Phase Transitions in liquid Helium 3

Markus Kindermann and Christof Wetterich

Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany

(October 31, 2018)

The phase transitions of liquid $^3$He are described by truncations of an exact nonperturbative renormalization group equation. The location of the first order transition lines and the jump in the order parameter are computed quantitatively. At the triple point we find indications for partially universal behaviour. We suggest experiments that could help to determine the effective interactions between fermion pairs.

05.30.Jp, 67.40.Kh, 03.65.Db

The low temperature phase transitions and the superfluid phases of liquid Helium 3 can be described by a field theory for complex $3 \times 3$ matrices $A$ representing fermion pairs. An approximation based on quartic polynomials in $A$, including the renormalization group running of the coupling constants, can account for the rough phase structure but not for details of the transitions. We will be concerned with the transitions from the normal liquid to superfluidity in vanishing external magnetic field. In a mean-field treatment these transitions are of second order, whereas early RG-calculations found hints for them to be fluctuation induced first order transitions. We will be concerned with the transitions from the normal liquid to superfluidity in vanishing external magnetic field. In a mean-field treatment those transitions are of second order, whereas early RG-calculations found hints for them to be fluctuation induced first order transitions. The BCS weak-coupling theory estimates the critical region where fluctuations invalidate the mean-field approximation to extend over a temperature interval $\Delta T \approx 10^{-8} K$. Based on experiments on zero sound absorption it has been conjectured, however, that the critical region might be up to a thousand times larger. The theoretical answers therefore hinge on two problems: first, the “microphysical” effective interactions at the scale of the fermion pairs ($\Lambda^{-1} \approx 120 \AA$) are poorly known and second, the transition from the microphysical interactions to the macrophysics of the phase transitions (i.e. the thermodynamic potentials) is very complex. Only if the second problem can be solved, the experimental observations of the macrophysics (phase structure, jump in the order parameter, etc.) can be used to constrain the microphysical interactions. In this letter we propose a quantitatively reliable mapping of the microphysics onto the properties of the free energy density. This is effected by means of extended truncations to an exact nonperturbative renormalization group equation, in contrast to largely uncontrolled approximations used earlier.

The different phases of $^3$He are characterized by expectation values $\langle A \rangle$ (order parameters) in different directions: the BW-state, $A_1 = 1$, the ABM-state, $A_2 = \sqrt{2} (\lambda_7 + i \lambda_6 + \lambda_4 - i \lambda_3)$, and the planar state, $A_3 = \sqrt{2} i \lambda_2$ represent the phases of the mean-field diagram. The $\lambda_i$ are the Gell-Mann matrices. Additionally, we examine the direction $A_4 = \sqrt{3} (\lambda_3 - i \lambda_1)$. Here, $\langle A \rangle$ corresponds to the minimum of the effective potential $U(A)$ for spatially homogeneous fields. Our main aim is therefore the computation of $U(A)$ for a given microscopic effective action (at the scale $\Lambda$) for which we assume the conventional form:

$$\Gamma_A = \int d^3x \left\{ Tr \; \bar{\nabla} A^I \nabla A + m_A + \sum_{i=0}^{4} b_i I_i + a K_D \right\}$$  \hspace{1cm} (1)

with $\rho = Tr \; A^I A$ and $I_i$ the invariants to fourth order allowed by the symmetry group $G = SO(3) \times SO(3) \times U(1)$:

$I_0 = \rho^2$, $I_1 = |Tr \; A^T A|^2$, $I_2 = Tr \left( A^T A \right) (A^T A)^*$, $I_3 = Tr \left( A^I A \right)^2$, and $I_4 = Tr \left( A^I A \right) (A^I A)^*$.

We include the dipole interaction

$$K_D = Tr \; A^I A \; Tr \; A + Tr \; (A^* A)$$  \hspace{1cm} (2)

as a symmetry breaking perturbation arising from the spin-orbit coupling of the atoms in a Cooper pair and neglect the strain gradient terms. All quantities are in units of a characteristic critical temperature $T_c = 2.6 \; mK$ and $m(T) = m(T_0) + \frac{T-T_0}{T_0} + \ldots$ reflects the temperature dependence. In the paramagnon theory the quartic couplings $b_i$ are given by

$$b_0 = c(2 + 2\delta), \quad b_1 = -c(1 + .1\delta), \quad b_2 = c(2 - .05\delta), \quad b_3 = c(2 - .55\delta), \quad b_4 = -c(2 + .7\delta)$$  \hspace{1cm} (3)

with $c = 0.001$ and $\delta$ parametrizing the pressure dependence. We study a pressure regime of $-2 < \delta < 1.8$. For $\delta > 1.9$ the microscopic potential becomes unbounded from below. This will be compared to a calculation where the couplings are ten times as strong.

Our approach is based on the effective average action $\Gamma_k[A(x)]$ which interpolates between the microphysics ($k = \Lambda$) and the macrophysics ($k = 0$) by means of an exact flow equation. We truncate the most general functional dependence on $A(x)$ by

$$\Gamma_k = \int d^3x \{ \int d^3x \; \bar{\nabla} A^I \nabla A + U_k(A) + a_k K_D \}$$  \hspace{1cm} (4)
where $U_k(A)$ is a function of the eighteen real fields $\varphi_y$ in $A$ respecting the symmetry group $G$. We can compute the effective potential $U(A) \equiv U_0(A)$ by a numerical integration of the flow equation

$$\partial_t U_k(A) = \frac{1}{2} \int \frac{d^3q}{2\pi^3} \partial_t R_k \; \text{Tr} \left( (Z_k q^2 + R_k) + M_k^2(A) \right)^{-1}$$

(5)

Here, $t = \ln k$. The infrared cut-off

$$R_k(q) = Z_k q^2 \left( 1 - e^{-\frac{q}{\Lambda}} \right)^{-1}$$

(6)

makes the flow equation ultraviolet and infrared finite. We need the mass matrix of second derivatives of $U$, $M_{\rho,\varphi} = \frac{\partial^2 U}{\partial \rho \partial \varphi}$. The flow equation for the wave function renormalization $Z_k$ can be found in [5].

In a first investigation we have used a sixth order polynomial approximation for $U_k$ and searched for minima in all the fifteen directions with different residual symmetries given in [5]. In this scheme we found a phase transition into the direction $A_4$ for large $\delta$. We have convinced ourselves, however, that polynomial approximations do not allow reliable statements about the phase diagram. This is why we deal with an arbitrary dependence of $U_k$ on the field in a given direction $A = \xi A_n$ for which we write $U_k(\xi A_n) = V_k^{(n)}(\rho)$ with $\rho = 3\xi^2$. We assume, that the directions $n$ include the absolute minimum of $U_k$. We get the flow equation for $V_k^{(n)}$ by evaluating (5) at points on the line $\xi A_n$. To do so, we need to know the second derivatives of $U_k$ with respect to all fields, however. That is why we include in our truncation four coupling functions $b_i^{(n)}(\rho)$ multiplying all fourth order invariants other than $\rho^2$:

$$U(\xi A_n + \epsilon) = \left\{ V^{(n)}(\rho) + \sum_{i=1}^{4} b_i^{(n)}(\rho) (I_i - f_i^{(n)}(\rho^2)) \right\} |_{\xi A_n + \epsilon}$$

(7)

The numbers $f_i^{(n)}$ are chosen such that the invariants $I_i - f_i^{(n)}(\rho^2)$ do not contribute on the line $\xi A_n$. Here $\epsilon$ denotes fields orthogonal to $\xi A_n$ and we expect that this truncation gives a good approximation to the true potential in a neighborhood of the line $\xi A_n$.

The flow equations of the coupling functions $b_i^{(n)}(\rho)$ are derived by forming appropriate linear combinations of second and third derivatives of (5) with respect to certain fields. At the origin we need to take fourth derivatives to extract the couplings from our potential truncation. We have computed these flow equations for the four field directions $A_n$. The resulting equations are very long, altogether about 4000 lines. They all derive, however, from compact expressions, namely field derivatives of (5), which can be evaluated by a computer. We consider this structural simplicity to be one of the major strengths of our formalism.

We have integrated the resulting equations numerically by laying the potentials and coupling functions on grids of ten points each. In a first run, we detected truncation errors in the quantities $\frac{\partial V^{(n)}}{\partial \rho}|_{A=0}$. They originate in an insufficient treatment of the symmetry constraints. We fight this problem in two steps. First, we infer the first potential derivatives at the origin from the term $m\rho$ and integrate a separate flow equation for $m$. Second, for the higher orders we employ for small $|t|$ a mixed truncation of the form

$$U(\xi A_n + \epsilon) = \left\{ m\rho + \sum_{i=0}^{4} b_i I_i + \sum_{j=0}^{10} c_j J_j + \rho^4 w^{(n)}(\rho) \right\} \frac{1}{\xi A_n + \epsilon}$$

(8)

Here $J_j$ are the eleven invariants of sixth order. $w^{(n)}$ and $B_i^{(n)}$ describe the deviation from a sixth order polynomial approximation. The symmetry is manifestly respected up to sixth order. We have estimated the accuracy of the sixth order polynomial approximation by calculating $\partial_t U(\xi A_n)$ to fourth and sixth order in $\xi$ in two ways: from the flow of the couplings in the polynomial part of truncation (8) and directly by equation (5). The discrepancy stays smaller than 20% as long as the flow parameter $|t|$ is not too big, $t > -5$. It grows to a few hundred percent, however, until the phase transition occurs at $t \approx -8$. This is due to terms of eighth and higher order that are truncated in the flow equations for the couplings $b_i$ and $c_j$ in (8). We conclude, that with a polynomial truncation of the free energy to realistic order, no conclusive statements can be obtained about the phase diagram. As soon as the errors exceed 20%, we switch to the truncation (5). The truncation error in the quantities $\partial_t V^{(n)}$ at the potential minima is estimated by comparing truncation (5) with a truncation in which we extract the orthogonal masses from the polynomial truncation in the origin. The errors induced by the inaccuracies of $\frac{\partial V^{(n)}}{\partial \rho}|_{A=0}$ are of the order of magnitude of the truncation errors and both are not bigger than a few percent. This truncation should therefore allow us to compute the phase structure reliably.

First, we study the case of large couplings, $c = 0.01$. We neglect the spin-orbit coupling, which is justified in this case, the other masses being much bigger. By computing the temperature dependence of the renormalization of the fourth order couplings, we find a width of the critical region of about $10^{-8}K$, in good agreement with the Ginzburg criterion. The resulting phase diagram is shown in figure[6]. In a pressure regime of $0.26 < \delta < 0.46$
a beak of A-phase between the symmetric and the B-phase is stabilized by the fluctuations of the order parameter. In the middle of this pressure interval the temperature width of this wedge is about $10^{-6}K$.

Towards the upper end it rises, however, to $10^{-5}K$ at $\delta = 0.41$. All transitions are of the first order. The potentials at the transitions look qualitatively like figure 

\[ \triangle \nu^2 \approx 6c \cdot 10^{10} \ H_z \cdot \triangle \rho \]  

(9)

which follows from the mean-field temperature dependence of $\triangle \nu$ given in [3]. We observe a substantial increase of the correlation length as we approach the triple point $\delta = 0.26$. This observation nourishes first hope for a fixed point in the vicinity of this point in parameter space which would give rise to universal behaviour of the system. Accurate predictions could then be made despite the uncertainty of our knowledge of the microscopic theory. Also an examination of the flow of the couplings at the origin looks promising for the triple point. We observe strong renormalization of the fourth order couplings $b_i$. For example, $\frac{b_4}{b_0} \approx 8$, whereas analogous ratios away from the triple point, say, at $\delta = 0$, grow no larger than 1.5 at the phase transition.

Figure 3 shows the dimensionless renormalized couplings $b_i$. We cannot find a true fixed point behaviour. However, ratios of couplings seem to tend to constant values, such as $\frac{b_1}{b_4} = 1$, corresponding to a partial fixed point.

The qualitative features for smaller couplings $c = 0.001$ look similar, with correspondingly smaller $\triangle T = 10^{-6}K$. In this case, however, the dipole term should be included ($a_A = 2 \cdot 10^{-6}$) which we do next. Figure 4 shows the
phase diagram with first order transition lines. The con-
siderations in [2] are confirmed by our calculation. The
width of the stripe of planar phase stabilized by the
dipole interaction is about $2 \cdot 10^{-9} K$ as it was expected
from mean-field considerations. We have not inspected
the phase structure much further than $10^{-9} K$ away from
the transition line. We expect the same features we found
above: stabilization of a wedge of A phase with a tem-
perature width of about $10^{-8} K$ by the fluctuations.

Tuning the temperature to the transition from the
symmetric to the planar phase we find a potential $U_{k_f}$(figure 5) which shows clearly that the transition is of
first order. (Fluctuations with momenta smaller than $k_f$
makes the potential convex without much influence on the
equation of state [10].) From figure 5 we can easily infer
the order parameter jump: $\Delta \rho \approx 8 \cdot 10^{-5}$. This turns out
to be almost independent of the pressure in the regime
examined by us. It corresponds to a discontinuity in the
NMR-frequency shift at the transition of about $50 Hz$.

Let us finally ask how experiments may constrain the
size of the couplings. We have seen that large couplings
lead to a widening of the critical region $\Delta T$. It should
be possible to exploit this fact by measuring the width
of the critical region. Varying the temperature at a pres-
sure right under the triple point, one should be able to
observe two distinct phase transitions . The width of
the intermediate phase relates to the coupling strength
via the Ginzburg criterion. By the described experiment
one could also measure the pressure corresponding to the
values $\delta = 0.26$ and $\delta = 0.46$. An approximate form of
the relation $\rho(\delta)$ could be obtained then by linearization.
Strictly speaking, these arguments hold only if the dipole
term can be neglected. However, the temperature range
for the planar phase, $10^{-9} K$, is so narrow, that it would
probably not even show up in experiments with realistic
temperature resolution.

Also the order parameter jump depends strongly on the
coupling strength. With the larger couplings $c = 0.01$ we
find $\Delta \rho \approx 0.01$ for the jump into the B-phase at the triple
point. This leads to a discontinuity in the frequency shift
of about $2000 Hz$ as opposed to e.g. $370 Hz$ for $c = 0.002$.
We think, that it would be even easier to infer the size of
the critical region from measuring this order parameter
jump. At least, this way it should be possible to get up-
per bounds for the coupling strength that are rather close
to the range discussed in this article. To facilitate the
interpretation of such NMR-experiments, we calculated
the jump of $\Delta \nu$ for various other values of the coupling
strength $c$ at $\delta = 0.15$. We show the result in figure 5.

In conclusion, our quantitative description of the phase
diagram of $^3He$ should permit to extract reliable informa-
tion about the system’s microphysical interactions from
future experiments.

* e–mail: M.Kindermann@ThPhys.Uni-Heidelberg.DE
† e–mail: C.Wetterich@ThPhys.Uni-Heidelberg.DE

[1] D. D. Osheroff, R. C. Richardson, and D. M. Lee, Phys.
Rev. Lett. 28, 885 (1972).
[2] D. R. T. Jones, A. Love, and M. A. Moore, J. Phys C 9,
743 (1976).
[3] A. I. Sokolov, JETP Letters 29, 590 (1979).
[4] Paulson and Wheatley, Phys. Rev. Lett. 41, 497 (1978).
[5] A. I. Sokolov, Sov. Phys. JETP 57, 798 (1983).
[6] C. Wetterich, Phys. Lett. B 301, 90 (1993).
[7] B. Bailin, A. Love, and M. A. Moore, J. Phys C 10, 1159
(1977).
[8] N. Tetradis and C. Wetterich, Nucl.Phys. B 422, 541
(1994).
[9] D. Vollhardt and P. Wölfle, The superfluid phases of He-
lium 3 (Taylor and Francis, London, 1990).
[10] J. Berges, N. Tetradis, C. Wetterich, hep-ph/0005122
(2000).