Boundary S-matrix of the $O(N)$-symmetric Non-linear Sigma Model

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Abstract
We conjecture that the $O(N)$-symmetric non-linear sigma model in the semi-infinite $(1+1)$-dimensional space is “integrable” with respect to the “free” and the “fixed” boundary conditions. We then derive, for both cases, the boundary S-matrix for the reflection of massive particles of this model off the boundary at $x=0$.

1. Introduction

Recently, two-dimensional integrable field theory with a reflecting boundary has been studied [1,2,3,4,5]. An essential ingredient of an integrable field theory in the infinite space is the existence of an infinite number of mutually commuting integrals of motion. In the presence of the boundary, for arbitrary boundary conditions, these “charges” no longer remain conserved. However, sometimes, it is possible to modify these charges with special “integrable” boundary conditions so that the modified charges are indeed conserved. Then such a theory may be called a “two-dimensional integrable field theory with a boundary”.

An integrable “bulk” field theory enjoys the property that its multi-particle S-matrix amplitude factorizes into a product of an appropriate number of two-particle S-matrix amplitudes. The latter satisfy several constraints, namely, Yang-Baxter equation, unitarity

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and crossing symmetry \[6\]. These constraints enable one to compute the exact S-matrix up to the so-called “CDD”-factors. It has been known for quite some time how to generalise this factorizable structure of the S-matrix in the presence of a reflecting boundary \[1\]. In addition to the “bulk” two-particle S-matrices one needs to introduce specific “boundary reflection amplitudes” describing reflections of various particles in the theory when they fall on the boundary. The latter have to satisfy appropriate generalisations of the constraints of the bulk theory — the “boundary Yang-Baxter equation”, the “boundary unitarity condition” and the “boundary cross-unitarity condition”. The last of these was introduced in \[5\]. Thus, one can in a way similar to the bulk case, pin down the factorizable boundary S-matrix, again, up to the “CDD”-factors.

In this paper we study the boundary S-matrix of the \(O(N)\)-symmetric non-linear sigma model. In section-2, we briefly introduce the model and conjecture that the model is integrable in the semi-infinite space with the “free” and the “fixed” boundary conditions. In section-3, we study the fully \(O(N)\)-symmetric boundary scattering which is the case for the “free” boundary conditions. In section-4, we consider the boundary scattering with the “fixed” boundary condition which enjoys \(O(N-1)\)-symmetry. Concluding remarks appear in section-5.

2. \(O(N)\)-symmetric Nonlinear Sigma Model

Let us collect some known facts about the \(O(N)\)-symmetric nonlinear sigma model in infinite \((1 + 1)\)-dimensional space. It is described by the Lagrangian density and the constraint

\[
\mathcal{L} = \frac{1}{2g_0} \sum_{i=1}^{N} (\partial_{\mu} n_i)^2; \quad \sum_{i=1}^{N} n_i^2 = 1 \tag{2.1}
\]

where \(n_i = n_i(x, y)\) are \(N\) scalar fields with \(N = 3, 4, \ldots\) and \(g_0\) is a (bare) coupling constant. The model is \(O(N)\)-symmetric, renormalizable and asymptotically free \([7,8]\). It has \(N\) massive particles in the \(O(N)\)-multiplet. The exact S-matrix of this model has been found in \([3]\). The scattering involves no particle production and the \(n\)-particle S-matrix factorizes into a product of two particle S-matrices. So, it is sufficient to obtain the two-particle scattering amplitude. Let us parametrize the energy and momentum of a particle of mass \(m\) in terms of the rapidity variable \(\theta\) in the usual manner

\[
p^0 = m \cosh \theta \quad ; \quad p^1 = m \sinh \theta \tag{2.2}
\]
Then the following commutation relation between the formal particle creating operators $A_i(\theta)$ describes the two-particle scattering:

$$A_i(\theta)A_j(\theta') = \delta_{ij} \sigma_1(\theta - \theta') + \sigma_2(\theta - \theta')A_j(\theta')A_i(\theta) + \sigma_3(\theta - \theta')A_i(\theta')A_j(\theta)$$  

(2.3)

where

$$\sigma_1(\theta) = -\frac{i\lambda}{\pi - \theta} \sigma_2(\theta)$$  

(2.4)

$$\sigma_3(\theta) = -\frac{i\lambda}{\theta} \sigma_2(\theta)$$  

(2.5)

$$\sigma_2(\theta) = \frac{\Gamma\left(\frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - i\frac{\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \frac{\lambda}{2\pi} + i\frac{\theta}{2\pi}\right)\Gamma\left(1 + i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(-i\frac{\theta}{2\pi}\right)\Gamma\left(1 + \frac{\lambda}{2\pi} + i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} + i\frac{\theta}{2\pi}\right)}$$  

(2.6)

and

$$\lambda = \frac{2\pi}{N - 2}$$  

(2.7)

The $O(N)$-sigma model in the semi-infinite space with a boundary at $x = 0$ can be described by the following action

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx \, \mathcal{L} + \int_{-\infty}^{\infty} dy \, \mathcal{L}_B$$  

(2.8)

where $\mathcal{L}_B$, the boundary Lagrangian density, encodes the boundary conditions for the theory. In general $\mathcal{L}_B$ may spoil the integrability of the original field theory because under general boundary conditions one does not expect the integrals of motion of the original field theory to survive in the boundary field theory. However, under specific choices of $\mathcal{L}_B$ these integrals of motion, although modified, continue to remain so. In that case we have a “boundary integrable field theory” [5]. In this paper we consider two simple boundary conditions

1. Free Boundary Condition: $\mathcal{L}_B = 0$  

(2.9)

2. Fixed Boundary Condition: $n_i(x, y)|_{x=0} = n_i^{(0)}$; $n_i^{(0)} = (0 0 0 \ldots 1)$  

(2.10)

The fixed boundary condition can also be recovered as the $h \to \infty$ limit of

$$A_B = \int_{-\infty}^{\infty} dy \, h_i n_i$$  

$$h_i = h(0 0 0 \ldots 1)$$  

(2.11)
One has to examine the integrals of motion of the $O(N)$-sigma model in order to claim integrability of the boundary field theory with the “free” or the “fixed” boundary conditions. Here, we simply conjecture that both of these boundary conditions preserve integrability.

In addition to the bulk S-matrices we need to consider the boundary reflection amplitudes in the boundary theory. The amplitudes for reflections of $N$ species of particles off the boundary can be derived using the boundary bootstrap approach as discussed in [3,5].

The reflection of a particle of the $i$-th kind (Fig.1) is formally encoded in

$$A_i(\theta)B = R^i_j(\theta)A_j(-\theta)B$$  \hspace{1cm} (2.12)

where $A_i(\theta)$ and $B$ formally represent the operators that create the $i$-th particle and the boundary respectively. Using (2.12) repeatedly one can obtain the “boundary unitarity condition” (Fig.2)

$$R^i_j(\theta)R^k_j(-\theta) = \delta^k_i$$  \hspace{1cm} (2.13)

where $\delta^k_i$ is the well-known Kroneckar delta simbol. The “boundary cross-unitarity” equation was introduced in [5]. It reads

$$R^i_j(\theta) = S^{ij}_{kl}(2\theta)R^l_k(\theta)$$  \hspace{1cm} (2.14)

where $S^{ij}_{kl}(2\theta)$ is the two particle scattering amplitude for $i + j \rightarrow k + l$ scattering. In addition, the boundary scattering amplitudes must also satisfy the “boundary Yang-Baxter equation” (illustrated in Fig.3)

$$R^m_j(\theta')S^{mp}_{im}(\theta + \theta')R^k_n(\theta)S^{lk}_{pq}(\theta - \theta') = S^{mn}_{ij}(\theta - \theta')R^m_n(\theta)S^{lk}_{np}(\theta + \theta')R^l_q(\theta')$$  \hspace{1cm} (2.15)

3. Boundary S-matrix For The Free Boundary Condition

In this case the boundary S-matrix must enjoy full $O(N)$-symmetry and is therefore described by the following ansatz:

$$R^i_j(\theta) = \delta^i_j R(\theta)$$  \hspace{1cm} (3.1)

Then the boundary unitarity condition (2.13) and the boundary cross-unitarity condition (2.14) for this case assume the following forms respectively

$$R(\theta)R(-\theta) = 1$$  \hspace{1cm} (3.2)

$$R(i\frac{\pi}{2} - \theta) = \sigma(2\theta)R(i\frac{\pi}{2} + \theta)$$  \hspace{1cm} (3.3)
where

\[
\sigma(\theta) = N\sigma_1(\theta) + \sigma_2(\theta) + \sigma_3(\theta) \tag{3.4}
\]

or using (2.4), (2.5), and (2.6)

\[
\sigma(\theta) = -\frac{\Gamma(1 + \frac{\lambda}{2\pi} - i\frac{\theta}{2\pi})\Gamma(-\frac{1}{2} - i\frac{\theta}{2\pi})\Gamma(\frac{1}{2} + \frac{\lambda}{2\pi} + i\frac{\theta}{2\pi})\Gamma(1 + i\frac{\theta}{2\pi})}{\Gamma(1 + \frac{\lambda}{2\pi} + i\frac{\theta}{2\pi})\Gamma(-\frac{1}{2} + i\frac{\theta}{2\pi})\Gamma(\frac{1}{2} + \frac{\lambda}{2\pi} - i\frac{\theta}{2\pi})\Gamma(1 - i\frac{\theta}{2\pi})} \tag{3.5}
\]

The boundary Yang-Baxter equation (2.13) turns out to be an identity for this fully \(O(N)\)-symmetric case. So, (3.2) and (3.3) constitute the set of equations that enable us to pin down the boundary reflection amplitudes. The solution is

\[
R(\theta) = -\frac{\Gamma(\frac{1}{2} + \frac{\lambda}{4\pi} - i\frac{\theta}{2\pi})\Gamma(1 + i\frac{\theta}{2\pi})\Gamma(\frac{3}{4} + \frac{\lambda}{4\pi} + i\frac{\theta}{2\pi})\Gamma(1 + \frac{\lambda}{4\pi} + i\frac{\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{\lambda}{4\pi} + i\frac{\theta}{2\pi})\Gamma(-\frac{1}{4} - i\frac{\theta}{2\pi})\Gamma(\frac{3}{4} + \frac{\lambda}{4\pi} - i\frac{\theta}{2\pi})\Gamma(\frac{1}{4} + i\frac{\theta}{2\pi})} \tag{3.6}
\]

There are no poles in \(R(\theta)\) in the “physical strip” \(0 < \theta < \frac{i\pi}{2}\). This is consistent with the fact that the repulsive interaction in the bulk theory does not allow any bound states. Any such bound state of the bulk theory would contribute a pole in \(R(\theta)\) \[4\]. Additional poles might still occur in \(R(\theta)\) due to boundary bound states. Because we do not have any particular reason to expect such boundary bound states, we conjecture that (3.6), without any additional “CDD” factors describes the boundary scattering for the “free” boundary condition.

4. Boundary S-matrix For The Fixed Boundary Condition

For the fixed boundary condition we do not expect the boundary scattering to respect the full \(O(N)\)-symmetry. However, with the field on the boundary pointing in a certain direction we have a \(O(N-1)\)-symmetric boundary scattering. Consistent with this symmetry the ansatz for the boundary S-matrix is

\[
R^i_j(\theta) = \delta^i_j R_1(\theta);
\]

\[
R^N_N(\theta) = R_2(\theta)
\]

\[
R^i_j(\theta) = R^N_N(\theta) = 0; \quad i, j = 1, 2, 3 ..., (N-1) \tag{4.1}
\]
The Boundary Yang-Baxter equation \((2.15)\) for this case results in two independent equations:

\[
R_2(\theta')R_1(\theta)\sigma_1(\theta + \theta')[(N - 1)\sigma_1(\theta - \theta') + \sigma_2(\theta - \theta') + \sigma_3(\theta - \theta')] \\
+ R_2(\theta')R_2(\theta)\sigma_1(\theta - \theta')[\sigma_1(\theta + \theta') + \sigma_2(\theta + \theta') + \sigma_3(\theta + \theta')] \\
= R_1(\theta')R_1(\theta)\sigma_1(\theta - \theta')[(N - 1)\sigma_1(\theta + \theta') + \sigma_2(\theta + \theta') + \sigma_3(\theta + \theta')] \\
+ R_1(\theta')R_2(\theta)\sigma_1(\theta + \theta')[\sigma_1(\theta - \theta') + \sigma_2(\theta - \theta') + \sigma_3(\theta - \theta')] \tag{4.2}
\]

and

\[
R_2(\theta')R_1(\theta)\sigma_2(\theta + \theta')\sigma_3(\theta - \theta') + R_2(\theta')R_2(\theta)\sigma_2(\theta - \theta')\sigma_3(\theta + \theta') \\
= R_1(\theta')R_2(\theta)\sigma_2(\theta + \theta')\sigma_3(\theta - \theta') + R_1(\theta')R_1(\theta)\sigma_2(\theta - \theta')\sigma_3(\theta + \theta') \tag{4.3}
\]

Solving these equations we get

\[
\frac{R_1(\theta)}{R_2(\theta)} = \frac{i\pi + 2\theta}{i\pi + 2\theta} \tag{4.4}
\]

The \((+)\) sign in the numerator implies \(R_1(\theta) = R_2(\theta)\) which is the case of full \(O(N)\)-symmetry discussed in the previous section. The \((-)\) sign distinguishes between \(R_1(\theta)\) and \(R_2(\theta)\) and therefore gives the solution for the present case. So, we can now write

\[
\frac{R_1(\theta)}{R_2(\theta)} = \frac{i\pi - 2\theta}{i\pi + 2\theta} \tag{4.5}
\]

The boundary unitarity equation \((2.13)\) in the present case translates into

\[
R_1(\theta)R_1(-\theta) = 1 ; \quad R_2(\theta)R_2(-\theta) = 1 \tag{4.6}
\]

while the boundary cross-unitarity equation \((2.14)\) becomes

\[
R_1(i\frac{\pi}{2} - \theta) = R_1(i\frac{\pi}{2} + \theta)[(N - 1)\sigma_1(2\theta) + \sigma_2(2\theta) + \sigma_3(2\theta)] + R_2(i\frac{\pi}{2} + \theta)\sigma_1(2\theta) \tag{4.7}
\]

\[
R_2(i\frac{\pi}{2} - \theta) = R_2(i\frac{\pi}{2} + \theta)[\sigma_1(2\theta) + \sigma_2(2\theta) + \sigma_3(2\theta)] + R_1(i\frac{\pi}{2} + \theta)(N - 1)\sigma_1(2\theta) \tag{4.8}
\]

Using \((4.5),(2.4),(2.6)\) and \((2.3)\) we see that \((4.7)\) and \((4.8)\) both reduce to the same following equation, indicating the consistency of the boundary bootstrap approach:

\[
R_1(i\frac{\pi}{2} - \theta) = -\frac{i\lambda - \theta}{i\lambda + \theta}\sigma(2\theta)R_1(i\frac{\pi}{2} + \theta) \tag{4.9}
\]
where $\sigma(\theta)$ is the same as in (3.5). We now have to solve (4.6) and (4.9) simultaneously to obtain $R_1(\theta)$. The solution reads

$$R_1(\theta) = \frac{\Gamma\left(\frac{1}{2} + \frac{\lambda}{4\pi} - i \frac{\theta}{2\pi}\right)\Gamma\left(1 - i \frac{\theta}{2\pi}\right)\Gamma\left(\frac{3}{4} - i \frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{4\pi} + i \frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{4} + \frac{\lambda}{4\pi} + i \frac{\theta}{2\pi}\right)\Gamma\left(\frac{3}{4} + i \frac{\theta}{2\pi}\right)}$$

(4.10)

Using (4.5) we have for $R_2(\theta)$ then

$$R_2(\theta) = \frac{\Gamma\left(\frac{1}{2} + \frac{\lambda}{4\pi} - i \frac{\theta}{2\pi}\right)\Gamma\left(1 + i \frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{4} + \frac{\lambda}{4\pi} + i \frac{\theta}{2\pi}\right)\Gamma\left(-\frac{1}{4} - i \frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{4\pi} + i \frac{\theta}{2\pi}\right)\Gamma\left(1 - i \frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{4} + \frac{\lambda}{4\pi} - i \frac{\theta}{2\pi}\right)\Gamma\left(-\frac{1}{4} + i \frac{\theta}{2\pi}\right)}$$

(4.11)

There are no poles in $R_1(\theta)$ in the “physical strip” $0 < \theta < \frac{\pi}{2}$. For $R_2(\theta)$ the same is true except for the existence of a pole at $\theta = \frac{\pi}{2}$. This pole signals the presence of the 1-particle contribution in the corresponding boundary state $|A_N\rangle$. Clearly, the $O(N - 1)$-symmetry of the “fixed” boundary condition allows the contribution of the state $|A_N\rangle$ in this boundary state.

5. Conclusion

In this paper we have considered the boundary scattering with the “free” and the “fixed” boundary conditions for the $O(N)$-symmetric non-linear sigma model and conjectured the boundary S-matrix in each case. Of course, our conjecture that for both of these boundary conditions the boundary field theory remains integrable will have to be supported by explicit demonstration that the integrals of motion for this theory do indeed survive in the boundary theory.

Also it would be interesting to analyse, in general, what boundary conditions for the $O(N)$-sigma model are consistent with integrability.

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\[ j \theta \quad = \quad R_i^j (\theta) \]

**FIG 1**

\[ i \theta \quad \]

\[ = \quad j_\theta \]

**FIG 2**

\[ i \theta \quad \]

\[ = \quad j_{-\theta} \]

**FIG 3**