A PRIORI ESTIMATES FOR DONALDSON’S EQUATION OVER
COMPACT HERMITIAN MANIFOLDS

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ABSTRACT. In this paper we prove a priori estimates for Donaldson’s equation
\[ \omega \wedge (\chi + \sqrt{-1} \partial \bar{\partial} \varphi)^{n-1} = e^F (\chi + \sqrt{-1} \partial \bar{\partial} \varphi)^n, \]
over a compact complex manifold \( X \) of complex dimension \( n \), where \( \omega \) and \( \chi \) are arbitrary Hermitian metrics. Our estimates answer a question of Tosatti-Weinkove in [17].

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1. INTRODUCTION

1.1. Donaldson’s equation over compact Kähler manifolds. Let \((X, \omega)\) be a compact Kähler manifold of the complex dimension \( n \), and \( \chi \) another Kähler metric on \( X \). In [3], Donaldson considered the following interesting equation
\[ \omega \wedge \eta^{n-1} = c\eta^n, \quad [\eta] = [\chi], \]
where \( c \) is a constant, depending only on the Kähler classes of \([\chi]\) and \([\omega]\), given by
\[ c = \frac{\int_X \omega \wedge \chi^{n-1}}{\int_X \chi^n}. \]
He noted that a necessary condition for equation (1.1) is
\[ nc\chi - \omega > 0, \]
and then conjectured that the condition (1.3) is also sufficient. For \( n = 2 \), Chen [11] observed that in this case the equation (1.1) reduces to a complex Monge-Ampère
equation completely solved by Yau on his celebrated work on Calabi’s conjecture [24].

1.2. $J$-flow and Donaldson’s equation. To better understand the equation (1.1), Donaldson [3] and Chen [1] independently discovered the $J$-flow whose critical point gives the equation (1.1), and Chen showed that such flow always exists for all time. Using the $J$-flow, Chen [2] proved that if $n = 2$ and the holomorphic bisectional curvature of $\omega$ is nonnegative then the $J$-flow converges to a critical metric. Later, the curvature assumption was removed by Weinkove [22] and hence gave an alternative proof of Donaldson’s conjecture on Kähler surfaces. For higher dimensional case, Weinkove [23] solved Donaldson’s conjecture on a slightly stronger condition

\[ nc\chi - (n - 1)\omega > 0 \]

using the $J$-flow. For more detailed discussions and related works, we refer to [4, 5, 6, 7, 15, 16].

1.3. Donaldson’s equation over compact Hermitian manifolds. Recently, the complex Monge-Ampère equation over compact Hermitian manifolds was solved Tosatti and Weinkove [17, 18]. Other interesting estimates can be found in [19, 23, 26]. A parabolic proof was later given by Gill [8] by considering a parabolic complex Monge-Ampère equation. Other parabolic flows over compact Hermitian manifolds were considered in [14, 19, 20, 21], where they obtained lots of interesting results parallel to those in Kähler case. By Tosatti-Weinkove’s work, the author considers Donaldson’s equation over compact Hermitian manifolds.

Let $(X, \omega)$ be a compact Hermitian manifold of the complex dimension $n$ and $\chi$ another Hermitian metric on $X$. We denote by $\mathcal{H}_\chi$ the set of all real-valued smooth functions $\varphi$ on $X$ such that $\chi_\varphi := \chi + \sqrt{-1} \partial \bar{\partial} \varphi > 0$. Locally we have

\[ \omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad \chi = \sqrt{-1} \chi_{i\bar{j}} dz^i \wedge d\bar{z}^j. \]

For any real positive $(1,1)$-form $\alpha := \sqrt{-1} \alpha_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and real $(1,1)$-form $\beta := \sqrt{-1} \beta_{i\bar{j}} dz^i \wedge d\bar{z}^j$ we set

\[ \text{tr}_\alpha \beta := \alpha^{i\bar{j}} \overline{\beta}_{i\bar{j}}. \]

We consider Donaldson’s equation

\[ \omega \wedge \chi_{n-1}^{n-1} = e^F \cdot \chi^n, \quad \varphi \in \mathcal{H}_\varphi \]

on $X$, where $F$ is a given smooth function on $X$.

The main result of this paper is the following a priori estimates.

**Theorem 1.1.** Let $(X, \omega)$ be a compact Hermitian manifold of the complex dimension $n$ and $\chi$ another Hermitian metric. Let $\varphi$ be a smooth solution of Donaldson’s equation (1.7). Assume that

\[ \chi - \frac{n-1}{ne^n} \omega > 0. \]

Then

1. there exist uniform constant $A > 0$ and $C > 0$, depending only on $X, \omega, \chi,$ and $F$, such that

\[ \text{tr}_\omega \chi \varphi \leq C \cdot e^{A(\varphi - \inf X \varphi)}; \]
(2) there exists a uniform constant $C > 0$, depending only on $X, \omega, \chi$, and $F$, such that

$$||\varphi||_{C^0} \leq C;$$

(3) there are uniform $C^\infty$ a priori estimates on $\varphi$ depending only on $X, \omega, \chi$, and $F$.

Meanwhile, Guan, Li and Sun [9, 11, 12, 13] considered a priori estimates for Donaldson’s equation over compact Hermitian manifolds under very general structure conditions rather than the condition (1.8).

Remark 1.2. As remarked in [17] (see page 22, line 27–28), to prove the zeroth estimate in Theorem 1.1 it suffices to show the second order estimate on $\varphi$. Our result gives an affirmative answer to the question in [17] (see page 22, line 28–30). Using the same argument in [17] (page 33), we can get a $C^\alpha$ estimate on $\varphi$ for some $\alpha \in (0, 1)$. Differentiating (1.7) and applying the standard local elliptic estimates imply uniform $C^\infty$ estimates on $\varphi$.

There are some natural questions about the equation (1.7). Is condition (1.8) sufficient to produce a solution to (1.7)? When $\omega$ and $\chi$ both are Kähler, it has been proved in [2, 22, 23] that this condition is sufficient. The second question is to consider a parabolic flow over compact Hermitian manifolds like the $J$-flow. Can we prove the long time existence and convergence of such a flow? Song and Weinkove [16] gave a necessary and sufficient condition for existence of solutions to the Donaldson’s equation over compact Kähler manifolds (and also for convergence of the $J$-flow over compact Kähler manifolds). The last question then is whether we can find an analogous of above Song-Weinkove’s condition. Those questions will be answered later.

Remark 1.3. Here and henceforth, when we say a “uniform constant” it should be understood to be a constant that depends only on $X, \omega, \chi$, and $F$. We will often write $C$ or $C'$ for such a constant, where the value of $C$ or $C'$ may differ from line to line. For the relation $P \leq CQ$ for a uniform constant $C$ in the above sense, we write it as $P \lesssim Q$. Re($P$) means the real part of $P$.

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2. The second order estimates

2.1. Basic facts and notions. Let $(X, \omega)$ be a complex Hermitian manifold of the complex dimension $n$ and $\chi$ another Hermitian metric on $X$. For a solution $\varphi$ of Donaldson’s equation (1.7), we denote by

$$\chi' := \chi + \sqrt{-1} \partial \bar{\partial} \varphi = \sqrt{-1}(\chi_{ij} + \varphi_{ij})dz^i \wedge dz^j.$$

Also, we set $\chi'_{ij} := \chi_{ij} + \varphi_{ij}$. Then we observe that

$$\text{tr}_{\chi'} \omega = n \frac{\omega \wedge (\chi')^{n-1}}{(\chi')^n} = nF.$$
Consequently, $\text{tr}_{\chi'}\omega$ is uniformly bounded away from zero and infinity. Let $\Delta_\omega$ denote the Laplacian operator of the Chern connection associated to the Hermitian metric $\omega$, and similarly for $\Delta_\chi$. Note that

$$\text{tr}_\omega \chi' = g^{ij}(\chi'_{ij} + \varphi_{ij}) = \text{tr}_\omega \chi + \Delta_\omega \varphi.$$  

**Remark 2.1.** $\text{tr}_\omega \chi'$ and $\text{tr}_\chi \omega$ are uniformly bounded from below away from zero. More precisely,

$$\text{tr}_\omega \chi' \geq \frac{n}{e^F}, \quad \text{tr}_\chi \omega = ne^F.$$

The second assertion follows from (2.2), while the first inequality is obtained as follows. We choose a normal coordinate system so that

$$g_{ij} = \delta_{ij}, \quad \chi'_{ij} = \lambda'_{ij}$$

for some $\lambda'_1, \cdots, \lambda'_n > 0$. Donaldson’s equation then yields

$$ne^F = \sum_{1 \leq i \leq n} \frac{1}{\lambda'_i}.$$

An elementary inequality shows that

$$\text{tr}_\omega \chi' = \sum_{1 \leq i \leq n} \lambda'_i \geq \frac{n^2}{\sum_{1 \leq i \leq n} \frac{1}{\lambda'_i}} = \frac{n^2}{ne^F} = \frac{n}{e^F}.$$  

We will frequently use the following

**Lemma 2.2.** (Guan-Li [10]) At any point $p \in X$ there exists a holomorphic coordinates system centered at $p$ such that, at $p$,

$$g_{ij} = \delta_{ij}, \quad \partial_j g_{ii} = 0$$

for all $i$ and $j$. Furthermore, we can assume that $\chi'_{ij}$ is diagonal.

Let $\tilde{\Delta}$ denote the Laplacian operator associated to the Hermitian metric $h_{i\bar{j}}$ whose inverse matrix is given by

$$h^{ij} := \chi^{i\bar{k}} \chi^{j\bar{k}} g_{k\bar{\ell}},$$

and $\tilde{\nabla}$ the associated covariant derivatives.

The basic idea to obtain the second order estimate, following from the method of Yau [24], is to consider the quantity

$$Q := \log(\text{tr}_\omega \chi') - A\varphi$$

for some suitable constant $A$. Our first step is to estimate the term $\tilde{\Delta} \log(\text{tr}_\omega \chi')$.

**Definition 2.3.** For convenience, we say that a term $E$ is of type I if

$$|E|_\omega \lesssim 1,$$

and is of type II if

$$|E|_\omega \lesssim \text{tr}_\omega \chi'.$$

It is easy to see that any uniform constant is of type I and any type I term is of type II. We will use $E_1$ and $E_2$ to denote a type I and type II term, respectively.
2.2. The estimate for $\tilde{\Delta}\log(tr_\omega \chi')$. Direct computation shows

\[
\tilde{\Delta}\log(tr_\omega \chi') = \frac{\tilde{\Delta}tr_\omega \chi'}{tr_\omega \chi'} - \frac{|\nabla tr_\omega \chi'|^2}{(tr_\omega \chi')^2}.
\]

By the definition, we have

\[
\tilde{\Delta}tr_\omega \chi' = h^{ij}\partial_i \partial_j (g^{k\ell} \chi'_{k\ell})
\]

\[
= h^{ij}\partial_i \left( -g^{kb} g^{af} \partial_j g_{ab} \cdot \chi'_{k\ell} + g^{k\ell} \partial_j \chi'_{k\ell} \right)
\]

\[
= h^{ij} \left[ g^{k\ell} \partial_i \partial_j \chi'_{k\ell} - g^{kb} g^{af} \partial_i g_{ab} \cdot \partial_j \chi'_{k\ell} = h^{k\ell} g^{af} \partial_i g_{ab} \cdot \partial_j \chi'_{k\ell} \right.
\]

\[
- \left( g^{kb} g^{af} \partial_i g_{ab} \cdot \partial_j \chi'_{k\ell} - g^{k\ell} g^{af} \partial_i g_{ab} \cdot \partial_j \chi'_{k\ell} \right.
\]

\[
+ g^{k\ell} g^{af} \partial_i \partial_j g_{ab} \right] \chi'_{k\ell}.
\]

Using the local coordinates in Lemma 2.2, we deduce that

\[
\tilde{\Delta}tr_\omega \chi' = \sum_{1 \leq i,j \leq n} h^{ij} \partial_i \partial_j \chi'_{k\ell} - \sum_{1 \leq i, k, \ell \leq n} h^{ij} \partial_i \partial_j g_{k\ell} \cdot \partial_i \chi'_{k\ell}
\]

\[
- \sum_{1 \leq i, k, \ell \leq n} h^{ij} \partial_i \partial_j g_{k\ell} \cdot \partial_i \chi'_{k\ell} + \sum_{1 \leq i, k, \ell \leq n} h^{ij} \partial_i g_{k\ell} \cdot \partial_i \chi'_{k\ell}
\]

\[
+ \sum_{1 \leq i, k, \ell \leq n} h^{ij} \partial_i g_{k\ell} \cdot \partial_i \chi'_{k\ell} + \sum_{1 \leq i, k, \ell \leq n} h^{ij} \partial_i \partial_j g_{k\ell} \cdot \chi'_{k\ell}
\]

\[
(2.11)
\]

\[
= \sum_{1 \leq i, k \leq n} h^{ij} \partial_i \partial_j \chi'_{k\ell} - 2 \cdot \text{Re} \left( \sum_{1 \leq i, j, k \leq n} h^{ij} \partial_i g_{jk} \cdot \partial_j \chi'_{k\ell} \right) + E_1,
\]

where

\[
E_1 = \sum_{1 \leq i, j, k \leq n} h^{ij} \partial_i g_{jk} \cdot \partial_j g_{jk} \cdot \chi'_{k\ell} + \sum_{1 \leq i, j, k \leq n} h^{ij} \partial_i g_{jk} \cdot \partial_j g_{jk} \cdot \chi'_{k\ell}
\]

\[
- \sum_{1 \leq i, k \leq n} h^{ij} \partial_i \partial_j g_{k\ell} \cdot \chi'_{k\ell}.
\]

Since under the above mentioned local coordinates $\chi'_{k\ell} = \chi' \delta_{ij}$, it follows that $h^{ij} = (\chi'_{k\ell})^2 = 1/\chi'_{k\ell}$; hence $h^{ij} \leq e^{2F}$ using Remark 2.4. Therefore we see that $E_1$ is of type II, i.e.,

\[
|E_1|_{\omega} \lesssim tr_\omega \chi'.
\]

The first term on the right hand side of (2.11) can be computed as follows: From Donaldson’s equation (1.7), we obtain

\[
n e^F = tr_\chi' \omega = \chi'_{ij} g_{ij}
\]

and, after taking the derivative with respect to $z^k$,

\[
n \partial_i F \cdot e^F = -\chi'_{ij} \chi'_{jk} \partial_i g_{jk} + \chi'_{ij} \partial_i g_{jk}.
\]
we conclude that
\[
\begin{align*}
\Delta F = \sum_{i,j,k,\ell,p,q} \partial_i \partial_j \partial_k \partial_\ell \chi_{ijkl} \partial_p \chi_{ijkl} + \sum_{i,j,k,\ell,a,b} \partial_i \partial_j \partial_k \partial_\ell \chi_{ijkl} \partial_a \chi_{ijkl} + \sum_{i,j,k,\ell,a,b} \partial_i \partial_j \partial_k \partial_\ell \chi_{ijkl} \partial_b \chi_{ijkl}.
\end{align*}
\]

Multiplying above by \(g^{i\ell}\) on both sides implies
\[
(\Delta_+ F + |\nabla F|_w^2) ne^F = - \sum_{1 \leq i,j,k,\ell \leq n} \left( \chi•^{i\ell} g^{i\ell} \partial_i \partial_j \partial_k \partial_\ell g_{ij} - \chi^{i\ell} g^{i\ell} \partial_i \partial_j \partial_k \partial_\ell g_{ij} \right)
\]
\[
- \sum_{1 \leq i,j,k,\ell,a,b \leq n} \chi^{i\ell} \chi•^{i\ell} \partial_i \partial_j \partial_k \partial_\ell \chi_{ijkl} - \sum_{1 \leq i,j,k,\ell,p,q \leq n} \chi^{i\ell} \chi•^{i\ell} \partial_i \partial_j \partial_k \partial_\ell \chi_{ijkl} - \sum_{1 \leq i,j,k,\ell,a,b \leq n} \chi^{i\ell} \chi•^{i\ell} \partial_i \partial_j \partial_k \partial_\ell \chi_{ijkl}.
\]

Using the local coordinates (2.13) we arrive at
\[
(\Delta_+ F + |\nabla F|_w^2) ne^F = - \sum_{1 \leq i,j,\ell \leq n} h^{i\ell} \partial_i \partial_j \chi^{ij}_{i\ell} + \sum_{1 \leq i,j \leq n} \chi^{i\ell} \partial_i \partial_j g_{ij} + \sum_{1 \leq i,j,\ell \leq n} h^{i\ell} \chi^{i\ell} \partial_i \partial_j \chi_{ij} - 2 \cdot \text{Re} \left( \sum_{1 \leq i,j,\ell \leq n} \chi^{i\ell} \chi^{i\ell} \partial_i \partial_j \chi_{ij} \right).
\]

Equivalently,
\[
\sum_{1 \leq i,j \leq n} h^{i\ell} \partial_i \partial_j \chi^{ij}_{i\ell} = \sum_{1 \leq i,j \leq n} h^{i\ell} \chi^{i\ell} \partial_i \partial_j \chi_{ij} + \sum_{1 \leq i,j,\ell \leq n} h^{i\ell} \chi^{i\ell} \partial_i \partial_j \chi_{ij} - 2 \cdot \text{Re} \left( \sum_{1 \leq i,j \leq n} \chi^{i\ell} \chi^{i\ell} \partial_i \partial_j \chi_{ij} \right) + \sum_{1 \leq i,j \leq n} \chi^{i\ell} \partial_i \partial_j g_{ij} - (\Delta_+ F + |\nabla F|_w^2) ne^F.
\]

Since
\[
\partial_k \partial_\ell \chi_{ij} = \partial_k \partial_\ell (\chi_{ij} + \varphi_{ij}) = \partial_k \partial_\ell \chi_{ij} + \partial_k \partial_\ell \varphi_{ij} = \partial_k \partial_\ell \chi_{ij} + \partial_\ell \partial_k \varphi_{ij} = \partial_k \partial_\ell \chi_{ij} + \partial_\ell \partial_k (\chi_{kk} - \chi_{kk}) = \partial_\ell \partial_k \chi_{kk} + (\partial_k \partial_\ell \chi_{ij} - \partial_i \partial_j \chi_{kk}),
\]
we conclude that
\[
\sum_{1 \leq i,j,\ell \leq n} h^{i\ell} \partial_i \partial_j \chi_{kk} = \sum_{1 \leq i,j \leq n} h^{i\ell} \partial_i \partial_\ell \chi_{ij} + \sum_{1 \leq i,j \leq n} h^{i\ell} (\partial_i \partial_j \chi_{kk} - \partial_i \partial_\ell \chi_{ij}).
\]
Combining (2.13) and (2.14) yields

\[
\sum_{1 \leq i, k \leq n} h^{ik} \partial_{ik} \chi'_{kk} = \sum_{1 \leq i,j,k \leq n} h^{ij} \chi'_{ij} \partial_{ik} \chi'_{ij} + \sum_{1 \leq i,j,k \leq n} h^{ij} \chi'_{ij} \partial_{ik} \chi'_{ij}
\]

(2.15)

\[
- 2 \cdot \text{Re} \left( \sum_{1 \leq i,j,k \leq n} \chi^{ik} \chi'_{ik} \partial_{ik} \chi'_{ik} \right) + E_2,
\]

where

\[
E_2 = \sum_{1 \leq i,k \leq n} \chi^{ik} \partial_{ik} g_{ii} + \sum_{1 \leq i,k \leq n} h^{ik} (\partial_i \chi_{kk} - \partial_k \chi_{ii}) - (\Delta \omega F + |\nabla F|^2) n e^F.
\]

By the same reason that \(\chi^{ik} \leq e^F\) and \(h^{ik} \leq e^{2F}\), we observe that \(E_2\) is of type I and

\[
|E_2|_\omega \lesssim 1.
\]

From (2.11) and (2.15), we get

\[
\tilde{\Delta} \text{tr} \omega' = \sum_{1 \leq i,j,k \leq n} h^{ij} \chi'_{ij} \partial_{ik} \chi'_{ij} + \sum_{1 \leq i,j,k \leq n} h^{ij} \chi'_{ij} \partial_{ik} \chi'_{ij}
\]

\[
- 2 \cdot \text{Re} \left( \sum_{1 \leq i,j,k \leq n} \chi^{ik} \chi'_{ik} \partial_{ik} \chi'_{ik} \right) + E_1 + E_2
\]

\[
= \sum_{1 \leq i,j,k \leq n} h^{ij} \chi'_{ij} \partial_{ik} \chi'_{ij} + \sum_{1 \leq i,j,k \leq n} h^{ij} \chi'_{ij} \partial_{ik} \chi'_{ij}
\]

\[
- 2 \cdot \text{Re} \left( \sum_{1 \leq i,j,k \leq n} \chi^{ik} \chi'_{ik} \partial_{ik} \chi'_{ik} \right)
\]

\[
- 2 \cdot \text{Re} \left( \sum_{1 \leq i,j,k \leq n} h^{ik} \partial_{ik} g_{ii} \partial_{ik} \chi'_{ik} \right) + E_2,
\]

since any type I term is also of type II.

2.3. The estimate for \(\tilde{\Delta} \log(\text{tr} \omega')\), continued: \(\omega\) is Kähler. In the case that \(\omega\) is Kähler, we in addition have \(\partial_k g_{ij} = 0\) for any \(i, j, k\) in Lemma 2.2 and we deduce from the above equation that

\[
\tilde{\Delta} \text{tr} \omega' = \sum_{1 \leq i,j,k \leq n} h^{ij} \chi'_{ij} \partial_{ik} \chi'_{ij} + \sum_{1 \leq i,j,k \leq n} h^{ij} \chi'_{ij} \partial_{ik} \chi'_{ij} + E_2.
\]

(2.17)

It remains to control the term \(\mid \nabla \text{tr} \omega' \mid^2 / (\text{tr} \omega')^2\). Notice that

\[
\partial_i (\text{tr} \omega') = \partial_i \left( g^{ik} \chi'_{ik} \right) = g^{ik} \partial_i \chi'_{ik} = \sum_{1 \leq k \leq n} \partial_i \chi'_{kk}.
\]
As in [17], we first give an inequality for $\frac{\nabla \text{tr}_\omega \chi^\prime_j}{\text{tr}_\omega \chi^\prime}$ and then we control the term $\text{Re}(\sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \partial_i \chi^{j^\prime} (\partial_i \chi^{j^\prime} - \partial_j \chi^{j^\prime}))$. From

$$\frac{|\nabla \text{tr}_\omega \chi^\prime_j|^2}{\text{tr}_\omega \chi^\prime} = \sum_{1 \leq i,j,k \leq n} \frac{h^{i^\prime} \partial_i \chi^\prime_j \partial_k \chi^\prime_k}{\text{tr}_\omega \chi^\prime} = \sum_{1 \leq j,k,i \leq n} \frac{\sqrt{h^{i^\prime} \partial_i \chi^{j^\prime}} \sqrt{h^{i^\prime} \partial_i \chi^{j^\prime}}}{\text{tr}_\omega \chi^\prime}$$

$$\leq \frac{1}{\text{tr}_\omega \chi^\prime} \sum_{1 \leq j,k \leq n} \left( \sum_{1 \leq i \leq n} h^{i^\prime} \left| \partial_i \chi^{j^\prime} \right|^2 \right)^{1/2} \left( \sum_{1 \leq i \leq n} h^{i^\prime} \left| \partial_i \chi^{j^\prime} \right|^2 \right)^{1/2}$$

$$= \frac{1}{\text{tr}_\omega \chi^\prime} \left[ \sum_{1 \leq j \leq n} \left( \sum_{1 \leq i \leq n} h^{i^\prime} \left| \partial_i \chi^{j^\prime} \right|^2 \right)^{1/2} \right]^2$$

$$= \frac{1}{\text{tr}_\omega \chi^\prime} \left[ \sum_{1 \leq j \leq n} \sqrt{\chi^{j^\prime}} \left( \sum_{1 \leq i \leq n} h^{i^\prime} \left| \partial_i \chi^{j^\prime} \right|^2 \right) \right]^{1/2}$$

$$\leq \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \left| \partial_i \chi^{j^\prime} \right|^2 = \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \partial_i \chi^{j^\prime} \partial_j \chi^{j^\prime}.$$ 

From

$$\partial_i \chi^{j^\prime} = \partial_i (\chi^{j^\prime} + \varphi^{j^\prime}) = \partial_i \chi^{j^\prime} + \partial_j \varphi^{j^\prime} = \partial_i \chi^{j^\prime} - \partial_j \chi^{j^\prime},$$

$$\partial_i \chi^{j^\prime} = \partial_i (\chi^{j^\prime} + \varphi^{j^\prime}) = \partial_i \chi^{j^\prime} + \partial_j \varphi^{j^\prime} = \partial_i \chi^{j^\prime} - \partial_j \chi^{j^\prime},$$

it follows that

$$\frac{|\nabla \text{tr}_\omega \chi^\prime_j|^2}{\text{tr}_\omega \chi^\prime} \leq \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \left( \partial_j \chi^{j^\prime} + \partial_i \chi^{j^\prime} - \partial_j \chi^{j^\prime} \right) \left( \partial_j \chi^{j^\prime} + \partial_i \chi^{j^\prime} - \partial_j \chi^{j^\prime} \right)$$

(2.18)

$$= \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \partial_j \chi^{j^\prime} \partial_i \chi^{j^\prime} + \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \left| \partial_i \chi^{j^\prime} - \partial_j \chi^{j^\prime} \right|^2$$

$$+ 2 \cdot \text{Re} \left[ \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \partial_j \chi^{j^\prime} \left( \partial_i \chi^{j^\prime} - \partial_j \chi^{j^\prime} \right) \right].$$

Note that

$$\partial_j \chi^{j^\prime} = \partial_j (\chi^{j^\prime} + \varphi^{j^\prime}) = \partial_j \chi^{j^\prime} + \partial_i \varphi^{j^\prime} = \partial_j \chi^{j^\prime} - \partial_i \chi^{j^\prime} + \partial_i \chi^{j^\prime}.$$

Substituting (2.19) into (2.18) we obtain

$$\frac{|\nabla \text{tr}_\omega \chi^\prime_j|^2}{\text{tr}_\omega \chi^\prime} \leq \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \partial_j \chi^{j^\prime} \partial_i \chi^{j^\prime} - \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \left| \partial_i \chi^{j^\prime} - \partial_j \chi^{j^\prime} \right|^2$$

(2.20)

$$+ 2 \cdot \text{Re} \left[ \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \partial_i \chi^{j^\prime} \left( \partial_i \chi^{j^\prime} - \partial_j \chi^{j^\prime} \right) \right].$$

$$\leq \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \partial_j \chi^{j^\prime} \partial_i \chi^{j^\prime} + 2 \cdot \text{Re} \left[ \sum_{1 \leq i,j \leq n} h^{i^\prime} \chi^{j^\prime} \partial_i \chi^{j^\prime} \left( \partial_i \chi^{j^\prime} - \partial_j \chi^{j^\prime} \right) \right].$$

Lemma 2.4. If $\omega$ is Kähler, then $\Delta \log(\text{tr}_\omega \chi^\prime) \gtrsim -1$. 


Proof. Calculate, since $h^{ij} = \chi^{ij} \chi^{ji}$,

$$
2 \cdot \text{Re} \left[ \sum_{1 \leq i, j \leq n} h^{ij} \chi^{ij} \partial_i \chi'_{jj} \left( \partial_j \chi_{ij} - \partial_j \chi_{ji} \right) \right]
$$

$$= 2 \cdot \text{Re} \left[ \sum_{1 \leq i, j \leq n} h^{ij} \left( \sqrt{h^{ij}} \partial_i \chi'_{jj} \cdot \sqrt{h^{jj}} h^{ji} \left( \partial_j \chi_{ij} - \partial_j \chi_{ji} \right) \right) \right]
$$

(2.21)

$$\leq \sum_{1 \leq i, j \leq n} h^{ij} \chi^{ij} \partial_i \chi'_{jj} \partial_k \chi'_{ji} + \sum_{1 \leq i, j \leq n} \chi'_{jj} (h^{ii})^2 \left| \partial_i \chi_{ij} - \partial_j \chi_{ji} \right|^2
$$

$$\leq \sum_{1 \leq i, j, k \leq n} h^{kk} \chi^{ij} \partial_i \chi'_{kj} \partial_k \chi'_{ij} + E_2
$$

$$= \sum_{1 \leq i, j, k \leq n} h^{ii} \chi^{ij} \partial_k \chi'_{ji} \partial_k \chi'_{ij} + E_2,$$

where $E_2$ is a term of type II:

$$E_2 = \sum_{1 \leq i, j \leq n} \chi'_{jj} (h^{ii})^2 \left| \partial_i \chi_{ij} - \partial_j \chi_{ji} \right|^2.$$

From (2.10), (2.17), (2.20), and (2.21), we have

$$\widetilde{\Delta} \log(\text{tr}_\omega \chi') \geq \frac{1}{\text{tr}_\omega \chi'} \left[ \sum_{1 \leq i, j, k \leq n} h^{ij} \chi^{ij} \partial_k \chi'_{ji} \partial_k \chi'_{jj} + E_2 \right]
$$

$$- \sum_{1 \leq i, j \leq n} h^{ij} \chi^{ij} \partial_j \chi'_{ij} \partial_j \chi'_{ji}]
$$

(2.22)

$$= \frac{1}{\text{tr}_\omega \chi'} \left( \sum_{1 \leq i, j \leq n} \sum_{1 \leq k \leq n} h^{ij} \chi^{ij} \partial_k \chi'_{ij} \partial_k \chi'_{ji} + E_2 \right)
$$

$$= \frac{1}{\text{tr}_\omega \chi'} \left( \sum_{1 \leq i, j \leq n} \sum_{1 \leq j \neq k \leq n} h^{ij} \chi^{ij} \left| \partial_k \chi'_{ij} \right|^2 + E_2 \right)
$$

$$\geq \frac{E_2}{\text{tr}_\omega \chi'}.
$$

By the definition of type II terms, there exists a positive universal constant $C$ satisfying $|E_2|_\omega \leq C \cdot \text{tr}_\omega \chi'$. Therefore

$$\widetilde{\Delta} \log(\text{tr}_\omega \chi') \geq -1.$$

Thus we complete the proof of the lemma. \qed

**Theorem 2.5.** Let $(X, \omega)$ be a compact Kähler manifold of complex dimension $n$, and $\chi$ a Hermitian metric. Let $\varphi$ be a smooth solution of Donaldson’s equation

$$\omega \wedge \chi_{\varphi}^{n-1} = e^F \chi_{\varphi}^n$$

where $F$ is a smooth function on $X$. Assume that

$$\chi \geq \frac{n-1}{ne^n} \omega > 0.$$
Then there are uniform constants $A > 0$ and $C > 0$, depending only on $X, \omega, \chi$, and $F$, such that

$$\text{tr}_\omega \chi \varphi \leq C \cdot e^{A(\varphi - \inf X \varphi)}.$$  

**Proof.** Use the local coordinates in Lemma 2.2. The proof is similar to that in \cite{22, 23}. By the definition, one has

$$\tilde{\Delta} \varphi = h^{kk} p_{kk} = (\chi^{kk})^2 (\chi^{kk} - \chi_{kk}) = \sum_{1 \leq k \leq n} \chi^{kk} - \text{tr}_h \chi = \text{tr}_\chi \omega - \text{tr}_h \chi.$$  

Lemma 2.4 and (2.7) imply that

$$\tilde{\Delta} \varphi = \Delta [\log (\text{tr}_\omega \chi') - A \varphi] \geq -C - A (\text{tr}_\chi \omega - \text{tr}_h \chi)$$

$$\geq -C - A \sum_{1 \leq i \leq n} \chi^{ii} + A \sum_{1 \leq i \leq n} \chi^{ii} \chi^{ii}.$$  

Since $\varphi$ is a solution of Donaldson’s equation, it follows that $\text{tr}_\chi \omega = ne^F$ by (2.4) and hence, for any given positive uniform constants $A$ and $B$ (we will chose those constants later),

$$\tilde{\Delta} \varphi \geq (Bne^F - C) - (A + B) \sum_{1 \leq i \leq n} \chi^{ii} + A \sum_{1 \leq i \leq n} \chi^{ii} \chi^{ii}.$$  

By the assumption we have $\chi \geq \frac{n-1}{ne^F} (1 + \epsilon) \omega$ for some suitable number $\epsilon$ such that $0 < \epsilon < \frac{1}{n-1}$. Let $p \in X$ be a point where $Q$ achieves its maximum; so $\tilde{\Delta} \varphi \leq 0$. At this point, we conclude that

$$0 \geq (Bne^F - C) - (A + B) \sum_{1 \leq i \leq n} \chi^{ii} + A \sum_{1 \leq i \leq n} \chi^{ii} \chi^{ii}$$

$$\geq (Bne^F - C) - (A + B) \sum_{1 \leq i \leq n} \chi^{ii} + A \frac{n-1}{ne^F} (1 + \epsilon) \sum_{1 \leq i \leq n} \chi^{ii}.$$

We denote by $\lambda_i'$ the eigenvalues of $\chi'$ at point $p$ such that $\lambda_1' \leq \cdots \leq \lambda_n'$. Hence

$$0 \geq (Bne^F - C) - (A + B) \sum_{1 \leq i \leq n} \frac{1}{\lambda_i'} + A \frac{n-1}{ne^F} (1 + \epsilon) \sum_{1 \leq i \leq n} \frac{1}{\lambda_i'^2}.$$  

In order to obtain the upper bound for $\lambda_i'$ we need the following

**Lemma 2.6.** Let $\lambda_1, \cdots, \lambda_n$ be a sequence of positive numbers. Suppose

$$0 \geq 1 - \alpha \sum_{1 \leq i \leq n} \frac{1}{\lambda_i} + \beta \sum_{1 \leq i \leq n} \frac{1}{\lambda_i^2}$$

for some $\alpha, \beta > 0$ and $n \geq 2$. If

$$\frac{4}{n} \leq \frac{\alpha^2}{\beta} < \frac{4}{n-1}$$

holds, then

$$\lambda_i \leq \frac{2\beta}{\alpha - \sqrt{n\alpha^2 - 4\beta}}$$

for each $i$.  

Proof. Note that $\alpha - \sqrt{na^2 - 4\beta} > 0$ by (2.23). Since

$$1 + \sum_{1 \leq i \leq n} \left( \frac{\alpha - \sqrt{\beta}}{2 \sqrt{\beta}} \right)^2 \leq \frac{na^2}{4\beta}$$

it implies that

$$\sum_{1 \leq i \leq n} \left( \frac{\alpha - \sqrt{\beta}}{2 \sqrt{\beta}} \right)^2 \leq \frac{na^2 - 4\beta}{4\beta}.$$  

The right hand side of the above inequality is nonnegative by (2.23). Consequently,

$$\frac{\alpha - \sqrt{na^2 - 4\beta}}{2 \sqrt{\beta}} \leq \frac{\sqrt{\beta}}{\lambda_i}.$$  

Hence we obtain (2.24).  

To apply Lemma 2.6, we assume

$$(2.25) Bne^F > C,$$

and set

$$(2.26) \alpha \doteq \frac{A + B}{Bne^F - C}, \quad \beta \doteq \frac{A_{n+1}}{Bne^F - C}.$$  

In the following we will find the explicit formulas for $A$ and $B$ in terms of $C$ such that the assumption (2.25) and the condition (2.23) are both satisfied.

We choose a real number $\eta$ satisfying

$$(2.27) \quad 0 \leq \eta < 1.$$  

Set

$$(2.28) \quad \frac{\alpha^2}{\beta} = \frac{4}{n - \eta},$$  

where $\alpha$ and $\beta$ are given in (2.26). If (2.28) was valid, then the condition (2.23) is true. Equations (2.26) and (2.28) imply

$$(A + B)^2 = \frac{4}{n - \eta}(1 + \epsilon)(Bne^F - C) \frac{n - 1}{ne^F}A$$

so that

$$A^2 + B^2 + 2 \left[ 1 - \frac{2(1 + \epsilon)(n - 1)}{n - \eta} \right] AB + \frac{4(1 + \epsilon)(n - 1)C}{(n - \eta)ne^F}A = 0.$$  

The above relation can be rewritten as

$$\left[ A + \left( 1 - \frac{2(1 + \epsilon)(n - 1)}{n - \eta} \right) B \right]^2 = \left[ \left( 1 - \frac{2(1 + \epsilon)(n - 1)}{n - \eta} \right)^2 - 1 \right] B^2 - \frac{4(1 + \epsilon)(n - 1)C}{(n - \eta)ne^F}A.$$  

Taking

$$(2.29) \quad A = \left( -1 + \frac{2(1 + \epsilon)(n - 1)}{n - \eta} \right) B$$
we have $A > B$ and

$$
(2.30) \quad B = \frac{4(1+\epsilon)(n-1)C}{n^e} \left( -1 + \frac{2(1+\epsilon)(n-1)}{n^e} \right) = \frac{C}{n^e} \left[ -1 + \frac{2(1+\epsilon)(n-1)}{n^e} \right] = \frac{C}{n^e} \cdot \frac{2(1+\epsilon)(n-1)}{n^e}.
$$

assuming

$$
(2.31) \quad (1 + \epsilon) > \frac{n - \eta}{n - 1}.
$$

From $2.30$ and $2.31$ we see that

$$
\frac{Bn^e}{C} = \frac{-1}{-1} + \frac{2(1+\epsilon)(n-1)}{n^e} > 1.
$$

From the assumption $0 < \epsilon < \frac{1}{n-1}$ we have $0 < n - (n-1)(1+\epsilon) < 1$ and then such a $\eta$ always exists. Hence Lemma 2.6 yields

$$
\lambda_i' \leq \frac{2\beta}{\alpha - \sqrt{n\alpha^2 - 4\beta}}
$$

where $\alpha$ and $\beta$ are determined by $2.26$, $2.29$, and $2.30$. Since $tr_\omega \chi' = \sum_{i=1}^n \lambda_i'$, it follows that, at $p \in X$, $tr_\omega \chi' \leq C$ for some uniform constant $C$ and, for any point $q \in X$,

$$
Q(q) \leq Q(p) \leq \log(tr_\omega \chi')(p) - A\varphi(p) \leq C - A \inf \varphi.
$$

Equivalently, $\log(tr_\omega \chi') \leq C + A(\varphi - \inf \varphi)$. \hfill $\square$

### 2.4. The estimate for $\bar{\Delta} \log(tr_\omega \chi')$, continued: general case.

Now we consider the general case that both $\omega$ and $\chi$ may not be Kähler. Using Lemma 2.2 we have

$$
\bar{\Delta} tr_\omega \chi' = \sum_{1 \leq i,j,k \leq n} h^{ij} \chi^{ij}_{k} \partial_k \chi^i_j + \sum_{1 \leq i,j,k \leq n} h^{i'} \chi^{i'j}_{k} \partial_k \chi^{j'}_i + E_2
$$

$$
- 2 \cdot \Re \left( \sum_{1 \leq i,j,k \leq n} h^{i'} \chi^{i'j}_{k} \partial_k g_{ji} \partial_k \chi^j_i \right)
$$

$$
- 2 \cdot \Re \left( \sum_{1 \leq i,j,k \leq n} h^{i'} \partial_ig_{jk} \partial_k \chi^j_i \right).
$$

As in [17] we deal with the last two terms by using the local coordinates in Lemma 2.2. Starting from the last term, we calculate

$$
\sum_{1 \leq i,j,k \leq n} h^{i'} \partial_i \chi^j_k \partial_j g_{jk} = \sum_{1 \leq i,j,k \leq n} h^{i'} \partial_i g_{jk} \partial_i \chi^j_k + \varphi_{kij}
$$

$$
= \sum_{1 \leq i,j,k \leq n} h^{i'} \partial_i g_{jk} \partial_i \chi^j_k + h^{i'} \partial_i g_{jk} \partial_k \chi^j_i + \varphi_{kij}
$$

$$
= \sum_{1 \leq i,j,k \leq n} h^{i'} \partial_i g_{jk} \partial_k \chi^j_i + \varphi_{kij}.
$$

$$
(2.33) \quad \bar{\Delta} tr_\omega \chi' = \sum_{1 \leq i,j,k \leq n} h^{i'} \partial_i g_{jk} \partial_k \chi^j_i + \varphi_{kij} + E_1.
$$
where $E_1$ is a term of type I and is given by

\begin{equation}
E_1 = \sum_{1 \leq i,j,k \leq n} h^{\bar{i}} \partial_{\bar{j}} g_{jk} \left( \partial_i \chi_{kj} - \partial_k \chi_{ij} \right).
\end{equation}

Taking the real part of (2.33) gives

\begin{equation}
\begin{aligned}
(2.35) \quad \left| 2 \cdot \text{Re} \left( \sum_{1 \leq i,j,k \leq n} h^{\bar{i}} \partial_{\bar{j}} g_{jk} \right) \right| &= \left| 2 \cdot \text{Re} \left( \sum_{1 \leq i \leq n \leq 1 \leq j \neq k \leq n} \sqrt{h^{\bar{i}}} \sqrt{\chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime \cdot \sqrt{h^{\bar{i}}} \sqrt{\chi_{ij}^{\bar{j}} \partial_k g_{jk}}} \right) + E_1 \right| \\
&\leq \sum_{1 \leq i \leq n \leq 1 \leq j \neq k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime + \sum_{i=1}^n \sum_{1 \leq j \neq k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k g_{jk} + E_1 \\
&\leq \sum_{1 \leq i \leq n \leq 1 \leq j \neq k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime + E_2,
\end{aligned}
\end{equation}

since $\sum_{i=1}^n \sum_{1 \leq j \neq k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k g_{jk} \partial_k g_{kj}$ is of type II. Similarly we have

\begin{equation}
(2.36) \quad \left| 2 \cdot \text{Re} \left( \sum_{1 \leq i,j,k \leq n} \chi^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime \partial_k g_{ij} \right) \right| = \left| 2 \cdot \text{Re} \left( \sum_{1 \leq i,j,k \leq n} \sqrt{h^{\bar{i}}} \sqrt{\chi^{\bar{i}} \partial_k \chi_{ij}^\prime \cdot \sqrt{\chi^{\bar{i}} \partial_k g_{ij}}} \right) \right| \\
\leq \frac{1}{2} \sum_{1 \leq i,j,k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime + E_1 = \frac{1}{2} \sum_{1 \leq i,j,k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime + E_1,
\end{equation}

where

\begin{equation}
E_1 = 2 \sum_{1 \leq i,j,k \leq n} \chi^{\bar{i}} \partial_k g_{ij} \partial_k g_{ji}
\end{equation}

is a term of type I.

From (2.32), (2.35), and (2.36), we conclude that

\begin{equation}
\Delta \text{tr}_\omega \chi^\prime \geq \sum_{1 \leq i,j,k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime + \sum_{1 \leq i,j,k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime \\
- \sum_{i=1}^n \sum_{1 \leq j \neq k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime \partial_k \chi_{ij}^\prime - \frac{1}{2} \sum_{1 \leq i,j,k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime \partial_k \chi_{ij}^\prime + E_2 \\
= \frac{1}{2} \sum_{1 \leq i,j,k \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime \partial_k \chi_{ij}^\prime + \sum_{1 \leq i,j \leq n} h^{\bar{i}} \chi_{ij}^{\bar{j}} \partial_k \chi_{ij}^\prime \partial_k \chi_{ij}^\prime + E_2.
\end{equation}
It remains to control the term $|\tilde{\nabla}\Tr_{\omega}\chi'|^2_h/(\Tr_{\omega}\chi')^2$. As in (2.20) one has

$$\frac{|\tilde{\nabla}\Tr_{\omega}\chi'|^2_h}{\Tr_{\omega}\chi'} \leq \sum_{1 \leq i, j \leq n} h^{i\bar{j}} \chi'^{i\bar{j}} \partial_i \chi'_j \partial_j \chi'_i$$

(2.39) \quad + 2 \cdot \text{Re} \left[ \sum_{1 \leq i, j \leq n} h^{i\bar{j}} \chi'^{i\bar{j}} \partial_i \chi'_j (\partial_j \chi_{ji} - \partial_j \chi_{ij}) \right].

**Lemma 2.7.** One has $\tilde{\Delta}\log(\Tr_{\omega}\chi') \gtrsim -1$.

**Proof.** As in the proof of Lemma 2.4 we have

$$2 \cdot \text{Re} \left[ \sum_{1 \leq i, j \leq n} h^{i\bar{j}} \chi'^{i\bar{j}} \partial_i \chi'_j \partial_j \chi'_i \right]$$

(2.40) \quad = \left| 2 \cdot \text{Re} \left[ \sum_{1 \leq i, j \leq n} \sqrt{h^{i\bar{j}}} \sqrt{\chi'^{i\bar{j}}} \partial_i \chi'_j \partial_j \chi'_i \right] \right| \leq \frac{1}{2} \sum_{1 \leq i, j \leq n} h^{i\bar{j}} \chi'^{i\bar{j}} \partial_i \chi'_j \partial_j \chi'_i + 2 \sum_{1 \leq i, j \leq n} \chi'_j (h^{i\bar{j}})² \left| \partial_i \chi_{ji} - \partial_j \chi_{ij} \right|^2 \leq \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{i\bar{j}} \chi'^{i\bar{j}} \partial_k \chi'_{ij} \partial_k \chi'_{ij} + E_2 = \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{i\bar{j}} \chi'^{i\bar{j}} \partial_k \chi'_{ij} \partial_k \chi'_{ij} + E_2,

where $E_2$ is a term of type II and given by

$$E_2 = 2 \sum_{1 \leq i, j \leq n} \chi'_j (h^{i\bar{j}})² \left| \partial_i \chi_{ji} - \partial_j \chi_{ij} \right|^2.$$

Combining (2.40) with (2.20), (2.38), and (2.39), we arrive at

$$\tilde{\Delta}\log(\Tr_{\omega}\chi') \geq \frac{1}{\Tr_{\omega}\chi'} \left[ \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{i\bar{j}} \chi'^{i\bar{j}} \partial_k \chi'_{ij} \partial_k \chi'_{ij} + \sum_{1 \leq i, j \leq n} h^{i\bar{j}} \chi'^{i\bar{j}} \partial_i \chi'_j \partial_j \chi'_i \right. - \left. \sum_{1 \leq i, j \leq n} h^{i\bar{j}} \chi'^{i\bar{j}} \partial_i \chi'_j \partial_i \chi'_j - \frac{1}{2} \sum_{1 \leq i, j, k \leq n} h^{i\bar{j}} \chi'^{i\bar{j}} \partial_k \chi'_{ij} \partial_k \chi'_{ij} + E_2 \right] = \frac{E_2}{\Tr_{\omega}\chi'}.$$

By the definition of type II terms, there exists a positive uniform constant $C$ satisfying $|E_2| \leq C \cdot \Tr_{\omega}\chi'$. Therefore

$$\tilde{\Delta}\log(\Tr_{\omega}\chi') \geq -C.$$

This complete the proof. \qed

By using the similar method as in the proof of Theorem 2.6 we have

**Theorem 2.8.** Let $(X, \omega)$ be a compact Hermitian manifold of the complex dimension $n$, and $\chi$ another Hermitian metric. Let $\varphi$ be a smooth solution of Donaldson’s equation

$$\omega \wedge \chi^{n-1}_\varphi = e^F \chi^n_\varphi,$$

where $F$ is a smooth function on $X$. Assume that

$$\chi - \frac{n-1}{ne^n} \omega > 0.$$
Then there are uniform constants $A > 0$ and $C > 0$, depending only on $X, \omega, \chi,$ and $F$, such that
\[
\text{tr}_\omega \chi \varphi \leq C \cdot e^{A(\varphi - \inf_X \varphi)}.
\]

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