Mathematical model of highways network optimization

R L Sakhapov1, R V Nikolaeva, M H Gatiyatullin and M M Makhmutov

Kazan State University of Architecture and Engineering, 420043, Kazan, Zelenaya str., 1, Russia

E-mail: rusakhapov@gmail.com

Abstract. The article deals with the issue of highways network design. Studies show that the main requirement from road transport for the road network is to ensure the realization of all the transport links served by it, with the least possible cost. The goal of optimizing the network of highways is to increase the efficiency of transport. It is necessary to take into account a large number of factors that make it difficult to quantify and qualify their impact on the road network. In this paper, we propose building an optimal variant for locating the road network on the basis of a mathematical model. The article defines the criteria for optimality and objective functions that reflect the requirements for the road network. The most fully satisfying condition for optimality is the minimization of road and transport costs. We adopted this indicator as a criterion of optimality in the economic-mathematical model of a network of highways. Studies have shown that each offset point in the optimal binding road network is associated with all other corresponding points in the directions providing the least financial costs necessary to move passengers and cargo from this point to the other corresponding points. The article presents general principles for constructing an optimal network of roads.

Keywords: road network, road transport, correspondence points, mathematical model, route, optimization

1. Introduction

The availability of a reliable and efficient transport system is an important factor in the development of the economy of any country. Today, transport is an important part of the world economy, as it is a material carrier between states. Presence of a well-developed transport system is one of the important factors of attracting people and production and is an important advantage for locating productive forces, which gives an integration effect.

The dominant type of transport around the world is automobile, which is rapidly growing every year. The importance of the road network can also be judged by the share of the total volume of passenger and freight traffic carried out by automobiles. Studies of foreign scientists have shown that road transport is one of the largest contributions to budget revenues. For example, road user taxes and fees in the US amounted to $ 78 billion in 1994 (6.2 percent of federal revenues of the state budget) and $ 33 billion in the United Kingdom in 1995-96 (of which only $ 10 billion was spent on roads). The share of the transport system in the main production assets of Russia is 27%, and a significant share of transportation services in the gross domestic product is 8%. Net financial flows from the road sector tend to be positively correlated with economic development [10].

1 To whom any correspondence should be addressed.
The road network is a complex, large-scale system. The road network is the basis for the functioning of the economy and society through the movement of people, goods and services. At present, in most developed countries of the world, there is a serious imbalance in supply and demand in the transport sector: traffic volumes, especially in large megacities, are approaching the limits of the capacity of existing highways, and sometimes surpass them in rush hour [11,15].

From the economic point of view, the construction of the road network should be based on the basic topology of the network, the demand of the users of the road network and the cost of its construction. Equilibrium occurs when the number of trips between the corresponding departure points (for example, the place of residence, the place of work) and the points of arrival (place of work, residence) are equal to the demand for travel taking into account the cost policy [9].

The study of the road network has been developed at the intersection of various fields of knowledge - physics, mathematics, economics, operating systems research, transportation processes, etc. Measurements, observations and modeling of individual processes in the transport sector and the totality of demographic and socioeconomic relations are important for the development of a transport infrastructure development strategy [4,8].

A consequence of the difficulty of optimizing the road network is a certain imbalance between the results of scientific research and mathematical calculations, on the one hand, and the practically observed results, on the other. That’s why today we have various approaches to mathematical modeling and a sufficiently large number of software products modeling transport flows.

In recent years, successful mathematical and computational methods have been developed to formulate the model of a dynamic transport system. Many of them generalize traditional statistical methods; advanced and realistic models use a traffic simulator, traffic flow dynamics. The number of software products for the dynamic transport systems, which are sufficiently tested in the developed countries of the world, includes: DYNASMART [2], Dynamic [7], AIMSUN [1]. Creating models of transport systems in these programs is a rather cumbersome task that involves the presence of a large number of elements and requires a long process of calibrating hundreds or thousands of coefficients.

The purpose of this article is to present a network of highways in the form of a mathematical model and to select the necessary conditions for optimizing the road network. The road network is represented as a set of corresponding points (departure and arrival points) connected by a set of routes. As a result of its use, it is possible to reduce the task of optimizing the road network to a multitude of similar tasks of distributing the transport flow between two nodes along the routes and obtaining the equilibrium distribution of the transport stream in an explicit form. In turn, the explicit form of such distributions will make it possible to substantially reduce the time spent on computations.

2. Fundamental solutions, concepts and models of road networks

The road network model is the basis for solving many transport tasks, such as optimal network planning, improvement of traffic organization, optimization of public transport routes system, etc.

In 1952, Wardrop suggested that any transport system after a certain time comes to an equilibrium state and formulated two principles of the equilibrium distribution of traffic flows [12]. According to the first principle, "the travel time for all used routes is the same for all participants of the traffic, and less time that any participant will spend changing their route," and according to the second, "the average travel time is minimal."

Beckmann, McGuire, and Winsten (1956) were first who mathematically formulated these conditions [13]. Specifically, Beckmann, McGuire, and Winsten (1956) established an equivalence between transport network equilibrium conditions that argue that all paths connecting a pair of correspondent points will have equal and minimum travel times (or costs) [13]. Consequently, in this case, the equilibrium of the transport network can be obtained by solving the problem of mathematical programming [4]. This approach allowed to formulate, analyze and then solve the transport problem on the basis of actual transport networks.

Dafermos and Sparrow (1969) came up with terms "optimization-user" and "optimization-system" for the transport network [14]. These terms distinguish two situations, in the first situation, users act
unilaterally in their own interests when choosing routes, in the second situation, users choose the optimal routes from a public point of view, taking into account the minimum cost in the transport system. Fig. 1 presents the two behavioral principles underlying transport networks.

![Figure 1. Behavioral principles of transport networks](image)

Traditional methods of transport planning, including methods of network design, focus on improving mobility based on parameters such as speed and travel time. However, a mobile-oriented approach to transportation planning can stimulate the demand for travel and lead to congestion [3]. While mobility planning can increase the availability of the network, the availability of planning extends the range of possible solutions compared to the mobile-oriented strategy [6]. One of the obstacles is the lack of methods to solve the problem of accessibility directly, rather than mobility in transporting the network design.

In this situation, the goal of optimizing the transport network is to determine the shortest distance between the corresponding points. The road network should be planned in such a way as to minimize travel time and financial costs in the transport system.

3. Requirements for the optimal location of the road network

The location, density and composition of the road network form under the influence of numerous factors, which can be conditionally divided into three groups: economic, social and natural. When designing roads, taking these factors into account leads to the need for a quantitative assessment of their impact. The variety of these factors makes this assessment difficult. The question arises of identifying the most significant factors that determine the development of the road network. For the construction of the optimal version of the road network, it is necessary to investigate the influence of each factor on the result of the solution. It is impossible to conduct such studies in real conditions, because for this it is necessary to firstly build roads and only then to make a comparison. In such cases, studies can be carried out on mathematical models. The mathematical model will allow describing the process under study in the form of equations and inequalities.

When creating a mathematical model of the road network, it is necessary to first of all determine the criterion of optimality and to choose an objective function that reflects the requirements for the road network in the design.

To model a transport network it’s necessary to have:

- cartographic material; Usually these are large-scale maps, since they allow you to measure with great accuracy the distances between points;
- information on the location of the main objects of the transport system and its environment (depending on the task to be solved: load-forming and load-absorbing enterprises, residential areas, places of employment, etc.);
- additional information from municipal and road organizations in the form of a list of streets with a characteristic of their carriageway;
• Information on the organization of traffic, i.e., traffic management schemes at intersections, squares and traffic intersections, as well as information on various traffic restrictions associated with established road signs.

4. Structure of the road network

To describe the structure of the transport network we apply graph theory, where the transport network is represented as an oriented graph consisting of a set of successively numbered vertices and a set of consecutively numbered edges (arcs) symbolizing the elements and their connections. The graph $G$ modeling the road network must necessarily be tied, so that there is always a path from any vertex to any other vertex. Numbers characterizing the links of such a graph usually express the length of the route, time or cost of travel.

Let’s introduce the notation: $m$ is the set of successively numbered nodes of the graph $G$; $Z$ is the set of sequentially numbered arcs of the graph $G$; $I$ is the set of nodes that are the departure points, $i \subseteq m$; $J$ is the set of nodes that are arrival points, $j \subseteq m$; It is assumed that $i \cap j = \emptyset$; $M_{ij}$ is the set of routes between the departure points $i \in I$ and the arrival points $j \in J$.

The graph $F$ modeling the road network is determined by the set of vertices and the set of pairs of vertices, between which there is a link (Fig. 2).

![Figure 2. Model of road network graph](image)

The task in this case is to decide: to include the $k$-th link in the projected network or not to include. The number of links of the $k$-th type is indivisible and can take only two values: 0 and 1. The definition of the unknown links of the road network ($Z_k$) must satisfy the condition:

$$Z_k = 0;1 \quad (k = 1,2,3,...),$$

or

$$Z_k = \begin{cases} 1, & \text{if arc } e \in Z \text{ is on the route } M_{ij} \\ 0, & \text{in the other case} \end{cases}.$$ (2)

Let’s suppose, to take the minimum of road costs as a criterion of optimality (costs for construction, maintenance and repair of the road network). In this case, the optimal option will be met by the network with the least total length, for which the amount of travel costs is minimal:

$$\sum_{k} D_k l_k = \min ,$$ (3)

where:

$D_k$ – road costs associated with the construction, maintenance and repair of the road network;

$l_k$ – length of road network;
k – number of sections (links of the road network).

If the road network is required to reduce the cost of transport work or the travel time of passengers or cargo, the only solution that meets these requirements is to ensure all transport links by independent roads along the shortest path, i.e. direct connection between each of the corresponding points (departure and arrival points). The solution of the problem is to determine the unknown routes between the points $i$ and $j$ $(M_{ij})$ satisfying the following condition:

$$M_{ij} \geq (i = 1,2,3,...,m; j = 1,2,3,...,m),$$

(4)

for which transport operation on the route $ij$ $(P_{ij})$ or connection time between $i$ and $j$ $(t_{ij})$ will be minimal:

$$\sum_{i \rightarrow j} P_{ij} = \min,$$

(5)

$$\sum_{i \rightarrow j} t_{ij} = \min.$$

(6)

Considering the complexity and practical limitations of numerical programming, combinatorial analysis methods can be used in solving such problems, viewing different combinations of variable values, but only a reasonably selected part of possible combinations.

The realization of transportation with the lowest possible financial costs corresponds to the reduction in the costs of the means of production consumed in transport, and the criterion for assessing the optimality in the design of the road network corresponds to the general criterion of optimality for the economy of the country. This most fully satisfies the condition of optimality of the minimum financial costs, changing with the change in the outline of the network:

$$\sum_{i \rightarrow j} E_{ij} = \min,$$

(7)

where $\sum_{i \rightarrow j} E_{ij}$ - total financial costs, i.e. costs associated with the construction, maintenance and repair of roads and with the movement of vehicles in the directions $ij$.

To meet the requirement (7), it is necessary and sufficient that at each of the directions of the projected network, a possible minimum financial cost is provided, i.e. to:

$$E_{1-2} = \min;$$
$$E_{1-3} = \min;$$
$$\vdots$$
$$E_{1-n} = \min;$$
$$\vdots$$
$$E_{ij} = \min.$$

(8)

The validity of condition (8) can be proved in the following way. Let's say that there is an optimal road network for which:

$$E_{1-2} = \min$$
$$E_{1-3} = \min$$
$$\vdots$$
$$E_{1-n} = \min$$
$$\vdots$$

but

$$E_{ij} \neq \min.$$

If we replace the route for which $E_{ij} \neq \min$, by such a route that $E'_{ij} = \min$, then the sum of all financial costs on the network will decrease by the amount:
\[ \Delta = E_{ij} - E'_{ij}, \]

and it appears that:

\[ \sum_i \sum_j E'_{ij} < \sum_i \sum_j E_{ij}. \]  

(10)

It follows that for the case under consideration \( \sum_i \sum_j E'_{ij} \not= \min \) the original proposal on the optimality of the road network, for which at least one route \( E_{ij} \not= \min \), is incorrect.

Thus, unlike the original proposal, the road network can not be optimal if inequality (8) is not respected.

5. Economic and mathematical model of road network optimization

The economic-mathematical model of a network of highways should be considered as a special case of the model of optimization of a unified transport network.

On the basis of the above considerations, it seems possible to draw up a system of equations characterizing the main regular relations between the formulated requirements of road transport and the deployment of a network of highways:

\[
\begin{align*}
\sum_i \sum_j E_{ij} &= \min; \quad E_{ij} = \min; \\
F(l_{ij}, Q_{ij}, D_{ij}, v_{ij}, E_{ij}) &= 0; \\
M_{ij} &\geq (i=1,2,3,...,m; \ j=1,2,3,...,m)
\end{align*}
\]  

(11)

where:
- \( l_{ij} \) – route length between the corresponding points \( i \) and \( j \);
- \( Q_{ij} \) – traffic between the corresponding points \( i \) and \( j \);
- \( D_{ij} \) – road costs that ensure the movement of vehicles in the direction;
- \( A_{ij} \) – transport costs in the same direction;
- \( v_{ij} \) – average vehicle speed.

The system (11) reflects the basic necessary condition for optimal road networks and the general principle of their construction in the mathematical formulation. Given mathematical model instructs on ways of the decision of a problem contains: definition of road and transport expenses; Selection of routes on which these costs are minimal, which leads to the fulfillment of the requirement expressed by the first ratio. The penultimate ratio in the system indicates that there is a functional dependence on a certain route between the objects of transportation, traffic speed, road and transport costs. The solution of this dependence with respect to \( E_{ij} \), i.e. determination of total road and transport costs, is of independent interest and is an essential element of the task as a whole.

The principal relationship between total costs and their road and transport components can be represented by the equation:

\[ E_{ij} = l_{ij} (D_{ij} + A_{ij}). \]

(12)

Considering the issue of optimal allocation of the road network, it is legitimate to assume that the most loaded links of the network should first of all be selected in the optimal network. If, during the selection of links, it becomes necessary to identify passenger transportations and consider them separately, this will require additional calculations and new principles for constructing optimal binding networks of roads.

The general principle of constructing the optimal binding network of roads is that any corresponding point in the optimal connecting road network is connected to all other corresponding
points of the links, which ensure the least specific road and transport costs for moving goods from this point to all other corresponding network points.

Drawing up a mathematical model of the road network, one should start from the following limiting conditions:

- the optimal network must be a binder, i.e. must connect all without exception the corresponding points (there must be at least one route between each pair of points);
- for various reasons, some link \( z_k \) must be excluded from the construction of the optimal network or, conversely, necessarily included in it.

This condition is expressed by equality (1), and the transport-operational level of the projected network must be such that the speed \( v_{ij} \) between points \( i \) and \( j \) can not be less than a certain predetermined speed, the traffic intensity \( N_{ij} \) is not more than \( \overline{N}_{ij} \), the safety factor of traffic \( K_{\delta_{ij}} \) is not less than \( \overline{K}_{\delta_{ij}} \).

As a result, the economic-mathematical model for optimizing the road network can be represented by the following system of equations and inequalities:

\[
M_{ij} \geq \begin{cases} i = 1, 2, 3, \ldots, m; & j = 1, 2, 3, \ldots, m \end{cases} \\
\sum_{i} z_k = 0; l; \\
L_{ij} = \frac{L_{ij}}{Q_{ij}} \left(D_{ij} + A_{ij}\right); \\
L_{ij} = \min; \\
\sum_{j} L_{ij} = \min; \\
v_{ij} \geq \overline{v}_{ij}; \quad N_{ij} \geq \overline{N}_{ij}; \quad K_{\delta_{ij}} \geq \overline{K}_{\delta_{ij}}; \\
\end{cases}
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As a result, the economic-mathematical model for optimizing the road network can be represented by the following system of equations and inequalities:

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\sum_{i} z_k = 0; l; \\
L_{ij} = \frac{L_{ij}}{Q_{ij}} \left(D_{ij} + A_{ij}\right); \\
L_{ij} = \min; \\
\sum_{j} L_{ij} = \min; \\
v_{ij} \geq \overline{v}_{ij}; \quad N_{ij} \geq \overline{N}_{ij}; \quad K_{\delta_{ij}} \geq \overline{K}_{\delta_{ij}}; \\
\end{cases}
\]

6. Conclusion
The goal of optimizing the road network is to improve the efficiency of transport. The most fully satisfying condition for optimality is the minimum financial costs associated with the construction, maintenance and repair of the road network and the movement of vehicles along them. In the study, this indicator was adopted as a criterion of optimality in the economic and mathematical model of the road network. Therefore, the main requirement for the road network from the road transport side is to ensure the realization of all the transport links served by it with the least cost required for this.

From the example considered, it can be concluded that the main condition for the optimal binding networks of roads is the following: each corresponding point in the optimal connecting road network is connected with all other corresponding points in the directions providing the least financial costs necessary for moving passengers and cargo from this point to other corresponding points.

This also implies the general principle of constructing an optimal connecting road network, which consists in the fact that any isolated point in the optimal connecting road network is connected to all other corresponding points, which ensure the least financial costs for the movement of passengers and goods from this point to all other corresponding Network point.

On the basis of the general principle, the task is to select such links, under which routes would be created that would ensure road and transport costs between two corresponding points.

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