Nonlinear dynamics of the interface of dielectric liquids in a strong electric field: Reduced equations of motion

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The evolution of the interface between two ideal dielectric liquids in a strong vertical electric field is studied. It is found that a particular flow regime, for which the velocity potential and the electric field potential are linearly dependent functions, is possible if the ratio of the permittivities of liquids is inversely proportional to the ratio of their densities. The corresponding reduced equations for interface motion are derived. In the limit of small density ratio, these equations coincide with the well-known equations describing the Laplacian growth.

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It is well known that the flat interface of two dielectric liquids is unstable in a sufficiently strong vertical electric field. The dispersion relation for the surface waves has the following form [1, 2]:

$$\omega^2 = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} g k - \frac{E_1 E_2 (\varepsilon_1 - \varepsilon_2)^2}{4\pi(\rho_1 + \rho_2)(\varepsilon_1 + \varepsilon_2)} k^2 + \frac{\alpha}{\rho_1 + \rho_2} k^3,$$

where $k$ is the wave number, $\omega$ is the frequency, $g$ is the acceleration of gravity, $\alpha$ is the surface tension coefficient, $\rho_1$ and $\rho_2$ are the mass densities of lower and of upper liquids ($\rho_1 > \rho_2$), $\varepsilon_1$ and $\varepsilon_2$ are the dielectric constants of fluids. The external electric field strengths under and above the interface, $E_1$ and $E_2$, are related by the expression

$$\varepsilon_1 E_1 = \varepsilon_2 E_2. \quad (1)$$

It is seen from the dispersion relation that, if the electric field is sufficiently strong,

$$E_1 E_2 \gg \frac{\varepsilon_1 + \varepsilon_2}{(\varepsilon_1 - \varepsilon_2)^2} \sqrt{g\alpha(\rho_1 - \rho_2)},$$

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the second term in right-hand side of the dispersion relation dominates for the waves with wave numbers in the range
\[
g(\varepsilon_1 + \varepsilon_2)(\rho_1 - \rho_2) \ll k \ll \frac{E_1 E_2(\varepsilon_1 - \varepsilon_2)^2}{\alpha(\varepsilon_1 + \varepsilon_2)}.
\]
Then \(\omega^2 \propto k^2\) and, hence, we can separate the dispersion relation into two branches
\[
\omega^{(\pm)} = \pm i ck, \quad \epsilon^2 = \frac{E_1 E_2(\varepsilon_1 - \varepsilon_2)^2}{4\pi(\rho_1 + \rho_2)(\varepsilon_1 + \varepsilon_2)}.
\] (2)
For one branch, small periodic perturbations of the surface increase exponentially with the characteristic times \((ck)^{-1}\), while, for the other branch, these perturbations attenuate. In such a situation, we can restrict our consideration to the increasing branch \(\omega^{(+)} = +i ck\), that essentially simplifies the problem of describing the evolution of the interface at the linear stage of the development of instability. The buildup of perturbations of the surface inevitably transforms the system to a state in which its evolution is determined by nonlinear processes. Then, in the general case, splitting into the branches becomes impossible.

In this paper we will show that, for the particular case \(\varepsilon_1 \rho_1 = \varepsilon_2 \rho_2\), we can extract the separate branches from the equations of motion. This makes it possible to reduce by half the number of equations required for describing the evolution of the boundary. The reduced equations coincide with the well-known equations describing the Laplacian growth in the limit of small ratio of liquid densities. An important point is that the Laplacian growth equations not only define a subclass of particular solutions of the problem, but they also describe the asymptotic behavior of the system.

It should be noted that the behavior of the interface of two fluids in normal electric or magnetic field (these problems are similar from the mathematical point of view) is usually investigated in the quasi-monochromatic approximation (see [3, 4, 5, 6] and the references therein). This approach allows one to obtain immediately an equation for the complex amplitude of surface waves. However, the applicability of such an equation is limited by the condition of the smallness of the slopes of the surface. The development of instability can violate this condition. In the strong-field limit, the approach developed in the present work provides a way of studying the interface behavior at essentially nonlinear stages of instability development.

Consider the evolution of the interface of two ideal liquids of infinite depth in an external vertical electric field. In the unperturbed state, the boundary of the liquid is a flat horizontal
surface. Let the $z$ axis of the Cartesian coordinate system is normal to the unperturbed interface. The function $\eta(x, y, t)$ specifies the shape of the deformed boundary, i.e., the liquids occupy the regions $z < \eta(x, y, t)$ and $z > \eta(x, y, t)$, respectively. It is convenient for the subsequent analysis to choose an origin of coordinates so that the level of liquids is determined by the expression $z = -vt$. In other words, the origin moves with respect to the interface at a certain constant velocity $v$.

Let us assume that the motion of both liquids is potential. The velocity potentials for incompressible liquids $\Phi_1$ and $\Phi_2$ satisfy the Laplace equations,

$$
\nabla^2 \Phi_1 = 0, \quad \nabla^2 \Phi_2 = 0,
$$

with the following conditions at the boundary and at infinity:

$$
\rho_1 \left[ \frac{\partial \Phi_1}{\partial t} + \frac{(\nabla \Phi_1)^2}{2} \right] - \rho_2 \left[ \frac{\partial \Phi_2}{\partial t} + \frac{(\nabla \Phi_2)^2}{2} \right] = \frac{\varepsilon_1 - \varepsilon_2}{8\pi} (\nabla \varphi_1 \cdot \nabla \varphi_2), \quad z = \eta(x, y, t),
$$

$$
\frac{\partial \Phi_1}{\partial n} = \frac{\partial \Phi_2}{\partial n}, \quad z = \eta(x, y, t),
$$

$$
\Phi_1 \to -vz, \quad z \to -\infty,
$$

$$
\Phi_2 \to -vz, \quad z \to +\infty,
$$

where $\varphi_1$ and $\varphi_2$ are the electric-field potentials in and above the liquid, and $\partial / \partial n$ denotes the derivative along the normal to the interface. The expression on the right-hand side of the dynamic boundary condition (nonstationary Bernoulli equation) is responsible for the electrostatic pressure at the interface between two ideal dielectric liquids in the absence of free electric charges. The evolution of the interface is determined by the kinematic relation,

$$
\frac{\partial \eta}{\partial t} = \frac{\partial \Phi_1}{\partial z} - (\nabla \cdot \nabla \Phi_1), \quad z = \eta(x, y, t).
$$

The electric potentials $\varphi_1$ and $\varphi_2$ satisfy the Laplace equations,

$$
\nabla^2 \varphi_1 = 0, \quad \nabla^2 \varphi_2 = 0.
$$

Since the electric field potential and normal component of the displacement vector have to be continuous at the interface, we should add the following conditions at the boundary:

$$
\varphi_1 = \varphi_2, \quad z = \eta(x, y, t),
$$

$$
\varepsilon_1 \frac{\partial \varphi_1}{\partial n} = \varepsilon_2 \frac{\partial \varphi_2}{\partial n}, \quad z = \eta(x, y, t).
$$
The system of equations is closed by the condition of the electric field uniformity at an infinite distance from the surface:

\[ \varphi_1 \to -E_1 z, \quad z \to -\infty, \quad (12) \]
\[ \varphi_2 \to -E_2 z, \quad z \to +\infty. \quad (13) \]

Let us show that a flow regime, wherein the harmonic potentials of velocity and of electric field are linearly dependent functions, is possible for certain relations between the problem parameters. Suppose that

\[ \varphi_1 = a \Phi_1 (4 \pi \rho_1 / \varepsilon_1)^{1/2}, \quad \varphi_2 = b \Phi_2 (4 \pi \rho_2 / \varepsilon_2)^{1/2}, \quad (14) \]

where \( a \) and \( b \) are unknown constants. It is necessary to verify that the initial equations of motion (3)–(13) are compatible with these relations. Substituting them into (4) and (10)–(13), we obtain

\[ \rho_1 \left[ \frac{\partial \Phi_1}{\partial t} + \frac{(\nabla \Phi_1)^2}{2} \right] - \rho_2 \left[ \frac{\partial \Phi_2}{\partial t} + \frac{(\nabla \Phi_2)^2}{2} \right] = ab(\varepsilon_1 - \varepsilon_2) \sqrt{\rho_1 \rho_2} \frac{(\nabla \Phi_1 \cdot \nabla \Phi_2)}{\sqrt{\varepsilon_1 \varepsilon_2}}, \quad z = \eta(x,y,t), \quad (15) \]

\[ \Phi_1 a (4 \pi \rho_1 / \varepsilon_1)^{1/2} = \Phi_2 b (4 \pi \rho_2 / \varepsilon_2)^{1/2}, \quad z = \eta(x,y,t), \quad (16) \]

\[ \frac{\partial \Phi_1}{\partial n} a (\rho_1 \varepsilon_1)^{1/2} = \frac{\partial \Phi_2}{\partial n} b (\rho_2 \varepsilon_2)^{1/2}, \quad z = \eta(x,y,t), \quad (17) \]

\[ \Phi_1 \to -z E_1 a^{-1} (4 \pi \rho_1 / \varepsilon_1)^{-1/2}, \quad z \to -\infty, \quad (18) \]

\[ \Phi_2 \to -z E_2 b^{-1} (4 \pi \rho_2 / \varepsilon_2)^{-1/2}, \quad z \to +\infty. \quad (19) \]

For the system of equations (3), (5)–(8) and (15)–(19) to be compatible (it is overdetermined in the general case), the conditions (5)–(7) must coincide with the conditions (17)–(19), and the condition (8) must coincide with the condition (15).

It is apparent that the conditions (5) and (17) coincide if

\[ a(\rho_1 \varepsilon_1)^{1/2} = b(\rho_2 \varepsilon_2)^{1/2}. \quad (20) \]

In view of Eqs. (1) and (20), the conditions at infinity (4), (7) and (18), (19) are consistent if the auxiliary parameter \( v \) takes the following value:

\[ v = a^{-1} v_0, \quad v_0 = E_1 (4 \pi \rho_1 / \varepsilon_1)^{-1/2} > 0. \]
Finally, we consider the condition under which the dynamic (15) and kinematic (8) relations coincide. Let us write Eq. (8) in the form which does not contain function $\eta$ explicitly.

With the help of the formula (20), the boundary condition (16) can be rewritten as follows:

$$\varepsilon_1^{-1}\Phi_1 = \varepsilon_2^{-1}\Phi_2, \quad z = \eta(x, y, t).$$

Differentiating this expression with respect to time or spatial variables, we arrive at

$$\frac{\partial \eta}{\partial t} \cdot \left[ \varepsilon \frac{\partial \Phi_1}{\partial z} - \frac{\partial \Phi_2}{\partial z} \right]_{z=\eta} = - \left[ \varepsilon \frac{\partial \Phi_1}{\partial t} - \frac{\partial \Phi_2}{\partial t} \right]_{z=\eta},$$

$$\nabla_\perp \eta \cdot \left[ \varepsilon \frac{\partial \Phi_1}{\partial z} - \frac{\partial \Phi_2}{\partial z} \right]_{z=\eta} = - \left[ \varepsilon \nabla_\perp \Phi_1 - \nabla_\perp \Phi_2 \right]_{z=\eta},$$

where $\varepsilon = \varepsilon_2/\varepsilon_1$ is the ratio of the permittivities. These relations allow us to eliminate $\eta$ from Eq. (3). We obtain from the kinematic boundary condition:

$$\varepsilon \frac{\partial \Phi_1}{\partial t} - \rho \frac{\partial \Phi_2}{\partial t} = -\varepsilon (\nabla \Phi_1)^2 + (\nabla \Phi_2 \cdot \nabla \Phi_1), \quad z = \eta(x, y, t) \quad (21)$$

Decomposing the velocities of fluids into the normal $(\partial \Phi/\partial n)$ and tangential $(\partial \Phi/\partial \tau)$ components in Eqs. (15) and (21), and taking into account Eqs. (3), (16) and (20), we get

$$\frac{\partial \Phi_1}{\partial t} - \rho \frac{\partial \Phi_2}{\partial t} + \frac{(1-\rho-a^2\varepsilon^{-1}+a^2)}{2} \left[ \frac{\partial \Phi_1}{\partial n} \right] + \frac{(1-\rho \varepsilon^2-a^2+a^2\varepsilon)}{2} \left[ \frac{\partial \Phi_1}{\partial \tau} \right]^2 = 0, \quad z = \eta(x, y, t)$$

$$\frac{\partial \Phi_1}{\partial t} - \varepsilon^{-1} \frac{\partial \Phi_2}{\partial t} + (1-\varepsilon^{-1}) \left[ \frac{\partial \Phi_1}{\partial n} \right]^2 = 0, \quad z = \eta(x, y, t),$$

where $\rho = \rho_2/\rho_1$. Clearly, these expressions can coincide only if the following conditions hold:

$$\rho = \varepsilon^{-1}, \quad 1 - \rho \varepsilon^2 - a^2 + a^2 \varepsilon = 0,$$

$$1 - \rho - a^2 \varepsilon^{-1} + a^2 = 2 - 2 \varepsilon^{-1}.$$ 

From this it is inferred that the equations are compatible provided that $a^2 = 1$ and also

$$\varepsilon_1 \rho_1 = \varepsilon_2 \rho_2. \quad (22)$$

The equation for the parameter $a$ has two roots, $a^{(\pm)} = \pm 1$, corresponding to different branches of solutions.

Thus, we have proved that the functional relation (14) can be compatible with the equations of motion if the condition (22) is valid. The corresponding flow regime is described by
the following equations:

\[ \nabla^2 \Phi_1 = 0, \quad \nabla^2 \Phi_2 = 0, \]  
\[ \frac{\partial \eta}{\partial t} = \frac{\partial \Phi_1}{\partial n} \sqrt{1 + (\nabla L \eta)^2}, \quad z = \eta(x, y, t), \]  
\[ \frac{\partial \Phi_1}{\partial n} = \frac{\partial \Phi_2}{\partial n}, \quad z = \eta(x, y, t), \]  
\[ \rho_1 \Phi_1 = \rho_2 \Phi_2, \quad z = \eta(x, y, t), \]  
\[ \Phi_1 \rightarrow -a^{(\pm)} v_0 z, \quad z \rightarrow -\infty, \]  
\[ \Phi_2 \rightarrow -a^{(\pm)} v_0 z, \quad z \rightarrow +\infty. \]

The reduction of the initial equations (3)–(13) to Eqs. (23)–(28) significantly simplifies the analysis of the interface motion. As will be discussed below, in the formal limit \( \rho_2 / \rho_1 \rightarrow 0 \), these equations describe the so-called Laplacian growth.

Let us find the dispersion relation for Eqs. (23)–(28). We will seek a solution in the form

\[ \Phi_1 = c_1 e^{i(kx - \omega t)} e^{a^{(\pm)} v_0 t} - a^{(\pm)} v_0 t - v_0^2 t, \]  
\[ \Phi_2 = c_2 e^{i(kx - \omega t)} e^{-a^{(\pm)} v_0 t} - a^{(\pm)} v_0 t - v_0^2 t, \]  
\[ \eta = c_3 e^{i(kx - \omega t)} - a^{(\pm)} v_0 t, \]

where \( c_1, c_2, \) and \( c_3 \) are small constants. These expressions correspond to a small-amplitude sinusoidal deformation of the initially plane liquid-liquid interface. After simple transformations, we obtain the following relation between the frequency \( \omega \) and the wave number \( k \):

\[ \omega = i a^{(\pm)} \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} v_0 k = a^{(\pm)} \frac{i \sqrt{\varepsilon_1 (\rho_1 - \rho_2)} \sqrt{4\pi \rho_1 (\rho_1 + \rho_2)} E_1 k. \]

It can be seen that, for the branch \( a = a^{(+) = +1} \), initial perturbation will increase and, for \( a = a^{(-) = -1} \), it will attenuate. It should be noted that, with regard to Eqs. (1) and (22), this expression coincides with the expression (2) specifying different branches of the dispersion relation for the unreduced equations of motion.

Thus, if the condition (22) is satisfied, the separation of two branches corresponding to solutions increasing and decreasing with time is possible not only in the linearized equations, but also in the initial nonlinear equations (3)–(13).

The question arises as to whether the flow regime under consideration is stable. In other words, whether or not Eqs. (23)–(28) describe the large-time asymptotic behavior
of the system. Stability of the increasing branch is evident at linear stages of the interface evolution, when the linearized equations of motion can be split into two independent systems. At the nonlinear stages, the equations do not split completely, and the stability problem becomes nontrivial.

It turns out that the stability can be proved in the limiting case $\rho_1 \gg \rho_2$ and $\varepsilon_1 \ll \varepsilon_2$ (the condition (22) can be violated). Then the evolution of the interface will be governed by the influence of the lower liquid. The equations determining the interface motion become:

$$
\nabla^2 \Phi_1 = 0, \quad \nabla^2 \varphi_1 = 0,
$$

$$
\frac{\partial \Phi_1}{\partial t} + \left(\frac{\nabla \Phi_1}{2} + \frac{\varepsilon_1 (\nabla \varphi_1)^2}{8\pi \rho_1}\right) = 0, \quad z = \eta(x, y, t),
$$

$$
\frac{\partial \varphi_1}{\partial t} + (\nabla \Phi_1 \cdot \nabla \varphi_1) = 0, \quad z = \eta(x, y, t),
$$

$$
\varphi_1 = 0, \quad z = \eta(x, y, t),
$$

$$
\Phi_1 \to -vz, \quad z \to -\infty,
$$

$$
\varphi_1 \to -E_1 z, \quad z \to -\infty,
$$

where the kinematic boundary condition is given in the implicit form. If we introduce a pair of auxiliary potentials,

$$
\Psi^{(\pm)} = 2^{-1}\Phi_1 \pm \left(16\pi \rho_1 / \varepsilon_1\right)^{-1/2} \varphi_1,
$$

these equations can be rewritten in the following symmetric form (compare with Refs. [8, 9]):

$$
\nabla^2 \Psi^{(\pm)} = 0, \quad (29)
$$

$$
\frac{\partial \Psi^{(\pm)}}{\partial t} + (\nabla \Psi^{(\pm)})^2 = 0, \quad z = \eta(x, y, t), \quad (30)
$$

$$
\Psi^{(+)} = \Psi^{(-)}, \quad z = \eta(x, y, t), \quad (31)
$$

$$
\Psi^{(+)} \to -v_0 z, \quad z \to -\infty, \quad (32)
$$

$$
\Psi^{(-)} \to 0 \quad z \to -\infty. \quad (33)
$$

Here, we set $a = +1$ and, as a consequence, $v = v_0$.

One can readily see that these equations are compatible with the condition $\Psi^{(-)} = 0$, which corresponds to the situation of interest, where the velocity potential and the electric field potential are functionally related. For $\Psi^{(-)} = 0$, the set of equations (29)-(33) reduces...
to
\[ \nabla^2 \Psi^+ = 0, \]  \tag{34}
\[ \frac{\partial \eta}{\partial t} = \frac{\partial \Psi^+}{\partial n} \sqrt{1 + (\nabla \perp \eta)^2}, \quad z = \eta(x, y, t), \]  \tag{35}
\[ \Psi^+ = 0, \quad z = \eta(x, y, t), \]  \tag{36}
\[ \Psi^+ \rightarrow -v_0 z, \quad z \rightarrow -\infty. \]  \tag{37}

The same equations can be immediately obtained from Eqs. (23)–(28) in the limit \( \rho \to 0 \).

They coincide with the equations describing the so-called Laplacian growth, viz., the motion of the phase boundary with a velocity directly proportional to the normal derivative of a certain harmonic scalar field (\( \Psi^+ \) in our case). Depending on the chosen frame of reference, this field may have the meaning of temperature (Stefan’s problem in the quasi-stationary limit), electrostatic potential (electrolytic deposition), or pressure (flow through a porous medium). It is important for us that there are many known exact solutions to Eqs. (34)–(37). They describe the evolution of the interface up to the formation of “fingers”, cuspidal dimples, and so on (see, for example, [10, 11, 12, 13]).

Let us prove that the class of solutions of the motion equations (29)–(33) corresponding to the reduced Eqs. (34)–(37) is stable to small perturbations of potential \( \Psi^-(\cdot) \). It should be noted that the motion of the liquid-liquid boundary described by Eqs. (34)–(37) is always directed towards the lower liquid; this is associated with the extremum principle for harmonic functions. Let function \( \eta \) at the initial instant \( t_0 \) be a single-valued function of variables \( x \) and \( y \). In this case, for \( t > t_0 \), the inequality
\[ \eta(x, y, t) \leq \eta(x, y, t_0) \]  \tag{38}
holds for any \( x \) and \( y \). This inequality remains valid for small perturbations of \( \Psi^-(\cdot) \) also, when the effect of potential \( \Psi^-(\cdot) \) in relation (31) can be disregarded as compared to the effect of potential \( \Psi^+(\cdot) \), and the motion of the boundary is described by the same Eqs. (34)–(37).

As regards the evolution of potential \( \Psi^-(\cdot) \), it is sufficient, for small \( |\nabla \Psi^-(\cdot)| \), to consider the boundary condition (30) in the linear approximation. It takes the trivial form:
\[ \Psi_t^-(\cdot) = 0, \quad z = \eta(x, y, t). \]  \tag{39}

This means that the potential does not change with time in the chosen reference frame (the origin moves relative to the interface with speed \( v_0 \)). In the simplest case of a periodic
perturbation, the solution to Eqs. (29), (33) and (39) is given by

$$\Psi^(-) = Ae^{\kappa z} \sin (\kappa x),$$

where $\kappa$ is the perturbation wave number, and $A$ is a constant small amplitude. Let us denote the potential at the boundary $z = \eta$ by $\psi$. We have

$$\psi(x, y, t) \equiv \Psi^(-)|_{z=\eta} = Ae^{\kappa \eta(x,y,t)} \sin (\kappa x).$$

Taking into account the inequality (38), we finally get

$$|\psi(x, y, t)| \leq |\psi(x, y, t_0)|$$

for any $x$ and $y$ at $t > t_0$, that is the value of the potential $\Psi^(-)$ at the interface does not increase with time. Furthermore, since the level of the interface (the value of function $\eta$ averaged over the spatial variables) moves downwards at a constant velocity, it is evident that the potential $\Psi^(-)$ relaxes to zero at the boundary. Thus, we have proved that Eqs. (34)–(37) describe the asymptotic behavior of the liquid-liquid interface in a strong vertical electric field.

It should be noted that the results of this work can be used to describe the motion of the interface of two dielectric liquids in an applied electric field for other geometries of the problem. All one has to do is to modify the conditions (6), (7), (12), and (13). This will allow us to consider the interface dynamics in an oblique or tangential electric field, and also the dynamics of closed interfaces in an external field.

In addition, the results of the above investigation can be extended to the case of two magnetic fluids in a vertical magnetic field. For this purpose one should replace the electric fields $E_{1,2}$ by the magnetic fields $H_{1,2}$ and the permittivities $\varepsilon_{1,2}$ by the magnetic permeabilities $\mu_{1,2}$.

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