ABSTRACT

We consider a colocated MIMO radar scenario, in which the receive antennas forward their measurements to a fusion center. Based on the received data, the fusion center formulates a matrix which is then used for target parameter estimation. When the receive antennas sample the target returns at Nyquist rate, and assuming that there are more receive antennas than targets, the data matrix at the fusion center is low-rank. When each receive antenna sends to the fusion center only a small number of samples, along with the sample index, the receive data matrix has missing elements, corresponding to the samples that were not forwarded. Under certain conditions, matrix completion techniques can be applied to recover the full receive data matrix, which can then be used in conjunction with array processing techniques, e.g., MUSIC, to obtain target information. Numerical results indicate that good target recovery can be achieved with occupancy of the receive data matrix as low as 50%.

Index Terms—Array processing, compressed sensing, matrix completion, MIMO radar, MUSIC

1. INTRODUCTION

Multiple-input and multiple-output (MIMO) radar systems have received considerable attention in recent years due to their superior target estimation performance. Colocated MIMO radar systems exploit waveform diversity to formulate a long virtual array with number of elements equal to the product of the number of transmit and receive antennas. As a result, they achieve higher resolution than traditional phased array radars for the same amount of data [1][2]. Compressed sensing (CS) enables MIMO radar systems to maintain their advantages while significantly reducing the required measurements per receive antenna [3][4]. In CS-based MIMO radar, target parameters are estimated by exploiting the sparsity of targets in the angle, Doppler and range space, referred to as the target space. For CS-based sparse target estimation, the target space needs to be discretized into a fine grid, based on which the CS sensing matrix is constructed. However, performance of CS-based MIMO radar degrades when targets fall between grid points, a case also known as basis mismatch [5] in the CS literature.

Another approach related in spirit to CS is that of matrix completion. Matrix completion aims to recover a low-rank data matrix from partial samples of its entries by solving a relaxed nuclear norm optimization problem [6][7]. Array signal processing with matrix completion has been studied in [8][9]. To the best of our knowledge, however, matrix completion has not been exploited for target estimation in colocated MIMO radar. Our paper is related to the ideas in [9] in the sense that matrix completion is applied to the received data matrix formed by an array. However, due to the unique structure of the received signal in MIMO radar, the problem formulation and treatment in here is different than that in [9].

The main idea of our work is as follows. We consider a colocated MIMO radar scenario in which receive antennas forward their measurements to a fusion center. Based on the received data, the fusion center formulates a matrix, which is then used for estimating the target parameters. When the receive antennas sample the target returns at Nyquist rate, and assuming that there are more receive antennas than targets, the data matrix at the fusion center is low-rank. When each receive antenna sends to the fusion center only a small number of samples, along with the sample index, the receive data matrix has missing elements, corresponding to the samples that were not forwarded. Under certain conditions, matrix completion techniques can be applied to recover the full receive data matrix, which can then be used in conjunction with parametric methods such as MUSIC to obtain target information. Compared to CS MIMO radar, our proposed method has the same advantage in terms of reduction of samples needed for accurate estimation but it avoids the basis mismatch issue inherent in CS-based approach.

The rest of the paper is organized as follows. The colocated MIMO radar system model is described in Section 2. Background of noisy matrix completion is introduced in Section 3. The applicability of matrix completion to MIMO radar is discussed in Section 4, and numerical results are given in Section 5. Finally, Section 6 provides concluding remarks.

2. COLOCATED MIMO RADAR SYSTEM

We consider a MIMO radar system that employs colocated transmit and receive antennas, shown in Fig. 1. We use
$M_t$ and $M_r$ to denote the numbers of transmit antennas and receive antennas, respectively. Although our results can be extended to an arbitrary antenna configuration, the results here are presented for the case in which the transmit and receive antennas form a uniform linear array (ULA) with inter-element spacing between transmit and receive antennas $d_t$ and $d_r$, respectively. We further assume $d_t = d_r = \lambda/2$, where $\lambda$ is the wavelength of the carrier signal. Further, the waveforms $s_k(t)$, $k = 1, ..., M_t$, transmitted from the transmit antennas are assumed to be narrow-band and orthogonal. Now suppose there are $K$ point targets in the far field at angles $\theta_k$, $k = 1, ..., K$, each moving with speed $\dot{\theta}_k$. The corresponding $M_t \times K$ dimensional transmit steering matrix can be expressed as $A(\theta) = [a(\theta_1), ..., a(\theta_K)]$, where

$$a(\theta_k) = \left[1, e^{i\frac{2\pi}{\lambda}d_t \sin(\theta_k)}, ..., e^{i\frac{2\pi}{\lambda}(M_t-1)d_t \sin(\theta_k)}\right]^T,$$

and the $M_r \times K$ dimensional receive steering matrix $B(\theta)$ can be defined in a similar fashion based on the receive steering vectors $b(\theta_k)$.

In order to estimate the speed of each target, multiple pulses need to be transmitted. Let $Q$ be the number of transmitted pulses, with $T_{PRI}$ being the pulse repetition interval. Assume the target reflection coefficients $\{\beta_k\}_{k = 1, ..., K}$ and remain constant during the $Q$ pulses. For slowly moving targets, $(2\dot{\theta}T_p/\lambda \ll 1)$, where $T_p$ is the pulse duration) the Doppler shift within a pulse can be ignored, while the Doppler changes from pulse to pulse.

Under the narrowband assumption the received signal at the $l$th receive antenna can be approximated as 

$$x_l(t) \approx \sum_{k=1}^{K} \beta_k e^{i\frac{2\pi}{\lambda}d_t \sin(\theta_k) t} a^T(\theta_k)t s(t) + w_l(t).$$

Here $s(t) = [s_1(t), ..., s_{M_t}(t)]^T$.

Suppose that each receive antenna samples the received signal at rate $L/T_q$ and forwards the samples to the fusion center. Here, $T_q$ is the sampling time, $L$ is the number of nonzero samples and we assume $L \gg K$. At the fusion center, received data during the $q$th pulse can be written as

$$X_q = B(\theta) \Sigma D_q A^T(\theta) S + W_q = Z_q + W_q,$$

where $\Sigma = \text{diag} ([\beta_1, ..., \beta_K])$; $D_q = \text{diag} (d_q)$, with $d_q = [e^{i2\pi 2\theta_1(q-1)T_{PRI}}, ..., e^{i2\pi 2\theta_K(q-1)T_{PRI}}]^T$; $S = [s(0T_q), ..., s((L-1)T_q)]$; and $W_q$ is a Gaussian noise matrix. Note that both matrices $\Sigma$ and $D_q$ are rank-$K$, while the rank of matrix $S$ is $\min \{M_t, L\}$. Thus, for $M_t > K$ the rank of the noise free data matrix $Z_q = B(\theta) \Sigma D_q A^T(\theta) S$ is $K$. In other words, the data matrix $Z_q$ is low-rank based on the assumption that $M_r \gg K$.

3. MATRIX COMPLETION WITH NOISE

We now provide a brief overview of the problem of recovering a rank $r$ matrix $M \in \mathbb{C}^{n_1 \times n_2}$ based on partial knowledge of its entries, possibly corrupted by noise, i.e., $[Y]_{ij} = [M]_{ij} + [E]_{ij}, \forall (i, j) \in \Omega$, where, $[E]_{ij}$ represents noise and $\Omega$ is the set of observed entries. This can also be expressed as $P_\Omega(Y) = P_\Omega(M) + P_\Omega(E)$, where $P_\Omega$ represents the sampling operation. According to [7], when $M$ is low-rank and its singular vectors are sufficiently spread, i.e., both the numbers of zero and large elements in the singular vectors are not large, $M$ can be recovered by solving a relaxed nuclear norm optimization problem, given by

$$\min \|X\|_* \quad \text{s.t.} \quad \|P_\Omega(X - Y)\|_F \leq \delta$$

where $\|\|_*$ denotes the nuclear norm, i.e., the sum of singular values of $X$, while $\delta > 0$ is a constant.

To test the ‘sufficiently spread’ requirement, the strong incoherence property of matrix $M$ with parameter $\mu$ has been introduced in [6]. Consider the singular value decomposition (SVD) of $M$, i.e., $M = \sum_{k=1}^{r} \sigma_k u_k v_k^H$, and define $P_U = \sum_{1 \leq i \leq r} u_i u_i^H$, $P_V = \sum_{1 \leq i \leq r} v_i v_i^H$, and $T = \sum_{1 \leq i \leq r} u_i v_i^H$. When the following conditions are satisfied, the matrix $M$ is said to satisfy the strong incoherence property with $\mu = \max (\mu_1, \mu_2)$.

A1) For all pairs $(a, a') \in [n_1] \times [n_1]$ and $(b, b') \in [n_2] \times [n_2]$, there is $\mu_1 > 0$ such that

$$\langle e_a, P_U e_{a'} \rangle - \frac{r}{n_1} 1_{a=a'} \leq \mu_1 \sqrt{\frac{r}{n_1}},$$

$$\langle e_b, P_V e_{b'} \rangle - \frac{r}{n_2} 1_{b=b'} \leq \mu_2 \sqrt{\frac{r}{n_2}},$$

where $e_a$ is the vector with the $a$th element equal to 1 and others being zero, while $1_{a=a'}$ indicates that it is equal to 1 when $a = a'$ and 0 otherwise.

A2) For all $(a, b) \in [n_1] \times [n_2]$, there exists a constant $\mu_2 > 0$ such that $|T_{ab}| \leq \mu_2 \sqrt{\frac{r}{n_1 n_2}}$, where $T_{ab}$ is the $(a, b)$ entry of the matrix $T$. 
Note that it has been shown in [6] that when the maximum (in magnitude) values of the \( K \) left and right singular vectors are bounded, i.e.,
\[
\|u_k\|_{\infty} \leq \sqrt{\frac{\mu_B}{n_1}}, \quad \|v_k\|_{\infty} \leq \sqrt{\frac{\mu_B}{n_2}} \tag{7}
\]
with \( \mu_B = O(1) \), then the strong incoherence property is guaranteed with \( \mu \leq \mu_B \sqrt{r} \).

Now define \( n = \max(n_1, n_2) \) and suppose \( M \) satisfies the strong incoherence property. Then [6] establishes in the noiseless case that, by observing \( N \) randomly selected entries with \( N \geq C \mu^2 \eta r \log^6 n \) for some constant \( C \), the matrix \( M \) can be recovered exactly with a probability of at least \( 1 - n^{-3} \). Further, [7] establishes that, when observations are corrupted with white noise \( [E]_{ij} \), that is zero-mean Gaussian with variance \( \sigma^2 \), the recovery error is bounded as \( \| M - \hat{M} \|_F \leq 4 \sqrt{\frac{p}{p(2 + p)} \min(n_1, n_2) \delta + 2 \delta} \), where \( p = \frac{N}{n_1 n_2} \) is the fraction of observed entries.

4. MATRIX COMPLETION FOR MIMO RADAR

The left singular vectors of \( Z_q \) defined in [5] are the eigenvectors of \( Z_q Z_q^H = H S S^H H^H \), where \( H = B(\theta) \Sigma D_2 A^T(\theta) \). The right singular vectors of \( Z_q \) are the eigenvectors of \( S S^H H^H H S \). Since \( S \) is orthogonal, it holds that \( SS^H = I \). Thus, the left singular vectors are only determined by matrix \( H \), while the right singular vectors are affected by both transmit waveforms and matrix \( H \).

For the problem considered in this paper, it is difficult to determine analytically the behavior of the entries of the left and right singular vectors of \( Z_q \). Instead, we get an idea of the behavior of the maximum values and the parameters that affect their spread using simulations. We consider a MIMO radar setup in which the target direction of arrival (DOA) angles are uniformly distributed in \([-90^\circ, 90^\circ]\) and the corresponding target speeds are uniformly distributed in the range \([150, 450] m/s\). In addition, \( \beta_k \) are following complex Gaussian distribution and kept unchanged for \( Q = 10 \) pulses. The pulse repetition interval is \( T_{PRI} = 1/4000 \) second, and the carrier frequency is \( f = 10^9 \) Hz, resulting in \( \lambda = c/f = 0.3 \) meter. Two types of orthogonal waveforms are considered: Hadamard and Gaussian orthogonal waveforms. Several cases of parameters are verified. Case I: \( M_r = 40, \ L = 128 \); Case II: \( M_r = 1000, \ L = 128 \); Case III: \( M_r = 40, \ L = 1024 \). Each case runs for 300 iterations.

Let \( m_1 \) and \( m_2 \) denote the maximum element value of \( K \) left and right singular vectors of \( Z_q \). The complementary cumulative distribution function (CDF) curves of \( m_1 \) and \( m_2 \), i.e., \( Pr(m > m_i), i = 1, 2 \) are plotted in Fig. 2 for Cases I and II. It can be seen from these plots that as \( M_r \) increases, the bounds of \( m_1 \) for both \( K = 2 \) and \( K = 10 \) decrease, while the distribution of \( m_2 \) does not significantly change when \( L \) is fixed. Figure 2(c) shows that as \( M_r \) gets large, \( m_1 \) gets bounded by a small number with high probability. Space limitations prevent us from displaying more figures of Case III, which show that \( m_2 \) is bounded by a small number with high probability as \( L \) gets large.

Extensive simulations also show that the bounds on \( m_1 \) and \( m_2 \) scale as \( 1/\sqrt{M_r} \) and \( 1/\sqrt{L} \), respectively, with some constant \( \sqrt{\mu_B} \). For Gaussian orthogonal waveform and \( K = 2, \mu_B \approx 2.4 \) for the bound of \( m_1 \) (see Fig. 3(a)) and \( \mu_B \approx 6.5 \) for the bound of \( m_2 \) (see Fig. 3(b)). For \( K = 10 \) and Hadamard waveform, the scaling laws also hold but with larger constants \( \mu_B \). Therefore, depending on the number of receive antennas \( M_r \) and the number of samples \( L \) in one pulse, \( m_1 \) and \( m_2 \) can be assumed to be bounded by small numbers. Based on [6], therefore, we conclude that the strong incoherence property is likely satisfied in our problem.

It is also worth noting that under Gaussian orthogonal waveforms, \( m_2 \) is concentrated in a smaller range as compared with the range for Hadamard waveforms. This indicates that the waveform indeed plays a role for the use of matrix completion, and perhaps the waveform can be optimally designed to result in low probability of large values for \( m_2 \).

Since the matrix completion conditions appear to be satisfied in our case, we propose that each antenna obtains and forwards to the fusion center a small number of samples during each pulse. Note that along with the samples, the antenna needs to inform the fusion center on how the sample was obtained so that the fusion center can determine where to position the received samples in the receive data matrix \( Z_q \). In a practical setting, a pseudo-random sampling ADC at each re-
The sampled covariance matrix of the 

reshaping and kept unchanged for Q pulses. For matrix completion, the TFOCS software package [11] is used.

First, we plot relative errors (averaged over 50 Monte Carlo runs) of the received data matrix \( \hat{X}_q \) for Hadamard and Gaussian orthogonal (G-Orth) waveforms. The relative error is defined as \( \frac{\|Z_q - \hat{X}_q\|_F}{\|Z_q\|_F} \). Under each waveform, \( K = 2 \) point targets in the far field are randomly generated. The result is shown in Fig. 4(a) for \( q = 1 \). It can be seen from this figure that, as \( p \) increases, the relative recovery error of the data matrix under Gaussian orthogonal waveform reduces to the reciprocal of the SNR faster than that under Hadamard waveform. A plausible reason for this is that under the Gaussian orthogonal waveform, the maximum value of elements in the singular vectors of \( Z_q \) is bounded by a smaller number with high probability, as compared with that under the Hadamard waveform (see Fig. 2).

Next, the probabilities of DOA estimation resolution under the two orthogonal waveforms are plotted in Fig. 4(b) for the following scenario. Two targets are randomly generated among DOA range \([-20^\circ, 20^\circ]\) with minimum DOA separations \( d_\theta = [0.2^\circ, 0.3^\circ, 0.4^\circ, 0.5^\circ, 0.7^\circ, 1^\circ] \) and the corresponding speeds are set to 150 and 400 m/s. The MUSIC algorithm is applied to obtain the target DOA information. If the DOA estimates \( \hat{\theta}_i, i = 1, 2 \) satisfy \( |\hat{\theta}_i - \theta_i| \leq \varepsilon d_\theta, \varepsilon = 0.1 \), we declare this as a success. The probability of DOA resolution is then defined as the fraction of successful events in 50 iterations. It can be seen from the figure that when \( p = 0.3 \), the Gaussian orthogonal waveform has a much better DOA estimation resolution compared with Hadamard waveform. As \( p \) increases to 0.5, the performance difference becomes small since the relative recovery errors under both waveforms are similar (see Fig. 4(a)). Figure 4(b) confirms that Gaussian orthogonal waveforms are better than Hadamard waveforms for matrix completion-based DOA estimation.
7. REFERENCES

[1] P. Stoica and J. Li, “MIMO radar with colocated antennas,” IEEE Signal Process. Magazine, vol. 24, no. 5, pp. 106-114, 2007.

[2] C. Y. Chen and P. P. Vaidyanathan, “MIMO radar space-time adaptive processing using prolate spheroidal wave functions,” IEEE Trans. on Signal Processing, vol. 56, no. 2, pp. 623-635, 2008.

[3] M. Herman and T. Strohmer, “High-resolution radar via compressed sensing,” IEEE Trans. on Signal Processing, vol. 57, no. 6, pp. 2275-2284, 2009.

[4] Y. Yu, A. P. Petropulu, and H. V. Poor, “MIMO radar using compressive sampling,” IEEE Journal of Selected Topics in Signal Processing, vol. 4, no. 1, pp. 146-163, 2010.

[5] Y. Chi, L. L. Scharf, A. Pezeshki, and A. R. Calderbank, “Sensitivity to basis mismatch in compressed sensing,” IEEE Trans. on Signal Processing, vol. 59, no. 5, pp. 2182-2195, 2011.

[6] E. J. Candès and T. Tao, “The power of convex relaxation: Near-optimal matrix completion,” IEEE Trans. on Information Theory, vol. 56, no. 5, pp. 2053-2080, 2010.

[7] E. J. Candès and Y. Plan, “Matrix completion with noise,” Proceedings of the IEEE, vol. 98, no. 6, pp. 925-936, 2010.

[8] A. Waters and V. Cevher, “Distributed bearing estimation via matrix completion,” in Proc. of IEEE ICASSP, 2010.

[9] Z. Weng and X. Wang, “Low-rank matrix completion for array signal processing,” in Proc. of IEEE ICASSP, 2012.

[10] D. Nion and N. D. Sidiropoulos, “Tensor algebra and multi-dimensional harmonic retrieval in signal processing for MIMO radar,” IEEE Trans. on Signal Processing, vol. 58, no. 11, pp. 5693-5705, 2010.

[11] S. Becker, E. J. Candès, and M. Grant, “Templates for convex cone problems with applications to sparse signal recovery,” Math. Prog. Comp. vol. 3, no. 3, pp. 165-218, 2011.