The Podolsky Propagator as a Tool for Hadron Physics

Bruno El-Bennich,1,* G. E. R. Zambrano,2,† and Eduardo Rojas2,‡

1Laboratório de Física Teórica e Computacional, Universidade Cidade de São Paulo,
Rua Galvão Bueno 868, 01506-000, São Paulo, SP, Brazil
2Departamento de Física, Universidad de Nariño, A.A. 1175, San Juan de Pasto, Colombia

Based on the Generalized Quantum Electrodynamics expressions for the Podolsky propagator, which preserves gauge invariance, we propose a model for the massive gluon propagator that reproduces well-known features of established strong-interaction models in the framework of the Dyson-Schwinger equation. By adjusting the Podolsky mass and the coupling strength we thus construct a model with simple analytical properties known from perturbative theory, yet well suited to describe a confining interaction. We obtain solutions of the Dyson-Schwinger equation for the quark at space-like momenta on the real axis as well as on the complex plane and solving the bound-state problem with the Bethe-Salpeter equation yields masses and weak decay constants of the π, K and ηc in excellent agreement with experimental values, while the D and Ds are reasonably well described. The analytical simplicity of this effective interaction has the potential to be useful for phenomenological applications and may facilitate calculations in Minkowski space.

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I. INTRODUCTION

Mandelstam’s seminal work [1] established that the rainbow-ladder truncation of the gap and bound-state equations is an adequate approximation to describe dynamical chiral symmetry breaking (DCSB) [2–4] in Quantum Chromodynamics (QCD). Later on, within this same truncation scheme, Munczek and Nemirovsky succeeded in reproducing the masses of pseudoscalar and vector meson ground states [5]. More sophisticated models followed in the following decades that satisfy theoretical and phenomenological constraints [6–15]. Their popularity owes to a wide range of successful applications to mesons, baryons, hyperons, their excited states and parity partners [12–38].

In this work we propose an interaction model that describes effectively the dressing functions of the gluon and quark-gluon vertex in the infrared region, where we are inspired by the functional structure of Generalized Quantum Electrodynamics (GQED) proposed long ago by Podolsky [39, 40]. Historically, this generalization aimed at remedying pathologies inherent to the Maxwell theory and consisted in introducing higher-order derivatives in the Lagrangian of electrodynamics, maintaining at the same time linearity of the equations of motion in the fields. In other words, the goal was to eliminate the infinities that arise in higher-order corrections of point charges and the associated coupling.

However, since this extension of the Lagrangian preserves gauge invariance in a consistent treatment [41, 42], GQED has come to be viewed more as a prototype of a theory that contains massless as well as massive photons that do not break gauge invariance. This is because Podolsky’s extension of electrodynamics is the only possible linear, Lorentz and U(1) invariant generalization of the Maxwell theory [43] and a consistent quantization of GQED was shown to require a generalized Landau gauge condition [42] while the proper covariant quantization of GQED in this generalized gauge was obtained with functional methods in Ref. [44]. More recently, the Podolsky approach to QED was also reinterpreted as a natural way of providing a Pauli-Villars regularization in ordinary QED [45]. GQED introduces therefore in a consistent manner a mass parameter \( m_P \) in the vector-boson propagator while preserving gauge invariance and acting as an effective ultraviolet cutoff in Landau gauge.

These features are clearly attractive for modeling the quark-gluon interaction in an Abelianized truncation, given the compelling body of work that evidence an infrared-finite gluon propagator [46–55] and which can be related to an effective gluon mass. It turns out that in Landau gauge and in the leading truncation of the quark’s Dyson-Schwinger equation (DSE) we may interpret the Podolsky propagator as a nonperturbative model for the gluon propagator, at least in the low-momentum region, where its massive “dressing functions” function also effectively accounts for the quark-gluon vertex dressing.

*Electronic address: bruno.bennich@cruzeiroidosul.edu.br
†Electronic address: gramoszge@gmail.com
‡Electronic address: eduro4000@gmail.com
In Section II we introduce the DSE that describes the quark-gap equation with a Podolsky propagator in Landau gauge and obtain its solutions for different flavors on the space-like real axis. The functional behavior of the quark’s mass and wave-renormalization function is very much reminiscent of that found with the Maris-Tandy [14] or Qin-Chang [15] models and the obvious question arises whether this interaction is useful for hadron phenomenology. We solve the Bethe-Salpeter equation (BSE) as usual in Euclidean space to find antiquark-quark bound states, which implies that the arguments of the quark propagators are complex-valued momenta. To obtain the quark propagators on the complex plane, we apply Cauchy’s integral theorem that requires DSE solutions on a contour defined by a parabola describing the complex-momentum distribution [56, 57]. We note that the convergence of the DSE on such a contour is not generally guaranteed for a given interaction regardless of its convergence on the real axis. Nonetheless, using the Podolsky propagator with an appropriate parameterizations the DSE converges rapidly on this contour and the BSE solutions reproduce the mass spectrum and weak decay constants of pseudoscalar mesons as we discuss in Section III. We finish with some concluding remarks about possible extensions and applications of the Podolsky propagator in Section IV.

II. DYSON-SCHWINGER EQUATION

The gap equation for a quark of flavor \( f \) can be expressed by a DSE for the inverse propagator in Minkowski space as,

\[
S_f^{-1}(p) = Z_2 \gamma \cdot p - Z_4 m_f(\mu) - Z_1 g^2 \int_\Lambda d^4k \, G^{\mu\nu}(q) \frac{\lambda^a_\mu}{2} \gamma_\mu S_f(k) \frac{\lambda^a_\nu}{2} \Gamma(k, p) ,
\]

where \( Z_2(\mu, \Lambda), Z_4(\mu, \Lambda) \) and \( Z_1(\mu, \Lambda) \) are the wave-function, mass and vertex renormalization constants, respectively. Moreover, \( \Gamma^a(k, p) = \frac{1}{2} \lambda^a \Gamma_\mu(k, p) \) is the quark-gluon vertex and \( \lambda^a \) are the SU(3) color matrices in the fundamental representation, while \( \Lambda \) is a Poincaré-invariant regularization scale, chosen such that \( \Lambda \gg \mu \). In GQED the vector-boson propagator in a covariant gauge with gauge fixing parameter \( \xi \) and momentum, \( q = p - k \), is given by,

\[
G_{\mu\nu} = -i \Delta(q^2) P_{\mu\nu}(q) ,
\]

\[
P_{\mu\nu}(q) = \Delta_{\mu\nu}(q) - \left[g_{\mu\nu} + (1 - \xi) \frac{\eta_{\mu\nu}}{q^2 - m^2_p}\right] \frac{1}{q^2 - m^2_p} + (1 - 2\xi) \frac{\eta_{\mu\nu}}{q^2} \frac{q_{\mu}q_{\nu}}{q^2 - m^2_p} + \frac{\eta_{\mu\nu}}{(q^2 - m^2_p)^2} ,
\]

with the standard gauge-boson propagator,

\[
\Delta_{\mu\nu} = \left[g_{\mu\nu} - (1 - \xi) \frac{\eta_{\mu\nu}}{q^2}\right] \frac{1}{q^2} .
\]

In general, the Dirac structure of the fermion propagator is fully defined by two covariants and associated scalar functions, the wave-function renormalization \( F_f(p) \) and the mass function \( M_f(p) \), so that,

\[
S_f(p) = \frac{F_f(p)}{\gamma \cdot p - M_f(p)} .
\]

In order to determine the renormalization constants and to make quantitative matching with pQCD, one imposes the renormalization conditions,

\[
F_f(p^2)|_{p^2 = \mu^2} = 1 , \quad S_f^{-1}(p)|_{p^2 = \mu^2} = \gamma \cdot p - m_f(\mu) ,
\]

where \( \mu^2 \gg \Lambda^2_{\text{QCD}} \) and \( m_f(\mu) \) is the renormalized running quark mass; in particular, \( m_f(\mu) \) is nothing else but the dressed-quark mass function evaluated at one particular deep space-like point, \( p^2 = \mu^2 \), namely \( m_f(\mu) = M_f(\mu) \).

The mass function \( M_f(p^2) \), and the the renormalization wave function \( F_f \) can be projected out from the DSE (1) and rewritten in Euclidean space one obtains\(^1\) in Landau gauge the two coupled, nonlinear integral equations,

\[
\frac{M_f(p_E)}{F_f(p_E)} = Z_4 m(\mu) + 3C_F \int \frac{d^4k_E \, M_f(k_E) F_f(k_E) \mathcal{G}(q_E^2)}{2 \pi^4 k^2_E + M_f^2(k_E)} \frac{G(q_E^2)}{q_E} ,
\]

\[
\frac{1}{F_f(p_E)} = Z_2 + \frac{C_F}{p^2_E} \int \frac{d^4k_E \, F_f(k_E) F_f(k_E)}{2 \pi^4 k^2_E + M_f^2(k_E)} \left[ 3(p_E \cdot k_E) + \frac{2[(p_E \cdot k_E)^2 - p^2_E k^2_E]}{q^2_E} \right] \frac{G(q_E^2)}{q_E} .
\]

\(^1\) See Appendix A for details of the calculation in arbitrary covariant gauge.
where,

\[ G(q_E^2) = 4\pi\alpha_{\text{eff}} \frac{m_P^2}{q_E^2 + m_P^2} \]  

where we define \( \alpha_{\text{eff}} = Z_1 q^2/4\pi = 7.69 \) and \( m_P^2 = 0.6 \text{ GeV}^2 \). These values have been chosen to achieve the light-quark mass function \( M_f \) and wave function \( F_f \) as known from phenomenological models [14, 15] and which yield the pion’s mass and weak decay constant \( f_\pi \). In our approach we choose \( \Delta(q^2) = 1 \), that is the perturbative value, such that the running gluon-dressing function is defined implicitly by Eq. (8) and thus parameterized by \( m_P \) and \( g^2 \).

We note that the value of the Podolsky mass, \( m_P \approx 0.77 \text{ GeV} \), which serves as an infrared mass scale in our model, agrees with other gluon mass scales [58, 59], whereas \( \alpha_{\text{eff}} \) can obviously not be directly compared with the running strong coupling at this scale. This is because we merely employ the rainbow truncation of the DSE in which \( \alpha_{\text{eff}} \) partially accounts for the lack of DCSB from a fully dressed quark-gluon vertex [60, 61]. We also stress that it is possible to modify the interaction (8) to include the perturbative \( \sim 1/q^2 \) running at larger momenta. However, our aim here merely consists in verifying the simple expression’s (8) capacity to yield the adequate DCSB observed in hadron phenomenology.

III. PSEUDOSCALAR MESONS MASSES AND DECAY CONSTANTS

The homogeneous BSE for a \( \bar{q}q \) bound state with relative momentum \( p \) and total momentum \( P \) can be written as,

\[ \Gamma_{mn}^f(p, P) = \int d^4k E \frac{k_{mn}(p, k, P)}{(2\pi)^4} K_{mn}^{kl}(p, k, P) [S_f(k_+)] \Gamma^{fg}(k, P) S_g(k_-) ]_{lk}, \]  

where \( m, n, k, l \) collect Dirac and color indices, \( f, g \) are flavor indices and \( k_+ = k + \eta_+ P, k_- = k - \eta_- P; \eta_+ + \eta_- = 1 \).

Since we work within the rainbow-ladder truncation, the BSE kernel is given by,

\[ K_{mn}^{kl}(p, k, P) = -\frac{G(q^2)}{q^2} \left( \frac{\lambda^a}{2} \gamma_\mu \right)_{mn} P_{\mu\nu}(q) \left( \frac{\lambda^a}{2} \gamma_\nu \right)_{kl}, \]

which satisfies the axial-vector WTI [13] and as a consequence ensures a massless pion in the chiral limit. As such, Eqs. (9) and (10) define an eigenvalue problem with physical solutions at the on-shell points, \( P^2 = -M^2 \), where \( M \) is the bound-state mass of the \( \bar{q}q \) pair. As in the DSE (1), the vertex renormalization is absorbed in \( \alpha_{\text{eff}} \).

The general Poincaré-invariant form of the Bethe-Salpeter amplitudes (BSA), i.e. the solutions of Eq. (9), for the pseudoscalar channel \( J^P = 0^- \) in a nonorthogonal base with respect to the Dirac trace, \( A^0(p, P) = \gamma_5 \{ i D, \gamma \cdot P, \gamma \cdot p (\cdot P), \sigma_{\mu\nu} p_{\mu} P_{\nu} \} \), is given by,

\[ \Gamma(p, P) = \gamma_5 \left[ i D E(p, P) + \gamma \cdot P F(p, P) + \gamma \cdot p P G(p, P) + \sigma_{\mu\nu} p_{\mu} P_{\nu} H(p, P) \right], \]

Figure 1: Mass function, \( M_f(p^2) \), and wave function renormalization, \( F_f(p^2) \), solutions for the light, strange and charm quarks obtained with Eqs. (6) and (7) using the gluon-interaction model (8) renormalized at \( \mu = 19 \text{ GeV} \).
where we suppress color, Dirac and flavor indices for the sake of readability and $\sigma_{\mu\nu} = i/2 [\gamma_{\mu}, \gamma_{\nu}]$. The functions $\mathcal{F}_J(p, P) = \{ E(p, P), F(p, P), G(p, P), H(p, P) \}$ are Lorentz-invariant scalar amplitudes. For sake of completeness, we note that all BSA are normalized canonically as,

$$
2P_\mu = \int^\Lambda d^4k_{E} \left( \frac{1}{2\pi} \right)^4 \text{Tr}_{\text{CD}} \left[ \Gamma(k, -P) \frac{\partial S(k_+)\partial S(k_-)}{\partial P_\mu} + \frac{\partial S(k_-)}{\partial P_\mu} \right],
$$

(12)

where we omit a third term that stems from the derivative of the kernel, $\partial k_{mn}^{\mu}(p, k, P)/\partial P_\mu$, since it does not contribute in the rainbow-ladder truncation of Eq. (10)$^2$. In Eq. (12), the charge-conjugated BSA is defined as $
abla(k_1, P) := C \nabla(-k_1, -P) C^T$, where $C$ is the charge conjugation operator and the trace is over Dirac and color indices. With this normalization we obtain the meson’s leptonic decay constants via the integral,

$$
f_{P}P_\mu = \int^\Lambda d^4k_{E} \left( \frac{1}{2\pi} \right)^4 \text{Tr}_{\text{CD}} [\gamma_5 \gamma_\mu S(k_+)\Gamma(k, P)S(k_-)].
$$

(13)

In Fig. 1, the mass functions and the wave-function renormalization solving Eq. (1) with the Podolsky propagator and the model in Eq. (8) are plotted, where the interaction parameters $\alpha_{\text{eff}}$ and $M_P$ as well as the quark masses have been adjusted to reproduce the experimental mass and decay constant values. Both functions compare well with those obtained with established interaction models [14, 15] or with more sophisticated ansätze for the quark-gluon vertex using Lattice-QCD dressing functions [64]. In this context, we remind that one can infer qualitative and analytic properties of the interaction kernel from the mass and wave-renormalization functions via an inversion process of the DSE [64, 65] which allows for the comparison of different models. The BSE (9) is calculated in Euclidean space and therefore the momenta $k_+$ and $k_-$ of the quark propagators are complex valued. We follow the numerical prescriptions introduced in Ref. [56] and refined in Refs. [19, 57] in computing solutions of the DSE in a parabola on the complex plane.

The potential of this effective interaction is illustrated for the pseudoscalar $q\bar{q}$ channel in Tab. I. We work in the isospin-symmetric limit $m_u = m_d$ and set the light quark’s mass scale at 19 GeV with the pion mass; analogously we fix the strange- and charm-quark masses with the kaon and the $\eta_c$. The resulting weak decay constant of the pion is within 2% of the experimentally extracted value and that of the kaon is within 5% of its reference value. The charmonium’s decay constant is about 15% larger than a calculation using lattice QCD. We also compare the weak decay constants obtained with Eq. (13) with those making use of the Gell-Mann-Oaks-Renner (GMOR) relation described in detail, for example, in Ref. [19] and find very good agreement.

\[\text{Table I: Mass spectrum and decay constants for flavor singlet and non-singlet} J^P = 0^- \text{ mesons in GeV. We adjusted the current masses to} m_u = m_d = 2.65 \text{ MeV,} m_s = 70 \text{ MeV and} m_c = 865 \text{ MeV in order to reproduce the ground-state masses of the} \pi, K \text{ and} \eta \text{ mesons, respectively. The weak decay constant is obtained with Eq. (13) and the appropriate GMOR relation.}\]

\[
\begin{array}{|c|c|c|}
\hline
\text{Meson} & \text{Podolsky Model} & \text{Reference} \\
\hline
m_\pi & 0.138 & 0.139 [66] \\
f_\pi & 0.133 & 0.130 [66] \\
f_{\pi\text{GMOR}} & 0.117 & \\
m_K & 0.494 & 0.493 [66] \\
f_K & 0.164 & 0.156 [66] \\
f_{K\text{GMOR}} & 0.162 & \\
m_{\eta_c(1S)} & 2.985 & 2.984 [66] \\
f_{\eta_c(1S)} & 0.454 & 0.395 [67] \\
f_{\eta_c(1S)\text{GMOR}} & 0.451 & \\
m_{\eta_c(2S)} & 2.100 & 1.870 [66] \\
f_{\eta_c(2S)} & 0.263 & 0.212 [68] \\
f_{\eta_c(2S)\text{GMOR}} & 0.304 & 0.250 [68] \\
\hline
\end{array}
\]

\[\text{We verify the values obtained with Eq. (12) with the equivalent normalization condition [62, 63]:} \ (d\ln\lambda_n/dP^2)^{-1} = \text{tr} \int^\Lambda d^4k_{E} 3\Gamma(k, -P)S(k_+)\Gamma(k, P)S(k_+). \]

$^2$ We verify the values obtained with Eq. (12) with the equivalent normalization condition [62, 63]: $(d\ln\lambda_n/dP^2)^{-1} = \text{tr} \int^\Lambda d^4k_{E} 3\Gamma(k, -P)S(k_+)\Gamma(k, P)S(k_+)$. 
This is a first step to establish the model’s potential to correctly describe light-meson and quarkonia ground states and as additional check we calculate the $D$ and $D_s$ meson’s masses and decay constants. As observed in Tab. I and previously in Ref. [19], their masses are overestimated by 12% for the $D$ and 8% for the $D_s$ with respect to experimental values. This is a consequence of the rainbow-ladder truncation which neglects the dramatically different impact of vertex dressing for heavy and light quarks and can be strongly improved by inclusion of this effect [25]. Likewise, the weak decay constants are also 22% larger than results reported by the FLAG collaboration [68]. Nonetheless, the Podolsky propagator facilitates the numerical treatment of the DSE with a large external Euclidean mass (using $\eta_+ = 0.8$ and $\eta_- = 0.2$) on the complex plane in comparison with the model of Ref. [15] and both the iterative treatments of the DSE and BSE mesons converge rapidly in case of the $D$ and $D_s$.

IV. SUMMARY AND CONCLUSIONS

We propose a novel interaction model for the nonperturbative quark-gluon interaction within the rainbow-ladder truncation of the DSE and BSE kernels. This model is based on the Podolsky propagator in GQED in which it preserves gauge invariance, though in our approach this massive propagator accounts for the non-perturbative dressing functions of the quark-gluon vertex and the gluon propagator. The associated mass scale we find is reminiscent of earlier DSE studies of the gluon. The Podolsky propagator has the same integrability properties as the perturbative propagator, a feature which is the practical motivation for our model, and when employed in the appropriate DSE we find well behaved solutions on the complex plane even for larger time-like momenta. Employing these complex solutions for the quark propagators in the BSE, we fix the quark masses and interaction parameters with the masses of the $\pi$, $K$, $\eta_c$ and find weak decay constants that agree very well with experimental reference values or results from lattice-QCD simulations. The $D$ mesons are also obtained within this framework and their masses are somewhat overestimated and the mass difference between the $D$ and $D_s$ is too small, a consequence of the too simplistic truncation for heavy-light systems.

Obviously, the Podolsky propagator along with the ansatz in Eq. (8) ought to be used in future calculations of radial excitations and other $J^{PC}$ channels. Since we are interested in static properties of the mesons, we here omit the perturbative term of the interaction commonly included in other models. As a consequence of this simple form, we established that the angular integration of the DSE, Eqs. (A11) and (A12), can be performed analytically. Likewise, the simple pole structure of the Podolsky propagator allows, in given cases, for analytical calculations as with any perturbative propagator. These features are attractive enough to rise one’s curiosity about possible solutions of the DSE and BSE in Minkowski space, which is of practical importance with regard to the study of parton distribution functions defined on the light front. Amongst other applications, the exploration of the QCD phase diagram in the strong coupling regime could be considered, replacing simple BSA models and quark propagators at finite density to understand the pion’s properties in a dense nuclear medium [69]. As we work in an Abelianized approximation of QCD, one may wonder what the actual Podolsky version of the strong-interaction SU(3) Lagrangian would be like.

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Appendix A: Dyson-Schwinger Equation with the Podolsky Propagator

For a quark propagator, the DSE in Minkowski space is described by a nonlinear integral equation,

$$S^{-1}(p) = Z_2 \gamma \cdot p - Z_4 m(\mu) + iZ_1 g^2 C_F \int \frac{d^4 k}{(2\pi)^4} \gamma_\mu S(k) \Gamma_\nu(k,p) G^{\mu\nu}(q),$$  \hspace{1cm} (A1)
with $C_F = 4/3$ in the fundamental representation of SU(3). The GQED gauge propagator in a covariant gauge, specified by the gauge parameter $\xi$, is given by,

$$G_{\mu\nu} = \Delta(q^2)P_{\mu\nu}(q), \quad (A2)$$

with $q = k - p$ and where,

$$P_{\mu\nu}(q) = \Delta_{\mu\nu}(q) - \left[ g_{\mu\nu} + (1 - \xi) \frac{q\mu q\nu}{q^2 - m^2_F} \right] \frac{1}{q^2 - m^2_F} + (1 - 2\xi) \frac{q\mu q\nu}{q^2 (q^2 - m^2_F)} + \frac{q\mu q\nu}{(q^2 - m^2_F)^2}. \quad (A3)$$

Here, $\Delta_{\mu\nu}(q)$ is the contribution of the Maxwell theory:

$$\Delta_{\mu\nu}(q) = \frac{1}{q^2} \left( g_{\mu\nu} - \frac{q\mu q\nu}{q^2} \right) + \xi \frac{q\mu q\nu}{q^4}. \quad (A4)$$

We employ for the vertex structure its bare form, which is the rainbow truncation:

$$\Gamma_\mu \rightarrow \gamma_\mu. \quad (A5)$$

In general, the Dirac structure of the fermion propagator depends on two independent functions, the wave function renormalization $F(p)$ and the mass function $M_F(p)$, such that:

$$S(p) = \frac{F(p)}{\gamma \cdot p - M(p)}. \quad (A6)$$

With this, expression (A1) can be rewritten as:

$$\gamma \cdot p - M(p) \over F(p) = Z_2 \gamma \cdot p - Z_4 m(\mu) + i Z_1 g^2 C_F \int \frac{d^4k}{(2\pi)^4} F(k) \gamma_\mu \gamma_\nu \frac{\gamma \cdot k + M(k)}{k^2 - M^2(k)} \Delta_{\mu\nu}(q) \gamma_\nu \cdot. \quad (A7)$$

Taking the trace of Eq. (A7) results in the expression,

$$\frac{\mathcal{M}(p)}{F(p)} = Z_4 m(\mu) - i Z_1 g^2 C_F \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{M}(k) F(k)}{k^2 - M^2(k)} \Delta_{\mu\nu}(q) \gamma_\nu. \quad (A8)$$

where $G^\mu_\nu(q) = \Delta(q^2)P^\mu_\nu(q)$ and,

$$P^\mu_\mu(q) = \Delta^\mu_\mu(q) - \left[ g^\mu_\mu + (1 - \xi) \frac{q^\mu q_\mu}{q^2 - m^2_F} \right] \frac{1}{q^2 - m^2_F} + (1 - 2\xi) \frac{q^\mu q_\mu}{q^2 (q^2 - m^2_F)} + \frac{q^\mu q_\mu}{(q^2 - m^2_F)^2}$$

$$= \Delta^\mu_\mu(q) - (3 + 2\xi) \frac{1}{q^2 - m^2_F} + \xi \frac{q^2}{(q^2 - m^2_F)^2}, \quad (A9)$$

with,

$$\Delta^\mu_\mu(q) = \frac{1}{q^2} \left( g^\mu_\mu - \frac{q^\mu q_\mu}{q^2} \right) + \xi \frac{q^\mu q_\mu}{q^4} = (3 + \xi) \frac{1}{q^2}. \quad \text{it follows that,}$$

$$\frac{\mathcal{M}(p)}{F(p)} = Z_4 m(\mu) - i Z_1 g^2 C_F \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{M}(k) F(k)}{k^2 - M^2(k)} \Delta(q^2) \left[ \Delta^\mu_\mu(q) - (3 + 2\xi) \frac{1}{q^2 - m^2_F} + \xi \frac{q^2}{(q^2 - m^2_F)^2} \right] \quad (A10)$$

where the two terms are,

$$\left. \frac{\mathcal{M}(p)}{F(p)} \right|_M = Z_4 m(\mu) - i Z_1 g^2 C_F \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{M}(k) F(k)}{k^2 - M^2(k)} \Delta(q^2) \Delta^\mu_\mu(q) \quad \text{and}$$

$$\left. \frac{\mathcal{M}(p)}{F(p)} \right|_\gamma = Z_4 m(\mu) - i Z_1 g^2 C_F (3 + \xi) \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{M}(k) F(k)}{k^2 - M^2(k)} \Delta(q^2) \gamma_\mu. \quad (A11)$$
which is the Maxwell contribution and,
\[
\frac{\mathcal{M}(p)}{\mathcal{F}(p)}\bigg|_p = (3 + 2\xi)iZ_1g^2C_F \left( \frac{d^4k}{(2\pi)^4} \frac{\mathcal{M}(k)\mathcal{F}(k)}{k^2 - M^2(k)} \Delta(q^2) - \frac{iZ_1g^2C_F\xi}{(2\pi)^4} \frac{d^4k}{k^2 - M^2(k)} \frac{q^2\Delta(q^2)}{(q^2 - m_p^2)^2} \right),
\]
the Podolsky contribution is the Maxwell contribution and,
\[
\frac{p^2}{\mathcal{F}(p)} = Z_2p^2 + iZ_1g^2C_F \int \frac{d^4k}{(2\pi)^4} \mathcal{F}(k) \frac{2p_{\mu}k_{\nu}P^{\mu\nu}(q) - p \cdot k P^{\mu}(q)}{k^2 - M^2(k)} \Delta(q^2),
\]
where the expression in the numerator is found to be:
\[
2p_{\mu}k_{\nu}P^{\mu\nu}(q) - p \cdot k P^{\mu}(q) = \left[ (\xi - 3) \frac{(p \cdot k)}{q^2} + 2(1 - \xi - 1) \frac{1}{q^4} [(p \cdot k)^2 - p^2k^2] \right]
+ \left[ (3 - 2\xi)(p \cdot k) + 2(1 - \xi) \frac{(p \cdot k)^2 - p^2k^2}{q^2} \right] \frac{1}{q^2 - m_p^2}
+ \left[ \xi (p \cdot k) q^2 + 2\xi \left( (p \cdot k)^2 - p^2k^2 \right) \right] \frac{1}{(q^2 - m_p^2)^2}.
\]
We can again separate the integral into a Maxwell and Podolsky contribution,
\[
\frac{1}{\mathcal{F}(p)} = \frac{1}{\mathcal{F}(p)} \bigg|_M + \frac{1}{\mathcal{F}(p)} \bigg|_P,
\]
where,
\[
\left. \frac{1}{\mathcal{F}(p)} \right|_M = Z_2 + \frac{iZ_1g^2C_F}{p^2} \int \frac{d^4k}{(2\pi)^4} \mathcal{F}(k) \Delta(q^2) \left[ (\xi - 3) \frac{(p \cdot k)}{q^2} + 2(1 - \xi - 1) \frac{1}{q^4} [(p \cdot k)^2 - p^2k^2] \right],
\]
and
\[
\left. \frac{1}{\mathcal{F}(p)} \right|_P = \frac{iZ_1g^2C_F}{p^2} \int \frac{d^4k}{(2\pi)^4} \mathcal{F}(k) \Delta(q^2) \left[ (3 - 2\xi)p \cdot k + 2(1 - \xi) \frac{(p \cdot k)^2 - p^2k^2}{q^2} \right] \frac{1}{q^2 - m_p^2}
+ \frac{iZ_1g^2C_F}{p^2} \xi \int \frac{d^4k}{(2\pi)^4} \mathcal{F}(k) \Delta(q^2) \left[ (p \cdot k) q^2 + 2 \left( (p \cdot k)^2 - p^2k^2 \right) \right] \frac{1}{(q^2 - m_p^2)^2}.
\]

1. Landau Gauge

In Landau gauge, Eq. (A10) takes the form,
\[
\left. \frac{\mathcal{M}(p)}{\mathcal{F}(p)} \right|_M = Z_4m(\mu) - 3iZ_1g^2C_F \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{M}(k)\mathcal{F}(k)}{k^2 - M^2(k)} \Delta(q^2),
\]
\[
\left. \frac{\mathcal{M}(p)}{\mathcal{F}(p)} \right|_P = 3iZ_1g^2C_F \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{M}(k)\mathcal{F}(k)}{k^2 - M^2(k)} \Delta(q^2),
\]
If we apply a Wick rotation we obtain the Euclidean space expressions for these integrals,
\[
\left. \frac{\mathcal{M}(p_E)}{\mathcal{F}(p_E)} \right|_M = Z_4m(\mu) + 3Z_1g^2C_F \int \frac{d^4k_E}{(2\pi)^4} \frac{\mathcal{M}(k_E)\mathcal{F}(k_E)}{k_E^2 + M^2(k_E)} \Delta(q_E^2),
\]
\[
\left. \frac{\mathcal{M}(p_E)}{\mathcal{F}(p_E)} \right|_P = -3Z_1g^2C_F \int \frac{d^4k_E}{(2\pi)^4} \frac{\mathcal{M}(k_E)\mathcal{F}(k_E)}{k_E^2 + M^2(k_E)} \Delta(q_E^2),
\]
and we obtain explicitly,
\[
\frac{\mathcal{M}_f(p_E)}{\mathcal{F}(p_E)} = Z_4m(\mu) + 3Z_1g^2C_F \int \frac{d^4k_E}{(2\pi)^4} \frac{\mathcal{M}(k_E)\mathcal{F}(k_E)}{k_E^2 + M^2(k_E)} \left( \frac{1}{q_E^2} - \frac{1}{q_E^2 + m_p^2} \right) \Delta(q_E^2).
\]
Similarly, Eq. (A14) becomes in Landau gauge,
\[
\frac{1}{F(p)} \bigg|_M = Z_2 - \frac{iZ_1 g^2 C_F}{p^2} \int \frac{d^4k}{(2\pi)^4} \frac{F(k)}{k^2 - M^2(k)} \left[ 3p \cdot k + 2 \frac{(p \cdot k)^2 - p^2 k^2}{q^2} \right] \frac{\Delta(q^2)}{q^2}, \tag{A22}
\]
\[
\frac{1}{F(p)} \bigg|_P = \frac{iZ_1 g^2 C_F}{p^2} \int \frac{d^4k}{(2\pi)^4} \frac{F(k)}{k^2 - M^2(k)} \left[ 3p \cdot k + 2 \frac{(p \cdot k)^2 - p^2 k^2}{q^2} \right] \frac{\Delta(q^2)}{q^2 - m_p^2}. \tag{A23}
\]
In Euclidean space these integrals are given by,
\[
\frac{1}{F(pE)} \bigg|_M = Z_2 + \frac{Z_1 g^2 C_F}{p_E^2} \int \frac{d^4k_E}{(2\pi)^4} \frac{F(k_E)}{k_E^2 + M^2(k_E)} \left[ 3p_E \cdot k_E + 2 \frac{(p_E \cdot k_E)^2 - p_E^2 k_E^2}{q_E^2} \right] \frac{\Delta(q_E^2)}{q_E^2}, \tag{A24}
\]
\[
\frac{1}{F(pE)} \bigg|_P = -\frac{Z_1 g^2 C_F}{p_E^2} \int \frac{d^4k_E}{(2\pi)^4} \frac{F(k_E)}{k_E^2 + M^2(k_E)} \left[ 3p_E \cdot k_E + 2 \frac{(p_E \cdot k_E)^2 - p_E^2 k_E^2}{q_E^2} \right] \frac{\Delta(q_E^2)}{q_E^2 + m_p^2}. \tag{A25}
\]
Adding the two contributions we finally arrive at the Euclidean-space integral equation,
\[
\frac{1}{F(pE)} = Z_2 + \frac{Z_1 g^2 C_F}{p_E^2} \int \frac{d^4k_E}{(2\pi)^4} \frac{F(k_E)}{k_E^2 + M^2(k_E)} \left[ 3p_E \cdot k_E + 2 \frac{(p_E \cdot k_E)^2 - p_E^2 k_E^2}{q_E^2} \right] \left[ \frac{1}{q_E^2} - \frac{1}{q_E^2 + m_p^2} \right] \Delta(q_E^2) \tag{A26}.
\]
It turns out that, the angular integration of Eqs. (A21) and (A26) can be performed analytically using,
\[
(p_E \cdot k_E)^2 - p_E^2 k_E^2 = -p_E^2 k_E^2 \sin^2 \theta_1,
\]
and working in the rest frame where \( p_E^\mu = (p_E, 0, 0, 0) \).

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