MIXING TIMES OF A BURNSIDE PROCESS MARKOV CHAIN ON SET PARTITIONS

J. E. PAGUYO

Abstract. Let $X$ be a finite set and let $G$ be a finite group acting on $X$. The group action splits $X$ into disjoint orbits. The Burnside process is a Markov chain on $X$ which has a uniform stationary distribution when the chain is lumped to orbits. We consider the case where $X = [k]^n$ with $k \geq n$ and $G = S_k$ is the symmetric group on $[k]$, such that $G$ acts on $X$ by permuting the value of each coordinate. The resulting Burnside process gives a novel algorithm for sampling a set partition of $[n]$ uniformly at random. We obtain bounds on the mixing time and show that the chain is rapidly mixing.

1. Introduction

Markov chain Monte Carlo algorithms are used as computational methods for sampling from complicated probability distributions and are a mainstay in a wide range of scientific fields. Let $X$ be a finite set and let $\pi$ be a probability distribution on $X$. The computational problem is to sample from $X$ according to the distribution $\pi$. Typically $X$ is large so that it is difficult to sample from $\pi$ directly. Sampling procedures have found many important applications, such as in approximating the size of large finite combinatorial structures and in generating physical configurations of systems arising from statistical mechanics.

The Markov chain Monte Carlo method provides an algorithmic solution to the sampling problem as follows. First, we construct a Markov chain with state space $X$ specified by a $|X| \times |X|$ transition matrix $K$, where $K(x, y)$ is the probability of moving from state $x$ to state $y$. We also define the chain to be irreducible and aperiodic, which ensures that it converges to its stationary distribution. Finally, the transition probabilities $K(x, y)$ are defined such that the stationary distribution is $\pi$. To sample from $X$ according to the distribution $\pi$, we start at an arbitrary state $x_0 \in X$ and simulate the Markov chain $K$ for some number of steps $T$, then output the final state. Since $K$ converges to $\pi$, taking $T$ to be sufficiently large ensures that the distribution of the final state will be close to $\pi$.

The efficiency of these algorithms rely on the mixing time of the Markov chain, which is the number of simulation steps needed in order to get sufficiently close to the stationary distribution $\pi$. Since $X$ is typically a large set, an efficient algorithm should have a mixing time that is much smaller than $|X|$. Obtaining bounds on

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mixing times is generally a difficult task, and mixing time analyses for many Monte Carlo algorithms remain open problems. We refer the reader to Diaconis [9, 10] for wonderful and accessible surveys on Markov chain Monte Carlo.

1.1. The Burnside Process. Let $X$ be a finite set and let $G$ be a finite group acting on $X$. The group action splits $X$ into disjoint orbits. Let $O_x = \{gx : g \in G\}$ be the orbit containing $x \in X$. The Burnside process is a Markov chain developed by Jerrum [19] as a practical and novel way of choosing an orbit uniformly at random. This provides an algorithm for the computational problem of sampling from unlabelled structures, which are combinatorial structures modulo a group of symmetries.

For motivation, let $X$ be the set of all trees on $n$ labeled vertices. By Cayley’s theorem, there are $|X| = n^{n-2}$ labeled trees. Let $G = S_n$ be the symmetric group acting on $X$ by permuting the labels of the vertices. The computational problem is to choose an unlabeled tree uniformly at random. This is a difficult problem in general. As another example, let $X = G$ with $G$ acting on itself by conjugation $gx := g^{-1}xg$. The orbits are the conjugacy classes of $G$. When $G = S_n$, the orbits are the partitions of $n$. The problem is to choose a random partition.

Let $X_g = \{y \in X : gy = y\}$ be the fixed set of $g \in G$ and let $G_x = \{g \in G : gx = x\}$ be the stabilizer of $x \in X$. The Burnside process is a Markov chain on $X$ whose transition between states $x$ and $y$ is:

- From $x \in X$, choose $g \in G_x$ uniformly at random;
- From $g$, choose $y \in X_g$ uniformly at random.

The transition matrix is given by

$$K(x, y) = \sum_{g \in G_x \cap G_y} \frac{1}{|G_x|} \frac{1}{|X_g|}$$

and the stationary distribution is

$$\pi(x) = \frac{1}{z|O_x|}$$

where $z$ is the number of orbits.

For $X = O_1 \cup O_2 \cup \cdots \cup O_z$ and $\{X_t\}_{t=0}^\infty$ a Markov chain on $X$, the lumped chain or projection is the chain $\{Y_t\}_{t=0}^\infty$, given by $Y_t = a$ if $X_t \in O_a$. It follows that the Burnside process has a uniform stationary distribution when lumped into orbits.

We conclude this section with a survey of previous results. The Burnside process was introduced by Jerrum in [19] as an algorithm for sampling unlabelled structures, and he showed that for many groups the Burnside process is rapidly mixing, which means the mixing time is upper bounded by a polynomial in the input size. In particular, Jerrum considered the process where $X = \{0, 1\}^n$ and $G = S_n$ is the symmetric group acting on $X$ by permuting coordinates, and proved a mixing time upper bound of order $\sqrt{n}$. Subsequently, Goldberg and Jerrum [15] showed that the Burnside process is not rapidly mixing in general by constructing a family of permutation groups for which the mixing time is exponential in the degree of the group.
Aldous and Fill [2] generalized Jerrum’s process by considering the case where \( X = [k]^n \), the set of \( n \)-tuples with entries from \([k]\), and \( G = S_n \) acts on \( X \) by permuting coordinates, and obtained a mixing time upper bound of order \( k \log n \) using coupling. Diaconis [8] found a connection to Bose-Einstein configurations and used minorization to obtain a mixing time upper bound which is independent of \( n \), for \( k \) fixed or growing slowly with \( n \). More recently, Diaconis and Zhong [12] considered the case where \( k = 2 \) and used spectral methods and connections to orthogonal polynomials to get sharp mixing time bounds.

In another direction, Rahmani [23] studied the commuting chain, where \( X = G \) is a finite group which acts on itself by conjugation, and used coupling, minorization, and spectral methods to show that the chain is rapidly mixing for various centralizer abelian groups. In computer science, Holtzen, Millstein, and Van den Broeck [17] found an application of the Burnside process to an approximate inference algorithm called the orbit-jump Markov chain Monte Carlo.

All other special cases of the Burnside process are open research problems. Working towards the general case, Chen [6] used minorization to obtain a mixing time upper bound which permutes coordinates by \( n \), and coupling to obtain a mixing time upper bound for the lumped chain of order \( |X| \), for any Burnside process with state space \( X \) and group \( G \) acting on \( X \). Although these bounds work for general Burnside processes, they often are not strong enough bounds to prove rapid mixing for many special cases. Thus the quantitative analysis of the rates of convergence for the Burnside process, in full generality, remains an open problem.

1.2. Main Results. Let \( [n] := \{1, 2, \ldots, n\} \). A set partition of \( [n] \) is a set of nonempty subsets of \( [n] \) such that \([n]\) is a disjoint union of the subsets. We refer to these subsets as blocks. Let \( \Pi_n \) denote the collection of all set partitions of \([n]\). The number of set partitions of \([n]\) is given by \( B_n \), the \( n \)th Bell number. The Stirling numbers of the second kind \( \{ \}_k \) count the number of set partitions of \([n]\) into \( k \) nonempty subsets. These numbers are connected through the sum \( B_n = \sum_{k=0}^{n} \{ \}_k \).

We consider a variant of the Burnside process studied by Aldous and Fill [2], Diaconis [8], and Diaconis and Zhong [12]. Let \( X = [k]^n \), where \( k \geq n \), and let \( G = S_k \). Consider the group action of \( S_k \) on \([k]^n\) which permutes coordinates by value. That is, if \( u = (u_1, \ldots, u_n) \in [k]^n \) and \( \sigma \in S_k \), then \( \sigma u = (\sigma(u_1), \ldots, \sigma(u_n)) \). Let \( X_\sigma = \{ v \in X : \sigma v = v \} \) and \( G_u = \{ \sigma \in S_k : \sigma u = u \} \). The Burnside process on \([k]^n\) is a Markov chain whose transition between states \( u, v \in [k]^n \) can be described explicitly as:

- From \( u \in [k]^n \), identify the set of distinct values \( J \subseteq [k] \) which appear in \( u \). Choose a uniformly random permutation \( \sigma \in S_k \) conditioned to have fixed points at positions given by \( J \). Let \( \text{FP}(\sigma) \) be the set of fixed points of \( \sigma \).
- From \( \sigma \), let \( v \in [k]^n \) be obtained by setting each coordinate independently with a uniform choice in \( \text{FP}(\sigma) \).

Observe that the orbit of \( u \in [k]^n \) is determined by the sets of coordinates whose values are equal. Indeed, if \( u_i = u_j \), then \( \sigma(u_i) = \sigma(u_j) \) for all \( \sigma \in S_k \). Given \( u \), we can define a corresponding set partition \( x \in \Pi_n \) such that \( i, j \in [n] \) are in the
same block of $x$ if and only if $u_i = u_j$. Thus if $x$ is the set partition corresponding to $u$, then $x$ also corresponds to $\sigma u$. It follows that the orbits are indexed by the set partitions of $[n]$.

When lumped into orbits, the Burnside process on $[k]^n$ defines a Markov chain on set partitions of $[n]$ with uniform stationary distribution. We refer to the lumped chain as the Burnside process on set partitions or the Burnside process on $\Pi_n$. Thus in order to study the mixing time of the Burnside process on $[k]^n$, it suffices to study the lumped chain.

The Burnside process on $[k]^n$ provides a novel Markov chain Monte Carlo algorithm for sampling a set partition of $[n]$ uniformly at random. Starting from an arbitrary state $u \in [k]^n$, we simulate the Burnside process for a sufficiently large number of steps and then output the final state. Then the set partition corresponding to the final state will be approximately uniformly distributed.

Our main result provides an upper bound on the mixing time of the Burnside process on $[k]^n$.

**Theorem 1.1.** Let $K(u, v)$ be the transition matrix of the Burnside process on $[k]^n$, where $k \geq n$. Let

$$\pi(u) = \frac{1}{B_u|O_u|}$$

be its stationary distribution. Then

$$\|K^t_u - \pi\|_{TV} \leq n \left(1 - \frac{1}{2k}\right)^t.$$ 

The upper bound is uniform in the starting state $u \in [k]^n$. Therefore the mixing time of $K$ satisfies

$$t_{mix}(\varepsilon) \leq 2k \log \left(\frac{n}{\varepsilon}\right).$$

This upper bound on the mixing time shows that the Burnside process on $[k]^n$ is rapidly mixing.

In the case where $k < n$, the Burnside process on $[k]^n$ gives a Markov chain Monte Carlo algorithm for sampling a set partition of $[n]$ of at most $k$ blocks uniformly at random. For this regime, we obtain a stronger mixing time upper bound in Theorem 4.5

$$t_{mix}(\varepsilon) \leq \lceil (k - 1)! \log (\varepsilon^{-1}) \rceil.$$ 

In particular, this shows that for fixed $k$, the mixing time is independent of the number of coordinates $n$.

Finally for $k \geq n \geq 6$, we also provide the following lower bound on the mixing time in Corollary 5.5

$$t_{mix}(\varepsilon) \geq \left(\frac{n}{5 \log n} - 1\right) \log \left(\frac{1}{2\varepsilon}\right).$$

The main difficulty in proving our results comes from the complicated formulas for the transition probabilities of both the Burnside process on $[k]^n$ and the lumped
chain. These make spectral methods intractable. To overcome these difficulties, we instead use probabilistic and geometric techniques. The upper bounds are proved using the probabilistic techniques of coupling and minorization, which turn out to be useful techniques for chains which make large jumps in a single step. The lower bound uses a Cheeger inequality combined with a relaxation time bound.

1.3. Outline. The paper is organized as follows. In Section 2 we give a brief introduction to Markov chains and present the various techniques that we use for bounding mixing times. Then we derive the transition probabilities for the Burnside process on \([k]^n\) and for the Burnside process on \(\Pi_n\) in Section 3.

In Section 4 we use coupling and minorization to obtain upper bounds on the mixing time of the Burnside process on \([k]^n\). We then use geometric methods in Section 5 to obtain upper and lower bounds on the second largest eigenvalue of the Burnside process on \(\Pi_n\). Finally, we obtain mixing time bounds for the Burnside process on \(\Pi_n\) via the relaxation time and connect these bounds to the mixing time of the Burnside process on \([k]^n\).

We conclude the paper with some final remarks and open problems in Section 6.

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2. Preliminaries

This section gives necessary background on Markov chain convergence, the Burnside process, and auxiliary variables algorithms, and presents the various techniques which we use to obtain bounds on mixing times. We aim to give an accessible introduction, along with several references to the literature.

2.1. Markov Chains and Mixing Times. Let \(X\) be a finite set. A Markov chain is specified by a transition matrix \(K(x, y) \geq 0\) with \(\sum_y K(x, y) = 1\), so that \(K(x, y)\) is the probability of moving from \(x\) to \(y\) in one step. Under mild conditions, there exists a unique stationary distribution \(\pi(x) \geq 0\) with \(\sum_x \pi(x) = 1\), such that \(\sum_x \pi(x) K(x, y) = \pi(y)\). Thus \(\pi\) is a left eigenvector of \(K\) with eigenvalue 1. The probabilistic interpretation is that if we choose \(x\) according to \(\pi\) and take one step in the chain, then the probability of being at \(y\) is \(\pi(y)\).

A Markov chain is reversible if it satisfies the detailed balance equation, given by \(\pi(x) K(x, y) = \pi(y) K(y, x)\), for all \(x, y \in X\). Let \(L^2(\pi) = \{f : X \to \mathbb{R}\}\) equipped with the inner product \(\langle f, g \rangle_\pi = \sum_x \pi(x) f(x) g(x)\). Then \(K\) acts on \(L^2(\pi)\) by \(Kf = \sum_y K(x, y) f(y)\). Reversibility is equivalent to \(\langle Kf, g \rangle = \langle f, Kg \rangle\), so that \(K\) is a bounded, self-adjoint operator on \(L^2(\pi)\). By the spectral theorem, there exists an orthogonal basis of eigenfunctions \(\{f_i\}\) and real eigenvalues \(\{\lambda_i\}\) such that \(1 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{|X| - 1} \geq -1\) and \(Kf_i(x) = \lambda_i f_i(x)\).
Let $K^t(x,y)$ be the probability of moving from $x$ to $y$ in $t$ steps. The chain $K$ is irreducible if for all $x,y \in X$ there exists $t \geq 1$ with $K^t(x,y) > 0$, and is aperiodic if $\gcd\{t \geq 1 : K^t(x,x) > 0\} = 1$ for all $x \in X$. The fundamental theorem of Markov chains states that if $K$ is irreducible and aperiodic, then $K^t_k(y) := K^t(x,y) \to \pi(y)$ as $t \to \infty$. The distance to stationarity can be measured in $L^1$ by the total variation distance,
\[
\|K^t_k - \pi\|_{TV} = \max_{A \subseteq X} |K^t(x,A) - \pi(A)| = \frac{1}{2} \sum_y |K^t(x,y) - \pi(y)|,
\]
or in $L^2$ by the chi-squared distance,
\[
\|K^t_k - \pi\|_2 = \sum_y \frac{(K^t(x,y) - \pi(y))^2}{\pi(y)} = \sum_{i=1}^{|X|} \lambda_i^t f_i^2(x).
\]
The Cauchy-Schwarz inequality gives the relationship between the two distances:
\[
4\|K^t_k - \pi\|_{TV}^2 \leq \|K^t_k - \pi\|_2^2.
\]
Thus a useful strategy to obtaining bounds on the convergence rate is to bound $L^1$ by $L^2$, then use eigenvalues to bound $L^2$.

Let $K$ be an irreducible and aperiodic Markov chain on $X$ with stationary distribution $\pi$, so that $\|K^t_k - \pi\|_{TV} \to 0$ as $t \to \infty$ for all $x \in X$. Let
\[
d(t) = \max_{x \in X} |K^t_k(x) - \pi|_{TV}
\]
be the distance to stationarity or distance function. Then the mixing time of the Markov chain $K$ is defined to be
\[
t_{\text{mix}}(\varepsilon) = \min\{t : d(t) \leq \varepsilon\}.
\]

2.2. Auxiliary Variables. The auxiliary variables Markov chain was introduced by Edwards and Sokal [13] as an abstraction of the Swendsen-Wang algorithm [33] for sampling from the Ising and Potts models from statistical mechanics. Auxiliary variables gives a method for constructing a reversible Markov chain with stationary distribution $\pi$, and is related to several classes of unifying algorithms, which include data augmentation [34] and the hit-and-run algorithm [3]. These are non-local chains which are able to make large moves in a single step. Thus they typically mix much faster than local or diffusive chains.

Let $\pi(x)$ be a probability distribution on $X$. Let $I$ be a set of auxiliary variables. For all $x \in X$, let $w_x(i)$ be a probability distribution on $I$, which gives the probability of moving from $x$ to $i$. These define a joint distribution $f(x,i) = \pi(x)w_x(i)$ and a marginal distribution $m(i) = \sum_x f(x,i)$. For all $i \in I$, we specify a Markov chain $K_i(x,y)$ with reversing measure $f(x|i) = f(x,i)/m(i)$. The auxiliary variables Markov chain transitions from states $x$ to $y$ as follows: from $x \in X$, choose $i \in I$ with probability $w_x(i)$, then from $i$, choose $y \in X$ with probability $K_i(x,y)$. The transition matrix is
\[
K(x,y) = \sum_i w_x(i)K_i(x,y)
\]
It is straightforward to check that $K$ is reversible with respect to $\pi$, and so $\pi$ is the stationary distribution. Currently, almost none of the auxiliary variables or hit-and-run algorithms have sharp running time analysis.

The Burnside process is the special case obtained by setting $I = G$, $w_x(i)$ the uniform distribution on $G_x$, and $K_g(x, y)$ the uniform distribution on $X_g$. The motivation is that the group structure should make mixing time analysis more tractable. However the Burnside process has only been studied for a few special cases and the mixing time analysis in full generality remains a difficult open problem.

2.3. Coupling. The method of coupling is one of the most powerful probabilistic techniques for bounding mixing times. A coupling of two probability distributions $\mu, \nu$ is a Markov chain $(X_t, Y_t)$ defined on a single probability space such that the marginal distribution of $X_t$ is $\mu$ and the marginal distribution of $Y_t$ is $\nu$.

We can always define a coupling of $\mu$ and $\nu$ by letting $(X_t)$ and $(Y_t)$ be independent of each other. However, in order to get good mixing time bounds, it is desirable to have some correlation that will bring $(X_t)$ and $(Y_t)$ closer together such that they will eventually coalesce so that $X_t = Y_t$ at some time $t$.

The coupling lemma states that for all couplings $(X, Y)$ of distributions $\mu$ and $\nu$, the total variation distance can between $\mu$ and $\nu$ can be upper bounded by the probability that the two couplings are not equal,

$$\|\mu - \nu\|_{TV} \leq P(X \neq Y).$$

Moreover, there always exists an optimal coupling which achieves equality.

2.4. Minorization. A Markov chain $K$ on a state space $X$ satisfies a minorization condition if there exists a probability measure $\nu$ on $X$, a positive integer $t_0$, and $\delta > 0$ such that

$$P^{t_0}(x, A) \geq \delta \nu(A),$$

for all $x \in X$ and for all measurable subsets $A \subseteq X$. Minorization is closely related to Harris recurrence [5] and we refer the reader to [25] for an accessible survey on minorization in Markov chains. The following proposition gives a bound on the distance to stationarity for chains which satisfy a minorization condition.

**Proposition 2.1** ([27], Proposition 2). Let $K$ be the transition matrix of a Markov chain on a finite state space $X$ with stationary distribution $\pi$. Suppose there exists a probability measure $\nu$ on $X$, a positive integer $t_0$, and some $\delta > 0$ such that $K^{t_0}(x, A) \geq \delta \nu(A)$ for all $x \in X$ and for all measurable subsets $A \subseteq X$. Then

$$\|K^t_x - \pi\|_{TV} \leq (1 - \delta)^{|t/t_0|}$$

for all $t$.

Minorization has previously been used to obtain convergence rates on Markov chain Monte Carlo algorithms, for example by Rosenthal for the data augmentation algorithm on finite sample spaces [26] and for the Gibbs sampler applied to a variance component model [28]. More recently, minorization was used by Diaconis [8] and Rahmani [23] to obtain mixing time bounds on Burnside processes.
2.5. Spectral Gap. Let $K$ be a reversible Markov chain. Recall by the spectral theorem that $K$ has real eigenvalues with $1 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{|X|-1} \geq -1$. The \textit{absolute spectral gap} of $K$ is defined to be $1 - \lambda_1$, where $\lambda_1 = \max \{ \lambda_1, |\lambda_{|X|-1}| \}$. If $K$ is also irreducible and aperiodic, then $1 - \lambda_1 > 0$. The \textit{spectral gap} of $K$ is defined as $1 - \lambda_1$. In practice, it suffices to consider $\lambda_1$ since the chain can be modified so that $\lambda_1 \geq |\lambda_{|X|-1}|$.

One way is to consider the \textit{continuous-time chain} with transition matrix $K$, where the waiting times between transitions are iid exponential rate 1 random variables. The transition kernel for this continuous time chain is given by

$$H_t(x, y) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n K^n(x, y)}{n!},$$

for all $x, y \in X$. The following proposition, which follows from Theorem 2.1.7 in [29] when $p = 1$, shows the relationship between $H_t$ and the spectral gap of $K$.

\textbf{Proposition 2.2} ([29], Theorem 2.1.7). \textit{Let $K$ be an irreducible and reversible Markov chain, and let $H_t$ be transition kernel for the continuous time chain with transition matrix $K$. Then the spectral gap of $K$ satisfies}

$$1 - \lambda_1 = \lim_{t \to \infty} -\frac{1}{t} \log \left( 2 \max_{x \in X} \| H_t(x, \cdot) - \pi(\cdot) \|_{TV} \right).$$

2.6. Geometric Bounds. Let $K$ be the transition matrix of an irreducible and reversible Markov chain on a finite state space $X$ with stationary distribution $\pi$. Recall that bounds on the spectral gap lead to bounds on the rate of convergence.

Let $Q(x, y) = \pi(x) K(x, y)$ for all $x, y \in X$ be the reversing measure. Consider the graph with vertex set $X$ and $(x, y)$ an edge if and only if $Q(x, y) > 0$. This is the \textit{underlying graph} of the Markov chain $K$. Note that this graph may contain self-loops, is always connected, and uniquely specifies the chain. For all pairs of distinct vertices $x, y \in X$, construct a path $\gamma_{xy}$ from $x$ to $y$, called the \textit{canonical path}. Let $\Gamma$ denote the collection of all canonical paths.

Diaconis and Stroock [11] and Sinclair [30] obtained bounds on $\lambda_1$ in terms of geometric quantities arising in the underlying graph. The following \textit{Poincaré bound} is based on the Poincaré inequality and uses the method of canonical paths.

\textbf{Proposition 2.3} ([11], Proposition 1'). \textit{For an irreducible Markov chain, the second largest eigenvalue satisfies}

$$\lambda_2 \leq 1 - \frac{1}{\kappa},$$

where

$$\kappa = \max_{e} \frac{1}{Q(e)} \sum_{\gamma_{xy} \ni e} |\gamma_{xy}| \pi(x) \pi(y)$$

and $|\gamma_{xy}|$ denotes the number of edges in the path $\gamma_{xy}$.

The geometric quantity $\kappa$ is a measure of \textit{bottlenecks} in the graph, and will be small if it is possible to pick paths that do not pass through any one edge too often.
Let $S \subseteq X$ be a proper subset and define
\[ Q(S, S^c) = \sum_{x \in S} \sum_{y \in S^c} Q(x, y) = \sum_{x \in S} \sum_{y \in S^c} \pi(x) K(x, y). \]

Bounds on $\lambda_1$ can also be derived in terms of the conductance
\[ \Phi = \min_{\pi(S) \leq 1/2} \frac{Q(S, S^c)}{\pi(S)}. \]

This geometric quantity measures the flow out of the set $S$ when the chain is at stationarity. If $\Phi$ is large for all $S$, then the chain should converge to $\pi$ quickly since this would indicate that there are no bottlenecks. Sinclair and Jerrum [31] obtained the Cheeger bound $\lambda_1 \leq 1 - \frac{\Phi^2}{2}$ based on the Cheeger inequality.

Computing $\Phi$ is difficult in general. Jerrum and Sinclair [21, 31] used canonical paths to define the geometric quantity $\eta = \max_e Q(e) - \sum_{\gamma \in e} \pi(x) \pi(y)$ and obtain a Cheeger bound in terms of $\eta$ as $\lambda_1 \leq 1 - \frac{1}{8\eta^2}$. Diaconis and Stroock [11] observed that for many Markov chains, the Poincaré bound beats the Cheeger bound. Fulman and Wilmer [14] showed that this is true for simple random walk on trees and random walks on finite groups with a symmetric generating set.

Next we turn to a lower bound on $\lambda_1$. The first inequality is written in a form given by Ingrassia [18].

**Proposition 2.4** ([11], Proposition 6). Let $S$ be a proper subset of $X$. Then
\[ \lambda_1 \geq 1 - \frac{Q(S, S^c)}{\pi(S) \pi(S^c)} \geq 1 - 2\Phi. \]

Finally recall that the relaxation time of a reversible Markov chain is related to the spectral gap by $t_{rel} = (1 - \lambda_1)^{-1}$. The following proposition gives bounds on the mixing time in terms of the relaxation time.

**Proposition 2.5** ([22], Theorems 12.4 and 12.5). Let $K$ be the transition matrix of a reversible, irreducible, and aperiodic Markov chain with state space $X$, and let $\pi_{\min} := \min_{x \in X} \pi(x)$. Then
\[
(t_{rel} - 1) \log \left( \frac{1}{2\varepsilon} \right) \leq t_{mix}(\varepsilon) \leq \left[ t_{rel} \log \left( \frac{1}{\varepsilon \pi_{\min}} \right) \right]
\]

The upper bound follows from Proposition 3 in [11] while the lower bound can be derived from a discrete time version of Proposition 8 in [1].

3. Transition Probabilities

In this section, we compute the transition probabilities for the Burnside process on $[k]^n$ and use this to derive the transition probabilities for the Burnside process on $\Pi_n$. We then show that in order to get total variation bounds on the Burnside process on $[k]^n$, it suffices to obtain bounds on the lumped chain.

Recall the definition of the Burnside process on $[k]^n$ from Section [12]. The following derivation of the transition matrix $M$ for the Burnside process on $[k]^n$ is due to Rahmani, Theorem 5.2.1, in their thesis [24].
Let \( u, v \in [k]^n \). Let \( G_u \) and \( G_v \) be the stabilizers of \( u \) and \( v \), respectively. Let \( j_u \) be the number of distinct entries of \( u \), \( j_v \) the number of distinct entries of \( v \), and \( j \) the number of distinct entries when the entries of \( u \) and \( v \) are combined. Observe that \( |G_u| = (k - j_u)! \). Then by definition of the Burnside process, the transition probability from \( u \) to \( v \) is

\[
K(u, v) = \sum_{\sigma \in G_u \cap G_v} \frac{1}{|G_u|} \frac{1}{X_\sigma} = \frac{1}{(k - j_u)!} \sum_{\sigma \in G_u \cap G_v} \frac{1}{X_\sigma}
\]

Let \( \sigma \in S_k \) and let \( X_\sigma \) be the fixed set of \( \sigma \). Let \( \text{fp}(\sigma) \) be the number of fixed points of \( \sigma \) and observe that \( |X_\sigma| = \text{fp}(\sigma)^n \). Note that if \( \sigma \in G_u \cap G_v \), we must have that \( j \leq \text{fp}(\sigma) \leq k \). Let \( D_{k,i} \) be the number of permutations in \( S_k \) with exactly \( i \) fixed points and let \( !k \) denote the number of derangements of \([k]\). Then

\[
\sum_{\sigma \in G_u \cap G_v} \frac{1}{|X_\sigma|} = \sum_{\sigma \in G_u \cap G_v} \frac{1}{\text{fp}(\sigma)^n} = \sum_{i=j}^k \frac{1}{i^n} D_{k,i} = \sum_{i=j}^k \frac{1}{i^n} \binom{k}{i} !((k - i))
\]

Combining the above gives

\[
K(u, v) = \frac{1}{(k - j_u)!} \sum_{i=j}^k \frac{1}{i^n} \binom{k}{i} !((k - i)) = \frac{(k - j)!}{(k - j_u)!} \sum_{i=0}^{k-j} \frac{1}{(i + j)^n} \binom{k-j}{i} !((k - j - i)) \frac{(k - j - i)!}{(k - j)!}
\]

Finally, observe that the sum is precisely equal to \( E \left[ \frac{1}{(Y+j)^n} \right] \), where \( Y \) is the number of fixed points of a uniformly random permutation \( \sigma \in S_{k-j} \). Therefore the transition probability of the Burnside process on \([k]^n\) is

\[
K(u, v) = \frac{(k - j)!}{(k - j_u)!} E \left[ \frac{1}{(Y+j)^n} \right]
\]

Observe that \( K(u, v) \) is completely determined by the values of \( j_u \) and \( j_v \).

It is straightforward to check that the \( K \) is irreducible, aperiodic, and reversible with respect to \( \pi(u) = \frac{1}{B_n|O_u|} = \frac{(k-j_u)!}{B_n k!} \), where \( |O_u| = \frac{k!}{(k-j_u)!} \) by the Orbit-Stabilizer Theorem.

Using the transition matrix \( K \), our new contribution in this paper is the following derivation of the transition matrix, \( \tilde{K} \), of the Burnside process on \( \Pi_n \). Let \( u, v \in [k]^n \). Let \( j_u \) and \( j_v \) be the number of distinct entries in \( u \) and \( v \), respectively. Assume \( j_u \leq j_v \). Then the probability of transitioning from the orbit \( O_u \) to \( O_v \) is given by \( K(O_u, O_v) = K(u, v) \). Since orbits are determined by entries with equal values, it suffices to add the transition probabilities \( K(u, v) \) as \( v \) ranges over all states in the orbit \( O_v \), so that \( K(O_u, O_v) = \sum_{y \in O_v} K(u, y) \).

Let \( j \) be the number of distinct entries when the entries of \( u \) and \( v \) are combined, and note that \( j_u \leq j_v \leq j \leq j_u + j_v \). We count the possible values that can occur in the distinct entries of \( v \). There are \( \binom{j_u + j_v - j}{j_u} \) ways to pick the values in \( u \) which
also appear in \( v \). Next there are \( \binom{k-j_u}{j_u-j_v} \) to pick the remaining values from \([k]\) which appear in \( v \). This gives \( j_v \) distinct values from \([k]\) which appears in \( v \). Finally there are \( j_v! \) ways to distribute these values among the \( j_v \) entries where the distinct values occur.

Combining the above yields the transition probabilities for the lumped chain.

\[
K(O_u, O_v) = K(u, O_v) = \sum_{j=j_u}^{j_u+j_v} \frac{j_u!j_v!}{(j-j_u)!((j-j_v)!(j_u+j_v-j)!)^E} \left[ \frac{1}{(Y+j)^n} \right]
\]

where \( Y \) is the number of fixed points of a uniformly random permutation \( \sigma \in S_{k-j} \).

If \( j_x > j_y \), then \( \tilde{K}(x, y) = \tilde{K}(y, x) \).

It is clear that \( \tilde{K} \) is irreducible, aperiodic, and reversible with respect to the uniform distribution on set partitions \( \pi(x) = \frac{1}{B_n} \).

Recall that the orbits of \([k]^n\) are indexed by set partitions of \([n]\). Let \( \tilde{K} \) be the \( B_n \times B_n \) transition matrix of the Burnside process on \( \Pi_n \). It will be useful to fix the notation for this matrix as follows. Let \( x_1, x_2, \ldots, x_{B_n} \) denote the \( B_n \) set partitions of \([n]\), such that \( x_1 \) is the unique set partition consisting of a single block, \( x_2, \ldots, x_{2+\binom{n}{2}} \) are the \( \binom{n}{2} \) set partitions consisting of two blocks (listed in a fixed but arbitrary order), and so on, until \( x_{B_n} \) which is the unique set partition consisting of \( B_n \) blocks. We refer to the set partitions in this ordering as \( \Pi_n \) ordered by block sizes. Then the transition matrix \( \tilde{K} \) is obtained by setting \( \tilde{K}_{ij} := \tilde{K}(x_i, x_j) \).

First note that \( \tilde{K} \) is a symmetric matrix and that \( \tilde{K}(x, y) \) is completely determined by the block sizes \( j_x \) and \( j_y \). Thus we may write \( \tilde{K}(x, y) := \tilde{K}(j_x, j_y) \), and we use \( \tilde{K}(j_x, j_y) \) whenever it is more convenient to describe the transition probability in terms of the block sizes \( j_x, j_y \) instead of the set partitions \( x, y \).

Next observe that \( \tilde{K} \) is a symmetric block matrix which can be written as

\[
\tilde{K} = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{pmatrix}
\]
where \( A_{ij} \) is a \( \{n\} \times \{n\} \) submatrix whose entries are all equal to \( \tilde{K}(i,j) \) and \( A_{ji} = A_{ij}^T \) for all \( 1 \leq i, j \leq n \). It follows that we can further project the chain \( K \) to a Markov chain \( L \), given by the \( n \times n \) matrix \( L(i,j) = \{n\} \tilde{K}(i,j) \), where \( L(i,j) \) is the transition probability of going from a set partition with \( i \) blocks to a set partition with \( j \) blocks. However \( L \) is no longer a symmetric matrix.

We conclude this section with a useful lemma which shows that the total variation distance between \( K^t \) and \( \pi \) is equal to the total variation distance between \( \tilde{K}^t \) and \( \bar{\pi} \). It follows that in order to obtain bounds on the mixing time of the Burnside process on \([k]_n\), it suffices to study the Burnside process on \( \Pi_n\).

**Lemma 3.2.** Let \( K \) and \( \pi \) be the transition matrix and stationary distribution of the Burnside process on \([k]^n\). Let \( \tilde{K} \) and \( \bar{\pi} \) be the transition matrix and stationary distribution of the Burnside process on \( \Pi_n\). Let \( x \in \Pi_n \) be the set partition corresponding to \( u \in [k]^n \). Then

\[
\|K^t_u - \pi\|_{TV} = \|\tilde{K}^t_x - \bar{\pi}\|_{TV}.
\]

**Proof.** Recall that we can write \([k]^n = \bigcup_{i=1}^{B_n} O_i \), where \( O_i \) are the orbits. Then

\[
\|K^t_u - \pi\|_{TV} = \frac{1}{2} \sum_{v \in [k]^n} |K^t_u(v) - \pi(v)| = \frac{1}{2} \sum_{i=1}^{B_n} \sum_{v \in O_i} |K^t_u(v) - \pi(v)|
\]

\[
= \frac{1}{2} \sum_{i=1}^{B_n} \left| \sum_{v \in O_i} K^t_u(v) - \frac{|O_i|}{B_n|O_i|} \right| = \frac{1}{2} \sum_{i=1}^{B_n} \left| \tilde{K}^t_x(O_i) - \frac{1}{B_n} \right|
\]

\[
= \|\tilde{K}^t_x - \bar{\pi}\|_{TV}. \qedhere
\]

4. **Mixing Time Upper Bounds**

In this section we prove our main result, Theorem 1.1, the upper bound on the mixing time of the Burnside process on \([k]^n\), for \( k \geq n \), using the method of coupling. We then prove a stronger upper bound in Theorem 4.5 for the \( k < n \) regime using minorization.

4.1. **Coupling Bound.** Our upper bound on the mixing time is obtained through the coupling method. We start with two useful lemmas which we use for our coupling constructions. We provide proofs of the lemmas for the convenience of the reader.

**Lemma 4.1** ([2], Lemma 12.4). Given finite sets \( F^1, F^2 \) we can construct (for \( u = 1, 2 \)) a uniformly random permutation \( \sigma^u \) of \( F^u \) with cycles \( (C^u_j, j \geq 1) \), where the cycles are labeled such that

\[
C^1_j \cap F^1 \cap F^2 = C^2_j \cap F^1 \cap F^2
\]

for all \( j \), where in the equality the \( C^u_j \) are interpreted as sets.

**Proof.** Let \( \sigma \) be a uniformly random permutation of a finite set \( A \), written in cycle notation. Then for all subsets \( B \subseteq A \), there exists an induced permutation \( \sigma|_B \)
on $B$ obtained by deleting the elements which do not appear in $B$ from the cycle representation of $\sigma$. Moreover $\sigma|_B$ is a uniformly random permutation of $B$.

Let $\sigma$ be a uniformly random permutation of $F^1 \cup F^2$. Let $\sigma^1$ and $\sigma^2$ be the induced permutations on $F^1$ and $F^2$, respectively. Then the equality follows since each side represents the cycles of the induced permutation of $F^1 \cap F^2$.

Lemma 4.2 ([20], Lemma 4.10). Let $U$ be a finite set and let $A, B \subseteq U$. Let $Z_a, Z_b$ be random variables taking values in $U$. Then there exists a joint distribution on $(Z_a, Z_b)$ such that $Z_a$ is uniform on $A$, $Z_b$ is uniform on $B$, and

$$P(Z_a = Z_b) = \frac{|A \cap B|}{\max\{|A|, |B|\}}.$$  

Proof. We can assume that $A \neq B$, since in this case we can simply take $P(Z_a = x, Z_b = y) = \frac{1}{|A|}$ for $x = y$ and 0 otherwise.

Without loss of generality assume $\max\{|A|, |B|\} = |A|$. We can construct a joint distribution on $(Z_a, Z_b)$ as

$$P(Z_a = x, Z_b = y) = \begin{cases} \frac{1}{|A|}, & x, y \in A \cap B \text{ and } x = y \\ \frac{|A| - |B|}{|A| |B| (|A| \cap |B|)}, & x \in A \setminus (A \cap B), y \in A \cap B \\ \frac{1}{|B| |A| (|A| \cap |B|)}, & x \in A \setminus (A \cap B), y \in B \setminus (A \cap B) \\ 0, & \text{otherwise.} \end{cases}$$

Then the joint distribution satisfies $P(Z_a = Z_b) = \frac{|A \cap B|}{|A|}$, and the marginal distributions satisfy $\sum_{y \in B} P(Z_a = x, Z_b = y) = P(Z_a = x) = \frac{1}{|A|}$ and $\sum_{x \in A} P(Z_a = x, Z_b = y) = P(Z_b = y) = \frac{1}{|B|}$ so that $Z_a$ and $Z_b$ are uniform on $A$ and $B$, respectively.

We are now in a position to prove our main theorem.

Proof of Theorem 1.1. Let $u, v \in [k]^n$. Let $J_u$ and $J_v$ be the set of distinct entries in $u$ and $v$, respectively. We construct a coupling $(U, V)$ of a step of the chain, started at $(u, v)$. We consider two cases.

Case 1: Suppose $|J_u \cap J_v| \neq 0$. Define the sets $A := [k] \setminus J_u$ and $B := [k] \setminus J_v$. Let $\sigma$ be a uniformly random permutation of the set $A \cup B$. Let $\sigma^1 := \sigma|_A$ and $\sigma^2 := \sigma|_B$ be the induced permutation of $\sigma$ on $A$ and $B$, respectively. By Lemma 4.1, $\sigma^1$ is a uniformly random permutation of $A$ and $\sigma^2$ is a uniformly random permutation of $B$. Finally define $\sigma^u \in S_k$ to be the permutation with fixed points at indices given by $J_u$ and such that the remaining indices $A$ are permuted according to $\sigma^1$. Similarly define $\sigma^v \in S_k$ to be the permutation with fixed points at indices given by $J_v$ and such that the remaining indices $B$ are permuted according to $\sigma^2$. By construction $\sigma^u$ and $\sigma^v$ are uniformly random permutations of $S_k$. This completes the first substep of the chain.

Let $\text{FP}(\sigma^u)$ (respectively, $\text{FP}(\sigma^v)$) be the set of fixed points of $\sigma^u$ (respectively, $\sigma^v$). Let $\text{fp}(\sigma^u) = |\text{FP}(\sigma^u)|$ and $\text{fp}(\sigma^v) = |\text{FP}(\sigma^v)|$. Without loss of generality, suppose $\text{fp}(\sigma^u) \geq \text{fp}(\sigma^v)$. Since $|J_u \cap J_v| \neq 0$, we have that $|\text{FP}(\sigma^u) \cap \text{FP}(\sigma^v)| \geq 1$.
by construction. Moreover, \( \text{fp}(\sigma^n) \leq k \) and \( \text{fp}(\sigma^v) \leq k \). By Lemma 4.2 there exists a coupling \((U_i, V_i)\) such that \( U_i \) is uniform on \( \text{FP}(\sigma^u) \), \( V_i \) is uniform on \( \text{FP}(\sigma^v) \), and
\[
P(U_i = V_i) = \frac{\text{card}(\text{FP}(\sigma^u) \cap \text{FP}(\sigma^v))}{\text{card}(\text{FP}(\sigma^u))} \geq \frac{1}{k},
\]
for all \( 1 \leq i \leq n \). Let \( U \in [k]^n \) (respectively, \( V \in [k]^n \)) be constructed by setting the \( i \)th coordinate equal to \( U_i \) (respectively, \( V_i \)) independently for all \( 1 \leq i \leq n \). This completes the second substep of the chain.

Therefore the coupled process \((U, V)\) satisfies
\[
P(U_i \neq V_i) \leq \left(1 - \frac{1}{k}\right).
\]
In particular, at time \( t \) we have that
\[
P(U_t \neq V_t) \leq n \left(1 - \frac{1}{k}\right)^t.
\]

**Case 2:** Suppose \( |J_u \cap J_v| = 0 \). We construct a coupling \((U, V)\) of a step of the chain in exactly the same way as Case 1. However, since \( |J_u \cap J_v| = 0 \), we have by construction that \( |\text{FP}(\sigma^u) \cap \text{FP}(\sigma^v)| \geq 1 \) if and only if \( \sigma \in S_{k-j_u-j_v} \) (as constructed in Case 1) has at least one fixed point.

Using the well-known formula for derangements, we get
\[
P(\sigma \text{ has at least one fixed point}) = 1 - P(\sigma \text{ has no fixed points})
= 1 - \sum_{i=0}^{k-j_u-j_v} \frac{(-1)^i}{i!}
\geq \frac{1}{2}.
\]
Therefore by conditioning on the first substep of the chain, we get
\[
P(U_i = V_i) = P(U_i = V_i, \sigma \text{ has at least one fixed point}) \geq \frac{1}{2k}.
\]
By the same argument as in Case 1, it follows that at time \( t \),
\[
P(U_t \neq V_t) \leq n \left(1 - \frac{1}{2k}\right)^t.
\]
Thus we have constructed a coupling \((U, V)\) of a step of the chain for all pairs of initial states \((u, v)\) such that
\[
P(U_t \neq V_t) \leq n \left(1 - \frac{1}{2k}\right)^t.
\]
Therefore by the coupling lemma,
\[
\|K_u^t - \pi\|_{TV} \leq n \left(1 - \frac{1}{2k}\right)^t
\]
uniformly over all starting states \( u \in [k]^n \).
Finally we obtain an upper bound on the mixing time via the upper bound on the total variation distance. We further bound the total variation distance as
\[
\|K^t_u - \pi\|_{TV} \leq n \left(1 - \frac{1}{2k}\right)^t \leq n \exp\left(-\frac{t}{2k}\right).
\]
Setting the right hand side equal to \(\varepsilon\) and solving for \(t\) yields the desired mixing time upper bound
\[
t_{\text{mix}}(\varepsilon) \leq 2k \log \left(\frac{n}{\varepsilon}\right).
\]

Since the mixing time is upper bounded by a polynomial in \(k\), Theorem 1.1 shows that the Burnside process on \([k]^n\) is rapidly mixing.

Using Theorem 1.1 we can also obtain an upper bound on the second largest eigenvalue of \(K\).

**Proposition 4.3.** The second largest eigenvalue of the transition matrix, \(K\), of the Burnside process on \([k]^n\) is upper bounded by
\[
\lambda_1 \leq 1 - \frac{1}{2k}.
\]

*Proof.* Let \(H_t\) be the continuous-time chain with transition matrix \(K\). Then for all \(u \in [k]^n\), we have that
\[
2\|H_t(u, \cdot) - \pi(\cdot)\|_{TV} = \sum_{v \in [k]^n} |H_t(u, v) - \pi(v)|
\]
\[
= \sum_{v \in [k]^n} \left| e^{-t} \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} [K^\ell(u, v) - \pi(v)] \right|
\]
\[
\leq e^{-t} \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} \sum_{v \in [k]^n} |K^\ell(u, v) - \pi(v)|
\]
\[
\leq 2e^{-t} \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} \|K^\ell_u - \pi\|_{TV}
\]
\[
\leq 2n e^{-t} \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} \left(1 - \frac{1}{2k}\right)^\ell
\]
\[
= 2n e^{-t/2k},
\]
where we used Theorem 1.1 in the third inequality.
Combining this upper bound with Proposition 2.2 yields

\[ 1 - \lambda_1 = \lim_{t \to \infty} -\frac{1}{t} \log \left( \frac{2 \max_{u \in [k]^n} \| H_t(u, \cdot) - \pi(\cdot) \|_{TV} }{t} \right) \geq \lim_{t \to \infty} -\frac{\log(2ne^{-t/2k})}{t} = \lim_{t \to \infty} \left( -\frac{\log(2n)}{t} + \frac{1}{2k} \right) = \frac{1}{2k}, \]

from which it follows that \( \lambda_1 \leq 1 - \frac{1}{2k} \). \( \square \)

### 4.2. Minorization Bound

Consider the Burnside process on \([k]^n\) in the \(k < n\) regime. This chain has the same transition probabilities \(K(u, v)\) as the \(k \geq n\) regime, but has a different stationary distribution given by

\[ \pi(u) = \frac{(k - j_u)!}{k! \sum_{i=1}^{k} \{ n \choose i \}} \]

where \( j_u \) is the number of distinct entries in \( u \).

The Burnside process on \([k]^n\) in the \(k < n\) regime provides a Markov chain Monte Carlo algorithm for sampling a set partition of \([n]\) of at most \(k\) blocks uniformly at random.

Using minorization, we can obtain a stronger mixing time upper bound in the \(k < n\) regime than the one given in Theorem 1.1. We begin with a lemma that lower bounds the transition probabilities \(K(u, v)\).

**Lemma 4.4.** For \(k < n\), the transition probability of the Burnside process on \([k]^n\) satisfies

\[ K(u, v) \geq \frac{1}{(k - 1)!k^n} \]

for all \( u, v \in [k]^n \).

**Proof.** By inspection of the transition probabilities, we find that \(K(u, v)\) is minimized when \( j_u = 1 \) and \( j_v = k \). This implies that there are \( j = k \) distinct entries in \( u \) and \( v \) combined. Let \( Y = \text{fp}(\sigma) \) where \( \sigma \in S_{k-j} \) is a uniformly random permutation. Then \( Y \leq k - j \), which implies \( E \left( \frac{1}{(Y+j)^n} \right) \geq \frac{1}{k^n} \). Therefore

\[ K(u, v) = \frac{(k - j)!}{(k - j_u)!} E \left( \frac{1}{(Y+j)^n} \right) \geq \frac{1}{(k - 1)!k^n}. \] \( \square \)

We now prove an upper bound on the mixing time for the Burnside process on \([k]^n\) in the \(k < n\) regime.

**Theorem 4.5.** For the Burnside process on \([k]^n\) in the \(k < n\) regime, the mixing time is upper bounded by

\[ t_{\text{mix}}(\varepsilon) \leq \left( (k - 1)! \log \left( \frac{1}{\varepsilon} \right) \right). \]
Proof. Let $\nu'(v) := \min_{u \in [k]^n} K(u, v)$ for all $v \in [k]^n$. Define $\nu(v) := \frac{\nu'(v)}{\nu'([k]^n)}$ and observe that $\nu$ is a probability distribution on $[k]^n$. Then

$$K(u, v) \geq \min_{u \in [k]^n} K(u, v) = \nu'(v) = \nu'([k]^n) \nu(v).$$

Set

$$c := \nu'([k]^n) = \sum_{v \in [k]^n} \min_{u \in [k]^n} K(u, v).$$

It follows that

$$K(u, A) \geq c \nu(A)$$

for all $u \in [k]^n$ and all measurable subsets $A \subseteq [k]^n$. This shows that $K$ satisfies a minorization condition. By Proposition 2.1,

$$\|K^t - \pi\|_{TV} \leq (1 - c)^t$$

for all $t$. Using the inequality $(1 - c)^t \leq e^{-ct}$ yields the following upper bound on the mixing time

$$t_{mix}(\varepsilon) \leq \lceil c^{-1} \log \left(\varepsilon^{-1}\right) \rceil.$$

By Lemma 4.4, we can lower bound $c$ by

$$c = \sum_{v \in [k]^n} \min_{u \in [k]^n} K(u, v) \geq \sum_{v \in [k]^n} \frac{1}{(k-1)!k^n} = \frac{1}{(k-1)!}.$$

Therefore we get the upper bound

$$t_{mix}(\varepsilon) \leq \lceil (k-1)! \log \left(\varepsilon^{-1}\right) \rceil.$$

Observe that for fixed $k$, the mixing time is independent of $n$. Moreover, for fixed $k$, or $k$ growing slowly with $n$, Theorem 1.1 gives a better upper bound than Theorem 4.5.

5. Geometric Bounds on the Second Largest Eigenvalue

In this section we obtain upper and lower bounds on the second largest eigenvalue of the lumped chain, $\bar{K}$. We then use these to obtain bounds on the relaxation time and mixing time. We start by using Poincare and Cheeger bounds to obtain bounds on the second largest eigenvalue.

Lemma 5.1. The transition probability of the Burnside process on $\Pi_n$ satisfies

$$\bar{K}(x, y) \geq \frac{1}{(n+1)^{n-1}}$$

for all $x, y \in \Pi_n$ and all $k \geq n$. 
Proof. The proof proceeds similarly as the proof of Lemma 4.4. By inspection of the transition probability in Proposition 3.1, \( \overline{K}(x, y) \) is minimized when \( j_x = 1, j_y = n, \) and \( k = n + 1. \) Plugging these values into the formula for \( \overline{K} \) yields
\[
\overline{K}(x, y) \geq \frac{n!}{(n-1)!} \cdot \frac{1}{(n+1)^n} + \frac{n!}{n!} \cdot \frac{1}{(n+1)^n} = \frac{n+1}{(n+1)^{n-1}} = \frac{1}{(n+1)^{n-1}}.
\]

The next theorem provides bounds on the second largest eigenvalue of the Burnside process on set partitions.

**Theorem 5.2.** The second largest eigenvalue of the transition matrix \( \overline{K} \), of the Burnside process on \( \Pi_n \) satisfies
\[
1 - \frac{5 \log n}{n} \leq \lambda_1 \leq 1 - \frac{B_n}{(n+1)^{n-1}}
\]
for all \( k \geq n \geq 6. \)

Proof. For the upper bound, we use the canonical paths method. The underlying graph of \( \overline{K} \) is the complete graph on \( B_n \) vertices, where each vertex has a self loop, since \( Q(x, y) > 0 \) for all \( x, y \in \Pi_n \). For all pairs \( x, y \in \Pi_n \), define the canonical path \( \gamma_{xy} \) to be the unique edge connecting \( x \) and \( y \). That is, the canonical paths are the geodesics of the underlying graph. Let \( \Gamma \) be the set of all canonical paths.

We compute the geometric quantity \( \tilde{\kappa} \) to be
\[
\tilde{\kappa} = \max_{e} Q(e)^{-1} \sum_{\gamma_{xy} \in e} |\gamma_{xy}| \frac{\overline{\pi}(x)}{\overline{\pi}(y)} = \max_{(x,y)} \frac{1}{\overline{\pi}(x) \overline{\pi}(y)} \overline{K}(x,y)
\]
\[
= \frac{1}{B_n \min_{(x,y)} K(x,y)} \leq \frac{(n+1)^{n-1}}{B_n},
\]
where we used Lemma 5.1 in the last inequality. Hence by Proposition 2.3
\[
\lambda_1 \leq 1 - \frac{1}{\tilde{\kappa}} \leq 1 - \frac{B_n}{(n+1)^{n-1}}.
\]

Next we turn to the lower bound. Let \( x_1, \ldots, x_{B_n} \) denote the set partitions in \( \Pi_n \) ordered by the number of blocks, as defined in Section 3. Define the set \( S = \{x_1, \ldots, x_{[B_n/2]}\} \). Then
\[
\frac{Q(S, S^c)}{\overline{\pi}(S) \overline{\pi}(S^c)} = \frac{\overline{\pi}(x) \sum_{x \in S} \sum_{y \in S^c} \overline{K}(x,y)}{\overline{\pi}(S) \overline{\pi}(S^c)} = \frac{1}{B_n} \sum_{x \in S} \sum_{y \in S^c} \overline{K}(x,y)
\]
\[
\leq \frac{5}{B_n} \sum_{x \in S} \sum_{y \in S^c} \overline{K}(x,y),
\]
where the sum satisfies
\[
\sum_{x \in S} \sum_{y \in S^c} \overline{K}(x,y) < B_{n-1},
\]
for all \( k \geq n \geq 6. \)
Therefore by Proposition 2.4,
\[ \lambda_1 \geq 1 - \frac{Q(S, S^c)}{\pi(S)\pi(S^c)} \geq 1 - \frac{5}{B_n} \sum_{x \in S} \sum_{y \in S^c} K(x, y) \geq 1 - \frac{5B_{n-1}}{B_n}. \]

Harper [16] showed that the quantity \( \frac{B_{n+1}}{B_n} \) is of the order \( \frac{n}{\log n} \). In fact, \( \frac{B_n}{B_{n-1}} \geq \frac{n}{\log n} \) for all \( n \geq 4 \). This establishes the lower bound
\[ \lambda_1 \geq 1 - \frac{5 \log n}{n}. \]

□

Taking the limit as \( n \) goes to infinity in the inequalities in Theorem 5.2 shows that the second largest eigenvalue converges to 1.

**Corollary 5.3.** Let \( \lambda_1 \) be the second largest value of the Burnside process on \( \Pi_n \). Then \( \lambda_1 \to 1 \) as \( n \to \infty \).

Our computer simulations suggest that the true value of \( \lambda_1 \) is much closer to the lower bound. We conjecture that the second largest eigenvalue of the Burnside process on \( \Pi_n \) satisfies \( \lambda_1 \leq 1 - \log n \frac{n}{n} \).

Finally, our bounds on the second largest eigenvalue directly translate to bounds on the relaxation time.

**Proposition 5.4.** For all \( k \geq n \geq 6 \), the mixing time of the Burnside process on \( \Pi_n \) satisfies
\[ \left( \frac{n}{5 \log n} - 1 \right) \log \left( \frac{1}{2\varepsilon} \right) \leq t_{\text{mix}}(\varepsilon) \leq \left\lceil \frac{(n + 1)^{n-1}}{B_n} \log \left( \frac{B_n}{\varepsilon} \right) \right\rceil. \]

**Proof.** By Theorem 5.2 the relaxation time satisfies
\[ \frac{n}{5 \log n} \leq t_{\text{rel}} \leq \frac{(n + 1)^{n-1}}{B_n}. \]

Applying Proposition 2.5 gives the desired bounds. □

Therefore combining Proposition 5.4 with Lemma 3.2 we obtain a lower bound on the mixing time of the Burnside process on \([k]^n\).

**Corollary 5.5.** For all \( k \geq n \geq 6 \), the mixing time of the Burnside process on \([k]^n\) satisfies
\[ t_{\text{mix}}(\varepsilon) \geq \left( \frac{n}{5 \log n} - 1 \right) \log \left( \frac{1}{2\varepsilon} \right). \]

Note that the lower bound depends only on \( n \) and not on \( k \). We remark that with a more careful analysis in the proof of Theorem 5.2 one should be able to obtain a lower bound which depends on both \( n \) and \( k \).
6. Final Remarks

6.1. There are other algorithms for uniformly sampling from set partitions of \([n]\) that are not based on a Markov chain. Perhaps the most well known is Stam’s algorithm \([32]\). The starting point is Dobinski’s formula,

\[
B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.
\]

This states that the \(n\)th moment of a Poisson random variable with mean 1 equals \(B_n\). Then for \(n \geq 0\),

\[
\mu_n(k) = \frac{1}{eB_n} \frac{k^n}{k!}
\]

defines a probability measure on \(k \in \mathbb{N}\).

Stam’s algorithm proceeds as follows. Pick an integer \(N\) according to the distribution \(\mu_n\). Then drop \(n\) labeled balls uniformly among \(N\) urns. Finally, form a set partition of \([n]\) such that \(i, j \in [n]\) are in the same block if and only if the balls labeled \(i\) and \(j\) are in the same urn.

Stam showed that the set partition generated by this algorithm is a uniformly random set partition of size \(n\). Moreover observe that after dropping the balls in the urns that some of the urns may be empty. Stam showed that the number of empty urns is Poisson distributed and is independent from the generated set partition.

Another algorithm for exact sampling is Arratia and DeSalvo’s probabilistic divide-and-conquer (PDC) algorithm \([4]\). The PDC algorithm divides the sample space into two parts, samples each part separately, then appropriately pieces them back together to form an exact sample from the target distribution.

More precisely, we start by decomposing the sample space \(\Omega\) by sets \(A\) and \(B\), such that \(\Omega = A \times B\). For a random variable \(X \in \Omega\) and an event \(E\), suppose that we can write \(X\) as \(X = (A, B)\) for some random variables \(A \in A\) and \(B \in B\), where \(A\) and \(B\) are independent, and \(\mathcal{L}(X | E) = \mathcal{L}((A, B) | E)\), where \(\mathcal{L}(X)\) denotes the distribution of \(X\). We sample from \(\mathcal{L}(A | E)\), say observing \(x\), then we sample from \(\mathcal{L}(B | E, A = x)\), say observing \(y\). Then the PDC lemma \((4, \text{Lemma } 2.1)\) states that \((x, y)\) is an exact sample from \(\mathcal{L}(X | E)\) when \(E\) has positive probability.

6.2. Spectral methods can be used to obtain sharp bounds on the mixing time of Markov chains. In our setting, the complicated formula for the transition probabilities \(K(x, y)\) makes spectral decomposition intractable. We have been unable to diagonalize the transition matrices \(K\) or \(\bar{K}\) for general values of \(n\) and \(k\), and we suspect that getting exact formulas for the characteristic polynomial, eigenvalues, and eigenvectors is a hard open problem.

It may be possible to extrapolate the behavior of the Burnside process from small \(n\) values. For example, let \(n = 2\) with \(k \geq 2\). The transition matrix is

\[
\bar{K} = \begin{pmatrix}
1 - \frac{1}{k}, & \frac{1}{k} \\
\frac{1}{k}, & 1 - \frac{1}{k}
\end{pmatrix}
\]
with stationary distribution is $\pi = (1/2, 1/2)$. The eigenvalues are $\lambda = 1$ and $\lambda = 1 - 2 \left(\frac{k}{k^2}\right)$. It follows that $c_\varepsilon \leq \tau_{\text{mix}}(\varepsilon) \leq C_\varepsilon$, where $c_\varepsilon, C_\varepsilon$ are constants which do not depend on $k$.

For $n = 3$ with $k \geq 3$, we can write the transition matrix in block matrix form as

$$
\begin{pmatrix}
S_{3,k,1} + S_{3,k,2} & 2S_{3,k,2} + S_{3,k,3} & 3S_{3,k,3} + S_{3,k,4} \\
2S_{3,k,2} + S_{3,k,3} & 2S_{3,k,2} + 4S_{3,k,3} + S_{3,k,4} & 6S_{3,k,3} + 6S_{3,k,4} + S_{3,k,5} \\
3S_{3,k,3} + S_{3,k,4} & 6S_{3,k,3} + 6S_{3,k,4} + S_{3,k,5} & 6S_{3,k,3} + 18S_{3,k,4} + 9S_{3,k,5} + S_{3,k,6}
\end{pmatrix}
$$

where the dashed lines denote the submatrices $A_{ij}$ as defined in Section 3 and

$$S_{n,k,j} := E \left[ \frac{1}{(Y + j)^n} \right]
$$

where $Y$ is the number of fixed points of a uniformly random permutation $\sigma \in S_{k-j}$.

Even for this small case the characteristic polynomial does not admit a nice formula. Setting $k = 3$ gives

$$K = \begin{pmatrix}
5/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
1/9 & 2/9 & 2/9 & 2/9 & 2/9 \\
1/9 & 2/9 & 2/9 & 2/9 & 2/9 \\
1/9 & 2/9 & 2/9 & 2/9 & 2/9 \\
1/9 & 2/9 & 2/9 & 2/9 & 2/9
\end{pmatrix}
$$

so that the eigenvalues are $\lambda = 1$, $\lambda = 4/9$, and $\lambda = 0$ (with multiplicity 3).

6.3. In [12], Diaconis and Zhong introduced a generalization of the original Burnside process, called the twisted Burnside process. Let $G$ be a finite group acting on a finite set $X$. Let $X_g$ be the fixed set of $g \in G$ and let $G_x$ be the stabilizer of $x \in X$. Let $w$ be a positive weight on $G$ such that $W(x) = \sum_{g \in G_x} w(g)$ and let $v$ be a positive weight on $X$ such that $V(g) = \sum_{x \in X_g} v(x)$.

The twisted Burnside process is a Markov chain on $X$ whose transition between states $x$ and $y$ is:

- From $x \in X$, choose $g \in G_x$ with probability $\frac{w(g)}{W(x)}$;
- From $g$, choose $y \in X_g$ with $\frac{v(y)}{V(g)}$.

The transition matrix is given by

$$K(x, y) = \frac{v(y)}{W(x)} \sum_{g \in G_x \cap G_y} \frac{w(g)}{V(g)}
$$

with stationary distribution $\pi(x) \propto W(x)v(x)$ for all $x \in X$.

In fact, if $w$ is constant on each conjugacy class of $G$ and $v$ is constant on each orbit of $X$ under the action of $G$, then the Markov chain can be lumped into orbits of $X$. The lumped chain is reversible with respect to $\tilde{\pi}(O_x) \propto \frac{W(x)v(x)}{|O_x|}$, where $O_x$ is the orbit containing $x$.

One direction of further study is to consider a twisted Burnside process on $[k]^n$ by replacing the uniform distribution on $S_k$ used in the first step of the Burnside process with a distribution that is constant on conjugacy classes of $S_k$. Many examples of
such distributions can be found in [7]. One can then obtain bounds on the mixing time of the twisted Burnside process.

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Department of Mathematics & Statistics, McMaster University, Hamilton, ON, L8S 4K1, Canada

E-mail address: paguyoj@mcmaster.ca