Certifying unknown genuine multipartite entanglement by neural networks

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Abstract
Suppose we have an unknown multipartite quantum state, how can we experimentally find out whether it is genuine multipartite entangled or not? Recall that even for a bipartite quantum state whose density matrix is known, it is already NP-Hard to determine whether it is entangled or not. Therefore, it is hard to efficiently solve the above problem generally. However, since genuine multipartite entanglement (GME) is such a fundamental concept that plays a crucial role in many-body physics and quantum information processing tasks, finding realistic approaches to certify GME is undoubtedly necessary. In this work, we show that neural networks can provide a nice solution to this problem, where measurement statistics data produced by measuring involved quantum states with local measurement devices serve as input features of neural networks. By testing our models on many specific multipartite quantum states, we show that they can certify GME very accurately, even including some new results unknown before. We also exhibit a possible way to improve the efficiency of our models by reducing the size of features. Lastly, we show that our models enjoy remarkable robustness against flaws in measurement devices, implying that they are very experiment-friendly.

1. Introduction
Quantum entanglement plays a central role in many quantum information processing tasks, including quantum communication [1], quantum cryptography [2] and quantum key distribution [3]. As a consequence, certifying the existence of quantum entanglement is a fundamental problem.

However, it has been proved that even for a bipartite quantum state that the density matrix is completely known, to determine whether it is entangled or not is already NP-Hard [4], implying that it is hard to generally solve this problem efficiently. Nevertheless, due to its importance, numerous approaches have been put forward to certify bipartite entanglement [5–9].

Meanwhile, we often face the situation that the target bipartite quantum state we would like to characterize is unknown to us, i.e. its density matrix is not available, making the task even tougher. For this, one may first reconstruct the density matrix by quantum state tomography [10–12], and then try to solve the problem accordingly. However, it is well-known that this procedure is extremely expensive and can only be implementable when the size of the target quantum state is small. To overcome this difficulty, some realistic alternative approaches can be utilized, which include entanglement witness [13–15] and device-independent schemes [16–19]. But they also suffer from other drawbacks, say being sensitive to operation errors or being easy to fail in providing valuable outcomes.

When it comes to multipartite quantum states, the problem becomes even more complicated, as the mathematical structures of multipartite quantum entanglement are much richer than the bipartite case. Particularly, as a special form of multipartite entanglement that is highly valuable, genuine multipartite entanglement (GME) has significant applications in quantum teleportation [20, 21], quantum state...
sharing [22], quantum metrology [23, 24], and even chemical and biological processes [25]. To certify it, many methods have been proposed. For example, when the full information of density matrices is known, quite a lot of mathematical criteria that can detect multipartite genuine entanglement have been proposed [26–32].

Similar to the bipartite case, we also have to handle the case that full information on target multipartite quantum states is not available, which is actually an extremely common and realistic problem from the viewpoint of engineering. For this, known methods such as quantum state tomography, entanglement witness, and device-independent schemes have been applied to certify GME [17, 33–38]. However, facing similar difficulties as in the bipartite case, it is not hard to understand that these approaches cannot provide satisfying solutions for this NP-Hard problem. As a result, due to its central role in quantum computing and quantum engineering, finding realistic and efficient approaches to certify unknown GME is a challenging yet urgent task, which is also the primary motivation of the current paper.

Recently, machine learning approaches have been employed to characterize quantum properties [39–46]. Different from analytic methods, machine learning is a data-driven approach which aims at making predictions on unseen data by learning from training data. In these works, following standard procedures of machine learning tasks, certain features are extracted from training quantum states, which are then fed into machine learning models. Then we train these models by adjusting the parameters they contain such that the models can make predictions on training quantum states with high accuracy, and if the model details are chosen properly, they can also make accurate predictions on target quantum states unseen before. Here, based on the chosen forms of features, different machine learning models can be designed to detect quantum properties.

In this work, we exploit the possibility of utilizing neural networks to certify GME for general quantum states. In fact, neural networks have been utilized to characterize quantum entanglement in many different approaches, but in most of these works it is full information on density matrices of target quantum states that is input into neural networks [39–41]. These approaches enjoy very high accuracy in predicting the existence of entanglement, but in real-life quantum experiments or quantum engineering applications, full information on density matrices is often not available, restricting the applications of these approaches severely.

To overcome this problem, one may apply machine learning models to detect entanglement based on the partial information of unknown target quantum states, say some expectation values of observables for target quantum states [42, 47]. Although being more experiment-friendly, these approaches suffer from the drawback of limited applicability, as they can only handle some particular classes of quantum states.

In the current manuscript, we choose measurement statistics data produced by measuring involved quantum states with local measurement devices as features for our neural network, where each subsystem of involved multipartite quantum states is measured by at least two devices. Inspired by the remarkable success of Bell experiments, we believe that this kind of measurement statistics data can reveal not only the existence of entanglement, but also its structure.

It turns out that our idea works very well. Particularly, we successfully train a series of neural network models that can certify unknown GME very accurately, where the target quantum states are quite diverse. Taking 4-qubit quantum states for example, we first train a proper model, and then we run the same trained model on four different classes of 4-qubit quantum states, on each of which our model successfully certifies genuine 4-qubit entanglement with accuracy over 99%. Interestingly, our model even reports some new results that are unknown before, indicating that machine learning models can be highly valuable in such a challenging task. We confirm the high performance of neural networks on many other test quantum states, which include quantum state sets that are sampled according to various GME criteria that are different from the ones utilized to train our models.

Meanwhile, we also proposed a modified scheme called k-correlation to reduce the cost of our approach, and we show that in some cases where certain specific prior knowledge is available, the cost of our approach can be sharply reduced while the prediction accuracy is still comparable.

Lastly, we provide evidence showing that our approach enjoys remarkable robustness against flaws in measurement devices, which implies that our approach is very experiment-friendly.

### 2. Deep learning and entanglement structure

In this work, the certification of GME is formulated as a supervised binary classification task fulfilled by neural networks. We call a machine learning method supervised if the training data provided to the model includes both inputs (called features) and the corresponding correct outputs (called labels). Following the standard procedure of supervised learning task, we need to gather a training dataset of the form...
\[ \{(x_1, y_1), \ldots, (x_N, y_N)\}, \]

where \( N \) is the size of the set, \( x_i \) is the feature of the \( i \)th sample, and \( y_i \) is the corresponding label. The labels of the training dataset are known to us, and the mission of the machine learning model is to learn useful patterns from the training dataset such that when new data unseen before come, their correct labels can be predicted based only on their features.

As the heart of the deep learning method, neural network models can enjoy great learning ability [48, 49] and have been widely utilized in a variety of fields such as image recognition [50], natural language processing [51], recommendation systems [52], and so on.

We now use a simple example to look into the mathematical structure of neural networks. Consider a neural network model consisting of one input layer and one output layer, and suppose each input \( x \) is characterized by \( x_i \) where \( i = 1, \ldots, D \) runs over all elements of \( x \). Each of the elements is assigned a weight variable \( w_i \in \mathbb{R} \) as parameters. The output can be expressed by

\[ x \mapsto \sigma \left( w_0 + \sum_{i=1}^{D} w_i x_i \right), \]

where \( \sigma \) is a non-linear function that is used to increase the representative power of the model, called the activation function. In the next step, we can enlarge the scale of the neural network by adding multiple middle layers (called hidden layers) composed of enormous neurons, forming a deep neural network structure. The neural network model is inspired by the human brain, mimicking how biological neurons signal to one another. When training the model, we input the features of the training dataset into the model and adjust its parameters such that it can reproduce correct labels for the training dataset, and then make accurate predictions for new datasets. For this, a proper loss function, a reasonable configuration for the neural network, and an efficient optimization method such as gradient-descent have to be chosen. For more details on neural networks, please see [53, 54].

In this paper, we apply neural networks to certify GME, which is also a typical binary-classification task. In general, a multipartite quantum state can involve many subsystems and thus its entanglement structure can be very complicated. An \( n \)-partite pure quantum state \( |\Psi_{k-sep}\rangle \) is called \( k \)-separable, where \( 1 \leq k \leq n \), if and only if it can be written as a tensor product of \( k \) substates:

\[ |\Psi_{k-sep}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \cdots \otimes |\Psi_k\rangle. \quad (1) \]

Correspondingly, a mixed state is called \( k \)-separable, if and only if it has a decomposition into \( k \)-separable pure states. A multipartite quantum state is called genuinely multipartite entangled if it can not be written as \( k \)-separable for \( k = 2 \), otherwise we call it biseparable state. In addition, a quantum state is said to be of entanglement intactness \( k \), if it is \( k \)-separable but not \((k + 1)\)-separable.

3. Detecting GME for qubit systems

3.1. 3-qubit case

3.1.1. The setup

As the simplest case, we first try to detect GME for 3-qubit quantum states. As mentioned above, since our approach is based on a neural network, we need to prepare a large number of quantum states to train (and test) the neural network. In this work, we always sample random \( d \)-dimensional quantum state \( \rho \in \mathcal{H}^d \) according to spectral decomposition

\[ \rho = \sum_{i=0}^{d-1} \lambda_i |u_i\rangle \langle u_i|. \quad (2) \]

Here we randomly choose nonnegative numbers \( \lambda_i \)'s such that they satisfy \( \sum_i \lambda_i = 1 \). Then we generate a \( d \times d \) Haar random unitary \( U \), and set \( |u_i\rangle \) to be the \( i \)th column of \( U \), which means that \( \{|u_i\rangle\} \) forms a set of orthonormal basis for \( \mathcal{H}^d \). Particularly, if \( \lambda_i = 1 \) for some \( 0 \leq i \leq d - 1 \), \( \rho \) will be a pure state. For now we need to sample 3-qubit quantum states, so we let \( d = 8 \).

After sampling 3-qubit quantum states, we need to prepare labels for them, i.e. indicating whether they are genuinely entangled or not. This will be done by the result given in [26], which proves that the following relation must be satisfied by any biseparable 3-qubit quantum states,
\[ |\varrho_{2,5}| + |\varrho_{3,5}| + |\varrho_{4,5}| \leq \sqrt{\varrho_{1,1}\varrho_{4,4}} + \sqrt{\varrho_{1,1}\varrho_{6,6}} + \sqrt{\varrho_{1,1}\varrho_{7,7}} + \frac{1}{2} (\varrho_{2,2} + \varrho_{3,3} + \varrho_{5,5}), \]  

where \( \varrho_{ij} \) is the \((i,j)\)th entry of the density matrix.

Therefore, violating equation (3) implies the existence of genuine tripartite entanglement. Utilizing this criterion, we sample 30 000 genuinely tripartite entangled states. Concretely, we generate a 3-qubit state by equation (2), then substitute the density matrix into equation (3). If the equation (3) does not hold, we keep this state, otherwise we drop it. Repeat this process until 30 000 genuinely tripartite entangled states are collected. We label each of them with a ‘true’ label.

Next, we construct states with ‘false’ labels by sampling 20 000 biseparable entangled quantum states and 20 000 fully separable states. When sampling biseparable entangled quantum states, all three possible partitions are included, that is, these quantum states are sampled according to the form

\[ p_1 \rho_{A|BC} + p_2 \rho_{B|AC} + p_3 \rho_{AB|C}. \]  

where \( p_1, p_2, \) and \( p_3 \) are sampled from a uniform distribution over the interval \([0, 1]\) and then normalized to satisfy \( \sum_{i=1}^3 p_i = 1 \). \( \rho_{A|BC} \) is a biseparable state according to partition \( A|BC \), similar for \( \rho_{B|AC} \) and \( \rho_{AB|C} \). That is to say, \( \rho_{A|BC} \) can be written as

\[ \rho_{A|BC} = \sum_j q_j \rho_A^{(j)} \otimes \rho_{BC}^{(j)} \]  

where \( q_j \geq 0 \) are sampled from a uniform distribution over the interval \([0, 1]\) and then normalized to meet \( \sum_j q_j = 1 \), and \( \rho_{BC}^{(j)} \) are bipartite entangled states whose partial transpositions are not positive semi-definite, implying the existence of partial entanglement [5]. According to the caratheodory theorem [55], for a quantum state belonging to the Hilbert space \( \mathcal{H}_d^4 \), the number of necessary terms in the convex hull is at most \( d^4 \). Due to limited computational resources, in this paper when sampling biseparable state we only produce \( d^4 \) terms, where \( d = 8 \).

Putting the 30 000 genuinely tripartite entangled states, 20 000 biseparable entangled states, and 20 000 fully separable states sampled above together, we obtain the set of training quantum states, for which the correct labels have been pinned down.

Then we extract the features for the training quantum states, which is essentially the measurement outcome statistics data when local quantum measurements are performed on each qubit. To choose measurement devices, we assign \( M \) Haar random unitary matrices \( U_1, U_2, \ldots, U_M \in \mathbb{C}^{2 \times 2} \) to the \( i \)-qubit. Let \( |U_{ij}^k\rangle \) be the \( k \)th column of \( U_{ij} \), then the \( j \)th measurement device belonging to the \( i \)th qubit can be expressed by the operators \( P_j = \{ |U_{ij}^k\rangle \langle U_{ij}^k| \}_{k=1}^2 \), \( j = 1, 2, \ldots, M \). We emphasize that once the measurement devices are sampled, they remain unchanged during the whole process of extracting features for all the training quantum states. What is more, when afterward we run the trained model on test quantum states, we also use the same measurement devices to extract features for the latter.

According to Born’s rule, the possibility that we obtain outcome \( a_i \in \{1, 2\} \) when measuring the \( i \)th qubit of \( \rho \) with measurement device \( P_{a_i} \) is given by

\[ p(a_1a_2\cdots a_n | x_1x_2\cdots x_n) = \text{Tr} \left( \left( \bigotimes_{i=1}^n P_{a_i} \right) \rho \right), \]  

where \( x_i \in \{1, 2, \ldots, M\}. \) All the joint possibility distributions \( p(a_1a_2\cdots a_n | x_1x_2\cdots x_n) \) combined are called a correlation, which is the input features for our neural networks.

In the 3-qubit case, we assign two measurement devices for each qubit, i.e. \( M = 2 \), thus the dimension of the input features turns out to be \( 2^3 \times 2^3 = 64 \).

The FNN we employ in this case has one input layer which stores all the data of the correlation, and four hidden layers which contain 40, 26, 26, 12 neurons respectively, where the activation functions are all rectified linear unit function. To avoid overfitting, after the first hidden layer we add one drop-out layer with a rate of 20% (this layer randomly sets input units to 0 with a rate of 20% at each step of the training). To obtain the output, the neurons in the last layer are connected to one single neuron by the sigmoid function

\[ S(x) = \frac{1}{1 + e^{-x}}, \]

which is widely used in the classification task, as it is a differentiable real function with output ranges from 0 to 1, making it suitable to various algorithms based on gradient descent. The loss function adopted for training is the binary cross-entropy, which is widely used in binary classification tasks and can be written as:
where $N$ is the number of the samples in the training set, $y_i$ is the $i$th label of the training set, and $\hat{y}_i$ is the $i$th output of the neural network. In the training process, the parameters of neural networks are constantly adjusted such that the loss function is optimized. In our problem, when $\hat{y}_i \geq 0.5$ the neural network outputs an outcome ‘true’, meaning that the corresponding quantum state is genuinely tripartite entangled, otherwise an outcome ‘false’, implying that the neural network predicts the corresponding quantum state to be biseparable.

### 3.1.2. Performance on the GHZ and W states mixed with white noise

After training the neural network, we now test its performance in certifying genuine tripartite entanglement. First, we choose two well-known classes of quantum states, which are the Greenberger–Horne–Zeilinger (GHZ) states mixed with white noise

$$\hat{\rho}_{\text{GW}} \equiv (1 - p) |\text{GHZ}_n\rangle \langle \text{GHZ}_n| + pI/2^n,$$

and the W states mixed with white noise

$$\hat{\rho}_{\text{WW}} \equiv (1 - p) |\text{W}_n\rangle \langle \text{W}_n| + pI/2^n,$$

where $|\text{GHZ}_n\rangle = (|0\rangle^n + |1\rangle^n)/\sqrt{2}$ and $|\text{W}_n\rangle = (|100\cdots0\rangle + |010\cdots0\rangle + \cdots + |000\cdots1\rangle)/\sqrt{n}$. In our case, $n = 3$.

In fact, these two classes of quantum states have been well-studied [26, 32]. It turns out that $\hat{\rho}_{\text{GW}}$ is genuinely $n$-partite entangled if and only if $0 \leq p < 1/2 (1 - 2^{-n})$, and is biseparable when $p \geq 1/2 (1 - 2^{-n})$. For convenience, we call the value $1/2 (1 - 2^{-n})$ the threshold for $\hat{\rho}_{\text{GW}}$ to be genuinely $n$-partite entangled. When $n = 3$, the threshold is 0.571. Similarly, $\hat{\rho}_{\text{WW}}$ is tripartite genuinely entangled if and only if $0 \leq p < 0.521$, thus the threshold is 0.521. And the threshold of $\hat{\rho}_{\text{WW}}$ with $n \geq 4$ remains open.

The above theoretical results provide us a very nice chance to examine the performance of the FNN model we have trained. For this, we first extract features for $\hat{\rho}_{\text{GW}}$ and $\hat{\rho}_{\text{WW}}$ by performing the measurement devices we sampled previously, where $p$ varies from 0 to 1 at intervals of size 0.001, and then we input these features into our neural network.

It turns out that the prediction accuracy of our model on both of the two classes of test quantum states is over 99%. Particularly, the FNN model predicts that the $\hat{\rho}_{\text{GW}}$ and $\hat{\rho}_{\text{WW}}$ are genuinely entangled if and only if $p < 0.570$ and $p < 0.512$, respectively (recall that the exact thresholds for these two classes are 0.571 and 0.521 respectively). It is interesting to see that the output of the FNN model also has a threshold pattern, i.e. it always gives ‘true’ predictions when $p$ is smaller than a certain value, and always gives ‘false’ otherwise. Our model only makes mistakes near the exact threshold for $\hat{\rho}_{\text{GW}}$ and $\hat{\rho}_{\text{WW}}$. Taking $\hat{\rho}_{\text{GW}}$, for example, the neural network provides wrong predictions only when $p = 0.570$.

We would like to highlight a very interesting phenomenon that the power of a neural network model in detecting GME can be remarkably stronger than the original criterion used to train the model. As the first evidence, note that the criterion in equation (3) could only detect genuine entanglement for W states mixed with white noise when $p < 0.471$, whereas the exact GME threshold is 0.521. However, our above neural network model trained with the criterion in equation (3) can successfully witness the existence of genuine tripartite entanglement as long as $p \leq 0.512$. We believe that this comparison provides compelling evidence showing that the neural networks learn some nontrivial structure of multipartite entanglement, as opposed to simply imitating the criteria that train them. This phenomenon is confirmed by later comparisons over and over.

Lastly, we would also stress a very important advantage of our neural network models over these GME criteria from the viewpoint of quantum engineering. Note that most GME criteria require the full density matrix of a target quantum state (for example, the criteria in equation (3)). As a result, if a target quantum state is unknown to us (the density matrix is not available, and this is very common in quantum labs), in order to apply these criteria one probably has to perform quantum state tomography, which needs extremely high costs. As a sharp comparison, the input of our neural networks is statistics data generated by some local quantum measurements, and thus we only need to collect partial information on target quantum states (the cost can be much less than quantum state tomography). This means that when handling unknown quantum states, our approach is much more efficient.
3.1.3. Performance on more general quantum states

In some sense, $\rho_{\text{GWS}}$ and $\rho_{\text{WWS}}$ are quite special 3-qubit quantum states. Therefore, to further assess the performance of the FNN model we have trained, we now test it on more general 3-qubit quantum states. For this, we now generate a test state set with the help of two GME criteria different from equation (3), the one utilized to train our model.

The first criterion comes from [26], which proves that if $\rho$ is a biseparable 3-qubit state, then the entries of its density matrix fulfill

$$|\theta_{1,3}| \leq \sqrt{\theta_{1,3}^{2} \theta_{1,7}^{2}} + \sqrt{\theta_{3,7}^{2} \theta_{4,6}^{2}} + \sqrt{\theta_{4,4} \theta_{5,5}},$$

and any violation of this inequality indicates the existence of genuine tripartite entanglement.

The second criterion comes from [27], where the concept of tripartite entanglement of formation denoted $E_f(\rho)$ was proposed to quantify genuine tripartite entanglement, which is defined as

$$E_f(\rho) \equiv \min_{|\psi_i\rangle} \left( \sum_i p_i \min \left\{ S_i(A), S_i(B), S_i(C) \right\} \right).$$

Here the first minimum is taken over all pure state decompositions of $\rho$ and $S_i(A), S_i(B), S_i(C)$ are the von Neumann entropies of the subsystem $A$, $B$, and $C$ of $|\psi_i\rangle$. It turns out that $E_f(\rho)$ equals zero for biseparable states [27], and can be lower bounded by $V_\rho$, defined as [56]

$$-S(A|BC) - S(B|AC) - S(C|AB) - 2\log(D_{\text{max}}),$$

where $S(A|BC)$ is the von Neumann conditional entropy, and $D_{\text{max}}$ is the maximum dimension of parties $A$, $B$, and $C$. $V_\rho$ is easy to compute when $\rho$ is given, and can also certify the existence of genuine tripartite entanglement if its value is positive.

By first generating random quantum states by equation (2) and then utilizing the criteria given in equations (10) and (12), we sample 600 genuine tripartite entangled states. Together with 200 biseparable entangled states and 200 fully separable states, we construct a set of test quantum states. The features for these test quantum states are also generated by equation (6), where the chosen measurement devices are the same as those utilized to prepare the training dataset.

We now run the neural network we have trained before on this test dataset. It turns out that our model again performs very well in this case, and the prediction accuracy is over 99%. Considering that the test dataset enjoys obvious diversity, it is fair to say that our model enjoys a very decent performance in the 3-qubit case.

Particularly, on the 1000 quantum states sampled above, our model accurately identifies all biseparable states, and only predicts 8 genuinely entangled states as biseparable. In contrast, the criterion in equation (3), with which we train our model, only witnesses 120 genuinely entangled states from the 600 ones. This comparison provides further evidence showing that the power of neural networks in detecting GME can remarkably surpass that of the original criterion used to train the model.

3.2. 4-qubit case

Now we apply our approach on larger quantum systems, and turn to certify genuine 4-partite entanglement. Again, to train the neural network properly, we need to prepare a lot of training quantum states for which we know entanglement structure accurately. However, the criteria we utilized to certify genuine tripartite entanglement can not detect genuine 4-partite entanglement very well. Instead, we now apply a new criterion provided by [29]. Particularly, the concept of concurrence $C(\rho)$ is a GME measure, and positive concurrence indicates the existence of GME (see [28, 29] for more details). Define

$$F(\rho, \psi) = \sum_{1 \leq i < j \leq n} \sqrt{\langle \psi_i | \rho^{\otimes 2} | \psi_j \rangle} - \sum_{1 \leq i \neq j \leq n} \sqrt{\langle \psi_i | p_i^{\dagger} \rho^{\otimes 2} p_j | \psi_j \rangle} - (n - 2) \sum_{1 \leq i < j \leq n} \sqrt{\langle \psi_i | p_i^{\dagger} \rho^{\otimes 2} p_j | \psi_j \rangle},$$

then it holds that $C(\rho) \geq \frac{1}{\sqrt{2(n-1)}} F(\rho, \psi)$. Here $|\psi\rangle = \bigotimes_{i=1}^{n} |x_i\rangle = |x_1 x_2 \cdots x_n\rangle$ is an arbitrary product state in Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, $|\psi_i\rangle = |x_1 x_2 \cdots x_{i-1} x'_i x_{i+1} \cdots x_n\rangle$ and $|\psi_j\rangle = |x_1 x_2 \cdots x_{j-1} x'_j x_{j+1} \cdots x_n\rangle$ are the product states obtained from $|\psi\rangle$ by applying (independently) local unitaries to
After training the neural network, we use known analytical results on genuine 4-partite entanglement to gather one test set of quantum states for each of the above four classes of quantum states, where \( p \) varies at intervals of size 0.001. Again, the features for these test quantum states labeled ‘false’ are constructed for each entanglement intactness of 4, 3, 2, and \( < \text{p} \) respectively, overall our model makes excellent predictions. Interestingly, the results for \( \text{p} = 0.611 \) and \( 0.614 \) for \( |\text{GHZ}_4 \rangle \) and \( |\text{Cl}_4 \rangle \) respectively, are 0.533 and 0.539, indicating the potential value of our neural network in identifying genuine 4-partite entangled states. We use known analytical results on genuine 4-partite entanglement to gather one test set of quantum states for each of the above four classes of quantum states, where \( p \) varies at intervals of size 0.001. Again, the features for these test quantum states labeled ‘false’ are constructed for each entanglement intactness of 4, 3, 2, and \( < \text{p} \) respectively, overall our model makes excellent predictions. Interestingly, the results for \( \text{p} = 0.611 \) and \( 0.614 \) for \( |\text{GHZ}_4 \rangle \) and \( |\text{Cl}_4 \rangle \) respectively, are 0.533 and 0.539, indicating the potential value of our neural network in identifying genuine 4-partite entangled states.

### Table 1. The configuration details of FNNs. In the first column, \( n,d \) denotes the number and the dimension of the subsystems, respectively.

| System size | Hidden layers |
|-------------|---------------|
| \( n = 3,d = 2 \) | 40, one drop-out layer, 26, 26, 12 |
| \( n = 3,d = 4 \) | 20, one drop-out layer, 12, 12, 12 |
| \( n = 4,d = 2 \) | 70, one drop-out layer, 46, 46, 46 |
| \( n = 5,d = 2 \) | 65, one drop-out layer, 42, 42, 42 |

### Table 2. Performance of the neural network for the 4-qubit case. The quantum states sampled according to the criteria equations (16) and (17) are called ‘more general’ states. For the classes marked by *, the exact threshold has been known. The last two columns denote the thresholds given by our model and the best-known noise tolerance for the existence of GME.

| States | Accuracy | Threshold (FNN) | Best-known |
|--------|----------|----------------|------------|
| \( |\text{GHZ}_4 \rangle^* \) | 99% | 0.523 | 0.533 [32] |
| \( |\text{W}_4 \rangle \) | 100% | 0.635 | 0.526 [32] |
| \( |\text{Cl}_4 \rangle^* \) | 99.7% | 0.611 | 0.614 [32] |
| \( |\text{D}_{24} \rangle \) | 100% | 0.585 | 0.539 [32] |
| More general | 99.6% | / | / |

We compute lower bounds for concurrence by randomly sampling a product state \( |\psi \rangle \) and local Haar random unitaries. By this way, we sample 30000 states with positive \( F(\rho, \psi) \) to serve as ‘true’ samples of our training set. Additional 20000 states labeled ‘false’ are constructed for each entanglement intactness of 4, 3, 2, and \( < \text{p} \) respectively, where as before all possible partitions are included for each intactness.

After picking up the quantum states for training, we extract the features by a similar approach as before, that is, randomly sampling a set of local measurement devices and fixing them, and then collecting the outcome statistics data when measuring the training quantum states with these measurement devices, where the only difference is that we now have four parties, resulting in that the dimension of the input feature of the neural network is 256 accordingly.

The neural network we employ here has a similar structure to the one used for the 3-qubit case, and the concrete configuration is listed in table 1. The loss function remains the binary cross-entropy.

After training the neural network, we use known analytical results on genuine 4-partite entanglement to evaluate the performance of our model. For this, we focus on four kinds of 4-qubit quantum states mixed with white noise, which are \( |\text{GHZ}_4 \rangle, |\text{W}_4 \rangle, |\text{Cl}_4 \rangle \) and \( |\text{D}_{24} \rangle \). The definition of \( |\text{GHZ}_4 \rangle \) and \( |\text{W}_4 \rangle \) have been given in equations (8) and (9), and \( |\text{Cl}_4 \rangle \) and \( |\text{D}_{24} \rangle \) are represented as

\[
|\text{Cl}_4 \rangle = (|0000 \rangle + |0011 \rangle + |1100 \rangle - |1111 \rangle)/2, \\
|\text{D}_{24} \rangle = \frac{1}{\sqrt{6}}(|0011 \rangle + |1100 \rangle + |0110 \rangle + |0101 \rangle + |1001 \rangle + |1010 \rangle).
\]

The exact thresholds for \( |\text{GHZ}_4 \rangle \) and \( |\text{Cl}_4 \rangle \) with white noise have been found to be 0.533 and 0.614 respectively [32]. That is to say, (1 - \( \text{p} \)) \( |\text{GHZ}_4 \rangle \) \( |\text{GHZ}_4 \rangle + \text{p}I/16 \) is genuine 4-partite entangled if and only if \( 0 \leq \text{p} < 0.533 \), and similar for \( |\text{Cl}_4 \rangle \). However, to our best knowledge the exact thresholds for \( |\text{W}_4 \rangle \) and \( |\text{D}_{24} \rangle \) mixed with white noise are unknown. And the best-known results for \( |\text{W}_4 \rangle \) and \( |\text{D}_{24} \rangle \) mixed with white noise to be genuine 4-partite entangled is that \( \text{p} < 0.526 \) and \( \text{p} < 0.539 \) respectively [32].

Similar to the 3-qubit case, we gather one test set of quantum states for each of the above four classes of 4-qubit quantum states, where \( \text{p} \) varies at intervals of size 0.001. Again, the features for these test quantum states are extracted by the same measurement devices as what we utilized in training the neural network. The labels for these test quantum states are largely known from the analytical results given in [32], and we use them to benchmark our model.

On the four test dataset, the results provided by our neural network are listed in table 2. It turns out that on all the instances that analytical results are known \( 0 \leq \text{p} < 1 \) for \( |\text{GHZ}_4 \rangle \) and \( |\text{Cl}_4 \rangle \), \( 0 \leq \text{p} < 0.526 \) for \( |\text{W}_4 \rangle \), and \( 0 \leq \text{p} < 0.539 \) for \( |\text{D}_{24} \rangle \), overall our model makes excellent predictions. Interestingly, the results given by the neural network again have a threshold pattern, which reveals that the thresholds for the \( |\text{W}_4 \rangle \) class and the \( |\text{D}_{24} \rangle \) class are possibly around 0.635 and 0.585 respectively, indicating the potential value of our neural network again, particularly in quantum engineering areas.
To provide more convincing evidence that the neural network we trained works well, we now test it on more general 4-qubit quantum states, where two different criteria are adopted to label the test states. First, it has been proven in [30] that for a 4-qubit quantum state $\rho$, if one of the inequalities

$$
\|M_{22}(T_{i\rightarrow i\oplus k})\|_k > \begin{cases} 
2\sqrt{k} & 1 \leq k \leq 3 \\
1 + 2k/3 & 4 \leq k \leq 9
\end{cases}
$$

(16)

holds, then the state must be genuine multipartite entangled. Here $T_{i\rightarrow i\oplus k} = \text{Tr}(\sigma_i \otimes \cdots \otimes \sigma_i \rho)$, $i = 1, 2, 3$ for each $j = 1, \cdots, 4$ and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrix $X, Y, Z$. $\|M_{22}(T_{i\rightarrow i\oplus k})\|_k = \left(\|T_{i\rightarrow i\oplus k}\|_k^2 + \|T_{2i\rightarrow 2i\oplus k}\|_k^2 + \|T_{2i\oplus 2i\rightarrow 2i\oplus k}\|_k^2\right)/3$, $T_{i\rightarrow i\oplus k} = \sum_{i_1=\ldots\rightarrow i_k} T_{i\rightarrow i\oplus k} |i_1i_2\rangle \langle i_2i_3|$, and $\|\cdot\|_k$ is the Ky Fan $k$ norms defined as the first $k$ largest singular values of the matrix [57].

The second criterion comes from [31], which says that any biseparable quantum state $\rho$ satisfies

$$
\sqrt{\langle \Phi | \rho^{\otimes 2} \Pi | \Phi \rangle} = \sum_{\gamma} \sqrt{\langle \Phi | \Pi_{\gamma} \rho^{\otimes 2} | \Phi \rangle} \leq 0,
$$

(17)

where $\gamma$ is taken over all possible subsets of $\{1, 2, \ldots, n\}$, $|\Phi\rangle$ is any state that is separable with respect to two copies of $n$-partite Hilbert spaces $H \otimes H$, and $\Pi_{\gamma}$ is the cyclic permutation operator that exchanges the states of the subsystems in $\gamma \subseteq \{1, 2, \ldots, n\}$ in the first $n$-partite Hilbert space with those in the second $n$-partite Hilbert space. Particularly, $\Pi = \Pi_{\{1, 2, \ldots, n\}}$.

Similar as before, we sample 600 genuine 4-partite entangled states by first sampling quantum states by equation (2), and then picking up those satisfying equation (16) or violating equation (17). Together with 600 biseparable states, we construct a nice test set of quantum states. The experimental result is also listed in table 2. As we can see, the prediction accuracy is over 99%, indicating a very decent and stable performance of our model.

Meanwhile, we also use the criterion in equation (13), with which we train our neural network model, to detect GME for the same test set. It turns out that only around 140 out of the 600 genuinely entangled states can be detected, which again indicates that our model can dramatically surpass the original criterion used to train our model.

### 4. Detecting GME for qudit states

Next we turn to more general multipartite quantum states, where each subsystem can be higher-dimensional, and the number of subsystems can also be larger.

We would like to point out that in this case training neural networks is more challenging. The reason is that after randomly sampling candidate quantum states for training, it is hard to provide correct labels for them. For example, if we apply the criteria that we have utilized previously, say equations (12), (13) and (17), on 5-qubit quantum states or higher dimensional quantum states, it is very difficult to certify the existence of GME. In fact, we also tried to use the technique introduced in [32] for the same task, which is actually an optimization approach based on semi-definite programming, and we find that applying this method is very time-consuming.

After careful comparisons, we realize that the approach proposed in [58] is very suitable to provide labels for candidate training quantum states. Basically, this approach points out that an $(n + 2)$-partite genuinely entangled state can be obtained by merging two $(n + 1)$-partite genuinely entangled states. Specifically, suppose $\rho_{\mathcal{H}_{A_{1},A_{2}..,A_{n}}} \otimes \tau_{\mathcal{B}_{1},B_{2}..,B_{n}}$ are two $(n + 1)$-partite genuinely entangled states in the Hilbert spaces $\mathcal{H}_{A_{1},A_{2}..,A_{n}}$ and $\mathcal{H}_{B_{1},B_{2}..,B_{n}}$, respectively. Then an $(n + 2)$-partite genuinely entangled state can be constructed by applying the operator $\otimes K_j$ as follows

$$
\mathcal{H}_{A_{1},A_{2}..,A_{n}} \otimes K_j \mathcal{H}_{B_{1},B_{2}..,B_{n}} := \mathcal{H}_{A_{1}} \otimes \mathcal{H}_{B_{1}} \otimes (\mathcal{H}_{C_{1},C_{2}..,C_{n}}),
$$

(18)

where $\mathcal{H}_{C_{j}} = \otimes K_j \mathcal{H}_{A_{1}}, 1 \leq j \leq n$. By definition, $\rho_{\mathcal{H}_{A_{1},A_{2}..,A_{n}}} \otimes K_j \tau_{\mathcal{B}_{1},B_{2}..,B_{n}}$ is an $(n + 2)$-partite state of systems $A, B$, and $C_j$‘s, where $C_j := (C_{1..j}jC_{j+1}..n)$ for $1 \leq j \leq n$.

By this method, we can use the small-scale genuine multipartite entangled states sampled previously to build larger genuine multipartite entangled states in an iterative manner, which in turn are utilized to train neural networks customized for large quantum systems.

Meanwhile, note that the entanglement dimension is enlarged after the merging operation introduced above. For example, if we have two bipartite entangled quantum states in Hilbert space $\mathcal{H}^4 \otimes \mathcal{H}^2$, after the merging operation we can obtain a genuine multipartite entangled state in $\mathcal{H}^4 \otimes \mathcal{H}^4 \otimes \mathcal{H}^4$. 
4.1. Tripartite qudit case

We first apply our approach on tripartite qudit systems. To sample genuine tripartite entangled quantum states for training, we first generate a lot of bipartite entangled states, which can be achieved by the positive partial transpose criterion. Then with the merging operation introduced in [58], we can generate many desirable genuine tripartite entangled states. In this way, we eventually collect 20 000 genuine tripartite entangled states in $\mathcal{H}^4 \otimes \mathcal{H}^4 \otimes \mathcal{H}^4$. Combining them with 40 000 biseparable states in the same Hilbert space, we finish preparing the set of training quantum states.

Next we extract features for the training quantum states, which is again the measurement statistics data produced by measuring these quantum states locally. Since the dimensions of subsystems are now larger, we increase the number of measurement devices for each party from two to three. Accordingly, we let $M = 3$ in equation (6), and the feature size of the neural network becomes 576. After extracting the features of training quantum states, we input them into the neural network to train our model. The configuration of the FNN for this case is also listed in table 1.

To test the model we just trained, we first consider the generalized GHZ state $|\text{GHZ}_{43}\rangle = \frac{1}{2}\sum_{i=0}^{3} |i\rangle^\otimes 3$ with white noise:

$$\rho = p|\text{GHZ}_{43}\rangle \langle \text{GHZ}_{43}| + (1-p)\frac{1}{4^3} I.$$

By running our model on this class of quantum states, we observe that it makes the ‘true’ prediction when $p > 0.161$, while it has been proved analytically that this class of quantum states is genuine tripartite entangled when $p > 0.157$ [31].

As the second class of test quantum states, we now turn to the qutrit state

$$\rho = \frac{1 - \alpha - \beta}{27} I + \frac{\alpha}{3} \rho_{\text{bisep}} + \beta |\text{GHZ}_{33}\rangle \langle \text{GHZ}_{33}|,$$

where $\rho_{\text{bisep}} = |0\rangle \langle 0| \otimes (|00\rangle + |11\rangle + |22\rangle)(|00\rangle + |11\rangle + |22\rangle)$. To employ our model for these qutrit states, we embed each qutrit into a 4-dimensional Hilbert space by padding zeros to the redundant bases to make these test quantum states have the same dimension as the training quantum states and the measurement devices that extract features.

The result given by the neural network is depicted in figure 1. Particularly, Area I is the parameter space that has been proved to be genuine tripartite entangled in [31], which is strictly smaller than the parameter space that the neural network reports to be genuine tripartite entangled, as the latter also contains Area II.

To double check the correctness of the results provided by the neural network, we perform independent numerical calculations to study the case that $\alpha = 0$. By brute-force search for all possible biseparable decompositions of $\rho$, we find that $\rho$ is genuine tripartite entangled only when $\beta$ is larger than 0.13, which is very close to the result given by our model. We believe that the neural network provides very reliable results in this case.

Lastly, let us assess the performance of our model by testing it on more general quantum states. For this, we first pick up 600 genuinely entangled tripartite states in Hilbert space $\mathcal{H}^4 \otimes \mathcal{H}^4 \otimes \mathcal{H}^4$ by the sampling
approach in equation (2) and the GME criterion in equation (12). Combined with 400 biseparable tripartite states, these quantum states serve as the test quantum states. After extracting the feature data by measuring these states with the same measurement devices utilized during the training phase, we complete the construction of the test dataset.

Running the trained model on this new test set, we observe that the prediction accuracy reaches 98.7% in this case, where the neural network misclassifies only 13 genuinely entangled states as biseparable. This provides strong evidence showing that our approach can be employed to detect GME for higher-dimensional quantum systems.

4.2. 5-partite qudit case
As the last demonstration of our approach, we now use it to detect genuine 5-qudit entanglement. Based on the 4-qubit genuinely entangled states sampled in section 3.2, we use the merging technique introduced in [58] to generate 20000 5-partite genuinely entangled states in $(\mathcal{H}^2)^{\otimes 2} \otimes (\mathcal{H}^4)^{\otimes 3}$. To unify the dimension of subsystems for simplicity, we embed these quantum states into a larger Hilbert space $(\mathcal{H}^4)^{\otimes 5}$ also by padding zeros to the redundant bases. Together with 60000 biseparable states with 'false' labels, we obtain the whole set of training quantum states. Then as usual, we extract features for these states by measuring them with local measurement devices, where each qudit has three devices. The configuration of the neural network we employ in this case can be found in table 1.

We now test the performance of this trained model. Due to the fact that quantum states which have been certified analytically to be genuine 5-qudit entanglement are rare, we first test our model with the $|\text{GHZ}_5\rangle$ with white noise, which are accordingly embedded into the Hilbert space $(\mathcal{H}^4)^{\otimes 5}$. Again, when testing we vary the value of $p$ at intervals of size 0.001. Recall that the 5-qubit GHZ state mixed with white noise has been proved to be genuinely entangled if and only if $0 < p < 0.516$, and it turns out that our model achieves over 98% prediction accuracy on this class of quantum states.

In order to further test the power of our model, we construct another test set composed of more general quantum states. For this, we sample 600 biseparable states labeled 'false', and 600 genuine 5-qudit entangled states labeled 'true' with the technique introduced in equation (18). Combining these 1200 quantum states, we obtain the new test set, on which our model certifies GME with accuracy over 99%.

With the successes of all the models we have trained above, we expect that if one extends the neural network approach to more complicated cases of target quantum states, the performance will also be decent.

5. Reducing the cost of quantum measurements
We have shown the great potential of detecting GME by machine learning methods, where the features are measurement statistics data. However, if the number of subsystems goes up, the cost of quantum measurements (or the size of features) will increase very quickly. With this in mind, we now propose the so-called $k$-correlation scheme to reduce the cost of quantum measurements.

5.1. $k$-correlation scheme
Recall that when extracting features for training and test quantum states, a set of local quantum measurements are performed on these quantum states to produce correlation data, where many cross terms are involved. For example, suppose Alice (Bob) possesses measurement devices $A_1, A_2$ ($B_1, B_2$), and she (he) measures the subsystem of a bipartite quantum state locally. The correlation data they produce comes from four different combinations of measurement devices, i.e. $A_1 B_1, A_1 B_2, A_2 B_1$ and $A_2 B_2$, where $A_1 B_1, A_2 B_2$ are called the cross terms. If we ignore these two cross terms, we can reduce the size of correlation data by half in this case. By generalizing this idea to $n$-partite quantum state $\rho$, we define the concept of $k$-correlation data, which is given by

\[ p(a_1 a_2 \cdots a_n | x) = \text{Tr} \left( \bigotimes_{i=1}^{n} p_{a_i}^x \rho \right), \tag{19} \]

where $a_i \in [d_i] \equiv \{1, \ldots, d_i\}$ represents the outcome for measurement device $P_{a_i}$, here $x \in \{1, 2, \cdots, k\}$. In this section, we will try to certify GME by neural networks with features being $k$-correlation data, and for convenience, we call this approach the $k$-correlation scheme.

5.2. Detecting genuine 4-qubit entanglement using the $k$-correlation scheme
Since analytical results on genuine 4-qubit entanglement are relatively rich, we now test the performance of the $k$-correlation scheme in this case. To be specific, we will reuse the same sets of training and test quantum states as before, and the only difference between the two models is the forms of features, which involves both the training and test stages.
Quantum computing and quantum information states are genuine multipartite entangled. It is well-known that graph states have significant applications in reported in clearly that prior knowledge can be very helpful in our approach. In fact, a similar result has also been reduced, where the number of measurement device combinations goes down dramatically. This indicates accuracy for each class are listed in Table 3.

As we can see, the performance of the new model is worse than the original one based on full correlation data. For example, the prediction accuracy for the GHZ class is lower compared with the old result. To extract more information from the training and test quantum states, we offer each party more measurement devices, and generate k-correlations with \( k \geq 6 \). It turns out that even if we let \( k = 16 \), which means the features now have the same size as the original model, the new model still can not certify genuine 4-qubit entanglement for the GHZ-Werner class with accuracy over 90%. This reveals an interesting fact that cross terms in correlation data are very crucial to extract quantum properties for underlying quantum states.

### 5.3. Detecting GME for graph states

Though the power of the \( k \)-correlation scheme is weaker than the original scheme based on complete correlation data, we now show that if certain prior knowledge on target quantum states is known, the \( k \)-correlation scheme still enjoys a nice performance.

To demonstrate this fact, in this subsection we utilize the \( k \)-correlation scheme to predict whether graph states are genuine multipartite entangled. It is well-known that graph states have significant applications in quantum computing and quantum information [59–62].

The motivation to apply the \( k \)-correlation scheme on graph states is very clear. Suppose we have a graph state consisting of 12 qubits, then if we let full correlation data serve as features for the neural network, the feature size will be \( 2^{12} \) even if each qubit is measured by only two measurement devices. As a comparison, if we apply the \( k \)-correlation scheme, the feature size will be \( 2^{12} \cdot k \), which can be much smaller. Interestingly, it turns out that the \( k \)-correlation scheme works well in this case.

Specifically, we first randomly generate 600 10- (11-, 12-) qubit graph states to train the neural network, where the corresponding labels (whether the graph states are genuine multipartite entangled) can be determined by the connectivity of the underlying graphs [63]. Here the FNN model we choose has 3 hidden layers that contain 50, 25, 10 neurons respectively. After training, we test the model by running it on another 200 random graph states. The result is that even 4-correlation data is already enough to achieve a prediction accuracy of 97%.

It can be seen that the cost of quantum measurement in the \( k \)-correlation scheme has been sharply reduced, where the number of measurement device combinations goes down dramatically. This indicates clearly that prior knowledge can be very helpful in our approach. In fact, a similar result has also been reported in [42], where a machine learning method is applied to certify the entanglement structure of pure and noisy generalized GHZ states based on expectation values of local observables, and a very nice performance is achieved due to the fact that the training and test quantum states there have a very similar configuration, which is essentially a kind of prior knowledge. We stress that in the current work our models can handle much more general quantum states.

We remark that the \( k \)-correlation scheme and the original scheme based on full correlation data are two extreme ends to apply our neural network approach. From the viewpoint of quantum engineering, we believe that there are many other ways to exploit correlation data with neural networks, and a proper tradeoff between efficiency and prediction accuracy can be found out based on specific application scenarios.

### 6. Robustness of our models

Recall that in each model we have trained, at the beginning we always randomly sample a set of measurement devices and fix them, then the features of all training and test quantum states are extracted based on these devices. Naturally, one may ask, if we repeat the whole process and sample a different set of measurement devices, can we obtain better prediction performance?

Interestingly, in all the models we have trained, we do make this kind of comparisons, and it turns out that measurement devices generated by different choices of Haar random matrices can only change the prediction performance very slightly.

| States | accuracy |
|--------|----------|
| \(|\text{GHZ}_4^*\rangle\) | 86% |
| \(|\text{W}_4\rangle\) | 100% |
| \(|\text{Cl}_4\rangle^*\) | 99.6% |
| \(|\text{D}_{2,4}\rangle\) | 100% |

After retraining our model based on 5-correlation data, we again test the model with the four classes of 4-qubit quantum states, i.e. \(|\text{GHZ}_4\rangle, |\text{W}_4\rangle, |\text{Cl}_4\rangle\) and \(|\text{D}_{2,4}\rangle\) mixed with white noise. The results on prediction accuracy for each class are listed in Table 3.

It turns out that even if we let \( k = 16 \), which means the features now have the same size as the original model, the new model still can not certify genuine 4-qubit entanglement for the GHZ-Werner class with accuracy over 90%. This reveals an interesting fact that cross terms in correlation data are very crucial to extract quantum properties for underlying quantum states.
Particularly, in the 4-qubit case we also try to extract features with the observables taken from [64], which is widely used to witness entanglement intactness. Specifically, we assign each party the following two observables

\[ A_1 = \sigma_z, \]  
\[ A_2 = A_+ = \cos \left( \frac{n+1}{2n} \phi \right) \sigma_x + \sin \left( \frac{n+1}{2n} \phi \right) \sigma_y. \]

In our case, \( n = 4 \) and \( \phi = 1.1055 \). After repeating the training and test procedures with the new way to extract features, we observe that the accuracy of detecting genuine 4-qubit entanglement keeps almost unchanged.

To provide further evidence for the above robustness, we also introduce the mechanism of ‘learnable measurement’ to find out what measurement devices are optimal in our models. For this, we parameterize involved local measurement devices, and search for the optimal family of parameters by optimization approaches [46]. Again, the result shows that even the optimized measurement devices still provide almost the same prediction accuracy.

We would like to stress that from the point of view of engineering, this kind of robustness is very valuable. This implies that, even if the quality of the measurement devices we utilize is poor, as long as their workings keep stable, our approach can still work very well.

7. Conclusion

To summarize, in this work we demonstrate that neural networks can certify GME very accurately based on the measurement statistics data produced by measuring involved quantum states with local measurements. Specifically, we successfully train neural networks to detect GME for many multipartite qubit and qudit systems. By testing these trained models on various multipartite quantum states, we observe that the prediction accuracy is very high. Particularly, in many cases the test quantum states are sampled according to various GME criteria that are different from the ones utilized to train our models, which provides convincing evidence showing that neural networks can work very well in such a challenging task.

In addition, to improve the efficiency of our models, we propose the \( k \)-correlation scheme to reduce the cost of quantum measurements. We show that when prior knowledge of target quantum states is known, the performance of this scheme can still be very good. This indicates that one can adjust model details to save computational resources according to application scenarios.

Meanwhile, we remark that our approach can also be combined with the idea of shadow tomography [65], by which one can generate correlation data based on the classical shadow of involved quantum states, and thus reduce the cost of quantum operations dramatically.

Lastly, we show that our models enjoy remarkable robustness against flaws in involved quantum measurements, implying that our models are very experiment-friendly. We expect that in the future our neural network approach can be applied to certify unknown GME, providing a realistic solution to this fundamental yet challenging problem.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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