THERE ARE NO EXOTIC ACTIONS OF DIFFEOMORPHISM GROUPS ON 1-MANIFOLDS

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Abstract. Let $M$ be a manifold, $N$ a 1-dimensional manifold. Assuming $r \neq \dim(M) + 1$, we show that any nontrivial homomorphism $\rho : \text{Diff}_r^c(M) \to \text{Homeo}(N)$ has a standard form: necessarily $M$ is 1-dimensional, and there are countably many embeddings $\phi_i : M \to N$ with disjoint images such that the action of $\rho$ is conjugate (via the product of the $\phi_i$) to the diagonal action of $\text{Diff}_r^c(M)$ on $M \times M \times \ldots$ on $\bigcup_i \phi_i(M)$, and trivial elsewhere. This solves a conjecture of Matsumoto. We also show that the groups $\text{Diff}_r^c(M)$ have no countable index subgroups.

1. Introduction

Let $\text{Diff}_r^c(M)$ denote the identity component of the group of compactly supported $C^r$ diffeomorphisms of a manifold $M$. In this paper, we prove the following statement.

Theorem 1.1. Let $M$ be a connected manifold, and suppose that $\rho : \text{Diff}_r^c(M) \to \text{Homeo}(N)$ is a homomorphism, where $N = S^1$ or $N = \mathbb{R}$, $r \neq \dim(M) + 1$. Then $\dim(M) = 1$ and there are countably many disjoint embeddings $\phi_i : M \to N$ such that $\rho(g)|_{\phi_i(M)} = \phi_i g \phi_i^{-1}$ and $N - \bigcup_i \phi_i(M)$ is globally fixed by the action.

This proves [12, Conjecture 1.3] and generalizes works of Mann [8], Militon [13] and Matsumoto [12], but with an independent proof. Matsumoto’s work [12] proves an analogous result when the target is $\text{Diff}^1(N)$ using rigidity theorems of [3] for solvable affine subgroups of $\text{Diff}^1(\mathbb{R})$. This generalized [8], which proved the result for homomorphisms to $\text{Diff}^2(N)$ using Kopell’s lemma. Militon [13] studies homomorphisms where the source is the group of homeomorphisms of $M$. Our proof here is comparatively short, and is self-contained modulo the standard but difficult result that $\text{Diff}^r_c(M)$, for $r \neq \dim(M) + 1$ is a simple group. Whether this holds for $r = \dim(M) + 1$ is an open question; this is responsible for our restrictions on dimension in the statement.

Theorem 1.1 is already known in the case where $\rho$ is assumed to be continuous; it is a consequence of the orbit classification theorem of [5], and was likely known to others before. In the case where the target is the group of smooth diffeomorphisms of $N$, this also follows from work of Hurtado [6] who proves additionally that any such homomorphism is necessarily (weakly) continuous. Here we make no assumptions on continuity, however, our proof suggests that diffeomorphism groups exhibit “automatic continuity”–like properties. Specifically, we show the following small index property.

Theorem 1.2 (The small index property of $\text{Diff}_r^c(M)$). If $r \neq \dim(M) + 1$, then $\text{Diff}_r^c(M)$ has no proper countable index subgroup. Equivalently, $\text{Diff}_r^c(M)$ has no nontrivial homomorphism to the permutation group $S_\infty$. 

1
This is in stark contrast with the case for finite dimensional Lie groups, where we have the following.

**Theorem 1.3** (Thomas [17] and Kallman [7]). There is an injective homomorphism $SL_n(\mathbb{R}) \to S_\infty$.

Thus, one consequence of Theorem 1.2 and 1.3 is that there is no nontrivial homomorphism from $\text{Diff}_c^r(M)$ into a linear group. Of course, this is nearly immediate if one considers only continuous homomorphisms, since $\text{Diff}_c^r(M)$ is infinite dimensional, and one may simply cite the invariance of domain theorem.

If $G$ is a group with a non-open subgroup $H$ of countable index, then the action of $G$ on the coset space $G/H$ gives a discontinuous homomorphism to $S_\infty$. This is one of very few known general recipes for producing discontinuous group homomorphisms (see [16]), so gives some (weak) evidence that $\text{Diff}_c^r(M)$ might have the automatic continuity property already known to hold for $\text{Homeo}(M)$ by [9], or for homomorphisms between groups of smooth diffeomorphisms as in [6].

Theorem 1.1 also gives new examples of left orderable groups that do not act on the line. It is a well known fact that any countable group with a left-invariant total order admits a faithful homomorphism to $\text{Homeo}_+(\mathbb{R})$. For $r > 0$, the groups $\text{Diff}_c^r(\mathbb{R}^n)$ for $r > 0$ are known to be left-orderable: the Thurston stability theorem [19] implies that they are locally indicable (any finitely generated subgroup surjects to $\mathbb{Z}$), which implies that they are left-orderable by the Burns-Hale theorem ([4], see also [14, Corollary 2]).

Thus, we have the following.

**Corollary 1.4.** For $r > 0$, the group $\text{Diff}_c^r(\mathbb{R}^n)$ is left-orderable but has no faithful action on the line or the circle.

The proof of Theorem 1.2 uses the idea from the first step of the proof of automatic continuity for homeomorphism groups of [9], following Rosendal [15]. This result is then used to prove Theorem 1.1 by constraining the supports and fixed sets of elements for the action on $N$. We are then able to use this information to build a map from $M$ to $N$.

## 2. Proof of the small index property

In this section, we prove Theorem 1.2. The proof follows the local version of the arguments in [9] and [15] for automatic continuity of $\text{Homeo}_0(M)$.

**Proof.** Let $M$ be a manifold and $r \neq \text{dim}(M) + 1$. Let $G = \text{Diff}_c^r(M)$, and for an open subset $U \subset M$, denote by $G_U$ the subgroup of $\text{Diff}_c^r(M)$ consisting of elements whose support is compactly contained in $U$. Thus, $G_U \cong \text{Diff}_c^r(U)$. Suppose for contradiction that $H \subset G$ is a countable index subgroup. We will show in step 1 that there is some ball $U$ in $M$ such that $G_U \subset H$. After this, we will show that $H$ acts transitively on $M$, thus every $x \in M$ is contained in some open set $U_x$ such that $G_{U_x} \subset H$. The fragmentation property gives that $\text{Diff}_c^r(M)$ is generated by the union of such sets $G_{U_x}$ (this is true for any collection of sets $U_x$ which form an open cover of $M$, see [2, Ch.1]), so this is sufficient to prove $H = G$.

**Step 1.** Let $g_1H, g_2H, \ldots$ denote the left cosets of $H$. Let $B \subset M$ be any ball, and take a sequence of disjoint balls $B_i \subset M$, with diameter tending to 0 and such that the sequence $B_i$ Hausdorff converges to a point.
We first claim that there exists some \( i \) such that every element \( f \in G_{B_i} \) sufficiently close to the identity agrees with the restriction of an element \( w_f \in g_i H \) to \( B_i \). Furthermore, we will have that \( w_f \) is supported on \( B \). To prove this, let \( U_i \) be an identity neighborhood of \( G_{B_i} \), chosen small enough so that for any sequence of diffeomorphisms \( f_i \in U_i \), the infinite composition \( \prod_i f_i \) is an element of \( G \). Equivalently, \( f_i \) must converge to the identity in \( G \) fast enough. Supposing our claim is not true, we can find \( h_i \in H \) such that the restriction of \( g_i h_i \) to \( B_i \) does not lie in \( U_i \). But \( \prod_i (g_i h_i) \in G \) so lies in some coset \( g_j H \) and restricts to \( g_j h_j \) on each \( B_j \), a contradiction. We have in fact shown something stronger, for if \( f \in G_{B_j} \), then \( w_{i_d}^{-1} w_f \in H \) and restricts to \( f \) on \( B_j \), so this shows that every element in \( U_j \) agrees with the restriction of an element of \( H \) to \( B_j \). Since \( U_j \) generates \( G_{B_j} \), we conclude that every element of \( G_{B_j} \) agrees with the restriction of an element of \( H \) to \( B_j \).

Now we use a commutator trick. Apply the same argument as above using \( B_j \) in place of \( B \). We find a smaller ball \( B' \subset B_j \) such that every element \( f \in G_{B'} \) agrees with the restriction to \( B' \) of an element \( v_f \in H \) and \( v_f \) is supported on \( B_j \). Since \( \text{Diff}_c(B') \) is perfect \([10, 11, 15]\), any element \( f \in \text{Diff}_c(B') \) may be written as a product of commutators \( f = \prod [a_i, b_i] \). But \([a_i, b_i] = [v_{a_i}, w_{b_i}] \) since the supports of \( v_{a_i} \) and \( w_{b_i} \) intersect only in \( B' \), and so \( f = \prod [v_{a_i}, w_{b_i}] \in H \). This ends the proof of the first step.

**Step 2: transitivity.** To prove transitivity, let \( B' \) be the ball from step 1, and let \( x \in B' \). Suppose \( y \in M \) is some point not in the orbit of \( x \). Let \( f_t \) be a flow such that \( f_t(y) \in B' \) for all \( t \in (1, 2) \). Such a flow can be defined to have support on a neighborhood of a path from \( x \) to \( y \). Since \( B' \) lies in the orbit of \( x \) under \( H \), we have that \( f_t \notin H \) for \( t \in (1, 2) \). We know that \( H \cap \{ f_t : t \in \mathbb{R} \} \) is a countable index subgroup of \( \{ f_t : t \in \mathbb{R} \} \cong \mathbb{R} \). In particular, it must intersect every open interval of \( \mathbb{R} \), this gives the desired contradiction. \( \square \)

As an immediate consequence, we can conclude that any fixed point free action of such a group on the line or circle is minimal.

**Corollary 2.1.** With the same restrictions on \( r \) as above, if \( \text{Diff}_c^r(M) \) acts on \( \mathbb{R} \) or \( S^1 \) without global fixed points, then there are no invariant open sets. In particular the action has a dense orbit.

**Proof.** Suppose the action has an invariant open set. Then \( \text{Diff}_c^r(M) \) permutes the (countably many) connected components of \( U \). The stabilizer of an interval is a countable index subgroup, so by Theorem 1.2 the permutation action is trivial. Thus each interval is fixed and their endpoints are global fixed points. \( \square \)

### 3. Proof of Theorem 1.1

For the proof Theorem 1.1 we set the following notation. As in the previous section we fix some \( r \neq \dim(M) + 1 \) and when \( U \subset M \) is an open set we denote by \( G_U \) the set of elements of \( \text{Diff}_c^r(M) \) supported on \( U \). Also, \( G^U \subset \text{Diff}_c^r(M) \) denotes the set of elements that pointwise fix \( U \). The open support of a homeomorphism \( g \) is the set \( \text{Osupp}(g) := N - \text{Fix}(g) \); as is standard, the support of \( g \) is defined to be the closure of \( \text{Osupp}(g) \).

**Proof.** We will assume the action on \( N \) has no global fixed points, since if the action does have fixed points, then \( N - \text{Fix}(\rho) \) is a union of open intervals, each with a fixed-point free action of \( \text{Diff}_c^r(M) \), so it suffices to understand such actions. In this case, we will show that there is a single homeomorphism \( \phi : M \to N \) such that the action on \( N \) is induced by conjugation by \( \phi \).
Lemma 3.1. For any action, if \( U \cap V = \emptyset \), then \( \text{Osupp}(\rho(G_U)) \cap \text{Osupp}(\rho(G_V)) = \emptyset \).

Proof. Since \( G_U \) and \( G_V \) commute, \( \rho(G_V) \) preserves \( \text{Osupp}(\rho(G_U)) \), permuting its connected components. By Theorem [1,2] this action is trivial. Let \( I \) be a connected component of \( \text{Osupp}(\rho(G_U)) \). Suppose \( \rho(G_V) \) acts nontrivially on \( I \). Since \( G_V \) is simple group, its action on \( I \) is faithful. Since \( G_V \) is not abelian, Hölder’s theorem implies that some nontrivial \( \rho(g) \in \rho(G_V) \) acts with a fixed point. But then \( \rho(G_V) \) permutes the connected components of \( I - \text{Osupp}(\rho(g)) \), and this permutation action is trivial. Thus, \( \rho(G_U) \) has a fixed point in \( I \), contradicting that \( I \subseteq \text{Osupp}(\rho(G_U)) \). \( \square \)

Also, if \( \bar{U} \cap \bar{V} = \emptyset \) then \( G^U \) and \( G^V \) generate \( \text{Diff}^r_c(M) \), so our assumption that there are no global fixed points for the action implies that \( \text{Fix}(\rho(G^U)) \cap \text{Fix}(\rho(G^V)) = \emptyset \) as well.

Our next goal is to define a map from \( M \) to \( N \). For each \( x \in M \) pick a neighborhood basis \( U_n \) of \( x \) so \( \bigcap_n U_n = \{x\} \). Let \( S_x = \bigcap_n \text{Osupp}(\rho(G_{U_n})) \) and let \( T_x = \bigcap_n \text{Fix}(\rho(G_{U_n}^x)) \). Note that this is independent of the choice of neighborhood basis.

Lemma 3.2. If \( x \neq y \), then \( S_x \cap S_y = \emptyset \) and \( T_x \cap T_y = \emptyset \). Also, \( S_x \) and \( T_x \) have empty interior.

Proof. The first assertion follows immediately from Lemma 3.1 and the second because \( T_x \cap T_y \) would be globally fixed by \( \rho \) by our observation above. Furthermore, if \( g(x) = y \), then it follows from the definition that
\[
\rho(g)S_x = \bigcap_n \rho(g)\text{Osupp}(\rho(G_{U_n})) = \bigcap_n \text{Osupp}(\rho(g)\rho(G_{U_n})\rho(g)^{-1}) = \bigcap_n \text{Osupp}(\rho(G_{U_n}^y)).
\]
Thus, \( \rho(g)S_x = S_y \). Similarly we have \( T_y = \rho(g)T_x \). Thus, if some \( S_x \) has nonempty interior, disjointness of \( S_x \) and \( S_y \) would give an uncountable family of disjoint open sets in \( N \), a contradiction. The same applies to the sets \( T_x \). \( \square \)

We next prove these sets, though defined differently, are in fact the same.

Lemma 3.3. \( S_x = T_x \)

Proof. Fix \( x \) and let \( U_n \) be a neighborhood basis of \( x \). Since \( G^{U_n} \subset G_{N - \overline{U_{n+1}}} \) and since \( N - \overline{U_{n+1}} \) and \( U_n \) are disjoint, by Lemma 3.1 we have that
\[
\text{Osupp}(\rho(G_{U_{n+1}})) = N - \text{Fix}(\rho(G_{U_{n+1}})) \subset \text{Fix}(\rho(G^{U_n})).
\]
Thus \( S_x \subset T_x \). For the reverse inclusion, suppose \( z \in T_x - S_x \). Then \( z \notin \text{Osupp}(\rho(G_{U_n})) \) for some \( n \); i.e., \( z \in \text{Fix}(\rho(G_{U_n})) \). Also \( z \in \text{Fix}(\rho(G^{U_{n+1}})) \) by the definition of \( T_x \). But \( G_{U_n} \) and \( G^{U_{n+1}} \) together generate \( \text{Diff}^r_c(M) \) (this again is the fragmentation property), so this implies that \( z \) is a global fixed point. \( \square \)

Lemma 3.4. \( S_x \) is nonempty.

Proof. If the action is on \( S^1 \), this follows immediately since \( S_x = T_x \) is the intersection of nested, nonempty closed sets. If \( N = \mathbb{R} \), the same is true provided that \( \text{Fix}(\rho(G^{U_n})) \), (or equivalently \( \text{Osupp}(\rho(G_{U_n})) \), does not leave every compact set as \( n \to \infty \). Supposing for contradiction that this is true, this means that for each \( x \in M \) and compact \( K \subset \mathbb{R} \) there is a neighborhood \( U(x) \) such that \( \text{Osupp}(\rho(G_{U(x)})) \cap K = \emptyset \). Fix any compact subset \( A \subseteq M \). Then finitely many of these neighborhoods \( U(x) \), for \( n \in A \) cover \( A \). But then the union of these finitely many subgroups \( G_{U(x)} \) generate \( \text{Diff}_c(A) \subset \text{Diff}_c(M) \),
hence $K \cap \text{Osupp}(\text{Diff}_c(A)) = \emptyset$. Since every element of $\text{Diff}_c(M)$ lies in $\text{Diff}_c(A)$ for some compact subset $A$ of $M$, we conclude that $K \cap \text{Osupp}(\text{Diff}_c(M)) = \emptyset$ contradicting that $\rho$ had no global fixed points in $N$. \hfill \square

**Construction of $\phi$.** To finish the proof, we wish to show that $S_x$ is a singleton, and the assignment $\phi : x \mapsto S_x$ is a homeomorphism conjugating $\rho$ with the standard action of $\text{Diff}(M)$ on $M$. We will actually show first that $x \mapsto S_x$ is a local homeomorphism, use this to *conclude* that $S_x$ is discrete, and proceed from there.

Let $I = (a, b)$ be a connected component of $N - S_x$, chosen so that $a \neq -\infty$ if $N = \mathbb{R}$. If $N = S^1$ and $S_x$ is a singleton, it is possible that both “endpoints” of this interval agree. For simplicity, we treat the case where $a \neq b$, the case $a = b$ on the circle can be handled exactly the same strategy, and in fact the argument simplifies quite a bit since $S_x$ is already a singleton. Let $U$ be a neighborhood of $x$ small enough so that $b \notin \text{Osupp}(\rho(G_U))$; this is possible by definition of $S_x$. Then for each $g \in G_U$, $g(a) < b$.

**Step 1: definition of $\phi$ locally**

Note that $a$ is not accumulated from the right in $S_x$. For any $n$, denote by $O_n$ the connected component of $\text{Osupp}(\rho(G_{U_n}))$ that contains $a$. Since $a$ cannot be accumulated to the right by points of $S_x$ (i.e. on at least the right side it looks like an isolated point of $S_x$), there exists $k \in \mathbb{N}$ such that $a$ is the right most point of $S_x \cap O_k$. We claim that, for $y \in U_k$, the set $S_y \cap O_k$ also has a rightmost point, in which case we define $\phi(y)$ to be this rightmost point.

To see such a rightmost point exists, take $g \in G_{U_k}$ with $g(x) = y$. Then $\rho(g)(S_x) = S_y$. Since $\rho(g)$ fixes endpoints of $O_k$ by definition, we know that $\rho(g)(a) \in S_y$, which is also the rightmost point of $S_y \cap O_k$. Thus, $\phi$ is a well defined function on $O_k$ with image contained in $U_k$. An equivalent definition of $\phi$ is that $\phi(y) := \rho(g)(a)$, where $g$ is any diffeomorphism in $G_{U_k}$ such that $g(x) = y$. Our argument above shows this is independent of choice of $g$.

**Step 2: local continuity of $\phi$ on $U_k$** We first show that $\phi$ is continuous at $x$. Suppose $x_n \to x$ is a convergent sequence. Passing to a subsequence and reindexing if needed, we may assume that $x_n \in U_n$. Then we may take $g \in G_{U_n}$ so that $g(x) = x_n$, so $\phi(x_n) = \rho(g)(x)$. Since the sequence of connected components of $\text{Osupp}(\rho(G_{U_n}))$ containing $x$ converges to $x$, we get that $\phi(x_n) \to x$.

Now we show continuity on $U_k$. Now take any point $x' \in U_k$, and a sequence $x'_n \to x$ in $U_k$. There exists $g \in G_{U_k}$ such that $g(x) = x'$ so $g^{-1}(x'_n)$ is a sequence converging to $x$. It follows from continuity at $x$ that $\phi(g^{-1}(x'_n))$ converges to $\phi(x)$. By definition, $\rho(g)\phi(g^{-1}(x'_n)) = \phi(x'_n)$, so we conclude that $\phi(x'_n)$ converges to $\phi(x)$.

Note also that $\phi$ is injective by Lemma \ref{lem:injective}. Thus, by invariance of domain, we conclude that $M$ is one-dimensional so equal to $\mathbb{R}$ or $S^1$, and $\phi$ gives a homeomorphism from $U_k$ onto an open neighborhood $A$ of $a$ in $N$. In particular, this shows that $a$ is an isolated point of $S_x$.

**Step 3: extension of $\phi$ globally**

The last step is to show that $\phi$ extends naturally to a globally defined homeomorphism $M \to N$. Note first that the orbit of $A$ under $\rho(G)$ is an open, $\rho(G)$-invariant set, so by Corollary \ref{cor:global_extension}, $\rho(G)(A) = N$.

This topological transitivity implies that, for all $x$, every point of $S_x$ is an isolated point, and the map $S_x \mapsto x$ is a covering map, and it is equivariant with respect to $\rho$. 
since $\rho(g)(S_x) = S_{g(x)}$. In the case $M = S^1$, it follows that $N = S^1$ and the cover is degree one, as can be seen by considering the subgroup of rotations. In the case $M = \mathbb{R}$ we immediately have that $N = \mathbb{R}$.

□

References

[1] R. D. Anderson. The algebraic simplicity of certain groups of homeomorphisms. Amer. J. Math., 80:955–963, 1958.
[2] Augustin Banyaga. The structure of classical diffeomorphism groups, volume 400 of Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1997.
[3] C. Bonatti, I. Monteverde, A. Navas, and C. Rivas. Rigidity for $C^1$ actions on the interval arising from hyperbolicity I: solvable groups. Math. Z., 286(3-4):919–949, 2017.
[4] R. G. Burns and V. W. D. Hale. A note on group rings of certain torsion-free groups. Canad. Math. Bull., 15:441–445, 1972.
[5] Lei Chen and Kathryn Mann. Structure theorems for actions of homeomorphism groups. Preprint: https://arxiv.org/abs/1902.05117, 2019.
[6] Sebastian Hurtado. Continuity of discrete homomorphisms of diffeomorphism groups. Geom. Topol., 19(4):2117–2154, 2015.
[7] Robert R. Kallman. Every reasonably sized matrix group is a subgroup of $S_\infty$. Fund. Math., 164(1):35–40, 2000.
[8] Kathryn Mann. Homomorphisms between diffeomorphism groups. Ergodic Theory Dynam. Systems, 35(1):192–214, 2015.
[9] Kathryn Mann. Automatic continuity for homeomorphism groups and applications. Geom. Topol., 20(5):3033–3056, 2016. With an appendix by Frédéric Le Roux and Mann.
[10] John N. Mather. Commutators of diffeomorphisms. Comment. Math. Helv., 49:512–528, 1974.
[11] John N. Mather. Commutators of diffeomorphisms. II. Comment. Math. Helv., 50:33–40, 1975.
[12] Shigenori Matsumoto. Actions of groups of diffeomorphisms on one-manifolds by $C^1$ diffeomorphisms. In Geometry, dynamics, and foliations 2013, volume 72 of Adv. Stud. Pure Math., pages 441–451. Math. Soc. Japan, Tokyo, 2017.
[13] Emmanuel Militon. Actions of groups of homeomorphisms on one-manifolds. Groups Geom. Dyn., 10(1):45–63, 2016.
[14] Dale Rolfsen. A topological view of ordered groups. In Knots in Poland III. Part III, volume 103 of Banach Center Publ., pages 357–369. Polish Acad. Sci. Inst. Math., Warsaw, 2014.
[15] Christian Rosendal. Automatic continuity in homeomorphism groups of compact 2-manifolds. Israel J. Math., 166:349–367, 2008.
[16] Christian Rosendal. Automatic continuity of group homomorphisms. Bull. Symbolic Logic, 15(2):184–214, 2009.
[17] Simon Thomas. Infinite products of finite simple groups. II. J. Group Theory, 2(4):401–434, 1999.
[18] William Thurston. Foliations and groups of diffeomorphisms. Bull. Amer. Math. Soc., 80:304–307, 1974.
[19] William P. Thurston. A generalization of the Reeb stability theorem. Topology, 13:347–352, 1974.

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