Improved Small Domain Estimation via Compromise Regression Weights

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\textbf{ABSTRACT}
Shrinkage estimates of small domain parameters typically use a combination of a noisy "direct" estimate that only uses data from a specific small domain and a more stable regression estimate. When the regression model is misspecified, estimation performance for the noisier domains can suffer due to substantial shrinkage toward a poorly estimated regression surface. In this article, we introduce a new class of robust, empirically-driven regression weights that target estimation of the small domain means under potential misspecification of the global regression model. Our regression weights are a convex combination of the model-based weights associated with the best linear unbiased predictor (BLUP) and those associated with the observed best predictor (OBP). The mixing parameter in this convex combination is found by minimizing a novel, unbiased estimate of the mean-squared prediction error for the small domain means, and we label the associated small domain estimates the "compromise best predictor," or CBP. Using a data-adaptive mixture for the regression weights enables the CBP to preserve the robustness of the OBP while retaining the main advantages of the EBLUP whenever the regression model is correct. We demonstrate the use of the CBP in an application estimating gait speed in older adults. Supplementary materials for this article are available online.

\section{Introduction}

Analyzing clustered data where the targets of estimation are the cluster, area, or "unit"-specific attributes is an important task that arises in a wide range of applied contexts. Common examples include estimating disease burden in specific geographic regions (e.g., Wakefield 2007), estimating subgroup-specific treatment effects in clinical trials (e.g., Jones et al. 2011), quantifying hospital performance (e.g., Normand et al. 2016), and analyzing measures of gene expression (e.g., Smyth 2004). A feature of many such applications is the availability of a "direct" estimate for each unit, large standard errors for many of these direct estimates, and considerable heterogeneity in estimation precision across units. When using hierarchical models to stabilize direct estimates and predictions for a collection of units, shrinkage estimates of unit-specific parameters often arise as a weighted combination of the direct, unit-specific estimate and a regression prediction for that unit. The direct estimates, while unbiased, typically have large variance, and while the regression estimates are biased, they are usually much more stable than the direct estimates. Typically, shrinkage estimates of the unit-specific parameters are obtained by taking a weighted average of the direct estimates and the regression prediction with more influence coming from the regression model for units with larger variance. In standard practice (i.e., maximum likelihood estimation), the regression model itself is estimated by using regression weights which place more importance on units with smaller estimation variance. Consequently, while shrinkage estimates for high-variance units are more influenced by the regression estimate, they play a relatively minor role in determining the form of the fitted regression model itself. In other words, as noted in Jiang, Nguyen, and Rao (2011), the units that should really "care about" the regression model have relatively little impact on its estimation.

When computing shrinkage estimates of unit-specific mean parameters, giving the relatively unstable units additional weight when estimating the regression model can substantially reduce overall bias while increasing variance. Overall, this may or may not reduce the mean-squared prediction error (MSPE) of the procedure. When the model is correctly specified, the MLE regression weights are optimal and cannot be improved upon. Under model misspecification however, regression weights targeting reduced prediction error can often result in substantial improvements in MSPE. Best predictive estimates (BPEs) of the regression coefficients (Jiang, Nguyen, and Rao 2011) target minimization of the MSPE for the resulting shrinkage estimates without relying on an assumption of correct model specification. In particular, the BPEs are found by minimizing an “observed” MSPE associated with a particular choice of regression coefficients. In contrast to the MLE regression weights which minimize estimation variance when the model is correctly specified, the regression weights used in the BPE instead minimize a squared bias term which depends on the degree of model misspecification.

A natural way of building upon the strengths of the MLE and BPE weighting schemes is to allow, in a limited way, the form...
of the regression weights to depend on the observed responses. This enables the regression weights to adapt to the extent of regression function misspecification and the magnitude of the variance associated with a given set of shrinkage estimates. In this article, we consider regression weights that are an empirically-determined convex combination of the MLE and BPE regression weights. Such adaptive “compromise” regression weights will automatically be closer to the MLE regression weights when the model is well-specified but will more closely resemble the BPE weights in scenarios with substantial model misspecification. Using such compromise regression weights to compute shrinkage estimates of small domain parameters can offer the robustness of the BPE while having MSPE performance which is close to the model-based estimates in cases where the model is well specified. In addition, our compromise regression weights only depend on a single additional tuning parameter, namely the mixture term in the convex combination of the MLE and BPE regression weights, and hence estimating this additional tuning parameter will not introduce substantial additional estimation variance.

The foregoing discussion sets the context for our estimation approach that uses compromise regression weights constructed with the main goal of providing effective estimation of unit-specific parameters that can adapt to varying degrees of mean function misspecification. Our compromise regression weights induce a class of shrinkage estimates that depends only on a mixing parameter and a variance component. To determine these terms empirically, we propose minimizing an unbiased estimate misspecification. Our compromise regression weights resemble the BPE weights in scenarios with substantial model misspecification. Using such compromise regression weights to compute shrinkage estimates of small domain parameters can offer the robustness of the BPE while having MSPE performance which is close to the model-based estimates in cases where the model is correctly, or nearly correctly specified. In this article, we consider regression weights that are an alternative form of the working assumptions, the maximum likelihood estimate of the parameter is computed and alternative values of the variance components are simply plugged in. This alternative form of the regression function misspecification and the magnitude of the variance associated with a given set of shrinkage estimates.

Section 2. Regression Weights for Small Domain Estimates

We let \(Y_1, \ldots, Y_K\) denote measurements made from \(K\) separate units with each \(Y_k\) representing a direct estimate of a corresponding parameters of interest \(\theta_k\). In addition to the direct estimate \(\hat{Y}_k\), each unit \(k\) has an associated \(p \times 1\) vector of covariates \(x_k\). The main goal here is to estimate each \(\theta_k\) by combining the direct estimate \(\hat{Y}_k\) with a regression prediction that uses covariate information \(x_k\). Rather than assume a particular regression model to describe the variation in the \(\theta_k\) we instead, as in Jiang, Nguyen, and Rao (2011), use a mixed model formulation which does not depend on an assumed regression structure for the unit-specific means. Specifically, we consider the following mixed model representation

\[
\begin{align*}
Y_k &= \mu_k + v_k + \epsilon_k, \quad k = 1, \ldots, K, \\
E(\epsilon_k) &= E(v_k) = 0, \\
\text{var}(v_k) &= \tau^2, \quad \text{var}(\epsilon_k) = \sigma^2_k, \quad v_k \perp \epsilon_k, \\
\end{align*}
\]

where the notation \(v_k \perp \epsilon_k\) means that \(v_k\) and \(\epsilon_k\) are independent random variables. In addition to assuming that \(v_k\) and \(\epsilon_k\) are independent, we assume the values of \(\sigma^2_k\) are known.

Of primary interest is estimation/prediction of the mixed effects

\[
\theta_k = \mu_k + v_k,
\]

with mean-squared prediction error (MSPE) serving as the main measure of performance. For a given estimate/predictor \(\hat{\theta}\) of the vector \(\theta = (\theta_1, \ldots, \theta_K)\) of mixed effects, the MSPE is defined as

\[
\text{MSPE}(\hat{\theta}) = E(\{(\hat{\theta} - \theta)^T(\hat{\theta} - \theta)\}) = \sum_{k=1}^K E((\hat{\theta}_k - \theta_k)^2).
\]

Note that the \(\theta_k\) are often referred to as predictors in the context of mixed models, but we will use the terms estimates and predictors interchangeably when referring to the \(\hat{\theta}_k\).

In estimating \(\theta_k\), it is often assumed, as in the well-known Fay-Herriot model (Fay and Herriot 1979), that the means \(\mu_k\) are related to the unit-specific covariates \(x_k\) via \(\mu_k = x_k^\top \beta\). We will make such an assumption when deriving the form of particular estimating procedures, but we evaluate MSPE under the more general mixed model formulation (1). For instance, if we add to model (1) the three additional working assumptions that \(\mu_k = x_k^\top \beta\), \(v_k \sim N(0, \tau^2)\), and \(\epsilon_k \sim N(0, \sigma^2_k)\), the “estimate” of \(\theta_k\) which minimizes MSPE for known values of \(\beta\) and \(\tau^2\) is

\[
\theta_k(\beta) = E(\theta_k | Y_k, \beta, \tau^2) = B_{k,\tau} x_k^\top \beta + (1 - B_{k,\tau}) Y_k,
\]

where \(B_{k,\tau} = \sigma^2_k / (\sigma^2_k + \tau^2)\). Additionally, under these three working assumptions, the maximum likelihood estimate of the
regression coefficients $\beta$ (for an assumed value of $\tau$) is the following quantity

$$
\hat{\beta}_{\text{MLE}} = (X^T V^{-1} X)^{-1} X^T V^{-1} Y,
$$

(4)

where $Y = (Y_1, \ldots, Y_K)$ is the vector containing the $K$ direct estimates, $X$ is the $K \times p$ matrix whose $k$th row is $x_k^T$, and $V = \tau I + V_{Y|\theta}$. Here, $V_{Y|\theta} = \text{diag}(\sigma_1^2, \ldots, \sigma_K^2)$ denotes the covariance matrix of $Y$ conditional on the vector of mixed effects $\theta$. If one plugs in $\beta_{\text{MLE}}$ into (3), the associated estimates of the $\theta_k$ are

$$
\hat{\theta}_k = \hat{\theta}_k(\hat{\beta}_{\text{MLE}}) = B_k^T x_k^T \hat{\beta}_{\text{MLE}} + (1 - B_k x_k^T) Y_k.
$$

(5)

The estimate in (5) is commonly referred to as the best linear unbiased predictor (BLUP) of the mixed effect $x_k^T \beta + v_k$ (see, e.g., Henderson 1975). The BLUPs $\hat{\theta}_k$ are optimal in the sense that, under the assumption that $\mu_k = x_k^T \beta$, they achieve the smallest MSPE within the class of linear unbiased estimators (see, e.g., Rao and Molina (2015) or Datta and Ghosh (2012)). Beyond the assumptions of model (1), the optimality of the BLUPs in (5) only relies on the assumption that $\mu_k = x_k^T \beta$ and that the variance of $v_k$ is correctly specified (i.e., $\tau^2 = \tau_k^2$) and does not rely on any normality assumptions. In practice, $\tau$ is not usually known and is estimated from the data. When $\hat{\beta}_{\text{MLE}}$ and the $B_k$ are computed using an estimated value of $\tau$, the resulting mixed-effects estimates in (5) are usually referred to as the empirical best linear unbiased estimates (EBLUPs).

As an alternative to the BLUPs of the mixed effects, Jiang, Nguyen, and Rao (2011) suggest plugging the following estimate of the regression coefficients into (3)

$$
\hat{\beta}_{\text{BPE}} = (X^T B^2_k X)^{-1} X^T B^2_k Y,
$$

(6)

where $B_t = \text{diag}(B_{1,t}, \ldots, B_{K,t})$. Jiang, Nguyen, and Rao (2011) refer to $\hat{\beta}_{\text{BPE}}$ as the best predictive estimator (BPE) of $\beta$, and they refer to the associated mixed effect estimates $\hat{\theta}_k = \hat{\theta}_k(\hat{\beta}_{\text{BPE}})$ as the observed best predictor (OBP). The BPE is the best estimator in the sense that it is the vector of regression coefficients minimizing the following estimator $\hat{Q}(\beta)$ of the MSPE associated with any predictor $\hat{\theta}_k(\beta)$ of the form (3)

$$
\hat{Q}(\beta) = C + \sum_{k=1}^K B^2_{k,t}(x_k^T \beta)^2 - 2 \sum_{k=1}^K B^2_{k,t}(x_k^T \beta) Y_k,
$$

(7)

where $C$ is a constant not depending on $\beta$. The quantity $\hat{Q}(\beta)$ is an unbiased estimate of the MSPE associated with the best predictor (3) when both $\beta$ and $\tau$ are assumed to be fixed. Upon inspection of (4) and (6), both the MLE and the BPE of $\beta$ may be viewed as weighted least-squares estimates with regression weights $w^\text{MLE}(\tau) \propto 1/(\tau^2 + \sigma_k^2)$ and $w^\text{BPE}(\tau) \propto (\sigma_k^2/(\tau^2 + \sigma_k^2))^2 = B^2_{k,t}$, respectively. Relative to the MLE, the BPE uses regression weights which assign greater weight to units with larger sampling variances $\sigma_k^2$. Hence, the BPE enables the higher variance units to have more influence in determining the form of the estimate of $\beta$. In the context of prediction, this is sensible because it is the units with the largest sampling variances that are shrunk more toward the fitted regression surface $x_k^T \hat{\beta}$ while the mixed-effects estimates for low-variance units are impacted much less from the fitted regression. In this sense, the fitted regression surface is more important for the highly variable units, and relative to the MLE, the BPE lets the more variable units play a larger role in fitting this regression surface.

### 2.1 Estimating the MSPE for Arbitrary Regression Weights

While both the BLUP and OBP possess specific optimality properties, these procedures may be potentially improved upon by examining the MSPE associated with an arbitrary weighted least-squares estimate of the regression coefficients. To this end, we consider a vector of unit-specific weights $w = (w_1, \ldots, w_K)$ with $w_k \geq 0$, $\sum_k w_k = 1$ and where each $w_k$ will usually depend on an assumed value of $\tau$. For the choice of regression weights $w$, the corresponding weighted least-squares estimate of $\beta$ is

$$
\hat{\beta}_w = (X^T W X)^{-1} X^T W Y,
$$

(8)

where $W = \text{diag}(w_1, \ldots, w_K)$. By plugging $\hat{\beta}_w$ into (3), one obtains that the mixed-effects estimates $\hat{\theta}_k(\hat{\beta}_w)$ associated with these weights are $\hat{\theta}_k(\hat{\beta}_w) = \hat{\theta}_k(\hat{\beta})$. Note that the vector of mixed-effects estimates $\hat{\theta}(w, \tau) = (\hat{\theta}_{1,w}, \ldots, \hat{\theta}_{K,w})^T$ is a linear predictor that can be expressed as

$$
\hat{\theta}(w, \tau) = (U_{w,\tau} + I) Y,
$$

(9)

where $U_{w,\tau}$ is the $K \times K$ matrix defined as

$$
U_{w,\tau} = B_t \left( X(X^T W X)^{-1} X^T W - I \right),
$$

(10)

and where $B_t$ is as defined in (6). The mixed-effects estimates defined in (9) can be thought of as defining a class of mixed-effects estimates indexed by both $\tau$ and the vector of unit-specific regression weights $w$.

For fixed weights $w$ and an assumed value of $\tau$, we let $\text{MSPE}(w, \tau) = \text{MSPE}(\hat{\theta}(w, \tau))$ denote the MSPE associated with $\hat{\theta}(w, \tau)$. This is given by

$$
\text{MSPE}(w, \tau) = \mu^T U_{w,\tau}^{-1} U_{w,\tau} \mu + \text{tr}\{U_{w,\tau}^{-1} I\} V_{Y|\theta} (U_{w,\tau} + I)^T + \tau_0^2 \text{tr}\{U_{w,\tau}^{-1} U_{w,\tau}\}.
$$

(11)

While MSPE($w$, $\tau$) is unobservable, one may use the fact that the vector $Y$ has mean $\mu$ and covariance matrix $\tau_0^2 I + V_{Y|\theta}$ to show that, for fixed weights $w$ and $\tau \geq 0$, the following quantity is an unbiased estimator of (11)

$$
\hat{\text{MSPE}}_{\tau}(w, \tau) = Y^T U_{w,\tau}^{-1} U_{w,\tau} Y - 2\text{tr}\{U_{w,\tau} V_{Y|\theta}\} + \text{tr}\{V_{Y|\theta}\}.
$$

(12)

It is worth emphasizing that $\hat{\text{MSPE}}_{\tau}(w, \tau)$ is an unbiased estimator of MSPE($w$, $\tau$) under the assumption that the weight vector $w$ and $\tau$ are fixed, and the unbiasedness does not hold when either $w$ or $\tau$ are determined from the data. The estimator $\hat{\text{MSPE}}_{\tau}(w, \tau)$ is equivalent to Stein's unbiased risk estimate (SURE) (Stein (1981)) when $\sigma_k^2 = \cdots = \sigma_1^2$, and hence, $\hat{\text{MSPE}}_{\tau}(w, \tau)$ may be viewed as a SURE-type estimator where heteroscedasticity is taken into consideration. In the special case of equal regression weights and an intercept-only model (i.e., $X$ only has an intercept term), $\hat{\text{MSPE}}_{\tau}(w, \tau)$ is equivalent to the unbiased risk estimate of the shrinkage toward the grand mean estimator described in Xie, Kou, and Brown (2012).
It is worth noting that the unbiasedness of \( \hat{M}_K(w, \tau) \) only relies on the assumptions of model (1) and does not require any further assumptions about the distributions of \( v_k \) or \( e_t \). Moreover, as stated in the following theorem, this unbiasedness holds even if one is interested evaluating the MSPE conditional on the unobserved \( \theta \) rather than marginally over \( \theta \).

**Theorem 1.** Under model (1), \( \hat{M}_K(w, \tau) \) is an unbiased estimator of MSPE\((w, \tau)\) in the sense that

$$E[\hat{M}_K(w, \tau)] = \text{MSPE}(w, \tau).$$

Moreover, the expectation of \( \hat{M}_K(w, \tau) \) conditional on \( \theta \) is equal to the conditional MSPE associated with \( \hat{\theta}(w, \tau) \)

$$E[\hat{M}_K(w, \tau)|\theta] = E\left[\left(\hat{\theta}(w, \tau) - \theta\right)^T \left(\hat{\theta}(w, \tau) - \theta\right) | \theta\right].$$

### 2.2. Compromise Regression Weights and the CBP

The quantity \( \hat{M}_K(w, \tau) \) is only guaranteed to be an unbiased estimate of the MSPE of \( \hat{\theta}(w, \tau) \) for fixed weights, and hence may be inappropriate for evaluating the MSPE associated with data-determined weights. Nevertheless, comparing \( \hat{M}_K(w, \tau) \) for different weights can be a useful way for choosing among different weighting schemes when such weights are indexed by a small number of hyperparameters. To allow the form of the weights to be partially driven by the observed value of \( \hat{M}_K(w, \tau) \) without spending many additional degrees of freedom, we consider a family of weights that are convex combinations of the MLE and BPE weights. This only requires that we estimate one additional hyperparameter (i.e., the mixing parameter) when compared to the EBLUP or the OBP. Though one could use other weights to form the components of a convex combination, the choice of the MLE and BPE as the “basis” weights is motivated by a particular decomposition of MSPE\((w, \tau)\) described in Jiang, Nguyen, and Rao (2011). In the following proposition, we state a version of Theorem 1 in Jiang, Nguyen, and Rao (2011) which specializes this theorem to our formulation of the mixed-effects prediction problem.

**Proposition 1.** (Due to Jiang, Nguyen, and Rao 2011) Consider the expression for the MSPE given in (11)

$$\text{MSPE}(w, \tau) = \mu^T U_{w, \tau}^T U_{w, \tau} \mu$$

$$+ \text{tr}\left[(U_{w, \tau} + I) V_{\theta}(U_{w, \tau} + I)^T + \tau_\beta^T U_{w, \tau}^T U_{w, \tau}\right].$$

For fixed \( \tau, \mu^T U_{w, \tau} U_{w, \tau} \mu \) is minimized when \( w \) are the BPE weights, and, when \( \tau \) is fixed at \( \tau_\beta \), the second term is minimized by the MLE weights. Moreover, \( \mu^T U_{w, \tau} U_{w, \tau} \mu = 0 \) whenever \( \mu = X\beta \) for some \( \beta \in \mathbb{R}^p \).

Proposition 1 states that the MSPE may be decomposed into a model misspecification term and a variance term which are minimized by the BPE weights and MLE weights, respectively. This suggests that using weights which compromise between these two weighting schemes can potentially lead to meaningful reductions in MSPE. Moreover, allowing the degree of compromise to be data dependent enables the compromise weights to adapt to the extent of model misspecification and of estimation variance.

To compute the empirically driven compromise weights, we adopt a direct approach which uses a convex combination of the MLE and BPE weights. Specifically, for \( \alpha \in [0, 1] \), we consider the family of compromise weights

$$w^\alpha(\alpha, \tau) = (w^\alpha_1(\alpha, \tau), \ldots, w^\alpha_K(\alpha, \tau))^T,$$

where the \( k \)th element of \( w^\alpha(\alpha, \tau) \) is a convex combination of the \( k \)th MLE weight \( w^\text{mle}_k(\alpha, \tau) \) and the \( k \)th BPE weight \( w^\text{bpe}_k(\alpha, \tau) \)

$$w^\alpha_k(\alpha, \tau) = \alpha w^\text{mle}_k(\alpha, \tau) + (1 - \alpha) w^\text{bpe}_k(\alpha, \tau)$$

$$= \frac{\alpha}{\sum_k (\sigma^2_k + \tau^2)} + \frac{(1 - \alpha)(\sigma^2_k/(\sigma^2_k + \tau^2))^2}{\sum_k (\sigma^2_k/(\sigma^2_k + \tau^2))^2}.$$

(13)

To determine the optimal values of \( \alpha \) and \( \tau \) for the compromise weights, we minimize the estimate \( \hat{M}_K(w^\alpha(\alpha, \tau), \tau) \) of the MSPE associated with this vector of regression weights. Because this estimator only depends on \( (\alpha, \tau) \) when using the family of weights (13), we henceforth use \( \hat{M}^\alpha(\alpha, \tau) = \hat{M}_K(w^\alpha(\alpha, \tau), \tau) \) to denote the unbiased MSPE estimator when using compromise weights \( w^\alpha(\alpha, \tau) \). Using \( \hat{M}^\alpha(\alpha, \tau) \), the optimal values \( (\alpha^*, \tau^*) \) for the compromise weights are determined empirically as

$$\alpha^*, \tau^* = \arg \min_{\alpha \in [0,1], \tau \geq 0} \hat{M}_K(w^\alpha(\alpha, \tau), \tau)$$

$$= \arg \min_{\alpha \in [0,1], \tau \geq 0} \hat{M}^\alpha(\alpha, \tau).$$

(14)

Recalling (8), the weights \( w^\alpha(\alpha^*, \tau^*) \) will generate the following empirically driven compromise estimate \( \hat{\beta}_\text{cure} \) of the fixed effects regression coefficients

$$\hat{\beta}_\text{cure} = (X^TW^\text{cure}_{\alpha^*, \tau^*} X)^{-1} X^T W^\text{cure}_{\alpha^*, \tau^*} Y,$$

(15)

where \( W^\text{cure}_{\alpha^*, \tau^*} = \text{diag}\{w^\alpha_1(\alpha^*, \tau^*), \ldots, w^\alpha_K(\alpha^*, \tau^*)\} \). We label \( \hat{\beta}_\text{cure} \) the “compromise unbiased risk estimator,” or CURE of the regression coefficients.

The mixed-effects estimates \( \hat{\theta}_k \) associated with the optimal compromise regression weights are then defined as

$$\hat{\theta}_k = B_{k, \tau^*} X_k^T \hat{\beta}_\text{cure} + (1 - B_{k, \tau^*}) Y_k.$$

(16)

We refer to the vector of mixed-effects estimates \( \hat{\theta}_\text{CBP} = (\hat{\theta}_1, \ldots, \hat{\theta}_K)^T \) as the “compromise best predictor” or CBP of \( \theta \). Recalling (9) and (10), we can also express \( \hat{\theta}_\text{CBP} \) as

$$\hat{\theta}_\text{CBP} = \hat{\theta}(w^\alpha(\alpha^*, \tau^*), \tau^*)$$

$$= Y + B_{\tau^*} (X (X^T W^\alpha_{\alpha^*, \tau^*} X)^{-1} X^T W^\alpha_{\alpha^*, \tau^*} - I) Y.$$

In practice, we compute the optimal values \( (\alpha^*, \tau^*) \) in (14) using the constraints \( (\alpha, \tau) \in [0, 1] \times [0, \tau_{\text{max}}] \). The maximal value of \( \tau \) is determined empirically and is set to \( \tau_{\text{max}} = 10 \sqrt{\frac{1}{K} \sum_{k=1}^K (Y_k - \bar{Y})^2} \). Because \( \tau^2 \) represents the variance of the random effects \( v_k \), the sample variance of the \( Y_k \) is likely to be an overestimate of the best value of \( \tau^2 \) as \( \tau^2 \) only accounts for a fraction of the variation in the \( Y_k \). Hence, setting \( \tau_{\text{max}} \) equal to 10 times the sample standard deviation can be interpreted.
as choosing an upper bound which is very likely to be a substantial overestimate of the optimal value of \( \tau \). Minimization of \( \hat{M}_k(\alpha, \tau) \) with respect to these box constraints is performed using the limited-memory BFGS algorithm (Byrd et al. 1995) which we have found to work quite well in this context.

It is also possible to construct compromise regression weights in the context of a nested-error regression model (Battese, Harter, and Fuller 1988) using a very similar approach to the one outlined above. Specifically, one would compute the optimal compromise regression weights by minimizing an unbiased estimator of the MSPE associated with a predictor of the mixed effects. Section F of the supplementary materials describes an unbiased estimator of the MSPE in the context of a nested-error regression model.

### 2.3. Variations of the CBP

We also consider two close variations of the CBP. The first of these, which we call the “plug-in” CBP, uses a restricted maximum likelihood (REML) and an OBP-based estimate of \( \tau \) as the starting point for defining the shrinkage and regression weights and then finds the best value of the convex combination mixing parameter \( \alpha \) assuming the REML and OBP-based estimates are fixed. Another alternative which we explore is the “multi-\( \tau \) CBP. In a variety of numerical experiments, we have observed that the plug-in version of the CBP often has better finite-sample performance than the CBP estimates defined in (16). For very large values of \( K \), we typically see similar performance between the CBP and plug-in CBP. See Section 5 for comparisons of the performance of the CBP using both the plug-in and multi-\( \tau \) approaches.

**The plug-in CBP.** For the plug-in CBP, we consider regression weights \( w_k^{\text{pli}}(\alpha) \) and shrinkage weights \( B_k^1(\alpha) \) of the form

\[
w_k^{\text{pli}}(\alpha) = \alpha w_k^\text{mle}(\hat{\tau}_\text{REML}) + (1 - \alpha) w_k^\text{bpe}(\hat{\tau}_\text{OBP})
\]

\[
B_k^1(\alpha) = \frac{\sigma_k^2}{\sigma_k^2 + \alpha \hat{\tau}_\text{REML}^2 + (1 - \alpha) \hat{\tau}_\text{OBP}^2},
\]

where \( \hat{\tau}_\text{REML} \) denotes the restricted maximum likelihood (REML) estimate of \( \tau \) while \( \hat{\tau}_\text{OBP} \) denotes the OBP-based estimate of \( \tau \). As described in Jiang, Nguyen, and Rao (2011), the OBP-based estimate \( \hat{\tau}_\text{OBP} \) maximizes the following objective function

\[
Q_{\text{OBP}}(\tau) = Y^T(B_k^2 - B_k^1(X^T B_k^1 X)^{-1}X^TB_k^1)Y + 2\tau^T\text{tr}(B_1).
\]

Because \( w_k^{\text{pli}}(\alpha) \) and \( B_k^1(\alpha) \) only depend on \( \alpha \), the unbiased MSPE estimate \( \hat{M}_k(w, \tau) \) defined in (12) will only depend on \( \alpha \). The plug-in CBP regression and shrinkage weights are then obtained by plugging in the value of \( \alpha \) which minimizes \( \hat{M}_k(w, \tau) \) into both \( w_k^{\text{pli}}(\alpha) \) and \( B_k^1(\alpha) \).

**The multi-\( \tau \) CBP.** For the multi-\( \tau \) CBP, we consider weights similar to those for the plug-in CBP except that we do not restrict the variance component terms in the MLE and BPE regression weights to be equal. In particular, the multi-\( \tau \) CBP considers regression weights \( w_k^{\text{mle}}(\tau_0, \tau_1) \) and shrinkage weights \( B_k^2(\alpha, \tau_0, \tau_1) \) of the form

\[
w_k^{\text{mle}}(\tau_0, \tau_1) = \alpha w_k^\text{mle}(\tau_1) + (1 - \alpha) w_k^\text{bpe}(\tau_0)
\]

\[
B_k^2(\alpha, \tau_0, \tau_1) = \frac{\sigma_k^2}{\sigma_k^2 + \alpha \tau_1^2 + (1 - \alpha) \tau_0^2},
\]

with the values of \( (\alpha, \tau_0, \tau_1) \) being chosen to minimize (12).

### 2.4. A More General CBP

In this section, we consider the following more general formulation of the linear mixed model

\[
Y = \mu + Zv + e,
\]

where \( Z \) is a known \( K \times q_v \) model matrix, \( v \) is a random effects vector of length \( q_v \), and \( \mu \) and \( e \) are both vectors of length \( K \). It will be further assumed that \( E(v) = E(e) = 0 \), \( \text{var}(v) = G_v \), and \( \text{var}(e) = \Sigma \), where \( G_v \) and \( \Sigma \) are \( q_v \times q_v \) and \( K \times K \) matrices, respectively. The covariance matrix \( \Sigma \) of \( e \) is assumed to be known, and the entries of the covariance matrix \( G_v \) of \( v \) are assumed to be determined by the \( g \times 1 \) parameter vector \( \lambda = (\lambda_1, \ldots, \lambda_g)^T \). The parameter vector \( \lambda \in \Lambda \) could, for example, represent parameters modeling spatial dependence. The marginal covariance matrix of \( Y \) under model (17) is given by \( \text{Var}(Y) = V_\lambda = ZG_v Z^T + \Sigma \).

We now consider the situation where one is interested in predicting the following \( p_q \times 1 \) vector of mixed effects

\[
\eta = A^T \mu + R^T v,
\]

where \( A \) and \( R \) are fixed \( K \times p_q \) and \( q_v \times p_q \) matrices, respectively. Under model (17) with the additional assumptions that \( \mu = X\beta \), \( v \sim N(0, G_v) \), and \( e \sim N(0, \Sigma) \), the best predictor of \( \eta \) for known \( \beta \) is given by \( \hat{\eta}_W(\beta) = A^T X \hat{\beta} + R^T G_v Z^T V_\lambda^{-1} P_W - A^T P_W \). (18)

Because \( G_v \) or \( \Sigma \) may be nondiagonal, we now consider estimates of \( \beta \) which depend on a potentially nondiagonal positive semidefinite weight matrix \( W \) rather than the diagonal matrix \( W \) considered in Sections 2.1 and 2.2. By plugging in the weighted least-squares estimate \( \hat{\beta}_W = (X^T W X)^{-1} X^T W Y \) into \( \eta(\beta) \) we obtain the class of estimates \( \hat{\eta}_W(\lambda) \) given by

\[
\hat{\eta}_W(\lambda) = \hat{\eta}_W(\lambda) = (L_{W,\lambda} + A^T) Y,
\]

where

\[
L_{W,\lambda} = R^T G_v Z^T V_\lambda^{-1} P_W - A^T P_W.
\]

Here, \( P_W \) denotes the matrix \( P_W = I - X(X^T W X)^{-1} X^T W \), and \( X^T W X \) denotes the Moore-Penrose inverse of \( X^T W X \). The MSPE associated with \( \hat{\eta}_W(\lambda) \) is \( \text{MSPE}(W, \lambda) = E[(\hat{\eta}_W(\lambda) - \eta)^T(\hat{\eta}_W(\lambda) - \eta)] \) which can be shown (see Section C of the supplementary materials) to equal

\[
\text{MSPE}(W, \lambda) = \mu^T L_{W,\lambda}^T L_{W,\lambda} \mu + \text{tr}\left( L_{W,\lambda} V_\lambda L_{W,\lambda}^T \right)
\]

\[
+ 2\text{tr}(L_{W,\lambda} (V_\lambda A - ZG_v R))
\]

\[
+ \text{tr}(A^T V_\lambda A) - 2\text{tr}(R^T G_v Z^T A)
\]

\[
+ \text{tr}(R^T G_v R)\}
\]

(20)

It can also be shown (see Section C of the supplementary materials) that, for fixed weight matrix \( W \) and \( \lambda \),
\[\hat{\beta}_{\text{CURE}}^g = (X^T W_{\alpha \lambda \gamma}^g X + X^T W_{\alpha \lambda \gamma}^q Y)^+ X^T W_{\alpha \lambda \gamma}^g X^T Y, \tag{22}\]

where \(W_{\alpha \lambda \gamma}^{\epsilon q}\) is the positive semidefinite matrix \(W_{\alpha \lambda \gamma}^{\epsilon q} = \alpha V_\lambda^{-1} + (1 - \alpha)(A^T - R^T G_k Z T V_\lambda^{-1})^T \times (A^T - R^T G_k Z T V_\lambda^{-1}).\)

The optimal compromise parameters \(\alpha^q\) and \(\lambda^q\) used in (22) are found by minimizing the unbiased risk estimate (21). Specifically, \((\alpha^q, \lambda^q) = \arg\min_{\alpha \in [0,1], \lambda \in \Lambda} E[\hat{\beta}_{\text{CURE}}^g(W_{\alpha \lambda \gamma}^g, \lambda)].\) Using \(\alpha^q\) and \(\lambda^q\), the CBR \(\tilde{\eta}\) of \(\eta\) is then defined as

\[\tilde{\eta}^\text{CBR} = (L_{W_{\alpha \lambda \gamma}^{\epsilon q} + A^T}) Y.\]

### 3. Estimating a Population Mean

We now consider the case where the primary target of inference is an average of unit-level attributes rather than the unit-level attributes themselves. Such a target of interest may arise, for example, if one is primarily interested in assessing the average test performance of randomly selected schools in a particular region rather than estimating the test performance of individual schools. For this type of goal, we take as the population estimate the equally weighted average of the unit-specific means

\[\mu_0 = \frac{1}{K} \sum_{k=1}^{K} \theta_k, \tag{23}\]

but more general weighted averages may be approached in a similar manner. Note that the single-parameter target (23) is a special case of (18), where \(\mu_0 = \eta, \ A = R, \) and \(A\) is the \(K \times 1\) column vector \(A = (1/K, \ldots, 1/K)^T.\)

An inferential target such as \(\mu_0 = K^{-1} \sum_k \theta_k\) arises often, for example, when one uses a stratified random sample to estimate a population mean. In such cases, the population is divided into \(K\) separate strata/units, and for each stratum/unit \(k, n_k\) responses \(Z_{ik}, i = 1, \ldots, n_k\) are drawn from stratum \(k.\)

The sample mean from stratum \(k, Y_k = \frac{1}{n_k} \sum_{i=1}^{n_k} Z_{ik},\) is an unbiased estimate of the stratum-specific mean \(\theta_k,\) and hence, it is sensible to focus on the “mean model” wherein \(E(Y_k|\theta_k) = \theta_k, \) \(\text{var}(Y_k|\theta_k) = \sigma^2/\tilde{n}_k,\) and where there are no unit-specific covariates used in the sense that the \(n_k\) have some association with the values of \(\theta_k.\)

In this setting, specializing (19) to the estimation of \(\mu_0\) leads to the following family of estimates

\[\hat{\mu}_0(\omega, \tau) = \frac{1}{K} \sum_{k=1}^{K} B_{k, \tau} \left(\sum_{k=1}^{K} \frac{Y_k w_k}{K} + \frac{1}{K} \sum_{k=1}^{K} (1 - B_{k, \tau}) Y_k\right), \tag{24}\]

where \(B_{k, \tau} = \sigma^2 / (\sigma^2 + \tau^2)\) and \(B_{\tau} = \sum_{k=1}^{K} B_{k, \tau}.\)

Following (21), an unbiased estimator of \(E\{\hat{\mu}_0(\omega, \tau) - \mu_0^2\}\) is given by

\[\hat{\mu}_{0, \text{MV}}(\omega, \tau) = \frac{1}{K} \sum_{k=1}^{K} B_{k, \tau} Y_k - \frac{B_{\tau}}{K} \sum_{k=1}^{K} w_k Y_k \right)^2 \tag{25}\]

It is also worth noting that when the \(\theta_k\) are treated deterministically (i.e., \(\tau = 0\)) the BPE weights are uniform (i.e., \(w_{k, \text{BPE}}^2 \sim 1\)), and the corresponding estimate of \(\mu_0\) is the same as the direct estimate. In other words, \(\hat{\mu}_0^\text{direct} = \hat{\mu}_0(w_{k, \text{BPE}}, 0)\) when the vector of BPE weights \(w_{k, \text{BPE}}\) are formed under the assumption that \(\tau = 0.\)

Weights \(w_{k, \text{MV}}\) minimizing the variance (conditional on the \(\theta_k\)) of \(\hat{\mu}_0(\omega, \tau)\) in (24) lead to the following estimator of \(\mu_0\)

\[\hat{\mu}_{0, \text{MV}} = \frac{1}{K \tilde{n}} \sum_{k=1}^{K} Y_k n_k, \tag{27}\]

where \(\tilde{n} = K^{-1} \sum_k n_k.\) When the \(\theta_k\) are treated deterministically so that \(\tau = 0\) and \(B_{k, \tau} = 1,\) the weights \(w_{k, \text{MV}}\) and \(w_{k, \text{MLE}}\) are equivalent. Hence, \(\hat{\mu}_{0, \text{MV}} = \hat{\mu}(w_{\text{MLE}})\) when the vector of MLE weights \(w_{\text{MLE}}\) are formed under the assumption that \(\tau = 0.\)

As in Section 2, we consider an arbitrary convex combination of two weighting schemes in order to improve estimation of \(\mu_0.\) Given two vectors of weights \(w_{0}(\tau)\) and \(w_{1}(\tau),\) compromise parameters \((\alpha, \tau)\) are obtained by minimizing \(E_k(\omega(\alpha, \tau))\), where \(\omega(\alpha, \tau) = \alpha w_{0}(\tau) + (1 - \alpha) w_{1}(\tau).\) As stated in the following theorem, for fixed \(\tau\) the optimal value of the mixing proportion \(\alpha\) actually has a closed-form expression.
Theorem 2. Let \( w_k^1(\tau) \) and \( w_k^0(\tau) \) be two weighting schemes such that \( \sum_k w_k^1(\tau) = \sum_k w_k^0(\tau) = 1 \), for all \( \tau \geq 0 \). Consider compromise weights \( w^c(\alpha, \tau) = (w_1^c(\alpha, \tau), \ldots, w_K^c(\alpha, \tau)) \) defined as \( w_k^c(\alpha, \tau) = \alpha w_k^1(\tau) + (1 - \alpha) w_k^0(\tau) \). If we consider \( \hat{M}_{K,0} (w^c(\alpha, \tau), \tau) \) as a function of \( \alpha \) and \( \tau \) where \( \hat{M}_{K,0} \) is as defined in (25), then, for a fixed \( \tau \) such that \( \sum_k w_k^0(\tau) Y_k \neq \sum_k w_k^1(\tau) Y_k \), the value of \( \alpha \in [0, 1] \) which minimizes \( \hat{M}_{K,0} (w^c(\alpha, \tau), \tau) \) is given by

\[
\alpha_{opt}(\tau) = \begin{cases} 
0, & \text{if } C_2(\tau) \leq 0 \\
1, & \text{if } C_1(\tau) \leq C_2(\tau) \\
C_2(\tau)/C_1(\tau), & \text{otherwise},
\end{cases}
\]

where \( C_1(\tau) \) and \( C_2(\tau) \) are defined as

\[
C_1(\tau) = \frac{B_\tau^2}{K^2} \left( \sum_{k=1}^K (w_k^1(\tau) - w_k^0(\tau)) Y_k \right)^2 \\
C_2(\tau) = \frac{B_\tau^2}{K^2} \left( \sum_{k=1}^K (w_k^1(\tau) - w_k^0(\tau)) Y_k \right) \\
\times \left( \sum_{k=1}^K B_{k,\tau} \left[ Y_k - \sum_{j=1}^K w_j^0(\tau) Y_j \right] \right) - \frac{B_\tau \sigma^2}{K} \sum_{k=1}^K w_k^1(\tau) - w_k^0(\tau) n_k.
\]

Example: Combining the Minimum Variance and Direct Estimates. Suppose we want a compromise estimate based on the minimum variance \( w_k^1(0) = n_k/K\hat{n} \) and the direct estimate weights \( w_k^0(\tau) = 1/K \) while assuming that \( \tau \) is fixed at zero. In this case, \( w_k^1(0) - w_k^0(\tau) = (n_k - \hat{n})/K\hat{n} \), and a direct computation (see Section C of the supplementary materials) shows that

\[
\alpha_{opt}(0) = \min \left\{ \max \left\{ \frac{\sigma^2(\hat{n} - \hat{n})}{K\hat{n}(\hat{\sigma}^2_{MV} - \hat{\sigma}^2_{direct}), 0} \right\}, 1 \right\}.
\]

An alternative approach for estimating a single population quantity such as \( \mu_0 \) is to construct a flexible regression model relating the unit-specific means and unit-specific sample sizes. As described in Zheng and Little (2005) and Little (2004) in the survey context, if sample inclusion is informative, bias can be reduced or removed by building a regression model that adjusts for the association and then uses weighted least squares. Though our concern in this context is informative sample size, the structure of the problem is essentially the same. Similar to Zheng and Little (2005), we consider the following model for the direct estimates \( Y_k \)

\[
Y_k|\theta_k \sim \text{Normal}(\theta_k, \sigma^2/n_k) \\
\theta_k = h(n_k; \beta^h),
\]

where the function \( h \) is a spline with coefficients \( \beta^h \) though more general models for \( h \) could be considered. After using weighted least squares to estimate the spline coefficients \( \beta^h \), the estimate \( \hat{\mu}_0^w \) of \( \mu_0 \) is then obtained by taking the average of the fitted values. That is, \( \hat{\mu}_0^w = K^{-1} \sum_{k=1}^K h(n_k, \beta^h) \). The estimator \( \hat{\mu}_0^w \) of the population average is also very similar to the approach described in Matloff (1981) where an estimator is developed to use additional covariate information when the goal is to estimate the unconditional mean of an outcome. Section 5.3 describes a simulation study comparing the performance of \( \hat{\mu}_0^w, \hat{\mu}_{opt}^w, \hat{\mu}_{MV}^w, \) and a direct compromise estimator which uses \( \alpha_{opt}(0) \) defined in (29) as the compromise parameter.

4. Asymptotic Risk of the CBP

In this section, we compare the mean-squared prediction error of the CBP with the MSPE obtained by an oracle who knows the true values of \( \theta_1, \ldots, \theta_K \) but is restricted to use an estimate of the form (9) with compromise weights of the form (13). To be more precise, first consider the following loss function

\[
\mathcal{L}_K(\hat{\theta}, \theta) = \frac{1}{K} \sum_{k=1}^K \left( \theta_k - \hat{\theta}_k \right)^2.
\]

The (pre-posterior) risk \( \mathcal{R}_K(\hat{\theta}) = E[\mathcal{L}_K(\hat{\theta}, \theta)] \) associated with \( \mathcal{L}_K \) is just the MSPE scaled by the number of units, that is, \( \mathcal{R}_K(\hat{\theta}) = K^{-1} \times \text{MSPE}(\hat{\theta}) \).

With respect to this loss function, the oracle "estimate" \( \hat{\theta}^{OR} \) of \( \theta \) is defined as \( \hat{\theta}^{OR} = \hat{\theta} \{ w^c(\alpha^{OR}, \tau^{OR}), \tau^{OR} \} \) where \( \hat{\theta} \{ w, \tau \} \) is as defined in (9) and (10). The oracle hyperparameters \( (\alpha^{OR}, \tau^{OR}) \) are found by minimizing the (unobservable) loss

\[
(\alpha^{OR}, \tau^{OR}) = \arg \min_{\alpha \in [0,1], \tau \geq 0} \frac{1}{K} \sum_{k=1}^K (\theta_k - \hat{\theta}_k(\alpha, \tau))^2,
\]

where \( \hat{\theta}_k(\alpha, \tau) \) denotes the \( k \)th component of \( \hat{\theta} \{ w^c(\alpha, \tau), \tau \} \). By definition, the oracle risk \( \mathcal{R}_K(\hat{\theta}^{OR}) \) is, for any \( K \), less than or equal to the risk associated with either the CBP, OBP, or EBLUP. Despite this, we show that, under appropriate conditions, the risk obtained by the CBP is asymptotically the same as the oracle risk. Specifically, the difference between the oracle and CBP risk goes to zero as the number of units \( K \) goes to infinity. To show this asymptotic equivalence, we assume that the following conditions hold.

(A1) There is a \( \delta \in (0, 1/2) \) such that

\[
\lim_{K \to \infty} \frac{\sigma^2_{max K}}{K^{1-\delta/4} \sigma^2_{min K}} = 0,
\]

where \( \sigma^2_{max K} = \max\{\sigma^2_1, \ldots, \sigma^2_K\} \) and \( \sigma^2_{min K} = \min\{\sigma^2_1, \ldots, \sigma^2_K\} \).

(A2) For the same \( \delta \in (0, 1/2) \) used in condition (A1),

\[
\lim_{K \to \infty} \frac{1}{K^{1+\delta/2}} \sum_{k=1}^K \mu_k^2 = 0 \quad \text{and} \quad \lim_{K \to \infty} \frac{1}{K^{1+\delta/2}} \sum_{k=1}^K \sigma_k^2 = 0.
\]
(A3) For each $k$, $E(e_k^4) < \infty$, and
\[
\lim_{K \to \infty} \frac{1}{K^2} \sum_{k=1}^{K} \mu_k^4 = 0,
\]
\[
\lim_{K \to \infty} \frac{1}{K^2} \sum_{k=1}^{K} \sigma_k^4 = 0,
\]
\[
\lim_{K \to \infty} \frac{1}{K^2} \sum_{k=1}^{K} E(e_k^4) = 0.
\]

(A4) For each $K$, the design matrix $X$ has full rank. Moreover, for the same $\delta \in (0, 1/2)$ used in conditions (A1) and (A2)
\[
\lim_{K \to \infty} K^{1-\delta/2} D_{\max}(P_X) = 0,
\]
where $D_{\max}(P_X)$ denotes the maximum value of the diagonal elements of the matrix $P_X = X(X^TX)^{-1}X^T$.

Condition (A1) places a fairly moderate restriction on the spread of the $\sigma_k^2$. For instance, condition (A1) would be satisfied if the $\sigma_k^2$ had the form $\sigma_k^2 = \sigma^2/n_k$ for positive integers $n_k$ that were bounded by some number $M$. Condition (A2) is a moderately unrestrictive condition and is one that would be automatically satisfied if both the partial averages $K^{-1} \sum_{k=1}^{K} \mu_k^2$ and $K^{-1} \sum_{k=1}^{K} \sigma_k^2$ converged to some finite limit. Condition (A3) is a very weak assumption about the convergence of the sum of fourth moments. Condition (A4) requires that, for each $K$, the design matrix has full rank. Additionally, condition (A4) places a restriction on the maximal “leverage” $D_{\max}(P_X)$ that any one unit may have. In a classic regression setting, one often expects that $D_{\max}(P_X) = O(1/K)$ to ensure no single observation has undue impact on the fitted regression line. Compared to this, condition (A4) makes the slightly weaker assumption that $D_{\max}(P_X) = o(K^{-1+\delta/2})$.

As in Li (1985), the key to the asymptotic risk optimality of $\hat{\theta}^{\text{CBP}}$ lies in the quality of $\hat{\theta}_M^k(\alpha, \tau)$ as an approximation of the loss function (not the risk). Specifically, the difference between $\hat{\theta}_M^k(\alpha, \tau)/K$ and the associated loss function approaches zero uniformly as $K$ goes to infinity. This result is stated by the following theorem.

**Theorem 3.** Consider the family of estimates $\hat{\theta}(w(\alpha, \tau), \tau)$ with $\hat{\theta}(w, \tau)$ as defined in (9). If conditions (A1)–(A4) hold, then $L(\hat{\theta}, \hat{\theta}(w(\alpha, \tau), \tau)) - L(\hat{\theta}_M^k(\alpha, \tau)/K)$ converges uniformly in $L^1$ to zero. That is,
\[
\lim_{K \to \infty} \mathbb{E} \left( \sup_{\alpha \in [0,1], \tau \geq 0} \left| L(\hat{\theta}, \hat{\theta}(w(\alpha, \tau), \tau)) - L(\hat{\theta}_M^k(\alpha, \tau)/K) \right| \right) = 0.
\]

Due to the close proximity of the loss function and $\hat{\theta}_M^k(\alpha, \tau)/K$, we should expect the performance of the oracle procedure and the CBP to be quite close as $\hat{\theta}^{\text{OR}}$ and $\hat{\theta}^{\text{CBP}}$ use values of $(\alpha, \tau)$ which minimize the loss (32) and $\hat{\theta}_M^k(\alpha, \tau)/K$, respectively. Indeed, as stated in the following theorem, the difference between these risks goes to zero as the number of units goes to infinity.

**Theorem 4.** If conditions (A1)–(A4) hold, then
\[
\lim_{K \to \infty} \left[ R_K(\hat{\theta}^{\text{CBP}}) - R_K(\hat{\theta}^{\text{OR}}) \right] = 0.
\]

Theorem 4 establishes that the risk associated with the CBP is as good (asymptotically) as the oracle risk. Additionally, it follows from Theorem 4 that the CBP risk is asymptotically at least as good as any other procedure using an estimate of the form $\hat{\theta}(w(\alpha, \tau), \tau)$—a class of estimates which includes the OBP, different versions of the EBLUP, or other procedures which might use alternative combinations of the OBP and EBLUP regression weights.

## 5. Simulation Studies

To evaluate the performance of the CBP and compare it with other methods, we conducted three main simulation studies. In the first two simulation studies, we compared the CBP and plug-in CBP with the following four approaches for predicting the mixed effects $\theta_k$: the OBP and three different versions of the EBLUP which vary according to how $\tau$ is estimated. For these three versions of the EBLUP, we consider the following approaches for estimating $\tau$: marginal maximum likelihood (MLE), restricted maximum likelihood (REML), and unbiased risk estimation (URE). For the EBLUP with the URE of $\tau$, the unbiased risk estimate $\hat{\tau}_{\text{URE}}$ is found by minimizing $\hat{M}_k(w(\alpha, \tau))$ with respect to $\tau$ when the regression weights are assumed to be the MLE weights $w_k^\text{MLE}(\tau) \propto 1/(\tau^2 + \sigma_k^2)$. To our knowledge, the use of such an unbiased risk estimate has not received substantial attention in the context of mixed models though, for example, Kou and Yang (2017) considers unbiased risk estimates of both the shrinkage weights and the target regression surface. We include $\hat{\tau}_{\text{URE}}$ in our simulations for two main reasons: to explore its use as an alternative approach to variance component estimation and to more clearly examine the benefits of combining the BPE and MLE weighting schemes. Because both EBLUP (URE) and the CBP are based on minimizing an unbiased risk criterion, comparing EBLUP (URE) and CBP provides a more direct way of examining the benefits of using compromise weights, since estimation of hyperparameters for both procedures is more closely related. The third simulation study concerns estimation of the population-average parameter discussed in Section 3. Here, we also compare our compromise estimators with the direct, minimum-variance, and regression spline-based estimators described in Section 3.

For each simulation setting, we estimate the MSPE with $\frac{1}{n_{\text{rep}}} \sum_{j=1}^{n_{\text{rep}}} \sum_{k=1}^{K} \left( \hat{\theta}_k^{(j)} - \theta_k \right)^2$, where $\hat{\theta}_k^{(j)}$ denotes the estimated value of $\theta_k$ in the $j$th simulation replication and $n_{\text{rep}}$ denotes the total number of replications used for that simulation setting. For every setting, we use $n_{\text{rep}} = 5000$.

### 5.1. Two Unmodeled Latent Groups

**Simulation Description**

We consider a scenario where units belong to two distinct clusters but such cluster membership is unmodeled in the
Moreover, the estimation precision in the analysis. Specifically, we consider responses generated as
\[ Y_k = \beta_0 + \beta_1 Z_k + \nu_k + \sigma_k \epsilon_k, \quad \text{for } k = 1, \ldots, K, \]  
where \( K \) is a positive integer and the \( Z_k \in \{0, 1\} \) are independent Bernoulli random variables with \( P(Z_k = 1) = 1/2 \). For these simulations, we assume that both \( \nu_k \sim \text{Normal}(0, 1) \) and \( \epsilon_k \sim \text{Normal}(0, 1) \). The residual variances \( \sigma_k^2 \) are assumed to take the form \( \sigma_k^2 = 1/n_k \) with the \( n_k \) being determined by \( n_k = 10Z_k + 2(1 - Z_k) \). Note that while the random effects \( \nu_k \) are simulated from a known distribution, each of the estimation procedures considered in this simulation study (i.e., EBLUP, OBP, and CBP) does not use this known value of \( \tau \) and uses a value of \( \tau \) which is estimated from the data.

Model (33) is meant to represent a situation involving two latent groups where the units in one group (i.e., the group where \( Z_k = 1 \)) tend to have larger means than the other group, and moreover, the estimation precision in the \( Z_k = 1 \) group is much greater than in the \( Z_k = 0 \) group. Specifically, the residual standard deviation is \( \sigma_0 = 1/\sqrt{10} \) for those in the \( Z_k = 1 \) group and \( \sigma_k = 1/\sqrt{2} \) in the \( Z_k = 0 \) group.

Performance under an assumed intercept-only model.

While (33) is the true data-generating model, we consider estimates of \( \beta_k \) which assume an intercept-only model. That is, the assumed design matrix \( X \) when computing the shrinkage estimates of \( \beta_k \) consists of a single \( K \times 1 \) column vector whose entries are all equal to 1. Hence, we have misspecification of the mean model whenever \( \beta_1 \neq 0 \) because the true \( \mu_k \) can take one of two values.

Figure 1 shows results for the MSPE in this simulation setting. The left-hand panel of Figure 1 compares the MSPE across different methods where \( \beta_1 \) is fixed at one and the number of units \( K \) varies from 5 to 50. When the number of units is very small (i.e., \( K \leq 10 \)), the EBLUPs generally perform very well due to the greater role of variance in driving estimation performance. However, for settings with more units, the systematic bias in the assumed mean structure becomes much more important, and hence, the OBP tends to clearly outperform every version of the EBLUP. Though never quite the top performer, the CBP is quite robust here in the sense that it always has MSPE near the best performer for all values of \( K \). Interestingly, the plug-in CBP is the top performer for all values of \( K \). Though not much better than the CBP and OBP for \( K \geq 30 \), for the ranges \( 5 \leq K \leq 25 \), the plug-in CBP provides a noticeable improvement in MSPE over the CBP.

The right-hand panel of Figure 1 shows how the MSPE changes when we consider a fixed number of units and vary the severity of mean model misspecification. When there is no model misspecification (i.e., when \( \beta_1 = 0 \)), the EBLUPs, as expected, have lower MSPE than the OBP with the two versions of the CBP falling in between the OBP and the EBLUPs. As \( \beta_1 \) increases however, the bias term in the MSPE grows substantially while the variance remains mostly unchanged. Hence, as \( \beta_1 \) increases, the OBP quickly dominates the EBLUPs due to the greater role that the model misspecification plays in impacting MSPE performance. Notably, the CBP never has poor MSPE performance regardless of the value of \( \beta_1 \). When \( \beta_1 = 0 \), the MSPE of the CBP is marginally worse than the MLE and REML versions of the EBLUP and is just as good as EBLUP (URE), and for larger values of \( \beta_1 \), the weights of the CBP adapt in such a way that its performance is very similar to that of the OBP.
Performance when also including irrelevant covariates.

The OBP generally has very good performance under an intercept-only assumption especially when the number of units is large and \( \beta_1 > 0 \). This is because, in these scenarios, the bias is the dominating factor in determining the MSPE. For scenarios having prominent roles for both bias and variance, the CBP can often clearly outperform both the OBP and the EBLUPs. We demonstrate this here by comparing MSPE when one also includes irrelevant covariates in the analysis of data simulated from model (33). More specifically, for these simulations the data are simulated from model (33), but when estimating the \( \theta_k \), the \( k \)th row of the design matrix \( X \) is assumed to take the form \( x^T_k = (1, x_{k1}, \ldots, x_{kq}) \) where the \( x_{kj} \) are standard normal random variables generated independently from the \( v_k \) and \( e_k \) in (33). Including such irrelevant covariates substantially increases the variance of each method while having a minimal impact on bias.

Figure 2 displays the estimated MSPE for different methods with the number of irrelevant covariates \( q \) ranging from 0 to 12 and with the number of units fixed at \( K = 50 \). Here, \( q = 0 \) corresponds to estimating the \( \theta_k \) assuming an intercept-only model. In the left-hand panel where \( \beta_1 = 2 \), the misspecification in the mean model is substantial which leads to strong performance of the OBP for \( q = 0 \). However, as more irrelevant covariates are added, the variance contribution to the MSPE grows which leads the OBP to perform even worse than all EBLUP methods for \( q > 10 \). Using weights which can adapt to different levels of bias and variance enables both versions of the CBP to perform very well. As shown in the left-hand panel of Figure 2, when \( \beta_1 = 2 \) the CBP has nearly identical MSPE to the OBP for \( q = 0 \), and it clearly dominates all other methods for \( q \geq 4 \). In the right-hand panel of Figure 2, \( \beta_1 \) is set to \( 1/2 \) so that model misspecification is much less severe than the scenario depicted in the left-hand panel. In this scenario, the EBLUPs generally have the best performance due the strong role of estimation variance in these scenarios. Despite this, the CBP has nearly identical performance to the EBLUP (URE) and is very competitive with the other EBLUP methods for all values of \( q \) considered. Moreover, the plug-in CBP has consistently better performance than the CBP, and it has lower MSPE than both EBLUP(MLE) and EBLUP(URE) for most values of \( q \) considered.

5.2. Sample Size as an Ignored Covariate

Simulation Description

We begin by examining the performance of the CBP when the mixed effects are related linearly to the unit-specific sample sizes and this dependency is not properly modeled. Specifically, in this simulation study we generate unit-specific responses \( Y_k \), \( k = 1, \ldots, K \)

\[
Y_k = x_k^T \beta + \rho \tau n_k / sd(n) + v_k \tau \sqrt{1 - \rho^2} + \sigma e_k / \sqrt{n_k}, \quad k = 1, \ldots, K, \tag{34}
\]

where \( sd(n) = \left( \frac{1}{K-1} \sum_{k=1}^K (n_k - \bar{n}) \right)^{1/2} \) is the standard deviation of the \( n_k \) and \( x_k \) is a \( p \times 1 \) vector of regression coefficients. For each \( k \), we draw \( v_k \) independently from the \( n_k \) and the distribution of \( v_k \) is chosen so that \( E(v_k) = 0 \) and \( \text{var}(v_k) = 1 \). Consequently, the sample correlation between the \( \theta_k \) and \( n_k \) will be approximately equal to \( \rho \) in each simulation replication. In these simulations, the dependence of the mean of \( Y_k \) on \( n_k \) is...
Figure 3. Estimated MSPE for various methods using data generated from model (34). In these simulations, $K = 50$ and $v_k \sim \text{Normal}(0, 1)$. In (a), $\sigma^2 = 0.5$ while in panel (b), $\sigma^2 = 1.5$. Both figures show the role that the correlation parameter $\rho$ plays in determining the relative performance of the methods considered. In both cases, the plug-in CBP performs very well as it has the lowest MSPE for larger values of $|\rho|$ and is not much worse than EBLUP (MLE) and EBLUP (REML) for small values of $|\rho|$ where the model is nearly correctly specified.

**Performance when varying $\sigma^2$.**

Figure 3 shows the results for simulations from model (34) where $K = 50$ and $v_k \sim \text{Normal}(0, 1)$, and the correlation parameter $\rho$ varies from $-0.9$ to $0.9$. The left-hand panel of Figure 3 corresponds to a simulation setting with $\sigma^2 = 0.5$ while the right-hand panel of Figure 3 corresponds to a simulation setting with $\sigma^2 = 1.5$. When $\rho = 0$, model (34) is correctly specified, and thus, in these cases, we should expect the EBLUP procedures to generally perform the best. As shown in Figure 3, this is indeed the case. For each plot shown, the EBLUP (MLE) and EBLUP (REML) procedures have the lowest MSPE whenever $\rho = 0$. For values of $\rho$ which are larger in absolute value however, both the CBP and the plug-in CBP can provide substantial improvements over the EBLUPs in terms of MSPE. For these simulations, both the CBP and the plug-in CBP exhibit the same downward facing parabola as $\rho$ varies from $-0.9$ to $0.9$, but the plug-in CBP clearly has better performance for both $\sigma^2 = 0.5$ and $\sigma^2 = 1.5$ settings. For both settings of $\sigma^2$, the plug-in CBP dominates the OBP for all values of $\rho$ while the CBP dominates the OBP except for very large values of $|\rho|$ where the model is highly misspecified.

**Performance for different distributions of $v_k$.**

Figure 4 shows the results for simulations corresponding to model (34) where $K = 50$ and $\sigma^2 = 1$. Again, the correlation parameter $\rho = \text{corr}(\theta_k, n_k)$ is varied from $-0.9$ to $0.9$. We consider two choices for the distribution of $v_k$: a Gaussian mixture distribution with two components and a uniform distribution. For the Gaussian mixture model, the $v_k$ are generated under the assumption that $v_k | Z_k = 0 \sim \text{Normal}(-\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $v_k | Z_k = 1 \sim \text{Normal}(\frac{1}{\sqrt{2}}, \frac{1}{2})$ with $P(Z_k = 0) = 1/2$. For the uniform distribution, we use $v_k \sim \text{Uniform}(-\sqrt{3}, \sqrt{3})$ so that $\text{var}(v_k) = 1$.

The left-hand panel of Figure 4 shows the estimated values of the MSPE when $v_k$ is generated from the Gaussian mixture distribution while the right-hand panel corresponds to the cases when $v_k$ is generated from a uniform distribution. In both panels of Figure 4, we see an overall pattern which is similar to that in Figure 3. Namely, the EBLUPs dominate for values of $\rho$ near $0$ while the OBP and the different versions of the CBP dominate for more extreme values of the correlation parameter $\rho$. Similar to the results shown in Figure 3, the plug-in CBP performs better here than the CBP across all simulation settings. Indeed, the plug-in CBP is the clear winner with respect to MSPE for values of $\rho$ such that $0.4 \leq |\rho| \leq 0.7$. Figure 4 also shows results for
the “multi-τ” CBP approach described in Section 2.3. As shown in this figure, the differences between the CBP and the multi-τ CBP were very minimal in these simulation scenarios.

5.3. Estimating a Population Average

This simulation study concerns estimation of the population-average parameter $μ_0 = K^{-1} \sum_{k=1}^{K} \theta_k$ discussed in Section 3. For this simulation study, we simulate the direct estimates $Y_k$ under the assumption that

$$Y_k|θ_k ∼ \text{Normal}(θ_k, σ^2/n_k),$$

where the $n_k$ can be thought of as unit-specific sample sizes though we do not constrain the $n_k$ to be integers in our simulations. The sample sizes $n_1, \ldots, n_K$ and the unit-specific parameters $θ_1, \ldots, θ_K$ are generated from the following scheme

$$n_k = \frac{K\tilde{n}}{\sum_{k=1}^{K} \exp\left\{\frac{a(2k-K-1)}{K-1}\right\}}, \quad \text{for } k = 1, \ldots, K \quad (35)$$

$$θ_k = c_1(ρ, ξ)f_1(n_k) + c_2(ρ, ξ)v_k, \quad \text{for } k = 1, \ldots, K \quad (36)$$

where $v_k ∼ \text{Normal}(0, 1)$, $μ_{ln} = \frac{1}{K} \sum_{k=1}^{K} \log(n_k)$, and $σ_{ln} = \left[\frac{1}{K-1} \sum_{k=1}^{K} (\log(n_k) - μ_{ln})^2\right]^{1/2}$, and $f_1(n_k)$ is defined as

$$f_1(n_k) = \Phi\left(\frac{2[\log(n_k) - μ_{ln}]}{σ_{ln}}\right) - \frac{1}{2}.$$

The sample sizes $n_k$ in this setting are equally spaced on the log scale, and the constant $a$ in (35) can be chosen to achieve a desired value for the standard deviation of the $n_k$, namely, $sd(n) = \left\{\frac{1}{K-1} \sum_{k=1}^{K}(n_k - \bar{n})\right\}^{1/2}$. The constants $c_1(ρ, ξ)$, $c_2(ρ, ξ)$ in (36) are defined as $c_1(ρ, ξ) = sd(n)ξρ/σ_n$ and $c_2(ρ, ξ) = \sqrt{ξ^2 - c_1(ρ, ξ)^2}$, where $\kappa_n = \frac{1}{K} \sum_{k=1}^{K} f_1(n_k)$ and $σ^2 = \frac{1}{K} \sum_{k=1}^{K} f_2(n_k)$. Defining the constants this way ensures that

$$\frac{1}{K} \sum_{k=1}^{K} \text{var}(θ_k) = ξ^2 \quad \text{and} \quad \frac{E(\frac{1}{K} \sum_{k=1}^{K} θ_k n_k)}{sd(n) \sqrt{\frac{1}{K} \sum_{k=1}^{K} \text{var}(θ_k)}} = ρ.$$

In other words, $ξ$ measures the standard deviation of the unit-specific means $θ_k$ while $ρ$ measures the correlation between the $θ_k$ and the unit-specific samples sizes $n_k$. Hence, larger values of $ρ$ correspond to settings where sample size is more informative for the magnitude of $θ_k$.

Table 1 shows simulation-based estimates of the MSPE for 8 estimation methods and different choices of $(K, σ^2, ρ)$. In each row of Table 1, the ratio between the MSPE and the minimum MSPE for that row is shown. For these simulations, we included the estimators $\hat{μ}_{\text{direct}}$ and $\hat{μ}_{\text{mv}}$ described in Section 3 along with the “direct compromise” estimator $\hat{μ}_{\text{compr}} = α_{\text{opt}}(0)\hat{μ}_{\text{mv}} + (1 - α_{\text{opt}}(0))\hat{μ}_{\text{direct}}$, where $α_{\text{opt}}(0)$ is as defined in (29). In our comparisons, we also included the nonparametric regression-based estimator $\hat{μ}^n_a = \frac{1}{K} \sum_{k=1}^{K} h(n_k, β^h)$ described in Section 3. For the function $h(n_k, β^h)$ in (30), we used a cubic smoothing spline with the smoothing parameter selected using generalized cross-validation (Craven and Wahba 1978).

The results in Table 1 show that either the EBLUP(REML) estimate of $μ_0$ or $\hat{μ}_{\text{mv}}$ generally perform the best whenever
Table 1. Estimates of the MSPE for estimates of the population-average parameter \( \mu_0 \).

| K | \( \sigma^2 \) | \( \rho \) | EBLUP(REML) | OBP | CBP | CBP(plug-in) | Direct | MinVar | Direct-Compr | SR |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 1 | 0.0 | 1.00 | 1.67 | 1.12 | 1.07 | 1.27 | 2.54 | 1.21 | 1.27 |
| 0.1 | 1.00 | 1.62 | 1.11 | 1.05 | 1.23 | 2.61 | 1.18 | 1.24 |
| 0.2 | 1.00 | 1.53 | 1.09 | 1.02 | 1.15 | 3.11 | 1.15 | 1.16 |
| 0.3 | 1.0 | 1.46 | 1.06 | 1.00 | 1.08 | 3.70 | 1.11 | 1.07 |
| 0.4 | 1.16 | 1.35 | 1.09 | 1.03 | 1.00 | 4.79 | 1.14 | 1.00 |
| 0.5 | 1.41 | 1.36 | 1.15 | 1.12 | 1.00 | 6.08 | 1.18 | 1.00 |
| 10 | 4 | 0.0 | 1.49 | 2.47 | 1.77 | 1.83 | 2.1 | 1.0 | 1.78 | 1.85 |
| 0.1 | 1.47 | 2.47 | 1.74 | 1.81 | 2.09 | 1.0 | 1.75 | 1.85 |
| 0.2 | 1.36 | 2.25 | 1.62 | 1.66 | 1.90 | 1.0 | 1.63 | 1.68 |
| 0.3 | 1.28 | 2.12 | 1.53 | 1.57 | 1.79 | 1.0 | 1.54 | 1.58 |
| 0.4 | 1.16 | 1.91 | 1.40 | 1.42 | 1.61 | 1.0 | 1.41 | 1.41 |
| 0.5 | 1.02 | 1.64 | 1.22 | 1.23 | 1.38 | 1.0 | 1.24 | 1.24 |
| 50 | 1 | 0.0 | 1.00 | 1.80 | 1.19 | 1.11 | 1.34 | 2.72 | 1.29 | 1.33 |
| 0.1 | 1.00 | 1.64 | 1.14 | 1.07 | 1.22 | 3.41 | 1.21 | 1.22 |
| 0.2 | 1.14 | 1.37 | 1.09 | 1.03 | 1.01 | 5.47 | 1.13 | 1.00 |
| 0.3 | 1.56 | 1.48 | 1.15 | 1.12 | 1.00 | 9.54 | 1.17 | 1.00 |
| 0.4 | 2.22 | 1.56 | 1.14 | 1.17 | 1.00 | 14.95 | 1.12 | 1.01 |
| 0.5 | 3.25 | 1.63 | 1.14 | 1.15 | 1.00 | 22.32 | 1.09 | 1.01 |
| 50 | 4 | 0.0 | 1.43 | 2.53 | 1.82 | 1.89 | 2.19 | 1.0 | 1.84 | 1.88 |
| 0.1 | 1.33 | 2.32 | 1.69 | 1.74 | 2.02 | 1.0 | 1.71 | 1.70 |
| 0.2 | 1.04 | 1.81 | 1.35 | 1.37 | 1.57 | 1.0 | 1.36 | 1.34 |
| 0.3 | 1.00 | 1.64 | 1.28 | 1.27 | 1.42 | 1.32 | 1.31 | 1.24 |
| 0.4 | 1.00 | 1.59 | 1.30 | 1.27 | 1.37 | 1.71 | 1.34 | 1.21 |
| 0.5 | 1.00 | 1.54 | 1.29 | 1.24 | 1.32 | 2.31 | 1.34 | 1.17 |

NOTE: In each row of the table, the ratio between the MSPE and the minimum MSPE for that row is shown. The “Direct” and “MinVar” estimates correspond to the estimates \( \hat{\mu}_{\text{direct}} \) and \( \hat{\mu}_{\text{MV}} \), respectively. The “Direct-Compr” estimate corresponds to the direct compromise estimate \( \hat{\mu}_{\text{comp}} = \hat{\mu}_{\text{opt}} \hat{\mu}_{\text{MV}} + (1 - \hat{\mu}_{\text{opt}}) \hat{\mu}_{\text{direct}} \). The “SR” estimate refers to the spline-based estimator \( \hat{\mu}_{SR} \) of the population-average parameter described in Section 3.

\[ \rho = 0. \] This is expected as \( \rho = 0 \) corresponds to a correctly specified regression model. The EBLUP estimator generally performs better than \( \hat{\mu}_{MV} \) in lower noise settings, that is, when \( \sigma^2 = 1 \). Moreover, for settings with high variance (i.e., \( \sigma^2 = 4 \)), the EBLUP and \( \hat{\mu}_{MV} \) both perform relatively well even for larger values of \( \rho \). For large values of \( \rho \), the direct estimator \( \hat{\mu}_{\text{direct}} \) generally does quite well though the estimator \( \hat{\mu}_{\text{MV}} \) does somewhat better in many of these settings, particularly when \( \sigma^2 = 4 \). The compromise estimators (CBP, CBP(plug-in), Direct-Compr) are quite robust across different settings in the sense that their MSPE performance is never especially poor when compared with the best method. For each setting, they lie somewhere between the best and worst performer. Indeed, the worst relative performance of 1.82 for the CBP occurs when \( \rho = 0.0 \) and \( \sigma^2 = 4. \) The regression-based estimator \( \hat{\mu}_{\text{opt}} \) is also quite robust in this sense. For settings with \( \sigma^2 = 1 \), the compromise estimators are very competitive with \( \hat{\mu}_{\text{opt}} \) and usually have lower MSPE when \( \rho \leq 0.1 \). This is despite the fact that no modeling is involved in implementing the compromise approaches whereas the regression approach requires modeling the dependence of \( \theta_k \) on the unit-specific sample sizes.

### 6. Estimation of Normative Gait Speed in Older Adults

In this section, we apply the CBP and plug-in CBP approaches to estimate gait speed within a collection of demographically defined strata of older adults. The data analyzed for this purpose come from round 8 of the National Health and Aging Trends Study (NHATS) public use data. NHATS is a nationally representative survey of adults from the United States aged 65 and older that is designed to track key measures of well-being related to the aging process. One such measure recorded by NHATS is gait speed. The ability to walk is essential for independent living, and gait speed is a simple measure of the ability to walk. It is a valid measure of the overall functional health of older adults. It is typically measured as the speed at which a person walks a specified, short distance at usual pace. Typically, two measurements are taken and averaged. Slower gait speed has been shown to be a powerful predictor of mortality in older adults (Studenski et al. 2011) and is sometimes referred to as the “sixth vital sign” (Middleton, Fritz, and Lusardi 2015). In NHATS, gait speed was measured by instructing participants to “walk at their usual pace” over a 3-meter course (distance measured using a 5 meters colored chain). Participants started from a standing position and time was marked when the last foot crossed over the 3-meter mark on the link-chain. This was done twice and the average of the two trials was taken.

Our aim is to estimate the average gait speed within key demographic strata recorded by NHATS. Being derived from a nationally representative sample, these estimates may be considered as “normative” values of gait speed in older adults. Specifically, we look at 48 strata created from the following demographic characteristics: sex (male and female), race (white non-Hispanic, black, Hispanic, and other), and age (65-69, 70-74, 75-79, 80-84, 85-89, 90+). In this context, we define \( Y_k \) to be the sample mean of gait speed within the \( k \)th demographically defined stratum, and \( \sigma_k^2 = \frac{s_k^2}{n_k} \), where \( s_k \) is the sample standard deviation of gait speed within the \( k \)th stratum. It has also been recognized that height can play an important role in gait speed (Bohannon 1997) and hence including height in our regression model can potentially improve the stratum-specific estimates of gait speed. We incorporate this into our analysis by defining \( x_k = u_k - 65 \), where \( u_k \) is the mean height (in inches) within stratum \( k \).
Figure 5. Sample means $Y_k$ of gait speed within demographically-defined strata and fitted regression lines $\hat{\beta}_0 + \hat{\beta}_1 (u_k - 65)$ estimated via CURE, CURE (plug-in), REML, or BPE, where $u_k$ represents the sample mean of height within stratum $k$. The size of the circles surrounding the within-stratum means are inversely proportional to the stratum-specific standard errors $\sigma_k = s_k / \sqrt{n_k}$, where $s_k$ is the sample standard deviation of gait speed within stratum $k$. In (a), sample means and fitted regression lines are displayed for all 48 of the strata defined by all combinations of the demographic categories of sex, race, and age. In (b), sample means and fitted regression lines are shown for a selected subset of 32 strata. The CURE (plug-in) and REML estimates are not shown in panel (a) because these are identical to the CURE estimates.

In addition to an analysis involving all 48 strata of interest, we also performed an analysis which only used data from a subset of 32 strata. This subset of 32 strata was created by excluding the 12 strata that contained the "other" race category and the 4 strata where the race was "hispanic" and the age category was either 85–89 or 90+. This analysis of the subset of 32 strata was conducted to better highlight differences between the CBP, OBP, and EBL UP that cannot often arise in practice.

Figure 5 shows the direct estimates $Y_k$ of mean gait speed for each of the 48 strata of interest in the left-hand panel and the subset of 32 strata in the right-hand panel. The sizes of the circles surrounding each direct estimate are inversely proportional to the standard error of the direct estimate. For the small area model $Y_k = \beta_0 + \beta_1 x_k + v_k + \epsilon_k$ using data from all 48 strata, the REML estimates of $\beta_0$, $\beta_1$, and $\tau$ were $\hat{\beta}_0 = 1.833$, $\hat{\beta}_1 = 0.062$, and $\hat{\tau} = 0.60$, respectively, and the BPE estimates of these parameters were $(\hat{\beta}_0, \hat{\beta}_1, \hat{\tau}) = (1.830, 0.062, 0.46)$ while the plug-in CURE estimates were $(\hat{\beta}_0, \hat{\beta}_1, \hat{\tau}) = (1.833, 0.062, 0.60)$. The value of $\alpha^*$ used in the CURE estimates was $\alpha^* = 1.0$, and the value $\alpha^*_{plug}$ of the mixing parameter used for the plug-in CURE estimates was 0.999. Because $\alpha^*_{plug}$ is very close to 1, the plug-in CBP regression weights are essentially the same as the REML regression weights, and hence the plug-in CURE and REML estimates of $\beta_0$ and $\beta_1$ are very similar.

As shown in the fitted regression lines of Figure 5, the CURE estimates of $(\hat{\beta}_0, \hat{\beta}_1, \hat{\tau})$ were substantially different from both the REML and BPE estimates when only looking at the 32-strata subset. Specifically, we obtained $(\hat{\beta}_0, \hat{\beta}_1, \hat{\tau}) = (2.056, 0.026, 0.73)$ for the CURE estimates while the REML and BPE estimates were $(1.902, 0.50, 0.64)$ and $(2.299, −0.10, 0.63)$, respectively. The optimal values of the mixing parameter $\alpha$ were $\alpha^* = 0.635$ and $\alpha^*_{plug} = 0.531$ for the CURE and plug-in CURE estimates, respectively. When comparing these results with the analysis of all 48 strata, you may note that the fitted regression lines associated with the CURE and BPE estimates have changed more dramatically than the fitted regression line associated with the REML estimates. This is mainly due to the fact that all of the larger strata present in the group of 48 are also present in the group of 32, and the removal of a number of smaller strata does not substantially change the values of the REML estimates. In contrast, both the CURE and BPE regression lines are much more sensitive to the presence/absence of the smaller strata.

The variation in the circle sizes in Figure 5 demonstrates the considerable variability in the stratum-specific sample sizes, and hence we should expect that many of the stratum-specific estimates of gait speed will have little impact from the overall regression fit $\hat{\beta}_0 + \hat{\beta}_1 x_k$ while others will have considerably more shrinkage toward the regression target. Figure 6 shows that this is indeed the case for both the analysis of the 48 strata and the
Figure 6. Estimates of mean gait speed using the CBP (plug-in), OBP, and EBLUP (REML) methods within (a) 48 demographically-defined strata, (b) a subset of 32 of these demographically-defined strata. The direct estimates are the within-stratum sample means $Y_k$. The regression estimates are given by $\hat{\beta}_0 + \hat{\beta}_1(u_k - 65)$, where $\hat{\beta}_0$, $\hat{\beta}_1$ are the REML estimates and $u_k$ represents the sample mean of height within stratum $k$. Stratum-specific estimates are ordered according to the stratum-specific standard errors $\sigma_k$.

32-strata subset. This figure shows the plug-in CBP, OBP, and EBLUP estimates of stratum-specific gait speed along with the fitted regression and direct estimates for each stratum. The fitted regression points are based on the REML estimates of $\beta_0$ and $\beta_1$. EBLUP (REML) estimates of gait speed are not shown in the 48 strata graph since the EBLUP estimates are essentially identical to the plug-in CBP estimates in this case. A table containing the stratum-specific estimates and the demographic characteristics which comprise each stratum is presented for both the 48 and 32 strata analyses is provided in the supplementary material. As shown in Figure 6(a), when $\sqrt{\sigma_k} \leq 0.4$ both the plug-in CBP and OBP provide almost no shrinkage of the direct estimates toward their corresponding regression targets. For $\sqrt{\sigma_k} \geq 0.5$, noticeable differences between the plug-in CBP and the OBP become more apparent in Figure 6(a). Specifically, the OBP appears to consistently shrink the direct estimates toward the REML-based estimates of the regression line more than the plug-in CBP though this is largely due to the fact that the OBP is shrinking the direct estimates toward a different regression target.

As in the 48 strata analysis, the CBP, EBLUP, and OBP all apply very little shrinkage to the direct estimates for $\sqrt{\sigma_k} \leq 0.4$ in the analysis of the 32 strata subset shown in Figure 6(b). Only for smaller strata where $\sqrt{\sigma_k} \geq 0.5$ do the differences between the OBP and CBP become more apparent, and there are only noticeable differences between the EBLUP and CBP for the very small strata. The greater similarity between the CBP and EBLUP estimates than between the EBLUP and OBP estimates mostly reflects the greater similarity between the CURE and REML estimates of the regression coefficients. The two strata with the smallest values of $\sigma_k$ shown in Figure 6(b) are the hispanic aged 65–69 strata. For these strata, the EBLUP and the plug-in CBP estimates were 3.08 and 3.05, respectively for the hispanic/65–69/male subgroup where $\sqrt{\sigma_k} = 0.58$ and $n_k = 6$, and the EBLUP and CBP estimates were 2.63 and 2.58, respectively for the hispanic/65–69/female subgroup where $\sqrt{\sigma_k} = 0.57$ and $n_k = 9$.

7. Discussion

In this article, we have introduced a new approach for choosing regression weights in contexts where a regression model and direct estimates are combined to estimate a collection of unit-specific mean parameters. In our approach, regression weights are expressed as a convex combination of the MLE and BPE regression weights with the values of the regression weights depending on a mixing parameter $\alpha \in [0,1]$ and a variance component parameter $\tau \geq 0$. The terms $(\alpha, \tau)$ are determined empirically so that the corresponding estimates of the small
domain means are competitive with the EBLUP in situations where the model is correctly specified. The adaptive nature of the regression weights can improve the robustness of the small domain estimates in situations where the mean model is misspecified particularly when unit-specific sample sizes have unmodeled association with the unit-specific mean parameters. While enriching the covariates or making the regression model more flexible can reduce the impact of such informative sample size and hence improve the likelihood-based approach, determining the correctness of a model is ultimately an imperfect process and using our approach for constructing regression weights provides an automatic extra layer of robustness.

As shown in the simulations studies described in Sections 5.1 and 5.2, the plug-in CBP typically performs as good or better than the CBP when evaluated by MSPE. We have found that this is generally the case in other simulation studies we have conducted. The discrepancy between CBP and plug-in CBP seems to be largely related to the differences in performance between EBLUP (REML) and EBLUP (URE). More specifically, EBLUP (REML) generally works better than EBLUP (URE) when the model is either correctly specified or nearly correctly specified, and for these cases, the plug-in CBP estimates of \( \theta_k \) will be very close to the corresponding EBLUP (REML) estimates. In contrast, the CBP estimates will more closely resemble the EBLUP (URE) estimates in cases with correct model specification. While EBLUP (URE) can perform better than EBLUP (REML) in many misspecified scenarios, this is irrelevant to the performance of either the CBP or the plug-in CBP as both versions of the CBP will be much closer to the OBP in such cases.

Our approach provides a more flexible procedure for constructing regression weights by using a weight function that uses a single additional hyperparameter \( \alpha \) to determine the relative weights given to the MLE and BPE regression weights. While this greater flexibility can offer better performance, there is certainly potential to devise even better weighting schemes by considering more flexible weight functions. The key challenge in constructing such more flexible weight functions would be how to tradeoff increased flexibility with the increased variance that would accompany the use of many additional hyperparameters to index a class of highly flexible weight functions. Constructing an appropriate definition of the “degrees of freedom” in this context would be one approach for addressing this tradeoff as one could then directly compare weight functions with differing number of hyperparameters by penalizing the additional degrees of freedom appropriately. A possible related approach that would not require one to directly define an appropriate measure of degrees of freedom would be to select the weights by minimizing a “penalized” version of the unbiased estimator \( \hat{M}_k(w, \tau) \) where one adds a penalty term that penalizes the complexity of the regression weights \( w \). The exploration of more flexible weighting schemes is beyond the scope of this article but is certainly an important issue to consider in future research.

We have primarily focused on the use and performance of the CBP for estimating unit-specific mean parameters and did not explore using the CBP framework to generate uncertainty intervals for the mixed effects \( \theta_k \). One approach for doing so would be to use the percentiles of the marginal posterior of \( \theta_k \) under an assumption that the \( \theta_k \) follow a Gaussian distribution. Rather than computing the percentiles of the marginal posterior by integrating with respect to a specific choice of hyperprior for \( \alpha \) and \( \tau \), the marginal posterior could be approximated using a bootstrap approach similar to that described in Laird and Louis (1987).

As is the case in many small domain estimation procedures, the CBP is derived under an assumption that the unit-specific sampling variances \( \sigma_k^2 \) are known. Though not explored in this article, one way of relaxing the assumption of known sampling variances assumption is to introduce a hierarchical model for both the direct estimates and the corresponding estimates of their variances as has been done in, for example, Dass et al. (2012) and Sugawara, Tamae, and Kubokawa (2017). With this approach, one would specify a conditional joint distribution for each direct estimate and its corresponding standard error and specify a distribution for the underlying sampling variances. Using this setup, one could then implement a type of two-stage procedure where one first computes shrunken estimates of the sampling variances based on their posterior means and then uses these shrunken estimates to construct the shrinkage weights \( B_{k,\tau} \). Using these alternative shrinkage weights, one could then find the variance component and mixing parameter estimates for the CBP using the unbiased risk estimate described in Section 2. A closely related alternative to this would be to use shrunken values of the sampling variances and assume a \( t \) distribution for the direct estimates as was suggested in Lu and Stephens (2019).

### Supplementary Materials

**Appendix:** Section A contains proofs of Proposition 1 and Theorems 1–2. Section B contains proofs of Theorems 3–4. Section C shows a derivation for the general unbiased MSPE estimator of Section 2.4 and a derivation for the unbiased estimator of the population mean-MSPE of Section 3. Section D contains a few additional derivations, and Section E contains values of the stratum-specific estimates from the gait speed application. Section F describes an unbiased estimator of the MSPE that can be used to find compromise regression weights in the context of the nested-error regression model. (PDF file)

**Replication files:** This zip file contains the R code needed to reproduce the simulation results described in Section 5, and it contains the R code used for the application described in Section 6. (Zip file)

**R package:** An R package entitled shrinkcbp which implements the methods discussed in this article. (GNU zipped tar file). This R package may also be retrieved from https://github.com/nchenderson/shrinkcbp.

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