Deterministic and stochastic models in population dynamics. 
General analysis and approximate results

N. Cufaro Petroni, S. De Martino, and S. De Siena

1 Dipartimento di Matematica and TIRES, Università di Bari (Ret) and INFN, Sezione di Bari
Via Edoardo Orabona, 4, 70126 Bari, Italy
2 Dipartimento di Ingegneria dell’Informazione ed Elettrica e Matematica applicata, Università di Salerno (Ret) and INFN, Sezione di Napoli, Gruppo Collegato di Salerno
Via Giovanni Paolo II, 132, 84084 Fisciano (SA), Italy
3 Dipartimento di Ingegneria Industriale, Università di Salerno (Ret)
Personal address: via Bastioni 15, 84122, Salerno, Italy

ABSTRACT: We analyze some widespread models of population dynamics (logistic, ϑ-logistic, and Gompertz model). In the first part of the article we discuss the deterministic versions of these models. The discussion is also instrumental, on the basis of some very simplified, but working, hypotheses, to frame the logistic and ϑ-logistic equations in an unified context, within which also the Gompertz model emerges from anomalous scaling. In the second half of the paper we pass then to deal with the logistic and ϑ-logistic stochastic models for which, but for the Gompertz case, many aspects are not completely defined. The solutions of the processes contain in fact some exponential functionals of Brownian motion, a rather complex kind of processes extensively studied in financial contexts. First, we recall a few known results, and then we exploit some integral representation that allow to establish simple estimates on the moments of the processes. Furthermore, we derive closed expressions for the transition probability density functions, on which too we obtain some easy estimate implying other bounds for all the moments

KEYWORDS: Population dynamics; Logistic equations, Stochastic growth models

I. INTRODUCTION

Investigation of population dynamics can be traced back to the Fibonacci series in thirteenth century, and have been then developed until the present day [1, 2] with the introduction of various models designed to describe a very large number of systems with both theoretical and practical relevance [3, 4]. Phenomenological equations have been proposed to account for the macroscopic behaviors resulting from a suitable averaging. All the systems under investigation, however, are made up of a large number of individuals (cells for biological systems, atoms or nucleons for solid state systems or stars, and so on), and an effective description requires selecting the right set of variables to represent a specific phenomenon on a chosen scale. For example, in growing cancers the existence of a multiscale structure is well established and this implies a specific approach for each given scale [5, 6].

*corresponding author: cufaro@ba.infn.it
Accordingly, the scientific investigations include statistical mechanics methods [6, 7, 9, 12], entropic techniques [12–14], and stochastic models [13–19].

On a macroscopic level, two approaches became very popular along the years and can now be considered as prototypical: the Verhulst (logistic) model [20], and the Gompertz model [21], both introduced in the first half of the nineteenth century, and then resumed and developed in the first half of the twentieth century. The \( \theta \)-logistic equation [22] was subsequently added as a generalization of the logistic evolution. The corresponding laws can indeed be obtained resorting to a proportionality between the differential increment of the size of a system and its current size, and then suitably correcting it by adding a nonlinear factor that prevents an un-physical (Malthusian) explosion allowed only in the first stage of the evolution: this will eventually drive the system toward a finite asymptotic dimension, namely to a stable equilibrium point. The said correction is in fact related to the finite amount of resources available for a given system, and to its growing density, two features both leading to a reduction of the individual resources. As a matter of fact, any growing organism is an open dynamical system getting resources in an exchange with the surrounding environment (e.g. metabolic exchanges in the case of biological systems), and only unbounded resources and no spatial limitations could allow for indefinite growth.

The paper is organized as follows: in the Section II we perform a preliminary analysis of the logistic, \( \theta \)-logistic and Gompertz equations, with the aim of getting first a perspicuous and unified interpretation of their structure, and then a more definite identification of the underlying hypotheses leading to the macroscopic evolutions. After a look to the form of the equations with a focus on the important role of time scales, we start again from the very beginning, i.e. from the task of describing how the average growth of a system, made up by many individuals, leads to the macroscopic laws. We show that this result can be deduced from extremely simplified – but working – assumptions, with macroscopic laws connecting percentage increments, and then realizing a self-controlled evolution. Within this framework we recognize a \( \theta \)-hierarchy in dissipating resources, and we also suggest a unifying procedure accounting for the emergence of the – seemingly eccentric – Gompertz term, by providing a more defined physical meaning to a known mathematical approach, and by including in so doing the Gompertz growth in the \( \theta \)-logistic frame as a limiting case.

The Section III is then devoted to the stochastic implementations of the said equations. Our attention will be mainly focused on the Logistic and \( \theta \)-Logistic instances, the Gompertz stochastic model being already rather well established: its distributions are indeed log-normal and it has been shown that its macroscopic evolution is properly described by the median of the process [26]. The same can not be said, instead, for the Logistic and \( \theta \)-Logistic models, whose solution procedure look much more involved. After summarizing the state of the art, including their explicit stationary distributions and path-wise solutions, we deal with the open question of determining the moments and the transition pdf’s (probability density functions) of the logistic and \( \theta \)-logistic processes. A few approximate results will be provided: upper bounds on the moments of the processes obtained by exploiting their integral representations, and a closed form for the transition pdf’s of Logistic and \( \theta \)-Logistic processes also with their upper bounds upper bounds resulting in further upper bounds on all the moments of the processes. Discussion and conclusions finally follow in the Section IV.
II. DETERMINISTIC EVOLUTION

A. An overview of known results

In this section we will briefly summarize the main features of the logistic and Gompertz equations, and we will find out their general structure in what we regard as their most revealing setting, a formulation that will provides a hint for later developments. At the same time we will also put in evidence the important role played by the time scales. In our models the main variable will be the macroscopic size of the system \( n(t) \), namely the (dimensionless) number of elementary components (e.g. the cells in a biological systems) at the instant \( t \). The \( \theta \)-logistic equation then usually takes the form

\[
\frac{dn(t)}{dt} = \omega_e n(t) - \omega_f n^{\theta+1}(t)
\]

(1)

(the simple logistic is recovered for \( \theta = 1 \)), while the Gompertz equation reads

\[
\frac{dn(t)}{dt} = \omega_e n(t) - \omega_f n(t) \ln n(t)
\]

(2)

where the constants \( \omega_e = 1/\tau_e \) and \( \omega_f = 1/\tau_f \) are the reciprocal of the characteristic times \( \tau_e \) and \( \tau_f \). The \( \theta \)-logistic equation can also be recast in the form

\[
\frac{dn(t)}{dt} = \omega_e n(t) \left[ 1 - \left( \frac{n(t)}{K} \right)^{\theta} \right], \quad K = \left( \frac{\tau_f}{\tau_e} \right)^{\frac{1}{\theta}} = \left( \frac{\omega_e}{\omega_f} \right)^{\frac{1}{\theta}}
\]

(3)

while in the Gompertz case we have

\[
\frac{dn(t)}{dt} = \omega_e n(t) \left( 1 - \frac{\ln n(t)}{\ln K} \right), \quad K = e^{\frac{\tau_f}{\tau_e}} = e^{\frac{\omega_f}{\omega_e}}
\]

(4)

that for later convenience can also be written as

\[
\frac{d\ln n(t)}{dt} = -\omega_f \ln \frac{n(t)}{K}
\]

(5)

The quantity \( K \) in the previous equations is the asymptotic value of \( n(t) \) when \( t \to \infty \), i.e. the value of \( n \) that nullifies its derivative, and that is also known as carrying capacity. It is known that the solutions of our equations for \( n(0) = n_0 \) respectively are (see for example [1], [2], [13])

\[
n(t) = \frac{K n_0}{n_0 + (K-n_0)e^{-\omega_e t}} \quad \text{(simple logistic)}
\]

(6)

\[
n(t) = \frac{K n_0}{\sqrt[n_0]{n_0} + (K^{\theta} - n_0^{\theta})e^{-\theta \omega_e t}} \quad \text{(\( \theta \)-logistic)}
\]

(7)

\[
n(t) = K \exp\{\alpha_0 e^{-\omega_f t}\} \quad \alpha_0 = \ln(n_0/K) \quad \text{(Gompertz)}
\]

(8)

Looking back now at the equations (3) and (4), we see that they are all of the general form

\[
\frac{dn(t)}{dt} = \omega_e n(t) [1 - h(n(t))]
\]

(9)
where $0 < h(n(t)) < 1$, and therefore also $0 < 1 - h(n(t)) < 1$, because we always have $n(t) < K$ if – as it is realistic in our investigation – we take $n_0 < K$. The second member in the equations is a product of two terms: the first term, that by himself would produce an exponential explosion $n_0 e^{\omega t}$, is corrected by the second one (a negative feedback, usually known as individual growth rate): it is this counteraction that drives the system toward its finite asymptotic size. Remark that, accordingly, one can assume almost vanishing values of $h(n(t))$ at the early stage of the evolution, the region of time where Malthusian growth dominates, while the value 1 is asymptotically approached for $t \to \infty$, when the number attains its maximum value and stops growing.

As for the two characteristic times, it is apparent that $\tau_e$ is the time scale of the purely exponential growth, while, as emerges from (1) and (2), $\tau_f$ characterizes the strength or speed of the correcting term. Obviously it will be $\tau_f > \tau_e$, and usually also $\tau_f \gg \tau_e$. The carrying capacity emerges from the competition between the correction and exponential trends, and it is in fact connected with their ratio: the slower the action of the feedback w.r.t. the explosion, the larger the carrying capacity. In the Gompertz case the carrying capacity is the exponential of the said ratio. Since moreover the whole growth is controlled by the individual growth rate, the braking mechanism must be linked to the decrease of resources available for an elementary component of the system.

Before concluding the section, it is useful for later convenience to introduce a rescaled variable $x(\tau) = x(\omega t) = n(t)/K$ and a rescaled time $\tau = \omega t$ so that the form of the logistic and $\theta$-logistic equations respectively become

\[
\dot{x}(\tau) = x(\tau) \left(1 - x(\tau)\right) \quad \quad \quad \quad \dot{x}(\tau) = x(\tau) \left(1 - x^\theta(\tau)\right)
\]

while the corresponding solutions with $x_0 = n_0/K$ are

\[
x(\tau) = \frac{x_0}{x_0 + (1 - x_0) e^{-\tau}} \quad \quad \quad \quad x(\tau) = \left(\frac{x_0^\theta}{x_0^\theta + (1 - x_0^\theta) e^{-\theta \tau}}\right)^{1/\theta}
\]

B. Merging the equations

1. General principles of a unified model

The nonlinear term $h(t)$ in (2) is usually chosen by resorting to phenomenological criteria depending on the specific system to be described. We propose instead to get a somewhat more meaningful description by deriving it from suitable, albeit still phenomenological, general assumptions. To this end we will reboot our procedure starting again from the beginning, i.e. from the generally recognized main goal of a population dynamics inquiry: taken an evolving natural system consisting, at a given time, of a large number of individuals components, address the problem of forecasting the growth of this number at later times. The realistic details of this evolutions could in fact be rather intricate, and therefore a macroscopic dynamics should emerge by retrieving suitable averaged quantities from a fully probabilistic setting. Of course this would require a very accurate description at a microscopic scale, namely an outright introduction of stochastic models (see the subsequent Section III). However, a preliminary intermediate approach can can help to shed some light on the whole of these phenomena, and we will go on here to show that such a kind of approach is possible and instrumental, in a way reminiscent of what happens to similar simplified models introduced in very different contexts.
Denoting with $n(t)$ the average number of the elementary components of our system at the generic instant $t$, the main point is to compute its increment $\Delta n(t) = n(t + \Delta t) - n(t)$ at a subsequent time $t + \Delta t$. Here $\Delta n(t)$ will be supposed to result from the accumulation of many microscopic increments produced by the possible occurrence of random events (the birth or death of one individual, one mitosis, and so on) between $t$ and $t + \Delta t$: at this stage of the inquiry, however, we will keep this underlying microscopic probabilistic mechanism only in the background. Without yet assuming a fully stochastic model, indeed, we will only surmise the existence of this random underworld as a background justification of our average deterministic equations. We will moreover assume the following, simplified hypotheses:

1. At each instant, the system can rely on a finite and fixed (mean) amount of resources that we will (conventionally) denote $E_T$. The specific nature of these resources, which can have different origins, is not relevant in our scheme, because eventually all the quantities will be translated in terms of number of components

2. Within the system the individuals exploit these resources both to survive and to grow, but survival takes precedence in the sense that, at each stage, the resources available for growth are what is left of $E_T$ once the resources for survival have been taken out. Furthermore, at each step every individual needs on average a quantity $\epsilon_s$ of resources to survive

3. Growth stops when the total amount of resources $E_T$ is only sufficient to the survival of all the individuals: in that case the population achieves its maximum, finite dimension $K$ a.k.a. carrying capacity

4. There is a constant, average rate of increment per unit time $\omega_e = \tau_e^{-1}$ of the number of individuals, so that the average rate of increase in $dt$ will be $\omega_e n dt$. In the literature $\omega_e$ is often called probability per unit time and has been already introduced in very different contexts as, for example, in the Drude simplified model of conduction [24]

Before further developing our model from the previous assumptions, we consider first an ideal case to provide some suggestions for the more realistic ones. We will suppose then that there are no limitations to the available resources ($E_T = \infty$) and to the available space. In this case, whatever the need for survival resources, at any instant the availability of growth resources would be boundless, and thus the population increment would be obtained by simply applying the average rate of increase to the whole number $n(t)$

$$dn(t) = \omega_e n(t) dt$$

with a resulting Malthusian explosion $n(t) = n_0 e^{\omega_e t}$. Here of course $n_0$ denotes the system size at time zero. The previous relation can however be also written as

$$\frac{dn(t)}{n(t)} = \omega_e dt \times 1$$

On the l.h.s. we find the (infinitesimal) percentage increment of the number, while from the r.h.s. we see that this increment results from the product of the average rate of increment in $dt$ and 1. Being in our case the available resources not bounded, the factor 1 can be simply interpreted as the fraction of resources available for growth at any instant. On the basis of this consideration we are led then to propose the following principle:
A growth equation is obtained by imposing that the percentage increment of a population in a small time interval $dt$ is equal to the product between the average rate of increment in the same time interval, and the percentage of resources (w.r.t. the total ones) that is left available after the survival resources have been used.

We will see soon that this latter percentage depends only on the population size.

Going now to more realistic instances, we start from the simplest case by supposing that at each instant the resources are evenly distributed among all the $n(t)$ individuals. Being $\epsilon_s$ the mean amount of resources exploited by an individual to survive, in our approximation we first of all have

$$E_T = \epsilon_s K$$

Then, according to our hypotheses, if $n(t) < K$ is the number of individuals at the instant $t$, the resources exploited for survival at that instant are $E_s(t) = \epsilon_s n(t) < E_T$, and those available for growth are $E_g(t) = E_T - E_s(t) = \epsilon_s (K - n(t))$ so that

$$\frac{dn(t)}{n(t)} = \omega_e dt \frac{E_g(t)}{E_T} = \omega_e dt \frac{K - n(t)}{K} = \omega_e dt \left(1 - \frac{n(t)}{K}\right)$$

and finally in terms of the reduced number and time

$$\frac{dx(\tau)}{x(\tau)} = d\tau \left(1 - x(\tau)\right)$$

that can be easily rearranged into the simple logistic equation $\theta$ ($\theta = 1$). The result can then be quickly retrieved by reintroducing the variable $n(t)$ and the characteristic time $\tau_e$, and defining the time $\tau_f = \tau_e K$.

On the other hand – according to whether the system has a coherent character, with consequent collective and synergistic behaviors, or on the contrary, it displays inefficiencies and non-collaborating elementary components – resource scalings different from the linear one are allowed. A generalized scaling $E_T = (\epsilon_s K)^\theta$ and $E_s(t) = \epsilon_s n^\theta(t)$ can thus be introduced, giving rise to the $\theta$-logistic equation

$$\frac{dx(\tau)}{x(\tau)} = d\tau \left(1 - x^\theta(\tau)\right)$$

In this formulation, however, the Gompertz model still seems to stand apart: would it be possible to recover even this equation within the framework of the previous scheme? In the next section we will provide a path to a positive answer.

2. Retrieving the Gompertz equation

To explain in the above context the eccentric logarithmic term of the Gompertz model, we must at once recognize that we can no longer start from some kind of proportionality between the percentage increase of $n(t)$ and the time interval $\Delta t$. We will instead suppose more in general for the reduced quantities

$$\frac{\Delta x(\tau)}{x(\tau)} = w(x(\tau), \Delta \tau)$$

(16)
where $w(x(\tau), \Delta \tau)$ is a function still to be determined. To this purpose we preliminarily remark that, to be consistent, the procedure we will establish must anyway lead to a final result that fulfills some obvious constraints:

- $w(x(\tau), \Delta \tau)$ must become small for large times, and must approach 1 for small times
- $w(x(\tau), \Delta \tau)$ must go to zero with $\Delta \tau$ as a continuity requirement

We also expect moreover that, at the end of our procedure, at the r.h.s. of the equation we will find again the product of an infinitesimal probability times a percentage term constraining the growth.

We go on now by assuming that $w(x(\tau), \Delta \tau)$ generalizes the $\theta$-logistic term with the anomalous scaling $\theta(\Delta \tau) = \omega_f \Delta \tau + o(\Delta \tau)$, where $\tau_f = \omega_f^{-1}$ is the characteristic time-scale.

We therefore take the function

$$w(x(\tau), \Delta \tau) = 1 - x(\tau)\omega_f \Delta \tau + o(\Delta \tau)$$

which apparently fulfills the required constraints: since indeed $K$ is the maximum asymptotic value of $n(t)$, for $t \to \infty$ we find $x(\tau) \to 1$ and the increment of the number (i.e. the correcting term) tends to become small, while in a very early stage of evolution $x(\tau) \ll 1$ and $w \approx 1$.

The requirement $w(x(\tau), \Delta \tau) \approx 0$ when $\Delta \tau \approx 0$, is clearly fulfilled as well. We can then take advantage of a power expansion to write

$$w(x(\tau), \Delta \tau) = 1 - e^{(\omega_f \Delta \tau + o(\Delta \tau)) \ln x(\tau)} = 1 - (1 + \omega_f \Delta \tau \ln x(\tau)) + o(\Delta \tau)$$

finding first

$$w(x(\tau), d\tau) = -\omega_f \ln x(\tau) d\tau$$

and then finally the Gompertz equation (5) for the reduced variables

$$\frac{dx(\tau)}{d\tau} = -\omega_f x(\tau) \ln x(\tau),$$

If we remember that $x(\tau) = n(t)/K$, and $\tau_e \equiv (\ln K)^{-1} \tau_f$, we can also retrace the factorized form of (14) as a product of the probability per unit time and a reduced percentage of available resources. This concludes the retrieval of the Gompertz model within the framework of our general scheme.

Remark that the Gompertz growth is obtained when $\theta \to 0$ in a suitable sense, justifying in this way its maximally coherent character. Moreover, some physical sense can be ascribed to the well known mathematical result $1 - x^\theta = -\theta \ln x + o(\theta)$ when $\theta \to 0$ often recalled in the literature when the Gompertz model is investigated: the meaning indeed is that scaling in the Gompertz growth depends on the microscopic scales (times) of the system. In turn this fact can clarify once again the origin of the extremely coherent character of Gompertz evolution, because the cooperation level extends on the microscopic domain.

### III. STOCHASTIC GROWTH MODELS

We will now discuss a few questions arising from the introduction of fluctuations and leading to stochastic growth models. Here, the reduced number $x(\tau)$ will be promoted to a full-fledged stochastic process $X(\tau)$ in the reduced, dimensionless time $\tau = \omega_e t$, but since
from now on there will be no risk of ambiguity we will revert in the following to the simpler notation X(t) where it will be always understood that t is the dimensionless time

In our scheme it will be rather natural to take fluctuations on the fraction

\[ Q_g = \frac{E_g}{E_T} = \frac{E_T - E_s}{E_T} \]

of the resources available for the growth. Considering indeed the general \( \theta \)-logistic case and following an usual procedure \[19\], we will simply add to \( Q_g \) a white noise \( B(t) \) (namely a process such that \( E [B(t)] = 0 \), \( E [B(t)B(s)] = 2D \delta(t-s) \), where \( D \) is a constant diffusion coefficient and \( E [\cdot] \) denotes an expectation) and therefore (15) will become

\[
\frac{dX(t)}{X(t)} = \left( Q_g + B(t) \right) dt = \left[ X(t)(1 - X^\theta(t)) + B(t) \right] dt
\]

giving rise finally to the stochastic differential equation (SDE)

\[
dX(t) = X(t)(1 - X^\theta(t)) dt + X(t) dW(t)
\]

where we exploited the well known fact that the white noise \( B(t) \) is the (distributional) derivative of a Wiener process \( W(t) \) in the sense that \( B(t) dt \) is in fact the increment \( dW(t) \) where \( E [dW(t)] = 0 \) and \( E [dW(t)dW(s)] = 2D \delta(t-s) dt ds \). Remark that with this procedure, whatever the growth law considered, the stochastic term is always given by \( X dW \): this term is widely adopted in the literature about the logistic and \( \theta \)-logistic cases, although multiplicative noises, or even more complex additive stochastic terms, have been introduced both in discrete and continuous time versions \[17 \text{–} 19, 27 \text{–} 34\]. In the Gompertz instance, adding this noise term directly leads to the a geometric Wiener process and, as pointed out in the introduction, in this case all the aspects of the model, and its connection with the macroscopic equation, are completely defined. For the stochastic logistic and \( \theta \)-logistic models instead only a few aspects have been completely elaborated, while others, and very important too, still are not. In the following, we first summarize the results already obtained in the literature, and then we discuss the main open questions providing also some approximated answers

### A. A few known results about the logistic models

Many aspects of the logistic and \( \theta \)-logistic stochastic models have been systematically discussed in \[40\] with several references to the existing literature: we will recall here just a few relevant results useful in the following sections. First the stationary distributions have been computed and their stability has been studied too \[29\]; also quasi-stationary distributions have been investigated in the discrete case \[28, 30, 31\]. The stationary distribution for the stochastic \( \theta \)-logistic equation is a generalized gamma law \( \mathcal{G}_\theta \left( \frac{1}{\theta D}, \frac{1}{(\theta D)^{1/\theta}} \right) \) with pdf

\[
f_s(x) = \frac{x e^{-x/\theta D}}{(\theta D)^{1/\theta} \Gamma \left( \frac{1}{\theta D} \right)}
\]

provided that \( D < 1 \). This last condition ensures normalization, and defines the region of stability of the system. The simple logistic case is obtained by choosing \( \theta = 1 \). (for computational details, see also \[40\]).
Even the path-wise solutions of the processes are explicitly known \[18, 40\]. If we define the following Wiener process with constant drift
\[
Z(t) = (1 - D)t + W(t)
\] (24)
it is possible to show that the solution of the \(\theta\)-logistic SDE (22) with initial condition \(X(0) = X_0, \ P\text{-a.s.}\) is
\[
X(t) = \left( \frac{X_0^\theta e^{\theta Z(t)}}{1 + \theta X_0^\theta \int_0^t e^{\theta Z(u)} du} \right)^{1/\theta}
\] (25)
that is correctly brought back to the deterministic solution (11) by switching off the noise \((D = 0 \text{ and } W(t) = 0, \ P\text{-a.s.})\), namely \(Z(t) = t\) and by taking a degenerate initial condition \(X_0 = x_0, \ P\text{-a.s.}\). The solution of the simple logistic SDE (22) with \(\theta = 1\) finally is
\[
X(t) = \frac{X_0 e^{Z(t)}}{1 + X_0 \int_0^t e^{Z(u)} du}
\] (26)

B. Advances on the open questions

The relevant problems of determining the distribution and the moments of these processes still remain without a comprehensive, final answer. Moreover, even the choice of a suitable averaged quantity of the process that could describe its macroscopic evolution deserves further attention: only in the case of the stochastic Gompertz process this quantity has been found to be the median \[26\], but for the logistic equations the answer is not yet there.

1. The moments of a logistic process

Despite the expressions (25) and (26) being fully explicit, to compute the expectation \(E[X(t)]\) and the higher moments \(E[X^k(t)]\) is not at all a simple task. Moreover not even a perturbative approach in terms of small noisy disturbances seems to be available \[35\], and thus fully non-perturbative tools will be required. Looking at expressions (25) and (26) we see on the other hand that the integrals in the denominators (the terms hardest to crack) are indeed processes usually called exponential functionals of Brownian motion (EFBM) of the type
\[
A_{a,b}(t) = \int_0^t e^{aW(u)+bu} du \quad \dot{A}_{a,b}(t) = e^{aW(t)+bt}
\] (27)
that have been extensively studied in the financial context. In our case, within the simplified notation
\[
A_\theta(t) = A_{\theta,\theta(1-D)}(t) = \int_0^t e^{\theta Z(u)} du \quad \dot{A}_\theta(t) = e^{\theta Z(t)}
\]
we find for (25)
\[
X(t) = \left( \frac{X_0^\theta e^{\theta Z(t)}}{1 + \theta X_0^\theta A_\theta(t)} \right)^{1/\theta} = \left( \frac{X_0^\theta \dot{A}_\theta(t)}{1 + \theta X_0^\theta A_\theta(t)} \right)^{1/\theta} = \left( \frac{1}{\theta} \frac{d}{dt} \ln \left( 1 + \theta X_0^\theta A_\theta(t) \right) \right)^{1/\theta}
\] (28)
Remark that since the Wiener process is Gaussian we have $W(t) \sim \mathcal{N}(0, 2Dt)$, and therefore it is also $\theta Z(t) \sim \mathcal{N}(\theta(1 - D)t, 2\theta^2Dt)$. As a consequence the integrand of our EFBM is log-normal $e^{\theta Z(t)} \sim \mathcal{L}(\theta(1 - D)t, 2\theta^2Dt)$ and the following expectations are easily calculated

$$E[e^{\theta Z(t)}] = e^{\theta[1+(\theta-1)D]t} \quad E[A_0(t)] = \frac{e^{\theta[1+(\theta-1)D]t} - 1}{\theta[1+(\theta-1)D]}$$

(29)

Many other results about these EFBM are collected in the literature [36–38], but unfortunately their exact distributions are rather convoluted, and on the other hand the determination of the moments of (25) and (26) requires precisely the utilization of these tangled joint distributions of $Z(t)$ (a.k.a. geometric Brownian motion) and of the corresponding EFBM. In the following we will instead confine ourselves to provide a few useful and easy formulas and estimates about the values of these moments.

We will consider first the simple logistic instance ($\theta = 1$) with a degenerate initial value $x_0$: from the integral representation

$$\int_0^\infty \beta e^{-\beta a} = \frac{1}{a} \quad \text{Re}[a] > 0.$$  

(30)

by taking $a = 1 + x_0 A_1(t)$ – an always positive quantity for any realization of the process $A_1(t)$ – we obtain

$$X(t) = x_0 e^{Z(t)} \int_0^\infty \beta e^{-\beta(1 + x_0 A_1(t))}$$

(31)

and thus for the first logistic moment

$$E[X(t)] = x_0 \int_0^\infty \beta e^{-\beta} E[e^{Z(t)} e^{-\beta x_0 A_1(t)}]$$

(32)

Resorting then to a more general integral representation (see [42] 3.478.1)

$$\frac{\nu}{\Gamma(\lambda)} \int_0^\infty \beta^\mu e^{-a\beta^\nu} = \frac{1}{a^\lambda} \quad \text{Re}[a] > 0, \text{Re}[\mu] > 0, \text{Re}[\nu] > 0, \quad \lambda = \frac{\mu + 1}{\nu}$$

(33)

and choosing $a = 1 + x_0 A_1(t)$, $\nu = 1$ and $\lambda \equiv \mu + 1 = k$, we can write

$$X^k(t) = \frac{x_0^k e^{kZ(t)}}{\Gamma(k)} \int_0^\infty \beta e^{-\beta k - \beta x_0 A_1(t)}$$

(34)

so that the $k$th logistic moment will be

$$E[X^k(t)] = \frac{x_0^k}{\Gamma(k)} \int_0^\infty \beta e^{-\beta k - \beta x_0 A_1(t)}$$

(35)

Finally, for the $\theta$-logistic $k$th moment, the generalized representation (33) with $a = 1 + \theta x_0^k A_0(t)$, $\nu = 1$ and $\lambda \equiv \mu + 1 = k/\theta$, will provides

$$X^k(t) = \frac{x_0^k e^{kZ(t)}}{\Gamma(k/\theta)} \int_0^\infty \beta e^{-\beta k - \theta x_0^k A_0(t)}$$

(36)

$$E[X^k(t)] = \frac{x_0^k}{\Gamma(k/\theta)} \int_0^\infty \beta e^{-\beta k - \theta x_0^k A_0(t)}$$

(37)
2. Estimates on the moments

We will find now some estimates for the moments of the logistic $X(t)$ by exploiting the previous formulas and the Jensen inequality. The well known Jensen inequality

$$\phi(\mathbb{E}[Y]) \leq \mathbb{E}[\phi(Y)]$$

holds for every random variable $Y$ when $\phi(y)$ is a convex function: therefore, since $\phi(y) = e^y$ is convex, the following inequality holds for the expectation

$$\mathbb{E}[X(t)] \geq x_0 e^{\mathbb{E}[Z(t)]} \int_0^\infty d\beta e^{-\beta(1+x_0 \mathbb{E}[A_1(t)])}$$

and exploiting in reverse the representation we get

$$\mathbb{E}[X(t)] \geq \frac{x_0 e^{\mathbb{E}[Z(t)]}}{1 + x_0 \mathbb{E}[A_1(t)]}$$

The expectations being easily calculated, from we finally have the following lower bounds on the expectation

$$\mathbb{E}[X(t)] \geq m(t) = \frac{x_0 e^{(1-D)t}}{1 + x_0(e^t - 1)}$$

In the Figure we have plotted $m(t)$ for $x_0 = 10^{-4}$, and for several values of $D$: it is apparent then that a noisy logistic process in average is found above $m(t)$, and shows an early growth that could be subsequently tamed and even reversed after a maximum has been reached when the noise surges. In particular for $D \geq 1$ the lower bound is an utterly monotone decreasing function as shown in the Figure. By summarizing, the process understandably approaches in average its deterministic ($D = 0$) growing behavior in the limit for decreasing...
values of \( D \), and shows instead a larger variability induced by the noise when \( D \) increases. Within the same procedure we also get the lower bounds for the \( k \)th moment of the \( \theta \)-logistic

\[
E \left[ X^k(t) \right] \geq m^k_\theta(t) = \frac{x_0^k e^{k E[Z(t)]}}{\left( 1 + \theta x_0^\theta E[A_\theta(t)] \right)^{\frac{k}{\theta}}} \left( 1 + \theta x_0^\theta e^{\theta[1+(\theta-1)D]t[-1]} \right)^{\frac{k}{\theta}}
\]

(42)

All these bounds show similar qualitative properties: they start from their initial value \( x_0^k \), they vanish for large times according to \( e^{-k\theta Dt} \), and they correctly return the \( k \)th power of the deterministic \( \theta \)-logistic \( (1) \) when the noise goes off and \( D \) tends to zero

3. Reduction to the deterministic equation

For stochastic systems that are either outright Gaussians (as for instance the Ornstein-Uhlenbeck process), or that can be traced back to some other Gaussian process (as the geometric Wiener process), it is simple enough to find the averaged quantity that abides by the associated deterministic equation, the key to all that being the linearity of the Fokker-Planck equation. Consider for example the Gompertz stochastic model (giving rise to a log-normal process, see [40]) that is described by the SDE

\[
dY(t) = \left( Y(t) - \alpha Y(t) \ln Y(t) \right) dt + Y(t) dW(t)
\]

As it is well known, with a simple Ito transformation \( X(t) = \ln Y(t) \) we find

\[
dX(t) = (1 - D - \alpha X(t)) + dW(t)
\]

namely a modified Ornstein-Uhlenbeck equation with Gaussian solutions, that, in turn, provides for the original process \( Y(t) \) a log-normal distribution. If we now take the median of \( Y(t) \) we find

\[
M \left[ Y(t) \right] = e^{E[\ln Y(t)]} = e^{E[X(t)]}
\]
and, from the properties of the process $X(t)$, it is immediately seen that this median is the solution of the deterministic Gompertz equation. On the other hand, when $D$ vanishes we of course find that $E \left[ \ln Y(t) \right]$ goes to $\ln y(t)$ ($y(t)$ is the solution of the deterministic equation), and then $M \left[ Y(t) \right]$ tends to $y(t)$ too: all is here very simple and exactly computable.

Not so, instead, for the stochastic logistic instance because the distributions of the solutions are not completely known. We will confine ourselves then to a few remarks taking into account the bounds proved above. These bounds provide indeed a description qualitatively quite reliable at least of the behavior of the first moment which likely follows the same path: it starts from an initial condition, attains a maximum and then asymptotically goes to zero. In the presence of stochastic fluctuations it cannot, therefore, satisfy the deterministic equation, except in an approximate way for very small fluctuations. On the other hand, this case is far to be Gaussian, and thus finding an exact averaged quantity that satisfies the deterministic equation is still an open problem.

4. A semi-explicit transition pdf

Also the computation of the logistic transition pdf's is a difficult task that stimulated numerical investigations too [19, 34]. By exploiting a general formula [41] we will provide here closed – but far from simple – expressions of the transition pdf's of the SDE (22) whose finalization however still requires the calculation of some residual, unmanageable expectation: for more details about these developments see [40]. For the simple logistic and the $\theta$-logistic processes we respectively have

$$f(x, t|y, s) = g(x, t; y, s) E \left[ G(x, t; y, s) \right]$$ \hspace{1cm} (43)

$$f_\theta(x, t|y, s) = g_\theta(x, t; y, s) E \left[ G_\theta(x, t; y, s) \right]$$ \hspace{1cm} (44)

where, by taking advantage of the Brownian bridge between $W_{st}(0) = 0$ and $W_{st}(1) = 0$, we have defined

$$g(x, t; y, s) = \frac{e^{-\frac{x-y}{4D^{1/2}(t-s)^{1/2}(1-D)(t-s)-\ln \frac{s}{y}}^2}}{x\sqrt{4\pi D(t-s)}}$$ \hspace{1cm} (46a)

$$g_\theta(x, t; y, s) = \frac{e^{-\frac{\theta^2-y^2}{2D^{1/2}(t-s)^{1/2}(1-D)(t-s)-\ln \frac{s}{y}}^2}}{x\sqrt{4\pi D(t-s)}}$$ \hspace{1cm} (46b)

$$G(x, t; y, s) = e^{-\frac{x-y}{4D}H(x, t, y, s)} \quad G_\theta(x, t; y, s) = e^{-\frac{\theta^2-y^2}{4D}H(x, t, y, s)}$$ \hspace{1cm} (48)

$$H(x, t; y, s) = y^2 \int_0^1 dr \left( \frac{x}{y} \right)^{2r} e^{2W_{st}(r)} - 2y \int_0^1 dr \left( \frac{x}{y} \right)^r e^{2W_{st}(r)}$$ \hspace{1cm} (49a)

$$H_\theta(x, t; y, s) = y^{2\theta} \int_0^1 dr \left( \frac{x}{y} \right)^{2\theta r} e^{2\theta W_{st}(r)} - 2 \left[ 1 + (\theta - 1)D \right] y^\theta \int_0^1 dr \left( \frac{x}{y} \right)^{\theta r} e^{\theta W_{st}(r)}$$ \hspace{1cm} (49b)

The expected values contained in the above formulas are hard to compute exactly, and therefore we will provide just an estimate deduced from the Jensen inequality and resulting in the following lower bound for the simple logistic

$$f(x, t|y, s) \geq \tilde{f}(x, t; y, s) = g(x, t; y, s) e^{-\frac{1}{4D}E[H(x, t, y, s)]}$$ \hspace{1cm} (51)
The details of the calculation of the second factor in $\hat{f}$ can be found in the Appendix A where it is shown indeed that

$$E[H(x, t; y, s)] = y^2 \frac{\sqrt{\pi}}{8A} e^{(2A+B)^2} \left[ \Phi(2A - B) + \Phi(2A + B) \right]$$

$$-2y \frac{\sqrt{\pi}}{4A} e^{(A+B)^2} \left[ \Phi(A - B) + \Phi(A + B) \right]$$

where it is understood that

$$\Phi(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-r^2} dr \quad A = \frac{\sqrt{D(t-s)}}{2} \quad B = \frac{\ln \frac{x}{y}}{2\sqrt{D(t-s)}}$$

Of course, being our random variables non negative, this implies another lower bound on the moments

$$E[X^k(t)] \geq \hat{m}(t) = \int_0^\infty dx x^k \hat{f}(x, t; y, s). \quad (52)$$

We must finally remark that, given a stochastic process, an important question is to determine the averaged quantity that describes the macroscopic (deterministic) behavior of the system, and the characteristics of fluctuations around this mean behavior. The answer to this question is very simple in the case of Gaussian processes: the quantity is the expectation, and the relevant amount of fluctuations is provided by considering the variance of the process. Also in the case of a geometric process (the Gompertz model for instance) the answer is simple: the quantity describing macroscopic evolution is the median, and fluctuations can be as well determined [26]. Unfortunately for the logistic and the $\theta$-Logistic models this answer is not yet available.

**IV. CONCLUSIONS**

In the present paper we have first performed a careful analysis of the laws describing a population dynamics. In the case of the macroscopic (deterministic) equations we have shown that they can all be discussed in an unified context, where the growth is produced by the proportionality between the relative increment of the number of elementary individuals, and the percentage of resources exceeding those needed for the simple subsistence. We guessed also that the parameter $\theta$ characterizes a growing level of correlation (classical coherence) among the individuals of a system, with increasing correlations as $\theta$ decreases, and with the Gompertz model – retrieved when $\theta$ goes to zero in a suitable sense – placed by an anomalous scaling at the top of the hierarchy as the more coherent one.

In the stochastic models, after introducing the fluctuations in accordance with our previous analysis, we summarized the known results about the stochastic logistic and $\theta$-logistic models and subsequently we confronted their open problems. We have shown that an exact calculation of the moments in these models would require an explicit utilization of the joint distribution – or of its generating function – of a geometric Brownian motion and of the associated EFBM process, but (being that computationally rather demanding) for the time being we confined ourselves to supply a few bounds on the general formulas of those moments. Furthermore we provided a semi-explicit closed form for the transition pdf of the corresponding SDE’s and, by exploiting the Jensen inequality, we proved more estimates on the moments. About the problem of the possible average quantities that could describe
the macroscopic behaviors, an exact answer can be given only for the systems associated to Gaussian distributions, but this is not the case for the logistic equations even if, in any case, it is possible to look for approximate solutions.

In the foreseeable future, by considering the results established in the present paper, we plan to give a complete solution for the logistic and $\theta$-logistic processes by taking advantage of the available explicit joint distribution of a geometric Brownian motion and its associated EFBM. Furthermore, we also plan to investigate a few possible control techniques enabling one to modify the behavior in time of these systems, for example to drive them toward some desirable suitable distribution.

[1] J. Müller and C. Kuttler, *Methods and Models in Mathematical Biology*, Lecture Notes on Mathematical Modelling in the Life Sciences (Springer-Verlag, Berlin Heidelberg, 2015); J. Müller, *Mathematical Models in Biology*, Lecture held in the Winter-Semester 2003/2004 at the Centre for Mathematical Sciences, Technical University Munich, www.bionica.info/Biblioteca/Muller2004MathematicalModelsInBiology.pdf.

[2] O. Ovaskainen and B. Meerson, Trends in ecology & evolution 25, 643 (2010).

[3] A. Salisbury, *Mathematical Models in Population Dynamics*, PhD Thesis, https://core.ac.uk/download/pdf/141995076.pdf.

[4] J. D. Murray, *Mathematical Biology I: An Introduction* (Springer-Verlag, New York Berlin Heidelberg, 2002).

[5] J. D. Murray, *Mathematical Biology II: Spatial Models and Biomedical Applications* (Springer-Verlag, New York Berlin Heidelberg, 2003).

[6] N. Bellomo, E. De Angelis and L. Preziosi, *Multi-scale Modeling and Mathematical Problems Related to Tumor Evolution and Medical Therapy*, Journal of Theoretical Medicine, 5(2), 111-136 (2003).

[7] N. Bellomo, N. K. Li and P. K. Maini, *On the foundations of cancer modelling: selected topics, speculations, and perspectives*, Mathematical Models and Methods in Applied Sciences 18, 593-646 (2008).

[8] J. S. Lowengrub, H. B. Frieboes, F. Jin, Y-L. Chuang, X. Li, P. Macklin, S. M. Wise, and V. Cristini, *Nonlinear modelling of cancer: bridging the gap between cells and tumours*, Nonlinearity 23(1), R1-R9 (2010).

[9] D. Drasdo, S. Hoehme, and M. Block, *On the Role of Physics in the Growth and Pattern Formation of Multi-Cellular Systems: What can we Learn from Individual-Cell Based Models?*, J Stat Phys (2007) 128: 287. https://doi.org/10.1007/s10955-007-9289-x.

[10] O. Alekseev and M. Mineev-Weinstein, *Statistical mechanics of stochastic growth phenomena*, Phys. Rev. E 96, 010103(R) (2017).

[11] J. West and P. K. Newton, *Cellular cooperation shapes tumor growth: a statistical mechanics mathematical model*, bioRxiv preprint first posted online Mar. 8, 2018, http://dx.doi.org/10.1101/278614.

[12] M. I. Riffi, *A Generalized Transmuted Gompertz-Makeham Distribution*, Journal of Scientific and Engineering Research, 5(8), 252-266 (2018).

[13] T. Yamano, *Statistical Ensemble Theory of Gompertz Growth Model*, Entropy 11, 807-819 (2009).

[14] T. F. Wyrzyca, *Entropy of the Gompertz-Makeham mortality model*, DEMOGRAPHIC RE-
SEARCH, 30, 13971404 (2014).

[15] R. Lande, S. Engen, and B.-E. Saether, Stochastic population dynamics in ecology and conservation (Oxford University Press, 2003)

[16] R. Gutierrez-Jaimez, P. Roman, D. Romero, J.J. Serrano, F. Torres, A new Gompertz-type diffusion process with application to random growth, Math. Biosci. 208, 147 (2007).

[17] H. Schurz, Modeling, analysis and discretization of stochastic logistic equations, International journal of numerical analysis and modeling, 4, 178-197 (2007).

[18] C. H. Skiadas, Exact Solutions of Stochastic Differential Equations: Gompertz, Generalized Logistic and Revised Exponential, Methodol Comput Appl Probab 12, 261270 (2010).

[19] M. Khodabin and N. Kiae, Stochastic Dynamical Theta-Logistic Population Growth Model, SOP TRANSACTIONS ON STATISTICS AND ANALYSIS, 1, 1 (2014).

[20] P.F. Verhulst, Notice sur la loi que la population suit dans son accroissement, Corr. Mat. et Phys. 10, 113121 (1838); P.F. Verhulst (1845) Nouveaux Memoires de l’Academie Royale des Sciences et Belles-Lettres de Bruxelles 18, pp. 1-38.

[21] B. Gompertz, On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies, Phil. Trans. R. Soc. 115, 513 (1825).

[22] M. E. Gilpin and F. J. Ayala, Global Models of Growth and Competition, PNAS, 70, 35903593 (1973).

[23] Y. Kuang, Lecture 2, https://math.la.asu.edu/ kuang/class/mmm/Lecture2.pdf.

[24] N. W. Ashcroft and N. D. Mermin, Solid State Physics (Acourt College Publishers, 1976).

[25] P. A. M. Dirac, The quantum theory of the emission and absorption of radiation, Proc. R. Soc. Lond. A 114 (1927), 243-265; P. A. M. Dirac, The Principles of Quantum Mechanics (Oxford University Press, 1958); E. Fermi, Nuclear Physics (University of Chicago Press, Chicago, 1950), page 142.

[26] S. De Martino and S. De Siena, Stochastic roots of growth phenomena, Physica A 401, 207213 (2014).

[27] Bartlett, M.S., Gower, J.S., Leslie, P.H., A comparison of theoretical and empirical results for some stochastic population models, Biometrika 47, 111 (1960).

[28] O. Ovaskainen, The quasistationary distribution of the stochastic logistic model, J. Appl. Prob. 38, 898-907 (2001).

[29] S. Pasquali, The stochastic logistic equation : stationary solutions and their stability, Rendiconti del Seminario Matematico della Università di Padova, tome 106, p. 165-183 (2001).

[30] I. Nasell, Extinction and quasi-stationarity in the Verhulst logistic model I, J. Theor. Biol. 211, 1127 (2001).

[31] I. Nasell, Extinction and quasi-stationarity in the Verhulst logistic model II, www.math.kth.se/ ingemar/forsk/verhulst/verhulst.html.

[32] I. Nasell, Moment closure and the stochastic logistic model, Theoretical Population Biology 63, 159168 (2003).

[33] B. Ramasubramanian, Stochastic Differential Equations in Population Dynamics: Numerical Analysis, Stability and Theoretical Perspectives, https://pdfs.semanticscholar.org/cfe9/ be4bf6e638b29b8c723cd3ba6d06225e1f48.pdf.

[34] L.-M. Tenkès, R. Hollerbach, and E. Kim, Time-dependent probability density functions and information geometry in stochastic logistic and Gompertz models, Journal of Statistical Mechanics: Theory and Experiment 17, 123201 (2017).
C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag Berlin Heidelberg New York, 1994).

M. Yor, *Exponential Functionals of Brownian Motion and Related Processes* (Springer, Berlin 2001).

H. Matsumoto and M. Yor, *Exponential functionals of Brownian motion, I: Probability laws at Fixed time*, Probability Surveys Vol. 2 (2005) 312-347.

H. Matsumoto and M. Yor, *Exponential functionals of Brownian motion, II: Some related diffusion processes*, Probability Surveys Vol. 2 (2005) 348-384.

A. Gulisashvili, *Analytically Tractable Stochastic Stock Price Models*, Series in Springer Finance (Springer-Verlag Berlin Heidelberg New York, 2012).

N. Cufaro Petroni, S. De Martino, and S. De Siena, *Gompertz and logistic stochastic dynamics: Advances in an ongoing quest*, arXiv:2002.06409 [math.PR].

I. I. Gihman and A.V. Skorohod, *Stochastic Differential Equations* (Springer, Berlin 1972).

I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Elsevier Academic Press publications, San Diego 2007).

---

**Appendix A: An estimate on the logistic transition pdf**

We will perform here the explicit calculation for the second term appearing in the estimation (51). From the Jensen inequality (38) and (48) we have

\[ E \left[ G(x, t; y, s) \right] \geq e^{-\frac{4D(t-s)}{4} E[H(x,t;y,z)]} \]  

\[ E \left[ H(x, t; y, s) \right] = y^2 \int_0^1 dr \left( \frac{x}{y} \right)^{2r} E \left[ e^{2W_{st}(r)} \right] - 2y \int_0^1 dr \left( \frac{x}{y} \right)^r E \left[ e^{W_{st}(r)} \right] \]  

and hence the problem is reduced indeed to that of calculating a few log-normal expectations. Since apparently for \(0 \leq r \leq 1\) it is also \(s \leq s + (t-s)r \leq t\), it is easy to see from (45) that

\[ W_{st}(r) = (1 - r)W_{sr} - rW_{rt} \]

results from the linear combination of the two independent Gaussian increments of the Wiener process

\[ W_{sr} = W(s + (t-s)r) - W(s) \sim \mathcal{N}(0, 2D(t-s)r) \]

\[ W_{rt} = W(t) - W(s + (t-s)r) \sim \mathcal{N}(0, 2D(t-s)(1-r)) \]

and hence from the usual log-normal expectations after a little algebra we get

\[ E \left[ e^{2W_{st}(r)} \right] = E \left[ e^{2(1-r)W_{sr}} \right] E \left[ e^{2rW_{rt}} \right] = e^{4D(t-s)r(1-r)} \]

\[ E \left[ e^{W_{st}(r)} \right] = E \left[ e^{(1-r)W_{sr}} \right] E \left[ e^{rW_{rt}} \right] = e^{D(t-s)r(1-r)} \]

Therefore we have

\[ \int_0^1 dr \left( \frac{x}{y} \right)^{2r} E \left[ e^{2W_{st}(r)} \right] = \int_0^1 dr e^{-4D(t-s)r^2} e^{[4D(t-s)+2 \ln \frac{x}{y}] r} \]  

\[ \int_0^1 dr \left( \frac{x}{y} \right)^r E \left[ e^{W_{st}(r)} \right] = \int_0^1 dr e^{-D(t-s)r^2} e^{[D(t-s)+\ln \frac{x}{y}] r} \]
Taking now into account the standard Gaussian integral (see [42] 3.322)

\[
\int_0^u dr e^{-r^2/4\pi} e^{-\gamma r} = \sqrt{\pi} \beta e^{\beta \gamma^2} \left[ \Phi \left( \gamma \sqrt{\beta} + \frac{u}{2 \sqrt{\beta}} \right) - \Phi \left( \gamma \sqrt{\beta} \right) \right]
\]

\[\Phi(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-r^2} dr \quad \Phi(-u) = -\Phi(u)\]

and defining for short

\[A = \frac{\sqrt{D(t-s)}}{2} \quad B = \frac{\ln \frac{\pi}{y}}{2\sqrt{D(t-s)}}\]

from (A3) and (A4) we find

\[
\int_0^1 dr \left( \frac{x}{y} \right)^{2r} E \left[ e^{2W(r)} \right] = \frac{\sqrt{\pi}}{8A} e^{(2A+B)^2} \left[ \Phi(2A - B) + \Phi(2A + B) \right] \tag{A5}
\]

\[
\int_0^1 dr \left( \frac{x}{y} \right)^{r} E \left[ e^{W(r)} \right] = \frac{\sqrt{\pi}}{4A} e^{(A+B)^2} \left[ \Phi(A - B) + \Phi(A + B) \right] \tag{A6}
\]

Therefore the expectation (A2) is

\[
E \left[ H(x, t; y, s) \right] = y^2 \frac{\sqrt{\pi}}{8A} e^{(2A+B)^2} \left[ \Phi(2A - B) + \Phi(2A + B) \right] - 2y \frac{\sqrt{\pi}}{4A} e^{(A+B)^2} \left[ \Phi(A - B) + \Phi(A + B) \right] \tag{A7}
\]