Sequences of lower bounds for entropic uncertainty relations from bistochastic maps

Paolo Giorda

Dip. Fisica, University of Pavia, via Bassi 6, I-27100 Pavia, Italy; Consorzio Nazionale Interuniversitario per la Scienze fisiche della Materia (CNISM), Italy

Given two orthonormal bases $\mathcal{A}$ and $\mathcal{B}$, the basic form of the entropic uncertainty principle is stated in terms of the sum of the Shannon entropies of the probabilities of measuring $\mathcal{A}$ and $\mathcal{B}$ onto a given quantum state. State independent lower bounds for this sum encapsulate the degree of incompatibility of the observables diagonal in the $\mathcal{A}$ and $\mathcal{B}$ bases, and are usually derived by extracting as much information as possible from the unitary operator $U$ connecting the two bases. Here we show a strategy to derive sequences of lower bounds based on alternating sequences of measurements onto $\mathcal{A}$ and $\mathcal{B}$. The problem can be mapped into the multiple application of bistochastic processes that can be described by the powers of the unistochastic matrices directly derivable from $U$. By means of several examples we study the applicability of the method. The results obtained show that the strategy can allow for an advantage both in the pure state and in the mixed state scenario. The sequence of lower bounds is obtained with resources which are polynomial in the dimension of the underlying Hilbert space, and it is thus suitable for studying high dimensional cases.

I. INTRODUCTION

Uncertainty relations are some of the ways in which the peculiar behavior of quantum systems with respect to classical ones is characterized. The same feature described by these relation, the uncertainty associated with the measurements results of distinct incompatible observables, can be casted in several different ways depending on the context and the aim. Originally, the product of two observables’ variances were used \([1, 2]\), more recently the relations have been stated in terms of sum of variances \([3]\) or, when the spectrum of the observables is not relevant, in terms of sum of entropic quantities \([4, 5]\). For discrete observables acting on finite dimensional Hilbert spaces $\mathcal{H}_M$, $M = \dim \mathcal{H}_M$, the elementary form of entropic uncertainty relations (EURs) is stated for two bases $\mathcal{A} = \{|a_n\rangle\}_{n=1}^{M}$ and $\mathcal{B} = \{|b_n\rangle\}_{n=1}^{M}$ as

$$H(\mathcal{A}, \mathcal{B}) = H(\mathcal{A}) + H(\mathcal{B}) \geq \mathcal{L}_B$$

where $H(\mathcal{A}) = -\sum_i p_i^\mathcal{A} \log p_i^\mathcal{A}$ is the Shannon entropy of the probability vector $\vec{p}^\mathcal{A} = (p_1^\mathcal{A}, \ldots, p_M^\mathcal{A})$ with $p_i^\mathcal{A} = Tr[\rho |a_i\rangle \langle a_i|]$; and analogously $H(\mathcal{B}) = -\sum_j p_j^\mathcal{B} \log p_j^\mathcal{B}$, with $p_j^\mathcal{B} = Tr[\rho |b_j\rangle \langle b_j|]$. Here $\mathcal{L}_B \geq 0$ is a constant that lower bounds the sum for a given set of quantum states $\rho$. When $\mathcal{L}_B$ is a function of the measurements $\mathcal{A}, \mathcal{B}$ only, it is termed as state independent and the relation \([1]\) and it is satisfied by all states $\rho \in \mathcal{B} (\mathcal{H})$. In such case $\mathcal{L}_B$ encodes the degree of incompatibility of the observables $\mathcal{A}, \mathcal{B}$, since it puts a limit to the irreducible amount of uncertainty, as measured by the sum of the Shannon entropies, of the experiments represented by $\mathcal{A}$ and $\mathcal{B}$. If instead $\mathcal{L}_B$ also depends on the von Neumann entropy of the set of states $\mathcal{S}_c = \{\rho |S(\rho) = c\}$, it provides a state dependent lower bound for all states in $\mathcal{S}_c$, and it therefore encodes the degree of incompatibility of $\mathcal{A}$ and $\mathcal{B}$ in presence of a given fixed value of entropy. Given the definition of coherence for a given basis $\mathcal{A}$ with respect to a state $\rho$ in terms of the relative entropy of coherence $C_\mathcal{A}(\rho) = -S(\rho) + H(\mathcal{A}) \[6\]$, the EURs can also be formulated as

$$C_\mathcal{A}(\rho) + C_\mathcal{B}(\rho) \geq \mathcal{L}_B - 2S(\rho)$$

If $\mathcal{L}_B - 2S(\rho) > 0$, the latter formula constitutes a non trivial lower bound for the sum of coherences. This connection between the EURs and the sum of coherences naturally provides an interpretation of complementarity property expressed by the EURs: the complementarity between $\mathcal{A}$ and $\mathcal{B}$, and thus the minimum uncertainty for the two experiments represented by $\mathcal{A}$ and $\mathcal{B}$, is rooted in minimum of the sum of their coherences. The formulation \([2]\) will be used in the following, it reduces to \([1]\) for pure states, and it may find application for special cases of entropic uncertainties with memory \([9]\) (see below).

Since the first formulation of the uncertainty principle in terms of entropic quantities \([4, 5]\), several methods have been developed to provide tighter bounds $\mathcal{L}_B$ for \([1]\), for its generalizations to mixed states \([9]\), for more than two bases \([10]\) and for generalized measurements \([11]\) (an extensive collection of methods, results, applications and

*Electronic address: magpaolo16@gmail.com*
citations can be found in the excellent recent review [7]. In many of these approaches, the goal has been to extract as much information as possible from the unitary operator $U$ connecting the two bases i.e., $|b_n⟩ = U|a_n⟩$, $∀n$. In the original fundamental works [3, 4], the authors initially gave a state independent lower bound in terms of the largest overlap between the elements of $A$ and $B$; in particular Maassen and Uffink provided the following lower bound $L_{MU} = -\log s_{MU}$, where $s_{MU} = \max_{i,j} |⟨a_i| b_j⟩|^2$ is the largest modulus square element of $U$ in the $A$ basis. Subsequent approaches have successfully managed to exploit more of the information contained in $U$ i.e., they have provided lower bounds that depends on two matrix elements $|⟨a_i| U|a_j⟩|$ [12, 13] or sub-matrices of $U$ [13, 16, 18].

In the following we show how a basic strategy developed in [9, 19] can be extended (see Figure 1 and below) to alternating chains of $n+1$ measurements on $A$ and $B$ in order to provide a sequence of lower bounds $L_n$. We show that the problem of finding a tighter lower bound can be mapped into finding the maximum element of a $n$-fold product of bistochastic matrices whose factors are given by $\bar{U}$ and its transposed $\bar{U}^T$. Here $\bar{U}$ is the bistochastic operator whose matrix elements $\bar{U}_{i,j} = |⟨a_i| U|a_j⟩|^2$ are given by the moduli squared elements of $U$. Indeed, the matrix $\bar{U}$ and its transposed $\bar{U}^T$, being bi-stochastic, can be seen as the realization of classical Markovian processes that transform the input probability vectors $\bar{p}^a = (p^a_1, ..., p^a_M)^T$ and $\bar{p}^b = (p^b_1, ..., p^b_M)^T$ into output ones i.e., $\bar{U} \bar{p}^a$ and $\bar{U}^T \bar{p}^b$. As we shall see, the multiple applications of such processes describe the alternating sequences of measurements on $A$ and $B$. In the most simple case where $\bar{U} = \bar{U}^T$ is symmetric, this amounts to implement the Markovian processes $(\bar{U})^n \bar{p}^a$ and $(\bar{U})^n \bar{p}^b$. The effects of these classical Markovian “filters” clearly depends on the existing relation between $A$ and $B$, as described by $\bar{U}$. By upper bounding the effects of such “filters” one can then obtained the desired state independent and state dependent lower bounds. Indeed, for any $n \geq 1$, by essentially comparing the output probabilities $(\bar{U})^n \bar{p}^a$ and $(\bar{U})^n \bar{p}^b$ with $\bar{p}^a$ or $\bar{p}^b$ by means of the classical relative entropy, one can obtain a lower bound of the form

$$L_n = \mathcal{U}_n + S_n$$

(3)

where $\mathcal{U}_n$ is a state independent term and $S_n$ is a term that solely depends on the von Neumann entropy of the state $S(\rho)$. As we shall see the sequence $\mathcal{U}_n$, $n > 1$ provides new expressions for state independent lower bounds, that in the simplest case $\bar{U} = \bar{U}^T$ can be written as

$$\mathcal{U}_n = -\frac{\log s_n}{n + 1}$$

(4)

where $s_n = \left[\max_{i,j} (\bar{U}^n)_{i,j}\right]^2$ is the squared largest matrix element in the $A$ basis of the bi-stochastic matrix $\bar{U}^n$. We show that the $s_n$ can be easily computed for each $n$ and for arbitrary dimensions in terms of the eigenvalues of $\bar{U}$, its eigenvectors and their overlap with the some of the elements of $A$. On the other hand, the term dependent on the von Neumann entropy can be simply written for each $n$ as

$$S_n = \frac{2n}{n + 1}S(\rho)$$

(5)

For fixed bases $A, B$ and $U$, the tightest expression of the lower bound provided by this method is given by $\mathcal{L} = \max_n \mathcal{L}_n$. As we shall see the method can also be easily applied when $\bar{U} \neq \bar{U}^T$. For $n = 1$ one recovers the basic result [9]

$$\mathcal{L}_1 = -\log s_{MU} + S(\rho)$$

(6)
which extended the Maassen-Uffink result [6] to mixed states. For $n > 1$ the various state independent bounds $U_n$ are shown to provide in some situations a significant improvement with respect to [9] and to bounds obtained with other strategies [12, 13, 25]. This is in particular true when the dimension $M$ of the underlying Hilbert space is large. The improvement is maintained and in certain cases enhanced by the $L_n$’s when one considers mixed states e.g., when one seeks for lower bound of the sum of coherences [2] or conditional entropic quantities. In this case, we show and example in which the lower bounds provided by the $L_n$ are the only non-trivial i.e., non-zero, and simply computable ones.

In the following we first show how the main result [7] can be obtained (section II) and discuss its interpretation. We then show how the terms $U_n$ can be easily expressed in terms of the eigenvalues and eigenvectors of $\hat{U}^\dagger \hat{U}$ and $\hat{U}^T \hat{U}$ (subsection IIB). In order to test the above outlined strategy, we then apply it to different examples and compare the results with those obtained with other strategies (section III). We finally recap and discuss the results obtained and give some perspective on the possible extensions of the method proposed (section IV).

II. TWO OBSERVABLES LOWER BOUNDS

In this section we describe how the sequence of lower bounds $L_n$ can be derived. The various $L_n$ are ultimately based on the use of (quantum) relative entropies in order to compare the output probability vectors of different experiments. We first show the case $n = 1$ (already obtained in [2, 9, 19]) and later we extend it to the case $n > 1$. For $n = 1$, in the first experiment one measures the observable defined by the basis $A$ and obtains the output probability vector $\vec{p}^i = (p^i_1, p^i_2, \ldots, p^i_N)$. The second probability vector $\vec{p}^{ab} = (p^{ab}_1, p^{ab}_2, \ldots, p^{ab}_N)$ is the output of a second experiment where first one measures $\rho$ onto the basis $B$ and then applies a measurement in the $A$ basis. Here

$$p^{ab}_i = \sum_j p^b_j |\langle b_j | a_i \rangle|^2$$

is the probability of obtaining $|a_i\rangle$ after the $B$ measurement process has occurred on $\rho$. The classical relative entropy (Kullback-Leibler divergence)

$$D(\vec{p}^a || \vec{p}^{ab}) = -\sum_i p^a_i \log \frac{p^a_i}{p^{ab}_i}$$

is a measure of the statistical distinguishability between the two probability vectors $\vec{p}^a, \vec{p}^{ab}$. In this case it measures how much the measurement process $A$ is “disturbed” when one first measures $\rho$ onto the observable defined by $B$. The disturbance clearly depends on the relation between the two basis $A, B$. Analogously, by exchanging the role of $A$ and $B$, one can use $D(\vec{p}^b || \vec{p}^{ba})$ to compare the output of two different measurement processes where in the first one directly measures $B$ on $\rho$, while in the second one first measures $A$ and then $B$. In order to obtain the lower bound, one then introduces the maps $A(\cdot) = \sum_i \Pi^a_i (\cdot) \Pi^a_i$ and $B(\cdot) = \sum_i \Pi^b_i (\cdot) \Pi^b_i$, that describe the measurement processes on $A$ and $B$ respectively; here $\Pi^a_i = |a_i\rangle\langle a_i|$ and $\Pi^b_j = |b_j\rangle\langle b_j|$. One can thus write the relative entropy $D(\vec{p}^b || \vec{p}^{ab}) = S(A(\rho)||A \cdot B(\rho)) \equiv S(A||AB)$ in terms of the quantum relative entropy $S(A||AB)$ between the states $A(\rho) = \sum_i \Pi^a_i \rho \Pi^a_i$ and $A \circ B(\rho) = \sum_i \Pi^b_i \rho \Pi^b_i$; analogously $D(\vec{p}^a || \vec{p}^{ba}) = S(B||BA)$ since $A, B$ are completely positive maps one can then use the data process inequality [7, 21, 22] to write

$$-S(\rho) + S(B(\rho)) = S(\rho||B) \geq S(A||AB)$$

where now $S(B(\rho)) = H(B)$ i.e., the von Neumann entropy of the state $B(\rho)$ coincides with the Shannon entropy of the probability vector $\vec{p}^b$. By exchanging the role of $A$ and $B$ one obtains the analogous relation $-S(\rho) + H(A) = S(\rho||A) \geq S(B||BA)$. Summing up the two relations one has

$$H(A) + H(B) \geq S(A||AB) + S(B||BA) + 2S(\rho) \quad (7)$$

This is the prototypical expression that will allow us to obtain the lower bounds $L_n$ for any $n > 1$. Since now $S(A||AB) = D(\vec{p}^a || \vec{p}^{ab}) = -H(A) + C(A||AB)$, where $C(A||AB) = -\sum_i p^a_i \log p^{ab}_i$ is the cross-entropy [22] between $\vec{p}^a$ and $\vec{p}^{ba}$, one finally obtains

$$H(A) + H(B) \geq \frac{C(A||AB) + C(B||BA)}{2} + S(\rho) \quad (8)$$

Since $\forall i, j$ the probabilities $p^{ab}_i, p^{ba}_j \leq \max_{h,k} \hat{U}_{hk} = s_{MU}$, then $C(A||BA), C(B||AB) \geq -\log s_{MU}$, and one obtains $L_1$ [10] in terms of the Maassen-Uffink state independent lower bound. The above strategy can be extended to more
the (quantum) relative entropy $D (\tilde{p}^a || \tilde{p}^{ba}) = S (A|ABA)$ now measures the statistical distinguishability between the probability of measuring $A$ directly on $\rho$ or onto $B \circ A (\rho)$. It thus is a measure of how much the interposition of a measurement stage $B$ between the two $A$ stages of measurements changes the initial $A$ outcome probability vector $\tilde{p}^a$. In other words it is a measure of the effect introduced by interposing a $B$ “filtering” operation between two $A$ consecutive measurements processes. The same strategy can be applied by exchanging the role of $A, B$, thus obtaining the vector $\tilde{p}^{bab}$ and $D (\tilde{p}^b || \tilde{p}^{bab}) = S (B|BAB)$. In order to obtain a bound analogous to (7), one makes a repeated use of the data process inequalities; indeed

$$S (A|ABA) \leq S (\rho||BA) = -S (\rho) + H (B) + S (B||BA) \leq$$

$$\leq -S (\rho) + H (B) + S (\rho||A) =$$

$$= -2S (\rho) + H (A) + H (B)$$

Applying the same arguments to $S (B||BAB)$ and summing the two results one thus obtains

$$H (A) + H (B) \geq \frac{S (A|ABA) + S (B||BAB)}{2} + 2S (\rho)$$

(10)

The same reasonings can then be applied to an arbitrary sequence of alternating measurement processes $ABA...$ and $BAB...$, see also Figure [left panel. The first, described by $\tilde{p}^a$ is the measurement $A (\rho)$ on $A$; the second is the sequence of measurements $A \circ B \circ A (\rho)$ described by the probability vector $\tilde{p}^{aba}$:

$$\tilde{p}_1^{aba} = \sum_k p_k^a \sum_i |\langle a_k | b_i \rangle|^2 |\langle b_i | a_k \rangle|^2$$

(9)

complex measurement processes. Indeed the bound $L_2$ can be obtained by comparing the following different processes, see also Figure [left panel. The first, described by $\tilde{p}^a$ is the measurement $A (\rho)$ on $A$; the second is the sequence of measurements $A \circ B \circ A (\rho)$ described by the probability vector $\tilde{p}^{aba}$:

Thus, by summing up the two results and dividing by $n$ one gets

$$H (A) + H (B) \geq \frac{S (A|ABA...A) + S (B||BAB...B)}{n} + 2S (\rho)$$

(11)

Analogously, it's easy to check that when $n$ is odd the final expressions obtained for the two sequences are $-nS (\rho) + \frac{n+1}{2} H (A) + \frac{n-1}{2} H (B)$ and $-nS (\rho) + \frac{n-1}{2} H (A) + \frac{n+1}{2} H (B)$ respectively. Thus, by summing them up and dividing by $n$ one again obtains (11). The latter expression allows now to obtain a lower bound simply by expressing the relative entropies $S (A||ABA...)$ in terms of the cross entropies $C (A||ABA...).$ By multiplying by $n$ both sides of the inequality (11) and collecting the terms $H (A) + H (B)$ one then has

$$H (A) + H (B) \geq \frac{C (A|ABA...A) + C (B||BAB...B)}{n + 1} + \frac{2n}{n + 1} S (\rho)$$

(12)
The above formula allows to obtain the lower bounds $L_n$ for any $n > 1$ in a compact way. We show how by starting from $L_2$. The probabilities $p^{aba}_i$ \[ (9) \] and $p^{bab}_j$ can be written in terms of the matrix elements of the product $UU^T$ and $U^TU$ respectively. Indeed, $p^{aba}_i$ \[ (13) \] is written in terms of the elements

$$
\sum_t |(a_k | b_t)|^2 |(b_t | a_i)|^2 = \sum_t |(a_k | U|a_t)|^2 |(a_t | U^T|a_i)|^2 
$$

$$
\sum_t |U_{kt}|^2 |U_{ti}^T|^2 = \langle a_k |U^T U| a_i \rangle 
$$

and analogously

$$
p^{bab}_j = \sum_k p^b_k \sum_t |(b_k | a_t)|^2 |(b_t | a_j)|^2 = \sum_k p^b_k \langle a_k |U^T U| a_j \rangle
$$

We now observe that since $U$ is bi-stochastic, $U^T$ and $U^TU$, $UU^T$ are bi-stochastic too \[ (29) \]. The lower bound $L_2$ can then be written in terms of the largest matrix elements

$$
s_2 = \left( \max_{k,i} \langle a_k |U^T U| a_i \rangle \right) \left( \max_{k,j} \langle a_k |U^T U| a_j \rangle \right)
$$

Since $C(A||ABA) + C(B||BAB) \geq - \log s_2$, using \[ (12) \] for $n = 2$ one has

$$
H(A) + H(B) \geq L_2 = - \frac{1}{3} \log s_2 + \frac{4}{3} S(\rho)
$$

The same strategy can be straightforwardly applied to higher order processes. One obtains the sequence of state independents lower bounds $\{L_n\}$ where for each $n$

$$
H(A) + H(B) \geq L_n = U_n + S_n = - \log s_n + \frac{2n}{n+1} S(\rho)
$$

The state independent part of the bound $U_n$ depends on the relation between $A, B$ through

$$
s_n = \left( \max_{k,i} \langle a_k |U^T U U^T U| a_i \rangle \right) \left( \max_{k,j} \langle a_k |U^T U U^T U| a_j \rangle \right)
$$

i.e., the largest element of the n-fold matrix product $\tilde{U}^T \tilde{U} U^T U$ or $\tilde{U} U^T \tilde{U} U^T$. Or if now $\tilde{U} = U^T$ is symmetric then the expression simplifies to $s_n = (\max_{i,j} |\langle a_k |U^n|a_i \rangle|^2)$. The term $S_n$ instead only depends on the von Neumann entropy of the state $\rho$. Overall, given two bases $A, B$ and for fixed level of entropy $S(\rho)$, the best lower bound provided by the above described strategy is given by $L = \max_n L_n$.

A. Discussion of the results: general considerations.

Before analyzing the method that one can use to evaluate $s_n$ in a simple way, we first comment on some properties of the found bounds that can be understood without explicitly computing them. We start with the $S_n$ part. If $A \equiv B$, then $U = I$, $s_n = 1$, $\forall n$ and the minimum for the sum of entropies is given by twice the von Neumann entropy of the state. Indeed, in this case $L_n = S_n \forall n$ and, since $S_{n+1} \geq S_n$, the best lower bound in the sequence is provided by $L_{n \to \infty} = S_{n \to \infty} = 2S(\rho)$. This term captures the obvious feature that for states with a given fixed von Neuman entropy and whatever the relation between $A$ and $B$, it holds $H(A), H(B) \geq S(\rho)$ and thus $H(A) + H(B) \geq 2S(\rho)$. Since $\forall n$, $S_n > S(\rho)$, we may expect that the bounds given by the $L_n$’s may provide an improvement with respect to other existing bounds, in particular $L_1$, when the entropic part plays a relevant role; which is for example the case of conditional entropic uncertainty relations and sum of coherences (see below).

As for the state independent $U_n$ part, the latter provide a state independent bound that is usable also for pure states. In the next section we will see how the $U_n$ depend on the properties (eigenvalues) of $U$. Here, in order to discuss
and eigenvectors of $\mathbf{p}$ can be easily computed in a compact form since $s_i$ is diagonal in the same basis for all $i$, and one needs to evaluate the maximum matrix element of $n$-fold products $\mathbf{p}^{\otimes n}$, with increasing $n$ the effect is to produce an increasing level of mixing. Being $s_m$ composed by the product of upper bounds to $p^{\otimes n}_{i_1i_2...i_n}$ and $\overline{p^{\otimes n}_{j_1j_2...j_n}}$, respectively, it’s information content is thus related to the amount of global mixing introduced by the $n$-steps bi-stochastic processes modeled by $UU^TUU^T...$ and $U^TU^TU^T...$. Thus one can expect that if $m > n$ then $s_m < s_n$ and $-\log s_m > -\log s_n$; this however in general does not imply that $U_m > U_n$ since

\[
\frac{U_m}{U_n} = \frac{-\log s_m \left( n + 1 \right)}{-\log s_n \left( m + 1 \right)}
\]

and $\frac{n+1}{m+1} < 1$. Aside from simple analytic examples (see the qubit case below), finding general conditions for determining whether and when $U_m > U_n$ seems difficult in the general case, and one must to resort to study case by case or check numerically. In the examples we provide below we show that indeed with growing $n$ one can improve the overall bound. The largest instance of $n > 1$ that one has to check is however limited, since the above reasoning lead to expect that for large $n$ the level of mixing introduced by the product of bi-stochastic maps e.g. $UU^TUU^T...$ reaches its maximum i.e., $p_i^{\otimes n}...p_j^{\otimes n} \approx 1/M$ and $\lim_{n\to\infty} U_n = \lim_{n\to\infty} -2\log M/\left( n + 1 \right) = 0$. Therefore, independently on the existing relation between $A, B$, for large $n$ the lower bound $U_n \to 0$. This consideration clearly limits the number of lower bounds $U_n$ that one needs to evaluate to determine the best bound $L$. This fact is in particular true for the situations in which $A, B$ are very close to being mutually unbiased, since in this case one should expect that $U^T \approx \bar{U} \approx U^*$ where $U^*$ is the van der Waerden matrix [26] i.e., the matrix whose elements are all equal to $1/M$. If this is the case then one expects that for pure states the best lower bound is provided by the Maassen-Uffink result $U_1$ since $\forall n \ s_n \approx 1/M, U_1 > U_n$, and $U_1$ as expected provides a tight lower bound. However, if $A, B$ are sufficiently far from being mutually unbiased the bound will be given by $U_n$, $n > 1$, and in the case of mixed states case, by $L_n$, $n > 1$. These general considerations will become clearer in the next sections, where we analyze in detail the state independent part $U_n$, and we discuss some examples.

B. Evaluation of the state independent lower bounds $U_n$

To evaluate the lower bounds $U_n$ one needs in principle to evaluate the maximum matrix element of $n$-fold products of the kind $UU^TUU^T...$ for $n > 1$. This can become a demanding computational task, especially when the dimension $M$ of the given Hilbert space is large. However, on the one hand, the computation complexity is that of matrix multiplication and thus is polynomial in the dimension $M$. On the other hand, by taking advantage of the properties of the matrix $\bar{U}$ one can drastically reduce the complexity, since the $s_n$ can be written in a compact form that depends on the eigenvalues of $\bar{U}$. We start by analyzing the simplest scenario i.e., when $U^T = \bar{U}$ is symmetric and has the properties detailed below. Since $\bar{U}$ is real and symmetric it can be diagonalized by means of a real orthogonal matrix $O$ such that $OD_{\bar{O}}O^T$ and $D_{\bar{O}} = \text{diag}(\bar{u}_1, \bar{u}_2, ..., \bar{u}_N)$ where $\bar{u}_i$ are the eigenvalues of $\bar{U}$ in decreasing order; since $\bar{U}$ is bi-stochastic one has that: the maximal eigenvalue is $\bar{u}_1 = 1$ and it corresponds to the uniform normalized eigenvector $|u_i\rangle = \sum_j |a_j\rangle / \sqrt{M} [26]$; furthermore $|\bar{u}_i\rangle \leq 1$, $\forall i > 1$. In the following we suppose for simplicity that that $1 > \bar{u}_i \geq 0$, $\forall i > 1$ (some other special cases, in particular when $\bar{U}$ is not symmetric, are treated in Appendix $A$). Suppose now that at the first order $n = 1$ the largest matrix element of $\bar{U}$ is given by $\langle a_i | U | a_j \rangle$ for some specific pair $(i, j)$, then $\sqrt{s_1} = \sum_k \bar{u}_k |a_i\rangle \langle \bar{u}_k | a_j \rangle$. Since $U^n = \sum_k \bar{u}_k^n |a_i\rangle \langle \bar{u}_k | a_j \rangle$ is diagonal in the same basis for all $n$, and since for all $k$ and for all $m \geq n$ one has that $\bar{u}_k^n \geq \bar{u}_k^m$, it obviously follows that at any order $n$ the lower bound will be given by the same matrix element $i, j$ i.e., $\sqrt{s_n} = \langle a_i | U^n | a_j \rangle = \sum_k \bar{u}_k^n |a_i\rangle \langle \bar{u}_k | a_j \rangle$. Therefore, in order to evaluate $s_n$ one only needs to: i) determine the pair $(i, j)$ that identifies $U$’s largest matrix element ii) find the eigenvalues and eigenvectors of $U$ iii) evaluate the coefficients $U_{i,j}^k = \langle a_i | \bar{u}_k | a_j \rangle$, $\forall k$. Then the lower bounds $U_n$ can then be easily computed in a compact form since $\forall n$

\[
s_n = \left( \sum_k \bar{u}_k^n U_{i,j}^k \right)^2
\]
This expression also shows that for symmetric $U$, the terms $s_n$ form a non-increasing sequence, and it allows to better understand the above described behaviour of the $U_n$ terms.

If $A$ is close to being mutually unbiased with $B$ then $|\langle a_i | U(a_j) \rangle| \approx 1/\sqrt{M}$ $\forall i,j$, such that $U_{i,j} \approx 1/M$. In this case $U \approx |\bar{u}_1 \rangle \langle \bar{u}_1 |$ i.e., the single relevant eigenvalue is the largest one i.e., $\bar{u}_1 = 1$, while for $k > 1$, $\bar{u}_k \approx 0$. Therefore $s_n \approx 1/M$ $\forall n$ and $U_n \gg U_n \forall n$. We thus expect that for bases approximately mutually unbiased, the dominant term is the Maassen-Uffink one $U$ and $B$ non-increasing sequence for $n \geq 2$ and indeed $ar{u}_n \gg U_n / U_1$ i.e., tighter than the Maassen-Uffink one $U_1$. As show in the following examples, and as confirmed by extensive simulations, there are indeed relevant cases in which indeed $L$ provides a bound tighter that Maassen-Uffink’s and other bounds available for pure states. And thus by extension, if the bases $A, B$ are not mutually unbiased and since $S_n > S_1$, one expects that the lower bounds $L_n$ to be tighter than $L_1$.

III. EXAMPLES

In order to assess the applicability and performance of the above outlined strategy we explicitly apply it to different examples. We will study both the pure and the mixed state situation. For the small dimension cases, in order to compare our results, we choose to evaluate the lower bounds obtainable for both pure and mixed states by means of direct sum majorization strategy [15], which is one of the most sophisticated one and it has been shown to provide the best (tighter) lower bounds in several examples with small $M \leq 4$. The full method is based on the determination of an $M$-dimensional vector

$$W = (w_1, w_2 - w_1, w_3 - w_2, ..., w_M - w_{M-1})$$

where each of the coefficients $w_k$ is the largest singular value of the sub-matrices obtainable from $U$ with $n_n + n_c = k + 1$, where $n_r/n_c$ is number of rows/columns of the sub-matrices. Once $W$ is determined the state independent lower bound is given by $L_{Maj} = -\sum_k w_k \log w_k$. In case of mixed states with spectrum $\bar{\lambda} = (\lambda_1, ..., \lambda_M)$ (in decreasing order), one needs to evaluate the $2M$ dimensional vector $W(\bar{\lambda}) = \Lambda W$ where $\Lambda$ is an appropriate matrix that depends on the $\lambda_i$’s (see [15] for details). The corresponding lower bound is then given by $L_{Maj} = -\sum_k w_k^\Lambda \log w_k^\Lambda$. For small dimensions $M$ the method can be easily applied in full. When the dimension grows exceedingly, the elements $w_k$ are the result an optimization problem that becomes gradually more difficult as $k$ increases, due to the large number $\sum_j (M)_{k+1-j}$ of sub-matrices that one has to consider. In such case, one can determine only the first few $k^*$ coefficients $w_{k^*}$ and use the $M$-dimensional vectors $W_{k^*} = (w_1, w_2 - w_1, ..., 1 - w_{k^* - 1}, 0, 0, ..., 0)$ and $W_{k^*}^\Lambda = \Lambda W_{k^*}$ instead. For pure states in the high-dimensional cases we also use other formulations of the lower bounds (see below).

As for the mixed state case, we have already mentioned that the form in which we test our approach is given by [2]. This formulation is equivalent to a specific case of uncertainty relations with memory. In this context the idea is that two parties share a pure state $|\psi\rangle = \sum \sqrt{\lambda_i} |i\rangle |^2 \rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, here written in its Schmidt decomposition. Then two different measurements $A, B$ are applied onto $\rho_1 = Tr_2(\rho_2)$, while the other subsystem is used as a memory. The uncertainty of the protocol can then be expressed in terms of the conditional entropies

$$H(A|2) + H(B|2) = H(A) + H(B) - 2S(\rho_2)$$

(17)

where $H(A|2) = S(\rho_{A2}) - S(\rho_2)$ is the conditional quantum entropy expressed in terms of the von Neumann entropy of the state $\rho_{A2} = \sum_i (|a_i\rangle |\bar{a}_i \rangle \otimes I_2) \rho_{12} (|a_i\rangle |\bar{a}_i \rangle \otimes I_2)$ and the von Neumann entropy of the state $\rho_2 = Tr_1[\rho]$. Given the above particular setting one has that $S(\rho_1) = S(\rho_2) = S(\rho)$, and the above expression reduces to [2]. Since both reduced density matrices have the same spectrum $\bar{\lambda} = (\lambda_1, ..., \lambda_M)$, given by squares of the Schmidt coefficients, by changing the $\lambda_i$’s, i.e., the entanglement of the bipartite state $|\psi\rangle$, one can fix the entropy of the state $\rho$ to a desired value. Given a lower bound for the sum $H(A) + H(B)$ one can then check its performance with different levels of entropy. In the following, we use for the elements of the spectrum the expressions $\lambda_k = \exp(\beta k) / \sum_k \exp(\beta k)$. By changing the value of $\beta$ we can thus vary $S(\rho) \in [0, \log M]$. 

A. Example I. Single qubit.

We start by analyzing a two dimensional quantum system. While as we now see, in this case the lower bound $L$ is always less tight that $L_{Maj}$, this example, thanks to its simplicity, allows us to review in detail the method and
some of the arguments exposed above. For a single qubit [27], the problem of finding a lower bound \( \mathcal{L}_B \) can always be reduced to the case where \( \mathcal{A} = \{\ket{0}, \ket{1}\} \) is given by the eigenstates of \( \sigma_z \) and \( \mathcal{B}(\theta) \) is obtained from \( \mathcal{A} \) by means of the unitary operator

\[
U = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

For \( \theta = 0 \) the two bases coincide, while for \( \theta = \pi/4 \) they are mutually unbiased. The unistochastic matrix \( \bar{U} \) is symmetric, and \((\bar{u}_1, \bar{u}_2) = (1, \cos^n \theta)\) are the positive eigenvalues of \( \bar{U}^n \). The largest matrix element for \( \bar{U}^n \) is given by \( \bra{0} \bar{U}^n \ket{0} \) or by \( \bra{1} \bar{U}^n \ket{1} \); one has \( \bra{0} \bar{u}_1 \ket{2} = \bra{0} \bar{u}_2 \ket{2} = 1/2 \). Thus we can analytically compute the \( s_n \) with (16) in terms of the eigenvalues of \( \bar{U}^n \) and the projections \( \bra{0} \bar{u}_i \ket{2} \) as

\[
s_n = \left( \frac{1 + \cos^n \theta}{2} \right)^2
\]

The state independent part of the lower bounds thus reads

\[
\mathcal{U}_n = -\log \left( \frac{1 + \cos^n \theta}{2} \right)^2
\]

Here it’s easy to see that for \( \theta > 0 \), as \( n \to \infty, \mathcal{U}_n \to 0 \). When \( \theta \leq \pi/4 \), \( \mathcal{A} \) and \( \mathcal{B}(\theta) \) are quasi mutually unbiased, the second eigenvalue \( \bar{u}_2 \approx 0 \) and thus \( \mathcal{U}_1 > \mathcal{U}_n \), \( \forall n > 1 \). When \( \theta \) is sufficiently smaller that \( \pi/4 \), \( \bar{u}_2 \) and thus its contribution to \( s_n \) becomes non-vanishing. Indeed, by using (15), one finds that if \( \theta \) and \( n \) are such that

\[
\left( \frac{1 + \cos \theta}{2} \right)^2 < \left( \frac{1 + \cos^n \theta}{2} \right)^{\frac{\pi}{\theta}}
\]

then \( \mathcal{U}_n > \mathcal{U}_1 \) and \( \mathcal{U}_n \) provides a better lower bound than the Maassen-Uffink one. For example, for \( n = 2 \) this happens for \( \theta \approx 0.592 \).

Figure 2: Single qubit. Plots of \( \mathcal{L} \) (blu dotted), \( \mathcal{L}_{\text{Maj}} \) (orange), \( \mathcal{L}_1 \) (green) Left panel: Pure states case Right panel: Lower bound for sum of coherences (2) for fixed Von Neumann entropy \( S(\rho) = 0.32 \)

In Figure 2 we plot the best lower bound \( \mathcal{L} = \max_n \{\mathcal{L}_n\} \), together with \( \mathcal{L}_1 \) and \( \mathcal{L}_{\text{Maj}} \). In the left panel, we show the results for the pure state case i.e., \( S(\rho) = 0 \). For \( \theta \leq \pi/4 \) the two bases are mutually unbiased such that, as expected, the best bound is provided by \( \mathcal{L}_1 = \mathcal{U}_1 \) i.e., the Maassen-Uffink result. When \( \theta \) is sufficiently smaller that \( \pi/4 \) the bound provided by the direct sum majorization approach \( \mathcal{L}_{\text{Maj}} \) is always better that \( \mathcal{L} \) and \( \mathcal{L}_1 \); while, as explained above, for \( \theta \leq 0.592 \) the lower bound \( \mathcal{L} \) is tighter with respect to \( \mathcal{L}_1 \). In right panel, we show the results obtained for the expression (2) when \( S(\rho) = 0.32 \). Some of the main features already discussed above are visually reproduced by the plots. When \( \theta \geq 0 \) the two bases \( \mathcal{A} \) and \( \mathcal{B}(\theta) \) are very close to each other and the dominant part of \( \mathcal{L} \) is given by \( \mathcal{L}_{n \geq 32} \approx S_{n \geq 32} \approx 2S(\rho) \). Then for a large part of the interval \( \theta > 0, \mathcal{L} \) is obtained for \( n > 1 \) and it provides a lower bound that is tighter than \( \mathcal{L}_1 \). However, when \( \theta \leq \pi/2, \mathcal{A} \) and \( \mathcal{B}(\theta) \) become mutually unbiased, \( \bar{U} \) closely approximates the \( 2 \times 2 \) van der Waerden matrix \( U^* \), such that \( \bar{U}_{i,j} \approx 1/2 \). Since \( (U^*)^n = U^* \) then \( \forall n > 1 \), \( s_n \approx 1/4 \) and consequently \( \mathcal{L}_n \approx 2 \log 2 / (n + 1) \); thus \( \forall n > 1 \), \( \mathcal{L}_n < \mathcal{L}_1 \approx \log 2 \) and the best lower bound is given by \( \mathcal{L}_1 \).
B. Example II. Three-dimensional system

In order to test the method for three dimensional system we choose to focus on the operator $F_3^\beta$ (also used in [18]), where $F_3$ is the three dimensional quantum Fourier transform i.e., $(F_3)_{k,h} = \exp(2i\pi kh/3)/\sqrt{3}$, and $\beta \in \{0, 2\}$. When $\beta = 1$ the bases connected by $F_3$ are mutually unbiased, thus in the range $\beta \in \{0, 2\}$ all possible values $H(A, B) \in \{0, \ln 3\}$ are achieved. In this case the direct sum majorization algorithm can be fully used, and the vector $W = (w_1, w_2 - w_1, 1 - w_2)$ and its counterpart for mixed states $\Lambda W$ can be easily determined. In Fig. 3 left panel we plot the lower bounds for pure states. In this case the best bound are always given by $L_{\text{Maj}}$ for $\beta$ sufficiently different from 1, and by $L_1$ for $\beta \approx 1$. The latter result, as described above, is not surprising, since for $\beta = 1$, $L_1$ provides a tight lower bound. For the same reason, $L$ obviously coincides with $L_1$ for $\beta \approx 1$. While away from the central region $L$ provides a slight improvement with respect to $L_1$.

The case of mixed states is described in Fig. 3 right panel, where the level of the Von Neumann entropy is fixed to $S(\rho) = 0.914$. While in general the main features discussed for pure state are again reproduced, we notice that in a symmetric region around $\beta = 1$ the bound $L$ provides an advantage with respect to the other two. This result has been confirmed by other simulations with other three dimensional operators, such that one can induce that for very low dimensional systems, the strategy proposed in this paper seems able to improve the existing bounds in particular when mixed state are considered.

![Figure 3: 3-dimensional QFT. Plots of $L_{\text{Maj}}$ (orange), $L_1$ (green), $L$ (blue dotted) Left panel: Pure states case Right panel: Lower bound for sum of coherences (2) for fixed Von Neumann entropy $S(\rho) = 0.914$ states case. In a region symmetric around $\beta = 1$, $L$ provides a lower bound tighter than the other ones considered.]

C. Example III. High dimensional systems

Being based on algorithms that have polynomial complexity in nature, the method introduced in this work is easily applicable to situations where other methods might be limited by their computational complexity i.e., large dimensions. In this context, in order to test the method, we applied it to two different operators and we fix the dimension of the Hilbert space to $M = 128$. On one hand we again use the quantum Fourier transform operator $F_{128}^\beta$, and on the other hand we use $U = \exp(-i2\theta J_y)$, where $J_y$ is the $y$-spin operator for $j = 127/2$. For high dimensions there is a small number of lower bounds that go beyond the Maassen-Uffink result for pure states and $L_1$ for the mixed ones. For pure states, there are the formulas described in [12, 13, 25] that depend on the largest element of $U$ and in [12, 13] that also depend on the second-largest matrix elements of $U$. For pure and mixed states we can again rely on $L_1$ and $L_{\text{Maj}}$. For the latter, as mentioned above, we opt to use for $W$ and $W^\Lambda$ their restricted versions $W_2 = (w_1, w_2 - w_1, 1 - w_2, 0, 0, \ldots, 0)$ and $W_2^\Lambda = \Lambda W_2$, that require the computation of $w_1, w_2$ only. We start by discussing the $F_{128}^\beta$ case. In Figure left panel we analyze the pure state case, and we report $L$, $L_{\text{Maj}}$, $L_1$ and the bound $L_{\text{deV}}$ given in [25] that reads:

$$L_{\text{deV}} = -2 \left[ \left( \frac{1 - \sqrt{s_{\text{MU}}}}{2} \right) \log \left( \frac{1 - \sqrt{s_{\text{MU}}}}{2} \right) + \left( \frac{1 + \sqrt{s_{\text{MU}}}}{2} \right) \log \left( \frac{1 + \sqrt{s_{\text{MU}}}}{2} \right) \right]$$

where again $s_{\text{MU}}$ equals the maximum matrix element of $U$. The other bounds [12, 13] are not reported since it turns out they only provide a marginal improvement with respect to the Maassen-Uffink result $L_1$. The plots show that again in the region $\beta \approx 1$, $L_1$ (green-dotted) gives the best lower bounds. Away from the $\beta \approx 1$, and the newly
introduced bound $\mathcal{L}$ (blue-dotted curve) is shown to give an improvement with respect to all the other tested bounds. In the vicinity of $\beta \gtrapprox 0$, the three functionals $\mathcal{L}$, $\mathcal{L}_{\text{Maj}}$ and $\mathcal{L}_{\text{dev}}$ give approximately the same result, with some advantage given by $\mathcal{L}_{\text{dev}}$ (see inset).

In Figure 4-right panel we instead consider the mixed case scenario, and we compare $\mathcal{L}$, $\mathcal{L}_{\text{Maj}}$ and $\mathcal{L}_1$ when $S(\rho) = 1.25$. The plot shows that, aside from a central region where $\mathcal{L}_1$ (green-dotted) dominates, for all other values the bound provided by $\mathcal{L}$ (blue-dotted) performs largely better than the other two. The same result can be obtained with arbitrary fixed values of $S(\rho)$.

Figure 4: 128-dimensional QFT. Plots of $\mathcal{L}$ (blue dotted), $\mathcal{L}_{\text{Maj}}$ (orange), $\mathcal{L}_1$ (green, dashed), $\mathcal{L}_{\text{dev}}$ (black, dashed). Left panel: Pure states case Right panel: Lower bound for sum of coherences $\left(2\right)$ for fixed Von Neumann entropy $S(\rho) = 1.25$ states case. Here $\mathcal{L}$ provides a bound that is tighter for a large portion of the interval considered.

The other example is based on the unitary operator $U = \exp (-i2\theta J_y)$. In Figure 5-left panel the pure state case shows that aside from a small region around $\theta = 0, \pi/2$ where $\mathcal{L}_{\text{dev}}$ and $\mathcal{L}_{\text{Maj}}$ provides some advantage (see inset for $\theta \gtrapprox 0$), for all other values of $\theta$ the best bound is given by $\mathcal{L}_1$. On the other hand, for the mixed state case, starting from $S(\rho) = 1$ (right panel, main plot), for all values of $\theta$, $\mathcal{L}$ is the tightest lower bound and it also provides a significant improvement with respect to $\mathcal{L}_1$. When the entropy of the state is increased i.e., $S(\rho) > 1$, the improvement becomes even more significant and for $S(\rho) \gtrapprox 2.4$ (see central inset) $\mathcal{L}$ becomes the only non-zero easily computable lower bound available.

Figure 5: 128-dimensional spin case. Plots of $\mathcal{L}$ (blue dotted), $\mathcal{L}_{\text{Maj}}$ (orange), $\mathcal{L}_1$ (green, dashed), $\mathcal{L}_{\text{dev}}$ (black, dashed). Left panel: Pure states case Right panel: Lower bound for sum of coherences $\left(2\right)$ for fixed Von Neumann entropy: main plot $S(\rho) = 1$; inset $S(\rho) = 1.25$

IV. DISCUSSION

In this paper we have introduced a strategy to derive a sequence of lower bounds for entropic uncertainty relations for two observables $A, B$. The strategy starts by recognizing that the output probabilities of sequences of alternating measurements onto the bases $A$ and $B$ can be expressed in terms of multiple applications of bi-stochastic maps $\bar{U}, \bar{U}^T$ that are derivable from the unitary operator $U$ connecting two bases $A, B$. By upper-bounding the degree of mixing...
induced by the $n$-fold application of such bi-stochastic maps on can derive lower bounds in terms of the eigenvalues and eigenvectors of $U, U^T$ that take the form $L_n = U_n + S_n$. While $U_n$ is the state independent part, $S_n$ only depends linearly on the Von Neumann entropy of the class of state considered. The lower bounds obtainable with such strategy can thus be applied to both pure and mixed states, and can be used for bounding certain schemes involving conditional entropies and sum of coherences. The method, being based on matrix multiplication, has a polynomial complexity in the dimension $M$ of the underlying Hilbert space and it thus can also be applied to high dimensional cases. We have shown how the complexity can be further reduced, by taking advantage of the symmetry of the operators involved.

We have applied the method to several different examples. The main results about the application of the method can be summarized as follows. For the smallest case, the single qubit, the method does not provide an appreciable advantage with respect to other existing strategies. Already for three dimensional cases, we have shown that the method introduced may provide some advantage in the case of mixed states. In the high dimensional case, the results show that the method performs better than the other scalable methods available both in the pure and in the mixed states scenario. In the latter case the strategy can provide non-trivial lower bounds even in the case where other scalable methods do not.

The method can be easily extended for multiple measurement in an obvious way i.e., by applying it to all possible pairs of bases and adding the corresponding lower bounds. On the other hand, further research is needed to extend the application of the method to other situations such as for example generalized measurements (POVM).

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Appendix A: Evaluation of $L_n$

We first examine the case where $\tilde{U}$ is symmetric but has some negative eigenvalues, it may happen that $s_1$ is provided by the matrix element $\tilde{U}_{i,j}$, while for $n = 2$ by $\tilde{U}_{s,t}$ with $(i,j) \neq (s,t)$. For the same reason seen in Section IIB, this means that the term corresponding to the $n$-th order is equal to

$$s_{n=2m+1} = \left( \sum_k \tilde{u}_k^{2m+1} \tilde{U}_{i,j}^k \right)^2, \quad n \text{ odd}$$

$$s_{n=2m} = \left( \sum_k \tilde{u}_k^{2m} (\tilde{U}^2)^k_{s,t} \right)^2, \quad n \text{ even} \quad (A.1)$$

In any case, if $\tilde{U}$ is symmetric, in order to evaluate the bounds $U_n$ one does not need to explicitly evaluate the powers $\tilde{U}_n$ in order to find the various $s_n$ for $n \geq 2$, but instead one only needs to find the eigenvalues and eigenvectors of $\tilde{U}$ and use the above analytical formulas.

If $\tilde{U}^T \neq \tilde{U}$ we first notice that both $\tilde{U} \tilde{U}^T$ and $\tilde{U}^T \tilde{U}$ are symmetric bi-stochastic matrices, and thus one can readily apply the above arguments to express each $s_n$ for $n = 2m$ even in terms of the eigenvectors and eigenvalues of $\tilde{U} \tilde{U}^T$ and $\tilde{U}^T \tilde{U}$, by using analytical formulas analogous to (A.1). For $n = 2m+1$ odd there seem to be no easily derivable formulas, however since the products containing an odd number of factors will be of the form $(\tilde{U} \tilde{U}^T)^m \tilde{U}$ or $(\tilde{U}^T \tilde{U})^m \tilde{U}^T$ one can use the decomposition $OV^T = diag(v_1, v_2, ..., v_N)$ to evaluate $OV^n O^T$ for the terms $V^m = (\tilde{U} \tilde{U}^T)^m \tilde{U}$ and $(\tilde{U}^T \tilde{U})^m \tilde{U}^T$. Aside from the diagonalization of $\tilde{U} \tilde{U}^T$ or $\tilde{U}^T \tilde{U}$ the procedure require two simple matrix multiplications to determine $V^m$ and then the given odd term $n = 2m + 1$. Clearly the determination of the odd part of the sequence $L_n$ is indeed more demanding from a computational point of view. However, by relying on the even part of the sequence one can obtain a sequence of lower bounds $L_{2m}$ that are representative of the whole sequence $L_n$. 