section, we present an uncertainty relation based on the skew information for multiple incompatible observables.

**Theorem 1.** For non-commutative observables $A_i$, $i = 1, 2, \ldots, n$, the following uncertainty inequalities hold

$$
\sum_{i} I_{\rho}(A_i) \geq \frac{1}{n} I_{\rho}(\sum_{i} A_i) + \frac{1}{n^2} \left( \sum_{1 \leq i < j \leq n} \sqrt{I_{\rho}(A_i - A_j)} \right)^2.
$$

(3)

If $A_i$s are mutually non-commutative, then the lower bound in (3) is non-zero.

**Proof.** On a Hilbert space the following identity holds [31]:

$$
n \sum_{i=1}^{n} \| u_i \|^2 = \left( \sum_{i=1}^{n} u_i \right)^2 + \sum_{1 \leq i < j \leq n} \| u_i - u_j \|^2,
$$

(4)

where $u_i$ is a vector in Hilbert space.

Also the inequality holds

$$
\sum_{1 \leq i < j \leq n} \| u_i - u_j \|^2 \geq \frac{1}{n} \left( \sum_{1 \leq i < j \leq n} \| u_i - u_j \| \right)^2.
$$

Therefore one has that

$$
\sum_{i=1}^{n} \| u_i \|^2 \geq \frac{1}{n} \left( \sum_{i=1}^{n} \| u_i \| \right)^2 + \frac{1}{n^2} \left( \sum_{1 \leq i < j \leq n} \| u_i - u_j \| \right)^2.
$$

(5)

Let $u_i = [\sqrt{\rho}, A_i]$, we obtain the uncertainty relation for skew information in the form (3). Moreover, both $I_{\rho}(\sum_{i} A_i)$ and $I_{\rho}(A_i - A_j)$ are equal to zero if the lower bound (3) is zero, which implies that $I_{\rho}(A_i) = 0$, then the observables $A_i$s are mutually commutative.

As the operators $A_i$s are mutually non-commutative, the inequality (3) is non-trivial, so the lower bound (3) is non-trivial.

**Remark 1.** In the case of pure state $\rho$ which is an eigenvector of observable $A_i$, the skew information $I_{\rho}(A_i)$ is zero. This means that the sum of skew information $I_{\rho}(A_i) + I_{\rho}(B)$ is non-trivial if $\rho$ is not a common eigenvector of observables $A$ and $B$. However, both Heisenberg–Robertson’s and Schrödinger’s uncertainty relations are trivial in that case.

**Remark 2.** In particular, if $\rho$ is a pure state, the skew information $I_{\rho}(H)$ happens to be the variance $(\Delta_{\rho} H)^2$. According to the definition of skew information $I_{\rho}(H) = \frac{1}{2} \text{Tr}((\sqrt{\rho}, H)^2) = \text{Tr}(\rho H^2) - \text{Tr}(\rho^{1/2} H \rho^{1/2})$, in case of $\rho = |\varphi\rangle \langle \varphi|$, then $I_{\rho}(H) = \langle \varphi | H^2 | \varphi \rangle - \langle \varphi | H | \varphi \rangle^2 = (\Delta_{\rho} H)^2$. Thus, our relation happens to be the inequality obtained by Song in [32],

$$
\sum_{i} \Delta_{\rho}(A_i)^2 \geq \frac{1}{n} \left( \sum_{i} \Delta_{\rho}(A_i) \right)^2 + \frac{1}{n^2} \left( \sum_{1 \leq i < j \leq n} \Delta_{\rho}(A_i - A_j) \right)^2.
$$

(6)

It means that the relation (3) can reduce to the inequality (6) in case of pure states.

When there are two non-commutative observables in theorem 1, we can get a corollary below.

**Corollary 1.** For non-commutative observables $A$ and $B$, we have

$$
I_{\rho}(A) + I_{\rho}(B) \geq \frac{1}{2} I_{\rho}(A + B) + \frac{1}{4} I_{\rho}(A - B)
$$

$$
\geq \frac{1}{2} I_{\rho}(A + B).
$$

(7)

In case $\rho$ is a pure state, we can rewrite the inequality (7) according to the relation $I_{\rho}(H) = (\Delta_{\rho} H)^2$, thus we obtain an inequality based on variance in the following

$$
(\Delta_{\rho} A)^2 + (\Delta_{\rho} B)^2 \geq \frac{1}{2} (\Delta_{\rho} (A + B))^2 + \frac{1}{4} (\Delta_{\rho} (A - B))^2.
$$

(8)

Also our relation (8) has a stronger lower bound, which is tighter than the uncertainty relation derived by Maccone and Pati in [33]:

$$
(\Delta_{\rho} A)^2 + (\Delta_{\rho} B)^2 \geq \frac{1}{2} (|\langle A^+ B| \rangle + |\langle B A^+| \rangle|^2)^2 = \frac{1}{2} (\Delta_{\rho} (A + B))^2.
$$

(9)

It is worth noting that Chen et al derived an uncertainty relation based on Wigner–Yanase skew information [27] which states that

$$
\sum_{i} I_{\rho}(A_i) \geq \frac{1}{n - 2} \left[ \sum_{1 \leq i < j \leq n} I_{\rho}(A_i + A_j) - \frac{1}{(n - 1)^2} \left( \sum_{1 \leq i < j \leq n} \sqrt{I_{\rho}(A_i + A_j)} \right)^2 \right].
$$

(10)
Also, inequality (3) has a stronger lower bound than the one in (10) for a qubit system [32]. As an example, we consider the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Let \( \rho = \frac{1}{2}(I + \vec{r}\vec{\sigma}) \), where the Bloch vector \( \vec{r} = (\frac{\cos \theta}{\sqrt{2}}, \frac{\sin \theta}{\sqrt{2}}, 0) \). Then \( I_0(\sigma_1 - \sigma_2) = \frac{1}{2}(1 + \frac{1}{2}\sin 2\theta) \), \( I_1(\sigma_1 - \sigma_3) = \frac{1}{2}(3 - \cos 2\theta) \), \( I_2(\sigma_2 - \sigma_3) = \frac{1}{2}(3 + \cos 2\theta) \), \( I_0(\sigma_1 + \sigma_2 + \sigma_3) = 1 - \frac{1}{2}\sin 2\theta \). The comparison between the two bounds (3) and (10) is given in Figure 1, where one sees clearly that our bound outperforms that of (10).

Furthermore, our relation (3) is stronger than the one derived from the uncertainty inequality for two observables [33]. As an example, we consider another two uncertainty relations derived from the parallelogram law in the Hilbert space: \( 2(||u||^2 + ||v||^2) = ||u + v||^2 + ||u - v||^2 \). Let \( u = [\sqrt{\rho}, A], v = [\sqrt{\rho}, B] \), \( A \) and \( B \) are two incompatible observables, then we get uncertainty relations based on Wigner–Yanase skew information

\[
I_p(A) + I_p(B) = \frac{1}{2}(I_p(A + B) + I_p(A - B)).
\]

Using the above uncertainty equality, one can obtain two inequalities for arbitrary \( n \) observables, namely,

\[
\sum_{i=1}^{n} I_p(A_i) \geq \frac{1}{2(n - 1)} \sum_{1 \leq i < j \leq n} I_p(A_i + A_j),
\]

\[
\sum_{i=1}^{n} I_p(A_i) \geq \frac{1}{2(n - 1)} \sum_{1 \leq i < j \leq n} I_p(A_i - A_j).
\]

There is an example of comparison between our relation (3) and ones (10), (12). We consider the spin-1 system with the pure state \( |\psi\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} |2\rangle, 0 \leq \theta < 2\pi \). Take the angular momentum operators [34] with \( \hbar = 1 \):

\[
J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, J_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.
\]

Direct calculation gives

\[
I_p(J_x) = \frac{1}{2}(1 + \sin \theta), \quad I_p(J_y) = \frac{1}{2}(1 - \sin \theta), \\
I_p(J_x + J_y) = \frac{1}{2}(1 + \sin \theta + \sin^2 \theta), \\
I_p(J_x - J_y) = \frac{1}{2}(1 - \sin \theta + \sin^2 \theta), \\
I_p(J_x + J_y) = 1 + \sin^2 \theta, \\
I_p(J_x - J_y) = \sin^2 \theta, \\
I_p(J_x - J_y) = 0.
\]

The comparison between the lower bounds (3), (10) and (12) is shown by figure 2. The results suggest that the relation (3) can give tighter bound than other ones (10) and (12) for a spin-1 particle and measurement of angular momentum operators \( J_x, J_y \) and \( J_z \).

3. Uncertainty relation in terms of skew information with weight

Additionally we can get many uncertainty relations if involving parameters, so we consider uncertainty relations with weight based on skew information for two non-commutative observables.
Theorem 2. For two non-commutativity observables $A$ and $B$, we have the uncertainty relation with weight

$$
I_p(A - B) + I_p\left(\frac{\lambda}{\lambda - 1} A - B\right) \leq \frac{1}{\lambda} I_p(A) + \frac{1}{1 - \lambda} I_p(B) + I_p\left(\frac{\lambda}{\lambda - 1} A - B\right),
$$

where $\frac{1}{2} \leq \lambda < 1$, and the equality holds when $\lambda = \frac{1}{2}$.

Proof. Recall that for bounded linear operators $U$ and $V$ in Hilbert space the following inequality holds [35]

$$
\left\|U - V\right\|^2 + \left\|(1 - p)U - V\right\|^2 \leq p\left\|U\right\|^2 + q\left\|V\right\|^2 \leq \left\|U - V\right\|^2 + \left\|U - (1 - q)V\right\|^2.
$$

for any $p < 2, \frac{1}{p} + \frac{1}{q} = 1$, and the equality holds if and only if $p = 2$ or $V = (1 - p)U$. Now set $p = \frac{1}{\lambda}, U = [\sqrt{\rho}, A]$ and $V = [\sqrt{\rho}, B]$, we obtain the uncertainty relation (15). □

Remark 3. In particular, in the case of $\lambda = \frac{1}{2}$, inequality (15) happens to the parallelogram law in terms of skew information (11).

The idea of weighted averaging is one of the popular techniques in both statistical mechanics and mathematical physics. Through the weighted averaging one may know better about the whole picture in an unbiased way. Also we consider a perturbation of $A$ and $B$, or $A' = \sqrt{1/\lambda}A$, $B' = \sqrt{1/(1 - \lambda)}B$, then

$$
I_p(A') + I_p(B') = \frac{1}{\lambda} I_p(A) + \frac{1}{1 - \lambda} I_p(B).
$$

This means that the lower bound of the sum of skew information can be obtained by scaled observables.

Our lower bound remains non-zero unless $\rho$ is a common eigenvector of $A$ and $B$, which means that besides having a non-trivial bound in almost all cases, our weighted uncertainty relations can also lead to a tighter bound for the sum of skew information.

Let us consider again the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ and the measured state given by the family of states with the Bloch vector $\vec{r} = \left(\frac{3\sqrt{3}}{2} \cos \theta, \frac{3\sqrt{3}}{2} \sin \theta, 0\right), \theta \in (0, \pi)$. It is shown in figure 3 that the sum of skew information with weight $\lambda I_p(\sigma_1) + \frac{1}{\lambda - 1} I_p(\sigma_2)$.

From figure 3, it follows that the uncertainty relation $\lambda I_p(\sigma_1) + \frac{1}{\lambda - 1} I_p(\sigma_2)$ attains the minimum value when the parameters $\lambda = \frac{1}{2}$ and $\theta = \frac{\pi}{2}$.

4. Entanglement detection via uncertainty relation based on skew information

The skew information $I_p(A)$, viewed as a quantum uncertainty of $A$ at the quantum state $\rho$, has been well studied by Lieb in [28]. Among various characteristic properties, the convexity and additivity are the most important ones.

Skew information entropy is a convex function in $\rho$, that is to say, if $\rho$ is a bipartite state on the quantum system $\mathcal{H}_1 \otimes \mathcal{H}_2$, $\rho = \sum_k p_k \rho_k$ is a convex combination (i.e. $p_k \geq 0$, $\sum_k p_k = 1$) of some states $\{\rho_k\}$, $\{M_i\}$ are some observables, then one has that

$$
\sum I_p(M_i) \leq \sum_k p_k \sum I_{p_k}(M_i).
$$

We call a state violating inequality (17) iff there are no states $\{\rho_k\}$ and no $\{p_k\}$ such that inequality (17) is fulfilled. That is different with the variance, which is concave in $\rho$ in contrast. Inequality (17) has an obvious physical interpretation: one cannot decrease the uncertainty of an observable by mixing several states. Moreover, in the case $\rho$ is separable, i.e. $\rho$ is a convex combination of product states, $\{\rho_k\}$ is a set of product states, violation of the inequality (17) implies entanglement of the state; therefore, entanglement can be detected with skew information uncertainties [29]. Furthermore, it can be used to define the correlation limit of separable states [28, 30].

The skew information entropy is fixed by the state $\rho$ and the observable $H$. Luo introduced a quantity according to skew information [36]

$$
Q(\rho) = \sum_{i=1}^{n} I_p(H_i),
$$

where $\{H_i\}$ is an orthonormal basis for Hilbert space $\mathcal{L}(\mathcal{H})$ consisting with all observables on quantum system $\mathcal{H}$ with dimensional $n$. Then $Q(\rho)$ is an intrinsic quantity only depending on state $\rho$, and it is independent of the choice of the orthonormal basis $\{H_i\}$. Also $Q(\rho)$ is both a measure of information content of $\rho$ and a measure of quantum uncertainty. Then we can obtain a separability criterion depending on $Q(\rho)$.  

![Figure 3](image-url)  

Figure 3. The sum uncertainty relations based on skew information with weight are satisfied by observables $\sigma_1$ and $\sigma_2$ with state $\rho$.  

Theorem 1. For two observables $A$ and $B$, we have the uncertainty relation

$$
I_p(A - B) + I_p\left(\frac{\lambda}{\lambda - 1} A - B\right) \leq \frac{1}{\lambda} I_p(A) + \frac{1}{1 - \lambda} I_p(B).
$$

where $\frac{1}{2} \leq \lambda < 1$, and the equality holds when $\lambda = \frac{1}{2}$.
Theorem 3. Let \( \rho \) be a bipartite state on the quantum system \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), if \( \rho \) is separable, then the following inequality holds
\[
Q(\rho) \leq \sum_i p_i Q(\rho_i),
\]
where \( \sum_i p_i = 1 \), \( p_i \geq 0 \), \( \{\rho_i\} \) are product states on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \).

The subadditivity of quantum entropy \( S(\rho, M) \) describes the correlation between the quantum state with its partial traces, and global measurement between the local measurement. The entropy \( S(\rho, M) \) used in this definition may be the standard Shannon entropy \( S(\rho, M) = -\sum_i p_i \ln(p_i) \), or, more generally any so-called entropic function \( S(\rho, M) = \sum_i s(p_i) \) where \( s: [0, 1] \to \mathbb{R} \) is a concave function, may be used. Let \( \rho \) be a bipartite quantum state on Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), the partial traces of quantum state \( \rho_1 = \text{Tr}_2(\rho) \) and \( \rho_2 = \text{Tr}_1(\rho) \) are operators on subsystems \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively. Subadditivity of quantum entropy is stated as follows,
\[
S(\rho, A \otimes I_2 + I_1 \otimes B) \leq S(\rho_1, A) + S(\rho_2, B).
\]

However, skew information is as an entropy, the subadditivity is not satisfied, that is to say, the inequality \( I_s(\rho, A \otimes I_2 + I_1 \otimes B) \leq I_s(\rho_1, A) + I_s(\rho_2, B) \) does not hold. The state \( \rho \) and that of the partial trace \( \rho_1 \) and \( \rho_2 \) have the relation [28]
\[
I_s(\rho, A \otimes I_2 + I_1 \otimes B) \geq I_s(\rho_1, A), I_s(\rho_1 \otimes I_2) \geq I_s(\rho_2, B),
\]
for arbitrary Hermitian operator \( A \) in \( \mathcal{H}_1 \), where \( I_2 \) denotes the identity operator in \( \mathcal{H}_2 \).

Particularly, the skew information entropy has additivity in this sense.

Lemma 1. Let \( \rho_1 \) and \( \rho_2 \) be two density operators of two subsystems, and \( A_1 \) (resp. \( A_2 \)) be a self-adjoint operator on subsystem \( \mathcal{H}_1 \) (resp. \( \mathcal{H}_2 \)). Let \( M = A_1 \otimes I_2 + I_1 \otimes A_2 \), then the skew information \( I_s(M) \) is additive in the sense that if \( \rho = \rho_1 \otimes \rho_2 \), then \( I_s(M) = I_s(\rho_1) + I_s(\rho_2) \), where \( I_1 \) and \( I_2 \) are the density matrices for the first and second systems, respectively.

Proof. Suppose \( \rho = \rho_1 \otimes \rho_2 \), then
\[
\sqrt{\rho} \sqrt{M} \sqrt{\rho} = \sqrt{\rho_1} A_1 \sqrt{\rho_1} \otimes \rho_2 + \sqrt{\rho_1} A_1 \sqrt{\rho_1} \otimes \rho_2 A_2 + I_1 \otimes \sqrt{\rho_2} A_2 \sqrt{\rho_2} + \rho_1 \otimes \sqrt{\rho_2} A_2 \sqrt{\rho_2} A_2,
\]
and
\[
\text{Tr}(\sqrt{\rho} \sqrt{M} \sqrt{\rho}) = \text{Tr}(\sqrt{\rho_1} A_1 \sqrt{\rho_1} A_1) + \text{Tr}(\sqrt{\rho_2} A_2 \sqrt{\rho_2} A_2) + 2 \text{Tr}(\rho_1 A_1) \text{Tr}(\rho_2 A_2),
\]
also,
\[
\text{Tr}(\rho M^2) = \text{Tr}(\rho_1 A_1^2) + \text{Tr}(\rho_2 A_2^2) + 2 \text{Tr}(\rho_1 A_1) \text{Tr}(\rho_2 A_2).
\]
It follows from (23)–(24) that
\[
I_s(M) = \text{Tr}(\rho M^2) - \text{Tr}(\sqrt{\rho} M \sqrt{\rho} M) = I_{s_1}(A_1) + I_{s_2}(A_2).
\]

Proof. Let \( \rho = \rho_1 \otimes \rho_2 \) be a product state, and \( \sum_i I_{s_i}(A_i) \geq c_A, \sum_i I_{s_i}(B_i) \geq c_B \), since the skew information is additive, the inequality (26) holds. Because of the convexity of skew information, this inequality also holds for all convex combinations of product states [28], i.e. for all separable states.

Inequality (26) manifests the correlation between the sum uncertainty and the local uncertainty for separable states. Any violation of the limit of the uncertainty therefore proves that the quantum state cannot be separated into a mixture of product states. The violation of any local uncertainty relation of the form (26) is therefore a sufficient condition for the existence of entanglement.

Furthermore, relation (26) is a spin-squeezing [37] criterion for the angular momentum measurements. As such, it requires the same experimental data as other spin-squeezing criteria, see [38, 39], namely, only a measurement of first and second moments of the total angular momentum. In contrast to entanglement criteria based on tomography, these are advantageous in typical experimental implementation.

5. Conclusions

Uncertainty relations are one of the central properties in quantum theory and quantum information. We have investigated the uncertainty relation based on the Wigner–Yanase skew information for multiple non-commutative observables. The corresponding lower bounds derived in this paper are...
shown to be tighter than the previous ones, and thus better capture the incompatibility of the observables. The results are expected to shed new light on investigating quantum tasks, as uncertainty relations are closely related to many quantum information processing like entanglement detection, security analysis of quantum key distribution in quantum cryptography and non-locality quantum tasks.

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