Braided Quantum Field Theory

Robert Oeckl*

Centre de Physique Théorique,
CNRS Luminy, 13288 Marseille, France
and
Department of Applied Mathematics and Mathematical Physics,
University of Cambridge, Cambridge CB3 0WA, UK

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Abstract

We develop a general framework for quantum field theory on non-commutative spaces, i.e., spaces with quantum group symmetry. We use the path integral approach to obtain expressions for \(n\)-point functions. Perturbation theory leads us to generalised Feynman diagrams which are braided, i.e., they have non-trivial over- and under-crossings. We demonstrate the power of our approach by applying it to \(\phi^4\)-theory on the quantum 2-sphere. We find that the basic divergent diagram of the theory is regularised.

*email: r.oeckl@damtp.cam.ac.uk
1 Introduction

The idea that space-time might not be accurately described by ordinary geometry was expressed already a long time ago. It was then motivated by the problems encountered in dealing with the divergences of quantum field theories. An early suggestion was that spatial coordinates might in fact be noncommuting observables [27]. For a long time development has been hampered by the lack of proper mathematical tools. Only with the advent of noncommutative geometry [5] and quantum groups have such ideas taken a more concrete form. Quantum groups emerged in fact from the theory of integrable models in physics and were connected from the beginning to the idea of noncommutative symmetries in physical systems [2, 11, 30]. It was then also suggested that they might play a role in physics at very short distances [18]. The idea that quantum symmetry or noncommutativity might serve as a regulator for quantum field theories was emphasised in [19] and [12]. The persistent inability to unite quantum field theory with gravity is a main motivation behind such considerations. In this context it is interesting to note that noncommutative geometric structures are emerging also in string theory [5]. Despite progress in describing various physical models on noncommutative spaces (see e.g. [17, 11, 4, 3]), an approach general enough to be independent of a particular choice of noncommutative space has been lacking. We aim at taking a step in this direction by providing a framework for doing quantum field theory on any noncommutative space with quantum group symmetry.

The basic underlying idea of our approach is to take ordinary quantum field theory, formulate it in a purely algebraic language and then generalise in this formulation to noncommutative spaces. It turns out that this generalisation is completely natural. It involves no arbitrary additional input and no further choices (except for trivial choices like taking left or right actions). We start with two fundamental ingredients of quantum field theory, namely the space of fields together with the group of symmetries acting on it. Generalising to the noncommutative context, this means that we have a vector space of fields coacted upon by a quantum group (which we take to mean coquasitriangular Hopf algebra) of symmetries. Thus, the space of fields becomes an object in the category of representations (comodules) of the quantum group, which is braided. I.e., we are naturally in the context of braided geometry [21, Chapter 10]. We emphasise that the braiding is forced on us by the requirement of covariance under the quantum group symmetry.

\footnote{Recall that a braiding means that for two representations $V, W$ the intertwiner of the tensor products $V \otimes W \rightarrow W \otimes V$ becomes nontrivial, i.e. different from the flip map.}
symmetry and not introduced by hand. It also turns out (at least for our example in Section 3) that the braiding rather than the noncommutativity itself is crucial to achieve regularisation of a conventional theory. This seems to have been missed out in previous works. For previous indications that noncommutativity is not necessarily sufficient for regularisation see e.g. [8].

We follow the path integral approach, going from Gaussian path integrals via perturbation theory to Feynman diagrams. In the noncommutative setting this procedure naturally leads us to generalised Feynman diagrams that are braid diagrams, i.e., they have nontrivial over- and under-crossings.

For an algebraically rigorous treatment we require the quantum group of symmetries to be cosemisimple corresponding to compactness in the commutative case. However, when aiming to regularise UV-divergences this is not necessarily a disadvantage, since they should not be affected by the global properties of a space.

We start out in Section 2 by defining normalised Gaussian integrals on braided spaces based on [13] naturally generalising Gaussian integration on commutative spaces. This provides us with the free $n$-point functions of a braided quantum field theory. Developing perturbation theory in analogy to ordinary quantum field theory we obtain the braided analogues of Feynman diagrams. It turns out that symmetry factors of ordinary Feynman diagrams are resolved into different (and not necessarily equivalent) diagrams in the braided case.

In Section 3 we consider the case where the space of fields is a quantum homogeneous space under the symmetry quantum group. Inspired by the conventional commutative case this gives us a more compact description of $n$-point functions. Furthermore, it allows for simplifications in braided Feynman diagrams.

While our approach is somewhat formal up to this point, Section 4 introduces a context that allows us to work algebraically rigorously in infinite dimensions. We need a further assumption to do this, which corresponds in the commutative case to the space-time being compact.

Finally, in Section 5 we deliver on the promise to perform $q$-regularisation within braided quantum field theory. To this end we consider $\phi^4$-theory on the standard quantum 2-sphere [20]. We make use of all the machinery developed up to this point to show that the only basic divergence of $\phi^4$-theory in two dimensions, the tadpole diagram, becomes finite at $q > 1$. We identify the divergence in $q$-space and suggest that it would not depend on the conventional degree of divergence of a diagram.

By a quantum group we generally mean a Hopf algebra equipped with a coquasitriangular structure (see e.g. [21]. We denote the coaction by $\Delta$, the counit by $\epsilon$, and the antipode by $S$. We use Sweedler’s notation [28].
\( \Delta a = a_{(1)} \otimes a_{(2)}, \) etc., with summation implied. We apply the same notation to Hopf algebras in braided categories. The braiding is denoted by \( \psi. \)

While working over a general field \( \mathbb{k} \) in Sections 2–4 we specialise to the complex numbers in Section 5.

2 Formal Braided Quantum Field Theory

We start out in this section by developing normalised Gaussian integration on braided spaces leading to a braided generalisation of Wick’s Theorem. The less algebraically minded reader may find it convenient to proceed with Section 2.2 where braided path integrals are discussed in quantum field theoretic language, and accept the main result of Section 2.1 (Theorem 2.1 and its corollary) as given.

2.1 Braided Gaussian Integration

Braided categories arise as the categories of modules or comodules over quantum groups (Hopf algebras) with quasitriangular respectively coquasitriangular structure (see e.g. \[21\]). The latter case will be the one of interest to us later. We consider rigid braided categories, where we have for every object \( X \) a dual object \( X^* \) and morphisms \( ev : X \otimes X^* \to \mathbb{k} \) (evaluation) and \( coev : \mathbb{k} \to X^* \otimes X \) (coevaluation) that compose to the identity in the obvious ways. Although rigidity usually implies finite dimensionality, we shall see later (Section 4) how we can deal with infinite dimensional objects. The differentiation and Gaussian integration on braided spaces that we require were developed by Majid \[20\] and Kempf and Majid \[13\] in an \( R \)-matrix setting. (The special case of \( \mathbb{R}^n_q \) was treated earlier in \[1\].) We need a more abstract and basis free formulation of the formalism so that we redevelop the notions here. Furthermore, our Theorem 2.1 goes beyond \[13, \text{Theorem 5.1}\].

Recall that a braiding on a category of vector spaces is an assignment to any pair of vector spaces \( V, W \) of an invertible morphism \( \psi_{V,W} : V \otimes W \to W \otimes V. \) These morphisms are required to be compatible with the tensor product such that \( \psi_{U,V \otimes W} = (\psi_{U,W} \otimes id) \circ (id \otimes \psi_{V,W}) \) and \( \psi_{U \otimes V,W} = (id \otimes \psi_{U,W}) \circ (\psi_{U,V} \otimes id). \) If the category is a category of modules or comodules of a quantum group the morphisms are the intertwiners. The braiding then generalises the trivial exchange map \( \psi_{V,W}(v \otimes w) = w \otimes v \) which is an intertwiner for representations of ordinary groups. In the following we simply write \( \psi \) for the braiding if no confusion can arise as to the spaces on which it is defined.

Suppose we have some rigid braided category \( \mathcal{B} \) and a vector space \( X \in \mathcal{B}. \)
Essentially, we want to define the (normalised) integral of functions \( \alpha \) in the “coordinate ring” on \( X \) multiplied by a Gaussian weight function \( w \), i.e., we want to define

\[
Z(\alpha) := \frac{\int \alpha w}{\int w}.
\]

(1)

First, we need to specify this “coordinate ring”. We identify the dual space \( X^* \in B \) as the space of “coordinate functions” on \( X \). This corresponds to the situation in \( \mathbb{R}^n \) where a coordinate function is just a linear map from \( \mathbb{R}^n \) into the real numbers. The polynomial functions on \( X \) are naturally elements of the free unital tensor algebra over \( X^* \),

\[
\hat{X}^* := \bigoplus_{n=0}^{\infty} X^{*n}, \quad \text{with} \quad X^{*0} := 1 \quad \text{and} \quad X^{*n} := X^* \otimes \cdots \otimes X^*, \quad n \text{ times}
\]

where 1 is the one-dimensional space generated by the identity. 1 plays the role of the constant functions and the tensor product corresponds to the product of functions. \( \hat{X}^* \) naturally has the structure of a braided Hopf algebra (a Hopf algebra in a braided category, see [21]) via

\[
\Delta a = a \otimes 1 + 1 \otimes a, \quad \epsilon(a) = 0, \quad S a = -a
\]

for \( a \in X^* \) and \( \Delta, \epsilon, S \) extend to \( \hat{X}^* \) as braided (anti-)algebra maps. Explicitly, the coproduct is defined inductively by the identity

\[
\Delta \circ \cdot = (\cdot \otimes \cdot) \circ (\text{id} \otimes \psi \otimes \text{id}) \circ (\Delta \otimes \Delta)
\]

of maps \( \hat{X}^* \otimes \hat{X}^* \to \hat{X}^* \otimes \hat{X}^* \). The braided Hopf algebra structure can be thought of as encoding translations on \( X \).

To make the notion of “coordinate ring” more precise, one could perhaps consider a kind of symmetrised quotient of \( \hat{X}^* \) in analogy with the observation that coordinates commute in ordinary geometry. There seems to be no obvious choice for such a quotient in the general braided case. Remarkably, however, such a choice is not necessary. In fact, the following discussion is entirely independent of any relations, as long as they preserve the (graded) braided Hopf algebra structure.

The next step is the introduction of differentials [20]. The space of coordinate differentials should be dual to the space \( X^* \) of coordinate functions. We just take \( X \) itself and define differentiation on \( X^* \) by the pairing \( \text{ev} : X \otimes X^* \to \mathbb{k} \) in \( B \). To extend differentiation to the whole “coordinate
ring” $\hat{X}^*$, we note that the coproduct encodes coordinate translation. This leads to the natural definition that

$$\text{diff} := (\hat{\text{ev}} \otimes \text{id}) \circ (\text{id} \otimes \Delta) : X \otimes \hat{X}^* \to \hat{X}^*$$

is differentiation on $\hat{X}^*$. Here, $\hat{\text{ev}}$ is the trivial extension of ev to $X \otimes \hat{X}^* \to \mathbb{k}$, i.e., $\hat{\text{ev}}|_{X \otimes X^*} = 0$ for $n \neq 1$. We also use the more intuitive notation $\partial(a) := \text{diff}(\partial \otimes a)$ for $\partial \in X$ and $a \in \hat{X}^*$. Let $\partial \in X$ and $\alpha, \beta \in \hat{X}^*$. The definition of $\hat{\text{ev}}$ gives at once

$$\hat{\text{ev}}(\partial \otimes \alpha \beta) = \hat{\text{ev}}(\partial \otimes \alpha) \epsilon(\beta) + \hat{\text{ev}}(\partial \otimes \beta) \epsilon(\alpha).$$

Using that the coproduct is a braided algebra map, we obtain the braided Leibniz rule

$$\partial(\alpha \beta) = \partial(\alpha) \beta + \psi^{-1}(\partial \otimes \alpha)(\beta). \quad (2)$$

Iteration yields

$$\partial(\alpha) = (\text{ev} \otimes \text{id}^{n-1})(\partial \otimes [n]_{\psi} \alpha),$$

where $n$ is the degree of $\alpha$ and

$$[n]_{\psi} := \text{id}^n + \psi \otimes \text{id}^{n-2} + \cdots + \psi_{n-2,1} \otimes \text{id} + \psi_{n-1,1}$$

is a braided integer. We adopt the convention of writing $\psi_{n,m}$ for the braiding between $X^*^n$ and $X^*^m$ (respectively $\psi_{n,m}^{-1}$ for the inverse braiding).

As in [13] we view the Gaussian weight $w$ formally as an element of $\hat{X}^*$ and define its differentiation via an isomorphism

$$\gamma : X \to X^* \quad \text{so that} \quad \partial(w) = -\gamma(\partial)w \quad \text{for} \quad \partial \in X. \quad (3)$$

This expresses the familiar notion that differentiating a Gaussian weight yields a coordinate function times the Gaussian weight. $\gamma$ should accordingly be thought of as defining a braided analogue of the quadratic form in the exponential of the weight.

Also familiar from ordinary Gaussian integration is the fact that integrals of total differentials vanish. That is, we require

$$\int \partial(\alpha w) = 0 \quad \text{for} \quad \partial \in X, \alpha \in \hat{X}^*. \quad (4)$$

It turns out that the three rules (2), (3), and (4) completely determine the integral (1).
Remarkably, the statement that the Gaussian integral of a polynomial function can be expressed solely in terms of Gaussian integrals of quadratic functions still holds true in the braided case. This generalises what is known in quantum field theory as Wick’s Theorem. To state it, we need another set of braided integers \([n]'_\psi : X^* \rightarrow X^*\) with
\[
[n]'_\psi := \id^n + \id^{n-2} \otimes \psi^{-1} + \cdots + \psi^{-1}_{1,n-1},
\]
which are related to the original ones by \([n]'_\psi = \psi^{-1}_{1,n-1} \circ [n]_\psi\). We also require the corresponding braided double factorials \([2n-1]'!!_\psi : X^{*2n} \rightarrow X^{*2n}\) with
\[
[2n-1]'!!_\psi := ([1]'_\psi \otimes \id^{2n-1}) \circ ([3]'_\psi \otimes \id^{2n-3}) \circ \cdots \circ ([2n-1]'_\psi \otimes \id).
\]

**Theorem 2.1 (Braided Wick Theorem).**

\[
\begin{align*}
\mathcal{Z}|_{X^*} &= \ev \circ \psi \circ (\id \otimes \gamma^{-1}), \\
\mathcal{Z}|_{X^{*2n}} &= (\mathcal{Z}|_{X^*})^{2n} \circ [2n-1]'!!_\psi, \quad \mathcal{Z}|_{X^{*2n-1}} = 0, \quad \forall n \in \mathbb{N}.
\end{align*}
\]

**Proof.** For \(\alpha \in \widetilde{X}^*\) and \(a \in X^*\) we have
\[
aaw = -\alpha \diff (\gamma^{-1}(a) \otimes w) = -\diff (\psi(\alpha \otimes \gamma^{-1}(a))w) + \diff \psi(\alpha \otimes \gamma^{-1}(a))w
\]
using the differential property (3) of \(w\) and the braided Leibniz rule (2).
Applying \(\mathcal{Z}\), we can ignore the total differential and obtain
\[
\mathcal{Z}(\alpha a) = \mathcal{Z}(\diff \psi(\alpha \otimes \gamma^{-1}(a))).
\]

This gives us immediately
\[
\mathcal{Z}(a) = 0 \quad \text{and} \quad \mathcal{Z}(ab) = \ev \circ \psi(a \otimes \gamma^{-1}(b))
\]
for \(b \in X^*\). We rewrite (6) to find
\[
\begin{align*}
\mathcal{Z}|_{X^{*n}} &= \mathcal{Z}|_{X^{*n-2}} \circ \diff \circ (\gamma^{-1} \otimes \id^{n-1}) \circ \psi_{n-1,1} \\
&= \mathcal{Z}|_{X^{*n-2}} \circ (\ev \otimes \id^{n-2}) \circ (\gamma^{-1} \otimes [n-1]_\psi) \circ \psi_{n-1,1} \\
&= (\ev \otimes \mathcal{Z}|_{X^{*n-2}}) \circ (\gamma^{-1} \otimes [n-1]_\psi) \circ \psi_{n-1,1} \\
&= (\ev \otimes \mathcal{Z}|_{X^{*n-2}}) \circ \psi_{n-1,1} \circ ([n-1]_\psi \otimes \gamma^{-1}) \\
&= (\mathcal{Z}|_{X^*} \otimes \mathcal{Z}|_{X^{*n-2}}) \circ (\id \otimes \psi_{n-2,1}) \circ ([n-1]_\psi \otimes \id) \\
&= (\mathcal{Z}|_{X^{*n-2}} \otimes \mathcal{Z}|_{X^*}) \circ \psi_{1,n-2}^{-1} \circ (\id \otimes \psi_{n-2,1}) \circ ([n-1]_\psi \otimes \id) \\
&= (\mathcal{Z}|_{X^{*n-2}} \otimes \mathcal{Z}|_{X^*}) \circ ([n-1]_\psi \otimes \id),
\end{align*}
\]
which gives us a recursive definition of \(\mathcal{Z}\) leading to the formulas stated. \(\blacksquare\)
Another set of the braided integers

\[ [n]_\psi'' := \text{id}^n + \psi^{-1} \otimes \text{id}^{n-2} + \cdots + \psi_{1,n-1}^{-1} \]

with \[ (2n-1)_\psi'' !! := (\text{id} \otimes [2n-1]_\psi'') \cdots (\text{id}^{2n-3} \otimes [3]_\psi'') (\text{id}^{2n-1} \otimes [1]_\psi'') \]

serves to formulate the dual version of the theorem.

**Corollary 2.2.** Let \( Z^k \in X^k \) denote the dual of \( Z \mid_{X^k} \). Then

\[
Z^2 = \psi \circ (\gamma^{-1} \otimes \text{id}) \circ \text{coev}, \\
Z^{2n} = [2n-1]_\psi'' !! (Z^2)^n, \quad Z^{2n-1} = 0, \quad \forall n \in \mathbb{N},
\]

**Proof.** This is obtained from Theorem 2.1 by reversing of arrows or equivalently by turning diagrams upside down in the diagrammatic language of braided categories.

\[ \square \]

### 2.2 Braided Path Integrals

The \( n \)-point function of an ordinary quantum field theory with action \( S \), evaluated at \((x_1, \ldots, x_n)\) is given by the path integral\(^2\)

\[
\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) e^{-S(\phi)}}{\int \mathcal{D}\phi e^{-S(\phi)}}.
\]

This is really the normalised integral of the functional \( \phi \mapsto \phi(x_1) \cdots \phi(x_n) \) with weight \( w(\phi) = e^{-S(\phi)} \) over the space \( X \) of classical fields of the theory. The parameters \( x_i \) denote here points in space-time as well as additional internal field indices.

For the non-interacting theory the action \( S \) is replaced by the free action \( S_0 \). The path integral is then a Gaussian integral and the decomposition of \( n \)-point functions into 2-point functions (propagators) is governed by Wick’s theorem. Generalising to braided spaces (when the symmetry group is allowed to be a quantum group) we are in the framework of Section 2.1. Then, the value of an \( n \)-point function is still given in terms of values of 2-point functions (propagators). This is the result of Theorem 2.1 which generalises Wick’s Theorem. The (unevaluated) \( n \)-point function \( Z^n \) itself is an element in the \( n \)-fold tensor product \( X^n \) of the space of fields \( X \) and we write

\[
Z^n(x_1, \ldots, x_n) = \langle \phi(x_1) \cdots \phi(x_n) \rangle_0,
\]

\(^2\)The Euclidean signature of the action is chosen for definiteness and does not imply a restriction to Euclidean field theory.
the index 0 indicating that we deal with the free theory. The decomposition of \( Z^n \) into propagators \( Z^2 \) is given by Corollary 2.2, which is Theorem 2.1 in dual form, i.e., for "unevaluated" functions.

The connection between the map \( \gamma \) determining the (unevaluated) propagator according to Theorem 2.1 (Corollary 2.2) and the free action in ordinary quantum field theory is as follows. Let \( \partial \) be some differential with respect to the space of fields. The definition of \( \gamma \) in (3) corresponds to

\[
(\partial(e^{-S_0})) = -(\gamma(\partial))(\phi)e^{-S_0(\phi)},
\]

in ordinary quantum field theory. Thus we obtain

\[
(\gamma(\partial))(\phi) = (\partial S_0)(\phi). \tag{8}
\]

To determine interacting \( n \)-point functions, we use the same perturbative techniques as in ordinary quantum field theory. For \( S = S_0 + \lambda S_{\text{int}} \) with coupling constant \( \lambda \), we expand

\[
Z^n_{\text{int}}(x_1, \ldots, x_n) = \langle \phi(x_1) \cdots \phi(x_n) \rangle = \int \mathcal{D}\phi \frac{\phi(x_1) \cdots \phi(x_n)(1 - \lambda S_{\text{int}}(\phi) + \ldots) e^{-S_0(\phi)}}{\int \mathcal{D}\phi (1 - \lambda S_{\text{int}}(\phi) + \ldots) e^{-S_0(\phi)}} = \frac{\langle \phi(x_1) \cdots \phi(x_n) \rangle_0 - \lambda \langle \phi(x_1) \cdots \phi(x_n) S_{\text{int}}(\phi) \rangle_0 + \ldots}{1 - \lambda \langle S_{\text{int}}(\phi) \rangle_0 + \ldots}.
\]

For \( S_{\text{int}} \) of degree \( k \) we can write

\[
\langle \phi(x_1) \cdots \phi(x_n) S_{\text{int}}(\phi) \rangle_0 = ((\text{id}^n \otimes S_{\text{int}}) Z^{n+k})(x_1, \ldots, x_n)
\]

etc. by viewing \( S_{\text{int}} \) as a map \( X^k \to k \). Then, removing the explicit evaluations we obtain

\[
Z^n_{\text{int}} = \frac{Z^n - \lambda(\text{id}^n \otimes S_{\text{int}})(Z^{n+k}) + \frac{1}{2}\lambda^2(\text{id}^n \otimes S_{\text{int}} \otimes S_{\text{int}})(Z^{n+2k}) + \ldots}{1 - \lambda S_{\text{int}}(Z^k) + \frac{1}{2}\lambda^2(S_{\text{int}} \otimes S_{\text{int}})(Z^{2k}) + \ldots}, \tag{9}
\]

an expression for the interacting \( n \)-point function valid in the general braided case. Vacuum contributions cancel as usual. Note that we have used the ordinary exponential expansion for the interaction and not, say, a certain braided version. The latter might be more natural if, e.g., one wants to look at identities between diagrams of different order. However, we shall not consider this issue here.
### 2.3 Braided Feynman Diagrams

We are now ready to generalise Feynman Diagrams to our braided setting. To do this we use and modify the diagrammatic language of braided categories appropriately:

- An $n$-point function is an element in $X \otimes \cdots \otimes X$ ($n$-fold). Thus, its diagram is closed to the top and ends in $n$ strands on the bottom. Any strand represents an element of $X$, i.e., a field.

- The propagator $Z^2 \in X \otimes X$ is represented by an arch, see Figure 1.a.

- An $n$-leg vertex is a map $X \otimes \cdots \otimes X \to \mathbb{k}$. It is represented by $n$ strands joining in a dot, see Figure 1.b. Notice that the order of incoming strands is relevant.

- Over- and under-crossings correspond to the braiding and its inverse, see Figure 2.

- Any Feynman diagram is built out of propagators, (possibly different kinds of) vertices, and strands with crossings, connecting the propagators and vertices, or ending at the bottom.

Otherwise the usual rules of braided diagrammatics apply. Notice that in contrast to ordinary Feynman diagrams all external legs end on one line (the bottom line of the diagram) and are ordered. This is necessary due to the possible non-trivial braid statistics in our setting. For the case of trivial braiding we can relax this and shift the external legs around as well as change the order of strands at vertices so as to obtain ordinary Feynman diagrams in more familiar form.

The diagrams for the free $2n$-point functions can be read off directly from Corollary 2.2. The crossings are encoded in the braided integers $[j]_\psi^\nu$. Figure 3 shows for example the free 4-point function and Figure 4 the free 6-point function. For the interacting $n$-point functions we use formula (9) to obtain the diagrams. $S_{\text{int}}$ gives us the vertices. Consider for example the 2-point function in Euclidean $\phi^4$-theory. To order $\lambda$ we get

$$Z^2_{\text{int}} = Z^2 - \lambda \left( (\text{id}^2 \otimes S_{\text{int}})(Z^6) - Z^2 \otimes S_{\text{int}}(Z^4) \right) + O(\lambda^2).$$

(10)

$S_{\text{int}}$ is just the map $\phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4 \mapsto \int \phi_1 \phi_2 \phi_3 \phi_4$. To obtain the diagrams at order $\lambda$ we start by drawing the free 6-point function (Figure 3) and attach to the 4 rightmost strands of each diagram a 4-leg vertex (Figure 4.b). Those diagrams are generated by the first term in brackets of (10). We realise that the first three of our diagrams are vacuum diagrams which are exactly
 cancelled by the second term in the brackets. The remaining 12 diagrams are shown in Figure 3. In ordinary quantum field theory they all correspond to the same diagram: The tadpole diagram, see Figure 6. However, not all of them are necessarily different, as we shall see in Section 3.2.

3 Braided QFT on Homogeneous Spaces

In ordinary quantum field theory fixing one point of an \( n \)-point function still allows to recover the whole \( n \)-point function. Thus, we can reduce an \( n \)-point function to a function of just \( n - 1 \) variables. This is simply due to the fact that any \( n \)-point function is invariant under the isometry group \( G \) of the space-time \( M \) and \( G \) acts transitively on \( M \). In this case \( M \) is a homogeneous space under \( G \) and we can make the above statement more precise in the following way.

Lemma 3.1. Let \( G \) be a group and \( K \) a subgroup of \( G \). For any \( n \in \mathbb{N} \) there is an isomorphism of coset spaces

\[
\rho_n : (K\backslash G \times \cdots \times K\backslash G)/G \cong (K\backslash G \times \cdots \times K\backslash G)/K
\]

Figure 3: Free 4-point function.
Figure 4: Free 6-point function.

Figure 5: Interacting 2-point function of $\phi^4$-theory at order 1.

Figure 6: Tadpole diagram of ordinary $\phi^4$-theory.
given by \( \rho_n : [a_1, \ldots, a_n] \mapsto [a_1a_n^{-1}, \ldots, a_{n-1}a_1^{-1}] \) for \( a_i \in K \setminus G \). Its inverse is given by \( \rho_n^{-1} : [b_1, \ldots, b_{n-1}] \mapsto [b_1, \ldots, b_{n-1}, e] \) for \( b_i \in K \setminus G \), where \( e \) denotes the equivalence class of the identity in \( K \setminus G \). If \( G \) is a topological group (i.e., it is a topological space and multiplication and inversion are continuous), then equipping the coset spaces with the induced topologies makes \( \rho_n \) into a homeomorphism.

If space-time is an ordinary manifold we can obviously do the same trick in braided quantum field theory. More interestingly, however, we can extend it to noncommutative space-times.

### 3.1 Quantum Homogeneous Spaces

Lemma 3.1 generalises to the quantum group case. To see this we first recall the notion of a quantum homogeneous space.

Suppose we have two Hopf algebras \( A \) and \( H \) together with a Hopf algebra surjection \( \pi : A \to H \). This induces coactions \( \beta_L = (\pi \otimes \text{id}) \circ \Delta \) and \( \beta_R = (\text{id} \otimes \pi) \circ \Delta \) of \( H \) on \( A \), making \( A \) into a left and right \( H \)-comodule algebra. Define \( H_A \) to be the left \( H \)-invariant subalgebra of \( A \), i.e., \( H_A = \{ a \in A | \beta_L(a) = 1 \otimes a \} \). We have \( \Delta H_A \subseteq H_A \otimes A \) since \( (\beta_L \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \beta_L \). This makes \( H_A \) into a right \( A \)-comodule (and \( H \)-comodule) algebra. Observe also that \( \pi(a) = \epsilon(a)1 \) for \( a \in H_A \). \( H_A \) is called a right quantum homogeneous space. Define the left quantum homogeneous space \( A^H \) correspondingly. Due to the anti-coalgebra property of the antipode we find \( S H_A \subseteq A^H \) and \( S A^H \subseteq H_A \). If the antipode is invertible, the inclusions become equalities.

**Proposition 3.2.** In the above setting with invertible antipode the map

\[
\rho_n : (H_A \otimes \cdots \otimes H_A)^A \to (H_A \otimes \cdots \otimes H_A)^H
\]

given by \( \rho_n = (\text{id}^{n-1} \otimes \epsilon) \) for \( n \in \mathbb{N} \) is an isomorphism. Its inverse is \( (\text{id}^{n-1} \otimes S) \circ \beta^{n-1} \), where \( \beta^{n-1} \) is the right coaction of \( A \) on \( ^H A \) extended to the \((n-1)\)-fold tensor product.

**Proof.** Let \( a^1 \otimes \cdots \otimes a^n \) be an element of \( (H_A \otimes \cdots \otimes H_A)^A \). In particular,

\[
a^1(1) \otimes \cdots \otimes a^n(1) \otimes a^1(2) \cdots a^n(2) = a^1 \otimes \cdots \otimes a^n \otimes 1.
\]

Applying the antipode to the last component and multiplying with the \( n \)-th component we obtain

\[
a^1(1) \otimes \cdots \otimes a^{n-1}(1) \otimes \epsilon(a^n) S(a^1(2) \cdots a^{n-1}(2)) = a^1 \otimes \cdots \otimes a^n.
\]
Thus, \((\text{id}^{n-1} \otimes S) \circ \beta^{n-1} \circ (\text{id}^{n-1} \otimes \epsilon)\) is the identity on \((HA \otimes \cdots \otimes HA)^A\).

On the other hand, applying the inverse antipode and then \(\pi\) to the last component of \((\text{Id})\) we get

\[
a^{(1)} \otimes \cdots \otimes a^{n-1}(1) \otimes \epsilon(a^n) \pi(a_{(2)} \cdots a^{n-1}(2)) = a^{(1)} \otimes \cdots \otimes a^{n-1} \otimes \epsilon(a^n) 1.
\]

This is to say that \(a^{(1)} \otimes \cdots \otimes a^{n-1} \otimes \epsilon(a^n)\) is indeed right \(H\)-invariant.

Conversely, it is clear that \((\text{id}^{n-1} \otimes \epsilon) \circ (\text{id}^{n-1} \otimes S) \circ \beta^{n-1} = (\text{id}^{n-1} \otimes \epsilon) \circ \beta^{n-1}\) is the identity. Now take \(b^{(1)} \otimes \cdots \otimes b^{n-1}\) in \((HA \otimes \cdots \otimes HA)^H\). Its image under \(\beta^{n-1}\) is

\[
b^{(1)} \otimes \cdots \otimes b^{n-1}(1) \otimes b^{(2)} \cdots b^{n-1}(2)
\]

(12)

Applying \(\pi\) to the last component we get

\[
b^{(1)} \otimes \cdots \otimes b^{n-1}(1) \otimes \pi(b_{(2)} \cdots b^{n-1}(2)) = b^{(1)} \otimes \cdots \otimes b^{n-1} \otimes 1
\]

by right \(H\)-invariance. Applying \(\beta^{n-1} \otimes \text{Id}\) we arrive at

\[
b^{(1)} \otimes \cdots \otimes b^{n-1}(1) \otimes b^{(2)} \cdots b^{n-1}(2) \otimes \pi(b_{(3)} \cdots b^{n-1}(3)) = b^{(1)} \otimes \cdots \otimes b^{n-1}(1) \otimes b^{(2)} \cdots b^{n-1}(2) \otimes 1.
\]

We observe that this is the same as applying \((\text{id}^{n-1} \otimes \beta_R)\) to (12). Thus, the last component of (12) lives in \(A^H\) and the application of the antipode sends it to \(HA\) as required. That the result is right \(A\)-invariant is also clear by the defining property of the antipode.

To make use of the result we assume our space \(X\) of fields to be a quantum homogeneous space under a quantum group (coquasitriangular Hopf algebra) \(A\) of symmetries. (Note that coquasitriangularity implies invertibility of the antipode.) That is, together with \(A\) we have another Hopf algebra \(H\) and a Hopf algebra surjection \(A \to H\). We then assume that the algebra of fields is the right quantum homogeneous space \(X = HA\) living in the braided category \(\mathcal{M}^A\) of right \(A\)-comodules.

### 3.2 Diagrammatic Techniques

Proposition 3.2, to which we shall refer as invariant reduction, is not only useful to express \(n\)-point functions in a more compact way, but can also be applied in the evaluation of braided Feynman diagrams. For this we note that any horizontal cut of a braided Feynman diagram lives in some tensor power of \(X\) (since the only allowed strand lives in \(X\)) and is invariant (since the diagram is closed at the top). Thus, we can apply invariant reduction to it.
We shall give three examples for this, assuming vertices that are evaluated by multiplication and subsequent integration. Here, any quantum group invariant linear map \( X \to k \) is admissable as the integral.

**Vertex evaluation.** Consider the evaluation of an \( n \)-leg vertex (the horizontal slice of an invariant diagram depicted in Figure 7) with incoming elements \( a_1 \otimes \cdots \otimes a_{k+n} \). By invariant reduction this can be expressed in two ways,

\[
\begin{align*}
    a_1 \otimes \cdots \otimes a_k \int a_{k+1} \cdots a_{k+n} \\
    = a_{1(1)} \otimes \cdots \otimes a_{k(1)} \epsilon(a_{k+1}) \cdots \epsilon(a_{k+n}) \int S(a_{1(2)} \cdots a_{k(2)})
\end{align*}
\]

Depending on the circumstances each side might be easier to evaluate.

**Loop extraction.** Assume that the integral on \( HA \) is normalised, \( \int 1 = 1 \). Consider the diagram in Figure 8 (left-hand side). It is obviously invariant. Thus, the single outgoing strand carries a multiple of the identity and we can replace it by the integral followed by the identity element (Figure 8, right-hand side).

**Loop separation.** We assume further that the coquasitriangular structure \( \mathcal{R}: H \otimes H \to k \) is trivial on \( HA^H \) in the sense

\[
\mathcal{R}(a \otimes b) = \epsilon(a) \epsilon(b), \quad \text{if } a \in HA^H \text{ or } b \in HA^H. \tag{13}
\]

Consider now the diagram in Figure 9 (left-hand side) as a horizontal slice of an invariant diagram. According to invariant reduction we apply the counit to the rightmost outgoing strand. This makes the braiding trivial due to the assumed property of \( \mathcal{R} \). We can push the counit up to each of the joining strands and disentangle them. Then proceeding as in the previous example leads to the diagram in Figure 9 (right-hand side). Note that this works the same way for an under-crossing.

Let us come back to the 2-point function of \( \phi^4 \) theory that we considered at the end of Section 2.3. Assuming \( \int 1 = 1 \) and property (13) we can use loop extraction and loop separation to simplify the order 1 diagrams of Figure 8 considerably. The result is shown in Figure 10. Instead of 12 different diagrams we only have 2 different and much simpler diagrams, each with a multiplicity of 6.
Figure 7: Vertex evaluation in a diagram slice.

Figure 8: Extracting a loop.

Figure 9: Separating a loop in an invariant slice.

Figure 10: Simplified 2-point function of $\phi^4$-theory at order 1.
4 Braided QFT on Compact Spaces

4.1 Braided Spaces of Infinite Dimension

Up to now we have developed our approach on a formal level insofar, that we have not addressed the question how an infinite dimensional space (of fields) can be treated in a braided category. This is certainly necessary if we want to do quantum field theory, i.e., deal with infinitely many degrees of freedom. An obvious problem is the definition of the coevaluation. It seems that we need at least a completed tensor product for this. However, instead of introducing heavy functional analytic machinery, we can stick with our algebraic approach given a further assumption.

Let us assume that the space of (regular) fields $X$ decomposes into a direct sum $\bigoplus X_i$ of countably many finite dimensional comodules under the symmetry quantum group $A$. This corresponds roughly to the classical case of the space-time manifold being compact. In particular, it is the case if the symmetry quantum group $A$ is cosemisimple (or classically the Lie group of symmetries is compact, see Section 4.2 below). Denote the projection $X \to X_i$ by $\tau_i$.

We now allow arbitrary sums of elements in $X$ given that any projection $\tau_i$ annihilates all but finitely many summands. Similarly, we allow infinite sums in the $n$-fold tensor product $X^n$ with the restriction that any projection $\tau_i \otimes \cdots \otimes \tau_i$ yields a finite sum. To define the dual of $X$, we take the dual of each $X_i$ and set $X^* = \bigoplus X_i^*$. For each component $X_i$ we have an evaluation map $ev_i : X_i \otimes X_i^* \to \mathbb{k}$ and a coevaluation map $coev_i : \mathbb{k} \to X_i \otimes X_i^*$ in the usual way. We then formally define $ev = \sum_i ev_i \circ (\tau_i \otimes \tau_i^*)$ and $coev = \sum_i coev_i$.

Our definition is invariant under coactions of $A$ as it should be, since the projections $\tau_i$ commute with the coaction of $A$. In particular, it is invariant under braidings.

4.2 Cosemisimplicity and Peter-Weyl Decomposition

We describe a context in which all comodules over a Hopf algebra decompose into finite dimensional (and even simple) pieces. The discussion here uses results of [28] but is more in the spirit of [2, II.9]. Assume $\mathbb{k}$ to be algebraically closed, e.g., $\mathbb{k} = \mathbb{C}$.

Let $C$ be a coalgebra, $V$ a simple right $C$-comodule (i.e. $V$ has no proper subcomodules) with coaction $\beta : V \to V \otimes C$. In particular, $V$ is finite dimensional. The dual space $V^*$ is canonically a (simple) left $C$-comodule. Denote a basis of $V$ by $\{e_i\}$, the dual basis of $V^*$ by $\{f^i\}$. Identify the
endomorphism algebra on $V$, $\text{End} \ V \cong \text{End} \ V^* \otimes \text{End} \ V^* \cong V \otimes \text{End} \ V \otimes V^* \cong V \otimes 1 
abla f^i \otimes f^j) = \delta^i_j (e^i \otimes f^j).$ We denote the dual coalgebra by $(\text{End} \ V)^*$ and identify $(\text{End} \ V)^* \cong V^* \otimes V^* \otimes V^* \otimes V.$

Now consider the map $(\text{End} \ V)^* \to C$ given by $f^i \otimes e_j \mapsto (f^i \otimes 1) \circ \beta(e_j).$ It is an injective (since $V$ is simple) coalgebra map. We extend this to the direct sum of all inequivalent simple comodules. The resulting map

$\bigoplus_V (\text{End} \ V)^* \to C$

is a coalgebra injection. It is an isomorphism of coalgebras if and only if all $C$-comodules are semisimple (i.e. they are direct sums of simple ones) or equivalently if $C$ is semisimple (i.e. it is a direct sum of simple coalgebras).

Assume now that $A$ is a cosemisimple Hopf algebra, i.e., $A$ is semisimple as a coalgebra. We write the above decomposition as

$A \cong \bigoplus_V (V^* \otimes V). \quad (14)$

It is also referred to as the Peter-Weyl decomposition, in analogy to the corresponding decomposition of the algebra of regular functions on a compact Lie group. There is a unique normalised left- and right-invariant integral (Haar measure) on $A$, given by the induced projection to the unit element in $A$. Note also that the antipode is invertible.

Consider a second Hopf algebra $H$ with a Hopf algebra surjection $\pi : A \to H.$ This induces a coaction of $H$ on each $A$-comodule. For the right quantum homogeneous space we have

$HA \cong \bigoplus_V (H(V^*) \otimes V). \quad (15)$

as right $H$-comodules.

5 $\phi^4$-Theory on the Quantum 2-Sphere

In accordance with the motivation of braided quantum field theory as a way of regularising ordinary quantum field theory, we replace Lie groups of symmetries by corresponding parametric deformations. In order to have a well defined theory in the sense of Section 4 we make use of the Peter-Weyl decomposition and thus restrict to compact Lie groups. A natural choice are the standard $q$-deformations of Lie groups with compact $*$-structure. We specialise to $\mathbb{k} = \mathbb{C}$, although the discussion of the free action in Section 2.2
was in the spirit of real-valued scalar field theory. This is necessary since the standard $q$-deformations viewed as deformations of complexifications of compact Lie groups do not restrict to real subalgebras for $q \neq 1$. However, viewing $q$-deformation purely as a mathematical tool we can always restrict to $\mathbb{R}$ when considering physical quantities living at $q = 1$.

In the following we consider perturbative $\phi^4$-theory on the quantum 2-sphere with $SU_q(2)$-symmetry as an example of a quantum field theory on a braided space. Ordinary $\phi^4$-theory in 2 dimensions is super-renormalisable and has just one basic divergence: The tadpole diagram (Figure 6). (See e.g. [31] for a treatment of ordinary $\phi^4$-theory.) We demonstrate that this diagram becomes finite for $q > 1$. Our Hopf algebra of symmetries is $SU_q(2)$ under which $S^2_q$ is a homogeneous space as a right comodule. (We adopt the convention to denote the Hopf algebra of regular functions by the name of the (quantum) group or space.)

5.1 The Decomposition of $SU_q(2)$ and $S^2_q$

To prepare the ground we need to recall the construction of $S^2_q$ as a quantum homogeneous space under $SU_q(2)$ and the Peter-Weyl decomposition of the latter [15, 25]. This will enable us to apply the machinery of the previous sections.

Recall that $SU_q(2)$ is the compact real form of $SL_q(2)$ for $q$ real which we assume in the following. (See Appendix A for the defining relations.) It is cosemisimple and there is one simple (right) comodule $V_l$ for each integer dimension, conventionally labelled by a half-integer $l$ such that the dimension is $2l + 1$. Thus, the Peter-Weyl decomposition (14) is

$$SU_q(2) \cong \bigoplus_{l \in \frac{1}{2} \mathbb{N}_0} (V^*_l \otimes V_l).$$

There is a Hopf $*$-algebra surjection $\pi : SU_q(2) \to U(1)$ corresponding to the diagonal inclusion in the commutative case. (See Appendix A for an explicit definition of $\pi$.) This defines the quantum 2-sphere $S^2_q$ as the right quantum homogeneous $*$-space $SU_q(2)$. Under the coaction of $U(1)$ induced by $\pi$ the comodules $V_l$ decompose into inequivalent one-dimensional comodules classified by integers. (This is the usual representation theory of $U(1)$.) This determines up to normalisation a basis $\{v_n^{(l)} \}$ for $V_l$ with half-integers $n$ taking values $-l, -l + 1, \ldots, l$. In particular, we find that $V_l^{U(1)}$ is one-dimensional if $l$ is integer and zero-dimensional otherwise. Thus, [15]
simplifies to

\[ S^2_q \cong \bigoplus_{l \in \mathbb{N}_0} V_l \]

as right $SU_q(2)$-comodules. We write the induced (normalisation independent) basis vectors of $SU_q(2)$ as $t^{(l)}_{ij} = (f^{(l)}_i \otimes \text{id}) \circ \beta(e^{(l)}_j)$ where $f^{(l)}_n$ is dual to $e^{(l)}_n$ and $\beta : V_l \to V_l \otimes SU_q(2)$ is the coaction of $SU_q(2)$ on $V_l$. As a subalgebra $S^2_q$ has the basis $\{t^{(l)}_{00}\}$. The bi-invariant subalgebra $S^2_q U(1) = U(1) SU_q(2) U(1)$ has the basis $\{t^{(l)}_{00}\}$.

Note that by construction

\[ \epsilon \left( t^{(l)}_{mn} \right) = \delta_{m,n} \quad \text{and} \quad \Delta t^{(l)}_{mn} = \sum_k t^{(l)}_{mk} \otimes t^{(l)}_{kn}. \]

The antipode and $\ast$-structure of $SU_q(2)$ in this basis are

\[ S t^{(l)}_{mn} = (-q)^{m-n} t^{(l)}_{-n-m}, \quad (t^{(l)}_{mn})^\ast = S t^{(l)}_{mn} = (-q)^{n-m} t^{(l)}_{-n-m}, \]

as can be verified by direct calculation from the formulas in [4, 4.2.4]. The normalised invariant integral (Haar measure) is simply $\int t^{(l)}_{ij} = \delta_{l,0}$. We also need its value on the product of two basis elements

\[ \int t^{(l)}_{mn} t^{(l')}_{m'n'} = \frac{(-1)^{m-n} q^{m+n}}{[2l+1]_q} \delta_{l,l'} \delta_{m+m',0} \delta_{n+n',0}. \quad (16) \]

This can be easily worked out considering the equation $\epsilon(a) = \int a_{(1)} S a_{(2)}$ and using the invariance of the integral in the form $b_{(1)} \int ab_{(2)} = S a_{(1)} \int a_{(2)} b$ and $S b_{(2)} \int ab_{(1)} = a_{(2)} \int a_{(1)} b$ on basis elements. The $q$-integers for $q \in \mathbb{C}^*$ are defined as

\[ [n]_q := \sum_{k=0}^{n-1} q^{n-2k-1} = \frac{q^n - q^{-n}}{q - q^{-1}}. \]

(The second expression is only defined for $q^2 \neq 1$).

Denoting a dual basis of $\{t^{(l)}_{mn}\}$ by $\{t^{(l)}_{mn}\}$, we observe that $SU_q(2)^\ast$ becomes an object in $\mathcal{M}^{SU_q(2)}$, the category of right comodules over $SU_q(2)$ by equipping it with the coaction $t^{(l)}_{mn} \mapsto \sum_k t^{(l)}_{mk} \otimes S^{-1} t^{(l)}_{nk}$. We then have an evaluation map $\text{ev} : SU_q(2) \otimes SU_q(2)^\ast \to \mathbb{C}$ and a coevaluation map $\text{coev} : \mathbb{C} \to SU_q(2)^\ast \otimes SU_q(2)$ in the obvious way.

In the commutative case $q = 1$, the basis $\{t^{(l)}_{mn}\}$ becomes the usual basis of regular functions (i.e., matrix elements of representations) on $SU(2)$ (see e.g.
to whose conventions we conform in this case). The restriction to \(\{t_{0n}^{(l)}\}\) recovers nothing but (a version of) the spherical harmonics on \(S^2\).

In particular, we notice that the zonal spherical functions can be expressed in terms of Legendre polynomials \(t_{00}^{(l)}(\phi, \theta, \psi) = P_l(\cos \theta)\), where \(\phi, \theta, \psi\) are the Euler angles on \(SU(2)\) (see [29, Chapter 6]). From the orthogonality relation of the Legendre polynomials, the fact that their only common value is at \(P_l(1) = 1\), and considering that \(\theta = 0\) denotes a pole of \(SU(2)\), we find that the delta function at the identity of \(SU(2)\) restricted to \(S^2\) can be represented as

\[
\delta_0(\phi, \theta) = \sum_l (2l + 1) P_l(\cos \theta) = \sum_l (2l + 1) t_{00}^{(l)}(\phi, \theta).
\]

(17)

Recall that a coquasitriangular structure \(\mathcal{R} : H \otimes H \to \mathbb{k}\) on a quantum group \(H\) determines a braiding between right comodules \(V\) and \(W\) via

\[
\psi(v \otimes w) = w(1) \otimes v(1) \mathcal{R}(v(2) \otimes w(2))
\]

for \(v \in V\) and \(w \in W\). (We use here Sweedler’s coproduct notation for the coaction.) For calculations we need the functionals \(u\) and \(v\) defined with \(\mathcal{R}\) as (see e.g. [21])

\[
u(a) := \mathcal{R}(a(2) \otimes S a(1)), \quad v(a) := \mathcal{R}(a(1) \otimes S a(2))
\]

(18)

for \(a \in H\). For \(H = SU_q(2)\) in our basis they are

\[
u(t^{(l)}_{m,n}) = \delta_{m,n} q^{-2l(l+1)+2m}, \quad v(t^{(l)}_{m,n}) = \delta_{m,n} q^{-2l(l+1)-2m}.
\]

(19)

We also note that property (13) is satisfied, i.e.,

\[
\mathcal{R} \left( t^{(l)}_{00} \otimes t^{(l)}_{ij} \right) = \delta_{i,j} = \mathcal{R} \left( t^{(l)}_{ij} \otimes t^{(l)}_{00} \right).
\]

(20)

See Appendix [3] for a derivation of (19) and (21).

5.2 The Free Propagator

In ordinary quantum field theory the free propagator is defined by the free action. For a Euclidean massive real scalar field theory on a manifold \(M\) it takes the form

\[
S_0(\phi) = \frac{1}{2} \int_M dx \phi(x)(m^2 - \Delta_M)\phi(x),
\]

21
where $\Delta_M$ is the Laplace operator on $M$ and $m$ is the mass of the field. Define $L := m^2 - \Delta_M$. Let $\{\phi_i\}$ be a basis of $X$ and $\{\phi^*_i\}$ a dual basis. Denote the differential with respect to $\phi_i$ by $\partial_i$. We have

$$ (\partial_i S_0)(\phi) = \int_M dx \phi(x) L \phi_i(x) = \sum_k \phi^*_k(\phi) \int_M dx \phi_k(x) L \phi_i(x). $$

Comparing with equation (8) we obtain in the more abstract notation of Section 2.1

$$ \gamma = \left( \text{id} \otimes \int_M \right) \circ (\text{id} \otimes \cdot) \circ (\text{coev} \otimes L), $$

which we take as the defining equation for $\gamma$. While initially well defined only at $q = 1$ we extend it to the noncommutative realm in the following.

First, note that at $q \neq 1$ we still have a well defined integral on our “manifold” $M = S^2_q$, namely the induced Haar measure of $SU_q(2)$. Next, we need an analogue of the Laplace operator. By the duality of $SU_q(2)$ with the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$, a central element of the latter defines an invariant operator on $SU_q(2)$-comodules. A natural choice is the quantum Casimir element which we define as

$$ C_q = EF + \frac{(K - 1)q^{-1} + (K^{-1} - 1)q}{(q - q^{-1})^2}. $$

Here $K$, $K^{-1}$, $E$, and $F$ are the generators of $U_q(\mathfrak{sl}_2)$ (see Appendix B). $C_q$ differs from quantum Casimir elements considered elsewhere (see e.g. [25] or [14]) only by a $q$-multiple of the identity. The eigenvalue of $C_q$ on $V_l$ is $[l]_q[l + 1]_q$ so that we get exactly the (negative of the) usual Laplace operator for $q = 1$. Including a mass term we set

$$ L = C_q + m^2. $$

Thus, the eigenvalue of $L$ on $V_l$ is

$$ L_l = [l]_q[l + 1]_q + m^2. $$

We determine $\gamma$ according to (21). Using (19) we find

$$ \gamma \left( \tilde{t}^{(l)}_{0,i} \right) = \sum_{m,j} \tilde{t}^{(m)}_{0,j} \int \tilde{t}^{(m)}_{0,j} L \left( \tilde{t}^{(l)}_{0,i} \right) = [2l + 1]_q^{-1} L_l (-q)^{-i} \tilde{t}^{(l)}_{0,-i}, $$

Inverting we obtain

$$ \gamma^{-1} \left( \tilde{t}^{(l)}_{0,i} \right) = [2l + 1]_q L_l^{-1} (-q)^{-i} \tilde{t}^{(l)}_{0,-i}. $$
Now we are ready to determine the free propagator according to Corollary 2.2.

\[ \mathcal{Z}^2 = \sum_{l,k} (\text{id} \otimes \gamma^{-1}) \circ \psi \left( \tilde{t}_{0,k}^{(l)} \otimes t_{0,k}^{(l)} \right) \]

\[ = \sum_{l,i,j,k} t_{0,i}^{(l)} \otimes \gamma^{-1} \left( \tilde{t}_{0,j}^{(l)} \right) \mathcal{R} \left( S^{-1} t_{k,j}^{(l)} \otimes t_{i,k}^{(l)} \right) \]

\[ = \sum_{l,i,j} [2l + 1]_q L_i^{-1} q^{-2(l+1)} (-q)^i \tilde{t}_{0,i}^{(l)} \otimes t_{0,-i}^{(l)}. \]

Using invariant reduction (Proposition 3.2) we find

\[ \tilde{\mathcal{Z}}^2 = \sum_l [2l + 1]_q L_i^{-1} q^{-2(l+1)} t_{0,0}^{(l)} \]

(22)

to be the reduced form of the propagator as an element of \( S^2_qU(1) \). In the commutative case \( q = 1 \) we can rewrite \( (22) \) as

\[ \tilde{\mathcal{Z}}^2 \big|_{q=1} = (m^2 - \Delta)^{-1} \delta_0 \]

by comparison with \((17)\). This is the familiar expression from ordinary quantum field theory.

### 5.3 Interactions

We proceed to evaluate the order 1 contribution of the \( \phi^4 \)-interaction to the 2-point function. The corresponding diagrams are depicted in Figure 3 (see Section 2.3). Since the property \((13)\) holds in \( SU_q(2) \) the diagrams simplify to those of Figure 10 (see Section 3.2). The disconnected loop comes out as

\[ \delta_{\text{loop}} := \sum_l [2l + 1]_q L_i^{-1} q^{-2(l+1)} t_{0,0}^{(l)}. \]

(23)

(Just apply the counit to \((22)\).) The connected diagram in the right-hand summand of Figure 14 is (in reduced form)

\[ \left( \text{id} \otimes \epsilon \otimes \int \right) \circ (\text{id}^2 \otimes \cdot) \circ (\text{id} \otimes \mathcal{Z}^2 \otimes \text{id}) \circ \mathcal{Z}^2 \]

23
\[\sum_{l,m,i,j} \alpha_l \alpha_m t_{0i}^{(l)} \epsilon \left(t_{0j}^{(m)}\right) \int S t_{j0}^{(m)} S t_{i0}^{(l)} = \sum_l \alpha_l^2 [2l + 1]_q^{-1} t_{00}^{(l)}.\]

with \(\alpha_l := [2l + 1]_q L_q^{-1} q^{-2l(l+1)}.\) We have used \(Z^2\) as reconstructed from its reduced form (22), the property \(\int \circ S = \int\) of the integral, and (16). The connected diagram in the left-hand summand of Figure 10 is (in reduced form)

\[
\begin{aligned}
\left( \id \otimes \epsilon \otimes \int \right) \circ (\id^2 \otimes \epsilon) \circ (\id \otimes \psi^{-1} \otimes \id) \circ (Z^2 \otimes Z^2) \\
= \sum_{l,m,i,j,n} \alpha_l \alpha_m t_{0i}^{(l)} \epsilon \left(t_{0j}^{(m)}\right) \int S t_{j0}^{(m)} S t_{i0}^{(l)} R^{-1} \left(t_{k0}^{(m)} \otimes S t_{i0}^{(l)}\right) \\
= \sum_{l,m,i,j,n} \alpha_l \alpha_m t_{0i}^{(l)} \int t_{j0}^{(m)} t_{i0}^{(l)} R \left(t_{0j}^{(m)} \otimes t_{i0}^{(l)}\right) \\
= \sum_{l,m,i,j,k} \alpha_l \alpha_m t_{0i}^{(l)} \int t_{j0}^{(m)} t_{i0}^{(l)} R \left(t_{0j}^{(m)} \otimes S t_{j0}^{(m)}\right) \\
= \sum_{l,m,i,k} \alpha_l \alpha_m t_{0i}^{(l)} \int t_{k0}^{(m)} t_{i0}^{(l)} \psi \left(t_{0k}^{(m)}\right) \\
= \sum_l \alpha_l^2 [2l + 1]_q^{-1} q^{-2l(l+1)} t_{00}^{(l)}.\]
\end{aligned}
\]

We have also used the invariance of the integral in the form \((\int ab_{(2)})b_{(1)} = (\int a_{(2)}b)S a_{(1)}\) in the third equality. Thus, the (reduced) 2-point function up to order 1 comes out as

\[
\tilde{Z}^2_{\text{int}} = \sum_l [2l + 1]_q L_q^{-1} q^{-2l(l+1)} t_{00}^{(l)} (1 - 6 \lambda \delta_{\text{loop}} L_q^{-1} q^{-2l(l+1)} (1 + q^{-2l(l+1)}) + O(\lambda^2)) .
\]

In the commutative case \((q = 1)\), we know that the order 1 contribution (given by the tadpole diagram in Figure 6) is divergent. We can easily see where this divergence comes from. The loop contribution (24)

\[
\delta_{\text{loop}}|_{q=1} = \sum_l \frac{2l + 1}{l(l + 1) + m^2}(25)
\]
is infinite. However, at $q > 1$ it becomes finite. We are truly able to regularise the tadpole diagram. Let us identify the divergence in $q$-space. For $q > 1$ we can find both an upper and a lower bound for (23) of the form

$$\text{const} + \int_1^{\infty} dl \frac{2}{l} q^{-2l^2},$$

where $\text{const}$ does not depend on $q$ (but may depend on $m^2$). Setting $q = e^{2h^2}$ with $h > 0$ we find

$$\delta_{\text{loop}}|_{q>1} = \frac{1}{h} + O(1).$$

The conventional divergence of (23) is only logarithmic in $l$. What would happen with higher divergences? It seems natural to assume that they would give rise to terms like

$$\sum_l [l]_q^n q^{-2(l+1)}.$$

But this converges in the domain $q > 1$ for any $n$. We can even apply the very same discussion of the divergence in $q$-space as above. The nature of the divergence in $q$-space does not seem to be affected by the degree of the ordinary (commutative) divergence at all. This suggests that $q$-regularisation in our framework is powerful indeed.

Reviewing our calculations of $Z^2$ and $Z^2_{\text{int}}$ we find that the crucial factor of $q^{-2(l+1)}$ is caused by the braiding. Thus, the braiding and not the mere noncommutativity appears to be essential for the regularisation.

### 5.4 Renormalisation

Ordinarily, $\phi^4$-theory in dimension 2 is super-renormalisable. The only basic divergent diagram is the tadpole (Figure 3). Our approach yields a simple and diagrammatic way to renormalise it. We have used above the loop separation technique of Section 3.2 (Figure 9) to factorise the single tadpole diagram(s) into $\phi^2$-vertex diagrams and the loop factor $\delta_{\text{loop}}$. For any given diagram we can perform the same operation for all tadpole subdiagrams appearing in it. The remaining diagram (with the loop factors removed) is finite at $q = 1$, since the commutative theory has no further divergences.

However, from a rigorous point of view this procedure can only be performed if the diagram we start out with is finite. While we have seen that the tadpole diagram alone becomes finite for $q > 1$, it is conceivable that certain diagrams that converge at $q = 1$ would diverge at $q > 1$. This might be due
to the introduction of factors like $q^{2l(l+1)}$ into summations over $l$. The expression (24) suggests, however, that this does not happen, but rather that all $q$-factors introduced in summations have negative exponent. We shall assume this in the following.

Let us perform the usual mass renormalisation in our framework. We introduce an extra perturbative mass term which generates diagrams with $\phi^2$-vertices. These diagrams are then used to cancel the corresponding diagrams where the $\phi^2$-vertices are the remnants of the factorisation of tadpole subdiagrams. To effect the cancellation the perturbative mass term must carry the same factor $\delta_{\text{loop}}$ as the factorised tadpoles. To compensate for the different combinatoric multiplicity of quadratic and quartic vertices we need an extra factor of 6 in front of the $\phi^2$-vertex. Since a mass term carries an overall factor of $1/2$ in the action, the effective mass shift is

$$m^2 \rightarrow m^2 - 12\lambda \delta_{\text{loop}}.$$ 

Performing this (finite) mass renormalisation at $q > 1$, only the divergence-free diagrams without tadpoles remain as $q \rightarrow 1$ at any given order in perturbation theory.

### 6 Concluding Remarks

We have presented a coherent framework for the treatment of quantum field theory on braided spaces. In particular, we have developed a quantum group covariant perturbation theory.

The example of $\phi^4$-theory on the quantum 2-sphere has shown that quantum deformations of symmetries do lead to the regularisation of divergences in our approach. This method is superior to regularisation methods such as using a lattice or fuzzy spaces in that it does not resort to discrete approximations with only finitely many degrees of freedom. On the other hand it does not suffer from the crude breaking of symmetries as many quantum field theoretic methods do (e.g. momentum cut-off, dimensional regularisation, lattice). However, symmetries are not preserved as such, but deformed to quantum group symmetries. Our results also suggest that divergences of arbitrary order could be regularised in this way.

A next step would be the investigation of quantum field theories on deformations of higher dimensional spaces to obtain more physically interesting models. We note in particular that quantum deformations of Minkowski space are available (see [1, 22] and [16, 23]). Further one would like to include internal (quantum group) symmetries as well. In particular, this might
open new possibilities for the old idea of unifying internal and external sym-
metries.

In a different direction, one might speculate that the braided Feynman di-
agrams obtained from theories with $q$-deformed symmetries have interesting
number theoretic properties related to modular functions. This is suggested
by the observation of such properties for the quantum rank of $q$-deformed
enveloping algebras [24].

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A Definition of $SU_q(2)$

This appendix recalls the defining relations of $SU_q(2)$ and the quantum Hopf
fibration, see e.g., [24] or [14].

The matrix Hopf algebra $SL_q(2)$ is defined over $\mathbb{C}$ with generators $a, b, c, d$
and relations

\[
ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb,
\]
\[
ad - da = (q - q^{-1})bc, \quad ad - qbc = 1,
\]
\[
\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}.
\]

Matrix multiplication is understood in the definition of the coproduct. The
*$*$-structure defining the real form $SU_q(2)$ for real $q$ is given by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -qc \\ -q^{-1}b & a \end{pmatrix}.
\]

As a Hopf *-algebra, $U(1)$ has one generator $g$ with inverse $g^{-1}$ and
relations and *-structure

\[
\Delta g = g \otimes g, \quad \epsilon g = 1, \quad S g = g^{-1}, \quad g^* = g^{-1}.
\]
There is a Hopf ∗-algebra surjection $\pi : SU_q(2) \to U(1)$ defined by
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}.
\]
This determines the quantum 2-sphere $S^2_q$ as a right quantum homogeneous ∗-space under $SU_q(2)$. At $q = 1$ we recover the ordinary Hopf fibration.

**B  Coquasitriangular Structure of $SU_q(2)$**

In this appendix we provide the formulas for the coquasitriangular structure of $SU_q(2)$ in the Peter-Weyl basis needed in Section 5. We use the context of Section 5.1. Definitions and results that are just stated are standard and can be found e.g. in [21] or [14].

The Hopf algebra $U_q(sl_2)$ is defined over $\mathbb{C}$ for $q \in \mathbb{C}^*$ and $q^2 \neq 1$ with generators $E, F, K, K^{-1}$ and relations
\[
KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,
\]
\[
KK^{-1} = K^{-1}K = 1, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}},
\]
\[
\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,
\]
\[
\Delta(K) = K \otimes K, \quad \epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0,
\]
\[
S(K) = K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.
\]

$U_q(sl_2)$ and $SU_q(2)$ are non-degenerately paired. Thus, actions of $U_q(sl_2)$ and coactions of $SU_q(2)$ on finite dimensional vector spaces are dual to each other. In particular, the simple comodule $V_l$ of $SU_q(2)$ is a simple module of $U_q(sl_2)$. By the representation theory of $U_q(sl_2)$ it has a basis $\{w_i\}, i = -l, -l + 1, \ldots, l$ such that
\[
K \triangleright w_m = q^{2m}w_m, \quad E \triangleright w_m = ([l - m]_q[l + m + 1]_q)^{1/2}w_{m+1},
\]
\[
F \triangleright w_m = ([l + m]_q[l - m + 1]_q)^{1/2}w_{m-1}.
\]

(26)

$U_q(sl_2)$ has an $h$-adic version $U_h(sl_2)$ defined over $\mathbb{C}[[h]]$ correspondingly with $q = e^h$ and an additional generator $H$ so that $q^H = K$. It has the quasitriangular structure
\[
R = q^{(H \otimes H)/2} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(1 - q^{-2})^n}{[n]_q!} E^n \otimes F^n.
\]

(27)
The elements \((\text{define } R^{(1)} \otimes R^{(2)} = R)\)

\[ u' = (S R^{(2)}) R^{(1)}, \quad v' = R^{(1)} S R^{(2)} \quad (28) \]

act on \(V_l\) as [27, Proposition 3.2.7]

\[ u' \triangleright w_m = q^{-2l(l+1)+2m} w_m, \quad v' \triangleright w_m = q^{-2l(l+1)-2m} w_m. \quad (29) \]

The coquasitriangular structure \(\mathcal{R}\) of \(SU_q(2)\) is given by the duality with \(U_q(\mathfrak{sl}_2)\) from the quasitriangular structure \(R\) of \(U_h(\mathfrak{sl}_2)\). Using

\[ u(a_{(1)}) a_{(2)} = S^2 a_{(1)} u(a_{(2)}) \quad \text{and} \quad v(a_{(1)}) S^2 a_{(2)} = a_{(1)} v(a_{(2)}) \]

we find

\[ u\left(t^{(l)}_{m,n}\right) = \delta_{m,n} q^{2(m-k)} u\left(t^{(l)}_{k,k}\right), \quad v\left(t^{(l)}_{m,n}\right) = \delta_{m,n} q^{2(k-m)} v\left(t^{(l)}_{k,k}\right). \quad (30) \]

Since the definitions (18) and (28) are dual to each other we can use

\[ g \triangleright v_n = \sum_m v_m \langle g, t^{(l)}_{m,n} \rangle, \quad g \in U_q(\mathfrak{sl}_2) \]

to compare (29) with (30). We find (19) and infer that \(w_i\) is (a multiple of) \(v_i\). With the latter, the pairing between \(U_q(\mathfrak{sl}_2)\) and \(SU_q(2)\) comes out from (27) as

\[ \langle K, t^{(l)}_{m,n} \rangle = \delta_{m,n} q^{2n}, \quad \langle E, t^{(l)}_{m,n} \rangle = \delta_{m,n+1} ([l - n]_q [l + n + 1]_q)^{1/2}, \]

\[ \langle F, t^{(l)}_{m,n} \rangle = \delta_{m,n-1} ([l + n]_q [l - n + 1]_q)^{1/2}. \]

Note also \(\langle H, t^{(l)}_{m,n} \rangle = \delta_{m,n} 2n\) in the \(h\)-adic version. With this pairing and (27) we easily verify the property (20).

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