In [2], Bost and Connes studied a particular Hecke $C^*$-algebra $C_Q$ arising in number theory. The algebra $C_Q$ can be realised as a semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \alpha \mathbb{N}^*$ by an endomorphic action $\alpha$ of the multiplicative semigroup $\mathbb{N}^*$ on the group $C^*$-algebra $C^*(\mathbb{Q}/\mathbb{Z})$ [7], and this realisation has provided useful insight into the analysis of $C_Q$ [5, 13]. Since individual elements of $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{N}^*$ involve only finitely many primes, $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \alpha \mathbb{N}^*$ is the direct limit of subalgebras $C^*(G_F) \rtimes \alpha \mathbb{N}^F$, where $F$ is a finite set of primes, $G_F$ is the subgroup of $\mathbb{Q}/\mathbb{Z}$ in which the denominators have all prime factors in $F$, and $\mathbb{N}^F$ acts through the embedding $(n_p) \mapsto \prod_{p \in F} p^{n_p}$ of $\mathbb{N}^F$ in $\mathbb{N}^*$ (see Section 1). One can therefore hope to understand the Hecke algebra $C_Q$ in terms of the finite-prime analogues $C^*(G_F) \rtimes \alpha \mathbb{N}^F$.

Our goal is to analyse the structure of these finite-prime analogues of the Bost-Connes algebra. We started this analysis in [10], where we described a composition series for the two-prime analogue and identified the subquotients in familiar terms: there is a large type I ideal, a commutative quotient isomorphic to $C(T^2)$, and the intermediate subquotient is isomorphic to a direct sum of Bunce-Deddens algebras. Here we describe a composition series for $C^*(G_F) \rtimes \alpha \mathbb{N}^F$. Again there are a large type I ideal and a commutative quotient, and the intermediate subquotients are direct sums of simple $C^*$-algebras. We can describe the simple summands as ordinary crossed products by actions of $\mathbb{Z}^S$ for $S \subset F$. When $|S| = 1$, these actions are odometers and the crossed products are Bunce-Deddens algebras; when $|S| > 1$, the actions are an apparently new class of higher-rank odometer actions, and the crossed products are an apparently new class of classifiable AT-algebras.

We begin with a short section in which we describe the algebras we intend to study. In [12] we describe our composition series for the semigroup crossed product $C^*(G_F) \rtimes \alpha \mathbb{N}^F$. It has $|F| + 1$ subquotients, and all but two of them are direct sums of algebras stably isomorphic to ordinary crossed products of the form $C(U(\mathbb{Z}_{F\setminus S})) \rtimes \mathbb{Z}^S$, where $S \subset F$ and $U(\mathbb{Z}_{F\setminus S})$ is the group of units in the ring $\prod_{p \in F\setminus S} \mathbb{Z}_p$. Our main tools are the analysis of invariant ideals in semigroup crossed products from [9] and some technical lemmas on sums and intersections of ideals in $C^*$-algebras. We also use the general results of [16] to see that the simple summands are classifiable.

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In §3 we show that when $S = \{q\}$ is a singleton, $C(\mathcal{U}(\mathbb{Z}_F\setminus S)) \rtimes \mathbb{Z}^S$ is a direct sum of finitely many Bunce-Deddens algebras; as in [10], the number of summands depends on the orders of $q$ in the finite groups $\prod_{p \neq q} \mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ for large $l \in \mathbb{N}$. We then consider the case where $S = \{q, r\}$. By computing the $K$-theory of $C(\mathcal{U}(\mathbb{Z}_F\setminus S)) \rtimes \mathbb{Z}^S$, we can see that they are not Bunce-Deddens algebras, for example. We expect these summands to be even harder to recognise when $|S| > 2$.

In §4 we use techniques like those of §2 to identify subquotients of the Bost-Connes algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$. These include algebras stably isomorphic to $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ when $S$ is a cofinite subset of the set $\mathcal{P}$ of all primes; in this case, though, these crossed products are themselves simple, and even though the general theory of [16] no longer applies, we can see using results from [1] that they are classifiable AT-algebras. We expect these are needed when we want to identify the number of simple summands in the various subquotients.

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1. Preliminaries

We denote by $\mathbb{N}^*$ the semigroup of positive integers under multiplication, and by $\mathcal{N}$ the semigroup of nonnegative integers under addition. It was shown in [7, Proposition 2.1] that there is an action $\alpha$ of $\mathbb{N}^*$ by endomorphisms of $C^*(\mathbb{Q}/\mathbb{Z})$ such that

$$\alpha_n(\delta_r) = \frac{1}{n} \sum_{ns=r} \delta_s \quad \text{for} \ r \in \mathbb{Q}/\mathbb{Z} \text{ and } n \in \mathbb{N}^*.$$ 

The corresponding semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ is isomorphic to the Hecke $C^*$-algebra $\mathcal{C}_Q$ of Bost and Connes [1 Corollary 2.10]. We denote by $(i_A, i_{\mathcal{N}^*})$ the canonical covariant representation of $(\mathcal{C}^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^*, \alpha)$ in $\mathcal{C}^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$.

Let $F$ be a set of prime numbers. The rational numbers of the form $k\left(\prod_{p \in F} p^m_p\right)^{-1}$ form a subgroup of $\mathbb{Q}$, whose image in $\mathbb{Q}/\mathbb{Z}$ we denote by $\mathcal{G}_F$. The integrated form of the map $r \mapsto \delta_r : \mathcal{G}_F \to UC^*(\mathbb{Q}/\mathbb{Z})$ is a homomorphism $i_F$ of $C^*(\mathcal{G}_F)$ into $C^*(\mathbb{Q}/\mathbb{Z})$; a standard duality argument shows that $i_F$ is injective, so that we can identify $C^*(\mathcal{G}_F)$ with the subalgebra $i_F(C^*(\mathcal{G}_F))$ of $C^*(\mathbb{Q}/\mathbb{Z})$. When $n$ has all its prime factors in $F$, $\alpha_n$ leaves this subalgebra invariant, and hence composing $\alpha$ with the map $(m_p)_{p \in F} \mapsto \prod_{p \in F} p^{m_p}$ gives an action of $\mathbb{N}^F$ on $C^*(\mathcal{G}_F)$, which we also denote by $\alpha$. The pair $(i_F, i_{\mathbb{N}^F})$ is a covariant representation of $(\mathcal{C}^*(\mathcal{G}_F), \mathbb{N}^F, \alpha)$ in $\mathcal{C}^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$. Since $i_F$ is injective, we can deduce from the main theorem of [11] (or by minor modifications to the argument in §3 of [7]) that the corresponding homomorphism

$$i_F \times i_{\mathbb{N}^F} |_{\mathbb{N}^F} : C^*(\mathcal{G}_F) \rtimes_\alpha \mathbb{N}^F \to C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$$

applies, we can see using results from [1] that they are classifiable AT-algebras.
is also an injection. We use this injection to identify $C^*(G_F) \rtimes \alpha \mathbb{N}^F$ with a subalgebra of $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \alpha \mathbb{N}^*$. The crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \alpha \mathbb{N}^*$ is spanned by the elements of the form $i_A(\delta_r) i_{\mathbb{N}^*}(m) i_{\mathbb{N}^*}(n)^*$. If $F$ contains all the prime factors of $m$, $n$ and the denominator of $r$, then $i_A(\delta_r) i_{\mathbb{N}^*}(m) i_{\mathbb{N}^*}(n)^*$ lies in $C^*(G_F) \rtimes \alpha \mathbb{N}^F$. Thus $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \alpha \mathbb{N}^*$ is the direct limit $\bigcup_F C^*(G_F) \rtimes \alpha \mathbb{N}^F$ over increasing finite subsets $F$ of the set $\mathcal{P}$ of prime numbers.

In the next section, we shall describe a composition series for $C^*(G_F) \rtimes \alpha \mathbb{N}^F$ when $F$ is a finite subset of $\mathcal{P}$, and identify the subquotients in terms of ordinary crossed products $C(X_S) \rtimes \mathbb{Z}^S$ associated to subsets $S$ of $F$. The underlying space $X_S$ is the group of units $\mathcal{U}(\mathbb{Z}_F\backslash S)$ in the ring $\mathbb{Z}_F\backslash S := \prod_{p \in F\backslash S} \mathbb{Z}_p$; as an additive group, $\mathbb{Z}_F\backslash S$ is the dual group of $G_F\backslash S$. The action of a prime $q \in S$ on $C(\mathcal{U}(\mathbb{Z}_F\backslash S)) \subset C(\mathbb{Z}_F\backslash S) \cong C^*(G_F\backslash S) \subset C^*(\mathbb{Q}/\mathbb{Z})$ induced by $\alpha_q$ is multiplication by $q$ on $\mathcal{U}(\mathbb{Z}_F\backslash S)$ (see [10, Lemma 1.1]), which because $q$ is a unit in $\mathbb{Z}_F\backslash S$ is an automorphism. Thus the action of $\mathbb{N}^S$ on $C(\mathcal{U}(\mathbb{Z}_F\backslash S))$ extends to an action $\sigma$ of $\mathbb{Z}^S$ such that

$$\sigma_{(m_p)}(f)(x) = f\left((\prod_{p \in S} \mathbb{Z}_p^{m_p})^{-1} x\right) \text{ for } (m_p) \in \mathbb{N}^S.$$ 

As a matter of notation, we shall view a crossed product $A \rtimes \beta G$ by an action of a group as the universal $C^*$-algebra generated by a copy of $A$ and a unitary representation $i_G : G \to U(A \rtimes \beta G)$ satisfying the covariance relation $\beta_s(a) = i_G(s)a_i_G(s)^*.$

### 2. Finitely many primes

The object of this section is to prove the following theorem. For the definitions of AT-algebra, real rank zero and stable rank one, see [20] and the references given there.

**Theorem 2.1.** Let $F$ be a finite set of primes. Then there is a composition series $\{I_k \mid 1 \leq k \leq |F|\}$ of ideals in $C^*(G_F) \rtimes \alpha \mathbb{N}^F$ such that

(a) $I_1 \cong C(\mathcal{U}(\mathbb{Z}_F), \mathcal{K}(\ell^2(\mathbb{N}^F)))$;

(b) $I_{k+1}/I_k \cong \bigoplus_{S \subseteq F, |S| = k} \left(C(\mathcal{U}(\mathbb{Z}_F\backslash S)) \rtimes \alpha \mathbb{Z}^S \right) \otimes \mathcal{K}(\ell^2(\mathbb{N}^F\backslash S))$;

(c) $C^*(G_F) \rtimes \mathbb{N}^F/I_F \cong C(\mathbb{T}^F)$.

Each $C(\mathcal{U}(\mathbb{Z}_F\backslash S)) \rtimes \alpha \mathbb{Z}^S$ is a finite direct sum of simple AT-algebras with real rank zero and a unique tracial state.

The proof of the theorem will occupy the rest of the section. We will need some notation and a number of preliminary results.

Under the Fourier transform $C^*(G_F) \cong C(\mathbb{Z}_F)$ the action $\alpha$ becomes

$$\alpha_{(n_p)}(f)(x) = \begin{cases} 
  f((\prod_{p \in F} \mathbb{Z}_p^{n_p})^{-1} x) & \text{if } x \in (\prod_{p \in F} \mathbb{Z}_p^{n_p}) \mathbb{Z}_F \\
  0 & \text{otherwise}
\end{cases}$$
Lemma 2.2. For $S \subset F$, $C_0(\mathbb{Z}_F \setminus Z_S)$ is an extendibly invariant ideal in $C(\mathbb{Z}_F)$.

Proof. It suffices by [9, Theorem 4.3] to show that for each $n \in \mathbb{N}^F$, the endomorphism $x \mapsto (\prod_{p \in F} p^{\alpha_p})x$ of $\mathbb{Z}_F$ leaves both $Z_S$ and $\mathbb{Z}_F \setminus Z_S$ invariant. Certainly $(\prod_{p \in F} p^{\alpha_p})Z_S$ is contained in $Z_S$. If $x \notin Z_S$, then $x_r \neq 0$ for some $r \in S$, $\prod_{p \in F} p^{\alpha_p}x_r \neq 0$ for this $r$, and $(\prod_{p \in F} p^{\alpha_p})x \notin Z_S$. □

Theorem 1.7 of [9] now allows us to identify $C_0(\mathbb{Z}_F \setminus Z_S) \rtimes \mathbb{N}^F$ with an ideal $J_S$ in $C(\mathbb{Z}_F) \rtimes \alpha \mathbb{N}^F$ such that $(C(\mathbb{Z}_F) \rtimes \alpha \mathbb{N}^F)/J_S = C(\mathbb{Z}_S) \rtimes \mathbb{N}^F$; we write $J_p$ for $J_{\{p\}}$.

Lemma 2.3. $J_S = \sum_{p \in S} J_p$.

Proof. Since $Z_S = \bigcap_{p \in S} Z_p$, we have $\mathbb{Z}_F \setminus Z_S = \bigcup_{p \in S} \mathbb{Z}_F \setminus Z_p$, and $C_0(\mathbb{Z}_F \setminus Z_S) = \sum_{p \in S} C_0(\mathbb{Z}_F \setminus Z_p)$. It follows from Lemma 1.3 of [9] that if $I$, $J$ and $I + J$ are extendibly invariant ideals in $(A, P)$, then $(I + J) \rtimes P = (I \ltimes P) + (J \rtimes P)$. Thus the result follows from Lemma 2.2. □

For $1 \leq k \leq |F|$, we define

\[(2.1) \quad I_k := \prod_{|S \subset F, |S| = k} J_S = \cap_{|S \subset F, |S| = k} J_S.\]

It follows from [9, Lemma 1.3] that if $I$ and $J$ are extendibly invariant ideals in $(A, P)$, then

\[(I \ltimes P)(J \ltimes P) = (IJ) \ltimes P,\]

and hence $I_k = C_0(\mathbb{Z}_F \setminus Z_S) \rtimes \mathbb{N}^F$. Therefore

\[I_1 = C_0(\mathbb{Z}_F \setminus Z_0) \rtimes \mathbb{N}^F = C_0(\prod_{p \in F} (\mathbb{Z}_p \setminus \{0\})) \rtimes \mathbb{N}^F;\]

since $\mathbb{Z}_0 \setminus \{0\}$ is homeomorphic to $\mathcal{U}(\mathbb{Z}_0) \times \mathbb{N}$ by [10, Lemma 2.3], part (a) of Theorem 2.1 follows from an argument similar to the one in the last paragraph of [10, page 176]. Similarly, we can prove part (c) by following the proof of (2.4) of [10], because $(C^*(G_F) \rtimes \alpha \mathbb{N}^F)/I_{|F|} = \mathbb{C} \times \mathbb{N}^F$.

To prove part (b) of Theorem 2.1, we need some lemmas. The first contains some general facts about families of ideals in $C^*$-algebras.

Lemma 2.4. Suppose that $I_1, \ldots, I_n$ are ideals in a $C^*$-algebra $B$.

(a) With $F_n = \{1, \ldots, n\}$, we have

\[(2.2) \quad \prod_{S \subset F_n, |S| = k} \left( \sum_{i \in S} I_i \right) = \sum_{R \subset F_n, |R| = n - k + 1} \left( \prod_{j \in R} I_j \right) \text{ for } 1 \leq k \leq n.\]

(b) Suppose that $K$ is an ideal such that $I_i I_j \subset K$ for all $i, j$. Then $(\sum_{i=1}^n I_i)/K$ is naturally isomorphic to $\bigoplus_{i=1}^n (I_i/I_i \cap K)$.\]
Proof. We prove (a) by induction on \( n \). The statement is trivial for \( n = 1, 2 \). Suppose it holds for \( n - 1 \). When \( k = 1 \), both sides of (2.2) are \( \prod_{i=1}^{n} I_i \), so we assume \( k \geq 2 \). Writing the left-hand side LHS of (2.2) as \( (\prod_{n \in S}(\prod_{n \notin S}) \) and applying the inductive hypothesis to \( F_{n-1} \) shows that

\[
(2.3) \quad \text{LHS} = \left( \prod_{|S|=k, n \in S} (I_n + \sum_{i \in S \setminus \{n\}} I_i) \right) \left( \sum_{R \subset F_{n-1}, |R| = n-k} (\prod_{j \in R} I_j) \right).
\]

Because \( I_n \) is an ideal and \( I_n^2 = I_n \), the first term of (2.3) simplifies to give

\[
\text{LHS} = \left( I_n + \prod_{S' \subset F_{n-1}, |S'| = k-1} (\sum_{i \in S'} I_i) \right) \left( \sum_{R \subset F_{n-1}, |R| = n-k} (\prod_{j \in R} I_j) \right).
\]

We can use the inductive hypothesis on \( F_{n-1} \) with \( k \) replaced by \( k - 1 \) to get

\[
(2.4) \quad \text{LHS} = \left( I_n + \sum_{R' \subset F_{n-1}, |R'| = n-k+1} (\prod_{j \in R'} I_j) \right) \left( \sum_{R \subset F_{n-1}, |R| = n-k} (\prod_{j \in R} I_j) \right),
\]

which is contained in

\[
(2.5) \quad \sum_{R \subset F_{n-1}, |R| = n-k} (\prod_{j \in R \setminus \{n\}} I_j) + \sum_{R' \subset F_{n-1}, |R'| = n-k+1} (\prod_{j \in R'} I_j).
\]

Since (2.5) is the same as the right-hand side RHS of (2.2), LHS \( \subset \text{RHS} \). On the other hand, every element of every \( \prod_{j \in R'} I_j \) arises in (2.4) because we can pick \( R \subset R' \), so RHS \( \subset \text{LHS} \).

To prove (b), note that the map \( \phi_i : a + I_i \cap K \mapsto a + K \) is an injection of \( I_i/(I_i \cap K) \) into \( (\sum_{i=1}^{n} I_i)/K \), and

\[
\phi_i(a + I_i \cap K)\phi_j(b + I_j \cap K) = ab + K = 0 \text{ for } i \neq j
\]

because \( ab \in I_i I_j \subset K \). So the \( \phi_j \) combine to give an injection \( \phi \) of \( \bigoplus (I_i/I_i \cap K) \) into \( (\sum_{i=1}^{n} I_i)/K \), which is clearly surjective. \( \square \)

Lemma 2.5. The ideals \( I_k \) of \( C^*(G_F) \times \alpha \mathbb{N}^F \) defined in (2.1) satisfy

\[
I_{k+1}/I_k = \bigoplus_{S \subset F, |S| = k} \left( \bigcap_{p \notin S} J_{S \cup \{p\}} \right)/J_S.
\]

Proof. Lemma 2.4 (a) gives \( I_{k+1} = \sum_{R \subset F, |R| = n-k} (\prod_{p \in R} J_p) \). The product of any two ideals \( \prod_{p \in R} J_p \) with \( |R| = n-k \) has at least \( n-k+1 \) factors \( J_p \), and hence is contained in \( I_k = \sum_{R \subset F, |R| = n-k+1} (\prod_{p \in R} J_p) \). Thus part (b) of Lemma 2.4 gives

\[
(2.6) \quad I_{k+1}/I_k = \bigoplus_{R \subset F, |R| = n-k} \frac{\prod_{p \in R} J_p}{I_k \cap (\prod_{p \in R} J_p)}.
\]

Now

\[
I_k \cap (\prod_{p \in R} J_p) = \sum_{|T| = n-k+1} (\prod_{q \in T} J_q)(\prod_{p \in R} J_p);
\]
each of these summands has at least one factor $J_q$ for $q \notin R$, and is then contained in $J_q(\prod_{p \in R} J_p)$. Using $I \cap J = IJ$ again gives

$$I_k \cap (\prod_{p \in R} J_p) = \sum_{q \notin R} J_q(\prod_{p \in R} J_p) = (\sum_{q \notin R} J_q)(\prod_{p \in R} J_p),$$

and using the isomorphism $(I + J)/I = J/(I \cap J)$ and Lemma 2.3 gives

$$\frac{\prod_{p \in R} J_p}{I_k \cap (\prod_{p \in R} J_p)} = \frac{J_{F \setminus R} + (\prod_{p \in R} J_p)}{J_{F \setminus R}}.$$ 

Finally we observe that

$$J_{F \setminus R} + (\prod_{p \in R} J_p) = \prod_{p \in R}(J_{F \setminus R} + J_p) = \prod_{p \in R}J_{(F \setminus R) \cup \{p\}}$$

and write $S = F \setminus R$ to deduce the result. \qed

Lemma 2.6. The ideals $J_S$ in $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$ satisfy

$$\left(\bigcap_{p \in F \setminus S} J_{S \cup \{p\}}\right)/J_S \cong \left(C(U(Z_{F \setminus S})) \rtimes_\sigma \mathbb{Z}^S\right) \otimes \mathcal{K}(l^2(\mathbb{N}^F \setminus S)).$$

Proof. We first realise $\left(\bigcap_{p \in F \setminus S} J_{S \cup \{p\}}\right)/J_S$ as a semigroup crossed product:

$$\left(\bigcap_{p \in F \setminus S} J_{S \cup \{p\}}\right) = C_0\left(\bigcap_{p \in F \setminus S}(Z_{F \setminus S} \setminus Z_{S \cup \{p\}})\right) \rtimes \mathbb{N}^F$$

$$= C_0\left(Z_F \setminus \bigcup_{p \in F \setminus S} Z_{S \cup \{p\}}\right) \rtimes \mathbb{N}^F.$$ 

Thus

$$\left(\bigcap_{p \in F \setminus S} J_{S \cup \{p\}}\right)/J_S = C_0\left(Z_S \setminus \bigcup_{p \in F \setminus S} Z_{S \cup \{p\}}\right) \rtimes \mathbb{N}^F$$

$$= C_0\left((\prod_{p \in F \setminus S} Z_p \setminus \{0\}) \times \prod_{p \in S} \{0\}\right) \rtimes \mathbb{N}^F.$$ 

The arguments of Corollary 2.4 and Lemma 2.5 of [10] show that this last crossed product is isomorphic to $\left(C(U(Z_{F \setminus S})) \rtimes_\sigma \mathbb{Z}^S\right) \otimes \mathcal{K}(l^2(\mathbb{N}^F \setminus S)).$ \qed

Part (b) of Theorem 2.1 follows immediately from Lemmas 2.5 and 2.6.

To finish the proof of Theorem 2.1 it remains to prove the statements about the structure of $C(U(Z_{F \setminus S})) \rtimes_\sigma \mathbb{Z}^S$. Corollary A.6 implies that $H := \overline{Z}^L$ has finite index in $U(Z_{F \setminus S})$. The argument at the end of the proof of [10] Theorem 3.1 shows that $C(U(Z_{F \setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is a finite direct sum of algebras isomorphic to $C(H) \rtimes_\sigma \mathbb{Z}^S$, which is simple because $\mathbb{Z}^S$ acts minimally and freely on $H$. Since $H$ is an open and closed subset of $U(Z_{F \setminus S})$, it is totally disconnected, and it follows from [10] Theorem 6.11 that $C(H) \rtimes_\sigma \mathbb{Z}^S$ has real rank zero and stable rank one.

The space $U(Z_{F \setminus S})$ is the inverse limit of the finite groups $U(Z/(\prod_{p \in F \setminus S} \mathbb{Z}))$ over $l = (l_p) \in \mathbb{N}^F \setminus S$. The diagonally embedded copy of $\mathbb{N}$ is cofinal in $\mathbb{N}^F \setminus S$, and hence

$$U(Z_{F \setminus S}) = \varprojlim U(Z/(\prod_{p \in F \setminus S} \mathbb{Z})).$$ 

Let $\pi_n$ denote the canonical surjection of $U(Z_{F \setminus S})$ onto $U(Z/(\prod_{p \in F \setminus S} \mathbb{Z}))$. 

Lemma 2.7. Let $H_n := \pi_n(H) \subset U(\mathbb{Z} / (\prod_{\ell \in F} \sigma^{n_\ell}) \mathbb{Z})$ and let $\mathbb{Z}^S$ act on $H_n$ via the embedding $(n_q) \mapsto \prod_{q \in F} q^{n_q}$ of $\mathbb{Z}^S$ in $\mathbb{Z}$. Then there are $C^*$-subalgebras $A_n$ of $C(H) \rtimes_{\sigma} \mathbb{Z}^S$ such that $A_n \cong C(H_n) \rtimes \mathbb{Z}^S$ and $C(H) \rtimes_{\sigma} \mathbb{Z}^S = \bigcup A_n$.

Proof. The homomorphism $\pi_n$ induces an injection $\pi_n^*$ of $C(H_n)$ into $C(H)$, and then $C(H) = \bigcup_{n \in \mathbb{Z}} \pi_n^*(C(H_n))$. On $\mathbb{Z}^S \subset H$, $\pi_n$ is reduction modulo $\prod_{\ell \in F \setminus \ell} p^n$, so $\pi_n^*$ converts the action into the canonical action of $\mathbb{Z}^S$ by multiplication on $H_n$. Thus $\pi_n^*$ induces a homomorphism $\pi_n^* \rtimes \text{id}$ of $C(H_n) \rtimes \mathbb{Z}^S$ into $C(H) \rtimes \sigma \mathbb{Z}^S$. The homomorphism $\pi_n^*$ is faithful on $C(H_n)$ and intertwines the dual actions, and hence a standard argument shows that $\pi_n^* \rtimes \text{id}$ is faithful on $C(H_n) \rtimes \mathbb{Z}^S$ (see, for example, [10] Lemma 4.2). Since $\bigcup_{n \in \mathbb{Z}} \pi_n^*(C(H_n))$ is dense in $C(H)$, we therefore have

$$C(H) \rtimes_{\sigma} \mathbb{Z}^S = \bigcup_{n \in \mathbb{Z}} \pi_n^* \rtimes \text{id}(C(H_n) \rtimes \mathbb{Z}^S),$$

as claimed. \qed

We can identify the subalgebras $A_n$ explicitly.

Proposition 2.8. Let $F$ be a finite quotient of $\mathbb{Z}^k$. Then $C(F) \rtimes \mathbb{Z}^k$ is isomorphic to $C(T^k, M_{|F|}(\mathbb{C}))$.

Proof. Let $H$ be the subgroup of $\mathbb{Z}^k$ with $F = \mathbb{Z}^k / H$. Then $H$ is itself a free abelian group of rank $k$, and hence has the form $A\mathbb{Z}^k$ for some $A \in M_k(\mathbb{Z}) \cap \text{GL}_k(\mathbb{Q})$. The matrix $A$ has a Smith normal form: there are matrices $P, Q \in \text{GL}_k(\mathbb{Z})$ such that $B := P^{-1}AQ^{-1}$ is diagonal [12, §3.22]. Then $H = A\mathbb{Z}^k = PBQ\mathbb{Z}^k = PBZ^k \cong B\mathbb{Z}^k = b_1\mathbb{Z} \oplus \cdots \oplus b_k\mathbb{Z}$. In other words, multiplying by $P^{-1}$ gives an automorphism of $\mathbb{Z}^k$ which carries $H$ into $\bigoplus b_ii\mathbb{Z}$. Thus

$$C(F) \rtimes \mathbb{Z}^k \cong C(\prod_{i=1}^k (\mathbb{Z}/b_ii\mathbb{Z})) \rtimes \mathbb{Z}^k \cong \bigotimes_{i=1}^k (C(\mathbb{Z}/b_ii\mathbb{Z}) \rtimes_{\tau} \mathbb{Z}),$$

where $\tau$ is the action of $\mathbb{Z}$ by translation.

By [13] Corollary 2.5, $C(\mathbb{Z}/b\mathbb{Z}) \rtimes_{\tau} \mathbb{Z}$ is isomorphic to the induced algebra

$$\text{Ind}_{(\mathbb{Z}/b\mathbb{Z})}^\mathbb{T} (C(\mathbb{Z}/b\mathbb{Z}) \rtimes_{\tau} (\mathbb{Z}/b\mathbb{Z}), \hat{\tau}),$$

which is described in terms of a generator $\beta$ of the dual action $\hat{\tau}$ as the mapping torus

$$(2.8) \quad MT(\beta) = \{ f : [0, 1] \rightarrow C(\mathbb{Z}/b\mathbb{Z}) \rtimes_{\text{tr}} (\mathbb{Z}/b\mathbb{Z}) \mid f(1) = \beta(f(0)) \}.$$

Since $C(\mathbb{Z}/b\mathbb{Z}) \rtimes_{\tau} (\mathbb{Z}/b\mathbb{Z}) \cong B(l^2(\mathbb{Z}/b\mathbb{Z})) = M_{|b|}(\mathbb{C})$, the automorphism $\beta$ is inner, and there is a continuous path $\beta_t$ in $\text{Aut} M_{|b|}(\mathbb{C})$ such that $\beta_0 = \text{id}$ and $\beta_1 = \beta$. Now $\phi(f)(t) = \beta_t^{-1}(f(t))$ defines an isomorphism $\phi$ of (2.8) onto $C(\mathbb{T}, M_{|b|}(\mathbb{C}))$. We therefore have

$$C(F) \rtimes \mathbb{Z}^k \cong \bigotimes_{i=1}^k C(\mathbb{T}, M_{|b_ii|}(\mathbb{C})) \cong C(\mathbb{T}^k, M_{\prod_{i} |b_ii|}(\mathbb{C})),$$

and the result follows on observing that $\prod_{i} |b_ii| = |\det B| = |\det A| = |F|$. \qed
It follows from Proposition 2.8 and the decomposition $C(H) \rtimes \sigma \mathbb{Z}^S = \bigcup A_n$ that $C(H) \rtimes \sigma \mathbb{Z}^S$ is an AH-algebra\(^1\). The K-theory of $C(H_n) \rtimes \mathbb{Z}^S$ is torsion-free and this property is preserved under inductive limits, so $C(H) \rtimes \sigma \mathbb{Z}^S$ has torsion-free K-theory. Since $C(H) \rtimes \sigma \mathbb{Z}^S$ is a simple AH-algebra with real rank zero and no dimension growth, it follows from results of Elliott and Lin that it is an AT-algebra (see [17, Lemma 7.5]).

We also use the decomposition $C(H) \rtimes \sigma \mathbb{Z}^S = \bigcup A_n$ to prove that $C(H) \rtimes \sigma \mathbb{Z}^S$ has a unique tracial state. Let $\mu$ denote the Haar measure on $H \subset \mathcal{U}(\mathbb{Z}_{F\setminus S})$. The action $\sigma$ permutes the cylinder sets $\{\pi_n^{-1}(m) \mid m \in H_n\}$, so every invariant probability measure agrees with $\mu$ on cylinder sets. Since the characteristic functions of such sets span a dense subspace of $C(H)$, it follows that $\mu$ is the only invariant probability measure, and $C(H) \rtimes \mathbb{Z}^S$ has a unique tracial state by [3 Corollary VIII.3.8].

This completes the proof of Theorem 2.1.

3. The structure of $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes \sigma \mathbb{Z}^S$

3.1. When $S$ contains just one prime. We consider $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes \sigma \mathbb{Z}^S$ when $S = \{q\}$. To simplify the notation, we relabel $F \setminus \{q\}$ as $F$. The following result generalises [10, Theorem 3.1] in two directions: to sets $F$ with $|F| > 1$ and to sets $F$ containing the even prime 2. If $l = (l_p) \in \mathbb{N}^F$ is a multi-index, we write $o_l(q)$ for the order of $q$ in $\prod_{p \in F} \mathcal{U}(\mathbb{Z}/p^{l_p}\mathbb{Z})$.

**Theorem 3.1.** Suppose $F$ is a finite set of primes and $q$ is a prime which does not belong to $F$. Then there are a multi-index $K = (K_p) \in \mathbb{N}^F$ and $d \in \mathbb{N}$ such that

$$o_{K+1}(q) = d(\prod_{p \in F} p^{l_p}) \text{ for every } l \in \mathbb{N}^F,$$

and $C(\mathcal{U}(\mathbb{Z}_F)) \rtimes \sigma \mathbb{Z}$ is the direct sum of $(\prod_{p \in F} (p - 1) p^{K_p - 1})/d$ copies of a Bunce-Deddens algebra with supernatural number $d(\prod_{p \in F} p^{\infty})$.

The existence of $K$ and $d$ satisfying (3.1) is established in Proposition 3.1. We saw in [2] that $C(\mathcal{U}(\mathbb{Z}_F)) \rtimes \sigma \mathbb{Z}$ is the direct sum of copies of the simple algebra $C(H) \rtimes \sigma \mathbb{Z}$, where $H$ is the closure of $q^{\mathbb{Z}}$ in $\mathcal{U}(\mathbb{Z}_F)$. It remains to prove that $C(H) \rtimes \sigma \mathbb{Z}$ is a Bunce-Deddens algebra and to calculate the index $[\mathcal{U}(\mathbb{Z}_F) : H]$, which is the number of simple direct summands.

Let $\{n_k\}$ be integers with $n_k \geq 2$, and let $X_k = \{0, 1, \ldots, n_k - 1\}$. Addition by 1 with carry over is a homeomorphism of the totally disconnected space $X := \prod_{k \geq 0} X_k$

\(^1\)To see that an inductive limit $\bigcup A_n$ is an AH-algebra, it suffices to show that each $A_n$ is a corner in a matrix algebra $M_N(C(X))$, or, equivalently, that $A_n$ is a homogeneous algebra with vanishing Dixmier-Douady class. Since the Dixmier-Douady class $\delta(A)$ of an $m$-homogeneous algebra satisfies $m\delta(A) = 0$, and $H^3(\mathbb{T}^k, \mathbb{Z})$ has no torsion, it suffices to prove that each $A_n$ is a homogeneous algebra with spectrum $\mathbb{T}^k$. In our situation we could prove this in several ways. However, Proposition 2.8 makes the stronger statement that $A_n$ is isomorphic to $M_{m_n}(C(\mathbb{T}^k))$.\)
called an odometer action, and the resulting crossed product \( C(X) \rtimes_\sigma \mathbb{Z} \) is a Bunce-Deddens algebra with supernatural number \( n := \prod_{k \geq 0} n_k \) (see \cite{3} Chapter VIII.4).

Our claim that \( C(H) \rtimes_\sigma \mathbb{Z} \) is a Bunce-Deddens algebra will follow from the next proposition, which generalises \cite{10} Proposition 3.6.

**Proposition 3.2.** Suppose \( \{G_l \mid \ell \in \mathbb{N}\} \) are finite groups and \( G = \varprojlim (G_l, \pi_l) \). Fix \( g \in G \) and let \( L \) denote the closed subgroup of \( G \) generated by \( g \). Consider the action \( \sigma : \mathbb{Z} \to \text{Aut} C(G) \) such that \( \sigma_n(f)(x) = f(g^{-n}x) \). Let \( \alpha_l(g) \) denote the order of \( \pi_l(g) \) in \( G_l \), and let

\[
d_l := \begin{cases} 
\alpha_0(g) & \text{if } l = 0 \\
\alpha_l(g)/\alpha_{l-1}(g) & \text{if } l \geq 1.
\end{cases}
\]

Then \( C(L) \rtimes_\sigma \mathbb{Z} \) is a Bunce-Deddens algebra with supernatural number \( \prod_{l \geq 0} d_l \).

**Proof.** Let \( X := \bigsqcup_{l \geq 0} \{0, 1, \ldots, d_l - 1\} \). The argument in the proof of \cite{10} Proposition 3.6] shows that the continuous maps \( h_l : X \to G_l \) given by

\[
h_l(\{a_0\}) = \pi_l(g^{a_0+a_1d_0+\cdots+a_{l-1}d_0\cdots d_{l-1}})
\]

combine to give an equivariant homeomorphism \( h : X \to L \) which induces the required isomorphism.

Our subgroup \( H \) of \( \mathcal{U}(\mathbb{Z}_F) \) is the inverse limit \( \varprojlim \pi_l(H) \), where \( \pi_l : \mathcal{U}(\mathbb{Z}_F) \to \mathcal{U}(\mathbb{Z}/(\prod_{p \in F} p^{K_p^l+1})\mathbb{Z}) \) is the canonical surjection. Then Proposition 3.2 and 3.4 imply that \( C(H) \rtimes_\sigma \mathbb{Z} \) is a Bunce-Deddens algebra with supernatural number \( d(\prod_{p \in F} p) \infty \) for \( d = \alpha_K(q) \). By Corollary A.6, we have that

\[
\mathcal{U}(\mathbb{Z}_F) : H = (\prod_{p \in F} (p - 1)p^{K_p-1})/d,
\]

which finishes the proof of Theorem 3.3.

3.2. **When \( S \) consists of two primes.** We now analyse \( C(\mathcal{U}(\mathbb{Z}_{F \setminus S})) \rtimes \mathbb{Z}^S \) when \( S = \{q, r\} \). For simplicity, we consider only the case \( F = \{p, q, r\} \), so that we are interested in the crossed product \( C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}^2 \), where

\[
\sigma_{m,n}(f)(x) = f(q^{-m}r^{-n}x).
\]

**Theorem 3.3.** The \( C^* \)-algebra \( C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}^2 \) is a finite direct sum of copies of a simple \( AT \)-algebra \( A \) which has real rank zero, a unique tracial state and \( K \)-theory satisfying two short exact sequences:

\[
0 \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow K_0(A) \longrightarrow \mathbb{Z} \longrightarrow 0
\]

\[
0 \longrightarrow \mathbb{Z} \longrightarrow K_1(A) \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow 0.
\]

Everything except the assertion about \( K \)-theory was proved in Theorem 2.1, the simple \( C^* \)-algebra \( A \) is \( C(H) \rtimes_\sigma \mathbb{Z}^2 \), where \( H \) is the closure of \( q^{-m}r^{-n}x \) in \( \mathcal{U}(\mathbb{Z}_p) \). We aim to analyse \( C(H) \rtimes_\sigma \mathbb{Z}^2 \) by writing it as an iterated crossed product \( (C(H) \rtimes_\sigma \mathbb{Z}) \rtimes \mathbb{Z} \).
The inside crossed product is not simple unless \( q^Z \) is dense in \( H \), and it is helpful to reduce to this case using the following lemma.

**Lemma 3.4.** Let \( H_q \) denote the closure of \( q^Z \) in \( \mathcal{U}(\mathbb{Z}_p) \). Then \( H_q \) has finite index \( I(q) \) in \( H \), and hence is an open and closed subset of \( H \). The inclusion of \( C(H_q) \) in \( C(H) \) induces an isomorphism of \( C(H_q) \times_{\sigma} (\mathbb{Z} \times I(q)\mathbb{Z}) \) onto the corner \( \chi_{H_q}(C(H) \times_{\sigma} \mathbb{Z}^2) \chi_{H_q} \).

**Proof.** Corollary [A.6] implies that \( H_q \) has finite index in \( \mathcal{U}(\mathbb{Z}_p) \), so it certainly has finite index in \( H \). The inclusion of \( C(H_q) \) in \( C(H) \) and the map

\[(m, I(q)n) \mapsto \chi_{H_q}(m, I(q)n)\chi_{H_q}\]

form a covariant representation of \( (C(H_q), \mathbb{Z} \times I(q)\mathbb{Z}, \sigma) \) in \( \chi_{H_q}(C(H) \times_{\sigma} \mathbb{Z}^2) \chi_{H_q} \), and hence give a homomorphism

\[\phi : C(H_q) \times_{\sigma} (\mathbb{Z} \times I(q)\mathbb{Z}) \to \chi_{H_q}(C(H) \times_{\sigma} \mathbb{Z}^2) \chi_{H_q}.\]

We can identify \((\mathbb{Z} \times I(q)\mathbb{Z})^\wedge\) with \( \mathbb{T}^2/(\mathbb{Z} \times I(q)\mathbb{Z})^\perp = \mathbb{T}^2/(1 \times C_{I(q)}) \), where \( C_n \) denotes the group of \( n \)-th roots of unity, and then \( \phi \) carries the dual action \( \hat{\sigma}_{(w,z)} \) into \( \hat{\sigma}_{(w,z)} \), now a standard argument implies that \( \phi \) is injective (or we could apply [19 Corollary 4.3], for example). We have

\[\chi_{H_q}(f i_{\mathbb{Z}^2}(m,n))\chi_{H_q} = (f \chi_{H_q}) i_{\mathbb{Z}^2}(m,n) \chi_{H_q} = i_{\mathbb{Z}^2}(m,n) \sigma_{m,n}^{-1}(f \chi_{H_q}) \chi_{H_q}.\]

Since the support of \( \sigma_{m,n}^{-1}(f \chi_{H_q}) \) is contained in \( q^{-m} r^{-n} H_q = r^{-n} H_q \), we have

\[\sigma_{m,n}(f \chi_{H_q}) \chi_{H_q} = \begin{cases} \sigma_{m,n}^{-1}(f \chi_{H_q}) & \text{if } I(q) \text{ divides } n \\ 0 & \text{otherwise,} \end{cases}\]

and

\[\chi_{H_q}(f i_{\mathbb{Z}^2}(m,n))\chi_{H_q} = \begin{cases} i_{\mathbb{Z}^2}(m,n) \sigma_{m,n}^{-1}(f \chi_{H_q}) & \text{if } I(q) \text{ divides } n \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \phi((f \chi_{H_q}) i_{\mathbb{Z}^2}(m,n)) & \text{if } I(q) \text{ divides } n \\ 0 & \text{otherwise.} \end{cases}\]

Thus every \( \chi_{H_q}(f i_{\mathbb{Z}^2}(m,n))\chi_{H_q} \) is in the range of \( \phi \), and \( \phi \) is surjective. \( \square \)

**Corollary 3.5.** Define \( \gamma : \mathbb{Z} \to \text{Aut}(C(H_q) \times_{\sigma_q} \mathbb{Z}) \) by

\[
(3.6)\quad \gamma_m(f i_{\mathbb{Z}}(n)) = \sigma_{I(q)m}^{-1}(f) i_{\mathbb{Z}}(n).
\]

Then \( (C(H_q) \times_{\sigma_q} \mathbb{Z}) \times_{\sigma} \mathbb{Z}^2 \) is isomorphic to a full corner in \( C(H) \times_{\sigma} \mathbb{Z}^2 \).

**Proof.** Theorem 4.1 of [15] gives \( C(H_q) \times_{\sigma} (\mathbb{Z} \times I(q)\mathbb{Z}) \cong (C(H_q) \times_{\sigma_q} \mathbb{Z}) \times I(q)\mathbb{Z} \), so the result follows from Lemma 3.4 on replacing \( I(q)\mathbb{Z} \) by the isomorphic group \( \mathbb{Z} \). \( \square \)
The analysis in [8, 1] shows that \( C(H_q) \rtimes_{\sigma} \mathbb{Z} \) is a Bunce-Deddens algebra. The \( K \)-theory of Bunce-Deddens algebras is well-known. To state the version we need, recall that if \( n = (n_k)_{k \geq 0} \) is a sequence with \( n_k \geq 2 \), then \( \mathbb{Z}[n^{-1}] \) denotes the set of rational numbers with denominator \( N_k = \prod_{i=0}^{k} n_i \) for some \( k \geq 0 \).

**Proposition 3.6.** Suppose \( n = (n_k)_{k \geq 0} \), \( X_k = \{0, \ldots, n_k - 1\} \), \( X = \bigoplus X_k \) and \( \tau : \mathbb{Z} \to \text{Aut} C(X) \) is the associated odometer. Then there are isomorphisms \( \phi_0 : \bigoplus (C(X) \rtimes_{\tau} \mathbb{Z}) \to \mathbb{Z}[n^{-1}] \) such that \( \phi_0(\{x_J(a_0, \ldots, a_k)\}) = n_k^{-1} \) for each cylinder set \( J(a_0, \ldots, a_k) \), and \( \phi_1 : \bigoplus (C(X) \rtimes_{\tau} \mathbb{Z}) \to \mathbb{Z} \) such that \( \phi_1(i_{\mathbb{Z}}(1)) = 1 \).

**Proof.** Because \( K_1(C(X)) = 0 \), the Pimsner-Voiculescu sequence for the system \( (C(X), \mathbb{Z}, \tau) \) reduces to

\[
0 \to K_1(C(X) \rtimes_{\tau} \mathbb{Z}) \xrightarrow{\delta} K_0(C(X)) \xrightarrow{\text{id} - \tau} K_0(C(X)) \xrightarrow{\text{id}} K_0(C(X) \rtimes_{\tau} \mathbb{Z}) \to 0.
\]

Now let \( C_k = \{J(a_0, \ldots, a_k)\} \) be the set of cylinder sets of length \( k + 1 \), and note that \( C(X) = \bigcup_{k=1}^{\infty} A_k \), where \( A_k = \text{span}\{\chi_J \mid J \in C_k\} \). Each \( \chi_J \) for \( J \in C_k \) is the sum of \( n_{k+1} \) basis elements of \( A_{k+1} \), so the maps \( \chi_J(a_0, \ldots, a_k) \mapsto N_k^{-1} \) of \( A_k \) into \( \mathbb{R} \) combine to give a homomorphism \( \psi_0 : K_0(C(X)) = \bigcup_{k=1}^{\infty} K_0(A_k) \to \mathbb{R} \) with range \( \mathbb{Z}[n^{-1}] \).

Since the generating automorphism \( \tau = \tau_1 \) permutes \( C_k \), the kernel of \( \psi_0 \) is the range of \( \text{id} - \tau_1 \), and hence \( \psi_0 \) induces the required isomorphism \( \phi_0 \) of \( K_0(C(X) \rtimes_{\tau} \mathbb{Z}) \) onto \( \mathbb{Z}[n^{-1}] \). To verify the statement about \( K_1 \), recall that \( \delta \) is the coboundary map for the Toeplitz extension of \( C(X) \rtimes_{\tau} \mathbb{Z} \) (see [13, \S 2]), and compute the index of \( [i_{\mathbb{Z}}(1)] \) in \( K_0(C(X) \otimes K) \cong K_0(C(X)) \).

**Proof of Theorem 3.3.** We saw in the proof of Proposition 3.2 and in the paragraph following it that the homeomorphism \( h \) of \( \prod_{k \geq 0} X_k \) onto the subgroup \( H_q \) of \( \mathcal{U}(\mathbb{Z}_p) \) satisfies

\[
\pi_k(h(\{a_n\})) = \pi_k(q^{a_0 + a_1 p(q) + \cdots + a_k p(q) p^{k-1}}) \quad \text{for } k \geq 0,
\]

and hence carries \( J(a_0, \ldots, a_k) \) onto \( Z(q^{a_0 + a_1 p(q) + \cdots + a_k p(q) p^{k-1}}) \), where

\[
Z_k(n) = \{x \in \mathcal{U}(\mathbb{Z}_p) \mid \pi_k(x) = \pi_k(n)\}.
\]

So we deduce from Proposition 3.6 that there is an isomorphism \( \phi_0 \) of \( K_0(C(H_q) \rtimes_{\sigma} \mathbb{Z}) \) onto \( \mathbb{Z}[p^{-1}] \) such that

\[
\phi_0(\{x_{Z_k(m)}\}) = \frac{1}{a_p(q)} \frac{1}{p^k}
\]

for every integer \( m \) which lies in \( H_q \).

Multiplying by the unit \( r^{-1(q)} \) carries \( Z_k(m) \) into \( Z_k(r^{-1(q)} m) \), and hence \( \phi_0 \circ (\gamma_1)_* = \phi_0 \). Thus \( (\gamma_1)_* \) is the identity on \( K_0(C(H_q) \rtimes_{\sigma} \mathbb{Z}) \). It is also the identity on \( K_1(C(H_q) \rtimes_{\sigma} \mathbb{Z}) \), and hence the Pimsner-Voiculescu sequence for \( ((C(H_q) \rtimes_{\sigma} \mathbb{Z}), \mathbb{Z}, \gamma) \)
collapses to the two short exact sequences

\[
0 \longrightarrow \frac{1}{o_p(q)} \mathbb{Z}[p^{-1}] \longrightarrow K_0(C(H_q) \rtimes \mathbb{Z}^2) \longrightarrow \mathbb{Z} \longrightarrow 0
\]

\[
0 \longrightarrow \mathbb{Z} \longrightarrow K_1(C(H_q) \rtimes \mathbb{Z}^2) \longrightarrow \frac{1}{o_p(q)} \mathbb{Z}[p^{-1}] \longrightarrow 0.
\]

From this and Corollary 3.5 we can deduce (3.5); since the isomorphism induced by Corollary 3.5 scales the class of \([1]\), we have removed the factor \(o_p(q)^{-1}\) by a further scaling to ensure that the final statement does not depend on the order of factors in our decomposition.

\[\square\]

Remark 3.7. The number of simple summands in Theorem 3.3 is \(|\mathcal{U}(\mathbb{Z}_p) : H|\), and we can compute this using [10, Lemma 3.7]. For example, if \(p\) is odd and \(l\) is large, we have from (A.1) that

\[
|\pi_l(H)| = \left[ o_p'(q), o_p'(r) \right] = \left[ p^{l - L_p(q)} o_p(q), q^{l - L_p(r)} o_p(r) \right] = p^{l - \min(L_p(q), L_p(r))} [o_p(q), o_p(r)].
\]

thus we deduce

\[
|\mathcal{U}(\mathbb{Z}_p) : H| = |\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z}) : \pi_l(H)| = \frac{(p - 1)^{p^{\min(L_p(q), L_p(r))} - 1}}{[o_p(q), o_p(r)]}.
\]

We could carry out a similar analysis when \(|F| > 1\), though it would not be so easy to work out some of the indices explicitly.

Remark 3.8. Theorem 2.1 implies in particular that \(C(H) \rtimes_\sigma \mathbb{Z}^2\) satisfies the hypotheses of the classification theorem of Elliott for AT-algebras [20, Theorem 3.2.6]. We can tell from the computation of \(K\)-theory in Theorem 3.3 that \(C(H) \rtimes_\sigma \mathbb{Z}^2\) is not a Bunce-Deddens algebra, but it is still closely related to an odometer. The homeomorphism of \(\prod_{k \geq 0} X_k\) onto \(H_q\) identifies the action of the first copy of \(\mathbb{Z}\) (multiplication by \(q\) on \(H_q\)) with an odometer (addition of 1 with carry over). The action of the second copy of \(\mathbb{Z}\) (multiplication by \(r\) on \(H_q\)) also acts as a permutation on each \(X_k\); it moves \(X_0\) around in a different order, and this action carries over into \(X_1\) when the marker in \(X_0\) returns to the starting point. So we can think of the action of \(\mathbb{Z}^2\) as two independent odometers on the same set. We can normalise the scale so that either copy of \(\mathbb{Z}\) acts by addition of 1 with carry over, but not so that both act this way at once.

4. The Bost-Connes algebra

The Hecke \(C^*-\)algebra \(C_\mathbb{Q}\) of Bost and Connes [2] is isomorphic to the semigroup crossed product \(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*\). The Fourier transform takes \(C^*(\mathbb{Q}/\mathbb{Z})\) onto the
algebra of continuous functions on the compact group $Z := \prod_{p \in \mathcal{P}} \mathbb{Z}_p$ and carries $\alpha$ into the action given by (see [3, §3.1])

$$\alpha_n(f)(x) = \begin{cases} f(x/n) & \text{if } n \text{ divides } x \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.2 is valid with $F$ replaced by $\mathcal{P}$ and $\mathbb{Z}_F$ by $Z$. Thus for $S \subset \mathcal{P}$, an application of [9 Theorem 1.7] gives that $J_S := C_0(Z \setminus Z_S) \rtimes_\alpha \mathbb{N}^*$ is an ideal of $C_Z = C(Z) \rtimes \mathbb{N}^*$, with quotient isomorphic to $C(Z_S) \rtimes \mathbb{N}^*$. Choose $a \in Z$ such that $a_p = 0 \iff p \in S$. Then $\{Q_+^*, a \cap Z\}$ has closure $Z_S$, so $C_0(Z \setminus Z_S)$ is the kernel of the representation $\pi_a$ considered in [8, page 440], and it follows from [10, Lemma 4.2] that $J_S$ is the kernel of the representation $\pi_a \times V$ described in [8, page 440]. We can now deduce that $S \mapsto J_S$, as $S$ runs through the proper subsets of $\mathcal{P}$, is the parametrisation of $(\text{Prim } C_Q) \setminus \hat{\mathbb{Q}}_+$ given in [8, Theorem 2.8].

**Theorem 4.1.** Suppose that $S$ is a proper subset of $\mathcal{P}$.

(a) If $\mathcal{P} \setminus S$ is infinite, then $J_S = \bigcap_{p \notin S} J_{S \cup \{p\}}$.

(b) If $0 < |\mathcal{P} \setminus S| < \infty$, then

$$\left(\bigcap_{p \notin S} J_{S \cup \{p\}}\right)/J_S \cong \left(C(\mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S)) \rtimes_\sigma \mathbb{Z}^S\right) \otimes K(l^2(\mathbb{N}^{P \setminus S})).$$

(c) $C_Q/J_\mathcal{P}$ is isomorphic to $C^*(Q_+^*) = C(\hat{\mathbb{Q}}_+)$. Moreover, $C(\mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S)) \rtimes_\sigma \mathbb{Z}^S$ is a simple AT-algebra with real rank zero and a unique tracial state.

It follows from [8, §2] that every basic open neighbourhood of $J_S$ has the form

$$U_G = \{J_T \mid T \subset \mathcal{P}, T \cap G = \emptyset\}$$

for some finite subset $G$ of $\mathcal{P} \setminus S$. When $\mathcal{P} \setminus S$ is infinite, there are always lots of $J_{S \cup \{p\}}$ in $U_G$, and thus $J_S \in \left\{J_{S \cup \{p\}} \mid p \notin S\right\}$; this says precisely that $\bigcap_{p \notin S} J_{S \cup \{p\}} \subset J_S$. The other inclusion is trivial, and (a) follows. Part (c) is true because $Z_\mathcal{P} = \{0\}$. To prove (b) we just need to replace $F$ by $\mathcal{P}$ and $\mathbb{Z}_F$ by $Z$ in the proof of Lemma 2.6.

It remains to prove the statements about $C(\mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S)) \rtimes_\sigma \mathbb{Z}^S$. The Chinese Remainder Theorem implies that $Z$ is dense in $\mathbb{Z}_\mathcal{P}\setminus S$, and hence $Z^S = Z \cap \mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S)$ is dense in $\mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S)$. Thus $Z^S$ acts minimally and freely on $\mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S)$, and $C(\mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S)) \rtimes_\sigma \mathbb{Z}^S$ is simple. However, since $|S| = \infty$, we cannot apply the results of [16] to conclude that $C(\mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S)) \rtimes_\sigma \mathbb{Z}^S$ has real rank zero and stable rank one, as we did in Section 2 for $C(H) \rtimes_\sigma \mathbb{Z}^S$. Instead we aim to use Theorems 1 and 2 of [9], and to do this we need to show that $C(\mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S)) \rtimes_\sigma \mathbb{Z}^S$ is an AH-algebra with the extra property of slow dimension growth.

Since $\mathcal{P} \setminus S$ is finite, we have as in (2.11) that $\mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S)$ is the inverse limit of the finite groups $\mathcal{U}(\mathbb{Z}/(\prod_{p \in \mathcal{P} \setminus S} p^n))$ over $l = (l_p) \in \mathbb{N}^{P \setminus S}$. Hence $\mathcal{U}(\mathbb{Z}_\mathcal{P}\setminus S) = \varprojlim F_n$, where
where

\[(4.1) \quad F_n := U \left( \mathbb{Z}/ \left( \prod_{p \in \mathcal{P} \setminus S} p^n \right) \mathbb{Z} \right). \]

We denote the canonical surjection of \( U(\mathbb{Z}_{\mathcal{P} \setminus S}) \) onto \( F_n \) by \( \pi_n \). The analogue of Lemma 2.7 for \( F_n \) and the canonical action of \( \mathbb{Z}^S \) by multiplication on \( F_n \) implies that \( C(U(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes_\sigma \mathbb{Z}^S \) is the closed union of \( C^* \)-subalgebras isomorphic to \( C(F_n) \rtimes \mathbb{Z}^S \).

Towards applying Proposition 2.8, we note that the infinite direct sum \( \bigoplus \limits_{n,i} \mathbb{Z} \) is a product of at most \( |\mathcal{P} \setminus S| + 1 \) cyclic groups (the +1 allows for the possibility that \( 2 \in \mathcal{P} \setminus S \)), and hence has a generating set \( \{x_{n,i}\} \) with at most \( |\mathcal{P} \setminus S| + 1 \) elements. By Dirichlet’s Theorem, there are primes \( q_{n,i} \) such that

\[q_{n,i} \equiv x_{n,i} \pmod{\prod_{p \in \mathcal{P} \setminus S} p^n} \]

and each \( q_{n,i} \) belongs to \( S \) because it is a unit modulo \( \prod_{p \in \mathcal{P} \setminus S} p^n \). Now let \( E'_n := \{q_{n,i}\} \), list the primes in \( S \) as \( \{r_n \mid n \in \mathbb{N}\} \), and take

\[E_n := \left( \bigcup_{m \leq n} E'_m \right) \cup \{r_1, \ldots, r_n\}.\]

We then have \( \pi_n(\mathbb{Z}_{E_n}) = F_n \), \( E_m \subset E_n \) for \( m \leq n \), and \( \bigcup E_n = S \).

We have now realised \( C(U(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes_\sigma \mathbb{Z}^S \) as the closure of an increasing union \( \bigcup_{n \in \mathbb{N}} B_n \), in which \( B_n \) is isomorphic to the crossed product \( C(F_n) \rtimes \mathbb{Z}^{E_n} \) by a transitive action of \( \mathbb{Z}^{E_n} \). By an argument identical to the one at the end of Section 2 we conclude that \( C(U(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes \mathbb{Z}^S \) has a unique tracial state.

We prove next that \( C(U(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes \mathbb{Z}^S \) is an AH-algebra with real rank zero. Proposition 2.8 implies that \( B_n \cong C(F_n) \rtimes \mathbb{Z}^{E_n} \cong C([\mathcal{T}^{E_n}], M_{|E_n|}(\mathbb{C})) \). But

\[ \frac{|E_n|}{|F_n|} \leq \frac{n(|\mathcal{P} \setminus S| + 2)}{\prod_p (p^1 - 1)^{p^{n-1}}} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \]

and thus the sequence \( B_n \cong C(F_n) \rtimes \mathbb{Z}^{E_n} \) of subalgebras of \( C(U(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes \mathbb{Z}^S \) has slow dimension growth. It now follows from \([1]\) Theorem 1] that \( C(U(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes \mathbb{Z}^S \) has topological stable rank one. Since the projections in \( C(U(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes \mathbb{Z}^S \) trivially separate the unique tracial state, \([1]\) Theorem 2] implies that \( C(U(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes \mathbb{Z}^S \) has real rank zero. The K-groups of \( C(U(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes \mathbb{Z}^S \) are inductive limits of torsion-free groups, and hence are themselves torsion-free, so it follows as in Section 2 that \( C(U(\mathbb{Z}_{\mathcal{P} \setminus S})) \rtimes \mathbb{Z}^S \) is an AT-algebra.

This completes the proof of Theorem 4.1.
APPENDIX A. THE ORDERS OF A PRIME IN GROUPS OF UNITS

For $p$ prime and $m \in \mathbb{N}$ such that $(m, p) = 1$, we denote by $o_p(m)$ the order of $m$ in $U(\mathbb{Z}/p^l\mathbb{Z})$. It was shown in [10, Theorem 3.1] that if $p$ is odd, there is a positive integer $L_p(m)$ such that

\begin{equation}
(A.1) 
\begin{array}{ll}
o_p(m) & \text{if } 1 \leq l \leq L_p(m) \\
\frac{p^l-1}{p^l-L_p(m)}o_p(m) & \text{if } l > L_p(m);
\end{array}
\end{equation}

the proof uses that the groups $U(\mathbb{Z}/p^l\mathbb{Z})$ are cyclic. We will now show how to modify the arguments of [10, §3] to obtain an analogue of (A.1) for $p = 2$, in which case $U(\mathbb{Z}/2^l\mathbb{Z})$ are no longer cyclic.

**Proposition A.1.** If $m$ is an odd integer and $m \equiv 1 \pmod{4}$, then there exists a positive integer $K = L_2(m)$ such that

\begin{equation}
(A.2) 
\begin{array}{ll}
o_2(m) = 1 & \text{if } 1 \leq l \leq K \\
2^{l-K} & \text{if } l > K;
\end{array}
\end{equation}

if $m \equiv 3 \pmod{4}$, then there exists a positive integer $L = L_2(m)$ such that

\begin{equation}
(A.3) 
\begin{array}{ll}
o_2(m) = 1 & \text{if } l = 1 \\
2 & \text{if } 1 < l \leq L \\
2^{l-(L-1)} & \text{if } l > L.
\end{array}
\end{equation}

To prove Proposition A.1 we use general properties of cyclic groups as in [10, §3]. We begin with a lemma.

**Lemma A.2.** Suppose $l \geq 3$. Then the group \( \{ n \in U(\mathbb{Z}/2^l\mathbb{Z}) \mid n \equiv 1 \pmod{4} \} \) is the cyclic subgroup \( \langle 5 \rangle_l \) of \( U(\mathbb{Z}/2^l\mathbb{Z}) \) generated by 5.

**Proof.** Theorem 2 of [4, Chapter 4.1] says that \( |\langle 5 \rangle_l| = 2^{l-2} \). For $k \geq 0$ we have

\[ 5^k = (4 + 1)^k = \sum_{n=0}^{k} \binom{k}{n} 4^n = 4 \sum_{n=1}^{k} \binom{k}{n} 4^{n-1} + 1, \]

so $5^k \equiv 1 \pmod{4}$. Hence, if $n \equiv 5^k \pmod{2^l}$ for some $0 \leq k < 2^{l-2}$, then $n \equiv 1 \pmod{4}$. Since the order of \( \{ n \in U(\mathbb{Z}/2^l\mathbb{Z}) \mid n \equiv 1 \pmod{4} \} \) is also $2^{l-2}$, the result follows.

**Corollary A.3.** An element of $U(\mathbb{Z}/2^l\mathbb{Z})$ is congruent to 3 (mod 4) if and only if it is congruent to $-5^k$ (mod $2^l$) for some $k$ satisfying $0 \leq k < 2^{l-2}$.

**Corollary A.4.** Suppose $m \in \mathbb{Z}$ satisfies $m \equiv 1 \pmod{4}$. Then for every $l > 0$ we have

\begin{equation}
(A.4) 
\begin{array}{ll}
o_{2^l}(m) = o_{2^{l+1}}(m) & \text{if } 2 \text{ does not divide } o_{2^{l+1}}(m) \\
o_{2^{l+1}}(m)/2 & \text{if } 2 \text{ divides } o_{2^{l+1}}(m).
\end{array}
\end{equation}
Proof. Since a number is coprime to $2^l$ if and only if it is coprime to $2^{l+1}$, the reduction map $\pi: \mathcal{U}(\mathbb{Z}/2^{l+1}\mathbb{Z}) \to \mathcal{U}(\mathbb{Z}/2^l\mathbb{Z})$ is a surjective homomorphism. Lemma A.2 implies that $m \equiv 5^r \pmod{2^{l+1}}$, where $r = o_{2^{l+1}}(5)/o_{2^{l+1}}(m) = 2^{l-1}/o_{2^{l+1}}(m)$. Hence, by applying Lemma 3.2 to the restriction of $\pi$ to a homomorphism of $\langle 5 \rangle_{l+1}$ onto $\langle 5 \rangle_l$, we have

$$o_{2^l}(m) = o(\pi(5^r)) = \begin{cases} 2^{l-1}/(2^{l-1}/o_{2^{l+1}}(m), 2^{l-1}) & \text{if } 2^{l-1} \text{ divides } 2^{l-1}/o_{2^{l+1}}(m) \\ 2^{l-1}/(2(2^{l-1}/o_{2^{l+1}}(m), 2^{l-1})) & \text{if } 2^{l-1} \text{ does not divide } 2^{l-1}/o_{2^{l+1}}(m), \end{cases}$$

which simplifies to (A.4).

Proof of Proposition A.5. Suppose first that $m \equiv 1 \pmod{4}$. For fixed $N$, there exists an $l \in \mathbb{N}$ satisfying $m^N < 2^l$. Then $o_{2^l}(m) > N$ and hence the sequence $\{o_{2^l}(m) \mid l \in \mathbb{N}\}$ must be unbounded. In particular, $\{o_{2^l}(m)\}$ is not a constant sequence. Let $K$ be the first integer such that $o_{2^K}(m) < o_{2^{K+1}}(m)$. Then $o_{2^l}(m) = o_{2^l}(m) = 1$ for $1 \leq l \leq K$, and by Corollary A.4 we have $o_{2^{K+1}}(m) = 2o_{2^l}(m) = 2$. Since $o_{2^{K+1}}(m)$ divides $o_{2^l}(m)$ for all $l > K$, it follows that 2 divides $o_{2^l}(m)$ for all $l > K$.

We now apply Corollary A.4 $l-K$ times to deduce that $o_{2^l}(m) = 2^{l-K}o_{2^{K}}(m) = 2^{l-K}$.

Now suppose that $m \equiv 3 \pmod{4}$. Certainly $o_{2^l}(m) = 1$. For $l > 1$, Corollary A.3 tells us that $m \equiv -5^k$ for some $0 \leq k < 2^{l-2}$. Thus $m^2 \equiv 5^{2k} \pmod{2^l}$, and therefore $m^2 \in \langle 5 \rangle_l$. Let $L$ be the first integer such that $o_{2^L}(m^2) < o_{2^{L+1}}(m^2)$. Applying Corollary A.4 to $m^2$ and repeating the argument of the preceding paragraph gives (A.3) because $o_{2^l}(m) = 2o_{2^l}(m^2)$.

We now need to extend these results to cover actions on $\mathcal{U}(\mathbb{Z}/p^k)$ for an arbitrary finite set $F$ of primes. We write $F = \{p_1, \ldots, p_n\}$ and fix a prime $q$ which is not in $F$. We denote by $o_{(l_1, \ldots, l_n)}(q)$ the order of $(q, \ldots, q)$ in $\prod_{i=1}^n \mathcal{U}(\mathbb{Z}/p_i^{l_i}\mathbb{Z})$.

Proposition A.5. There exist positive integers $K_1, \ldots, K_n$ and $d$ such that

(A.5) \hspace{1cm} o_{(K_1+l_1, \ldots, K_n+l_n)}(q) = dp_1^{l_1} \cdots p_n^{l_n}

for every $(l_1, \ldots, l_n) \in \mathbb{N}^F$.

Proof. Suppose first that $p_1, \ldots, p_n$ are distinct odd primes, and let $L_{p_i}(q)$ be as in (A.1). Let

$$z_i := \max\{z \mid p_i^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{1, \ldots, n\}\},$$

and define $K_i := L_{p_i}(q) + z_i$ and $d := [o_{p_1}(q), \ldots, o_{p_n}(q)]$, where $[r_1, \ldots, r_n]$ is the least common multiple of the integers $r_i$. In general, if $g_i$ are elements of order $r_i$ in finite groups $G_i$, then the order of $(g_1, \ldots, g_n)$ in $G_1 \times \cdots \times G_n$ is $[r_1, \ldots, r_n]$. Thus from
the properties of \( L_{p_i}(q) \) we obtain

\[
(A.6) \quad o_{(K_1+t_1, \ldots, K_n+t_n)}(q) = [p_{1}^{(K_1+t_1) - L_{p_1}(q)}, \ldots, p_{n}^{(K_n+t_n) - L_{p_n}(q)} o_{p_n}(q)]
\]

\[
= [p_{1}^{z_1+t_1} o_{p_1}(q), \ldots, p_{n}^{z_n+t_n} o_{p_n}(q)]
\]

\[
= p_{1}^{t_1} \cdots p_{n}^{t_n} [o_{p_1}(q), \ldots, o_{p_n}(q)],
\]

which is \((A.5)\).

Now suppose that \( 2 \in F \), say \( p_1 = 2 \). If \( q \equiv 1 \pmod{4} \), we let

\[
z_i := \max \{ \{z | p_i^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{2, \ldots, n\} \} \}
\]

and define \( K_1 := L_2(q) + z_1 \), \( K_i := L_{p_i}(q) + z_i \) for \( i > 1 \), and \( d := [o_{p_2}(q), \ldots, o_{p_n}(q)] \).

Reasoning as in \((A.6)\) gives \((A.5)\).

If \( q \equiv 3 \pmod{4} \), we let

\[
z_i := \max \{ \{z | p_i^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{2, \ldots, n\} \} \}
\]

for \( i > 1 \), and define \( K_1 := L_2(q) + z_1 - 1 \), \( K_i := L_{p_i}(q) + z_i \) for \( i > 1 \) and \( d := [2, o_{p_2}(q), \ldots, o_{p_n}(q)] \).

Now reasoning as in \((A.6)\) gives \((A.5)\). \(\square\)

**Corollary A.6.** The closure \( H \) of \( \mathbb{Q}^\mathbb{Z} \) in \( U(\mathbb{Z}_F) \) is a subgroup of finite index

\[
|U(\mathbb{Z}_F) : H| = (\prod_{i=1}^{n}(p_i - 1)p_i^{K_i - 1})/d.
\]

**Proof.** Apply Proposition \((A.5)\) to \( l = (l, l, \ldots, l) \) to see that \( |\pi_l(H)| = d(\prod_{i=1}^{n} p_i^l) \) for large \( l \), and the result follows from \([10] \) Lemma 3.7. \(\square\)

**References**

[1] B. Blackadar, M. Dadarlat and M. Rørdam, *The real rank of inductive limit C*-algebras*, Math. Scand. 69 (1991), 211–216.
[2] J.-B. Bost and A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series) 1 (1995), 411–457.
[3] K. R. Davidson, *C*-Algebras by Example*, Fields Institute Monographs, vol. 6, Amer. Math. Soc., Providence, 1996.
[4] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Second Ed., Graduate Texts in Math., vol. 84, Springer-Verlag, Berlin, 1993.
[5] M. Laca, *Semigroups of *-endomorphisms, Dirichlet series, and phase transitions in number theory*, J. Funct. Anal. 152 (1998), 330–378.
[6] M. Laca, *From endomorphisms to automorphisms and back: dilations and full corners*, J. London Math. Soc. 61 (2000), 893–904.
[7] M. Laca and I. Raeburn, *A semigroup crossed product arising in number theory*, J. London Math. Soc. 59 (1999), 330–344.
[8] M. Laca and I. Raeburn, *The ideal structure of the Hecke C*-algebra of Bost and Connes*, Math. Ann. 318 (2000), 433–451.
[9] N. S. Larsen, *Nonunital semigroup crossed products*, Math. Proc. Royal Irish Acad. 100A (2000), 205–218.
[10] N. S. Larsen, I. F. Putnam and I. Raeburn, The two-prime analogue of the Hecke C*-algebra of Bost and Connes, Indiana Univ. Math. J. 51 (2002), 171–186.

[11] N. S. Larsen and I. Raeburn, Faithful representations of crossed products by actions of $\mathbb{N}^k$, Math. Scand. 89 (2001), 283–296.

[12] M. Marcus and H. Minc, A Survey of Matrix Theory and Matrix Inequalities, Allyn and Bacon, Inc., Boston, 1964.

[13] S. Neshveyev, Ergodicity of the action of the positive rationals on the group of finite adeles and the Bost-Connes phase transition theorem, Proc. Amer. Math. Soc. 130 (2002), 2999–3003.

[14] D. Olesen and G. K. Pedersen, Partially inner C*-dynamical systems, J. Funct. Anal. 66 (1986), 262–281.

[15] J. A. Packer and I. Raeburn, Twisted crossed products of C*-algebras, Math. Proc. Camb. Phil. Soc. 160 (1989), 293–311.

[16] N. C. Phillips, Crossed products of the Cantor set by minimal actions of $\mathbb{Z}^d$, ArXiv: math.OA/0208085

[17] N. C. Phillips, Crossed products by finite cyclic group actions with the tracial Rokhlin property, ArXiv: math.OA/0306410

[18] M. Pimsner and D. Voiculescu, Exact sequences for $K$-groups and Ext-groups of certain crossed product C*-algebras, J. Operator Theory 4 (1980), 93–118.

[19] I. Raeburn, On crossed products by coactions and their representation theory, Proc. London Math. Soc. 64 (1992), 625–652.

[20] M. Rørdam, Classification of nuclear simple C*-algebras, in Encyclopaedia Math. Sci., vol. 126, Springer-Verlag, Berlin, 2002, pages 1–145.

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