ON THE MOMENTS OF THE RIEMANN ZETA-FUNCTION IN SHORT INTERVALS

ALEKSANDAR IVIĆ

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Abstract. Assuming the Riemann Hypothesis it is proved that, for fixed $k > 0$ and $H = T^\theta$ with fixed $0 < \theta \leq 1$,

$$\int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll H (\log T)^{k^2(1+O(1/\log_3 T))},$$

where $\log_j T = \log(\log_{j-1} T)$. The proof is based on the method of K. Soundararajan [8] for counting the occurrence of large values of $\log |\zeta(\frac{1}{2} + it)|$, who proved that

$$\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll T (\log T)^{k^2 + \varepsilon}.$$

1. Introduction

Power moments of $|\zeta(\frac{1}{2} + it)|$ are a central problem in the theory of the Riemann zeta-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ ($\sigma = \Re s > 1$) and a vast literature exists on this subject (see e.g., the monographs [3], [4], [6] and [9]). If

\begin{equation}
I_k(T, H) := \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} \, dt \quad (k > 0, 1 \ll H \leq T),
\end{equation}

then naturally one seeks asymptotic formulas for $I_k(T, H)$. It is only in the cases $k = 1$ and $k = 2$ that we have precise expressions for $I_k(T, T)$, due to the well-known works of F.V. Atkinson [1] and Y. Motohashi (see e.g. [6]), respectively. Although with the use of methods relating to random matrix theory (see J.B. Conrey et al. [2]) it is possible to make plausible conjectures for the asymptotic
formulas for $I_k(T, T)$ when $k \in \mathbb{N}$ is fixed, no one has proved yet such an asymptotic formula for $k \geq 3$, even assuming the Riemann Hypothesis (RH, that all complex zeros of $\zeta(s)$ have real parts equal to 1/2). Unconditional lower bounds for $I_k(T, H)$ and similar expressions involving the derivatives of $\zeta(s)$ have been obtained in a series of papers by R. Balasubramanian and K. Ramachandra. These results, which are of a general nature and involve various convexity techniques, are expounded in Ramachandra’s monograph [7]. For example, one has unconditionally

\begin{equation}
I_k(T, H) \gg H (\log H)^{k^2} \quad (k \in \mathbb{N}, \log_2 T \ll H \leq T)
\end{equation}

when $k$ is fixed and $\log_j T = \log(\log_{j-1} T)$ is the $j$-th iteration of the natural logarithm. Under the RH it is known that (1.2) holds for any fixed $k > 0$.

Furthermore a classical result of J.E. Littlewood states (see e.g., [9] for a proof) that, under the RH,

\begin{equation}
|\zeta(\frac{1}{2} + it)| \ll \exp \left( C \frac{\log t}{\log_2 t} \right) \quad (C > 0),
\end{equation}

which can be used to provide a trivial upper bound for $I_k(T, H)$. However, recently K. Soundararajan [8] complemented (1.2) in the case $H = T$ by obtaining, under the RH, the non-trivial upper bound

\begin{equation}
\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll \varepsilon T (\log T)^{k^2 + \varepsilon},
\end{equation}

which is valid for any fixed $k > 0$ and any given $\varepsilon > 0$. This result, apart from $'\varepsilon'$, is therefore best possible. His method of proof is based on a large values estimate for $\log |\zeta(\frac{1}{2} + it)|$, which gives as a corollary (under the RH) the bound (1.3) with the explicit constant $C = 3/8$.

The aim of this note is twofold. The main objective is to generalize (under the RH) (1.4) to upper bounds for $I_k(T, H)$. The second aim is to replace $'\varepsilon'$ by an explicit function of $T$ which is $o(1)$ as $T \to \infty$. The result is contained in

**THEOREM 1.** Let $H = T^\theta$ where $0 < \theta \leq 1$ is a fixed number, and let $k$ be a fixed positive number. Then, under the RH, we have

\begin{equation}
I_k(T, H) = \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll H (\log T)^{k^2 \left(1 + O(1/\log_3 T)\right)}.
\end{equation}

Note that, since $H = T^\theta$ and $\theta$ is fixed, the right-hand side of (1.5) is

\begin{equation}
\ll H (\log H)^{k^2 \left(1 + O(1/\log_3 T)\right)},
\end{equation}
which is more in tune with the lower bound in (1.2), although it does not seem possible to reach the range \(\log_2 T \ll H \leq T\) by the present method or to remove \(O(1/\log T)\) from the exponent in (1.5). As already noted, upper bounds of the form \(I_k(T, H) \ll H(\log T)^k\) can be derived unconditionally in the cases \(k = 1\) and \(k = 2\). They are known to hold for \(\theta > 1/3\) (and even for some slightly smaller values of \(\theta\)) when \(k = 1\), and for \(\theta > 2/3\) when \(k = 2\). In the case when \(k = 1/2\) it is known (see K. Ramachandra [7]) that this bound holds unconditionally when \(\theta > 1/2\) and for \(\theta > 1/4\) under the RH. No other results of this type seem to be known for other values of \(\theta\).

Theorem 1 will be deduced from a large values estimate for \(\log |\zeta(1/2 + it)|\), based on Soundararajan’s method [8]. This is

**THEOREM 2.** Let \(H = T^\theta\) where \(0 < \theta \leq 1\) is a fixed number, and let \(\mu(T, H, V)\) denote the measure of points \(t\) from \([T, T + H]\) such that

\[
(1.6) \quad \log |\zeta(1/2 + it)| \geq V, \quad 10\sqrt{\log_2 T} \leq V \leq \frac{3\log 2T}{8\log_2(2T)}.
\]

Then, under the RH, for \(10\sqrt{\log_2 T} \leq V \leq \log_2 T\) we have

\[
(1.7) \quad \mu(T, H, V) \ll H \frac{V}{\sqrt{\log_2 T}} \exp\left( -\frac{V^2}{\log_2 T}\left(1 - \frac{7}{2\theta \log_3 T}\right)\right),
\]

for \(\log_2 T \leq V \leq \frac{1}{2} \theta \log_2 T \log_3 T\) we have

\[
(1.8) \quad \mu(T, H, V) \ll H \exp\left( -\frac{V^2}{\log_2 T}\left(1 - \frac{7V}{4\theta \log_2 T \log_3 T}\right)\right),
\]

and for \(\frac{1}{2} \theta \log_2 T \log_3 T \leq V \leq \frac{3\log 2T}{8\log_2(2T)}\) we have

\[
(1.9) \quad \mu(T, H, V) \ll H \exp\left( -\frac{1}{20} \theta V \log V\right).
\]

2. The necessary Lemmas

In this section we shall state the necessary lemmas for the proof of Theorem 1 and Theorem 2.

**LEMMA 1.** Assume the RH. Let \(T \leq t \leq 2T, T \geq T_0, 2 \leq x \leq T^2\). If \(\lambda_0 = 0.4912\ldots\) denotes the unique positive real number satisfying \(e^{-\lambda_0} = \lambda_0 + \frac{1}{2}\lambda_0^2\), then for \(\lambda \geq \lambda_0\) we have

\[
(2.1) \quad \log |\zeta(1/2 + it)| \leq \Re \sum_{2 \leq n \leq x} \Lambda(n) \frac{\log(x/n)}{\log n} + \frac{(1 + \lambda) \log T}{2 \log x} + O\left(\frac{1}{\log x}\right).
\]
Lemma 1 is due to K. Soundararajan [8]. It is based on Selberg’s classical method (see e.g., E.C. Titchmarsh [9, Th. 14.20]) of the use of an explicit expression for \( \zeta'(s)/\zeta(s) \) by means of a sum containing the familiar von Mangoldt function \( \Lambda(n) \) (equal to \( \log p \) if \( n = p^\alpha \), where \( p \) denotes primes, and \( \Lambda(n) = 0 \) otherwise). Soundararajan’s main innovation is the observation that

\[
F(s) := \Re \sum_{\rho} \frac{1}{s-\rho} = \sum_{\rho} \frac{\frac{\sigma}{2} - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \geq 0 \quad (s = \sigma + it)
\]

if \( \sigma \geq \frac{1}{2} \), since all the complex zeros \( \rho \) of \( \zeta(s) \) are (this is the RH) of the form \( \rho = \frac{1}{2} + i\gamma \) \((\gamma \in \mathbb{R})\). Note that \( F(s) \) appears in the classical expression (this is unconditional, valid when \( s = \sigma + it \), \( t \) is not an ordinate of any \( \rho \) and \( T \leq t \leq 2T \))

\[
-\Re \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2} \log T - F(s) + O(1).
\]

This is integrated over \( \sigma \) from \( \frac{1}{2} \) to \( \sigma_0 (> \frac{1}{2}) \) to produce \((s_0 = \sigma_0 + it)\)

\[
\log |\zeta(\frac{1}{2} + it)| - \log |\zeta(s_0)| \leq (\sigma_0 - \frac{1}{2}) \left( \frac{1}{2} \log T - \frac{1}{2} F(s_0) + O(1) \right),
\]

at which point an expression similar to [9, Th. 14.20] for \( \log |\zeta(s_0)| \) is used, and the non-negativity of \( F(s) \) can be put to advantage.

**Lemma 2.** Assume the RH. If \( T \leq t \leq 2T, 2 \leq x \leq T^2, \sigma \geq \frac{1}{2}, \) then

\[
(2.2) \quad \sum_{2 \leq n \leq x, n \neq p} \frac{\Lambda(n)}{n^{\sigma+iit}} \frac{\log(x/n)}{\log x} \ll \log_3 T.
\]

This is Lemma 2 from Soundararajan [8], where a brief sketch of the proof is indicated. The details are to be found in the work of M.B. Milinovich [5].

**Lemma 3.** Let \( 2 \leq x \leq T, T \geq T_0 \). Let \( 1 \ll H \leq T \) and \( r \in \mathbb{N} \) satisfy \( x^r \leq H \). For any complex numbers \( a(p) \) we have

\[
(2.3) \quad \int_T^{T+H} \left| \sum_{p \leq x} \frac{a(p)}{p^{\frac{1}{2}+it}} \right|^{2r} dt \ll H r! \left( \sum_{p \leq x} \frac{|a(p)|^2}{p} \right)^r.
\]

**Proof.** This lemma is unconditional and is a generalization of Lemma 3 of Soundararajan [8] (when \( H = T \)), but he had the more stringent condition \( x^r \leq T/\log T \) (we have changed his notation from \( k \) to \( r \) to avoid confusion with \( k \) in Theorem 1). Write

\[
\left( \sum_{p \leq x} \frac{a(p)}{p^{\frac{1}{2}+it}} \right)^r = \sum_{n \leq x^r} \frac{a_{r,n}(n)}{n^{\frac{1}{2}+it}},
\]
where \( a_{r,x}(n) = 0 \) unless \( n \) is a product of \( r \) primes factors, each of which is \( \leq x \).

By the mean value theorem for Dirichlet polynomials (see e.g., [3, Chapter 4]) the left-hand side of (2.3) is equal to

\[
(2.4) \quad H \sum_{n \leq x^r} \frac{|a_{r,x}(n)|^2}{n} + O \left( \sum_{n \leq x^r} |a_{r,x}(n)|^2 \right) \ll (H + x^r) \sum_{n \leq x^r} \frac{|a_{r,x}(n)|^2}{n}.
\]

But, as shown in detail in [8], it is not difficult to see that

\[
\sum_{n \leq x^r} \frac{|a_{r,x}(n)|^2}{n} \ll r! \left( \sum_{p \leq x} \frac{|a(p)|^2}{p} \right)^r,
\]

hence (2.3) follows from (2.4), since our assumption is that \( x^r \leq H \).

3. PROOF OF THEOREM 1

The contribution of \( t \) satisfying \( \log |\zeta(\frac{1}{2} + it)| \leq \frac{1}{2} k \log_2 T \) to \( I_k(T, H) \) is trivially

\[
(3.1) \quad \leq H \left\{ (\log T)^{k/2} \right\}^{2k} = H(\log T)^{k^2}.
\]

Likewise the bound (1.5) holds, by (1.7) and (1.8), for the contribution of \( t \) satisfying \( \log |\zeta(\frac{1}{2} + it)| \geq 10k \log_2 T \). Thus we can consider only the range

\[
(3.2) \quad V + \frac{j - 1}{\log_3 T} \leq \log |\zeta(\frac{1}{2} + it)| \leq V + \frac{j}{\log_3 T},
\]

where \( 1 \leq j \ll \log_3 T, V = 2^\ell - \frac{1}{2} k \log_3 T, 1 \leq \ell \leq \frac{3}{2} + \left[ \frac{\log 10}{\log 2} \right] \). If we set

\[
(3.3) \quad U = U(V, j; T) := V + \frac{j - 1}{\log_3 T},
\]

then we have

\[
(3.4) \quad I_k(T, H) \ll H(\log T)^{k^2} + \log_3 T \max_U \mu(T, H, U) \exp \left( 2k(U + 1/\log_3 T) \right),
\]

where \( \mu(T, H, U) \) is the measure of \( t \in [T, T + H] \) for which \( \log |\zeta(\frac{1}{2} + it)| \geq U \), and the maximum is over \( U \) satisfying (3.2)–(3.3). If we use (1.7) and (1.8) of Theorem 2, then in the relevant range for \( U \) we obtain

\[
\mu(T, H, U) \exp(2k(U + 1/\log_3 T)) \ll H \log_2 T \exp \left( 2kU - U^2 G(T) \right),
\]

\[
G(T) := \frac{1}{\log_2 T} \left( 1 + O(1/\log_3 T) \right).
\]

Since \( 2kU - U^2 G(T) \) attains its maximal value at \( U = k/G(T) \), we have

\[
\mu(T, H, U) \exp(2k(U + 1/\log_3 T)) \ll H \log_2 T \exp \left( k^2 \log_2 T(1 + O(1/\log_3 T)) \right)
\]

\[
= H(\log T)^{k^2(1 + O(1/\log_3 T))},
\]

so that (3.4) yields then (1.5) of Theorem 1.
4. Proof of Theorem 2

We let

\[ x = H^{A/V}, \quad z = x^{1/\log_2 T}, \quad A = A(T, V) \geq 1, \]

where \( A \) will be suitably chosen below. We follow the method of proof of [8] and accordingly consider three cases.

**Case 1.** When \( 10 \sqrt{\log_2 T} \leq V \leq \log_2 T \), we take \( A = \frac{1}{2} \log_3 T \).

**Case 2.** When \( \log_2 T \leq V \leq \frac{1}{2} \theta \log_2 T \log_3 T \), we take \( A = \log_2 T \log_3 T \cdot \frac{2}{V} \).

**Case 3.** When \( \frac{1}{2} \theta \log_2 T \log_3 T \leq V \leq \frac{3 \log 2 T}{8 \log_2 2 T} \) we take \( A = \frac{2}{\theta} \).

Note that the last bound for \( V \) comes from the bound (1.3) with \( C = 3/8 \) (under the RH). Suppose that \( \log |\zeta(\frac{1}{2} + it)| \geq V \geq 10 \sqrt{\log_2 T} \) holds. Then Lemma 1 and Lemma 2 yield

\[ V \leq S_1(t) + S_2(t) + \frac{1 + \lambda_0}{2A\theta} V + O(\log_3 T), \]

where we set

\[ S_1(t) := \left| \sum_{p \leq z} \frac{\log(x/p)}{\log x} p^{-\frac{1}{2} - \frac{\lambda_0}{\log x} - it} \right|, \quad S_2(t) := \left| \sum_{z < p \leq x} \frac{\log(x/p)}{\log x} p^{-\frac{1}{2} - \frac{\lambda_0}{\log x} - it} \right|. \]

This means that either

\[ S_1(t) \geq V_1 = V \left( 1 - \frac{7}{8A\theta} \right) \]

or

\[ S_2(t) \geq \frac{V}{8A\theta}. \]

Namely, if neither (4.4) nor (4.5) is true, then (4.2) implies that for some constant \( C > 0 \)

\[ V \leq V \left( 1 - \frac{7}{8A\theta} \right) + \frac{V}{8A\theta} + C \log_3 T. \]

Therefore we should have

\[ \frac{V}{A} \ll \log_3 T, \]

but (4.6) obviously cannot hold in view of the ranges of \( V \) and the choice of \( A \) in Cases 1-3. Let now \( \mu_i(T, H, V) \) \( (i = 1, 2) \) denote the measure of the set of points...
$t \in [T, T + H]$ for which (4.4) and (4.5) hold, respectively. Supposing that (4.4) holds then, by using Lemma 3 with $a(p) = \frac{\log(x/p)}{\log x} p^{-\lambda_0/\log x}$, we obtain

\begin{equation}
\mu_1(T, H, V) V_1^{2r} \leq \int_T^{T+H} |S_1(t)|^{2r} \, dt \ll H r! \left( \sum_{p \leq z} \frac{1}{p} \right)^r.
\end{equation}

The condition in Lemma 3 ($x^r \leq H$ with $x = z$) is equivalent to

\begin{equation}
\frac{Ar}{V \log_2 T} \leq 1.
\end{equation}

Recalling that

$$\sum_{p \leq x} \frac{1}{p} = \log_2 X + O(1),$$

it follows that

$$\log z = \frac{\log x}{\log_2 T} = \frac{A\theta}{V \log_2 T} \log T \leq \frac{\log T}{\log_2 T},$$

since $A \leq V$ in all cases. Therefore we have

\begin{equation}
\sum_{p \leq z} \frac{1}{p} \leq \log_2 T \quad (T \geq T_0).
\end{equation}

Noting that Stirling’s formula yields $r! \ll r^r \sqrt{e}^{-r}$, we infer from (4.7) and (4.9) that

\begin{equation}
\mu_1(T, H, V) \ll H \sqrt{T} \left( \frac{r \log_2 T}{e V_1^2} \right)^r.
\end{equation}

In the Cases 1. and 2. and also in the Case 3. when $V \leq \frac{2}{\theta} \log_2^2 T$, one chooses

$$r = \left[ \frac{V_1^2}{\log_2 T} \right] \geq 1.$$

With this choice of $r$ it is readily seen that (4.8) is satisfied, and (4.10) gives

\begin{equation}
\mu_1(T, H, V) \ll H \sqrt{T} \exp \left( - \frac{V_1^2}{\log_2 T} \right).
\end{equation}

Finally in the Case 3. when $\frac{2}{\theta} \log_2^2 T \leq V \leq (3 \log 2T)/(8 \log_2 2T)$ and $A = 2/\theta$, we have

$$V_1 = V \left( 1 - \frac{7}{8A\theta} \right) = V \left( 1 - \frac{7}{16} \right) > \frac{V}{2},$$
so that with the choice $r = \lfloor V/2 \rfloor$ we see that (4.8) is again satisfied and
\[ \sqrt{r} \left( \frac{r \log_2 T}{eV^2} \right)^r \leq \sqrt{V} \left( \frac{2 \log_2 T}{eV} \right)^r \leq V^{\frac{1}{2}-\frac{r}{2}} \ll \exp\left(\frac{1}{10} V \log V\right), \]
giving in this case
\begin{equation} \mu_1(T, H, V) \ll H \exp\left(\frac{1}{10} V \log V\right). \end{equation}

We bound $\mu_2(T, H, V)$ in a similar way by using (4.5). It follows, again by Lemma 3, that
\[ \left( \frac{V}{8A^2} \right)^{2r} \mu_2(T, H, V) \leq \int_T^{T+H} |S_2(t)|^{2r} \, dt \]
\[ \ll H r! \left( \sum_{z < p \leq x} \frac{1}{p} \right)^r = H r! (\log_2 x - \log_2 z + O(1))^r \]
\[ \ll H \left( r (\log_3 T + O(1)) \right)^r. \]

We obtain
\begin{equation} \mu_2(T, H, V) \ll H \left( \frac{8A}{V} \right)^{2r} (2r \log_3 T)^r \ll H \exp\left(\frac{-V}{2A} \log V\right). \end{equation}

Namely the second inequality in (4.13) is equivalent to
\begin{equation} \left( \frac{A}{V} \right)^2 r \log_3 T \ll \exp\left(\frac{-V}{2rA} \log V\right). \end{equation}

In all Cases 1.-3. we take
\[ r = \left\lfloor \frac{V}{A} - 1 \right\rfloor \geq 1. \]

The condition $x^r \leq H$ in Lemma 3 is equivalent to $rA \leq V$, which is trivial with the above choice of $r$. To establish (4.14) note first that
\begin{equation} \left( \frac{A}{V} \right)^2 r \log_3 T \leq \frac{A}{V} \log_3 T. \end{equation}

In the Case 1. the second expression in (4.15) equals $\log_3^2 T/(2V)$, while
\[ \exp\left(\frac{-V}{2rA} \log V\right) = \exp\left( -(\frac{1}{2} + o(1)) \log V\right) = V^{-1/2+o(1)}. \]
Therefore it suffices to have
\[ \frac{\log^2 T}{V} \ll V^{-1/2+o(1)}, \]
which is true since \( 10 \sqrt{\log_2 T} \leq V \). In the Case 2. the analysis is similar. In the Case 3. we have \( A = 2/\theta \), hence \((A \log_3 T)/V \ll (\log_3 T)/V\) and
\[ \exp \left( -\frac{V}{2rA} \log V \right) = \exp \left( -\left( \frac{1}{2} \theta + o(1) \right) \log V \right) = V^{-\left( \frac{1}{2} \theta + o(1) \right)}, \]
so that (4.14) follows again. Thus we have shown that in all cases
\[ \mu_2(T, H, V) \ll H \exp \left( -\frac{V}{2A} \log V \right). \]

Theorem 2 follows now from (4.10), (4.11) and (4.16). Namely in the Case 1. we have
\[ \frac{V^2}{\log_2 T} = \frac{V^2 \left( 1 - \frac{7}{4\theta \log_3 T} \right)^2}{\log_2 T} \leq V \frac{\log V}{\log_3 T} = \frac{V \log V}{2A}, \]
which gives (1.7). If the Case 2. holds we have again
\[ \frac{V^2_1}{\log_2 T} = \frac{V^2 \left( 1 - \frac{7V}{4\theta \log_2 T \log_3 T} \right)^2}{\log_2 T \log_3 T} \leq \frac{V^2 \log V}{\log_2 T \log_3 T} = \frac{V \log V}{2A}, \]
and (1.8) follows. In the Case 3. when \( \frac{1}{2} \theta \log_2 T \log_3 T \leq V \leq \frac{3}{\theta} \log_2^2 T \) we have
\[ \mu(T, H, V) \ll H \exp \left( -\frac{V^2}{\log_2 T} \right) + H \exp(-\theta V \log V) \]
\[ \ll H \exp(-\frac{\theta}{20} V \log V), \]
since
\[ \frac{V^2_1}{\log_2 T} \geq \frac{V^2}{4 \log_2 T} \geq \frac{\theta V \log_2 T \log_3 T}{8 \log_2 T} \geq \frac{\theta}{20} V \log V. \]

In the remaining range of Case 3. we have
\[ \mu(T, H, V) \ll H \exp(-\frac{1}{10} V \log V) + H \exp(-\frac{V}{2A} \log V) \]
\[ \ll H \exp(-\frac{1}{10} V \log V) + \exp(-\frac{\theta}{4} V \log V), \]
and (1.9) follows. The proof of Theorem 2 is complete.
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Katedra Matematike RGF-a, Universitet u Beogradu, Dušina 7, 11000 Beograd, Serbia.

E-mail address: ivic@rgf.bg.ac.rs, aivic@matf.bg.ac.rs