Symmetric Orbifolds and Entanglement Entropy for Primary Excitations in Two Dimensional CFT

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We use the techniques in symmetric orbifolding to calculate the Entanglement Entropy of a single interval in a two dimensional conformal field theory on a circle which is excited to a pure highest weight state. This is achieved by calculating the Rényi Entropy which is found in terms of a 2n-point function of primary operators, n being the replica number.

Entanglement Entropy (EE) has been the subject of extensive research in the last few years. Early interests in the subject, [1,2,3], came through the similarities of this quantity with the entropy of black holes through the area law [4,5]. However, EE proved to be a powerful tool on its own for understanding the quantum nature of physical systems (see for example [6–8] for reviews and references). Apart from being an important quantity in the subject, [1][2][3], came through the similarities of this quantity with the entropy of black holes through the area law [4,5].

The present work grew out of an attempt to study this quantity when the system is in a pure state. Most of the research on this subject has focused on the case where this pure state is the ground state of the system. In such situations EE is a quantification of his lack of information about the subsystem that is accessible to him.

Lack of information can also be caused by statistical distribution of states in a system such as thermal ensembles. In such situations EE will no longer be a useful measure of quantum entanglement and thus one usually studies this quantity when the system is in a pure state. Most of the research on this subject has focused on the case where this pure state is the ground state of the theory. In this article we are interested in excited pure states.

After the proposal of [9] (see also [10–12] for reviews and references), which gives a holographic interpretation of EE, there has been an even greater interest in the subject. The present work grew out of an attempt to find the holographic description of the EE of a single interval in a two dimensional CFT which has been excited by primary operators. The field theory side of this problem has already been addressed by two different methods in [13] and [14]. Here we present a third method, symmetric orbifolding [15], to address this problem which proves useful in finding its holographic description [16]. But first some preliminaries.

**Entanglement Entropy in QFT**

Suppose a physical system consists of two subsystems A and B and that the whole system is in a pure quantum state with the density operator \( \hat{\rho} \). Once we take the trace of the density operator over the B degrees of freedom, the resultant operator is called the reduced density operator and is denoted by \( \hat{\rho}_A(\equiv tr_B\hat{\rho}) \). Generically \( \hat{\rho}_A \) will no longer be pure and one can associate entropy to it. The EE for the subsystem A is denoted by \( S_A \) and is defined as the Von Neuman entropy of the reduced density operator, \( S_A \equiv -tr_A \hat{\rho}_A \ln \hat{\rho}_A \).

A useful mathematical quantity, called the Rényi Entropy (RE), is defined by the replica trick as

\[
S_A^{(n)} \equiv \frac{1}{n} \ln tr_A \hat{\rho}_A^n \quad \text{such that} \quad S_A = \lim_{n \to 1} S_A^{(n)}.
\]

Here \( n \) is a positive integer and is called the replica number. In Quantum Field Theory (QFT) this quantity can be represented in terms of a path integral. This is achieved by considering \( n \) copies of the world volume of the original theory, \( M \), and glueing them along the entangling subspaces in a cyclic order. This results in a space, which we denote by \( R_n \), and which has singularities on the boundaries of entangling subspaces. The path integral on \( R_n \) denoted by \( Z_{R_n} \), and is defined as

\[
Z_{R_n} = \int [d\varphi(x)] e^{-S[\varphi]} , \quad x \in R_n .
\]

This expression is proportional to RE. Except for some rare examples it is extremely difficult, if not impossible, to calculate \( Z_{R_n} \) directly. One can go around this by transferring the geometric complexities of the world volume into the geometry of target space. That is, one considers the original nonsingular world volume but instead introduces \( n \) copies of the target space fields, \( \varphi_i \ (i = 1, 2, ..., n) \), on that. Instead of gluing the world
volulmes one now restricts the fields to satisfy certain conditions along the entangling subspaces

$$Z_{\text{res}} = \int [d^{} \varphi (x)] e^{S[\varphi_1 , \ldots , \varphi_n]} , \quad x \in \mathcal{M} ,$$

where the subscript res stands for restrictions on fields. Note that these restrictions replace the nontrivial geometry of $\mathcal{R}_n$. One way to impose the restrictions is to insert the so called twist operators at the boundaries of entangling subspaces and calculate an unrestricted integral

$$Z_{\text{Twist}} = \int_{\text{unres}} [d^{} \varphi (x)] e^{S[\varphi_1 , \ldots , \varphi_n]} \prod \sigma_k , \ldots , x \in \mathcal{M} ,$$

where $\sigma_k$ are the twist operators that enforce restrictions through their Operator Product Expansion (OPE) with fields. Another way of imposing the restriction is to move over to the covering space of the fields, denoted by $\mathcal{M}_C$, with a suitable coordinate transformation and perform the calculations on this smooth manifold

$$Z_{\mathcal{M}_C} = \int [d^{} \varphi (x)] e^{-S[\varphi]} , \quad x \in \mathcal{M}_C .$$

On the covering space the restrictions on fields in the integration are taken care of by the geometry of $\mathcal{M}_C$. In the following we use this last way of imposing restrictions by orbifolding techniques. These restrictions amount to identifications of target space fields under subgroups of the symmetric group, $S_n$, and hence the name symmetric orbifolding. This process has been worked out in full detail in [13] for two dimensional theories in their ground state. The new ingredients in our case are the primary operators which excite the theory out of its ground state. In what follows, we focus on a generic two dimensional conformal field theory and consider a single entangling subspace. First a short outline of the method.

**Symmetric Orbifolding**

The outline is as follows[13]; suppose we start with a theory on sphere, parametrized by $(z, \bar{z})$, with a flat metric and with two branch points of order $n$ at $u$ and $v$. Here $n$ is the replica number and $u$ and $v$ are the endpoints of the entangling interval. By a coordinate transformation to $(w(z), \bar{w}(\bar{z}))$, which behaves as $w \approx z^{1/n}$ at branch points, one moves over to the covering sphere with the same line element (but a different metric). By a Weyl transformation with a conformal factor $|d^2 \varphi|^2$, one ends up with a third sphere with a fiducial metric $dz^2$ which we have chosen to be flat.

As the first two spheres are related by diffeomorphism, partition functions on the two are equal. This in turn is related to the partition on the third sphere by the exponent of the Liouville action imposed by Weyl anomaly. A careful calculation of this action results in the known expression for RE.

**The case for excitations**

We now wish to calculate the RE for a CFT on a sphere with a single branch cut. We further assume that the state we start with is a highest weight state with weights $(h, \bar{h})$. To proceed we use the following parameterisation for the sphere

$$ds^2 = dz \, d\bar{z} , \quad z < \frac{1}{\delta} ,$$

or $d\bar{z} \, dz , \quad \bar{z} < \frac{1}{\delta} ,

$$\bar{z} = \frac{1}{\delta} \frac{1}{z} .$$

We call this the $z$-sphere for which we have chosen a flat metric with a regularisation parameter $\delta$. Without loss of generality we choose the branch points as follows

$$u = ae^{i(\pi + \theta)} , \quad v = ae^{i(\pi - \theta)} .$$

In order for the theory to be in a highest weight excitation, we create the asymptotic in and out states by putting the corresponding operator $\mathcal{O}(z, \bar{z})$ at $z = \bar{z} = 0$ and $\mathcal{O}(\bar{z}, z)$ at $\bar{z} = z = 0$ where

$$\mathcal{O}(\bar{z}, z) = \mathcal{O}(z, \bar{z}) z^{2h} \bar{z}^{2\bar{h}} \delta^{2(h + \bar{h})} ,$$

and $\mathcal{O}$ is a primary operator with weights $(h, \bar{h})$. Note that the operator $\mathcal{O}$ is defined on the north pole cap which is parameterised by $\bar{z}$ and $\bar{z}$. One can equivalently create the out state at the north pole by introducing the adjoint operator as

$$\lim_{z, \bar{z} \to 0} \mathcal{O}(\bar{z}, z) = \lim_{\bar{z}, z \to 0} \mathcal{O}(\bar{z}, z) = \lim_{\bar{z}, z \to \infty} \mathcal{O}(\bar{z}, z) z^{2h} \bar{z}^{2\bar{h}} \delta^{2(h + \bar{h})} .$$

We should now make an $n$-sheeted Riemann sphere by appropriately gluing $n$ copies of the above spheres along the branch cuts and calculate the path integral on the resulting manifold. As stated before, one can equivalently calculate a restricted path integral of $n$ copies of the theory on a single sphere. We denote such a quantity by $\text{trp}^n(\theta)$ which is defined as

$$\text{trp}^n(\theta) \equiv \frac{\int_{\text{res}} [d^{} \varphi] e^{-S[\varphi_1 , \ldots , \varphi_n]} \prod_{i=1}^{n} \mathcal{O}_i(0) \mathcal{O}_i(\infty)}{\left[ \int [d^{} \varphi] e^{-S[\varphi]} \mathcal{O}(0) \mathcal{O}(\infty) \right]^n} ,$$

1 In the following, as a shorthand, we will be rather sloppy with this notation and denote the out state by $\mathcal{O}$ in the $z$ and $\bar{z}$ argument. So, for example, an out state in the north pole will be denoted by $\mathcal{O}(\infty).$
where \( \theta \) determines the entangling interval and the expression in the denominator is a normalization factor which ensures \( \text{tr} \rho_\theta = 1 \).

As explained in [15], the above path integral can be properly defined by cutting out circular holes of radius, say, \( \epsilon \) around the branch points as well as the infinity. One should then specify the proper boundary conditions for fields along the edges of the holes. The precise way of performing this procedure has been carried out in [15] with full details which we mostly skip. There are two new ingredients in our case which should be addressed. The first one is that unlike the case of [15] where the unity operator is inserted in \( z = \infty \), we insert the primary operator \( \mathcal{O} \) at \( z = 0 \). We address these issues in the following.

Let us define the covering space, the \( w \)-sphere, through the map

\[
\frac{z-u}{z-v} = \frac{1}{1 - \left( \frac{w+1}{w+1} \right)^n}. \tag{6}
\]

Near the branch points, the map behaves as \( z \approx t^n \). On this sphere the metric is induced through the map as

\[
ds^2 = \frac{dz}{dw} d\bar{z} dwd\bar{w}, \tag{7}
\]

and the regularisation parameter is also found as \( \delta' = a\delta \sin (\theta/2)/n \). There are several holes on this sphere. Two of these are the images of the branch points. There are also \( n \) holes coming from the images of \( z = \infty \) and \( n \) holes from \( z = 0 \).

The prescription of filling these holes should be such that upon path integration inside the holes we should end up with the desired states at the edges. As for the branch points, one such prescription is given in [15]. This is roughly filling the holes with disks of a flat metric which is continuously matched with the metric outside the hole over the edge. There is however a curvature concentration along the edge which is expected because around each branch point there is a cone with an excess angle.

The holes coming from \( z = \infty \) are naturally filled in by the images of \( |\bar{z}| < 1/\delta \) and those coming from \( z = 0 \) by the images of \( |z| < \epsilon \). The vertex operators \( \hat{O} \) and \( \hat{O} \) guarantee the desired wave functionals at the edges.

We now have a closed surface, the \( w \)-sphere, on which we want to calculate a certain path integral in presence of operator insertions. As usual, we choose a fiducial metric, which for simplicity, we take it to be a flat metric with an arbitrary regularisation parameter \( \delta' \). This is achieved by performing a Weyl transformation on the \( w \)-sphere, with a factor \( |dz/dw|^2 \), to make the metric flat.

We then introduce the coordinate \( t = w \) but choose \( \bar{\delta} \) as the regularisation parameter which defines \( \bar{t} = 1/(\bar{\delta} t) \).

The \( t \)-sphere thus defined is where we perform our calculations. Note that the path integrals on the different spheres are schematically related as

\[
\int_{\text{res}} [d^n\varphi(z, \bar{z})] e^{-S[\varphi_1, \ldots, \varphi_n]} = \int [d\varphi(w, \bar{w})] e^{-S[\varphi]} \tag{8}
\]

where dots stand for possible insertions, in case of which, the appropriate transformation factors should be included.

\[
S_L = \frac{c}{96\pi} \int dt^2 \sqrt{g} \partial_\mu \varphi \partial_\nu \varphi \gamma^{\mu\nu} + 2R\varphi \tag{9}
\]

where \( \varphi \) is the central charge of the theory and

\[
e^{\phi} = \frac{dz}{dt} \tag{10}
\]

The transformation turns the restricted path integral into an unrestricted one. This is so because the \( w \)-sphere, being an \( n \)-fold cover of the \( z \)-sphere, automatically enforces the desired identifications of fields along the branch cut.

As for the insertions, we note that the sequence of the \( \text{Diff} \times \text{Weyl} \) transformations, keeping the metric invariant, is a conformal transformation on the whole. Therefore for the primary fields \( \mathcal{O} \), we have

\[
\mathcal{O}(z, \bar{z}) = \left( \frac{dt}{dz} \right)^h \left( \frac{d\bar{t}}{d\bar{z}} \right)^{\bar{h}} \mathcal{O}(t, \bar{t}) \tag{11}
\]

and a similar one for \( \hat{O} \).

Putting everything together we find that

\[
\text{tr} \rho_\theta(\theta) = e^{S_L} \frac{Z_t}{Z_n} T \frac{\prod_{k=0}^{n-1} \langle \mathcal{O}(t_k) \hat{O}(t'_k) \rangle_t}{\langle \mathcal{O}(0) \hat{O}(\infty) \rangle^n_\bar{t}} \tag{12}
\]

where \( T \) is the transformation factor for operators, a product of those appearing in [15], and \( t_k \) and \( t'_k \) are the images of \( z = 0 \) and \( \bar{z} = 0 \) respectively. \( Z \) stands for partition function and the subscripts \( z \) and \( t \) denote on which sphere the corresponding quantities are calculated.

We now have all the ingredients to perform our calculations. First start with the Liouville part. It turns out that there are three different contributions to \( S_L \) (see [15] for details). One is coming from the kinetic term in the \( 1/\delta' < |t| < 1/\bar{\delta} \) region and the other comes from the curvature ring at \( |t| = 1/\bar{\delta} \). These two sum up to an

\[2\] We take \( \bar{\delta} < \delta' \) as a convenient choice for reasons to follow.

\[3\] In [15] these are denoted by \( S_L^{(2)} \) and \( S_L^{(3)} \) respectively.
expression which only depends on regulators $\delta$ and $\hat{\delta}$ and which cancels out in the end.

The main contribution to $S_L$, which we call $S_L^{(1)}$, comes from the region which is bounded between the edges of the holes, on the one hand, and $|t| = 1/\delta'$ on the other. Given that the metric is flat in this region, the only contribution to $\mathcal{W}$ comes from the Kinetic term which we can turn into a boundary term

$$S_L^{(1)} = \frac{c}{96\pi} \left[ i \int dt \, \phi \partial_t \phi + c.c. \right] ,$$

where the integration is along the boundaries of the region. The calculation of this integral for the branch points as well as the images of $z = \infty$ is identical to those in \cite{12} and nothing changes. As for the images of $z = 0$, it is straightforward to show that near such points

$$\phi = \log \frac{dz}{dt} \approx \log (c + a(t - t_0)) \; , \; \partial_t \phi \approx \frac{1}{c + a(t - t_0)} ,$$

where $a$ and $c$ are constants and $t_0$ is any of the images of $z = 0$. The integral \cite{13} for these values in the limit $t \rightarrow t_0$ will obviously be zero and there is no contribution to $S_L$ from the images of $z = 0$. The upshot is that as far as $S_L$ is concerned, operator insertions have no effect. Recalling that $tr \rho^n = e^{S_L} Z_t / Z^n$, and consulting \cite{12}, this statement leads to

$$\frac{tr \rho^n \phi}{tr \rho^n \phi} = T \left( \prod_{k=0}^{n-1} \frac{\langle O(t_k) \tilde{O}(t'_k) \rangle }{\langle O(0) \tilde{O}(\infty) \rangle} \right) \equiv \mathcal{F}_O^{(n)}(\theta) .$$

We now find the effect of vertex operators, i.e., calculate the factor $T$ appearing in \cite{12}. On t-sphere the branch points $z = (u, v)$ are mapped to $t = (z, \tilde{z})$. The point $z = 0$, on the other hand, is mapped to $n$ points which we denote by $t_k$

$$t_k = -i \cot \left( \frac{\theta + 2\pi k}{2n} \right) , \; \theta = 0, 1, ..., n - 1 .$$

The images of $z = \infty$ are denoted by $t'_k$ where

$$t'_k = -i \cot \left( \frac{\pi k}{n} \right) , \; k = 0, 1, ..., n - 1 .$$

Note that $t'_0 = \infty$.

To find $T$, we need to calculated $dt / dz$ at $t = t_k$ ($k = 0, 1, ..., n - 1$), $dt / dz$ at $t = t'_k$ ($k = 1, ..., n - 1$) and finally $dt / dz$ at $t = t'_0$. One can then write $T$ as a product of

$$T = \left( \frac{dz}{dt} \right)_{t_{k_0}} \times \prod_{k=1}^{n-1} \frac{dz}{dt} \left. \right|_{t'_k} \times \prod_{k=0}^{n-1} \frac{dz}{dt} \left. \right|_{t'_k} ,$$

times a similar expression but with $(z, t, h)$ replaced with $(\tilde{z}, \tilde{t}, \tilde{h})$. One finds that

$$\frac{dz}{dt} \bigg|_{t_k} = \frac{an}{\sin(\theta/2)} \sin^2 \left( \frac{\theta + 2\pi k}{2n} \right) , \; k = 0, 1, ..., n - 1 ,$$

$$\frac{dz}{dt} \bigg|_{t'_k} = \frac{n}{\delta^2 a \sin(\theta/2)} \sin^2 \left( \frac{\pi k}{n} \right) , \; k = 1, 2, ..., n - 1 ,$$

$$\frac{dz}{dt} \bigg|_{t'_0} = \left( \frac{\hat{\delta}}{\delta} \right)^2 \frac{n}{a \sin(\theta/2)} ,$$

which gives the final result as

$$\mathcal{F}_O^{(n)}(\theta) = \left( \frac{\hat{\delta} \, n \sin(\theta/2) / a}{2(n-1) \sin^2(\theta/2n)} \right)^{2(h+\tilde{h})} \times \langle \prod_{k=0}^{n-1} \frac{O(t_k) \tilde{O}(t'_k)}{(O(0) \tilde{O}(\infty))} \rangle .$$

This is our main result. Let us find an approximation to this formula in the limit $\theta \ll 2\pi$. Recall that

$$\langle O(t, \tilde{t}) \tilde{O}(0, 0) \rangle = \frac{1}{t^{2n+2} \hat{\delta}^2} [1 + O(\Delta, \tilde{\Delta}) t^{\Delta} \tilde{t}^{\tilde{\Delta}} + ...] ,$$

where dots in the second line stand for higher powers of $t$ and $\bar{t}$ and $O, \tilde{O}$ is the operator with smallest dimensions, $(\Delta, \tilde{\Delta})$, in the OPE. This gives

$$\prod_{k=0}^{n-1} \frac{O(t_k) \tilde{O}(t'_k)}{(O(0) \tilde{O}(\infty))} \equiv \mathcal{F}_O^{(n)}(\theta) .$$

We also recall that

$$\langle \tilde{O}(0) \rangle \equiv \mathcal{F}_O^{(n)}(\theta) \cdot$$

Putting everything together

$$\mathcal{F}_O^{(n)}(\theta) = 1 + \left( \frac{\theta}{3} \right)^2 + O(\theta^{\Delta+\tilde{\Delta}}) ,$$

which is in complete agreement with the result in \cite{13} in the same limit.

To further compare our result to those in \cite{13} and \cite{14}, we define yet another coordinate $s$ by

$$\frac{t + 1}{t - 1} = e^{is} ,$$

which maps the t-sphere into a cylinder with unit radius. Upon this transformation, the expression in \cite{19} will find the following form

$$\mathcal{F}_O^{(n)}(\theta) = \left( \frac{\prod_{k=0}^{n-1} \frac{O(t_k) \tilde{O}(t'_k)}{O(0) \tilde{O}(\infty))}^{cy} }{\langle \tilde{O}(0) \tilde{O}(\infty) \rangle} \right) ,$$

$$= \frac{1}{n^{2n(h+\tilde{h})} \prod_{k=0}^{n-1} \frac{O(t_k) \tilde{O}(t'_k)}{(O(0) \tilde{O}(\infty))}^{cy} } ,$$

(25)
where in the second line we have scaled the arguments in the denominator by a factor $n$. This is the result found in [13] and [14].

$EE$ is now immediately obtained by calculating $S = \frac{\partial}{\partial n} tr_\Omega(\theta)$ at $n = 1$. Noting that $tr_\Omega(\theta) = F^{(1)}_\Omega(\theta) = 1$, we find

$$S_\Omega(\theta) = S_{GS}(\theta) - \frac{\partial}{\partial n} F^{(n)}_\Omega(\theta)|_{n=1}. \quad (26)$$

Given the Gauge/Gravity duality, [17][18][19], it is a fair question to ask for a holographic analogue of the above calculations and results. This will amount to identifying the gravitational counterparts of the procedure used here.

In the original proposal of [9], the distinct role of the branch points on the boundary result in a desirable geometric realisation of $EE$ in terms of minimal lengths. This is intuitively understood by extending the curvature concentration at the branch points into the bulk. Using the covering space, as is the case in this letter, this geometric intuition is lost because the singularities at branch points are smoothed out. One should then find the bulk geometry which corresponds to the covering space in field theory. Exciting the theory into primary states will then be a matter of turning on appropriate bulk fields. This is the subject of [16].

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