ALGEBRAIC PROPERTIES OF BOUNDED KILLING VECTOR FIELDS

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Abstract. In this paper, we consider a connected Riemannian manifold $M$ where a connected Lie group $G$ acts effectively and isometrically. Assume $X \in \mathfrak{g} = \text{Lie}(G)$ defines a bounded Killing vector field, we find some crucial algebraic properties of the decomposition $X = X_r + X_s$ according to a Levi decomposition $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}$, where $\mathfrak{r}(\mathfrak{g})$ is the radical, and $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$ is a Levi subalgebra. The decomposition $X = X_r + X_s$ coincides with the abstract Jordan decomposition of $X$, and is unique in the sense that it does not depend on the choice of $\mathfrak{s}$. By these properties, we prove that the eigenvalues of $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ are all imaginary. Furthermore, when $M = G/H$ is a Riemannian homogeneous space, we can completely determine all bounded Killing vector fields induced by vectors in $\mathfrak{g}$. We prove that the space of all these bounded Killing vector fields, or equivalently the space of all bounded vectors in $\mathfrak{g}$ for $G/H$, is a compact Lie subalgebra, such that its semi-simple part is the ideal $\mathfrak{r}(\mathfrak{r}(\mathfrak{g}))$ of $\mathfrak{g}$, and its Abelian part is the sum of $\mathfrak{r}(\mathfrak{g})$ and all two-dimensional irreducible $\text{ad}(\mathfrak{r}(\mathfrak{g}))$-representations in $\mathfrak{r}(\mathfrak{g})$ corresponding to nonzero imaginary weights, i.e. $\mathbb{R}$-linear functionals $\lambda : \mathfrak{r}(\mathfrak{g}) \to \mathfrak{r}(\mathfrak{g})/\mathfrak{n}(\mathfrak{g}) \to \mathbb{R} \sqrt{-1}$, where $\mathfrak{n}(\mathfrak{g})$ is the nilradical.

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1. Introduction

In a recent paper [16], the second author considered a Riemannian manifold $M$, which permits the effective isometric action of a Lie group $G$. He studied the Killing vector field of constant length induced by a vector $X \in \mathfrak{g} = \text{Lie}(G)$, and proved that the eigenvalues of $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ are all imaginary. This discovery inspired him to propose the following conjecture (see Conjecture 1 in [16]).

Conjecture 1.1. Assume that a semi-simple Lie group $G$ acts effectively and isometrically on a connected Riemannian manifold $M$, and the vector $X \in \mathfrak{g}$ defines a Killing vector field of constant length. Then $X$ is a compact vector in $\mathfrak{g}$, i.e. the subalgebra $\mathbb{R}X$ is compactly imbedded in $\mathfrak{g}$.

See Section 2.1 for the notions of compact vector and compactly imbedded subalgebra.

Our initial motivation is to prove Conjecture 1.1. Our approach is different from that in [16], which depends on the Riemannian structure and the constant length condition. Here we only need to assume that the vector $X \in \mathfrak{g}$ defines a bounded Killing vector field.

Recall that a Killing vector field on a Riemannian manifold is called bounded if its length function with respect to the given metric is a bounded function. This condition is relatively weak. For example, any Killing vector field on a compact Riemannian
manifold is bounded. The special case, Killing vector fields of constant length, is intrinsically related to Clifford–Wolf translations [4, 8]. See [14, 20, 21] for some recent progress on this subject. On the other hand, curvature conditions may provide obstacles or rigidities for bounded Killing vector fields. For example, on a complete negatively curved Riemannian manifold, bounded Killing vector field must be zero [18]. On a complete non-positively curved Riemannian manifold, a bounded Killing vector field must be parallel [3].

We first prove the following theorem solving Conjecture 1.1, not only for Killing vector fields of constant length, but also for all bounded Killing vector fields.

**Theorem 1.2.** Let $M$ be a connected Riemannian manifold on which a connected semi-simple Lie group acts effectively and isometrically. Assume $X \in \mathfrak{g}$ defines a bounded Killing vector field. Then $X$ is contained in a compact ideal in $\mathfrak{g}$.

As a compact ideal in the semi-simple Lie algebra $\mathfrak{g}$ generates a compact semi-simple subgroup of $G$, we see immediately after Theorem 1.2 that $X$ is a compact vector when it is bounded, and hence $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ has only imaginary eigenvalues.

It is then natural to further study this spectral property of bounded Killing vector fields when $G$ is not semi-simple. For this purpose, we take a Levi decomposition $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}$ for $\mathfrak{g} = \text{Lie}(G)$ (see Section 2.1), and then we have the decomposition $X = X_r + X_s$ accordingly. Applying the argument for Theorem 1.2, some technique in the proof of Lemma 2.3 and Lemma 2.4 in [19], and more Lie algebraic discussion from Lemma 3.1 and Corollary 3.2 (see Lemma 3 in [17] for similar argument for bounded automorphisms of Lie groups), we prove the following crucial algebraic properties for the bounded Killing vector field $X$.

**Theorem 1.3.** Let $M$ be a connected Riemannian manifold on which the connected Lie group $G$ acts effectively and isometrically. Assume that $X \in \mathfrak{g}$ defines a bounded Killing vector field, and $X = X_r + X_s$ according to the Levi decomposition $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}$, where $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$. Then we have the following:

1. The vector $X_s \in \mathfrak{s}$ is contained in the compact semi-simple ideal $\mathfrak{c}_s(\mathfrak{r}(\mathfrak{g}))$ of $\mathfrak{g}$;
2. The vector $X_r \in \mathfrak{r}$ is contained in the center $\mathfrak{c}(\mathfrak{n})$ of $\mathfrak{n}$.

Here the centralizer $\mathfrak{c}_a(\mathfrak{b})$ of the subalgebra $\mathfrak{b} \subset \mathfrak{g}$ in the subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is defined as $\mathfrak{c}_a(\mathfrak{b}) = \{ u \in \mathfrak{a} \mid [u, \mathfrak{b}] = 0 \}$. In particular, the center $\mathfrak{c}(\mathfrak{a})$ of $\mathfrak{a} \subset \mathfrak{g}$ coincides with $\mathfrak{c}_a(\mathfrak{a})$.

Theorem 1.3 helps us find more algebraic properties for bounded Killing vector fields. In particular, $X = X_r + X_s$ is an abstract Jordan decomposition which is irrelevant to the choice of the Levi subalgebra $\mathfrak{s}$, and the eigenvalues of $\text{ad}(X)$ coincide with those of $\text{ad}(X_s)$, which are all imaginary (see Theorem 4.1). As a direct corollary, we have proved the following spectral property.

**Corollary 1.4.** Let $M$ be a connected Riemannian manifold on which the connected Lie group $G$ acts effectively and isometrically. Assume that $X \in \mathfrak{g}$ defines a bounded Killing vector field. Then all eigenvalues of $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ are imaginary.

When $M = G/H$ is a Riemannian homogeneous space on which the connected Lie group $G$ acts effectively, we can apply Theorem 1.3 to prove the following theorem, which completely determine all bounded vectors in $\mathfrak{g}$ for $G/H$, or equivalently all bounded Killing vector fields induced by vectors in $\mathfrak{g}$ (see Section 2.3 for the notion of bounded vectors for a coset space, and Lemma 2.3 for the equivalence).
Theorem 1.5. Let $G/H$ be a Riemannian homogeneous space on which the connected Lie group $G$ acts effectively. Let $\mathfrak{r}(g)$, $\mathfrak{n}(g)$ and $\mathfrak{s} = \mathfrak{s}_{sc} \oplus \mathfrak{s}_{nc}$ be the radical, the nilradical, and the Levi subalgebra respectively. Then the space of all bounded vectors in $\mathfrak{g}$ for $G/H$ is a compact subalgebra. Its semi-simple part coincides with the ideal $\mathfrak{c}_{sc}(\mathfrak{r}(g))$ of $\mathfrak{g}$, which is independent of the choice of the Levi subalgebra $\mathfrak{s}$, and its Abelian part $\mathfrak{v}$ is contained in $\mathfrak{c}(\mathfrak{n}(g))$, which coincides with the sum of $\mathfrak{c}_{c}(\mathfrak{r}(g))(\mathfrak{s}_{nc})$ and all two-dimensional irreducible representations of $\text{ad}(\mathfrak{r}(g))$ in $\mathfrak{c}_{c}(\mathfrak{n}(g))(\mathfrak{s}_{nc})$ corresponding to nonzero imaginary weights, i.e. $\mathbb{R}$-linear functionals $\lambda : \mathfrak{r} \rightarrow \mathfrak{r}/\mathfrak{n} \rightarrow \mathbb{R}\sqrt{-1}$.

Theorem 1.5 is a summarization of Theorem 4.3 and Theorem 4.8.

Note that $\mathfrak{c}_{sc}(\mathfrak{r}(g))$ is a compact semi-simple summand in the Lie algebra direct sum decomposition of $\mathfrak{g}$, which can be easily determined. For the other, the Abelian factor $\mathfrak{v}$, we propose a theoretic algorithm which explicitly describes all bounded vectors in $\mathfrak{c}(\mathfrak{n}(g))$.

Theorem 1.5 provides a simple and self contained proof of the following theorem.

Theorem 1.6. The space of bounded vectors in $\mathfrak{g}$ for a Riemannian homogeneous space $G/H$ on which the connected Lie group $G$ acts effectively is irrelevant to the choice of $H$.

Notice that the arguments in [17] indicate that the subset of all bounded isometries in $G$ is irrelevant to the choice of $H$. So Theorem 1.6 can also be proved by J. Tits’ Theorem 1 in [17], which implies that all bounded isometries in $G$ are generated by bounded vectors in $\mathfrak{g}$.

Meanwhile, Theorem 1.5 provides an alternative explanation why in some special cases, the much stronger constant length condition for Killing vector fields or Clifford–Wolf condition for translations may be implied by the boundedness condition [12, 19].

At the end, we remark that all lemmas, theorems and corollaries are still valid when $M$ is a Finsler manifold. The Finsler metric on a smooth manifold is a natural generalization of the Riemannian metric, which satisfies the properties of the smoothness, positiveness, homogeneity of degree one and strong convexity, but not the quadratic property in general. See [2] for its precise definition and more details. The proofs for all the results of this work in the Finsler context only need an add-on from the following well-known fact. The isometry group of a Finsler manifold is a Lie group [7] with a compact isotropy subgroup at any point.

This work is organized as following. In Section 2, we summarize some basic knowledge on Lie theory and homogeneous geometry which are necessary for later discussions. We define the bounded vector in $\mathfrak{g}$ for a smooth coset space $G/H$ and discuss its basic properties and relation to the bounded Killing vector field. In Section 3, we prove Theorem 1.2 and Theorem 1.3. In Section 4, we discuss two applications of Theorem 1.3. One is to prove the Jordan decomposition and spectral properties for bounded Killing vector fields. The other is to study the Lie algebra of all bounded vectors in $\mathfrak{g}$ for a Riemannian homogeneous space $G/H$, on which $G$ acts effectively. We will provide explicit description for this compact Lie algebra and completely determine all bounded Killing vector fields for a Riemannian homogeneous space.
2. Preliminaries in Lie Theory and Homogeneous Geometry

2.1. Some fundamental facts in Lie theory. Let \( \mathfrak{g} \) be a real Lie algebra. Its radical \( \mathfrak{r}(\mathfrak{g}) \) and nilradical (or nilpotent radical) \( \mathfrak{n}(\mathfrak{g}) \) are the unique largest solvable and nilpotent ideals of \( \mathfrak{g} \) respectively. By Corollary 5.4.15 in [10], we have

\[
[\mathfrak{r}(\mathfrak{g}), \mathfrak{r}(\mathfrak{g})] \subset [\mathfrak{r}(\mathfrak{g}), \mathfrak{g}] \subset \mathfrak{n}(\mathfrak{g}) \subset \mathfrak{r}(\mathfrak{g}).
\]

By Levi’s theorem, we can find a semi-simple subalgebra \( \mathfrak{s} \subset \mathfrak{g} \), which is the complement of the radical \( \mathfrak{r}(\mathfrak{g}) \) in \( \mathfrak{g} \). We will further decompose the \( \mathfrak{s} \) as \( \mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc} \), where \( \mathfrak{s}_c \) and \( \mathfrak{s}_{nc} \) are the compact and noncompact parts of \( \mathfrak{s} \) respectively. We will call

\[
\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s} \tag{2.1}
\]
a Levi decomposition, and the semi-simple subalgebra \( \mathfrak{s} \) in (2.1) a Levi subalgebra. By Malcev’s Theorem (see Theorem 5.6.13 in [10]), the Levi subalgebra \( \mathfrak{s} \) is unique up to \( \text{Ad}(\exp([\mathfrak{g}, \mathfrak{r}(\mathfrak{g})])) \)-actions.

If \( G \) is a connected Lie group with \( \text{Lie}(G) = \mathfrak{g} \), \( \mathfrak{r}(\mathfrak{g}) \) and \( \mathfrak{n}(\mathfrak{g}) \) generate closed solvable and nilpotent normal subgroups respectively.

A subalgebra \( \mathfrak{t} \subset \mathfrak{g} \) is called compactly imbedded if after taking closure it generates a compact subgroup in the inner automorphism group \( \text{Inn}(\mathfrak{g}) = G/Z(G) \). A vector \( X \in \mathfrak{g} \) is called a compact vector if \( \mathbb{R}X \) is a compactly imbedded subalgebra of \( \mathfrak{g} \).

Assume that \( G \) is a connected Lie group with \( \text{Lie}(G) = \mathfrak{g} \), and \( H \) the connected subgroup generated by a subalgebra \( \mathfrak{h} \). Obviously if \( H \) is compact, then any subalgebra of \( \mathfrak{h} \) is compactly imbedded. The converse statement is not true in general. We call a subgroup \( H \) of \( G \) compactly imbedded if the closure of \( \text{Ad}_g(H) \subset \text{Aut}(\mathfrak{g}) \) is compact.

Any compactly imbedded subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) is contained in a maximal compactly imbedded subalgebra. A maximal compactly imbedded subalgebra can be presented as the pre-image in \( \mathfrak{g} \) for the subalgebra of \( \mathfrak{g}/c(\mathfrak{g}) \) generating a maximal compact subgroup in \( G/Z(G) \). As an immediate corollary for the conjugation theorem for maximal compact connected subgroups (see Theorem 14.1.3 in [10]), the maximal compactly imbedded subalgebra is unique up to \( \text{Ad}(G) \)-actions.

2.2. Homogeneous metric and reductive decomposition. Let \( M \) be a Riemannian homogeneous space on which the connected Lie group \( G \) acts effectively and isometrically. The effectiveness implies that \( G \) is a subgroup of the isometry group \( I(M) \). When \( G \) is a closed subgroup of \( I(M) \), then \( H \) is compact. When \( G \) is not closed in \( I(M) \), then we still have the following consequence from the discussion in [11].

**Lemma 2.1.** Let \( M \) be a Riemannian homogeneous space on which the connected Lie group \( G \) acts effectively and isometrically. Then the isotropy subgroup \( H \) at any \( x \in M \) and its Lie algebra \( \mathfrak{h} \) are compactly imbedded.

To be more self contained, we propose a direct proof here.

**Proof.** Let \( \overline{G} \) be the closure of \( G \) in \( I(M) \) and \( \overline{\mathfrak{g}} \) be the isotropy subgroup at \( x \in M \) for the \( \overline{G} \)-action on \( M \). Then \( \overline{\mathfrak{g}} \) is compact. On the other hand, the property that \( \text{Ad}(G) \)-actions preserve \( \mathfrak{g} \) can be passed by continuity to \( \overline{G} \), i.e. \( \mathfrak{g} \) is an ideal of \( \overline{\mathfrak{g}} = \text{Lie}(\overline{G}) \). Denote \( \text{Ad}_g \) the restriction of \( \text{Ad}(\overline{G}) \)-actions from \( \mathfrak{g} \) to \( \mathfrak{g} \), then the subgroup \( \text{Ad}_g(H) \) of \( \text{Aut}(\mathfrak{g}) \) (which is contained in \( \text{Inn}(\mathfrak{g}) \) because of the connectedness of \( G \)) is contained in the compact subgroup \( \text{Ad}_g(\overline{\mathfrak{g}}) \). From this argument, we also see that both \( H \) and \( \mathfrak{h} \) are compactly imbedded. \( \blacksquare \)

Now we further assume \( M = G/H \) is a Riemannian homogeneous space.
The $H$-action on $T_o(G/H)$ at $o = eH$ is called the *isotropy action*. A linear direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\text{Lie}(H) = \mathfrak{h}$, is called a *reductive decomposition* for $G/H$, if it is $\text{Ad}(H)$-invariant. We can identify $T_o(G/H)$ with $\mathfrak{m}$ such that the isotropy action coincides with the $\text{Ad}(H)$-action on $\mathfrak{m}$.

Generally speaking, there exist many different reductive decompositions for a Riemannian homogeneous space $G/H$. A canonical one can be constructed by the following lemma, which summarizes Lemma 2 and Remark 1 in [15].

**Lemma 2.2.** Let $G/H$ be a Riemannian homogeneous space on which $G$ acts effectively. Then we have the following:

1. The restriction of the Killing form $B_\mathfrak{g}$ of $\mathfrak{g}$ to $\mathfrak{h}$ is negative definite;
2. The $B_\mathfrak{g}$-orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is reductive and $\mathfrak{n}(\mathfrak{g}) \subset \mathfrak{m}$.

### 2.3. Bounded vector for a coset space.

For any smooth coset space $G/H$, where $H$ is a closed subgroup of $G$, $\text{Lie}(G) = \mathfrak{g}$ and $\text{Lie}(H) = \mathfrak{h}$, we denote $\text{pr}_{\mathfrak{g}/\mathfrak{h}}$ the natural linear projection from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{h}$. We call any vector $X \in \mathfrak{g}$ a *bounded vector* for $G/H$, if

$$f(g) = \|\text{pr}_{\mathfrak{g}/\mathfrak{h}}(\text{Ad}(g)X)\|, \quad \forall g \in G,$$

is a bounded function, where $\| \cdot \|$ is any norm on $\mathfrak{g}/\mathfrak{h}$.

Since $\mathfrak{g}/\mathfrak{h}$ has a finite dimension, any two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on it are equivalent in the sense that

$$c_1\|u\|_1 \leq \|u\|_2 \leq c_2\|u\|_1, \quad \forall u \in \mathfrak{g}/\mathfrak{h},$$

where $c_1$ and $c_2$ are some positive constants. So the boundedness of $X \in \mathfrak{g}$ for $G/H$ is not relevant to the choice of the norm.

When $\| \cdot \|$ is an $\text{Ad}(H)$-invariant quadratic norm, which defines a $G$-invariant Riemannian metric on $G/H$, the function $f(\cdot)$ on $G$ defined in (2.2) is right $H$-invariant, so it can be descended to $G/H$, and coincides with the length function of the Killing vector field induced by $X$. Summarizing this observation, we have the following lemma.

**Lemma 2.3.** If $X \in \mathfrak{g}$ is a bounded vector for $G/H$, then it defines a bounded Killing vector field for any $G$-invariant Riemannian metric on $G/H$. Conversely, if $G/H$ is endowed with a $G$-invariant Riemannian metric and $X \in \mathfrak{g}$ induces a bounded Killing vector field, then $X$ is a bounded vector for $G/H$.

The boundedness condition may be kept when we change the coset space. By definition, it is obvious to see

**Lemma 2.4.** A vector $X \in \mathfrak{g}$ is bounded for the smooth coset space $G/H$ iff it is bounded for the universal covering $\tilde{G}/\tilde{H}$ of $G/H$, where $\tilde{G}$ is the universal covering group of $G$, and $\tilde{H}$ is closed connected subgroup which $\text{Lie}(H) = \mathfrak{h}$ generates in $\tilde{G}$.

For any chain of subalgebras $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$, the natural linear projection $\text{pr} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{k}$ is continuous with respect to standard topologies. So it maps bounded sets to bounded sets, with respect to any norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{k}$ respectively. Obviously $\text{pr} \circ \text{pr}_{\mathfrak{g}/\mathfrak{h}} = \text{pr}_{\mathfrak{g}/\mathfrak{k}}$, so

$$\text{pr}(\text{pr}_{\mathfrak{g}/\mathfrak{h}}(\text{Ad}(G)X)) = \text{pr}_{\mathfrak{g}/\mathfrak{k}}(\text{Ad}(G)X).$$

By these observations, it is easy to prove the following lemma.

**Lemma 2.5.** Assume $K$ is a closed subgroup of $G$ which $\text{Lie}$ algebra $\mathfrak{k}$ satisfies $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$. If $X \in \mathfrak{g}$ is bounded for $G/H$, then it is bounded for $G/K$ as well.
To summarize, the boundedness of Lie algebra vectors for a coset space is originated and intrinsically related to the boundedness of Killing vector fields for a homogeneous metric. However it is an algebraic condition, which can be discussed more generally and is not relevant to the choice or existence of homogeneous metrics.

3. Proof of Theorem 1.2 and Theorem 1.3

3.1. A key lemma for proving Theorem 1.3.

**Lemma 3.1.** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}$ be a Levi decomposition. Then we have the following Lie algebra direct sum for the centralizer $c_{\mathfrak{g}}(\mathfrak{n}(\mathfrak{g}))$ of the nilradical $\mathfrak{n}(\mathfrak{g})$ in $\mathfrak{g}$:

$$c_{\mathfrak{g}}(\mathfrak{n}(\mathfrak{g})) = (c_{\mathfrak{g}}(\mathfrak{n}(\mathfrak{g})) \cap \mathfrak{r}(\mathfrak{g})) \oplus (c_{\mathfrak{g}}(\mathfrak{n}(\mathfrak{g})) \cap \mathfrak{s}) = c_{\mathfrak{r}(\mathfrak{g})}(\mathfrak{n}(\mathfrak{g})) \oplus c_{\mathfrak{s}}(\mathfrak{n}(\mathfrak{g})). \quad (3.3)$$

Moreover we have the following:

1. The two summands $c_{\mathfrak{r}(\mathfrak{g})}(\mathfrak{n}(\mathfrak{g})) = c(\mathfrak{n}(\mathfrak{g}))$ and $c_{\mathfrak{s}}(\mathfrak{n}(\mathfrak{g})) = c_{\mathfrak{s}}(\mathfrak{r}(\mathfrak{g}))$ are Abelian and semi-simple ideals of $\mathfrak{g}$ respectively.
2. The summand $c_{\mathfrak{s}}(\mathfrak{r}(\mathfrak{g}))$ is contained in the intersection of all Levi subalgebras, so it does not depend on the choice of the Levi subalgebra $\mathfrak{s}$.

**Proof.** Firstly, we prove (3.3) as a linear decomposition.

Assume conversely that this is not true, then we can find a vector $X \in \mathfrak{g}$ such that $[X, \mathfrak{n}(\mathfrak{g})] = 0$ and $[X, \mathfrak{s}(\mathfrak{g})] \neq 0$. Denote $\text{ad}_{\mathfrak{n}(\mathfrak{g})}$ the restriction of the ad-action from $\mathfrak{n}(\mathfrak{g})$ to $\mathfrak{n}(\mathfrak{g})$. Then $\text{ad}_{\mathfrak{n}(\mathfrak{g})}(X) = -\text{ad}_{\mathfrak{n}(\mathfrak{g})}(X)$ is a nonzero linear endomorphism in the general linear Lie algebra $\mathfrak{gl}(\mathfrak{n}(\mathfrak{g})) = \text{Lie}(\text{GL}(\mathfrak{n}(\mathfrak{g})))$ where $\mathfrak{n}(\mathfrak{g})$ as well as its subspaces are viewed as real vector spaces.

The map $\text{ad}_{\mathfrak{n}(\mathfrak{g})}$ is a Lie algebra endomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(\mathfrak{n}(\mathfrak{g}))$. The vector $X_s$ generates a semi-simple ideal $\mathfrak{s}_1$ of $\mathfrak{s}$, which can be presented as

$$\mathfrak{s}_1 = \mathbb{R}X_s + [\mathfrak{s}, X_s] + [\mathfrak{s}, [\mathfrak{s}, X_s]] + [\mathfrak{s}, [\mathfrak{s}, [\mathfrak{s}, X_s]]] + \cdots. \quad (3.4)$$

Meanwhile, $X_r$ generates a sub-representation space $\mathfrak{v}_1$ in $\mathfrak{r}(\mathfrak{g})$ for the ad($\mathfrak{s}$)-actions, i.e.

$$\mathfrak{v}_1 = \mathbb{R}X_r + [\mathfrak{s}, X_r] + [\mathfrak{s}, [\mathfrak{s}, X_r]] + [\mathfrak{s}, [\mathfrak{s}, [\mathfrak{s}, X_r]]] + \cdots. \quad (3.5)$$

Compare (3.4) and (3.5), we can see that $\text{ad}_{\mathfrak{n}(\mathfrak{g})}(\mathfrak{s}_1)$ and $\text{ad}_{\mathfrak{n}(\mathfrak{g})}(\mathfrak{v}_1)$ have the same image in $\mathfrak{gl}(\mathfrak{n}(\mathfrak{g}))$, i.e.

$$\text{ad}_{\mathfrak{n}(\mathfrak{g})}(\mathfrak{s}_1) = \text{ad}_{\mathfrak{n}(\mathfrak{g})}(\mathfrak{v}_1) = \mathbb{R}A + [\text{ad}_{\mathfrak{n}(\mathfrak{g})}(\mathfrak{s}), A] + [\text{ad}_{\mathfrak{n}(\mathfrak{g})}(\mathfrak{s}), [\text{ad}_{\mathfrak{n}(\mathfrak{g})}(\mathfrak{s}), A]] + \cdots. \quad (3.6)$$

Denote $\mathfrak{u}_1 = \text{ad}_{\mathfrak{n}(\mathfrak{g})}(\mathfrak{s}_1)$ and $\mathfrak{u}_2 = \text{ad}_{\mathfrak{n}(\mathfrak{g})}(\mathfrak{r}(\mathfrak{g}))$. We have just showed $0 \neq \mathfrak{u}_1 \subset \mathfrak{u}_2$. Since $\text{ad}_{\mathfrak{n}(\mathfrak{g})} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{n}(\mathfrak{g}))$ is a Lie algebra endomorphism, $\mathfrak{u}_1$ is semi-simple and $\mathfrak{u}_2$ is solvable. But this is impossible, so (3.3) is a linear direct sum decomposition.

Further, we prove that $c_{\mathfrak{r}(\mathfrak{g})}(\mathfrak{n}(\mathfrak{g})) = c(\mathfrak{n}(\mathfrak{g}))$ is an Abelian ideal of $\mathfrak{g}$.

The summand $c_{\mathfrak{r}(\mathfrak{g})}(\mathfrak{n}(\mathfrak{g}))$ in (3.3) is an ideal of $\mathfrak{g}$ contained in the radical $\mathfrak{r}(\mathfrak{g})$. It is not hard to check that $c_{\mathfrak{r}(\mathfrak{g})}(\mathfrak{n}(\mathfrak{g})) + \mathfrak{n}(\mathfrak{g})$ is a nilpotent ideal of $\mathfrak{g}$. By the definition of the nilradical, we must have $c_{\mathfrak{r}(\mathfrak{g})}(\mathfrak{n}(\mathfrak{g})) \subset \mathfrak{n}(\mathfrak{g})$, i.e. $c_{\mathfrak{r}(\mathfrak{g})}(\mathfrak{n}(\mathfrak{g})) = c(\mathfrak{n}(\mathfrak{g}))$. So it is an Abelian ideal.

Finally, we prove that $c_{\mathfrak{s}}(\mathfrak{n}(\mathfrak{g})) = c_{\mathfrak{s}}(\mathfrak{r}(\mathfrak{g}))$ is a semi-simple ideal of $\mathfrak{g}$ contained in the intersection of all Levi subalgebras.
Obviously $c_s(n(g))$ is an ideal of $\mathfrak{s}$. It is a semi-simple Lie algebra itself, so we have $[c_s(n(g)), c_s(n(g))] = c_s(n(g))$. It commutes with $\mathfrak{r}(g)$ because
\[
[r(g), c_s(n(g))] \subset [r(g), [c_s(n(g)), c_s(n(g))]] = [[r(g), c_s(n(g))], c_s(n(g))]
\subset [n(g), c_s(n(g))] = 0.
\]
So we get $c_s(n(g)) = c_s(r(g))$.

It is an ideal of $\mathfrak{g}$ because $[\mathfrak{g}, c_s(r(g))] = [\mathfrak{s}, c_s(r(g))] \subset c_s(r(g))$. Therefore, we have $c_s(r(g)) = \text{Ad}(g)c_s(r(g)) \subset \text{Ad}(g)s$ for all $g \in G$. So by Malcev’s Theorem, $c_s(r(g))$ is contained in all Levi subalgebras, and thus independent of the choice of the Levi subalgebra.

We have proved all statements and finished the proof of the lemma. ■

By similar arguments as above, we can also establish the Lie algebra direct sum $c_s(r(g)) = c_s(r(g)) \oplus c_{s_{nc}}(r(g))$ in which each summand is a semi-simple ideal of $\mathfrak{g}$. So we get the following corollary.

**Corollary 3.2.** Keep all relevant notations and assumptions, then we have the following Lie algebra direct sum decomposition,
\[ c_s(n(g)) = c_s(r(g)) \oplus c_{s_{nc}}(r(g)) \oplus c(n(g)), \]
in which each summand is an ideal of $\mathfrak{g}$.

### 3.2. Proof of Theorem 1.2
Fix any $x \in M$, and denote $H$ the isotropy subgroup of $G$ at $x$. The smooth coset space $G/H$ can be identified with an immersed submanifold in $M$. The submanifold metric on $G/H$ is $G$-invariant. The restriction of the bounded Killing vector field induced by $X \in \mathfrak{g}$ to $G \cdot x = G/H$ is still a bounded Killing vector field, induced by the same $X$. By Lemma 2.3, $X \in \mathfrak{g}$ is a bounded vector for $G/H$.

By Lemma 2.1, $\mathfrak{h}$ is compactly imbedded. We can find a maximal compactly imbedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$. Denote $\tilde{G}$ the universal cover of $G$, $\tilde{H}$, and $\tilde{K}$ the connected subgroup of $\tilde{G}$ generated by $\mathfrak{h}$ and $\mathfrak{k}$ respectively. The subgroup $\tilde{K}$ is closed because it is the identity component of the pre-image in $\tilde{G}$ for a maximal compact subgroup of $G/Z(G)$. By Lemma 2.4 and Lemma 2.5, $X \in \mathfrak{g}$ is also bounded for $G/\tilde{H}$ and $\tilde{G}/\tilde{K}$.

Since $G$ is semi-simple, we have $\tilde{G} = \tilde{G}_c \times \tilde{G}_{nc}$, where $\tilde{G}_c$ and $\tilde{G}_{nc}$ are the compact and non-compact parts of $\tilde{G}$ respectively, $\tilde{K} = \tilde{G}_c \times \tilde{K}_{nc}$, and $X = X_c + X_{nc}$ with $X_c \in \mathfrak{g}_c = \text{Lie}(\tilde{G}_c)$ and $X_{nc} \in \mathfrak{g}_{nc} = \text{Lie}(\tilde{G}_{nc})$ accordingly. The coset space $\tilde{G}/\tilde{K} = \tilde{G}_{nc}/\tilde{K}_{nc}$ is a symmetric space of non-compact type. The vector $X_{nc} \in \mathfrak{g}_{nc}$ defines the same Killing vector field as $X$ on $\tilde{G}/\tilde{K}$, so it is bounded as well. Since the Riemannian symmetric metric on $\tilde{G}/\tilde{K}$ has negative Ricci curvature and non-positive sectional curvature, the bounded vector $X_{nc}$ must vanish [18]. So $X = X_c$ is contained in the compact ideal $\mathfrak{g}_c$ in $\mathfrak{g}$.

This completes the proof of Theorem 1.2.

### 3.3. Proof of Theorem 1.3
The key steps are summarized as the following two claims.

**Claim 1:** $X_s$ is contained in a compact ideal of $\mathfrak{s}$.

The proof of Claim 1 applies a similar method as for Theorem 1.2.

By similar argument as in the proof of Theorem 1.2, we can restrict our discussion to any orbit $G \cdot x = G/H$ in $M$, where the isotropy subgroup $H$ has a compactly imbedded Lie algebra. The vector $X$ indicated in Theorem 1.3 is bounded for $G/H$. 
The radical \( \mathfrak{r}(\mathfrak{g}) \) generates a closed normal subgroup \( R \) of \( G \) and its product \( RH \) with the compact subgroup \( H \) is also a closed subgroup. By Lemma 2.5, \( X \in \mathfrak{g} \) is bounded for \( G/HR \). We can identify \( G/HR \) as the orbit space for the left \( R \)-actions on \( G/H \). So \( G/HR \) admits a \( G \)-invariant metric induced by submersion. On the other hand, the coset space \( G/HR \) can be identified as \( S/H_S = (G/R)/(HR/R) \), where the Lie algebra of \( S = G/R \) can be identified with \( \mathfrak{s} \) by Levi’s Theorem, and \( \text{Lie}(H_S) \) is a compactly imbedded subgroup because it is the image of the compactly imbedded \( \mathfrak{h} \) in \( \mathfrak{g}/\mathfrak{r}(\mathfrak{g}) \). With this identification, \( X \) defines the same Killing vector field as \( X_s \) on \( S/H_S \). By Lemma 2.3, \( X_s \) is bounded for \( S/H_S \). Now we have the semi-simplicity for \( S \) and the compactly imbedded property for \( \text{Lie}(H_S) \), so we can apply a similar argument as for Theorem 1.2 to prove \( X_s \) is contained in a compact ideal of \( \mathfrak{s} \).

This completes the proof of Claim 1.

**Claim 2:** \( X \) commutes with the nilradical \( \mathfrak{n} \).

To prove this claim, we still restrict our discussion to a single \( G \)-orbit. But we need to be careful because the effectiveness is required in later discussion. The following lemma guarantees that suitable \( G \)-orbits with effective \( G \)-actions can be found.

**Lemma 3.3.** Let \( M \) be a connected Riemannian homogeneous space on which a connected Lie group \( G \) acts effectively. Then there exists \( x \in M \), such that \( G \) acts effectively on \( G \cdot x \).

**Proof.** Denote \( \overline{G} \) the closure of \( G \) in \( I(M) \). Then the \( \overline{G} \)-action on \( M \) is proper (see Proposition 3.62 in [1]). By the Principal Orbit Theorem (see Theorem 3.82 in [1]), the principal orbit type for the \( \overline{G} \)-action is unique up to conjugations, and the union \( U \) of all principal orbits is open dense in \( M \).

Let \( G \cdot x \) be any \( G \)-orbit in \( U \), and assume \( g \in G \) acts trivially on \( G \cdot x \). Because \( G \cdot x \) is dense in \( \overline{G} \cdot x \), \( g \) acts trivially on \( \overline{G} \cdot x \) as well. Now we consider any other orbit \( \overline{G} \cdot y \) in \( U \). The point \( y \) can be suitably chosen such that \( x \) and \( y \) have the same isotropy subgroups \( \overline{G}_x = \overline{G}_y \) in \( \overline{G} \). Then their isotropy subgroups in \( G \) are the same because \( G_x = \overline{G}_x \cap G = \overline{G}_y \cap G = G_y \). The \( g \)-action on \( G \cdot y \) is trivial, and by continuity, that on \( \overline{G} \cdot y \) is trivial as well. This argument proves that \( g \) acts trivially on the dense open subset \( U \) in \( M \), so it acts trivially on \( M \). Due to the effectiveness of the \( G \)-action, we must have \( g = e \in G \).

To summarize, the \( G \)-action on \( G \cdot x \subseteq U \) is effective, which completes the proof of this lemma. \( \blacksquare \)

Take the orbit \( G \cdot x = G/H \) indicated in Lemma 3.3, endowed with the invariant submanifold metric. Since \( X \in \mathfrak{g} \) defines a bounded Killing vector field on the whole manifold, it also defines a bounded Killing vector field when restricted to \( G \cdot x \). So by Lemma 2.3, \( X \) is a bounded vector for \( G/H \).

By Lemma 2.2, we have \( B_{\mathfrak{g}} \)-orthogonal reductive decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \) with \( \mathfrak{n}(\mathfrak{g}) \subseteq \mathfrak{m} \). Denote \( X_m \) the \( \mathfrak{m} \)-component of \( X \). For any \( Y \in \mathfrak{n}(\mathfrak{g}) \),

\[
\text{pr}_m(\text{Ad}(\exp(tY))X) = X_m + t[Y,X] + \frac{t^2}{2!}[Y,[Y,X]] + \cdots, \tag{3.6}
\]

in which all terms except the first one in the right side are contained in \( \mathfrak{n}(\mathfrak{g}) \). Since \( \mathfrak{n}(\mathfrak{g}) \) is nilpotent, the right side of (3.6) is in fact a vector-valued polynomial with respect to \( t \). If it has a positive degree, we can get

\[
\lim_{t \to \infty} \| \text{pr}_m(\text{Ad}(\exp(tY))X) \| = +\infty,
\]
for any norm \( \| \cdot \| \) on \( \mathfrak{m} \). This is a contradiction to the boundedness of \( X \) for \( G/H \). So we get \( [X,Y]=0 \) for any \( Y \in \mathfrak{n}(\mathfrak{g}) \) which proves Claim 2.

Finally, we finish the proof of Theorem 1.3. Claim 2 indicates that \( X \in \mathfrak{c}_g(\mathfrak{n}(\mathfrak{g})) \). By Lemma 3.1 or Corollary 3.2, we have \( X_r \in \mathfrak{c}_{\tau(\mathfrak{g})} = \mathfrak{c}(\mathfrak{n}(\mathfrak{g})) \) and \( X_s \in \mathfrak{c}_s(\mathfrak{n}(\mathfrak{g})) = \mathfrak{c}_{s_\tau}(\tau(\mathfrak{g})) \oplus \mathfrak{c}_{s_{\tau c}}(\tau(\mathfrak{g})) \). Claim 1 indicates \( X_s \) is contained in the compact semi-simple ideal \( \mathfrak{c}_{s_\tau}(\tau(\mathfrak{g})) \) of \( \mathfrak{g} \).

This finishes the proof of Theorem 1.3.

4. Applications of Theorem 1.3

4.1. Jordan decomposition and spectral property for bounded Killing vector fields. Theorem 1.3 and Lemma 3.1 provide the following obvious observations for \( X = X_r + X_s \in \mathfrak{g} \) which defines a bounded Killing vector field:

1. The linear endomorphism \( \text{ad}(X_s) \in \mathfrak{gl}(\mathfrak{g}) \) is semi-simple with only imaginary eigenvalues;
2. The linear endomorphism \( \text{ad}(X_r) \in \mathfrak{gl}(\mathfrak{g}) \) is nilpotent, i.e. it has only zero eigenvalues;
3. These two endomorphisms commute because \([X_r,X_s]=0\).
4. By a suitable conjugation, we can present \( \text{ad}(X) \), \( \text{ad}(X_r) \) and \( \text{ad}(X_s) \) as upper triangular, strict upper triangular and diagonal matrices respectively. So \( \text{ad}(X) \in \mathfrak{gl}(\mathfrak{g}) \) has the same eigenvalues (counting multiples) as \( \text{ad}(X_s) \).
5. The centralizer \( \mathfrak{c}_{s_\tau}(\tau(\mathfrak{g})) \) containing \( X_s \) is an compact semi-simple ideal of \( \mathfrak{g} \) contained in the intersection of all Levi subalgebras.

The observations (1)–(3) imply \( \text{ad}(X) = \text{ad}(X_s) + \text{ad}(X_r) \) is a Jordan–Chevalley decomposition, and hence \( X = X_s + X_r \) is an abstract Jordan decomposition. See 4.2 and 5.4 in [9] for a comprehensive discussion of these notions.

The observation (4) explains why \( \text{ad}(X) \) has only imaginary eigenvalues, which solves our spectral problem for bounded Killing vector fields.

Notice that the decomposition \( \text{ad}(X) = \text{ad}(X_r) + \text{ad}(X_s) \) is unique by the uniqueness of Jordan–Chevalley decomposition, while the abstract Jordan decomposition may not be because of the center \( \mathfrak{c}(\mathfrak{g}) \). However, by the observation (5), the decomposition \( X = X_r + X_s \) is unique in the sense that it does not depends on the choice of the Levi subalgebra.

Above observations and discussions can be summarized to the following theorem.

**Theorem 4.1.** Let \( M \) be a connected Riemannian manifold on which the connected Lie group \( G \) acts effectively and isometrically. Assume that \( X \in \mathfrak{g} \) defines a bounded Killing vector field. Let \( X \) be decomposed as \( X = X_r + X_s \) according to any Levi decomposition \( \mathfrak{g} = \tau(\mathfrak{g}) + \mathfrak{s} \), then we have the following:

1. The decomposition \( \text{ad}(X) = \text{ad}(X_r) + \text{ad}(X_s) \) is the unique Jordan–Chevalley decomposition for \( \text{ad}(X) \) in \( \mathfrak{gl}(\mathfrak{g}) \);
2. The decomposition \( X = X_r + X_s \) is the abstract Jordan decomposition which is unique in the sense that \( X_s \) is contained in all Levi subalgebras, i.e. this decomposition is irrelevant to the choice of the Levi subalgebra;
3. The eigenvalues of \( \text{ad}(X) \) coincide with those of those of \( \text{ad}(X_s) \), counting multiples.
4.2. Bounded Killing vectors on a connected Riemannian homogeneous space.

In this section, we will always assume that $M = G/H$ is a Riemannian homogeneous space on which the connected Lie group $G$ acts effectively. Applying Theorem 1.3 and some argument in its proof, we can completely determine all the bounded vectors for $G/H$ as following.

Let $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}$ be a Levi decomposition, and $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$ be a Lie algebra direct sum decomposition. By Lemma 2.2, we have a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that the nilradical $\mathfrak{n}(\mathfrak{g})$ is contained in $\mathfrak{m}$.

We have mentioned that $\mathfrak{c}_s(\mathfrak{r}(\mathfrak{g}))$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{s}_c$. Denote $\mathfrak{s}_c'$ the ideal of $\mathfrak{s}_c$ such that $\mathfrak{s}_c = \mathfrak{c}_s(\mathfrak{r}(\mathfrak{g})) \oplus \mathfrak{s}_c'$. Then we have a Lie algebra direct sum decomposition

$$\mathfrak{g} = \mathfrak{c}_s(\mathfrak{r}(\mathfrak{g})) \oplus (\mathfrak{s}_c' + \mathfrak{s}_{nc} + \mathfrak{r}(\mathfrak{g})).$$

(4.7)

By this observation, we find the following lemma.

Lemma 4.2. Keep all assumptions and notations of this section, then any vector in $\mathfrak{c}_s(\mathfrak{r}(\mathfrak{g}))$ is bounded for $G/H$.

Proof. The ideal $\mathfrak{c}_s(\mathfrak{r}(\mathfrak{g}))$ generates a compact semi-simple subgroup in $G$. So for any $X \in \mathfrak{c}_s(\mathfrak{r}(\mathfrak{g}))$, the orbit

$$\text{Ad}(G)X = \text{Ad}(\exp \mathfrak{c}_s(\mathfrak{r}(\mathfrak{g})))$$

is a compact set, which projection in $\mathfrak{g}/\mathfrak{h}$ is obviously bounded with respect to any norm. So any vector $X \in \mathfrak{c}_s(\mathfrak{r}(\mathfrak{g}))$ is bounded for $G/H$, which proves this lemma. ■

Obviously linear combinations of bounded vectors for $G/H$ are still bounded vectors for $G/H$, i.e. the set of all bounded vectors for $G/H$ is a real linear subspace of $\mathfrak{g}$. It is preserved by all $\text{Ad}(G)$-actions. So it is an ideal of $\mathfrak{g}$. Applying Theorem 1.3 and Lemma 4.2, we get the following immediate consequence.

Theorem 4.3. Assume $G/H$ is a Riemannian homogeneous space on which the connected Lie group $G$ acts effectively. Then the space of all bounded vectors for $G/H$ is a compact ideal of $\mathfrak{g}$. Its semi-simple part is coincides with $\mathfrak{c}_s(\mathfrak{r}(\mathfrak{g}))$. Its Abelian part $\mathfrak{v}$ is contained in $\mathfrak{c}(\mathfrak{n}(\mathfrak{g}))$.

Before we continue to determine all the bounded vectors, there are several remarks.

For some Riemannian homogeneous spaces, bounded Killing vector fields can only be found from $\mathfrak{c}(\mathfrak{n}(\mathfrak{g}))$. For example,

Corollary 4.4. Let $G/H$ be a Riemannian homogeneous space which is diffeomorphic to an Euclidean space on which the connected Lie group $G$ acts effectively. Assume that $X \in \mathfrak{g}$ defines a bounded Killing vector field, then $X \in \mathfrak{c}(\mathfrak{n}(\mathfrak{g}))$.

Proof. Since $G/H$ is diffeomorphic to an Euclidean space, the subgroup $H$ is a maximal compact subgroup of $G$. Assume conversely that $X$ is not contained in $\mathfrak{c}(\mathfrak{n}(\mathfrak{g}))$, then by Theorem 1.3 or Theorem 4.3, there exists a non-trivial compact semi-simple normal subgroup $H'$ of $G$. We can get $H' \subset H$ by the conjugation theorem for maximal compact subgroups (i.e. Theorem 14.1.3 in [10]). This is a contradiction to the effectiveness of the $G$-action. ■

When $G/H$ is a geodesic orbit space (that means that every geodesic is an orbit of some one-parameter isometry group from $G$), the second author have proved that any vector in $\mathfrak{c}(\mathfrak{n}(\mathfrak{g}))$ defines a Killing vector field of constant length (see Theorem 1 in [13] or Theorem 5 in [16]). By Theorem 4.3, it implies an equivalence between the boundedness and the constant length condition for Killing vector fields $X \in \mathfrak{n}(\mathfrak{g})$ for
a geodesic orbit space. Similar phenomenon can also be seen from Corollary 3.4 in [19], for exponential solvable Lie groups endowed with left invariant metrics.

Applying a similar style for defining the restrictive Clifford–Wolf homogeneity [5] and the δ-homogeneity (which is equivalent to the notion of the generalized normal homogeneity) [4, 6], we can use bounded Killing vector fields in order to define the following condition for Riemannian homogeneous spaces.

**Definition 4.5.** Let $G/H$ be a Riemannian homogeneous space on which the connected Lie group $G$ acts effectively. Then it satisfies Condition (BH) if for any $x \in G/H$ and any $v \in T_x(G/H)$, there exists a bounded vector $X \in \mathfrak{g}$ such that $X(x) = v$.

Then Theorem 4.3 provides the following criterion for Condition (BH).

**Corollary 4.6.** Let $G/H$ be a Riemannian homogeneous space on which the connected Lie group $G$ acts effectively. Then it satisfies Condition (BH) iff there exists a connected subgroup $K$ of $G$ such that its Lie algebra is compact and the $K$-action on $G/H$ is transitive.

**Proof.** If $G/H$ satisfies Condition (BH), then the space of all bounded vectors for $G/H$ generated a connected quasi-compact subgroup $K$ of $G$ which acts transitively on $G/H$.

Conversely, if such a quasi-compact subgroup exists, all vectors in it are bounded for $G/H$. The Condition (BH) is satisfied because the exponential map from Lie($K$) to $K$ is surjective.

This completes the proof of the corollary. ■

To completely determine all bounded vectors for $G/H$, we just need to determine the subspace $\mathfrak{v}$ of all bounded vectors $X \in \mathfrak{c}(n)$ for $G/H$. Obviously the Ad($G$)-actions preserve $\mathfrak{v}$, which is contained in the summand $\mathfrak{m}$ in the reductive decomposition. The condition that $X \in \mathfrak{v}$, i.e. $X \in \mathfrak{c}(n)$ is bounded for $G/H$, is equivalent to that Ad($G$)$X$ is a bounded set in $\mathfrak{c}(n)$ with respect to any norm.

The restriction of the Ad($G$)-actions defines a Lie group endomorphism Ad$_v$ from $G/N$ to the general linear group GL($\mathfrak{v}$), where $N$ is the closed connected normal subgroup generated by $n(g)$. The tangent map at $e$ for Ad$_v$ induces the Lie algebra endomorphism which coincides with ad$_v$ defined by restricting the ad-action from $\mathfrak{v}$ to $\mathfrak{v}$.

The following key lemma helps us determine the subspace $\mathfrak{v}$.

**Lemma 4.7.** Let $G/H$ be a Riemannian homogeneous space on which the connected Lie group $G$ acts effectively. Keep all relevant assumptions and notations. Then the image Ad$_v(G)$ has a compact closure in GL($\mathfrak{v}$).

**Proof.** Fix a quadratic norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ on $\mathfrak{v}$ and an orthonormal basis $\{v_1, \ldots, v_k\}$ for $\mathfrak{v}$. By the boundedness of each $v_i$ for $G/H$, and the speciality of the reductive decomposition, we can find a positive $c_i > 0$, such that

$$\|\text{Ad}(g)v_i\| < c_i, \quad \forall g \in G, \quad \forall i = 1, \ldots, k.$$  

For any $v \in \mathfrak{v}$ with $\|v\| = 1$, we can present it as $v = \sum_{i=1}^k a_i v_i$ with $\sum_{i=1}^k a_i^2 = 1$, then for any $g \in G$ we have

$$\|\text{Ad}(g)v\| \leq \sum_{i=1}^k |a_i| \cdot \|\text{Ad}(g)v_i\| \leq C = c_1 + \cdots + c_k.$$
So we get
\[ C^{-1} \|v\| \leq \|\Ad(g)v\| \leq C \|v\|, \quad \forall g \in G, \quad \forall v \in \mathfrak{v}. \quad (4.8) \]

For any sequence \(\Ad_{g_i}(g_i)\) with \(g_i \in G\), we can find a subsequence \(\Ad_{g_i}(g'_i)\) such that \(\lim_{i \to \infty} \Ad_{g_i}(g'_i)v_j\) exists for each \(j\), so \(\Ad_{g_i}(g'_i)\) converges to a \(\mathbb{R}\)-linear endomorphism \(A\). By continuity, the estimates (4.8) for each \(\Ad_{g_i}(g'_i)\) can be inherited by \(A\), from which we see that \(A \in \text{GL}(\mathfrak{v})\). So \(\Ad_{g_i}(G)\) has a compact closure in \(\text{GL}(\mathfrak{v})\), which proves this lemma.

By Lemma 4.7, \(\text{ad}_\mathfrak{g}\) maps the reductive Lie algebra \(\text{Lie}(G/N) = \mathfrak{s}_c \oplus \mathfrak{s}_{nc} \oplus \mathfrak{a}\), where \(\mathfrak{a} = \mathfrak{r}/\mathfrak{n}\), to a compact subalgebra. The summand \(\mathfrak{s}_{nc}\) must be mapped to 0, from which we get \([\mathfrak{s}_{nc}, \mathfrak{v}] = 0\). Then it is easy to see that
\[ \mathfrak{v} \subset \mathfrak{c}_{(n)}(\mathfrak{s}_{nc}) \quad \text{and} \quad [\tau(\mathfrak{g}), \mathfrak{c}_{(n)}(\mathfrak{s}_{nc})] \subset \mathfrak{c}_{(n)}(\mathfrak{s}_{nc}). \]

Moreover, the Abelian summand \(\mathfrak{a}\) in \(\text{Lie}(G/N)\) is mapped to a space of semi-simple matrices with imaginary eigenvalues, so \(\mathfrak{v}\) can be decomposed as a sum
\[ \mathfrak{v} = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_k \]
of irreducible representations of \(\mathfrak{a}\), each of which is either one-dimensional or two-dimensional. Any one-dimensional \(\mathfrak{v}_i\) must be a trivial representation of \(\mathfrak{a}\), and hence \(\mathfrak{v}_1 \subset \mathfrak{c}_{(n)}(\mathfrak{s}_{nc})\). Any two-dimensional \(\mathfrak{v}_i\) corresponds to a pair of imaginary weights in \(\mathfrak{a}^* \otimes \mathbb{C}\), i.e. \(\mathbb{R}\)-linear functionals \(\pm \lambda : \mathfrak{a} \to \mathbb{R}\sqrt{-1}\), such that the eigenvalues of \(\text{ad}_\mathfrak{g}(u) : \mathfrak{v}_i \to \mathfrak{v}_i\) are \(\pm \lambda(u)\).

Conversely, we consider the sum \(\mathfrak{v}'\) of the centralizer \(\mathfrak{c}_{(n)}(\mathfrak{s}_{nc})\) and all two-dimensional irreducible \(\text{ad}(\mathfrak{r})\)-representations in \(\mathfrak{c}_{(n)}(\mathfrak{s}_{nc})\) corresponding to imaginary weights of \(\mathfrak{a} = \mathfrak{r}(\mathfrak{g})/\mathfrak{n}(\mathfrak{g})\). Then \(\mathfrak{v}'\) is \(\Ad(G)\)-invariant, and \(\mathfrak{v} \subset \mathfrak{v}'\). Denote \(\Ad_{\mathfrak{g}'}\) the restriction of the \(\Ad_{\mathfrak{g}}\)-action from \(\mathfrak{v}'\) to \(\mathfrak{v}'\). The subspace \(\mathfrak{v}'\) satisfies similar descriptions as given above for \(\mathfrak{v}\). The image group \(\Ad_{\mathfrak{g}'}(R/N)\) is contained in a torus which commutes with the image \(\Ad_{\mathfrak{g}'}(S_c) \subset \text{GL}(\mathfrak{v}')\) of the compact subgroup \(S_c = \exp \mathfrak{s}_c\), so
\[ \Ad_{\mathfrak{g}'}(G) = \Ad_{\mathfrak{g}'}(S_c) \cdot \Ad_{\mathfrak{g}'}(R/N) \subset \Ad_{\mathfrak{g}'}(S_c) \cdot \Ad_{\mathfrak{g}'}(R/N) \]
has a compact closure in \(\text{GL}(\mathfrak{v}')\). This implies all vectors \(X \in \mathfrak{v}'\) are bounded for \(G/H\), i.e. \(\mathfrak{v} = \mathfrak{v}'\).

Summarizing above argument, we get the following theorem which determines all bounded vectors \(X \in \mathfrak{c}(n(\mathfrak{g}))\) for a Riemannian homogeneous space \(G/H\).

**Theorem 4.8.** Let \(G/H\) be a Riemannian homogeneous space on which the connected Lie group \(G\) acts effectively. Keep all relevant assumptions and notations. Then the space \(\mathfrak{v}\) of all bounded vectors \(X \in \mathfrak{c}(n)\) for \(G/H\) is the sum of \(\mathfrak{c}_{(n)}(\mathfrak{s}_{nc})\) and all two-dimensional irreducible representations in \(\mathfrak{c}_{(n)}(\mathfrak{s}_{nc})\) for the \(\mathfrak{r}(\mathfrak{g})\)-actions, which corresponds to nonzero imaginary weights in \(\mathfrak{a}^* \otimes \mathbb{C}\), i.e. nonzero \(\mathbb{R}\)-linear functionals \(\lambda : \mathfrak{r}(\mathfrak{g}) \to \mathfrak{r}(\mathfrak{g})/\mathfrak{n}(\mathfrak{g}) \to \mathbb{R}\sqrt{-1}\).

A theoretical algorithm presenting all vectors in \(\mathfrak{v}\) can be given as follows. Let us consider the complex representation for \(\text{ad}(\mathfrak{r}(\mathfrak{g}))\)-actions on \(\mathfrak{c}_{(n)}(\mathfrak{s}_{nc}) \otimes \mathbb{C}\), then we can find distinct real weights \(\lambda_i \in \mathfrak{a}^* \subset \mathfrak{r}^*\), \(1 \leq i \leq n_1\), and non-real complex weights \(a_j \pm b_j \sqrt{-1} \in \mathfrak{a}^* \otimes \mathbb{C} \subset \mathfrak{r}^* \otimes \mathbb{C}\), with \(1 \leq i \leq n_2\) and \(b_j > 0\), such that we have the direct sum decomposition
\[ \mathfrak{c}_{(n)}(\mathfrak{s}_{nc}) \otimes \mathbb{C} = \bigoplus_{i=1}^{n_1} u_{\lambda_i}^C \oplus \bigoplus_{j=1}^{n_2} (u_{a_j + b_j \sqrt{-1}}^C + u_{a_j - b_j \sqrt{-1}}^C), \]
where the complex subspace $u^C_\alpha$ for any real weight $\alpha \in a^*$ or any complex weight $\alpha \in a^* \otimes \mathbb{C}$ is defined as

$$u^C_\alpha = \left\{ X \in c_{c(n)}(s_{nc}) \otimes \mathbb{C} \mid (\text{ad}(u) - \alpha(u)\text{Id})^k X = 0, \forall u \in r(g) \right\}, \text{ for some } k > 0.$$ 

We assume $a_j = 0 \in a^*$ iff $1 \leq j \leq m$, where $m$ can be zero. Denote $v^C_\alpha$ be the eigenvector subspace in $u^C_\alpha$ for the $\text{ad}(r(g))$-actions, i.e.

$$v^C_\alpha = \left\{ X \in u^C_\alpha \mid \text{ad}(u)X = \alpha(u)X, \forall u \in r(g) \right\}.$$ 

We take any basis $\{v_{j,1}, \ldots, v_{j,k_j}\}$ for $v^C_{b_j \sqrt{-1}}$, then the space $v$ of all bounded vectors in $c(n)$ for $G/H$ can be presented as

$$v = \left( v^C_0 \cap v \right) \oplus \bigoplus_{j=1}^m \left( v^C_{b_j \sqrt{-1}} + v^C_{-b_j \sqrt{-1}} \right) \cap v$$

$$= c_{c(r(g))}(s_{nc}) \oplus \text{span}^\mathbb{R} \{ v_{j,k} + \overline{v_{j,k}}, \sqrt{-1}(v_{j,k} - \overline{v_{j,k}}), \forall 1 \leq j \leq m, 1 \leq k \leq k_j \}.$$

We hope that all the above results will be useful in the study of related topics.

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