Dynamic monopolies with randomized starting configuration

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Abstract

Properties of systems with majority voting rules have been exhaustingly studied. In this work we focus on the randomized case - where the system is initialized by randomized initial set of seeds. Our main aim is to give an asymptotic estimate for sampling probability, such that the initial set of seeds is (is not) a dynamic monopoly almost surely. After presenting some trivial examples, we present exhaustive results for toroidal mesh and random 4-regular graph under simple majority scenario.

1 Introduction

Idea of majority voting is commonly used to resolve many problems related to achieving consistency between different parts of distributed computation. For example, majority voting is used to preserve data consistency when updating copies of the same data. Also, it is quite common to use majority voting to resolve inconsistencies in distributed database management. Majority based systems were also successfully used by Agur to examine the plasticity and precision of the immune response in [17].

The model for the system is as follows: Let $G = (V, E)$ be simple undirected graph of size $n$. Every vertex has its color which is either black or white, and represents the state of the node (for example black = faulty, white = not faulty). By $S$ we shall denote the set of black vertices at the beginning of the process. These vertices are also called seeds. By evolution of such system (or, the coloring process) we shall mean the synchronous process where in each step each vertex adjusts its color according to colors of its neighbors and its internal contamination and decontamination rules. (De)contamination rules determine under what configuration of its neighbors the white (black) vertex turns black (white). In this work we focus on the case when there is no decontamination rule, and the contamination rule is the simple majority rule. This means that the white vertex turns black if at least half of its neighbors are black and there is no possibility for black vertex to turn white. Set of seeds $S$ is called dynamic monopoly, (or dynamo for short) if the corresponding coloring process leads to monochromatic black graph. For more rigorous definition, see section 2.1.

A significant attention was given to this model, and many interesting results were obtained. Probably the most basic (but certainly not trivial) question asks for determining the minimal cardinality of a dynamo on a fixed graph $G$ [4, 5]. Another interesting parameter of a dynamo besides its cardinality is the time that is needed for contamination to spread. This was analyzed for the first time in the literature in [4, 5]. Several works are related to more advanced topics such...
as decontamination of the system by external agents [7]. Finally, [9] defines the terminology of immune subgraphs and asks how the immune subgraph of a certain graph looks like.

All these tasks were solved for small class of graphs. Close attention is given to the ring and its several modifications [4], as well as to the torus and its modifications [5] (toroidal mesh, torus cordalis, torus serpentinus). Many results exist also for hypercube and binary tree.

Other possible questions ask about the minimal cardinality of a dynamo on arbitrary graphs. It is proved that on general directed graph the minimal dynamo has at most $0.727|V|$ vertices. It is likely that this result will be improved. On the other hand, in the case of general undirected graphs the minimal dynamo consists of at most $|V/2| (+1)$ vertices (depending on the exact specification of contaminating rules), this result is proved to be sharp [18].

There exist some interesting results about such systems even in the ”most general” case, where contamination rule is that the vertex is contaminated if at least fraction $\alpha$ of its neighbors are black [20, 21, 22]. These works ask under what condition at least some fraction $\delta$ of all vertices is turned black.

Finally, we mention several works that we find most related to the paper. For a randomly chosen set of initial black vertices, Gleeson and Cahalane [21] gave an exact formula for the expected fraction of black vertices at the end in tree-like graphs. In [20, 23] authors gave their estimate for minimal number of black vertices needed for re-coloring of at least fraction $\delta$ of all vertices on Erdős-Rényi random graph. For more thorough survey, see [10, 8].

Motivation for studying coloring process induced by random set of seeds is quite straightforward. For example, considering vertices as computing nodes, each node can fail with probability $p$, independent on failure of other nodes. Although this looks like a very common scenario, there exists only few results about systems initialized with random initial coloring. Given graph $G$ We consider random initial set of seeds $S^p = S^p(G)$ as the set containing any vertex of $G$ with probability $p$ (independently on other vertices). Naturally, for every fixed graph $G$ this gives us some probability that the random set of seeds is a dynamo. Determining these probabilities analytically is quite hard (if not impossible) and moreover, it can be done numerically with sufficient accuracy. Therefore, we shall try to obtain asymptotic results of the form: assuming 4-regular graph with $n$ vertices, random set of seeds $S^{0.12}$ is dynamo with high probability (w.h.p.). On the other hand, $S^{0.10}$ is not a dynamo (w.h.p.).

In many situations, to determine the minimal value of $p$ that $S^p$ is (w.h.p) dynamo is trivial. For example, assuming Erdős-Rényi random graph $G(n, p')$ for fixed $p'$, $S^p$ forms a dynamo almost surely (a.s.) if $p > 1/2 + \varepsilon$ (where $\varepsilon > 0$ is arbitrarily small constant) and does not form a dynamo a.s. if $p < 1/2 - \varepsilon$ (this follows from Chernoff bounds and Markov inequality). If in the same model we allow $p'$ to be dependent on $n$, but not to be significantly decreasing, the same results can be easily derived. Finally, if $p'$ decreases significantly with $n$ (for example $p' = c/n$), the graph contains (with probability tending to some $\gamma > 0$) some isolated vertices. Therefore until $p$ is not quite close to 1 (so close that all isolated vertices are a.s. seeds) we can not say that $S^p$ is w.h.p. dynamo. Another trivial example is the toroidal mesh under strict majority contamination rule (that is the white vertex turns black only if it has at least three black neighbors). In this case, any square of size $2 \times 2$ forms an immune subgraph (that is such subgraph that turns black only if some of its vertices are seeds). Therefore, the sampling probability that would form dynamic monopoly must prevent all such squares from being colored entirely white. Once again, this can be done only for $p$ very close to 1.

However, the motivation presented in previous text leads to considering $p$ to be a fault probability of a node in the network (or something alike). Therefore we can assume $1 - p$ not to be very small. This makes the cases where $p \to 1$ non-realistic.

In this work we shall examine dynamic monopolies with random initial condition on two types of underlying graphs. In the section 2 we focus on the toroidal mesh. The results are quite
surprising - $S^p$ containing only a fraction $o(1)$ of all the vertices a.s. form a dynamo. To be more concrete, we shall show that there exist constants $\alpha, \beta$ such that if $p > \alpha / \ln(n)$, then $S^p$ a.s. is a dynamo. Similarly, when $p < \beta / \ln(n)$, then $S^p$ a.s. is not a dynamo. At the end of the section we present our attempt to solve the problem numerically and we show ”measured” $\alpha$ and $\beta$. As it was said earlier, to form a dynamo on random graph $G(n, p)$ we need $p$ to be close to $1/2$. This makes the results about the toroidal mesh even more interesting. The natural question arises, whether the reason that such low values of $p$ are needed to form a dynamo comes from specific topology of toroidal mesh, or whether it is just implied by the fact, that the degree of the vertices is constant and low. This is the motivation for the section 3, where we investigate the same questions but on random 4-regular graphs, that is the graphs with all the vertices having degree 4. We show that if $p \geq 0.12$, then the random initial coloring a.s. forms a dynamo. Similarly, if $p \leq 0.10$, then the random initial coloring a.s. does not form a dynamo.

2 Toroidal mesh

2.1 Preliminaries

By toroidal mesh of size $n$ we shall consider an undirected graph $G = G(V, E)$ consisting of $n^2$ vertices labeled as $V[i, j]$ for $0 \leq i, j \leq n$. The set of edges consists of all pairs $(V[i, j], V[(i + 1) \mod n, j])$ and $(V[i, j], V[i, (j + 1) \mod n])$ for all $0 \leq i, j < n$. By rectangle of size $w \times h$ located at position $x, y$ we denote the subgraph of $G$ induced by vertices $V[i, j]$ for $x \leq i < x + w$ and $y \leq j < y + h$. Such rectangle will be denoted as $V[x, y, w, h]$. The set of all vertices of the rectangle $R$ having degree (in $R$) lower than 4 is called circumference of $R$. Rectangles with size $w \times 1$ or $1 \times h$ are called lines, rectangles with $w = n$ or $h = n$ are called vertical or horizontal stripes. Note that circumference of a stripe consists of two closed lines.

Let $\psi: V \to \mathbb{N}$ and $S \subseteq V$ be the set of seeds. By coloring process induced by $S$ we shall denote progression of sets $\mathcal{N}(S, G, \psi) = S_0, S_1, \ldots$ where $S_0 = S$ and $S_i \subseteq V$ represents the set of black vertices in the $i$-th step of the coloring process. The relation between $S_i$ and $S_{i+1}$ is that $S_{i+1} = S_i \cup B_{i+1}$, where $B_{i+1} \subseteq V$ is the set of all vertices $v$ that are white in the $i$-th step of the coloring process (that is, $v \notin S_i$) and they have at least $\psi(v)$ neighbors that are black in the $i$-th step of the coloring process (that is, they belong to $S_i$). Furthermore, we shall set $B_0 = S$ and we shall say that vertices from $B_i$ are turned black in time $i$. Vertex $v$ is said to be turned black (by the coloring process) if it is turned black in some time. Since the object of our interest is the 4-regular graph under the simple majority scenario, we shall put throughout the paper $\psi(v) = 2$ for all $v \in V$. If for some $i$, the coloring $S_i$ contains all vertices $V$, then we say that the corresponding $S$ is dynamic monopole or dynamo (with respect to $G$ and $\psi$).

Vertex that belongs (does not belong) to set of seeds $S$ shall be called $S$-vertex (non-$S$-vertex). Similarly, rectangle (line) consisting of $S$ vertices (non-$S$-vertices) is called $S$-rectangle ($S$-line, non-$S$-rectangle, non-$S$-line).

We want to analyze the coloring process with random initial condition. For this purpose, we define the random set of seeds $S^p = S^p(G)$. This is obtained as follows: Every vertex $v$ of $G$ is contained in $S^p$ with probability $p$, independently on the other vertices. All possible set of seeds form together with adequate probabilistic measure a probabilistic space of random set of seeds $S^p = S(G)^p$. We put $S^p \in S^p$ throughout the text. Naturally, probability $p$ can depend on $n$. Since our aim is to obtain an asymptotic description of the behavior of the system, all limits we use are for the case $n \to \infty$. We shall say that event $A$ happens with high probability (w.h.p.), if $\lim_{n \to \infty} \mathbb{P}(A) \to 1$. Alternatively we shall say that $A$ happens almost surely (a.s.).

For abbreviation we shall put $l = l(n) = \lfloor \ln(n) \rfloor$ and $q = 1 - p$. 


2.2 Lower bound

In this section we show that unless \( p \) is high enough, there w.h.p. exists covering of \( G \) by cages (for the definition of cage see below), that prevents the black vertices from spreading. The result of this section is stated in the following lemma and proved at the end of the section.

**Theorem 2.1.** If \( p \leq 0.012 / \ln(n) \), then the random set of seeds \( S^p \) w.h.p. is not a dynamo on graph \( G \).

In this section we shall assume \( n \) to be divisible by \( 2l \). We shall show that this is only a technical complication at the end of this section.

Let \( G \) be a toroidal mesh of size \( n \times n \) and let \( S^p \in S \) be random set of seeds. From this graph we can derive corresponding graph \( G' \) and corresponding set of seeds \( S' \) as follows: \( G' \) is a toroidal mesh of a size \( n/2 \times n/2 \). We shall say that \( V'[x, y] \) corresponds to foursome of vertices \( V[x, 2y], V[2x + 1, 2y], V[2x, 2y + 1], V[2x + 1, 2y + 1] \). Vertex \( v' \in V' \) belongs to \( S' \) if at least one of its corresponding vertices is \( S \)-vertex. Note that probability \( p' \) that the vertex in \( G' \) is \( S' \)-vertex is \( p' = 1 - (1 - p)^4 \leq 4p \) and for \( p \to 0 \) we get \( p' \sim 4p \).

**Definition 2.2.** By cage we denote the circumference of some rectangle \( R \) in \( G' \) consisting of non-\( S' \)-vertices only. In more detail, for any integers \( x, y, w \leq n, h \leq n \) if the vertex of \( G' \) on the coordinates \( [x+i, y], [x+i, y+h], [x, y+j], [x+w, y+j] \) is not member of \( S' \) for all \( 0 \leq i < w \) and \( 0 \leq j < h \), then these vertices form cage in \( G' \) and vertices corresponding to these form cage in \( G \).

By the interior of the cage we denote those vertices of \( R \) that do not lie on the circumference.

**Lemma 2.3.** If there exists a set \( K \) of cages such that

- every \( v' \in S^p \) belongs to the interior of some cage \( K \in K \);
- for \( K_i, K_j \in K \) and \( K_i \neq K_j \) the interiors of the cages \( K_i \) and \( K_j \) are disjoint,

then \( S^p \) is not dynamic monopoly on \( G \).

**Proof.** We can assume all seeds to be ”trapped” inside cages which they can not escape from. More rigorous proof can be done by induction on the number of steps of the coloring process. \( \Box \)

Let us consider rectangle \( R = V'[x, y, w, h] \) of \( G' \). We shall call this rectangle good, if

- \( w \geq h \) and there exists a non-\( S' \)-line \( V'[x, y+i, w, 1] \) for some \( 0 \leq i < h \) of \( R \), or
- \( w < h \) and there exists a non-\( S' \)-line \( V'[x+i, y, 1, h] \) for some \( 0 \leq i < w \) of \( R \).

If none of the above holds, we shall call \( R \) bad.

**Lemma 2.4.** Let \( \alpha \) be arbitrarily low positive number, \( \beta \) the root of the equation

\[
\ln(1 - e^{-12\beta}) = -2
\]

and let \( p \) be equal to \( (\beta - \alpha) / \ln(n) \). Then \( G' \) a.s. does not contain bad rectangle of size \( l \times 3l \) or \( 3l \times l \).
Proof. As was mentioned before, $p' \sim 4p = 4(\beta - \alpha)/\ln(n)$ The probability that the rectangle with width $w$ and height $h$, $w > h$ is bad equals to:

$$p_{\text{one rec}} = (1 - (1 - p')^w)^h.$$ 

There are $2n^2$ different rectangles of given dimensions, therefore the probability that there will be at least one bad rectangle of given dimensions can be upper-bounded using union bound by:

$$p_{\text{all recs}} \leq 2n^2 \cdot p_{\text{one rec}}.$$ 

By letting $w = 3l$, $h = l$, and using $(1 - p')^w \sim e^{-12(\beta - \alpha)}$ we immediately get

$$p_{\text{all recs}} \to 0 \quad \text{as} \quad n \to \infty.$$

Now we show, how it is possible to satisfy preconditions of Lemma 2.3. Assuming that $n$ is divisible by $2l$ we divide $G'$ to $n/(2l)$ vertical stripes of width $l$:

$$Z_0 = V'[0, 0, l, n/2]$$
$$Z_1 = V'[l - 1, 0, l, n/2]$$
$$Z_2 = V'[2l - 1, 0, l, n/2]$$
$$\ldots$$

In each of these stripes we find closed path consisting of non $S'$ vertices that goes ”around” this strip. This will be done as follows: In every stripe $Z_i$ we construct a sequence of rectangles $R_{i,1}, R_{i,2}, \ldots$ as in the top-left part of Figure 1. Due to Lemma 2.4 in each $R_{i,j}$ there is a non-$S'$-line $L_{i,j}$. Suitable parts of these lines form desired path as is shown in the figure. Let us denote the path that goes around the stripe $Z_i$ by $P_i$ and let $Q_i = \{L_{i,j} | j \in \mathbb{N}\}$. Let us now consider all horizontal lines from $Q_i \cup Q_{i+1}$. These clearly divide the whole area between paths $P_1$ and $P_2$ to cages with disjunct interiors. By repetition of this argument we can show that the area between any two paths $P_i$ and $P_{(i+1) \mod n}$ can be covered by cages with respect to preconditions of Lemma 2.3. For proof of Theorem 2.1 it is therefore sufficient to calculate appropriate $\beta$ and $p$ that would satisfy precondition of Lemma 2.4.

The very last thing needed to finish the proof of Theorem 2.1 is to deal with such toroidal meshes that do not have its size divisible by $2l$. We shall use the following intuitive lemma:

**Lemma 2.5.** Let $H$ be the toroidal mesh, $S_H$ the set of seeds, and let $t$ be such a time after which no vertex changes its color to black. Let $S_H^*$ be another set of seeds satisfying $S_H \subset S_H^*$ and let us denote $D := S_H^* - S_H$. Then every vertex that turns black in $N(S_H^*, H, 2)$ after time $u > t$ is distant at most $2u$ from some vertex of $D$.

We assume that $n = 2lk + r$ for some $k$ and $r < 2l$. As before, we divide $G$ to vertical stripes with width $2l$ (note that since we do not assume $n$ even as before, we are working with $G$ instead of $G'$) and one special stripe with width $r$. As above, whole graph $G$ up to the special stripe can be covered by cages. Every such a cage has the size of its interior limited to $(2l) \cdot (6l) = 12l^2$. Therefore, if the special stripe contains only non-$S^p$-vertices, then in time higher than $12l^2$ no vertex is turned black. Therefore the preconditions of the Lemma 2.5 are satisfied with $H = G$, $D$ equal to the set of all $S^p$-vertices from the special stripe, $S_H^* = S^p$, $S_H = S_H^* - D$ and $t = 12l^2$. Therefore we know that every vertex that turns black after time $t$ must lie within the distance $2t$ from some black vertex from the special stripe. Note that dividing $G$ to the stripes can be
Figure 1: Left: Three consecutive stripes. In every (dashed) rectangle there exists a non $S'$ line (bold lines). These lines form path around the stripe. Right-top: Black rectangle $R$ will expand itself to $ABCD$ because of incrementing vertices $v_i$. Right-middle: Every square $2l \times 2l$ (dashed) w.h.p. contains this configuration of non $S'$ lines (bold lines). Bottom: results of a numerical study.
done in many ways - the special stripe can be located at arbitrary position. If we assume that special stripe equals to $V[0,0,r,n]$, then every vertex that turns black after time $t$ must has its $x$ coordinate within the interval $(-2t, 2t + r)$. On the other hand, if the special stripe equals to $V[n/2], 0, r, n]$, then every vertex that turns black after time $t$ must has its $x$ coordinate within the interval $([n/2] - 2t, [n/2] + 2t + r)$. Since $t = o(n)$ and $r = o(n)$, these two intervals do not overlap and therefore no vertex turns black in time $t + 1$. This implies that no vertex turns black anymore and therefore the initial coloring $S^p$ is not a dynamo.

2.3 Upper bound

The result of this section is stated in the following theorem:

**Theorem 2.6.** If $p \geq 1.65/\ln(n)$, then the random set of seeds $S^p$ w.h.p. is a dynamo on graph $G$.

We want to illustrate the main idea of the proof first. The rigorous proof is presented after few technical lemmas. Let $R$ be such square that in some step of the coloring process consists of black vertices only. If in each one of the four dashed lines (Figure 1 top-right) there exists at least one seed (as are $v_1, v_2, v_3, v_4$ in the figure), then also the square $ABCD$ will be black at some time. We shall call vertices $v_1, v_2, v_3, v_4$ incrementing with respect to $R$. Expansion of $R$ caused by incrementing vertices can possibly continue until whole $G$ turns black. We shall show that we can choose $R$ such that with probability tending to one this really will happen.

Let $h = h(n)$ such that $h(n)$ is odd. Let us divide $G$ to the set of $[n/h]^2$ disjoint squares $U_i$ such that the size of $U_i$ is $h \times h$. The event that $G$ turns black at some point in the coloring process is implied by existence of specific square $U_k$ that satisfies:

- $\Gamma_1$: A square consisting of single vertex $v$ in the middle of $U_k$ can grow due to the existence of incrementing vertices from $S^p$ in such a way that in some time of the coloring process the whole $U_k$ will be black. Note that the square of size $1 \times 1$ can grow due to incrementing vertices to the square of size $3 \times 3$ even if it is not black.

- $\Gamma_2$: Square $U_k$ will grow due to the existence of incrementing vertices from $S^p$, and at some time whole $G$ turns black.

Let us note that although it is tempting to combine the conditions $\Gamma_1$ and $\Gamma_2$, it can not be done easily. The reason is that for two different $U_i, U_j$, the corresponding $\Gamma_1$’s are independent events. On the other hand, the ”spreading” of the square $U_i$ to the whole $G$ is influenced by (not) spreading of other squares $U_j$.

We shall estimate probabilities of conditions $\Gamma_1$ and $\Gamma_2$ right after stating following technical lemma.

**Lemma 2.7.** Let $p > c/l$ for some constant $c$. Then every line of length $L := l^3$ contains w.h.p. at least one $S^p$-vertex.

**Proof.** The probability that the fixed line of length $L$ is non-$S^p$-line is

$$p_{\text{one line}} = q^L \sim e^{-pL} = e^{-c \cdot l^2}.$$  

With the use of the union bound we get that the probability that at least one line of length $L$ is non-$S^p$-line can be bounded as

$$p_{\text{all lines}} \leq 2n^2 \cdot p_{\text{one line}} \to 0.$$
Lemma 2.8. Let us assume that $h \to \infty$. Then the condition

$$\ln(n) - \frac{q \cdot \pi^2}{6p} - \ln(1/p) - \ln(h) \to \infty$$

w.h.p. implies that $\Gamma_1$ happens for arbitrarily large number of $U_k$'s.

Proof. Let $A_i$ denote the event that for given $U_i$ the condition $\Gamma_1$ holds. Let $X$ be the random variable that equals to the number of $A_i$'s holding. There are at least $\lfloor n/h \rfloor^2 \sim (n/h)^2$ different $U_i$'s and therefore the expected value of $X$ satisfies the relation $\mathbb{E}(X) \sim (n/h)^2 \mathbb{P}(A_i)$. Since the $A_i$’s are independent, $\mathbb{E}(X) \to \infty$ implies that w.h.p. at least one of the $A_i$’s holds. Let us estimate $\mathbb{P}(A_i)$ (for fixed $U_i$) as follows:

$$\mathbb{P}(A_i) \geq V_{\text{odd}}$$

where

$$V_{\text{odd}} = \prod_{i \geq 1, i \text{ is odd}} (1 - q^i)^4$$

Furthermore we define

$$V_{\text{even}} = \prod_{i \geq 1, i \text{ is even}} (1 - q^i)^4 \quad \text{and} \quad V_{\text{all}} = V_{\text{even}} \cdot V_{\text{odd}}$$

Note that $V_{\text{odd}} < V_{\text{even}}$ but $V_{\text{odd}} > V_{\text{even}} \cdot (1 - q)^4 = V_{\text{even}} \cdot p^4$. Therefore $V_{\text{odd}} \geq \sqrt{V_{\text{all}}} \cdot p^2$.

Further, we compute:

$$V_{\text{all}} = \prod_{i=1}^{\infty} (1 - q^i)^4 = \exp\left(4 \sum_{i=1}^{\infty} \ln(1 - q^i)\right) = \exp\left(-4 \sum_{i=1}^{\infty} q^i + q^2/2 + q^3/3 + \ldots \right) =$$

$$= \exp\left(-4 \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{q^{ki}}{k}\right) = \exp\left(-4 \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{q^{ki}}{k}\right) =$$

$$= \exp\left(-4 \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{q^k}{1 - q^k}\right) = \exp\left(-4 \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{q^{-k} - 1}\right)$$

where in the second step we used Taylor’s expansion for $\ln(1+x)$. Furthermore, we use Bernoulli’s inequality:

$$q^{-k} - 1 = (1+z)^k - 1 \geq k z$$

where we put $1+z = 1/q$. Using this we obtain

$$\ln(V_{\text{all}})/4 \geq - \sum_{k=1}^{\infty} \frac{1}{z k^2} = - \frac{1}{z} \sum_{k=1}^{\infty} \frac{1}{k^2} = - \pi^2/6 \cdot \frac{1}{z} = - \pi^2/6 \cdot \frac{q}{p}.$$ 

Finally, $\mathbb{E}(X) \to \infty$ can be written as:

$$(n/h)^2 \cdot \mathbb{P}(A_i) \geq (n/h)^2 \sqrt{\exp\left(-4 \cdot \frac{\pi^2 \cdot q}{6p}\right) \cdot p^2} \to \infty$$

Calculating logarithm of the last relation proves the lemma. \( \Box \)

Proof. (Theorem 2.6). Let us choose $h = 2l^3 + 1$ and let $p$ be as in the statement of Theorem 2.6. The direct calculation shows that the preconditions of Lemma 2.8 are satisfied, and therefore there a.s. exists $U_k$ that satisfies the condition $\Gamma_1$. Moreover, since $h > l^3$, Lemma 2.7 implies that there w.h.p. exist all incrementing vertices needed for the condition $\Gamma_2$ to hold. \( \Box \)
Let us note that both Theorems 2.6 and 2.1 can be sharpened a little. When improving Theorem 2.1 we can show (with some additional effort) that for \( p \leq 0.018/\ln(n) \) every square of size \( 2l \times 2l \) a.s. contains four non-\( S^2 \)-lines such as in the Figure 1 (middle-right). Existence of such structures is sufficient for showing the existence of covering of \( G \) by cages as required by Lemma 2.3. For improvement of Theorem 2.6, let us note that we do not actually need all incrementing vertices whose existence is included in (1) in Lemma 2.8. For successful ”spreading” of black vertices it is sufficient if only approximately half of them exist. Let us recall that in Lemma 2.8 we started with \( U \) whose existence is included in (1) in Lemma 2.8. For successful “spreading” of black vertices it is improvement of Theorem 2.6, let us note that we do not actually need all incrementing vertices \( U \) whose existence is included in (1) in Lemma 2.8. For successful “spreading” of black vertices it is sufficient if only approximately half of them exist. Let us recall that in Lemma 2.8 we started with \( 1 \times 1 \) square and this square potentially grew until the whole \( S_k \) became black. The expansion of this square of size \( i \times i \) to \( (i + 2) \times (i + 2) \) requires the existence of the set of four incrementing vertices \( U_1 \). Another set of incrementing vertices \( U_2 \) then causes expansion to size \( (i + 4) \times (i + 4) \). But in almost all situations, \( U_2 \) can cause expansion from \( i \times i \) to \( (i + 4) \times (i + 4) \) directly - without need of \( U_1 \). This leads to \( p \geq 0.83/\ln(p) \).

This considerations sharpen the results, but it is still an interesting question, whether there exists a threshold \( t \) such that if \( t' > t \) and \( p > t'/\ln(n) \), then the initial random set of seeds is a dynamo and vice-versa. Sadly, we are not able to prove the (non)existence of such threshold and determine its possible value. Therefore we present a numerical study that tries to answer these questions at least partially. By \( p_z(n) \) let us denote the probability \( p \) such that the random set of seeds \( S^p \) is a dynamo with probability \( z \). Results of the numerical study are presented in the Figure 1 (bottom). The circles present values of \( p_{0.5}(n) \). The top and the bottom points of the ”error bars” correspond to values \( p_{0.05}(n) \) and \( p_{0.95}(n) \). Size of 90% confidence intervals for measured values is comparable with the diameter of the circles.

We conclude that threshold \( t \) is likely to exist and its value is near 0.25. Proving this statement rigorously remains an open problem.

## 3 Dynamos on random 4-regular graph

In the previous section we analyzed a particular 4-regular graph. In this section we try to solve a similar task for a random 4-regular graph under simple majority scenario.

Random regular graphs are very well studied. Pioneering studies in this field were brought by Bender and Canfield [12], Bollobás [13] and Wormald [14, 15]. A systematic research in this area grew enormously since then, partly driven by applications in many areas such as computer science. For exhaustive survey see [16]. To make this paper self-contained, we now present the definition and basic properties that are necessary for further reading.

We use \( G_{n,d} \) to denote the uniform probability space of \( d \)-regular graphs on the \( n \) vertices \( \{1, 2, \ldots, n\} \) (where \( dn \) is even). Sampling from \( G_{n,d} \) is therefore equivalent to taking such a graph uniformly at random (u.a.r.). Another possible way how to define (the same) probabilistic space is this: We construct graph with \( n \) vertices (named as before) and no edges. Then, for every vertex \( i \) we construct \( d \) slots \( v_{i,1}, \ldots, v_{i,d} \). A perfect matching of these slots into \( dn/2 \) pairs is called a pairing. A pairing \( P \) corresponds to a multigraph (with loops and multiple edges permitted), in which two vertices \( i \) and \( j \) are connected, if there exist two different slots \( v_{i,x} \) and \( v_{j,y} \) that form a pair from \( P \). Although this process can lead to graph that is not simple, it is quite easy to show that the probabilities of obtaining two simple graphs \( G_1 \) and \( G_2 \) are equal. Therefore if we reject every graph that is not simple and repeat the whole process until simple graph is found, we obtain the same probabilistic space \( G_{n,d} \) as before.

A pairing can be selected u.a.r. in many different ways. In particular, the slots in the pairs can be chosen sequentially. At any stage, the first slot in the next random chosen pair can be selected using any rule whatsoever, as long as the second slot in that pair is chosen u.a.r. from the remaining slots. For example, one can insist that the next chosen slot is the next one available
in any pre-specified ordering of the slots, or comes from a cell containing one of the slots in the previous chosen pair (if any such slots are still unpaired). We shall call this the independence property of the pairing model.

In what follows \( G \in G_{n,A} \) will be random 4-regular graph and \( l = \lfloor \ln(n) \rfloor \) as before. Also, we consider the same coloring process as the one described in preliminaries to section 2.

Let us note that we do not need to restrict our interest to simple graphs. The coloring process is well-defined also for multi-graphs. Bender and Canfield showed that the probability of obtaining a simple graph by the pairing-slot process is asymptotically \( \exp((1 - d^2)/4) \) (\( d \) being the degree of a vertex), which is a value close to 2% for 4-regular graph and high number of vertices. Therefore if we prove that \( S^p \) is not a dynamo w.h.p. on random 4-regular multi-graph, it implies that \( S^p \) is not w.h.p. dynamo on random 4-regular graph.

In the future text, let \( G \) be random 4-regular (simple) graph of size \( n \) and \( H \) be random 4-regular (not necessarily simple) graph of size \( n \).

**Theorem 3.1.** Let \( p \geq 0.12 \). Then \( S^p \) w.h.p. is a dynamo on \( G \).

**Proof.** As was justified above, we shall work with \( H \) instead of \( G \). We use some properties of the random regular graph that are summarized in [16]. The graph has only small number of short cycles. Therefore, the neighborhood of any vertex looks like a part of an infinite tree. For \( v \in V(H) \), let \( T_v \) be a rooted tree that is obtained as follows: Vertex set of \( T \) is a subset of \( V(H) \), vertex \( v \) is the root. Tree \( T \) consists of \( h = \lceil 2 \cdot \lg \log(n) \rceil \) levels. Vertex \( v \) forms 0-th level of the tree. The \((i+1)\)-th level consists of all vertices that are neighbors in \( H \) to some vertex in the \( i \)-th level and that are not already contained in the \( j < i \)-th level for any \( j \). If \( u, v \) are two vertices of \((i+1)\)-th level respectively, then there is an edge between \( u \) and \( v \) if there is an edge between \( u \) and \( v \) in \( H \) and furthermore, there is no other vertex \( w \) than \( u \) from \( i \)-th level of \( T \) that is adjacent to \( v \) in \( H \). There are no edges within one level of \( T_v \). Note that although some edges from \( H \) are not present in \( T_v \), with the use of the independence property we can easily calculate that the probability that some vertex in \( i \)-th level \((i < h)\) has degree lower than 4 is \( o(1) \).

It is clear that if we run the coloring process on \( T_v \) instead of \( H \) (but we still set the threshold \( \psi \) to 2 for all vertices), then the probability that the root of \( T_v \) turns black in coloring process induced by \( S^p(T_v) \) is lower than the probability that the same vertex \( v \) turns black in \( H \) (in the coloring process induced by \( S^p(H) \)). By \( Y(i) \) let us denote the probability that the vertex of the \( i \)-th level of \( T_v \) turns black (in the coloring process induced by \( S^p(T_v) \)). We shall start analyzing the bottom level of the tree and proceed to the top. It obviously holds that \( Y(h) = p \). For the \((l - i)\)-th level \((i > 0)\) the situation is more complicated. In order to obtain some estimate for \( Y(l - i) \) we shall assume that the color of some vertex \( u \) from \((l - i)\)-th level can be changed only because of (at least) two black neighbors of \( u \) belonging to \((l - i + 1)\)-th level. This gives us the following estimation:

\[
Y(i - 1) \geq p + (1 - p) \cdot f_{3, \geq 2}(Y(i))
\]

where

\[
f_{n,=k}(p) = \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k} \quad \text{and} \quad f_{n,\geq k}(p) = \sum_{i=k}^{n} f_{n,=i}
\]

Numerical calculation shows that given \( p \geq 0.12 \) the condition \( Y(h - 100) \geq 0.999 \) holds. Here we change our approach and analyze higher levels analytically. We easily prove that for any \( p \), \( q = 1 - p \), it holds that:

\[
1 - f_{3,2}(p) \leq 5q^3
\]
Let us denote $\varepsilon_i = 1 - Y(i)$. Then we can rewrite (2) using (3) to the form:

$$Y(i - 1) \geq p + (1 - p) \cdot (1 - 5 \cdot \varepsilon_i^3)$$

And finally, we get: $\varepsilon_{i-1} \leq 5\varepsilon_i^3$. Last inequality together with numerically obtained result $\varepsilon_{h-100} < 0.001$ implies, that

$$\log(\varepsilon_{i-1}) \leq \log(5) + 3 \log(\varepsilon_i) \leq 2 \log(\varepsilon_i).$$

Therefore

$$\log(\varepsilon_0) \leq -(2^h - \Theta(1)) = -\log(n)^2/\Theta(1) \leq -2 \log(n)$$

for sufficiently big $n$ and finally $\varepsilon_0 \leq n^{-2}$. This means that the expected value of number of vertices from $H$ that do not turn black is $o(1)$. \qed

**Theorem 3.2.** Let $p \leq 0.10$. Then $S^p$ w.h.p. is not a dynamo on $G$.

Before we present the actual proof of this theorem, let us recall several facts concerning the Galton-Watson branching process.

Let $Z$ be a probability distribution on the nonnegative integers. The Galton-Watson branching process with offspring distribution $Z$ is (loosely) defined as follows. The 0-th generation consists of $N$ particles. For $t \geq 0$, the $(t+1)$-th generation consists of the children of all particles in the $t$-th generation. For each particle, the number of its children has distribution $Z$, and is independent of all the other particles.

Let $X$ be a (Galton-Watson) branching process with $N$ particles in the beginning, distribution $Z$ and we shall denote the number of particles in generation $t$ by $X_t$ and the average number of children of a particle by $\lambda (= \mathbb{E}(Z))$. Moreover, let

$$X_{\text{sum}} := \sum_{i=0}^{\infty} X_i.$$

**Lemma 3.3.** For all $t \geq 0$,

$$\mathbb{E}(X_t) = N \cdot \lambda^t \quad \text{and if } \lambda < 1 \text{ then } \mathbb{E}(X_{\text{sum}}) = \frac{N}{1 - \lambda}.$$

Furthermore, if $\lambda < 1$, and $\sigma^2$ is the finite variance of $Z$, then

$$\mathbb{P}(X_{\text{sum}} > 2 \cdot \mathbb{E}(X_{\text{sum}})) = o(1).$$

**Proof.** The proof of the first and the second statement is trivial. For the proof of the last statement, we use the first part of the lemma and variance estimation $\text{var}(X_{\text{sum}}) \leq N\sigma^2/(1 - \lambda)^2$ (see for instance [11]). Therefore Chebyshev inequality is applicable, leading to the required result. \qed

Now we are ready to prove the main result of this section.

**Proof.** (Theorem 3.2) As was justified above, we shall work with $H$ instead of $G$. Although an approach quite different from that used in previous section is to be used for proving the Theorem 3.1, there is one similarity: we first numerically compute how the coloring process evolves (w.h.p.) up to the certain time and only then we solve the problem analytically by finding analogy with easily solvable branching process.

In Table 3 we present a randomized algorithm that simulates the coloring process on random 4-regular multi-graph. The algorithm returns 1 with the same probability as is the probability that $S^p$ is a dynamo on random 4-regular multi-graph. Let us recall that the analysis of the coloring process on multi-graphs is sufficient for proving the statement of this theorem.
### Table 1: Sampling-Coloring (SC) Algorithm

| Step | Description |
|------|-------------|
| 1.   | Sample the set \( B_0 \) and let \( R_0 := V(H) - B_0 \). |
| 2.   | Sample the neighbors of all vertices from \( B_0 \). Let \( M_1 \) be those vertices from \( R_0 \) that have one neighbor in \( B_0 \) and let \( B_1 \) be those vertices that have two or more neighbors. Let \( R_1 := R_0 - B_1 - M_1 \). |
| 3.   | Let \( i := 1 \). |
| 4.   | Sample the neighbors of \( B_i \) and denote by \( T_i \) the set of them. |
| 5.   | Let \( T_i^1 \subseteq T_i \cap R_i \) be the vertices that have at least two neighbors in \( B_i \). |
| 6.   | Let \( T_i^2 := T_i \cap M_i \). |
| 7.   | Let \( T_i^3 := T_i - T_i^1 - T_i^2 \). |
| 8.   | Let \( B_{i+1} := T_i^1 \cup T_i^2 \). |
| 9.   | Let \( M_{i+1} := (M_i - T_i^2) \cup T_i^3 \). |
| 10.  | Let \( R_{i+1} := R_i - B_{i+1} - M_{i+1} \). |
| 11.  | Let \( i := i + 1 \). |
| 12.  | If \( B_i = \emptyset \) and \( R_i = \emptyset \) return 1; if \( B_i = \emptyset \) and \( R_i \neq \emptyset \) return 0. |
| 13.  | Go to line 4. |

The main idea behind the algorithm is that the edges in \( H \) are not sampled at once. Instead, in the \( i \)-th step of the coloring process we sample only edges corresponding to slots that belong to black vertices and have not yet been sampled (that is, \( B_i \)). By doing this, the sampling of the graph \( H \) takes place "in the same time" as the black color spreads. Because of independence property, this leads to the same probabilistic space of coloring processes as if we first sample the whole graph \( H \), then the random initial coloring and only then start with the coloring process.

The meaning of used symbols is: \( B_i \) (as defined in the preliminaries) is the set of vertices that turns black in the \( i \)-th step of the coloring process; \( M_i \) is the set of white vertices that have exactly one black neighbor in the \( i \)-th step of the coloring process (excluding \( B_i \)); \( T_i \) are neighbors of \( B_i \) and they are split up to the sets \( T_i^k \) \((k = 1, 2, 3)\) according to their future contribution to one of following sets: \( B_{i+1}, B_{i+1}, M_{i+1} \).

We now calculate some estimations for the cardinalities of \( B_i, R_i \) and \( T_i^k \). For abbreviation we shall use \( x \) for \( |X| \) for all used sets. We shall restrict the considerations to \( i \leq 60 \) and \( p = 0.10 \). Note, that under these assumptions all cardinalities are \( \Theta(n) \). We define \( z_i \) as the ratio of number of empty slots of the vertices from \( B_i \) to the total number of empty slots in the \( i \)-th step, that is:

$$ z_i \leq \frac{2 \cdot b_i}{4 \cdot r_i + 3 \cdot m_i + 2 \cdot b_i} \quad (4) $$

for \( i > 0 \) and \( z_0 = |B_0|/|V(H)| \). By \( f_{n \geq k}, f_{n=k} \) we shall mean the functions defined in the proof of Theorem 2.6. Now the following estimations hold:

**Lemma 3.4.** For abbreviation, let \( X := t_i^k \). Then, if \( r_k, b_k = \Theta(n) \), then

$$ \mathbb{E}(X) = r_k f_{k \geq 2}(z_k) (1 + o(1)) \quad \text{and} \quad \mathbb{E}(X) < E(X)\Theta(1) + \mathbb{E}^2(X) o(1). $$

**Proof.** Let \( X = \sum_i X_i \) where \( X_i \) is an indicator random variable indicating that \( i \)-th vertex from \( R_k \) has at least two neighbors from \( B_k \), that is, it belongs to \( T_k^1 \) (and consequently to \( B_{k+1} \)). The probability that this happens (that is, \( X_i = 1 \)) is clearly \( f_{k \geq 2}(z_k) (1 + o(1)) \). There are \( r_k \) such vertices, which proves the first part of the lemma. For the second part, let us compute:

$$ \mathbb{E}(X^2) - \mathbb{E}^2(X) = \sum_{i,j} \mathbb{E}(X_i X_j) - \mathbb{E}^2(X) = \sum_{i \neq j} \mathbb{E}(X_i X_j) + \mathbb{E}(X) - \mathbb{E}^2(X). $$

$$ \mathbb{E}(X^2) - \mathbb{E}^2(X) = \sum_{i,j} \mathbb{E}(X_i X_j) - \mathbb{E}^2(X) = \sum_{i \neq j} \mathbb{E}(X_i X_j) + \mathbb{E}(X) - \mathbb{E}^2(X). $$
Clearly for \( i \neq j \) holds \( \mathbb{E}(X_iX_j) = (f_{4,2}(z_k))^2(1 + o(1)) \). There are \( r_k(r_k - 1) \) such terms in the sum. The second part of the lemma follows by straightforward calculation.

From this lemma and from Chebyshev inequality we have, that if \( r_k, b_k = \Theta(n) \), then with high probability \( t_k^1 = r_k f_{4,2}(z_k) \gamma \), where \( \gamma \) is some constant that is arbitrarily close to 1. Quite similarly, we can derive such relations for cardinalities of all sets of SC algorithm. This gives us:

\[
\begin{align*}
    b_0 &= p \cdot n \cdot \gamma \\
    r_0 &= n - b_0 \\
    b_1 &= r_0 \cdot f_{4,2}(z_0) \cdot \gamma \\
    r_1 &= r_0 - b_1 - m_1 \\
    t_i^1 &= r_i \cdot f_{4,2}(z_i) \cdot \gamma \\
    t_i^2 &= m_i \cdot f_{3,1}(z_i) \cdot \gamma \\
    m_i &= r_0 \cdot f_{4,1}(z_0) \cdot \gamma \\
    m_{i+1} &= m_i + t_i^3 - t_i^{12} \\
    r_{i+1} &= r_i - b_{i+1} - (m_{i+1} - m_i)
\end{align*}
\]

For proving that the coloring does not form a dynamo we need to upper bound \( m_i \) and \( b_i \) and lower bound \( r_i \). Therefore we put \( z_i \) as big as possible (see (4)), and furthermore we put \( \gamma = 1.0001 \) in all equations above except for the case of \( t_i^2 \), where we use \( \gamma = 0.9999 \). With the use of this approach we obtain the following results:

\[
m_{60} < 0.36n \quad b_{60} < 10^{-8}n \quad r_{60} > 0.41n \quad \left| \bigcap_{i=0}^{60} B_i \right| < 0.24n
\]

The statement of the theorem looks now very persuasive - the fraction of vertices that turn black in the 60-th step of the coloring process is lower than \( 10^{-8} \). However, we can not continue with evaluation of the above equations indefinitely, since we need all quantities to be at least \( \Theta(n) \). Otherwise, it would be hard to bound \( \gamma \) present in these equations. However, we can use the analogy with the branching process discussed by Lemma 3.3. Let us now consider the following conditions:

\[
m_k > 0.37n \quad b_k > 10^{-6}n \quad r_k < 0.40n \quad \left| \bigcap_{i=0}^{k} B_i \right| > 0.25n
\] (5)

We prove that w.h.p. there is no such \( k \) that would satisfy a condition from (5). We already concluded that this is true for \( k \leq 60 \). Let us fix some coloring process on the graph \( H \) such that at least one condition from (5) holds and let \( k' \) be the minimal \( k \) for which this happens. It is easy to calculate that until no condition from (5) holds, the expected number of vertices that turn black in the \((60 + k)\)-th step \((k \geq 0)\) can be upper bounded by the expected number of particles that exists in the \( k \)-th generation of branching process with parameters \( N = 10^{-8}n \) and \( \lambda = 0.9 \). This gives us the equality \( \mathbb{E}(X_{\text{sum}}) = 10^{-7} \). Note also that any of the conditions from (5) implies that the number of the particles in the branching process exceeds \( 100N = 10 \cdot \mathbb{E}(X_{\text{sum}}) \), which by Lemma 3.3 happens with probability \( o(1) \). Therefore, w.h.p. less than 40% from all vertices turn black.

Finally, there are rounding errors in our numerical analysis. Influence of these errors can be estimated by multiplication (or division) of the result of each numerical operation by some number close to 1. In the proof of Theorem 3.2 this is clearly hidden in our choice of \( \gamma \); in the proof of Theorem 3.1 we can simple multiply the right side of (2) by \( 1 - 10^{-6} \) which produces more
significant "error" than the error of the machine. However, the result $Y(h - 100) \geq 0.999$ still holds.

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