NON-SYMMETRIC JACOBI AND WILSON TYPE POLYNOMIALS

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Abstract. Consider a root system of type $BC_1$ on the real line $\mathbb{R}$ with general positive multiplicities. The Cherednik-Opdam transform defines a unitary operator from an $L^2$-space on $\mathbb{R}$ to a $L^2$-space of $\mathbb{C}^2$-valued functions on $\mathbb{R}^+$ with the Harish-Chandra measure $|c(\lambda)|^{-2}d\lambda$. By introducing a weight function of the form $\cosh^{-\sigma}(t) \tanh^{2k} t$ on $\mathbb{R}$ we find an orthogonal basis for the $L^2$-space on $\mathbb{R}$ consisting of even and odd functions expressed in terms of the Jacobi polynomials (for each fixed $\sigma$ and $k$). We find a Rodrigues type formula for the functions in terms of the Cherednik operator. We compute explicitly their Cherednik-Opdam transforms. We discover thus a new family of $\mathbb{C}^2$-valued orthogonal polynomials. In the special case when $k = 0$ the even polynomials become Wilson polynomials, and the corresponding result was proved earlier by Koornwinder.

1. Introduction and Main result

The Hermite polynomials and functions on the Euclidean space $\mathbb{R}$ play an important role in Fourier analysis and in the theory of special functions. The Hermite functions are a product of the Gaussian function and the Hermite polynomials, they form an orthogonal basis of the space $L^2(\mathbb{R})$ and their Fourier transforms are also Hermite type functions. A natural generalization of the Fourier transform is the Jacobi transform for even functions on $\mathbb{R}$. In his paper \cite{Koornwinder} Koornwinder computed the Jacobi transforms of certain even orthogonal function of Jacobi type and proved that they are, up to some factor of Gamma functions, the Wilson hypergeometric orthogonal polynomials. The Jacobi transform is, for special parameters, the Harish-Chandra spherical transform in rank one. Recently \cite{Opdam} Opdam introduced a generalization of the Harish-Chandra spherical transform for general non-symmetric functions on any root system $R$ in $\mathbb{R}^r$ with general positive root multiplicities, and proved the corresponding transform is unitary from certain $L^2$-space to a space of vector-valued functions on the positive Weyl chamber in $\mathbb{R}^r$ with the Harish-Chandra measure $|c(\lambda)|^{-2}d\lambda$. The transform is also called the Cherednik-Opdam transform. When restricted to symmetric (i.e. Weyl group invariant) functions on root systems of a non-compact symmetric space $G/K$, the transform reduces to the spherical transform for $K$-invariant functions.

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There appears thus a natural question, namely to find an orthogonal basis on $\mathbb{R}^r$ and to compute its Cherednik-Opdam transform. Motivated by the study of Berezin transform [14], [13], and branching rule [15], Zhang introduces [16] certain weight function, also called canonical weight $w_\sigma(t)$ on root system of Type BC of arbitrary rank, computes the spherical transform $\tilde{w}_\sigma(\lambda)$ and proves that the (Weyl group-invariant) Jacobi polynomials $P^J$ multiplied by the weight function, have their spherical transform being multi-variable Wilson polynomials $P^W$ [12] multiplied by the spherical transform $\tilde{w}_\sigma(\lambda)$ of $w_\sigma$, thus generalizing the classical theory of Fourier transform of Gaussian and Hermite functions. The rank one case has been done earlier by Koornwinder [7] where the spherical transform is the classical Jacobi transform. In the present paper we will study the non-symmetric analogue of the above result of Koornwinder in rank one in the setup of Opdam-Cherednik transform.

Let $R = \{\pm 2\varepsilon, \pm 4\varepsilon\} \subset \mathbb{R}\varepsilon$ be a root system on $\mathbb{R}$ of type BC with general positive multiplicities. Consider the $L^2$-space $L^2(\mathbb{R}, d\mu)$ with $d\mu$ given in (2.1). Consider the weight function $w_{\sigma,k}(t) = (\cosh^{-\sigma} t) \tanh^{2k} t$ for fixed $\sigma$ and integer $k$ and the system $\{w_{\sigma,k}(t)(\tanh t)^n\}_{n=0}^\infty$; the latter forms a basis of the $L^2$-space and we find an orthogonal basis of the form $w_{\sigma,k}(t)p_n(\tanh t)$ where $p_n$ can be written in terms of Jacobi polynomial of degree $n$ (see Lemma 3.1). We find a Rodrigues type formula for the functions in Theorem 3.5, and their Cherednik-Opdam transforms are computed in Theorem 4.1 and Theorem 4.3, written in terms of certain $3F_2$-hypergeometric series. The functions and their transform can be viewed as non-symmetric analogues of the Jacobi and Wilson polynomials explaining the title of this paper.

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2. Preliminaries. Cherednik-Opdam transform

We recall briefly the Plancherel formula for non-symmetric functions proved by Opdam [22] applied to the root system of type $BC_1$ (in which case it can also be deduced from the known Plancherel formula for the Jacobi transform for symmetric functions). We use the same notations as in [16]. Let $R = \{\pm 2\varepsilon, \pm 4\varepsilon\}$ be a root system of type $BC_1$ with multiplicities $k_{2\varepsilon} = b$ and $k_{4\varepsilon} = \frac{1}{2}$ respectively. We normalize $\varepsilon$ as a unit vector and identify $\mathbb{R}\varepsilon$ with $\mathbb{R}$ and with its dual space. The half sum of positive roots is then $\rho = b + \iota$.

Let $d\mu$ be the measure

$$d\mu(t) = \prod_{\alpha=2\varepsilon,4\varepsilon} |2 \sinh(\frac{1}{2} \alpha(t))|^{2k_\alpha} dt = 2^{2b+\iota}|\sinh t|^{2b}|\sinh(2t)|^\iota dt$$
on $\mathbb{R}$, and let $L^2(\mathbb{R}, d\mu)$ be the corresponding $L^2$-space. The Weyl group $\{\pm 1\}$ acts on $L^2(\mathbb{R}, d\mu)$ via $f(t) \mapsto f(\pm t)$, and under this action the space is decomposed as

$$L^2(\mathbb{R}, d\mu) = L^2(\mathbb{R}, d\mu)_1 \oplus L^2(\mathbb{R}, d\mu)_{-1},$$

a sum of subspaces of even and respectively odd functions.

Let

$$D := \partial + 2i \frac{1}{1 - e^{-4t}}(1 - s) + 2b \frac{1}{1 - e^{-2t}}(1 - s) - \rho,$$

be the Cherednik operator on $L^2(\mathbb{R}, d\mu)$. Here $sf(t) = f(-t)$ is the reflection. The eigenvalue problem of $D$ is solved in [9]; the function

$$G(\lambda, t) = _2F_1\left(\frac{\lambda + \rho}{2}, -\frac{\lambda + \rho}{2}; \frac{1 + t}{2}; b, -\sinh^2 t\right)$$

$$+ \frac{1}{\lambda + \rho} \sinh(2t) _2F_1\left(\frac{\lambda + \rho}{2}, -\frac{\lambda + \rho}{2}; \frac{1 + t}{2}; b, -\sinh^2 t\right)$$

is an eigenfunction of $D$,

$$(2.3) \quad DG(\lambda, t) = \lambda G(\lambda, t).$$

Here $_2F_1(a_1, a_2; b, x)$ is the Gauss hypergeometric function. For later purpose we recall that the hypergeometric function is defined by

$$pF_q(a_1, \ldots, a_p, b_1, \ldots, b_q, x) = \sum_{m=0}^{\infty} \left(\frac{a_1}{b_1}\right)_m \cdots \left(\frac{a_p}{b_q}\right)_m x^m m!$$

where $(a)_m = a(a+1) \cdots (a+m-1)$ is the Pochhammer symbol; see e. g. [2].

The Cherednik-Opdam transform is defined by, for $f \in C_0^\infty(\mathbb{R})$, as a $\mathbb{C}^2$-valued function, $\mathcal{F}f(\lambda) = (\mathcal{F}_1f(\lambda), \mathcal{F}_{-1}f(\lambda))$ in $\lambda$, with

$$\mathcal{F}_{\pm 1}f(\lambda) = \int_{\mathbb{R}} f(t)G(\lambda, \pm t)d\mu(t).$$

Let $d\tilde{\mu}(\lambda)$ be the measure

$$d\tilde{\mu}(\lambda) = (2\pi)^{-1} \frac{c_{-1}(\rho)^2}{c(\lambda)c(-\lambda)} d\lambda$$

on the imaginary half axis $i\mathbb{R}^+$, with

$$c(\lambda) = \frac{\Gamma(\lambda)\Gamma\left(\frac{\lambda}{2} + b\right)}{\Gamma(\lambda + b)\Gamma\left(\frac{\lambda}{2} + b + \frac{1}{2}\right)}$$

and

$$c_{-1}(\lambda) = \frac{\Gamma(\lambda + 1)\Gamma\left(\frac{\lambda}{2} + b + 1\right)}{\Gamma(\lambda + b + 1)\Gamma\left(\frac{\lambda}{2} + b + \frac{3}{2} + 1\right)}.$$

Then the transform $\mathcal{F}$ extends to a unitary operator from $L^2(\mathbb{R}, d\mu(t))$ onto $L^2(i\mathbb{R}^+, d\tilde{\mu}(\lambda)) \otimes \mathbb{C}^2$, namely

$$\|f\|^2 = \|\mathcal{F}f\|^2_{L^2(i\mathbb{R}^+, d\tilde{\mu}(\lambda)) \otimes \mathbb{C}^2}.$$
When $f$ is an even function $F_1 f(\lambda) = F_{-1} f(\lambda) := \tilde{f}(\lambda)$ is the spherical transform of $f$; see \cite{9} and \cite{11}.

3. Jacobi type functions and Rodrigues type formulas

In this section we will find an orthogonal basis of the space $L^2(\mathbb{R}, d\mu)$ consisting of Jacobi type functions and find certain Rodrigues type formulas. We denote

\begin{equation}
(3.1) \quad w_{\sigma,k}(t) := (\cosh^{-\sigma} t) \tanh^{2k} t, \quad w_\sigma(t) := w_{\sigma,0}(t) = \cosh^{-\sigma} t.
\end{equation}

The function $w_\sigma$ will play the role of the Gaussian function.

Let $P_n^{\alpha,\beta}$ be the Jacobi polynomials \cite{2}. We define the even and odd Jacobi-type functions as follows,

\begin{equation}
(3.2) \quad Q_{n,1}^{(k)}(t) = w_{\sigma,k}(t) P_n^{\sigma_0,\delta_0 + 2k}(2 \tanh^2 t - 1), \quad n = 0, 1, \cdots,
\end{equation}

and respectively

\begin{equation}
(3.3) \quad Q_{n,-1}^{(k)}(t) = w_{\sigma,k}(t) P_n^{\sigma_0,\delta_1 + 2k}(2 \tanh^2 t - 1) \tanh t, \quad n = 0, 1, \cdots
\end{equation}

where

\begin{equation}
(3.4) \quad \sigma_0 := \sigma - (\iota + b + 1), \quad \delta_0 := \frac{\iota - 1}{2} + b, \quad \delta_1 := \delta_0 + 1 = \frac{\iota + 1}{2} + b.
\end{equation}

**Lemma 3.1.** Suppose $\sigma > \iota + b$. The functions $\{Q_{n,1}^{(k)}(t)\}$ and $\{Q_{n,-1}^{(k)}(t)\}$ form orthogonal bases for $L^2(\mathbb{R}, d\mu(t))_{1}$ and $L^2(\mathbb{R}, d\mu(t))_{-1}$ respectively. Their norms in the $L^2$-space are given by

\begin{equation}
||Q_{n,1}^{(k)}||^2 = 2^{2(\iota + b)} \frac{\Gamma(n + \sigma - \iota - b - 1) \Gamma(n + 2k + \frac{\iota + 2b + 1}{2})}{n!(2n + 2k + \sigma - \frac{\iota + 1}{2}) \Gamma(n + 2k + \sigma - \frac{\iota + 1}{2})},
\end{equation}

and

\begin{equation}
||Q_{n,-1}^{(k)}||^2 = 2^{2(\iota + b)} \frac{\Gamma(n + \sigma - \iota - b - 1) \Gamma(n + 2k + \frac{\iota + 2b + 1}{2} + 1)}{n!(2n + 2k + \sigma - \frac{\iota + 1}{2}) \Gamma(n + 2k + \sigma - \frac{\iota + 1}{2})}.
\end{equation}

**Proof.** We perform first the change of variables $t \in \mathbb{R} \mapsto z = \tanh t \in (-1, 1)$. Then the measure $d\mu(t)$, written in the variable $z$, is (with some abuse of notation)

\begin{equation}
d\mu(z) := d\mu(t) = 2^{2b+2|} \sinh^{2b} t \cosh t |dt = 2^{2b+2}| z^{\iota + 2b} (1 - z^2)^{-(\iota + b + 1)} |dz.
\end{equation}

Thus the map $g(t) \mapsto f(z) := f(\tanh t) = g(t)$ is unitary from the space $L^2(\mathbb{R}, d\mu(t))_{\pm 1}$ onto $L^2((-1, 1), d\mu(z))_{\pm 1}$ and we will thus identify the two spaces. We consider further the space $L^2((-1, 1), \mu_{\sigma,k})$ on $(-1, 1)$ with the measure

\begin{equation}
d\mu_{\sigma,k}(x) := 2^{2(\iota + b) - \sigma - \frac{\iota + 2b - 1}{2} - 2k(1 - x)^{\sigma - (\iota + b + 1)} (1 + x)^{\frac{\iota + 2b - 1}{2} + 2k} dx,
\end{equation}

and the operator $U : L^2((-1, 1), d\mu(z))_{1} \mapsto L^2((-1, 1), \mu_{\sigma,k})$ defined by

\begin{equation}
F(x) := (U f)(x) := f(z)(1 - z^2)^{-\frac{\iota}{2} - 2k} = f(\tanh t) w_{\sigma,k}(t)^{-1}, \quad x = 2z^2 - 1 = 2 \tanh^2 t - 1, z = \tanh t;
\end{equation}
its inverse is given by

\[ f(tanh \, t) = f(tanh(-t)) = F(2 \tanh^2 t - 1) w_{\sigma,k}(t). \]

Then

\[
\|f\|^2 = \int_{-1}^{1} |f(z)|^2 d\mu(z) = 2 \int_{0}^{1} |f(z)|^2 d\mu(z)
\]
\[
= 2^{3(\iota + b) - \sigma - 2(\iota + b) - 2k} \int_{-1}^{1} |F(x)|^2 (1 - x)^{\sigma - (\iota + b + 1)} (1 + x)^{\iota + 2b - 1 + 2k} dx,
\]

namely the map \( f \rightarrow F = Uf \) is an isometry, and it is clearly onto. Now the Jacobi polynomials form an orthogonal basis of the later space, whose norm can be computed by using the result that

\[
\int_{-1}^{1} (1 - x)^{\alpha} (1 + x)^{\beta} P_{m}^{(\alpha, \beta)}(x) P_{n}^{(\alpha, \beta)}(x) dx = \delta_{m,n} \frac{2^{\alpha + \beta + 1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{n! (2n + \alpha + \beta + 1)!};
\]

see e.g. [2]. Our result for \( Q_{n,1}^{(k)} \) follows immediately. The claim for \( L^2(\mathbb{R}, \mu(t)dt)_{-1} \) follows by considering the map \( f(t) \mapsto \frac{f(t)}{\tanh t} \), \( z = \tanh t \) and by similar computation as above; we omit the elementary computations. \( \square \)

We shall now find certain Rodrigues type formula for the functions \( \{Q_{n,\pm 1}^{(k)}\} \) in terms of the weight \( w_{\sigma}(t) = w_{\sigma,0}(t) \) and the Cherednik operator \( D \). We consider first the case \( k = 0 \) for which we have rather compact formulas, and the case for general \( k \geq 0 \) is somewhat different.

We recall the formula for the spherical transform of the weight function \( w_{\sigma}(t) \) and certain Bernstein-Sato type formula proved in [16], formulated here slightly differently using the Pochhammer symbol.

**Lemma 3.2.**

(i) Let \( B_{m,\sigma}(x) \) and \( b_{m,\sigma} \) be defined by

\[
B_{m,\sigma}(x) = \prod_{\epsilon = \pm} \left( \frac{\sigma - \rho + \epsilon x}{2} \right)_{m}, \quad b_{m,\sigma} = \left( \frac{\sigma}{2} \right)_{m} \left( \frac{\sigma + 1 - \iota}{2} \right)_{m}.
\]

Then we have a Bernstein-Sato type formula for the weight function \( w_{\sigma}(t) = w_{\sigma}(t) \),

\[
(3.5) \quad B_{m,\sigma}(D)w_{\sigma}(t) = b_{m,\sigma}w_{\sigma + 2m}(t).
\]

(ii) Suppose \( \sigma > 2(\iota + b) \). The spherical transform \( \tilde{w}_{\sigma} = \mathcal{F}_{\pm}w_{\sigma} \) of \( w_{\sigma}(t) \) is given by

\[
\tilde{w}_{\sigma}(\lambda) = 2^{2b + 2\iota} \Gamma\left( \frac{1}{2} + 1 + 2b \right) \Gamma\left( \frac{\sigma}{2} + \iota + b \right) \prod_{\epsilon = \pm} \Gamma\left( \frac{1}{2}(\sigma - \rho) + \frac{\epsilon}{2} \iota \right) \Gamma\left( \frac{1}{2}(\sigma - \rho) + \frac{\epsilon}{2}(\iota + b) \right), \quad \lambda \in i\mathbb{R}.
\]

**Remark 3.3.** Note that the conditions on \( \sigma \) in Lemma 3.1 and in Lemma 3.2(ii) are different. The weaker condition \( \sigma > (\iota + b) \) in Lemma 3.1 is necessary and sufficient for the functions \( w_{\sigma} \) in \( L^2(\mathbb{R}, d\mu(t)) \) whereas the condition \( \sigma > 2(\iota + b) \) is so that the function \( w_{\sigma} \) is \( L^1(\mathbb{R}, d\mu(t)) \). Lemma 3.2 (ii) and Theorem 4.1 below can be proved to
true for the weaker condition $\sigma > (\iota + b)$ using some certain arguments on analytic continuation and we will not present them there.

Applying the operator $D + \rho$ on the above identity and using the fact that

$$(D + \rho) \cosh^{-\sigma - 2m} t = (-\sigma - 2m)(\cosh^{-\sigma - 2m} t) \tanh t$$

we find a similar Bernstein-Sato type formula.

**Lemma 3.4.** Let $\sigma \in \mathbb{C}$. The following formula holds

$$(D + \rho) \prod_{\epsilon = \pm} \left( \frac{\sigma - \rho + \epsilon D}{2} \right)_m w_{\sigma}(t)$$

$$= (-\sigma) \left( \frac{\sigma}{2} + 1 \right)_m \left( \frac{\sigma + 1 - \iota}{2} \right)_m w_{\sigma+2m}(t) \tanh t.$$  

This lemma is quite similar to (3.5) except that there is now an extra factor of $D + \rho$ in the left hand side and a factor $\tanh t$ in the right hand, and that the factor $(\frac{\sigma}{2})_m$ is changed to $(\frac{\sigma}{2} + 1)_m$. This difference will appear in all the formulas below between the even polynomials $Q_{n,k}^{(0)}$ and the odd polynomials $Q_{n,-1}^{(0)}$.

The orthogonal functions $Q_{n,\pm 1}^{(0)}(t)$ can now be obtained by some polynomials of $D$ acting on the weight function $w_{\sigma}$.

**Theorem 3.5.** Let $Q_{n,1}^{(0)}(x)$ and $Q_{n,-1}^{(0)}(x)$ be the following polynomial

$Q_{n,1}^{(0)}(x) = \left( \frac{\sigma_0 + 1}{n!} \right) x$ $4F_3(-n, n + \sigma_0 + \delta_0 + 1, \sigma - \rho + x, \sigma - \rho - x; \sigma_0 + 1, \sigma + 1 - \iota, \sigma, 1),$  

$Q_{n,-1}^{(0)}(x) = \left( \frac{\sigma_0 + 1}{n!(-\sigma)} \right) (\rho + x) x$ $4F_3(-n, n + \sigma_0 + \delta_1 + 1, \sigma - \rho + x, \sigma - \rho - x; \sigma_0 + 1, \sigma + 1 - \iota, \sigma, 2 + 1, 1).$

We have the following Rodrigue’s type formulas

$$Q_{n,1}^{(0)}(t) = Q_{n,1}^{(0)}(D) w_{\sigma}(t),$$

and

$$Q_{n,-1}^{(0)}(t) = Q_{n,-1}^{(0)}(D) w_{\sigma}(t).$$

**Proof.** The function $Q_{n,1}^{(0)}$ is

$$Q_{n,1}^{(0)}(t) = \left( \frac{\sigma_0 + 1}{n!} \right) x 2F_1(-n, n + \sigma_0 + \delta_0 + 1; \sigma_0 + 1; \cosh^{-2} t) \cosh^{-\sigma} t$$

$$= \left( \frac{\sigma_0 + 1}{n!} \right) \sum_{m=0}^{n} (-n)_m (n + \sigma_0 + \delta_0 + 1)_m (\sigma_0 + 1)_m \cosh^{-\sigma - 2m} t.$$
We rewrite Lemma 3.2 as

\begin{equation}
(3.6) \quad w_{\sigma+2m}(t) = \cosh^{-\sigma-2m} t = \frac{B_{m,\sigma}(D) \cosh^{-\sigma} t}{b_{m,\sigma}}.
\end{equation}

Our first formula follows then immediately by rewriting the sum as a hypergeometric function.

Similarly,

\[
Q_{n,-1}^{(0)}(t) = \frac{(\sigma_0 + 1)_n}{n!} {}_2F_1(-n, n + \sigma_0 + \delta_1 + 1; \sigma_0 + 1; \cosh^{-2} t) \cosh^{-\sigma} t \tanh t
\]

\[
= \frac{(\sigma_0 + 1)_n}{n!} \sum_{m=0}^{n} \frac{(-n)_m(n + \sigma_0 + \delta_1 + 1)_m}{(\sigma_0 + 1)_m} \cosh^{-\sigma-2m} t \tanh t,
\]

and by Lemma 3.4 each term \((\cosh^{-\sigma-2m} t) \tanh t\) in the sum can be rewritten as a polynomial of \(D\) acting on \(w_{\sigma}(t)\). This proves our claim. \(\square\)

We consider now the general case of \(k \geq 0\). We shall first generalize Lemma 3.2 and Lemma 3.4 and find a Bernstein Sato type formula expressing the functions \(w_{\sigma+2m,k}(t)\) and \(w_{\sigma+2m,k}(t) \tanh t\) as a polynomial of the operator \(D\) acting on \(w_{\sigma}(t)\). This proves our claim.

**Proposition 3.6.** Let \(L_{k,m}(x)\) and \(M_{k,m}(x)\) be the polynomials

\begin{equation}
(3.7) \quad L_{k,m}(x) = {}_3F_2(-k, \frac{\sigma - \rho +}{2}, \frac{\sigma - \rho - D}{2}; \frac{\sigma}{2} + m, \frac{\sigma + 1 - \iota}{2}, 1)
\end{equation}

\begin{equation}
(3.8) \quad M_{k,m}(x) = {}_3F_2(-k, \frac{\sigma - \rho +}{2}, \frac{\sigma - \rho - D}{2}; \frac{\sigma}{2} + 1 + m, \frac{\sigma + 1 - \iota}{2}, 1).
\end{equation}

Then the following Bernstein-Sato type formulas hold

\begin{equation}
(3.9) \quad B_{m,\sigma}(D)L_{k,m}(D)w_{\sigma}(t) = b_{m,\sigma}w_{\sigma+2m,k}(t),
\end{equation}

\begin{equation}
(3.10) \quad (D + \rho)B_{m,\sigma}(D)M_{k,m}(D)w_{\sigma}(t) = (-\sigma)(\frac{\sigma}{2} + 1)_m(\frac{\sigma + 1 - \iota}{2} + 1)_mw_{\sigma+2m,k}(t) \tanh t.
\end{equation}

**Proof.** We write the weight function \(w_{\sigma+2m,k}(t)\) as

\[
w_{\sigma+2m,k}(t) \cosh^{-\sigma+2m} t \tanh^{2k} t = \cosh^{-\sigma+2m}(1 - \cosh^{-2})^k
\]

\[
= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \cosh^{-\sigma+2m+2k} t
\]

\[
= \sum_{j=0}^{k} \frac{(-k)_j}{j!} \cosh^{-\sigma+2m+2j} t.
\]
By the Lemma 3.2 each term $\cosh^{-(\sigma+2m+2j)}$ is
\[
\cosh^{-(\sigma+2m+2j)} t = \frac{(\frac{\sigma-\rho+D}{2})_{m+j}(\frac{\sigma-\rho-D}{2})_{m+j}}{(\frac{\sigma}{2})_{m+j}(\frac{\sigma+1-\lambda}{2})_{m+j}} \cosh^{-\sigma} t
\]
\[
= \frac{(\frac{\sigma-\rho+D}{2})_{m}(\frac{\sigma-\rho-D}{2})_{m}}{(\frac{\sigma}{2})_{m}(\frac{\sigma+1-\lambda}{2})_{m}} \left(\frac{\cosh(\frac{\sigma-\rho+D}{2}) + m}{\frac{\sigma}{2} + m}\right)j(\frac{\sigma-\rho-D}{2} + m) j \cosh^{-\sigma} t.
\]
Our first result now follows by writing the sum as a hypergeometric series. The second formula is proved by similar computations using Lemma 3.4. □

4. Cherednik-Opdam transform of the Jacobi type functions

We compute now the Cherednik-Opdam transform of the orthogonal functions \(\{Q^k_{n,\pm}\}\). Consider the case \(k = 0\) first.

**Theorem 4.1.** Suppose \(\sigma > 2(\nu + b)\). The Cherednik-Opdam transform of the Jacobi type functions \(Q^{(0)}_{n,1}\) and \(Q^{(0)}_{n,-1}\) are given by
\[
\mathcal{F}_{\pm 1}(Q^{(0)}_{n,1}) = \frac{(\sigma_0 + 1)_n \tilde{w}_\sigma(\lambda)}{n!} \times
\]
\[
_{4}F_{3}(-n, n + \sigma_0 + \delta_0 + 1, \frac{\sigma - \rho + \lambda}{2}, \frac{\sigma - \rho - \lambda}{2}; \sigma_0 + 1, \frac{\sigma + 1 - \nu}{2}, 1),
\]
\[
\mathcal{F}_{\pm 1}(Q^{(0)}_{n,-1}) = -\frac{(\sigma_0 + 1)_n \tilde{w}_\sigma(\lambda)(\pm \lambda + \rho)}{n!\sigma} \times
\]
\[
_{4}F_{3}(-n, n + \sigma_0 + \delta_1 + 1, \frac{\sigma - \rho + \lambda}{2}, \frac{\sigma - \rho - \lambda}{2}; \sigma_0 + 1, \frac{\sigma + 1 - \nu}{2}, 1).
\]

**Proof.** The first result follows immediately by using the Rodrigue’s type formula. Indeed as in the proof of Theorem 3.3 the function \(Q_{n,1}\) is a linear combination of the functions \(\tilde{w}_{\sigma+2m}(t)\), whose spherical transform is, by (3.6),
\[
\tilde{w}_{\sigma+2m}(\lambda) = \frac{B_{m,\sigma}(\lambda)\tilde{w}_\sigma(\lambda)}{b_{m,\sigma}}.
\]
Here we use the equation (2.3) and the formal self-adjointness of \(B_{m,\sigma}(D)\) acting on functions of the form \(\cosh^{-\sigma-2j} t\), which can be easily justified by direct computation or by the Plancherel formula, since the function \(\tilde{w}_\sigma(i\lambda)\) has exponential decay for \(\lambda \to \infty\).

Our first formula follows then immediately by rewriting the sum as a hypergeometric function. For the second result we need to compute the Cherednik-Opdam transform of \((D + \rho)B_{m,\sigma}(D)\tilde{w}_\sigma(t)\). By the inversion formula [11, Theorem 9.13], we have,
\[
B_{m,\sigma}(D)\tilde{w}_\sigma(t) = \int_{i\mathbb{R}^+} (\tilde{w}_\sigma(\lambda)G(\lambda, t) + \tilde{w}_\sigma(\lambda)\lambda t G(-\lambda, t))d\mu(\lambda).
\]
We let \(D + \rho\) acts on both side,
\[
(D + \rho)B_{m,\sigma}(D)\tilde{w}_\sigma(t) = \int_{i\mathbb{R}^+} (\tilde{w}_\sigma(\lambda)(D + \rho)G(\lambda, t) + \tilde{w}_\sigma(\lambda)(D + \rho)G(-\lambda, t))d\mu(\lambda).
\]
Rewriting the integral using (2.3) we have
$$(D + \rho)B_{m,\sigma}(D)w_\sigma(t) \int_{i\mathbb{R}^+} (\bar{w}_\sigma(\lambda)(\lambda + \rho)G(\lambda, t) + \bar{w}_\sigma(\lambda)(-\lambda + \rho)G(-\lambda, t))d\hat{\mu}(\lambda),$$
which is equivalent to that
$$\mathcal{F}_{\pm 1}((D + \rho)B_{m,\sigma}(D)w_\sigma)(\lambda) = \bar{w}_\sigma(\lambda)(\pm \lambda + \rho).$$
The rest follows by elementary computations.

The Plancherel formula for the transform $\mathcal{F}$ implies that

**Corollary 4.2.** The functions \{\(\mathcal{F}Q_{n,\pm 1}^{(0)}\)\} form an orthogonal basis of $L^2(i\mathbb{R}^+, d\hat{\lambda}(\lambda)) \otimes \mathbb{C}^2$, and their norms are given by
$$\|\mathcal{F}Q_{n,\pm 1}^{(0)}\| = \|Q_{n,\pm 1}^{(0)}\|.$$

The general case of $k \geq 0$ can then be obtained by the same method using Proposition 3.6.

**Theorem 4.3.** Suppose $\sigma > 2(\iota + b)$. The Cherednik-Opdam transforms of the Jacobi type functions $Q_{n,1}^{(k)}$ and $Q_{n,-1}^{(k)}$ are given by
$$\mathcal{F}_{\pm 1}(Q_{n,1}^{(k)}) = \frac{(\sigma_0 + 1)n}{n!} \bar{w}_\sigma(\lambda) \times \sum_{m=0}^{n} \frac{(-n)_m(n + \sigma_0 + \delta_0 + 2k + 1)_m}{(\sigma_0 + 1)_m} \frac{(-\frac{\sigma - \rho + \lambda}{2})_m}{\frac{\sigma}{2}_m} \frac{(-\frac{\sigma - \rho - \lambda}{2})_m}{\frac{\sigma}{2}_m} L_{k,m}(\lambda),$$
and
$$\mathcal{F}_{\pm 1}(Q_{n,-1}^{(k)}) = -\frac{(\sigma_0 + 1)n}{n!\sigma} (\pm \lambda + \rho) \bar{w}_\sigma(\lambda) \times \sum_{m=0}^{n} \frac{(-n)_m(n + \sigma_0 + \delta_0 + 2k + 1)_m}{(\sigma_0 + 1)_m} \frac{(-\frac{\sigma - \rho + \lambda}{2})_m}{\frac{\sigma}{2}_m} \frac{(-\frac{\sigma - \rho - \lambda}{2})_m}{\frac{\sigma}{2}_m} M_{k,m}(\lambda),$$
where $L_{k,m}$ and $M_{k,m}$ are the polynomials in (3.7) and (3.8). For each fixed $k \geq 0$ the $\mathbb{C}^2$-valued polynomials $\mathcal{F}(Q_{n,-1}^{(k)})(\lambda)\mathcal{F}_1(Q_{n,-1}^{(k)}(\lambda)), \mathcal{F}_1(Q_{n,-1}^{(k)}(\lambda))$ form an orthogonal basis for the space $L^2(i\mathbb{R}^+, d\hat{\mu}) \otimes \mathbb{C}^2$ and
$$\|\mathcal{F}(Q_{n,-1}^{(k)}))\| = \|Q_{n,-1}^{(k)}\|.$$

**Remark 4.4.** The functions $\mathcal{F}_{\pm 1}(Q_{n,\pm 1}^{(k)})$ are, apart from the common factor $\bar{w}_\sigma(\lambda)$, polynomials of degree $2n + 2k$ or $2n + 2k + 1$. However they are not the orthogonal polynomials obtained by the usual Gram-Schmidt procedure by ordering the monomials $1, \lambda, \lambda^2, \cdots$ according to the degrees. Thus we have discovered some new orthogonal polynomials of hypergeometric type.
Remark 4.5. As an application, we recall that [10] we can find an orthogonal basis 
\( \{ P_{n}^{(a,k)}(2|z|^2 - 1)|z|^k \}_{n=0}^{\infty} \), for each fixed \( k \), of the \( L^2 \)-space of radial functions on the unit disk with the invariant measure 
\( d\nu(z) = (1 - |z|^2)^{-2}dz \wedge d\bar{z} \), as a symmetric space \( SU(1,1)/U(1) \). They are just the functions \( Q_{n,\pm 1}^{(k)}(t) \) in variable \( \tanh t = |z| \). So Theorem 4.3 gives their Helgason transform [6] on the unit disk.

Finally we mention that the result obtained in this paper can also be used to find a family of polynomials of Wilson type orthogonal with respect to the so-called asymmetric Harish-Chandra \( c \)-function [11], see also [3].

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