Abstract. We prove existence, multiplicity, and bifurcation results for $p$-Laplacian problems involving critical Hardy–Sobolev exponents. Our results are mainly for the case $\lambda \geq \lambda_1$ and extend results in the literature for $0 < \lambda < \lambda_1$. In the absence of a direct sum decomposition, we use critical point theorems based on a cohomological index and a related pseudo-index.

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1. Introduction

Consider the critical $p$-Laplacian problem
\begin{equation}
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u + \frac{|u|^{p^*(s)-2}}{|x|^s} u \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{equation}
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ containing the origin, $1 < p < N$, $\lambda > 0$ is a parameter, $0 < s < p$, and $p^*(s) = (N-s)p/(N-p)$ is the critical Hardy–Sobolev exponent. Ghoussoub and Yuan [6] showed, among other things, that this problem has a positive solution when $N \geq p^2$ and $0 < \lambda < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the eigenvalue problem
\begin{equation}
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega.
\end{cases}
\end{equation}
In the present paper we mainly consider the case $\lambda \geq \lambda_1$. Our existence results are the following.
Theorem 1.1. If \( N \geq p^2 \) and \( 0 < \lambda < \lambda_1 \), then problem (1.1) has a positive ground state solution.

Theorem 1.2. If \( N \geq p^2 \) and \( \lambda > \lambda_1 \) is not an eigenvalue of problem (1.2), then problem (1.1) has a nontrivial solution.

Theorem 1.3. If
\[
(N - p^2)(N - s) > (p - s) p
\]
and \( \lambda \geq \lambda_1 \), then problem (1.1) has a nontrivial solution.

Remark 1.4. We note that (1.3) implies \( N > p^2 \).

Remark 1.5. In the nonsingular case \( s = 0 \), related results can be found in Degiovanni and Lancelotti [4] for the \( p \)-Laplacian and in Mosconi et al. [7] for the fractional \( p \)-Laplacian.

Weak solutions of problem (1.1) coincide with critical points of the \( C^1 \)-functional
\[
I_\lambda(u) = \int_\Omega \left[ \frac{1}{p} (|\nabla u|^p - \lambda |u|^p) - \frac{1}{p^*(s)} \frac{|u|^{p^*(s)}}{|x|^s} \right] dx, \quad u \in W^{1,p}_0(\Omega).
\]

Recall that \( I_\lambda \) satisfies the Palais-Smale compactness condition at the level \( c \in \mathbb{R} \), or the \((PS)_c\) condition for short, if every sequence \( (u_j) \subset W^{1,p}_0(\Omega) \) such that \( I_\lambda(u_j) \to c \) and \( I_\lambda'(u_j) \to 0 \) has a convergent subsequence. Let
\[
\mu_s = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\left( \int_\Omega |u|^{p^*(s)} \frac{dx}{|x|^s} \right)^{p/p^*(s)}} \tag{1.4}
\]
be the best constant in the Hardy–Sobolev inequality, which is independent of \( \Omega \) (see [6, Theorem 3.1.(1)]). It was shown in [6, Theorem 4.1.(2)] that \( I_\lambda \) satisfies the \((PS)_c\) condition for all
\[
c < \frac{p - s}{(N - s) p} \mu_s^{(N-s)/(p-s)}
\]
for any \( \lambda > 0 \). We will prove Theorems 1.1 – 1.3 by constructing suitable minimax levels below this threshold for compactness. When \( 0 < \lambda < \lambda_1 \), we will show that the infimum of \( I_\lambda \) on the Nehari manifold is below this level. When \( \lambda \geq \lambda_1 \), \( I_\lambda \) no longer has the mountain pass geometry and a linking type argument is needed. However, the classical linking theorem cannot be used here since the nonlinear operator \(-\Delta_p\) does not have linear eigenspaces. We will use a nonstandard linking construction based on sublevel sets as in Perera and Szulkin [11] (see also Perera et al. [9, Proposition 3.23]). Moreover, the standard sequence of eigenvalues of \(-\Delta_p\) based on the genus does not give enough information about the structure of the sublevel sets to carry out this construction. Therefore, we will use a different sequence of eigenvalues introduced in Perera [8] that is based on a cohomological index.
For $1 < p < \infty$, eigenvalues of problem (1.2) coincide with critical values of the functional
\[
\Psi(u) = \frac{1}{\int_{\Omega} |u|^p \, dx}, \quad u \in \mathcal{M} = \left\{ u \in W^{1,p}_0(\Omega) : \int_{\Omega} |\nabla u|^p \, dx = 1 \right\}.
\]

Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$, let $i(M)$ denote the $\mathbb{Z}_2$-cohomological index of $M \in \mathcal{F}$ (see Sect. 2.1), and set
\[
\lambda_k := \inf_{M \in \mathcal{F}, i(M) \geq k} \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.
\]
Then $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$ is a sequence of eigenvalues of (1.2) and
\[
\lambda_k < \lambda_{k+1} \implies i(\Psi^{\lambda_k}) = i(M \setminus \Psi^{\lambda_{k+1}}) = k,
\]
where $\Psi^a = \{ u \in \mathcal{M} : \Psi(u) \leq a \}$ and $\Psi_a = \{ u \in \mathcal{M} : \Psi(u) \geq a \}$ for $a \in \mathbb{R}$ (see Perera et al. [9, Propositions 3.52 and 3.53]). We also prove the following bifurcation and multiplicity results for problem (1.1) that do not require $N \geq p^2$. Set
\[
V_s(\Omega) = \int_{\Omega} |x|^{(N-p)s/(p-s)} \, dx,
\]
and note that
\[
\int_{\Omega} |u|^p \, dx \leq V_s(\Omega)^{(p-s)/(N-s)} \left( \int_{\Omega} |u|^{p^*(s)} \, dx \right)^{p/p^*(s)} \quad \forall u \in W^{1,p}_0(\Omega) \tag{1.6}
\]
by the Hölder inequality.

**Theorem 1.6.** If
\[
\lambda_1 - \frac{\mu_s}{V_s(\Omega)^{(p-s)/(N-s)}} < \lambda < \lambda_1,
\]
then problem (1.1) has a pair of nontrivial solutions $\pm u^\lambda$ such that
\[
\int_{\Omega} |\nabla u^\lambda|^p \, dx \leq \lambda_1 (\lambda_1 - \lambda)^{(N-p)/(p-s)} V_s(\Omega).
\]

**Theorem 1.7.** If $\lambda_k \leq \lambda < \lambda_{k+1} = \cdots = \lambda_{k+m} < \lambda_{k+m+1}$ for some $k, m \in \mathbb{N}$ and
\[
\lambda > \lambda_{k+1} - \frac{\mu_s}{V_s(\Omega)^{(p-s)/(N-s)}},
\]
then problem (1.1) has $m$ distinct pairs of nontrivial solutions $\pm u_j^\lambda$, $j = 1, \ldots, m$ such that
\[
\int_{\Omega} |\nabla u_j^\lambda|^p \, dx \leq \lambda_{k+1} (\lambda_{k+1} - \lambda)^{(N-p)/(p-s)} V_s(\Omega). \tag{1.8}
\]

In particular, we have the following existence result that is new when $N < p^2$.

**Corollary 1.8.** If
\[
\lambda_k - \frac{\mu_s}{V_s(\Omega)^{(p-s)/(N-s)}} < \lambda < \lambda_k
\]
for some $k \in \mathbb{N}$, then problem (1.1) has a nontrivial solution.
Remark 1.9. We note that $\lambda_1 \geq \mu_s / V_s(\Omega)^{(p-s)/(N-s)}$. Indeed, let $\varphi_1 > 0$ be an eigenfunction associated with $\lambda_1$. Then

$$\lambda_1 = \frac{\int_{\Omega} |\nabla \varphi_1|^p \, dx}{\int_{\Omega} \varphi_1^p \, dx} \geq \frac{\mu_s \left( \int_{\Omega} \varphi_1^{p^*(s)} / |x|^s \, dx \right)^{p/p^*(s)}}{\int_{\Omega} \varphi_1^p \, dx} \geq \frac{\mu_s}{V_s(\Omega)^{(p-s)/(N-s)}}$$

by (1.4) and (1.6).

Remark 1.10. Since $V_0(\Omega)$ is the volume of $\Omega$, in the nonsingular case $s = 0$, Theorems 1.6 & 1.7 and Corollary 1.8 reduce to Perera et al. [10, Theorem 1.1 and Corollary 1.2], respectively.

2. Preliminaries

2.1. Cohomological index

The $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [5] is defined as follows. Let $W$ be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\mathcal{A} = A/\mathbb{Z}_2$ be the quotient space of $A$ with each $u$ and $-u$ identified, let $f : \mathcal{A} \to \mathbb{RP}^{-1}$ be the classifying map of $\mathcal{A}$, and let $f^* : H^*(\mathbb{RP}^{-1}) \to H^*(\mathcal{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of $A$ is defined by

$$i(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \sup \left\{ m \geq 1 : f^*(\omega^{m-1}) \neq 0 \right\} & \text{if } A \neq \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{RP}^{-1})$ is the generator of the polynomial ring $H^*(\mathbb{RP}^{-1}) = \mathbb{Z}_2[\omega]$.

Example 2.1. The classifying map of the unit sphere $S^{m-1}$ in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{RP}^{m-1} \subset \mathbb{RP}^{-1}$, which induces isomorphisms on the cohomology groups $H^q$ for $q \leq m - 1$, so $i(S^{m-1}) = m$.

The following proposition summarizes the basic properties of this index.

Proposition 2.2. (Fadell–Rabinowitz [5]) The index $i : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ has the following properties:

(i_1) Definiteness: $i(A) = 0$ if and only if $A = \emptyset$.

(i_2) Monotonicity: If there is an odd continuous map from $A$ to $B$ (in particular, if $A \subset B$), then $i(A) \leq i(B)$. Thus, equality holds when the map is an odd homeomorphism.

(i_3) Dimension: $i(A) \leq \dim W$.

(i_4) Continuity: If $A$ is closed, then there is a closed neighborhood $N \in \mathcal{A}$ of $A$ such that $i(N) = i(A)$. When $A$ is compact, $N$ may be chosen to be a $\delta$-neighborhood $N_\delta(A) = \{ u \in W : \text{dist}(u, A) \leq \delta \}$.

(i_5) Subadditivity: If $A$ and $B$ are closed, then $i(A \cup B) \leq i(A) + i(B)$.
(i₆) Stability: If $SA$ is the suspension of $A \neq \emptyset$, obtained as the quotient space of $A \times [-1,1]$ with $A \times \{1\}$ and $A \times \{-1\}$ collapsed to different points, then $i(SA) = i(A) + 1$.

(i₇) Piercing property: If $A$, $A₀$ and $A₁$ are closed, and $\varphi : A \times [0,1] \to A₀ \cup A₁$ is a continuous map such that $\varphi(-u,t) = -\varphi(u,t)$ for all $(u,t) \in A \times [0,1]$, $\varphi(A \times [0,1])$ is closed, $\varphi(A \times \{0\}) \subset A₀$ and $\varphi(A \times \{1\}) \subset A₁$, then $i(\varphi(A \times [0,1]) \cap A₀ \cap A₁) \geq i(A)$.

(i₈) Neighborhood of zero: If $U$ is a bounded closed symmetric neighborhood of the origin, then $i(\partial U) = \dim W$.

2.2. Abstract critical point theorems

We will prove Theorems 1.2 and 1.3 using the following abstract critical point theorem proved in Yang and Perera [13], which generalizes the well-known linking theorem of Rabinowitz [12].

**Theorem 2.3.** Let $I$ be a $C¹$-functional defined on a Banach space $W$, and let $A₀$ and $B₀$ be disjoint nonempty closed symmetric subsets of the unit sphere $S = \{u \in W : \|u\| = 1\}$ such that

$$i(A₀) = i(S \setminus B₀) < \infty.$$  

Assume that there exist $R > r > 0$ and $v \in S \setminus A₀$ such that

$$\sup I(A) \leq \inf I(B), \quad \sup I(X) < \infty,$$

where

$$A = \{tu : u \in A₀, 0 \leq t \leq R\} \cup \{R\pi((1-t)u+tv) : u \in A₀, 0 \leq t \leq 1\},$$

$$B = \{ru : u \in B₀\},$$

$$X = \{tu : u \in A, \|u\| = R, 0 \leq t \leq 1\},$$

and $\pi : W \setminus \{0\} \to S$, $u \mapsto u/\|u\|$ is the radial projection onto $S$. Let $Γ = \{γ \in C(X,W) : γ(X) is closed and γ|A = id_A\}$, and set

$$c := \inf_{γ \in Γ} \sup_{u \in γ(X)} I(u).$$

Then

$$\inf I(B) \leq c \leq \sup I(X),$$  \hspace{1cm} (2.1)

in particular, $c$ is finite. If, in addition, $I$ satisfies the $(PS)₂$ condition, then $c$ is a critical value of $I$.

**Remark 2.4.** The linking construction used in the proof of Theorem 2.3 in [13] has also been used in Perera and Szulkin [11] to obtain nontrivial solutions of $p$-Laplacian problems with nonlinearities that cross an eigenvalue. A similar construction based on the notion of cohomological linking was given in Degiovanni and Lancelotti [3]. See also Perera et al. [9, Proposition 3.23].

Now let $I$ be an even $C¹$-functional defined on a Banach space $W$, and let $A²$ denote the class of symmetric subsets of $W$. Let $r > 0$, let $S_r = \{u \in W : \|u\| = r\}$, let $0 < b \leq +\infty$, and let $Γ$ denote the group of odd
homeomorphisms of $W$ that are the identity outside $I^{-1}(0,b)$. The pseudo-index of $M \in \mathcal{A}^*$ related to $i$, $S_r$, and $\Gamma$ is defined by

$$i^*(M) = \min_{\gamma \in \Gamma} i(\gamma(M) \cap S_r)$$

(see Benci [2]). We will prove Theorems 1.6 and 1.7 using the following critical point theorem proved in Yang and Perera [13], which generalizes Bartolo et al. [1, Theorem 2.4].

**Theorem 2.5.** Let $A_0$ and $B_0$ be symmetric subsets of $S$ such that $A_0$ is compact, $B_0$ is closed, and

$$i(A_0) \geq k + m, \quad i(S \setminus B_0) \leq k$$

for some integers $k \geq 0$ and $m \geq 1$. Assume that there exists $R > r$ such that

$$\sup I(A) \leq 0 < \inf I(B), \quad \sup I(X) < b,$$

where $A = \{Ru : u \in A_0\}$, $B = \{ru : u \in B_0\}$, and $X = \{tu : u \in A, 0 \leq t \leq 1\}$. For $j = k + 1, \ldots, k + m$, let

$$\mathcal{A}_j^* = \{M \in \mathcal{A}^* : M \text{ is compact and } i^*(M) \geq j\},$$

and set

$$c_j^* := \inf_{M \in \mathcal{A}_j^*} \max_{u \in M} I(u).$$

Then

$$\inf I(B) \leq c_{k+1}^* \leq \cdots \leq c_{k+m}^* \leq \sup I(X),$$

in particular, $0 < c_j^* < b$. If, in addition, $I$ satisfies the $(PS)_c$ condition for all $c \in (0, b)$, then each $c_j^*$ is a critical value of $I$ and there are $m$ distinct pairs of associated critical points.

**Remark 2.6.** Constructions similar to the one used in the proof of Theorem 2.5 in [13] have also been used in Fadell and Rabinowitz [5] to prove bifurcation results for Hamiltonian systems and in Perera and Szulkin [11] to prove multiplicity results for $p$-Laplacian problems. See also Perera et al. [9, Proposition 3.44].

**2.3. Some estimates**

It was shown in [6, Theorem 3.1.(2)] that the infimum in (1.4) is attained by the family of functions

$$u_\varepsilon(x) = \frac{C_{N,p,s} \varepsilon^{(N-p)/(p-s)} p}{[\varepsilon + |x|^{(p-s)/(p-1)}]^{(N-p)/(p-s)}}, \quad \varepsilon > 0$$

when $\Omega = \mathbb{R}^N$, where $C_{N,p,s} > 0$ is chosen so that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} \frac{u_\varepsilon^{p^*(s)}}{|x|^s} \, dx = \mu_s^{(N-s)/(p-s)}.$$
Take a smooth function $\eta : [0, \infty) \to [0, 1]$ such that $\eta(s) = 1$ for $s \leq 1/4$ and $\eta(s) = 0$ for $s \geq 1/2$, and set

$$u_{\varepsilon, \delta}(x) = \eta \left( \frac{|x|}{\delta} \right) u_{\varepsilon}(x), \quad v_{\varepsilon, \delta}(x) = \frac{u_{\varepsilon, \delta}(x)}{\left( \int_{\mathbb{R}^N} \frac{u_{\varepsilon, \delta}^{p^*(s)}}{|x|^s} \, dx \right)^{1/p^*(s)}}, \quad \varepsilon, \delta > 0,$$

so that

$$\int_{\mathbb{R}^N} \frac{v_{\varepsilon, \delta}^{p^*(s)}}{|x|^s} \, dx = 1. \quad (2.2)$$

The following estimates were obtained in [6, Lemma 11.1.(1),(3),(4)]:

$$\int_{\mathbb{R}^N} |\nabla v_{\varepsilon, \delta}|^p dx \leq \mu_s + C\varepsilon^{(N-p)/(p-s)}, \quad (2.3)$$

$$\int_{\mathbb{R}^N} v_{\varepsilon, \delta}^p \, dx \geq \begin{cases} \frac{1}{C} \varepsilon^{(p-1)p/(p-s)} & \text{if } N > p^2 \\ \frac{1}{C} \varepsilon^{(p-1)p/(p-s)} \log \varepsilon & \text{if } N = p^2, \end{cases} \quad (2.4)$$

where $C = C(N, p, s, \delta) > 0$ is a constant. While these estimates are sufficient for the proof of Theorem 1.2, we will need the following finer estimates in order to prove Theorem 1.3.

**Lemma 2.7.** There exists a constant $C = C(N, p, s) > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v_{\varepsilon, \delta}|^p dx \leq \mu_s + C\Theta_{\varepsilon, \delta}^{(N-p)/(p-s)}, \quad (2.5)$$

$$\int_{\mathbb{R}^N} v_{\varepsilon, \delta}^p \, dx \geq \begin{cases} \frac{1}{C} \varepsilon^{(p-1)p/(p-s)} & \text{if } N > p^2 \\ \frac{1}{C} \varepsilon^{(p-1)p/(p-s)} \log \Theta_{\varepsilon, \delta} & \text{if } N = p^2, \end{cases} \quad (2.6)$$

where $\Theta_{\varepsilon, \delta} = \varepsilon \delta^{-(p-s)/(p-1)}$.

**Proof.** We have

$$u_{\varepsilon, \delta}(\delta x) = \delta^{-(N-p)/p} u_{\Theta_{\varepsilon, \delta}, 1}(x)$$

and

$$\int_{\mathbb{R}^N} \frac{u_{\varepsilon, \delta}^{p^*(s)}}{|x|^s} \, dx = \int_{\mathbb{R}^N} \frac{u_{\Theta_{\varepsilon, \delta}, 1}^{p^*(s)}}{|x|^s} \, dx.$$ 

So

$$v_{\varepsilon, \delta}(\delta x) = \delta^{-(N-p)/p} v_{\Theta_{\varepsilon, \delta}, 1}(x)$$

and hence

$$\nabla v_{\varepsilon, \delta}(\delta x) = \delta^{-N/p} \nabla v_{\Theta_{\varepsilon, \delta}, 1}(x).$$

Then

$$\int_{\mathbb{R}^N} |\nabla v_{\varepsilon, \delta}(\delta x)|^p \, dx = \delta^N \int_{\mathbb{R}^N} |\nabla v_{\varepsilon, \delta}(\delta x)|^p \, dx = \int_{\mathbb{R}^N} |\nabla v_{\Theta_{\varepsilon, \delta}, 1}(x)|^p \, dx.$$
and
\[ \int_{\mathbb{R}^N} v_{\varepsilon, \delta}^p (x) \, dx = \delta^N \int_{\mathbb{R}^N} v_{\varepsilon, \delta}^p (\delta x) \, dx = \delta^p \int_{\mathbb{R}^N} v_{\Theta, \delta, 1}^p (x) \, dx, \]
so (2.5) and (2.6) follow from (2.3) and (2.4), respectively. \( \square \)

Let \( i, \mathcal{M}, \Psi, \) and \( \lambda_k \) be as in the introduction, and suppose that \( \lambda_k < \lambda_{k+1} \). Then the sublevel set \( \Psi^{\lambda_k} \) has a compact symmetric subset \( E \) of index \( k \) that is bounded in \( L^\infty(\Omega) \cap C^{1, \alpha}_{\text{loc}}(\Omega) \) (see Degiovanni and Lancelotti [4, Theorem 2.3]). Let \( \delta_0 = \text{dist} (0, \partial \Omega) \), take a smooth function \( \theta : [0, \infty) \to [0, 1] \) such that \( \theta(s) = 0 \) for \( s \leq 3/4 \) and \( \theta(s) = 1 \) for \( s \geq 1 \), and set
\[ v_\delta(x) = \theta \left( \frac{|x|}{\delta} \right) v(x), \quad v \in E, \quad 0 < \delta \leq \frac{\delta_0}{2}. \]

Since \( E \subset \Psi^{\lambda_k} \) is bounded in \( C^{1}(B_{\delta_0/2}(0)) \),
\[ \int_\Omega |\nabla v_\delta|^p \, dx \leq \int_{\Omega \setminus B_\delta(0)} |\nabla v|^p \, dx + C \int_{B_\delta(0)} \left( |\nabla v|^p + \frac{|v|^p}{\delta^p} \right) \, dx \leq 1 + C \delta^{N-p} \]
and
\[ \int_\Omega |v_\delta|^p \, dx \geq \int_{\Omega \setminus B_\delta(0)} |v|^p \, dx = \int_\Omega |v|^p \, dx - \int_{B_\delta(0)} |v|^p \, dx \geq \frac{1}{\lambda_k} - C \delta^N, \]
where \( C = C(N, p, s, \Omega, k) > 0 \) is a constant. By (1.6) and (2.8),
\[ \int_\Omega \frac{|v_\delta|^{p^*(s)}}{|x|^s} \, dx \geq \frac{1}{C} \]
if \( \delta > 0 \) is sufficiently small.

Now let \( \pi : W_0^{1,p}(\Omega) \setminus \{0\} \to \mathcal{M}, \ u \mapsto u/\|u\| \) be the radial projection onto \( \mathcal{M} \), and set
\[ w = \pi(v_\delta), \quad v \in E. \]
If \( \delta > 0 \) is sufficiently small,
\[ \Psi(w) = \frac{\int_\Omega |\nabla v_\delta|^p \, dx}{\int_\Omega |v_\delta|^p \, dx} \leq \lambda_k + C \delta^{N-p} < \lambda_{k+1} \]
by (2.7) and (2.8), and
\[ \int_\Omega \frac{|w|^{p^*(s)}}{|x|^s} \, dx = \frac{\int_\Omega \frac{|v_\delta|^{p^*(s)}}{|x|^s} \, dx}{\left( \int_\Omega |\nabla v_\delta|^p \, dx \right)^{p^*(s)/p}} \geq \frac{1}{C} \]
by (2.7) and (2.9). Since \( \text{supp} \ w = \text{supp} \ v_\delta \subset \Omega \setminus B_{3\delta/4}(0) \) and \( \text{supp} \pi(v_{\varepsilon, \delta}) = \text{supp} \ v_{\varepsilon, \delta} \subset B_{\delta/2}(0) \),
\[ \text{supp} \ w \cap \text{supp} \pi(v_{\varepsilon, \delta}) = \emptyset. \]
Set

\[ E_\delta = \{ w : v \in E \} \, . \]

**Lemma 2.8.** For all sufficiently small \( \delta > 0 \),

(i) \( E_\delta \cap \Psi_{\lambda_{k+1}} = \emptyset \),
(ii) \( i(E_\delta) = k \),
(iii) \( \pi(v_{\varepsilon, \delta}) \notin E_\delta \).

**Proof.** (i) follows from (2.10). By (i), \( E_\delta \subset \mathcal{M} \setminus \Psi_{\lambda_{k+1}} \) and hence

\[ i(E_\delta) \leq i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}}) = k \]

by the monotonicity of the index and (1.5). On the other hand, since \( E \to \tilde{E}_\delta \), \( v \mapsto \pi(v_\delta) \) is an odd continuous map,

\[ i(E_\delta) \geq i(E) = k \, . \]

(ii) follows. (iii) is immediate from (2.12). \( \square \)

3. **Proofs**

3.1. **Proof of Theorem 1.1**

All nontrivial critical points of \( I_\lambda \) lie on the Nehari manifold

\[ \mathcal{N} = \left\{ u \in W^{1,p}_0(\Omega) \setminus \{ 0 \} : I_\lambda'(u) u = 0 \right\} . \]

We will show that \( I_\lambda \) attains the ground state energy

\[ c := \inf_{u \in \mathcal{N}} I_\lambda(u) \]

at a positive critical point.

Since \( 0 < \lambda < \lambda_1 \), \( \mathcal{N} \) is closed, bounded away from the origin, and for \( u \in W^{1,p}_0(\Omega) \setminus \{ 0 \} \) and \( t > 0 \), \( tu \in \mathcal{N} \) if and only if \( t = t_u \), where

\[ t_u = \left[ \frac{\int_\Omega (|\nabla u|^p - \lambda |u|^p) \, dx}{\left( \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} \, dx \right)^{p/(p-s)}} \right]^{(N-p)/(p-s)} \, . \]

Moreover,

\[ I_\lambda(t_u u) = \sup_{t > 0} I_\lambda(tu) = \frac{p-s}{(N-s)p} \psi_\lambda(u)^{(N-s)/(p-s)} \, , \]

where

\[ \psi_\lambda(u) = \frac{\int_\Omega (|\nabla u|^p - \lambda |u|^p) \, dx}{\left( \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} \, dx \right)^{p/p^*(s)}} \, . \]
By (2.2)–(2.4),
\[
\psi_\lambda(v_{\varepsilon, \delta}) \leq \begin{cases} 
\mu_s - \frac{\varepsilon (p-1)p/(p-s)}{C} + C\varepsilon (N-p)/(p-s) & \text{if } N > p^2 \\
\mu_s - \frac{\varepsilon (p-1)p/(p-s)}{C} \log |\varepsilon| + C\varepsilon (p-1)p/(p-s) & \text{if } N = p^2,
\end{cases}
\]
and in both cases the last expression is strictly less than \(\mu_s\) if \(\varepsilon > 0\) is sufficiently small, so
\[
c \leq I_\lambda(t_{v_{\varepsilon, \delta}}v_{\varepsilon, \delta}) < \frac{p - s}{(N - s)p} \mu_s (N-s)/(p-s).
\]
Then \(I_\lambda\) satisfies the \((PS)\) condition by \([6, \text{Theorem 4.1.}(2)]\), and hence \(I_\lambda|_N\) has a minimizer \(u_0\) by a standard argument. Then \(|u_0|\) is also a minimizer, which is positive by the strong maximum principle.

### 3.2. Proof of Theorem 1.2

We will show that problem (1.1) has a nontrivial solution as long as \(\lambda > \lambda_1 \) is not an eigenvalue from the sequence \((\lambda_k)\). Then we have \(\lambda_k < \lambda < \lambda_{k+1}\) for some \(k \in \mathbb{N}\). Fix \(\delta > 0\) so small that the first inequality in (2.10) implies
\[
\Psi(w) \leq \lambda \quad \forall w \in E_\delta\quad (3.1)
\]
and the conclusions of Lemma 2.8 hold. Then let \(A_0 = E_\delta\) and \(B_0 = \Psi_{\lambda, k+1}\), and note that \(A_0\) and \(B_0\) are disjoint nonempty closed symmetric subsets of \(\mathcal{M}\) such that
\[
i(A_0) = i(\mathcal{M} \setminus B_0) = k\quad (3.2)
\]
by Lemma 2.8 (i), (ii) and (1.5). Now let \(R > r > 0\), let \(v_0 = \pi(v_{\varepsilon, \delta})\), which is in \(\mathcal{M} \setminus A_0\) by Lemma 2.8 (iii), and let \(A, B\) and \(X\) be as in Theorem 2.3.

For \(u \in B_0\),
\[
I_\lambda(ru) \geq \frac{1}{p} \left( 1 - \frac{\lambda}{\lambda_{k+1}} \right) r^p - \frac{r^{p^*(s)}}{p^*(s) \mu_s^{p^*(s)/p}}.
\]
Since \(\lambda < \lambda_{k+1}\), and \(s < p\) implies \(p^*(s) > p\), it follows that \(\inf I_\lambda(B) > 0\) if \(r\) is sufficiently small.

Next we show that \(I_\lambda \leq 0\) on \(A\) if \(R\) is sufficiently large. For \(w \in A_0\) and \(t \geq 0\),
\[
I_\lambda(tw) \leq \frac{t^p}{p} \left( 1 - \frac{\lambda}{\Psi(w)} \right) \leq 0
\]
by (3.1). Now let \(w \in A_0\) and \(0 \leq t \leq 1\), and set \(u = \pi((1 - t)w + tv_0)\). Clearly, \(\|(1 - t)w + tv_0\| \leq 1\), and since the supports of \(w\) and \(v_0\) are disjoint by (2.12),
\[
\int_\Omega \frac{|(1 - t)w + tv_0|^{p^*(s)}}{|x|^s} dx = (1 - t)p^*(s) \int_\Omega \frac{|w|^{p^*(s)}}{|x|^s} dx + tp^*(s) \int_\Omega \frac{|v_0|^{p^*(s)}}{|x|^s} dx.
\]
In view of (2.11), and since
\[
\int_{\Omega} \frac{v_0^p(s)}{|x|^s} \, dx = \frac{\int_{\Omega} \frac{v_{\varepsilon, \delta}^p(s)}{|x|^s} \, dx}{\left( \int_{\Omega} |\nabla v_{\varepsilon, \delta}|^p \, dx \right)^{p^*(s)/p}} \geq \frac{1}{C}
\]
by (2.2) and (2.3) if \( \varepsilon > 0 \) is sufficiently small, it follows that
\[
\int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} \, dx = \frac{\int_{\Omega} \frac{|(1-t) w + tv_0|^{p^*(s)}}{|x|^s} \, dx}{\| (1-t) w + tv_0 \|^{p^*(s)}} \geq \frac{1}{C}.
\]
Then
\[
I_{\lambda}(Ru) \leq \frac{R^p}{p} - \frac{R^p}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} \, dx \leq 0
\]
if \( R \) is sufficiently large.

Now we show that
\[
\sup I_{\lambda}(X) < \frac{p - s}{(N - s) p} \mu_{s(N-s)/(p-s)}^{(N-s)/(p-s)} \tag{3.3}
\]
if \( \varepsilon > 0 \) is sufficiently small. Noting that
\[
X = \{ \rho \pi((1-t) w + tv_0) : w \in E_\delta, 0 \leq t \leq 1, 0 \leq \rho \leq R \},
\]
let \( w \in E_\delta \) and \( 0 \leq t \leq 1 \), and set \( u = \pi((1-t) w + tv_0) \). Then
\[
\sup_{0 \leq \rho \leq R} I_{\lambda}(\rho u) \leq \sup_{\rho \geq 0} \left[ \frac{\rho^p}{p} \left( 1 - \lambda \int_{\Omega} |u|^p \, dx \right) - \frac{\rho^{p^*(s)}}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} \, dx \right]
\]
\[
= \frac{p - s}{(N - s) p} \psi_{\lambda}(u)^{(N-s)/(p-s)}, \tag{3.4}
\]
where
\[
\psi_{\lambda}(u) = \left( 1 - \lambda \int_{\Omega} |u|^p \, dx \right)^+ \left( \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} \, dx \right)^{p/p^*(s)}
\]
\[
= \left( \int_{\Omega} \frac{|(1-t) \nabla w + t \nabla v_0|^p - \lambda |(1-t) w + tv_0|^p}{|x|^s} \, dx \right)^+ \left( \int_{\Omega} \frac{|(1-t) w + tv_0|^{p^*(s)}}{|x|^s} \, dx \right)^{p/p^*(s)}
\]
\[
\leq \frac{(1-t)^p}{(1-t)^{p^*(s)}} \left( 1 - \lambda \int_{\Omega} |w|^p \, dx \right)^+ + t^p \left( 1 - \lambda \int_{\Omega} v_0^p \, dx \right)^+ \left( \int_{\Omega} \frac{|w|^{p^*(s)}}{|x|^s} \, dx + t^{p^*(s)} \int_{\Omega} \frac{v_0^{p^*(s)}}{|x|^s} \, dx \right)^{p/p^*(s)} \tag{3.5}
\]
since the supports of \( w \) and \( v_0 \) are disjoint. Since

\[
1 - \lambda \int_\Omega |w|^p \, dx = 1 - \frac{\lambda}{\Psi(w)} \leq 0
\]

by (3.1),

\[
\psi_\lambda(u) \leq \psi_\lambda(v_0) = (\int_\Omega \left[ |\nabla v_{\varepsilon, \delta}|^p - \lambda v_{\varepsilon, \delta}^p \right] \, dx)^{\frac{1}{p}} + \left( \int_\Omega \frac{v_{\varepsilon, \delta}^{p^*(s)}}{|x|^s} \, dx \right)^{\frac{p}{p^*(s)}} \leq \left\{ \begin{array}{ll}
\mu_s - C \varepsilon^{\frac{(p-1)p/(p-s)}{s}} & \text{if } N > p^2 \\
\mu_s - C \varepsilon^{\frac{(p-1)p/(p-s)}{s}} & \text{if } N = p^2
\end{array} \right.
\]

by (2.2)–(2.4). In both cases the last expression is strictly less than \( \mu_s \) if \( \varepsilon > 0 \) is sufficiently small, so (3.3) follows from (3.4).

The inequalities (2.1) now imply that

\[
0 < c < \frac{p-s}{(N-s)p} \mu_s^{(N-s)/(p-s)}.
\]

Then \( I_\lambda \) satisfies the (PS)\(_c\) condition by [6, Theorem 4.1.(2)], and hence \( c \) is a positive critical value of \( I_\lambda \) by Theorem 2.3.

### 3.3. Proof of Theorem 1.3

The case where \( \lambda > \lambda_1 \) is an eigenvalue, but not from the sequence \((\lambda_k)\), was covered in the proof of Theorem 1.2, so we may assume that \( \lambda = \lambda_k < \lambda_{k+1} \) for some \( k \in \mathbb{N} \). Take \( \delta > 0 \) so small that (2.10) and the conclusions of Lemma 2.8 hold, let \( A_0, B_0 \) and \( v_0 \) be as in the proof of Theorem 1.2, and let \( A, B \) and \( X \) be as in Theorem 2.3.

As before, \( \inf I_\lambda(B) > 0 \) if \( r \) is sufficiently small, and

\[
I_\lambda(R \pi((1 - t) w + tv_0)) \leq 0 \quad \forall w \in A_0, \ 0 \leq t \leq 1
\]

if \( \Theta_{\varepsilon, \delta} \) is sufficiently small and \( R \) is sufficiently large. On the other hand,

\[
I_\lambda(tw) \leq \frac{t^p}{p} \left( 1 - \frac{\lambda_k}{\Psi(w)} \right) \leq CR^p \delta^{N-p} \quad \forall w \in A_0, \ 0 \leq t \leq R
\]

by (2.10). It follows that \( \sup I_\lambda(A) < \inf I_\lambda(B) \) if \( \delta \) is also sufficiently small.

It only remains to verify (3.3) for suitable choice of \( \delta(\varepsilon) \) and small \( \varepsilon \). Maximizing the last expression in (3.5) over \( 0 \leq t \leq 1 \) gives

\[
\psi_\lambda(u) \leq \left[ \psi_\lambda(v_0)^{(N-s)/(p-s)} + \psi_\lambda(w)^{(N-s)/(p-s)} \right]^{(p-s)/(N-s)}. \quad (3.6)
\]
We only give the proof of Theorem 1.7. Proof of Theorem 1.6 is similar and simpler. By [6, Theorem 4.1.(2)],

By (2.2), (2.5), and (2.6),

and by (2.10) and (2.11),

Recalling that $\Theta_{\varepsilon,\delta} = \varepsilon^{-\delta-(p-s)/(p-1)}$, if there exist $\alpha \in (0, (p-1)/(p-s))$ and a sequence $\varepsilon_j \to 0$ such that, for $\varepsilon = \varepsilon_j$ and $\delta = \varepsilon_j^\alpha$, $\psi_\lambda(v_0) < \mu_s/3$, then $\psi_\lambda(u) \leq 2\mu_s/3$ for sufficiently large $j$ by (3.6) and (3.8), which together with (3.4) gives the desired result. So we may assume that for all $\alpha \in (0, (p-1)/(p-s))$, $\psi_\lambda(v_0) \geq \mu_s/3$ for all sufficiently small $\varepsilon$ and $\delta = \varepsilon^\alpha$. Since $(p-s)/(N-s) < 1$, then (3.6)--(3.8) with $\delta = \varepsilon^\alpha$ yield

where

by (1.3). Taking $\alpha \in (\alpha_1, \alpha_2)$ now gives the desired conclusion.

### 3.4. Proofs of Theorems 1.6 and 1.7

We only give the proof of Theorem 1.7. Proof of Theorem 1.6 is similar and simpler. By [6, Theorem 4.1.(2)], $I_\lambda$ satisfies the (PS)$_c$ condition for all

so we apply Theorem 2.5 with $b$ equal to the right-hand side.

By Degiovanni and Lancelotti [4, Theorem 2.3], the sublevel set $\Psi^{\lambda_k+m}$ has a compact symmetric subset $A_0$ with

We take $B_0 = \Psi^{\lambda_k+1}$, so that

We take $B_0 = \Psi^{\lambda_k+1}$, so that
by (1.5). Let $R > r > 0$ and let $A$, $B$ and $X$ be as in Theorem 2.5. For $u \in \Psi_{\lambda_k+1}$, 
\[ I_\lambda(ru) \geq \frac{r^p}{p} \left( 1 - \frac{\lambda}{\lambda_k+1} \right) - \frac{Rp^*(s)}{p^*(s) \mu_s^{p^*(s)/p}} \]
by (1.4). Since $\lambda < \lambda_k+1$, and $s < p$ implies $p^*(s) > p$, it follows that $\inf I_\lambda(B) > 0$ if $r$ is sufficiently small. For $u \in A_0 \subset \Psi_{\lambda_k+1}$, 
\[ I_\lambda(Ru) \leq \frac{Rp}{p} \left( 1 - \frac{\lambda}{\lambda_k+1} \right) - \frac{Rp^*(s)}{p^*(s) \lambda_k^{p^*(s)/p} V_s(\Omega)(p-s)/(N-p)} \]
by (1.6), so there exists $R > r$ such that $I_\lambda \leq 0$ on $A$. For $u \in X$,
\[ I_\lambda(u) \leq \frac{\lambda_k+1 - \lambda}{p} \int_\Omega |u|^p \, dx - \frac{1}{p^*(s) V_s(\Omega)(p-s)/(N-p)} \left( \int_\Omega |u|^p \, dx \right)^{p^*(s)/p} \]
\[ \leq \sup_{\rho \geq 0} \left[ \frac{(\lambda_k+1 - \lambda) \rho}{p} - \frac{\rho p^*(s)/p}{p^*(s) V_s(\Omega)(p-s)/(N-p)} \right] \]
\[ = \frac{p - s}{(N-s)p} (\lambda_k+1 - \lambda)^{(N-s)/(p-s)} V_s(\Omega). \]
So 
\[ \sup I_\lambda(X) \leq \frac{p - s}{(N-s)p} \lambda_k^{p^*(s)/(p-s)} V_s(\Omega) < \frac{p - s}{(N-s)p} \mu_s^{p^*(s)/(p-s)} \]
by (1.7). Theorem 2.5 now gives $m$ distinct pairs of (nontrivial) critical points $\pm u_j^\lambda$, $j = 1, \ldots, m$ of $I_\lambda$ such that 
\[ 0 < I_\lambda(u_j^\lambda) \leq \frac{p - s}{(N-s)p} \lambda_k^{p^*(s)/(p-s)} V_s(\Omega). \]
Since 
\[ \int_\Omega |\nabla u_j^\lambda|^p \, dx = p I_\lambda(u_j^\lambda) + \lambda \int_\Omega |u_j^\lambda|^p \, dx + \frac{p}{p^*(s)} \int_\Omega \frac{|u_j^\lambda|^{p^*(s)}}{|x|^s} \, dx, \]
\[ \int_\Omega |u_j^\lambda|^p \, dx \leq V_s(\Omega)^{(p-s)/(N-s)} \left( \int_\Omega \frac{|u_j^\lambda|^{p^*(s)}}{|x|^s} \, dx \right)^{p/p^*(s)} \]
by (1.6), and 
\[ \int_\Omega \frac{|u_j^\lambda|^{p^*(s)}}{|x|^s} \, dx = \frac{(N-s)p}{p-s} \left[ I_\lambda(u_j^\lambda) - \frac{1}{p} I_\lambda'(u_j^\lambda) u_j^\lambda \right] = \frac{(N-s)p}{p-s} I_\lambda(u_j^\lambda), \]
(1.8) follows. This completes the proof of Theorem 1.7.

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