S-duality and loop operators in canonical formalism

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Abstract

We study the gauge invariant t’ Hooft operator in canonical formalism for Yang-Mills theory as well as the $\mathcal{N} = 4$ Super Yang-Mills theory. It is shown that the spectrum of the t’ Hooft operator labeled by the dual representation of the gauge group is the same as the spectrum of the Wilson operator labeled by the same representation. So it is possible to construct a unitary operator $S$ making the two kinds of loop operators transformed into each other. S-duality transformation could be realized by the operator $S$. We compute the supersymmetry variations of the loop operators with the fermionic couplings turned off. The result is consistent with the expectation that the action of $S$ should make the supercharges transform with a $U(1)_Y$ phase.
1. INTRODUCTION

In $U(1)$ gauge theory, S-duality \cite{1,3} has a simple realization in canonical formalism. The canonical coordinates are $A_i$ with the conjugate momentum $\Pi^i, i = 1, 2, 3$. In temporal gauge,
\[ \partial_i \Pi^i = 0. \] The unitary operator \( S \) with
\[ S^{-1} \Pi_i S = \frac{B_i}{2\pi} \quad S^{-1} B_i S = -2\pi \Pi_i \quad (1.1) \]
gives the S-transformation of the theory. For the gauge potential eigenstate \( |A\rangle \),
\[ S |A\rangle = \int DA' \exp\left\{ \frac{i}{2\pi} \int d^3 x \, \epsilon^{ijk} A'_i \partial_j A_k \right\} |A'\rangle \quad (1.2) \]
is the eigenstate of \( \Pi_i \). In \( U(1) \) gauge theory, Wilson and t’ Hooft operators for the spacial loop \( C \) are given by
\[ W(C) = \exp\left\{ i \oint_C ds A_i \dot{x}^i \right\} = \exp\left\{ i \int\int_{\Sigma_C} d\sigma^i \, B_i \right\} \quad T(C) = \exp\left\{ 2\pi i \int\int_{\Sigma_C} d\sigma^i \, \Pi_i \right\} . \quad (1.3) \]
Under the action of \( S \),
\[ S^{-1} T(C) S = W(C) \quad S^{-1} W(C) S = T^+(C) . \quad (1.4) \]
In canonical formalism, S-duality transformation rule is determined by the field content with no dynamical information involved. \((1.1)-(1.4)\) apply for \( U(1) \) gauge theory with the arbitrary Hamiltonian. If the Hamiltonian is invariant under the action of \( S \), the theory will be S-duality invariant.

The purpose of this paper is to extend \((1.4)\) into the nonabelian gauge theories. For Yang-Mills theory with the gauge group \( G \), we give a definition for the gauge invariant t’ Hooft operator \( T_R(C) \) in the arbitrary representation \( R \). When \( R \) is the dual representation of \( G \), or equivalently, the representation of the GNO dual group \( G^* \) thus could be denoted as \( R^* \), we show that the spectrum of \( T_{R^*}(C) \) is the same as the spectrum of the Wilson operator \( W_{R^*}(C) \) in the same representation. For the arbitrary \( |A\rangle \) with
\[ W_{R^*}(C) |A\rangle = W_{R^*}(A; C) |A\rangle , \quad (1.5) \]
where \( W_{R^*}(A; C) \) is the Wilson loop of \( A \) labeled by \( R^* \), there is a corresponding physical state \( |D\rangle_A \) with
\[ T_{R^*}(C) |D\rangle_A = W_{R^*}(A; C) |D\rangle_A \quad (1.6) \]
for the arbitrary \( C \). Eigenstates of \( T_{R^*}(C) \) with the different eigenvalues are orthogonal. Suppose \( |A\rangle_{ph} \) is the projection of \( |A\rangle \) in the physical Hilbert space, then the mapping between
\[ |A \rangle_{ph} \text{ and } |D \rangle_A \] gives a unitary operator \( S \) with

\[
S^{-1} \mathcal{T}_{R^*}(C) S = \mathcal{W}_{R^*}(C) \quad \text{and} \quad S^{-1} \mathcal{W}_{R^*}(C) S = \mathcal{T}_{R^*}(C) \ .
\] (1.7)

For \( \mathcal{N} = 4 \) Super Yang-Mills (SYM) theory with the gauge group \( G \) and the coupling constant \( \tau \), Wilson and t’ Hooft operators in the dual representation of \( G \) are \( \mathcal{W}_{R^*}(\tau; \lambda^I, \lambda^a, C) \) and \( \mathcal{T}_{R^*}(\tau; \lambda^I, \lambda^a, C) \) with \( \lambda^I \) and \( \lambda^a \) the parameters characterizing the scalar and the fermionic couplings. We show that it is possible to construct \( S \) with

\[
S^{-1} \mathcal{T}_{R^*}(\tau; \lambda^I, \lambda^a, C) S = \mathcal{W}_{R^*}(\frac{-1}{\tau}; \lambda^I, e^{-i\theta} \lambda^a, C) ,
\] (1.8)

\[
S^{-1} \mathcal{W}_{R^*}(\tau; \lambda^I, \lambda^a, C) S = \mathcal{T}_{R^*}^+(\frac{-1}{\tau}; \lambda^I, e^{-i\theta} \lambda^a, C) ,
\] (1.9)

where \( e^{i\theta} = (\tau/\bar{\tau})^{1/4} \). This is consistent with the expectation that the S-transformation will make the t’ Hooft operator labeled by \( R^* \) in theory with the coupling constant \( \tau \) and the gauge group \( G \) mapped into the Wilson operator labeled by \( R^* \) in theory with the coupling constant \( -1/\tau \) and the gauge group \( G^* \). [5]

With the S-duality transformation operator \( S \) given, we may study whether the theory is duality invariant. For \( \mathcal{N} = 4 \) SYM theory, if the supercharges transform as

\[
SQ^a_\beta(\tau) S^{-1} = e^{i\frac{\theta}{\tau}} Q^a_\beta(-\frac{1}{\tau}) , \quad S\bar{Q}_{a\dot{\beta}}(\tau) S^{-1} = e^{-i\frac{\theta}{\tau}} \bar{Q}_{a\dot{\beta}}(-\frac{1}{\tau}) ,
\] (1.10)

\[
SS^{a\dot{\beta}}(\tau) S^{-1} = e^{-i\frac{\theta}{\tau}} S^{a\dot{\beta}}(-\frac{1}{\tau}) , \quad S\bar{S}^{a\dot{\beta}}(\tau) S^{-1} = e^{i\frac{\theta}{\tau}} \bar{S}^{a\dot{\beta}}(-\frac{1}{\tau}) ,
\] (1.11)

the theory will be duality invariant [6]. We calculate the supersymmetry variations of the loop operators with \( \lambda^a = 0 \) and show that the constructed \( S \) is consistent with (1.10). To prove (1.10), the situation with \( \lambda^a \neq 0 \) should also be considered, which is left for future study.

t’ Hooft operator is usually defined in path integral formalism [7]. We investigate the relation between these two kinds of definitions and show that it is possible to extract \( T_R(\tau; \lambda^I, \lambda^a, C) \) in canonical formulation from the path integral.

The rest of the paper is organized as follows: section 2 is a review for the electric and the magnetic weight lattices of the gauge group; in section 3, we give a definition for the gauge invariant t’ Hooft operator in canonical formalism and compute the generic commutation relations for t’ Hooft and Wilson operators in the arbitrary representations; in section 4, we consider the T-transformation rule for loop operators; in section 5, we study the S-transformation of the loop operators; the discussion is in section 6.
2. THE ELECTRIC AND MAGNETIC WEIGHT LATTICES

For a semi-simple and simply connected group \( G \) with the rank \( r \), simple roots and simple coroots are given by the \( r \)-dimensional vectors \( \{ \tilde{\alpha}_A | A = 1, 2, \cdots, r \} \) and \( \{ \tilde{\alpha}_A^* = \tilde{\alpha}_A / |\tilde{\alpha}_A|^2 | A = 1, 2, \cdots, r \} \). The fundamental roots are \( \{ \tilde{\lambda}_A | A = 1, 2, \cdots, r \} \) satisfying

\[
2 \tilde{\alpha}_A^* \cdot \tilde{\lambda}_B = \frac{2 \tilde{\alpha}_A \cdot \tilde{\lambda}_B}{|\tilde{\alpha}_A|^2} = \delta_{AB} .
\] (2.1)

The electric weight lattice \( \Upsilon(G) \) and the magnetic weight lattice \( \Upsilon^*(G) \) are generated by the fundamental roots \( \{ \tilde{\lambda}_A \} \) and the simple coroots \( \{ \tilde{\alpha}_A^* \} \):

\[
\forall \tilde{\Upsilon} \in \Upsilon(G) , \quad \tilde{\Upsilon} = \sum_{A=1}^{r} m_A \tilde{\lambda}_A , \quad \forall \tilde{\Upsilon}^* \in \Upsilon^*(G) , \quad \tilde{\Upsilon}^* = \sum_{A=1}^{r} m_A \tilde{\alpha}_A^* , \quad m_A \in \mathbb{Z} . \] (2.2)

The magnetic weight lattice \( \Upsilon^*(G) \) can be identified with the electric weight lattice of the GNO dual group \( G^* \) \cite{4}. \( \Upsilon^*(G) = \Upsilon(G^*) \), \( \Upsilon^*(G^*) = \Upsilon(G) \). The center and the fundamental group of \( G^* \) are isomorphic to the fundamental group and the center of \( G \).

An irreducible representation \( R \) of the group \( G \) is labeled by the highest weight \( \tilde{\Upsilon} = \sum_{A=1}^{r} m_A \tilde{\lambda}_A \) with \( m_A \geq 0 \). Suppose \( \{ t_I | I = 1, 2, \cdots, \dim G \} \) are generators for the Lie algebra of \( G \) in fundamental representation, among which, \( \{ H_A | A = 1, 2, \cdots, r \} \) are generators of the Cartan subalgebra. \( \text{tr}(t_I t_J) = \frac{1}{2} \delta_{IJ} \). For \( \tilde{m} = (m_1, m_2, \cdots, m_r) \), \( \tilde{H} = (H_1, H_2, \cdots, H_r) \), let

\[
H_{\tilde{m}} = \tilde{\Upsilon} \cdot \tilde{H} = \sum_{A=1}^{r} m_A \tilde{\lambda}_A \cdot \tilde{H} = \sum_{A=1}^{r} m_A H_{\lambda_A} , \] (2.3)

\( R \) could also be represented by \( H_{\tilde{m}} \) with \( m_A \geq 0 \). \( \exp \{ 4 \pi i H_{\tilde{m}} \} = Z \), where \( Z \) is a center element of the group. The dual representation \( R^* \) is characterized by \( \tilde{\Upsilon}^* = \sum_{A=1}^{r} m_A \tilde{\alpha}_A^* \) in \( \Upsilon^*(G) \) with \( m_A \geq 0 \). Suppose

\[
H_{\tilde{m}}^* = \tilde{\Upsilon}^* \cdot \tilde{H} = \sum_{A=1}^{r} m_A \tilde{\alpha}_A^* \cdot \tilde{H} = \sum_{A=1}^{r} m_A H_{\alpha_A}^* , \] (2.4)

then according to (2.1), \( \exp \{ 4 \pi i H_{\tilde{m}}^* \} = I \), and

\[
\text{tr}(H_{\tilde{m}} H_{\tilde{m}}^* \cdot H_{\tilde{m}}') = \frac{1}{4} \sum_{A=1}^{r} m_A m'_A . \] (2.5)
Moreover, 
\[
tr(H_{\vec{m}}H_{\vec{m}'}) = \frac{1}{2} \sum_{A=1}^{r} \sum_{B=1}^{r} m_{A}m'_{B} \vec{\lambda}_{A} \cdot \vec{\lambda}_{B}.
\] 
(2.6)

When \(G = SU(2)\), \(r = 1\), \(H_{1} = diag(1/2, -1/2)\), \(\alpha_{1} = \alpha_{1}^{*} = 1\), \(\lambda_{1} = 1/2\). \(H_{m} = diag(m/4, -m/4)\) gives the \(SU(2)\) representation with the spin \(m/2\). \(m = 1, 2, \cdots G^{*} = SO(3)\), \(H^{*}_{m} = diag(m/2, -m/2)\) corresponds to the representation with the integer spin \(m\).

When \(G = SU(N)\), \(r = N - 1\), generators of the Cartan subalgebra in fundamental representation are selected as
\[
H_{A} = \frac{1}{\sqrt{2A(A + 1)}} diag(1, \cdots, 1, -A, 0, \cdots, 0),
\]
(2.7)

\[A = 1, 2, \cdots, N - 1.\] \(|\vec{\alpha}_{A}|^{2} = 1\), \(\vec{\alpha}^{*}_{A} = \vec{\alpha}_{A}\), \(|\vec{\lambda}_{A}|^{2} = \frac{N-1}{2N}\), \(\vec{\lambda}_{A} \cdot \vec{\lambda}_{B} = -\frac{1}{2N}\) for \(A \neq B\). \(\Upsilon^{*}(SU(N)) \subset \Upsilon(SU(N))\), the magnetic weight lattice is a sub-lattice of the electric weight lattice. For \(\vec{\Upsilon} = \sum_{A=1}^{r} m_{A} \vec{\lambda}_{A} \in \Upsilon(SU(N))\), \(\vec{\Upsilon} \in \Upsilon^{*}(SU(N))\) if and only if \(\sum_{A=1}^{r} m_{A} = pN\) for some \(p \in \mathbb{Z}\).

For the arbitrary representation \(R\) and \(R'\) labeled by \(\vec{m}\) and \(\vec{m}'\), from (2.6),
\[
tr(H_{\vec{m}}H_{\vec{m}'}) = \frac{1}{4} \sum_{A=1}^{r} m_{A}m'_{A} - \frac{1}{4N} \sum_{A=1}^{r} \sum_{B=1}^{r} m_{A}m'_{B}.
\]
(2.8)

When \(R\) is the fundamental representation, \(\vec{m} = (0, \cdots, 0, 1, 0, \cdots, 0), k = 0, 1, \cdots, N - 2\),
\[
H_{\vec{m}} = diag(-\frac{1}{2N}, \cdots, -\frac{1}{2N}, \frac{1}{2}, -\frac{1}{2N}, \cdots, -\frac{1}{2N})
\]
(2.9)

and \(\exp\{4\pi iH_{\vec{m}}\} = e^{-2\pi i/N}I\);

\[
H^{*}_{m} = diag(0, \cdots, 0, \frac{1}{2}, -\frac{1}{2}, 0, \cdots, 0)
\]
(2.10)

and \(\exp\{4\pi iH^{*}_{m}\} = I\). \(\Upsilon^{*}(SU(N)) = \Upsilon(SU(N)/Z_{N})\), \(\Upsilon^{*}(SU(N)/Z_{N}) = \Upsilon(SU(N))\). For definiteness, we will use
\[
H = diag(\frac{1}{2}, -\frac{1}{2N}, -\frac{1}{2N}, \cdots, -\frac{1}{2N})
\]
(2.11)
and
\[ H^* = \text{diag}(\frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0) \] (2.12)
to represent \( R \) and \( R^* \) when \( R \) is the fundamental representation.

When \( G = U(N) \) that is not semi-simple but could be locally decomposed as \( SU(N) \times U(1) \),
\[ r = N, \]
\[ H_A = \text{diag}(0, \ldots, 0, \frac{1}{\sqrt{2}}, 0, \ldots, 0) \] (2.13)
with \( A = 1, 2, \ldots, N \). The fundamental weights and the simple coroots are
\[ \tilde{\lambda}_A = \tilde{\alpha}_A = (0, \ldots, 0, \frac{1}{\sqrt{2}}, 0, \ldots, 0). \] (2.14)

The electric and the magnetic weight lattices are identical, \( \Upsilon(U(N)) = \Upsilon^*(U(N)) \) and \( U(N) = U(N)^* \). \( \exp\{4\pi i H_{\tilde{m}}\} = \exp\{4\pi i H_{\tilde{m}}^*\} = I \). For the representation \( R \) with \( \tilde{m} = (m_1, \ldots, m_N) \),
\[ H_{\tilde{m}} = H_{\tilde{m}}^* = \text{diag}(\frac{m_1}{2}, \ldots, \frac{m_N}{2}). \] (2.15)

When \( R \) is the fundamental representation and \( \tilde{m} = (1, 0, \ldots, 0) \),
\[ H = H^* = \text{diag}(\frac{1}{2}, 0, \ldots, 0). \] (2.16)

3. \textbf{T’ HOOFT OPERATOR IN CANONICAL FORMALISM}

In canonical quantization formulation, Yang-Mills (YM) theory with the gauge group \( G \) has
the canonical coordinate \( A_i \) and the conjugate momentum \( \Pi_i \), \( i = 1, 2, 3, A_i = A_i^t t^t, \Pi_i = \Pi_i^t t^t \).
The complete orthogonal bases of the Hilbert space \( \mathcal{H} \) can be selected as \( \{|A_i\rangle \text{ } \forall \text{ } A\} \). Local
gauge transformation operators compose the group \( \mathcal{G} \). \( \forall \text{ } U \in \mathcal{G}, \)
\[ U|A_i\rangle = |UA_i\rangle = |u^{-1}A_iu - iu^{-1}\partial_i u\rangle. \] (3.1)

The physical Hilbert space \( \mathcal{H}_{ph} \) is composed by states invariant under the action of \( U \). \( \forall \text{ } |\psi\rangle \in \mathcal{H}_{ph}, \forall \text{ } U \in \mathcal{G}, U|\psi\rangle = |\psi\rangle \). \( \forall \text{ } |A_i\rangle \), the corresponding \( |A_i\rangle_{ph} \in \mathcal{H}_{ph} \) is obtained as
\[ |A_i\rangle_{ph} = \int DU U|A_i\rangle. \] (3.2)
For the arbitrary spatial loop $C$, the Wilson loop in representation $R$ labeled by $\vec{m}$ is given by
\begin{equation}
W_R(A; C) = \frac{1}{d_R} tr P \exp \left\{ i \oint_C ds A_R^i \dot{x}_i \right\} ,
\end{equation}
(3.3)
where $d_R$ is the dimension of the representation and $A_R^i$ is the gauge potential in representation $R$. For the fundamental representation, we will use $A^i$ and $W$ with the omitted subscript.

Instead of (3.3), the Wilson loop in representation $R$ also has an equivalent definition as a path integral over all gauge transformations periodic along the loop $C$ [8, 9]:
\begin{equation}
W_R(A; C) = \int DU \exp \left\{ i \oint_C ds 2 tr [H_{\vec{m}}(u^{-1}A_i u - iu^{-1}\partial_i u)] \dot{x}_i \right\} .
\end{equation}
(3.4)
In (3.4), all fields are in the fundamental representation with the information on $R$ encoded in $H_{\vec{m}}$. The action of the Wilson operator $W_R(C)$ is given by
\begin{equation}
W_R(C)|A_i \rangle = W_R(A; C)|A_i \rangle .
\end{equation}
(3.5)
Based on (3.4), another operator $W_R(C)$ can be introduced with
\begin{equation}
W_R(C)|A_i \rangle = W_R(A; C)|A_i \rangle ,
\end{equation}
(3.6)
where
\begin{equation}
W_R(A; C) = \exp \left\{ i \oint_C ds 2 tr (H_{\vec{m}}A_i) \dot{x}_i \right\} .
\end{equation}
(3.7)
$W_R(C)$ can be written in terms of $W_R(C)$:
\begin{equation}
W_R(C) = \int DU \, U W_R(C) U^{-1} .
\end{equation}
(3.8)
The integration is taken over all of the local gauge transformation operators $U \in \mathcal{G}$. Obviously, if $u = I$ at the loop $C$, $U W_R(C) U^{-1} = W_R(C)$. $W_R(A; C)$ is only affected by the gauge transformation at $C$.

To see the equivalence between (3.4) and (3.8), consider the situation when $G = SU(N)$. The gauge transformation periodic on $C$ gives a closed curve in $SU(N)$, $\Pi_1(SU(N)) = 0$, so the curve can be continuously deformed to $I$ and the gauge transformation on $C$ can be covered by $U \in \mathcal{G}$. When $G = U(N)$, $\Pi_1(U(N)) = Z$, however, the $U(1)$ part of the transformation periodic on $C$ will make $W_R(C) \to W_R(C) \exp \{4k\pi i \, tr(H_{\vec{m}})\} = W_R(C) \exp \{2km\pi i\} = W_R(C)$ thus could be neglected. (3.8) is still equivalent to (3.4).
In [10], t’ Hooft operator $\mathcal{T}(C)$ is introduced satisfying the commutation relation
\[
\mathcal{T}(C_1)\mathcal{W}(C_2) = Z^{-l(C_1,C_2)}\mathcal{W}(C_2)\mathcal{T}(C_1)
\tag{3.9}
\]
with the Wilson operator $\mathcal{W}(C)$ in fundamental representation. When $G = SU(N)$, the center element $Z = \exp\{2\pi i/N\}$. $l(C_1, C_2)$ is the linking number of the two spatial loops $C_1$ and $C_2$.

In [11], an operator $T_R(C)$ satisfying (3.9) was explicitly constructed. The action of $T_R(C)$ on $|A_i\rangle$ is given by
\[
T_R(C)|A_i\rangle = |T_R(C)A_i\rangle = |\Omega^{-1}_{\vec{m}}(\Sigma_C)A_i\Omega_{\vec{m}}(\Sigma_C) - H_{\vec{m}}a_i(C)\rangle,
\tag{3.10}
\]
where $\Omega_{\vec{m}}(\Sigma_C, x) = e^{-iH_{\vec{m}}\omega(\Sigma_C, x)}$ with $H_{\vec{m}}$ defined in [2,3] charactering the representation $R$.

\[
a_i(C, x) = 4\pi \int_C d\vec{x}_k \epsilon_{kij} \partial_j D(x - \vec{x}).
\tag{3.11}
\]

$D(x)$ denotes the Greens function of the 3-dimensional Laplacian, $-\partial^2 D(x) = \delta^3(x)$. $\Sigma_C$ is an arbitrary surface with the boundary $C$, $\partial \Sigma_C = C$. $\omega(\Sigma_C, x)$ is the solid angle subtended by the loop $C$ seen from the point $x$, and is smooth everywhere except at the surface $\Sigma_C$.

\[
\omega(\Sigma_C, x) = 4\pi \int_{\Sigma_C} d^2\sigma_i \partial_i^x D(x - \vec{x}(\sigma)).
\tag{3.12}
\]

When $x$ crosses $\Sigma_C$, $\omega(\Sigma_C, x) \to \omega(\Sigma_C, x) \pm 4\pi$ and accordingly, $\Omega_{\vec{m}}(\Sigma_C, x) \to Z^{\pm 1}\Omega_{\vec{m}}(\Sigma_C, x)$, since $e^{-4\pi iH_{\vec{m}}\omega} = Z$. Depending on the different $\Sigma_C$ selected, $\Omega_{\vec{m}}(\Sigma_C, x)$ is determined up to the multiplication of $Z$. Even though, $\Omega^{-1}_{\vec{m}}(\Sigma_C)A_i\Omega_{\vec{m}}(\Sigma_C)$ is only $C$-dependent.

Away from $\Sigma_C$, $\Omega_{\vec{m}}(\Sigma_C, x)$ is smooth with $i\Omega^{-1}_{\vec{m}}(\Sigma_C)\partial_i\Omega_{\vec{m}}(\Sigma_C) = H_{\vec{m}}a_i(C)$. Locally, $T_R(C)A_i$ and $A_i$ are related by a gauge transformation. So the path-ordered integrations
\[
I(A_i; C') = \frac{1}{N}P \exp\{i \int_0^1 ds A_i\dot{x}^i\}
\tag{3.13}
\]
and
\[
I(T_R(C)A_i; C') = \frac{1}{N}P \exp\{i \int_0^1 ds T_R(C)A_i\dot{x}^i\}
\tag{3.14}
\]
starting and ending at the point $x \in \Sigma_C$, moving along a loop $C'$ intersecting $\Sigma_C$ once at $x$ will differ by $Z$, i.e. $I(T_R(C)A_i; C') = ZI(A_i; C')$. As a result, $\mathcal{W}(T_R(C)A_i; C') = Z\mathcal{W}(A_i; C')$. 

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\( T_R(C) \) is an operator satisfying
\[
T_R(C_1) \mathcal{W}(C_2) = Z^{-i(C_1 \cdot C_2)} \mathcal{W}(C_2) T_R(C_1) .
\] (3.15)

When \( G = SU(N) \) and \( R \) is the fundamental representation, \( T_R \) is abbreviated as \( T \), \( Z = \exp\{2\pi i/N\} \),
\[
T(C_1) \mathcal{W}(C_2) = \exp\{-\frac{2\pi i(C_1 \cdot C_2)}{N}\} \mathcal{W}(C_2) T(C_1) .
\] (3.16)

\( \Omega_{\vec{m}}(\Sigma_C, x) \) is a \( \Sigma_C \)-dependent function with the value jumping \( Z \) at \( \Sigma_C \). Depending on
the group \( G \), \( Z^k = 1 \) for some integer \( K \), so \( Z^k \Omega_{\vec{m}}(\Sigma_C, x) \) with \( k = 0, 1, \ldots, K - 1 \) glued

When \( \kappa\) transformation if \( \bar{\Omega}_{\vec{m}} \)
the action of
\( R_2 \) is an operator satisfying
\( \Omega_{\vec{m}}(\Sigma_C) \) is not a physical operator. For
\(\bar{\Omega}_{\vec{m}} \) is not a pure gauge and \( \Omega_{\vec{m}}(C) \) is not a
multi-valued, \( \bar{\Omega}_{\vec{m}} \) is single-valued.

When \( e^{-4\pi iH_{\vec{m}}} = I \), which is the situation for \( G = U(N) \) and \( R \) the arbitrary representation
or \( G = SU(N) \) and \( R \) a dual representation, \( \Omega_{\vec{m}}(\Sigma_C, x) = \Omega_{\vec{m}}(C, x) \) is a \( C \)-dependent continues
function with \( i\bar{\Omega}_{\vec{m}}^{-1}(C)\partial_i \Omega_{\vec{m}}(C) = H_{\vec{m}}a_i(C) \) everywhere. Even though, due to the singularity
at \( C \), \( H_{\vec{m}}a_i(C) \) is not a pure gauge and \( T_R(C) \) is not a gauge transformation. For example,
when \( G = U(1) \), \( H_{\vec{m}} = 1/2 \), \( T(C) |A_i \rangle = |A_i - 1/2a_i(C) \rangle \). The magnetic field transforms as
\( B_i \rightarrow B_i - \frac{1}{2}b_i(C) \), where \( \frac{1}{2}b_i(C) \) is a unit magnetic field loop located at \( C \), i.e.
\[
\frac{1}{4\pi} b^i(C, x) = \frac{1}{4\pi} \epsilon^{ijk} \partial_j a_k(C, x) = \int_C d\vec{x}\cdot \delta^3(\vec{x} - \vec{\nu}) .
\] (3.18)

\( T(C_1) \mathcal{W}(C_2) = \mathcal{W}(C_2) T(C_1) \). The Wilson loop of \( A_i \) and \( A_i - \frac{1}{2}a_i(C) \) are the same, but \( A_i \)
and \( A_i - \frac{1}{2}a_i(C) \) are not gauge equivalent. \( T(C) \) defined in this way is identical to \( (3.13) \).

\( T_R(C) \) is not a physical operator. For \( |\psi\rangle \in \mathcal{H}_{ph} \), \( T_R(C) |\psi\rangle \) may not be a state in \( \mathcal{H}_{ph} \). Consider
the action of \( T_R(C) \) on a gauge transformation operator \( U \) giving by \( (3.1) \), \( \bar{\Omega}_{\vec{m}}(C)u\bar{\Omega}_{\vec{m}}^{-1}(C) \)
is the transformation matrix related with \( T_R^{-1}(C) U T_R(C) \). \( T_R^{-1}(C) U T_R(C) \) will be a gauge transformation if \( \bar{\Omega}_{\vec{m}}(C)u\bar{\Omega}_{\vec{m}}^{-1}(C) \) is still a single-valued continuous function taking values in \( G \)
and approaching \( I \) at infinity, which requires \( [u, H_{\vec{m}}] = 0 \) at \( C \), since \( \bar{\Omega}_{\vec{m}}(C) \) is singular at \( C \).
In this case, \( \forall |\psi\rangle \in \mathcal{H}_{ph} \)
\[
U T_R(C) |\psi\rangle = T_R(C) T_R^{-1}(C) U T_R(C) |\psi\rangle = T_R(C) U' |\psi\rangle = T_R(C) |\psi\rangle .
\] (3.19)
Otherwise, \( U_T(C)\psi \neq T_R(C)\psi \). \( T_R(C)\psi \) is a state that would only be affected by the gauge transformation at \( C \). In this respect, \( T_R(C) \) is quite similar with \( W_R(C) \).

The gauge invariant t’ Hooft operator could be constructed as

\[
T_R(C) = \int DU \ U_T(C)U^{-1}.
\]  
(3.20)

From (3.15),

\[
T_R(C_1)W(C_2) = Z^{-l(C_1,C_2)}W(C_2)T_R(C_1).
\]  
(3.21)

\( T_R(C) \) defined in (3.17) is labeled by \( H_{\bar{m}} \) related with the highest weight \( \bar{\lambda} \in \Upsilon(G) \). \( e^{-4\pi i H_{\bar{m}}} = Z \). For \( W_{R'}(C) \) in representation \( R' \) labeled by \( H_{\bar{m}'} \), consider \( C_1 \) and \( C_2 \) with the linking number \( l(C_1,C_2) \),

\[
T_{R'}^{-1}(C_1)W_{R'}(C_2)T_R(C_1) = \exp\{i \oint C_2 \ ds \ 2tr[H_{\bar{m}'}(A) - H_{\bar{m}}a_i(C_1)]\hat{x}^i\}
\]
\[
= W_{R'}(C_2) \exp\{-8\pi il(C_1,C_2)tr[H_{\bar{m}'}H_{\bar{m}}]\}, \quad (3.22)
\]

since

\[
\oint C_2 \ ds \ a_i(C_1)\hat{x}^i = 4\pi l(C_1,C_2) . \quad (3.23)
\]

We have

\[
T_R(C_1)W_{R'}(C_2) = W_{R'}(C_2)T_R(C_1) \exp\{8\pi il(C_1,C_2)tr[H_{\bar{m}'}H_{\bar{m}}]\} . \quad (3.24)
\]

Instead of (3.8), the Wilson operator \( W_{R'}(C_2) \) can also be written as

\[
W_{R'}(C_2) = \int DU(C_2) \ U(C_2)W_{R'}(C_2)U^{-1}(C_2) , \quad (3.25)
\]

where \( U(C_2) \) are gauge transformations equal to \( I \) away from a torus surrounding \( C_2 \). (3.8) and (3.25) are equivalent if the proper normalization are assumed for the integration over \( U(C_2) \).

\[
T_R(C_1)W_{R'}(C_2) = \int DU(C_2) \ T_R(C_1)U(C_2)T_{R'}^{-1}(C_1)T_R(C_1)W_{R'}(C_2)U^{-1}(C_2)
\]
\[
= \int DU(C_2) \ T_R(C_1)U(C_2)T_{R'}^{-1}(C_1)W_{R'}(C_2)T_R(C_1)U^{-1}(C_2)T_{R'}^{-1}(C_1)T_R(C_1)
\]
\[
\exp\{8\pi il(C_1,C_2)tr[H_{\bar{m}'}H_{\bar{m}}]\}
\]
\[
= W_{R'}(C_2)T_R(C_1) \exp\{8\pi il(C_1,C_2)tr[H_{\bar{m}'}H_{\bar{m}}]\} , \quad (3.26)
\]

where we have used the fact that \( T_R(C_1)U(C_2)T_{R'}^{-1}(C_1) \) is still a gauge transformation equal to
I away from the torus surrounding $C_2$. From (3.26),

$$T_R(C_1)W_{R'}(C_2) = W_{R'}(C_2)T_R(C_1) \exp\{8\pi il(C_1, C_2)tr[H_{\bar{n}^\prime}H_{\bar{m}}]\} = W_{R'}(C_2)T_R(C_1) \exp\{iL(R, R'; C_1, C_2)\} ,$$

(3.27)

where $\exp\{iL(R, R'; C_1, C_2)\} := \exp\{8\pi il(C_1, C_2)tr[H_{\bar{n}^\prime}H_{\bar{m}}]\}$. This is the generic commutation relation for loop operators in the arbitrary representation.

When $H_{\bar{n}^\prime}$ and $H_{\bar{m}}$ are in the electric and the magnetic weight lattices $\Upsilon(G)$ and $\Upsilon^*(G)$, from (2.5), $\exp\{iL(R, R'; C_1, C_2)\} = 1$,

$$T_R(C_1)W_{R'}(C_2) = W_{R'}(C_2)T_R(C_1) .$$

(3.28)

When $G = SU(N)$, according to (2.8),

$$T_R(C_1)W_{R'}(C_2) = W_{R'}(C_2)T_R(C_1) \exp\{-\frac{2\pi il(C_1, C_2)}{N} \sum_{A=1}^{r} \sum_{B=1}^{r} m_A m_B'\} .$$

(3.29)

Especially, when $R$ and $R'$ are fundamental representation characterized by $H$,

$$T(C_1)W(C_2) = W(C_2)T(C_1) \exp\{-\frac{2\pi il(C_1, C_2)}{N}\} .$$

(3.30)

When $G = U(N)$, for the arbitrary $R$ and $R'$, from (2.15), $\exp\{iL(R, R'; C_1, C_2)\} = 1$,

$$T_R(C_1)W_{R'}(C_2) = W_{R'}(C_2)T_R(C_1) .$$

(3.31)

4. T-TRANSFORMATION OF LOOP OPERATORS

S-duality transformation is generated by the T-transformation and the S-transformation. In canonical quantization formalism, T-transformation could be realized by a unitary operator $g(A) = \exp\{-\frac{iX(A)}{2\pi}\}$ [12] with

$$X(A) = \frac{1}{2} \int d^3x \epsilon^{ijk} tr(A_i \partial_j A_k + \frac{2i}{3} A_i A_j A_k)$$

(4.1)

the Chern-Simons term.

$$g(A)\Pi_i g^{-1}(A) = \Pi_i + \frac{B_i}{2\pi} \quad g(A)A_i g^{-1}(A) = A_i ,$$

(4.2)
\[ B^i = \frac{\epsilon^{ijk}}{2} F_{jk}. \]

Obviously, Wilson operator is invariant under the action of \( g, \ g(A)\mathcal{W}_R(C)g^{-1}(A) = \mathcal{W}_R(C) \). For the T-transformation of the t’ Hooft operator, we can compute the action of \( TR(C) \) on \( X(A) \):

\[ TR^{-1}(C)X(A)TR(C) = X(TR(C)A), \tag{4.3} \]

where \( TR(C)A_i = \bar{\Omega}_{\bar{m}}^{-1}(C)A_i\bar{\Omega}_{\bar{m}}(C) - H_{\bar{m}}a_i(C) \). Direct calculation gives

\[ X(TR(C)A) = X(A) - 4\pi \int_C ds \text{tr}(H_{\bar{m}}A_i)x^i, \tag{4.4} \]

so

\[ TR^{-1}(C)g(A)TR(C) = \mathcal{W}_R(C)g(A), \tag{4.5} \]

or equivalently,

\[ g(A)TR(C)g^{-1}(A) = TR(C)\mathcal{W}_R(C). \tag{4.6} \]

Under the T-transformation, \( TR(C) \) is multiplied by \( \mathcal{W}_R(C) \) in the same representation. On the other hand, T-transformation for the gauge invariant t’ Hooft operator \( TR(C) \) is given by

\[ g(A)TR(C)g^{-1}(A) = \int DU U[TR(C)\mathcal{W}_R(C)]U^{-1} := [TW]_R(C), \tag{4.7} \]

where \([TW]_R(C) \) could be taken as the Wilson-t Hooft operator originally proposed in path integral formulation \([5]\). Here, \( W \) and \( T \) in \([TW]_R \) are both labeled by the representation \( R \).

\([TW]_R(C) \) is different from the double trace operator \( TR(C)\mathcal{W}_R(C) \) unless the gauge group is \( U(1) \) so that \( \mathcal{W}(C) = W(C), T(C) = T(C) \).

5. S-TRANSFORMATION OF LOOP OPERATORS

S-transformation is expected to make the t’ Hooft operator \( TR^{-1}(C) \) in theory with the gauge group \( G \) and the coupling \(-1/\tau \) mapped into the Wilson operator \( \mathcal{W}_{R^*}(C) \) in theory with the gauge group \( G^* \) and the coupling \( \tau \) \([5]\). For it to be possible, two kinds of operators should be equivalent in physical Hilbert space. They should have the same spectrum and degeneracy and could be related by a unitary transformation. In this section, we will study the spectrum and eigenstates of the t’ Hooft operator in YM theory as well as the \( N = 4 \) SYM theory. We will show that it is possible to construct a unitary operator \( S \) relating the Wilson operator \( \mathcal{W}_{R^*}(-1/\tau; C) \) and the t’ Hooft operator \( TR^{-1}(-1/\tau; C) \).

So S-transformation could be realized at the kinematical level. At the dynamical level, if
S could also make the Hamiltonian with the coupling $-1/\tau$ transformed into the Hamiltonian with the coupling $\tau$, the theory is S-duality invariant. For $\mathcal{N} = 4$ SYM theory, the condition for the S-duality invariance is that the supercharges should transform with a $U(1)_Y$ phase. We will calculate the supersymmetry variations of the loop operators and provide the evidence for the $U(1)_Y$ transformation of the supercharges under the action of $S$.

5.1. Spectrum and eigenstates of t’ Hooft operator in YM theory

In YM theory with the gauge group $G$, the complete orthogonal bases of the Hilbert space $\mathcal{H}$ can be selected as $\{|A\rangle| \forall A\}$, eigenstates of the gauge potential. The action of the Wilson operator in representation $R$ on $|A\rangle$ is given by

$$W_R(C)|A\rangle = W_R(A; C)|A\rangle . \quad (5.1)$$

For the arbitrary spacial loop $C$ and the arbitrary local gauge transformation operator $U \in G$, the action of $T(\pm C)$ and $U$ could make $\{|A\rangle| \forall A\}$ divided into the equivalent classes, where $T(\pm C)$ stands for $T_R(\pm C)$ with $R$ the fundamental representation of $G$, $\pm$ gives the orientation of $C$, $T(-C) = T^{-1}(C)$. For $|A\rangle$ and $|A'\rangle$, if

$$U_nT(\pm C_{n-1})U_{n-1}\cdots T(\pm C_2)U_2T(\pm C_1)U_1|A\rangle = |A'\rangle \quad (5.2)$$

for some $U_i$ and $C_i$, $|A\rangle$ and $|A'\rangle$ belong to the same equivalent class. In other words, for the group $\mathcal{L}$ defined as

$$\mathcal{L} := \{U_nT(\pm C_{n-1})U_{n-1}\cdots T(\pm C_2)U_2T(\pm C_1)U_1| \forall C_i, U_i, n\} , \quad (5.3)$$

$|A'\rangle = L|A\rangle$ for some $L \in \mathcal{L}$.

$$\{|A\rangle| \forall A\} = \bigcup_{\hat{A}} E(\hat{A}) , \quad (5.4)$$

where $E(\hat{A})$ is an equivalent class with $|\hat{A}\rangle$ an arbitrary element in it. $E(\hat{A}) \cap E(\hat{A}') = \emptyset$, if $|\hat{A}\rangle$ and $|\hat{A}'\rangle$ are not in the same class. Starting from $|\hat{A}\rangle$, the action of the group $\mathcal{L}$ generates

$$E(\hat{A}) := \{L|\hat{A}\rangle| \forall L \in \mathcal{L}\} . \quad (5.5)$$

$T(C)\mathcal{L} = \mathcal{L}$, $U\mathcal{L} = \mathcal{L}$, so $T(C)E(\hat{A}) = E(\hat{A})$, $UE(\hat{A}) = E(\hat{A})$, where

$$T(C)E(\hat{A}) := \{T(C)|A\rangle| \forall A \in E(\hat{A})\} , \quad UE(\hat{A}) := \{U|A\rangle| \forall A \in E(\hat{A})\} . \quad (5.6)$$
Moreover, for the arbitrary representation $R$, $T_R(C)$ can always be decomposed as the products of $T(C)$ and $U$, $T_R(C) \in \mathcal{L}$, so $T_R(C)\mathcal{L} = \mathcal{L}$, $T_R(C) E(\hat{A}) = E(\hat{A})$. As a matrix in $\{|A| \forall A\}$ representation, $T_R(C)$ is block diagonal with respect to the decomposition (5.4).

Eigenstates of $T_R(C)$ can be constructed in the sub Hilbert space $\mathcal{H}[E(\hat{A})]$ generated by $E(\hat{A})$. Let

$$|D\rangle_{(\hat{A},\hat{A})} = \sum_{|A\rangle \in E(\hat{A})} g^{-1}(A)|A\rangle,$$  \hspace{1cm} (5.7)

where the first $\hat{A}$ in the subscript $(\hat{A}, \hat{A})$ means it is a state in $\mathcal{H}[E(\hat{A})]$ and the second indicates it is to be constructed as an eigenstate with the eigenvalue $\mathcal{W}_R(\hat{A}; C)$. $|D\rangle_{(\hat{A},\hat{A})}$ is $R$-independent. $g(A) = e^{-\frac{iX(A)}{2\kappa}}$ with $X(A)$ the Chern-Simons term.

$$g(T_R(C)A) = W_R(A; C)g(A).$$  \hspace{1cm} (5.8)

$$g(UA) = g(A), \text{ } U|D\rangle_{(\hat{A},\hat{A})} = |D\rangle_{(\hat{A},\hat{A})}, \text{ } |D\rangle_{(\hat{A},\hat{A})} \in \mathcal{H}_{ph}.$$

$$\forall |A\rangle \in E(\hat{A}), \text{ } |A\rangle = U_n T(\pm C_{n-1}) \cdots T(\pm C_2) U_2 T(\pm C_1) U_1 |\hat{A}\rangle,$$

$$\mathcal{W}_R(A; C) = e^{-iL(1,R;\pm C_1,C)} \cdots e^{-iL(1,R;\pm C_{n-1},C)} \mathcal{W}_R(\hat{A}; C),$$  \hspace{1cm} (5.9)

where we use 1 in $L(1,R;\pm C_k,C)$ to stand for $T(\pm C_k)$ in fundamental representation. The action of $T_R(C)$ on $|D\rangle_{(\hat{A},\hat{A})}$ is given by

$$T_R(C)|D\rangle_{(\hat{A},\hat{A})} = \int DU \text{ } U T_R(C) |D\rangle_{(\hat{A},\hat{A})}$$

$$= \int DU \text{ } U \sum_{|A\rangle \in E(\hat{A})} g^{-1}(A)|T_R(C)A\rangle$$

$$= \int DU \text{ } U \sum_{|A\rangle \in E(\hat{A})} W_R(A; C) g^{-1}(T_R(C)A)|T_R(C)A\rangle$$

$$= \int DU \text{ } \sum_{|A\rangle \in E(\hat{A})} W_R(U^{-1}A; C) g^{-1}(A)|A\rangle$$

$$= \sum_{|A\rangle \in E(\hat{A})} W_R(A; C) g^{-1}(A)|A\rangle$$

$$= \mathcal{W}_R(\hat{A}; C) \sum_{|A\rangle \in E(\hat{A})} e^{-iL(1,R;\pm C_1,C)} \cdots e^{-iL(1,R;\pm C_{n-1},C)} g^{-1}(A)|A\rangle. \hspace{1cm} (5.10)$$
When $G = SU(N)$, for $R$ characterized by $\bar{m}$, from (3.29),

$$e^{-iL(1,R;\pm C_k,C)} = \exp\left\{ \frac{2\pi i l(C, \pm C_k)}{N} \sum_{A=1}^{r} m_A \right\} = (Z^{\sum_{A=1}^{r} m_A})^{l(C, \pm C_k)}, \quad (5.11)$$

$$T_R(C)|D\rangle_{(\hat{A},\hat{A})} = W_R(\hat{A};C) \sum_{|A\rangle \in E(\hat{A})} (Z^{\sum_{A=1}^{r} m_A})^{\sum_{k=1}^{n-1} l(C, \pm C_k)} g^{-1}(A)|A\rangle. \quad (5.12)$$

$|D\rangle_{(\hat{A},\hat{A})}$ is the eigenstate of $T_R(C)$ with the eigenvalue $W_R(\hat{A};C)$ if and only if $\sum_{A=1}^{r} m_A = pN$ for $p \in \mathbb{Z}$. In this case, the highest weight of $R$ is also in the magnetic lattice $\mathcal{Y}^*(SU(N)) = \mathcal{Y}(SU(N)/Z_N) \subset \mathcal{Y}(SU(N))$. $R$ is also the dual representation of $SU(N)$, or equivalently, the representation of $SU(N)/Z_N$ and could be written as $R^*$. For example, when $G = SU(2)$, the Hooft operator labeled by the integer spin could have the same eigenvalue as the Wilson operator labeled by the same integer. When $G = U(N)$, $\mathcal{Y}^*(U(N)) = \mathcal{Y}(U(N))$, the representation is also the dual representation and $e^{-iL(1,R;\pm C_k,C)} = 1$ for the arbitrary $R$.

S-transformation is supposed to make $T_R(C)$ in theory with the group $G$ mapped into $W_R(C)$ in theory with the group $G^*$. The necessity to replace $T_R(C)$ by $T_R^*(C)$ is also demonstrated by (5.12). From (3.28), $e^{-iL(1,R;\pm C_k,C)} = 1$, so

$$W_{R^*}(A;C) = W_{R^*}(\hat{A};C), \quad \forall \ |A\rangle \in E(\hat{A}). \quad (5.13)$$

$$T_{R^*}(C)|D\rangle_{(\hat{A},\hat{A})} = W_{R^*}(\hat{A};C)|D\rangle_{(\hat{A},\hat{A})}. \quad (5.14)$$

$|D\rangle_{(\hat{A},\hat{A})}$ is the eigenstate of $T_{R^*}(C)$ with the eigenvalue $W_{R^*}(\hat{A};C)$.

With the reference state $|\hat{A}\rangle$ chosen, $|A\rangle \in E(\hat{A})$ can be written as $|A\rangle = L|\hat{A}\rangle$, $L \in \mathcal{L}$. If $L|\hat{A}\rangle = |\hat{A}\rangle$ only when $L = I$, there is a one-to-one correspondence between $|A\rangle$ and $L$, so $E(\hat{A})$ can be identified with the group manifold $\mathcal{L}$. $|D\rangle_{(A,\hat{A})}$ can be rewritten as

$$|D\rangle_{(\hat{A},\hat{A})} = \sum_{L \in \mathcal{L}} g^{-1}(L\hat{A})|L\hat{A}\rangle. \quad (5.15)$$

If $L|\hat{A}\rangle = |\hat{A}\rangle$ also for some $L \neq I$, such $L$ can only be a gauge transformation, under which, $g$ is invariant. (5.15) is still valid, provided that the integration over $\mathcal{L}$ is normalized. $g^{-1}(L\hat{A})$ is a wave function on $\mathcal{L}$, so with the reference state modified to the arbitrary $\hat{A}'$, one may get

$$|D\rangle_{(\hat{A}',\hat{A})} = \sum_{L \in \mathcal{L}} g^{-1}(L\hat{A})|L\hat{A}'\rangle. \quad (5.16)$$

$|D\rangle_{(\hat{A}',\hat{A})} \in \mathcal{H}[E(\hat{A}')]$ is an eigenstate of $T_{R^*}(C)$ with the eigenvalue $W_{R^*}(\hat{A};C)$ for the arbitrary
gauge potential \( \hat{A} \) that does not necessarily belong to \( E(\hat{A}) \).

The conjugation of \( T_{R^*}(C) \) is \( T_{R^*}^+(C) = \int DU U T_{R^*}^{-1}(C) U^{-1}, \)

\[
T_{R^*}^+(C)|D\rangle_{(\hat{A}',\hat{A})} = \sum_{L \in \mathcal{L}} \int DU W_{R^*}^{-1}(U^{-1}L\hat{A};C)g^{-1}(L\hat{A})|L\hat{A}'
\]

\[
= \sum_{L \in \mathcal{L}} W_{R^*}^*(L\hat{A};C)g^{-1}(L\hat{A})|L\hat{A}'
\]

\[
= W_{R^*}^*(\hat{A};C)|D\rangle_{(\hat{A}',\hat{A})},
\]

(5.17)

where \( W_{R^*}^*(A;C) = \int DU W_{R^*}^{-1}(U^{-1}A;C) \). For \( |D\rangle_{(\hat{A},A')}, |D\rangle_{(\hat{A},A'')} \in \mathcal{H}[E(\hat{A})], \)

\[
(\hat{A},A')(D|T_{R^*}(C)|D\rangle_{(\hat{A},A'')} = W_{R^*}(A'';C)_{(\hat{A},A')} \langle D|D\rangle_{(\hat{A},A'')} = W_{R^*}(A';C)_{(\hat{A},A')} \langle D|D\rangle_{(\hat{A},A')}.
\]

(5.18)

When \( W_{R^*}(A'';C) \neq W_{R^*}(A';C) \), \( (\hat{A},A') \langle D|D\rangle_{(\hat{A},A'')} = 0. \)

To conclude, in \( \mathcal{H}[E(\hat{A})] \), one may construct the eigenstate of \( T_{R^*}(C) \) with the eigenvalue \( W_{R^*}(A';C) \) for the arbitrary gauge potential \( A' \). Eigenstates with the different eigenvalues are orthogonal.

Moreover, since \([T_{R^*}(C_1), W_{R^*}(C_2)] = 0, |D\rangle_{(\hat{A},A')} \) is in fact the common eigenstate of \( T_{R^*}(C) \) and \( W_{R^*}(C) \),

\[
W_{R^*}(C)|D\rangle_{(\hat{A},A')} = W_{R^*}(A;C)|D\rangle_{(\hat{A},A')}, \quad T_{R^*}(C)|D\rangle_{(\hat{A},A')} = W_{R^*}(A';C)|D\rangle_{(\hat{A},A')}.
\]

(5.19)

The spectrum of \( W_{R^*}(C) \) is highly degenerate, making the Hilbert space \( \mathcal{H} \) decomposed as

\[
\mathcal{H} = \bigoplus_{\hat{A}} \mathcal{H}[E(\hat{A})]
\]

(5.20)

with \( \mathcal{H}[E(\hat{A})] \) the eigenspace with the eigenvalue \( W_{R^*}(\hat{A};C) \). It is expected that for \( \mathcal{H}[E(\hat{A})] \neq \mathcal{H}[E(\hat{A}')], W_{R^*}(\hat{A};C) \neq W_{R^*}(\hat{A}';C) \), since \( U \) and \( T \) are minimum elements commuting with \( W_{R^*}(C) \). In each \( \mathcal{H}[E(\hat{A})] \), we can further construct the eigenstate of \( T_{R^*}(C) \) with the eigenvalue \( W_{R^*}(\hat{A}';C) \) for the arbitrary \( \hat{A}' \), which may still be degenerate. For example, when \( G = SU(N) \), \( |D\rangle_{(\hat{A},\hat{A})} \) and \( W(C)|D\rangle_{(\hat{A},\hat{A})} \) are degenerate eigenstates since \([W(C), T_{R^*}(C)] = 0 \) but \([W(C), T(C)] \neq 0. \)

When \( G = U(N) \), for \( T \) and \( W \) in fundamental representation, \([T(C_1), W(C_2)] = 0, \)

\[
W(C)|D\rangle_{(\hat{A},\hat{A})} = W(\hat{A};C)|D\rangle_{(\hat{A},\hat{A})}, \quad T(C)|D\rangle_{(\hat{A},\hat{A})} = W(\hat{A}';C)|D\rangle_{(\hat{A},\hat{A})}.
\]

(5.21)

We may expect \( |D\rangle_{(\hat{A},\hat{A})} \) determined by (5.21) is unique with no degeneracy. If there is a gauge
invariant operator \( F(A) \) with \( [F(A), T(C)] = 0 \), it will mostly also commute with \( T(C) \) thus will be a constant in \( E(\hat{A}) \). \( |D\rangle_{(A,A')} \) and \( F|D\rangle_{(A,A')} \) can only differ by a constant.

When the gauge group is \( U(1) \), \( T(C) = T(C), W(C) = W(C) \),

\[
g(A) = \exp\{-\frac{i}{2\pi} \int d^3x \frac{1}{2} \epsilon^{ijk} A_i \partial_j A_k\} = \exp\{-\frac{i}{2\pi} \int d^3x \frac{1}{2} A_i B^i\} \quad (5.22)
\]

For \( L = U_n T(\pm C_{n-1}) \cdots U_2 T(\pm C_1) U_1, |A\rangle = L|\hat{A}\rangle \), \( B_i = \hat{B}_i - \frac{1}{2} \sum_{k=1}^{n-1} b_i(\pm C_k) \),

\[
g(L\hat{A}) = \prod_{k=1}^{n-1} W(\hat{A}; \pm C_k) g(\hat{A}) = g^{-1}(\hat{A}) \exp\{\frac{i}{2\pi} \int d^3x \hat{A}^i (\frac{1}{2} \sum_{k=1}^{n-1} b_i(\pm C_k) - \hat{B}_i)\} \]
\[
= g^{-1}(\hat{A}) \exp\{-\frac{i}{2\pi} \int d^3x \hat{A}^i B_i\} = g^{-1}(\hat{A}) \exp\{-\frac{i}{2\pi} \int d^3x \hat{A}^i \hat{B}_i\} \quad (5.23)
\]

where we have used

\[
W(A; \pm C) = \exp\{i \int_{\pm C} ds A_i \dot{x}^i\} = \exp\{\frac{i}{4\pi} \int d^3x A_i b^i(\pm C)\} \quad (5.24)
\]

Finally,

\[
|D\rangle_{(\hat{A},\hat{A})} = g(\hat{A}) \exp\{\frac{i}{2\pi} \int d^3x (\hat{A}_i - \hat{A}'_i) \hat{B}^i\} \sum_{|A'\rangle \in E(\hat{A})} \exp\{\frac{i}{2\pi} \int d^3x A'_i \hat{B}^i\} |A'\rangle \quad (5.25)
\]

5.2. Mapping of the loop operators

(5.19) exhibits the symmetry between \( T_{R^*}(C) \) and \( W_{R^*}(C) \). One may construct the operator \( S \) with

\[
S|D\rangle_{(A,A')} = |D\rangle_{(\hat{A},\hat{A})} \quad (5.26)
\]

where \( \hat{A} \) is the gauge potential satisfying \( W_{R^*}(\hat{A}; C) = W_{R^*}(A; C) \). \( \hat{A} = -A \) when \( G = U(1) \).

\[
S^2|D\rangle_{(A,A')} = |D\rangle_{(\hat{A},\hat{A})} \quad (5.27)
\]

(5.19) together with (5.26) indicates

\[
S^{-1}T_{R^*}(C)S = W_{R^*}(C) \quad \text{and} \quad S^{-1}W_{R^*}(C)S = T_{R^*}(C) \quad (5.28)
\]

Especially, when \( G = U(N) \), \( |D\rangle_{(A,A')} \) is uniquely determined by (5.21) with no degeneracy and the action of \( S \) in (5.26) can be fixed up to the multiplication of a phase.
As for the action of $S$ on $|A\rangle$, consider the $U(1)$ theory, for which the S-transformation rule is known. For $|A\rangle_{ph}$ with $\mathcal{W}(C)|A\rangle_{ph} = \mathcal{W}(A; C)|A\rangle_{ph}$, one may construct

$$|\mathcal{D}\rangle_A = \sum_{\hat{A}'} e^{ih(\hat{A}',A)}|\mathcal{D}\rangle_{(\hat{A'},A)}$$

(5.29)
satisfying $\mathcal{T}(C)|\mathcal{D}\rangle_A = \mathcal{W}(A; C)|\mathcal{D}\rangle_A$, where $e^{ih(\hat{A}',A)}$ is the arbitrary coefficient and the summation is taken over all equivalent classes. When $e^{ih(\hat{A}',A)} = g^{-1}(A) \exp\{\frac{i}{2\pi} \int d^3x (\hat{A}'_i - A_i)B^i\}$,

$$|\mathcal{D}\rangle_A = \int DA' \exp\{\frac{i}{2\pi} \int d^3x \epsilon^{ijk} A'_i \partial_j A_k\} |A'\rangle$$

(5.30)
is the dual state of $|A\rangle_{ph}$ under the S-transformation. $\{|A\rangle_{ph} \forall A\}$ and $\{|\mathcal{D}\rangle_A \forall A\}$ compose two sets of complete orthogonal bases for $\mathcal{H}_{ph}$. The unitary operator $S$ with $S|A\rangle_{ph} = |\mathcal{D}\rangle_A$, $S|\mathcal{D}\rangle_A = | - A\rangle_{ph}$ will make $S^{-1}\mathcal{T}(C)S = \mathcal{W}(C)$, $S^{-1}\mathcal{W}(C)S = \mathcal{T}^+(C)$.

In theory with the group $G$, let

$$|\mathcal{D}\rangle_A = \sum_{\hat{A}'} e^{ih(\hat{A}',A)}|\mathcal{D}\rangle_{(\hat{A'},A)}$$

(5.31)
for some $e^{ih(\hat{A}',A)}$,

$$\mathcal{T}_{R^*}(C)|\mathcal{D}\rangle_A = \mathcal{W}_{R^*}(A; C)|\mathcal{D}\rangle_A .$$

(5.32)

$|\mathcal{D}\rangle_A$ is not unique due to the ambiguity in $h(\hat{A}', A)$. S-transformation will select a particular $h(\hat{A}', A)$ with $\{|A\rangle_{ph} \forall A\}$ and $\{|\mathcal{D}\rangle_A \forall A\}$ composing two sets of complete orthogonal bases for $\mathcal{H}_{ph}$. The unitary operator $S$ with

$$S|A\rangle_{ph} = |\mathcal{D}\rangle_A , \quad S|\mathcal{D}\rangle_A = |\hat{A}\rangle_{ph}$$

(5.33)
will make

$$S^{-1}\mathcal{T}_{R^*}(C)S = \mathcal{W}_{R^*}(C) , \quad S^{-1}\mathcal{W}_{R^*}(C)S = \mathcal{T}_{R^*}^+(C) .$$

(5.34)

The spectrum of $\mathcal{T}_{R^*}(C)$ and $\mathcal{W}_{R^*}(C)$ are highly degenerate, so (5.34) can only make $S$ determined up to $S \sim VS$ with $[V, \mathcal{T}_{R^*}(C)] = [V, \mathcal{W}_{R^*}(C)] = 0$, which is also reflected in the ambiguity of $h(\hat{A}', A)$ in (5.31). The degeneracies in loop operators may be reduced in flux operators. Suppose $\mathcal{T}_{R^*}(C) = e^{it_{R^*}(C)}$, $\mathcal{W}_{R^*}(C) = e^{iw_{R^*}(C)}$ with $t_{R^*}(C)$ and $w_{R^*}(C)$ the corresponding flux operators, the mapping

$$S^{-1}t_{R^*}(C)S = w_{R^*}(C) , \quad S^{-1}w_{R^*}(C)S = -t_{R^*}^+(C)$$

(5.35)
could fix $S$ further as is in $U(1)$ case.

The above discussion on loop operators has the simple analogy in one dimensional quantum mechanics. The position and the momentum operators are $X$ and $P$, $[X, P] = i$, $e^{iX} e^{2\pi iP} = e^{2\pi iP} e^{iX}$. \{$|x\rangle \in \mathbb{R}$\} can be divided into the equivalent classes generated by $e^{2\pi iP}$. \{$|x\rangle \in \mathbb{R}$\} = $\cup_{a \in [0, 2\pi]} E(a)$. $E(a) := \{|a - 2\pi k\rangle \forall k \in \mathbb{Z}\}$. In $\mathcal{H}[E(a)]$, the eigenvalue of $e^{iX}$ can only be $e^{ia}$, but the eigenvalue of $e^{2\pi iP}$ can be $e^{ib}$, $\forall b \in [0, 2\pi)$. For

$$|D\rangle_{(a,b)} = e^{-i\frac{h(a-a)}{2\pi}} \sum_{|x\rangle \in E(a)} e^{ibx} |x\rangle ,$$

$$e^{2\pi iP}|D\rangle_{(a,b)} = e^{ib}|D\rangle_{(a,b)} .$$

The discrete \{$|a - 2\pi k\rangle \forall k \in \mathbb{Z}\}$ and the continuous \{$|D\rangle_{(a,b)} | \forall b \in [0, 2\pi)\}$ compose two sets of orthogonal bases for $\mathcal{H}[E(a)]$. The mapping between $e^{iX}$ and $e^{2\pi iP}$ is realized in $\mathcal{H}$. $\forall x \in \mathbb{R}$, one may define

$$|D\rangle_x = \sum_{a \in [0, 2\pi)} e^{ih(a,x)} |D\rangle_{(a,x)} ,$$

and the unitary operator $U$ with $U|x\rangle = |D\rangle_x$, $U|D\rangle_x = |-x\rangle$ will make

$$U^{-1} e^{2\pi iP} U = e^{iX}, \quad U^{-1} e^{iX} U = e^{-2\pi iP} .$$

The requirement \eqref{5.39} cannot uniquely fix $U$ due to the degeneracies in $e^{iX}$ and $e^{2\pi iP}$. When $h(a, x) = e^{-i\frac{h(a-a)}{2\pi}}$,

$$|D\rangle_x = \int_{-\infty}^{\infty} dx' e^{i\frac{h(a,a')}{2\pi}} |x'\rangle$$

is the momentum eigenstate, and the operator $U$ making

$$U^{-1} PU = \frac{X}{2\pi}, \quad U^{-1} XU = -2\pi P$$

is the duality operator in quantum mechanics.

$[e^{iX}, e^{2\pi iP}] = 0$, as the common eigenstate of $e^{iX}$ and $e^{2\pi iP}$, $|D\rangle_{(a,b)}$ is determined up to a phase.

$$e^{iX}|D\rangle_{(a,b)} = e^{ia}|D\rangle_{(a,b)} , \quad e^{2\pi iP}|D\rangle_{(a,b)} = e^{ib}|D\rangle_{(a,b)} .$$

Aside from \{$|x\rangle \in \mathbb{R}$\} and \{$|p\rangle \in \mathbb{R}$\}, we get another set of complete orthogonal bases \{$|D\rangle_{(a,b)} \forall a, b \in [0, 2\pi)\}$ for $\mathcal{H}$. The three sets of bases have the same degrees of freedom. Since \{$b \in [0, 2\pi)\} \sim \{k \in \mathbb{Z}\}$, we have \{|(a, b) | a, b \in [0, 2\pi)\} \sim \{|(a, k) | a \in [0, 2\pi), k \in \mathbb{Z}\} \sim \{|a + 2\pi k \in \mathbb{R}\}$.  

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5.3. Modified t’ Hooft operator

The standard t’ Hooft operator \( T_R(C) \) satisfies the commutation relation

\[
T_R(C_1)W_{R'}(C_2) = W_{R'}(C_2)T_R(C_1) \exp\{iL(R, R'; C_1, C_2)\}. \tag{5.43}
\]

For an operator \( Y(C) \) built from \( A \), if \( T_R(C)Y(C) = Y(C)T_R(C) \), \( Y^{-1}(C) = Y^+(C) \), suppose

\[
T_R'(C) := \int DU [T_R(C)Y(C)]U^{-1} = \int DU UT_R'(C)U^{-1}, \tag{5.44}
\]

then

\[
T_R'(C_1)W_{R'}(C_2) = W_{R'}(C_2)T_R'(C_1) \exp\{iL(R, R'; C_1, C_2)\}. \tag{5.45}
\]

\( Y(C) \) does not need to be gauge invariant.

If there is a gauge invariant operator \( K(A) \) with

\[
T_R^{-1}(C)K(A)T_R(C) = K(T_R(C)A) = Y(A, C)K(A), \tag{5.46}
\]

then

\[
K(A)T_R(C)K^{-1}(A) = T_R(C)Y(C) \tag{5.47}
\]

and

\[
K(A)T_R(C)K^{-1}(A) = \int DU [T_R(C)Y(C)]U^{-1} = T_R'(C). \tag{5.48}
\]

\( T_R'(C) \) and \( T_R(C) \) are equivalent.

(5.46) could be taken as an integration equation for \( K(A) \) with the integrable condition

\[
Y(T_R(C_2)A, C_1)Y^{-1}(A, C_1) = Y(T_R(C_1)A, C_2)Y^{-1}(A, C_2) \tag{5.49}
\]

coming from the commutation relation \([T_R'(C_1), T_R'(C_2)] = 0\). If (5.49) is not satisfied, \( T_R'(C_1) \) and \( T_R'(C_2) \) do not commute. Even though, \( T_R'(C) \) and \( T_R(C) \) may still be equivalent, but are related by a \( C \)-dependent unitary operator \( K(A, C) \).

When \( Y(A, C) = W_R(A, C) \), \( K(A) = g(A) \), (5.47) and (5.48) become

\[
g(A)T_R(C)g^{-1}(A) = T_R(C)W_R(C) \tag{5.50}
\]

and

\[
g(A)T_R(C)g^{-1}(A) = T_R'(C), \tag{5.51}
\]
where
\[
\mathcal{T}_R(C) = \int DU \ U[T_R(C)W_R(C)]U^{-1}.
\] (5.52)
This is the T-transformation of the t’ Hooft operator.

5.4. t’ Hooft operator in path integral formalism and canonical formalism

In path integral formulation, t’ Hooft operator is introduced by expanding the quantum fields around the singular configurations. For the globally defined gauge potential \( A_\mu \), the Bianchi identity is automatically satisfied:
\[
\epsilon^\mu{}_{\nu\rho\sigma} D_\nu F_{\rho\sigma} = 0 ,
\] (5.53)
where
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] .
\] (5.54)
For the given \( j^\mu \),
\[
\epsilon^\mu{}_{\nu\rho\sigma} D_\nu F_{\rho\sigma} = j^\mu
\] (5.55)
also has the solution denoted as \( G_\mu \), which is not globally defined. (5.55) can be reduced to
\[
2\epsilon^\mu{}_{\nu\rho\sigma} \partial_\nu \partial_\rho G_\sigma = j^\mu .
\] (5.56)
To get the spatial t’ Hooft operator \( \mathcal{T}_{R^*(C)} \) located at the time \( t = t_0 \), \( j^\mu \) is taken to be
\[
j^0 = 0 , \quad j^i = 2H^*_m b^i \delta(t - t_0) .
\] (5.57)
The corresponding \( G_\mu \) can be selected as
\[
G_i = 0 , \quad G_0 = -H^*_m \omega \delta(t - t_0) .
\] (5.58)
\( b^i = \epsilon^{ijk} \partial_j a_k \). \( b^i, \omega \) and \( a_i \) are given by (3.18), (3.12) and (3.11). \( \omega(\Sigma_C, x) \) jumps \( 4\pi \) when \( x \) crosses \( \Sigma_C \), but \( e^{i\omega} \) is continues except for the singularity at \( C \). To solve for (5.56), note that
\[
2H^*_m \epsilon^{ijk} \partial_j a_k \delta(t - t_0) = 2H^*_m b^i \delta(t - t_0) = j^i .
\] (5.59)
\( a_i \) is not a pure gauge, but \( a_i = -ie^{-i\omega} \partial_i e^{i\omega} \), so locally, we have \( a_i = \partial_i \omega \). (5.56) becomes
\[
2H^*_m \epsilon^{ijk} \partial_j \partial_k \omega \delta(t - t_0) = j^i = 2H^*_m b^i \delta(t - t_0)
\] (5.60)
with the non-globally defined $\omega$.

In (5.56), $G_\mu$ is determined up to the addition of an arbitrary globally defined gauge potential $\tilde{A}_\mu$. The generic solution of (5.55) is $A_\mu = G_\mu + \tilde{A}_\mu$.

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} + \tilde{D}_\mu G_\nu - \tilde{D}_\nu G_\mu + i[G_\mu, G_\nu] , \quad (5.61)$$

where $\tilde{F}_{\mu\nu} = \partial_\nu \tilde{A}_\mu - \partial_\mu \tilde{A}_\nu + i[\tilde{A}_\mu, \tilde{A}_\nu]$, $\tilde{D}_\mu f = \partial_\mu f + i[\tilde{A}_\mu, f]$. For (5.58),

$$F_{ij} = \tilde{F}_{ij} , \quad F_{i0} = \tilde{F}_{i0} + \tilde{D}_i G_0 . \quad (5.62)$$

Consider the YM theory with the gauge group $G$ and the coupling constant $\tau$, $\tau = \tau_1 + i \tau_2$. According to the relation between the YM theory and $D3$ branes in type IIB string theory, $\tau_2 = 1/g_s$, where $g_s$ is the type IIB string coupling constant. The gauge coupling is $g = \sqrt{2\pi g_s}$. $1/g^2 = \frac{\tau_1}{2\pi}$. The action of the YM theory is

$$S = \frac{\tau_2}{2\pi} \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\tau_1}{8\tau_2} \epsilon^{\mu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right)$$

$$= \frac{\tau_2}{2\pi} \int d^4x \left( -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{\tau_1}{8\tau_2} \epsilon^{\mu\rho\sigma} \tilde{F}_{\mu\nu} \tilde{F}_{\rho\sigma} + \tilde{F}_{i0} \tilde{D}^i G_0 + \frac{1}{2} \tilde{D}_i G_0 \tilde{D}^i G_0 - \frac{\tau_1}{2\tau_2} \epsilon^{ijk} \tilde{F}_{ij} \tilde{D}_k G_0 \right)$$

$$= \tilde{S} + \frac{1}{2\pi} \int d^3x \left[ -\tau_2 \tilde{F}_{i0} \tilde{D}^i (H^*_m \omega) + \frac{\tau_2}{2} \delta(0) \tilde{D}_i (H^*_m \omega) \tilde{D}^i (H^*_m \omega) + \tau_1 \tilde{B}_i \tilde{D}^i (H^*_m \omega) \right] . \quad (5.63)$$

In temporal gauge, $\tilde{\Pi}_i = -\frac{\tau_2}{2\pi} \tilde{F}_{i0} + \frac{\tau_1}{2\pi} \tilde{B}_i$, so replacing $\tilde{S}$ by $S$ amounts to adding the operator

$$T'_{R^*}(C) = \exp\{ i \int d^3x \left[ \tilde{\Pi}_i \tilde{D}^i (H^*_m \omega) + \frac{\tau_2}{4\pi} \delta(0) \tilde{D}_i (H^*_m \omega) \tilde{D}^i (H^*_m \omega) \right] \} \quad (5.64)$$

into the path integral. $T'_{R^*}(C) = T_{R^*}(C) Y(C)$, where

$$T_{R^*}(C) = \exp\{ i \int d^3x \left[ \tilde{\Pi}_i \tilde{D}^i (H^*_m \omega) \right] \} \exp\{ i \int d^3x \left[ H^*_m \tilde{\Pi}_i a_i + i[\tilde{\Pi}_i, \tilde{A}_i] H^*_m \omega \right] \} \quad (5.65)$$

is the standard t’ Hooft operator in the dual representation $R^* [\Pi]$, $Y(C) = \exp\{ i \int d^3x \left[ \frac{\tau_2}{4\pi} \delta(0) \tilde{D}_i (H^*_m \omega) \tilde{D}^i (H^*_m \omega) \right] \}$

is an operator constructed from $\tilde{A}$. $T'_{R^*}(C)$ and $T_{R^*}(C)$ are equivalent if the suitable $K(A)$ can be obtained as is in (5.46).
For \( A_\mu = G_\mu + \tilde{A}_\mu \), under the gauge transformation \( U \),

\[
G_\mu \to U G_\mu U^{-1} \quad \tilde{A}_i \to U \tilde{A}_i U^{-1} - iU \partial_i U^{-1} \quad j_i \to U j_i U^{-1}.
\] (5.67)

To preserve the gauge invariance, the integration should cover the background \( U G_\mu U^{-1} \), or more concretely, \( U(H^*_m(\omega)) U^{-1} \) for the arbitrary \( U \). The obtained gauge invariant operator is

\[
T'_{R*}(C) = \int DU U T'_{R*}(C) U^{-1}.
\] (5.68)

Under the T-transformation, \( \tilde{\Pi}_i \to \tilde{\Pi}_i + \tilde{\Pi}_i \). According to (5.65),

\[
T_{R*}(C) \to \exp \{ i \int d^3 x \text{tr} [\tilde{\Pi}_i \dot{D}^i (H^*_m(\omega))] + \frac{1}{2\pi} \text{tr} [\tilde{B}_i \dot{D}^i (H^*_m(\omega))] \} = \exp \{ i \int d^3 x \text{tr} [\tilde{\Pi}_i \dot{D}^i (H^*_m(\omega))] \} \exp \{ i \oint_C ds 2\text{tr} (H^*_m \dot{A}_i) \dot{x}^i \} = T_{R*}(C) W_{R*}(C)
\] (5.69)

and then

\[
T_{R*}(C) \to \int DU U [T_{R*}(C) W_{R*}(C)] U^{-1}.
\] (5.70)

This is the manifestation of the T-transformation rule in path integral formalism.

In canonical formulation, t’ Hooft operator, in parallel with the Wilson operator, is determined by the field content with no dynamical information involved. But in path integral formalism, t’ Hooft operator is action-dependent. \( T'_{R*}(C) = \int DU U [T_{R*}(C) Y(C)] U^{-1} \), where \( Y(C) \) depends on action. We may consider YM theories with the arbitrary higher order interactions, and for each theory, there is a corresponding \( Y(C) \). It is expected that \( T'_{R*}(C) \) and \( T_{R*}(C) \) are equivalent.

The Lagrangian for \( N = 4 \) SYM theory is

\[
L = \frac{\tau_2}{2\pi} \text{tr} \{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\tau_1}{8\tau_2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} - i \bar{\Psi}^a \tilde{\sigma}^\mu D_\mu \Psi_a - \frac{1}{2} D_\mu \Phi^I D^\mu \Phi^I
\]
\[
+ \frac{1}{2} C_I^{ab} \bar{\Psi}_a [\Phi^I, \Psi_b] + \frac{1}{2} \tilde{C}_{Iab} \bar{\Psi}^a [\Phi^I, \bar{\Psi}^b] + \frac{1}{4} [\Phi^I, \Phi^J]^2 \}.
\] (5.71)

\( \mu, \nu = 0, 1, 2, 3; \ a, b = 1, 2, 3, 4; \ I, J = 1, 2, \cdots, 6 \). In addition to the gauge potential, background fields for the scalar \( \Phi^I \) and fermion \( \Psi^a \) can also be introduced [7].

\[
A_\mu = G_\mu + \tilde{A}_\mu \quad \Phi^I = \phi^I + \tilde{\phi}^I \quad \Psi^a = \psi^a + \tilde{\psi}^a.
\] (5.72)
and $\psi^a$ satisfy
\[ \partial^\mu \partial_\mu \phi^I(x) = 2(\frac{\tau_2}{2\pi})^{-\frac{1}{2}} H_{m}^* \int_C ds \lambda^I \delta^4(x - \bar{x}_C) \] (5.73)
\[ \text{and} \]
\[ \sigma^\mu \partial_\mu \psi^a(x) = -2(\frac{\tau_2}{2\pi})^{-\frac{1}{2}} H_{m}^* \int_C ds \lambda^a \delta^4(x - \bar{x}_C) . \] (5.74)

The Lagrangian can be expanded as
\[ L = \tilde{L} + tr\{\frac{1}{2\pi}(\tau_2 \tilde{F}_{i0} - \tau_1 \tilde{B}_i) \tilde{D}^i G_0 + \frac{\tau_2}{2\pi}(\tilde{\Psi}^a \bar{\sigma}^0[\tilde{G}_0, \tilde{\Psi}_a] + i\partial_0 \tilde{\Phi}^I[\tilde{G}_0, \tilde{\Phi}^I]) \]
\[ - \frac{\tau_2}{2\pi} \partial_\mu \phi^I \partial^\mu \tilde{\Phi}^I + \frac{i\tau_2}{2\pi} \partial_\mu \tilde{\psi}^a \bar{\sigma}^\mu \tilde{\Psi}_a - \frac{i\tau_2}{2\pi} \tilde{\Psi}^a \bar{\sigma}^\mu \partial_\mu \psi_a + r(\tau; \tilde{A}_\mu, \tilde{\Phi}^I, \tilde{\Psi}^a; G_\mu, \phi^I, \psi^a) \}
\[ \sim \tilde{L} + tr\{\frac{1}{2\pi}(\tau_2 \tilde{F}_{i0} - \tau_1 \tilde{B}_i) \tilde{D}^i G_0 + \frac{\tau_2}{2\pi}(\tilde{\Psi}^a \bar{\sigma}^0[\tilde{G}_0, \tilde{\Psi}_a] + i\partial_0 \tilde{\Phi}^I[\tilde{G}_0, \tilde{\Phi}^I]) \]
\[ + 2(\frac{\tau_2}{2\pi})^{\frac{1}{2}} \int_C ds H_{m}^* \tilde{\Phi}^I \lambda^I \delta^4(x - \bar{x}_C) + 2i(\frac{\tau_2}{2\pi})^{\frac{1}{2}} \int_C ds H_{m}^*(\tilde{\Psi}^a_+ - \tilde{\Psi}_a^a) \lambda^a \delta^4(x - \bar{x}_C) \]
\[ + r(\tau; \tilde{A}_\mu, \tilde{\Phi}^I, \tilde{\Psi}^a; G_\mu, \phi^I, \psi^a) \} . \] (5.75)

Canonical quantization of $\tilde{L}$ in temporal gauge gives
\[ \tilde{\Pi}_i = -\frac{\tau_2}{2\pi} \tilde{F}_{i0} + \frac{\tau_1}{2\pi} \tilde{B}_i , \quad \tilde{\Pi}^I = \frac{\tau_2}{2\pi} \partial_0 \tilde{\Phi}^I , \quad \tilde{\Pi}^a = \frac{i\tau_2}{2\pi} \tilde{\Psi}^a \bar{\sigma}^0 . \] (5.76)

So adding the background fields amounts to adding the operator
\[ T_{R^*}(\tau; \lambda^I, \lambda^a, C) \]
\[ = \exp\{i \int d^4x tr\{\tilde{\Pi}_i \tilde{D}^i G_0 - i\tilde{\Pi}^a[\tilde{G}_0, \tilde{\Psi}_a] + i\tilde{\Pi}^I[\tilde{G}_0, \tilde{\Phi}^I] + 2(\frac{\tau_2}{2\pi})^{\frac{1}{2}} \int_C ds H_{m}^* \tilde{\Phi}^I \lambda^I \delta^4(x - \bar{x}_C) \]
\[ + 2i(\frac{\tau_2}{2\pi})^{\frac{1}{2}} \int_C ds H_{m}^*(\tilde{\Psi}^a_+ - \tilde{\Psi}_a^a) \lambda^a \delta^4(x - \bar{x}_C) + r(\tau; \tilde{A}_\mu, \tilde{\Phi}^I, \tilde{\Psi}^a; G_\mu, \phi^I, \psi^a) \} \]
\[ = \exp\{i \int d^4x tr\{\tilde{\Pi}_i \tilde{D}^i (H_{m}^* \omega) + i\tilde{\Pi}^a[H_{m}^* \omega, \tilde{\Psi}_a] - i\tilde{\Pi}^I[H_{m}^* \omega, \tilde{\Phi}^I] \} \exp\{-\frac{(\tau_2)}{2\pi} \int_C ds 2tr(H_{m}^*(\tilde{\Psi}^a_+ - \tilde{\Psi}_a^a)) \lambda^a \}
\[ \exp\{i \int d^4x r(\tau; \tilde{A}_\mu, \tilde{\Phi}^I, \tilde{\Psi}^a; G_\mu, \phi^I, \psi^a) \} \]
\[ = T_{R^*}(\tau; \lambda^I, C)W_{R^*}(\tau; \lambda^a, C)Y(\tau; \lambda^I, \lambda^a, C) \] (5.77)
into the path integral.
\[ T_{R^*}(C) = \exp\{i \int d^3x tr\{\tilde{\Pi}_i \tilde{D}^i (H_{m}^* \omega) + i\tilde{\Pi}^a[H_{m}^* \omega, \tilde{\Psi}_a] - i\tilde{\Pi}^I[H_{m}^* \omega, \tilde{\Phi}^I] \} \} \] (5.78)
is the standard t’ Hooft operator generating the singular gauge transformation in \( \mathcal{N} = 4 \) SYM theory.

\[
W^\Phi_{R^*}(\tau; \lambda^I, C) = \exp\left\{ i \frac{\tau_2}{2\pi} \oint_C ds \, 2tr(H^s_{m}^* \Phi_f)\lambda^I \right\}
\]

(5.79)

and

\[
W^\Psi_{R^*}(\tau; \lambda^a, C) = \exp\left\{ -i \frac{\tau_2}{2\pi} \oint_C ds \, 2tr[H^s_{m}(\Psi^a_f - \bar{\Psi}_a)]\lambda^a \right\}
\]

(5.80)

are Wilson loops for \( \Phi \) and \( \Psi \).

\[
Y(\tau; \lambda^I, \lambda^a, C) = \exp\left\{ i \int d^4 x \, r(\tau; A_\mu, \Phi^I, \Psi^a, G_\mu, \phi^I, \psi^a) \right\}
\]

(5.81)

is an operator constructed from \( \tilde{A}_i, \tilde{\Phi}^I, \tilde{\Psi}^a \), whose form depends on action as well as the explicit solutions for \( \phi^I \) and \( \psi^a \).

Under the local gauge transformation,

\[
H^s_{m}\omega \rightarrow U(H^s_{m}\omega)U^{-1} \quad \phi^I \rightarrow U\phi^I U^{-1} \quad \psi^a \rightarrow U\psi^a U^{-1}
\]

\[
\tilde{A}_i \rightarrow U\tilde{A}_i U^{-1} - iU\partial_i U^{-1} \quad \tilde{\Phi}^I \rightarrow U\tilde{\Phi}^I U^{-1} \quad \tilde{\Psi}^I \rightarrow U\tilde{\Psi}^I U^{-1}.
\]

(5.82)

The path integral should cover all of \( \{U(H^s_{m}\omega)U^{-1}, U\phi^I U^{-1}, U\psi^a U^{-1}\} \) and the final gauge invariant t’ Hooft operator is

\[
T'_{R^*}(\tau; \lambda^I, \lambda^a, C) = \int DU \, UT'_{R^*}(\tau; \lambda^I, \lambda^a, C)U^{-1}.
\]

(5.83)

### 5.5. Spectrum and eigenstates of t’ Hooft operator in \( \mathcal{N} = 4 \) SYM theory

In canonical quantization formalism, for \( \mathcal{N} = 4 \) SYM theory with the coupling constant \( \tau = \tau_1 + i\tau_2 \) and the gauge group \( G \), the canonical coordinates are \( \Lambda := (A_i, \Phi^I, \Psi^a) \) with the conjugate momentum \( (\Pi^I, \Pi_f, \Pi_a) \). The generic supersymmetric Wilson operator in dual representation \( R^* \) can be defined as

\[
W_{R^*}(\tau; \lambda^I, \lambda^a, C) = \int DU \, U W_{R^*}(\tau; \lambda^I, \lambda^a, C)U^{-1},
\]

(5.84)

where

\[
W_{R^*}(\tau; \lambda^I, \lambda^a, C) = W^A_{R^*}(C)W^\Phi_{R^*}(\tau; \lambda^I, C)W^\Psi_{R^*}(\tau; \lambda^a, C),
\]

(5.85)
\[ W_{R^*}(C) = \exp\{i \oint_C ds \text{tr}(H^*_m A_i) \hat{x}^i \}, \]
\[ W^\Phi_{R^*}(\tau; \lambda^I, C) = \exp\{i(\frac{\tau_2}{2\pi})^{\frac{1}{2}} \oint_C ds \text{tr}(H^*_m \Phi_I) \lambda^I \}, \]
\[ W^\Psi_{R^*}(\tau; \lambda^a, C) = \exp\{i(\frac{\tau_2}{2\pi})^{\frac{1}{2}} \oint_C ds \text{tr}(H^*_m \Psi_a) \lambda^a \}. \] (5.86)

\[ W_{R^*}(\tau; \lambda^I, \lambda^a, C) \] can also be written as
\[ W_{R^*}(\tau; \lambda^I, \lambda^a, C) = \frac{1}{d_{R^*}} \text{tr} \exp\{i \oint_C ds (A^I_{R^*} \dot{x}_i + (\frac{\tau_2}{2\pi})^{\frac{1}{2}} \Phi_I^I \lambda_I + (\frac{\tau_2}{2\pi})^{\frac{1}{2}} \Psi_a^a \lambda_a) \}, \] (5.87)

where \( A_{R^*}, \Phi_{R^*}, \Psi_{R^*} \) are fields in representation \( R^* \). Suppose \( |\Lambda\rangle := |A_i, \Phi^I, \Psi^a\rangle \) is the common eigenstate of \( (A_i, \Phi^I, \Psi^a) \).

\[ W_{R^*}(\tau; \lambda^I, \lambda^a, C)|\Lambda\rangle = W_{R^*}(\Lambda; \tau; \lambda^I, \lambda^a, C)|\Lambda\rangle. \] (5.88)

The corresponding supersymmetric t’ Hooft operator is
\[ T_{R^*}(\tau; \lambda^I, \lambda^a, C) = \int DU \text{UT}_{R^*}(\tau; \lambda^I, \lambda^a, C)U^{-1} \] (5.89)

with
\[ T_{R^*}(\tau; \lambda^I, \lambda^a, C) = T_{R^*}(C)W^\Phi_{R^*}(\tau; \lambda^I, C)W^\Psi_{R^*}(\tau; \lambda^a, C). \] (5.90)

\( T_R(C) \) can be defined via its action on \( |A_i, \Phi^I, \Psi^a\rangle \):
\[ T_R(C)|A_i, \Phi^I, \Psi^a\rangle = |\tilde{\Omega}^{-1}_m(C)A_i \tilde{\Omega}_m(C) - H_m a_i(C), \tilde{\Omega}^{-1}_m(C)\Phi^I \tilde{\Omega}_m(C), \tilde{\Omega}^{-1}_m(C)\Psi^a \tilde{\Omega}_m(C)\rangle. \] (5.91)

Loop operators satisfy the commutation relation
\[ T_R(C_1)W_{R^*}(\tau; \lambda^I_2, \lambda^a_2, C_2) = W_{R^*}(\tau; \lambda^I_2, \lambda^a_2, C_2)T_R(C_1) \exp\{iL(R, R'; C_1, C_2)\} \] (5.92)

and then
\[ T_R(\tau; \lambda^I_2, \lambda^a_2, C_1)W_{R^*}(\tau; \lambda^I_2, \lambda^a_2, C_2) = W_{R^*}(\tau; \lambda^I_2, \lambda^a_2, C_2)T_R(\tau; \lambda^I_1, \lambda^a_1, C_1) \exp\{iL(R, R'; C_1, C_2)\}. \] (5.93)

Especially,
\[ T_{R^*}(C_1)W_{R^*}(\tau; \lambda^I_2, \lambda^a_2, C_2) = W_{R^*}(\tau; \lambda^I_2, \lambda^a_2, C_2)T_{R^*}(C_1), \] (5.94)
\[ T_R^*(\tau; \lambda_1^I, \lambda_2^a, C_1)W_R^*(\tau; \lambda_2^I, C_2) = W_R^*(\tau; \lambda_2^I, \lambda_2^a, C_2)T_R^*(\tau; \lambda_1^I, \lambda_1^a, C_1) . \]  

(5.95)

Similar with the previous discussion, complete orthogonal bases of the Hilbert space \( \mathcal{H} \) can be selected as \( \{|\Lambda\rangle \forall \Lambda\} \), and \( \{|\Lambda\rangle \forall \Lambda\} = \bigcup \Lambda \in E(\hat{\Lambda}) \), where

\[ E(\hat{\Lambda}) := \{U_n T(\pm C_{n-1}) U_{n-1} \cdots T(\pm C_2) U_2 T(\pm C_1) U_1 |\hat{\Lambda}\rangle \forall C_k, U_k, n\} \]  

(5.96)

is the equivalent class generated by the action of \( T \) and \( U \). \( T \) stands for \( T_R \) with \( R \) the fundamental representation of \( G \).

\[ [T_R(\tau; \lambda_1^I, \lambda_1^a, C_1), T_R(\tau; \lambda_2^I, \lambda_2^a, C_2)] = 0 . \]  

(5.97)

Common eigenstates of \( T_R^*(\tau; \lambda^I, \lambda^a, C) \) can be constructed as

\[ |D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} = \sum_{|\Lambda\rangle \in E(\hat{\Lambda})} g^{-1}(A)|\Lambda\rangle , \]  

(5.98)

where the first \( \hat{\Lambda} \) in the subscript \( (\hat{\Lambda}, \hat{\Lambda}) \) indicates \( |D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} \in \mathcal{H}[E(\hat{\Lambda})] \) and the second means \( |D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} \) is to be constructed as an eigenstate with the eigenvalue \( W_R^*(\hat{\Lambda}; \tau; \lambda^I, \lambda^a, C) \). \( \forall |\Lambda\rangle \in E(\hat{\Lambda}) \),

\[ W_R^*(\Lambda; \tau; \lambda^I, \lambda^a, C) = W_R^*(\hat{\Lambda}; \tau; \lambda^I, \lambda^a, C) . \]  

(5.99)

\[ U|D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} = |D\rangle_{(\hat{\Lambda}, \hat{\Lambda})}, |D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} \in \mathcal{H}_{ph}. \]  

\[ T_R^*(\tau; \lambda^I, \lambda^a, C)|D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} = \int DU U T_R^*(C) W_R^*(\tau; \lambda^I, C) W_R^*(\tau; \lambda^a, C)|D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} = \int DU U \sum_{|\Lambda\rangle \in E(\hat{\Lambda})} W_R^*(\Phi^I; \tau; \lambda^I, \lambda^a, C)|D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} = \int DU U \sum_{|\Lambda\rangle \in E(\hat{\Lambda})} W_R^*(\Phi^I; \tau; \lambda^I, \lambda^a, C)|D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} = \sum_{|\Lambda\rangle \in E(\hat{\Lambda})} W_R^*(\Lambda; \tau; \lambda^I, \lambda^a, C)|D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} = W_R^*(\Lambda; \tau; \lambda^I, \lambda^a, C)|D\rangle_{(\hat{\Lambda}, \hat{\Lambda})} . \]  

(5.100)
Written in terms of the group $L$,

$$|D\rangle_{(\hat{\Lambda},\Lambda)} = \sum_{L \in \mathcal{L}} g^{-1}(L\hat{\Lambda})|L\hat{\Lambda}\rangle,$$

where $\hat{\Lambda} = (\hat{A}_i, \hat{\Phi}_I, \hat{\Psi}_a)$, $\Lambda = L\hat{\Lambda} = (A_i, \Phi^I, \Psi^a)$. Again, we are free to select another reference state $|\hat{\Lambda}'\rangle$ to get

$$|D\rangle_{(\hat{\Lambda}',\Lambda)} = \sum_{L \in \mathcal{L}} g^{-1}(L\hat{\Lambda}')|L\hat{\Lambda}'\rangle,$$

where $\hat{\Lambda}' = (\hat{A}_i', \hat{\Phi}_I, \hat{\Psi}_a)$ with $\hat{A}_i'$ an arbitrary gauge potential. $\Lambda' = L\hat{\Lambda}' = (A_i', \Phi^I, \Psi^a)$.

The action of $\mathcal{T}_R^+ = \mathcal{T}^+(\tau; \lambda^a, \lambda^a, C)$ is given by

$$\mathcal{T}_R^+(\tau; \lambda^a, \lambda^a, C)|D\rangle_{(\hat{\Lambda},\Lambda)} = \mathcal{W}_R(\hat{\Lambda}; \tau; \lambda^a, \lambda^a, C)|D\rangle_{(\hat{\Lambda},\Lambda)}.$$  

(5.103)

The action of $\mathcal{T}_R^+$ is given by

$$\mathcal{T}_R^+(\tau; \lambda^a, \lambda^a, C)|D\rangle_{(\hat{\Lambda},\Lambda)} = \mathcal{W}_R^*(\hat{\Lambda}; \tau; \lambda^a, \lambda^a, C)|D\rangle_{(\hat{\Lambda},\Lambda)}.$$

(5.104)

So for $|D\rangle_{(\hat{\Lambda},\Lambda')}$ and $|D\rangle_{(\hat{\Lambda},\Lambda'')}$ with $\hat{\Lambda}'' = (\hat{A}_i'', \hat{\Phi}_I, \hat{\Psi}_a)$,

$$(\hat{\Lambda},\Lambda')\langle D|D\rangle_{(\hat{\Lambda},\Lambda'')} = 0$$

(5.105)

if $\mathcal{W}_R(\hat{\Lambda}'; \tau; \lambda^a, \lambda^a, C) \neq \mathcal{W}_R(\hat{\Lambda}''; \tau; \lambda^a, \lambda^a, C)$.

To summarize, in the sub-Hilbert space $\mathcal{H}[E(A_i, \Phi^I, \Psi^a)]$, one may construct the eigenstate of $\mathcal{T}_R^+(\tau; \lambda^a, \lambda^a, C)$ with the eigenvalue $\mathcal{W}_R^*(\hat{\Lambda}''; \tau; \lambda^a, \lambda^a, C)$ for the arbitrary gauge potential $A'$. Eigenstates with the different eigenvalues are orthogonal.

In $U(1)$ case,

$$\mathcal{T}(\tau; \lambda^a, \lambda^a, C) = T(\tau; \lambda^a, \lambda^a, C) = T(C)W^\Phi(\tau; \lambda^a, C)W^\Psi(\tau; \lambda^a, C).$$

(5.106)

$T(C)$ only acts on $A$, so $|D\rangle_{(\hat{\Lambda},\Lambda)} = |D\rangle_{(\hat{\Lambda}',\Lambda)}|\hat{\Phi}, \hat{\Psi}\rangle$ with $|D\rangle_{(\hat{\Lambda}',\Lambda)}$ given by (5.25).

### 5.6. Modified supersymmetric t’ Hooft operator

There is no exact distinction between the spectrum of

$$\mathcal{W}_R(\tau; \lambda^a, \lambda^a, C) = \frac{1}{d_R} tr P \exp\left\{ i \int_C \left[ A_R^i dx_i + \left( \frac{T_2}{2\pi} \right)^{1/2} \Phi^I R d\tau + \left( \frac{T_2}{2\pi} \right)^{1/2} \Psi^a R d\lambda_d \right] \right\}$$

(5.107)
and the spectrum of
\[ W_R(C) = \frac{1}{d_R} tr P \exp \{ i \oint_C (A_R^i dx_i) \} . \] (5.108)

Suppose \( y'(x) \) and \( z^a(x) \) are functions specifying a three dimensional hypersurface in superspace \((x^i, y^I, z^a)\), if \( \lambda'^i = \partial_i y^I \dot{x}^i, \lambda'^a = \partial_i z^a \dot{x}^i \), then
\[ A_R^i dx_i + \left( \frac{\tau_2}{2\pi} \right)^i \Phi_R^I \lambda_I ds + \left( \frac{\tau_2}{2\pi} \right)^a \Psi_R^a \lambda_a ds = A_R^i dx_i + \left( \frac{\tau_2}{2\pi} \right)^i \Phi_R^I dy_I + \left( \frac{\tau_2}{2\pi} \right)^a \Psi_R^a dz_a , \] (5.109)
and
\[ W_R(\tau; \lambda', \lambda^a, C) = \frac{1}{d_R} tr P \exp \{ i \oint_C ds \left[ A_R^I + \left( \frac{\tau_2}{2\pi} \right)^i \Phi_R^I \partial_i y_I + \left( \frac{\tau_2}{2\pi} \right)^a \Psi_R^a \partial_i z_a \right] \} . \] (5.110)

\( W_R(\tau; \lambda', \lambda^a, C) \) could be taken as the Wilson loop of the gauge potential
\[ A_R^i = A_R^i + \left( \frac{\tau_2}{2\pi} \right)^i \Phi_R^I \partial_i y_I + \left( \frac{\tau_2}{2\pi} \right)^a \Psi_R^a \partial_i z_a . \] (5.111)

Let
\[ p(\tau; \Lambda; y, z) = \exp \{ i \int d^3 x \left( \frac{\tau_2}{2\pi} \right)^I tr [ (\Phi_I \partial_I y^I + \Psi_a \partial_i z^a) \Pi^I ] \} , \] (5.112)
for \( \lambda'^i = \partial_i y^I \dot{x}^i, \lambda'^a = \partial_i z^a \dot{x}^i \), there will be
\[ p(\tau; \Lambda; y, z) W_R^A(C) p^{-1}(\tau; \Lambda; y, z) = W_R^A(C) W_R^\Phi(\tau; \lambda', C) W_R^\Psi(\tau; \lambda^a, C) \] (5.113)
and
\[ p(\tau; \Lambda; y, z) W_R(C) p^{-1}(\tau; \Lambda; y, z) = W_R(\tau; \lambda', \lambda^a, C) . \] (5.114)

The corresponding supersymmetric t’ Hooft operator \( T_R(\tau; \lambda', \lambda^a, C) \) and \( T_R(C) \) can also be related by a unitary transformation. For operator
\[ q(\tau; \Lambda; y, z) = \exp \left\{ - \frac{i}{2\pi} \int d^3 x \left( \frac{\tau_2}{2\pi} \right)^I tr [ (\Phi_I \partial_I y^I + \Psi_a \partial_i z^a) B^I ] \right\} , \] (5.115)
\[ T_R^{-1}(C) q(\tau; \Lambda; y, z) T_R(C) = \exp \left\{ \frac{i}{2\pi} \int d^3 x \left( \frac{\tau_2}{2\pi} \right)^I tr [ H_{\bar m} (\Phi_I \partial_I y^I + \Psi_a \partial_i z^a) B^I ] \right\} q(\tau; \Lambda; y, z) , \] (5.116)
where we have used
\[ T_R^{-1}(C) B_i T_R(C) = \bar{\Omega}_{\bar m}^{-1}(C) B_i \bar{\Omega}_{\bar m}(C) - H_{\bar m} b_i (C) . \] (5.117)
Therefore,
\[
q(\tau; \Lambda; y, z) T_R(C) q^{-1}(\tau; \Lambda; y, z) = T_R(C) \exp \left\{ \frac{i}{2\pi} \int d^3x \left( \frac{\tau_2}{2\pi} \right)^\frac{1}{2} tr[H_m(\Phi_I \partial_i y^I + \Psi_a \partial_i z^a)b^j] \right\}
\]
\[
= T_R(C) \exp \left\{ i \oint_C ds \left( \frac{\tau_2}{2\pi} \right)^\frac{1}{2} 2tr[H_m(\Phi_I \partial_i y^I + \Psi_a \partial_i z^a)]\dot{x}^j \right\}
\]
\[
= T_R(C) W_R^\Phi(\tau; \lambda^I, C) W_R^\Psi(\tau; \lambda^a, C),
\] (5.118)
and
\[
q(\tau; \Lambda; y, z) T_R(C) q^{-1}(\tau; \Lambda; y, z) = T_R(\tau; \lambda^I, \lambda^a, C). \] (5.119)

More generically, with \( g(A) \) taken into account, suppose
\[
q(\tau; \Lambda; y, z, m) = \exp \left\{ -\frac{i}{2\pi} \int d^3x \left( \frac{\tau_2}{2\pi} \right)^\frac{1}{2} tr[(\Phi_I \partial_i y^I + \Psi_a \partial_i z^a)B^j] \right\} g^m(A),
\] (5.120)
we will have
\[
q(\tau; \Lambda; y, z, m) T_R(C) q^{-1}(\tau; \Lambda; y, z, m) = T_R(C)(W_R^A(C))^m W_R^\Phi(\tau; \lambda^I, C) W_R^\Psi(\tau; \lambda^a, C) \] (5.121)
and
\[
q(\tau; \Lambda; y, z, m) T_R(C) q^{-1}(\tau; \Lambda; y, z, m) = T_R(\tau; m, \lambda^I, \lambda^a, C)
\] (5.122)
with
\[
T_R(\tau; m, \lambda^I, \lambda^a, C) = \int DU [T_R(C)(W_R^A(C))^m W_R^\Phi(\tau; \lambda^I, C) W_R^\Psi(\tau; \lambda^a, C)] U^{-1}.
\] (5.123)

This is the generalized T-transformation inserting a supersymmetric Wilson loop into the t’ Hooft operator.

In path integral formulation, the obtained t’ Hooft operator is
\[
T'_{R^*}(\tau; \lambda^I, \lambda^a, C) = \int DU \left[ T_{R^*}(\tau; \lambda^I, C) W_{R^*}^\Phi(\tau; \lambda^a, C) \right] Y(\tau; \lambda^I, \lambda^a, C) U^{-1} \] (5.124)
with \( Y(\tau; \lambda^I, \lambda^a, C) \) a unitary operator constructed from \( \Lambda \). It is expected that \( T_{R^*}(\tau; \lambda^I, \lambda^a, C) \) and \( T'_{R^*}(\tau; \lambda^I, \lambda^a, C) \) are equivalent.
5.7. S-transformation of the supersymmetric loop operators

\[
[T_{R^*}(\tau; \lambda^I, \lambda^a, C_1), \mathcal{W}_{R^*}(\tau; \lambda^I, \lambda^a, C_2)] = 0 .
\] (5.125)

\( |D\rangle_{(N', \Lambda)} \) is the common eigenstate of \( T_{R^*}(\tau; \lambda^I, \lambda^a, C) \) and \( \mathcal{W}_{R^*}(\tau; \lambda^I, \lambda^a, C) \) with

\[
T_{R^*}(\tau; \lambda^I, \lambda^a, C)|D\rangle_{(N', \Lambda)} = \mathcal{W}_{R^*}(\Lambda; \tau; \lambda^I, \lambda^a, C)|D\rangle_{(N', \Lambda)} ,
\] (5.126)

\[
\mathcal{W}_{R^*}(\tau; \lambda^I, \lambda^a, C)|D\rangle_{(N', \Lambda)} = \mathcal{W}_{R^*}(\Lambda'; \tau; \lambda^I, \lambda^a, C)|D\rangle_{(N', \Lambda)} .
\] (5.127)

\( \Lambda = (A_i, \Phi^I, \Psi^a), \Lambda' = (A'_i, \Phi^I, \Psi^a), |D\rangle_{(N', \Lambda)} \in \mathcal{H}[E(\Lambda')] \).

For \( |\Lambda\rangle = |A_i, \Phi^I, \Psi^a\rangle \), the dual state of \( |\Lambda\rangle_{ph} \) is the superposition of \( |D\rangle_{(N', \Lambda)} \):

\[
|\mathcal{D}\rangle_{\Lambda} = \sum_{\tilde{\Lambda}} e^{ih(\tilde{\Lambda}', \Lambda)}|\mathcal{D}\rangle_{(N', \Lambda)} ,
\] (5.128)

where the summation is taken over all of the equivalent classes \( E(\tilde{\Lambda}') \) with \( |\tilde{\Lambda}'\rangle = |\tilde{\Lambda}'_i, \tilde{\Phi}^I, \tilde{\Psi}^a\rangle \).

\( e^{ih(\tilde{\Lambda}', \Lambda)} \) is the particular coefficient.

\[
T_{R^*}(\tau; \lambda^I, \lambda^a, C)|\mathcal{D}\rangle_{\Lambda} = \mathcal{W}_{R^*}(\Lambda; \tau; \lambda^I, \lambda^a, C)|\mathcal{D}\rangle_{\Lambda} .
\] (5.129)

When \( G = U(1) \), the gauge potential part is factorized. \( |\mathcal{D}\rangle_{\Lambda} = |\mathcal{D}\rangle_{\Lambda} |\tilde{\Phi}, \tilde{\Psi}\rangle \) with \( |\mathcal{D}\rangle_{\Lambda} \) given by (5.30).

Both \( |\Lambda\rangle_{ph} \) and \( |\mathcal{D}\rangle_{\Lambda} \) are \( (R, \tau, \lambda^I, \lambda^a, C) \)-independent. \( \{ |\Lambda\rangle_{ph} \forall \Lambda \} \) and \( \{ |\mathcal{D}\rangle_{\Lambda} \forall \Lambda \} \) compose two sets of complete orthogonal bases for \( \mathcal{H}_{ph} \). The unitary operator \( S_1 \) with

\[
S_1|\Lambda\rangle_{ph} = |\mathcal{D}\rangle_{\Lambda} , \quad S_1|\mathcal{D}\rangle_{\Lambda} = |\tilde{\Lambda}\rangle_{ph}
\] (5.130)

will make

\[
S_1^{-1}T_{R^*}(\tau; \lambda^I, \lambda^a, C)S_1 = \mathcal{W}_{R^*}(\tau; \lambda^I, \lambda^a, C) , \quad S_1^{-1}\mathcal{W}_{R^*}(\tau; \lambda^I, \lambda^a, C)S_1 = T_{R^*}^+(\tau; \lambda^I, \lambda^a, C) .
\] (5.131)

\( \tilde{\Lambda} \) is the configuration satisfying \( \mathcal{W}_{R^*}(\tilde{\Lambda}; \tau; \lambda^I, \lambda^a, C) = \mathcal{W}_{R^*}^*(\Lambda; \tau; \lambda^I, \lambda^a, C) \).

In (5.131),

\[
\mathcal{W}_{R^*}(\tau; \lambda^I, \lambda^a, C) = \int DU U \exp\{ i \int_{C} ds 2\tau tr[H^*_{\tilde{m}}(A_i\dot{x}^i + (\frac{\tau_2}{2\pi})^{\frac{3}{2}}\Phi_I\lambda^I + (\frac{\tau_2}{2\pi})^{\frac{3}{2}}\Psi_a\lambda^a)]\} U^{-1} ,
\] (5.132)
\[
\mathcal{T}_{R^*}(\tau; \lambda^I, \lambda^a, C) = \int DU\, U T_{R^*}(C) \exp\{i \int_C ds\, 2t r[H^*_m((\frac{T_2}{2\pi})^2 \Phi_I \lambda^I + (\frac{T_2}{2\pi})^2 \Psi_a \lambda^a)]\} U^{-1},
\]

(5.133)

\(S_1\) is independent of \((\lambda^I, \lambda^a, C)\), so the variation of \(\lambda^I\) and \(\lambda^a\) gives

\[
S_1 \delta \mathcal{W}_{R^*}(\tau; \lambda^I, \lambda^a, C) S_1^{-1} = \delta \mathcal{T}_{R^*}(\tau; \lambda^I, \lambda^a, C),
\]

(5.134)

where

\[
\delta \mathcal{W}_{R^*}(\tau; \lambda^I, \lambda^a, C) = \int DU\, U \{i \int_C ds\, 2(\frac{T_2}{2\pi})^2 t r[H^*_m(\Phi_I \delta \lambda^I + \Psi_a \delta \lambda^a)]\} W_{R^*}(\tau; \lambda^I, \lambda^a, C) U^{-1}
\]

(5.135)
is the S-dual of

\[
\delta \mathcal{T}_{R^*}(\tau; \lambda^I, \lambda^a, C) = \int DU\, U \{i \int_C ds\, 2(\frac{T_2}{2\pi})^2 t r[H^*_m(\Phi_I \delta \lambda^I + \Psi_a \delta \lambda^a)]\} T_{R^*}(\tau; \lambda^I, \lambda^a, C) U^{-1}.
\]

(5.136)

In the above discussion, bases of the Hilbert space are selected to be \(|\Lambda\rangle := |A_i, \Phi^I, \Psi^a\rangle\) \(\forall \Lambda\), but they can also be taken as \(|\Xi\rangle := |A_i, \Phi^I, \Pi^a\rangle\) \(\forall \Xi\) or \(|\Gamma\rangle := |A_i, \Pi^I, \Psi^a\rangle\) \(\forall \Gamma\). Suppose

\[
E(\hat{\Xi}) = \{L|\hat{\Xi}\rangle \mid \forall L \in \mathcal{L}\} \quad E(\hat{\Gamma}) = \{L|\hat{\Gamma}\rangle \mid \forall L \in \mathcal{L}\}
\]

(5.137)

are equivalent classes generated by the action of \(\mathcal{L}\) on \(|\Xi\rangle \forall \Xi\) and \(|\Gamma\rangle \forall \Gamma\), since

\[
|A_i, \Phi^I, \Pi^a\rangle = \int D\Psi_a \exp\{i \int d^3 x\, tr(\Pi^a \Psi_a)|A_i, \Phi^I, \Psi^a\}
\]

(5.138)

\[
|A_i, \Pi^I, \Psi^a\rangle = \int D\Phi_I \exp\{i \int d^3 x\, tr(\Pi^I \Phi_I)|A_i, \Phi^I, \Psi^a\}
\]

(5.139)

and \(tr(\Pi^a \Psi_a) = tr(\hat{\Pi}^a \hat{\Psi}_a), tr(\Pi^I \Phi_I) = tr(\hat{\Pi}^I \hat{\Phi}_I)\) in \(E(\hat{\Xi})\) and \(E(\hat{\Gamma})\), the action of \(S_1\) on \(|\Xi\rangle_{ph}\) and \(|\Gamma\rangle_{ph}\) becomes

\[
S_1|\Xi\rangle_{ph} = |D\rangle_{\Xi} = \sum_{\hat{\Xi}} e^{i h(\hat{\Xi}'\Xi)}|D\rangle_{(\hat{\Xi}', \Xi)}, \quad S_1|\Gamma\rangle_{ph} = |D\rangle_{\Gamma} = \sum_{\hat{\Gamma}} e^{i h(\hat{\Gamma}'\Gamma)}|D\rangle_{(\hat{\Gamma}', \Gamma)},
\]

(5.140)

where \(h(\hat{\Xi}', \Xi) = h(\hat{\Gamma}', \Gamma) = h(\hat{\Lambda}', \Lambda), |D\rangle_{(\hat{\Xi}', \Xi)}\) and \(|D\rangle_{(\hat{\Gamma}', \Gamma)}\) are defined in the same way as \(|D\rangle_{(\hat{\Lambda}', \Lambda)}\) with \(\Psi^a\) and \(\Phi^I\) replaced by \(\Pi^a\) and \(\Pi^I\) respectively.

\[
S_1 \mathcal{W}_{R^*}^\Xi(\tau; \lambda^I, \lambda^a, C) S_1^{-1} = \mathcal{T}_{R^*}^\Xi(\tau; \lambda^I, \lambda^a, C), \quad S_1 \mathcal{W}_{R^*}^\Gamma(\tau; \lambda^I, \lambda^a, C) S_1^{-1} = \mathcal{T}_{R^*}^\Gamma(\tau; \lambda^I, \lambda^a, C),
\]
\[ S_1 T^\Xi_R (\tau; \lambda^I, \lambda^a, C) S_1^{-1} = W^\Xi_R (\tau; \lambda^I, \lambda^a, C), \quad S_1 T^\Gamma_R (\tau; \lambda^I, \lambda^a, C) S_1^{-1} = W^\Gamma_R (\tau; \lambda^I, \lambda^a, C), \]

where \( W^\Xi, W^\Gamma, T^\Xi \) and \( T^\Gamma \) are Wilson and t’ Hooft operators constructed from \( \Xi \) and \( \Gamma \).

In addition to \( S_1 \), another unitary physical operator \( S_2 \) inducing a rescaling can also be introduced:

\[
\begin{align*}
S_2 A_i S_2^{-1} &= A_i \\
S_2 \Phi^I S_2^{-1} &= \frac{\Phi^I}{|\tau|} \\
S_2 \Psi^a S_2^{-1} &= \frac{e^{i\theta} \Psi^a}{|\tau|} \\
S_2 \Pi_i S_2^{-1} &= \Pi_i \\
S_2 \Pi^I S_2^{-1} &= |\tau| \Pi^I \\
S_2 \Pi^a S_2^{-1} &= e^{-i\theta} |\tau| \Pi^a,
\end{align*}
\]

where \( e^{i\theta} = (\tau/\bar{\tau})^3 \).

\[
\begin{align*}
S_2 W^R_R (\tau; \lambda^I, \lambda^a, C) S_2^{-1} &= W^R_R (\frac{1}{\tau}; \lambda^I, e^{i\theta} \lambda^a, C), \\
S_2 T^R_R (\tau; \lambda^I, \lambda^a, C) S_2^{-1} &= T^R_R (\frac{1}{\tau}; \lambda^I, e^{i\theta} \lambda^a, C).
\end{align*}
\]

S-transformation operator is taken to be \( S = S_1 S_2 \) with

\[
\begin{align*}
S W^R_R (\tau; \lambda^I, \lambda^a, C) S^{-1} &= T^R_R (\frac{1}{\tau}; \lambda^I, e^{i\theta} \lambda^a, C), \\
S T^R_R (\tau; \lambda^I, \lambda^a, C) S^{-1} &= W^R_R (\frac{1}{\tau}; \lambda^I, e^{i\theta} \lambda^a, C),
\end{align*}
\]

which is the expected transformation rule for supersymmetric loop operators.

5.8. Supersymmetry transformation of loop operators

In canonical quantization formulation, the 32 supercharges of \( \mathcal{N} = 4 \) SYM theory are

\[
\begin{align*}
Q^a_\alpha &= \int d^3 x \ J^a_{0\alpha} (x) \\
\bar{Q}^a_\dot{\alpha} &= \int d^3 x \ \bar{J}^a_{0\dot{\alpha}} (x) \\
S^{a\dot{a}} &= \int d^3 x \ x^\mu \sigma^{a\dot{a}} J^a_{0\alpha} (x) \\
\bar{S}^{\dot{a}a} &= \int d^3 x \ x^\mu \sigma^{\dot{a}a} \bar{J}^a_{0\dot{\alpha}} (x),
\end{align*}
\]

where the supercurrent density is

\[
\begin{align*}
J^a_{0\beta} (\tau) &= tr \left\{ \frac{2\pi}{\tau_2} \Pi_{\alpha\beta} \Pi^{\alpha\alpha} + \frac{\tau}{\tau_2} B_{\alpha\beta} \Pi^{\alpha\alpha} + \epsilon_{\alpha\beta} \Pi^{ba} [X_{bc}, X^{ac}] + \frac{\tau_2}{2\pi} \psi_{ba} \sigma^{0a\dot{a}} \sigma^{\dot{a}b} D_i X^{ab} \right\} \\
\bar{J}_{0a\dot{\beta}} (\tau) &= tr \left\{ \psi^a_{\alpha} \sigma^{0b} \Pi_{\alpha\beta} + \frac{\tau}{2\pi} \psi^a_{\alpha} \sigma^{0b} B_{\alpha\beta} + \frac{2\pi}{\tau} \Pi^{b\dot{a}} \sigma^{\dot{a}b} \Pi_{ab} - \frac{\tau_2}{2\pi} \epsilon_{\dot{\alpha}\dot{b}} \psi_{ba} \sigma^{0a\dot{a}} [X_{bc}, X^{ac}] \\
&\quad + \Pi^{b\dot{a}} \sigma^{\dot{a}b} D_i X_{ab} \right\}.
\end{align*}
\]
The actions of $W_{R^*}^A(C)$ and $T_{R^*}(C)$ on supercharges both bring a loop integration:

$$W_{R^*}^A(C)\theta^a_\beta Q^a_\beta(\tau)W_{R^*}^{A-1}(C) = \theta^a_\beta Q^a_\beta(\tau) + \frac{4\pi}{\tau_2} \int_C ds \, tr(H^*_m \Pi^{a\alpha}) \theta^a_\beta \bar{x}_{\alpha\beta}$$

$$W_{R^*}^A(C)\bar{\theta}^{a\bar{\beta}} \bar{Q}_{a\bar{\beta}}(\tau)W_{R^*}^{A-1}(C) = \bar{\theta}^{a\bar{\beta}} \bar{Q}_{a\bar{\beta}}(\tau) + 2 \int_C ds \, tr(H^*_m \Psi^a_\alpha) \bar{\theta}^{a\bar{\beta}} \sigma^{0\beta}_\bar{\beta} \bar{x}_{\alpha\beta}$$

$$T_{R^*}(C)\theta^a_\beta Q^a_\beta(\tau)T_{R^*}^{-1}(C) = \theta^a_\beta Q^a_\beta(\tau) - \frac{4\pi \tau}{\tau_2} \int_C ds \, tr(H^*_m \Pi^{a\alpha}) \theta^a_\beta \bar{x}_{\alpha\beta}$$

$$T_{R^*}(C)\bar{\theta}^{a\bar{\beta}} \bar{Q}_{a\bar{\beta}}(\tau)T_{R^*}^{-1}(C) = \bar{\theta}^{a\bar{\beta}} \bar{Q}_{a\bar{\beta}}(\tau) - 2\tau \int_C ds \, tr(H^*_m \Psi^a_\alpha) \bar{\theta}^{a\bar{\beta}} \sigma^{0\beta}_\bar{\beta} \bar{x}_{\alpha\beta} \ . \ (5.150)$$

Consider the supersymmetry variations of the loop operators and for simplicity, suppose $\lambda^a = 0$, then

$$W_{R^*}(\tau; \lambda^{ab}, C) = W_{R^*}^A(C)W_{R^*}^\Phi(\tau; \lambda^{ab}, C) = \exp\{i \int_C ds \, 2tr[H^*_m (A_{\alpha\beta} \bar{x}^{\alpha\beta} + \frac{\tau_2}{2\pi}) \frac{1}{2} \Phi_{ab}\lambda^{ab}]\}$$

$$W_{R^*}(\tau; \lambda^{ab}, C) = \int DU \, UW_{R^*}(\tau; \lambda^{ab}, C)U^{-1} \ , \ (5.151)$$

$$T_{R^*}(\tau; \lambda^{ab}, C) = T_{R^*}^A(C)T_{R^*}^\Phi(\tau; \lambda^{ab}, C) = T_{R^*}(C) \exp\{i \int_C ds \, 2tr[H^*_m (\frac{\tau_2}{2\pi}) \frac{1}{2} \Phi_{ab}\lambda^{ab}]\}$$

$$T_{R^*}(\tau; \lambda^{ab}, C) = \int DU \, UT_{R^*}(\tau; \lambda^{ab}, C)U^{-1} \ . \ (5.152)$$

Direct calculation gives

$$[\theta^a_\beta Q^a_\beta(\tau), T_{R^*}(C)] = \left[\frac{4\pi \tau}{\tau_2} \int_C ds \, tr(H^*_m \Pi^{a\alpha}) \theta^a_\beta \bar{x}_{\alpha\beta} \right] T_{R^*}(C)$$

$$[\bar{\theta}^{a\bar{\beta}} \bar{Q}_{a\bar{\beta}}(\tau), T_{R^*}(C)] = \left[2\bar{\tau} \int_C ds \, tr(H^*_m \Psi^a_\alpha) \bar{\theta}^{a\bar{\beta}} \sigma^{0\beta}_\bar{\beta} \bar{x}_{\alpha\beta} \right] T_{R^*}(C) \ (5.153)$$

$$[\theta^a_\beta Q^a_\beta(\tau), W_{R^*}^A(C)] = -\left[-\frac{4\pi}{\tau_2} \int_C ds \, tr(H^*_m \Pi^{a\alpha}) \theta^a_\beta \bar{x}_{\alpha\beta} \right] W_{R^*}^A(C)$$

$$[\bar{\theta}^{a\bar{\beta}} \bar{Q}_{a\bar{\beta}}(\tau), W_{R^*}^A(C)] = -\left[2 \int_C ds \, tr(H^*_m \Psi^a_\alpha) \bar{\theta}^{a\bar{\beta}} \sigma^{0\beta}_\bar{\beta} \bar{x}_{\alpha\beta} \right] W_{R^*}^A(C) \ (5.154)$$

$$[\theta^a_\beta Q^a_\beta(\tau), W_{R^*}^\Phi(\tau; \lambda^{ab}, C)] = -\left[2(\frac{\tau_2}{2\pi})^{\frac{1}{2}} \int_C ds \, tr(H^*_m \Psi^a_\alpha) \theta^{b\alpha} \lambda_{ab} \right] W_{R^*}^\Phi(\tau; \lambda^{ab}, C)$$

$$[\bar{\theta}^{a\bar{\beta}} \bar{Q}_{a\bar{\beta}}(\tau), W_{R^*}^\Phi(\tau; \lambda^{ab}, C)] = -\left[2(\frac{\tau_2}{2\pi})^{\frac{1}{2}} \int_C ds \, tr(H^*_m \Pi^{a\alpha}) \sigma^{0\beta}_{aa} \bar{\theta}^{a\bar{\beta}} \lambda_{ab} \right] W_{R^*}^\Phi(\tau; \lambda^{ab}, C) \ . \ (5.155)$$
So the supersymmetry variations of $W_{R^*}(\tau; \lambda^{ab}, C)$ and $T_{R^*}(\tau; \lambda^{ab}, C)$ are

$$[\theta^\beta Q^\alpha_\beta(\tau) + \bar{\theta}^{\dot{a}\dot{\beta}} Q_{\dot{a}\dot{\beta}}(\tau), W_{R^*(\tau; \lambda^{ab}, C)}]$$

$$= \int DU U[ \int_C ds \left\{ H^*_m[\Pi^{\alpha\dot{a}}(-\frac{4\pi}{T^2})\theta^\beta \hat{x}_a^\alpha \beta - 2(\frac{T_2}{2\pi})\tilde{\sigma}^a^{0\alpha} \hat{\theta}^a \lambda^{ab})
+ \Psi^0_a(-2\sigma^{0\beta} \bar{\theta}^{\dot{a}\dot{\beta}} \hat{x}_a^\alpha \beta - 2(\frac{T_2}{2\pi})\tilde{\sigma}^a \theta_{ba} \lambda^{ab})\right\}]W_{R^*(\tau; \lambda^{ab}, C)}U^{-1}, \quad (5.156)$$

$$[\bar{\theta}^\beta \bar{Q}^\alpha_\beta(\tau) + \theta^{\dot{a}\dot{\beta}} \bar{Q}_{\dot{a}\dot{\beta}}(\tau), T_{R^*(\tau; \lambda^{ab}, C)}]$$

$$= \int DU U[ \int_C ds \left\{ H^*_m[\Pi^{\alpha\dot{a}}\bar{\theta}^\beta \hat{x}_a^\alpha \beta - 2(\frac{T_2}{2\pi})\tilde{\sigma}^a^{0\alpha} \bar{\theta}^a \lambda^{ab})
+ \bar{\Psi}^0_a(2\bar{\theta}^{\dot{a}\dot{\beta}} \sigma^{0\beta} \hat{x}_a^\alpha \beta - 2(\frac{T_2}{2\pi})\tilde{\sigma}^a \theta_{ba} \lambda^{ab})\right\}]T_{R^*(\tau; \lambda^{ab}, C)}U^{-1}. \quad (5.157)$$

Acted on by the duality transformation operator $S = S_1 S_2$, (5.156) becomes

$$[S \theta^\beta Q^\alpha_\beta(\tau)S^{-1} + S \bar{\theta}^{\dot{a}\dot{\beta}} \bar{Q}_{\dot{a}\dot{\beta}}(\tau)S^{-1}, SW_{R^*(\tau; \lambda^{ab}, C)}S^{-1}]$$

$$= S_1 \int DU U[ \int_C ds \left\{ H^*_m[\Pi^{\alpha\dot{a}}\frac{e^{-i\theta}}{\tau}(\frac{4\pi}{T^2})\theta^\beta \hat{x}_a^\alpha \beta - 2(\frac{T_2}{2\pi})\tilde{\sigma}^a \bar{\theta}^a \lambda^{ab})
+ \frac{e^{i\theta}}{\tau\tau}(\frac{2\sigma^{0\beta} \bar{\theta}^{\dot{a}\dot{\beta}} \hat{x}_a^\alpha \beta - 2(\frac{T_2}{2\pi})\tilde{\sigma}^a \theta_{ba} \lambda^{ab})\right\}]W_{R^*(\tau; \lambda^{ab}, C)}U^{-1}S_1^{-1}$$

$$= S_1 \int DU U[ \int_C ds \left\{ H^*_m[\Pi^{\alpha\dot{a}}\delta \lambda'_a + \bar{\Psi}^0_a \delta \lambda_{\alpha}]\right\}]W_{R^*(\tau; \lambda^{ab}, C)}U^{-1}S_1^{-1}, \quad (5.158)$$

where

$$\delta \lambda'_a = e^{-i\theta} \left\{ -\frac{4\pi}{T^2} \hat{x}_a^\alpha \beta - 2(\frac{T_2}{2\pi})\tilde{\sigma}^a \bar{\theta}^a \lambda^{ab} \right\}, \quad (5.159)$$

$$\delta \lambda _{\alpha} = \frac{e^{i\theta}}{\tau\tau}\left\{ -2\sigma^{0\beta} \bar{\theta}^{\dot{a}\dot{\beta}} \hat{x}_a^\alpha \beta - 2(\frac{T_2}{2\pi})\tilde{\sigma}^a \theta_{ba} \lambda^{ab} \right\}. \quad (5.160)$$

On the other hand, with $\tau$ replaced by $-1/\tau$, and $\theta^\beta, \bar{\theta}^{\dot{a}\dot{\beta}}$ replaced by $e^{i\theta} \theta^\beta, e^{-i\theta} \bar{\theta}^{\dot{a}\dot{\beta}}$, (5.157) becomes

$$[e^{i\theta} \theta^\beta Q^\alpha_\beta(-\frac{1}{\tau}) + e^{-i\theta} \bar{\theta}^{\dot{a}\dot{\beta}} \bar{Q}_{\dot{a}\dot{\beta}}(-\frac{1}{\tau}), T_{R^*(-\frac{1}{\tau}; \lambda^{ab}, C)}]$$

$$= \int DU U[ \int_C ds \left\{ H^*_m[\Pi^{\alpha\dot{a}}\delta \lambda'_a + \bar{\Psi}^0_a \delta \lambda_{\alpha}]\right\}]T_{R^*(-\frac{1}{\tau}; \lambda^{ab}, C)}U^{-1}. \quad (5.161)$$
With (5.158) compared with (5.161), from (5.134)-(5.136), for

\[ \Delta_{a}^{\alpha}(\tau) := SQ_{a}^{\alpha}(\tau)S^{-1} - e^{\frac{ia}{2}} Q_{a}^{\alpha}(-\frac{1}{\tau}) , \quad \bar{\Delta}_{a\beta}(\tau) := S\bar{Q}_{a\beta}(\tau)S^{-1} - e^{-\frac{ia}{2}} \bar{Q}_{a\beta}(-\frac{1}{\tau}) , \]  

(5.162)

we have

\[ [\theta_{a}^{\beta}\Delta_{a}^{\alpha}(\tau) + \bar{\theta}^{a\beta}\bar{\Delta}_{a\beta}(\tau), T_{R^{\ast}}(-\frac{1}{\tau}; \lambda^{ab}, C)] = 0 \]  

or equivalently,

\[ [\theta_{a}^{\beta}S^{-1}\Delta_{a}^{\alpha}(\tau)S + \bar{\theta}^{a\beta}S^{-1}\bar{\Delta}_{a\beta}(\tau)S, W_{R^{\ast}}(\tau; \lambda^{ab}, C)] = 0 \]  

(5.163)

(5.164)

for the arbitrary \( \theta_{a}^{\beta}, \bar{\theta}^{a\beta}, \lambda^{ab}, C \). This is consistent with the S-duality transformation rule

\[ SQ_{a}^{\alpha}(\tau)S^{-1} = e^{\frac{ia}{2}} Q_{a}^{\alpha}(-\frac{1}{\tau}) , \quad S\bar{Q}_{a\beta}(\tau)S^{-1} = e^{-\frac{ia}{2}} \bar{Q}_{a\beta}(-\frac{1}{\tau}) \]  

(5.165)

for supercharges [13, 16].

When \( \theta_{a}^{\beta} \tilde{x}_{a\beta} + (\frac{\tau_{2}}{2\pi})^{1} \delta_{a\alpha} \bar{\theta}^{ab} \lambda_{ab} = 0, \delta \lambda_{a}^{\prime} = \delta \lambda_{a}^{0} = 0, \)

\[ [\theta_{a}^{\beta}Q_{a}^{\alpha}(\tau) + \bar{\theta}^{a\beta}\bar{Q}_{a\beta}(\tau), W_{R^{\ast}}(\tau; \lambda^{ab}, C)] = 0 \]  

(5.166)

\[ [e^{\frac{ia}{2}} \theta_{a}^{\beta}Q_{a}^{\alpha}(-\frac{1}{\tau}) + e^{-\frac{ia}{2}} \bar{\theta}^{a\beta}\bar{Q}_{a\beta}(-\frac{1}{\tau}), T_{R^{\ast}}(-\frac{1}{\tau}; \lambda^{ab}, C)] = 0 \]  

(5.167)

So the supersymmetries preserved by \( W_{R^{\ast}}(\tau; \lambda^{ab}, C) \) in theory with the coupling constant \( \tau \) and those preserved by \( T_{R^{\ast}}(-1/\tau; \lambda^{ab}, C) \) in theory with the coupling constant \(-1/\tau \) are related by a \( U(1)_{Y} \) phase, as is already shown in path integral formalism [17].

(5.163) or (5.164) is still not enough to guarantee (5.165). In fact, only the first three terms of \( J, \bar{J} \) in (5.149) play a role in our calculation. To prove (5.165), we should further check the validity of

\[ [\theta_{a}^{\beta}\Delta_{a}^{\alpha}(\tau) + \bar{\theta}^{a\beta}\bar{\Delta}_{a\beta}(\tau), T_{R^{\ast}}(-\frac{1}{\tau}; \lambda^{ab}, \lambda^{a\alpha}, C)] = 0 \]  

(5.168)

or

\[ [\theta_{a}^{\beta}S^{-1}\Delta_{a}^{\alpha}(\tau)S + \bar{\theta}^{a\beta}S^{-1}\bar{\Delta}_{a\beta}(\tau)S, W_{R^{\ast}}(\tau; \lambda^{ab}, \lambda^{a\alpha}, C)] = 0 \]  

(5.169)

with the fermionic couplings also turned on. The Wilson loop of \( \Psi \) is

\[ W_{R_{m}^{\Psi}}^{\Psi}(\tau; \lambda^{a}, C) = \exp \{ i \left( \frac{\tau_{2}}{2\pi} \right)^{\frac{1}{2}} \int_{C} ds \, 2tr(H_{m}^{a}\Psi_{a})\lambda^{a} \} \]  

(5.170)
with

\[
[\theta^a Q^a_{\beta}(\tau), W^\Psi_{\beta} (\tau; \lambda^a, C)] = \left( \frac{\tau_2}{2\pi} \right)^{\frac{1}{2}} \oint C ds 2 \text{tr} \left[ H^a_m (\frac{2\pi}{\tau_2} \Pi_{\alpha \beta} \lambda^{a \alpha} \theta^a_{\beta} + \frac{\tau}{\tau_2} B_{\alpha \beta} \lambda^{a \alpha} \theta^a_{\beta} + \lambda^{b a} \theta^a_{\beta} \epsilon_{\alpha \beta} [X_{bc}, X^{ac}]) \right] W^\Psi_{\beta} (\tau; \lambda^a, C)
\]

\[
[\bar{\theta}^0 \bar{Q}^0 (\tau), W^\Psi (\tau; \lambda^a, C)] = \left( \frac{\tau_2}{2\pi} \right)^{\frac{1}{2}} \oint C ds 2 \text{tr} \left[ H^a_m (\lambda^{b a} \bar{\theta}^i \sigma_{\alpha \beta} D_i X_{ab} + \frac{2\pi}{\tau_2} \lambda^{b b} \bar{\theta}^i \sigma_{\alpha \beta} \sigma_0 \Pi_{ab}) \right] W^\Psi (\tau; \lambda^a, C) .
\]

(5.171)

The supersymmetry variation of \( W^\Psi_{\beta} (\tau; \lambda^a, C) \) contains the conjugate momentum \( \Pi_{\alpha \beta} \) and \( \Pi_{ab} \).

For completeness, we should consider the Wilson loops constructed from both \( A_i, \Phi^I, \Psi^a \) and \( \Pi^i, \Pi_I, \Pi_a \). For such operators, \( \Lambda \rangle \) is not the eigenstate, so it is not straightforward to determine their transformation under \( S_1 \) which is defined in (5.130) through the action on \( \Lambda \rangle_{ph} \).

6. CONCLUSION AND DISCUSSION

In this paper, we studied the gauge invariant t’ Hooft operator in canonical formalism. The generic commutation relations for t’ Hooft and Wilson operators in the arbitrary representations are obtained. In particular, Wilson and t’ Hooft operators labeled by the representation and the dual representation of the gauge group commute. Under the T-transformation, t’ Hooft operator \( T_R^R(C) \) becomes the Wilson-t Hooft operator \( [T^W]_{R}^R(C) \) with \( W \) and \( T \) both in representation \( R \).

For S-transformation of the loop operators, it is showed that the spectrum of the t’ Hooft operator labeled by the dual representation of the gauge group is the same as the spectrum of the Wilson operator labeled by the same representation. This gives the possibility to construct a unitary operator \( S \) making the two kinds of loop operators transformed into each other. S-transformation of the theory is realized by the operator \( S \). However, the spectrum of the loop operators are highly degenerate, so the S-transformation rule for loop operators cannot uniquely determine \( S \). To fix \( S \), we need to get the coefficient \( e^{ih(\Lambda', \Lambda)} \) in (5.31) and \( e^{ih(\Lambda', \Lambda)} \) in (5.128), which are known only in \( U(1) \) case.

For \( N = 4 \) SYM theory, with \( S \) given, if the supercharges transform with a \( U(1)_Y \) phase, the theory will be S-duality invariant. We compute the supersymmetry variations of the loop operators with the fermionic couplings turned off and get the evidence for the \( U(1)_Y \) transformation of the supercharges. Especially, supersymmetries preserved by t’ Hooft operators labeled by the dual representation in theory with the coupling constant \( \tau \) and those preserved by Wilson operators labeled by the same representation in theory with the coupling constant
$-1/\tau$ are related by a $U(1)_Y$ phase. Nevertheless, to prove the $U(1)_Y$ transformation rule of the supercharges, it is also necessary to consider the action of $S$ on the supersymmetry variations of the loop operators with the fermionic couplings turned on. This requires the further knowledge of $S$, for example, the explicit form of the coefficient $e^{ih(\hat{\Lambda}', \Lambda)}$ in (5.128).

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