Kink Stability of Self-Similar Solutions of Scalar Field in $2 + 1$ Gravity

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The kink stability of self-similar solutions of a massless scalar field with circular symmetry in $2 + 1$ gravity is studied, and found that such solutions are unstable against the kink perturbations along the sonic line (self-similar horizon). However, when perturbations outside the sonic line are considered, and taking the ones along the sonic line as their boundary conditions, we find that non-trivial perturbations do not exist. In other words, the consideration of perturbations outside the sonic line limits the unstable mode of the perturbations found along the sonic line. As a result, the critical solution for the scalar collapse remains critical even after the kink perturbations are taken into account.

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I. INTRODUCTION

The studies of non-linearity of the Einstein field equations near the threshold of black hole formation reveal very rich phenomena [1], which are quite similar to critical phenomena in statistical mechanics and quantum field theory [2]. In particular, by numerically studying the gravitational collapse of a massless scalar field in 3 + 1-dimensional spherically symmetric spacetimes, Choptuik found that the mass of such formed black holes takes the form [1],

$$M_{BH} = C(p)(p - p^*)^\gamma,$$

where $C(p)$ is a finite constant with $C(p^*) \neq 0$, and $p$ parameterizes a family of initial data in such a way that when $p > p^*$ black holes are formed, and when $p < p^*$ no black holes are formed. It was shown that, in contrast to $C(p)$, the exponent $\gamma$ is universal to all the families of initial data studied, and was numerically determined as $\gamma \sim 0.37$. The solution with $p = p^*$, usually called the critical solution, is found also universal. Choptuik’s studies were soon generalized to other matter fields [3]. From all the work done so far, the collapse in general falls into two different types, depending on whether the black hole mass takes the scaling form (1.1) or not. When it takes the form, the corresponding collapse is called Type II collapse, and when it does not it is called Type I collapse. In the type II collapse, all the critical solutions found so far have either discrete self-similarity (DSS) or homothetic self-similarity (HSS), depending on the matter fields. In the type I collapse, the critical solutions have neither DSS nor HSS. For certain matter fields, these two types of collapse can co-exist. A critical solution in both two types has one and only one unstable mode. This now is considered as one of the main criteria for a solution to be critical.

The studies of critical collapse have been mainly numerical so far, and analytical ones are still highly hindered by the complexity of the problem, even after imposing some symmetries. Lately, some progress has been achieved in the studies of critical collapse of a scalar field in an anti-de Sitter background in 2 + 1-dimensional spacetimes both numerically [4,5] and analytically [6–9]. This serves as the first analytical model in critical collapse. In particular, Garfinkle [6] first found a class of exact solutions to Einstein-scalar field equations, denoted by $S[n]$, and later Garfinkle and Gundlach (GG) studied their linear perturbations and found that the solution with $n = 2$ has only one unstable mode [8]. By definition this is a critical solution, and the corresponding exponent $\gamma$ in Eq.(1.1) can be read off from the expression [10]

$$\gamma = \frac{1}{|k_1|},$$

where $k_1$
where it was found $\gamma = 4/3$, where $k_1$ denotes the unstable mode. Although the exponent $\gamma$ is close to that found numerically by Pretorius and Choptuik [4], $\gamma \sim 1.2 \pm 0.05$ (but not to the one of Husain and Olivier, $\gamma \sim 0.81$), this solution is different from the numerical critical solution [6]. Using different boundary conditions, Hirschmann, Wang and Wu (HWW) found that the solution with $n = 4$ has only one unstable mode [9]. As first noted by Garfinkle [8], this $n = 4$ solution matches extremely well with the numerical critical solution found by Pretorius and Choptuik [4]. However, the corresponding exponent $\gamma$ now is given by $\gamma = |k_1|^{-1} = 4$, which is significantly different from the numerical ones. The boundary conditions used by HWW are [9]: (a) The perturbations must be free of spacetime singularity on the symmetry axis; (b) They are analytical across the self-similarity horizon, as the background solutions do; (c) No matter field come out of the already formed trapped region [11]. GG considered only Conditions (a) and (b) [8].

In this paper we shall study another important issue of critical collapse for a scalar field, the kink stability. The kink modes result from the existence of critical characteristic lines (they are also referred to as self-similarity horizons, and sonic lines), along which discontinuities of (higher order) derivatives of some physical quantities can be developed and propagate. The instability is characterized by the divergence of the discontinuity, and the blow-up may imply the formation of shock waves [12]. An example that discontinuities of derivatives can propagate along a sonic line is given by the linear perturbation, $\delta \varphi (\tau, z) = \varphi_1 (z)e^{k\tau}$, of the massless scalar field in $2+1$ gravity, which satisfies the following equation [9],

$$z (1 - z) \varphi_1'' + \frac{1}{2} [(1 + 2k) - z(3 + 2k)] \varphi_1' - \frac{1}{2} k \varphi_1 = f(z), \quad (1.3)$$

where a prime denotes the ordinary differentiation with respect to the indicated argument, $f(z)$ is a smooth function of $z$, and $z = 1$ is the location of the sonic line (cf. Eq.(112) in [9]). From the above equation we can see that it is possible for $\varphi_1$ to have discontinuous derivatives only across the line $z = 1$. In fact, assume that $\varphi_1$ is continuous across $z = 1$ (but not its first-order derivative), we can write it in the form

$$\varphi_1(z) = \varphi^+_1(z)H(z - 1) + \varphi^+_1(z) [1 - H(z - 1)], \quad (1.4)$$

where $H(x)$ denotes the Heaviside (step) function, defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \quad (1.5)$$

Then, we find that

$$\varphi'_1 = \varphi^+_1'H(z - 1) + \varphi^+_1'[1 - H(z - 1)],$$

$$\varphi''_1 = \varphi^+_1''H(z - 1) + \varphi^+_1'' [1 - H(z - 1)] + [\varphi'_1']^- \delta(z - 1), \quad (1.6)$$

where $\delta(x)$ denotes the Dirac delta function, and

$$[\varphi'_1']^- \equiv \lim_{z \to 1^+} \left( \frac{d\varphi^+_1(z)}{dz} \right) - \lim_{z \to 1^-} \left( \frac{d\varphi^-_1(z)}{dz} \right). \quad (1.7)$$

Substituting Eq.(1.6) into Eq.(1.3) and considering the facts

$$H^m(x) = H(x), \quad [1 - H(x)]^m = [1 - H(x)],$$

$$[1 - H(x)]H(x) = 0, \quad x\delta(x) = 0, \quad (1.8)$$

where $m$ is an integer, one can see that Eq.(1.3) holds also on the horizon $z = 1$ even when $[\varphi'_1']^- \neq 0$. This is because $(1 - z) [\varphi'_1']^- \delta(z - 1) = 0$, as long as $[\varphi'_1']^-$ is finite. Thus, we have

$$[\delta \varphi, z]^+ = [\varphi'_1]^- e^{k\tau}, \quad (1.9)$$

where $(\quad, z) = \partial(\quad)/\partial z$. The above expression shows clearly how the discontinuity of the first derivative of the perturbation $\delta \varphi (\tau, z)$ propagate along the sonic line $z = 1$. When $Re(k) > 0$ the perturbation grows exponentially as $\tau \to \infty$, and is said unstable with respect to the kink perturbation. When $Re(k) < 0$ the perturbation decays exponentially and is said stable.

Note that if the discontinuity happened on other places, say, $z = z_0 \neq 1$, clearly Eq.(1.3) would not hold on $z = z_0$, because now $(1 - z) [\varphi'_1']^- \delta(z - z_0) \neq 0$. This explains why the discontinuities are allowed only along the sonic lines.
The above analysis also shows that the kink perturbations are different from the ones considered in [8] and [9], because there it was required that \( \varphi_1(z) \) is analytical across \( z = 1 \), that is,

\[
\varphi_1^{(m)} = 0, \quad (m = 1, 2, \ldots)
\]

where \( \varphi_1^{(m)} \) denotes the \( m \)-th order derivative of \( \varphi_1 \). Therefore, kink perturbations were excluded in the studies of linear perturbations of [8] and [9].

Ori and Piran first studied kink stability of self-similar solutions in newtonian gravity [13], and lately Harada generalized such a study to the relativistic case and found that the critical self-similar solutions of a perfect fluid with the equation of state \( P = k\rho \) are not stable against kink perturbations for \( k \geq 0.89 \), where \( P \) and \( \rho \) denote, respectively, the pressure and energy density of the fluid [14]. More recently, Harada and Maeda showed that in four-dimensional spherically symmetric case the self-similar massless scalar solution found lately by Brady et al [15] is also not stable against kink perturbations [16].

In this paper, we study the kink stability of the scalar field in \( 2 + 1 \) gravity. Instead of assuming that \( \delta \varphi(\tau, z) \) is \( \delta \) across the sonic line, as we did in the above example, following Harada [14], and Harada and Maeda [16], we shall assume that \( \delta \varphi(\tau, z) \) is \( c^1 \), that is, \( \delta \varphi(\tau, z) \) and its first-order derivative with respect to \( z \) are continuous across the sonic line, but not its second-order derivative. We shall first show that perturbations obtained along the sonic line allow the existence of unstable modes. However, when we consider perturbations outside the sonic line and take the ones obtained along the sonic line as their boundary conditions, we find that these conditions together with the ones on the symmetry axis do not allow any non-trivial perturbations in the regions outside the sonic line. Therefore, the consideration of perturbations in the whole spacetime limits the unstable mode found along the sonic line. Thus, all the self-similar solutions of the massless scalar field are stable against kink perturbations in \( 2 + 1 \) gravity. As a result, the critical solution for the scalar collapse remains critical, even after the kink perturbations are taken into account.

Specifically, the paper is organized as follows: In Sec. II we give a brief review of the self-similar solution, which is needed in the studies of linear perturbations in Sec. III, in which we first consider the linear perturbations of the self-similar solutions along the sonic line, and then the perturbations outside the sonic line. In Sec. IV, we summarize the main results obtained in this paper and then present our concluding remarks.

II. THE EINSTEIN-SCALAR FIELD EQUATIONS

The general form of metric for a \( (2 + 1) \)-dimensional spacetime with circular symmetry can be cast in the form,

\[
ds^2 = -2e^{2\sigma(u,v)}du dv + r^2(u,v) d\theta^2,
\]

where \( (u, v) \) is a pair of null coordinates varying in the range \( (-\infty, \infty) \), and \( \theta \) is the usual angular coordinate with the hypersurfaces \( \theta = 0 \), \( 2\pi \) being identified. \( \xi(\theta) = \partial_\theta \) is a Killing vector. It should be noted that the form of the metric is unchanged under the coordinate transformations,

\[
u = v(\bar{v}).
\]

To have circular symmetry, some conditions on the symmetry axis needed to be imposed. In general this is not trivial. As a matter of fact, only when the axis is free of spacetime singularity, do we know how to impose these conditions. Since in this paper we are mainly interested in gravitational collapse, we shall assume that the axis is regular at the beginning of the collapse. In particular, we impose the following conditions:

(i) There must exist a symmetry axis, which can be expressed as

\[
X = \left[ \xi_{(\theta)}^\mu \xi_\mu \right] \to 0,
\]

as \( v \to f(u) \), where we assumed that the axis is located at \( r(v = f(u), u) = 0 \).

(ii) The spacetime near the symmetry axis is locally flat, which can be written as

\[
\frac{X_{\alpha\beta} X_{\alpha\beta}}{4X} \to 1,
\]

as \( v \to f(u) \), where \( \left( \right)_\alpha = \partial(\ )/\partial x^\alpha \).

The corresponding Einstein-scalar field equations for the metric (2.1) take the form,
where scalar field are given by

\[ \begin{align*}
    r_{,uu} - 2\sigma_u r_{,u} &= -8\pi Gr\phi_{,u}^2, \\
    r_{,uv} - 2\sigma_v r_{,u} &= -8\pi Gr\phi_{,v}^2, \\
    r_{,uv} + 2\sigma_s u_{,u} &= -8\pi Gr\phi_{,u}\phi_{,v}, \\
    r_{,uv} &= 0,
\end{align*} \]

while the equation of motion for the scalar field is given by

\[ 2\phi_{,uv} + \frac{1}{r} (r_{,u} \phi_{,v} + r_{,v} \phi_{,u}) = 0. \]  

To study self-similar solutions, we first introduce the dimensionless variables, \( u \) and \( v \), via the relations

\[ z = \frac{v}{(-u)}, \quad \tau = -\ln \left( \frac{(-u)}{u_0} \right), \]

where \( u_0 \) is a dimensional constant with the dimension of length, and the above relations are assumed to be valid only in the region \( v \geq 0, u \leq 0 \). We will refer to this region as Region I [cf. Fig. 1].

Self-similar solutions are given by

\[ F(\tau, z) = F_{ss}(z), \]

where \( F \equiv \{ \sigma, s, \varphi \} \), and

\[ r(u, v) \equiv (-u)s(\tau, z), \]
\[ \phi(u, v) \equiv c \ln |u| + \varphi(\tau, z), \]

with \( c \) being an arbitrary constant. A class of such solutions was first found by Garfinkle [6], which can be written as [9]

\[ \sigma_{ss}(u, v) = \frac{1}{2} \ln \left\{ \frac{[v^{1/2} + \epsilon(-u)^{1/2}]^4}{(-uv)^{\chi}} \right\} + \sigma_0^1, \]
\[ r_{ss}(u, v) = (-u) - v, \]
\[ \phi_{ss}(u, v) = 2c \ln \left| v^{1/2} + \epsilon(-u)^{1/2} \right| + \phi_0^1, \]

where \( \epsilon = \pm 1, \sigma_0^1 \) and \( \varphi_0^1 \) are integration constants, and \( \chi \equiv 8\pi Gc^2 \). As shown in [9], the hypersurface \( v = 0 \) for the solutions with \( 1 > \chi \geq 1/2 \) represents a sonic line, and the solutions can be extended across the hypersurface, whereby they can be interpreted as representing the gravitational collapse of a scalar field, in which a black hole is finally formed. The extension can be realized by introducing two new coordinates \( \bar{u} \) and \( \bar{v} \) via the relations

\[ \bar{u} = -(-u)^{1/2n}, \quad \bar{v} = v^{1/2n}, \]

where \( n \equiv 1/[2(1 - \chi)] \geq 1 \). In order to have the extension unique, we require that it be analytical across the hypersurface \( v = 0 \), which, in turn, requires \( n \) to be an integer and satisfy the condition,

\[ n = \frac{1}{2(1 - \chi)} = \begin{cases} 2l, & \epsilon = 1, \\ 2l + 1, & \epsilon = -1, \end{cases} \]

where \( l \) is another integer. For the detail, we refer readers to [9]. In these new coordinates, the metric and the massless scalar field are given by

\[ ds^2 = -2e^{2\sigma_{ss}(\bar{u}, \bar{v})} \bar{u} d\bar{u} d\bar{v} + r_{ss}^2(\bar{u}, \bar{v}) d\theta^2, \]
\[ \bar{\sigma}_{ss}(\bar{u}, \bar{v}) = \sigma_{ss}(u, v) + \frac{1}{2} \ln \left\{ 4n^2 (-\bar{u} \bar{v})^{2n-1} \right\} \]
\[ = \frac{1}{2} \ln \left\{ 4n^2 |f(\bar{u}, \bar{v})|^{4\chi} \right\} + \sigma_0^1, \]
\[ r_{ss}(\bar{u}, \bar{v}) = (-\bar{u})^{2n} - \bar{v}^{2n} \]
\[ \phi_{ss}(\bar{u}, \bar{v}) = 2c \ln |f(\bar{u}, \bar{v})| + \phi_0^1, \]

where \( f(\bar{u}, \bar{v}) \equiv \bar{v}^n + \epsilon(-\bar{u})^n \). Note that the symmetry axis (the vertical line \( r = 0 \) in Fig. 1) is located at \( \bar{v} = \bar{u} \), for which conditions (2.3) requires \( \sigma_0^1 = \frac{1}{4}(1 - 4\chi) \ln(2) \). The corresponding Penrose diagram is given by Fig. 1.
FIG. 1. The Penrose diagram for the solutions given by Eq.(2.16). $\phi, \alpha$ is timelike in Region II and null on the hypersurface $v = 0$. In Region I it is spacelike for $n = 2l$ and timelike for $n = 2l + 1$. The horizontal line $r = 0$ is singular for $n = 2l + 1$ but not for $n = 2l$. In Region I all the rings of $u, v = $ Constant are trapped, while in Region II none of them is trapped.

III. LINEAR PERTURBATIONS OF SELF-SIMILAR SOLUTIONS: KINK STABILITY

In this section, we consider the linear perturbations of the self-similar solutions given by Eq.(2.16). For the sake of simplicity, we shall drop all the bars from $\bar{\sigma}, \bar{u}$ and $\bar{v}$, so that the background solutions can be written as,

$$ds^2 = -2e^{2\sigma_{ss}(u,v)}dudv + r_{ss}^2(u,v)d\theta^2,$$

$$\sigma_{ss}(u,v) = \frac{1}{2} \ln \left( \frac{4n^2 |f(u,v)|^4v}{\chi} \right) + \sigma_0^1,$$

$$r_{ss}(u,v) = (-u)^{2n} - v^{2n},$$

$$\phi_{ss}(u,v) = 2c \ln |f(u,v)| + \phi_0^1,$$

$$f(u,v) \equiv v^n + \epsilon(-v)^n.$$ (3.1)

Let us first divide the spacetime in Fig. 1 into three different regions, $\Omega^\pm$ and $\Sigma$, defined, respectively, by $\Omega^+ = \{x^\alpha : u \leq 0, v \geq 0, v \leq |u|\}$, $\Omega^- = \{x^\alpha : u \leq 0, v \geq 0, v \geq |u|\}$, and $\Sigma = \{x^\alpha : v = 0\}$. Then, for any given $C^1$ function $f(u,v)$, we can write it as

$$f(u,v) = f^+(u,v) [1 - H(v)] + f^-(u,v)H(v),$$ (3.2)

where $f^\pm$ denote the functions, defined, respectively, in the regions $\Omega^\pm$. In the present case, we have

$$f^\pm(u,v) = f_{ss}^\pm(u,v) + \delta f^\pm(u,v),$$ (3.3)

where $f_{ss} \equiv \{\sigma_{ss}, r_{ss}, \phi_{ss}\}$ denotes the background solutions given by Eq.(3.1), which are analytical across $v = 0$,

$$\lim_{v \to 0^-} \frac{\partial^m f_{ss}^+(u,v)}{\partial v^m} = \lim_{v \to 0^+} \frac{\partial^m f_{ss}^-(u,v)}{\partial v^m}, \quad (m = 0, 1, 2,...).$$ (3.4)

Since $f(u,v)$ is $C^1$, we must have

$$\lim_{v \to 0^-} \delta f^+(u,v) = \lim_{v \to 0^+} \delta f^-(u,v) \equiv \delta f_c(u),$$

$$\lim_{v \to 0^-} \delta f^+_{,v}(u,v) = \lim_{v \to 0^+} \delta f^-_{,v}(u,v) \equiv \delta f^{(1)}(u).$$ (3.5)

Then, we find
\[ f_v(u,v) = f^+_v(u,v) [1 - H(v)] + f^-_v(u,v) H(v) , \]
\[ f_{vv}(u,v) = f^+_{vv}(u,v) [1 - H(v)] + f^-_{vv}(u,v) H(v). \]  
(3.6)

Inserting Eqs.(3.3)-(3.6) into Eqs.(2.5)-(2.9) and considering Eq.(1.8), to the first order of \( \delta f \), we obtain

\[
\begin{align*}
\delta r_{uu} - 2 ( \sigma_{ss,u} \delta r_u + r_{ss,u} \delta \sigma_u ) &= -8\pi G \left( 2 r_{ss} \phi_{ss,uu} \delta \phi_u + \phi_{ss,u}^2 \delta r \right), \\
\delta r_{vv} - 2 ( \sigma_{ss,v} \delta r_v + r_{ss,v} \delta \sigma_v ) &= -8\pi G \left( 2 r_{ss} \phi_{ss,vv} \delta \phi_v + \phi_{ss,v}^2 \delta r \right), \\
2 ( r_{ss} \delta \sigma_{uu} + \sigma_{ss,uu} \delta r ) &= -8\pi G \left[ r_{ss} \left( \phi_{ss,uu} \delta \phi_u + \phi_{ss,v} \delta \phi_v + \phi_{ss,uu} \delta r \right) \right], \\
\delta r_{uv} &= 0,
\end{align*}
\]
(3.7)

(3.8)

(3.9)

(3.10)

(3.11)

where the quantities \( f_{ss} \) and \( \delta f \) should be understood as \( f_{ss}^+ \) and \( \delta f^+ \) in \( \Omega^+ \), and \( f_{ss}^- \) and \( \delta f^- \) in \( \Omega^- \).

### A. Kink Stability

Kink stability is the study of the linear perturbations of Eqs.(3.7) - (3.11) along the sonic line \( v = 0 \). To solve these equations for \( \delta f_v(u) \), following Ori and Piran [13] (See also [14,16]), we impose the following conditions: Assume that the perturbations turn on at the moment, say, \( u = u_0 \), then we require

(A) the perturbations initially vanish in the interior,

\[ \delta f^-(u_0,v) = 0, \quad v \in \Omega^-, \quad (3.12) \]

(B) the perturbations and their first-order derivatives be continuous everywhere, and in particular across the sonic line,

\[ [\delta f]^- = 0, \quad [\delta f_v]^- = 0, \quad (v = 0), \quad (3.13) \]

(C) \( \delta \phi_{,,vv} \) and \( \delta \sigma_{,,vv} \) be discontinuous across the sonic line,

\[ \delta \phi''_c(u) \equiv [\delta \phi_{vv}]^- \neq 0, \quad \delta \sigma''_c(u) \equiv [\delta \sigma_{vv}]^- \neq 0, \quad (v = 0). \quad (3.14) \]

From the above we first note that Eq.(3.12) remains true for all \( u > u_0 \). In fact, \( \delta f^-(u,v) = 0 \) are indeed solutions of Eqs.(3.7) - (3.11) in \( \Omega^- \). Then, from Eqs.(3.12) and (3.13) we find

\[
\begin{align*}
\delta f^+(u,0) &= 0, \quad \delta f^+_v(u,0) = 0, \\
\delta \phi''_c(u) &= \delta \phi_{vv}^+(u,0), \quad \delta \sigma''_c(u) = \delta \sigma_{vv}^+(u,0).
\end{align*}
\]
(3.15)

Taking the limit \( v \to 0^- \) in Eqs.(3.7)-(3.11) and considering the above equation we find that

\[ [\delta r_{vv}]^- = \delta r_{vv}^+(u,0) = 0. \quad (3.16) \]

On the other hand, taking derivatives of Eqs.(3.11) and (3.9) with respect to \( v \), and then taking the limit \( v \to 0^- \), we obtain

\[
\begin{align*}
2 r_{ss} (\delta \phi''_c)_{,u} + r_{ss,u} \delta \phi''_c = 0, \quad (3.17) \\
(\delta \sigma''_c)_{,u} = -4\pi G \phi_{ss,uu} \delta \phi''_c, \quad (3.18)
\end{align*}
\]
along the sonic line \( v = 0 \). Substituting Eq.(3.1) into the above equations and then integrating them, we obtain

\[
\begin{align*}
\delta \phi''_c(u) &= \frac{A}{(-u)^n} = \frac{A}{u_0^{1/2} e^{\tau/2}}, \\
\delta \sigma''_c(u) &= -\frac{8\pi G c A}{(-u)^n} = -\frac{8\pi G c A}{u_0^{1/2} e^{\tau/2}}, \quad (3.19)
\end{align*}
\]

where \( A \) is an integration constant. Since \( n \geq 1 \), from the above expressions we can see that both \( \delta \phi''_c(u) \) and \( \delta \sigma''_c(u) \) diverge as \( u \to 0^- \) (or \( \tau \to \infty \)), or in other words, the self-similar solutions are not stable against the kink perturbations.
It should be noted that \( \delta f^+(u,v) \) cannot be zero identically in \( \Omega^+ \), because we already have \( \delta f^-(u,v) = 0 \) in \( \Omega^- \) and
\[
\delta f''(u) = \delta f_{,uv}^+(u,0^-) \neq 0. \tag{3.20}
\]
Then, a natural question rises: Do the perturbations given by Eq.(3.19) match to the ones in region \( \Omega^+ \)? To answer this question, in the next subsection we shall consider the linear perturbations of Eqs.(3.7)-(3.11) in region \( \Omega^+ \), by considering Eqs.(3.15), (3.16) and (3.19) as their boundary conditions at \( v = 0 \).

**B. Linear Perturbations in \( \Omega^+ \)**

To study the linear perturbations in \( \Omega^+ \), it is found convenient to use the dimensionless variables \( \tau \) and \( z \), defined by Eq.(2.10). However, they are valid only in region \( \Omega^- \). In region \( \Omega^+ \) we define them as
\[
\tilde{\tau} = -\ln \left( -\frac{\tilde{\tilde{u}}}{u_0} \right), \quad \tilde{z} = \frac{\tilde{\tilde{\nu}}}{u},
\tag{3.21}
\]
where \( \tilde{\tilde{u}}, \tilde{\tilde{\nu}} \leq 0 \) in \( \Omega^+ \), and
\[
\tilde{\tilde{u}} \equiv -(u)^{2n}, \quad \tilde{\tilde{\nu}} \equiv -(-v)^{2n}.
\tag{3.22}
\]
The null coordinates \( u \) and \( v \) in Eq.(3.22) should be understood as the ones, \( \tilde{u} \) and \( \tilde{\nu} \), defined by Eq.(2.14). In terms of \( \tilde{\tau} \) and \( \tilde{z} \), the background solutions (3.1) in \( \Omega^+ \) can be written in the form,
\[
s_0(\tilde{z}) = 1 - \tilde{z},
\]
\[
\sigma_0(\tilde{z}) = 2\chi \ln \left( \tilde{\tilde{z}}^{1/4} + \tilde{\tilde{z}}^{-1/4} \right) + \sigma_0^1,
\]
\[
\varphi_0(\tilde{z}) = 2c \ln \left( 1 + \tilde{\tilde{z}}^{1/2} \right) + \varphi_0^1,
\tag{3.23}
\]
with
\[
\sigma_{ss}(u,v) = \sigma_0(\tilde{z}) + \frac{1}{2} \ln \left[ 4n^2(1-uv)^{2n-1} \right],
\]
\[
r_{ss}(u,v) = (-u)^{2n}s_0(\tilde{z}),
\]
\[
\phi_{ss}(u,v) = \varphi_0(\tilde{z}) + c \ln \left[ (-u)^{2n} \right].
\tag{3.24}
\]
Again, \( u \) and \( v \) in Eq.(3.24) should be understood as \( \tilde{u} \) and \( \tilde{\nu} \) defined by Eq.(2.14). For detail, we refer readers to Eqs.(57)-(59) in [9]. Without causing any confusions, in the following we shall drop the tildes from \( \tilde{\tau} \) and \( \tilde{z} \). Then, writing the perturbations as
\[
\delta r = (-\tilde{u})s_1(\tilde{z})e^{k\tau},
\]
\[
\delta \sigma = \sigma_1(\tilde{z})e^{k\tau},
\]
\[
\delta \phi = \varphi_1(\tilde{z})e^{k\tau},
\tag{3.25}
\]
it can be shown that the linearized perturbations given by Eqs.(3.7)-(3.11) reduce exactly to the ones of (67)-(71) of [9], the general solutions of which are Eqs.(110)-(118) for \( k = 1 \), and Eqs.(120)-(125) for \( k \neq 1 \), given in [9]. In particular, \( s_1(\tilde{z}) \) is given by
\[
s_1(\tilde{z}) = \begin{cases} 
\beta \ln(z) + s_1^0, & k = 1, \\
\beta z^{1-k} + s_1^0, & k \neq 1,
\end{cases}
\tag{3.26}
\]
where \( \beta \) and \( s_1^0 \) are the integration constants.

However, since here we consider the kink stability, the boundary conditions are different from the ones used in [8,9]. In particular, in [8,9] it was required that the perturbations be analytical across the surface \( v = 0 \), while in the present case these conditions should be replaced by Eqs.(3.15), (3.16) and (3.19), which can be written as
\begin{align*}
k &= \frac{1}{2} \\
s_1(z) &\approx O\left(z^3\right) \\
\sigma_1(z) &\approx -\frac{4\pi GcA}{2u_0^{1/2}}z^2 + O\left(z^3\right) \\
\varphi_1(z) &\approx \frac{A}{2u_0^{1/2}}z^2 + O\left(z^3\right) \\
\end{align*}
(3.27)
as \to 0. \text{ Thus, the instability of the perturbations along the hypersurface } v = 0 \text{ found in the last subsection is due to a single mode, } k = 1/2. \text{ In addition to the above conditions, we also need to impose some conditions on the symmetry axis } r = 0, \text{ so that the local-flatness conditions (2.3) and (2.4) are satisfied. In terms of } f_1(z), \text{ these conditions are exactly the ones given by Eq.(105) in [9],}
\begin{align*}
s_1(z)|_{z=1} &= 0 \\
\sigma_1(z)|_{z=1} &\sim \text{ finite} \\
\left\{(1-z)\frac{d\varphi_1(z)}{dz} - 2k\varphi_1(z)\right\}|_{z=1} &\sim \text{ finite}, \ (r = 0). \\
\end{align*}
(3.28)
From Eqs.(3.26) and (3.28) we find that \( s_1^0 = -\beta \), while Eq.(3.27) requires \( \beta = 0 \), for which the solutions of \( \sigma_1(z) \) and \( \varphi_1(z) \) with \( k = 1/2 \) are given by [9]
\begin{align*}
\sigma_1(z) &= \frac{2\chi}{c}(1-z^{1/2}) \left[ z^{1/2}(1 + z^{1/2}) \frac{d\varphi_1(z)}{dz} + \frac{1}{2}\varphi_1 \right], \\
\varphi_1(z) &= c_1 F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) + c_2 F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - z\right), \\
\end{align*}
(3.29)
(3.30)
where \( c_1 \) and \( c_2 \) are two arbitrary constants, and \( F(a, b; c; z) \) denotes the ordinary hypergeometric function with \( F(a, b; c; 0) = 1 \). From the expression [17],
\begin{align*}
F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) &= \frac{1}{\pi^2} \sum_{n=0}^{\infty} \frac{2\Gamma^2\left(\frac{4}{3} + n\right)}{(n!)^2} \left\{ \psi(n + 1) - \psi\left(n + \frac{1}{2}\right) - \frac{1}{2}\ln(1-z) \right\} (1-z)^n, \\
\end{align*}
(3.31)
we find
\begin{align*}
F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) &\to -\frac{1}{\pi}\ln(1-z), \\
\end{align*}
(3.32)
as \( z \to 1 \). Then, Eq.(3.28) requires \( c_1 = 0 \). On the other hand, from Eq.(3.31) we also find
\begin{align*}
F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - z\right) &\to -\frac{1}{\pi}\ln(z), \\
\end{align*}
(3.33)
as \( z \to 0 \). Thus, the conditions of Eq.(3.27) yield \( c_2 = 0 \). In review of all the above, we find that the boundary conditions (3.27) and (3.28) require
\begin{align*}
s_1(z) = \sigma_1(z) = \varphi_1(z) = 0. \\
\end{align*}
(3.34)
That is, non-trivial perturbations in \( \Omega^+ \) are not allowed by the boundary conditions (3.27) and (3.28). Then, we must have \( \delta f''(u) = 0 \). In other words, the consideration of the perturbations in \( \Omega^+ \) limits the unstable mode of the perturbations along the sonic line \( v = 0 \).

IV. CONCLUSIONS

In this paper, we have studied the kink stability of the self-similar solutions of a massless scalar field in 2+1 gravity, and found that perturbations along the sonic line (self-similar horizon) indeed allow the existence of an unstable mode.
In the study of kink stability, it is assumed that the spacetime inside the sonic line is not perturbed, that is, \( \delta f^{-}(u, v) = 0 \) identically [13,14,16]. Then, \( \delta f^{+}(u, v) \) must be non-zero outside the sonic line, in order to have non-vanishing perturbations along the sonic line. However, the perturbations outside the sonic line cannot be arbitrary. In particular, they have to match to the ones along the sonic line. In addition, they need also satisfy some physical/geometrical conditions, such as, the local-flatness conditions on the symmetry axis. A natural question now is: After considering all these, does the spectrum of the perturbations obtained along the sonic line still remain the same?

To answer this question, in Sec. III we have studied the perturbations outside the sonic line, by taking the ones obtained along the sonic line as their boundary conditions. We have shown explicitly that these conditions, together with the ones on the symmetry axis, indeed alter the spectrum of the perturbations along the sonic line, and in particular, they limit all the unstable modes. Thus, \textit{all the self-similar solutions of the massless scalar field in } 2 + 1 \textit{ gravity is stable against kink perturbations.} As a result, \textit{the critical solution for the scalar collapse remains critical even after the kink perturbations are taken into account.}

Finally, we note that in the newtonian gravity the spectrum of the perturbations along the sonic line remains the same, even after the perturbations outside the sonic line are taken into account [18]. It would be very interesting to see if this is still the case in four-dimensional spacetimes in the framework of Einstein’s theory of gravity.

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\[1\] M. W. Choptuik, Phys. Rev. Lett. \textbf{70}, 9 (1993); “Critical Behavior in Massless Scalar Field Collapse,” in \textit{Approaches to Numerical Relativity, Proceedings of the International Workshop on Numerical Relativity}, Southampton, December, 1991, Edited by Ray d’Invern (Cambridge University Press, Cambridge 1994); “Critical Behavior in Scalar Field Collapse,” in \textit{Deterministic Chaos in General Relativity}, Edited by D. Hobill et al. (Plenum Press, New York, 1994).

\[2\] G.I. Barenblatt, \textit{Similarity, Self-Similarity, and Intermediate Asymptotics} (Consultants Bureau, New York, 1979); N. Goldenfeld, \textit{Lectures on Phase Transitions and the Renormalization Group} (Addison Wesley Publishing Company, New York, 1992).

\[3\] A. Wang, Braz. J. Phys. \textbf{31}, 188 (2001) [arXiv:gr-qc/0104073]; C. Gundlach, Phys. Rep. \textbf{376}, 339 (2003); and references therein.

\[4\] F. Pretorius and M. W. Choptuik, Phys. Rev. \textbf{D62}, 124012 (2000).

\[5\] V. Husain and M. Olivier, Class. Quantum Grav. \textbf{18}, L1 (2001).

\[6\] D. Garfinkle, Phys. Rev. \textbf{D63}, 044007 (2001).

\[7\] G. Clément and A. Fabbri, Class. Quantum Grav. \textbf{18}, 3665 (2001); Nucl. Phys. \textbf{B630}, 269 (2002); M. Cavaglia, G. Clement, and A. Fabbri, Phys. Rev. \textbf{D70}, 044010 (2004).

\[8\] D. Garfinkle and C Gundlach, Phys. Rev. \textbf{D66}, 044015 (2002).

\[9\] E.W. Hirschmann, A. Wang, and Y. Wu, Class. Quantum Grav. \textbf{21}, 1791 (2004) [arXiv:gr-qc/0207121].

\[10\] C.R. Evans and J. S. Coleman, Phys. Rev. Lett. \textbf{72}, 1782 (1994); T. Koike, T. Hara, and S. Adachi, \textit{ibid}., \textbf{74}, 5170 (1995); C. Gundlach, \textit{ibid}., \textbf{75}, 3214 (1995); E.W. Hirschmann and D. M. Eardley, Phys. Rev. \textbf{D52}, 5850 (1995).

\[11\] S. Chandrasekhar, \textit{The Mathematical Theory of Black Holes} (Clarendon Press, Oxford University Press, Oxford, 1983).

\[12\] R. Courant and K.O. Friedrichs, \textit{Supersonic Flow and Shock Waves} (Springer-Verlag, 1948), Sections 25 - 29; L.D. Landau and E.M. Lifshitz, \textit{Fluid Mechanics}, Second Edition (Pergamon Press, New York, 1987), Section 96; I.I. Lipatov and V.V. Teshukov, Fluid Dynamics, \textbf{39}, 97 (2004).

\[13\] A. Ori and T. Piran, Mon. Not. R. Astron. Soc. (London) \textbf{214}, 1 (1988).

\[14\] T. Harada, Class. Quantum Grav. \textbf{18}, 4549 (2001).

\[15\] P.R. Brady, M. W. Choptuik, C. Gundlach, and D. W. Neilsen, Class. Quantum Grav. \textbf{19}, 6359 (2002).

\[16\] T. Harada and H. Maeda, Class. Quantum Grav. \textbf{21}, 371 (2004).

\[17\] M. Abramowitz and I.A. Stegun, \textit{Handbook of Mathematical Functions}, (Dover Publications, INC., New York, 1972), pp.555-566.

\[18\] A. Wang and Y. Wu, “Kink Stability of Isothermal Spherical Self-Similar Flow Revisited,” arXiv:astro-ph/0504451 (2005).