Simultaneous recovery of a locally rough interface and the embedded obstacle with its surrounding medium

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Abstract
Consider the problem of scattering of time-harmonic point sources by an infinite locally rough interface with bounded obstacles embedded in the lower half-space. The model problem is first reduced to an equivalent integral equation formulation defined in a bounded domain, where the well-posedness is obtained in \(L^p\) by the classical Fredholm theory. Then a global uniqueness theorem is proved for the inverse problem of recovering the locally rough interface, the embedded obstacles and the wave number in the lower-half space by means of near-field measurements above the interface.

Keywords: inverse acoustic scattering, Lippmann–Schwinger equation, uniqueness, rough interface, embedded obstacle

(Some figures may appear in colour only in the online journal)

1. Introduction
This paper is concerned with the two-dimensional inverse scattering of time-harmonic acoustic point sources by a locally rough interface with obstacles embedded in the lower half-space. This type of problems can find applications in diverse scientific areas such as radar, underwater exploration and non-destructive testing, where the shape, location and boundary conditions of...
for a continuous impedance function measurement of the scattered field made on some sub-domain of the upper-half space. Both the interface and embedded obstacles need to be simultaneously reconstructed from the measurements of the scattered field made on some sub-domain of the upper-half space.

Let the scattering interface be denoted by a smooth curve \( \Gamma \) \( \{ (x_1, x_2) \in \mathbb{R}^2 : x_2 = f(x_1) \} \), where \( f \) is assumed to be a Lipschitz continuous function with compact support. This means that \( \Gamma \) is just a local perturbation of the planar interface \( \Gamma_0 := \{ (x_1, x_2) \in \mathbb{R}^2 : x_2 = f(x_1) \} \) and the lower half-space \( \Omega_2 := \{ (x_1, x_2) \in \mathbb{R}^2 : x_2 < f(x_1) \} \), where a bounded obstacle \( D \) with the \( C^2 \)-boundary \( \partial D \) is assumed to be embedded into \( \Omega_2 \). We refer the reader to Figure 1 for the problem geometry.

Consider the incident field \( u^{inc}(\cdot, x_s) \) to be generated by a point source

\[
\Phi_{s_1}(x, x_s) := \frac{1}{4\pi} H_0^{(1)}(\kappa_1 |x - x_s|), \quad x_s \in \Omega_1,
\]

which is the fundamental solution of the two-dimensional Helmholtz equation satisfying \( \Delta \Phi_{s_1}(x, x_s) + \kappa_1^2 \Phi_{s_1}(x, x_s) = -\delta_s(x) \) in the distributional sense. Here, \( \delta_s(x) := \delta(x - x_s) \) is the Dirac delta distribution. Then the scattering of \( u^{inc}(\cdot, x_s) \) by the scatterers \( (\Gamma, D) \) can be reduced to the problem of seeking a scattered field \( u^s(\cdot, x_s) \) in \( \Omega_1 \) and a transmitted field \( u^t(\cdot, x_s) \) in \( \Omega_2 \setminus D \) satisfying that

\[
\begin{align*}
\Delta u^s + \kappa_2^2 u^s &= 0 & \text{in } \Omega_1, \\
\Delta u^t + \kappa_2^2 u^t &= 0 & \text{in } \Omega_2 \setminus \overline{D}, \\
u^s|_+ - u^t|_- &= -i u^{inc} & \text{on } \Gamma, \\
\partial_n u^s|_+ - \partial_n u^t|_- &= -\partial_n u^{inc} & \text{on } \Gamma, \\
Bu^s &= 0 & \text{on } \partial D, \\
\lim_{r \to \infty} \left( \frac{\partial u^s}{\partial r} - \kappa u^t \right) &= 0 & \text{for } r = |x|,
\end{align*}
\]

where \( \cdot \) indicates the limits of \( \cdot \) approaching \( \Gamma \) from \( \Omega_1 \) and \( \Omega_2 \setminus \overline{D} \), respectively, \( \kappa \) is the wavenumber defined by \( \kappa := \kappa_1 > 0 \) in \( \Omega_1 \) and \( \kappa := \kappa_2 > 0 \) in \( \Omega_2 \), and \( B \) stands for the boundary condition on \( \partial D \) satisfying \( Bu^s := u^s \) if \( D \) is a sound-soft obstacle, and \( Bu^s := \partial_n u^s + i\kappa u^s \) for a continuous impedance function \( \lambda(x) \geq 0 \) if \( D \) is an imperfect obstacle. Here, \( \nu = \nu(x) \) is the upward normal vector directing into \( \Omega_1 \) for \( x \in \Gamma \), and is the outward normal vector directing into \( \Omega_2 \setminus \overline{D} \) for \( x \in \partial D \), and \( \partial_n \) stands for the normal derivative. Moreover, the last
condition in (1.1) is the well-known Sommerfeld radiation condition which holds uniformly for all directions \( \hat{x} = x/|x| \in S^1 := \{ x \in \mathbb{R}^2 : |x| = 1 \} \).

If \( D \) is a penetrable obstacle, then the scattering of \( u^{inc}(\cdot, x_0) \) by \((\Gamma, D)\) can be formulated as finding the scattered field \( u^s(\cdot, x_0) \) in \( \Omega_1 \) and the transmitted field \( u^t(\cdot, x_0) \) in \( \Omega_2 \) satisfying that

\[
\begin{align*}
\Delta u^t + \kappa_2^2 u^t &= 0 & \text{in } \Omega_1, \\
\Delta u^s + \kappa_2^2 n u^s &= 0 & \text{in } \Omega_2, \\
|u^s|^+ - |u^s|^-= -u^{inc} & \quad \text{on } \Gamma, \\
\partial_n u^s|^+ - \partial_n u^{inc} &= 0 & \quad \text{on } \Gamma, \\
\lim_{r \to \infty} r^2 \left( \frac{\partial u^s}{\partial r} - i \kappa u^s \right) &= 0 & \quad \text{for } r = |x|, 
\end{align*}
\]

where \( n = n(x) \in L^\infty(\mathbb{R}^2) \) is the refractive index with \( \text{Re}(n) > 0, \text{Im}(n) \geq 0 \) and \( n = 1 \) in \( \mathbb{R}^2 \backslash \overline{D} \), and \( u^s \) satisfies the Sommerfeld radiation condition uniformly for all directions \( \hat{x} \in S^1 \).

For convenience, for problems (1.1) and (1.2), let \( u(\cdot, x_0) \) denote the total field consisting of the point source \( u^{inc}(\cdot, x_0) \) and the scattered field \( u^s(\cdot, x_0) \) in \( \Omega_1 \), and \( u(\cdot, x_0) := u^s(\cdot, x_0) \) denote the transmitted field in \( \Omega_2 \setminus D \).

Given the interface \( \Gamma \), the embedded obstacle \( D \) or the refractive index \( n \), and the incident wave \( u^{inc} \), the forward problem is to determine the distribution of the scattered field \( u^s \) in \( \mathbb{R}^2 \). If \( D = \emptyset \), many works have been done on the well-posedness of problem (1.1) or (1.2) in the literature; see, e.g., [9, 37] for the variational method and [14, 15, 26, 32, 40] for the integral equation method. In particular, we refer to [1, 8, 16, 19, 20, 25, 30, 31, 34] for the case with a planar surface \( \Gamma \). Different from all previous works, in the first part of the present paper we will propose a novel technique to establish the existence of a unique solution to problem (1.1) or (1.2) with \( D = \emptyset \) into an equivalent Lippmann–Schwinger type integral equation defined in a bounded domain for which the well-posedness of the problem follows from a direct application of the classical Fredholm alternative. Then the existence of solutions to problem (1.1) or (1.2) follows by using the integral equation technique based on the background Green’s function in two-layered medium. One advantage of the method is to avoid the discussion of integral operators in an unbounded domain, which can lead further to the \( L^p \) estimate of solutions of problem (1.1) or (1.2) when the incident field is induced by a family of hyper-singular point sources.

In the second part of the present paper, we study the inverse problem of determining the locally rough interface \( \Gamma \), the wavenumber \( \kappa_2 \) and the embedded obstacle \((D, B)\) or the refractive index \( n \) from the measurements made in \( \Omega_1 \). To the best of our knowledge, no result is available in the literature for the simultaneous recovery of \( \Gamma \) and \((D, B)\). If \( \Gamma \) is a planar interface with \( D \) embedded in \( \Omega_2 \), or \( \Gamma \) is a locally rough surface without \( D \), or \( \Gamma \) is an impenetrable surface, many works have been done for the inverse problems. We refer to [1, 3–6, 8, 10, 16–20, 25, 27–31, 34, 38] and reference therein there. Precisely, it was shown [13] that a uniqueness result was first established for determining a Dirichlet surface by assuming that the homogeneous medium above the surface is lossy. It was then shown in [31] that an embedded electromagnetic obstacle was uniquely determined in a two-layered lossy medium separated by a planar surface. A similar result was obtained in [29] for Maxwell’s equations on determining an interface with a perfectly conducting obstacle embedded in \( \Omega_1 \) if the background medium is lossy. Furthermore, the uniqueness result was also obtained in [38] for inverse acoustic scattering by an embedded penetrable obstacle in the lower-half space, where the background
medium is allowed to be lossless. Moreover, many numerical methods have also been proposed for the inverse problems such as the MUSIC-type method [1], the algorithms based on transformed field expansions [3, 4], Newton-type algorithms [5, 36, 41], the Kirsch–Kress scheme [6, 28], qualitative methods [17, 22], the asymptotic factorization method [19], the time-domain singular source method in [24], and direct sampling methods [25].

Partially motivated by [39] for the inverse scattering by bounded obstacles, we investigate the unique determination of \((\Gamma, D, B, \kappa_2)\) and \((\Gamma, n, \kappa_2)\) by the scattered fields taken on a finite line segment of \(\Omega_1\). To this end, we will first establish uniform a priori estimates of solutions of problem (1.1) or (1.2) when \(u^{\text{inc}}\) are induced by a family of hyper-singular point sources \(\partial\Phi_{\kappa_1}(\cdot, x_\ell)\) for \(\ell = 1, 2\) with \(x_\ell\) approaching the interface. Then the two group of solutions of problem (1.1) or (1.2) associated with two different locally interfaces are coupled in a sufficiently small domain as an interior transmission problem (ITP). The uniform \(L^2\)-regularity of the solutions will be obtained as \(x_\ell\) approaches the interface by the well-posedness of the ITP, which will leads to a contradiction, whence the uniqueness of \(\Gamma\) follows. One advantage of this technique is that the medium surrounding \(D\) can also be simultaneously recovered in view of same measurements. Finally, the inverse problem is reduced to the case of recovering the embedded obstacle into a known layered medium, where the uniqueness result directly follows from the standard discussion (cf [2, 12]).

The remaining of this paper is built up as follows. In section 2, we briefly introduce some necessary function spaces, some useful notations and the background Green’s functions associated with two special rough interfaces. In section 3, we establish the well-posedness of problem (1.1) and (1.2), based on a novel technique of reducing the model problem into a Lippmann–Schwinger type integral equation in a bounded domain. In section 4, we prove a global uniqueness theorem for the inverse problem of simultaneously recovering locally rough interfaces and the embedded obstacles with its surrounding homogeneous medium.

2. Preliminaries

2.1. Some useful function spaces

Let \(\Omega\) be a bounded domain of \(\mathbb{R}^2\) with a Lipschitz boundary \(\partial\Omega\). Let \(W^{m,p}(\Omega)\) denote the usual Sobolev space with index \(m \in \mathbb{N}\) and \(p \in [1, \infty)\), equipped with the norm

\[
\|u\|_{m,p} := \left( \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}.
\]

For \(p = 2\), we also write \(H^m(\Omega)\) for \(W^{m,2}(\Omega)\), which is a Hilbert space under the inner product

\[
(u, v)_m := \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}.
\]

For \(m = 0\), \(W^{0,p}(\Omega)\) is reduced to the \(L^p(\Omega)\) space consisting of all \(L^p\)-integrable functions on \(\Omega\). Moreover, we also introduce the following function space

\[
H^1_\Delta(\Omega) := \{u \in \mathcal{D}'(\Omega) | u \in H^1(\Omega), \Delta u \in L^2(\Omega)\},
\]

which is a Hilbert space with respect to the inner product

\[
(u, v)_{H^1_\Delta(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} + (\Delta u, \Delta v)_{L^2(\Omega)} \quad \text{for } u, v \in H^1_\Delta(\Omega),
\]

\[
4
\]
where $\mathcal{D}'(\Omega)$ denotes the set consisting of all distributions defined on $C_0^\infty(\Omega)$.

2.2. Background Green’s functions

In this subsection, the Green’s functions in a two-layered medium will be introduced. We first define three special rough surfaces $\Gamma_0$, $\Gamma_R$ and $\Gamma$ (see figure 2), where

$$\Gamma_R := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ for } |x_1| \geq R, \text{ and } x_2 = -\sqrt{R^2 - x_1^2} \text{ for } |x_1| < R\}.$$

Moreover, we also define the domains above and below $\Gamma_R$ by $\Omega_R^\uparrow$ and $\Omega_R^\downarrow$, respectively.

2.2.1. The Green’s functions $G$ and $\Pi$ for $\Gamma_0$. We first introduce the background Green’s function $G(x, y)$ associated with $\Gamma_0$ which solves

$$\begin{align*}
\Delta_x G(x, y) + \kappa_0^2(x)G(x, y) &= -\delta_y(x) \quad \text{in } \mathbb{R}^2, \\
\lim_{r \to \infty} r^{-\frac{1}{2}} \left( \frac{\partial G(x, y)}{\partial r} - i\kappa_0(x)G(x, y) \right) &= 0 \quad \text{for } r = |x|,
\end{align*}$$

in the distributional sense with the Sommerfeld radiation condition uniformly for all $\hat{x} \in S^1$. Here, the wavenumber $\kappa_0(\cdot)$ is defined by $\kappa_0 := \kappa_1$ in $\mathbb{R}_+^2$ and $\kappa_0 := \kappa_2$ in $\mathbb{R}_-^2$.

It follows from [23] that $G(x, y)$ has the following form

$$G(x, y) = \begin{cases}
\Phi_{\kappa_1}(x, y) + \frac{i}{4\pi} \int_{-\infty}^{+\infty} \frac{1}{\beta_1 + \beta_2} e^{i(\beta_1 x_2 + \beta_2 y_2)} e^{i(\xi_1 x_1 - \xi_2 y_1)} d\xi_1 d\xi_2 & \text{for } x \in \mathbb{R}_+^2, \ y \in \mathbb{R}_+^2, \\
\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\beta_1 + \beta_2} e^{i(\beta_1 x_2 - \beta_2 y_2)} e^{i(\xi_1 x_1 - \xi_2 y_1)} d\xi_1 d\xi_2 & \text{for } x \in \mathbb{R}_-^2, \ y \in \mathbb{R}_+^2, \\
\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\beta_1 + \beta_2} e^{i(\beta_1 x_2 - \beta_2 y_2)} e^{i(\xi_1 x_1 - \xi_2 y_1)} d\xi_1 d\xi_2 & \text{for } x \in \mathbb{R}_+^2, \ y \in \mathbb{R}_-^2, \\
\Phi_{\kappa_2}(x, y) + \frac{i}{4\pi} \int_{-\infty}^{+\infty} \frac{1}{\beta_1 + \beta_2} e^{-i(\beta_1 x_2 + \beta_2 y_2)} e^{i(\xi_1 x_1 - \xi_2 y_1)} d\xi_1 d\xi_2 & \text{for } x \in \mathbb{R}_-^2, \ y \in \mathbb{R}_-^2,
\end{cases}$$

Figure 2. Three special locally rough interfaces.
where $\Phi_s(x,y) := \frac{1}{2} H_0^{(1)}(s|x-y|)$ for $s := \kappa_1, \kappa_2$, and $\beta_1, \beta_2$ are defined by

$$
\beta_1 = \begin{cases} 
\sqrt{\kappa_1^2 - \xi^2} & \text{for } |\kappa_1| > |\xi|, \\
i\sqrt{\xi^2 - \kappa_1^2} & \text{for } |\kappa_1| < |\xi|,
\end{cases}
$$

$$
\beta_2 = \begin{cases} 
\sqrt{\kappa_2^2 - \xi^2} & \text{for } |\kappa_2| > |\xi|, \\
i\sqrt{\xi^2 - \kappa_2^2} & \text{for } |\kappa_2| < |\xi|.
\end{cases}
$$

Next, we introduce the hyper-singular Green’s function $\Pi_j(x,y)$, $j = 1, 2$, associated with $\Gamma_0$ which solves the problem

$$
\begin{align*}
\Delta_j \Pi_j(x,y) + \kappa_0^2(x) \Pi_j(x,y) &= -\partial_j \delta(x) & \text{in } \mathbb{R}^2, \\
\lim_{r \to \infty} \left( \frac{\partial \Pi_j(x,y)}{\partial r} - i\kappa_0(x) \Pi_j(x,y) \right) &= 0 & \text{for } r = |x|,
\end{align*}
$$

(2.1)

in the distributional sense with the Sommerfeld radiation condition uniformly for all $\hat{x} \in \mathbb{S}^1$. Here, $\partial_j$ stands for $\partial / \partial x_j$ for $j = 1, 2$. Following the lines of [23] with the Fourier transform and its inverse, we also have the forms:

$$
\Pi_1(x,y) = \begin{cases} 
\frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{1}{\beta_1 + \beta_2} e^{i\beta_1(x_2+y_2)} e^{i\beta_1(x_1-y_1)} d\xi & \text{for } x \in \mathbb{R}^2_+, \ y \in \mathbb{R}^2_+, \\
\quad - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\beta_1 + \beta_2} e^{i\beta_1(x_2+y_2)} e^{i\beta_1(x_1-y_1)} d\xi & \text{for } x \in \mathbb{R}^2_+, \ y \in \mathbb{R}^2_+, \\
\quad - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\beta_1 + \beta_2} e^{i\beta_1(x_2+y_2)} e^{i\beta_1(x_1-y_1)} d\xi & \text{for } x \in \mathbb{R}^2_+, \ y \in \mathbb{R}^2_+, \\
\end{cases}
$$

for $j = 1$, and

$$
\Pi_2(x,y) = \begin{cases} 
\frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{1}{\beta_2 + \beta_1} e^{i\beta_2(x_2+y_2)} e^{i\beta_2(x_1-y_1)} d\xi & \text{for } x \in \mathbb{R}^2_+, \ y \in \mathbb{R}^2_+, \\
\quad + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\beta_1}{\beta_1 + \beta_2} e^{i\beta_1(x_2+y_2)} e^{i\beta_1(x_1-y_1)} d\xi & \text{for } x \in \mathbb{R}^2_+, \ y \in \mathbb{R}^2_+, \\
\quad - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\beta_2}{\beta_1 + \beta_2} e^{i\beta_2(x_2+y_2)} e^{i\beta_2(x_1-y_1)} d\xi & \text{for } x \in \mathbb{R}^2_+, \ y \in \mathbb{R}^2_+, \\
\end{cases}
$$

for $j = 2$.

It is known by the dominated convergence theorem that for $y \in \mathbb{R}^2 \setminus \Gamma_0$, $G(x,y) \in C^\infty(\mathbb{R}^2_+ \setminus \{y\})$ and $\Pi_j(x,y) \in C^\infty(\mathbb{R}^2_+ \setminus \{y\})$ for $j = 1, 2$. 


2.2.2. The Green’s functions $\mathcal{G}_R$ and $\Pi_R$ for $\Gamma_R$. This subsection is devoted to the existence of the Green’s function $\mathcal{G}_R$ and $\Pi_R$ associated with the two-dimensional Helmholtz equation in a two-layered medium separated by $\Gamma_R$. For simplicity, we only consider the hyper-singular Green’s function $\Pi_R^{(j)}$, $j = 1, 2$, which satisfies

\[
\begin{cases}
\Delta_r \Pi_R^{(j)}(x, y) + \kappa_R^2(x) \Pi_R^{(j)}(x, y) = -\partial_y \delta_y(x) & \text{in } \mathbb{R}^2, \\
\lim_{r \to \infty} r^2 \left( \frac{\partial \Pi_R^{(j)}(x, y)}{\partial r} - i\kappa_R(x) \Pi_R^{(j)}(x, y) \right) = 0 & \text{for } r = |x|,
\end{cases}
\tag{2.2}
\]

in the distributional sense and the Sommerfeld radiation condition uniformly for all directions $\hat{x} \in S^1$. Here, $y \in \mathbb{R}^2 \setminus \Gamma_R$ and the wavenumber $\kappa_R(\cdot)$ is defined by $\kappa_R := \kappa_1$ in $\Omega_R^{(1)}$ and $\kappa_R := \kappa_2$ in $\Omega_R^{(2)}$. The results obtained can be easily extended to the case of the Green’s function $\mathcal{G}_R(x, y)$.

For convenience, let $\eta := \kappa_1^2 - \kappa_2^2$ and $B_R := \{ x \in \mathbb{R}^2 : -\sqrt{r^2 - x_1^2} < x_2 < 0 \text{ for } |x_1| < R \}$. To obtain the existence of $\Pi_R^{(j)}(y, y)$ for $y \in \mathbb{R}^2 \setminus \Gamma_R$, we first have the following theorem.

**Theorem 2.1.** For $j = 1, 2$ and $p \in (1, 2)$, if $\Pi_R^{(j)}(\cdot, y) \in L^p(\mathbb{R}^2)$ is the solution of problem (2.2), then $\Pi_R^{(j)}(\cdot, y)|_{B_R}$ is a solution to the Lippmann–Schwinger equation

\[
\Pi_R^{(j)}(x, y) + \eta \int_{B_R} \mathcal{G}(x, z) \Pi_R^{(j)}(z, y) dz = \Pi_\delta(x, y).
\tag{2.3}
\]

Conversely, if $\Pi_R^{(j)}(\cdot, y)|_{B_R} \in L^p(B_R)$ is a solution to equation (2.3), then $\Pi_R^{(j)}(\cdot, y)$ can be extended to be a solution to problem (2.2) with $\Pi_R^{(j)}(\cdot, y) - \Pi_\delta(\cdot, y) \in W^{2,p}(\mathbb{R}^2)$.

**Proof.** For simplicity, we only consider the case $y \in B_R$. Other cases can be handled similarly with a slight modification.

First, let $\Pi_R^{(j)}(\cdot, y)$ be the solution to problem (2.2) for $j = 1, 2$. Then define the difference, denoted by $U(\cdot, y)$, between $\Pi_R^{(1)}(\cdot, y)$ and $\Pi_R^{(2)}(\cdot, y)$, i.e., $U(\cdot, y) = \Pi_R^{(1)}(\cdot, y) - \Pi_R^{(2)}(\cdot, y)$ in $\mathbb{R}^2$. It is verified that $\Pi_\delta(\cdot, y)$ solves the problem

\[
\begin{cases}
\Delta_r U(\cdot, y) + \kappa_0^2 U(\cdot, y) = \varphi(\cdot) & \text{in } \mathbb{R}^2, \\
\lim_{r \to \infty} r^2 \left( \frac{\partial U(\cdot, y)}{\partial r} - i\kappa_0 U(\cdot, y) \right) = 0 & \text{for } r = |x|,
\end{cases}
\tag{2.4}
\]

with the right term $\varphi(\cdot)$ given by

\[
\varphi(\cdot) := \begin{cases}
\eta \Pi_R^{(1)}(\cdot, y) & \text{in } B_R, \\
0 & \text{in } \mathbb{R}^2 \setminus B_R.
\end{cases}
\]

For $x, y \in B_R$, let $B_r(x)$ and $B_r(y)$ be two balls centered at $x$ and $y$, respectively, with radius $\varepsilon > 0$ such that $B_r(x) \subset B_R$ and $B_r(y) \subset B_R$. Notice that $\varepsilon > 0$ can be also chosen to be sufficiently small such that $B_r(x) \cap B_r(y) = \emptyset$ if $x \neq y$. 


For \( x \neq y \), by the Green’s theorem for \( U(\cdot, y) \) and \( G(\cdot, x) \) in \( \bar{B}_R \backslash (\bar{B}_x \cup \bar{B}_y) \) with (2.1) and (2.4), we have

\[
\eta \int_{\partial B_R \backslash (\partial B_x \cup \partial B_y)} G(z, x) \Pi_R^{0,j}(z, y) \, dz = \left\{ \int_{\partial B_R} - \int_{\partial B_x} - \int_{\partial B_y} \right\} \left( G(z, x) \frac{\partial U(z, y)}{\partial \nu(z)} - \frac{\partial G(z, x)}{\partial \nu(z)} U(z, y) \right) \, ds(z)
= : I_1 - I_2(\varepsilon) - I_3(\varepsilon),
\]

where \( \nu \) denotes the exterior unit normal vector to \( B_R \), \( B_x \) and \( B_y \). Since both \( U(z, y) \) and \( G(z, x) - \Phi_{x,y}(z, x) \) are smooth at \( z = x \), it follows by the mean value theorem that \( \lim_{\varepsilon \to 0} I_2(\varepsilon) = U(x, y) \). Moreover, it follows from the definition of \( \Pi_R^{0,j}(\cdot, \cdot) \), \( j = 1, 2 \), that \( \Pi_R^{0,j}(\cdot, \cdot) - \partial \Phi_{x,y}(\cdot, \cdot) \) is sufficiently smooth in \( \bar{B}_y \). This, together with the estimate \( \partial \Phi_{x,y}(\cdot, \cdot) \in L^p(B_y) \) for any \( p \in (1, 2) \), gives \( \varphi \in L^p(B_y) \). Thus, one has \( U(\cdot, y) \in W^{2,p}(B_y) \), \( p \in (1, 2) \), from the interior regularity result of elliptic equations of second order, which means \( U(\cdot, y) \in C(B_y) \) and \( \partial_s U(\cdot, y) \in L^p(\partial B_y) \) from the imbedding theorem and the trace theorem. Furthermore, it is also known that \( G(\cdot, x) \) is a smooth function in \( \bar{B}_y \) due to \( B_x \cap B_y = \emptyset \). With these results, we conclude from the Cauchy–Schwarz inequality that \( \lim_{\varepsilon \to 0} I_3(\varepsilon) = 0 \).

For \( x = y \), similarly to the equality (2.5), we have

\[
\eta \int_{\partial B_R \backslash (\partial B_x \cup \partial B_y)} G(z, x) \Pi_R^{0,j}(z, y) \, dz = \left\{ \int_{\partial B_R} - \int_{\partial B_x} - \int_{\partial B_y} \right\} \left( G(z, x) \frac{\partial U(z, y)}{\partial \nu(z)} - \frac{\partial G(z, x)}{\partial \nu(z)} U(z, y) \right) \, ds(z)
= : I_1' - I_2'(\varepsilon),
\]

from the Green’s theorem. Since \( \partial_s U(\cdot, y) \in L^p(\partial B_y) \) for any \( p \in (1, 2) \), one deduces by the Hölder inequality that

\[
\int_{\partial B_y} \left| G(z, x) \frac{\partial U(z, y)}{\partial \nu(z)} \right| \, ds(z) \leq \| G(\cdot, x) \|_{L^p(\partial B_y)} \cdot \left\| \frac{\partial U(\cdot, y)}{\partial \nu} \right\|_{L^p(\partial B_y)} \to 0,
\]
as \( \varepsilon \to 0 \), where \( \frac{1}{p} + \frac{1}{q} = 1 \), and \( G(z, x) \) has the singularity of \( \ln |x - z| \) at \( z = x \), which leads to \( G(\cdot, x) \in L^q(\partial B_y) \). Notice that \( U(\cdot, y) \) is also a continuous function in \( B_y \), it follows by the mean value theorem that

\[
- \int_{\partial B_y} \frac{\partial G(z, x)}{\partial \nu(z)} U(z, y) \, ds(z) \to U(x, y) \quad \text{as } \varepsilon \to 0.
\]

Combining these two inequalities yields \( \lim_{\varepsilon \to 0} I_2'(\varepsilon) = U(x, y) \) in the case of \( x = y \).
Next, we estimate $I_1$ of (2.5) or $I_1'$ of (2.6). Let $B_\rho$ be a ball centered at 0 with radius $\rho > R$. Using the Green’s theorem again, it is then concluded that

$$
I_1 = \int_{\partial B_\rho} \left[ \left( \frac{\partial U(z,y)}{\partial \nu(z)} \right)^2 - i\kappa_0 U(z,y) \right] G(z,x) - \left( \frac{\partial G(z,x)}{\partial \nu(z)} \right)^2 - i\kappa_0 G(z,x) \right] U(z,y) \right] ds(z).$

(2.7)

Now, we claim that

$$\int_{\partial B_\rho} |U(z,y)|^2 ds(z) + \int_{\partial B_\rho} |G(z,x)|^2 ds(z) = O(1) \quad \text{as } \rho \to \infty. \quad (2.8)$$

To see this, it is found by the Sommerfeld radiation condition that

$$\int_{\partial B_\rho} \left[ \frac{\partial U(z,y)}{\partial \nu(z)} \right]^2 + \kappa_1^2 |U(z,y)|^2 + 2\kappa_1 \text{Im} \left( U(z,y) \frac{\partial U(z,y)}{\partial \nu(z)} \right) \right] ds(z) \to 0, \quad (2.9)$$

and

$$\int_{\partial B_\rho} \left[ \frac{\kappa_1}{\kappa_2} \frac{\partial U(z,y)}{\partial \nu(z)} \right]^2 + \kappa_1 \kappa_2 |U(z,y)|^2 + 2\kappa_1 \text{Im} \left( U(z,y) \frac{\partial U(z,y)}{\partial \nu(z)} \right) \right] ds(z) \to 0, \quad (2.10)$$

as $\rho \to \infty$. The same limits also hold in (2.9) and (2.10) for the Green’s function $G(z,x)$.

Notice by the Green’s theorem for $U(.,y)$ in $B_\rho \setminus B_R$ that

$$\text{Im} \left\{ \int_{\partial B_\rho} - \int_{\partial B_R^-} \left( U(z,y) \frac{\partial U(z,y)}{\partial \nu(z)} \right) \right] ds(z) = 0,$$

which, together with (2.9) and (2.10), gives

$$\int_{\partial B_\rho \cap \partial B_R} \left[ \frac{\partial U(z,y)}{\partial \nu(z)} \right]^2 + \kappa_1^2 |U(z,y)|^2 \right] ds(z)$$

$$+ \int_{\partial B_\rho \cap \partial B_R} \left[ \frac{\kappa_1}{\kappa_2} \frac{\partial U(z,y)}{\partial \nu(z)} \right]^2 + \kappa_1 \kappa_2 |U(z,y)|^2 \right] ds(z)$$

$$\to 2\kappa_1 \text{Im} \int_{\partial B_R} \left( U(z,y) \frac{\partial U(z,y)}{\partial \nu(z)} \right) ds(z), \quad \text{as } \rho \to \infty. \quad (2.11)$$

In a similar way, it can be also obtained that (2.11) holds with $U(.,y)$ replaced by $G(.,x)$. With these results, we thus have (2.8), which further gives $I_1 \to 0$ as $\rho \to \infty$ in (2.7) from the Cauchy–Schwarz inequality. Hence,

$$U(x,y) = -\eta \int_{B_R} G(z,x) \Pi^0_R(z,y) dz.$$
from (2.6) by letting \( \varepsilon \to 0 \). This proved (2.3) by recalling \( U(\cdot, y) = \Pi_\kappa^0(\cdot, y) - \Pi(\cdot, y) \).

Conversely, let \( \Pi_\kappa^0(\cdot, y) \) be a solution of equation (2.3). We then extend \( \Pi_\kappa^0(\cdot, y) \) into the exterior domain of \( B_\kappa \) by

\[
\Pi_\kappa^0(x, y) := \Pi_\kappa(x, y) - \eta \int_{B_\kappa} G(x, z) \Pi_\kappa^0(z, y) dz \quad \text{for } x \in \mathbb{R}^2. \tag{2.12}
\]

It is easily checked that \( \Pi_\kappa^0(\cdot, y) - \Pi(\cdot, y) \in W^{2,p}_\text{loc}(\mathbb{R}^2) \) (cf. [21]) due to the fact that \( \Pi_\kappa^0(\cdot, y) \in L^p(B_\kappa) \) and \( \Pi_\kappa^0(\cdot, y) \) satisfies the two-dimensional Helmholtz equation \( \Delta \Pi_\kappa^0(\cdot, y) + \kappa_\kappa^2 \Pi_\kappa^0(\cdot, y) = - \partial_\kappa \delta(y) \) in the distributional sense with the Sommerfeld radiation condition. Therefore, \( \Pi_\kappa^0(\cdot, y) \) is a solution of problem (2.2), which completes the proof.

Based on theorem 2.1, we introduce the integral operator \( T : L^p(B_\kappa) \to L^p(B_\kappa) \) with \( p > 1 \) by

\[
(T\varphi)(x) := \int_{B_\kappa} G(x, z) \varphi(z) dz \quad \text{for } x \in B_\kappa,
\]

and equation (2.3) can be then rewritten in the following form

\[
(I + \eta T) \Pi_\kappa^0(\cdot, y) = \Pi_\kappa(\cdot, y) \quad \text{in } L^p(B_\kappa), \tag{2.13}
\]

for \( j = 1, 2 \) where \( I : L^p(B_\kappa) \to L^p(B_\kappa) \) is the identity operator. Now we are able to obtain the existence result for the hyper-singular Green’s function \( \Pi_\kappa^0, j = 1, 2 \).

**Theorem 2.2.** For \( 1 < p < 2 \) and \( j = 1, 2 \), there exists a unique solution \( \Pi_\kappa^0 \in L^p(B_\kappa) \) to problem (2.13) such that

\[
\| \Pi_\kappa^0 \|_{L^p(B_\kappa)} \lesssim \| \Pi_\kappa(\cdot, y) \|_{L^p(B_\kappa)}, \tag{2.14}
\]

where the notation \( \lesssim \) means \( a \leq Cb \) for \( a, b \) with \( C > 0 \) is a generic constant.

**Proof.** Since \( T \) is a bounded operator from \( L^p(B_\kappa) \) into \( W^{2,p}(B_\kappa) \), it follows from the Sobolev imbedding theorem that \( T \) is a compact operator on \( L^p(B_\kappa) \). Hence, \( I + \eta T \) is a Fredholm operator of index 0. By the Fredholm alternative, the existence of solutions to problem (2.13) follows from the uniqueness of problem (2.13).

Let \( (I + \eta T) \varphi = 0 \) for some \( \varphi \in L^p(B_\kappa) \). One then has

\[
\varphi(x) = -\eta \int_{B_\kappa} G(x, z) \varphi(z) dz \quad \text{for } x \in B_\kappa, \tag{2.15}
\]

which implies that \( \Delta \varphi + \kappa_\kappa^2 \varphi = 0 \) in \( B_\kappa \). Furthermore, \( \varphi \) can be also extended into \( \mathbb{R}^2 \setminus B_\kappa \) by the right term of (2.15). Thus, \( \varphi \) satisfies the Helmholtz equation \( \Delta \varphi + \kappa_\kappa^2 \varphi = 0 \) in \( \mathbb{R}^2 \) with the Sommerfeld radiation condition. We then conclude by the uniqueness of problem (1.1) or (1.2) (see theorem 3.1) that \( \varphi = 0 \). Therefore, \( I + \eta T \) is bijective with a bounded inverse \( (I + \eta T)^{-1} \), leading to the estimate (2.14), which completes the proof. \( \square \)
3. The well-posedness in \( L^p \) for \( 1 < p < 2 \)

In this section, we aim to prove the existence of a unique solution of problems (1.1) and (1.2). The uniqueness follows directly from a simple application of proposition 2.1 of [11]. To show the existence, the case \( D = \emptyset \) will be first considered for which problem (1.1) will be reduced to a Lippmann–Schwinger integral equation defined in a bounded domain. With this result, we then study the case \( D \neq \emptyset \) via the boundary integral equation method.

**Theorem 3.1.** There exists at most one solution to problems (1.1) and (1.2).

**Proof.** Let \( u^inc = 0 \). Then \( u = u^s \) and satisfies the Sommerfeld radiation condition. It is enough to prove that \( u = 0 \). To do this, by proposition 2.1 of [11] we need to show that

\[
\lim_{r \to \infty} \int_{\partial B_r} \left( \frac{|\partial u}{\partial r} \right)^2 + |u|^2 \, ds = 0,
\]

(3.1)

for the scattering solution \( u \) of the homogeneous problems (1.1) or (1.2), where \( B_r := \{ x \in \mathbb{R}^2 : |x| < r \} \) with sufficiently large \( r > 0 \) such that \( \overline{D} \subset B_r \).

First, a direct calculation with using Green’s theorem in \( \Omega_1 \cap B_r \) shows that

\[
\int_{\partial B_r \cap \Omega_1} \left( \frac{|\partial u}{\partial r} \right)^2 - i\kappa_1 u \, ds = \int_{\partial B_r \cap \Omega_1} \left( \frac{|\partial u}{\partial r} \right)^2 + \kappa_1^2 |u|^2 \, ds - 2\kappa_1 \text{Im} \int_{\Gamma \cap \Omega_1} \frac{\partial u}{\partial \nu} \, ds.
\]

(3.2)

Similarly, we have

\[
\int_{\partial B_r \cap \Omega_2} \left( \frac{|\partial u}{\partial r} \right)^2 + i\kappa_2 u \, ds = \int_{\partial B_r \cap \Omega_2} \left( \frac{|\partial u}{\partial r} \right)^2 + \kappa_2^2 |u|^2 \, ds
\]

\[
+ 2\kappa_2 \text{Im} \left( \oint_{\Gamma \cap \Omega_2} - \oint_{\partial D} \right) \frac{\partial u}{\partial \nu} \, ds.
\]

(3.3)

Here, \( \nu(x) \) is directed into \( \Omega_1 \) for \( x \in \Gamma \), and is directed into the exterior of \( D \) for \( x \in \partial D \).

Combining (3.2) and (3.3) with continuous transmission conditions on \( \Gamma \) leads to

\[
\int_{\partial B_r \cap \Omega_1} \frac{1}{\kappa} \left( \frac{|\partial u}{\partial r} \right)^2 - i\kappa u \, ds = \int_{\partial B_r \cap \Omega_2} \frac{1}{\kappa} \left( \frac{|\partial u}{\partial r} \right)^2 + \kappa^2 |u|^2 \, ds
\]

\[
- 2\text{Im} \left( \oint_{\partial D} \frac{\partial u}{\partial \nu} \, ds \right),
\]

whence (3.1) directly follows from the Sommerfeld radiation condition, if \( u \) satisfies a homogeneous Dirichlet or Neumann boundary condition on \( \partial D \). For an impedance boundary condition, i.e., \( \partial_r u + i\lambda u = 0 \) on \( \partial D \), since

\[
- 2\text{Im} \left( \oint_{\partial D} \frac{\partial u}{\partial \nu} \, ds \right) = 2 \int_{\partial D} \lambda |u|^2 \, ds \geq 0,
\]

for \( \lambda(x) \geq 0 \), it is also seen by the above equality with the Sommerfeld radiation condition that (3.1) holds.

If \( D \) is a penetrable obstacle with the refractive index \( n(x) \), we use the Green’s theorem in \( D \) to obtain

\[
- 2\text{Im} \left( \oint_{\partial D} \frac{\partial u}{\partial \nu} \, ds \right) = 2 \int_D \kappa_2^2 \text{Im}(n)|u|^2 \, dx \geq 0,
\]
since \( \text{Im}(\kappa) \geq 0 \), which gives (3.1) in a similar manner.

Finally, it follows by proposition 2.1 of [11] with the unique continuation principle that \( u = 0 \). This completes the proof. \( \square \)

### 3.1. The case \( D = \emptyset \)

In this subsection, we study the interface scattering problem in a two-layered medium. Different from all existing works, we will propose a novel technique to prove the existence of a \( L^p \)-solution of problem (1.1) or (1.2) for \( D = \emptyset \), based on the background Green’s functions \( \mathcal{G}_R(\cdot, \cdot) \).

Given an incident wave \( u^{\text{inc}} \) induced by a point source \( \Phi_{\kappa_j}(\cdot, x_j) \) or a hyper-singular source \( \partial_j \Phi_{\kappa_j}(\cdot, x_j) \) for \( j = 1, 2 \) and \( x_j \in \Omega_j \), problem (1.1) or (1.2) is reduced to finding a scattering solution \( u^s(\cdot, x_j) \) satisfying that

\[
\begin{cases}
\Delta u^s + \kappa_1^2 u^s = 0 & \text{in } \Omega_1 \\
\Delta u^s + \kappa_2^2 u^s = 0 & \text{in } \Omega_2 \\
|u^s|_+ - |u^s|_- = -u^{\text{inc}} & \text{on } \Gamma \\
\partial_\nu u^s|_+ - \partial_\nu u^s|_- = -\partial_\nu u^{\text{inc}} & \text{on } \Gamma \\
\lim_{r \to \infty} r^{\frac{1}{2}} \left( \frac{\partial u^s}{\partial r} - i\kappa u^s \right) = 0 & \text{for } r = |x|.
\end{cases}
\]  

(3.4)

Define the total field \( u \) in \( \mathbb{R}^2 \) by \( u(\cdot, x_j) := u^{\text{inc}}(\cdot, x_j) + u^s(\cdot, x_j) \) in \( \Omega_1 \) and \( u(\cdot, x_j) := u^s(\cdot, x_j) \) in \( \Omega_2 \). For \( R > 0 \), let \( B := \{(x_1, x_2) \in \mathbb{R}^2 : -\sqrt{R^2 - x_1} < x_2 < f(x_1) \text{ for } |x_1| < R\} \). Choose sufficiently large \( R \) such that the local perturbation of \( \Gamma \) lies totally above \( \Gamma_R \), and let \( P(\cdot, x_j) \) denote the total field of problem (3.4) with \( \Gamma \) replaced by \( \Gamma_R \), induced by the incident field \( u^{\text{inc}}(\cdot, x_j) \). It follows from theorems 2.1 and 2.2 that such \( P(\cdot, x_j) \) is well-defined. Similarly to theorems 2.1 and 2.2, we have the following theorems without essential difficulties.

**Theorem 3.2.** For \( x_j \in \Omega_j \), let \( u^{\text{inc}}(\cdot, x_j) \) be given by \( \Phi_{\kappa_j}(\cdot, x_j) \) or \( \partial_j \Phi_{\kappa_j}(\cdot, x_j) \) for \( j = 1, 2 \). If \( u^s(\cdot, x_j) \in L^p_{\text{loc}}(\mathbb{R}^2) \) is the scattering solution of problem (3.4), then \( u^s(\cdot, x_j) \) is a solution of the Lippmann–Schwinger equation

\[
u(x_j) - \eta \int_B \mathcal{G}_R(x, z)u(z, x_j)dz = P(x, x_j) \quad \text{for } x \in B.
\]  

(3.5)

Conversely, if \( u(\cdot, x_j) \in L^p(B) \) is a solution to equation (3.5), then \( u(\cdot, x_j) \) can be extended into a total field to problem (3.4) such that \( u(\cdot, x_j) - P(\cdot, x_j) \in W^1_{\text{loc}}(\mathbb{R}^2) \).

**Theorem 3.3.** For \( 1 < p < 2 \), there exists a unique solution \( u(\cdot, x_j) \in L^p(B) \) to (3.5) such that

\[
\|u(\cdot, x_j)\|_{L^p(B)} \lesssim \|P(\cdot, x_j)\|_{L^p(B)}.
\]  

(3.6)

It is remarked that theorems 3.2 and 3.3 still hold for \( x_j \in \Omega_j \) by a slight modification. Moreover, it is clear that the total field \( u \) of problem (3.4) defines the Green’s function \( \mathcal{G}_T(\cdot, \cdot) \) and the hyper-singular Green’s function \( \Pi^j_T(\cdot, \cdot), j = 1, 2 \), respectively, in a two-layered medium separated by \( \Gamma \).
3.2. The case $D \neq \emptyset$

Based on theorems 3.2 and 3.3 for $D = \emptyset$, we now study the case $D \neq \emptyset$. Without loss of generality, we consider problems (1.1) and (1.2) with the incident wave $u^{inc}(\cdot, x_j) = \Phi_j(\cdot, x_j)$ for $j = 1, 2, \gamma = \kappa_1, \kappa_2$, and $x_j \in \mathbb{R}^2 \setminus \Gamma$. The results also holds for the incident wave $u^{inc}(\cdot, x_j) = \Phi_j(\cdot, x_j)$.

Consider the difference $V(\cdot, x_j) := u(\cdot, x_j) - \Pi^{(j)}_{\Gamma}(\cdot, x_j)$ in $\mathbb{R}^2 \setminus D$. Then the solvability of problem (1.1) is reduced to finding a unique solution $V(\cdot, x_j)$ such that

$$
\begin{aligned}
\Delta V(\cdot, x_j) + \kappa^2 V(\cdot, x_j) &= 0 & &\text{in } \mathbb{R}^2 \setminus D \\
BV(\cdot, x_j) &= -B\Pi^{(j)}_{\Gamma}(\cdot, x_j) & &\text{on } \partial D \\
\lim_{r \to 0} r^2 \left( \frac{\partial V(\cdot, x_j)}{\partial r} - i\kappa V(\cdot, x_j) \right) &= 0 & &\text{for } r = |x|.
\end{aligned}
$$

(3.7)

If $D$ is a penetrable obstacle with $n \in L^\infty(D)$, the solvability of problem (1.2) is reduced to finding a unique solution $V(\cdot, x_j)$ such that

$$
\begin{aligned}
\Delta V(\cdot, x_j) + \kappa^2 n V(\cdot, x_j) &= g(\cdot) & &\text{in } \mathbb{R}^2 \\
\lim_{r \to 0} r^2 \left( \frac{\partial V(\cdot, x_j)}{\partial r} - i\kappa V(\cdot, x_j) \right) &= 0 & &\text{for } r = |x|,
\end{aligned}
$$

(3.8)

with $g(\cdot) := \kappa^2 (1 - n)\Pi^{(j)}_{\Gamma}(\cdot, x_j)$.

**Theorem 3.4.** For $1 < p < 2$, there exists a unique solution $u \in L^p_{loc}(\mathbb{R}^2 \setminus D)$ to problem (1.1).

**Proof.** By theorem 3.1 and (3.7), it is sufficient to prove the existence of $V(\cdot, x_j)$ of problem (3.7). For simplicity, we only consider problem (3.7) with a Dirichlet boundary condition on $\partial D$. Other cases can be handled similarly with a slight modification.

In this case, we seek the solution $V(\cdot, x_j)$ in the form

$$
V(x, x_j) = \int_{\partial D} \left( \frac{\partial \Phi_j(x, z)}{\partial \nu(z)} - i\kappa \Phi_j(x, z) \right) \psi(z) ds(z) & &\text{for } x \in \mathbb{R}^2 \setminus D,
$$

(3.9)

for $\psi \in L^p(\partial D)$. By the Dirichlet boundary condition, it is seen that the $V(\cdot, x_j)$ solves problem (3.7), provided $\psi$ is a solution of a second-kind integral equation

$$(I + K - iS)\psi = -2\Pi^{(j)}_{\Gamma}(\cdot, x_j) & &\text{on } \partial D,$

where $S$ and $K$ are the single- and double-layer operators, respectively, given by

$$(S\psi)(x) := 2 \int_{\partial D} \Phi_j(x, z) \psi(z) ds(z) & &x \in \partial D,$$

$$(K\psi)(x) := 2 \int_{\partial D} \frac{\partial \Phi_j(x, z)}{\partial \nu(z)} \psi(z) ds(z) & &x \in \partial D.$$
the Fredholm alternative, standard arguments (cf [12, theorem 3.11]) shows \((I + K - iS)^{-1}\) is a bounded operator on \(L^p(\partial D)\), which yields
\[
\|\psi\|_{L^p(\partial D)} \lesssim \|\Pi^{(D)}_1(\cdot, x_s)\|_{L^p(\partial D)}. \tag{3.10}
\]
This means that there exists a unique solution \(u \in L^p_{\text{loc}}(\mathbb{R}^2 \setminus \mathcal{D})\) to problem (1.1). The proof is thus complete.

**Theorem 3.5.** For \(1 < p < 2\), there exists a unique solution \(u \in L^p_{\text{loc}}(\mathbb{R}^2)\) to problem (1.2).

**Proof.** Using the Green’s theorem, we first reduce problem (3.8) to solving the Lippmann–Schwinger equation
\[
V(x, x_s) = -\kappa^2 \int_D G_\Gamma(x, z)(1 - n(z))(V(z, x_s) + \Pi^{(D)}_1(z, x_s))dz,
\]
on \(L^p(D)\)
Since \(V(\cdot, x_s) = u(\cdot, x_s) - \Pi^{(D)}_1(\cdot, x_s)\), it is seen that problem (1.2) is equivalent to
\[
u(x, x_s) + \kappa^2 \int_D G_\Gamma(x, z)(1 - n(z))u(z, x_s)dz = \Pi^{(D)}_1(x, x_s). \tag{3.11}
\]
Following the arguments for the proof of theorem 2.2, it is then concluded that equation (3.11) is uniquely solvable on \(L^p(D)\) with the estimate
\[
\|u(\cdot, x_s)\|_{L^p(D)} \lesssim \|\Pi^{(D)}_1(\cdot, x_s)\|_{L^p(D)},
\]
which completes the proof.

**4. Uniqueness of the inverse problem**

Based on the well-posedness of problems (1.1) and (1.2), we investigate in this section the inverse problem of recovering the locally rough surface \(\Gamma\) and the embedded obstacle \(D\) (or the refractive index \(n\)) with its surrounding homogeneous medium \(\kappa^2\) from the measurements generalized by incident point sources \(u^{inc}(x, x_s) = \Phi_{\kappa^2}(x, x_s)\), \(x_s \in \Omega_1\).

More precisely, the measurements are taken only on a line segment
\[
\Gamma_{b,a} := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq a, \ x_2 = b\},
\]
where \(a > 0\) denotes the measurement width and \(b > 0\) denotes the measurement height satisfying that \(b > \|f\|_{L^\infty(\mathbb{R})}\). Our purpose is to prove two global uniqueness theorems for determining all unknowns in the inverse problem. The method is mainly based on reconstructing a well-defined ITP associated with the Helmholtz equations in a sufficiently small domain as well as the uniform apriori estimates of solutions of problems (1.1) and (1.2).

For \(j = 1, 2\), let \(u^j(\cdot, x_s)\) and \(u(\cdot, x_s)\) denote the scattered field and transmitted field of problem (1.1) (or problem (1.2)) in \(\Omega_1, \Gamma_j\) and \(\Omega_2, \Gamma_j\), respectively, with respect to the scatterer \((\Gamma_j, \kappa_2, D_j, B_j)\) (or \((\Gamma_j, \kappa_2, n_j)\)) and incident waves \(u^{inc}(\cdot, x_s) = \Phi_{\kappa_2}(\cdot, x_s)\) for \(x_s \in \Omega_1\). Here, \(\Omega_1, \Gamma_j\), and \(\Omega_2, \Gamma_j\), denote the upper and lower half-spaces, respectively, separated by the interface \(\Gamma_j\).
4.1. The case of an impenetrable obstacle D

**Theorem 4.1.** If \( u(x, x_0) = u_0(x, x_0) \) for all \( x, x_0 \in \Gamma_b \), then \( \Gamma = \Gamma_2, \kappa_{2,1} = \kappa_{2,2}, D_1 = D_2 \) and \( B_1 = B_2 \).

**Proof.** The proof is divided into four steps.

- **Step 1.** Firstly, we will establish a generalized reciprocity relation for the solutions of problem (1.1) with respect to two kinds of incident fields \( U^{inc} \) and \( V^{inc} \). More precisely, let \( U(\cdot, x_i^j) \) and \( V(\cdot, x_i^j) \) be the solutions to problem (1.1) associated with \( (\Gamma, D, B, \kappa) \), which are induced by the incident fields \( U^{inc}(\cdot, x_i^j) \) and \( V^{inc}(\cdot, x_i^j) \), respectively, for \( x_i^1, x_i^2 \in \set{R}^2 \setminus (D \cup \Gamma) \). Define two functions \( E_i(x_i^j) \) and \( F_i(x_i^j) \) by

\[
E_i(x_i^j) := \int_{\partial B_i(x_i^j)} \left[ \frac{\partial U^{inc}(y, x_i^j)}{\partial \nu(y)} V(y, x_i^j) - \frac{\partial V(y, x_i^j)}{\partial \nu(y)} U^{inc}(y, x_i^j) \right] \mathrm{d}s(y), \tag{4.1}
\]

\[
F_i(x_i^j) := \int_{\partial B_i(x_i^j)} \left[ \frac{\partial V^{inc}(y, x_i^j)}{\partial \nu(y)} U(y, x_i^j) - \frac{\partial U(y, x_i^j)}{\partial \nu(y)} V^{inc}(y, x_i^j) \right] \mathrm{d}s(y), \tag{4.2}
\]

where \( B_i(x_i^j), j = 1, 2 \), are two balls centered at \( x_i^j \) with radius \( \varepsilon \) which is small enough such that \( B_i(x_i^j) \cap B_i(x_i^j) = \emptyset \). Using the Green’s theorem shows

\[
\lim_{\varepsilon \to 0} (E_i(x_i^j) - F_i(x_i^j)) = 0, \tag{4.3}
\]

with \( x_i^1 \neq x_i^2 \), if \( U(\cdot, x_i^1) \in L^p(B_i(x_i^1)) \) and \( V(\cdot, x_i^2) \in L^p(B_i(x_i^2)) \) for \( p > 1 \). Here, we refer to [35] where a similar equality (4.3) has been established in the case of an inhomogeneous cavity. Especially, if we choose the incident fields \( U(\cdot, x_i^1) \) and \( V(\cdot, x_i^2) \) of the forms

\[
U^{inc}(\cdot, x_i^1) := \Phi_{s_1}(\cdot, x_i^1), \quad V^{inc}(\cdot, x_i^2) := \nabla \Phi_{s_1}(\cdot, x_i^2) \cdot \vec{c},
\]

in (4.1) and (4.2) with \( x_i^1 \neq x_i^2 \), it can be deduced by the smoothness of \( V(\cdot, x_i^2) \) in \( B_i(x_i^1) \) for sufficiently small \( \varepsilon > 0 \) that \( \lim_{\varepsilon \to 0} E_i(x_i^j) = V(x_i^1, x_i^2) \), which means that

\[
V(x_i^1, x_i^2) = \lim_{\varepsilon \to 0} \int_{\partial B_i(x_i^j)} \left[ \frac{\partial V^{inc}(y, x_i^j)}{\partial \nu(y)} U(y, x_i^j) - \frac{\partial U(y, x_i^j)}{\partial \nu(y)} V^{inc}(y, x_i^j) \right] \mathrm{d}s(y), \tag{4.4}
\]

follows from (4.3). Here, \( \vec{c} \in \set{R}^2 \) is a fixed constant vector.

- **Step 2.** In this part, we will prove the uniqueness of the interface \( \Gamma \) by contradiction. Suppose \( \Gamma_1 \neq \Gamma_2 \). Note that \( \Gamma_1 \) and \( \Gamma_2 \) are characterized by Lipschitz continuous functions \( f_1 \) and \( f_2 \), respectively. Without loss of generality, we can choose \( z^* \in \Gamma_1 \setminus \Gamma_2 \) such that \( f_1(z^*) > f_2(z^*) \); see figure 3. In this case, there exists \( \varepsilon_0 > 0 \) and \( \delta_0 > 0 \) such that \( B_{\varepsilon_0}(z^*) \cap \Gamma_2 = \emptyset \) and

\[
z_j := z^* + \frac{\delta_0}{f_j} \nu(z^*) \in B_{\varepsilon_0}(z^*) \quad \text{for} \quad j = 1, 2, \ldots \ . \tag{4.5}
\]

Here, \( \nu(z^*) \) is the unit normal vector directed into the interior of the domain \( \Omega_{1, \Gamma_1} \).
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Define \( x \in y \) \( \ell \in \mathbb{N} \) and each \( j \in \mathbb{N} \), we deduce by (4.4) that

\[
\Psi_1(y, z_j) = \lim_{\varepsilon \to 0} \int_{\partial B(z_j)} \left[ \frac{\partial p_{\text{inc}}(x, z_j)}{\partial \nu(x)} u_1(x, y) - p_{\text{inc}}(x, z_j) \frac{\partial u_1(x, y)}{\partial \nu(x)} \right] \text{d}s(x),
\]

(4.6)

\[
\Psi_2(y, z_j) = \lim_{\varepsilon \to 0} \int_{\partial B(z_j)} \left[ \frac{\partial p_{\text{inc}}(x, z_j)}{\partial \nu(x)} u_2(x, y) - p_{\text{inc}}(x, z_j) \frac{\partial u_2(x, y)}{\partial \nu(x)} \right] \text{d}s(x).
\]

(4.7)

Noticing \( u_1'((x, y) = u_2'((x, y) \) for all \( x \in \Gamma_{b,a} \), we have \( u_1'((x, y) = u_2'((x, y) \) for all \( x \in \Gamma_{b} \) \( \Gamma_{b} := \{ (x_1, x_2) \in \mathbb{R}^2 : x_2 = b \} \), since both \( u_1'((x, y) \) and \( u_2'((x, y) \) are analytic functions of the \( x \) variable. It then follows from the uniqueness of the Dirichlet problem in \( \Omega_2 := \{ x \in \mathbb{R}^2 : x_2 > b \} \) and the unique continuation principle that \( u_1'((x, y) = u_2'((x, y) \) in \( \Omega_{1,1} \cap \Omega_{1,2} \). Hence, one has \( u_1((x, y) = u_2((x, y) \) in \( \Omega_{1,1} \cap \Omega_{1,2} \{ y \} \). Moreover, for each \( j \in \mathbb{N} \), we can choose sufficient small \( \varepsilon > 0 \) such that \( \partial B_\varepsilon(z_j) \subset (\Omega_{1,1} \cap \Omega_{1,2}) \), leading to \( u_1((x, y) = u_2((x, y) \) and \( \partial_\nu u_1((x, y) = \partial_\nu u_2((x, y) \) on \( \partial \Omega_2 \). This, combined with (4.6) and (4.7), gives \( \Psi_1(y, z_j) = \Psi_2(y, z_j) \) for each \( j \in \mathbb{N} \) and all \( y \in \Gamma_{b,a} \). It is then deduced by the above arguments again that

\[
\Psi_1(y, z_j) = \Psi_2(y, z_j) \quad \text{for all } y \in (\Omega_{1,1} \cap \Omega_{1,2}) \{ z_j \},
\]

(4.8)

for each \( j \in \mathbb{N} \).

Define \( D_{\varepsilon} := B_{\varepsilon}(z_j) \cap \Omega_{2,1} \) for sufficiently small \( \varepsilon > 0 \) such that \( \overline{D_{\varepsilon}} \subset \Omega_{1,2} \). We then construct the following ITP in \( D_{\varepsilon} \):

\[
\begin{align*}
\Delta v_1 + \kappa_{1,1}^2 v_1 &= 0 \quad \text{in } D_{\varepsilon}, \\
\Delta v_2 + \kappa_{1,2}^2 v_2 &= 0 \quad \text{in } D_{\varepsilon}, \\
v_2 - v_1 &= g_j \quad \text{on } \partial D_{\varepsilon}, \\
\partial_\nu v_2 - \partial_\nu v_1 &= g_j \quad \text{on } \partial D_{\varepsilon},
\end{align*}
\]

(4.9)

where \( g_j := \Psi_2((., z_j) - \Psi_1((., z_j) \) and \( g_j(x) := \partial_\nu \Psi_2((., z_j) - \partial_\nu \Psi_1((., z_j) \) \) It first follows by (4.8) that \( g_j = 0 \) and \( g_j = 0 \) on \( B_{\varepsilon}(0) \cap \Gamma_{1} \) for all \( j \in \mathbb{N} \). Next, we will construct one function \( h_j(\cdot) \in H^1_0(D_{\varepsilon}) \) such that \( \text{Tr}_0 h_j = g_j \) and \( \text{Tr}_1 h_j = g_j \) on \( \partial D_{\varepsilon} \), where \( \text{Tr}_0 \) and \( \text{Tr}_1 \) are 0- and 1-order traces defined by \( \text{Tr}_0 \omega = \omega|_{\partial D_{\varepsilon}} \) and \( \text{Tr}_1 \omega = \partial_\nu \omega|_{\partial D_{\varepsilon}} \) for a smooth function \( \omega \in C^1(D_{\varepsilon}) \). To this end, we introduce

\[
h_j(\cdot) := (1 - \chi(\cdot))(\Psi_2((., z_j) - \Psi_1((., z_j)) \quad \text{in } D_{\varepsilon}.
\]

(4.10)
where $\chi(\cdot) \in C^2(\mathbb{R}^2)$ is a cut-off function satisfying $\chi(x) = 1$ for $x \in B_1(\zeta^*)$ ($\varepsilon_1 < \varepsilon_0$) and $\chi(x) = 0$ in $\mathbb{R}^2 \setminus B_1(\zeta^*)$.

Since $g_j \equiv 0$ and $h_j \equiv 0$ on $B_1(\zeta^*) \cap \Gamma_1$, we deduce by (4.8) that for each $j \in \mathbb{N}$, such $h_j$ satisfies $\text{Tr}_0 h_j = g_j$ and $\text{Tr}_1 h_j = g_j$ on $\partial D_{\zeta}$. In the following, we claim that there exists a fixed constant $C > 0$ such that

$$
\|h_j\|_{H^1(D_{\zeta})} = \|h_j\|_{H^1(D_{\zeta})} + \|\Delta h_j\|_{L^2(D_{\zeta})} \leq C,
$$

which is equivalent to showing that

$$
\|\Psi_1(\cdot, z_j)\|_{H^1(D_0)} + \|\Psi_2(\cdot, z_j)\|_{H^1(D_0)} \lesssim C,
$$

since $\Psi(x, z_j), \ell = 1, 2$, satisfies the Helmholtz equation, where $D_0 := D_{\zeta} \setminus B_1(\zeta^*)$. Indeed, by theorems 3.2, 3.3 and 3.4, it is concluded that the estimate (4.12) holds, since $P_{\text{inc}}(\cdot, z_j)$ are uniformly bounded in $L^p(D_0)$ for each $p \in (1, 2)$ and the embedding mapping of $W^{1, p}(D_0)$ into $H^1(D_0)$ is bounded for $p > 1$ in the two-dimensional case. Therefore, we have proved (4.11).

Furthermore, it is easily found that $(\Psi_1(\cdot, z_j), \Psi_2(\cdot, z_j))|_{D_{\zeta}}$ is the solution of the ITP (4.9). It follows from [7] with the estimate (4.11) that the ITP (4.9) is well-posed in $L^2(D_{\zeta}) \times L^2(D_{\zeta})$ satisfying that

$$
\|\Psi_1(\cdot, z_j)\|_{L^2(D_{\zeta})} + \|\Psi_2(\cdot, z_j)\|_{L^2(D_{\zeta})} \lesssim \|h_j\|_{H^1(D_{\zeta})} \lesssim C,
$$

if $\varepsilon_0 > 0$ is chosen to be sufficiently small so that the smallest real eigenvalue of the homogeneous ITP is bigger than $\kappa_2 + \kappa_2^2$. However, this is a contradiction since

$$
C \gtrsim \|\Psi_j(\cdot, z_j)\|_{L^2(D_{\zeta})} \gtrsim \|P_{\text{inc}}(\cdot, z_j)\|_{L^2(D_{\zeta})} - \|\Psi_2(\cdot, z_j)
$$

$$
= P_{\text{inc}}(\cdot, z_j)|_{L^2(D_{\zeta})} \rightarrow \infty,
$$

as $j \rightarrow \infty$, where we have used the fact that $\Psi_j(\cdot, z_j) = P_{\text{inc}}(\cdot, z_j)$ are uniformly bounded in $L^2(D_{\zeta})$ for all $j \in \mathbb{N}$ due to a positive distance between $\zeta^*$ and $\Gamma_2$. Hence, $\Gamma_1 = \Gamma_2$.

- **Step 3.** Let $\Gamma := \Gamma_1 = \Gamma_2$ and $\Omega_\ell := \Omega_\ell \setminus \{z_j\}$ for $\ell = 1, 2$. We are now at a position to prove $\kappa_{2,1} = \kappa_{2,2}$ in a similar manner. Suppose $\kappa_{2,1} \neq \kappa_{2,2}$, and we have from (4.8) that $\Psi_j(y, z_j) = \Psi_{2}(y, z_j)$ for all $y \in \Omega_1 \setminus \{z_j\}$. Moreover, it is easily seen that $\Psi_1(\cdot, z_j), \Psi_2(\cdot, z_j)|_{D_{\zeta}}$ solves another ITP in $D_{\zeta}$:

$$
\begin{cases}
\Delta w_1 + \kappa_{2,1}^2 w_1 = 0 & \text{in } D_{\zeta}, \\
\Delta w_2 + \kappa_{2,2}^2 w_2 = 0 & \text{in } D_{\zeta}, \\
w_2 - w_1 = \varrho_j & \text{on } \partial D_{\zeta}, \\
\partial_{\nu} w_2 - \partial_{\nu} w_1 = g_j & \text{on } \partial D_{\zeta},
\end{cases}
$$

(4.15)

where $\varrho_j(\cdot) := \Psi_2(\cdot, z_j) - \Psi_1(\cdot, z_j)$ and $g_j(\cdot) := \partial_{\nu} \Psi_2(\cdot, z_j) - \partial_{\nu} \Psi_1(\cdot, z_j)$ which satisfy $\varrho_j = 0$ and $g_j = 0$ on $B_1(\zeta^*) \cap \Gamma$ for all $j \in \mathbb{N}$. Following the lines of step 2 with the $L^2(D_{\zeta}) \times L^2(D_{\zeta})$-well-posedness of the ITP (4.15), we still have a contradiction (4.14). Therefore, $\kappa_{2,1} = \kappa_{2,2}$.

- **Step 4.** Let $\kappa_2 := \kappa_{2,1} = \kappa_{2,2}$. Having proved the uniqueness of $\Gamma$ and $\kappa_2$, the inverse problem is reduced to determine an embedded obstacle $D$ with its boundary condition $B$ in a known two-layered medium. The proof is similar to the case of a homogeneous
Suppose $D_1 \neq D_2$. Without loss of generality, we can choose $z^* \in \partial D_1$ but $z^* \notin \overline{D_2}$ such that the sequence $z_j \in (B_{\varepsilon_0}(z^*) \cap \mathcal{C}) \neq \emptyset$, where $z_j$ is defined as in $(4.5)$, $\varepsilon_0 > 0$ is chosen to be sufficiently small such that $B_{\varepsilon_0}(z^*) \cap \overline{D_2} = \emptyset$, and $\mathcal{C}$ is the unbounded, connected component of $\mathbb{R}^2 \setminus \overline{D_1} \cup \overline{D_2}$. Since $u_1(x, y) = u_2(x, y)$ for all $x, y \in \Gamma_{ba}$, it follows from the standard arguments (see the contents between (4.7) and (4.8)) that $u_1(x, y) = u_2(x, y)$ for $x \in \Omega_1$ and $y \in \Gamma_{ba}$. Combining the transmission conditions on $\Gamma$ with Holmgren’s theorem (cf [12, theorem 2.3]), we have $u_1(x, y) = u_2(x, y)$ for $x \in \mathcal{C}$ and $y \in \Gamma_{ba}$. Especially, it holds that $u_1(z_j, y) = u_2(z_j, y)$ for each $j \in \mathbb{N}$ and $y \in \Gamma_{ba}$. Using the reciprocity relation (4.3) with the above analysis again, we can obtain $u_1(y, z_j) = u_2(y, z_j)$ for each $j \in \mathbb{N}$ and $y \in \mathcal{C}$. However, this leads to a contradiction since

$$C \geq B_1 u_1^j(z^*, z_j) = B_1 u_1^j(z^*, z_j) = -B_1 \Phi_{a_2}(z^*, z_j) \to \infty,$$

as $j \to \infty$, where the first inequality is due to the positive distance between $z^*$ and $\overline{D_2}$. Therefore, $D_1 = D_2$.

Finally, we prove $B_1 = B_2$ by contradiction again. Let $D := D_1 = D_2$, and assume $B_1 \neq B_2$. We first have $u(y, x) := u_1(y, x) = u_2(y, x)$ for all $x, y \in \Omega_2 \setminus \overline{\mathcal{D}}$. If $B_1$ represents a Dirichlet boundary condition and $B_2$ represents an impedance boundary condition with $\lambda_2(\cdot) \geq 0$, we then have $u = \partial_\nu u = 0$ on $\partial D$, which gives $u(y, x) = 0$ for $x, y \in \Omega_2 \setminus \overline{\mathcal{D}}$, $x \neq y$, from Holmgren’s theorem. This is impossible since the incident wave $u^{inc}(y, x) = \Phi_{a_2}(y, x)$ has a singularity at $x = y$. On the other hand, if both $B_1$ and $B_2$ represent impedance boundary conditions on $\partial D$ but with different impedance coefficients $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$, that is, $\lambda_1(\cdot) \neq \lambda_2(\cdot)$, it is then derived that $(\lambda_1 - \lambda_2) u = 0$ on $\partial D$, which means $u = \partial_\nu u = 0$ on a nonempty open subset $\Sigma$ of $\partial D$ due to $\lambda_\ell \in C(\partial D)$ for $\ell = 1, 2$. This also leads to a contradiction by the above analysis. Therefore, $B_1 = B_2$, which ends the proof.

4.2. The case of an inhomogeneous medium $n(x)$

In this subsection, we investigate the inverse problem for problem $(1.2)$ to determine the locally rough surface $\Gamma$, the wavenumber $\kappa_2$ and the refractive index $n$ by taking the measurements on $\Gamma_{ba}$. We formulate the main result as follows.

**Theorem 4.2.** If $u_1^j(x, x_s) = u_2^j(x, x_s)$ for all $x, x_s \in \Gamma_{ba}$, then $\Gamma_1 = \Gamma_2$, $n_{2,1} = \kappa_{2,2}$ and $n_1 = n_2$.

**Proof.** It follows from the same arguments as theorem 4.1 that one has $\Gamma_1 = \Gamma_2$ and $\kappa_{2,1} = \kappa_{2,2}$. Thus, the inverse problem is reduced to determine an inhomogeneous medium in a known two-layered medium, where the equality $n_1 = n_2$ directly follows from a standard discussion [2].

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Data availability statement

No new data were created or analysed in this study.

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