ON SPECTRAL NORM OF LARGE BAND RANDOM MATRICES

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Abstract

We consider the ensemble of $N \times N$ hermitian random matrices $H^{(N,b)}$ whose entries are equal to zero outside of the band of width $b$ along the principal diagonal. Inside this band the entries $\{H_{ij}, i \leq j\}$ are given by independent identically distributed gaussian random variables with zero mean value and variance $\nu^2/b$.

We study asymptotic behavior of the spectral norm $\|H^{(N,b)}\|$ in the limit $N \to \infty$ when the band width $b$ is much smaller than the matrix size $N$ but also tends to infinity. Our main result is that if $b/(\log N)^3 \to \infty$, then $\limsup_{N,b} \|H^{(N,b)}\|$ is bounded by $2\nu$ with probability 1. To prove this, we derive a system of recurrent relations for the moments $M_{2k}^{(N,b)}$ and analyze these relations in the limit when $k$ goes to infinity as $N \to \infty$.

1 Random matrices and eigenvalue distribution

Random matrices of infinitely increasing dimensions $N \to \infty$ were considered first in theoretical physics. Physics serves as the source of new questions and results here and the development of the random matrix theory reveals many relations between random matrices and other branches of mathematics (see the special issue on physics and mathematics of random matrix theory of the Journal of Physics A: Mathematical and General, vol. 36 (2003)). In these studies, the special attention is paid to the extreme eigenvalues of $N \times N$ hermitian random matrices with unitary invariant probability distribution. Recent progress in this field relates random matrix theory with nonlinear differential equations and integrable systems (the pioneering mathematical results are obtained in [23]); from another hand the probability law of the maximal eigenvalue of random hermitian matrices $l_{\max}^{(N)}$ is asymptotically equivalent to that of the longest increasing subsequence in the random permutation of $1, \ldots, N$.
These results motivate extended studies of extreme values of random matrix ensembles.

In present paper we study asymptotic behavior of $l_{\text{max}}$ of the ensemble of band random matrices. This ensemble plays an important role in the quantum chaos theory (see e.g. [12]). The probability distribution of this ensemble is not invariant with respect to the unitary transformations.

1.1 Wigner ensemble and the semicircle law

The spectral theory of random matrices was started half a century ago by E. Wigner (see e.g. [24]). He studied the eigenvalue distribution of the ensemble of $N \times N$ real symmetric random matrices $W^{(N)}$ of the form

$$\left(W^{(N)}\right)_{ij} = \frac{1}{\sqrt{N}} w_{ij}, \quad (1.1)$$

where $\{w_{ij}, i \leq j\}$ are independent random variables. The main proposition proved in [24] is that if $w_{ij}$ are centered random variables of the variance $v^2$, odd moments zero and all even moments finite, then the normalized eigenvalue counting function of $W^{(N)}$ defined by the formula

$$\sigma_N(l) = \frac{\# \{l_j^{(N)} \leq l\}}{N},$$

where $l_1^{(N)} \leq \ldots \leq l_N^{(N)}$ are eigenvalues of $W^{(N)}$, weakly converges in average to the limiting function $\sigma(l; 2v)$

$$\lim_{N \to \infty} \sigma_N(l) = \sigma(l; 2v) \quad (1.2)$$

with the density of the semicircle form;

$$\sigma'(l; 2v) = \rho(l; 2v) = \frac{1}{2\pi v^2} \left\{ \begin{array}{ll} \sqrt{4v^2 - l^2}, & \text{if } |l| \leq 2v, \\ 0, & \text{otherwise.} \end{array} \right. \quad (1.3)$$

This statement is known as the semicircle (or Wigner) law. In particular, Wigner has shown that the moments of the limiting distribution $m_{2k}$ are determined by recurrent relations

$$m_{2k} = v^2 \sum_{j=0}^{k-1} m_{2k-2-2j} m_{2j}, \quad m_0 = 1. \quad (1.4)$$

Obviously, $m_{2k+1} = 0$. 

\begin{align*}
(1.1) & \left(W^{(N)}\right)_{ij} = \frac{1}{\sqrt{N}} w_{ij}, \\
(1.2) & \lim_{N \to \infty} \sigma_N(l) = \sigma(l; 2v) \\
(1.3) & \sigma'(l; 2v) = \rho(l; 2v) = \frac{1}{2\pi v^2} \left\{ \begin{array}{ll} \sqrt{4v^2 - l^2}, & \text{if } |l| \leq 2v, \\ 0, & \text{otherwise.} \end{array} \right. \\
(1.4) & m_{2k} = v^2 \sum_{j=0}^{k-1} m_{2k-2-2j} m_{2j}, \quad m_0 = 1.
\end{align*}
1.2 Gaussian Unitary (Invariant) Ensemble

If one considers the ensemble of $N \times N$ hermitian matrices $(H^{(N)})_{ij}$

$$\left( H^{(N)} \right)_{ij} = \frac{1}{\sqrt{N}} h_{ij},$$  \hfill (1.5)

where $h_{ij} = \alpha_{ij} + i \beta_{ij}, i \leq j$ are jointly independent complex gaussian random variables with zero mean values and variance $\nu^2$, the density of the probability distribution of $H^{(N)} = H$ can be written in the form

$$Z_N^{-1} \exp \left\{ - \frac{N}{2\nu^2} \text{Tr} \, H^2 \right\},$$  \hfill (1.6)

where $Z_N$ is the normalization constant. Note that in this case random variables $\alpha$ and $\beta$ are jointly independent (for more details, see section 5 of the present article).

The probability distribution (1.6) is invariant with respect to the unitary transformations. Therefore the random matrix ensemble (1.5)-(1.6) is referred to as the Gaussian Unitary Ensemble (GUE) of random matrices. It plays the central role in the spectral theory of random matrices (see e.g. [20]).

It is easy to see that $\{H^{(N)}\}$ verifies the conditions imposed on $\{W^{(N)}\}$ and therefore the semicircle law (1.2)-(1.3) is valid for GUE. Moreover, convergence (1.2) holds with probability 1 in this case. The first study of the maximal eigenvalue $l_{\text{max}}(H^{(N)}) = \|H^{(N)}\|$ was carried out by S. Geman [11] for the ensemble of real symmetric matrices having the probability distribution of the form (1.6). This ensemble is known as the Gaussian Orthogonal Ensemble (GOE) of random matrices. Geman has proved that $l_{\text{max}}(H^{(N)})$ converges with probability 1 as $N \to \infty$ to the border $2\nu$ of the support of $d\sigma(l; 2\nu)$.

2 Band random matrices

Given a parameter $b$, let us consider hermitian random matrices $H^{(N,b)}$ of the form

$$H_{xy}^{(N,b)} = h_{xy} \sqrt{\Psi_{xy}^{(b)}}, \quad x, y = 1, \ldots, N,$$  \hfill (2.1)

where $\{h_{xy}, x \leq y\}$ are the same as in (1.5) with the law (1.6) and

$$\Psi_{xy}^{(b)} = \frac{1}{b} \psi \left( \frac{x-y}{b} \right),$$

where

$$\psi(t) = \begin{cases} 1, & \text{if } |t| \leq 1/2, \\ 0, & \text{otherwise} \end{cases} \hfill (2.2)$$
The ensemble of band random matrices (2.1)-(2.2) was considered in relation to the quantum chaos theory and solid state physics (see [8, 12, 10]). The most intriguing question was that the ratio \( b^2/N \) separates two major asymptotic regimes that characterize behavior of certain spectral characteristics of random matrices.

In paper [17], the real symmetric analog of (2.1) is considered. It is proved that in the limit \( b \to \infty, b = o(N) \) the Wigner law is valid, i.e. the normalized eigenvalue counting function \( \sigma_{N,b}(l) \) of \( H^{(N,b)} \) weakly converges in probability to the semi-circle distribution (see also papers [7, 18, 21] for this and related result). The same proposition can be easily proved for the ensemble of hermitian band random matrices. If \( b = O(N^\gamma) \) with some \( \gamma > 0 \), then the convergence of \( \sigma_{N,b} \) to the semicircle distribution holds with probability 1.

Much less is known about the limiting behavior of the spectral norm \( \|H^{(N,b)}\| = \lambda_{\text{max}}^{(N,b)} \) of band random matrices. Up to our knowledge, this question was addressed in paper [3] only. If one looks at the computations presented in [3], one can easily see that \( \|H^{(N,b)}\| \to \infty \) as \( b = o(\log N), b \to \infty \). In present paper we show that if \( b \) goes to infinity faster than \( (\log N)^3 \), then the spectral norm of band random matrices remains bounded by \( 2v \).

Our main result is given by the following proposition.

**Theorem 2.1.**

If \( N \) and \( b \) go to infinity in the way such that \( b/(\log N)^3 \to \infty \), then the spectral norm \( \|H^{(N,b)}\| \) remains bounded with probability 1;

\[
\limsup_{N,b \to \infty} \|H^{(N,b)}\| \leq 2v. \tag{2.4}
\]

If \( b = O(N^\gamma) \) with \( \gamma > 0 \), then (2.4) turns to equality.

To study the spectral norm of random matrices \( H \), we employ the general approach used first by S. Geman by suggestion of U. Grenander [11]. Later it was employed by many authors in applications to various random matrix ensembles with jointly independent entries [2, 9, 15] and also for random matrices whose elements are statistically dependent random variables [4]. The key observation is that if one considers the moments

\[
M_{2k}^{(N)} = \mathbb{E} \left\{ \frac{1}{N} \text{Tr} \left[ H^{(N)} \right]^{2k} \right\}
\]

in the limit \( N \to \infty, k = O(\log N) \), then the leading contribution to \( M_{2k}^{(N)} \) will be given by the maximal eigenvalue of \( \lambda_{\text{max}}(H) \).

The next important step was made in [5]. It was observed that to study the limiting behavior of the moments \( M_{2k}^{(N)} \) of the Wigner ensemble (1.1), it is not
necessary to compute them explicitly. It is sufficient to show that they verify a
system of recurrent relations that converges as $N, k \to \infty$ to the system of equalities
(1.4) that determines the moments of the semicircle law.

In paper [4] we have developed a method of derivation and study the recurrent
relations for the moments $M_{2k}$ in the case when the matrix elements are gaussian
correlated random variables. In present paper we develop a new version of this
method that we adapt to the band random matrix ensemble. Also we present more
precise analysis of these relations aiming the best possible estimates of the moments
$M^{(N,b)}_{2k}$ of $H^{(N,b)}$.

We start with the basic example of the GUE (see Section 3). We pay much
attention to this ensemble because the relations and estimates of the moments $M^{(N,b)}_{2k}$
are very similar to those of GUE. Thus, the estimates we get in the GUE case are
true for the moments of the band random matrices. We did not manage to get the
optimal estimates for $M^{(N)}_{2k}$ obtained by other methods. The benefit of the approach
developed is that it can be applied for random matrix ensembles different from GUE.

In Section 4 we modify our approach and study the band random matrices. These
computations lead to the proof of Theorem 2.1. In Section 5 we prove auxiliary
statements.

3 Moments of GUE matrices

3.1 Recurrent relations

In this section we derive two main recurrent relations for the moments of $H^{(N,b)}$ and
their variances. Let us consider the normalized trace $L_a = \frac{1}{N} \text{Tr} H^a$ and compute
the mathematical expectation with respect to the Gaussian measure (1.4) of the
following expression, where $G$ denotes some regular function of $H$:

$$
E \{ L_a G \} = \frac{1}{N} \sum_{x,s=1}^{N} E \{ H_{xs} H_{sx}^{a-1} G \}.
$$

Regarding the last average, we can use the integration by parts formula

$$
E \{ H_{xs} H_{sx}^{a-1} G \} = E \{ |H_{xs}|^2 \} \ E \left\{ \frac{\partial H_{sx}^{a-1} G}{\partial H_{sx}} \right\}
$$

(3.1)

The use of the partial derivative symbol is explained in section 5.

Then we can write that

$$
E \{ L_a G \} = \frac{v^2}{N} \sum_{x,s=1}^{N} \sum_{j=0}^{N} \left[ E \{ H_{ss}^{a-2-j} H_{sx}^j R \} + E \{ H_{sx}^{a-1} \partial G / \partial H_{sx} \} \right]
$$

5
v^2 \sum_{j=0}^{a-2} \mathbb{E} \{ L_{a-2-j} L_j G \} + \frac{v^2}{N^2} \sum_{x,s=1}^N \mathbb{E} \{ H_{sx}^{a-1} \partial G / \partial H_{sx} \}.

If \( G = 1 \) and \( a = 2k \), then we get the following relation

\[
\mathbb{E} \{ L_{2k} \} = v^2 \sum_{j=0}^{2k-2} \mathbb{E} \{ L_{2k-2-j} \}
\]

Denoting \( L^o = L - \mathbb{E} L \), we can write that

\[
\mathbb{E} \{ L_{a_1} L_{a_2} \} = \mathbb{E} \{ L_{a_1} E L_{a_2} + D_{a_1,a_2}^{(2)} \} = \mathbb{E} \{ L_{a_1} L_{a_2}^o \}
\]

Then using the fact that \( M_{2k+1}^{(N)} = \mathbb{E} \{ L_{2k+1} \} = 0 \) (see Section 5), we obtain our first equality

\[
M_{2k}^{(N)} = v^2 \sum_{j=0}^{k-1} M_{2k-2-j}^{(N)} M_{2j}^{(N)} + v^2 \sum_{a_1 + a_2 = 2k-2} D_{a_1,a_2}^{(2)}
\]

that form the system of recurrent relations. Let us note that for two random variables we always have \( \mathbb{E} \{ L_{1Q}^o \} = \mathbb{E} \{ L_1 L_2^o \} \).

Following our general scheme, we introduce variables

\[
D_{a_1,...,a_q}^{(q)} = \mathbb{E} \{ L_{a_1}^o L_{a_2}^o \cdots L_{a_q}^o \} = \mathbb{E} \{ L_{a_1} [ L_{a_2}^o \cdots L_{a_q}^o ]^o \}
\]

and apply (3.1) to this expression with \( G = [L_{a_2}^o \cdots L_{a_q}^o]^o \). Then we obtain relation

\[
D_{a_1,...,a_q}^{(q)} = v^2 \sum_{j=0}^{a_1-2} \mathbb{E} \{ L_{a_1-2-j} L_j L_{a_2}^o \cdots L_{a_q}^o \} + \frac{v^2}{N^2} \sum_{i=2}^q \mathbb{E} \{ L_{a_2}^o \cdots L_{a_{i-1}}^o a_i L_{a_i+a_1-2} \cdots L_{a_q}^o \}.
\]

The last term arises because of equality

\[
\sum_{t=1}^N \frac{\partial (L_{a}^{tt})}{\partial W_{sx}} = \sum_{j=0}^{a-1} \sum_{t=1}^N W_{ts}^{a-1-j} W_{xt}^j = (a-1) \left( W^{a-1} \right)_{xs}.
\]

Now we use two times the identity

\[
\mathbb{E} \{ L_1 L_2 Q \} = \mathbb{E} \{ L_1 L_2^o Q \} + \mathbb{E} \{ L_1^o L_2 Q \} + \mathbb{E} \{ L_1^o L_2^o Q \} - \mathbb{E} \{ L_1^o L_2^o \} \mathbb{E} \{ Q \}
\]

and obtain our second recurrent relation

\[
D_{a_1,...,a_q}^{(q)} = v^2 \sum_{j=0}^{a_1-2} M_{j}^{(N)} D_{a_1-2-j,a_2,...,a_q}^{(q)} + v^2 \sum_{j=0}^{a_1-2} M_{a_1-2-j}^{(N)} D_{j,a_2,...,a_q}^{(q)} +
\]

6
\[
v^2 \sum_{j=0}^{a_1-2} D_{j,a_1-2-j,a_2,...,a_q}^{(q+1)} - v^2 \sum_{j=0}^{a_1-2} D_{j,a_1-2-j,a_2,...,a_q}^{(2)} D_{a_2,...,a_q}^{(q-1)} + \]

\[
\frac{v^2}{N^2} \sum_{i=2}^{q} a_i M_{a_1+a_i-2,a_2,...,a_i-1,a_{i+1},...,a_q}^{(N)} + v^2 \sum_{i=2}^{q} a_i D_{a_2,...,a_i-1,a_i+1,...,a_q}^{(q-1)}
\]

(3.3)

When deriving (3.3), we assumed \( q > 3 \). For \( q = 2 \) we have equality

\[
D_{a_1,a_2}^{(2)} = 2v^2 \sum_{j=0}^{a_1-2} M_{j,a_1-2-j,a_2}^{(N)} + v^2 \sum_{j=1}^{a_1-3} D_{j,a_1-2-j,a_2}^{(3)} + \frac{a_2}{N^2} v^2 M_{a_1,a_2}. \quad (3.4)
\]

In the case of \( q = 3 \) we adopt relation (3.3) assuming that \( D^{(0)} = 1 \) and \( D^{(1)} = 0 \). Also one has to remember that \( M_{2l+1}^{(N)} = 0 \).

### 3.2 Recurrent estimates

Let us prove the following proposition.

**Lemma 3.1** If \( N \geq 4 \) and \( k \leq N^{1/3} \), then

\[
M_{2k}^{(N)} \leq \left( 1 + \frac{4k^3}{N^2} \right)^k m_{2k}
\]

and

\[
|D_{a_1,...,a_q}^{(q)}| \leq \frac{a_1 a_2 \cdots a_q}{N^q} \left( 1 + \frac{4k^3}{N^2} \right)^k m_{2k},
\]

where \( 2k = a_1 + \ldots + a_q \).

We prove this statement following the procedure of recurrent estimates proposed in [4]. Let us consider the plane of positive integers, i.e. the set of points \( B = \{(l,p) : l,p \in \mathbb{N}\} \). Actually, \( B \) is a quarter-plane.

We say that the random variable \( D_{a_1,...,a_q}^{(q)} \) with given \( A_q = (\alpha_1, \ldots, \alpha_q) \) belongs to the point \((2k,q) \in B\), if the sum of \( a \)'s is equal to \( 2k \). We call such variables \( D \) the elements of \((2k,q)\). Obviously, \( 2 \leq q \leq l, l = 2k \) and we restrict ourselves with the corresponding subdomain \( \tilde{B} \) of \( B \).

It follows from relations (3.3) that the elements \( D \) of \((2k,q)\) are expressed in terms of the elements \( D \) belonging to one of the four other points: \((2k-2,q)\), \((2k-2,q+1)\), \((2k-2,q-1)\) and \((2k-2,q-2)\). If any of these points is situated outside of the domain \( \tilde{B} \), then the corresponding term in (3.3) is equal to zero and gives no contribution. On figure 1 we represent three possible situations. The fact
that the element of \((2k, q)\) is expressed in terms of other elements is reflected by a flesh. Let us note that the right-hand side of (3.3) uses also the moments \(M_{2l}\) with \(2l \leq 2k - 2\). So, the flesh lines directed from points \((2k, \cdot)\) to \((2k - 2, \cdot)\) also indicate the dependence of \(D\) on \(M\).

Suppose we want to prove the estimate (3.6) for the elements \(D\) of the point \((2k', q')\) with given \(k'\) and \(q'\). On Figure 1 this is the point \((10, 5)\). We assume that these estimates are true for the elements \(D\) of all points \((2k'', q'')\) that belong to the triangle \(2k'' + q'' < 2k' + q'\) and of those situated on the line \(2k'' + q'' = 2k' + q'\) with \(2k'' \leq 2k' - 2\). On Figure 1 we marked these points by fat circles. Also we assume that (3.5) holds for all \(M_{2l}\) with \(2l \leq 2k' - 2 + q'\). Then we use relations (3.3) to deduce the estimate (3.6) for the point \((2k', q')\).

So, starting with the point \((2, 2)\), we obtain the estimates (3.6) for the elements of \((4, 3)\) and \((4, 2)\). Then we go step by step over the points of \(\tilde{B}\) until we reach the point \((2k', q') = (10, 5)\). The next points are \((12, 4)\), \((14, 3)\) and so on. On this way we proceed along the line \(2k'' + q'' = 2k' + q'\) until \(q'' = 2\) (the point \((16, 2)\) on Figure 1). Using (3.2), we prove the estimate (3.2) for \(M_{2k' + q'}\). Then we complete the next line starting from the point \((10, 6)\).

Computationally this recurrent procedure means that we assume that all terms of the right-hand side of (3.3) verify inequalities (3.5) and (3.6) and then estimate their sum to show that it is less than the expression corresponding to the term of the left-hand side of (3.3). The same is true for (3.2) with the initial point of the recurrence \(M_0^{(N)} = 1\). We note also that \(M_2^{(N)} = v^2\) that obviously verifies (3.5) with \(k = 1\) because \(m_2 = v^2\). Let us have a look at \(D_{1,1}^{(2)}\) that serves as the initial
point for (3.3). It is easy to see that

\[
D^{(2)}_{1,1} = \mathcal{E} \left\{ \frac{1}{N} \text{Tr} H \frac{1}{N} \text{Tr} H \right\} = \frac{v^2}{N^2}
\]

that certainly satisfies (3.6).

Now let us consider (3.2). We want to show that if (3.5) and (3.6) are valid for the terms of the right-hand side of (3.2), then (3.5) is valid for \( M^{(N)}_{2k} \). The first term of the right-hand side of (3.2) is bounded by

\[
v^2 \sum_{j=0}^{k-1} \left( 1 + \frac{4(k - 1 - j)^3}{N^2} \right)^{k-1-j} \left( 1 + \frac{4j^3}{N^2} \right)^j m_{2k-2-2j}m_{2j+1} \leq \left( 1 + \frac{4(k - 1)^3}{N^2} \right)^{k-1} m_{2k}.
\]

The second term of the right-hand side of (3.2) is bounded by

\[
v^2 m_{2k-2} \left( 1 + \frac{4(k - 1)^3}{N^2} \right)^{k-1} \sum_{a_1+a_2=2k-2} \frac{a_1a_2}{N^2} \leq \left( 1 + \frac{4k^3}{N^2} \right)^{k-1} \frac{(2k-2)^3}{2N^2} m_{2k}
\]

because

\[
\sum_{j=1}^{2k-3} (2k - 2 - j)j = \sum_{i=1}^{k-1} (2i - 1)^2 < (2k - 2)^3/2.
\]

Gathering these two estimates, one obtains that

\[
M^{(N)}_{2k} \leq \left( 1 + \frac{4k^3}{N^2} \right)^{k-1} \left[ 1 + \frac{4(k - 1)^3}{N^2} \right] m_{2k}
\]

and (3.5) obviously follows.

Now let us turn to (3.3). Let us denote

\[
\frac{a_1a_2 \cdots a_q}{N^q} \left( 1 + \frac{4k^3}{N^2} \right)^k \equiv \Pi(A_q; N).
\]

We assume that (3.5) and (3.6) are valid for the all terms of the right-hand side of (3.4) and show that their sum is bounded by the expression standing at the right-hand side of (3.6). Let us consider two first terms of the right-hand side of (3.4).
Repeating calculations that lead to (3.7), we can write that the sum of the first two terms is estimated by

$$\frac{v^2 a_2 \cdots a_q}{N^q} \left( 1 + \frac{4k^3}{N^2} \right)^{k-1} \left[ \sum_{j=1}^{a_1-3} (a_1 - 2 - j) m_{2k-2-j} + \sum_{j=1}^{a_1-3} j m_{a_1-2-j} m_{2k-a_1+j} \right]$$

The terms in square brackets represent the first and the third parts of the sum

$$[(a_1 - 3) \cdot m_1 m_{2k-3} + \ldots + 2m_{a_1-4}m_{2k-a_1+2} + 1 \cdot m_{a_1-3}m_{2k-a_1+1}] +$$

$$\{m_{a_1-2}m_{2k-a_1} + m_{a_1-1}m_{2k-a_1-1} + \ldots + m_{2k-a_1}m_{a_1-2}\} +$$

$$[1 \cdot m_{2k-a_1+1}a_1-3 + 2m_{2k-a_1+2}m_{a_1-4} + \ldots + (a_1 - 3)m_{2k-3}m_1] \leq$$

$$(a_1 - 3) \sum_{j'=1}^{2k-3} m_{j'} m_{2k-2-j'}.$$  

Certainly, all the sums run over even numbers, but we do not care about this because $m_{2j+1} = 0$ and all relations are still true when regarding the sums over even and odd numbers. Thus we obtain that the sum of two first terms of the right-hand side of (3.4) is bounded by

$$\Pi(A_q;N) \left( 1 - \frac{3}{a_1} \right) m_{2k}. \quad (3.8)$$

Now let us consider the third term of the right-hand side of (3.4). Assuming that (3.5) and (3.6) hold for corresponding $M$ and $D$, we can write that

$$v^2 a_1^{-3} \sum_{j=1}^{a_1-3} |D^{(q+1)}(j, a_1 - 2 - j, a_2, \ldots, a_q)| \leq$$

$$\Pi(A_q;N) v^2 m_{2k-2} \sum_{j=1}^{a_1-3} j (a_1 - 2 - j) \leq \Pi(A_q;N) \frac{a_1^2}{2N} m_{2k}. \quad (3.9)$$

The fourth term of the right-hand side of (3.4) is estimated by the same expression.

The fifth term is estimated by

$$\frac{v^2 a_2 \cdots a_q}{N^{q+1}} \sum_{i=2}^{q} \left( 1 + \frac{(a_1 + a_i - 2)^3}{2N^2} \right)^{a_{i+1}-a_i} \left( 1 + \frac{(2k - a_1 - a_i)^3}{2N^2} \right)^{2k-a_i-a_{i-1}} m_{a_1+a_i-2m_{2k-a_1-a_i}} \leq$$

$$\Pi(A_q;N) \frac{1}{a_1} v^2 \sum_{i=2}^{q} m_{a_1+a_i-2m_{2k-a_1-a_i}} \leq \Pi(A_q;N) \frac{1}{a_1} m_{2k}. \quad (3.10)$$
Finally, the last term of (3.4) is less or equal to
\[
\frac{v^2}{N^{q+1}} \sum_{i=2}^{q} (a_1 + a_i - 2) a_2 \cdots a_q \left( 1 + \frac{4(k-1)^3}{N^2} \right)^{k-1} m_{2k-2} \leq
\]
\[
\Pi(A_q; N) v^2 m_{2k-2} \sum_{i=2}^{q} \frac{a_1 + a_i - 2}{a_1 N} \leq \Pi(A_q; N) \frac{2k - 2}{N} m_{2k}. \tag{3.11}
\]

The sum of all expressions (3.6)-(3.9) gives us inequality
\[
|D_{a_1, \ldots, a_q}(q)| \leq \Pi(A_q; N) \left[ \left( 1 - \frac{2}{a_1} \right) + \frac{a_1^2}{N} + \frac{2k - 2}{N} \right] m_{2k}.
\]

Taking into account that \(a_1 \leq 2k - 2\), it is easy to show that the sum of the terms in square brackets is strictly less than 1 provided \(k \leq N^{1/3}\). This implies inequality (3.6).

Lemma 3.1 is proved.

### 3.3 Asymptotic behavior of the moments of GUE matrices

#### 3.3.1 Convergence of the maximal eigenvalue

It is very well known the numbers \(m_{2k} = m_{2k}(2v)\) determined by (3.12) given by Catalan numbers
\[
m_{2k} = v^{2k} \frac{1}{k+1} \binom{2k}{k}. \tag{3.12}
\]

Elementary computations imply that \(m_{2k}(2v) \leq (2v)^{2k}\). Then it follows from Lemma 3.1 that given \(\varepsilon > 0\), we have estimate
\[
M^{(N)}_{2k} \leq \left( 2v \sqrt{1 + \varepsilon} \right)^{2k} \tag{3.13}
\]
for all \(k \leq N^{1/3}\) provided \(4k^3/N^2 \leq \varepsilon\).

Taking into account inequality \((l_{\max})^{2k} \leq \sum_{i=1}^{N} l_i^{2k} = \text{Tr} H^{2k}\), we can write that
\[
\text{Prob}\{l_{\max} \geq 2v(1 + \varepsilon)\} \leq \frac{N M^{(N)}_{2k}}{(2v(1 + \varepsilon))^{2k}} \leq \frac{N}{(1 + \varepsilon)^k} \quad \text{for all} \quad k \leq N^{1/3}. \tag{3.14}
\]

Regarding \(k \gg \log N\) and using the Borel-Cantelli lemma, it is easy to deduce from (3.14) that \(\limsup_{N \to \infty} l^{(\text{GUE})}_{\max} \leq 2v\) with probability 1. This estimate together with the convergence (1.2) with probability 1 implies that \(l_{\max} \to 2v\) with probability 1 as \(N \to \infty\).
3.3.2 High moments and the scale at the spectral edge

Let us compare our results with those already known for the moments of GUE with $\nu^2 = 1/4$. It is known that the moments $M_{2k}^{(N)}$ are given by the following recurrent relation [13]:

$$M_{2k}^{(N)} = \frac{2k - 1}{2k + 2} M_{2k-2}^{(N)} + \frac{2k - 1}{2k + 2} \cdot \frac{2k - 3}{2k} \cdot \frac{k(k - 1)}{4N^2} \cdot M_{2k-4}^{(N)}. \quad (3.15)$$

Explicit expression for $m'_{2k} = m_{2k}(1)$ (3.12) implies that

$$m'_{2k} = \frac{2k - 1}{2k + 2} m'_{2k-2} = \frac{2k - 1}{2k + 2} \cdot \frac{2k - 3}{2k} m'_{2k-4}. \quad (3.16)$$

Then it is easy to deduce from (3.15) that $M_{2k}^{(N)}$ admit the following estimates (see, for example [19]):

$$M_{2k}^{(N)} \leq \left(1 + \frac{k^3}{4N^2}\right) m'_{2k} \text{ for all } k, N. \quad (3.16)$$

Also one can write that

$$M_{2k}^{(N)} \leq \left(1 + \frac{k^2}{4N^2}\right)^k m'_{2k} \text{ for all } k, N. \quad (3.17)$$

Inequality (3.16) means that the moments $M_{2k}^{(N)}$ admit the power-like estimates by $(2\nu)^{2k}(1 + o(1))$ in the limit $1 \ll k \ll N^{2/3}$ and that this behavior can change on the regime $k = tN^{2/3}$.

Inequality (3.16) can be obtained by using the orthogonal polynomial approach (see the early paper by Bronk [6], where the scaling at the edge of the semicircle distribution has been determined for the first time); explicit asymptotic expressions were found in the seminal paper by Tracy and Widom [23]. It was shown that the fraction $k^3/N^2$ is really the optimal one in the sense that one cannot decrease the exponent of $k$ and increase that of $N$ in (3.16).

Basing on the relations (3.2) and (3.3), we did not manage to obtain estimates as precise as (3.16). Our result (3.5) implies that

$$M_{2k}^{(N)} \leq \left(1 + C \frac{k^4}{N^2}\right) m_{2k} \text{ for } k \leq N^{1/3}. \quad (3.18)$$

Inequalities of this type are sufficiently powerful to estimate the maximal eigenvalue of $H$ but they do not reflect the real scale of the eigenvalue distribution at the edge of the limiting spectrum.
From another hand, we did not used the orthogonal polynomial approach to obtain (3.2) and (3.4). Therefore the positive counterpart is that our approach is applicable for more general ensembles of random matrices than the GUE. In the next section we show how our approach works in the case of band random matrices.

4 Moments of band random matrices

4.1 Main technical proposition and proof of Theorem 2.1

Our main goal is to study the moments

$$M^{(N)}_k = \mathbb{E} \left\{ \frac{1}{N} \text{Tr} \left[ H^{(N,b)} \right]^k \right\},$$

(4.1)

where $H^{(N,b)}$ are given by matrices (2.1). The first observation is that $M^{(N,b)}_{2k+1} = 0$ (see Lemma 5.2 of section 5). Thus we can consider the even moments $M^{(N,b)}_{2k}$ only. The next observation is that in the case of band random matrices we cannot derive recurrent relations for $M_{2k}$ themselves as it was for the case of GUE. Instead we find a family of random variables that make a close system. Let us start with the random variables

$$L_k(x) = L_x^{(N,b)} = \left[ H^{(N,b)} \right]_{xx}^k.$$

Using integration by parts formula (see section 5), it is not hard to show that the mathematical expectation of $L$ verifies the following identity

$$\mathbb{E} L_{2k}(x) = v^2 \sum_{j=0}^{2k-1} \mathbb{E} \{ L_{2k-2-j}(x) L_j[x] \},$$

(4.2)

where we denoted

$$L_j[x] = \frac{1}{b} \sum_{s=1}^{N} L_j(s) \psi\left( \frac{s-x}{b} \right).$$

One can say that $L_j[x]$ represents a partial trace of $H^j$ normalized by $b$. So, one can expect that the variance of $L$ goes to zero when $b \to \infty$ and that the mathematical expectation in the right-hand side of (4.2) factorizes. In what follows, we prove this factorization and estimate the variance of $L[x]$.

Regarding mathematical expectation $M^{(N,b)}_{2k}(x) = \mathbb{E} L_x(k)$, we obtain equality

$$M^{(N,b)}_{2k}(x) = v^2 \sum_{j=0}^{k-1} M^{(N,b)}_{2k-2-2j}(x) M^{(N,b)}_{2j}[x] + v^2 D^{(2)}_{2k-2}(x),$$

(4.3)
where
\[ D_{2k-2}^{(2)}(x) = \sum_{\alpha_1 + \alpha_2 = 2k-2} \mathbb{E} \{ L_{\alpha_1}^o(x)L_{\alpha_2}^o[x] \}. \]

We recall that \( L^o = L - \mathbb{E} L \).

In the next subsection we will prove Lemma 4.1 that implies the following estimates that are true for all \( 2k \leq b^{1/3} \):

\[ \sup_{x=1,\ldots,N} M_{2k}^{(N,b)}(x) \leq \left( 1 + \frac{4k^3}{b^2} \right)^k m_{2k}, \quad (4.4) \]

where the family \( \{ m_{2k} \}_{k \in \mathbb{N}} \) determines the semicircle law (1.2).

It follows from (4.4) that

\[ M_{2k}^{(N,b)} = \frac{1}{N} \sum_{x=1}^{N} M_{2k}^{(N)}(x) \leq \left( 1 + \frac{4k^3}{b^2} \right)^k m_{2k}, \quad (4.5) \]

Using the estimate (3.14), we see that

\[ \text{Prob}\{l_{\text{max}}^{(N,b)} \geq 2v(1+\varepsilon)\} \leq \frac{N}{(1+\varepsilon)^k} \]

for all \( k \leq b^{1/3} \) provided \( 4k^3/b^2 \leq \varepsilon \). Choosing \( k \) such that \( k/\log N \to \infty \), we obtain the bound

\[ \limsup_{N \to \infty} \max_{j=1,\ldots,N} |l_{i}^{(N,b)}| \leq 2v \quad (4.6) \]

with probability 1 provided \( b \geq (\log N)^3 \). Theorem 2.1 follows from (4.6).

4.2 Recurrent relations for the generalized moments

Let us introduce random variables that serve as the elementary blocks for the closed system of recurrent relations. These are the products

\[ L_{R(a)}(x) = \left( H^{\alpha_1} \Psi^{(y_1)} H^{\alpha_2} \ldots H^{\alpha_{r-1}} \Psi^{(y_{r-1})} H^{\alpha_r} \right)_{xx}, \]

where \( R \) is given by two \( r \)-dimensional vectors \( A \) and \( Y; A_r = (\alpha_1, \ldots, \alpha_r), \alpha_1 + \ldots + \alpha_r = a, Y_r = (y_1, \ldots, y_r), y_i \in \{1, \ldots, N\} \), and \( \Psi^{(y)} \) represents a diagonal \( N \)-dimensional matrix \( \Psi^{(y)}_{st} = \delta_{st}\psi((s-y)/b) \).

In section 5 (see Lemma 5.2) we prove that

\[ \mathbb{E} L_{R(a)} = 0 \quad \text{if} \quad a = 2k + 1. \quad (4.7) \]
In present section we prove the following proposition.

**Lemma 4.1** If \(2k \leq b^{1/3}\), then

\[
0 \leq \mathbf{E} L_{R(a)} \leq \left(1 + \frac{4k^3}{b^2}\right)^k m_{2k} \quad \text{if} \quad a = 2k,
\]

and

\[
\sup_{x,y} \mathbf{E} \left\{ L_{R_1(a_1)}(y) L_{R_2(a_2)}[x] \cdots L_{R_q(a_q)}[x] \right\} \leq \begin{cases} 
\Pi(A_q; b) m_{2k}, & \text{if} \ |A_q| = 2k, \\
0, & \text{if} \ |A_q| = 2k + 1,
\end{cases}
\]

where

\[
\Pi(A_q; b) = \frac{a_1 a_2 \cdots a_q}{b^q} \left(1 + \frac{4k^3}{b^2}\right)^k
\]

and

\[
|A_q| = a_1 + \ldots + a_q.
\]

To prove estimates (4.8) and (4.9), we derive a system of recurrent relations that resemble very much relations (3.2), (3.3), and (4.3). Then we use the recurrent procedure described in subsection 3.2. So, we do not describe the details of the derivation but give the general description.

Regarding \(L_{R(a)}\), one can always assume that \(\alpha_1 > 1\). Therefore one can write that

\[
\mathbf{E} \left\{ \sum_{s=1}^{N} H_{xs} \left(H^{\alpha_1-1} \Psi_1 \cdots \Psi_{r-1} H^{\alpha_r}\right)_{sx} \right\}
\]

\[
v^2 \sum_{l=1}^{r} \sum_{j=0}^{\alpha'_l-1} \mathbf{E} \left\{ \sum_{s=1}^{N} (H^{\alpha_1-1} \Psi_1 \cdots \Psi_{l-1} H^j)_{ss} \frac{1}{b} \psi \left(\frac{s-x}{b}\right)(H^{\alpha_{l-1}-j} \Psi_l \cdots \Psi_{r-1} H^{\alpha_r})_{xx} \right\},
\]

where we denoted \(\Psi_i = \Psi(y_i)\) and

\[
\alpha'_l = \begin{cases} 
\alpha_l - 2, & \text{if} \ l = 1 \\
\alpha_l - 1, & \text{if} \ l \neq 1.
\end{cases}
\]

Now, factorizing the mathematical expectation, one gets

\[
\mathbf{E} L_{R(a)}(x) = v^2 \sum_{l=1}^{r} \sum_{j=0}^{\alpha'_l-1} \mathbf{E} \left\{ (H^{\alpha_1-1} \Psi_1 \cdots \Psi_{l-1} H^j)[x] \right\} \mathbf{E} \left\{ (H^{\alpha_{l-1}-j} \Psi_l \cdots \Psi_{r-1} H^{\alpha_r})_{xx} \right\}
\]

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\[ v^2 \sum_{l=1}^{r} \sum_{j=0}^{\alpha'_l-1} E \left\{ (H^{\alpha_1-1} \Psi_1 \ldots \Psi_{l-1} H^j)^o[x](H^{\alpha_{l-1}j} \Psi_l \ldots \Psi_{r-1} H^{\alpha_r})^o_{xx} \right\}. \quad (4.9) \]

If one accepts the symbolic denotation \( \sum_{l=1}^{r} \sum_{j=0}^{\alpha'_l-1} = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor - 1} \), then we can rewrite (4.9) in the form close to (4.2)

\[ E L_{R(a)}(x) = \sum_{j=0}^{a-2} E \{ L_J[x] \} E \{ L_{a-2-j}(x) \} + \sum_{j=0}^{a-2} E \{ L_J[x] L_{a-2-j}^o(x) \}. \]

Now let us derive recurrent relations for \( D ; \) to simplify the formulas, we accept denotation \( D^{(q)}(y, x) = E \{ L_J[y] L_2^o[x] \ldots L_q^o[x] \} \) with obvious agreement that the vectors \( A_i, Y_i \) that correspond to \( L_i \) are not necessarily the same for different \( i \).

Denoting \( T_{2 \ldots q} = L_2^o \ldots L_q^o \) and mimicking computations of section 3, and those presented above, we obtain that

\[ D^{(q)}(y, x) = v^2 \sum_{j=1}^{a_1-3} E \{ L_J[x] L_{a_1-2-j} T_{2 \ldots q}^o \} + \]

\[ \frac{v^2}{b^2} \sum_{s, t=1}^{N} \Psi_s(y) \Psi_t(x) \sum_{p=2}^{q} \sum_{l=1}^{p} \sum_{j=0}^{\alpha'_{p-1}} E \{ L_2^o \ldots L_{p-1}^o \left( H^{\alpha_{p-1}} \Psi \ldots \Psi H^{\alpha_{1} - 1 - j} \right)_{ts} \}
\times
\left( H^j \Psi \ldots \Psi H^{\alpha_{p-1}} \right)_{ys} L_{p+1}^o \ldots L_{q}^o \left( H^{\alpha_{1}-1} \Psi \ldots \Psi H^{\alpha_{1}} \right)_{syt}. \quad (4.10) \]

After factorization of the mathematical expectation, the first term of the right-hand side of (4.9) produces four terms

\[ E \{ L_J^o[x] L^o_2[x] \ldots L^o_q[x] \} E \{ L_{a_1-2-J}(y) \} + E \{ L_{a_1-2-j}^o(y) L_2^o[x] \ldots L_q^o[x] \} E \{ L_J[x] \} + \]

\[ E \{ L_{a_1-2-j}^o(y) L_2^o[x] \ldots L_q^o[x] \} - E \{ L_2^o[x] \ldots L_q^o[x] \} E \{ L_{a_1-2-j}(y) L_J^o[x] \}. \quad (4.11) \]

The estimates of corresponding sums \( v^2 \sum_{j=0}^{\alpha'_l-1} \) are the same as those obtained in section 3 with the only difference that we take \( \sup_{x,y} \) of the left-hand sides and replace in (3.8) and (3.9) the factor \( \Pi(A_q; N) \) by \( \Pi(A_q; b) \).

Let us consider the last term of (4.9). As one can see, it produces the terms of the forms \( \frac{1}{p^2} MD^{(q-2)} \) and \( \frac{1}{b^2} D^{(q-1)} \). Let us consider them in more details. For the first term we can write that

\[ \frac{v^2}{b^2} \sum_{p=2}^{q} \sum_{l=2}^{p} \sum_{j=0}^{\alpha'_{p-1}} E \{ L_2^o \ldots L_p^o \} E \{ L_{p+1} \ldots L_q^o \}. \]
$$\mathbb{E} \{ [H^j \Psi \cdots \Psi H^\alpha_{(p)} \Psi(x) H^\alpha_{(p)} \cdots \Psi H^\alpha_{(1)} - j \Psi(y) H^\alpha_{(1)} - 1 \cdots H^\alpha_{(1)} ]_{yy} \} \leq$$

$$\frac{v^2}{b^{q+1}} \sum_{p=2}^{q} \sum_{l=2}^{p} \sum_{j=0}^{\alpha_{(p)}-1} \frac{a_2 \cdots a_q}{a_p} \left( 1 + \frac{(|A_q| - a_1 - a_p)^3}{b^2} \right) \left( 1 + \frac{(a_1 + a_q - 2)^3}{b^2} \right)^{a_1 + a_q - 1} \times$$

$$m|A_q| - a_1 - a_p m^{-1} + a_q - 2 \leq \Pi(A_q; b) \frac{1}{a_1} m^{-2k}.$$  \hspace{1cm} (4.12)

Here we have used the fact that $\sum_{i=2}^{p} \sum_{j=0}^{\alpha_{(p)}-1} 1 = a_p - 1$. The remaining estimate is

$$\frac{v^2}{b^2} \sum_{p=2}^{q} \sum_{l=2}^{p} \sum_{j=0}^{\alpha_{(p)}-1} \mathbb{E} \{ L_2^\alpha \cdots L_{q-1}^\alpha [H^j \cdots H^\alpha_{(p)} \Psi(x) H^\alpha_{(p)} \cdots H^\alpha_{(1)} - j \Psi(y) H^\alpha_{(1)} - 1 \cdots H^\alpha_{(1)} ]_{yy} L_{p+1} \cdots L_q^\alpha \} \leq$$

$$\Pi(A_q; b) \sum_{p=2}^{q} \frac{a_1 + a_p - 2}{a_1} \left( 1 + \frac{(2k - 2)^3}{b^2} \right)^{-2} m^{-2k}.$$ \hspace{1cm} (4.13)

Now it is easy to gather the estimates of (4.11) with expressions (4.12) and (4.13) and obtain inequality

$$\sup_{x,y} |D^{(q)}(x,y)| \leq \Pi(A_q; b) \left[ \left( 1 - \frac{2}{a_1} \right) + \frac{a_1^2}{b^2} + \frac{2k - 2}{b^2} \right] m^{-2k}.$$  

To complete the proof of Lemma 4.1, it remains to check that inequalities (4.8) and (4.9) are true for $L_{R(a)}$ and $D^{(2)}$ (see (3.7)).

### 5 Auxiliary propositions

#### 5.1 Integration by parts formula for GUE

Let us consider random matrices

$$H_{ij} = \alpha_{ij} + \imath \beta_{ij},$$

where $\alpha$ and $\beta$ are real independent gaussian random variables with zero mean value and variances

$$\mathbb{E} \alpha_{ij}^2 = v^2/2, \quad \mathbb{E} \beta_{ij}^2 = v^2/2.$$ \hspace{1cm} (5.1)

We consider also the symmetric continuation $A$:

$$A_{ij} = \begin{cases} \alpha_{ij}, & \text{if } i \leq j, \\ \alpha_{ji}, & \text{if } i > j \end{cases}$$
and anti-symmetric continuation of $B$. Then $H = A + iB$ is hermitian matrix with the probability distribution (1.6). Integration by parts formula implies that

$$E H_{xy} G(H) = E A_{xy}^2 E \frac{\partial G(H)}{\partial A_{xy}} + i E B_{xy}^2 E \frac{\partial G(H)}{\partial B_{xy}}. \quad (5.2)$$

Let us consider $G(H) = (H^l)_{st}$. Then we can write that

$$\frac{\partial G}{\partial A_{xy}} = \lim_{\Delta \to 0} \frac{1}{\Delta} \sum_{j=1}^{l} \sum_{u,v=1}^{N} (H^{j-1})_{su} [\delta_{ux} \delta_{vy} \Delta + \delta_{uy} \delta_{vx} \Delta] (H^{l-j})_{vt} =$$

$$\sum_{j=1}^{l} \left \{ (H^{j-1})_{sx} (H^{l-j})_{yt} + (H^{j})_{sy} (H^{l-j})_{xt} \right \}.$$

Similarly we get

$$\frac{\partial G}{\partial B_{xy}} = 1 \sum_{j=1}^{l} \left \{ (H^{j-1})_{sx} (H^{l-j})_{yt} - (H^{j-1})_{sy} (H^{l-j})_{xt} \right \}.$$

Substituting these relations into (5.2) and remembering (5.1), we obtain that

$$E \left \{ H_{xy} (H^l)_{st} \right \} = E |H_{xy}|^2 E \left \{ \sum_{j=1}^{l} (H^{j-1})_{sy} (H^{l-j})_{xt} \right \}. \quad (5.3)$$

Regarding this formula, one can say that formally

$$E \left \{ H_{xy} (H^l)_{st} \right \} = E |H_{xy}|^2 E \left \{ \frac{\partial (H^l)_{st}}{\partial H_{yx}} \right \}.$$ 

That is the way that we use the partial derivatives in sections 3 and 4.

### 5.2 Even and odd moments

To prove equality (4.7), we consider the mathematical expectation of the product

$$E \left \{ H_{x_1} \cdots H_{s_0 \alpha_1} \psi_{t_0}^{(1)} H_{t_0 t_1} \cdots H_{t_0 \alpha_2} \psi_{u_0}^{(2)} \cdots H_{z_0 \cdots \alpha_r} \psi_{z_0 \cdots \alpha_r x} \right \} \quad (5.4)$$

for fixed values of $s_i, t_j, \ldots$ and odd sum of $\alpha_i$’s. It is clear that the number of factors $H$ is odd and since we have gaussian random variables, the mathematical expectation is equal to zero. The same reasoning shows that the left inequality of (4.8) is true.
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