The affine ambitwistor space as the moduli space of SUYM in $AdS_5 \otimes S^5$

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Abstract

By extending the dressing symmetric action of IIB string in $AdS_5 \times S^5$ to the $D_3$ brane, we find a gauged WZW action of Higgs Yang-Mills field including the 2-cocycle of axial anomaly. The left and right twistor structure of left and right $\alpha$-planes glue into an ambitwistor. The symmetry group of Nahm equations is central extended to an affine group, thus we explain why the spectral curve is given by affine Toda.

1 Introduction

Bena, Polchinsky and Roiban [1] and Polyakov [2] show the integrability of IIB string in $AdS_5 \times S^5$. Using Metsaev Tseytlin’s action [3], we find in the $\kappa$ symmetric Killing gauge, the $S$ duality between geometric Maurer Cartan equations and dynamic equations of motion [4]. Then the reparametrization symmetry permits a dual twisted transformation, so dictates the dressing symmetry. The left and right moving worldsheet of string are chiral embedded in $AdS_5 \times S^5$. By imposing the 2-cocycle of axial

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anomaly on Roiban Siegel’s action \[5\], we get the gauged WZW action \[6\], which dictates the conformal affine Toda model\[6\]. We find the solitonic string solutions, its moduli space realize the moduli of IIB string.

In this paper, we extended this to the super Yang-Mills theory in \(D_3\) brane world sheet. At first, we shortly review the action for ADHM’s anti self-dual Yang-Mills and the action for Nahm-Drinfeld-Hitchin-Atiyah’s construction of BPS monopole. The twistor structure is manifest, since in all the \(\alpha\) plane, the anti self-dual Yang-Mills field becomes a pure gauge. Next, from the Hamiltonian reduction induced by chiral vacuum expectation value of Higgs field, we obtain a gauged WZW with ambitwistor structure, the \(TCP_3 \times \overline{TCP}_3\) bundle. It happens that the \(D_3\) brane and IIB string shares a common covariant constant quaternion 4-bein field embedded in the hyper-Kähler structure inherited by \(AdS_5\) background. This is exhibited by a common Robinson congruence, its dilation and twist spin coefficients are given by the modulus and phase of the level \(\mu\) of hamiltonian reduction. So the zweibein of string is uniquely completed to the 4-bein of brane. At last, we very shortly sketch the original NDHA construction with a finite Toda type spectral curve and suggest the way of affiniz ation to a conformal affine Toda. And how the moduli spaces of stretched string soliton corresponds to the moduli of twisted monopole.

2 Action of self-dual YM and twistor

Firstly we consider the Lagrangian formulation of self-dual Yang-Mills theory in four dimensional Euclidean space. We start from the \(sp(4)\) gauge field

\[
A_\mu = A_\mu^a t^a / 2i, \quad F_{\mu\nu} = F_{\mu\nu}^a t^a / 2i = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],
\]

where \(t^a\) are Hermitian generators of \(sp(4)\). As in Yang’s gauge \[7\], we adapt the complexified coordinates of Minkowski space

\[
\sqrt{2}y = x_1 + ix_2 \equiv x^{11}; \quad \sqrt{2}\bar{y} = x_1 - ix_2 \equiv x^{22};
\]

\[
\sqrt{2}z = -x_3 + ix_4 \equiv x^{12}; \quad \sqrt{2}\bar{z} = x_3 + ix_4 \equiv x^{21}.
\]

and the metric becomes

\[
ds^2 = 2dyd\bar{y} - 2dzd\bar{z},
\]

The Kähler form is self-dual (i.e. spans an \(\alpha\)-plane) and is given by

\[
-2i\Omega_1 = dy \wedge d\bar{y} - dz \wedge d\bar{z},
\]

There are two other self-dual planes (i.e. \(\alpha\)-plane)

\[
\alpha = dy \wedge dz = \Omega_2 + i\Omega_3,
\]

\[
\bar{\alpha} = d\bar{y} \wedge d\bar{z} = \Omega_2 - i\Omega_3,
\]

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and three anti-self-dual planes ($\beta$-planes)

$$\beta = dy \wedge d\bar{z} = \bar{\Omega}_2 - i \bar{\Omega}_3,$$
$$\bar{\beta} = d\bar{y} \wedge dz = \bar{\Omega}_2 + i \bar{\Omega}_3,$$
$$-2i \bar{\Omega}_1 = dy \wedge d\bar{y} - dz \wedge d\bar{z}. \quad (8)$$

Then the equation of anti self-dual Yang-Mills

$$F_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \quad (9)$$

becomes

$$F_{yz} = 0, \quad (10a)$$
$$F_{\bar{y}\bar{z}} = 0, \quad (10b)$$
$$F_{y\bar{y}} + F_{z\bar{z}} = 0. \quad (11)$$

Since (10a) and (10b) implies vanishing in $\alpha$ and $\bar{\alpha}$-plane, one may introduce two pure gauge restricted to $\alpha$ and $\bar{\alpha}$-plane,

$$A_y = h^{-1} \partial_y h, \quad A_z = h^{-1} \partial_z h$$
$$A_{\bar{y}} = \bar{h}^{-1} \partial_{\bar{y}} \bar{h}, \quad A_{\bar{z}} = \bar{h}^{-1} \partial_{\bar{z}} \bar{h} \quad (12)$$

where $h$ and $\bar{h}$ are $SL(4, c)$ matrix depending on $y, \bar{y}, z$ and $\bar{z}$. Then (10a) and (10b) becomes

$$[D_y, D_z] = 0, [D_{\bar{y}}, D_{\bar{z}}] = 0. \quad (13)$$

For real $x$, $A_{\mu}^a \doteq$ real gauge field, then

$$\bar{h} \doteq (h^\dagger)^{-1}. \quad (14)$$

here $\doteq$ imply restricted to real $x$. A gauge transformation is characterized by the replacement

$$h \rightarrow hg, \quad \bar{h} \rightarrow \bar{h}g \quad (15)$$

with $g = g(y, \bar{y}, z, \bar{z}) \in SL(4, c), h^\dagger h \doteq I =$ unit $4 \times 4$ matrix, under which the gauge potential $A_{\mu}$ and the field strength $F_{\mu\nu}$ transform as

$$A_{\mu} \rightarrow g^{-1} (A_{\mu} + \partial_{\mu}) g, \quad F_{\mu\nu} \rightarrow g^{-1} F_{\mu\nu} g, \quad (16)$$

respectively. We introduce a Hermitian matrix

$$U = h\bar{h}^{-1} \doteq hh^\dagger. \quad (17)$$
which has the important property of being invariant under the gauge transformation (13). Then (11) becomes

\[(U^{-1}U_y)\bar{y} + (U^{-1}U_z)\bar{z} = 0,\]  

(18)

where \(U\) is \(sp(4)\) matrix. It can also be written in a covariant form

\[\bar{\partial}(U^{-1}\partial U) \wedge \Omega_1 = 0,\]  

(19)

where

\[\partial = dy\partial_y + dz\partial_z,\]
\[\bar{\partial} = d\bar{y}\partial_{\bar{y}} + d\bar{z}\partial_{\bar{z}}.\]  

(20)

Hou and Song [8] discover that Eq.(11) (equivalently to (18), (19)) can be obtained from a variation of the action,

\[S = -\frac{4\pi}{2} \int_{E_4} Tr(dU^{-1} \wedge^* dU) + \frac{4\pi}{3} \int_{M_5} Tr[(U^{-1}dU)^3] \wedge \Omega_1, \quad \partial M_5 = E_4\]  

(21)

The Euler-Lagrange eq. of (21) is (18). The variation of this action is suggested by Donaldson in 1985 [9],

\[\delta S = - \int Tr(U^{-1}\delta U\partial(U^{-1}\bar{\partial}U)) \wedge \Omega_1.\]  

(22)

Now we may combine (10a), (10b) and (11) together

\[D_1 = D_y - \zeta D_{\bar{z}},\]  

(23)
\[D_2 = D_z - \zeta D_{\bar{y}},\]  

(24)

where \(\zeta\) is the \(\text{CP}(1)\) fibre of \(\text{CP}(3)\) bundle on base \(S^4\) i.e. compactified \(E_4\), and they satisfy

\[[D_1, D_2] = 0.\]  

(25)

When \(\zeta = 0, \infty\) and 1, (25) becomes (10a), (10b) and (11) respectively. So in self-dual plane the anti-self-dual connection is trivial, that is to say, they are sections of holomorphic bundle on the self-dual plane

\[\Omega = (\Omega_2 + i\Omega_3) - 2\Omega_1\zeta + (\Omega_2 - i\Omega_3)\zeta^2
\]
\[= \alpha - 2\Omega_1\zeta + \bar{\alpha}\zeta^2\]

here \(\zeta\) describes the \(\text{CP}_1\) set of complex structures.
3 Dressing symmetric action of Higgs YM fields

The D3-brane moving in $AdS_5 \otimes S^5$ background should be described by DBI action as by Metsaev and Tseytlin \[10\]. But as point out by Kallosh and Rajaraman \[11\] on the maximally supersymmetric $AdS_5 \otimes S^5$ background, the basic scalar superfield are covariant constant and the fermionic superfield $\Lambda^\alpha$ (related with the S-duality $U(1) \subset SU(1,1)$) vanishes. So in $\kappa$ symmetric static Killing gauge, the background is pure geometrical. The only field remains essentially are the $R_{\alpha\beta} \sim \eta^{\alpha}_{\gamma} \eta^{\beta}_{\delta} - \eta^{\alpha}_{\delta} \eta^{\beta}_{\gamma}$ and 5-form field $\sim \epsilon^{abcde}$, all others can be expressed by the geometrical quantity such as torsion. Thus Kallosh and Rajaraman \[11\] argued that D3-brane may be simply written as the nonlinear sigma models. To display the hidden symmetry of the moduli, it is sufficient to restrict to the bosonic case. So to describe the chiral embedding of D3-brane in $AdS_5 \otimes S^5$, we will construct the nonlinear sigma models by gauging the WZW form of last section. But now instead of Euclidean world sheet $E_4$, we will change to Minkowski world sheet $M_4(t,x_1,x_2,x_3)$ with topology $S^1 \times \mathbb{R}^3$. Now we define

$$\sqrt{2}y = t + x_1; \sqrt{2}\bar{y} = t - x_1;$$
$$\sqrt{2}z = -x_2 + ix_3; \sqrt{2}\bar{z} = x_2 + ix_3.$$ \hspace{1cm} (26)

The $SU(2,2)$ symmetry is extended to $GL(4)^{(1)}$ as \[11\]. Then adding the chiral anomaly 2 cocycles term, the action is given as

$$S = -\frac{4\pi}{2} \int_{M_4} Tr(dU^{-1} \wedge^* dU)$$
$$+ \frac{4\pi}{3} \int_{M_5} Tr[(U^{-1}dU)^3] \wedge \Omega_1$$
$$+ \frac{4\pi}{2} \int_{M_4} d^2y d^2z Tr\{A_y (U^{-1} \partial_y U - \mu) + (\partial_y U U^{-1} - \nu) \tilde{A}_y + \tilde{A}_y U A_y U^{-1}\}$$
$$- \frac{4\pi}{2} \int_{M_4} d^2y d^2z Tr\{A_z (U^{-1} \partial_z U - \mu) + (\partial_z U U^{-1} - \nu) \tilde{A}_z + \tilde{A}_z U A_z U^{-1}\}.$$ \hspace{1cm} (27)

here

$$\mu = \mu E_+ \equiv \mu \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \nu = \nu E_- \equiv \nu \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$ \hspace{1cm} (28)

in principal representation. Later we will always choose $\mu = \nu^{-1}$, so the $4\pi$ before 3rd and 4th integral is the mass \[12\] of monopole. Here if we consider $SP(4)$ as $SL_2(H)$, then the left $A$ correspond to the algebra of left multiplication group of quaternion, the $\tilde{A}$ to the right conjugate one. So left $\alpha$-plane correspond to that of the twistor $Z^a$ and right $\bar{\alpha}$-plane for the anti-twistor googly $\bar{Z}^a$. Thus the model are section of
ambitwistor bundle with base \( \mathbb{C}M_4 \) and the two left right \( \mathbb{C}P_1 \) fibre are mutually conjugate and trivialized by \( \zeta \) and \( \bar{\zeta} \) respectively. Varying the action with respect to \( A_1 = A_y - \zeta A_z, A_2 = A_z - \zeta A_y \), etc. and \( \delta U = \epsilon_L U, \delta \bar{U} = U \epsilon_R \) respectively, we obtain the equation of motion of \( A, \tilde{A} \) and \( U \),

\[
(\partial(U^{-1} \tilde{\partial} U + U^{-1} \tilde{A} U) - [A, U^{-1} \tilde{\partial} U + U^{-1} \tilde{A} U] + \tilde{\partial} A) \wedge \Omega_1 = 0
\]

(29)

\[
(\tilde{\partial}(\partial U U^{-1} + U A U^{-1}) - [\tilde{A}, \partial U U^{-1} + U A U^{-1}] + \partial A) \wedge \Omega_1 = 0
\]

(30)

\[
Tr[E_{\alpha_+}(U^{-1} \partial y U + U^{-1} \tilde{A} y U - \nu)] = 0
\]

(31)

\[
Tr[E_{\alpha_+}(U^{-1} \partial x U + U^{-1} \tilde{A} x U - \nu)] = 0
\]

(32)

\[
Tr[E_{\alpha_-}(\partial y U U^{-1} + U A_y U^{-1} - \mu)] = 0
\]

(33)

\[
Tr[E_{\alpha_-}(\partial x U U^{-1} + U A_x U^{-1} - \mu)] = 0
\]

(34)

Here \( U \) is the transfer matrix in two different order of Gauss decomposition (cf. [6]). After acting on the highest (lowest) weight vector of level one affine algebra, only the upper and lower Borel part remain and we will get sections of holomorphic and anti-holomorphic bundle respectively.

4 Nahm Donaldson Hitchin Atiyah’s monopole moduli

In next section, we will argue that the gauged WZW action (27) are the axially twisted affinize generalization of Nahm, Donaldson, Hitchin, Atiyah’s theory of monopole. The holomorphic structure of the left and the right anti-holomorphic structure are glued in an axially twisted way.

Nahm’s formalism for the BPS monopoles is obtained by Fourier transform the ADHM construction of multi-instantons to a Hilbert space.

4.1 ADHM construction of instanton[13, 14]

In su(n) self-dual Yang-Mills theory, the potential is

\[
A_\mu = v^\dagger \partial_\mu v,
\]

(35)

\[
v^\dagger v = 1,
\]

(36)

here \( v \) is \((2k + n) \times (2k)\) matrices with instanton number \( k \). Properly choose a \((2k + n) \times (2k)\) matrices

\[
\triangle(x) = a + b x,
\]

(37)

\[
x = x_\mu q^\mu,
\]

(38)
where \( q \) is quaternion. Then the solution \( \upsilon \) of
\[
\Delta^\dagger \upsilon = 0, \tag{39}
\]
and (36) give the potential (33).

### 4.2 Nahm equation [15]

For the \( \text{su}(2) \) BPS monopole with magnetic charge \( k \), Nahm introduce a \( 2k \times 2k \) matrix function \( \upsilon(s) \) of \( s \) satisfying
\[
\left( i \frac{d}{ds} + x^\dagger + T^\dagger \right) \upsilon = 0,
\]
\[
\int_0^2 \upsilon^\dagger \upsilon ds = 1. \tag{40}
\]
where \( x^\dagger = x^i q^i \) (\( i = 1, 2, 3 \)) and \( s \) is the Fourier transformation of \( x_0 \). Nahm has show that the matrix \( T = T_\mu q^\mu \) satisfy the self-dual Nahm equation (equivalent to Donaldson’s [16] complex representation)
\[
\frac{dT_i}{ds} = \frac{1}{2} \sum \epsilon_{ijk} [T_j, T_k] + [T_0, T_i],
\]
\[
T^\dagger_\mu = \eta_{\mu\nu} T_\nu. \tag{41}
\]
The \( T_0 \) can be gauge transformed into zero, by \( g(s) \) with \( g(s) \xrightarrow{s \to 0} 1 \), at last we have the Nahm equation,
\[
\frac{dT_i}{ds} + \frac{1}{2} \sum \epsilon_{ijk} [T_j, T_k] = 0, \tag{42}
\]
\[
T_i^*(s) = -T_i(s), \tag{43}
\]
\[
T_i(2 - s) = T_i(s)^T, \tag{44}
\]
The \( T^i \) are analytic over \((0, 2)\), with simple poles at \( 0, 2 \).
\[
\text{The residues of } T_i \text{ at } s = 0 \text{ form an irreducible representation of } \text{su}(2). \tag{45}
\]

### 4.3 The Donaldson’s rational map and Hitchin’s spectral curve

Donaldson has established a one to one correspondence between the equivalence class of Nahm’s complex (12)-(16) and a rational function \( S(z) \). Donaldson’s \( S(z) \) is a rational function of degree \( k \), regular at \( \infty \), \( S(\infty) = 0 \).

\[
S(z) = \langle s | (z I - B)^{-1} | s \rangle \in \mathbb{C} \cup \infty \tag{47}
\]
where (i) $B$ is a symmetric $k \times k$ matrix and $|s> \in \mathbb{C}^k$ is a column vector.
(ii) $|s>$ generates $\mathbb{C}^k$ as a $\mathbb{C}[B]$ module.

Hitchin established the correspondence between the holomorphic vector bundle and the solution of BPS eq. and explain the $S(z)$ as scattering function. He consider the differential operator

$$\nabla_U - i\phi, \nabla_i = \partial_i + iA_i, A, \phi \text{ solution of BPS eq.}$$  \hspace{1cm} (48)

acting on section $s(t) \in TP_1(C)$ over a fixed oriented line $U$ with parameter $t$. If there is a solution $s_0$ decays at both $t \to +\infty$ and $t \to -\infty$. A line $U$ with this property then is called in [12] a spectral line. Thus the poles of $S(z)$ represent the $k$ spectral lines parallel to the $x_1$-axis passing through the $x_1 = 0$ plane at $z = x_2 + ix_3$. The space of all oriented straight lines in $\mathbb{R}^3$ has identified with the tangent bundle of the complex projective line $TP_1[17]$. The subspace consisting of all spectral lines is called the spectral curve $S[18]$.

5 The Ambitwistor description of moduli space

5.1 Mini-twistor $\mathbb{T}P_1$ and Robinson congruence of $TCP_3 \times \overline{TCP_3}$

Using the Jacobi vector field, Hitchin [17] define the complex structure for the space $TP_1$ of the oriented straight lines in $\mathbb{R}^3$. The real structure $\tau$ is given by reversing the orientation. The conformal structure is induced on $\mathbb{R}^3$ with the metric of 3 dimensional Euclidean space $E_3$. Hitchin consider the geometry of the complexification $\mathbb{C}^3$ of $\mathbb{R}^3$ define a mini-twistor representation. Mini-twistor is the quotient space of $CP_3/CP_1$ by the action of time translation which is the real part of the action of $C$. A point $(a, b, c)$ in $\mathbb{C}^3$ is represented by a holomorphic action $(a\zeta^2 + b\zeta + c) \frac{d}{d\zeta}$ of $TP_1$. The sections through a fixed point $(\eta_0, \zeta_0) \in TP_1$. \{(a, b, c) \in C^3|a\zeta_0^2 + b\zeta_0 + c\}$ is a null plane in $C^3$. His null planes give the usual null geodesic congruence of real $M_4[19]$.

Now in our action (27) restricted to the null cone component $dy = \frac{dt + dx_1}{\sqrt{2}}, \overline{dy} = \frac{dt - dx_1}{\sqrt{2}}$, it turns to be the same as the affine Toda case in [1]. Here in D3 brane case, the left moving and right moving world-sheet has been extended to $CM_4$, So the zweibein $dy, \overline{dy}$ (the corresponding tangent vectors are null momentum) corresponds to spinor $\lambda^\alpha, \overline{\lambda}^\dot{\alpha}$, and can be completed to the full 4-bein corresponding to the twistor

$$X = (\lambda^\alpha, \mu_\dot{\alpha}) \text{ and } \overline{X} = (\mu^\alpha, \lambda_\dot{\alpha})$$

which is the Robinson congruence, that is it intersect with a non-null twistor $Z$

$$\overline{Z}_a X^\alpha = 0 \text{ with } \overline{Z}_a Z^\alpha = m^2 \neq 0.$$
Now we will explain why dressing symmetric string embedding and $D_3$ brane embedding are described by the same Robinson congruence. Both string and $D_3$ brane share the same quaternion basis inherited by the target space $AdS_5 \cong \frac{GL(4)}{Sp(4) \otimes U(1)}$. The Serret Frenet equation of complex curve spanned by the string world sheet implies a covariant invariant Jacobi field with covariant constant quaternion basis—the moving frame. Since the embedding of $E_\pm$ in $AdS_5$ is principal, so the moving of zweibein of tangent vector will determine a complete 4-bein by the normal, binormal etc of this complex curve, this will be the same covariant constant Jacobi field for the moving frame” of quaternion curve”, which describes the embedding of brane. That is the same Robinson congruence with the same parameter $\mu = \nu^{-1}$ (m has been fixed). It is crucial that the Robinson congruence is shearfree [19]. So $|\mu|$ = the dilation spin coefficient and $\frac{\mu}{|\mu|}$ = the twist uniquely determine and is determined by the common dilation and rotation of 3 space component for the Jacobi field. From the quaternion $\eta$ and Jacobi field, one may construct a hyper kähler structure [12] which is covariant constant [12]. These constitute the 3 family of $\alpha$ planes. So from Robinson congruence of one set $\alpha$ planes, one may get the whole hyper kähler structure. The Ward transition function $U$ and factorized $W$ [20] may be constructed also. Now the $W$ is realized by the Riemann Hilbert transformation of affine Toda.

5.2 NDHA moduli of monopole—Toda; moduli of dressing affinized case—affine Toda

Now we turn to argue that as the NDHA’s moduli space is described by spectral curve of Toda, the moduli space of gauged WZW [27] is described by spectral curve of affine Toda. The group $GL(4)^{(1)}$ in action (39) is the central extension of the $SL(4)$ in action (21). The Lie algebra structure in Nahm’s eq. will be replaced by an affine one. The section $|s\rangle \in L^2 \times O(k)$ in the kernel of left $\Delta$ operator and section $\langle s|$ in the kernel of right $\Delta^*$ operator will be replaced by the heighest weight space vector $|\hat{s}\rangle$ and $\langle \hat{s}|$ of affine algebra. The Toda type [21]

$$U^{-1} \partial_+ U = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & 0 \end{pmatrix}$$

$$U^{-1} \partial_- U = \begin{pmatrix} 0 & 1 & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

(49)

will be replaced by the affine one [28]. Let us illustrate these modification in the following equivalence class of [18].

A. A solution to the Bogomolny equations $D\Phi = *F$ on $\mathbb{R}^3$ with boundary conditions as $r \to \infty$. 

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1. \(||\Phi|| = 1 - \frac{2}{r} + O (r^{-2})\),
2. \(\frac{\partial ||\Phi||}{\partial \Omega} = O (r^{-2})\),
3. \(||D\Phi|| = O (r^{-2}).\)

**B.** A compact algebraic curve \(S \in TP_1\) in the linear system \(|O(2k)|\).

**C.** The Nahm complex eq.(12)-(16)

### 5.3 Spectral curve \((B \rightarrow C)\)

As in Hitchin [17] theorem (6.3) and P156 in [18], let \(\tilde{E}\) be the holomorphic bundle on \(TP_1\) represented by the exact sequence

\[
0 \to L (-k) \to \tilde{E} \to L^* (k) \to 0,
\]

There are two distinguished bundle \(L^+\) and \(L^-\)

\[
L^\pm = \{u \in \tilde{E}|u (t) \to 0, t = \pm \infty\}
\]

where \(L^+ \simeq L (-k), L^-\) are the conjugate of \(L^+\) by real structure. the spectral curve \(S\) is defined by

\[
S = \{u \in TP_1|L^+_u = L^-_u\}.
\]

Then as in proposition (7.3) in [17] and P156 in [18], \(S\) is the divisor of a section of \(\psi \in H (TP_1, O (2k))\) which is defined by the equation

\[
S = \{(\eta, \zeta) \in TP_1|\psi = \eta^k + a_1 (\zeta) \eta^{k-1} + \cdots + a_k (\zeta) = 0\},
\]

(50)

where \(a_i (\zeta)\) is a polynomial of degree \(2i\) in \(\zeta\).

And this spectral curve has been identified as p170 in [18]

\[
S = \{(\eta, \zeta) \in TP_1|\det (\eta + A (\zeta)) = 0\},
\]

(51)

\[
A = A_0 + \zeta A_1 + \zeta^2 A_2.
\]

(52)

As shown by [18], the equation (48) is the spectral curve of “open chain” Toda, \(det L_{T\text{oda}} = 0\). This can be explained by Gauss decomposition of the \(U\) in WZW action [24]. As in [22], the upper and lower triangle part becomes holomorphic and anti holomorphic respectively, thus corresponds to \(L^+\) and \(L^-\).

\[\text{For example Krichever and Vanisky [23] show that the spectral curve of open Toda chain is that as in (50) with } \psi \to \xi . \text{ Marshakov [24] prove it is the perturbative limit of the affine Toda type spectral curve of quantum moduli space for the 4D pure SUYM.}\]
After find the independent of the eigenvector $\eta$, Hintchin shows the Toda type Lax pair,

$$[A_+, A] + \frac{dA}{ds} = 0,$$

(A_{+})

and $A_+ \equiv \frac{1}{2} A_1 + \zeta A_2$. Let $A_0 = T_1 + iT_2$, $A_1 = 2iT_3$, $A_2 = T_1 - iT_2$, equation (53) turn to be the Nahm equation.

In the process of determine the surface $S$ by the spectral eq. the essential point is to use the series of exact sequence of inclusion $i$, restriction $\rho$ and the coboundary $\delta$. In the ambitwistor case, this $\delta$ of Cech double cohomology will be given by chiral anomaly as shown by [25, 26], realized in our affine action (27) where $U$ turns to be an element of the affine group.

The $\hat{S}$ becomes

$$\hat{S} = \{ u \in \mathbb{CP}_3 \times \overline{\mathbb{CP}}_3 \mid L_u^+ = (L_u^-)^* \},$$

where $L^- = e^{k-} M_-$, $L^+ = e^{k+} M_+$ is the upper (lower) triangle part in two different order of Gauss decomposition $U = e^{k-} N_+ M_- = e^{k+} N_- M_+$, with $k_\pm$ diagonal, $N_+, M_+$ upper triangle and $N_-, M_-$ lower triangle. So $\hat{S}$ is hyper-elliptic, symmetrical under $\zeta \rightarrow \frac{1}{\zeta}$.

$$\hat{S} = \{ (\eta, \zeta) : \zeta + \frac{1}{\zeta} = \eta^k + a_1\eta^{k-1} + \cdots + a_k = 0 \},$$

which is the spectral curve of affine Toda. For affine Toda case, $\hat{L}$ is determined by the vanishing of double pole of Akhizer-Baker operator, $\hat{A}_+$ is by vanishing of the simple pole, which is independent of $\eta$ by the dispersion phase of AB function. So in the affine Toda case, Nahm equation is satisfied also.

### 5.4 The boundary behaviour of Higgs $\Phi$ of BPS equation and Nahm equation ($C \longleftrightarrow A$)

The $\Phi(\infty)$ up to $\frac{1}{\mu} \in$ Sobolev space $H'$, is given by coboundary, which is determined by the residue of Nahm’s $T_i(s)$ at $s = 0$. Hitchin show $A$ by consider the pole of $\langle$ kernel of $\Delta^*$, kernel of $\Delta$ $\rangle$, Hurtbise [22] find the coboundary by using Lagrange interpolation, to get the transition matrix, so that the $\frac{1}{\mu^2}$ order is given by the level $\mu$, which describes the central position of the monopole, i.e. the scale of the Jacobi field, given by solution of the linearized BPS equation [17, 12].

In affine Toda case, we know from the action (27), the $\mu \sim < \Phi > \longleftrightarrow \Phi(\infty)$. Now the usual $SL(2)$ algebra has been complexified, So $< \Phi >$ will be complex.

The kernel of affine $\hat{\Delta}, \hat{\Delta}^*$ is given by affine vertex operator $V(\xi)$ with pole and zero at $\zeta = \mu$. Here $\mu$ is the rapidity, i.e. the scale and phase of the Jacobi field along the null direction. It is the common scale of all three space components, three Jacobi fields. This is in consistent with the well known fact, that the radial and transverse
component of Higgs and Yang-Mills field satisfying BPS equation will give asymptotic value as \( A \) (e.g. [12] (4.5)), the radial part gives the monopole value \( k \) and scale \( \mu \).

It worth mention that the quantum \( k \), the topological charge of S.G. contributed by boundary flow is extended to the charge of monopoles, the magnetic flux which actually realized the coboundary, given by the 1st Chern class.

5.5 Donaldson’s classification

Hurtbise [27] has chosen an trivialization for the solution \( s(t) \) of Nahm equation. Such that both \( s(0) \) and \( s(1) \) becomes \( \cong (1, 1, \cdots, 1) \). Then as Donaldson (47), the residue of \( B(0) \) becomes

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & \cdots & 0
\end{pmatrix}
\]

In the affine case, corresponding to the constrain (47), the residue should be replaced by the cyclic element in affine algebra. [28] \[
\begin{pmatrix}
\mu & \mu & \mu \\
\mu & \mu & \mu \\
\mu^{-1} & \mu^{-1} & \mu^{-1} \\
\mu^{-1} & \mu^{-1} & \mu^{-1}
\end{pmatrix}
\]

acting as permutations on left highest weight space and \[
\begin{pmatrix}
\mu^{-1} & \mu^{-1} & \mu^{-1} \\
\mu^{-1} & \mu^{-1} & \mu^{-1}
\end{pmatrix}
\]

acting on right conjugate space. This is really the behaviour at the multiple pole and zero at \( \zeta \rightarrow \pm \infty \) on spectral curve of affine Toda [29, 30].

5.6 Solution space of twisted monopole

The utility of gauged WZW with affinization is that it implies dressing transformation and Riemann Hilbert problem. Thus we may find soliton (twisted monopole) solution and describe the moduli space by \( \mu \) as following.

The affine algebra with dressing symmetry will enable us to find the soliton solution. One may find the stretched IIB string solution for the conformal affine Toda [31]. [32]. This turns to be the left and right ray in the Robinson congruence. Meantime the other two orthogonal component, the Jacobi field is given also [33, 34], as in the Robinson congruence. So it determines the four dimension monopole solution. Here, it is important that to obtain an soliton solution we have start from the highest weight vector with nontrivial dependence of \( z_+, z_- \) (cf [31]), then acted by the left \( g^+ \) and right \( g^- \) part of the dressing group \( g \), which can be expressed by the \( \pm \) frequency part of vertex operator of affine algebra. To obtain multi-soliton, one may simply use
the Wick theorem of normal order. The Riemann Hilbert problem with zero and pole give the classification of moduli space, implies implicitly the commutative rule of multi Backlund transformation.

It should stressed that the dressing symmetry, the Poisson Lie structure of the bi-algebra are important, since the classical $r$ matrix can be quantized to quantum $R$ matrix. The quantized Yang-Baxter representation operator $L$ and vertex operators (type I and type II) has been given. And the quantum affine algebra, its currents and q-deformed Virasoro ($W$-algebra.) is given, including the affine Toda and sine Gordan. This may provide the formulation of covariant quantization of string and SUYM explicitly in the Green-Schwarz formulation.

5.7 SUYM and brane

The level $\mu$ in hyper kähler quotient are realized as the vacuum expectation value $\langle \phi \rangle$, usually in the following way

1. Appears in D term as FI term and superpotential associate with anomaly. Then for brane set D3-D1, N5-D3, the equation of motion becomes the Nahm eq. or the Hitchin system. For example.

2. The Seiberg Witten curve is related with integrable system. For pure SU(N) super Yang-Mills theory, it is given by the spectrum curve of affine Toda.

It is plausible that this two fact will unified by the hidden symmetry in gauged WZW embedding in twistor space $\mathbb{CP}_3$, further to $\mathbb{CP}^{3|4}$. This implies we may simply affinize the Nahm eq. with an axial definition of conjugation. Then the vacuum moduli may be classified by the moduli space of its soliton monopole solution, the twisted monopole and the Seiberg Witten curve of the moduli of SUYM will be identified with the spectral curve $|\eta + L(\zeta)|$.

In summary, the expert will find that the added term couple to $A$ are the familiar 2 cocycle of axial anomaly i.e. D term and FI term as, so the level $\mu$ equals the vacuum eigenvalues of Higgs field in the $N = 1$ chiral multiplet of $N = 2$ vector multiplet,

$$\mu = \langle \phi \rangle.$$  \hspace{1cm} (54)

Really as shown by Diaconescu the $\mu$ will be the VEV of $\phi$ or its asymptotic value. Our twist of the parameter $\mu$ will change the module and phase of it which correspond to dilation of monopole and dual rotation of electric and magnetic (topologic) charge, or equivalently change the phase $t$ and $r$ of the $\theta$ term and $D$ term as, which is the charge of Hodge structure of the A, B twist as Witten describes in paper. All are related to S duality i.e. $U(1)$ in the $S_{\mathfrak{L}(2|\mathfrak{R})}$ of SUGRA and M theory which is restricted to the
$SL(2\mathbb{Z})$ by Dirac quantization. Now we have continuous $SL(2\mathbb{R})$ to describe different point (vacuum) in moduli space, where the phase of $\langle \phi \rangle$, just implies a redefinition of the $U(1)$ charge ("electric") minimally coupled to $\phi$. It would be interesting to study the continuous effect of this affine $gl(1\mathbb{C})$ such as: 1. the $U(1)$ current in CFT, related to the spectral flow for chiral ring [41]; 2. the special geometry [42]; 3. the Hanny Witten effect [37]; 4. the blow up for A brane to B brane [43]. We expect that this hidden symmetry will be helpful to clarify the singularity, the cut of moduli space, the phase diagram of super Yang-Mills theory.

Recently there are a lot of papers on (super)twistor formulation of superstring [20]. It would be interesting to study its dressing symmetric form with affine symmetry $gl(4|4)^{(1)}$ and to study the CFT/AdS correspondence, by using the quantum double covariant quantization of the classical double ambitwistor.

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**Note**

This paper gives the conjecture for the affinization of NADHM construction. Recently, we find a left (right) moving, holomorphic (anti), selfdual(anti) spherical symmetrical solution for affine Higgs Yang-Mills field, and the Dirac eq. on this background. Thus realize the NADHM construction explicitly.

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