Measurement of the Boltzmann constant by Johnson noise thermometry using a superconducting integrated circuit

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Abstract
We report on our measurement of the Boltzmann constant by Johnson noise thermometry (JNT) using an integrated quantum voltage noise source (IQVNS) that is fully implemented with superconducting integrated circuit technology. The IQVNS generates calculable pseudo white noise voltages to calibrate the JNT system. The thermal noise of a sensing resistor placed at the temperature of the triple point of water was measured precisely by the IQVNS-based JNT. We accumulated data of more than 429 200 s in total (over 6 d) and used the Akaike information criterion to estimate the fitting frequency range for the quadratic model to calculate the Boltzmann constant. Upon detailed evaluation of the uncertainty components, the experimentally obtained Boltzmann constant was \( k = 1.380\,6436 \times 10^{-23} \) J K\(^{-1}\) with a relative combined uncertainty of \( 10.22 \times 10^{-6} \). The value of \( k \) is relatively \(-3.56 \times 10^{-6}\) lower than the CODATA 2014 value (Mohr et al 2016 Rev. Mod. Phys. 88 035009).

Keywords: Boltzmann constant, SI unit, redefinition, Johnson noise thermometry, quantum voltage noise source, Josephson junction, thermal noise

(Some figures may appear in colour only in the online journal)

1. Introduction

According to the resolution of the 25th General Conference on Weights and Measures (CGPM), four of the SI (international system of units) base units—namely the kilogram, the ampere, the kelvin and the mole are supposed to be redefined based on fixed numerical values of fundamental constants at the next CGPM in 2018 [1]. Therefore, toward the ‘Revised SI’, the determination of the Boltzmann constant \( k \) is an important topic in the field of metrology and precision temperature measurements. There are several methods of determining the Boltzmann constant based on thermodynamic temperature measurement, such as acoustic gas thermometry (AGT), dielectric constant gas thermometry (DCGT), Doppler broadening thermometry (DBT) and Johnson noise thermometry (JNT) [2]. At the moment, the results obtained by AGT have the lowest uncertainty of \( 0.56 \times 10^{-6} \) [3]. Recently, the total uncertainty obtained by DCGT and JNT reached \( 1.9 \times 10^{-6} \) and \( 2.7 \times 10^{-6} \), respectively [4, 21], which satisfy the requirements for the determination of the Boltzmann constant [5].

Inspired by the pioneering works on JNT using a quantum voltage noise source [6–9], AIST has been trying to contribute to the determination of the Boltzmann constant with JNT based on an integrated quantum voltage noise source (IQVNS) [12–18]. The IQVNS fully consists of superconducting circuits including a pseudo random number generator based on a maximum length sequence algorithm, enabling us to eliminate concerns related to bit code generators such as input-output coupling, electromagnetic interference, and so on. Previously, the power spectral ratio was analyzed with a quadratic fitting model in [16], but the judgement of setting the optimal lower
and upper cutoff frequencies were not discussed and the uncertainty budget was not given. Therefore, the Boltzmann constant was not determined in [16]. In this paper, we compared various fitting models using the Akaike information criterion (AIC) to estimate the lower- and the upper-cutoff frequencies for the quadratic fitting model, which, we believe, should be the best-describing model for the low frequency limit. The uncertainty of the Boltzmann constant accompanied by the uncertainty of the determination of the cutoff frequencies are evaluated for the quadratic model. Other uncertainty components are also evaluated by experimental data and available information. Finally, the Boltzmann constant obtained by JNT at NMIJ is officially shown for the first time with combined uncertainty.

2. Experimental equipment of IQVNS based JNT

Figure 1 shows a schematic diagram of the NMIJ JNT system. This system includes two isolated measurement channels, A and B, for calculating the cross-power spectrum. Each measurement channel consists of an ultra-low-noise pre-amplifier (PreAmp), a low pass filter (LPF), a buffer amplifier (BufAmp) and an analog-to-digital (A/D) converter. The A/D converters are connected to a computer via optical fibers to eliminate ground loops and to reduce the noise from the computer. The calculation of the power spectra is done on the computer. During the JNT measurements, the sensing resistor and the IQVNS are alternately selected and connected to the measurement channels by a mechanical switch board (SW) controlled by the computer. Optical fibers were also utilized for the connection between the computer and the switch board. All the instruments except the microwave signal generator (SG) are driven by batteries to eliminate ground loops. The measurement equipment was installed in a normal test lab. A shield room was not used for the experiments. Data was collected over 6 d from June 13 to 20, 2016.

2.1. Integrated quantum voltage noise source

Figure 2 shows a simplified schematic diagram of the IQVNS. A simplified schematic diagram of the IQVNS is depicted in figure 2. The IQVNS is clocked at a frequency, $f_{\text{clk}}$, using an external SG. The IQVNS contains a digital circuit block and a bipolar voltage multiplier (BP-VM). The digital circuit block is fully implemented with superconducting electronics. The digital circuit block generates pseudo random noise signals based on the 21 bit maximum length sequence at a clock
rate of \( f_{LF} = f_{ck}/4096 \). The BP-VM comprises a superconducting quantum interference device (SQUID) array. One of the remarkable features in the structure of the IQVNS is the inductive decoupling technique between grounds of the superconducting digital circuit block and the BP-VM. Details of the IQVNS circuit design, fabrication, and its operation are given in [17]. Using the IQVNS design parameters of \( M = 2^{21} - 1 \), code length), \( N_1 (=74) \) and \( N_2 (=4) \), the power spectrum density (PSD) of the IQVNS at the DC limit is given by equation (1) [13]

\[
\text{PSD}_{IQ}(0) = \frac{2}{M} \left( \frac{h}{2e} \right)^2 f_{LF} N_1^2 N_2^2,
\]

where \( h \) and \( e \) stand for the Planck constant and the elementary charge, respectively. By setting the clock frequency \( f_{ck} \) at 8.192 GHz \( (f_{LF} = 2.000 \text{ MHz}) \), we obtain a \( \text{PSD}_{IQ}(0) \approx 1.499 \times 10^{-18} \text{ V}^2 \text{ Hz}^{-1} \), closely matching the PSD of a 100 \( \Omega \) sensing resistor at the temperature of the triple point of water \( (T_{TPW}) \). The tone spacing of the IQVNS, \( f_{LF}/M \), is 0.953 Hz.

2.2. Sensing resistor, resistor probe and triple point of water cell

A pair of custom-made 50 \( \Omega \) thick-film resistors connected in series was used as a 100 \( \Omega \) sensing resistor. The sensing resistor has two equivalent output channels for the cross-spectrum measurement and was installed in a 5 mm \( \times \) 5 mm \( \times \) 1 mm ceramic package covered with a metallic cap. It was firmly connected to a copper block at the tip of a laboratory fabricated probe, which was placed at the center of a commercial triple point of water (TPW) cell.

2.3. Amplifiers and filters

The low noise amplifier was composed of four stages: a low noise input amplifier, a high-gain instrumentation amplifier, a first buffer amplifier and a second buffer amplifier, which were integrated on a printed circuit board. The amplifier was the same as, or very similar to, those used in the JNT systems at the National Institute of Standards and Technology (NIST), USA [8] and National Institute of Metrology (NIM), China [10].

An 11 pole Butterworth filter with a cutoff frequency of 650kHz was connected between the first and second buffer stages to suppress aliasing signals. The signal-to-aliasing noise ratio is suppressed to 70 dB at 600kHz for a Nyquist frequency of 1 MHz, corresponding to \( 0.1 \times 10^{-6} \) in the Boltzmann constant measurements. At higher frequencies, the contribution of the aliasing signals becomes significant and may lead to crucial errors in the estimate of the Boltzmann constant. We define 600kHz as the physical upper limit for analysis to avoid uncertainty associated with the low pass filters.

2.4. Analog-to-digital converter

Commercial low distortion A/D converters were used for data acquisition. To make the tone spacing of equation (1) an integer multiple of the fast Fourier transform (FFT) bin spacing, \( D_s = f_s/N_s = 0.477 \text{ Hz} \), the sampling frequency, \( f_s \) was set to 2.000 MHz and the number of data points \( N_s \) to 2.000 MHz and \( 2M = 2(2^{21} - 1) \), respectively. According to the A/D converter’s specifications, the resolution of the A/D converter was at least 20 bits below 2 MHz.

2.5. Calculation of cross power spectra

Cross-power spectra were calculated using FFTs of a set of data consisting of \( N_s \) samples measured in each channel. We define a ‘chop’ as a set of 100 FFT results. The cross spectrum was calculated to eliminate uncorrelated noise picked up by the readout cables in two different channels and noise from the amplifiers.

2.6. Spectral ratio obtained using IQVNS based JNT

Figure 3 shows the ratio of spectral power densities of the thermal noise of the 100 \( \Omega \) sensing resistor and the IQVNS waveform accumulated for 2146 chops [16]. Each data point of the ratio, \( P_R/P_{IQ} \), is obtained by a mean average over 200 bins corresponding to 94.5 Hz. It is desirable to adjust the spectral ratio close to 1 to avoid the effect of non-linearity. Due to operating limitations of the commercial A/D converters and the IQVNS, it was not possible to adjust the spectral ratio to 1.000 at the frequency of \( f = 0 \). However, the ratio of spectral power densities always stays within \( 1 \pm 0.006 \) over a wide frequency range below \( f \leq 600 \text{ kHz} \).

3. Determination of fitting model and fitting frequency range

The physical model for the spectral ratio of the sensing resistor and the IQVNS can generally be expressed by equation (2) [10, 19]

\[
\frac{P_R}{P_{IQ}} = r_d(f) = a_0 + a_2 f^2 + \cdots + a_d f^d.
\]

The Boltzmann constant is calculated using the constant term \( a_0 \) of this fitting model. It is considered that the high-order terms are mainly due to the difference between the transfer functions of the wirings of the two different noise sources. In general, the quadratic model \((d = 2)\) should best describe the
physics in the low frequency limit [7, 8]. However, there is always the possibility to have small contributions from higher-order terms ($d > 2$) than $f^2$ even if the transfer functions of the wirings in both of the probes are expressed with up to 2nd order. The influences of the higher order terms cannot be ignored as the frequency increases. Therefore, it is necessary to quantitatively determine the frequency range that can be described with the quadratic model, if we employ a quadratic model to calculate the Boltzmann constant.

To find the optimal frequency range for fitting is indeed a complicated issue. Qu et al and Mohr et al [10, 19] determined the optimal fitting model using the cross-validation method. The fitting bandwidth was selected by an uncertainty minimization criterion. In this paper, we introduce various fitting models up to $d = 10$ to find the optimal fitting frequency range described by the quadratic model ($d = 2$), which is describing the low frequency limit.

For this purpose, the AIC was used to compare various order fitting models [20]. In our case, the AIC value is defined as follows,

$$AIC = n \ln \left( \frac{1}{2\pi} \sum \frac{(r(f_i) - r_d(f_i))^2}{n} \right) + 2 (d + 2),$$

where $r(f_i)$ denotes the experimentally obtained spectral ratio at $f = f_i$ and $r_d(f_i)$ is the spectral ratio calculated with the $d$th order fitting model. $n$ stands for the total number of data points. In this equation, the first term decreases with increasing $d$.

The AIC values for the models with $d = 2, 4, 6, 8$ and 10 were calculated as a function of the lower cutoff frequency, $F_{LC}$. To find the optimal and the widest frequency range with the model with $d = 2$, first, we changed the lower cutoff frequency, $F_{LC}$, from 2.5 kHz to 200 kHz, and fit the data from the $F_{LC}$ up to the cutoff frequency of the Butterworth filter of 650 kHz. Figure 4 shows the frequency range best-fitted by the model with $d = 2$ as a function of $F_{LC}$. Note that figure 4 shows the result only for $F_{LC} \leq 100$ kHz and the result with higher $F_{LC}$ showed a narrower best-fitting frequency range. We found that the data is best-fitted with the model with $d = 2$ and with the widest frequency range when we set $F_{LC} = 40$ kHz [16] and $F_{UC} = 380$ kHz. Figure 5(a) depicts the optimal fitting order $d$ as a function of $F_{UC}$ obtained by fixing $F_{LC} = 40$ kHz. The quadratic model with $d = 2$ is optimal for $F_{UC} \leq 380$ kHz, while the model with $d = 4$ is the best one for $F_{UC} \geq 460$ kHz, i.e. the order $d$ of the optimal model was changed from $d = 2$ to $d = 4$ at $F_{UC} = 380$ kHz. The optimal order is unstable in the frequency range of $380$ kHz $< F_{UC} < 460$ kHz. This widest and maximized frequency range led us to minimize the statistical uncertainty of the Boltzmann constant determination.

The uncertainty of the Boltzmann constant associated with the uncertainty of the determination of $F_{LC}$ and $F_{UC}$ for the quadratic model are discussed in the next chapter.

As shown in figure 5(b), the statistical uncertainty of the estimate of $a_0$, $\sigma_{a_0}$ monotonically decreases along the model with $d = 2$ down to $\sigma_{a_0} = 9.85 \times 10^{-6}$ at $F_{UC} = 380$ kHz. After showing unstable behavior in $380$ kHz $< F_{UC} < 460$ kHz, the value of $\sigma_{a_0}$ decreases monotonically along the model with $d = 4$ at $F_{UC} \geq 460$ kHz. As the order of the optimal $d$ increases, statistical uncertainty increases due to increase of fitting parameters. For this reason, in spite of the increasing fitting bandwidth, the statistical uncertainty obtained for the model with $d = 4$ at the maximum $F_{UC}$ is almost the best result for the model with $d = 2$.

Here, we define $a_{2014}$ as $a_{2014} = k_{2014} F_{TPW} R_{PSDQ}$, where $k_{2014}$ is the 2014 CODATA value of the Boltzmann constant, $k_{2014} = 1.38064852 \times 10^{-23}$ J K$^{-1}$ [11]. As displayed in figure 5(c), $(a_0 - a_{2014})$ gradually approaches to zero with increasing $F_{UC}$, $(a_0 - a_{2014})$ is $-3.56 \times 10^{-6}$ at $F_{UC} = 380$ kHz, which is the upper limit for the model with $d = 2$. At the frequency range of $380$ kHz $< F_{UC} < 600$ kHz, the value of $(a_0 - a_{2014})$ is within the range of $\pm 5 \times 10^{-6}$. The change is well within the statistical uncertainty at $F_{UC} = 380$ kHz, as discussed above.

In summary, there is no significant difference between the best result obtained for the quadratic model and the result obtained for maximum frequency range, at least at the current statistical uncertainty level. We adopt the simplest quadratic model, which is supposed to best describe the behavior of the spectral ratio at the low frequency limit. We obtained $(a_0 - a_{2014}) = -3.56 \times 10^{-6}$ and $\sigma_{a_0} = 9.85 \times 10^{-6}$ for the quadratic fitting model in the frequency from $F_{LC} = 40$ kHz to $F_{UC} = 380$ kHz. The values of $a_0$ and the statistical uncertainty of the estimate of $a_0$, $\sigma_{a_0}$, are listed in tables 1 and 2, respectively.

4. Uncertainty evaluation and the Boltzmann constant

Evaluation of uncertainty of the JNT using quantum voltage noise source has been extensively investigated by preceding studies [8, 10, 21, 22]. In the case of JNT using IQVNS the Boltzmann constant is expressed using equation (1) as

$$k = \frac{P_R \ PSDQ}{P_{IQ} 4 F_{TPW} R},$$

Therefore, the uncertainty of JNT using a quantum voltage noise source can be divided into four categories: the uncertainty of the measured spectral ratio $u_r(P_R/P_{IQ})$, the uncertainty of realization of TPW $u_t(F_{TPW})$, the uncertainty of the
The combined uncertainty $u_c$ is calculated as follows.

$$u_c^2 = u_r^2 \left( \frac{P_R}{P_{IQ}} \right) + u_r^2 (T_{TPW}) + u_r^2 (R) + u_r^2 (\text{PSDIQ}).$$

(5)

4.1. Spectral power ratio $u_r \left( \frac{P_R}{P_{IQ}} \right)$

4.1.1. Statistical uncertainty. The relative statistical uncertainty of $\sigma_{a_0}$ decreases with chop number $N$ as $1/\sqrt{N}$ without obvious signature of saturation for the fitting model $d = 2$ with a fitting frequency range of $40 \text{kHz} \leq F_{UC} \leq 380 \text{kHz}$ up to $N = 2146$, as shown in figure 6. This suggests that the influence of electromagnetic interference (EMI) is considerably lower than $10 \times 10^{-6}$.  

Table 1. Values used for calculation of the Boltzmann constant.

| Parameter | Value | Note |
|-----------|-------|------|
| $a_0 \left( = \frac{P_R}{P_{IQ}} \right)$ | 1.00569512 | Least square fitting with 2nd-order model from 40kHz to 380kHz |
| $T_{TPW}$ | 273.15993 K | Compared with NMIJ standard TPW cell |
| $R$ | 99.90395 Ω | Calculated with $K_{90}$ |
| $\text{PSDIQ}$ | $1.49856232 \times 10^{-18}$ | Calculated with $K_{90}$ |
| Chop number $N$ | 2146 | 1 chop $= 100 \text{s} \times 2$ |

4.1.2. Model ambiguity. As discussed in section 3, there are possibilities that other fitting models were selected depending on the upper cutoff frequency, resulting in a different answer. We assume the model ambiguity is not so different from the
We assign $1.8 \times 10^{-6}$ as the uncertainty of the model ambiguity (table 2).

4.1.3. Cutoff frequency dependence. Using AIC, we have roughly determined $F_{LC}$ and $F_{UC}$ as 40 kHz and 380 kHz, respectively. To investigate the cutoff frequency dependence of $a_0$, $a_0$ was calculated by fixing one of the cutoff frequencies, $F_{LC} = 40\, \text{kHz}$ or $F_{UC} = 380\, \text{kHz}$, and varying the other. The ambiguity of determination of the cutoff frequency is 5 kHz, because the frequency resolution of the calculation shown in figure 5 was 10.0 kHz. This ambiguity of $\pm 5\, \text{kHz}$ for $F_{LC}$ and $F_{UC}$ can lead to ambiguity of $1.9 \times 10^{-6}$ for $a_0$ in both cases. We account for this ambiguity by assigning a rectangular probability distribution of half-width $1.9 \times 10^{-6}$ for each case. Combining the uncertainties of $F_{LC}$ and $F_{UC}$, we assign $1.5 \times 10^{-6}$ for the cutoff frequency dependence (table 2).

4.1.4. Electromagnetic interference, EMI. In order to discuss the influences of EMI on the correlated signals of the IQVNS, we compared the noise floor of the IQVNS waveform appearing at odd bins to the IQVNS tones observed at even bins. The noise floor of the IQVNS waveform is expected to converge to zero with accumulation of data, since it is equivalent to measuring the correlated signal of a superconductor whose thermal noise is zero. It was found that the average value of the noise floor of the IQVNS waveform over the IQVNS tones was at most $0.1 \times 10^{-6}$. In order to see the influences of EMI in the resistance probe, the 100 $\Omega$ sensing resistor was removed and short circuited instead, and the correlated signal was accumulated for two days. We assign $0.5 \times 10^{-6}$ in total as the influence of EMI on the correlated signals of the IQVNS probe and the resistance probe (table 2).

4.1.5. Nonlinearity of noise thermometer. Ideally, the ratio of the power spectral density of thermal noise of the sensing resistor to that of the IQVNS is close to unity, in order to avoid the influence of the nonlinearity of the amplifier. As shown in figure 3, the spectral ratio is always within $1 \pm 0.006$ in the wide frequency range below 600 kHz. Instead of performing the Boltzmann constant measurement with several mismatches of the power spectral ratio of the sensing resistor to IQVNS [10], the IQVNS waveforms with different PSDs were directly compared by supplying different reference clock frequencies in order to eliminate the effect of the difference in the transfer functions of the wirings in the sensing resistance probe and the IQVNS probe. As a result, the ratio of power spectrum densities becomes flat against frequency. This comparison is possible for the IQVNS, because PSD can
be switched quickly by changing microwave frequency and power at the same time. During the measurement switchboards were turned off so that the amplifiers are always connected to the IQVNS. The microwave frequency and power for the IQVNS were changed every time after accumulating spectra for one chop (100 s) like a comparison between the sensing resistor and IQVNS. Spectral ratios, $R_{\text{Exp}} = P_{\text{IQVNS}} / P_{\text{IQ2}}$, for various nominal values were obtained by half day measurements. Figure 7 displays nominal spectral ratio, $R_{\text{nom}}$, versus relative deviation of $R_{\text{Exp}}$ from the nominal values $R_{\text{nom}}$. $(R_{\text{Exp}} - R_{\text{nom}}) / R_{\text{nom}}$. The nominal spectral ratio is equivalent to the ratio of clock frequencies to the IQVNS. The error bar is negligibly small for the nominal ratio of unity, because all the even chops and odd chops of the IQVNS collected so far were used. The slope obtained by a weighted fitting is $-2 \times 10^{-5}$, which is of the same order as [10], although the sign is opposite. The influence of the nonlinearity is estimated to be about $0.1 \times 10^{-6}$ for the nominal PSD ratio of 1.0057 at around $f = 0$. An uncertainty of $0.1 \times 10^{-6}$ is assigned to the nonlinearity effect of the noise thermometer (table 2).

4.1.6. Dielectric losses. It is not practical to actually measure dielectric losses of insulating materials of wirings and printed circuit boards (PCB) in the experimental equipment of the JNT system. Instead of measuring those quantities, we roughly estimated the influence of dielectric loss, $\tan \delta$, on power spectral densities. The calculations were based on the physical dimensions and properties of the insulating materials of the wirings in the noise sources and the PCB of the switch boards. The insulating materials of the wirings and the PCB are PTFE and FR4, respectively. We assigned $1 \times 10^{-6}$ as an influence on the spectrum intensity due to the dielectric loss (table 2).

4.2. Triple point of water temperature $u_4(T_{\text{TPW}})$

The thermowell of the TPW cell, in which the resistance probe is inserted, is filled with water so that the surface of the water in the thermowell has the same level as that of the TPW cell. The TPW temperature, $T_{\text{TPW}}$, is defined as 273.16 K in the present SI. In practice, however, there is always a difference from the definition in the actual TPW cell. The TPW cell was calibrated to be traceable to the national standard of temperature maintained at NMIJ/AIST. The temperature of the bottom of the thermowell was 273.15974 K. Using hydrostatic pressure correction, we determined the temperature of the bottom of the thermowell was 273.15974 K (table 1), which is static pressure correction, we determined the temperature of the bottom of the thermowell was 273.15974 K. Using hydrostatic pressure correction coefficient of 0.73 mK m$^{-1}$, which is about twice as large as hydrostatic pressure correction coefficient of 0.73 mK m$^{-1}$. We assigned $1 \times 10^{-7}$ as the uncertainty for the temperature of the sensing resistor placed at 30 mm from the bottom of the thermowell due to immersion effect (table 2).

4.3. Sensing resistor $u_4(R)$

The resistance across the pair of 50 $\Omega$ resistors, that form the sensing resistor, was found to be 99.90395 $\Omega$ (table 1). The resistance value was measured every day before measurement of the thermal noise. The statistical uncertainty was $0.7 \times 10^{-7}$ (table 2). The resistance of the thermometer was measured using a commercial resistance thermometry bridge, Fluke 1594A. This bridge measures the resistance ratio of a device under test with respect to a 100 $\Omega$ reference standard resistor. The reference resistor was calibrated using a resistance calibration system traceable to the quantized Hall resistance standard (the national standard of resistance) of NMIJ/AIST. The uncertainty of resistance calibration is $5.6 \times 10^{-8}$ ($k = 2$). We assigned $2.8 \times 10^{-8}$ ($k = 1$) as the uncertainty associated with the long-term drift of the sensing resistor (table 2). The ratio error of the resistance thermometry bridge was investigated using two calibrated 100 $\Omega$ standard resistors. The uncertainty due to the ratio error of the resistance thermometry bridge was found to be $6 \times 10^{-8}$ (table 2). The uncertainty due to leakage resistance was found to be of the order of $1 \times 10^{-10}$ (table 2). The temperature dependence of the resistance was measured in the temperature range from 273.6 K to 303.2 K. Extrapolating the result to 273.16 K, the temperature coefficient of the sensing resistor was estimated to be $2 \times 10^{-6}$ K$^{-1}$. Since temperature distribution in the thermowell is about 0.2 mK, the upper limit
of the uncertainty due to the temperature dependence of the sensing resistor is $2 \times 10^{-10}$ (table 2).

Precise measurement of the frequency dependence of the sensing resistor is not straightforward, especially when the relative resistance change caused by frequency change is below the $1 \times 10^{-6}$ level. Instead of actually measuring the frequency dependence of the sensing resistor installed on the resistance probe, we estimated the uncertainty associated with frequency dependence of the sensing resistor using data on the specification sheet of a similar thick-film resistor. The resistance change of a $100 \Omega$ thick-film resistor is $2\%$ at $500 \text{ MHz}$. Assuming that the frequency dependence is quadratic, we estimated that the resistance change is estimated to be of the order of $0.02 \times 10^{-6}$ at $500 \text{ kHz}$. We assigned $0.02 \times 10^{-6}$ to the uncertainty associated with frequency dependence of the sensing resistor (table 2).

### 4.4. IQVNS waveform $u_t(\text{PSD}_{\text{IQ}})$

The clock frequency of the pseudo random number generator in the superconducting digital circuit block $f_{\text{CLK}}$ is related to the clock frequency supplied to the IQVNS from an external SQUID $f_{\text{CLK}}$ as $f_{\text{CLK}} = f_{\text{CLK}}/0.096$. Since the uncertainty of the external SQUID is governed by the uncertainty of the $10 \text{ MHz}$ reference signal from GPS, which is $1 \times 10^{-11}$, relative uncertainty of PSD$_{\text{IQ}}$ given by equation (1) is also expected to be of the order of $10^{-11}$, which is negligibly small compared to other uncertainty components (table 2).

The IQVNS does not utilize any modulation techniques with quantization error. The pseudo random number sequence is generated based on a primitive polynomial. The SQUID arrays in the BP-VM guarantee quantization of voltage pulses. We have investigated bit error probability of the IQVNS, and estimated that the error in the spectrum density of the IQVNS is as small as $1 \times 10^{-7}$ (table 2).

### 4.5. The Boltzmann constant and combined uncertainty

The Boltzmann constant $k_{00}$ determined by the actual experiment is obtained to be $k_{00} = 1.3806434 \times 10^{-23} \text{ J K}^{-1}$ using the parameters, $a_0 = \text{PSD}_{\text{IQ}}$, $T_{\text{TPW}}$, $R$ and PSD$_{\text{IQ}}$ summarized in table 1. The value of $k_{00}$ and the Boltzmann constant $k$ to be evaluated are related as $k/h = k_{00}/h_{00}$, where the Planck constant $h$ and $h_{00} \equiv 4/(k_{1,0} R_{\text{K},90})$ are defined as $h = 6.626 \times 10^{-40}(s) \times 10^{-34} \text{ J} \cdot \text{s}$ and $h_{00} = 6.626 \times 854 \times 10^{-34} \text{ J} \cdot \text{s}$, respectively [11]. Therefore, $k$ is finally determined as $k = 1.3806436 \times 10^{-23} \text{ J K}^{-1}$. The value of $k$ is relatively $3.56 \times 10^{-6}$ lower than the CODATA 2014 value [11]. The combined relative uncertainty in the measurement of $k$ is calculated as $10.22 \times 10^{-6}$ using equation(5). The details of the uncertainties are listed in table 2.

### 5. Conclusion

We evaluated the Boltzmann constant $k$ by JNT calibrated with an IQVNS. To obtain the Boltzmann constant, the ratio of power spectral densities of the sensing resistor over that of the IQVNS were fitted with a quadratic fitting function. The Akaike information criterion was employed to roughly estimate the upper- and the lower-cutoff frequency for the quadratic fitting model. Based on experiments and theoretical considerations, the uncertainty budget was given. The Boltzmann constant evaluated for 2146 chops is determined to be $k = 1.3806436 \times 10^{-23} \text{ J K}^{-1}$ with a relative combined uncertainty of $10.22 \times 10^{-6}$. Considering that the noise generation employing the IQVNS differs from the established QVNS, our result reported here has significance as an independent result for the JNT.

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