PROOF OF THE PARSHIN’S CONJECTURE

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Abstract. We prove the Parshin’s conjecture on the rational triviality of the higher algebraic $K$-theory of smooth projective varieties over finite fields. This is known to imply the Beilinson-Soulé conjecture for the fields of positive characteristic. Especially it implies that for a field $F$ of char $p$ and $n > 0$, the only rationally non-trivial weight appearing in $K_n(F) \otimes \mathbb{Q}$ can be $n$, thus $K_n(F) \otimes \mathbb{Q} = K_n^M(F) \otimes \mathbb{Q}$ where $K_n^M(F)$ is the Milnor $K$-theory.

1. Introduction

In [Gra82] it is proved that the higher algebraic $K$-theory of smooth projective curves over finite fields are finitely generated. Harder [Har77a] proved that these groups are torsion. It was conjectured by Parshin that the higher algebraic $K$-theory of smooth projective varieties over finite fields are torsion. This is known as the Parshin’s conjecture. In this paper we prove this conjecture.

Theorem 1.1 (Parshin’s Conjecture). Let $X$ be a smooth projective variety over a finite field then $K_i(X) \otimes \mathbb{Q} = 0$ for $i \geq 1$.

In [Gei98], it is proved that the Parshin’s conjecture implies the following corollary:

Corollary 1.2 (Beilinson’s Conjecture). Let $k$ be a field of characteristic $p \neq 0$ then for $i \geq 1$ we have $K_i(k) \otimes \mathbb{Q} = K_i^M(k) \otimes \mathbb{Q}$ and if $d$ is the transcendental degree of $k$ then $K_i(k) \otimes \mathbb{Q} = 0$ for $i > d$.

The proof can be broken into two parts. The first part is to utilize the Grayson filtration [Gra95] to obtain a fibration. The total space in the fibration sequence will be the $K$-theory space or a space closely related to it. The objective is to prove that the total space is rationally trivial. To do so we will prove that the fiber is rationally trivial and the base is trivial. Triviality of the fiber follows from the fact that we are working on a projective variety over a finite field (it does not require smoothness). On the other hand triviality of the base, follows from the fact that the variety is finite type and regular over a perfect field (or more generally, finite type and smooth over any field.). It does not require projectivity or being defined over a finite field.

The two main gadgets we will use are the binary complexes for defining higher algebraic $K$-theory [Gra12] and the Grayson motivic filtration [Gra95, Sus03]. In the second section we will give an overview of algebraic $K$-theory via binary complexes. In the third section we will state the critical fibration sequence obtained from the Grayson filtration and prove why the fiber is rationally trivial. In the final section we will prove the triviality of the base.
Proving the Theorem 1.1 for $i$-th algebraic $K$-group will require working with $i-1$ dimensional binary complexes. So the case $K_1$ is the simplest case which requires only working with the exact category of vector bundles. In this paper we will try to give a uniform proof such that $K_i$ for $i > 1$ does not seem detached from $K_1$. The proof for $i > 1$ is more or less similar to the $i = 1$ but of course it requires the binary complexes.

2. Binary Complexes

We start this section by giving an overview of the construction of higher algebraic $K$-groups via binary complexes [Gra12]. In this section $N$ is an exact category. All chain complexes are considered to be bounded. The category of chain complexes in $N$ is denoted by $CN$.

**Definition 2.1.** A chain complex $(P_*, d)$ in $N$ is called acyclic if each differential epimorphism followed by an admissible monomorphism:

$$d_n : P_n \rightarrow J_{n-1} \rightarrow P_{n-1}$$

such that $J_n \rightarrow P_n \rightarrow J_{n-1}$ is a short exact sequence.

The full subcategory of acyclic chain complexes is denoted by $C^aN \subseteq CN$.

**Definition 2.2.** A binary acyclic complex $(P_*, d, d')$ is a graded object $P_*$ over $N$ with two degree $-1$ differentials $d, d' : P_* \rightarrow P_*$ such that both $(P_*, d)$ and $(P_*, d')$ are acyclic chain complexes. The differentials $d$ and $d'$ are called top and bottom differentials.

A morphism of binary acyclic complexes is a degree 0 map of the underlying graded objects that commutes with both of the differentials. The category of binary acyclic complexes is denoted by $B^aN$. If in the definition of the binary acyclic complex we drop the acyclicity condition with call it a binary complex and denote the corresponding category by $BN$. There is an exact functor $\Delta : C^a(N) \rightarrow B^a(N)$ which duplicates the differential of a given acyclic chain complex.

Both of $C^aN$ and $B^aN$ are exact categories and these constructions can be iterated. Assume $n > 0$. For any sequence $W = (W_1, \ldots, W_n)$ in $\{B, C\}$, the category $W_1 \ldots W_nN$ is denoted by $W^aN$. If $W$ is the constant sequence on the letter $B$, we also denote it by $B^nN$. We call this the category of $n$-dimensional binary acyclic complexes . In the case $n = 0$ we simply assume that the 0-dimensional binary acyclic complexes are the same as the exact category $N$. Letting $W$ vary over all possible choices defines a commutative $n$-cube of exact categories. This induces a commutative $n$-cube of spectra after taking the algebraic $K$-theory. The spectrum $K(\Omega^nN)$ is defined to be the total homotopy cofiber of this cube.

The major result of [Gra12] is the following Theorem:

**Theorem 2.3.** The abelian groups $K_n(N)$ are naturally isomorphic to $K_0(\Omega^nN)$.

This theorem gives us a generator and relation representation for higher algebraic $K$-groups (see [Gra12, Corollary 7.4]). Generators of $K_n(N)$ are $n$-dimensional acyclic binary complexes. There are two types of relations. First type are those that are coming from the short exact sequence of binary complexes and the second type are those that the two differential at some direction coincide with each other. Now let’s compare $K_n(N) \cong K_0(\Omega^nN)$ and $K_1((B^n)^{n-1}N) \cong K_0(\Omega((B^n)^{n-1})N)$. We see that both have the same generators which are the $n$-dimensional acyclic
Proof. The aforementioned pullbacks realize \( K_1((B^n)^{n-1}N) \) are subset of the relations in \( K_n(N) \). So \( K_n(N) \) is a quotient of \( K_1((B^n)^{n-1}N) \). Therefore in order to prove Theorem 1.1 we will prove the following Theorem:

**Theorem 2.4.** Let’s \( X \) be a smooth projective variety over a finite field. Then for \( n \geq 1 \) the groups \( K_1((B^n)^n X) \) are torsion. Here \( Vect(X) \) is the exact category of vector bundles on \( X \).

More generally, by [Gra12, Corollary 7.4], we can see that \( K_n(N) \) is a quotient of \( K_1((B^n)^{n-1}N) \). The relations or the kernel of this surjection is generated by the image of \( K_i((B^n)^{n-i-1}C^qN) \) under various \( \Delta \) functors in different directions \( (n - i) \) directions more precisely. We also have a surjection from \( K_i((B^n)^{n-i-1}N) \) to \( K_i((B^n)^{n-i-1}C^qN) \). In order to see the surjection you need to first note that any bounded acyclic chain complex of \( n - 1 \)-dimensional binary complexes can be broken down to the sum of chain complexes that are either of length 2 (short exact sequence) or length 1 (an isomorphism which is the identity map). The length 2 chain complexes can also be broken to the sum of two length 1 chain complexes. The length 1 chain complexes can be identified with elements of \( K_i((B^n)^{n-i-1}N) \). So we have an exact sequence in the following form:

\[
\bigoplus_{i=1}^{n-i} K_i((B^n)^{n-i-1}N) \to K_1((B^n)^{n-i}N) \to K_n(N) \to 0
\]

(2.1)

Since \( X \) is smooth then its algebraic \( K \)-theory space is \( \mathbb{A}^1 \) invariant. In the last portion of this section we aim to prove that the \( K \)-theory space of \( (B^n)^n X) \) is also \( \mathbb{A}^1 \) invariant.

**Proposition 2.5.** The maps between \( K((B^n)^n X) \) and \( K((B^n)^n Vect(X \times \mathbb{A}^1)) \) induced by pullbacks along the zero section and projection are homotopy equivalence. Here \( n \geq 0 \) is an integer and \( X \) is any regular Noetherian scheme.

**Proof.** The aforementioned pullbacks realize \( K_i((B^n)^n X) \) as a direct summand of \( K_i((B^n)^n Vect(X \times \mathbb{A}^1)) \). So we need to prove that the complementary summand is trivial. To do so we induct on \( n \). For the case \( n = 0 \), this follows immediately from the \( \mathbb{A}^1 \) invariance of algebraic \( K \)-theory. Take \( n \) to be \( n + i \) and \( N \) to be \( Vect(X) \) in (2.1). The term on the right hand-side is \( \mathbb{A}^1 \) invariant since it is the algebraic \( K \)-group. The term on the left hand-side is \( \mathbb{A}^1 \) invariant by the induction hypothesis. This implies that the term in the middle is also \( \mathbb{A}^1 \) invariant and finishes the proof of the Proposition. \( \square \)

### 3. Grayson Filtration

We start this section by giving an overview of the Grayson motivic filtration. In [Gra95] Grayson gave filtration of algebraic \( K \)-theory space of a regular affine scheme \( \text{Spec}(R) \). To do so he constructed spaces \( W^i \), for \( i \geq 0 \) an integer, such that:

\[
K(R) = W^0 \leftarrow W^1 \leftarrow W^2 \leftarrow \ldots
\]

(3.1)

He also gave a description of the spaces \( W^i/W^{i+1} \) as a simplicial abelian group, where \( W^i/W^{i+1} \) fits into a fibration in the following form:

\[
W^{i+1} \to W^i \to W^i/W^{i+1}.
\]
The direct sum $K$-theory of an additive category $\mathcal{M}$, denoted by $K^\oplus(\mathcal{M})$ is defined as the $K$-theory space associated to the additive category, where $\mathcal{M}$ is considered to be an exact category such that the short exact sequences are precisely the split exact sequences.

Let $[1]$ denote the ordered set $\{0 < 1\}$ as a category. An object of this category is denoted by $\epsilon$. By an $n$-dimensional cube in a category $\mathcal{C}$ we mean a functor from $[1]^n$ to $\mathcal{C}$. An object of the category $\mathcal{C}$ like $C$ is considered to be a zero dimensional cube and is denoted by $[C]$. If products in the category $\mathcal{C}$ exist we can define the external product of cubes in the following manner: Let $X$ and $Y$ be $n$-dimensional and $n'$-dimensional cubes in $\mathcal{C}$ respectively. $X \boxtimes Y$ denote the $n + n'$-dimensional cube defined by $(X \boxtimes Y)(\epsilon_1, \ldots, \epsilon_{n+n'}) = X(\epsilon_1, \ldots, \epsilon_n) \times Y(\epsilon_{n+1}, \ldots, \epsilon_{n+n'})$. If $\mathcal{M}$ is an exact category and $X$ is a cube of affine schemes, then we denote by $\mathcal{M}(X)$ the cube of exact categories defined by $\mathcal{M}(X)(\epsilon_1, \ldots, \epsilon_n) = \mathcal{M}(X(\epsilon_1, \ldots, \epsilon_n))$. Let $G^A_m$ be the cube of affine schemes constructed by the external product of $t$ copies of $[1 \to G_m]$. This cube gives rise to the cube of exact categories $\mathcal{M}(G^A_m)$. In ([Gra92, Section 4]) there is a construction that turns a cube of exact categories into a multisimplicial exact category. Its $K$-theory is the iterated cofiber of the $K$-theory corresponding to the cube. The spaces $K^\oplus(\mathcal{M}(G^A_m))$ and $K(\mathcal{M}(G^A_m))$ are the cofiber space associated to the direct sum and the exact $K$-theory space, respectively.

One of the major results of the paper is the following Corollary 9.6 [Gra95]:

**Theorem 3.1.** Let $R$ be a simplicial ring, and $\mathcal{M}$ an $R$-linear simplicial additive category. If $R$ is contractible then there is the following filtration:

$$K^\oplus(\mathcal{M}) = W^0 \leftarrow W^1 \leftarrow \ldots$$

with

$$W^t = \Omega^{-t}|d \mapsto K^\oplus(\mathcal{M}_d(G^A_m))|$$

and

$$W^t/W^{t+1} = \Omega^{-t}|d \mapsto K^\oplus_0(\mathcal{M}_d(G^A_m))|.$$ 

Another important result of the paper is the following [Gra95, Theorem 10.5]:

**Theorem 3.2.** Let $R$ be a contractible simplicial ring, and let $\mathcal{M}$ be an $R$-linear exact simplicial category. Then the map $K^\oplus \mathcal{M} \to K\mathcal{M}$ is a homotopy equivalence. 

For our purpose in this paper we will only need the first layer of the filtration i.e. when $t = 1$. Let $k\Delta^\bullet$ be the simplicial ring that at level $d$ it is given by $k[T_0, \ldots, T_d]/(T_0 + \ldots + T_d - 1)$ where $k$ is the base field that our variety is defined on (a finite field in our case). According to ([Gra95, p. 4]) this is a contractible simplicial ring and will play the role of $R$ in Theorem 3.1. Consider the cosimplicial scheme $X \times \text{Spec}(k\Delta^\bullet)$. To this we can assign a simplicial exact category where at level $d$ is given by $(B^d)^n\text{Vect}(X \times \mathbb{A}^d)$ ($n \geq 0$ is an integer). This simplicial exact category will play the role of $\mathcal{M}$ in the Theorems 3.1 and 3.2. Now with the aforementioned substitutions and using the Theorems 3.1 and 3.2 we get the following fibrations:

$$\Omega^{-1}|d \mapsto K((B^d)^n\text{Vect}(X \times \mathbb{A}^d), G^A_m)| \to |d \mapsto K((B^d)^n\text{Vect}(X \times \mathbb{A}^d))| \to |d \mapsto K_0^\oplus((B^d)^n\text{Vect}(X \times \mathbb{A}^d))|.$$
According to Proposition 2.5 we can replace the middle term and re-write the fibration as:

\[
\Omega^{-1}d \mapsto K((B^q)^n Vect(X \times \mathbb{A}^d), \mathbb{G}^\lambda_m)) \rightarrow K((B^q)^n Vect(X)) \rightarrow
\]

\[
|d \mapsto K_0^0((B^q)^n Vect(X \times \mathbb{A}^d))).
\]

(3.3)

Taking the homotopy groups and looking at the end of the long exact sequence gives us the following exact sequence:

\[
\pi_0(|d \mapsto K((B^q)^n Vect(X \times \mathbb{A}^d), \mathbb{G}^\lambda_m))) \rightarrow K_1((B^q)^n Vect(X)) \rightarrow \]

\[
\pi_1(|d \mapsto K_0^0((B^q)^n Vect(X \times \mathbb{A}^d))) \rightarrow 0.
\]

(3.4)

Note that here \( n \geq 0 \). Our goal is to prove Theorem 2.4, hence we want to prove the middle group is torsion. The major result of this section is that we prove the group on the left-hand side is torsion. In the next section we will prove that the group on the right-hand side is trivial and thus finishing the proof of Theorems 2.4 and 1.1.

If \( F \) is a functor from rings to spaces, you can consider the simplicial space \( d \mapsto F(\mathbb{A}^d) \) or more succinctly \( F(\mathbb{A}^\bullet) \). Then the group \( \pi_0(F(\mathbb{A}^\bullet)) \) is the quotient of \( \pi_0(F(\text{Spec}(k))) \) (the is the base field) by equivalence relations generated by the requirement that \( x(0) \sim x(1) \) for every \( x = x(T) \in \pi_0(F(\text{Spec}(k[T]))) = \pi_0(F(\mathbb{A}^1)) \). Using this fact we see \( \pi_0(|d \mapsto K((B^q)^n Vect(X \times \mathbb{A}^d), \mathbb{G}^\lambda_m))) \) is generated by pairs \((b, \Theta)\) such that \( b \) is an object of \((B^q)^n Vect(X)\) and \( \Theta \) is a non-identity automorphism of \( b \). If \( \Theta \) becomes identity it implies that the class of \([b, \Theta]\) is trivial. There are various relations among the generators, some of which come from being in \( K_0((B^q)^n Vect(X), \mathbb{G}^\lambda_m)) \), like additivity on short exact sequences and some come from the homotopy invariance i.e. identifying \( x(0) \) and \( x(1) \) for \( x(T) \) in \( K_0((B^q)^n Vect(X \times \mathbb{A}^1), \mathbb{G}^\lambda_m)) \).

**Proposition 3.3.** For \([b, \Theta_1], [b, \Theta_2] \in \pi_0(|d \mapsto K((B^q)^n Vect(X \times \mathbb{A}^d), \mathbb{G}^\lambda_m)))\), we have:

\[
[b, \Theta_1] + [b, \Theta_2] = [(b, \Theta_1 \circ \Theta_2)]
\]

Proof: The proof is same as the proof that appears in [Gre18, Proposition 3.5.2]. We modify the proof for our setting. The first step is to show

\[
[(b, \Theta)] + [(b, \Theta^{-1})] = 0.
\]

(3.5)

Given endomorphisms \( \Phi_{i,j} \) of \( b \) for \( 1 \leq i, j \leq m \), where \( m \) is an integer. The \( m \times m \) matrix that its \((i, j)\) element is given by \( \Phi_{i,j} \) gives an endomorphism of \( b^{\oplus m} \). Furthermore if this matrix is upper or lower triangular and the diagonal elements are invertible this turns out to be an automorphism. This is even easier to check for \( m = 2 \), which is the case we need. In order to show 3.5 we need to show that \([b \oplus b, \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix}] = 0 \). We denote by \( b[T] \) the pullback of \( b \) to \((B^q)^n Vect(X \times \mathbb{A})\) under the projection. Then the following is an automorphism of \( b[T] \):

\[
\Theta(T) = \begin{pmatrix} 1 & T(\Theta) & -T \Theta^{-1} \\
T \Theta^{-1} & 1 & -T \Theta \\
0 & 1 & 1 \end{pmatrix}
\]

The element \([b[T] \oplus b[T], \Theta(T)] \) is in \( K_0((B^q)^n Vect(X \times \mathbb{A}^1), \mathbb{G}^\lambda_m)) \). By plugging \( T = 0 \) and \( T = 1 \) we get the following equality:

\[
[(b \oplus b, \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix})] = [(b \oplus b, id \oplus id)] = 0.
\]
This proves 3.5. In order to finish the proof of the Proposition it suffices to prove
the following:

\[ [[(b, \Theta_1 \circ \Theta_2)] + [(b, \Theta_1^{-1})] + [(b, \Theta_2^{-1})] = 0 \]

By additivity this is equivalent to:

\[ [(b \oplus 3, \begin{pmatrix} \Theta_1 \circ \Theta_2 & 0 & 0 \\ 0 & \Theta_1^{-1} & 0 \\ 0 & 0 & \Theta_2^{-1} \end{pmatrix})] = 0 \]

The $3 \times 3$ matrix can be factorized as the product of two matrices and this gives us:

\[ [(b \oplus 3, \begin{pmatrix} \Theta_1 \circ \Theta_2 & 0 & 0 \\ 0 & \Theta_1^{-1} & 0 \\ 0 & 0 & \Theta_2^{-1} \end{pmatrix})] = [(b \oplus 3, \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_1^{-1} & 0 \\ 0 & 0 & \Theta_2^{-1} \end{pmatrix})] + [(b \oplus 3, \begin{pmatrix} \Theta_2 & 0 & 0 \\ 0 & 0 & \Theta_1^{-1} \end{pmatrix})] \]

Triviality of each of the summands above can be done in the same way we did
\[ [(b \oplus b, \begin{pmatrix} \Theta & 0 \\ 0 & \Theta^{-1} \end{pmatrix})] = 0. \]

\[ \square \]

**Proposition 3.4.** For any coherent sheaf $E$ on $X$, $\text{Aut}(E)$ is a finite group. Here
$X$ is any projective variety over a finite field.

**Proof.** The endomorphism sheaf $\text{End}(E)$ is also a coherent sheaf on $X$. We can
identify the endomorphisms of $E$ with the global sections $\Gamma(\text{End}(E), X)$. On the
other hand we know the global sections of a coherent sheaf on a projective variety
is a finite dimensional vector space over the global sections of the structure sheaf
[Har77b, Thm III.5.2]. Due to our assumption on $X$ the global sections form a
finite field. This implies that $\text{End}(E)$ is finite dimensional over a finite field, hence
it is finite. \[ \square \]

**Corollary 3.5.** The group $\pi_0([(d \mapsto K((B^n)^n \text{Vect}(X \times \mathbb{A}^d), \mathbb{G}_m^1)])$ is torsion.

**Proof.** This group is generated by the elements $[(b, \Theta)]$. According to Propositions
3.3 and 3.4 these elements are torsion, thus the whole group is torsion. \[ \square \]

4. PROOF OF THE MAIN THEOREM

Our goal in this section is to prove that the group $\pi_1([d \mapsto K_0^0((B^n)^n \text{Vect}(X \times
\mathbb{A}^d))])$ is trivial. By Dold-Kan correspondence, the homotopy groups of a simplicial
abelian group can be expressed as a homology of a certain chain complex. Let $\partial_0$ and
$\partial_1$ be the pullbacks from $(B^n)^n \text{Vect}(X \times \mathbb{A}^1)$ to $(B^n)^n \text{Vect}(X)$ along the sections

\[ T = 0 \text{ and } T = 1 \text{ respectively. Each element in the group can be represented in the form of } [\alpha] - [\beta] \text{ where } \alpha \text{ and } \beta \text{ are objects of the category } (B^n)^n \text{Vect}(X \times \mathbb{A}^1).

The bracket means the class of the object in the group $K_0^0((B^n)^n \text{Vect}(X \times \mathbb{A}^d))$.
Furthermore we have $\partial_i([\alpha]) = \partial_i([\beta])$ for $i = 0, 1$. Let $[\alpha]^*$ and $[\beta]^*$ be the images of $[\alpha]$ and $[\beta]$ in the quotient $K_0((B^n)^n \text{Vect}(X \times \mathbb{A}^d))$ (Note that we removed the
direct sum). Because of $\mathbb{A}^1$ invariance of $K_0$ by 2.5, we have $[\alpha]^* = [\beta]^*$.

**Lemma 4.1** (Heller). Let $\mathcal{N}$ be an exact category and let $J, J', K, K' \in \text{mathcal{N}}$.

Then $[J] - [J'] = [K] - [K'] \in K_0(\mathcal{N})$ if and only if there exists $S, A, B \in \mathcal{N}$ such that the following short exact sequences hold:

\[ A \rightarrow J \oplus K' \rightarrow B, A \rightarrow K \oplus J' \rightarrow B. \]

**Proof.** See [KW20, Lemma 3.2]. \[ \square \]
Now if we apply this Lemma for $N = (B^q)^n \text{Vect}(X \times \mathbb{A}^1)$ we can find an object $\gamma$ such that $\alpha \oplus \gamma$ and $\beta \oplus \gamma$ are extensions of the same two objects. Without loss of generality we can replace $\alpha$ with $\alpha \oplus \gamma$ and $\beta$ with $\beta \oplus \gamma$ and this is not going to change the element $[\alpha] - [\beta]$. So we have the following short exact sequences in $(B^q)^n \text{Vect}(X \times \mathbb{A}^1)$:

$$A \mapsto \alpha \rightarrow B, A \mapsto \beta \rightarrow B.$$ 

Furthermore if the Heller’s lemma is applied to the direct sum $K$-theory, it implies that if two binary complex of vector bundles have the same class in the direct sum $K_0^B$, then you can find an object $\gamma$, that after taking the direct sum with both of them they become isomorphic. This especially implies that the conditions $\partial_i([\alpha]) = \partial_i([\beta])$ for $i = 0, 1$, after adding the same direct summand to both of $\alpha$ and $\beta$ can be re-stated as the following conditions:

$$\alpha|_{T=0} \equiv \beta|_{T=0} \text{ and } \alpha|_{T=1} \equiv \beta|_{T=1}.$$ 

Since $\pi_1([d \mapsto K_0^B((B^q)^n \text{Vect}(X \times \mathbb{A}^d))]) \cong H_1([d \mapsto K_0^B((B^q)^n \text{Vect}(X \times \mathbb{A}^d))])$ (Because the simplicial abelian group is an H-space), we need to show that the element $[\alpha] - [\beta]$ is zero by the relations that come from the homology. The relations that we are interested to use are the following:

For $C \in (B^q)^n \text{Vect}(X \times \mathbb{A}^2) : C|_{S=0} + C|_{T+S=1} + C|_{T=0}$

and $C|_{S=0} + C|_{T=1} + C|_{S=1} + C|_{T=0}$

(4.1)

Here $T$ is parameter corresponding to the first $\mathbb{A}^1$ and $S$ corresponds to the second one. The vertical line means restriction or pullback to various lines. We assume a counter-clockwise orientation on our simplices. It is also important to mention that the second relations which come from a square rather than a triangle, can be deduced from the relations of the triangle type, since each square can be broken into two triangles.

**Definition 4.2.** Given an element in $(B^q)^n \text{Vect}(X \times \mathbb{A}^2)$, it generates two types of relations according to 4.1. We call the first type the *triangular* relations and the second type the *square* relations.

**Definition 4.3.** Given two objects $A$ and $B$ in $(B^q)^n \text{Vect}(X \times \mathbb{A}^1)$, we call them *mirror* of each other iff one becomes isomorphic to the other one by flipping the affine line that switches 0 and 1. We call an object *symmetric* iff it is the mirror of itself. We call an object *extended* iff it is the pullback of an object $A$ in $(B^q)^n \text{Vect}(X)$ under the projection. We denote the extended object from $A$ by $\tilde{A}$. In particular extended objects are symmetric.

**Lemma 4.4.** Given two mirror objects $A$ and $B$ in $(B^q)^n \text{Vect}(X \times \mathbb{A}^1)$, we have $[A] = -[B]$ in $\pi_1([d \mapsto K_0^B((B^q)^n \text{Vect}(X \times \mathbb{A}^d))])$. Specially the image of symmetric objects are rationally trivial and image of extended objects are trivial.

**Proof.** We first prove that for an extended object $\tilde{A}$, $[\tilde{A}]$ is trivial. For this just extend $\tilde{A}$ to $(B^q)^n \text{Vect}(X \times \mathbb{A}^2)$ by pullback along the projection and look at the triangular or square relations. Triangular relations imply that $3[\tilde{A}] = 0$. Square relations imply that $4[\tilde{A}] = 0$, so $[\tilde{A}] = 0$. Now consider the map $\mathbb{A}^2 \to \mathbb{A}^1$ given by the homomorphism $k[T] \to k[T, S]$ that sends $T$ to $T + S$. The pullback along this morphism sends $\tilde{A}$ to an object $A' \in (B^q)^n \text{Vect}(X \times \mathbb{A}^2)$. Consider the triangular relations associated to $A'$. Since $A'|_{S+T=1}$ is extended, this implies that $[A] = -[B]$. \qed
Definition 4.5. For an arbitrary scheme $X$ let’s assume we are given a short exact sequence in $(B^q)^q \text{Vect}(X)$ of the form $0 \to A \to B \to C \to 0$. Then you can pullback this under projection to a short exact sequence in $(B^q)^q \text{Vect}(X \times \mathbb{A}^1)$. We denote it by $0 \to \tilde{A} \to \tilde{B} \to \tilde{C} \to 0$. Let’s $t$ be the parameter corresponding to $\mathbb{A}^1$. You can consider the pushout of the injection $\tilde{A} \to \tilde{B}$ along the map from $\tilde{A}$ to itself which is multiplication by $t$. Because of the properties of exact categories the pushout will give rise to another short exact sequence in $(B^q)^q \text{Vect}(X \times \mathbb{A}^1)$ of the form $0 \to A \to B' \to C \to 0$. By restricting $B'$ to $t = 1$ we get the exact sequence $0 \to A \to B \to C \to 0$ and by restricting to $t = 0$ we get the split exact sequence $0 \to A \to A \oplus C \to C \to 0$. We call this process deforming to the split extension. We refer to $B'$ as the deformation.

Remark 4.6. Now consider the element $[\alpha] - [\beta]$. Both of $\alpha$ and $\beta$ are extensions of $A$ and $B$. We can assume that $A$ and $B$ are non-trivial, because otherwise it implies that $\alpha$ and $\beta$ are extended, thus their class becomes trivial. Now let’s deform both extensions to the split extension. Consider the square relations coming from both of the deformations. It implies the following equality:

$$[\alpha] - [\beta] = [\alpha_1] - [\beta_1] + [\beta_2] - [\alpha_2]$$

where $\alpha_1$, $\beta_1$, $\alpha_2$ and $\beta_2$ are the deformations of $\alpha|_{T=1}$, $\beta|_{T=1}$, $\alpha|_{T=0}$ and $\beta|_{T=0}$ to split extensions, respectively. Here we have used Lemma 4.4 to assign negative signs to some of the summands. This reduces the problem to showing that elements of the form $[\alpha] - [\beta]$, such that both of $\alpha$ and $\beta$ are deformations to the split extensions of two possibly different extensions of $A$, $B \in (B^q)^q \text{Vect}(X)$, are trivial. So without loss of generality we assume that $\alpha$ and $\beta$ are deformations of two extensions in $\text{Ext}^1(B, A)$, for $A, B \in (B^q)^q \text{Vect}(X)$, to the split extensions. Note that there is also an isomorphism $\alpha|_{T=1} \cong \beta|_{T=1}$. (This is because the image of $[\alpha]$ and $[\beta]$ under $\partial_i$s are isomorphic for $i = 0, 1$.) Hence this implies that the two extensions in $\text{Ext}^1(B, A)$ are isomorphic as objects of $(B^q)^q \text{Vect}(X)$ (not necessarily as extensions).

Theorem 4.7. Let’s $X$ be a smooth variety of finite type over a field and $U$ is an open subset such that $\text{codim}(X \setminus U) \geq 2$. Then any vector bundle on $U$ extends uniquely to a coherent sheaf on $X$.

Proof. See [VA.10].

Theorem 4.8 (Horrocks). Let $A$ be a regular, local and Noetherian ring such that $\text{dim} A = 2$. Let $m$ be the maximal ideal of $A$. Denote the punctured spectrum $\text{Spec}(A) \setminus m$ with $Y$. Then any vector bundle on $Y$ is a trivial vector bundle.

Proof. See [Hor64, Corollary 4.1.1].

Theorem 4.9 (Geometric Bass-Quillen). Let $A$ be a regular, finite type algebra over a perfect field. Then any vector bundle on $\text{Spec}(A) \times \mathbb{A}^k$ is an extended vector bundle from $\text{Spec}(A)$.

Proof. This follows from [Lin81].

Corollary 4.10. Let $A$ be regular, finite type algebra over a field. Let $\mathbb{A}^2 \setminus \{(0, 0)\}$ be the punctured plane. Then any vector bundle on $\text{Spec}(A) \times (\mathbb{A}^2 \setminus \{(0, 0)\})$ is extended i.e. it is a pullback of a vector bundle on $\text{Spec}(A)$ under the projection map.
Proof. Let $E$ be a vector bundle on $\text{Spec}(A) \times (\mathbb{A}^2 \setminus \{(0,0)\})$. By Theorem 4.7, this extends uniquely to a coherent sheaf $\tilde{E}$ on $\text{Spec}(A) \times \mathbb{A}^2$. We claim that $\tilde{E}$ is a vector bundle. In order to prove this claim, it suffices to prove that the localization of $\tilde{E}$ at the maximal ideal $m$, corresponding to the closed point $(0,0)$, is free. By Theorem 4.8, restriction of $\tilde{E}$ to $\text{Spec}(A_m) \setminus m$ is free. So its restriction to $\text{Spec}(A_m)$, which is the unique extension from $\text{Spec}(A_m) \setminus m$ (By Theorem 4.7), is also free. This implies our claim. Since $\tilde{E}$ is a vector bundle, by Theorem 4.9, it is extended. This implies that $E$ is also extended. □

Let $A,B,C \in (B^q)^n \text{Vector}(X)$. Assume that there are admissible injections $i_1: A \to C$ and $i_2: B \to C$. Consider the morphism $i_1 \oplus i_2: A \oplus B \to C$, where on each component is the admissible injection. Let’s denote the pullback of this morphism to $X \times \mathbb{A}^2$ by $i_1 \oplus i_2: A \oplus B \to C$. Consider the morphism $f: \tilde{A} \oplus \tilde{B} \to \tilde{C}$ given by the multiplication by $T$ on the first component and $S$ on the second one. We denote the pushout of $i_1 \oplus i_2$ along $f$ by $C_{A,B}$. $C_{A,B}$ is an object in $B^n \text{Coh}(X \times \mathbb{A}^2)$. Note that here $\text{Coh}(X)$ means the category of coherent sheaves on $X$ and we dropped the $q$ superscript from $B$. This means that the binary complex is not necessarily acyclic. In the next proposition we will prove that in fact, $C_{A,B} \in (B^n)^n \text{Vector}(X \times \mathbb{A}^2)$.

Definition 4.11. Let’s $A,B,C \in (B^q)^n \text{Vector}(X)$ with admissible injections from $A$ and $B$ to $C$. We call the pushout construction $C_{A,B}$ above, the double deformation of $C$ with respect to $A$ and $B$.

Remark 4.12. Assume $A_1, A_2, B_1, B_2, C \in (B^q)^n \text{Vector}(X)$, where the following short exact sequences hold:

$$l_1: A_1 \to C \to A_2, \quad l_2: B_1 \to C \to B_2.$$ 

Then $C_{A_1,B_1}|_{T=1}$ is the deformation of $l_2$ to a split exact sequence. $C_{A_1,B_1}|_{S=1}$ is the deformation of $l_1$ to a split exact sequence. $C_{A_1,B_1}|_{T=0} = A_1 \oplus A_2^2$, where $A_1$ is the pullback of $A_1$ to $(B^q)^n \text{Vector}(X \times \mathbb{A})$ and $A_2^2$ is a deformation of $A_2$ with respect to a sub-object, to a split sequence. Note that the latter deformation is happening in $B^n \text{Coh}(X \times \mathbb{A})$ rather than $(B^q)^n \text{Vector}(X \times \mathbb{A})$. The same construction of Definition 4.5, works in the exact category $B^n \text{Coh}(X)$. You can identify the cokernel of the short exact sequence corresponding to $A_2^2$ with the quotient of $C$ by the sub-object generated by the images of both $A_1$ and $B_1$. Similarly you see $C_{A_1,B_1}|_{S=0} = B_1 \oplus B_2^2$.

Remark 4.13. Let’s take a look at $C_{A,B}|_{T \neq 0}$. Since $T$ is invertible in this restriction, it becomes isomorphic to the deformation of $l_2$ to the split exact sequence. Here $l_2$ is the pullback of $l_2$ to $(B^q)^n \text{Vector}(X \times \mathbb{A}^1)$. So $C_{A,B}|_{T \neq 0} \in (B^q)^n \text{Vector}(X \times \mathbb{A}^2)$. We can similarly show that $C_{A,B}|_{S \neq 0} \in (B^q)^n \text{Vector}(X \times \mathbb{A}^2)$. Therefore the restriction of $C_{A,B}$ to the open set where $(T,S) \neq (0,0)$ is already in $(B^q)^n \text{Vector}(X \times \mathbb{A}^2)$.

Proposition 4.14. With the notation and assumptions of the Definition 4.11, we have $C_{A,B} \in (B^n)^n \text{Vector}(X \times \mathbb{A}^2)$.

Proof. By Remark 4.13, the restriction of $C_{A,B}$ to $(T,S) \neq (0,0)$ is already in $(B^q)^n \text{Vector}(X \times \mathbb{A}^2)$. We first prove that $C_{A,B} \in B^n \text{Vector}(X \times \mathbb{A}^2)$. In order to prove that each graded object in $C_{A,B}$ is a vector bundle, it suffices to prove it for affine charts. Consequently we need to consider an affine scheme $\text{Spec}(A) \times \mathbb{A}^2$. Each graded object on $\text{Spec}(A) \times \mathbb{A}^2$ is the unique coherent sheaf that is extended from $\text{Spec}(A) \times (\mathbb{A}^2 \setminus \{(0,0)\})$ according to Theorem 4.7 (Because we know the
restriction of each graded object to \(\text{Spec}(A) \times (\mathbb{A}^2 \setminus \{(0,0)\})\) is a vector bundle by Remark 4.13. By Corollary 4.10, this coherent sheaf is an extended vector bundle. This proves that each graded object is a vector bundle and \(C_{A,B} \in B^n\text{Vect}(X \times \mathbb{A}^2)\).

Now we prove that the binary complex is acyclic. Looking at the differentials of the binary complex at any direction and restricting it to \(\text{Spec}(A) \times (\mathbb{A}^2 \setminus \{(0,0)\})\), we can split the long exact sequences to short split exact sequence of the form \(0 \to Z_1 \to P \to Z_2 \to 0\). Here \(P\) is the restriction of a vector bundle appearing in the binary complex, to the affine chart. \(Z_1\) corresponds to the kernel of the chosen differential and \(Z_2\) is the image. We have \(P \cong Z_1 \oplus Z_2\). This isomorphism extends uniquely to \(\text{Spec}(A) \times \mathbb{A}^2\) by Theorem 4.7 and Corollary 4.10. Therefore the short exact sequence \(0 \to Z_1 \to P \to Z_2 \to 0\) extends uniquely to \(\text{Spec}(A) \times \mathbb{A}^2\). This implies the acyclicity of the binary complex, hence \(C_{A,B} \in (B^n)\text{Vect}(X \times \mathbb{A}^2)\).

\((\text{Hom}(P_1, P_2)\) is a vector bundle for two vector bundles \(P_1\) and \(P_2\) on \(\text{Spec}(A) \times (\mathbb{A}^2 \setminus \{(0,0)\})\); so by Corollary 4.10 for a section of this \(\text{Hom}\) vector bundle if it is extendable to the whole \(\text{Spec}(A) \times \mathbb{A}^2\), the extension has to be unique. Hence it is an easy exercise to see that isomorphisms which are extendable to a morphism, extend to an isomorphism and this implies that the split short exact sequences with extendable morphisms, also extend uniquely to another short split exact sequence.

\[\square\]

**Proposition 4.15.** The group \(\pi_1([d \mapsto K_0^B((B^n)\text{Vect}(X \times \mathbb{A}^2))])\) is trivial.

**Proof.** By Remark 4.6, we need to prove that elements of the form \([\alpha] - [\beta]\), where \(\alpha\) and \(\beta\) are deformations of two elements in \(\text{Ext}^1(B, A)\) to the split exact sequence, are trivial. The extensions in \(\text{Ext}^1(B, A)\) are isomorphic as objects of \((B^n)\text{Vect}(X)\). We denote that object by \(C\). Let’s also denote the short exact sequences corresponding to the extensions by \(l_1\) and \(l_2\). Consider the double deformation \(C_{A,A}\). Writing the square relations for \(C_{A,A}\) and using the notation of the Remark 4.12, we arrive to the following equality:

\[ [\alpha] - [\beta] = -[\overline{A} \oplus B^1] + [\overline{A} \oplus B^2] \]

Note that all the summands above are in \((B^n)\text{Vect}(X \times \mathbb{A}^1)\), by Proposition 4.14. Since \(\overline{A}\) and \(\overline{B}\) are extended, by Lemma 4.4 their class are trivial. So we obtain \([B^2] - [B^1]\). Both of \(B^i\)'s for \(i = 1, 2\) are deformations of short exact sequences of the form \(l'_{1i} : 0 \to M \to B \to N \to 0\) and \(l'_{2i} : 0 \to M' \to B \to N \to 0\) to split exact sequences. Note that the middle and right term of the exact sequences \(l'_{1i}\) and \(l'_{2i}\) coincide and both are in \(B^n\text{Coh}(X)\). By Proposition 4.14, we can see that in fact, \(l'_{2i}\) for \(i = 1, 2\) are in \((B^n)\text{Vect}(X)\). Now consider the double deformation \(B_{M,M'}\). Repeating the same argument for \(B_{M,M'}\), yields an equality of the form \([B^2] - [B^1] = [N^2] - [N^1]\). This will lead to two exact sequences \(l''_{1i}\) and \(l''_{2i}\) where their middle and right term match. So we can repeat the same process. In each stage by Proposition 4.14, we are dealing with bounded binary complexes of vector bundles. Hence at each stage the rank of vector bundles is decreasing. This process has to stop. When it stops, for example let’s say it stopped at \([B^2] - [B^1]\), then either both of \(M\) and \(M'\) should be rank zero vector bundles or \(N\) should be rank zero. This implies that the vector bundles \(N^i\) for \(i = 1, 2\) are both extended, thus their class is zero. This finishes the proof.  \(\square\)
Proof of the Parshin’s Conjecture. Considering the exact sequence 3.4, Corollary 3.5 and Proposition 4.15, we deduce that $K_1((B^q)^n\text{Vect}(X))$ is rationally trivial. Hence implying Theorem 2.4 and subsequently Theorem 1.1. □

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