NICE LATENT VARIABLE MODELS HAVE LOG-RANK

MADELEINE UDELL AND ALEX TOWNSEND

Abstract. Matrices of low rank are pervasive in big data, appearing in rec-
ommender systems, movie preferences, topic models, medical records, and ge-
nomics. While there is a vast literature on how to exploit low rank structure in
these datasets, there is less attention on explaining why the low rank structure
appears in the first place. We explain the abundance of low rank matrices in
big data by proving that certain latent variable models associated to piecewise
analytic functions are of log-rank. A large matrix from such a latent variable
model can be approximated, up to a small error, by a low rank matrix.

1. Introduction

Low rank matrices appear throughout the sciences in computational mathemat-
ics [Beb08], statistics [Gir14], and machine learning [KSD06]. Numerous tech-
niques have been developed over the last 50 years to exploit low rank structure whenever
it appears, whether in movie preferences [Fun06, BK07], social networks [LNK07,
ME11], genomics [BTGM04, GC05, KP07, WTH09], medical records [SLW+16], or
text documents [DDF+90, Dhi01, PSM14].

It is useful to know when a data set can be approximated by a low rank matrix.
A low rank approximation can be used to make filtering and statistics either computa-
tionally feasible or more efficient. In machine learning, low rank approximations
to data tables are often employed to impute missing data, denoise noisy data, or
for feature extraction [UHZB16]. These techniques are also fundamental for many
algorithms in recommender systems [KBY09].

The broad applicability of low rank techniques is at first rather puzzling. Since
the set of singular matrices is nowhere dense, random (“average”) matrices are
almost surely of full rank. In addition, the singular values of random Gaussian
matrices are large with extraordinarily high probability [Ede88]. We must conclude
that matrices and datasets that appear in the real-world must be far from average.
We would like to understand the underlying phenomena that generate compressible
datasets.

Here, we give an explanation for the prevalence of low rank matrices in data
science. Our result holds for any sufficiently large data set generated by sampling
columns and rows from a so-called nice latent variable model (intuitively, smooth; see
Definition 3 for a formal definition), or a piecewise nice model. A nice latent
variable model has a simple parametrization, but not a linear parametrization.
One might worry that a low rank approximation could overlook this structure, and
a more complicated approximation scheme is required to compress such datasets.
However, our main theorem suggests that low rank approximation is a remarkably
powerful technique for approximating datasets from nice latent variable models.

We state our main theorem informally now. (We formally define “nice” LVMs
and log-rank in §2)

Theorem 1 (Informal statement). Nice latent variables models are of log-rank.
Theorem 2 extends this result to piecewise nice latent variable models, while Theorem 3 considers symmetric latent variable models, i.e., graphons.

Our main tool is the Johnson–Lindenstrauss Lemma (see Lemma 1), which says that given any point cloud in a high dimensional Euclidean space there exists an embedding onto a low dimensional Euclidean space that approximately preserves pairwise distances between points.

This result has ramifications for how to interpret an underlying low rank structure in datasets. In particular, we have good news for those designing algorithms: if your model requires an assumption of low rank structure, then that assumption is not particularly strong. Conversely, we have bad news for those who attempt to find meaning in low rank structure. Researchers often give post-hoc explanations for why a particular data set is approximately of low rank. For example, typical arguments are: customers’ movie preferences are low rank because movies are well parametrized by a few meaningful genres or that word document matrices are low rank because they are well parametrized by a handful of meaningful topics. Our main theorem shows that even without an underlying physical reason, low rank structure can persist. In particular, a dataset from a nice latent variable model has an \( \epsilon \)-rank that grows slowly with its dimensions; no matter how many genres or topics describe the origin of the data.

Throughout we use \( \|v\| = \left(\sum_{i=1}^{N} v_i^2\right)^{1/2} \) to denote the Euclidean length of a vector \( v \in \mathbb{R}^N \) and \( \|f\| = \sup_{x \in \Omega} |f(x)| \) to denote the supremum norm of \( f : \Omega \to \mathbb{R} \) over its domain \( \Omega \).

2. Background material

We briefly review some necessary background material.

**Rank.** A nonzero matrix \( X \in \mathbb{R}^{m \times n} \) is said to be of rank 1 if \( X \) can be written as an outer-product of two column vectors, i.e., \( X = uv^T \) for \( u \in \mathbb{R}^{m \times 1} \) and \( v \in \mathbb{R}^{n \times 1} \). Moreover, a matrix \( X \) is of rank \( k \) if \( k \) is the smallest integer so that \( X \) can be written as a sum of \( k \) rank 1 matrices. That is,

\[
X = u_1 v_1^T + \cdots + u_k v_k^T, \quad u_1, \ldots, u_k \in \mathbb{R}^{m \times 1}, \quad v_1, \ldots, v_k \in \mathbb{R}^{n \times 1}.
\]

Generically, a matrix is of full rank; however, we find in data science that a full rank matrix can often be well-approximated by a low rank matrix in the sense that \( X \approx u_1 v_1^T + \cdots + u_k v_k^T \). If one finds that a matrix \( X \) can be well-approximated by a rank \( k \) matrix, \( X_k \), then one can perform diagnostics directly on \( X_k \), instead of \( X \).

**The \( \epsilon \)-rank of a matrix.** A matrix \( X \) can be approximated by a rank \( k \) matrix, up to an absolute accuracy of \( \epsilon > 0 \), if the \( \epsilon \)-rank of \( X \) is less than equal to \( k \).

**Definition 1** (\( \epsilon \)-rank). Let \( X \in \mathbb{R}^{m \times n} \) be a matrix and \( \epsilon > 0 \) a tolerance. The (absolute) \( \epsilon \)-rank of \( X \) is given by

\[
\text{rank}_\epsilon(X) = \min \{ \text{rank}(A) : A \in \mathbb{R}^{m \times n}, \|X - A\|_{\text{max}} \leq \epsilon \},
\]

where \( \| \cdot \|_{\text{max}} \) is the absolute maximum matrix entry. That is, \( k = \text{rank}_\epsilon(X) \) is the smallest integer for which \( X \) can be approximated by a rank \( k \) matrix, up to an accuracy of \( \epsilon \).

There are several alternative definitions of \( \epsilon \)-rank in the literature [BT16].
A log-rank family of matrices. We are particularly interested in a family of matrices \( \mathcal{X} = \{X^{(m \times n)}\}_{m,n \geq 1} \), where the \( \epsilon \)-rank of \( X^{(m \times n)} \in \mathbb{R}^{m \times n} \) grows slower than a polylogarithm in \( m \) and \( n \). We use the notation \( X^{(m \times n)} \in \mathcal{X} \) to emphasize that \( X^{(m \times n)} \) is a matrix of size \( m \times n \).

**Definition 2.** An infinite family of matrices \( \mathcal{X} = \{X^{(m \times n)}\}_{m,n \geq 1} \) is of log-rank if there is a finite degree polynomial \( p \) such that for any fixed \( \epsilon > 0 \),

\[
\text{rank}_\epsilon(X^{(m \times n)}) = \mathcal{O}(p(\log(m + n))).
\]

Here, the implicit constant in the big-O notation is allowed to depend on \( \epsilon \). In many interesting applications (including the results in this paper), the polynomial \( p \) is simply \( p(x) = x \).

In machine learning, \( \mathcal{X} \) might represent a family of datasets for which one can generate datasets of varying dimensions by sampling more examples (rows \( m \)) or features (columns \( n \)) from a data distribution: say, by collecting more text documents, patient records, customer preferences, or movie reviews. A log-rank family of matrices contains datasets for which the \( \epsilon \)-rank grows only modestly as we collect more examples and more features. Low rank techniques often lead to algorithms that have near-optimal complexity for log-rank families of matrices.

**Latent variable models.** Latent variable models (LVMs) are a particularly interesting class of families of matrices. A latent variable model is parametrized by a continuous function \( f \) and two distributions \( \mathcal{A} \) and \( \mathcal{B} \). A family of matrices \( \mathcal{X}_{f,\mathcal{A},\mathcal{B}} = \{X^{(m \times n)}\}_{m,n \geq 1} \) is a latent variable model (depending on \( f \), \( \mathcal{A} \), and \( \mathcal{B} \)) if for every \( X^{(m \times n)} \in \mathcal{X} \),

\[
(X^{(m \times n)})_{ij} = f(\alpha_i, \beta_j), \quad 1 \leq i \leq m, \quad 1 \leq j \leq n,
\]

where \( \alpha_i \) and \( \beta_j \) are independent random variables from the distributions \( \mathcal{A} \) and \( \mathcal{B} \), respectively.

Latent variable models have a natural relationship to low rank matrices. If \( \mathcal{A} \) and \( \mathcal{B} \) are distributions over vectors in \( \mathbb{R}^r \) and \( f(\alpha, \beta) = \alpha^T \beta \), then the rank of any matrix in the family \( \mathcal{X}_{f,\mathcal{A},\mathcal{B}} \) is at most \( r \). However, latent variable models can be used to model more complex distributions. For example, \( f \) might be a kernel function, and \( \mathcal{A} \) and \( \mathcal{B} \) might be distributions over very high-dimensional spaces.

**The Johnson–Lindenstrauss Lemma.** The Johnson–Lindenstrauss Lemma \cite{JL84} is a key tool in theoretical computer science. Roughly, it says that a high dimensional point cloud can be projected onto a low-dimensional space while approximately preserving all pairwise distances between the points. There are several alternative forms and proofs \cite{Mat08}.

**Lemma 1** (The Johnson–Lindenstrauss Lemma). Let \( 0 < \epsilon_{JL} < 1 \), \( x_1, \ldots, x_n \) be \( n \) points in \( \mathbb{R}^N \), and \( r = \lceil 8(\log n)/\epsilon_{JL}^2 \rceil \). Then, there is a linear map \( Q : \mathbb{R}^N \to \mathbb{R}^r \) such that

\[
(1 - \epsilon_{JL})\|x_i - x_j\|^2 \leq \|Q(x_i - x_j)\|^2 \leq (1 + \epsilon_{JL})\|x_i - x_j\|^2, \quad 1 \leq i,j \leq n.
\]

Here, \( \lceil a \rceil \) is the smallest integer larger than \( a \).

**Proof.** See \cite{Mat08} Thm. 1.1. Also, see \cite{JL84}. \( \square \)
3. Related work

The majority of the literature focuses on either how to find low rank matrices or how to exploit low rank structure after it has been found. This trend is set to continue with the emerging field of multilinear algebra, and the increasing use of tensor factorizations in machine learning and data analysis [OGA07, HGS14b, HGS+14a]. This keen practical interest in low rank structure lends urgency to our quest to understand why and when low rank techniques work well on real datasets.

Bounds on \( \epsilon \)-rank. The work of Alon and his coauthors is closest in spirit to our paper [Alo09, ALSV13]. These papers use the Johnson–Lindenstrauss Lemma to show that the identity matrix, and any positive semidefinite matrix, has an \( \epsilon \)-rank that grows logarithmically with the number of columns and rows.

Chatterjee [Cha15] shows that any matrix with bounded entries can be well-approximated by thresholding all singular values lower than a given value to 0. His main theorem implies that the \( \epsilon \)-rank of a matrix of size \( n \times n \) grows like \( O(\sqrt{n}) \).

Our theorem improves this result to \( O(\log n) \) when the matrix comes from a nice latent variable model.

In [BT16], bounds were derived on a slightly different \( \epsilon \)-rank of certain matrices \( X \in \mathbb{R}^{m\times n} \) with displacement structure, i.e., a matrix that satisfies \( AX - XB = F \). For example, [BT16, Thm. 3.1] showed that all \( n \times n \) positive-definite Hankel matrices, \( (H_n)_{ij} = h_{i+j} \), have an \( \epsilon \)-rank that grows logarithmically in \( n \). These matrices have more rapidly decaying singular values for each fixed \( n \) than the LVMs we study here.

Exchangeable families of matrices. Latent variable models are related to so-called exchangeable families of matrices. We say that an infinite matrix \( \mathcal{X} \) is exchangeable if for any permutations \( \sigma \) and \( \pi \) on \( \mathbb{N} \), we have

\[
X_{i,j} \sim X_{\sigma(i),\pi(j)}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n,
\]

where \( \sim \) denotes equality in distribution. A celebrated result by Aldous [Ald81] states that if \( \mathcal{X} \) is exchangeable, then

\[
X_{ij} \sim f(\omega, \alpha_i, \beta_j, \eta_{ij}),
\]

where \( f \) is a measurable function, \( \omega, \alpha_i, \beta_j, \eta_{ij} \) are scalar-valued, and the \( \omega, \alpha_i, s, \beta_j, s, \) and \( \eta_{ij}, s \) are mutually independent and uniformly distributed random variables on \([0,1]\). One can generate a family of matrices from \( \mathcal{X} \) by taking the leading \( m \times n \) principal submatrices.

There is some resemblance here to the latent variable model. There are two significant differences: (1) There is an intrinsic noise term \( \eta_{ij} \) and (2) The latent variables \( \omega, \alpha_i, \) and \( \beta_j \) are scalar-valued and uniform random variables on \([0,1]\). Our result on latent variable models can be extended to exchangeable families of matrices, under additional smoothness assumptions on \( f \).

The symmetric analogue of an exchangeable array is a graphon. Graphons can be seen as the continuous limit of a sequence of (dense) graphs [LS06]. Many authors have proposed methods for graphon estimation from samples of the entries [CWA12, ACC13, WO13, CA14]. For example, Airoldi et al. required that the graphon be piecewise Lipshitz, and provided an approximate graphon that gives a complexity that grows linearly in the number of pieces [ACC13]. Our theory shows that this procedure overestimates the complexity required to model a graphon when the
A latent variable model is nice. Indeed, Theorem 2 shows that the $\epsilon$-rank of a nice graphon grows with the maximum complexity of each piece. For reasonable distributions, the maximum complexity grows sublinearly in the number of pieces. Choi et al. showed that it is possible to find a consistent estimator for the graphon when the number of classes in a stochastic block model grows at most like the square root of the dimension \([\text{CWA12}]\). Our theory shows that a low rank model for the graphon (which generalizes a stochastic block model) only requires a rank that grows like the logarithm of the dimension. Whether it is possible to find statistically consistent estimators that obtain this threshold is an important question for future research.

The theory of exchangeable matrices has been used to motivate the use of latent variable models for collaborative filtering and other applications in machine learning. For example, many authors have used the assumption that the latent variable model is Lipschitz to design efficient estimators for symmetric and asymmetric distributions of data \([\text{SLLS16, LS17, LLSS17}]\). We show a connection between this approach and the standard low rank model.

4. Any Nice Latent Variable Model is Log-Rank

Our result applies to any nice latent variable model, which we now define.

**Definition 3.** A latent variable model $X = X_{f,A,B}$ is called nice if the following conditions hold:

- The associated distributions $A$ and $B$ are supported on a closed ball $B_R \subset \mathbb{R}^N$ for some $N \geq 1$ of radius $R > 0$, i.e., $B_R = \{ x \in \mathbb{R}^N : \| x \| \leq R \}$. Here, $N$ is allowed to be extremely large.
- The associated function $f : B_R \times B_R \to \mathbb{R}$ is bounded and sufficiently smooth in the sense that $f(\alpha, \cdot)$ is uniformly analytic in $B_R$ for every $\alpha \in B_R$ and for all $\mu \in \mathbb{N}^N$ we have
  $$\| D^\mu f(\alpha, \beta) \| \leq C M^{\| \mu \|} \| f \|.$$  
  Here, $\mu = (\mu_1, \ldots, \mu_N)$ is a multi-index, $| \mu | = \sum_{i=1}^N \mu_i$, $D^\mu f = \frac{\partial | \mu |}{\partial x_1^{\mu_1} \cdots \partial x_N^{\mu_N}}$, and $C \geq 0$ and $M \geq 0$ are positive constants.

Nice latent variable models are common in machine learning and data analysis. Functions that give rise to nice latent variable models include:

- **Linear functions.** If $f(\alpha, \beta) = \alpha^T \beta$ and the distributions $A$ and $B$ have bounded support, then $X_{f,A,B}$ is a nice LVM with $M = C = 1$. In this case, $X_{f,A,B}$ has a rank bounded by $N$. Theorem 4 shows that when $N$ is sufficiently large the $\epsilon$-rank is actually smaller than $N$ for $\epsilon > 0$.
- **Polynomials.** If $f$ is a polynomial in $2N$-variables, then there is a constant $M$ that depends on $N$, $R$, and the degree of the polynomial so that $\| D^\mu f(\alpha, \beta) \| \leq C M^{\| \mu \|} \| f \|$. For simplicity, consider $N = 1$ and $f(\alpha, \beta) = \beta^d$. Then, for $k < d$ we have
  $$\| D^k f(\alpha, \beta) \| = d(d-1) \cdots (d-k+1) \sup_{|\beta| \leq R} |\beta|^{d-k} \leq d^k R^{-k} \| f \|.$$  
  So, $M = d/R$ and $C = 1$ suffices.
- **Kernels.** If $f(\alpha, \beta) = \exp(p(\alpha, \beta))$ for some $2N$-variable polynomial $p$, then $\| D^\mu f(\alpha, \beta) \| \leq C M^{\| \mu \|} \| f \|$ for some constants $C$ and $M$. This includes most kernels typically used in machine learning. For example, consider the
radial basis function kernel $f(\alpha, \beta) = \exp(-||\alpha - \beta||^2)$ with $R > 1/2$. Then, 
$$||D^\mu f(\alpha, \beta)|| \leq N(4R)^{N+|\mu|}||f||.$$ 
We see that the bound on the derivatives of $f$ allows for many relevant examples. Our framework can also handle the case of piecwise nice LVMs, which we treat below in Theorem 2.

We are now ready to formally state our main result. An alternative theorem with the analytic assumptions of $f$ on the first variable is also possible with an analogous proof.

**Theorem 1** (Formal statement). Let $\mathcal{X}_{f,A,B}$ be a nice latent variable model and $0 < \epsilon < 1$. Then, for each $X^{(m \times n)} \in \mathcal{X}_{f,A,B}$, the $\epsilon\|f\|$-rank of $X^{(m \times n)}$ is no more than 
$$r = \left\lfloor 8\log(m + n + 1) \left(1 + \frac{2(C_u + C_v + 1)}{\epsilon}\right)^2 \right\rfloor,$$
where $C_u$ and $C_v$ are constants defined below that depend on the latent variable model $\mathcal{X}_{f,A,B}$.

We state Theorem 1 in terms of the $\epsilon\|f\|$-rank to show that we achieve a natural sort of relative-error guarantee. Consider the LVM $\mathcal{X}'_{f,A,B}$ where $f' = cf$ for some constant $c$. The entries of a matrix drawn from $\mathcal{X}'_{f,A,B}$ will be about a factor of $c$ larger in expectation than the entries of a matrix drawn from $\mathcal{X}_{f,A,B}$. It is natural to compare the $c$-rank of a matrix from $\mathcal{X}'_{f,A,B}$ with the 1-rank of a matrix from $\mathcal{X}_{f,A,B}$. Theorem 1 shows both satisfy the same bound, since $\|f\| = c\|f\|$.

**Proof of Theorem 1.** The proof proceeds in three main steps. Suppose $X \in \mathcal{X}_{f,A,B} \cap \mathbb{R}^{m \times n}$ has entries $X_{ij} = f(\alpha_i, \beta_j)$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$. First, we use a Taylor expansion of the function $f(\alpha, \cdot)$ about 0 to show that $f(\alpha_i, \beta_j) \approx u_i^T v_j$. That is, $f$ can be well-approximated as the inner product between two (extremely high dimensional) vectors, $u_i$ and $v_j$. Second, we bound the Euclidean norms of $u_i$ and $v_j$. Third, we use the Johnson–Lindenstrauss Lemma to reduce the dimensionality of $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ while approximating preserving the quantities $u_i^T v_j$.

(1) **Showing that** $f(\alpha_i, \beta_j) \approx u_i^T v_j$. By Taylor expanding $f(\alpha_i, \beta_j)$ in the second variable about 0 with $K$ terms, we find that 
$$|X_{ij} - \hat{X}_{ij}| \leq \frac{(N + K)^N R^{K+1}}{(K + 1)!} \max_{|\tau| = K + 1, z \in B_\mu} |D^\tau f(\alpha_i, z)|, \quad \hat{X}_{ij} = \sum_{|\mu| \leq K} \frac{D^\mu f(\alpha_i, 0)}{\mu!} \beta_j^\mu,$$
where $D^\mu f = \frac{\partial^{ |\mu|} f}{\partial \beta_1^{\mu_1} \cdots \partial \beta_N^{\mu_N}}$, $\mu! = \mu_1! \cdots \mu_N!$, and $\beta_j^\mu = (\beta_j)^{\mu_1}_1 \cdots (\beta_j)^{\mu_N}_N$. Here, the $(N + K)^N$ term in the Taylor error comes from the fact that there are $(\frac{N+K}{K+1})^N$ $\mu$’s with $|\mu| = K + 1$ and $(\frac{N+K}{K+1}) \leq (N + K)^N$. From the formula for $\hat{X}_{ij}$, there are vectors $u_i$ and $v_j$ with $\sum_{|\mu| \leq K} 1 = \hat{N}$ entries, such that $\hat{X}_{ij} = u_i^T v_j$. The vectors $u_i$ and $v_j$ are indexed by $|\mu| \leq K$ and can be taken to be 
$$(u_i)_\mu = \frac{1}{\sqrt{\mu!} \sqrt{\|f\|}} D^\mu f(\alpha_i, 0), \quad (v_j)_\mu = \frac{1}{\sqrt{\mu!} \sqrt{\|f\|}} \beta_j^\mu.$$ 
Hence, we write $\hat{A} = UV$, $U = [u_1 \cdots u_m]^T$, $V = [v_1 \cdots v_n]$. 
Now, select $K$ sufficiently large so that
\[
|X_{ij} - \hat{X}_{ij}| \leq \frac{(N + K)^N R^{K+1}}{(K + 1)!} \max_{\tau = K + 1} \sup_{z \in B_R} |D^\tau f(\alpha, z)| \\
\leq C \frac{(N + K)^N R^{K+1}M^{K+1}}{(K + 1)!} \|f\| \\
\leq \epsilon / 2 \|f\|.
\]
Since the denominator grows superexponentially in $K$, there is always a sufficiently large $K$ for the bound above for any $0 < \epsilon < 1$. Therefore, we have the approximation
\[
|X_{ij} - \hat{X}_{ij}| \leq \epsilon / 2 \|f\|,
\]
where $u_i \in \mathbb{R}^N$ and $v_j \in \mathbb{R}^N$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Notice that the vectors $u_i$ and $v_j$ may have an extremely large number of entries!

(2) Bounding the norms of $u_i$ and $v_j$. Let us remark on the norms of $u_i$ and of $v_j$. We will suppress the indices $i$ and $j$ in this discussion.

Let $u^{(\infty)} = (u_{\mu})_{|\mu| \geq 0}$ and $v^{(\infty)} = (v_{\mu})_{|\mu| \geq 0}$ be infinite dimensional vectors. Then,
\[
\|u\|^2 \leq \|u^{(\infty)}\|^2 = C_u \|f\| < \infty, \\
\|v\|^2 \leq \|v^{(\infty)}\|^2 = C_v \|f\| < \infty,
\]
where $C_u$ and $C_v$ are constants that depend only on the properties of the nice LVM.

For $C_v$ we have
\[
\|v^{(\infty)}\|^2 \leq \sum_{\mu} \frac{1}{|\mu|!} \|v^{(\infty)}\|^2 \leq \sum_{s=0}^{\infty} \frac{1}{s!} (N + s)^N \|f\|^2 \leq C_v \|f\|,
\]
showing that $C_v$ is finite.

The constant $C_u$ depends on how quickly the derivatives of $f$ grow; it is bounded so long as they grow no faster than exponentially. Since $\mu! \geq (|\mu|/N)!$, we have
\[
\|u^{(\infty)}\|^2 \leq \sum_{s=0}^{\infty} (N + s)^N \|f\|^2 \leq C_u \|f\|,
\]
showing that $C_u$ is finite.

(3) Showing that $u_i^T v_j \approx (Qu_i)^T Q v_j$ for $Q \in \mathbb{R}^{r \times \hat{N}}$. Consider the collection of $m + n + 1$ vectors \{0, $u_1, \ldots, u_m, v_1, \ldots, v_n$\} $\in \mathbb{R}^\hat{N}$. By the Johnson–Lindenstrauss lemma and given any $0 < \epsilon_{\text{JL}} < 1$, we can find a linear map $Q \in \mathbb{R}^{r \times \hat{N}}$ with $r = \lfloor 8\log(m + n + 1)/\epsilon_{\text{JL}}^2 \rfloor$ so that the distances between each of these vectors are approximately preserved. More precisely, for any $u \in \{u_1, \ldots, u_m\}$ and $v \in \{v_1, \ldots, v_n\}$, we have
\[
(1 - \epsilon_{\text{JL}})\|u - v\|^2 \leq \|Q(u - v)\|^2 \leq (1 + \epsilon_{\text{JL}})\|u - v\|^2,
\]
\[
(1 - \epsilon_{\text{JL}})\|u\|^2 \leq \|Q u\|^2 \leq (1 + \epsilon_{\text{JL}})\|u\|^2,
\]
\[
(1 - \epsilon_{\text{JL}})\|v\|^2 \leq \|Q v\|^2 \leq (1 + \epsilon_{\text{JL}})\|v\|^2,
\]
where for the last two displays we have compared the vectors $u$ and $v$, respectively, with 0.
Let \( x = Qu \) and \( y = Qv \). To derive a bound on \(|u^Tv - x^Ty|\), we will use the basic identity \( \|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2a^Tb \). By adding the last two displayed equations together and subtracting the first, we find that
\[
(1 - \epsilon_{JL})(\|u\|^2 + \|v\|^2) - (1 + \epsilon_{JL})\|u - v\|^2 \leq 2x^Tv \leq (1 + \epsilon_{JL})(\|u\|^2 + \|v\|^2) - (1 - \epsilon_{JL})\|u - v\|^2,
\]
\[
2u^Tv - \epsilon_{JL}(\|u\|^2 + \|v\|^2 + \|u - v\|^2) \leq 2x^Tv \leq 2u^Tv + \epsilon_{JL}(\|u\|^2 + \|v\|^2 + \|u - v\|^2),
\]
\[
-\frac{\epsilon_{JL}}{2}(\|u\|^2 + \|v\|^2 + \|u - v\|^2) \leq x^Tv - u^Tv \leq \frac{\epsilon_{JL}}{2}(\|u\|^2 + \|v\|^2 + \|u - v\|^2),
\]
where for the second displayed equations we have again used the basic identity \( \|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2a^Tb \). Using this basic identity one more time, we find that
\[
|u^Tv - x^Tv| \leq \frac{\epsilon_{JL}}{2}(\|u\|^2 + \|v\|^2 + \|u - v\|^2)
\]
\[
= \epsilon_{JL}(\|u\|^2 + \|v\|^2 - u^Tv).
\]
Now, using our bound on \( \|u\|^2 \) and \( \|v\|^2 \) from above, we obtain the following inequalities for every \( u \in \{u_1, \ldots, u_m\} \) and \( v \in \{v_1, \ldots, v_n\} \):
\[
|u^Tv - x^Tv| \leq \epsilon_{JL}(\|u\|^2 + \|v\|^2 - u^Tv)
\]
\[
\leq \epsilon_{JL} \left( \|u(\infty)\|^2 + \|v(\infty)\|^2 + |f(\alpha, \beta)| + \frac{\epsilon}{2}\|f\| \right)
\]
\[
\leq \epsilon_{JL}(\|C_u + C_v\||f\| + (1 + \frac{\epsilon}{2})\|f\|),
\]
where we have used the fact that \( |f(\alpha, \beta)| \leq \|f\| \) and \( |u^Tv - f(\alpha, \beta)| \leq \epsilon/2\|f\| \).

The total error in each entry of our approximation is thus
\[
|f(\alpha_i, \beta_j) - x^Tv_{y_j}| \leq |f(\alpha_i, \beta_j) - u^Tv_j| + |u^Tv_j - x^Tv_{y_j}|
\]
\[
\leq \frac{\epsilon}{2}\|f\| + \epsilon_{JL}(\|C_u + C_v\| + 1 + \frac{\epsilon}{2})\|f\|.
\]
Thus, if we select \( \epsilon_{JL} \) to be
\[
\epsilon_{JL} = \frac{\epsilon/2}{C_u + C_v + 1 + \epsilon/2},
\]
then we have \( |f(\alpha_i, \beta_j) - x^Tv_{y_j}| \leq \epsilon\|f\| \), as desired.

Therefore, the \( \epsilon\|f\| \)-rank of \( X \) is at most the rank of the matrix \( \tilde{X}_{ij} = x^Tv_{y_j} \), which is of rank at most \( r \). Here, \( r \) is the integer given by
\[
r = \left\lceil 8\log(m + n + 1) \left(1 + \frac{2(C_u + C_v + 1)}{\epsilon}\right)^2 \right\rceil.
\]

\[\Box\]

Remark 1. Note that Theorem [1] is only interesting when
\[
\min(m, n) > \left\lceil 8\log(m + n + 1) \left(1 + \frac{2(C_u + C_v + 1)}{\epsilon}\right)^2 \right\rceil,
\]
since the rank of a matrix is always bounded by its smallest dimension. Hence, we see Theorem [1] is interesting for sufficiently large matrices.
4.1. Piecewise nice latent variable models. The requirement that the function $f$ associated to the LVM be analytic can be relaxed to piecewise analytic. We call such models piecewise nice LVMs.

**Definition 4.** The family of matrices $\mathcal{X}_{f,A,B}$ is call a piecewise nice LVM if there exists a finite partition of the distributions

$$A \times B = \bigcup_{\ell=1}^{P} (A_{\ell} \times B_{\ell}), \quad (A_{\ell} \times B_{\ell}) \cap (A_{\ell'} \times B_{\ell'}) = \emptyset, \quad \ell \neq \ell'$$

so that

$$f(\alpha, \beta) = f_{\ell}(\alpha, \beta), \quad (\alpha, \beta) \in A_{\ell} \times B_{\ell}$$

with $\mathcal{X}_{f_{\ell},A_{\ell},B_{\ell}}$ being nice LVMs for $1 \leq \ell \leq P$.

We find that any piecewise nice LVM is also of log-rank.

**Theorem 2.** Let $\mathcal{X}_{f,A,B}$ be a piecewise nice latent variable model with distributions of $A$ and $B$ of bounded support. Then, for each $0 < \epsilon < 1$ and for any $X^{(m \times n)} \in \mathcal{X}_{f,A,B}$ the $\epsilon\|f\|$-rank of $X^{(m \times n)}$ is no more than

$$r = \left\lceil 8 \log(m + n + 1) \left(1 + \frac{2(C_u + C_v + 1)}{\epsilon} \right)^2 \right\rceil,$$

where $C_u$ and $C_v$ are constants that depend on properties of the latent variable model $\mathcal{X}_{f,A,B}$.

The proof of this theorem is an easy modification of the proof of Theorem 1 because the dimension of the projected vectors in the Johnson–Lindenstrauss Lemma is independent of the dimension of the original vectors. For example, we can take

$$u_i = (0, \ldots, 0, u_{i}^{(\ell)}, 0, \ldots, 0), \quad v_{i} = (0, \ldots, 0, v_{j}^{(\ell)}, 0, \ldots, 0)$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. (Note it is possible that $\alpha_i \in A_{\ell}$ for multiple $\ell$s, so $u_i$ may have more than one nonzero block; ditto for $v_j$.) We can also take

$$\hat{X}_{ij} = u_{i}^{T}v_{j} = \sum_{\ell: (\alpha, \beta) \in A_{\ell} \times B_{\ell}} u_{i}^{(\ell)}v_{j}^{(\ell)} = \left(u_{i}^{(\ell)}\right)^{T}v_{j}^{(\ell)}$$

where $\ell_{ij}$ is the unique $\ell$ so that $(\alpha_i, \beta_j) \in A_{\ell} \times B_{\ell}$. (It is unique because $\{A_{\ell} \times B_{\ell}\}_{\ell=1}^{P}$ partitions $A \times B$.) Lastly, the norms of $u_i$ and $v_j$ are just the sum of the norms of $u_{i}^{(\ell)}$ and $v_{j}^{(\ell)}$ so the constants $C_u$ and $C_v$ in the proof are replaced by $\max_{\alpha} \sum_{\ell: \alpha \in A_{\ell}} C_u^{(\ell)}$ and $\max_{\beta} \sum_{\ell: \beta \in B_{\ell}} C_v^{(\ell)}$.

4.2. Symmetric latent variable models. Above, we noticed a connection between latent variable models and exchangeable families of matrices. To understand the rank of symmetric exchangeable families of matrices (e.g., graphons), and the rank of symmetric matrices, we define a symmetric notion of latent variable models:

**Definition 5.** The family of matrices $\mathcal{X}_{f,A}$ is a symmetric latent variable model (depending on $f$ and $A$) if for every $X^{(n \times n)} \in \mathcal{X}_{f,A}$,

$$(X^{(n \times n)})_{ij} = f(\alpha_i, \alpha_j), \quad 1 \leq i, j \leq n.$$
If $A$ is compact and $|D^\mu f(\alpha, \alpha')| \leq CM^{||\mu||} \|f\|$, we say the symmetric LVM is nice.

If $A = [0, 1]$ and $f : [0, 1] \times [0, 1] \to [0, 1]$, then $X_{f,A}$ is a graphon [LS06].
Graphons are often used to model processes that generate random graphs, by interpreting the entries of $X^{(n \times n)} \in X_{f,A}$ as the probability that a graph on $n$ nodes has an edge between node $i$ and node $j$.

We show any symmetric LVM is of log-rank.

**Theorem 3.** Let $X_{f,A}$ be a nice symmetric latent variable model and let $0 < \epsilon < 1$. Then, for $X^{(n \times n)} \in X_{f,A}$, the $\epsilon\|f\|$-rank of $X^{(n \times n)}$ is no more than

$$r = \left\lceil 8 \log(2n + 1) \left( 1 + \frac{2(C_u + C_v + 1)}{\epsilon} \right)^2 \right\rceil,$$

where $C_u$ and $C_v$ are constants which depend on the latent variable model $X_{f,A}$.

The proof of this theorem is nearly identical to the proof of Theorem 1 since we never use independence of $\alpha_i$ and $\beta_j$.

5. Conclusion

This paper seeks to answer the question: “Why are low rank techniques so effective for solving problems in data analysis and machine learning?” Theorem 1 provides an explanation for this effectiveness: when rows and columns of the data are drawn from a nice and consistent distribution, the rank of the resulting matrix cannot increase very quickly. Formally, we have shown that nice latent variable models give rise to matrices that have an $\epsilon$-rank that grows only logarithmically with the matrix dimensions. This suggests that low rank structure in large datasets is a universal feature and provides a broad motivation for low rank techniques in data science and machine learning.

Many questions remain open. How does the rank of the best approximation change in the presence of noisy or partial observations from these models? What is the sample complexity required for consistent estimation of these models? Are higher order tensor latent variable models also of low (tensor) rank, and what does that rank depend on? Are the technical assumptions we use here tight, or can they be relaxed?

We expect these questions to spur exciting developments in both theoretical and applied research.

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Department of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14853. This work is supported by DARPA Award No. FA8750-17-2-0101.

E-mail address: udell@cornell.edu

Department of Mathematics, Cornell University, Ithaca, NY 14853. This work is supported by National Science Foundation grant No. 1522577.

E-mail address: townsend@cornell.edu