Derivatives of the $L^p$-cosine transform

Yossi Lonke

Abstract

The $L^p$-cosine transform of an even, continuous function $f \in C_e(S^{n-1})$ is defined by:

$$H(x) = \int_{S^{n-1}} |\langle x, \xi \rangle|^p f(\xi) \, d\xi, \quad x \in \mathbb{R}^n.$$ 

It is shown that if $p$ is not an even integer then all partial derivatives of even order of $H(x)$ up to order $p + 1$ (including $p + 1$ if $p$ is an odd integer) exist and are continuous everywhere in $\mathbb{R}^n \setminus \{0\}$. As a result of the corresponding differentiation formula, we show that if $f$ is a positive bounded function and $p > 1$ then $H^{1/p}$ is a support function of a convex body whose boundary has everywhere positive Gauss-Kronecker curvature.

1 Introduction

Recent research in convex geometry has repeatedly utilized two important integral transforms of functions defined on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. These are the cosine transform and the spherical Radon transform, both acting on $C^\infty_e(S^{n-1})$, the space of infinitely differentiable even functions on $S^{n-1}$, by:

$$Tf(x) = \int_{S^{n-1}} |\langle x, \xi \rangle| f(\xi) \, d\xi, \quad \text{(cosine transform)}$$

$$Rf(x) = \int_{S^{n-1} \cap x^\perp} f(\xi) \, d\xi, \quad \text{(spherical Radon transform)}$$

where $\langle , \rangle$ denotes the scalar product, $d\xi$ the spherical Lebesgue measure, and $x^\perp$ the $n-1$ dimensional subspace orthogonal to $x$. It is well known that $T$ and $R$ are both continuous bijections of $C^\infty_e(S^{n-1})$ onto itself, (the topology on $C^\infty_e(S^{n-1})$ taken as uniform convergence of all derivatives). This fact
allows an extension of both transforms, by duality, to bi-continuous bijections of the dual space $D_e(\mathbb{S}^{n-1})$ of even distributions on $\mathbb{S}^{n-1}$. A pleasant consequence of this extension is that we may assign precise meanings to the symbols $R\rho, R^{-1}\rho, T\rho, T^{-1}\rho$, for a given even distribution $\rho \in D_e(\mathbb{S}^{n-1})$. For example, one has

$$(T^{-1}\rho)(f) = \rho(T^{-1}f), \quad \forall \rho \in D_e(\mathbb{S}^{n-1}), \quad \forall f \in C^\infty_e(\mathbb{S}^{n-1}).$$

In particular, one talks about the cosine transform of an $L^1$ function, or the spherical Radon transform of a measure. These purely analytic manipulations turned out it to have surprisingly far reaching consequences. For example, the key to the ultimate solution of the Busemann-Petty problem, (which was one of the most intriguing unsolved problems of convex geometry) was uncovered by Lutwak in [17], where the notion of intersection body was invented. An origin symmetric convex body is called an intersection body if its radial function is realized as a spherical Radon transform of a positive measure on $\mathbb{S}^{n-1}$. Lutwak reduced the Busemann-Petty problem to the analytic question of whether $R^{-1}\rho$ is a positive measure whenever $\rho$ is a radial function of a centrally symmetric convex body. The answer is yes, if and only if the dimension is at most 4. Although in general it was known that for sufficiently large $n$ the Busemann-Petty problem has a negative answer in $\mathbb{R}^n$ (see [3]), the curious dependence on the dimension and the precise role of convexity were not understood until they were revealed by means of sophisticated analysis in [12].

The relevance of the cosine transform to convex geometry becomes clear through the concept of zonoids, also called projection bodies. These are bodies that can be approximated to any degree of accuracy, in the Hausdorff metric sense, by finite vector sums of intervals, called zonotopes. Every zonotope has a center of symmetry (namely, the sum of the centers of the intervals). Up to translation, every zonotope $Z$ has therefore the form $Z = \sum_i^n \lambda_i[-u_i, u_i]$, for some positive numbers $\lambda_i$ and $u_i \in \mathbb{S}^{n-1}$. Here $[-u_i, u_i]$ denotes the convex hull of $\{-u_i, u_i\}$. The support function of $Z$ is then $h_z(x) = \sum_i^n \lambda_i |\langle u_i, x \rangle|$. Let $\delta_u$ denote the unit-mass measure concentrated
at \( u \in \mathbb{S}^{n-1} \). Put \( \mu = \sum_{i=1}^{m} \lambda_i \delta_{u_i} + \delta_{-u} \). Then
\[
 h_Z(x) = \int_{\mathbb{S}^{n-1}} |\langle x, u \rangle| \, d\mu = T\mu(x).
\] (1)

In other words, the support function of a zonotope is a cosine transform of a positive, discrete measure. A standard approximation argument yields a fundamental theorem: A centrally symmetric convex body is a zonoid if and only if its support function is a cosine transform of a positive measure.

The measure \( \mu \) in (1) is called the generating measure of \( Z \). Generalizing this concept, Weil [21] proved that to every centrally symmetric convex body \( K \subset \mathbb{R}^n \) corresponds a unique generating distribution, that is, a continuous linear functional \( \rho_K \) on the space \( C^\infty_c(\mathbb{S}^{n-1}) \), whose domain can be extended as to include the functions \( |\langle u, \cdot \rangle| \) with \( u \in \mathbb{S}^{n-1} \), such that \( \rho_K(|\langle u, \cdot \rangle|) = h_K(u) \) for every \( u \in \mathbb{S}^{n-1} \). Recall that positive distributions are in fact positive measures. Thus in the context of zonoids Weil’s result is particularly useful — it provides a-priori a functional, namely \( T^{-1}h_K \), whose positivity is to be checked. Interestingly, the cosine and spherical Radon transforms are related by:
\[
 T^{-1} = c_n (\Delta_n + n - 1) R^{-1},
\] (2)

where \( \Delta_n \) is the spherical Laplace operator on \( \mathbb{S}^{n-1} \), and \( c_n > 0 \) (see [4]). The inversion formula (2) proved a useful analytic tool in constructing examples of non-smooth zonoids whose polars are zonoids [16], and of convex bodies whose generating distributions have large degree [15].

Often one thinks of \( h_Z(x) \) in (1) as representing the norm of some space, which in this case is isometric to a subspace of \( L^1(\mathbb{S}^{n-1}, \mu) \). A natural generalization is then to look at functions of the form
\[
 H^p(x) = \int_{\mathbb{S}^{n-1}} |\langle x, \xi \rangle|^p \, d\mu, \quad (p \geq 1)
\] (3)

If \( \mu \) is positive, \( H \) is continuous, convex and 1-homogeneous, hence a support function of some convex body, and also the norm of some normed space, which is evidently isometric to a subspace of \( L^p(\mathbb{S}^{n-1}, \mu) \). The r.h.s of (3) is called the \( L^p \)-cosine transform of the measure \( \mu \), and is denoted by \( T_p\mu \). If \( p \) is not an even integer, the measure \( \mu \) in (3) is uniquely determined by the
norm on the left hand side. For $p = 1$, this was first proved by Alexandrov \[1\] and rediscovered several times since. In \[19\], Neyman proved that if $p$ is not an even integer, the linear span of the functions $|\langle x, \cdot \rangle|^p$, defined on $S^{n-1}$ and indexed by $x \in \mathbb{R}^n$, is dense in the space $C_e(S^{n-1})$ of continuous even functions on $S^{n-1}$. In particular, $\mu$ in (3) is uniquely determined. If $p$ is an even integer, the functions $|\langle x, \cdot \rangle|^p$ span precisely the subspace of homogeneous (even) polynomials of degree $p$ (see \[19\]), so that there is no longer uniqueness in the representation (3). The inversion problem for the $L^p$-cosine transform of $L^1$ functions has been treated in \[7\] in several important special cases. The general case of inversion has apparently been neglected.

In a recent paper \[18\], the cosine transform of a continuous function was shown to be a $C^2$ function. In the first section below, this result is generalized in two ways. First, it is proved that for a nonnegative integer $k$, the $2k+1$-cosine transform of a continuous function is of class $C^{2k+2}$. The proof below invokes Fourier transform techniques developed by Koldobksy in a series of papers (\[6, 7, 8, 9, 10\]). Then, we deal with the $L^p$-cosine transform where $p > 1$ is not an integer, and show that if $f$ is a bounded function, then $T_p f$ has continuous derivatives of the largest even order smaller than $p + 1$. For second order derivatives, this was done in a more general setting in \[11\], using other methods. The first section is concluded with an additional result, asserting that the cosine transform carries $L^1(S^{n-1})$ into $C^1(S^{n-1})$. Our main application is expounded in section 2, where we show that if $H^p = T_p f$ with $f$ positive and bounded, then $H$ is a support function of a centrally symmetric $C^2_+$ convex body. That is, the boundary of the body has everywhere positive Gauss-Kronecker curvature. This should be compared to Theorem 2 of \[18\], which asserts that zonoids (i.e, the $p = 1$ case) whose generating measures are continuous functions may fail to have positive Gauss-Kronecker curvature at some boundary point only because all the principal radii of curvature evaluated at the corresponding outward unit normal are zero (whereas in general the curvature may not exist due to just one vanishing principal radius of curvature).
2 Differentiation of the $L^p$-cosine transform

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote a multi-index ( $\alpha_k$ are nonnegative integers ). $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Given $\alpha$, $D^\alpha$ denotes the differential operator

$$D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} (\frac{\partial}{\partial x_2})^{\alpha_2} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$$

In what follows, we denote $C_t = \frac{2^{t+1} \pi^{(t+1)/2}}{\Gamma((-t)/2)}$. Our first result is a generalization of Th. 1 in [18].

**Theorem 2.1** Let $n \geq 2$ and suppose that

$$H(x) = \int_{S^{n-1}} |\langle x, \xi \rangle|^{2k+1} f(\xi) d\xi$$

where $k$ is a nonnegative integer, and $f \in C_c(S^{n-1})$. Then $H \in C^{2k+2}(\mathbb{R}^n \setminus \{0\})$ and for every multi-index $\alpha$ with $|\alpha| = 2k + 2$, one has for each $x \in \mathbb{R}^n \setminus \{0\}$

$$D^\alpha H(x) = C_{2k+1}^{0} (\frac{-1}{|x|})^{k+1} \int_{S^{n-1} \cap x \bot} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} f(\xi) d\xi \quad (4)$$

In case the differentiation-order $|\alpha|$ strictly smaller than $p + 1$, (and even) the assumptions on $f$ can be somewhat relaxed, and the corresponding differentiation formula is different. For these reasons the result is formulated separately.

**Theorem 2.2** Let $n \geq 2$ and suppose that

$$H(x) = \int_{S^{n-1}} |\langle x, \xi \rangle|^p f(\xi) d\xi$$

where $p > 1, p \neq 2k$ and $f \in L^\infty(S^{n-1})$. Let $\alpha$ be a multi-index such that $|\alpha|$ is even and $|\alpha| < p + 1$. Then $H \in C^{|\alpha|}(\mathbb{R}^n \setminus \{0\})$ and

$$D^\alpha H(x) = i^{\alpha} \frac{C_p}{C_{p-|\alpha|}} \int_{S^{n-1}} |\langle x, \xi \rangle|^{p-|\alpha|} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} f(\xi) d\xi. \quad (5)$$

For the proofs, we use distribution theory and Fourier transforms. As usual, let $S(\mathbb{R}^n)$ denote the space of rapidly decreasing infinitely differentiable functions (test functions) in $\mathbb{R}^n$, and $S'(\mathbb{R}^n)$ is the space of distributions over $S(\mathbb{R}^n)$. The Fourier transform of a distribution $f \in S'(\mathbb{R}^n)$ is defined by $(\hat{f}, \phi) = (2\pi)^n (f, \phi)$, for every test function $\phi$. 

Proof of Theorem 2.1 For every test function $\phi(x)$ supported in $\mathbb{R}^n \setminus \{0\}$ consider another test function $\psi(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \phi(x)$. Since $|\alpha|$ is even, so is $\psi$. From lemma 2.2 of [7] we have

$$ (\hat{H}, \psi) = (2\pi)^{-n+1} C_{2k+1} \int_{\mathbb{S}^{n-1}} f(\xi) d\xi \int_{\mathbb{R}} t^{-2k-2} \psi(t\xi) dt. \quad (6) $$

Therefore,

$$ (\prod_{k=1}^{n} x_k^{\alpha_k} \hat{H}, \phi) = (2\pi)^{-n+1} C_{2k+1} \int_{\mathbb{S}^{n-1}} \prod_{k=1}^{n} \xi_k^{\alpha_k} f(\xi) d\xi \int_{\mathbb{R}} \phi(t\xi) dt. $$

By the well-known connection between the Fourier transform and the Radon transform (see [8]), the function $t \to (2\pi)^n \phi(-t\xi)$ is the Fourier transform of the function $z \to \int_{\langle x, \xi \rangle = z} \hat{\phi}(x) dx$. Therefore, $\int_{\mathbb{R}} \phi(t\xi) dt = (2\pi)^{-(n+1)} \int_{\mathbb{S}^{n-1}} \hat{\phi}(x) dx$, so we have

$$ (\prod_{k=1}^{n} x_k^{\alpha_k} \hat{H}, \phi) = C_{2k+1} \int_{\mathbb{S}^{n-1}} \prod_{k=1}^{n} \xi_k^{\alpha_k} f(\xi) d\xi \int_{\mathbb{S}^{n-1}} \hat{\phi}(x) dx $$

Put $g(\xi) = \prod_{k=1}^{n} \xi_k^{\alpha_k} f(\xi)$, and let $R$ denote the spherical Radon transform. Since for $n \geq 2$ the function $|x|_2^{-1}$ is locally integrable, we have

$$ \int_{\mathbb{R}^n} |x|_2^{-1} \phi(x) Rg(x/|x|_2) dx = \int_{0}^{\infty} r^{n-2} \left( \int_{\mathbb{S}^{n-1}} \hat{\phi}(r\xi) Rg(\xi) d\xi \right) dr $$

$$ = \int_{0}^{\infty} r^{n-2} \int_{\mathbb{S}^{n-1}} g(\xi) d\xi \int_{\mathbb{S}^{n-1}} \hat{\phi}(ru) du dr $$

$$ = \int_{\mathbb{S}^{n-1}} g(\xi) d\xi \int_{\mathbb{S}^{n-1}} \hat{\phi}(x) dx $$

Self-duality of the spherical Radon transform was used here. Consequently,

$$ (\prod_{k=1}^{n} x_k^{\alpha_k} \hat{H}, \phi) = C_{2k+1} \int_{\mathbb{R}^n} |x|_2^{-1} \phi(x) Rg(x/|x|_2) dx \quad (7) $$

On the other hand, the well known connection between differentiation and Fourier transforms gives:

$$ (\prod_{k=1}^{n} x_k^{\alpha_k} \hat{H}, \phi) = i^{-|\alpha|} (D^\alpha H, \hat{\phi}) \quad (8) $$

Recall that $\hat{\phi} = (2\pi)^n \phi(-x)$. Therefore, for every distribution $f$ and an even test function $\phi$, one has $(f, \hat{\phi}) = (\hat{f}, \phi)$. Since $\phi(x)$ is an arbitrary
even test function (with \(0 \notin \text{supp} \phi\)). (8) together imply that the Fourier
transforms of the distributions
\[
D^\alpha H(x) \quad \text{and} \quad C_{2k+1} \frac{(-1)^{k+1}}{|x|_2^{2k+1}} \text{Rg} \left( \frac{x}{|x|_2} \right)
\]
are equal distributions in \(\mathbb{R}^n \setminus \{0\}\). Therefore, the distributions in (3) can
differ by a polynomial only \([5], \text{p. 119}\). Since both distributions are even
and homogeneous of degree \(-1\), the polynomial must be identically zero.
Hence the distributions in (3) are equal. To show that \(H\) is a \(C|\alpha|\) function
we must show that \(D^\alpha H\) exists also in the classical sense and is continuous.
As is well known in the theory of distributions, classical and distributional
derivatives coincide if the distributional derivative in question happens to be
a continuous function. \([13], \text{p. 136}\). Since \(f\) is continuous, so is the spher-
éical Radon transform of \(\prod_{k=1}^{n} \xi_k^\alpha f(\xi)\). Therefore \(D^\alpha H(x)\) is a continuous
function, and we have (4).

The proof of Theorem 2.2 uses the same technique. Instead of (3) we
now have:
\[
\langle \hat{H}, \psi \rangle = (2\pi)^{n-1} C_p \int_{S^{n-1}} f(\xi) d\xi \int_{\mathbb{R}} |t|^{-|\alpha|-p-1} \psi(t\xi) dt
\]

Therefore,
\[
\langle \prod_{k=1}^{n} x_k^\alpha \hat{H}, \phi \rangle = (2\pi)^{n-1} C_p \int_{S^{n-1}} \prod_{k=1}^{n} \xi_k^\alpha f(\xi) d\xi \int_{\mathbb{R}} |t|^{-|\alpha|-p-1} \phi(t\xi) dt.
\]

Since \(p - |\alpha| > -1\), and \(p - |\alpha|\) is not an even integer, we can apply Lemma
2.1 of \([13]\):
\[
\int_{\mathbb{R}} |t|^{-|\alpha|-p-1} \phi(t\xi) dt = \frac{1}{(2\pi)^{n-1} C_{p-|\alpha|}} \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^{-p-1} \hat{\phi}(x) dx
\]

Consequently,
\[
\langle \prod_{k=1}^{n} x_k^\alpha \hat{H}, \phi \rangle = \frac{C_p}{C_{p-|\alpha|}} \int_{\mathbb{R}^n} \left[ \int_{S^{n-1}} |\langle x, \xi \rangle|^{-p-|\alpha|} \prod_{k=1}^{n} \xi_k^\alpha f(\xi) d\xi \right] \hat{\phi}(x) dx
\]

The connection between differentiation and the Fourier transform yields in
this case:
\[
\langle \prod_{k=1}^{n} x_k^\alpha \hat{H}, \phi \rangle = i^{-|\alpha|} \langle D^\alpha H, \hat{\phi} \rangle
\]
Together, (11) and (12) imply that the Fourier transforms of the distributions

\[ D^\alpha H(x) \quad \text{and} \quad i^{[\alpha]} \frac{C_p}{C_{p-|\alpha|}} \int_{S^{n-1}} |\langle x, \xi \rangle|^{p-|\alpha|} \prod_{k=1}^{n} \xi_k^{\alpha_k} f(\xi) d\xi \]  

are equal distributions in \( \mathbb{R}^n \backslash \{0\} \). As before, the distributions in (13) are equal. It remains to check that the right hand side of (13) is a continuous function. It is obviously continuous in \( x \) if \( p > |\alpha| \). To see that it is also continuous in the case \( |\alpha| - 1 < p < |\alpha| \), pick a sequence \( x_m \neq 0 \) such that \( \lim_{m \to \infty} x_m = x_0 \neq 0 \). For sufficiently large \( m \), we have \( |\langle x_m, \xi \rangle| \geq |\langle x_0, \xi \rangle|/2 \) for each \( \xi \in S^{n-1} \). Therefore, the integrand in the right hand side of (13) is almost everywhere bounded above by the function \( \xi \to (|\langle x_0, \xi \rangle|/2)^{p-|\alpha| \|f\|_\infty} \), which is in \( L^1(S^{n-1}) \) since \( p - |\alpha| > -1 \). The desired continuity now follows from Lebesgue’s bounded convergence theorem.

We conclude this section with a supplementary result, related to the \( k = 0 \) case of Th. 2.1 above.

**Proposition 2.3** The cosine transform of an \( L^1 \) function is continuously differentiable in \( \mathbb{R}^n \backslash \{0\} \).

**Proof.** Let \( f \in L^1 \). By linearity of \( T \) and by writing \( f = f_+ - f_- \), where \( f_+ \), \( f_- \) are the positive and negative parts of \( f \) respectively, we may assume \( f \geq 0 \). In that case, \( Tf \) is a support function of a zonoid \( Z \). Put \( h_Z = Tf \). A convex body is strictly convex (i.e., contains no line-segments in its boundary) if and only if its support function differentiable in \( \mathbb{R}^n \backslash \{0\} \) ([21], 1.7.3, p. 40). If a zonoid \( Z \) is not strictly convex, its boundary has some lower dimensional face, which must be a translate of a zonoid of lower dimension that is a summand of \( Z \). ([2], Th. 3.2). This means that \( Z \) can be decomposed as \( Z = Z_1 + Z_2 \) where at least one of the summands has lower dimension. It follows that the generating measure of \( Z \) is a sum \( \mu_1 + \mu_2 \) of the generating measures of \( Z_1, Z_2 \), and at least one of these measures is supported on a proper subspace. In particular, \( \mu_1 + \mu_2 \) is not absolutely continuous; but the generating measure of \( Z \) is. Thus if \( h_Z = Tf \) with \( f \in L^1 \) and \( f > 0 \), then \( Z \) is a strictly convex zonoid, so \( h_Z \) is
differentiable in $\mathbb{R}^n\backslash\{0\}$. The proof is completed by noting that support functions differentiable in $\mathbb{R}^n\backslash\{0\}$ are already continuously differentiable there.

3 Application to curvature and convexity

The main result in this section is the following

**Theorem 3.1** Suppose $n \geq 2$ and

$$H^p(x) = \int_{S^{n-1}} |\langle x, \xi \rangle|^p f(\xi) \, d\xi$$

where $p > 1, p \neq 2k$ and $f$ is a positive element of $L^\infty(S^{n-1})$. Then $H(x)$ is a support function of a centrally symmetric convex body that has everywhere positive Gauss-Kronecker curvature.

The proof is largely based upon the next lemma.

**Lemma 3.2** Assume $H^p = T_p f$, where $p$ and $f$ are as in Theorem 3.1. For every unit vector $u \in S^{n-1}$, and every $v \neq u$, the second directional derivative of $H$ in the direction of $u$, evaluated at $v$, is positive.

**Proof.** Differentiating, one finds:

$$\frac{\partial^2 H}{\partial x_i \partial x_j} = \frac{1}{H^{p-1}} \frac{1}{p} \left( \frac{\partial^2 H^p}{\partial x_i \partial x_j} - \frac{p-1}{p} \frac{\partial H^p}{\partial x_i} \frac{\partial H^p}{\partial x_j} \right)$$

(14)

If $H^p = T_p f$, then by Theorem 2.2

$$\frac{\partial^2 H^p}{\partial x_i} = p(p-1) \int_{S^{n-1}} |\langle u, \xi \rangle|^{p-2} \xi_i f(\xi) \, d\xi$$

Moreover, differentiation under the integral sign can easily be justified and

$$\frac{\partial H^p}{\partial x_1} = p \int_{S^{n-1}} |\langle u, \xi \rangle|^{p-1} \text{sgn}\langle u, \xi \rangle \xi_1 f(\xi) \, d\xi$$

Next, applying the triangle inequality and the Cauchy-Schwartz inequality:

$$\left| \int_{S^{n-1}} |\langle u, \xi \rangle|^{p-1} \text{sgn}\langle u, \xi \rangle \xi_1 f(\xi) \, d\xi \right| \leq \int_{S^{n-1}} |\langle u, \xi \rangle|^{p-1} |\xi_1 f(\xi) \, d\xi |

\leq \left( \int_{S^{n-1}} |\langle u, \xi \rangle|^{p-2} \xi_i^2 f(\xi) \, d\xi \right)^{1/2} \left( \int_{S^{n-1}} |\langle u, \xi \rangle|^p f(\xi) \right)^{1/2}

= \left( \frac{1}{p(p-1)} \frac{\partial^2 H}{\partial x_1^2} \right)^{1/2} H^{p/2}$$

9
Therefore,
\[
\left( \frac{\partial H}{\partial x_1} \right)^2 \leq \frac{p}{p-1} \frac{\partial^2 H}{\partial x_1^2} H,
\]
which implies \( \frac{\partial^2 H}{\partial x_1^2} \geq 0 \). In case of equality, we have equality in the triangle inequality, and in the Cauchy-Schwartz inequality, applied to the functions \( |\langle u, \cdot \rangle|^{\frac{p-2}{p}} |\xi_1| \) and \( |\langle u, \cdot \rangle|^\frac{p}{2} \). Therefore, for every \( \xi \in \text{supp}\ f \), we have for some real constants \( s, t \) not both zero:

(i) \( \text{sgn}(u, \xi) \xi_1 = |\xi_1| \),

(ii) \( s|\langle u, \xi \rangle|^{p-2} \xi_2 = t|\langle u, \xi \rangle|^p \)

(i) implies \( s\xi_2 = t(u, \xi)^2 \). We cannot have \( s = 0 \) (resp. \( t = 0 \)), for then the support of \( f \) would have to be contained in \( u^\perp \) (resp. \( e_1^\perp \)), which contradicts \( \int_{S^{n-1}} f \, d\xi > 0. \) Hence both \( s, t \) are non zero, and have the same sign, so that with \( \lambda = (s/t)^{1/2} \) we have \( \lambda|\langle u, \xi \rangle| = |\langle e_1, \xi \rangle| \), and we can drop the absolute values, because \( \langle u, \xi \rangle, \langle e_1, \xi \rangle \) have the same sign. Consequently,

\[ \langle \xi, \lambda e_1 - u \rangle = 0 \quad \forall \xi \in \text{supp}\ f, \]

which unless \( u = e_1 \), contradicts the fact that \( \int_{S^{n-1}} f \, d\xi > 0. \) Therefore, unless \( u = e_1 \), one has \( \frac{\partial^2 H}{\partial x_1^2} > 0. \)

Now let \( u \) be any direction, and let \( U \) be an orthonormal matrix such that \( Ue_1 = u \). Let \( D_u \) denote differentiation in the \( u \) direction. A simple calculation yields:

\[ D_u(D_u H)(Uv) = \frac{\partial^2 H \circ U}{\partial x_1^2}(v). \]

Since \( H^p = T_p f \), one has \( (H \circ U)^p = T_p(f \circ U) \). By the first part of the proof, applied to \( H \circ U \) and \( f \circ U \) in place of \( H \) and \( f \), we get:

\[ \frac{\partial^2 H \circ U}{\partial x_1^2}(v) > 0 \]

whenever \( v \in S^{n-1} \) and \( v \neq e_1 \). Therefore \( D_u^2 H(v) > 0 \) whenever \( \xi \neq u \), as was asserted.

Proof of Theorem 3.1 By Theorem 2.2, \( H^p \), and therefore \( H \), are \( C^2 \) functions in \( \mathbb{R}^n \setminus \{0\} \). Since \( f \) is positive, \( H \) is a support function of some (strictly) convex body, say, \( K \). To show that \( K \) is of class \( C^2_+ \), it suffices to show that \( K \) has everywhere positive principal radii of curvature. (20, p. 111). Let \( T_u \) denote the tangent space to \( S^{n-1} \) at \( u \). The principal radii of curvature are eigenvalues of the reverse Weingarten map \( W_u : T_u \to T_u \),
where $\mathcal{W} u$ is $d(\nabla H) u$. (Note that since the gradient $\nabla H(u)$ is the unique point on the boundary of $K$ at which $u$ is an outer normal vector, its gradient $d(\nabla H) u$ maps the tangent space $T_u$ into itself). By [20], p. 108, Lemma 2.5.1,

$$d^2 H_u(v, w) = \langle \mathcal{W} u v, w \rangle, \quad \forall v, w \in T_u.$$ 

Therefore, if $\lambda$ is an eigenvalue of $\mathcal{W} u$ with an eigenvector $v$, then $\lambda = d^2 H_u(v, v)$. As explained in [20] p. 110, $d^2 H_u(v, v) = D^2 u H(v)$, which by Lemma 3.2 is positive.

**Remark** Theorem 3.1 no longer holds for $p = 1$. In fact, we can have $h = T f$ with $f \in C^\infty(\mathbb{S}^{n-1})$ and $f > 0$, but nonetheless $h$ is not $C^2_\pm$. Any zonoid whose support function is $C^\infty$ but not $C^2_\pm$ will do.

A special case of Theorem 2.1, for $k = 0$, was proved (in an elementary way) recently in [18]. Clearly, the $L^p$-cosine transform of a positive measure is a convex function, if $p \geq 1$. However, there are also $L^p$-cosine transforms of signed measures, possibly not positive, that are convex functions. A theorem by Lindquist [14] asserts that the cosine transform $T f(x)$ defines a support function if and only if

$$\int_{\mathbb{S}^{n-1} \cap u^+} (\xi, x)^2 f(\xi) \, d\xi \geq 0$$

for all $u \in \mathbb{S}^{n-1}$ and all $x \in \mathbb{S}^{n-1} \cap u^\perp$. As was observed in [18], the expression in (15) is precisely $d^2 H_u(x, x)$, where $H = T f(x)$. Thus, *a-posteriori* Lindquist’s criterion reduces to the classical assertion that a positively 1-homogeneous function (i.e, $T f$ ) is a support function if and only if its second differential is positive semidefinite at every point. In this case, homogeneity of $H$ permits consideration of $d^2 H_u(x, x)$ only for $x \perp u$.

Put $k = 0$ in (4). The result is:

$$\frac{\partial^2 H}{\partial x_i \partial x_j}(u) = \frac{2}{||u||^2} \int_{\mathbb{S}^{n-1} \cap u^\perp} \xi_i \xi_j f(\xi) \, d\xi \quad (u \in \mathbb{R}^n \setminus \{0\})$$

This in turn implies that for $u \in \mathbb{S}^{n-1}$, the Hessian matrix $H''$ evaluated at $u$ is given by:

$$\langle H''_u x, y \rangle = 2 \int_{\mathbb{S}^{n-1} \cap u^\perp} \langle x, \xi \rangle \langle y, \xi \rangle f(\xi) \, d\xi$$
Therefore \( \langle H''_u x, x \rangle \) becomes the integral in \([17]\). All this was pointed out in \([18]\). Applying the same reasoning to \((3)\), we get for \( p > 1 \) (\( p \) not an even integer):

\[
\frac{\partial^2 H}{\partial x_i \partial x_j}(u) = p(p - 1) \int_{S^{n-1}} \langle u, \xi \rangle|^{p-2} \xi_i \xi_j f(\xi) \, d\xi
\]

(18)

Hence we derive the following result – a \( p \)-version of Lindquist’s criterion, which is an immediate consequence of the previous equation.

**Theorem 3.3** Suppose \( n \geq 2 \) and

\[
H(x) = \int_{S^{n-1}} |\langle x, \xi \rangle|^p f(\xi) \, d\xi
\]

where \( p > 1, p \neq 2k \) and \( f \in C_e(S^{n-1}) \). Then \( H(x) \) is convex if and only if for all \( u \in S^{n-1}, x \in \mathbb{R}^n \)

\[
\int_{S^{n-1}} |\langle u, \xi \rangle|^{p-2} \langle x, \xi \rangle^2 f(\xi) \, d\xi \geq 0
\]

(19)

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**References**

[1] A. D. Alexandrov. On the theory of mixed volumes of convex bodies II. New inequalities between mixed volumes and their applications. *Mat. Sbornik N.S.*, 2:1205–1238, 1937.

[2] E. D. Bolker. A class of convex bodies. *Trans. Amer. Math. Soc.*, 145:323–346, 1969.

[3] K. Ball. Some remarks on the geometry of convex sets. In *Geometric aspects of Functional Analysis*, volume 1317, pages 224–231. Springer, 1988.

[4] P. Goodey and W. Weil. Centrally symmetric convex bodies and the spherical Radon transform. *Journal of Differential Geometry*, 35:675–688, 1992.
[5] I.M.Gelfand and G.E.Shilov. Generalized functions 2. Spaces of fundamental and generalized functions. Academic Press, New York, 1964.

[6] A. Koldobsky. Intersection bodies and the busemann-petty problem. C. R. Acad. Sci. Paris Sér. I Math., 325, no. 11:1181–1186, 1997.

[7] A. Koldobsky. Inverse formula for the Blaschke-Levy representation. Houston J.Math., 23, no. 1:95–108, 1997.

[8] A. Koldobsky. Intersection bodies in $\mathbb{R}^4$. Advances in Matematics, 136 no. 1:1–14, 1998.

[9] A. Koldobsky. Intersection bodies, positive definite distributions, and the busemann-petty problem. Amer. J. Math., 120, no. 4:827–840, 1998.

[10] A. Koldobsky. Second derivative test for intersection bodies. Advances in Mathematics, 136 no. 1:15–25, 1998.

[11] A. L. Koldobsky. Isometries of $L_p(X;L_q)$ and equimeasurability. Indiana Univ. Math. Journal, 40, no. 2:677–705, 1991.

[12] R. J. Gardner A. Koldobsky and T. Schlumprecht. An analytic solution to the Busemann-Petty problem on sections of convex bodies. Ann. of Math. (2), 149:691–703, 1999.

[13] E. H. Lieb and M. Loss. Analysis. American Mathematical Society, 1997.

[14] N. Lindquist. Support functions of central convex bodies. Port. Math., 34:241–252, 1975.

[15] Y. Lonke. On the degree of generating distributions of centrally symmetric convex bodies. Arch. Math. (Basel), 69, no. 4:343–349, 1997.

[16] Y. Lonke. On zonoids whose polars are zonoids. Israel Journal of Mathematics, 102:1–12, 1997.

[17] E. Lutwak. Intersection bodies and dual mixed volumes. Advances in Math., 71:232–261, 1988.
[18] Y. Martinez-Maure. Hedgehogs and Zonoids. *Advances in Math.*, 158:1–17, 2001.

[19] A. Neyman. Representation of $L_p$-norms and isometric embedding in $L_p$-spaces. *Isr. J. Math.*, 48:129–138, 1984.

[20] R. Schneider. *Convex bodies: The Brunn–Minkowski Theory*. Cambridge University Press, 1993.

[21] W. Weil. Centrally symmetric convex bodies and distributions. *Israel Journal of Mathematics*, 24:352–367, 1976.

Email address: yossi11@mac.com