The martingale problem for a class of stable-like processes

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Abstract

Let $\alpha \in (0, 2)$ and consider the operator

$$\mathcal{L} f(x) = \int [f(x + h) - f(x) - 1_{|h| \leq 1} \nabla f(x) \cdot h] \frac{A(x, h)}{|h|^{d+\alpha}} dh,$$

where the $\nabla f(x) \cdot h$ term is omitted if $\alpha < 1$. We consider the martingale problem corresponding to the operator $\mathcal{L}$ and under mild conditions on the function $A$ prove that there exists a unique solution.

Keywords: martingale problem, stable-like processes, symmetric stable process, stochastic differential equation, jump process, Poisson point process, Harnack inequality.

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1 Introduction

A stable-like process is a pure jump process where the jump intensity kernel is comparable in some sense to that of one or more stable processes. The term was introduced in [3] for processes whose associated operators were of the form

$$\int_{\mathbb{R}} [f(x + h) - f(x) - 1_{|h| \leq 1} \nabla f(x) \cdot h] \frac{dh}{|h|^{1+\alpha(x)}},$$

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and the use of the term was extended in [8] to refer to symmetric Markov processes whose jump kernels $J(x, y)$ were comparable to $|x - y|^{-d-\alpha}$ for a fixed $\alpha$.

In this paper we fix $\alpha \in (0, 2)$. For $\alpha \in [1, 2)$ we consider jump processes associated to the operator

$$
\mathcal{L} f(x) = \int [f(x + h) - f(x) - 1_{|h| \leq 1} \nabla f(x) \cdot h] \frac{A(x, h)}{|h|^{d+\alpha}} dh,
$$

(1.1)

and for $\alpha \in (0, 1)$ associated to the operator

$$
\mathcal{L} f(x) = \int [f(x + h) - f(x)] \frac{A(x, h)}{|h|^{d+\alpha}} dh,
$$

(1.2)

where $A(x, h)$ is bounded above and below by positive constants not depending on $x$ or $h$. For the domain of $\mathcal{L}$ we take the class of $C^2$ functions such that the function and its first and second partial derivatives are bounded. These jump processes, when at a point $x$, jump to $x + h$ with intensity given by $A(x, h)|h|^{-d-\alpha}$. These processes stand in the same relationship to symmetric stable processes of index $\alpha$ as uniformly elliptic operators in non-divergence form do to Brownian motion.

For $\alpha \geq 1$ the $\nabla f(x) \cdot h$ term is needed to guarantee convergence of the integral, while for $\alpha < 1$ the $\nabla f(x) \cdot h$ term cannot be present, or else the jumps of the process will be dominated by the drift.

Processes corresponding to $\mathcal{L}$ given by (1.1) or (1.2) were considered in [7] and [13], where Harnack inequalities and regularity of harmonic functions were proved. It is natural to ask whether there exists a process corresponding to $\mathcal{L}$, and if so, is there only one.

We view this question as a martingale problem. Let $\Omega = D([0, \infty))$, the set of paths that are right continuous with left limits, endowed with the Skorokhod topology. Set $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$, define $\theta_t : \Omega \to \Omega$ by $\theta_t(\omega)(s) = \omega(s + t)$, and let $\mathcal{F}_t$ be the right continuous filtration generated by the process $X$. A probability measure $\mathbb{P}$ is a solution to the martingale problem for $\mathcal{L}$ started at $x$ if $\mathbb{P}(X_0 = x) = 1$ and $f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_s) ds$ is a martingale whenever $f$ is a $C^2$ function such that $f$ and its first and second partial derivatives are bounded. The question to be answered is the existence and uniqueness of a solution to the martingale problem for $\mathcal{L}$. 

Our results on existence are merely an application of techniques used in [4] and the novelty in the current paper is a sufficient condition for uniqueness. Let \( \eta > 0 \) and set
\[
\psi_\eta(x) = (1 + \log^+(1/x))^{1+\eta}, \quad x > 0.
\]
We require continuity in \( x \) of the function \( A(x, h) \), with more continuity the smaller \( h \) is. More specifically, let
\[
\overline{A}(x, h) = A(x, h)\psi_\eta(|h|).
\]
Our main assumption is that \( \overline{A}(x, h) \) be continuous in \( x \), uniformly in \( h \).

We assume

**Assumption 1.1**  
(a) There exist \( c_1, c_2 > 0 \) such that \( c_1 \leq A(x, h) \leq c_2 \) for all \( x \) and \( h \).
(b) There exists \( \eta > 0 \) such that for every \( y \in \mathbb{R}^d \) and every \( b > 0 \)
\[
\limsup_{x \to y, |h| \leq b} |\overline{A}(x, h) - \overline{A}(y, h)| = 0.
\]

Part (a) of Assumption 1.1 may be regarded as the jump process equivalent of uniform ellipticity.

We then have

**Theorem 1.2** Suppose Assumption 1.1 holds and \( x_0 \in \mathbb{R}^d \). Then there is one and only one solution to the martingale problem for \( \mathcal{L} \) started at \( x_0 \).

As we alluded to above, existence is already known and can in fact be proved under slightly weaker hypotheses. Some other generalizations are possible; see Remarks 4.7 and 4.10. Our theorem also extends some of the results obtained in [10]; see Remark 4.9.

We do not know if our theorem is still true if \( \overline{A} \) is replaced by \( A \) in Assumption 1.1. We point out that uniqueness for the martingale problem for jump processes does not always hold; see [2, Section 6].

In the next section we establish some estimates. An approximation is given in Section 3 and Theorem 1.2 is proved in Section 4.
2 Estimates

Let \( B(x,r) = \{ y \in \mathbb{R}^d : |y - x| < r \} \). Let \( C^k \) be the functions which are \( k \) times continuously differentiable, \( C^k_b \) the elements of \( C^k \) such that the function and its partial derivatives up to order \( k \) are bounded, and \( C^k_K \) the functions in \( C^k \) that have compact support. We use the probabilist’s version of the Fourier transform:

\[
\hat{f}(u) = \int e^{iux} f(x) \, dx.
\]

For processes whose paths are right continuous with left limits, we set \( X_{t-} = \lim_{s \to t, s \downarrow t} X_s \) and \( \Delta X_t = X_t - X_{t-} \). We use the letter \( c \) with or without subscripts to denote constants whose value is unimportant and may change from line to line.

We suppose throughout the remainder of the paper that Assumption 1.1 holds.

**Definition 2.1** We say a collection \( \{ \mathbb{P}^x \} \) of probability measures is a strong Markov family of solutions to the martingale problem for \( \mathcal{L} \) if for each \( x \in \mathbb{R}^d \), \( \mathbb{P}^x \) is a solution to the martingale problem for \( \mathcal{L} \) started at \( x \) and in addition the strong Markov property holds: for any finite stopping time \( T \), any \( Y \) bounded and \( \mathcal{F}_\infty \)-measurable, and any \( x \in \mathbb{R}^d \),

\[
\mathbb{E}^x[Y \circ \theta_T \mid \mathcal{F}_T] = \mathbb{E}^{X_T}[Y], \quad \mathbb{P}^x - \text{a.s.}
\]

**Proposition 2.2** Suppose \( r < 1 \), \( x \in \mathbb{R}^d \), \( \tau_r = \inf \{ t : |X_t - x| \geq r \} \), and \( \mathbb{P} \) is a solution to the martingale problem for \( \mathcal{L} \) started at \( x \). There exists \( c_1 \) not depending on \( x \) such that

\[
\mathbb{P}(\sup_{s \leq t} |X_s - x| \geq r) \leq c_1 t/r^2, \quad t > 0.
\]

**Proof.** Let \( f : \mathbb{R}^d \to [0,1] \) be a \( C^2 \) function such that \( f(0) = 0 \) and \( f(y) = 1 \) if \( |y| > 1 \). Let \( f_{rx}(y) = f((y - x)/r) \). There exists a constant \( c \) such that the first derivatives of \( f_{rx} \) are bounded by \( c/r \) and the second derivatives are bounded by \( c/r^2 \). By Taylor’s theorem,

\[
|f_{rx}(z + h) - f_{rx}(z) - \nabla f_{rx}(z) \cdot h| \leq c|h|^2/r^2
\]
\[ |f_{rx}(z + h) - f_{rx}(z)| \leq c|h|/r. \]

Suppose \( \alpha \geq 1. \) Then
\[
|\mathcal{L} f_{rx}(z)| \leq \int_{|h| \leq r} |f_{rx}(z + h) - f_{rx}(z) - \nabla f_{rx}(z) \cdot h| \frac{A(x, h)}{|h|^{d+\alpha}} \, dh \\
+ \int_{1 \geq |h| > r} |f_{rx}(z + h) - f_{rx}(z) - \nabla f_{rx}(z) \cdot h| \frac{A(x, h)}{|h|^{d+\alpha}} \, dh \\
+ \int_{|h| > 1} |f_{rx}(z + h) - f_{rx}(z)| \frac{A(x, h)}{|h|^{d+\alpha}} \, dh \\
\leq \frac{c}{r^2} \int_{|h| \leq r} \frac{|h|^2}{|h|^{d+\alpha}} \, dh + \frac{c}{r} \int_{|h| > r} \frac{|h|}{|h|^{d+\alpha}} \, dh \\
\leq cr^{-\alpha}.
\]

Therefore by Doob’s optional stopping theorem
\[
P(\tau_r \leq t) \leq \mathbb{E} f_{rx}(X_{\tau_r \wedge t}) - f_{rx}(x) \\
= \mathbb{E} \int_0^{\tau_r \wedge t} \mathcal{L} f_{rx}(X_s) \, ds \\
\leq ct/r^\alpha.
\]

The case \( \alpha < 1 \) is similar. \( \square \)

**Proposition 2.3** If \( f \in C^2_b \), then \( \mathcal{L} f \) is continuous.

**Proof.** Let \( \varepsilon > 0 \) and suppose that \( \alpha \geq 1 \), the case when \( \alpha < 1 \) being very similar. Let \( \delta \in (0, 1) \) and write
\[
\mathcal{L} f(x) = \int_{|h| \leq \delta} [f(x + h) - f(x) - \nabla f(x) \cdot h] \frac{A(x, h)}{|h|^{d+\alpha}} \, dh \\
+ \int_{\delta < |h| \leq 1} [f(x + h) - f(x) - \nabla f(x) \cdot h] \frac{A(x, h)}{|h|^{d+\alpha}} \, dh \\
+ \int_{1 < |h| \leq \delta^{-1}} [f(x + h) - f(x)] \frac{A(x, h)}{|h|^{d+\alpha}} \, dh \\
+ \int_{\delta^{-1} < |h|} [f(x + h) - f(x)] \frac{A(x, h)}{|h|^{d+\alpha}} \, dh.
\]
The first term is bounded by
\[ c \int_{|h| \leq \delta} \frac{|h|^2}{|h|^{d+\alpha}} \, dh, \]
where \( c \) depends on \( f \). This is less than \( \varepsilon \) if \( \delta \) is sufficiently small. The fourth term is bounded by
\[ c \int_{|h| > \delta^{-1}} \frac{dh}{|h|^{d+\alpha}}, \]
where again \( c \) depends on \( f \). This will also be less than \( \varepsilon \) if \( \delta \) is sufficiently small. The second and third terms are continuous in \( x \) by dominated convergence and the continuity of \( A(x, h) \) in \( x \).

\[ \square \]

**Proposition 2.4** Suppose \( \{P^x\} \) is a strong Markov family of solutions to the martingale problem for \( \mathcal{L} \). Let \( x_0 \in \mathbb{R}^d \), suppose \( r < 1 \), and \( \tau_r = \inf\{t : |X_t - x_0| > r\} \).

(a) If \( \varepsilon > 0 \), there exists \( c_1 \) (depending on \( \varepsilon \)) such that
\[ \inf_{z \in B(x_0, (1-\varepsilon)r)} E^{\tau_r \tau} \geq c_1 r^{-\alpha}. \]

(b) There exists \( c_2 \) such that
\[ \sup_{z} E^{\tau_r \tau} \leq c_2 r^\alpha. \]

**Proof.** The proof consists of minor modifications to the proofs of [3] Lemmas 3.2 and 3.3.

\[ \square \]

**Proposition 2.5** Let \( \mathbb{P} \) be a solution to the martingale problem for \( \mathcal{L} \) started at some point \( x_0 \). If \( B \) and \( C \) are Borel sets whose closures are disjoint, then
\[ \sum_{s \leq t} 1_B(X_s) 1_C(X_s) - \int_0^t 1_B(X_s) \int_C \frac{A(X_s, u - X_s)}{|u - X_s|^{d+\alpha}} \, du \, ds \]
is a martingale with respect to \( \mathbb{P} \).

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Proof. Suppose \( B \) and \( C \) are disjoint compact sets, \( f \in C^2_b \) is 0 on \( B \) and 1 on \( C \), and \( \nabla f \) is 0 on \( B \). Then

\[
f(X_t) - f(X_0) = M_t + \int_0^t \mathcal{L}f(X_s) \, ds,
\]

where \( M_t \) is a martingale. It follows that \( \int_0^t 1_{B(X_s)} \, dM_s \) is also a martingale.

By Ito’s formula

\[
f(X_t) - f(X_0) = \int_0^t \nabla f(X_s) \cdot dX_s + \sum_{s \leq t} [f(X_s) - f(X_s^{-}) - \nabla f(X_s^{-}) \cdot \Delta X_s].
\]

Hence

\[
\int_0^t 1_{B(X_s)} \nabla f(X_s) \cdot \Delta X_s + \sum_{s \leq t} 1_{B(X_s)}[f(X_s) - f(X_s^{-}) - \nabla f(X_s^{-}) \cdot \Delta X_s]
\]

\[
- \int_0^t 1_{B(X_s)} \mathcal{L}f(X_s) \, ds
\]

is a martingale. Since \( f \in C^2 \) and both \( f \) and \( \nabla f \) are 0 on \( B \), the first term of (2.1) is equal to 0 and the second term of (2.1) is

\[
\sum_{s \leq t} 1_{B(X_s)} f(X_s).
\]

We have

\[
1_B(x) \mathcal{L}f(x) = 1_B(x) \left[ f(x + h) - f(x) - 1_{|h| \leq 1} \nabla f(x) \cdot h \right] \frac{A(x, h)}{|h|^{d+\alpha}} \, dh
\]

\[
= 1_B(x) \int f(x + h) \frac{A(x, h)}{|h|^{d+\alpha}} \, dh
\]

\[
= 1_B(x) \int f(u) \frac{A(x, u - x)}{|u - x|^{d+\alpha}} \, du.
\]

Putting this in (2.1), and using the fact that \( X_s \) differs from \( X_s^{-} \) on a set of times having Lebesgue measure 0, the last term in (2.1) is

\[
\int_0^t 1_B(X_s) \int f(u) \frac{A(X_s, u - X_s)}{|u - X_s|^{d+\alpha}} \, du \, ds.
\]

Our result follows by using a limit argument. \(\square\)
Proposition 2.6  Suppose \( \{\mathbb{P}^x\} \) is a strong Markov family of solutions to the martingale problem for \( \mathcal{L} \). Suppose \( g \) is bounded and \( \lambda > 0 \). Let

\[
S_\lambda g(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} g(X_t) \, dt.
\]

Then \( S_\lambda g \) is Hölder continuous in \( x \).

Proof. The proof follows by [7, Theorem 4.3] and the arguments leading up to it. See also [13].

Let \( \mathcal{M}^z \) be the operator on \( C^2_b \) functions defined by

\[
\mathcal{M}^z f(x) = \int \left[ f(x + h) - f(x) - \nabla f(x) \cdot h 1_{|h| \leq 1} \right] \frac{A(z, h)}{|h|^{d+\alpha}} \, dh, \tag{2.2}
\]

where the \( \nabla f(x) \cdot h \) term is missing if \( \alpha < 1 \). Let \( R^z_\lambda \) be the resolvent for the Lévy process whose infinitesimal generator is \( \mathcal{M}^z \) and let \( P^z_t \) be the corresponding transition operator.

Proposition 2.7  With \( \mathcal{M}^z \) as above,

\[
\Re (\hat{\mathcal{M}}^z(u)) \leq \begin{cases} 0, & |u| \leq 1; \\ -c|u|^\alpha, & |u| > 1. \end{cases}
\]

Proof. \( \hat{\mathcal{M}}^z(u) = \int \left[ e^{iu \cdot h} - 1 - iu \cdot h 1_{|h| \leq 1} \right] \frac{A(z, h)}{|h|^{d+\alpha}} \, dh, \) with the \( iu \cdot h 1_{|h| \leq 1} \) term missing if \( \alpha < 1 \). So

\[
-\Re (\hat{\mathcal{M}}^z(u)) = \int \left[ 1 - \cos(u \cdot h) \right] \frac{A(z, h)}{|h|^{d+\alpha}} \, dh,
\]

and the assertion in the case \( |u| \leq 1 \) is immediate.
If $|u| > 1$, setting $u = rv$, where $|v| = 1$ and $r \in (1, \infty)$,

$$-\text{Re} \left( \hat{M}^z(u) \right) \geq c \int_{|h| \leq 1} \left[ 1 - \cos(u \cdot h) \right] \frac{1}{|h|^{d+\alpha}} \, dh$$

$$= c \int_{|h| \leq 1} \left[ 1 - \cos(r(v \cdot h)) \right] \frac{1}{|h|^{d+\alpha}} \, dh$$

$$= \frac{c}{r^\alpha} \int_{|h| \leq r} \left[ 1 - \cos(v \cdot h) \right] \frac{1}{|h|^{d+\alpha}} \, dh$$

$$\geq \frac{c}{r^\alpha} \int_{|h| \leq 1} \left[ 1 - \cos(v \cdot h) \right] \frac{1}{|h|^{d+\alpha}} \, dh,$$

using a change of variables. The integral in the last line is bounded below by a constant, using rotational invariance, and the result follows on noting $r = |u|$.

\[ \blacksquare \]

**Corollary 2.8** If $p^z(t, x, y) = \mathbb{P}_x^z (t - y)$ is the transition density for the Lévy process with generator $\mathcal{M}^z$, then for each $t$, $\sup_z \|p^z_t\|_2 \leq c(t) < \infty$.

Moreover, if $r^\lambda_t(x) = \int_0^\infty e^{-\lambda s} \mathbb{P}_x^z (s) \, ds$, then

$$|\hat{r}^\lambda_t(u)| \leq \frac{c}{\lambda + |u|^{\alpha}}.$$

**Proof.** The Fourier transform of $\mathbb{P}_x^z$ is $e^{t \hat{M}^z(u)}$, and so

$$|e^{t \hat{M}^z(u)}| = e^{t \text{Re} \left( \hat{M}^z(u) \right)}.$$

With the estimates from Proposition 2.7, this is less than or equal to 1 if $|u| \leq 1$ and less than or equal to $e^{-c|u|^{\alpha}}$ if $|u| > 1$, where $c$ does not depend on $z$. So the Fourier transform of $\mathbb{P}_x^z$ is in $L^2$, hence $\mathbb{P}_x^z$ is in $L^2$ by Plancherel’s theorem, with a bound not depending on $z$.

Now

$$|\hat{r}^\lambda_t(u)| = \left| \frac{1}{\lambda - \hat{M}^z(u)} \right| \leq \frac{1}{\text{Re} \left( \lambda - \hat{M}^z(u) \right)}.$$

This is less than or equal to $1/\lambda$ if $|u| \leq 1$ and less than $1/(\lambda + c|u|^{\alpha})$ if $|u| > 1$.

\[ \blacksquare \]
Proposition 2.9 If $f \in L^2$, $\|R_\lambda^zf\|_2 \leq \frac{1}{\lambda}\|f\|_2$.

Proof. By Corollary [2.8] $P_t^z$ has a density $p^z(t, x, y) = \mathfrak{p}_t^z(x - y)$ for some function $\mathfrak{p}_t^z$ in $L^1$. Then

$$\int \mathfrak{p}_t^z(x - y) \, dx = \int \mathfrak{p}_t^z(x - y) \, dy = 1$$

by a change of variables. Hence $\|P_t^zf\|_2 \leq \|f\|_2$ by [5, Theorem IV.5.1]. We now apply Minkowski’s inequality for integrals. \qed

Proposition 2.10 Let $R_\lambda^z$ be as above, $f \in L^2 \cap C^2_K$.

(a) If $\alpha < 1$ and $g(x) = R_\lambda^zf(x + h) - R_\lambda^zf(x)$, then

$$\|g\|_2 \leq c|h|^\alpha\|f\|_2.$$ 

(b) If $\alpha \in (0, 2)$, then

$$\|g\|_2 \leq \frac{c}{\lambda}\|f\|_2.$$ 

(c) If $\alpha \in [1, 2)$ and

$$G(x) = R_\lambda^zf(x + h) - R_\lambda^zf(x) - \nabla R_\lambda^zf(x) \cdot h,$$

then

$$\|G\|_2 \leq c|h|^\alpha\|f\|_2.$$ 

(d) If $\alpha \in [1, 2)$, then

$$\|G\|_2 \leq c\left(\frac{1}{\lambda} + |h|\right)\|f\|_2.$$ 

Proof. First of all, if $f \in L^2 \cap C^2_K$, then $R_\lambda^zf \in L^2 \cap C^2_b$ by Proposition [2.9] and translation invariance. So $\nabla R_\lambda^zf$ is well defined. By translation invariance, $\frac{\partial R_\lambda^zf}{\partial x_i} = R_\lambda^z\left(\frac{\partial f}{\partial x_i}\right)$, and $\frac{\partial f}{\partial x_i} \in C^1_K \subset L^2$, so $R_\lambda^z\left(\frac{\partial f}{\partial x_i}\right) \in C^1_b$, and is in $L^2$. Therefore to prove the proposition it suffices to look at Fourier transforms and to use Plancherel’s theorem.

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(a) We have $$\hat{g}(u) = \hat{f}(u)\hat{r}_\lambda(u)[e^{iu\cdot h} - 1],$$ so using Corollary 2.8

$$|\hat{g}(u)| \leq \frac{c|\hat{f}(u)| |h|^{\alpha} |u|^{\alpha}}{\lambda + |u|^{\alpha}} \leq c|\hat{f}(u)| |h|^{\alpha}.$$ Therefore

$$\|\hat{g}\|_2 \leq c|h|^{\alpha} \|\hat{f}\|_2,$$

and the result follows by Plancherel’s theorem.

(b) As in (a), but using $|e^{iu\cdot h} - 1| \leq 2$, we have

$$|\hat{g}(u)| \leq \frac{2c|\hat{f}(u)|}{\lambda},$$

and we use Plancherel’s theorem as in (a).

(c) $$\hat{G}(u) = \hat{f}(u)\hat{r}_\lambda(u)[e^{iu\cdot h} - 1 - iu \cdot h].$$

Now

$$|e^{iu\cdot h} - 1 - iu \cdot h| = \left| \int_0^{u\cdot h} [ie^{is} - i] \, ds \right|$$

$$\leq c \int_0^{u\cdot h} |s|^{\alpha - 1} \, ds$$

$$\leq c|u \cdot h|^{\alpha}.$$ Hence

$$|\hat{G}(u)| \leq c\frac{|\hat{f}(u)| |h|^{\alpha} |u|^{\alpha}}{\lambda + |u|^{\alpha}} \leq c|h|^{\alpha} |\hat{f}(u)|.$$ 

(d) Similarly to the proofs of (b) and (c),

$$|\hat{G}(u)| \leq c\frac{|\hat{f}(u)|}{\lambda + |u|^{\alpha}}(2 + |u \cdot h|).$$ 

If $|u| \leq 1$, then

$$|\hat{G}(u)| \leq c\frac{|\hat{f}(u)|}{\lambda}(2 + |h|).$$
On the other hand, if $|u| > 1$, then since $\alpha \geq 1$ and
\[
\frac{|u \cdot h|}{\lambda + |u|^{\alpha}} \leq \frac{|u| |h|}{|u|^{\alpha}} \leq |h|,
\]
we have
\[
|\hat{G}(u)| \leq c|h||\hat{f}(u)|.
\]
\[
\square
\]

Using this proposition we can extend the definition of the functions $g, G$ and extend the above estimates to every $f \in L^2$.

3 Approximation

A key step in the uniqueness proof is to get a bound on the resolvent for an arbitrary solution to the martingale problem for $L$. We do that by an approximation procedure.

We begin with

Definition 3.1 Let $(S, \mathcal{S}, \lambda)$ be a measure space, where $\lambda$ is a $\sigma$-finite measure. A random measure $\mu([0, t] \times A)(\omega)$ is a Poisson point process with intensity measure $\lambda$ if

(a) whenever $\lambda(A) < \infty$, $N_t(A) = \mu([0, t] \times A)$ is a Poisson process with intensity $\lambda(A)$ and

(b) If $n \geq 1$ and $A_1, \ldots, A_n$ are disjoint with $\lambda(A_1), \ldots, \lambda(A_n) < \infty$ for each $i$, then the processes $N_t(A_i), i = 1, \ldots, n$, are independent.

Proposition 3.2 Suppose whenever $\lambda(A) < \infty$, $N_t(A)$ is a process starting at 0 with paths that are right continuous and left limits that are constant except for jumps that are of size one. If $N_t(A) - \lambda(A)t$ is a martingale for each such $A$ and $N_t(A)$ and $N_t(B)$ have no jumps in common when $A$ and $B$ are disjoint, then $\mu([0, t] \times A) = N_t(A)$ is a Poisson point process.

Proof. Property (a) of the definition of Poisson point process follows from a very slight modification of [12, III.T12]. In addition, that theorem shows that $\sigma(N_t(A) - N_s(A) : A \in \mathcal{S})$ is independent of $\mathcal{F}_s$. 12
We next prove that if \( A_i, i = 1, \ldots, n \), are disjoint sets of finite \( \lambda \)-measure and \( t_0 > 0 \), then

\[
N_{t_0}(A_1), \ldots, N_{t_0}(A_n) \text{ are independent random variables.} \tag{3.1}
\]

To prove (3.1), we do the case when \( n = 2 \), the general case being very similar. Let \( u_1, u_2 \) be two reals and define

\[
M^j_t = \exp \left( iu_j N_{t \wedge t_0}(A_j) - \lambda(A_j)(t \wedge t_0)(e^{iu_j} - 1) \right), \quad j = 1, 2.
\]

Because \( N_t(A_j) \) is a Poisson process with intensity \( \lambda(A_j) \), each \( M^j_t \) is a martingale with \( M^j_0 = 0 \).

Since \( N_{t_0}(A_1) \) and \( N_{t_0}(A_2) \) have no jumps in common and are non-decreasing, the quadratic variation process \([N_t(A_1), N_t(A_2)]_t\) is zero. So by Ito’s product formula,

\[
M^1_t M^2_t = M^1_0 M^2_0 + \int_0^t M^1_s \, dM^2_s + \int_0^t M^2_s \, dM^1_s,
\]

or \( \mathbb{E}[M^1_t M^2_t] = 1 \). It follows that

\[
\mathbb{E} \left[ e^{iu_1 N_{t_0}(A_1)} e^{iu_2 N_{t_0}(A_2)} \right] = e^{\lambda(A_1)t_0(e^{iu_1} - 1)} e^{\lambda(A_2)t_0(e^{iu_2} - 1)} = \mathbb{E} \left[ e^{iu_1 N_{t_0}(A_1)} \right] \mathbb{E} \left[ e^{iu_2 N_{t_0}(A_2)} \right].
\]

This holds for every \( u_1, u_2 \), so \( N_{t_0}(A_1) \) and \( N_{t_0}(A_2) \) are independent.

A very similar argument shows that if \( 0 < s_0 < t_0 \), then \( N_{t_0}(A_1) - N_{s_0}(A_1), \ldots, N_{t_0}(A_n) - N_{s_0}(A_n) \) are independent random variables. This and the independence of \( \sigma(N_t(A) - N_s(A) : A \in \mathcal{S}) \) from \( \mathcal{F}_s \) implies part (b) of Definition 3.1. \( \square \)

We next construct a function \( F(x, u) \) such that for every Borel set \( B \) and every \( x \in \mathbb{R}^d \)

\[
\int 1_B(F(x, u)) \frac{du}{|u|^{d+a}} = \int_B A(x, h) \frac{dh}{|h|^{d+a}}. \tag{3.2}
\]

Such constructions are known (see [11] or [9]), but we want our \( F \) to be continuous in \( x \) as well, and the existing constructions do not necessarily
possess this property. (When $d > 1$, $F$ satisfying (3.2) are by no means unique.) We will do the case $d = 2$ for simplicity of notation, but the idea for higher dimensions is essentially the same.

Fix $x$. We define $F$ for $u$ in the first quadrant, and the other quadrants are done similarly. Set $r_0 = 0$ and choose $r_1 > r_2 > \cdots > 0$ such that

$$\int_{[r_{i+1}, r_i] \times [0, \infty)} \frac{A(x, h)}{|h|^{d+\alpha}} dh = 2^{-1}, \quad i = 0, 1, \ldots$$

For each strip $[r_{i+1}, r_i] \times [0, \infty)$, let $s_0 = \infty$ and choose $s_1 > s_2 > \cdots > 0$ such that

$$\int_{[r_{i+1}, r_i] \times [s_{j+1}, s_j]} \frac{A(x, h)}{|h|^{d+\alpha}} dh = 2^{-2}, \quad j = 0, 1, \ldots$$

Let $\mathcal{R}_1 = \mathcal{R}_1(x)$ be the collection of such rectangles. Note that each rectangle in $\mathcal{R}_1$ has the same mass with respect to the measure $A(x, h)/|h|^{d+\alpha} dh$, but the rectangles are not congruent in shape.

Set $v_0 = \infty$ and choose $v_1 > v_2 > \cdots > 0$ such that

$$\int_{[v_{i+1}, v_i] \times [0, \infty)} \frac{du}{|u|^{d+\alpha}} = 2^{-1}, \quad i = 0, 1, \ldots$$

For each strip $[v_{i+1}, v_i] \times [0, \infty)$, let $w_0 = \infty$ and choose $w_1 > w_2 > \cdots > 0$ such that

$$\int_{[v_{i+1}, v_i] \times [w_{j+1}, w_j]} \frac{du}{|u|^{d+\alpha}} = 2^{-2}, \quad j = 0, 1, \ldots$$

Let $\mathcal{V}_1 = \mathcal{V}_1(x)$ be the collection of such rectangles.

Let $\Gamma_1$ be the map from $\mathcal{V}_1$ to $\mathcal{R}_1$ taking the element $[v_{i+1}, v_i] \times [w_{j+1}, w_j]$ of $\mathcal{V}_1$ to the element $[r_{i+1}, r_i] \times [s_{j+1}, s_j]$ of $\mathcal{R}_1$.

If $[r, r'] \times [s, s']$ is an element of $\mathcal{R}_1$, choose $r''$ such that

$$\int_{[r, r''] \times [s, s']} \frac{A(x, h)}{|h|^{d+\alpha}} dh = 2^{-3},$$

and then $s'', s'''$ such that the integrals of $A(x, h)/|h|^{d+\alpha}$ over $[r, r''] \times [s, s'']$ and over $[r'', r'] \times [s'', s''']$ are both equal to $2^{-4}$. We put the 4 rectangles $[r, r''] \times [s, s''], [r, r'] \times [s'', s']$, $[r'', r'] \times [s', s''']$, and $[r'', r'] \times [s', s'']$ into
$R_2 = R_2(x)$ and do this for each rectangle in $R_1$. We divide each rectangle of $V_1$ similarly into 4 rectangles of mass $2^{-4}$ with respect to the measure $du/|u|^{d+\alpha}$ and let $V_2$ be the collection of such subrectangles. We define the map $\Gamma_2$ from $V_2$ into $R_2$ that takes a rectangle of $V_2$ into the corresponding rectangle of $R_2$.

We continue by dividing each rectangle of $R_2$ and $V_2$ into 4 subrectangles, and so on. Now define $F_m(x, u) : (0, \infty)^2 \to (0, \infty)^2$ by setting $F_m(x, u)$ to be the lower left hand point of $\Gamma_m(U)$ if $u \in U \in V_m$. Recall $x$ is fixed. It is easy to check that $F_m(x, u)$ converges uniformly over $u$ in compact subsets of $(0, \infty)^2$, and if we call the limit $F(x, u)$, then $u \to F(x, u)$ will be one-to-one.

The construction shows that the equality (3.2) holds if $m \geq 1$ and $B \in V_m$, and therefore it holds for every Borel set contained in $(, \infty)^2$.

Moreover, if $x_0 \in \mathbb{R}^d$ is fixed and $m \geq 1$, then by taking $x$ sufficiently close to $x_0$ the boundaries of each rectangle in $R_i(x)$, $i \leq m$, can be made as close as we please to the boundary of the corresponding rectangle of $R_i(x_0)$; we are using the continuity of $A(x, h)$ here. It follows that $F(x, u)$ is continuous in $x$, uniformly over $u$ in compact subsets of $(0, \infty)^2$.

Using Assumption 1.1(a), the construction also tells us that there exists $\beta$ such that

$$\beta |u| \leq |F(x, u)| \leq \beta^{-1} |u|, \quad x \in \mathbb{R}^d, \quad u \neq 0. \quad (3.3)$$

For each $x$, let $G(x, \cdot)$ be the inverse of $F(x, \cdot)$. Define

$$N_t(C) = \sum_{s \leq t} 1_{G(X_s, \Delta X_s) \in C}$$

and

$$\lambda(C) = \int_C \frac{du}{|u|^{d+\alpha}}.$$

**Proposition 3.3** Let $x_0 \in \mathbb{R}^d$ and let $P$ be a solution to the martingale problem for $L$ started at $x_0$. Then with respect to $P$, $N_t(\cdot)$ is a Poisson point process with intensity measure $\lambda$.

**Proof.** Let $F(x, C) = \{F(x, z) : z \in C\}$. Then

$$N_t(C) = \sum_{s \leq t} 1_{\Delta X_s \in F(x, \cdot, C)}.$$
By Proposition 2.5 and a limit argument, the right hand side is equal to a martingale plus
\[ \int_0^t \int_{F(X_s, C)} \frac{A(X_s, h)}{|h|^{d+\alpha}} \, dh \, ds. \]

By (3.2) and the fact that $X$ has only countably many jumps, this in turn is equal to
\[ \int_0^t \int_{C} \frac{du}{|u|^{d+\alpha}} \, ds = \int_0^t \int_{C} \frac{du}{|u|^{d+\alpha}} \, ds = \lambda(C)t. \]

Therefore by Proposition 3.2 we see that $N_t(\cdot)$ is a Poisson point process. □

Set $\mu([0, t] \times C) = N_t(C)$. Note the definition of $\mu$ does not depend on $\mathbb{P}$.

**Proposition 3.4** $X_t$ solves the stochastic differential equation

\[ X_t = X_0 + \int_0^t \int_{|F(X_s, z)| \leq 1} F(X_s, z) \left( \mu(dz) \, ds - \lambda(dz) \, ds \right) \]
\[ + \int_0^t \int_{|F(X_s, z)| > 1} F(X_s, z) \, \mu(dz) \, ds. \]  

**Proof.** Let $\delta > 0$. Set $G(x, D) = \{G(x, w) : w \in D\}$, $D^\delta = \{y : |y| > \delta\}$,

\[ H_t^\delta = \sum_{s \leq t} \Delta X_s 1_{(1 \geq |\Delta X_s| > \delta)}, \]
\[ \tilde{H}_t^\delta = \int_0^t \int_{G(X_s, D^\delta \setminus D^1)} F(X_s, z) \lambda(dz) \, ds, \]
\[ K_t = \sum_{s \leq t} \Delta X_s 1_{(|\Delta X_s| > 1)}. \]

If $\Delta X_s \neq 0$, the definition of $\mu$ via $N_t$ shows that $\mu$ assigns unit mass to some point $(z, s)$ satisfying $z = G(X_s, \Delta X_s)$, or with $\Delta X_s = F(X_s, z)$. Hence

\[ H_t^\delta = \int_0^t \int_{G(X_s, D^\delta \setminus D^1)} F(X_s, z) \, \mu(dz) \, ds \]  

(3.5)
and

\[ K_t = \int_0^t \int_{G(X_{s-}, D^1)} F(X_{s-}, z) \mu(dz ds). \]  

(3.6)

By [9] or [11, Theorem II.10], there exists a function \( F(x, z) \) satisfying

\[ \int 1_B(F(x, u)) \frac{du}{|u|^{d+\alpha}} = \int_B A(x, h) \frac{dh}{|h|^{d+\alpha}}, \]  

(3.7)

for \( B \) Borel and a Poisson point process \( \mu \) such that \( X_t \) solves

\[ X_t = X_0 + \int_0^t \int_{|F(X_{s-}, z)| \leq 1} F(X_{s-}, z) (\mu(dz ds) - \lambda(dz) ds) \]
\[ + \int_0^t \int_{|F(X_{s-}, z)| > 1} F(X_{s-}, z) \mu(dz ds). \]

From this equation we see that \( \mu \) gives unit mass to a point \((z, s)\) if and only if \( \Delta X_s = F(X_{s-}, z) \). It follows that

\[ V^\delta_t = X_t - X_0 - K_t - (H^\delta_t - \tilde{H}^\delta_t) \]
\[ = \int_0^t \int_{|F(X_{s-}, z)| \leq \delta} F(X_{s-}, z) (\mu(dz ds) - \lambda(dz) ds). \]

A limit argument and (3.7) show that

\[ \int |F(x, u)|^2 \frac{du}{|u|^{d+\alpha}} = \int |h|^2 A(x, h) \frac{dh}{|h|^{d+\alpha}} \]

which is bounded uniformly in \( x \). Consequently each component of \( V^\delta \) is a pure jump martingale and \( \mathbb{E} \sup_{s \leq t} |V^\delta_t|^2 \to 0 \) as \( \delta \to 0 \).

On the other hand, using (3.5) and (3.6),

\[ V^\delta_t = X_t - \left( X_0 + \int_0^t \int_{|F(X_{s-}, z)| \leq 1} F(X_{s-}, z) (\mu(dz ds) - \lambda(dz) ds) \right) \]
\[ + \int_0^t \int_{|F(X_{s-}, z)| > 1} F(X_{s-}, z) \mu(dz ds) \].

Our conclusion follows. \( \square \)
Define \( Y^n_s \) to be equal to \( x_0 \) if \( s < 1/n \) and equal to \( X_{(k-1)/n} \) if \( k/n \leq s < (k+1)/n \). The reason for the \( 1/n \) delay will appear in (4.12). Let

\[
X^n_t = X_0 + \int_0^t \int_{|F(Y^n_s, z)| \leq 1} F(Y^n_s, z)(\mu(dz \, ds) - \lambda(dz) \, ds) \\
+ \int_0^t \int_{|F(Y^n_s, z)| > 1} F(Y^n_s, z) \mu(dz \, ds).
\]  

(3.8)

**Proposition 3.5** Let \( x_0 \in \mathbb{R}^d \) and let \( \mathbb{P} \) be a solution to the martingale problem for \( \mathcal{L} \) started at \( x_0 \). For each \( t_0 \)

\[
\sup_{t \leq t_0} |X_t - X^n_t| \to 0
\]

in probability as \( n \to 0 \).

**Proof.** Except for \( s = 0 \), notice \( Y^n_s \to X_{s-} \) a.s. under \( \mathbb{P} \), using the fact that the paths of \( X_t \) have left limits. Except for \( z \) in the boundary of any of the \( 2^d \) orthants, \( F(Y^n_s, z) \to F(X_{s-}, z) \) a.s. if \( s > 0 \).

Let \( X^{n,I}_t \) be the first double integral on the right hand side of (3.8) and \( X^{n,II}_t \) the second. Similarly let \( X^I_t \) be the first double integral on the right in (3.4) and \( X^{II}_t \) the second. Let

\[
Z^n_t = X_0 + \int_0^t \int_{|F(X^n_s, z)| \leq 1} F(X^n_s, z)(\mu(dz \, ds) - \lambda(dz) \, ds) \\
+ \int_0^t \int_{|F(X^n_s, z)| > 1} F(X^n_s, z) \mu(dz \, ds) \\
= X_0 + Z^{n,I}_t + Z^{n,II}_t.
\]

Using Doob’s inequality on each component and basic properties of stochastic integrals with respect to Poisson point processes (see, e.g., [9]),

\[
\mathbb{E} \sup_{t \leq t_0} |X^I_t - Z^{n,I}_t|^2 \leq c \mathbb{E} |X^I_{t_0} - Z^{n,I}_{t_0}|^2 \\
= c \mathbb{E} \int_0^{t_0} \int_{|F(X_{s-}, z)| \leq 1} |F(X_{s-}, z) - F(Y^n_s, z)|^2 \lambda(dz) \, ds.
\]
Using (3.3), the integrand is bounded by
\[ c|z|^21_{(|z|\leq \beta-1)}, \]
which is integrable with respect to \( \lambda(dz) \, ds \). For \( s > 0 \) and all \( z \) not on the boundary of any of the orthants, and hence for almost every \( z \) with respect to \( \lambda \), the integrand tends to 0, a.s. Therefore by dominated convergence
\[ \mathbb{E} \sup_{t \leq t_0} |X^I_t - Z^{n,I}_t|^2 \to 0. \]

Since \( |F(X_{s-}, z)| > 1 \) implies \( |z| \geq \beta \) by (3.3), with probability one, there are only finitely many points \( (z, s) \) with \( |z| \geq \beta \) charged by \( \mu \) before time \( t_0 \). Also with probability one, none of the \( z \) values will lie on the boundary of any of the orthants. It follows then that \( \sup_{t \leq t_0} |X^I_t - Z^{n,I}_t| \to 0 \) as \( n \to \infty \).

Notice
\[ X^n_t - Z^n_t = \int_0^t \int_{C(n, s)} F(Y^n_{s-}, z) \lambda(dz) \, ds, \]
where
\[ C(n, s) = \{|F(Y^n_{s-}, z)| \leq 1, |F(X_{s-}, z)| > 1\} \]
\[ \quad \cup \{ |F(X_{s-}, z)| \leq 1, |F(Y^n_{s-}, z)| > 1\}. \]

Therefore
\[ \mathbb{E} \sup_{t \leq t_0} |X^n_t - Z^n_t| \leq \sum_{i=1}^5 \mathbb{E} \int_0^{t_0} \int_{D^i(n, s)} |F(Y^n_{s-}, z)| \lambda(dz) \, ds, \]
where \( \gamma > 0 \) will be chosen in a moment and
\[ D^1(n, s) = \{|F(Y^n_{s-}, z)| \leq 1 - \gamma, |F(X_{s-}, z)| > 1\}, \]
\[ D^2(n, s) = \{|F(Y^n_{s-}, z)| \leq 1, |F(X_{s-}, z)| \geq 1 + \gamma\}, \]
\[ D^3(n, s) = \{|F(Y^n_{s-}, z)| \geq 1 + \gamma, |F(X_{s-}, z)| \leq 1\}, \]
\[ D^4(n, s) = \{|F(Y^n_{s-}, z)| > 1, |F(X_{s-}, z)| \leq 1 - \gamma\}, \]
\[ D^5(n, s) = \{1 - \gamma \leq |F(Y^n_{s-}, z)|, |F(X_{s-}, z)| < 1 + \gamma\}. \]
By (3.3), if $|F(X_{s-}, z)| \geq 1 - \gamma$, then $|z| \geq c$. So using (3.2) and Assumption 1.1

$$\mathbb{E} \int_0^t \int_{D_5(n,s)} |F(Y^n_s, z)| \lambda(dz) \, ds$$

$$\leq (1 + \gamma) \mathbb{E} \int_0^t \int_{B(0,1+\gamma) \setminus B(0,1-\gamma)} |F(X_{s-}, z)| \lambda(dz) \, ds$$

$$= (1 + \gamma) \mathbb{E} \int_0^t \int_{B(0,1+\gamma) \setminus B(0,1-\gamma)} A(X_{s-}, h) \frac{1}{|h|^{d+\alpha}} dh \, ds$$

$$\leq c \gamma t_0.$$  

So the integral over $D_5(n,s)$ can be made as small as we like by taking $\gamma$ sufficiently small. Once $\gamma$ is chosen, observe that $1_{D_1(n,s)} \to 0$ a.s. for every $s > 0$ because $Y^n_s \to X_{s-}$. Also, on $D_1(n,s)$, we have $|F(X_{s-}, z)| > 1$, and as above $|z| > c$, so $|F(Y^n_s, z)| 1_{D_1(n,s)}$ is dominated by $(1 + \gamma) 1_{(|z| \geq c)}$, which is integrable with respect to $\lambda(dz) \, ds$. So by dominated convergence,

$$\mathbb{E} \int_0^t \int_{D_1(n,s)} |F(Y^n_s, z)| \lambda(dz) \, ds \to 0.$$  

The argument for $D^2(n,s), D^3(n,s),$ and $D^4(n,s)$ is the same. Hence

$$\mathbb{E} \sup_{t \leq t_0} |X^n_t - Z^n_t| \to 0.$$  

\hfill $\square$

**Proposition 3.6** Let $x_0 \in \mathbb{R}^d$ and let $\mathbb{P}$ be a solution to the martingale problem for $\mathcal{L}$ started at $x_0$. If $f \in C^2_b$, then

$$f(X^n_t) - f(X^n_0) - \int_0^t \mathcal{M} Y^n_s f(X^n_s) \, ds$$

is a martingale under $\mathbb{P}$, where $\mathcal{M} y$ is defined in (2.2).

**Proof.** If $\mu$ assigns unit mass to $(z,s)$, then $\Delta X^n_s = F(Y^n_s, z)$. By Itô's
formula
\[ f(X^n_t) - f(X^n_0) = \text{martingale} + \int_0^t \int_{|F(X^n_s, z)| > 1} \nabla f(X^n_s) \cdot F(Y^n_s, z) \mu(dz \, ds) \]
\[ + \sum_{s \leq t} [f(X^n_s) - f(X^n_{s-}) - \nabla f(X^n_{s-}) \cdot \Delta X^n_s] \]
\[ = \text{martingale} + \int_0^t \int \left[ f(X^n_{s-} + F(Y^n_s, z)) - f(X^n_{s-}) \right. \]
\[ - \nabla f(X^n_{s-}) \cdot F(Y^n_s, z) 1_{(|F(Y^n_s, z)| \leq 1)} \mu(dz \, ds) \]
\[ = \text{martingale} + \int_0^t \int \left[ f(X^n_{s-} + F(Y^n_s, z)) - f(X^n_{s-}) \right. \]
\[ - \nabla f(X^n_{s-}) \cdot F(Y^n_s, z) 1_{(|F(Y^n_s, z)| \leq 1)} \right] |z|^{-(d+\alpha)} \, dz \, ds. \]

Fix \( y \) and if \( \alpha \geq 1 \), let
\[ g(v) = f(y + v) - f(y) - \nabla f(y) \cdot v 1_{(|v| \leq 1)}. \]
A limit argument using (3.2) shows
\[ \int g(F(x, z)) \frac{1}{|z|^{d+\alpha}} \, dz = \int g(h) \frac{A(x, h)}{|h|^{d+\alpha}} \, dh. \]

Now taking \( y = X^n_{s-} \) and \( x = Y^n_s \) shows that \( f(X^n_t) - f(X^n_0) \) is equal to a martingale plus
\[ \int_0^t \int \left[ f(X^n_{s-} + h) - f(X^n_{s-}) - \nabla f(X^n_{s-}) \cdot h 1_{(|h| \leq 1)} \right] \frac{A(Y^n_s, h)}{|h|^{d+\alpha}} \, dh \, ds, \]
which proves the proposition when \( \alpha \geq 1 \). The case \( \alpha < 1 \) is similar. \( \square \)

4 Existence and uniqueness

Theorem 4.1 Suppose Assumption L1 holds. Then for each \( x \) there exists a solution to the martingale problem for \( \mathcal{L} \) started at \( x \).
Proof. In view of Propositions 2.2 and 2.3, existence of a solution follows by the proof in [4, Section 3], with minor modifications to handle the case of $d$ dimensions. 

Remark 4.2 It is easy to see by the same arguments that existence holds if $A(x, h)$ is bounded above and below by positive constants and for each $h$, $A(x, h)$ is continuous in $x$.

We now turn to the proof of uniqueness. Fix $x_0 \in \mathbb{R}^d$. If $G$ is the set of solutions to the martingale problem for $L$ started at $x_0$, then $G$ is a tight family by the proof in [4, Section 3]. Any subsequential limit point of $G$ is in $G$ by the arguments in that same section, and therefore $G$ is compact. Hence by the proofs in [14, Chapter 12], it suffices to consider uniqueness of strong Markov families of solutions $\{P^x\}$ to the martingale problem for $L$.

We will sometimes make the following temporary assumption, where we will choose $\zeta$ later:

Assumption 4.3 There exists $\zeta$ such that

$$|A(x, h) - A(x_0, h)| \leq \frac{\zeta}{\psi_\eta(|h|)}$$

$x \in \mathbb{R}^d$, $|h| \leq 1$.

For the rest of this section we take $\alpha \geq 1$, the case $\alpha < 1$ being similar.

Let $M^z f$ be defined by (2.2) and let $R^z_\lambda$ be the corresponding resolvent. Define an operator $H$ by

$$Hf(x) = \int_{|h| \leq 1} |f(x + h) - f(x) - \nabla f(x) \cdot h| \frac{\zeta}{\psi_\eta(|h|)|h|^{d+\alpha}} dh$$

$$+ \int_{|h| > 1} |f(x + h) - f(x)| \frac{dh}{|h|^{d+\alpha}}, \quad f \in C^2_b.$$ 

Proposition 4.4 There exists a constant $c_1$ not depending on $x_0$ such that

$$\|HR^z_{\lambda} f\|_2 \leq c_1(\zeta + \lambda^{-1})\|f\|_2, \quad f \in L^2 \cap C^2_b.$$ 

(4.1)
Proof. By Minkowski’s inequality for integrals,
\[
\|\mathcal{H} R^x_\lambda f\|_2 \leq \int_{|h| \leq 1} \| R^x_\lambda f(x + h) - R^x_\lambda f(x) \\
- \nabla R^x_\lambda f(x) \cdot h \|_2 \frac{\zeta}{\psi_\eta(h) |h|^{d+\alpha}} \, dh \\
+ \int_{|h| > 1} \| R^x_\lambda f(x + h) - R^x_\lambda f(x)\|_2 \frac{c}{|h|^{d+\alpha}} \, dh. \tag{4.2}
\]
By Proposition 2.10(c), the first term on the right of (4.2) is bounded by
\[
c \int_{|h| \leq 1} |h|^\alpha \frac{\zeta}{\psi_\eta(h) |h|^{d+\alpha}} \, dh \|f\|_2 \leq c \zeta \|f\|_2. \tag{4.3}
\]
By Proposition 2.10(b) the second term on the right of (4.2) is bounded by
\[
\frac{c}{\lambda} \int_{|h| > 1} \frac{dh}{|h|^{d+\alpha}} \|f\|_2. \tag{4.4}
\]
\]

\textbf{Corollary 4.5} Suppose Assumption 4.3 holds. There exists \( \kappa \) such that
\[
\|(\mathcal{L} - \mathcal{M}^x_0) R^x_\lambda f\|_2 \leq \kappa (\zeta + \lambda^{-1}) \|f\|_2, \quad f \in L^2 \cap C^2_b, \tag{4.5}
\]
and
\[
\| \sup_{w \in \mathbb{R}^d} |\mathcal{M}^w R^x_\lambda f(\cdot) - \mathcal{M}^x_0 R^x_\lambda f(\cdot)|_2 \leq \kappa (\zeta + \lambda^{-1}) \|f\|_2, \quad f \in L^2 \cap C^2_b. \tag{4.6}
\]

\textbf{Proof.} If Assumption 4.3 holds, then
\[
\|(\mathcal{L} - \mathcal{M}^x_0) R^x_\lambda f(x)\|
= \left| \int [R^x_\lambda f(x + h) - R^x_\lambda f(x) \\
- \nabla R^x_\lambda f(x) \cdot h 1_{|h| \leq 1}] \frac{A(x, h) - A(x_0, h)}{|h|^{d+\alpha}} \, dh \right| \\
\leq c \mathcal{H} R^x_\lambda f(x)
\]
and for each \( w \)

\[
| (\mathcal{M}^w R^x_\lambda f(x, y) - \mathcal{M}^x_\lambda R^x_\lambda f(x) | \\
\leq \left| \int_{|h|\leq 1} [R^x_\lambda f(x + h) - R^x_\lambda f(x)] A(w, h) - A(x_0, h) \lambda f(x) \right| \\
+ \left| \int_{|h|\geq 1} [R^x_\lambda f(x + h) - R^x_\lambda f(x)] \frac{A(w, h) + A(x_0, h)}{|h|^{d+\alpha}} dh \right| \\
\leq c HR^x_\lambda f(x).
\]  

(4.8)

Now combine Proposition 4.4, (4.7), and (4.8).

Proposition 4.6 Let \( \{P^x\} \) be a strong Markov family of solutions to the martingale problem for \( \mathcal{L} \). Set

\[ S_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t) dt. \]

Suppose Assumption 4.3 holds with \( \zeta \) and \( \lambda \) chosen so that \( \kappa (\zeta + \lambda^{-1}) \leq 1/2 \), where \( \kappa \) is as in Corollary 4.5. Let \( \rho \in L^2 \) be non-negative with compact support. Then

\[
\sup_{\|g\|_2 \leq 1} \left| \int S_\lambda g(x) \rho(x) dx \right| < \infty.
\]

Proof. Define \( X^n_t \) as in Section 3 and define

\[ S^n_\lambda g(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} g(X^n_t) dt, \quad g \in C^2_b. \]

Step 1: Our first goal is to show that if

\[
\Lambda_n = \sup_{\|g\|_2 \leq 1} \left| \int S^n_\lambda g(x) \rho(x) dx \right|,
\]

then \( \Lambda_n < \infty \). The value of \( \Lambda_n \) will depend on \( \rho \).
To prove (4.9) it suffices to suppose $g \geq 0$ since we can write an arbitrary $g$ as the difference of its positive and negative parts. Suppose $g \in C^2_K$ and write

$$S^n_\lambda g(x) = \mathbb{E}^x \int_0^{1/n} e^{-\lambda t} g(X^0_t) dt + \sum_{k=1}^{\infty} \mathbb{E}^x \int_{k/n}^{(k+1)/n} e^{-\lambda t} g(X^0_t) dt.$$  \hspace{1cm} (4.10)

Over the time interval $[0, 1/n)$, the process $X^n_t$ behaves like the Lévy process corresponding to $\mathcal{M}^{x_0}$ started at $x$. So the first term on the right hand side of (4.10) is bounded by $R^{x_0}_\lambda g(x)$. By the Cauchy-Schwarz inequality and Proposition 2.9,

$$\left| \int R^{x_0}_\lambda g(x) \rho(x) \, dx \right| \leq \| R^{x_0}_\lambda g \|_2 \| \rho \|_2 \leq \frac{c}{\lambda} \| g \|_2.$$  \hspace{1cm} (4.11)

The $k^{th}$ term on the right hand side of (4.10) is

$$e^{-\lambda(k-1)/n} \mathbb{E}^x \int_{k/n}^{(k+1)/n} e^{-\lambda(t-(k-1)/n)} g(X^0_t) dt
\leq ce^{-\lambda k/n} \mathbb{E}^x \left[ \mathbb{E}^x \left[ \int_{1/n}^{2/n} g(X^0_{t+\frac{k-1}{n}}) dt \mid \mathcal{F}_{(k-1)/n} \right] \right].$$

Let us temporarily write $\bar{Y}$ for $Y^{x_0}_{(k-1)/n}$. Conditional on $\mathcal{F}_{(k-1)/n}$, the process $X^n_t$ over the time interval $[k/n, (k+1)/n)$ behaves like the Lévy process corresponding to $\mathcal{M}^{\bar{Y}}$ started at $X^n_{(k-1)/n}$ and run over the time interval $[1/n, 2/n]$. Therefore

$$\mathbb{E}^x \left[ \int_{1/n}^{2/n} g(X^0_{t+\frac{k-1}{n}}) dt \mid \mathcal{F}_{(k-1)/n} \right]
\leq \int_{1/n}^{2/n} P_t^{\bar{Y}} g(X^0_{(k-1)/n}) dt
\leq e^{\lambda/n} P_{1/n}^{\bar{Y}} R_\lambda^{y_0} g(X^n_{(k-1)/n}).$$  \hspace{1cm} (4.12)

Using Corollary 2.8 we have

$$\mathbb{P}^{\omega}_{1/n} R_\lambda^{v_0} g(v) = \int \bar{\mathbb{P}}^{\omega}(1/n, v-z) R_\lambda^{\omega} g(z) \, dz
\leq \| \bar{\mathbb{P}}^{\omega}(1/n, \cdot) \|_2 \| R_\lambda^{\omega} g \|_2
\leq c_n \frac{1}{\lambda} \| g \|_2,$$
where \(c_n\) depends on \(n\). Hence the \(k^{th}\) term on the right hand side of (4.10) is bounded by \(ce^{-\lambda k/n} \|g\|_2\). Because \(\rho\) is in \(L^2\) with compact support,

\[
\int \mathbb{E}^x \int_{k/n}^{(k+1)/n} g(X_t) \, dt \, \rho(x) \, dx \leq c \|g\|_2 \int \rho(x) \, dx \leq c \|g\|_2.
\]

Combining this with (4.11) and taking the supremum over \(g \in C^2_\mathcal{K}\) with \(\|g\|_2 \leq 1\) proves (4.9).

**Step 2:** Next we show there exists a constant \(\Lambda < \infty\) independent of \(n\) such that

\[
\sup_{\|g\|_2 \leq 1} \left| \int S^n_\lambda g(x) \rho(x) \, dx \right| \leq \Lambda. \tag{4.13}
\]

Let \(f \in C^2_0\). By Proposition 3.6

\[
\mathbb{E}^x f(X^n_t) - f(x) = \mathbb{E}^x \int_0^t \mathcal{M}^x f(X^n_s) \, ds.
\]

Multiplying by \(e^{-\lambda t}\) and integrating over \(t\) from 0 to \(\infty\)

\[
S^n_\lambda f(x) - \frac{1}{\lambda} f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} \int_0^t \mathcal{M}^x f(X^n_s) \, ds \, dt \tag{4.14}
\]

\[
= \mathbb{E}^x \int_0^\infty \mathcal{M}^x f(X^n_s) \int_s^\infty e^{-\lambda t} \, dt \, ds,
\]

\[
= \frac{1}{\lambda} \mathbb{E}^x \int_0^\infty e^{-\lambda s} \mathcal{M}^x f(X^n_s) \, ds,
\]

\[
= \frac{1}{\lambda} S^n_\lambda \mathcal{M}^x f(x) + \frac{1}{\lambda} \mathbb{E}^x \int_0^\infty e^{-\lambda s} (\mathcal{M}^x f - \mathcal{M}^x f(X^n_s)) \, ds.
\]

If \(g \in C^2_\mathcal{K}\), set \(f = R^x \lambda g\). By translation invariance, \(f \in C^2_0\). Standard semigroup manipulations show

\[
\mathcal{M}^x f = \mathcal{M}^x R^x \lambda g = \lambda R^x \lambda g - g.
\]

Therefore

\[
S^n_\lambda R^x \lambda g(x) - \frac{1}{\lambda} R^x \lambda g(x) \leq S^n_\lambda R^x \lambda g(x) - \frac{1}{\lambda} S^n_\lambda g(x) + \frac{1}{\lambda} S^n_\lambda H(x),
\]

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where
\[ H(y) = \sup_{w \in \mathbb{R}^d} |\mathcal{M}^w R_{\lambda}^0 g(y) - \mathcal{M}^x R_{\lambda}^0 g(x)|. \]

We thus have
\[ S^n_{\lambda} g(x) \leq R^n_{\lambda} g(x) + S^n_{\lambda} H(x). \] (4.15)

By Corollary 4.5 and our choice of \( \zeta \) and \( \lambda \),
\[ \|H\|_2 \leq \kappa (\zeta + \lambda^{-1}) \|g\|_2 \leq \frac{1}{2} \|g\|_2. \]

Multiplying (4.15) by \( \rho(x) \) and integrating,
\[
\left| \int S^n_{\lambda} g(x) \rho(x) \, dx \right| \leq \left| \int R^n_{\lambda} g(x) \rho(x) \, dx \right| + \left| \int S^n_{\lambda} H(x) \rho(x) \, dx \right|
\leq \|R^n_{\lambda} g\|_2 \|\rho\|_2 + \Lambda_n \|H\|_2
\leq \frac{1}{\lambda} \|\rho\|_2 \|g\|_2 + \frac{1}{2} \Lambda_n \|g\|_2,
\]
where \( \Lambda_n \) is defined in Step 1. Taking the supremum over \( g \in C^2_K \) with \( \|g\|_2 \leq 1 \), we thus have
\[ \Lambda_n \leq \frac{\|\rho\|_2}{\lambda} + \frac{1}{2} \Lambda_n. \]

In Step 1 we proved \( \Lambda_n < \infty \), and we conclude
\[ \Lambda_n \leq \frac{2}{\lambda} \|\rho\|_2. \]

Step 3: We now pass to the limit in \( n \). By Step 1 and Step 2, if \( g \in C^2_K \) with \( \|g\|_2 \leq 1 \), then
\[ \left| \int S^n_{\lambda} g(x) \rho(x) \, dx \right| \leq \frac{2\|\rho\|_2}{\lambda}. \]

On the other hand,
\[ S^n_{\lambda} g(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} g(X^n_t) \, dt \to \mathbb{E}^x \int_0^\infty e^{-\lambda t} g(X_t) \, dt = S_{\lambda} g(x) \]
by dominated convergence. We thus see that
\[ \left| \int S_{\lambda} g(x) \rho(x) \, dx \right| \leq \frac{2\|\rho\|_2}{\lambda}. \]
Our result follows by taking the supremum over \( g \in C^2_K \) with \( \|g\|_2 \leq 1 \).

**Proof of Theorem 1.2:** Let \( x_0 \in \mathbb{R}^d \). Let \( \rho \in L^2 \) with compact support. We have seen that it suffices to prove uniqueness when we have a strong Markov family of solutions to the martingale problem for \( \mathcal{L} \), so suppose we have two such families \( \{\mathbb{P}^x_i\} \), \( i = 1, 2 \). Define

\[
S^i_\lambda f(x) = \mathbb{E}^x_i \int_0^\infty e^{-\lambda t} \, dt, \quad i = 1, 2,
\]

and let

\[
S^\Delta = S^1_\lambda - S^2_\lambda.
\]

Suppose \( \lambda_0 \) and \( \zeta \) are chosen so that \( \kappa(\zeta + \lambda_0^{-1}) \leq \frac{1}{2} \) and \( \lambda > \lambda_0 \), and suppose Assumption 4.3 holds with this choice of \( \zeta \).

Since \( \mathbb{P}^x_i \) is a solution to the martingale problem for \( \mathcal{L} \) started at \( x \), for \( f \in C^2_b \)

\[
\mathbb{E}^x_i f(X_t) - f(x) = \mathbb{E}^x_i \int_0^t \mathcal{L} f(X_s) \, ds.
\]

Multiplying by \( e^{-\lambda t} \) and integrating over \( t \) from 0 to \( \infty \),

\[
S^i_\lambda f(x) - \frac{1}{\lambda} f(x) = \mathbb{E}^x_i \int_0^\infty e^{-\lambda t} \int_0^t \mathcal{L} f(X_s) \, ds \, dt
\]

\[
= \mathbb{E}^x_i \int_0^\infty \mathcal{L} f(X_s) \int_s^\infty e^{-\lambda t} \, dt \, ds
\]

\[
= \frac{1}{\lambda} S^\Delta S^i_\lambda f(x)
\]

\[
= \frac{1}{\lambda} S^\Delta S^i_\lambda \mathcal{M}^{x_0} f(x) + \frac{1}{\lambda} S^i_\lambda (\mathcal{L} - \mathcal{M}^{x_0}) f(x).
\]

Now take \( g \in C^2_K \) and set \( f = R^{x_0}_\lambda g \). Then \( f \in C^2_b \) and \( \mathcal{M}^{x_0} f = \lambda R^{x_0}_\lambda g - g \).

Hence

\[
S^i_\lambda R^{x_0}_\lambda g(x) - \frac{1}{\lambda} R^{x_0}_\lambda g(x) = S^i_\lambda R^{x_0}_\lambda g(x) - \frac{1}{\lambda} S^\Delta S^i_\lambda f(x) + \frac{1}{\lambda} S^i_\lambda (\mathcal{L} - \mathcal{M}^{x_0}) R^{x_0}_\lambda g(x),
\]

or

\[
S^i_\lambda g(x) = R^{x_0}_\lambda g(x) + S^i_\lambda (\mathcal{L} - \mathcal{M}^{x_0}) R^{x_0}_\lambda g(x).
\]

(4.16)

Let

\[
\Theta = \sup_{\|g\|_2 \leq 1} \left| \int S^\Delta S^i_\lambda g(x) \rho(x) \, dx \right|.
\]

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By Proposition 4.6, we know that $\Theta < \infty$. From (4.16)

\[ S^\Delta \lambda g(x) = S^\Delta \lambda (\mathcal{L} - \mathcal{M}^{x_0}) R^{x_0}_\lambda g(x). \]

Multiplying by $\rho(x)$ and integrating,

\[ \left| \int S^\Delta \lambda g(x) \rho(x) \, dx \right| = \left| \int S^\Delta \lambda (\mathcal{L} - \mathcal{M}^{x_0}) R^{x_0}_\lambda g(x) \rho(x) \, dx \right| \leq \Theta \| (\mathcal{L} - \mathcal{M}^{x_0}) R^{x_0}_\lambda g \|_2. \]

By Corollary 4.5 this is bounded by $\frac{1}{2} \| g \|_2$. Taking the supremum over $g \in C^2_K$ with $\| g \|_2 \leq 1$, we then obtain $\Theta \leq \frac{1}{2} \Theta$. Since $\Theta < \infty$, this implies $\Theta = 0$. This can be rewritten as

\[ \int S^1 \lambda g(x) \rho(x) \, dx = \int S^2 \lambda g(x) \rho(x) \, dx. \]

This is true for each $\rho \in L^2$ with compact support, and we conclude $S^1 \lambda g(x) = S^2 \lambda g(x)$ for almost every $x$. By Proposition 2.6 $S^i \lambda g(x)$ is continuous in $x$, so we have equality for all $x$. By the uniqueness of the Laplace transform and the right continuity of $X_t$, we conclude

\[ \mathbb{E}^x_t g(X_t) = \mathbb{E}^x_t g(X_t) \]

for all $x$ and all $t$ whenever $g$ is continuous and bounded. By a limit argument this equality holds for all bounded $g$. Finally, by using the Markov property, the finite dimensional distributions under $\mathbb{P}^x_1$ and $\mathbb{P}^x_2$ are the same for each $x$.

The last step is to remove the use of Assumption 4.3. This is a standard localization argument. Because of Assumption 1.1 there exists $\tilde{A}(x, h)$ such that $\tilde{A}$ agrees with $A$ in a neighborhood of $x_0$ and such that Assumption 1.1 holds for $\tilde{A}$. If $\tilde{\mathcal{L}}$ is the operator defined in terms of $\tilde{A}$ in the same way as $\mathcal{L}$ is defined in terms of $A$, the above shows we have uniqueness for the martingale problem for $\tilde{\mathcal{L}}$ started at $x_0$. From this point on, we proceed exactly as in the diffusion case; see [6, Chapter VI]. This completes the proof of Theorem 1.2.

\[ \square \]

**Remark 4.7** It is clear that $\psi_\eta$ can be replaced by any decreasing function $\psi$ such that

\[ \int_{|h| \leq 1} \frac{1}{\psi(|h|)|h|^{d+\alpha}} \, dh < \infty. \]
Remark 4.8 Just as in the case of diffusions, we do not really need continuity of $A(x, h)$ in $x$, just that each point $x_0$ has a neighborhood in which $\overline{A(x, \cdot)}$ is sufficiently close to $\overline{A(x_0, \cdot)}$.

Remark 4.9 In [10] Komatsu considers uniqueness for operators of the form $\mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_1$ is a stable process of index $\alpha$ (not necessarily symmetric, but he requires that the jump kernel for $\mathcal{L}_1$ be $d$ times continuously differentiable in $h$ away from the origin) and

$$\mathcal{L}_2 f(x) = \int [f(x + h) - f(x)] n(x, dh)$$

(with the appropriate modification when $\alpha \geq 1$) where there exists a measure $n^*$ such that $|n(x, dh)| \leq n^*(dh)$ and $\int (1 \wedge |h|^\alpha) n^*(dh) < \infty$. If we write the kernel for $\mathcal{L}_1$ as $A_0(h)/|h|^{d+\alpha}$ and if in addition we assume $n^*$ has a density with respect to Lebesgue measure, we can fit his framework into ours by setting

$$A(x, h) = A_0(h) + \frac{n(x, dh)}{dh} |h|^{d+\alpha}.$$

Remark 4.10 We have not tried to find the weakest possible conditions possible, particularly with regard to “large jumps.” One will still have uniqueness with minimal assumptions on the intensity of the jumps above some size $\delta$. This is apparent from the stochastic differential equations representation of $X$: there are only finitely many jumps of size larger than $\delta$ in any finite time interval, and so one can consider them sequentially. Our results will imply uniqueness up to the time of the first jump of size larger than $\delta$, the law of that jump is uniquely determined by the location the process jumps from, and one then has uniqueness up to the time of the second large jump, and so on.

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