Prograde and retrograde precession of a fluid-filled cylinder

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Abstract

We numerically study precession driven flows in a cylindrical container whose nutation angle varies between 60 and 90 degrees for prograde and retrograde precession. For prograde precession we observe sharp transitions between a laminar and a turbulent flow state with low and high geostrophic axisymmetric flow components related with a centrifugal instability, while for retrograde precession a rather smooth transition between a low state and a high state occurs. At the same time prograde and perpendicular precession shows an abrupt breakdown of the flow directly excited by the forcing mechanism, which is not the case for retrograde motion. We characterize the corresponding flow states in terms of the directly driven, non-axisymmetric Kelvin mode, the axisymmetric geostrophic mode, and an axisymmetric poloidal flow which is promising for precession-driven dynamo action. The latter issue is discussed with particular view on an optimal parameter choice for the DRESDYN dynamo project.

1. Introduction

1.1. Context and motivation

Precession denotes the motion of an object rotating around one axis which, in turn, rotates around a second axis fixed in a Galilean reference frame. Precession driven flows are ubiquitous natural phenomena occurring, e.g. in the cyclical structures of the Earth’s atmosphere or the fluid layers in the cores of planets, moons and asteroids [1]. Fluids enclosed in precessing containers are also interesting in a wide range of technical applications, in particular in spacecrafts, rockets and satellites [2, 3] where the stability of the payloads is of primary importance.

The theoretical investigation of precession-driven flows started with the case of an inviscid fluid in a spheroidal cavity [4, 5] whose analytical solution was obtained assuming a uniform vorticity. This Poincaré solution was later extended by Busse [6] to the weakly nonlinear regime including the viscous effects in boundary layers. Meanwhile, several experiments have shown that precession is an efficient mechanism to drive flows without making use of any propellers or pumps [7, 8]. With view on that, precessing-driven flows were also proposed as alternative or at least complementary energy sources for magnetic field self-excitation in the Earth’s core [9, 10], the ancient Moon [11–13], and even asteroids [14]. Indeed, numerical works have demonstrated that precession is able to generate a magnetic field in spherical shells [15, 16], full spheres [17], spheroids [18], cylinders [19–21], and cubes [22, 23]. The amplification of an applied magnetic field by a factor of 3 has been observed previously in the precession experiment of Gans [24].

Further laboratory experiments in different geometries were dedicated to understand particular features of precession-driven flows such as flow instabilities [25–28], large scale vortex formation [29], transitions to and sustaining mechanisms of turbulence [30–32].

In frame of the project DRESDYN (DResden Sodium facility for DYNamo and thermohydraulic studies) at Helmholtz-Zentrum Dresden-Rossendorf, a large-scale liquid sodium experiment is under construction whose final purpose is to show dynamo action in a precession driven flow [33–35]. The core of the experiment consists of a cylinder with radius \( R = 1 \) m and height \( H = 2 \) m that can achieve a
rotation frequency \( f_c = 10 \text{ Hz} \) and a precession frequency \( f_p = 1 \text{ Hz} \). In the first instance, the cylindrical geometry has been chosen because the presence of corners allows a more efficient injection of kinetic energy into the fluid than a corresponding spherical shell. Even if the cylindrical geometry seems to be far from a geophysical application, remarkable commonalities have been revealed between the characteristic flow transition in cylinders [32], spheroids [30, 31] and ellipsoids [36] while the spherical case is in some sense exceptional.

Precession-driven flows in cylinders show a non-trivial behavior already in the weakly forced regime being governed by a 3D flow field consisting of inertial waves which are particularly excited when approaching the resonance condition. While outside the resonance the amplitudes of the inertial waves can be predicted by a linear-inviscid model [37], at resonance the viscosity must be taken into account in order to find the saturated amplitude. Various models have been proposed which include viscous effects [38, 39] and even weakly nonlinear interactions [40]. At resonance the flow is highly unstable, tending to degenerate into a state of chaotic and fine-scale motion called ‘resonant collapse’ [25, 26]. Hysteresis phenomena, in terms of relaminarization-breakdown cycles, have been observed [25, 26, 32], too. Several instability mechanisms were identified [40–43] and particular attention was laid on the emergence of a geostrophic motion when the nonlinear effects become important. Both numerical [40, 44–47] and experimental [27] studies have revealed a geostrophic flow whose structure is prevalently axisymmetric and dominated by an azimuthal velocity opposite to the container’s rotation. The generation of an axisymmetric geostrophic mode in a rotating fluid has been reported for various forcing mechanisms, like libration [48] or tidal forces as well as in case of instabilities (e.g. elliptical instability [49]). Although its impact for rotating turbulence is well established [49, 50] there is no unique scenario to explain its generation. Since Greenspan’s theorem [51] forbids the possibility of wave-to-geostrophic transfer (at least for a first order nonlinear problem) several authors [40, 52] argued that the nonlinear interactions in the viscous boundary layers are able to trigger it. However, experimental works have evidenced that also interior shear layers could be a source for geostrophic flows [53]. More recent models take into account the possibility of instabilities as driving mechanisms for the geostrophic flow, e.g. in form of quartic wave interaction [54], or as a sequence of consecutive destabilization mechanisms [55]. The emergence of a geostrophic azimuthal circulation opposite to the cylinder rotation eventually goes along with the braking of solid body rotation (SBR) for large enough forcing and could result in a centrifugal unstable flow as proposed by Kobine [27].

Another remarkable phenomenon which has gained attention in recent theoretical [40, 56] and numerical [57] works is the appearance of a zonal flow (also called streaming flow) which raises from nonlinear interaction of inertial modes and which may not be invariant along the axial direction (it is non-geostrophic).

1.2. Scope of this study

Preccession driven flows are governed by four key parameters: (i) the Reynolds number \( \text{Re} \), i.e. the ratio of Coriolis force to viscosity; the precession ratio (also called Poincaré number) \( \text{Po} \), i.e. the ratio of precession frequency to rotation frequency; the geometric aspect ratio \( \Gamma \), i.e. the ratio of height to radius of the container and finally the nutation angle \( \alpha \) defined as the angle between the precession and the rotation axis. So far, most studies of fluid-filled precessing cylinders focused on the impact of Reynolds number and/or precession ratio, were carried out for small nutation angles. While for cylindrical geometry the prograde precession (i.e. when the container and the turntable rotate in the same direction) [26, 40, 44, 46] and retrograde [57–59] precession were investigated separately, a direct comparison between these two motions is still elusive. The present numerical investigation aims at assessing the role of prograde and/or retrograde motion for comparably large nutation angles for the future large scale precession experiment in frame of the DRESYN project. In particular, we will extend the work of Giesecke et al. [21, 60] where precession ratios for efficient dynamo action were found only for perpendicular nutation angle, by identifying the optimal parameters in terms of nutation angle and pro- or retrograde precession. The main idea is that the optimal range is characterized by the emergence of axisymmetric large scale rolls which resemble the flow structure studied in a spherical kinematic dynamo model by Dudley and James [61] and in a cylindrical model by Xu et al [62].

The paper is organized as follows: in section 2 we formulate the hydrodynamic problem and give a brief description of the numerical method. The results concerning the different flow responses to the nutation angle and the difference between prograde and retrograde motion in terms of inertial modes and flow structures are presented in section 3. The stability of the flow is analyzed in detail in section 3.2, and the poloidal structures emerging for certain precession ratios are discussed in section 3.3. All results of this work are summarized, together with their implications, in section 4.
2. Mathematical formulation and numerical methods

2.1. Navier–Stokes equation

We consider an incompressible fluid of kinematic viscosity $\nu$ enclosed in a cylinder of radius $R$ and height $H$. The container rotates and precesses with angular velocities $\Omega_c$ and $\Omega_p$ for prograde (retrograde) motion, with $\alpha$ denoting the nutation angle, as illustrated in figure 1(a). While $\alpha$ could also be considered to run between $0^\circ$ and $180^\circ$, we restrict it here to the range between $0^\circ$ and $90^\circ$, and differentiate instead between pro- and retrograde precession.

The fluid motion inside the precessing cylinder is governed by the Navier–Stokes equation \[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} - 2\mathbf{\Omega} \times \mathbf{u} + \frac{d\mathbf{\Omega}}{dt} \times \mathbf{r}, \] and \( \nabla \cdot \mathbf{u} = 0 \), (1)

which obeys no-slip boundary conditions $\mathbf{u} = 0$ at all walls. Here $\mathbf{u}$ is the velocity flow field, $\mathbf{\Omega} = \mathbf{\Omega}_c + \mathbf{\Omega}_p$ is the total rotation vector and $\mathbf{r}$ is the position vector with respect to the origin $O$. $P$ is the reduced pressure which includes the hydrostatic pressure and the gradient terms (including the centrifugal force) that do not change the dynamical behavior of the flow. The last two terms on the right-hand side are the Coriolis and the Poincaré force, respectively. The cylindrical coordinates, fixed in the origin $O$, are the axial ($z$), radial ($r$) and azimuthal ($\phi$) ones, respectively, as shown in figure 1(b). In order to non-dimensionalize the Navier–Stokes equation we use the radius $R$ as length scale and $\left| \mathbf{\Omega}_c + \mathbf{\Omega}_p \cos \alpha \right|^{-1}$ as time scale, which represents the projection of total angular velocity on the cylinder axis, i.e. $(\mathbf{\Omega}_c + \mathbf{\Omega}_p) \cdot \hat{z}$.

The key parameters governing precession-driven flows, the Reynolds number $\text{Re}$, the Poincaré number $\text{Po}$ and the aspect ratio of the container $\Gamma$ are defined as

\[ \text{Re} = \frac{R^2 \left| \mathbf{\Omega}_c + \mathbf{\Omega}_p \cos \alpha \right|}{\nu}, \quad \text{Po} = \frac{\Omega_p}{\Omega_c}, \quad \Gamma = \frac{H}{R}. \] (2)

2.2. Numerical methods

For studying precession-driven flows in cylinders, two frames of reference are usually employed: the mantle frame (attached to the cylinder wall) and the turntable frame in which the cylinder walls rotate at $\Omega_c$ and the total vector $\mathbf{\Omega}$ is fixed. We perform our simulations, using the DNS code SEMTEX [63], in the turntable frame where $\partial \mathbf{\Omega} / \partial t = 0$ so that in equation (1) the Poincaré force disappears and both the rotation vector $\mathbf{\Omega}_c$ as well the precession vector $\mathbf{\Omega}_p$ are stationary. For the numerical simulations, we use 300 quadrilateral elements to mesh the meridional half plane (see figure 1(b)) and 128 Fourier modes in azimuthal direction. The initial conditions at $t = 0$ correspond to a pure SBR state, given by $\mathbf{u} = (\mathbf{\Omega}_c r) \phi$. 

Figure 1. Schematic representation of the precessing cylinder. (a) Visualization of the cylinder with the rotation angular velocity $\Omega_c$ and the precession angular velocity $\Omega_p$ (prograde) or $-\Omega_p$ (retrograde); the nutation angle $\alpha$ is measured from the turntable axis $\hat{k}$ to the cylinder axis $\hat{z}$. (b) Cylinder body of radius $R$ and height $H$. The origin $O$ corresponds to the center of the cylinder with the cylindrical coordinate system $(z, r, \phi)$. The meridional half plane in blue shows the 2D mesh with 300 elements.
The parameter space to be investigated in this work is the following: the Reynolds number varies in the range $[5 \times 10^5, 10^6]$ and the Poincaré number in the range $[10^{-3}, 3.5 \times 10^{-1}]$. The nutation angles are $\alpha = 60^\circ$, $\alpha = 75^\circ$ (in appendix A we show results also for other angles), both for prograde and retrograde precession, and $\alpha = 90^\circ$ for which there is no difference between prograde and retrograde precession. The aspect ratio will be fixed at $\Gamma = 2$ which is quite close to the resonance point $\Gamma = 1.989$ of the first inertial mode with $(m, n, k) = (1, 1, 1)$ which represents a gyroscopic motion resulting from the tendency of the fluid flow to align the flow rotation and the precession axis. A summary of all simulations in the parameter space (Re, Po) is shown in the stability diagram of figure 5(a).

Although the simulations are performed in the precessing (i.e. turntable) frame of reference, almost all results will be presented and discussed in the mantle frame in which the inertial and geostrophic modes are more intuitive (the only exception are the plots of the angular momentum and the Rayleigh criterion shown in figures 4 and 5). The inertial modes are the eigenfunctions $u_{mkn}$ of the inviscid-linearized form of (1), explicitly given in the appendix B, which are characterized by three integers $(m, k, n)$ indicating azimuthal, axial and radial wave numbers.

A useful tool to analyze the numerical results is the projection of the DNS flow field onto a basis given directly forced mode and the SBR energy (as reference) as follows:

$$\mathbf{u}(r, t) = \sum_{n=1}^{N} A_{00n}(t) \mathbf{u}_{00n}(r) + \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{2} [A_{m0n}(t) \mathbf{u}_{m0n}(r, \varphi) + \text{c.c.}]$$

$$+ \sum_{k=1}^{K} \sum_{n=1}^{N} \frac{1}{2} [A_{0kn}(t) \mathbf{u}_{0kn}(z, r) + \text{c.c.}]$$

$$+ \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{n=1}^{2N} \frac{1}{2} [A_{mkn}(t) \mathbf{u}_{mkn}(z, r, \varphi) + \text{c.c.}]$$

$$+ \mathbf{u}^b(r)$$

where c.c. means complex conjugate, and $\mathbf{u}^b$ is the boundary layer velocity. The amplitude of each inertial mode is computed as

$$A_{mkn} = \frac{2}{(\int_{V} \mathbf{u}_{mkn} \cdot \mathbf{u}_{mkn}^* dV)^{1/2}}.$$

### 3. Results

#### 3.1. Impact of precession on the base flow

It is well-known that a precession-driven flow in cylindrical geometry changes with increasing forcing due to the emergence of an axisymmetric azimuthal flow and the related modification of the non-axisymmetric poloidal flow [40, 44–46]. Here we focus on the specific influence of the nutation angle on this behavior.

In figure 2, we show the time-averaged axial velocity $u_z$ which is a good representative for the base state. The three selected configurations $\alpha = 60^\circ (p)$, $\alpha = 90^\circ$ and $\alpha = 60^\circ (r)$, as examples for prograde, perpendicular and retrograde precession, respond in different ways to the increase of the precession ratio $Po$. Only the prograde and the perpendicular cases show a reduction of the axial flow to negligible values inside the bulk region whose extension increases with the precession ratio. By contrast, the retrograde case (bottom row) remains essentially unchanged in magnitude and shape, but with counterclockwise phase shift from low to large $Po$.

For the smallest precession ratio $Po = 0.010$ (left column) the 3 configurations appear very similar. The flow magnitude is quite weak and the three-dimensional structures (indicated by the $\pm 0.15$ levels) are symmetrical with respect to $\varphi$ (the shape for $\alpha = 90^\circ$ is a little more elongated). The plots for $Po = 0.100$ (central column) prove that in the bulk $u_z$ vanishes at lower $Po$ for prograde precession. Finally, at $Po = 0.200$, $u_z$ is confined close to the sidewall for $\alpha = 60^\circ (p)$ and $\alpha = 90^\circ$, with this region being thinner for the latter case.

In order to quantify the flow response, we investigate the energy densities of the main inertial modes in dependence of the precession ratio. We define the energy densities of the geostrophic-axisymmetric, the directly forced mode and the SBR energy (as reference) as follows:

$$e_{00} = \frac{1}{2V} \sum_n |A_{00n}|^2, \quad e_{11} = \frac{1}{2V} \sum_n |A_{11n}|^2, \quad e_{sbr} = \frac{1}{2V} \int_V (\Omega r)^2 dV.$$  

(5)
Figure 2. Flow structures for prograde, perpendicular and retrograde precession. 3D isosurfaces and 2D contours (at \( z = 0 \)) of the time averaged axial velocity \( u_z \) for \( \text{Re} = 6500 \) and three representative precession ratios \( Po = 0.010, Po = 0.100, Po = 0.200 \) (from left to right). From top to bottom: prograde \( \alpha = 60^\circ \); perpendicular \( \alpha = 90^\circ \); retrograde \( \alpha = 60^\circ \). In the equatorial 2D polar plane 100 equispaced levels are shown; the isosurfaces are drawn at levels \( \pm 0.15 \), and the meridional semiplanes of cylinder frame represent \( \varphi = 0 \).

The energies for the axisymmetric-poloidal structures are:

\[
e_{02} = \frac{1}{2V} \sum_n (\Im A_{02n})^2, \quad e_{0k} = \frac{1}{2V} \sum_n (\Im A_{0kn})^2
\]

with \( \Im \) denoting the imaginary part (see appendix B for descriptions of this type of inertial modes which are called axisymmetric oscillations [44, 52]).
Figure 3. Energy density of various inertial modes versus precession ratio $P_0$ for different nutation angles. (a) Geostrophic-axisymmetric energy $e_{00}$; (b) energy contained in the directly forced mode $e_{11}$; (c) axisymmetric poloidal energy $e_{02}$ for the second axial wave number; (d) remaining part of the axisymmetric poloidal energy with $k \neq 2$. The energies are scaled by the solid body energy $e_{sb}$, all quantities are defined in equations (5) and (6).

Figure 3 shows the energies as defined in equations (5) and (6) versus the precession ratio $P_0$. The differences between different nutation angles (and prograde/retrograde precession) become stronger with increasing $P_0$. The prograde and $\alpha = 90^\circ$ cases display an abrupt transition from a state with low energy in the geostrophic axisymmetric mode (low state) to a state with high energy (high state). The retrograde cases (orange and red curves) reveal a much smoother increase of $e_{00}$ whose final level is lower than for the prograde counterparts (figure 3(a)). While in the low state region (for $P_0 < 0.075$) the prograde curves are almost overlapping, from the transition onward to the maximum precession ratio they diverge, e.g. the blue and green curves ($\alpha = 60^\circ$ and $\alpha = 75^\circ$) show the jump at lower $P_0$ with the saturated levels differing by 10%. Interestingly, the linear increase of $\alpha = 90^\circ$ case in the large precession region indicates the growth of the geostrophic azimuthal circulation which illustrates why the axial velocity contour (in figure 2) is thin and concentrated close to the sidewall boundary for $P_0 = 0.200$. This is consistent with the outcome of Kong et al [44] who found that the larger the nutation angle the larger the bulk region that is occupied by the geostrophic-axisymmetric flow. The behavior of the forced mode energy $e_{11}$ is closely related to that of $e_{00}$. In figure 3(b) the prograde curves increase until $P_0 \approx 0.08$ followed by a sharp breakdown of more
than 60%. We notice two particular aspects: (i) the breakdown occurs at the same $P_0$ as does the jump of $e_0$; (ii) the smaller the angle the earlier the transition occurs. After the breakdown the curves for $\alpha = 60^\circ, 75^\circ$ remain flat, while the curve for $\alpha = 90^\circ$ continues to decrease. In accordance with the linear increase of $e_0$ as discussed above, the two retrograde curves do not show any sharp transition.

Also related to the breakdown of $e_1$, for the prograde and perpendicular cases, and the weaker decrease of $e_1$ for the retrograde case, we observe energy peaks for the poloidal modes. Figure 3(c) shows $e_{02} = f(P_0)$ which is the energy associated with the axisymmetric vortex structure in the cylinder’s meridional semi-plane (see also figure 6). The nutation angle and the prograde or retrograde configuration are decisive in this respect. The largest value of $e_{02}$, as found for the case $\alpha = 75^\circ$ (r), is almost 50% larger than for $\alpha = 90^\circ$ (and more than double than those of the other cases). It has a maximum around $P_0 = 0.130$ with a remarkable wide range of $\Delta P_0 \approx 0.020$. By contrast, the $\alpha = 60^\circ$ (r) case (orange curve) shows the smallest magnitude whose maximum values are situated at still larger values of $P_0$. Quite generally, the
Figure 5. Regime diagram in the (Po, Re) space with respect to the centrifugal stability. (a) Plane showing all the simulations performed for prograde (Po > 0) and retrograde (Po < 0) precession at three different nutation angles (195 simulations). Black and red symbols represent stable and unstable solutions in the sense of Rayleigh’s criterion (equation (7)). The zoom plot shows a focus on larger Re to underline the impact of $\alpha$ for the occurrence of instability. (b) Diagram for $\alpha = 90^\circ$. Green symbols are the stable solutions, black symbols represent the solutions already gone through the breakdown of the directly forced mode but yet centrifugally stable and red symbols denote the centrifugal unstable solutions. The grey dashed-dotted line marks the critical precession ratio $Po_{(1)}$ for the breakdown of the (1, 1) mode and the blue solid line shows the critical threshold $Po_{(2)}$ for the violation of the Rayleigh criterion.

peaks for the retrograde profiles are shifted to larger precession ratios compared to the prograde ones. The prograde and perpendicular precession show curves with similar values for Po < 0.090 but while $\alpha = 90^\circ$ has a clear peak followed by a steep decrease, the other cases remain at non-negligible level of $e_0^2$. Similar features are shown by the complementary part of the poloidal energy $\sum_{k \neq 2} e_0^k$, figure 3(d), where $\alpha = 90^\circ$ and $\alpha = 75^\circ$ retrograde have the peak at the same Po as $e_0^2$.

If we sum up the energies contained in the inertial modes considered in figure 3 we obtain that, for prograde and perpendicular precession, their summation contributes more than 90% of the total energy of the flow, whereas the summation for the retrograde cases stays around 60%. This fact indicates that the prograde precession is essentially characterized by these inertial modes while the retrograde cases are characterized by a more complex flow structure. However, other contributions of inertial modes are outside the scope of the present work because they are less relevant for possible dynamo action.

3.2. Role of the centrifugal instability

We have shown that for the prograde and perpendicular cases three main regions can be identified: (i) a low state dominated by the forced $m = 1$ Kelvin mode, (ii) a transition region and (iii) a high state dominated by an axisymmetric-geostrophic flow. The increase of the axisymmetric-geostrophic mode results in the
dominance of an azimuthal circulation and the near-absence of axial flow in the bulk region as shown in figure 2. The strong azimuthal rotation is opposed to the container rotation leading to the braking of SBR and, for large enough precession ratio, possibly to a centrifugally unstable flow. This topic will be our next focus.

As a common discriminant to evaluate the hydrodynamic stability of rotating flows the centrifugal stability criterion, or Rayleigh criterion, defined as

$$\frac{\partial(L)^2}{\partial r} > 0 \quad \text{with } L = r L^m,$$

(7)

is employed. In equation (7), $L$ is the angular momentum and $u_r^L = u_r + \Omega_r r$ denotes the azimuthal velocity in the turntable reference frame. Strictly speaking this criterion holds only for purely rotational shear flows. Nevertheless, the application in the present case is supported by the prevalently azimuthal nature of the flow once the high state is achieved, and is also consistent with the description of the experimental observations by Kobine [27].

The radial profile of the angular momentum and the radial derivative of $L^2$, averaged both in azimuthal and axial direction, are shown in figure 4. The left column of figures 4(a1)–(e1) shows the impact of the precession ratio on $L$. With increasing Po the flow deviates more and more from the SBR profile, eventually developing rather flat profiles in the bulk region (for $r < 0.8$, say). For prograde precession and $\alpha = 90^\circ$, at large enough Po the angular momentum becomes negative, indicating that in this region the flow rotates opposite to the container. For large Po the deviation from SBR goes along with an emergence of a huge velocity gradient between $0.90 < r < 1.0$, owing to a marked sidewall boundary layer [47]. The retrograde cases do not show any negative $L$ for Po $\leq 0.20$.

The right column, figures 4(a2)–(e2), shows the radial derivative of $L^2$. For prograde precession and $\alpha = 90^\circ$, at large enough Po the slope of $L^2$ becomes negative indicating a centrifugal unstable flow. We find that for $\alpha = 60^\circ$ (p) and $\alpha = 75^\circ$ (p) the violation of Rayleigh’s criterion occurs at $Po \approx 0.100$ (green curve in (a2) and (b2)) while for $\alpha = 90^\circ$ the flow becomes unstable above $Po \approx 0.125$ developing a marked ‘nose’ shape with a positive peak at $r \approx 0.8$ and negative peak at $r \approx 0.9$.

Up to this point, the analysis has been carried out for Re $= 6500$, so our goal is to extend the range to other Reynolds numbers.

In figure 5, we present regime diagrams for the instabilities described so far, i.e. the breakdown of the directly forced mode and the violation of Rayleigh’s criterion. Figure 5(a) shows the parameter space (Po, Re) that includes all simulated cases accessible in our numerical simulations. The flow is defined as stable (black symbols) or unstable (red symbols) according to the Rayleigh criterion (7). From the preponderance of the red symbols in the upper half plane, it is obvious that the prograde motion is substantially more prone to become unstable. It is noteworthy that in order to find an unstable flow for retrograde motion we must achieve more than twice the precession ratio of the prograde counterpart, e.g. the first unstable solution at Re $= 6500$ for $\alpha = 75^\circ$ (r) is found at $Po = 0.250$ while for prograde it is at $Po = 0.100$ (see zoom plot). This discrepancy is even more pronounced for the case $\alpha = 60^\circ$ where no centrifugal instability is found at all for retrograde precession. In the zoom panel of figure 5(a), we observe that for the prograde cases the unstable points appear at smaller Po for smaller $\alpha$ (see for instance at Re $= 6500$ and Re $= 10 000$ the asterisks for $\alpha = 60^\circ$ and the triangles $\alpha = 90^\circ$). However, we do not find a systematic dependence of the critical precession ratio for the onset of the instability with respect to $\alpha$. In particular it is not possible to unify the precession ratio and the nutation angle into a general forcing parameter $Po \sin \alpha$.

Figure 5(b) focuses on the case $\alpha = 90^\circ$ for which we have simulations for several Reynolds numbers. In addition to the violation of Rayleigh’s criterion we illustrate also the breakdown of the energy of the directly forced mode. For the sake of clarity we show only the last stable point where $e_{11}$ is maximum (green symbols), the points included between the two stability curves (black symbols), and then the first centrifugally unstable point (red symbols). The grey dashed-dotted curve marks the derived scaling law for the breakdown of $e_{11}$ whose expression is $Po^{(e1)} = 0.025 + 0.40 \text{ Re}^{-1/3}$. Note the absence of points for Re $= 500$ since this Reynolds number does not show a clear breakdown of $e_{11}$ (see figure B1 in the appendix B). The blue curve is the fit marking the scaling for the critical threshold, $Po^{(c2)} = 0.033 + 0.7 \text{ Re}^{-1/4}$, above which the flow is centrifugally unstable. This kind of scaling law is reminiscent of the instability threshold found by Lin et al [28] for the regime of strongly non-linear flow in a precessing cylindrical annulus.

We should remark that the parameter space studied is quite limited in terms of Reynolds numbers, therefore any extrapolation for geophysical phenomena and the DRESDYN experiment should be taken with a grain of salt. Formally, the expressions for $Po^{(e1)}$ and $Po^{(c2)}$ would cross around Re $\sim 10^3$, but the applicability for such large Reynolds numbers is questionable. Comparing these results with figure 3(a), we
Figure 6. Dependence of the poloidal flow field on the nutation angle. Contours of the meridional semi-plane for $Re = 6500$ and five cases (corresponding to the maximum of $\epsilon_{02}$ in figure 3(c)). Vector field for $[u_z, u_r]_{m=0}$ and color scheme for the azimuthal vorticity $\omega_m = 0$; the lines in the colorbar represent color-levels which are not uniform. (a) $\alpha = 60^\circ$ (p), $Po = 0.105$; (b) $\alpha = 75^\circ$ (p), $Po = 0.095$; (c) $\alpha = 90^\circ$, $Po = 0.105$; (d) $\alpha = 75^\circ$ (r), $Po = 0.130$; (e) $\alpha = 60^\circ$ (r), $Po = 0.135$. (f) Velocity profiles for $u_z$ at $r = 0.5$ and (g) $u_r$ at $z = 0$.

see that, for the prograde cases, the centrifugal instability appears close to the value of $Po$ where $\epsilon_{02}$ achieves the high state. We conclude that the emergence of the axisymmetric-geostrophic flow (essentially an azimuthal flow which counteracts the SBR) is responsible for the decrease of the angular momentum $L$ and consequently its negative radial derivative (i.e. violation of Rayleigh criterion). If this is the case our results...
indicate a hierarchical relation between the centrifugal instability and the secondary-geostrophic instability theory proposed by Kerswell [55] whose occurrence would scale $\propto \text{Re}^{-1/4}$.

3.3. Poloidal flow field: promising flow structures for dynamo action

In this section we focus our attention on the poloidal flow structure whose energy was plotted in figures 3(c) and (d). The interest in this particular kind of inertial mode is mainly related to their suitability for dynamo action in a precessing cylinder, as discussed in [21, 60].
Figure 8. Time-evolution of quadruple rolls topology and the amplitudes $A_{021,2}$ and $A_{041,2}$ for $\alpha = 60^\circ$ (p) and $\text{Po} = 0.105$. 1–2–3 are characterized by $A_{021} > 0$ (solid black curve) and $A_{041} < 0$ (solid red curve). Step 4 is the cross which causes the sign inversion and the emergence of the larger rolls inside the bulk. 5–6 are the final state characterized by $A_{021} < 0$ and $A_{041} \approx 0$. Vector field for $[u_z, u_r]_{m=0}$ and color scheme for the azimuthal vorticity $\omega_{m=0}$.

Figure 6 shows the poloidal flow structure for various nutation angles, taken at the respective values of Po where the energy density $e_{22}$ is maximum (see figure 3(c)). The vector field comprises the azimuthally averaged radial and axial velocities $[u_z, u_r]_{m=0}$, while the color scale represents the magnitude of the azimuthal vorticity (again azimuthally averaged) which is a measure of the rotatory behavior of $[u_z, u_r]$ in the meridional half plane:

$$\langle \omega \cdot \hat{\varphi} \rangle_{m=0} = \omega_{m=0} = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) d\varphi.$$  

(8)
The nutation angle plays a major role both for the topology and the magnitude of the poloidal flow field. Remarkably, the prograde case with \( \alpha = 60^\circ \) (figure 6(a)) shows an opposite orientation of the double rolls compared to those in the other cases and includes smaller vortices in the corners. The colors illustrate how the azimuthal vorticity is distributed in the plane, including zones characterized by an alternation of signs. The strength and the extension of the larger rolls present an increase from \( \alpha = 75^\circ \) (p) to \( \alpha = 90^\circ \), finally achieving a maximum at \( \alpha = 75^\circ \) (r). The case \( \alpha = 60^\circ \) (r), figure 6(e), shows the weakest and smallest vortices which additionally are centered more towards the corners. This behavior is in accordance with the maximum level of \( \theta_{92} \) shown in figure 3(c).

In order to have a direct quantitative comparison, we plot also the axial velocity \( u_z^{m=0} \) at \( r = 0.5 \) and the radial velocity profile \( u_r^{m=0} \) at the equator \( z = 0 \), in figures 6(f) and (g), respectively. Again, the largest values are achieved for \( \alpha = 75^\circ \) (r) (red solid-dashed curve) both for radial and axial velocities, while \( \alpha = 60^\circ \) (r) is the weakest case. The case \( \alpha = 60^\circ \) (p) has opposite values with respect to all other cases, consistent with the inverse rotation of the vortices in the bulk.

### 3.4. Transient behavior of double and quadruple rolls

In the following, we will discuss the time dependence of the poloidal rolls. Specifically, we will work out the difference between \( \alpha = 90^\circ \) (double rolls) and \( \alpha = 60^\circ \) (quadruple rolls), i.e. the two paradigmatic cases of double and four vortices. We choose to present the amplitude since it is directly related to the velocity and we include also the dependence on the radial wave number because it is a useful characterization of the flow field’s radial distribution in the half plane.

The two cases show a different evolution of the poloidal structure as we can see in figures 7 and 8 where snapshots of the instantaneous flow field at different time-steps are presented together with the corresponding amplitudes.

The case \( \alpha = 90^\circ \) (figure 7) exhibits a double-vortex structure that remains rather stable during the entire period showing only a little enlargement from \( t = 38 \) to \( t = 145 \). The vortex close to the top endwall has a counterclockwise direction associated with a positive azimuthal vorticity and the opposite direction for the bottom vortex. This behavior is also reflected in the flow amplitude and indeed, the \( A_{921} \) component is dominant and always positive while the others are of minor importance.

The case \( \alpha = 60^\circ \) (p) (figure 8) shows a much more complex evolution. The first part of the transient (panel 1) is characterized by a double vortex structure (green arrows) analogous to the case \( \alpha = 90^\circ \). The time \( t = 38 \) coincides with a local peak of \( A_{921} \), which in the subsequent period decreases while \( A_{922} \) (red curve) increases till \( t = 79 \) when the original double rolls are confined in the corner and a radial inward flow appears around the equator (panel 2, purple arrows). The next step (panel 3) is characterized by the organization of vortical flow around \( z \approx 0 \) which evolves into a clockwise vortex, denoted by the purple arrow in the blue central region, followed by the formation of another roll of opposite rotation (panel 4).

At \( t = 90 \) (blue dashed line in the amplitude plot) we observe the crossing of \( A_{921} \) and \( A_{922} \), which is accompanied by the formation of 4 asymmetric vortices. In the remaining time, the central vortices migrate towards the endwalls (panel 5): the large bottom roll has acquired the final extension and inclination while the top vortex is stretched. The last panel at \( t = 145 \) shows the final setup where the corner rolls and bulk rolls lie next to each other with opposite rotation in a sort of ‘gear’ interaction. The final level of \( A_{921} < 0 \) denotes an opposite rotation of the larger vortices with respect to the case \( \alpha = 90^\circ \) shown in figure 7.

### 4. Conclusions and perspectives

In this study we have numerically investigated the influence of the nutation angle on the precession-driven flow in cylindrical geometry with special focus on prograde and retrograde cases.

Our results show that for large precession ratio the angle and the orientation of motion (prograde/retrograde) result in different flow structures and magnitudes. The main outcomes can be summarized as follows:

(a) Prograde and perpendicular precession show an abrupt transition of the flow state (dominance of geostrophic-axisymmetric flow together with breakdown of the directly forced mode) around a critical precession ratio; the smaller the angle the smaller is the critical precession ratio at which the transition occurs. The resulting flow structure is characterized by a bulk region with negligible axial velocity and a prevalently azimuthal circulation which nearly compensates the original SBR. The transition found here is not unique for the cylinder but is indeed reminiscent of the steep jump between laminar and turbulent regime observed in ellipsoids or spheroids [30]. This mechanism could also be related to a hysteresis cycle [9, 10], but since our simulations have been performed separately for each precession...
ratio, it is not possible to deduce any hysteretical behavior. However, previous experimental studies [24, 32] indeed found hysteresis for the transition between laminar and turbulent states in a precessing cylinder. Retrograde precession does not exhibit a clear breakdown of the directly forced mode but it shows a smoother increase of the geostrophic-axisymmetric flow. In all case the growth of geostrophic-axisymmetric flow modifies the radial distribution of the angular momentum. At large precession ratio the prograde and perpendicular cases, being dominated by the zonal-geostrophic flow, show a violation of the Rayleigh criterion whereas the retrograde motion is much more stable against a centrifugal instability in the considered parameter space. For \( \alpha = 90^\circ \) two marginal stability curves were found. The first one, related to the breakdown of the directly forced mode, seems to scale \( \propto \Re^{-1/5} \). This kind of phenomenon has also been investigated by e.g. by Manasseh [26] even if his experimental work was focused on small nutation angle. Remarkably, our results present commonalities with the so-called type A breakdown due to its occurrence at large precession ratio with nearly no bulk motion which is indeed our case.

The second threshold, which denotes the separation between centrifugally stable/unstable flow seems to scale \( \propto \Re^{-1/4} \) which indicates a connection with the geostrophic instability proposed by Kerswell [55]. Moreover, this is the scaling of the sidewall boundary layer thickness of a geostrophic flow on a vertical wall, the so called Stewartson layer, found in cylinders at very large precession ratio in previous works [44, 47]. Even if the geostrophic and the centrifugal instabilities are two different phenomena, our results suggest that the centrifugal unstable flows are a consequence of the geostrophic instability, therefore the scaling should be the same. The difference with Kerswell’s theory is that our fitting expression is asymptotic, therefore the geostrophic instability observed here could remain over certain range of Po for Re \( \rightarrow \infty \).

The most striking connection of the centrifugal instability scaling law is with the experimental results of Lin et al [28] for precessing cylindrical annulus where the threshold for the so called strongly non-linear regime was found to scale as \( \Po^2 = 0.67(\pm 0.31)\Re^{-0.24(\pm 0.04)} \). There are several commonalities between our results and this work: first of all the exponent and the coefficient in front of the Reynolds number are very similar. Second, the marginal stability curve in [28] marks the threshold to a secondary instability which indeed applies to our case since the centrifugal instability represents a secondary instability (the first is the breakdown of the directly forced mode). From a physical point of view the fluid filled precessing cylinder studied in our work becomes analogous to a cylindrical annulus when the bulk is dominated by the zonal geostrophic flow which is similar to an internal fluid cylinder at rest. The instabilities described in the present work should differ from the threshold of unstable flow observed experimentally by Goto et al [31] who found the transition between steady and unsteady flow in a precessing prolate spheroid to scale \( \propto \Re^{-4a} \) with \( 0.4 < a < 0.5 \).

The modification of the radial distribution of angular momentum is also related with the emergence of poloidal flow structures. Almost all cases analyzed show a double vortex dominated by the inertial modes \((m,k) = (0,2)\). The only exception is \( \alpha = 60^\circ \) prograde which has four vortices whose formation is rather complex: we observe an evolution from double vortices to quadruple vortices with an opposite sense of rotation compared to the rolls emerged at other nutation angles. This kind of ‘exceptional’ state can occur in strongly non-linear regimes in accordance, e.g. with the precessing cube [23].

The largest poloidal energy appears for the retrograde case with \( \alpha = 75^\circ \) which, in terms of dynamo applications, is the most promising case because the \((1,1)\) energy remains at high level also at large Po (no breakdown of the directly forced mode). As a consequence any violation of the Rayleigh criterion appears to be shifted to quite large precession ratios so that the flow remains centrifugally stable in an extended range (twice the level of the corresponding prograde case). Furthermore the double rolls are the strongest ones. At this precession ratio, the flow forcing is most efficient and we may speculate that it is possible to inject more energy into the flow without breaking the base state. The case \( \alpha = 75^\circ \) \((r)\) proves that, to obtain a strong poloidal flow, it is not necessary to violate the Rayleigh criterion.

The remarkable difference between prograde/perpendicular and retrograde precession for large forcing can be discussed in the mathematical formulation of the problem. In the turntable reference frame the rotation vector \( \pmb{\Omega} \) reads (Albrecht et al 2021)

\[
\pmb{\Omega} = (\pmb{\Omega}_p \cos \alpha) \hat{\pmb{z}} + (\pmb{\Omega}_p \sin \alpha) \hat{\pmb{x}},
\]

(9)

with \( \hat{\pmb{x}} \) being the equatorial coordinate, with the following specifications:

- For perpendicular precession \( \rightarrow \pmb{\Omega} = 0\hat{\pmb{z}} + |\pmb{\Omega}_p|\hat{\pmb{x}} \)
• For prograde precession → \( \Omega = (\Omega_p \cos \alpha) \mathbf{\hat{z}} + |\Omega_p| \sin \alpha \mathbf{\hat{x}} \)

• For retrograde precession → \( \Omega = (-\Omega_p \cos \alpha) \mathbf{\hat{z}} + |\Omega_p| \sin \alpha \mathbf{\hat{x}} \).

We can conclude that the combinations of these two components are responsible for the difference in the flow structures (being the only difference in the Navier–Stokes equation): the equatorial component is sufficient to trigger the breakdown of the forced mode and the rising of the geostrophic zonal flow (perpendicular precession). However the background rotation can either amplify this effect if its sign is positive (prograde motion) or reduce the effect if it has a negative sign (retrograde motion).

The scenario which emerges from the present study is far from trivial. The flow field enclosed in a precessing cylinder evolves through several stages: the emergent geostrophic axisymmetric flow grows with Po modifying the distribution of the flow’s angular momentum. Together with the increase of the geostrophic flow the breakdown/reduction of the directly forced mode occurs. Those two effect can be considered as the first kind of instability discussed in the present work.\(^3\) This process completes with the violation of Rayleigh’s criterion, quite ‘easily’ for prograde precession and hardly for retrograde. Before the onset of the centrifugal instability we observe a peak of the axisymmetric poloidal field. We argue that there is no direct cause-effect relation between the centrifugal instability and the poloidal flow, rather the rising of poloidal flow is attributed to the nonlinear self-interaction of the directly forced mode. Such mechanism was shown by Waleffe \([64]\) and discussed in the nonlinear theory of Meunier et al. \([40]\).\(^3\)

Let us finish with some final remarks and perspectives. Having identified the most interesting configuration in terms of \((\pm \text{Po}, \alpha)\) the next step will be the assessment of the resulting flow using a kinematic dynamo code to test whether the flow field indeed provides an efficient source for dynamo action. Furthermore, the present work should be extended to more extreme regimes; for instance the stability diagram (figure 5) shows a limited range of Reynolds number due to the computational constraints. The use of experimental campaigns could help to overcome this limitations. If we use the scaling law \(\propto \text{Re}^{-1/4}\) as guideline, at what critical precession ratio \(\text{Po}(2\text{c})\) the flow field in the future precession experiment (in the framework of DRESDYN project) will become centrifugally unstable, the braking of the base flow (followed by centrifugal instability) should occur at \(\text{Po}(2\text{c}) \sim [0.045, 0.040]\), when assuming \(\text{Re} \in O(10^7, 10^8)\).

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**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors.

**Appendix A. Impact of \(\alpha\) for low precession ratio: comparison with analytical theory**

In this appendix we focus on the mode \((1, 1, 1)\) which represents a gyroscopic motion resulting from the tendency of the fluid flow to align the flow rotation and the precession axis. This mode belongs to the inertial wave group described in appendix B. We investigate the flow behavior with respect to the nutation angle \(\alpha\) for a range between \([30^\circ, 90^\circ]\) for prograde and retrograde cases. In figure A1(a) the axial velocity for the forced mode \(u_{11}\) increases only in magnitude with \(\alpha\) and for prograde motion it is always larger than for the corresponding retrograde motion. The dominance of the first radial wave number \(n = 1\) is clearly

\(^3\) In general the first type of instability occurring in precessing flows is due to the triadic interaction of the waves which is, however, not in the focus of this paper.
visible since all profiles have the typical shape of the first order Bessel function (diamond-black curve) whose argument is the first root of the dispersion relation $\xi_{111} = 2.7346$ confirming that the (1, 1, 1) mode is substantially dominant for low Po. Next, we analyze the behavior of the directly forced mode’s amplitude in terms of magnitude and phase. Figure A1(b) shows the phase $\psi = \arctan(\mathcal{J}(A_{111})/\Re(A_{111}))$ vs $\alpha$ for two different Reynolds numbers. The increase in the nutation angle provokes a phase shift quite pronounced for the prograde profiles (blue and green curve) while the retrograde profiles are much flatter indicating minor changes in the orientation of the flow structure. This features seem to be rather independent of Re. Then, we focus on the amplitude’s magnitude defining the scaled value as $a_{111} = |A_{111}|/\sqrt{Re|Po|\sin\alpha}$ and compare them with the analytical viscous solution close to resonance [39]. In figure A1(c) are shown the results: our numerical results have smaller values than the analytical viscous theory. Only the prograde case for $Re = 3500$ is close and parallel to the analytical model for $\alpha < 30^\circ$. The reason for this quantitative discrepancy is due to the fact that our results are beyond the limit of validity for the analytical model, i.e. $Po\sqrt{Re \sin \alpha} < O(1)$.

Appendix B. Inertial modes in rotating cylinder

Following the definition by Liao et al [39] we divide the inertial modes (solution of the inviscid-linear Navier Stokes) into three categories.

(a) Inertial waves $(m, k, n) \neq (0, 0, 0)$:

$$\hat{\mathbf{z}} \cdot \mathbf{u}_{mkn} = -\frac{ik\pi}{\omega_{mkn}} J_m(\xi_{mkn}r) \sin(k\pi z/\Gamma) e^{im\phi},$$

$$\hat{\mathbf{r}} \cdot \mathbf{u}_{mkn} = \frac{\Gamma \xi_{mkn}}{4 - \omega_{mkn}^2} \times [(1 + \omega_{mkn}/2) J_{m-1}(\xi_{mkn}r)$$

$$+ (1 - \omega_{mkn}/2) J_{m+1}(\xi_{mkn}r)] \times \cos(k\pi z/\Gamma) e^{im\phi},$$

$$\hat{\mathbf{\phi}} \cdot \mathbf{u}_{mkn} = \Gamma \xi_{mkn} \times [(1 + \omega_{mkn}/2) J_{m-1}(\xi_{mkn}r)$$

$$- (1 - \omega_{mkn}/2) J_{m+1}(\xi_{mkn}r)] \times \cos(k\pi z/\Gamma) e^{im\phi},$$

where $J_m$ is the Bessel function of order $m$ and $\omega_{mkn}$ is the frequency of the inviscid inertial modes which is calculated from

$$\omega_{mkn} = \pm 2 \times \left(1 + \left(\frac{\Gamma \xi_{mkn}}{k\pi}\right)^2\right)^{-1/2},$$

(b) Plot at $z = 0$ of the radial profile of axial velocity $u_1^z$ for $Re = 6500$ and several nutation angles. Diamond curve shows the Bessel function profile for the first radial wave number. (b) Plot of the phase of (1, 1, 1) mode vs nutation angle for two Reynolds number (symbols). (c) Comparison between the general asymptotic theory by Liao et al [39] with our numerical results function of the nutation angle.

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Figure B1. Behavior of the directly forced energy $e_1$ for different Reynolds numbers and $\alpha = 90^\circ$. Notice that the breakdown is clearly expressed only for the largest Reynolds number curves.

$$\xi_{mkn} J_{m-1} (\xi_{mkn}) + \left( \pm 1 + \left( \frac{\Gamma \xi_{mkn}}{k\pi} \right)^2 \right)^{1/2} - 1 \times m J_m (\xi_{mkn}) = 0. \quad (B.3)$$

(b) Geostrophic modes $(m, 0, n)$ as solutions of the Taylor–Proudman problem. They can be sub-divided into: axisymmetric $(m = 0)$ modes with

$$\hat{z} \cdot u_{00n} = 0,$$
$$\hat{r} \cdot u_{00n} = 0, \quad (B.4)$$

and non-axisymmetric $(m \neq 0)$ modes, which may correspond e.g. to the low-frequency mode found by Herault [43]

$$\hat{\varphi} \cdot u_{00n} = J_1 (\xi_{00n}) \quad \text{with } J_1 (\xi_{00n}) = 0$$

$$\hat{z} \cdot u_{mn0} = 0,$$
$$\hat{r} \cdot u_{mn0} = -\frac{i\Gamma}{2r} J_m (\xi_{mn0}) e^{i\varphi}, \quad (B.5)$$
$$\hat{\varphi} \cdot u_{mn0} = \frac{1}{2} \left[ \Gamma \xi_{mn0} J_{m-1} (\xi_{mn0}) - \frac{\Gamma m}{r} J_m (\xi_{mn0}) \right] e^{i\varphi}. $$

The corresponding dispersion relation is

$$J_m (\xi_{mn0}) = 0.$$  

(c) Axisymmetric oscillations $(0, k, n)$ with $k \neq 0$. In contrast to the inertial waves with $(m, k, n) > (0, 0, 0)$ the inertial oscillations’ solution presents the following mathematical structure: poloidal $u_z, u_r$ are purely imaginary and toroidal $u_\varphi$ is purely real

$$\hat{z} \cdot u_{0kn} = -i\frac{n \pi}{\omega_{0kn}} J_0 (\xi_{0kn}) \sin \left( k\pi z / \Gamma \right),$$
$$\hat{r} \cdot u_{0kn} = -i \frac{\Gamma \xi_{0kn}}{4 - \omega_{0kn}^2} J_1 (\xi_{0kn}) \cos \left( k\pi z / \Gamma \right), \quad (B.6)$$
$$\hat{\varphi} \cdot u_{0kn} = -i \frac{\Gamma \xi_{0kn}}{4 - \omega_{0kn}^2} J_1 (\xi_{0kn}) \cos \left( k\pi z / \Gamma \right),$$

where $\omega_{0kn}$ is the frequency of the inviscid inertial modes which is calculated from

$$\omega_{0kn} = \pm 2 \times \left( 1 + \left( \frac{\Gamma \xi_{0kn}}{k\pi} \right)^2 \right)^{-1/2}$$
$$J_1 (\xi_{0kn}) = 0 \quad \text{and} \quad u_{0kn} (\omega_{0kn}) = u_{0kn}^* (-\omega_{0kn}) .$$
This third group is particularly important for the present study. Indeed the imaginary part (only the \(\tau_c, \tau_p\)) represents the axisymmetric double/quadruple roll structure found around the transition (see e.g. figures 6–8).

In figure B1 we show the energy contained in the \((m, k) = (1, 1)\) mode function of Po for several Reynolds numbers. It is straightforward noticing that the steep decrease in the energy does not occur for the smallest Reynolds i.e. \(Re = 500\) and \(Re = 1000\).

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**References**

[1] Le Bars M, Cébron D and Le Gal P 2015 *Annu. Rev. Fluid Mech.* **47** 163–93

[2] Gans R F 1984 *AIAA J.* **22** 1465–71

[3] Stewartson K 1959 *J. Fluid Mech.* **5** 577–92

[4] Sloudsky T 1895 *De la rotation de la terre supposée fluide à son intérieur Bull. Soc. Imp. Natur. Mosc.* IX 285–318

[5] Poincaré H 1910 Sur la précession des corps deformables *Bull. Astron.* **27** 321–56

[6] Busse F H 1968 *J. Fluid Mech.* **33** 739–51

[7] Léorat J 2006 *Magnetohydrodynamics* **42** 143–51

[8] Léorat J, Rigaud F, Vitry R and Herpe G 2003 *Magnetohydrodynamics* **39** 321–6

[9] Malkus W V R 1968 *Science* **160** 3825

[10] Vanyo J P 1991 *Geophys. Astrophys. Fluid Dyn.* **59** 209–34

[11] Dwyer C A, Stevenson D J and Nimmo F 2011 *Nature* **479** 212–4

[12] Noir J and Cébron D 2013 *J. Fluid Mech.* **737** 412–39

[13] Cébron D, Laguerre R, Noir J and Schaeffer N 2019 *Geophys. J. Int.* **219** 534–57

[14] Fu R R, Weiss B P, Shuster D L, Gattacceca J, Grose T L, Suzuet C, Lima E A, Li L and Kuan A T 2012 *Science* **338** 238–41

[15] Tilgner A 2005 *Phys. Fluids* **17** 034104

[16] Tilgner A 2007 *Geophys. Astrophys. Fluid Dyn.* **101** 1–9

[17] Lin Y, Marti P, Noir J and Jackson A 2016 *Phys. Fluids* **28** 066601

[18] Wu C-C and Roberts P H 2009 *Geophys. Astrophys. Fluid Dyn.* **103** 467–501

[19] Cappanera L, Guermond J L and Luddens F 2011 *Phys. Rev. E* **84** 016317

[20] Komoda K and Goto S 2018 *Phys. Rev. Fluids* **3** 043104

[21] Goto S, Matsunaga A, Fujiwara M, Nishioka M, Kida S, Yamato M and Tsuda S 2014 *Phys. Fluids* **26** 055107

[22] Krauze A 2010 *Magnetohydrodynamics* **46** 271–80

[23] Giesekar D, Laguerre R, Noir J and Shuster D L 2013 *Geophys. Astrophys. Fluid Dyn.* **107** 345–67

[24] Giesekar D, Laguerre R, Noir J and Schaeffer N 2019 *Phys. Fluids* **31** 014603

[25] Giesekar D, Laguerre R, Noir J and Schaeffer N 2019 *Phys. Fluids* **31** 014603

[26] Manasseh R 1992 *J. Fluid Mech.* **233** 261–96

[27] Kobine J J 1996 *J. Fluid Mech.* **319** 387–406

[28] Lin Y, Noir J and Jackson A 2014 *Phys. Fluids* **26** 046604

[29] Mouhali W, Lehner T, Léorat J and Vitry R 2012 *Exp. Fluids* **53** 1693–700

[30] Komoda K and Goto S 2018 *Phys. Rev. Fluids* **3** 043104

[31] Goto S, Matsunaga A, Fujiwara M, Nishioka M, Kida S, Yamato M and Tsuda S 2014 *Phys. Fluids* **26** 055107

[32] Herault J, Gundrum T, Giesekar A and Stefani F 2015 *Phys. Fluids* **27** 124102

[33] Stefani F, Eckert S, Gerbeth G, Giesekar A, Gundrum T, Steglich C, Weier T and Wustmann B 2012 *Magnetohydrodynamics* **48** 103–14

[34] Stefani F, Albrecht T, Gerbeth G, Giesekar A, Gundrum T, Herault J, Noir C and Steglich C 2015 *Magnetohydrodynamics* **51** 275–84

[35] Stefani F, Gailitis A, Gerbeth G, Giesekar A, Gundrum T, Rüdiger G, Seilmayer M and Vogt T 2019 *Geophys. Astrophys. Fluid Dyn.* **113** 51–70

[36] Burmann F 2020 *PhD Thesis* ETH Zurich

[37] Greenspan H 2017 *The Theory of Rotating Flows* (Cambridge: Cambridge University Press)

[38] Gans R F 1970 *J. Fluid Mech.* **41** 865–72

[39] Liao X and Zhang K 2012 *J. Fluid Mech.* **709** 610–21

[40] Meunier P, Eloy C, Lagrange R and Nadal F 2008 *J. Fluid Mech.* **599** 405–40

[41] Giesekar A, Albrecht T, Gundrum T, Herault J and Stefani F 2015 *New J. Phys.* **17** 113044

[42] Lagrange R, Meunier P, Nadal F and Eloy C 2011 *J. Fluid Mech.* **666** 104–45

[43] Herault J, Gundrum T, Giesekar A and Stefani F 2019 *Phys. Rev. Fluids* **4** 033901

[44] Kong D, Cui Z, Liao X and Zhang K 2015 *Geophys. Astrophys. Fluid Dyn.* **109** 62–83

[45] Kong D, Liao X and Zhang K 2014 *Phys. Fluids* **26** 051703

[46] Jiang J, Kong D, Zhu R and Zhang K 2015 *Phys. Rev. E* **92** 033302

[47] Pizzi F, Giesekar A and Stefani F 2021 *AIP Adv.* **11** 035023

[48] Le Reun T, Favier B and Le Bars M 2019 *J. Fluid Mech.* **879** 296–326

[49] Le Reun T, Favier B, Barker A J and Le Bars M 2017 *Phys. Rev. Lett.* **119** 035002

[50] Seshasayanan K and Gallet B 2020 *J. Fluid Mech.* **901** R5
[51] Greenspan H P 1969 J. Fluid Mech. 36 257–64
[52] Zhang K and Liao X 2017 Theory and Modeling of Rotating Fluids: Convection, Inertial Waves and Precession (Cambridge: Cambridge University Press)
[53] Morize C, Le Bars M, Le Gal P and Tilgner A 2010 Phys. Rev. Lett. 104 214501
[54] Brunet M, Gallet B and Cortet P-P 2020 Phys. Rev. Lett. 124 124501
[55] Kerswell R R 1999 J. Fluid Mech. 382 283–306
[56] Gao D, Meunier P, Dizès S L and Eloy C 2021 J. Fluid Mech. 923 A29
[57] Albrecht T, Blackburn H M, Lopez J M, Manasseh R and Meunier P 2021 J. Fluid Mech. 910 A51
[58] Marques F and Lopez J M 2015 J. Fluid Mech. 782 63–98
[59] Lopez J M and Marques F 2018 J. Fluid Mech. 839 239–70
[60] Giesecke A, Vogt T, Gundrum T and Stefani F 2019 Geophys. Astrophys. Fluid Dyn. 113 235–55
[61] Dudley M L and James R 1989 Proc. R. Soc. A 425 407–29
[62] Xu M, Stefani F and Gerbeth G 2008 J. Comput. Phys. 227 8130–44
[63] Blackburn H M, Lee D, Albrecht T and Singh J 2019 Comput. Phys. Commun. 245 106804
[64] Waleffe F 1989 PhD Thesis Massachusetts Institute of Technology