Influence of interactions on the anomalous quantum Hall effect

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(Dated: May 28, 2019)

We investigate the influence of interactions on the anomalous quantum Hall conductivity within the framework of the particular tight-binding models of the 2 + 1 D topological insulator and the 3 + 1 D Weyl semimetal. Several types of interactions are considered including the contact four-fermion interactions, Yukawa and Coulomb interactions. It is shown that when the considered interactions are taken into account in the one-loop approximation, the Hall conductivity for the insulator is the topological invariant in momentum space composed of the complete two-point Green function of the interacting model. It remains robust to the smooth modification of the system. For the Weyl semimetal the Hall conductivity is given by the similar expression composed of the two-point interacting Green function. It inherits the algebraic structure of the corresponding topological invariant of an insulator, but may vary continuously under smooth modifications of the system. We also demonstrate that the interactions may lead to the topological phase transitions accompanied by the change of Hall conductivity.

PACS numbers: 73.43.-f

I. INTRODUCTION

The topological invariant responsible for the Anomalous Quantum Hall effect (AQHE) in the ideal two-dimensional non-interacting condensed matter systems has been proposed in [1]. It is called now the TKNN invariant and is given by the integral of Berry curvature over the occupied electronic states [2, 3, 5]. An extension of this approach to the three dimensional topological insulators was considered, in particular, in [6]. It is widely believed, that the introduction of weak interactions does not affect Hall conductivity. Therefore, it is important to express it through the quantities defined within the interacting theory. The two-point Green function is such a quantity. The expression of the AQHE conductivity for the 2 + 1 D systems through the topological invariant composed of the Green functions has been proposed in [3, 8] (see also Chapter 21.2.1 in [9]). The extension of this construction to various three-dimensional systems has been given in [10], where, in particular, the description of the AQHE in topological Weyl semimetals [11–16] was given. We notice here also the discussion of the similar topological invariants in [17, 18]. It is widely believed that the AQHE conductivity is given by the expressions of [7–10] expressed through the two-point Green functions of interacting systems. However, there is still no direct proof that the other interaction contributions to the AQHE are absent.

In the recent paper written with the participation of one of the present authors the construction of [10] was further extended to the non-homogeneous systems [19]. This construction allows to give a relatively simple alternative proof that weak disorder does not affect the AQHE conductivity. This complements the previous consideration of the role of disorder in the QHE (see, for example, [2, 3, 20–22]). In the same paper the indications were presented that the AQHE conductivity is robust to the introduction of weak electromagnetic interactions, i.e. that in the presence of interactions the AQHE conductivity is given by its expression through the two-point Green functions presented in [10]. The absence of corrections (due to weak Coulomb interactions) to the QHE in the disordered ferromagnetic metal was shown in [23]. The absence of such corrections to the QHE existing in the presence of magnetic field has been discussed long time ago (see, for example, [24, 25] and references therein). It is also worth mentioning, that the strong interactions are able to lead the fermionic system in the presence of external magnetic field to the fractional QHE phases [3]. The similar phenomenon for the AQHE existing without magnetic field has been discussed as well (see [4] and references therein).

In [26] the corrections due to interactions to the AQHE in the Weyl semimetals were considered. The authors of [26] restricted themselves to the simple Hubbard interactions in the particular tight-binding model (the same model will also be used here). The interaction corrections in Weyl semimetals were also discussed in [27, 28] without any relation to the AQHE. Interaction effects in the 2D topological insulators were discussed, for example, in [29]

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(see also references therein). Recently the effects of electron-electron interactions were investigated in graphene-like systems, and the renormalization of Fermi velocity was studied taking into account Coulomb interactions \cite{30}. The corrections due to the simple on-site interactions to the AQHE in the 2D insulators were discussed, for example, in \cite{31, 32}. However, to the best of our knowledge, the corrections to Hall conductivity due to the most relevant corrections due to the simple on-site interactions to the AQHE in the 2D insulators were discussed, for example, in \cite{31, 31, 38–40}. Those models give qualitative description of the really existing systems. We will show here, that both the four-fermion interactions and the interactions due to exchange by scalar bosons (including the Coulomb interactions) do not give any corrections to the AQHE conductivities of \cite{10} expressed through the (interacting) Green functions, at least in the one-loop approximation. We also have found, that the sufficiently strong interactions may lead to the topological phase transitions.

On the technical side we use the version of Wigner-Weyl technique \cite{41–44} adapted in \cite{10} to the lattice models of solid state physics combined with the ordinary perturbation theory (see also \cite{45}). We expect that the obtained result may be extended further to the consideration of the other non-dissipative transport phenomena (for the review of the latter see \cite{46} and references therein). The main advantage of the Wigner-Weyl formalism is that it allows to express the Hall conductivity directly through the interacting two-point Green function. This expression is the topological invariant for the topological insulators and as such it is robust to the weak modification of interactions. For the Weyl semimetals, the expression for the Hall conductivity obtained in this way inherits the algebraic structure of the expression for the topological insulators, is expressed through the two-point interacting Green function, and does not contain the higher order ones.

The paper is organized as follows. In Sect. \textbf{II} we start from the consideration of the tight-binding model of the 2D topological insulator with the four-fermion interactions between electrons. In Sect. \textbf{III} we consider the 3 + 1 D Weyl semimetals in the presence of the four-fermion interactions. In Sect. \textbf{IV} we discuss the 2 + 1 D insulators in \textbf{V} we discuss the 3 + 1 D Weyl semimetals in the presence of Yukawa and Coulomb interactions. In Sect. \textbf{VI} we discuss the 3 + 1 D Weyl semimetals in the presence of Coulomb interactions. In Sect. \textbf{VII} we end with the conclusions.

\section{AQHE in the 2 + 1 D Tight-Binding Model with the Four Fermion Interactions}

\subsection{A. Neglecting Interactions}

Let us start from the non-interacting 2 + 1 D lattice model with the fermion Green function of the form $G^{-1} = \omega - H(p)$, where $H$ is the one-particle Hamiltonian. This is the Green function in Euclidean momentum space (i.e. the dynamics in imaginary time is considered). In \cite{10} it has been shown that the term linear in the field strength in electric current may be written as

$$j^{(1)k}(x) = \frac{1}{4\pi} \epsilon^{ijk} M_{ikj}(x), \quad (1)$$

Here $M_{ij}$ is the Euclidean field strength $A_{ij} = \partial_x A_j - \partial_y A_i$. In the present paper we define the components $A_k = A^k$ for $k = 1, 2$ as equal to the space components of real external electromagnetic potential $A$ in Minkowski space - time. Correspondingly, $A_3 = -A^3 = -iA^0$, where $A^0$ is the external electric potential. In order to make the present paper self-contained we give the derivation of Eq. \textbf{14} in Appendix A. The generalization to the case of the 3 + 1 D models is straightforward. It is worth mentioning, that the derivation of Eq. \textbf{11} requires that the field $A$ does not vary fast, i.e. its variation on the distance of the order of lattice spacing may be neglected.

We suppose, that the fermions are gapped and the Green function $G(p)$ depends on the three-vector $p = (p_1, p_2, p_3)$ of Euclidean momentum. In order to obtain expression for the Hall current let us introduce into Eq. \textbf{11} the external electric field $E = (E_1, E_2)$ as $A_{3k} = -iE_k$ (the third component of vector corresponds to imaginary time). This results in the following expression for the Hall current

$$j_{Hall}^{k} = \frac{1}{2\pi} N \epsilon^{ki} E_i, \quad (2)$$
where the topological invariant denoted by \( \mathcal{N} \) is to be calculated for the original system with vanishing background gauge field:

\[
\mathcal{N} = -\frac{1}{24\pi^2} \text{Tr} \int G^{-1} dG \wedge dG^{-1} \wedge dG .
\]  

Eq. (3) defines the topological invariant (this is proved, in particular, in Appendix B of [10]). Recall, that for the given lattice model \( \mathcal{G} \) is the Green function in momentum space, i.e. the Fourier transformation of the two point Green function in coordinate space (it is assumed that the original model without external gauge field is translation invariant).

Let us consider the example with the Green function of the form \( G^{-1} = i\omega - H(p) \), where the Hamiltonian is

\[
H = \sin p_1 \sigma^2 - \sin p_2 \sigma^1 - (m + \sum_{i=1,2} \cos p_i) \sigma^3
\]  

For \( m \in (-2,0) \) we have \( \mathcal{N} = 1 \) while \( \mathcal{N} = -1 \) for \( m \in (-4,-2) \) and \( \mathcal{N} = 0 \) for \( m \in (-\infty,-4) \cup (0,\infty) \). The calculation is given in Appendix B.

### B. The four - fermion interaction.

We add to the model discussed above the four - fermion interaction term, which gives the Euclidean action

\[
S_\lambda = \int d\tau \sum_{x,x'} \bar{\psi}(x) \left( i(i\partial_\tau - A_3(-i\tau,x))\delta_{x,x'} - iD_{x,x'} \right) \psi_x 
+ \int d\tau \sum_x \frac{\lambda}{2} \left( \bar{\psi}(\tau,x)\psi(\tau,x) \right)^2
\]  

where \( \bar{\psi} \) is the Hermitian conjugation of \( \psi \), i.e. \( \bar{\psi} = \psi^\dagger \), \( \tau \) is imaginary time (\( t = -i\tau \)), and

\[
D_{x,x'} = \left\{ \begin{array}{ll}
\frac{i}{2} \sum_{i=1,2} [(1 + \sigma^i)\delta_{x+e_i,x'} e^{iA_{x+e_i,x'}} ]& + (1 - \sigma^i)\delta_{x-e_i,x'} e^{iA_{x-e_i,x'}} \sigma_3 + i(m + 2)\delta_{x,x'} \sigma_3
\end{array} \right.
\]

\( A_3 \) is expressed through the external electric potential \( \phi(t,x) \) as \( A_3 = -i\phi(-i\tau,x) \). Space coordinates \( x \) are discrete while the values of \( \tau \) are continuous. We denote the Euclidean three - momentum by \( p = (\omega,\mathbf{p}) \). In 3D Euclidean coordinate space, a point is denoted by \( x = (\tau,\mathbf{x}) \), and the Euclidean 3 - potential is \( A = (-i\phi,\mathbf{A}) \).

Therefore, the partition function and the two-point Green function are given by

\[
Z_\lambda = \int D\bar{\psi} D\psi e^{S_\lambda}
\]

and

\[
G = G_\lambda(x_1,x_2) = \frac{1}{Z_\lambda} \int D\bar{\psi} D\psi \bar{\psi}(x_1)\psi(x_2) e^{S_\lambda},
\]

Let us define

\[
Q(p,x) = Q(\omega,\mathbf{p},\tau,\mathbf{x}) = i(\omega - \phi(-i\tau,\mathbf{x})) - H(\mathbf{p} - \mathbf{A}(i\tau,\mathbf{x}))
\]

When the four - fermion interaction is turned off, i.e. \( \lambda = 0 \), the free Green function in momentum space \( \tilde{G}_0(p_1,p_2) \) is defined as

\[
\tilde{G}_0(p_2,p_1) = \frac{1}{Z_0} \int \frac{D\bar{\psi} D\psi}{(2\pi)^3} \bar{\psi}(p_2)\psi(p_1) e^{\frac{i\phi_{\mathbf{p},\mathbf{x}}}{2\pi} \bar{\psi}(p)Q(p,i\partial_p)\psi(p)},
\]
which satisfies equation
\[ Q(p_1, i\partial_{p_1})\tilde{G}_0(p_1, p_2) = \delta^3(p_1 - p_2). \]  
(9)
The free Green function in coordinate space \( G_0(x_1, x_2) \) is related to \( \tilde{G}_0 \) by the Fourier transformation
\[ G_0(x_1, x_2) = \int \frac{d^3p_1}{(2\pi)^{3/2}} \int \frac{d^3p_2}{(2\pi)^{3/2}} e^{ip_1x_1} i\tilde{G}_0(p_1, p_2)e^{-ip_2x_2} \]  
(10)
Contrary to the definition of Eq. (7), all components of variable \( x_i \) in the above equation may take continuous values. Similar to Eq. (9), \( G_0(x_1, x_2) \) satisfies
\[ Q(-i\partial_{x_1}, x_1)G_0(x_1, x_2) \bigg|_{x_1=(\tau_1, x_1)} = \delta(\tau_1 - \tau_2)\delta_{x_1,x_2}, \]  
(11)
where \( x_1 \) and \( x_2 \) take discrete lattice values. This may be proved directly:
\[ Q(-i\partial_{x_1}, x_1)G_0(x_1, x_2) = \int \frac{d^3p_1d^3p_2}{(2\pi)^3} Q(-i\partial_{x_1}, x_1)e^{ip_1x_1}\tilde{G}_0(p_1, p_2)e^{-ip_2x_2} \]
\[ = \int \frac{d^3p_1d^3p_2}{(2\pi)^3} Q(-i\partial_{x_1}, -i\partial_{p_1})e^{ip_1x_1}\tilde{G}_0(p_1, p_2)e^{-ip_2x_2} \]
\[ = \int \frac{d^3p_1d^3p_2}{(2\pi)^3} e^{ip_1x_1}[Q(p_1, i\partial_{p_1})\tilde{G}_0(p_1, p_2)]e^{-ip_2x_2} \]
\[ = \int \frac{d^3p_1d^3p_2}{(2\pi)^3} e^{ip_1x_1}\delta(p_1 - p_2)e^{-ip_2x_2} \]
\[ = \delta(\tau_1 - \tau_2)\delta_{x_1,x_2}, \]
where \( \int d^3p = \int_{-\infty}^{\infty} d\omega \int_{-\pi}^{\pi} d^2p. \)

Applying Wigner transformation, one obtains (assuming that the field \( A \) is slowly varying, i.e. when its variations on the distances of the order of the lattice spacing may be neglected)
\[ Q_W(x, p) \star G_{0,W}(x, p) = 1 \]  
(13)
where \( Q_W(x, p) \) and \( G_{0,W} \) are the Wigner transformations of \( Q \) and \( G_0 \), respectively:
\[ G_{0,W}(x, p) = \int d^3q e^{iqx} \tilde{G}_0(p + q/2, p - q/2) \]
\[ Q_W(x, p) = \int d^3q e^{ixq} \hat{Q}(p + q/2, p - q/2), \]
where
\[ \hat{Q}(p_1, p_2) = \int d^3k\delta^{(3)}(p_1 - k)Q(k, i\partial_k)\delta^{(3)}(p_2 - k) \]
represents the matrix elements of operator \( \hat{Q} \). Star product \( \star \) is the operation \( e^{\Delta} \), with \( \Delta = i(\partial_x \partial_p - \partial_p \partial_x)/2 \).

The gradient expansion further gives \( G_{0,W} = G_{0,W}^{(0)} + G_{0,W}^{(1)} + \ldots \) with \( G_{0,W}^{(n)} \sim O(\partial^p) \). \( G_{0,W}^{(0)} \) is given by \( G_{0,W}^{(0)}(x, p) = g(p - A(x)) \), in which \( g(p) = [i\omega - H(p)]^{-1} \), and (\( \mu = 1, 2 \))
\[ A_\mu(x) = \int \left[ \frac{\sin(k_{\mu}/2)}{k_{\mu}/2} \hat{A}_\mu(k)e^{ikx} + c.c. \right] dk \]  
(14)
that is
\[ A_1(x) = \int_{x-e_1/2}^{x+e_1/2} A_1(y_1, x_2)dy_1 \]
\[ A_2(x) = \int_{x-e_2/2}^{x+e_2/2} A_2(x_1, y_2)dy_2 \]  
(15)
where \( e_\mu \) is the unit lattice vector directed along the \( \mu \)-th axis. (The original electromagnetic field itself may be represented in the form: \( A_\mu(x) = \int [\hat{A}_\mu(k)e^{ikx} + c.c.]dk \).) For the slowly varying electromagnetic fields we may substitute \( A \) by \( A \), which will be done further.
C. Influence of interactions on the Hall current

In the presence of the 4-fermion interaction and the external field $A_\mu$, the Green function $G(x_1, x_2)$ can be expressed as

$$G(x_1, x_2) = G_0(x_1, x_2) + \lambda \int d\tau_y \sum_y G_0(x_1, y) H_0(y) G_0(y, x_2)$$

$$- \lambda \int d\tau_y \sum_y (\text{Tr} H_0(y)) G_0(x_1, y) G_0(y, x_2)$$

$$+ O(\lambda^2) = G_0 + W * G$$  \hspace{1cm} (16)$$

according to the Feynman diagrams, where $H_0(y) = G_0(y, y)$. For an arbitrary function $f(x_1, x_2)$, the convolution $W * f$ is defined as $\lambda \int d\tau_y \sum_y f(x_1, y) \Xi_0(y) f(y, x_2)$, with $\Xi_0(y) = H_0(y) - \text{Tr} H_0(y)$. From Eq. (16), one obtains $(1 - W) * G = G_0$, in which corrections in $\lambda^2$ order and higher have been neglected. Applying $Q$ to both sides, we get

$$Q(i \partial_x, x) G(x, y) - \lambda \Xi_0(x) G(x, y) = \delta(x_3 - y_3) \delta_{xy}$$  \hspace{1cm} (17)$$

Wigner transformation gives

$$(Q_W(x, p) - \lambda \Xi_0(x)) * G_W(x, p) = 1$$  \hspace{1cm} (18)$$

where $G_W$ is the Wigner transformation of $G$. In the presence of the 4-fermion interaction, the gradient expansion of the Green function is similar:

$$G_W(x, p) = G_W^{(0)} + G_W^{(1)} + ...$$  \hspace{1cm} (19)$$

where $G_W^{(n)} \sim O(\partial^n)$. The leading order term $G_W^{(0)}(R, p)$ contains the correction from the 4-fermion interaction

$$G_W^{(0)}(x, p) = [i(\omega - A_3(x)) - H(p - A(x)) - \lambda \Xi_0^{(0)}(x)]^{-1},$$

in which

$$\Xi_0^{(0)}(x) = \int \frac{d^3 p}{(2\pi)^3} \xi^{(0)}_{0, W}(x, p)$$  \hspace{1cm} (20)$$

$$= \int \frac{d^2 p}{(2\pi)^2} \int \frac{d\omega}{2\pi} \frac{-H(p - A(x))}{\omega^2 + H(p - A(x))^2}$$

$$= \int \frac{d^2 p}{(2\pi)^2} \int \frac{d\omega}{2\pi} \frac{m + 2 - \cos p_1 - \cos p_2}{\omega^2 + H(p)^2} \sigma_3$$

does not depend on the space coordinate, and $\xi^{(0)}_0$ can be labelled as $\xi_0 \sigma_3$. Quantity $\lambda \xi$ in the Green function $G_W^{(0)}(x, p)$ may be considered as the correction to the fermion mass. To obtain the value of $\xi$, we compute numerically the integral of Eq. (20). The dependence of $\xi$ on $m$ is shown on Fig. [1]. If coupling constant $\lambda$ is sufficiently large, the 4-fermion interaction will change essentially the value of the effective mass parameter $m - \lambda \xi$. As a result the system drops into the phase with the value of $N_3$ different from that of the model without interactions. The expression for the electric current, which follows from the relation $\delta \log Z = J^k(x) \delta A_k(x)$, gives the Hall current

$$J_{Hall}^k = \frac{1}{2\pi} N \epsilon^{ki} E_i,$$  \hspace{1cm} (21)$$

where the topological invariant $N$ is to be calculated using the Green function $G_\lambda = [i \omega - H(p) - \lambda \Xi_0^{(0)}]^{-1}$ as follows

$$N = - \frac{1}{24\pi^2} \text{Tr} \int G_\lambda^{-1} dG_\lambda \wedge dG_\lambda^{-1} \wedge dG_\lambda$$  \hspace{1cm} (22)$$
III. AQHE IN THE $3 + 1$ D WEYL SEMIMETAL WITH THE 4-FERMION INTERACTION

A. Modification of mass parameter due to the interactions

In this section, we consider the particular model of Weyl semimetals with the four-fermion interaction in $3 + 1$ D space-time. The Euclidean action is

$$S_\lambda = \int d\tau \sum_x \left[ \bar{\psi} \left( i(i\partial_\tau - A_4(-i\tau, x)) \right) - H(-i\partial_x - A(x)) \right] \psi + \frac{\lambda}{2} \left( \bar{\psi}(\tau, x)\psi(\tau, x) \right)^2,$$

where

$$H(p) = \sin p_1 \sigma^2 - \sin p_2 \sigma^1 - (m - \cos p_3 + \sum_{i=1,2} (1 - \cos p_i)) \sigma^3$$

with $m \in (-1, 1)$. This system contains the two Fermi points

$$K_+ = (0, 0, \pm \beta, 0), \quad \beta = \arccos m.$$  

Here $K_+$ is the right-handed Weyl point while $K_-$ is the left-handed one. As in the $2 + 1$ D model, the electric current in the $3 + 1$ D Weyl semimetal is given by

$$J^k(x) = -\int \frac{d^4p}{(2\pi)^4} TrG_{\lambda,W}(x, p) \frac{\partial}{\partial p_k} [G_{0,W}^{(0)}(x, p)]^{-1} = J^{(0),k} + J^{(1),k} + ...$$
In the leading order

\[ J^{(0),k} = - \int \frac{d^4p}{(2\pi)^4} Tr G_{\lambda,W}^{(0)}(x,p) \frac{\partial}{\partial p_k} [G_{\lambda,W}^{(0)}(x,p)]^{-1} \]
\[ = - \int \frac{d^4p}{(2\pi)^4} Tr G_{\lambda,W}^{(0)}(x,p) \frac{\partial}{\partial p_k} [G_{\lambda,W}^{(0)}(x,p)]^{-1} \]
\[ = 0 \quad (27) \]

where

\[ G_{\lambda,W}^{(0)}(x,p) = \left[ i(\omega - A_4(x)) - H(p - A(x)) + \lambda K_0^{(0)}(x) \right]^{-1} \]

with the corrections of the order of \( \lambda^2 \) neglected. \( K_0^{(0)} \) in the above equation is given by

\[ K_0^{(0)} = - \int \frac{d^4p}{(2\pi)^4} \left( G_{0,W}^{(0)}(x,p) - Tr \left[ G_{0,W}^{(0)}(x,p) \right] \right) \]
\[ = - \int \frac{d^3p d\omega}{(2\pi)^4} \frac{m - \cos p_3 + 2 - \cos p_1 - \cos p_2}{\omega^2 + H(p)^2} \sigma_3, \quad (28) \]

which can be expressed by \( \kappa \sigma_3 \). \( \kappa \) does not depend on the space coordinates and can be calculated numerically (Fig. 2). Comparing with Eq.(24), we find that \( \lambda \kappa \) is the correction to mass parameter. As for the next to leading order, following [10] we obtain

\[ J^{(1),k} = - \int \frac{d^4p}{(2\pi)^4} Tr G_{\lambda,W}^{(0)}(x,p) \frac{\partial}{\partial p_i} [G_{\lambda,W}^{(0)}(x,p)]^{-1} \]
\[ \frac{\partial}{\partial p_j} [G_{\lambda,W}^{(0)}(x,p)] A_{ij} \frac{\partial}{\partial p_k} [G_{\lambda,W}^{(0)}(x,p)]^{-1} \]
\[ = - \int \frac{d^4p}{(2\pi)^4} Tr G_{\lambda,W}^{(0)}(x,p) \frac{\partial}{\partial p_i} [G_{\lambda,W}^{(0)}(x,p)]^{-1} \]
\[ \frac{\partial}{\partial p_j} [G_{\lambda,W}^{(0)}(x,p)] A_{ij} \frac{\partial}{\partial p_k} [G_{\lambda,W}^{(0)}(x,p)]^{-1} \]
\[ = 0 \quad (29) \]

Up to the terms linear in \( A \) we obtain for the Hall current in 3 + 1 D the result similar to that of [10]:

\[ j_Hall^k = \frac{1}{4\pi^2} N_i \epsilon^{ijkl} E_j, \quad (31) \]

where

\[ N_i = - \frac{1}{24\pi^2} \epsilon^{ijkl} \int d^4p Tr G_{\lambda} \frac{\partial G_{\lambda}^{-1}}{\partial p_l} \frac{\partial G_{\lambda}}{\partial p_j} \frac{\partial G_{\lambda}^{-1}}{\partial p_k}. \quad (32) \]

Without interactions (at \( \lambda = 0 \) the values of \( N_i \) were calculated in [10]. We repeat this calculation in Appendix C for completeness. Notice, that unlike the case of the insulators for the Weyl semimetals the values of \( N_i \) are not topological invariants. However, unlike the case of ordinary metals with Fermi surfaces the corresponding integrals over momenta are convergent and the values of \( N_i \) are well - defined.

**B. States of the Weyl semimetals that correspond to the unconventional expressions for the Hall conductivity**

From [10] we know that \( N_1 = N_2 = N_4 = 0 \), while \( N_3 \) depends on mass parameter \( m \). When \( m \) belongs to the interval \((-1, 1)\) we get

\[ N_3 = 2\beta = |K_- - K_+| \quad (33) \]
FIG. 2: The dependence of $\kappa$ on mass parameter $m$. The horizontal axis represents the dimensionless mass parameter, i.e. the physical value of $m$ multiplied by the lattice spacing.

(see Appendix B). This gives the conventional expression for the Hall conductivity according to Eq. (31):

$$\sigma_{xy} = -N_3 = \frac{|K_+ - K_-|}{4\pi^2}$$

Notice, that we define here the conductivity $\sigma_{xy}$ through relation $j_x = -\sigma_{xy}E_y$. The sign minus originates from the two-dimensional notation relation between the conductivity tensor and resistivity $\rho_{xy}$:

$$\begin{pmatrix} 0 & -\sigma_{xy} \\ \sigma_{xy} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \rho_{xy} \\ -\rho_{xy} & 0 \end{pmatrix}^{-1}$$

that gives $\sigma_{xy} = 1/\rho_{xy}$.

If $m$ is larger than 1, then $N_3 = 0$. On the other hand, when $m$ approaches $-1$, the two Fermi points tend to $\pm \pi$ (which represent actually the same point). At $m = -1$ the value of $N_3$ achieves its maximum equal to $2\pi$. When $m$ is decreased further into the interval $(-3, -1)$, the new positions of the Fermi points are

$$K'_\pm = (0, \pi, \pm \beta', 0), \tilde{K}'_\pm = (\pi, 0, \pm \beta', 0)$$

with $\beta' = \arccos(m + 2)$. $K'_\pm, \tilde{K}'_\pm$ are the right-handed Weyl points while $K'_-, \tilde{K}'_-$ are the left-handed ones. Using the machinery described in Appendix C, we come to the conclusion that

$$N_3 = -4\beta' + 2\pi$$

(34)

When $m$ crosses the value $-1$, the two Fermi points of Eq. (25) disappear and the new two pairs of the Fermi points appear. However, the value of $N_3$ is changed continuously. Nevertheless, the conventional expression for the Hall conductivity corresponding to Eq. (33) is broken. Instead we may represent the value of $N_3$ as

$$N_3 = (K'_{-,3} - K'_{+,3}) + (\tilde{K}'_{-,3} - \tilde{K}'_{+,3}) + 2\pi$$

(35)
and obtain the unconventional expression for the Hall conductivity

$$\sigma_{xy} = \frac{\left( K'_{\perp,3} - K''_{\perp,3} \right) + \left( \tilde{K}'_{\perp,3} - \tilde{K}''_{\perp,3} \right)}{4\pi^2} - \frac{1}{2\pi a} \tag{36}$$

Here we restore in our expressions the lattice spacing $a$ and take the distance between the two Fermi points $\left( K'_{\perp} - K''_{\perp} \right)_{\perp} = \left( \tilde{K}'_{\perp} - \tilde{K}''_{\perp} \right)_{\perp} = -2\beta'$ corresponding to the part of the straight line connecting them. Eq. (35) may be interpreted as the sum of three contributions: the first two contributions correspond to the two pairs of the Weyl points while the last one corresponds to the contribution of the pair that has disappeared. If there would be no two new pairs, the system would become the topological insulator possessing the AQHE with the value of $N_3$ equal to $2\pi$.

The further decrease of $m$ leads to the disappearance of the two pairs $K'_{\pm}, \tilde{K}'_{\pm}$. The new pair appears for $m \in (-5, -3)$:

$$K''_{\pm} = (\pi, \pi, \pm \beta'', 0)$$

The value of $N_3$ reads:

$$N_3 = 2\pi - 2\pi - 2\pi + \left( K''_{\perp} - K''_{\perp} \right)_{\perp} = 2\beta'' - 2\pi \tag{37}$$

and gives

$$\sigma_{xy} = \frac{\left( K''_{\perp,3} - K''_{\perp,3} \right) + \frac{1}{2\pi a}}{4\pi^2} \tag{38}$$

At $m = -5$ the system undergoes transition to the insulator state, and for $m < -5$

$$N_3 = 2\pi - 2\pi - 2\pi + 2\pi = 0$$

that is $\sigma_{xy} = 0$.

We conclude, that the sufficiently large values of $\lambda$ lead to the essential change of the effective mass parameter and thus to the transition into the semimetal states with different configurations of the Weyl points as well as the different expressions for the Hall conductivity.

### IV. AQHE IN THE 2+1 D TOPOLOGICAL INSULATORS IN THE PRESENCE OF YUKAWA AND COULOMB INTERACTIONS

#### A. AQHE conductivity as the topological invariant in momentum space

In this section, we consider the 2+1 D tight-binding model with interactions caused by scalar excitations. Let us start by consideration of Yukawa interactions, but our conclusions remain valid for the exchange by the wide class of excitations (including the most relevant case of Coulomb interactions to be discussed further). The Euclidean action is

$$S_\eta = \int d\tau \sum_{x,x'} \left[ \bar{\psi}_x \left( i \left( \partial_\tau + A_3(i\tau, x) \right) \delta_x, x' - i \partial_{x, x'} \right) \psi_x 
+ \phi_x \left( \partial_\tau^2 \delta_x, x' + B_{x', x} \right) \phi_x 
- \eta \bar{\psi}(\tau, x) \psi(\tau, x) \phi(\tau, x) \right]. \tag{39}$$

where matrix

$$B_{x', x} = \sum_{i=1,2} \left( \delta_{x', x+e_i} + \delta_{x', x-e_i} \right) + (M^2 - 2) \delta_{x', x} \tag{40}$$

corresponds to the boson $\phi$ with mass $M$. Eq. (55) gives rise to Yukawa interaction. It contributes to the self-energy of the fermions, the leading order contribution is proportional to $\eta^2$. In the present work we consider its effect up to the order of $\eta^2$. 
The electric current is given by

\[ J^k(x) = -\int \frac{d^3p}{(2\pi)^3} \text{Tr} G_{\eta,W}(x,p) \frac{\partial}{\partial p_k} [G_{0,W}^{(0)}(x,p)]^{-1} \]  

(41)

where \( G_{\eta,W}(x,p) = G_{n,W}^{(0)}(x,p) + G_{n,W}^{(1)}(x,p) + \ldots \) is the full Green function with the interactions taken into account. The term \( G_{n,W}^{(k)}(x,p) \) is proportional to the product of \( k \) derivatives \( \frac{\partial}{\partial p} \). In particular,

\[ G_{n,W}^{(0)}(x,p) = [i(\omega - A_3(x)) - H(p - A(x)) - \eta^2\Sigma(p,x)]^{-1} \]

The self-energy function is given by

\[ \Sigma(x,p) = -\int G_{0,W}(x,q) D(p-q) \frac{d^3q}{(2\pi)^3} = \Sigma^{(0)} + \Sigma^{(1)} + \ldots \]  

(42)

and is also expanded in powers of \( \frac{\partial}{\partial x} \). The bosonic Green function is

\[ D(p) = \frac{1}{\omega^2 + \sin^2 p_1 + \sin^2 p_2 + M^2}. \]  

(43)

Contrary to the leading-order contribution of the 4 - fermion interaction (which is constant), the contribution of Yukawa interactions to the self-energy depends both on momenta and space coordinates. Let us consider the difference \( J^k_n - J^k_{n\eta} \), where

\[ J^k_n = -\int d^3p \text{Tr} G_{\eta,W}^{(0)}(x,p) \frac{\partial}{\partial p_k} [G_{\eta,W}^{(0)}(x,p)]^{-1} \]

Then

\[ \delta J^k = J^k_n - J^k_{n\eta} \]

\[ \quad = -\int \frac{d^3p}{(2\pi)^3} \left( \text{Tr} G_{\eta,W}(x,p) \frac{\partial}{\partial p_k} [G_{0,W}^{(0)}(x,p)]^{-1} \right) \]

\[ \quad - \text{Tr} G_{0,W}(x,p) \frac{\partial}{\partial p_k} [G_{0,W}^{(0)}(x,p)]^{-1} \]

\[ \quad = -\int \frac{d^3p}{(2\pi)^3} \text{Tr} G_{\eta,W}(x,p) \frac{\partial}{\partial p_k} [G_{0,W}^{(0)}(x,p)]^{-1} \]

\[ \quad - G_{\eta,W}^{(0)}(x,p)]^{-1} \]

\[ \quad = -\eta^2 \int \frac{d^3p}{(2\pi)^3} \text{Tr} G_{\eta,W}(x,p) \frac{\partial}{\partial p_k} \Sigma(p) \]

\[ \quad = -\eta^2 \int \frac{d^3p}{(2\pi)^3} \text{Tr} G_{0,W}(x,p) \frac{\partial}{\partial p_k} \Sigma(p) + O(\eta^4). \]  

(44)

Let us represent \( G_{0,W}(x,p) = G_{0,W}^{(0)}(x,p) + G_{0,W}^{(1)}(x,p) + \ldots \) with \( G_{n,W}^{(n)} \sim O(\partial x^n) \), and \( \delta J^k = \delta J^{k,(0)} + \delta J^{k,(1)} + \ldots \) correspondingly. Let us first consider \( \delta J^{k,(0)} \) and denote for simplicity \( G_{0,W}^{(0)}(x,p) \) by \( g(p) \):

\[ \delta J^{k,(0)} = \eta^2 \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ G_{0,W}^{(0)}(x,p) \frac{\partial}{\partial p_k} \left( G_{0,W}^{(0)}(x,p) \frac{d^3q}{(2\pi)^3} \right) \right] g(p) \frac{\partial}{\partial p_k} \int g(p-q) D(q) \frac{d^3q}{(2\pi)^3} \]
We also denote \( \delta J^{k,(0)} = \eta^2 I \), where

\[
I = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[ g(p) \frac{\partial}{\partial p_k} g(p-q) \right] D(q)
\]

\[
= - \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[ \frac{\partial g(p)}{\partial p_k} g(p-q) \right] D(q)
\]

\[
= - \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[ g(p-q) \frac{\partial g(p)}{\partial p_k} \right] D(q)
\]

\[
= - \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[ g(s) \frac{\partial g(s+q)}{\partial s_k} \right] D(q)
\]

\[
= - \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[ g(s) \frac{\partial g(s-t)}{\partial s_k} \right] D(-t).
\]

Since \( D(-t) = D(t) \), one finds that \( I = -I \), therefore \( I = 0 \), which implies \( \delta J^{k,(0)} = 0 \).

Next, we consider the next order in the derivatives of the gauge field.

\[
\delta J^{k,(1)} = \eta^2 \int \frac{d^3p}{(2\pi)^3} \left[ \text{Tr} G^{(1)}(x,p) \frac{\partial}{\partial p_k} \Sigma^{(0)}(x,p) \right]
\]

\[
+ \text{Tr} G^{(0)}(x,p) \frac{\partial}{\partial p_k} \Sigma^{(1)}(x,p)
\]

\[
= \eta^2 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[ \text{Tr} G^{(1)}(x,p)G^{(0)}(x,q) \frac{\partial}{\partial p_k} D(p-q) \right]
\]

\[
+ \text{Tr} G^{(0)}(x,p)G^{(1)}(x,q) \frac{\partial}{\partial p_k} D(p-q)
\]

\[
= \eta^2 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[ \text{Tr} G^{(1)}(x,p)G^{(0)}(x,q) \frac{\partial}{\partial p_k} D(p-q) \right]
\]

\[
+ \text{Tr} G^{(1)}(x,q)G^{(0)}(x,p) \frac{\partial}{\partial q_k} D(p-q)
\]

\[
= \eta^2 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[ \text{Tr} G^{(1)}(x,q)G^{(0)}(x,p) \frac{\partial}{\partial q_k} D(q-p) \right]
\]

\[
+ \text{Tr} G^{(1)}(x,q)G^{(0)}(x,p) \frac{\partial}{\partial p_k} D(q-p)
\]

\[
= 0
\]

We used here that \( D(-t) = D(t) \), and \( D'(-t) = -D'(t) \). Therefore, we obtain \( \delta J^{y,(1)} = 0 \) and conclude, that, at least, in the one - loop approximation Yukawa interactions do not affect the expression for the Hall conductivity through the (interacting) Green function. As it was mentioned above, in the same way it may be proved that the Hall conductivity is not affected (up to the term \( \sim \eta^2 \)) by the exchange by scalar boson with arbitrary propagator \( D(p) \) obeying \( D(p) = D(-p) \). The important particular case is when \( D(p) \) is the (three - dimensional) Coulomb interaction. It is resulted from the interactions due to the exchange by real photons between the Bloch electrons of the given 2D material. The Hall current is given by

\[
J_{Hall}^k = \frac{1}{2\pi} \mathcal{N} e^{ki} E_i,
\]

where the topological invariant \( \mathcal{N} \) is to be calculated using the interacting Green function \( G_{\eta} = \frac{1}{i\omega - H(p) - \eta^2 \Sigma(p)} \), (with \( \Sigma(p) = -\int G_0(q) D(p-q) d^3\omega/(2\pi)^3 \)):

\[
\mathcal{N} = \frac{1}{24\pi^2} \text{Tr} \int G_{\eta}^{-1} dG_{\eta} \wedge dG_{\eta}^{-1} \wedge dG_{\eta}
\]

The extension of this result to the three - dimensional materials is also straightforward and will be considered in the next section.
B. Topological phase transitions due to interactions

In this subsection, we consider the 2+1 D tight-binding model with Coulomb interaction. The Euclidean action is

\[ S = \int d\tau \sum_{x,x'} \left[ \bar{\psi}_{x'} \left( i (i \partial_\tau - A_3(i \tau, x)) \delta_{x,x'} - i D_{x,x'} \right) \psi_x \right. 
\[ - \alpha \bar{\psi}(\tau, x) \psi(\tau, x) \psi(\tau, x') \bar{\psi}(\tau, x') \left. \right], \] (48)

where \( \psi \) is the external electrical field. We chose the gauge with vanishing interactions the topological phase transitions occur at \( m \) with \( \alpha \). The electric current is given by Eq. (40) with \( N \) given by Eq. (17). The Green function entering this expression is

\[ \hat{G}_\alpha(p) = [i \omega - H(p) - \alpha \Sigma(p)]^{-1} \]

with

\[ \Sigma(p) = - \int \hat{G}_{\alpha=0}(q) \hat{V}(p - q) \frac{d^3 q}{(2\pi)^3} \] (49)

Coulomb interaction in momentum space is given by

\[ \hat{V}(p) = \sum_x \frac{e^{i p \cdot x}}{\sqrt{x_1^2 + x_2^2}} \] (50)

Therefore, \( \Sigma(p) \) depends only on \( p_1 \) and \( p_2 \). According to the results of Sect. IV, Coulomb interactions do not change the Hall conductivity until the topological phase transition is encountered, at least to the leading order in \( \alpha \). Without interactions the topological phase transitions occur at \( m = 0, -2, -4 \). The self-energy function contribution to the Green function modifies those critical values. Therefore, the values of \( m \) that give rise to a certain value of \( N \) without interactions may lead to the different value of \( N \) in the presence of interactions.

In order to investigate the effect of Coulomb interactions on the critical values \( m', m'', m''' \) of \( m \) we consider the effect of \( \Sigma \) on the poles of the Green’s function. Let us denote \( \hat{G}_{\alpha=1}^{-1}(p) = i \sigma^3 (\sigma^k g_k(p) - ig_4(p)) \) with \( k = 1, 2, 3 \). Recall that [10] (see also Appendix B)

\[ N' = \frac{1}{2} \sum_l \text{sign}(g_4(y^{(l)})) \text{Res}(y^{(l)}) \] (51)

where

\[ \text{Res}(y) = \frac{1}{8\pi} \epsilon^{ijk} \int_{\partial \Omega(y)} v_i dv_j \wedge dv_k \] (52)

and \( v_k = g_k/\sqrt{q_1^2 + q_2^2 + q_3^2} \) with \( k = 1, 2, 3 \). By \( y^{(l)} \) we denote the positions of the zeros of function \( g_4^2 + g_2^2 + g_3^2 \).

It may be shown that in the first order in \( \alpha \), the Coulomb interactions do not change the positions of \( y^{(l)} \), that are

\[ y^{(1)} = (0, 0, 0), \quad y^{(2)} = (0, 0, \pi), \]
\[ y^{(3)} = (0, \pi, 0), \quad y^{(4)} = (0, \pi, \pi) \]

Let us denote \( \Sigma(p) = -i \sigma^3 (\sigma^k f_k(p) - i f_4(p)) \). It is easy to find that \( f_k(y^{(m)}) = 0 \) for \( k = 1, 2, 3 \) because \( \hat{V}(p - y^{(l)}) = \hat{V}(-p - y^{(l)}) \). Therefore, the self energy affects the Green function at \( p = y^{(l)} \) only through the modification of \( g_4 \) by \( f_4 \), which is given by

\[ f_4(p_1, p_2|m) = \frac{m + 2 - \cos q_1 - \cos q_2}{\sqrt{\sin^2 q_1 + \sin^2 q_2 + (m + 2 - \cos q_1 - \cos q_2)^2}} \hat{V}(p_1 - q_1, p_2 - q_2) \frac{d^2 q}{2(2\pi)^2}. \] (53)
The critical values of $m$ appear as the solutions of equations:

\[ m' + \alpha f_4(0,0|m') = 0, \]
\[ m'' + 2 + \alpha f_4(0,\pi|m'') = 0, \]
\[ m''' + 4 + \alpha f_4(\pi,\pi|m''') = 0 \]

To the leading order in $\alpha$, these critical values are given by

\[ m' = -\alpha f_4(0,0|m = 0), \]
\[ m'' = -2 - \alpha f_4(0,\pi|m = -2) \]
\[ m''' = -4 - \alpha f_4(\pi,\pi|m = -4) \]

We found analytically that $f_4(\pi,\pi|m = -4) = -f_4(0,0|m = 0)$ and $f_4(0,\pi|m = -2) = 0$. As for the numerical value of $f_4(0,0|m = 0)$, we computed the related integral on the $40 \times 40$ lattice, and the numerical result is -0.28.

![Graph](image)

**FIG. 3**: The dependence of $\mathcal{N}$ (y-axis) on mass parameter $m$ (x-axis). The blue solid line represents $\mathcal{N}(m)$ without interaction, while the red dashed line is $\mathcal{N}(m)$ with Coulomb interactions taken into account to the first order in $\alpha$, with $\alpha = 0.3$.

Up to the leading order in $\alpha$, the topological number $\mathcal{N}$ is represented in Fig 3 as a function of $m$ for $\alpha = 0.3$.

**V. THE 3 + 1 D WEYL SEMIMETALS IN THE PRESENCE OF COULOMB INTERACTIONS**

**A. AQHE conductivity expressed through the two-point Green functions**

Let us now discuss the three-dimensional model of Weyl semimetal with Coulomb interactions. In particular, the above considered particular model of Weyl semimetals corresponds to the action

\[
S = \int d\tau \sum_x \left[ \bar{\psi} \left( i \partial_\tau - A_4(\tau, x) \right) - H(\tau) \psi \right] + \int d\tau \sum_{x,x'} \left[ \frac{1}{8\pi\alpha} \phi_\xi \cdot \mathcal{U}_{\xi'} \cdot \phi_\xi - \bar{\psi}(\tau, x) \psi(\tau, x) \phi(\tau, x) \right]
\]
where $A_4$ is the Euclidean scalar potential that corresponds to external electric field while

$$H(p) = \sin p_1 \sigma^2 - \sin p_2 \sigma^1 - (m - \cos p_3 + \sum_{i=1,2} (1 - \cos p_i)) \sigma^3$$

with $m \in (-1, 1)$. With the interactions neglected this system contains the two Fermi points

$$K_\mp = (0, 0, \pm \beta, 0), \quad \beta = \arccos m. \quad (54)$$

The interactions correspond to matrix $U_{k',k}$ that is inverse to the matrix of lattice Coulomb potential $D_{k',k}$:

$$\sum_k U_{k,k'} D_{k'y} = \delta_{z,y} \quad (55)$$

$D_{k,k'}$ is caused by the photon exchange in medium, which in the leading order is reduced to the Coulomb interaction because of the relative smallness of the Fermi velocity in medium. In the leading order it is given by the continuum Coulomb potential $1/|x-x'|$, where $|x-x'|$ is the real distance between the lattice points $x',x$. There are also certain corrections to this potential that will not be discussed here but that may be relevant for certain physical phenomena.

We will need only, that the Fourier transform $D(p)$ of $D$ obeys $D(p) = D(-p)$. Then repeating all the steps of Sect. IV.A we come to the conclusion that the Coulomb interactions between electrons in Weyl semimetals do not affect the expression for the Hall conductivity, at least, in the first order in the effective fine structure constant in medium $\alpha$, that is the Hall conductivity is given by the expression of [10] expressed through the two point Green function of the interacting system:

$$j_{Hall}^k = \frac{1}{4\pi^2} N_i e^{ijkl} E_j, \quad (56)$$

where

$$N_i = -\frac{1}{24\pi^2} \epsilon^{ijkl} \int d^4p \text{Tr} \frac{\partial G_\alpha^{-1}}{\partial p_i} \frac{\partial G_\alpha}{\partial p_j} \frac{\partial G_\alpha^{-1}}{\partial p_k}. \quad (57)$$

Here $G_\alpha$ is the two point Green function with the Coulomb interactions taken into account. It has the form

$$G_\alpha(p) = [i\omega - H(p) - \alpha \Sigma(p)]^{-1}. \quad (58)$$

The self-energy function is given by

$$\Sigma(p) = -\int G_{\alpha=0}(q) D(p-q) \frac{d^3q}{(2\pi)^3} \quad (59)$$

**B. Transitions between the states of Weyl semimetals with various patterns of the Fermi points**

For the above considered particular model of Weyl semimetals the action has the form

$$S = \int d\tau \sum_x \left[ \bar{\psi} \left( i(\partial_\tau - A_4(i\tau, x)) - H(-i\partial_x) \right) \psi 
- \alpha \bar{\psi}(\tau,x) \psi(\tau,x) V(x-x') \bar{\psi}(\tau,x') \psi(\tau,x') \right]. \quad (60)$$

where $A_4$ is the Euclidean scalar potential that corresponds to the external electric field while

$$H(p) = \sin p_1 \sigma^2 - \sin p_2 \sigma^1 - (m - \cos p_3 + \sum_{i=1,2} (1 - \cos p_i)) \sigma^3$$

with $m \in (-1, 1)$. With the interactions neglected this system contains the two Fermi points

$$K_\mp = (0, 0, \pm \beta, 0), \quad \beta = \arccos m. \quad (61)$$
The electric current is given by Eq. (31) with $N_3$ given by Eq. (57). The Green function entering this expression is given by Eq. (58). But the self-energy function is

$$
\Sigma(p) = -\int G_{\alpha=0}(q) \tilde{V}(p-q) \frac{d^3q}{(2\pi)^3}
$$

(62)
in which the Coulomb interaction in momentum space is

$$
\tilde{V}(p) = \sum_x \frac{e^{i p \cdot x}}{\sqrt{x_1^2 + x_2^2 + x_3^2}}.
$$

(63)
The leading order of the self-energy function is given by

$$
\Sigma^{(0)}(x,p) = -\int \frac{1}{iq_4 - H(q)} \tilde{V}(p-q) \frac{d^4q}{(2\pi)^4}
$$

(64)

$\Sigma^{(0)}(x,p)$ depends on $p_1$, $p_2$ and $p_3$. The Coulomb interactions will change the value of Hall conductivity through the self energy function. To see this more explicitly, let us denote $G^{-1} = i\sigma^3(\sigma_k g_k(p) - ig_4(p))$ and represent the Hall conductivity as follows (Eq.(55) in [10]):

$$
\sigma_H = -\frac{1}{4\pi^2} N_3
$$

(65)
where

$$
N_3 = -\frac{1}{2} \sum_l \int_{y^{(l)}} \text{sign}(g_4(y^{(l)})) \text{Res}(y^{(l)}) dp_3
$$

(66)
y^{(l)}(s) is the line in 4D momentum space, where $g_1 = g_2 = g_3 = 0$, and

$$
\text{Res}(y) = \frac{1}{8\pi} \epsilon^{ijk} \int_{\partial\Omega(y)} v_i dv_j \wedge dv_k
$$

(67)
(see Appendix C for details).

To the first order in $\alpha$, the Coulomb interactions do not change the positions of the lines of $y^{(l)}$, which are given by

$$
y^{(1)}(p_3) = (p_1 = 0, p_2 = 0, p_3, p_4 = 0),
$$

$$
y^{(2)}(p_3) = (0, \pi, p_3, 0),
$$

$$
y^{(3)}(p_3) = (\pi, 0, p_3, 0), \quad y^{(4)}(p_3) = (\pi, \pi, p_3, 0),
$$

but it can change the position of the point when $g_4 = 0$ on the line, where $g_4$ changes its sign. Therefore, it changes the value of $N_3$ in Eq. (66).

Let us take as an example $y^{(1)}$. We represent $\Sigma(p) = -i\sigma^3(\sigma_k f_k(p_1, p_2, p_3|m) - if_4(p_1, p_2, p_3|m))$. Then $\Sigma(y^{(1)}) = -f_4(0, 0, p_3|m)\sigma^3$ and

$$
f_4(0, 0, p_3|m) = \int \frac{m + 2 - \cos q_1 - \cos q_2 - \cos q_3}{\sqrt{\sin^2 q_1 + \sin^2 q_2 + (m + 2 - \cos q_1 - \cos q_2 - \cos q_3)^2}} \tilde{V}(q_1, q_2, p_3 - q_3) \frac{d^3q}{2(2\pi)^3},
$$

(68)
The zero of $g_4(p_3)$ is the solution of equation

$$
g_4(p_3) = m - \cos p_3 + \alpha f_4(0, 0, p_3|m) = 0.
$$
The solution is \( p_3 = \arccos (m + \alpha f_4(0, 0, \beta |m|)) + O(\alpha^2) \). The value of \( N_3 \) is modified (it is equal to \( 2\arccos (m) \) in the absence of Coulomb interactions): \( N_3 = 2\arccos (m + \alpha f_4(0, 0, \beta |m|)) \), which is still equal to the difference between the positions of the Weyl points. Their positions are modified due to the interactions.

Besides, the pattern of the AQHE may be modified in a more complicated way. Namely, without interactions for \(-1 < m < 1\), the signs of \( g_4 \) along the lines \( y^{(2)} \), \( y^{(3)} \) and \( y^{(4)} \) remain positive. However, say, if \( m \) remains slightly larger than \(-1\), the Coulomb interactions may lead to the appearance of the zero of \( g_4(0, \pi, p_3) \) that is the solution of equation:

\[
g_4(p_3) = 2 + m - \cos p_3 + \alpha f_4(0, \pi, p_3|m) = 0
\]

This equation may have a solution \( p_3 = \beta' \) if \( f_4(0, \pi, p_3|m) \) is able to become negative at small \( p_3 \). Our numerical calculations on a \( 40 \times 40 \times 40 \) lattice show that \( f_4(0, \pi, 0|m = -1) \approx -0.15 \). Therefore, the zero of \( g_4 \) really may appear along the line \( y^{(2)} \) (near \( p_3 = 0 \)), when parameter \( m \) is close to \(-1\). The new Fermi points will be located at \( K_\pm = (0, \pi, \pm \beta', 0); K'_\pm = (\pi, 0, \pm \beta', 0) \)

On the other hand, if in addition \( f_4(0, 0, p_3|m) \) is able to become negative at \( p_3 \) close to \( \pi \), then the zero of \( g_4(0, 0, p_3) \) may disappear together with the conventional pair of Weyl points of Eq. (61). Our numerical results show that \( f_4(0, 0, \pi|m = -1) \approx -1.4 \). Therefore, this may really occur.

Since \( f_4(0, 0, \pi|m = -1) < f_4(0, \pi, 0|m = -1) < 0 \), the interval \( m \in (-1, 1) \) may be divided into 3 regions (when \( \alpha << 1 \)):

I) \(-1 - \alpha f_4(0, 0, \pi|m = -1) < m < 1 \): \( g_4 \) has one zero along the line \( y^{(1)} \), but there is no zero along \( y^{(2)} \), and \( y^{(3)} \).

This case is similar to the original one without interaction.

II) \(-1 - \alpha f_4(0, \pi, 0|m = -1) < m < -1 - \alpha f_4(0, 0, \pi|m = -1): g_4 \) does not have zero along \( y^{(1)} \), \( y^{(2)} \), and \( y^{(3)} \).

Therefore,

\[
\sigma_{xy} = -\frac{1}{2\pi a} \tag{69}
\]

This is the case, when interactions bring the system to the insulator phase. The insulator has a nontrivial topology that causes the AQHE according to the mechanism discussed in Sect. 9 of [10].

III) \(-1 < m < -1 - \alpha f_4(0, \pi, 0|m = -1): g_4 \) does not have zero along \( y^{(1)} \), but has one zero along each of the lines \( y^{(2)} \) and \( y^{(3)} \). The Hall conductivity is then given by

\[
\sigma_{xy} = \frac{(K_{+,3} - K_{-,3}) + (K'_{+,3} - K'_{-,3})}{4\pi^2} - \frac{1}{2\pi a} \tag{70}
\]

as in the case of the four - fermion interactions discussed above (see Eq. (35)).

VI. CONCLUSIONS AND DISCUSSION

In the present paper we investigated the influence of interactions on the AQHE. We consider the particular tight - binding models of the \( 2 + 1 \) D topological insulator discussed in [10, 33–40] and of the \( 3 + 1 \) D Weyl semimetals (discussed, for example, in [33, 37]). We consider the effect of the four - fermion interactions, Yukawa and Coulomb interactions, and the other interactions caused by the exchange by scalar bosons. As expected, we obtain, that in all considered cases the AQHE conductivities are given by expressions discussed in [10]. Those expressions are composed of the two point Green functions (defined in momentum space) of the interacting systems. For the case of the insulators the given expressions are the topological invariants and are not changed when the system is modified smoothly. For the case of Weyl semimetals strictly speaking the obtained expressions are not the topological invariants. The smooth modification of the system may lead to the smooth change of their values. However, certain features are inherited by these expressions from the case of the topological insulators. At least, up to the one - loop order the corrections to the conductivities due to interactions may be taken into account only through the modification of the mentioned one - particle Green functions. The other corrections do not appear. We expect that this result remains valid to all orders of the perturbation expansion (at least, for the sufficiently small values of coupling constants).

In the case of the four - fermion interactions at the one loop level we observe the renormalization of the parameters of the considered tight - binding models that may bring the \( 2 + 1 \) D topological insulator to the phase with the value of the Hall conductivity different from that of the non - interacting model. Thus we deal with the topological phase transition caused by interactions. Due to the four - fermion interactions the \( 3 + 1 \) D Weyl semimetal may
drop to the different state of the semimetal with the unconventional expression for the AQHE conductivity (the conventional AQHE conductivity in the Weyl semimetal is proportional to the distance between the Weyl points of opposite chiralities). This is not a topological phase transition. However, the Weyl semimetal may also undergo the true topological phase transition to the insulator phase.

Coulomb interactions in one loop already affect the systems more strongly. Not only the parameters are renormalized, the very dependence of the propagators on momenta becomes different. We demonstrate, that this modification also may bring the 2 + 1 D topological insulator to the new phase with the value of Hall conductivity that differs from that of the non-interacting model. We also show that the Coulomb interactions similar to the four-fermion interactions may bring the 3 + 1 D Weyl semimetals to the states with the unconventional expression for the Hall conductivity. In the latter case we may deal both with the true phase transition to the topological insulator exhibiting the AQHE and with the smooth modification of the Weyl point pattern that gives rise to the unconventional AQHE conductivity.

Our present consideration was essentially limited to the one-loop corrections. It is generally believed, however, that the anomalous quantum Hall conductivity is the topological property of materials at zero temperature. The meaning of this hypothesis for the 2 + 1 D topological insulators is that the Hall conductivity is not changed when the system is modified smoothly. Let us repeat here the reasoning of [13] (Appendix C) in favor of this hypothesis. We consider the case, when the fermions are in the presence of Euclidean vector potential $A_k$. Let us consider the average total integrated current (i.e. the integral of the current density over all space averaged with respect to time)

$$\langle J^k \rangle = T \int d^3x J^k = -T \int d^3x \int \frac{d^3p}{(2\pi)^3} \text{Tr} G_W(p,x) \partial_{p_k} Q_W(p-A(x))$$

$$= -T \int d^3x \int \frac{d^3p}{(2\pi)^3} \text{Tr} G_W(p,x) * \partial_{p_k} Q_W(p-A(x))$$

Here $T \to 0$ is temperature. The last expression is the topological invariant in phase space (for the detailed discussion of this concept and the derivations see [13]). Its value is not changed if the system is modified smoothly. In principle, such a modification may contain the modification of $A_k(x)$. However, for the configurations of $A_k(x)$ that correspond to different values at different points of spatial infinity $|x| \to \infty$ we should be careful with the variation $\delta A_k$. Actually, we should restrict ourselves by the case, when $\delta A_k(x) \to 0$ at $|x| \to \infty$. Next, let us expand $J$ in powers of the field strength $A_{ij}(x)$ and its derivatives. For the homogeneous system (this is the case of the AQHE considered in the present paper) the coefficient in the term proportional to the field strength $A_{ij}$ does not depend on coordinates. Then this coefficient is in itself the topological invariant because the integral $\int d^Dx A_{ij}$ (with $D = 3$) is not changed, when the gauge potential $A$ is modified smoothly in a finite region of space:

$$\delta \int d^3x A_{ij} = \int d^3x \delta A_{ij} = 0$$

because $\delta A \to 0$ at infinity. This way for $D = 3$ we come to the following result for the part of the current proportional to $A_{ij}$:

$$\langle J^k \rangle \approx \frac{1}{4\pi} M^{ij} A_{ij} \tag{71}$$

where tensor $M^{ij}$ is the topological invariant in momentum space. We should take into account that $A_{ij} = -iE_j$ in the presence of external electric field $E_k$. This way we come to the conclusion, that the AQHE conductivity is the topological invariant. In the present paper we have shown, that up to the one-loop approximation it is given by Eq. [20] with $N$ given by Eq. [3] expressed through the interacting two-point Green function.

Is that possible, that the higher order Green functions (those with more than two legs) compose the other topological invariants and give the contributions to the Hall conductivity of insulator? We suppose, that the answer is "no". The reason for this is the following. Let us take for definiteness the 2D insulating system with Coulomb interactions and effective fine structure constant $\alpha$. We consider the general structure of the expected answer:

$$\sigma_{xy} = \frac{1}{2\pi} N|\mathcal{G}(p)| + \frac{\alpha}{2\pi} \mathcal{F}[\mathcal{G}(p_1, ..., p_n)] + ... \tag{72}$$

The first term here is proportional to $N$ expressed through the two-point interacting Green function according to Eq. [3]. The second term contains the functional $\mathcal{F}$ of the multi-leg interacting Green functions. The factor $\alpha$
appears because the second term exists only for the nonzero coupling constant. In turn, we may expand $F$ in powers of $\alpha$:

$$F[\mathcal{G}(p_1, \ldots, p_n)] = F^{(0)}[\mathcal{G}_{\alpha=0}(p_1, \ldots, p_n)] + \alpha F^{(1)}[\mathcal{G}_{\alpha=0}(p_1, \ldots, p_n)] + \ldots$$

The terms in this expansion proportional to the negative powers of $\alpha$ cannot appear because in the limit $\alpha \to 0$ we should come back to the expression for the noninteracting system. Then looking at Eq. (72) we see the contradiction to the supposition that $\sigma_{xy}$ remains the topological invariant: this expression is changed smoothly when alpha is changed.

For the Weyl semimetals the situation is more involved. The AQHE conductivity is not a topological invariant even for the noninteracting systems. The popular explanation of the topological nature of the AQHE in Weyl semimetals is that it is related in these materials to the chiral anomaly. For example, the authors of [15] considered the (naive) effective continuum field theory of Weyl semimetals. In this naive continuum theory the axial gauge transformation $\Psi \to e^{i\gamma_5 \beta x^3} \Psi$ results in the separation of Weyl points in momentum space. The distance between them is given by $2\beta$. The fermionic measure acquires the following contribution

$$D\bar{\Psi}D\Psi \to e^{\frac{i}{\beta} \int dt dx (2\beta x^3)e^{\epsilon_{\mu\nu\rho\sigma} A_{\mu\nu} A_{\rho\sigma}}$$

The argument of this exponent becomes the anomalous contribution to the effective action of the electromagnetic field after integration over the fermions. In turn, the variation of this term in the effective action with respect to $A$ gives the electric current of the AQHE given by Eq. (31) with $N_3$ equal to $2\beta$. It is well known that the chiral anomaly is not renormalized by interactions (this follows from the fact that the chiral anomaly appears via the above mentioned transformation of the integration measure over fermions [54]). Therefore, the conventional expression for the Hall conductivity in Weyl semimetals remains valid as soon as we believe that the effective low energy continuum theory describes the AQHE well enough. It is worth mentioning, however, that the chiral anomaly in real Dirac/Weyl semimetals receives an extra contribution compared to the naive continuum field theory and to its conventional regularizations [47].

It is worth mentioning, that in all our considerations we neglected completely the finite temperature effects and assume that the given system remains at zero temperature. Correspondingly, speaking of the topological nature of the AQHE conductivity we assume the case of zero temperature. In turn, several publications discuss the corrections due to interactions to the AQHE that manifest themselves at nonzero temperatures [48]. In [49] the experimental results on their observation were reported. Those finite temperature corrections do not contradict our findings.

The authors kindly acknowledge useful discussions with I.Fialkovsky, Xi Wu, and M.Suleymanov.

Appendix A. AQHE conductivity in the noninteracting lattice model

Here we repeat for the completeness some of the results reported earlier in [11, 19, 45] that are used in the main text of the present paper. We consider the lattice $2 + 1$ D tight - binding fermionic model with the following partition function

$$Z = \int D\bar{\Psi}D\Psi \exp \left( \int dt \sum_{r_n,r_m} \bar{\Psi}^T(r_m, \tau) \right) \left( -i D(\tau, \tau') r_n, r_m \right) \Psi(r_n, \tau') \right) \right)$$

\[ (73) \]

\textsuperscript{1} The conventional regularizations of continuum field theory are those that obey conditions given in [47]. For the rectangular lattices those conditions assume, in particular, the absence of the distinguished spacial direction. To the best of our knowledge the regularizations normally used in practise by lattice theorists (Wilson fermions, overlap fermions, staggered fermions etc) belong to this class.
$D_{x,y}(\tau, \tau')$ depends on $2D$ lattice sites $x, y$ and on the points along the axis of imaginary time $\tau, \tau'$. The lattice is assumed to be rectangular for simplicity. Variables $\Psi, \bar{\Psi}$ are the anti-commuting Grassmann-valued fields. We may rewrite this partition function in momentum space:

$$Z = \int D\bar{\psi} D\psi \exp \left( \int_{\mathcal{M}} \frac{d^3 p}{(2\pi)^3} \bar{\psi}^T(p) Q(p) \psi(p) \right),$$

(74)

Here integration is over the fields $\bar{\psi}$ and $\psi$ that are functions of momenta. The partition function of Eq. (75) allows to describe the non-interacting fermions. $Q(p)$ is the Fourier transform of $D_{xy}(\tau, \tau')$ that has the meaning of the inverse fermion propagator.

External gauge field $A(x)$ may be introduced to the model through the Peierls substitution [1 0, 45]:

$$Z = \int D\bar{\psi} D\psi \exp \left( \int_{\mathcal{M}} \frac{d^3 p}{(2\pi)^3} \bar{\psi}^T(p) \left( Q(p - A(i\partial_\mu)) \psi(p) \right) \right),$$

(75)

We assume in this expression the symmetrization of the products of operators within $Q(p - A(i\partial_\mu))$).

The matrix elements of $\hat{Q} = Q(p - A(i\partial_\mu))$ and its inverse $\hat{G} = \hat{Q}^{-1}$ are denoted here by $Q(p, q)$ and $G(p, q)$ correspondingly:

$$Q(p, q) = \langle p | \hat{Q} | q \rangle, \quad G(p, q) = \langle p | \hat{G} | q \rangle$$

It is assumed that the basis elements $|q\rangle$ of the space of functions are normalized as $\langle p | q \rangle = \delta^{(3)}(p - q)$. We also have

$$\langle p | \hat{Q} \hat{G} | q \rangle = \delta(p - q)$$

Eq. (75) may be rewritten as follows

$$Z = \int D\bar{\psi} D\psi \exp \left( \int_{\mathcal{M}} \frac{d^3 p_1}{\sqrt{(2\pi)^3}} \int_{\mathcal{M}} \frac{d^3 p_2}{\sqrt{(2\pi)^3}} \bar{\psi}^T(p_1) Q(p_1, p_2) \psi(p_2) \right),$$

(76)

while the propagator of fermion is given by

$$G_{ab}(k_1, k_2) = \frac{1}{Z} \int D\bar{\psi} D\psi \exp \left( \int_{\mathcal{M}} \frac{d^3 p_1}{\sqrt{(2\pi)^3}} \int_{\mathcal{M}} \frac{d^3 p_2}{\sqrt{(2\pi)^3}} \bar{\psi}^T(p_1) Q(p_1, p_2) \psi(p_2) \right) \bar{\psi}_b(k_1) \psi_a(k_2) \sqrt{(2\pi)^3} \sqrt{(2\pi)^3}$$

(77)

Components of $\psi$ are enumerated by $a, b$. Below we omit those indices.

Wigner transformation of $G$ is equal to the Weyl symbol of $\hat{G}$:

$$G_W(x, p) = \int dq e^{iq(x - y)} G(p + q/2, p - q/2)$$

(78)

The Groenewold equation [10] relates them as follows

$$G_W(x, p) \ast Q_W(x, p) = 1$$

(79)

that is

$$1 = G_W(x, p) e^{i \left( \frac{i}{\hbar} p \cdot \partial_x - \frac{i}{\hbar} p \cdot \partial_y \right)} Q_W(x, p)$$

(80)

By $x$ we denote the coordinates that are composed of the discrete lattice coordinates and the continuous imaginary time. The differentiation over $x$ may be defined if the definition of the functions of coordinates is extended to
continuous values. Eq. (80) is valid as long as the variation of the field \( A(x) \) on the distance of the order of the lattice spacing may be neglected.

For an arbitrary lattice model the calculation of Weyl symbol \( Q_W \) of \( \hat{Q} = Q(p - A(i\partial_p)) \) is technically complicated. This problem has been solved in \[45\] for the particular case of lattice Wilson fermions. In general case if the field \( A \) is slowly varying, then up to the terms linear in the field strength \( Q_W(p, x) = Q_W(p - A(x)) \equiv Q(p - A(x)) \) (see \[10\]).

Electric current may be expressed as follows

\[
J^k(x) = -\int d^3 p \, \text{Tr} \, G_W(p, x) \partial_{p_k} Q_W(p - A(x)) \tag{81}
\]

The solution of Groenewold equation may be obtained via the expansion in powers of derivatives. The first two terms in this expansion are given by \[45\]:

\[
G_W(x, p) = G_W^{(0)}(x, p) + G_W^{(1)}(x, p) + \ldots \tag{82}
\]

\[
G_W^{(1)} = -i 2 G_W^{(0)} \frac{\partial}{\partial p_i} G_W^{(0)} \frac{\partial}{\partial p_j} G_W^{(0)} A_{ij}(x) \tag{83}
\]

Here \( G_W^{(0)}(x, p) \) is defined as the Green function with the field strength \( A_{ij} = \partial_i A_j - \partial_j A_i \) neglected. It is given by

\[
G_W^{(0)}(x, p) = Q^{-1}(p - A(x)) \tag{84}
\]

Substituting Eq. (82) to Eq. (81) gives the following result for the part of the current proportional to \( A_{ij} \):

\[
\langle J^k \rangle \approx \frac{1}{4\pi} M^{ijk} A_{ij} \tag{84}
\]

Tensor \( M^{ijk} \) is the topological invariant in momentum space given by

\[
M^{ijk} = \epsilon^{ijk} M, \quad M = \int \text{Tr} \, \nu \, d^3 p \tag{85}
\]

\[
\nu = \frac{i}{3! 4\pi^2} \epsilon^{ijk} \left[ Q(p) \frac{\partial Q^{-1}(p)}{\partial p_i} \frac{\partial Q(p)}{\partial p_j} \frac{\partial Q^{-1}(p)}{\partial p_k} \right] \tag{86}
\]

Appendix B. Calculation of \( \mathcal{N} \) for the 2 + 1 D systems

Here we repeat the calculation presented in Appendix C of \[10\]. We calculate the topological invariant \( \mathcal{N} \) in the case, when the Green function has the form

\[
G^{-1}(p) = i\sigma^3 \left( \sum_k \sigma^k g_k(p) - ig_4(p) \right) \tag{87}
\]

where \( \sigma^k \) are Pauli matrices while \( g_k(p) \) and \( g_4(p) \) are the real-valued functions, \( k = 1, 2, 3 \). Let us define

\[
\mathcal{H}(p) = \left( \sum_k \sigma^k \hat{g}_k(p) - i\hat{g}_4(p) \right) \tag{88}
\]

where \( \hat{g}_k = \frac{g_k}{g} \), and \( g = \sqrt{\sum_{k=1,2,3,4} g_k^2} \). Then

\[
\mathcal{N} = -\frac{1}{24\pi^2} \text{Tr} \int G^{-1} \, d\mathcal{G} \wedge d\mathcal{G}^{-1} \wedge d\mathcal{G} \tag{89}
\]

\[
= -\frac{1}{24\pi^2} \text{Tr} \int \mathcal{H} d\mathcal{H}^+ \wedge d\mathcal{H} \wedge d\mathcal{H}^+ \tag{90}
\]

\[
= -\frac{1}{48\pi^2} \text{Tr} \, \gamma^5 \int \mathcal{H} d\mathcal{H} \wedge d\mathcal{H} \wedge d\mathcal{H} \tag{91}
\]
where

$$\tilde{H}(p) = \sum_{k=1,2,3,4} \gamma^k \tilde{g}_k(p) = \text{diag}(H, -H^+) \gamma^4$$  \hspace{1cm} (89)$$

and $\gamma^k$ are Euclidean Dirac matrices in chiral representation, that is

$$\gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (90)$$

with $i = 1, 2, 3$. $\gamma^5$ in chiral representation is given by $\text{diag}(1, 1, -1, -1)$. This gives

$$N = \frac{1}{12\pi^2} \epsilon^{ijkl} \int \tilde{g}_i d\tilde{g}_j \wedge d\tilde{g}_k \wedge d\tilde{g}_l$$  \hspace{1cm} (91)$$

Let us introduce the parametrization

$$\tilde{g}_4 = \sin \alpha, \quad \tilde{g}_i = k_i \cos \alpha$$  \hspace{1cm} (92)$$

where $i = 1, 2, 3$ while $\sum_i k^2 = 1$, and $\alpha \in [-\pi/2, \pi/2]$. Let us suppose, that $\tilde{g}_4(p) = 0$ on the boundary of momentum space $p \in \partial M$. This gives

$$N = \frac{1}{4\pi^2} \epsilon^{ijkl} \int_{\mathcal{M}} \cos^2 \alpha k_i d\alpha \wedge dk_j \wedge dk_k$$  \hspace{1cm} (93)$$

Let us illustrate the above calculation by the consideration of the particular example of the system with the Green function $G^{-1} = i\omega - H(p)$, where the Hamiltonian has the form

$$H = \sin p_1 \sigma^2 - \sin p_2 \sigma^1 - (m + \sum_{i=1,2} (1 - \cos p_i)) \sigma^3$$  \hspace{1cm} (96)$$

This gives

$$-i \sigma^3 G^{-1} = \sin p_1 \sigma^1 + \sin p_2 \sigma^2 + \omega \sigma^3 - i(m + \sum_{i=1,2} (1 - \cos p_i))$$  \hspace{1cm} (97)$$

The boundary of momentum space corresponds to $\omega = \pm \infty$. We have

$$\tilde{g}_4(p) = \frac{m + \sum_{i=1,2} (1 - \cos p_i)}{\sqrt{(m + \sum_{i=1,2} (1 - \cos p_i))^2 + \sin^2 p_1 + \sin^2 p_2 + \omega^2}}$$
For example, for \( m \in (-2, 0) \) we have
\[
\begin{align*}
\hat{g}_4(p) &= 0, \quad p \in \partial \mathcal{M} \\
\hat{g}_4(p) &= -1, \quad \hat{g}_l(p) = 0 \quad (i = 1, 2, 3), \quad p = (0, 0, 0) \\
\hat{g}_4(p) &= 1, \quad \hat{g}_l(p) = 0 \quad (i = 1, 2, 3), \quad p = (0, \pi, 0) \\
\hat{g}_4(p) &= 1, \quad \hat{g}_l(p) = 0 \quad (i = 1, 2, 3), \quad p = (\pi, 0, 0) \\
\hat{g}_4(p) &= 1, \quad \hat{g}_l(p) = 0 \quad (i = 1, 2, 3), \quad p = (\pi, \pi, 0)
\end{align*}
\] (98)

Therefore, we get immediately
\[
\mathcal{N} = \frac{1}{2} - \frac{1}{2}(-1) - \frac{1}{2}(-1) - \frac{1}{2} = 1
\] (99)

In the similar way \( \mathcal{N} = -1 \) for \( m \in (-4, -2) \) and \( \mathcal{N} = 0 \) for \( m \in (-\infty, -4) \cup (0, \infty) \).

**Appendix C. AQHE in the considered toy model of \( 3 + 1 \)D Weyl semimetal**

Here we repeat the calculation of the AQHE conductivity given in Sect. 7.1 and Sect. 10.1 of [10]. The considered toy model of Weyl semimetal corresponds to the Green function \( G^{-1} = i\omega - H(p) \) and the Hamiltonian of the form
\[
H = \sin p_1 \sigma^2 - \sin p_2 \sigma^1 - (m - \cos p_3 + \sum_{i=1,2} (1 - \cos p_i)) \sigma^3
\] (100)

For \( m \in (0, 1) \) the system contains the two Fermi points
\[
K_{\pm} = (0, 0, \pm \beta, 0), \quad \beta = \arccos m
\] (101)

Although the Green function contains singularities, the integral in the expression for \( \mathcal{N}_3 \) is convergent. We first integrate over momentum space with the small vicinities of the poles subtracted, and then consider the limit, when those vicinities are infinitely small. We represent the Green function as follows
\[
-i\sigma^3 G^{-1} = \sin p_1 \sigma^1 + \sin p_2 \sigma^2 + \omega \sigma^3
\]
\[
-i(m - \cos p_3 + \sum_{i=1,2} (1 - \cos p_i))
\] (102)

The Hall current is given by
\[
\hat{j}^k_{Hall} = \frac{1}{4\pi^2} \mathcal{N}_l \epsilon^{kjl} E_j,
\] (103)

where \( \mathcal{N}_l \) is given by
\[
\mathcal{N}_l = -\frac{1}{3!4\pi^2} \epsilon_{ijkl} \int d^4p \text{Tr} \left[ G \frac{\partial G^{-1}}{\partial p_i} \frac{\partial G}{\partial p_j} \frac{\partial G^{-1}}{\partial p_k} \right]
\] (104)

Let us denote
\[
G^{-1}(p) = i\sigma^3 \left( \sum_k \sigma^k g_k(p) - ig_4(p) \right)
\] (105)

and \( \hat{g}_k = \frac{\alpha_k}{g} \), \( g = \sqrt{\sum_{k=1,2,3,4} \hat{g}_k^2} \). Next, we introduce the parametrization
\[
\hat{g}_4 = \sin \alpha, \quad \hat{g}_a = k_a \cos \alpha
\] (106)

where \( a = 1, 2, 3 \) while \( \sum_{a=1,2,3} k_a^2 = 1 \), and \( \alpha \in [-\pi/2, \pi/2] \). One can check that \( \hat{g}_4(p) = 0 \) on the boundary of momentum space. This gives
\[
\mathcal{N}_n = -\frac{1}{4\pi^2} \epsilon^{ijk} \int_{\mathcal{M}} \cos^2 \alpha \ k_n \partial_i \alpha \partial_j k_b \partial_k k_c \ d^4p
\]
\[
= -\frac{1}{4\pi^2} \epsilon^{ijk} \int_{\mathcal{M}} k_n \ d(\alpha/2 + \frac{1}{4} \sin 2\alpha) \wedge dk_b \wedge dk_c \wedge dp_n
\]
\[
= \sum_{l} \frac{1}{4\pi^2} \epsilon^{ijk} \int_{\partial M(\alpha)} k_n \left( (\alpha/2 + \frac{1}{4} \sin 2\alpha) dk_b \wedge dk_c \wedge dp_n \right)
\]
Now $\Omega(y^{(l)})$ is the small vicinity of line $y^{(l)}(s)$ in momentum space, where vector $k_i$ is undefined. Along these lines $\alpha \to \pm \pi/2$. We have

$$N_j = \frac{1}{2} \sum_l \int_{y^{(l)}(s)} \text{sign}(g_4(y^{(l)})) \text{Res} (y^{(l)}) \wedge dp_j$$ \hspace{1cm} \text{(107)}$$

Here we denote

$$\text{Res} (y^{(l)}) = \frac{1}{8\pi} \epsilon^{ijk} \int_{\Sigma(y^{(l)})} k_idk_j \wedge dk_k$$ \hspace{1cm} \text{(108)}$$

the corresponding integral is along the infinitely small surface $\Sigma$, which is wrapped around the line $y^{(j)}(s)$ near to the given point of this line. Notice, that in general case $\text{Res} (y^{(l)})$ enters the expression for the differential form. However, if we chose $\Sigma$ that belongs to three-dimensional hypersurface orthogonal to the curve $y^{(j)}(s)$, then $\text{Res} (y^{(l)})$ and $dp_j$ are factorized. $\text{Res} (y^{(l)})$ becomes the integer number, and we get

$$N_j = \frac{1}{2} \sum_l \int_{y^{(l)}(s)} \text{sign}(g_4(y^{(l)})) \text{Res} (y^{(l)}) \times dp_j (-1)^{N[\Sigma;y^{(j)}(s)]}$$ \hspace{1cm} \text{(109)}$$

Here $(-1)^{N[\Sigma;y^{(j)}(s)]} = \pm 1$ depending on the mutual orientation of the surface $\Sigma$ and the curve $y^{(j)}(s)$. For example, if $\Sigma$ belongs to the hyperplane $(x_1, x_2, x_3)$ while the line $y^{(j)}(s)$ coincides with the third axis, then $(-1)^{N[\Sigma;y^{(j)}(s)]} = -1$.

For the given particular Hamiltonian

$$\hat{g}_4(p) = \frac{m - \cos p_3 + \sum_{i=1,2} (1 - \cos p_i)}{\sqrt{(m - \cos p_3 + \sum_{i=1,2} (1 - \cos p_i))^2 + \sum_{i=1,2} \sin^2 p_i + \omega^2}}$$

and

$$\hat{g}_4(p) = 0, \quad p \in \partial M$$
$$\hat{g}_4(p) = -1, \quad \hat{g}_4(p) = 0 \quad (i = 1, 2, 3), \quad p = (0, 0, p_3, 0), \quad \text{for} \quad p_3 \in (-\beta, \beta)$$
$$\hat{g}_4(p) = 1, \quad \hat{g}_4(p) = 0 \quad (i = 1, 2, 3), \quad p = (0, 0, p_3, 0), \quad \text{for} \quad p_3 \in (-\pi, -\beta) \cup (\beta, \pi)$$
$$\hat{g}_4(p) = 1, \quad \hat{g}_4(p) = 0 \quad (i = 1, 2, 3), \quad p = (0, \pi, p_3, 0), \quad \text{for} \quad p_3 \in (-\pi, -\pi)$$
$$\hat{g}_4(p) = 1, \quad \hat{g}_4(p) = 0 \quad (i = 1, 2, 3), \quad p = (\pi, 0, p_3, 0), \quad \text{for} \quad p_3 \in (-\pi, -\pi)$$
$$\hat{g}_4(p) = 1, \quad \hat{g}_4(p) = 0 \quad (i = 1, 2, 3), \quad p = (\pi, \pi, p_3, 0), \quad \text{for} \quad p_3 \in (-\pi, -\pi)$$ \hspace{1cm} \text{(110)}$$

Therefore $N_1 = N_2 = N_4 = 0$ while

$$N_3 = \frac{-2\pi - 2\beta}{2} + \frac{2\beta}{2} - \frac{2\pi}{2} (-1) - \frac{2\pi}{2} (-1) - \frac{2\pi}{2}$$
$$= \frac{2\beta}{2}$$ \hspace{1cm} \text{(111)}$$

Thus we come to the expression for the AQHE current

$$J_{Hall}^k = \frac{\beta}{2\pi^2} \epsilon^{kij} E_j$$ \hspace{1cm} \text{(112)}$$

This result coincides with the one of the naive low energy effective field theory [15].

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