Geometrical structures of the instantaneous current and their macroscopic effects: vortices and perspectives in non-gradient models

Leonardo De Carlo

Abstract First we discuss the definition of the instantaneous current in interacting particle systems, in particular in mass-energy systems and we point out its role in the derivation of the hydrodynamics. Later we present some geometrical structures of the instantaneous current when the rates of stochastic models satisfy a very common symmetry. These structures give some new idea in non-gradient models and show a new phenomenology in diffusive interacting particle systems. Specifically, we introduce models with vorticity and present some new perspectives on the link between the Green-Kubo’s formula and the hydrodynamics of non-gradient models.

Key words: Stochastic lattice gases, non-gradient models, discrete Hodge decomposition

AMS 2010 Subject Classification: 60K35, 82C22, 82C20

1 Introduction and Results

When many interacting particles are modeled by Newton’s equations the rigorous derivation of hydrodynamics equations, consisting in some PDEs and describing the evolution of thermodynamic quantities, is often a too optimistic programme, mainly because of the lack of good ergodic property of the system. To overcome the mathematical problem two assumptions are traditionally made: or modeling the problem with a stochastic microscopic evolution or assuming a low density of particles. In the present framework we are interested in the first assumption and we are not having a complete rigorous point of view. For a rigorous and didactic treatment traditional references are [14] [18]. The microscopic dynamics consists of random
walks of particles on a lattice \( V_N \) that are constrained to some rule expressing the local interaction, these are the so-called \textit{interacting particle systems} introduced by Spitzer [17].

In this paper we focus on the instantaneous current which is the bridge from the microscopic description to the macroscopic description of interacting particle systems. In section 2 we give some definitions that we will use throughout all the paper. In sections 3 and 4 we present the models and describe the instantaneous current, in particular its definitions it is clarified in mass-energy systems like KMP [6, 13]. In section 5 we recall the functional Hodge decomposition obtained in [5] in dimension one and two and we apply it to some interacting particle models. The expert reader can skip the first four sections and section 6, where well known notions of the literature are presented with a general flavour, and refer these sections just for notation if necessary.

In the work the attention is on diffusive models. Some new models with vorticity are introduced in section 8. After reviewing the qualitative theory of scaling limits in diffusive systems in section 7 for the first time we study the macroscopic consequences of this decomposition. This leads us to some new phenomenology in particle systems, that is we show in a non-rigorous way that the hydrodynamics of the macroscopic current can present zero divergence terms that are not observed in the hydrodynamics of the density. This extend the usual Fick’s law (58) to the a new picture (68) where the diffusion matrix is a positive non-symmetric matrix.

In diffusive non gradient systems a derivation of the hydrodynamics with an explicit diffusion coefficient is an open problem. The relative PDEs are in term of a variational expression of the diffusion coefficient equivalent to the Green-Kubo’s formula, see [15, 18]. In the last section 9 we try to give some perspectives coming from our Hodge decomposition. We give a possible explicit description of the minimizer of Green-Kubo’s formula using our functional Hodge decomposition and describe a scheme that connects this minimizer with an explicit hydrodynamics.

## 2 Definitions

Interacting particle systems are stochastic models evolving on a lattice along a continuous time Markov dynamics. For the purposes of the paper, we are going to consider only periodic boundary conditions for the lattice where particles move, i.e. the set of vertices \( V_N \) of the lattice will be the \( n \)-dimensional discrete torus

\[
\mathbb{T}_N^n = \mathbb{Z}^n / \mathbb{L}^n \quad \text{or} \quad \mathbb{T}^n_\epsilon = \epsilon \mathbb{Z}^n / \mathbb{L}^n, \quad \epsilon = 1 / N
\]

along the space scale we want to consider. We denote with \( \delta_N \) the set of all couples of vertices \( \{x, y\} \) of \( V_N \) such that

\[
y = x \pm \delta e_i
\]

where \( e_i \) is the canonical versor in \( \mathbb{Z}^n \) along the direction \( i \) and \( \delta \) is equals to 1 on \( \mathbb{T}_N^n \) and to \( 1 / N \) on \( \mathbb{T}^n_\epsilon \). The elements of \( \delta_N \) are named non-oriented edges or simply edges. In this way we have an non-oriented graph \( (V_N, \delta_N) \). To every non-oriented graph \( (V_N, \delta_N) \) we associate canonically an oriented graph \( (V_N, E_N) \) such that the set of oriented edges \( E_N \) contains all the ordered pairs \( \{x, y\} \) such that
\{x,y\} \in \mathcal{E}_N. Note that if \( (x,y) \in E_N \) then also \( (y,x) \in E_N \). If \( e = (x,y) \in E_N \) we denote \( e^- := x \) and \( e^+ := y \) and we call \( e := \{x,y\} \) the non-oriented edge.

The microscopic configurations of our particle models are given by the collection of variables \( \eta(x) \) representing the number of particles, the energy or mass at \( x \in \mathcal{V}_N \) along the model. When the variables \( \eta(x) \) are discrete we interpret them as number particles and when continuous as mass-energy. Calling \( \Sigma \) the state space at \( x \) we define the configuration state space as \( \Sigma_N := \Sigma^{|\mathcal{V}_N|} \). The microscopic dynamics is a Markov process \( \{\eta_t\}_{t \in \mathbb{R}} \) where particles or masses interact along rules encoded in the generator \( \mathcal{L}_N \), i.e.

\[
\mathcal{L}_N f(\eta) = \sum_{\eta' \in \Sigma_N} c(\eta, \eta')[f(\eta') - f(\eta)],
\]

where \( f \) is an observable and \( c(\eta, \eta') \) the transition rates from \( \eta \) to \( \eta' \).

Let \( \tau_z \) be the shift by \( z \) on \( \mathbb{Z}^n \) defined by the relation \( \tau_z \eta(x) := \eta(x-z) \) with \( z \in \mathbb{Z}^n \) and for a function \( h : \eta \to h(\eta) \in \mathbb{R} \) we define \( \tau_z h(\eta) := h(\tau_z \eta) \), moreover for a domain \( B \subseteq \mathcal{V}_N \) we define \( \tau_B := B + z. \) A function \( h : \Sigma_N \to \mathbb{R} \) is called \textit{local} if it depends only through the configuration in a finite domain \( B \subseteq \mathcal{V}_N \) denoted \( D(f) \).

Let \( \lceil \cdot \rceil_+ \) be the positive part function.

### 3 Particle models and instantaneous current

We treat only nearest neighbour conservative dynamics, that is \( \mathcal{L}_N \) becomes

\[
\mathcal{L}_N f(\eta) = \sum_{(x,y) \in E_N} c_{x,y}(\eta)(f(\eta^{x,y}) - f(\eta)), \quad \eta^{x,y}(z) := \begin{cases} 
\eta(x) - 1 & \text{if } z = x \\
\eta(y) + 1 & \text{if } z = y \\
\eta(z) & \text{if } z \neq x,y
\end{cases}.
\]

We study \textit{translational covariant models}, i.e. \( c_{x,x+e^i} \in \mathcal{L}_N \) for \( i \in \mathbb{Z} \) and \( x \in \mathcal{V}_N \).

#### 3.1 Exclusion process and the 2-SEP

In an \textit{exclusion process} particles move according to a conservative dynamics of independent random walks with the exclusion rule that there cannot be more than one particle in a single lattice site (hard core interaction). The rates of \( \mathcal{L}_N \) have the general form

\[
c_{x,y}(\eta) = \eta(x)(1 - \eta(y))c_{x,y}(\eta),
\]

where \( c_{x,y}(\eta) \) is the jump rates when \( \eta \) has a particle in \( x \) and an empty site in \( y \).

The next example of \( \mathcal{L}_N \) is the \textit{2-SEP} (2-simple exclusion process), in this model the interaction is simply hardcore but in every site there can be at most 2 particles.
The state space is $\Sigma_N = \{0, 1, 2\}$ and the dynamics is defined by

$$\mathcal{L}_N^{2-\text{SEP}} f(\eta) = \sum_{(x,y) \in E_N} c_{x,y}(\eta) \left( f(\eta^{x,y}) - f(\eta) \right), \quad c_{x,y}(\eta) = \chi^+(\eta(x)) \chi^-(\eta(y)),$$

where $\chi^+(\alpha) = 1$ if $\alpha > 0$ and zero otherwise while $\chi^-(\alpha) = 1$ if $\alpha < 2$ and zero otherwise. (4)

### 3.2 Instantaneous current in particle systems

In interacting particle systems there are deep underlying geometrical structures that reflects in the hydrodynamics of lattice models as we will discuss later, see also [5]. The basis is the fact that the instantaneous current is a discrete vector field and closely related to a microscopic mass conservation law leading to the hydrodynamics.

**Definition 1.** A discrete vector field is a function $\varphi : E_N \to \mathbb{R}$ that is *antisymmetric*, i.e. $\varphi(x,y) = -\varphi(y,x)$ for any $(x,y) \in E_N$.

The *instantaneous current* for our particle models is defined as

$$j_\eta(x,y) := c_{x,y}(\eta) - c_{y,x}(\eta),$$

which is a discrete vector field for each fixed configuration $\eta$. The intuitive interpretation of the instantaneous current is the rate at which particles cross the bond $(x,y)$. Let $\mathcal{N}_t(x,y)$ be the number of particles that jumped from site $x$ to site $y$ up to time $t$. The *current flow* across the bond $(x,y)$ up to time $t$ is defined as

$$J_t(x,y) := \mathcal{N}_t(x,y) - \mathcal{N}_t(y,x).$$

This is a discrete vector field $(J_t(x,y) = -J_t(y,x))$ depending on the trajectory $\{\eta_t\}_t$. Between the instantaneous current $j_\eta(x,y)$ and the current flow $J_t(x,y)$ there is a strict connection given by the key observation (see for example [18] section 2.3 in part II) that

$$\mathcal{M}_t(x,y) = J_t(x,y) - \int_0^t j_{\eta(s)}(x,y)ds$$

is a martingale. This allows to treat the difference between $J_t(x,y)$ and the integral $\int_0^t ds j_{\eta(s)}(x,y)$ as a microscopic fluctuation term. It also gives a more physical definition of $j_\eta(x,y)$ as follows. Consider an initial configuration $\eta_0 = \eta$, the explicit expression of the instantaneous current can be defined as

$$j_\eta(x,y) := \lim_{t \to 0} \frac{\mathbb{E}^\eta(J_t(x,y))}{t}.$$ (8)

The expectation is $\mathbb{E}^\eta(J_t(x,y)) = \int \mathbb{P}^\eta(d\{\eta_t\}_t) J_t(x,y)$, where the integration is over all trajectories $\{\eta_t\}_t$ starting from $\eta$ at time 0 and $\mathbb{P}^\eta$ the probability induced by the
Geometrical structures of the instantaneous current and their macroscopic effects

Markov process. For a trajectory \( \{ \eta_t \} \), the probability to observe more than one jump goes like \( O(t^2) \), then it is negligible since we are interested in an infinitesimal time interval. Since \( c(\eta, \eta') = \lim_{t \to 0} \frac{p_\eta(\eta = \eta')}{t} = \lim_{t \to 0} \frac{p_\eta(\eta, \eta')}{t} \), where \( p_\eta(\eta, \eta') \) are the transition probability, when \( t \) goes to zero \( J_t(x, y) \) takes value +1 if a jump from \( x \) to \( y \) happens, −1 in the opposite case and 0 in the other cases. So the current defined in (8) becomes \( J_0(x, y) = c_{x,y}(\eta) - c_{y,x}(\eta) \) as in (5).

The discrete divergence for a discrete vector field \( \varphi \) on \( E_N \) is \( \nabla \cdot \varphi(x) = \sum_{y \sim x} \varphi(x, y) \), where the sum is on the nearest neighbours \( y \sim x \) of \( x \). For convenience of notation, we will use the symbol \( \nabla \cdot \) both for the discrete case and the continuous one, therefore we recommend to the reader to pay attention about this. The local microscopic conservation law of the number of particles is then given by

\[
\eta_t(x) - \eta_0(x) + \nabla \cdot J_t(x) = 0.
\]

Using (7) in (9) we get

\[
\eta_t(x) - \eta_0(x) + \int_0^t ds \nabla \cdot j_s(x) + \nabla \cdot M_s(x) = 0. \tag{10}
\]

We can deduce that at the equilibrium, that is when for a measure \( \mu_N \) on \( \Sigma_N \) the detailed balance condition is true, i.e. \( \mu_N(\eta)c(\eta, \eta^{+z}) = \mu_N(\eta^{+z})c(\eta^{+z}, \eta) \) for all \( (x, y) \in E_N \), the average flow \( \mathbb{E}_{\mu_N}^\eta(J_t(x, y)) \) is constantly zero, where the subscript \( \mu_N \) indicates the average respect to the equilibrium measure \( \mu_N \). For a small time interval \( \Delta t \) from (7), (8) and the detailed balance we have \( \mathbb{E}_{\mu_N}^\eta(J_{\Delta t}(x, y)) \sim \mathbb{E}_{\mu_N}^\eta(j_0(x, y)) \Delta t = 0. \) Since this is true for any time interval \( \Delta t \) and the current flow \( J_t(x, y) \) is additive we conclude that \( \mathbb{E}_{\mu_N}^\eta(J_t(x, y)) = 0. \) More generally for a stationary measure \( \mu_N \), that is \( \mu_N(\mathcal{L}_N f) = 0 \) for any \( f \), we have that

\[
\mathbb{E}_{\mu_N}^\eta(J_t(x, y)) = \mathbb{E}_{\mu_N}(j_\eta(x, y)) t. \tag{11}
\]

Remark 1. For a translational covariant model, i.e. \( c_{x,y}(\eta) = c_{x+z,y+z}(\tau_z \eta) \) for any \( z \in V_N \), then the instantaneous current is translational covariant too, namely it satisfies the symmetry relation

\[
j_\eta(x, y) = j_{\eta}(x+z, y+z). \tag{12}
\]

4 Energy-mass models

In this section we adapt the concepts of the previous section to the continuous case, where we consider models that exchange continuous quantity between sites. The lattice variables are interpreted as energy or mass along the context and the configuration is denoted with \( \xi = \{ \xi(x) \}_{x \in \mathbb{V}_N} \). The first model to be described is the most
famous model of this class, namely the Kipnis-Marchioro-Presutti (KMP) model [13].

4.1 KMP model and generalization, dual KMP, gaussian model

The **KMP dynamics** is a generalized stochastic lattice gas on which energies or masses are associated to oscillators at the vertices $V_N$. The stochastic evolution is of the type

$$\mathcal{L}_N f(\xi) = \sum_{\{x,y\} \in \delta_N} \mathcal{L}_{\{x,y\}} f(\xi), \text{ with}$$

$$\mathcal{L}_{\{x,y\}} f(\xi) := \int_{-\xi(y)}^{\xi(x)} \frac{dq}{\xi(x) + \xi(y)} [f(\xi - q(e^x - e^y)) - f(\xi)].$$

where $e^x = \{e^x(y)\}_{y \in V_N}$ is the configuration of mass with all the sites different from $x$ empty and having unitary mass at site $x$, this means that $e^x(y) = \delta_{x,y}$ where $\delta$ is the Kronecker symbol. Formula (14) define the model as a uniform distributed random current model.

The dynamics (14) can be generalized substituting the uniform distribution on $[-\xi(y), \xi(x)]$ for a different probability measure (or just positive measure) $\Gamma^{\xi}_{x,y}(dq)$, i.e.

$$\mathcal{L}_{\{x,y\}} f(\xi) := \int \Gamma^{\xi}_{x,y}(dq) [f(\xi - q(e^x - e^y)) - f(\xi)].$$

with the symmetry $\Gamma^{\xi}_{x,y}(q) = \Gamma^{\xi}_{y,x}(-q)$ so that (13) is a sum over unordered edges. When considering a discrete state space, a natural choice for $\Gamma^{\xi}_{x,y}(dq)$ in (15) is the discrete uniform distribution on the integer points in $[-\xi(y), \xi(x)]$. This means that if $\xi$ is a configuration of mass assuming only integer values then

$$\Gamma^{\xi}_{x,y}(dq) = \frac{1}{\xi(x) + \xi(y) + 1} \sum_{i \in [-\xi(y), \xi(x)]} \delta_i(dq)$$

where $\delta_i(dq)$ is the delta measure at $i$ and the sum is over the integer values belonging to the interval. This is exactly the dual model of KMP [13] called also KMPd.

Another interesting model could be the following **Gaussian model**. In this case the interpretation in terms of mass is missing since the variables can assume also negative values and it could be interpreted as a charge model. The bulk dynamics is defined by a distribution of current having support on all the real line

$$\Gamma^{\xi}_{x,y}(dq) = \frac{1}{\sqrt{2 \pi \gamma^2}} e^{-\frac{(\xi(x) - \xi(y))^2}{2 \gamma^2}} dq.$$
4.2 Weakly asymmetric energy-mass models

We consider dynamics perturbed by a space and time dependent discrete external field $F$ defined as follows. Let $F : T \rightarrow \mathbb{R}^n$ be a smooth vector field with components $F(x) = (F_1, \ldots, F_n)$, describing the force acting on the masses of the systems. We associate to $F$ a discrete vector field $F(x,y)$ defined by

\[ F(x,y) = \int_{(x,y)} F(z) \cdot dz, \tag{18} \]

$(x,y)$ is an oriented edge and the integral is a line integral that corresponds to the work done by the vector field $F$ when a particle moves from $x$ to $y$. So we think about $F(x,y)$ as work done per particle. We want to change the random distribution of the current on each bond according to a perturbed measure $\Gamma^F$, that is

\[ \mathcal{L}_{\{x,y\}}^F f(\xi) := \int \Gamma_{x,y}^F (dq) \left[ f(\xi - q (\varepsilon^x - \varepsilon^y)) - f(\xi) \right], \quad \Gamma_{x,y}^F (dq) = \Gamma_{x,y}^F (dq) e^{F_{x,y}(q)}q. \tag{19} \]

The effect of an external field is modelled by perturbing the rates and giving a net drift toward a specified direction. When the size of $|y-x|$ is of order $1/N$ we obtain a weakly asymmetric model, the discrete vector field (18) is of order $1/N$ too and the hydrodynamics is studied considering a perturbative expansion of $\Gamma_{x,y}^F (dq)$ for the orders that will give a macroscopic effect. We will see that for weakly asymmetric diffusive models this expansion is necessary up to the order two. If $F = -\nabla H$ is a gradient vector field, then $F(x,y) = H(x) - H(y)$ and $\Gamma_{x,y}^F (dq) = \Gamma_{x,y}^F (dq) e^{H(x) - H(y)}q$.

By the symmetry of the measure $\Gamma$ and the antisymmetry of the discrete vector field $F$ we have that $\Gamma_{x,y}^F (q) = \Gamma_{y,x}^F (-q)$ and we can define the generator considering sums over unordered bonds

\[ \mathcal{L}_N f(\xi) = \sum_{\{x,y\} \in \mathcal{E}_N} \mathcal{L}_{\{x,y\}}^F f(\xi). \tag{20} \]

4.3 Instantaneous current of energy-mass systems

Here we adapt the definition of instantaneous current to the formalism of the interacting nearest neighbour energy-mass models. The generator is (19), the case $F = 0$ is treated as a subcase and we omit the index when the external field is zero. The instantaneous current for the bulk dynamics is defined as

\[ j_{\xi}^F (x,y) := \int \Gamma_{x,y}^F (dq)q. \tag{21} \]
Its interpretation is the rate at which masses-energies cross the bond \((x, y)\) and it is still a discrete vector field. The \textit{current flow} now is indicated with \(\mathcal{J}_t(x, y)\) and it is the net total amount of mass-energy that has flown from \(x\) to \(y\) in the time window \([0, t]\). It can be defined as sum of all the differences between the mass-energy measured in \(x\) before and after of every jump on the bond \(\{x, y\}\). Let \(\tau_i\) be the time of the \(i^{-th}\) jump on the bond \(\{x, y\}\) for some \(i\), we write the current flow as follows

\[
\mathcal{J}_t(x, y) := \sum_{\tau : \tau \in [0, t]} J_{\tau}(x, y),
\]

where \(J_{\tau}(x, y)\) is the \textit{present flow} defined as the current flowing from \(x\) to \(y\) jump time \(\tau\)

\[
J_{\tau}(x, y) := \lim_{h \downarrow 0} \xi_{\tau - h}(x) - \lim_{h \downarrow 0} \xi_{\tau + h}(x).
\]

Defining \(J_{\tau}(y, x) := \lim_{h \downarrow 0} \xi_{\tau - h}(y) - \lim_{h \downarrow 0} \xi_{\tau + h}(y)\), the flow \(\mathcal{J}_t(x, y)\) is still an anti-symmetric vector field depending on the trajectory \(\{\xi_t\}\), i.e. \(J_{\tau}(y, x) := -J_{\tau}(x, y)\). As in the particles case \(\mathcal{J}_t(x, y)\) is a function on the trajectory space, while the instantaneous current \(\bar{j}_{\xi}^F(x, y)\) is a function on the configuration space and the difference

\[
M_t(x, y) = \mathcal{J}_t(x, y) - \int_0^t ds j_{\xi}^F(x, y).
\]

is a martingale. Repeating what we did in subsection 3.2 (with a formalism suitable to energy-mass models) the instantaneous current \((21)\) can be obtained as

\[
\bar{j}_{\xi}^F(x, y) := \lim_{t \to 0} \frac{\xi_t^F(\mathcal{J}_t(x, y))}{t}.
\]

As we did in subsection 4.2 from the local discrete conservation of the mass-energy \(\xi_t(x) - \xi_0(x) + \nabla \cdot \mathcal{J}_t(x) = 0\) we have

\[
\xi_t(x) - \xi_0(x) + \int_0^t ds \nabla \cdot \bar{j}_{\xi}^F(x) + \nabla \cdot M_t(x) = 0.
\]

The microscopic fluctuation \((24)\) has mean zero and \((11)\) can be obtained similarly to conclude that the average currents are zero in the equilibrium case, i.e. when detailed balance conditions (DBC) hold.

The natural scaling limit for this class of processes is the diffusive one, where the rates have to be multiplied by \(N^2\) to get a non trivial scaling limit. So, instead of \((24)\), we will consider in the macroscopic theory the speeded up martingale \(M_t(x, y) = \mathcal{J}_t(x, y) - N^2 \int_0^t ds \bar{j}_{\xi}^F(x, y)\).

**Example 1.** For example the instantaneous current across the edge \((x, y)\) for the KMP process is given by

\[
\int_{\nu(y)}^{\nu(x)} \frac{qdq}{\nu(x) + \nu(y)} = \frac{1}{2} (\xi(x) - \xi(y)).
\]
This computation shows that the KMP model is of gradient type, see definition (35), with $h(\xi) = -\frac{\xi(0)}{2}$. Also the KMPd is gradient with respect to the same function $h$.

**Example 2.** For the weakly asymmetric KMP in the case of a constant external field $F = E$ in the direction from $x$ to $y$ the discrete field $\mathcal{F}(x, y)$ is given by $E/N$ on $\mathbb{T}_e^n$ and

$$\Gamma_{x, y}^{E}(q) = \frac{1 + \frac{k}{N}q}{\xi(x) + \xi(y)} + o(N)$$

Then the instantaneous current is

$$j_{\xi}^{E}(x, y) = \int_{-\xi(y)}^{\xi(x)} \Gamma_{x, y}^{E}(q) dq = \frac{2N}{E(\xi(x) + \xi(y))} \left[ e^{\frac{E}{N}\xi(x)} + e^{\frac{E}{N}\xi(y)} \frac{\xi(y) - \xi(x)}{E} \right] = \frac{1}{2}(\xi(x) - \xi(0)) + \frac{E}{N} \left[ 2(\xi(x))^2 + \xi(y)^2 - \xi(x)\xi(y) \right] + o(N). \quad (28)$$

The hydrodynamic behavior of the model under the action of an external field in the weakly asymmetric regime, i.e. when the external field $E$ is of order $1/N$, is determined by the first two orders in the expansion (28). In particular any perturbed KMP model having the same expansion as in (28) will have the same hydrodynamics.

While for the KMPd model we get

$$j_{\xi}^{E}(x, y) = \frac{1}{2}(\xi(x) - \xi(y)) + \frac{E}{N} \left[ 2(\xi(x))^2 + (\xi(y))^2 - \xi(x)\xi(y) + 3\xi(x) + 3\xi(y) \right] + o(N). \quad (29)$$

## 5 Discrete Hodge decomposition in interacting particle systems

In the first section we defined the graph $(V_N, \partial N)$. Now we enter into the detail of the discrete mathematics we need to study the geometrical structures of the current. We consider the case when the graph $(V_N, \partial N)$ is on $\mathbb{T}_e^2$.

A sequence $(z_0, z_1, \ldots, z_k)$ of elements of $V_N$ such that $(z_i, z_{i+1}) \in E_N$, $i = 0, \ldots, k-1$, is called oriented path, or simply a path. A cycle $C = (z_0, z_1, \ldots, z_k)$ is a path with distinct vertices except $z_0 = z_k$ and it is defined as an equivalence class modulo cyclic permutations. If $C$ is a cycle and there exists an $i$ such that $(x, y) = (z_i, z_{i+1})$ we write $(x, y) \in C$. Likewise if there exists an $i$ such that $x = z_i$ we write $x \in C$. A discrete vector field $\varphi$ on $(V_N, E_N)$ is a map $\varphi : E_N \to \mathbb{R}$ such that $\varphi(x, y) = -\varphi(y, x)$. A discrete vector field is of gradient type if there exists a function $h : V_N \to \mathbb{R}$ such that $\varphi(x, y) = [\nabla h](x, y) := h(y) - h(x)$. The divergence of a discrete vector field $\varphi$ at $x \in V_N$ is defined by

$$\nabla \cdot \varphi(x) := \sum_{y : (x, y) \in \partial N} \varphi(x, y). \quad (30)$$
We call $\Lambda^1$ the $|\delta_N|$-dimensional vector space of discrete vector fields. We endow $\Lambda^1$ with the scalar product

$$
\langle \nu, \psi \rangle := \frac{1}{2} \sum_{(x,y) \in E_N} \nu(x,y) \psi(x,y), \quad \nu, \psi \in \Lambda^1.
$$

We recall briefly the Hodge decomposition for discrete vector fields. We call $\Lambda^0$ the collection of real valued function defined on the set of vertices $\Lambda^0 := \{ g : V_N \rightarrow \mathbb{R} \}$. Finally we call $\Lambda^2$ the vector space of 2-forms defined on the faces of the lattice $\mathbb{Z}_2^N$. Let us define this precisely. An oriented face is for example an elementary cycle in the graph of the type $(x, x + e(1), x + e(1) + e(2), x + e(2), x)$. In this case we have an *anticlockwise oriented face*. This corresponds geometrically to a square having vertices $x, x + e(1), x + e(1) + e(2), x + e(3)$ plus an orientation in the anticlockwise sense. The same elementary face can be oriented *clockwise* and this corresponds to the elementary cycle $(x, x + e(2), x + e(1) + e(2), x + e(1), x)$. If $f$ is a given oriented face we denote by $-f$ the oriented face corresponding to the same geometric square but having opposite orientation. A 2-form is a map $\psi$ from the set of oriented faces $F_N$ to $\mathbb{R}$ that is antisymmetric with respect to the change of orientation, i.e. such that $\psi(-f) = -\psi(f)$. The boundary $\delta \psi$ of $\psi$ is a discrete vector field defined by

$$
\delta \psi(e) := \sum_{f, e \in f} \psi(f).
$$

Since a face is a cycle the meaning of $e \in f$ has been just discussed above. Note that (32) is a discrete orthogonal gradient, the orthogonal gradient $\nabla^\perp f$ of a smooth function $f$ is defined as $(-\partial_x f, \partial_y f)$. In higher dimension this a discrete curl.

By construction $\nabla \cdot \delta \psi = 0$ for any $\psi$. The 2-dimensional discrete Hodge decomposition is written as the direct sum

$$
\Lambda^1 = \nabla \Lambda^0 \oplus \delta \Lambda^2 \oplus \Lambda^1_H,
$$

where the orthogonality is with respect to the scalar product (31). The discrete vector fields on $\nabla \Lambda^0$ are the gradient ones. The dimension of $\nabla \Lambda^0$ is $N^2 - 1$. The vector subspace $\delta \Lambda^2$ contains all the discrete vector fields that can be obtained by (32) from a given 2-form $\psi$. The dimension of $\delta \Lambda^2$ is $N^2 - 1$. Elements of $\delta \Lambda^2$ are called *circulations*. The dimension of $\Lambda^1_H$ is simply 2. Discrete vector fields in $\Lambda^1_H$ are called *harmonic*. A basis in $\Lambda^1_H$ is given by the vector fields $\phi^{(1)}$ and $\phi^{(2)}$ defined by

$$
\phi^{(i)}(x, x + e^{(j)}) := \delta_{ij}, \quad i, j = 1, 2.
$$

Given a vector field $\phi \in \Lambda^1$, we write $\phi = \phi^V + \phi^S + \phi^H$ to denote the unique splitting in the three orthogonal components. This decomposition can be computed as follows. The harmonic part is determined writing $\phi^H = c_1 \phi^{(1)} + c_2 \phi^{(2)}$, with the coefficients $c_i$ determined by $c_i = \frac{1}{N^2} \sum_{x \in V_N} \phi^{(i)}(x, x + e^{(i)})$. To determine the gradient component $\phi^V$ we need to determine a function $h$ for which $\phi^V(x, y) = \frac{1}{2} \sum_{(x,y) \in E_N} \phi^{(i)}(x, y)$. This corresponds geometrically to a square having vertices $x, x + e^{(1)}, x + e^{(1)} + e^{(2)}, x + e^{(2)}$ plus an orientation in the clockwise sense.
\[ \nabla h(x, y) = h(y) - h(x) \]. This is done by taking the divergence on both side of \( \varphi = \varphi^V + \varphi^\delta + \varphi^H \) and obtaining the \( h \) solving the discrete Poisson equation \( \nabla \cdot \nabla h = \nabla \cdot \varphi \). The remaining component \( \varphi^\delta \) is computed just by difference \( \varphi^\delta = \varphi - \varphi^V - \varphi^H \).

We refer to [4, 10] for a version of discrete calculus with cubic cells and to [8] for a version of discrete calculus with simplexes.

Given an oriented edge \( e \) or an oriented face \( f \) we denote respectively by \( e, \bar{f} \) the corresponding un-oriented edge and face. Note that both \( f \) and \( -f \) are associated with the same un-oriented face \( \bar{f} \). Given an oriented edge \( e \in E_N \) of the lattice there is only one anticlockwise oriented face to which \( e \) belongs that we call it \( f^- (e) \). There is also an unique anticlockwise face, that we call \( f^+ (e) \), such that \( e \in -f^- (e) \) (see Figure 1).

It is useful to define \( \tau_f \) for an un-oriented face \( f \). If \( \bar{f} = \{ x, x + e^{(1)} , x + e^{(2)} , x + e^{(1)} + e^{(2)} \} \) then we define \( \tau_{\bar{f}} := \tau_e \). For \( e = \{ x, x + e^{(1)} \} \) we define \( \tau_e := \tau_x \). We use also the notation \( f^- \) for an anticlockwise face and \( f^+ \) for a clockwise one.

![Fig. 1](image-url) On discrete two dimensional torus, given \((x, y) = e\) we draw the faces \( f^- (e) \) and \( f^+ (e) \).

### 5.1 Functional discrete Hodge decomposition and lattice gases

A relevant notion in the derivation of the hydrodynamic behavior for diffusive particle systems is the definition of gradient particle system. A particle system is called of gradient type if there exists a local function \( h \) such that

\[ j_\eta (x, y) = \tau_y h(\eta) - \tau_x h(\eta) \] for all \((\eta, (x, y)) \in (\Sigma_N, E_N)\).
The relevance of this notion is on the fact that the proof of the hydrodynamic limit for gradient systems is extremely simplified. Moreover for gradient and reversible models it is possible to obtain explicit expressions of the transport coefficients.

Here we show that (35) is a subcase of general geometrical structures for the instantaneous current. In next sections, we will try to understand the consequences of these structures in the hydrodynamic limits and how it could be useful in understanding the hydrodynamics of non-gradient models. We present a functional Hodge decomposition of translational covariant discrete vector fields. This means vector fields $j_\eta(x,y)$ depending on the configuration $\eta \in \Sigma_N$ and satisfying (12). Vector fields of the form (35) play the role of the gradient vector fields. Circulations will also be suitably defined in the context of particle systems.

5.2 The one dimensional case

On the one dimensional torus $V_N$, we have the following theorem.

**Theorem 1.** Let $j_\eta$ be a translational covariant discrete vector field. Then there exists a function $h(\eta)$ and a translational invariant function $C(\eta)$ such that

$$j_\eta(x,x+1) = \tau_{x+1}h(\eta) - \tau_xh(\eta) + C(\eta).$$

(36)

The function $C$ is uniquely identified and coincides with

$$C(\eta) = \frac{1}{N} \sum_{x \in V_N} j_\eta(x,x+1).$$

(37)

The function $h$ is uniquely identified up to an arbitrary additive translational invariant function and coincides with

$$h(\eta) = \sum_{x=1}^{N-1} \frac{x}{N} j_\eta(x,x+1).$$

(38)

**Proof.** The basic idea of the theorem is the usual strategy to construct the potential of a gradient discrete vector field plus a subtle use of the translational covariance of the model. For the details of the proof see [5].

Observe that a one dimensional system of particles is of gradient type (with a possibly not local $h$) if and only if $C(\eta) = 0$. This corresponds to say that for any fixed configuration $\eta$ then $j_\eta(x,y)$ is a gradient vector field. This was already observed in [2][15]. Now we compute the decomposition (36) in some examples. Later we will discuss how it can be related to the hydrodynamics of non-gradient systems.

**Example 3.** On the one-dimensional discrete torus, the symmetric exclusion process with rates $c_{x,x+1}(\eta) = \eta(x)(1-\eta(x+1))[1 + \alpha \eta(x-1)]$ and $c_{x+1,x}(\eta) = \eta(x+1)(1-\eta(x))[1 + \alpha \eta(x-1)]$, with the constant $\alpha \in (0,1)$, is reversible with
respect to the Bernoulli measure. This is a non-gradient systems, expected to have a diffusive scaling limits, where the instantaneous current is given by

$$j_\eta(x, x+1) = (\eta(x) - \eta(x+1)) + (\eta(x) - \eta(x+1))\alpha \eta(x-1).$$  \hspace{1cm} (39)$$

Therefore its functional Hodge decomposition \ref{eq36} is

$$h(\eta) = -\eta(0) + \sum_{x=1}^{N-1} \frac{1}{N}(\eta(x) - \eta(x+1))\alpha \eta(x-1).$$  \hspace{1cm} (40)$$

Example 4 (The 2-SEP). The model we are considering is the 2-SEP, see its definition in subsection \ref{sec3.1}. We denote by $D^+_\eta(x, x+1)$ the local functions associated with the presence on the bond $(x, x+1)$ of what we call respectively a positive or negative discrepancy. More precisely $D^+_\eta(x, x+1) = 1$ if $\eta(x) = 2$ and $\eta(x+1) = 1$ and zero otherwise. We have instead $D^-_\eta(x, x+1) = 1$ if $\eta(x+1) = 2$ and $\eta(x) = 1$ and zero otherwise. We define also $D_\eta := D^+_\eta - D^-_\eta$. The instantaneous current across the edge $(x, x+1)$ associated with the configuration $\eta$ is

$$j_\eta(x, x+1) := \chi^+(\eta(x)) - \chi^+(\eta(x+1)) + D_\eta(x, x+1).$$  \hspace{1cm} (41)$$

For this specific model formulas \ref{eq37} and \ref{eq38} become

$$h(\eta) = -\chi^+(\eta(0)) + \sum_{x=1}^{N-1} \frac{1}{N} D_\eta(x, x+1),$$  \hspace{1cm} (42)$$

Remark 2. Both formulas \ref{eq39} and \ref{eq41} are written in the form $j_\eta(x, y) = j^+_{\eta}(x, y) + j^-_{\eta}(x, y)$, namely they are given by the sum of a local gradient current $j^+_{\eta}(x, y) = \tau_j h(\eta) - \tau_j h(\eta)$ and a single net contribution $j^-_{\eta}(x, y)$ to the harmonic function $C(\eta)$. We will refer to $j^-_{\eta}(x, y)$ as single harmonic contribution on $(x, y)$. In section \ref{sec9} we will discuss that we think from this way of rewriting the current it has to start both the study of an explicit hydrodynamics for the case of non-gradient diffusive model and the computation of Green-Kubo’s formula.

Our decomposition is motivated by the study of diffusive models where the current can not be written in the gradient form \ref{eq35}, but it can be computed also in not diffusive models when the hypothesis of theorem \ref{th1} hold. For example for the asymmetric simple exclusion process it is as follows.

Example 5 (ASEP). The asymmetric simple exclusion process is characterized by the rates $c_{x,x+1}(\eta) = p\eta(x)(1 - \eta(x+1))$ and $c_{x,x-1}(\eta) = q\eta(x)(1 - \eta(x-1))$. Given a configuration of particles $\eta \in \Sigma$, we call $\mathcal{C}(\eta)$ the collection of clusters of particles that is induced on $V_N$. A cluster $c \in \mathcal{C}(\eta)$ is a subgraph of $(V_N, \mathcal{E}_N)$. Two sites $x, y \in V_N$ belong to the same cluster $c$ if $\eta(x) = \eta(y) = 1$ and there exists an un-oriented path $(z_0, z_1, \ldots, z_k)$ such that $\eta(z_i) = 1$ and $(z_i, z_{i+1}) \in \mathcal{E}_N$. Given a cluster $c \in \mathcal{C}$ we call $\partial^c c$ and $\partial^c c \in V_N$ respectively the first element on the left of the leftmost site of $c$ and the rightmost one. The decomposition \ref{eq36} holds with
\[ h(\eta) = \frac{1}{N} \sum_{c \in \mathcal{C}(\eta)} \left[ p\partial^c e - q\partial^c c \right], \quad C(\eta) = \frac{(p - q)|\mathcal{C}(\eta)|}{N}. \]  

where \(|\mathcal{C}(\eta)|\) denotes the number of clusters.

### 5.3 The two dimensional case

On the two dimensional torus \(V_N\) the decomposition is as follows.

**Theorem 2.** Let \(j_\eta\) be a covariant discrete vector field. Then there exist four functions \(h, g, C^{(1)}, C^{(2)}\) on configurations of particles such that for an edge of the type \(e = (x, x \pm e^{(i)})\) we have

\[ j_\eta(e) = \left[ \tau_{e^+}^{(i)} h(\eta) - \tau_{e^-}^{(i)} h(\eta) \right] + \left[ \tau_{e^+}^{(i)} g(\eta) - \tau_{e^-}^{(i)} g(\eta) \right] \pm C^{(i)}(\eta). \]  

The functions \(C^{(i)} = \frac{1}{N} \sum_{x \in V_N} j_\eta(x, x + e^{(i)})\) are translational invariant and uniquely identified. The functions \(h\) and \(g\) are uniquely identified up to additive arbitrary translational invariant functions.

**Proof.** see \([5]\).

We remark that the proof in \([5]\) is constructive, that is the function \(h(\eta), g(\eta)\) and \(C^{(i)}(\eta)\) have explicit expressions. In analogy to gradient systems we can say a particle system is of *circulation type* when there exist a local function \(g\) such that

\[ j_\eta(e) = \tau_{e^+}^{(i)} g(\eta) - \tau_{e^-}^{(i)} g(\eta), \]  

for all edges \(e \in E_N\) and \(\eta \in \Sigma_N\). We will see that for these systems the hydrodynamics can be treated with the same method of gradient systems. In particular later in section \([8]\) we study the scaling limits of systems where gradient and circulation dynamics are superposed. Now we introduce some examples of this kind.

**Example 6 (A non gradient lattice gas with local decomposition).** We construct a model of particles satisfying an exclusion rule, with jumps only through nearest neighbour sites and having a non trivial decomposition of the instantaneous current \([44]\) with \(C^{(i)} = 0\) and \(h\) and \(g\) local functions. The functions \(h\) and \(g\) have to be chosen suitably in such a way that the instantaneous current is always zero inside cluster of particles and empty clusters and has to be always such that \(j_\eta(x, y) \geq 0\) when \(\eta(x) = 1\) and \(\eta(y) = 0\). A possible choice is the following perturbation of the SEP. We fix \(h(\eta) = -\eta(0)\) and \(g(\eta)\) with \(D(g) = \{0, e^{(1)}, e^{(2)}, e^{(1)} + e^{(2)}\}\) (we denote by 0 the vertex \((0, 0)\)) defined as follows. We have \(g(\eta) = \alpha\) if \(\eta(0) = \eta(e^{(1)} + e^{(2)}) = 1\) and \(\eta(e^{(1)}) = \eta(e^{(2)}) = 0\). We have also \(g(\eta) = \beta\) if \(\eta(0) = \eta(e^{(1)} + e^{(2)}) = 0\) and \(\eta(e^{(1)}) = \eta(e^{(2)}) = 1\). The real numbers \(\alpha, \beta\) are such that \(|\alpha| + |\beta| < 1\). For all the remaining configurations we have \(g(\eta) = 0\). Since \(\Sigma = \{0, 1\}\) the rates of jump are uniquely determined by \(c_{x,y}(\eta) = \{j_\eta(x, y)\}_+\).
Example 7 (A perturbed zero range dynamics). A face $f = \{0, e^{(1)}, e^{(2)}, e^{(1)} + e^{(2)}\}$ is occupied in the configuration $\eta \in \mathbb{N}^{1}\mathbb{V}$ if $\eta(x) \neq 0$ for some $x \in f$. Consider two non-negative functions $h^\pm$ that are identically zero when the face $f$ is not occupied. Given a positive function $\overline{h} : \mathbb{N} \to \mathbb{R}^+$, we define the rates of jump as

$$c_{e^-, e^+}(\eta) = \overline{h}(\eta(e^-)) + \tau_{f^+(e)}w^+ + \tau_{f^-(e)}w^-.$$  

(46)

This corresponds to a perturbation of a zero range dynamics such that one particle jumps from one site with $k$ particles with a rate $\overline{h}(k)$. The perturbation increases the rates of jump if the jump is on the edge of a full face. The gain depends on the orientation and the effect of different faces is additive. For such a model the instantaneous current has a local decomposition (44) with $\overline{h}(\eta) = -\overline{h}(\eta(0))$ and $g(\eta) = w^+(\eta) - w^-(\eta)$.

The decomposition can be extended to higher dimensions. For the three dimensional case we refer to [3].

6 Interacting particle systems with vorticity

The models presented in examples [6] and [7] are superposition of a gradient system and a circulation one, see definition (45). This kind of models are not gradient along the classical definition. Here we want to study them from the microscopic point of view and giving some physical motivation why we talk about them as *interacting particle systems with vorticity,* this will become more clear at the end of section [7]. A better discussion with graphical examples will appear in [7].

Let us consider the instantaneous current (5) with a decomposition (44) as

$$j_{\eta}(x, y) = [\tau_{h}(\eta) - \tau_{e}(\eta)] + [\tau_{f^+(x, y)}g(\eta) - \tau_{f^-(x, y)}g(\eta)] = j_{\eta}^{h}(x, y) + j_{\eta}^{g}(x, y),$$  

(47)

with $h$ and $g$ local functions. We are defining $j_{\eta}^{h}(x, y) := \tau_{h}(\eta) - \tau_{e}(\eta)$ and $j_{\eta}^{g}(x, y) := \tau_{f^+(x, y)}g(\eta) - \tau_{f^-(x, y)}g(\eta)$. For example, taking an exclusion process with rates

$$c_{x, y}(\eta) = \eta(x)(1 - \eta(y)) + \eta(x)[\tau_{f^+(x, y)}g(\eta) - \tau_{f^-(x, y)}g(\eta)],$$  

(48)

we have $j_{\eta}(x, y)\text{ as in (47) with } h(\eta) = -\eta(0),$ note that example [1] is of this form.

Models with $j_{\eta}(x, y)\text{ as in (47)}$ can be thought as a generalization of the gradient case $j_{\eta}(x, y) = [\tau_{h}(\eta) - \tau_{e}(\eta)]$, indeed the current is a gradient part plus an orthogonal gradient part (discrete bidimensional curl). Because of the presence of this discrete curl we use the terminology of ”exclusion process with vorticity”.

When the rates satisfies (47), we will see that the hydrodynamics for the empirical measure (50) works exactly as if only the gradient part was present because

$$\nabla \cdot j_{\eta}^{h}(x) = 0, \ \forall x \in V_N,$$  

(49)
that is the part of the dynamics related to the current \(j^N_\eta(x,y)\) does not give any macroscopic effect to the hydrodynamics of the particles density because its contribution to the microscopic conservation law \(\mathcal{L}\) is already zero. To observe macroscopically the effect of the discrete curl we have to consider the scaling limits of the current flow \(J(x,y)\) of formula \(\mathcal{L}\). In section \(\mathcal{L}\) we derive the macroscopic current \(J(\rho)\) that will appear in the hydrodynamics \(\partial\rho = \nabla \cdot (-J(\rho))\). Another physical phenomena of this kind of dynamics \(\mathcal{L}\) is that they are diffusive even if in general they are not reversible on the torus \(\mathbb{T}^n_N\), namely this means that at the stationary state there is a non-zero macroscopic current \(\mathcal{L}\). For an explicit example see \(\mathcal{L}\).

### 7 Scaling limits and transport coefficients of diffusive models

To derive the hydrodynamics of diffusive systems the rates are multiplied by a factor \(N^2\) (diffusive time scale) and the space scale \(\varepsilon = 1/N\) is considered. The particles jump on the discrete torus \(\mathbb{T}^n_\varepsilon := \varepsilon \mathbb{Z}/\mathbb{Z}\) with mesh of size \(\varepsilon\). When \(N\) goes to infinity \(\mathbb{T}^n_\varepsilon\) approximates the continuous torus \(\mathbb{T}^n = [0,1)^n\). A very general class of diffusive systems are models that are reversible with respect to a Gibbs measure when no boundary conditions are imposed. Reversibility with respect to a measure \(\mu_N\) means \(\langle f, \mathcal{L}_N g \rangle_{\mu_N} = \langle \mathcal{L}_N f, g \rangle_{\mu_N}\) for all functions \(f, g\) while stationarity means \(\langle \mathcal{L}_N f \rangle_{\mu_N} = 0\). \(\langle \cdot \rangle\) is the expectation on \(\Sigma_N\) respect to \(\mu_N\) and \(\langle \cdot, \cdot \rangle_{\mu_N}\) is the scalar product respect to \(\mu_N\). We assume \(\mu_N\) to be a grand-canonical measure parametrized by the density \(\rho\), i.e. \(\mathbb{E}_{\mu_N}(\eta(x)) = \rho\). For this reason instead of \(\mu_N\) we are going to use the notation \(\mu_N^\rho\).

The macroscopic evolution of the mass is described by the empirical measure. This is a positive measure on the continuous torus \(\mathbb{T}^n\) associated to any fixed microscopic configuration \(\eta\), defined as a convex combination of delta measures

\[
\pi_N(\eta):= \frac{1}{N} \sum_{x \in V_N} \eta(x) \delta_x .
\]

It represents a mass density or an energy density along the interpretation of the model. Integrating a continuous function \(f : \mathbb{T}^n \to \mathbb{R}\) with respect to \(\pi_N(\eta)\) we get \(\int_{\mathbb{T}^n} f d\pi_N(\eta) = \frac{1}{N} \sum_{x \in V_N} f(x) \eta(x)\). In the hydrodynamic scaling limit the empirical measure becomes deterministic and absolutely continuous for suitable initial conditions \(\xi_0\) associated to a given density profile \(\gamma(x)dx\), in the sense that in probability

\[
\lim_{N \to +\infty} \int_{\mathbb{T}^n} f d\pi_N(\xi_0) = \int_{\mathbb{T}^n} f(x) \gamma(x)dx.
\]

Let \(P^\gamma_N\) be the distribution of the Markov chain of the energy-mass/particle interacting model with initial condition associated to \(\gamma\) as in \(\mathcal{L}\). On \(D([0,T];\mathcal{M}(\mathbb{T}^n))\) the space of trajectories from \([0,T]\) to the space of positive measure \(\mathcal{M}(\mathbb{T}^n)\), \(\mathbb{P}_N^\gamma := P^\gamma_N \circ \pi_N^{-1}\) is the measure induced by the empirical measure. We have that \(\pi_N(\eta)\) is associated to the density profile \(\rho(x,t)dx\) where \(\rho\) is the weak solution.
to a diffusive equation with initial condition $\gamma$, i.e. $P_N^\epsilon \xrightarrow{\epsilon \to 0} \delta_\rho$ where and $\delta_\rho$ is the distribution concentrated on the unique weak solution of a Cauchy problem

$$
\begin{aligned}
\partial_t \rho &= \nabla \cdot (D(\rho) \nabla \rho) \\
\rho(x,0) &= \gamma(x).
\end{aligned}
$$

(52)

This is a space-time law of large numbers, where $D(\sigma)$ is a positive symmetric matrix called diffusion matrix.

### 7.1 Qualitative derivation of hydrodynamics

In this subsection we illustrate the general structure of the proof of the hydrodynamic limit for reversible gradient models on the torus $\mathbb{T}^n_N$. We use the notion $\xi$ of section 4 of energy-mass models because for them we gave some example of weakly asymmetric model and we want to emphasize that the KMP model is gradient. But the whole scheme apply to particle models in the same way.

The starting point for the hydrodynamic description is the continuity equation

$$
\xi_t(x) - \xi_0(x) = -\nabla \cdot \mathcal{J}_t(x),
$$

(53)

where $\mathcal{J}_t$ has been defined in subsection 4.3 and $\nabla \cdot$ denotes the discrete divergence defined in (30). Using (24) we can rewrite (53) as (26) with $F = 0$. Multiplying (26) by a test function $\psi$, dividing by $N$ and summing over $x$ we obtain

$$
\int_{T^N} \psi \, d\pi_N(\xi_t) - \int_{T^N} \psi \, d\pi_N(\xi_0) = -N \int_0^t \sum_{x \in V_N} \nabla \cdot j_{\xi_t}(x) \, \psi(x) \, ds + o(1).
$$

(54)

The infinitesimal term $o(1)$ comes from the martingale term. The idea is that the martingales $M_t(x,y)$ in (7) describe some microscopic fluctuations whose additive macroscopic contributions vanishes as $N \to \infty$ as they are mean zero martingales and are almost independent for different bonds. This contribution can be shown to be negligible (in probability) in the limit of large $N$ with the methods of [11, 14]. Using the gradient condition $j_{\xi}(x,y) = \tau_x h(\xi) - \tau_y h(\xi)$, for example for the KMP (14) and KMPd (16) we have $h(\xi) = \frac{\xi(0)}{2}$, and performing a double discrete integration by part, up to the infinitesimal term, one has that the right hand side of (53) is

$$
\frac{1}{N} \sum_{x \in V_N} \int_0^t \tau_x h(\xi_s) \left[ N^2 \left( \psi(x + \frac{\epsilon}{N}) + \psi(x - \frac{\epsilon}{N}) - 2\psi(x) \right) \right] ds.
$$

Considering a $C^2$ test function $\psi$, the term inside squared parenthesis coincides with $A\psi(x)$ up to an uniformly infinitesimal term.

At this point the main issue in proving hydrodynamic behavior is to prove the validity of a local equilibrium property. Let us define

$$
A(\rho) = \mathbb{E}_{\mu_\rho} (h(\xi)),
$$

(55)
where $\mu_N^\rho$ is the invariant measure characterized by a density profile $\rho$, that is $\mathbb{E}_{\mu_N^\rho}(\xi(x)) = \rho$. The local equilibrium property is explicitly stated through a replacement lemma that shows that (in probability)

$$
\frac{1}{N} \sum_{x \in V_N} \int_0^T \tau_x h(\xi_x) \Delta \psi(x) \, ds \simeq \frac{1}{N} \sum_{x \in V_N} \int_0^T A \left( \frac{\int_{B_x} d\pi_N(\xi_x)}{|B_x|} \right) \Delta \psi(x) \, ds \quad (56)
$$

where $B_x$ is a microscopically large but macroscopically small volume around the point $x \in V_N$. For a precise formulation of (56) see lemma 1.10 and corollary 1.3 respectively in chapter 5 and in chapter 6 of [14] or chapter 2 in [11]. This allows to write (up to infinitesimal corrections) equation (54) in terms only of the empirical measure. Substituting the r.h.s. of (56) in the place of the r.h.s. of (54), we obtain that in the limit of large $N$ the empirical measure $\pi_N(\eta_t)$ converges in weak sense to $\rho(x,t)dx$ satisfying for any $C^2$ test function $\psi$

$$
\int_{\mathbb{T}^d} \psi(x) \rho(x,t) \, dx - \int_{\mathbb{T}^d} \psi(x) \rho(x,0) \, dx = \int_0^t \int_{\mathbb{T}^d} A(\rho(x,s)) \Delta \psi(x) \, dx. \quad (57)
$$

Equation (57) is a weak form of (52) with diagonal diffusion matrix $D(\rho)$ with each term in the diagonal equal to $D(\rho) = \frac{dk(\rho)}{d\rho}$. We are calling $D(\rho)$ both the number and the diagonal matrix $D(\rho) \mathbb{I}$. For $h(\xi) = \frac{\xi(0)}{2}$ it is $A(\rho) = \frac{\rho}{2}$. To have an unitary diffusion matrix we multiply all the rates of transition by a factor of 2 and correspondingly the diffusion matrix is the identity matrix. Equation (52) can be written in the form

$$
\partial_t \rho + \nabla \cdot (J(\rho)) = 0, \text{ with } J(\rho) = -D(\rho) \nabla \rho, \quad (58)
$$

where the macroscopic current $J(\rho)$ associated to $\rho$ satisfies the Fick’s law. The hydrodynamics for weakly asymmetric diffusive models of subsection (4.2) is

$$
\partial_t \rho = \nabla \cdot (-J_E(\rho)) \text{ with } J_E(\rho) := D(\rho) \nabla \rho - \sigma(\rho) E. \quad (59)
$$

The positive definite matrix $\sigma$ is called the mobility. For the weakly asymmetric versions of the KMP and the KMPd, in subsection (4.2) it is respectively $\sigma(\rho) = 2E_{\mu_N^\rho}[g(\eta)] = \rho^2$ and $\rho + \rho^2$, where respectively $g(\xi) = \frac{1}{2} (\xi(0)^2 + \xi(1/N)^2 - \xi(0)\xi(1/N))$ and $g(\xi) = 12 (2\xi(x)^2 + 2\xi(y)^2 - 2\xi(x)\xi(y) + 3\xi(x) + 3\xi(y))$. For a discussion on the computations of these kind of expectations see [9].

The hydrodynamics was derived with periodic boundary conditions but in the bulk it is still the same for a boundary driven version of the system, see [9].
8 Scaling limit of an exclusion process with vorticity

In this section we want to show how to compute the scaling limit of the macroscopic current \( J(\rho) \) for diffusive models with vorticity of section 6 namely having the instantaneous current with an expression like (47). Here for the purpose of the paper the treatment will be qualitative. It is the first time that the hydrodynamics of this kind of models is discussed. A complete rigorous treatment of the problem is now being developed in the work in progress[7], where a generalized picture of the Fick’s law is under construction. Here we will discuss its main ideas. We consider a discrete torus of mesh \( \varepsilon = 1/N \) but specifically in dimension 2, i.e. \( V_N = \mathbb{T}^2_\varepsilon \).

If the current has an Hodge decomposition (44) only the gradient part contributes to the hydrodynamics (53), indeed \( \nabla \cdot j_\eta \equiv \nabla \cdot j_\eta^g \) since \( \nabla \cdot j_\eta^H = 0 \) with \( j_\eta^H(x,y) = \mathcal{C}^1(\eta) \varphi^1(x,y) + \mathcal{C}^2(\eta) \varphi^2(x,y) \). So if the gradient part of the current \( j_\eta^g \) is diffusive with respect to a local gradient function \( h(\cdot) \), the hydrodynamics of \( \pi_N(\eta) \) works exactly as if we considered a model with \( j_\eta(x,y) = j_\eta^g(x,y) \) along the scheme in section 7.

Now we want to study the scaling limits of the current \( J(\rho) \) appearing in (58), as model of reference for what we are going to present, the reader should keep in mind the exclusion process of example 6 but with \( \alpha = \beta \). More precisely the model has the rates (48) with the local function \( g(\eta) \) defined as

\[
g(\eta) := \begin{cases} 
\alpha & \text{if } \eta(0) = \eta(\frac{1}{N} + \frac{2}{N}) = 1 \text{ and } \eta(\frac{1}{N}) = \eta(\frac{2}{N}) = 0 , \\
\alpha & \text{if } \eta(\frac{1}{N}) = \eta(\frac{2}{N}) = 1 \text{ and } \eta(0) = \eta(\frac{1}{N} + \frac{2}{N}) = 0 , \\
0 & \text{otherwise} ,
\end{cases} \tag{60}
\]

where \( \alpha \) is a real parameter such that \( |\alpha| < 1 \). The informal and intuitive description of the dynamics associated to the rates (48) is the following. Particles perform a simple exclusion process, but the faces containing exactly 2 particles located at sites which are not nearest neighbors let the particles rotate anticlockwise when \( \alpha > 0 \) and clockwise when \( \alpha < 0 \) with a rate equal to \( |\alpha| \). For this choice of the parameters, the model of example 5 can be proven to be a non-reversible stationary dynamics with respect to Bernoulli measures of density parameter \( \rho \). In this section, the language will be general for a model that is invariant with respect to a measure \( \mu_N^\rho \) parametrized by a density \( \rho \), having the decomposition (47) and hydrodynamics for the empirical measure of the form (58), while the results will be made explicit for the toy model (60).

The scaling limits for the current \( J(\rho) \) it is obtained from the empirical current measure \( J_N \) in the space of the vector signed measure \( \mathcal{M}(\mathbb{T}^2, \mathbb{R}^2) \) defined as

\[
\int_{\mathbb{T}^2} H \cdot dJ_N := \frac{1}{N^2} \sum_{\{x,y\} \in \delta_N} J_t(x,y) \mathbb{H}(x,y) \text{ where } \mathbb{H}(x,y) = \int_x^y H(z) \cdot dz . \tag{61}
\]
The family \((J_N(t))_{t \in [0,T]}\) belongs to the space \(D([0,T], \mathcal{M}(\mathbb{T}^2, \mathbb{R}^2))\) of trajectories from \([0,T]\) to \(\mathcal{M}(\mathbb{T}^2, \mathbb{R}^2)\). Calling \(\mathbb{P}_{\beta_N} := P^T_N \circ \mathcal{L}^{-1}_N\) the measure induced by empirical current measure on \(D([0,T], \mathcal{M}(\mathbb{T}^2, \mathbb{R}^2))\), we have that \(J_N(t)\) is associated to a vector signed measure \(J(\rho)\) on weak sense, that is in probability for any \(C^1\) vector field on \(\mathbb{T}^2\) we have

\[
\lim_{N \to +\infty} \int \nabla \cdot d\mathbb{P}_N(t) = \int \nabla \cdot \rho \, dt \, J(\rho(x)) \cdot \mathcal{H}(x),
\]

\[
J(\rho) = -D(\rho \mathcal{H}^2) \mathcal{H} \rho - D^2(\rho \mathcal{H}^2) \mathcal{H} \rho
\]

where \(D(\rho)\) and \(D^2(\rho)\) are two real coefficients depending on \(\rho\). \(\rho(0)\) is the solution of the Cauchy problem \((62)\) and \(J(\rho(0))\) is equal 0 by definition. This means that \(\mathbb{P}_{\beta_N} \overset{d}{\to} \delta(\rho)\) where \(\delta(\rho)\) is the distribution concentrated on the measure \(J(\rho)\) on weak sense, that we have just described. For the model \((60)\) we will show that \(D(\rho) = 1\) and \(D^2(\rho) = \frac{1}{2\alpha} (2\alpha(\rho \mathcal{H}^2))^2\).

The derivation of the hydrodynamics starts from the model \((60)\) its limit converges (in probability) to

\[
M(t) = \frac{1}{N^d} \sum_{\{x, y\} \in \delta_N} J(x,y) \mathbb{H}(x,y) - N^{2-d} \int ds \sum_{\{x, y\} \in \delta_N} J_{\eta}(x,y) \mathbb{H}(x,y),
\]

where \(N^{2-d}\) is the diffusive scaling and the factor \(N\) it is a normalization. By the antisymmetry of the discrete vector fields there is no ambiguity in this definition. Therefore \(\frac{1}{N^d} \sum_{\{x, y\} \in \delta_N} J(x,y) \mathbb{H}(x,y) = N^{2-d} \int ds \sum_{\{x, y\} \in \delta_N} J_{\eta}(x,y) \mathbb{H}(x,y) + o(1)\), where \(o(1)\) is a negligible (in probability) martingale term for large \(N\), for which holds a discussion like that one about the martingale in \((54)\). From \((17)\)

\[
\sum_{\{x, y\} \in \delta_N} J_{\eta}(x,y) \mathbb{H}(x,y) = \int_0^t \left[ \sum_{\{x, y\} \in \delta_N} \tau_h(\eta) \mathbb{H}(x,y) + \sum_{\{x, y\} \in \delta_N} \tau_{\eta}(\eta) \sum_{\{x, y\} \in \delta_N} \mathbb{H}(x,y) \right],
\]

where \(N^2 \mathbb{H}(x,y) = \nabla \cdot \mathcal{H}(x,y) + o(1)\) and \(N^2 \sum_{\{x, y\} \in \delta_N} \mathbb{H}(x,y) = \nabla \cdot \mathcal{H}(z) + o(1)\).

In the above formula \(z\) is any point belonging to the face, while given a \(C^1\) vector field \(H = (H_1, H_2)\) we used the notation \(\nabla \cdot \mathcal{H}(z) := -\partial_1 H_1(z) + \partial_2 H_2(z)\).

When \(N\) is diverging, we assume the local equilibrium hypothesis with respect to the grand-canonical measure \(\mu^p_N\) to prove with a replacement lemma as discussed in section\([\ref{section:replacement}]\) this means that \((54)\) converges (in probability) to

\[
\int_0^t ds \int_\mathbb{T}^2 dx \left[ a(\rho(x,s)) \nabla \cdot H(x) + a^+(\rho(x,s)) \nabla \rho \cdot \nabla \mathcal{H}(x) \right],
\]

applying the replacement lemmas \(a(\rho) = \mathbb{E}_{\mu^p_N}[h(\eta)]\) and \(a^+(\rho) = \mathbb{E}_{\mu^p_N}[g(\eta)]\), for the model of reference \((60)\) its \(a(\rho) = \rho\) and \(a^+(\rho) = 2\alpha(\rho(1 - \rho))^2\). Formula \((65)\) is a weak form of \(\int_0^t ds \int_\mathbb{T}^2 dx J(\rho(x,s)) \mathcal{H} dx\) with

\[
J(\rho) = -\nabla a(\rho) - \nabla a^+(\rho) = -D(\rho) \nabla \rho - D^2(\rho) \nabla \rho,
\]
where \( D(\rho) = d(a(\rho))/d\rho \), \( D^\perp(\rho) = d(a^\perp(\rho))/d\rho \) and \( \nabla^\perp f := (-\partial_x f, \partial_y f) \). As we expected from the microscopic argument (49) we have \( \nabla \cdot (-D(\rho)\nabla \rho - D^\perp(\rho)\nabla^\perp \rho) = \nabla \cdot (-D(\rho)\nabla \rho) \), hence the hydrodynamics is left unchanged with respect to the usual gradient case. For the model (60) we obtain \( D(\rho) = 1 \) and \( D^\perp(\rho) = \frac{d}{d\rho}(2\alpha(\rho - \rho^2)^2) \) as anticipated above. Hence formula (66) can be rewritten in the form

\[
J(\rho) = -D(\rho) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nabla \rho.
\] (67)

From the computations presented here and other considerations in [7] we think that the Fick’s law (58) for particle models has to be replaced by the general picture

\[
J(\rho) = -\mathcal{D}(\rho) \nabla \rho,
\] (68)

where the diffusion matrix \( \mathcal{D}(\rho) \) is positive but not necessarily symmetric and in general with respect to (67) the terms on the same diagonal can have different coefficients.

Remark 3. An important question is to understand if there exist models with vorticity that are also reversible. At the present stage, we are not able neither to find reversible models of this kind neither to prove that this property is a genuine microscopic non equilibrium property, if this is case, it looks that typically they will be diffusive models with a non-zero average microscopic current at the stationary state.

9 Green-Kubo’s formula and perspectives in non-gradient particles systems

Scaling limits of non gradient particles systems can be proved to be diffusive with the methods developed in [16, 19] if the spectral gap of the generator satisfies suitable conditions [14], but even in one dimension when the instantaneous current is not gradient there are no explicit PDEs. For what we know, the only case where there is an explicit PDE is [20], where the author consider a spatial inhomogeneous simple symmetric exclusion process where particles jump with two different constant along an edge is even or odd. The model of Wick is translational covariant, see [36], with respect to translations on two sites instead of one. To look at Wick model into the context of this paper and in particular of this section, one has to generalize the decomposition (36) for translational covariant models on two sites, this is done considering a renormalized current on two sites and rewriting the decomposition for it. Then, taking a lattice with a even number of sites, the discrete hydrodynamics (54) can be written with respect to this current.

We start to explore if the decompositions (36) and (44) can tell something about this problem. Let us consider exclusion processes in one dimension on the torus with nearest neighbours interaction, reversible with respect to \( \mu^N_N \) and non gradient,
a case is example[5]. For these models the hydrodynamics is expected to be diffusive with the diffusion coefficient having the following variational expression

\[
D(\rho) = \frac{1}{2\chi(\rho)} \inf_{f} \mathbb{E}_{\mu_{N}}\left[c_{0,1}(\eta)\left((\eta(0) - \eta(0,1)) + \sum_{x \in V_{N}} \langle S_{0,1}(x, f) \rangle \right)^{2}\right],
\]

where \(\chi(\rho)\) is the mobility \(\mathbb{E}_{\mu_{N}}(\eta^{2}(0)) - \rho^{2}\), \(\langle S_{x,y} f \rangle = f(\eta^{x,y}) - f(\eta)\) and the inf is over all functions \(f : \Sigma_{N} \rightarrow \mathbb{R}\). This is discussed in chapter 2 of part 2 in [18] and has been proved for the 2-SEP in chapter 7 of [14]. The variational formula (69) is proved to be equal to the Green-Kubo’s formula for interacting particle systems

\[
D(\rho) = \frac{1}{2\chi(\rho)} \left[\mathbb{E}_{\mu_{N}}(c_{0,1}(\eta)) - 2 \int_{0}^{+\infty} \mathbb{E}_{\mu_{N}}(j_{\eta}(0,1)e^{-\mathcal{L}_{N}(\xi)} f_{\eta}(0,1))\right],
\]

where \(e^{\mathcal{L}_{N}t}\) is the evolution operator of the Markov process. We consider translational covariant rates (remark [1]) to have the decomposition (36), plugging this one in (70) we find

\[
D(\rho) = \frac{1}{2\chi(\rho)} \left[\mathbb{E}_{\mu_{N}}(c_{0,1}(\eta)) - 2N\mathbb{E}_{\mu_{N}}(C(\eta) \mathcal{L}_{N}^{-1} C(\eta))\right],
\]

where \(\mathcal{L}_{N}^{-1}\) is the generalized inverse operator of \(\mathcal{L}_{N}\) (for \(f(\eta)\) constant function \(\mathcal{L}_{N} f(\eta) = 0\)), for this definition see [12]. Formula (71) tells us that just the harmonic part of the current \(C(\eta)\) contributes to the second term of the Green-Kubo’s formula and for gradient systems we have \(D(\rho) = \frac{1}{2\chi(\rho)} \mathbb{E}_{\mu_{N}}(c_{0,1}(\eta))\) even if the \(h\) is not local, admitting that such models exist. Expression (71) has an equivalent variational formulation with a minimizer that is computable in principle. The term \(\mathbb{E}_{\mu_{N}}(C(\eta) \mathcal{L}_{N}^{-1} C(\eta))\) can be seen as the scalar product \(\langle C(\eta), \mathcal{L}_{N}^{-1} C(\eta) \rangle_{\mu_{N}}\), where \(\langle f, g \rangle_{\mu_{N}} = \sum_{\eta} f(\eta) g(\eta) \mu_{N}(\eta)\). Since \(\mathcal{L}_{N}\) is symmetric with respect to this scalar product we have

\[
\langle C(\eta), \mathcal{L}_{N}^{-1} C(\eta) \rangle_{\mu_{N}} = \inf_{f} \left\{-\langle f, \mathcal{L}_{N} f \rangle_{\mu_{N}} - 2 \langle C(\eta), f \rangle_{\mu_{N}}\right\}
\]

where the minimizer is over all function \(f : \Sigma_{N} \rightarrow \mathbb{R}\) and a solution is given by

\[
\mathcal{L}_{N} f(\eta) = -C(\eta)\text{ for all } \eta \in \Sigma_{N}.
\]

The solution (73) is well posed since \(C(\eta)\) is orthogonal to the eigenspace of eigenvalue zero. This minimizer looks to us more simple to solve than the one of the expression in [13], for example interpreting the model of Wick as explained at the beginning of the section this minimum can be solved within the framework we are going to explain in next paragraphs. We think that the solution of this minimizer is equivalent to rewrite the discrete hydrodynamics in a form such that the only macroscopic relevant terms are reduced to the usual case of section 7 of gradient systems.
In some special non-gradient cases (as Wick [20]) we expect a simplified scheme, that is an exact case of a more general scheme briefly described at the end.

The idea starts from the observation that a natural attempt to solve (73) is to look for a $f(\eta)$ of the form

$$f(\eta) = \sum_{x \in V_N} \tau_x g(\eta),$$

(74)

where $g(\eta)$ is a local function. Note that the left-hand side of (73) is invariant by translation as it has to be. Remark 2 gives a connection between the minimizer and the conservation law leading to the hydrodynamics, there we discussed that in reversible non-gradient model the current can be rewritten in the form $j_\eta(x,y) = j_\eta^0(x,y) + j_\eta^0(x,y)$, where the single harmonic contributions are such that $C(\eta) = \frac{1}{N} \sum_{x \in V_N} j_\eta(x,x+1) = \frac{1}{N} \sum_{x \in V_N} j_\eta^0(x,x+1).$ So to the part of the current denoted $j_\eta^0(x,y)$ we can apply the scheme of section 7 with respect to a gradient function $h(\eta)$, while it is not possible for the part $j_\eta^0(x,y)$.

But if we are able to find a local function $g(\eta)$ such that

$$\mathcal{L}_\eta \tilde{g}(\eta) = j_\eta^0(0,1) + \tau \tilde{h}(\eta) = \tilde{h}(\eta),$$

(75)

where $\tilde{h}(\eta)$ is another local function, then we are done both with the solution (74) and the discrete form of the hydrodynamics (54). Indeed respectively taking $f(\eta) = \sum_{x \in V_N} \tau_x g(\eta)$ with $g(\eta) = -\tilde{g}(\eta)$ we solve (73) and with the replacements (75) of the harmonic contributions we will be able to treat the hydrodynamics. This is because considering the translations of relation (75), in the discrete hydrodynamics (54) will contribute only the local function $h'(\eta) = -\tilde{h}(\eta)$ since with the local equilibrium hypothesis (55) the terms $\sum_{x \in V_N} \tau_x g(\eta)$ will be negligible in the scaling limit as they are time derivatives and from the time integral they will give a contribution of order $O(1/N^2)$ each one. At the end, the hydrodynamics will follow section 7 with respect to the local function $H(\eta) := h(\eta) + h'(\eta)$.

In the non-local decomposition (55) the part $C(\eta)$ is divergence free and therefore will not appear in the hydrodynamics (54). We expect that writing this last one with respect to the non-local $h^a(\eta)$ of the Hodge decomposition $j_\eta^0(x,y) = \tau_x h^a(\eta) = \tau_x h^a(\eta) + h^a(\eta)$, doing the substitution (75) in $h^a = \sum_{x \in V_N} \frac{1}{N} j_\eta^0(x,x+1)$, with proper cancellations the hydrodynamics will still reduce to the one related to $H(\eta)$.

In general solving (73) with an $f(\eta)$ of the form (74) with the property (75) will be not possible. But for models like example 3 we expect a generalization of this case where (73) is solved unless of (non-local) gradients (which will not contribute in the computation of the scalar products in (72)) with a solution as (74) where $g(\eta)$ satisfies (75) unless extra terms on the right-hand side that in probabilistic sense will be of order $o(1/N)$ and will not contribute to hydrodynamics.
Similarly to (71), in dimension higher than one, the extra terms for non-gradient systems will come only from the harmonic part, i.e. \( C^1(\eta)\varphi_1(x, y) \) and \( C^1(\eta)\varphi_1(x, y) \) in dimension two.

Acknowledgements The author thanks prof. Davide Gabrielli of Università degli studi di L’Aquila for a plenty of discussions and ideas in the paper. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovative programme (grant agreement No 715734).

References

1. L. Bertini, D. Gabrielli, J.L. Lebowitz, Large Deviations for a Stochastic Model of Heat Flow Journ. Stat. Phys 121, 843-885 (2005).
2. L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Stochastic inter-acting particle systems out of equilibrium, J. Stat. Mech, P07014 (2007).
3. L. De Carlo, Microscopic and macroscopic perspectives on stationary nonequilibrium states arXiv:1906.05763 (2019).
4. L. De Carlo, Discrete calculus with cubic cells on discrete manifolds, arXiv:1906.07054 (2019).
5. L. De Carlo, D. Gabrielli, Gibbsian stationary nonequilibrium states J Stat Phys (2017) 168 (2017), 1191-1222.
6. L. De Carlo, D. Gabrielli, Totally Asymmetric Limit for Models of Heat Conduction J Stat Phys 168, 508-534 (2017).
7. L. De Carlo, D. Gabrielli, P. Gonçalves, Scaling limit of an exclusion process with vorticity, in preparation.
8. M. Desbrun, E. Kanso, Y. Tong, Discrete Differential Forms for Computational Modeling, in: Bobenko A.I., Sullivan J.M., Schröder P., Ziegler G.M. (eds) Discrete Differential Geometry. Oberwolfach Seminars, vol 38. Birkhäuser Basel (2008).
9. G.L. Eyink, J.L. Lebowitz, H. Spohn Hydrodynamics and fluctuations outside of local equilibrium: driven diffusive systems J. Statist. Phys. 83, no. 3-4, 385-472 (1986).
10. D. Gabrielli, C. Valente Which random walks are cyclic? ALEA, Lat. Am. J. Probab. Math. Stat. 9, 231-267 (2012).
11. P. Gonçalves, Equilibrium Fluctuations for Totally Asymmetric Particle Systems: exclusion and zero-range processes, VDM Verlag Dr. Müller (2010).
12. J. Hunter Generalized inverses of Markovian kernels in terms of properties of the Markov chain, Linear Algebra and its Applications 447 (2014) 38-55.
13. C. Kipnis, C. Marchioro, E. Presutti Heat flow in an exactly solvable model J. Stat. Phys. 27, 65 (1982).
14. C. Kipnis and C. Landim, Scaling Limits of Interacting Particle Systems (Springer, New York, 1999).
15. Y. Nagahata Regularity of the diffusion coefficient matrix for generalized exclusion process Stochastic Processes and their Applications, 116, 957-982 (2016).
16. J. Quastel : Diffusion of color in the simple exclusion process. Comm. Pure Appl. Math. XLV, 623-679 (1992).
17. F. Spitzer, Interaction of Markov processes, Adv. Math. 5, 246-290 (1970).
18. H. Spohn, Large Scale Dynamics of Interacting Particles (Springer-Verlag, New York, 1991).
19. S.R.S. Varadhan, Nonlineardiffusion limit for a system with nearest neighbor interactions II. Asymptotic Problems in Probability Theory: Stochastic Models and Diffusion on Fractals, 283, 75-128 (1994a).
20. W.D. Wick Hydrodynamic Limit of a Nongradient Interacting Particle Process, J. of Stat. Phys. 54, 873-892 (1989).