ON A QUESTION OF MOSHE ROITMAN AND EULER CLASS OF STABLY FREE MODULE

MANOJ K. KESHARI AND SOUMI TIKADER

2020 Mathematics Subject Classification: 13C10, 13B25, 19A13

Keywords: Projective modules, affine algebra, unimodular elements.

Abstract. Let $A$ be a ring of dimension $d$ containing an infinite field $k$, $T_1, \ldots, T_r$ be variables over $A$ and $P$ be a projective $A[T_1, \ldots, T_r]$-module of rank $n$. Assume one of the following conditions hold.

1. $2n \geq d + 3$ and $P$ is extended from $A$.
2. $2n \geq d + 2$, $A$ is an affine $P_p$-algebra and $P$ is extended from $A$.
3. $2n \geq d + 3$ and singular locus of $Spec(A)$ is a closed set $V(J)$ with $ht J \geq d - n + 2$.

Assume $Um(P_f) \neq \emptyset$ for some monic polynomial $f(T_r) \in A[T_1, \ldots, T_r]$. Then $Um(P) \neq \emptyset$ (see 6.1).

1. Introduction

All rings are commutative noetherian with unity and all projective modules are finitely generated of constant rank.

Let $A$ be a ring of dimension $d$ and $P$ be a projective $A[T]$-module of rank $n$. We say that $p \in P$ is a unimodular element if there exist $\phi \in Hom(P, A)$ such that $\phi(p) = 1$. We write $Um(P)$ for the set of unimodular elements of $P$. When $n > d$, then Plumstead [17] proved that $Um(P) \neq \emptyset$. Further, there are well known examples in the case $n \leq d$ with $Um(P) = \emptyset$. For example, let $d = 2m$, $R = \mathbb{R}[X_0, \ldots, X_d]/(X_0^2 + \ldots + X_d^2 - 1)$, $\phi : R^{d+1} \rightarrow R$ defined by $e_i \mapsto X_i$ and $Q = ker(\phi)$. Then $Q$ is stably free $R$-module of rank $d$ and $Um(Q) = \emptyset$. Hence if $P = Q[T]$, then $Um(P) = \emptyset$. Thus we need some further conditions in the case $n \leq d$ to get $Um(P) \neq \emptyset$.

Let $f \in A[T]$ be a monic polynomial and $P_f = P \otimes A[T]/f$. Roitman [20] Lemma 10] proved that if $A$ is a local ring, then $Um(P_f) \neq \emptyset$ implies $Um(P) \neq \emptyset$. He asked whether this result holds for arbitrary ring $A$.

Question 1.1 (Roitman’s question). Let $A$ be a ring of dimension $d$ and $P$ be a projective $A[T]$-module of rank $n \leq d$. Let $f \in A[T]$ be a monic polynomial such that $Um(P_f) \neq \emptyset$. Does this imply $Um(P) \neq \emptyset$?

If $A$ contains an infinite field $k$, then an affirmative answer is given in the following cases.

1. Bhatwadekar [2] Proposition 3.3] for $n = 2$, arbitrary $d$ and $A$ need not contain any field.
2. Bhatwadekar-Sridharan [4] Theorem 3.4] for $n = d$.
3. Bhatwadekar-Keshari [3] Theorem 5.3] for $2n \geq d + 3$ when $P$ is extended from $A$.
4. Bhatwadekar-Keshari [5] Corollary 5.4] for $2n \geq d + 3$ when $A$ is regular.

We will follow the proof of Bhatwadekar-Keshari [3] Theorem 5.3] with suitable modification and prove the following generalisation of (4) (see 5.1).
Theorem 1.2. Let \( A \) be a ring of dimension \( d \) containing an infinite field \( k \) and \( P \) be a projective \( A[T] \)-module of rank \( n \) with \( 2n \geq d + 3 \). Assume the singular locus of \( \text{Spec}(A) \) is a closed set \( V(J) \) with \( \text{ht}(J) \geq d - n + 2 \). Let \( f \in A[T] \) be a monic polynomial such that \( Um(P_f) \neq \emptyset \). Then \( Um(P) \neq \emptyset \). 

If \( A \) is an affine algebra over an infinite field \( k \), then singular locus of \( \text{Spec}(A) \) is a closed set. In particular, if \( A \) is an affine \( k \)-algebra having an isolated singularity, then \( \text{ht} J = d \) and (1.2) is applicable.

The following result is proved in Mandal-Murthy [13, Theorem 3.2] when \( L = I^2 \). We will essentially follow their proof (see 4.1).

Theorem 1.3. Let \( A \) be an affine \( \mathbb{F}_p \)-algebra of dimension \( d \geq 2 \) and \( I, L \) be ideals of \( A \) with \( L \subset I^2 \). Let \( P \) be a projective \( A \)-module of rank \( d \) and \( \phi : P \rightarrow I/L \) be a surjection. Then \( \phi \) can be lifted to a surjection \( \Phi : P \rightarrow I \).

When \( A \) is an affine algebra over \( \mathbb{F}_p \), then using (1.3), we can improve Bhatwadekar-Keshari [3, Theorem 5.3] as follows (see 4.1).

Theorem 1.4. Let \( A \) be an affine \( \mathbb{F}_p \)-algebra of dimension \( d \) and \( P \) be a projective \( A[T] \)-module of rank \( n \) which is extended from \( A \) with \( 2n \geq d + 2 \). Let \( f \in A[T] \) be a monic polynomial such that \( Um(P_f) \neq \emptyset \). Then \( Um(P) \neq \emptyset \).

If \( A \) is a smooth affine \( \mathbb{F}_p \)-algebra of dimension \( d \) and \( P \) is a projective \( A[T] \)-module of rank \( n \), then \( P \) is extended from \( A \) by Popescu [13]. Thus if \( 2n \geq d + 2 \) and \( Um(P_f) \neq \emptyset \) for some monic polynomial \( f \in A[T] \), then \( Um(P) \neq \emptyset \) by (1.4).

Using above results, we derive the results stated in the abstract (see 6.1) and the following results (see 8.1, 8.2, 9.3, 9.4). When \( A \) is regular, (1) is due to Das-Sridharan [8, Theorem 3.9], (2) is due to Bhatwadekar-Keshari [3, Theorem 4.13], (3) and (4) are due to Bhatwadekar-Sridharan [6, Theorem 5.4] when \( r = 1 \), \( P \oplus R = R^{n+1} \) and \( 2n \geq d + 4 \).

Theorem 1.5. Let \( R \) be a ring of dimension \( d \) containing a field \( k \) and \( n \) be an integer with \( 2n \geq d + 3 \). Assume the singular locus of \( \text{Spec}(R) \) is a closed set \( V(J) \) with \( \text{ht} J \geq d - n + 2 \). Then following holds.

1. Assume \( k \) is infinite. The following sequence of Euler class groups is exact, where \( m \) runs over all the maximal ideals of \( R \).

\[
0 \rightarrow E^n(R) \rightarrow E^n(R[T]) \rightarrow \prod_mE^n(R_m[T]).
\]

2. Assume \( k \) is infinite. Let \( P \) be a projective \( R \)-module of rank \( n \) and \( I \) be an ideal of \( R[T] \) of height \( n \). Let \( \phi : P[T] \rightarrow I/I^2T \) be a surjection. Then \( \phi \) can be lifted to a surjection \( \Phi : P[T] \rightarrow I \) if and only if \( \phi \otimes R_m[T] \) can be lifted to a surjection \( P_m[T] \rightarrow IR_m[T] \) for all maximal ideals \( m \) of \( A \).

3. There exist a well defined Euler class map \( e : Um_{r,n+r}(R[T]) \rightarrow E^n(R[T]) \).

4. Assume \( k \) is infinite. Let \( P \) be a stably free \( R[T] \)-module of rank \( n \). Then \( Um(P) \neq \emptyset \) if and only if \( e(P) = 0 \in E^n(R[T]) \).
2. Preliminaries

The following result is due to Keshari-Zinna [11 Proposition 1.1].

**Proposition 2.1.** Let $R$ be a ring and $P$ be a projective $R[T]$-module. Let $J \subset R$ be an ideal such that $P_s$ is extended from $R_s$ for every $s \in J$. Suppose that

1. $P/JP$ contains a unimodular element.
2. If $I$ is an ideal of $(R/J)[T]$ of height $\text{rank}(P) - 1$, then there exist $\sigma \in \text{Aut}((R/J)[T])$ with $\sigma(T) = T$ and a lift $\sigma' \in \text{Aut}(R[T])$ of $\sigma$ with $\sigma(T) = T$ such that $\sigma'(I)$ contains a monic polynomial in $T$.
3. $\text{EL}(P/(T, J)P)$ acts transitively on $Um(P/(T, J)P)$.
4. There exists a monic polynomial $f \in R[T]$ such that $Um(P_f) \neq \emptyset$.

Then the natural map $Um(P) \rightarrow Um(P/TP)$ is surjective. In particular, $Um(P/TP) \neq \emptyset$ implies $Um(P) \neq \emptyset$.

**Definition 2.2.** Let $A$ be a ring and $P$ be a projective $A$-module. Then $\sigma \in \text{Aut}(P)$ is called isotopic to identity if there exist $\Phi(W) \in \text{Aut}(P[W])$ such that $\Phi(0) = \text{Id}$ and $\Phi(1) = \sigma$. Here $P[W] = P \otimes_A A[W]$.

The proof of the following result is contained in Quillen [19, Theorem 1].

**Lemma 2.3.** Let $B$ be a ring and $P$ be a projective $B$-module. Let $a, b \in B$ generate the ideal $B$ and $\sigma \in \text{Aut}_{B_{ab}}(P_{ab})$ which is isotopic to identity. Then $\sigma = \tau_a \circ \theta_b$ where $\tau \in \text{Aut}_{B_{ab}}(P_{ab})$ with $\tau = \text{Id}$ modulo $Ba$ and $\theta \in \text{Aut}_{B_{ab}}(P_{ab})$ with $\theta = \text{Id}$ modulo $Bb$.

Next two results are due to Bhatwadekar-Sridharan [4, Lemma 3.1, Lemma 3.2], respectively.

**Lemma 2.4.** Let $A$ be a ring containing an infinite field $k$ and $\tilde{P}$ be a projective $A[T]$-module of rank $n$. If $Um(\tilde{P}) \neq \emptyset$ for some monic polynomial $f \in A[T]$, then there exists a surjection $\alpha : \tilde{P} \rightarrow I$ where $I$ is an ideal of $A[T]$ of height $\geq n$ containing a monic polynomial.

**Lemma 2.5.** Let $R$ be a ring and $Q$ be a projective $R$-module. Let $(\alpha(T), f(T)) : Q[T] \oplus R[T] \rightarrow R[T]$ be a surjection with $f(T)$ monic. Let $pr_2 : Q[T] \oplus R[T] \rightarrow R[T]$ be the second projection. Then there exists $\sigma(T) \in \text{Aut}(Q[T] \oplus R[T])$ which is isotopic to identity and $pr_2 \circ \sigma(T) = (\alpha(T), f(T))$.

Next result is due to Bhatwadekar-Keshari [3, Lemma 4.5].

**Lemma 2.6.** Let $A$ be a ring with $\text{dim } A/J(A) = r$, where $J(A)$ is the Jacobson radical of $A$. Let $I$ and $L$ be ideals of $A[T]$ such that $L \subset I^2$ and $L$ contains a monic polynomial. Let $P'$ be a projective $A[T]$-module of rank $\geq r + 1$. Then any surjection $\phi : P' \oplus A[T] \rightarrow I/L$ can be lifted to a surjection $\Phi : P' \oplus A[T] \rightarrow I$ with $\Phi(0, 1)$ a monic polynomial.

The following result is due to Mandal-Murthy [13, Lemma 3.1].

**Lemma 2.7.** Let $A$ be an affine $\mathbb{F}_p$-algebra of dimension $\geq 1$ and $J$ be a complete intersection ideal of height $d$. If $J = (a_1, \ldots, a_{d+1})$, then there exist $(\lambda_1, \ldots, \lambda_{d+1}) \in Um_{d+1}(A)$ such that $\sum_{i=1}^{d+1} \lambda_i a_i = 0$. 


3. Roitman’s Question : Proof of Theorem 1.2

Theorem 3.1. Let $A$ be a ring of dimension $d$ containing an infinite field $k$ and $P$ be a projective $A[T]$-module of rank $n$ with $2n \geq d + 3$. Assume that the singular locus of Spec($A$) is a closed set $V(f)$ with $ht(f) \geq d - n + 2$. Let $f \in A[T]$ be a monic polynomial such that $Um(P_f) \neq \emptyset$. Then $Um(P) \neq \emptyset$.

Proof. Since $Um(P_f) \neq \emptyset$ and $A$ contains an infinite field $k$, by (2.4), we get a surjection $\Phi : P \rightarrow I$, where $I$ is an ideal of $A[T]$ of height $\geq n$ containing a monic polynomial. If $ht I > n$, then $I = A[T]$. Hence $Um(P) \neq \emptyset$ and we are done. So we assume $ht I = n$. The map $\Phi$ induces two surjections $\phi : P \rightarrow I/(I^2T)$ and $\overline{\Phi} : P/TP \rightarrow I(0)$. Let $J = (I \cap A) \cap J$ and write $R = A_{1+J}$ and $\overline{P} = P_{1+J}$.

Step 1. We will show that $Um(\overline{P}) \neq \emptyset$ by verifying the conditions of (2.1) for the ideal $JR$ of $R$.

1. For any $s \in J$, $A_s$ is a regular ring containing an infinite field $k$. Thus by Popescu [13], $\overline{P}_s$ is extended from $R_s$.
2. Since $n - 1 \geq d - n + 2$, height of $J$ is $\geq d - n + 2$, so $dim(R/J) \leq d - (d - n + 2) = n - 2$.
   By Plumstead [17], $\overline{P}/J\overline{P}$ has a unimodular element.
3. Since $dim(R/J) \leq n - 2$, any ideal of $(R/J)[T]$ of height $\geq n - 1$ contains a monic polynomial in $T$.
4. $EL(\overline{P}/(T, J)\overline{P})$ acts transitively on $Um(\overline{P}/(T, J)\overline{P})$, by Bass cancellation theorem [1].
5. By hypothesis, there exists a monic polynomial $f \in R[T]$ such that $Um(\overline{P}) \neq \emptyset$.

By (2.4), $Um(\overline{P}) \rightarrow Um(\overline{P}/TP)$ is surjective. Since $\overline{P}/TP$ is a projective $R$-module of rank $n$ and $J$ is contained in the Jacobson radical of $R$ with $dim(R/J) \leq n - 2$, we get $Um(\overline{P}/TP) \neq \emptyset$. Therefore, $Um(\overline{P} = P_{1+J}) \neq \emptyset$.

Step 2. Write $P_{1+J} = Q \oplus A_{1+J}[T]$. By (2.6), the surjection $\phi \otimes A_{1+J}[T] : Q \oplus A_{1+J}[T] \rightarrow I_{1+J}/(I^2T)_{1+J}$ can be lifted to a surjection $\Psi = (\psi, h(T)) : Q \oplus A_{1+J}[T] \rightarrow I_{1+J}$ such that $h(T)$ is a monic polynomial in $A_{1+J}[T]$. The surjection $\Psi$ induces the surjection $\overline{\Psi} : Q/TP \oplus A_{1+J} \rightarrow I(0)_{1+J}$. Hence $\overline{\Psi} \otimes A_{1+J} = \overline{\Phi}$. We can find $a \in J$ such that if $b = 1 + a$, then there exist a projective $A_b[T]$-module $Q_1$ with $h(T) \in A_{b}[T]$ monic such that

1. $Q_1 \otimes A_{1+J} = Q$,
2. $P_b = Q_1 \oplus A_b[T]$,
3. $\Psi = (\psi, h(T)) : P_b \rightarrow I_b$ is a surjection,
4. $\overline{\Psi} \otimes A_b = \Psi \otimes A_b[T]/(T)$.

Let $pr_2 : Q_1 \rightarrow A_b[T] \rightarrow A_0[T]$ be the second projection. Since $a \in J$, $I(0)_a = A_a$ and hence $\overline{\Psi}_a : P_a/TP_a \rightarrow A_a$ is a surjection. Further, since $A_a$ is a regular ring containing a field $k$, by Popescu [13], $(Q_1)_a$ is extended from $A_{ab}$. Thus there exist a projective $A_{ab}$ module $Q_2$ such that $(Q_1)_a = Q_2[T]$. Thus $P_{ab} = Q_2[T] \oplus A_{ab}[T]$.

Consider two unimodular elements $\Psi_a = (\psi, h(T))_a$ and $(pr_2)_a$ of $P_{ab}^* = (Q_2[T] \oplus A_{ab}[T])^*$. Since $h(T)$ is a monic polynomial in $A_{ab}[T]$, by (2.5), $\Psi_a = (\psi, h(T))_a$ and $(pr_2)_a$ are isotopically connected. Since $A_a$ is regular, by Popescu [13], kernel of $\Psi_a$ is an extended projective module. Therefore, there exist $\Theta \in Aut(Q_2[T] \oplus A_{ab}[T])$ such that $\Theta(0) = Id$ and $\Psi_a \circ \Theta = \Psi_a(0) \otimes A_{ab}[T] = \overline{\Psi}_{ab} \otimes A_{ab}[T]$. Thus
ψ_a and \(\overline{\Theta}_{ab} \otimes A_{ab}[T]\) are isotopically connected. Therefore, there exist \(\Gamma \in Aut(Q_2[T] \oplus A_{ab}[T])\) such that \(\Gamma\) is isotopic to identity and \((\overline{\Theta}_{ab} \otimes A_{ab}[T]) \circ \Gamma = (pr_2)_a\).

By (2.2), \(\Gamma = \Omega_0 \otimes \Omega_a\) where \(\Omega \in Aut(P_a)\) and \(\Omega' \in Aut(P_a)\). Hence, we have surjections \(\Delta_1 = pr_2 \circ \Omega^{-1} : P_b \to A_{ab}[T]\) and \(\Delta_2 : P_a \to A_{ab}[T]\) such that \((\Delta_1)_a = (\Delta_2)_a\). Therefore, they patch up to yield a surjection \(\Delta : P \to A[T]\). This proves the result. \(\square\)

4. Generalization of Mandal-Murthy : Proof of Theorem 1.3

**Theorem 4.1.** Let \(A\) be an affine \(\mathbb{F}_p\)-algebra of dimension \(d \geq 2\) and \(I, L\) be ideals of \(A\) with \(L \subset I^2\). Let \(P\) be a projective \(A\)-module of rank \(d\) and \(\phi : P \to I/L\) be a surjection. Then \(\phi\) can be lifted to a surjection \(\Phi : P \to I\).

**Proof.** If \(R\) is a ring and \(Q\) is a projective \(R\)-module, then \(Um(Q) \neq \emptyset\) if and only if \(Um(Q_{red}) \neq \emptyset\), where \(Q_{red} = Q \otimes A_{red}\) and \(A_{red} = A/mil(A)\) is the reduced ring. Therefore, it is enough to assume that the ring \(A\) is reduced. We will give the proof in steps.

**Step 1.** If \(\phi' : P \to I\) is a lift of \(\phi\), then \(\phi'(P) + L = I\). There exist \(c \in L\) such that \(c(1-c) \in \phi'(P)\) and \((\phi'(P), c) = I\). By Swan Bertini’s theorem [25, Theorem 1.3, 1.4], there exist \(\psi \in P^*\) such that \(ht(\phi' + c\psi)(P)_c = d\) and \((A/(\phi' + c\psi))(P)_c\) is reduced. Replacing \(\phi'\) with \(\phi' + c\psi\), we may assume that \(ht(\phi'(P), c) = d\) and \((A/\phi'(P))(P)_c\) is reduced. If \(J = (\phi'(P), 1-c)\), then \(\phi'(P) = I \cap J = IJ\). Since \(J_c = \phi'(P)_c\) and \((J, c) = A\), thus we get \(htJ = htJ_c = d\) and \(A/J\) is reduced. Hence \(J\) is a product of distinct maximal ideals, say \(J = \prod_i m_i\), where \(m_i\)'s are maximal ideals of \(A\).

The surjection \(\phi' : P \to IJ\) induces a surjection \(\overline{\phi'} : P/JP \to J/J^2\). If \(p \supset J\) is a prime ideal, then \(\overline{\phi'}\) lifts to a surjection \(P_p \to J_p\). Since \(A\) is a reduced ring, using prime avoidance lemma, we may assume that \(J_p\) is generated by a regular sequence of \(d\) elements. Consequently, \(J\) is a local complete intersection ideal. Hence \(J\) is a product of distinct smooth maximal ideals \(m_i\). If \(J\) is the ideal defining the singular locus of \(A\), then \(J + \mathcal{J} = A\).

By Mohan Kumar-Murthy-Roy [15, Theorem 3.6] for \(d \geq 3\) and Murthy [16, Corollary 3.4] for \(d = 2\), \(J\) is a complete intersection ideal. By Mohan Kumar-Murthy-Roy [15, Theorem 3.7] for \(d \geq 3\) and Krishna-Srinivas [10] for \(d = 2\), \(P\) has a unimodular element, i.e. \(P \simeq P' \oplus Ap_d\) with rank \(P' = d-1\). As \(\phi'\) induces an isomorphism of \(P/JP \simeq J/J^2\), we have \(\phi'(p_1), \ldots, \phi'(p_{d-1}), \phi'(p_d)\) is a base for \(J\) modulo \(J^2\) for some \(p_1, \ldots, p_d \in P'\).

Let \(\mathcal{O}\) denotes the image of the map \(p_1 \wedge p_2 \ldots \wedge p_d : \wedge^n P^* \to A\). Then \(\mathcal{O}/(J \cap \mathcal{O}) \simeq A/J\). Thus \(J + \mathcal{O} = A\). Since \(J\) is comaximal with \(L, \mathcal{O}\) and \(\mathcal{J}\), if \(\mathcal{T} = LOJ\), then \(J + \mathcal{T} = A\).

**Step 2.** We will prove the followings.

1. There exist \(h \in J\) with \(h - 1 \in \mathcal{T}\) such that \(A/Ah\) is a smooth of dimension \(d - 1\).
2. If \(\mathcal{T}\) denote going modulo the ideal \((h)\), then \(\overline{\mathcal{T}}\) is a complete intersection ideal of height \(d - 1\).
3. \(\overline{\mathcal{T}}\) is free with basis \(\overline{p_1}, \ldots, \overline{p_d}\).

**Case 1.** \(d \geq 3\). Since \(J + \mathcal{T} = A\), we get \(J^2 + \mathcal{T}J = J\). Since \(\phi'(p_1), \ldots, \phi'(p_d)\) is a basis for \(J\) modulo \(J^2\), then there exist \(g_1, \ldots, g_d \in \mathcal{T}J\) which forms a basis for \(J\) modulo \(J^2\) and \(\phi'(p_i) - g_i \in J^2\). Let \(c \in J^2\) be such that \(c - 1 \in \mathcal{T}\). If \(l_d = g_d + c \in J\), then \(l_d - 1 \in \mathcal{T}\). Applying Swan’s Bertini [25],
there exist \( a \in J^2T \) such that if \( h = l_d + a \), then \( A/Ah \) is smooth outside the singular locus of \( A \) and \( \dim A/Ah = d - 1 \). Note \( h - 1 = l_d - 1 + a \in T \subset L \subset I \), hence \( h \) is comaximal with the ideal \( J \) defining the singular locus of \( A \). Therefore \( A/Ah \) is smooth. This proves (1).

Since \( J \) is a product of distinct smooth maximal ideals and \( \overline{A} \) is smooth, we get \( \overline{J} \) is also product of distinct smooth maximal ideal. Therefore, by Mohan Kumar-Murthy-Roy [15] Theorem 3.6], \( \overline{J} \) is a complete intersection of height \( d - 1 \geq 2 \). Moreover, \( \phi'(P) = I \cap J \) and \( h - 1 \in I \) implies \( \phi'(\overline{P}) = \overline{J} \). This proves (2).

The construction of \( h \) gives a surjection \( f : P' \oplus Apd \rightarrow J/J^2 \) defined by \( f = \phi' \) on \( P' \) and \( f(pd) = h \). Since \( \phi'(pd) \equiv h \mod J^2 \), \( f \) induces a surjection \( \overline{f} : \overline{P'} \rightarrow \overline{J/J^2} \), where \( \overline{f} \) denote modulo \( h \).

Since \( P' = J/J^2 \), we get \( \overline{J} \) is a smooth complete intersection curve. Let \( \overline{\text{tilde}} \) denote going modulo \( (\overline{g}_1, \ldots, \overline{g}_{d-2}) \). By Murthy [16] Lemma 2.10, \( \text{pic}(\overline{A}) = F^1K_0(\overline{A}) \) is a divisible group. Since \( \overline{J} \) a principal (invertible) ideal, we get \( \overline{J} = \overline{K}^{d-2} \) for a complete intersection ideal \( \overline{K} \subset \overline{A} \) with \( \text{ht} \overline{K} = d - 2 \). Therefore, we have \( \overline{J} = (\overline{g}_1, \ldots, \overline{g}_{d-2}) + \overline{K}^{d-2} \) and a surjection \( \overline{P'} \rightarrow \overline{J} \). Then invoking the arguments in the proof of Murthy [16] Theorem 2.2], we get \( (\overline{P'}) - (\overline{A}^{d-1}) = (\overline{A}^{d-1}) \). Since \( \overline{K} \) is a complete intersection, by [16] Corollary 3.4, we have \( (\overline{P'}) - (\overline{A}^{d-1}) = 0 \). Thus \( \overline{P'} \) is stably free \( \overline{A} \) module of rank \( d - 1 \). By Suslin’s cancellation theorem [22], \( \overline{P'} \) is free. Hence, \( \overline{P} \) is free. This proves (3).

**Case 2.** \( d = 2 \). Given that \( J \) is a complete intersection ideal of \( \text{ht} J = 2 \). Let \( J = (h_1, h_2) \). Since \( \text{dim} J = \text{dim} A = 2 \), \( (h_1, h_2) \in U m_2(A) \), where \( A = A/T \). Applying Mohan Kumar-Murthy-Roy [15] Theorem 2.4], we get \( (h_1', h_2') \in U m_2(A) \) such that \( h_1 = h_1' \) and \( h_2 = h_2' \). Let \( \beta \in SL_2(A) \) be such that \( (h_1', h_2') \beta = (0, 1) \). Replacing \( (h_1, h_2) \) by \( (h_1, h_2) \beta \), we may assume \( h_2 = 1 \). Thus \( h_2 - 1 \in T \subset L \). Further, we may assume \( h = h_2 = 1 \) is non-zero divisor by Suslin-Vaserstein [24] Lemma 9.2]. Note that \( \overline{P'} = P/hP \) is free \( \overline{A} \)-module with basis \( \overline{p}_1, \overline{p}_2 \) and \( \overline{J} = (h_1, h_2) = (\phi'(\overline{p}_1), \phi'(\overline{p}_2)) \).

**Step 3.** We have \( \overline{J} \) is a smooth affine \( \mathbb{F}_p \)-algebra of dimension \( d - 1 \). \( \overline{J} \) is complete intersection of height \( d - 1 \) and \( \overline{P'} \) is free, so \( \overline{J} = (\phi'(\overline{p}_1), \ldots, \phi'(\overline{p}_d)) \). By [27], there exist \( (\overline{X}_1, \ldots, \overline{X}_d) \in U m_d(\overline{A}) \) such that \( \sum_{i=1}^d \overline{X}_i \phi'(\overline{p}_i) = 0 \). Since \( \sum_{i=1}^d \overline{X}_i \overline{p}_i \in U m(\overline{P}) \), by Mohan Kumar-Murthy-Roy [15] Theorem 2.4], there exist a \( p' \in U m(P) \) with \( p'' = \sum_{i=1}^d \overline{X}_i \overline{p}_i \) \( \in U m(\overline{P}) \). Thus \( P = P'' \oplus Ap' \) and \( \phi(p') = ah \) for some \( a \in A \). If \( m \) is a smooth maximal ideal containing \( (J, a) \), then \( ah \in m^2 \), as \( h \in J \subset m \). By Chinese remainder theorem, a basis of \( J/J^2 \) induces a basis of \( m/m^2 \). Since \( ah \in J \) is an element of basis of \( J/J^2 \), image of \( ah \) is zero modulo \( m^2 \). Thus we get a contradiction and hence \( Aa + J = A \).

Now \( ah = \phi'(p') \in IJ \) and \( h - 1 \in L \subset I \) gives \( a \in I \). In view of \( I/\phi'(P) = I/IJ = A/J \), we have \( I = (\phi'(P), Aa) \). Define \( \Phi : P = P'' \oplus Ap' \rightarrow I \) by letting \( \Phi = \phi' \) on \( P'' \) and \( \Phi(p') = a \). Then \( \Phi \) is surjective lift of \( \phi' \). This completes the proof.

As an application of [41], we prove the following result which extends Bhawadekar-Keshari [3] Lemma 4.4] in case of an affine \( \mathbb{F}_p \)-algebra.

**Corollary 4.2.** Let \( C \) be an affine \( \mathbb{F}_p \)-algebra and \( A = C_{1 + K} \), where \( K \) is an ideal of \( C \) with \( \text{dim} A/K = r \geq 2 \). Let \( P \) be a projective \( A \)-module of rank \( m \geq r \). Let \( I \) and \( L \) be ideals of \( A \) with \( L \subset I^2 \). Let
\( \phi : P \to I/L \) be a surjection. Then \( \phi \) can be lifted to a surjection \( \Psi : P \to I \).

**Proof.** Follow the proof of Bhatwadekar-Keshari [3, Lemma 3.4] where it is proved for general ring in the case \( m \geq r + 1 \). Let \( \overline{A} = A/K \) and let \( \overline{\phi} \) denote reduction modulo \( K \). Then \( \overline{\phi} \) induces a surjection \( \overline{\phi} : \overline{P} \to \overline{I}/L \). By [4, Theorem 3.4], \( \overline{\phi} \) can be lifted to a surjection \( \overline{\Phi} : \overline{P} \to \overline{I} \). Let \( \overline{\Phi} \) be a lift of \( \overline{\phi} \). Since \( T = (I + K)/K = I/(I \cap K) \), we have \( \Phi(P) + (I \cap K) = 0 \). Also \( \Phi(P) + L = I \). Since \( K \) is contained in Jacobson radical of \( A \), any maximal ideal of \( A \) containing \( \Phi(I) \) contains \( I \). Thus \( \Phi(P) = I \) and \( \Phi \) is a surjective lift of \( \phi \).

**Corollary 4.3.** Let \( C \) be an affine \( \mathbb{F}_p \)-algebra and \( A = C_{1+k} \), where \( K \) is an ideal of \( C \) with \( \dim A/K = r \geq 2 \). Let \( I \) and \( L \) be ideals of \( A[T] \) such that \( L \subset I^2 \) and \( L \) contains a monic polynomial. Let \( P' \) be a projective \( A[T] \)-module of rank \( m \geq r \). Let \( \phi : P' \to A[T] \) be a surjection. Then we can lift \( \phi \) to a surjection \( \Phi : P' \to A[T] \) with \( \Phi(0,1) \) a monic polynomial.

**Proof.** Follow the proof of Bhatwadekar-Keshari [3, Lemma 4.5] where it is proved for general ring in the case \( m \geq r + 1 \) and use (4.2).

5. **Roitman’s question for \( \mathbb{F}_p \)-algebras:** Proof of Theorem 1.4

**Theorem 5.1.** Let \( A \) be an affine \( \mathbb{F}_p \)-algebra of dimension \( d \) and \( P \) be a projective \( A[T] \)-module of rank \( n \) which is extended from \( A \) with \( 2n \geq d + 2 \). Let \( f \in A[T] \) be a monic polynomial such that \( Um(P_f) \neq \emptyset \). Then \( Um(P) \neq \emptyset \).

**Proof.** We may assume \( n \leq d \) by Plumstead [17]. When \( d = 2 \), then \( n = 2 \) and we are done by Bhatwadekar [2, Proposition 3.3]. When \( d = 3 \), then \( n = 3 \) and we are done by Bhatwadekar-Sridharan [4, Theorem 3.4]. Therefore, we can assume \( d \geq 4 \) and \( n \geq 3 \). The proof is exactly the same as that of Bhatwadekar-Keshari [3, Theorem 5.3], just use (4.3) instead of (2.6).

6. Proof of results stated in the abstract

**Theorem 6.1.** Let \( A \) be a ring of dimension \( d \) containing an infinite field \( k \) and \( P \) be a projective \( A[T_1, \ldots, T_r] \)-module of rank \( n \). Assume one of the following holds.

1. \( 2n \geq d + 3 \) and \( P \) is extended from \( A \).
2. \( 2n \geq d + 2 \), \( A \) is an affine \( \mathbb{F}_p \)-algebra and \( P \) is extended from \( A \).
3. \( 2n \geq d + 3 \) and the singular locus of \( \text{Spec}(A) \) is a closed set \( V(J) \) with \( \text{ht}(J) \geq d - n + 2 \).

If \( Um(P_f) \neq \emptyset \) for some monic polynomial \( f(T_r) \in A[T_1, \ldots, T_r] \), then \( Um(P) \neq \emptyset \).

**Proof.** Case 1, 2. If \( \overline{P} = P/(T_1, \ldots, T_{r-1}) \), then \( \overline{f} \in A[T_r] \) is monic and \( Um(\overline{P}_f) \neq \emptyset \). By Bhatwadekar-Keshari [3, Theorem 5.3] in case 1 and by [5, 1] in case 2, \( Um(\overline{P}) \neq \emptyset \). Since \( P \) is extended from \( A \), we get \( Um(P) \neq \emptyset \).

**Case 3.** The case \( r = 1 \) is [3.1]. Assume \( r > 1 \) and write \( R = A[T_2, \ldots, T_r] \). Since \( Um(P_f) \neq \emptyset \) implies \( Um((P/T_1P)_f) \neq \emptyset \), by induction on \( r \), we get \( Um(P/T_1P) \neq \emptyset \). Further if \( S \) is the set of monic polynomials in \( k[T_1] \), then \( S^{-1}P \) is projective \( B[T_2, \ldots, T_r] \)-module where \( B = S^{-1}A[T_1] \) is of dimension \( d \) with height of \( JB \geq d - n + 2 \). Thus again by induction on \( r \), \( Um(S^{-1}P_f) \neq \emptyset \) implies
\( Um(S^{-1}P) \neq \emptyset. \) Thus \( Um(P_g) \neq \emptyset \) for some monic \( g \in k[T_1]. \) Now we will verify the conditions of \eqref{2.1} to show that the map \( Um(P) \to Um(P/T_1P) \) is surjective. In particular, \( Um(P) \neq \emptyset. \)

1. Since \( J = FR \) is the ideal defining singular locus of \( R, R_s \) is regular for every \( s \in J. \) Hence \( P_s \) is extended from \( R_s, \) by Popescu \[18].

2. Since \( R[T_1]/(J) = (A/J)[T_1, \ldots , T_r] \) and \( \text{dim}(A/J) \leq d-(d-n+2) \leq n-2, \) by Bhatwadekar-Roy \[7], we get \( Um(P/IP) \neq \emptyset. \)

3. Since \( \text{dim}(A/J) \leq n-2, \) if \( I \) is an ideal of \( R[T_1]/(J) \) of height \( n-1, \) then by Suslin \[22] Lemma 6.2, there exist \( \sigma \in \text{Aut}_{(A/J)[T_1]}((A/J)[T_1, \ldots , T_i]) \) of the form \( \sigma(T_i) = T_i + T_i^n \) for \( i = 2, \ldots , r \) such that \( \sigma(I) \) contains a monic polynomial. Clearly, \( \sigma \) lifts to an automorphism of \( R[T_1]. \)

4. By Lindel \[12], \( EL(P/(T_1, J)P) \) acts transitively on \( Um(P/(T_1, J)P). \)

5. \( Um(P_g) \neq \emptyset \) for some monic \( g \in R[T_1]. \)

Now we are done by \eqref{2.1}.

**Remark 6.3.** Let \( A \) be a ring of dimension \( d \) containing an infinite field \( k \) of positive characteristic \( p > d \) and \( P \) be a projective \( A[T_1, \ldots , T_r]-\)module of rank \( n \) which is stably extended from \( A. \) Assume one of the following holds:

1. \( 2n \geq d + 3. \)
2. \( 2n \geq d + 2 \) and \( A \) is an affine \( \overline{\mathbb{F}}_p \)-algebra.

If \( Um(P_f) \neq \emptyset \) for some monic polynomial \( f(T_i) \in A[T_1, \ldots , T_r], \) then \( Um(P) \neq \emptyset. \)

**Proof.** By Roitman \[21] Corollary 6], \( P \) is extended from \( A. \) Use \eqref{6.1} to complete the proof. \[21\]

**Remark 6.4.** Let \( A \) be a normal affine \( \overline{\mathbb{F}}_p \)-algebra of dimension \( d \geq 4 \) and \( P \) be a projective \( A[T]-\)module of rank \( d-1. \) Let \( f \in A[T] \) be a monic polynomial such that \( Um(P_f) \neq \emptyset. \) Following the proof of \eqref{5.1}, we get a surjection \( \Phi : P \to I \) where \( I \) is a height \( n \) ideal of \( A[T] \) containing a monic polynomial. Let \( J = (I \cap A) \cap J. \) If we assume that \( Um(P_{1+j}) \neq \emptyset, \) then following the proof of \eqref{5.1} and using \eqref{4.3} instead of \eqref{2.6}, we get \( Um(P) \neq \emptyset. \) Therefore, the natural question under above assumptions is whether \( Um(P_{1+j}) \neq \emptyset? \)

Let \( A \) be a ring of dimension \( d \) and \( n \) be an integer with \( 2n \geq d + 3. \) Bhatwadekar-Sridharan \[6\] defined \( n^{th} \) Euler class group \( E^n(A) \) of \( A. \) An element of \( E^n(A) \) is a pair \( (I, \omega_I), \) where \( I \subset A \) is an ideal of height \( n \) and \( \omega : (A/I)^n \to I/I^2 \) is a surjection. It is proved in \[6\] Theorem 4.2] that \( (I, \omega_I) \) is zero in \( E^n(A) \) if and only if \( \omega_I \) can be lifted to a surjection \( : A^n \to I. \) Mandal-Yang \[14\] extended the definition of Euler class group \( E^n(A) \) for any \( 1 \leq s \leq d. \)
Lemma 7.1. Let $A$ be a ring containing a field $k$ and $P$ be a projective $A$-module. Assume the singular locus of $\text{Spec}(A)$ is a closed set $V(J)$. Let $I \subset A[T]$ be an ideal, $J = I \cap A \cap J$ and $\overline{\theta}: P[T] \rightarrow I/I^2T$ be a surjection. Assume $\overline{\theta} \otimes A_{1+j}[T]$ can be lifted to a surjection $\theta : P_{1+j}[T] \rightarrow I_{1+j}$. Then $\overline{\theta}$ can be lifted to a surjection $\Phi : P[T] \rightarrow I$.

Proof. Follow the proof of Bhatwadekar-Sridharan [5, Lemma 3.5] where it is proved for regular $A$. Here $s \in J$, implies $A_s$ is a regular ring containing a field. Thus by Popescu [18], projective $A_{s(1+s)}[T]$-modules are extended from $A_{s(1+s)}$. Rest of the proof is same as in [5, Lemma 3.5]. □

Lemma 7.2. Let $A$ be a ring of dimension $d$ and $n$ be an integer with $2n \geq d + 3$. Let $I \subset A[T]$ be an ideal of height $n$ such that $I + J(A)A[T] = A[T]$, where $J(A)$ denotes the Jacobson radical of $A$. Assume $\text{ht}J(A) \geq d - n + 2$. Let $P$ be a projective $A$-module of rank $n$ and $\phi : P[T] \rightarrow I/I^2$ be a surjection. If the surjection $\phi \otimes A(T) : P(T) \rightarrow IA(T)/I^2A(T)$ can be lifted to a surjection from $P(T)$ to $IA(T)$, then $\psi$ can be lifted to a surjection $\Lambda : P[T] \rightarrow I$ such that $(\phi - \Lambda)(P[T]) \subset I^2T$.

Proof. Follow the proof of Bhatwadekar-Keshari [3, Lemma 4.6] where it is proved with the condition $\text{ht}J(A) \geq n - 1$. The similar proof works in our case. □

Lemma 7.3. Let $A$ be a ring of dimension $d$ and $n$ be an integer with $2n \geq d + 3$. Let $L \subset A$ be an ideal of height $\geq d - n + 2$ and $I_1 \subset A[T]$ be ideals of height $n$. Let $P = P_1 \oplus A$ be a projective $A$-module of rank $n$. Assume $J = I \cap A \cap L \subset J(A)$, where $J(A)$ denotes the Jacobson radical of $A$ and $I_1 + (J^2T) = A[T]$. Let $\Phi : P[T] \rightarrow I \cap I_1$ and $\Psi : P[T] \rightarrow I_1$ be two surjection with $\Phi \otimes A[T]/I_1 = \Psi \otimes A[T]/I_1$. Then we get a surjection $\Lambda : P[T] \rightarrow I$ such that $(\Phi - \Lambda)(T) \subset I^2T$.

Proof. Follow the proof of Bhatwadekar-Keshari [3, Lemma 4.7] where it is proved with the implicit condition $\text{ht}J(A) \geq n - 1$. The same proof works in our case. □

When $A$ is regular containing a field $k$, the following result is proved in Das-Sridharan [8, Theorem 2.11] when $P$ is free and in Bhatwadekar-Keshari [3, Proposition 4.9] in general case.

Proposition 7.4. Let $A$ be a ring of dimension $d$ containing a field $k$ and $n$ be an integer with $2n \geq d + 3$. Assume the singular locus of $\text{Spec}(A)$ is a closed set $V(J)$ with $\text{ht}J \geq d - n + 2$. Let $I \subset A[T]$ be an ideal of height $n$. Let $P$ be a projective $A$-module of rank $n$ and $\psi : P[T] \rightarrow I/I^2T$ be a surjection. Assume there exist a surjection $\Psi' : P[T] \otimes A(T) \rightarrow IA(T)$ which is a lift of $\psi \otimes A(T)$. Then $\psi$ can be lifted to a surjection $\Psi : P[T] \rightarrow I$.

Proof. Let $J = I \cap A \cap J$. Then $\text{ht}J \geq d - n + 2$ and $\text{dim} A/J \leq n - 2$. By [7,1], we may replace $A$ by $A_{1+j}$ and assume that $J \subset J(A)$. Since $n > \text{dim} A/J(A)$, we may assume that $P$ has unimodular element i.e. $P \cong P_1 \oplus A$.

Applying the moving lemma of Das-Keshari [9, Lemma 3.1], the surjection $\psi : P[T] \rightarrow I/I^2T$ can be lifted to a surjection $\theta : P[T] \rightarrow I''$ of $\psi$ such that

1. $I = I'' + (J^2T)$,
2. $I'' = I \cap I'$, where $\text{ht} I' = n$ and
3. $I' + (J^2T) = A[T]$. 

The surjection $\Theta(= \theta \otimes A(T)) : P[T] \otimes A(T) \rightarrow IA(T) \cap I'A(T)$ satisfies $\Psi' \otimes A(T)/IA(T) = \Theta \otimes A(T)/IA(T)$. Since $\dim A(T) = d$, applying Bhatwadekar-Sridharan [3] Proposition 3.2 to $\Theta$ and $\Psi'$, we get a surjection $\Phi' : P(T) \rightarrow I'A(T)$ such that $\Phi' \otimes A(T)/I'A(T) = \Theta \otimes A(T)/I'A(T)$. Since $I' + J(A) = A[T]$, where $J(A)$ is the Jacobson radical of $A$, applying (7.2) to the surjections $\phi \otimes A[T]/I' : P[T]/I'P[T] \rightarrow I'/I'^2$ and $\Phi'$, we get a surjection $\Theta : P[T] \rightarrow I'$ which is a lift of $\phi$. Applying (7.2) for the surjections $\phi$ and $\Phi$, we get our desired result. 

**Corollary 7.5.** Let $A$ be a ring of dimension $d$ containing an infinite field $k$ and $n$ be an integer with $2n \geq d + 3$. Assume the singular locus of $\text{Spec}(A)$ is a closed set $V(J)$ with $\text{ht}J \geq d - n + 2$. Let $I, I'$ be comaximal ideals of height $n$. Let $P = P_1 \oplus A$ be a projective $A$-module of rank $n$. Suppose we have surjections $\Gamma : P[T] \rightarrow I$ and $\Theta : P[T] \rightarrow I' \cap I'$ satisfying $\Gamma \otimes A[T]/I = \Theta \otimes A[T]/I$. Then we have a surjection $\Psi : P[T] \rightarrow I'$ such that $\Psi \otimes A[T]/I' = \Theta \otimes A[T]/I'$.

**Proof.** Follow the proof of the subtraction principle of Bhatwadekar-Keshari [3] Corollary 4.11 where it is proved for regular $A$ and use (7.4).

**Theorem 7.6.** Let $A$ be a ring of dimension $d$ containing an infinite field $k$ and $n$ be an integer with $2n \geq d + 3$. Assume the singular locus of $\text{Spec}(A)$ is a closed set $V(J)$ with $\text{ht}J \geq d - n + 2$. Let $I \subset A[T]$ be an ideal of height $n$ and $\omega_1 : (A[T]/I)^n \rightarrow I/I'^2$ be a surjection. Then the element $(I, \omega_1) \in E^n(A[T])$ is zero if and only if $\omega_1$ can be lifted to a surjection $\Psi : A[T]^n \rightarrow I$.

**Proof.** Follow the proof of Das-Sridharan [3] Theorem 3.1 where it is proved for regular $A$ and use (7.4) instead of (3) Theorem 2.11.

**Proposition 7.7.** Let $A$ be a ring of dimension $d$ containing an infinite field $k$ and $n$ be an integer with $2n \geq d + 3$. Assume the singular locus of $\text{Spec}(A)$ is a closed set $V(J)$ with $\text{ht}J \geq d - n + 2$. Let $I \subset A[T]$ be an ideal of height $n$ and $\phi : A[T]^n \rightarrow I/I'^2$ be a surjection. Assume $N(I; \phi)$ be the set of all $s \in A$ such that $\phi \otimes A_s[T]$ can be lifted to a surjection $\Phi : A_s[T]^n \rightarrow I_s$. Then $N(I; \phi)$ is an ideal of $A$.

**Proof.** Let $J = I \cap A \cap J$ and $B = A_{1+j}$. Note $J$ is contained in the Jacobson radical $J(B)$ of $B$, $\text{ht}J(B) \geq d - n + 2$ and $\dim B/J(B) \leq n - 1$. Follow the proof of Das-Sridharan [3] Theorem 2.8] where it is proved for regular $A$ and use (7.4) instead of (3) Lemma 2.6, Proposition 2.2].

**Corollary 7.8.** Let $A$ be a ring of dimension $d$ containing an infinite field $k$ and $n$ be an integer with $2n \geq d + 3$. Assume the singular locus of $\text{Spec}(A)$ is a closed set $V(J)$ with $\text{ht}J \geq d - n + 2$. Let $I \subset A[T]$ be an ideal of height $n$ and $A[T]^n \rightarrow I/I'^2$ be a surjection. Assume $\phi \otimes A_m[T]$ can be lifted to a surjection from $A_m[T]^n \rightarrow IA_m[T]$ for all maximal ideals $m$ of $A$. Then $\phi$ can be lifted to a surjection $\Phi : A[T]^n \rightarrow I$.

**Proof.** Following the proof of Das-Sridharan [3] Theorem 2.11 where it is proved for regular $A$ and use (7.4).
have the following exact sequence of groups

\[ 0 \to E^n(A) \to E^n(A[T]) \to \prod_mE^n(A_m[T]) \]

where the product runs over all maximal ideals \( m \) of \( A \).

**Proof.** Follow the proof of local global principle of Das-Sridharan [8] Theorem 3.9] where it is proved for regular \( A \) and use (7.7). \( \square \)

**Theorem 8.2.** Let \( A \) be a ring of dimension \( d \) containing an infinite field \( k \) and \( n \) be an integer with \( 2n \geq d + 3 \). Assume the singular locus of \( \text{Spec}(A) \) is a closed set \( V(J) \) with \( \text{ht}J \geq d - n + 2 \). Let \( I \subset A[T] \) be an ideal of height \( n \) and \( P \) be a projective \( A \)-module of rank \( n \). Let \( \phi : P[T] \to I/I^2T \) be a surjection. Assume \( \phi \otimes A_m[T] \) can be lifted to a surjection from \( P_m[T] \to IA_m[T] \) for all maximal ideals \( m \) of \( A \). Then \( \phi \) can be lifted to a surjection \( \Phi : P[T] \to I \).

**Proof.** Let \( S \) be the set of all \( s \in A \) such that \( \phi \otimes A_s[T] \) can be lifted to a surjection \( \Phi : P_s[T] \to I_s \). Our aim is to show that \( 1 \in S \). If \( t \in S \) and \( a \in A \), then \( at \in S \). For every maximal ideal \( \mathfrak{m} \) of \( A \), \( \phi \otimes A_{\mathfrak{m}}[T] \) has a surjective lift. Thus there exist \( s \in A - \mathfrak{m} \) such that \( P_s \) is free and \( s \in S \). Thus we can find \( s_1, \ldots, s_r \in S \) such that \( P_{s_i} \) is free and \( s_1 + \ldots + s_r = 1 \). Therefore, using induction, it is enough to show that if \( s, t \in S \) and \( P_s \) is free, then \( s + t \in S \). Replacing \( A \) by \( A_{x+y} \), \( x \) by \( x/(x+y) \) and \( y \) by \( y/(x+y) \), we may assume \( s + t = 1 \). We will follow the proof of Bhatwadekar-Keshari [3] Theorem 4.13 and indicate only the necessary changes.

In step 1, if \( J = I \cap A \cap J \), then by (7.1), we may replace \( A \) by \( A_{1,J} \) and assume that \( J \) is contained in the Jacobson radical of \( A \). Note \( \text{ht}J \geq d - n + 2 \), thus \( \dim(A/J) \leq n - 2 \). Rest of the arguments of step 1 is same, just use (7.3). In step 2, just replace [3] Corollary 4.11 with the subtraction principle (7.5).

In step 3, the arguments are same. In step 4, use (7.5), (3.4) to complete the proof. \( \square \)

**Remark 8.3.** The following example is due to Bhatwadekar-Mohan Kumar-Srinivas [8] Example 6.4]. Let \( B = \mathbb{C}[X, Y, Z, W]/(X^5 + Y^5 + Z^5 + W^5) \). Then \( B \) has an isolated singularity at the origin. Thus singular locus of \( \text{Spec}(B) \) is defined by the maximal ideal \( J = m = (x, y, z, w) \). Here \( \dim(B) = 3 = \text{ht}J \). There exist \( a \in B - m \), an ideal \( I \subset A[T] \) of height 3 and a surjection \( \phi : A[T]^3 \to I/I^2T \) which does not have a surjective lift from \( A[T]^3 \to I \). In fact \( \phi \otimes A_m[T] \) does not have a surjective lift : \( A_m[T]^3 \to IA_m[T] \). Note that if \( n \neq m \) is another maximal ideal of \( A \), then \( \phi \otimes A_n[T] \) does have a surjective lift : \( A_n[T]^3 \to IA_n[T] \), by [3] Theorem 4.13 as \( A_n \) is regular. Our result (8.2) shows that lifting \( \phi \) locally is precisely the obstruction for lifting \( \phi \) globally. \( \square \)

**Remark 8.4.** (Segre class) Let \( A \) be a ring of dimension \( d \) containing an infinite field \( k \) and \( n \) be an integer with \( 2n \geq d + 3 \). Assume the singular locus of \( \text{Spec}(A) \) is a closed set \( V(J) \) with \( \text{ht}J \geq d - n + 2 \). Following the proofs of Das-Keshari [9] Section 4], where they are proved for regular \( A \), we can prove the following results.

1. Let \( I \subset A[T] \) be an ideal such that \( \mu(I/I^2) = n \) and \( n + \text{ht}I = \dim A[T] + 2 \). Let \( \omega_I : (A[T]/I)^n \to I/I^2 \) be a surjection. Following [9] section 4], we can define the \( n \)-th Segre class \( s^n(I, \omega_I) \) of \( (I, \omega_I) \) as an element of \( E^n(A[T]) \).
9. Euler class of stably free module - Proof of Theorem 3.4

Let $R$ be regular ring of dimension $d$ containing a field $k$ and $P$ be a stably free $R$-module of rank $n$ with $P \oplus R = R^{n+1}$, where $2n \geq d + 3$. Bhatwadekar-Sridharan [6] Theorem 5.4] associated an element $e(P) \in E^n(R)$ and proved that $Um(P) \neq \emptyset$ if and only if $e(P)$ is zero in $E^n(R)$. We will extend this result by relaxing the regularity assumption on $R$ by the condition $ht J \geq d - n + 2$, where $J$ is the ideal defining the singular locus of $R$. Further we will extend this result to arbitrary stably free $R$ and $R[T]$-modules of rank $n$.

**Proposition 9.1.** Let $R$ be a ring of dimension $d$ containing a field and $I \subset R[W]$ be an ideal of height $n$ with $J = I(0)$ a proper ideal of $R$. Let $P$ be a projective $R$-module of rank $n$ and $\alpha(W) : P[W] \rightarrow I$ be a surjection. Assume the singular locus of $Spec(R)$ is a closed set $V(J)$ and $P/NP$ is free, where $N = (I \cap R \cap J)^2$. Let $p_1, \ldots, p_n$ be elements of $P$ whose reduction modulo $N$ form a basis of $P/NP$ and let $\alpha(0)(p_i) = a_i \in J$. Then there exists an ideal $K \subset R$ of height $\geq n$ and comaximal with $N$ such that:

1. $I \cap KR[W] = (F_1(W), \ldots, F_n(W))$.
2. $F_i(0) - F_i(1) \in K^2$.
3. $\alpha(W)(p_i) - F_i(W) \in I^2$.
4. $F_i(0) - a_i \in J^2$.

**Proof.** Follow the proof of Bhatwadekar-Sridharan [6] Proposition 5.2] where it is proved for regular ring $R$. Note that $a \in N \subset J$, hence $R_a$ is a regular ring containing a field. Hence by Popescu [13], projective modules over $R_{a(1+a)}$ are extended from $R_{a(1+a)}$. This is the only place where regularity hypothesis was used in [6].

9.1. Euler class of stably free $R[T]$-module $P$. Let $R$ be a ring of dimension $d$ containing a field $k$, $R = R[T]$ and $n$ be an integer with $2n \geq d + 3$. Assume the singular locus of $Spec(R)$ is a closed set $V(J)$ with $ht J \geq d - n + 2$. Let $P$ be a stably free $R$-module of rank $n$. We will define the Euler class $e(P) \in E^n(R)$ of $P$ and prove that $Um(P) \neq \emptyset$ if and only if $e(P) = 0$ in $E^n(R)$.

Let $r \geq 1$ and $Um_{r,n+r}(R)$ be the set of all $r \times (n + r)$ matrices $\sigma$ in $M_{r,n+r}(R)$ which has a right inverse, i.e. there exists $\tau \in M_{n+r,r}(R)$ such that $\sigma \circ \tau = Id_r$. For any $\sigma \in Um_{r,n+r}(R)$, we have an exact sequence

$$0 \rightarrow R^r \rightarrow \sigma \rightarrow \sigma^{n+r} \rightarrow P \rightarrow 0$$

where $\sigma(v) = \nu \sigma$ for $v \in R^r$ and $P$ is a stably free $R$-module of rank $n$. Hence every element of $Um_{r,n+r}(R)$ corresponds to a stably free $R$-module of rank $n$ and conversely, any stably free $R$-module $P$ of rank $n$ will give rise to an element of $Um_{r,n+r}(R)$ for some $r$. We will define a map

$$e : Um_{r,n+r}(R) \rightarrow E^n(R)$$

(2) Let $\omega : (A[T]/I)^n \rightarrow I/I^2$ be a surjection, where $n \geq \dim A - \text{ht} I + 3$. If $s^n(I, \omega, I) = 0$ in $E^n(A[T])$, then $\omega$ can be lifted to a surjection $\Theta : A[T]^n \rightarrow I$.  


which is a natural generalization of the map $U_{m_{n+1}}(\mathcal{R}) \to E^n(\mathcal{R})$ defined in [6]. Let $\sigma$ be an element of $U_{m_{r,n+r}}(\mathcal{R})$, then

$$\sigma = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n+r} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,n+r} \end{bmatrix}$$

Let $e_1, \ldots, e_{n+r}$ be the standard basis of $\mathcal{R}^{n+r}$ and let

$$P = \mathcal{R}^{n+r}/\left( \sum_{i=1}^{n+r} a_{1,i}e_i, \ldots, \sum_{i=1}^{n+r} a_{r,i}e_i \right) \mathcal{R}.$$  

Let $p_1, \ldots, p_{n+r}$ be the images of $e_1, \ldots, e_{n+r}$ respectively in $P$. Then

$$P = \sum_{i=1}^{n+r} \mathcal{R}p_i \text{ with relations } \sum_{i=1}^{n+r} a_{1,i}p_i = 0, \ldots, \sum_{i=1}^{n+r} a_{r,i}p_i = 0.$$  

To the triple $(P, (p_1, \ldots, p_{n+r}), \sigma)$, we associate an element $e(P, (p_1, \ldots, p_{n+r}), \sigma)$ of $E^n(\mathcal{R})$ as follows: Let $\lambda : P \rightarrow J$ be a generic surjection, i.e. $J \subset \mathcal{R}$ is an ideal of height $n$. Since $P \oplus \mathcal{R}^r = \mathcal{R}^{n+r}$ and $\dim \mathcal{R}/J = d + 1 - n \leq n - 2$, we get $P/JP$ is a free $\mathcal{R}/J$ module of rank $n$, by Bass [1]. Since $J/J^2$ is surjective image of $P/JP$, $J/J^2$ is generated by $n$ elements.

Let “bar” denote reduction modulo $J$. By Bass [1], there exist $\Theta \in E_{n+r}(\overline{\mathcal{R}})$ such that

$$[a_{1,1}, \ldots, a_{1,n+r}]\Theta = [1, 0, \ldots, 0]$$

i.e. the first row of $\Theta^{-1}$ is $[a_{1,1}, \ldots, a_{1,n+r}]$. Let $\sigma \circ \Theta$ be given by

$$\sigma \circ \Theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n+r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r,1} & b_{r,2} & \cdots & b_{r,n+r} \end{bmatrix}.$$  

Note that $[b_{2,2}, \ldots, b_{2,n+r}] \in U_{m_{n+r-1}}(\overline{\mathcal{R}})$. By Bass [1], there exist $\Theta_1 \in E_{n+r-1}(\overline{\mathcal{R}})$ such that

$$[b_{2,2}, \ldots, b_{2,n+r}]\Theta_1 = [1, 0, \ldots, 0].$$  

If $\Phi \in E_m(\mathcal{R})$, then $\begin{bmatrix} Id_{r+1} & 0 \\ 0 & \Phi \end{bmatrix} \in E_{m+1}(\mathcal{R})$. Let

$$\sigma \circ \Theta \circ \Theta_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_{2,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{r,1} & b_{r,2} & \cdots & b_{r,n+r} \end{bmatrix}.$$
Counting in this way, we get \( \hat{\Theta} \in E_{n+r}(\overline{R}) \) such that

\[
\varpi \circ \hat{\Theta} = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}.
\]

We can find elementary matrix \( \Psi \in E_{n+r}(\overline{R}) \) such that

\[\varpi \circ \hat{\Theta} \circ \Psi = [Id_r, \hat{0}]\]

where \( \hat{0} \) is \( r \times n \) zero matrix. Let \( \Delta = (\hat{\Theta} \circ \Psi)^{-1} \in E_{n+r}(\overline{R}) \), then \( \varpi \) is the first \( r \) rows of \( \Delta \), i.e., \( \varpi \) can be completed to an elementary matrix \( \Delta \). Since

\[
\sum_{i=1}^{n+r} a_{1,i}p_i = 0, \ldots, \sum_{i=1}^{n+r} a_{r,i}p_i = 0.
\]

we get

\[\Delta[p_1, \ldots, p_{n+r}]^t = [0, \ldots, 0, q_1, \ldots, q_n]^t,\]

where \( t \) stands for transpose. Thus \( (q_1, \ldots, q_n) \) is a basis of the free module \( P/JP \).

Let \( \omega_f : (\mathcal{R}/J)^n \rightarrow J/J^2 \) be the surjection given by the set of generators of \( \lambda(q_1), \ldots, \lambda(q_n) \) of \( J/J^2 \). We define

\[e(P, (p_1, \ldots, p_{n+r}), \sigma) = (J, \omega_f) \in E^n(\mathcal{R}).\]

We need to show that \( e(P, (p_1, \ldots, p_{n+r}), \sigma) \) is independent of the choice of the elementary completion of \( \varpi \) and the choice of the generic surjection \( \lambda \).

We begin with the following result which shows that \( e(P, (p_1, \ldots, p_{n+r}), \sigma) \) is independent of the choice of the elementary completion \( \varpi \).

**Lemma 9.2.** Suppose \( \Gamma \in E_{n+r}(\overline{R}) \) is chosen so that its first \( r \) rows are \( \varpi \). Let \( \Gamma[p_1, \ldots, p_{n+r}]^t = [0, \ldots, 0, q_1', \ldots, q_n']^t \). Then there exist \( \Psi \in E_n(\overline{R}) \) such that \( \Psi[q_1, \ldots, q_n]^t = [q_1', \ldots, q_n'] \).

**Proof.** The Matrix \( \Gamma \circ \Delta^{-1} \in E_{n+r}(\overline{R}) \) is such that its first \( r \) rows are \( [Id_r, \hat{0}] \). Therefore, there exists \( \Psi \in SL_n(\overline{R}) \cap E_{n+r}(\overline{R}) \) such that \( \Psi[q_1, \ldots, q_n]^t = [q_1', \ldots, q_n'] \). Since \( n > \text{dim} R + 1 \), by Suslin-Vaserstein [24], \( \Psi \in E_n(\overline{R}) \).

Let \( \hat{\omega}_J : (\mathcal{R}/J)^n \rightarrow J/J^2 \) be the surjection given by the set of generators \( \lambda(q_1'), \ldots, \lambda(q_n') \) of \( J/J^2 \). Then, by [22], \( (J, \omega_f) = (J, \hat{\omega}_J) \in E^n(\mathcal{R}) \). Thus for a given surjection \( \lambda : P \rightarrow J \), the element \( e(P, (p_1, \ldots, p_{n+r}), \sigma) \) is independent of the choice of the elementary completion of \( \varpi \).

Now we have to show that \( e(P, (p_1, \ldots, p_{n+r}), \sigma) \) is independent of the choice of the generic surjection \( \lambda \). In other words, we have to show that if \( \lambda' : P \rightarrow J' \) is another generic surjection, where \( J' \) is an ideal of \( \mathcal{R} \) of height \( n \) and \( \omega'_f : (\mathcal{R}/J')^n \rightarrow J'/J'^2 \) is a surjection obtained as above by completion of \( \sigma \) modulo \( J' \) to an element of \( E_{n+r}(\mathcal{R}/J') \), then \( (J, \omega_f) = (J', \omega'_f) \in E^n(\mathcal{R}) \).

By Bhatwadekar-Sridharan [6] Lemma 5.1, there exist an ideal \( I \subset \mathcal{R}[W] \) of height \( n \) and a surjection \( \alpha(W) : P[W] \rightarrow I \) such that \( I(0) = J, \alpha(0) = \lambda \) and \( I(1) = J', \alpha(1) = \lambda' \). Let \( N = (I \cap \mathcal{R} \cap J)^2 \).
Note that \(\mathcal{J}R\) is the ideal defining singular locus of \(R\). Since \(\text{ht} N \geq d - n + 2\) and \(\dim \mathcal{R}/N \leq d + 1 - (d - n + 2) \leq n - 1\), by Bass \([1]\), \(P/NP\) is free.

Using \([7]\), rest of the proof of well definedness of \(e(P, (p_1, \cdots, p_{n+r}), \sigma)\) is same as in \([6]\). We denote the element \(e(P, (p_1, \cdots, p_{n+r}), \sigma)\) of \(E^n(\mathcal{R})\) by \(e(P)\) or \(e(\sigma)\). Following the above arguments for a stably free \(R\)-module \(Q\), we get a well defined element \(e(Q)\) of \(E^n(R)\). Therefore, we have proved the following result.

**Theorem 9.3.** Let \(R\) be a ring of dimension \(d\) containing a field \(k\) and \(n\) be an integer with \(2n \geq d + 3\). Assume the singular locus of \(\text{Spec}(R)\) is a closed set \(V(\mathcal{J})\) with \(\text{ht} \mathcal{J} \geq d - n + 2\). Then we have well defined maps

\[
e : Um_{r,n+r}(R) \to E^n(R), \quad e : Um_{r,n+r}(R[T]) \to E^n(R[T])
\]

In particular, given stably free \(R\) (resp. \(R[T]\)) module \(Q\) (resp. \(P\)) of rank \(n\), we can associate an element \(e(Q) \in E^n(R)\) (resp. \(e(P) \in E^n(R[T])\)).

**Theorem 9.4.** Let \(R\) be a ring of dimension \(d\) containing a field \(k\) and \(n\) be an integer with \(2n \geq d + 3\). Assume the singular locus of \(\text{Spec}(R)\) is a closed set \(V(\mathcal{J})\) with \(\text{ht} \mathcal{J} \geq d - n + 2\). Let \(Q\) (resp. \(P\)) be stably free \(R\) (resp. \(R[T]\))-modules of rank \(n\). Then

1. \(Um(Q) \neq \emptyset\) if and only if \(e(Q) = 0 \in E^n(R)\).
2. Assume \(k\) is infinite. Then \(Um(P) \neq \emptyset\) if and only if \(e(P) = 0 \in E^n(R[T])\).

**Proof.**

(1) Following the proof of Bhatwadekar-Sridharan \([6]\) Theorem 5.4).

(2) Assume \(e(P) = 0\). Then \(e(P \otimes R(T)) = 0 \in E^n(R(T))\). By (1), \(Um(P \otimes R(T)) \neq \emptyset\). By (3.1), \(Um(P) \neq \emptyset\).

Conversely, assume \(Um(P) \neq \emptyset\). Let \(e(P) = (I, \omega_I)\) for some height \(n\) ideal \(I \subset R[T]\) and \(\omega_I : (R[T]/I)^n \to I/I^2\) a surjection. Since \(R\) contains an infinite field, by Bhatwadekar-Sridharan \([5]\) Lemma 3.2], we may assume that either \(I(0) = R\) or \(I(0)\) is height \(n\) ideal of \(R\).

When \(I(0) = R\), \(\omega_I\) can be lifted to a surjection \(\overline{\phi} : (R[T]/I)^n \to I/I^2\).

When \(I(0)\) has height \(n\), then using \(Um(P/TP) \neq \emptyset\) and \(e(P/TP) = (I(0), \omega_{I(0)}) = 0 \in E^n(R)\), by first part of these theorem. Thus by Bhatwadekar-Sridharan \([6]\) Theorem 4.2], the surjection \(\omega_{I(0)} : (R/I(0))^n \to I(0)/I(0)^2\) can be lifted to a surjection \(\phi_1 : R^n \to I(0)\). Patching \(\phi_1\) and \(\omega_I\), we get a surjection \(\Phi : (R[T]/I)^n \to I/I^2\) which is a lift of \(\omega_I\).

Since \(Um(P \otimes R(T)) \neq \emptyset\) and \(\dim R(T) = d\), by Bhatwadekar-Sridharan \([6]\) Theorem 4.2], \(\omega_I \otimes R(T)\) can be lifted to a surjection \(\Phi : R(T)^n \to IR(T)\) which is also a lift of \(\overline{\phi}\). By (3.4), \(\overline{\phi}\) has a surjective lift \(\theta : R[T]^n \to I\) which is also a lift of \(\omega_I\). Thus \(e(P) = (I, \omega_I) = 0 \in E^n(R[T])\).

**References**

[1] H. Bass, K-Theory and stable algebra, IHES 22 (1964), 5-60.
[2] S.M. Bhatwadekar, Inversion of monic polynomials and existence of unimodular elements (II), Math. Z. 200 (1989), 233-238 (1989).
[3] S.M. Bhatwadekar and M.K. Keshari, A question of Nori: projective generation of ideals, K-Theory 28 (2003), 329-351.
[4] S.M. Bhatwadekar and Raja Sridharan, On a question of Roitman, J. of Ramanujan Math. Soc., 16(1) (2001), 45-61.
[5] S.M. Bhatwadekar and Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain and a question of Nori, Invent. Math. 133 (1998), 161-192.

[6] S.M. Bhatwadekar and Raja Sridharan, On Euler classes and stably free projective modules, Proceedings of the international colloquium on Algebra, Arithmetic and Geometry, Mumbai 2000, Narosa Publishing House, 139-158.

[7] S.M. Bhatwadekar and A. Roy, Some theorems about projective modules over polynomial rings, J. Algebra 86 (1984), 150-158.

[8] M.K. Das, Raja Sridharan, Euler class groups and a theorem of Roitman, J. Pure and Applied Algebra 215 (2011), 1340-1347.

[9] M.K. Das and M.K.Keshari, A question of Nori, Segre classes of ideals and other applications, J. Pure and Applied Algebra 216 (2012), 2193-2203.

[10] A. Krishna, V. Srinivas, Zero cycles on singular varieties, in Algebraic Cycles and Motives I, London Mathematical Society Lecture note ser., Cambridge University Press 343 (2007), 264-277.

[11] M.K. Keshari and Md. Ali Zinna, Unimodular elements in projective modules and an analogue of a result of Mandal, J. Commut. Algebra 10(3) (2018), 359-373.

[12] H. Lindel, Unimodular elements in projective modules, J. Algebra 172 (1995), 301-319.

[13] S. Mandal and M.P. Murthy, Ideals as sections of projective modules, J. Ramanujan Math. Soc. 13(1) (1998), 51-62.

[14] S. Mandal and Y. Yang, Intersection theory of algebraic obstructions, J. Pure and Applied Algebra 214 (2010), 2279-2293.

[15] N. Mohan Kumar, M.P. Murthy and A. Roy, A cancellation theorem for projective modules over finitely generated rings, Algebraic geometry and commutative algebra in honour of Masayoshi Nagata, vol 1 (1987), 281-297, Tokyo.

[16] M.P. Murthy, Zero cycles and projective modules, Ann. Math. 140 (1994), 405-434.

[17] B. Plumstead, The conjecture of Eisenbud and Evans, Amer. J. Math 105 (1983), 1417-1433.

[18] D. Popescu, Polynomial rings and their projective modules, Nagoya Math J. 113 (1989), 121-128.

[19] D. Quillen, Projective modules over polynomial rings, Invent. Math. 46 (1976), 167-171.

[20] M. Roitman, Projective modules over polynomial rings, J. Algebra 58 (1979), 51-63.

[21] M. Roitman, On stably extended projective modules over polynomial rings, Proc. AMS 97(4) (1986), 585-589.

[22] A.A. Suslin, On structure of special linear group over polynomial rings, Math USSR Izvest. 11 (1977), 221-238.

[23] A.A. Suslin, A cancellation theorem for projective modules over affine algebras, Sov. Math. Dokl. 18 (1977), 1281-1284.

[24] A.A. Suslin and L.N. Vaserstein, Serre’s problem on projective modules over polynomial rings and algebraic K-theory, Izv. Akad. Nauk SSSR Ser. Mat. 40(5) (1976), 993-1054.

[25] R.G. Swan, A cancellation theorem for projective modules in the metastable range, Invent. Math. 27 (1974), 23-43.

E-mail: Manoj K. Keshari <keshari@math.iitb.ac.in>
E-mail: Soumi Tikader <tikadersoumi@gmail.com>