ON SYSTEMS OF POLYNOMIALS WITH AT LEAST ONE POSITIVE REAL ZERO

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ABSTRACT. In this paper, we prove several theorems on systems of polynomial equations with at least one positive real zero, which can be viewed as a generalization of Birch's theorem. Moreover, we give a class of polynomials attaining their minimums, which is useful in polynomial optimization.

1. INTRODUCTION

An important problem in real algebraic geometry is bounding the numbers of positive real zeros of polynomials or polynomial systems, which has applications in many fields, such as polynomial optimization, algebraic statistics, chemical reaction networks and so on. In the univariate case, the well-known Descartes’ rule gives upper bounds of positive real zeros.

Descartes’ rule

Given a univariate real polynomial $f(x)$ such that the terms of $f(x)$ are ordered by descending variable exponent, the number of positive real roots of $f$ (counted with multiplicity) is bounded from above by the number of sign variations between consecutive nonzero coefficients. Additionally, the difference between these two numbers (the number of positive real roots and the number of sign variations) is even.

However, no complete multivariate generalization of Descartes’ rule is known, except for a conjecture proposed by Itenberg and Roy in 1996 ([5]) and subsequently disproven by T.Y. Li in 1998 ([8]). In [6], a special case for polynomial systems with at most one positive real zero was considered through the theory of oriented matroids. Based on this method, a partially multivariate generalization of Descartes’ rule for polynomial systems supported on circuits can be found in [2, 3].

While lower bounds guarantee the existence of positive real zeros, there are few results on lower bounds for general polynomial systems in the literature. In this paper, by virtue of the connection with nonnegative polynomials, we prove several theorems on systems of polynomial equations with at least one positive real zero, which can be viewed as a generalization of Birch’s theorem (Remark 4.11). Moreover, we give a class of polynomials attaining their minimums, since in polynomial optimization, the existence of minimizers of objective polynomials is often formulated as an assumption for some of the algorithmic approaches ([11, 9]).
2. Preliminaries

2.1. Nonnegative polynomials. Let \( \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n] \) be the ring of real \( n \)-variate polynomial, and \( \mathbb{N}^* = \mathbb{N}\setminus\{0\} \). Let \( \mathbb{R}_+ \) be the set of positive real numbers. For a finite set \( \mathcal{A} \subseteq \mathbb{N}^n \), we denote by \( \text{conv}(\mathcal{A}) \) the convex hull of \( \mathcal{A} \), and by \( V(\mathcal{A}) \) the vertices of the convex hull of \( \mathcal{A} \). Also we denote by \( V(P) \) the vertex set of a polytope \( P \). For a polynomial \( f \in \mathbb{R}[x] \) of the form \( f(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha \) with \( c_\alpha \in \mathbb{R}, x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \) the support of \( f \) is \( \text{supp}(f) := \{ \alpha \in \mathcal{A} \mid c_\alpha \neq 0 \} \) and the Newton polytope of \( f \) is defined as \( \text{New}(f) = \text{conv}(\text{supp}(f)) \). For a polytope \( P \), we use \( P^0 \) to denote the interior of \( P \).

A polynomial \( f \in \mathbb{R}[x] \) which is nonnegative over \( \mathbb{R}^n \) is called a nonnegative polynomial. A nonnegative polynomial must satisfy the following necessary conditions.

**Proposition 2.1.** ([10] Theorem 3.6) Let \( \mathcal{A} \subseteq \mathbb{N}^n \) and \( f = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha \in \mathbb{R}[x] \) with \( \text{supp}(f) = \mathcal{A} \). Then \( f \) is nonnegative only if the following hold:

1. \( V(\mathcal{A}) \subseteq (2\mathbb{N})^n \);
2. If \( \alpha \in V(\mathcal{A}) \), then the corresponding coefficient \( c_\alpha \) is positive.

2.2. Coercive polynomials. A polynomial \( f \in \mathbb{R}[x] \) is called a coercive polynomial, if \( \lim_{\|x\| \to +\infty} f(x) = +\infty \) holds whenever \( \|x\| \to +\infty \), where \( \|\cdot\| \) denotes some norm on \( \mathbb{R}^n \). Obviously the coercivity of \( f \) implies the existence of minimizers of \( f \) over \( \mathbb{R}^n \). Necessary conditions ([1 Theorem 2.8]) and sufficient conditions ([1 Theorem 3.4]) for a polynomial to be coercive were given in [1].

**Theorem 2.2.** ([1] Theorem 2.8) Let \( f = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha \in \mathbb{R}[x] \) with \( \text{supp}(f) = \mathcal{A} \) be a coercive polynomial and \( c_0 > 0 \). Then the following three conditions hold:

1. \( V(\mathcal{A}) \subseteq (2\mathbb{N})^n \);
2. If \( \alpha \in V(\mathcal{A}) \), then the corresponding coefficient \( c_\alpha \) is positive;
3. For every \( i, 1 \leq i \leq n \), there exists a vector \( 2k_i e_i \in V(\mathcal{A}) \) with \( k_i \in \mathbb{N}^* \), where \( e_i \) is the standard basis vector.

2.3. Circuit polynomials. A subset \( \mathcal{A} \subseteq (2\mathbb{N})^n \) is called a trellis if \( \mathcal{A} \) comprises the vertices of a simplex.

**Definition 2.3.** Let \( \mathcal{A} \) be a trellis and \( f \in \mathbb{R}[x] \). Then \( f \) is called a circuit polynomial if it is of the form

\[
(2.1) \quad f(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - dx^\beta,
\]

with \( c_\alpha > 0 \) and \( \beta \in \text{conv}(\mathcal{A})^0 \). Assume

\[
(2.2) \quad \beta = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \alpha \quad \text{with} \quad \lambda_\alpha > 0 \quad \text{and} \quad \sum_{\alpha \in \mathcal{A}} \lambda_\alpha = 1.
\]

For every circuit polynomial \( f \), we define the corresponding circuit number as \( \Theta_f := \prod_{\alpha \in \mathcal{A}} (c_\alpha/\lambda_\alpha)^{\lambda_\alpha} \).

The nonnegativity of a circuit polynomial \( f \) is decided by its circuit number alone.

**Theorem 2.4.** ([1] Theorem 3.8) Let \( f = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - dx^\beta \in \mathbb{R}[x] \) be a circuit polynomial and \( \Theta_f \) its circuit number. Then \( f \) is nonnegative if and only if \( \beta \notin (2\mathbb{N})^n \) and \( |d| \leq \Theta_f \), or \( \beta \in (2\mathbb{N})^n \) and \( d \leq \Theta_f \).
3. Polynomials attaining minimums

In this section, we prove some lemmas and a theorem on the existence of global minimizers in $\mathbb{R}_+^n$ of polynomials.

Let $\mathbb{R}[x^\pm]$ denote the Laurent polynomial ring and $g(x) = \sum c_\alpha x^\alpha \in \mathbb{R}[x^\pm]$. For an invertible matrix $T \in \text{GL}_n(\mathbb{Q})$, the polynomial obtained by applying $T$ to the exponent vectors of $g$ is denoted by $g^T = \sum c_\alpha x^{T\alpha}$.

Let $\Delta$ be a polytope of dimension $d$. For a vertex $\alpha$ of $\Delta$, if $\alpha$ is the intersection of precisely $d$ edges, then we say $\Delta$ is simple at $\alpha$. Obviously, a polygon is simple at any vertex.

**Lemma 3.1.** Suppose $f = c_0 + \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta \in \mathbb{R}[x^\pm]$ such that $\dim(\text{New}(f)) = n$, $\mathcal{B} \subseteq \text{New}(f)^\circ$, $0 \in V(\text{New}(f))$ and $\text{New}(f)$ is simple at $0$. Then there exists $\mathcal{A}_0 = \{\alpha_1, \ldots, \alpha_n\} \subseteq V(\text{New}(f))$ and $T \in \text{GL}_n(\mathbb{Q})$ such that $f^T = c_0 + \sum_{i=1}^n c_{\alpha_i} x_i^2 k_i + \sum_{\alpha \in \mathcal{A}' \setminus \mathcal{A}_0} c_\alpha x^{\alpha T \alpha} - \sum_{\beta \in \mathcal{B}} d_\beta x^{T \beta}$, where $k_i \in \mathbb{N}^*$, $T \alpha \in (2\mathbb{N})^n$ for each $\alpha \in \mathcal{A}' \setminus \mathcal{A}_0$ and $T \beta \in \text{New}(f^T)^\circ \cap \mathbb{N}^n$ for each $\beta \in \mathcal{B}$.

**Proof.** Since $\text{New}(f)$ is simple at $0$, $0$ is the intersection of precisely $n$ edges. Let $\mathcal{A}_0 = \{\alpha_1, \ldots, \alpha_n\} \subseteq V(\text{New}(f))$ be the other extreme points of these $n$ edges. Let $T' \in \text{GL}_n(\mathbb{Q})$ such that $T'(\alpha_1, \ldots, \alpha_n) = \text{diag}(k_1', \ldots, k_n')$, where $k_i' \in \mathbb{N}^*$. Suppose $\mu \in \mathbb{N}^*$ is the least common multiple of the denominators of the coordinates of $T \alpha$ and $T \beta$ for $\alpha \in \mathcal{A}' \setminus \mathcal{A}_0$ and $\beta \in \mathcal{B}$. Let $T = 2\mu T'$. Then $T \alpha \in (2\mathbb{N})^n$ for each $\alpha \in \mathcal{A}' \setminus \mathcal{A}_0$ and $T \beta \in \mathbb{Z}^n$ for each $\beta \in \mathcal{B}$. Moreover, since $T$ keeps convex combinations, we have $T \alpha \in (2\mathbb{N})^n$ and $T \beta \in \text{New}(f^T)^\circ \cap \mathbb{N}^n$. Thus $T$ meets the requirement with $k_i = 2\mu k_i'$, $i = 1, \ldots, n$. □

Consider the bijective componentwise exponential map

$$\exp : \mathbb{R}^n \to \mathbb{R}^n_+, \ x = (x_1, \ldots, x_n) \mapsto e^x = (e^{x_1}, \ldots, e^{x_n}).$$

The image of $g(x) = \sum c_\alpha x^\alpha$ under the map $\exp$ is $g(e^x) = \sum c_\alpha e^{\langle \alpha, x \rangle}$, where $\langle \alpha, x \rangle = \alpha^T x$ is the inner product of $\alpha$ and $x$. Obviously, the range of $g(x)$ over $\mathbb{R}^n_+$ is same as the range of $g(e^x)$ over $\mathbb{R}^n$.

**Lemma 3.2.** Let $g(x) = \sum c_\alpha x^\alpha \in \mathbb{R}[x^\pm]$ and $T \in \text{GL}_n(\mathbb{Q})$ such that $g^T(x) \in \mathbb{R}[x^\pm]$. Then the infimums of $g(x)$ and $g^T(x)$ over $\mathbb{R}^n_+$ are the same. The minimizers (and the zeros) of $g(x)$ and $g^T(x)$ over $\mathbb{R}^n_+$ are in a one-to-one correspondence.

**Proof.** We only need to show that the same conclusions hold for $g(e^x)$ and $g^T(e^x)$ over $\mathbb{R}^n$, which follow from the equalities $\exp(g(x)) = \sum c_\alpha e^{\langle \alpha, x \rangle} = \sum c_\alpha e^{\langle T \alpha, T^T x \rangle} = g^T(e^{T^T x})$ and $\exp(g^T(x)) = \sum c_\alpha e^{\langle T \alpha, x \rangle} = \sum c_\alpha e^{\langle \alpha, T^T x \rangle} = g(e^{T^T x})$, where $T^* = (T^{-1})^T = (T^T)^{-1}$. □

**Lemma 3.3.** Suppose $f = c_0 + \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta \in \mathbb{R}[x^\pm]$, $c_0, c_\alpha, d_\beta > 0$ such that $\dim(\text{New}(f)) = n$, $\mathcal{B} \subseteq \text{New}(f)^\circ$, $0 \in V(\text{New}(f))$ and $\text{New}(f)$ is simple at $0$. Assume $\sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta$ is not nonnegative over $\mathbb{R}^n_+$. Then $f$ has a minimizer over $\mathbb{R}_+^n$.

**Proof.** By Lemma 3.1, there exists $\mathcal{A}_0 = \{\alpha_1, \ldots, \alpha_n\} \subseteq V(\text{New}(f))$ and $T \in \text{GL}_n(\mathbb{Q})$ such that $f^T = c_0 + \sum_{i=1}^n c_{\alpha_i} x_i^{2k_i} + \sum_{\alpha \in \mathcal{A}' \setminus \mathcal{A}_0} c_\alpha x^{\alpha T \alpha} - \sum_{\beta \in \mathcal{B}} d_\beta x^{T \beta} \in \mathbb{R}[x]$, where $T \alpha \in (2\mathbb{N})^n$ for each $\alpha \in \mathcal{A}' \setminus \mathcal{A}_0$ and $T \beta \in \text{New}(f^T)^\circ$ for each $\beta \in \mathcal{B}$. By Theorem 3.4 in [1], $f^T$ is a coercive polynomial, and hence has a global minimizer over $\mathbb{R}_+^n$. Note that $f^T(|x|) = c_0 + \sum_{i=1}^n c_{\alpha_i} |x_i|^{2k_i} + \sum_{\alpha \in \mathcal{A}' \setminus \mathcal{A}_0} c_\alpha |x|^{T \alpha} - \sum_{\beta \in \mathcal{B}} d_\beta |x|^{T \beta} \geq \sum_{\alpha \in \mathcal{A}' \setminus \mathcal{A}_0} c_\alpha |x|^{T \alpha} - \sum_{\beta \in \mathcal{B}} d_\beta |x|^{T \beta}$.
\[ \sum_{\beta \in \mathcal{B}} d_\beta |x|^{\beta} \leq f^T(x), \text{ where } |x| = (|x_1|, \ldots, |x_n|). \] So \( f^T \) has a global minimizer \( x^* \) in \( \mathbb{R}^n_{\geq 0} \). Since \( f - c_0 \) is not nonnegative over \( \mathbb{R}^n_+ \), by Lemma 3.2, \( f^T - c_0 \) is not nonnegative over \( \mathbb{R}^n_+ \). It follows that the global minimum of \( f^T \) is lower than \( c_0 \), and since for \( x \in \mathbb{R}^n_{\geq 0} \setminus \mathbb{R}^n_+ \), \( f^T(x) \geq c_0 \), we have \( x^* \in \mathbb{R}^n_+ \). Thus \( f^T \) has a minimizer over \( \mathbb{R}^n_+ \) and so does \( f \) by Lemma 3.2.

**Theorem 3.4.** Suppose \( f = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta \in \mathbb{R}[x] \), \( c_\alpha, d_\beta > 0 \) such that \( \dim(\text{New}(f)) = n \), \( \mathcal{A} \subseteq (2\mathbb{N})^n \), \( \mathcal{B} \subseteq \text{New}(f)^\circ \). Assume that \( \text{conv}(\mathcal{A} \cup \{0\}) \) is simple at 0. If 0 is not a global minimizer of \( f \), then \( f \) has a global minimizer in \( \mathbb{R}^n_+ \).

**Proof.** Since \( f(|x|) = \sum_{\alpha \in \mathcal{A}} c_\alpha |x|^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta |x|^\beta \leq f(x) \), we only need to search the global minimizer of \( f \) in \( \mathbb{R}^n_{\geq 0} \), or equivalently in \( \{0\} \cup \mathbb{R}^n_+ \). If \( 0 \in \mathcal{A} \) and \( f - c_0 \) is nonnegative, then 0 is a global minimizer of \( f \). If \( 0 \in \mathcal{A} \) and \( f - c_0 \) is not nonnegative, then by Lemma 3.3, \( f \) has a minimizer over \( \mathbb{R}^n_+ \), which is also a global minimizer. If \( 0 \notin \mathcal{A} \) and \( f \) is nonnegative, then 0 is a global minimizer of \( f \). If \( 0 \notin \mathcal{A} \) and \( f \) is not nonnegative, consider the polynomial \( f + c, c > 0 \). By Lemma 3.3, \( f + c \) has a minimizer over \( \mathbb{R}^n_+ \). It follows \( f \) has a minimizer over \( \mathbb{R}^n_+ \), which is also a global minimizer.

4. Systems of Polynomial Equations with One Positive Real Zero

A positive real zero of a polynomial or a system of polynomial equations is a real zero with positive coordinates. Note that the positive real zeros of the polynomials \( f(x_1, \ldots, x_n) \) and \( f(x_1^2, \ldots, x_n^2) \) are in a one-to-one correspondence. Since we only consider positive real zeros in this paper, we can apply the map \( x_i \mapsto x_i^2 \) and assume that the supports of polynomials are in \((2\mathbb{N})^n \) if necessary.

**Proposition 4.1.** Let \( F \) be the following system of polynomial equations

\[
(4.1) \quad \sum_{\alpha \in \mathcal{A}} c_\alpha (\alpha - \gamma) x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta (\beta - \gamma) x^\beta = 0,
\]

where \( c_\alpha, d_\beta > 0 \) and \( \gamma \in V(\Delta) \), \( \mathcal{B} \subseteq \Delta^\circ \) with \( \Delta = \text{conv}(\mathcal{A} \cup \{\gamma\}) \). Assume that \( \dim(\Delta) = n \), \( \Delta \) is simple at \( \gamma \) and \( \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta \) is not nonnegative over \( \mathbb{R}^n_+ \). Then \( F \) has at least one positive real zero.

**Proof.** Consider the polynomial \( f = dx^\gamma + \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta - \gamma \). Let \( f' = f/\gamma^\gamma = d + \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta - \gamma \). Then by Lemma 3.3, \( f' \) has a minimizer over \( \mathbb{R}^n_+ \). Assume the minimum of \( f' \) over \( \mathbb{R}^n_+ \) is \( \xi \). Then \( f'(x) - \xi \) is nonnegative over \( \mathbb{R}^n_+ \) and has a positive real zero. It follows that \( f - \xi x^\gamma = (d - \xi) x^\gamma + \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta \) is nonnegative over \( \mathbb{R}^n_+ \) and has a positive real zero, which implies that the system of \( f - \xi x^\gamma = 0 \) and \( \nabla(f - \xi x^\gamma) = 0 \) has a positive real zero. Multiplying \( f - \xi x^\gamma = 0 \) by \( \gamma \) gives

\[
(4.2) \quad (d - \xi) \gamma x^\gamma + \sum_{\alpha \in \mathcal{A}} c_\alpha \gamma x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta \gamma x^\beta = 0.
\]

Multiplying the \( i \)-th equation of \( \nabla(f - \xi x^\gamma) = 0 \) by \( x_i \) gives

\[
(4.3) \quad (d - \xi) \gamma x_i^\gamma + \sum_{\alpha \in \mathcal{A}} c_\alpha x_i^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x_i^\beta = 0.
\]
From (4.3)–(4.2), we obtain

\[ \sum_{\alpha \in A} c_{\alpha} (\alpha - \gamma) x^\alpha - \sum_{\beta \in \Delta} d_{\beta} (\beta - \gamma) x^\beta = 0, \]

which is \( F \). Thus \( F \) has a positive real zero.

\[ \square \]

**Example 4.2.** The following system of polynomial equations satisfies the conditions of Theorem 4.1 with \( \gamma = (8, 8) \), and hence has a positive real zero.

\[ \begin{align*}
-8y^8 - 4x^4 y^4 - 8 + 18x^2 y + 5x^3 y^2 &= 0 \\
-8x^8 - 4x^4 y^4 - 8 + 21x^2 y + 6x^3 y^2 &= 0
\end{align*} \]

**Lemma 4.3.** Suppose \( f_d = \sum_{\alpha \in A} c_{\alpha} x^\alpha + d x^\gamma - \sum_{\beta \in \Delta} d_{\beta} x^\beta \in \mathbb{R}[x] \), \( c_{\alpha}, d_{\beta} > 0 \), \( d \geq 0 \) such that \( \mathcal{B} \subseteq \Delta^o \) with \( \Delta = \text{conv}(\mathcal{A} \cup \{\gamma\}) \). Assume that \( \dim(\Delta) = n \), \( \Delta \) is simple at some vertex \( \alpha_0 \) \( (\alpha_0 \neq \gamma) \) and \( \sum_{\alpha \in A} c_{\alpha} x^\alpha - \sum_{\beta \in \Delta} d_{\beta} x^\beta \) is not nonnegative over \( \mathbb{R}^n_+ \). Let \( d^* = \inf \{d \mid f_d \text{ is nonnegative over } \mathbb{R}^n_+ \} \). Then \( f_{d^*} \) has a positive real zero.

**Proof.** Let \( |\mathcal{B}| = l \). For each \( \beta \in \mathcal{B} \), since \( \beta \subseteq \Delta^o \), then there must exist a subset \( A_\beta \) of \( \mathcal{A} \) such that \( A_\beta \cup \{\gamma\} \) comprises the vertices of a simplex \( \Delta_\beta \) containing \( \beta \) as an interior point. For each \( \alpha \in \cup_{\beta \in \mathcal{B}} A_\beta \), count how many simplices contain \( \alpha \) and even distribute \( c_{\alpha} \). Then we can write

\[ f_d = \sum_{\beta \in \mathcal{B}} \left( \sum_{\alpha \in A_\beta} c_{\beta} x^\alpha + \frac{d}{l} x^\gamma - d_{\beta} x^\beta \right) + \sum_{\alpha \notin \cup_{\beta \in \mathcal{B}} A_\beta} c_{\alpha} x^\alpha \]

as a sum of circuit polynomials. Observe that if \( d \) is large enough, then every circuit polynomial in (4.5) is nonnegative by Theorem 2.3 and hence \( f \) is nonnegative. So the set in the definition of \( d^* \) is nonempty and obviously has lower bounds. It follows \( d^* \) exists.

Let \( f_d' = f_d(x^\alpha) = c_{\alpha_0} + \sum_{\alpha \in \mathcal{A} \setminus \{\alpha_0\}} c_{\alpha} x^{\alpha - \alpha_0} + d x^\gamma - \sum_{\beta \in \Delta} d_{\beta} x^{\beta - \alpha_0} \). By Lemma 3.3 there exists \( \mathcal{A}_0 = \{\alpha_1, \ldots, \alpha_n\} \subseteq V(\mathcal{A} \setminus \{\alpha_0\}) \) and \( T \in \text{GL}_n(\mathbb{Q}) \) such that \( f_d'' = c_{\alpha_0} + \sum_{i=1}^n c_{\alpha_i} x^{\alpha_i} + \sum_{\alpha \in \mathcal{A} \setminus \{\alpha_0\}} c_{\alpha} x^{T(\alpha - \alpha_0)} + d x^T(\gamma - \alpha_0) - \sum_{\beta \in \Delta} d_{\beta} x^{T(\beta - \alpha_0)} \in \mathbb{R}[x] \), where \( T(\alpha - \alpha_0), T(\gamma - \alpha_0) \in (2\mathbb{N})^n \), \( T(\beta - \alpha_0) \in \text{New}(f_d'') \) (we assume \( \gamma \notin \mathcal{A}_0 \) for simplicity). By Lemma 3.2 the nonnegativity of \( f_d'' \) over \( \mathbb{R}^n_+ \) is the same as the nonnegativity of \( f_d' \) over \( \mathbb{R}^n_+ \), and hence is the same as the nonnegativity of \( f_d \) over \( \mathbb{R}^n_+ \). Let \( d < d^* \) and by the definition of \( d^* \), \( f_d \) and hence \( f_d'' \) is not nonnegative over \( \mathbb{R}^n_+ \). That is to say, there exists \( x_d \in \mathbb{R}^n_+ \) such that \( f_d''(x_d) < 0 \). On the other hand, by Theorem 3.4 in [1], \( f_d'' \) is a coercive polynomial. Hence there exists \( N_d > 0 \) such that for \( \|x\|_\infty > N_d \), \( f_d''(x) > 0 \). It follows \( \|x_d\|_\infty \leq N_d \). Let \( \varepsilon > d \) and \( d' \leq d^* \). Since \( f_d''(x) - f_d''(x) = (d' - d) x^T(\gamma - \alpha_0) > 0 \) over \( \mathbb{R}^n_+ \), we have \( f_d''(x) > f_d''(x) \) over \( \mathbb{R}^n_+ \). Thus for \( \|x\|_\infty > N_d \) and \( x \in \mathbb{R}^n_+ \), \( f_d''(x) > 0 \). It follows \( \|x_d\|_\infty \leq N_d \). Let \( d \to d^* \). Then we have \( f_d''(x_d) - f_d''(x_d) = (d - d^*) x^T(\gamma - \alpha_0) \to 0 \). Since \( f_d''(x_d) \geq 0 \) and \( f_d''(x_d) < 0 \), we must have \( f_d''(x_d) \to 0 \). Thus the infimum of \( f_d'' \) over \( \mathbb{R}^n_+ \) is 0. It follows that \( f_d'' \) is not nonnegative over \( \mathbb{R}^n_+ \). So by Lemma 3.1, \( f_d'' \) is a minimizer over \( \mathbb{R}^n_+ \), which is also a zero. As a consequence, \( f_d' \) has a positive real zero by Lemma 3.2 and so does \( f_d'' \). \[ \square \]
Lemma 4.7. Suppose $R$ is simple at some vertex and $\alpha \in A$. Let $\Delta = \text{conv}(A \cup \{\gamma\})$. Assume that $\dim(\Delta) = n$, $\Delta$ is simple at some vertex $\alpha_0$ ($\alpha_0 \neq \gamma$) and $\sum_{\alpha \in A} c_\alpha x^\alpha - \sum_{\beta \in B} d_\beta x^\beta$ is not nonnegative over $\mathbb{R}^n_+$. Then $F$ has at least one positive real zero.

Proof. Consider the polynomial $f_d = \sum_{\alpha \in A} c_\alpha x^\alpha + dx^\gamma - \sum_{\beta \in B} d_\beta x^\beta$. Define $d^*$ as in Lemma 4.3. Then by Lemma 4.3, $f_{d^*}$ has a positive real zero as a minimizer, which implies that the system of $f_{d^*} = 0$ and $\nabla(f_{d^*}) = 0$ has a positive real zero. Multiplying $f_{d^*} = 0$ by $\gamma$ gives

$$
\sum_{\alpha \in A} c_\alpha \gamma x^\alpha + d^* \gamma x^\gamma - \sum_{\beta \in B} d_\beta \gamma x^\beta = 0.
$$

Multiplying the $i$-th equation of $\nabla(f_{d^*}) = 0$ by $x_i$ gives

$$
\sum_{\alpha \in A} c_\alpha x^\alpha + d^* \gamma x^\gamma - \sum_{\beta \in B} d_\beta x^\beta = 0.
$$

From (4.8)–(4.7), we obtain

$$
\sum_{\alpha \in A} c_\alpha (\alpha - \gamma) x^\alpha - \sum_{\beta \in B} d_\beta (\beta - \gamma) x^\beta = 0,
$$

which is $F$. Thus $F$ has a positive real zero. \hfill \Box

Example 4.5. The following system of polynomial equations satisfies the conditions of Theorem 4.4 with $\gamma = (4, 4)$, and hence has a positive real zero.

$$
\begin{align*}
4x^8y^8 + 4x^8 - 4y^8 - 4 + 6x^2y + x^3y^2 &= 0, \\
4x^8y^8 - 4x^8 + 4y^8 - 4 + 9x^2y + 2x^3y^2 &= 0.
\end{align*}
$$

Combine Proposition 4.1 and Proposition 4.4, we have

Theorem 4.6. Let $F$ be the following system of polynomial equations

$$
\sum_{\alpha \in A} c_\alpha (\alpha - \gamma) x^\alpha - \sum_{\beta \in B} d_\beta (\beta - \gamma) x^\beta = 0,
$$

where $c_\alpha, d_\beta > 0$ and $B \subseteq \Delta^\circ$ with $\Delta = \text{conv}(A \cup \{\gamma\})$. Assume that $\dim(\Delta) = n$, $\Delta$ is simple at some vertex and $\sum_{\alpha \in A} c_\alpha x^\alpha - \sum_{\beta \in B} d_\beta x^\beta$ is not nonnegative over $\mathbb{R}^n_+$. Then $F$ has at least one positive real zero.

Lemma 4.7. Suppose $f_d = \sum_{\alpha \in A} c_\alpha x^\alpha - \sum_{\beta \in B} d_\beta x^\beta - dx^\gamma \in \mathbb{R}[x]$, $c_\alpha, d_\beta > 0$, $d \geq 0$ such that $B \cup \{\gamma\} \subseteq \Delta^\circ$ with $\Delta = \text{conv}(A)$. Assume that $\dim(\Delta) = n$, $\Delta$ is simple at some vertex $\alpha_0$ and $\sum_{\alpha \in A} c_\alpha x^\alpha - \sum_{\beta \in B} d_\beta x^\beta$ is nonnegative over $\mathbb{R}^n_+$. Let $d^* = \sup\{d \mid f_d \text{ is nonnegative over } \mathbb{R}^n_+\}$. Then $f_{d^*}$ has a positive real zero.

Proof. It is clear that the set in the definition of $d^*$ is nonempty and has upper bounds. So $d^*$ exists. Let $f'_{d^*} = f'_d = c_{\alpha_0} x^{\alpha_0} = c_{\alpha_0} + \sum_{\alpha \in A \setminus \{\alpha_0\}} c_\alpha x^{\alpha_0} - \sum_{\beta \in B} d_\beta x^{\beta - \alpha_0} - dx^{\gamma - \alpha_0}$. By Lemma 4.1, there exists $\alpha_0 \in \mathbb{A} \setminus \{\alpha_1, \ldots, \alpha_n\} \subseteq V(A) \setminus \{\alpha_0\}$ and $T \in \text{GL}_n(\mathbb{Q})$ such that $f'_{d^*} = c_{\alpha_0} + \sum_{i=1}^n c_{\alpha_0} x^{2k_i} + \sum_{\alpha \in A \setminus (A \cup \{\alpha_0\})} c_\alpha x^{(\alpha - \alpha_0)} - \sum_{\beta \in B} d_\beta x^{T(\beta - \alpha_0)} - dx^{T(\gamma - \alpha_0)} \in \mathbb{R}[x]$, where $T(\alpha - \alpha_0) \in (2\mathbb{N})^n$,

\end{document}
Consider the polynomial \( R \), which implies that the system of equations satisfies the conditions of Theorem 3.4 in [1]. \( f_d^T \) is a coercive polynomial. Hence there exists \( N_d > 0 \) such that for \( ||x||_\infty > N_d \), \( f_d^T(x) > 0 \). It follows that \( ||x_d||_\infty \leq N_d \). Let \( d' < d \) and \( d' > d \). Since \( f_d^T(x) - f_d^T(x) = (d - d')x^T(\gamma - \alpha_0) > 0 \) over \( \mathbb{R}_+^n \), we have \( f_d^T(x) > f_d^T(x) \) over \( \mathbb{R}_+^n \). Thus for \( ||x||_\infty > N_d \) and \( x \in \mathbb{R}_+^n \), \( f_d^T(x) > 0 \). It follows that \( ||x_d||_\infty \leq N_d \). Let \( d \to d' \). Then we have \( f_d^T(x_d) - f_d^T(x_d) = (d' - d)x_d^T(\gamma - \alpha_0) \to 0 \). Since \( f_d^T(x_d) \geq 0 \) and \( f_d^T(x_d) < 0 \), we must have \( f_d^T(x_d) \to 0 \). Thus the infimum of \( f_d^T \) over \( \mathbb{R}_+^n \) is 0. It follows that \( f_d^T - c_\alpha \) is non-negative over \( \mathbb{R}_+^n \). So by Lemma 3.3, \( f_d^T \) has a minimizer over \( \mathbb{R}_+^n \), which is also a zero. As a consequence, \( f_d^T \) has a positive real zero by Lemma 3.2 and so does \( f_d^T \).

Theorem 4.8. Let \( F \) be the following system of polynomial equations

\[
\sum_{\alpha \in \mathcal{A}} c_\alpha (\alpha - \gamma) x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta (\beta - \gamma) x^\beta = 0.
\]

where \( c_\alpha, d_\beta > 0 \) and \( \mathcal{B} \cup \{ \gamma \} \subseteq \Delta^c \) with \( \Delta = \text{conv}(\mathcal{A}) \). Assume that \( \dim(\Delta) = n \), \( \Delta \) is simple at some vertex and \( \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta \) is non-negative over \( \mathbb{R}_+^n \). Then \( F \) has at least one positive real zero.

Proof. Consider the polynomial \( f_d = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta - d' x^\gamma \). Define \( d^* \) as in Lemma 4.7. Then by Lemma 4.7, \( f_d^* \) has a positive real zero as a minimizer, which implies that the system of \( f_d^* = 0 \) and \( \nabla(f_d^*) = 0 \) has a positive real zero.

Multiplying \( f_d^* = 0 \) by \( \gamma \) gives

\[
\sum_{\alpha \in \mathcal{A}} c_\alpha \gamma x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta \gamma x^\beta - d^* \gamma x^\gamma = 0.
\]

Multiplying the \( i \)-th equation of \( \nabla(f_d^*) = 0 \) by \( x_i \) gives

\[
\sum_{\alpha \in \mathcal{A}} c_\alpha \alpha x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta x^\beta - d^* \gamma x^\gamma = 0.
\]

From (4.13) - (4.12), we obtain

\[
\sum_{\alpha \in \mathcal{A}} c_\alpha (\alpha - \gamma) x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta (\beta - \gamma) x^\beta = 0,
\]

which is \( F \). Thus \( F \) has a positive real zero.

Example 4.9. The following system of polynomial equations satisfies the conditions of Theorem 4.8 with \( \gamma = (2, 1) \), and hence has a positive real zero.

\[
\begin{align*}
6x^8 + 6x^8 & - 2y^8 + 2x^4y^4 - 2 - x^3y^2 = 0 \\
7x^8 - 3y^8 + 3x^4y^4 - 1 - x^3y^2 &= 0
\end{align*}
\]

Combine Theorem 4.6 and Theorem 4.8 we have

Theorem 4.10. Let \( F \) be the following system of polynomial equations

\[
\sum_{\alpha \in \mathcal{A}} c_\alpha (\alpha - \gamma) x^\alpha - \sum_{\beta \in \mathcal{B}} d_\beta (\beta - \gamma) x^\beta = 0,
\]
where $c_\alpha, d_\beta > 0$ and $\mathcal{S} \cup \{\gamma\} \subseteq \Delta^2$ with $\Delta = \text{conv}(\mathcal{S})$. Assume that $\dim(\Delta) = n$ and $\Delta$ is simple at some vertex. Then $F$ has at least one positive real zero.

**Remark 4.11.** Birch’s theorem ([6]) in statistics states that the following system of polynomial equations

\[
\sum_{\alpha \in \mathcal{S}} c_\alpha (\alpha - \gamma) x^\alpha = 0,
\]

where $c_\alpha > 0$ for $\alpha \in \mathcal{S}$, $\gamma \in \text{conv}(\mathcal{S})^\circ$, $\dim(\text{conv}(\mathcal{S})) = n$, has exactly one positive real zero. Our theorems hence can be viewed as a generalization of Birch’s theorem.

**Remark 4.12.** In above theorems, we always assume that the Newton polytope $\Delta$ is simple at some vertex since we need to exploit the property of coercive polynomials in the proofs. It is not clear whether this condition can be dropped.

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