NONLOCALITY AND THE CENTRAL GEOMETRY OF DIMER ALGEBRAS

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Abstract. Let $A$ be a dimer algebra and $Z$ its center. It is well known that if $A$ is cancellative, then $A$ and $Z$ are noetherian and $A$ is a finitely generated $Z$-module. Here we show the converse: if $A$ is non-cancellative (as almost all dimer algebras are), then $A$ and $Z$ are nonnoetherian and $A$ is an infinitely generated $Z$-module. Although $Z$ is nonnoetherian, we show that it nonetheless has Krull dimension 3 and is generically noetherian. Furthermore, we show that the reduced center is the coordinate ring for a Gorenstein algebraic variety with the strange property that it contains precisely one 'smeared-out' point of positive geometric dimension. In our proofs we introduce formalized notions of Higgsing and the mesonic chiral ring from quiver gauge theory.

Contents

1. Introduction 1
2. Preliminary results 4
3. Cancellative dimer algebras 12
4. Non-cancellative dimer algebras and their homotopy algebras 17
   4.1. Cyclic contractions 17
   4.2. Reduced and homotopy centers 25
   4.3. Homotopy dimer algebras 35
   4.4. Nonnoetherian and nonlocal 37
   4.5. Integral closure 53
References 56

1. Introduction

We begin by recalling the definition of a dimer algebra, which is a type of quiver with potential whose quiver is dual to a dimer model.

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Definition 1.1.

- Let $Q$ be a quiver whose underlying graph $ar{Q}$ embeds into a two-dimensional real torus $T^2$, such that each connected component of $T^2 \setminus ar{Q}$ is simply connected and bounded by an oriented cycle of length at least 2, called a unit cycle. The dimer algebra of $Q$ is the quiver algebra $A = kQ/I$ with relations

$$ I := \langle p - q \mid \exists a \in Q_1 \text{ such that } pa \text{ and } qa \text{ are unit cycles} \rangle \subset kQ. $$

- $A$ and $Q$ are called cancellative if for all paths $p, q, r \in A$,

$$ p = q \quad \text{whenever} \quad pr = qr \neq 0 \quad \text{or} \quad rp = rq \neq 0. $$

Cancellative dimer algebras are now well understood: they are 3-Calabi-Yau algebras and noncommutative crepant resolutions of 3-dimensional toric Gorenstein singularities (e.g., [Bo, Theorem 10.2], [Br], [D, Theorem 4.3], [MR, Theorem 6.3]). Non-cancellative dimer algebras, on the other hand, are much less understood. However, almost all dimer algebras are non-cancellative, and so it is of great interest to understand them.

Our first main theorem is the following.

Theorem 1.2. Let $A$ be a dimer algebra which is 2-cycle free (Definition 4.4), and denote by $Z$ its center. Then the following are equivalent (Corollary 4.39):

1. $A$ is cancellative.
2. $A$ is noetherian.
3. $Z$ is noetherian.
4. $A$ is a finitely generated $Z$-module.
5. The vertex corner rings $e_i Ae_i$ are all isomorphic, and isomorphic to $Z$.

We then use the notion of depiction and geometric dimension, introduced in [B2], to make sense of the central geometry of non-cancellative dimer algebras. A depiction of a nonnoetherian integral domain $R$ is a closely related noetherian overring $S$ that provides a way of visualizing the geometry of $R$ (Definition 4.41). The underlying idea is that nonnoetherian geometry is the geometry of nonlocal algebraic varieties and schemes. In this framework, (closed) points can be ‘smeared-out’, and thus have positive dimension. Such points are therefore inherently nonlocal. We use the term ‘nonlocal’ in the sense that two widely separated points may somehow be very near to each other.

Our second main theorem characterizes the central geometry of non-cancellative dimer algebras.

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1. A ring $A$ which is a finitely-generated module over a central normal Gorenstein subdomain $R$ is Calabi-Yau of dimension $n$ if (i) $\text{gl. dim } A = \dim R = n$; (ii) $A$ is a maximal Cohen-Macaulay module over $R$; and (iii) $\text{Hom}_R(A, R) \cong A$, as $A$-bimodules [Br, Introduction].

2. A dimer algebra is cancellative if and only if it satisfies any one of the various combinatorial consistency conditions; see [Bo, D, U].
Theorem 1.3. Let $A$ be a non-cancellative dimer algebra which is 2-cycle free. Then

1. $Z$ and $Z_{\text{red}} := Z/\nil Z$ each have Krull dimension 3 (Theorem 4.56).
2. $Z_{\text{red}}$ is a nonnoetherian integral domain depicted by the cycle algebra of $A$ (Corollary 4.26 and Theorem 4.58).
3. The reduced induced scheme structure of $\Spec Z$ is birational to a noetherian affine scheme, and contains precisely one closed point of positive geometric dimension. Furthermore, the maximal ideal spectrum $\Max Z_{\text{red}}$ may be viewed as a Gorenstein algebraic variety $X$ with a positive dimensional subvariety $Y \subset X$ identified as a single ‘smeared-out’ point (Theorem 4.58).

Let $A$ and $A'$ be non-cancellative and cancellative dimer algebras respectively. In order to prove Theorem 1.2, we introduce the notion of a cyclic contraction (Definitions 4.1 and 4.2),

$$
\psi : A \to A', \quad \text{where } S := k \left[ \cup_{i \in Q_0} \tilde{\psi}(e_i A e_i) \right] = k \left[ \cup_{i \in Q_0'} \bar{\tau}(e_i A' e_i) \right];
$$

the associated homotopy dimer algebra of $A$ (Definition 4.32),

$$
\tilde{A} := A/\langle p - q \mid \exists \text{ path } r \text{ such that } pr = qr \neq 0 \text{ or } rp = rq \neq 0 \rangle;
$$

and the associated cycle algebra and homotopy center of $A$ (Definitions 4.2 and 4.24),

$$
S := k \left[ \cup_{i \in Q_0} \tilde{\psi}(e_i A e_i) \right] \quad \text{and} \quad R := k \left[ \cap_{i \in Q_0} \tilde{\psi}(e_i A e_i) \right].
$$

Here $\tilde{\psi} := \bar{\tau} \psi$ and $\bar{\tau}$ are special $k$-linear maps from the respective corner rings of $A$ and $A'$ to the polynomial ring generated by the simple matchings of $A'$ (Definition 2.6, Lemma 2.7, and Notation 2.8). Contractions formalize the notion of Higgsing in abelian quiver gauge theories (Remark 4.6), and the cycle algebra formalizes the mesonic chiral ring (Remark 4.7).

We show that the center of the homotopy dimer algebra $\tilde{A}$ is isomorphic to the homotopy center $R$ of $A$ (Theorem 4.34). Furthermore, the reduced center $Z_{\text{red}}$ of $A$ is a subalgebra of its homotopy center $R$ (Theorem 4.25). In particular, the kernel of $\psi$, restricted to $Z$, coincides with the nilradical of $Z$ (Theorem 4.22).

In general, the containment $Z_{\text{red}} \subseteq R$ may be proper (Theorem 4.30). Even so, $Z_{\text{red}}$ and $R$ determine the same nonlocal variety (Theorem 4.58). Furthermore, their integral closures coincide (Theorem 4.65). Finally, we give necessary and sufficient conditions for $R$ to be normal; for instance, $R$ is normal if and only if there is an ideal $J$ of $S$ such that $R = k + J$ (Theorem 4.64).

Conventions: Throughout, $k$ is an uncountable algebraically closed field of characteristic zero. We will denote by $\dim R$ the Krull dimension of $R$; by $\Frac R$ the ring of fractions of $R$; by $\Max R$ the set of maximal ideals of $R$; by $\Spec R$ either the set of prime ideals of $R$ or the affine $k$-scheme with global sections $R$; by $R_p$ the localization of $R$ at $p \in \Spec R$; and by $Z(a)$ the closed set $\{ m \in \Max R \mid m \supseteq a \}$ of $\Max R$ defined by the subset $a \subseteq R$. 
We will denote by $Q = (Q_0, Q_1, t, h)$ a quiver with vertex set $Q_0$, arrow set $Q_1$, and head and tail maps $h, t : Q_1 \to Q_0$. We will denote by $kQ$ the path algebra of $Q$, and by $e_i$ the idempotent corresponding to vertex $i \in Q_0$. Multiplication of paths is read right to left, following the composition of maps. We say a noncommutative ring is noetherian if it satisfies the ascending chain condition on both left ideals and right ideals. By module we mean left module. By infinitely generated $R$-module, we mean an $R$-module that is not finitely generated. By non-constant monomial, we mean a monomial that is not in $k$.

2. Preliminary results

Suppose $A = kQ/I$ is a dimer algebra (cancellative or not). Unless otherwise specified, by path or cycle we mean path or cycle modulo $I$.

**Notation 2.1.** Let $\pi : \mathbb{R}^2 \to T^2$ be a covering map such that for some $i \in Q_0$,

$$\pi(Z^2) = i \in Q_0.$$ 
Denote by $Q^+ := \pi^{-1}(Q) \subset \mathbb{R}^2$ the covering quiver of $Q$. For each path $p$ in $Q$, denote by $p^+$ the unique path in $Q^+$ with tail in the unit square $[0, 1) \times [0, 1) \subset \mathbb{R}^2$ satisfying $\pi(p^+) = p$.

By a cyclic subpath of a path $p$, we mean a proper subpath of $p$ that is a non-vertex cycle.

**Notation 2.2.** We consider the following sets of cycles in $A$.

- For $u \in \mathbb{Z}^2$, let $C^u$ be the set of cycles $p \in A$ such that $h(p^+) = t(p^+) + u \in Q_0^+$. 
- For $i \in Q_0$, let $C_i$ be the set of cycles in the vertex corner ring $e_iAe_i$. 
- Let $\hat{C}$ be the set of cycles $p \in A$ such that $(p^2)^+$ has no cyclic subpaths. 

We denote the intersection $\hat{C} \cap C^u \cap C_i$, for example, by $\hat{C}_i^u$. Note that $C^0$ is the set of cycles whose lifts are cycles in $Q^+$. In particular, $\hat{C}^0 = \emptyset$. Furthermore, the lift of any cycle $p$ in $\hat{C}$ has no cyclic subpaths, although $p$ itself may have cyclic subpaths.

The following two lemmas, which characterize the unit cycles, are clear (see e.g. [MR Section 4]).

**Lemma 2.3.** Let $A = kQ/I$ be a dimer algebra. If $\sigma_i, \sigma'_i$ are two unit cycles at $i \in Q_0$, then $\sigma_i = \sigma'_i$. Furthermore, the element $\sum_{i \in Q_0} \sigma_i$ is in the center of $A$.

Henceforth, we denote by $\sigma_i \in A$ the unit cycle at vertex $i$.

**Lemma 2.4.** Let $A$ be a dimer algebra, and let $p, q \in e_jAe_i$ be paths.

1. If $t(p^+) = t(q^+)$ and $h(p^+) = h(q^+)$, then $pq_i^m = q\sigma_i^n$ for some $m, n \geq 0$.
2. If $A$ is cancellative and $p^+$ is a cycle in $Q^+$, then $p = \sigma_i^m$ for some $m \geq 0$. 

Proof. We prove Claim (2). Suppose \( p^+ \) is a cycle in \( Q^+ \). By Claim (1) there is some \( m, n \geq 0 \) such that \( \sigma_i^m p = \sigma_i^n \). Therefore if \( A \) is cancellative, then \( p = \sigma_i^{n-m} \). \( \square \)

The following definition, introduced in \[B\], captures a useful matrix ring embedding.

**Definition 2.5.** \[B, Definition 2.1\] An *impression* \((\tau, B)\) of a finitely generated noncommutative algebra \( A \) is a finitely generated commutative algebra \( B \) and an algebra monomorphism \( \tau : A \hookrightarrow M_d(B) \) such that

- for generic \( b \in \text{Max } B \), the composition
  \[
  A \xrightarrow{\tau} M_d(B) \xrightarrow{\iota_b} M_d(B/b) \cong M_d(k)
  \]
  is surjective; and
- \( \text{Max } B \to \text{Max } \tau(Z), b \mapsto b \cap \tau(Z), \) is surjective.

An impression determines the center of \( A \) explicitly \[B, Lemma 2.1\],

\[
Z \cong \{ f \in B \mid f1_d \in \text{im } \tau \} \subseteq B.
\]

Furthermore, if \( A \) is a finitely generated module over its center, then its impression classifies all simple \( A \)-module isoclasses of maximal \( k \)-dimension \[B, Proposition 2.5\]: for each such module \( V \), there is some \( b \in \text{Max } B \) such that

\[
V \cong (B/b)^d,
\]

where \( av := \epsilon_b(\tau(a))v \) for each \( a \in A, v \in V \). If \( A \) is nonnoetherian, then its impression may characterize the central geometry \( A \) through the technology of depictions \[B_2, Section 3\]. We will use this technology in Section 4.4.

**Definition 2.6.**

- A *perfect matching* \( D \subset Q_1 \) is a set of arrows such that each unit cycle contains precisely one arrow in \( D \).
- A *simple matching* \( D \subset Q_1 \) is a perfect matching such that \( Q \setminus D \) supports a simple \( A \)-module of dimension \( 1^{Q_0} \).

Denote by \( \mathcal{P} \) and \( \mathcal{S} \) the sets of perfect and simple matchings of \( A \), respectively.

Henceforth we will only consider dimer algebras that admit at least one perfect matching, \( \mathcal{P} \neq \emptyset \).

Denote by \( E_{ij} \in M_d(k) \) the \( d \times d \) matrix with a 1 in the \( ij \)-th slot and zeros elsewhere.

**Lemma 2.7.** Let \( A \) be a dimer algebra. Consider the maps

\[
\tau : A \to M_{|Q_0|}(k[x_D \mid D \in \mathcal{S}]) \quad \text{and} \quad \eta : A \to M_{|Q_0|}(k[x_D \mid D \in \mathcal{P}])
\]
defined on \( i \in Q_0 \) and \( a \in Q_1 \) by

\[
\begin{align*}
\tau(e_i) &:= E_{ii}, \quad \tau(a) := E_{h(a),t(a)} \prod_{a \in D \in \mathcal{S}} x_D, \\
\eta(e_i) &:= E_{ii}, \quad \eta(a) := E_{h(a),t(a)} \prod_{a \in D \in \mathcal{P}} x_D.
\end{align*}
\]
and extended multiplicatively and \( k \)-linearly to \( A \). Then \( \tau \) and \( \eta \) are algebra homomorphisms. Furthermore, each unit cycle \( \sigma_i \in e_i A e_i \) satisfies
\[
(5) \quad \tau(\sigma_i) = E_{ii} \prod_{D \in S} x_D \quad \text{and} \quad \eta(\sigma_i) = E_{ii} \prod_{D \in \mathcal{P}} x_D.
\]

Proof. If \( a \in Q_1 \) and \( pa, qa \) are unit cycles, then
\[
\tau(p) = E_{t(a),h(a)} \prod_{\alpha \notin D \in S} x_D = \tau(q).
\]
\( \tau \) is therefore well-defined by (1), and thus clearly an algebra homomorphism. Similarly \( \eta \) is also an algebra homomorphism.

By definition, each perfect matching contains precisely one arrow in each unit cycle. Therefore (5) holds. \( \square \)

In Theorem 3.3 we will show that \((\tau, k[\{x_D | D \in S\}]) \) in (3) is an impression of \( A \) if \( A \) is cancellative. Furthermore, in Corollary 3.5 we will show that \( A \) is cancellative if and only if \( A \) admits an impression \((\tau, B)\) where \( B \) is an integral domain and \( \tau(e_i) = E_{ii} \) for each \( i \in Q_0 \).

Notation 2.8. Set
\[
B := k[\{x_D | D \in S\}] \quad \text{and} \quad \sigma := \prod_{D \in S} x_D.
\]
Furthermore, for each \( i, j \in Q_0 \), denote by
\[
\bar{\tau} : e_j A e_i \to B
\]
the \( k \)-linear map defined on \( p \in e_j A e_i \) by
\[
\tau(p) = \bar{\tau}(p) E_{ji}.
\]
For brevity, we also denote \( \bar{\tau}(p) \) by \( \bar{p} \). In particular, \( \bar{p} \) is the single nonzero matrix entry of \( \tau(p) \). Furthermore, \( \sigma_i = \sigma \) for each \( i \in Q_0 \).

Lemma 2.9. If \( p^+ \) is a non-vertex cycle in \( Q^+ \), then there is an \( m \geq 1 \) such that
\[
\bar{p} = \sigma^m.
\]

Proof. Set \( i := t(p) \). By Lemma 2.4.1, there is some \( n > m \geq 0 \) such that
\[
p \sigma_i^n = \sigma_i^m.
\]
Furthermore, \( \tau \) is an algebra homomorphism by Lemma 2.7. Thus \( \bar{\tau} \) is an algebra homomorphism on \( e_i A e_i \). Whence
\[
\bar{p} \sigma^m = \sigma^n.
\]
Therefore \( \bar{p} = \sigma^{n-m} \) since \( B \) is an integral domain. \( \square \)

Lemma 2.10. Let \( A \) be a dimer algebra, and suppose paths \( p, q \) are equal modulo \( I \). Then
(1) The lifts \( p^+ \) and \( q^+ \) bound a compact region \( R_{p,q} \) in \( \mathbb{R}^2 \).

(2) If \( i^+ \) is a vertex in \( R_{p,q} \), then there is a path \( r^+ \) from \( t(p^+) \) to \( h(p^+) \) that is contained in \( R_{p,q} \), passes through \( i^+ \), and satisfies
\[
p = r = q \quad \text{(modulo } I)\.
\]

**Proof.** Claim (1) is clear.

(2) The relations generated by the ideal \( I \) in (1) lift to homotopy relations on the paths of \( Q^+ \). In particular, \( I \) is generated by relations of the form \( s = t \), where for some arrow \( a, sa \) and \( ta \) are unit cycles. Furthermore, each vertex subpath of the unit cycle \( sa \) (resp. \( ta \)) is a vertex subpath of \( s \) (resp. \( t \)). Therefore each vertex in each unit cycle contained in \( R_{p,q} \) is a vertex subpath of some \( r^+ \).

**Lemma 2.11.** Suppose \( A \) is a dimer algebra. Let \( u \in \mathbb{Z}^2 \setminus 0 \).

(1) If \( p, q \in C_u^+ \) and \( \sigma \upharpoonright \tilde{p}, \sigma \upharpoonright \tilde{q} \), then \( \tilde{p} = \tilde{q} \).

Furthermore, if \( A \) is cancellative, then the following hold:

(2) Each vertex corner ring \( e_i A e_i \) is commutative.

(3) If \( p, q \in C_u^+ \), then \( p = q \).

(4) If \( a \in Q_1, p \in C_u^+ \), and \( q \in \hat{C}_u^\sigma \), then \( ap = qa \).

(5) If \( p, q \in C_u^+ \), then \( \tilde{p} = \tilde{q} \).

Note that in Claim (3) the cycles \( p \) and \( q \) are based at the same vertex \( i \), whereas in Claim (5) \( p \) and \( q \) may be based at different vertices.

**Proof.** (1) Suppose the hypotheses of Claim (1). Since \( Q \) is a dimer quiver, there is a path \( r \) from \( t(p) \) to \( t(q) \). By Lemma 2.4.1, there is some \( m, n \geq 0 \) such that
\[
rpq^m_{t(p)} = qrp^m_{t(p)}.
\]
Furthermore, by Lemma 2.7, \( \tau \) is an algebra homomorphism. Thus
\[
\bar{p} \sigma^m = \bar{r} \psi \left( rpa^m_{t(p)} \right) = \bar{r} \psi \left( qra^m_{t(p)} \right) = \bar{q} \sigma^m.
\]
Therefore \( \bar{p} = \bar{q} \) since \( B \) is an integral domain and \( \sigma \) does not divide \( p \) or \( q \).

(2) Since \( I \) is generated by binomials, it suffices to show that if \( p, q \in e_i A e_i \) are cycles, then \( qp = pq \). Let \( r^+ \) be a path in \( Q^+ \) from \( h((qp)^+) \) to \( t((qp)^+) \). Set \( r := \pi(r^+) \). Then \( (rqp)^+ \) and \( (rpq)^+ \) are cycles in \( Q^+ \). Thus by Lemma 2.4.2, there is some \( m, n \geq 1 \) such that
\[
rqp = \sigma^m_i \quad \text{and} \quad rpq = \sigma^n_i.
\]
Furthermore, since \( \tau \) is an algebra homomorphism,
\[
\sigma^m_i = \bar{r} \psi \left( \sigma^m_i \right) = \bar{r} \bar{q} \bar{p} = \bar{r} \bar{q} \bar{p} = \bar{r} \bar{p} \bar{q} = \bar{r} \psi \left( \sigma_i^n \right) = \sigma^n_i.
\]
Whence \( m = n \) since \( B \) is a polynomial ring. Thus \( rqp = \sigma^m_i = rpq \). Therefore \( qp = pq \) since \( A \) is cancellative.
(3) Suppose the hypotheses of Claim (3). Let \( r^+ \) be a path in \( Q^+ \) from \( h(p^+) \) to \( t(p^+) \). Then by Lemma 2.4.2, there is some \( m,n \geq 1 \) such that

\[
rp = \sigma^m_i \quad \text{and} \quad rq = \sigma^n_i.
\]

Suppose \( m \leq n \). Then \( rpσ^{n-m}_i = \sigma^n_i = rq \). Thus \( pσ^{n-m}_i = q \) since \( A \) is cancellative. Whence \( m = n \) since by assumption \( p \) and \( q \) are in \( \hat{C}^u \). Therefore \( p = q \).

(4) Suppose the hypotheses of Claim (4). Then

\[
\begin{align*}
\text{t(}(ap)^+) &= \text{t(}(qa)^+) \quad \text{and} \quad \text{h(}(ap)^+) = \text{h(}(qa)^+) .
\end{align*}
\]

Let \( r^+ \) be a path in \( Q^+ \) from \( h((ap)^+) \) to \( t((ap)^+) \). Then by Lemma 2.4.2, there is some \( m,n \geq 1 \) such that

\[
\begin{align*}
rap &= \sigma^m_{t(a)} \quad \text{and} \quad rqa &= \sigma^n_{t(a)}.
\end{align*}
\]

Assume to the contrary that \( m < n \). Then \( qa = apσ^{n-m}_{t(a)} \) since \( A \) is cancellative. Let \( b \) be a path such that \( ab \) is a unit cycle. Then

\[
qσ_{h(a)} = qab = apσ^{n-m}_{t(a)}b = apbσ^{n-m}_{h(a)}.
\]

Thus, since \( A \) is cancellative and \( n - m \geq 1 \),

\[
q = apbσ^{n-m-1}_{h(a)}.
\]

Furthermore, \((ba)^+\) is cycle in \( Q^+ \) since \( ba \) is a unit cycle. But this is a contradiction since \( q \) is in \( \hat{C}^u \). Whence \( m = n \). Therefore \( qa = ap \).

(5) Suppose the hypotheses of Claim (5). Let \( r \) be a path from \( t(p) \) to \( t(q) \). Then by Claim (4), \( rp = qr \). Thus

\[
\tilde{r}\tilde{p} = \tilde{r}\tilde{p} = \tilde{q}\tilde{r} = \tilde{q}\tilde{r} = \tilde{r}\tilde{q}.
\]

Therefore \( p = q \) since \( B \) is an integral domain. \( \square \)

**Lemma 2.12.** Suppose \( A \) is a dimer algebra. Let \( p,q \) be paths such that

\[
\text{t}(p^+) = \text{t}(q^+), \quad \text{h}(p^+) = \text{h}(q^+), \quad \text{and} \quad \bar{\eta}(p) = \bar{\eta}(q).
\]

Then there is a path \( r \) such that \( rp = rq \neq 0 \).

**Proof.** In the following, denote by \( \sigma := \prod_{D \in \mathcal{P}} x_D \) the product of variables indexed by the perfect matchings of \( Q \) (rather than the simple matchings).

Suppose the hypotheses hold. If \( p = q \), then the lemma trivially holds, so suppose \( p \neq q \). It suffices to suppose that the area of the region \( \mathcal{R}_{p,q} \) is minimal among all such pairs of paths satisfying (6) such that \( p \neq q \).

Factor \( p \) and \( q \) into subpaths that are maximal length subpaths of unit cycles,

\[
p = p_m p_{m-1} \cdots p_1 \quad \text{and} \quad q = q_n q_{n-1} \cdots q_1.
\]

Then for each \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), there are paths \( p'_i \) and \( q'_j \) such that

\[
p'_i p_i \quad \text{and} \quad q'_j q_j
\]
are unit cycles, and $p_j^+$ and $q_j^+$ lie in the region $\mathcal{R}_{p,q}$. See Figure 1. Note that if $p_j$ is itself a unit cycle, then $p_j'$ is the vertex $t(p_j)$, and similarly for $q_j$.

Consider the paths

$$p' := p_1'p_2'\cdots p_m' \quad \text{and} \quad q' := q_1'q_2'\cdots q_n'.$$

Then by Lemma 2.3 there is some $s,t \geq 1$ such that

$$(7) \quad p'p = \sigma_t^{(p)} \quad \text{and} \quad q'q = \sigma_t^{(p)}.$$

We claim that $s = t$. Let $D$ be a perfect matching of $Q$. Suppose $x_D$ divides the $\bar{\eta}$-image of $\ell$ of the subpaths $p_1,\ldots,p_m$ of $p$. Then $x_D$ will divide the $\bar{\eta}$-image of $m - \ell$ of the subpaths $p_1',\ldots,p_m'$ of $p'$. Similarly for $q$ and $q'$. But by assumption $\bar{\eta}(p) = \bar{\eta}(q)$. Thus $\ell$ is maximal such that $x_D^\ell | \bar{\eta}(q)$. Therefore $n - \ell$ is maximal such that $x_D^{n-\ell} | \bar{\eta}(q')$.

Since $D \in \mathcal{P}$ was arbitrary, this implies

$$(8) \quad \bar{\eta}(p') = \bar{\eta}(q').$$

Therefore by (7),

$$\sigma^* = \bar{\eta}(p'p) = \bar{\eta}(p')\bar{\eta}(p) = \bar{\eta}(q')\bar{\eta}(q) = \bar{\eta}(q'q) = \sigma^t.$$

Since $Q$ admits at least one perfect matching, $\sigma = \prod_{D \in \mathcal{P}} x_D \neq 1$. Whence $s = t$, proving our claim.

The relation (8) also implies $p'$ and $q'$ satisfy the conditions (6). Furthermore, since $p_j^+$ and $q_j^+$ lie in the region $\mathcal{R}_{p,q}$, $\mathcal{R}_{p',q'}$ is properly contained in $\mathcal{R}_{p,q}$. But $p$ and $q$ were chosen so that $\mathcal{R}_{p,q}$ has minimal area such that $p \neq q$. Therefore $p' = q'$.

Set $r = p'$. Then

$$rp = p'p = p'p = \sigma_t^{(p)} = \sigma_t^{(p)} = q'q = p'q = rq.$$ 

Indeed, (i) and (iii) hold by (7), (ii) holds since $s = t$, and (iv) holds since $p' = q'$.

**Definition 2.13.** We call distinct paths $p,q$ non-cancellative if there is some path $r$ such that $rp = rq \neq 0$ or $pr = qr \neq 0$.

By the definition of $\hat{C}$, each cycle $p \in \mathcal{C}_i \setminus \hat{C}$ has a representative $\tilde{p}$ that factors into subpaths $\tilde{p} = \tilde{p}_3\tilde{p}_2\tilde{p}_1$, where

$$(9) \quad p_1p_3 \in \mathcal{C}^0 \quad \text{and} \quad p_2 \in \hat{C}.$$

**Proposition 2.14.** If $\hat{C} = \emptyset$ for some $u \in \mathbb{Z}^2 \setminus 0$ and $i \in Q_0$, then there is a pair of non-cancellative cycles in $\mathcal{C}_i^2u$. In particular, $A$ is non-cancellative.

**Proof.** In the following, we will use the notation (9), and we will denote by $\sigma := \prod_{D \in \mathcal{P}} x_D$ the product of variables indexed by the perfect matchings of $Q$.

Suppose $\hat{C} = \emptyset$. Let $p,q \in \mathcal{C}_i^u$ be cycles such that the region

$$\mathcal{R}_{\tilde{p}_3\tilde{p}_2\tilde{p}_1,\tilde{q}_3\tilde{q}_2\tilde{q}_1}$$
Figure 1. Setup for Lemma 2.12. The paths $p = p_m \cdots p_1$, $q = q_n \cdots q_1$, $p' = p'_1 \cdots p'_m$, and $q' = q'_1 \cdots q'_n$ are respectively drawn in red, blue, purple, and violet. Each product $p'_i p_i$ and $q'_j q_j$ is a unit cycle. Note that the region $R_{p',q'}$ is properly contained in the region $R_{p,q}$.

contains the vertex $i^+ + u \in Q_0^+$. Furthermore, suppose $p$ and $q$ admit representatives $\tilde{p}$ and $\tilde{q}$ (possibly distinct from $p$ and $q$) such that the region $R_{\tilde{p},\tilde{q}}$ has minimal area among all such pairs of cycles $p,q$. See Figure 2.

By Lemmas 2.4.1 and 2.7, there is some $m \in \mathbb{Z}$ such that

$$\bar{\eta}(p_3p_2p_1) = \bar{\eta}(q_3q_2q_1)\sigma^m.$$  

Suppose $m \geq 0$. Set

$$s := p_3p_2p_1 \quad \text{and} \quad t := q_3q_2q_1\sigma^m.$$  

Then

$$\bar{\eta}(s) = \bar{\eta}(t).$$

Assume to the contrary that $s = t$. Then by Lemma 2.10.2, there is a path $r^+$ in $R_{\tilde{z},\tilde{q}}$ which passes through the vertex $i^+ + u \in Q_0^+$, and is homotopic to $s^+$ (by the relations $I$). In particular, $r$ factors into paths $r = r_2r_1$ where

$$r_1, r_2 \in C_i^u.$$  

But $p$ and $q$ were chosen so that the area of $R_{\tilde{p}',\tilde{q}'}$ is minimal. Thus there is some $\ell_1, \ell_2 \geq 0$ such that

$$\tilde{r}_1 = \tilde{p}'\sigma_i^{\ell_1} \quad \text{and} \quad \tilde{r}_2 = \tilde{p}'\sigma_i^{\ell_2} \quad \text{(modulo $I$).}$$

Set $\ell := \ell_1 + \ell_2$. Then

$$r = r_2r_1 \equiv p_2^2\sigma_i^{\ell_1 + \ell_2} = p_2^2\sigma_i^\ell.$$  

Moreover, the $\tilde{\eta}$-image of any non-vertex cycle in $Q^+$ is a power of $\sigma$ by Lemma 2.9 (with $\tilde{\eta}$ in place of $\tilde{\tau}$). Thus, since $(p_1p_3)^+$ is a non-vertex cycle, there is an $n \geq 1$
such that
\[(12) \bar{\eta}(p_1p_3) = \sigma^n.\]

Hence
\[
\bar{\eta}(p)\bar{\eta}(p_2) \overset{(i)}{=} \bar{\eta}(s) = \bar{\eta}(r) \overset{(ii)}{=} \bar{\eta}(p^2)\sigma^\ell \overset{(iii)}{=} \bar{\eta}(p)\bar{\eta}(p_2)\bar{\eta}(p_1p_3)\sigma^\ell = \bar{\eta}(p)\bar{\eta}(p_2)\sigma^{n+\ell}.
\]
Indeed, (i) and (iii) hold by Lemma 2.7, (ii) holds by (11), and (iv) holds by (12). Furthermore, the image of \(\bar{\eta}\) is in the polynomial ring \(k[x_D \mid D \in \mathcal{P}]\). Thus \(\sigma^{n+\ell} = 1\). But \(n \geq 1\) and \(\ell \geq 0\). Whence \(\sigma = 1\). Therefore \(Q\) has no perfect matchings, contrary to our standing assumption that \(Q\) has at least one perfect matching. Consequently,

\(s \neq t\).

Furthermore, (10) implies that there is a path \(r\) such that \(rs = rt\) by Lemma 2.12. Therefore \(s\) and \(t\) are non-cancellative paths.

**Definition 2.15.** A 2-cycle in \(Q\) is **removable** if the two arrows it is composed of are redundant generators for \(A\), and otherwise the 2-cycle is **non-removable**.

**Lemma 2.16.** Dimer algebras with non-removable 2-cycles exist.

**Proof.** Follows from the two examples of non-removable 2-cycles in Figure 3. □
Figure 3. Cases for Lemma 2.16. In each case, $a$ and $b$ are arrows, and $p$ and $q$ are paths of positive length. In case (i) $ap$, $bq$, and $ab$ are unit cycles; in case (ii) $abp$ and $ab$ are unit cycles; in case (iii) $qbpa$ and $ab$ are unit cycles, and $p$ and $q$ are cycles. In case (i) $ab$ is a removable 2-cycle, and in cases (ii) and (iii) $ab$ is a non-removable 2-cycle.

3. Cancellative dimer algebras

In this section we give an explicit impression of all cancellative dimer algebras.

It is well known that if $A$ is cancellative, then $A$ is a 3-Calabi-Yau algebra [D, MR]. In particular, the center $Z$ of $A$ is noetherian and reduced, and $A$ is a finitely generated $Z$-module. In the following lemma we give independent proofs of these facts so that Theorem 3.3 may be self-contained. Furthermore, we prove the converse in Corollary 4.39 below.

Theorem 3.1. Suppose $A$ is cancellative, and let $i, j \in Q_0$. Then

1. $e_iAe_i = Ze_i \cong Z$.
2. $\bar{\psi}(e_iAe_i) = \bar{\psi}(e_jAe_j)$.
3. $A$ is a finitely generated $Z$-module, and $Z$ is a finitely generated $k$-algebra.
4. $Z$ is reduced.

Proof. (1) For each $i \in Q_0$ and $u \in \mathbb{Z}^2$, there exists a unique cycle $c_{ui} \in \hat{C}_i^u$ (modulo $I$) by Lemma 2.11 and Proposition 2.14. Then by Lemma 2.11, the sum

$$\sum_{i \in Q_0} c_{ui} \in \bigoplus_{i \in Q_0} e_iAe_i$$

is in $Z$. Thus $e_iAe_i \subseteq Ze_i$. Furthermore, $Ze_i = Ze_i^2 = e_iZe_i \subseteq e_iAe_i$. Therefore $Ze_i = e_iAe_i$.

We now claim that there is an algebra isomorphism $Z \cong Ze_i$ for each $i \in Q_0$. Indeed, fix $i \in Q_0$ and suppose $z \in Z$ is nonzero. Then there is some $j \in Q_0$ such that $ze_j \neq 0$. Furthermore, by the construction of $Q$ there is a path $p$ from $i$ to $j$. 
Assume to the contrary that \( ze_jp = 0 \). Thus, since \( I \) is generated by binomials, it suffices to suppose \( ze_j = c_1 - c_2 \) where \( c_1 \) and \( c_2 \) are paths. But since \( A \) is cancellative, \( ze_jp = 0 \) implies \( c_1 = c_2 \). Whence \( ze_j = 0 \), a contradiction. Therefore \( ze_jp \neq 0 \). Consequently,

\[ pe_i z = pz = zp = ze_j p \neq 0. \]

Whence \( ze_i \neq 0 \). Thus the algebra homomorphism \( Z \rightarrow \mathcal{Z}e \), \( z \mapsto ze_i \), is injective, hence an isomorphism. This proves our claim.

(2) By Claim (1), there are isomorphisms \( Z \rightarrow e_i A e_i \) and \( Z \rightarrow e_j A e_j \) given by multiplication by the vertex idempotents \( e_i \) and \( e_j \) respectively. Therefore \( \bar{\tau} (e_i A e_i) = \bar{\tau} (e_j A e_j) \) by Lemma 2.11.5.

(3) \( A \) is generated as a \( Z \)-module by all paths of length at most \( |Q_0| \) by Claim (1) and [B, second paragraph of proof of Theorem 2.11]. Thus \( A \) is a finitely generated \( Z \)-module. Furthermore, \( A \) is a finitely generated \( k \)-algebra since \( Q \) is finite. Therefore \( Z \) is also a finitely generated \( k \)-algebra [McR, 1.1.3].

(4) Suppose \( z \in \mathcal{Z} \) and \( z^n = 0 \). Fix \( i \in Q_0 \). Then

\[ (ze_i)^n = z^n e_i = 0. \]

Furthermore, we may write

\[ ze_i = \sum_{j=1}^{\ell} p_j \in e_i A e_i, \]

where each \( p_j \) is a cycle with a scalar coefficient. For \( 1 \leq j \leq \ell \), let \( u_j \in \mathbb{Z}^2 \) be such that \( p_j \in C^{u_j} \). Then the path summand \( p_{j_1} \cdots p_{j_\ell} \) of \( (ze_i)^n \) is in \( C^{\sum_{k=1}^{\ell} u_{j_k}} \). In particular, if \( u_k, 1 \leq k \leq n \), has maximal length, then \( p_{j_k}^{n_k} \) is the only path summand of \( (ze_i)^n \) which is in \( C^{n u_k} \). But no path is zero modulo \( I \), and so \( z = 0 \). Therefore \( Z \) is reduced. \( \square \)

**Proposition 3.2.** Let \( A \) be a cancellative dimer algebra. If \( p \in A \) is a cycle without cyclic subpaths such that \( p^+ \) is not a cycle, then \( \sigma \nmid \bar{p} \). In particular, if \( u \in \mathbb{Z}^2 \setminus \{0\} \) and \( p \in \hat{C}^u \), then \( \sigma \nmid \bar{p} \).

**Proof.** (i) Recall that if \( B \) is a finitely generated PI algebra over a commutative Jacobson ring \( R \), then the intersection of a primitive ideal of \( B \) with \( R \) is a maximal ideal of \( R \) [L, Theorem 6.3.3]. Furthermore, if \( B \) is module-finite over a noetherian central subring \( R \), then the primitive and maximal ideal spectrums of \( B \) coincide, \( \text{Prim} B = \text{Max} B \). Thus by Theorem 3.1.3, there is a map

\[ \text{Prim} A = \text{Max} A \to \text{Max} \mathcal{Z}, \quad n \mapsto n \cap \mathcal{Z}. \]

(ii) We claim that this map is surjective. Indeed, let \( m \in \text{Max} \mathcal{Z} \). Since \( \text{Max} A = \text{Prim} A \), there is some \( n \in \text{Prim} A \) such that \( n \supseteq mA \). Thus

\[ m \subseteq mA \cap \mathcal{Z} \subseteq n \cap \mathcal{Z}. \]

Whence \( m = n \cap \mathcal{Z} \) since \( m \) is maximal, proving our claim.
(iii) Suppose to the contrary that there is a cycle $p \in e_iAe_i$, without cyclic subpaths, such that $\sigma \mid p$. Then by Definition 2.6, $p$ annihilates every simple $A$-module that is annihilated by an arrow. In particular, if $n \in \text{Prim } A$, then

$$\sigma_i \in n \implies p \in n.$$  

By Claim (ii), the map $\text{Prim } A \to \text{Max } Z$, $n \mapsto n \cap Z$, is surjective. Thus by Theorem 3.1.1, the map

$$\text{Prim } A \to \text{Max } (Ze_i), \quad n \mapsto n \cap Ze_i,$$

is also surjective.

By Theorem 3.1.1, $e_iAe_i = Ze_i$. Thus $p \in Ze_i$. Let $m \in \text{Max } (Ze_i)$. Then by (13) and the surjectivity of (14),

$$\sigma_i \in m \implies p \in m.$$

Consequently, the zero locus of $\sigma_i$ is contained in (whence equals) the zero locus of $p$ in $\text{Max } (Ze_i),$

$$Z(\sigma_i) \subseteq Z(p).$$

By Theorems 3.1.3 and 3.1.4, $Ze_i \cong Z$ is a reduced finitely generated $k$-algebra. Thus by Hilbert’s Nullstellensatz, (15) implies

$$\sqrt{pZ} = \mathcal{I}(Z(p)) \subseteq \mathcal{I}(Z(\sigma_i)) = \sqrt{\sigma_iZ}.$$

Whence there is some $n \geq 1$ and $q \in Ze_i$ such that

$$p^n = \sigma_iq.$$

Since $I$ is generated by binomials, we may assume $q$ is a cycle.

If $n = 1$, then $p$ has a cyclic subpath, contrary to assumption. Thus $n \geq 2$. Since $\sigma_i^+$ is a cycle in $Q^+$, $(p^n)^+$ intersects itself (modulo $I$). Thus $(p^2)^+$ also intersects itself, say at vertex $j \in Q_0$. The unit cycle $\sigma_j$ is therefore a subpath of $p^2$ by Lemma 2.4.2.

Let $m \geq 1$ be the largest integer such that $\sigma_j^m$ is a subpath of $p^2$. Then $p$ is the product of paths

$$p = q_2rq_1,$$

where $(q_1q_2)^+$ is a cycle satisfying $q_1q_2 = \sigma_j^m$:

$$p^2 = (q_2rq_1)(q_2rq_1) = q_2r(q_1q_2)rq_1 = q_2r\sigma_j^mraq_1 \overset{(i)}{=} \sigma_i^mq_2r^2q_1,$$

where (i) holds by Lemma 2.3. Furthermore, by Lemma 2.4.2, $r^+$ has no cyclic subpaths since $m \geq 1$ is maximal.

Let $u \in \mathbb{Z}^2$ be such that $p \in \hat{C}^u$. Then $r$ is a cycle in $\hat{C}^u$ since $(q_1q_2)^+$ is a cycle in $Q^+$ and $r^+$ has no cyclic subpaths. Therefore by Lemma 2.11.5,

$$\tilde{r} = \tilde{p}.$$
Let \( t \geq m \) be such that
\[
\sigma^t \mid p^2 \quad \text{and} \quad \sigma^{t+1} \nmid p^2.
\]
Then by (16),
\[
\sigma^{t-m} \mid q_2 r^2 q_1 \quad \text{and} \quad \sigma^{t-m+1} \nmid q_2 r^2 q_1.
\]
Thus, since \( \sigma \) is a non-unit in \( B \),
\[
\sigma^{t-m+1} \nmid r^2.
\]
Therefore by (17),
\[
\sigma^{t-m+1} \nmid p^2.
\]
But this is a contradiction to (18) since \( m \geq 1 \).
\[\square\]

We now prove our main theorem for this section.

**Theorem 3.3.** Let \( A = kQ/I \) be a cancellative dimer algebra. Then the algebra homomorphism
\[
\tau : A \to M_{|Q_0|}(B),
\]
defined in (4), is an impression of \( A \). Therefore \( \tau \) classifies all simple \( A \)-module isoclasses of maximal \( k \)-dimension. Furthermore, the following holds:
\[
Z \cong k \left[ \cap_{i \in Q_0} \bar{\tau} (e_i A e_i) \right] = k \left[ \cup_{i \in Q_0} \bar{\tau} (e_i A e_i) \right].
\]

**Proof.** (i) We first show that \( \tau \) is injective.

(i.a) We claim that \( \tau \) is injective on the corner rings \( e_i A e_i, i \in Q_0 \).

Fix a vertex \( i \in Q_0 \) and let \( p, q \in e_i A e_i \) be cycles satisfying \( p = q \). Let \( r \) be a path such that \( r^+ \) is path from \( h(p^+) \) to \( t(p^+) \). Then \( (rp)^+ \) is a cycle in \( Q^+ \). Thus by Lemma 2.4.2, there is some \( m \geq 1 \) such that
\[
rp = \sigma_i^m.
\]
Whence
\[
\bar{r}q = \bar{r}q = \bar{r}p = \bar{r}p = \bar{\sigma}_i^m = \bar{\sigma}_i^m = \sigma^m.
\]
By Proposition 3.2 if \( (rq)^+ \) were not a cycle in \( Q^+ \), then there would be a simple matching \( D \) such that \( x_D \nmid \bar{r}q \). Thus \( (rq)^+ \) is a cycle. Whence \( rq = \sigma_i^m = rp \) by Lemma 2.4.2. Therefore \( p = q \) since \( A \) is cancellative.

(i.b) We now claim that \( \tau \) is injective on paths.

Consider two paths \( p, q \) in \( Q \) that satisfy \( \bar{p} = \bar{q} \). Then
\[
\bar{p} = \bar{q}, \quad h(p) = h(q), \quad \text{and} \quad t(p) = t(q).
\]
Let \( r \) be a path from \( h(p) \) to \( t(p) \). The two cycles \( pr \) and \( qr \) at \( h(p) \) then satisfy \( \bar{pr} = \bar{qr} \). Thus \( pr = qr \) since \( \tau \) is injective on the corner ring \( e_{h(p)} A e_{h(p)} \) by Claim (i.a). Therefore \( p = q \) since \( A \) is cancellative.

Since \( A \) is generated by paths and \( \tau \) is injective on paths, it follows that \( \tau \) is injective.
For each \( b \in \mathbb{Z} (\sigma)^c \subset \text{Max } B = \mathbb{A}^{|S|}_k \), the composition \( \epsilon_b \tau \) defined in (2) is a representation of \( A \) where, when viewed as a vector space diagram on \( Q \), each arrow is represented by a non-zero scalar. Thus \( \epsilon_b \tau \) is simple since there is an oriented path between any two vertices in \( Q \). Therefore \( \epsilon_b \tau \) is surjective.

(iii) We claim that the morphism
\[
\text{Max } B \to \text{Max } \tau(Z), \quad b \mapsto b |_{Q_0} \cap \tau(Z),
\]
is surjective. Indeed, for any \( m \in \text{Max } R \), \( Bm \) is a (nonzero) proper ideal of \( B \). Thus there is a maximal ideal \( b \in \text{Max } B \) containing \( Bm \) since \( B \) is a noetherian. Furthermore, since \( B \) is a finitely generated \( k \)-algebra and \( k \) is algebraically closed, the intersection \( b \cap R =: m' \) is a maximal ideal of \( R \). Whence
\[
m \subseteq Bm \cap R \subseteq b \cap R = m'.
\]
But \( m \) and \( m' \) are both maximal ideals of \( R \). Thus \( m = m' \). Therefore \( b \cap R = m \), proving our claim.

Claims (i), (ii), (iii) imply that \( (\tau, B) \) is an impression of \( A \).

(iv) Since \( (\tau, B) \) is an impression of \( A \), \( Z \) is isomorphic to \( R \) by [B, Lemma 2.1 (2)]. Furthermore, \( R \) is equal to \( S \) by Lemma 2.11.5. Therefore (19) holds. \( \square \)

**Remark 3.4.** The labeling of arrows we obtain, namely (4), agrees with the labeling of arrows in the toric construction of [CQ, Proposition 5.3]. We note, however, that impressions are defined more generally for non-toric algebras and have different implications than the toric construction of [CQ].

**Corollary 3.5.** A dimer algebra \( A \) is cancellative if and only if it admits an impression \( (\tau, B) \) where \( B \) is an integral domain and \( \tau(e_i) = E_{ii} \) for each \( i \in Q_0 \).

**Proof.** Suppose \( A \) admits an impression \( (\tau, B) \) such that \( B \) is an integral domain and \( \tau(e_i) = E_{ii} \) for each \( i \in Q_0 \). Then \( \bar{\tau}(p) = \bar{b} \in B \) for each path \( p \).

Consider paths \( p, q, r \) satisfying \( pr = qr \neq 0 \). Then
\[
\bar{p} r = \bar{p} \bar{r} = \bar{q} \bar{r} = \bar{q} r.
\]
Thus \( \bar{p} = \bar{q} \) since \( B \) is an integral domain. Whence
\[
\tau(p) = p E_{h(p),h(r)} = \bar{q} E_{h(p),h(r)} = \tau(q).
\]
Therefore \( p = q \) by the injectivity of \( \tau \).

The converse follows from Theorem 3.3. \( \square \)

**Remark 3.6.** Although non-cancellative dimer algebras do not admit impressions \( (\tau, B) \) with \( B \) an integral domain by Corollary 3.5, we will find that their homotopy algebras do; see Definition 4.32 and Theorem 4.34.
Example 3.7. A dimer algebra $A = kQ/I$ is square if the underlying graph of its cover $Q^+$ is a square grid graph with vertex set $\mathbb{Z} \times \mathbb{Z}$, and with at most one diagonal edge in each unit square. Examples of square dimer algebras are given in Figure 5 (i.b, ii.b, iii.c, iv.c).

By [B, Theorem 3.7], any square dimer algebra $A$ admits an impression $(\tau, B = k[x, y, z, w])$, where for each arrow $a \in Q^1$, $\bar{\tau}(a)$ is the monomial corresponding to the orientation of $a$ given in Figure 4. Specifically, $\tau : A \rightarrow M_{|Q_0|}(B)$ is the algebra homomorphism defined by

$$
\tau(e_i) = E_{ii} \text{ and } \tau(a) = \bar{\tau}(a) E_{h(a), t(a)}
$$

for each $i \in Q_0$ and $a \in Q_1$. If $Q$ only possesses three arrow orientations, say up, left, and right-down, then we may label the respective arrows by $x, y,$ and $z$, and obtain an impression $(\tau, k[x, y, z])$. In either case, $A$ is cancellative by Corollary 3.5.

4. Non-cancellative dimer algebras and their homotopy algebras

4.1. Cyclic contractions. In this section we introduce a new method for studying non-cancellative dimer algebras that is based on the notion of Higgsing, or more generally symmetry breaking, in physics. In this strategy, we gain information about a non-cancellative dimer algebra by relating it to a cancellative dimer algebra with certain similar structure. Throughout, $A = kQ/I$ denotes a dimer algebra, usually non-cancellative.

The following definition formalizes the notion of Higgsing in abelian quiver gauge theories.

Definition 4.1. Consider a dimer algebra $A = kQ/I$ with quiver $Q = (Q_0, Q_1, t, h)$. Fix a subset of arrows $Q^*_1 \subset Q_1$. Form the quiver $Q' = (Q'_0, Q'_1, t', h')$ by identifying the three paths $a, h(a), t(a)$ in $Q$ for each $a \in Q^*_1$. More precisely, set

$$Q'_0 := Q_0 / \{ h(a) \sim t(a) \mid a \in Q^*_1 \}, \quad Q'_1 = Q_1 \setminus Q^*_1,$$

and $t'(a) = t(a), h'(a) = h(a)$ for each $a \in Q'_1$.

Consider the $k$-linear map $\psi : kQ \rightarrow kQ'$ defined by sending a path in $kQ$ to the corresponding path in $kQ'$. If the algebra $A' = kQ'/\psi(I)$ is also a dimer algebra, then we call the induced $k$-linear map

$$
\psi : A = kQ/I \rightarrow A' = kQ'/\psi(I)
$$
a contraction of dimer algebras.

A contraction must preserve some structure in order to be a useful operation. Indeed, every dimer algebra can be contracted to the cancellative dimer algebra whose quiver consists of one vertex and three loops, which is the simplest possible dimer quiver. To specify the structure we wish to preserve, we introduce the following commutative algebras.

**Definition 4.2.** Let $\psi : A \to A'$ be a contraction to a cancellative dimer algebra, and let $(\tau, B)$ be an impression of $A'$. For each path $p \in A$, set

$$\bar{\psi}(p) := \bar{\tau}(\psi(p)).$$

The *cycle algebras* of $A$ and $A'$ are the respective commutative rings

$$S := k \left[ \bigcup_{i \in Q_0} \bar{\psi} (e_i A e_i) \right] \quad \text{and} \quad S' := k \left[ \bigcup_{i \in Q_0'} \bar{\tau} (e_i A' e_i) \right].$$

We say $\psi$ is *cyclic* if $S = S'$.

Henceforth we will consider cyclic contractions

$$\psi : A \to A',$$

where $A$ is non-cancellative and $A'$ is cancellative.

**Notation 4.3.** For $g, h \in B$, by $g \mid h$ we mean $g$ divides $h$ in $B$, even if $g$ or $h$ is assumed to be in $S$.

An immediate question is whether all non-cancellative dimer algebras admit cyclic contractions. Set

$$Q_1^\dagger := \{ a \in Q_1 \mid a \not\in D \text{ for all } D \in S \},$$

where $S$ is the set of simple matchings of $Q$. Recall the definition of a non-removable 2-cycle (Definition 2.15).

**Definition 4.4.** A dimer algebra $A = kQ/I$ is *2-cycle free* if $Q$ does not obtain a non-removable 2-cycle by contracting a set of arrows in $Q_1^\dagger$.

**Theorem 4.5.** [BIU] Suppose $A$ is a dimer algebra which is 2-cycle free. Then there is a cyclic contraction $\psi : A \to A'$ to a cancellative dimer algebra $A'$.

**Physics Remark 4.6.** In the context of a 4-dimensional $\mathcal{N} = 1$ quiver gauge theory with quiver $Q$, the Higgsing considered here is presumably related to RG flow. We start with a non-superconformal (strongly coupled) quiver theory $Q$, give nonzero vev’s to a set of bifundamental fields $Q_1^\star$, and obtain a new theory $Q'$ that lies at a conformal fixed point.

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Physics Remark 4.7. The cycle algebra $S$ has similar structure to the mesonic chiral ring in a quiver gauge theory. In such a theory, the mesonic operators, which are the gauge invariant operators, are generated by the cycles in the quiver. These operators are viewed as functions on the classical vacuum moduli space, which is the representation space of the corresponding superpotential algebra with dimension vector specified by the gauge group, modulo isomorphism. If the gauge group is abelian, then the dimension vector is $(1, \ldots, 1)$. In the case of a dimer theory with abelian gauge group, two separated cycles may share the same $\psi$-image, but take different values on some point of the vacuum moduli space. These cycles would then be distinct elements in the mesonic chiral ring, although they are identified in the cycle algebra $S$.

Example 4.8. Consider the four examples of cyclic contractions given in Figure 5. In each example, the quiver labeled (a) is non-cancellative and contracts to the cancellative quiver $Q'$ labeled (b). In examples (iii) and (iv), quiver (c) is obtained from quiver (b) by removing all 2-cycles. The corresponding dimer algebras are equal. Each cancellative quiver $Q'$ is a square dimer (Example 3.7) with an impression given by the arrow orientations in Figure 4. The non-cancellative quivers (a) first appeared respectively in [DHP, Table 5, 2.3]; [FHPR, Section 4], [FKR]; [Bo, Example 3.2]; and [DHP] Table 6, 2.6 (each in a different form from what is shown here).

In example (i), the impression ring $B$ is the polynomial ring $k[x, y, z]$ since there are only three orientations of arrows in $Q'$, and in examples (ii)-(iv), $B$ is the polynomial ring $k[x, y, z, w]$. It is easy to check that $S = S'$ in each example. Explicitly, $S$ is given by:

(i): $S = k[x^2, y^2, xy, z]$  
(ii): $S = k[xz, xw, yz, yw]$  
(iii): $S = k[xz, yz, xw, yw]$  
(iv): $S = k[xz, yw, x^2w^2, y^2z^2]$  

$R$ is defined in Definition 4.24 below.

Lemma 4.9. Consider a contraction $\psi : A \to A'$ to a cancellative dimer algebra. Let $p$ be a cycle in $Q$ whose lift $p^+$ is not a cycle. Then $\psi(p^+)$ is not a cycle. In particular, $\psi(p^+)$ is not a vertex.

Proof. The number of vertices, edges, and faces in the underlying graphs $Q$ and $Q'$ of $Q$ and $Q'$ are given by

$V = |Q_0|$,  
$E = |Q_1|$,  
$F = \#$ of connected components of $T^2 \setminus Q$,

$V' = |Q'_0|$,  
$E' = |Q'_1|$,  
$F' = \#$ of connected components of $T^2 \setminus Q'$.

\footnote{The algebra $R$ is defined in Definition 4.24 below.}
Figure 5. Nonnoetherian deformations of some square dimer algebras. Each quiver is drawn on a torus, and the contracted arrows are drawn in green.

Since $\overline{Q}$ and $\overline{Q}'$ each embed into a two-torus, their respective Euler characteristics vanish:

\[
V - E + F = 0, \quad V' - E' + F' = 0.
\]

Assume to the contrary that there is a cycle $p$ whose lift $p^+$ is not a cycle in $Q^+$, but $\psi(p^+)$ is a cycle in $Q'^+$. Then there is an unoriented path in $Q^+$ between $t(p^+)$ and $h(p^+)$ that is contracted to the vertex $t(\psi(p^+))$ in $Q'^+$.

In any cycle, the number of vertex subpaths equals the number of arrow subpaths. Thus if an unoriented cycle of length $m$ is contracted to a vertex, then the total number of arrows in the quiver drops by $m$ whereas the total number of vertices drops by $m - 1$:

\[
V' = V - m + 1, \quad E' = E - m.
\]
Similarly, suppose two intersecting unoriented cycles of lengths \( m \) and \( n \) are contracted to a single vertex. Then the total number of arrows drops by \( m \), whereas the total number of vertices drops by \( m + (n - 1) - 1 = m + n - 2 \):

\[
V' = V - (m + n) + 2, \quad E' = E - (m + n).
\]

Therefore if an unoriented cycle is contracted to a vertex, then \( F \neq F' \) by \([21]\). In particular, some unit cycle \( \sigma_j \) in \( Q \) is contracted to a vertex.

Since each connected component of \( T^2 \setminus Q' \) is simply connected, the arrow set \( Q'_1 \) is nonempty. Fix \( a \in Q'_1 \). By the definition of a dimer, \( a \) is contained in some unit cycle \( \sigma' \). Let \( \sigma_i \) be the unit cycle in \( Q \) such that \( \psi(\sigma_i) = \sigma' \), and let \( q \) be a path from \( i \) to \( j \). Then

\[
\psi(q)\sigma' = \psi(q)\psi(\sigma_i) = \psi(q\sigma_i) \overset{(1)}{=} \psi(q\sigma_j) = \psi(\sigma_j)\psi(q) = e_{\psi(i)}\psi(q) = \psi(q),
\]

where (t) holds by Lemma \([2,3]\). But by assumption \( Q' \) is cancellative. Therefore \( \sigma' = e_{\psi(i)} \), a contradiction. \( \square \)

Recall that if \( p \) and \( q \) are paths satisfying

\[
t(p^+) = t(q^+) \quad \text{and} \quad h(p^+) = h(q^+),
\]

then representatives \( \bar{p}^+ \) and \( \bar{q}^+ \) of their lifts bound a compact region \( R_{\bar{p},\bar{q}} \) in \( \mathbb{R}^2 \). If only one pair of representatives is considered, then by abuse of notation we denote \( R_{\bar{p},\bar{q}} \) by \( R_{p,q} \). Furthermore, we denote the interior of \( R_{p,q} \) by \( R_{p,q}^o \).

**Lemma 4.10.** Let \( p, q \in e_j A e_i \) be paths such that \( \psi(p) = \psi(q) \).

1. The lifts \( p^+ \) and \( q^+ \) bound a compact region \( R_{p,q} \) in the cover \( \mathbb{R}^2 \supset Q^+ \) of \( Q \).
2. Suppose \( p \neq q \), and there are no proper subpaths \( p', q' \) (modulo \( I \)) of \( p \) and \( q \) satisfying \( \psi(p') = \psi(q') \). If \( r \) is a path of minimal length such that \( rp = rq \neq 0 \) or \( pr = qr \neq 0 \), then \( r^+ \) intersects the interior of \( R_{p,q} \) (modulo \( I \)).

**Proof.**

1. Since \( \psi(p) = \psi(q) \), their lifts \( \psi(p)^+ \) and \( \psi(q)^+ \) have coincident heads and tails by Lemma \([2,10,1]\). By Lemma \([4,9]\) there is no non-vertex cycle \( c \in e_i A e_i \) satisfying \( \psi(c) = e_{\psi(\bar{c})} \). Therefore \( p^+ \) and \( q^+ \) have coincident heads and tails as well, so \( p^+ \) and \( q^+ \) bound a compact region in \( \mathbb{R}^2 \).

2. Suppose \( rp = rq \neq 0 \), and assume to the contrary that \( r^+ \) does not intersect the interior of \( R_{p,q} \); see Figure \([6]\). Since \( p \neq q \), there is a rightmost positive-length subpath \( p_2 \) of \( p \) and a leftmost positive-length subpath \( r_1 \) of \( r \) (modulo \( I \)) such that for some arrow \( s \), \( r_1 p_2 s \) is a unit cycle. Let \( ts \) be the complementary unit cycle containing \( s \); then \( t = r_1 p_2 \) modulo \( I \). Since \( rp \) homotopes to \( rq \), \( t^+ \) intersects the interior of \( R_{p,q} \). But then \( q^+ \) intersects the interiors of the unit cycles \( (r_1 p_2)^+ \) and \( (ts)^+ \) since \( h(q^+) = h(p_2^+) \), a contradiction. Therefore \( r^+ \) intersects the interior of \( R_{p,q} \). \( \square \)
Figure 6. Setup for Lemma 4.10.2. The paths $p = p_2p_1$, $q$, and $r = r_2r_1$ are drawn in red, blue, and green respectively. The orange path $s$ is an arrow, and the cycles $st$ and $sr_1p_2$ are unit cycles. This leads to a contradiction since $q$ passes through their interior of these unit cycles.

Remark 4.11. The assumption in Lemma 4.10.2 that $p$ and $q$ do not have proper subpaths $p'$ and $q'$ such that $\psi(p') = \psi(q')$ is necessary. Indeed, consider the setup given in Figure 7. Here $p$ is the product of paths $p = p_3p_2p_1$, and $r^+$ does not intersect the interior of $R_{p,q}$. Furthermore, $a$ is an arrow, the cycles $arp_3$ and $at_{n+1}t_n\cdots t_1$ are unit cycles, and $t_1,\ldots,t_n$ each bound a unit cycle with a subpath of $p_2$. Then by Lemma 2.3,

$$rp = t_{n+1}t_{n+1}\cdots t_1p_2p_1 = t_{n+1}p_1\sigma_i^n.$$ 

Thus $rp$ has picked up the factor $\sigma_i^n$ by homotoping over the arrow $a$. However, $(t_{2}p_1)^+$ is farther from $(r_{q})^+$ than $(rp)^+$ is, rather than closer. In order to homotope closer to $(r_{q})^+$, $(t_{2}p_1)^+$ must first 'give back' the factor $\sigma_i^n$, and is thus forced to homotope back to $(rp)^+$.

Indeed, consider the two explicit examples given in Figure 7. In each case, $\psi$ contracts the arrows whose tails are at vertices with in-out degree 1. Case (i) differs from case (ii) by a single extra vertex, drawn in bold, and an arrow at that vertex. In case (i), $rp$ is not equal to $r_{q}$ because of this extra vertex, whereas in case (ii) we have $rp = r_{q}$. However, in case (ii) $p$ and $q$ have proper subpaths $p'$ and $q'$ modulo $I$ satisfying $\psi(p') = \psi(q')$, and in case (i) there are no such subpaths.

Lemma 4.12. Let $p,q \in e_jAe_i$ be paths such that

$$t(p^+) = t(q^+) \quad \text{and} \quad h(p^+) = h(q^+) \neq t(p^+).$$

If $p^+$ and $q^+$ have no cyclic subpaths and $R^o_{p,q}$ contains no vertices, then $p = q$.

Proof. Suppose the hypotheses hold. Let $a^+$ be a path of minimal length that lies in $R_{p,q}$, such that its head lies on $p^+$ or $q^+$, and its tail lies on $p^+$ or $q^+$. Then $a$ is an arrow since $R^o_{p,q}$ contains no vertices. Furthermore, suppose the head and tail of $a^+$ both lie on $p^+$ (resp. $q^+$), and at vertices distinct from $i^+$ and $j^+$. Then $p^+$ (resp. $q^+$) contains a unit cycle subpath (modulo $I$), contrary to assumption. (Such an example is given in Figure 8.iii, where the heads and tails of the green arrows all lie on $q^+$ at
Figure 7. The general setup for Remark 4.11 is given in the top figure, and explicit examples are given in the bottom two figures. Case (i) differs from case (ii) by an additional vertex, drawn in bold, and an additional arrow at this vertex. In each case, the paths $p$, $q$, $r$ are drawn in red, blue, green respectively. Note that $r^+$ does not intersect the interior of $R_{p,q}$. In case (i), $rp \neq rq$, whereas in case (ii) $rp = rq \neq 0$. However, in case (ii) $p$ and $q$ have proper subpaths $p'$ and $q'$ modulo $I$ such that $\psi(p') = \psi(q')$, and in case (i) there are no such subpaths.

vertices distinct from $i^+$ and $j^+$. Consequently, $q^+$ has a cyclic subpath.) Thus each arrow $a^+$ in $R_{p,q}$, which is not a subpath of $p^+$ or $q^+$, meets both $p^+$ and $q^+$.

We therefore have one of the two cases (i) and (ii) given in Figure 8, possibly with $p$ and $q$ swapped. However, in case (ii) $q^+$ has a cyclic subpath, again contrary to assumption. Hence $p$ and $q$ are as in case (i). Therefore $p = q$. □

**Remark 4.13.** The assumption in Lemma 4.12 that $p^+$ and $q^+$ are not cycles is necessary. Indeed, consider the dimer algebra with quiver $Q$ given in Figure 11.i. The $\psi$-image of the red cycle equals the $\psi$-image of the blue unit cycle in its interior (modulo $I'$, since four pairs of removable 2-cycles form under $\psi$). Furthermore, there are no vertices in the interior of the annulus bounded by the red and blue cycles. However, these two cycles are not equal (modulo $I$).
Figure 8. Cases for Lemma 4.12. The paths $a_1, \ldots, a_\ell, b_1, \ldots, b_\ell$ are arrows, and the paths $p = p_\ell \cdots p_1$ and $q = q_\ell \cdots q_1$ are drawn in red and blue respectively. In case (iii), the green paths are also arrows. Note that $p = q$ (modulo $I$) in case (i), and $q^+$ has a cyclic subpath in cases (ii) and (iii).

We conclude with the following generalization of Lemmas 2.11.4 and 2.11.5 to the case where $A$ is non-cancellative.

**Lemma 4.14.** Let $u \in \mathbb{Z}^2 \setminus \{0\}$. Suppose $a \in Q_1$, $p \in \hat{C}_u t(a)$, and $q \in \hat{C}_u h(a)$. If $\mathcal{R}_{ap,qa}$ contains no vertices, then $ap = qa$. Consequently, $p = q$.

**Proof.** Suppose the hypotheses hold. If $(ap)^+$ and $(qa)^+$ have no cyclic subpaths (modulo $I$), then $ap = qa$ by Lemma 4.12.

So suppose $(qa)^+$ contains a cyclic subpath. The path $q^+$ has no cyclic subpaths since $q$ is in $\hat{C}$. Thus $q$ factors into paths $q = q_2 q_1$, where $(q_1 a)^+$ is a cycle. In particular,

$$t(p^+) = t((q_1 q_2)^+) \quad \text{and} \quad h(p^+) = h((q_1 q_2)^+).$$

Whence $p$ and $q_1 q_2$ bound a compact region $\mathcal{R}_{p,q_1 q_2}$. Furthermore, its interior $\mathcal{R}_{p,q_1 q_2}^\circ$ contains no vertices since $\mathcal{R}_{ap,qa}^\circ$ contains no vertices.

The path $(q_2^2)^+$ has no cyclic subpaths, again since $q$ is in $\hat{C}$. Thus $(q_1 q_2)^+$ also has no cyclic subpaths. Furthermore, $p^+$ has no cyclic subpaths since $p$ is in $\hat{C}$. Therefore $p = q_1 q_2$ by Lemma 4.12.

Since there are no vertices in $\mathcal{R}_{ap,qa}^\circ$, there are also no vertices in the interior of the region bounded by the cycle $(aq_1)^+$. Thus there is some $\ell \geq 1$ such that

$$aq_1 = a_{h(a)}^\ell \quad \text{and} \quad q_1 a = a_{t(a)}^\ell.$$
Figure 9. A dimer algebra $A$ for which $\text{nil} Z \neq 0$. A fundamental domain of $Q$ is shown on the left, and a larger region of $Q^+$ is shown on the right. The paths $p$, $q$, $a$ are drawn in red, blue, and green respectively. The element $(p - q)a + a(p - q)$ is central and squares to zero.

Therefore

$$ap = aq_1q_2 = \sigma_{h(a)}^\ell q_2 \overset{(1)}{=} q_2\sigma_{t(a)}^\ell = q_2q_1a = qa,$$

where (1) holds by Lemma 2.3.

It thus follows that $\bar{p} = \bar{q}$ by the proof of Lemma 2.11.5 with $r = a$. □

4.2. Reduced and homotopy centers. Throughout, $A$ is a non-cancellative dimer algebra and $\psi : A \to A'$ is a cyclic contraction to a cancellative dimer algebra. We will denote by $Z$ and $Z'$ the respective centers of $A$ and $A'$.

4.2.1. The central nilradical. If $A$ is a cancellative dimer algebra, then its central nilradical $\text{nil} Z$ is zero by Theorem 3.1.4. However, this no longer holds if $A$ is non-cancellative.

Theorem 4.15. Dimer algebras with non-vanishing central nilradical exist.

Proof. Consider the non-cancellative dimer algebra $A$ with quiver $Q$ given in Figure 9. Then the paths $p$, $q$, $a$ satisfy

$$(p - q)a + a(p - q) \in \text{nil} Z.$$

In particular, $\text{nil} Z \neq 0$. (A cyclic contraction of $A$ is given in Example 4.8.i.) □

Question 4.16. Is there a necessary and/or sufficient combinatorial condition for the central nilradical of a non-cancellative dimer algebra to vanish?

In the following, we characterize the central nilradical in terms of the cyclic contraction $\psi$, and show that it is a prime ideal.

Lemma 4.17. Suppose $p^+$ is a cycle in $Q^+$. If $p$ is not equal to a product of unit cycles (modulo $I$), then $p \not\in Ze_{t(p)}$. 

Proof. Suppose \( p \in C^0_i \) satisfies \( p \neq \sigma^n_i \) for all \( n \geq 1 \). Two examples of such a cycle are given by the red cycles in Figures 11.i and 11.ii.

Let \( r \) be a path with tail at \( i \) whose lift \( r^+ \) does not intersect the interior of the region bounded by \( p^+ \) in \( \mathbb{R}^2 \), modulo \( I \). Suppose \( q \) is a cycle satisfying \( rp = qr \). Then the region bounded by \( q^+ \) in \( \mathbb{R}^2 \) contains \( p^+ \). Furthermore, \( rp = qr \) implies \( \bar{p} = \bar{q} \) since \( B \) is an integral domain.

But for a sufficiently long path \( r^+ \), the \( \bar{\psi} \)-image of any cycle at \( h(r) \) whose lift contains \( p^+ \) in its interior will clearly not equal \( \bar{p} \). Thus for such a path \( r \), \( rp \neq qr \) for all cycles \( q \). Therefore \( p \) is not in \( Ze_i \). \( \square \)

Remark 4.18. It is possible for two cycles in \( Q^+ \), one of which is properly contained in the region bounded by the other, to have equal \( \bar{\psi} \)-images. Indeed, consider Figure 11.i: the red cycle and the unit cycle in its interior both have \( \bar{\psi} \)-image \( \sigma \).

Lemma 4.19. Let \( u \in \mathbb{Z}^2 \setminus 0 \). Suppose \( p, q \in C^*_i \) are cycles such that

\[ pq = qp \quad \text{and} \quad \psi(p) = \psi(q). \]

Then

\[ p^2 = qp = q^2. \]

Consequently

\[ (p - q)^2 = 0. \]

Proof. For brevity, if \( sa \) and \( ta \) are unit cycles with \( a \in Q_1 \), then we refer to \( s \) as an arc and \( t \) as its complementary arc.

Suppose the hypotheses hold. If \( p = q \), then the lemma trivially holds, so suppose \( p \neq q \). Since \( pq = qp \), there are subpaths \( q', q'', a, c \) of \( q \) and a path \( b \) such that

\[ pq = p(cq') = (pc)q' = (q''b)q' = q''(bq') = q''(ap) = q''(a)p = qp. \]

In particular,

\[ q = cq' = q''a, \quad ap = bq', \quad pc = q''b. \]

See Figure 10, where \( p \) is drawn in red, \( q \) is drawn in blue, and their common subpaths are drawn in violet. In case (i) \( a \) is a rightmost subpath of \( q \) alone, and in case (ii) \( a \) is a rightmost subpath of both \( p \) and \( q \).

(i) First suppose \( a \) is a rightmost subpath of \( q \) alone.

We claim that \( p^+ \) and \( q^+ \) have no cycle subpaths. Indeed, if \( p^+ \) has a unit cycle subpath \( \sigma_i^+ \), then \( \sigma_i^+ \) can be homotoped to a unit cycle at \( h(p) \) which contains \( a \) by Lemma 2.3, yielding case (ii). Otherwise, if \( p^+ \) contains a cyclic subpath \( r^+ \) which is not equal to a product of unit cycles, then \( pq \neq qp \) by Lemma 4.17. This proves our claim.

Now suppose to the contrary that \( ac \neq b \). Then, since \( p \neq q \) and \( pc = q''b \), there is an arc subpath of \( pc \) (drawn in green in Figure 10) containing the rightmost arrow subpath of \( c \) and rightmost arrow subpath of \( p \), whose complementary arc (also drawn in green) lifts to a path that lies in the region bounded by \( (pc)^+ \) and \( (q''b)^+ \).
In particular, the two cycles formed from the green paths and the orange arrow are unit cycles. But this is not possible since \( t(a) = t(p) \), a contradiction. Thus \( ac = b \), yielding

\[
(22) \quad qp = q''ap = q''bq' = q''acq' = q^2.
\]

Similarly \( pq = p^2 \).

(ii) Now suppose \( a \) is a rightmost subpath of both \( p \) and \( q \). Then the relation \( pc = q''b \) implies \( ac = b \) by the contrapositive of Lemma 4.10.2. Therefore (22) holds.

**Remark 4.20.** The assumption in Lemma 4.19 that \( u \neq 0 \) is necessary. Indeed, if \( p, q \in C^0 \) (that is, \( p^+ \) and \( q^+ \) are cycles in \( Q^+ \)), then it is possible for \( pq = qp \), \( \psi(p) = \psi(q) \), and \( (p - q)^2 \neq 0 \). For example, consider Figure 11: the cycle \( p \in e_3 A e_3 \) formed from the red arrows satisfies

\[
p\sigma_3^2 = \sigma_3^2p \quad \text{and} \quad \psi(p) = \psi(\sigma_3^2).
\]

However, \( (p - \sigma_3^2)^m \neq 0 \) for each \( m \geq 1 \).

**Lemma 4.21.** Consider a central element

\[
z = \sum_{i \in Q_0} (p_i - q_i),
\]

where \( p_i, q_i \) are elements in \( e_i A e_i \). Then for each \( i \in Q_0 \),

\[
p_i q_i = q_i p_i.
\]

**Proof.** For each \( i \in Q_0 \) we have

\[
p^2_i - p_i q_i = p_i (p_i - q_i) = p_i z = z p_i = (p_i - q_i) p_i = p^2_i - q_i p_i.
\]

Whence \( p_i q_i = q_i p_i \).
Theorem 4.22. Let $A$ be a non-cancellative dimer algebra satisfying \((\ref{20})\). Then the central elements in the kernel of $\psi$ are precisely the nilpotent central elements of $A$,

$$Z \cap \ker \psi = \text{nil} Z.$$  

Proof. (i) We first claim that if $z \in Z \cap \ker \psi$, then $z^2 = 0$, and in particular $z \in \text{nil} Z$.

Consider a central element $z$ in $\ker \psi$. Since $z$ is central it commutes with the vertex idempotents, and so $z$ is a $k$-linear combination of cycles. Therefore, since $\psi$ sends paths to paths and $I'$ is generated by differences of paths, it suffices to suppose $z$ is of the form

$$z = \sum_{i \in Q_0} (p_i - q_i),$$

where $p_i, q_i$ are cycles in $e_i A e_i$ with equal $\psi$-images modulo $I'$. Note that there may be vertices $i \in Q_0$ for which $p_i = q_i = 0$.

By Lemma 4.21, for each $i \in Q_0$ we have

$$p_i q_i = q_i p_i.$$

Furthermore, by Lemmas 2.3 and 4.17 it suffices to suppose that the lifts $p_i^+, q_i^+$ are not cycles in $Q^+$. Therefore by Lemma 4.19

$$z^2 = \left( \sum_{i \in Q_0} (p_i - q_i) \right)^2 = \sum_{i \in Q_0} (p_i - q_i)^2 = 0.$$

Whence $z \in \text{nil} Z$.

(ii) We now claim that if $z \in \text{nil} Z$, then $z \in \ker \psi$.

Suppose $z^n = 0$. Then for each $i \in Q_0$,

$$\bar{\psi}(z e_i)^n \overset{(\text{i})}{=} \bar{\psi}((z e_i)^n) \overset{(\text{ii})}{=} \bar{\psi}(z^n e_i) = 0,$$

where $(\text{i})$ holds since $\bar{\psi}$ is an algebra homomorphism on $e_i A e_i$, and $(\text{ii})$ holds since $z$ is central. But $\bar{\psi}(e_i A e_i)$ is contained in the integral domain $B$, and so $\bar{\psi}(z e_i) = 0$.

Whence

$$\bar{\tau} \bar{\psi}(z e_i) = \bar{\psi}(z e_i) = 0.$$

Therefore $\psi(z e_i) = 0$ since $\bar{\tau}$ is injective by Theorem 3.3.

Furthermore,

$$\psi(z) \overset{(\text{i})}{=} \psi \left( z \sum_{i \in Q_0} e_i \right) \overset{(\text{ii})}{=} \sum_{i \in Q_0} \psi(z e_i) = 0,$$

where $(\text{i})$ holds since the vertex idempotents form a complete set, and $(\text{ii})$ holds since $\psi$ is a $k$-linear map. Therefore $\psi(z) = 0$.  

\hfill \Box
Figure 11. For Claims (i.a) and (i.b) in the proof of Theorem 4.22. The quivers are drawn on a torus, the contracted arrows are drawn in green, and the 2-cycles have been removed from $Q'$. In each example, the cycle in $Q$ formed from the red arrows is not equal to a product of unit cycles (modulo $I$). However, in example (i) this cycle is mapped to a unit cycle in $Q'$ under $\psi$.

4.2.2. The reduced center as a subalgebra. Let $\psi : A \rightarrow A'$ be a cyclic contraction to a cancellative dimer algebra. Consider the map

$$\tilde{\tau} : A \rightarrow M_{|Q_0|}(B)$$

defined on $i \in Q_0$ and $a \in Q_1$ by

$$(23) \quad \tilde{\tau}(e_i) = E_{ii} \quad \text{and} \quad \tilde{\tau}(a) = \bar{\psi}(a)E_{h(a),t(a)},$$

and extended multiplicatively and $k$-linearly to $A$.

Lemma 4.23. The map $\tilde{\tau} : A \rightarrow M_{|Q_0|}(B)$ is an algebra homomorphism.

Proof. By Lemma 2.7, $\tau : A' \rightarrow M_{|Q'_0|}(B)$ is an algebra homomorphism. Furthermore, $\psi$ is $k$-linear map, and clearly an algebra homomorphism when restricted to each vertex corner ring $e_iAe_i$. □

In addition to the cycle algebra $S := k \left[ \bigcup_{i \in Q_0} \bar{\psi}(e_iAe_i) \right]$, which is generated by the union of the $\bar{\psi}$-images of the vertex corner rings of $A$, we will also consider the algebra generated by their intersection:

Definition 4.24. The homotopy center of $A$ is the algebra

$$R := k \left[ \bigcap_{i \in Q_0} \bar{\psi}(e_iAe_i) \right] = \bigcap_{i \in Q_0} \bar{\psi}(e_iAe_i).$$
Theorem 4.25. Let $A$ be a non-cancellative dimer algebra satisfying (20). Then there is an exact sequence

$$0 \longrightarrow \text{nil } Z \longrightarrow Z \xrightarrow{\bar{\psi}} R,$$

where $\bar{\psi}$ is an algebra homomorphism. Therefore the reduction $Z_{\text{red}} := Z / \text{nil } Z$ of $Z$ is isomorphic to a subalgebra of $R$.

Proof. (i) We first claim that the composition

$$\tau \psi : A \rightarrow A' \rightarrow M_{|Q'_0|}(B)$$

induces an algebra homomorphism

$$\bar{\psi} : Z \rightarrow R, \quad z \mapsto \bar{\psi}(ze_i) = \bar{\psi}(ze_i) = ze_i,$$

where $i \in Q_0$ is any vertex.

Consider a central element $z \in Z$ and vertices $j,k \in Q_0$. By the construction of $Q$, there is a path $p$ from $j$ to $k$. For $i \in Q_0$, set $z_i := ze_i \in e_iAe_i$. By Lemma 4.23 $\psi$ is an algebra homomorphism on each vertex corner ring $e_iAe_i$. Thus

$$pz_j = p\bar{z}_j = p\bar{z} = z_k\bar{p} = \bar{z}_k\bar{p} \in B.$$

But the image $\bar{p}$ is nonzero since $\tau$ is an impression of $A'$ and the $\psi$-image of any path is nonzero. Thus, since $B$ is an integral domain,

$$z_j = \bar{z}_k.$$

Therefore, since $j,k \in Q_0$ were arbitrary,

$$z_j \in k \left[ \cap_{i \in Q_0} \bar{\psi}(e_iAe_i) \right] = R.$$

(ii.a) Let $z \in Z$ and $i \in Q_0$. We claim that $\psi(ze_i) = 0$ implies $\psi(z) = 0$. For each $i \in Q_0'$, denote by

$$c_j := |\psi^{-1}(j) \cap Q_0|$$

the number of vertices in $\psi^{-1}(j)$. Since $\psi$ maps $Q_0$ surjectively onto $Q_0'$, we have $c_j \geq 1$. Furthermore, if $k \in \psi^{-1}(j)$, then

$$\psi(z)e_j = c_j \psi(ze_k).$$

Set

$$z'_j := e_j^{-1}\psi(z)e_j.$$

Then the sum

$$z' := \sum_{j \in Q_0'} z'_j$$

is in the center $Z'$ of $A'$ by (26) and (19) in Theorem 3.3. Therefore

$$\bar{\tau}(z'_j) = \bar{\tau}(z'e_j) = \bar{\tau}(z'e_{\psi(i)}) = \bar{\tau}(z'_{\psi(i)}) = \bar{\tau}(e^{-1}_{\psi(i)}\psi(z)e_{\psi(i)}) = \bar{\tau}(\psi(ze_i)) = 0,$$

Note that $\psi(z)$ is not in $Z'$ if there are vertices $i,j \in Q_0'$ for which $c_i \neq c_j$. Therefore in general $\psi(Z)$ is not contained in $Z'$. 

where (i) holds by (27). Thus \( z_j' = 0 \) since \( \bar{\tau} \) is injective. Whence
\[
\psi(z)e_j = c_jz_j' = 0.
\]
But this holds for each \( j \in Q_0' \). Therefore \( \psi(z) = 0 \), proving our claim.

(ii.b) We now claim that the homomorphism (25) can be extended to the exact sequence (24). Indeed, \( \bar{\psi} : Z \to R \) factors into the homomorphisms
\[
Z \xrightarrow{\bar{\tau}} Z e_i \xrightarrow{\psi} \psi(Z e_i) \xrightarrow{\bar{\tau}} R.
\]
Suppose \( z \in Z \) is in the kernel of \( \bar{\psi} \),
\[
\bar{\tau}\psi(ze_i) = \bar{\psi}(ze_i) = 0.
\]
By Theorem 3.3, \( \bar{\tau} \) is injective on each vertex corner ring \( e_i A e_i \). Thus \( \psi(ze_i) = 0 \). Whence \( \psi(z) = 0 \) by Claim (ii.a). Therefore \( z \in \text{nil} Z \) by Theorem 4.22. This proves our claim.

**Corollary 4.26.** The algebras \( Z_{\text{red}}, R, \) and \( S \) are integral domains. Therefore the central nilradical \( \text{nil} Z \) of \( A \) is a prime ideal, and the schemes
\[
\text{Spec } Z \quad \text{and} \quad \text{Spec } Z_{\text{red}}
\]
are irreducible.

**Proof.** \( R \) and \( S \) are domains since they are subalgebras of the domain \( B \). Therefore \( Z_{\text{red}} \) is a domain since it isomorphic to a subalgebra of \( R \) by Theorem 4.25. □

The following is a converse to Lemma 2.11.1.

**Lemma 4.27.** Let \( u, v \in Z^2 \setminus 0 \). If \( p \in C^u \) and \( q \in C^v \) satisfy \( p = q \), then \( u = v \).

**Proof.** Suppose to the contrary that \( u \neq v \). Then \( p \) and \( q \) intersect at some vertex \( i \) since \( u \) and \( v \) are both nonzero. Let \( p_i \) and \( q_i \) be the respective cyclic permutations of \( p \) and \( q \) with tails at \( i \). Then
\[
\bar{\tau}\psi(p_i) = p_i = p = q = q_i = \bar{\tau}\psi(q_i).
\]
Thus, since \( \bar{\tau} : e_i A e_i \to S \) is injective,
\[
\psi(p_i) = \psi(q_i).
\]
Whence
\[
\psi(h(p_i^+)) = \psi(h(q_i^+)) \in Q^+_0.
\]
Let \( r^+ \) be a path from \( h(p_i^+) = i^+ + u \) to \( h(q_i^+) = i^+ + v \). Then (28) implies that \( \psi(r^+) \) is a cycle. Hence \( r^+ \) is a cycle by Lemma 4.9. But \( r^+ \) is not a cycle since \( u \neq v \), a contradiction. □

**Lemma 4.28.** If \( g \) is a monomial in \( B \) and \( g\sigma \) is in \( S \), then \( g \) is also in \( S \).
Proof. Suppose \( g \) is a non-constant monomial in the polynomial ring \( B \). Then there is a cycle \( p \) in \( Q \) such that \( p = g \). Let \( u = (u_1, u_2) \in \mathbb{Z}^2 \) be such that \( \psi(p) \in \mathcal{C}^u \). It suffices to suppose \( u_1 \) and \( u_2 \) are relatively prime.

Construct the path \( q^+ \) in \( Q^+ \) by removing all the cyclic subpaths of \( \psi(p)^+ \), modulo \( I \). Thus, since \( u_1 \) and \( u_2 \) are relatively prime, \( q = \pi(q^+) \) also has no cyclic subpaths, modulo \( I \). Whence \( \sigma \nmid \bar{\tau}(q) \) by Proposition 3.2. Thus there is an \( n \geq 1 \) such that

\[
\bar{\tau}(q)^n = \bar{\tau}(\psi(p)),
\]

by Lemma 2.4.1. Therefore, since \( \psi \) is cyclic,

\[
g = (g\sigma)^{n^{-1}} = \bar{\tau}(\psi(p))^{n^{-1}} = \bar{\tau}(q)^{n^{-1}} \in \mathcal{S}' = S.
\]

\[\square\]

Proposition 4.29.

(1) If \( g \in R \) and \( \sigma \nmid g \), then \( g \in \bar{\psi}(Z) \).

(2) If \( g \in S \), then there is some \( N \geq 0 \) such that for each \( n \geq 1 \), \( g^n\sigma^N \in \bar{\psi}(Z) \).

(3) If \( g \in R \), then there is some \( N \geq 1 \) such that \( g^N \in \bar{\psi}(Z) \).

Proof. Since \( R \) is generated by monomials, it suffices to consider a non-constant monomial \( g \in R \). Then for each \( i \in Q_0 \), there is a cycle \( c_i \in e_i A e_i \) satisfying \( c_i = g \).

(1) Suppose \( \sigma \nmid g \). Fix \( a \in \mathbb{Q}_1 \), and set

\[
p := c_i(a) \quad \text{and} \quad q := c_h(a).
\]

See Figure 12. We claim that \( ap = qa \).

Let \( u, v \in \mathbb{Z}^2 \) be such that

\[
p \in \mathcal{C}^u \quad \text{and} \quad q \in \mathcal{C}^v.
\]

Since \( \sigma \nmid g = \bar{p} = \bar{q} \), \( u \) and \( v \) are both nonzero by Lemma 2.9. Thus \( u = v \) by Lemma 4.27. Therefore \((ap)^+ \) and \((qa)^+ \) bound a compact region \( \mathcal{R}_{ap,qa} \) in \( \mathbb{R}^2 \).

We proceed by induction on the number of vertices in the interior \( \mathcal{R}_{ap,qa}^o \). First suppose there are no vertices in \( \mathcal{R}_{ap,qa}^o \). Since

\[
\sigma \nmid g \quad \text{and} \quad \sigma = \prod_{D \in \mathcal{S'}} x_D,
\]

we have

\[
\sigma \nmid g^2 = \bar{p}^2 = \bar{q}^2.
\]

Thus \((p^2)^+ \) and \((q^2)^+ \) have no cyclic subpaths by Lemma 2.9. Hence \( p \) and \( q \) are in \( \hat{\mathcal{C}} \). Therefore \( ap = qa \) by Lemma 4.14.

So suppose \( \mathcal{R}_{ap,qa}^o \) contains at least one vertex \( i^+ \). Let \( w \in \mathbb{Z}^2 \) be such that \( c_i \in \mathcal{C}^w \). Then \( w = u = v \), again by Lemma 4.27. Therefore \( c_i \) intersects \( p \) at least twice or \( q \) at least twice. Suppose \( c_i \) intersects \( p \) at vertices \( j \) and \( k \). Then \( p \) factors into paths

\[
p = p_2 e_k e_j p_1 = p_2 t p_1.
\]
Let \( s^+ \) be the subpath of \((c_i^2)^+\) from \( j^+ \) to \( k^+ \). Then 
\[
t(s^+) = t(t^+) \quad \text{and} \quad h(s^+) = h(t^+).
\]
In particular, \( s^+ \) and \( t^+ \) bound a compact region \( R_{s,t} \).

Since we are free to choose the vertex \( i^+ \) in \( R_{ap,q} \), we may suppose \( R_{s,t} \) contains no vertices. Furthermore, \( c_i^+ \) and \( p^+ \) have no cyclic subpaths since \( \sigma \nmid g \), again by Lemma 2.9. Thus their respective subpaths \( s^+ \) and \( t^+ \) have no cyclic subpaths. Whence \( s = t \) by Lemma 4.12.

Furthermore, since \( R_{ap^2sp_1,qa} \) contains less vertices than \( R_{ap,q} \), it follows by induction that 
\[
ap_2sp_1 = qa.
\]
Therefore 
\[
ap = a(p_2tp_1) = a(p_2sp_1) = qa.
\]

Since \( a \in Q_1 \) was arbitrary, the sum \( \sum_{i \in Q_0} c_i \) is central in \( A \).

(2) Fix an arrow \( a \in Q_1 \). Set \( i := t(a) \) and \( j := h(a) \). Let \( r^+ \) be a path in \( Q^+ \) from \( h((ac_i)^+) \) to \( t((ac_i)^+) \). Then by Lemma 2.4.1, there is some \( \ell, m, n \geq 0 \) such that
\[
\sigma_i^m = rc_ja \sigma_j^\ell \quad \text{and} \quad ac_i r \sigma_j^\ell = \sigma_j^n.
\]
Thus 
\[
\sigma = \bar{\psi}(rc_ja \sigma_j^\ell) = \bar{\psi}(rc_ja \sigma_j^\ell) = \bar{\psi}(ac_i r \sigma_j^\ell) = \sigma^n.
\]
Furthermore, \( \sigma \neq 1 \) since \( \bar{\tau} \) is injective. Whence \( n = m \) since \( B \) is an integral domain. Therefore 
\[
\sum_{i \in Q_0} c_i \sigma_i^N \text{ is central.}
\]

Then (29) implies that the element \( \sum_{i \in Q_0} c_i \sigma_i^N \) is central.

Now fix \( n \geq 2 \) and an arrow \( a \in Q_1 \). Again set \( i := t(a) \) and \( j := h(a) \). Then 
\[
ac_i^a \sigma_i^N = ac_i^{a-1} (c_i \sigma_i^N) = (c_j \sigma_j^N) ac_i^{a-1} (c_i \sigma_i^N) = \cdots = c_j^a \sigma_i^N a,
\]
where (i) holds by Lemma 2.3. Therefore, for each \( n \geq 1 \), the element 
\[
\sum_{i \in Q_0} c_i^a \sigma_i^N
\]
is central. But its \( \bar{\psi} \)-image is \( g^n \sigma_i^N \), proving Claim (2).

(3) By Claim (1), it suffices to suppose \( \sigma \mid g \). Then there is a monomial \( h \in B \) such that 
\[
g = h \sigma.
\]
Figure 12. Setup for Proposition 4.29.1. The cycles $p = p_2tp_1$, $q$, $c_i$, are respectively drawn in red, blue, and green. The path $a$ is an arrow.

By Lemma 4.28, $h$ is in $S$. Therefore by Claim (2), there is some $N \geq 1$ such that $g^N = h^N \sigma^N \in \tilde{\psi}(Z)$.

The following theorem shows that the reduced center $Z_{\text{red}}$ and the homotopy center $R$ are not isomorphic algebras in general. However, we will show that they determine the same nonlocal variety in Theorem 4.58 and that their integral closures are isomorphic in Theorem 4.65.

**Theorem 4.30.** In general, the containment $Z_{\text{red}} \hookrightarrow R$ is proper.

**Proof.** Consider the contraction given in Figure 13, we claim that the reduced center $Z_{\text{red}}$ of $A = kQ/I$ is not isomorphic to $R$. By the exact sequence (24), it suffices to show that the homomorphism $\bar{\psi} : Z \hookrightarrow R$ is not surjective.

We claim that the monomial $z\sigma$ is in $R$, but is not in the image $\tilde{\psi}(Z)$. It is clear that $z\sigma$ is in $R$ from the $\tilde{\psi}$ labeling of arrows given in the lower left quiver of Figure 13.

Assume to the contrary that $z\sigma \in \tilde{\psi}(Z)$. Then by (26), for each $j \in Q_0$ there is an element $c_j$ in $Ze_j$ whose $\tilde{\psi}$-image is $z\sigma$. Consider the vertex $i$ given in the upper left quiver of Figure 13. The set of cycles in $e_iAe_i$ with $\tilde{\psi}$-image $z\sigma$ are as follows:

\[
\begin{align*}
p_1 &:= \delta_5a_4a_3(\delta_1a_3)\delta_6b_1\delta_3, \\
p_2 &:= \delta_3b_1(\delta_1a_3)\delta_7b_2, \\
p_3 &:= \delta_2b_4(\delta_1a_3)\delta_1a_2\delta_4\delta_3 \\
p_4 &:= \delta_5a_6a_3(\delta_1a_3)\delta_7b_2, \\
p_5 &:= \delta_5a_6a_3(\delta_1a_3)\delta_1a_2\delta_4\delta_3.
\end{align*}
\]
Thus for some coefficients $\alpha_1, \ldots, \alpha_5 \in k$,

$$c_i = \sum_{\ell=1}^{5} \alpha_\ell p_\ell.$$  

Since $c_i = z\sigma$, there is some $\ell$ for which $\alpha_\ell$ is nonzero. In particular, there are cycles $p'_\ell$ and $p''_\ell$ satisfying

$$b_2p_\ell = p'_\ell b_2 \quad \text{and} \quad \delta_3p_\ell = p''_\ell \delta_3.$$  

However, there are no cycles $p'_1, p'_2, p'_3, p'_4, p'_5$, for which

$$b_2p_1 = p'_1 b_2, \quad \delta_3p_2 = p''_2 \delta_3, \quad b_2p_3 = p'_3 b_2, \quad \delta_3p_4 = p''_4 \delta_3, \quad b_2p_5 = p'_5 b_2.$$  

Thus no such element $c_i \in Ze_i$ can exist, a contradiction. Therefore $Z_{\text{red}} \not\cong R$.  

Theorem 4.30 raises the following question.

**Question 4.31.** Are there necessary and sufficient conditions for the isomorphism $Z_{\text{red}} \cong R$ to hold?

### 4.3. Homotopy dimer algebras

We introduce the following class of algebras. Recall the definition of non-cancellative paths (Definition 2.13).

**Definition 4.32.** Let $\psi : A \to A'$ be a cyclic contraction to a cancellative dimer algebra. We call the quotient algebra $\tilde{A} := A/\langle p - q \mid p, q \text{ non-cancellative paths} \rangle$ the *homotopy (dimer) algebra* of $A$.

Note that a dimer algebra $A$ equals its homotopy algebra $\tilde{A}$ if and only if $A$ is cancellative.

Recall the algebra homomorphism $\tilde{\tau} : A \to M_{|Q_0|}(B)$ defined in (23) (Lemma 4.23). This homomorphism induces an algebra homomorphism on the quotient $\tilde{A}$,

$$\tilde{\tau} : \tilde{A} \to M_{|Q_0|}(B).$$

**Remark 4.33.** The ideal $\langle p - q \mid p, q \text{ non-cancellative paths} \rangle \subseteq A$ is contained in the kernel $\ker \psi$, but not conversely. Indeed, if $\psi$ contracts an arrow $\delta$, then $\delta - e_t(\delta)$ is in the kernel of $\psi$, but $\delta$ and $e_t(\delta)$ do not form a pair of non-cancellative paths.

**Theorem 4.34.** The algebra homomorphism $\tilde{\tau} : \tilde{A} \to M_{|Q_0|}(B)$ is an impression of the homotopy dimer algebra $\tilde{A}$. Consequently, the center of $\tilde{A}$ is isomorphic to the homotopy center $R$ of $A$.

**Proof.** Denote by $\tilde{Z}$ the center of $\tilde{A}$.

(i) We first claim that $\tilde{\tau}$ is injective. Indeed, $\tau : A' \to M_{|Q_0|}(B)$ is injective since $A'$ is cancellative, by Theorem 3.3. Thus for each $i, j \in Q_0^\prime$, the $k$-linear map $\tilde{\tau} : e_iA'e_j \to B$ is injective. But $\psi := \tilde{\tau}\psi$. Therefore the kernel of the algebra homomorphism

$$\tilde{\tau} : A \to M_{|Q_0|}(B),$$

is $\ker\psi$.
Figure 13. A cyclic contraction \( \psi : A \to A' \) for which \( Z_{\text{red}} \not\sim R \). In the top two quivers: \( Q \) and \( Q' \) are drawn on a torus, and arrows with the same label are identified. \( \psi \) contracts the \( \delta_j \) arrows in \( Q \) to vertices in \( Q' \). There are two removable 2-cycles that have been removed from \( Q' \), formed from the \( b_j \) arrows in \( Q \). Only the \( a_j \) arrows in \( Q \) remain in \( Q' \). In the bottom two quivers: The arrows in \( Q \) are labeled by their \( \bar{\psi} \)-images, and the arrows in \( Q' \) are labeled by their \( \bar{\tau} \)-images. Each arrow in \( Q \) labeled 1 is contracted to a vertex.

\[ \langle p - q \mid p, q \text{ non-cancellative paths} \rangle \]

(ii) For generic \( b \in \text{Max } B \), the composition \( \epsilon_b \bar{\tau} \) (defined in (2)) is surjective by Claim (ii) in the proof of Theorem 3.3.
Furthermore, the morphism
\[ \text{Max } B \to \text{Max } \tau(\tilde{Z}), \quad b \mapsto b_1|_{Q_0} \cap \tau(\tilde{Z}), \]
is surjective by Claim (iii) in the proof of Theorem \[3.3.\]

Claims (i) and (ii) imply that \((\tilde{\tau}, B)\) is an impression of \(\tilde{A}\).

(iii) Since \((\tilde{\tau}, B)\) is an impression of \(\tilde{A}\), \(\tilde{Z}\) is isomorphic to \(R\) by [B, Lemma 2.1 (2)]. \(\square\)

4.4. Nonnoetherian and nonlocal. Throughout, \(A\) is a non-cancellative dimer algebra and \(\psi : A \to A'\) is a cyclic contraction to a cancellative dimer algebra. Let \(\tilde{A} := A/\langle p - q \mid p, q \text{ non-cancellative paths} \rangle\) and \(R = Z(\tilde{A})\) be the homotopy algebra and homotopy center of \(A\), respectively.

**Proposition 4.35.** The homotopy center \(R\) of \(A\) is properly contained in the cycle algebra \(S\),
\[ R \subsetneq S. \]

**Proof.** (i) Assume to the contrary that \(R = S\). Let \(p, q \in e_j Ae_i\) be non-cancellative paths. Then \(\psi(p) = \psi(q)\) since \(A'\) is cancellative. Thus by Lemma 4.10.1, \(p^+\) and \(q^+\) bound a compact region \(R_{p,q} \subset \mathbb{R}^2\). Suppose this region has minimal area among all pairs of non-cancellative paths (not modulo \(I\)).

Furthermore, we may suppose the vertices \(i^+\) and \(j^+\) have rational coordinates in \(\mathbb{R}^2\). Let \(u \in \mathbb{Z}^2\) satisfy
\[ u \cdot (j^+ - i^+) = 0, \]
where \(\cdot\) is the usual Euclidean dot product on \(\mathbb{R}^2\). Since \(R = S = S'\), Proposition 2.14 and Lemma 4.9 imply that there are cycles \(s, t \in C^u\) such that
\[ \sigma \nmid s = t. \]

In Claims (ii.a), (ii.b), and (ii.c) below, we will show that \(s^+\) does not intersect the interior \(R_{p,q}^o\) (modulo \(I\)). (Recall that \(p^+\) denotes the unique path in \(\pi^{-1}(p)\) with tail in the fundamental domain of \(Q^+\) containing \((0, 0)\) \in \(\mathbb{Z}^2\), and so \(s^+\) may intersect other paths in \(\pi^{-1}(p)\) or \(\pi^{-1}(q)\).)

(ii) Assume to the contrary that \(s^+\) intersects the interior \(R_{p,q}^o\). Then \(s^+\) intersects \(q^+\) at a vertex \(\ell^+\) other than the head or tail of \(q^+\). \(s\) and \(q\) therefore factor into paths
\[ s = s_2e_\ell s_1 \quad \text{and} \quad q = q_2e_\ell q_1. \]
It suffices to chose \(s\) (not modulo \(I\)) so that the area of \(R_{p,q}^o \cap R_{s_1,q_1}\) is minimal.

(ii.a) We claim that the interior of the region
\[ R_{p,q}^o \cap R_{s_1,q_1} \]
contains at least one vertex \(k^+\). Indeed, suppose otherwise; see Figure 14a.

Specifically, if \((j^+ - i^+) = (\frac{a}{b}, \frac{c}{d}) \in Q^2\), then we may take \(u = (bc, -ad) \in \mathbb{Z}^2\).
By Lemma 2.4.1, there is some $m \in \mathbb{Z}$ such that $q_1 = s_1 \sigma^m$.

Furthermore, $m \geq 0$ since $\sigma \nmid s_1$. Whence

$$(30) \quad \tilde{\psi}(q_2 s_1 \sigma^m) = p = \bar{q}.$$  

But $\mathcal{R}_{p,q}$ has minimal area among all pairs of non-cancellative paths, and $(q_2 s_1)^+$ lies in $\mathcal{R}_{p,q}$. Thus $(30)$ implies

$$p = q_2 s_1 \sigma^m = q.$$  

But then $p = q$, a contradiction. Therefore there is a vertex $k^+$ in $\mathcal{R}_{p,q} \cap \mathcal{R}_{s_1,q_1}$.

Since $R = S$, there is a cycle $r \in e_k A e_k$ such that $\bar{r} = \bar{s}$. There are two cases to consider:

(ii.b) Suppose $r^+$ intersects $s^+$; see Figure 14.b. Then a path $w^+$ from $t(s^+)$ to $h(s^+)$ can be constructed from subpaths of $r^+$ and $s^+$, which passes through $k^+$ (drawn in green in the figure). In particular, $x_D \nmid \bar{w}$ since $x_D \nmid \bar{r}$ and $x_D \nmid \bar{s}$. Thus $\sigma \nmid \bar{w}$. Whence by Lemma 2.11.1, $\bar{w} = s$.

But $k^+$ is contained in the interior of $\mathcal{R}_{p,q} \cap \mathcal{R}_{s_1,q_1}$. Thus the existence of $w$ contradicts the minimality of the area of $\mathcal{R}_{p,q} \cap \mathcal{R}_{s_1,q_1}$.

(ii.c) Therefore $r^+$ does not intersect $s^+$; see Figure 14.c. In particular, there is a subpath $r'$ of $r$ (drawn in green) and a subpath $q'_2$ of $q_1 = q'_3 q'_2 q'_1$ such that $t(r^+) = t(q'_2^+)$ and $h(r^+) = h(q'_2^+)$.

By Lemma 2.4.1, there is some $m \in \mathbb{Z}$ such that $\bar{r}' = \bar{q}'_2 \sigma^m$.

Furthermore, $\sigma \nmid \bar{s} = \bar{r}$, and so $\sigma \nmid \bar{r}'$. Whence $m \geq 0$. Thus

$$(31) \quad \tilde{\psi}(q_2 q'_3 r' q'_1 \sigma^m) = \bar{p} = \bar{q}.$$  

But $\mathcal{R}_{p,q}$ has minimal area among all pairs of non-cancellative paths, and $r'^+$ lies in $\mathcal{R}_{p,q}$. Thus $(31)$ implies

$$p = q_2 q'_3 r' q'_1 \sigma^m = q.$$  

But then $p = q$, a contradiction.

(iii) By Claims (ii.a), (ii.b), and (ii.c), $s^+$ does not intersect $\mathcal{R}_{p,q}$; see Figure 14.d. In particular, the path $(ps)^+$ lies in the region $\mathcal{R}_{tq,qs}$ (modulo $I$). Furthermore, by Proposition 4.29.1, $\sigma \nmid \bar{s} = \bar{t} \in R$ implies $tq = qs$. Thus, since $I$ generates homotopy relations on the paths of $Q^+$ and $tq = ps = \bar{q}s$, we have

$$tq = ps = qs.$$  

But again $s^+$ does not intersect $\mathcal{R}_{p,q}$. Therefore $p = q$ by Lemma 4.10.2, again a contradiction. \qed
Figure 14. Cases for Proposition 4.35. In the respective cases (a), (b), (c), the path $s_1$, the cycle $w$, and the path $r'$ are drawn in green.

Lemma 4.36. If $p$ is a cycle such that $\bar{p} \not\in R$ and $\sigma \nmid \bar{p}$, then for each $n \geq 1$,
$$p^n \not\in R.$$  
Furthermore, if $R \neq S$, then such a cycle exists.

Proof. (i) Assume to the contrary that there is a cycle $p \in e_i A e_i$ such that $\bar{p} \not\in R$, $\sigma \nmid \bar{p}$, and $p^n \in R$ for some $n \geq 2$. Let $u \in \mathbb{Z}^2$ be such that $p \in C^u_i$. Since $\bar{p}$ is not in
Figure 15. Setup for Lemma 4.36. The path $r^+$ is drawn in red. Its projection $r = \pi(r^+)$ to $Q$ is a cycle at $j$.

In $R$, there is a vertex $j \in Q_0$ such that
\begin{equation}
\bar{p} \notin \bar{\psi}(e_j A e_j).
\end{equation}
Furthermore, since $\bar{p}^n$ is in $R$, $p^n$ homotopes to a cycle $q \in e_i A e_i$ that passes through $j$,
\[ q = p^n \quad \text{(modulo $I$)}. \]
For $v \in \mathbb{Z}^2$, denote by $t^+_v \in \pi^{-1}(q)$ the preimage with tail
\[ t(q^+_v) = t(q^+) + v \in Q^+_0. \]
It is clear (see Figure 15) that there is a path $r^+$ from $j^+$ to $j^+ + u$, constructed from subpaths of $q^+$, $q^+_m$, and possibly $q^+_{mu}$ for some $m \in \mathbb{Z}$. In particular, the cycle $r := \pi(r^+) \in e_j A e_j$ is in $C^u_j$.

Furthermore, since $\sigma \nmid \bar{p}$, there is a simple matching $D$ such that $x_D \nmid \bar{p}$. Whence $x_D \nmid \bar{q}$. Thus $x_D \nmid \bar{r}$, and so $\sigma \nmid \bar{r}$. In particular,
\begin{align*}
\sigma \nmid p, \quad &\sigma \nmid r, \quad \text{and} \quad p, r \in C^u.
\end{align*}
Therefore $r = \bar{p}$ by Lemma 2.11. But then
\[ \bar{p} = r \in \bar{\psi}(e_j A e_j), \]
contrary to (32).

(ii) Now suppose $R \neq S$. We claim that there exists a cycle $p$ as in Claim (i). Indeed, assume to the contrary that for each cycle $p$ satisfying $\bar{p} \notin R$, we have $\sigma \nmid \bar{p}$. Then by the contrapositive of this assumption, for each cycle $q$ satisfying $\sigma \nmid \bar{q}$, we have $\bar{q} \in R$. But $S$ is generated over $k$ by $\sigma$ and monomials in $S$ not divisible by $\sigma$. Therefore $S \subseteq R$ since $\sigma \in R$. Whence $S = R$, contrary to assumption.

Theorem 4.37. Let $A$ be a non-cancellative dimer algebra satisfying (20). Then each algebra
\[ A, \ Z, \ Z_{\text{red}}, \ R, \]
is nonnoetherian.
Proof. (i) We first claim that $R$ is nonnoetherian. Indeed, $R \neq S$ by Proposition 4.35. Thus there is a cycle $p \in A$ such that for each $n \geq 1,$

$$\bar{p}^n \notin R$$

by Lemma 4.36. Whence there is some $N \geq 1$ such that for each $n \geq 1,$

$$\bar{p}^n \sigma^N \in \bar{\psi}(Z)$$

by Proposition 4.29.2. Therefore

$$\bar{p}^n \sigma^N \in R$$

by Theorem 4.25. Thus there is an ascending chain of ideals of $R,$

$$(34) \langle \bar{p} \sigma^N \rangle \subseteq \langle \bar{p} \sigma^N, \bar{p}^2 \sigma^N \rangle \subseteq \langle \bar{p} \sigma^N, \bar{p}^2 \sigma^N, \bar{p}^3 \sigma^N \rangle \subseteq \cdots$$

Assume to the contrary that for some $\ell \geq 2,$ there are elements $g_1, \ldots, g_{\ell-1} \in R$ such that

$$z_{\ell} = \sum_{n=1}^{\ell-1} g_n \bar{p}^n \sigma^N. \quad (36)$$

Then since $R$ is a domain,

$$\bar{p}^\ell = \sum_{n=1}^{\ell-1} g_n \bar{p}^n.$$

Furthermore, since $R$ is generated by monomials in the polynomial ring $B,$ there is some $1 \leq n \leq \ell - 1$ such that $g_n$ is a monomial and

$$\bar{p}^\ell = g_n \bar{p}^n.$$

Whence $g_n = \bar{p}^{\ell-n},$ again since $R$ is a domain. But then $\bar{p}^{\ell-n}$ is in $R,$ a contradiction. Thus each inclusion in the chain (34) is proper. Therefore $R$ is nonnoetherian, proving our claim.

(ii) We now claim that $A$ and $Z$ are nonnoetherian. Again consider the elements (33) in $\tau(Z).$ Denote by $z_n \in Z$ the central element whose $\bar{\psi}$-image is $\bar{p}^n \sigma^N.$ Consider the ascending chain of (two-sided) ideals of $A,$

$$(35) \langle z_1 \rangle \subseteq \langle z_1, z_2 \rangle \subseteq \langle z_1, z_2, z_3 \rangle \subseteq \cdots.$$

Assume to the contrary that for some $\ell \geq 2,$ there are elements $a_1, \ldots, a_{\ell-1} \in A$ such that

$$z_\ell = \sum_{n=1}^{\ell-1} a_n z_n. \quad (37)$$

Since $p \notin R,$ there is a vertex $i \in Q_0$ such that $p$ is not in $\bar{\psi}(e_i A e_i).$ From (36) we obtain

$$\bar{p}^\ell \sigma^N = \bar{\psi}(z_\ell e_i) = \sum \bar{\psi}(a_n e_i) \bar{p}^n \sigma^N.$$

Whence

$$\bar{p}^\ell = \sum \bar{\psi}(a_n e_i) \bar{p}^n. \quad (37)$$
Furthermore, since each $z_n$ is central,
\[ \sum (e_i a_n)(z_n e_i) = e_i z_\ell = z_\ell e_i = \sum (a_n e_i)(z_n e_i). \]
Thus each product $a_n e_i$ is in the corner ring $e_i A e_i$. Therefore each image $\bar{\psi}(a_n e_i)$ is in $\bar{\psi}(e_i A e_i)$. In particular, $\bar{\psi}(a_n e_i)$ cannot equal $p^{\ell-n}$ for any $\ell - n \geq 1$ by our choice of vertex $i$. But $R$ is generated by monomials in the polynomial ring $B$. It follows that (37) cannot hold. Thus each inclusion in the chain (35) is proper. Therefore $A$ is nonnoetherian.

Since the elements $z_1, z_2, z_3, \ldots$ are in $Z$, we may consider the ascending chain of ideals of $Z$,
\[ (z_1) \subseteq (z_1, z_2) \subseteq (z_1, z_2, z_3) \subseteq \cdots. \]
A similar argument then shows that $Z$ is also nonnoetherian.

(iii) Finally, we claim that $Z_{\text{red}}$ is nonnoetherian. For each $i \in Q_0$, there are no cycles in $(\text{nil } Z) e_i$ by Theorem 4.22. Thus the cycle $z_n e_i$ is not in $(\text{nil } Z) e_i$. Whence $z_n$ is not in nil $Z$. The claim then follows similar to Claim (ii). \qed

Although $Z_{\text{red}}$ and $R$ are nonnoetherian, we will show in Theorem 4.56 that they nonetheless have Krull dimension 3 and are generically noetherian. Furthermore, we will show in Theorem 4.55 that the homotopy algebra $\tilde{A}$ of $A$ is also nonnoetherian.

**Theorem 4.38.** Let $A$ be a non-cancellative dimer algebra satisfying (20). Then $A$ is an infinitely generated $Z$-module.

**Proof.** Assume to the contrary that there are elements $a_1, \ldots, a_N \in A$ such that
\[ A = \sum_{n=1}^{N} Z a_n. \]
It suffices to suppose that each $a_n$ is in some corner ring $e_j A e_i$, for otherwise we could instead consider the finite generating set
\[ \{ e_j a_n e_i \mid i, j \in Q_0, \ 1 \leq n \leq N \}. \]
For each cycle $p \in C$, there is thus a subset $J_p \subseteq \{1, \ldots, N\}$ and nonzero central elements $z_n \in Z$ such that
\[ p = \sum_{n \in J_p} z_n a_n. \]
Therefore by Theorem 4.25
\[ p = \sum_{n \in J_p} \bar{\psi}(z_n) \bar{a}_n \in \sum_{n \in J_p} R \bar{a}_n. \]
Set
\[ J := \bigcup_{p \in C} J_p \subseteq \{1, \ldots, N\}. \]
Then (39) implies that

\[ S \subseteq \sum_{n \in J} R\alpha_n. \]  

Again consider (38), and suppose \( n \in J_p \). Then \( a_n \) is in \( e_i Ae_i \) since \( p \) is in \( e_i Ae_i \) and \( z_n \) is central. Whence \( \alpha_n \) is in \( S \). Thus

\[ S \supseteq \sum_{n \in J} R\alpha_n. \]

Therefore, together with (40), we obtain

\[ S = \sum_{n \in J} R\alpha_n. \]  

In particular, \( S \) is a finitely generated \( R \)-module.

But \( R \) is an infinitely generated \( k \)-algebra by Theorem 4.37. Furthermore, \( S \) is a finitely generated \( k \)-algebra by Theorems 3.1 and 3.3. Therefore \( S \) is an infinitely generated \( R \)-module by the Artin-Tate lemma, in contradiction to (41). \( \square \)

In Theorem 4.55, we will show that the homotopy algebra \( \tilde{A} \) is also an infinitely generated module over its center \( R \).

As a corollary, we obtain the following equivalences.

**Corollary 4.39.** Let \( A \) be a dimer algebra which is 2-cycle free. Then the following are equivalent:

1. \( A \) is cancellative.
2. \( A \) is noetherian.
3. \( Z \) is noetherian.
4. \( A \) is a finitely generated \( Z \)-module.
5. The vertex corner rings \( e_i Ae_i \) are all isomorphic, and isomorphic to \( Z \).

**Proof.** First suppose \( A \) is cancellative. Then \( Z \) is noetherian and \( A \) is a finitely generated \( Z \)-module by Theorem 3.1.3. Therefore \( A \) is noetherian. Furthermore, the vertex corner rings are pairwise isomorphic and isomorphic to \( Z \) by Theorem 3.1.1.

Conversely, suppose \( A \) is noncancellative. Then by Theorem 4.5 \( A \) admits a cyclic contraction \( \psi : A \to A' \) to a cancellative dimer algebra. Therefore \( A \) and \( Z \) are nonnoetherian by Theorem 4.37 \( A \) is an infinitely generated \( Z \)-module by Theorem 4.38, and the vertex corner rings are not all isomorphic by Proposition 4.35. \( \square \)

**Notation 4.40.** In the remainder of this section, we identify \( Z_{\text{red}} \) with its isomorphic \( \tilde{\psi} \)-image in \( R \) (Theorem 4.25), and thus write \( Z_{\text{red}} \subseteq R \).

The following definition aims to capture the geometry of nonnoetherian algebras with finite Krull dimension, introduced in [B2].
Definition 4.41. [B2, Definition 2.11.] Let $\mathcal{S}$ be an integral domain and a noetherian $k$-algebra. Let $\mathcal{R}$ be a nonnoetherian subalgebra of $\mathcal{S}$ which contains $\mathcal{S}$ in its fraction field, and suppose there is a point $m \in \text{Max} \mathcal{R}$ such that $\mathcal{R}_m$ is noetherian.

(1) We say $\mathcal{R}$ is depicted by $\mathcal{S}$ if
   (a) the morphism $\iota_{\mathcal{R},\mathcal{S}} : \text{Spec} \mathcal{S} \to \text{Spec} \mathcal{R}, \ q \mapsto q \cap \mathcal{R},$ is surjective, and
   (b) for each $n \in \text{Max} \mathcal{S},$ if $\mathcal{R}_n \cap \mathcal{R}$ is noetherian, then $\mathcal{R}_n \cap \mathcal{R} = \mathcal{S}_n.$

(2) The geometric codimension or geometric height of $p \in \text{Spec} \mathcal{R}$ is the infimum
   \[ \text{ght}(p) := \inf \{ \text{ht}(q) \mid q \in (\iota_{\mathcal{S}})^{-1}(p), \text{ } \mathcal{S} \text{ a depiction of } \mathcal{R} \}. \]

The geometric dimension of $p$ is the difference
\[ \text{gdim} p := \dim \mathcal{R} - \text{ght}(p). \]

We will consider the following subsets of the variety $\text{Max} \mathcal{S},$
\begin{align*}
U^*_{\mathcal{R},\mathcal{S}} &:= \{ n \in \text{Max} \mathcal{S} \mid \mathcal{R}_n \cap \mathcal{R} \text{ is noetherian} \} \\
U_{\mathcal{R},\mathcal{S}} &:= \{ n \in \text{Max} \mathcal{S} \mid \mathcal{R}_n \cap \mathcal{R} = \mathcal{S}_n \}.
\end{align*}

Note that the locus $U_{\mathcal{R},\mathcal{S}}$ specifies where $\text{Max} \mathcal{R}$ ‘looks like’ the variety $\text{Max} \mathcal{S}.$ Furthermore, condition (1.b) in Definition 4.41 is equivalent to
\[ U^*_{\mathcal{R},\mathcal{S}} = U_{\mathcal{R},\mathcal{S}}. \]

Denote by $U^c_{\mathcal{R},\mathcal{S}}$ and $U^*_c_{\mathcal{R},\mathcal{S}}$ the respective complements of $U_{\mathcal{R},\mathcal{S}}$ and $U^*_{\mathcal{R},\mathcal{S}}$ in $\text{Max} \mathcal{S}.$ We will show that both the reduced and homotopy centers of $A,$ $Z_{\text{red}}$ and $R,$ are depicted by its cycle algebra $S.$ Furthermore, we will find that $Z_{\text{red}}$ and $R$ determine the same nonlocal variety.

Lemma 4.42. The cycle algebra $S$ is a finitely generated $k$-algebra and a normal domain of Krull dimension 3. Furthermore, $\text{Max } S$ is a toric Gorenstein singularity.

Proof. Since $A'$ is a cancellative dimer algebra, it is well known that its center $Z'$ is a finitely generated $k$-algebra (Theorem 3.1.3), and a normal toric Gorenstein domain (Corollary 4.26) of Krull dimension 3. Furthermore,
\[ Z' \overset{(i)}{=} S' \overset{(ii)}{=} S, \]
where (i) holds by Theorem 3.3 and (ii) holds by our assumption that $\psi$ is cyclic. □

Lemma 4.43. The morphisms
\begin{align*}
\kappa_{Z,S} : \text{Max } S &\to \text{Max } Z_{\text{red}}, \quad n \to n \cap Z_{\text{red}}, \\
\kappa_{R,S} : \text{Max } S &\to \text{Max } R, \quad n \to n \cap R,
\end{align*}
and
\begin{align*}
\iota_{Z,S} : \text{Spec } S &\to \text{Spec } Z_{\text{red}}, \quad q \to q \cap Z_{\text{red}}, \\
\iota_{R,S} : \text{Spec } S &\to \text{Spec } R, \quad q \to q \cap R,
\end{align*}
are well-defined and surjective.
Proof. Let \( n \in \text{Max} \, S \). By Lemma 4.42, \( S \) is a finitely generated \( k \)-algebra, and by assumption \( k \) is an algebraically closed field. Therefore the intersections \( n \cap Z_{\text{red}} \) and \( n \cap R \) are maximal ideals of \( Z_{\text{red}} \) and \( R \) respectively.

Surjectivity of \( \kappa_{Z,S} \) (resp. \( \kappa_{R,S} \)) follows from Claim (iii) in the proof of Theorem 3.3, with \( S \) in place of \( B \), and \( Z_{\text{red}} \) (resp. \( R \)) in place of \( \tau(Z) \).

By assumption \( k \) is also uncountable. Surjectivity of \( \iota_{Z,S} \) (resp. \( \iota_{R,S} \)) then follows from the surjectivity of \( \kappa_{Z,S} \) (resp. \( \kappa_{R,S} \)), by [B2, Lemma 2.15]. \( \square \)

Lemma 4.44. If \( p \in \text{Spec} \, Z_{\text{red}} \) contains a monomial, then \( p \) contains \( \sigma \).

Proof. Suppose \( p \) contains a monomial \( g \). Since \( p \) is a proper ideal, \( g \) is not in \( k \). Thus there is a non-vertex cycle \( p \) such that \( \overline{p} = g \).

Let \( q^+ \) be a path from \( h(p^+) \) to \( t(p^+) \). Then \( (pq)^+ \) is a cycle in \( Q^+ \). Whence \( \psi(pq)^+ \) is a cycle in \( Q^+ \). Thus there is some \( n \geq 1 \) such that \( \overline{pq} = \sigma^n \) by Lemma 2.4.2.

By Lemma 4.43, there is a prime ideal \( q \in \text{Max} \, S \) such that \( q \cap R = p \). Then \( \overline{pq} = \sigma^n \) is in \( n \) since \( \overline{q} \in S \) and \( \overline{p} = g \in \mathfrak{m} \). Thus \( \sigma \) is also in \( q \) since \( q \) is a prime ideal. Therefore by Lemma 2.3,

\[
\sigma \in q \cap Z_{\text{red}} = p.
\]

Denote the origin of Max \( S \) by

\[
n_0 := (s \in S \mid s \text{ a non-vertex cycle}) \, S \in \text{Max} \, S.
\]

Consider the maximal ideals of \( Z_{\text{red}} \) and \( R \) respectively,

\[
j_0 := n_0 \cap Z_{\text{red}} \quad \text{and} \quad m_0 := n_0 \cap R.
\]

Lemma 4.45. The localizations \( (Z_{\text{red}})_{j_0} \) and \( R_{m_0} \) are nonnoetherian.

Proof. Let \( \overline{p} \in S \setminus Z_{\text{red}} \) be as in Claim (i) in the proof of Theorem 4.37. Since \( Z_{\text{red}} \) is generated by monomials in the polynomial ring \( B \), the monomial \( \overline{p^n} \) is not in the localization \( (Z_{\text{red}})_{j_0} \) for any \( n \geq 1 \). Whence the chain (34) does not terminate in \( (Z_{\text{red}})_{j_0} \). Therefore \( (Z_{\text{red}})_{j_0} \) is nonnoetherian. Similarly \( R_{m_0} \) is nonnoetherian. \( \square \)

Lemma 4.46. Suppose that each non-constant monomial in \( Z_{\text{red}} \) is divisible (in \( B \)) by \( \sigma \). If \( p \in \text{Spec} \, Z_{\text{red}} \) contains a monomial, then \( p = j_0 \).

Proof. Suppose \( p \) contains a monomial. Then \( \sigma \) is in \( p \) by Lemma 4.44. Furthermore, there is some \( q \in \text{Spec} \, S \) such that \( q \cap Z_{\text{red}} = p \) by Lemma 4.43.

Suppose \( f \) is a non-constant monomial in \( Z_{\text{red}} \). Then by assumption, there is a monomial \( h \) in \( B \) such that \( f = \sigma h \). By Lemma 4.28, \( h \) is also in \( S \). Therefore \( f = \sigma h \in q \) since \( \sigma \in p \subseteq q \). But \( f \in Z_{\text{red}} \). Whence

\[
f \in q \cap Z_{\text{red}} = p.
\]

Since \( f \) was arbitrary, \( p \) contains all non-constant monomials in \( Z_{\text{red}} \). \( \square \)
Remark 4.47. In Lemma 4.46 we assumed \( \sigma \) divides all non-constant monomials in \( Z_{\text{red}} \). This can indeed happen; for example, consider the dimer algebra with quiver given in Figure 16. In this example, \( R = k + \sigma S \) and \( S = k[zx, xw, yz, yw] \) (with impression given by the arrow orientations in Figure 4).

Lemma 4.48. Suppose that there is a non-constant monomial in \( Z_{\text{red}} \) which is not divisible (in \( B \)) by \( \sigma \). Let \( m \in \Max Z_{\text{red}} \setminus \{ \mathfrak{n}_0 \} \). Then there is a non-constant monomial \( g \in Z_{\text{red}} \) such that \( g \not\in m \) and \( \sigma \nmid g \).

Proof. Let \( m \in \Max Z_{\text{red}} \setminus \{ \mathfrak{n}_0 \} \).

(i) We first claim that there is a non-constant monomial in \( Z_{\text{red}} \) which is not in \( m \). Assume otherwise. Then

\[
\mathfrak{n}_0 \cap Z_{\text{red}} \subseteq m.
\]

But \( \mathfrak{n}_0 \cap Z_{\text{red}} \) is a maximal ideal by Lemma 4.43. Thus \( \mathfrak{n}_0 = \mathfrak{n}_0 \cap Z_{\text{red}} = m \), a contradiction.

(ii) We now claim that there is a non-constant monomial in \( Z_{\text{red}} \setminus \{ m \} \) which is not divisible by \( \sigma \). Assume otherwise; that is, assume that every non-constant monomial in \( Z_{\text{red}} \), which is not divisible by \( \sigma \), is in \( m \). By assumption, \( \sigma \) does not divide all non-constant monomials in \( Z_{\text{red}} \). Thus there is at least one monomial in \( m \). Therefore \( \sigma \) is in \( m \) by Lemma 4.44.

By Lemma 4.43 there is an \( n \in \Max S \) such that \( n \cap Z_{\text{red}} = m \). Then \( \sigma \in n \) since \( \sigma \in m \). Suppose \( \sigma \) divides the monomial \( g \in Z_{\text{red}} \); say \( g = \sigma h \) for some monomial \( h \in B \). Then \( h \in S \) by Lemma 4.28. Thus \( g = \sigma h \in n \). Whence

\[
g \in n \cap Z_{\text{red}} = m.
\]

Thus every non-constant monomial in \( Z_{\text{red}} \), which is divisible by \( \sigma \), is also in \( m \). Therefore every non-constant monomial in \( Z_{\text{red}} \) is in \( m \). But this contradicts our choice of \( m \) by Claim (i).

Recall the subsets (42) of \( \Max S \) and the morphisms (43). For brevity we will write \( U_{Z, S}, U^*_{Z, S} \), and \( \kappa_{Z, S} \) for \( U_{Z_{\text{red}}, S}, U^*_{Z_{\text{red}}, S} \), and \( \kappa_{Z_{\text{red}}, S} \) respectively.
Proposition 4.49. Let $n \in \text{Max } S$. Then
\begin{equation}
(n \cap Z_{\text{red}} \neq 0 \text{ if and only if } (Z_{\text{red}})_n \cap Z_{\text{red}} = S_n).
\end{equation}
Consequently,
\[\kappa_{Z,S}(U_{Z,S}) = \text{Max } Z_{\text{red}} \setminus \{0\}.
\]
Furthermore,
\[\kappa_{R,S}(U_{R,S}) = \text{Max } R \setminus \{0\}.
\]

Proof. (i) Let $n \in \text{Max } S$ and set $m := n \cap Z_{\text{red}}$. We first want to show that if $m \neq 0$, then $(Z_{\text{red}})_m = S_n$.

Consider $g \in S \setminus Z_{\text{red}}$. It suffices to show that $g$ is in the localization $(Z_{\text{red}})_m$. But $S$ is generated by $\sigma$ and monomials in $B$ not divisible by $\sigma$, by Proposition 3.2. Furthermore, $\sigma$ is in $Z_{\text{red}}$ by Lemma 2.3. Thus it suffices to suppose $g$ is a non-constant monomial which is not divisible by $\sigma$. Let $u \in \mathbb{Z}^2$ and $p \in C^u$ be such that $\bar{p} = g$.

We claim that $u \neq 0$. Indeed, suppose otherwise. Then $p^+$ is a cycle in $Q^+$. Whence $\psi(p)^+$ is a cycle in $Q^+$. Thus $\bar{p} = \sigma^n$ for some $n \geq 1$ by Lemma 2.4.2. But $\sigma^n$ is in $Z_{\text{red}}$. Consequently, $\bar{p} = g$ is in $Z_{\text{red}}$, contrary to our choice of $g$. Therefore $u \neq 0$.

(i.a) First suppose $\sigma$ does not divide all non-constant monomials in $Z_{\text{red}}$. Fix $i \in Q_0$. By Lemma 4.48, there is a non-vertex cycle $q \in e_i A e_i$ such that $\bar{q} \in Z_{\text{red}} \setminus m$ and $\sigma \nmid \bar{q}$.

Let $v \in \mathbb{Z}^2$ be such that $q \in C^v$. Then $v \neq 0$ since $\sigma \nmid \bar{q}$.

We claim that $u \neq v$. Assume to the contrary that $u = v$. Then by Lemma 2.4.1, $\bar{p} = \bar{q}$ since $\sigma \nmid \bar{p}$ and $\sigma \nmid \bar{q}$. But $\bar{q}$ is in $Z_{\text{red}}$, whereas $\bar{p}$ is not, a contradiction. Therefore $u \neq v$.

Since $u \neq v$ are nonzero, the paths $p^+$ and $q^+$ are transverse in $Q^+$. Thus $p^+$ and $q^+$ intersect at a vertex $j^+$. Therefore $p$ and $q$ factor into the product of paths $p = p_2 e_j p_1$ and $q = q_2 e_j q_1$.

Consider the cycle
\[r := q_2 p_1 p_2 q_1 \in e_i A e_i.
\]

Let $\ell \geq 0$ be such that $\sigma^\ell \mid \bar{r}$ and $\sigma^{\ell+1} \nmid \bar{r}$. Since $\sigma \nmid \bar{p}$ and $\sigma \nmid \bar{q}$, there is a cyclic subpath $r_1^+$ of $(p_1 p_2 q_1)^+$ that is not a subpath of $(p_1 p_2)^+$ or $q_1^+$, or a cyclic subpath $r_2^+$ of $(q_2 p_1 p_2)^+$ that is not a subpath of $q_2^+$ or $(p_1 p_2)^+$, such that
\[\bar{r}_1 = \sigma^{m_1} \text{ and } \bar{r}_2 = \sigma^{m_2},
\]
where $m_1 + m_2 = \ell$. Consider the cycle $t$ obtained from $r$ by removing the cyclic subpaths $r_1$ and $r_2$, modulo $I$. Then $\sigma \nmid t$.

Since $i \in Q_0$ was arbitrary, we have $t \in R$. But $\sigma \nmid t$. Thus by Proposition 4.29.1.,
\[i \in Z_{\text{red}}.
\]
Therefore, since \( q \in Z_{\text{red}} \setminus m \),
\[
g = p = r^{-1} = \sigma^{-1} q^{-1} \in (Z_{\text{red}})_m.
\]
But \( g \) was an arbitrary non-constant monomial. Thus, since \( S \) is generated by monomials,
\[
S \subseteq (Z_{\text{red}})_m.
\]
Therefore\footnote{Denote by \( \bar{m} := m(Z_{\text{red}})_m \) the maximal ideal of \( (Z_{\text{red}})_m \). Then, since \( Z_{\text{red}} \subseteq S \),
\[
(Z_{\text{red}})_m = (Z_{\text{red}})_{\bar{m} \cap Z_{\text{red}}} \subseteq S \bar{m} \subseteq (Z_{\text{red}})_m \bar{m} \cap (Z_{\text{red}})_m = (Z_{\text{red}})_m.
\]}
\[
S_n = (Z_{\text{red}})_m.
\]
(i.b) Now suppose \( \sigma \) divides all non-constant monomials in \( Z_{\text{red}} \). Further suppose \( m \neq \bar{z}_0 \). Then by Lemma \[4.46\] \( m \) does not contain any monomials. In particular, \( \sigma \notin m \). By Lemma \[4.29\] \[2\], there is an \( N \geq 0 \) such that \( g\sigma^N \in Z_{\text{red}} \). Thus
\[
g = (g\sigma^N)\sigma^{-N} \in (Z_{\text{red}})_m.
\]
But \( g \) was an arbitrary non-constant monomial. Therefore
\[
S \subseteq (Z_{\text{red}})_m.
\]
It follows that
\[
S_n = (Z_{\text{red}})_m.
\]
Therefore, in either case (a) or (b), Claim (i) holds.
(ii) Now suppose \( n \in \text{Max} \, S \) satisfies \( n \cap R \neq m \). We claim that \( R_{n \cap R} = S_n \).
Since \( n \cap R \neq m \), there is a monomial \( g \in S \setminus n \). By Proposition \[4.29\] \[3\], there is some \( N \geq 1 \) such that \( g^N \in Z_{\text{red}} \). But \( g^N \notin n \) since \( n \) is a prime ideal. Thus
\[
g^N \in Z_{\text{red}} \setminus (n \cap Z_{\text{red}}).
\]
Whence
\[
n \cap Z_{\text{red}} \neq \bar{z}_0.
\]
Therefore
\[
S_n \stackrel{(i)}{=} (Z_{\text{red}})_{n \cap Z_{\text{red}}} \subseteq R_{n \cap R} \subseteq S_n,
\]
where (i) holds by Claim (i), and (ii) follows from Theorem \[4.25\]. Consequently \( R_{n \cap R} = S_n \), proving our claim.
(iii) Finally, we claim that
\[
(Z_{\text{red}})_{\bar{z}_0} \neq S_{\bar{z}_0} \quad \text{and} \quad R_{m_0} \neq S_{m_0}.
\]
These inequalities hold since the local algebras \( (Z_{\text{red}})_{\bar{z}_0} \) and \( R_{m_0} \) are nonnoetherian by Lemma \[4.45\] whereas \( S_n \) is noetherian by Lemma \[4.42\]. \( \square \)

**Lemma 4.50.** Let \( q \) and \( q' \) be prime ideals of \( S \). Then
\[
q \cap Z_{\text{red}} = q' \cap Z_{\text{red}} \quad \text{if and only if} \quad q \cap R = q' \cap R.
\]
Proof. (i) Suppose \( q \cap Z_{\text{red}} = q' \cap Z_{\text{red}} \), and let \( s \in n \cap R \). Then \( s \in R \). Whence there is some \( n \geq 1 \) such that \( s^n \in Z_{\text{red}} \) by Lemma 4.29.3. Thus
\[
 s^n \in q \cap Z_{\text{red}} = q' \cap Z_{\text{red}}.
\]
Therefore \( s^n \in n' \). Thus \( s \in q' \) since \( q' \) is prime. Consequently \( s \in q' \cap R \). Therefore \( q \cap R \subseteq q' \cap R \). Similarly \( q \cap R \supseteq q' \cap R \).

(ii) Now suppose \( q \cap R = q' \cap R \), and let \( s \in q \cap Z_{\text{red}} \). Then \( s \in Z_{\text{red}} \subseteq R \). Thus
\[
 s \in q \cap R = q' \cap R.
\]
Whence \( s \in q' \cap Z_{\text{red}} \). Therefore \( q \cap Z_{\text{red}} \subseteq q' \cap Z_{\text{red}} \). Similarly \( q \cap Z_{\text{red}} \supseteq q' \cap Z_{\text{red}} \).

\( \square \)

**Proposition 4.51.** The subsets \( U_{Z,S} \) and \( U_{R,S} \) of Max \( S \) coincide,
\[
 U_{Z,S} = U_{R,S}.
\]

**Proof.** (i) We first claim that
\[
 U_{Z,S} \subseteq U_{R,S}.
\]
Indeed, suppose \( n \in U_{Z,S} \). Then since \( Z_{\text{red}} \subseteq R \subseteq S \),
\[
 S_n = (Z_{\text{red}})_{n \cap Z_{\text{red}}} \subseteq R_{n \cap R} \subseteq S_n.
\]
Thus
\[
 R_{n \cap R} = S_n.
\]
Therefore \( n \in U_{R,S} \), proving our claim.

(ii) We now claim that
\[
 U_{R,S} \subseteq U_{Z,S}.
\]
Let \( n \in U_{R,S} \). Then \( R_{n \cap R} = S_n \). Thus by Proposition 4.49,
\[
 n \cap R \neq n_0 \cap R.
\]
Therefore by Lemma 4.50
\[
 n \cap Z_{\text{red}} \neq n_0 \cap Z_{\text{red}}.
\]
But then again by Proposition 4.49,
\[
 (Z_{\text{red}})_{n \cap Z_{\text{red}}} = S_n.
\]
Whence \( n \in U_{Z,S} \), proving our claim. \( \square \)

**Definition 4.52.** We say an integral domain \( R \) is *generically noetherian* if there is an open dense set \( W \subset \text{Max} \ R \) such that for each \( m \in W \), the localization \( R_m \) is noetherian.

**Theorem 4.53.** The following subsets of Max \( S \) are nonempty and coincide:

\[
 U_{Z,S}^* = U_{Z,S} = U_{R,S}^* = U_{R,S} = \kappa_{Z,S}^{-1} (\text{Max} \ Z_{\text{red}} \setminus \{z_0\}) = \kappa_{R,S}^{-1} (\text{Max} \ R \setminus \{m_0\}).
\]

In particular, \( Z_{\text{red}} \) and \( R \) are isolated nonnoetherian singularities and generically noetherian.
Proof. The equalities (45) hold since:
- By Proposition 4.51, \( U_{R,S} = U_{Z,S} \).
- By (44) in Proposition 4.49,
  \[
  U_{Z,S} = \kappa_{Z,S}^{-1} (\text{Max } Z_{\text{red}} \setminus \{m_0\}) \quad \text{and} \quad U_{R,S} = \kappa_{R,S}^{-1} (\text{Max } R \setminus \{m_0\}).
  \]
- By Lemma 4.42, the localization \( S_n \) is noetherian for each \( n \in \text{Max } S \). Therefore again by (44),
  \[
  U_{Z,S}^* = U_{Z,S} \quad \text{and} \quad U_{R,S}^* = U_{R,S}.
  \]

Furthermore, \( \kappa_{R,S}^{-1} (\text{Max } R \setminus \{m_0\}) \) is nonempty since there is a maximal ideal of \( R \) distinct from \( m_0 \), and \( \kappa_{R,S} \) is surjective by Lemma 4.43. \( \square \)

Proposition 4.54. The locus \( U_{R,S} \subset \text{Max } S \) is an open set.

Proof. We claim that the complement of \( U_{R,S} \subset \text{Max } S \) is the closed subvariety

\[
U_{R,S}^c = \{ n \in \text{Max } S \mid n \supseteq m_0 S \} =: Z(m_0 S).
\]

Indeed, let \( n \in \text{Max } S \). First suppose \( m_0 S \subseteq n \). Then

\[
(46) \quad m_0 \subseteq m_0 S \cap R \subseteq n \cap R.
\]

Whence \( n \cap R = m_0 \) since \( m_0 \) is a maximal ideal of \( R \). Thus \( n \not\subseteq U_{R,S} \) by Theorem 4.53. Therefore \( U_{R,S}^c \supseteq Z(m_0 S) \).

Conversely, suppose \( n \not\in U_{R,S} \). Assume to the contrary that \( m_0 S \subseteq n \). Then (46) implies that \( n \cap R = m_0 \), contrary to Theorem 4.53. Therefore \( U_{R,S}^c \subseteq Z(m_0 S) \). \( \square \)

Theorem 4.55. The homotopy algebra \( \tilde{A} \) of \( A \) is nonnoetherian and an infinitely generated module over its center \( R \).

Proof. The following hold:
- \((\tau, B)\) is an impression of \( \tilde{A} \) by Theorem 4.34
- \( U_{R,S}^* = U_{R,S} \) by Theorem 4.53
- \( R \neq S \) by Proposition 4.35

Therefore \( \tilde{A} \) is nonnoetherian and an infinitely generated \( R \)-module by [12, Theorem 3.2.2]. \( \square \)

Theorem 4.56. Let \( A \) be a non-cancellative dimer algebra satisfying (20). Then the center \( Z \), reduced center \( Z_{\text{red}} \), and homotopy center \( R \) each have Krull dimension 3,

\[
\dim Z = \dim Z_{\text{red}} = \dim R = \dim S = 3.
\]

Furthermore, the fraction fields of \( Z_{\text{red}}, R, S \), coincide,

\[
(47) \quad \text{Frac } Z_{\text{red}} = \text{Frac } R = \text{Frac } S.
\]
Proof. Recall that $Z_{\text{red}}$, $R$, and $S$ are domains by Corollary 4.26. Furthermore, the subsets $U_{Z,S}$ and $U_{R,S}$ of Max $S$ are nonempty by Theorem 4.53. Since $U_{Z,S}$ and $U_{R,S}$ are nonempty, the equalities (47) hold by \cite[Lemma 2.4]{B2}. Furthermore, $\dim Z_{\text{red}} = \dim S = \dim R$ by \cite[Theorem 2.5.4]{B2}. Finally, let $p \in \text{Spec } Z$. Then nil $Z \subseteq p$ since $0 \in p$. Therefore $\dim Z = \dim Z_{\text{red}}$. □

Lemma 4.57. Let $g, h \in S$ be non-constant monomials such that $\sigma \nmid gh$. If gh $\not\in R$, then $g \not\in R$ and $h \not\in R$.

Proof. Suppose the hypotheses hold, and assume to the contrary that $g \in R$. Fix $i \in Q_0$. Since $g$ is a non-constant monomial in $R$ and $h$ is a non-constant monomial in $S$, there is some $u, v \in \mathbb{Z}^2$ and cycles

$$p \in C^u_i \quad \text{and} \quad q \in C^v$$

such that

$$\bar{p} = g \quad \text{and} \quad \bar{q} = h.$$

By assumption $\sigma \nmid gh = \bar{p} \bar{q}$. Thus $\sigma \nmid \bar{p}$ and $\sigma \nmid \bar{q}$. Therefore $u \neq 0$ and $v \neq 0$ by Lemma 2.11.2.

If $u = v$, then $\bar{p} = \bar{q}$ by Lemma 2.11.1. Whence $\bar{q} \in R$ since $\bar{p} \in R$. But then $\bar{p} \bar{q} \in R$, contrary to assumption. Therefore $u \neq v$. It follows that the lifts $p^+$ and $q^+$ are transverse cycles in $Q^+$. Thus there is a vertex $j^+ \in Q^+_0$ where $p^+$ and $q^+$ intersect. We may therefore write $p$ and $q$ as products of cycles

$$p = p_2e_jp_1 \quad \text{and} \quad q = q_2e_jq_1.$$

Consider the cycle

$$r = p_2q_1q_2p_1 \in e_iAe_i.$$

Then

$$\bar{r} = \bar{p} \bar{q} = gh.$$

Therefore, since $i \in Q_0$ was arbitrary, $r = gh$ is in $R$, a contradiction. □

Recall that the reduction $X_{\text{red}}$ of a scheme $X$, or its reduced induced scheme structure, is the closed subspace of $X$ associated to the sheaf of ideals $\mathcal{I}$, where for each open set $U \subset X$,

$$\mathcal{I}(U) := \{ f \in \mathcal{O}_X(U) \mid f(p) = 0 \ \text{for all} \ p \in U \}.$$

$X_{\text{red}}$ is the unique reduced scheme whose underlying topological space equals that of $X$. If $R = \mathcal{O}_X(X)$, then $\mathcal{O}_{X_{\text{red}}}(X_{\text{red}}) = R/\text{nil } R$, where nil $R$ is the nilradical of $R$ (that is, the radical of the zero ideal of $R$).
Theorem 4.58. Let $A$ be a non-cancellative dimer algebra satisfying (20), and let $\tilde{A}$ be the homotopy algebra of $A$.

(1) The reduced center $Z_{\text{red}}$ and homotopy center $R$ of $A$ are both depicted by the center $Z' \cong S$ of $A'$.

(2) The reduced induced scheme structure of $\text{Spec } Z$ and the scheme $\text{Spec } R$ are birational to the noetherian scheme $\text{Spec } S$, and each contain precisely one closed point of positive geometric dimension.

(3) The spectra $\text{Max } Z_{\text{red}}$ and $\text{Max } R$ may both be viewed as the Gorenstein algebraic variety $\text{Max } S$ with the subvariety $U_{R,S}^c$ identified as a single ‘smeared-out’ point.

Proof. (1) We first claim that $Z_{\text{red}}$ and $R$ are depicted by $S$. By Theorem 4.53

$$U^*_{Z,S} = U_{Z,S} \neq \emptyset \quad \text{and} \quad U^*_{R,S} = U_{R,S} \neq \emptyset.$$ 

Furthermore, by Lemma 4.43 the morphisms $\iota_{Z,S}$ and $\iota_{R,S}$ are surjective.\footnote{The fact that $S$ is a depiction of $R$ also follows from [B2, Theorem 3.2.1], since $(\tilde{\tau}, B)$ is an impression of $\tilde{A}$ by Theorem 4.34.}

(2.i) Claim (1) and [B2, Theorem 2.5.3] together imply that the schemes $\text{Spec } Z_{\text{red}}$ and $\text{Spec } R$ are birational to $\text{Spec } S$, and are isomorphic on $U_{Z,S} = U_{R,S}$. By Lemma 4.42 $S$ is a normal toric Gorenstein domain. By Theorem 4.53 $\text{Max } Z_{\text{red}}$ and $\text{Max } R$ each contain precisely one point where the localizations of $Z_{\text{red}}$ and $R$ are nonnoetherian, namely $z_0$ and $m_0$.

(2.ii) We claim that the closed points $z_0 \in \text{Spec } Z_{\text{red}}$ and $m_0 \in \text{Spec } R$ have positive geometric dimension.

Indeed, since $A$ is non-cancellative, $R \neq S$ by Proposition 4.35. Thus there is a cycle $p$ such that $\sigma \nmid \tilde{p}$ and $\tilde{p}^n \in S \setminus R$ for each $n \geq 1$, by Lemma 4.36. In particular, $\tilde{p}$ is not a product $\tilde{p} = gh$, where $g \in R$ or $h \in R$, by Lemma 4.57. Therefore

$$\tilde{p} \not\in m_0 S.$$ 

Thus for each $c \in k$, there is a maximal ideal $n_c \in \text{Max } S$ such that

$$(\tilde{p} - c, m_0)S \subseteq n_c.$$ 

Consequently,

$$m_0 \subseteq (\tilde{p} - c, m_0)S \cap R \subseteq n_c \cap R.$$ 

Whence $n_c \cap R = m_0$ since $m_0$ is maximal. Therefore by Theorem 4.53

$$n_c \in U_{R,S}^c.$$ 

Set

$$q := \bigcap_{c \in k} n_c.$$
Then \( q \) is a radical ideal since it is the intersection of radical ideals. Thus, since \( S \) is noetherian, the Lasker-Noether theorem implies that there are minimal primes \( q_1, \ldots, q_\ell \in \text{Spec} \ S \) over \( q \) such that
\[
q = q_1 \cap \cdots \cap q_\ell.
\]
Since \( \ell < \infty \), at least one \( q_i \) is a non-maximal prime, say \( q_1 \). Then
\[
m_0 = \bigcap_{c \in k} (n_c \cap R) = \bigcap_{c \in k} n_c \cap R = q \cap R \subseteq q_1 \cap R.
\]
Whence \( q_1 \cap R = m_0 \) since \( m_0 \) is maximal.
Since \( q_1 \) is a non-maximal prime ideal of \( S \),
\[
\text{ht}(q_1) < \dim S.
\]
Furthermore, \( S \) is a depiction of \( R \) by Claim (1). Thus
\[
\text{ght}(m_0) \leq \text{ht}(q_1) < \dim S \quad (1)
\]
where (1) holds by Theorem 4.56. Therefore
\[
gdim m_0 = \dim R - \text{ght}(m_0) \geq 1,
\]
proving our claim.
(3) Follows from Claims (1), (2), and Theorem 4.53. The locus \( U_{R,S}^c \) is a (closed) subvariety by Proposition 4.54. □

Remark 4.59. Although \( Z_{\text{red}} \) and \( R \) determine the same nonlocal variety using depictions, their associated affine schemes
\[(\text{Spec} \ Z_{\text{red}}, \mathcal{O}_{Z_{\text{red}}}) \quad \text{and} \quad (\text{Spec} \ R, \mathcal{O}_R)\]
will not be isomorphic if their rings of global sections, \( Z_{\text{red}} \) and \( R \), are not isomorphic.

4.5. Integral closure. Throughout, \( A \) is a non-cancellative dimer algebra and \( \psi : A \to A' \) is a cyclic contraction to a cancellative dimer algebra. Let \( \tilde{A} \) and \( R = Z(\tilde{A}) \) be the homotopy algebra and homotopy center of \( A \), respectively. It is well known that the center of a cancellative dimer algebra is normal. In this section, we characterize the central normality of non-cancellative dimer algebras.

We denote by \( Z_{\text{red}} \) and \( R \) the respective integral closures of \( Z_{\text{red}} \) and \( R \).

Proposition 4.60. The homotopy center \( R \) is normal if and only if \( \sigma S \subseteq R \).

Proof. (1) First suppose \( \sigma S \subseteq R \).
(1.i) By Lemma 4.42, \( S \) is normal. Therefore, since \( R \) is a subalgebra of \( S \),
\[
R \subseteq S.
\]
(1.ii) Now let \( s \in S \setminus R \). We claim that \( s \) is not in \( R \). Indeed, assume otherwise. Since \( S \) is generated by monomials in the polynomial ring \( B \), there are monomials \( s_1, \ldots, s_\ell \in S \) such that
\[
s = s_1 + \cdots + s_\ell.
\]
Since $s \notin R$, there is some $1 \leq k \leq \ell$ such that $s_k \notin R$. Choose $s_k$ to have maximal degree among the subset of monomials in $\{s_1, \ldots, s_\ell\}$ which are not in $R$.

By assumption, $\sigma S \subset R$. Thus

$$\sigma \nmid s_k.$$  

Since $s \in \overline{R}$, there is some $n \geq 1$ and $r_0, \ldots, r_{n-1} \in R$ such that

$$s^n + r_{n-1}s^{n-1} + \cdots + r_1s = -r_0 \in R.$$  

By Lemma 4.36, the summand $s^n_k$ of $s^n$ is not in $R$ since $\sigma \nmid s_k$. Thus $-s^n_k$ is a summand of the left-hand side of (49). In particular, for some $1 \leq m \leq n$, there are monomial summands $r'$ of $r_m$ and $s' = s_{j_1} \cdots s_{j_m}$ of $s^m$, and a nonzero scalar $c \in k$, such that

$$r's' = cs^n_k.$$  

By Lemma 4.57, $r'$ is a nonzero scalar since $r' \in R$, $s^n_k \notin R$, and $\sigma \nmid s^n_k$. Furthermore, $s'$ is a non-constant monomial since $r' \in R$ and $s^n_k \notin R$. Therefore

$$s_{j_1} \cdots s_{j_m} = s' = (c/r')s^n_k.$$  

By Lemma 4.57, each monomial factor $s_{j_1}, \ldots, s_{j_m}$ is not in $R$. But the monomial $s_k$ was chosen to have maximal degree, a contradiction. Therefore

$$R \cap S = R.$$  

It follows from (48) and (50) that

$$R = R \cap S = R.$$  

(2) Now suppose $\sigma S \notin R$. Then there are monomials $s \in S \setminus R$ and $t \in S$ such that $s = t\sigma$. Let $n \geq 2$ be sufficiently large so that the product of $n$ unit cycles $\sigma^n_i \in A$ is equal (modulo $I$) to a cycle that contains each vertex in $Q$. Then

$$\sigma^n S \subset R.$$  

In particular, the product $s^n = t^n\sigma^n$ is in $R$. But then $s \in \text{Frac } S \setminus R$ is a root of the monic polynomial

$$x^n - s^n \in R[x].$$  

Thus $s$ is in $R \setminus R$. Therefore $R$ is not normal.

**Corollary 4.61.**

(1) If the head or tail of each contracted arrow has in-out degree 1, then $R$ is normal.

(2) If $\psi$ contracts precisely one arrow, then $R$ is normal.

**Proof.** In both cases (1) and (2), clearly $\sigma S \subset R$. □

**Proposition 4.62.** For each $n \geq 1$, there are dimer algebras for which

$$\sigma^n S \notin R \quad \text{and} \quad \sigma^{n+1} S \subset R.$$  

Consequently, there are dimer algebras for which $R$ is not normal.
Proof. Recall the conifold quiver $Q$ with 1 nested square given in Figure 11.1. Clearly $\sigma S \subset R$. More generally, the conifold quiver with $n \geq 1$ nested squares satisfies

$\sigma^{n-1} S \not\subset R$ and $\sigma^n S \subset R$.

The corresponding homotopy center $R$ is therefore not normal for $n \geq 2$ by Proposition 4.60.

Let $\hat{m}_0 \subset m_0$ be the ideal of $R$ generated by all non-constant monomials in $R$ which are not powers of $\sigma$.

Proposition 4.63. Let $n \geq 0$, and suppose $\sigma^n S \not\subset R$ and $\sigma^{n+1} S \subset R$. Then

\begin{equation}
R = k[\sigma] + (\hat{m}_0, \sigma^{n+1})S.
\end{equation}

Proof. Suppose $p \in C^u$ is a non-vertex cycle such that for each $n \geq 1$,

$p := \tau(\psi(p)) \neq \sigma^n$.

Then $\psi(p) \not\in C^0$ by Lemma 2.4.2. Whence $p \not\in C^0$. Therefore $u \neq 0$. The equality \eqref{51} then follows similar to the proof of Lemma 4.57.

Theorem 4.64. The following are equivalent:

1. $R$ is normal.
2. $\sigma S \subset R$.
3. $R = k + m_0 S$.
4. $R = k + J$ for some ideal $J$ in $S$.

Proof. We have

- (1) $\Leftrightarrow$ (2) holds by Proposition 4.60.
- (2) $\Leftrightarrow$ (3) holds by Proposition 4.63.
- (3) $\Leftrightarrow$ (4) holds again by Proposition 4.63.

Theorem 4.65. The integral closures of the reduced and homotopy centers are isomorphic,

$\overline{Z_{\text{red}}} \cong \overline{R}$.

Proof. For brevity, we identify $Z_{\text{red}}$ with its isomorphic $\overline{\psi}$-image in $R$ (Theorem 4.25), and thus write $Z_{\text{red}} \subseteq R$. This inclusion implies

\begin{equation}
Z_{\text{red}} \subseteq \overline{R}.
\end{equation}

To show the reverse inclusion, recall that by Theorem 4.56.3,

$\text{Frac} Z_{\text{red}} = \text{Frac} R = \text{Frac} S$.

(i) First suppose $r \in R$. Then there is some $n \geq 1$ such that $r^n \in Z_{\text{red}}$ by Proposition 4.29.3. Whence $r$ is a root of the monic polynomial

$x^n - r^n \in Z_{\text{red}}[x]$. 
Thus \( r \) is in \( \mathbb{Z}_{\text{red}} \). Therefore
\[
\text{(53)} \quad R \subseteq \mathbb{Z}_{\text{red}}.
\]

(ii) Now suppose \( s \in R \setminus R \). Since \( R \) is generated by monomials in \( S \), it suffices to suppose \( s \) is a monomial. Then by Claim (2) in the proof of Theorem 4.60, there is a monomial \( t \in S \) such that \( s = t\sigma \). Furthermore, by Proposition 4.29.2, there is some \( N \geq 0 \) such that for each \( m \geq 1 \),
\[
 t^m \sigma^N \in \mathbb{Z}_{\text{red}}.
\]
In particular,
\[
 s^N = t^N \sigma^N \in \mathbb{Z}_{\text{red}}.
\]
Whence \( s \) is a root of the monic polynomial
\[
 x^N - s^N \in \mathbb{Z}_{\text{red}}[x].
\]
Thus \( s \) is in \( \mathbb{Z}_{\text{red}} \). Therefore, together with (53), we obtain
\[
\text{(54)} \quad \overline{R} \subseteq \mathbb{Z}_{\text{red}}.
\]
The theorem then follows from (52) and (54).

\begin{proof}

Proposition 4.66. The integral closures \( \overline{R} \cong \mathbb{Z}_{\text{red}} \) are nonnoetherian and properly contained in the cycle algebra \( S \).

\begin{proof}

Since \( A \) is non-cancellative, \( R \neq S \) by Proposition 4.35. Hence there is some \( s \in S \) such that \( \sigma \nmid s \) and \( s^n \notin R \) for each \( n \geq 1 \), by Lemma 4.36. In particular, \( \sigma \nmid s^n \) for each \( n \geq 1 \) since \( \sigma = \prod_{D \in S} x_D \). Thus \( s \) is not the root of a monic binomial in \( R[x] \) by Lemma 4.57. Therefore \( s \notin \overline{R} \) since \( R \) is generated by monomials in the polynomial ring \( B \).

Similarly, \( s^m \notin \overline{R} \) for each \( m \geq 1 \). It follows that \( \overline{R} \) is nonnoetherian by Claim (i) in the proof of Theorem 4.37.
\end{proof}

\end{proof}

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