Bonds with volatilities proportional to forward rates

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Abstract
The problem of existence of solution for the Heath-Jarrow-Morton equation with linear volatility and purely jump random factor is studied. Sufficient conditions for existence and non-existence of the solution in the class of bounded fields are formulated. It is shown that if the first derivative of the Lévy-Khinchin exponent grows slower than logarithmic function then the answer is positive and if it is bounded from below by a fractional power function of any positive order then the answer is negative. Numerous examples including models with Lévy measures of stable type are presented.

Key words: bond market, HJM condition, linear volatitlity.

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JEL Classification Numbers: G10, G12.

1 Introduction

We are concerned with the bond market model, on a fixed time interval $[0, T^*)$, $T^* < \infty$, in which the bond prices $P(t, T)$, $0 \leq t \leq T \leq T^*$, are represented in the form,

$$P(t, T) = e^{-\int_t^T f(t,u)du}, \quad t \leq T \leq T^*.$$ 

Moreover, forward curves processes $f(t, T)$, $0 \leq t \leq T \leq T^*$, are Itô processes with stochastic differentials:

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dL(t), \quad (t, T) \in T,$$

where

$$T := \{(t, T) \in \mathbb{R}^2 : 0 \leq t \leq T \leq T^*\}.$$ 

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The random factor process $L$ is a real Lévy process defined on a fixed probability space $(\Omega, \mathcal{F}, P)$.

One can extend the definition of $f$ given by (1.1) on the set $[0, T^*] \times [0, T^*]$ by putting

$$\alpha(t, T) = 0, \quad \sigma(t, T) = 0 \quad \text{for } t \in (T, T^*].$$

(1.3)

Let $\mathcal{P}$ and $\mathcal{O}$ denote the predictable and optional $\sigma$-field on $\Omega \times [0, T^*]$ respectively. We follow [5] in imposing assumptions on the drift and volatility coefficients in (1.1):

$$(\omega, t, T) \rightarrow \alpha(\omega, t, T), \sigma(\omega, t, T)$$ are $\mathcal{P} \otimes \mathcal{B}([0, T^*])$ measurable

(1.4)

$$\sup_{0 \leq t, T \leq T^*} \left\{ |\alpha(t, T)| + |\sigma(t, T)| \right\} < \infty.$$  

(1.5)

Conditions (1.3)-(1.5) provide that we can find a version of $f$ such that for each $T \in [0, T^*]$ 

$$(\omega, t, T) \rightarrow f(\omega, t, T), \quad t \leq T \leq T^*$$ is $\mathcal{O} \otimes \mathcal{B}([0, T^*])$ measurable.

(1.6)

Condition (1.3) implies that

$$f(t, T) = f(T, T), \quad \text{for } t \in (T, T^*]$$

and consequently that the discounted bond price process defined by

$$\hat{P}(t, T) := e^{-\int_t^T r(s)ds} \cdot P(t, T), \quad (t, T) \in [0, T^*] \times [0, T^*],$$

with a short rate $r(t) := f(t, t)$, is given by the formula

$$\hat{P}(t, T) = e^{-\int_t^T f(t, u)du}, \quad (t, T) \in [0, T^*] \times [0, T^*].$$

If one assumes in addition that $\hat{P}(\cdot, T), T \in [0, T^*]$ are local martingales then for each $T \in [0, T^*]$, see [4], [8],

$$\int_t^T \alpha(t, u)du = J \left( \int_t^T \sigma(t, u)du \right)$$

(1.7)

for almost all $t \in [0, T]$, where the function $J$ is the Lévy - Khinchin exponent determined by the Laplace transform:

$$E(e^{-zL(t)}) = e^{tJ(z)}, \quad t \in [0, T^*], \ z \in \mathbb{R}.$$  

Of prime interest is to find out under what conditions one can model bond prices with volatility proportional to forward curves:

$$\sigma(t, T) = \lambda(t, T)f(t-, T), \quad (t, T) \in T,$$

(1.8)

where $\lambda$ is a continuous deterministic function on $T$ bounded from below and from above by positive constants $\underline{\lambda}$, $\overline{\lambda}$:

$$0 < \underline{\lambda} \leq \lambda(t, T) \leq \overline{\lambda} < +\infty, \ (t, T) \in T.$$  

Obviously one can choose $\overline{\lambda}$ arbitrarily large. For technical reasons we assume that $\overline{\lambda} \geq 1$.

This problem has been first stated in [9] in the case when $L$ is a Wiener process and solved with a negative answer: linearity of volatility implies explosion of forward rates, see
Differentiating the identity (1.7) with respect to $T$ and taking into account the condition (1.8) we see that proportionality of the volatility implies that the forward curve satisfies the following equation on $T$,

$$df(t, T) = J'(\int_t^T \lambda(t, u)f(t-, u)du) \lambda(t, T)f(t-, T)dt + \lambda(t, T)f(t-, T)dL(t).$$  \hspace{1cm} (1.9)

with initial condition,

$$f(0, T) = f_0(T), \quad T \in [0, T^*].$$  \hspace{1cm} (1.10)

In particular if $L$ is a Wiener process then $J(z) = \frac{1}{2}z^2$ and if $\sigma(t, T) = f(t, T)$ then (1.9) becomes

$$df(t, T) = \left(\int_t^T f(t, u)du\right) f(t, T)dt + f(t, T)dL(t), \quad (t, T) \in T.$$

This equation has been studied in [9].

Taking into account (1.3)-(1.6) we assume that $\lambda(t, T) = 0$ for $t \in (T, T^*]$ and we search for a solution $f$ of (1.9) in the class of random fields satisfying the following conditions

$$(\omega, t, T) \rightarrow f(\omega, t, T), \quad 0 \leq t \leq T \leq T^* \text{ is } \mathcal{O} \times \mathcal{B}([0, T^*]) \text{ measurable},$$  \hspace{1cm} (1.11)

$$f(\cdot, T) \text{ is càdlàg on } [0, T] \text{ for each } T \in [0, T^*]$$  \hspace{1cm} (1.12)

$$(\omega, t, T) \rightarrow f(\omega, t-, T) \text{ is } \mathcal{P} \times \mathcal{B}([0, T^*]) \text{ measurable},$$  \hspace{1cm} (1.13)

$$\sup_{(t,T) \in T} f(t, T) < \infty, \quad P - \text{a.s..}$$  \hspace{1cm} (1.14)

Requirement (1.14) states that the function $f(\omega, \cdot, \cdot)$ is bounded on $T$ but notice that the bounds may depend on $\omega$. Random fields satisfying (1.11)-(1.14) will be called the class of bounded fields on $T$.

We also examine explosions of solutions from the class of locally bounded fields. For $0 < x \leq T^*$, $0 < y \leq T^*$ consider a family of subsets of $T$ given by

$$T_{x,y} := \{(t, T) \in T : 0 \leq t \leq x, 0 \leq T \leq y\}.$$  \hspace{1cm} (1.15)

A random field is locally bounded if it is bounded on $T_{T^*-\delta, T^*-\delta}$ for each $0 < \delta < T^*$.

The main question of the paper is concerned with existence or non-existence of solutions to (1.9) - (1.10). We derive conditions on the Lévy process $L$ under which there exists a bounded field solving (1.9), see Theorem 3.1 and conditions under which such solutions do not exist, see Theorem 3.2. In the latter case we assume that $\lambda$ is equal to 1. Under assumptions of Theorem 3.2 we also show that if there exists a locally bounded field $f$ solving (1.9) then it explodes, i.e.

$$\lim_{(t,T) \rightarrow (T^*, T^*)} f(t, T) = +\infty,$$
see, Theorem 3.3. From general characterizations explicit conditions on the jumps of the random factor are deduced implying existence or non-existence of models with proportional volatilities. Results for models with negative jumps are stated as Theorem 4.1 and Theorem 4.3 and with strictly positive jumps in Theorem 4.5, Theorem 4.6 and Theorem 4.7. Note that models with positive jumps are very attractive from the practical point of view. In fact typical shocks shift forward curves upwards what is equivalent to drops in bond prices. Special cases of our existence results can be deduced, via Musiela parametrization, from results presented in [10]. The method of establishing the results on non-existence was inspired by the idea of Morton in [9], where the solution is being compared with a deterministic exploding function.

The paper is organized as follows. Section 2 contains preliminaries necessary to formulation and proofs of the main results of the paper. Section 3 is devoted to the formulation of the main general theorems. Specific families of bond market models are examined in Section 4. Proofs are postponed to Section 5.

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2 Preliminaries

We fix here some notation and definitions needed in the sequel. We also formulate our basic equation in a form easier to investigate.

If $L$ is a Lévy process with the Laplace transform

$$
E(e^{-zL(t)}) = e^{tJ(z)}, \quad t \in [0, T^*], \quad z \in \mathbb{R},
$$

then function $J$ is given by, see [2], [13], [10],

$$
J(z) = -az + \frac{1}{2}qz^2 + \int_{\mathbb{R}}(e^{-zy} - 1 + zy\mathbf{1}_{(-1,1)}(y)) \nu(dy),
$$

where $a \in \mathbb{R}, q \geq 0$ and $\nu$ is a measure which satisfies integrability condition

$$
\int_{\mathbb{R}}y^2 \wedge 1 \nu(dy) < \infty.
$$

In this paper we examine the equation (1.9) with noise being a purely discontinuous Lévy process, without a drift nor a Gaussian part. Thus $L$ is of the form

$$
L(t) := \int_0^t \int_{|y|<1} y \tilde{\pi}(ds, dy) + \int_0^t \int_{|y|\geq 1} y \pi(ds, dy),
$$

where $\pi$ is the Poisson random measure of jumps of $L$ and $\tilde{\pi}$ is the measure $\pi$ compensated by $dt \times \nu(dy)$.

Let us notice, that for each $T$ the solution $f(t, T), \ t \in [0, T]$ of (1.9) is a stochastic exponential and therefore (see Theorem 37 in [11]), equation (1.9) can be equivalently written as:

$$
f(t, T) = f_0(T) \ e^{\int_0^T \int_s^T \lambda(s,u)\mathbf{1}_{(0,T)}(u)ds} \mathbf{1}_{(0,T)}(s)ds + \int_0^T \lambda(s,T)dL(s)
 \prod_{s \leq t}(1 + \lambda(s, T)\Delta L(s)) e^{-\lambda(s, T)\Delta L(s)}, \quad (t, T) \in T,
$$

(2.19)
where $\triangle L(s) = L(s) - L(s-)$. To limit our considerations to models with non-negative forward rates, we impose the following natural assumptions.

**Standing assumptions:**

(K1) The initial curve $f_0$ is positive on $[0, T^*)$.

(K2) The support of the Lévy measure is contained in the interval $(-1/\lambda, +\infty) \subseteq (-1, \infty)$.

Under assumptions (K1) and (K2) we can write equation (2.19) in the form

\[
f(t, T) = f_0(T) \ e^{\int_0^T J^\prime \left( \int_s^T \lambda(s, u) f(s-, u) du \right) ds + \int_0^T \lambda(s, T) dL(s)} \cdot \ e^{\int_0^t \int_{-1/\lambda}^{+\infty} \left( \ln(1 + \lambda(s, T)y) - \lambda(s, T)y \right) \pi(ds, dy),}
\]

\[ (t, T) \in T. \tag{2.20} \]

For brevity denote

\[
a(t, T) := f_0(T) \ e^{\int_0^T \lambda(s, T) dL(s) + \int_0^t \int_{-1/\lambda}^{+\infty} \left( \ln(1 + \lambda(s, T)y) - \lambda(s, T)y \right) \pi(ds, dy)}.
\]

Thus

\[
f(t, T) = a(t, T) \ e^{\int_0^T J^\prime \left( \int_s^T \lambda(s, u) f(s-, u) du \right) \lambda(s, T) ds}, \quad (t, T) \in T. \tag{2.21}
\]

Since, for each $T$ the process, $\tilde{L}(t, T)$, $t \in [0, T]$:

\[
\tilde{L}(t, T) = \int_0^t \lambda(s, T) dL(s) + \int_0^t \int_{-1/\lambda}^{+\infty} \left( \ln(1 + \lambda(s, T)y) - \lambda(s, T)y \right) \pi(ds, dy), \quad t \in [0, T],
\]

has càdlàg trajectories

\[
\sup_{t \in [0, T^*]} \tilde{L}(t, T) < \infty, \quad a.s.,
\]

and therefore $a(\cdot, T)$ is bounded on $[0, T]$ with probability 1.

It turns out that due to the special form of the coefficient $a$ given by (2.21) we can replace $f(s-, u)$ in (2.22) by $f(s, u)$.

**Proposition 2.1** Assume that $f$ is a bounded field. Then $f$ is a solution of (2.22) if and only if

\[
f(t, T) = a(t, T) \ e^{\int_0^T J^\prime \left( \int_s^T \lambda(s, u) f(s, u) du \right) \lambda(s, T) ds}, \quad (t, T) \in T. \tag{2.24}
\]

**Proof:** We will show that for each $(t, T) \in T$

\[
\int_0^t J^\prime \left( \int_s^T \lambda(s, u) f(s, u) du \right) \lambda(s, T) ds = \int_0^t J^\prime \left( \int_s^T \lambda(s, u) f(s-, u) du \right) \lambda(s, T) ds.
\]

Let us start with the observation that for $T \in [0, T^*]$ moments of jumps of the process $f(\cdot, T)$ are the same as for $a(\cdot, T)$. Moreover, it follows from (2.21) that the set of jumps of $a(\cdot, T)$ is independent of $T$ and is contained in the set

\[
Z := \{ t \in [0, T^*] : \triangle L(t) \neq 0 \}.
\]

Thus if $s \notin Z$ then

\[
J^\prime \left( \int_s^T \lambda(s, u) f(s, u) du \right) \lambda(s, T) = J^\prime \left( \int_s^T \lambda(s, u) f(s-, u) du \right) \lambda(s, T).
\]

By Th. 2.8 in [1] the set $Z$ is at most countable, so the assertion follows. \qed

In the sequel we will examine equation (2.22) with $f(s-, u)$ replaced by $f(s, u)$.
2.1 Properties of $J$

In virtue of (2.16), (2.18) and the standing assumption (K2) the function $J$ is given by the formula

$$J(z) = \int_{\mathbb{R}} (e^{-zy} - 1 + zy\mathbf{1}_{(-1,1)}(y)) \, \nu(dy)$$

$$= \int_{-1/\lambda}^{1} (e^{-zy} - 1 + zy) \, \nu(dy) + \int_{1}^{\infty} (e^{-zy} - 1) \, \nu(dy).$$

Taking into account (2.17) we see that the function $J$ is well defined for $z \geq 0$. Let us notice that in our setting we do not have to consider $J$ on the set $(-\infty, 0)$. Indeed, the assumptions (K1) and (K2) imply that $f$ is positive, so the form of the equation (1.7) together with the condition (1.8) allow us to focus on the properties of the function $J$ and its derivatives on the interval $[0, \infty)$. Moreover, the condition (2.17) implies that for $z > 0$ the function $J$ has derivatives of any order and the following formulas hold, see Lemma 8.1 and 8.2 in [12],

$$J'(z) = \int_{-1/\lambda}^{1} y(1 - e^{-zy}) \, \nu(dy) - \int_{1}^{\infty} ye^{-zy} \, \nu(dy), \quad J'(0) = -\int_{1}^{\infty} y \nu(dy)$$

$$J''(z) = \int_{-1/\lambda}^{1} y^2 e^{-zy} \, \nu(dy), \quad J'''(z) = -\int_{1/\lambda}^{\infty} y^3 e^{-zy} \, \nu(dy).$$

Thus the objective of this paper is to examine existence of a bounded solution for the equation

$$f(t, T) = a(t, T) e^{\int_{0}^{t} J'(s) \lambda(s, u) f(s, u) du} \lambda(s, T) ds, \quad (t, T) \in T,$$

where

$$J'(z) = \int_{-1/\lambda}^{1} y(1 - e^{-zy}) \nu(dy) - \int_{1}^{\infty} ye^{-zy} \nu(dy), \quad z \geq 0,$$

and the jump intensity measure $\nu$ is concentrated on $(-1/\lambda, 0) \cup (0, +\infty)$ and satisfies

$$\int_{(-1/\lambda, 1]} y^2 \nu(dy) + \int_{1}^{\infty} y \nu(dy) < \infty.$$

Note that the function $J'$ in the basic equation is increasing on the whole interval $[0, +\infty)$ and $J'(0)$ is either 0, if all jumps of $L$ are of size smaller or equal than 1, or is strictly negative. The latter integral in (2.29) is required to be finite to imply that $J'(0)$ is finite. Moreover, if

$$\int_{1}^{\infty} y^2 \nu(dy) < \infty,$$

then $J''$ is a bounded function on $[0, +\infty)$ and therefore $J'$ is a Lipschitz function on $[0, +\infty)$. In fact, (2.30) is also a necessary condition for $J'$ to be Lipschitz. The conditions (2.29) and (2.30) are equivalent to the, respectively, integrability and square integrability of the process $L$, see [13].

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3 Main results

In this section we present formulation of the main theorems which provide sufficient conditions for existence and non-existence solution of the problem stated in Section 2. Their proofs are contained in Section 5 and are preceded by a sequence of auxiliary results.

The following result provides sufficient conditions for existence of a bounded solution.

**Theorem 3.1** Assume (2.29) and that
\[
\limsup_{z \to \infty} \left( \ln z - \bar{\lambda} T^* J'(z) \right) = \infty. \tag{3.31}
\]
i) If the initial forward curve \( f_0 \) is bounded almost surely then there exists a solution \( f : T \to \mathbb{R}_+ \) of (2.28) which is also bounded almost surely.

ii) If, in addition, (2.30) holds then the solution \( f \) is unique in the class of bounded fields.

The next results provide conditions which imply non-existence of solution in the class of bounded fields and explosions of locally bounded fields.

**Theorem 3.2** Assume (2.29), that \( \lambda \equiv 1 \) and for some \( \alpha > 0, \beta \in \mathbb{R}, \gamma \in (0,1), \)
\[
J'(z) \geq \alpha z^\gamma + \beta, \quad \forall z \geq 0. \tag{3.32}
\]
For arbitrary \( \kappa \in (0,1) \), there exists a positive constant \( K \) such that if
\[
f_0(T) > K, \quad \forall T \in [0,T^*], \tag{3.33}
\]
then there is no solution \( f : T \to \mathbb{R}_+ \) of the equation (2.28) which is bounded with probability greater or equal than \( \kappa \).

**Theorem 3.3** Assume that there exists a locally bounded solution of (2.28) and that all the assumptions of Theorem 3.2 are satisfied. Then
\[
\lim_{(t,T) \to (T^*,T^*)} f(t,T) = +\infty
\]
with probability greater or equal than \( \kappa \).

In the case when \( \lambda \equiv 1 \) and there is no solution of equation (2.28) in the class of bounded fields then one may ask if the solution does exist in a wider class of fields satisfying some integrability conditions. However, in some situations these two classes are the same. Assume, for example, that the solution is supposed to satisfy condition:
\[
\int_0^{T^*} J' \left( \int_s^{T^*} f(s,u)du \right) ds < \infty.
\]
Then, due to the fact that \( J'(\cdot) \) is increasing, we see that \( f \) is well defined for any \( (t,T) \in T \). Moreover, if \( f_0 \) is bounded, then for any \( (t,T) \in T \)
\[
f(t,T) = e^{\int_t^T J' \left( \int_s^T f(s,u)du \right) ds} \cdot a(t,T)
\]
\[
\leq e^{\int_0^{T^*} J' \left( \int_s^{T^*} f(s,u)du \right) ds} \sup_{T \in [0,T^*]} f_0(T) \cdot \sup_{t \in [0,T^*]} e^{L(t)+f_0^* \int_t^{T^*} \left( \ln(1+y) - y \right) \pi(ds,dy)} < \infty,
\]
and as a consequence \( f \) is bounded.
Remark 3.4 Let the assumptions of Theorem 3.2 be satisfied and that \( f \) is a random field solving (2.28) and for which
\[
\sup_{(t,T) \in \mathcal{T}_{x,y-\delta}} f(t,T) < \infty \quad P - \text{a.s.,}
\]
for some \( 0 < x \leq y \leq T^* \) and each \( 0 < \delta < y \). Then following the proof of Theorem 3.3 one can show that if \( f_0 \) is sufficiently large, then
\[
\lim_{(t,T) \to (x,y)} f(t,T) = +\infty,
\]
with probability arbitrarily close to 1.

4 Specific models

The crucial properties which imply existence or non-existence of solution of the equation (2.28) are (3.32) and (3.31). If (3.32) holds then there is no solution and if (3.31) is satisfied then there is a solution. It turns out that models with negative jumps do not allow bounded solutions. For models with positive jumps the answer does depend on the growth of the measure \( \nu \) near 0.

4.1 Models with negative jumps

Theorem 4.1 If the measure \( \nu \) has support in \((-1,0)\) then the equation (2.28) with \( \lambda \equiv 1 \) has no bounded solutions.

Proof: Since
\[
J''(z) = -\int_{-1}^{0} y^3 e^{-zy} \nu(dy) \geq 0, \quad \forall z \geq 0,
\]
the function \( J' \) is convex and due to Lemma 4.2 below the condition (3.32) is satisfied and it is enough to apply Theorem 3.2.

Lemma 4.2 If \( J' \) is a convex function on \([0,\infty)\) then (3.32) is satisfied.

Proof: In virtue of the inequality \( z \geq \sqrt{z} - 1, \) for \( z \geq 0, \) we have
\[
J'(z) \geq J''(0)z + J'(0) \geq J''(0)(\sqrt{z} - 1) + J'(0), \quad \forall z \geq 0.
\]

Theorem 4.3 Let \( \nu \) be given by
\[
\nu(dy) = \frac{1}{\left| y \right|^{1+\rho}} \mathbb{1}_{(-1,1)}(y) dy, \quad \rho \in (0,2) \quad \text{or} \quad \nu(dy) = \frac{1}{\left| y \right|^{1+\rho}} \mathbb{1}_{(-1,\infty)}(y) dy, \quad \rho \in (1,2),
\]
then equation (2.28) with \( \lambda \equiv 1 \) has no bounded solutions.

Proof: We will show that
\[
J'(z) \geq \frac{2}{2 - \rho} z, \quad z \geq 0.
\]
in the first case and for some $\beta$,

$$J'(z) \geq \frac{2}{2 - \rho} z - \beta, \quad z \geq 0,$$

in the second case. By Theorem 3.2 the result will follow. In virtue of (2.26) we have

$$J'(z) = \int_{-1}^{1} y(1 - e^{-zy}) \frac{1}{y^{1+\rho}} dy = \int_{-1}^{0} y(1 - e^{-zy}) \frac{1}{y^{1+\rho}} dy + \int_{0}^{1} y(1 - e^{-zy}) \frac{1}{y^{1+\rho}} dy = -z^{\rho-1} \int_{0}^{z} \frac{1 - e^v}{v^\rho} dv + z^{\rho-1} \int_{0}^{z} \frac{1 - e^{-v}}{v^\rho} dv = z^{\rho-1} \int_{0}^{z} \frac{e^v - e^{-v}}{v^\rho} dv.$$

We use the series expansion

$$e^v - e^{-v} = 2 \sum_{k=0}^{\infty} \frac{v^{2k+1}}{(2k+1)!}.$$

As a consequence we have

$$\int_{0}^{z} \frac{e^v - e^{-v}}{v^\rho} dv = 2 \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = 2 \sum_{k=0}^{\infty} \frac{z^{2k+2-\rho}}{(2k + 2 - \rho)(2k + 1)!}$$

and

$$J'(z) = z^{\rho-1} \int_{0}^{z} \frac{e^v - e^{-v}}{v^\rho} dv = z^{\rho-1} \int_{0}^{z} \frac{e^v - e^{-v}}{v^\rho} dv = 2 \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+2-\rho)(2k+1)!} \geq \frac{2}{2 - \rho} z.$$

We pass now to the second case.

Using (2.26) and calculating as above we have

$$J'(z) = z^{\rho-1} \int_{0}^{z} \frac{e^v - e^{-v}}{v^\rho} dv - \int_{1}^{\infty} \frac{e^{-zy}}{y^\rho} dy.$$

For $z \geq 0$ we have the following estimation

$$\int_{1}^{\infty} \frac{e^{-zy}}{y^\rho} dy \leq \int_{1}^{\infty} \frac{1}{y^\rho} dy =: \beta < \infty$$

and as a consequence

$$J'(z) \geq \frac{2}{2 - \rho} z - \beta, \quad z \geq 0.$$

The proof is complete in virtue of the inequality $z \geq \sqrt{z} - 1$ for $z \geq 0$. □

**Remark 4.4** We restricted $\rho$ to the interval $(1, 2)$ to satisfy (2.29).
4.2 Models with positive jumps only

We pass now to models which generate bounded solutions and therefore might be attractive for applications.

We start from the following theorem which covers many interesting cases with finite and infinite measure $\nu$.

**Theorem 4.5** Let $\nu$ be a Lévy measure on $(0, \infty)$ satisfying

$$\int_0^\infty y \nu(dy) < \infty,$$

Then the equation (2.28) has a bounded solution.

**Proof:** It is enough to prove that $J'$ is a bounded function. In virtue of (2.26) we have

$$J'(z) = \int_0^1 y(1 - e^{-zy})\nu(dy) - \int_1^\infty ye^{-zy}\nu(dy).$$

Since $J'(z) \leq \int_0^1 y \nu(dy)$ the boundedness follows.

**Theorem 4.6** Let $\nu$ be given by

$$\nu(dy) = \frac{1}{y^{1+\rho}}1_{(0,1)}(y) dy, \quad \rho \in (0, 2).$$

Then

1) if $\rho \in (1, 2)$ then equation (2.28) with $\lambda \equiv 1$ has no bounded solutions

2) if $\rho \in (0, 1)$ or

3) $\rho = 1$ and $\bar{T}^* < 1$ then equation (2.28) has a bounded solution.

**Proof:** In virtue of (2.26) we have

$$J'(z) = \int_0^1 y(1 - e^{-zy})\frac{1}{y^{1+\rho}}dy = \int_0^z \frac{1 - e^{-v}}{(\frac{v}{z})^\rho} \frac{1}{z} dv = z^{\rho-1} \int_0^z \frac{1 - e^{-v}}{v^\rho} dv. \quad (4.34)$$

Let us consider the following cases.

1) $\rho \in (1, 2)$

Then for $\alpha := \int_0^1 \frac{1 - e^{-v}}{v^\rho} dv > 0$ we have

$$J'(z) \geq \alpha z^{\rho-1} \quad \text{for} \quad z \geq 1.$$

The function $J'$ is nonnegative on $[0, \infty)$ and thus

$$J'(z) \geq \alpha z^\gamma - \alpha \quad \text{for} \quad z \geq 0,$$

with $\gamma := \rho - 1 \in (0, 1)$. As a consequence (3.32) is satisfied with $\beta = -\alpha$. 

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2) $\rho \in (0, 1)$

We will show that $\lim_{z \to \infty} \tilde{\lambda} T^* J'(z) < \infty$, what implies (3.31). We have

$$\lim_{z \to \infty} J'(z) = \lim_{z \to \infty} z^{\rho-1} \int_0^1 \frac{1 - e^{-v}}{v^{\rho}} dv$$

$$= \lim_{z \to \infty} \int_0^z \frac{1 - e^{-v}}{v^{1-\rho}} dv \frac{1}{z^{1-\rho}} = \lim_{z \to \infty} \frac{1 - e^{-v}}{(1 - \rho)z^{\rho}}$$

$$= \lim_{z \to \infty} \frac{1 - e^{-z}}{1 - \rho} = \frac{1}{1 - \rho}.$$

3) $\rho = 1$ and $\tilde{\lambda} T^*< 1$

One can check that in this case $J'$ is unbounded and we can show that

$$\lim_{z \to \infty} \frac{\ln z}{\tilde{\lambda} T^* J'(z)} > 1.$$

This condition clearly implies (3.31). We have

$$\lim_{z \to \infty} \frac{\ln z}{\tilde{\lambda} T^* J'(z)} = \lim_{z \to \infty} \frac{-e^{-z}}{z(1 - e^{-z})} = \frac{1}{\lambda T^*} > 1.$$  

□

Our final class of examples is with large jumps.

**Theorem 4.7** Let $\nu$ be given by

$$\nu(dy) = \frac{1}{y^{1+\rho}} 1_{(0, \infty)}(y) dy, \quad \rho \in (1, 2).$$

Then the equation (2.28) with $\lambda \equiv 1$ has no bounded solutions.

**Proof:** In virtue of (2.26) we have

$$J'(z) = \int_0^1 y(1 - e^{-zy}) \frac{1}{y^{1+\rho}} dy - \int_1^{\infty} ye^{-zy} \frac{1}{y^{1+\rho}} dy$$

$$= z^{\rho-1} \int_0^z \frac{1 - e^{-v}}{v^{\rho}} dv - \int_1^{\infty} e^{-zy} \frac{1}{y^{\rho}} dy. \quad (4.35)$$

Due to the inequality

$$\int_1^{\infty} \frac{e^{-zy}}{y^{\rho}} dy \leq \int_1^{\infty} \frac{1}{y^{\rho}} dy < \infty, \quad z \geq 0,$$

and the estimation from the proof of Th. 4.6 (1) we have

$$J'(z) \geq \alpha z^\gamma - \alpha - \int_1^{\infty} \frac{1}{y^{\rho}} dy, \quad z \geq 0,$$

so (3.32) holds with $\alpha = \int_0^1 \frac{1 - e^{-v}}{v^{\rho}} dv, \quad \gamma = \rho - 1$, $\beta = -\alpha - \int_1^{\infty} \frac{1}{y^{\rho}} dy$. □

5 Proofs of the main theorems

This section is divided into two parts containing proofs of Theorems 3.2, 3.3 and 3.1 respectively with all auxiliary lemmas and propositions.
5.1 Non-existence

Recall that the sets $T$ and $T_{x,y}$, where $0 < x \leq T^*$, $0 < y \leq T^*$ are given by (1.2) and (1.15). In the sequel we will use the notation: $\mathbb{R}_+ := \mathbb{R}_+ \cup \{+\infty\}$.

Lemma 5.1 Let $f: [a,b] \rightarrow \mathbb{R}_+$, where $a, b \in \mathbb{R}, a < b$, be a continuous function. For any $\gamma \in (0,1)$ we have

$$
\int_a^b f^\gamma(x)dx \leq (b-a)^{1-\gamma} \left(\int_a^b f(x)dx\right)^\gamma.
$$

(5.36)

Proof: If $z_1, z_2, ..., z_n$ are positive reals and $\gamma \in (0,1)$ then

$$
\left(\frac{1}{n} \sum_{i=1}^n z_i^\gamma\right)^\frac{1}{\gamma} \leq \left(\frac{1}{n} \sum_{i=1}^n z_i\right)^\frac{1}{\gamma}.
$$

(5.37)

In fact, by Hölder inequality with $p = \frac{1}{\gamma}$ and $q = \frac{1}{1-\gamma}$,

$$
\sum_{i=1}^n z_i^\gamma \leq \left(\sum_{i=1}^n (z_i^\gamma)^\frac{1}{\gamma}\right)^\gamma \left(\sum_{i=1}^n 1^\frac{1}{1-\gamma}\right)^{1-\gamma}
$$

and rearranging terms one gets (5.37).

Let us consider an equidistant partition of the interval $[a,b]$ with $x_i = a + i \cdot \frac{b-a}{n}$, $i = 1, 2, ..., n$. Using (5.37) with $z_i = f(x_i)$, $i = 1, 2, ..., n$, we obtain

$$
\sum_{i=1}^n \frac{b-a}{n} f^\gamma(x_i) = (b-a) \left(\frac{1}{n} \sum_{i=1}^n f^\gamma(x_i)\right) \leq (b-a) \left(\frac{1}{n} \sum_{i=1}^n f(x_i)\right)^\gamma
$$

$$
= (b-a)^{1-\gamma} \left(\sum_{i=1}^n \frac{b-a}{n} f(x_i)\right)^\gamma.
$$

(5.38)

Letting $n \rightarrow \infty$ in (5.38) we obtain (5.36). \qed

In the following, for any $\alpha > 0$ and $\gamma \in (0,1)$ and $0 < x \leq y \leq T^*$, we will consider the function $h: T_{x,y} \rightarrow \mathbb{R}_+$ given by

$$
h(t,T) := \begin{cases} 
\left(\frac{1}{x-t+y-z}\right)^\frac{1}{\gamma} & \text{for } (t,T) \neq (x,y) \\
\infty & \text{for } (t,T) = (x,y),
\end{cases}
$$

(5.39)

and the function $R: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined as

$$
R(z) := \alpha z 1_{[0,1]}(z) + \alpha z^\gamma 1_{(1,\infty)}(z) \quad z \in \mathbb{R}_+.
$$

(5.40)

The following properties of the function $R$ can be easily verified

$$
\alpha z^\gamma \geq R(z) \geq \alpha z^\gamma - 1, \quad z \in \mathbb{R}_+.
$$

(5.41)

$$
|R(z_1) - R(z_2)| \leq \alpha |z_1 - z_2|, \quad z_1, z_2 \in \mathbb{R}_+.
$$

(5.42)

Proposition 5.2 Let $\alpha > 0$, $\gamma \in (0,1)$, $0 < x \leq y \leq T^*$ and the functions $h, R$ be given by (5.39) and (5.40) respectively. The function $g: T_{x,y} \rightarrow \mathbb{R}_+$ defined by the formula

$$
g(t,T) := \begin{cases} 
eq f_0^T R(s, h(s,u))du) \cdot h(t,T) & \text{for } (t,T) \neq (x,y) \\
0 & \text{for } (t,T) = (x,y)
\end{cases}
$$

(5.43)

is continuous.
**Proof:** Let us start with an auxiliary calculation and estimation. One can check that
\[
\int_0^t \int_s^T \frac{1}{(x-s+y-u)^3} \, du \, ds = \frac{t}{2} \cdot \frac{-T^2 - Tt - ty + 2Ty + 2Tx - tx}{(x-t+y-T)(x+y-T)(x+y)} \tag{5.44}
\]
for any \((t, T) \in T_{x,y}\).

In virtue of Lemma 5.1 we have
\[
\int_0^t \left( \int_s^T h(s,u) \, du \right)^\gamma \, ds \geq \int_0^t \left( (T-s)^{\gamma-1} \int_s^T h^\gamma(s,u) \, du \right) \, ds \\
\geq T^{\gamma-1} \int_0^t \left( \int_s^T h^\gamma(s,u) \, du \right) \, ds \\
\geq T^{\gamma-1} \int_0^t \int_s^T \frac{1}{(x-s+y-u)^3} \, du \, ds \tag{5.45}
\]

As a consequence of (5.41), (5.45) and (5.44) we have
\[
e^{-\int_0^t \int_s^T h(s,u) \, du \, ds} \cdot h(t, T) \leq e^{-\int_0^t \left\{ \alpha \left( \int_s^T h(s,u) \, du \right)^{\gamma-1} \right\} \, ds} \cdot h(t, T) \\
= e^{-\alpha \int_0^t \left( \int_s^T h(s,u) \, du \right)^\gamma} \cdot e^t \cdot h(t, T) \\
\leq e^{-\alpha T^{\gamma-1} \int_0^t \int_s^T \frac{1}{(x-s+y-u)^3} \, du \, ds} \cdot e^t \cdot \left( \frac{1}{x-t+y-T} \right)^\frac{\gamma}{2} \\
\leq e^{-\frac{\alpha T^2}{2} \frac{T^2 - (y-2t)(x+y-2t)}{(x-t+y-T)(x+y-T)(x+y)} \cdot e^t \cdot \left( \frac{1}{x-t+y-T} \right)^\frac{\gamma}{2} \}.
\]

We need to show continuity of \(g\) only in the point \((x,y)\). We have
\[
\lim_{t \to x,T \to y} (-T^2 - Tt - ty + 2Ty + 2Tx - tx) = y^2 - x^2 > 0 \\
\lim_{t \to x,T \to y} (x + y - 2t) = y - x > 0 \\
\lim_{t \to x,T \to y} (x + y - T) = x > 0.
\]

Thus to show that
\[
\lim_{t \to x,T \to y} g(t, T) = 0
\]

it is enough to notice that
\[
\lim_{z \to -\infty} e^{-cz} z^{\frac{\gamma}{2}} = 0, \quad \text{for } c > 0.
\]

\[\square\]

**Remark 5.3** Let \(\alpha > 0, \gamma \in (0, 1), 0 < x \leq y \leq T^*\). The functions \(h, R, g\) given by (5.39), (5.40), (5.43) satisfy the following equation
\[
h(t, T) = e^{\int_0^t R(\int_s^T h(s,u) \, du) \, ds} \cdot g(t, T), \quad \forall(t,T) \in T_{x,y}.
\]
Proof: We have
\[ h(t, T) = e^{\int_{0}^{t} R(f_{s}^{T} h(s, u)du)ds} \cdot e^{-\int_{0}^{t} R(f_{s}^{T} h(s, u)du)ds} \cdot h(t, T) \]
\[ = e^{\int_{0}^{t} R(f_{s}^{T} h(s, u)du)ds} \cdot g(t, T), \quad \forall (t, T) \in T_{x,y}. \]
\[ \square \]

Lemma 5.4 Let \( 0 < t_{0} \leq T_{0} < \infty \) and define a set
\[ A := \{ (t, T) : t \leq T, \ 0 \leq t \leq t_{0}, \ t \leq T \leq T_{0} \}. \]
If \( d : A \to \mathbb{R}_{+} \) is a bounded function satisfying
\[ d(t, T) \leq K \int_{0}^{t} \int_{s}^{T} d(s, u)duds \quad \forall (t, T) \in A \] (5.46)
where \( 0 < K < \infty \) then \( d(t, T) \equiv 0 \) on \( A \).

Proof: Assume that \( d \) is bounded by a constant \( M > 0 \) on \( A \). We show inductively that
\[ d(t, T) \leq MK^{n} \frac{(tT)^{n}}{(n!)^{2}} \quad \forall (t, T) \in A. \] (5.47)
The formula (5.47) is valid for \( n = 0 \). Assume that it is true for some \( n \) and show that it is true for \( n + 1 \). We have the following estimation
\[ d(t, T) \leq K \int_{0}^{t} \int_{s}^{T} MK^{n} \frac{(su)^{n}}{(n!)^{2}}duds = MK^{n+1} \frac{1}{(n!)^{2}} \int_{0}^{t} s^{n}(\int_{s}^{T} u^{n}du)ds \]
\[ = MK^{n+1} \frac{1}{(n!)^{2}} \int_{0}^{t} s^{n} \frac{T^{n+1} - s^{n+1}}{n + 1} ds \leq MK^{n+1} \frac{1}{(n!)^{2}} \int_{0}^{t} s^{n} \frac{T^{n+1}}{n + 1} ds \]
\[ = MK^{n+1} \frac{1}{(n!)^{2}} \frac{T^{n+1}}{(n + 1)(n + 1)} = MK^{n+1} \frac{(tT)^{n+1}}{((n + 1)!)^{2}}. \]
Letting \( n \to \infty \) in (5.47) we see that \( d(t, T) = 0. \) \( \square \)

Proposition 5.5 Let \( 0 < x \leq y \leq T^{\ast}, \ 0 < \delta < y \) and \( g : T_{x,y-\delta} \to \mathbb{R}_{+} \) be a bounded function. Assume that there exists a bounded function \( h : T_{x,y-\delta} \to \mathbb{R}_{+} \) which solves the following equation
\[ h(t, T) = e^{\int_{0}^{t} R(f_{s}^{T} h(s, u)du)ds} \cdot g(t, T), \quad \forall (t, T) \in T_{x,y-\delta}. \] (5.48)
where \( R \) is given by (5.30). Then \( h \) is uniquely determined in the class of bounded functions on \( T_{x,y-\delta} \).

Proof: Assume that \( h_{1}, h_{2} : T_{x,y-\delta} \to \mathbb{R}_{+} \) are bounded solutions of (5.48). Then the function \( | h_{1} - h_{2} | \) is bounded and satisfies
\[ | h_{1}(t, T) - h_{2}(t, T) | \leq \| g \| \cdot \left| e^{\int_{0}^{t} R(f_{s}^{T} h_{1}(s, u)du)ds} - e^{\int_{0}^{t} R(f_{s}^{T} h_{2}(s, u)du)ds} \right|, \quad \forall (t, T) \in T_{x,y-\delta}, \]
where
\[ \| g \| = \sup_{(t,T)\in T_{x,y-\delta}} | g(t, T) |. \]
As a consequence of the inequality $|e^x - e^y| \leq \max\{e^x, e^y\} |x - y|$ for $x, y \in \mathbb{R}$ we have

$$|h_1(t, T) - h_2(t, T)| \leq K \int_0^t \left| R\left(\int_s^T h_1(s, u)du\right) - R\left(\int_s^T h_2(s, u)du\right)\right| ds, \quad \forall (t, T) \in T_{x,y-\delta},$$

where

$$K := \|g\| \sup_{(t,T)\in T_{x,y-\delta}} \max_{i=1,2} \{e^{f_i^T} R(f_{i(s,u)}du)\} < \infty.$$ 

In virtue of (5.42) we have

$$|h_1(t, T) - h_2(t, T)| \leq \alpha K \int_0^t \int_s^T |h_1(s, u) - h_2(s, u)| \, du \, ds, \quad \forall (t, T) \in T_{x,y-\delta}.$$ 

In view of Lemma 5.4, with $t_0 = \min\{x, y - \delta\}$, $T_0 = y - \delta$, we have $h_1(t, T) = h_2(t, T)$ for all $(t, T) \in T_{x,y-\delta}$. \[\square\]

**Proposition 5.6** Let $\alpha > 0$, $\gamma \in (0,1)$ and function $R$ be given by (5.40). Let $f_1 : T_{x,y-\delta} \rightarrow \mathbb{R}_+$, where $0 < x \leq y \leq T^*$; $0 < \delta < y - x$, be a bounded function satisfying inequality

$$f_1(t, T) \geq e^{\int_0^T R(f_{1(s,u)}du)} \cdot g_1(t, T), \quad \forall (t, T) \in T_{x,y-\delta},$$

(5.49)

where $g_1 : T_{x,y-\delta} \rightarrow \mathbb{R}_+$. Let $f_2 : T_{x,y-\delta} \rightarrow \mathbb{R}_+$ be a bounded function solving equation

$$f_2(t, T) = e^{\int_0^T R(f_{2(s,u)}du)} \cdot g_2(t, T), \quad \forall (t, T) \in T_{x,y-\delta},$$

(5.50)

where $g_2 : T_{x,y-\delta} \rightarrow \mathbb{R}_+$ is a bounded function. Moreover, assume that

$$g_1(t, T) \geq g_2(t, T) \geq 0, \quad \forall (t, T) \in T_{x,y-\delta}.$$ 

(5.51)

Then $f_1(t, T) \geq f_2(t, T)$ for all $(t, T) \in T_{x,y-\delta}$.

**Proof:** Let us define the operator $\mathcal{K}$ acting on bounded functions on $T_{x,y-\delta}$ by

$$\mathcal{K}k(t, T) := e^{\int_0^T R(f_{k(s,u)}du)} \cdot g_2(t, T), \quad (t, T) \in T_{x,y-\delta}.$$ 

(5.52)

Let us notice that in view of (5.49), (5.51) and (5.52) we have

$$\mathcal{K}f_1(t, T) \leq e^{\int_0^T R(f_{1(s,u)}du)} \cdot g_1(t, T) \leq f_1(t, T), \quad \forall (t, T) \in T_{x,y-\delta}.$$ 

(5.53)

It is clear that the operator $\mathcal{K}$ is monotonic, i.e.

$$k_1(t, T) \leq k_2(t, T) \quad \forall (t, T) \in T_{x,y-\delta} \implies \mathcal{K}k_1(t, T) \leq \mathcal{K}k_2(t, T) \quad \forall (t, T) \in T_{x,y-\delta}.$$ 

(5.54)

Let us consider the sequence of functions: $f_1, \mathcal{K}f_1, \mathcal{K}^2 f_1, \ldots$. In virtue of (5.53) and (5.54) we see that $f_1 \geq \mathcal{K}f_1 \geq \mathcal{K}^2 f_1 \geq \ldots$. Thus this sequence is pointwise convergent to some function $\bar{f}$ and it is bounded by $f_1$, so applying the dominated convergence theorem in the formula

$$\mathcal{K}^{n+1} f_1(t, T) = e^{\int_0^T R(f_{n+1(s,u)}du)} \cdot g_2(t, T), \quad (t, T) \in T_{x,y-\delta}.$$
Assume that there exists a bounded solution of (2.28). Fix any $(\cdot, \cdot)$,

**Proof of Theorem 3.2**

Assume that there exists a bounded solution of (2.28). Fix any $(x, y) \in \mathcal{T}$ such that $x > 0$ and three deterministic functions $h : \mathcal{T} \rightarrow \mathbb{R}_+$, $R : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $g : \mathcal{T} \rightarrow \mathbb{R}$ given by (5.39), (5.40) and (5.43) respectively. Recall that, due to Remark 5.3, they satisfy the equation

\[ h(t, T) = e^{\int_0^T R(f(s, u)du)ds} \cdot g(t, T), \quad \forall (t, T) \in \mathcal{T}_{x,y}. \]  

Due to (3.32) and (5.41), the forward rate $f$ satisfies the following inequality

\[ f(t, T) = e^{\int_0^T R(f(s, u)du)ds} \cdot a(t, T) \]

In virtue of Proposition 5.2 the function $g$ is continuous on $\mathcal{T}_{x,y}$ and thus bounded. Thus, see (2.23), if the constant $K$ is sufficiently large, with a probability arbitrarily close to 1,

\[ e^{\beta t} a(t, T) \geq g(t, T), \quad \forall (t, T) \in \mathcal{T}_{x,y}. \]  

Let us fix $0 < \delta < y$ and consider inequality (5.56) and equality (5.55) on the set $\mathcal{T}_{x,y-\delta}$. Then the function $h$ is continuous. In virtue of Proposition 5.6 we have

\[ f(t, T) \geq h(t, T), \quad \forall (t, T) \in \mathcal{T}_{x,y-\delta}. \]

As a consequence we have

\[ f(t, T) \geq h(t, T) = \frac{1}{(x - t + y - T)^\gamma}, \quad \forall (t, T) \in \mathcal{T}_{x,y-\delta}. \]

For any sequence $(t_n, T_n) \in \mathcal{T}_{x,y}$ satisfying $t_n \uparrow x$, $T_n \uparrow y$ define a sequence $\delta_n := \frac{y - T_n}{2}$. Then

\[ f(t, T) \geq \frac{c}{(x - t + y - T)^\gamma}, \quad \forall (t, T) \in \mathcal{T}_{x,y-\delta_n}, \]

and in particular $f(t_n, T_n) \geq \frac{c}{(x - t_n + y - T_n)^\gamma}$. As a consequence $\lim_{n \to \infty} f(t_n, T_n) = +\infty$ what is a contradiction with the assumption that $f$ is bounded.

The proof of Th 3.3 can be deduced from the proof of Th 3.2

**Proof of Theorem 3.3**

We follow the proof of Th 3.2 with $x = T^*$, $y = T^*$. From the fact that $f$ is locally bounded we have

\[ f(t, T) \geq h(t, T), \quad \forall (t, T) \in \mathcal{T}_{T^*, T^*-\delta}, \]

for each $0 < \delta < T^*$. As a consequence

\[ \lim_{(t,T) \to (T^*, T^*)} f(t, T) = +\infty. \]
5.2 Existence

We can write (2.28) in the form

\[ f = A f, \]

where

\[ A h(t, T) := a(t, T) \cdot \int_{t}^{T} J'(s) \lambda(s, T) h(s, T) ds, \]

\( (t, T) \in T. \)

The proof of Theorem 3.1 is based on the properties of the operator \( A \). If we fix \( \omega \in \Omega \) then we can treat \( A \) as a purely deterministic transformation with the function \( a \) positive and bounded.

**Proposition 5.7** Assume that the function \( J' \) satisfies (3.31) and \( a \) is a nonnegative function bounded from above by some constant \( K \). Then there exists a positive constant \( c \) such that if

\[ h(t, T) \leq c, \quad \forall (t, T) \in T \]

for a non-negative function \( h \), then

\[ A h(t, T) \leq c, \quad \forall (t, T) \in T. \]

**Proof:** Let us assume that \( h(t, T) \leq c \) for all \((t, T) \in T\) for some positive \( c \). Using the fact that \( J' \) is increasing and \( \lambda \) positive, we have

\[ A h(t, T) \leq a(t, T) \cdot J'(\tilde{\lambda}cT^*) \int_{t}^{T} \lambda(s, T) ds \]

Since \( a \) is bounded by a constant \( K \) we arrive at the following inequality

\[ A h(t, T) \leq Ke^{J'(\tilde{\lambda}cT^*)} \int_{t}^{T} \lambda(s, T) ds, \quad (t, T) \in T. \]

It is therefore enough to find a positive constant \( c \) such that

\[ \ln K + J'(\tilde{\lambda}cT^*) \cdot \int_{t}^{T} \lambda(s, T) ds \leq \ln c, \quad (t, T) \in T. \]

If the function \( J' \) is negative on \([0, +\infty)\) then it is enough to take \( c = K \). If \( J' \) takes positive values then it is enough to find a positive an arbitrarily large constant \( c \) such that

\[ \ln K + \tilde{\lambda}T^* \cdot J'(\tilde{\lambda}cT^*) \leq \ln c, \quad (t, T) \in T. \]

Existence of such \( c \) is an immediate consequence of the assumption (3.31).

**Proof of Theorem 3.1**

Part i). The operator \( A \) is monotonic, i.e.

\[ h_1 \leq h_2 \implies A h_1 \leq A h_2. \]

The sequence \( h_0 \equiv 0, \ h_{n+1} := A h_n \) is thus monotonically increasing to \( \bar{h} \) and by the monotone convergence theorem we have

\[ \bar{h}(t, T) = A \bar{h}(t, T), \quad \forall (t, T) \in T. \]

Moreover, since \( h_0 \leq c \), where \( c = c(\omega) \) is given by Proposition 5.7 \( \bar{h} \) is bounded. From the form of the operator \( A \) it follows that \( \bar{h}(\cdot, T) \) is càdlàg for each \( T \in [0, T^*] \). Conditions (1.11) and (1.13) follows from the fact that \( \bar{h} \) is a pointwise limit.

Part ii). The function \( J' \) is Lipschitz on \([0, +\infty)\) and therefore we can repeat all arguments from the proof of Proposition 5.5 and the result follows. \( \square \)
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