A note on Einstein–Sasaki metrics in $D \geq 7$

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Abstract

In this paper, we obtain new non-singular Einstein–Sasaki spaces in dimensions $D \geq 7$. The local construction involves taking a circle bundle over a $(D - 1)$-dimensional Einstein–Kähler metric that is itself constructed as a complex line bundle over a product of Einstein–Kähler spaces. In general, the resulting Einstein–Sasaki spaces are singular, but if parameters in the local solutions satisfy appropriate rationality conditions, the metrics extend smoothly onto complete and non-singular compact manifolds. The seven-dimensional space, whose base is a complex line bundle over $S^2 \times S^2$, is discussed in detail since it has relevance in terms of the AdS/CFT correspondence.

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1. Introduction

Einstein metrics admitting Killing spinors are of considerable interest in string theory and M-theory, since they can provide supersymmetric backgrounds of relevance to the AdS/CFT correspondence [1]. For the case of an Einstein–Sasaki space $X_{2n+3}$, a solution with the geometry $\text{AdS}_d \times X_{2n+3}$ is expected to be dual to a $(d - 1)$-dimensional superconformal field theory with reduced supersymmetry. Such solutions arise in the near-horizon limit of certain $p$-branes located at the tip of the corresponding Calabi–Yau cone $C(X_{2n+3})$ [2–6]. For example, an M2-brane on a cone with special holonomy $SU(4)$ interpolates between $\text{AdS}_3 \times X_7$ and $\text{Minkowski}_3 \times C(X_7)$, which implies that there is an RG flow in the quantum field theoretical picture [5].

Until recently, the known explicit Einstein–Sasaki metrics were relatively sparse. Well-known examples are the round sphere in any odd dimension, and the sphere with the non-standard ‘squashed’ Einstein metric in dimensions $D = 4n - 1$, described as a coset $Sp(n + 1)/Sp(n)$. Other examples include the five-dimensional $T^{1,1}$ space (which is topologically $S^2 \times S^3$) and higher-dimensional analogues. Aside from these isolated examples, which are all homogeneous, there were various existence proofs for further
inhomogeneous Einstein–Sasaki metrics, including, for example, 13 on $S^2 \times S^3$ [7]. The collection of examples increased dramatically recently, with the explicit construction of infinitely many inhomogeneous non-singular Einstein–Sasaki metrics in all odd dimensions $D = 2n + 3 \geq 5$ [8, 9].

An Einstein–Sasaki metric can always be viewed as a circle bundle over an Einstein–Kähler base space, written as
\[
d\hat{s}^2 = (d\psi' + 2A_{(1)})^2 + ds^2,
\]
where $dA_{(1)}$ is proportional to the Kähler form for $ds^2$. (See, for example, [10] for an explicit discussion of this.) The Einstein–Kähler bases $ds^2$ used in [8, 9] are the class of such metrics that were constructed in [13, 14]. These Einstein–Kähler metrics were themselves obtained as two-dimensional bundles over Einstein–Kähler base metrics $d\tilde{s}^2$ of dimension $2n$:
\[
d\hat{s}^2 = \frac{\rho^2}{U(\rho)} + \rho^2 U(\rho)(d\tau' + B_{(1)})^2 + \rho^2 d\tilde{s}^2,
\]
where $dB_{(1)}$ is proportional to the Kähler form for $d\tilde{s}^2$.

The $(2n + 2)$-dimensional Einstein–Kähler metrics obtained in [13, 14] are generally singular. However, this need not necessarily imply that the Einstein–Sasaki metrics on circle bundles over them are singular. Indeed, the main subtlety in the construction of [8, 9] consists in showing that the Einstein–Sasaki metrics can, for suitable choices of parameters, be extended smoothly onto compact manifolds, even though the Einstein–Kähler base spaces by themselves are singular\(^4\).

In this paper, we obtain further examples of non-singular Einstein–Sasaki metrics, by generalizing the construction described above. Specifically, we do this by extending the construction of $(2n + 2)$-dimensional Einstein–Kähler metrics to cases where the $2n$-dimensional base metric is a product of Einstein–Kähler factors $d\tilde{s}^2_i$, rather than a single one:
\[
ds^2 = dr^2 + c^2(d\tau' + B_{(1)})^2 + \sum_i a_i^2 d\tilde{s}^2_i,
\]
where $c$ and $a_i$ are functions of the radial variable $t$. Although we find that the metrics $ds^2$ are generally singular, in certain cases the Einstein–Sasaki metric $d\hat{s}^2$ given by (1) can extend smoothly onto a non-singular manifold, even though the Einstein–Kähler base space is singular.

This paper is organized as follows. In section 2, we give a detailed exposition of the construction for the case where $ds^2$ is a six-dimensional Einstein–Kähler metric constructed as a two-dimensional bundle over a product $S^2 \times S^2$ base. We obtain seven-dimensional Einstein–Sasaki metrics, which is the lowest dimensionality for which the generalization extends beyond the results in [9]. In section 3, we generalize the construction to higher dimensions by using a base space composed of a product of an arbitrary number of Einstein–Kähler spaces, each of arbitrary even dimensionality. Conclusions are presented in section 4.

2. Seven-dimensional Einstein–Sasaki metrics

2.1. The six-dimensional Einstein–Kähler base

We begin by constructing six-dimensional Einstein–Kähler metrics of the form
\[
ds^2_6 = dr^2 + c^2(d\tau' + B_{(1)})^2 + a^2 d\Omega^2_2 + b^2 d\tilde{\Omega}^2_2,
\]
\[^4\] An analogous approach can be used to demonstrate that the $G_2$ holonomy metrics constructed in [11] as $SU(2)$ bundles over singular self-dual Einstein 4-spaces are also complete and non-singular [12].
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where
\[ d\Omega_1^2 = d\theta^2 + \sin^2 \theta \, d\phi^2, \quad d\tilde{\Omega}_2^2 = d\tilde{\theta}^2 + \sin^2 \tilde{\theta} \, d\tilde{\phi}^2, \] (5)

and the connection $\mathcal{B}_{(1)}$ is such that
\[ d\mathcal{B}_{(1)} = p\Omega_{(2)} + q\tilde{\Omega}_{(2)} = -p \, dA_{(1)} - q \, d\tilde{A}_{(2)}, \] (6)

where $\Omega_{(2)}$ and $\tilde{\Omega}_{(2)}$ are the volume forms of the two unit 2-spheres. We shall take
\[ A_{(1)} = \cos \theta \, d\phi, \quad \tilde{A}_{(1)} = \cos \tilde{\theta} \, d\tilde{\phi}. \] (7)

In order that the circle bundle over $S^2 \times S^2$ be well defined, the ratio $p/q$ must be rational, so that the periods dictated for $\tau'$ by the consideration of the bundle over each $S^2$ factor are commensurate. By a rescaling of $\tau'$, one can then, without loss of generality, choose $p$ and $q$ to be relatively prime integers. They characterize the winding numbers of the circle bundle over the two 2-spheres of the base. Without loss of generality, $p$ and $q$ can be taken to be positive. In fact, when we construct the Einstein–Sasaki 7-metrics as circle bundles over these 6-metrics, it will turn out that the ratio $p/q$ no longer needs to be rational.

To impose the Kähler condition on (4), we begin by choosing a complex structure for which the 2-form is
\[ \omega = \frac{1}{2} \left( \Omega_{(2)} - \tilde{\Omega}_{(2)} \right). \] (8)

A necessary condition for Kählerity is then that $d\omega = 0$, implying
\[ \frac{\hat{a}}{a} = \frac{pc}{2a^2}, \quad \frac{b}{b} = \frac{qc}{2b^2}. \] (9)

In fact, in this case it is easily established that this already implies that $\omega$ is covariantly constant, $\nabla_k \omega_j = 0$, and thus (9) constitutes necessary and sufficient conditions for (4) to be Kähler.

Now, we impose the additional requirement that (4) be an Einstein metric. We shall choose the normalization $R_{\mu
u} = 5g^2g_{\mu\nu}$. Calculating the Ricci tensor for (4), we find that the Einstein condition implies
\[
\begin{align*}
-\frac{\ddot{a}}{a} - \frac{a^2}{a} - \frac{2ab}{ab} - \frac{ac}{ac} &= -\frac{p^2c^2}{2a^4} + \frac{1}{a^2} = 5g^2, \\
-\frac{\ddot{b}}{b} - \frac{b^2}{b} - \frac{2ab}{ab} - \frac{bc}{bc} &= -\frac{q^2c^2}{2b^4} + \frac{1}{b^2} = 5g^2, \\
-\frac{\ddot{c}}{c} - \frac{2ac}{ac} &= \frac{p^2c^2}{2a^4} + \frac{q^2c^2}{2b^4} = 5g^2, \\
-\frac{2\ddot{a}}{a} - \frac{2\ddot{b}}{b} - \frac{2\ddot{c}}{c} &= 5g^2.
\end{align*}
\] (10)

Substituting (9) into (10), we find that
\[ \frac{\ddot{c}}{c} = -\frac{pc}{2a^2} - \frac{qc}{2b^2} + \frac{1 - 5g^2a^2}{pc}, \] (11)

together with an algebraic constraint
\[ p - q + 5g^2(qa^2 - pb^2) = 0. \] (12)

Summarizing our results so far, we have shown that (4) is an Einstein–Kähler metric if the following system of equations is satisfied:
\[
\begin{align*}
\frac{\dot{a}}{a} &= \frac{pc}{2a^2}, \quad \frac{\dot{b}}{b} = \frac{qc}{2b^2}, \quad \frac{\ddot{c}}{c} = -\frac{pc}{2a^2} - \frac{qc}{2b^2} + \frac{1 - 5g^2a^2}{pc}, \\
p - q + 5g^2(qa^2 - pb^2) &= 0.
\end{align*}
\] (13)
To solve (13), we introduce a new radial variable \( r \) such that \( dr = c \, dr' \), leading straightforwardly to
\[
a^2 = pr + \ell_1, \quad b^2 = qr + \ell_2, \\
c^2 = \frac{2}{a^2b^2} \int_0^r a^2b^2 \left( \frac{1-5g^2a^2}{1} \right) \, dr' \\
= -\frac{1}{12p(p\ell_1 + \ell_2)(q\ell_1 + \ell_2)} \left[ 15g^2p^2\tau^3 + 4p(5g^2p\ell_2 + 2q\ell_1 - q)\tau^2 \\
+ 6(5g^2\ell_1(q\ell_1 + 2p\ell_2) - (q\ell_1 + p\ell_2))\tau + 60g^2\ell_1^2\ell_2 - 12\ell_1\ell_2 \right]. \tag{14}
\]
The algebraic constraint in (13) becomes
\[
p - q + 5g^2(q\ell_1 - p\ell_2) = 0. \tag{15}
\]
Note that the choice of origin for \( r \) is arbitrary, since \( r \) does not appear explicitly in the equations. We have exploited this by choosing the lower limit of the integration in (14) to be at \( r = 0 \). This implies that the two integration constants \( \ell_1 \) and \( \ell_2 \) in (14) are non-trivial parameters.

### 2.2. The seven-dimensional Einstein–Sasaki metrics

Having obtained the six-dimensional Einstein–Kähler base metrics \( ds^2 \), we can now proceed to the construction, via (1), of the seven-dimensional Einstein–Sasaki metrics. With the Kähler form \( J \) for \( ds^2 \) given by (8), we can introduce the following potential \( A_1(1) \), such that \( J = dA_1(1) \):
\[
A_1(1) = r \, dr' - a^2A(1) - b^2A(1). \tag{16}
\]
The Einstein–Sasaki 7-metric is then given by
\[
d\tilde{s}^2 = k^2(d\psi' + 2A_1(1))^2 + ds^2_6, \tag{17}
\]
where the Einstein condition implies that we must have
\[
k^2 = \frac{5}{8}g^2. \tag{18}
\]

The Ricci tensor of \( d\tilde{s}^2 \) then satisfies \( \tilde{R}_{\alpha\beta} = \frac{15}{4} \tilde{g}^{\alpha\beta} \tilde{g}_{\alpha\beta} \). It is convenient to choose a normalization such that \( \tilde{R}_{\alpha\beta} = 6\tilde{g}_{\alpha\beta} \), implying that \( g^2 = 8/5 \). Hence, \( k = 1 \) and the six-dimensional Einstein–Kähler metric satisfies \( R_{ij} = 8g_{ij} \).

The six-dimensional Einstein–Kähler metrics \( ds^2_6 \) that we obtained in section 2.1 generally do not extend smoothly onto complete non-singular manifolds. We take the radial coordinate \( r \) to range between two zeros of the metric function \( c(r) \), \( r_0 \leq r \leq r_\infty \), for which \( a(r) \) and \( b(r) \) remain non-vanishing. In order for the metric to have a smooth extension, \( c(r) \) must approach zero at the two endpoints at the appropriate rate. This rate determines the period required for \( \tau' \) in order that the metric in the \((r, \tau')\) plane extend smoothly onto \( \mathbb{R}^2 \) at the ‘origin’ \( r = r_\infty \) or \( r = r_0 \). In order for the metric to extend globally onto a smooth manifold, the periods for \( \tau' \) at the two endpoints need to be identical, and must be consistent with that allowed by the requirement of well definedness of the 1-form \( (d\tau' + B_{(1)}) \). These multiple criteria are, in fact, not fulfilled for the six-dimensional Einstein–Kähler metrics \( ds^2_6 \).

Nevertheless, as we mentioned earlier, this does not necessarily imply that the seven-dimensional Einstein–Sasaki metric \( d\tilde{s}^2 \) on the circle bundle over \( ds^2_6 \) is singular. We therefore need to study the global structure of \( d\tilde{s}^2 \) carefully, using techniques of the kind described in [8, 9]. We find that it is appropriate to define new fibre coordinates \( \tau \) and \( \psi \), related to \( \tau' \) and \( \psi' \) by
\[
\psi' = 2\psi, \quad \tau' = \frac{8\tau - \psi}{8\beta}. \tag{19}
\]
In terms of these, we can re-express the Einstein–Sasaki metric (17) as
\[
\begin{align*}
d\tilde{s}^2 = \frac{dr}{c^2} + \frac{c^2}{16(c^2 + 4(\beta + r)^2)} & \left( d\psi - A_{(1)} - \tilde{A}_{(1)} \right)^2 + a^2 d\Omega_2^2 + b^2 d\Omega_2^2 \\
+ \frac{c^2 + 4(\beta + r)^2}{\beta^2} & \left( d\tau - \ell_1 A_{(1)} - \ell_2 \tilde{A}_{(1)} - \frac{c^2 + 4r(\beta + r)}{8(c^2 + 4(\beta + r)^2)} (d\psi - A_{(1)} - \tilde{A}_{(1)}) \right)^2,
\end{align*}
\]
where
\[
\beta = \frac{8\ell_1 - 1}{8p} = \frac{8\ell_2 - 1}{8q}.
\]
(This equation, which follows from a global consideration, will be discussed below.) Note that (21) is consistent with the algebraic constraint (15), since we have made the normalization choice \(g^2 = 8/5\).

The metric has a rescaling symmetry under which \(p \rightarrow \lambda p, \ q \rightarrow \lambda q\) and \(r \rightarrow r/\lambda\). Thus, only the ratio \(p/q\) of the parameters \(p\) and \(q\) is non-trivial. This ratio is determined in terms of \(\ell_1\) and \(\ell_2\) by the algebraic constraint (15), which is rewritten, after setting \(g^2 = 8/5\), in (21). It should be emphasized that in the discussion of the six-dimensional Einstein–Kähler space in section 2.1, \(p/q\) had to be rational in order that the circle bundle over \(S^2 \times S^2\) was non-singular, but that requirement is, as we shall see, no longer necessary when considering the regularity of the Einstein–Sasaki space. The parameters \(p\) and \(q\) need only satisfy the constraint (21). Thus as far as local considerations are concerned, we have a family of Einstein–Sasaki metrics described by the two non-trivial real parameters \(\ell_1\) and \(\ell_2\). They characterize the size of the \(S^2\) bolts.

In our solution (14) for the metric functions \(a, b\) and \(c\), we choose an integration constant so that \(r = 0\) is one of the zeros of the function \(c(r)\). Thus, we take \(r\) to lie in the range \(0 \leq r \leq r_0\), where \(r_0\) is the smallest positive zero of \(c(r)\). (We can always, without loss of generality, choose to consider \(r\) non-negative.) The functions \(a(r)\) and \(b(r)\) should remain non-zero in the entire range \(0 \leq r \leq r_0\). We therefore have the following conditions:
\[
\begin{align*}
\ell_1 > 0, & \quad \ell_2 > 0, \\
r_+ > 0, & \quad c(r_+) = 0, \\
(c^2)'(0) > 0.
\end{align*}
\]
Note that from the last condition, it follows that \(c^2(r) > 0\) for \(0 < r < r_0\), and that \((c^2)'(r_+) < 0\).

To study the global structure, we first consider the base manifold, whose metric is given by the terms appearing in the first line of (20) (i.e. the terms orthogonal to the \(\tau\) fibres). It can be viewed as an \(S^2\) bundle over \(S^2 \times S^2\), where the \(S^2\) bundle is coordinatized by \(r\) and \(\psi\). Without loss of generality, we may assume that \(0 \leq p \leq q\), in which case the positivity of \(a^2\) and \(b^2\), together with the conditions (22), implies that
\[
\frac{1}{8} \left( 1 - \frac{p}{q} \right) < \ell_1 < \frac{1}{8}, \quad 0 < \ell_2 < \frac{1}{8}.
\]

The Killing vector \(\partial/\partial \psi\) degenerates at the points \(r = 0\) and \(r = r_+\) where \(c(r)\) vanishes. In order for the metric to extend smoothly onto these points, it is necessary that the period of \(\psi\) be commensurate with the slope of \(c^2(r)\). Specifically, if \(c^2(r)\) has slope \((c^2)'(r_0) = K(r_0)\) at one of these endpoints, say \(r = r_0\), then writing \(c^2(r) \sim K(r_0)(r - r_0)\) nearby, and defining \(\rho^2 = r - r_0\), we see from (20) that in the \((r, \psi)\) frame we shall have
\[
ds^2 \sim \frac{4}{K} \left( d\rho^2 + \frac{K^2 \rho^2}{256(\beta + r_0)^2} d\psi^2 \right).
\]
This implies that $\psi$ must have period

$$\Delta \psi = \left| \frac{32\pi (\beta + r_0)}{K(r_0)} \right|.$$  \hfill (25)

The periods determined by these conditions at $r_0 = 0$ and $r_0 = r_*$ will agree if (see (14))

$$\beta \left( r_* + \frac{8\ell_1 - 1}{8\rho} \right) = (\beta + r_*) \left( \frac{8\ell_1 - 1}{8\rho} \right),$$  \hfill (26)

which is satisfied if

$$\beta = \frac{8\ell_1 - 1}{8\rho}. \hfill (27)$$

This, together with the relation in terms of $\ell_2$ implied by (15), gives the conditions appearing in (21). Note that (25) now implies that $\psi$ has period $2\pi$.5

The $U(1)$ fibre parametrized by the coordinate $\tau$ in (20) never collapses, and so it follows that the period of $\tau$ is governed only by the connection on the fibre, given by

$$d\tau - \ell_1 A_{(1)} - \ell_2 \tilde{A}_{(1)} = \frac{c^2 + 4r(\beta + r)}{8(c^2 + 4(\beta + r)^2)} (d\psi - A_{(1)} - \tilde{A}_{(1)}). \hfill (28)$$

The global structure can be examined by looking at all the cycles at $r = 0$ and $r = r_*$ where $c^2$ vanishes. They are given by

$$r = 0 : A_{(1)} : 2\pi \ell_1, \quad \tilde{A}_{(1)} : 2\pi \ell_2,$$

$$r = r_* : A_{(1)} : 2\pi \left( \frac{r_*}{8(\beta + r_*)} - \ell_1 \right), \quad \tilde{A}_{(1)} : 2\pi \left( \frac{r_*}{8(\beta + r_*)} - \ell_2 \right). \hfill (29)$$

For the expression in (28) to be globally extendible, the ratios of the above quantities must all be rational. Thus there are two independent requirements, namely

$$\frac{\ell_1}{\ell_2} = \alpha \equiv \text{rational number}, \quad \frac{r_*}{(\beta + r_*)\ell_1} = \gamma \equiv \text{rational number}. \hfill (30)$$

One then solves the cubic polynomial for $r_*$ that follows from setting $c(r_*)^2 = 0$ in (14). Using the two rationality conditions (30), together with (15), enables us to express $\ell_2$ purely in terms of $\alpha$ and $\gamma$:

$$0 = 1536\alpha^2 \gamma^3 \ell_1^3 + 64\alpha \gamma^2 (\alpha(\gamma - 96) - 30\gamma) \ell_1^2 + 8\gamma (32\alpha^2(36 - \gamma) + 27\gamma^2$$

$$+ 4\alpha \gamma (72 + 7\gamma)) \ell_1 \ell_2 + (384\alpha^2(\gamma - 16) - 32\alpha \gamma (24 + 7\gamma)$$

$$- \gamma^2(192 + 29\gamma)) \ell_2 + 48\alpha (16 - \gamma) + 16\gamma (2\gamma - 3). \hfill (31)$$

Appropriate choices of rational values for $\alpha$ and $\gamma$ lead to a countable infinity of solutions for $\ell_2$, which in general is real but not necessarily rational, satisfying the condition $0 < \ell_2 < 1/8$ specified in (23).

5 One might think that different linear coordinate transformations from $(\tau', \psi')$ to $(\tau, \psi)$ could lead to inequivalent results, but it is easy to show that (19) is the unique possibility, up to trivial scalings and shiftings.
2.3. Further remarks

The regular Einstein–Sasaki metrics that we have obtained are parametrized by the two rational numbers $\alpha$ and $\gamma$, subject only to the condition that $\ell_2$ following from (31) satisfy $0 < \ell_2 < 1/8$. In the case where $\ell_1 = \ell_2$, the solutions are included within those discussed in [9].

Although in general $\ell_2$ need not be, and indeed is not, rational, special cases can arise where $\ell_2$ is rational. Since $\ell_1 = \alpha \ell_2$, where $\alpha$ must be rational, it follows from (21) that if $\ell_2$ is rational then $p/q$ is rational. It also follows from (30) that $\beta$, and hence $r_+$, must then be rational too. Using the scaling symmetry discussed previously, one can then choose $p$ and $q$ to be relatively prime integers. In the special case with $(p, q) = (1, 2)$, the polynomial expression for $c^2$ in (14) factorizes, giving

$$c^2 = -\frac{r(r + \ell_2)(128r^2 + 128r\ell_2 + 64\ell_2^2 - 1)}{(2r + \ell_2)(16r + 8\ell_2 + 1)},$$

and hence

$$r_+ = \frac{\sqrt{2 - 64\ell_2^2} - 8\ell_2}{16}.\quad (33)$$

For $r_+$ to be rational, it is necessary that $\ell$, defined by $64\ell^2 + 64\ell_2^2 = 2$, be rational, in which case $r_+$ is given by

$$r_+ = \frac{1}{2}(\ell - \ell_2).\quad (34)$$

Thus, the existence of a rational solution amounts to finding rational solutions for $64\ell^2 + 64\ell_2^2 = 2$, in which one of $\ell$ and $\ell_2$ must be less than $\frac{1}{8}$, and the other greater than $\frac{1}{8}$. Let $\ell_2$ be less than $\frac{1}{8}$. Having a rational solution for $64\ell^2 + 64\ell_2^2 = 2$ is then equivalent to having integer-valued solutions to $x^2 + y^2 = 2z^2$. One can find many integer solutions, by using a computer enumeration, and presumably there are infinitely many. Here, we present a few explicit examples:

$$(\ell_1, \ell_2) = \left(\frac{30}{40}, \frac{1}{40}\right), \quad c^2 = \frac{4r(3 - 40r)(1 + 10r)(1 + 40r)}{5(3 + 40r)(1 + 80r)},$$

$$(\ell_1, \ell_2) = \left(\frac{3}{34}, \frac{7}{136}\right), \quad c^2 = \frac{2r(1 - 17r)(7 + 136r)(15 + 136r)}{17(3 + 34r)(7 + 272r)},$$

$$(\ell_1, \ell_2) = \left(\frac{5}{52}, \frac{7}{104}\right), \quad c^2 = \frac{2r(5 - 104r)(3 + 26r)(7 + 104r)}{13(5 + 52r)(7 + 208r)}.\quad (35)$$

For the cases with $p \neq 2q$, the analysis is much more complicated. For $(p, q) = (1, 3)$, we did not find any rational solutions. It is not clear whether such solutions are intrinsically absent or whether our search was insufficiently exhaustive. For some other values of integer $(p, q)$, we have found isolated rational solutions.

We should again emphasize, however, that the parameters $p$ and $q$ do not need to be rationally related in order that the Einstein–Sasaki metric can be complete.

3. A general class of solutions

In this section, we consider a more general class of Einstein–Sasaki metrics in dimension $D = d + 1$, constructed as circle bundles over $d$-dimensional Einstein–Kähler spaces. The $d$-dimensional Einstein–Kähler space is itself constructed as a complex line bundle over
a product of $N$ Einstein–Kähler spaces, with dimensions $n_i$ and metrics $d\Sigma_{n_i}^2$. Thus $d = 2 + \sum_{i=1}^{N} n_i$, and the $d$-dimensional Einstein–Kähler metric will be written as

$$ds_d^2 = dr^2 + c^2\left( \frac{d\tau}{-\frac{N}{p_1} \sum_{i=1}^{N} p_i A_i^{(1)}} \right)^2 + \sum_{i=1}^{N} a_i^2 d\Sigma_{n_i}^2,$$  \hspace{1cm} (36)

where $J_i^{(2)} = dA_i^{(1)}$ is the Kähler form for the Einstein–Kähler metric $d\Sigma_{n_i}^2$, with cosmological constant $\lambda_i$. The metric (36) is Einstein–Kähler with cosmological constant $\Lambda$, provided that the functions $c$ and $a_i$ satisfy the first-order equations

$$\frac{\dot{a}_i}{a_i} = \frac{p_i c}{2a_i^2}, \quad \frac{\dot{c}}{c} = \frac{\lambda_i - \Lambda a_i^2}{p_1 c} - \frac{1}{2} \sum_{i=1}^{N} n_i \dot{a}_i \dot{a}_i, \hspace{1cm} (37)$$

and the set of independent constraints

$$\lambda_i p_i - \lambda_j p_i + \Lambda (\lambda_j a_i^2 - \lambda_i a_j^2) = 0. \hspace{1cm} (38)$$

Note that there are $(N - 1)$ independent constraints. The solutions can be obtained straightforwardly, given by

$$a_i^2 = p_i r + \ell_i, \quad c^2 = \frac{2}{\prod a_i^{n_i}} \int_0^r \frac{\lambda_i - \Lambda a_i^2}{p_i} \prod a_i^{n_i}, \hspace{1cm} (39)$$

where the coordinate $r$ is defined by $dr = c \, dt$. The integration constants $\ell_i$ satisfy the constraints

$$\beta = \frac{\Lambda p_i - \lambda_i}{\Lambda p_i} = \text{constant}, \quad \text{for all } i. \hspace{1cm} (40)$$

The $D = d + 1 = 3 + \sum n_i$ dimensional Einstein–Sasaki metric is given by

$$ds_D^2 = (d\psi' + 2A_i^{(1)})^2 + ds_d^2, \hspace{1cm} (41)$$

with $A_i^{(1)}$ given by

$$A_i^{(1)} = r \, dt' - \sum_{i=1}^{N} a_i^2 A_i^{(1)}. \hspace{1cm} (42)$$

For the solution to be Einstein, we must have $\Lambda = 4 + \sum n_i$ (after choosing, without loss of generality, $\lambda_i = 1$).

To study the global structure of the metrics, it is appropriate to make the coordinate transformation $\psi' = 2\tau$ and $t' = e^{-1}(\tau - \Lambda^{-1}\psi)$. The metric becomes

$$ds_D^2 = \frac{dr^2}{\frac{4c^2}{\Lambda(2c^2 + 4(\beta + r)^2)}(d\psi - \sum \lambda_i A_i^{(1)})^2 + \sum a_i^2 d\Sigma_{n_i}^2} + \frac{c^2 + 4(\beta + r)^2}{\beta^2}(d\tau + \sum e_i A_i^{(1)}) - \frac{c^2 + 4r(\beta + r)}{\Lambda(2c^2 + 4(\beta + r)^2)}(d\psi - \sum \lambda_i A_i^{(1)})^2. \hspace{1cm} (43)$$

As in the $D = 7$ case we discussed in the previous section, the rate of collapse of the circle parametrized by $\psi$ is the same at all the roots of $c^2(r) = 0$. The period of $\psi$ must then be $2\pi$. Consideration of the connection on the fibres parametrized by $\tau$ (which never shrink to zero) implies the conditions

$$\frac{\ell_i}{\ell_N} = \frac{\alpha_i}{\gamma} \equiv \text{rational number}, \quad i = 1, 2, \ldots, N - 1,$$

$$\frac{r_s}{\Lambda(\beta + r_s)/\ell_N} = \gamma \equiv \text{rational number}. \hspace{1cm} (44)$$
Note that the $p_i$ do not have to be rationally related, but they satisfy the conditions (40). Substituting (44) and (40) into the $c^2 = 0$ equation, we obtain a polynomial equation in $\ell_N$ of order $1 + \sum n_i$, with rational coefficients that are polynomials in $\alpha_i$ and $\gamma$. Without loss of generality, we can choose $0 \leq p_1 \leq p_2 \leq \cdots \leq p_N$ and $\lambda_i = 1$. The constant $\ell_N$ must lie in the range $0 < \ell_N < \Lambda^{-1}$. Thus provided $\ell_N$ satisfies this condition, the corresponding set of rational numbers $(\alpha_i, \gamma)$ gives a non-singular Einstein–Sasaki metric.

It is also possible to find special solutions where $\ell_N$ is rational too. These correspond to cases where the parameters $p_i$ are all relatively prime integers (after rescaling). Such solutions occur sporadically and their significance is unclear.

4. Conclusions

In this paper, we obtained an infinite number of Einstein–Sasaki metrics in $D = 2n + 3$ dimensions, which are circle bundles over Einstein–Kähler $(2n + 2)$-spaces. These spaces are themselves complex line bundles over a product of $N$ Einstein–Kähler manifolds of diverse dimensions $n_i$. Locally, the Einstein–Sasaki metrics are characterized by $N$ real parameters. Global considerations for non-singular metrics that extend smoothly onto complete compact manifolds restrict these $N$ parameters to be rational within a certain region. We focused our attention principally on seven-dimensional examples, which provide natural supersymmetric compactifying manifolds for M-theory.

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Note added. After this work was completed, a paper appeared that also obtained the local form of the Einstein–Sasaki metrics that we have constructed in this paper [16].

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