TORSION GENERATORS OF THE TWIST SUBGROUP

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Abstract. We showed that the twist subgroup of the mapping class group of a closed connected nonorientable surface of genus $g \geq 13$ can be generated by two involutions and an element of order $g$ or $g - 1$ depending on $g$ is odd or even.

1. Introduction

Let $N_g$ denote a closed connected nonorientable surface of genus $g$. The mapping class group of $N_g$ is defined to be the group of the isotopy classes of all diffeomorphisms of $N_g$. Throughout the paper this group will be denoted by $\text{Mod}(N_g)$. Let $\Sigma_g$ denote a closed connected orientable surface of genus $g$. The mapping class group of $\Sigma_g$ is the group of the isotopy classes of orientation preserving diffeomorphisms and is denoted by $\text{Mod}(\Sigma_g)$.

In the orientable case, it is a classical result that $\text{Mod}(\Sigma_g)$ is generated by finitely many Dehn twists about nonseparating simple closed curves [4, 6, 11]. The study of algebraic properties of mapping class group, finding small generating sets, generating sets with particular properties, is an active one leading to interesting developments. Wajnryb [20] showed that $\text{Mod}(\Sigma_g)$ can be generated by two elements given as a product of Dehn twists. As the group is not abelian, this is the smallest possible. Later, Korkmaz [8] showed that one of these generators can be taken as a Dehn twist, he also proved that $\text{Mod}(\Sigma_g)$ can be generated by two torsion elements.

Recently, the third author showed that $\text{Mod}(\Sigma_g)$ is generated by two torsions of small orders [21].

Generating $\text{Mod}(\Sigma_g)$ by involutions was first considered by McCarthy and Papadopoulus [14]. They showed that the group can be generated by infinitely many conjugates of a single involution (element of order two) for $g \geq 3$. In terms of generating by finitely many involutions, Luo [13] showed that any Dehn twist about a nonseparating simple closed curve can be written as a product six involutions, which in turn implies that $\text{Mod}(\Sigma_g)$ can be generated by $12g + 6$ involutions. Brendle and Farb [2] obtained a generating set of six involutions for $g \geq 3$. Following their work, Kassabov [7] showed that $\text{Mod}(\Sigma_g)$ can be generated by four involutions if $g \geq 7$. Recently, Korkmaz [9] showed that $\text{Mod}(\Sigma_g)$ is generated by three involutions if $g \geq 8$ and four involutions if $g \geq 3$. Also, the third author improved his result showing that it is generated by three involutions if $g \geq 6$ [22].

Compared to orientable surfaces less is known about $\text{Mod}(N_g)$. Lickorish [10, 12] showed that it is generated by Dehn twists about two-sided simple closed curves and a so-called $Y$-homeomorphism (or a crosscap slide). Chillingworth [3] gave a
finite generating set for $\text{Mod}(N_g)$ that linearly depends on $g$. Szepietowski [19] proved that $\text{Mod}(N_g)$ is generated by three elements and by four involutions.

The twist subgroup $T_g$ of $\text{Mod}(N_g)$ is the group generated by Dehn twists about two-sided simple closed curves. The group $T_g$ is a subgroup of index 2 in $\text{Mod}(N_g)$ [12]. Chillingworth [3] showed that $T_g$ can be generated by finitely many Dehn twists. Stukow [17] obtained a finite presentation for $T_g$ with $(g + 2)$ Dehn twist generators. Later Omori [15] reduced the number of Dehn twist generators to $(g + 1)$ for $g \geq 4$. If it is not required that all generators are Dehn twists, Du [5] obtained a generating set consisting of three elements, two involutions and an element of order $g$ whenever $g \geq 5$ and odd. Recently, Yoshihara [23] was interested in the problem of finding generating sets for $T_g$ consisting of only involutions. He proved that $T_g$ can be generated by six involutions for $g \geq 14$ and by eight involutions if $g \geq 8$. The authors proved that $T_g$ can be generated by four involutions (see [1] for details and conditions on $g$). Since any group generated by two involutions is isomorphic to a quotient of a dihedral group, $T_g$ cannot be generated by two involutions. We are not sure whether $T_g$ can be generated by three involutions. Finally, in this paper we prove that it can also be generated by two involutions and an element of order $g$ or $g - 1$ depending on $g$ is odd or even, respectively.

After quickly reviewing the basics of the mapping class group, we give a generating set for $T_g$ consisting of Dehn twists and an element of finite order (Theorem 2.3). We also prove that it can be generated by two involutions and an element of order $g$ or $g - 1$ depending on $g$ is odd or even (Theorem 2.4 and 2.5).

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2. Background and Results on Mapping Class Groups

Let $N_g$ be a closed connected nonorientable surface of genus $g$. Note that the genus for a nonorientable surface is the number of projective planes in a connected sum decomposition. The mapping class group $\text{Mod}(N_g)$ of the surface $N_g$ is defined to be the group of the isotopy classes of diffeomorphisms $N_g \to N_g$. Throughout the paper we do not distinguish a diffeomorphism from its isotopy class. For the composition of two diffeomorphisms, we use the functional notation; if $g$ and $h$ are two-sided simple closed curves on $N_g$ and $f \in \text{Mod}(N_g)$.

- **Commutativity:** If $a$ and $b$ are disjoint, then $AB = BA$.
- **Conjugation:** If $f(a) = b$, then $fA f^{-1} = B^s$, where $s = \pm 1$ depending on whether $f$ is orientation preserving or orientation reversing on a neighbourhood of $a$ with respect to the chosen orientation.
Consider the surface $N_g$ shown in Figure 1. The Dehn twist generators of Omori can be given as follows (note that we do not have the curve $d_r$ when $g$ is odd).

Theorem 2.1. [15] The twist subgroup $T_g$ is generated by the following $(g + 1)$ Dehn twists

1. $A_1, A_2, B_1, \ldots, B_r, C_1, \ldots, C_{r-1}$ and $E$ if $g = 2r + 1$ and
2. $A_1, A_2, B_1, \ldots, B_r, C_1, \ldots, C_{r-1}, D_r$ and $E$ if $g = 2r + 2$.

Consider a basis $\{x_1, x_2, \ldots, x_{g-1}\}$ for $H_1(N_g; \mathbb{R})$ such that the curves $x_i$ are one-sided and disjoint as in Figure 2. It is known that every diffeomorphism $f : N_g \to N_g$ induces a linear map $f_* : H_1(N_g; \mathbb{R}) \to H_1(N_g; \mathbb{R})$. Therefore, one can define a homomorphism $D : \text{Mod}(N_g) \to \mathbb{Z}_2$ by $D(f) = \det(f_*)$. The following lemma from [10] tells when a mapping class falls into the twist subgroup $T_g$.

Lemma 2.2. Let $f \in \text{Mod}(N_g)$. Then $D(f) = 1$ if $f \in T_g$ and $D(f) = -1$ if $f \notin T_g$.

2.1. A generating set for $T_g$. We start with presenting a generating set for $T_g$. The diffeomorphism $T$ is the rotation by $\frac{2\pi}{g}$ or $\frac{2\pi}{g-1}$ as shown on the right hand sides of Figures 3 and 4, respectively. Note that the rotation $T$ satisfies $D(T) = 1$, which implies that $T$ belongs to $T_g$. 
Theorem 2.3. The twist subgroup $T_g$ is generated by the elements

1. $T, A_1A_2^{-1}, B_1B_2^{-1}$ and $E$ if $g = 2r + 1$ and $r \geq 3$,
2. $T, A_1A_2^{-1}, B_1B_2^{-1}, D_r$ and $E$ if $g = 2r + 2$ and $r \geq 3$.

Proof. Let $G$ be the subgroup of $T_g$ generated by the following set

$$G = \begin{cases} 
\{T, A_1A_2^{-1}, B_1B_2^{-1}, E\} & \text{if } g = 2r + 1, \\
\{T, A_1A_2^{-1}, B_1B_2^{-1}, D_r, E\} & \text{if } g = 2r + 2,
\end{cases}$$

where $r \geq 3$. It follows from Theorem 2.1 that we only need to prove that $G$ contains the elements $A_1, A_2, B_i$ and $C_j$ shown in Figure 3 and 4 where $i = 1, \ldots, r$ and $j = 1, \ldots, r - 1$. (We use the explicit homeomorphism constructed in [16, Section 3] to identify the models in these figures.)

Let $S$ denote the set of isotopy classes of two-sided non-separating simple closed
curves on $N_2$. Define a subset $\mathcal{G}$ of $S \times S$ as

$$\mathcal{G} = \{(a, b) : AB^{-1} \in G\}.$$ 

Using the arguments similar to the proof of [9, Theorem 5], the set $\mathcal{G}$ satisfies

- if $(a, b) \in \mathcal{G}$, then $(b, a) \in \mathcal{G}$ (symmetry),
- if $(a, b)$ and $(b, c) \in \mathcal{G}$, then $(a, c) \in \mathcal{G}$ (transitivity) and
- if $(a, b) \in \mathcal{G}$ and $H \in G$ then $(H(a), H(b)) \in \mathcal{G}$ ($G$-invariance).

Hence, $\mathcal{G}$ defines an equivalence relation on $S$.

We begin by showing that $B_jC_j^{-1}$ is contained in $G$ for all $i, j$. It will follow from the definition of $G$ and from the fact that $T(b_1, b_2) = (c_1, c_2)$, we have $C_1C_2^{-1}$ is in $G$ (here, we use the notation $f(a, b)$ to denote $(f(a), f(b))$). Also, by conjugating $C_1C_2^{-1}$ with powers of $T$, one can show that the elements $B_iB_i^{-1}$ and $C_iC_i^{-1}$ are contained in $G$. Moreover, the subgroup $G$ contains the elements $B_jC_j^{-1}$ and $C_jC_j^{-1}$ by the transitivity. To start with, since $B_2B_3^{-1} \in G$ and it is easy to check that

$$B_2B_3^{-1}A_1^{-1}(b_2, b_3) = (a_2, b_3),$$

so that $A_2B_3^{-1}$ is contained in the subgroup $G$. We have

$$(A_1A_2^{-1})(A_2B_3^{-1})(B_3B_2^{-1}) = A_1B_2^{-1} \in G,$$

since each of the factors is contained in $G$. Hence, $T(a_1, b_2) = (b_1, c_2)$ implies that $B_1C_2^{-1}$ is also in $G$. Now, the subgroup $G$ contains the element

$$B_1C_2^{-1} = (B_1C_2^{-1})(C_2C_1^{-1}).$$

Therefore, the elements $B_iC_i^{-1}$ is contained in $G$ by conjugating with powers of $T$ for all $i = 1, \ldots, r - 1$. It follows from the transitivity that $B_iC_i^{-1}$ is in $G$. Note that, we have

- $(A_1B_2^{-1})(B_2C_1^{-1}) = A_1C_1^{-1},$
- $(C_1A_1^{-1})(A_2A_2^{-1}) = C_1A_2^{-1},$ and
- $(C_2C_1^{-1})(C_1A_1^{-1}) = C_2A_1^{-1}$

from which it follows that the elements $A_1C_1^{-1}, C_1A_2^{-1}$ and $C_2A_1^{-1}$ belong to $G$. It can also be shown that

$$(B_2A_1^{-1})(C_1A_2^{-1})(C_2A_1^{-1})(b_2, a_1) = (d_1, a_1)$$

and

$$(A_1B_2^{-1})(A_1C_1^{-1})(A_1C_2^{-1})(A_1B_2^{-1})(a_2, a_1) = (d_2, a_1)$$

so that $G$ contains $D_1A_1^{-1}$ and $D_2A_1^{-1}$ (here, the curves $d_1$ and $d_2$ are shown in [9, Figure 1]). Also, we have

$$(D_2A_1^{-1})(A_1C_1^{-1}) = D_2C_1^{-1} \in G.$$ 

By similar arguments as in the proof of [9, Theorem 5], the lantern relation implies the following identity

$$A_3 = (A_2C_2^{-1})(D_1A_1^{-1})(D_2C_1^{-1}).$$

Since the subgroup $G$ contains each factor on the right hand side, the element $A_3$ belongs to $G$. It follows from

$$B_3 = A_3(B_3B_1^{-1})A_3(B_1B_3^{-1})A_3^{-1}$$
that $B_3$ is also contained in $G$. By conjugating $B_3$ with the powers of $T$, we get $A_1, B_1, C_1, \ldots, B_{r-1}, C_{r-1}$ and $B_r$ are all contained in $G$. Moreover,

$$A_2 = (A_2 A_1^{-1}) A_1 \in G.$$ 

Therefore, we conclude that $G = T_g$. \qed

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**Figure 5.** The involution $\sigma$ if $g = 2r + 1$.

**Figure 6.** The involution $\sigma$ if $g = 2r + 2$.

We consider the surface $N_g$ where $g$-crosscaps are distributed on the sphere as in Figure 5 and 6. If $g = 2r + 1 \geq 13$, there is a reflection, $\sigma$, of the surface $N_g$ in the $xy$-plane such that

- $\sigma(a_1) = f, \sigma(b_r) = c_{r-1}$,
- $\sigma(x_2) = x_3, \sigma(x_g) = x_{g-2}$ and
- $\sigma(x_i) = x_i$ if $i = 4, \ldots, g - 3$ or $i = 1, g - 1$.

with reverse orientation as in Figure 5. (Recall that $x_i$'s are the generators of $H_1(N_g; \mathbb{R})$ as shown in Figure 2.) If $g = 2r + 2 \geq 14$, there is a reflection, $\sigma$, of the surface $N_g$ in the $xy$-plane such that

- $\sigma(a_1) = f, \sigma(b_r) = d_r$,
- $\sigma(x_2) = x_3, \sigma(x_3) = x_4, \sigma(x_g) = x_{g-2}$ and
- $\sigma(x_i) = x_i$ if $i = 6, \ldots, g - 3$ or $i = 1, g - 1$.

with reverse orientation as in Figure 6. Note that in both cases the reflection $\sigma$ in $T_g$ since $D(\sigma) = 1$ for $g \geq 13$.

Now, for the remaining part of the paper, let $\Gamma_i$ denote the right handed Dehn twist about the curve $\gamma_i$ shown in Figure 1.
Theorem 2.4. For odd $g \geq 13$, the twist subgroup $T_g$ is generated by the three elements $T, \sigma$ and $\sigma\Gamma_{g-3}C_{\frac{g-2}{2}}^{-1}$.

Proof. Consider the surface $N_g$ as in Figure 5. Since

$$\sigma(\gamma_{g-3}) = \gamma_{g-3} \text{ and } \sigma(c_{\frac{g-2}{2}}) = c_{\frac{g-2}{2}},$$

and $\sigma$ reverses the orientation of a neighbourhood of a two-sided simple closed curve, we have

$$\sigma\Gamma_{g-3}\sigma = \Gamma_{g-3}^{-1} \text{ and } \sigma C_{\frac{g-2}{2}}^{-1}\sigma = C_{\frac{g-2}{2}}^{-1}.$$ 

Hence, it is easy to verify that $\sigma\Gamma_{g-3}C_{\frac{g-2}{2}}^{-1}$ is an involution. Let $H$ be the subgroup of $T_g$ generated by the following set

$$\{T, \sigma, \sigma\Gamma_{g-3}C_{\frac{g-2}{2}}^{-1}\},$$

where $g \geq 13$ and odd. It follows from Theorem 2.3 that we only need to prove that the elements $A_2^{-1}, B_1B_2^{-1}$ and $E$ are contained in the subgroup $H$. Since

$$\Gamma_{g-3}C_{\frac{g-2}{2}}^{-1} = (\sigma)(\sigma\Gamma_{g-3}C_{\frac{g-2}{2}}^{-1}),$$

the element $\Gamma_{g-3}C_{\frac{g-2}{2}}^{-1}$ belongs to $H$.

It follows from

- $T^{13-g}(\gamma_{g-3}, c_{\frac{g-2}{2}}) = (\gamma_{10}, c_2)$,
- $T^{-4}(\gamma_{10}, c_2) = (\gamma_{6}, a_1)$ and
- $T^2(\gamma_{6}, a_1) = (\gamma_{8}, c_1),$

that the elements $\Gamma_{10}C_{2}^{-1}, \Gamma_{6}A_{1}^{-1}$ and $\Gamma_{8}C_{1}^{-1}$ are in $H$. Since

$$\Gamma_{6}A_{1}^{-1})(C_{2}^{-1})(\gamma_{6}, a_1) = (c_2, a_1),$$

the element $C_2A_{1}^{-1}$ is in $H$. Since

- $(\Gamma_{6}A_{1}^{-1})(A_{1}C_{2}^{-1}) = \Gamma_{6}C_{2}^{-1} \in H$ and
- $T^4(\gamma_{6}, c_2) = (\gamma_{10}, c_4),$ \n
$\Gamma_{10}C_{4}^{-1}$ is contained in $H$. Also, we have

$$(C_{4}^{-1}C_{10}^{-1}) = C_{4}C_{2}^{-1} \in H.$$ 

Thus, $C_3C_{1}^{-1}, C_3C_{5}^{-1}, B_4B_2^{-1}$ and $C_2A_{1}^{-1}$ are contained in $H$ by conjugating $C_4C_{2}^{-1}$ with some powers of $T$. Then, since

$$(C_{4}^{-1}C_{2}^{-1})(B_{4}^{-1}B_{2}^{-1})(c_2, a_1) = (b_2, a_1),$$

$H$ contains $B_2A_{1}^{-1}$. Also, we get

$$(C_{2}^{-1}A_{1})(A_{1}B_{2}^{-1}) = C_{2}B_{2}^{-1} \in H.$$ 

Since

$$(C_{1}^{-1}B_{2}^{-1}B_{4}^{-1})(c_3, c_5) = (b_4, c_5),$$

$H$ contains $B_4C_{5}^{-1}$. Then,

$$T^{-4}(b_4, c_5) = (b_2, c_3)$$ 

implies that $B_2C_{3}^{-1} \in H$. It follows from the following equalities

- $C_{2}^{-1}C_{3}^{-1} = (C_{2}B_{2}^{-1})(B_{2}C_{3}^{-1})$ and
- $T^{-2}(b_2, c_3) = (b_1, b_2)$.
that $B_1B_2^{-1}$ belongs to $H$. On the other hand, since

1. $(\Gamma_8C_7^{-1})(C_1C_3^{-1})(C_3C_5^{-1})(C_5B_4^{-1}) = \Gamma_8B_4^{-1}$ and
2. $T^{-7}(\gamma_8, b_4) = (\gamma_1, a_1) = (a_2, a_1)$

that $A_1A_2^{-1}$ is contained in $H$.

Since the elements $T, A_1A_2^{-1}$ and $B_1B_2^{-1}$ are contained in the subgroup $H$, the generators $A_1, B_1, C_1, \ldots, B_{r-1}, C_{r-1}, B_r$ are in $H$ by the proof of Theorem 2.1. Moreover, $A_1\sigma(a_1) = A_1(f) = e$ leads to $E \in H$. This completes the proof.

**Theorem 2.5.** For even $g \geq 14$, the twist subgroup $T_g$ is generated by the three elements $T, \sigma$ and $\sigma\Gamma_{g-3}C_{z/2}^{-1}$.

**Proof.** Consider the surface $N_g$ as in Figure 6. Since

$$\sigma(\gamma_{g-3}) = \gamma_{g-3} \text{ and } \sigma(c_{z/2}) = c_{z/2},$$

and $\sigma$ reverses the orientation of a neighbourhood of a two-sided simple closed curve, we have

$$\sigma\Gamma_{g-3}\sigma = \Gamma_{g-3}^{-1} \text{ and } \sigma C_{z/2}\sigma = C_{z/2}^{-1}.$$

Hence, it is easy to show that $\sigma\Gamma_{g-3}C_{z/2}^{-1}$ is an involution. Let $K$ be the subgroup of $T_g$ generated by the following set

$$\{T, \sigma, \sigma\Gamma_{g-3}C_{z/2}^{-1}\},$$

where $g \geq 14$ and even.

Since

$$\Gamma_{g-3}C_{z/2}^{-1} = (\sigma)(\sigma\Gamma_{g-3}C_{z/2}^{-1}),$$

the element $\Gamma_{g-3}C_{z/2}^{-1}$ is contained in $K$.

It follows from

$$T^{13-g}(\gamma_{g-3}, c_{z/2}) = (\gamma_{10}, c_2)$$

that the elements $\Gamma_{10}C_{2}^{-1}$ is in the subgroup $K$. Recall that this element also appears in the proof of Theorem 2.4. The remaining part of the proof follows as in the proof of the previous theorem. Also, note that the element $D_r$ belongs to $K$ since $\sigma(b_r) = d_r$. This finishes the proof.

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