The Real-Space Renormalization Group
Applied to Diffusion in Inhomogeneous Media

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The real-space renormalization group technique is introduced to evaluate the effective
diffusion constant for diffusion in inhomogeneous media, which has been obtained by
singular perturbation methods. Our method is formulated on a discretized real space and
hence it can be easily combined with numerical studies for partial differential equations.

KEYWORDS: real-space renormalization group, effective diffusion constant, inhomogeneous media

§1. Introduction

The renormalization group (RG) is one of general techniques to study macroscopic (or self-
similar) and universal characteristics of equilibrium and non-equilibrium systems whose microscopic
models have been known.¹)

In the present paper, the RG technique is applied to the problem of diffusion in inhomogeneous
media. It can be considered as one of pedagogical examples of the RG method. Furthermore, since
our real-space RG method is formulated on a discrete space, our solution can provide a simple
example to show that the RG method and numerical simulations of partial differential equations
can be easily combined.

For example, there are some attempts to apply the RG technique to Navier-Stokes equation.²)
It is well known that the scale of turbulent flows of any practical significance are expected to be
much larger than the Kolmogorov scale, which is the smallest scale of activated eddies. Hence, in
order to save computer resource, only some large-scale eddies are computed explicitly and effects
from other smaller eddies are modeled as a eddy viscosity.³) The numerical simulations based on
the idea of sub-grid modeling are called the large eddy simulations. The eddy viscosity used in
large eddy simulations for turbulent fluids is expected to be evaluated with the RG applied to
Navier-Stokes equation.⁴)

The only difference between the applications of the RG to Navier-Stokes equation and the
present application to diffusion is whether the equations to be solved for decimation are non-linear
or linear. So, our solution can be expected to help investigate applicability of the RG technique to
solve other partial differential equations with aid of numerical computation.

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This introductory section is concluded with note that the problem has been solved with other singular perturbation methods like multi-scale methods.\textsuperscript{5)}

The present paper is organized as follows: After the problem to be studied is explained in § 2, the RG is formulated in § 3 step by step. At first, the diffusion equation is extended formally to close the renormalization transform in § 3.1. The construction of the renormalization transform is divided into two procedures, decimation explained in § 3.2 and rescaling in § 3.3. The formulae of renormalization transform are obtained in § 3.4. With the formulae, the fixed points are evaluated in § 3.5. Finally, the effective diffusion equation is obtained in § 4. The results are summarized and discussed in § 5.

\section{Formulation of the Problem}

A one-dimensional inhomogeneous medium where the diffusion constant $D(x)$ is a spatially periodic function is assumed. The period is denoted as $l$ and hence $D(x + l) = D(x)$. The objective of our study is evaluation of the effective diffusion constant $D_e$ when the spatial resolution of observation of diffusion process is much larger than the period $l$. The scale of observation is denoted as $L_0$. The diffusion equation is written as

$$\frac{\partial}{\partial t} \rho(x, t) = \frac{\partial}{\partial x} D(x) \frac{\partial}{\partial x} \rho(x, t), \quad D(x + l) = D(x). \quad (1)$$

In order to make formulation of the real-space RG easier, we discretize the continuum space into a chain.\textsuperscript{8)} The final results for the effective diffusion constant are obtained in the continuum limit in § 4.

Grid points on the chain are numbered with the integers $\{\cdots, i-1, i, i+1, \cdots\}$ and the lattice constant is assumed to be $a \ll l$, i.e., $x = \lim_{a \to 0} i a$. There are $N_l$ sites in one period $l$, i.e., $N_l \equiv l/a$. The probability $P_i(t)$ that a random walker is found at site $i$, or the number of the random walkers, is related to the probability density $\rho(x, t)$ as

$$\rho(x, t) \equiv \lim_{a \to 0} \frac{P_i(t)}{a}. \quad (2)$$

Then, the diffusion equation (eq. (1)) is discretized as

$$\frac{dP_i(t)}{dt} = \frac{1}{a^2} [D_{i-1} P_{i-1}(t) - (D_i + D_{i+1})P_i(t) + D_{i+1} P_{i+1}(t)]. \quad (3)$$

\section{Renormalization Group}

In this section, we formulate the renormalization transform for the discretized diffusion equation (eq. (3)) and obtain the fixed point. The construction of the renormalization transform is explained by being divided into two steps, decimation and rescaling.
3.1 Formal extension of the diffusion equation

Before construction of the renormalization transform, new parameters are introduced in the discretized diffusion equation (the master equation) in order to obtain the closed renormalization transform as follows:

$$\frac{dP_i(t)}{dt} = \frac{1}{a^2} \int_0^t d\tau [D_{i-1}(t-\tau)P_{i-1}(\tau) - (D_{i-1}(t-\tau) + D_i(t-\tau))P_i(\tau) + U_i(t-\tau)P_i(\tau) + D_i(t-\tau)P_{i+1}(\tau)].$$

(4)

Although the memory effect is introduced, the effect does not exist in fact. Hence, the memory function $D_i(t)$ is defined with the delta function as

$$D_i(t) = \delta(t)D_i \quad \text{where} \quad \int_0^t d\tau \delta(t-\tau)P_i(\tau) \equiv P_i(t).$$

(5)

Furthermore, since $U_i(t)$ is introduced only for closing the renormalization transform, $U_i(t) = 0$.

In order to make decimation procedure easier, the integro-differential equation, eq. (4), is Laplace transformed as

$$sP_i(s) - P_i(0) = \frac{1}{a^2} [D_{i-1}(s)P_{i-1}(s) - (D_{i-1}(s) + D_i(s) + U_i(s))P_i(s) + D_i(s)P_{i+1}(s)].$$

(6)

Here, we denote the Laplace transform of $P_i(t), D_i(t), U_i(t)$ as $P_i(s), D_i(s), U_i(s)$ respectively. The Laplace variable $s$ is explicitly written for prevention of confusing. For simpler notation, eq. (6) is rewritten as

$$sP_i(s) - P_i(0) = \left[ w_{i-1}(s)P_{i-1}(s) - (w_{i-1}(s) + w_i(s) + v_i(s))P_i(s) + w_i(s)P_{i+1}(s) \right],$$

(7)

where the Laplace transformed jump rates are defined as

$$w_i(s) \equiv D_i(s)/a^2, \quad v_i(s) \equiv U_i(s)/a^2.$$  

(8)

3.2 Decimation

Decimation procedure, i.e., reduction of the degrees of freedom, is performed in this subsection. It is the first step of the renormalization transform.

Especially in spin systems on lattices, the procedure is often implemented as averaging the variables in small blocks which the whole system is divided into.

On the other hand, for diffusion phenomena, development for long time is regarded as a kind of average process for distribution of the random walkers. Hence, we expect that the explicit average is not needed and reduce the degrees of freedom with algebraic elimination explained below. It means that the effective development equation for the non-averaged probability density $\rho(x, t)$, instead of the averaged density, is obtained at last.
Although it appears that probability is annihilated with decimation, the renormalization transform for the initial condition, eq. (32), shows that the total probability (or the total amount of matter) is conserved in the renormalization transform.

Decimation process is performed by just eliminating the probability of the nearest neighbor sites, $P_{i-1}(s)$ and $P_{i+1}(s)$, from the development equation for the probability $P_i(s)$, eq. (7). Decimation process as well as rescaling is explained schematically in Fig. 1.

Elimination of $P_{i-1}(s)$ is done by solving eq. (7) where the site index is changed as $i \rightarrow i-1$. $P_{i-1}(s)$ is given as

$$P_{i-1}(s) = \frac{w_{i-2}(s)P_{i-2}(s) + w_{i-1}(s)P_i(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)} + \frac{P_{i-1}(0)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)}. \quad (9)$$

The probability $P_{i+1}(s)$ is obtained in the same way. By inserting $P_{i-1}(s)$ and $P_{i+1}(s)$ in eq. (7), the decimated master equation is obtained as

$$sP_i(s) - P_i(0) - \frac{w_{i-1}(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)}P_{i-1}(0)$$

$$- \frac{w_i(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)}P_{i+1}(0)$$

$$= \frac{w_{i-2}(s)w_{i-1}(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)}P_{i-2}(s)$$

$$- \left[ \frac{w_{i-2}(s)w_{i-1}(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)} \right] P_{i-2}(s)$$

$$+ \frac{w_i(s)w_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)}P_{i+1}(0)$$

$$+ v_i(s) + \frac{sw_{i-1}(s) + w_{i-1}(s)v_{i-1}(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)}$$

$$+ \frac{sw_i(s) + w_i(s)v_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)}$$

$$+ \frac{sw_{i+1}(s) + w_i(s)v_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)} P_{i+2}(s)$$

$$+ \frac{sw_{i+1}(s) + w_i(s)v_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)} P_{i+2}(s)$$

(10)
Only the probabilities of the nearest neighbor sites appear in the master equation (eq. (7)). This is the reason why decimation is performed easily and it is locality of the renormalization transform.

3.3 Rescaling

Next, we perform rescaling procedure, which is the second step of the renormalization transform. Since one of every two neighboring sites is eliminated, the sparse grid points are renamed as

\[ i \rightarrow i' \equiv i/2, \]

(11)

where \( i \) is assumed even. In the continuum limit, the rescaling eq. (11) corresponds to \( x \rightarrow x' \equiv x/2 \), since the lattice constant \( a \) is not rescaled.

Since the spatial scale is rescaled, time development slows down accordingly. It is expressed as

\[ t \rightarrow t' \equiv 2^{-\mu} t, \]

(12)

where the exponent \( \mu \) is a positive number to be determined later. It means that the Laplace variable is rescaled as

\[ s \rightarrow s' \equiv 2^\mu s. \]

(13)

Properly speaking, the probabilities \( \{P_i(t)\} \) are also to be rescaled. However, the diffusion equation that we consider is linear and hence it does not matter whether the probabilities are rescaled. For simplicity, we do not rescale the quantities;

\[ P_i(t) \rightarrow P_i'(t') \equiv P_i(t). \]

(14)

Then, by using rescaling of the Laplace variable (eq. (13)) and the definition of the Laplace transform, the Laplace transformed probability \( P_i(s) \) is rescaled as

\[ P_i(s) \rightarrow P_i'(s') \equiv 2^{-\mu} P_i(s). \]

(15)

3.4 The renormalization transform

The renormalization transformed jump rates and the initial condition, \( w_i'(s'), v_i'(s'), P_i'(0) \), are derived by using the rescaling formulae obtained above. By inserting the rescaling formulae (eqs. (11, 13, 15)) in the decimated master equation (eq. (10)), the renormalized master equation
is given as
\[
s' P_{i'}(s') - P_{i'}(0) = 2^\mu \frac{w_{i-2}(s)w_{i-1}(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)} P_{i'-1}(s') \\
- 2^\mu \frac{w_{i-2}(s)w_{i-1}(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)} \frac{w_i(s)w_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)} \frac{v_i(s) + sw_{i-1}(s) + v_{i-1}(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)} \frac{sw_i(s) + w_i(s)v_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)} P_{i'}(s') \\
+ 2^\mu \frac{w_i(s)w_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)} \frac{v_i(s) + sw_{i-1}(s) + v_{i-1}(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)} \frac{sw_i(s) + w_i(s)v_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)} P_{i'+1}(s').
\] (16)

Since the jump rates are the coefficients of the probability \( P_{i'}(s') \) in the master equation, the renormalized jump rates \( w_{i'}(s'), v_{i'}(s') \) are given as
\[
w_{i'}(s') = 2^\mu \frac{w_i(s)w_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)},
\] (17)
\[
v_{i'}(s') = 2^\mu \left( v_i(s) + \frac{sw_{i-1}(s) + v_{i-1}(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)} \frac{sw_i(s) + w_i(s)v_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)} \right)
\] (18)

These formulae are the renormalization transform for the jump rates.

In the renormalized master equation, eq. (16), \( P_{i'}(0) \) is defined as
\[
P_{i'}(0) \equiv P_i(0) + \frac{w_{i-1}(s)}{s + w_{i-2}(s) + w_{i-1}(s) + v_{i-1}(s)} P_{i-1}(0) \\
+ \frac{w_i(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)} P_{i+1}(0).
\] (19)

This is the renormalization transform for the initial condition. It is weird that the renormalized initial condition, which should be real, depends on the Laplace variable \( s \). However, it is shown in eq. (31) that the \( s \)-dependence vanishes in the limit \( l/L_0 \to 0 \). Furthermore, for complete rigorous deduction, a new parameter should be introduced to absorb the \( s \)-dependence. If the parameter was introduced, the \( s \)-dependence of the renormalized initial condition would be eliminated and the newly introduced parameter would become zero in the limit.

By remembering that we are interested in the behavior whose spatial scale \( L_0 \) is much larger than \( l \), the renormalization transform formulae obtained above are simplified. The characteristic time scale which corresponds to the spatial scale \( L_0 \) is given by the transit time that it takes for the random walker to pass the region of length \( L_0 \). The transit time is of the order of \( L_0^2/D_i \). Converted into the scale of the Laplace regime, \( s \sim D_i/L_0^2 \). From the consideration, the order of
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$s/w_i(s)$ is estimated as

$$\frac{s}{w_i(s)} \sim \frac{a^2}{D_i} s \sim \frac{a^2 D_i}{D_i L_0^2} \sim \left( \frac{a}{L_0} \right)^2. \tag{20}$$

Since $L_0 \gg a$, $s/w_i(s)$ is a small quantity, $s/w_i(s) \ll 1$.

The estimation of the order is not valid when the renormalization transform is performed many times. Here, we show that $s/w_i(s)$ is small even at the fixed point, i.e., $s^*/w^* i^*(s^*) \ll 1$. Afterwards, the parameter $\mu$ is determined as $\mu = 1$ and hence $s' = 2s$. After the renormalization transform is performed $\log_2 N_i$ times, the fixed point is reached. Hence, the order is estimated as $s^*/w^* i^*(s^*) \sim O(al/L_0^2)$. Since $l \ll L_0$, $s^*/w^* i^*(s^*)$ is small. It means that $s^{(n)}/w_{i(n)}^{(n)}(s^{(n)})$, which is the quantity renormalized $n$ times, is small for arbitrary $n$, since the rescaled Laplace variable grows monotonically.

The result of the order estimation means that the jump rates $w_i(s), v_i(s)$ can be expand in terms of relatively small parameter $s$. The jump rates are expanded up to the first order $O(s)$;

$$w_i(s) = w_i + r_i s + O(s^2), \quad v_i(s) = v_i s + O(s^2). \tag{21}$$

The ratio of the neglected terms to the terms considered explicitly is of order of $(a/L_0)^2$ and hence contribution from these terms can vanish at the continuum limit. Furthermore, the neglected terms at the fixed point are relatively of order of $(l/L_0)^2$. It means that the second order quantities can be eliminated even at the fixed point, since $L_0 \gg l$. In addition, note that the initial values for $w_i, r_i, v_i$ are given by $w_i \equiv D_i/a^2, r_i = v_i = 0$.

By inserting the expansion (eq. (21)) in the renormalization transform for $w_i(s), v_i(s)$ (eqs. (17, 18)), the renormalization transform formulae for $w_i, r_i, v_i$ are derived. At first, the transform for $w_i(s)$ is expanded in terms of $s$ as

$$w_{i'}^{(s')} = w_{i'}(s') = 2^{\mu} \frac{w_i(s)w_{i+1}(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)} \]$$

On the other hand, $w_{i'}^{(s')}$ is expanded with eq. (21) in terms of $s'$ as

$$w_{i'}^{(s')} \equiv w_{i'} + s' r_{i'} = w_{i'} + 2^{\mu} s r_{i'}. \tag{23}$$

By comparison, the renormalization transform formulae for $w_i, r_i$ are given as

$$w_{i'} = 2^{\mu} \frac{w_i w_{i+1}}{w_i + w_{i+1}}, \quad \tag{24}$$

$$r_{i'} = \frac{r_{i+1} w_i^2 + r_i w_{i+1}^2 - w_i w_{i+1}(1 + v_{i+1})}{(w_i + w_{i+1})^2}. \quad \tag{25}$$
Next, by inserting the expansion, eq. (21), in the transform for \( v_i(s) \), eq. (18), the renormalization transform for \( v_i \) is given as

\[
v_i'(s') = \left[ v_i + \frac{w_i-1(1+v_{i-1})}{w_{i-2} + w_{i-1}} + \frac{w_i(1+v_{i+1})}{w_i + w_{i+1}} \right] s + O(s^2)
\]

\[
\equiv v_i's'^\mu
\]

\[
= v_i's.
\]  

(26)

By comparison, the renormalization transform of \( v_i \) is shown to be

\[
v_i' = v_i + \frac{w_i-1(1+v_{i-1})}{w_{i-2} + w_{i-1}} + \frac{w_i(1+v_{i+1})}{w_i + w_{i+1}}.
\]  

(27)

The renormalization transform for the initial condition is also simplified in the similar way. With eq. (21), the coefficients in the transform (eq. (19)) is expanded as

\[
\frac{w_i(s)}{s + w_i(s) + w_{i+1}(s) + v_{i+1}(s)} = \frac{w_i}{w_i + w_{i+1}} + \frac{r_iw_{i+1}-w_i(1+r_{i+1} + v_{i+1})}{(w_i + w_{i+1})^2} s + O(s^2).
\]  

(28)

We consider the case that the transform has been performed \( n \) times. Since \( s \sim D_i/L_0^2, v_i^{(n)}(s) \sim 2^n \), the second term on the right hand side of eq. (28) is estimated as

\[
\frac{r_i^{(n)}w_i^{(n)}(s') + w_i^{(n)}(1+r_i^{(n)}(s) + v_i^{(n)}(s+1))}{(w_i^{(n)} + w_i^{(n)}(s+1))} \sim \frac{v_i^{(n)}(s)}{w_i^{(n)}}
\]

\[
\sim \frac{2^n a^2 v_i^{(n)}(s)}{L_0^2}
\]

\[
\sim \frac{2^{2n} a^2}{L_0^2}
\]

\[
\leq \left( \frac{L}{L_0} \right)^2
\]  

(29)

By neglecting the quantities of \( O((a/L_0)^2), O(al/L_0^3), O(l/L_0)^2 \), the approximation

\[
\frac{w_i^{(n)}(s)}{s + w_i^{(n)}(s) + w_i^{(n)}(s+1)} \sim \frac{w_i^{(n)}(s)}{w_i^{(n)} + w_i^{(n)}(s+1)}
\]  

(30)

is justified. Hence, the renormalization transform for the initial condition eq. (19) is simplified as

\[
P_i^{(n+1)}(s) \sim P_i^{(n)}(s) + \frac{w_i^{(n)}(s) - 1}{w_i^{(n)}(s) + w_i^{(n)}(s+1)} P_i^{(n)}(s) + \frac{w_i^{(n)}(s)}{w_i^{(n)} + w_i^{(n)}(s+1)} P_i^{(n+1)}(s).
\]  

(31)
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Even though some variables for probability $P_i(s)$ in decimation procedure are eliminated, it is shown from eq. (31) that the total probability is conserved as

$$\sum P^{(n+1)}_{i(n+1)}(0) = \sum P^{(n)}_{i(n)}(0).$$

(32)

Here, we consider the renormalized initial condition. From eq. (31), the renormalization for the initial distribution localized at the site $mN_l$ where $m$ is arbitrary integer is given as

$$P^{(n)}_{i(n)}(0) = \delta_{i(n),mN_l/2^n}.$$  

(33)

The result is equivalent to the limiting form at the limit $t \to 0$ of the rescaling of $P_i(t)$ (eq. (14)). However, the result of the transform for general initial distribution is inconsistent with the limit $t \to 0$ of rescaling of the probability. In order to solve the inconsistency, we would had to reformulate decimation procedure with averaging of the neighboring probabilities. In the rest of the present paper, we restrict our interest in the localized initial distribution like eq. (33) for its simplicity. In the case, there is not any inconsistency.

Since all the formulae are obtained, the value of $\mu$ is determined. We consider condition for existence of the non-trivial fixed point $w^*$ for the jump rate $w_i$. From eq. (24), assumption that the fixed point $w^*$ exists is written as

$$w^* = 2^{\mu-1}w^*.$$  

(34)

For $w^*$ is non-zero, it is necessary and sufficient that $\mu = 1$.

Since the number of parameters is increased, we summarize definition and the renormalization transform formulae:

- The master equation

$$sP_i(s) - P_i(0) = [w_{i-1}(s)P_{i-1}(s) - (w_{i-1}(s) + w_i(s) + v_i(s))P_i(s) + w_i(s)P_{i+1}(s)],$$

(35)

where $w_i(s) \equiv w_i + r_is, v_i(s) \equiv v_is, D_i \equiv a^2w_i$.

- The renormalization transform

$$s' = 2s \quad (t' = t/2)$$

(36)

$$i' = i/2 \quad (x' = x/2)$$

(37)

$$P'_{i'}(s') = P_i(s)/2 \quad (P'_{i'}(t') = P_i(t))$$

(38)

$$w'_{i'} = 2\frac{w_iw_{i+1}}{w_i + w_{i+1}}$$

(39)

$$r'_{i'} = \frac{r_{i+1}w_i^2 + r_iw_{i+1}^2 - w_iw_{i+1}(1 + v_{i+1})}{(w_i + w_{i+1})^2}$$

(40)

$$v'_{i'} = v_i + \frac{w_{i-1}(v_{i-1} + 1)}{w_{i-2} + w_{i-1}} + \frac{w_i(v_i + 1)}{w_i + w_{i+1}}$$

(41)

$$P'_{i'}(0) = P_i(0) + \frac{w_{i-1}}{w_{i-2} + w_{i-1}}P_{i-1}(0) + \frac{w_i}{w_i + w_{i+1}}P_{i+1}(0).$$

(42)
3.5 Evaluation of the fixed point

At first, we consider the fixed point of the jump rate \( w_i \). Equation (39) is converted into another form as

\[
\frac{1}{w'_i} = \frac{1}{2} \left( \frac{1}{w_i} + \frac{1}{w_{i+1}} \right).
\]

(43)

It means that the renormalization transform is equivalent to averaging of the inverse of the neighboring jump rates. Hence, the global stability of the fixed point is obvious. One period of the diffusion constant with \( N_l \) sites is renormalized into one grid point and the fixed point is reached, when the renormalization transform is iterated \( n_l = \log_2 l/a = \log_2 N_l \) times. At the fixed point, the medium is uniform and the fixed point value of the jump rate, \( w^* \), is independent of the site. The value is given by

\[
w^* = \frac{1}{\frac{1}{N_l} \sum_{i=0}^{N_l-1} \frac{1}{w_i}}.
\]

(44)

Next, we calculate the value of \( v_i \) transformed \( n_l \) times, \( v^* \). The list of the renormalization transform of \( v_i \) for one period is given as

\[
\begin{align*}
v'_1 &= v_2 + \frac{w_1(v_1+1)}{w_N + w_1} + \frac{w_2(v_3+1)}{w_2+w_3}, \\
& \vdots \\
v'_{i'} &= v_i + \frac{w_{i-1}(v_{i-1}+1)}{w_{i-2}+w_{i-1}} + \frac{w_i(v_{i+1}+1)}{w_i+w_{i+1}}, \\
v'_{i'+1} &= v_{i+2} + \frac{w_{i+1}(v_{i+1}+1)}{w_{i}+w_{i+1}} + \frac{w_{i+2}(v_{i+3}+1)}{w_{i+2}+w_{i+3}}, \\
& \vdots \\
v'_{N_l/2} &= v_{N_l} + \frac{w_{N_l-1}(v_{N_l-1}+1)}{w_{N_l-2}+w_{N_l-1}} + \frac{w_{N_l}(v_{1}+1)}{w_{N_l}+w_{1}}.
\end{align*}
\]

(45)

Summing up the both sides respectively, the conservation law that the sum of \( v_i + 1 \) over one period does not change by the renormalization transform is obtained;

\[
\sum_{i'=1}^{N_l/2} (v'_{i'} + 1) = \sum_{i=1}^{N_l} (v_i + 1).
\]

(46)

Application of the conservation law \( n_l \) times gives

\[
v^* + 1 = \sum_{i=1}^{N_l} (v_{i^2(n_l-1)} + 1) = \cdots = \sum_{i=1}^{N_l} (v_i + 1) = \sum_{i=1}^{N_l} 1 = N_l,
\]

(47)

where the fact that the initial value for \( v_i \) is given as \( v_i = 0 \) is used. The fixed point value is given as

\[
v^* = N_l - 1.
\]

(48)

There is another parameter \( r_i \). After iteration of the transform with \( n_l \) times, the value of the parameter, \( r^* \), is independent of location of the grid point. Hence, the jump rate \( w_i(s) \) is
transformed into $w^*(s^*) = w^* + r^*s^*$. We show below that the second term can be neglected. Since $s^* = N_is, s/w^* \sim (a/L_0)^2$ from eqs. (20, 36), the order of $r^*s^*$ is estimated as

$$
\frac{r^*s^*}{w^*} \sim r^*N_i \left( \frac{a}{L_0} \right)^2 \sim r^* \frac{l}{L_0} \frac{a}{L_0}.
$$

(49)

Furthermore, the transform formula of $r_i$ and the fixed point of $v_i$ (eqs. (40, 48)) show that

$$
r^* \sim v^* \sim \frac{l}{a}.
$$

(50)

Hence,

$$
\frac{r^*s^*}{w^*} \sim \left( \frac{l}{L_0} \right)^2.
$$

(51)

It means that $r^*s^*$ can be neglected compared to $w^*$ when the macroscopic limit $l/L_0 \to 0$ is taken.

We discuss the initial condition renormalized $n_i$ times. Here, for simplicity, we assume that the initial distribution is localized at the origin of the chain; $P_i(0) = \delta_{i,0}$. As a special case of eq. (33), the fixed point is given as

$$
P^*_{i^*}(0) \equiv P^{(n_i)}_{i^*(n_i)}(0) = \delta_{i^*,0}.
$$

(52)

From eqs. (36, 37, 38), the values of $i, s, P_i(s)$ rescaled $n_i$ times are given as

$$
i^* = i/N_i
$$

$$s^* = N_is
$$

$$
P^*_{i^*}(s^*) = P_i(s)/N_i
$$

(53)

§4. The Effective Diffusion Equation

The master equation at the fixed point is written as

$$(v^* + 1)s^*P^*_{i^*}(s^*) - P^*_{i^*}(0) \simeq w^* \left[ P^*_{i^*+1}(s^*) - 2P^*_{i^*}(s^*) + P^*_{i^*-1}(s^*) \right],
$$

(54)

where the symbol $\simeq$ denotes equality when the limits $a/L_0 \to 0, l/L_0 \to 0$ are taken. The former limit is the continuum limit and the latter corresponds to the large spatial scale and the long time limit $D(x)t/l^2 \to \infty$.

With the fixed point values of eqs. (48, 52, 53), the master equation is rewritten in terms of the non-renormalized parameters as

$$
N_isP_i(s) - P_i(0) \simeq \frac{w^*}{N_i} \left[ P_{i-N_i}(s) - 2P_i(s) + P_{i+N_i} \right].
$$

(55)

The inverse Laplace transform gives

$$
\frac{dP_i(t)}{dt} \simeq \frac{w^*}{N_i^2} \left[ P_{i-N_i}(t) - 2P_i(t) + P_{i+N_i}(t) \right]
$$

$$
\simeq \frac{a^2w^*}{l^2} \left[ P_{i-N_i}(t) - 2P_i(t) + P_{i+N_i}(t) \right]
$$

(56)
The continuum limit is taken by dividing the both sides by \( a \) and the limit \( a \to 0 \) is taken. By using the definition \( \rho(x, s) \equiv \lim_{a \to 0} P_t(s)/a \) (eq. (2)) in the procedure,

\[
\frac{\partial \rho(x, t)}{\partial t} \simeq \frac{D_e}{l^2} [\rho(x - l, t) - 2\rho(x, t) + \rho(x + l, t)]
\]

(57)
is obtained. Here, we introduced the new parameter

\[
D_e \equiv \lim_{a \to 0} a^2 w^*.
\]

(58)

Since spatial scale of observation is of order of \( L_0 \), the spatial coordinate \( x \) is changed to \( \bar{x} \equiv x/L_0 \) as

\[
\frac{\partial \rho(\bar{x}, t)}{\partial t} \simeq \frac{D_e}{L_0^2} \left[ \frac{\rho(\bar{x} - l/L_0, t) - 2\rho(\bar{x}, t) + \rho(\bar{x} + l/L_0, t)}{(l/L_0)^2} \right].
\]

(59)
The limit \( l/L_0 \to 0 \) is taken with the observation scale \( L_0 \) is fixed at a finite value. The result of the limit is given as

\[
\frac{\partial \rho(\bar{x}, t)}{\partial t} = \frac{D_e}{L_0^2} \frac{\partial^2 \rho(\bar{x}, t)}{\partial \bar{x}^2},
\]

(60)

where we assume that the distribution of the random walker after long-time development is smoothed away and the derivative on the right hand side exists. Since \( L_0 \) is finite and the spatial variable can be changed back to \( x \equiv L_0 \bar{x} \), the effective diffusion equation is obtained as

\[
\frac{\partial \rho(x, t)}{\partial t} = D_e \frac{\partial^2 \rho(x, t)}{\partial x^2}.
\]

(61)

It is important to note that the solution of the effective diffusion equation, \( \rho(x, t) \), is the non-averaged density. It means that the solution to the original diffusion equation (eq. (1)) at time \( t \gg l^2/D(x) \) satisfies the effective diffusion equation without any data-processing like averaging.

Finally, we evaluate the effective diffusion constant defined by eq. (44). The continuum limit gives

\[
D_e = \lim_{a \to 0} a^2 w^* = \lim_{a \to 0} \frac{l}{N T \sum_{i=0}^{N_T-1} \frac{1}{D_i}} = \frac{l}{\int_0^l dx \frac{1}{D(x)}}.
\]

(62)

The formula, eq. (62), implies that the effective diffusion constant is mainly determined by regions where the value of the diffusion constant is small. In other words, the speed of the diffusion process is limited by the region where diffusion is slow. It can be called the bottle-neck effect.

§5. Discussion and Summary

In this paper, the effective diffusion constant of diffusion process in periodic inhomogeneous media is evaluated analytically with the real-space RG method.
As pointed out in § 1, the problem has been solved with the multi-scale method. We compare the two methods.

In the multi-scale method, different space-time scales $x, t; \epsilon x, \epsilon^2 t; \ldots$ characterized by arbitrary small number $\epsilon$ are introduced and the perturbation expansion in terms of $\epsilon$ is performed. The parameter $\epsilon$ describes separation of the two characteristic scales, the scale of the microscopic structure of the medium and the observation scale. Hence the parameter $\epsilon$ corresponds to the parameter $L_0$ used to characterize the scale of observation in the RG method. The development equations of the slow variations are obtained as conditions for absence of singular and secular terms, which are called the solvability conditions and compatibility conditions.

In general, the proper slow variables, $\epsilon x, \epsilon^2 t; \epsilon^2 x, \epsilon^4 t; \ldots$, cannot be selected automatically and selection needs trial and error. On the other hand, in the RG methods the slow variables are obtained automatically as rescaled variables $x^*, t^*$ at the fixed points. However, the extension as explained in § 3.1 and the decimation procedure have to be performed in a proper way so that the renormalization transform is as simple as possible. Selection of the best way of extension and decimation needs some trial and error.

Furthermore, the renormalization transform is formulated as recursion formulae of parameters. Hence, even if the transform is so complicated that evaluating the fixed point analytically is difficult, it would be easier to compute the fixed points numerically. Especially, the real-space RG on a discrete lattice space can be converted to numerical algorithms easily.

Although the RG method has some advantages, for the problem studied in the present paper, calculation with the multi-scale method is much easier than that with the RG method.

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