EXTENSIONS OF POSITIVE DEFINITE FUNCTIONS ON FREE GROUPS

M. BAKONYI AND D. TIMOTIN

Abstract. An analogue of Krein’s extension theorem is proved for operator-valued positive definite functions on free groups. The proof gives also the parametrization of all extensions by means of a generalized type of Szegő parameters. One singles out a distinguished completion, called central, which is related to quasi-multiplicative positive definite functions. An application is given to factorization of noncommutative polynomials.

1. Introduction

Positive definite functions are an important object of study in relation to group theory and $C^*$-algebras since the basic work of Godement and Eymard [20, 14]. In the case of abelian groups, Bochner’s theorem ensures that such a function is the Fourier transform of a positive measure on the dual group, and much of the theory develops along this line. On nonabelian groups Fourier analysis is no more available, but the focus is now on the relation to group representations. The theory of positive definite functions is more intricate; for some notable results, see also [8, 9, 12, 23].

The starting point of our investigation was the extension problem for positive definite functions. The earliest result seems to be Krein’s extension theorem [27], which says that every positive definite continuous scalar function on a real interval $(-a,a)$ admits a continuous positive definite extension to $\mathbb{R}$; a recent generalization to totally ordered groups can be found in [3].

The analogue of Krein’s theorem in $\mathbb{Z}^2$ is no more true: a positive definite function defined on a rectangle symmetric with respect to the origin may have no positive definite extension [33]. The sets with the property that every positive definite function defined on them can be extended are characterized in [5]. A good reference for extension results (including extensions from subgroups) is [34, Chapter 4].

In the current paper we consider positive definite functions on the most basic nonabelian group, namely the free group with an arbitrary number of generators. (For general facts about Fourier analysis on free groups, see [15].) We obtain an analogue of Krein’s theorem: a positive definite function defined on the set of words of length bounded by a fixed constant can be extended to the whole group. This is rather surprising in view of the result concerning $\mathbb{Z}^2$ discussed above; it is an illustration of a principle that has appeared recently, namely that noncommutative objects might behave better than their commutative analogues (see, for instance, [24]). Actually, our result is connected to [24] or [28], and in the last...
section we deduce a factorization theorem for certain polynomials in noncommutative variables. This is an area that has received much interest in the last years [28, 24, 25].

The extension property is closely related to a parametrization of all operator-valued positive definite functions by means of sequences of contractions. These are an analogue of the choice sequences of contractions, a generalization of the Szegő parameters in the theory of orthogonal polynomials [19], that has been developed by Foias and his coauthors in connection with intertwining liftings (see [16] and the references within). By using some nontrivial graph theory, we are able to parallel the theory of positive matrices as developed, for instance, in [10][11].

One should note that results in a close area of investigation have been obtained by Popescu [30, 31, 32]. Most notably, in [32] a similar extension problem is proved for the free semigroup, together with a description of all solutions. The group case that we have considered requires however new arguments.

The plan of the paper is the following: in Section 2 we present preliminary material concerning graphs and matrices. Section 3 introduces the free group, and proves a basic result about its Cayley graph. The most important part of the paper is Section 4, where the main theorems concerning the structure of positive definite functions and their extension properties are proved. In Section 5 one discusses the existence of a distinguished extension, which is called the central one; it plays a role similar to that of central lifting in the theory of intertwining liftings (or maximum entropy in data analysis)—see [6][17][18]. It is shown in Section 6 that some well known positive definite functions on the free group may be obtained as central extensions. Finally, in Section 7 we present the application to factorization of noncommutative polynomials.

2. Preliminaries

2.1. Graphs. A basic reference for graph theory is [7]. We consider undirected graphs $G = (V, E)$, where $V = V(G)$ denotes the set of vertices and $E = E(G)$ the set of edges. If $\{v, w\} \in E(G)$, we say that $v$ and $w$ are adjacent. $G$ is called complete if any two vertices are adjacent. $G' = (V', E')$ is a subgraph of $G$ if $V' \subset V$ and $E' \subset E$; we write then $G' \subset G$. If all edges of $G$ connecting two vertices in $V'$ are also edges of $G'$, we say that $G'$ is the subgraph of $G$ induced by $G'$; we will write in this case $G' = G|_{V'}$. A set $C \subset V(G)$ is a clique if the induced subgraph is complete.

A graph is called chordal if it contains no minimal cycles of length $> 3$. An induced subgraph of a chordal graph is chordal. We have the following basic result about chordal graphs [22].

Lemma 2.1. If $G$ is chordal, then for any two nonadjacent vertices $v_1, v_2 \in V(G)$ the set of vertices adjacent both to $v_1$ and to $v_2$ is a clique.

A tree is a connected graph with no cycles. If $G$ is a tree, and $v, w \in V(G)$, there is a unique path $P(v, w)$ joining $v$ and $w$ which passes at most once through each vertex. We will call it the minimal path joining $v$ and $w$, and define $d(v, w)$ to be its length (the number of edges it contains). For $n \geq 1$, we will denote by $\hat{G}_n$ the graph that has the same vertices as $G$, while $E(\hat{G}_n) = \{(v, w) : d(v, w) \leq n\}$ (in particular, $G = \hat{G}_1$).

Lemma 2.2. If $G$ is a tree, then $\hat{G}_n$ is chordal for any $n \geq 1$. 
Proof. Take a minimal cycle $C$ of length $> 3$ in $\hat{G}_n$. Suppose $x,y$ are elements of $C$ at a maximal distance. If $d(x,y) \leq n$, then $C$ is actually a clique, which is a contradiction. Thus $x$ and $y$ are not adjacent in $\hat{G}_n$. Suppose $v, w$ are the two vertices of $\hat{G}_n$ adjacent to $x$ in the cycle $C$. Now $P(x, v)$ has to pass through a vertex which is on $P(x, y)$, since otherwise the union of these two paths would be the minimal path connecting $y$ and $v$, and it would have length strictly larger than $d(x, y)$. Denote by $v_0$ the element of $P(x, v) \cap P(x, y)$ which has the largest distance to $x$; since $d(y, v) = d(y, v_0) + d(v_0, v) \leq d(y, x) = d(y, v_0) + d(v_0, x)$, it follows that $d(v_0, v) \leq d(v_0, x)$.

Similarly, if $w_0$ is the element of $P(x, w) \cap P(x, y)$ which has the largest distance to $x$, it follows that $d(w_0, w) \leq d(w_0, x)$.

Suppose now that $d(v_0, x) \leq d(w_0, x)$. Then
\[
d(v, w) = d(v, v_0) + d(v_0, w_0) + d(w_0, w)^
\leq d(x, v_0) + d(v_0, w_0) + d(w_0, w) = d(x, w) \leq n,
\]
since $w$ is adjacent to $x$. Then $(v, w) \in E$, and $C$ is not minimal: a contradiction. Thus $\hat{G}_n$ is chordal. \qed

Corollary 2.3. Suppose $G$ is a tree, and $x, y \in V(G)$, $d(x, y) = n + 1$. Then the set
\[
\{z \in V(G) : \max(d(z, x), d(z, y)) \leq n\}
\]
is a clique in $\hat{G}_n$.

Proof. We use Lemmas 2.1 and 2.2. \qed

2.2. Matrices and operators. Suppose $A = (A_{ij})_{i,j \in I}$ is a block operator matrix, with entries $A_{ij} \in L(\mathcal{H})$, and $A \geq 0$ (as an operator on $\bigoplus_{i \in I} \mathcal{H}$). There is an essentially unique Hilbert space, denoted by $\mathcal{R}(A)$, together with isometries $\omega_i(A) : \mathcal{H} \to \mathcal{R}(A)$, such that $A_{ij} = \omega_i^* \omega_j$. (If $\iota_i$ denotes the canonical embedding of $\mathcal{H}$ into the $i$th coordinate of $\bigoplus_{i \in I} \mathcal{H}$, we may take, for instance, $\mathcal{R}(A)$ to be the subspace of $\bigoplus_{i \in I} \mathcal{H}$ spanned by all $\omega_i(A)\mathcal{H}$, with $\omega_i(A) = A^{1/2}\iota_i$.) We will call $(\mathcal{R}(A), \omega_i(A))$ a realization of $A$.

If $J \subset I$, and $A' = A|J$ is the submatrix of $A$ obtained by taking only rows and columns in $J$, then one can embed isometrically $\mathcal{R}(A')$ into $\mathcal{R}(A)$, such that $\omega_i(A)$ is just $\omega_i(A')$ followed by this embedding.

Suppose now that we are given only certain entries $A_{ij} \in L(\mathcal{H})$ of a $p \times p$ block operator matrix, and we are interested in completing the remaining entries in order to obtain a positive matrix. We assume that all $A_{ii}$ ($i \in I$) are specified, and, since we are interested in symmetric matrices, that $A_{ij}$ and $A_{ji}$ are simultaneously specified. The matrix is called partially positive if all fully specified principal submatrices are positive semidefinite. A similar definition applies to infinite matrices.

The pattern of given entries can be specified by a graph $G$, whose set of vertices is $I$, while $\{i, j\}$ ($i \neq j$) is an edge if $A_{ij}$ is specified. An important role in completion problems is then played by chordal graphs: if $G$ is chordal, then every partially positive matrix with pattern defined by $G$ has a positive semidefinite completion. \cite{22}. A general reference for matrix completions is \cite{20}.

In Section 4 we will use similar arguments adapted to the group structure in order to extend positive definite functions on the free group. But we will only rely
on the following basic result concerning completions by a single element (see, for instance, [1] for a similar result).

**Lemma 2.4.** Suppose $A = (A_{ij})_{i,j \in I}$ is a partially positive block operator matrix whose corresponding pattern is the graph whose only missing edge is $\{k, l\}$. Then

(i) There exists a positive semidefinite completion of $A$.

(ii) All such completions are in one-to-one correspondence with contractions

$$\gamma : \mathcal{R}(A[I \setminus \{k\}] \oplus \mathcal{R}(A[I \setminus \{k, l\}]) \rightarrow \mathcal{R}(A[I \setminus \{l\}] \oplus \mathcal{R}(A[I \setminus \{k, l\}]).$$

The correspondence is given by associating to each completion $\hat{A}$ the contraction

$$\gamma = P_{\mathcal{R}(\hat{A}[I \setminus \{i\}] \oplus \mathcal{R}(\hat{A}[I \setminus \{k, l\}]}} \mathcal{R}(\hat{A}[I \setminus \{k\}] \oplus \mathcal{R}(\hat{A}[I \setminus \{k, l\}]).$$

In particular, there is a distinguished completion, called central, which corresponds to $\gamma = 0$.

**Remark 2.5.** Although we intend to avoid computational details related to matrix completion, below are sketched briefly some details concerning the correspondence stated in Lemma 2.4. Denote

$$\mathcal{E} = \mathcal{R}(A[I \setminus \{k, l\}], \quad \mathcal{F} = \mathcal{R}(A[I \setminus \{k\}] \oplus \mathcal{R}(A[I \setminus \{k, l\}],$$

$$\mathcal{G} = \mathcal{R}(A[I \setminus \{l\}] \oplus \mathcal{R}(A[I \setminus \{k, l\}],$$

and suppose $\omega_i : \mathcal{H} \rightarrow \mathcal{E}, i \in I \setminus \{k, l\}$ are the embeddings defining the realization space of $A[I \setminus \{k, l\}]$. We may identify $\mathcal{R}(A[I \setminus \{k\}])$ with $\mathcal{E} \oplus \mathcal{F}$: with respect to this decomposition the embeddings $\omega_i'$ are given by $\omega_i' = \left(\begin{array}{c} \alpha_i' \\ \beta_i' \end{array}\right)$ for $i \neq l$, while we will denote $\omega_i' = \left(\begin{array}{c} \alpha_i' \\ \beta_i' \end{array}\right)$. Similarly, $\mathcal{R}(A[I \setminus \{l\}])$ is identified with $\mathcal{E} \oplus \mathcal{G}$, and if $\omega_i''$ are the embeddings, then $\omega_i'' = \left(\begin{array}{c} \alpha_i'' \\ \beta_i'' \end{array}\right)$ for $i \neq k$, and $\omega_k'' = \left(\begin{array}{c} \alpha'' \\ \beta'' \end{array}\right)$.

Suppose now that we have a contraction $\gamma : \mathcal{F} \rightarrow \mathcal{G}$, and denote $D_\gamma = (I_\mathcal{F} - \gamma^* \gamma)^1/2$, and $D_\gamma = D_\gamma^* \mathcal{F}$. A concrete form for the representing space of the associate completion $\hat{A}$ is then

$$\mathcal{R}(\hat{A}) = \mathcal{E} \oplus \mathcal{G} \oplus D_\gamma,$$

where the embeddings $\hat{\omega}_i : \mathcal{H} \rightarrow \mathcal{R}(\hat{A})$ are given by

$$\hat{\omega}_i = \left(\begin{array}{c} \omega_i \\ 0 \\ 0 \end{array}\right) \text{ for } i \neq k, l, \quad \hat{\omega}_k = \left(\begin{array}{c} \alpha'' \\ \beta'' \\ 0 \\ 0 \end{array}\right), \quad \hat{\omega}_l = \left(\begin{array}{c} \alpha' \\ \gamma \beta' \\ 0 \\ 0 \end{array}\right).$$

Consequently, the formula for the completed entry is

$$A_{kl} = \hat{\omega}_k^* \hat{\omega}_l = \alpha''^* \alpha' + \beta''^* \gamma \beta.$$

Alternate formulas can be obtained by “passing to the adjoint”: we have then $\mathcal{R}(\hat{A}) = \mathcal{E} \oplus \mathcal{F} \oplus D_\gamma^*$, with corresponding embeddings. The essential uniqueness of the space $\mathcal{R}(\hat{A})$ is reflected by a unitary $U_\gamma : \mathcal{E} \oplus \mathcal{G} \oplus D_\gamma^* \rightarrow \mathcal{E} \oplus \mathcal{G} \oplus D_\gamma$ intertwining the two embeddings; this is given by

$$U_\gamma = \left(\begin{array}{ccc} I_\mathcal{E} & 0 & 0 \\ 0 & \gamma & D_\gamma^* \\ 0 & D_\gamma & -\gamma^* \end{array}\right).$$
The operator appearing in the lower right corner is the well known “Julia operator” related to \( \gamma \).

3. Graphs and groups

We consider in the sequel the group \( \Gamma = F_m \), the free group with \( m \) generators \( a_1, \ldots, a_m \). Denote by \( e \) the unit and by \( A \) the set of generators of \( F \). Elements in \( \Gamma \) are therefore words \( s = b_1 \ldots b_n \), with letters \( b_i \in A \cup A^{-1} \); each word is usually written in its reduced form, obtained after cancelling all products \( aa^{-1} \).

The length of a word \( s \), denoted by \( |s| \), is the number of generators which appear in (the reduced form of) \( s \). For a positive integer \( n \), define \( S_n \) to be the set of all words of length \( \leq n \) in \( \Gamma \) and \( S_n \subset S_n \) the set of words of length exactly \( n \); the number of elements in \( S_n \) is \( 2m(2m - 1)^{n-1} \).

There is a notion that we will use repeatedly, and so we prefer to introduce a common beginning by induction:

\[
\text{CB}(s_1, \ldots, s_p) = \text{CB}(\text{CB}(s_1, \ldots, s_{p-1}), s_p).
\]

It can be seen that \( \text{CB}(s_1, \ldots, s_p) \) does not depend on the order of the elements \( s_1, \ldots, s_p \), and is formed by their first common letters.

We include for completeness the proof of the following lemma.

**Lemma 3.1.** If \( s \neq e \), then \( |s^2| > |s| \); in particular, \( s^2 \neq e \).

**Proof.** When we write the word \( s^2 \), it may happen that some of the letters at the beginning of \( s \) cancel with some at the end of \( s \). Let us group the former in \( s_1 \) and the latter in \( s_3 \); thus \( s = s_1s_2s_3 \), with \( s_i \) in reduced form, \( s_3s_1 = e \), and the reduced form of \( s^2 \) is \( s_1s_2^2s_3 \). We have then \( |s| = |s_1| + |s_2| + |s_3| \) and \( |s^2| = |s_1| + 2|s_2| + |s_3| \). If \( |s^2| \leq |s| \), we must have \( s_2 = e \). Therefore \( s = s_1s_3 = e \). \( \square \)

A total order \( \preceq \) on \( \Gamma \) will be called **lexicographic** if, for two elements \( s_1, s_2 \in \Gamma \), we have \( s_1 \preceq s_2 \) whenever one of the following holds:

1. \( |s_1| < |s_2| \);
2. \( |s_1| = |s_2| \), and, if \( t = \text{CB}(s_1, s_2) \), and \( a_i \) is the first letter of \( t^{-1}s_i \) (for \( i = 1, 2 \)), then \( a_1 \preceq a_2 \).

We will write \( s < t \) if \( s \preceq t \) and \( s \neq t \). One can see that lexicographic orders are in one-to-one correspondence with their restrictions to \( A \cup A^{-1} \). For the remainder of this section and for Section 4 a lexicographic order \( \preceq \) on \( \Gamma \) will be fixed.

Let \( \sim \) be the equivalence relation on \( \Gamma \) obtained by having the equivalence classes \( \hat{s} = \{ s, s^{-1} \} \). We will denote also by \( \preceq \) the order relation on \( \hat{\Gamma} = \Gamma / \sim \) defined by \( \hat{s} \preceq \hat{t} \) iff \( \min\{s, s^{-1}\} \preceq \min\{t, t^{-1}\} \). If \( \nu \in \hat{\Gamma} \), then \( \nu^- \) and \( \nu^+ \) will be respectively the predecessor and the successor of \( \nu \) with respect to \( \preceq \).

There is a graph \( \Gamma \) naturally associated to \( \Gamma \): the Cayley graph corresponding to the set of generators \( A \). Namely, \( V(\Gamma) \) are the elements of \( \Gamma \), while \( s \) and \( t \) are connected by an edge if \( |s^{-1}t| = 1 \). Moreover, \( \Gamma \) is easily seen to be a tree: any cycle would correspond to a nontrivial relation satisfied by the generators of the group.

The distance \( d \) between vertices of a tree defined in Section 3 is \( d(s, t) = |s^{-1}t| \).
As a consequence of Lemma 3.2, $\hat{\Gamma}_n$ is chordal for any $n \geq 1$. We will introduce a sequence of intermediate graphs $\Gamma_\nu$, with $\nu \in \mathbb{F}$, as follows. We have $V(\Gamma_\nu) = \mathbb{F}$ for all $\nu$, while $\{s,t\} \in E(\Gamma_\nu)$ iff $\tilde{s}^{-1}t = \nu'$ for some $\nu' \leq \nu$. Obviously $\Gamma_\nu \subseteq \Gamma_{\nu'}$ for $\nu \leq \nu'$, and $E(\Gamma_{\nu'})$ is obtained by adding to $E(\Gamma_\nu)$ all edges $\{s,t\}$ with $\tilde{s}^{-1}t = \nu^+$. Each $\Gamma_\nu$ is invariant with respect to translations, and $\hat{\Gamma}_n = \Gamma_{\nu_n}$, where $\nu_n$ is the last element in $\mathbb{S}_n$.

The next proposition is the main technical ingredient of the paper.

**Proposition 3.2.** With the above notation, $\Gamma_\nu$ is chordal for all $\nu$.

**Proof.** If $\Gamma_\nu$ is not chordal, and $n = |\nu|$, we may assume that $\nu$ is the last element in $\mathbb{F}$ of length $n$ with this property. Since $\mathbb{F}_n$ is chordal, $\nu \neq \nu_n$, and thus $|\nu^+| = n$.

Suppose then that $\Gamma_\nu$ contains the cycle $(s_1, \ldots, s_q)$, with $q \geq 4$. At least one of $\{s_1,s_3\}$ or $\{s_2,s_4\}$ must be an edge of $\Gamma_{\nu^+}$, since otherwise $\{s_1,s_2,s_3,s_4\}$ is a part of a cycle of length $\geq 4$ in $\Gamma_{\nu^+}$. We may assume that $\{s_1,s_3\} \in V(\Gamma_{\nu^+})$; if we denote $t = s_1^{-1}s_3$, then, since $\{s_1,s_3\} \notin V(\Gamma_\nu)$, we must have $t = \nu^+$.

Suppose that $q > 4$. Then $\{s_1,s_4\} \notin V(\Gamma_\nu)$. If $\{s_1,s_4\} \in V(\Gamma_{\nu^+})$, then $s_1^{-1}s_4 = \nu^+$, and thus either $s_1^{-1}s_4 = t$ or $s_1^{-1}s_4 = t^{-1}$. The first equality is impossible since it implies $s_3 = s_4$. As for the second, it would lead to $t^2 = s_4^{-1}s_3$. But $|t| = |\nu^+| = n$, and thus, by Lemma 3.1, $|s_4^{-1}s_3| = |t^2| > n$. This contradicts $\{s_1,s_3\} \in V(\Gamma_\nu)$; consequently, we must have $q = 4$.

Performing, if necessary, a translation, we may suppose that the four cycle is $(e,s,t,r)$, and that $\{e,t\} \in E(\Gamma_{\nu^+})$, (thus $t \in \nu^+$, and $t \prec t^{-1}$). Thus, $e,s,r,t$ are all different, $\{e,s\}, \{s,t\}, \{r,t\}, \{e,r\}$ are edges of $G_\nu$, while $\{e,t\}, \{s,r\}$ are not, and $\{e,t\}$ is an edge of $G_{\nu^+}$. These assumptions imply that:

\begin{equation}
|t| = n, \ |s| \leq n, \ |r| \leq n, \ |t^{-1}s| \leq n, \ |t^{-1}r| \leq n, \ |r^{-1}s| \geq n.
\end{equation}

Let us denote $u = CB(s,r,t)$. We have $r = ur_1, t = ut_1, s = us_1$, and at least two among the elements $CB(s_1,r_1), CB(r_1,t_1), CB(s_1,t_1)$ are equal to $e$.

Suppose first that this happens with $(s_1,t_1)$ and $(r_1,t_1)$. If $v = CB(s_1,r_1)$, then $s_1 = v s_2, r_1 = v r_2$. The inequalities (3.1) imply

\begin{equation}
|u| + |t_1| = n,
\end{equation}

\begin{equation}
|u| + |v| + |s_2| \leq n, \ |u| + |v| + |r_2| \leq n,
\end{equation}

\begin{equation}
|t_1| + |v| + |s_2| \leq n, \ |t_1| + |v| + |r_2| \leq n,
\end{equation}

\begin{equation}
|s_2| + |r_2| \geq n.
\end{equation}

Replacing $|t_1| = n - |u|$ from (3.2), one obtains from (3.3) $|v| + |s_2| \leq |u|, |v| + |r_2| \leq |u|$. Then (3.3) imply $|v| + |s_2| \leq n/2, |v| + |r_2| \leq n/2$. Comparing the last two inequalities with (3.5), one obtains $|v| = 0$. This means that all pairs $(s_1,r_1), (r_1,t_1)$ and $(s_1,t_1)$ have as common beginning $e$, and so we may as well assume from the start that $(s_1,r_1)$ and $(s_1,t_1)$ have as common beginning $e$ (the case $(s_1,r_1)$ and $(r_1,t_1)$ is symmetrical, and can be treated likewise; note that the whole situation is symmetric in $s$ and $r$).

Then we will assume in the sequel that $w = CB(t_1,r_1)$; thus $r_1 = wr_2$ and $t_1 = wt_2$ (see Figure 1). We have then

\begin{equation}
|u| + |w| + |t_2| = n, \ |u| + |w| + |r_2| \leq n, \ |s_1| + |w| + |t_2| \leq n,
\end{equation}
Suppose \( s = us_1 \)

\[
\begin{array}{cccccc}
& o & s = us_1 & \ \ \ & \ & \ \\
\vdots & & s_1 & \ \ \ & \ & \ \\
\ & e & \ldots & w & uw & t_2 \\
\ & u & \ldots & o & \ & \ \\
\ & \ & \ & r_2 & \ & \ \\
\ & \ & \ & o & r = uwr_2 \ \\
\end{array}
\]

\( \text{Figure 1.} \)

whence \( |s_1| \leq |u|, |r_2| \leq |t_2| \). But, since \( \{s, r\} \) is not an edge of \( \Gamma_\nu \), we have \( |s_1| + |w| + |r_2| \geq n \). Comparing this last inequality with the first inequality in (3.6), it follows that

\[ |s_1| = |u|, \quad |r_2| = |t_2| . \]

Now, \( s^{-1}t \) is a word of length \( n \) different from \( t \); thus \( s_1^{-1}w \neq uw \), and \( |s_1^{-1}w| = |uw| \).

On the other hand, \( t \in \nu^+ \) begins with \( uw \), and \( t \preceq t^{-1} \). Suppose \( t' \in \mathcal{F} \) with \( |t'| = n \), begins with \( t_1 \), with \( |t_1| = |uw| \). If \( t_1' \prec uw \), then the definition of the lexicographic order implies that \( t' \prec t \) and \( t' \prec t \). Applying this argument to \( t' = s^{-1}r \), \( t_1' = s_1^{-1}w \), it follows that, if \( s_1^{-1}w \prec uw \), then \( s^{-1}r \prec \nu^+ \). Then \( s^{-1}r \prec \nu \), and therefore \( \{s, r\} \in E(\Gamma_\nu) \): a contradiction. Therefore \( uw \prec s_1^{-1}w \), and \( t \prec s^{-1}t \).

Since, however, \( \{s, t\} \in \Gamma_\nu \), it follows that \( t^{-1}s \preceq \nu \), and thus

\[ t^{-1}s \preceq t \preceq t^{-1} . \]

Also, \( \{1, s\} \in E(\Gamma_\nu) \) implies \( |s| = |u| + |s_1| = 2|u| \leq n \), and thus \( |u| \leq n/2 \). Therefore \( |t_2^{-1}w^{-1}| = |wt_2| \geq n/2 \).

Now \( t_2^{-1}w^{-1} = \text{CB}(t^{-1}s, t^{-1}) \), and (3.7) implies that \( t_2^{-1}w^{-1} \) is also the beginning of \( t \). Thus \( \text{CB}(t, t^{-1}) \) has length \( t \geq n/2 \). Writing then \( t^2 = (t^{-1})^{-1}t \), \( \text{CB}(t, t^{-1}) \) cancels, and we obtain \( |t^2| \leq 2n - 2t \leq n = |t| \). By Lemma 3.1 this would imply \( t = e \): a contradiction, since the elements \( e, s, r, t \) of the assumed 4-cycle must be distinct. The proposition is proved.

If \( \nu \in \bar{\mathcal{F}} \), then there exist cliques in \( \Gamma_\nu^+ \) which are not cliques in \( \Gamma_\nu \); we may start with any edge of \( \Gamma_\nu^+ \) which is not an edge of \( \Gamma_\nu \) and take a maximal clique \( \text{in } \Gamma_\nu^+ \) which contains it. Such a clique is necessarily finite, since the length of edges is bounded by \( |\nu^+| \).

**Corollary 3.3.** Suppose \( \nu \in \bar{\mathcal{F}} \), and \( C \) is a clique in \( \Gamma_\nu^+ \) which is not a clique in \( \Gamma_\nu \). Then \( C \) contains a single edge in \( \Gamma_\nu^+ \) which is not an edge in \( \Gamma_\nu \).

**Proof.** Suppose \( C \) contains \( \{s, t\}, \{s', t'\} \) with \( s^{-1}t = s'^{-1}t' = v \in \nu^+ \). Obviously \( s \neq s' \) and \( t \neq t' \). If \( s = t' \), then \( s'^{-1}t = v^2 \), and thus the edge \( \{s', t\} \) has length
strictly larger than $n = |v|$ by Lemma 3.4. One shows similarly that $s' \neq t$. Then $(s, t, t', s')$ is a 4-cycle in $\Gamma_\nu$, which contradicts Proposition 3.2.

4. Positive definite functions

A function $\Phi : F \to L(\mathcal{H})$ is positive definite if and for every $s_1, \ldots, s_k \in F$ the operator matrix $A(\Phi; \{s_1, \ldots, s_k\}) := [\Phi(s_i^{-1}s_j)]_{i,j=1}^k$ is positive semidefinite.

In general, for a finite set $S \subset F$, we will use the notation $A(\Phi; S) := [\Phi(s_i^{-1}s_j)]_{s,t \in S}$. There is here a slight abuse of notation, since the matrix in the right hand side of the equality depends on the order of the elements of $S$, and changing the order amounts to intertwining the rows and columns of $A(\Phi; S)$; however, the reader can easily check that this ambiguity is irrelevant in all instances where this notation is used below.

Positive definite functions are connected to representations of $F$. The relation is given by Naimark’s Dilation Theorem [29, 34].

Naimark’s Theorem. The function $\Phi : F \to L(\mathcal{H})$ is positive definite if and only if there exists a representation $\pi$ of $F$ on a Hilbert space $\mathcal{K}$, and an operator $V : \mathcal{H} \to \mathcal{K}$, such that, for all $s, t \in F$, $\Phi(s^{-1}t) = V^* \pi(s^{-1}t)V$.

Such a representation $\pi$ is called a dilation of $\Phi$; it is uniquely determined, up to unitary isomorphism, by the minimality condition $\mathcal{K} = \bigvee_{s \in F} \pi(s)VH$.

It is immediate from the definition of positive definiteness that $\Phi(e)$ is a positive operator. A standard argument shows that, if $\mathcal{H}_0 = \overline{\Phi(e)\mathcal{H}}$, then there exists a positive definite function $\Phi_0 : F \to L(\mathcal{H}_0)$, such that $\Phi_0(e) = I_{\mathcal{H}_0}$, and $\Phi(s)h = \Phi_0(s)\Phi(e)^{1/2}h$ for all $h \in \mathcal{H}$. We will suppose in the sequel that $\Phi(e) = I_{\mathcal{H}}$; equivalently, the operator $V$ in Naimark’s Theorem is an isometry.

We will also consider positive definite functions defined on subsets of $F$. If $\Sigma \subset F$ such that $\Sigma = \Sigma^{-1}$, then a function $\varphi : \Sigma \to L(\mathcal{H})$ is called positive definite if for every $s_1, \ldots, s_k \in F$ such that $s_i^{-1}s_j \in \Sigma$ for every $i, j = 1, \ldots, k$, the operator matrix $A(\Phi; \{s_1, \ldots, s_k\})$ is positive semidefinite. Obviously, if $\Phi : F \to L(\mathcal{H})$ is positive definite, then $\Phi|\Sigma$ is positive definite for all $\Sigma \subset F$.

Remember now that we have a fixed lexicographic order on $\mathring{F}$, and consequently on $F$. Define $\Sigma_\nu = \bigcup_{\nu \preceq \nu'} \nu'$; the following lemma is then obvious, on closer inspection.

Lemma 4.1. Suppose $\mu \in F$, and $\varphi : \Sigma_\mu \to L(\mathcal{H})$ is positive definite. Then the map $\Phi \mapsto (\Phi|\Sigma_\nu)_{\nu \leq \mu}$ defines a one-to-one correspondence between:

1) positive definite functions on $F$ whose restriction to $\Sigma_\mu$ is $\varphi$; and

2) sequences $(\varphi_\nu)_{\nu \in \mathcal{F}, \mu \leq \nu}$, $\varphi_\nu$ positive definite function defined on $\Sigma_\nu$, such that, if $\nu \preceq \nu'$, then $\varphi_\nu = \varphi_\nu'|\Sigma_\nu$.

Note that the lemma may be applied to the case $\mu = \{e\}$ and $\varphi(e) = I_{\mathcal{H}}$, when we obtain the correspondence between a positive definite function on the whole group $F$ and the sequence of its restrictions.

Let us now fix $\nu \in \mathring{F}, \nu \neq \{e\}$; thus $\nu = \{s_\nu, s_\nu^{-1}\}$, with $s_\nu \preceq s_\nu^{-1}$. Suppose that $\varphi_\nu$ is a positive definite function on $\Sigma_\nu$. Consider the maximal clique $C_\nu$ in $\Gamma_\nu$ that contains $\{e, s_\nu\}$; by Corollary 3.3, $\{e, s_\nu\}$ is the unique edge of $C_\nu$ that is not
in \( E(\Gamma_{\nu^-}) \). Consequently, the following definitions make sense:

\[
\begin{align*}
D(\varphi_{\nu^-}) &= R(A(\varphi_{\nu^-}; C_{\nu} \setminus \{s_{\nu}\})) \ominus R(A(\varphi_{\nu^-}; C_{\nu} \setminus \{e, \nu\})) \\
R(\varphi_{\nu^-}) &= R(A(\varphi_{\nu^-}; C_{\nu} \setminus \{e\})) \ominus R(A(\varphi_{\nu^-}; C_{\nu} \setminus \{e, \nu\}))
\end{align*}
\] (4.1)

**Lemma 4.2.** With the above notations, if \( \varphi_{\nu^-} \) is a positive definite function on \( \Sigma_{\nu^-} \), then all positive extensions \( \varphi_{\nu} \) of \( \varphi_{\nu^-} \) are in one-to-one correspondence with contractions \( \gamma_{\nu} : D(\varphi_{\nu^-}) \to R(\varphi_{\nu^-}) \).

**Proof.** Note first that

\[
(\varphi_{\nu^-}(s^{-1}t))_{\{s,t\} \subset C_{\nu}, \{s,t\} \in E(\Gamma_{\nu^-})}
\]

is a partially positive matrix whose corresponding pattern is the graph induced by \( \Gamma_{\nu^-} \) on \( C_{\nu} \). By applying Lemma 2.4, we obtain that there exists a positive definite completion of this matrix, and, moreover, that all completions are in one-to-one correspondence with contractions \( \gamma_{\nu} : D(\varphi_{\nu^-}) \to R(\varphi_{\nu^-}) \). Denote the completed entry by \( B(\gamma_{\nu}) \).

Now, any maximal clique in \( \Gamma_{\nu} \) that is not a clique in \( \Gamma_{\nu^-} \) is a translate of \( C_{\nu} \), and the corresponding partially defined matrix determined by \( \varphi_{\nu^-} \) is the same. If we define then

\[
\varphi_{\nu}(s) = \begin{cases} 
\varphi_{\nu^-}(s) & \text{for } s \in \Sigma_{\nu^-}, \\
B(\gamma_{\nu}) & \text{for } s = s_{\nu}, \\
B(\gamma_{\nu})^* & \text{for } s = s_{\nu}^{-1}
\end{cases}
\]

we obtain a positive definite function on \( \Sigma_{\nu} \) that extends \( \varphi_{\nu^-} \). It is easy to see that this correspondence is one-to-one. \( \blacksquare \)

In particular, such extensions exist, and thus any positive definite function on \( \Sigma_{\nu^-} \) can be extended to a positive definite function on \( \Sigma_{\nu} \). Among them there exists a distinguished one, the **central extension**, obtained by taking \( \gamma_{\nu} = 0 \).

Now, combining Lemmas 4.1 and 4.2, we obtain an extension theorem for positive definite functions.

**Theorem 4.3.** If \( \varphi : \Sigma_{\nu} \to \mathcal{L}(\mathcal{H}) \) is a positive definite function, then \( \varphi \) has a positive definite extension \( \Phi : \mathbb{F} \to \mathcal{L}(\mathcal{H}) \). There is a one-to-one correspondence between the set of all positive extensions and the set of sequences of contractions \( (\gamma_{\mu})_{\nu \leq \mu} \), where \( \gamma_{\mu} : D(\Phi|\Sigma_{\mu^-}) \to R(\Phi|\Sigma_{\mu^-}) \).

Note that in this correspondence \( \Phi|\Sigma_{\mu} \) depends only on \( \gamma_{\mu'} \) with \( \nu \leq \mu' \leq \mu \), and thus the domain and range of \( \gamma_{\mu} \) are well defined by the previous \( \gamma_{\mu'} \). Then at step \( \mu \) one can choose \( \gamma_{\mu} \) to be an arbitrary element of the corresponding operator unit ball. In particular, one obtains the **central extension** by choosing at each step \( \gamma_{\mu} = 0 \).

The most important consequence is an extension result for positive definite functions on \( S_n \). It should be noted that, contrary to \( \Sigma_{\nu} \), the sets \( S_n \) do not depend on the particular lexicographic order \( \leq \).

**Proposition 4.4.** Every positive definite function \( \varphi \) on \( S_n \) has a positive definite extension \( \Phi \) on \( \mathbb{F} \). If \( \varphi \) is radial (that is, \( \varphi(s) \) depends only on \( |s| \)), then one can choose also \( \Phi \) radial.
Corollary 4.6. Suppose they have order $N$. Theorem 4.3, analogue to the choice sequences can thus obtain explicit, but complicated formulas for $M$ will call elements. Denote by $\gamma$ defect spaces of some formulas identifying the realization spaces appearing in (4.1) with direct sums of details. However, using (2.1), (2.2) and (2.3), one can also write down precise formulas for $\gamma$ since $\gamma$ is a representation $\pi$ such that, for all $s,t \in S$, such that $s^{-1}t$ has length $\leq n$, we have

$$\varphi(s^{-1}t) = V^{*}\pi(s^{-1})V.$$  

There is an alternate way of looking at Theorem 4.3 by using Naimark’s Theorem.  

Corollary 4.6. Suppose $\varphi : S_n \to \mathcal{L}(\mathcal{H})$ is a positive definite function. Then there exists a representation $\pi$ of $\mathbb{F}$ on a Hilbert space $\mathcal{K}$, and an operator $V : \mathcal{H} \to \mathcal{K}$, such that, for any $s,t \in \mathbb{F}$ such that $s^{-1}t$ has length $\leq n$, we have

$$\varphi(s^{-1}t) = V^{*}\pi(s^{-1})V.$$  

Proof: The first part is an immediate consequence of Theorem 4.3, obtained by taking $\Sigma_{\mu} = S_n$. Suppose $\Phi$ is positive definite; it is proved in 16. 3.1 that, if $\bar{\Phi}(s)$ is the average of $\Phi$ on words of length $|s|$, then $\bar{\Phi}$ is also positive definite. The second part of the proposition follows.  

Another consequence is the parametrization of all positive definite functions.

Theorem 4.5. There is a one-to-one correspondence between the set of all positive definite functions $\Phi : \mathbb{F} \to \mathcal{L}(\mathcal{H})$ and the set of sequences of contractions $(\gamma_{\mu})_{\mu \neq (e)}$, where $\gamma_{\mu} : D(\Phi | \Sigma_{\mu^{-}}) \to R(\Phi | \Sigma_{\mu^{-}})$.

The parametrization obtained is an analogue, for the free group, of the structure theory for classical Toeplitz matrices as presented, for instance, in 10. We have chosen here to concentrate on the conceptual meaning, avoiding computational details. However, using (2.1), (2.2) and (2.3), one can also write down precise formulas identifying the realization spaces appearing in (4.1) with direct sums of defect spaces of some $\gamma_{\mu}$‘s, with $\mu \preceq \mu^{-}$, while repeated applications of (2.4) yield unitaries connecting the different expressions for the same realization space. One can thus obtain explicit, but complicated formulas for $D(\Phi | \Sigma_{\mu^{-}})$ and $R(\Phi | \Sigma_{\mu^{-}})$, that would allow a recurrent definition of the set of all families $\gamma_{\mu}$ appearing in Theorem 4.3, analogue to the choice sequences of the classical case [16].

We may also rephrase Theorem 4.5 by using Naimark’s Theorem.

Corollary 4.6. Suppose $\varphi : S_n \to \mathcal{L}(\mathcal{H})$ is a positive definite function. Then there exists a representation $\pi$ of $\mathbb{F}$ on a Hilbert space $\mathcal{K}$, and an operator $V : \mathcal{H} \to \mathcal{K}$, such that, for any $s,t \in \mathbb{F}$ such that $s^{-1}t$ has length $\leq n$, we have

$$\varphi(s^{-1}t) = V^{*}\pi(s^{-1})V.$$  

There is an alternate way of looking at Theorem 4.3 that will be useful in Section 7. Denote by $\mathfrak{M}_n(\mathcal{H})$ the block operator matrices indexed by elements in $S_n$; they have order $N(n)$ equal to the cardinality of $S_n$. To any function $\varphi$ on $S_{2n}$ corresponds then a matrix $M(\varphi) \in \mathfrak{M}_n(\mathcal{H})$, defined by $M(\varphi)_{\epsilon,\eta} = \varphi(s^{-1}t)$. We will call elements $M = M_{\epsilon,\eta} \in \mathfrak{M}_n$ whose entries depend only on $s^{-1}t$ Toeplitz, and denote by $\mathfrak{T}_n$ the space of Toeplitz matrices in $\mathfrak{M}_n$. It is spanned by the matrices $\epsilon(\sigma), \sigma \in S_n$, where $\epsilon(\sigma)_{\epsilon,\eta} = \delta_{\sigma,\eta}$. The above definition shows that any $M_{\varphi}$ is Toeplitz. The converse is also true, as shown by the next Lemma.

Lemma 4.7. If $M \in \mathfrak{M}_n$ is a Toeplitz matrix, then there exists a uniquely defined $\varphi : S_{2n} \to \mathcal{L}(\mathcal{H})$, such that $M = M_{\varphi}$.

Proof: If $s \in S_{2n}$, we may write $s = t^{-1}r$, with $t,r \in S_n$. We define then $\varphi(s) = M_{t,r}$. The Toeplitz condition implies that this definition does not depend on the particular decomposition of $s$. In order to show that $\varphi$ is indeed positive definite, take $S \in \mathbb{F}$, such that for any $s,t \in S$, $s^{-1}t \in S_{2n}$. By adding elements to $S$, if necessary, we may assume that the maximum of the lengths $|s^{-1}t|$, for $s,t \in S$, is actually equal to $2n$. Take then $s_0,t_0 \in S$, such that $|s_0^{-1}t_0| = n$, and let $\hat{s}$ be the “middle” of the path (in the Cayley graph of $\mathbb{F}$) going from $s_0$ to $t_0$; that is, $\hat{s}$ is uniquely defined by the conditions $|s_0^{-1}\hat{s}| = |s_0^{-1}t_0| = n$. Using the tree structure of the Cayley graph, it is then easy to see that $|s_0^{-1}\hat{s}| \leq n$ for any $s \in S$. So

$$A(\varphi : S) = [\varphi(t^{-1}r)]_{t,r \in \hat{s}^{-1}S}$$
is a submatrix of
\[ M = [\varphi(t^{-1}r)]_{t,r \in S_n}, \]
and is thus positive definite. The uniqueness of \( \varphi \) is immediate. \( \square \)

**Corollary 4.8.** If \( T \in \mathfrak{S}_n \), then there exists a representation \( \pi \) of \( F \) and an operator \( V \), such that
\[ T_{s,t} = V^* \pi(s^{-1}t)V. \]

**Remark 4.9.** Theorems 4.3 and 4.5 and their corollaries extend to the case when \( F \) is a free group with an infinite number of generators. The main ideas are similar, but the details are more cumbersome, since we have to use in several instances transfinite induction, along the lines of [35].

5. The central extension

Suppose \( \varphi : S_n \to \mathcal{L}(\mathcal{H}) \) is positive definite. We have defined in Section 4, for a fixed lexicographic order \( \preceq \) on \( F \), the central extension of \( \varphi \) to the whole of \( F \). We intend to prove in this section that this central extension does not actually depend on the lexicographic order.

If \( \Phi : F \to \mathcal{L}(\mathcal{H}) \) is positive definite, denote \( \Phi_m = \Phi|_{S_m} \). As in Lemma 4.1, the map \( \Phi \mapsto (\Phi|_{S_m})_{m\geq n} \) gives a one-to-one correspondence between positive definite extensions \( \Phi \) of \( \varphi \) and sequences \((\Phi_m)_{m\geq n}\), with the property that \( \Phi_n = \varphi \), and, if \( m \leq m' \), then \( \Phi_{m'}|_{S_m} = \Phi_m \). Then, in order to show that the central extension does not depend on the order, it is enough to show that \( \Phi_{n+1} \) does not depend, since then an induction argument finishes the proof.

If \( \Phi \) is an extension of \( \varphi \), suppose \( (K_\Phi, \pi_\Phi, V_\Phi) \) is its minimal Naimark dilation. For \( \Sigma \subset F \), denote
\[ K_\Phi(\Sigma) = \bigvee_{\sigma \in \Sigma} \pi_\Phi(\sigma)V_\Phi \mathcal{H}. \]

Note, for further use, that for any \( s \in F \), \( \Sigma \subset F \) we have
\[ K_\Phi(s\Sigma) = \pi_\Phi(s)K_\Phi(\Sigma). \]

We will say that \( \Phi \) is a maximal \( n+1 \) orthogonal extension of \( \varphi \) if the following property is satisfied: whenever \( s, t \in F \), \( |s^{-1}t| = n+1 \), and
\[ \Sigma = \{ r \in F : \max(|r^{-1}s|, |r^{-1}t|) \leq n \}, \]
we have
\[ (K_\Phi(\Sigma \cup \{s\}) \ominus K_\Phi(\Sigma)) \perp (K_\Phi(\Sigma \cup \{t\}) \ominus K_\Phi(\Sigma)). \]
Note that it is enough to check this property for \( s = e \). Also, (5.2) is equivalent to any of the equations
\[ (K_\Phi(\Sigma \cup \{s\}) \ominus K_\Phi(\Sigma)) \perp (K_\Phi(\Sigma \cup \{t\}) \ominus K_\Phi(\Sigma)); \]
\[ (K_\Phi(\Sigma \cup \{t\}) \ominus K_\Phi(\Sigma)) \perp (K_\Phi(\Sigma \cup \{s\}) \ominus K_\Phi(\Sigma)). \]

**Lemma 5.1.** If \( \Phi, \Phi' \) are two maximal \( n+1 \) orthogonal extensions of \( \varphi \), then \( \Phi_{n+1} = \Phi'_{n+1} \).
Proof. Take $s \in \mathbb{F}$, with $|s| = n + 1$. We have to show that $\Phi(s) = \Phi'(s)$.

Denote by $(K, \pi, V)$, $(K', \pi', V')$ the minimal Naimark dilations of $\Phi, \Phi'$ respectively. If

$$\Sigma = \{r \in \mathbb{F} : |r| \leq n, |r^{-1}s| \leq n\},$$

let $P_\Sigma, Q_e, Q_s$ be the orthogonal projections (in $K$) onto $K(\Sigma), K(\Sigma \cup \{e\}) \ominus K(\Sigma)$, and $K(\Sigma \cup \{s\}) \ominus K(\Sigma)$ respectively, and similarly $P_{\Sigma}', Q_e', Q_s'$. The maximal orthogonality assumption (5.2) implies that $P_\Sigma, Q_e, Q_s$ are mutually orthogonal projections, as well as $P_{\Sigma}', Q_e', Q_s'$.

By Corollary 2.3 for any $r, t \in \Sigma \cup \{s\}$ we have $|r^{-1}t| \leq n$, and thus $\Phi(r^{-1}t) = \Phi'(r^{-1}t) = \varphi(r^{-1}t)$. Thus there is a unitary operator $\Omega : K(\Sigma \cup \{s\}) \rightarrow K'(\Sigma \cup \{s\})$, such that $\Omega \pi(r)V = \pi'(r)V'$ for $r \in \Sigma \cup \{s\}$. A similar argument gives the existence of $\Xi : K(\Sigma \cup \{e\}) \rightarrow K'(\Sigma \cup \{e\})$, such that $\Xi \pi(r)V = \pi'(r)V'$ for $r \in \Sigma \cup \{e\}$. Moreover, $\Omega K(\Sigma) = \Xi K(\Sigma)$, and $\Omega P_\Sigma \Omega^* = P_{\Sigma'}, \Xi P_\Sigma \Xi^* = P_{\Sigma'}$.

Now, suppose $h \in H$. We have $V = (P_\Sigma + Q_e)V$ and $\pi(s)V = (P_\Sigma + Q_s)\pi(s)V$, whence

$$\Phi(s) = V^* \pi(s)V h = V^* (P_\Sigma + Q_e)(P_\Sigma + Q_s)\pi(s)V = V^* P_{\Sigma} \pi(s)V$$

$$= V^* \Xi \Xi P_{\Sigma} \Xi^* \Omega P_{\Sigma} \Omega^* \Omega \pi(s)V = V^* P_{\Sigma} \pi'(s)V' = \Phi'(s).$$

The proof is finished. □

Proposition 5.2. The central extension of $\varphi$ corresponding to a given lexicographic order is maximal $n + 1$ orthogonal.

Proof. We have to prove (5.5) for all pairs $\{s, t\}$. We will use induction with respect to $\nu \in \bar{\mathbb{F}}^+$ (note that the lexicographic total order $\preceq$ is fixed): assuming that (5.5) is true whenever $s^{-1}t < \nu = \{r, r^{-1}\}$, we will show that it is also true when $s^{-1}t = \nu$. As noted above, this is equivalent to show it for $s = e, t = r$.

Let us then fix, for the rest of this proof, $r \in \mathbb{F}$, and assume $|r| = n + 1$. Denote by $C$ the set of elements which are at distance at most $n$ to both $e$ and $r$ and by $D$ the set of elements which are adjacent to $e$ and $r$ in $\Gamma_{\nu^{-}}$: they are both cliques in $\Gamma_{\nu^{-}}$: $C$ by Corollary 2.3 (note that $\Gamma_{\nu^{-}}$ is a subgraph of $\Gamma_{\nu^{-}}$), and $D$ by Lemma 2.1 and Proposition 5.2. We know by the definition of the central extension with respect to the given lexicographic order that

$$\left(\mathcal{K}(D \cup \{e\}) \ominus \mathcal{K}(D)\right) \perp \left(\mathcal{K}(D \cup \{r\}) \ominus \mathcal{K}(D)\right),$$

and we want to show that

$$\left(\mathcal{K}(C \cup \{e\}) \ominus \mathcal{K}(C)\right) \perp \left(\mathcal{K}(C \cup \{r\}) \ominus \mathcal{K}(C)\right)$$

Obviously $C \subset D$; and for each $s \in D \setminus C$ we have either $|s| = n + 1$ or $|r^{-1}s| = n + 1$. Let us denote $S_1 = \{s \in D \cup \{e\} : |r^{-1}s| = n + 1\}$, $S_2 = \{s \in D \cup \{r\} : |s| = n + 1\}$.

We need another notation, also relative to the pair $\{e, r\}$. For any $s \in \mathbb{F}$, we define an element $\beta(s)$, as follows. In the Cayley graph of $\mathbb{F}$, suppose $P, P_1, P_2$ are the minimal paths connecting $e$ and $r$, $e$ and $s$, and $s$ and $r$ respectively. If $P \cap P_1 = \{e\}$, we put $\beta(s) = e$, and if $P \cap P_2 = \{r\}$, then $\beta(s) = r$. (Note that at most one of these equalities can be true, since otherwise the union $P \cup P_1 \cup P_2$ would be a cycle.) In the remaining case the element on $P \cap P_1$ farthest from $e$ is also the element on $P \cap P_2$ farthest from $r$; we will denote it by $\beta(s)$. One sees that $\beta : \mathbb{F} \rightarrow P$, and $\beta(s) = s$ for all $s \in P$. 


Suppose $s \in D$; if $\beta(s) = e$, then $|r^{-1}s| > n + 1$, which is a contradiction. Similarly one cannot have $\beta(s) = r$, and therefore $\beta(s) \in P \setminus \{e, r\}$.

Let us now take $t \in C$, and $s \in S$. Suppose, for instance, that $\beta(t)$ is closer to $r$ than $\beta(s)$. Since $|s^{-1}r| = |s^{-1}\beta(s)| + |\beta(s)^{-1}r|$, $|r| = |\beta(s)| + |\beta(s)^{-1}r|$, and $|s^{-1}r| \leq n + 1 = |r|$, we have $|s^{-1}\beta(s)| \leq |\beta(s)|$. Then

$$|s^{-1}t| = |s^{-1}\beta(s)| + |\beta(s)^{-1}t| \leq |\beta(s)| + |\beta(s)^{-1}t| = |t| = n.$$ 

Therefore $t \in C$ implies that $|t^{-1}s| \leq n$ for all $s \in D \cup \{e, r\}$.

Take now $s_1 \in S_1$, $s_2 \in S_2$. Since

$$|s_1^{-1}\beta(s_1)| + |\beta(s_1)^{-1}r| = |s_1^{-1}r| = n + 1 = |r| = |\beta(s_1)| + |\beta(s_1)^{-1}r|,$$

we have $|s_1^{-1}\beta(s_1)| = |\beta(s_1)|$. Similarly, $|s_2^{-1}\beta(s_2)| = |r^{-1}\beta(s_2)|$. Also, $\beta(s_1)$ is closer to $e$ than $\beta(r)$. Therefore

$$|s_1^{-1}s_2| = |s_1^{-1}\beta(s_1)| + |\beta(s_1)^{-1}\beta(s_2)| + |\beta(s_2)^{-1}r|$$

$$= |\beta(s_1)| + |\beta(s_1)^{-1}\beta(s_2)| + |r^{-1}\beta(s_2)| = |r| = n + 1.$$ 

Suppose now that there exists a vertex $t \notin C$, such that $|t^{-1}s_1| \leq n$ and $|t^{-1}s_2| \leq n$. As shown above for elements of $D$, in this case also $\beta(t) \in P\{e, r\}$. If $\beta(s_1)$ is closer to $e$ than $\beta(t)$, or if $\beta(s_1) = \beta(t)$ then

$$|t| = |\beta(s_1)| + |\beta(s_1)^{-1}\beta(t)| + |\beta(t)^{-1}t|$$

$$= |s_1^{-1}\beta(s_1)| + |\beta(s_1)^{-1}\beta(t)| + |\beta(t)^{-1}t| = |s_1^{-1}t| = n.$$ 

In case $\beta(t)$ is closer to $e$ than $\beta(s_1)$, we obtain $|w| < n$.

Similarly, one proves that $|r^{-1}t| \leq n$. Consequently, $t \in C$. It follows then that

$$C = \{t \in F : d(t,s_1) \leq n, \ d(t,s_2) \leq n\}.$$ 

Now, if the pair $\{s_1, s_2\}$ is different from $\{e, r\}$, then, since it is an edge of $\Gamma_{r\rightarrow e}$, we must have $s_1^{-1}s_2 = \mu$ for some $\mu \preceq \nu$. By applying the induction hypothesis, it follows that

\begin{equation}
(5.7) \quad (K(C \cup \{s_1\}) \ominus K(C)) \perp (K(C \cup \{s_2\}) \ominus K(C)).
\end{equation}

Therefore $(K(S_1 \cup C) \ominus K(C)) \perp (K(S_2 \cup C) \ominus K(C))$, and thus

\begin{equation}
(5.8) \quad K(D) = K(C) \ominus (K(S_1 \cup C) \ominus K(C)) \ominus (K(S_2 \cup C) \ominus K(C))
\end{equation}

Since $e \in S_1$, (5.7) implies in particular that $K(\{e\}) \perp K(S_2 \cup C) \ominus K(C)$. If $\xi \in K(\{e\})$, we can write the orthogonal sum

$$\xi = \xi_\perp + \xi_C + \xi_1,$$

with $\xi_\perp \perp K(D)$, $\xi_C \in K(C)$, and $\xi_1 \in K(S_1 \cup C) \ominus K(C)$.

Again by (5.7), using the fact that $r \in S_2$, we have

$$K(\{r\}) \perp (K(S_1 \cup C) \ominus K(C)),$$

and thus $K(\{r\}) \perp \xi_1$. On the other hand, the centrality condition says that $\xi_\perp \perp \xi$. Therefore

$$\xi - \xi_C = \xi_\perp + \xi_1 \perp K(\{r\}),$$

which is exactly what have to prove. \qed
As a consequence of Lemma 5.1 and Proposition 5.2 we obtain the main result of this section.

**Theorem 5.3.** The central extension of \( \varphi : S_n \to \mathcal{L}(\mathcal{H}) \) does not depend on the lexicographic order considered and is maximal orthogonal.

We may then speak about the central extension of a positive definite function defined on words of finite length, with no reference to a lexicographic order on \( \mathbb{F} \). Note that central liftings and central extensions have been extensively studied (see, for instance, [6, 17, 21])

6. Quasi-multiplicative functions

Haagerup has considered in [23] the functions \( s \mapsto e^{-|s|} \), and has proved that they are positive definite for \( t > 0 \). In [12] a larger class is defined: a function \( u : \mathbb{F} \to \mathbb{C} \) is called a Haagerup function if \( u(e) = 1, |u(s)| \leq 1, u(s^{-1}) = \overline{u(s)} \), and \( u(st) = u(s)u(t) \) whenever \( |st| = |s| + |t| \); it is proved therein that any Haagerup function is positive definite. The result is generalized in [8], where Božek introduces the analogue operator-valued functions (called quasi-multiplicative) and proves that they are positive definite. Thus, we say that \( \Phi : \mathbb{F} \to \mathcal{L}(\mathcal{H}) \) is quasi-multiplicative if \( \Phi(e) = I, \|\Phi(s)\| \leq 1, \Phi(s^{-1}) = \Phi(s)^* \), and \( \Phi(st) = \Phi(s)\Phi(t) \) whenever \( |st| = |s| + |t| \).

One may say more about these functions in our context. We start with a preparatory lemma.

**Lemma 6.1.** Suppose \( \Phi : \mathbb{F} \to \mathcal{L}(\mathcal{H}) \) is quasi-multiplicative, and \((\mathcal{K}, \pi, V)\) is the minimal Naimark dilation of \( \Phi \). If \( |st| = |s| + |t| \), then

\[
\langle \mathcal{K}(\{e\}) \rangle \ominus \mathcal{K}(\{s\}) \perp \mathcal{K}(\{st\}).
\]

**Proof.** Take \( \xi \in \mathcal{K}(\{e\}) \) and \( \eta \in \mathcal{K}(\{st\}) \). By definition \( \xi = Vh \) and \( \eta = \pi(st)Vk \) for some \( h, k \in \mathcal{H} \), and we have

\[
\langle \xi, \eta \rangle = \langle Vh, \pi(st)Vk \rangle = \langle h, V^*\pi(st)Vk \rangle = \langle h, \Phi(st)k \rangle.
\]

On the other hand, the orthogonal projection \( P_s \) onto \( \mathcal{K}(\{s\}) \) is given by the formula \( P_s = \pi(s)VV^*\pi(s)^* \). Therefore

\[
\langle P_s\xi, P_s\eta \rangle = \langle \pi(s)VV^*\pi(s)^*Vh, \pi(s)VV^*\pi(s)^*\pi(st)Vk \rangle
=
\langle V^*\pi(s)^*Vh, V^*\pi(s)^*\pi(st)Vk \rangle
=
\langle V^*\pi(s)^*Vh, V^*\pi(s)VtVk \rangle
=
\langle \Phi(s)^*h, \Phi(t)k \rangle = \langle h, \Phi(s)\Phi(t)k \rangle = \langle h, \Phi(st)k \rangle
\]

(we have applied quasi-multiplicity for the last equality).

Comparing the last relation with (5.2), it follows that \( \langle \xi, \eta \rangle = \langle P_s\xi, P_s\eta \rangle \). Since

\[
\langle \xi, \eta \rangle = \langle P_{\mathcal{K}(\{s\})}\xi, P_{\mathcal{K}(\{s\})}\eta \rangle + \langle \xi - P_{\mathcal{K}(\{s\})}\xi, \eta - P_{\mathcal{K}(\{s\})}\eta \rangle,
\]

we obtain \( \langle \xi - P_{\mathcal{K}(\{s\})}\xi, \eta - P_{\mathcal{K}(\{s\})}\eta \rangle = 0 \). But the space on the left hand side of (5.11) is spanned by the vectors \( \xi - P_{\mathcal{K}(\{s\})}\xi \), with \( \xi \in \mathcal{K}(\{e\}) \); thus the lemma is proved.

One sees easily that a function \( \varphi : S_1 \to \mathcal{L}(\mathcal{H}) \) with \( \varphi(e) = I \) is positive definite iff \( \varphi(s^{-1}) = \varphi(s)^* \) and \( |\varphi(s)| \leq 1 \).

**Theorem 6.2.** A quasi-multiplicative function \( \Phi : \mathbb{F} \to \mathcal{L}(\mathcal{H}) \) is the central extension of its restriction to \( S_1 \).
Proof. Denote by $\varphi$ the restriction of $\Phi$ to $S_1$, and by $(K, \pi, V)$ the minimal Naimark dilation of $\Phi$. According to Proposition 5.2 we have to prove that $\Phi$ is maximal $n + 1$ orthogonal for each $n \geq 1$.

Consider then $s, t \in F$, with $|s^{-1}t| = n + 1$, and suppose $\Sigma$ is defined by (5.1). We will show that equality (5.3) is true.

If $P$ is the minimal path connecting $s$ and $t$, suppose $x$ is the element on $P$ between $s$ and $t$ adjacent to $s$. For any $r \in \Sigma \cup \{t\}$ we have $|s^{-1}r| = |s^{-1}x| + |x^{-1}r|$. Then Lemma 6.1 implies that
\[
(K(\{e, s^{-1}x\}) \ominus K(\{s^{-1}x\})) \perp K(\{s^{-1}r\}).
\]
Since $(K(\{s, x\}) \ominus K(\{sx\})) = \pi(s)(K(\{e, s^{-1}x\}) \ominus K(\{s^{-1}x\}))$ and $\pi(s)(K(\{s^{-1}r\})) = K(\{r\})$, it follows that
\[
(K(\{s, x\}) \ominus K(\{x\})) \perp K(\{r\}).
\]
Denote by $P_x$ the orthogonal projection onto $K(\{x\})$ and by $P_{S_x}$ the orthogonal projection onto $K(\Sigma)$. The last equality says that for any $\xi \in K(\{s\})$ we have $\xi - P_{S_x} \xi \perp K(\{r\})$. Therefore
\[
(6.3) \quad \xi - P_{S_x} \xi \perp K(\{s\} \cup \{t\}).
\]
In particular $\xi - P_{S_x} \xi \perp K(\Sigma)$; since obviously $P_{S_x} \xi \in K(\Sigma)$, it follows that $P_{S_x} \xi = P_{S_x} \xi$. But then (6.3) says that $\xi - P_{S_x} \xi \perp K(\Sigma \cup \{t\})$. Since the vectors of the form $\xi - P_{S_x} \xi$, with $\xi \in K(\{s\})$, span $K(\{s\} \cup \Sigma) \ominus K(\Sigma)$, relation (6.3) follows; this ends the proof of the theorem.

7. Noncommutative Factorization

We will apply in this section the above result to noncommutative factorization problems, in the line of [28] and [24]. It is worth mentioning that the connection between extension problems of positive definite functions and factorization has been noted and used in the commutative case already in [33]. The relation in the noncommutative case appears in [28] and [24]. The technique used below is adapted from [25]. A few preliminaries are in order.

The elements of $F$ can also be considered as “monomials” in the indeterminates $X_1, \ldots, X_m$ and $X_1^{-1}, \ldots, X_m^{-1}$; the monomial $X(s)$ corresponding to $s \in F$ is obtained by replacing $a_i$ by $X_i$ and $a_i^{-1}$ by $X_i^{-1}$. It is then possible to consider also polynomials in these indeterminates; that is, formal finite sums
\[
(7.1) \quad p(X) = \sum_s A_s X(s),
\]
where we assume the coefficients $A_s$ to be operators on a fixed Hilbert space $\mathcal{C}$. We denote by $\deg(p)$ (the degree of $p$) the maximum length of the words appearing in the sum in (7.1). We introduce also an involution on polynomials by defining, for $p$ as in (7.1), $p(X)^* = \sum_s A_s^* X(s^{-1})$.

If $U_1, \ldots, U_m$ are unitary (not necessarily commuting) operators acting on a separable Hilbert space $\mathcal{X}$, then, for $s \in F$, $U(s)$ is the operator on $\mathcal{X}$ obtained by replacing $X_i$ with $U_i$, and $X_i^{-1}$ by $U_i^{-1}$. Then $p(U) \in \mathcal{L}(\mathcal{C} \otimes \mathcal{X})$ is defined by $p(U) = \sum_s A_s \otimes U(s)$. We say that such a polynomial is positive if, for any choice of unitary operators $U_1, \ldots, U_m$, the operator $p(U)$ is positive.
Theorem 7.1. If \( p \) is a positive polynomial, then there exists an auxiliary Hilbert space \( \mathcal{E} \) and operators \( B_s : \mathcal{C} \to \mathcal{E} \), defined for words \( s \) of length \( \leq \deg(p) \) such that, if \( q(X) = \sum_s B_s X(s) \), then
\[
(7.2) \quad p(X) = q(X)^* q(X).
\]

Proof. As noted above, the proof follows the line of Theorem 0.1 in [28]; the argument uses at a crucial point Theorem 4.3.

Denote \( d = \deg(p) \), and consider the space of Toeplitz matrices \( \mathfrak{T}_d \subset \mathfrak{M}_d \); it is a finite dimensional operator system. We define a map \( \psi : \mathfrak{T}_d \to \mathcal{L}(\mathcal{C}) \) by the formula \( \psi(\varepsilon(\sigma)) = A_\sigma \).

Suppose that an element \( T \in \mathfrak{T}_d \) is a positive matrix. By considering \( T = \tau_{s^{-1} t} \) as an element of \( \mathfrak{T}_d(M_k) \), we can apply Corollary 4.8 and obtain a representation \( \pi_F : \mathfrak{F} \to \mathcal{L}(\mathfrak{H}_\pi) \) and an operator \( V_F : \mathbb{C}^k \to \mathfrak{H}_\pi \), such that
\[
\tau_{s^{-1} t} = V_\phi^* \pi_F(s^{-1} t)V_\phi.
\]

Therefore
\[
(\psi \otimes 1_k)(T) = (\psi \otimes 1_k) \left( \sum_s \varepsilon(s) \otimes \tau_s \right) = \sum_s A_s \otimes \tau_s = (1_c \otimes V_\phi)^* \left( \sum_s A_h \otimes \pi_F(s) \right) (1_c \otimes V_\phi).
\]

Since \( p \) is positive, \( \left( \sum_s A_h \otimes \pi_F(s) \right) \) is a positive operator, and therefore the same is true about \( (\psi \otimes 1_k)(T) \). It follows then that \( \psi \) is completely positive. Applying Arveson’s Extension Theorem [2] [29], we can extend \( \psi \) to a completely positive map \( \tilde{\psi} : \mathfrak{M}_n \to \mathcal{L}(\mathcal{C}) \).

Suppose \( E_{s,t} \in \mathfrak{M}_n \) has 1 in the \((s, t)\) position and 0 everywhere else. The block operator matrix \( (E_{s,t})_{s,t} \) is positive; by Choi’s Theorem [29] ch. 3 \( \left( \tilde{\psi}(E_{s,t}) \right)_{s,t} \in \mathcal{L}(\bigoplus^N \mathcal{C}) \) is positive, where \( N = N(d) \). Therefore, there exist operators \( B_s : \mathcal{C} \to \bigoplus^N \mathcal{C} \), such that \( B_s^* B_t = \tilde{\psi}(E_{s,t}) \). Consequently
\[
A_x = \psi(\varepsilon(x)) = \psi(\sum_{x=s^{-1} t} E_{s,t}) = \sum_{x=s^{-1} t} \tilde{\psi}(E_{s,t}) = \sum_{x=s^{-1} t} B_s^* B_t.
\]

If we define \( q(X) = \sum_s B_s X(s) \), then the last equality is equivalent to \( p(X) = q(X)^* q(X) \). \( \blacksquare \)

Note that the factorization of \( p \) is usually not unique. The theorem produces a factor \( B \) which has degree at most equal to \( \deg(p) \). Also, in case \( \mathcal{C} \) is finite dimensional, the resulting space \( \mathcal{E} \) can also be taken finite dimensional, with \( \dim \mathcal{E} = N(d) \times \dim \mathcal{C} \).

One can rephrase the result of Theorem 7.1 as a decomposition into sum of squares (making thus the connection with [21]).

Corollary 7.2. If \( p \) is a positive polynomial, then there exist a finite number of polynomials \( Q_1, \ldots, Q_N \), with coefficients in \( \mathcal{L}(\mathcal{C}) \), such that
\[
(7.3) \quad p(X) = \sum_{j=1}^N Q_j(X)^* Q_j(X).
\]
Proof. Since $B_s : C \to \bigoplus^N C$, we can write $B_s = (B_s^{(1)} \ldots B_s^{(N)})^t$. We consider then $Q_j(X) = \sum_s B_s^{(j)} X(s)$, and obtain the required decomposition. 

It is worth comparing the factorization obtained in Theorem 7.1 with the results of [24] and [28]. In [24] the author considers real polynomials in the indeterminates $X_1, \ldots, X_m$ and $X_1^t, \ldots, X_m^t$; $p$ is called positive when any replacement of $X_i$ with real matrices $A_i$ and of $X_i^t$ with the transpose of $A_i$ leads to a positive semidefinite matrix. The main result is an analogue of decomposition (7.3) for such polynomials.

The analogue of Theorem 7.1 in [28] deals with the case when the words appearing in the polynomial are of the form $s_i^{-1} + t_i + s_i$, where $s_i, t_i$ are elements in the free semigroup with $m$ generators. The positivity condition replaces the indeterminates with unitary matrices, and the consequence is a corresponding decomposition.

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References

[1] Gr. Arsene, Z. Ceaușescu, T. Constantinescu, Schur analysis of some completion problems. Linear Algebra Appl. 109 (1988), 1–35.
[2] W.B. Arveson, Subalgebras of C*-algebras. Acta Math. 123 (1969), 141–224.
[3] M. Bakonyi, The extension of positive definite operator-valued functions defined on a symmetric interval of an ordered group. Proc. Amer. Math. Soc. 130 (2002), 1401–1406.
[4] M. Bakonyi, T. Constantinescu, Inheritance principles for chordal graphs. Linear Algebra Appl. 148 (1991), 125–143.
[5] M. Bakonyi, G. Naevdal, The finite subsets of Z^2 having the extension property. J. London Math. Soc. (2) 62 (2000), 904–916.
[6] H. Berovici, C. Foias, A. Frazho, Central commutant liftings in the coupling approach. Libertas Math. 14 (1994), 159–169.
[7] C. Berge, The theory of graphs. Dover Publications, Inc., Mineola, NY, 2001.
[8] M. Bożejko, Positive-definite kernels, length functions on groups and a noncommutative von Neumann inequality. Studia Math. 95 (1989), 107–118.
[9] M. Bożejko, R. Speicher, Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. Math. Ann. 300 (1994), 97–120.
[10] T. Constantinescu, On the structure of positive Toeplitz forms. Dilation Theory, Toeplitz Operators, and Other Topics, 127–149, Oper. Theory Adv. Appl., 11, Birkhäuser, Basel, 1983.
[11] T. Constantinescu, Schur analysis of positive block-matrices. I.Schur Methods in Operator Theory and Signal Processing, 191–206, Oper. Theory Adv. Appl., 18, Birkhäuser, Basel, 1986.
[12] L. De-Michele, A. Figá-Talamanca, Positive definite functions on free groups. Amer. J. Math. 102 (1980), 503–509.
[13] J. Dixmier, Les C*-algèbres et leurs représentations. Gauthier-Villars, Paris, 1969.
[14] P. Eymard, L’algèbre de Fourier d’un groupe localement compact. Bull. Soc. Math. France 92 (1964), 181–236.
[15] A. Figá-Talamanca, M.A. Picardello, Harmonic Analysis on Free Groups. Marcel Dekker, New York, 1983.
[16] C. Foias, A.E. Frazho, The Commutant Lifting Approach To Interpolation Problems. Operator Theory: Advances and Applications 44, Birkhäuser Verlag, Basel, 1990.
[17] C. Foias, A.E. Frazho, I. Gohberg, Central intertwining lifting, maximum entropy and their permanence. Integral Equations Operator Theory 18 (1994), 166–201.
[18] C. Foias, A.E. Frazho, I. Gohberg, M.A. Kaashoek, Metric constrained interpolation, commutant lifting and systems. Operator Theory: Advances and Applications 100, Birkhäuser Verlag, Basel, 1998.
[19] Yu. L. Geronimus, Orthogonal Polynomials. Consultants Bureau, New York, 1961.
[20] R. Godement, Les fonctions de type positif et la théorie des groupes. Trans. Amer. Math. Soc. 63 (1948), 1–84.
[21] I. Gohberg, M.A. Kaashoek, H.J. Woerdeman, A maximum entropy principle in the general framework of the band method. J. Funct. Anal. 95 (1991), 231–254.
[22] R. Grone, Ch.R. Johnson, E.M. de Sá, H. Wolkowicz, Positive definite completions of partial Hermitian matrices. Linear Algebra Appl. 58 (1984), 109–124.
[23] U. Haagerup, An example of a nonnuclear C∗-algebra, which has the metric approximation property. Invent. Math. 50 (1978/79), 279–293.
[24] W.J. Helton, “Positive” noncommutative polynomials are sums of squares. Annals of Math. 156 (2002), 675–694.
[25] W.J. Helton, S.A. McCullough, A Positivstellensatz for non-commutative polynomials. Trans. Amer. Math. Soc. 356 (2004), 3721–3737.
[26] Ch.R. Johnson, Matrix completion problems: a survey. Matrix theory and applications (Phoenix, AZ, 1989), 171–198, Proc. Sympos. Appl. Math., 40, Amer. Math. Soc., Providence, RI, 1990.
[27] M.G. Krein: Sur le problème de prolongement des functions hermitiennes positives et continues. Dokl. Akad. Nauk. SSSR, 26 (1940), 17–22.
[28] S. McCullough, Factorization of operator-valued polynomials in several non-commuting variables. Linear Algebra Appl. 326 (2001), 193–203.
[29] V.I. Paulsen, Completely Bounded Maps and Dilations. Longman, New York, 1986.
[30] Gelu Popescu, Positive-definite functions on free semigroups. Canad. J. Math. 48 (1996), 887–896.
[31] Gelu Popescu, Structure and entropy for Toeplitz kernels. C. R. Acad. Sci. Paris Sr. I Math. 329 (1999), 129–134.
[32] Gelu Popescu, Structure and entropy for positive-definite Toeplitz kernels on free semigroups. J. Math. Anal. Appl. 254 (2001), 191–218.
[33] W. Rudin, The extension problem for positive-definite functions. Illinois J. Math. 7 (1963), 532–539.
[34] Z. Sasvári, Positive Definite and Definitizable Functions. Akademie Verlag, Berlin, 1994.
[35] D. Timotin, Completions of matrices and the commutant lifting theorem. J. Funct. Anal. 104 (1992), 291–298.