SPRAGUE-GRUNDY FUNCTION OF MATROIDS AND RELATED
HYPERGRAPHS

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Abstract. We consider a generalization of the classical game of NIM called hypergraph
NIM. Given a hypergraph \( \mathcal{H} \) on the ground set \( V = \{1, \ldots, n\} \) of \( n \) piles of stones, two
players alternate in choosing a hyperedge \( H \in \mathcal{H} \) and strictly decreasing all piles \( i \in H \).
The player who makes the last move is the winner. In this paper we give an explicit
formula that describes the Sprague-Grundy function of hypergraph NIM for several classes
of hypergraphs. In particular we characterize all 2-uniform hypergraphs (that is graphs) and
all matroids for which the formula works. We show that all self-dual matroids are included
in this class.

1. Introduction

In the classical game of NIM there are \( n \) piles of stones and two players move alternating.
A move consists of choosing a nonempty pile and taking some positive number of stones from
it. The player who cannot move is the looser. Bouton [10] analyzed this game and described
the winning strategy for it.

In this paper we consider the following generalization of NIM. Given a hypergraph
\( \mathcal{H} \subseteq 2^V \), where \( V = \{1, \ldots, n\} \), two players alternate in choosing a hyperedge \( H \in \mathcal{H} \) and
strictly decreasing all piles \( i \in H \). We assume in this paper that \( \mathcal{H} \neq \emptyset \) and \( \emptyset \not\in \mathcal{H} \) for all
considered hypergraphs \( \mathcal{H} \). In other words, every move strictly decreases some of the piles.
Similarly to NIM, the player who cannot move is loosing. This game is called \( NIM_{\mathcal{H}} \) and
some special cases of it were considered in [8, 9].

\( NIM_{\mathcal{H}} \) is an impartial game. In this paper we do not need to immerse in the theory of
impartial games. We will need to recall only a few basic facts to explain and motivate our
research. We refer the reader to [1, 3] for more details.

It is known that the set of positions of an impartial game can uniquely be partitioned
into sets of winning and loosing positions. Every move from a loosing position goes to a
winning one, while from a winning position always there exists a move to a loosing one. This
partition shows how to win the game, whenever possible. The so-called Sprague-Grundy
(SG) function \( G_\Gamma \) of an impartial game \( \Gamma \) is a refinement of the above partition. Namely,
\( G_\Gamma(x) = 0 \) if and only if \( x \) is a loosing position. The notion of the SG function for impartial

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games was introduced by Sprague and Grundy [17, 18, 11] and it plays a fundamental role in determining the winning-loosing partition of disjunctive sums of impartial games.

Finding a formula for the SG function of an impartial game remains a challenge. Closed form descriptions are known only for some special classes of impartial games. We recall below some known results. The purpose of our research is to extend these results and to describe classes of hypergraphs for which we can provide a closed formula for the SG function of \( NIM_\mathcal{H} \). To follow our proofs, we need to recall the precise definition of the SG function, which we will do in Section 2.

The game \( NIM_\mathcal{H} \) is a common generalization of several families of impartial games considered in the literature. For instance, if \( \mathcal{H} = \{\{1\}, \ldots, \{n\}\} \) then \( NIM_\mathcal{H} \) is the classical \( NIM \), which was analyzed and solved by Bouton [10]. The case of \( \mathcal{H} = \{S \subseteq V \mid 1 \leq |S| \leq k\} \), where \( k < n \), was considered by Moore [15]. He characterized for these games the set of loosing positions, that is those with SG value 0. Jenkyns and Mayberry [14] described also the set of positions in which the SG value is 1 and provided an explicit formula for the SG function in the subcase of \( k = n - 1 \). This result was extended in [7]. In [8] the game \( NIM_\mathcal{H} \) was considered in the case of \( \mathcal{H} = \{S \subseteq V \mid |S| = k\} \) and the corresponding SG function was determined when \( 2k \geq n \).

To state our main result we need to introduce some additional notation. We denote by \( \mathbb{Z}_+ \) the set of nonnegative integers and use \( x \in \mathbb{Z}_+^V \) to describe a position, where coordinate \( x_i \) denotes the number of stones in pile \( i \in V \). Given a hypergraph \( \mathcal{H} \) and position \( x \in \mathbb{Z}_+^V \), we denote by \( \mathcal{G}_\mathcal{H}(x) \) the SG value of \( x \) in \( NIM_\mathcal{H} \). The Tetris value \( T_{\mathcal{H}}(x) \) was defined in [8] as the maximum number of consecutive moves that the players can make in \( NIM_\mathcal{H} \) starting from position \( x \).

To a position \( x \in \mathbb{Z}_+^V \) of \( NIM_\mathcal{H} \) let us associate the following quantities:

\[
\begin{align*}
(1a) \quad m(x) & = \min_{i \in V} x_i \\
(1b) \quad y_{\mathcal{H}}(x) & = T_{\mathcal{H}}(x - m(x)e) + 1 \\
(1c) \quad v_{\mathcal{H}}(x) & = \left( y_{\mathcal{H}}(x) \right)^2 + \left( m(x) - \left( y_{\mathcal{H}}(x) \right) \right) \mod y_{\mathcal{H}}(x),
\end{align*}
\]

where \( e \) is the \( n \)-vector of full ones. Finally, we define

\[
\begin{align*}
(2a) \quad \mathcal{U}_\mathcal{H}(x) & = \begin{cases} 
T_{\mathcal{H}}(x) & \text{if } m(x) \leq \left( y_{\mathcal{H}}(x) \right)/2 \\
v_{\mathcal{H}}(x) & \text{otherwise.}
\end{cases}
\end{align*}
\]

With this notation the results of [7, 8, 14] can be stated as the SG function of the considered games is defined by (2a)-(2b), that is, \( \mathcal{G} = \mathcal{U} \). It was a surprise to see that the “same” formula works for seemingly very different games. In view of this, we call the expression (2a)-(2b) the JM formula, in honor of the results of Jenkyns and Mayberry [14]. We call a hypergraph \( \mathcal{H} \) a JM hypergraph if this formula describes the SG function of \( NIM_\mathcal{H} \).

Let us add that the formula looks the same but it depends on \( T_{\mathcal{H}} \) and, hence, the actual values depend on the hypergraph \( \mathcal{H} \). In fact, function \( T_{\mathcal{H}} \) may be difficult to compute [9], even for cases when the JM formula is valid.

Given a hypergraph \( \mathcal{H} \subseteq 2^V \) and a subset \( S \subseteq V \), we denote by \( \mathcal{H}_S \) the induced subhypergraph, defined as

\[ \mathcal{H}_S = \{H \in \mathcal{H} \mid H \subseteq S\}. \]
A set \( T \subseteq V \) is called a transversal if \( T \cap H \neq \emptyset \) for all \( H \in \mathcal{H} \). A hypergraph \( \mathcal{H} \) is called transversal-free if no hyperedge \( H \in \mathcal{H} \) is a transversal of \( \mathcal{H} \). Finally, we say that \( \mathcal{H} \) is minimal transversal-free if it is transversal-free and every nonempty proper induced subhypergraph of it is not. A hypergraph \( \mathcal{H} \) is \( k \)-uniform if \( |H| = k \) for all \( H \in \mathcal{H} \).

We assume that the readers are familiar with the notion of matroid; see, e.g., [19, 20]. A matroid hypergraph \( \mathcal{H} \subseteq 2^V \) is formed by the family of bases of a matroid on the ground set \( V \). It is self-dual if \( V \setminus H \in \mathcal{H} \) for all \( H \in \mathcal{H} \), that is, if the corresponding matroid is self-dual. Let us remark that in some papers self-dual matroids are called identically self-dual.

In this paper we provide some necessary and some sufficient conditions for a hypergraph to be JM. We summarize our main results as follows.

(i) A JM hypergraph is minimal transversal-free.

(ii) A graph (that is, a 2-uniform hypergraph) is JM if and only if it is connected and minimal transversal-free. We provide a complete list of JM graphs.

(iii) A matroid hypergraph is JM if and only if it is transversal-free. This implies that all self-dual matroid hypergraphs are JM.

(iv) Hypergraphs defined by connected \( k \)-edge subgraphs of a given graph are JM under certain conditions.

(v) For every integer \( k \), the number of vertices of a \( k \)-uniform JM hypergraph is bounded by \( k^{(2k)} \).

For instance, \( \binom{V}{k} = \{ H \subseteq V \mid |H| = k \} \) is a self-dual matroid hypergraph if \( n = 2k \). This example shows that (iii) generalizes the main result of [8]. Another example for a self-dual matroid with \( n = 2k \) is the hypergraph \( \mathcal{H}_{2k} = \{ H \subseteq V \mid |H \cap \{i, i + k\}| = 1, 1 \leq i \leq k \} \), that is, the family of \( 2^k \) minimal transversals of a family of \( k \) pairs. It was proved in [6] that any self-dual matroid on \( n = 2k \) elements must have at least \( 2^k \) bases. Thus, the latter construction is extremal in this respect.

We remark that [6] showed also the existence of self-dual matroids on \( n = 2k \) elements whenever certain type of symmetric block designs exists on \( k \) points. Since many families of such block designs are known, the above cited result shows that numerous other families of self-dual matroids (and JM hypergraphs) exist.

For (iv) we can mention the following circulant hypergraphs defined by consecutive \( k \) edges of simple cycles on \( n = 2k \) or \( n = 2k + 1 \) vertices. Another example is defined by connected \( k \)-edge subgraphs of a rooted tree, where the root has degree \( k + 1 \) and each of the subtrees connected to the root have exactly \( k \) edges; see Section 5 for precise definitions and details.

The structure of the paper is as follows. In Section 2 we fix our notation and define basic concepts. We also prove several properties of JM hypergraphs that are needed for our proofs later. In Section 3 we show necessary conditions for a hypergraph to be JM. In Section 4 we provide a general sufficient condition for a hypergraph to be JM. In Section 5 we apply the general sufficient condition, and show that several other families of hypergraphs are JM. Among them, we provide a complete characterization of JM graphs. In Section 6 we discuss the size of \( k \)-uniform JM hypergraphs. Finally, in Section 7 we present further examples of JM hypergraphs and discuss related topics.

2. Basic Concepts and Notation

In this section we introduce the basic notation and definitions. We prove some of the basic properties of \( NIM_{\mathcal{H}} \) games, the Tetris functions and the JM formula.
2.1. \( \textit{NIM}_H \) Games. We need to recall first the precise definition of impartial games and the SG function.

To a subset \( S \subseteq \mathbb{Z}_+ \) of nonnegative integers let us associate its \textit{minimal excludant} \( \text{mex}(S) = \min\{i \in \mathbb{Z}_+ \mid i \not\in S\} \), that is, the smallest nonnegative integer that is not included in \( S \). Note that \( \text{mex}(0) = 0 \).

An impartial game \( \Gamma \) is played by two players over a (possibly infinite) set \( X \) of positions. They take turns to move, and the one who cannot move is the looser. For a position \( x \in X \) let us denote by \( N(x) \subseteq X \) the set of positions \( y \in N(x) \) that are reachable from \( x \) by a single move. For \( y \in N(x) \) we denote by \( x \to y \) such a move. We assume that the same set of moves are available for both players from every position. We also assume that no matter how the players play and which position they start, the game ends in a finite number of moves. The SG function \( \mathcal{G}_\Gamma \) of the game is a mapping \( \mathcal{G}_\Gamma : X \mapsto \mathbb{Z}_+ \) that associates a nonnegative integer to every position, defined by the following recursive formula:

\[
\mathcal{G}_\Gamma(x) = \text{mex}\{\mathcal{G}(y) \mid y \in X, \text{ s.t. } \exists x \to y\}.
\]

In our proofs we shall use the following, more combinatorial characterization of SG functions that can be derived easily from the above definition. Assume that \( \Gamma \) is an impartial game over the set of positions \( X \), and \( g : X \to \mathbb{Z}_+ \) is a given function. Then, \( g \) is the SG function of \( \Gamma \) if and only if the following two conditions hold:

(A) For all positions \( x \in X \) and moves \( x \to y \) in \( \Gamma \) we have \( g(x) \neq g(y) \).

(B) For all positions \( x \in X \) and integers \( 0 \leq z < g(x) \) there exists a move \( x \to y \) in \( \Gamma \) such that \( g(y) = z \).

It is easy now to verify that for a hypergraph \( H \subseteq 2^V \) the game \( \textit{NIM}_H \) is indeed an impartial game over the infinite set \( X = \mathbb{Z}_+^V \) of positions.

Let us note that all quantities used in (1a)–(2b), \( m(x), y_H(x), v_H(x), T_H(x) \) as well as \( \mathcal{U}_H \) are well defined for an arbitrary hypergraph \( \mathcal{H} \). Let us also note that the values \( m(x) \) and \( y_H(x) \) determine completely the value of \( v_H(x) \).

To simplify our language and notation in the sequel, let us call a position \( x \in \mathbb{Z}_+^V \) long (in \( \textit{NIM}_H \)) if \( m(x) \leq \left(\frac{v_H(x)}{2}\right) \) (that is, if \( \mathcal{U}_H(x) = T_H(x) \)) and call it short if \( m(x) > \left(\frac{v_H(x)}{2}\right) \) (that is, if \( \mathcal{U}_H(x) = v_H(x) \)).

According to the rules of \( \textit{NIM}_H \), if \( x \to x' \) is a move then for the set \( H = \{i \mid i \in V, x_i > x_i'\} \) we must have \( H \in \mathcal{H} \). We call such a move an \textit{H-move}. For a subset \( S \subseteq V \) let us denote by \( \chi(S) \) the characteristic vector of \( S \), that is, \( \chi(S)_j = 1 \) if \( j \in S \) and \( \chi(S)_j = 0 \) if \( j \not\in S \). We denote for a position \( x \in \mathbb{Z}_+^V \) and hyperedge \( H \in \mathcal{H} \) by \( x^H \) the vector \( x - \chi(H) \). Thus, \( x \to x^H \) is an \textit{H-move} in \( \textit{NIM}_H \). We call such a move also a \textit{slow move}, since we have \( x' \leq x^H \) for all \textit{H-moves} \( x \to x' \).

2.2. \textit{Tetris Function}. Let us recall that the Tetris function value of a position \( x \) is defined as the maximum number of consecutive moves one can take starting with \( x \). Let us first observe some basic properties of the Tetris function that will be instrumental in our proofs. We assume in the sequel that a hypergraph \( H \subseteq 2^V \) is fixed, and all positions mentioned are assumed to be from \( \mathbb{Z}_+^V \).

**Lemma 1.** For every position \( x \in \mathbb{Z}_+^V \) we have

\[
\mathcal{G}_H(x) \leq T_H(x).
\]
Lemma 5. Let $T$ thus the claim follows from Lemma 2. □

Lemma 4. Proof. Since $T$ one. From this the second inequality follows. □

Furthermore decreasing one of the piles by one unit can decrease the tetris value by at most $T$. If $Corollary 1. If $x, x' \in \mathbb{Z}_+^n$ are two positions such that

$$x'_i \geq x_i \text{ for all } i \neq j, \text{ and } x'_j = x_j - 1$$

for some index $j$, then we have $T(x') \geq T(x) - 1$.

Proof. Follows from Lemma 2. □

Lemma 3. Assume $x \rightarrow x'$ is an $H$-move. Then for every integer $z$ such that $T_H(x') \leq z \leq T_H(x^{s(H)})$ there exists an $H$-move $x \rightarrow x''$ for which $T_H(x'') = z$ and $x' \leq x'' \leq x^{s(H)}$.

Proof. Consider a sequence of positions $x^0 = x^{s(H)} \geq x^1 \geq \cdots \geq x^p = x'$, where $\sum_{j=1}^p (x^{i-1}_j - x^i_j) = 1$ for all $i = 1, \ldots, p$. By Lemma 2 we have $T(x^{i-1}) \geq T(x^i) \geq T(x^{i-1}) - 1$ and all of these positions are reachable from $x$ by an $H$-move. Thus, the statement follows. □

Lemma 4. Consider an arbitrary position $x \in \mathbb{Z}_+^n$ and hyperedge $H \in \mathcal{H}$ such that $T_H(x^{s(H)}) = T_H(x) - 1$ and $m(x^{s(H)}) = m(x) - 1$. Then we have $y_H(x^{s(H)}) \geq y_H(x)$.

Proof. Since $m(x^{s(H)}) = m(x) - 1$ we have the inequality $x^{s(H)} - m(x^{s(H)})e \geq x - m(x)e$ and thus the claim follows from Lemma 2. □

Lemma 5. Let $\mathcal{H} \subseteq \tilde{\mathcal{H}} \subseteq 2^V$ be two (nested) hypergraphs. Then for every position $x \in \mathbb{Z}_+^V$ we have $T_H(x) \leq T_{\tilde{H}}(x)$.

Proof. Since $\mathcal{H} \subseteq \tilde{\mathcal{H}}$, any move in $NIM_{\mathcal{H}}$ is also a move in $NIM_{\tilde{\mathcal{H}}}$. Thus, the claim follows by the definition of the Tetris function. □

2.3. JM Formula. For a positive integer $\eta \in \mathbb{Z}_+$ let us associate the set

$$Z(\eta) = \left\{ i \in \mathbb{Z}_+ \left| \left\lfloor \frac{\eta}{2} \right\rfloor \leq i < \left\lfloor \frac{\eta + 1}{2} \right\rfloor \right. \right\}.

(3)

It is immediate to see the following properties:

Lemma 6. If $\eta \neq \eta'$ then $Z(\eta) \cap Z(\eta') = \emptyset$. Furthermore, we have

$$\mathbb{Z}_+ = \bigcup_{\eta=1}^{\infty} Z(\eta).$$
Let us recall next that by the definitions of the quantities in (I) the value $v_{H}(x)$ depends only on the pair of integers $m(x)$ and $y_{H}(x)$.

**Lemma 7.** For an arbitrary positive integer $\eta \in \mathbb{Z}_{+}$ we have

$$\{v_{H}(x) \mid x \in \mathbb{Z}_{+}^{V}, \ y_{H}(x) = \eta\} = Z(\eta).$$

**Proof.** Follows by (1c) and the fact that $m(x)$ can take arbitrary integer values modulo $y_{H}(x) = \eta$. \hfill $\square$

**Lemma 8.** For an arbitrary position $x \in \mathbb{Z}_{+}^{V}$ and move $x \rightarrow x'$ in NIM$_{H}$ we have $(m(x), y_{H}(x)) \neq (m(x'), y_{H}(x'))$.

**Proof.** If $m(x) = m(x')$ and $x \rightarrow x'$ is an $H$-move for a hyperedge $H \in \mathcal{H}$, then we have the inequality $x - \chi(H) \geq x'$, where $\chi(H)$ is the characteristic vector of $H$. This implies

$$x - m(x)e \geq \chi(H) + x' - m(x')e$$

from which $y_{H}(x') \geq y_{H}(x) + 1$ follows. \hfill $\square$

**Lemma 9.** A position $x \in \mathbb{Z}_{+}^{V}$ is long if and only if $v_{H}(x) \geq m(x)$.

**Proof.** Note first that if $x$ is long then $m(x) \leq \left(\frac{y_{H}(x)}{2}\right)$ by (2a), and thus, by Lemma 7 it follows that $m(x) \leq v_{H}(x)$. On the other hand, if $x$ is short then we have $m(x) > \left(\frac{y_{H}(x)}{2}\right)$ by (2b), and thus, $m(x) - \left(\frac{y_{H}(x)}{2}\right) - 1 \geq 0$, implying

$$m(x) - \left(\frac{y_{H}(x)}{2}\right) - 1 \geq \left(\left(\frac{m(x) - \left(\frac{y_{H}(x)}{2}\right)}{2}\right) - 1\right) \mod y_{H}(x)$$

from which by (2b) it follows that $m(x) - 1 \geq v_{H}(x)$. \hfill $\square$

**Lemma 10.** For an arbitrary position $x \in \mathbb{Z}_{+}^{V}$ and move $x \rightarrow x'$ such that $m(x) \leq \mathcal{U}_{H}(x') < \mathcal{T}_{H}(x)$ position $x'$ is long.

**Proof.** If $x'$ were short then by Lemma 9 we would get $\mathcal{U}_{H}(x') = v_{H}(x') \leq m(x')$, contradicting $m(x) \leq \mathcal{U}_{H}(x')$. \hfill $\square$

**Lemma 11.** Let $\mathcal{H} \subseteq 2^{V}$ and $\tilde{\mathcal{H}} \subseteq 2^{V}$ be two hypergraphs, and $x, \tilde{x} \in \mathbb{Z}_{+}^{V}$ be two positions such that $m(x) = m(\tilde{x})$ and $y_{H}(x) = y_{\tilde{H}}(\tilde{x})$. Then we have $v_{H}(x) = v_{\tilde{H}}(\tilde{x})$. Furthermore, $x$ is short in NIM$_{H}$ if and only if $\tilde{x}$ is short in NIM$_{\tilde{H}}$.

**Proof.** Recall that by (2b) the $v(x)$-value of a position $x$ depends only on the $m(x)$ and $y(x)$ values, and do not depend on any other parameters of the underlying hypergraphs. Similarly, the type of a position (long or short) also depends only on these two integer values. \hfill $\square$

**Lemma 12.** Let $\mathcal{H}, \tilde{\mathcal{H}} \subseteq 2^{V}$ be two hypergraphs such that $\tilde{\mathcal{H}}$ contains a hyperedge different from $V$. Then for every position $x \in \mathbb{Z}_{+}^{V}$ there exists a position $\tilde{x} \in \mathbb{Z}_{+}^{V}$ such that $m(x) = m(\tilde{x})$ and $y_{H}(x) = y_{\tilde{H}}(\tilde{x})$.

**Proof.** Choose a minimal hyperedge $H \in \tilde{\mathcal{H}}$, and consider the position $\tilde{x} = m(x)e + (y_{H}(x) - 1)\chi(H)$. Since $\tilde{\mathcal{H}}$ is assumed to have a hyperedge different from $V$, we can choose $H$ such that $H \neq V$. Therefore, we have $m(\tilde{x}) = m(x)$, and $\tilde{x} - m(\tilde{x})e = (y_{H}(x) - 1)\chi(H)$, implying $y_{H}(x) = y_{\tilde{H}}(\tilde{x})$. \hfill $\square$
3. Necessary Conditions

In this section we prove some properties of JM hypergraphs.

For every hypergraph $\mathcal{H} \subseteq 2^V$, we assume that $V = \bigcup_{H \in \mathcal{H}} H$.

Let us recall that a hypergraph $\mathcal{H} \subseteq 2^V$ is not connected if $V$ can be partitioned into two nonempty subsets $V_1$ and $V_2$ such that every hyperedge of $\mathcal{H}$ is contained in either $V_1$ or $V_2$. Otherwise $\mathcal{H}$ is called connected.

**Lemma 13.** A JM hypergraph $\mathcal{H}$ is connected.

*Proof.* Let us assume indirectly that $\mathcal{H}$ is not connected. Let $H_1, H_2 \in \mathcal{H}$ be two minimal hyperedges in two different connected components of $\mathcal{H}$. Consider the position $x$ defined as follows:

$$
x_i = \begin{cases}
1 & \text{if } i \in H_1 \\
3 & \text{if } i \in H_2 \\
0 & \text{otherwise}.
\end{cases}
$$

This position has $G_\mathcal{H}(x) = 2$ (the NIM sum of 1 and 3). It has $m(x) \leq 1$ and $3 \leq y_\mathcal{H}(x) \leq 4$. Therefore, $x$ is long Since $T_\mathcal{H}(x) = 4$ we can conclude $U_\mathcal{H}(x) \neq G_\mathcal{H}(x)$, which contradicts the assumption.

Given a hypergraph $\mathcal{H} \subseteq 2^V$ and a subset $S \subseteq V$, we denote by $\mathcal{H}_S$ the induced subhypergraph, defined as

$$
\mathcal{H}_S = \{ H \in \mathcal{H} \mid H \subseteq S \}.
$$

A set $T \subseteq V$ is called a transversal if $T \cap H \neq \emptyset$ for all $H \in \mathcal{H}$. A hypergraph $\mathcal{H}$ is called transversal-free if it contains no transversal, i.e., no hyperedge $H \in \mathcal{H}$ satisfies $H \cap H' \neq \emptyset$ for all $H' \in \mathcal{H}$. Finally, we say that $\mathcal{H}$ is minimal transversal-free if it is transversal-free and every proper induced subhypergraph $\mathcal{H}_S$ is either empty, or contains a transversal hyperedge $H \in \mathcal{H}_S$.

**Lemma 14.** If $\mathcal{H}$ is a JM hypergraph, then it is minimal transversal-free.

*Proof.* Let us assume first indirectly that $H_0 \in \mathcal{H}$ is a transversal of $\mathcal{H}$, say $H_0 \in \mathcal{H}$ intersects all hyperedges of $\mathcal{H}$. Consider the position $x \in \mathbb{Z}^V_+$ defined by $x_i = 1$, $i \in V$. For this position we have $m(x) = 1$, $y_\mathcal{H}(x) = 1$, and $v_\mathcal{H}(x) = 0$. Thus, $x$ is short and $U_\mathcal{H}(x) = 0$. On the other hand, a slow $H_0$-move $x \to x'$ takes us into a position with $x_i' = 0$, $i \in H_0$. Since $H_0$ intersects all hyperedges of $\mathcal{H}$, we must have $T_\mathcal{H}(x') = 0$, which implies by property (A) of the SG function that $G_\mathcal{H}(x) \neq 0$, or in other words that $G_\mathcal{H}(x) \neq U_\mathcal{H}(x)$, which implies on its turn that $\mathcal{H}$ is not JM. This contradiction implies that $\mathcal{H}$ is transversal-free.

To see minimality with respect the transversal-freeness, let us consider an arbitrary proper subset $S \subset V$ for which the induced subhypergraph $\mathcal{H}_S$ is not empty, and a position with

$x_i = 0$ for all $i \in V \setminus S$, and $x_i > 0$ for all $i \in S$. Every move from $x$ is an $H$-move for some $H \in \mathcal{H}_S$. Furthermore, $T_\mathcal{H}(x) > 0$, $m(x) = 0$ (since $V \setminus S \neq \emptyset$), and thus, all such positions are long, implying (by our assumption that $\mathcal{H}$ is JM) that $G_\mathcal{H}(x) = U_\mathcal{H}(x) = T_\mathcal{H}(x) > 0$ for all such positions. Thus, we must have a move $x \to x'$ such that $G_\mathcal{H}(x') = T_\mathcal{H}(x') = 0$. This is possible only if this move is an $H$-move for a hyperedge $H \in \mathcal{H}_S$ that intersects all hyperedges of $\mathcal{H}$. □

In the rest of this section, we study further properties of transversal-free hypergraphs. This is used to obtain sufficient conditions for a hypergraph to be JM.
Lemma 15. If \( \mathcal{H} \) is transversal-free and \( x \to x' \) is a move in NIM\(_{\mathcal{H}}\), then we have \( T_{\mathcal{H}}(x') \geq m(x) \).

**Proof.** Since \( \mathcal{H} \) is transversal-free, for every hyperedge \( H \in \mathcal{H} \) there exists an \( H' \in \mathcal{H} \) such that \( H \cap H' = \emptyset \). Consequently, for every move \( x \to x' \) we must have \( T_{\mathcal{H}}(x') \geq m(x) \). This is because if \( x \to x' \) is an \( H' \)-move and \( H' \in \mathcal{H} \) is disjoint from \( H \), then we have \( x'_i = x_i \geq m(x) \) for all \( i \in H' \). \( \square \)

Lemma 16. If \( \mathcal{H} \) is a transversal-free hypergraph, \( x \in \mathbb{Z}_+^\mathcal{H} \) is a long position, and \( x \to x' \) is a move such that \( 0 \leq U_{\mathcal{H}}(x') < m(x) \), then \( x' \) is a short position, for which we have \( m(x') \leq m(x) \) and \( m(x) - m(x') + 1 \leq y_{\mathcal{H}}(x') < y_{\mathcal{H}}(x) \).

**Proof.** By Lemma 15 we have \( T_{\mathcal{H}}(x') \geq m(x) \). Now, if \( x' \) were long then \( U_{\mathcal{H}}(x') = T_{\mathcal{H}}(x') \geq m(x) \) would follow, contradicting our assumption that \( U_{\mathcal{H}}(x') < m(x) \). The inequality \( m(x') \leq m(x) \) holds for any move \( x \to x' \). Since \( x \) is long and \( x' \) is short, we have the inequalities

\[
\left( \frac{y_{\mathcal{H}}(x')}{2} \right) \geq m(x) \geq m(x') > \left( \frac{y_{\mathcal{H}}(x)}{2} \right)
\]

from which \( y_{\mathcal{H}}(x') < y_{\mathcal{H}}(x) \) follows. Assume next that \( x \to x' \) is an \( H \)-move for some hyperedge \( H \in \mathcal{H} \). Since \( \mathcal{H} \) is transversal-free by Lemma 14 there exists \( H' \in \mathcal{H} \) such that \( H \cap H' = \emptyset \). Then we have \( x'_i = x_i \geq m(x) \) for all \( i \in H' \), and thus, \( y_{\mathcal{H}}(x') \geq m(x) - m(x') + 1 \) follows by (11).

Lemma 17. If \( \mathcal{H} \) is a transversal-free hypergraph, \( x \in \mathbb{Z}_+^\mathcal{H} \) is a short position, and \( x \to x' \) is a move such that \( 0 \leq U_{\mathcal{H}}(x') < U_{\mathcal{H}}(x) \), then \( x' \) must also be a short position, for which we have \( m(x') \leq m(x) \) and \( m(x) - m(x') + 1 \leq y_{\mathcal{H}}(x') \leq y_{\mathcal{H}}(x) \) with \( (m(x), y_{\mathcal{H}}(x)) \neq (m(x'), y_{\mathcal{H}}(x')) \).

**Proof.** Since \( x \) is short, we have \( U_{\mathcal{H}}(x) = v_{\mathcal{H}}(x) < m(x) \) by Lemma 9. We also have \( T_{\mathcal{H}}(x') \geq m(x) \) by Lemma 15. Thus, \( U_{\mathcal{H}}(x') < U_{\mathcal{H}}(x) = v_{\mathcal{H}}(x) < m(x) \leq T_{\mathcal{H}}(x') \) follows by our assumption, implying \( U_{\mathcal{H}}(x') \neq T_{\mathcal{H}}(x') \). Thus, \( x' \) is not long. The inequality \( m(x') \leq m(x) \) holds for any move \( x \to x' \). Lemma 4 and \( v_{\mathcal{H}}(x') < v_{\mathcal{H}}(x) \) implies \( y_{\mathcal{H}}(x') \geq y_{\mathcal{H}}(x) \). Furthermore, \( v_{\mathcal{H}}(x') \leq v_{\mathcal{H}}(x) \) also implies \( (m(x), y_{\mathcal{H}}(x)) \neq (m(x'), y_{\mathcal{H}}(x')) \) since the pair of integers \( (m(x), y_{\mathcal{H}}(x)) \) determines uniquely the value \( v_{\mathcal{H}}(x) \). Assume next that \( x \to x' \) is an \( H \)-move for some hyperedge \( H \in \mathcal{H} \). Since \( \mathcal{H} \) is transversal-free by Lemma 14 there exists \( H' \in \mathcal{H} \) such that \( H \cap H' = \emptyset \). Then we have \( x'_i = x_i \geq m(x) \) for all \( i \in H' \), and thus, \( y_{\mathcal{H}}(x') \geq m(x) - m(x') + 1 \) follows by (11).

Lemma 18. If a hypergraph \( \mathcal{H} \subseteq 2^\mathcal{V} \) is transversal-free, then the function \( U_{\mathcal{H}} \) satisfies property (A), that is, for all moves \( x \to x' \) in NIM\(_{\mathcal{H}}\) we have \( U_{\mathcal{H}}(x) \neq U_{\mathcal{H}}(x') \).

**Proof.** To prove this statement, we consider four cases, depending on the types of the positions \( x \) and \( x' \), which can be long or short.

If both \( x \) and \( x' \) are long then \( U_{\mathcal{H}}(x) = T_{\mathcal{H}}(x) \neq T_{\mathcal{H}}(x') = U_{\mathcal{H}}(x') \), since every move strictly decreases the Tetris value by its definition.

If \( x \) is long and \( x' \) is short then we have \( U_{\mathcal{H}}(x) = T_{\mathcal{H}}(x) > m(x) \geq m(x') > v_{\mathcal{H}}(x') = U_{\mathcal{H}}(x') \), proving the claim. Here the first strict inequality is implied by the fact that every move strictly decreases the Tetris value and by Lemma 15 yielding \( T_{\mathcal{H}}(x) > T_{\mathcal{H}}(x') \geq m(x) \). The inequality \( m(x) \geq m(x') \) holds for every move \( x \to x' \). Finally \( m(x') > v_{\mathcal{H}}(x') \) is implied by Lemma 9.
If \( x \) is short and \( x' \) is long then we have \( U_H(x') = T_H(x') \geq m(x) \) by Lemma 15 and \( m(x) > v_H(x) = U_H(x) \) by Lemma 8 which together imply the claim.

Finally, if both \( x \) and \( x' \) are short then we have \((m(x), y_H(x)) \neq (m(x'), y_H(x'))\) by Lemma 8 if \( y_H(x) \neq y_H(x') \), then \( v_H(x) \neq v_H(x') \) follows by Lemma 7 since in this case \( Z(y_H(x)) \cap Z(y_H(x')) = \emptyset \). If \( y_H(x) = y_H(x') \) then by (2b) we have \( v_H(x) = v_H(x') \) if and only if \( m(x') = m(x) - \alpha y_H(x) \) for some positive integer \( \alpha \). Thus, we must have \( m(x') \leq m(x) - y_H(x) \). This implies, by (1b), that \( y_H(x') \geq y_H(x) + 1 \), which contradicts \( y_H(x) = y_H(x') \), completing the proof of our statement. To see the last implication, recall that \( H \) is transversal-free. Thus, if \( x \to x' \) is an \( H \)-move, then there is a hyperedge \( H' \in H \) such that \( H \cap H' = \emptyset \). For this hyperedge we have \( x_i = x_i \geq m(x) \) for all \( i \in H' \), from which the claim follows by (1b).

The above lemma has the following consequence.

**Theorem 1.** Let \( H \) be a JM hypergraph, \( H, H' \in H \), \( H \cap H' = \emptyset \), and \( H \subseteq S \subseteq V \setminus H' \). Then \( H^+ = H \cup \{ \emptyset \} \) is also a JM hypergraph with \( G_{H^+} = G_H \).

**Proof.** Let note first that \( H \subseteq S \) implies that \( T_{H^+} = T_H \) and consequently \( y_{H^+} = y_H \), \( v_{H^+} = v_H \), and thus \( U_{H^+} = U_H \). Furthermore, any move in \( NIM_H \) is still a move in \( NIM_{H^+} \). Therefore, for every \( 0 \leq v < U_{H^+} (x) \) there exists a move \( x \to x' \) in \( NIM_{H^+} \) such that \( U_{H^+} (x') = v \). Finally, by \( S \cap H' = \emptyset \) the hypergraph \( H^+ \) is also transversal-free, and thus by Lemma 18 we have \( U_{H^+} (x') \neq U_{H^+} (x) \) for all moves \( x \to x' \) of \( NIM_{H^+} \). \( \square \)

4. Sufficient Conditions

Let us first recall that properties (A) and (B) characterize the SG function of an impartial game. We can reformulate these now for \( NIM_H \), and obtain the following necessary and sufficient condition for \( H \) to be JM:

**Lemma 19.** A hypergraph \( H \subseteq 2^V \) is JM if and only if the following conditions hold:

(A0) \( H \) is transversal-free.

(B1) For every long position \( x \in \mathbb{Z}_+^V \) and integer \( m(x) \leq z < T_H(x) \) there exists a move \( x \to x' \) such that \( x' \) is long and \( T_H(x') = z \).

(B2) For every long position \( x \in \mathbb{Z}_+^V \) and integer \( 0 \leq z < m(x) \) there exists a move \( x \to x' \) such that \( x' \) is short and \( v_H(x') = z \).

(B3) For every short position \( x \in \mathbb{Z}_+^V \) and integer \( 0 \leq z < v_H(x) \) there exists a move \( x \to x' \) such that \( x' \) is short and \( v_H(x') = z \).

**Proof.** It is easy to see by Lemmas 10, 16 and 17 that conditions (B1), (B2), and (B3) are simple and straightforward reformulations of condition (B) for the case of \( NIM_H \) and the function \( g = U_H \) defined in (2a)-(2b).

Finally, Lemma 18 shows that condition (A0) implies (A), while Lemma 14 shows that if \( H \) is JM, then it is also transversal-free. \( \square \)

4.1. General Sufficient Conditions. Let us next replace conditions (B2) and (B3) with somewhat simpler sufficient conditions.

**Lemma 20.** If a hypergraph \( H \subseteq 2^V \) satisfies the following two conditions then it also satisfies (B2) and (B3).

(C2) For every position \( x \in \mathbb{Z}_+^V \) and integer \( 1 \leq \eta < y_H(x) \) there exists a move \( x \to x' \) such that \( m(x') = m(x) \) and \( y_H(x') = \eta \).
(C3) For every position \( x \in \mathbb{Z}_+^V \) and integers \( 0 \leq \mu < m(x) \) and \( m(x) - \mu + 1 \leq \eta \leq y_H(x) \) there exists a move \( x \rightarrow x' \) such that \( m(x') = \mu \) and \( y_H(x') = \eta \).

**Proof.** Let us consider first another hypergraph \( \tilde{H} = \{ S \subseteq V \mid 1 \leq |S| \leq n - 1 \} \). Then by the earlier cited result of Jenkyns and Mayberry [14] \( \tilde{H} \) is a JM hypergraph. Note also that the games \( NIM_H \) and \( NIM_{\tilde{H}} \) are both played over the same set of positions \( \mathbb{Z}_+^V \).

Let us now consider a position \( x \in \mathbb{Z}_+^V \). By Lemma 12 there exists a position \( \tilde{x} \in \mathbb{Z}_+^V \) such that \( m(x) = m(\tilde{x}) \) and \( y_H(x) = y_{\tilde{H}}(\tilde{x}) \). Now, let us observe that since \( \tilde{H} \) is JM, properties (B2) and (B3) are satisfied by Lemma 19. Let us also note that if \( \tilde{x} \rightarrow \tilde{x}' \) is a move in \( NIM_{\tilde{H}} \), guaranteed to exist by properties (B2) and (B3) then Lemmas 16 and 17 show that \( (m(\tilde{x}'), y_{\tilde{H}}(\tilde{x}')) \in S(m(\tilde{x}), y_{\tilde{H}}(\tilde{x})) = S(m(x), y_H(x)) \), where the set \( S(\alpha, \beta) \) is defined as

\[
S(\alpha, \beta) = \left\{ (\mu, \eta) \mid 0 \leq \mu \leq \alpha, \alpha - \mu + 1 \leq \eta \leq \beta \right\} \setminus \{(\alpha, \beta)\}.
\]

Let us observe next that properties (C2) and (C3) imply that for every \( (\mu, \eta) \in S(m(x), y_H(x)) \) there exists a move \( x \rightarrow x' \) in \( NIM_H \) such that \( m(x) = \mu \) and \( y_H(x) = \eta \). Thus, for every move \( \tilde{x} \rightarrow \tilde{x}' \) in \( NIM_{\tilde{H}} \) that validates properties (B2) and (B3) for \( \tilde{H} \) we have a corresponding move \( x \rightarrow x' \) in \( NIM_H \) such that \( m(\tilde{x}') = m(x') \) and \( y_H(x') = y_{\tilde{H}}(\tilde{x}') \), implying by Lemma 11 that \( v_H(x') = v_{\tilde{H}}(\tilde{x}') \) and that \( x' \) and \( \tilde{x}' \) are of the same type, that is, both are long or both are short.

Consequently, properties (C2) and (C3) do imply properties (B2) and (B3), as claimed. \( \Box \)

**Corollary 2.** If a hypergraph \( H \) satisfies properties (A0), (B1), (C2) and (C3), then it is JM.

**Proof.** The claim follows by Lemmas 20 and 19. \( \Box \)

### 4.2. Simplified Sufficient Conditions

The conditions in Corollary 2 still involve the existence of moves with certain properties. In this section we further weaken those conditions, and replace them with easier to check properties of the hypergraph itself.

Given a hypergraph \( H \subseteq 2^V \), we say that a subfamily \( \{H_0, H_1, \ldots, H_p\} \subseteq H \) forms a **chain**

\[
H_{k+1} \cap H_k \neq \emptyset \quad \text{and} \quad |H_{k+1} \setminus H_k| = 1 \quad \text{for all} \quad k = 0, \ldots, p - 1.
\]

For convenience, \( p = 0 \) is possible, that is, a single set is considered to be a chain.

For a subhypergraph \( F \subseteq H \subseteq 2^V \) we denote by \( V(F) \) the set of vertices of \( F \), that is, \( V(F) = \bigcup_{F \in H} F \). In particular we have \( V(H) = V \).

We shall consider the following properties:

(A1) \( H \) is minimal transversal-free.

(D1) For every pair of hyperedges \( H, H' \in H \) there exists a chain \( \mathcal{C} = \{H_0, H_1, \ldots, H_p\} \subseteq H \) such that \( H = H_0 \) and \( H' = H_p \).

(D2) For every subhypergraph \( F \subseteq H \subseteq 2^V \) such that \( V(F) \neq V \) there exist hyperedges \( F \in F \) and \( S \in H \) such that \( \emptyset \neq (S \setminus F) \subseteq V \setminus V(F) \).

Our main claim in this section is that the above properties are sufficient for a hypergraph to be JM.

**Theorem 2.** If a hypergraph \( H \) satisfies properties (A1), (D1), and (D2), then it is JM.
To arrive to a proof of this theorem, we need to prove several consequences of the above properties. In particular, property (D2) we need only to show that it implies the existence of a particular tetris move in NIM_H.

**Lemma 21.** If a hypergraph \( H \subseteq 2^V \) satisfies property (D2), then it satisfies the following property as well:

(D3) For every position \( x \in \mathbb{Z}_+ \) with \( m(x) > 0 \) there exists a hyperedge \( H \in H \) such that \( T_H(x^{s(H)}) = T_H(x) - 1 \) and \( m(x^{s(H)}) = m(x) - 1 \).

**Proof.** Let us fix a position \( x \in \mathbb{Z}_+ \) for which \( m(x) > 0 \) holds. An equivalent way of saying property (D3) is that there exists a hyperedge \( H \in H \) such that \( x \rightarrow x^{s(H)} \) is a tetris move and that for some \( i \in H \) we have \( x_i = m(x) \).

To prove the lemma let us assume indirectly that this is not the case. In other words, let us introduce

\[
F = \{ H \in H \mid T_H(x^{s(H)}) = T_H(x) - 1 \},
\]

and assume indirectly that \( V(F) \neq V \) (in particular, if \( x_i = m(x) \) then \( i \notin V(F) \)).

By property (D2) we have hyperedges \( F \in F \) and \( S \) such that \( S \setminus F \neq \emptyset \) and \( S \setminus F \subseteq V \setminus V(F) \). Let us now consider a mapping \( \mu : H \rightarrow \mathbb{Z}_+ \) that defines a tetris sequence for position \( x \), that is, we have \( \sum_{H \in H} \mu(H) = T_H(x) \) and \( x' = \sum_{H \in H} \mu(H) \chi(H) \leq x \). Since \( F \in F \), we can choose \( \mu \) such that \( \mu(F) > 0 \). Note that by our assumption, we have for any \( i \in V \setminus V(F) \) that \( x'_i = 0 < m(x) \leq x \). Thus, if we define

\[
\mu'(H) = \begin{cases} 
\mu(F) - 1 & \text{if } H = F \\
1 & \text{if } H = S \\
\mu(H) & \text{otherwise,}
\end{cases}
\]

then \( \mu' \) also defines a tetris sequence, with \( \mu'(S) > 0 \) in contradiction with the fact that \( S \notin V(F) \). This contradiction proves that our indirect assumption is not true, that is, \( V(F) = V \), from which the claim of the lemma follows.

Let us remark that condition (D3) may be necessary for a hypergraph to be JM, but we cannot prove this.

In particular, we are going to show that properties (A1), (D1), and (D3) imply properties (A0), (B1), (C2) and (C3), and thus, Theorem 2 will follow by Corollary 2.

Clearly (A1) implies (A0). The other three implications we will show separately.

**Lemma 22.** If a hypergraph \( H \subseteq 2^V \) satisfies properties (A1), (D1), and (D3), then it also satisfies property (B1), that is, for all long positions \( x \in \mathbb{Z}^V_+ \) and for all integer values \( m(x) \leq z < T_H(x) \) there exists a move \( x \rightarrow y \) such that \( y \) is long and \( T_H(y) = z \).

**Proof.** Let us fix a position \( x \in \mathbb{Z}^V_+ \). By Lemma 21 there exist \( j \in H \in H \) such that \( T_H(x^{s(H)}) = T_H(x) - 1 \) and \( x_j = m(x) \). By property (A1) the subhypergraph \( H \setminus \{j\} \) contains a transversal \( H' \in H \). By property (D2) we have then a chain \( \{H_0, H_1, \ldots, H_p\} \) such that \( H_0 = H, H_p = H' \) and \( |H_{k+1} \setminus H_k| \leq 1 \) for all \( k = 0, 1, \ldots, p - 1 \). Let us then define positions \( x^{\alpha,k} \) and \( x^{\omega,k} \) for \( k = 1, \ldots, p \) by

\[
x^{\alpha,k}_{i} = \begin{cases} 
0 & \text{if } i \in H_{k-1} \cap H_k \\
x_i - 1 & \text{if } i \in H_k \setminus H_{k-1} \\
x_i & \text{if } i \notin H_k,
\end{cases}
\]

and

\[
x^{\omega,k}_{i} = \begin{cases} 
0 & \text{if } i \in H_k \\
x_i & \text{if } i \notin H_k.
\end{cases}
\]
Set \( x^{\alpha,0} = x^{s(H)} \), and define \( x^{\mu,0} \) and \( x^{\omega,0} \) by
\[
x^{\mu,0}_i = \begin{cases} 0 & \text{if } i = j \\ x_i - 1 & \text{if } i \in H_0 \setminus \{j\} \\ x_i & \text{if } i \notin H_0, \end{cases}
\]
\[
x^{\omega,0}_i = \begin{cases} 0 & \text{if } i \in H_0 \\ x_i & \text{if } i \notin H_0. \end{cases}
\]

We claim first that all positions \( x^{\alpha,k} \geq y \geq x^{\omega,k} \) are long and are reachable from \( x \) by an \( H_k \)-move \( x \rightarrow y \), for \( k = 1, \ldots, p \). This is because \( m(y) = 0 \) for all these positions since they have a zero (namely \( y_i = 0 \) for all \( i \in H_{k-1} \cap H_k \), which is not an empty set by (B)). The analogous claim holds for positions \( x^{\mu,0} \geq y \geq x^{\omega,0} \) since \( y_j = 0 \) for all these positions. We also claim that positions \( x^{\alpha,0} \geq y \geq x^{\mu,0} \) are also long whenever \( x \) is long (and they are reachable from \( x \) by an \( H_0 \)-move \( x \rightarrow y \)). The last claim is true because we have \( m(y) \leq m(x^{\alpha,0}) = m(x) - 1 \), and \( y_H(y) \geq y_H(x^{\alpha,0}) \geq y_H(x) \) by Lemma 4, and thus, the fact that \( x \) is long implies \( m(y) < m(x) \leq \left( \frac{y_H(x)}{2} \right) \leq \left( \frac{y_H(y)}{2} \right) \).

Let us observe next that the sets of Tetris values for these ranges of positions form intervals by Lemma 3. Namely, we have
\[
\{ T_H(y) \mid x^{\alpha,0} \geq y \geq x^{\mu,0}\} = \left[ T_H(x^{\mu,0}), T_H(x) - 1 \right],
\]
\[
\{ T_H(y) \mid x^{\mu,0} \geq y \geq x^{\omega,0}\} = \left[ T_H(x^{\omega,0}), T_H(x^{\mu,0}) \right], \text{ and}
\]
\[
\{ T_H(y) \mid x^{\alpha,k} \geq y \geq x^{\omega,k}\} = \left[ T_H(x^{\omega,k}), T_H(x^{\alpha,k}) \right] \text{ for } k = 1, \ldots, p.
\]
We claim that these intervals cover all values in the interval \([m(x), T_H(x) - 1]\), as stated in the lemma. To see this claim, we show the following inequalities:
\[
T_H(x^{\alpha,k}) \geq T_H(x^{\omega,k-1}) - 1 \text{ for } 1 \leq k \leq p, \text{ and }
\]
\[
T_H(x^{\omega,p}) \leq m(x).
\]
The first group of inequalities follow by Corollary 1. For the second inequality observe that by our choice the set \( H' = H_0 \) intersects every hyperedge that does not contain \( j \in V \). Thus, the only possible moves from \( x^{\omega,p} \) are \( H \)-moves for hyperedges \( H \in \mathcal{H} \) that contain element \( j \). Since \( x_j = m(x) \), the total number of such moves is limited by \( m(x) \), as stated.

**Lemma 23.** If a hypergraph \( \mathcal{H} \subseteq 2^V \) satisfies properties (A1) and (D1), then it also satisfies property (C2), that is, for every position \( x \in \mathbb{Z}_+^V \) and integer \( 1 \leq \eta < y_H(x) \) there exists a move \( x \rightarrow x' \) such that \( m(x') = m(x) \) and \( y_H(x') = \eta \).

**Proof.** Let us fix a position \( x \in \mathbb{Z}_+^V \) and assume that \( x_j = m(x) \). Since \( y_H(x) + 1 = T_H(x - m(x)e) \) is a Tetris value, there exists a hyperedge \( H \in \mathcal{H} \) such that \( y_H(x^{s(H)}) = y_H(x) - 1 \). By property (A1) there exists a hyperedge \( H' \in \mathcal{H} \) that intersects all hyperedges of \( \mathcal{H} \) that do not contain element \( j \in V \). Then by property (D1) there exists a chain \( \mathcal{C} = \{ H_0, \ldots, H_p \} \) such that \( H_0 = H \) and \( H_p = H' \).

Then let \( x^{\alpha,0} = x^{s(H_0)} \), and define \( x^{\omega,0} \) by
\[
x^{\omega,0}_i = \begin{cases} m(x) & \text{for } i \in H_0 \\ x_i & \text{otherwise}, \end{cases}
\]
and positions \( x^{\alpha,k} \) and \( x^{\omega,k} \) for \( k = 1, \ldots, p \) by
\[
x^{\alpha,k}_i = \begin{cases} m(x) & \text{if } i \in H_{k-1} \cap H_k \\ x_i - 1 & \text{if } i \in H_k \setminus H_{k-1} \text{ and } x^{\omega,k}_i = \begin{cases} m(x) & \text{if } i \in H_k \\ x_i & \text{if } i \notin H_k, \end{cases} \end{cases}
\]
Let us observe next that the sets of \( y_H(x') \) values for the ranges \( x^{\alpha,k} \geq x' \geq x^{\omega,k} \), \( k = 0,1,\ldots,p \) form intervals by Lemma 3. Namely, we have
\[
\{ y_H(x') \mid x^{\alpha,k} \geq x' \geq x^{\omega,k} \} = [y_H(x^{\omega,k}), y_H(x^{\alpha,k})] \quad \text{for } k = 0,1,\ldots,p.
\]
We claim that these intervals cover all values in the interval \([1, y_H(x) - 1]\). To see this claim, we show the following relations:
\[
y_H(x^{\omega,p}) = 1,
\]
\[
y_H(x^{\alpha,k}) \geq y_H(x^{\omega,k-1}) - 1 \quad \text{for } 1 \leq k \leq p, \quad \text{and}
\]
\[
y_H(x^{\alpha,0}) = y_H(x) - 1.
\]
The second group of inequalities follow by Corollary 1 since these \( y_H \)-values are essentially Tetris values by their definition (1b). The first equality holds since \( H_p = H' \) intersects all hyperedges that avoids element \( j \in V \), and thus, we have \( T_H(x^{\omega,p} - m(x)e) = 0 \). The last equality \( y_H(x^{\alpha,0}) = y_H(x^{s(H)}) = y_H(x) - 1 \) follows by our choice of the set \( H \).

**Lemma 24.** If a hypergraph \( H \subseteq 2^V \) satisfies properties (A1) and (D1), then it also satisfies property (C3), that is, for every position \( x \in \mathbb{Z}_+^V \) and integers \( 0 \leq \mu < m(x) \) and \( m(x) - \mu \leq \eta \leq y_H(x) \) there exists a move \( x \to x' \) such that \( m(x') = \mu \) and \( y_H(x') = \eta \).

**Proof.** Let us fix a position \( x \in \mathbb{Z}_+^V \) and assume that \( x_j = m(x) \). Let us further fix an integer \( 0 \leq \mu < m(x) \).

Let us first choose an arbitrary hyperedge \( H \in H \) such that \( j \in H \). By property (A1) there exists another hyperedge \( H' \in H \) that intersects all hyperedges of \( H \) that do not contain element \( j \in V \). Then by property (D1) there exists a chain \( C = \{H_0,\ldots,H_p\} \) such that \( H_0 = H \) and \( H_p = H' \).

Let us then define \( x^{\alpha,0} \) and \( x^{\omega,0} \) by
\[
x^{\alpha,0}_i = \begin{cases} 
\mu & \text{if } i = j \\
x_i - 1 & \text{if } i \in H_0 \setminus \{j\} \\
x_i & \text{otherwise},
\end{cases}
\]
and define positions \( x^{\alpha,k} \) and \( x^{\omega,k} \) for \( k = 1,\ldots,p \) by
\[
x^{\alpha,k}_i = \begin{cases} 
\mu & \text{if } i \in H_{k-1} \cap H_k \\
x_i - 1 & \text{if } i \in H_k \setminus H_{k-1} \\
x_i & \text{if } i \notin H_k,
\end{cases}
\]
Let us observe next that the sets of \( y_H(x') \) values for the ranges \( x^{\alpha,k} \geq x' \geq x^{\omega,k} \), \( k = 0,1,\ldots,p \) form intervals by Lemma 3. Namely, we have
\[
\{ y_H(x') \mid x^{\alpha,k} \geq x' \geq x^{\omega,k} \} = [y_H(x^{\omega,k}), y_H(x^{\alpha,k})] \quad \text{for } k = 0,1,\ldots,p.
\]
We claim that these intervals cover all values in the interval \([1, y_H(x) - 1]\), as stated in the lemma. To see this claim, we prove the following relations.
\[
y_H(x^{\omega,p}) = 1,
\]
\[
y_H(x^{\alpha,k}) \geq y_H(x^{\omega,k-1}) - 1 \quad \text{for } 1 \leq k \leq p, \quad \text{and}
\]
\[
y_H(x^{\alpha,0}) \geq y_H(x).
\]
The second group of inequalities follow by Corollary 1 since these \( y_H \)-values are essentially Tetris values by their definition (1b). The first equality holds since \( H_p = H' \) intersects
all hyperedges that avoids element \( j \in V \), and thus we have \( T_H(x^{\omega_p} - \mu_e) = 0 \). The last inequality holds since \( x^{\omega_p} - \mu_e \geq x - m(x)e \).

\[ \square \]

**Proof of Theorem 2** Clearly, property (A1) is stronger than property (A0). Lemmas 22 and 24 imply that properties (B1), (C2), and (C3) hold. Thus, the statement follows by Corollary 2.

5. Classes of JM Hypergraphs

In this section we apply Theorem 2 to a variety of hypergraphs classes and show that they are JM.

5.1. Matroid hypergraphs. In this section, we study JM matroid hypergraphs.

Let us call \( \mathcal{H} \subseteq 2^V \) a matroid hypergraph if the following exchange property holds for all pairs of hyperedges \( H, H' \in \mathcal{H} \):

\[
\forall i \in H \setminus H' \exists j \in H' \setminus H : (H \setminus \{i\}) \cup \{j\} \in \mathcal{H}.
\]

In other words, \( \mathcal{H} \) is a matroid hypergraph if it is the set of bases of a matroid (see [19, 20]).

**Lemma 25.** Matroid hypergraphs satisfy (D1) and (D2).

**Proof.** For property (D1), let us fix two arbitrary hyperedges \( H, H' \in \mathcal{H} \), and let us consider a chain \( C = \{H_0, \ldots, H_p\} \subseteq \mathcal{H} \) such that \( H_0 = H \) and \( d(C) = |H' \setminus H_p| \) is as small as possible. Since there are only finitely many different chains in \( \mathcal{H} \), the quantity \( d(C) \) is well defined. We claim that \( d(C) = 0 \), which implies property (D1), since this applies to any two hyperedges. To see this claim, assume indirectly that \( d(C) > 0 \), and apply the exchange axiom for sets \( H_p \) and \( H' \). Since \( d(C) > 0 \) we have an element \( i \in H_p \setminus H' \), and by axiom (M) there exists an element \( j \in H' \setminus H_p \) such that \( H_{p+1} = (H_p \setminus \{i\}) \cup \{j\} \in \mathcal{H} \). Then for \( C' = \{H_0, \ldots, H_p, H_{p+1}\} \) it follows that \( d(C') = d(C) - 1 \), contradicting the fact that \( d(C) \) is as small as possible. This contradiction proves our claim.

For property (D2), let us consider an arbitrary subfamily \( F \subseteq \mathcal{H} \) such that \( V(F) \neq V \). Let us choose two distinct sets \( S \in \mathcal{H} \) and \( F \in \mathcal{F} \) with minimum \( |(S \setminus F) \cap V(F)| \). We claim that \( |(S \setminus F) \cap V(F)| = 0 \), and hence these sets \( S \) and \( F \) show property (D2).

Assume a contrary that there exists an element \( i \) in \( (S \setminus F) \cap V(F) \). By the exchange axiom there exists a \( j \in F \setminus S \), such that \( S' = (S \setminus \{i\}) \cup \{j\} \in \mathcal{H} \). Then we have \( |(S' \setminus F) \cap V(F)| < |(S \setminus F) \cap V(F)| \), which contradicts our assumption.

**Theorem 3.** Let \( \mathcal{H} \) be a matroid hypergraph. Then \( \mathcal{H} \) is a JM hypergraph if and only if it is minimal transversal-free.

**Proof.** It follows from Theorem 2 and Lemmas 14 and 25.

**Corollary 3.** Self-dual matroid hypergraphs are JM.

**Proof.** We apply Theorem 3 and show that self-dual matroid hypergraphs are minimal transversal-free.

Let \( \mathcal{H} \) be a self-dual matroid. Since for every hyperedge in \( \mathcal{H} \), its complement is also a hyperedge, no \( H \in \mathcal{H} \) is a transversal of \( \mathcal{H} \). Furthermore, by self-duality of \( \mathcal{H} \), any hyperedge \( H \in \mathcal{H} \) has size \( k = n/2 \). In any proper induced subhypergraph on at most \( 2k - 1 \) elements any two hyperedges of size \( k \) must intersect. Therefore \( \mathcal{H} \) is minimal transversal-free. □

Recall that any matroid hypergraph is \( k \)-uniform for some \( k \). If \( k > n/2 \), then it is not transversal-free, and hence not JM. If \( k = n/2 \), then we can see that a matroid hypergraph
is JM if and only if it is self-dual. For $k < n/2$, we remark that no matroid hypergraph is self-dual. However, the following discussion shows that there are a number of JM matroid hypergraphs.

Let $V = \{1, \ldots, 7\}$, and define $\mathcal{H} \subseteq 2^V$ by $\mathcal{H} = \binom{V}{3} \setminus \mathcal{F}$, where $\mathcal{F}$ denotes Fano plane, i.e.,

$$\mathcal{F} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{1, 6, 7\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}.$$  

Then we can see that $\mathcal{H}$ is a matroid hypergraph and minimal transversal-free.

We extend the above example and show that there exists a large family of matroid hypergraphs with $n = 2k + \delta$ for $\delta > 0$, which are minimal transversal-free and hence JM.

Let $\delta$ and $k$ be integers such that $0 < \delta \leq k - 2$, and assume that $V = \{1, \ldots, n\}$ where $n = 2k + \delta$. Let us consider a $(k + \delta - 1)$-uniform hypergraph $\mathcal{K} \subseteq 2^V$ satisfying the following three conditions.

(K1) $|K \cap K'| \geq \delta$ for all hyperedges $K, K' \in \mathcal{K}$.

(K2) $|K \cap K'| \leq k - 2$ for all distinct hyperedges $K, K' \in \mathcal{K}$.

(K3) No singleton is a transversal of $\mathcal{K}$.

Define

$$\mathcal{H} = \binom{V}{k} \setminus \binom{\binom{V}{k}}{K} | K \in \mathcal{K}.$$  

Lemma 26. Let $\mathcal{K}$ be a $(k + \delta - 1)$-uniform hypergraph that satisfies (K2), $S \subseteq V$ be a set of size $|S| = k - 1$, and $W \subseteq V \setminus S$. Assume that for any $v \in W$, $S \cup \{v\}$ is not contained in $\mathcal{H}$. Then we have $|W| \leq \delta$.

Proof. Since $S \cup \{v\}$ is not contained in $\mathcal{H}$ for every $v \in W$, there exists $K_v$ in $\mathcal{K}$ such that $K_v \supseteq S \cup \{v\}$. Note that the union of the sets $K_v$ is of size at least $k - 1 + |W|$, and thus if $|W| \geq \delta + 1$, we have two elements $v$ and $v'$ in $W$ such that $K_v \neq K_{v'}$. However, since $K_v \cap K_{v'} \supseteq S$, this contradicts (K2). \hfill $\Box$

Lemma 27. If a $(k + \delta - 1)$-uniform hypergraph $\mathcal{K}$ satisfies (K1), (K2), and (K3), then $\mathcal{H}$ is minimal transversal-free.

Proof. We first show that $\mathcal{H}$ is transversal-free, i.e., any hyperedge $H \in \mathcal{H}$ has a hyperedge $H' \in \mathcal{H}$ such that $H \cap H' = \emptyset$. Let $S \subseteq V \setminus H$ with $|S| = k - 1$, and $W = V \setminus (H \cup S)$. Since $|W| > \delta$, by Lemma 26 we must have at least one $v \in W$ such that $H' = S \cup \{v\}$ belongs to $\mathcal{H}$.

We next show that for any nonempty set $R \subseteq V$, the induced subhypergraph $\mathcal{H}_{V \setminus R}$ is either empty or it has a transversal $T \in \mathcal{H}_{V \setminus R}$.

If $|R| \geq \delta + 1$, then any two hyperedges in $\mathcal{H}_{V \setminus R}$ intersect. Thus it remains to consider the case of $|R| \leq \delta$. Let $K \in \mathcal{K}$ be a hyperedge that maximizes $|R \setminus K|$. Then by (K3) we have $R \nsubseteq K$, and thus $|V \setminus (R \cup K)| \leq k$ is implied.

If $|V \setminus (R \cup K)| < k$, let $S$ be a set such that $V \setminus (R \cup K) \subseteq S \subseteq V \setminus R$ and $|S| = k - 1$, and let $W = K \setminus (R \cup S)$. Since $|W| = k + \delta + 1 - |R| > \delta$, by Lemma 26 we must have at least one $v \in W$ such that $H' = S \cup \{v\}$ belongs to $\mathcal{H}$. We claim that $H'$ is a transversal of $\mathcal{H}_{V \setminus R}$. Indeed, $V \setminus (H' \cup R)$ is a subset of $K$, and thus no subset of it (of size $k$) is contained in $\mathcal{H}$.

On the other hand, if $|V \setminus (R \cup K)| = k$, then $H' = V \setminus (R \cup K)$ is a transversal hyperedge in $\mathcal{H}_{V \setminus R}$. Indeed, $H' \in \mathcal{H}_{V \setminus R}$, since otherwise there exists a $K' \in \mathcal{K}$ such that $H' \subseteq K'$. This implies $|K \cap K'| < \delta$, contradicting (K1). Furthermore, $\mathcal{H}_{V \setminus R}$ contains no hyperedge disjoint from $H'$, since $V \setminus (R \cup H')$ is a subset of $K \in \mathcal{K}$.
Lemma 28. If a \((k + \delta - 1)\)-uniform hypergraph \(K\) satisfies \((K2)\), then \(H\) is a matroid hypergraph.

Proof. Consider two distinct hyperedges \(H, H' \in H\). We assume that \((M)\) does not hold for \(H\) and \(H'\), and derive a contradiction.

If \(|H \cap H'| = k - 1\) then \((M)\) clearly holds, and hence we have \(|H \cap H'| \leq k - 2\). By our assumption, there exists an element \(i \in H \setminus H'\) such that any \(j \in H' \setminus H\) satisfies \((H \setminus \{i\}) \cup \{j\} \notin H\). This means that for any \(j \in H' \setminus H\), we have \(K_j \in K\) such that \(K_j \supseteq (H \setminus \{i\}) \cup \{j\}\). We note that \(H' \setminus H\) contains two elements \(j, \ell\) that satisfy \(K_j \neq K_\ell\), since otherwise \(K_j\) contains \(H'\), a contradiction on its own. Since for these indices \(j\) and \(\ell\), we have \(|K_j \cap K_\ell| \geq k - 1\), we get a contradiction with \((K2)\). \(\square\)

Theorem 4. If a \((k + \delta - 1)\)-uniform hypergraph \(K\) satisfies \((K1)\), \((K2)\), and \((K3)\), then \(H\), defined by \((5)\), is a JM and matroid hypergraph.

Proof. Follows from Lemmas 27 and 28. \(\square\)

To complete this section, let us construct a hypergraph \(K\) with the desired properties.

Let \(\delta\) and \(k\) be integers such that \(0 < \delta \leq k - 2\). Define \(V\) by \(V = W \cup U\), where \(W = \{1, \ldots, k+\delta-1\}\), and \(U = \{1', \ldots, (k+1)\}'\). Note that \(|V| = (k+\delta-1)+(k+1) = 2k+\delta\). Let

\[ K = \{W, \{1, \ldots, \delta\} \cup \{1', \ldots, (k-1)\}', \{\delta + 1, \ldots, 2\delta\} \cup \{3', \ldots, (k+1)\}'\}. \]

It is not difficult to see that \(K \subseteq 2^U\) is a \((k + \delta - 1)\)-uniform hypergraph satisfying \((K1)\), \((K2)\), and \((K3)\).

5.2. JM Hypergraphs Arising from Graphs. In this section we use the standard terminology of graph theory, see e.g., \([13]\). We shall consider simple undirected graphs, and use their set of edges or set of vertices as the base set to define additional families of JM hypergraphs.

Given a graph \(G = (U, E)\) and a subset \(F \subseteq E\) of the edges, we denote by \(d_F(u)\) the degree of vertex \(u \in U\) with respect to the subgraph \((U, F)\). In other words, \(d_F(u)\) is the number of edges in \(F\) that are incident with vertex \(u\). If \(d_F(u) > 0\) then we call \(u\) a supporting vertex of subset \(F\), and we denote by \(U(F) \subseteq U\) the set of supporting vertices of \(F\), that is, \(U(F) = \{u \in U \mid d_F(u) > 0\}\).

Given an integer \(k < |E|\), let us define

\[
\begin{align*}
(6) & \quad F_{e,c}(G, k) = \{F \subseteq E \mid |F| = k, (U(F), F) \text{ is connected}\} \\
(7) & \quad F_{v,c}(G, k) = \left\{ U(F) \subseteq U \mid F \subseteq E, |F| = k, (U(F), F) \text{ is connected} \right\} \\
(8) & \quad F_{c,t}(G, k) = \{F \subseteq E \mid |F| = k, (U(F), F) \text{ is a tree}\} \\
(9) & \quad F_{v,t}(G, k) = \left\{ U(F) \subseteq U \mid F \subseteq E, |F| = k, (U(F), F) \text{ is a tree} \right\}.
\end{align*}
\]

Lemma 29. If \(G = (U, E)\) is a connected graph and \(k < |E|\), then the hypergraphs \(F_{e,c}(G, k)\), \(F_{c,t}(G, k)\) for \(k \geq 2\) and \(F_{v,c}(G, k)\), \(F_{v,t}(G, k)\) for \(k \geq 1\) satisfy property \((D1)\).

Proof. We are going to prove the statement for \(F_{e,c}(G, k)\). For the others similar proofs work. Let \(A, B \in F_{e,c}(G, k)\) and define

\[ d(A, B) = -\mu(A, B) + \rho(A, B), \]
Lemma 30. If \( A \) does not belong to any of the sets in \( \mathcal{F} \) then there exists \( f \in \mathcal{F} \) such that \( A \cup \{ e \} \) is a tree, and therefore it must have a leaf edge \( f \neq e \).

Proof. We are going to prove the statement for \( \mathcal{F}_{e,c}(G, k) \). For the others similar proofs work.

Assume that \( \mathcal{F} \subseteq \mathcal{F}_{e,c}(G, k) \) is a subfamily such that \( \bigcup_{F \in \mathcal{F}} F \neq \emptyset \). Let us denote by \( W \subseteq \emptyset \) the set of vertices incident to some of the sets in \( \mathcal{F} \). We claim that if \( A \neq B \) then there exists a \( D \in \mathcal{F}_{e,c}(G, k) \) such that \( A \cap D \neq \emptyset, |D \setminus A| = 1, d(A, B) > d(D, B) \). By repeatedly applying this claim, we can construct a chain from \( A \) to \( B \).

Case 1: \( U(A) \cap U(B) = \emptyset \). Let \( P \subseteq E \) be a shortest path connecting \( U(A) \) with \( U(B) \), and \( e \in P \) be the first edge incident with \( U(A) \). There exists an edge \( f \in A \) such that \( D = (A \cup \{ e \}) \setminus \{ f \} \) is connected: If \( A \) contains a cycle then \( f \) could be any edge of this cycle, otherwise \( A \cup \{ e \} \) is a tree, and therefore it must have a leaf edge \( f \neq e \).

Case 2: \( U(A) \cap U(B) \neq \emptyset \). Choose a maximum connected component \( K \subseteq A \cap B \). Choose \( e \in B \setminus A \) incident with \( K \). Such an edge exists since \( B \) is connected and \( B \neq A \). Then there exists \( f \in A \setminus K \) such that \( D = (A \cup \{ e \}) \setminus \{ f \} \) is connected. To see this contract edges \( K \cup \{ e \} \). If \( A \setminus K \) contains a cycle after this contraction, then any edge of this cycle can be chosen. Otherwise we choose a leaf edge \( f \in A \setminus K \).

Thus, the set \( D \) satisfies the claim in both cases. \( \square \)

Lemma 30. If \( G = (U, E) \) is a connected graph and \( 1 \leq k < |E| \), then the hypergraphs \( \mathcal{F}_{e,c}(G, k), \mathcal{F}_{e,t}(G, k), \mathcal{F}_{v,c}(G, k) \), and \( \mathcal{F}_{v,t}(G, k) \) satisfy property (D2).

Proof. Property (A1) follows by our assumption, while properties (D1) and (D2) follow by Lemmas 29 and 30. Thus, the statement is implied by Theorem 2. \( \square \)

There are several infinite families of graphs for which we can apply Theorem 5. We use standard graph theoretical notation, see e.g., [13]. We denote by \( C_n \) the simple cycle on \( n \) vertices, \( K_{a,b} \) the complete bipartite graph with \( a \) and \( b \) vertices in the two classes, etc.

Circulant hypergraphs: For any given value of \( k \geq 2 \), it is easy to see that the graphs \( C_{2k} \) and \( C_{2k+1} \) yield JM hypergraphs with all four definitions. In fact, they are isomorphic families for both \( C_{2k} \) and \( C_{2k+1} \).

Additional self-dual matroids: Graphs \( K_{2,k} \) are also good example for \( k \geq 2 \) with both definitions [3] and [3]. Interestingly, both hypergraphs are self-dual matroids, and they are not isomorphic with one another for \( k \geq 4 \).

Trees: For any \( k \geq 2 \) the following subfamily of trees on \( k^2 + k \) edges provide good examples with both definitions involving edge subsets. Let \( T_i, i = 1, \ldots, k + 1 \) be an arbitrary trees of \( k \) edges each on distinct sets of vertices, and let \( v_i \) be a leaf vertex
of \( T_i \) for all \( i = 1, \ldots, k + 1 \). Then we can get a tree \( T \) by identifying these leaf vertices. \( T \) has \( k^2 + k \) edges, and the family \( \mathcal{F}_{e,c}(T, k) \) is minimal transversal-free. (Note that definitions (6) and (8) yield isomorphic hypergraphs in this case.)

**Star of cliques:** Another example is a graph \( G \) formed by \( k + 1 \) cliques on \( k \) vertices each, joined by one-one edges to a common root vertex. In this case only definition (8) yields a hypergraph that is, \( \mathcal{F}_{e,t}(G, k) \) minimal transversal-free and has \( (k+1)((k\choose 2)+1) \) vertices.

**Petersen:** Finally, a singular example is provided by the Petersen graph \( P \) for which the family \( \mathcal{F}_{e,c}(P, 7) \) is minimal transversal-free.

**Figure 1.** The nine forbidden induced subgraphs characterizing line graphs, see [2].
In Section 6 we show that the number of JM hypergraphs defined by (6), (7), (8), and (9) is bounded by a function of \( k \).

5.3. JM Graphs. In this section we provide a complete characterization of JM graphs, i.e., 2-uniform JM hypergraphs. Note that

\[ E = \mathcal{F}_{e,c}(G, 1) = \mathcal{F}_{v,t}(G, 1) \]

holds for any graph \( G = (V, E) \), which implies the following result.

**Theorem 6.** A graph \( G \) is JM if and only if it is connected and minimal transversal-free.

**Proof.** The necessity follows from Lemmas 31 and 32 and the sufficiency follows from Theorem 5 and (10).

In the sequel we characterize all connected minimal transversal-free graphs.

**Lemma 31.** If a graph \( G = (V, E) \) is connected and minimal transversal-free, then it is the line graph of a simple graph.

**Proof.** Let us indirectly assume that \( G \) is not a line graph. Then \( G \) must contain one of the 9 forbidden induced subgraphs shown in [2], see Figure 1. We claim that none of these 9 graphs can be an induced subgraph of \( G \), since we assumed that \( G \) is minimal transversal-free. For this, note that in each of the graphs \( G_i \), \( i = 2, \ldots, 9 \) contains as a proper induced subgraph at least one of \( C_4, C_5, K_4 \), or \( 2K_2 \). In all cases we do not have an edge that would intersect all others.

We claim that the claw \( G_1 \) cannot be an induced subgraph of \( G \). Since \( G_1 \) is not transversal-free, we assume that \( G_1 \) is a proper induced subgraph of \( G \), and derive a contradiction. Remove from \( G \) a leaf of the claw, say \( v_2 \). Then \( G \setminus v_2 \) has an intersecting edge \( e \), i.e., an edge \( e \) in \( G \setminus v_2 \) intersects all edges in \( G \setminus v_2 \). We note that \( e \) is incident with the center of the claw \( v_1 \).

Suppose that \( e \) is an edge of the claw, say \( e = (v_1, v_3) \). Then \( v_4 \) is a vertex of degree 1 in \( G \). Let us denote by \( e' \) the intersecting edge in \( G \setminus v_4 \). The edge \( e' \) must be incident with the center \( v_1 \), and hence \( e' \) is an intersecting edge of \( G \), which contradicts that \( G \) is transversal-free.

Suppose now that \( e \) is not contained in the claw, say \( e = (v_1, v_5) \) where \( v_5 \) is not in \( G_1 \). If \( v_3 \) or \( v_4 \) is of degree 1 then by the same argument \( G \) has an intersecting edge. Otherwise \( G \) contains edge \( e_1 = (v_3, v_5) \). Since \( G \) is transversal-free, there must exist an edge \( e_2 \) disjoint from \( e \) through \( v_2 \), say \( e_2 = (v_2, v_6) \). Consider again \( G \setminus v_4 \). By our assumption it has an intersecting edge \( e_3 \). As we know, \( e_3 \) is incident with \( v_1 \). However, no such edge can intersect both \( e_1 \) and \( e_2 \). This contradicts that \( e_3 \) is an intersecting edge in \( G \setminus v_4 \).

The following statement is straightforward from the definition.

**Lemma 32.** For every simple graph \( G \) we have that \( \mathcal{F}_{e,c}(G, 2) \) is the edge set of the line graph of \( G \).

**Lemma 33.** If \( G = (V, E) \) is a JM graph then we have \( 4 \leq |V| \leq 6 \).

**Proof.** It follows from Theorem 6 that \( G \) has at least two disjoint edges. Hence we have \( |V| \geq 4 \). By Theorem 6 and Lemmas 31 and 32 there exists a graph \( G^* = (V^*, E^*) \) such that \( E = \mathcal{F}_{e,c}(G^*, 2) \). By Theorem 6 and Lemma 33 we have \( |E^*| \leq 2^2 + 2 = 6 \), where Lemma 35 can be found in the next section. Since \( G \) is the line graph of \( G^* \), we have \( E^* = V \), implying \( |V| \leq 6 \).
Figure 2. The six JM graphs

**Theorem 7.** Among all graphs, only six graphs in Figure 2 are JM.

**Proof.** It is easy to see that all six graphs are connected and minimal transversal-free. Since graphs correspond to \( F_v,c(G,1) \), by Theorem 5 they are JM graphs.

To show that no other JM graph exists, it is sufficient to check all graphs \( G = (V,E) \) with \( 4 \leq |V| \leq 6 \) by Lemma 33; see e.g., [13] for a complete list of graphs with up to 6 vertices.

Before concluding this section, we remark that Lemma 33 provides a tighter bound than the one in (iii) of Lemma 35, when \( k = 1 \).

### 6. Size of \( k \)-Uniform JM Hypergraphs

In this section we study the bound for the size of \( k \)-uniform JM hypergraphs. We first provide upper bound for the size of \( k \)-uniform minimal transversal-free hypergraphs, implying upper bound for the size of \( k \)-uniform JM hypergraphs.

**Lemma 34 ([4]).** Assume that \( \mathcal{H} \subseteq 2^V \) is a \( k \)-uniform minimal transversal-free hypergraph. Then, we have

\[
|V| \leq k \binom{2k}{k}.
\]

**Proof.** Since \( \mathcal{H} \) is a minimal transversal-free, for every \( i \in V \) we have a hyperedge \( H_i \in \mathcal{H} \) such that \( H_i \cap H' \neq \emptyset \) for all \( H' \in \mathcal{H} \) with \( i \not\in H' \). Let us denote by \( \mathcal{H}' = \{ H_i \mid i \in V \} \) the family of these hyperedges. By the transversal-freeness we also have for every hyperedge \( H \in \mathcal{H}' \) a disjoint hyperedge \( B(H) \in \mathcal{H} \), \( H \cap B(H) = \emptyset \). Let us now choose a minimal subhypergraph \( \mathcal{B} \subseteq \mathcal{H} \) such that

\[
\forall H \in \mathcal{H}' \exists B \in \mathcal{B} : H \cap B = \emptyset.
\]

Let us note first that such a \( \mathcal{B} \) must form a cover of \( V \), i.e., \( V = \bigcup_{B \in \mathcal{B}} B \). This is because for all \( H_i \in \mathcal{H}' \) we have a \( B \in \mathcal{B} \) such that \( H_i \cap B = \emptyset \), and consequently, \( i \in B \). Let us observe next that for all \( B \in \mathcal{B} \) we have at least one \( A(B) \in \mathcal{H}' \) such that \( A(B) \cap B = \emptyset \) and \( A(B) \cap B' \neq \emptyset \) for all \( B' \in \mathcal{B} \setminus \{B\} \). This is because we choose \( \mathcal{B} \) to be a minimal family with
respect to \( (\text{i}) \). Let us now define \( \mathcal{A} = \{ A(B) \in \mathcal{H}' \mid B \in \mathcal{B} \} \). The pair \( \mathcal{A}, \mathcal{B} \) of hypergraphs now satisfies the conditions of a classical theorem of Bollobás \([5]\), which then implies that

\[
|\mathcal{A}| = |\mathcal{B}| \leq \binom{2k}{k}.
\]

Since \( \mathcal{B} \) is a \( k \)-uniform hypergraph that covers \( V \), our claim follows.

\( \square \)

This clearly implies that \( |\mathcal{H}| \leq 2k^{2k} \).

An example, provided by D. Pávölgyi \([16]\), almost matches the upper bound above on the size of \( V \), and we recall it here for completeness.

Let \( V = U \cup W \), where \( |U| = 2k - 2 \), \( |W| = \frac{1}{2} \binom{2k}{k-1} \), and \( U \cap W = \emptyset \).

Consider all \( (k - 1) \) subsets of \( U \), labeled as \( A_{i} \) and \( B_{i} \) such that \( A_{i} \cap B_{i} = \emptyset \) for \( i = 0, \ldots, r - 1 \), where \( r = \frac{1}{2} \binom{2k}{k-1} \). Assume further that \( W = \{ w_{0}, w_{1}, \ldots, w_{r-1} \} \), and define

\[
\mathcal{H} = \{ B_{i} \cup \{ w_{i} \}, A_{i+1} \cup \{ w_{i} \} \mid i = 0, \ldots, r - 1 \},
\]

where indices are taken modulo \( r \). The hypergraph \( \mathcal{H} \) is \( k \)-uniform.

Easy to see that it is a minimal transversal-free hypergraph. Namely, if we delete some points from \( U \) then all remaining hyperedges are intersecting already in \( U \). If we delete some points from \( W \) but not \( U \) then consider an index \( i \) such that we deleted \( w_{i} \) and not \( w_{i+1} \). Then \( B_{i+1} \cup \{ w_{i+1} \} \) intersects all remaining hyperedges.

We next consider the size of JM hypergraphs discussed in Section \( 4 \). As mentioned in the introduction, self-dual matroid hypergraphs \( \mathcal{H} \) are \( k \)-uniform for \( k = n/2 \), and satisfy

\[
2^{k} \leq |\mathcal{H}| \leq \binom{2k}{k}.
\]

Since we characterize JM graphs in Section \( 5.3 \) in the rest of this section, we provide an upper bound for the size of JM hypergraphs defined by \((\text{6}), (\text{7}), (\text{8}), \) and \((\text{9})\). For this, we prove that the size of a graph, for which definitions \((\text{6}), (\text{7}), (\text{8}), \) and \((\text{9})\) yield minimal transversal-free hypergraphs, is bounded by a function of \( k \).

Lemma 35. Let \( G = (U, E) \) be a connected graph.

(i) If \( F_{e,c}(G, k) \) is minimal transversal-free, then \( |E| \leq k^{2} + k \).

(ii) If \( G \) is simple and \( F_{e,d}(G, k) \) is minimal transversal-free, then \( |E| \leq \frac{k^{3}}{2} + \frac{k}{2} + 1 \).

(iii) If \( F_{v,c}(G, k) \) or \( F_{e,f}(G, k) \) is minimal transversal-free, then \( |U| \leq 2k^{3} + 4k^{2} + k + 2 \).

Proof. We prove first (i). Let us choose an edge \( e \), such that the deletion of \( e \) does not disconnect the graph \( G \). Such an edge always exists, since we can pick an edge on a cycle or a leaf edge. Then, by the minimal transversal-freeness, after the deletion of \( e \) we must have a connected subgraph \( T \) of \( k \) edges such that no disjoint connected subgraph of \( k \) edges exists. This means that if we delete in addition the edges of \( T \), then the graph decomposes into connected subgraphs, each having at most \( k - 1 \) edges. Since these connected subgraphs intersect the vertex set of \( T \) in disjoint sets, and since \( T \) has at most \( k + 1 \) vertices, we cannot have more than \( (k - 1)(k + 1) + k + 1 = k^{2} + k \) edges in \( G \).

For (ii) let us repeat the same argument and note that in each connected component now we cannot have a tree of \( k \) edges. This means that each connected component has at most \( k \) vertices, that is, at most \( \binom{k}{2} \) edges, since \( G \) is assumed to be simple. Thus, in total we get that

\[
|E| \leq 1 + k + (k + 1) \binom{k}{2} = \frac{k^{3}}{2} + \frac{k}{2} + 1.
\]
For (iii) we provide a proof for $\mathcal{F}_{v,c}(G,k)$. The same proof works for $\mathcal{F}_{v,t}(G,k)$, as well. Let $v$ be a vertex in $G$ such that $G - v$ is connected. If $G - v$ contain no connected $k$-edge subgraph, then we have $|U| \leq k + 1$. Otherwise, since $\mathcal{F}_{v,c}(G,k)$ is minimal transversal-free, there exists a hyperedge $F_v \in \mathcal{F}_{v,c}(G,k)$ that intersects all $F \in \mathcal{F}_{v,c}(G,k)$ with $v \notin F$. Let $C_i$ ($i = 1, \ldots, p$) be connected components of $G - (\{v\} \cup F_v)$. Then we have $|V(C_i)| \leq k$, since $C_i$ contains at most $k - 1$ edges by the definition of $F_v$ and the hypergraph $\mathcal{F}_{v,c}(G,k)$. Furthermore, we have $N_G(C_i) \subseteq \{v\} \cup H_v$ for all $i$, where $N_G(C_i)$ denotes the set of neighbors of $C_i$ in $G$.

For any component $C_i$, let $w$ be a vertex in $C_i$. We first claim that a hyperedge $F_w$ satisfies

$$\tag{12} N_G(C_i) \not\subseteq F_w,$$

where we recall that $F_w$ is a hyperedge in $\mathcal{F}_{v,c}(G,k)$ that intersects all $F \in \mathcal{F}_{v,c}(G,k)$ with $w \notin F$. Since $\mathcal{F}_{v,c}(G,k)$ is minimal transversal-free, $\mathcal{F}_{v,c}(G,k)$ contains a hyperedge that is disjoint from $F_w$ and contains $w$. This implies the claim.

Let $u$ be a vertex in $N_G(C_i) \setminus F_w$, Then it holds that

$$\tag{13} |N_G(u) \cap (\bigcup_j C_j)| \leq 2k,$$

since otherwise, $|N_G(u) \setminus (\{w\} \cup F_w)| \geq k$, implying that $\mathcal{F}_{v,c}(G,k)$ contains a hyperedge $F$ disjoint from $\{w\} \cup F_w$, a contradiction.

By (12) and (13), the number of connected components $C_i$ is bounded by $2k|\{v\} \cup F_v| = 4k^2 + 2k$. Thus, the number of vertices of $G$ is bounded by

$$(4k^2 + 2k)k + k + 2 = 4k^3 + 2k^2 + k + 2.$$

The above bounds imply that for any given $k$ we have only finitely many different such JM hypergraphs, with all four definitions. The examples derived from trees and stars of cliques show that bounds (i) and (ii) are sharp.

7. Further Examples and Concluding Remarks

Let us first show a small example for which property (A1) holds, but both properties (D1) and (D3) fail. This example is not JM, showing that not all minimal transversal-free hypergraphs are JM. Unfortunately, we cannot prove the necessity of properties (D1) or (D3), though we conjecture that property (D3) may be necessary for a hypergraph to be JM.

Our example is $\mathcal{H}_{\text{cube}}$ formed by the facets of the 3-dimensional unit cube. The vertex set is $V = \{0,1\}^3$, and the six hyperedges of $\mathcal{H}_{\text{cube}} \subseteq 2^V$ are the subsets $H_{i,\alpha} = \{\sigma \in V \mid \sigma_i = \alpha\}$ for $i = 1, 2, 3$ and $\alpha = 0, 1$. This is a 4-uniform hypergraph on 8 vertices, and it is clearly minimal transversal-free. On the other hand it does not satisfy any of three properties (D1), (D2), (D3). To see that it is not a JM hypergraph, assume that $m = \binom{3p+1}{2}$ and $q = m + p$ for some positive integer $p$, and consider the position $x \in \mathbb{Z}_+^V$ defined as $x_000 = m$, $x_{100} = x_{010} = x_{001} = q$, $x_{110} = x_{101} = x_{001} = 2q$ and $x_{111} = 3q$. It is easy to see that for this position we have $m(x) = m$, $y(x) = 3p + 1$ and $T(x) = 3q$, consequently this is a long position. Furthermore, every Tetris move $x \rightarrow x'$ is an $H_{i,1}$-move for some $i = 1, 2, 3$ and we must have $m(x') = m(x)$ and $y(x') < y(x)$. Consequently, $x'$ is always a short position. Hence, the necessary property (B1) with $z = T(x) - 1$ is violated, and thus, $\mathcal{H}_{\text{cube}}$ cannot be JM. We show a smallest such position with $p = 1$ in Figure 3.
Let us remark that by its definition, a JM hypergraph $\mathcal{H} \subseteq 2^V$ is minimal non-Tetris, in the sense that for any proper induced subhypergraph $\mathcal{H}_S, S \subset V$ the SG function of $\text{NIM}_{\mathcal{H}_S}$ is the Tetris function of $\mathcal{T}_{\mathcal{H}_S}$. We refer the reader to the companion paper [9] for more information on Tetris functions.

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