DOMAIN WALLS: MOMENTUM CONSERVATION IN ABSENCE OF ASYMPTOTIC STATES

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Gravitational potentials of the domain walls in the linearized gravity are growing with distance, so the particle scattering by the wall can not be described in terms of free asymptotic states. In the non-relativistic case this problem is solved using the concept of the potential energy. We show that in the relativistic case one is able to introduce gravitationally dressed momenta the sum of which is conserved up to the momentum flux through the lateral surface of the world tube describing losses due to excitation of the branon waves.

Keywords: Collision theory, branes, large extra dimensions.

1. Introduction

In the standard theory of particle collisions, both classical and quantum, one assumes the existence of asymptotic states in which the particles can be regarded as non-interacting. For this picture to be valid, the interaction force between the colliding objects has to fall down sufficiently fast with the distance. Meanwhile, in various physical systems, like quarks joined by the gluon strings, this is not so, and the question arises, whether one can sensibly define the notion of the potential energy within the classical relativistic two-body problem. With this motivation, we consider the scattering problem for a point particle impinging onto the brane imbedded into space-time with the codimension one, in which case the interaction force does not fall down asymptotically. We will be interested in the two-body problem, accounting for the brane back-reaction on equal footing with the particle. This problem may have physical applications in cosmology¹², in particular, perforation of the domain walls by black holes was suggested as a novel mechanism of domain walls destruction in the Early Universe³⁵. It may be of interest also in the study of the black hole escape from the Randall-Sundrum brane, in the dynamical description of colliding branes in supergravity/string theory and so on.

Recently we have shown⁶ that the perforation of the domain wall by a point particle can be described within linearized gravity in terms of distributions. Here we concentrate on the the energy-momentum balance in this process, extending the treatment to the second order in gravitational constant. This allows to introduce the effective gravitational stress-tensor which has to be taken into account in the energy-momentum balance. This tensor generically is non-local, but we show that it can be still unambiguously divided between the two objects leading to the notion of gravitationally dressed momenta whose balance involves an extra momentum flux through the lateral surface of the world tube.
2. The setup

Our system consists of a point particle moving along the world-line $x^M = z^M(\tau)$ and an infinitely thin domain wall filling the world-volume $V_{D-1}$ given by the embedding equations $x^M = X^M(\sigma^\mu)$ in $D-$ dimensional space-time ($D \geq 4$) with the metric $g_{MN}$, $M = 0, ..., D - 1$, $\mu = 0, ... D - 2$ of the signature $(+,-,...,-)$. The action reads:

$$ S = -\frac{1}{2} \int \left( e g_{MN} \dot{z}^M \dot{z}^N + \frac{m^2}{e} \right) d\tau $$

$$ -\frac{\mu}{2} \int \left[ X^M_X \dot{X}^N g_{MN} \gamma^{\mu \nu} - (D - 3) \right] \sqrt{-\gamma} d^{D-1} \sigma - \frac{1}{\varkappa^2} \int R_D \sqrt{-g} d^D x $$

where the first term is the particle action in the Polyakov form ($e(\tau)$ is the einbein on the particle world-line), the second one is the domain wall geometrical action, $X^M = \partial X^M / \partial \sigma^\mu$ are the tangent vectors and $\gamma^{\mu \nu} = X^M_X X^N g_{MN} \big|_{x=X}$ is the inverse metric on the domain wall world-volume $V_{D-1}$, $\gamma = \det \gamma^{\mu \nu}$. The last term is the Einstein-Hilbert action, $\varkappa^2 \equiv 16\pi G_D$.

To treat the problem perturbatively we expand all variables in powers of $\varkappa$ and derive the system of iterative equations. The $D-$dimensional cartesian coordinates of the embedding space-time are split as $x^M = (x^\mu, z)$, $x^\mu = (t, r)$, and the particle is assumed to move along $z$, i.e. normally to the domain wall. In the zeroth order the particle is assumed to move with the constant velocity $u^M = \gamma(1,0,0,0,v)$, $\gamma = 1 / \sqrt{1 - v^2}$, so the world-line is $z^M(\tau) = u^M \tau$ while the einbein reads $e = \text{const} = m$, corresponding to the parametrization in terms of the proper time. The wall in the zeroth order is assumed to be plane, unexcited and being at rest at $z = 0$ in the chosen Lorentz frame: $X^M = \Sigma^M_{\mu} \sigma^\mu$, where $\Sigma^M_{\mu} = \delta^M_{\mu}$ are $(D - 1)$ constant Minkowski vectors normalized as $\gamma^{\mu \nu} = \eta^{\mu \nu}$. The moment of perforation of the wall by the particle that occurs at $z = 0$ is $t = \tau = 0$.

The metric deviation must be further expanded in $\varkappa$:

$$ H^{MN} = h^{MN} + \bar{h}^{MN} + \delta H^{MN}, $$

where the first order term is split into the sum of contributions of the wall

$$ h_{MN} = \frac{2k|z|}{\varkappa} \text{ diag } (-1, 1, ..., 1, D - 1), \quad k = \frac{\varkappa^2 \mu}{4(D - 2)} $$

and of the particle

$$ \bar{h}_{MN} = -\frac{\mu \Gamma \left( \frac{D-3}{2} \right)}{4^\frac{D-3}{2} \varkappa^2} \left( u_M u_N - \frac{1}{D - 2} \eta_{MN} \right) \frac{1}{[\gamma^2(z - vt)^2 + v^2]^{\frac{D-3}{2}}}. \quad (3) $$

The second order metric deviation $\delta H^{MN}$ does not split anymore on separate contributions and obeys (in the same gauge) the d’Alembert equation

$$ \Box \left( \delta H^{MN} - \frac{1}{2} \eta^{MN} \delta H \right) = -\varkappa \left( \delta T^{MN} + \delta \bar{T}^{MN} + S^{MN}(h, \bar{h}) \right), \quad (4) $$
where the perturbations of the particle (noted with the bar) and the brane stress-tensors:

$$\delta T^{MN}(x) = \frac{\mu}{2} \int \left[ 4 \delta^M \delta u^N - \kappa u^M u^N \left( \frac{\partial}{\partial \sigma} - 2 \delta z P \partial P \right) \right] (x - ut) \, d\tau, \quad (5)$$

with $h$ being the trace of the first order metric deviation due to the wall $\{2\}$; the symmetrization $(MN)$ over the indices is defined with $1/2$. The delta-function indicates on the localization of the integrand at the non-perturbed particle world-line.

$$\delta T^{MN}(x) = \frac{\mu}{2} \int \left[ 4 \delta^M \delta X^N \eta_{\mu\nu} - 2 \delta^M \delta X^N \left( \frac{\partial}{\partial \sigma} - 2 \delta z P \partial P \right) \right] +$$

$$+ \frac{\delta^M \delta N}{\delta \lambda} \eta_{\mu\nu} \left( \frac{\lambda}{\lambda} - \bar{h} + 2 \delta X^4 \delta L - 2 \delta X^4 \partial L \right) \delta^{D-1}(x - \sigma) \delta(\mathbf{z}) \, d^{D-1}\sigma, \quad (6)$$

Again, the delta-functions in the integrand indicate on its localization on the unperturbed wall world-volume. Finally $S^{MN}(h, \bar{h})$ stands for the quadratic form

$$S^{MN} = 2 H^{MP, Q} H^{N, P, Q} + H_{P Q} \left( H^{M Q, P} + H^{N P, M Q} - H^{M N, P Q} \right)$$

$$- 2 H^{(M} \partial^{N)} P - \frac{1}{2} H_{P Q} H^{P Q, N} + \frac{1}{2} H^{M N} \partial H$$

$$+ \frac{1}{2} \eta^{MN} \left( 2 H^{P Q} \partial H_{P Q} - H_{P Q, L} H^{P Q, L} + \frac{1}{2} H_{P Q, L} H^{P Q, L} \right), \quad (7)$$

in which $H^{MN}$ should be taken as the sum $H^{MN} = h^{MN} + \bar{h}^{MN}$, keeping only the crossed terms in $h^{MN}$, $\bar{h}^{MN}$.

Perturbation of the domain wall embedding functions $\delta X^M$ due to gravitational interaction with the particle in the aligned coordinates on the brane $\sigma^\mu = (t, \mathbf{r})$ is described by a single component $\Phi = \delta X^z$ as follows:

$$\square_{D-1} \Phi = \kappa \left( \frac{1}{2} \eta_{\mu\nu} \bar{h}^{\mu\nu; z} - \bar{h}^{z; 0} \right)_{z=0}; \quad (8)$$

where $\square_{D-1} \equiv \partial_\mu \partial^\mu$. The retarded solution of this equation consists of two parts $\Phi = \Phi_a + \Phi_b$, where the first is antisymmetric in time and represents an eventual deformation of the wall correlated with the particle motion. The second part is the spherical branon wave starting at the moment of perforation ($\Phi_b \sim \theta(t)$) and propagating to infinity with the velocity of light.

$$\Phi_a \equiv - \Lambda \text{sgn}(t) \, I_a, \quad \Phi_b \equiv 2 \Lambda \theta(t) \, I_b, \quad \Lambda = \frac{\kappa^2 m^2 \gamma^2 \gamma^2 + 1}{4(2\pi)^D/2-1}, \quad (9)$$

$$I_{a, b}(t, r) = \frac{1}{r^{D/2-1}} \int_0^\infty dk \, J_{\frac{D-6}{2}}(kr) \frac{D-6}{2} w_{a, b}(t, k), \quad w_{a, b} = \left\{ \begin{array}{l} e^{-k \gamma|t|} \\ \cos kt \end{array} \right\}, \quad (10)$$

where $J_{\nu}(z)$ is a Bessel function of the first kind.
3. Conservation of the energy-momentum

In the first order in $\kappa$ the total energy-momentum tensor consists of three contributions (4) and satisfies the conservation equation $\partial_N \tau^{MN} = 0$. To convert the latter into the energy-momentum balance equation one has to integrate over the world-tube $\Omega$:

$$0 = \int_\Omega \partial_N \tau^{MN} = \int_{\partial \Omega} \tau^{MN} d\Sigma_N,$$

bounded by the closed hypersurface, consisting of two space-like hypersurfaces associated with the moments of time $t_0$, $t_f$ (usually chosen orthogonal to the time-axis), and the closing lateral hypersurface $\Sigma_\infty$ at spatial infinity.

Since the wall is infinite its total energy-momentum diverges both in zero and the first order in $\kappa$, so some subtraction is required. Another problem is the non-zero lateral flux through $\Sigma_\infty$ from the wall. In accord with the splitting of the total energy-momentum tensor (4) one can write

$$P^M_{\text{tot}}(t) = \delta \bar{P}^M(t) + \delta P^M(t) + S^M(t), \quad (9)$$

where

$$\delta \bar{P}^M(t) = \int \delta \bar{T}^{MO} d\Sigma_{D-2}, \quad \delta P^M(t) = \int \delta T^{MO} d\Sigma_{D-2}, \quad \delta S^M(t) = \int \delta S^{MO} d\Sigma_{D-2} \quad (10)$$

are the first-order kinetic momenta carried by the particle and the wall, and

$$S^M(t) = \int \delta S^{MO} d\Sigma_{D-2}, \quad (11)$$

is the momentum carried by the gravitational field. The lateral flux can be split into similar three contributions, revealing that only the wall $\delta T^{Mr}$ does not vanish. To get rid of infinities, we pass to the time derivatives of the partial momenta which are all finite:

$$\frac{d}{dt} \left( \delta \bar{P}^M(t) + \delta P^M(t) + S^M(t) \right) = -\lim_{R \to \infty} \left( \int_{-\infty}^{\infty} d\Sigma_{D-3} \int_{S_{D-3}}^{\infty} d\Sigma_{D-3} \delta T^{Mr} R^{D-3} d^{D-3}\Omega \right), \quad (12)$$

where the integral at the right hand side represents the lateral momentum flux. This term looks like an external force acting upon the system.

The first-order particle stress tensor is determined by the world-line perturbation and the metric deviation due to the wall. Integrating over the spatial volume, one obtains the derivative of the particle momentum $\delta \bar{P}^M$ (only $z$ and 0-components are non-zero):

$$\delta \bar{P}^z = mk \left[(3D-2)\gamma v^2 + \gamma^{-1} \right] \text{sgn}(t), \quad \delta \bar{P}^0 = 2Dmk\gamma v \text{sgn}(t).$$

The first order stress-tensor of the wall is obtained by using the first-order metric deviation due to the particle and by the first-order perturbations of the wall world-
volume. The corresponding derivatives read:

\[
\delta \dot{P}_0 = -\gamma v \left( (D - 2)\gamma^2 v^2 + 2D - 7 \right) mk \text{sgn}(t),
\]

\[
\delta \dot{P}_a = -mk \left( (D - 2)\gamma^2 v^2 + 1 \right) \gamma v^2 \text{sgn}(t)
\]

\[
\delta \dot{P}_b = -\frac{2km}{\gamma} \left( (D - 2)\gamma^2 v^2 + 1 \right) \theta(t).
\]

where \(z\)-components are split into \(a, b\)-parts.

The lateral flux in right part of Eq. (12) is due to \((z, r)\)-component of the energy-momentum \(f_z^r \equiv \frac{d}{dt} \int T^{zr} dS_r\) and consists of antisymmetric and branon part:

\[
f_a^z = -km\gamma \text{sgn}(t) \left( (D - 3) v^2 + 1 \right),
\]

\[
f_b^z = -\delta \dot{P}_b^z.
\]

All the momenta derivatives are constant before and after the moment of piercing \(t = 0\) when they change the sign. The sum \(\delta \dot{P}_M + \delta \dot{P}_{\bar{M}}\) does not vanish for both values of \(M\). This is not surprising since we still need to add contribution of the gravitational stresses.

**Gravitational dressing.** Analysing different terms in \(S^{MN}(h, \bar{h})\) obtained by substituting the metric deviations \(h_{MN}\) and \(\bar{h}_{MN}\) one detects presence of contributions of two types: containing the delta-function \(\delta(z)\) localized in the wall world-volume \(S\), and containing the delta-function \(\delta^D(x - u\tau)\) localized at the particle world-line \(\bar{S}\). The corresponding momentum time derivatives for the particle are:

\[
f_a^z = -km\gamma \text{sgn}(t) \left( (D - 3) v^2 + 1 \right),
\]

\[
f_b^z = -\delta \dot{P}_b^z.
\]

The dressed particle energy term is \(\dot{\bar{P}}_0 = \delta \dot{\bar{P}}_0 + f_0^z\), while the dressed brane energy term is \(\dot{\bar{P}}^0 = \delta \dot{\bar{P}}^0 + f^0\). It is easy to establish the conservation equation \(\dot{\bar{P}}_0 + \dot{\bar{P}}^0 = 0\). Note that \(\delta \dot{\bar{P}}_0 + \delta \dot{\bar{P}}^0 \neq 0\). Thus treating gravity perturbatively in Minkowski space, one is able to introduce, instead of the potential energy, the dressed quantities for the particle in the gravitational field of the domain wall, and respectively, of the domain wall in the gravitational field of the particle such that their sum is conserved.

The situation of the spatial \(z\)-component is more involved. The dressed brane \(z\)-momentum terms can be defined analogously as \(\dot{\bar{P}}_z = \delta \dot{\bar{P}}_z + f_z^z\) for the brane and \(\dot{\bar{P}}_z = \delta \dot{\bar{P}}_z + f_z^z\) for the particle. Their sum, however, is not zero. The balance
equation contains the lateral flux:

\[ \dot{P}_z + \bar{\dot{P}}_z = -f^z. \]

Note that, since \( \text{sgn}(t) \) and \( \theta(t) \) are linearly independent, this extra contribution holds separately for \( a \) and \( b \) branon components.

4. Concluding remarks

We have shown that non-local gravitational stresses effectively localize within perturbative treatment of gravity enabling us to establish the local energy-momentum conservation for the scattering problem without asymptotical free states. The unusual feature of the balance equation is the existence of non-zero flux of \( z \)-component of the momentum density through the lateral surface of the world-tube due to the branon. Strictly speaking, in this case the conserved momentum can not be defined as an integral over the space only. But if one still keeps such a definition, the imbalance will be exactly accounted for by the lateral flux playing the role of an external force. This takes place separately for the bound part of the branon, and for the free branon wave.

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