Abstract. For a Lipschitz $\mathbb{Z}$-periodic function $\phi : \mathbb{R} \to \mathbb{R}^2$ satisfied that $\mathbb{R}^2 \setminus \{\phi(x) : x \in \mathbb{R}\}$ is not connected, an integer $b \geq 2$ and $\lambda \in (c/b^2, 1)$, we prove the following for the generalized Weierstrass-type function $W(x) = \sum_{n=0}^{\infty} \lambda^n \phi(b^n x)$: the box dimension of its graph is equal to $3 + 2 \log_b \lambda$, where $c$ is a constant depending on $\phi$.

1. Introduction

This paper concerns the box dimension of the graphs of the generalized Weierstrass-type functions

\begin{equation}
W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \phi(b^n x), \quad x \in \mathbb{R}
\end{equation}

where $b > 1$, $1/b < \lambda < 1$ and $\phi(x) : \mathbb{R} \to \mathbb{R}^d$ is a non-constant $\mathbb{Z}$-periodic Lipschitz function. The first famous example

$$\sum_{n=0}^{\infty} \lambda^n \cos(b^n x)$$

was introduced by Weierstrass as a example of a continuous but nowhere differentiable function, see [2]. The graphs of $W_{\lambda,b}$ and related functions were studied as fractal curves in fractal geometry starting from the work of Besicovitch and Ursell [3].

For $d = 1$ it is easy to check that $W_{\lambda,b}$ exhibits approximate self-affine with scales $\lambda$ and $b$, which suggests that its dimension should be equal to

$$D_1 = 2 + \frac{\log \lambda}{\log b}.$$ 

Indeed, Kaplan et al. [4] proved that in the case that $\phi$ is a trigonometric polynomial, either $W_{\lambda,b}$ is a $C^1$ curve or the box dimension of the graph of $W_{\lambda,b}$ is equal to $D_1$ (see also [5] [6]). In the literature there have been many works on or related to the Hausdorff dimension of the graph of the
Weierstrass type functions (see e.g. [8, 5, 9, 10, 11, 12, 18, 13, 14, 16, 17]). Recently, for the case φ is real analytic, thanks to the new theories in [11, 18], Shen and the author [19] proved the following dichotomy for $W_{\lambda, b}^\phi$:

Either $W_{\lambda, b}^\phi$ is real analytic, or the Hausdorff dimension of its graph is equal to $D_1$.

In this paper we consider the case $d = 2$, i.e. $\phi : \mathbb{R} \to \mathbb{R}^2$ is a Lipschitz $\mathbb{Z}$-periodic function. By the similar observation as $d = 1$, it is natural to conjecture that, unless $W_{\lambda, b}^\phi$ is Lipschitz, the box and Hausdorff dimension of its graph should be equal to

$$D_2 = \begin{cases} \frac{\log b}{\log \lambda}, & b\lambda^2 < 1; \\ 3 + 2\frac{\log \lambda}{\log b}, & b\lambda^2 \geq 1. \end{cases}$$

Barański [1] proved that for $\phi(x) = e^{2\pi i x}$ the box dimension of the graph of the function $W_{\lambda, b}^\phi$ is equal to $3 + 2\frac{\log \lambda}{\log b}$, provided $b \in \mathbb{N}$, $b \geq 2$ and $\lambda < 1$ is sufficiently close to 1. Inspired by the approach in [1], we generalize this result to more general functions $\phi$ by some planar topological methods. More precisely, we proved that

**Main Theorem.** Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be a $\mathbb{Z}$-periodic Lipschitz function. If the set $\mathbb{R}^2 \setminus \{ \phi(s) : s \in \mathbb{R} \}$ is not connected, then there is a number $c > 0$ depending only on $\phi$ such that the following holds. For any integer $b > 1$ and $0 < \lambda < 1$ with $b\lambda^2 > c$, the graph of $W_{\lambda, b}^\phi$ has box dimension equal to $3 + 2\frac{\log \lambda}{\log b}$.

However we can’t expect that the above $c$ equal to 1 for every such $\phi$. Since for any integer $b > 1$, $0 < \lambda < 1$ and $b\lambda^2 > 1$, we can consider $W_0(x) = e^{2\pi i x}$ and $\phi(x) = W_0(x) - \lambda W_0(bx)$, one has $W_{\lambda, b}^\phi(x) = W_0(x)$ and the set $\mathbb{R}^2 \setminus \{ \phi(s) : s \in \mathbb{R} \}$ is not connected. For the case $\phi(x) = e^{2\pi i x}$, we believe the box dimension of the graph of complex Weierstrass function $W_{\lambda, b}^{e^{2\pi i x}}$ is $3 + 2\frac{\log \lambda}{\log b}$ for $b\lambda^2 > 1$, unfortunately our method can only prove it for $b\lambda^2 > 8\pi$. For simplicity we just write $W_{\lambda, b}^\phi$ as $W$ in the rest of this paper.

2. Preliminaries

This section is devoted to give the basic formula for proving our Main Theorem. The observations are from [1], but we rewrite here for the reader’s convenience. Let $L = L(\phi)$ be the Lipschitz constant of $\phi$, which means

$$L = \sup_{a \neq b \in \mathbb{R}} \frac{|\phi(a) - \phi(b)|}{|a - b|}.$$ 

To give the specific value of the constant $c$ decided by $\phi$ in Main Theorem, we need the following notation, for any $t > 0$ and $z \in \mathbb{R}^2$, let
mean that $|z| \leq t$. For any $n \in \mathbb{N}$, $k = 0, 1, \ldots, b^n - 1$, let

$$z_{n,k} = \frac{k}{b^n}.$$ 

Let $\Lambda = \{0, 1, \ldots, b - 1\}$.

**Lemma 2.1.** For integer $n \geq 1$, $k = 0, 1, \ldots, b^n - 1$ and $j \in \Lambda$, we have

\begin{equation}
W(z_{n+1, kb+j}) - W(z_{n,k}) = \phi\left(\frac{j}{b}\right) - \phi(0) + O\left(\frac{L}{1 - \gamma}\right)
\end{equation}

\begin{equation}
W(z_{n+1, kb+j}) - W(z_{n,k}) = \phi\left(\frac{j}{b}\right) - \phi(0) + \lambda^{-1}\left(\phi\left(\frac{k}{b} + \frac{j}{b^n}\right) - \phi\left(\frac{k}{b}\right)\right) + O\left(\frac{L \gamma^2}{1 - \gamma}\right)
\end{equation}

where $\gamma = \frac{1}{b^n}$.

**Proof.** We only prove (2.1), since the proof of (2.2) is similar. By the definition of $W$ and the $\mathbb{Z}$-periodicity of $\phi$, we have

\begin{equation}
W(z_{n+1, kb+j}) - W(z_{n,k}) = \sum_{t=0}^{n} \lambda^t \left(\phi\left(\frac{kb + j}{b^n-t}\right) - \phi\left(\frac{k}{b^{n-t}}\right)\right)
\end{equation}

\begin{equation}
= \lambda^n \sum_{t=0}^{n} \lambda^{-t} \left(\phi\left(\frac{kb + j}{b^{n-t}+1}\right) - \phi\left(\frac{k}{b^t}\right)\right)
\end{equation}

\begin{equation}
= \lambda^n \left(\phi\left(\frac{j}{b}\right) - \phi(0) + O\left(\frac{L \gamma}{1 - \gamma}\right)\right).
\end{equation}

In the last equation we used that $\phi$ is a Lipschitz function for $t \geq 1$, which implies

$$\left|\phi\left(\frac{kb + j}{b^{n-t}+1}\right) - \phi\left(\frac{k}{b^t}\right)\right| \leq \frac{L}{b^t}.$$ \hfill \Box

3. **The Case When $\lambda^3 b > c_0$**

In this section, we will find a constant $c_0 > 0$ depending on $\phi$ and prove a result similar to the Main Theorem for $\lambda^3 b > c_0$, since our method can’t deal with the case $\lambda^2 b > c$ directly when $\lambda$ is sufficiently close to 1. For set $A, B \subset \mathbb{R}^2$ and $\mu \in \mathbb{R}^2$, let

$$A + B = \{a + b : a \in A, b \in B\}$$

$$\mu A = \{\mu a : a \in A\}.$$
We write $B(z, r)$ for the open disc centred at $z \in \mathbb{R}^2$ of radius $r$. For a function $f : E \subset \mathbb{R} \to \mathbb{R}^2$, let $\hat{f}$ be the image set of $f$ and let $\Delta(f)$ be the oscillation of $f$ defined as

$$\Delta(f) = \sup_{a, b \in E} |f(a) - f(b)|.$$ 

and

$$\varepsilon = \varepsilon(f) = \sup\{r > 0 : B(0, r) \subset (\mathbb{R}^2 \setminus \hat{f}) \setminus G_\infty\}$$

where $G_\infty$ is the unbounded component of $\mathbb{R}^2 \setminus \hat{f}$ (If the above set is empty, let $\varepsilon = 0$.)

**Lemma 3.1.** Let $\phi$ be a $\mathbb{Z}$-periodic complex Lipschitz function satisfied that $\hat{\phi}$ separates $B(0, \varepsilon)$ from $\infty$, then

$$\hat{\phi} + B(0, \varepsilon \lambda) \subset \bigcup_{s \in [0, 1)} \left(\phi(s) + \lambda \hat{\phi}\right) \forall \lambda \in (0, 1).$$

**Proof.** If (3.1) fails, then there exists $\lambda_0 \in (0, 1)$, $p_0 \in B(0, \varepsilon \lambda_0)$ and $s_0 \in [0, 1)$ such that

$$\phi(s_0) + p_0 \notin \bigcup_{s \in [0, 1)} \left(\phi(s) + \lambda_0 \hat{\phi}\right).$$

Thus

$$\frac{\phi(s_0) + p_0 - \phi(s)}{\lambda_0} \notin \hat{\phi} \quad \forall s \in [0, 1).$$

Since $\frac{\phi(s_0) + p_0 - \phi(s)}{\lambda_0}$ is a continuous function, the set of the image is contained in a connected component of $\mathbb{R}^2 \setminus \hat{\phi}$. Since we also observe that

$$\frac{\phi(s_0) + p_0 - \phi(s_0)}{\lambda_0} \in B(0, \varepsilon),$$

we have

$$\frac{\phi(s_0) + p_0 - \phi(s)}{\lambda_0} \in G \quad \forall s \in [0, 1)$$

where $G$ is the connected component of $\mathbb{R}^2 \setminus \hat{\phi}$ that contains 0 .

Since $G$ is bounded open set and $\partial G \subset \hat{\phi}$, (3.3) yield that

$$\Delta \left(\frac{\phi(s_0) + p_0 - \phi}{\lambda_0}\right) \leq \Delta(\phi).$$

By the definition of $\Delta$, we also have

$$\Delta \left(\frac{\phi(s_0) + p_0 - \phi}{\lambda_0}\right) = \frac{1}{\lambda_0} \Delta(\phi).$$

We combine this with (3.4) to get the contradiction.

\[\square\]
In the next lemma we apply lemma 3.1 repeatedly to find a specific open set contained in the image of the set \{W(s) : s \in \left[\frac{k}{b^n}, \frac{k+1}{b^n}\right]\} for integer \(n \geq 1\) and \(k = 0, 1, \ldots, b^n - 1\). The strategy of the proof is similar to Barański [1]. Let \(c_0 > 1\) be the solution of the following equation

\[
c_0 = \frac{L}{\varepsilon}\left(2 + \frac{1}{c_0 - 1}\right).
\]

**Lemma 3.2.** Assume \(\hat{\phi}\) separates \(B(0, \varepsilon)\) from \(\infty\). Let \(b \geq 2\) be an integer, \(\lambda \in (\frac{1}{b}, 1)\) and \(\lambda^3 b > c_0\). Then, for integer \(n \geq 1\) and \(k = 0, 1, \ldots, b^n - 1\), the following holds:

\[
W(z_{n,k}) = \frac{\lambda^n}{1 - \lambda}\phi(0) + \lambda^n(\hat{\phi} + B(0, \varepsilon, \lambda)) \subseteq \left\{\left.W(s) : s \in \left[\frac{k}{b^n}, \frac{k+1}{b^n}\right]\right\}.
\]

**Proof.** For \(s \in [0, 1]\), there exists \(j, s \in \Lambda \) such that \(|s - \frac{j_0}{b}| < \frac{1}{b}\). Thus

\[
\phi(s) \in \phi\left(\frac{j_0}{b}\right) + B\left(0, \frac{L}{b}\right).
\]

Combining (3.1) we obtain

\[
\hat{\phi} + B(0, \varepsilon, \lambda) \subseteq \bigcup_{j \in \Lambda} \left(\phi\left(\frac{j}{b}\right) + \lambda\phi + B\left(0, \frac{L}{b}\right)\right).
\]

Combining with (2.1), this implies

\[
\hat{\phi} + B(0, \varepsilon, \lambda) \subseteq \bigcup_{j \in \Lambda} \left(W(z_{n+1,kb+j}) - \frac{W(z_{n,k})}{\lambda^n} + \phi(0) + \lambda\phi + B\left(0, \frac{L}{b} + \frac{L\gamma}{1 - \gamma}\right)\right).
\]

Hence,

\[
W(z_{n,k}) + \lambda^n\hat{\phi} + B(0, \varepsilon\lambda^{n+1}) - \lambda^n\phi(0) \subseteq \bigcup_{j \in \Lambda} \left(W(z_{n+1,kb+j}) + \lambda^{n+1}\hat{\phi} + B\left(0, \left(\frac{1}{b} + \frac{\gamma}{1 - \gamma}\right)\lambda^{n}\right)\right).
\]

We also observe that

\[
bl^3 > c_0 = \frac{L}{\varepsilon}\left(2 + \frac{1}{c_0 - 1}\right) > \frac{L}{\varepsilon}\left(1 + \frac{1}{1 - \gamma}\right) > \frac{L}{\varepsilon}\left(\lambda + \frac{1}{1 - \gamma}\right),
\]

since \(\gamma < \frac{1}{bl} < \frac{1}{c_0}\) and \(\lambda \in (0, 1)\), which implies

\[
\varepsilon\lambda^2 > L\left(\frac{1}{b} + \frac{\gamma}{1 - \gamma}\right).
\]

We combine (3.7) and (3.8):

\[
W(z_{n,k}) + \lambda^n\hat{\phi} + B(0, \varepsilon\lambda^{n+1}) - \lambda^n\phi(0) \subset \bigcup_{j \in \Lambda} \left(W(z_{n+1,kb+j}) + \lambda^{n+1}\hat{\phi} + B(0, \varepsilon\lambda^{n+2})\right).
\]
Since (3.9) holds for any integer $n \geq 1$ and $k = 0, 1, \ldots, b^n - 1$, we also have
\begin{equation*}
W(z_{n+1,kb^j}) + \lambda^{n+1} \hat{\phi} + B(0, \varepsilon \lambda^{n+2}) - \lambda^{n+1} \phi(0) \subset \bigcup_{j \in A} \left( W(z_{n+2,kb^j+1}) + \lambda^{n+2} \hat{\phi} + B(0, \varepsilon \lambda^{n+3}) \right),
\end{equation*}
for any $j \in \Lambda$, which implies
\begin{equation*}
W(z_{n,k}) + \lambda^n \hat{\phi} + B(0, \varepsilon \lambda^{n+1}) - (\lambda^n + \lambda^{n+1}) \phi(0) \subset \bigcup_{j \in \{0,1,\ldots,b^n-1\}} \left( W(z_{n+2,kb^j+1}) + \lambda^{n+2} \hat{\phi} + B(0, \varepsilon \lambda^{n+4}) \right).
\end{equation*}
For any $m \in \mathbb{N}$, repeating the this process for $m$ times,
\begin{equation*}
W(z_{n,k}) + \lambda^n \hat{\phi} + B(0, \varepsilon \lambda^{n+1}) - \lambda^n {1 - \lambda^m \over 1 - \lambda} \phi(0) \subset \bigcup_{j \in \{0,1,\ldots,b^n-1\}} \left( W(z_{n+m,kb^j+1}) + \lambda^{n+m} \hat{\phi} + B(0, \varepsilon \lambda^{n+m+2}) \right).
\end{equation*}
Letting $m \to \infty$, we obtain (3.5).

\begin{definition}
Let $b \geq 2$, $d \geq 1$ and $n$ be an integer. The $n$ generation partition of $\mathbb{R}^d$ into $b$-adic intervals is
\begin{equation*}
\mathcal{L}_{n}^{b,d} = \left\{ \left[ k_1/b^n, k_1 + 1/b^n \right) \times \left[ k_2/b^n, k_2 + 1/b^n \right) \times \cdots \times \left[ k_d/b^n, k_d + 1/b^n \right) : k_1, k_2, \ldots, k_d \in \mathbb{Z} \right\}
\end{equation*}
The $n$ generation covering number of $A \subset \mathbb{R}^d$ by $b$-adic cubes is
\begin{equation*}
N(A, \mathcal{L}_n^{b,d}) = \#\{ d \in \mathcal{L}_n^{b,d} : D \cap A \neq \emptyset \}.
\end{equation*}
Recall that the lower and upper box dimension are defined as
\begin{align*}
\underline{\dim}_b(A) &= \liminf_{n \to \infty} \frac{\log N(A, \mathcal{L}_n^{b,d})}{\log b}, \quad \overline{\dim}_b(A) = \limsup_{n \to \infty} \frac{\log N(A, \mathcal{L}_n^{b,d})}{\log b}.
\end{align*}
By using Lemma 3.2, the following proof is standard, but we give the proof for the reader’s convenience.

\begin{theorem}
Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be a $\mathbb{Z}$-periodic Lipschitz function such that the set $\mathbb{R}^2 \setminus \hat{\phi}$ is not connected, then there is a number $c_0 \geq 1$ depending only on $\phi$ such that, for integer $b > 1$, $0 < \lambda < 1$ and $b \lambda^3 > c_0$, the graph of $W_{\Lambda,b}^{\phi}$ has box dimension $3 + 2 \log_b \lambda$.
\end{theorem}
\begin{proof}
Without loss of generality, we may assume that $\hat{\phi}$ separates $B(0, \varepsilon)$ from $\infty$. (Since there exists a dot $z_0 \in \mathbb{R}^2$ such that $B(z_0, \varepsilon) \subset \mathbb{R}^2 \setminus \hat{\phi}$.) We could consider $\phi - z_0$ instead of $\phi$.) Since $W$ is Hölder continuous with exponent $\log_b \lambda$, we obtain by [1] Lemma 2.2
\begin{equation*}
\overline{\dim}_b(\Gamma W) \leq 3 + 2 \log_b \lambda
\end{equation*}
where $\Gamma W = \{(s, W(s)) : s \in [0, 1)\}$ is the graph of $W$. 
For the other direction, by the definition of covering number, we can write
\[(3.10) \quad N(\Gamma W, L_n^{b, 2}) = \sum_{k=0}^{b^n-1} N\left(\left\{ W(s) : s \in \left[ \frac{k}{b^n}, \frac{k-1}{b^n} \right) \right\}, L_n^{b, 2} \right) \quad \forall n \in \mathbb{N}.\]

For sufficiently large \(n\), Lemma 3.2 implies
\[N\left(\left\{ W(s) : s \in \left[ \frac{k}{b^n}, \frac{k-1}{b^n} \right) \right\}, L_n^{b, 2} \right) \geq \pi b^{2+2n} \lambda^2 + \frac{2n b^2}{2} \]
We combine this with (3.10) to get
\[\dim_{\beta} \Gamma W \geq 3 + 2 \log_b \lambda.\]

4. The case when \(b \lambda^2 > c\)

In this section, we shall complete the proof of the Main Theorem under the additional assumption that \(\lambda \in (0, 1/2)\). Otherwise let \(c \geq 2c_0\) and the result holds by Theorem 3.1 since \(b \lambda^2 \geq c\) implies \(b \lambda^3 \geq c_0\). The idea of the proof is similar to what we did in section 3, but the argument is more complicated. For \(\beta \in \mathbb{R}\), let
\[\ell_\beta(s) = \phi(s) + \lambda^{-1} \left( \phi(\beta + \frac{s}{b}) - \phi(\beta) \right) \quad \forall s \in [0, 1).\]

**Lemma 4.1.** Let \(\phi : \mathbb{R} \to \mathbb{R}^2\) be a \(\mathbb{Z}\)-periodic Lipschitz function satisfied that \(\hat{\phi}\) separates \(B(0, \varepsilon)\) from \(\infty\). For \(\beta \in (0, 1)\) and \(b \lambda > \frac{k}{D(\phi(1-\lambda))}\), the following holds
\[(4.1) \quad \ell_\beta + B(0, \varepsilon \lambda) \subset \bigcup_{s \in [0, 1)} \left( \ell_\beta(s) + \lambda \hat{\phi} \right) \quad \forall \lambda \in (0, 1).\]

**Proof.** Arguing by contradiction, assume (4.1) fails. Then there exists \(\lambda_0 \in (0, 1), p_0 \in B(0, \varepsilon \lambda_0)\) and \(s_0 \in [0, 1)\) such that
\[(4.2) \quad \ell_\beta(s_0) + p_0 \notin \bigcup_{s \in [0, 1)} \left( \ell_\beta(s) + \lambda_0 \hat{\phi} \right).\]

Similar to the proof in Lemma 3.1, (4.2) implies
\[(4.3) \quad \Delta \left( \frac{\ell_\beta(s_0) + p_0 - \ell_\beta}{\lambda_0} \right) \leq \Delta(\phi).\]

We also observe that, for \(s_1, s_2 \in [0, 1),\n\[\left| \ell_\beta(s_1) - \ell_\beta(s_2) \right| \geq \left| \phi(s_1) - \phi(s_2) \right| - \lambda^{-1} \left| \phi(\beta + \frac{s_1}{b}) - \phi(\beta + \frac{s_2}{b}) \right| \geq \left| \phi(s_1) - \phi(s_2) \right| - Ly.\]

Therefore
\[(4.4) \quad \Delta(\ell_\beta(s_0) + p_0 - \ell_\beta) = \Delta(\ell_\beta) \geq \Delta(\phi) - Ly.\]
We combine (4.3) and (4.4) and obtain
\[ \lambda_0 \Delta(\phi) \geq \Delta(\phi) - L \gamma, \]
which contradicts our conditions. \[\square\]

Let \( c_1 > 1 \) be the solution of the following equation
\[
c_1 = \frac{L}{\varepsilon} \left( 4 + \frac{1}{c_1 - 1} \right).\]

**Lemma 4.2.** Assume \( \hat{\phi} \) separates \( B(0, \varepsilon) \) from \( \infty \), Let \( b \geq 2 \) be an integer, \( \lambda \in (0, 1/2) \) and \( \lambda^2 b > c_2 = \max(c_1, \frac{2L}{\Delta(\phi)}) \). Then, for integer \( n \geq 1 \) and \( k = 0, 1, \ldots, b^n - 1 \), the following holds.

(4.5) \( W(z_n, k) - \lambda_n \Delta(\phi) \geq \lambda_n (\hat{\ell}_k + B(0, \varepsilon \lambda)) \subset \left\{ W(s) : s \in \left[ \frac{k}{b^n}, \frac{k + 1}{b^n} \right] \right\} \).

**Proof.** For \( \beta, s \in [0, 1) \), there exists \( j_s \in \Lambda \) such that \( |s - j_s b| < \frac{1}{b} \). Note

(4.6) \[ \left| \ell_\beta \left( \frac{j_s}{b} \right) - \ell_\beta(s) \right| \leq \frac{L}{b} + \frac{L \gamma}{b} \leq \frac{2L}{b}. \]

We also have

(4.7) \( \hat{\phi} \subset \hat{\ell}_s + B(0, L \gamma) \quad \forall s \in \mathbb{R} \).

Combine (4.6), (4.1) and (4.7) with \( b \lambda > b \lambda^2 > c_1 \geq \frac{2L}{\Delta(\phi)} > \frac{L}{(1 - \lambda)\Delta(\phi)} \) to obtain

(4.8) \[ \hat{\ell}_\beta + B(0, \varepsilon \lambda) \subset \left\{ \ell_\beta \left( \frac{j}{b} \right) + \lambda \hat{\ell}_\frac{\lambda}{\phi} + B \left( 0, \frac{3L}{b} \right) \right\}. \]

Therefore

(4.9) \[ \hat{\ell}_\beta + B(0, \varepsilon \lambda) \subset \left\{ \ell_\beta \left( \frac{j}{b} \right) + \lambda \hat{\ell}_s + B(0, \varepsilon \lambda^2) \right\} \]

by using \( b \lambda^2 > c_1 = \frac{L}{\varepsilon} \left( 4 + \frac{1}{c_1 - 1} \right) > 3 \frac{L}{\varepsilon} \). Combine (4.9) with \( \beta = k/b \) and (2.2), we obtain

(4.10) \[ \hat{\ell}_\frac{\lambda}{\phi} + B(0, \varepsilon \lambda) \subset \left\{ \frac{W(z_{n+1,j} + b)}{A^n} - W(z_n, k) + \phi(0) + \lambda \hat{\ell}_{\lambda} + B \left( 0, \frac{3L}{b} + \frac{L \gamma}{1 - \gamma} \right) \right\} \]

We also observe that

(4.11) \[ b \lambda^2 > c_1 = \frac{L}{\varepsilon} \left( 4 + \frac{1}{c_1 - 1} \right) > \frac{L}{\varepsilon} \left( 3 + \frac{1}{1 - \gamma} \right) > \frac{L}{\varepsilon} \left( 3 + \frac{1}{c_1(1 - \gamma)} \right) > \frac{L}{\varepsilon} \left( 3 + \frac{by^2}{1 - \gamma} \right), \]
since \( \gamma < \frac{1}{b \lambda r} < \frac{1}{c_1} \) and \( \frac{b}{\lambda} > \gamma^2 \). We combine (4.10) and (4.11):

\[
W(z_n,k) + \lambda^n \hat{\ell}_k b + B(0, \varepsilon \lambda^{n+1}) - \lambda^n \phi(0) \subset \bigcup_{j \in \Lambda} \left( W(z_{n+1,k}b^j) + \lambda^{n+1} \hat{\ell}_{b^j} + B(0, \varepsilon \lambda^{n+2}) \right).
\]

The rest of the proof is the same as we did in Lemma 3.2, so (4.5) holds.

\[\square\]

\textbf{Proof.} Without loss generation we may assume \( \hat{\phi} \) separates \( B(0, \varepsilon) \) from \( \infty \). Let \( c = \max\{2c_0, c_1\} \). For the case \( \lambda \in (1/2, 1) \), we can use \( c \geq 2c_0 \) and finish the proof by Theorem 3.1 as explained at the beginning of this section. For the other case, the proof is the same as Theorem 3.1 by Lemma 4.2.

\[\square\]

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