Maximizing curves for the charged-particle action in globally hyperbolic spacetimes

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Abstract

In a globally hyperbolic spacetime any pair of chronologically related events admits a connecting geodesic. We present two theorems which prove that, more generally, under weak assumptions, given a charge-to-mass ratio there is always a connecting solution of the Lorentz force equation having that ratio. A geometrical interpretation of the charged-particle action is given which shows that the constructed solutions are maximizing.

1 Introduction

Over the last years there has been considerable interest in the connectability of spacetime events through solutions of the Lorentz force equation [1,3–9, 11,12,14,20,21,25]. The works on this topic are mainly concerned with the problem of determining whether a charged particle can reach an event from another on its past. In mathematical terms the problems is

Given a globally hyperbolic spacetime $M$, an event $x_0$ and a second event $x_1 \in I^+(x_0)$ determine whether the Lorentz force equation (cf. [17,22])

$$D_s \left( \frac{dx}{ds} \right) = \frac{q}{m} \tilde{F}(x) \left[ \frac{dx}{ds} \right], \quad (1)$$

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admits connecting solutions. Here $x = x(s)$ is the worldline of the particle parameterized with respect to the proper length, $\frac{dx}{ds}$ is the 4-velocity, $D_s\left(\frac{dx}{ds}\right)$ is the covariant derivative of $\frac{dx}{ds}$ along $x(s)$ associated to the Levi-Civita connection of $g$, and $\hat{F}(x)[\cdot]$ is the linear map on $T_xM$ obtained raising the first index of $F$. Conventions are such that $c = 1$ and the metric $g$ has signature $(+\ -\ -\ -)$. Actually, the references cited above studied the connectability of space-time through solutions of the equation

$$D_\lambda \left(\frac{dx}{d\lambda}\right) = Q\hat{F}(x)\left[\frac{dx}{d\lambda}\right], \quad (2)$$

where $\lambda$ is a generic parameter. Although apparently of the same form of the Lorentz force equation this equation is considerably weaker. Every solution of the Lorentz force equation is, suitably parametrized and independently of the charge-to-mass ratio, a solution of this last equation. As a consequence, as the space of solutions of this last equation is infinitely larger than the one of the Lorentz force equation, its is easier to find connecting solutions for it. What makes the Lorentz force equation more restrictive is the condition that the 4-velocity should be a priori normalized while the same condition on $\frac{dx}{d\lambda}$ is not imposed.

Substantial progress was made using a Kaluza-Klein approach [10,15,19]. The only connectability results up to now available on the Lorentz force equation are indeed those in [15,19]. We shall here review the basic ideas behind these two references. We shall skip the more technical proofs. Instead we shall focus on the geometrical meaning of some results that has not been previously pointed out.

The problem we are going to study is therefore that of generalizing the Avez-Seifert theorem [2,26]

**Theorem 1.1.** In a globally hyperbolic spacetime let $x_1 \in J^+(x_0)$ then there is a geodesic connecting the two events.

to each choice of $q/m \neq 0$ (the Avez-Seifert theorem corresponds to the case $q/m = 0$). We restrict to the case $x_1 \in I^+(x_0)$ since only timelike connecting solutions can be interpreted as charged particles.

Before going on we would like to point out that an affirmative solution to the problem would imply that it is impossible to construct an “electromagnetic barrier”. Eventually we shall see that the problem admits indeed
an affirmative answer. Since it is impossible to isolate an event from charged particles emitted from an event in its past it is also impossible to isolate an entire spacetime region. Figure 1 dramatically shows the problem of protecting ourselves from the charged particles emitted from a nuclear explosion by means of a suitable electromagnetic field around us. No electromagnetic field can protect us from the emitted particles. A consequence of our results is that one has to place walls between the nuclear explosion and the observer, i.e. one has to involve the absorption process of quantum physics to screen the particles.

The problem considered can be formulated as a variational one. Let $\gamma : [0,1] \rightarrow M$, be a $C^1$ curve, if timelike solutions of the Lorentz force equation connecting $x_0$ and $x_1$ exist, then they are extremals of

$$I_{x_0,x_1}[\gamma] = \int_{\gamma} (ds + \frac{q}{m} \omega), \quad \gamma(0) = x_0, \; \gamma(1) = x_1.$$ 

where $ds = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$ and $\omega$ is the 1-form potential over $M$. The electromagnetic field is

$$F = d\omega,$$

thus we are here considering only the case of exact electromagnetic field. This is not a strong requirement. Up to now there has been no experimental evidence of non-exact electromagnetic fields on spacetime although a lot of

Figure 1: Is it possible to screen charged particles with an electromagnetic barrier?
effort was devoted to the search of magnetic monopoles. Although mathematicians are used to consider the electromagnetic field as a closed 2-form, as observation is concerned, the electromagnetic field is an exact 2-form. In particular this implies that the electromagnetic field can be regarded as the curvature of a necessarily trivial \((\mathbb{R}, +)\) bundle over \(M\).

Our problem of connectability can be stated as follows: Does the action functional above have timelike extremals?, and maximizing extremals in a suitable set?

2 Basic definitions

We need some basic facts and definitions that we recall in this section. The spacetime \((M, g)\) is said to be time-orientable if it has a timelike continuous vector field \(v\). We assume \(M\) time-orientable and make a choice of past and future. We recall that a \(C^1\) curve \(\gamma : \mathbb{R} \rightarrow M\) is said

- timelike if: \(g(\dot{x}, \dot{x}) > 0\),
- non-spacelike if: \(g(\dot{x}, \dot{x}) \geq 0\),
- null if: \(g(\dot{x}, \dot{x}) = 0\),
- future-directed if: \(g(v, \dot{x}) > 0\).

for any \(\lambda \in \mathbb{R}\). We shall always consider future-directed curves. We can now recall the definitions of the future sets

- Chronological future: \(I^+(p) = \{ q \in M \text{ that are connected to } p \text{ through a future-directed timelike } C^1 \text{ curve} \} \)
- Causal future: \(J^+(p) = \{ q \in M \text{ that are connected to } p \text{ through a future-directed non-spacelike } C^1 \text{ curve} \} \)

The following properties hold. A proof can be found, for instance, in \([16, 23]\).

- \(I^+(p) \subset J^+(p)\)
- \(I^+(p)\) is open
- \(\bar{I}^+(p) = J^+(p)\)
\[ \dot{I}^+(p) = \dot{J}^+(p) \]

where with \( \dot{K} = \bar{K} \cap (M - K) \) we denoted the boundary of \( K \). Note that \( J^+(p) \) is not necessarily closed. Figure 2 shows an example where an event has been removed from Minkowski spacetime. We shall see below that this

Figure 2: \( J^+(p) \) is not necessarily closed.

spacetime is not globally hyperbolic.

The future horismos of \( p \in M \) is the set

\[ E^+(p) = J^+(p) - I^+(p) \]

The following theorem holds [16]

**Theorem 2.1.** Any non-spacelike curve between two events which is not a null geodesic can be deformed into a timelike curve between the same points.

Then

**Corollary 2.2.** Any point \( q \in E^+(p) \) is connected to \( p \) by a null geodesic

But the converse is not true. There can be events in null geodesics from \( p \) that do not belong to \( E^+(p) \), see figure 3 where an example is given in a cylindrical spacetime.

We now recall that a curve \( \gamma \) has **future endpoint** \( p \) if for any neighborhood \( U \) of \( p \) there is a \( \lambda_1 \) such that \( \gamma(\lambda) \in U \) for \( \lambda > \lambda_1 \). A curve is said **future-inextendible** if \( \gamma \) has no future endpoints. A curve is said **inextendible** if \( \gamma \) is both past and future-inextendible. A **Cauchy surface** is a spacelike hypersurface which every non-spacelike inextendible curve intersects exactly once. We are ready to recall a basic definition that we shall use throughout the work. A spacetime \( M \) is said **globally hyperbolic** if it admits a Cauchy hypersurface. We also recall that a spacetime \( M \) satisfies the **strong causality condition** if any event \( p \) has a neighborhood that no non-spacelike curve intersects more than once. The following remarkable results holds [16]
Figure 3: There can be events in null geodesics from \( p \) that do not belong to \( E^+(p) \).

**Theorem 2.3.** The Lorentzian manifold \( M \) is globally hyperbolic iff the strong causality condition holds and for any pair of events \( J^+(p) \cap J^-(q) \) is compact.

From this it follows [16, Proposition 6.6.1] a result that will be central in our study

**Corollary 2.4.** If \( M \) is globally hyperbolic then

\[
J^+(p) = \tilde{J}^+(p), \quad E^+(p) = \hat{J}^+(p) = \hat{I}^+(p)
\]

Thus by a previous observation

**Corollary 2.5.** If \( M \) is globally hyperbolic then events in \( \hat{I}^+(p) \) are connected to \( p \) by null geodesics.

## 3 Kaluza-Klein spacetime

Consider a trivial 5-dimensional principal bundle \( P = M \times \mathbb{R} \) of structure group \( T_1 = (\mathbb{R},+) \), and projection \( \pi : P \to M \) such that \( (m, y) \to m \). Here \( y \) is the (dimensionless) coordinate over \( \mathbb{R} \). We can define the real-valued connection 1-form \( \tilde{\omega} \) on \( P \):

\[
\tilde{\omega} = (dy + \beta \omega).
\]
The coefficient $\beta$ is introduced for dimensional reasons but will be otherwise arbitrary. Over $P$ we introduce the Kaluza-Klein metric

$$\tilde{g} = g - a^2(dy + \beta \omega)^2 \quad (4)$$

The constant $a$ represents the scale factor of the extra dimension and will be later fixed.

**Remark 3.1.** The scale factor $a$ is interpreted in the $U(1)$ version as the radius of the extra dimension. The field equations for general relativity in 5-dimensions are seen in 4-dimensional spacetime, as 4-dimensional general relativity plus electromagnetism. In fact the 5d Einstein-Hilbert Lagrangian is, if one chooses $\beta = \sqrt{\frac{16 \pi G}{a}}$ with $G$ the Newton constant,

$$\tilde{R} = R + \frac{16 \pi G}{4} F_{\mu\nu} F^{\mu\nu}.$$  

as it should be in order to reproduce the correct coupling between electromagnetism and gravity. Here we use the Kaluza-Klein spacetime only as a technical tool. We never use this physical interpretation so we do not need to relate $\beta$ and $a$ as above.

The projection of a 5d geodesic is a 4d solution to the Lorentz force equation for a suitable charge-to-mass ratio. Indeed geodesics $z(\lambda) = (x(\lambda), y(\lambda))$ are extremals of the functional

$$S = \int_0^1 \frac{1}{2} \tilde{g}[\dot{z}(\lambda), \dot{z}(\lambda)]d\lambda,$$

where with a dot we have denoted derivation with respect to $\lambda$. The Lagrangian is independent of $y$ thus the vertical conjugated momentum

$$p = \frac{\partial L}{\partial \dot{y}} = -a^2(\dot{y} + \beta \omega[\dot{x}]),$$

is conserved. The other Euler-Lagrange equation is

$$D_\lambda \dot{x} = p \beta \hat{F}[\dot{x}], \quad (5)$$

which is quite similar to the Lorentz force equation. The previous equation implies that $g(\dot{x}, \dot{x}) = C^2$ is conserved (contract equation (5) with $\dot{x}$), and
since $z$ is a geodesic $\tilde{g}(\dot{z}, \dot{z})$ is conserved too. We have that the conserved quantities are related by

$$\tilde{g}(\dot{z}, \dot{z}) = C^2 - \frac{p^2}{a^2}.$$ 

There follows

**Remark 3.2.** If $z$ is a null geodesic then $C^2 = p^2/a^2$ and in this case $x = \pi(z)$ is timelike iff $p \neq 0$.

Suppose we want to study the connectability of spacetime through solutions of the Lorentz force equation of charge-to-mass ratio $q/m$. Our strategy in order to find such connecting solutions is based on the following

**Remark 3.3.** Consider on $P$ the Kaluza-Klein metric with $a = \frac{1}{\beta |\frac{q}{m}|}$. Let $z$ be a null geodesic on $P$. If $x = \pi(z)$ is timelike then it satisfies the Lorentz force equation of charge-to-mass ratio $\pm |q/m|$ where the sign is that of $p$.

This remark can be proved easily by substituting $d\lambda = ds/C$ in (5). Our strategy will be therefore the following. Chosen $a$ as above and a point $p_0 \in \pi^{-1}(x_0)$ we look for two null geodesics starting from $p_0$ and ending ($\lambda = 1$) on $x_1$’s fiber (see figure 4). If their projections are timelike and the

\[ p \text{ sign is respectively } + \text{ and } - \text{ then their projections solve affirmatively the problem of connectability for the charge-to-mass ratios } +|\frac{q}{m}| \text{ and } -|\frac{q}{m}|. \]

Let us now show that the two null geodesics above indeed exist. The set $J^+(p_0) \cap \pi^{-1}(x_1)$ is a compact subset of $x_1$’s fiber (real line). The maximum

![Figure 4: Constructing the solution using Kaluza-Klein.](image-url)
\( \bar{p} \) and the minimum \( \hat{p} \) of this set belong to \( J^+(p_0) = I^+(p_0) \). Then by corollary \([2,5]\) if we shown that the Kaluza-Klein spacetime is itself globally hyperbolic, there are null geodesics on \( P \) that reach them starting from \( p_0 \). Their projections if timelike can be shown to have opposite signs \([19]\) of \( p \) as required. We are therefore left only with two open questions

- If \( M \) is globally hyperbolic is \( P \) globally hyperbolic?
- What can we say if the projection of the null geodesics so constructed in \( P \) are null curves?

It is not difficult to show that the first question admits a positive answer \([10, 15]\). The proof has a clear geometrical meaning shown in figure 5. If \( M \) is globally hyperbolic then it has a Cauchy hypersurface \( C \). The proof goes on showing that \( \pi^{-1}(C) \) is a Cauchy hypersurface for \( P \). Indeed if \( z \) is an inextendible timelike curve in \( P \) its projection \( x \) is timelike and inextendible in \( M \) (this is the only technical point in the proof) and thus intersects \( C \) in some point. Therefore \( z \) intersects \( \pi^{-1}(C) \) at some point i.e. \( P \) is globally hyperbolic.

The answer to the second question is that by Eq. \([5]\) and remark \([3,2]\) \( p = 0 \) and therefore the projection is a null geodesic. If there are no connecting null geodesics between the events \( x_0 \) and \( x_1 \) this case is excluded. A typical example is Minkowski spacetime: if \( x_1 \in I^+(x_0) \) then there is no null geodesics connecting the events.

We are ready to state the first result obtained but first let us see what can be said about the associated variational problem.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{\( P \) is globally hyperbolic.}
\end{figure}
4 A geometrical interpretation for the action

In [19] the author found a simple geometrical interpretation for the functional \( I \) which is somewhat reminiscent of the time of arrival functional in the Fermat principle of general relativity [18, 24].

Given a (future-directed) causal connecting curve \( \sigma(\lambda) : [0, 1] \to M \), consider two lifts of the curve (dependent on the sign) in null curves \( \tilde{\sigma}^\pm(\lambda) \) of \( P \) starting at \( p_0 = (x_0, y_0) \), through the following definition

\[
\tilde{\sigma}^\pm(\lambda) = (\sigma(\lambda), y^\pm(\lambda)) = \left( \sigma(\lambda), y_0 \mp \int_{\sigma(\lambda)} \omega \right).
\]

These are the curve “light lifts” to be distinguished from the usual “horizontal lift”. The fiber coordinate \( y^\pm_1 \) of the final point of this curve is, essentially,

\[
y^\pm_1 = y_0 \mp \frac{1}{a} \left( \int_{\sigma} ds + (\pm |q/m|) \int_{\sigma} \omega \right).
\]

Figure 6: The light lift.

the action functional:

\[
y^\pm_1 = y_0 \mp \frac{1}{a} \left( \int_{\sigma} ds + (\pm |q/m|) \int_{\sigma} \omega \right).
\]

thus a maximization on the space of \( C^1 \) connecting causal curves, \( N_{x_0,x_1} \), of the functional \( I_{x_0,x_1} \) relative to the ratio \( +|q/m| \) (resp. \( -|q/m| \)), corresponds to a minimization (resp. maximization) of \( y^+_1(\sigma) \) (resp. \( y^-_1(\sigma) \)). The maximizing curve is exactly the one constructed in the previous section in order to find a solution to the Lorentz force equation. In summary we have the following [19]:

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Theorem 4.1. Let \((M, g)\) be a globally hyperbolic spacetime, \(\omega\) be a \((C^2)\) 1-form on \(M\), and \(F = d\omega\). Let \(x_1\) be an event in the chronological future of \(x_0\) and \(q/m \in \mathbb{R} - \{0\}\) any charge-to-mass ratio. Let \(P = M \times \mathbb{R}\) be the Kaluza-Klein spacetime of metric (4), with \(a = \frac{1}{\beta|q/m|}\). Let \(p_0\) be a point in the fibre of \(x_0\) and let the compact set \(J^+(p_0) \cap \pi^{-1}(x_1)\) have endpoints \(\tilde{p}_1 = (x_1, \tilde{y}_1), \hat{p}_1 = (x_1, \hat{y}_1)\) with \(\tilde{y}_1 \geq \hat{y}_1\). Let the curve \(\gamma_0\) be the projection of a null geodesic \(\bar{\gamma}\) that connects \(p_0\) to \(\tilde{p}_1\) if \(q/m < 0\) or the projection of a null geodesic \(\hat{\gamma}\) that connects \(p_0\) to \(\hat{p}_1\) if \(q/m > 0\).

The null geodesics \(\bar{\gamma}\) and \(\hat{\gamma}\) exist, and the future-directed causal curve \(\gamma_0\) connecting \(x_0\) and \(x_1\) maximizes the functional \(I[\gamma](x_0, x_1)\) on the space \(\mathcal{N}_{x_0, x_1}\). Moreover, \(\gamma_0\) being the projection of a null geodesic is everywhere timelike or null. In the former case, the reparametrization of \(\gamma_0\) with respect to proper time is a solution of the Lorentz force equation (7); in the latter case, \(\gamma_0\) is a null geodesic.

Corollary 4.2. Let \((M, \eta)\) be the Minkowski spacetime. Let \(F\) be an electromagnetic tensor field (closed 2-form). Let \(x_1\) be an event in the chronological future of \(x_0\) and \(q/m\) a charge-to-mass ratio, then there exists at least one future-directed timelike solution to the Lorentz force equation connecting \(x_0\) and \(x_1\).

Proof. Since \(M\) is contractible \(F\) is exact. Moreover, in Minkowski spacetime, if \(x_1 \in I^+(x_0)\) there is no null geodesic connecting \(x_0\) with \(x_1\).

\[\square\]

5 A complete answer in the classical case

Unfortunately the previous theorem leaves open the possibility that \(\gamma_0\) is a null geodesic. We ask whether it is possible to discard this possibility. If the charge-to-mass ratio is smaller in absolute value of a certain real number the answer is affirmative [15]. Define

\[R = \sup_{x \in \mathcal{T}_{x_0, x_1}} \left( \frac{\int_x ds}{\sup_{w \in \mathcal{N}_{x_0, x_1}} |\int_w \omega - \int_x \omega|} \right) \tag{8}\]

where \(\mathcal{T}_{x_0, x_1}\) is the space of \(C^1\) timelike connecting curves. Note that \(R\) does not depend on the gauge chosen. It can be shown that \(R\) is strictly positive and moreover the following theorem holds [15].
Theorem 5.1. Let $(M, g)$ be a time-oriented Lorentzian manifold. Let $\omega$ be a 1-form ($C^2$) on $M$ (an electromagnetic potential) and $F = d\omega$ (the electromagnetic tensor field). Assume that $(M, g)$ is a globally hyperbolic manifold. Let $x_1$ be an event in the chronological future of $x_0$ and let $R$ be defined as in (8), then there exists at least one future-directed timelike solution to the Lorentz force equation connecting $x_0$ and $x_1$, for any charge-to-mass ratio satisfying $|\frac{q}{m}| < R$.

Proof. Let $z = (x, y)$ be the null geodesic that connects $p_0$ with $\bar{p}$ (notations of the previous theorem). From $p = -a^2(y + \beta \omega [\dot{x}])$, integrating and introducing an arbitrary timelike connecting curve $\sigma$

\[
p = -a^2(y_1 - y_0 + \beta \int_x \omega) = -a^2(y_1 - y_0 + \beta \int_\sigma \omega) - a^2 \beta(\int_x \omega - \int_\sigma \omega) = -a^2(y_1 - y_1^-[\sigma] + \frac{1}{a} \int_\sigma ds) - a^2 \beta(\int_x \omega - \int_\sigma \omega)
\]

Since $z$ connects $p_0$ with $\bar{p}$, by construction it is $y_1 - y_1^-[\sigma] \geq 0$, thus

\[
p \leq -a \int_\sigma ds - a^2 \beta(\int_x \omega - \int_\sigma \omega) = - \left(1 + a\beta \frac{\int_x \omega - \int_\sigma \omega}{\int_\sigma ds}\right) a \int_\sigma ds
\]

If

\[a\beta = |\frac{q}{m}| < \frac{\int_\sigma ds}{\int_x \omega - \int_\sigma \omega}\]

for some $\sigma$ then $p < 0$ and the projection of $z$ is timelike as we want to prove. If $|q/m| < R$ there is indeed a timelike curve $\sigma$ such that

\[|\frac{q}{m}| < \frac{\int_\sigma ds}{\sup_{w \in N_{x_0,x_1}} |\int_w \omega - \int_\sigma \omega|} \leq \frac{\int_\sigma ds}{\int_x \omega - \int_\sigma \omega|},\]

and therefore $p < 0$. An analogous reasoning holds with $z$ connecting $p_0$ and $\hat{p}$ and leads to a conserved vertical momentum of opposite sign $p > 0$. This concludes the proof. \qed
5.1 Physical meaning of theorem 5.1

In quantum mechanics a particle behaves as it could experience all the possible paths (quantum histories) between two events. The condition $|q/m| < R$ can be written

$$\sup_{x \in T_{x_0,x_1}} \frac{m}{q \Delta \Phi_x} > 1$$

the denominator

$$q \Delta \Phi_x = \frac{N_{x_0,x_1}|\int_w q \omega - \int_x q \omega|}{\int_x ds}$$

is the maximum mean (with respect to proper time) variation of energy potential that a particle moving on $x$ experiences with respect to the other alternatives (the curves $w$). Now, if the particle moves on $x$ it must be

$$q \Delta \Phi_x < m$$

otherwise there would be pair creations. If the particle moves without pair creation effects then there must be at least a curve such that the previous equation holds

$$\inf_{x \in T_{x_0,x_1}} q \Delta \Phi_x < m$$

which coincides with the condition of the theorem. We conclude that the condition $|q/m| < R$ means classical regime. If this condition is not satisfied there can be pair creation effects (like in the so called Klein paradox) and the Lorentz force equation is no longer valid in the quantum regime. If the condition is satisfied we are in a classical regime and the theorem states that there are indeed connecting solutions to the Lorentz force equation. Note that in the limit $\omega \to 0$, we have $R \to \infty$ which means that in a weak electromagnetic field we are always in a classical regime an the theorem can be applied (notice that the word weak here does not mean that the theorem holds in some approximation or in some limit).

6 Conclusions

We have presented two theorems that answer affirmatively to the existence of connecting solutions to the Lorentz force equation (1). One of them implies that there is a connecting solution if there is no null connecting geodesic as it happens, for instance, in Minkowski spacetime. The other shows that
even if this is not the case, if the absolute value of the charge-to-mass ratio is less than a certain geometrical gauge invariant number $R > 0$, there are connecting solutions. This last condition has been shown to correspond to the classical limit, i.e. to the case where there are no quantum mechanical pair creations effects and it is therefore completely satisfactory from the physical point of view. Indeed, in a non-classical (quantum mechanical) regime the very Lorentz force equation becomes meaningless. Finally, thanks to a geometrical interpretation of the charged-particle action the solutions above have been shown to maximize the action.

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