CORDES-NIRENBERG TYPE ESTIMATES FOR NONLOCAL PARABOLIC EQUATIONS

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Abstract. In this paper, we obtain Cordes-Nirenberg type estimates for nonlocal parabolic equations on the more flexible solution space $L_\infty^\alpha(L^1_\omega)$ than the classical solution space $B(\mathbb{R}^n_T)$ consisting of all bounded functions on $\mathbb{R}^n_T$.

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1. INTRODUCTION

1.1. Nonlocal parabolic equations. In this paper, we study Cordes-Nirenberg type estimates on the more flexible solution space $L_\infty^\alpha(L^1_\omega)$ for nonlocal parabolic integro-differential equations. In [KL], we obtained interior $C^{1,\alpha}$-estimates on the solution space $B(\mathbb{R}^n_T)$ for nonlocal parabolic translation-invariant equations, and also the reader can refer to [CS1] and [CS2] for the elliptic case.

Throughout this paper, we consider the purely nonlocal parabolic Isaacs equations of the form

$$\mathbf{I}u(x,t) - \partial_t u(x,t) := \inf_{a \in A} \sup_{b \in B} (L_{ab}u(x,t) - \partial_t u(x,t))$$

(1.1)

$$= \inf_{a \in A} \sup_{b \in B} \left( \int_{\mathbb{R}^n} \mu_t(u,x,y)(2 - \sigma) \frac{c_{ab}(x,y,t)}{|y|^n + \sigma} dy - \partial_t u(x,t) \right)$$

$$= f(x,t) \text{ in } \Omega \times (-\tau,0] := \Omega_{\tau}, \; 0 < \tau \leq T,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $\mu_t(u,x,y) = u(x+y,t) + u(x-y,t) - 2u(x,t)$ and $\{c_{ab}\}_{(a,b) \in A \times B}$ is a family of nonnegative functions with indexes $a$ and $b$ in.

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arbitrary sets $A$ and $B$, respectively. We call such $L_{ab}$ a \textit{linear integro-differential operator} and also we simply write $L$ without indices.

We denote by $\omega_\sigma(y) = 1/(1 + |y|^{n+\sigma})$ for $\sigma \in (0, 2)$ and we write $\omega := \omega_\sigma_0$ for some $\sigma_0 \in (1, 2)$, and also we denote by $\omega(Q_r) = \int_{Q_r} \omega(y) dy$. Let $\mathfrak{F}$ denote the family of all real-valued measurable functions defined on $\mathbb{R}^n_+ := \mathbb{R}^n \times (-T, 0]$. For $u \in \mathfrak{F}$ and $t \in (-T, 0]$, we define the weighted norm $\|u(\cdot, t)\|_{L^1_\omega}$ by

$$\|u(\cdot, t)\|_{L^1_\omega} = \int_{\mathbb{R}^n} |u(x, t)| \omega(x) \, dx.$$ 

We consider the function space $L^\infty_\omega(L^1_\omega)$ of all continuous $L^1_\omega$-valued functions $u \in \mathfrak{F}$ given by the family

$$\{ u \in \mathfrak{F} : \sup_{s \in (-T, 0]} \|u(\cdot, s)\|_{L^1_\omega} < \infty, \lim_{s \to t^-} \|u(\cdot, s) - u(\cdot, t)\|_{L^1_\omega} = 0 \text{ for any } t \in (-T, 0) \}$$ 

with the norm $\|u\|_{L^\infty_\omega(L^1_\omega)} = \sup_{t \in (-T, 0]} \|u(\cdot, t)\|_{L^1_\omega}$, which is separable with respect to the topology given by the norm.

A mapping $\mathbb{I} : \mathfrak{F} \to \mathfrak{F}$ given by $u \mapsto \mathbb{I} u$ is called a \textit{nonlocal parabolic operator} if

(a) $\mathbb{I} u(x, t)$ is well-defined for any $u \in C^2_\infty(x, t) \cap L^\infty_\omega(L^1_\omega)$,

(b) $\mathbb{I} u$ is continuous on $\Omega \subset \mathbb{R}^n_+$, whenever $u \in C^2_\infty(\Omega) \cap L^\infty_\omega(L^1_\omega)$, where $C^2_\infty(x, t)$ is the class of all $u \in \mathfrak{F}$ whose second derivatives $D^2 u$ in space variables exist at $(x, t)$ and $C^2_\infty(\Omega)$ denotes the class of all $u \in \mathfrak{F}$ such that $u \in C^2_\infty(x, t)$ for any $(x, t) \in \Omega$ and $\sup_{(x, t) \in \Omega} |D^2 u(x, t)| < \infty$. Such a nonlocal operator $\mathbb{I}$ is said to be \textit{uniformly elliptic} with respect to a class $\mathcal{L}$ of linear integro-differential operators if

$$M_{\mathcal{L}}^+ v(x, t) \leq \mathbb{I}(u + v)(x, t) - \mathbb{I} u(x, t) \leq M_{\mathcal{L}}^- v(x, t)$$

where $M_{\mathcal{L}}^+ v(x, t) := \sup_{L \in \mathcal{L}} L v(x, t)$ and $M_{\mathcal{L}}^- v(x, t) := \inf_{L \in \mathcal{L}} L v(x, t)$.

We say that an operator $L$ belongs to $\mathcal{L}_0$ if its corresponding kernel $K \in K_0$ satisfies the uniform ellipticity assumption

$$\frac{(2 - \sigma)}{|y|^{n+\sigma}} \leq K(x, y, t) \leq \frac{\Lambda}{|y|^{n+\sigma}}, \quad 0 < \sigma < 2.$$ 

We consider the corresponding maximal and minimal operators

$$M_{\mathcal{L}_0}^+ u(x, t) = \sup_{L \in \mathcal{L}_0} Lu(x, t) = \int_{\mathbb{R}^n} \frac{\Lambda \mu_t(u, x, y)^+ - \lambda \mu_t(u, x, y)^-}{|y|^{n+\sigma}} \, dy,$$

$$M_{\mathcal{L}_0}^- u(x, t) = \inf_{L \in \mathcal{L}_0} Lu(x, t) = \int_{\mathbb{R}^n} \frac{\lambda \mu_t(u, x, y)^+ - \Lambda \mu_t(u, x, y)^-}{|y|^{n+\sigma}} \, dy.$$ 

In the final section, we shall consider some of the most interesting applications as follows:

- if $0 < \lambda \leq c_{ab} \leq \Lambda$ and $|\nabla_y c_{ab}| \leq C/|y|$, and $c_{ab}(x, y, t)$ is continuous in $(x, t)$ for a modulus of continuity independent of $y$, then there is some $\varepsilon > 0$ (with an estimate depending on $\|u\|_{L^\infty_\omega(L^1_\omega)}$) such that a solution $u$ of (1.1) is $C^{1, \varepsilon}$.

- if $0 < \lambda \leq c_{ab} \leq \Lambda$ and $|\nabla_y c_{ab}| \leq C/|y|$, $c_{ab}$ is constant in $(x, t)$, and $\Lambda - \delta \leq 1 \leq \lambda + \delta$ for a small enough $\delta > 0$, then there is some $\varepsilon > 0$ (with an estimate depending on $\|u\|_{L^\infty_\omega(L^1_\omega)}$) such that the solution $u$ of (1.1) is $C^{2, \varepsilon}$. 

We consider the corresponding maximal and minimal operators
1.2. Outline. In Section 2, we get various parabolic interpolation inequalities which facilitate the required estimates for viscosity solutions for nonlocal parabolic equations. In Section 3, we obtain Hölder regularities and interior $C^{1,\alpha}$-estimates of such viscosity solutions by applying the result of [KL] (refer to [CS1] for the elliptic case). In Section 4, we get boundary estimates and global estimates by certain parabolic adaptation of the barrier function which was used in [CS1] for the elliptic case. In Section 5, we establish stability properties of viscosity solutions by applying the result of [KL], we denote by $\| \cdot \|_{C^\alpha(Q_r)}$ the gradient and derivatives of $u$ in space variable, respectively.

For $u \in C(Q_r)$, we define $\| u \|_{C^\alpha(Q_r)} = \sup_{(x,t) \in Q_r} |u(x,t)|$. For $\alpha \in (0,1]$, $\sigma \in (0,2)$ and $r > 0$, we define the parabolic $\alpha$th Hölder seminorm of $u$ by

$$[u]_{C^\alpha(Q_r)} = \sup_{(x,t), (y,s) \in Q_r} \frac{|u(x,t) - u(y,s)|}{(|x-y|^{\sigma} + |t-s|)^{\alpha/\sigma}}.$$  

Given $u \in C(Q_r) \cap L^\infty_{\tau T}(L^1_\omega)$ and $r > 0$, we say that $u \in C^{0,1}(I_r^\tau; L^1_\omega)$, if it satisfies

$$\sup_{-r^\tau < \tau \leq 0} \sup_{\tau \in (-r^\tau,0] - t} \frac{\|u(\cdot, t + \tau) - u(\cdot, t)\|_{L^1_\omega}}{|\tau|} < C_\sigma < \infty.$$  

We denote by $\| u \|_{C^{0,1}(I_r^\tau; L^1_\omega)}$ the smallest number of such constants $C_\sigma$ in the above, and denote by $C^{0,1}(I_r^\tau; L^1_\omega) = \{ u \in C(Q_r) \cap L^\infty_{\tau T}(L^1_\omega) : \| u \|_{C^{0,1}(I_r^\tau; L^1_\omega)} < \infty \}$.  

1.3. Notations and Definitions. We write the notations and definitions briefly for the reader.

- $B_r = B_r(0)$ and $\mathbb{R}^n_r = \mathbb{R}^n \times (-T, 0]$ for $r > 0$ and $T > 0$.
- $Q_r = B_r \times I_r$ and $Q_r(x,t) = Q_r + (x,t)$ for $r > 0$, $(x,t) \in \mathbb{R}^n_T$ and $I_r^\tau = (-r^{\sigma}, 0]$ with $\sigma \in (0,2)$.
- $\partial_x \Omega_r := \partial_x \Omega \cap (0, -\tau] \cup (0, \tau] \cap \{ x \}$ for a bounded domain $\Omega \subset \mathbb{R}^n$ and $\tau \in (0,T)$.
- For $(x,t), (y,s) \in \mathbb{R}^n_T$, we define the parabolic distance $d$ by
  $$d((x,t), (y,s)) = \begin{cases} \left( |x-y|^\sigma + |t-s| \right)^{1/\sigma}, & t \leq s, \\ \infty, & t > s. \end{cases}$$  

- Denote by $\omega_n$ the surface measure of the unit sphere $S^{n-1}$ of $\mathbb{R}^n$.
- For $z \in \mathbb{R}^n_T$, we denote the translation operators $\tau_z$, $\tau^s$ and $\tau^s_z$ by $\tau_z u(x,t) = u(x+z, t)$, $\tau^s u(x,t) = u(x,t+s)$ and $\tau^s_z u(x,t) = u(x+z, t+s)$, respectively.
- Denote by $\eta$ any fixed sufficiently small positive number.
- Denote by $\nabla u$ and $Du$ the gradient and derivatives of $u$ in space variable, respectively.
- For $u \in C(Q_r)$, we define $\| u \|_{C^\alpha(Q_r)} = \sup_{(x,t) \in Q_r} |u(x,t)|$. For $\alpha \in (0,1]$, $\sigma \in (0,2)$ and $r > 0$, we define the parabolic $\alpha$th Hölder seminorm of $u$ by
  $$[u]_{C^\alpha(Q_r)} = \sup_{(x,t), (y,s) \in Q_r} \frac{|u(x,t) - u(y,s)|}{(|x-y|^{\sigma} + |t-s|)^{\alpha/\sigma}}.$$  

- Given $u \in C(Q_r) \cap L^\infty_{\tau T}(L^1_\omega)$ and $r > 0$, we say that $u \in C^{0,1}(I_r^\tau; L^1_\omega)$, if it satisfies
  $$\sup_{-r^\tau < \tau \leq 0} \sup_{\tau \in (-r^\tau,0] - t} \frac{\|u(\cdot, t + \tau) - u(\cdot, t)\|_{L^1_\omega}}{|\tau|} \leq C_\sigma < \infty.$$  

We denote by $\| u \|_{C^{0,1}(I_r^\tau; L^1_\omega)}$ the smallest number of such constants $C_\sigma$ in the above, and denote by $C^{0,1}(I_r^\tau; L^1_\omega) = \{ u \in C(Q_r) \cap L^\infty_{\tau T}(L^1_\omega) : \| u \|_{C^{0,1}(I_r^\tau; L^1_\omega)} < \infty \}$.
Let $\mathcal{I}$ be a uniformly elliptic operator in the sense of (1.2) with respect to some class $\mathcal{L}$ and let $f: \mathbb{R}^n_T \to \mathbb{R}$ be a continuous function. Then a function $u \in \mathcal{G}$ upper (lower) semicontinuous on $\overline{\Omega} \times J$ where $J := (a, b) \subset (-T, 0]$ is said to be a viscosity subsolution (viscosity supersolution) of an equation $\mathcal{I}u - \partial_t u = f$ on $\Omega \times J$ and we write $\mathcal{I}u - \partial_t u \geq f$ ( $\mathcal{I}u - \partial_t u \leq f$) on $\Omega \times J$ in the viscosity sense, if for each $(x, t) \in \Omega \times J$ there is some neighborhood $Q_T(x, t) \subset \Omega \times J$ of $(x, t)$ such that $\mathcal{I}v(x, t) - \partial_t v(x, t) \geq f(x, t)$ ( $\mathcal{I}v(x, t) - \partial_t v(x, t) \leq f(x, t)$) for $v = \varphi_{Q_T(x, t)} + u\mathbb{1}_{\mathbb{R}^n_T \setminus Q_T(x, t)}$ whenever $\varphi \in C^2(Q_T(x, t))$ with $\varphi(x, t) = u(x, t)$ and $\varphi > u$ ( $\varphi < u$) on $Q_T(x, t) \setminus \{(x, t)\}$ exists. Also a function $u$ is called as a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution to $\mathcal{I}u - \partial_t u = f$ on $\Omega \times J$.

2. Parabolic interpolation inequalities

Let $u \in C(Q_T)$. For $0 < \alpha \leq 1$ and $\sigma \in (0, 2)$, we define the $\alpha$th Hölder seminorms of $u$ in the space and time variable, respectively;

(i) $[u]_{C^\alpha(Q_T)} = \sup_{t \in (-\tau^2, 0]} \sup_{(x,t), (y,t) \in Q_T} \frac{|u(x,t) - u(y,t)|}{|x - y|^\alpha}$,

(ii) $[u]_{C^\alpha_t(Q_T)} = \sup_{x \in B_r} \sup_{(t,x), (s,x) \in Q_T} \frac{|u(x,t) - u(x,s)|}{|t - s|^\alpha}$.

If $0 < \alpha/\sigma \leq 1$, then it is easy to check that the seminorms $[\cdot]_{C_t^\alpha(Q_T)} + [\cdot]_{C^\alpha(Q_T)}$ and $[\cdot]_{C_t^\alpha(Q_T)}$ are equivalent.

We give an useful parabolic interpolation inequality associated with our equations.

**Theorem 2.1.** Let $\mathcal{L}$ be a family of linear integro-differential operators for $\sigma \in (\sigma_0, 2)$ with $\sigma_0 \in (1, 2)$. If $u \in L^\infty_T(L^\sigma_{x,t})$ is a viscosity solution of the nonlocal parabolic equation

$$\mathcal{I}u - \partial_t u = 0 \text{ in } Q_T,$$

where $\mathcal{I}$ is uniformly elliptic with respect to $\mathcal{L}$, then there exist a universal constant $c > 0$ such that

$$\|u\|_{C(Q_T)} \leq c \|u\|_{L^\infty_T(L^\sigma_{x,t})}$$

for any $r \in (0, 2)$.

**Proof.** Without loss of generality, we may assume that $u$ is bounded on $\mathbb{R}^n_T$. Indeed, if we write $u = u_1 + u_2$ where $u_1 = u\mathbb{1}_{Q_r}$, then it easily follows from the uniform ellipticity of $\mathcal{I}$ that

(i) $M_{\mathbb{R}^n_T}^+ u_1 - \partial_t u_1 \geq -\|u\|_{L^\infty_T(L^\sigma_{x,t})}$ and (ii) $M_{\mathbb{R}^n_T}^- u_1 - \partial_t u_1 \leq \|u\|_{L^\infty_T(L^\sigma_{x,t})}$ in $Q_r$.

In case of the first equation (i), the estimate $\sup_{Q_T} u \leq c \|u\|_{L^\infty_T(L^\sigma_{x,t})}$ can be obtained as in Theorem 5.1 [KL] by rescaling argument and applying a parabolic Harnack inequality [KL].

Since $-u$ is another solution of the given equation, the second equation (ii) can be transformed equivalently to the equation

$$M_{\mathbb{R}^n_T}^- (-u_1) - \partial_t (-u_1) \geq -\|u\|_{L^\infty_T(L^\sigma_{x,t})}$$

in $Q_T$.

This and the upper bound estimate in the above imply that $\inf_{Q_T} u \geq -c \|u\|_{L^\infty_T(L^\sigma_{x,t})}$. Therefore we complete the proof.

Next we prove various lemmas which furnish parabolic interpolation inequalities.
Lemma 2.2. If $u \in L^\infty_{\gamma}(L^1_y)$ is a function with $u(\cdot, t) \in C^k(B_r)$ for $t \in (-r^\alpha, 0]$ and $[D^\beta u]_{C^2_{\gamma}(Q_r)} < \infty$ for some $\alpha \in (0, 1)$, then for each $t \in (-r^\alpha, 0]$ and multiindex $\beta$ with $|\beta| = k \in \mathbb{N}$, there exists some $z_0^t \in B_r$ (depending on $t$) such that

$$
|D^\beta u(z_0^t, t)| \leq \left( \frac{3r^\alpha}{2} \right)^{\alpha} [D^\beta u]_{C^2_{\gamma}(Q_r)} + \frac{2(4k)^{k^2}}{\omega(B_{r/2}) r^k} \|u\|_{L^\infty_{\gamma}(L^1_y)}.
$$

Proof. Take $h = \frac{r}{2}$ and any multiindex $\beta$ with $|\beta| = k$. For $(y, t) \in B_{r/2} \times (-T, 0]$, we consider the finite difference operator $D_h^\beta u(y, t) = D_{h,1}^\beta D_{h,2}^\beta \cdots D_{h,n}^\beta u(y, t)$ where

$$
D_{h,i} u(y, t) = \frac{1}{h} [u(y + he_i, t) - u(y, t)].
$$

By the mean value theorem, we see that there are some $z_1^t \in B_r(y)$ and $z_2^t \in B_{2r}(y)$ such that $D_{h,j} D_{h,j} u(y, t) = \partial_{y_j} D_{h,j} u(z_1^t, t) = \partial_{y_j} D_{h,j} u(z_2^t, t)$. This implies that $D_{h,i}^\beta u(y, t) = D^\beta u(z_1^t, t)$ for some $z_1^t \in B_{r/2}(y)$. Thus it follows from this and (2.1) that

$$
\omega(B_{r/2}) |D^\beta u(z_0^t, t)| \leq \omega(B_{r/2}) |D^\beta u(z_1^t, t) - \int_{\mathbb{R}^n} D^\beta_h u(y, t) \omega(y) dy| + \frac{2k}{h^k} \|u\|_{L^\infty_{\gamma}(L^1_y)}
$$

$$
\leq \int_{B_{r/2}} |D^\beta u(z_1^t, t) - D^\beta u(z_2^t, t)| \omega(y) dy + \frac{2^{k+1}}{h^k} \|u\|_{L^\infty_{\gamma}(L^1_y)}
$$

$$
\leq |D^\beta u|_{C^2_{\gamma}(Q_r)} \left( \frac{3r^\alpha}{2} \right)^{\alpha} \omega(B_{r/2}) + \frac{2(4k)^{k^2}}{r^k} \|u\|_{L^\infty_{\gamma}(L^1_y)}.
$$

Therefore, this completes the proof. \qed

In order to understand the parabolic Hölder spaces $C^{k,\gamma}(Q_r)$ with $k \in \mathbb{N}$ and $\gamma \in (0, 1)$, we define the Hölder spaces $C^{k,\gamma}_x(Q_r)$ and $C^{k,\gamma}_t(Q_r)$ in the space and time variable, respectively. For $u \in C(Q_r)$, we define the norms

$$
\|u\|_{C^{k,\gamma}_x(Q_r)} = \|u\|_{C(Q_r)} + \sum_{i=1}^k \|D^i u\|_{C(Q_r)} + [D^k u]_{C^2_{\gamma}(Q_r)},
$$

$$
\|u\|_{C^{k,\gamma}_t(Q_r)} = \|u\|_{C(Q_r)} + \sum_{i=1}^k \|\partial_t^i u\|_{C(Q_r)} + [\partial_t^k u]_{C^2_{\gamma}(Q_r)},
$$

where $\|D^i u\|_{C(Q_r)} = \sum_{|\beta| = i} \|D^\beta u\|_{C(Q_r)}$ and $[D^k u]_{C^2_{\gamma}(Q_r)} = \sum_{|\beta| = k} [D^\beta u]_{C^2_{\gamma}(Q_r)}$ for $i, k \in \mathbb{N}$. And we denote by $C^{k,\gamma}_x(Q_r) = \{ u \in \mathcal{S}(\mathbb{R}^n_{x}) : \|u\|_{C^{k,\gamma}_x(Q_r)} < \infty \}$ and $C^{k,\gamma}_t(Q_r) = \{ u \in \mathcal{S}(\mathbb{R}^n_T) : \|u\|_{C^{k,\gamma}_t(Q_r)} < \infty \}$.

Theorem 2.3. If $u \in L^\infty_{\gamma}(L^1_y)$ is a function such that $u(\cdot, t) \in C^k(B_r)$ for each $t \in (-r^\alpha, 0]$ and $\sup_{|\beta| = k} [D^\beta u]_{C^2_{\gamma}(Q_r)} < \infty$ for some $\alpha \in (0, 1)$, then we have that

$$
\|D^k u\|_{C(Q_r)} \leq 2 \left( \frac{3r^\alpha}{2} \right)^{\alpha} [D^k u]_{C^2_{\gamma}(Q_r)} + \frac{2c_k(4k)^k}{\omega(B_{r/2}) r^k} \|u\|_{L^\infty_{\gamma}(L^1_y)},
$$

where $c_k = \sum_{|\beta| = k} 1$. 


Proof. From Lemma 2.2, for any \((x, t) \in Q_r\), we obtain that
\[
|D^\beta u(x, t)| \leq |D^\beta u(x, t) - D^\beta u(z_0^t, t)| + |D^\beta u(z_0^t, t)|
\leq 2|D^\beta u|_{C^2(Q_r)} \left( \frac{3r^2}{2} \right)^{\alpha} + \frac{2(4k)^k}{\omega(B_{r/2})} r^k \|u\|_{L^\infty(I_L^\infty)}.
\]
Taking the supremum on \(Q_r\) and adding up on the multiindices \(\beta\) with \(|\beta| = k\) in the above, we easily obtain the required result.

If \(\sigma \in (\sigma_0, 2)\) for \(\sigma_0 \in (1, 2)\) and \(\alpha \in (0, \sigma_0 - 1)\), then \(0 < \alpha < 2 + \alpha - \sigma < 1\) and
\[
(2.2)
\]
then we define the **parabolic Hölder space** \(C^{2, \alpha}(Q_r)\) endowed with the norm
\[
\|u\|_{C^{2, \alpha}(Q_r)} = \|u\|_{C(Q_r)} + \sum_{i=1}^{2} \|D^i u\|_{C(Q_r)} + \|\partial_t u\|_{C(Q_r)}
+ |D^2 u|_{C^\infty(Q_r)} + |\partial_t u|_{C^{2+\sigma-\sigma}(Q_r)}.
\]

In the same case as the above, we can learn from Theorem 2.1 and Theorem 2.3 that the estimates on the norm \(\|u\|_{C^{2, \alpha}(Q_r)}\) must be controlled by those on the seminorms \([\partial_t u]_{C^{2+\sigma-\sigma}(Q_r)} \sim [\partial_t u]_{C^{2+\alpha-\sigma}(Q_r)} + [\partial_t u]_{C^{1, \sigma/\sigma}(Q_r)}\) and \([D^2 u]_{C^0(Q_r)} \sim [D^2 u]_{C^0(Q_r)} + [D^2 u]_{C^0(Q_r)}\). Similarly, the other parabolic Hölder spaces can be defined along this line.

**Lemma 2.4.** Let \(\sigma \in [\sigma_0, 2)\) for some \(\sigma_0 \in (1, 2)\) and \(\alpha \in (0, \sigma_0 - 1)\) as in (2.2). If \(u \in L^\infty_t(L^1_x)\) is a function with \(u(x, \cdot) \in C^1(-r^\sigma, 0)\) for \(x \in B_r\) and \(\|\partial_t u\|_{C^{2+2\sigma-\sigma}(Q_r)} < \infty\), then we have that
\[
\|\partial_t u\|_{C(Q_r)} \leq r^{2+\sigma-\sigma}[\partial_t u]_{C^{2+2\sigma-\sigma}(Q_r)} + \frac{4}{r^\sigma}\|u\|_{C(Q_r)}.
\]

**Proof.** Take any \(r \in (0, 2)\) and \((x, t) \in Q_r\). Then there is some \(t_0 \in (-r^\sigma, 0)\) such that \(|t - t_0| = \frac{1}{2} r^\sigma\), and by the mean value theorem, there is some \(t_0^\sigma\) between \(t\) and \(t_0\) such that \(u(x, t_0) - u(x, t) = \frac{1}{2} r^\sigma \partial_t u(x, t_0^\sigma)\). Thus we have the estimate
\[
\frac{1}{2} r^\sigma |\partial_t u(x, t)| \leq \frac{1}{2} r^\sigma |\partial_t u(x, t) - \partial_t u(x, t_0^\sigma)| + 2\|u\|_{C(Q_r)}
\leq \frac{1}{2} r^\sigma |\partial_t u(x, t) - \partial_t u(x, t_0)| + 2\|u\|_{C(Q_r)}
\leq \frac{1}{2} r^{2+\sigma}[\partial_t u]_{C^{2+2\sigma-\sigma}(Q_r)} + 2\|u\|_{C(Q_r)}.
\]
Hence this implies the required inequality. \(\square\)

**Lemma 2.5.** Let \(\sigma \in [\sigma_0, 2)\) for some \(\sigma_0 \in (1, 2)\), and let \(u \in C^{0, 1}(I_r^\sigma; L^1_x)\) be a viscosity solution of the equation
\[
Iu - \partial_t u = 0 \quad \text{in } Q_2
\]
where \(I\) is defined on \(\Sigma_2(\sigma)\). If \(u \in C^{2, \alpha}(Q_2)\), then there is a universal constant \(C > 0\) such that
\[
[D^2 u]_{C^\infty(Q_2)} \leq C(\|D^2 u\|_{C(Q_2)} + \|u\|_{L^\infty(I_r^\infty)}),
\]
\[
[\partial_t u]_{C^{2+\alpha-\sigma}(Q_2)} \leq C(\|\partial_t u\|_{C(Q_2)} + \|u\|_{C^{0, 1}(I_r^\sigma; L^1_x)}), \quad \text{for any } r \in (0, 2).
\]
Proof. Take any \( r \in (0, 2) \) and \((x, t) \in Q_r\). We consider the difference quotients in the \( x \)-direction
\[
u^h(x, t) = \frac{u(x + h, t) - u(x, t)}{|h|}.
\]
Write \( u^h = u_1^h + u_2^h \) where \( u_k^h = u^h |_{Q_k} \). By Theorem 2.4 [KL], we have that
\[
\begin{aligned}
\mathbf{M}^+_{\Omega_2} u^h - \partial_t u^h &\geq 0 \\
\mathbf{M}^-_{\Omega_2} u^h - \partial_t u^h &\leq 0
\end{aligned}
\]
respect to \( \Omega_2 \) that
\[
\begin{aligned}
\mathbf{M}^+_{\Omega_0} u^h - \partial_t u^h &\geq -\mathbf{M}^+_2 u^h_2 \\
\mathbf{M}^-_{\Omega_0} u^h - \partial_t u^h &\leq -\mathbf{M}^-_2 u^h_2
\end{aligned}
\]
in \( Q_r \). Then it is easy to show that \(|\mathbf{M}^+_{\Omega_2} u^h_2| \vee |\mathbf{M}^-_{\Omega_2} u^h_2| \leq c \|u\|_{L^\infty(Q_r)}\) for a universal constant \( c > 0 \). So we have that
\[
\begin{aligned}
\mathbf{M}^+_{\Omega_0} u^h_1 - \partial_t u^h_1 &\geq -c \|u\|_{L^\infty(Q_r)} \\
\mathbf{M}^-_{\Omega_0} u^h_1 - \partial_t u^h_1 &\leq c \|u\|_{L^\infty(Q_r)}
\end{aligned}
\]
and
\[
\mathbf{M}^+_{\Omega_0} u^h - \partial_t u^h \leq 2c \|u\|_{L^\infty(Q_r)}\]
from the Hölder estimate(Theorem 3.5) below, we get the estimate
\[
[w^h]^{\frac{1}{p}}_{C^\theta(Q_r)} \leq C\left\{ \|w^h\|_{C(Q_r)} + \|w^h\|_{L^\infty(Q_r)} + \|u\|_{L^\infty(Q_r)} \right\}
\]
with some universal constant \( C > 0 \). By the mean value theorem, we easily have that \( \|w^h\|_{C(Q_r)} \leq \|D^2 u\|_{C(Q_r)} \). Since \( \|D^2 \mathbf{w}(y, s)\| + \|D^2 \mathbf{w}(y, s)\| \leq \mathbf{c} \|\mathbf{w}(y)\| \), it follows from the integration by parts that
\[
\|w^h\|_{L^\infty(Q_r)} \leq \|u\|_{L^\infty(Q_r)}.
\]
Thus we obtain that
\[
[w^h]^{\frac{1}{p}}_{C^\theta(Q_r)} \leq C\left\{ \|D^2 u\|_{C(Q_r)} + \|u\|_{L^\infty(Q_r)} \right\}.
\]
Taking the limit \(|h| \to 0\), we conclude that the first inequality holds.

To show the second inequality, for \( \tau \) with \( t + \tau \in (-r^\alpha, 0) \) we consider the difference quotient in the \( t \)-direction
\[
u^\tau(x, t) = \frac{u(x, t + \tau) - u(x, t)}{\tau}, \quad (x, t) \in \mathbb{R}^n \times (-r^\alpha, 0].
\]
Let us write \( u^\tau = u_1^\tau + u_2^\tau \) where \( u_k^\tau = u^\tau |_{Q_k} \). From Theorem 2.4 [KL], we obtain that
\[
\begin{aligned}
\mathbf{M}^+_{\Omega_2} u^\tau - \partial_t u^\tau &\geq 0 \\
\mathbf{M}^-_{\Omega_2} u^\tau - \partial_t u^\tau &\leq 0
\end{aligned}
\]
in \( Q_r \). Since \( \partial_t u^\tau \equiv 0 \) in \( Q_r \), it follows from the uniform ellipticity of \( \mathbf{M}^+_{\Omega_2} \) and \( \mathbf{M}^-_{\Omega_2} \) with respect to \( \Omega_2 \) that
\[
\begin{aligned}
\mathbf{M}^+_{\Omega_0} u^\tau_1 - \partial_t u^\tau_1 &\geq -\mathbf{M}^+_2 u^\tau_2 \\
\mathbf{M}^-_{\Omega_0} u^\tau_1 - \partial_t u^\tau_1 &\leq -\mathbf{M}^-_2 u^\tau_2
\end{aligned}
\]
in \( Q_r \). Then it is easy to check that \(|\mathbf{M}^+_{\Omega_2} u^\tau_2| \vee |\mathbf{M}^-_{\Omega_2} u^\tau_2| \leq c \|u\|_{C^0(\Omega_r)}\) in \( Q_r \) for a universal constant \( c > 0 \). So we have that
\[
\begin{aligned}
\mathbf{M}^+_{\Omega_0} u^\tau_1 - \partial_t u^\tau_1 &\geq -c \|u\|_{C^0(\Omega_r)} \\
\mathbf{M}^-_{\Omega_0} u^\tau_1 - \partial_t u^\tau_1 &\leq c \|u\|_{C^0(\Omega_r)}
\end{aligned}
\]
in \( Q_r \).
By the mean value theorem and Theorem 3.4 below, we have that
\[
[u^\tau]_{C^2,\alpha,\sigma}(Q_\tau) \leq [u^\tau]_{C^{2+\alpha,\sigma}(Q_\tau)} \leq C \left( \|\partial_t u\|_{C(\sigma)} + \|u\|_{C^{0,1}(I_\tau;L_\tau)} \right).
\]
Taking the limit \(|\tau| \to 0\), we can conclude that
\[
[\partial_t u]_{C^2,\alpha,\sigma}(Q_\tau) \leq C \left( \|\partial_t u\|_{C(\sigma)} + \|u\|_{C^{0,1}(I_\tau;L_\tau)} \right).
\]
Hence we complete the proof.

**Remark 2.1.** In order to show Theorem 1.1, we learned from the interpolation results obtained in this section that the norm \(\|u\|_{C^{2,\alpha}(Q_\tau)}\) of viscosity solutions \(u \in C^{0,1}(I_\tau;L^1_\tau)\) of (1.1) is controlled by only two seminorms \([\partial_t u]_{C^{2+\alpha,\sigma}(Q_\tau)}\) and \([D^2 u]_{C^2(Q_\tau)}\), and so only two norms \(\|u\|_{C^{2,\alpha}(Q_\tau)}\) and \(\|u\|_{C^2(Q_\tau)}\).

Finally, we are going to define another parabolic Hölder space \(C^{1,\alpha}(Q_\tau)\) in case that \(1 < \sigma < 2\). From \([K1]\), such \(\alpha > 0\) could be chosen so that \(\alpha < \sigma - 1\), i.e. \(0 < \theta = \theta(\sigma, \alpha) = \frac{1+\alpha}{\sigma} < 1\). We learned from the definition of \(C^{2,\alpha}(Q_\tau)\) that one derivative in time variable amounts to two derivatives in space variable. Since there is only one derivative in space variable on \(C^{1,\alpha}(Q_\tau)\), the space should be defined as the family of all functions \(u \in \mathcal{F}\) with the norm
\[
\|u\|_{C^{1,\alpha}(Q_\tau)} = \|u\|_{C(\alpha)} + \|Du\|_{C(\alpha)} + [Du]_{C^{0,\theta}(Q_\tau)} + [u]_{C^\theta(Q_\tau)}.
\]
We define the class \(\Sigma_\alpha\) of operators \(L\) with kernels \(K \in \mathcal{K}_\alpha\) satisfying (1.3) such that there are some \(\gamma_0 > 0\) and a constant \(C > 0\) such that
\[
\sup_{(x,t) \in \mathbb{R}^n} |\nabla_y K(x,y,t)| \leq C \omega(y) \quad \text{for any } y \in \mathbb{R}^n \setminus B_{\gamma_0}.
\]

**Theorem 2.6.** Let \(\sigma \in (\sigma_0,2)\) for some \(\sigma_0 \in (1,2)\). Then there is some \(\gamma_0 > 0\) (depending on \(\lambda, \Lambda, \sigma_0\) and \(n\)) so that if \(I\) is a nonlocal, translation-invariant and uniformly elliptic operator with respect to \(\Sigma_\alpha\) and \(u \in L^\infty_\tau(L^1_\tau)\) satisfies \(Lu - \partial_t u = 0\) in \(Q_\tau\), then there are some \(\alpha > 0\) and \(C > 0\) (depending only on \(\lambda, \Lambda, n, \eta\) and \(\sigma_0\), but not on \(\sigma\)) such that
\[
\|Du\|_{C^\theta(Q_\tau)} \leq C \left( \|Du\|_{C(\alpha)} + \|u\|_{L^\infty_\tau(L^1_\tau)} \right)
\]
for any \(r \in (0,2)\), where the constant \(C\) also depends on the constant in (2.4).

**Proof.** We proceed the proof by applying Theorem 3.4 below to the difference quotients in the \(x\)-direction
\[
w_h(x,t) = \frac{u(x+h,t) - u(x,t)}{|h|}.
\]
Take any \(r \in (0,2)\). Then we write \(w_h = w_h^1 + w_h^2\) where \(w_h^1 = w_h^1 \mathbb{1}_{Q_\tau}\). From Theorem 2.4 \([KL]\), we have that \(M^+_w w_h^1 - \partial_t w_h^1 \geq 0\) and \(M^-_w w_h^2 - \partial_t w_h^2 \leq 0\) in \(Q_\tau\). Because \(\partial_t w_h^2 \equiv 0\) in \(Q_\tau\), it follows from the uniform ellipticity with respect to \(\Sigma_\alpha\) that we get that
\[
M^+_\Sigma_0 w_h^1 - \partial_t w_h^1 \geq M^+_\Sigma_0 w_h^1 - \partial_t w_h^1 \geq M^+_\Sigma_0 w_h^1 - M^-_\Sigma_0 w_h^2 - \partial_t w_h^2 \geq -M^-_\Sigma_0 w_h^2 \quad \text{in } Q_\tau,
\]
\[
M^-_\Sigma_0 w_h^1 - \partial_t w_h^1 \leq M^-_\Sigma_0 w_h^1 - \partial_t w_h^1 \leq M^-_\Sigma_0 w_h^1 - M^+_\Sigma_0 w_h^2 - \partial_t w_h^2 \leq -M^+_\Sigma_0 w_h^2 \quad \text{in } Q_\tau.
\]
If we can show that \(|M^+_\Sigma_0 w_h^2| + |M^-_\Sigma_0 w_h^2| \leq c\|u\|_{L^\infty_\tau(L^1_\tau)}\) in \(Q_\tau\), then we have that
\[
M^+_\Sigma_0 w_h^1 - \partial_t w_h^1 \geq -c\|u\|_{L^\infty_\tau(L^1_\tau)} \quad \text{and } M^-_\Sigma_0 w_h^1 - \partial_t w_h^1 \leq c\|u\|_{L^\infty_\tau(L^1_\tau)} \quad \text{in } Q_\tau.
\]
for \( h \) with a sufficiently small \( |h| \). Indeed, by using (2.4), it can be obtained from the fact that
\[
\int_{\mathbb{R}^n \setminus B_r} |u(x + y, t)| \frac{|K(x, y, t) - K(x, y - h, t)|}{|h|} dy + \int_{\mathbb{R}^n \setminus B_r} |u(x + y + h, t)| K(x, y, t) dy \leq C \|u\|_{L^p_c(Q_r)}
\]
for some \( \rho > 0 \). Hence \( u_1^h \) admits the Hölder estimate (Theorem 3.4) below on \( Q_r \), and thus applying the mean value theorem and integration by parts with (2.4) gives the estimate
\[
\|u_1^h\|_{C^{1,\alpha}(Q_r)} \leq \|Du\|_{C^\alpha(Q_r)} + \|u\|_{L^p_c(Q_r)}.
\]
Finally, taking the limit \( |h| \to 0 \), we obtain the required result. \( \square \)

**Remark 2.2.** From Theorem 2.5, we saw that the norm \( \|u\|_{C^{1,\alpha}(Q_r)} \) of viscosity solutions \( u \in L^p_c(L^1_r) \) of the equation (1.1) given on operators \( \mathcal{L} \) uniformly elliptic with respect to \( \mathcal{L}_\tau \) is controlled by only two seminorms \( [Du]_{C^\alpha(Q_r)} \) and \( [u]_{C^\alpha_c(Q_r)} \) with \( \theta = \frac{1+\alpha}{2} \). Thus the norm \( \|u\|_{C^{1,\alpha}(Q_r)} \) is completely governed by only two norms \( \|u\|_{C^{1,\alpha}(Q_r)} \) and \( \|u\|_{C^\alpha_c(Q_r)} \).

### 3. Preliminary estimates

In this paper, we always impose the following assumptions on \( \omega \):

(3.1) \[ 1 + |y| \in L^1_\omega, \]

(3.2) \[ \sup_{B_r(y)} \omega \leq C_r \omega(y). \]

The uniform ellipticity (1.2) depends on a class \( \mathcal{L} \) of *linear integro-differential operators*. Such an operator \( L \) in \( \mathcal{L} \) is of the form \( Lu(x, t) = \int_{\mathbb{R}^n} \mu_t(u, x, y)K(x, y, t) dy \) for a nonnegative symmetric kernel \( K \) satisfying
\[
\sup_{(x, t) \in \mathbb{R}^n \setminus \mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |y|^2) K(x, y, t) dy \leq C < \infty.
\]
Here the symmetric property means that for each \( (x, t) \in \mathbb{R}^n, \ K(x, -y, t) = K(x, y, t) \) for all \( y \in \mathbb{R}^n \).

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Then we say that a function \( u : \mathbb{R}^n_\tau \to \mathbb{R} \) is Lipschitz in space on \( \Omega_\tau = \Omega \times (-\tau, 0], \tau \in (0, T) \) (we write \( u \in C^{0,1}(\Omega_\tau) \)), if there is some constant \( C > 0 \) (independent of \( x, y \)) such that
\[
\sup_{t \in (-\tau, 0]} |u(x, t) - u(y, t)| \leq C |x - y|
\]
for any \( x, y \in \Omega \). We denote by \( [u]_{C^{0,1}(\Omega_\tau)} \) the smallest \( C \) satisfying (3.3).

We say that a function \( u : \mathbb{R}^n_\tau \to \mathbb{R} \) is in \( C^{1,1}_x(\Omega_\tau) \), if there is a constant \( C_0 > 0 \) (independent of \( x, t \) and \( y, t \)) such that
\[
|u(y, t) - u(x, t) - (y - x) \cdot \nabla u(x, t)| \leq C_0 |y - x|^2
\]
for all \( (x, t), (y, t) \in \Omega_\tau \). We denote by the norm \( \|u\|_{C^{1,1}_x(\Omega_\tau)} \) the smallest \( C_0 \) satisfying (3.4).

The following definition is the parabolic setting of that [CS1] of the elliptic case.
Lemma 3.5. Morrey's inequality.

Assumption 3.3 implies that

\[ \sup_{(y,s) \in \Omega} u(y,s) \]

is the supremum of all kernels corresponding to operators in the class \( \mathcal{L} \), then for each \( r > 0 \) there is a constant \( C_r > 0 \) such that \( \sup_{(x,t) \in \mathbb{R}^n} K_{\mathcal{F}}(x,y,t) \leq C_r \) \( \omega(y) \) for any \( y \in \mathbb{R}^n \setminus B_r \).

Assumption 3.3. There is some \( C > 0 \) such that \( \sup_{L \in \mathcal{E}} \| L \| \leq C < \infty \).

Remark. Assumption 3.3 implies that \( \| M^+_\alpha \| \leq C \) and \( \| M^-_\alpha \| \leq C \).

Theorem 3.4. Let \( \sigma \in (\sigma_0, 2) \) for some \( \sigma_0 \in (1, 2) \). If \( u \in L^\infty_\mathcal{F}(L^1_\mathcal{L}) \) is a function satisfying

\[
M^+_{\sigma_0} u - \partial_t u \geq -C_0 \quad \text{and} \quad M^-_{\sigma_0} u - \partial_t u \leq C_0 \quad \text{in} \ Q_{1+\eta},
\]

then there are some \( \alpha > 0 \) and \( C > 0 \) (depending only on \( \lambda, \Lambda, n, \eta \) and \( \sigma_0 \), but not on \( \sigma \)) such that

\[
\| u \|_{C^{\alpha}(Q_1)} \leq C \left( \| u \|_{C(Q_{1+\eta})} + \| u \|_{L^\infty_\mathcal{F}(L^1_\mathcal{L})} + C_0 \right).
\]

Proof. We note that if \( (x,t) \in Q_1 \), then \( |Lu(x,t)| \leq C \| u \|_{L^\infty_\mathcal{F}(L^1_\mathcal{L})} \) for any \( L \in \mathcal{L}_0 \), i.e. we have only to show that if \( (x,t) \in Q_1 \), then

\[
\int_{|y| \geq 1+\eta} u(y,t)K(x, x \pm y, t) \, dy \leq C \| u \|_{L^\infty_\mathcal{F}(L^1_\mathcal{L})}
\]

for any \( L \in \mathcal{L}_0 \). Indeed, we note that \( |y| > (1+\eta)|x| \) for any \( x \in B_1 \) and \( y \in \mathbb{R}^n \setminus B_1 \), and so \( |x \pm y| \geq |y| - |x| \geq \frac{\eta}{1+\eta}|y| \). Thus we have the estimate

\[
\int_{|y| \geq 1+\eta} u(y,t)K(x, x \pm y, t) \, dy \leq C \int_{|y| \geq 1+\eta} \frac{|u(y,t)|}{|x \pm y|^{n+\alpha}} \, dy \leq C \| u \|_{L^\infty_\mathcal{F}(L^1_\mathcal{L})}.
\]

This implies that \( M^+_{\sigma_0} v - \partial_tv \geq -C_0 - C \| u \|_{L^\infty_\mathcal{F}(L^1_\mathcal{L})} \) and \( M^-_{\sigma_0} v - \partial_tv \leq C_0 + C \| u \|_{L^\infty_\mathcal{F}(L^1_\mathcal{L})} \) in \( Q_1 \).

Hence we complete the proof by applying Theorem 5.2 [KL] to \( v \).

In the following lemma, we get a useful estimate which can be derived from Morrey's inequality.

Lemma 3.5. If \( u \in L^\infty_\mathcal{F}(L^1_\mathcal{L}) \) is a function with \( u(\cdot,t) \in C^1(B_r) \) for \( t \in (-r^\sigma,0] \) and \( [u]_{C^{1,\alpha}(Q_\cdot)} < \infty \) for some \( \alpha \in (0,1) \), then we have that

\[
[u]_{C^{1,\alpha}(Q_\cdot)} \leq \frac{(2^\alpha + 3^\alpha)^{\frac{1}{\alpha}}} {2^\alpha} [u]_{C^{1,\alpha}(Q_\cdot)} + \frac{8}{\omega(B_{r/2})^r} \| u \|_{L^\infty_\mathcal{F}(L^1_\mathcal{L})}.
\]
Proof. From Lemma 2.2, there is some \( z_0^t \in B_r \) (depending on \( t \)) such that
\[
|Du(z_0^t, t)| \leq \left( \frac{3r}{2} \right) \alpha u_{C^1, \alpha}(Q_r) + \frac{8}{\omega(B_{r/2})} \|u\|_{L_\infty(L_{\infty}^\lambda)}.
\]
Thus it follows from this fact and Morrey’s inequality that
\[
[u]_{C^{0,1}_x(Q_r)} \leq \frac{(2 + 3\alpha) r\alpha}{2\alpha} [u]_{C^1, \alpha}(Q_r) + \frac{8}{\omega(B_{r/2})} \|u\|_{L_\infty(L_{\infty}^\lambda)}.
\]
Hence we complete the proof.

**Theorem 3.6.** Let \( \sigma \in [\sigma_0, 2) \) for some \( \sigma_0 \in (1, 2) \). Then there is some \( \rho > 0 \) (depending on \( \lambda, \Lambda, \sigma_0 \) and \( n \)) so that if \( \mathbb{L} \) is a nonlocal, translation-invariant and uniformly elliptic operator with respect to \( \Sigma_\lambda \) and \( u \in L_\infty^\sigma(L_{\infty}^\lambda) \) satisfies \( u - \partial_t u = 0 \) in \( Q_{1+\eta} \), then there are some \( \alpha > 0 \) and \( C > 0 \) (depending only on \( \lambda, \Lambda, n, \eta \) and \( \sigma_0 \), but not on \( \sigma \)) such that
\[
\|u\|_{C^{1, \alpha}(Q_1)} \leq C \|u\|_{L_\infty(L_{\infty}^\lambda)}
\]
where the constant \( C \) also depends on the constant in (2.4).

**Remark.** We can derive from Theorem 2.1 and Theorem 3.4 that
\[
[u]_{C^{0,1}_x(Q_1)} \leq C \|u\|_{L_\infty(L_{\infty}^\lambda)}.
\]
Also it follows from the standard telescopic sum argument \( \text{[CC]} \) and (3.5) that
\[
[u]_{C^{0,1}_x(Q_1)} \leq C \|u\|_{L_\infty(L_{\infty}^\lambda)}.
\]

**Proof.** The proof of this theorem goes along the lines of the proof of Theorem 12.1 in \( \text{[CS2]} \) by applying Theorem 3.4 to the difference quotients in the x-direction
\[
w^h(x, t) = \frac{u(x + h, t) - u(x, t)}{|h|^{\beta}}
\]
for \( \beta = \alpha, 2\alpha, \cdots, 1 \). Write \( w^h = w^h_1 + w^h_2 \) where \( w^h_1 = w^h \mathbb{L} \). By Theorem 2.4 \( \text{[KL]} \), we have that \( M^+_{\sigma_0} w^h - \partial_t w^h \geq 0 \) and \( M^-_{\sigma_0} w^h - \partial_t w^h \leq 0 \) in \( Q_1 \). Since \( \partial_t w^h_2 \equiv 0 \) in \( Q_1 \) for \( h \) with \( |h| < \eta \), it follows from the uniform ellipticity with respect to \( \Sigma_\lambda \) that we have that
\[
M^+_{\sigma_0} w^h_1 - \partial_t w^h_1 \geq M^+_{\sigma_0} w^h_1 - \partial_t w^h_1 \geq M^+_{\sigma_0} w^h - M^-_{\sigma_0} w^h_2 - \partial_t w^h \geq -M^+_{\sigma_0} w^h_2 \quad \text{in} \quad Q_1,
\]
\[
M^-_{\sigma_0} w^h_1 - \partial_t w^h_1 \leq M^-_{\sigma_0} w^h_1 - \partial_t w^h_1 \leq M^-_{\sigma_0} w^h - M^+_{\sigma_0} w^h_2 - \partial_t w^h \leq -M^-_{\sigma_0} w^h_2 \quad \text{in} \quad Q_1.
\]
If we can show that \( |M^+_{\sigma_0} w^h_2| \leq \|u\|_{L_\infty(L^\lambda)} \) in \( Q_1 \), then we have that
\[
M^-_{\sigma_0} w^h_1 - \partial_t w^h_1 \geq -c \|u\|_{L_\infty(L^\lambda)} \quad \text{and} \quad M^-_{\sigma_0} w^h_1 - \partial_t w^h_1 \leq c \|u\|_{L_\infty(L^\lambda)} \quad \text{in} \quad Q_1
\]
for \( h \) with \( |h| < \eta \). Indeed, it can be obtained from the fact that
\[
\int_{\mathbb{R}^n \setminus B_\rho} |u(x + y, t)| \left| \frac{|K(x, y, t) - K(x, y - h, t)|}{|h|} \right| dy + \int_{\mathbb{R}^n \setminus B_\rho} |u(x + y + h, t)| K(x, y, t) dy \leq C \|u\|_{L_\infty(L^\lambda)}
\]
for some \( \rho > 0 \) (this can be seen by using (2.4) and (3.2)). Hence, by (2.3) and (3.5), \( u \) admits the required \( C^{1, \alpha} \)-estimates on \( Q_1 \); more precisely,
\[
\|u\|_{C^{1, \alpha}(Q_1)} \leq C \|u\|_{L_\infty(L_{\infty}^\lambda)}.
\]
Now we are going to show that \( u \) is \( C^{1,\alpha}_t \)-Hölder continuous in \( Q_1 \), following Lemma 2 in [CW]. For \((x_0, t_0) \in Q_1\), we consider
\[
w(x, t) = \frac{u(rx + x_0, r^\alpha t + t_0) - u(x_0, t_0) - r \nabla u(x_0, t_0) \cdot x}{r^{1+\alpha}}
\]
for any sufficiently small \( r > 0 \). Then \( w \) solves the given parabolic equation.

Without loss of generality, by (3.6) let us assume that \( 0 < [u]_{C^{0,1}_x(Q_1)} < \infty \). Then \( C^{1,\alpha}_x \)-regularity of \( u \) on \( Q_1 \) and Lemma 3.5 imply the estimate
\[
0 < [u]_{C^{0,1}_x(Q_1)} \leq 5 [u]_{C^{1,\alpha}_x(Q_1)} + \frac{8}{\omega(B_{1/2})} \| u \|_{L^\infty(L^1)} < \infty.
\]
So, by dividing \( u \) by the right-hand side in the above, we assume that \([u]_{C^{0,1}_x(Q_1)} \leq 1\).

We consider the function \( \phi \) given by
\[
\phi(y) = \begin{cases} |y|^2, & y \in B_{1+\eta}, \\ (1 + \eta)^2, & y \in \mathbb{R}^n \setminus B_{1+\eta}. \end{cases}
\]
If \( x \in B_{1} \), then we have that
\[
\begin{align*}
L \phi(x) &= \int_{|y| < \eta} \left[ \phi(x + y) + \phi(x - y) - 2\phi(x) \right] K(x, y, t) \, dy \\
&\quad + \int_{|y| \geq \eta} \left[ \phi(x + y) + \phi(x - y) - 2\phi(x) \right] K(x, y, t) \, dy \\
&\leq 2\alpha \omega_n \eta^{2-\sigma} + \frac{4(2-\sigma)}{\sigma} \alpha \omega_n \eta^{-\sigma} \leq 6\alpha \omega_n \eta^{-\sigma}
\end{align*}
\]
for any \( L \in \mathcal{L}_x \), and we have that \( I \phi \leq 6\alpha \omega_n \eta^{-\sigma} \) on \( B_1 \). If \( M = \sup_{B_t \times (-1,t_1)} w = w(x_0, t) \) for some \( x_0 \in B_1 \) and \( t \in (-1, t_1) \) where \( t_1 = -1 + \frac{1}{12\alpha \omega_n \eta^{-\sigma}} \), then it is easy to check that the functions
\[
\phi_1(x, t) = M \left( t + 1 + \frac{[u]_{C^{0,1}_x(Q_1)} \phi(x)}{6\alpha \omega_n \eta^{-\sigma}(1 + \eta)^2} \right),
\]
\[
\phi_2(x, t) = 6\alpha \omega_n \eta^{-\sigma} M \left( t + 1 + \frac{\phi(x) - \phi(x_M)}{6\alpha \omega_n \eta^{-\sigma}} \right) + [u]_{C^{0,1}_x(Q_1)}
\]
are supersolutions of the given equation on \( Q_1 \). So it follows from comparison principle [KL] that \( w(x, t) \leq \phi_1(x, t) \wedge \phi_2(x, t) \) for any \((x, t) \in Q_1 \). We now claim that (a) \( w \leq M + [u]_{C^{0,1}_x(Q_1)} \) on \( Q_1 \) and (b) \( M \leq 4[u]_{C^{0,1}_x(Q_1)} \). Indeed, since \( w \leq \phi_1 \) on \( Q_1 \) and we could assume that \( 6\alpha \omega_n \eta^{-\sigma} > 1 \) by the smallness of \( \eta \), we can easily derive (a). For the proof of (b), if we suppose that \( M > 4[u]_{C^{0,1}_x(Q_1)} \), then the inequality \( w \leq \phi_2 \) on \( Q_1 \) implies that
\[
w(x_0, t_0) \leq \frac{1}{2} M + [u]_{C^{0,1}_x(Q_1)} < M - [u]_{C^{0,1}_x(Q_1)},
\]
which is a contradiction. Hence by (3.7) and (3.8) we can get that
\[
w \leq 5[u]_{C^{0,1}_x(Q_1)} \leq 25[u]_{C^{0,1}_x(Q_1)} + \frac{48}{\omega(B_{1/2})} \| u \|_{L^\infty(L^1)} \| u \|_{L^\infty(L^1)} \leq C \| u \|_{L^\infty(L^1)}
\]
on \( Q_1 \), with a universal constant \( C > 0 \). In a similar way, we can show that \( w \geq -C \| u \|_{L^\infty(L^1)} \) by constructing subsolutions corresponding \( \phi_1 \) and \( \phi_2 \). Thus
we obtain the estimate
\begin{equation}
\|u\|_{C^1_a(Q_1)} \leq C \|u\|_{L^\infty(L^1_T)}
\end{equation}
Therefore by (3.7) and (3.9) we obtain the required estimate. \hfill \Box

**Theorem 3.7.** Let $\mathbf{I}$ be the nonlocal operator as in (1.1). Then the operator $\mathbf{I}$ satisfies the following properties:

(a) $\mathbf{I}u(x,t)$ is well-defined for any $u \in C^{1,1}_x(\Omega_T) \cap L^\infty_T(L^1_\omega)$.

(b) $\mathbf{I}u$ is continuous in $\Omega_T$, whenver $u \in C^{1,1}_x(\Omega_T) \cap L^\infty_T(L^1_\omega)$.

**Proof.** (a) It can be shown as in the elliptic case.

(b) Take any $u \in L^\infty_T(L^1_\omega)$ and $\varepsilon > 0$. Then, for any $t \in [-\tau, 0] \subset (-T, 0)$, there is some $g_t \in C^\infty_c(\mathbb{R}^n)$ with $\text{supp}(g_t) \supset \Omega$ such that $\|g_t - u(\cdot,t)\|_{L^1_\omega} < \varepsilon$. We consider a function $g \in L^\infty(\mathbb{R}^n_T)$ (i.e. $g \in L^\infty(\mathbb{R}^n_T)$ with compact support) so that
\[
g(x,t) = \begin{cases}
g_t(x), & (x,t) \in \mathbb{R}^n \times [-\tau, 0], \\
0, & (x,t) \in \mathbb{R}^n \times (-T, -\tau - 1)
\end{cases}
\]
and $\sup_{t \in [-\tau, 0]} \|g(\cdot,t) - u(\cdot,t)\|_{L^1_\omega} < \varepsilon$. So we may assume that $u \in L^\infty(\mathbb{R}^n_T) \cap C^{1,1}_x(\Omega_T) \cap L^\infty_T(L^1_\omega)$. Thus it easily follows from the continuity of $u(\cdot,t)$ in time variable on the norm $\|\cdot\|_{L^1_\omega}$ and the proof of the elliptic case (see \cite{CS1}, \cite{CS2} and \cite{KL}). \hfill \Box

### 4. Boundary estimates and Global estimates

In this section, we realize that a modulus of continuity on the parabolic boundary of the domain of some equation makes it possible to obtain another modulus of continuity inside the domain. This can be established by controlling the growth of $u$ away from its parabolic boundary values via barriers, scaling and interior regularity.

We use a barrier function which was used in \cite{CS1} for the elliptic case and adapted to our parabolic setting. This barrier function is appropriate as a supersolution of $M^+_\sigma \psi - \partial_\tau \psi \leq 0$ for all values of $\sigma$ greater than a given $\sigma_0$, where $M^+_\sigma$ denotes the maximal operator $M^{1,1}_a(\sigma)$. Another way to say this would be to define a larger class $\mathcal{L}$ which is the union of all classes $\mathcal{L}_a(\sigma)$ for $\sigma \in (\sigma_0, 2)$, then $M^+_\sigma \psi - \partial_\tau \psi \leq 0$. The proof of the following lemma can be achieved by a little modification to our parabolic setting (refer to \cite{CS1}), so we leave the proof for the reader.

**Lemma 4.1.** Let $\sigma_0 \in (0,2)$ be given. Then, for any $\sigma \in (\sigma_0, 2)$ and $\gamma \in (0,1)$, there are some $\alpha > 0$ and $r > 0$ so small that the function $g_\alpha(x,t) = (|x| - 1)^\sigma_\alpha$ satisfies $M^+_\sigma g_\alpha \leq -1/[(2\sigma - 1)\gamma^\alpha]$ in $(B_1 \setminus B_\alpha) \times (-T, 0)$.

**Corollary 4.2.** Let $\sigma_0 \in (0,2)$ be given. Then, for any $\sigma \in (\sigma_0, 2)$ and $\gamma \in (0,1)$, there is a continuous function $\psi$ defined on $\mathbb{R}^n_T$ such that $(a)$ $\psi = 0$ in $Q_1$, $(b)$ $\psi \geq 0$ in $\mathbb{R}^n_T$, $(c)$ $\psi \geq \gamma - \sigma$ in $\mathbb{R}^n_T \setminus Q_2$, $(d)$ $M^+_\sigma \psi - \partial_\tau \psi \leq 0$ and $\partial_\tau \psi \geq -[(2\sigma - 1)\gamma^\alpha]^{-1}$ in $\mathbb{R}^n_T \setminus Q_1$.

**Proof.** We consider $\psi(x,t) = \min\{\gamma - \sigma, C(|x| - 1)^{\alpha_\sigma} + (t + 1)/[(2\sigma - 1)\gamma^\alpha]\}$ for some large constant $C > 0$ and apply Lemma 4.1. \hfill \Box

The function $\psi$ obtained in Corollary 4.2 shall be utilized as a barrier to prove the boundary continuity of solutions to nonlocal parabolic equations. We observe that $\psi$ is a supersolution outside the parabolic cube $Q_1$.
Theorem 4.3. Let \( \sigma \in (\sigma_0, 2) \) for \( \sigma_0 \in (1, 2) \). If \( u \in L_{\infty}^{\frac{n}{\sigma}}(L_1^{\frac{1}{\sigma}}) \) satisfies that
\[
M^+_\sigma u - \partial_t u \geq -C \quad \text{and} \quad M^-\sigma u - \partial_t u \leq C \quad \text{in} \quad Q_{1+\eta},
\]
for every \((x, t) \in \partial_P Q_1 \) and \((y, s) \in \mathbb{R}_T^n \setminus Q_1 \), where \( \rho \) is a modulus of continuity, then there is another modulus of continuity \( \bar{\rho} \) (depending only on \( \rho, \Lambda, \sigma_0, \eta, \|u\|_{L_{\infty}^{\frac{n}{\sigma}}(L_1^{\frac{1}{\sigma}})} \), \( \eta \) and \( C \), but not on \( \sigma \)) such that
\[
|u(y, s) - u(x, t)| \leq \bar{\rho}((|x - y|^\sigma + |t - s|)^{1/\sigma})
\]
for every \((x, t) \in \overline{Q}_1 \) and \((y, s) \in \mathbb{R}_T^n \).

Lemma 4.4. Let \( \sigma \in (\sigma_0, 2) \) for \( \sigma_0 \in (1, 2) \). If \( u \in L_{\infty}^{\frac{n}{\sigma}}(L_1^{\frac{1}{\sigma}}) \) is a function such that
\[
\begin{align*}
M^+_\sigma u - \partial_t u &\geq -C \\
M^-\sigma u - \partial_t u &\leq C
\end{align*}
\]
in \( Q_{1+\eta} \),
for every \((x, t) \in \partial_P Q_1 \) and \((x, t) \in \mathbb{R}_T^n \setminus Q_1 \), where \( \rho \) is a modulus of continuity, then there is another modulus of continuity \( \bar{\rho} \) (depending only on \( \rho, \Lambda, \sigma_0, \eta, \|u\|_{L_{\infty}^{\frac{n}{\sigma}}(L_1^{\frac{1}{\sigma}})} \), \( \eta \) and \( C \), but not on \( \sigma \)) such that
\[
|u(x, t) - u(x_0, t_0)| \leq \bar{\rho}((|x - x_0|^\sigma + |t - t_0|)^{1/\sigma})
\]
for every \((x_0, t_0) \in \partial_P Q_1 \) and \((x, t) \in \mathbb{R}_T^n \).

Proof. If we write \( v = u\|_{Q_{1+\eta}} \), then as before we have that
\[
\begin{align*}
M^+_\sigma v - \partial_t v &\geq -c_\eta - \|v\|_{L_{\infty}^{\frac{n}{\sigma}}(L_1^{\frac{1}{\sigma}})} \\
M^-\sigma v - \partial_t v &\leq c_\eta + \|u\|_{L_{\infty}^{\frac{n}{\sigma}}(L_1^{\frac{1}{\sigma}})}
\end{align*}
\]
in \( Q_{1+\eta} \), since \( u \) is continuous on \( \overline{Q}_{1+\eta} \), we may assume that \( u \in B(\mathbb{R}_T^n) \).

Since \( \sigma \geq \sigma_0 > 0 \), the function
\[
p_0(x, t) = \frac{1}{4}(0 \vee (4 - |x|^2)) + (C + 4\Lambda\omega_n(2 - \sigma)\frac{1 - 3^{-\sigma}}{\sigma})t
\]
satisfies \( M^+_\sigma p_0 \leq -\lambda\omega_n + 4\Lambda\omega_n(2 - \sigma)\frac{1 - 3^{-\sigma}}{\sigma} \) in \( Q_1 \), because
\[
\begin{align*}
Lp_0(x, t) &= -\int_{B_1} |y|^2 K(x, y, t) \, dy + \int_{B_1 \setminus B_{3}} \mu_t(p_0, x, y) K(x, y, t) \, dy \\
&\leq -\frac{\lambda\omega_n}{2} + 4\Lambda\omega_n(2 - \sigma)\frac{1 - 3^{-\sigma}}{\sigma}
\end{align*}
\]
for any \( L \in \mathcal{L}_0(\sigma) \) and all \((x, t) \in Q_1 \). Since \( \partial_t p_0 = C + 4\Lambda\omega_n(2 - \sigma)\frac{1 - 3^{-\sigma}}{\sigma} \) in \( Q_1 \), we have that
\[
(4.1) \quad M^+_\sigma (u - p_0) - \partial_t (u - p_0) \geq M^+_\sigma u - \partial_t u - M^+_\sigma p_0 + \partial_t p_0 \geq \frac{\lambda\omega_n}{2} \geq 0 \quad \text{in} \quad Q_1.
\]
Let \( \rho_1 \) be the modulus of continuity of the function \( \psi \) in Corollary 4.2 and let \( \rho_1 \) be the modulus of continuity of the function \( p \). By the assumption, we see that
\[
(4.2) \quad u(x, t) - p_0(x, t) = u(x_0, t_0) + p_0(x_0, t_0) - u(x_0, t_0) \\
\leq \rho((|x - x_0|^\sigma + |t - t_0|)^{1/\sigma}) + \rho_1((|x - x_0|^\sigma + |t - t_0|)^{1/\sigma}),
\]
for every \((x_0, t_0) \in \partial_P Q_1 \) and \((x, t) \in \mathbb{R}_T^n \setminus Q_1 \).

Fix any \((x_0, t_0) \in \partial_P Q_1 \). For \( r > 0 \), we define \( \bar{\rho} \) by
\[
\bar{\rho}(r) = \inf_{\gamma \in (0, 1)} \left( \rho(3\gamma) + \rho_1(3\gamma) + \|u - p_0 - u(x_0, t_0) + p_0(x_0, t_0)\|_{L_{\infty}} \rho_0 \left( \frac{r}{\gamma} \right) \right).
\]
Then we must show that $\bar{\rho}$ is a modulus of continuity. We easily see that $\bar{\rho}$ is clearly monotonically increasing because $\rho_0$ is. So we have only to show that for any $\varepsilon > 0$, there is some $r > 0$ such that $\bar{\rho}(r) < \varepsilon$. Indeed, we choose some $\gamma \in (0,1)$ such that $\rho(3\gamma) + \rho_1(3\gamma) \leq \varepsilon/2$, and then choose some $r > 0$ so that $\|u - p_0 - u(x_0,t_0) + p_0(x_0,t_0)\|_{L^\infty} = \rho_0(r/\gamma) < \varepsilon/2$. Finally, we show that there is a modulus of continuity $\bar{\rho}$ such that $u(x,t) - u(x_0,t_0) \leq \bar{\rho}(|x - x_0|^\sigma + |t - t_0|)^{1/\sigma}$ for any $(x,t) \in \mathbb{R}_+^d$. Take any $(x,t) \in \mathbb{R}_+^d$. For $\gamma > 0$, we consider a barrier function $B(x,t) = u(x_0,t_0) - p_0(x_0,t_0) + \rho(3\gamma) + \rho_1(3\gamma)$

By (4.2) and the definition of $\psi$, we have that $B(x,t) \geq u(x_0,t_0) - p_0(x_0,t_0) + \rho(3\gamma) + \rho_1(3\gamma) \geq u(x,t) - p_0(x,t)$ for any $(x,t) \in Q_{3\gamma}(x_0,t_0) \cap (\mathbb{R}_+^d \setminus Q_1)$. Also by the definition of $\psi$, we obtain that $B(x,t) \geq u(x,t) - p_0(x,t)$ for any $(x,t) \in Q_{3\gamma}(x_0,t_0) \cap (\mathbb{R}_+^d \setminus Q_1)$. Thus we have that $B \geq u - p_0$ on $\mathbb{R}_+^d \setminus Q_1$. By (d) of Corollary 4.2, we see that $M_\gamma^+ \psi - \partial_t \psi \leq 0$ in $Q_{1/\gamma}(\frac{\gamma + 1}{\gamma}x_0,0)$ because $Q_{1/\gamma}(\frac{\gamma + 1}{\gamma}x_0,0) \subset \mathbb{R}_+^d \setminus Q_1$. We observe that $M_\gamma^+ \psi - \partial_t \psi \leq 0$ in $Q_{1/\gamma}(\frac{\gamma + 1}{\gamma}x_0,0) \Leftrightarrow M_\gamma^+ B - \partial_t B \leq 0$ in $Q_1$.

Hence we have that $u(x,t) - p_0(x,t) \leq B(x,t) \leq u(x_0,t_0) - p_0(x_0,t_0) + \bar{\rho}(|x - x_0|^\sigma + |t - t_0|)^{1/\sigma}$ for all $(x,t) \in \mathbb{R}_+^d$, because $\psi(x,t) \leq |\psi(x,t) - \psi(x_0,t_0)| \leq \rho_0(|x - x_0|^\sigma + |t - t_0|)^{1/\sigma}$, $\forall(x,t) \in \mathbb{R}_+^d$. Hence we have that $u(x,t) - u(x_0,t_0) \leq \bar{\rho}(|x - x_0|^\sigma + |t - t_0|)^{1/\sigma}$ for all $(x,t) \in \mathbb{R}_+^d$, where $\bar{\rho} = \rho_1 + \bar{\rho}$. Therefore we complete the proof.

**Lemma 4.5.** Let $\sigma \in (\sigma_0,2)$ for $\sigma_0 \in (1,2)$. If $u \in L^\infty_{0}(L^1_{\omega})$ satisfies that $M_\sigma^+ u - \partial_t u \geq -C$ and $M_\sigma^- u - \partial_t u \leq C$ in $Q_{1+\eta}$, $|u(y,s) - u(x,t)| \leq \rho(|x - y|^\sigma + |t - s|)^{1/\sigma}$ for every $(x,t) \in \partial_p Q_1$ and $(y,s) \in \mathbb{R}_+^d$, where $\rho$ is a modulus of continuity, then there is another modulus of continuity $\bar{\rho}$ (depending only on $\rho, \lambda, \sigma, \sigma_0, n, ||u||_{L^\infty_{0}(L^1_{\omega})}$, $\eta$ and $C$, but not on $\sigma$) such that $|u(y,s) - u(x,t)| \leq \bar{\rho}(|x - y|^\sigma + |t - s|)^{1/\sigma}$ for every $(x,t) \in \overline{Q}_1$ and $(y,s) \in \mathbb{R}_+^d$.

**Proof.** If we set $v = u|_{Q_{1+\eta}}$, then as before we have that $M_\sigma^+ v - \partial_t v \geq -c_\eta - ||u||_{L^\infty_{0}(L^1_{\omega})}$ and $M_\sigma^- v - \partial_t v \leq c_\eta + ||u||_{L^\infty_{0}(L^1_{\omega})}$ in $Q_1$. Since $u$ is continuous on $\overline{Q}_{1+\eta}$, we may assume that $u \in B(\mathbb{R}_+^d)$. Hence it can easily be obtained by an adaptation of Lemma 3 in [CS1] to our parabolic setting.

**Proof of Theorem 4.3.** We apply Lemma 4.4 to both $u$ and $-u$ to obtain a modulus of continuity that applies from any point on $\partial_p Q_1$ to any point in $\mathbb{R}_+^d$. Then we use Lemma 4.5 to finish the proof.
5. Some results by approximation

In this section, we show that two equations which are very close to each other in some appropriate way have their solutions which are close by each other on the unit cube $Q_1$.

In what follows, for a function $u : \mathbb{R}^n \to \mathbb{R}$ and a parabolic quadratic polynomial $p$ we denote by $u^p_{Q_r(x_0,t_0)} \doteq p(I_{Q_r(x_0,t_0)} + u|_{\mathbb{R}^n_+ \setminus Q_r(x_0,t_0)})$. The following lemma is an usual result in analysis on viscosity solutions, and so we will skip the proof.

**Lemma 5.1.** Let $L$ be a uniformly elliptic operator in the sense of (1.2) with respect to some class $\mathfrak{L}$ and let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a function which is upper semicontinuous on $\overline{Q}_T$. Then the followings are equivalent.

(a) $u$ is a viscosity subsolution of $\mathcal{L} u - \partial_t u = f$ in $\Omega_T$, i.e. $\mathcal{L} u - \partial_t u \geq f$ in $\Omega_T$.

(b) If $p$ is a parabolic quadratic polynomial satisfying $u(x_0,t_0) = p(x_0,t_0)$ and $u \leq p$ in $Q_{r}(x_0,t_0)$ where $Q_{r}(x_0,t_0) \subset \Omega_T$ for some $r > 0$ and $(x_0,t_0) \in \Omega_T$, then we have that $\mathcal{L} u^p_{Q_r(x_0,t_0)} - \partial_t u^p_{Q_r(x_0,t_0)} \geq f(x_0,t_0)$ for $u^p_{Q_r(x_0,t_0)}$.

We want to show that if $\mathcal{L} u_k(x,t) = f_k(x,t)$ and $\mathcal{L} u \to \mathcal{L} u_k \to u$ and $f_k \to f$ in some appropriate way, then $\mathcal{L} u(x,t) = f(x,t)$.

In the elliptic case [CSII], the solution space $L^1(\Omega)$ is enough for the weakly convergence of operators $\mathcal{L} u_k$. In the parabolic case, the possible substitute for the solution space $L^1(\Omega)$ is $L^\infty_{T}(L^1_\omega)$. This makes it possible to obtain the stability properties for the nonlocal parabolic case.

**Definition 5.2.** We say that $\mathcal{L} u$ converges weakly to $\mathcal{L} u_k$ in $\Omega_T$ with respect to $\omega$ (and we denote by $\lim_{k \to \infty} \mathcal{L} u_k = \omega \mathcal{L} u$ in $\Omega_T$), if for any $(x_0,t_0) \in \Omega_T$ there is some $Q_{r}(x_0,t_0) \subset \Omega_T$ such that

\begin{equation}
\lim_{k \to \infty} \mathcal{L} u_k(x_0,t_0) = \omega \mathcal{L} u(x_0,t_0)
\end{equation}

uniformly in $Q_{r/2}(x_0,t_0)$ for any function $u^p_{Q_r(x_0,t_0)}$ of the form $u^p_{Q_r(x_0,t_0)} = u^p_{Q_r(x_0,t_0)}$ where $p$ is a parabolic quadratic polynomial and $u \in L^\infty_{T}(L^1_\omega)$.

**Lemma 5.3.** Let $\mathcal{L} u$ be a uniformly elliptic operator with respect to a class $\mathfrak{L}$ of linear integro-differential operators. If $u^p = u^p_{Q_r(x_0,t_0)}$ where $p$ is a parabolic quadratic polynomial and $u \in L^\infty_{T}(L^1_\omega)$, then $u^p$ is continuous in $Q_r(x_0,t_0)$.

**Proof.** Since $u^p \in C^{1,1}_{x_0}(Q_r(x_0,t_0)) \cap L^\infty_{T}(L^1_\omega)$, the required result easily follows from Theorem 3.7. \hfill \Box

**Lemma 5.4.** Let $\{\mathcal{L} u_k\}$ be a sequence of uniformly elliptic operators with respect to some class $\mathfrak{L}$. Assume that Assumption 3.3 holds. Let $\{u_k\} \subset L^\infty_{T}(L^1_\omega)$ be a sequence of lower semicontinuous functions in $\Omega_T$ such that

(a) $\mathcal{L} u_k - \partial_t u_k \leq f_k$ in $\Omega_T$, (b) $\lim_{k \to \infty} \mathcal{L} u_k = u$ in the $\mathcal{L}$ sense in $\Omega_T$,

(c) $\lim_{k \to \infty} \|u_k - u\|_{L^\infty_{T}(L^1_\omega)} = 0$, (d) $\lim_{k \to \infty} \mathcal{L} u_k = \omega \mathcal{L} u$ in $\Omega_T$,

(e) $\lim_{k \to \infty} f_k = f$ locally uniformly in $\Omega_T$,

(f) $\sup_{k \in \mathbb{N}} \sup_{\Omega_T} |u_k| \leq C < \infty$.

Then we have that $\mathcal{L} u - \partial_t u \leq f$ in $\Omega_T$.

**Proof.** Let $p$ be a parabolic quadratic polynomial touching $u$ from below at a point $(x,t)$ in a neighborhood $V \subset \Omega_T$. Since $\{u_k\}$ $\Gamma$-converges to $u$ in $\Omega_T$, there are a cube $Q_{r}(x,t) \subset V$ and a sequence $\{(x_k,t_k)\} \subset Q_{r}(x,t)$ with $\lim_{k \to \infty} d((x_k,t_k),(x,t)) = 0$. Then $\mathcal{L} u_k - \partial_t u_k \leq p$ in $Q_{r}(x,t)$ as well and $\mathcal{L} u_k - \partial_t u_k \leq f_k$ in $Q_{r}(x,t)$.
0 such that $p$ touches $u_k$ from below at $(x_k, t_k)$ (refer to [GD]). Without loss of
generality, we assume that $Q_r(x, t)$ is a cube so that (5.1) holds for the point $(x, t)$.
If $(u_k^p)_p = (u_k^p)^p_{Q_r(x, t)}$, then we have $I(u_k^p)_{Q_r(x, t)} - \partial_t (u_k^p)_{Q_r(x, t)} \leq f(x_k, t_k)$.
If we set $u^p_k = u^p_{Q_r(x, t)}$, then we see that $u^p_k(z, q) = (u_k^p)_p(z, q)$ and $\partial_t (u_k^p)_{Q_r(x, t)} = \partial_t u^p_k(z, q)$ for any $k \in \mathbb{N}$ and $(z, q) \in Q_r(x, t)$. Take any $(z, q) \in Q_{r/4}(x, t)$. Then we have that
\[
\begin{align*}
&|I_k(u_k^p)_{Q_r(x, t)}(z, q) - \partial_t (u_k^p)_{Q_r(x, t)}(z, q) - Iu^p_k(z, q) + \partial_t (z, q)| \\
&\leq |I_k(u_k^p)_{Q_r(x, t)}(z, q) - Iu^p_k(z, q)| + |I_k(u_k^p)_{Q_r(x, t)}(z, q) - Iu^p_k(z, q)| \\
&\leq |M^+_{\mathcal{L}}((u_k^p)_p - u^p_k)(z, q)| \lor |M^+_{\mathcal{L}}((u_k^p)_p - (u_k^p)^p_{Q_r(x, t)})(z, q)| + |I_k(u^p_k(z, q) - Iu^p_k(z, q)| \\
&\leq \sup_{L \in \Sigma} |L((u_k^p)_p - u^p_k)(z, q)| + |I_k(u^p_k(z, q) - Iu^p_k(z, q)| \\
&\leq \int_{\mathbb{R}^n \setminus B_{r/2}} |\mu_q((u_k^p)_p - u^p_k, z, y)| K(x, y, t) \, dy + |I_k u^p_k(z, q) - Iu^p_k(z, q)| \\
&\leq C \int_{\mathbb{R}^n \setminus B_{r/2}} \{(u_k^p)_p - u^p_k)(z + y, q)\} \omega(y) \, dy \\
&\quad \quad \quad + |I_k u^p_k(z, q) - Iu^p_k(z, q)| \\
&\leq C \int_{\mathbb{R}^n} 2|(u_k^p)_p(y, q) - u^p_k(y, q)| \sup_{z \in B_{r/4}} \omega(y + z) \, dy + |I_k u^p_k(z, q) - Iu^p_k(z, q)| \\
&\leq C \|u_k - u\|_{L^p(\Omega)} + |I_k u^p_k(z, q) - Iu^p_k(z, q)| \to 0
\end{align*}
\]
as $k \to \infty$, by using Assumption 3.3 and (3.2). Since $(u_k^p)_p \in C^2(Q_r(x, t)) \cap L^\infty_\omega(L^1_\omega)$
for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} \|u_k - u\|_{L^\infty(L^1_\omega)} = 0$, we see $u^p_k \in C^2(Q_r(x, t)) \cap L^\infty_\omega(L^1_\omega)$
and thus $u^p_k$ is continuous in $Q_r(x, t)$ (by Lemma 5.3). Thus by (5.1) we have that
\[
\begin{align*}
&|I_k(u_k^p(x_k, t_k) - \partial_t (u_k^p)(x_k, t_k) - Iu^p_k(x, t) + \partial_t (x, t)| \\
&\leq |I_k(u_k^p(x_k, t_k) - \partial_t (u_k^p)(x_k, t_k) - Iu^p_k(x, t) + \partial_t (x, t)| \\
&\quad + |I_k u^p_k(x_k, t_k) - Iu^p_k(x, t)| + |Iu^p_k(x, t) - Iu^p_k(x, t)| \\
&\quad + |\partial_t u^p_k(x_k, t_k) - \partial_t u^p_k(x, t)| \to 0 \quad \text{as } k \to \infty.
\end{align*}
\]
Since $\lim_{k \to \infty} d((x_k, t_k), (x, t)) = 0$ and $\lim_{k \to \infty} f_k = f$ locally uniformly in $\Omega_r$, we have that $f_k(x_k, t_k) \to f(x, t)$. Thus this implies that $Iu^p_k(x, t) - \partial_t u^p_k(x, t) \leq f(x, t)$. Hence we conclude that $\lim u - \partial_t u \leq f$ in $\Omega_r$. \hfill \Box

**Lemma 5.5.** Let $u^p_k = u^p_{Q_r}$, where $p$ is a quadratic polynomial and $u \in L^\infty_\omega(L^1_\omega)$.
If $\{(x_k)\}$ is a sequence of uniformly elliptic operators with respect to some class $\Sigma$
satisfying Assumptions 2.2 and 2.3, then there is a subsequence $\{x_k\}$ such that $\forall_k u^p_k$ converges uniformly in $Q_{r/2}$.

**Proof.** We have only to find a uniform modulus of continuity for $\forall_k u^p_k$ in $Q_r$ so
that the lemma follows from Arzela-Ascoli Theorem.
Take any two $(x, t), (y, s) \in Q_{r/2}$ with $d((x, t), (y, s)) < r/8$. By the uniform
eellipticity of $\forall_k$, we have that
\[
\begin{align*}
&\forall_k u^p_k(x, t) - \forall_k u^p_k(y, t) \leq M^+_{\mathcal{L}}(u^p_k - \tau_{-y} u^p_k(x, t).
\end{align*}
\]
Also we see that $u^p_k - \tau_{-y} u^p_k = 0$ in $B_{r/4}(x) \times \{t\}$, because
\[
p(x + z, t) + p(x - z, t) - 2p(x, t) - p(y + z, t) - p(y - z, t) + 2p(y, t) = 0
\]
for any $z \in B_{r/4}$. From (3.2) and Assumption 3.3, for any $L \in \mathfrak{L}$ we have that
\[
L(u^p_r - \tau_{y,z} u^p_r)(x,t) = \int_{\mathbb{R}^n \setminus B_{r/4}} \mu^1(u^p_r - \tau_{y,z} u^p_r, x, z) K(x, z, t) \, dz
\]
\[
= \int_{\mathbb{R}^n \setminus B_{r/4}} \left[ u^p_r(x + z, t) + u^p_r(x - z, t) - 2u^p_r(x, t) \right] K(x, z, t) \, dz
\]
\[
- \int_{\mathbb{R}^n \setminus B_{r/4}} \left[ u^p_r(y + z, t) + u^p_r(y - z, t) - 2u^p_r(y, t) \right] K(x, z, t) \, dz
\]
\[
\leq C_r \left( |p(x, t) - p(y, t)| + \|\tau_x u^p_r - \tau_y u^p_r\|_{L^\infty(L^1)} \right)
\]
\[
\leq C_r \sup_{[\xi - \eta] \leq |x - y|} \left( |p(\xi, t) - p(\eta, t)| + \int_{\mathbb{R}^n} |\tau_{\xi - \eta} u^p_r(z, t) - u^p_r(z, t)| \left( \sup_{B_r(z)} \omega \right) \, dz \right)
\]
\[
\leq m_1(|x - y|)
\]
where $m_1$ is defined as $m_1(q) = \sup_{t \in (-r/2, r/2]} m_t(q)$ and
\[
m_t(q) = C_r \sup_{[\xi - \eta] \leq \varepsilon} \left( |p(\xi, t) - p(\eta, t)| + \int_{\mathbb{R}^n} |\tau_{\xi - \eta} u^p_r(z, t) - u^p_r(z, t)| \omega(z) \, dz \right).
\]
Thus by (5.2) we obtain that
\[
\Pi_k u^p_r(x,t) - \Pi_k u^p_r(y,t) \leq m_1(|x - y|).
\]

On the other hand, we now estimate $\Pi_k u^p_r(y,t) - \Pi_k u^p_r(y,s)$. By the uniform ellipticity of $\Pi_k$, we have that
\[
\Pi_k u^p_r(y,t) - \Pi_k u^p_r(y,s) \leq M^+_{L^\infty}(u^p_r - \tau^{s-t} u^p_r)(x,t).
\]
Observing $u^p_r - \tau^{s-t} u^p_r = 0$ in $B_{r/4}(x) \times \{t\}$ from the fact that
\[
p(x + z, t) + p(x - z, t) - 2p(x, t) - p(x + z, s) - p(x - z, s) + 2p(x, s) = 0
\]
for any $z \in B_{r/4}$, as in the above we can obtain that
\[
\Pi_k u^p_r(y,t) - \Pi_k u^p_r(y,s) \leq m_2(|t - s|^{1/\sigma})
\]
for some modulus of continuity $m_2$ depending on $u$ but not on $\Pi_k$. Hence by (5.3) and (5.4) we conclude that
\[
\Pi_k u^p_r(x,t) - \Pi_k u^p_r(y,s) \leq m((|x - y|^{\sigma} + |t - s|)^{1/\sigma})
\]
where $m((|x - y|^{\sigma} + |t - s|)^{1/\sigma}) = m_1(|x - y|) + m_2(|t - s|^{1/\sigma})$. Here it is clear that $m(q)$ is a modulus of continuity depending on $u$ but not on $\Pi_k$. Therefore there is a subsequence that converges uniformly by Arzela-Ascoli Theorem.

**Theorem 5.6.** Let $\{\Pi_k\}$ be a sequence of uniformly elliptic operators with respect to some class $\mathfrak{L}$ satisfying Assumptions 3.2 and 3.3 Then there is a subsequence $\{\Pi_{k_j}\}$ that converges weakly.

**Proof.** Since the space $L^\infty_{\mathcal{F}}(L^1_\omega)$ is separable with respect to the norm $\|\cdot\|_{L^\infty_{\mathcal{F}}(L^1_\omega)}$, we can take a countable dense subset $\mathcal{D} := \{u_j\}$ of $L^\infty_{\mathcal{F}}(L^1_\omega)$. We note that the set $\Pi_{n+1}$ of all parabolic quadratic polynomials is a finite dimensional space which has a countable dense subset $\{p_j\}$. For each $k \in \mathbb{N}$, we set
\[
v_{k,j_1,j_2} = p_{j_1} \mathbb{I}_{Q_{2-k}} + u_{j_2} \mathbb{I}_{\mathbb{R}^n \setminus Q_{2-k}}.
\]
Take any $\varepsilon > 0$ and any $v$ as in (5.1), i.e. $v = u^p_Q$, for some $p \in \Pi_{n+1}$ and $r > 0$. Then we choose $k$ so that $2^{-k} < r < 2^{-k+1}$ and select some $j_1$ and $j_2$ such that $\|u_{j_2} - u\|_{L^p_\infty(T^l_k)} < \varepsilon$, and $|D^2_p j_1 - D^2_p j_2| < \varepsilon$, $|D_p j_1 - D_p j_2| < \varepsilon$ and $|p_{j_1} - p| < \varepsilon$ in $Q_{2^{-k}}$. Since the set $\mathcal{E} = \{v_{k,j_1,j_2}\}$ is countable and dense, we can arrange it in a sequence $v_j$ of the form $v_j = p_j \mathbb{I}_{Q_{r_j}} + u_j \mathbb{I}_{\mathbb{R}^n \setminus Q_{r_j}}$ so that for each $v$ as in (5.1) there is some $v_j$ such that

$$
\|v - v_j\|_{L^p_\infty(T^l_k)} < \varepsilon,
$$

(5.5)

$$
\sup_{Q_{r_j}} |D^k v - D^k v_j| < \varepsilon \text{ for all } k = 0, 1, 2.
$$

By Lemma 5.5, for each $v_j \in \mathcal{E}$ there exists a subsequence $\mathbb{I}_{k_i}$ such that $\mathbb{I}_{k_i}(v_j)$ converges uniformly in $Q_{r_{j_i}/2}$.

By a standard diagonalization process, there is a subsequence $\{\mathbb{I}_{k_i}\}$ such that for each $v_j, \{\mathbb{I}_{k_i}v_j\}$ converges uniformly in $Q_{r_{j_i}/2}$. We call this limit $\mathbb{I}_x v_j(x,t)$. If $v$ is any test function, then there is some $j$ such that $v$ is close enough to $v_j$ in the sense of (5.2). Take any $(x,t) \in Q_{r/2}$. By the mean value theorem, we see that

$$
\mu_i(v - v_j, x, y) = \int_0^1 \int_0^1 (D^2_i(v - v_j))(x + s\tau y, t) y, y) ds d\tau
$$

for any $y \in B_{r/2}$. Thus it follows from (3.2), (5.2), (5.5) and Assumption 3.3 that

$$
\mathbb{I}_k v(x,t) - \mathbb{I}_k v_j(x,t) \leq \left( \int_{B_{r/2}} + \int_{\mathbb{R}^n \setminus B_{r/2}} \right) \mu_i(v - v_j, x, y) K_\mathbb{I}_k(x,y,t) dy
$$

$$
\leq C \varepsilon + C \|v - v_j\|_{L^p_\infty(T^l_k)} \leq C \varepsilon.
$$

for any $(x,t) \in Q_{r/2}$, uniformly in $k$. Taking $i$ large enough, we thus have that

$$
\|\mathbb{I}_k v(x,t) - \mathbb{I}_x v_j(x,t)\| < 2\varepsilon,
$$

and thus $\{\mathbb{I}_k v(x,t)\}$ is a Cauchy sequence in $L^\infty(Q_{r/2})$. We define $I_x v(x,t)$ to be the uniform limit of this sequence in $Q_{r/2}$. Thus we have shown that $\{\mathbb{I}_k v(x,t)\}$ converges uniformly to $I_x v(x,t)$ in $Q_{r/2}$.

To finish the proof, we must show that the operator $I_x$ can be extended to a uniformly elliptic operator for all test functions $\varphi$. We note that for any two test functions $v_1, v_2 \in \mathcal{T}$, we have that

$$
M^\mathbb{I}_x(v_1 - v_2)(x,t) \leq \mathbb{I}_k(v_1 - v_2)(x,t) \leq M^\mathbb{I}_x(v_1 - v_2)(x,t).
$$

Passing to the limit in this inequality, we obtain that

$$
M^\mathbb{I}_x(v_1 - v_2)(x,t) \leq I_x(v_1 - v_2)(x,t) \leq M^\mathbb{I}_x(v_1 - v_2)(x,t).
$$

Approximating on an arbitrary test function $\varphi$ as in the proof of Lemma 5.1, we can extend $I_x$ in a unique way to all test functions $\varphi$ such that $I_x$ is uniformly elliptic with respect to $\mathcal{L}$.

\[\square\]

**Lemma 5.7.** For some $\sigma \geq \sigma_0 > 1$ and $\gamma \in (0, \sigma_0 - 1)$, let $I_0, I_1$ and $I_2$ be nonlocal uniformly elliptic operators with respect to $\mathcal{L}_0(\sigma)$ satisfying Assumptions 3.2 and 3.3. Suppose that the boundary value problem

$$\begin{cases}
\mathbb{I}_0 u - \partial_t u = 0 & \text{in } Q_1, \\
u = h & \text{in } \mathbb{R}^n \setminus Q_1
\end{cases}$$
has at most one solution \( u \) for any \( h \in L_T^\infty(L_1^1) \). Given a modulus of continuity \( \rho \) and \( \varepsilon > 0 \), there are a small \( \delta > 0 \) and a large \( R > 0 \) so that if \( u, v, I_0, I_1 \) and \( I_2 \) satisfy

\[
I_0v - \partial_t v = 0, \quad I_1u - \partial_t u \geq -\delta, \quad \|I_2u - \partial_t u\| \leq \delta, \quad \|I_1 - I_0\| + \|I_2 - I_0\| \leq \delta \text{ in } Q_1,
\]

\[
u = v \text{ in } \mathbb{R}_T^n \setminus Q_1,
\]

\[
|u(x, t) - u(y, s)| + |v(x, t) - v(y, s)| \leq \rho((|x - y| + |t - s|)^{1/\sigma})
\]

for any \( x \in Q_R \setminus Q_1 \) and \( y \in \mathbb{R}_T^n \setminus Q_1 \),

\[
|u(x, t)| \leq M(1 \vee (|x|^\sigma + |t|)^{1/\sigma}),
\]

then we have that \( |u - v| < \varepsilon \) in \( Q_1 \).

**Proof.** Assume that the result was not true. Then there would be sequences \( \{R_k\}, \{\|I_0^{(k)}\|\}, \{\|I_1^{(k)}\|\}, \{\|I_2^{(k)}\|\}, \{\delta_k\}, \{u_k\}, \{v_k\}, \{I_k\} \) and \( \{\varepsilon_k\} \) such that \( R_k \to \infty \), \( \delta_k \to 0 \) and all the assumptions of the lemma hold, but \( \sup_{Q_1} |u_k - v_k| \geq \varepsilon \).

Since \( \{I_0^{(k)}\} \) is a sequence of uniformly elliptic operators, it follows from Theorem 5.6 that there is a subsequence that converges weakly to some nonlocal operator \( I_0 \) which is uniformly elliptic with respect to the same class \( \mathcal{L}_0(\sigma) \). Moreover we see that \( \{I_1^{(k)}\} \) and \( \{I_2^{(k)}\} \) converge to \( I_0 \) weakly, because \( \|I_0^{(k)} - I_1^{(k)}\| \to 0 \) and \( \|I_0^{(k)} - I_2^{(k)}\| \to 0 \).

Since \( \rho \) is a modulus of continuity on \( \partial_p Q_1 \) of both \( \{u_k\} \) and \( \{v_k\} \), by Theorem 4.3 there is a modulus of continuity \( \hat{\rho} \) which extends to the full unit cube \( \overline{Q_1} \). Thus \( \{u_k\} \) and \( \{v_k\} \) have a modulus of continuity on \( QR_k \) with \( R_k \to \infty \). We can find subsequences \( \{u_{k_j}\} \) and \( \{v_{k_j}\} \) which converges uniformly on compact sets in \( \mathbb{R}_T^n \) to \( u \) and \( v \), respectively. Since \( |u_{k_j}| \lor |v_{k_j}| \leq \gamma \in L_T^\infty(L^1) \) for all \( j \) where

\[
g(x, t) = M(1 \lor (|x|^\sigma + |t|)^{1/\sigma}),
\]

it follows from the Lebesgue’dominated convergence theorem that \( u, v \in L_T^\infty(L^1) \) and moreover

\[
\lim_{j \to \infty} \|u_{k_j} - u\|_{L_T^\infty(L^1)} = 0 \quad \text{and} \quad \lim_{j \to \infty} \|v_{k_j} - v\|_{L_T^\infty(L^1)} = 0.
\]

Since \( \sup_{Q_1} |u_{k_j} - v_{k_j}| \geq \varepsilon \), \( u \) and \( v \) must be different. By Lemma 5.4, we see that \( u \) and \( v \) solve the same equation \( I_0u - \partial_t u = I_0v - \partial_t v = 0 \) in \( Q_1 \). Thus by the assumption, we have \( u = v \), which is a contradiction.

\[\square\]

**Remark.** We will apply Lemma 5.7 to a translation-invariant operator \( I_0 \). In case that \( I_0 \) is a translation-invariant elliptic operator, the uniqueness for the viscosity solution of the boundary value problem was discussed in [CS2].

We also obtain the following simplified one of Lemma 5.7. The difference between this and Lemma 5.7 is that in Lemma 5.8 below we fix the boundary value \( h \), but we do not need a modulus of continuity in \( QR \setminus Q_1 \) and also on \( \partial_p Q_1 \).

**Lemma 5.8.** For some \( \sigma \geq \sigma_0 > 1 \), let \( \mathbb{I}_0, \mathbb{I}_1 \) and \( \mathbb{I}_2 \) be nonlocal uniformly elliptic operators with respect to \( \mathcal{L}_0(\sigma) \) satisfying Assumptions 3.2 and 3.3. Suppose that the boundary value problem

\[
\begin{cases}
\mathbb{I}_0u - \partial_t u = 0 & \text{in } Q_1, \\
u = h & \text{in } \mathbb{R}_T^n \setminus Q_1
\end{cases}
\]
has at most one solution $u$ for any $h \in L^\infty_r(L^1_1)$. Assume that $h$ is continuous on $\partial_t Q_1$. Given any $\varepsilon > 0$, there is some small $\delta > 0$ so that if $u, v$, $I_0, I_1$ and $I_2$ satisfy
\[ I_0 v - \partial_t v = 0, \quad I_1 u - \partial_t u \geq -\delta, \quad I_2 u - \partial_t u \leq \delta, \quad \|I_1 - I_0\| \vee \|I_2 - I_0\| \leq \delta \text{ in } Q_1, \]
\[ u = v = h \text{ in } \mathbb{R}^n \setminus Q_1, \]
then we have that $|u - v| < \varepsilon$ in $Q_1$.

Proof. We proceed the proof along the same line as that of Lemma 5.7. Assuming that the result was not true, we finish up the proof it by getting a contradiction. Assume that there are sequences $\{h_0^{(k)}\}, \{I_1^{(k)}\}, \{\delta_k\}, \{u_k\}, \{v_k\}$ and $I_k$ such that $\delta_k \to 0$ and all the assumptions of the lemma hold, but $\sup_{Q_1} |u_k - v_k| \geq \varepsilon$. The functions $u_k$ and $v_k$ have a fixed value $h$ outside $Q_1$. Since $h$ is continuous on $\partial_t Q_1$, by Theorem 4.3 we see that $\{u_k\}$ and $\{v_k\}$ are equicontinuous in $\overline{Q}_1$. So by Arzela-Ascoli Theorem, there is a subsequence which converges uniformly in $\overline{Q}_1$.

Continuing as in the proof of Lemma 5.7, we can take a subsequence such that $\{I_1^{(k)}\}$ and $\{I_0^{(k)}\}$ converges to $I_0$ weakly. Let $u$ and $v$ be the uniform limits of $u_k$ and $v_k$ in $Q_1$, respectively. Then we have that $\sup_{Q_1} |u_k - v_k| \geq \varepsilon$. But by Lemma 5.4, $u$ and $v$ must solve the same equation $I_0 u - \partial_t u = I_0 v - \partial_t v = 0$ in $Q_1$. Thus we conclude that $u = v$, which is a contradiction. \hfill \Box

6. $C^{1,\alpha}$-Regularity for Nonlocal Parabolic Equations with Variable Coefficients

The main concern of this section is to obtain $C^{1,\alpha}$ estimates for nonlocal parabolic equations which are not necessarily translation-invariant. Since our proofs rely on rescaling argument repeatedly, a kind of scale invariance will be needed. Even if we do not require a particular equation to be scale invariant, we will consider our equations within a whole class of equations that is scale invariant for which our regularity result up to the boundary is supposed to apply. Our proof on the parabolic case that will be given in this section is based on that of the elliptic case and the results of the parabolic case, but the main difference between them is to extend the solution space $B(\mathbb{R}^n)$ on the elliptic one to the more flexible space $L^\infty_r(L^1_1)$ on the parabolic one and to use the more wider class of kernels involving variables $(x, t) \in \mathbb{R}^n_t$.

The class $\mathcal{L}$ is said to have scale $\sigma$ if whenever the integro-differential operator with kernel $K(x, y, t)$ is in $\mathcal{L}$, its rescaled kernel $K_\lambda(x, y, t) := \lambda^{n+\sigma} K(x, \lambda y, t)$ is also in $\mathcal{L}$ for any $\lambda \in (0, 1)$. For example, the class $\mathcal{L}_0$ defined in (1.3) has scale $\sigma$, but the class $\mathcal{L}_n$ defined in (2.4) does not.

It is easy to check that if $\mathcal{L}$ has scale $\sigma$ and $u$ solves an equation $I_3 u(x, t) - \partial_t u(x, t) = f(x, t)$ in $Q_3$ that is elliptic with respect to $\mathcal{L}$, then the function $w_\mu(x, t) = u(\lambda x, \lambda^\sigma t)$ solves a uniformly elliptic equation
\[ \mathbb{I}_{\mu, \lambda} w_\mu(x, t) - \partial_t w_\mu(x, t) = \lambda^\sigma \mu f(\lambda x, \lambda^\sigma t) \text{ in } Q_3 \]
with respect to the same class $\mathcal{L}$. Equivalently, this condition becomes
\[ \mathbb{I}_{1, \lambda} w_1(x, t) - \partial_t w_1(x, t) = \lambda^\sigma f(\lambda x, \lambda^\sigma t) \text{ in } Q_3; \]
that is, $\mathbb{I}_{1, \lambda} u - \partial_t u = \lambda^\sigma f$ in $Q_{3\lambda}$. For instance, if $u(x, t) = \int_{\mathbb{R}^n} \mu_t(x, y, t) K(x, y, t) dy$, then $\mathbb{I}_{\mu, \lambda}$ is given by
\[ \mathbb{I}_{\mu, \lambda} u(x, t) = \int_{\mathbb{R}^n} [u(x + y, t) + u(x - y, t) - 2u(x, t)] \lambda^{n+\sigma} K(x, \lambda y, t) dy. \]
Here note that the coefficient $\mu$ does not have any effect on a linear operator.

We will call $\mathcal{L}_1$ the largest scale invariant class contained in the class $\mathcal{L}_0$ satisfying (2.4). This is the class of integro-differential operators with kernels $K$ satisfying (1.3) such that

\begin{equation}
\sup_{(x,t)\in \mathbb{R}^n_+} |\nabla_y K(x,y,t)| \leq \frac{C_1}{|y|^{n+\sigma+1}} \text{ for any } y \in \mathbb{R}^n \setminus \{0\}.
\end{equation}

Then it follows from Theorem 3.6 that an equation $\mathbb{I}_0 u - \partial_t u = 0$ has interior $C^{1,\beta}$-estimates for some $\beta > 0$, provided that $\mathbb{I}_0$ is uniformly elliptic with respect to the class $\mathcal{L}_1$.

Our main result in this section is to obtain that if an equation $\mathbb{I}_0 u - \partial_t u = f$ is uniformly elliptic with respect to a scale invariant class with interior $C^{1,\beta}$-estimates and have another equation $\mathbb{I}_q u - \partial_t u = f$ for a little perturbation $\mathbb{I}$ of $\mathbb{I}_0$, then this equation also has interior $C^{1,\alpha}$-estimates for any $\alpha \in (0, \beta \wedge (\sigma - 1))$.

**Definition 6.1.** For $\sigma \in (0,2)$ and an operator $\mathbb{I}$, we define the rescaled operator $\mathbb{I}_{\mu,\lambda}$ as in (6.1). Then the norm of scale $\sigma$ is defined as

$$\|\mathbb{I}^{(1)} - \mathbb{I}^{(2)}\|_\sigma = \sup_{(\mu,\lambda) \in [1,\infty) \times (0,1)} \|\mathbb{I}^{(1)}_{\mu,\lambda} - \mathbb{I}^{(2)}_{\mu,\lambda}\|$$

where $\|\cdot\|$ is the norm defined in Definition 2.1.

**Remark 6.2.** From (6.2), we see that $\|\mathbb{I}^{(1)} - \mathbb{I}^{(2)}\|_\sigma \leq \sup_{\lambda \in (0,1)} \|\mathbb{I}^{(1)}_{1,\lambda} - \mathbb{I}^{(2)}_{1,\lambda}\|$.

The rescaled operator implies that if $u$ solves the equation $\mathbb{I}_0 u - \partial_t u = f$ in $Q_\lambda$, then the rescaled function $u_{\mu,\lambda}(x,t) = \mu u(\lambda x, \lambda^\sigma t)$ solves an equation of the same ellipticity type $\mathbb{I}_{\mu,\lambda} w_{\mu,\lambda}(x,t) - \partial_t w_{\mu,\lambda}(x,t) = \lambda^\sigma \mu f(\lambda x, \lambda^\sigma t)$ in $Q_1$.

The following theorem is the main result of this paper.

**Theorem 6.3.** Let $\sigma \in (\sigma_0,2)$ for $\sigma_0 \in (1,2)$ and let $\mathbb{I}^{(0)}$ be a fixed translation-invariant nonlocal operator in a class $\mathcal{L} \subset \mathcal{L}_0(\sigma)$ with scale $\sigma$. Suppose that the equation $\mathbb{I}^{(0)} u - \partial_t u = 0$ in $Q_{1+\eta}$ has interior $C^{1,\beta}$-estimates. Let $\mathbb{I}^{(1)}$ and $\mathbb{I}^{(2)}$ be two nonlocal operators which are uniformly elliptic with respect to $\mathcal{L}_0(\sigma)$ and assume that $\|\mathbb{I}^{(0)} - \mathbb{I}^{(k)}\|_\sigma < \delta$ for some $\delta > 0$ small enough and $k = 1, 2$. If $u \in L^\infty_T(L^1)$ solves the equations

$$\mathbb{I}^{(1)} u - \partial_t u \geq f_1 \quad \text{and} \quad \mathbb{I}^{(2)} u - \partial_t u \leq f_2 \quad \text{in } Q_{1+\eta}$$

for functions $f_1, f_2 \in \Theta(Q_{1+\eta})$, then $u \in C^{1,\alpha}(Q_1)$ for any $\alpha \in (0, \beta \wedge (\sigma_0 - 1))$ and there is a constant $C > 0$ (depending only on $\sigma_0, \Lambda, \eta$, and the dimension but not on $\sigma$) such that

\begin{equation}
\|u\|_{C^{1,\alpha}(Q_1)} \leq C \left( \|u\|_{C(Q_{1+\eta})} + \|u\|_{L^\infty_T(L^1)} + \left( \sup_{Q_{1+\eta}} |f_1| \right) \lor \left( \sup_{Q_{1+\eta}} |f_2| \right) \right).
\end{equation}

**Proof.** We note that $u$ is continuous on $\overline{Q}_{1+\eta}$. We write $u = v + w$ where $v = u_{\|Q_{1+\eta}}$ and $w = u_{\|Q_{1+\eta}}$. Then by the uniform ellipticity of $\mathbb{I}^{(1)}$ we easily have that

$$M^+ v - \partial_t v \geq -\|u\|_{L^\infty_T(L^1)} - \left( \sup_{Q_{1+\eta}} |f_1| \right) \lor \left( \sup_{Q_{1+\eta}} |f_2| \right) \quad \text{in } Q_1.$$ 

Similarly, we have that

$$M^- v - \partial_t v \leq \|u\|_{L^\infty_T(L^1)} + \left( \sup_{Q_{1+\eta}} |f_1| \right) \lor \left( \sup_{Q_{1+\eta}} |f_2| \right) \quad \text{in } Q_1.$$ 

So we might use $v$ instead of $u$. 

We now select some $\lambda > 0$ small enough so that
\[
\lambda^{\beta-\alpha} + C_\theta 2^{\frac{(\frac{1}{\sigma}-(\alpha-\beta))}{\sigma}} \lambda^{\alpha_1-\alpha_2} 2^{\frac{1+\alpha_1}{\sigma}} < 1,
\]
\[
\lambda^{\beta-\alpha} + 2^{\frac{(\frac{1}{\sigma}-(\alpha-\beta))}{\sigma}} \lambda^{\alpha_1-\alpha_2}(3 + C_\theta) < 1.
\]
(6.5)

Take any $\varepsilon > 0$ with $\varepsilon < \lambda^{1+\beta}$. Then we choose $\delta = \delta(\varepsilon) > 0$ small enough as in Lemma 5.7. By scaling, without loss of generality, we may assume that
\[
\forall u \in L^\infty([0,T]) \quad \text{and} \quad u\big|_{R^\infty} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus Q_1.
\]

We have that $\forall u \in L^\infty([0,T]) \quad \text{and} \quad u\big|_{R^\infty} + \frac{1}{2} \sup_{Q_{1+\eta}} |f_1| \leq \frac{1}{2} \sup_{Q_{1+\eta}} |f_2| < \delta$ and $\sup_{\mathbb{R}^n} |u| \leq 1$

and $u$ solves the equation in some large cube $Q_R$.

By [C], it suffices to show that there are some $\lambda \in (0,1)$ and a sequence of linear functions
\[
\ell_k(x,t) = a_k + \langle b_k, x \rangle
\]
such that
\[
\sup_{Q_{1+\eta}} |u - \ell_k| \leq \lambda^{k(1+\alpha)}
\]
(6.6)
\[
|a_{k+1} - a_k| \leq \lambda^{k(1+\alpha)}
\]
\[
\lambda^k |b_{k+1} - b_k| \leq c_2 \lambda^{k(1+\alpha)}.
\]

Set $\ell_0 = 0$. Then we note that $|u_0| \leq 1$ in $Q_1$ and $|u_0(x,t)| \leq (|x| + |t|)^{\frac{1+\alpha}{\sigma}}$ for any $(x,t) \in \mathbb{R}^n \setminus Q_1$. We now continue the proof by the mathematical induction.

Assume that (6.6) holds for $k$-step. We shall show that they are still working for $(k+1)$-step. We set
\[
u_k(x,t) = \frac{u(\lambda^k x, \lambda^{k\sigma} t) - \ell_k(\lambda^k x, \lambda^{k\sigma} t)}{\lambda^{k(1+\alpha)}}.
\]

Since the class $\mathcal{L}$ has scale $\sigma$, $u_k$ solves equations of the same ellipticity type as follows;
\[
\Pi^{(1)}_k u_k(x,t) := \Pi^{(1)}_k(\lambda^{k(1+\alpha)} x, \lambda^k t) u_k(x,t) \geq \lambda^{k(\sigma-1-\alpha)} f_1(\lambda^k x, \lambda^{k\sigma} t),
\]
\[
\Pi^{(2)}_k u_k(x,t) := \Pi^{(2)}_k(\lambda^{k(1+\alpha)} x, \lambda^k t) u_k(x,t) \leq \lambda^{k(\sigma-1-\alpha)} f_2(\lambda^k x, \lambda^{k\sigma} t).
\]

We observe that the right hand side (6.7) is getting smaller as $k$ increases. Thus we have that
\[
\|\Pi^{(1)}_k - \Pi^{(0)}_k\| \leq \|\Pi^{(1)} - \Pi^{(0)}\|_{\sigma} < \delta.
\]

Let $\alpha_1 \in (\alpha, \beta \wedge (\sigma_0 - 1))$ be given. By the inductive assumption, we see that $|u_k| \leq 1$ in $Q_1$ and $|u_k(x,t)| \leq (|x| + |t|)^{\frac{1+\alpha}{\sigma}}$ for any $(x,t) \in \mathbb{R}^n \setminus Q_1$. Then we shall construct functions $\ell_{k+1}(x,t)$ and $u_{k+1}(x,t)$ so that
\[
|u_{k+1}(x,t)| \leq (|x| + |t|)^{\frac{1+\alpha}{\sigma}}, \quad \text{for any} \quad (x,t) \in \mathbb{R}^n \setminus Q_1.
\]

Since $u$ is uniformly continuous on $Q_{1+\eta}$, we may take some $R = R(\varepsilon) > 0$ (as in Lemma 5.7) so that $u$ admits a modulus of continuity $\varrho$ satisfying
\[
|u(x,t) - u(y,s)| \leq \varrho((|x-y| + |t-s|)^{1/\sigma})
\]
for any $(x,t) \in (Q_R \setminus Q_1)$ and $(y,s) \in \mathbb{R}^n \setminus Q_1$. Then we apply Lemma 5.7 to the function $g$ which solves
\[
\begin{cases}
\Pi^{(1)}_k g - \partial_\tau g = 0 & \text{in} \quad Q_1, \\
g = u_k & \text{in} \quad \mathbb{R}^n \setminus Q_1
\end{cases}
\]
to obtain that $\sup_{Q_1} |u_k - g| < \varepsilon$. By the assumption, we note that $u_k^{(0)}$ has interior $C^{1, \alpha}$-estimates. Let $\hat{g}(x, t) = \hat{a} + (\hat{b}, x)$ be the linear part of $g$ at the origin. Then we see that $|\hat{a}| < 1 + \varepsilon$, because $\sup_{Q_1} |g| \leq 1 + \varepsilon$. By the $C^{1, \alpha}$-estimates of $g$, we have that

$$|g(x, t) - \hat{g}(x, t)| \leq C_0(|x|^\sigma + |t|)\frac{1+\beta}{\sigma}$$

for any $(x, t) \in Q_{1/2}$. Since $\hat{g}(\frac{\hat{b}}{2|\hat{b}|}, 0) = \hat{a} + \frac{1}{2}|\hat{b}|$, by (6.8) we have the upper bound of $\hat{b}$ as follows;

$$|\hat{b}| \leq 2|\hat{g}(\frac{\hat{b}}{2|\hat{b}|}, 0) - g(\frac{\hat{b}}{2|\hat{b}|}, 0)| + 2|g(\frac{\hat{b}}{2|\hat{b}|}, 0)| + 2|\hat{a}|$$

$$= C_0 2^{-\beta} + 4(1 + \varepsilon) := C_\beta.$$

Then we can derive the estimates as follows;

$$|u_k(x, t) - \hat{g}(x, t)| \leq \begin{cases} 
\varepsilon + C_0(|x|^\sigma + |t|)^\frac{1+\beta}{\sigma}, & (x, t) \in Q_{1/2}, \\
\varepsilon + 2 + C_\beta, & (x, t) \in Q_1 \setminus Q_{1/2}, \\
\varepsilon + (2 + C_\beta)(|x|^\sigma + |t|)^\frac{1+\beta}{\sigma}, & (x, t) \in \mathbb{R}_T^n \setminus Q_1.
\end{cases}$$

We now set

$$\ell_{k+1}(x, t) = \ell_k(x, t) + \lambda^{k+1}(x, \lambda^{k+1}t),$$

$$u_{k+1}(x, t) := \frac{u(\lambda^{k+1}x, \lambda^{k+1}t) - \ell_{k+1}(\lambda^{k+1}x, \lambda^{k+1}t)}{\lambda^{k+1}(1+\alpha)}.$$  

Then it follows from (6.6) and (6.10) that

$$|u_{k+1}(x, t)| \leq \begin{cases} 
\lambda^{\beta-\alpha} + C^2_0 |x|^\sigma + |t|^{\frac{1+\alpha}{\sigma}} & (x, t) \in Q_{\lambda^{-1}/2}, \\
2 \lambda^{\alpha-\beta+1}(3 + C_\beta)|x|^\sigma + |t|^{\frac{1+\alpha}{\sigma}} & (x, t) \in Q_{\lambda^{-1}} \setminus Q_{\lambda^{-1}/2}, \\
\lambda^\beta + \lambda^{\alpha-\beta}(2 + C_\beta)|x|^\sigma + |t|^{\frac{1+\alpha}{\sigma}} & (x, t) \in \mathbb{R}_T^n \setminus Q_1 
\end{cases}$$

and moreover $|u_{k+1}| \leq 1$ on $Q_1$ and $|u_{k+1}(x, t)| \leq (|x|^\sigma + |t|)^{\frac{1+\alpha}{\sigma}}$ for any $(x, t) \in \mathbb{R}_T^n \setminus Q_1$. Finally, it follows from (6.11) that

$$u(x, t) - \ell_{k+1}(x, t) = \lambda^{(k+1)(1+\alpha)}u_{k+1}(\lambda^{-(k+1)}x, \lambda^{-(k+1)}t)$$

for any $(x, t) \in Q_{\lambda^{k+1}}$, and thus we conclude that

$$\sup_{Q_{\lambda^{k+1}}} |u - \ell_{k+1}| \leq \lambda^{(k+1)(1+\alpha)} \sup_{Q_1} |u_{k+1}| \leq \lambda^{(k+1)(1+\alpha)}.$$

Hence we complete the proof.

7. Cordes-Nirenberg Type Estimates and Applications

We furnish various concrete applications of the previous results in this section. Our proofs on the parabolic case are based on that [CST] of the elliptic case and the results [KL] of the parabolic case, but the main difference between them is in that we extend the solution space $B(\mathbb{R}^n)$ on the elliptic one to the more flexible
space $L^\infty_t(L^1_x)$ on the parabolic one and use the more wider class of kernels involving variables $(x,t) \in \mathbb{R}^n_t$, and moreover the parabolic case requires more careful consideration due to the time shift. Contrary to the elliptic case, we had better mention on the difficulty in the parabolic case; for instance, "time shift".

7.1. A parabolic version of the integral Cordes-Nirenberg type estimates

When the equation is linear and close to an operator in $\mathcal{L}_1$ in an appropriate way, we shall obtain the regularity results of its viscosity solutions. This is a parabolic version of the integral Cordes-Nirenberg type estimates.

**Theorem 7.1.** For $\sigma \in (1,2)$, let $u \in L^\infty_t(L^1_x)$ be a viscosity solution of the equation

$$Iu - \partial_t u := \int_{\mathbb{R}^n} \mu_t(u, \cdot, y) \frac{(2 - \sigma)a(x, y, \cdot)}{|y|^{n+\sigma}} \, dy - \partial_t u = f \text{ in } Q_{1+\eta},$$

where $f \in \mathcal{B}(Q_{1+\eta})$. Suppose that there is some $\delta > 0$ small enough such that

$$\sup_{x \in \mathbb{R}^n} \sup_{(t, \xi) \in Q_3} |a(x, y, t) - a_0(x, y, t)| < \delta,$$

where $a_0$ is a bounded function so that $(2 - \sigma)a_0(x, y, t)/|y|^{n+\sigma}$ satisfies (6.3).

If $u \in L^\infty_t(L^1_x)$ is a viscosity solution of the equation

$$\int_{\mathbb{R}^n} \mu_t(u, \cdot, y) \frac{(2 - \sigma)a_0(x, y, \cdot)}{|y|^{n+\sigma}} \, dy - \partial_t u = 0 \text{ in } Q_{1+\eta},$$

then there is some $\beta \in (0,1)$ so that $u \in C^{1,\alpha}(Q_1)$ for any $\alpha \in (0, \beta \wedge (\sigma - 1))$ and there is a constant $C > 0$ (depending only on $\sigma_0, \lambda, \Lambda, \eta$ and the dimension $n$, but not on $\sigma$) such that

$$\|u\|_{C^{1,\alpha}(Q_1)} \leq C \left( \|u\|_{C(Q_{1+\eta})} + \|u\|_{L^\infty_t(L^1_x)} + \sup_{Q_{1+\eta}} |f| \right).$$

**Proof.** Without loss of generality, we may assume that $\|u\|_{C(Q_{1+\eta})} \leq 1$ by dividing $u$ by $\|u\|_{C(Q_{1+\eta})} + \|u\|_{L^\infty_t(L^1_x)} + \sup_{Q_{1+\eta}} |f|$.

We apply Theorem 6.3. In this case, $\mathcal{P}^{(0)}$ is given by

$$\mathcal{P}^{(0)}(x, t) = \int_{\mathbb{R}^n} \mu_t(u, x, y) \frac{(2 - \sigma)a_0(x, y, t)}{|y|^{n+\sigma}} \, dy.$$

By the assumption, the operator $\mathcal{P}^{(0)}$ is translation-invariant and belongs to $\mathcal{L}_1$, which is a scale invariant class. By Theorem 3.6, the viscosity solution $u \in L^\infty_t(L^1_x)$ of $\mathcal{P}^{(0)}u - \partial_t u = 0$ in $Q_{1+\eta}$ has interior $C^{1,\beta}(Q_1)$-estimates for some $\beta \in (0,1)$. In this case, it is easy to see that since the equation is linear and the coefficients do not depend on $(x,t)$, the derivatives $Du$ of the solution $u$ of the equation solve the same equation, so that the solutions are actually $C^{1,\beta}_x(Q_1)$. So, moreover, the solutions $u$ of $\mathcal{P}^{(0)}u - \partial_t u = 0$ has interior $C^{1,1}_x(Q_1)$-estimates.

Now we estimate $||\mathcal{P} - \mathcal{P}^{(0)}||_\sigma := \sup_{\lambda \in (0,1)} \|\mathcal{P}_{1,\lambda} - \mathcal{P}^{(0)}_{1,\lambda}\|$. We take any $\lambda \in (0,1)$ and set $w_1 = u(\lambda \cdot, \lambda^* \cdot)$ in $Q_{1+\eta}$. Take any $(y, s) \in Q_{(1+\eta)\lambda}$. Then we compute

$$\langle \mathcal{P}_{1,\lambda} - \mathcal{P}^{(0)}_{1,\lambda} \rangle (y, s) = \int_{\mathbb{R}^n} \mu_s(u, y, z) \frac{(2 - \sigma)(a(y, \lambda z, s) - a_0(y, \lambda z, s))}{|z|^{n+\sigma}} \, dz$$
by applying (7.1). By Definition 3.1, we have that

\[
\|I_{1,\lambda} - I_{1,\lambda}^{(0)}\| = \sup_{(y,s) \in Q_{(1+\eta)}\lambda} \sup_{u \in F_{y,s}^M} \frac{|I_{1,\lambda}u(y,s) - I_{1,\lambda}^{(0)}u(y,s)|}{1 + \|u\|_{L^\infty(L^1)} + \|u\|_{C^{1,1}(Q_1(y,s))}}
\]

(7.2)

\[
\leq \sup_{(y,s) \in Q_{(1+\eta)}\lambda} \sup_{u \in F_{y,s}^M} \int_{\mathbb{R}^n} |\mu_x(u, y, z)| \frac{2 - \sigma}{|z|^{n+\sigma}} \, dz
\]

We take any function \(u \in F_{y,s}^M\), that is, \(\|u\|_{L^\infty(L^1)} + \|u\|_{C^{1,1}(Q_1(y,s))} \leq M\) for \(M > 0\) and \(u \in \mathcal{F} \cap C^2(y, s)\). Since \(y \pm z \in B_1(y)\) if \(\pm z \in B_1\), we obtain that

\[
\int_{B_1} |\mu_x(u, y, z)| \frac{2 - \sigma}{|z|^{n+\sigma}} \, dz \leq \int_{B_1} \|u\|_{C^{1,1}(Q_1(y,s))}|z|^2 \frac{2 - \sigma}{|z|^{n+\sigma}} \, dz \leq C\|u\|_{C^{1,1}(Q_1(y,s))}.
\]

Since \((y, s) \in Q_{(1+\eta)}\lambda\) and \(|y \pm z| \geq |z| - |y| \geq (1 - \lambda - \lambda\eta)|z|\) for \(y \in B_{(1+\eta)}\lambda\) and \(z \in \mathbb{R}^n \setminus B_1\), we have that

\[
\int_{\mathbb{R}^n \setminus B_1} |\mu_x(u, y, z)| \frac{2 - \sigma}{|z|^{n+\sigma}} \, dz
\]

\[
\leq \int_{\mathbb{R}^n \setminus B_1} \left( |u(y + z, s)| + |u(y - z, s)| + 2|u(y, s)| \right) \frac{2 - \sigma}{|z|^{n+\sigma}} \, dz
\]

\[
= \int_{|z| \geq 1 - \lambda - \lambda\eta} |u(z, s)| \left( \frac{2 - \sigma}{|y + z|^{n+\sigma}} + \frac{2 - \sigma}{|y - z|^{n+\sigma}} \right) \, dz + C
\]

\[
\leq C \int_{|z| \geq 1 - \lambda - \lambda\eta} |u(z, s)| \frac{2 - \sigma}{|z|^{n+\sigma}} \, dz + C \leq C(1 + \|u\|_{L^\infty(L^1)}).
\]

By (7.2), we conclude that \(\|I_{1,\lambda} - I_{1,\lambda}^{(0)}\| \leq C\delta\) for any \(\lambda \in (0, 1)\), and thus we have that \(\|I - I^{(0)}\|_\sigma \leq C\delta\). If we choose \(\eta\) small enough, we can apply Theorem 6.3 and conclude that the equation \(Lu - \partial_t u = f\) has interior \(C^{1,\alpha}\)-estimates for any \(\alpha \in (0, \beta \land (\sigma - 1))\). \(\square\)

### 7.2. Nonlinear equations

From Theorem 6.3 and Theorem 3.6, we can easily derive the following result.

**Theorem 7.2.** Let \(\sigma \in [\sigma_0, 2)\) for \(\sigma_0 \in (1, 2)\) and let \(\Sigma_1(\sigma)\) be the class satisfying (6.3). Suppose that \(u \in L^\infty(L^1)\) is a viscosity solution of the equation

\[\Pi^{(0)}u - \partial_t u = 0 \quad \text{in} \quad Q_{1+\eta}\]

and \(\|\Pi - \Pi^{(0)}\|_\sigma < \delta\) for some small \(\delta > 0\), where \(\Pi^{(0)}\) is a translation-invariant non-local operator which is uniformly elliptic with respect to \(\Sigma_1(\sigma)\) and \(\Pi\) is an operator which is uniformly elliptic with respect to \(\Sigma_0\). If \(u \in L^\infty(L^1)\) is a viscosity solution of the equation

\[Lu - \partial_t u = f \quad \text{in} \quad Q_{1+\eta}\]

where \(f \in B(Q_{1+\eta})\), then \(u \in C^{1,\alpha}(Q_1)\) for some small \(\alpha > 0\) and there is a constant \(C > 0\) (depending only on \(\sigma_0, \lambda, \Lambda, \eta, C_1\) and the dimension \(n\), but not on \(\sigma\)) such that

\[\|u\|_{C^{1,\alpha}(Q_1)} \leq C(\|u\|_{C(Q_{1+\eta})} + \|u\|_{L^\infty(L^1)} + \sup_{Q_{1+\eta}} |f|).\]
Remark 7.3. We consider the following operator $I$ given by
\[
Iu := \inf_{\alpha} \sup_{\beta} \int_{\mathbb{R}^n} \mu(u, \cdot, y) \frac{(2-\sigma)(a_0(y, \cdot) + a_{\alpha\beta}(\cdot, y, \cdot))}{|y|^{n+\sigma}} dy
\]
where $a_0$ and $a_{\alpha\beta}$ are functions satisfying
\[
\lambda \leq a_0(x, y, t) \leq \Lambda, \quad |\nabla_y a_0(x, y, t)| \leq \frac{C}{|y|} \text{ for any } y \in \mathbb{R}^n \setminus \{0\} \text{ and } (x, t) \in \mathbb{R}^n,
\]
\[
\sup_{\alpha, \beta} \sup_{(x, t) \in \mathbb{R}^n} |a_{\alpha\beta}(x, y, t)| < \delta \quad \text{for some small } \delta > 0.
\]
Then we see that this is a nonlinear operator which exemplifies Theorem 7.2.

Theorem 7.4. Let $\sigma \in [\sigma_0, 2)$ for $\sigma_0 \in (1, 2)$ and let $Iu$ be given by
\[
Iu = \inf_{\alpha} \sup_{\beta} \int_{\mathbb{R}^n} \mu(u, \cdot, y) \frac{(2-\sigma)a_{\alpha\beta}(\cdot, y, \cdot)}{|y|^{n+\sigma}} dy
\]
where $\lambda < a_{\alpha\beta}(x, y, t) < \Lambda$, $|\nabla_y a_{\alpha\beta}(x, y, t)| \leq C_2/|y|$ and
\[
\sup_{\alpha, \beta} |a_{\alpha\beta}(x_1, y_1, t_1) - a_{\alpha\beta}(x_2, y_2, t_2)| = o(1) \quad \text{as } d((x_1, t_1), (x_2, t_2)) \to 0
\]
with the parabolic distance $d$. If $u \in L^\infty_\infty(L^1_{Q_1})$ is a viscosity solution of the equation
\[
Iu - \partial_t u = f \quad \text{in } Q_{1+\eta}
\]
where $f \in B(Q_{1+\eta})$, then there are a small $\alpha > 0$ and a constant $C > 0$ (depending only on $\sigma_0, \lambda, \Lambda, \eta, C_2$, the modulus of continuity $\varrho$ and the dimension $n$, but not on $\sigma$) such that
\[
\|u\|_{C^{1,\alpha}(Q_1)} \leq C\left(\|u\|_{C(Q_{1+\eta})} + \|u\|_{L^\infty_\infty(L^1_{Q_1})} + \sup_{Q_{1+\eta}} |f|\right).
\]

Proof. For each $(x_0, t_0) \in Q_1$, we can find a ball $Q_r(x_0, t_0) \subset Q_{1+\eta}$ ($r > 0$ is independent of $x_0$ and $t_0$) so that
\[
\sup_{(x, t) \in Q_r(x_0, t_0)} |a_{\alpha\beta}(x, y, t) - a_{\alpha\beta}(x_0, y, t_0)| < \delta
\]
for some small $\delta > 0$. This implies that $\|I - I(x_0, t_0)\|_\sigma < C\delta$ on $Q_r(x_0, t_0)$ as in the proof of Theorem 7.2, where
\[
I(x_0, t_0)u(x, t) := \inf_{\alpha} \sup_{\beta} \int_{\mathbb{R}^n} \mu(u, x, y) \frac{(2-\sigma)a_{\alpha\beta}(x_0, y, t_0)}{|y|^{n+\sigma}} dy
\]
\[
= \bar{I}(x_0, t_0)u(x_0, t_0).
\]
We now apply Theorem 7.2 with $I^{(0)} = I(x_0, t_0)$ scaled in $Q_r(x_0, t_0)$. Let $N$ be the minimal number of such open balls $Q_r(x_0, t_0)$ covering $Q_{1+\epsilon}$. Then we have that
\[
\|u\|_{C^{1,\alpha}(Q_{1+\epsilon})} \leq CN\left(\|u\|_{C(Q_{1+\eta})} + \|u\|_{L^\infty_\infty(L^1_{Q_1})} + \sup_{Q_{1+\eta}} |f|\right).
\]
Hence we complete the proof. □

7.3. Nonlinear equations with non-differentiable kernels
We note that Theorem 6.3 makes it possible to obtain certain results even in the translation-invariant case. It was crucial in Theorem 3.6 that every kernel must be differentiable away from the origin. This condition can be weakened in the following
way. We establish $C^{1,\alpha}$-estimates for nonlocal equations that are uniformly elliptic with respect to the class $\mathcal{L}$ consisting of operators with kernels $K \in \mathcal{K}$ given by

$$K(x, y, t) = (2 - \sigma) \frac{a_1(x, y, t) + a_2(x, y, t)}{|y|^{n+\sigma}}$$

where $\lambda \leq a_1 \leq \Lambda$.

$$\sup_{y \in \mathbb{R}^n} \sup_{(x, t) \in \mathbb{R}^{n+1}} |a_2(x, y, t)| < \delta,$$  

$$\sup_{(x, t) \in \mathbb{R}^{n+1}} |\nabla_y a_1(x, y, t)| \leq \frac{C_2}{|y|} \text{ for any } y \in \mathbb{R}^n \setminus \{0\}.$$  

**Theorem 7.5.** Let $\sigma \in [\sigma_0, 2)$ for $\sigma_0 \in (1, 2)$ and let $\delta > 0$ be a small enough number (depending only on $\lambda, \Lambda, C_2$ and the dimension $n$, but not on $\sigma$) as in the above. If $u \in L^\infty_{\mathcal{L}}(L^1_{\mathcal{L}})$ solves the nonlocal equation

$$\mathbb{I}u - \partial_t u := \inf_\alpha \sup_\beta \int_{\mathbb{R}^n} \mu(u, \cdot, y)K_{\alpha\beta}(y, \cdot)dy - \partial_t u = f \text{ in } Q_{1+\eta}$$

for $f \in B(Q_{1+\eta})$ and $\{K_{\alpha\beta}\} \subset \mathcal{K}$, then there are some $\alpha > 0$ and $C > 0$ (depending only on $\sigma_0, \lambda, \Lambda, \eta$ and the dimension $n$ but not on $\sigma$) such that

$$\|u\|_{C^{1,\alpha}(Q_1)} \leq C\left(\|u\|_{C(Q_{1+\eta})} + \|u\|_{L^\infty_{\mathcal{L}}(L^1_{\mathcal{L}})} + \sup_{Q_{1+\eta}} |f|\right).$$

**Proof.** Let $L \in \mathcal{L}$ be an operator with kernel $K$. We write $K = K_1 + K_2$ where $K_1 = (2 - \sigma)c_1(x, y, t)/|y|^{n+\sigma}$ and set $L = L^1 + L^2$ where $L^1$ and $L^2$ are operators with kernels $K_1$ and $K_2$, respectively. Then we see that $\|L - L^2\|_\sigma < c\delta$. If we set $\mathbb{I}^{(0)}u = \inf_\alpha \sup_\beta L^1_{\alpha\beta}u$, then we have that $\|\mathbb{I} - \mathbb{I}^{(0)}\|_\sigma < c\delta$, and hence we can apply Theorem 6.3 to complete the proof. \[\square\]

**Remark.** This theorem works for a class which is still much smaller than $\mathcal{L}_0$. It would be very interesting to determine whether the class $\mathcal{L}_0$ has interior $C^{1,\alpha}$-estimates or not. This problem is still left open even for elliptic cases as mentioned in [CS]. Also it would be interesting to answer this problem on the parabolic case.

### 7.4. Nonlinear equations near the fractional Laplacian

We obtain another result in translation-invariant case by applying Theorem 6.3. In fact, we obtain $C^{2,\alpha}$-estimates for nonlinear translation-invariant nonlocal parabolic equations which are sufficiently close to the parabolic fractional Laplacian and their ellipticity constants are sufficiently close to each other. This is to improve Theorem 3.6 under these conditions.

**Theorem 7.6.** Let $\sigma \in [\sigma_0, 2)$ for $\sigma_0 \in (1, 2)$. Then there are some $\delta > 0$ and $\rho_0 > 0$ so that if $-\delta < \lambda - 1 < \Lambda - 1 < \delta$, $\mathbb{I}$ is a nonlocal translation-invariant uniformly elliptic operator with respect to $\mathcal{L}_\sigma$ and $u \in C^{0,1}((-1, 0]; L^1_{\mathcal{L}})$ satisfies $\mathbb{I}u - \partial_t u = 0 \text{ in } Q_{1+\eta}$, then $u \in C^{2,\alpha}(Q_1)$ for a constant $\alpha > 0$ (depending only on $n$ and $\sigma_0$) and there is a universal constant $C > 0$ (depending only on $\sigma_0, n, \eta$, and the constant in (2.5)) such that

$$\|u\|_{C^{2,\alpha}(Q_1)} \leq C\left(\|\partial_t u\|_{C(Q_{1+\eta})} + \|u\|_{L^{\infty}_{\mathcal{L}}(L^1_{\mathcal{L}})} + \|u\|_{C^{0,1}((-1, 0]; L^1_{\mathcal{L}})} + \|f\|\right)$$

where we denote by $\|f\|$ the value we obtain when we apply $\mathbb{I}$ to the constant function that is equal to zero.

**Proof.** From Theorem 3.6, we see that $u \in C^{1,\alpha}(Q_1)$. Thus the function $u$ is differentiable in $x$ on $Q_1$. Let $w = e \cdot \nabla u$ be a directional derivative for $e \in S^{n-1}$.
We write \( w = w_1 + w_2 \) where \( w_1 = w \|_{Q_{1+\eta}} \). Then by using the uniform ellipticity with respect to \( \mathfrak{L}_\ast \) we easily see that \( w_1 \) solves

\[
M^+_\mathfrak{L}_\ast w_1 - \partial_t w_1 \geq -\|u\|_{L^\infty_\ast(L^1_\ast)^*} - \|0\|, \quad M^-\mathfrak{L}_\ast w_1 - \partial_t w_1 \leq \|u\|_{L^\infty_\ast(L^1_\ast)^*} + \|0\| \text{ in } Q_{1+2\eta/3}.
\]

We now apply Theorem 6.3 instead of Theorem 3.4. Since \( 1 - \delta < \lambda < 1 + \delta \), as in (7.2) we easily obtain that

\[
\|M^+_\mathfrak{L}_\ast + (-\Delta)^{\sigma/2}\|_\sigma < c\delta \quad \text{and} \quad \|M^-\mathfrak{L}_\ast + (-\Delta)^{\sigma/2}\|_\sigma < c\delta.
\]

Thus Theorem 6.3 tells us that \( w = e \cdot \nabla u \) is in \( C^{1,\alpha}(Q_1) \). From (3.6) and the local equivalence between \( W^{1,\infty} \) and Lipschitz continuity, we see that \( \sup_{Q_{1+\eta}} |\nabla u| \leq \|u\|_{C^{0,\alpha}(Q_{1+\eta})} \leq C \|u\|_{L^\infty_\ast(L^1_\ast)} \). Also by (3.7) and integration by parts, we have that \( \|\nabla u\|_{L^\infty_\ast(L^1_\ast)^*} \leq C \|u\|_{L^\infty_\ast(L^1_\ast)} \). Moreover, if we take \( c = \nabla u/|\nabla u| \) in the above, then we have that

\[
\|\nabla u\|_{C^{1,\alpha}(Q_{1+\eta})} \leq \|\nabla u\|_{C^{1,\alpha}(Q_1)} \leq C \left( \|\nabla u\|_{C(Q_{1+\eta})} + \|\nabla u\|_{L^\infty_\ast(L^1_\ast)} + \|0\| \right) \leq C \left( \|u\|_{L^\infty_\ast(L^1_\ast)} + \|0\| \right).
\]

This implies that \( \|u\|_{C^{2,\alpha}(Q_1)} \leq C \left( \|u\|_{L^\infty_\ast(L^1_\ast)} + \|0\| \right) \).

As in the proof of Theorem 3.6, we proceed \( C^{2,\alpha}(Q_1) \) regularity of \( u \) in the \( t \)-direction; that is, from Remark 2.1, we have only to obtain \( C^{1,\frac{2+\alpha-\sigma}{\sigma}}(Q_1) \) regularity of \( u \). Since \( (2 + \alpha)/\sigma > 1 \) for such \( \alpha > 0 \), we see that

\[
\frac{2 + \alpha - \sigma}{\sigma} + 1 = \frac{2 + \alpha}{\sigma}
\]

and \( 0 < \alpha < 2 + \alpha - \sigma < 1 \). By Theorem 3.4 and the standard telescopic sum argument argument \([\mathbb{C}, \mathbb{C}]\), we can obtain that \( u \) is \( C^{2,\frac{2+\alpha-\sigma}{\sigma}} \) Hölder continuous in the \( t \)-direction. We consider the difference quotients in the \( t \)-direction

\[
w^\tau(x,t) = \frac{u(x,t+\tau) - u(x,t)}{\tau}.
\]

Let us write \( w^\tau = w^\tau_1 + w^\tau_2 \) where \( w^\tau_1 = w^\tau \|_{Q_{1+\eta}} \). By Theorem 2.4 \([\mathbb{K}, \mathbb{L}]\), we have that

\[
M^+_\mathfrak{L}_\ast w^\tau - \partial_t w^\tau \geq 0 \quad \text{and} \quad M^-\mathfrak{L}_\ast w^\tau - \partial_t w^\tau \leq 0 \quad \text{in } Q_1.
\]

Since \( \partial_t w^\tau \equiv 0 \) in \( Q_1 \), the uniform ellipticity of \( M^+_\mathfrak{L}_\ast \) and \( M^-\mathfrak{L}_\ast \) yield

\[
M^+_\mathfrak{L}_\ast w^\tau_1 - \partial_t w^\tau_1 \geq -M^+_\mathfrak{L}_\ast w^\tau_2 \quad \text{and} \quad M^-\mathfrak{L}_\ast w^\tau_1 - \partial_t w^\tau_1 \leq -M^-\mathfrak{L}_\ast w^\tau_2 \quad \text{in } Q_1.
\]

Then it is easy to check that \( \|M^+_\mathfrak{L}_\ast w^\tau_2 \| \vee \|M^-\mathfrak{L}_\ast w^\tau_2 \| \leq c \|u\|_{C^{0,1}((-1,0);L^1_\ast)} \) in \( Q_1 \) for a universal constant \( c > 0 \). Thus we have that

\[
M^+_\mathfrak{L}_\ast w^\tau_1 - \partial_t w^\tau_1 \geq -c\|u\|_{C^{0,1}((-1,0);L^1_\ast)} \quad \text{and} \quad M^-\mathfrak{L}_\ast w^\tau_1 - \partial_t w^\tau_1 \leq -c\|u\|_{C^{0,1}((-1,0);L^1_\ast)} \quad \text{in } Q_1.
\]

So we can now apply \( C^{\frac{2+\alpha-\sigma}{\sigma}} \) Hölder regularity(Theorem 3.4) to \( w^\tau \).

By the mean value theorem, we obtain the estimate

\[
\|w^\tau\|_{C^{\frac{2+\alpha-\sigma}{\sigma}}(Q_1)} \leq C \left( \|\partial_t u\|_{C(Q_{1+\eta})} + \|u\|_{C^{0,1}((-1,0);L^1_\ast)} \right)
\]

Taking the limit \( |\tau| \to 0 \) in the above, we have that

\[
\|u\|_{C^{\frac{2+\alpha-\sigma}{\sigma}}(Q_1)} \leq C \left( \|\partial_t u\|_{C(Q_{1+\eta})} + \|u\|_{C^{0,1}((-1,0);L^1_\ast)} \right).
\]
Therefore we can conclude that $u \in C^{2,\alpha}(Q_1)$ by obtaining the required estimate.

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