Some notes on the superintuitionistic logic of chequered subsets of $\mathbb{R}^\infty$

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Note 2018: This paper was originally published in the Bulletin of the Section of Logic, vol. 33(2), pp. 81–86, 2004. This version (prepared soon after the paper was published and previously only available on my webpage) has been slightly extended; in particular, the proof of main theorem is hopefully more readable than in the Bulletin version. Very interesting follow-up results, settling in the negative questions posed in the final paragraph of this paper, have been announced by Gaëlle Fontaine and Timofei Shatrov. The former author shows in her AiML 2006 paper that $ML$ is not finitely axiomatizable over $Cheq$. The latter author claims to have settled negatively the issue of finite axiomatizability of $Cheq$ itself; however, to the best of my knowledge, this has never been published. Speaking of unpublished work, I also cannot find any final journal version of Grigolia’s results referred to in this paper. If his claims have never been verified, this also leaves open the status of the last sentence in the statement of Corollary.

Abstract

We are going to investigate the superintuitionistic analogue of the modal logic of chequered subsets of $\mathbb{R}^\infty$ introduced by van Benthem et al. It will be observed that this logic possesses the disjunction property, contains the Scott axiom, fails to contain the Kreisel-Putnam axiom and is not structurally complete. We will prove that it is a sublogic of the Medvedev logic $ML$.

In recent years, there seems to be growing interest in modal logics determined by various topological spaces and particular families of their subsets. Bezhanishvili et al. improved on a classical result by McKinsey and Tarski that $S4$ is complete with respect to the real line by showing that it is actually enough to consider only countable unions of convex subsets. On the other hand, Aiello et al. proved that the modal logic determined by finite unions of convex subsets of $\mathbb{R}$ is a very strong tabular extension of $Grz$ complete with respect to 2-fork Kripke frame $\mathcal{F}_1$. $\mathcal{F}_1 = \langle W_1, R_1 \rangle$ is the first frame in Figure $W_1 = \{w_0, w_-, w_+\}$, all points are $R_1$-reflexive, $w_0$ $R_1$-sees all the other
points. Van Benthem et al. [2] investigated logics determined by finite unions of products of convex subsets of $\mathbb{R}$ in $\mathbb{R}^\alpha$ (where $\alpha \in \mathbb{N} \cup \{\infty\}$); such subsets were called chequered. It was established that for $\alpha = n$, the modal logic in question corresponds to the logic determined by $\mathcal{F}_n = \mathcal{F}_{n-1} \times \mathcal{F}$, the order being the standard product order. In case of $\alpha = \infty$, the respective modal logic is determined by infinite sequence of frames $\{\mathcal{F}_n\}_{n \geq 1}$.

It is, however, worth recalling that there exists another, simpler tool well-tailored for describing topological spaces: it is the language of intuitionistic propositional logic and its extensions. In particular, there is a strict correspondence between normal extensions of $\text{Grz}$ and intermediate logics, as follows e.g., from Blok-Esakia theorem and transfer results of Chagrov and Zakharyaschev (cf., e.g., Chagrov et al. [4]). Thus, the results of van Benthem et al. [2] describe semantically superintuitionistic logics determined by finite unions of products of open intervals. We will follow this line of investigation, denoting the logic determined by $\{\mathcal{F}_n\}_{n \geq 1}$ as Cheq. It will be proven that this logic possesses the disjunction property, contains Scott axiom $\text{sa}$ and fails to contain the Kreisel-Putnam axiom $\text{kp}$.

This perspective allows us to compare Cheq with other superintuitionistic systems. Perhaps the most famous semantically defined one is Medvedev logic ML. It is determined by the class of all Boolean cubes with top element deleted — i.e., by the sequence of frames $\{\mathcal{M}_n\}_{n \geq 1}$ where each $\mathcal{M}_n$ is the set of all proper subsets of $\{1, \ldots, n + 1\}$ ordered by the standard inclusion

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{frames.png}
\caption{Frames $\mathcal{F}_1$ and $\mathcal{F}_2$}
\end{figure}
relation (Figure 2). It is known from Maksimova et al. [6] that this logic is not finitely axiomatizable, its decidability being still one of the most famous open problems in the field. As perhaps the most interesting part of this short note, we are going to prove that \textbf{Cheq} is a proper sublogic of \textbf{ML}.

By At\((n)\) we shall denote the set of immediate successors of the root of \(\mathfrak{F}_n\). For example, \(At(1) = \{w_-, w_+\}\), \(At(2) = \{w_- w_0, w_+ w_0, w_0 w_-, w_0 w_+\}\) and so on. The only coordinate at which \(\bar{x}\) is distinct from \(w_0\) will be denoted as \(\uparrow \bar{x}\) i.e., for any \(\bar{x} \in At(n)\), \(\uparrow \bar{x} = i\) iff \(x_i \neq w_0\). For example, \(\uparrow w_0 w_0 w_- = 3\).

By \(\bar{x}|_n\) we will denote \(\bar{x}\) with \(n\) leftmost coordinates deleted; analogously, \(\bar{x}|_n\) will denote \(\bar{x}\) with \(n\) rightmost coordinates deleted. If \(\bar{x}\) belongs to \(\mathfrak{F}_n\), then both \(\bar{x}|_n\) and \(\bar{x}|_n\) belong to \(\mathfrak{F}_{n-m}\).

**Theorem 1** \textbf{Cheq} has the disjunction property.

**Proof.** Exactly as in the case of \textbf{ML}; it is known that a logic \(L\) has the disjunction property if it is characterized by a class \(\mathcal{C}\) of descriptive rooted frames s.t. the disjoint union of any two rooted frames from \(\mathcal{C}\) is a generated subframe of a frame for \(L\) (cf. Chagrov et al. \[4, Theorem 15.5\]). It may be easily seen that \(\mathfrak{F}_n + \mathfrak{F}_m\) is a generated subframe of \(\mathfrak{F}_{n+m}\). Indeed, one of the facts that make comparison of \textbf{Cheq} and \textbf{ML} interesting is that frames for both systems have the property of self-resemblance; any rooted generated subframe of a frame in the sequence is isomorphic to some frame earlier in the list. This fact will be used continually from now on.

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Figure 2: Frames \(\mathfrak{M}_1\) and \(\mathfrak{M}_2\)
Theorem 2 Cheq contains the Scott axiom

\[ \text{sa} = ((\neg p \rightarrow p) \rightarrow (p \lor \neg p)) \rightarrow (\neg p \lor \neg \neg p) \]

Proof. By induction; we are going to use the above-mentioned self-resemblance property.

1. The Scott axiom holds in \( F_1 \).

2. Assume that the Scott axiom holds in \( F_1, \ldots, F_n \). Hence, (by the self-resemblance property) the only point of \( F_{n+1} \) where it can fail under some valuation \( V \) is the root \( \bar{r} = w_0 \ldots w_0 \). We get that

\[
\begin{align*}
  r &\models_{V} (\neg\neg p \rightarrow p) \rightarrow (p \lor \neg p), \quad (1) \\
  r &\not\models_{V} \neg p \lor \neg\neg p \quad \text{but} \quad (2) \\
  x &\models_{V} \neg p \lor \neg\neg p \quad \text{for every other } x. \quad (3)
\end{align*}
\]

Statement (3) is obtained from (1) by the above-mentioned fact that no proper successor of the root can fail the Scott axiom. This statement implies that any immediate successor of \( r \) (i.e. each element of \( At(n+1) \)) satisfies either \( \neg p \) or \( \neg\neg p \) under \( V \). If all of them satisfy \( \neg p \), then it would contradict (2) for the same reason, at least one point in \( At(n+1) \) must refute \( \neg p \) (and hence verify \( p \)). Hence, we have that there are \( s, s' \in At(n+1) \) s.t. \( s \models_{V} \neg p \) and \( s' \models_{V} \neg p \). Every maximal successor of \( s \) must then verify \( \neg p \) and every maximal successor of \( s' \) must verify \( p \). But now the following fact holds:

For any \( u, u' \in At(n+1) \), there exist \( v(u, u') \in At(n+1) \) s.t. \( v(u, u') \) has a common (maximal) successor both with \( u \) and \( u' \).

Hence, \( v(s, s') \) fails \( \neg p \lor \neg\neg p \), a contradiction with (3). \( \dashv \)

Theorem 3 Cheq does not contain the Kreisel-Putnam axiom

\[ \text{kp} = (\neg p \rightarrow q \lor r) \rightarrow (\neg p \rightarrow q) \lor (\neg p \rightarrow r) \]

Proof. It fails in frame \( \mathfrak{F}_2 \) under valuation \( \mathfrak{V} \) defined as follows:

\[
\begin{align*}
  \mathfrak{V}(p) &:= \{w^-w_+, w_+w_-, w_+w_+\}, \quad \mathfrak{V}(q) := \{w^-w_-, w_+w_+\}, \quad \mathfrak{V}(r) := \{w_+w_+\}.
\end{align*}
\]

\( \dashv \)
Theorem 4 For every \( n \geq 0 \), \( \mathcal{M}_n \) is a (generated subframe of a) \( p \)-morphic image of some \( \mathfrak{F}_k \).

Proof. We are going to prove it by showing that for every
\( n = 2^m - 1 \ (m \geq 1) \) there exists a \( p \)-morphism \( f_n \) from \( \mathfrak{F}_n \) onto \( \mathcal{M}_n \). The restriction on \( n \) is motivated only by reasons of compactness and convenience; it is possible to prove Theorem 4 for any \( n \).

1. For \( m = 1 \), \( f_1 \) is just an isomorphism.
2. Assume that \( f_{2^m-1} \) is defined, \( p = m + 1 \), \( n = 2^p - 1 \).

\[
f_n(\bar{x}) = \begin{cases}
  f_{2^m-1}(\bar{x}) : & \bar{x} \in \text{At}(n), \uparrow \bar{x} < 2^m \\
  \{k + 2^m \mid k \in f_{2^m-1}(2^m-1|\bar{x}|)\} : & \bar{x} \in \text{At}(n), 2^m \leq \uparrow \bar{x} < n \\
  \{1, \ldots, 2^m\} : & \bar{x} \in \text{At}(n), x_n = w_- \\
  \{2^m + 1, \ldots, 2^p\} : & \bar{x} \in \text{At}(n), x_n = w_+ \\
  \bigcup \{f_n(\bar{y}) \mid \bar{y} \in \text{At}(n), \bar{y}R_n\bar{x}\} : & \bar{x} \notin \text{At}(n)
\end{cases}
\]

We will sketch why this is a \( p \)-morphism onto \( \mathcal{M}_n \). The proof is by induction; assume that \( m > 1 \) and \( f_{2^m-1} \) is a \( p \)-morphism.

- The image of \( \mathfrak{F}_n \) via \( f_n, f_n[\mathfrak{F}_n] \), is contained in \( \mathcal{M}_n \). As by definition \( f_n(\bar{y}) \) is a subset of \( \{1, \ldots, 2^p\} \) for any \( \bar{y} \), we only have to show that for no \( \bar{y}, f_n(\bar{y}) = \{1, \ldots, 2^p\} \). Assume that \( f_n(\bar{y}) \supseteq \{1, \ldots, 2^m\} \). \( f_n(\bar{y}) = \bigcup \{f_n(\bar{x}) \mid \bar{x} \in \text{At}(n), \bar{x}R_n\bar{y}\} \). For \( \bar{x} \in \text{At}(n), f_n(\bar{x}) \cap \{1, \ldots, 2^m\} \neq \emptyset \) only if \( \uparrow \bar{x} < 2^m \) or \( \uparrow \bar{x} = n \). But by the induction assumption \( f_{2^m-1} \) is a morphism into \( \mathfrak{F}_{2^m-1} \) and hence no sum of \( f_{2^m-1}(\bar{x}) \) for \( \uparrow \bar{x} < 2^m \) can give \( \{1, \ldots, 2^m\} \). Thus, it is necessary that \( \bar{y} \) is above the \( \bar{x} \in \text{At}(n) \) s.t. \( x_n = w_- \). This means, however, that the \( \bar{x} \in \text{At}(n) \) s.t. \( x_n = w_+ \) is not below \( \bar{y} \) and thus we obtain (by repeating the previous argument) that \( f_n(\bar{y}) \nsubseteq \{2^m + 1, \ldots, 2^p\} \).

- The forth condition — \( \bar{x}R_n\bar{y} \) implies \( f_n(\bar{x}) \subseteq f_n(\bar{y}) \) — follows directly from definition of \( f_n \); if \( \bar{z} \in \text{At}(n) \) and \( \bar{z}R_n\bar{x} \) then \( \bar{z}R_n\bar{y} \).

- That \( f_n \) is onto and that it satisfies the back condition — \( f_n(\bar{x}) \subseteq Y \) implies the existence of \( \bar{z} \) s.t. \( \bar{x}R_n\bar{z} \) and \( f_n(\bar{z}) = Y \) — may be established as follows. Assume that \( Y \nsubseteq \{1, \ldots, 2^m\} \). It means that \( f_n(\bar{x}) \nsubseteq \{1, \ldots, 2^m\} \); let us denote by \( \bar{u}_1 \) the supremum of \( \bar{y} \in \text{At}(n) \) s.t. \( \uparrow \bar{y} < 2^m \) and \( \bar{y}R_n\bar{x} \) (we use here the fact that for any element of \( \mathfrak{F}_n \), the
set of all its predecessors forms a lattice). Thus, \( f_n(\bar{x}) \cap \{1, \ldots, 2^m\} = f_n(\bar{u}_1), \; f_n(\bar{u}_1) = f_{2^m-1}(\bar{u}_1 | 2^m) \) and \( f_n(\bar{u}_1) \subseteq Y \cap \{1, \ldots, 2^m\} \). By the induction assumption that \( f_{2^m-1} \) satisfies the back condition, there exists \( \bar{v}_1 \in \mathfrak{F}_{2^m-1} \) s.t. \( \bar{u}_1 R_{2^m-1} \bar{v}_1 \) and \( f(\bar{v}_1) = Y \cap \{1, \ldots, 2^m\} \). Now, if \( Y \supseteq \{2^m+1, \ldots, 2^p\} \), then \( \bar{u}_2 \) will be defined as the supremum of all \( \bar{y} \in \mathfrak{A}(n) \) s.t. \( 2^m \leq \bar{y} < n \) and \( \bar{y} R_n \bar{x} \) and the \( \bar{y} \in \mathfrak{A}(n) \) s.t. \( \bar{y}_n = w_0 \); then define \( \bar{v}_2 =_{2^m-1} \bar{u}_2 \). Otherwise, \( \bar{v}_2 \) is constructed analogously as \( \bar{v}_1 \) with the addition of \( w_0 \) in the last coordinate. Now, concatenation of \( \bar{v}_1 \) and \( \bar{v}_2 \) is the desired \( \bar{z} \). The case when \( Y \supseteq \{1, \ldots, 2^m\} \) but \( Y \not\supseteq \{2^m+1, \ldots, 2^p\} \) is dealt with analogously.

Corollary 5 Cheq is a proper sublogic of ML. Thus, it is also a sublogic of KS — the logic of weak law of excluded middle. Cheq is not structurally complete.

Proof. The fact that Cheq is a sublogic of ML follows from Theorem 4. The fact that it is a proper sublogic follows from Theorem 3 as ML contains the Kreisel-Putnam axiom kp. It may be proven also in a more direct way: the Jankov formula of \( \mathfrak{F}_2 \) belongs to ML, which may be proven inductively, using self-resemblance property. However, this would not give us any new information: Theorem 3 implies that ML \( \not\supseteq \mathfrak{A} \) and that is equivalent to the fact that the Jankov formula of \( \mathfrak{F}_2 \) belongs to ML.

The fact that Cheq cannot be structurally complete follows from Theorem 3 and the recent result of Grigolia [5], according to which ML is the only intermediate logic which is both structurally complete and has the disjunction property.

Theorem 6 There exists a p-morphic image of \( \mathfrak{F}_2 \) which verifies kp but is not a p-morphic image of any \( \mathfrak{M}_n \).

Proof. Consider the frame \( \mathcal{Y} \) depicted in Figure 4. We will prove by induction that this is not a p-morphic image of \( \mathfrak{M}_n \). For \( n = 2 \) it is obvious by a
cardinality argument. For \( n = m + 1 \), by (IH) there is no \( p \)-morphism from any proper generated subframe of \( \mathfrak{M}_n \) onto \( \mathfrak{H} \), so the only set mapped onto \( r \) is the empty set, and there must be distinct singletons \( \{x\} \) and \( \{y\} \) mapped onto \( a \) and \( d \), respectively. The complement of \( \{y\} \) (denoted by \( Y \)) and the complement of \( \{x\} \) (denoted by \( X \)) must then be mapped onto \( e \) and \( f \), respectively. It means then that \( X \cap Y \) has to be mapped either onto \( b \) or \( c \). Wlog assume it is mapped onto \( b \). Then there has to be \( Z \not\subseteq X \cap Y \) mapped onto \( c \). However, it must be either the case that \( \{x\} \subseteq Z \) or \( \{y\} \subseteq Z \). Assume \( \{x\} \subseteq Z \). Then, by applying the forth condition, the image of \( \{x\} \) (equal to \( a \)) is below \( c \), a contradiction.

\[ \therefore \]

Figure 3: The frame \( \mathfrak{H} \).

Thus, to investigate further the connection between \( \text{Cheq} \) and \( \text{ML} \) it is crucial to know whether the techniques of Maksimova et al. can be applied in case of \( \text{Cheq} \). Let us recall that the above-mentioned paper used a sequence of frames, which we will denote here as \( \{ \mathfrak{G}_n \}_{n \geq 1} \) (in the original paper they were denoted as \( \{ \Phi_k \}_{k \geq 1} \)). It was proven that (A) none of \( \mathfrak{G}_n \) validates \( \text{ML} \), and hence the Jankov formula of \( \mathfrak{G}_n \) belongs to \( \text{ML} \) for every \( n \); (B) that for every \( n \) and every \( i \leq n \), a modification of \( \mathfrak{G}_n \) denoted as \( \mathfrak{G}^i_n \) validates \( \text{ML} \) and hence canonical formula of \( \mathfrak{G}^i_n \) does not belong to \( \text{ML} \); (C) that for a \( n \)-formula \( \varphi \) (an \( n \)-formula is a formula in \( n \)-variables), \( \varphi \) entails the canonical formula of \( \mathfrak{G}_n \) only if it entails the canonical formula of \( \mathfrak{G}^i_n \) for some \( i \leq n \).

Now, the question is whether the frames from the sequence \( \{ \mathfrak{G}_n \}_{n \geq 1} \) verify \( \text{Cheq} \). If so, it means that \( \text{ML} \) is not finitely axiomatizable over \( \text{Cheq} \).
and the connection between the two systems is much weaker than Theorem 4 would seem to suggest. If there are no suitable \( p \)-morphisms, it means that \( \text{Cheq} \), like \( \text{ML} \), has no axiomatization in finitely many variables. The author has not been able to generalize Lemma 6 from Maksimova et al. [6], which was crucial for the main result. On the other hand, he has not been able to define suitable \( p \)-morphisms either. Anyway, the answer to this question would make clear how similar (or how different) \( \text{Cheq} \) and \( \text{ML} \) are.

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