L-FUNCTIONS AND SUM-FREE SETS

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Dedicated to the 80th birthday of Endre Szemerédi

Abstract. For set $A \subset \mathbb{F}_p^*$ define by $\text{sf}(A)$ the size of the largest sum-free subset of $A$. Alon and Kleitman [3] showed that $\text{sf}(A) \geq |A|/3 + O(|A|/p)$. We prove that if $\text{sf}(A) - |A|/3$ is small then the set $A$ must be uniformly distributed on cosets of each large multiplicative subgroup. Our argument relies on irregularity of distribution of multiplicative subgroups on certain intervals in $\mathbb{F}_p$.

1. Introduction

Let $G$ be an additive group. For a finite set $A \subseteq G \setminus \{e_G\}$ denote by $\text{sf}^G_k(A)$ the maximal cardinality of a subset of $A$ without any solution to the equation

$$x_1 + \cdots + x_k = y.$$ (1)

If $k = 2$, then sets without solutions to (1) are called sum-free and hence $\text{sf}^G_2(A)$ is just the maximal size of a sum-free subset of $A$. Erdős [6] proved that for every finite set $A \subseteq \mathbb{Z} \setminus \{0\}$ we always have

$$\text{sf}^Z_2(A) \geq |A|/3.$$
This estimate was slightly improved by Alon and Kleitman [3]: \( \text{sf}^2_{\mathbb{Z}}(A) \geq \frac{|A| + 1}{3} \). In fact, they showed that \( \text{sf}^p_{\mathbb{Z}}(A) \geq \frac{|A| + 1}{3} \) for any \( A \subseteq \mathbb{F}_p^* \) provided that \( p \) is a prime of the form \( 3k + 2 \). The best known result was obtained by [4] by Bourgain, who proved
\[
\text{sf}^2_{\mathbb{Z}}(A) \geq \frac{|A| + 2}{3}
\]
for \( n \geq 3 \) using sophisticated Fourier analytical argument. A sequence of results (see for example [3], [17], [1]) provides upper estimates on \( \text{sf}^2_{\mathbb{Z}}(A) \).

However, a breakthrough theorem was obtained by Eberhard, Green and Manners [8] who showed that the constant \( 1/3 \) is optimal by using a very elaborate technique. In a subsequent paper Eberhard [7] gave a simpler proof of this result that also holds for every \( k \geq 2 \) i.e. the optimal constant for \( k \)-sum-free sets is \( 1/(k+1) \). Although it is still unknown whether \( \text{sf}^2_{\mathbb{Z}}(A) = |A|/3 + \psi(|A|) \) for \( \psi(|A|) \to +\infty \) as \( |A| \to +\infty \) it is widely believed to be true. Let us also mention that Bourgain [4] showed
\[
\text{sf}^3_{\mathbb{Z}}(A) \geq \frac{|A|}{4} + \frac{c \log |A|}{\log \log |A|},
\]
where \( c > 0 \) is an absolute constant.

Our main purpose is to study the quantity \( \text{sf}^p_{\mathbb{Z}}(A) \) for \( A \subseteq \mathbb{F}_p^* \) which we denote by \( \text{sf}(A) \).

The main result concerns distribution of sets with small value of \( \text{sf}(A) - |A|/3 \). We show that such sets are equally distributed among cosets of multiplicative subgroups of \( \mathbb{F}_p^* \). We formulate below a slightly weaker theorem, which is directly implied by Theorem 3 and Theorem 8.

THEOREM 1. Let \( p > 3 \) be a prime number and let \( \Gamma \subseteq \mathbb{F}_p^* \) be a multiplicative subgroup with \( -1 \in \Gamma \). Let \( A \) be a subset of \( \mathbb{F}_p \) and suppose that \( \text{sf}(A) = |A|/3 + \psi \). Then for any \( \varepsilon > 0 \) one has
\[
\sum_{\xi \in \mathbb{F}_p^*/\Gamma} \left| |A \cap \xi \Gamma| - \frac{|A||\Gamma|}{p-1} \right|^2 \ll_{\varepsilon} \psi^2 p^{1+\varepsilon}.
\]

It immediately follows from Theorem 1 that if \( A \) is a subset of quadratic residues then it contains a sum-free subset of size \( |A|/3 + \Omega_{\varepsilon}(|A| p^{-1/2-\varepsilon}) \), which improves Bourgain’s bound for \( |A| \gg_{\varepsilon} p^{1/2+2\varepsilon} \).

2. Definitions and preliminaries

Let \( p \) be an odd prime number. We denote by \( \mathcal{Q}, \mathcal{N} \subseteq \mathbb{F}_p^* \) the set of quadratic residues and quadratic non-residues, respectively. Let \( \rho(x) := (\frac{x}{p}) \)
be the Legendre symbol of \( x \in \mathbb{F}_p \). We denote by \( \chi_0 \) the principal (trivial) multiplicative character modulo \( p \). Given a non-trivial character \( \chi \), we write
\[
G(\chi) = \sum_{x \in \mathbb{F}_p} \chi(x)e^{2\pi ix/p}
\]
for the corresponding \textit{Gauss sum} and let
\[
L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}
\]
be the \textit{Dirichlet series} at point \( s \in \mathbb{C} \). By the famous result of Siegel for real multiplicative characters \( \chi \), see, e.g. \[ 18 \] or \[ 9 \], we know that
\[
L(1, \chi) \geq C \varepsilon p^{-\varepsilon},
\]
where \( C \varepsilon > 0 \) is an ineffective constant. The best known effective lower bound is
\[
L(1, \chi) \gg \log p/\sqrt{p},
\]
see \[ 10 \], \[ 14 \] and an excellent survey \[ 11 \]. Using partial summation method one can show a general inequality
\[
\left| L(1, \chi) \right| \ll \log p
\]
which holds for any non-trivial character \( \chi \). Furthermore, it is well-known that
\[
\left| G(\chi) \right| = \sqrt{p}
\]
and
\[
\sum_{x \in \mathbb{F}_p} \chi(x)e^{2\pi imx/p} = \overline{\chi(m)}G(\chi).
\]
Using the Fourier expansion it is not hard to obtain a formula for sums with a character \( \chi \) over the interval \([1, \alpha p]\), where \( \alpha \in (0, 1) \), namely,
\[
S(\alpha) := \sum_{1 \leq x \leq \alpha p} \chi(x) = \frac{G(\chi)}{2\pi i} \sum_{m \neq 0} \frac{\overline{\chi(m)}}{m} (1 - e^{-2\pi im\alpha}).
\]
Let \( \Gamma \subseteq \mathbb{F}_p^* \) be a multiplicative subgroup and put \( n = (p - 1)/|\Gamma| \). Define by \( \mathcal{X} \) the set of all characters \( \chi \) such that \( \chi^n = \chi_0 \), and put \( \mathcal{X}^* = \mathcal{X} \setminus \{\chi_0\} \). Note that a character \( \chi \) belongs to \( \mathcal{X} \) if and only if \( \chi \) equals one on \( \Gamma \). For interval \( I = [p/3, 2p/3] \subset \mathbb{F}_p \) and \( \xi \in \mathbb{F}_p^*/\Gamma \) put
\[
\Delta_\xi = \Delta_\xi(\Gamma) = |\xi \Gamma \cap I| - \frac{|I|}{n}.
\]
Clearly, \( \sum_\xi \Delta_\xi = 0 \). From now on we do not underline that the summation over \( \xi \) is taken over \( \xi \in \mathbb{F}_p^*/\Gamma \). Since \([p/3, 2p/3] = -[p/3, 2p/3]\) modulo \( p \) it follows that \( \Delta_{-\xi} = \Delta_\xi \).

The notation \( A(\cdot) \) always means the characteristic function of set \( A \subseteq \mathbb{F}_p \).

We denote the Fourier transform of a function \( f : \mathbb{F}_p \to \mathbb{C} \) by \( \hat{f} \),
\[
\hat{f}(\xi) = \sum_{x \in \mathbb{F}_p} f(x)e^{-2\pi i \xi x/p}.
\]
The Parseval formula states that

$$\sum_{x \in \mathbb{F}_p} |f(x)|^2 = \frac{1}{p} \sum_{\xi \in \mathbb{F}_p} |\hat{f}(\xi)|^2.$$  

All logarithms are to base 2 and the signs $\ll, \gg$ are the usual Vinogradov symbols.

3. The case of quadratic residues

We begin with a result on irregularity of the distribution of quadratic residues on the largest sum-free subset of $\mathbb{F}_p$.

**Lemma 2.** Let $p$ be a prime number, $p > 3$ and $p \equiv 1 \pmod{4}$. Then for any $\varepsilon \in (0, 1/2)$ there is a constant $C_\varepsilon > 0$ such that

$$|Q \cap [p/3, 2p/3]| - \frac{p}{6} \leq -C_\varepsilon p^{1/2-\varepsilon}.$$  

Furthermore, if $p \equiv 3 \pmod{4}$ then

$$|Q \cap [p/8, 3p/8]| - \frac{p}{8} \leq -C_\varepsilon p^{1/2-\varepsilon}.$$  

**Proof.** Put $\sigma := |Q \cap [p/3, 2p/3]| - \frac{1}{2}([2p/3] - [p/3])$. Clearly, by

$$|Q \cap [p/3, 2p/3]| - |N \cap [p/3, 2p/3]| = \sum_{p/3 \leq x \leq 2p/3} \left( \frac{x}{p} \right)$$  

and

$$|Q \cap [p/3, 2p/3]| + |N \cap [p/3, 2p/3]| = [2p/3] - [p/3]$$  

we have $\sigma = \frac{1}{2} \sum_{p/3 \leq x \leq 2p/3} \left( \frac{x}{p} \right)$. Note that since $p \equiv 1 \pmod{4}$ we have $Q = -Q$, therefore it follows that $\sum_{x \in \mathbb{F}_p} \left( \frac{x}{p} \right) = 0$ and

$$\sigma = - \sum_{1 \leq x < p/3} \left( \frac{x}{p} \right).$$  

It is well-known [20, Theorem 7.4(i)] that

$$\sum_{1 \leq x < p/3} \left( \frac{x}{p} \right) = \frac{\sqrt{3p}}{2\pi} L(1, \rho \chi_3),$$  

where $L(1, \rho \chi_3)$ is the value of the Dirichlet series with product of the Legendre symbols $\rho$ and $\chi_3$ at $s = 1$, where $\chi_3$ is the non-principal character.
modulo 3 (in other words $\chi_3$ is an integer function with period three such that $\chi_3(1) = 1$, $\chi_3(-1) = -1$ and $\chi_3(0) = 0$). By Siegel’s theorem for real multiplicative characters $\chi$, we know that $L(1, \chi) \gg \varepsilon p^{-\varepsilon}$, which completes the proof of (6).

Next, assume that $p \equiv 3 \pmod{4}$. Let $\chi_8$ be the non-principal real character modulo 8 (in other words $\chi_8$ is an integer function with period eight such that $\chi_8(1) = \chi_8(-1) = 1$, $\chi_8(3) = \chi_8(-3) = -1$ and $\chi_8$ vanishes otherwise) and let $\delta_4$ be a function such that $\delta_4(n) = 1$ if and only if $n \equiv 4 \pmod{8}$ and zero otherwise. We have

$$\cos \frac{\pi n}{4} = \frac{\chi_8(n)}{\sqrt{2}} + \delta_0(n) - \delta_4(n), \quad \cos \frac{3\pi n}{4} = -\frac{\chi_8(n)}{\sqrt{2}} + \delta_0(n) - \delta_4(n)$$

By using the Fourier expansion (3) of the characteristic function of interval $[0, \alpha p]$ one can easily obtain

$$S(\alpha) = \frac{\sqrt{p}}{\pi} \left( L(1, \rho) - \sum_{n \geq 1} \left( \frac{n}{p} \right) \frac{\cos 2\pi n \alpha}{n} \right).$$

By (9) and (10), we derive

$$S(3/8) - S(1/8) = \frac{\sqrt{2p}}{\pi} L(1, \rho \chi_8).$$

Again, to conclude the proof it is enough to apply Siegel’s theorem. $\square$

Now we are ready to prove our first main result. We make use of an observation that if $Q$ or $N$ has a large sum-free set and since $Q$ is a multiplicative subgroup then the same should be true for all relatively large subsets of $Q$ or $N$.

**Theorem 3.** Let $A \subseteq \mathbb{F}_p^*$, $p \equiv 1 \pmod{4}$ and suppose that $\text{sf}(A) = \frac{|A|}{3} + \psi$. Then for any $\varepsilon > 0$ one has

$$\left| \sum_{x \in A} \left( \frac{x}{p} \right) \right| \ll_\varepsilon \psi p^{1/2+\varepsilon}$$

or equivalently $\left| |A \cap Q| - |A \cap N| \right| \ll_\varepsilon \psi p^{1/2+\varepsilon}$. Furthermore, if $p \equiv 3 \pmod{4}$ and $\text{sf}_3(A) = |A|/4 + \psi$, then (11) holds as well.

**Proof.** Let $A_Q = A \cap Q$, $A_N = A \cap N$ and put $I = [p/3, 2p/3]$. Let us define $\Delta$ by

$$\Delta := \frac{1}{2} \left( \left| \frac{2p}{3} \right| - \left| \frac{p}{3} \right| \right) - \left| Q \cap I \right| = \left| N \cap I \right| - \frac{1}{2} \left( \left| \frac{2p}{3} \right| - \left| \frac{p}{3} \right| \right).$$
From the previous lemma we know that $\Delta$ is positive and $\Delta \gg \varepsilon p^{1/2-\varepsilon}$. We have

$$\sigma_1 := |N|^{-1} \sum_{x \in N} |xA \cap I| = |N|^{-1} \sum_{x \in N} (|xA_Q \cap I| + |xA_N \cap I|)$$

$$= \frac{|A_Q|}{|N|} |N \cap I| + \frac{|A_N|}{|N|} |Q \cap I| = \frac{|A|}{3} + \Delta \frac{|A_Q| - |A_N|}{|N|}$$

$$= \frac{|A|}{3} + \frac{\Delta}{|N|} \sum_{x \in A} \left( \frac{x}{p} \right) + O(1).$$

Similarly,

$$\sigma_2 := |Q|^{-1} \sum_{x \in Q} |xA \cap I| = |Q|^{-1} \sum_{x \in Q} (|xA_Q \cap I| + |xA_N \cap I|)$$

$$= \frac{|A_Q|}{|Q|} |Q \cap I| + \frac{|A_N|}{|Q|} |N \cap I| = \frac{|A|}{3} + \Delta \frac{|A_N| - |A_Q|}{|Q|}$$

$$= \frac{|A|}{3} - \frac{\Delta}{|Q|} \sum_{x \in A} \left( \frac{x}{p} \right) + O(1).$$

However, $I$ is a sum-free set, hence for any $x \in \mathbb{F}_p^*$, $|xA \cap I|$ does not exceed $\text{sf}(A)$. Therefore, both $\sigma_1, \sigma_2$ are at most $\text{sf}(A)$, which proves (11). The second assertion can be shown in a similar manner with the interval $J = [p/8, 3p/8)$, that has no solutions to $x_1 + x_2 + x_3 = y$. \hfill \Box

**Corollary 4.** Let $p \equiv 1 \pmod{4}$ and let $A \subseteq Q$ or $A \subseteq N$. Then for every $\varepsilon > 0$ a positive constant $C_\varepsilon$ exists such that

$$\text{sf}(A) \geq \frac{|A|}{3} + C_\varepsilon |A| p^{-1/2-\varepsilon}. \quad (12)$$

4. The case of arbitrary multiplicative subgroups

The study of distribution of an arbitrary multiplicative subgroup $\Gamma$ on sum-free intervals must be handled in a different way. It is difficult to determine the sign of the discrepancy of intersections of $\Gamma$ the intervals $I, J$ as we did in Lemma 2. Nevertheless, it is possible to find $L_2$-norm of such discrepancy on cosets of $\Gamma$. The intervals $I$ and $J$ suit our problem perfectly because we can deal with intervals that have rational number endpoints with small specific denominators, see discussion in [15, Section 4].
Let us recall that $n = (p - 1)/|\Gamma|$, $\mathcal{X} = \{\chi : \chi^n = \chi_0\}$, $\mathcal{X}^* = \mathcal{X} \setminus \{\chi_0\}$ and

$$\Delta_\xi = \Delta_\xi(\Gamma) = |\xi \Gamma \cap I| - \frac{|I|}{n}.$$ 

**Proposition 5.** Let $\Gamma \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup and let $\varepsilon \in (0,1)$ be any number. Then

$$p \log^2 p \gg \sum_\xi \Delta^2_\xi \gg \varepsilon \left(1 - \frac{2}{n}\right)p^{1 - \varepsilon}.$$ 

Moreover, if $n \neq 2, 4$ then

$$\sum_\xi \Delta^2_\xi \gg \frac{p}{\log^2 p}.$$ 

The above inequalities hold for the discrepancy on the interval $J$ provided that $-1 \not\in \Gamma$.

**Proof.** We have

$$\Gamma(x) = \frac{1}{n} \sum_{\chi \in \mathcal{X}} \chi(x) = \frac{1}{n} \sum_{\chi \in \mathcal{X}^*} \chi(x) + \frac{\chi_0(x)}{n},$$

hence

$$\Delta_\xi = |\xi \Gamma \cap I| - \frac{|I|}{n} = \frac{1}{n} \sum_{\chi \in \mathcal{X}^*} \overline{\chi}(\xi) \sum_{p/3 \leq x \leq 2p/3} \chi(x).$$

Using the expansion (3), we obtain

$$\sum_{p/3 \leq x \leq 2p/3} \chi(x) = S(2/3) - S(1/3) = \frac{G(\chi)}{2\pi i} \sum_{m \neq 0} \frac{\overline{\chi}(m)}{m} \left(e^{-2\pi im/3} - e^{-4\pi im/3}\right)$$

$$= -\sqrt{3}G(\chi) \sum_{m \neq 0} \frac{\overline{\chi}(m)\chi_3(m)}{m} = -\sqrt{3}G(\chi) L(1, \overline{\chi_3})(1 + \overline{\chi}(-1)),$$

hence by (15)

$$\Delta_\xi = -\frac{\sqrt{3}}{2\pi n} \sum_{\chi \in \mathcal{X}^*} (1 + \overline{\chi}(-1))G(\chi)L(1, \overline{\chi_3})\overline{\chi}(\xi).$$
Thus, by the Parseval formula applied for the quotient group $\mathbb{F}_p^*/\Gamma$

\begin{equation}
\sum_{\xi} \Delta_{\xi}^2 = \frac{3}{4\pi^2 n} \sum_{\chi \in \mathcal{X}^*} |1 + \overline{\chi}(-1)|^2 |G(\chi)|^2 |L(1, \overline{\chi}\chi)|^2
\end{equation}

\begin{equation}
= \frac{3p}{4\pi^2 n} \sum_{\chi \in \mathcal{X}^*} |1 + \overline{\chi}(-1)|^2 |L(1, \overline{\chi}\chi)|^2.
\end{equation}

By Siegel’s Theorem, we have

\begin{equation}
\sum_{\xi} \Delta_{\xi}^2 \gg \frac{p^{1-\varepsilon}}{n} \sum_{\chi \in \mathcal{X}^*} |1 + \overline{\chi}(-1)|^2
\end{equation}

\begin{equation}
= \frac{p^{1-\varepsilon}}{n} \left(2n - 2 + \sum_{\chi \in \mathcal{X}^*} (\overline{\chi}(-1) + \chi(-1)) \right) \geq \left(2 - \frac{4}{n}\right)p^{1-\varepsilon},
\end{equation}

where the last inequality follows from equations $\sum_{\chi \in \mathcal{X}^*} \chi(-1) = -1$ if $-1 \not\in \Gamma$ and $\sum_{\chi \in \mathcal{X}^*} \chi(-1) = n - 1$ if $-1 \in \Gamma$.

The upper bound in (13) follows from (17) and from a general inequality $|L(1, \chi)| \ll \log p$, which holds for every non-trivial character $\chi$.

Now let us assume that $n \not= 2, 4$. We only bound from below the subsum of (17) over non-quadratic characters. It was proven in [18, Theorems 11.4 and 11.11] that $|L(1, \chi)| \gg \frac{1}{\log p}$ for any complex character $\chi$, so

\begin{equation}
\sum_{\xi} \Delta_{\xi}^2 \gg \frac{p}{n \log^2 p} (2n - 4 - 4) \gg \frac{p}{\log^2 p} \left(1 - \frac{4}{n}\right).
\end{equation}

If $n = 3$ then we have $-1 \in \Gamma$ hence $\sum_{\chi \in \mathcal{X}^*} \chi(-1) = n - 1 = 2$ and thus (14) is also satisfied.

Next, we prove the last part of our proposition. Applying formula (3) again (one can use identity (9) as well), we obtain

\begin{equation}
\sum_{p/8 \leq x \leq 3p/8} \chi(x) = S(3/8) - S(1/8) = \frac{G(\chi)}{2\pi i} \sum_{m \not= 0} \overline{\chi}(m) \frac{\pi i}{m} \left( e^{-\pi im/4} - e^{-3\pi im/4} \right)
\end{equation}

\begin{equation}
= \frac{G(\chi)}{\sqrt{2\pi i}} \sum_{m \not= 0} \overline{\chi}(m) \chi_8(m) + \frac{G(\chi)}{\pi} \left( \sum_{m \equiv -2 \pmod{8}} \overline{\chi}(m) - \sum_{m \equiv 2 \pmod{8}} \overline{\chi}(m) \right)
\end{equation}

\begin{equation}
= \frac{G(\chi)}{\sqrt{2\pi i}} (1 - \overline{\chi}(-1)) \sum_{m \geq 1} \overline{\chi}(m) \chi_8(m) - \frac{G(\chi)}{\pi} (1 + \overline{\chi}(-1)) \sum_{m \equiv 2 \pmod{8}} \overline{\chi}(m)
\end{equation}
\[
\frac{G(\chi)}{\sqrt{2\pi i}} \left( 1 - \overline{\chi}(-1) \right) \sum_{m \geq 1} \frac{\overline{\chi}(m)\chi_8(m)}{m} \\
- \frac{G(\chi)\overline{\chi}(2)}{2\pi} \left( 1 + \overline{\chi}(-1) \right) \sum_{m \equiv 1 \pmod{4}} \frac{\overline{\chi}(m)}{m}.
\]

Let \( \mathcal{X}_o^* \) be the set of all odd characters from \( \mathcal{X}^* \), then from \( -1 \not\in \Gamma \) it follows that \( |\mathcal{X}_o^*| = n/2 \). Hence for all \( \chi \in \mathcal{X}_o^* \), we have

\[
(19) \quad \sum_{x=p/8 \leq x \leq 3p/8} \chi(x) = \frac{G(\chi)\sqrt{2}}{\pi i} \sum_{m \geq 1} \frac{\overline{\chi}(m)\chi_8(m)}{m} = \frac{G(\chi)\sqrt{2}}{\pi i} L(1, \overline{\chi}\chi_8).
\]

Thus, by (19) one obtains

\[
\sum_{\xi} \left| \xi \Gamma \cap J \right| - \left| J \right| \frac{1}{n} \gg \frac{p}{n} \sum_{\chi \in \mathcal{X}_o^*} |L(1, \overline{\chi}\chi_8)|^2
= \frac{p}{n} \left( |L(1, \rho\chi_8)|^2 + \sum_{\chi \in \mathcal{X}_o^* \setminus \{\rho\}} |L(1, \overline{\chi}\chi_8)|^2 \right) \gg \varepsilon \frac{1}{p^{1-\varepsilon}} \quad \square
\]

**Remark 6.** It was proven in [18, p. 366] that for odd characters \( \chi \in \mathcal{X}^* \) one has \( |L(1, \chi)| \gg (\log p)^{-\cos(\pi/n)} \). Thus, for small \( n \) the estimate (14) can be further improved. It is also well-known that under the Generalized Riemann Hypothesis a stronger inequality is satisfied

\[
(\log \log p)^{-1} \ll |L(1, \chi)| \ll \log \log p.
\]

For any non-trivial character \( \chi \) one has \( |L(1, \chi)| \ll \log p \), therefore by (16) it follows that \( |\Delta_\xi| \ll \sqrt{p} \log p \). Thus, we can derive the following corollary from Proposition 5.

**Corollary 7.** If \( n \neq 2, 4 \), then there exist \( \xi, \eta, \omega \in \mathbb{F}_p^*/\Gamma \) such that \( |\Delta_\xi| \gg \sqrt{p}/(n \log^3 p) \), \( |\Delta_\eta| \ll -\sqrt{p}/(n \log^3 p) \) and \( |\Delta_\omega| \gg \sqrt{p}/(\sqrt{n} \log p) \).

Our next theorem provides an analogous estimate to (11) for an arbitrary multiplicative subgroup.

**Theorem 8.** Let \( \Gamma \subseteq \mathbb{F}_p^* \) be a multiplicative subgroup, let \( A \) be a subset of \( \mathbb{F}_p \) and suppose that \( sf(A) = |A|/3 + \psi \). Then for any \( \varepsilon > 0 \) and for every even character \( \eta \in \mathcal{X}^* \) one has

\[
|\sum_{x \in A} \eta(x)| \ll \varepsilon \psi n^{1/2} p^{1/2+\varepsilon}.
\]

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Proof. Without losing generality, we can assume that 0 \notin A. For \xi \in \mathbb{F}_p^*/\Gamma put A_\xi = \xi \Gamma \cap A and

\[ a_\xi = |A_\xi| - \frac{|A||\Gamma|}{p-1} = |\xi \Gamma \cap A| - \frac{|A|}{n}. \]

Clearly, \sum_\xi a_\xi = 0. First, let us assume that \(-1 \notin \Gamma\), then the quotient group \mathbb{F}_p^*/\Gamma can be written as \mathbb{F}_p^*/\Gamma = H \sqcup (-H) for some set \( H \subset \mathbb{F}_p^*/\Gamma \) with \( H \cap (-H) = \emptyset \). By our assumption \( \eta(-1) = 1 \), we deduce that

\[ \sum_{x \in A} \eta(x) = \sum_{\xi} \sum_{x \in A_\xi} \eta(x) = \sum_{\xi} \eta(\xi)|A_\xi| = \sum_{\xi} \eta(\xi)a_\xi \]

\[ = \sum_{\xi \in H} \eta(\xi)(a_\xi + \eta(-1)a_{-\xi}) = \sum_{\xi \in H} \eta(\xi)(a_\xi + a_{-\xi}). \]

and by the Cauchy–Schwarz inequality

\begin{equation}
\left| \sum_{x \in A} \eta(x) \right|^2 \leq \frac{1}{2} n \cdot \sum_{\xi} (a_\xi + a_{-\xi})^2.
\end{equation}

If \(-1 \in \Gamma\) then \( a_\xi = a_{-\xi} \) and the inequality (21) holds as well.

Again let \( I = [p/3, 2p/3) \). For any \( \alpha \in \mathbb{F}_p^*/\Gamma \) in view of \( \sum_\xi a_\xi = \sum_\xi \Delta_\xi = 0 \), we get

\begin{equation}
|\Gamma|^{-1} \sum_{x \in \alpha \Gamma} |xA \cap I| = |\Gamma|^{-1} \sum_{\xi} \sum_{x \in \alpha \Gamma} |xA_\xi \cap I|
\end{equation}

\[ = |\Gamma|^{-1} \sum_{\xi} |A_\xi|(\Delta_\alpha + |I|/n) = \frac{|A||I|}{p-1} + |\Gamma|^{-1} \sum_{\xi} |A_\xi|\Delta_\alpha \xi \]

\[ = \frac{|A||I|}{p-1} + |\Gamma|^{-1} \sum_{\xi} a_\xi \Delta_\alpha_\xi. \]

By \( sf(A) = |A|/3 + \psi \) the left-hand side of (22) does not exceed \( |A|/3 + \psi \), which implies that for any \( \alpha \in \mathbb{F}_p^*/\Gamma \) the following inequality holds

\[ \sum_\xi a_\xi \Delta_\alpha_\xi \leq \psi|\Gamma| + O\left(|A||\Gamma|/p\right) . \]

Since \( \sum_\xi a_\xi = \sum_\xi \Delta_\xi = 0 \) it follows that \( \sum_\alpha \sum_\xi a_\xi \Delta_\alpha_\xi = 0 \) and

\[ \sum_\alpha \left| \sum_\xi a_\xi \Delta_\alpha_\xi \right| \leq 2\psi|\Gamma|n + O(|A|/p) \]

\[ \leq 2\psi p + O(|A|) \ll \psi p \]
because $\psi \gg 1$. Splitting the set $\mathbb{F}_p^*/\Gamma = H \cup (-H)$ if it is possible and using the property $\Delta_{\xi} = \Delta_{-\xi}$ and the last inequality we deduce that

$$
(23) \quad \sum_{\alpha} \left| \sum_{\xi} a_{\xi} \Delta_{\alpha} \xi \right|^2 = \sum_{\alpha \in H} \left| \sum_{\xi} \Delta_{\alpha} \xi (a_{\xi} + a_{-\xi}) \right|^2
$$

$$
\ll \psi p \cdot \max_{\alpha \in H} \left| \sum_{\xi} \Delta_{\alpha} \xi (a_{\xi} + a_{-\xi}) \right|.
$$

Again, if $-1 \in \Gamma$, then (23) also takes place. Therefore, by the upper bound (13) and (23) we have

$$
(24) \quad \sum_{\alpha} \left| \sum_{\xi} a_{\xi} \Delta_{\alpha} \xi \right|^2 \ll \psi \left( \sum_{\xi} (a_{\xi} + a_{-\xi})^2 \right)^{1/2} p^{3/2} \log p.
$$

On the other hand, from (16) and the Parseval identity, we obtain

$$
(25) \quad \sum_{\alpha} \left| \sum_{\xi} a_{\xi} \Delta_{\alpha} \xi \right|^2 = \frac{3}{4\pi^2 n} \sum_{\chi \in \chi^*} \left| 1 + \chi(-1) \right|^2 |G(\chi)|^2 |L(1, \chi \chi_3)| \left| \sum_{\xi} a_{\xi} \chi(\xi) \right|^2.
$$

Applying Siegel’s Theorem one more time and using $\sum_{\xi} a_{\xi} = 0$, we get

$$
(26) \quad \sum_{\alpha} \left| \sum_{\xi} a_{\xi} \Delta_{\alpha} \xi \right|^2 \gg \epsilon \frac{p^{1-\epsilon}}{n} \sum_{\chi \in \chi^*} \left| 1 + \chi(-1) \right|^2 \left| \sum_{\xi} a_{\xi} \chi(\xi) \right|^2
$$

$$
= \frac{p^{1-\epsilon}}{n} \sum_{\chi \in \chi^*} \left| 1 + \chi(-1) \right|^2 \left| \sum_{\xi} a_{\xi} \chi(\xi) \right|^2.
$$

Expanding the last sum, we derive

$$
(27) \quad \sum_{\alpha} \left| \sum_{\xi} a_{\xi} \Delta_{\alpha} \xi \right|^2 \gg \epsilon \frac{p^{1-\epsilon}}{n} \sum_{\xi} (a_{\xi} + a_{-\xi})^2.
$$

Combining (24) and (27) we have

$$
\sum_{\xi} (a_{\xi} + a_{-\xi})^2 \ll \epsilon \psi^2 p^{1+3\epsilon}.
$$

Substituting the last formula in (21), we obtain

$$
\left| \sum_{x \in A} \chi(x) \right|^2 \ll \epsilon n \psi^2 p^{1+3\epsilon}
$$

as required.  \(\square\)
The last result in this section provides an estimate of $\text{sf}(A)$ for sets with a very small product set.

**Corollary 9.** Let $A \subseteq \mathbb{F}_p$ be a set such that $|A| \leq \delta p/16$ and $|AA| \leq (2 - \delta)|A|$ for some $\delta \in (0, 1]$. Suppose that $\text{sf}(A) = |A|/3 + \psi$ then for any $\varepsilon > 0$ one has

$$\psi \gg_\varepsilon \delta^{1/2}|A|p^{-1/2-\varepsilon}.$$

**Proof.** Kneser’s theorem [16] implies that there is a subgroup $\Gamma \subseteq \mathbb{F}_p^*$ with $|\Gamma| \leq (2 - \delta)|A|$ such that $AA$ is covered by at most $\frac{2}{\delta} - 1 := t$ translates of $\Gamma$. Put $\Gamma_* = \Gamma \cup (-\Gamma)$ then clearly $A \subseteq \bigcup_{x \in X} x\Gamma_*$ for some set $X$ of size at most $t$. Using the Cauchy–Schwarz inequality, Theorem 1 and our assumption $|A| \leq \delta p/16$, we have

$$\delta|A|^2 8 \leq \frac{|A|^2}{4|X|} \leq \frac{1}{|X|} \left( |A| - \frac{|A||X||\Gamma_*|}{p - 1} \right)^2 \leq \sum_{\xi \in X} \left| A \cap \xi\Gamma_* \right| - \frac{|A||\Gamma_*|}{p - 1} \ll_\varepsilon \psi^2 p^{1+\varepsilon}. \quad \square$$

5. **Sum-free subsets in multiplicative subgroups**

In view of the results in the previous section a natural problem of determining $\text{sf}(\Gamma)$ for a multiplicative subgroup $\Gamma \subseteq \mathbb{F}_p^*$ arises. It is well-known that large multiplicative subgroups are pseudo-random sets, as they have small Fourier coefficients. Therefore, one can expect that $\text{sf}(\Gamma) \leq (1/3 + o(1))|\Gamma|$. Our next theorem will confirm this intuition. The idea behind the proof is that if $\Gamma$ contains a large sum-free set, then we show that there is a sum-free set of $\mathbb{F}_p$ roughly with the same density. Our argument is based on Fourier approximation method.

**Theorem 10.** Let $\Gamma \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup such that $|\Gamma| \gg (\log \log p)^2 / \sqrt{\log p} p$. Then

$$\text{sf}(\Gamma) \leq (1/3 + o(1))|\Gamma|.$$

**Proof.** Let us recall a simple bound on the Fourier coefficients of a multiplicative subgroup. For every $\xi_0 \in \mathbb{F}_p^*$, we have

$$|\Gamma||\hat{\Gamma}(\xi_0)|^2 \leq \sum |\hat{\Gamma}(\xi)|^2 = p|\Gamma|,$$

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hence $|\hat{\Gamma}(\xi_0)| \leq \sqrt{p}$, and it follows that

$$|m\hat{\Gamma}(\xi) - \hat{\mathbb{F}_p}(\xi)| \leq m \max_{\xi \neq 0} |\hat{\Gamma}(\xi)| \leq m\sqrt{p},$$

where $m := p/|\Gamma|$. Let $A$ be a sum-free subset of $\Gamma$ of the maximum size and put $|A| = \gamma|\Gamma|$. We show that there exists a sum-free subset in $\mathbb{F}_p$ roughly with the same density. For $\theta > 0$ define by

$$\text{Spec} = \text{Spec}_{\theta}(A) := \{ \xi : |\hat{A}(\xi)| \geq \theta|A| \}$$

the $\theta$-spectrum of $A$ and let

$$B = B(\text{Spec}, \theta) = \{ b \in \mathbb{F}_p : ||b\xi/p|| \leq \theta \text{ for every } \xi \in \text{Spec} \}.$$

Furthermore, let us denote $\beta(x) = \frac{1}{|B|}B(x)$, $a(x) = m \cdot A(x)$, $g(x) = m \cdot \Gamma(x)$ and define

$$f = a \ast \beta \ast \beta.$$

Notice that $\hat{f} = \hat{a} \cdot \hat{\beta}^2$, hence

$$(28) \quad |\hat{a}(\xi) - \hat{f}(\xi)| = |\hat{a}(\xi)||1 - \hat{\beta}(\xi)^2| \ll \theta m|A|,$$

for any $\xi \in \mathbb{F}_p$. We will use the function $f$ to construct a subset of $\mathbb{F}_p$ with similar properties and therefore we need to estimate $\|f\|_1$ and bound $f$ from above. By Fourier inversion and by (28), we have

$$\left| \sum_t f(t) - m|A| \right| = \left| \sum_t f(t) - \sum_t a(t) \right| \ll \theta m|A|,$$

and

$$f(t) = \frac{1}{p} \sum_\xi \hat{f}(\xi)e(-\xi t/p)$$

$$= \frac{1}{p} \sum_\xi \hat{a}(\xi)\hat{\beta}^2(\xi)e(-\xi t/p) \leq \frac{1}{p} \sum_\xi \hat{g}(\xi)\hat{\beta}^2(\xi)e(-\xi t/p)$$

$$\leq \frac{1}{p} \sum_\xi \hat{\mathbb{F}_p}(\xi)\hat{\beta}^2(\xi)e(-\xi t/p) + \frac{1}{p} m\sqrt{p} \sum_\xi |\hat{\beta}(\xi)|^2 \leq 1 + \frac{m\sqrt{p}}{|B|} =: 1 + \delta.$$
For a function \( w : \mathbb{F}_p \to \mathbb{R} \) put
\[
T(w) := \sum_{x+y=z} w(x)w(y)w(z).
\]
We show by (28) that \( T(f) \) is small
\[
T(f) = T(f) - T(a) = \frac{1}{p} \sum_{\xi} \hat{f}(\xi)|\hat{f}(\xi)|^2 - \frac{1}{p} \sum_{\xi} \hat{a}(\xi)|\hat{a}(\xi)|^2
\]
\[
\ll \frac{1}{p} \sum_{\xi} \hat{a}(\xi)(|\hat{f}(\xi)|^2 - |\hat{a}(\xi)|^2) + \frac{1}{p} \theta m |A| \sum_{\xi} |\hat{f}(\xi)|^2 \ll \theta m^3 |A|^2.
\]
Let
\[
h := \frac{1}{1+\delta} \cdot f
\]
then we see that \( h : \mathbb{F}_p \to [0,1], \sum_t h(t) = m|A| + O(\theta m|A|) \) and \( T(h) \ll \theta m^3 |A|^2 \). By a corollary to Beck–Spencer theorem (see [13]) there is a set \( S \subseteq \mathbb{F}_p \) such that
\[
|S| = \left\lfloor \sum_t h(t) \right\rfloor = m|A| + O(\theta m|A|)
\]
and for every \( \xi \in \mathbb{F}_p \)
\[
|\widehat{S}(\xi) - \hat{h}(\xi)| \ll \sqrt{p}.
\]
By the above property, we have
\[
|T(S) - T(a)| \leq |T(S) - T(h)| + (1+\delta)^{-3}|T(f) - T(a)| \ll \theta m|S|^2.
\]
Now it is sufficient to apply arithmetic removal lemma [12] to find set \( S' \subseteq S \) such that
\[
|S'| \geq (1-\varepsilon)|S| \geq (1-\varepsilon + O(\theta))m \text{sf}(\Gamma)
\]
and \( S' \) does not contain any solution to \( x + y = z \), where \( \varepsilon \to 0 \) as \( \theta m \to 0 \).
However, it immediately follows from the Cauchy–Davenport theorem that every sum-free set in \( \mathbb{F}_p \) is of size at most \((p+1)/3\), so
\[
\text{sf}(\Gamma) \leq (1 + O(\varepsilon)) \frac{|S'|}{m} \leq (1/3 + O(\varepsilon + \theta))|\Gamma|.
\]
To finish the proof it is enough to find a choice of \( \theta \) such that \( \theta m \to 0 \) and \( \delta \leq \theta \). We can assume that \( |A| \geq \frac{|\Gamma|}{4} \geq \frac{p}{4m} \) hence by Chang’s spectral
lemma [5] and by a well-known lower bound for the size of a Bohr set, we have
\[ |B| \geq (\theta/C \log m)^{C\theta^{-2}\log m/p}, \]
for some constant $C > 0$. We show that one can take $\theta = c\log \log p / (\log \log p)^{\gamma}$ for a sufficiently small constant $c > 0$. By our assumption $m \ll \sqrt[\gamma]{\log p} (\log \log p)^2$, so $\theta m \to 0$ and
\[ \delta = \frac{m\sqrt{p}}{|B|} \leq \frac{m}{(\theta/C \log m)^{C\theta^{-2}\log m/p}} \leq \theta, \]
provided that $c$ is small enough. □

Remark 11. To keep the statement of Theorem 10 simple we pick a suboptimal $m$; the optimal choice is $m = o(\theta^{-1})$. It is worth noting that Alon and Bourgain [2] constructed a sum-free subgroup of $\mathbb{F}_p^*$ of size $\gg p^{1/3}$.

6. Concluding remarks

Bourgain showed [4] that
\[(29) \quad \text{sf}_2^\mathbb{Z}(A) \geq \frac{|A|}{3} + \frac{c\|A\|_{RW}}{\log |A|}, \]
where $c > 0$ is an absolute constant and $\|A\|_{RW} = \int_0^1 |\sum_{a \in A} \cos(2\pi ax)| \, dx$. One can verify that the same argument works in the finite fields and the inequality (29) holds for $\text{sf}(A)$ with
\[ \|A\|_{RW} = p^{-1} \sum_x \left| \sum_{a \in A} \cos(2\pi ax/p) \right|. \]
Note that $\|A\|_{RW}$ can differ a lot from the usual Wiener norm $\|A\|_W = p^{-1} \sum_x |\hat{A}(x)|$, for example if $p \equiv 3 \pmod{4}$ then $\|Q\|_{RW} < 1$, while $\|Q\|_W \gg \sqrt{p}$. However, for symmetric sets $A = -A$ we have $\|A\|_{RW} = \|A\|_W$ and whence if $\text{sf}(A) = |A|/3 + \psi$ then
\[ \left| \xi \Gamma \cap A - \frac{|A||\Gamma|}{p} \right| = \frac{1}{p} \sum_{x \neq 0} \hat{A}(x) \hat{\Gamma}(x\xi) \leq \|A\|_W \cdot \max_{x \neq 0} |\hat{\Gamma}(x)|, \]
so
\[ \left| \xi \Gamma \cap A - \frac{|A||\Gamma|}{p} \right| \ll \psi \log |A| \cdot \max_{x \neq 0} |\hat{\Gamma}(x)|. \]
Nevertheless, the inequality above does not improve the $L_2$-bound given in (2).
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