Quantum Gravity of a Brane-like Universe

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Quantum gravity of a brane-like Universe is formulated, and its Einstein limit is approached. Regge-Teitelboim embedding of Arnowitt-Deser-Misner formalism, parameterized by the coordinates $y^A(t, x^i)$, is governed by some $\rho_{AB}(y, y', y'')$. Invoking a novel Lagrange multiplier $\lambda$, accompanying the lapse function $N$ and the shift vector $N^i$, we derive the quadratic Hamiltonian

$$H = \frac{1}{2} N \left[ P_A \left( (\rho - \lambda I)^{-1} \right)^{AB} P_B + \lambda \right] + N^i y^A_i P_A.$$  

The inclusion of matter resembles minimal coupling. Setting $P_A = -\frac{\delta}{\delta y^A}$, we derive a bifurcated Wheeler-Dewitt-like equation. Einstein gravity, associated with $\lambda$ being a certain 4-fold degenerate eigenvalue of $\rho_{AB}$, is characterized by a vanishing center-of-mass momentum $\int P_A d^3x = 0$. Troublesome $(\rho - \lambda I)^{-1}$ is replaced then by regular $M^{-1}$, such that $M^{-1}(\rho - \lambda I)$ defines a projection operator, modifying the Hamiltonian accordingly.

A prevailing theory is always seeded by a remarkably simple idea. Regge-Teitelboim gravity, a criticized rival of Einstein gravity, may eventually fall into such a category. After all, who can resist the philosophy that the first principle which governs the evolution of the entire Universe is essentially the one which determines the world-manifold behavior of particles, strings and membranes. Following such a viewpoint, the Universe, to be referred to as a brane-like Universe, is viewed as a 4-dim extended object floating in some (say) 10-dim flat Minkowski background. Some cosmological fingerprints of such a brane-like Universe have already been revealed. Staying on practical grounds, however, Regge-Teitelboim gravity needs not be considered a target by itself. In fact, recalling its original underlying motivation, this theory attempted to establish a viable mathematical trail towards the unification of quantum mechanics with Einstein gravity. This conjecture was driven by several remarkable facts:

- Regge-Teitelboim gravity is, by construction, a continuation of string theory. Unlike in Einstein gravity, the metric tensor $g_{\mu\nu}(x)$ does not serve as a canonical field; this role has been taken over by the embedding vector $y^A(x)$.
- Although Einstein equations are traded for $[(G^{\mu\nu} - T^{\mu\nu})y^M_{\mu\nu}] = 0$, energy/momentum conservation is still automatic.
- Regge-Teitelboim gravity exhibits a built-in Einstein limit. In turn, every solution of Einstein equations is automatically a solution of Regge-Teitelboim equations.

It has been speculated, relying on the structural similarity to string/membrane theory, that quantum Regge-Teitelboim gravity may be a somewhat easier task to achieve than quantum Einstein gravity. The real target is then the Einstein limit of the theory, which in principle may call for additional first-class geometric constraints. The trouble is, however, that the parent Regge-Teitelboim Hamiltonian has never been derived!

In this short essay, by deriving the quadratic Hamiltonian of a gravitating brane-like Universe, we have overcome the dead-end reached by Regge-Teitelboim, thereby opening the door for the quantum Einstein gravity limit. A key role in our formalism is played by a novel non-dynamical field $\lambda$ which accompanies the standard Lagrange multipliers, the lapse function $N$ and the shift vector $N^i$. Starting from the purely gravitational case, the inclusion of arbitrary matter serendipitously resembles minimal gauge coupling. Altogether, the quantum theory prescribes a Virasoro-type momentum constraint equation followed by a bifurcated Wheeler-Dewitt-like equation. Appealing to Poincare invariance of the embedding spacetime, a generic Regge-Teitelboim configuration is parameterized by $\mu^2 > 0$, recognized as the analogue of (mass)$^2$. Quite surprisingly, an Einstein configuration turns out to be characterized by $\mu^2 = 0$. In this language, Einstein gravity can be interpreted as the ‘massless’ limit of Regge-Teitelboim gravity.

Given the background Minkowski metric $\eta_{AB}$ and some embedding vector $y^A(t, x^i)$, the induced 4-dim line-element can be put in the Arnowitt-Deser-Misner (ADM) form

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$  

provided the 3-metric $h_{ij}$, the shift vector $N_i$, and the lapse function $N$ are identified with

*Honorable mentioned, Gravity Research Foundation (1998)
\[ h_{ij} = \eta_{AB} y^A_i y^B_j , \quad N_i = \eta_{AB} y^A_i \dot{y}^B , \quad N^2 = N_i N^i - \eta_{AB} \dot{y}^A \dot{y}^B . \tag{2} \]

Notice the time-like unit vector \( n^A \equiv \frac{1}{N} (\dot{y}^A - N^i y^A_i) \) orthogonal to \( y^A_i \).

The gravitational Regge-Teitelboim Lagrangian density is the standard one (the canonical fields are not). Up to a surface term, it can be written in the form

\[ \mathcal{L} = -\sqrt{h} \left[ N R^{(3)} - \frac{1}{N} (K_{ij} K^{ij} - K^2) + 2N \Lambda \right] , \tag{3} \]

where \( R^{(3)} \) denotes the 3-dim Ricci scalar constructed by means of the 3-metric \( h_{ij} \), and \( K_{ij} \equiv NK_{ij} \) is the extrinsic curvature \( K_{ij} \) factorized by the lapse function \( N \). \( K_{ij} \) is free of mixed derivative \( \dot{y}^A_i \)-terms, and since \( \dot{y}^A_i \)-terms are absent in the first place, the Lagrangian \( \mathcal{L}(y, \dot{y}, y_j, y_{ij}, \ldots) \) is apparently ripe for the Hamiltonian formalism.

The fact that the 3-metric \( h_{ij} \) is \( \dot{y}^A \)-independent helps us to derive the momenta \( P_A \) conjugate to \( y^A \), that is

\[ P_A = \frac{\delta \mathcal{L}}{\delta \dot{y}^A} = \sqrt{h} \left\{ [R^{(3)} + \frac{1}{N} (K_{ij} K^{ij} - K^2) + 2\Lambda] n^A + \frac{2}{N} (K^{ij} - h^{ij} K) y^A_{ij} \right\} . \tag{4} \]

To simplify the algebraic structure of \( P^A \), define the \( \dot{y}^A \)-independent tensor

\[ \rho^{AB} \equiv 2\sqrt{h} \left[ (h^{ab} h^{ij} - h^{ij} h^{ab}) y^A_{ab} y^B_{ij} + (R^{(3)} + 2\Lambda) \eta^{AB} \right] , \tag{5} \]

to finally arrive at

\[ P^A = \frac{1}{2} (\rho \dot{n}) n^A + \rho_B n^B \tag{6} \]

One can immediately verify, in analogy with Wheeler-DeWitt theory and string theory, that the Hamiltonian \( \mathcal{H} \) vanishes

\[ \mathcal{H} = \dot{y} A P_A - L = N \left( n^A P_A - \frac{1}{N} L \right) + N^i y^A_i P_A = 0 , \tag{7} \]

and thus can be interpreted as a sum of constraints. Invoking the powerful embedding identity \( \eta_{AB} y^A_i y^B_k \equiv 0 \), the first constraint \( y^A_i P_A = 0 \) is easily extracted, reflecting the fact that \( y^A_i n_A = 0 \). The second constraint is hidden within \( n^A P_A - \frac{1}{N} L = 0 \). A naive attempt to solve \( n^A (\rho, P) \) and substitute into \( n^2 + 1 = 0 \), falls short. The cubic equation involved rarely admits simple solutions, and even in cases it does, the resulting constraint is anything but a quadratic form in the momenta.

The way out involves the definition of a quantity \( \lambda \), such that

\[ P^A = (\rho - \lambda I) n^B \tag{8} \]

The price for an independent \( \lambda \) being an additional constraint \( n \rho + 2\lambda = 0 \). Assuming that \( \lambda \) is not an eigenvalue of \( \rho_B \), we can solve for \( n^A (\rho, P, \lambda) \) and find

\[ n^A = \left[ (\rho - \lambda I)^{-1} \right]^A_B P^B . \tag{9} \]

The leftover constraints can then be grouped into

\[ P (\rho - \lambda I)^{-2} P + 1 = 0 , \quad P (\rho - \lambda I)^{-1} P + \lambda = 0 . \tag{10} \]

The first of which, owing to \( \frac{d}{d\lambda} (\rho - \lambda I)^{-1} = (\rho - \lambda I)^{-2} \), can be regarded superfluous provided we elevate \( \lambda \) to the level of a canonical non-dynamical variable. Note in passing that the special case \( \rho_B \sim \delta_B \) corresponds to

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a Nambu-Goto string. Explicitly, $\rho = 4\Lambda \sqrt{h}$ fixes $\lambda = 2\Lambda \sqrt{h}$, and gives rise to the familiar Virasoro constraint $P^2 + 4\Lambda^2 \eta_{AB} y^A_i y^B_i = 0$.

Altogether, the Regge-Teitelboim Hamiltonian acquires the quadratic form

$$H = \frac{1}{2} N \left[ P_A \left( (\rho - \lambda I)^{-1} \right)^{AB} P_B + \lambda \right] + N^i y^A_i P_A$$  \hspace{1cm} (11)

with $N, N^i$, and notably $\lambda$ serving as Lagrange multipliers. $(\rho - \lambda I)^{-1}$ plays a role analogous to the Wheeler-DeWitt metric on superspace. Here, however, superspace has been traded for the embedding spacetime itself, and $(\rho - \lambda I)^{-1}_{AB}$ needs not be confused with the metric $\eta_{AB}$. Once matter is included, the momenta $P_A$ conjugate to $y^A$ receives an extra contribution $\Delta P_A = \delta L_{\text{matter}} \delta \dot{y}^A = \frac{1}{2} \sqrt{h} T_{\mu \nu} \delta g_{\mu \nu}$. Using the notations

$$T_{nn} \equiv (T^{\mu \nu} y^A_{,\mu} y^B_{,\nu}) n_A n_B, \quad T_{ni} \equiv (T^{\mu \nu} y^A_{,\mu} y^B_{,\nu}) n_A y^B_i, \quad (\rho - \lambda \eta_{AB})^{-1}$$  \hspace{1cm} (12)

and bearing in mind that $T_{nn}(h_{ij}, \Phi, \Pi_{\Phi}, \Phi, i)$ and $T_{ni}(h_{ij}, \Phi, \Pi_{\Phi}, \Phi, i)$, the general Hamiltonian is derivable from the purely gravitational Hamiltonian by means of

$$P_A \rightarrow P_A + \sqrt{h} T_{ni} y^A_i, \quad \rho^A_B \rightarrow \rho^A_B + 2\sqrt{h} T_{nn} \delta^A_B$$  \hspace{1cm} (13)

To be more specific, consider the case where $\Phi(x)$ stands for a scalar field. The corresponding energy/momentum projections are

$$T_{nn} = \frac{1}{2} \left( \frac{1}{h} \Pi^2 + h^{ij} \Phi,_{i} \Phi,_{j} \right) + V, \quad T_{ni} = \frac{1}{\sqrt{h}} \Pi^{ij} \Phi,_{j}.$$  \hspace{1cm} (14)

In a more general case, e.g. for a gauge field $A_{\mu}$, the door is open for non-gravitational constraints to enter the Hamiltonian.

At the quantum level, we set $P_A \equiv -i \frac{\delta}{\delta y^A}$. Up to order ambiguities, the wave functional $\Psi$ of an empty brane-like Universe is subject to three Virasoro-type constraints: The momentum constraint equation

$$y^A_i \frac{\delta \Psi}{\delta y^A} = 0,$$  \hspace{1cm} (15)

is accompanied by the bifurcated Wheeler-Dewitt-like equation

$$\frac{\delta}{\delta y^A} \left( (\rho - \lambda I)^{-1} \right)^{AB} \frac{\delta}{\delta y^B} \Psi = \lambda \Psi$$  \hspace{1cm} (16)

Upon the inclusion of matter, the ordinary functional derivatives are replaced by covariant functional derivatives (and $\rho$ gets modified) according to the above prescription.

The Einstein limit of Regge-Teitelboim gravity has two faces:

- First, using the purely geometric relation

$$2 G_{nn} = R^{(3)} + \frac{1}{N^2} (K_{ij} K^{ij} - K^2),$$  \hspace{1cm} (17)

we infer that

$$\rho_{AB} - \lambda \eta_{AB} = 2\sqrt{h} \left[ (h^{ab} h_{ib} - h^{ij} h_{ab}) y_A |ab y_B |ij + (G_{nn} - T_{nn}) \eta_{AB} \right].$$  \hspace{1cm} (18)

Appealing now to the embedding identity $\eta_{AB} y^A_i y^B_k = 0$, one concludes that Einstein equation $G_{nn} = T_{nn}$ can be satisfied if and only if
(\rho_{AB} - \lambda n_{AB})y^B_{ji} = 0. \tag{19}

We have learned that the Einstein case is characterized by \lambda being a 4-fold degenerate eigenvalue of \rho_{AB}. In turn, \((\rho - \lambda I)^{-1}\) does not make sense, and we face the unpleasant consequence that not all components of \(n^A\) are expressible in terms of momenta. This is, however, a curable situation. The residual \(n\)'s are treated as non-dynamical variables, and the troublesome \((\rho - \lambda I)^{-1}\) is replaced by some regular \(M^{-1}\), such that \(M^{-1}(\rho - \lambda I)\) defines the proper projection operator.

• Second, using the dynamical relation

\[
P_A = \sqrt{\hbar} \left[ (G_{nn} - T_{nn})n^A - (G_{ni} - T_{ni})h^{ij}y^A_{ij} + \left( y^A_{ij}n_B y^B_{kl}(h^{ik}h^{jl} - h^{ij}h^{kl}) \right) \right], \tag{20}
\]

one observes that if Einstein equations \(G_{ni} = T_{ni}\) and \(G_{nn} = T_{nn}\) are both satisfied, \(P^A\) makes a total derivative. On the other hand, reflecting the Poincare invariance of the embedding spacetime, we know that the center-of-mass momentum \(\mu^A \equiv \int d^3x P^A\) is a Noether conserved vector. And since the Arnowitt-Deser-Misner formalism exclusively involves compact 3-spaces, \(\mu^A\) must vanish if Einstein equations are to be respected. Whereas a generic Regge-Teitelboim configuration exhibits a non-vanishing Casimir \(\mu^2 = \eta_{AB}\mu^A\mu^B\), easily recognized as the analogue of \((\text{mass})^2\), Einstein configurations come with \(\mu^2 = 0\). In this language, Einstein gravity can be interpreted as the ‘massless’ limit of Regge-Teitelboim gravity.

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