On the log-convexity of combinatorial sequences *

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Abstract

This paper is devoted to the study of the log-convexity of combinatorial sequences. We show that the log-convexity is preserved under componentwise sum, under binomial convolution, and by the linear transformations given by the matrices of binomial coefficients and Stirling numbers of two kinds. We develop techniques for dealing with the log-convexity of sequences satisfying a three-term recurrence. We also introduce the concept of $q$-log-convexity and establish the connection with linear transformations preserving the log-convexity. As applications of our results, we prove the log-convexity and $q$-log-convexity of many famous combinatorial sequences of numbers and polynomials.

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1 Introduction

Let $a_0, a_1, a_2, \ldots$ be a sequence of nonnegative numbers. The sequence is called convex (resp. concave) if for $k \geq 1$, $a_{k-1} + a_{k+1} \geq 2a_k$ (resp. $a_{k-1} + a_{k+1} \leq 2a_k$). The sequence is called log-convex (resp. log-concave) if for all $k \geq 1$, $a_{k-1}a_{k+1} \geq a_k^2$ (resp. $a_{k-1}a_{k+1} \leq a_k^2$). By the arithmetic-geometric mean inequality, the log-convexity implies the convexity and the concavity implies the log-concavity. Clearly, a sequence $\{a_k\}_{k \geq 0}$ of positive numbers is log-convex (resp. log-concave) if and only if the sequence $\{a_{k+1}/a_k\}_{k \geq 0}$ is increasing (resp. decreasing).

It is well known that the binomial coefficients $\binom{n}{k}$, the Eulerian numbers $A(n, k)$, the Stirling numbers $c(n, k)$ and $S(n, k)$ of two kinds are log-concave in $k$ for fixed $n$ respectively (see [45] for instance). In contrast, it is not so well known that many famous sequences in combinatorics, including the Bell numbers, the Catalan numbers and the Motzkin numbers, are log-convex respectively. Although the log-convexity of a sequence

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of positive numbers is equivalent to the log-concavity of its reciprocal sequence, they are also fundamentally different. For example, the log-concavity of sequences is preserved by both ordinary and binomial convolutions (see Wang and Yeh \[46\] for instance). But the ordinary convolution of two log-convex sequences need not be log-convex. Even the sequence of partial sums of a log-convex sequence is not log-convex in general. On the other hand, Davenport and Pólya \[10\] showed that the log-convexity is preserved under the binomial convolution.

There have been quite a few papers concerned with the log-concavity of sequences (see the survey articles \[32, 5\] and some recent developments \[23, 41, 42, 43, 44, 45, 46\]). However, there is no systematic study of the log-convexity of sequences. Log-convexity is, in a sense, more challenging property than log-concavity. One possible reason for this is that the log-concavity of sequences is implied by the Pólya frequency property. An infinite sequence \(a_0, a_1, a_2, \ldots\) is called a Pólya frequency (PF, for short) sequence if all minors of the infinite Toeplitz matrix \((a_{i-j})_{i,j\geq 0}\) are nonnegative, where \(a_k = 0\) if \(k < 0\). A finite sequence \(a_0, a_1, \ldots, a_{n-1}, a_n\) is PF if the infinite sequence \(a_0, a_1, \ldots, a_{n-1}, a_n, 0, 0, \ldots\) is PF. Clearly, a PF sequence is log-concave. PF sequences are much better behaved and have been studied deeply in the theory of total positivity (Karlin \[21\]). For example, the fundamental representation theorem of Schoenberg and Edrei states that a sequence \(a_0 = 1, a_1, a_2, \ldots\) of real numbers is PF if and only if its generating function has the form

\[
\sum_{n\geq 0} a_n z^n = \prod_{j\geq 1} \frac{(1 + \alpha_j z)}{(1 - \beta_j z)} e^{\gamma z}
\]

in some open disk centered at the origin, where \(\alpha_j, \beta_j, \gamma \geq 0\) and \(\sum_{j\geq 1} (\alpha_j + \beta_j) < +\infty\) (see Karlin \[21\] p. 412 for instance). In particular, a finite sequence of nonnegative numbers is PF if and only if its generating function has only real zeros (\[21\] p. 399)). So it is often more convenient to show that a sequence is PF even if our original interest is only in the log-concavity. Indeed, many log-concave sequences arising in combinatorics are actually PF sequences. Brenti \[4, 6, 7\] has successfully applied total positivity techniques and results to study the log-concavity problems.

This paper is devoted to the study of the log-convexity of combinatorial sequences and is organized as follows. In \(\S 2\) we consider various operators on sequences that preserve the log-convexity. We show that log-convexity is preserved under componentwise sum, under binomial convolution, and by the linear transformations given by the matrices of binomial coefficients, Stirling numbers of the second kind, and signless Stirling numbers of the first kind. In \(\S 3\) we discuss the log-convexity of sequences satisfying a three-term recurrence. As consequences, some famous combinatorial sequences, including the central binomial coefficients, the Catalan numbers, the Motzkin numbers, the Fine numbers, the central Delannoy numbers, the little and large Schröder numbers, are log-convex. And also, each of bisections of combinatorial sequences satisfying three-term linear recurrences, including the Fibonacci, Lucas and Pell numbers, must be log-convex or log-concave respectively. In \(\S 4\) we introduce the concept of the \(q\)-log-convexity of sequences of polynomials in \(q\) and show the \(q\)-log-convexity of the Bell polynomials, the Eulerian polynomials, the \(q\)-
Schröder numbers and the $q$-central Delannoy numbers. We also present certain linear transformations preserving the log-convexity of sequences and establish the connection with the $q$-log-convexity. Finally, in §5, we present some conjectures and open problems.

2 Operators preserving log-convexity

In this section we consider operators on sequences that preserve the log-convexity. A similar problem for the log-concavity has been studied (see [4, 43, 46] for instance). However, there are fundamentally different between them. For example, it is somewhat surprising that the log-convexity is preserved under componentwise sum.

\textbf{Proposition 2.1.} If both $\{x_n\}$ and $\{y_n\}$ are log-convex, then so is the sequence $\{x_n + y_n\}$.

\textbf{Proof.} It follows immediately that

$$(x_{n-1} + y_{n-1})(x_{n+1} + y_{n+1}) \geq (\sqrt{x_{n-1}x_{n+1}} + \sqrt{y_{n-1}y_{n+1}})^2 \geq (x_{n} + y_{n})^2$$

from the well-known Cauchy’s inequality and the log-convexity of $\{x_n\}$ and $\{y_n\}$. □

Given two sequences $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$, define their ordinary convolution by

$$z_n = \sum_{k=0}^{n} x_k y_{n-k}$$

and binomial convolution by

$$z_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k},$$

respectively. It is known that the log-concavity of sequences is preserved by both ordinary and binomial convolutions (see Wang and Yeh [46] for instance). However, the ordinary convolution of two log-convex sequences need not be log-convex. On the other hand, the binomial convolution of two log-convex sequences is log-convex by Davenport and Pólya [10]. Proposition 2.1 can provide an interpretation of this result.

\textbf{Davenport-Pólya Theorem.} If both $\{x_n\}$ and $\{y_n\}$ are log-convex, then so is their binomial convolution

$$z_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k}, \quad n = 0, 1, 2, \ldots.$$

\textbf{Proof.} It is easy to verify that $z_0 z_2 = (x_0 y_0)(x_0 y_2 + 2x_1 y_1 + x_2 y_0) \geq (x_0 y_1 + x_1 y_0)^2 = z_1^2$. We proceed by induction on $n$. Note that

$$z_n = \sum_{k=0}^{n-1} \binom{n-1}{k} x_k y_{n-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x_{k+1} y_{n-k-1}.$$

Two sums in the right hand side are the binomial convolutions of $\{x_k\}_{0 \leq k \leq n-1}$ with $\{y_k\}_{1 \leq k \leq n}$ and $\{x_k\}_{1 \leq k \leq n}$ with $\{y_k\}_{0 \leq k \leq n-1}$ respectively. Hence both are log-convex by the induction hypothesis. Thus the sequence $\{z_n\}$ is log-convex by Proposition 2.1 □
Example 2.2. A permutation \( \pi \) of the \( n \)-element set \([n] = \{1, 2, \ldots, n\} \) is alternating if \( \pi(1) > \pi(2) < \pi(3) > \pi(4) < \cdots \pi(n) \). The number \( E_n \) of alternating permutations of \([n]\) is known as an Euler number. The sequence has the exponential generating function
\[
\sum_{k=0}^{n} E_n x^n / n! = \tan x + \sec x
\]
and satisfies the recurrence
\[
2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}
\]
with \( E_0 = 1, E_1 = 1, E_2 = 1 \) and \( E_3 = 2 \) (see Comtet [9, p. 258] and Stanley [33, p. 149] for instance). Let \( z_n = E_n / 2 \). Then \( z_{n+1} = \sum_{k=0}^{n} \binom{n}{k} z_k z_{n-k} \). Clearly, \( z_0, z_1, z_2, z_3 \) is log-convex. Thus the sequence \( \{z_n\} \) is log-convex by induction and Davenport-Pólya Theorem, and so is the sequence \( \{E_n\} \).

The following is a special case of Davenport-Pólya Theorem.

Proposition 2.3. The binomial transformation \( z_n = \sum_{k=0}^{n} \binom{n}{k} x_k \) preserves the log-convexity.

The Stirling number \( S(n, k) \) of the second kind is the number of partitions of the set \([n]\) having exactly \( k \) blocks. We have the following.

Proposition 2.4. The Stirling transformation of the second kind \( z_n = \sum_{k=0}^{n} S(n, k) x_k \) preserves the log-convexity.

Proof. Let \( \{x_k\}_{k \geq 0} \) be a log-convex sequence. We need to show that the sequence \( \{z_n\}_{n \geq 0} \) is log-convex. We proceed by induction on \( n \). It is easy to verify that \( z_1^2 \leq z_0 z_2 \). Now assume that \( n \geq 3 \) and \( z_0, z_1, \ldots, z_{n-1} \) is log-convex. Recall that
\[
S(n, k) = \sum_{j=k}^{n} \binom{n-1}{j-1} S(j-1, k-1)
\]
([9, p. 209]). Hence
\[
z_n = \sum_{k=1}^{n} \sum_{j=k}^{n} \binom{n-1}{j-1} S(j-1, k-1) x_k = \sum_{j=0}^{n-1} \binom{n-1}{j} \left[ \sum_{k=0}^{j} S(j, k) x_{k+1} \right].
\]
Let \( y_j = \sum_{k=0}^{j} S(j, k) x_{k+1} \) for \( 0 \leq j \leq n - 1 \). Then the sequence \( y_0, y_1, \ldots, y_{n-1} \) is log-convex by the induction hypothesis, so is the sequence \( z_0, z_1, \ldots, z_{n-1}, z_n \) by Proposition 2.3. This completes the proof.

The Bell number \( B_n \) is the total number of partitions of \([n]\), i.e.,
\[
B_n = \sum_{k=0}^{n} S(n, k).
\]

The log-convexity of Bell numbers was first obtained by Engel [17]. Then Bender and Canfield [2] gave a proof by means of the exponential generating function of the Bell
numbers. Another interesting proof is to use Dobinski formula (30). We can also obtain the log-convexity of the Bell numbers by Proposition 2.3 and the well-known recurrence

\[ B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \]

Proposition 2.4 gives a natural interpretation to the log-convexity of the Bell numbers. The following are some more applications of Proposition 2.4.

Example 2.5. The Bell number \( B_n \) can be viewed as the number of ways of placing \( n \) labeled balls into \( n \) indistinguishable boxes. Let \( S_n \) be the number of ways of placing \( n \) labeled balls into \( n \) unlabeled (but 2-colored) boxes. Clearly, \( S_n = \sum_{k=1}^{n} 2^k S(n, k) \). Hence the sequence \( \{S_n\}_{n \geq 0} \) is log-convex by Proposition 2.4. Note that \( S_n = \sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k} \).

Hence the log-convexity of \( \{S_n\} \) can also be followed from Davenport-Pólya Theorem and the log-convexity of \( \{B_n\} \).

Example 2.6. Consider the ordered Bell number \( c(n) \), i.e., the number of ordered partitions of the set \( [n] \). Note that \( c(n) = \sum_{k=0}^{n} k! S(n, k) \) (Stanley [33, p. 146]). The sequence \( \{c(n)\}_{n \geq 0} \) is therefore log-convex by Proposition 2.4.

Let \( c(n, k) \) be the signless Stirling number of the first kind, i.e., the number of permutations of \( [n] \) which contain exactly \( k \) permutation cycles. Then

\[ c(n, k) = \sum_{j=k}^{n} \binom{n-1}{j-1} (n-j)! c(j-1, k-1) \]

for \( 1 \leq k \leq n \) ([9, p. 215]). By a similar method used in the proof of Proposition 2.4, we can obtain the following result.

Proposition 2.7. The Stirling transformation of the first kind \( z_n = \sum_{k=0}^{n} c(n, k) x_k \) preserves the log-convexity.

It is possible to study the log-convexity problems using the theory of total positivity. Following Karlin [21], a matrix \( M = (m_{ij})_{i,j \geq 0} \) of nonnegative numbers is said to be **sign-regular of order** \( r \) (\( SR_r \) matrix, for short) if, for all \( 1 \leq t \leq r \), all minors of order \( t \) have the same sign. Denoting with \( \varepsilon_t \) (\( = \pm 1 \)) the common sign of these determinants, the vector \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r) \) is called the sign sequence of \( M \). The matrix is said to be **totally positive of order** \( r \) (or a \( TP_r \) matrix, for short) if it is an \( SR_r \) matrix with \( \varepsilon_t = 1 \) for all \( 1 \leq t \leq r \). A sequence \( a_0, a_1, a_2, \ldots \) of nonnegative numbers is called \( SR_r \) if its Hankel matrix \( A = (a_{i+j})_{i,j \geq 0} \) is \( SR_r \). Clearly, the sequence is log-convex (resp. log-concave) if and only if \( a_n a_m \leq a_{m-k} a_{n+k} \) (resp. \( a_m a_n \geq a_{m-k} a_{n+k} \)) for \( 1 \leq k \leq m \leq n \). In other words, a sequence is log-convex (resp. log-concave) if and only if it is \( SR_2 \) with sign sequence \( (1, 1) \) (resp. \( (1, -1) \)). As an application of total positivity techniques to the log-convexity problems, we demonstrate the following proposition, which is a special case of Brenti [4, Theorem 2.2.5].

Given an infinite lower triangular matrix \( A = (a_{n,k})_{n,k \geq 0} \), let \( A_n(u) = \sum_{k=0}^{n} a_{n,k} u^k \) denote the \( n \)-th row generating function of \( A \).
Proposition 2.8. Let $A, B, C$ be three infinite lower triangular matrices satisfying the following conditions.

(i) Both $B$ and $C$ are TP$_2$ matrices.

(ii) $A_{i+j}(u) = B_i(u)C_j(u)$ for all $i, j \in \mathbb{N}$.

Then the linear transformation

$$z_n = \sum_{k=0}^{n} a_{n,k} x_k, \quad n = 0, 1, 2, \ldots$$

preserves both the log-convexity and the log-concavity.

In particular, let $P = (p_{n,k})_{n,k \geq 0}$ be the Pascal matrix, where $p_{n,k} = \binom{n}{k}$ is the binomial coefficients. It is known that the matrix $P$ is TP$_2$ (all minors of $P$ are actually nonnegative, see Gessel and Viennot [18] for a combinatorial proof of this fact). Also, the row generating function $P_n(u) = (1 + u)^n$. Taking $A = B = C = P$ in Proposition 2.8, we obtain that the binomial transformation $z_n = \sum_{k=0}^{n} \binom{n}{k} x_k$ preserves both the log-convexity and the log-concavity.

3 Sequences satisfying three-term recurrences

In this section we consider the log-convexity of certain famous combinatorial numbers, including the central binomial coefficients $b(n) = \binom{2n}{n}$, the Catalan numbers, the Motzkin numbers, the Fine numbers, the central Delannoy numbers, the little and large Schröder numbers. These numbers play an important role in enumerative combinatorics and count various combinatorial objects. We review some basic facts about these numbers from the viewpoint of the enumeration of lattice paths in the $(x, y)$ plane.

The central binomial coefficient $b(n)$ counts the number of lattice paths from $(0, 0)$ to $(n, n)$ with steps $(0, 1)$ and $(1, 0)$ in the first quadrant. It is clear that $b(n) = \binom{2n}{n}$ for $n \geq 0$. The central binomial coefficients satisfy the recurrence $(n+1)b(n+1) = 2(2n+1)b(n)$. The sequence $\{b(n)\}_{n \geq 0}$ is log-convex since $\frac{b(n+1)}{b(n)} = \frac{2(2n+1)}{n+1}$ is increasing.

The Catalan number $C_n$ counts the number of lattice paths, Dyck Paths, from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$, never falling below the x-axis, or equivalently, the number of lattice paths from $(0, 0)$ to $(n, n)$ with steps $(0, 1)$ or $(1, 0)$, never rising above the line $y = x$. The Catalan numbers have an explicit expression $C_n = \frac{1}{n+1}\binom{2n}{n}$ and satisfy the recurrence $(n+2)C_{n+1} = 2(2n+1)C_n$. The sequence $\{C_n\}_{n \geq 0}$ is log-convex since $\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+1}$ is increasing.

The Motzkin number $M_n$ counts the number of lattice paths, Motzkin paths, from $(0, 0)$ to $(n, 0)$ with steps $(1, 0), (1, 1)$ and $(1, -1)$, never going below the x-axis, or equivalently, the number of lattice paths from $(0, 0)$ to $(n, n)$, with steps $(0, 2), (2, 0)$ and $(1, 1)$, never rising above the line $y = x$. It is known that the Motzkin numbers satisfy the recurrence

$$(n+3)M_{n+1} = (2n+3)M_n + 3nM_{n-1}, \quad (3.1)$$
with $M_0 = M_1 = 1$ (see [36] for a bijective proof).

The Fine number $f_n$ is the number of Dyck paths from $(0, 0)$ to $(2n, 0)$ with no hills. (A hill in a Dyck path is a pair of consecutive steps giving a peak of height 1). It is known that the Fine numbers satisfy the recurrence

$$2(n + 1)f_n = (7n - 5)f_{n-1} + 2(2n - 1)f_{n-2},$$

(3.2)

with $f_0 = 1$ and $f_1 = 0$ (see [24] for a bijective proof).

The central Delannoy number $D(n)$ is the number of lattice paths, king walks, from $(0, 0)$ to $(n, n)$ with steps $(1, 0), (0, 1)$ and $(1, 1)$ in the first quadrant. Clearly, the number of king walks with $n - k$ diagonal steps is $\binom{n+k}{n-k}b(k)$. Hence $D(n) = \sum_{k=0}^{n} \binom{n+k}{n-k}b(k)$ (38). It is known that the central Delannoy numbers satisfy the recurrence

$$nD(n) = 3(2n - 1)D(n - 1) - (n - 1)D(n - 2),$$

(3.3)

with $D(0) = 1$ and $D(1) = 3$ (see [25] for a bijective proof).

The (large) Schröder number $r_n$ is the number of king walks, Schröder paths, from $(0, 0)$ to $(n, n)$, and never rising above the line $y = x$. The large Schröder numbers bear the same relation to the Catalan numbers as the central Delannoy numbers do to the central binomial coefficients. Hence we have $r_n = \sum_{k=0}^{n} \binom{n+k}{n-k}C_k$ (16 47). The Schröder paths consist of two classes: those with steps on the main diagonal and those without. These two classes are equinumerous, and the number of paths in either class is the little Schröder number $s_n$ (half the large Schröder number). It is known that the Schröder numbers of two kinds satisfy the recurrence

$$(n + 2)z_{n+1} = 3(2n + 1)z_n - (n - 1)z_{n-1},$$

(3.4)

with $s_0 = s_1 = r_0 = 1$ and $r_1 = 2$ (see Foata and Zeilberger [15] for a combinatorial proof and Sulanke [35] for another one).

All these numbers presented previously satisfy a three-term recurrence. Aigner [1] first established algebraically the log-convexity of the Motzkin numbers and then Callan [8] gave a combinatorial proof. Recently, Došlić et al [11, 12] showed the log-convexity of the Motzkin numbers, the Fine numbers, the central Delannoy numbers, the large and little Schröder numbers using calculus. (We appreciate Došlić for acquainting us with his very recent work [13, 14], in which the techniques developed in [11, 12] are extended and more examples are presented.) Motivated by these results, we investigate the log-convexity problem of combinatorial sequences satisfying a three-term recurrence by an algebraic approach. We distinguish two cases according to the sign of coefficients in the recurrence relations.

### 3.1 The recurrence $a_nz_{n+1} = b_nz_n + c_nz_{n-1}$

Let $\{z_n\}_{n \geq 0}$ be a sequence of positive numbers satisfying the recurrence

$$a_nz_{n+1} = b_nz_n + c_nz_{n-1}$$

(3.5)
for \( n \geq 1 \), where \( a_n, b_n, c_n \) are all positive. Consider the quadratic equation

\[
a_n \lambda^2 - b_n \lambda - c_n = 0
\]

associated with the recurrence (3.5). Clearly, the equation has a unique positive root

\[
\lambda_n = \frac{b_n + \sqrt{b_n^2 + 4a_n c_n}}{2a_n}.
\]  

(3.6)

Define \( x_n = z_{n+1}/z_n \) for \( n \geq 0 \). Then the sequence \( \{z_n\}_{n \geq 0} \) is log-convex if and only if the sequence \( \{x_n\}_{n \geq 0} \) is increasing. By (3.5), the sequence \( \{x_n\}_{n \geq 0} \) satisfies the recurrence

\[
a_n x_n = b_n + \frac{c_n}{x_{n-1}}
\]  

(3.7)

for \( n \geq 1 \). It follows that \( x_n \geq x_{n-1} \) is equivalent to \( x_{n-1} \leq \lambda_n \), and is also equivalent to \( x_n \geq \lambda_n \). Thus the sequence \( \{z_n\}_{n \geq 0} \) is log-convex if and only if the sequence \( \{x_n\}_{n \geq 0} \) can be separated by the sequence \( \{\lambda_n\}_{n \geq 1} \):

\[
x_0 \leq \lambda_1 \leq x_1 \leq \lambda_2 \leq \cdots \leq x_{n-1} \leq \lambda_n \leq x_n \leq \lambda_{n+1} \leq \cdots
\]  

(3.8)

**Theorem 3.1.** Let \( \{z_n\}_{n \geq 0} \) and \( \{\lambda_n\}_{n \geq 1} \) be as above. Suppose that \( z_0, z_1, z_2, z_3 \) is log-convex and that the inequality

\[
a_n \lambda_{n-1} \lambda_{n+1} - b_n \lambda_{n-1} - c_n \geq 0
\]  

(3.9)

holds for \( n \geq 2 \). Then the sequence \( \{z_n\}_{n \geq 0} \) is log-convex.

**Proof.** Let \( x_n = z_{n+1}/z_n \) for \( n \geq 0 \). We prove the interlacing inequalities (3.8) by induction. By the assumption that \( z_0, z_1, z_2, z_3 \) is log-convex, we have \( x_0 \leq \lambda_1 \leq x_1 \leq \lambda_2 \leq x_2 \). Now assume that \( \lambda_{n-1} \leq x_{n-1} \leq \lambda_n \). Note that \( x_{n-1} \leq \lambda_n \) is equivalent to \( x_n \geq \lambda_n \). On the other hand, \( x_{n-1} \geq \lambda_{n-1} \) implies that

\[
x_n = \frac{b_n}{a_n} + \frac{c_n}{a_n x_{n-1}} \leq \frac{b_n}{a_n} + \frac{c_n}{a_n \lambda_{n-1}} \leq \lambda_{n+1}
\]

by the inequality (3.9). Hence we have \( \lambda_n \leq x_n \leq \lambda_{n+1} \). Thus (3.8) holds by induction. \( \square \)

**Corollary 3.2.** The Fine sequence \( \{f_n\}_{n \geq 2} \) is log-convex.

**Proof.** Let \( z_n = f_{n+2} \) for \( n \geq 0 \). Then \( z_0 = 1, z_1 = 2, z_2 = 6, z_3 = 18 \) and

\[
2(n + 4) z_{n+1} = (7n + 16) z_n + 2(2n + 5) z_{n-1}
\]

by (3.2). Solve the equation \( 2(n + 4) \lambda^2 - (7n + 16) \lambda - 2(2n + 5) = 0 \) to obtain

\[
\lambda_n = \frac{2(2n + 5)}{(n + 4)}.
\]

It is easy to verify that

\[
2(n + 4) \lambda_{n-1} \lambda_{n+1} - (7n + 6) \lambda_{n-1} - 2(2n + 5) = \frac{2(20n^2 + 151n + 171)}{(n + 3)(n + 5)} \geq 0.
\]

Hence the sequence \( \{z_n\}_{n \geq 0} \), i.e., \( \{f_n\}_{n \geq 2} \), is log-convex by Theorem 3.1. \( \square \)
Because of the expression (3.6) of $\lambda_n$, sometimes it is inconvenient to directly check the inequality (3.9). However, the inequality can be verified by means of Maple. For example, for the Motzkin sequence satisfying the recurrence (3.1), we have

$$\lambda_n = \frac{2n + 3 + \sqrt{16n^2 + 48n + 9}}{2(n + 3)}.$$  

Using Maple it is easy to verify the inequality

$$(n + 3)\lambda_{n-1}\lambda_{n+1} - (2n + 3)\lambda_{n-1} - 3n \geq 0.$$  

Thus the log-convexity of the Motzkin numbers follows from Theorem 3.1.

**Corollary 3.3.** The Motzkin sequence $\{M_n\}_{n \geq 0}$ is log-convex.

We can also give another criterion for the log-convexity of the sequence $\{z_n\}$ satisfying the recurrence (3.5).

**Theorem 3.4.** Let $\{z_n\}_{n \geq 0}$ and $\{\lambda_n\}_{n \geq 1}$ be defined by (3.5) and (3.6). Suppose that there exists a sequence $\{\mu_n\}_{n \geq 1}$ of positive numbers such that the following three conditions hold.

(i) $\mu_n \leq \lambda_n$ for all $n \geq 1$.

(ii) $z_1 \leq \mu_1 z_0$ and $z_2 \leq \mu_2 z_1$.

(iii) $a_n \mu_{n-1} \mu_{n+1} \geq b_n \mu_{n-1} + c_n$ for $n \geq 2$.

Then the sequence $\{z_n\}_{n \geq 0}$ is log-convex.

**Proof.** Let $x_n = z_{n+1}/z_n$ for $n \geq 0$. Then it suffices to show that the sequence $\{x_n\}$ is increasing. We prove this by showing the interlacing inequalities

$$x_0 \leq \mu_1 \leq x_1 \leq \mu_2 \leq \cdots \leq x_{n-1} \leq \mu_n \leq x_n \leq \mu_{n+1} \leq \cdots.$$  

(3.10)

The condition (ii) is equivalent to $x_0 \leq \mu_1$ and $x_1 \leq \mu_2$. However, $\mu_1 \leq \lambda_1$. Hence $x_0 \leq \lambda_1$, and so $x_1 \geq \lambda_1 \geq \mu_1$. Thus we have $\mu_1 \leq x_1 \leq \mu_2$. Now assume that $\mu_{n-1} \leq x_{n-1} \leq \mu_n$. Then $x_{n-1} \leq \mu_n$ implies $x_n \geq \mu_n$ since $\mu_n \leq \lambda_n$. On the other hand, $x_{n-1} \geq \mu_{n-1}$ implies

$$x_n = \frac{b_n}{a_n} + \frac{c_n}{a_n} \frac{1}{x_{n-1}} \leq \frac{b_n}{a_n} + \frac{c_n}{a_n} \frac{1}{\mu_{n-1}} \leq \mu_{n+1}$$

by the condition (iii). Hence we have $\mu_n \leq x_n \leq \mu_{n+1}$. Thus (3.10) holds by induction. 

For convenience, we may choose $\mu_n$ in the theorem as an appropriate rational approximation to $\lambda_n$. We present two examples to demonstrate this approach.

The derangements number $d_n$ is the number of permutations of $n$ elements with no fixed points. It is well known that the sequence $\{d_n\}_{n \geq 0}$ satisfies the recurrence

$$d_{n+1} = n(d_n + d_{n-1}),$$

with $d_0 = 1, d_1 = 0, d_2 = 1, d_3 = 2$ and $d_4 = 9$ (Comtet [9, p. 182]).
Corollary 3.5. The sequence of the derangements numbers \( \{d_n\}_{n \geq 2} \) is log-convex.

Proof. Let \( z_n = d_{n+2} \) for \( n \geq 0 \). Then the sequence satisfies the recurrence
\[
z_{n+1} = (n+2)(z_n + z_{n-1}),
\]
with \( z_0 = 1, z_1 = 2 \) and \( z_2 = 9 \). We have
\[
\lambda_n = \frac{(n+2) + \sqrt{n^2 + 8n + 12}}{2} \geq \frac{(n+2) + (n+3)}{2} = \frac{2n+5}{2}.
\]
Set \( \mu_n = (2n+5)/2 \). Then \( \lambda_n \leq \mu_n \). Also, \( z_1/z_0 = 2 < 7/2 = \mu_1 \) and \( z_2/z_1 = 9/2 = \mu_2 \). Furthermore,
\[
\mu_{n-1}\mu_{n+1} - (n+2)\mu_{n-1} - (n+2) = \frac{2n+1}{4} \geq 0.
\]
Thus the sequence \( \{z_n\}_{n \geq 0} \), i.e., \( \{d_n\}_{n \geq 2} \), is log-convex by Theorem 3.4.

Remark 3.6. Generally, if \( a_n \) takes a constant value, both \( b_n \) and \( c_n \) are linear functions in \( n \) respectively, then we can show that the sequence \( \{z_n\}_{n \geq 0} \) satisfying the recurrence (3.5) is asymptotically log-convex by means of Theorem 3.1. In other words, there exists an index \( N \) such that \( \{z_n\}_{n \geq N} \) is log-convex. We leave the proof of this result to the reader as an exercise.

Let \( A_n \) be the number of directed animals of size \( n \) (Stanley [34, Exercise 6.46]). The sequence \( \{A_n\}_{n \geq 0} \) is Sloane’s A005773 ([31]) and satisfies the recurrence
\[
(n+1)A_{n+1} = 2(n+1)A_n + 3(n-1)A_{n-1},
\]
with \( A_0 = 1, A_1 = 1 \) and \( A_2 = 2 \).

Corollary 3.7. The sequence \( \{A_n\}_{n \geq 0} \) counting directed animals is log-convex.

Proof. Note that
\[
\lambda_n = 1 + \sqrt{\frac{4n-2}{n+1}} = 1 + \sqrt{\frac{4n-2}{2n} \cdot \frac{4n}{2n+2}} \geq 1 + \frac{4n-1}{2n+1} = \frac{6n}{2n+1}
\]
for \( n \geq 1 \), where the inequality follows from \( \sqrt{\frac{x-1}{y-1}} \geq \frac{x}{y} \) when \( x \geq y > 1 \). Set \( \mu_n = \frac{6n}{2n+1} \). Then \( \mu_n \leq \lambda_n \), \( A_1/A_0 = 1 < 2 = \mu_1 \) and \( A_2/A_1 = 2 < 2.4 = \mu_2 \). Also,
\[
(n+1)\mu_{n-1}\mu_{n+1} - 2(n+1)\mu_{n-1} - 3(n-1) = \frac{9(n-1)}{(2n-1)(2n+3)} \geq 0.
\]
Thus the sequence \( \{A_n\}_{n \geq 0} \) is log-convex by Theorem 3.4.

The techniques developed in this subsection can also be used to study the log-concavity of the sequences \( \{z_n\}_{n \geq 0} \) satisfying the recurrence (3.5). For example, it is clear that \( \{z_n\}_{n \geq 0} \) is log-concave if and only if
\[
x_0 \geq \lambda_1 \geq x_1 \geq \lambda_2 \geq \cdots \geq x_{n-1} \geq \lambda_n \geq x_n \geq \lambda_{n+1} \geq \cdots.
\]
Then we have the following result similar to Theorem 3.1.
Theorem 3.8. Let \( \{z_n\}_{n \geq 0} \) and \( \{\lambda_n\}_{n \geq 1} \) be defined by (3.5) and (3.6). Suppose that \( z_0, z_1, z_2, z_3 \) is log-concave and that the inequality
\[
a_n \lambda_{n-1} \lambda_{n+1} - b_n \lambda_{n-1} - c_n \leq 0
\]
holds for \( n \geq 2 \). Then the sequence \( \{z_n\}_{n \geq 0} \) is log-concave.

Remark 3.9. When coefficients \( a_n, b_n, c_n \) in the recurrence (3.5) take constant values respectively, the sequence \( \{z_n\}_{n \geq 0} \) is neither log-convex nor log-concave since \( \lambda_n \) takes a constant value. For example, the Fibonacci numbers satisfy the recurrence \( F_{n+1} = F_n + F_{n-1} \) with \( F_0 = F_1 = 1 \). The sequence \( \{F_n\}_{n \geq 0} \) is neither log-convex nor log-concave. Actually, \( F_{n-1}F_{n+1} - F_n^2 = (-1)^{n-1} \). However, the bisection \( \{F_{2n}\}_{n \geq 0} \) with even index is log-convex and the bisection \( \{F_{2n+1}\}_{n \geq 0} \) with odd index is log-concave since \( F_{n-2}F_{n+2} - F_n^2 = (-1)^n \). We will give a general result about such sequences in Corollary 3.17.

3.2 The recurrence \( a_n z_{n+1} = b_n z_n - c_n z_{n-1} \)

In this part we consider the log-convexity of the sequence \( \{z_n\} \) of positive numbers satisfying the recurrence
\[
a_n z_{n+1} = b_n z_n - c_n z_{n-1}
\]
for \( n \geq 1 \), where \( a_n, b_n, c_n \) are all positive.

Let \( x_n = z_{n+1}/z_n \) for \( n \geq 0 \). Then we need to check whether the sequence \( \{x_n\}_{n \geq 0} \) is increasing.

By the recurrence (3.11), we have
\[
x_n = \frac{b_n}{a_n} - c_n \frac{1}{x_{n-1}}.
\]
Hence
\[
x_{n+1} - x_n = \left[ \left( \frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} \right) + \left( \frac{c_n}{a_n} - \frac{c_{n+1}}{a_{n+1}} \right) \frac{1}{x_n} \right] + \frac{c_n}{a_n} \left( \frac{1}{x_{n-1}} - \frac{1}{x_n} \right).
\]
(3.12)

Observe that if
\[
\begin{vmatrix}
a_n & a_{n+1} \\
b_n & b_{n+1}
\end{vmatrix} x_n + \begin{vmatrix} c_n \ c_{n+1} \end{vmatrix} \geq 0,
\]
(3.13)
then \( x_{n-1} \leq x_n \) implies \( x_n \leq x_{n+1} \) from (3.12). Hence we can conclude that if \( x_0 \leq x_1 \) and the inequality (3.13) holds for \( n \geq 1 \), then the sequence \( \{x_n\}_{n \geq 0} \) is increasing, and the sequence \( \{z_n\} \) is therefore log-convex.

Suppose now that \( a_n, b_n, c_n \) are all linear functions in \( n \). In this case, the inequality (3.13) is easily checked since two determinants take constant values respectively. More precisely, let
\[
a_n = \alpha_1 n + \alpha_0, \quad b_n = \beta_1 n + \beta_0, \quad c_n = \gamma_1 n + \gamma_0
\]
and denote
\[ A = \begin{vmatrix} \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{vmatrix}, \quad B = \begin{vmatrix} \gamma_0 & \gamma_1 \\ \alpha_0 & \alpha_1 \end{vmatrix}, \quad C = \begin{vmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{vmatrix}. \]

Then it is easy to see that two determinants in the inequality (3.13) are equal to \( C \) and \( B \) respectively. Thus we have the following criterion.

**Theorem 3.10.** Let \( \{z_n\}_{n \geq 0} \) be a sequence of positive numbers and satisfy the three-term recurrence

\[ (\alpha_1 n + \alpha_0)z_{n+1} = (\beta_1 n + \beta_0)z_n - (\gamma_1 n + \gamma_0)z_{n-1} \quad (3.14) \]

for \( n \geq 1 \), where \( \alpha_1 n + \alpha_0, \beta_1 n + \beta_0, \gamma_1 n + \gamma_0 \) are positive for \( n \geq 1 \). Suppose that \( z_0, z_1, z_2 \) is log-convex. Then the full sequence \( \{z_n\}_{n \geq 0} \) is log-convex if one of the following conditions holds.

1. \( B, C \geq 0 \).
2. \( B < 0, C > 0, AC \geq B^2 \) and \( z_0 B + z_1 C \geq 0 \).
3. \( B > 0, C < 0, AC \leq B^2 \) and \( z_0 B + z_1 C \geq 0 \).

**Proof.** Let \( x_n = z_{n+1}/z_n \) for \( n \geq 0 \). Then \( x_0 \leq x_1 \) since \( z_0 z_2 \geq z_1^2 \). Thus it suffices to show that the inequality \( Cx_n + B \geq 0 \) holds for \( n \geq 0 \). If \( B \geq 0 \) and \( C \geq 0 \), then the inequality is obvious. Next we assume that \( BC < 0 \) and show that \( Cx_n + B \geq 0 \) by induction on \( n \). We do it only for the case (ii) since the case (iii) is similar. Clearly, \( Cx_0 + B \geq 0 \) by the condition \( z_0 B + z_1 C \geq 0 \). Now assume that \( Cx_{n-1} + B \geq 0 \) for \( n \geq 1 \). Then

\[
Cx_n + B = C \left( \frac{b_n}{a_n} - \frac{c_n}{a_n x_{n-1}} \right) + B \\
\geq C \left( \frac{b_n + c_n C}{B} \right) + B \\
= \frac{C}{a_n B}(b_n B + c_n C) + B.
\]

Note that \( b_n B + c_n C = -a_n A \) since

\[
a_n A + b_n B + c_n C = \begin{vmatrix} a_n & \alpha_1 & \alpha_0 \\ b_n & \beta_1 & \beta_0 \\ c_n & \gamma_1 & \gamma_0 \end{vmatrix} = \begin{vmatrix} \alpha_1 n + \alpha_0 & \alpha_1 & \alpha_0 \\ \beta_1 n + \beta_0 & \beta_1 & \beta_0 \\ \gamma_1 n + \gamma_0 & \gamma_1 & \gamma_0 \end{vmatrix} = 0.
\]

Hence

\[
Cx_n + B \geq \frac{AC}{B} + B \geq 0
\]

by the condition \( AC \geq B^2 \), as desired. This completes our proof. \( \square \)

The techniques developed in Theorem 3.10 can also be used to study the log-concavity of the sequences \( \{z_n\}_{n \geq 0} \) satisfying the recurrence (3.14). We demonstrate the result without proof.
Theorem 3.11. Let \( \{z_n\}_{n \geq 0} \) be a sequence of positive numbers and satisfy the three-term recurrence (3.14). Suppose that \( z_0, z_1, z_2 \) is log-concave. Then the full sequence \( \{z_n\}_{n \geq 0} \) is log-concave if one of the following conditions holds.

(i) \( B, C \leq 0 \).

(ii) \( B < 0, C > 0, AC \leq B^2 \) and \( z_0B + z_1C \leq 0 \).

(iii) \( B > 0, C < 0, AC \geq B^2 \) and \( z_0B + z_1C \leq 0 \).

Corollary 3.12. The central Delannoy sequence \( \{D(n)\}_{n \geq 0} \) is log-convex.

Proof. By the recurrence (3.3), we have \( A = 3, B = -1, C = 3 \). Also, \( D(0) = 1, D(1) = 3, D(2) = 13 \). Thus the log-convexity of \( \{D(n)\}_{n \geq 0} \) follows from Theorem 3.10.

Corollary 3.13. The little and large Schröder numbers are log-convex respectively.

Proof. It suffices to show that the little Schröder numbers \( \{s_n\}_{n \geq 0} \) is log-convex since the large Schröder numbers \( r_n = 2s_n \) for \( n \geq 1 \).

By the recurrence (3.4), we have \( A = 9, B = -3, C = 9 \). Also, \( s_0 = s_1 = 1, s_2 = 3 \). Thus the log-convexity of \( \{s_n\}_{n \geq 0} \) follows from Theorem 3.10.

Let \( h_n \) be the number of the set of all tree-like polyhexes with \( n + 1 \) hexagons (Harary and Read [20]). It is known that \( h_n \) counts the number of lattice paths, from \((0,0)\) to \((2n,0)\) with steps \((1,1), (1,-1)\) and \((2,0)\), never falling below the \(x\)-axis and with no peaks at odd level. The sequence \( \{h_n\}_{n \geq 0} \) is Sloane’s A002212 and satisfies the recurrence

\[
(n+1)h_n = 3(2n-1)h_{n-1} - 5(n-2)h_{n-2}
\]

with \( h_0 = h_1 = 1 \) and \( h_2 = 3 \). Thus the following corollary is an immediate consequence of Theorem 3.10.

Corollary 3.14. The sequence \( \{h_n\}_{n \geq 0} \) counting tree-like polyhexes is log-convex.

Let \( w_n \) be the number of walks on cubic lattice with \( n \) steps, starting and finishing on the \( xy \) plane and never going below it (Guy [19]). The sequence \( \{w_n\}_{n \geq 0} \) is Sloane’s A005572 and satisfies the recurrence

\[
(n+2)w_n = 4(2n+1)w_{n-1} - 12(n-1)w_{n-2},
\]

with \( w_0 = 1, w_1 = 4 \) and \( w_2 = 17 \). Thus the following corollary is immediate from Theorem 3.10.

Corollary 3.15. The sequence \( \{w_n\}_{n \geq 0} \) counting walks on cubic lattice is log-convex.

A special interesting case of Theorem 3.10 (i) and Theorem 3.11 (i) is the following.

Corollary 3.16. Suppose that the sequence \( \{z_n\}_{n \geq 0} \) of positive numbers satisfies the recurrence \( az_{n+1} = bz_n - cz_{n-1} \) for \( n \geq 1 \), where \( a, b, c \) are positive constants. If \( z_0, z_1, z_2 \) is log-convex (resp. log-concave), then so is the full sequence \( \{z_n\}_{n \geq 0} \).
Corollary 3.17. Suppose that the sequence \( \{z_n\}_{n \geq 0} \) of positive numbers satisfies the recurrence \( az_{n+1} = bz_n + cz_{n-1} \) for \( n \geq 1 \), where \( a, b, c \) are positive constants. If \( z_0, z_1, z_2 \) is log-convex (resp. log-concave), then the bisection \( \{z_{2n}\} \) is log-convex (resp. log-concave) and the bisection \( \{z_{2n+1}\} \) is log-concave (resp. log-convex).

Proof. By the recurrence \( az_{n+1} = bz_n + cz_{n-1} \) for \( n \geq 1 \), we can obtain the recurrence
\[
a^2z_{n+2} = (b^2 + 2ac)z_n - c^2z_{n-2}
\]
for \( n \geq 2 \). It is not difficult to verify that
\[
a^2(z_0z_4 - z_2^2) = b^2(z_0z_2 - z_1^2), \quad a^3(z_1z_5 - z_3^2) = b^2c(z_1^2 - z_0z_2).
\]
So the statement follows from Corollary 3.16.

The Fibonacci numbers \( F_n \) satisfy the recurrence \( F_{n+1} = F_n + F_{n-1} \) with \( F_0 = F_1 = 1 \) and \( F_2 = 2 \). The Lucas numbers \( L_n \) satisfy the recurrence \( L_{n+1} = L_n + L_{n-1} \) with \( L_0 = 1, L_1 = 3 \) and \( L_2 = 4 \). And the Pell numbers \( P_n \) satisfy the recurrence \( P_{n+1} = 2P_n + P_{n-1} \) with \( P_0 = 1, P_1 = 2 \) and \( P_3 = 5 \). Thus we can conclude the following result from Corollary 3.17.

Corollary 3.18. The bisections \( \{F_{2n+1}\}, \{L_{2n}\}, \{P_{2n+1}\} \) are log-concave and the bisections \( \{F_{2n}\}, \{L_{2n+1}\}, \{P_{2n}\} \) are log-convex.

4 \( q \)-log-convexity

In this section we first introduce the concept of the \( q \)-log-convexity of polynomial sequences and then prove the \( q \)-log-convexity of certain well-known polynomial sequences, including the Bell polynomials, the Eulerian polynomials, the \( q \)-Schröder numbers and the \( q \)-central Delannoy numbers. We also present certain linear transformations preserving the log-convexity of sequences and establish the connection with the \( q \)-log-convexity.

Let \( q \) be an indeterminate. Given two real polynomials \( f(q) \) and \( g(q) \), write \( f(q) \leq_q g(q) \) if and only if \( g(q) - f(q) \) has nonnegative coefficients as a polynomial in \( q \). A sequence of real polynomials \( \{P_n(q)\}_{n \geq 0} \) is called \( q \)-log-convex if
\[
P_n^2(q) \leq_q P_{n-1}(q)P_{n+1}(q)
\]
for all \( n \geq 1 \). Clearly, if the sequence \( \{P_n(q)\}_{n \geq 0} \) is \( q \)-log-convex, then for each fixed positive number \( q \), the sequence \( \{P_n(q)\}_{n \geq 0} \) is log-convex. The converse is not true in general. If the opposite inequality in (4.1) holds, then the sequence \( \{P_n(q)\}_{n \geq 0} \) is called \( q \)-log-concave. The concept of the \( q \)-log-concavity was first suggested by Stanley and there has been much interest in this subject. We refer the reader to Sagan \[28, 29\] for further information about the \( q \)-log-concavity.

Perhaps the simplest example of \( q \)-log-convex polynomials is the \( q \)-factorial. It is well known that the factorial \( n! \) is log-convex. The standard \( q \)-analogue of an integer \( n \) is \( (n)_q = 1 + q + q^2 + \cdots + q^{n-1} \) and the associated \( q \)-factorial is \( (n)_q! = \prod_{k=1}^{n}(k)_q \). It is easy to verify that the \( q \)-factorial \( (n)_q! \) is \( q \)-log-convex by a direct calculation. We next provide more examples of \( q \)-log-convex sequences.
4.1 Bell polynomials and Eulerian polynomials

The Bell polynomial, or the exponential polynomial, is the generating function $B_n(q) = \sum_{k=0}^{n} S(n,k)q^k$ of the Stirling numbers of the second kind. It can be viewed as a $q$-analog of the Bell number and has many fascinating properties (see Roman [26, §4.1.3] for instance). Note that the Stirling numbers of the second kind satisfy the recurrence

$$S(n+1,k) = kS(n,k) + S(n,k-1)$$

Hence the Bell polynomials satisfy the recurrence

$$B_{n+1}(q) = qB_n(q) + qB'_n(q).$$

It is well known that the Bell polynomials $B_n(q)$ have only real zeros (see [45] for instance). In §2 we have shown that the linear transformation $z_n = \sum_{k=0}^{n} S(n,k)x_k$ can preserve the log-convexity of sequences. Therefore, for each positive number $q$, the sequence $\{B_n(q)\}_{n \geq 0}$ is log-convex. A further problem is whether the sequence $\{B_n(q)\}_{n \geq 0}$ is $q$-log-convex.

Let $\pi = a_1a_2\cdots a_n$ be a permutation of $[n]$. An element $i \in [n-1]$ is called a descent of $\pi$ if $a_i > a_{i+1}$. The Eulerian number $A(n,k)$ is defined as the number of permutations of $[n]$ having $k-1$ descents. The Eulerian numbers satisfy the recurrence

$$A(n,k) = kA(n-1,k) + (n-k+1)A(n-1,k-1).$$

Let $A_n(q) = \sum_{k=0}^{n} A(n,k)q^k$ be the Eulerian polynomial. Then

$$A_n(q) = nqA_{n-1}(q) + q(1-q)A'_{n-1}(q).$$

It is well known that $A_n(q)$ has only real zeros and $A(n,k)$ is therefore log-concave in $k$ for each fixed $n$ (see [45] for instance). By Frobenius formula

$$A_n(q) = q \sum_{k=1}^{n} k!S(n,k)(q-1)^{n-k}$$

and Proposition [27.1], the sequence $\{A_n(q)\}_{n \geq 0}$ is log-convex for each fixed positive number $q \geq 1$. We refer the reader to Comtet [9] for further information about the Eulerian numbers and the Eulerian polynomials.

To show the $q$-log-convexity of both the Bell polynomials and the Eulerian polynomials, we establish the following more general result.

Let $\{T(n,k)\}_{n,k \geq 0}$ be an array of nonnegative numbers satisfying the recurrence

$$T(n,k) = (a_1n + a_2k + a_3)T(n-1,k) + (b_1n + b_2k + b_3)T(n-1,k-1)$$

with $T(n,k) = 0$ unless $0 \leq k \leq n$. It is natural to assume that $a_1n + a_2k + a_3 \geq 0$ for $0 \leq k < n$ and $b_1n + b_2k + b_3 \geq 0$ for $0 < k \leq n$. Note that the former is equivalent to $a_1 \geq 0$, $a_1 + a_2 \geq 0$, $a_3 + a_3 \geq 0$ and the latter is equivalent to $b_1 \geq 0$, $b_1 + b_2 \geq 0$, $b_1 + b_2 + b_3 \geq 0$.

It is known that for each fixed $n$, the sequence $\{T(n,k)\}_{0 \leq k \leq n}$ is log-concave (Kurtz [22]) and further, is PF if $a_2b_1 \geq a_1b_2$ and $a_2(b_1 + b_2 + b_3) \geq (a_1 + a_3)b_2$ (Wang and Yeh [45, Corollary 3]).
Theorem 4.1. Let \( \{T(n,k)\}_{n,k \geq 0} \) be as above and the row generating function \( T_n(q) = \sum_{k=0}^{n} T(n,k)q^k \). Suppose that for \( 0 < k \leq n, \)

\[
(a_2b_1 - a_1b_2)n + a_2b_2k + (a_2b_3 - a_3b_2) \geq 0.
\]  \quad (4.3)

Then the sequence \( \{T_n(q)\}_{n \geq 0} \) is \( q \)-log-convex.

Remark 4.2. The condition (4.3) is equivalent to

\[
a_2b_1 - a_1b_2, a_2(b_1 + b_2) - a_1b_2, a_2(b_1 + b_2 + b_3) - (a_1 + a_3)b_2
\]

are all nonnegative. Hence the polynomial \( T_n(q) \) in Theorem 4.1 has only real zeros for each \( n \geq 0 \) by Wang and Yeh [45, Corollary 3].

Remark 4.3. If \( a_2 = b_2 = 0 \), then the condition (4.3) is trivially satisfied.

Proof of Theorem 4.1. Let \( T_{n-1}(q)T_{n+1}(q) - T_n^2(q) = \sum_{t=0}^{2n} A_t q^t \). We need to show that \( A_t \geq 0 \) for \( 0 \leq t \leq 2n \). Note that the recurrence (4.2) is equivalent to

\[
T_n(q) = (a_1n + a_3 + b_1nq + b_2q + b_3q)T_{n-1}(q) + (a_2 + b_2q)qT_{n-1}'(q).
\]

Hence

\[
\sum_{t=0}^{2n} A_t q^t = T_{n-1}(q)[(a_1n + a_1 + a_3 + b_1nq + b_1q + b_2q + b_3q)T_n(q) + (a_2 + b_2q)qT_n'(q)]
\]

\[
- T_n(q)[(a_1n + a_3 + b_1nq + b_2q + b_3q)T_{n-1}(q) + (a_2 + b_2q)qT_{n-1}'(q)]
\]

\[
= (a_1 + b_1q)T_{n-1}(q)T_n(q) + (a_2 + b_2q)q[T_{n-1}(q)T_n'(q) - T_{n-1}'(q)T_n(q)].
\]

Thus \( A_t = \sum_{k=0}^{t} c_k(n,t) \), where

\[
c_k = T(n,t-k)[a_1T(n-1,k) + b_1T(n-1,k-1) + a_2(t-k)T(n-1,k)]
\]

\[
- a_2kT(n-1,k) + b_2(t-k)T(n-1,k-1) - b_2(k-1)T(n-1,k-1)]
\]

\[
= T(n,t-k)[(a_1 + ta_2 - 2ka_2)T(n-1,k) + (b_1 + b_2 + tb_2 - 2kb_2)T(n-1,k-1)].
\]

Clearly, \( c_k \geq 0 \) if \( t \) is even and \( k = t/2 \). So, in order to prove that \( A_t \geq 0 \), it suffices to prove that \( c_k + c_{t-k} \geq 0 \) for \( 0 \leq k < t-k \leq n \). Let \( u_k = T(n-1,k) \) if \( 0 \leq k \leq n-1 \) and \( u_k = 0 \) otherwise. Then the sequence \( \{u_k\}_{k \geq 0} \) is log-concave. In what follows, we always assume that \( 0 \leq k < t-k \leq n \). By the recurrence (4.2), we have

\[
c_k + c_{t-k} = [(a_1n + a_2(t-k) + a_3)u_{t-k} + (b_1n + b_2(t-k) + b_3)u_{t-k-1}]
\]

\[
\times [(a_1 + ta_2 - 2ka_2)u_k + (b_1 + b_2 + tb_2 - 2kb_2)u_{k-1}]
\]

\[
+ [(a_1n + a_2k + a_3)u_k + (b_1n + b_2k + b_3)u_{k-1}]
\]

\[
\times [(a_1 - ta_2 + 2ka_2)u_{t-k} + (b_1 + b_2 - tb_2 + 2kb_2)u_{t-k-1}]
\]

\[
= xu_ku_{t-k} + yu_{k-1}u_{t-k-1} + zu_ku_{t-k-1} + wu_{k-1}u_{t-k},
\]
Remark 4.7. Proposition 4.6. The Eulerian polynomials $A_n^k(t)$ form a $q$-log-convex sequence.

Remark 4.8. An immediate consequence of Proposition 4.4 is the log-convexity of the Bell polynomials. Also, note that 2-colored Bell number $S_n = B_n(2)$ in Example 2.5 and the Frobenius formula. Hence the log-convexity of $\{S_n\}_{n \geq 0}$ follows from Proposition 4.4.

Remark 4.7. Note that the ordered Bell number $c(n) = A_n^k(t)/2$ in Example 2.6 by the Frobenius formula. Hence the log-convexity of $\{c(n)\}_{n \geq 0}$ follows from Proposition 4.4.
4.2 Linear transformations preserving log-convexity

In [46], Wang and Yeh established the connection between linear transformations preserving the log-concavity and the q-log-concavity. This method is also effective for the log-convexity.

Given a triangle \( \{a(n, k)\}_{0 \leq k \leq n} \) of nonnegative real numbers, consider the linear transformation

\[
    z_n = \sum_{k=0}^{n} a(n, k)x_k, \quad n = 0, 1, 2, \ldots .
\]

(4.4)

For convenience, let \( a(n, k) = 0 \) unless \( 0 \leq k \leq n \). For \( 0 \leq t \leq 2n \), define

\[
    a_k(n, t) = a(n-1, k)a(n+1, t-k) + a(n+1, k)a(n-1, t-k) - 2a(n, k)a(n, t-k)
\]

if \( 0 \leq k < t/2 \), and

\[
    a_k(n, t) = a(n-1, k)a(n+1, k) - a^2(n, k)
\]

if \( t \) is even and \( k = t/2 \). Also, define

\[
    \mathcal{A}_n(q) = \sum_{k=0}^{n} a(n, k)q^k, \quad n = 0, 1, 2, \ldots .
\]

It is clear that if the linear transformation \([4.4]\) preserves the log-convexity, then for each positive number \( q \), the sequence \( \{\mathcal{A}_n(q)\} \) is log-convex. On the other hand, we have the following.

**Theorem 4.8.** Suppose that the triangle \( \{a(n, k)\} \) of nonnegative real numbers satisfies the following two conditions.

(C1) The sequence of polynomials \( \{\mathcal{A}_n(q)\}_{n \geq 0} \) is q-log-convex.

(C2) There exists an index \( r = r(n, t) \) such that \( a_k(n, t) \geq 0 \) for \( k \leq r \) and \( a_k(n, t) < 0 \) for \( k > r \).

Then the following two results hold.

(R1) The linear transformation \( z_n = \sum_{k=0}^{n} a(n, k)x_k \) preserves the log-convexity.

(R2) If the sequence \( \{u_k\}_{k \geq 0} \) is log-convex and \( b(n, k) = a(n, k)u_k \) for \( 0 \leq k \leq n \), then the triangle \( \{b(n, k)\}_{0 \leq k \leq n} \) also satisfies the conditions (C1) and (C2).

**Proof.** Note that

\[
    z_{n-1}z_{n+1} - z_n^2 = \sum_{t=0}^{2n} \left[ \sum_{k=0}^{\lfloor t/2 \rfloor} a_k(n, t)x_kx_{t-k} \right]
\]

and

\[
    \mathcal{A}_{n-1}(q)\mathcal{A}_{n+1}(q) - \mathcal{A}_n^2(q) = \sum_{t=0}^{2n} \left[ \sum_{k=0}^{\lfloor t/2 \rfloor} a_k(n, t) \right] q^t.
\]
Denote $A(n,t) = \sum_{k=0}^{[t/2]} a_k(n,t)$. Then the condition (C1) is equivalent to $A(n,t) \geq 0$ for $0 \leq t \leq 2n$. Assume that $\{x_k\}_{k \geq 0}$ is log-convex. Then $x_0 x_t \geq x_1 x_{t-1} \geq x_2 x_{t-2} \geq \cdots$. It follows that
\[\sum_{k=0}^{[t/2]} a_k(n,t)x_k x_{t-k} \geq \sum_{k=0}^{[t/2]} a_k(n,t)x_x x_{t-r} = A(n,t)x_x x_{t-r}\]
by the condition (C2). Thus $z_{n-1} z_{n+1} - z_n^2 \geq \sum_{t=0}^{2n} A(n,t)x_x x_{t-r} \geq 0$, and $\{z_n\}_{n \geq 0}$ is therefore log-convex. This proves (R1).

Note that $b_k(n,t) = a_k(n,t)u_k u_{t-k}$ by the definition. Hence the triangle $\{b(n,k)\}_{0 \leq k \leq n}$ satisfies the condition (C2). On the other hand, we have
\[B(n,t) = \sum_{k=0}^{[t/2]} b_k(n,t) = \sum_{k=0}^{[t/2]} a_k(n,t)u_k u_{t-k} \geq \sum_{k=0}^{[t/2]} a_k(n,t)u_r u_{t-r} = A(n,t)u_r u_{t-r} \geq 0\]
by the log-convexity of $u_k$ and the condition (C2). So the triangle $\{b(n,k)\}_{0 \leq k \leq n}$ satisfies the condition (C1). This proves (R2).

**Proposition 4.9.** The linear transformation
\[z_n = \sum_{k=0}^{n} \binom{n+k}{n-k} x_k, \quad n = 0, 1, 2, \ldots\]
preserves the log-convexity of sequences.

**Proof.** Let $a(n,k) = \binom{n+k}{n-k}$ for $0 \leq k \leq n$. Then by Theorem 4.8 it suffices to show that the triangle $\{a(n,k)\}$ satisfies the conditions (C1) and (C2).

Let $\mathcal{A}_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} q^k$, which is the $n$-th Morgan-Voyce polynomial [39]. By the recurrence relation of the binomial coefficients, we can obtain
\[\binom{n+1+k}{n+1-k} = 2 \binom{n+k}{n-k} + \binom{n+k-1}{n-k+1} - \binom{n-1+k}{n-1-k}.\]
From this it follows that $\mathcal{A}_{n+1}(q) = (2+q)\mathcal{A}_n(q) - \mathcal{A}_{n-1}(q)$, which is equivalent to
\[\begin{pmatrix} \mathcal{A}_{n+1} \\ \mathcal{A}_n \end{pmatrix} = \begin{pmatrix} 2+q & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{A}_n \\ \mathcal{A}_{n-1} \end{pmatrix}.\]
Now consider the determinants on the two sides of the equality. Then we have
\[\mathcal{A}_{n-1}(q)\mathcal{A}_{n+1}(q) - \mathcal{A}_n^2(q) = \mathcal{A}_{n-2}(q)\mathcal{A}_n(q) - \mathcal{A}_{n-1}^2(q) = \cdots = \mathcal{A}_0(q)\mathcal{A}_2(q) - \mathcal{A}_1^2(q) = q\]
by the initial conditions $\mathcal{A}_0(q) = 1, \mathcal{A}_1(q) = 1 + q$ and $\mathcal{A}_2(q) = 1 + 3q + q^2$. Thus the sequence $\{\mathcal{A}_n(q)\}_{n \geq 0}$ is $q$-log-convex, and so the condition (C1) is satisfied.

By the definition, we have
\[a_k(n,t) = \frac{\binom{n-1+k}{n-1-k} \binom{n+1+t-k}{n+1-t+k} + \binom{n+1+k}{n+1-t+k} \binom{n-1+t-k}{n-1-t+k} - 2 \binom{n+k}{n-t+k} \binom{n+t-k}{n-t+k}}{(2k)!(2t-2k)!(n+1-k)!(n+1-t+k)!} = \tilde{a}_k(n,t)\]
when \( k < t/2 \), and
\[
a_k(n, t) = \left( \frac{n - 1 + k}{n - 1 - k} \right) \left( \frac{n + 1 + k}{n + 1 - k} \right) - \left( \frac{n + k}{n - k} \right)^2 - \frac{1}{2} \left[ \frac{(n - 1 + k)!}{(2k)!((n + 1 - k)!)} \right]^2 \tilde{a}_k(n, t)
\]
when \( t \) even and \( k = t/2 \), where
\[
\tilde{a}_k(n, t) = (n + k)(n + 1 + k)(n + t + k)(n + t - k + 1) + (n - k)(n - k + 1)(n + t - k)(n + t - k + 1) - 2(n + k)(n - k + 1)(n + t - k)(n - t + k + 1).
\]

Clearly, \( a_k(n, t) \) has the same sign as that of \( \tilde{a}_k(n, t) \) for each \( k \). Using Maple, we obtain that the derivative of \( \tilde{a}_k(n, t) \) with respect to \( k \) is
\[
-2(t - 2k)[2(2n + 1)^2 - t] \leq 0.
\]
Thus \( \tilde{a}_k(n, t) \) changes sign at most once (from nonnegative to nonpositive), and so does \( a_k(n, t) \). Thus the condition (C2) is also satisfied. This completes our proof.

Note that the even-indexed Fibonacci numbers \( F_{2n} = \sum_{k=0}^{n} \binom{n+k}{n-k} \). Hence the log-convexity of the numbers \( F_{2n} \) follows immediately from Proposition 4.9. It is also known that the large Schröder numbers \( r_n = \sum_{k=0}^{n} \binom{n+k}{n-k} C_k \) and the central Delannoy numbers \( D(n) = \sum_{k=0}^{n} \binom{n+k}{n-k} b(k) \). So the log-convexity of the numbers \( r_n \) and \( D(n) \) is implied by the log-convexity of the Catalan numbers \( C_k \) and the central binomial coefficients \( b(k) \) respectively.

The \( q \)-Schröder number \( r_n(q) \), introduced by Bonin, Shapiro and Simion [3], is defined as the \( q \)-analog of the large Schröder number \( r_n \):
\[
r_n(q) = \sum_{P} q^\text{diag}(P),
\]
where \( P \) takes over all Schröder paths from \((0,0)\) to \((n,n)\) and \( \text{diag}(P) \) denotes the number of diagonal steps in the path \( P \). Clearly, \( r_n(1) = r_n \), the large Schröder numbers. Also,
\[
r_n(q) = \sum_{k=0}^{n} \left( \frac{n + k}{n - k} \right) C_k q^{n-k}.
\]

From Theorem 4.8 and Proposition 4.9 we can obtain the \( q \)-log-convexity of the \( q \)-Schröder numbers.

**Corollary 4.10.** The \( q \)-Schröder numbers \( r_n(q) \) form a \( q \)-log-convex sequence.

Similarly, consider the \( q \)-central Delannoy numbers
\[
D_n(q) = \sum_{k=0}^{n} \left( \frac{n + k}{n - k} \right) b(k) q^{n-k}
\]
(Sagan [30]). We have the following.

**Corollary 4.11.** The \( q \)-central Delannoy numbers \( D_n(q) \) form a \( q \)-log-convex sequence.
5 Concluding remarks and open problems

In this paper we have explored the log-convexity of some combinatorial sequences by algebraic and analytic approaches. It is natural to look for combinatorial interpretations for the log-convexity of these sequences since their strong background in combinatorics. Callan [8] gave an injective proof for the log-convexity of the Motzkin numbers. It is possible to give combinatorial interpretations for the log-convexity of more combinatorial numbers. We feel that the lattice path techniques of Wilf [48] and Gessel-Viennot [18] are useful. As an example, we give an injective proof for the log-convexity of the Catalan numbers.

Recall that the Catalan number $C_n$ is the number of lattice paths from $(i, i)$ to $(n + i, n + i)$ with steps $(0, 1)$ and $(1, 0)$ and never rising above the line $y = x$ (see Stanley [34, Exercise 6.19 (h)] for instance). Let $C_n(i)$ be the set of such paths. We next show that $C^2_n \leq C_{n+1}C_{n-1}$ by constructing an injection

$$\phi : C_n(0) \times C_n(1) \rightarrow C_{n+1}(0) \times C_{n-1}(1).$$

Consider a path pair $(p, q) \in C_n(0) \times C_n(1)$. Clearly, $p$ and $q$ must intersect. Let $C$ be their first intersect point. Then $C$ splits $p$ into two parts $p_1$ and $p_2$, and splits $q$ into two parts $q_1$ and $q_2$. Thus the concatenation $p'$ of $p_1$ and $q_2$ is a path in $C_{n+1}(0)$, and the concatenation $q'$ of $q_1$ and $p_2$ is a path in $C_{n-1}(1)$. Define $\phi(p, q) = (p', q')$. Then the image of $\phi$ consists of precisely $(p', q') \in C_{n+1}(0) \times C_{n-1}(1)$ such that $p'$ and $q'$ intersect. It is easy to see that if $\phi(p, q) = (p', q')$, then applying the same algorithm to $(p', q')$ recovers $(p, q)$. Thus $\phi$ is injective, as desired.

It would be interesting to have a combinatorial interpretation for the log-convexity of combinatorial sequences satisfying a three-term recurrence. We refer the reader to Sagan [27] for combinatorial proofs for the log-concavity of combinatorial sequences satisfying a three-term recurrence.

The log-convexity of the Bell numbers has been shown by several different approaches in §2. An intriguing problem is to find a combinatorial interpretation for the log-convexity of the Bell numbers.

The Narayana number $N(n, k)$ is defined as the number of Dyck paths of length $2n$ with exactly $k$ peaks (a peak of a path is a place at which the step $(1, 1)$ is directly followed by the step $(1, -1)$). The Narayana numbers have an explicit expression $N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$. The Narayana polynomials $N_n(q) = \sum_{k=0}^{n} N(n, k)q^k$ are the generating function of the Narayana numbers and satisfy the recurrence

$$(n + 1)N_n(q) = (2n - 1)(1 + q)N_{n-1}(q) - (n - 2)(1 - q)^2N_{n-2}(q)$$

(see Sulanke [37] for a combinatorial proof). The Narayana polynomials $N_n(q)$ are closely related to the $q$-Schröder numbers $r_n(q)$. It is known that $r_n(q) = N_n(1 + q)$ [37]. We have showed the $q$-log-convexity of the $q$-Schröder numbers $r_n(q)$ in Corollary 4.10. We also propose the following stronger conjecture.
Conjecture 5.1. The Narayana polynomials $N_n(q)$ form a $q$-log-convex sequence.

This conjecture has been verified for $n \leq 100$ using Maple. It can also be shown that for each fixed nonnegative number $q$, the sequence $\{N_n(q)\}_{n \geq 0}$ is log-convex by means of Theorem 3.10 and the recurrence (5.1). As consequences, the Catalan numbers $C_n = \sum_{k=0}^{n} N(n, k)$ and the large Schröder number $r_n = \sum_{k=0}^{n} N(n, k)2^k$ (37) form log-convex sequences respectively. A problem naturally arises.

Conjecture 5.2. The Narayana transformation $z_n = \sum_{k=0}^{n} N(n, k)x_k$ preserves the log-convexity.

It is also known that the central binomial coefficients $b(n) = \sum_{k=0}^{n} \binom{n}{k}^2$ and the central Delannoy numbers $D(n) = \sum_{k=0}^{n} \binom{n}{k}^22^k$ (38). So we propose the following.

Conjecture 5.3. The triangle $\left\{ \binom{n}{k}^2 \right\}$ satisfies the conditions (C1) and (C2) in Theorem 4.8.

In Proposition 4.6 we obtain the $q$-log-convexity of the Eulerian polynomials. A closely related problem is the following.

Conjecture 5.4. The Eulerian transformation $z_n = \sum_{k=0}^{n} A(n, k)x_k$ preserves the log-convexity.

In this paper we show that the Bell polynomials, the Eulerian polynomials, the Morgan-Voyce polynomials, the Narayana polynomials, the $q$-central Delannoy numbers and the $q$-Schröder numbers are $q$-log-convex respectively. All these polynomials can be shown, using the methods established in [23, 45], to have only real zeros. It seems that this relation deserves further study and investigation.

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