Abstract. The main purpose of our paper is a new approach to design of algorithms of Kaczmarz type in the framework of operators in Hilbert space. Our applications include a diverse list of optimization problems, new Karhunen-Loève transforms, and Principal Component Analysis (PCA) for digital images. A key feature of our algorithms is our use of recursive systems of projection operators. Specifically, we apply our recursive projection algorithms for new computations of PCA probabilities and of variance data. For this we also make use of specific reproducing kernel Hilbert spaces, factorization for kernels, and finite-dimensional approximations. Our projection algorithms are designed with view to maximum likelihood solutions, minimization of “cost” problems, identification of principal components, and data-dimension reduction.

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1. Introduction

While positive-definite (p.d.) kernels date back to the 1950ties, recently they have found striking and new applications in applied mathematics. Indeed, there are by now multiple and diverse applications, most of quite recent vintage. In outline, a p.d. kernel is a generalization of the notion of positive-definite function, or p.d. matrix. Initially, p.d. kernels were introduced with view to the use for the solution of integral operator equations arising in PDE theory (especially boundary value problems), potential theory, and in optimization problems. Since their inception, positive-definite kernels and their associated Hilbert spaces, reproducing kernel Hilbert spaces (RKHSs) have come up in entirely new areas of mathematics. A RKHS is a Hilbert space $\mathcal{H}$ of functions on a set, say $X$, with the property that for all $x \in X$ and all $F \in \mathcal{H}$,

$F(x) = \langle K(\cdot, x), F \rangle_{\mathcal{H}_K}$

holds for all $x \in X$ and all $F \in \mathcal{H}$. Eq (1.1) is referred to as the reproducing property.

A kernel $K: X \times X \rightarrow \mathbb{C}$ is called positive definite (p.d.) if for all $n \in \mathbb{N}$, $\forall \{\alpha_i\}_{i=1}^n \subseteq \mathbb{C}$, $\forall \{x_i\}_{i=1}^n \subseteq X$, we have

$\sum_i \sum_j \alpha_i \alpha_j K(x_i, x_j) \geq 0$. (1.2)

Given $K$ p.d., a realization of the RKHS $\mathcal{H}_K$ is to take the completion of the linear span of $\{K(\cdot, x)\}_{x \in X}$ with respect to the norm

$\left\| \sum_i \alpha_i K(\cdot, x_i) \right\|^2_{\mathcal{H}_K} := \sum_i \sum_j \alpha_i \alpha_j K(x_i, x_j)$. (1.3)

Organization. The paper is organized as follows: in sections 2 and 3, we present systems $(K, \mu)$ in duality, where $K$ is a fixed p.d. kernel, and choices of $\mu$ runs through an associated family of measures. This duality will play an important role, for example in a unified approach to (i) a class of optimization problems, and to (ii) realization of p.d. kernels as integral operators. In particular, we show how properties of $\mu$ reflect themselves in spectral theory for the corresponding integral operator. Section 4 offers a framework for sampling. The starting point is a p.d. kernel defined on $X \times X$ for a given set $X$. The questions we address are: What countably discrete subsets $V$ of $X$ provide sets of sample points for suitable classes of functions on $X$, and algorithms of Kaczmarz type, sect. 7. The answer to this, and related, questions depends on configuration of systems of finite subsets of $X$, addressed in section 5; with examples in sections 6 and 8. In section 9, we then present a new harmonic analysis of general Gaussian processes, with the use of our RKHS-analysis, and projection based algorithms, from inside the paper.
Our projection approach to algorithms is motivated in part by principal component analysis (PCA), and its applications in dimension reduction and manifold learning, which in turn, extends earlier work on Monte Carlo, and Karhunen–Loève analysis, also known as the Kosambi–Karhunen–Loève approach. See, e.g., [Son08, JS07, LHN18, GK19, JKST19].

**Background material.** Our paper is interdisciplinary, and we have aimed for multiple target audiences. Inside our paper, we have therefore added explanations for the benefit of readers from neighboring areas. Specifically, background material is included for such notions as Karhunen-Loève transforms, Principal Component Analysis (PCA), Monte-Carlo simulations, and Paley-Wiener spaces. Monte-Carlo refers to the use randomness in order to solve problems that might be deterministic in principle. In rough outline, Karhunen-Loève transforms, and PCAs, are algorithmic tools, used data analysis. Our present emphasis is that of digital image algorithms. Such algorithms serve to produce the best possible bases for image expansions. This area is combined with Monte-Carlo simulations, i.e., random simulations that make use of computer-generated sequences of independent, centered real stochastic process (typically, Gaussian $N(0, 1)$); as well as related Paley-Wiener spaces. A key use of the Karhunen–Loève theorem is algorithmic designs of canonical and orthogonal representations of images and signals. Paley-Wiener spaces offer a useful framework for such reconstruction algorithms.

## 2. Positive definite kernels and measures

The notion of *positive definite* (p.d.) functions, usually called kernels, adapts well to many diverse optimization problems, arising for example in *machine learning*, where training data are chosen to best accommodate features which serve as input in the “learning.” The notion of *feature spaces* and *feature maps* are defined from specific Hilbert spaces (which we will call the feature space); details below. For most application a suitable infinite-dimensional framework is essential. So in the general case, if a p.d. kernel function $K$ is given on a $X \times X$, where $X$ is a set, then there is a rich variety of *factorizations*. More precisely, this means that there are many choices of feature maps from $X$ into some Hilbert space (a choice of feature space) which yield back $K$ in a factorization. Among the feature spaces, the RKHS $\mathcal{H}_K$ associated with $K$ is universal (minimal) in the sense that $\mathcal{H}_K$ is isometrically “included” in any of the feature spaces which arise in factorizations for $K$. For many optimization problems it will be helpful to adapt factorizations in such a way that the feature Hilbert space is an $L^2$ space, and this will be the present focus.

Motivated by related applications to the study of stochastic processes, it is of special significance to focus on the cases when the family of feature spaces may be chosen in the form $L^2(\mu)$. This raises the question of which measures $\mu$ are right for a particular kernel $K$ and its associated RKHS. The answer to this depends on the particular application at hand.

We begin with the correspondence between reproducing kernel Hilbert spaces (RKHSs) and integral operators and their spectral resolutions.

To make precise the notion of Borel measures, we must first introduce a suitable sigma-algebra of Borel subsets of $X$. The idea is as follows: Starting with a p.d. kernel $K$ on $X \times X$, we note below that there is then an induced metric $d_K$ on $X$ which makes $K$ jointly continuous as a function on $X \times X$. But of course, $d_K$ also introduces a (metric) topology on $X$, and therefore the associated sigma-algebra
\( \mathcal{B}_K \) generated by the \( d_K \) neighborhoods; see details below. For some purposes, it may be convenient to pass to the metric completion of \( (X, d_K) \).

Let \( X \times X \xrightarrow{K} \mathbb{R} \) be a positive definite (p.d.) kernel on a set \( X \). Let \( \mathcal{H}_K \) be the associated RKHS with norm \( \| \cdot \|_{\mathcal{H}_K} \). Let \( \mathcal{M}(X) \) be the set of all Borel measures on \( X \).

**Lemma 2.1.** Given \( X \times X \xrightarrow{K} \mathbb{R} \) as above, the set \( X \) can be equipped with a metric

\[
d_K(x, y) := \sqrt{K(x, x) + K(y, y) - 2K(x, y)}, \quad \forall x, y \in X.
\]

**Proof.** One checks that \( d_K(x, y) = \|K_x - K_y\|_{\mathcal{H}_K} \), where \( K_s = K(\cdot, s) \) for all \( s \in X \).

**Lemma 2.2.** The kernel \( K \) is continuous on \( X \times X \) with respect to the product topology induced by \( d_K \times d_K \).

**Proof.** Indeed, for all \( a, b, c, d \in X \), it holds that

\[
|K(a, b) - K(c, d)| = |\langle K_a, K_b \rangle_{\mathcal{H}_K} - \langle K_c, K_d \rangle_{\mathcal{H}_K}|
\leq |\langle K_a, K_b - K_d \rangle_{\mathcal{H}_K}| + |\langle K_a - K_c, K_d \rangle_{\mathcal{H}_K}|
\leq \|K_a\|_{\mathcal{H}_K} \|K_b - K_d\|_{\mathcal{H}_K} + \|K_d\|_{\mathcal{H}_K} \|K_a - K_c\|_{\mathcal{H}_K},
\]

so the assertion follows.

**Definition 2.3.** Let \( \mu \in \mathcal{M}(X) \), i.e., a Borel measure on \( (X, \mathcal{F}) \). The measure \( \mu \) is said to be in \( \text{Reg}(K) \) if

\[
K_x := K(\cdot, x) \in L^2(X, \mathcal{F}, \mu), \quad \forall x \in X.
\]

If (2.2) is assumed to hold for some \( \sigma \)-finite measure \( \mu \) on \( (X, \mathcal{F}) \) and \( \mathcal{F} = \mathcal{B}_K \), the Borel \( \sigma \)-algebra defined from the open sets with respect to the metric \( d_K \), then we get a well defined linear operator \( T_\mu : \mathcal{H}_K \to L^2(\mu) \), by

\[
T_\mu \left( \sum_j c_j K(\cdot, x_j) \right) := \sum_j c_j K(\cdot, x_j)
\]

where \( c_j \in \mathbb{R}, x_j \in X \) with a finite sum. However, the operator \( T_\mu \) may be unbounded. It is always closable (Lemma 2.5). We now turn to the adjoint operator \( T^*_\mu : L^2(\mu) \to \mathcal{H}_K \).

**Lemma 2.4.** Fix \( K \), and \( \mu \in \text{Reg}(K) \) (see Definition 2.3). Let \( T_\mu \) be as in (2.3), then

\[
(T^*_\mu \varphi)(x) = \int_X K(y, x) \varphi(y) \, d\mu(y), \quad \forall \varphi \in \text{dom} (T^*_\mu).
\]

**Proof.** For all \( x \in X \), and all \( \varphi \in \text{dom} (T^*_\mu) \),

\[
\langle T^*_\mu \varphi, K(\cdot, x) \rangle_{\mathcal{H}_K} = \langle \varphi, T_\mu(K(\cdot, x)) \rangle_{L^2(\mu)} = \int_X \varphi(y) K(y, x) \, \mu(dy).
\]

**Lemma 2.5.** Let \( T^*_\mu \) be as above. Then \( \varphi \in \text{dom} (T^*_\mu) \subset L^2(\mu) \) if and only if

\[
\int_X \int_X \varphi(y) \varphi(z) K(y, z) \, \mu(dy) \, \mu(dz) < \infty.
\]

Moreover, \( \text{dom} (T^*_\mu) \) is dense in \( L^2(\mu) \), and so \( T \) is closable.
Then, to produce splines from sample points; and to create best spline-fits. In statistics, machine learning. In numerical analysis, a popular version of the method is used kernel-optimization. It refers to training-data and feature spaces in the context of

Remark. Recall that the functions

\[ \langle \int_X K(y, \cdot) \varphi(y) \mu(dy), \int_X K(z, \cdot) \varphi(z) \mu(dz) \rangle_{\mathcal{H}_K} \]

which is (2.5).

Proof. Let \( \varphi \in L^2(\mu) \). If (2.5) holds, then \( \int_X K(y, \cdot) \varphi(y) \mu(dy) \) belongs to \( \mathcal{H}_K \), and so \( T_\mu^* \varphi \) is well-defined, i.e., \( \varphi \in \text{dom} (T_\mu^*) \). Conversely, if \( \varphi \in \text{dom} (T_\mu^*) \), then

\[
\| T_\mu^* \varphi \|^2_{\mathcal{H}_K} = \left< \int_X K(y, \cdot) \varphi(y) \mu(dy), \int_X K(z, \cdot) \varphi(z) \mu(dz) \right>_{\mathcal{H}_K} = \int_X \int_X \varphi(y) \varphi(z) K(y, z) \mu(dy) \mu(dz) < \infty
\]

Moreover, \( T_\mu T_\mu^* \varphi = \int \varphi(y) K(y, \cdot) \mu(dy) \) (2.7) with \( \varphi \in \text{dom} (T_\mu^*) \), the domain of the operator \( T_\mu^* \); i.e., \( \varphi \in \text{dom} (T_\mu^*) \) if the RHS of (2.7) is in \( \mathcal{H}_K \).

Corollary 2.6. Let \( \mu \in \text{Reg}(K) \) as above. For all \( x, z \in X \), then

\[
(T_\mu T_\mu^* K(\cdot, x))(z) = \int_X K(z, y) K(y, x) \mu(dy) = \langle K(\cdot, z), K(\cdot, x) \rangle_{L^2(\mu)}. \tag{2.6}
\]

Moreover,

\[
T_\mu T_\mu^* \varphi = \int \varphi(y) K(y, \cdot) \mu(dy) \tag{2.7}
\]

with \( \varphi \in \text{dom} (T_\mu^*) \), the associated operator, see Lemma 2.4. Then for \( \varphi \in \text{dom}(T_\mu^*) \subset L^2(\mu) \), and \( \alpha \in \mathbb{R}_+ \), consider the following optimization problem:

\[
\mathcal{H}_K \ni f = \arg\min \left\{ \| \varphi - T_\mu f \|^2_{L^2(\mu)} + \alpha \| f \|^2_{\mathcal{H}_K} \right\}. \tag{2.8}
\]

The solution is

\[
f = (\alpha I + T_\mu T_\mu^*)^{-1} \left( T_\mu^* \varphi \right). \]

Proof. This is a direct computation; see also [JT19b, JKST19, JT17].

Sketch. Define \( W : \mathcal{H}_K \to \mathcal{H}_K \times L^2(\mu) \), \( Wf = (f, T_\mu f) \), where

\[
\| (f, g) \|^2_{\mathcal{H}_K \times L^2(\mu)} := \alpha \| f \|^2_{\mathcal{H}_K} + \| g \|^2_{L^2(\mu)}.
\]

Then, \( W^* (f, g) = \alpha f + T_\mu^* g \), and by the standard least square approximation,

\[
\text{RHS}_{(2.8)} = \arg\min \left\{ \| Wf - (0, \varphi) \|^2_{\mathcal{H}_K \times L^2(\mu)} \right\} = (W^* W)^{-1} W^* ((0, \varphi)) = (\alpha I + T_\mu T_\mu^*)^{-1} T_\mu^* \varphi.
\]

Remark. Recall that the functions

\[
(x_1, x_2, \cdots, x_n) \mapsto (K(\cdot, x_1), K(\cdot, x_2), \cdots, K(\cdot, x_n)) \tag{2.9}
\]

are often called feature functions, and \( \mathcal{H}_K \) is interpreted as a feature space.

Kernel-optimization. One of the more recent applications of RKHSs is the kernel-optimization. It refers to training-data and feature spaces in the context of machine learning. In numerical analysis, a popular version of the method is used to produce splines from sample points; and to create best spline-fits. In statistics,
there are analogous optimization problems going by the names “least-square fitting,” and “maximum-likelihood” estimation.

What these methods have in common is a minimization (or a max problem) involving a “quadratic” expression $Q$ (see (2.8)) with two terms: (i) a $L^2$-square applied to a difference, and (ii) a penalty term which is a RKHS norm-squared. In the application to determination of splines, the penalty term may be a suitable Sobolev norm-squared; i.e., $L^2$ norm-squared applied to a chosen number of derivatives. Hence non-differentiable choices will be “penalized.”

**Definition 2.8.** Let $\mathcal{H}$ be a Hilbert space with inner product denoted $\langle \cdot, \cdot \rangle$, or $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ when there is more than one possibility to consider. Let $J$ be a countable index set, and let $\{w_j\}_{j \in J}$ be an indexed family of non-zero vectors in $\mathcal{H}$. We say that $\{w_j\}_{j \in J}$ is a frame for $\mathcal{H}$ iff (Def.) there are two finite positive constants $A$ and $B$ such that

$$A \|u\|^2_{\mathcal{H}} \leq \sum_{j \in J} |\langle w_j, u \rangle_{\mathcal{H}}|^2 \leq B \|u\|^2_{\mathcal{H}} \quad (2.10)$$

holds for all $u \in \mathcal{H}$. We say that it is a Parseval frame if $A = B = 1$.

For references to the theory and application of frames, see e.g., [HJL+13, KLZ09, CM13].

**Lemma 2.9.** If $\{w_j\}_{j \in J}$ is a Parseval frame in $\mathcal{H}$, then the (analysis) operator $T : \mathcal{H} \rightarrow l^2(J)$,

$$Tu = (\langle w_j, u \rangle_{\mathcal{H}})_{j \in J} \quad (2.11)$$

is well-defined and isometric. Its adjoint $T^* : l^2(J) \rightarrow \mathcal{H}$ is given by

$$T^* (\gamma_j)_{j \in J} := \sum_{j \in J} \gamma_j w_j \quad (2.12)$$

and the following hold:

1. The sum on the RHS in (2.12) is norm-convergent;
2. $T^* : l^2(J) \rightarrow \mathcal{H}$ is co-isometric; and for all $u \in \mathcal{H}$, we have

$$u = T^* Tu = \sum_{j \in J} \langle w_j, u \rangle w_j \quad (2.13)$$

where the RHS in (2.13) is norm-convergent.

**Proof.** The details are standard in the theory of frames; see the cited papers above. Note that (2.10) for $A = B = 1$ simply states that $V$ in (2.11) is isometric, and so $T^* T = I_{\mathcal{H}} = \text{the identity operator in } \mathcal{H}$, and $TT^*$ = the projection onto the range of $V$. \(\square\)

**Remark 2.10.** We may always get an orthonormal basis (ONB) or a Parseval frame $\{f_i\}_{i \in J}$ in $\mathcal{H}_K$, where the index $J$ is usually $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots \}$.

**Lemma 2.11.** Given a Parseval frame $\{f_i\}_{i \in \mathbb{N}_0}$ in $\mathcal{H}_K$, then

$$K(x, y) = \sum_{i \in \mathbb{N}_0} f_i(x) f_i(y), \quad x, y \in X. \quad (2.14)$$

If $\mu \in \text{Reg}(K)$, then

$$\phi \in L^2(\mu) \iff \left( \langle \phi, f_i \rangle_{L^2(\mu)} \right)_i \in l^2(\mathbb{N}_0) \quad (2.15)$$
Proof. Given \( \{ f_i \}_{i \in \mathbb{N}_0} \) in \( \mathcal{H}_K \), one checks that

\[
K(\cdot, y) = K_y(\cdot) = \sum_{i \in \mathbb{N}_0} \langle f_i, K_y \rangle f_i(\cdot) = \sum_{i \in \mathbb{N}_0} f_i(y) f_i(\cdot)
\]

which is (2.14).

\[\square\]

Question 2.12. Find the spectrum of \( T_{\mu} \) as \( \mu \) varies in \( \text{Reg}(K) \). Actually it is the selfadjoint operator \( T_{\mu} T_{\mu}^* : L^2(\mu) \rightarrow L^2(\mu) \), whose spectrum we want.

It is important to keep in mind that we get a family of operators \( T_{\mu} \) indexed by \( \mu \in \text{Reg}(K) \). They are densely defined operators in the Hilbert space \( \mathcal{H}_K \), but the spectra may vary with \( \mu \).

Fix a p.d. kernel \( X \times X \xrightarrow{K} \mathbb{R} \) on a set \( X \). Assign the Borel \( \sigma \)-algebra \( \mathcal{B}_K \) and measure \( \mu \), and get \( (X, \mathcal{B}_K) \) as above. What is the interconnection between the following three conditions?

1. \( K(\cdot, x) \in L^2(\mu) \), for all \( x \in X \);
2. \( \{ f_i \} \subset L^2(\mu) \), for all ONB \( \{ f_i \} \) in \( \mathcal{H}_K \);
3. How is the system \( \{ f_i \} \) related to the spectral properties of \( T_{\mu} T_{\mu}^* : L^2(\mu) \rightarrow L^2(\mu) \)?

A related question: Consider a p.d. kernel \( K \) on \( X \times X \). Suppose \( X \) is discrete and countable, equipped with the counting measure \( \gamma \) on \( X \), i.e.,

\[
\gamma(B) = \#(B), \quad \forall B \subset X.
\]

When is \( \gamma \in \text{Reg}(K) \)?

We shall give a precise solution to (2.17) in the following section.

3. Operators from p.d. kernels, and some of their applications

Here, we consider a special case in the setting of Section 2, where a p.d. kernel admits a factorization in some \( L^2(\mu) \)-space with \( \mu \in \text{Reg}(K) \).

Let \( X \times X \xrightarrow{K} \mathbb{R} \) be a p.d. kernel on a set \( X \). Let \( \mathcal{H}_K \) be the associated RKHS.

Assume:

1. \( \mu \in \text{Reg}(K) \);
2. \( \{ K_x \}_{x \in X} \) is dense in \( L^2(\mu) \);
3. for all \( F \in \mathcal{H}_K \),

\[
\| F \|_{\mathcal{H}_K} \geq \| F \|_{L^2(\mu)}.
\]

Lemma 3.1. Let \( (\mathcal{H}_j, \| \cdot \|_j) \), \( j = 1, 2 \), be a pair of Hilbert spaces. Suppose the inclusion map \( J : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) has dense image, and \( \| x \|_1 \geq \| x \|_2 \), for all \( x \in \mathcal{H}_1 \). Then there exists a unique positive (selfadjoint) operator \( A \geq 1 \), such that \( \mathcal{H}_1 = \text{dom}(A^{1/2}) \) and

\[
\langle x, y \rangle_1 = \left\langle A^{1/2}x, A^{1/2}y \right\rangle_2
\]

for all \( x, y \in \mathcal{H}_1 \).

Proof. It can be verified that

\[
A := (JJ^*)^{-1}
\]

is the unique positive operator having the desired properties. For more details, see e.g., [Nel69, DS88, AG93, RS75] and [JT15].
Corollary 3.2. Let $\mathcal{H}_1 = \mathcal{H}_K$ and $\mathcal{H}_2 = L^2(\mu)$. In view of Lemma 3.1, then $\Phi : X \rightarrow L^2(\mu)$, defined as $\Phi (x) = A^{1/2}Kx, \ x \in X,$ is a feature map for the p.d. kernel $K$, and $L^2(\mu)$ is the corresponding feature space.

Corollary 3.3. Let $A$ be as in (3.1). Suppose $A^{-1}$ is compact with spectral decomposition $A^{-1} = \sum_{i=1}^{\infty} \lambda_i |u_i\rangle \langle u_i|$, where $1 \geq \lambda_i \geq \lambda_{i+1} > 0$, $\lambda_i \rightarrow 0$ as $i \rightarrow \infty$, and $\{u_i\}$ is an ONB in $L^2(\mu)$.

Then, for all $x, y \in X$, it holds that $K(x, y) = \sum_{i=1}^{\infty} \lambda_i^{-1} \langle Kx, u_i \rangle_{L^2} \langle u_i, Ky \rangle_{L^2}$.

Proof. One checks that $K(x, y) = \langle Kx, Ky \rangle_{\mathcal{H}_K} = \langle \Phi (x), \Phi (y) \rangle_{L^2} = \langle A^{1/2}Kx, A^{1/2}Ky \rangle_{L^2} = \sum_{i=1}^{\infty} \langle A^{1/2}Kx, u_i \rangle_{L^2} \langle u_i, A^{1/2}Ky \rangle_{L^2} = \sum_{i=1}^{\infty} \langle Kx, A^{1/2}u_i \rangle_{L^2} \langle A^{1/2}u_i, Ky \rangle_{L^2} = \sum_{i=1}^{\infty} \lambda_i^{-1} \langle Kx, u_i \rangle_{L^2} \langle u_i, Ky \rangle_{L^2}$.

\[\square\]

4. Restrictions of Positive Definite Kernels

Let $K : X \times X \rightarrow \mathbb{C}$ be a positive definite kernel defined on $X \times X$ (where $X$ is a fixed set); i.e., it is assumed that for all finite subset $F \subset X$, and scalars $(c(x))_{x \in F}$ in $\mathbb{C}^{|F|}$ we have

$$\sum_{F \times F} c(x)c(y) K(x, y) \geq 0. \quad (4.1)$$

As before the RKHS will be denoted $\mathcal{H}_K$.

If $V \subset X$ is a countably infinite subset, then $K_V := K(\cdot, \cdot) |_{V \times V}$ (4.2) may be considered as an $\infty \times \infty$ matrix with $V$ serving as row \times column index.

Consider the standard $l^2$-space $l^2(V)$ with dense subspace $l_0^2(V)$ consisting of finitely supported sequences, i.e.,

$$c \in l_0^2(V) \iff \exists F \text{ finite s.t. } c \equiv 0 \text{ in } V \setminus F. \quad (4.3)$$
We now turn to the discretized version of the condition we introduced in Definition 2.3 in the general context of sigma-finite measures. Since we are now making selection of countably discrete subsets $V$ of $X$, we will have a condition for each choice of $V$. See details below:

**Remark 4.1.** When $K$, $X$, and $V \subseteq X$ are specified as in (4.2), we shall impose the following $l^2(V)$ condition, which restricts the possible choice of countable discrete subsets $V \subseteq X$:

We shall assume that for all $y \in V$,

$$\sum_{x \in V} |K(x, y)|^2 \leq C^V_y < \infty,$$

with the constant $C^V_y$ depending on $y$, and choice of $V$.

**Example 4.2.** If $X = \mathbb{R}$, and $K$ = the Shannon sinc kernel, i.e.,

$$K(x, y) = \frac{\sin \pi (x - y)}{\pi (x - y)}, \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R}$$

then one checks that condition (4.4) is satisfied when $V = (x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus \{0\}$ satisfying

$$\sum_{n \in \mathbb{N}} |x_n|^{-2} < \infty.$$

The verification is a direct computation; it is illustrated by the graph below (Figure 4.1).

![Figure 4.1](image)

**Figure 4.1.** Illustration of condition (4.4) for the Shannon sinc kernel. Assume $c > 1$, then the inequality in (4.4) is satisfied when $|x_n|$ is large. In the shaded region, $x_n^2 \leq c(x_n - y)^2$.

We now return to the general case. Note that the action on $l^2(V)$ given by

$$(K_V c)(x) = \sum_{y \in F} K_V(x, y) c(y)$$

is well defined. Note the finite set $F$ from (4.3) depends on $c$.

Hence, when $V$ is given, assumed countably infinite, it is of interest to consider the case when $K_V$, as in (4.6), defines a bounded operator in $l^2(V)$; and when this operator is invertible.
Proposition 4.3. Let $K$, $X$, and $V$ be as above, then the operator $K_V$ (see (4.6)) is bounded (relative to $l^2 (V)$) with bounded inverse if and only if there are constants $A, B$ (depending on $K, V$) such that

$$0 < A \leq B < \infty,$$

and one of the following two conditions hold:

1. $A c \|l\|_{\ell^2(V)}^2 \leq \sum_{x \in V} \left| \sum_{y \in V} K(x, y) c(y) \right|^2 \leq B c \|l\|_{\ell^2(V)}^2$, for all $c \in l^2 (V)$;

2. $A \|f\|_{\mathcal{H}_K}^2 \leq \sum_{x \in V} |f(x)|^2 \leq B \|f\|_{\mathcal{H}(K)}^2$, for all $f \in \mathcal{H}_K$.

Proof. From general facts about operators in Hilbert space, we have the equivalence of the following conditions on a densely defined operator $\mathcal{H}_1 \overset{T}{\to} \mathcal{H}_2$ acting from one Hilbert space $\mathcal{H}_1$ into another. First $T$ is said to be bounded iff

$$\|T\|_{\mathcal{H}_1 \to \mathcal{H}_2} = \sup_{u \in \mathcal{H}_1, \|u\| \leq 1} \|Tu\|_2 < \infty. \quad (4.7)$$

Note that

$$\|T^*T\|_{\mathcal{H}_1 \to \mathcal{H}_1} = \|TT^*\|_{\mathcal{H}_2 \to \mathcal{H}_2} = \|T\|_{\mathcal{H}_1 \to \mathcal{H}_2}^2. \quad (4.8)$$

Let $K$, $X$, $V \subset X$ be as specified in the proposition, and consider

$$\text{span} \{K(\cdot, x) : x \in V\} \quad (4.9)$$

as a dense subspace in $\mathcal{H}(K_V) := \text{RKHS}(K|_{V \times V})$. Then we have a naturally defined operator $\mathcal{H}(K_V) \to l^2 (V)$ with (4.9) as its dense domain: Set $T_V : \mathcal{H}(K_V) \to l^2 (V) ,

T = T_V : \mathcal{H}(K_V) \ni f \rightarrow \sum_{x \in V} (f(x)) \in l^2 (V). \quad (4.10)$

Considering the respective inner products in the two Hilbert spaces $\mathcal{H}(K_V)$ and $l^2 (V)$, it follows that the adjoint operator $T_V^* : l^2 (V) \to \mathcal{H}(K_V)$, to (4.10) is well defined with dense domain $l^2_0 (V)$, and

$$(T_V^* (c))(x) = \sum_{y \in V} K_V(x, y) c(y), \quad \forall c \in l^2_0 (V). \quad (4.11)$$

Now the conclusion of the proposition follows once we verify that the operator

$$T_V T_V^* : l^2 (V) \to l^2 (V) \quad (4.12)$$

(possibly unbounded) is given by the $\infty \times \infty$ matrix $K_V := K|_{V \times V}$. Indeed, for $c \in l^2_0 (V)$ we have the following computation of $T_V T_V^*$ in (4.11):

$$l^2_0 (V) \ni c \overset{T_V}{\rightarrow} \sum_{y \in V} c(y) K(\cdot, y) \quad (\text{as a function on } V); \quad (4.13)$$

and so

$$(T_V T_V^* c)(x) = \sum_{y \in V} K(x, y) c(y), \quad (4.14)$$

which is the desired conclusion. Note that we used that, for fixed $y$, we have

$$T_V (K(\cdot, y)) = (K(x, y))_{x \in V} \in l^2 (V). \quad (4.15)$$

See assumption (4.4) above. □
Corollary 4.4. Let $K$, $X$, and $V$ be as specified. Denote by $K_V$ the operator in $l^2(V)$ which have $K(\cdot, \cdot)_{|V \times V}$ as its $\infty \times \infty$ matrix realization, see Proposition 4.3.

Then there are constants $A, B, 0 < A \leq B < \infty$ (depending on $V$) such that

$$A \|f\|^2_{\mathcal{H}(K_V)} \leq \sum_{x \in V} |f(x)|^2 \leq B \|f\|^2_{\mathcal{H}(K_V)}$$

for all $f \in \mathcal{H}(K_V)$, if and only if both $K_V$ and $(K_V)^{-1}$ define bounded operators in $l^2(V)$.

5. Finite point-configurations governed by p.d. kernels

Given $X \times X \overset{K}{\rightarrow} \mathbb{R}$ a fixed positive definite (p.d.) kernel, we consider the case such that for every finite subset $F \subset X$, the restriction $K_F := K(\cdot, \cdot)_{|F \times F}$ is invertible. Denote by $K_F^{-1}$ the inverse matrix.

Lemma 5.1. Let $P_F :=$ the $\mathcal{H}_K$-orthogonal projection onto

$$\mathcal{H}_K(F) := \text{span}\{K(\cdot, x)\}_{x \in F} \quad (5.1)$$

then

$$(P_F f)(\cdot) = \sum_{x \in F} K_F^{-1}(f|_F)_x K(\cdot, x). \quad (5.2)$$

Note $f|_F := [f(x_1) \ f(x_2) \ \cdots \ f(x_n)]^T$ as a column vector, if $F = \{x_1, x_2, \cdots, x_n\}$, and so $K_F^{-1}(f|_F)$ refers to matrix multiplication

$$K_F^{-1}(f|_F) = \sum_{y \in F} (K_F^{-1})_{xy} f(y). \quad (5.3)$$

Proof. Step 1. $P_F^* = P_F$. Indeed,

$$P_F^* \ = \ P_F$$

$$\d|

\langle K_F^{-1} f|_F, g \rangle_{\mathcal{H}_K} \ = \ \langle f, K_F^{-1} g \rangle_{\mathcal{H}_K}$$

$$\d|$$

$$\sum_{F \times F} (K_F^{-1})_{xy} f(y) g(x) \ = \ \sum_{F \times F} f(x) (K_F^{-1})_{xy} g(y)$$

$$\d|$$

$$(K_F^{-1})_{xy} \ = \ (K_F^{-1})_{yx} \quad (5.4)$$

but (5.4) follows from the fact that $K$ is p.d. and so in particular $K(x, y) = K(y, x)$, i.e., symmetric.

Remark. If $K$ is complex valued, the same conclusion holds but with $K(x, y) = \overline{K(y, x)}$, $\forall x, y \in X$.

Step 2. $P_F^2 = P_F$. Now this follows from the definition (5.2) of the finite rank operator $P_F : \mathcal{H}_F \rightarrow \mathcal{H}_F$. So we iterate the formula (5.2) which defines the action of $P_F$:

$$f \overset{P_F}{\longrightarrow} \sum_{x \in F} (K_F^{-1})_x (f|_F)_x K(\cdot, x)$$
$$P_F \rightarrow \sum_{F \times F} K^{-1}_F(f|_F)_x (K^{-1}_F)_y K(y, x) K(\cdot, y)$$

$$= \sum_{F \times F} K^{-1}_F(f|_F)_x \delta_{xy} K(\cdot, y)$$

$$= \sum_{F} K^{-1}_F(f|_F)_x K(\cdot, x)$$

(by (5.2) $$(P_F f)(\cdot)$$)

so $P^2_F = P_F$, and we proved that $P_F$ is the desired finite rank projection in $\mathcal{H}_F$ onto the subspace $\mathcal{H}_F(F)$.

**Remark.** We work with the case that for every $F \subset X$ (discrete points)

$$K_F = K(\cdot, \cdot)|_{F \times F}$$

is an invertible matrix. (5.5)

But

$$\left(\text{5.5}\right) \iff \{K(\cdot, x_1), \ldots, K(\cdot, x_n)\} \text{ is linearly independent}, \quad (5.6)$$

where $F = (x_i)$, $x_i \neq x_j$ if $i \neq j$. To see this, fix $c = (c_1, \ldots, c_n)$, then

$$\sum_i c_i K(\cdot, x_i) = 0 \text{ in } \mathcal{H}_F$$

$$\updownarrow$$

$$\left\|\sum_i c_i K(\cdot, x_i)\right\|_{\mathcal{H}_F}^2 = 0$$

$$\downarrow$$

$$\sum_{F \times F} c_i c_j K(x_i, x_j) = 0$$

$$\downarrow$$

$$c^T K_F c = 0 \iff c = 0$$

and so (5.6) holds, i.e., linear independence.

**Corollary 5.2.** If $F_0 = \{x_0\}$ is a singleton, then

$$(P_{F_0} f)(\cdot) = \frac{f(x_0)}{K(x_0, x_0)} K(\cdot, x_0),$$

and

$$P_n \cdots P_2 P_1 P_0 : f \mapsto f(x_0) \frac{K(x_0, x_1) K(x_1, x_2) \cdots K(x_n, \cdot)}{K(x_0, x_0) K(x_1, x_1) \cdots K(x_n, x_n)}.$$
where one family $P_f(\cdot)$ indexed by $f \in \mathcal{H}$, and a second family $P_T$ indexed by a class of operators $T : \mathcal{H} \to \mathcal{H}$.

**Definition 6.1.** Let $\mathcal{H}$ be a Hilbert space. Let $(\psi_i)$ and $(\phi_i)$ be orthonormal bases (ONB), with index set $I$. Usually

$$I = \mathbb{N} = \{1, 2, \ldots\}.$$  \hfill (6.1)

If $(\psi_i)_{i \in I}$ is an ONB, we set $Q_n := \text{the orthogonal projection onto } \text{span}\{\psi_1, \ldots, \psi_n\}$.

**Proposition 6.2.** Consider an ensemble of a large number $N$ of objects of similar type such as a set of data, of which $Nw^\alpha, \alpha = 1, 2, \ldots, \nu$ where the relative frequency $w^\alpha$ satisfies the probability axioms:

$$w^\alpha \geq 0, \sum_{\alpha=1}^\nu w^\alpha = 1.$$  

Assume that each type specified by a value of the index $\alpha$ is represented by $f^\alpha(\xi)$ in a real domain $[a, b]$, which we can normalize as

$$\int_a^b |f^\alpha(\xi)|^2 d\xi = 1.$$  

Let $\{\psi_i(\xi)\}, i = 1, 2, \ldots, \nu$ be a complete set of orthonormal base functions defined on $[a, b]$. Then any function (or data) $f^\alpha(\xi)$ can be expanded as

$$f^\alpha(\xi) = \sum_{i=1}^\infty x_i^{(\alpha)} \psi_i(\xi)$$  \hfill (6.2)

with

$$x_i^{\alpha} = \int_a^b \psi_i^*(\xi) f^\alpha(\xi) d\xi.$$  \hfill (6.3)

Here, $x_i^{\alpha}$ is the component of $f^\alpha$ in $\psi_i$ coordinate system. With the normalization of $f^\alpha$ we have

$$\sum_{i=1}^\infty |x_i^{\alpha}|^2 = 1.$$  

**Proof.** If we substitute (6.3) into (6.2) we have

$$f^\alpha(\xi) = \int_a^b f^\alpha(\xi) \left[\sum_{i=1}^\infty \psi_i^*(\xi) \psi_i(\xi)\right] d\xi$$

$$= \sum_{i=1}^\infty (\psi_i, f^\alpha) \psi_i(\xi)$$

by definition of ONB. Note this involves orthogonal projection. \hfill $\square$

We here give mathematical background of PCA.

Let $\mathcal{H} = L^2(a, b), \psi_i : \mathcal{H} \to L^2(\mathbb{Z})$ and $U : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ where $U$ is a unitary operator.

Notice that the distance is invariant under a unitary transformation. Thus, using another coordinate system (principal axis) $\{\phi_j\}$ in place of $\{\psi_i\}$, would preserve the distance. The idea is that when PCA transform is applied on a set of data, the set of data $\{x_i^{\alpha}\}$ in the feature space represented in $\{\psi_i\}$ basis are now represented in another coordinate system $\{\phi_j\}$. 


Let \( \{ \phi_j \}, j = 1, 2, \ldots, \) be another set of orthonormal basis (ONB) functions instead of \( \{ \psi_i(\xi) \}, i = 1, 2, \ldots. \) Let \( y_j^\alpha \) be the component of \( f^\alpha \) in \( \{ \phi_j \} \) where it can be expressed in terms of \( x_i^\alpha \) by a linear relation

\[
y_j^\alpha = \sum_{i=1}^{\infty} \langle \phi_j, \psi_i \rangle x_i^\alpha = \sum_{i=1}^{\infty} U_{i,j} x_i^\alpha
\]

where \( U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \) is the unitary operator

\[
U_{i,j} = \langle \phi_j, \psi_i \rangle = \int_{a}^{b} \phi_j^\ast (\xi) \psi_i (\xi) \, d\xi.
\]

Also, \( x_i^\alpha \) can be written in terms of \( y_j^\alpha \) under the following relation

\[
x_i^\alpha = \sum_{j=1}^{\infty} \langle \psi_i, \phi_j \rangle y_j^\alpha = \sum_{j=1}^{\infty} U^{-1}_{i,j} y_j^\alpha
\]

where \( U^{-1}_{i,j} = \overline{U_{i,j}} \) and \( U^{-1}_{i,j} = U_{j,i}^\ast. \) Thus,

\[
f^\alpha(\xi) = \sum_{i=1}^{\infty} x_i^\alpha(\xi) \psi_i (\xi) = \sum_{j=1}^{\infty} y_j^\alpha (\xi) \phi_j (\xi).
\]

So \( U(x_i) = (y_i) \) which is coordinate change, and \( \sum_{i=1}^{\infty} x_i^\alpha \psi_i (\xi) = \sum_{j=1}^{\infty} y_j^\alpha \phi_j (\xi), \) and

\[
x_i^\alpha = \langle \psi_i, f^\alpha \rangle = \int_{a}^{b} \psi_i^\ast (\xi) f(\xi) \, d\xi.
\]

The squared magnitude \(|x_i^{(\alpha)}|^2\) of the coefficient for \( \psi_i \) in the expansion of \( f^{(\alpha)} \) can be considered as a good measure of the average in the ensemble

\[
Q_i = \sum_{\alpha=1}^{n} w^{(\alpha)} |x_i^{(\alpha)}|^2,
\]

and as a measure of importance of \( \{ \psi_i \}. \) Notice,

\[
Q_i \geq 0, \sum_i Q_i = 1.
\]

See also [Wat67, JKST19].

Let \( G (\xi, \xi') = \sum_{\alpha} w^{\alpha} f^{\alpha} (\xi) f^{\alpha*} (\xi'). \) Then \( G \) is a Hermitian matrix that is the covariance matrix and \( Q_i = G(i, i) = \sum_{\alpha} w^{\alpha} x_i^{\alpha} x_i^{\alpha*}. \) Here, \( Q_i = G(i, i) \) is the variance and \( G(i, j) \) determines the covariance between \( x_i \) and \( x_j. \) The normalization \( \sum Q_i = 1 \) gives us trace \( G = 1, \) where the trace means the diagonal sum.

Then define a special function system \( \{ \Theta_k(\xi) \} \) as the set of eigenfunctions of \( G, \) i.e.,

\[
\int_{a}^{b} G (\xi, \xi') \Theta_k (\xi') \, d\xi' = \lambda_k \Theta_k (\xi).
\]

(6.4)

So \( G \Theta_k (\xi) = \lambda_k \Theta_k (\xi). \) Also, \( U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \) is the unitary operator consisting of eigenfunctions of \( G \) in its columns. These eigenfunctions represent the directions of the largest variance of the data and the corresponding eigenvalues represent the magnitude of the variance in the directions. PCA allows us to choose the principal components so that the covariance matrix \( G \) of the projected data is as large as possible. The largest eigenfunction of the covariance matrix points to the direction
of the largest variance of the data and the magnitude of this function is equal to the corresponding eigenvalue. The subsequent eigenfunctions are always orthogonal to the largest eigenfunctions.

**Principal eigenfunctions and detecting largest variance.** When the data are not functions but vectors \(v^\alpha\)'s whose components are \(x_i^{(\alpha)}\) in the \(\psi_i\) coordinate system, we have
\[
\sum_{i'} G(i,i') t^i_{i'} = \lambda_k t^i_k
\]
where \(t^i_k\) is the \(i^{th}\) component of the vector \(\Theta_k\) in the coordinate system \(\{\psi_i\}\). So we get \(\psi : H \rightarrow (x_i)\) and also \(\Theta : H \rightarrow (t_i)\). The two ONBs result in
\[
x_i^\alpha = \sum_k c^\alpha_k t^i_k \text{ for all } i, \quad c^\alpha_k = \sum_i t^{i*}_k x_i^\alpha,
\]
which is the Karhunen-Loève expansion of \(f^\alpha(\xi)\) or vector \(v^\alpha\). Hence \(\{\Theta_k(\xi)\}\) is the K-L coordinate system dependent on \(\{w^\alpha\}\) and \(\{f^\alpha(\xi)\}\). Then we arrange the corresponding eigenfunctions or eigenvectors in the order of eigenvalues \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{k-1} \geq \lambda_k \geq \ldots\) in the columns of \(U\).

Now, \(Q_i = G_{i,i} = \langle \psi_i, G \psi_i \rangle = \sum_k A_{ik} \lambda_k\) where \(A_{ik} = t^{i*}_k t^i_k\) which is a double stochastic matrix. Then we have the following eigendecomposition of the covariance matrix (operator), \(G\)
\[
G = U \begin{pmatrix} 
\lambda_1 & \cdots & 0 \\
0 & \ddots & 0 \\
0 & \cdots & \lambda_k
\end{pmatrix} U^{-1}.
\]

### 6.1. Principal Component Analysis and Maximal Variance.

In this subsection, we discuss the orthonormal bases of Karhunen-Loève transform or PCA where it captures the maximal variance in the linear data to effectively perform dimensionality reduction. In [JKST19] these results to our Principal Component Analysis (PCA) on data, and dimension reduction algorithms for both linear and nonlinear data sets were shown. We shall recall here some definitions and results from [BJ02, JS07, JKST19].

The following definitions, lemmas and theorems are results from [JKST19, JS07]. Let \(H\) be a Hilbert space which realizes trace class \(G\) as a self-adjoint operator.

**Definition 6.3.** \(T \in B(H)\) is said to be trace class if and only if the series \(\sum \langle \psi_i, T \psi_i \rangle\), with \(|T| = \sqrt{T^*T}\), is convergent for some ONB \(\{\psi_i\}\). In this case, set
\[
tr(T) := \sum \langle \psi_i, T \psi_i \rangle.
\]

**Definition 6.4.** A sequence \((h_\alpha)_{\alpha \in A}\) in \(H\) is called a *frame* if there are constants \(0 < c_1 \leq c_2 < \infty\) such that
\[
c_1 \|f\|^2 \leq \sum_{\alpha \in A} |\langle h_\alpha, f \rangle|^2 \leq c_2 \|f\|^2 \text{ for all } f \in H.
\]

Also see [BJ02, HKLW07, HW19, BH19, CCEL15, CCK13, JKST19].

**Lemma 6.5.** Let \((h_\alpha)_{\alpha \in A}\) be a frame in \(H\). Set \(L : H \rightarrow l^2\),
\[
L : f \mapsto (\langle h_\alpha, f \rangle)_{\alpha \in A}.
\]
Then \( L^* : l^2 \to \mathcal{H} \) is given by

\[
L^*((c_\alpha)) = \sum_{\alpha \in A} c_\alpha h_\alpha
\]  

(6.10)

where \( (c_\alpha) \in l^2 \); and

\[
L^* L = \sum_{\alpha \in A} |h_\alpha\rangle \langle h_\alpha|
\]  

(6.11)

**Definition 6.6.** Suppose we are given \((f_\alpha)_{\alpha \in A}\), a frame, non-negative numbers \(\{w_\alpha\}_{\alpha \in A}\), where \(A\) is an index set, with \(\|f_\alpha\| = 1\), for all \(\alpha \in A\).

\[
G := \sum_{\alpha \in A} w_\alpha |f_\alpha\rangle \langle f_\alpha|
\]  

(6.12)

is called a frame operator associated to \((f_\alpha)\).

**Remark 6.7.** If we take vectors \((f_\alpha)\) from a frame \((h_\alpha)\) and normalize them such that \(h_\alpha = \|h_\alpha\| f_\alpha\), and \(w_\alpha := \|h_\alpha\|^2\), then \(L^* L\) has the form (6.12) and it becomes to covariance matrix \(G\) above. Thus \(G = L^* L : \mathcal{H} \to \mathcal{H}\).

**Lemma 6.8.** Let \(G\) be as in (6.12). Then \(G\) is trace class if and only if \(\sum_\alpha w_\alpha < \infty\); and then

\[
\text{tr}(G) = \sum_{\alpha \in A} w_\alpha.
\]  

(6.13)

**Definition 6.9.** Suppose we are given a frame operator

\[
G = \sum_{\alpha \in A} w_\alpha |f_\alpha\rangle \langle f_\alpha|
\]  

(6.14)

and an ONB \((\psi_i)\). Then for each \(n\), the numbers

\[
E^\psi_n = \sum_{\alpha \in A} w_\alpha \|f_\alpha\| - \sum_{i=1}^n \langle \psi_i, f_\alpha \rangle \psi_i \| \iota^2
\]  

(6.15)

are called the error or the residual of the projection.

**Lemma 6.10.** When \((\psi_i)\) is given, set \(Q_n := \sum_{i=1}^n |\psi_i\rangle \langle \psi_i|\) and \(Q_n^\perp = I - Q_n\) where \(I\) is the identity operator in \(\mathcal{H}\). Then (see (6.15))

\[
E^\psi_n = \text{tr}(GQ_n^\perp).
\]  

(6.16)

The more general frame operators are as follows: Let

\[
G = \sum_{\alpha \in A} w_\alpha P_\alpha
\]  

(6.17)

where \((P_\alpha)\) is an indexed family of projection operators in \(\mathcal{H}\), i.e., \(P_\alpha = P^*_\alpha = P^2_\alpha\), for all \(\alpha \in A\). \(P_\alpha\) is trace class if and only if it is finite-dimensional, i.e., if and only if the subspace \(P_\alpha \mathcal{H} = \{x \in \mathcal{H} | P_\alpha x = x\}\) is finite-dimensional.

PCA, is the scheme involves a choice of “principal components,” often realized as a finite-dimensional subspace of a global (called latent) data set. There are two views of principal components: The simplest case of consideration of covariance operators, and in [JKST19] kernel PCA which refers to a class of reproducing kernels, as used in learning theory is discussed. In the latter case, one identifies principal features for the machine learning algorithm.
In can be observed that the simplest way to identify a PCA subspace is to turn to a covariance operator, namely $G$, acting on the global data. With the use of a suitable Karhunen-Loève transform or PCA, and via a system of i.i.d. standard Gaussians, a covariance operator which is of trace class may be obtained. An application of the spectral theorem to this associated operator $G$ (see (6.18) below), we then the algorithm for computing eigenspaces corresponding to the top of the spectrum of $G$, i.e., the subspace spanned by the eigenvectors for the top $n$ eigenvalues can be obtained; see (6.19). These subspaces will then be principal components of order $n$ since the contribution from the span of the remaining eigenspaces will be negligible. The algorithm and example will be given in the next subsections.

A second approach to PCA is based on an analogous identification of principal component subspaces, but with the optimization involving maximum likelihood, or minimization of “cost.”

Now, although PCA is used popularly in linear data dimension reductions as PCA decorrelates data, it is noted that the decorrelation only corresponds to statistical independence in the Gaussian case. So PCA is not generally the optimal choice for linear data dimension reduction. However, PCA captures maximal variability in the data. The reader may find more details in [JKST19, XCM09, SSA+14].

PCA enables finding projections which maximize the variance: The first principal component is the direction in the feature space along which gives projections with the largest variance. The second principal component is the variance maximizing direction to all directions orthogonal to the first principal component. The $i^{th}$ component is the direction which maximizes variance orthogonal to the $i-1$ previous components. Thus, PCA captures maximal variability then projects a set of data in higher dimensional feature space to a lower dimensional feature space orthogonally and this was proved in Theorem 2.12 in [JKST19] which is the following theorem. In the proof of the theorem, we can observe how the orthogonal projection operators play the key part in capturing the maximal variance for PCA application. The example will be shown in the next subsections with a matrix and with PCA image compression with principal components which are obtained by using the projection operators.

**Theorem 6.11.** The Karhunen-Loève ONB with respect to the frame operator $G = L^*L$ gives the smallest error in the approximation to a frame operator and the covariance operator $G$ gives maximum variance.

In the proof of this theorem in [JKST19], we use the covariance operator $G$ which is trace class and positive semidefinite, applying the spectral theorem to $G$ results is a discrete spectrum, with the natural order $\lambda_1 \geq \lambda_2 \geq ...$ and a corresponding ONB $(\phi_k)$ consisting of eigenvectors, i.e.,

$$G\phi_k = \lambda_k \phi_k, \quad k \in \mathbb{N},$$  

(6.18)

called the Karhunen-Loève data or principal components. The spectral data is constructed recursively starting with

$$\lambda_1 = \sup_{\phi \in \mathcal{H}, \|\phi\|=1} \langle \phi, G\phi \rangle = \langle \phi_1, G\phi_1 \rangle, \quad \text{and}$$

$$\lambda_{k+1} = \sup_{\phi \in \mathcal{H}, \|\phi\|=1, \phi \perp \phi_1, \phi_2, ..., \phi_k} \langle \phi, G\phi \rangle = \langle \phi_{k+1}, G\phi_{k+1} \rangle. \quad (6.19)$$
This way, the maximal variance is achieved. Now by applying [AK06] we have
\[ \sum_{k=1}^{n} \lambda_k \geq \text{tr} (Q_n^\psi G) = \sum_{k=1}^{n} \langle \psi_k, G \psi_k \rangle \quad \text{for all } n, \tag{6.20} \]
where \( Q_n^\psi \) is the sequence of projections, deriving from some ONB \( (\psi_i) \) and are arranged such that the following holds:
\[ \langle \psi_1, G \psi_1 \rangle \geq \langle \psi_2, G \psi_2 \rangle \geq \ldots \]
Hence we are comparing ordered sequences of eigenvalues with sequences of diagonal matrix entries. So here, the sequence of projections \( Q_n^\psi \) derived from deriving from some ONB \( (\psi_i) \) play the key role in selection of principal components.

Lastly, we have
\[ \text{tr} (G) = \sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} \langle \psi_k, G \psi_k \rangle < \infty. \]

The assertion in Theorem 6.11 is the validity of
\[ E_n^\psi \leq E_n^{\psi_n} \tag{6.21} \]
for all \( (\psi_i) \in ONB(\mathcal{H}) \), and all \( n = 1, 2, \ldots \); and moreover, that the infimum on the RHS in (6.21) is attained for the KL-ONB \( (\phi_k) \). But in view of our lemma for \( E_n^\psi \) (6.10), we see that (6.21) is equivalent to the system (6.20) in the Arveson-Kadison theorem.

6.2. The Algorithm for a Digital Image Application. Here, we show how the above PCA theory with orthogonal projections \( Q_n^\psi \) are implemented as algorithm for application. Our aim is to reduce the number of bits needed to represent an image by removing redundancies as much as possible. Karhunen-Loève transform or PCA is a transform of \( m \) vectors with the length \( n \) formed into \( m \)-dimensional vector \( X = [X_1, \ldots, X_m] \) into a vector \( Y \) according to
\[ Y = A (X - m_X), \tag{6.22} \]
where matrix \( A \) is obtained by eigenvectors of the covariance matrix \( C \) as in (6.24) below.

The algorithm for Karhunen-Loève transform or PCA can be described as follows:

1. Take an image or data matrix \( X \), and compute the mean of the column vectors of \( X \)
\[ m_X = E(X) = \frac{1}{n} \sum_{i=1}^{n} X_i. \tag{6.23} \]
2. Subtract the mean: Subtract the mean, \( m_X \) in (6.23) from each column vector of \( X \). This produces a data set matrix \( B \) whose mean is zero, and it is called centering the data.
3. Compute the covariance matrix from the matrix in the previous step
\[ C = \text{cov}(X) = E \left( (X - m_X)(X - m_X)^T \right) \tag{6.24} \]
\[ = \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T - m_X m_X^T. \tag{6.25} \]
Here $X - m_X$ can be interpreted as subtracting $m_X$ from each column of $X$. $C(i, i)$ lying in the main diagonal are the variances of

$$C(i, i) = E((X_i - m_{X_i})^2).$$  \hspace{1cm} (6.26)

Also, $C(i, j) = E((X_i - m_{X_i})(X_j - m_{X_j}))$ is the covariance between $X_i$ and $X_j$.

4. Compute the eigenvectors and eigenvalues, $\lambda_i$ of the covariance matrix.
5. Choose components and form a feature vector (matrix of vectors),

$$A = (eig_1, ..., eig_n).$$  \hspace{1cm} (6.27)

List the eigenvectors in decreasing order of the magnitude of their eigenvalues. This matrix $A$ is called the row feature matrix. By normalizing the column vectors of matrix $A$, this new matrix $P$ becomes an orthogonal matrix. Eigenvalues found in step 4 are different in values. The eigenvector with highest eigenvalue is the principle component of the data set. Here, the eigenvectors of eigenvalues that are not up to certain specific values can be dropped thus creating a data matrix with less dimension value.

6. Derive the new data set.

Final Data = Row Feature Matrix $\times$ Row Data Adjust.

The rows of the feature matrix $A$ are orthogonal so the inversion of PCA can be done on equation (6.22) by

$$X = A^T Y + m_X.$$  \hspace{1cm} (6.28)

With the $l$ largest eigenvalues with more variance are used instead of $n$eigenvalues, the matrix $A_l$ is formed using the $l$ corresponding eigenvectors. This yields the newly constructed data or image $X'$ as follows:

$$X' = A_l^T Y + m_X.$$  \hspace{1cm} (6.29)

Row Feature Matrix is the matrix that has the eigenvectors in its rows with the most significant eigenvector (i.e., with the greatest eigenvalue) at the top row of the matrix. Row Data Adjust is the matrix with mean-adjusted data transposed. That is, the matrix contains the data items in each column with each row having a separate dimension (see e.g., [MP05, AW12, Mar14, JKST19, Tha20, Reb20, SSA+14, XCM09, JC16]).

In PCA, image compression occurs by the method of dimension reduction. Here, we need to determine how to choose the right axes. PCA gives a linear subspace of dimension that is lower than the dimension of the original image data in such a way that the image data points lie mainly in the linear subspace with the lower dimension. PCA creates a new feature-space (subspace) that captures as much variance in the original image data as possible. The linear subspace is spanned by the orthogonal vectors that form a basis. These orthogonal vectors give principal axes, i.e., directions in the data with the largest variations. As in section 6.2, the PCA algorithm performs the centering of the image data by subtracting off the mean, and then determines the direction with the largest variation of the data and chooses an axis in that direction, and then further explores the remaining variation and locates another axis that is orthogonal to the first and explores as much of the remaining variation as possible. This iteration is performed until all possible axes are exhausted. Once we have a principal axis, we subtract the variance along
this principal axis to obtain the remaining variance. Then the same procedure is applied again to obtain the next principal axis from the residual variance. In addition to being the direction of maximum variance, the next principal axis must be orthogonal to the other principal axes. When all the principal axes are obtained, the data set is projected onto these axes. These new orthogonal coordinate axes are also called principal components.

The outcome is all the variation along the axes of the coordinate set, and this makes the covariance matrix diagonal which means each new variable is uncorrelated with the rest of the variables except itself. As for some of the axes that are obtained towards last have very little variation. So they don’t contribute much, thus, can be discarded without affecting the variability in the image data, hence reducing the dimension (see e.g., [Mar14]).

In PCA image compression, feature selection method is used where we go through the available features of an image and select useful features such as variables or predictors, i.e., correlation of pixel values to the output variables.

PCA removes redundancies and describe the image data with less properties in a way that it performs a linear transformation moving the original image data to a new space spanned by principal component. This done by constructing a new set of properties based on combination of the old properties. The properties that present low variance are considered not useful. PCA looks for properties that has maximal variation across the data to make the principal component space. The eigenvectors found in PCA algorithm are the new set of axes of the principal component. Dimension reduction occurs when the eigenvectors with more variance are chosen but those with less variance are discarded. [Mar14, MP05, JKST19, XCM09, SSA +14, JC16]

6.3. Principal Component Analysis in a Digital Image. We would like to use a color digital image PCA to illustrate dimension change in this section, so we introduce a color digital image. A color digital image is read into a matrix of pixels. We would like to use Karhunen-Loève transform or PCA applied to a digital image data illustrate dimension reduction. Here, an image is represented as a matrix of functions where the entries are pixel values. The following is an example of a matrix representation of a digital image:

\[
\begin{pmatrix}
  f(0,0) & f(0,1) & \cdots & f(0,N-1) \\
  f(1,0) & f(1,1) & \cdots & f(1,N-1) \\
  \vdots & \vdots & \ddots & \vdots \\
  f(M-1,0) & f(M-1,1) & \cdots & f(M-1,N-1)
\end{pmatrix}
\]  

(6.30)

A color image has three components. Thus a color image matrix has three of above image pixel matrices for red, green and blue components and they all appear black and white when viewed “individually.” We begin with the following duality principle, (i) spatial vs (ii) spectral, and we illustrate its role for the redundancy, and for correlation of variables, in the resolution-refinement algorithm for images. Specifically:

(i) **Spatial Redundancy:** correlation between neighboring pixel values.

(ii) **Spectral Redundancy:** correlation between different color planes or spectral bands.

We are interested in removing these redundancies using correlations.
Starting with a matrix representation for a particular image, we then compute the covariance matrix using the steps from (3) and (4) in algorithm above. We then compute the Karhunen-Loève eigenvalues. Next, the eigenvalues are arranged in decreasing order. The corresponding eigenvectors are arranged to match the eigenvalues with multiplicity. The eigenvalues mentioned here are the same eigenvalues \( \lambda_i \) in step 4 above, thus yielding smallest error and smallest entropy in the computation (see e.g., [Son08, JKST19, XCM09, SSA\(^+\)14, JC16]).

The following figure shows the principal components of an image in increasing eigenvalues where the original image is a color png file.

![Figure 6.1. The Original Jorgensen Image](image)

The original file is in red, green and blue color image which had three R, G, B color components. So if \( I \) is the original image it can be represented as

\[
I = w_1 R + w_2 G + w_3 B = f_R + f_G + f_B
\]

(6.31)

where \( w_1, w_2 \) and \( w_3 \) are weights which are determined for different light intensity for color, and \( f_R, f_G \) and \( f_B \) are the three R, G, B components of the form equation (6.30). Each matrix appears black and white when viewed individually. Now, when PCA is performed on the image \( I \), it gives alternative components. Here the original image \( I \) is 6.1. The original image used for 6.1, is in red, green and blue color components which are \( f_R, f_G \), and \( f_B \). Please see [JKST19, MP05].

Here, after PCA transformation, instead of RGB components a new three components are used and this is shown in the Figures 6.2. The principal components of the image are in the order of increasing eigenvalues.

Once we see the principal components of the image, we can notice the significance of the eigenvalue \( \lambda \). With PCA, the principal component corresponding to the largest eigenvalue picks up all the dominant features of the original image in comparison to the second principal component corresponding to the second largest eigenvalue and so on. So the more prominent features of the image is captured in the principal components in the order of decreasing eigenvalues with the correlation of pixels of images.
6.4. A Matrix Example. In this section, we provide a matrix example of PCA to illustrated how the PCA algorithm in section 6.2 works.

Let,

$$X = \begin{bmatrix} 1 & 0 & 0 & 3 \\ -1 & 1 & 1 & 0 \\ -1 & -2 & 4 & -5 \\ 0 & 3 & -1 & 0 \end{bmatrix}$$

be an image matrix. \( (6.32) \)

Following the algorithm of section 6.2, we compute the mean of each column of the above matrix, \( X \). Then we subtract the mean of each column. For matrix \( X \), column 1 has mean of -0.25, column 2 has mean of 0.5, column 3 has mean of 1, and column 4 has mean of -0.5.

Next we compute the covariance matrix of \( X \) namely \( C = \text{cov}(X) \).

$$C = \text{cov}(X) = \begin{bmatrix} 0.9167 & 0.5000 & -1.3333 & 2.5000 \\ 0.5000 & 4.3333 & -4.0000 & 3.6667 \\ -1.3333 & -4.0000 & 4.6667 & -6.0000 \\ 2.5000 & 3.6667 & -6.0000 & 11.0000 \end{bmatrix}$$ \( (6.33) \)

The eigenvalues of the covariance matrix, \( C = \text{cov}(X) \) are \( \lambda_1 = 0 \), \( \lambda_2 = 0.3551 \), \( \lambda_3 = 3.2692 \), and \( \lambda_4 = 17.2924 \). The corresponding eigenvectors of the covariance matrix, \( C = \text{cov}(X) \) are as follows:

For \( \lambda_1 = 0 \), the corresponding eigenvector is

$$v_1 = \begin{bmatrix} 0.4295 \\ 0.5154 \\ 0.7302 \\ 0.1289 \end{bmatrix}$$
For \( \lambda_2 = 0.3551 \), the corresponding eigenvector is

\[
v_2 = \begin{bmatrix} 0.8584 \\ -0.1339 \\ -0.3484 \\ -0.3519 \end{bmatrix}
\]

For \( \lambda_3 = 3.2692 \), the corresponding eigenvector is

\[
v_3 = \begin{bmatrix} 0.2240 \\ -0.7582 \\ 0.3102 \\ 0.5279 \end{bmatrix}
\]

For \( \lambda_4 = 17.2924 \), the corresponding eigenvector is

\[
v_4 = \begin{bmatrix} 0.1685 \\ 0.3762 \\ -0.4992 \\ 0.7622 \end{bmatrix}
\]

We then form a matrix \( A \) which consists of eigenvectors in the columns of the matrix. The eigenvectors point to the direction of the principal components represented by eigenvectors with magnitude of eigenvalues. These \( \lambda \) eigenvalues are the variance of the image data and the magnitude of the eigenvector direction or principal component direction. The first column of matrix \( A \) is the eigenvector \( v_4 \) corresponding to the largest eigenvalue \( \lambda_4 \) and the column of matrix \( A \) is the eigenvector \( v_3 \) corresponding to the largest eigenvalue \( \lambda_3 \) and so on in decreasing order of eigenvalues from the largest to smallest respectively.

\[
A = \begin{bmatrix} 0.1685 & 0.2240 & 0.8584 & 0.4295 \\ 0.3762 & -0.7582 & -0.1339 & 0.5154 \\ -0.4992 & 0.3102 & -0.3484 & 0.7302 \\ 0.7622 & 0.5279 & -0.3519 & 0.1289 \end{bmatrix}
\]

Then we can put the corresponding eigenvalues as diagonal entries in the \( D \) matrix as follows,

\[
D = \begin{bmatrix} 17.2924 & 0 & 0 & 0 \\ 0 & 3.2692 & 0 & 0 \\ 0 & 0 & 0.3551 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

which is the diagonal matrix of eigenvalues. [JS07]

Performing Principal Component Analysis on our matrix \( X \) will result in putting the eigenvectors of \( C = \text{cov}(X) \) in column vector form in decreasing order of eigenvalue. After removing the least significant eigenvector \( v_4 \) we obtain \( A_l \) matrix

\[
A_l = \begin{bmatrix} 0.1685 & 0.2240 & 0.8584 \\ 0.3762 & -0.7582 & -0.1339 \\ -0.4992 & 0.3102 & -0.3484 \\ 0.7622 & 0.5279 & -0.3519 \end{bmatrix}
\]

Then using equation (6.29) we can obtain compressed image \( X' \).
Table 6.1. Table for Barbara PCA compression using different number of principal components used to reconstruct back the image.

| No. of P. C.s | CR  | File Size  | MSE    |
|---------------|-----|------------|--------|
| [0.5ex] 10    | 10.0392 | 160.642 KB | 9.3646 |
| [0.5ex] 20    | 5.0196  | 172.250 KB | 5.5603 |
| [0.5ex] 30    | 3.3464  | 175.908 KB | 3.3438 |
| [0.5ex] 40    | 2.5098  | 176.492 KB | 1.9192 |
| [0.5ex] 50    | 2      | 176.436 KB | 1.0752 |
| [0.5ex] 60    | 1.6678  | 176.949 KB | 0.6133 |
| [0.5ex] 70    | 1.4302  | 178 KB     | 0.3315 |
| [0.5ex] 80    | 1.2518  | 178 KB     | 0.1477 |
| [0.5ex] 90    | 1.1130  | 178 KB     | 0.0370 |
| [0.5ex] 100   | 1      | 178 KB     | 1.0363e-10 |

6.5. **Digital Image Compression Using Principal Component Analysis.**
In this section, we will do PCA image compression using black and white Barbara image using different number of principal components to reconstruct the image. For more information please see, [Tha20, Reb20].

![Figure 6.3. The original Barbara image from the Test image pool in Google.](image)

7. **Frames, projections, and Kaczmarz algorithms**

In this section, we consider certain infinite products of projections. Our framework is motivated by problems in approximation theory, in harmonic analysis, in frame theory, and the context of the classical Kaczmarz algorithm [Kac37]. Traditionally, the infinite-dimensional Kaczmarz algorithm is stated for sequences of vectors in a specified Hilbert space $\mathcal{H}$, (typically, $\mathcal{H}$ is an $L^2$-space.) We shall here formulate it instead for sequences of projections. As a corollary, we get explicit
and algorithmic criteria for convergence of certain infinite products of projections in $\mathcal{H}$.

**Motivation.** Our extension of the Kaczmarz algorithm to sequences of projections is highly nontrivial: while in general convergence questions for infinite products of projections (in Hilbert space) is difficult (see e.g., [Aro50, Rue82, Rue04],

![Reconstructed images of Barbara using 10, 20, 30, 40, 50, 60, 70, 80, 90 and 100 principal components respectively from top to bottom, left to right.](image)
our projection-valued formulation of Kaczmarz’ algorithm yields an answer to this convergence question, as well as a number of applications to stochastic analysis, and to frame-approximation questions in the Hilbert space $L^2(\mu)$, where $\mu$ is in a class of iterated function system (IFS) measures (see [Hut81, Hut95, DJ07, HJW19, JS18]).

**Literature guide:** In addition to Kaczmarz’ pioneering paper [Kac37], there are also the following more recent developments of relevance to our present discussion [Bau95, EP01, Pop01, HS05, KM06, Szw07, Pop10, EN11, CT13, IZ13, LZ15, NSW16, Che18, Pop18, Zha19], as well as [HJW19, HJW18a, HJW18b].

The classical Kaczmarz algorithm is an iterative method for solving systems of linear equations, for example, $Ax = b$, where $A$ is an $m \times n$ matrix. Assume the system is consistent. Let $x_0$ be an arbitrary vector in $\mathbb{R}^n$, and set

$$x_k := \text{argmin} \|x - x_{k-1}\|^2, \ k \in \mathbb{N}; \quad (7.1)$$

where $j = k \mod m$, and $a_j$ denotes the $j^{th}$ row of $A$. At each iteration, the minimizer is given by

$$x_k = x_{k-1} + b_j - \langle a_j, x_{k-1} \rangle \frac{1}{\|a_j\|^2} a_j. \quad (7.2)$$

That is, the algorithm recursively projects the current state onto the hyperplane determined by the next row vector of $A$.

There is a stochastic version of (7.2), where the row vectors of $A$ are selected randomly [SV09].

The Kaczmarz algorithm can be formulated in the Hilbert space setting as follows:

**Definition 7.1.** Let $\{e_j\}_{j \in \mathbb{N}_0}$ be a spanning set of unit vectors in a Hilbert space $\mathcal{H}$, i.e., span $\{e_j\}$ is dense in $\mathcal{H}$. For all $x \in \mathcal{H}$, let $x_0 = e_0$, and set

$$x_k := x_{k-1} + e_k \langle e_k, x - x_{k-1} \rangle . \quad (7.3)$$

We say the sequence $\{e_j\}_{j \in \mathbb{N}_0}$ is effective if $\|x_k - x\| \to 0$ as $k \to \infty$, for all $x \in \mathcal{H}$.

**Remark 7.2.** A key motivation for our present analysis is an important result by Stanisław Kwapień and Jan Mycielski [KM06], giving a criterion for stationary sequences (referring to a suitable $L^2(\mu)$) to be effective.

**Observation.** Equation (7.3) yields, by forward induction:

$$x - x_k = (1 - P_k) (x - x_{k-1}) = (1 - P_k) (1 - P_{k-1}) (x - x_{k-2}) \ldots = (1 - P_k) (1 - P_{k-1}) \cdots (1 - P_0) x,$$

where $P_j$ is the orthogonal projection onto $e_j$. Throughout, we shall use “$1$” also for the identity operator. This motivates the following:

**Definition 7.3.** A system $\{P_j\}_{j \in \mathbb{N}_0}$ of orthogonal projections in $\mathcal{H}$ is said to be effective if

$$T_n := (1 - P_n) (1 - P_{n-1}) \cdots (1 - P_0) \xrightarrow{\text{strong}} 0, \quad (7.4)$$

i.e., $T_n$ converges to zero as $n \to \infty$ in the strong operator topology.
Assume (7.5) holds. Then, for all $y \in \mathcal{H}$, and all $j \in \mathbb{N}$. Then the system $\{P_j\}_{j \in \mathbb{N}_0}$ is effective.

**Proof.** Assume (7.5) holds. Then, for all $x \in \mathcal{H}$,

\[
\| (1 - P_n) (1 - P_{n-1}) \cdots (1 - P_0) x \|^2 \\
= \| (1 - P_{n-1}) \cdots (1 - P_0) x \|^2 - \| P_n (1 - P_{n-1}) \cdots (1 - P_0) x \|^2 \\
\leq \| (1 - P_{n-1}) \cdots (1 - P_0) x \|^2 - c \| (1 - P_{n-1}) \cdots (1 - P_0) x \|^2 \\
= (1 - c) \| (1 - P_{n-1}) \cdots (1 - P_0) x \|^2 \\
\leq (1 - c)^2 \| (1 - P_{n-2}) \cdots (1 - P_0) x \|^2 \\
\vdots \\
\leq (1 - c)^n \| (1 - P_0) x \|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

Remark 7.5. Condition (7.5) may be replaced by

\[
\| P_j (1 - P_{j-1}) y \|^2 \geq c_j \| (1 - P_{j-1}) y \|^2, \quad \forall y \in \mathcal{H},
\]
with $0 < c_j < 1$. Then the proof of Proposition 7.4 is modified as:

\[
\| (1 - P_n) (1 - P_{n-1}) \cdots (1 - P_0) x \|^2 \\
\leq (1 - c_n) \| (1 - P_{n-1}) \cdots (1 - P_0) x \|^2 \\
\vdots \\
\leq (1 - c_n) \cdots (1 - c_1) \| (1 - P_0) x \|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

**Theorem 7.6.** Let $\{P_j\}_{j \in \mathbb{N}_0}$ be a sequence of orthogonal projections in a Hilbert space $\mathcal{H}$. Set

\[
T_n = (1 - P_n) (1 - P_{n-1}) \cdots (1 - P_0), \\
Q_n = P_n (1 - P_{n-1}) \cdots (1 - P_0),
\]

where $Q_0 = P_0$.

For all $n \in \mathbb{N}$, we have

\[
\| x \|^2 = \| T_n x \|^2 + \sum_{k=0}^{n} \| Q_k x \|^2, \quad x \in \mathcal{H}.
\]

Hence $T_n \xrightarrow{\ast} 0$ if and only if

\[
\| x \|^2 = \sum_{j \in \mathbb{N}_0} \| Q_j x \|^2, \quad x \in \mathcal{H}.
\]

More precisely, (7.8) means that,

\[
\langle x, y \rangle = \sum_{j \in \mathbb{N}_0} \langle Q_j x, Q_j y \rangle, \quad x, y \in \mathcal{H}.
\]

In particular,

\[
\| x \|^2 = \sum_{j \in \mathbb{N}_0} \| Q_j x \|^2, \quad x \in \mathcal{H}.
\]
Proof. Note that
\[
\|T_n x\|^2 = \|(1 - P_n) (1 - P_{n-1}) \cdots (1 - P_0) x\|^2 \\
= \|(1 - P_{n-1}) \cdots (1 - P_0) x\|^2 - \|P_n (1 - P_{n-1}) \cdots (1 - P_0) x\|^2 \\
= \|T_{n-1} x\|^2 - \|Q_n x\|^2 \\
= \|T_{n-2} x\|^2 - \|Q_{n-1} x\|^2 - \|Q_n x\|^2 \\
\vdots \\
= \|(1 - P_0) x\|^2 - \|Q_1 x\|^2 - \cdots - \|Q_{n-1} x\|^2 - \|Q_n x\|^2 \\
= \|x\|^2 - \|Q_0 x\|^2 - \|Q_1 x\|^2 - \cdots - \|Q_{n-1} x\|^2 - \|Q_n x\|^2.
\]
Therefore
\[
T_n \xrightarrow{s} 0 \iff \|x\|^2 = \sum_{j \in \mathbb{N}_0} \|Q_j x\|^2.
\]
\[\square\]

Remark 7.7. The system of operators \(\{Q_j\}_{j \in \mathbb{N}_0}\) in Theorem 7.6 has frame-like properties, see (7.8)–(7.10). Specifically, the mapping
\[
\mathcal{H} \ni x \xmapsto{V} (Q_j x) \in l^2(\mathbb{N}_0) \otimes \mathcal{H}
\]
plays the role of an analysis operator, and the synthesis operator \(V^*\) is given by
\[
\begin{align*}
\forall \xi \in l^2(\mathbb{N}_0) \otimes \mathcal{H} & \ni \xi \xmapsto{V^*} \sum_{j \in \mathbb{N}_0} Q_j^* \xi_j.
\end{align*}
\]
Note that \(1 = V^* V\), and (7.10) is the generalized Parseval identity.

7.1. The case of rank-1 projections in \(\mathcal{H}\). Let \(\{P_j\}_{j \in \mathbb{N}_0}\) be a system of rank-1 projections, i.e., \(P_j = |e_j\rangle\langle e_j|\), where \(\{e_j\}_{j \in \mathbb{N}_0}\) is a set of unit vectors in \(\mathcal{H}\). When the system \(\{e_j\}\) is independent, then the corresponding family of projections \(P_j = |e_j\rangle\langle e_j|\) is non-commutative.

Corollary 7.8. Suppose all the \(P_j\)'s are of rank-1, i.e., \(P_j = |e_j\rangle\langle e_j|\) where \(\{e_j\}_{j \in \mathbb{N}_0}\) is a set of unit vectors in \(\mathcal{H}\). Assume \(\{P_j\}_{j \in \mathbb{N}_0}\) is effective (Definition 7.3). Set
\[
\begin{align*}
g_0 &= e_0, \\
g_n &= (1 - P_0) (1 - P_1) \cdots (1 - P_{n-1}) e_n, \quad n \in \mathbb{N}.
\end{align*}
\]
Then \(\{g_n\}_{n \in \mathbb{N}_0}\) is a Parseval frame in \(\mathcal{H}\).

Specifically, we have
\[
\begin{align*}
g_0 &= e_0 \\
g_n &= e_n - \sum_{j=0}^{n-1} \langle e_j, e_n \rangle g_j, \quad n \in \mathbb{N}.
\end{align*}
\]
(7.11)

Proof. Recall that
\[
Q_n = P_n (1 - P_{n-1}) \cdots (1 - P_0),
\]
see (7.7).

Since \(P_n\) has rank-1, it follows that \(Q_n\) has the form
\[
Q_n = |e_n\rangle\langle g_n|.
\]
for some \( g_n \in \mathcal{H} \), which in turn is given by
\[
g_n = Q_n^* e_n = (1 - P_0) (1 - P_1) \cdots (1 - P_{n-1}) P_n e_n
\]
\[
= (1 - P_0) (1 - P_1) \cdots (1 - P_{n-1}) e_n
\]
\[
= (1 - P_0) (1 - P_1) \cdots (1 - P_{n-2}) (e_n - (e_{n-1}, e_n) e_{n-1})
\]
\[
= (1 - P_0) (1 - P_1) \cdots (1 - P_{n-3}) e_n - (e_{n-2}, e_n) g_{n-2} - (e_{n-1}, e_n) g_{n-1}
\]
\[
\vdots
\]
\[
= e_n - \sum_{j=0}^{n-1} (e_j, e_n) g_j.
\]

Now,
\[
\|x\|^2 = \sum_{j \in \mathbb{N}_0} \|Q_j x\|^2 = \sum_{j \in \mathbb{N}_0} |(g_j, x)|^2, \quad x \in \mathcal{H}.
\]

**Corollary 7.9.** The system \( \{|e_j\rangle, \langle e_j|\}_{j \in \mathbb{N}_0} \) is effective iff \( \{g_j\}_{j \in \mathbb{N}_0} \) (see (7.11)) is a Parseval frame in \( \mathcal{H} \).

**Example 7.10.** Consider a positive definite function \( G \times G \xrightarrow{K} \mathbb{C} \) on a lca group \( G \), where \( K(x, y) = K(x - y) \) and \( K(0) = 1 \). Let \( \mathcal{H}_K \) be the associated RKHS.

Fix a discrete subset \( \{x_j\}_{j \in \mathbb{N}_0} \subset G \), and define
\[
P_j = 1 - |K_{x_j}\rangle \langle K_{x_j}|, \quad j \in \mathbb{N}_0.
\]

**Corollary 7.11.** Assume there exists \( 0 < c < 1 \) such that
\[
K(x_j - x_{j-1})^2 \leq 1 - c, \quad j \in \mathbb{N}.
\]

Then the system \( \{P_j\}_{j \in \mathbb{N}_0} \) in (7.12) is effective.

For the operator valued frame \( \{Q_j\}_{j \in \mathbb{N}_0} \) in Theorem 7.6, it holds that
\[
\text{rank}(Q_j) \leq 2, \quad j \in \mathbb{N}_0.
\]

**Proof.** In the current setting, condition (7.5) in Proposition 7.4 translates to
\[
\| (1 - |K_{x_j}\rangle \langle K_{x_j}|) K_{x_{j-1}} \|_{\mathcal{H}_K}^2 \geq c \| K_{x_{j-1}} \|_{\mathcal{H}_K}^2 = c
\]
\[
\downarrow
\]
\[
\| K_{x_{j-1}} - K(x_j - x_{j-1}) K_{x_j} \|_{\mathcal{H}_K}^2 \geq c
\]
\[
\downarrow
\]
\[
K(x_j - x_{j-1})^2 \leq 1 - c.
\]

Therefore \( \{P_j\}_{j \in \mathbb{N}_0} \) is effective, provided that (7.13) is satisfied.

Let \( Q_j \) be as in Theorem 7.6. That is, for all \( f \in \mathcal{H}_K \) (see Corollary 5.2),
\[
Q_j f = P_j (1 - P_{j+1}) \cdots (1 - P_0) f
\]
\[
= (1 - |K_{x_j}\rangle \langle K_{x_j}|) (|K_{x_{j-1}}\rangle \langle K_{x_{j-1}}|) \cdots (|K_{x_0}\rangle \langle K_{x_0}|) f
\]
\[
= K_{x_{j-1}} K(x_j - x_{j-1}) K(x_j - x_{j-1}) \cdots K(x_1 - x_0) f(x_0)
\]
\[
- K_{x_j} K(x_{j-1} - x_{j-2}) K(x_{j-1} - x_{j-2}) \cdots K(x_1 - x_0) f(x_0).
\]
In particular,

\[ Q_j f \in \text{span} \{ K_{x_{j-1}}, K_{x_j} \} \]

and so \( \text{rank} (Q_j) \leq 2. \) \( \square \)

## 8. Paley Wiener Spaces

**Definition 8.1** (Paley Wiener spaces (see, e.g., [DS17])). Let \( \Omega \) be a bounded subset of \( \mathbb{R}^d \), and let \( L^2 = L^2 (\lambda_d) = L^2 (\mathbb{R}^d, \lambda_d) \) with \( \lambda_d = \) the usual Lebesgue measure on \( \mathbb{R}^d \). Set

\[ \text{PW} (\Omega) := \{ f \in L^2 : \text{supp}(\hat{f}) \subseteq \Omega \} \]

where

\[ \hat{f} (\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f (x) \, dx \] (8.1)

is the Fourier transform, and \( x = (x_1, \cdots, x_d) \in \mathbb{R}^d, \, dx = \lambda_d (x) \).

**Lemma 8.2.** \( \text{PW} (\Omega) \) is a RKHS.

**Proof.** Let \( f \in \text{PW} (\Omega) \). Since \( \Omega \) is bounded and \( \lambda_d (\Omega) < \infty \), then

\[ f (x) = \frac{1}{(2\pi)^d} \int_{\Omega} e^{i\xi \cdot x} \hat{f} (\xi) \, d\xi \] (8.2)

where \( \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d, \, d\xi = \lambda_d (\xi) \).

For \( f \in \text{PW} (\Omega) \), we have

\[ \| f \|_{\text{PW}(\Omega)}^2 := \| f \|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} |f (x)|^2 \, dx \]

\[ = \int_{\mathbb{R}^d} |\hat{f} (\xi)|^2 \, d\xi = \int_{\Omega} |\hat{f}|^2 \, d\lambda < \infty. \] (8.3)

From (8.2),

\[ |f (x)| \leq \frac{1}{(2\pi)^d} \lambda_d (\Omega) \left( \int_{\Omega} |\hat{f}|^2 \, d\lambda \right)^{1/2} = C_d \| f \|_{\text{PW}(\Omega)} \]

where \( C_d := \frac{\lambda_d (\Omega)}{(2\pi)^d} \). That is, point evaluation at every \( x \in \mathbb{R}^d \) is a bounded linear functional on \( \text{PW} (\Omega) \).

Moreover, the reproducing kernel is given by

\[ K_{\text{PW}(\Omega)} (x, y) := \hat{\chi}_\Omega (x-y), \quad x, y \in \mathbb{R}^d, \]

where \( \hat{\chi}_\Omega \) denotes the Fourier transform of the indicator function

\[ \chi_\Omega (\xi) = \begin{cases} 1 & \xi \in \Omega \\ 0 & \xi \in \mathbb{R}^d \setminus \Omega. \end{cases} \]

\( \square \)

Proposed duality:

\[ \begin{array}{c}
\text{countably discrete subset } V \subset \mathbb{R}^d \text{ with} \\
\text{generalized interpolation properties} \\
\hline
\end{array} \quad \begin{array}{c}
\Omega \subset \mathbb{R}^d \\
\lambda_d (\Omega) < \infty
\end{array} \quad (8.4) \]
When $\Omega$ and $V$ are as above, see (8.4), consider the question of existence of finite constants $0 < A \leq B < \infty$ such that
\[ A \|f\|_{PW(\Omega)}^2 \leq \sum_{x \in V} |f(x)|^2 \leq B \|f\|_{PW(\Omega)}^2, \quad \forall f \in PW(\Omega); \tag{8.5} \]
or with a weight function $w : V \to \mathbb{R}_+$,
\[ A \|f\|_{PW(\Omega)}^2 \leq \sum_{x \in V} w(x) |f(x)|^2 \leq B \|f\|_{PW(\Omega)}^2, \quad \forall f \in PW(\Omega). \tag{8.6} \]

**Definition 8.3.** Give $\Omega \subset \mathbb{R}^d$, $\lambda_d(\Omega) < \infty$, set
\[ SAMP(\Omega) := \{ V : V \subset \mathbb{R}^d \text{ countably discrete, and } (8.5) \text{ or } (8.6) \text{ holds for some finite constants } A, B, 0 < A \leq B < \infty \}; \]
\[ FREBD(V) := \{ \Omega : \lambda_d(\Omega) < \infty, \text{ and } (8.5) \text{ or } (8.6) \text{ holds for some constants } A, B, 0 < A \leq B < \infty \}. \]

**Problem.** (i) Given $\Omega$, find $SAMP(\Omega)$; (ii) Given $V$, find $FREBD(V)$.

### 9. Gaussian Processes and Gaussian Hilbert Spaces

By a probability space, we mean a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where
- $\Omega$: set of sample points,
- $\mathcal{F}$: $\sigma$-algebra of events (subsets of $\Omega$),
- $\mathbb{P}$: a probability measure defined on $\mathcal{F}$.

A random variable
\[ K : \Omega \to \mathbb{R} \ (\mathbb{C}, \text{ or a Hilbert space}) \tag{9.1} \]
is a measurable function defined on $(\Omega, \mathcal{F})$, i.e., we require that for Borel sets $B$ (in $\mathbb{R}$, or $\mathbb{C}$), and cylinder sets (referring to a fixed Hilbert space $\mathcal{H}$) we have $K^{-1}(B) \in \mathcal{F}$ where
\[ K^{-1}(B) = \{ \omega \in \Omega : K(\omega) \in B \}. \tag{9.2} \]

The distribution $\mu_K$ of $K$ is the measure
\[ \mu_K := \mathbb{P} \circ K^{-1}. \tag{9.3} \]

If $\mu_K$ is Gaussian, we say that $K$ is a Gaussian random variable. A Gaussian process is a system $\{K_x : x \in X\}$ of random variables (refer to $(\Omega, \mathcal{F}, \mathbb{P})$), indexed by some set $X$, in this case $X$.

Here we shall restrict to the case of a Gaussian process, and we shall assume
\[ \mathbb{E}(K_x) = 0, \quad \forall x \in X; \tag{9.4} \]
where
\[ \mathbb{E}(\cdot) = \int_{\Omega} \cdot \, d\mathbb{P} \tag{9.5} \]
denotes expectation w.r.t. $\mathbb{P}$.

If $\mu_K \in N(0, 1)$, i.e.,
\[ \mu_K(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \quad t \in \mathbb{R}, \tag{9.6} \]
we say that $K$ (or $\mu_K$) is a standard Gaussian.
Lemma 9.1. Let \( \{ Z_n \}_{n \in \mathbb{N}_0} \) be a system of independent identically distributed \( N(0,1) \)s (i.i.d \( N(0,1) \)). Let \( \mathcal{H} \) be a Hilbert space, and \( \{ Q_n \}_{n \in \mathbb{N}_0} \) a system of projections as in Theorem 7.6, i.e.,

\[
\sum_{n \in \mathbb{N}_0} \langle Q_n u, Q_n v \rangle_{\mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}, \quad \forall u, v \in \mathcal{H}.
\]

Then

\[
W(\cdot) = W(Q,\mathcal{H})(\cdot) := \sum_{n \in \mathbb{N}_0} Q_n Z_n(\cdot)
\]

defines an operator valued Gaussian process, and

\[
E(\langle W(\cdot) u, W(\cdot) v \rangle_{\mathcal{H}}) = \langle u, v \rangle_{\mathcal{H}}, \quad \forall u, v \in \mathcal{H}.
\]

Proof sketch. Fix \( u, v \in \mathcal{H} \); then

\[
\text{LHS}_{(9.9)} = \sum_{n_0 \times N_0} \sum_{n_0 \times N_0} \langle Q_n u, Q_m v \rangle_{\mathcal{H}} E(Z_n Z_m) = \delta_{n,m}
\]

by (9.7)

\[
= \sum_{n \in \mathbb{N}_0} \langle Q_n u, Q_n v \rangle_{\mathcal{H}}
\]

by (9.7)

\[
= \langle u, v \rangle_{\mathcal{H}}.
\]

\( \square \)

Corollary 9.2. Let \( \{ Q_n \}_{n \in \mathbb{N}_0} \) be an effective system in \( \mathcal{H}_K \), where \( K : X \times X \to \mathbb{R} \) is a given p.d. kernel and \( \mathcal{H}_K \) the associated RKHS. Then \( W \) from (9.8) has the property that

\[
K(x,y) = E\left( \langle W(\cdot) K_x, W(\cdot) K_y \rangle_{\mathcal{H}_K} \right).
\]

We recall the following theorem of Kolmogorov. It states that there is a 1-1 correspondence between p.d. kernels on a set and mean zero Gaussian processes indexed by the set. One direction is easy, and the other is the deep part:

Theorem 9.3 (Kolmogorov). Let \( X \) be a set. A function \( K : X \times X \to \mathbb{C} \) is positive definite if and only if there is a Gaussian process \( \{ W_x \}_{x \in X} \) realized in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) with mean zero, such that

\[
K(x,y) = E\left[ \sum_{i=1}^n c_i W_{x_i} \right].
\]

Proof. We refer to [PS75] for the non-trivial direction. To stress the idea, we include a proof of the easy part of the theorem: Assume (9.10). Let \( \{ c_i \}_{i=1}^n \subset \mathbb{C} \) and \( \{ x_i \}_{i=1}^n \subset X \), then we have

\[
\sum_i \sum_{j} c_i c_j K(x_i, x_j) = E\left[ \left| \sum_{i=1}^n c_i W_{x_i} \right|^2 \right] \geq 0,
\]

i.e., \( K \) is p.d. \( \square \)

Let \( (X, \mathcal{B}, \nu) \) be a \( \sigma \)-finite measure space, and let \( \mathcal{B}_{\text{fin}} = \{ E \in \mathcal{B} : \nu(E) < \infty \} \). Below we consider the following kernel \( K \) on \( \mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}} \):

\[
K(A,B) = \nu(A \cap B), \quad A, B \in \mathcal{B}_{\text{fin}}
\]

and let \( \mathcal{H}_{K(\nu)} \) denote the associated RKHS.

Proposition 9.4.

1. \( K = K(\nu) \) in (9.11) is positive definite.
(2) $K^{(\nu)}$ is the covariance kernel for the stationary Wiener process $W = W^{(\nu)}$ indexed by $\mathcal{B}_{\text{fin}},$ i.e., Gaussian, mean zero, and
\[ \mathbb{E}(W_A W_B) = K^{(\nu)}(A, B) = \nu(A \cap B). \] (9.12)

(3) If $f \in L^2(\nu),$ and $W_f = \int_X f(x) \, dW_x$ denotes the corresponding Itô-integral, then
\[ \mathbb{E}(|W_f|^2) = \int_X |f|^2 \, d\nu; \]
in particular, if $f = \sum_i \alpha_i \chi_{A_i},$ then
\[ \sum_i \sum_j \alpha_i \alpha_j K^{(\nu)}(A_i, A_j) = \int_X \left| \sum_i \alpha_i \chi_{A_i} \right|^2 \, d\nu. \]

(4) The RKHS $\mathcal{H}_{K^{(\nu)}}$ of the positive definite kernel in (9.11) consists of functions $F$ on $\mathcal{B}_{\text{fin}}$ represented by $f \in L^2(\nu)$ via
\[ F(A) = F_f(A) = \int_A f \, d\nu, \quad A \in \mathcal{B}_{\text{fin}}; \] (9.13)
and
\[ \|F_f\|_{\mathcal{H}_{K^{(\nu)}}}^2 = \|f\|_{L^2(\nu)}^2 = \int_X |f|^2 \, d\nu. \] (9.14)

(5) The map specified by
\[ \Psi \left( K^{(\nu)}(\cdot, A) \right) = \Psi(\nu((\cdot) \cap A)) = \chi_A, \quad \forall A \in \mathcal{B}_{\text{fin}} \] (9.15)
extends by linearity and by limits to an isometry
\[ \Psi : \mathcal{H}_{K^{(\nu)}} \rightarrow L^2(\nu). \] (9.16)
More generally if $F_f \in \mathcal{H}(K^{(\nu)})$ is as in (9.13), then $\Psi(F_f) = f \in L^2(\nu).$

Proof. The details can be found at various places in the literature; see e.g., [AJ12, JS18, JT19b, JT19a].

\[ \square \]

Proposition 9.5. Let $\mathcal{B}_{\text{fin}} \times \mathcal{B}_{\text{fin}} \xrightarrow{K} \mathbb{R}$ be the p.d. kernel as in (9.11), and $\mathcal{H}_{K^{(\nu)}}$ the RKHS of $K.$ Suppose $\{Q_n\}_{n \in \mathbb{N}_0}$ is an effective system in $\mathcal{H}_{K^{(\nu)}},$ and let $\Psi$ be the isometry specified in (9.15)–(9.16). Then
\[ \{\Psi Q_n\}_{n \in \mathbb{N}_0} \]
is effective in the closed subspace
\[ \Psi(\mathcal{H}_{K^{(\nu)}}) \subset L^2(\Omega, \mathbb{P}). \]

Proof. For all $F, G \in \mathcal{H}_{K^{(\nu)}},$ we have
\[ \mathbb{E}[\Psi(F) \Psi(G)] = \langle F, G \rangle_{\mathcal{H}_{K^{(\nu)}}} \]
\[ = \sum_{n \in \mathbb{N}_0} \langle Q_n F, Q_n G \rangle_{\mathcal{H}_{K^{(\nu)}}} \]
\[ = \sum_{n \in \mathbb{N}_0} \mathbb{E}[\Psi(Q_n F) \Psi(Q_n G)]. \]
\[ \square \]
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