Convergence of densities of some functionals of Gaussian processes

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Abstract
The aim of this paper is to establish the uniform convergence of the densities of a sequence of random variables, which are functionals of an underlying Gaussian process, to a normal density. Precise estimates for the uniform distance are derived by using the techniques of Malliavin calculus, combined with Stein’s method for normal approximation. We need to assume some non-degeneracy conditions. First, the study is focused on random variables in a fixed Wiener chaos, and later, the results are extended to the uniform convergence of the derivatives of the densities and to the case of random vectors in some fixed chaos, which are uniformly non-degenerate in the sense of Malliavin calculus. Explicit upper bounds for the uniform norm are obtained for random variables in the second Wiener chaos, and an application to the convergence of densities of the least square estimator for the drift parameter in Ornstein-Uhlenbeck processes is discussed.

Keywords: Multiple Wiener-Itô integrals; Wiener chaos; Malliavin calculus; integration by parts; Stein’s method; convergence of densities; Ornstein-Uhlenbeck process; least squares estimator; small deviation.

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1. Introduction

There has been a recent interest in studying normal approximations for sequences of multiple stochastic integrals. Consider a sequence of multiple stochastic integrals of order $q \geq 2$, $F_{n} = I_{q}(f_{n})$, with variance $\sigma^{2} > 0$, with respect to an isonormal Gaussian process $X = \{X(h), h \in \mathcal{S}\}$ associated with a Hilbert space $\mathcal{S}$. It was proved by Nualart and Peccati [24] and Nualart and Ortiz-Latorre [23] that $F_{n}$ converges in distribution to the normal law $N(0, \sigma^{2})$ as $n \to \infty$ if and only if one of the following three equivalent conditions holds:

(i) $\lim_{n \to \infty} E[F_{n}^{4}] = 3\sigma^{4}$ (convergence of the fourth moments).

(ii) For all $1 \leq r \leq q - 1$, $f_{n} \otimes r f_{n}$ converges to zero, where $\otimes_{r}$ denotes the contraction of order $r$ (see equation (2.5)).

(iii) $\|DF_{n}\|_{\mathcal{S}}^{2}$ (see definition in Section 2) converges to $q\sigma^{2}$ in $L^{2}(\Omega)$ as $n$ tends to infinity.

A new methodology to study normal approximations and to derive quantitative results combining Stein’s method with Malliavin calculus was introduced by Nourdin and Peccati [15] (see also Nourdin and Peccati [16]). As an illustration of the power of this method, let us mention the following estimate for the total variation distance between the law $\mathcal{L}(F)$ of $F = I_{q}(f)$ and distribution $\gamma = N(0, \sigma^{2})$, where $\sigma^{2} = E[F^{2}]$:

$$d_{TV}(\mathcal{L}(F), \gamma) \leq \frac{2}{q\sigma^{2}} \sqrt{\text{Var} \left(\|DF\|_{\mathcal{S}}^{2}\right)} \leq \frac{2\sqrt{q - 1}}{\sigma^{2}\sqrt{3q}} \sqrt{E[F^{4}]} - 3\sigma^{4}.$$ 

This inequality can be used to show the above equivalence (i)-(iii). A recent result of Nourdin and Poly [20] says that the convergence in law for a sequence of multiple stochastic integrals of order $q \geq 2$ is equivalent to the convergence in total variation if the limit is not constant. As a consequence, for a sequence $F_{n}$ of nonzero multiple stochastic integrals of order $q \geq 2$, the limit in law to is equivalent to the limit of the densities in $L^{1}(\mathbb{R})$, provided the limit is not constant. A multivariate extension of this result has been derived in [14].

The aim of this paper is to study the uniform convergence of the densities of a sequence of random vectors $F_{n}$ to the normal density using the techniques of Malliavin calculus, combined
with Stein’s method for normal approximation. It is well-known that to guarantee that each $F_n$ has a density we need to assume that the norm of the Malliavin derivative of $F_n$ has negative moments. Thus, a natural assumption to obtain uniform convergence of densities is to assume uniform boundedness of the negative moments of the corresponding Malliavin derivatives. Our first result (Theorem 4.1) says that if $F$ is a multiple stochastic integral of order $q \geq 2$ such that $E[F^2] = \sigma^2$ and $M := E(\|DF\|_{\beta}^6) < \infty$, we have

$$\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq C \sqrt{E[F^4]} - 3\sigma^4, \quad (1.1)$$

where $f_F$ is the density of $F$, $\phi$ is the density of the normal law $N(0, \sigma^2)$ and the constant $C$ depends on $q$, $\sigma$ and $M$. We can also replace the expression in the right-hand side of (1.1) by $\sqrt{\text{Var}(\|DF\|_{\beta}^2)}$. The main idea to prove this result is to express the density of $F$ using Malliavin calculus:

$$f_F(x) = E[\mathbf{1}_{\{F > x\}} q \|DF\|_{\beta}^2 F] - E[\mathbf{1}_{\{F > x\}} \langle DF, D(\|DF\|_{\beta}^2) \rangle_{\beta}].$$

Then, one can find an estimate of the form (1.1) for the terms $E[\|DF, D(\|DF\|_{\beta}^2)\|_{\beta}]$ and $E[q \|DF\|_{\beta}^2 - \sigma^{-2}]$. On the other hand, taking into account that

$$\phi(x) = \sigma^{-2} E[\mathbf{1}_{\{N > x\}} N],$$

it suffices to estimate the difference

$$E[\mathbf{1}_{\{F > x\}} F] - E[\mathbf{1}_{\{N > x\}} N],$$

which can be done by Stein’s method. The estimate (1.1) leads to the uniform convergence of the densities in the above equivalence of conditions (i) to (iii) if we assume that $\sup_n E(\|DF_n\|_{\beta}^6) < \infty$.

This methodology is extended in the paper in several directions. We consider the uniform approximation of the $m$th derivative of the density of $F$ by the corresponding densities $\phi^{(m)}$, in the case of random variables in a fixed chaos of order $q \geq 2$. In Theorem 4.4 we obtain an inequality similar to (1.1) assuming that $E(\|DF\|_{\beta}^\beta) < \infty$ for some $\beta > 6m + 6 \left(\left\lfloor \frac{m}{2} \right\rfloor \vee 1\right)$. Again the proof is obtained by a combination of Malliavin calculus and the Stein’s method. Here we need to consider Stein’s equation for functions of the form of $h(x) = 1_{\{x > a\}} p(x)$, where $p$ is a polynomial.

For a $d$ dimensional random vector $F = (F^1, \ldots, F^d)$ whose components are multiple stochastic integrals of orders $q_1, \ldots, q_d$, $q_i \geq 2$, we assume non degeneracy condition $E[\text{det} \gamma_F^p] < \infty$ for all $p \geq 1$, where $\gamma_F = (\langle DF, DF \rangle)_{1 \leq i, j \leq d}$ denotes the Malliavin matrix of $F$. Then, for any multi-index $\beta = (\beta_1, \ldots, \beta_k), 1 \leq \beta_i \leq d$, we obtain the estimate (see Theorem 5.2)

$$\sup_{x \in \mathbb{R}^d} |\partial_\beta f_F(x) - \partial_\beta \phi(x)| \leq C \left( |V - I|^{1/2} + \sum_{j=1}^d \sqrt{E[F_j^4] - 3(E[F_j^2])^2} \right), \quad (1.2)$$

where $V$ is the covariance matrix of $F$, $\phi$ is the standard $d$ dimensional normal density, and $\partial_\beta = \frac{\partial^k}{\partial x_{\beta_1} \cdots \partial x_{\beta_k}}$. As a consequence, we derive the uniform convergence of the densities and their derivatives for a sequence of vectors of multiple stochastic integrals, under the
assumption $\sup_n E[\det \gamma_{F_n}^{-p}] < \infty$ for all $p \geq 1$. A multivariate extension of Stein’s method is required for noncontinuous functions with polynomial growth (see Proposition 5.10). While univariate Stein’s equations with non-smooth test functions have been extensively studied, relatively few results are available for the multivariate case, see [4, 5, 12, 19, 26, 27], so this result has its own interest.

We also consider the case of random variables $F$ such that $E[F] = 0$ and $E[F^2] = \sigma^2$, belonging to the Sobolev space $\mathbb{D}^{2,s}$ for some $s > 4$. In this case, under a non-degeneracy assumption of the form $E[|\langle DF, -DL^{-1}F\rangle_{\mathbb{B}}|^r] < \infty$ for some $r > 2$, we derive an estimate for the uniform distance between the density of $F$ and the density of the normal law $N(0, \sigma^2)$.

In a recent paper [21], Nourdin, Peccati and Swan have obtained an upper bound on the total variation distance between the law of a vector of multiple stochastic integrals and a normal distribution, using a combination of entropy techniques and Malliavin calculus. Their main result can be briefly stated as follows. Let $F = (F_1, \ldots, F_d)$ be a $d$ dimensional random vector whose components are multiple stochastic integrals of orders $q_1, \ldots, q_d, q_i \geq 2$, respectively. Suppose the covariance of $F$ is the identity matrix. Denote $\phi(x)$ the density of $N \sim N(0, \text{Id})$. Then the relative entropy $D(F||N)$ of $F$ satisfies $D(F||N) := E[\log f_F(F) - \log \phi(F)] \leq C \Delta \log \Delta$, where $C > 0$ is a constant and $\Delta = E[|F|^4 - |N|^4]$. This leads to the bound

$$\|f_F - \phi\|_{L^1(\mathbb{R}^d)} \leq \sqrt{2D(F||N)} \leq C\sqrt{\Delta \log \Delta}.$$  

This result refines some estimates obtained in [14]. In the case $d = 1$, this result also refines our estimate (4.12), where by taking $p = 1$ and $\alpha$ close to $\frac{1}{2}$ we can only get $\Delta^{\frac{2}{3} - \epsilon}$ with $\epsilon > 0$ arbitrarily small.

Convergence of densities in uniform distance (also in total variation) has also been studied using Fisher information theory via Shimizu’s inequality (see for instance, [29, 2, 3, 8])

$$\sup_{x \in \mathbb{R}} |f_{F}(x) - \phi(x)| \leq C \sqrt{I(F||N)}, \quad (1.3)$$

where $F$ is a random variable with density $f \in C^1(\mathbb{R})$, $\phi$ is the density of $N \sim N(0, 1)$, and $I(F||N) := \int_{\mathbb{R}} \left( \frac{f_F(x)}{f_{\phi}(x)} - \frac{\phi'(x)}{\phi(x)} \right)^2 f(x)dx$ is the relative Fisher information. Recently, Bobkov, Chistyakov and Götze [2] studied the rate of convergence to 0 of $I(F_n||N)$ for $F_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$, where $\{X_i\}_{i \geq a}$ are i.i.d. random variables with mean 0 and variance 1, assuming that $f_{F_{n_0}} \in L^1(\mathbb{R})$ for some $n_0$.

In general, when studying uniform convergence of densities, one is necessarily led to introduce some stringent assumptions on the regularity of the laws of the underlying random variables. Here we showed that these assumptions can be reduced to requirements about the finiteness of the negative moments of Malliavin matrices.

The paper is organized as follows. Section 2 introduces some preliminary results of Gaussian analysis, Malliavin calculus and Stein’s method for normal approximations. Section 3 is devoted to density formulae with elementary estimates using Malliavin calculus. The density formulae themselves are well-known results, but we present explicit formulae with useful estimates, such as the Hölder continuity and boundedness estimates in theorems 3.1 and 3.3. The boundedness estimates enable us to prove the $L^p$ convergence of the densities (see (4.12)). The Hölder continuity estimates can be used to provide a short proof for the convergence of densities based on a compactness argument, assuming convergence in law (see
Theorem 6.5). Section 4 proves the convergence of densities of random variables in a fixed Wiener chaos, and Section 5 discusses convergence of densities for random vectors. In Section 6, the convergence of densities for sequences of general centered square integrable random variables are studied.

The main difficulty in the application of the above results is to verify the existence of negative moments for the determinant of the Malliavin matrix. We provide explicit sufficient conditions for this condition for random variables in the second Wiener chaos in Section 7. As an application we derive the uniform convergence of the densities and their derivatives to the normal distribution, as time goes to infinity, for the least squares estimator of the parameter \( \theta \) in the Ornstein-Uhlenbeck process: \( dX_t = -\theta X_t dt + \gamma dB_t \), where \( B = \{ B_t, t \geq 0 \} \) is a standard Brownian motion. Some technical results and proofs are included in Section 8.

Along this paper, we denote by \( C \) (maybe with subindexes) a generic constant that might depend on quantities such as the order of multiple stochastic integrals \( q \), the order of the derivatives \( m \), the variance \( \sigma^2 \) or the negative moments of the Malliavin derivative. We denote by \( \| \cdot \|_p \) the norm in the space \( L^p(\Omega) \).

2. Preliminaries

In the first two subsections, we introduce some basic elements of Gaussian analysis and Malliavin calculus, for which we refer to [22, 16] for further details. In the last subsection, we shall introduce some basic estimates from the univariate Stein’s method.

2.1. Isonormal Gaussian process and multiple integrals

Let \( \mathcal{H} \) be a real separable Hilbert space (with its inner product and norm denoted by \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and \( \| \cdot \|_{\mathcal{H}} \), respectively). For any integer \( q \geq 1 \), let \( \mathcal{H}^{\otimes q}(\mathcal{H}^{\otimes q}) \) be the \( q \)th tensor product (symmetric tensor product) of \( \mathcal{H} \). Let \( X = \{ X(h), h \in \mathcal{H} \} \) be an isonormal Gaussian process associated with the Hilbert space \( \mathcal{H} \), defined on a complete probability space \( (\Omega, \mathcal{F}, P) \). That is, \( X \) is a centered Gaussian family of random variables such that \( E[X(h)X(g)] = \langle h, g \rangle_{\mathcal{H}} \) for all \( h, g \in \mathcal{H} \). We assume that the \( \sigma \)-field \( \mathcal{F} \) is generated by \( X \).

For every integer \( q \geq 0 \), the \( q \)th Wiener chaos (denoted by \( \mathcal{H}_q \)) of \( X \) is the closed linear subspace of \( L^2(\Omega) \) generated by the random variables \( \{ H_q(X(h)) : h \in \mathcal{H}, \| h \|_{\mathcal{H}} = 1 \} \), where \( H_q \) is the \( q \)th Hermite polynomial recursively defined by \( H_0(x) = 1 \), \( H_1(x) = x \) and

\[
H_{q+1}(x) = xH_q(x) - qH_{q-1}(x), \quad q \geq 1. \tag{2.1}
\]

For every integer \( q \geq 1 \), the mapping \( I_q(h^{\otimes q}) = H_q(X(h)) \), where \( \| h \|_{\mathcal{H}} = 1 \), can be extended to a linear isometry between \( \mathcal{H}^{\otimes q} \) (equipped with norm \( \sqrt{q!} \| \cdot \|_{\mathcal{H}^{\otimes q}} \)) and \( \mathcal{H}_q \) (equipped with \( L^2(\Omega) \) norm). For \( q = 0 \), \( \mathcal{H}_0 = \mathbb{R} \), and \( I_0 \) is the identity map. The mapping \( I_q \) is called the multiple stochastic integral of order \( q \).

It is well-known (Wiener chaos expansion) that \( L^2(\Omega) \) can be decomposed into the infinite orthogonal sum of the spaces \( \mathcal{H}_q \). That is, any random variable \( F \in L^2(\Omega) \) has the following chaos expansion:

\[
F = \sum_{q=0}^{\infty} I_q(f_q), \tag{2.2}
\]

where \( f_0 = E[F] \), and \( f_q \in \mathcal{H}_{\mathcal{H}}^{\otimes q}, q \geq 1 \), are uniquely determined by \( F \). For every \( q \geq 0 \) we denote by \( J_q \) the orthogonal projection on the \( q \)th Wiener chaos \( \mathcal{H}_q \), so \( I_q(f_q) = J_q F \).
Let \( \{e_n, n \geq 1\} \) be a complete orthonormal basis of \( \mathcal{H} \). Given \( f \in \mathcal{H}_q^{\otimes q} \) and \( g \in \mathcal{H}_p^{\otimes p} \), for \( r = 0, \ldots, p \wedge q \) the \( r \)-th contraction of \( f \) and \( g \) is the element of \( \mathcal{H}_r^{\otimes (p+q-2r)} \) defined by

\[
 f \otimes_r g = \sum_{i_1, \ldots, i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H}^{\otimes r}}. \tag{2.3}
\]

Notice that \( f \otimes_r g \) is not necessarily symmetric. We denote by \( \tilde{f} \otimes_r g \) its symmetrization. Moreover, \( f \otimes_0 g = f \otimes g \), and for \( p = q \), \( f \otimes_q g = \langle f, g \rangle_{\mathcal{H}^{\otimes q}} \). For the product of two multiple stochastic integrals we have the multiplication formula

\[
 I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g). \tag{2.4}
\]

In the particular case \( \mathcal{H} = L^2(A, \mathcal{A}, \mu) \), where \( (A, \mathcal{A}) \) is a measurable space and \( \mu \) is a \( \sigma \)-finite and nonatomic measure, one has that \( \mathcal{H}_q^{\otimes q} = L^2(A^q, \mathcal{A}^q, \mu^q) \) and \( \mathcal{H}_r^{\otimes q} \) is the space of symmetric and square-integrable functions on \( A^q \). Moreover, for every \( f \in \mathcal{H}_q^{\otimes q} \), \( I_q(f) \) coincides with the \( q \)-th multiple Wiener–Itô integral of \( f \) with respect to \( X \), and (2.3) can be written as

\[
 f \otimes_r g (t_1, \ldots, t_{p+q-2r}) = \int_{A^r} f(t_1, \ldots, t_{q-2r}, s_1, \ldots, s_r) g(t_{1+q-2r}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r) d\mu(s_1) \ldots d\mu(s_r). \tag{2.5}
\]

### 2.2. Malliavin operators

We introduce some basic facts on Malliavin calculus with respect to the Gaussian process \( X \). Let \( \mathcal{S} \) denote the class of smooth random variables of the form \( F = f(X(h_1), \ldots, X(h_n)) \), where \( h_1, \ldots, h_n \) are in \( \mathcal{H} \), \( n \geq 1 \), and \( f \in C^\infty_p(\mathbb{R}^n) \), the set of smooth functions \( f \) such that \( f \) itself and all its partial derivatives have at most polynomial growth. Given \( F = f(X(h_1), \ldots, X(h_n)) \) in \( \mathcal{S} \), its Malliavin derivative \( DF \) is the \( \mathcal{H} \)-valued random variable given by

\[
 DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(h_1), \ldots, X(h_n)) h_i.
\]

The derivative operator \( D \) is a closable and unbounded operator on \( L^2(\Omega) \) taking values in \( L^2(\Omega; \mathcal{H}) \). By iteration one can define higher order derivatives \( D^k F \in L^2(\Omega; \mathcal{H}^{\otimes k}) \). For any integer \( k \geq 0 \) and any \( p \geq 1 \) and we denote by \( \mathcal{D}^{k,p} \) the closure of \( \mathcal{S} \) with respect to the norm \( \| \cdot \|_{k,p} \) given by:

\[
 \| F \|_{k,p}^p = \sum_{i=0}^{k} E(\| D^i F \|_{\mathcal{H}^{\otimes i}}^p).
\]

For \( k = 0 \) we simply write \( \| F \|_{0,p} = \| F \|_p \). For any \( p \geq 1 \) and \( k \geq 0 \), we set \( \mathcal{D}^{\infty,p} = \cap_{k \geq 0} \mathcal{D}^{k,p} \), \( \mathcal{D}^{k,\infty} = \cap_{p \geq 1} \mathcal{D}^{k,p} \) and \( \mathcal{D}^{\infty} = \cap_{k \geq 0} \mathcal{D}^{k,\infty} \).

We denote by \( \delta \) (the divergence operator) the adjoint operator of \( D \), which is an unbounded operator from a domain in \( L^2(\Omega; \mathcal{H}) \) to \( L^2(\Omega) \). An element \( u \in L^2(\Omega; \mathcal{H}) \) belongs to the domain of \( \delta \) if and only if it verifies

\[
 E(\langle DF, u \rangle_{\mathcal{H}}) \leq c_u \sqrt{E[F^2]}
\]
for any $F \in \mathbb{D}^{1,2}$, where $c_u$ is a constant depending only on $u$. In particular, if $u \in \text{Dom}\, \delta$, then $\delta(u)$ is characterized by the following duality relationship

$$E(\delta(u)F) = E(\langle DF, u \rangle_{\mathcal{B}})$$

(2.6)

for any $F \in \mathbb{D}^{1,2}$. This formula extends to the multiple integral $\delta^q$, that is, for $u \in \text{Dom}\, \delta^q$ and $F \in \mathbb{D}^{q,2}$ we have

$$E(\delta^q(u)F) = E(\langle D^q F, u \rangle_{\mathcal{B}^{\otimes q}}).$$

We can factor out a scalar random variable in the divergence in the following sense. Let $F \in \mathbb{D}^{1,2}$ and $u \in \text{Dom}\, \delta$ such that $Fu \in L^2(\Omega; \mathfrak{F})$. Then $Fu \in \text{Dom}\, \delta$ and

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle_{\mathcal{B}},$$

(2.7)

provided the right-hand side is square integrable. The operators $\delta$ and $D$ have the following commutation relationship

$$D\delta(u) = u + \delta(Du)$$

(2.8)

for any $u \in \mathbb{D}^{2,2}(\mathfrak{F})$ (see [22, page 37]).

The following version of Meyer’s inequality (see [22, Proposition 1.5.7]) will be used frequently in this paper. Let $V$ be a real separable Hilbert space. We can also introduce Sobolev spaces $\mathbb{D}^{k,p}(V)$ of $V$-valued random variables for $p \geq 1$ and integer $k \geq 1$. Then, for any $p > 1$ and $k \geq 1$, the operator $\delta$ is continuous from $\mathbb{D}^{k,p}(V \otimes \mathfrak{F})$ into $\mathbb{D}^{k-1,p}(V)$. That is,

$$\|\delta(u)\|_{k-1,p} \leq C_{k,p}\|u\|_{k,p}.$$  

(2.9)

The operator $L$ defined on the Wiener chaos expansion as $L = \sum_{q=0}^{\infty} (-q) J_q$ is the infinitesimal generator of the Ornstein–Uhlenbeck semigroup $T_t = \sum_{q=0}^{\infty} e^{-qt} J_q$. Its domain in $L^2(\Omega)$ is

$$\text{Dom}\, L = \left\{ F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2\|J_q F\|_2^2 < \infty \right\} = \mathbb{D}^{2,2}.$$  

The relation between the operators $D$, $\delta$ and $L$ is explained in the following formula (see [22, Proposition 1.4.3]). For $F \in L^2(\Omega)$, $F \in \text{Dom}\, L$ if and only if $F \in \text{Dom}(\delta D)$ (i.e., $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom}\, \delta$), and in this case

$$\delta DF = -LF.$$  

(2.10)

For any $F \in L^2(\Omega)$, we define $L^{-1}F = -\sum_{q=1}^{\infty} q^{-1} J_q(F)$. The operator $L^{-1}$ is called the pseudo-inverse of $L$. Indeed, for any $F \in L^2(\Omega)$, we have that $L^{-1}F \in \text{Dom}\, L$, and

$$LL^{-1}F = L^{-1}LF = F - E[F].$$

We list here some properties of multiple integrals which will be used in Section 4. Fix $q \geq 1$ and let $f \in \mathfrak{F}^{\otimes q}$. We have $I_q(f) = \delta^q(f)$ and $DI_q(f) = qI_{q-1}(f)$, and hence $I_q(f) \in \mathbb{D}^{\infty,2}$. The multiple stochastic integral $I_q(f)$ satisfies hypercontractivity property:

$$\|I_q(f)\|_p \leq C_{q,p}\|I_q(f)\|_2$$

for any $p \in [2, \infty)$.

(2.11)

This easily implies that $I_q(f) \in \mathbb{D}^{\infty}$ and for any $1 \leq k \leq q$ and $p \geq 2$,

$$\|I_q(f)\|_{k,p} \leq C_{q,k,p}\|I_q(f)\|_2.$$
As a consequence, for any \( F \in \bigoplus_{t=1}^{q} \mathcal{H}_t \), we have
\[
\|F\|_{k,p} \leq C_{q,k,p} \|F\|_2. \tag{2.12}
\]
For any random variable \( F \) in the chaos of order \( q \geq 2 \), we have (see [16], Equation (5.2.7))
\[
\frac{1}{q^2} \text{Var} \left( \|DF\|_2^2 \right) \leq \frac{q-1}{3q} \left( E[F^4] - (E[F^2])^2 \right) \leq (q-1) \text{Var} \left( \|DF\|_2^2 \right). \tag{2.13}
\]

In the case where \( \mathcal{F} \) is \( L^2(\mathcal{A}, \mathcal{A}, \mu) \), for an integrable random variable \( F = \sum_{q=0}^{\infty} I_q(f_q) \in \mathbb{D}^{1,2} \), its derivative can be represented as an element in of \( L^2(\mathcal{A} \times \Omega) \) given by
\[
D_tF = \sum_{q=1}^{\infty} qI_q(f_q(\cdot, t)), \quad t \in A.
\]

2.3. Stein’s method of normal approximation

We shall now give a brief account of Stein’s method of univariate normal approximation and its connection with Malliavin calculus. For a more detailed exposition we refer to [4, 16, 30].

Let \( F \) be an arbitrary random variable and let \( N \) be a \( N(0, \sigma^2) \) distributed random variable, where \( \sigma^2 > 0 \). Consider the distance between the law of \( F \) and the law of \( N \) given by
\[
d_{\mathcal{H}}(F, N) = \sup_{h \in \mathcal{H}} |E[h(F) - h(N)]| \tag{2.14}
\]
for a class of functions \( \mathcal{H} \) such that \( E[h(F)] \) and \( E[h(N)] \) are well-defined for \( h \in \mathcal{H} \). Notice first the following fact (which is usually referred as Stein’s lemma): a random variable \( N \) is \( N(0, \sigma^2) \) distributed if and only if \( E[\sigma^2 f'(N) - Nf(N)] = 0 \) for all absolutely continuous functions \( f \) such that \( E[|f'(N)|] < \infty \). This suggests that the distance of \( E[\sigma^2 f'(F) - Ff(F)] \) from zero may quantify the distance between the law of \( F \) and the law of \( N \). To see this, for each function \( h \) such that \( E[|h(N)|] < \infty \), Stein [30] introduced the Stein’s equation:
\[
f'(x) - \frac{x}{\sigma^2} f(x) = h(x) - E[h(N)] \tag{2.15}
\]
for all \( x \in \mathbb{R} \). For a random variable \( F \) such that \( E[|h(F)|] < \infty \), any solution \( f_h \) to Equation (2.15) verifies
\[
\frac{1}{\sigma^2} E[\sigma^2 f'_h(F) - Ff_h(F)] = E[h(F) - h(N)], \tag{2.16}
\]
and the distance defined in (2.14) can be written as
\[
d_{\mathcal{H}}(F, N) = \frac{1}{\sigma^2} \sup_{h \in \mathcal{H}} \left| E[\sigma^2 f'_h(F) - Ff_h(F)] \right|. \tag{2.17}
\]
The unique solution to (2.15) verifying \( \lim_{x \to \pm \infty} e^{-x^2/(2\sigma^2)} f(x) = 0 \) is
\[
f_h(x) = e^{x^2/(2\sigma^2)} \int_{-\infty}^{x} \{h(y) - E[h(N)]\} e^{-y^2/(2\sigma^2)} dy. \tag{2.18}
\]

From (2.17) and (2.18), one can get bounds for probability distances like the total variation distance, where we let \( \mathcal{H} \) consist of all indicator functions of measurable sets, Kolmogorov
distance, where we consider all the half-line indicator functions and Wasserstein distance, where we take \( \mathcal{H} \) to be the set of all Lipschitz-continuous functions with Lipschitz constant equal to 1.

In the present paper, we shall consider the case when \( h : \mathbb{R} \to \mathbb{R} \) is given by \( h(x) = 1_{\{x > z\}}H_k(x) \) for any integer \( k \geq 1 \) and \( z \in \mathbb{R} \), where \( H_k(x) \) is the \( k \)th Hermite polynomial. More generally, we have the following lemma whose proof can be found in the Appendix. It should be pointed out that the univariate Stein’s equations have been extensively studied. For example, we refer to [4, Section 2.2] and [18, Lemma 8.2] when the test functions have sub-polynomial growth.

**Lemma 2.1.** Suppose \( |h(x)| \leq a|x|^k + b \) for some integer \( k \geq 0 \) and some nonnegative numbers \( a, b \). Then, the solution \( f_h \) to the Stein’s equation (2.15) given by (2.18) satisfies

\[
|f'_h(x)| \leq aC_k \sum_{i=0}^{k} \sigma^{k-i} |x|^i + 4b
\]

for all \( x \in \mathbb{R} \), where \( C_k \) is a constant depending only on \( k \).

Nourdin and Peccati [15, 16] combined Stein’s method with Malliavin calculus to estimate the distance between the distributions of regular functionals of an isonormal Gaussian process and the normal distribution \( N(0, \sigma^2) \). The basic ingredient is the following integration by parts formula. For \( F \in \mathbb{D}^{1,2} \) with \( E[F] = 0 \) and any function \( f \in C^1 \) such that \( E[|f'(F)|] < \infty \), using (2.10) and (2.6) we have

\[
E[Ff(F)] = E[LL^{-1}Ff(F)] = E[-\delta DL^{-1}Ff(F)]
\]

\[
= E[(\langle -DL^{-1}F, Df(F) \rangle)] = E[f'(F)\langle -DL^{-1}F, DF \rangle_{\mathcal{H}}],
\]

Then, it follows that

\[
E[\sigma^2 f'(F) - Ff(F)] = E[f'(F)(\sigma^2 - \langle DF, -DL^{-1}F \rangle_{\mathcal{H}})]. \tag{2.19}
\]

Combining Equation (2.19) with (2.16) and Lemma 2.1 we obtain the following result.

**Lemma 2.2.** Suppose \( h : \mathbb{R} \to \mathbb{R} \) verifies \( |h(x)| \leq a|x|^k + b \) for some \( a, b \geq 0 \) and some integer \( k \geq 0 \). Let \( N \sim N(0, \sigma^2) \) and let \( F \in \mathbb{D}^{1,2k} \) with \( \|F\|_{2k} \leq c\sigma \) for some \( c > 0 \). Then there exists a constant \( C_{k,c} \) depending only on \( k \) and \( c \) such that

\[
|E[h(F) - h(N)]| \leq \sigma^{-2}[aC_{k,c}\sigma^k + 4b] \left\| \sigma^2 - \langle DF, -DL^{-1}F \rangle_{\mathcal{H}} \right\|_2.
\]

**Proof.** From (2.16), (2.19) and Lemma (2.1), it suffices to notice that \( \left\| \sum_{i=0}^{k} \sigma^{k-i} |F|^i \right\|_2 \leq \sum_{i=0}^{k} \|F\|_{2k}^{i} \sigma^{k-i} \leq C_{k,c}\sigma^k. \)

3. Density formulae

In this section, we present explicit formulae for the density of a random variable and its derivatives, using the techniques of Malliavin calculus.
3.1. Density formulae

We shall present two explicit formulae for the density of a random variable, with estimates of its uniform and Hölder norms.

**Theorem 3.1.** Let \( F \in \mathbb{D}^{2,s} \) such that \( E[|F|^{2p}] < \infty \) and \( E[\|DF\|^{-2r}_{\mathcal{B}}] < \infty \) for \( p, r, s > 1 \) satisfying \( \frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1 \). Denote

\[
w = \|DF\|_{\mathcal{B}}^2, \quad u = w^{-1}DF.
\]

Then \( u \in \mathbb{D}^{1,p'} \) with \( p' = \frac{p}{p - 1} \) and \( F \) has a density given by

\[
f_F(x) = E \left[ 1_{\{F > x\}} \delta(u) \right].
\]

Furthermore, \( f_F(x) \) is bounded and Hölder continuous of order \( \frac{1}{p} \), that is

\[
f_F(x) \leq C_p \|w^{-1}\|_r \|F\|_{2,s} \left( 1 + (|x|^{-2} \|F\|_{2,p}^2) \right),
\]

\[
|f_F(x) - f_F(y)| \leq C_p \|w^{-1}\|_{1, p'} \|F\|_{2,s, p}^2 |x - y|^\frac{1}{p}
\]

for any \( x, y \in \mathbb{R} \), where \( C_p \) is a constant depending only on \( p \).

**Proof.** Note that

\[
Du = w^{-1}D^2F - 2w^{-2}(D^2F \otimes 1DF) \otimes DF.
\]

Applying Meyer’s inequality (2.9) and Hölder’s inequality we have

\[
\|\delta(u)\|_{p'} \leq C_p \|u\|_{1, p'} \leq C_p(\|u\|_{p'} + \|Du\|_{p'})
\]

\[
\leq C_p(\|w^{-1}\|_r \|DF\|_{\mathcal{B}}) + 3 \|w^{-1}\|_p \|D^2F\|_{\mathcal{B}} + \|D^2F\|_s + \|D^2F\|_r)
\]

\[
\leq 3C_p \|w^{-1}\|_r \|DF\|_{\mathcal{B}} + \|D^2F\|_s + \|D^2F\|_r.
\]

Then \( u \in \mathbb{D}^{1,p'} \) and the density formula (3.1) holds (see, for instance, Nualart [22, Proposition 2.1.1]). From \( E[\delta(u)] = 0 \) and Hölder’s inequality it follows that

\[
|E \left[ 1_{\{F > x\}} \delta(u) \right]| \leq P(|F| > |x|)^{\frac{1}{p'}} \|\delta(u)\|_{p'} \leq \left( 1 + (|x|^{-2p} \|F\|_{2,p}^2) \right)^{\frac{1}{p'}} \|\delta(u)\|_{p'}.
\]

Then (3.2) follows from (3.5) and (3.4).

Finally, for \( x < y \in \mathbb{R} \), noticing that \( 1_{\{F > x\}} - 1_{\{F > y\}} = 1_{\{x < F \leq y\}} \), we have

\[
|f_F(x) - f_F(y)| \leq \left( E[1_{\{x < F \leq y\}}] \right)^{\frac{1}{p'}} \|\delta(u)\|_{p'}.
\]

Applying (3.2) and (3.4) with the fact that \( E[1_{\{x < F \leq y\}}] = \int_x^y f_F(z) dz \) one gets (3.3). \( \blacksquare \)

With the exact proof of [22, Propositions 2.1.1], one can prove the following slightly more general result.

**Proposition 3.2.** Let \( F \in \mathbb{D}^{1,p} \) and \( h : \Omega \to \mathcal{B} \), and suppose that \( \langle DF, h \rangle_{\mathcal{B}} \neq 0 \) a.s. and \( \frac{h}{\langle DF, h \rangle_{\mathcal{B}}} \in \mathbb{D}^{1,q(\mathcal{B})} \) for some \( p, q > 1 \). Then the law of \( F \) has a density given by

\[
f_F(x) = E \left[ 1_{\{F > x\}} \delta \left( \frac{h}{\langle DF, h \rangle_{\mathcal{B}}} \right) \right].
\]

(3.6)
Our next goal is to take $h$ to be $-DL^{-1}F$ in formula (3.6) and get a result similar to Theorem 3.1. First, to get a sufficient condition for $\langle DF, -DL^{-1}F, s \rangle \in \mathbb{D}^1_{p'}$ for some $p' > 1$, we need some technical estimates on $DL^{-1}F$ and $D^2L^{-1}F$. Estimates of this type have been obtained by Nourdin, Peccati and Reinert [17] (see also Nourdin and Peccati’s book [16, Lemma 5.3.8]), when proving an infinite-dimensional Poincaré inequality. More precisely, by using Mehler’s formula, they proved that for any $p \geq 1$, if $F \in \mathbb{D}^2_{p}$, then

$$E[\|DL^{-1}F\|_p^p] \leq E[\|DF\|_p^p], \quad (3.7)$$

$$E[\|D^2L^{-1}F\|_{op}^p] \leq 2^{-p}E[\|D^2F\|_{op}^p], \quad (3.8)$$

where $\|D^2F\|_{op}$ denotes the operator norm of the Hilbert–Schmidt operator from $\mathcal{F}$ to $\mathcal{F}$: $f \mapsto f \otimes_1 D^2F$. Furthermore, the operator norm $\|D^2F\|_{op}$ satisfies the following “random contraction inequality”

$$\|D^2F\|_{op}^4 \leq \|D^2F \otimes_1 D^2F\|_{\mathcal{F}\otimes_2}^2 \leq \|D^2F\|_{\mathcal{F}\otimes_2}^4. \quad (3.9)$$

Sometimes in application, the use of $L^{-1}$ in the integration by parts formula is useful. The next proposition gives a density formula with estimates similar to Theorem 3.1 with the use of $L^{-1}$. Let

$$\bar{w} = \langle DF, -DL^{-1}F, s \rangle, \quad \bar{u} = -\bar{w}^{-1}DL^{-1}F.$$ 

**Proposition 3.3.** Let $F \in \mathbb{D}^{2,s}$, $E[|F|^{2p}] < \infty$ and suppose that $E[|\bar{w}|^{-r}] < \infty$, where $p > 1$, $r > 2$, $s > 3$ satisfy $\frac{1}{p} + \frac{2}{r} + \frac{3}{s} = 1$. Then $\bar{u} \in \mathbb{D}^{1,p'}$ with $p' = \frac{p}{p-1}$ and the law of $F$ has a density given by

$$f_F(x) = E\left[1_{\{F > x\}}(\bar{u})\right]. \quad (3.10)$$

Furthermore, $f_F(x)$ is bounded and Hölder continuous of order $\frac{1}{p}$, that is

$$f_F(x) \leq K_0 \left(1 \wedge |x|^{-2} \|F\|_{2p}^2\right), \quad (3.11)$$

$$|f_F(x) - f_F(y)| \leq K_0^{1 + \frac{1}{p'}} |x - y|^{\frac{1}{p'}} \quad (3.12)$$

for any $x, y \in \mathbb{R}$, where $K_0 = C_p \|\bar{w}^{-1}\|_{r} \|F\|_{2,s} (\|\bar{w}^{-1}\|_{r} \|DF\|_{s}^2 + 1)$, and $C_p$ depends only on $p$.

**Proof.** Note that $D\bar{w} = -D^2F \otimes_1 DL^{-1}F - DF \otimes_1 D^2L^{-1}F$. Then, applying (3.7) and (3.8) we obtain

$$\|D\bar{w}\|_{\frac{p}{2}} \leq (1 + 2^{-s}) \|D^2F\|_{op} \|DF\|_{s}. \quad (3.13)$$

From $\bar{u} = -\bar{w}^{-1}DL^{-1}F$ we get $D\bar{u} = -\bar{w}^{-1}D^2L^{-1}F + \bar{w}^{-2}D\bar{w} \otimes DL^{-1}F$. Then, using (3.7)–(3.9) we have for $t > 0$ satisfying $\frac{1}{p'} = \frac{1}{r} + \frac{1}{r}$,

$$\|\bar{u}\|_{p'} \leq \||\bar{w}^{-1}DL^{-1}F\|_{\mathcal{F}} \|DF\|_{t},$$

and

$$\|D\bar{u}\|_{p'} \leq \||\bar{w}^{-1}D^2L^{-1}F\|_{\mathcal{F} \otimes \mathcal{F}} \|DF\|_{p'} + \||\bar{w}^{-2}D\bar{w}\|_{\mathcal{F}} \|DL^{-1}F\|_{\mathcal{F}} \|DF\|_{p'},$$

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2.9 \leq \|w^{-1}\|_{\ell} \|D^2F\|_{\ell} + \|w^{-2}\|_{\ell} \|D\overline{w}\|_{\ell} \frac{1}{2} \|DF\|_{s}.

Noticing that \(\|D^2F\|_{t} \leq \|D^2F\|_{s}\) because \(t < s\), and applying Meyer’s inequality (2.9) with (3.13) and (3.9) we obtain

\[\|\hat{\delta}(\overline{u})\|_{\nu'} \leq C_{p} \|\overline{u}\|_{1,\nu'} \leq K_{0}. \quad (3.14)\]

Then \(u \in \mathbb{D}^{1,\nu'}\) and the density formula (3.10) holds. As in the proof of Theorem 3.1, (3.11) and (3.12) follow from (3.14) and

\[\left|E\left[1_{\{|F| > \bar{y}\}}\delta(\overline{u})\right]\right| \leq P(\{|F| > \bar{y}\})^{\frac{1}{2}} \|\delta(\overline{u})\|_{\nu'} \leq \left(1 \wedge (2F)^{-2}\|F\|_{2p}^{2}\right)^{\frac{1}{p}} \|\delta(\overline{u})\|_{\nu'},\]

\[|f_{F}(x) - f_{F}(y)| \leq \left(E\left[1_{\{|x| \leq s\}}\right]\right)^{\frac{1}{2}} \|\delta(u)\|_{\nu'}.


3.2. Derivatives of the density

Next we present a formula for the derivatives of the density function, under additional conditions. A sequence of recursively defined random variables given by \(G_{0} = 1\) and \(G_{k+1} = \delta(G_{k}u)\) where \(u\) is an \(\mathcal{F}\)-valued process, plays an essential role in the formula. The following technical lemma gives an explicit formula for the sequence \(G_{k}\), relating it to Hermite polynomials. To simplify the notation, for an \(\mathcal{F}\)-valued random variable \(u\), we denote

\[\delta_{u} = \delta(u), \quad D_{u}G = \langle DG, u\rangle_{\mathcal{F}}, \quad D_{u}^{k}G = \langle D(D_{u}^{k-1}G), u\rangle_{\mathcal{F}}. \quad (3.15)\]

Recall \(H_{k}(x)\) denotes the \(k\)th Hermite polynomial. For \(\lambda > 0\) and \(x \in \mathbb{R}\), we define the generalized \(k\)th Hermite polynomial as

\[H_{k}(\lambda, x) = \lambda^{k}H_{k}\left(x, \frac{x}{\lambda}\right). \quad (3.16)\]

From the property \(H'_{k}(x) = kH_{k-1}(x)\) it follows by induction that the \(k\)th Hermite polynomials has the form

\[H_{k}(x) = \sum_{0 \leq i \leq [k/2]} c_{k,i}x^{k-2i},\]

where we denote by \([k/2]\) the largest integer less than or equal to \(k/2\). Then (3.16) implies

\[H_{k}(\lambda, x) = \sum_{0 \leq i \leq [k/2]} c_{k,i}x^{k-2i} \lambda^{i}. \quad (3.17)\]

**Lemma 3.4.** Fix an integer \(m \geq 1\) and a number \(p > m\). Suppose \(u \in \mathbb{D}^{m,p}(\mathcal{F})\). We define recursively a sequence \(\{G_{k}\}_{k=0}^{m}\) by \(G_{0} = 1\) and \(G_{k+1} = \delta(G_{k}u)\). Then, these variables are well-defined and for \(k = 1, 2, \ldots, m, \ G_{k} \in \mathbb{D}^{m-k, \frac{p}{k}} \) and

\[G_{k} = H_{k}(D_{u}\delta_{u}, \delta_{u}) + T_{k}, \quad (3.18)\]

where we denote by \(T_{k}\) the higher order derivative terms which can be defined recursively as follows: \(T_{1} = T_{2} = 0\) and for \(k \geq 2,\)

\[T_{k+1} = \delta_{u}T_{k} - D_{u}T_{k} - \partial_{\lambda}H_{k}(D_{u}\delta_{u}, \delta_{u})D_{u}^{2}\delta_{u}. \quad (3.19)\]

The following lemma is proved in the Appendix.
Lemma 3.5. From (3.19) we can deduce that for \( k \geq 3 \)
\[
T_k = \sum_{(i_0, i_1, \ldots, i_{k-1}) \in J_k} a_{i_0, i_1, \ldots, i_{k-1}} \delta_u \left( D_u \delta_u \right)^{i_1} \left( D_u^2 \delta_u \right)^{i_2} \cdots \left( D_u^{k-1} \delta_u \right)^{i_{k-1}},
\]
(3.20)
where the coefficients \( a_{i_0, i_1, \ldots, i_{k-1}} \) are real numbers and \( J_k \) is the set of multi-indices \((i_0, i_1, \ldots, i_{k-1}) \in \mathbb{N}^k\) satisfying the following three conditions
\[
\begin{align*}
(a) & \ i_0 + \sum_{j=1}^{k-1} j i_j \leq k - 1; \\
(b) & \ i_2 + \cdots + i_{k-1} \geq 1; \\
(c) & \ \sum_{j=1}^{k-1} i_j \leq \left\lfloor \frac{k-1}{2} \right\rfloor.
\end{align*}
\]
From (b) we see that every term in \( T_k \) contains at least one factor of the form \( D_u^j \delta_u \) with some \( j \geq 2 \). We shall show this type of factors will converge to zero. For this reason we call these terms high order terms.

Proof of Lemma 3.4. First, we prove by induction on \( k \) that the above sequence \( G_k \) is well-defined and \( G_k \in \mathbb{D}^{m-k, \frac{p}{p+1}} \). Suppose first that \( k = 1 \). Then, Meyer’s inequality implies that \( G_1 = \delta_u \in \mathbb{D}^{m-1, p} \). Assume now that for \( k \leq m-1, G_k \in \mathbb{D}^{m-k, \frac{p}{p+1}} \). Then it follows from Meyer’s and Hölder’s inequalities (see [22, Proposition 1.5.6]) that
\[
\|G_{k+1}\|_{m-k-1, \frac{p}{p+1}} \leq C_{m,p} \|G_k u\|_{m-k, \frac{p}{p+1}} \leq C'_{m,p} \|G_k\|_{m-k, \frac{p}{p+1}} \|u\|_{m-k, p} < \infty.
\]
Let us now show, by induction, the decomposition (3.18). When \( k = 1 \) (3.18) is true because \( G_1 = \delta_u \) and \( T_1 = 0 \). Assume now (3.18) holds for \( k \leq m-1 \). Noticing that \( \partial_x H_k(\lambda, x) = k H_{k-1}(\lambda, x) \) (since \( H'_k(x) = k H_{k-1}(x) \)), we get
\[
D_u H_k(D_u \delta_u, \delta_u) = k H_{k-1}(D_u \delta_u, \delta_u) D_u \delta_u + \partial \lambda H_k(D_u \delta_u, \delta_u) D_u^2 \delta_u.
\]
Hence, applying the operator \( D_u \) to both sides of (3.18),
\[
D_u G_k = k H_{k-1}(D_u \delta_u, \delta_u) D_u \delta_u + \tilde{T}_{k+1},
\]
where
\[
\tilde{T}_{k+1} = D_u T_k + \partial \lambda H_k(D_u \delta_u, \delta_u) D_u^2 \delta_u.
\]
(3.21)
From the definition of \( G_{k+1} \) and using (2.7) we obtain
\[
G_{k+1} = \delta(u G_k) = G_k \delta_u - D_u G_k = \delta_u H_k(D_u \delta_u, \delta_u) + D_u T_k - k H_{k-1}(D_u \delta_u, \delta_u) D_u \delta_u - \tilde{T}_{k+1}.
\]
Note that \( H_{k+1}(x) = x H_k(x) - k H_{k-1}(x) \) implies \( x H_k(\lambda, x) - k \lambda H_{k-1}(\lambda, x) = H_{k+1}(\lambda, x) \).
Hence,
\[
G_{k+1} = H_{k+1}(D_u \delta_u, \delta_u) + \delta_u T_k - \tilde{T}_{k+1}.
\]
The term \( T_{k+1} = \delta_u T_k - \tilde{T}_{k+1} \) has the form given in (3.19). This completes the proof. 

Now we are ready to present some formulae for the derivatives of the density function under certain sufficient conditions on the random variable \( F \). For a random variable \( F \) in \( \mathbb{D}^{1,2} \) and for any \( \beta \geq 1 \) we are going to use the notation
\[
M_\beta(F) = \left( E[\|DF\|_{\beta}^{-\beta}] \right)^{\frac{1}{\beta}}.
\]
(3.22)
**Proposition 3.6.** Fix an integer $m \geq 1$. Let $F$ be a random variable in $\mathbb{D}^{m+2,\infty}$ such that $M_{\beta}(F) < \infty$ for some $\beta > 3m + 3(\lceil \frac{3}{2} \rceil \lor 1)$. Denote $w = \|DF\|^2_{\mathbb{R}}$ and $u = \frac{DF}{w}$. Then, $u \in \mathbb{D}^{m+1,p}(\mathbb{S})$ for some $p > 1$, and the random variables $\{G_k\}_{k=0}^{m+1}$ introduced in Lemma 3.4 are well-defined. Under these assumptions, $F$ has a density $f$ of class $C^m$ with derivatives given by

$$f_F^{(k)}(x) = (-1)^k E[1_{\{F > x\}} G_{k+1}]$$

for $k = 1, \ldots, m$.

**Proof.** It is enough to show that $\{G_k\}_{k=0}^{m+1}$ are well-defined, since it follows from [22, Exercise 2.1.4] that the $k$th derivative of the density of $F$ is given by (3.23). To do this we will show that $G_k$ defined in (3.18) are in $L^1(\Omega)$ for all $k = 1, \ldots, m + 1$. From (3.18) we can write

$$E[|G_k|] \leq E[|H_k(D_u \delta_u, \delta_u)|] + E[|T_k|].$$

Recall the explicit expression of $H_k(\lambda, x)$ in (3.17). Since $\beta > 3(m + 1)$, we can choose $r_0 < \frac{\beta}{3}, r_1 < \frac{\beta}{6}$ such that

$$1 \geq \frac{k - 2i}{r_0} + \frac{i}{r_1} > \frac{3(k - 2i)}{\beta} + \frac{6i}{\beta} = \frac{3k}{\beta},$$

for any $0 \leq i \leq \lfloor k/2 \rfloor$ and $1 \leq k \leq m + 1$. Then, applying Hölder’s inequality with (3.17), (8.11) and (8.12) we have

$$E[|H_k(D_u \delta_u, \delta_u)|] \leq C_k \sum_{0 \leq i \leq \lfloor k/2 \rfloor} \|\delta_u\|_{\mathbb{R}}^{k - 2i} \|D_u \delta_u\|_{r_1}^{i} < \infty.$$

To prove that $E[|T_k|] < \infty$, applying Hölder’s inequality to the expression (3.20) and choosing $r_j > 0$ for $0 \leq j \leq k - 1$ such that

$$1 \geq \frac{i_0}{r_0} + \sum_{j=1}^{k-1} \frac{i_j}{r_j} > \frac{3i_0}{\beta} + \sum_{j=1}^{k-1} \frac{(3j + 3)i_j}{\beta},$$

we obtain that, (assuming $k \geq 3$, otherwise $T_k = 0$)

$$E[|T_k|] \leq C \sum_{(i_0, \ldots, i_k) \in J_k} \|\delta_u\|_{r_0}^{i_0} \prod_{j=1}^{k-1} \|D_u \delta_u\|_{r_j}^{i_j}.$$  

Due to (8.11) and (8.12), this expression is finite, provided $r_j < \frac{\beta}{3j+3}$ for $0 \leq j \leq k - 1$. We can choose $(r_j, 0 \leq j \leq k - 1)$ satisfying the above conditions because $\beta > 3(k - 1) + 3 \lceil \frac{k}{2} \rceil$ for all $1 \leq k \leq m + 1$, and from properties (a) and (c) of $J_k$ in Lemma 3.5 we have

$$\frac{3i_0}{\beta} + \sum_{j=1}^{k-1} \frac{(3j + 3)i_j}{\beta} \leq \frac{3(k - 1) + 3 \lfloor \frac{k}{2} \rfloor}{\beta}.$$

This completes the proof. \[\blacksquare\]
Example 3.7. Consider a random variable in the first Wiener chaos \( N = I_1(h) \), where \( h \in \mathcal{F} \) with \( \|h\|_{\mathcal{F}} = \sigma \). Then \( N \) has the normal distribution \( N \sim N(0, \sigma^2) \) with density denoted by \( \phi(x) \). Clearly \( \|DN\|_{\mathcal{F}} = \sigma \), \( u = \frac{h}{\sigma^2} \), \( \delta_u = \frac{N}{\sigma^2} \) and \( D_u\delta_u = \frac{h}{\sigma^2} \). Then \( G_k = H_k(\frac{1}{\sigma^2}, \frac{N}{\sigma^2}) \) and from (3.23) we obtain the formula

\[
\phi^{(k)}(x) = (-1)^k E \left[ 1_{\{N>0\}} H_{k+1} \left( \frac{1}{\sigma^2}, \frac{N}{\sigma^2} \right) \right],
\]

(3.24)

which can also be obtained by analytic arguments.

Remark 3.8. Let \( g \) be a function differentiable of order \( m \), and denote \( u_j(g, x) = \sup_{|t| \leq x} |g^{(j)}(t) - g^{(j)}(0)| \). Let \( F \) be a random variable with \( E(F) = 0 \), \( E(F^2) = 1 \) and \( E \{ |F|^{m+1} u_m(g, F) \} < \infty \). It is proved in [1] that the following expansion holds:

\[
E(Fg(F)) - E(g'(F)) = \sum_{j=2}^{m} \frac{\gamma_j+1}{j!} E g^{(j)}(F) + R,
\]

where \( \gamma_j \) is the \( j \)-th cumulant of \( X \) and \( |R| \leq CE \{ |F|^{m+1} u_m(g, F) \} \) for some constant \( C > 0 \). For any function \( h \), let \( f \) be the solution of the Stein’s equation (2.15) given by (2.18). Then

\[
E[h(F)] - E[h(N)] = E[f'(F)] - E[Ff(F)] = - \sum_{j=2}^{m} \frac{\gamma_j+1}{j!} E[f^{(j)}(F)] - R.
\]

This is the so-called Edgeworth expansion (see also [28] and references therein). Equation (3.23) can also be used to compute \( E[f^{(j)}(F)] \). We have easily

\[
E[h(F)] - E[h(N)] = - \sum_{j=2}^{m} \frac{\gamma_j+1}{j!} E[f(F)G_k] - R,
\]

(3.25)

where \( G_k \) is given in Lemma 3.4. Thus, it is possible to use Malliavin calculus to obtain the full Edgeworth expansion without assuming the differentiability of \( f \). However, we shall not pursue this aspect in the present work.

Remark 3.9. The recursive algorithms used in Lemma 3.4 have some similarities with the recursive formula developed by Privault in [25] to compute \( E(F[\delta(u)]^n) \).

4. Random variables in the \( q \)th Wiener chaos

In this section we establish our main results on uniform estimates and uniform convergence of densities and their derivatives. We shall deal first with the convergence of densities and later we consider their derivatives.

4.1. Uniform estimates of densities

Let \( F = I_q(f) \) for some \( f \in \mathcal{H}^\infty \) and \( q \geq 2 \). To simplify the notation, along this section we denote

\[
w = \|DF\|_{\mathcal{H}}^2, \quad u = w^{-1}DF.
\]

Note that \( LF = -qF \) and using (2.7) and (2.10) we can write

\[
\delta_u = \delta(u) = qFw^{-1} - \langle Dw^{-1}, DF \rangle_{\mathcal{H}}.
\]

(4.1)
Theorem 4.1. Let $F = I_q(f)$, $q \geq 2$, for some $f \in \mathcal{H}^{\sigma_q}$ be a random variable in the $q$th Wiener chaos with $E[F^2] = \sigma^2$. Assume that $M_6(F) < \infty$, where $M_6(F)$ is defined in (3.22). Let $\phi(x)$ be the density of $N \sim N(0, \sigma^2)$. Then $F$ has a density $f_F(x)$ given by (3.1). Furthermore,

$$
\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq C \sqrt{E[F^4]} - 3\sigma^4,
$$

where the constant $C$ has the form $C = C_q(\sigma^{-1}M_6(F)^2 + M_6(F)^3 + \sigma^{-3})$ and $C_q$ depends only on $q$.

We begin with a lemma giving an estimate for the contraction $D^k F \otimes_1 D^l F$ with $k + l \geq 3$.

Lemma 4.2. Let $F = I_q(f)$ be a random variable in the $q$th Wiener chaos with $E[F^2] = \sigma^2$. Then for any integers $k \geq l \geq 1$ satisfying $k + l \geq 3$, there exists a constant $C_{k,l,q}$ depending only on $k, l, q$ such that

$$
\|D^k F \otimes_1 D^l F\|_2 \leq C_{k,l,q} \|q\sigma^2 - \|DF\|_D^2\|_2.
$$

Proof. Note that $D^k F = q(q - 1) \cdots (q - k + 1)I_{q-k}(f)$. Applying (2.4), we get

$$
D^k F \otimes_1 D^l F = q^2(q - 1)^2 \cdots (q - l + 1)^2(q - l) \cdots (q - k + 1)
\times \sum_{r=0}^{q-k} r! \left(\begin{array}{c} q - k \\ r \end{array}\right) \left(\begin{array}{c} q - l \\ r \end{array}\right) I_{2q - k - l - 2r}(f \otimes_{r+1} f).
$$

Taking into account the orthogonality of multiple integrals of different orders, we obtain

$$
E[\|D^k F \otimes_1 D^l F\|_{\mathcal{H}^{\otimes(k+l-2)}}^2] = \frac{(q!)^4}{(q - l)!^2 (q - k)!^2}
\times \sum_{r=0}^{q-k} r!^2 \left(\begin{array}{c} q - k \\ r \end{array}\right)^2 \left(\begin{array}{c} q - l \\ r \end{array}\right)^2 (2q - k - l - 2r)! \|f \otimes_{r+1} f\|_{\mathcal{H}^{\otimes2q-2-2r}}^2.
$$

Applying (4.4) with $k = l = 1$, we obtain

$$
E[\|DF\|_D^4] = E[\|DF \otimes_1 DF\|_D^2]
= q^4 \sum_{r=0}^{q-1} r!^2 \left(\begin{array}{c} q - 1 \\ r \end{array}\right)^4 (2q - 2 - 2r)! \|f \otimes_{r+1} f\|_{\mathcal{H}^{\otimes2q-2-2r}}^2
= q^4 \sum_{r=0}^{q-2} r!^2 \left(\begin{array}{c} q - 1 \\ r \end{array}\right)^4 (2q - 2 - 2r)! \|f \otimes_{r+1} f\|_{\mathcal{H}^{\otimes2q-2-2r}}^2 + q^2 q!^2 \|f\|_{\mathcal{H}^{\otimes q}}^4.
$$

Taking into account that $\sigma^2 = E[F^2] = q! \|f\|_{\mathcal{H}^{\otimes q}}^2$, we obtain that for any $k + l \geq 3$, there exists a constant $C_{k,l,q}$ such that

$$
E[\|D^k F \otimes_1 D^l F\|_{\mathcal{H}^{\otimes(k+l-2)}}^2] \leq C_{k,l,q} E[\|DF\|_D^4 - q^2 \sigma^4].
$$

Meanwhile, it follows from $E[\|DF\|_D^2] = q \|f\|_{\mathcal{H}^{\otimes q}}^2 = q\sigma^2$ that

$$
E[\|DF\|_D^4 - q^2 \sigma^4] = E[\|DF\|_D^4 - 2q\sigma^2 \|DF\|_D^2 + q^2 \sigma^4] = E[(\|DF\|_D^4 - q\sigma^2)^2].
$$
Combining (4.4), (4.5) and (4.6) we have
\[ E[\|D^k f \otimes_1 D^l f\|_{\mathcal{S}_k}^2] \leq C_{k,l,q}^2 E[\|\sigma^2 - q\|^2], \]
which completes the proof. \(\blacksquare\)

**Proof of Theorem 4.1.** It follows from Theorem 3.1 that \(F\) admits a density \(f_F(x) = E[\mathbf{1}_{\{x > x\}} \delta(u)]\). By (3.24) with \(k = 1\) we can write \(\phi(x) = \frac{1}{\sigma^2} E[\mathbf{1}_{\{N > x\}}]\). Then, using (4.1), for all \(x \in \mathbb{R}\) we obtain
\[
\begin{aligned}
  f_F(x) - \phi(x) &= E[\mathbf{1}_{\{x > x\}} \delta(u)] - \sigma^{-2} E[\mathbf{1}_{\{N > x\}}]
  \\
  &= E[\mathbf{1}_{\{x > x\}} (F(\frac{q}{w} - \sigma^{-2}) - \langle Dw^{-1}, DF\rangle_{\mathcal{S}})] + \sigma^{-2} E[F\mathbf{1}_{\{x > x\}} - N\mathbf{1}_{\{N > x\}}]
  \\
  &= A_1 + A_2. 
\end{aligned}
\]

For the first term \(A_1\), Hölder’s inequality implies
\[
\begin{aligned}
|A_1| &= \left| E[\mathbf{1}_{\{x > x\}} (F(\frac{q}{w} - \sigma^{-2}) - \langle Dw^{-1}, DF\rangle_{\mathcal{S}})] \right|
  \\
&\leq \sigma^{-2} E[|Fw^{-1}(w - q\sigma^2)|] + 2E[w^{-\frac{3}{2}} \|D^2 F \otimes_1 DF\|_{\mathcal{S}}]
  \\
&\leq \sigma^{-2} \|w^{-1}\|_3 \|F\|_3 \|w - q\sigma^2\|_3 + 2 \|w^{-\frac{3}{2}}\|_2 \|D^2 F \otimes_1 DF\|_{\mathcal{S}}. 
\end{aligned}
\]

Note that (2.12) implies
\[
\|w - q\sigma^2\|_3 \leq C \|w - q\sigma^2\|_2
\]
and \(\|F\|_3 \leq C \|F\|_2 = C\sigma\). Combining these estimates with (4.3) we obtain
\[
|A_1| \leq C(\sigma^{-1} \|w^{-1}\|_3 + \|w^{-1}\|_3^\frac{3}{2}) \|w - q\sigma^2\|_2. 
\]

For the second term \(A_2\), applying Lemma 2.2 to the function \(h(z) = z\mathbf{1}_{\{z > x\}}\), which satisfies \(|h(z)| \leq |z|\), we have
\[
\begin{aligned}
|A_2| &= \sigma^{-2} \left| E[DF - DL^{-1} F] \right|
  \\
&\leq C\sigma^{-3} \left\|\sigma^2 - \langle DF, DL^{-1} F\rangle_{\mathcal{S}} \right\|_2 
  \leq C\sigma^{-3} \|q\sigma^2 - w\|_2. 
\end{aligned}
\]

Combining (4.7) with (4.8)–(4.9) we obtain
\[
\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq C(\sigma^{-1} \|w^{-1}\|_3 + \|w^{-1}\|_3^\frac{3}{2} + \sigma^{-3}) \|w - q\sigma^2\|_2.
\]

Then (4.2) follows from (2.13). This completes the proof. \(\blacksquare\)

Using the estimates shown in Theorem 4.1 we can deduce the following uniform convergence and convergence in \(L^p\) of densities for a sequence of random variables in a fixed \(q\)th Wiener chaos.

**Corollary 4.3.** Let \(\{F_n\}_{n \in \mathbb{N}}\) be a sequence of random variables in the \(q\)th Wiener chaos with \(q \geq 2\). Set \(\sigma_n^2 = E[F_n^2]\) and assume that \(\lim_{n \to \infty} \sigma_n^2 = \sigma^2\), \(0 < \delta \leq \sigma_n^2 \leq K\) for all \(n\), \(\lim_{n \to \infty} E[F_n^q] = 3\sigma^4\) and
\[
M := \sup_n \left( \mathbb{E}[\|DF_n\|_{\mathcal{S}}^6] \right)^{1/6} < \infty. 
\]
Let $\phi(x)$ be the density of the law $N(0, \sigma^2)$. Then, each $F_n$ admits a density $f_{F_n} \in C(\mathbb{R})$ and there exists a constant $C$ depending only on $q, \sigma, \delta$ and $M$ such that

$$\sup_{x \in \mathbb{R}} |f_{F_n}(x) - \phi(x)| \leq C \left( |E[F_n^q] - 3\sigma_n^4|^{\frac{1}{2}} + |\sigma_n - \sigma| \right). \tag{4.11}$$

Furthermore, for any $p \geq 1$ and $\alpha \in (\frac{1}{2}, p)$,

$$\|f_{F_n} - \phi\|_{L^p(\mathbb{R})} \leq C \left( |E[F_n^4] - 3\sigma_n^4|^{\frac{1}{2}} + |\sigma_n - \sigma| \right)^{\frac{p-\alpha}{p}}, \tag{4.12}$$

where $C$ is a constant depending on $q, \sigma, M, p, \alpha$ and $K$.

**Proof.** Let $\phi_n(x)$ be the density of $N(0, \sigma_n^2)$. Then Theorem 4.1 implies that

$$\sup_{x \in \mathbb{R}} |f_{F_n}(x) - \phi_n(x)| \leq C |E[F_n^q] - 3\sigma_n^4|^{\frac{1}{2}}.$$ 

On the other hand, if $N_n \sim N(0, \sigma_n^2)$, it is easy to see that

$$\sup_{x \in \mathbb{R}} |\phi_n(x) - \phi(x)| \leq C |\sigma_n - \sigma|.$$ 

Then (4.11) follows from triangle inequality. To show (4.12), first notice that (3.2) implies

$$f_{F_n}(x) \leq C(1 \wedge |x|^{-2}).$$

Therefore, if $\alpha > \frac{1}{2}$ the function $(f_{F_n}(x) + \phi(x))^{\alpha}$ is integrable. Then, (4.12) follows from (4.11) and the inequality

$$|f_{F_n}(x) - \phi(x)|^p \leq |f_{F_n}(x) - \phi(x)|^{p-\alpha} (f_{F_n}(x) + \phi(x))^{\alpha}.$$

\[\blacksquare\]

**4.2. Uniform estimation of derivatives of densities**

In this subsection, we establish the uniform convergence for derivatives of densities of random variables to a normal distribution. We begin with the following theorem which estimates the uniform distance between the derivatives of the densities of a random variable $F$ in the $q$th Wiener chaos and the normal law $N(0, E[F^2])$.

**Theorem 4.4.** Let $m \geq 1$ be an integer. Let $F$ be a random variable in the $q$th Wiener chaos, $q \geq 2$, with $E[F^2] = \sigma^2$ and $M_{\beta} := M_{\beta}(F) < \infty$ for some $\beta > 6m + 6(\lfloor \frac{q}{2} \rfloor \vee 1)$ (Recall the definition of $M_{\beta}(F)$ in (3.22)). Let $\phi(x)$ be the density of $N \sim N(0, \sigma^2)$. Then $F$ has a density $f_F(x) \in C^m(\mathbb{R})$ with derivatives given by (3.23). Moreover, for any $k = 1, \ldots, m$

$$\sup_{x \in \mathbb{R}} \left| f_F^{(k)}(x) - \phi^{(k)}(x) \right| \leq \sigma^{-k-3} C \sqrt{E[F^4]} - 3\sigma^2,$$

where the constant $C$ depends on $q, \beta, m, \sigma$ and $M_{\beta}$ with polynomial growth in $\sigma$ and $M_{\beta}$.

To prove Theorem 4.4, we need some technical results. Recall the notation we introduced in (3.15), where we denote $\delta_u = \delta(u)$, $D_u\delta_u = \langle D\delta_u, u \rangle_{\mathcal{B}}$.
Lemma 4.5. Let $F$ be a random variable in the $q$th Wiener chaos with $E[F^2] = \sigma^2$. Let $w = \|DF\|_2^2$ and $u = w^{-1}DF$.

(i) If $M_\beta(F) < \infty$ for some $\beta > 6$, then for any $1 < r \leq \frac{2\beta}{\beta+6}$

$$
\|\delta_u - \sigma^{-2}F\|_r \leq C\sigma^{-1}(M_\beta^3 \lor 1)\|q\sigma^2 - w\|_2.
$$

(ii) If $M_\beta(F) < \infty$ for some $\beta > 12$, then for any $1 < r < \frac{2\beta}{\beta+12}$

$$
\|D_u \delta_u - \sigma^{-2}\|_r \leq C\sigma^{-2}(M_\beta^6 \lor 1)\|q\sigma^2 - w\|_2,
$$

where the constant $C$ depends on $\sigma$.

Proof. Recall that $\delta_u = qFw^{-1} - DDFw^{-1}$. Using Hölder's inequality and (8.3) we can write

$$
\|\delta_u - \sigma^{-2}F\|_r \leq \|\sigma^{-2}Fw^{-1}(q\sigma^2 - w)\|_r + \|DDFw^{-1}\|_r
\leq \ C(\sigma^{-2}\|Fw^{-1}\|_s + (M_\beta^3 \lor 1))\|q\sigma^2 - w\|_2,
$$

provided $\frac{1}{r} = \frac{1}{s} + \frac{1}{2}$. By the hypercontractivity property (2.11) $\|F\|_\gamma \leq C_{q,\gamma}\|F\|_2$ for any $\gamma \geq 2$. Thus, by Hölder’s inequality, if $\frac{1}{s} = \frac{1}{\gamma} + \frac{1}{p}$

$$
\|Fw^{-1}\|_s \leq \|F\|_\gamma\|w^{-1}\|_p \leq C_{q,\gamma}\sigma M_\beta^2 2^p.
$$

Choosing $p$ such that $2p < \beta$ we get (4.13).

We can compute $D_u \delta_u$ as

$$
D_u \delta_u = qw^{-1} + qFw^{-1}DDFW^{-1} - w^{-1}D^2DFw^{-1} - w^{-1}(D^2F, DF \otimes Dw^{-1})_{\mathcal{B}}.
$$

Applying Hölder’s inequality we obtain

$$
\|D_u \delta_u - \sigma^{-2}\|_r \leq \|w^{-1}\|_r \left[\|\sigma^{-2}(q\sigma^2 - w) + qFDw^{-1} - D^2DFw^{-1}\|_r\right]
\leq \ \sigma^{-2}\|w^{-1}\|_r \left[\|q\sigma^2 - w\|_2 + \sigma\|w^{-1}\|_p \left(\|Daw^{-1}\|_s + \|D^2DFw^{-1}\|_s\right)\right],
$$

if $\frac{1}{r} > \frac{1}{p} + \frac{1}{s}$. Then, using (8.3) and (8.4) with $k = 2$ and assuming that $s < \frac{2\beta}{\beta+8}$ and that $2p < \beta$ we obtain (4.14). ■

Proof of Theorem 4.4. Proposition 3.6 implies that $f_F(x) \in C^{m-1}(\mathbb{R})$ and for $k = 0, 1, \ldots, m - 1$,

$$
f_F^{(k)}(x) = (-1)^k E[1_{\{F > x\}} G_{k+1}],
$$

where $G_0 = 1$ and $G_{k+1} = \delta(G_k u) = G_k \delta(u) - \langle DG_k, u \rangle_{\mathcal{B}}$. From (3.24),

$$
\phi^{(k)}(x) = (-1)^k E[1_{\{N > x\}} H_{k+1}(\sigma^{-2}, \sigma^{-2} N)].
$$

Then, the identity $G_{k+1} = H_{k+1}(D_u \delta_u, \delta_u) + T_{k+1}$ (see formula (3.18)), suggests the following triangle inequality

$$
\left| f_F^{(k)}(x) - \phi^{(k)}(x) \right| = \left| E[1_{\{F > x\}} G_{k+1} - 1_{\{N > x\}} H_{k+1}(\sigma^{-2}, \sigma^{-2} N)] \right| 
\leq \left| E[1_{\{F > x\}} G_{k+1} - 1_{\{F > x\}} H_{k+1}(\sigma^{-2}, \sigma^{-2} F)] \right|
$$
\[ A_2 = |E[h(F) - h(N)]| \]
\[ \leq C_{q,k} \sigma^{-2} |\sigma^k + 4\sigma^{-k-1}| \left\| \sigma^2 - \langle DF, -DL^{-1}F \rangle_B \right\|_2 \]
\[ \leq C_{q,k} \sigma^{-k-3} \| q\sigma^2 - w \|_2, \quad (4.15) \]

where in the second inequality we used the fact that \( \langle DF, -DL^{-1}F \rangle_B = \frac{w}{q} \).

For the term \( A_1 \), Lemma 3.4 implies
\[ A_1 \leq E[|H_{k+1}(D_u\delta_u, \delta_u) - H_{k+1}(\sigma^{-2}, \sigma^{-2}F)|] + E[|T_{k+1}|]. \quad (4.16) \]

To proceed with the first term above, applying (3.17) we have
\[ |H_{k+1}(D_u\delta_u, \delta_u) - H_{k+1}(\sigma^{-2}, \sigma^{-2}F)| \leq 2C_k |x - y| \sum_{0 \leq j \leq k-1} |x|^{k-j} |y|^j \]
and applying Hölder’s inequality and the hypercontractivity property (2.11) we obtain
\[ E \left[ |\delta_u^{k+1-2i} - (\sigma^{-2}F)^{k+1-2i}| \right] \leq C_k E \left[ |\delta_u - \sigma^{-2}F| \right] \sum_{0 \leq j \leq k-2i} \| \delta_u \|_{p}^{k-2i-j} \sigma^{-j}, \quad (4.18) \]

provided \( 1 \geq \frac{1}{r} + \frac{i}{s} + \frac{k-2i-j}{p} \), which is implied by \( 1 \geq \frac{1}{r} + \frac{i}{s} + \frac{k-2i}{p} \). In order to apply the estimates (4.13), (8.12) (with \( k = 1 \)) and (8.11) we need \( \frac{1}{r} > \frac{2}{3} + \frac{1}{2}, \frac{1}{s} > \frac{4}{3} \) and \( \frac{1}{p} > \frac{3}{\beta} \), respectively. These are possible because \( \beta > 6k+6 \). Then we obtain an estimate of the form
\[ E \left[ |\delta_u^{k+1-2i} - (\sigma^{-2}F)^{k+1-2i}| \right] \leq C_{q,k} \sigma^{-k} (M_{3k+3}^{\beta} \vee 1) \| q\sigma^2 - w \|_2. \quad (4.19) \]

Similarly,
\[
C_{q,k} \sigma^{-(k-1)} \left\| D_u \delta_u - \sigma^{-2} \right\|_r \sum_{0 \leq j \leq i-1} \left\| D_u \delta_u \right\|_s^j, \tag{4.20}
\]

provided \(1 > \frac{1}{r} + \frac{j}{s} \). In order to apply the estimates (4.14) and (8.12) (with \(k = 1\)) we need \(\frac{1}{r} > \frac{6}{\beta} + \frac{1}{2} \) and \(\frac{1}{s} > \frac{6}{\beta} \), respectively. This implies

\[
\frac{1}{r} + \frac{j}{s} > \frac{6 + 6j}{\beta} + \frac{1}{2}.
\]

Notice that \(6 + 6j \leq 6i \leq 3k + 3 \). So, we need \(1 > \frac{1}{r} + \frac{3k+3}{\beta} \). The above \(r, s \) and \(p \) exist because \(\beta > 6k + 6 \). Thus, we obtain an estimate of the form

\[
E \left[ |\sigma^{-2} F|^{k+1-2i} \left( D_u \delta_u \right)^i - \sigma^{-2i} \right] \leq C_{q,k,\sigma,\beta} \sigma^{-(k-1)} (M_{\beta}^{3k+3} \lor 1) \left\| q\sigma^2 - w \right\|_2. \tag{4.21}
\]

Combining (4.19) and (4.21) we have

\[
E \left[ |H_{k+1}(D_u \delta_u, \delta_u) - H_{k+1}(\sigma^{-2}, \sigma^{-2} F) | \right] \leq C_{q,k,\sigma,\beta} \sigma^{-k} (M_{\beta}^{3k+3} \lor 1) \left\| q\sigma^2 - w \right\|_2. \tag{4.22}
\]

Applying Hölder’s inequality to the expression (3.20) we obtain, (assuming \(k \geq 2 \), otherwise \(T_{k+1} = 0\))

\[
E \|T_{k+1}\| \leq C_{q,k,\sigma,\beta} \sum_{(i_0, \ldots, i_k) \in J_{k+1}} \|\delta_u\|_w^{i_0} \prod_{j=1}^k \| D_u \delta_u \|_{r_j}^{i_j},
\]

where \(1 = \frac{i_0}{r_0} + \sum_{j=1}^{k} \frac{i_j}{r_j} \). From property (b) in Lemma 3.5 there is at least one factor of the form \(\| D_u \delta_u \|_{s_j} \) with \(j \geq 2 \). We apply the estimate (8.13) to one of these factors, and the estimate (8.12) to all the remaining factors. We also use the estimate (8.11) to control \(\|\delta_u\|_w \). Notice that

\[
1 = \frac{i_0}{r_0} + \sum_{j=1}^{k} \frac{i_j}{r_j} > \frac{3i_0}{\beta} + \sum_{j=1}^{k} \frac{i_j(3j + 3)}{\beta} + \frac{1}{2},
\]

and, on the other hand, using properties (a) and (c) in Lemma 3.5

\[
\frac{3i_0}{\beta} + \sum_{j=1}^{k} \frac{i_j(3j + 3)}{\beta} + \frac{1}{2} \leq \frac{3k + 3}{\beta} + \frac{1}{2}.
\]

We can choose the \(r_j\)’s satisfying the above properties because \(\beta > 6k + 6\lfloor \frac{k}{2} \rfloor \), and we obtain

\[
E \|T_{k+1}\| \leq C_{q,k,\sigma,\beta} (M_{\beta}^{3k+3} \lor 1) \left\| q\sigma^2 - w \right\|_2. \tag{4.23}
\]

Combining (4.22) and (4.23) we complete the proof. ■

**Corollary 4.6.** Fix an integer \(m \geq 1 \). Let \(\{F_n\}_{n \in \mathbb{N}} \) be a sequence of random variables in the \(q\)th Wiener chaos with \(q \geq 2 \) and \(E[F_n^2] = \sigma_n^2 \). Assume \(\lim_{n \to \infty} \sigma_n = \sigma, 0 < \delta \leq \sigma_n^2 \leq K \) for all \(n \), \(\lim_{n \to \infty} E[F_n^3] = 3\sigma^4 \) and

\[
M := \sup_n \left( E[\|DF_n\|_{\beta}^{-\beta}] \right)^{\frac{1}{\beta}} < \infty \tag{4.24}
\]
for some $\beta > 6m + 6(\lfloor \frac{m}{2} \rfloor \vee 1)$. Let $\phi(x)$ be the density of $N(0, \sigma^2)$. Then, each $F_n$ admits a probability density function $f_{F_n} \in C^m(\mathbb{R})$ with derivatives given by (3.23) and for any $k = 1, \ldots, m,$

$$\sup_{x \in \mathbb{R}} \left| f_{F_n}^{(k)}(x) - \phi^{(k)}(x) \right| \leq C \left( \sqrt{E[F_n^4]} - 3\sigma_n^4 + |\sigma_n - \sigma| \right),$$

where the constant $C$ depends only on $q, m, \beta, M, \sigma, \delta$ and $K$.

**Proof.** Let $\phi_n(x)$ be the density of $N(0, \sigma_n^2)$. Then Theorem 4.4 implies that

$$\sup_{x \in \mathbb{R}} \left| f_{F_n}^{(k)}(x) - \phi_n^{(k)}(x) \right| \leq C_{q, m, \beta, M, \sigma} \sqrt{E[F_n^4]} - 3\sigma_n^4.$$

On the other hand, by the mean value theorem we can write

$$\left| \phi_n^{(k)}(x) - \phi^{(k)}(x) \right| \leq |\sigma_n - \sigma| \sup_{\gamma \in [\frac{\sigma}{2}, 2\sigma]} |\partial_\gamma \phi^{(k)}(x)| = \frac{1}{2} |\sigma_n - \sigma| \sup_{\gamma \in [\frac{\sigma}{2}, 2\sigma]} |\phi^{(k+2)}(x)|,$$

where $\phi_\gamma(x)$ is the density of the law $N(0, \gamma^2)$. Then, using the expression

$$\phi^{(k+2)}(x) = E[1_{N > z} H_{k+3}(\gamma^{-2}, \gamma^{-2} Z)],$$

where $Z \sim N(0, \gamma^2)$ and the explicit form of $H_{k+3}(\lambda, x)$, we obtain

$$\sup_{\gamma \in [\frac{\sigma}{2}, 2\sigma]} |\phi^{(k+2)}(x)| \leq C_{k, \sigma}.$$

Therefore,

$$\sup_{x \in \mathbb{R}} \left| \phi_n^{(k)}(x) - \phi^{(k)}(x) \right| \leq C_{k, \sigma} |\sigma_n - \sigma|.$$

This completes the proof.  

5. Random vectors in Wiener chaos

5.1. Main result

In this section, we study the multidimensional counterpart of Theorem 4.6. We begin with a density formula for a smooth random vector.

A random vector $F = (F_1, \ldots, F_d)$ in $\mathbb{D}^\infty$ is called *non-degenerate* if its Malliavin matrix $\gamma_F = (\langle DF_i, DF_j \rangle_\beta)_{1 \leq i, j \leq d}$ is invertible a.s. and $(\det \gamma_F)^{-1} \in \cap_{p \geq 1} L^p(\Omega)$. For any multi-index

$$\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \{1, 2, \ldots, d\}^k$$

of length $k \geq 1$, the symbol $\partial_\beta$ stands for the partial derivative $\frac{\partial^k}{\partial x_{\beta_1} \cdots \partial x_{\beta_k}}$. For $\beta$ of length 0 we make the convention that $\partial_\beta f = f$. We denote by $S(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing smooth functions, that is, the space of all infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\sup_{x \in \mathbb{R}^d} |x|^m |\partial_\beta f(x)| < \infty$ for any nonnegative integer $m$ and for all multi-index $\beta$. The following lemma (see Nualart [22, Proposition 2.1.5]) gives an explicit formula for the density of $F$. 

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Lemma 5.1. Let \( F = (F_1, \ldots, F_d) \) be a non-degenerate random vector. Then, \( F \) has a density \( f_F \in S(\mathbb{R}^d) \), and \( f_F \) and its partial derivative \( \partial_\beta f_F \), for any multi-index \( \beta = (\beta_1, \beta_2, \ldots, \beta_k) \) of length \( k \geq 0 \), are given by

\[
f_F(x) = E[1_{\{F > x\}}H_{(1,2,\ldots,d)}(F)],
\]

where \( 1_{\{F > x\}} = \prod_{i=1}^d 1_{\{F_i > x_i\}} \) and the elements \( H_\beta(F) \) are recursively defined by

\[
\begin{aligned}
H_\beta(F) &= 1, & \text{if } k = 0; \\
H_{(\beta_1,\beta_2,\ldots,\beta_k)}(F) &= \sum_{j=1}^d \delta(H_{(\beta_1,\beta_2,\ldots,\beta_{k-1})}(F)(\gamma_F^{-1})^{\beta_1 j}DF_j), & \text{if } k \geq 1.
\end{aligned}
\]

Fix \( d \) natural numbers \( 1 \leq q_1 \leq \cdots \leq q_d \). We will consider a random vector of multiple stochastic integrals: \( F = (F_1, \ldots, F_d) = (I_{q_1}(f_1), \ldots, I_{q_d}(f_d)) \), where \( f_i \in \mathcal{S}^{\otimes q_i} \). Denote

\[
V = (E[F_iF_j])_{1 \leq i,j \leq d}, \quad Q = \text{diag}(q_1, \ldots, q_d) \quad \text{(diagonal matrix of elements } q_1, \ldots, q_d).\]

Along this section, we denote by \( N = (N_1, \ldots, N_d) \) a standard normal vector given by \( N_i = I_1(h_i) \), where \( h_i \in \mathcal{F} \) are orthonormal. We denote by \( I \) the \( d \) dimensional identity matrix, and by \( |\cdot| \) the Hilbert-Schmidt norm of a matrix. The following is the main theorem of this section.

Theorem 5.2. Let \( F = (F_1, \ldots, F_d) = (I_{q_1}(f_1), \ldots, I_{q_d}(f_d)) \) be non-degenerate and let \( \phi \) be the density of \( N \). Then for any multi-index \( \beta \) of length \( k \geq 0 \), the density \( f_F \) of \( F \) satisfies

\[
\sup_{x \in \mathbb{R}^d} |\partial_\beta f_F(x) - \partial_\beta \phi(x)| \leq C \left( |V - I| + \sum_{1 \leq j \leq d} \sqrt{E[F_j^4]} - 3(E[F_j^2])^2 \right),
\]

where the constant \( C \) depends on \( d, V, Q, k \) and \( \|(\det \gamma_F)^{-1}\|_{(k+4)^{2k+3}} \).

Proof. Note that \( \partial_\beta \phi(x) = (-1)^k E[1_{\{F > x\}}H_{(1,2,\ldots,d,\beta_1,\beta_2,\ldots,\beta_k)}(N)] \). Then, in order to estimate the difference between \( \partial_\beta f_F \) and \( \partial_\beta \phi \), it suffices to estimate

\[
E[1_{\{F > x\}}H_\beta(F)] - E[1_{\{N > x\}}H_\beta(N)]
\]

for all multi-index \( \beta \) of length \( k \) for all \( k \geq d \).

Fix a multi-index \( \beta \) of length \( k \) for some \( k \geq d \). For the above standard normal random vector \( N \), we have \( \gamma_N = I \) and \( \delta(DN_i) = N_i \). We can deduce from the expression (5.3) that \( H_\beta(N) = g_\beta(N) \), where \( g_\beta(x) \) is a polynomial on \( \mathbb{R}^d \) (see Remark 5.4). Then,

\[
\begin{aligned}
&|E[1_{\{F > x\}}H_\beta(F)] - E[1_{\{N > x\}}H_\beta(N)]| \\
&\leq |E[1_{\{F > x\}}g_\beta(F)] - E[1_{\{N > x\}}g_\beta(N)]| + E[|H_\beta(F) - g_\beta(F)|] \\
&= A_1 + A_2.
\end{aligned}
\]

The term \( A_1 = |E[1_{\{F > x\}}g_\beta(F)] - 1_{\{N > x\}}g_\beta(N)| \) will be studied in Subsection 5.3 by using the multivariate Stein’s method. Proposition 5.10 will imply that \( A_1 \) is bounded by the right-hand side of (5.5).
Consider the term $A_2 = E[|H_\beta(F) - g_\beta(F)|]$. We introduce an auxiliary term $K_\beta(F)$, which is defined similar to $H_\beta(F)$ with $\gamma_F^{-1}$ replaced by $(VQ)^{-1}$. That is, for any multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ of length $k \geq 0$, we define

$$
\begin{align*}
K_\beta(F) &= 1 \quad \text{if } k = 0; \\
K_\beta(F) &= \delta\left(K_{(\beta_1, \beta_2, \ldots, \beta_{k-1})}(F) \left((VQ)^{-1}DF\right)_{\beta_k}\right) \quad \text{if } k \geq 1.
\end{align*}
$$

We have

$$
A_2 \leq E[|H_\beta(F) - K_\beta(F)|] + E[|K_\beta(F) - g_\beta(F)|] =: A_3 + A_4. \tag{5.8}
$$

Lemma 5.11 below shows that the term $A_3 = E[|H_\beta(F) - K_\beta(F)|]$ is bounded by the right-hand of (5.5).

It remains to estimate $A_4$. For this we need the following lemma which provides an explicit expression for the term $K_\beta(F)$. Before stating this lemma we need to introduce some notation. For any multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_{k})$, $k \geq 1$, denote by $\hat{\beta}_{i_1 \ldots i_m}$ the multi-index obtained from $\beta$ after taking away the elements $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_m}$. For example, $\hat{\beta}_{14} = (\beta_2, \beta_3, \beta_5, \ldots, \beta_k)$. For any $d$ dimensional vector $G$ we denote by $G_\beta$ the product $G_{\beta_1}G_{\beta_2}\cdots G_{\beta_k}$ and set $G_\beta = 1$ if the length of $\beta$ is 0. Denote by $(S^m_k, 0 \leq m \leq \lfloor \frac{k}{2} \rfloor)$ the following sets

$$
\begin{align*}
S^{-1}_k &= S^0_k = \emptyset \\
S^m_k &= \left\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m}) \in \{1, 2, \ldots, k\}^{2m} : \text{i_{2l-1} \neq i_{2l} for } 1 \leq l \leq m \text{ and } i_l \neq i_j \text{ if } l \neq j\right\}. \tag{5.9}
\end{align*}
$$

For each element $\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S^m_k$, we emphasize that the $m$ pairs of indices are unordered. In other words, for $m \geq 1$, the set $S^m_k$ can be viewed as the set of all partitions of $\{1, 2, \ldots, k\}$ into $m$ pairs and $k - 2m$ singletons.

Denote $M = V^{-1}\gamma_FV^{-1}Q^{-1}$ for $V$ and $Q$ given in (5.4) and denote $M_{ij}$ the $(i, j)$-th entry of $M$. Denote by $D_{\beta_i}$ the Malliavin derivative in the direction of $V^{-1}Q^{-1}DF_{\beta_i}$, that is,

$$
D_{\beta_i}G = \left(DG, (V^{-1}Q^{-1}DF)_{\beta_i}\right)_\delta \tag{5.10}
$$

for any random variable $G \in \mathbb{D}^{1,2}$.

**Lemma 5.3.** Let $F$ be a non-degenerate random vector. For a multi-index $\beta = (\beta_1, \ldots, \beta_k)$ of length $k \geq 0$, $K_\beta(F)$ defined by (5.7) can be computed as follows:

$$
K_\beta(F) = G_\beta(F) + T_\beta(F), \tag{5.11}
$$

where

$$
G_\beta(F) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \sum_{\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S^m_k} (V^{-1}F)^{\hat{\beta}_{k_{1 \ldots i_2}}_{1 \ldots 2m}} M_{\beta_{1 \ldots i_2}} \cdots M_{\beta_{2m-1} \beta_{2m}}, \tag{5.12}
$$

and $T_\beta(F)$ are defined recursively by

$$
T_\beta(F) = (V^{-1}F)^{\beta_k} T_{\hat{\beta}_k}(F) - D_{\beta_k}T_{\hat{\beta}_k}(F) - \sum_{\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S^m_{k-1}} (V^{-1}F)^{\hat{\beta}_{k_{1 \ldots i_2}}_{k_{1 \ldots 2m}}} D_{\beta_k} \left(M_{\beta_{1 \ldots i_2}} \cdots M_{\beta_{2m-1} \beta_{2m}}\right), \tag{5.13}
$$

for $k \geq 2$ and $T_1(F) = T_2(F) = 0$.  

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Proof. For simplicity, we write $K_\beta$, $G_\beta$ and $T_\beta$ for $K_\beta(F)$, $G_\beta(F)$ and $T_\beta(F)$, respectively. By using the fact that $\delta \left( ((VQ)^{-1}DF)_{\beta_l} \right) = (V^{-1}F)_{\beta_l}$ we obtain from (5.7) that

$$K_\beta = (V^{-1}F)_{\beta_k} K_{\beta_k} - D_{\beta_k} K_{\beta_k}. \tag{5.14}$$

If if $k = 1$, namely, $\beta = (\beta_1)$, then

$$K_\beta = (V^{-1}F)_{\beta_1} = G_\beta. \tag{5.15}$$

If $k = 2$, namely, $\beta = (\beta_1, \beta_2)$, then

$$K_\beta = (V^{-1}F)_{\beta} - M_{\beta_1\beta_2} = G_\beta. \tag{5.16}$$

Hence, the identity (5.11) is true for $k = 1, 2$. Assume now (5.11) is true for all multi-index of length less than or equal to $k$. Let $\beta = (\beta_1, \ldots, \beta_{k+1})$. Then, (5.14) implies

$$K_\beta = (V^{-1}F)_{\beta_{k+1}} \left( G_{\beta_{k+1}} + T_{\beta_{k+1}} \right) - D_{\beta_{k+1}} \left( G_{\beta_{k+1}} + T_{\beta_{k+1}} \right). \tag{5.17}$$

Noticing that

$$D_{\beta_{k+1}} (V^{-1}F)_{\beta_{k+1}} = \sum_{j \in \{1, \ldots, k\} \setminus \{i_1, \ldots, i_{2m}\}} (V^{-1}F)_{\beta_{k+1}} D_{\beta_{k+1}} \left( M_{\beta_{i_1} \beta_{i_2} \cdots \beta_{i_{2m-1}} \beta_{i_{2m}}} \right),$$

we have

$$D_{\beta_{k+1}} G_{\beta_{k+1}} = B_{\beta_{k+1}} \tag{5.18}$$

where we let

$$B_{\beta} = \sum_{m=0}^{[k/2]} (-1)^m \sum_{(i_1, i_2, \ldots, (i_{2m-1}, i_{2m}) \in S_m} (V^{-1}F)_{\beta_{j(k+1)i_1 \cdots i_{2m}}} M_{\beta_{i_1} \beta_{i_2} \cdots \beta_{i_{2m-1}} \beta_{i_{2m}}}.$$

Substituting the expression (5.16) for $D_{\beta_{k+1}} G_{\beta_{k+1}}$ into (5.15) and using (5.13) we obtain

$$K_\beta = (V^{-1}F)_{\beta_{k+1}} G_{\beta_{k+1}} - B_{\beta} + T_{\beta}.$$

To arrive at (5.11) it remains to verify

$$G_{\beta} = (V^{-1}F)_{\beta_k} G_{\beta_k} - B_{\beta}. \tag{5.17}$$

Introduce the following notation

$$C_{k+1}^m = \{((i_1, i_2), \ldots, (i_{2m-3}, i_{2m-2}), (j, k + 1)) : ((i_1, i_2), \ldots, (i_{2m-3}, i_{2m-2}) \in S_k^{m-1} \} \tag{5.18}$$
for \(1 \leq m \leq \lfloor \frac{k}{2} \rfloor\). Then, \(S_{k+1}^m\) can be decomposed as follows
\[
S_{k+1}^m = S_k^m \cup C_{k+1}^m.
\] (5.19)

We consider first the case when \(k\) is even. In this case, noticing that for any element in
\[\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S_{k+1}^m, \{1, \ldots, k\} \setminus \{i_1, \ldots, i_{2m}\} = \emptyset.\] For any collection of indices \(i_1, \ldots, i_{2m} \subset \{1, 2, \ldots, k\}\), we set
\[
\Phi_{i_1 \ldots i_{2m}} = (V^{-1}F)_{\beta_{i_1 \ldots i_{2m}}} M_{\beta_{i_1} \beta_{i_2}} \cdots M_{\beta_{i_{2m-1}} \beta_{i_{2m}}}.\]

Then, we have
\[
-B_\beta = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^{m+1} \sum_{\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S_k^m, j \in \{1, \ldots, k\} \setminus \{i_1, \ldots, i_{2m}\}} \Phi_{j(k+1)i_1 \ldots i_{2m}}
\]
\[
= \sum_{m=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \sum_{\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S_k^{m-1}, j \in \{1, \ldots, k\} \setminus \{i_1, \ldots, i_{2m}\}} \Phi_{j(k+1)i_1 \ldots i_{2m-2}}
\] (5.20)
\[
= \sum_{m=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \sum_{\{(i_1, i_2), \ldots, (i_{2m-3}, i_{2m-2}), j, k+1\}} \Phi_{j(k+1)i_1 \ldots i_{2m-2}}
\]

where in the last equality we used (5.18) and the fact that \(\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k+1}{2} \rfloor\) since \(k\) is even. Taking into account that \((V^{-1}F)_{\beta_{k+1}} (V^{-1}F)_{\beta_{(k+1)}i_1 \ldots i_{2m}} = (V^{-1}F)_{\beta_{i_1 \ldots i_{2m}}},\) we obtain from (5.12) that
\[
(V^{-1}F)_{\beta_{k+1}} G_{\beta_{k+1}} = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \sum_{\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S_k^m} (V^{-1}F)_{\beta_{i_1 \ldots i_{2m}}} M_{\beta_{i_1} \beta_{i_2}} \cdots M_{\beta_{i_{2m-1}} \beta_{i_{2m}}}.
\] (5.21)

Now combining (5.20) and (5.21) with (5.19) and using again \(\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k+1}{2} \rfloor\) we obtain
\[
(V^{-1}F)_{\beta_{k+1}} G_{\beta_{k+1}} - B_\beta = \sum_{m=0}^{\lfloor \frac{(k+1)}{2} \rfloor} (-1)^m \sum_{\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S_k^m} \Phi_{i_1 \ldots i_{2m}}
\]
\[
+ \sum_{m=1}^{\lfloor \frac{(k+1)}{2} \rfloor} (-1)^m \sum_{\{(i_1, i_2), \ldots, (i_{2m-3}, i_{2m-2}), j, k+1\}} \Phi_{i_1 \ldots i_{2m-2}}
\]
\[
= \sum_{m=0}^{\lfloor \frac{(k+1)}{2} \rfloor} (-1)^m \sum_{\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S_k^{m}} \Phi_{i_1 \ldots i_{2m}}
\]
\[
= G_\beta
\]
as desired. This verifies (5.17) for the case \(k\) is even. The case when \(k\) is odd can be verified similarly. Thus, we have proved (5.11) by induction.
Remark 5.4. For the random vector \(N \sim N(0, I)\), we have \(\gamma_N = VQ = I\), so \(H_\beta(N) = K_\beta(N)\). Then, it follows from Lemma 5.3 that \(H_\beta(N) = K_\beta(N) = g_\beta(N)\) with the function \(g_\beta(x) : \mathbb{R}^d \to \mathbb{R}\) given by

\[
g_\beta(x) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \sum_{\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S_k^m} x_{\tilde{\beta}_1 \cdots \tilde{\beta}_m} \delta_{\beta_1 \beta_2} \cdots \delta_{\beta_{2m-1} \beta_{2m}},
\]

where we used \(\delta_{ij}\) to denote the Kronecker symbol (without confusion with the divergence operator). Notice that

\[
g_\beta(x) = \prod_{i=1}^d H_{k_i}(x_i),
\]

where \(H_{k_i}\) is the \(i\)th Hermite polynomial and for each \(i = 1, \ldots, d\), \(k_i\) is the number of components of \(\beta\) equal to \(i\).

Let us return to the proof of Theorem 5.2 of estimating the term \(A_4\). From (5.11) we can write

\[
A_4 = E[\|K_\beta(F) - g_\beta(F)\| \leq E[\|G_\beta(F) - g_\beta(F)\| + E[\|T_\beta(F)\|].
\]

Observe from the expression (5.13) that \(T_\beta(F)\) is the sum of terms of the following form

\[
(V^{-1}F)_{\beta_1 \beta_2 \cdots \beta_s} D_{\beta_1} D_{\beta_2} \cdots D_{\beta_s} \left(\prod_{i} M_{\beta_i \beta_i}\right)
\]

for some \(\{i_1, \ldots, i_s, k_1, \ldots, k_t, j_1, l_1, \ldots, j_r, l_r\} \subset \{1, 2, \ldots, k\}\) and \(t \geq 1\). Applying Lemma 5.5 with (2.11) and (2.12) we obtain

\[
E[\|T_\beta(F)\|] \leq C \sum_{1 \leq l \leq d} \|\|DF_l\|^{2}_{\hat{\beta}_l} - q_l E[F_l^2]\|_{1/2}^{2}.
\]

In order to compare \(g_\beta(F)\) with \(G_\beta(F)\), from (5.22) we can write \(g_\beta(F)\) as

\[
g_\beta(F) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \sum_{\{(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})\} \in S_k^m} F_{\tilde{\beta}_1 \cdots \tilde{\beta}_m} \delta_{\beta_1 \beta_2} \cdots \delta_{\beta_{2m-1} \beta_{2m}}.
\]

Then, it follows from hypercontractivity property (2.11) that

\[
E[\|G_\beta(F) - g_\beta(F)\|] \leq C \left(\|V^{-1} - I\|_2 + \|M - I\|_2\right),
\]

where the constant \(C\) depends on \(k, V\) and \(Q\). From \(V^{-1} - I = V^{-1}(I - V)\) we have \(\|V^{-1} - I\| \leq C \|V - I\|\), where \(C\) depends on \(V\). We also have \(M - I = V^{-1}(\gamma_F - VQ) V^{-1}Q^{-1} + V^{-1} - I\). Then, Lemma 5.5 implies that

\[
\|M - I\|_2 \leq C \left(\|\gamma_F - VQ\|_2 + \|V^{-1} - I\|\right)
\]

\[
\leq C \left(\sum_{1 \leq l \leq d} \|\|DF_l\|^{2}_{\hat{\beta}_l} - q_l E[F_l^2]\|_{1/2}^{2} + \|V - I\|\right),
\]
where the constant $C$ depends on $k, V$ and $Q$. Therefore
\[
E \left[ |G_\beta(F) - g_\beta(F)| \right] \leq C \left( |V - I| + \sum_{1 \leq i \leq d} \left\| DF_i \right\|^2_\delta - q_t EF^2_t \right)^{\frac{1}{2}}.
\] (5.26)

Combining it with (5.25) we obtain from (5.23) that
\[
A_4 \leq C \left( |V - I| + \sum_{1 \leq i \leq d} \left\| DF_i \right\|^2_\delta - q_t EF^2_t \right)
\]
where the constant $C$ depends on $d, V, Q$. This completes the estimation of the term $A_4$. □

5.2. Sobolev norms of $\gamma_F^{-1}$

In this subsection we estimate the Sobolev norms of $\gamma_F^{-1}$, the inverse of the Malliavin matrix $\gamma_F$ for a random variable $F$ of multiple stochastic integrals. We begin with the following estimate on the variance and Sobolev norms of $(\gamma_F)_{ij} = \langle DF_i, DF_j \rangle_\delta$, $1 \leq i, j \leq d$, following the approach of [13, 16, 19].

**Lemma 5.5.** Let $F = I_p(f)$ and $G = I_q(g)$ with $f \in \mathcal{S}^{\otimes p}$ and $g \in \mathcal{S}^{\otimes q}$ for $p, q \geq 1$. Then for all $k \geq 0$ there exists a constant $C_{p,q,k}$ such that
\[
\left\| D^k (\langle DF, DG \rangle_\delta - \sqrt{pq}E[FG]) \right\|_2 \leq C_{p,q,k}(\|F\|^2 + \|G\|^2)(\|DF\|_\delta^2 - pE[F^2] + \|DG\|_\delta^2 - pE[G^2])^{\frac{1}{2}}.
\] (5.27)

**Proof.** Without lost of generality, we assume $p \leq q$. Applying (2.4) with the fact that $DI_p(f) = pI_{p-1}(f)$ we have
\[
\langle DF, DG \rangle_\delta = pq \langle I_{p-1}(f), I_{q-1}(g) \rangle_\delta
\] (5.28)
\[
= pq \sum_{r=0}^{p-1} r! \binom{p-1}{r} \binom{q-1}{r} I_{p+q-2-2r} (f \widetilde{\otimes} r+1 g)
\]
\[
= pq \sum_{r=1}^{p} (r-1)! \binom{p-1}{r-1} \binom{q-1}{r-1} I_{p+q-2r} (f \widetilde{\otimes} r g).
\]

Note that $E[FG] = 0$ if $p < q$ and $E[FG] = \langle f, g \rangle_\delta \otimes p = f \widetilde{\otimes} p g$ if $p = q$. Then
\[
\langle DF, DG \rangle_\delta - \sqrt{pq}E[FG] = pq \sum_{r=1}^{p} (1 - \delta_{qr}) (r-1)! \binom{p-1}{r-1} \binom{q-1}{r-1} I_{p+q-2r} (f \widetilde{\otimes} r g),
\]
where $\delta_{qr}$ is again the Kronecker symbol. It follows that
\[
E \left[ (\langle DF, DG \rangle_\delta - \sqrt{pq}E[FG])^2 \right] = p^2 q^2 \sum_{r=1}^{p} (1 - \delta_{qr}) (r-1)!^2 \binom{p-1}{r-1}^2 \binom{q-1}{r-1}^2 (p + q - 2r)! \left\| f \widetilde{\otimes} r g \right\|_{\delta^{\otimes(p+q-2r)}}^2.
\] (5.29)

Note that if $r < p \leq q$, then (see also [16, (6.2.7)])
\[
\left\| f \widetilde{\otimes} r g \right\|_{\delta^{\otimes(p+q-2r)}}^2 \leq \left\| f \otimes r g \right\|_{\delta^{\otimes(p+q-2r)}}^2 = \langle f \otimes p-r f, g \otimes q-r g \rangle_{\delta^{\otimes 2r}}.
\]
where the constant \( C \) depends on \( q_1, \ldots, q_d, d, p \) and \( V \). Moreover, for any integer \( k \geq 1 \) and any real number \( p > 1 \)

\[
\| \gamma_F^{-1} \|_{k,p} \leq C \| (\det \gamma_F)^{-1} \|_{(k+2)p} \sum_{i=1}^{d} \| \| D F_i \|_{\mathcal{B}_p}^2 - q_i E \| F_i^2 \|_2, \tag{5.34}
\]

where the constant \( C \) depends on \( q_1, \ldots, q_d, d, p, k \) and \( V \).
Proof. Let \( \gamma_F^* \) be the adjugate matrix of \( \gamma_F \). Note that H"older inequality and \((2.12)\) imply
\[
\| (DF_i, DF_j)_S \|_p \leq \|DF_i\|_{2p} \|DF_j\|_{2p} \leq C_{V,p}
\]
for all \( 1 \leq i, j \leq d, p \geq 1 \). Applying Holder’s inequality we obtain that the \( p \) norm of \( \gamma_F^* \) is also bounded by a constant. A further application of Holder’s inequality to \( \gamma_F^{-1} = (\det \gamma_F)^{-1} \gamma_F^* \) yields
\[
\| \gamma_F^{-1} \|_p \leq \| (\det \gamma_F)^{-1} \|_{2p} \| \gamma_F^* \|_{2p} \leq C_{V,p} \| (\det \gamma_F)^{-1} \|_{2p},
\]
which implies \((5.33)\).

Since \( F \) is non-degenerate, then (see \cite{22, Lemma 2.1.6}) \( (\gamma_F^{-1})_{ij} \) belongs to \( D_\infty \) for all \( i, j \) and
\[
D (\gamma_F^{-1})_{ij} = - \sum_{m,n=1}^d (\gamma_F^{-1})_{im} (\gamma_F^{-1})_{nj} D (\gamma_F)_{mn}.
\]

Then, applying Hölder’s inequality we obtain
\[
\| D (\gamma_F^{-1}) \|_p \leq \| \gamma_F^{-1} \|_{3p}^2 \| D \gamma_F \|_{3p}.
\]
\[
\leq C_{V,p} \| (\det \gamma_F)^{-1} \|_{3p}^2 \sum_{i=1}^d \|DF_i\|_{S}^2 - q_i E \left[ F_i^2 \right]_2,
\]
where in the second inequality we used \((5.33)\) and
\[
\|D \gamma_F\|_{3p} \leq C_{V,p} \|D \gamma_F\|_2 \leq C_{V,p} \sum_{i=1}^d \|DF_i\|_{S}^2 - q_i E \left[ F_i^2 \right]_2
\]
for all \( p \geq 1 \), which follows from \((2.12)\) and \((5.27)\). This implies \((5.34)\) with \( k = 1 \). For higher order derivatives, \((5.34)\) follows from repeating the use of \((5.36), (2.12)\) and \((5.27)\). ■

The following lemma estimates the difference \( \gamma_F^{-1} - V^{-1}Q^{-1} \).

**Lemma 5.7.** Let \( F = (F_1, \ldots, F_d) = (I_{q_1}(f_1), \ldots, I_{q_d}(f_d)) \) be a non-degenerate random vector with \( 1 \leq q_1 \leq \cdots \leq q_d \) and \( f_i \in \mathcal{F}_{\mathbb{R}^{q_i}} \). Let \( \gamma_F \) be the Malliavin matrix of \( F \). Recall the notation of \( V \) and \( Q \) in \((5.4)\). Then, for every integer \( k \geq 1 \) and any real number \( p > 1 \) we have
\[
\| \gamma_F^{-1} - V^{-1}Q^{-1} \|_{k,p} \leq C \| (\det \gamma_F)^{-1} \|_{(k+2)2p} \sum_{1 \leq l \leq d} \|DF_l\|_{S}^2 - q_l E \left[ F_l^2 \right]_2^{\frac{1}{2}},
\]
where the constant \( C \) depends on \( d, V, Q, p \) and \( k \).

**Proof.** In view of Lemma 5.6, we only need to consider the case when \( k = 0 \) because \( V \) and \( Q \) are deterministic matrices. Note that
\[
\gamma_F^{-1} - V^{-1}Q^{-1} = \gamma_F^{-1} (VQ - \gamma_F)V^{-1}Q^{-1}.
\]

Then, applying Holder’s inequality we have
\[
\| \gamma_F^{-1} - V^{-1}Q^{-1} \|_p \leq C_{V,Q} \| \gamma_F^{-1} \|_{2p} \|VQ - \gamma_F\|_{2p}.
\]

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Note that \((2.12)\) and \((5.27)\) with \(k = 0\) imply
\[
\|VQ - \gamma_F\|_{2p} \leq C_{V,Q,p} \|VQ - \gamma_F\|_2 \leq C_{V,Q,p} \sum_{i=1}^{d} \left\|\|DF_i\|_{\delta}^2 - q_i E\left[F_i^2\right]\right\|_2^{\frac{1}{2}}.
\]

Then, applying \((5.35)\) we obtain
\[
\|\gamma_F^{-1} - V^{-1}Q^{-1}\|_p \leq C_{d,V,Q,p} \sum_{i=1}^{d} \left\|\|DF_i\|_{\delta}^2 - q_i E\left[F_i^2\right]\right\|_2^{\frac{1}{2}} \tag{5.38}
\]
as desired. \(\blacksquare\)

\subsection{5.3 Technical estimates}

In this subsection, we study the terms \(A_1 = |E[h(F)] - E[h(N)]|\) in Equation \((5.6)\) and \(A_3 = E||H_\beta(F) - K_\beta(F)||\) in \((5.8)\). For \(A_1\), we shall use the multivariate Stein’s method to give an estimate for a large class of non-smooth test functions \(h\).

\textbf{Lemma 5.8.} Let \(h : \mathbb{R}^d \to \mathbb{R}\) be an almost everywhere continuous function such that \(|h(x)| \leq c (|x|^m + 1)\) for some \(m, c > 0\). Let \(F = (F_1, \ldots, F_d)\) be non-degenerate with \(E[F_i] = 0, 1 \leq i \leq d\) and denote \(N \sim N(0, I)\). Then there exists a constant \(C_{m,c}\) depending on \(m\) and \(c\) such that
\[
|E[h(F)] - E[h(N)]| \leq C_{m,c} (\|F\|_{2m}^m + 1) \sum_{i,j,k=1}^{d} \left\|\delta \left(A_{ij} (\gamma_F^{-1})_{jk} DF_k\right)\right\|_2, \tag{5.39}
\]
where \(\gamma_F^{-1}\) is the inverse of the Malliavin matrix of \(F\) and
\[
A_{ij} = \delta_{ij} - \langle DF_j, -DL^{-1}F_i\rangle_{\delta}. \tag{5.40}
\]

\textit{Proof.} For \(\varepsilon > 0\), let
\[
h_\varepsilon(x) = \left(1_{\{|x| < \frac{1}{\varepsilon}\}}\right)h \ast \rho_\varepsilon(x) = \int_{\mathbb{R}^d} 1_{|y| < \frac{1}{\varepsilon}} h(y) \rho_\varepsilon(x-y) dy.
\]
where \(\rho_\varepsilon\) is the standard mollifier. That is, \(\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right)\), where \(\rho(x) = C1_{\{|x| < 1\}} \exp\left(-\frac{1}{|x|^2 - 1}\right)\) and the constant \(C\) is such that \(\int_{\mathbb{R}^d} \rho(x) dx = 1\). Then \(h_\varepsilon\) is Lipschitz continuous. Hence, the solution \(f_\varepsilon\) to the following \textit{Stein’s equation:}
\[
\Delta f_\varepsilon(x) - \langle x, \nabla f_\varepsilon(x)\rangle_{\mathbb{R}^d} = h_\varepsilon(x) - E[h_\varepsilon(N)] \tag{5.41}
\]
extists and its derivative has the following expression \cite[Page 82]
\[
\partial_i f_\varepsilon(x) = \frac{\partial}{\partial x_i} \int_0^1 \frac{1}{2t} E[h_\varepsilon(\sqrt{t}x + \sqrt{1-t}N)] dt
\]
\[
= \int_0^1 E[h_\varepsilon(\sqrt{t}x + \sqrt{1-t}N)] \frac{1}{2\sqrt{t}\sqrt{1-t}} dt. \tag{5.42}
\]
It follows directly from the polynomial growth of \(h\) that
\[
|h_\varepsilon(x)| \leq C_1 |x|^m + C_2 \tag{5.43}
\]

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for all $\varepsilon < 1$, where $C_1, C_2 > 0$ are two constants depending on $c$ and $m$. Then, from (5.41) we can write
\[ |\partial_i f_\varepsilon(x)| \leq C_1 |x|^m + C_2, \]
with two possibly different constants $C_1$, and $C_2$. Hence,
\[ \|\partial_i f_\varepsilon(F)\|_2 \leq C_1 \|F\|_2^m + C_2. \] (5.44)

Meanwhile, note that for $1 \leq i \leq d$,
\[ E[F_i \partial_i f_\varepsilon(F)] = E[LL^{-1} F_i \partial_i f_\varepsilon(F)] = E[-DL^{-1} F_i, D\partial_i f_\varepsilon(F)] \]
\[ = \sum_{j=1}^d E[-DL^{-1} F_i, \partial_{ij} f_\varepsilon(F)DF_j]. \]

Then, replacing $x$ by $F$ and taking expectation in Equation (5.41) yields
\[ |E[h_\varepsilon(F)] - E[h_\varepsilon(N)]| = \sum_{i,j=1}^d E \left[ \partial_{ij}^2 f_\varepsilon(F)A_{ij} \right]. \] (5.45)

Notice that
\[ \langle DF_i, D\partial_i f_\varepsilon(F)\rangle_{\delta} = \left\langle DF_i, \sum_{j=1}^d \partial_{ij}^2 f_\varepsilon(F)DF_j \right\rangle_{\delta} = \sum_{j=1}^d \partial_{ij}^2 f_\varepsilon(F) \langle DF_i, DF_j\rangle_{\delta} \]
for all $1 \leq i, k \leq d$, which implies
\[ \partial_{ij} f_\varepsilon(F) = \sum_{k=1}^d (\gamma_F^{-1})_{jk} \langle DF_k, D\partial_i f_\varepsilon(F)\rangle_{\delta}, \]
and hence
\[ \sum_{i,j=1}^d E \left[ \partial_{ij}^2 f_\varepsilon(F)A_{ij} \right] = \sum_{i,j=1}^d E \left[ A_{ij} \left( \sum_{k=1}^d (\gamma_F^{-1})_{jk} DF_k, D\partial_i f_\varepsilon(F) \right) \right] \]
\[ = \sum_{i,j=1}^d E \left[ \partial_i f_\varepsilon(F) \delta \left( A_{ij} \sum_{k=1}^d (\gamma_F^{-1})_{jk} DF_k \right) \right]. \]

Substituting this expression in (5.45) and using (5.44) we obtain
\[ |E[h_\varepsilon(F)] - E[h_\varepsilon(N)]| \leq \sum_{i,j,k=1}^d E \left[ \partial_i f_\varepsilon(F) \delta \left( A_{ij} (\gamma_F^{-1})_{jk} DF_k \right) \right] \]
\[ \leq \sum_{i,j,k=1}^d \|\partial_i f_\varepsilon(F)\|_2 \left\| \delta \left( A_{ij} (\gamma_F^{-1})_{jk} DF_k \right) \right\|_2 \]
\[ \leq (C_1 \|F\|_2^m + C_2) \sum_{i,j,k=1}^d \left\| \delta \left( A_{ij} (\gamma_F^{-1})_{jk} DF_k \right) \right\|_2. \]
Then, we can conclude the proof by observing that
\[
\lim_{\varepsilon \to 0} |E[h_\varepsilon(F)] - E[h_\varepsilon(N)]| = |E[h(F)] - E[h(N)]|,
\]
which follows from (5.43) and the fact that \(h_\varepsilon \to h\) almost everywhere. \qed

The next lemma gives an estimate for \(\|\delta A_{ij} (\gamma_F^{-1})_{jk} DF_k\|_2\) when \(F\) is a vector of multiple stochastic integrals.

**Lemma 5.9.** Let \(F = (F_1, \ldots, F_d) = (I_{q_1}(f_1), \ldots, I_{q_d}(f_d))\), where \(f_i \in \mathcal{M}^\infty\), be non-degenerate and denote \(N \sim N(0, I)\). Recall the notation of \(V\) and \(Q\) in (5.4) and \(A_{ij}\) in (5.40). Then, for all \(1 \leq i, j, k \leq d\) we have
\[
\|\delta A_{ij} (\gamma_F^{-1})_{jk} DF_k\|_2 \leq C \|\det \gamma_F^{-1}\|_{12}^3 \times \left(\|V - I\| + \sum_{i=1}^d \|DF_i\|_{ij}^2 - q_i E[F_i^2]\right),
\]
where the constant \(C\) depends on \(d, V, Q\).

**Proof.** Applying Meyer’s inequality (2.9) we have
\[
\|\delta A_{ij} (\gamma_F^{-1})_{jk} DF_k\|_2 \leq \|A_{ij} (\gamma_F^{-1})_{jk} DF_k\|_2 + \|D A_{ij} (\gamma_F^{-1})_{jk} DF_k\|_2.
\]
Applying Holder’s inequality and (2.12) we have
\[
\|A_{ij} (\gamma_F^{-1})_{jk} DF_k\|_2 \leq \|A_{ij}\|_2 \|\gamma_F^{-1}\|_4 \|DF_k\|_4 \leq C_{d,V,Q} \|A_{ij}\|_2 \|\gamma_F^{-1}\|_4.
\]
Similarly, Holder’s inequality and (2.12) imply
\[
\|D A_{ij} (\gamma_F^{-1})_{jk} DF_k\|_2 \leq C_{d,V,Q} \|DA_{ij}\|_2 \|\gamma_F^{-1}\|_4 + \|A_{ij}\|_2 \|D (\gamma_F^{-1})_{jk}\|_4 + \|A_{ij}\|_2 \|\gamma_F^{-1}\|_4.
\]
Combining the above inequalities we obtain
\[
\|\delta A_{ij} (\gamma_F^{-1})_{jk} DF_k\|_2 \leq C_{d,V,Q} \|A_{ij}\|_2 \|\gamma_F^{-1}\|_{14}.
\]
Note that
\[
A_{ij} = \delta_{ij} - \langle DF_j, -DL^{-1}F_i\rangle_{\eta} = \delta_{ij} - V_{ij} + V_{ij} - \frac{1}{q_i} \langle DF_j, -DF_i\rangle_{\eta}.
\]
Then, it follows from Lemma 5.5 that
\[
\|A_{ij}\|_{12} \leq C_{d,V,Q} \left(\|V - I\| + \sum_{i=1}^d \|DF_i\|_{ij}^2 - q_i E[F_i^2]\right).\]
Then, the lemma follows by taking into account of (5.34) with \(k = 1\). \qed

As a consequence of the above two lemmas, we have the following result.
Proposition 5.10. Let $h : \mathbb{R}^d \rightarrow \mathbb{R}$ be an almost everywhere continuous function such that $|h(x)| \leq c (|x|^m + 1)$ for some $m, c > 0$. Let $F = (F_1, \ldots, F_d) = (I_{q_1}(f_1), \ldots, I_{q_d}(f_d))$, where $f_i \in \mathcal{S}^{\infty,q_i}$, be non-degenerate and denote $N \sim N(0, I)$. Recall the notation of $V$ and $Q$ in (5.4). Then
\[
|E[h(F)] - E[h(N)]| \leq C \left\| (\det \gamma_F)^{-1} \right\|_{12}^3 \times \left( |V - I| + \sum_{i=1}^d \left\| D F_i \right\|_2^2 - q_i E \left[ F_i^2 \right] \right)^{\frac{1}{2}},
\]

where the constant $C$ depends on $d, V, Q, m, c$.

In the following, we estimate the term $A_3 = E \left[ \| H_\beta(F) - K_\beta(F) \| \right]$ in (5.8), where $H_\beta(F)$ and $K_\beta(F)$ are defined in (5.3) and (5.7), respectively.

Lemma 5.11. Let $F = (F_1, \ldots, F_d) = (I_{q_1}(f_1), \ldots, I_{q_d}(f_d))$ be non-degenerate. Let $\beta = (\beta_1, \ldots, \beta_k)$ be a multi-index of length $k \geq 1$. Let $H_\beta(F)$ and $K_\beta(F)$ be defined by (5.3) and (5.7), respectively. Then there exists a constant $C$ depending on $d, V, Q, k$ such that
\[
E \left[ \| H_\beta(F) - K_\beta(F) \| \right] \leq C \left\| (\det \gamma_F)^{-1} \right\|_{(k+2)(k+4)2^{k+3}} \times \sum_{i=1}^d \left\| D F_i \right\|_2^2 - q_i E \left[ F_i^2 \right] \right)^{\frac{1}{2}}.
\]

Proof. To simplify notation, we write $H_\beta$ and $K_\beta$ for $H_\beta(F)$ and $K_\beta(F)$, respectively. From (5.3) and (5.7) we see that
\[
H_\beta - K_\beta = \delta \left( H_{\beta_k} \left( \gamma_F^{-1} DF \right) \beta_k - K_{\beta_k} \left( (VQ)^{-1} DF \right) \beta_k \right),
\]

where $\hat{\beta}_k = (\beta_1, \ldots, \beta_{k-1})$. For any $s \geq 0, p > 1$, using Meyer's inequality (2.9) we obtain
\[
\| H_\beta - K_\beta \|_{s,p} \leq C_{s,p} \left\| H_{\beta_k} \left( \gamma_F^{-1} DF \right) \beta_k - K_{\beta_k} \left( (VQ)^{-1} DF \right) \beta_k \right\|_{s+1,p}.
\]

Then, Hölder's inequality yields
\[
\| H_\beta - K_\beta \|_{s,p} \leq \left\| H_{\beta_k} - K_{\beta_k} \right\|_{s+1,2p} \left\| (VQ)^{-1} DF \right\|_{s+1,2p} + \left\| H_{\beta_k} \right\|_{s+1,2p} \left\| (\gamma_F^{-1} - (VQ)^{-1}) DF \right\|_{s+1,2p}.
\]

Note that (2.12) implies $\left\| (VQ)^{-1} DF \right\|_{s+1,2p} \leq C_{d,V,Q,s,p}$. Also note that (2.12), Hölder’s inequality and (5.37) indicate
\[
\left\| (\gamma_F^{-1} - (VQ)^{-1}) DF \right\|_{s+1,2p} \leq C_{d,V,Q,s,p} \Delta \left\| (\det \gamma_F)^{-1} \right\|_{(s+3)8p}.
\]
where we denote
$$
\Delta := \sum_{1 \leq l \leq d} \|DF_l\|_2^2 - q_l E\left[F_l^2\right] \|\frac{1}{2}
$$
to simplify notation. Thus we obtain
$$
\|H_\beta - K_\beta\|_{s,p} \leq C_{d,V,Q,s,p} \left\| H_{\beta_k} - K_{\beta_k} \right\|_{s+1,2p} + C_{d,V,Q,s,p} \Delta \left\| H_{\beta_k} \right\|_{s+1,2p} \left(\det \gamma_F\right)^{-1} \|s+2\|_{(s+3)8p}.
$$

Similarly, from Meyer’s inequality (2.9), Hölder’s inequality and (2.12) we obtain by iteration
$$
\|H_\beta\|_{s,p} \leq C_{s,p} \left\| H_{\beta_k} \left(\gamma_F^{-1} DF\right)_{\beta_k} \right\|_{s+1,p} \leq C_{d,V,Q,s,p} \left\| H_{\beta_k} \right\|_{s+1,2p} \left(\det \gamma_F\right)^{-1} \|s+2\|_{(s+3)8p} \ldots \leq C_{d,V,Q,s,p,k} \left(\det \gamma_F\right)^{-1} \|k(s+k)\|_{(s+k+1)2^{k+2}p}.
$$

Applying (5.50) into (5.49) and by iteration we can obtain
$$
\|H_\beta - K_\beta\|_{s,p} \leq C_{d,V,Q,s,p,k} \left(\det \gamma_F\right)^{-1} \|k(2s+k+2)\|_{(2s+k+4)2^{k+2}p} \Delta.
$$

Now (5.48) follows by taking $s = 0, p = 2$ in the above inequality.

\section{Uniform estimates for densities of general random variables}

In this section, we study the uniform convergence of densities for general random variables. We first characterize the convergence of densities with quantitative bounds for a sequence of centered random variables, using the density formula (3.10). In the second part of this section, a short proof of the uniform convergence of densities (without quantitative bounds) is given, using a compactness argument based on the assumption that the sequence converges in law.

\subsection{Convergence of densities with quantitative bounds}

In this subsection, we estimate the rate of uniform convergence for densities of general random variables. The idea is to use the density formula (3.10).

We use the following notations throughout this section.
$$
\bar{w} = \langle DF, -DL^{-1}F\rangle_S, \quad \vec{w} = -\bar{w}^{-1}DL^{-1}F.
$$

The following technical lemma is useful.

\textbf{Lemma 6.1.} Let $F \in \mathbb{D}^{2,s}$ with $s \geq 4$ such that $E[F] = 0$ and $E[F^2] = \sigma^2$. Let $m$ be the largest even integer less than or equal to $\frac{s}{2}$. Then there is a positive constant $C_m$ such that for any $t \leq m$,
$$
\|\bar{w} - \sigma^2\|_t \leq \|\bar{w} - \sigma^2\|_m \leq C_m \|D\bar{w}\|_m \leq C_m \|D\bar{w}\|_{s/2}.
$$

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That in Malliavin calculus it follows Theorem 6.2. Let variable \( f \) Proof. It suffices to show the above second inequality. From the integration by parts formula in Malliavin calculus it follows

\[
\sigma^2 = E[F^2] = E \left[ \langle DF, - DL^{-1} F \rangle_{\mathcal{B}} \right] = E[w].
\]

Note that from (3.9) and (3.13) we have \( w \in \mathbb{D}^{1,\frac{3}{2}} \). Then the lemma follows from the following infinite-dimensional Poincaré inequality [16, Lemma 5.3.8]:

\[
E[(G - E[G])^m] \leq (m - 1)^{m/2} E \left[ \|DG\|^m_{\mathcal{B}} \right],
\]

for any even integer \( m \) and \( G \in \mathbb{D}^{1,m} \).

The next theorem gives a bound for the uniform distance between the density of a random variable \( F \) and the normal density.

**Theorem 6.2.** Let \( F \in \mathbb{D}^{2,s} \) with \( s \geq 8 \) such that \( E[F] = 0, E[F^2] = \sigma^2 \). Suppose \( M := E[|w|^r] < \infty \), where \( w = \langle DF, - DL^{-1} F \rangle_{\mathcal{B}} \) and \( r > 2 \). Assume \( \frac{2}{r} + \frac{4}{s} = 1 \). Then \( F \) admits a density \( f_F(x) \) and there is a constant \( C_{r,s,\sigma,M} \) depending on \( r, s, \sigma \) and \( M \) such that

\[
\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq C_{r,s,\sigma,M} \|F\|^2_{1,s} \|D^2 F\|_{op}^2_{0,s},
\]

(6.2)

where \( \phi(x) \) is the density of \( N \sim N(0, \sigma^2) \) and \( D^2 F\|_{op} \) indicates the operator norm of \( D^2 F \) introduced in (3.9).

**Proof.** It follows from Proposition 3.3 that \( F \) admits a density given by \( f_F(x) = E[1_{\{F > x\}} \delta(w)] \). Then

\[
\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| = \sup_{x \in \mathbb{R}} \left| E\left[1_{\{F > x\}} \delta(w)\right] - \sigma^{-2} E[1_{\{N > x\}} N]\right|.
\]

(6.3)

Note that, from (2.7)

\[
\delta(w) = \delta(-DL^{-1} F w^{-1}) = F w^{-1} + \langle D w^{-1}, DL^{-1} F \rangle_{\mathcal{B}}.
\]

Then

\[
\left| E\left[\sigma^2 1_{\{F > x\}} \delta(w)\right] - E[1_{\{N > x\}} N]\right| \\
\leq E\left[|F w^{-1}(\sigma^2 - w)|\right] + \sigma^2 E\left[|\langle D w^{-1}, DL^{-1} F \rangle_{\mathcal{B}}\right] \\
+ E\left[|F 1_{\{F > x\}} - N 1_{\{N > x\}}|\right].
\]

(6.4)

Note that for \( t = \left(\frac{1}{r} + \frac{3}{s}\right)^{-1} \), we have \( \frac{1}{t} - t \geq 2 \), so there exists an even integer \( m \in [t, \frac{3}{2}] \). Also, we have \( \frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1 \). Then, we can apply Hölder’s inequality and (6.1) to obtain

\[
E\left[|F w^{-1}(w - \sigma^2)|\right] \leq \|F\|_s \|w^{-1}\|_r \|w - \sigma^2\|_t \\
\leq C_{r,s} \|F\|_s \|w^{-1}\|_r \|D w\|_{s/2}.
\]

(6.5)

Meanwhile, applying Hölder’s inequality and (3.7) we have

\[
E\left[|w^{-2} \langle D \bar{w}, - DL^{-1} F \rangle_{\mathcal{B}}|\right] \leq \|w^{-1}\|_r^2 \|D \bar{w}\|_{\frac{r}{2}} \|DL^{-1} F\|_{\frac{s}{2}} \\
\leq \|\bar{w}^{-1}\|_r^2 \|D \bar{w}\|_{\frac{r}{2}} \|DF\|_s.
\]

(6.6)
Also, applying Lemma 2.2 for \( h(y) = y 1_{\{y > x\}} \) and (6.1) we have

\[
|E[1_{F > x} - N 1_{N > x}]| \leq C_{\sigma} \|\sigma^2 - \bar{\omega}\|_2 \leq C_{\sigma} \|D\bar{\omega}\|_{s/2}.
\] (6.7)

Applying the estimates (6.5)-(6.7) to (6.4) we have

\[
|E[\sigma^2 1_{F > x} (\bar{\omega})] - E[1_{N > x} N]| \leq C_{r,s,\sigma,M} \|F\|_{1,s} \|D\bar{\omega}\|_{s/2}.
\] (6.8)

Combining (6.3), (6.8) and (3.13) one gets

\[
\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq C_{r,s,\sigma,M} \|F\|_{2}^2 \left\|\|D^2 F\|_{op}\right\|_s.
\]

This completes the proof. \( \blacksquare \)

**Corollary 6.3.** Let \( \{F_n\}_{n \in \mathbb{N}} \subset \mathbb{D}^{2,s} \) with \( s \geq 8 \) such that \( E[F_n] = 0 \) and \( \lim_{n \to \infty} E[F_n^2] = \sigma^2 \).

Assume \( E[F_n^2] \geq \delta > 0 \) for all \( n \). For \( r > 2 \) such that \( \frac{2}{r} + \frac{s}{r} = 1 \), assume

(i) \( M_1 = \sup_n \|F_n\|_{1,s} < \infty \).

(ii) \( M_2 = \sup_n E \left| (DF_n, -DL^{-1} F_n)_\beta \right|^{-r} < \infty \).

(iii) \( E\|D^2 F_n\|_{op}^s \to 0 \) as \( n \to \infty \).

Then each \( F_n \) admits a density \( f_{F_n}(x) \) and,

\[
\sup_{x \in \mathbb{R}} |f_{F_n}(x) - \phi(x)| \leq C \left( \left\|\|D^2 F_n\|_{op}\right\|_s + |E[F_n^2] - \sigma^2| \right),
\] (6.9)

where the constant \( C \) depends on \( \sigma, M_1, M_2 \) and \( \delta \). Moreover, if \( M_3 = \sup_n \|F_n\|_{2s} < \infty \), then for any \( k \geq 1 \) and \( \alpha \in (\frac{1}{2}, k) \),

\[
\|f_{F_n} - \phi\|_{L^k(\mathbb{R})} \leq C \left( \left\|\|D^2 F_n\|_{op}\right\|_s + |E[F_n^2] - \sigma^2| \right)^{\frac{k-\alpha}{k}},
\]

where the constant \( C \) depends on \( \sigma, M_1, M_2, M_3, \alpha \) and \( \delta \).

**Remark 6.4.** By the “random contraction inequality” (3.9), a sufficient condition for (iii) is \( E\|D^2 F_n \otimes_1 D^2 F_n\|_{\delta \otimes 2}^s \to 0 \) or \( E\|D^2 F_n\|_{\delta \otimes 2}^s \to 0 \).

**Proof of Corollary 6.3.** It follows from Theorem 6.2 and Proposition 3.3 with an argument similar to Corollary 4.3. \( \blacksquare \)

6.2. **Compactness argument**

In general, convergence in law does not imply convergence of the corresponding densities even if they exist. The following theorem specifies some additional conditions which ensure that convergence in law will imply convergence of densities.
Theorem 6.5. Let \( \{F_n\}_{n \in \mathbb{N}} \) be a sequence of random variables in \( \mathbb{D}^{2,s} \) satisfying any one of the following two conditions:

\[
\sup_n \|F_n\|_{2,s} + \sup_n \|F_n\|_{2p} + \sup_n \|DF_n\|_{r}^{-2} < \infty \tag{6.10}
\]

for some \( p, r, s > 1 \) satisfying \( \frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1 \), or

\[
\sup_n \|F_n\|_{2,s} + \sup_n \|\langle DF_n, -DL^{-1}F_n \rangle_{\mathcal{H}}\|_{r}^{-1} < \infty \tag{6.11}
\]

for some \( r, s > 1 \) satisfying \( \frac{2}{r} + \frac{4}{s} = 1 \).

Suppose in addition that \( F_n \to N \sim N(0, \sigma^2) \) in law. Then each \( F_n \) admits a density \( f_{F_n} \in C(\mathbb{R}) \) given by either (3.1) or (3.10), and

\[
\sup_{x \in \mathbb{R}} |f_{F_n}(x) - \phi(x)| \to 0
\]

as \( n \to \infty \), where \( \phi \) is the density of \( N \).

Proof. We assume (6.10). The other condition can be treated identically. From Theorem 3.1 it follows that the density formula (3.1) holds for each \( n \) and for all \( x, y \in \mathbb{R} \)

\[
|f_{F_n}(x)| \leq C(1 \wedge x^{-2}),
\]

\[
|f_{F_n}(x) - f_{F_n}(y)| \leq C |x - y|^\frac{1}{r}.
\]

Hence the sequence \( \{f_{F_n}\} \subset C(\mathbb{R}) \) is uniformly bounded and equi-continuous. Then applying Azelà-Ascoli theorem, we obtain a subsequence \( \{f_{F_{n_k}}\} \) which converges uniformly to a continuous function \( f \) on \( \mathbb{R} \) such that \( 0 \leq f(x) \leq C(1 \wedge x^{-2}) \). Then \( f_{F_{n_k}} \to f \) in \( L^1(\mathbb{R}) \) as \( k \to \infty \) with \( \|f\|_{L^1(\mathbb{R})} = \lim_k \|f_{F_{n_k}}\|_{L^1(\mathbb{R})} = 1 \). This implies that \( f \) is a density function. Then \( f \) must be \( \phi \) because \( F_n \) converges to \( N \) in law. Since the limit is unique for any subsequence, we get the uniform convergence of \( f_{F_n} \) to \( \phi \).

Corollary 6.6. Let \( \{F_n\}_{n \in \mathbb{N}} \) be a sequence of centered random variables in \( \mathbb{D}^{2,4} \) with the following Wiener chaos expansions: \( F_n = \sum_{q=1}^{\infty} J_q F_n \). Suppose that

(i) \( \lim_{Q \to \infty} \limsup_{n \to \infty} \sum_{q=Q+1}^{\infty} E[|J_q F_n|^2] = 0 \).

(ii) for every \( q \geq 1 \), \( \lim_{n \to \infty} E[(J_q F_n)^2] = \sigma_q^2 \).

(iii) \( \sum_{q=1}^{\infty} \sigma_q^2 = \sigma^2 \).

(iv) for all \( q \geq 1 \), \( \langle D(J_q F_n), D(J_q F_n) \rangle_{\mathcal{H}} \to q \sigma_q^2 \), in \( L^2(\Omega) \) as \( n \to \infty \).

(v) \( \sup_n \|F_n\|_{2,4} + \sup_n E[\|DF_n\|_{r}^{-8}] \leq \infty \).

Then each \( F_n \) admits a density \( f_{F_n}(x) \) and

\[
\sup_{x \in \mathbb{R}} |f_{F_n}(x) - \phi(x)| \to 0
\]

as \( n \to \infty \), where \( \phi \) is the density of \( N(0, \sigma^2) \).

Proof. It has been proved by Nualart and Ortiz-Latorre in [23, Theorem 8] that under conditions (i)–(iv), \( F_n \) converges to \( N \sim N(0, \sigma^2) \) in law. The condition (v) implies (6.10) with \( s = 4, p = 2, r = 4 \). Then we can conclude from Theorem 6.5.
7. Applications

The main difficulty in applying Theorem 4.1 or Theorem 5.2 is the verification of the non-degeneracy condition of the Malliavin matrix: \( \sup_n E[\|DF_n\|_Y^p] < \infty \) or \( \sup_n E[|\det \gamma_n|^{-p}] < \infty \), respectively. In this section we consider the particular case of random variables in the second Wiener chaos and we find sufficient conditions for \( \sup_n E[\|DF_n\|_Y^p] < \infty \). As an application we consider the problem of estimating the drift parameter in an Ornstein-Uhlenbeck process.

A general approach to verify \( E[G−p] < \infty \) for some positive random variable and for some \( p \geq 1 \) is to obtain a small ball probability estimate of the form

\[
P(G \leq \epsilon) \leq C \epsilon^\alpha \quad \text{for some } \alpha > p \text{ and for all } \epsilon \in (0, \epsilon_0),
\]

where \( \epsilon_0 > 0 \) and \( C > 0 \) is a constant that may depend on \( \epsilon_0 \) and \( \alpha \). We refer to the paper by Li and Shao [11] for a survey on this topic. However, finding upper bounds of this type is a challenging topic, and the application of small ball probabilities to Malliavin calculus is still an under-explored domain.

7.1. Random variables in the second Wiener chaos

A random variable \( F \) in the second Wiener chaos can always be written as \( F = I_2(f) \) where \( f \in \mathfrak{H} \otimes^2 \). Without loss of generality we can assume that

\[
f = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i, \tag{7.2}
\]

where \( \{\lambda_i, i \geq 1\} \) verifying \( |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \geq \cdots \) are the eigenvalues of the Hilbert-Schmidt operator corresponding to \( f \) and \( \{e_i, i \geq 1\} \) are the corresponding eigenvectors forming an orthonormal basis of \( \mathfrak{H} \). Then, we have \( F = I_2(f) = \sum_{i=1}^{\infty} \lambda_i (I_1(e_i)^2 - 1) \),

\[
DF = 2 \sum_{i=1}^{\infty} \lambda_i I_1(e_i)e_i \tag{7.3}
\]

and

\[
\|DF\|_Y^2 = 4 \sum_{i=1}^{\infty} \lambda_i^2 I_1(e_i)^2. \tag{7.4}
\]

For random variables of the form in (7.4), i.e., \( G = (\sum_{i=1}^{\infty} \lambda_i^2 X_i^2)^{\frac{1}{2}} \), Hoffmann-Jørgensen, Shepp and Dudley [6] used the volume of the small ball \( B_n(0, \epsilon) \) (the \( \mathbb{R}^n \) ball centered at 0 with radius \( \epsilon \)) to control \( P(G \leq \epsilon) \) as

\[
P(G \leq \epsilon) \leq P\left(\sum_{i=1}^{n} \lambda_i^2 X_i^2 \leq \epsilon^2\right) \leq (2\pi)^{-\frac{n}{2}} \epsilon^n |B_n(0, 1)| \prod_{i=1}^{n} \lambda_i^{-1}. \tag{7.5}
\]

They proved that \( P(G \leq \epsilon) \) converges to zero at the rate \( O(\epsilon^n) \) for all \( n \) as \( \epsilon \to 0 \), under some implicit conditions on \( \{\lambda_i, i \geq 1\} \). This idea can be used here to prove inequality (7.6) in the following lemma. However, our case is much simpler, and we shall use the Gamma function to give an alternative proof which leads to a necessary and sufficient condition for \( E[G−p] < \infty \).
Lemma 7.1. Let $G = \left( \sum_{i=1}^{\infty} \lambda_i^2 X_i^2 \right)^{\frac{1}{2}}$, where $\{\lambda_i\}_{i \geq 1}$ satisfies $|\lambda_i| \geq |\lambda_{i+1}|$ for all $i \geq 1$ and $\{X_i\}_{i \geq 1}$ are i.i.d standard normal. Fix an $\alpha > 1$. Then, $E[G^{-2\alpha}] < \infty$ if and only if there exists an integer $N > 2\alpha$ such that $|\lambda_N| > 0$ and in this case there exists a constant $C_\alpha$ depending only on $\alpha$ such that

$$
E[G^{-2\alpha}] \leq C_\alpha N^{-\alpha} |\lambda_N|^{-2\alpha}. \tag{7.6}
$$

Proof. Notice $\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda y} y^{-\alpha} dy$ and $E[e^{-tX^2}] = \frac{1}{\sqrt{1 + 2t}}$ for all $t > 0$. If there exists $N > 2\alpha$ such that $|\lambda_N| > 0$, then

$$
E[G^{-2\alpha}] \leq E \left[ \left( \sum_{i=1}^{N} \lambda_i^2 X_i^2 \right)^{-\alpha} \right] = \frac{1}{\Gamma(\alpha)} E \left[ \int_0^\infty e^{-y \sum_{i=1}^{N} \lambda_i^2 X_i^2 y^{-\alpha} dy} \right]
= \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{-\alpha-1} \prod_{i=1}^{N} (1 + 2\lambda_i^2 y)^{-\frac{\alpha}{2}} dy. \tag{7.7}
$$

Since $\lambda_i^2$ is non-increasing in $i$ and $N > 2\alpha$, using the change of variables $1 + 2\lambda_N^2 y = z$ we have

$$
\int_0^\infty y^{-\alpha-1} \prod_{i=1}^{N} (1 + 2\lambda_i^2 y)^{-\frac{\alpha}{2}} dy \leq \int_0^\infty y^{-\alpha-1} (1 + 2\lambda_N^2 y)^{-\frac{\alpha}{2}} dy
= (2\lambda_N^2)^{-\alpha} \int_1^\infty (z-1)^{-\alpha} z^{-\frac{\alpha}{2}} dz = (2\lambda_N^2)^{-\alpha} \int_1^\infty \frac{(z-1)^{1-\alpha}}{z^{1+\alpha}} z^{-\frac{\alpha}{2}} dz
= (2\lambda_N^2)^{-\alpha} \int_0^1 (1-x)^{-\alpha} x^{-\frac{\alpha}{2}} dx = (2\lambda_N^2)^{-\alpha} \frac{\Gamma(\alpha) \Gamma(\frac{N}{2} - \alpha)}{\Gamma(N/2)},
$$

which implies (7.6).

On the other hand, if $|\lambda_i| = 0$ for all $i > 2\alpha$, let $N \leq 2\alpha$ be the largest nonnegative integer such that $|\lambda_N| > 0$. Then, the inequality in (7.7) becomes an equality. Using again that $\{\lambda_i^2\}_{i \geq 1}$ is a decreasing sequence we have

$$
\int_0^\infty y^{-\alpha-1} \prod_{i=1}^{N} (1 + 2\lambda_i^2 y)^{-\frac{\alpha}{2}} dy \geq (1 + 2\lambda_1^2 y)^{-\frac{\alpha}{2}} \left( \int_0^1 y^{-\alpha-1} dy + \int_1^\infty y^{-\alpha-1} \frac{N}{2} dy \right) = \infty,
$$

and we conclude that $E[G^{-2\alpha}] = \infty$. This completes the proof. $\blacksquare$

The following theorem describes the distance between the densities of $F = I_2(f)$ and $N(0, E[F^2])$.

Theorem 7.2. Let $F = I_2(f)$ with $f \in S_{2,1}^{\infty}$ given in (7.2). Assume that there exists $N > 6m + 6 \left( \left\lfloor \frac{m}{2} \right\rfloor \vee 1 \right)$, for some integer $m \geq 0$, such that $\lambda_N \neq 0$. Then $F$ admits an $m$ times continuously differentiable density $f_F$. Furthermore, if $\phi(x)$ denotes the density of $N(0, E[F^2])$, then for $k = 0, 1, \ldots, m$,

$$
\sup_{x \in \mathbb{R}} \left| f_F^{(k)}(x) - \phi^{(k)}(x) \right| \leq C \left( \sum_{i=1}^{\infty} \lambda_i^4 \right)^{\frac{1}{2}} \leq C \left( E[F^4] - 3 \left( E[F^2] \right)^2 \right)^{\frac{1}{2}},
$$

where the constant $C$ depends on $N$ and $\lambda_N$. 

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Proof. Taking into account of (7.4), we have
\[ \text{Var} \left( \|DF\|_B^2 \right) = E \left| 4 \sum_{i=1}^{\infty} \lambda_i^2 (I_1(e_i)^2 - 1) \right|^2 = 32 \sum_{i=1}^{\infty} \lambda_i^4. \] (7.8)

From (7.4) and Lemma 7.1 it follows that
\[ E[\|DF\|_B - \beta]^2 \leq C_{\beta/2} N^{-\beta/2} |\lambda_N|^{-\beta}, \] (7.9)
for all \( \beta < N \). Then, the theorem follows from Theorem 4.4, taking into account (7.8).

Now we are ready to prove convergence of densities of random variables in the second Wiener chaos. Consider a sequence \( F_n = I_2(f_n) \) with \( f_n \in \mathcal{H}^{\otimes 2} \), which can be written as
\[ f_n = \sum_{i=1}^{\infty} \lambda_{n,i} e_{n,i} \otimes e_{n,i}, \] (7.10)
where \( \{\lambda_{n,i}, i \geq 1\} \) verifies \( |\lambda_{n,i}| \geq |\lambda_{n,i+1}| \) for all \( i \geq 1 \) and \( \{e_{n,i}, i \geq 1\} \) are the corresponding eigenvectors.

**Theorem 7.3.** Let \( F_n = I_2(f_n) \) with \( f_n \in \mathcal{H}^{\otimes 2} \) given by (7.10). Assume that \( \{\lambda_{n,i}\}_{n,i \in \mathbb{N}} \) satisfies

(i) \( \sigma^2 := 2 \lim_{n \to \infty} \sum_{i=1}^{\infty} \lambda_{n,i}^2 > 0 \);
(ii) \( \lim_{n \to \infty} \sum_{i=1}^{\infty} \lambda_{n,i}^4 = 0 \);
(iii) \( \inf_n \left( \sup_{i > 6m + 6 (\lceil \frac{m}{2} \rceil \vee 1)} |\lambda_{n,i}| \sqrt{i} \right) > 0 \) for some integer \( m \geq 0 \).

Then, each \( F_n \) admits a density function \( f_{F_n} \in C^m(\mathbb{R}) \). Furthermore, for \( k = 0, 1, \ldots, m \) and if \( \phi \) denotes the density of the law \( N(0, \sigma^2) \), the derivatives of \( f_{F_n}^{(k)} \) converge uniformly to the derivatives of \( \phi \) with a rate given by
\[ \sup_{x \in \mathbb{R}} \left| f_{F_n}^{(k)}(x) - \phi^{(k)}(x) \right| \leq C \left[ \left( \sum_{i=1}^{\infty} \lambda_{n,i}^4 \right)^{\frac{1}{2}} + \left| 2 \sum_{i=1}^{\infty} \lambda_{n,i}^2 - \sigma^2 \right|^{\frac{1}{2}} \right], \]
where \( C \) is a constant depending only on \( m \) and the infimum appearing in condition (iii).

**Proof of Theorem 7.3.** Note that \( E[(I_1(e_{n,i})^2 - 1)(I_1(e_{n,j})^2 - 1)] = 2\delta_{ij} \). Thus,
\[ \sum_{i=1}^{\infty} \lambda_{n,i}^2 = \|f_n\|_{\mathcal{H}^{\otimes 2}}^2 = \frac{1}{2} E[f_n^2]. \]
Then, the result follows from (7.8), (7.9) and Corollary 4.6.

Condition (iii) in Theorem 7.3 means that there exist a positive constant \( \delta > 0 \) such that for each \( n \) we can find an index \( i(n) > 6m + 6 (\lceil \frac{m}{2} \rceil \vee 1) \) with \( |\lambda_{n,i(n)}| \sqrt{i(n)} \geq \delta \).
Remark 7.4. It is interesting to compare Theorem 7.3 with the case when

\[ \lambda_{n,i} = \begin{cases} \frac{1}{\sqrt{n}} & \text{if } 1 \leq i \leq n; \\ 0 & \text{if } i \geq n + 1, \end{cases} \]

which corresponds to classical case of sum of independent and identically distributed random variables. In this case all the conditions of Theorem 7.3 are satisfied with \( \sigma^2 = 2 \). Moreover we have \( \sum_{i=1}^{\infty} \lambda_{n,i}^4 = \frac{1}{n} \) and \( \sum_{i=1}^{\infty} \lambda_{n,i}^2 = 1 \). Then we obtain a Berry–Essen type bound for the derivatives of the density. Namely, we have \( \sup_{x\in\mathbb{R}} \left| f_{F_n}^{(k)}(x) - \phi^{(k)}(x) \right| \leq \frac{C}{\sqrt{n}} \) for sufficiently large \( n \), which provides the right rate of convergence.

7.2. Parameter estimation in Ornstein-Uhlenbeck processes

Consider the following Ornstein-Uhlenbeck process

\[ X_t = -\theta \int_0^t X_s ds + \gamma B_t, \]

where \( \theta > 0 \) is an unknown parameter, \( \gamma > 0 \) is known and \( B = \{B_t, 0 \leq t < \infty\} \) is a standard Brownian motion. Assume that the process \( X = \{X_t, 0 \leq t \leq T\} \) can be observed continuously in the time interval \([0, T]\). Then the least squares estimator (or the maximum likelihood estimator) of \( \theta \) is given by \( \hat{\theta}_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} \). It is known (see for example, [9], [10]) that, as \( T \) tends to infinity, \( \sqrt{T}(\hat{\theta}_T - \theta) \) converges to \( N(0, 2\theta) \) almost surely as

\[ \sqrt{T}(\hat{\theta}_T - \theta) = -\frac{TF_T}{\int_0^T X_t^2 dt} \xrightarrow{\mathcal{L}} N(0, 2\theta), \quad (7.11) \]

where

\[ F_T = I_2(f_T) = \int_0^T \int_0^T f_T(t, s) dB_t dB_s, \quad (7.12) \]

with

\[ f_T(t, s) = \frac{\gamma^2}{2\sqrt{T}} e^{-\theta|t-s|}. \quad (7.13) \]

Recently, Hu and Nualart [7] extended this result to the case where \( B \) is a fractional Brownian motion with Hurst parameter \( H \in \left[ \frac{1}{2}, \frac{3}{4} \right] \), which includes the standard Brownian motion case. Since \( \frac{1}{T} \int_0^T X_t^2 dt \rightarrow \frac{1}{2} \gamma^2 \theta^{-1} \) almost surely as \( T \) tends to infinity, the main effort in proving (7.11) is to show the convergence in law of \( F_T \) to the normal law \( N(0, \frac{\gamma^4}{2\theta}) \). We shall prove that the density of \( F_T \) converges as \( T \) tends to infinity to the density of the normal distribution \( N(0, \frac{\gamma^4}{2\theta}) \).

Theorem 7.5. Let \( F_T \) be given by (7.13) and let \( \phi \) be the density of the law \( N(0, \sigma^2) \), where \( \sigma^2 = \frac{\gamma^4}{2\theta} \). Then for each \( T > 0 \), \( F_T \) has a smooth probability density \( f_{F_T} \) and for any \( k \geq 0 \),

\[ \sup_{x\in\mathbb{R}} \left| f_{F_T}^{(k)}(x) - \phi^{(k)}(x) \right| \leq C T^{-\frac{k}{2}}, \]

where the constant \( C \) depends on \( k, \gamma \) and \( \theta \).
Before proving the theorem, let us first analyze the asymptotic behavior of the eigenvalues of $f_T$. The Hilbert space corresponding to Brownian motion $B$ is $\mathcal{H} = L^2([0, T])$. Let $Q_T : L^2([0, T]) \rightarrow L^2([0, T])$ be the Hilbert-Schmidt operator associated to $f_T$, that is,

$$(Q_T \varphi)(t) = \int_0^T f_T(t, s) \varphi(s) \, ds$$

(7.14)

for $\varphi \in L^2[0, T]$. The operator $Q_T$ has eigenvalues $\lambda_{T,1} > \lambda_{T,2} > \cdots \geq 0$ and $\sum_{i=1}^{\infty} \lambda_{T,i}^2 < \infty$. The following lemma provides upper and lower bounds for these eigenvalues.

**Lemma 7.6.** Fix $T > 0$. Let $f_T$ be given by (7.13) and $Q_T$ be given by (7.14). The eigenvalues $\lambda_{T,i}$ of $Q_T$ (except maybe one) satisfy the following estimates

$$\frac{\gamma^2 \theta}{\sqrt{T} \left( \theta^2 + \left( \frac{i \pi + \frac{\pi}{2}}{T} \right)^2 \right)} < \lambda_{T,i} < \frac{\gamma^2 \theta}{\sqrt{T} \left( \theta^2 + \left( \frac{i \pi - \frac{\pi}{2}}{T} \right)^2 \right)}.$$  

(7.15)

**Proof.** Consider the eigenvalue problem $Q_T \varphi = \lambda \varphi$, that is,

$$\int_0^T f_T(t, s) \varphi(s) \, ds = \frac{\gamma^2}{2\sqrt{T}} \left( \int_0^t e^{-\theta(t-s)} \varphi(s) \, ds + \int_t^T e^{-\theta(s-t)} \varphi(s) \, ds \right) = \lambda \varphi(t).$$

(7.16)

Then, $\varphi$ is differentiable and

$$\frac{\gamma^2 \theta}{2\sqrt{T}} \left( -2 \varphi(t) + \theta \int_0^t e^{-\theta(t-s)} \varphi(s) \, ds + \theta \int_t^T e^{-\theta(s-t)} \varphi(s) \, ds \right) = \lambda \varphi'(t).$$

(7.17)

Differentiating again we have

$$\frac{\gamma^2 \theta}{2\sqrt{T}} \left( -2 \varphi(t) + \theta \int_0^t e^{-\theta(t-s)} \varphi(s) \, ds + \theta \int_t^T e^{-\theta(s-t)} \varphi(s) \, ds \right) = \lambda \varphi''(t).$$

Comparing this expression with (7.16), we obtain

$$(-\theta^2 \frac{\gamma^2 \theta}{\sqrt{T} \lambda}) \varphi(t) = \varphi''(t).$$

(7.18)

Also, from (7.16) and (7.17) it follows that

$$\varphi(0) = \theta \varphi'(0), \; \varphi(T) = -\theta \varphi'(T).$$

(7.19)

Equations (7.18) and (7.19) form a Sturm-Liouville system. Its general solution is of the form

$$\varphi(t) = C_1 \sin \mu t + C_2 \cos \mu t,$$

where $C_1$ and $C_2$ are constants, and $\mu > 0$ is an eigenvalue of the Sturm-Liouville system. By eliminating the constants $C_1$ and $C_2$ from (7.18) and (7.19) we obtain

$$-\mu^2 = \theta^2 - \frac{\gamma^2 \theta}{\sqrt{T} \lambda}. $$

(7.20)
Then, the desired estimates on the eigenvalues of \( Q_T \varphi = \lambda \varphi \) follow from estimates on \( \mu \). Note that the Neumann condition (7.19) yields

\[
(\mu^2 \theta^2 - 1) \sin \mu T = 2\mu \theta \cos \mu T.
\]

If we write \( x = \mu \theta > 0 \) (since \( \mu, \theta > 0 \)), the above equation becomes

\[
(x^2 - 1) \sin \frac{x}{\theta} T = 2x \cos \frac{x}{\theta} T.
\]

The solution \( x = 1 \) corresponds to the eigenvalue \( \mu = \frac{1}{\theta} \). If \( x \neq 1 \), then \( \cos \frac{x}{\theta} T \neq 0 \) and

\[
\tan \frac{x}{\theta} T = \frac{2x}{x^2 - 1}. \tag{7.21}
\]

For any \( i \in \mathbb{Z}_+ \), there is exactly one solution \( x_i \) to (7.21) such that \( \frac{x_i}{\theta} T \in (i\pi - \frac{\pi}{2}, i\pi + \frac{\pi}{2}) \). Corresponding to each \( x_i \) is an eigenvalue \( \mu_i = \frac{x_i}{\theta} \) of the Sturm-Liouville system, satisfying \( \frac{i\pi - \frac{\pi}{2}}{T} < \mu_i < \frac{i\pi + \frac{\pi}{2}}{T} \). The corresponding eigenvalue \( \lambda_i \) of \( Q_T \) obtained from Equation (7.20) satisfies the estimate (7.15). \( \blacksquare \)

**Proof of Theorem 7.5.** For each \( T \), let us compute the second moment of \( F_T \).

\[
E \left[ F_T^2 \right] = \| f_T \|_{\mathcal{S}^2}^2 = \int_0^T \int_0^T f_T(t, s)^2 ds dt
\]

\[
= \frac{\gamma^4}{4T} \int_0^T \int_0^t e^{-2\theta(t-s)} ds dt = \frac{\gamma^4}{2\theta} - \frac{\gamma^4}{8\theta T} (1 - e^{-2\theta T}).
\]

Also, noticing that \( F_T = I_2(f_T) = \delta^2(f_T) \) and

\[
D_s D_t F_T^3 = 3F_T^2 f_T(t, s) + 6F_T I_1(f(\cdot, t)) \otimes I_1(f(\cdot, s)),
\]

and using the duality between \( \delta \) and \( D \), we can write

\[
E \left[ F_T^4 \right] = E \left[ \langle f_T, D^2 F_T^3 \rangle_{\mathcal{S}^2} \right] = 3E \left[ F_T^2 \langle f_T, f_T \rangle_{\mathcal{S}^2} \right]
\]

\[
+ 6E \left[ F_T \langle f_T(t, s), I_1(f(\cdot, t)) \otimes I_1(f(\cdot, s)) \rangle_{\mathcal{S}^2} \right]
\]

\[
= 3 \left( E \left[ F_T^2 \right] \right)^2 + 6A,
\]

where

\[
A = E \left[ F_T \langle f_T(t, s), I_1(f(\cdot, t)) \otimes I_1(f(\cdot, s)) \rangle_{\mathcal{S}^2} \right]
\]

\[
= \langle f_T(u, v), f_T(t, s), f_T(u, t) \otimes f_T(v, s) \rangle_{\mathcal{S}^2}
\]

\[
= \frac{\gamma^8}{16T^2} \int_0^T \int_0^T \int_0^T \int_0^T e^{-\theta|u-v|+|t-s|+|u-t|+|v-s|} du dv dt ds.
\]

Because the integrand is symmetric, we have

\[
A = \frac{\gamma^8}{16T^2} 4! \int_0^T du \int_0^u dv \int_0^v ds \int_0^s dt \ e^{-2\theta(u-t)} \leq CT^{-1}.
\]

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Then, in order to complete the proof by applying Corollary 4.6, we only need to verify that condition (iii) of Theorem 7.3 holds for any integer \( m \geq 1 \), which implies the uniform boundedness of the negative moments

\[
\sup_{T>0} E \left[ \| D F_T \|^{-\beta}_H \right] < \infty
\]

for any \( \beta > 0 \). Fix \( \beta > 0 \), and for each \( T \), let \( i(T) = \lfloor \beta + 1 \rfloor + \lfloor T \rfloor \). Then, the lower bound in (7.15) yields

\[
\sqrt{i(T)} \lambda_{T,i(T)} \geq \frac{\sqrt{i(T)} \gamma^2 / \theta}{\sqrt{T} \left( 1 + \left( \frac{i(T)}{T} \right)^2 \right)} \geq \frac{\gamma^2 / \theta}{\max_{\beta+2^{-1} \leq r \leq 1} g(r)} > 0,
\]

where in the last inequality we made the substitution \( r^{-1} = \frac{i(T)}{T} \) and set

\[
g(r) := \sqrt{r} \left( 1 + r^{-2} 4 \frac{\pi^2}{\theta^2} \right).
\]

This implies condition (iii) and the proof of the theorem is complete. □

7.3. Multidimensional Case

Now we give an example for random vectors. Let \( X = \{X(h), h \in \mathcal{H}\} \) be an isonormal Gaussian process associated with the Hilbert space \( \mathcal{H} \). Suppose that \( \{e_{ij}, 1 \leq i \leq d, j \geq 1\} \) is a sequence of orthonormal elements in \( \mathcal{H} \). Set \( e_k = (e_{1k}, \ldots, e_{dk}) \) for any \( k \geq 1 \). Let \( A_n \) be a sequence of \( d \times d \) invertible matrices such that \( A_n \to I \) as \( n \to \infty \). For any \( k \geq 1 \) define

\[
\xi_{nk} = \begin{pmatrix} \xi_{ink} \\ \vdots \\ \xi_{dnk} \end{pmatrix} = A_n \begin{pmatrix} e_{1k} \\ \vdots \\ e_{dk} \end{pmatrix}
\]

and, for any \( j = 1, \ldots, d \) set

\[
F_{jn} = \sum_{k=1}^{\infty} \lambda_{jnk} I_2(\xi_{nk}^2) = \sum_{k=1}^{\infty} \lambda_{jnk} \left[ \tilde{\xi}_{jnk}^2 - \| \xi_{jnk} \|^2 \right],
\]

where \( \tilde{\xi}_{jnk} = I_1(\xi_{jnk}) = X(\xi_{jnk}) \) and \( \lambda_{jnk} \) are real numbers which will be specified later. We plan to use Theorem 5.2 to study the convergence of the random vectors \( F_n = (F_{1n}, \ldots, F_{dn}) \). For this we can follow the approach of Section 7.1, the main extra work being to prove the existence of a uniform bound for the negative moments of the Malliavin covariance matrices. We have

\[
DF_{jn} = 2 \sum_{k=1}^{\infty} \lambda_{jnk} \tilde{\xi}_{jnk} \xi_{jnk}.
\]

Thus

\[
\langle DF_{in}, DF_{jn} \rangle_{\mathcal{H}} = 4 \sum_{k=1}^{\infty} \lambda_{ink} \lambda_{jnk} \tilde{\xi}_{ink} \xi_{jnk} \alpha_{ijn}, \quad (7.22)
\]
where $\alpha_{ijn}$ is the $(i,j)$th entry of the matrix $\alpha_n = A_nA_n^T$. Consider the matrix $\beta_n := (\beta_{ijn})_{1 \leq i,j \leq d}$ given by

$$\beta_{ijn} := 4 \sum_{k=1}^{\infty} \lambda_{ijnk} \lambda_{jn}\xi_{ijnk} \xi_{jnk}.$$ 

Then from (7.22), we see that $\langle DF_n, DF_n \rangle = (\langle DF_n, DF_n \rangle_{ij})_{1 \leq i,j \leq d}$ is the Hadamard product of the nonnegative definite matrices $\alpha_n$ and $\beta_n$. By the Oppenheim’s inequality for Hadamard product, and taking into account that $\det(\alpha_n)$ converges to one, there exists a constant $c > 0$ such that

$$\det(\langle DF_n, DF_n \rangle) \geq \det(\alpha_n) \prod_{j=1}^{d} \beta_{jjn} \geq c \prod_{j=1}^{d} \beta_{jjn},$$

for all $n$. Note $\beta_{jjn} = 4 \sum_{k=1}^{\infty} \lambda_{ijnk}^2 \left(\xi_{ijnk}\right)^2$ and $\xi_{ijnk} \rightarrow e_{jk}$. Thus we can follow Section 7.1 to verify the conditions that allow us to apply Theorem 5.2. We will write down the theorem and omit the details. In the following we denote by $\phi_{\alpha}$ the density of the law $N(0, \text{diag}(\sigma_1^2, \ldots, \sigma_d^2))$ and $\partial_{\alpha_1} \cdots \partial_{\alpha_d} f(x) = \frac{\partial^{(|\alpha|)}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f(x)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

**Theorem 7.7.** Let $A_n$ be a sequence of $d \times d$ invertible matrices such that $A_n \rightarrow I$ and let $F_n = (F_{1n}, \ldots, F_{dn})$ be defined as above. We assume the $\lambda_{ijnk}$ satisfy the following conditions for any $1 \leq j \leq d$.

(i) $\sigma_j^2 := \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \lambda_{ijnk}^2 > 0$;

(ii) $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \lambda_{ijnk}^4 = 0$;

(iii) $\inf_{j,n} \left(\sup_{k \geq 6m+6(|\frac{m}{2} |\sqrt{k})} |\lambda_{ijnk}| \sqrt{k}\right) > 0$ for some integer $m \geq 0$.

Then, each $F_n$ admits a density function $f_{F_n} \in C^m(\mathbb{R}^d)$. Furthermore, for any $\alpha = (\alpha_1, \ldots, \alpha_d)$, with $|\alpha| \leq m$, the derivatives of $\partial_{\alpha_1} \cdots \partial_{\alpha_d} f_{F_n}$ converge uniformly to the derivatives of $\partial_{\alpha_1} \cdots \partial_{\alpha_d} \phi_{\alpha}$ with a rate given by

$$\sup_{x \in \mathbb{R}} |\partial_{\alpha_1} \cdots \partial_{\alpha_d} f_{F_n}(x) - \partial_{\alpha_1} \cdots \partial_{\alpha_d} \phi_{\alpha}(x)| \leq C \sum_{j=1}^{d} \left[ \left( \sum_{k=1}^{\infty} \lambda_{ijnk}^4 \right)^{\frac{1}{2}} + \left( \sum_{i=1}^{\infty} \lambda_{ijnk}^2 - \sigma_j^2 \right)^{\frac{1}{2}} \right],$$

where $C$ is a constant depending only on $m$ and the infimum appearing in condition (iii).

8. **Appendix**

In this section, we present the omitted proofs and some technical results.

**Proof of Lemma 2.1.** Since $\int_{-\infty}^{x} \{ h(y) - E[h(N)] \} e^{-y^2/(2\sigma^2)} dy = 0$, we have

$$\int_{-\infty}^{x} \{ h(y) - E[h(N)] \} e^{-y^2/(2\sigma^2)} dy = - \int_{x}^{\infty} \{ h(y) - E[h(N)] \} e^{-y^2/(2\sigma^2)} dy.$$ 

Hence

$$\left| \int_{-\infty}^{x} \{ h(y) - E[h(N)] \} e^{-y^2/(2\sigma^2)} dy \right| \leq \int_{|x|}^{\infty} [ay_k + b + E(|h(N)|)] e^{-y^2/(2\sigma^2)} dy.$$
By using the representation (2.18) of $f_h$ and Stein’s equation (2.15) we have

$$ |f'_h(x)| \leq |h(x) - E[h(N)]| + \frac{|x|}{\sigma^2} e^{x^2/(2\sigma^2)} \left| \int_{-\infty}^{x} \{h(y) - E[h(N)]\} e^{-y^2/(2\sigma^2)} dy \right| $$

$$ \leq a |x|^k + b + E[h(N)] + \frac{1}{\sigma^2} e^{x^2/(2\sigma^2)} \int_{|x|}^{\infty} y[ay^k + b + E|h(N)|] e^{-y^2/(2\sigma^2)} dy $$

$$ = a |x|^k + (b + E|h(N)|) \left( 1 + \frac{1}{\sigma^2} s_1(x) \right) + \frac{a}{\sigma^2} s_{k+1}(x), \quad (8.1) $$

where we let $s_k(x) = e^{x^2/(2\sigma^2)} \int_{|x|}^{\infty} y^k e^{-y^2/(2\sigma^2)} dy$ for any integer $k \geq 0$.

Note that $E|h(N)| \leq aE|N|^k + b \leq C_k a \sigma^k + b$ and

$$ s_1(x) = e^{x^2/(2\sigma^2)} \int_{x}^{\infty} ye^{-y^2/(2\sigma^2)} dy = \sigma^2 $$

for all $x \in \mathbb{R}$. Using integration by parts, we see by induction that for any integer $k \geq 1$,

$$ s_{k+1}(x) = e^{x^2/(2\sigma^2)} \int_{|x|}^{\infty} y^{k+1} e^{-y^2/(2\sigma^2)} dy $$

$$ = \sigma^2 e^{x^2/(2\sigma^2)} \int_{|x|}^{\infty} y^k d(-e^{-y^2/(2\sigma^2)}) = \sigma^2 [x^k + k s_{k-1}(x)]. $$

Then if $k \geq 1$ is even, we have

$$ s_{k+1}(x) \leq C_k \sigma^2 [x^k + \sigma^2 |x|^{k-2} + \ldots + \sigma^2 s_1(x)] \leq C_k \sigma^2 \sum_{i=0}^{k} \sigma^{k-i} |x|^i. $$

If $k \geq 1$ is odd, we have

$$ s_{k+1}(x) \leq C_k \sigma^2 [x^k + \sigma^2 |x|^{k-2} + \ldots + \sigma^2 (|x| + s_0(x))] \leq C_k \sigma^2 \sum_{i=0}^{k} \sigma^{k-i} |x|^i, $$

where we used the fact that $s_0(x) \leq s_0(0) = \sqrt{\frac{\pi}{2}} \sigma$ for all $x \in \mathbb{R}$ (indeed, when $x \geq 0$ we have $s_0'(x) = \frac{x}{\sigma^2} e^{x^2/(2\sigma^2)} \int_{x}^{\infty} e^{-y^2/(2\sigma^2)} dy - 1 \leq e^{x^2/(2\sigma^2)} \int_{x}^{\infty} \frac{y}{\sigma^2} e^{-y^2/(2\sigma^2)} dy - 1 = 0$; similarly when $x < 0$, $s_0'(x) \geq 0$). Putting the above estimates into (8.1) we complete the proof. □

**Proof of Lemma 3.5.** We shall prove these properties by induction. From $T_1 = T_2 = 0$, (3.17) and (3.19) we know that $T_3 = D_u^2 \delta_u$, with $J_3 = \{(0,0,1)\}$; and $T_4 = \delta_u D_u^2 \delta_u + D_u^3 \delta_u$, with $J_4 = \{(1,0,1,0),(0,0,0,1)\}$. Now suppose the statement is true for all $T_l$ with $l \leq k - 1$ for $k \geq 5$. We want to prove the multi-indices of $T_k$ satisfy (a)–(c). This will be done by studying the three operations, $\delta_u T_{k-1}$, $D_u T_{k-1}$ and $\delta_u H_{k-1}(D_u \delta_u, \delta_u) D_u^2 \delta_u$, in expression (3.19).

For the term $\delta_u H_{k-1}(D_u \delta_u, \delta_u) D_u^2 \delta_u$, we observe from (3.17) that

$$ \delta_u H_{k-1}(D_u \delta_u, \delta_u) D_u^2 \delta_u = D_u^2 \delta_u \sum_{1 \leq i \leq [(k-1)/2]} ic_{k-1,i} \delta_u^{k-1-2i}(D_u \delta_u)^{i-1}, $$

whose terms have multi-indices $(k - 1 - 2i, i - 1, 1, 0, \ldots, 0) \in \mathbb{N}^k$ for $1 \leq i \leq \lfloor k-1 \rfloor$. Then, it is straightforward to check that these multi-indices satisfy (a), (b) and (c).
The term $\delta_u T_{k-1}$ shifts the multi-index $(i_0, i_1, \ldots, i_{k-2}) \in J_{k-1}$ to $(i_0 + 1, i_1, \ldots, i_{k-2}, 0) \in \mathbb{N}^k$, which obviously satisfies (a), (b) and (c), due to the induction hypothesis.

The third term $D_u T_{k-1}$ shifts the multi-index $(i_0, i_1, \ldots, i_{k-2}) \in J_{k-1}$ to either $\alpha = (i_0 - 1, i_1 + 1, \ldots, i_{k-2}, 0) \in \mathbb{N}^k$ if $i_0 \geq 1$, or to

$$\beta = \begin{cases} (i_0, i_1, \ldots, i_{j_0} - 1, i_{j_0+1} + 1, \ldots, i_{k-2}, 0), & \text{for } 1 \leq j_0 \leq k - 3; \\ (i_0, i_1, \ldots, i_{j_0} - 1, 1), & \text{for } j_0 = k - 2, \end{cases}$$

if $i_{j_0} \geq 1$. It is easy to check that $\beta$ satisfies properties (a), (b) and (c) and $\alpha$ satisfies properties (b) and (c). We are left to verify that $\alpha$ satisfies property (c). That is, we want to show that

$$1 + \sum_{j=1}^{k-2} i_j \leq \left\lfloor \frac{k-1}{2} \right\rfloor. \quad (8.2)$$

If $k$ is odd, say $k = 2m+1$ for some $m \geq 2$, (8.2) is true because $(i_0, i_1, \ldots, i_{k-2}) \in J_{k-1}$, which implies by induction hypothesis that $\sum_{j=1}^{k-2} i_j \leq \left\lfloor \frac{k-2}{2} \right\rfloor = m - 1$. If $k$ is even, say $k = 2m + 2$, (8.2) is true because the following claim asserts that if $i_0 \geq 1$, then $\sum_{j=1}^{k-2} i_j < \left\lfloor \frac{k-2}{2} \right\rfloor = m$.

**Claim:** For $(i_0, i_1, \ldots, i_{2m}) \in J_{2m+1}$ with $m \geq 1$, if $\sum_{j=1}^{2m} i_j = m$ then $i_0 = 0$.

Indeed, suppose $(i_0, i_1, \ldots, i_{2m}) \in J_{2m+1}$, $\sum_{j=1}^{2m} i_j = m$ and $i_0 \geq 1$. We are going to show that leads to a contradiction. First notice that $i_1 \geq 1$, otherwise $i_1 = 0$ and $\sum_{j=2}^{2m} i_j = m$, which is not possible because

$$i_0 + 2m \leq i_0 + \sum_{j=1}^{2m} ji_j \leq 2m.$$ 

Also, we must have $i_{2m} = 0$, because otherwise property (a) implies $i_{2m} = 1$ and $i_0 = i_1 = \cdots = i_{2m-1} = 0$. Now we trace back to its parent multi-indices in $J_{2m}$ by reversing the three operations. Of the three operations, we can exclude $\partial_\lambda H_{2m}(D_u \delta_u, \delta_u)D_u^2 \delta_u$ and $\delta_u T_{2m}$, because $\partial_\lambda H_{2m}(D_u \delta_u, \delta_u)D_u^2 \delta_u$ generates $(2m - 2j, j - 1, 1, 0, \ldots, 0)$ with $1 \leq j \leq m$, where $j$ must be $m$; and $\delta_u T_{2m}$ traces it back to $(i_0 - 1, i_1, \ldots, i_{2m-1}) \in J_{2m}$, where $i_1 + \cdots + i_{2m-1} = m > \left\lfloor \frac{2m-1}{2} \right\rfloor$. Therefore, its parent multi-index in $J_{2m}$ must come from the operation $D_u T_{2m}$ and hence must be $(i_0 + 1, i_1 - 1, \ldots, i_{2m-1}) \in J_{2m}$. Note that for this multi-index, $i_1 - 1 + \cdots + i_{2m-1} = m - 1$. Repeating the above process we will end up at $(i_0 + i_1, 0, i_2, \ldots, i_{2m-i_1}) \in J_{2m+1-i_1}$ with $i_2 + \cdots + i_{2m-i_1} = m - i_1$, which contradicts the property (b) of $J_{2m+1-i_1}$ because

$$i_0 + 2m - i_1 \leq i_0 + i_1 + \sum_{j=2}^{2m-i_1} ji_j \leq 2m - i_1.$$ 

Recall that we denote $D_{DF}w^{-1} = \langle Dw^{-1}, DF \rangle_{\delta_1}$ and $D_{DF}^{k}w^{-1} = \langle D(D_{DF}^{k-1}w^{-1}), DF \rangle_{\delta_1}$ for any $k \geq 2$. The following lemma estimates the $L^p(\Omega)$ norms of $D_{DF}^{k}w^{-1}$.

**Lemma 8.1.** Let $F = I_q(f)$ with $q \geq 2$ satisfying $E[F^2] = \sigma^2$. For any $\beta \geq 1$ we define and

$$M_\beta = \left( E\|DF\|_{\delta_1}^{-\beta} \right)^{1/\beta}.$$ 

Set $w = \|DF\|_{\delta_1}^2$. 

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(i) If $M_\beta < \infty$ for some $\beta \geq 6$, then for any $1 \leq r \leq \frac{2\beta}{\beta+6}$
\[ \|D_{DF}w^{-1}\|_r \leq CM_\beta^3 \|q\sigma^2 - w\|_2. \]  
(8.3)

(ii) If $k \geq 2$ and $M_\beta < \infty$ for some $\beta \geq 2k + 4$, then for any $1 < r < \frac{2\beta}{\beta+2k+4}$
\[ \|D_{DF}^kw^{-1}\|_r \leq C \left( \sigma^{2k-2} \vee 1 \right) \left( M_\beta^{k+2} \vee 1 \right) \|q\sigma^2 - w\|_2. \]  
(8.4)

(iii) If $k \geq 1$ and $M_\beta < \infty$ for any $\beta > k + 2$, then for any $1 < r < \frac{\beta}{k+2}$
\[ \|D_{DF}^kw^{-1}\|_r \leq C \left( \sigma^{2k} \vee 1 \right) \left( M_\beta^{k+2} \vee 1 \right). \]  
(8.5)

Proof. Note that $D_{DF}w^{-1} = \langle D^{-1}, DF \rangle_{\delta} = -2w^{-2} \langle D^2 F \otimes_1 DF, DF \rangle$. Then
\[ |D_{DF}w^{-1}| \leq 2w^{-\frac{2}{r}} \|D^2 F \otimes_1 DF\|_{\delta}. \]

Applying Hölder’s inequality with $\frac{1}{r} = \frac{1}{p} + \frac{1}{2}$, yields
\[ \|D_{DF}w^{-1}\|_r \leq 2 \left( E(w^{-\frac{2}{r}}) \right)^{\frac{1}{p}} \|D^2 F \otimes_1 DF\|_2, \]

which implies (8.3) by choosing $p \leq \beta/3$ and taking into account (4.3). Notice that we need
\[ 1 \geq \frac{1}{r} \geq \frac{3}{\beta+6} + \frac{1}{2} = \frac{\beta+6}{2\beta}. \]

Consider now the case $k \geq 2$. From the pattern indicated by the first three terms,
\[
\begin{align*}
D_{DF}w^{-1} & = \langle D^{-1}, DF \rangle_{\delta}, \\
D_{DF}^2w^{-1} & = \langle D^{-1}, (DF) \otimes_2 \rangle_{\delta \otimes_2} + \langle D^{-1} \otimes DF, D^2 F \rangle_{\delta \otimes_2}, \\
D_{DF}^3w^{-1} & = \langle D^{-1}, (DF) \otimes_3 \rangle_{\delta \otimes_3} + 3 \langle D^{-1} \otimes DF, D^2 F \otimes DF \rangle_{\delta \otimes_3} \\
& \quad + \langle D^{-1} \otimes D^2 F, D^2 F \otimes DF \rangle_{\delta \otimes_3} + \langle D^{-1} \otimes (DF) \otimes_2, D^3 F \rangle_{\delta \otimes_3},
\end{align*}
\]

we can prove by induction that
\[ |D_{DF}^kw^{-1}| \leq C \sum_{i=1}^k \|D^i w^{-1}\|_{\delta \otimes i} \|DF\|_i \left( \sum_{\sum_{j=1}^k i_j = k-i} \prod_{j=1}^k \|D^j F\|_{\delta \otimes j} \right). \]

By (2.12), for any $p > 1$, $\|D^j F\|_p \leq C \|F\|_2 = C\sigma$. Applying Hölder’s inequality and assuming that $s > r$, we have,
\[ \|D_{DF}^kw^{-1}\|_r \leq C \sum_{i=1}^k \left| \|D^i w^{-1}\|_{\delta \otimes i} \|DF\|_i \right|_{s} \sigma^{k-i}. \]  
(8.6)

We are going to see that $\|DF\|_s$ will contribute to compensate the singularity of $\|D^i w^{-1}\|_{\delta \otimes i}$. First by induction one can prove that for $1 \leq i \leq m$, $D^iw^{-1}$ has the following expression
\[ D^iw^{-1} = \sum_{l=1}^i (-1)^l \sum_{(\alpha, \beta) \in I_{i,l}} w^{-(l+1)} \bigotimes_{j=1}^l (D^\alpha_j F \otimes_1 D^\beta_j F), \]  
(8.7)
where \(I_{i,l} = \{ (\alpha, \beta) \in \mathbb{N}^{2l} : \alpha_j + \beta_j \geq 3, \sum_{j=1}^{l} (\alpha_j + \beta_j) = i + 2l \}. \) In fact, for \(i = 1,\)

\[
Dw^{-1} = -2w^{-2}D^2 F \otimes_1 DF,
\]

which is of the above form because \(I_{1,1} = \{(1,2), (2,1)\}. \) Suppose that (8.7) holds for some \(i \leq m - 1. \) Then,

\[
D^{i+1}w^{-1} = \sum_{l=1}^{i} (-1)^{l+1}2(l+1) \sum_{(\alpha, \beta) \in I_{i,l}} w^{-(l+2)}(D^2 F \otimes_1 DF) \bigotimes_j (D^{\alpha_j} F \otimes_1 D^{\beta_j} F)
\]

\[
+ \sum_{l=1}^{i} (-1)^{l} \sum_{(\alpha, \beta) \in I_{i,l}} w^{-(l+1)} \sum_{h=1}^{l} (D^{\alpha_j+1} F \otimes_1 D^{\beta_j} F + D^{\alpha_j} F \otimes_1 D^{\beta_j+1} F)
\]

\[
\times \bigotimes_{j=1, j\neq h} (D^{\alpha_j} F \otimes_1 D^{\beta_j} F),
\]

which is equal to

\[
\sum_{l=1}^{i+1} (-1)^{l} \sum_{(\alpha, \beta) \in I_{i+1,l}} w^{-(l+1)} \bigotimes_j (D^{\alpha_j} F \otimes_1 D^{\beta_j} F).
\]

From (8.7) for any \(i = 1, \ldots, k\) we can write

\[
\left\| D^i w^{-1} \right\|_{\mathcal{B}^i} \left\| DF \right\|_{\mathcal{B}}^i \leq \sum_{l=1}^{i} w^{-(l+1) + \frac{s}{2}} \sum_{(\alpha, \beta) \in I_{i,l}} \prod_{j=1}^{l} \left\| D^{\alpha_j} F \otimes_1 D^{\beta_j} F \right\|_{\mathcal{B}^{\alpha_j+\beta_j-2}}^2,
\]

(8.8) where \(I_{i,l} = \{ (\alpha, \beta) \in \mathbb{N}^{l} \times \mathbb{N}^{l} : \alpha_j + \beta_j \geq 3, \sum_{j=1}^{l} (\alpha_j + \beta_j) = i + 2l \}. \) Note that by (2.12),

\[
\left\| D^{\alpha_j} F \otimes_1 D^{\beta_j} F \right\|_{p} \leq C \left\| F \right\|_{2}^2 = C\sigma^2
\]

for all \(p \geq 1\) and all \(\alpha_j, \beta_j. \) This inequality will be applied to all but one of the contraction terms in the product \(\prod_{j=1}^{i} \left\| D^{\alpha_j} F \otimes_1 D^{\beta_j} F \right\|_{\mathcal{B}^{\alpha_j+\beta_j-2}}. \) We decompose the sum in (8.8) into two parts. If the index \(l \) satisfies \(l \leq \frac{i}{2} - 1, \) then the exponent of \(w \) is nonnegative, and the \(p\)-norm of \(w \) can be estimated by a constant times \(\sigma^2, \) while for \(\frac{i}{2} - 1 < l \) this exponent is negative. Then, using Hölder’s inequality and assuming that \(\frac{1}{s} = \frac{1}{p} + \frac{1}{2}, \) we obtain

\[
\begin{align*}
\left\| D^i w^{-1} \right\|_{\mathcal{B}^i} \left\| DF \right\|_{\mathcal{B}}^i &\leq C \left( 1_{i \geq 2} \sigma^{i-2} + \sum_{\frac{i}{2}-1 < l \leq i} \left\| w^{-(l+1) + \frac{s}{2}} \right\|_{p}^{\sigma^{2(l-1)}} \right) \left\| D^{\alpha_1} F \otimes_1 D^{\beta_1} F \right\|_{2}^2.
\end{align*}
\]

(8.9) Note that for \(l \leq i \leq k, \ l + 1 - \frac{i}{2} = \frac{k}{2} + 1. \) Therefore, for \(\frac{i}{2} - 1 < l \leq i\)

\[
\left\| w^{-(l+1) + \frac{s}{2}} \right\|_{p} = M_{2(l+1-\frac{i}{2})p}^{2l+2-\frac{i}{2}} \leq M_{(k+2)p}^{2l+2-\frac{i}{2}} \leq M_{(k+2)p}^{k+2},
\]

\(\forall 1. \) Therefore, using (4.3) we obtain

\[
\left\| D^i w^{-1} \right\|_{\mathcal{B}^i} \left\| DF \right\|_{\mathcal{B}}^i \leq C \left( \sigma^{2i-2} \vee 1 \right) (M_{(k+2)p}^{k+2} \vee 1) \left\| q \sigma^2 - w \right\|_{2}.
\]

(8.10)
Combining (8.10) and (8.6) and choosing \( p \) such that \((k + 2)p \leq \beta\) we get (8.4). Note that we need
\[
1 > \frac{1}{r} > \frac{k + 2}{\beta} + \frac{1}{2} = \frac{\beta + 2k + 4}{2\beta},
\]
which holds if \(1 < r < \frac{2\beta}{\beta + 2k + 4}\). The proof of part (iii) is similar and omitted.

The next lemma gives estimates on \( D_u^k \delta_u \) for \( k \geq 0 \).

**Lemma 8.2.** Let \( F = I_q(f) \) with \( q \geq 2 \) satisfying \( E[F^2] = \sigma^2 \). For any \( \beta \geq 1 \) we define
\[
M_{\beta} = \left( E \|DF\|_{\beta}^{\beta} \right)^{1/\beta}
\]
and denote \( w = \|DF\|_{\beta}^2 \).

(i) If \( M_{\beta} < \infty \) for some \( \beta > 3 \), then for any \( 1 < s < \frac{\beta}{3} \),
\[
\|\delta_u\|_s \leq C(\sigma^2 \vee 1)(M_{\beta}^{\beta} \vee 1). \tag{8.11}
\]

(ii) If \( k \geq 1 \) and \( M_{\beta} < \infty \) for some \( \beta > 3k + 3 \), then for any \( 1 < s < \frac{\beta}{3k + 3} \),
\[
\|D_u^k \delta_u\|_s \leq C_{\sigma}(M_{\beta}^{3k+3} \vee 1). \tag{8.12}
\]

(iii) If \( k \geq 2 \) and \( M_{\beta} < \infty \) for some \( \beta > 6k + 6 \), then for any \( 1 < s < \frac{2\beta}{\beta + 6k + 6} \),
\[
\|D_u^k \delta_u\|_s \leq C_{\sigma}(M_{\beta}^{3k+3} \vee 1)\|\sigma^2 - w\|_2. \tag{8.13}
\]

**Proof.** Recall that \( \delta_u = q F w^{-1} - D_DF w^{-1} \). Then for any \( r > s \),
\[
\|\delta_u\|_s \leq C(\sigma \|w^{-1}\|_r + \|D_DF w^{-1}\|_s).
\]

Then, \( \|w^{-1}\|_r = M_{2r}^2 \), and the result follows by applying Lemma 8.1 (iii) with \( k = 1 \) and by choosing \( r < \frac{\beta}{3} \).

To show (ii) and (iii) we need to find a useful expression for \( D_u^k \delta_u \). Consider the operator \( D_u = w^{-1} D_DF \). We claim that for any \( k \geq 1 \) the iterated operator \( D_u^k \) can be expressed as
\[
D_u^k = \sum_{l=1}^{k} w^{-l} \sum_{i \in I_{l,k}} b_i \left[ \prod_{j=1}^{k-l} D_{DF}^{i_j} w^{-1} \right] D_{DF}^{i_0}, \tag{8.14}
\]
where \( b_i > 0 \) are real numbers and
\[
I_{l,k} = \{ i = (i_0, i_1, \ldots, i_l) : i_0 \geq 1, i_j \geq 0 \ \forall \ j = 1, \ldots, l, \sum_{j=0}^{k-l} i_j = k \}.
\]
In fact, this is clearly true for \( k = 1 \). Assume (8.14) holds for a given \( k \). Then
\[
D_u^{k+1} = w^{-1} D_DF D_u^k = \sum_{l=1}^{k} lw^{-l} D_DF w^{-1} \sum_{i \in I_{l,k}} b_i \left[ \prod_{j=1}^{k-l} D_{DF}^{i_j} w^{-1} \right] D_{DF}^{i_0}.
\]
Shifting the indexes, this can be written as

\[ D^{k+1}_u = \sum_{l=1}^{k} lw^{-l} D_{DF}^{w^{-1}} \sum_{i \in I_{i,k}} b_i \left[ \prod_{j=1}^{k-l} D^{i_j w^{-1}}_{DF} \right] D^{i_0}_D \]

\[ + \sum_{l=2}^{k+1} w^{-l} \sum_{i \in I_{i-1,k}} b_i \left[ \prod_{j=1}^{k-l} D^{i_j w^{-1}}_{DF} \right] D^{i_0}_D \]

\[ + \sum_{l=2}^{k+1} w^{-l} \sum_{i \in I_{i-1,k}} b_i \left[ \prod_{j=1}^{k-l} D^{i_j w^{-1}}_{DF} \right] D^{i_0+1}_D. \]

It is easy to check that this coincides with

\[ \sum_{l=1}^{k+1} w^{-l} \sum_{i \in I_{i,k+1}} b_i \left[ \prod_{j=1}^{k-l} D^{i_j w^{-1}}_{DF} \right] D^{i_0}_D. \]

Also, note that \( \delta_u = q F w^{-1} + D_{DF} w^{-1} \) and

\[ D_{DF} \delta_u = q + q F D_{DF} w^{-1} + D_{DF}^2 w^{-1}. \]

By induction we can show that for any \( i_0 \geq 1 \)

\[ D_{DF}^{i_0} \delta_u = q \delta_{1i_0} + q \sum_{j=1}^{i_0-1} c_{i,j} D_{DF}^{i_0-j} w D^{j}_{DF} w^{-1} + q F D_{DF}^{i_0} w^{-1} + D_{DF}^{i_0+1} w^{-1}, \tag{8.15} \]

where \( \delta_{1i_0} \) is the Kronecker symbol. Combining (8.14) and (8.15) we obtain

\[ D^{k}_u \delta_u = \sum_{l=1}^{k} w^{-l} \sum_{i \in I_{i,k}} b_i \left[ \prod_{j=1}^{k-l} D^{i_j w^{-1}}_{DF} \right] \times \left[ q \delta_{1i_0} \right. \]

\[ \left. + q \sum_{j=1}^{i_0-1} c_{i,0j} D_{DF}^{i_0-j} w D^{j}_{DF} w^{-1} + q F D_{DF}^{i_0} w^{-1} + D_{DF}^{i_0+1} w^{-1} \right]. \]

Next we shall apply Hölder’s inequality to estimate \( \| D^{k}_u \delta_u \|_{s} \). Notice that for \( l = k, i_0 = k \geq 2 \). Therefore,

\[ \| D^{k}_u \delta_u \|_{s} \leq C \sum_{l=1}^{k-1} \sum_{i \in I_{i,k}} \| w^{-l} \|_p \prod_{j=1}^{k-l} \| D^{i_j w^{-1}}_{DF} \|_{r_j} \left( \delta_{1i_0} + \max_{1 \leq h \leq k+1} \| D^{h}_{DF} w^{-1} \|_{r_0} \right) \]

\[ + C \| w^{-k} \|_p \max_{1 \leq h \leq k+1} \| D^{h}_{DF} w^{-1} \|_{r_0} = B_1 + B_2, \]
assuming that for \( l = 1, \ldots, k - 1, \frac{1}{s} > \frac{1}{p} + \sum_{j=0}^{k-l} \frac{1}{r_j} \) and \( \frac{1}{s} > \frac{1}{p} + \frac{1}{p_0} \), and where \( C_\sigma \) denotes a function of \( \sigma \) of the form \( C(1 + \sigma^M) \).

Let us consider first the term \( B_1 \). Note that if \( i_0 = 1 \) there is at least one factor of the form \( \| D_{r_j} w^{-1} \|_{r_j} \) in the above product, because \( \sum_{j=1}^{k-l} i_j = k - 1 \geq 1 \). Then, we will apply the inequality (8.4) to one of these factors and the inequality (8.5) to the remaining ones. The estimate (8.5) requires \( \frac{1}{r_j} > \frac{i_j + 2}{\beta} \) for \( j = 1, \ldots, k - l \) and \( \frac{1}{r_0} > \frac{i_0 + 3}{\beta} \). On the other hand, the estimate (8.4) requires \( \frac{1}{r_j} > \frac{i_j + 2}{\beta} + \frac{1}{2} \) for \( j = 1, \ldots, k - l \) and \( \frac{1}{r_0} > \frac{i_0 + 3}{\beta} + \frac{1}{2} \). Then, choosing \( p \) such that \( 2pl < \beta \), and taking into account that \( \sum_{j=0}^{k-l} i_j = k \) we obtain the inequalities

\[
\frac{1}{s} > \frac{1}{p} + \sum_{j=1}^{k-l} \frac{i_j + 2}{\beta} + \frac{i_0 + 3}{\beta} + \frac{1}{2} > \frac{3k + 3}{\beta} + \frac{1}{2}.
\]

Hence, if \( s < \frac{2\beta}{\beta + 6k + 6} \) we can write

\[
B_1 \leq C_\sigma \sum_{l=1}^{k-1} M^{2l}_\beta \prod_{j=1}^{k-l} (M^{i_j+2}_\beta \lor 1)(M^{i_0+3}_\beta \lor 1) \| q_{\sigma^2} - w^{-1} \|_2 \leq C_\sigma (M^{3k+3}_\beta \lor 1) \| q_{\sigma^2} - w^{-1} \|_2.
\]

For the term \( B_2 \) we use the estimate (8.4) assuming \( 2pk < \beta \) and

\[
\frac{1}{s} > \frac{1}{p} + \frac{k + 3}{\beta} + \frac{1}{2} > \frac{3k + 3}{\beta} + \frac{1}{2}.
\]

This leads to the same estimate and the proof of (8.13) is complete. To show the estimate (8.12) we proceed as before but using the inequality (8.5) for all the factors. In this case the summand \( \frac{1}{2} \) does not appear and we obtain (8.12). \( \blacksquare \)

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