ON THE ENTROPY OF PARABOLIC ALLEN-CAHN EQUATION

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ABSTRACT. In this paper we define the entropy of Radon measures, especially the measures associated to the parabolic Allen-Cahn equation. We show that when the entropy of the initial data is small enough (less than twice of the energy of the one dimensional standing wave), the limit measure of the parabolic Allen-Cahn equation has unit density.

1. INTRODUCTION

The parabolic Allen-Cahn equation

\[
\frac{\partial}{\partial t} u^\epsilon = \Delta u^\epsilon - \frac{1}{\epsilon^2} f(u^\epsilon)
\]

is introduced by Allen and Cahn in 1979. It is the gradient flow of the energy functional

\[
M^\epsilon(u) = \int_{\mathbb{R}^n} \frac{\epsilon}{2} |Du|^2 + \frac{1}{\epsilon} F(u) \, dx
\]

with a speed up factor $1/\epsilon$. Here $F(u)$ is the potential function known to be the “double well potential”, and $f$ is the derivative of the potential function, see Section 2.

The solutions to the Allen-Cahn equation are the models to the motion of phase boundaries by surface tension. When $\epsilon \to 0$, the term $-\frac{1}{\epsilon^2} f(u^\epsilon)$ makes the phase boundary sharp, and the limit should be the motion of surfaces by mean curvature. In [Ilm93], Ilmanen studied the measures $\mu^\epsilon_t$ associated to the solution $u^\epsilon$, which is defined by

\[
d\mu^\epsilon_t = \left( \frac{\epsilon}{2} |Du^\epsilon(\cdot, t)|^2 + \frac{1}{\epsilon} F(u^\epsilon(\cdot, t)) \right) \, dx.
\]

Ilmanen proved that under certain reasonable initial conditions, as $\epsilon \to 0$, $\mu^\epsilon_t$ converge to a rectifiable measure $\mu_t$. Moreover $\mu_t$ is a Brakke flow, a geometric measure theoretic weak solution of mean curvature flow. In [Ilm93, Section 13], Ilmanen asked the following question: when does $\mu_t$ have unit density?

In this paper, we define the entropy of Radon measures (see Definition 3.1) to partially answer this question. Let $\alpha$ be the Energy of the 1-dimensional standing wave (see (2.3)).

**Theorem 1.1.** If there exists $\kappa > 0, \epsilon_0 > 0$ such that the entropy $\lambda(\mu^\epsilon_0) < 2\alpha - \kappa$ for $\epsilon < \epsilon_0$, then the limit measure $\mu_t$ has unit density, i.e. the density of $\mu_t$ is $\alpha$ almost everywhere.
Bronsard-Stoth \cite{BS96} constructed examples such that the limits do not have unit density. The limit of the initial data constructed by Bronsard-Stoth has density $2\alpha$, this implies the entropy of their initial data is at least $2\alpha$ (see Lemma 3.4). Thus our theorem is sharp in some sense.

Entropy is used by Colding-Minicozzi \cite{CM12} to study mean curvature flow. It is a quantity that characterize a submanifold/measure from all scales. There are much research on entropy, especially surfaces with small entropy, see Colding-Ilmanen-Minicozzi-White \cite{CIMW13}, Bernstein-Wang \cite{BW18} and Zhu \cite{Zhu16}.

Since the density is a local quantity, we can use the local entropy developed in \cite{Sun18} to have a localized version of the main theorem:

Theorem 1.2. If there exists $\kappa > 0, \epsilon_0 > 0$ and $\alpha > 0$ such that the local entropy $\lambda^{(0,T)}(\mu^\epsilon_0) < 2\alpha - \kappa$ for $\epsilon < \epsilon_0$, then the limit measure $\mu_t$ has unit density for $t \in [0,T)$.

In particular, let $T = \infty$ we get our main Theorem 1.1.

There is a deep connection between the unit density problem of the limit of Allen-Cahn equation and the multiplicity one problem in mean curvature flow. In \cite{Ilm95}, Ilmanen conjectured that the tangent flow of mean curvature flow of smooth embedded surfaces in $\mathbb{R}^3$ has multiplicity one. It is known that the Allen-Cahn equation will converge to the smooth mean curvature flow at least before the first singular time (for instance, see \cite{Ilm93}). Thus the unit density of the limit of the parabolic Allen-Cahn equation would implies the multiplicity one conjecture.

As far as we know, the multiplicity one conjecture is still open. However, under some additional assumption the multiplicity one of the tangent flow is true. Small entropy is one of these assumptions. Thus it is reasonable that small entropy also implies that the limit measure of Allen-Cahn equation has unit density.

There is another connection between the unit density problem of the limit of Allen-Cahn equation and the unit density problem of Brakke flow. In \cite{Ilm94} Appendix E, Ilmanen asked that can we show the unit density is preserved for the Brakke flow starting from the boundaries of sets. Ilmanen also constructed an example called “double spoon” to illustrate that in general unit density may fail under Brakke flow, even when the initial measure has unit density. Thus the assumption that the Brakke flow starts from the boundaries of sets is necessary.

It is not too hard to realize the boundaries of sets by the limit of Allen-Cahn equation, see \cite{Ilm93} Section 1.4. In particular, the double spoon example can not be realized by Allen-Cahn equation. So the unit density conjecture on Brakke flow is true if the unit density problem for the limit of Allen-Cahn equation is true. As a special case, our result implies that for a Brakke flow, if the entropy of the initial measure is small (less than 2), then the unit density is preserved under the flow.

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2. Preliminaries of Allen-Cahn Equation

Let $u^{\varepsilon}$ be the unique smooth solutions of the equation

\begin{equation}
\frac{\partial}{\partial t} u^{\varepsilon} = \Delta u^{\varepsilon} - \frac{1}{\varepsilon^2} f(u^{\varepsilon}) \text{ on } \mathbb{R}^n \times [0, \infty)
\end{equation}

\begin{equation}
u^{\varepsilon}(\cdot,0) = u_0^{\varepsilon}(\cdot) \text{ on } \mathbb{R}^n \times \{0\}.
\end{equation}

where $u_0^{\varepsilon}$ is the initial data. The potential function $F: \mathbb{R} \to \mathbb{R}$ satisfies

\begin{equation}
f = F', \quad F = \frac{1}{2} g^2,
\end{equation}

where

\begin{equation}
\begin{cases}
f(-1) = f(0) = f(1) = 0, \\
f > 0 \text{ on } (-1,0), \quad f < 0 \text{ on } (0,1), \\
f'(1) > 0, \quad f'(1) > 0, \quad f'(0) < 0 \\
g(-1) = g(1) = 0, \quad g > 0 \text{ on } (-1,1).
\end{cases}
\end{equation}

The standard model is

\begin{equation}
F(u) = \frac{1}{2} (1 - u^2)^2, \quad f(u) = 2u(u^2 - 1), \quad g(u) = 1 - u^2.
\end{equation}

The one-dimensional standing wave $q^{\varepsilon}$ is defined to be the solution to

\begin{equation}q^{\varepsilon}_{xx} - \frac{1}{\varepsilon^2} f(q(x)) = 0, \quad x \in \mathbb{R}
\end{equation}

with the assumption that $q_x^{\varepsilon} > 0$, $q^{\varepsilon}(\pm \infty) = \pm 1$ and $q^{\varepsilon}(0) = 0$. By ODE theory we can solve $q^{\varepsilon}$ for any potential function satisfying (2.2). We define $\alpha$ to be its energy

\begin{equation}\alpha = \int_{-\infty}^{\infty} \frac{\varepsilon}{2} (q^{\varepsilon}(x))^2 + \frac{1}{\varepsilon} F(q^{\varepsilon}(x)) dx.
\end{equation}

By change of variable we know that $\alpha$ is independent of $\varepsilon$. One can also check that

\begin{equation}\alpha = \int_{-1}^{1} \sqrt{F(s)/2} ds.
\end{equation}

In particular, for the model case $F(u) = \frac{1}{2} (1 - u^2)^2$, we have $q^{\varepsilon}(x) = \tanh(x/\varepsilon)$, $\alpha = 4/3$.

We define the Radon measure $\mu_t^{\varepsilon}$ by

\begin{equation}d\mu_t^{\varepsilon} = \left(\frac{\varepsilon}{2} |Du^{\varepsilon}(\cdot, t)| + \frac{1}{\varepsilon} F(u^{\varepsilon}(\cdot, t))\right) dx.
\end{equation}

We say $\mu_t^{\varepsilon}$ is associated to the solution to the Allen-Cahn equation, or for brevity say $\mu_t^{\varepsilon}$ is associated to the Allen-Cahn equation.

Ilmanen [Ilm93] proved that, under certain technical requirement, there exits $\varepsilon_i \to 0$ such that $\mu_t^{\varepsilon_i}$ converge to a $(n - 1)$-rectifiable Radon measure $\mu_t$ for a.e. $t > 0$. Moreover, $\mu_t$ is a mean curvature flow in the sense of Brakke.

One motivation of Ilmanen to study the limit behavior of Allen-Cahn equation is to understand the weak mean curvature flow, see [Ilm93, Section 12]. Thus Ilmanen had some
technical requirement on the initial data $\mu_0$, see [Ilm93, p.423]. In our case we only require the initial data have uniform small entropy. This can be viewed as a restriction on (iv) on [Ilm93, p.423], cf. Lemma 3.2.

3. Entropy of Parabolic Allen-Cahn Equation

We follow the idea of Colding-Minicozzi [CM12] to define the entropy of a Radon measure. Let $\rho_{y,s}$ be the backward heat kernel

$$\rho_{y,s}(x,t) = \frac{1}{(4\pi(s-t))^{(n-1)/2}} e^{-\frac{|x-y|^2}{4(s-t)}}.$$  

**Definition 3.1.** Given a Radon measure $\mu$, we define the entropy $\lambda(\mu)$ to be

$$\lambda(\mu) = \sup_{(y,s)\in\mathbb{R}^n\times(0,\infty)} \int \rho_{y,s}(x,0) d\mu(x).$$

Entropy is translation invariant and dilation invariant. It is a quantity that characterizes the measure from all scales. For example, the following lemma indicates that the entropy is equivalent to the volume growth bound of a measure:

**Lemma 3.2.** There exists $C > 0$ only depending on $n$ such that for any Radon measure $\mu$ with bounded entropy, we have

$$C^{-1} \lambda(\mu) \leq \sup_{x\in\mathbb{R}^n, R > 0} \frac{\mu(B_R(x))}{R^{n-1}} \leq C \lambda(\mu).$$

**Proof.** On one hand, for any $x \in \mathbb{R}^n$ and $R > 0$, we have

$$\lambda(\mu) \geq \frac{1}{(4\pi t)^{n/2}} \int e^{-\frac{|y-x|^2}{4t}} \chi_{B_t(x)} d\mu(y) \geq \frac{1}{(4\pi t)^{n/2}} e^{-R^2/4t} \mu(B_R(x)).$$

Then by choosing $t = R^2$ we have $\frac{\mu(B_R(x))}{R^n} \leq C \lambda(\mu)$, where $C$ is a constant.

On the other hand, we only need to prove $\int e^{-|x|^2} d\mu(x) \leq C \sup_{x\in\mathbb{R}^n, R > 0} \frac{\mu(B_R(x))}{R^{n-1}}$ to conclude the second statement. Let $\chi_{B_t(x)}$ be the characteristic function of the set $B_t(x)$.

$$\int e^{-|x|^2} d\mu(x) \leq \sum_{y \in \mathbb{Z}^n} \int e^{-|x|^2} \chi_{B_2(y)} d\mu \leq C \sum_{y \in \mathbb{Z}^n} e^{-|y|^2} \mu(B_2(y))$$

$$\leq C \sup_{x \in \mathbb{R}^{n+1}, R > 0} \frac{\mu(B_R(x))}{R^{n-1}} \sum_{y \in \mathbb{Z}^n} e^{-|y|^2} \leq C \sup_{x \in \mathbb{R}^{n+1}, R > 0} \frac{\mu(B_R(x))}{R^{n-1}}.$$  

Here $C$ varies from line to line, but it is always a constant only depending on $n$. $\mathbb{Z}^n$ consists of all the integer points in $\mathbb{R}^n$. Then we conclude this Lemma.

If we only take the supremum over a subset of $\mathbb{R}^n \times (0,\infty)$, we get a localized quantity called local entropy, see [Sun18].
Definition 3.3. Given a Radon measure $\mu$ and $U \subset \mathbb{R}^n$, $I \subset (0, \infty)$, we define the local entropy $\lambda^I_U(\mu)$ to be
\[
\lambda^I_U(\mu) = \sup_{(y,s) \in U \times I} \int \rho_{y,s}(x,0) d\mu(x).
\]
We will omit $U$ if $U = \mathbb{R}^n$.

If we only care about the local property, the entropy/local entropy actually gives a bound on the density. We define the $(n-1)$-dimensional density $\theta(x)$ by
\[
\theta(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\omega_{n-1} r^{n-1}}
\]
whenever the limit exists. Here $\omega_{n-1}$ is the volume of the unit ball in $\mathbb{R}^{n-1}$.

Lemma 3.4. Suppose $\mu$ is an integral $(n-1)$-rectifiable Radon measure, then for $T \in (0, \infty)$ or $T = \infty$,
\[
\theta(x) \leq \lambda^{(0,T)}(\mu).
\]
Proof. Since $\mu$ is integral rectifiable, $\theta(x)$ just counts the multiplicity of the approximate tangent plane through $x$. Note the integral of the backward heat kernel on a hyperplane is 1. Thus
\[
\theta(x) = \lim_{s \to 0} \frac{1}{(4\pi s)^{(n-1)/2}} e^{-\frac{|y-x|^2}{4s}} d\mu(y) \leq \lambda^{(0,T)}(\mu).
\]

Based on Huisken [Hui90], Ilmanen [Ill93, Section 3] proved a monotonicity formula for the measures which are associated to the solutions to the Allen-Cahn equation, with a technical assumption on the initial data (see Ilmanen [Ill93, 1.4 (i)]). Later Soner [Son97] removed this assumption. In conclusion we have the following monotonicity formula for the limit measure.

Theorem 3.5. ([Ill93, 3.3],[Son97, Corollary 5.1]) For every $0 < t_1 \leq t_2 < s$, we have
\[
\int \rho_{y,s}(t_2,x) d\mu_{t_2}(x) \leq \int \rho_{y,s}(t_1,x) d\mu_{t_1}(x).
\]

Taking supremum of the monotonicity formula among all $(y,s)$ leads to the monotonicity of entropy and local entropy:

Corollary 3.6. For every $0 < t_1 \leq t_2$, we have
\[
\lambda(\mu_{t_2}) \leq \lambda(\mu_{t_1}).
\]

More generally, given $T > 0$, we have
\[
\lambda^{(0,T)}(\mu_{t_2}) \leq \lambda^{(0,T+(t_2-t_1))}(\mu_{t_1}).
\]

Proof. Note $\rho_{y,s}(t,x) = \rho_{y,s-t}(0,x)$. Taking supremum among all $(y,s) \in \mathbb{R}^n \times (t_2, \infty)$ on the left hand side of (3.5) gives us the monotonicity of entropy, and taking supremum among all $(y,s) \in \mathbb{R}^n \times (t_2, t_2 + T)$ on the left hand side of (3.5) gives us the monotonicity of local entropy.
4. Unit Density of Limit Measure

Let \( \mu_t \) be the limit of the measures which are associated to the parabolic Allen-Cahn equation. We say \( \mu_t \) has unit density if \( \theta(x) = \alpha \) for almost all time and almost all \( x \) in the support of \( \mu_t \). Note \( \alpha \) appears because of the nature of Allen-Cahn equation. In this section we prove that for \( \mu_0 \) with small entropy, the limit measure \( \mu_t \) has unit density.

**Theorem 4.1.** Suppose \( \lambda^{(0,T)}(\mu_0) < 2\alpha \), then \( \mu_t \) has unit density for \( t \in (0,T) \). Here \( T \in (0, \infty) \) or \( T = \infty \).

*Proof.* By Corollary 3.6, \( \lambda^{(0,T-t)}(\mu_t) \leq \lambda^{(0,T)}(\mu_0) < 2\alpha \). Then Lemma 3.4 implies that the density of \( \mu_t \) is less than \( 2\alpha \). Tonegawa [Ton03] proved that \( \alpha^{-1}\mu_t \) is integral, thus \( \mu_t \) has density \( \alpha \), i.e. \( \mu_t \) has unit density. \( \square \)

*Proof of Theorem 1.1 and Theorem 1.2.* By the lower semi-continuity of entropy and local entropy, \( \lambda^{(0,T)}(\mu^\epsilon_0) < 2\alpha - \kappa \) implies that \( \lambda^{(0,T)}(\mu_0) \leq 2\alpha - \kappa < 2\alpha \). Then Theorem 4.1 implies that \( \mu_t \) has unit density when \( t \in [0,T) \). \( \square \)

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