Abstract. In this paper, we consider the numerical approximation of time-fractional parabolic problems involving Caputo derivatives in time of order $\alpha$, $0 < \alpha < 1$. We derive optimal error estimates for semidiscrete Galerkin FE type approximations for problems with smooth and nonsmooth initial data. Our analysis relies on energy arguments and exploits the properties of the inverse of the associated elliptic operator. We present the analysis in a general setting so that it is easily applicable to various spatial approximations such as conforming and nonconforming FEMs, and FEM on nonconvex domains. The finite element approximation in mixed form is also presented and new error estimates are established for smooth and nonsmooth initial data. Finally, an extension of our analysis to a multi-term time-fractional model is discussed.

Key words. time-fractional parabolic equation, multi-term fractional diffusion, semidiscrete finite element scheme, optimal error estimates, mixed method, nonsmooth initial data

AMS subject classifications. 65M60, 65M12, 65M15

1. Introduction. Let $\Omega$ be a bounded, convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial \Omega$ and let $T > 0$ be a fixed value. We are interested in the numerical approximation of the solution $u(x,t)$ of the following time-fractional initial-boundary value problem:

$$\partial_t^\alpha u + Lu = f \text{ in } \Omega \times (0,T], \quad u(0) = u_0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \times (0,T], \quad (1.1)$$

where $Lu = -\text{div} [A(x)\nabla u] + c(x)u$, $f(x,t)$ is the forcing function and $u_0(x)$ is the initial data. Here, $A(x) = [a_{ij}(x)]_{1 \leq i,j \leq 2}$ is a $2 \times 2$ symmetric and uniformly positive definite in $\Omega$ matrix with smooth coefficients, and $c(x) \in L^\infty(\Omega)$ is nonnegative in $\Omega$. In (1.1), $\partial_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha$ ($0 < \alpha < 1$) with respect to $t$ defined by

$$\partial_t^\alpha \varphi(t) := \mathcal{I}^{1-\alpha} \varphi'(t) := \int_0^t \omega_{1-\alpha}(t-s)\varphi'(s) \, ds \quad \text{with} \quad \omega_{\alpha}(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad (1.2)$$

where $\varphi'$ is the time derivative of $\varphi$ and $\mathcal{I}^\nu$ is the Riemann–Liouville time-fractional integral of order $\nu$. As $\alpha \to 1^-$, $\partial_t^\alpha$ converges to $u'$, and thus, problem (1.1) reduces to the standard parabolic problem. In analogy with Brownian motion of normal diffusion, the equation in (1.1) with $0 < \alpha < 1$ represents a macroscopic counterpart of time continuous random walk [11, 29].

In recent years, the model (1.1) has received considerable attention, due to its great efficiency in capturing the dynamics of physical processes involving anomalous transport phenomena. Several numerical schemes have then been proposed with different types of spatial discretizations including finite difference, FE or spectral element methods, see [9, 6, 10, 38, 18, 13, 31, 22], and most recently, the finite volume element method [18, 20].

The main technical difficulty in designing robust numerical schemes and in carrying out a rigorous error analysis stems from the limited smoothing properties of the
problem. Specifically, for an initial data \( u_0 \in L^2(\Omega) \) and \( f = 0 \), the following estimate \[35\]:

\[
\|\partial_\alpha_t u(t)\|_{L^2(\Omega)} \leq C t^{-\alpha} \|u_0\|_{L^2(\Omega)}, \quad t > 0,
\]

shows the singular behaviour of the solution \( u \) near \( t = 0 \). Assuming high regularity on \( u \) imposes additional conditions on the given data \( u_0 \) and \( f \), which are not, in general, reasonable. Note also that in the case of fractional-order evolution problems, the solution operators do not form a semigroup, as in the parabolic case, so some useful techniques cannot be utilized. Attempts have then been made, using various techniques, including, spectral decomposition approach, Laplace transforms with the semigroup type theory, and novel energy arguments, to derive sharp error estimates for problem \( (1.1) \) under reasonable assumptions on the solution \( u \).

Early papers dealing with optimal (with respect to data regularity) error estimates for time-fractional order problems consider the following subdiffusion equation

\[
\partial_t^\alpha u(x,t) - R^{\partial_\alpha} \Delta u(x,t) = f(x,t), \quad (1.3)
\]

which is closely related to but different from the model in \((1.1)\). Here, \( R^{\partial_\alpha} \) is the Riemann-Liouville fractional derivative in time defined by \( R^{\partial_\alpha} \varphi(t) := \frac{d}{dt} I^{1-\alpha} \varphi(t) \). In \[27\], McLean and Thomeé established the first optimal \( L^2(\Omega) \)-error estimates for the Galerkin FE solution of \((1.3)\) with respect to the regularity of initial data using Laplace transform technique. Thus, they extended the classical results obtained in \[3\] for the standard parabolic problem. In \[28\], the authors derived convergence rates in the stronger \( L^\infty(\Omega) \)-norm. Recently, a delicate energy analysis has been developed in \[19\] to obtain similar estimates.

In recent papers \[14, 13, 16\], Jin et al. established optimal error estimates for the subdiffusion problem \((1.1)\), with respect to the solution smoothness expressed through the problem data, \( f \) and \( u_0 \). In \[14\], an approach based on Laplace transform and eigenfunction expansion of the solution has been exploited to derive \( a \ priori \) error estimates for the semidiscrete FEM applied to \((1.1)\) with \( f = 0 \). The semidiscrete FEM for the inhomogeneous equation with a weak right-hand side data \( f \) has been considered in \[13\]. In \[16\], fully discrete schemes based on convolution quadrature in time are derived and analyzed for problems with smooth and nonsmooth data.

The first motivation of this work is to derive optimal error estimates for semidiscrete Galerkin FE type approximations to the problem \((1.1)\) on convex and nonconvex domains with both smooth and nonsmooth initial data, using energy arguments combined with a technique developed in \[3\], which is based on the inverse of the associated elliptic operator. We shall present our method in a general setting so that it can be extended to various discretizations in space, and can be easily adopted to different time-fractional problems. Thereby, we extend known results of the parabolic case to the fractional-order case \( (0 < \alpha < 1) \). Our analysis depends on known properties of the associated elliptic problems, in contrast to the standard Laplace transform technique which relies on writing the corresponding semidiscrete problem in operator form. This procedure is not always feasible, e.g., in the analysis of the mixed form of problem \((1.1)\), and can be complicated in other cases, such as, in the case of nonconforming FE approximations.

The second aim is to investigate a mixed form of problem \((1.1)\), and derive optimal error estimates for the semidiscrete problem, using a standard Galerkin mixed FE method in space, for cases with smooth and nonsmooth initial data. To the best of our knowledge, there is hardly any result for the mixed form of \((1.1)\) except for a
recent paper \[39\]. In \[39\], a non-standard mixed FE method is proposed and analyzed assuming higher order regularity on the solution. Another related analysis for mixed method applied to the time-fractional Navier-Stokes equations is presented in \[24\] where high regularity assumptions on the exact solution are also made.

The rest of the paper is organized as follows. In Section 2 we recall regularity properties of the solution \(u\), and state some technical results. In Section 3 we present our error analysis for the initial-boundary value problem (1.1). In Section 4 applications are presented and optimal \(L^r(\Omega)\)-error estimates are established. The applications include the standard \(C^0\)-conforming FE method defined on convex and nonconvex domains, and some nonconforming methods. In section 5 we introduce the mixed form of problem (1.1) and derive new error estimates for cases with smooth and nonsmooth data. Particularly relevant to this a priori error analysis is the appropriate use of several properties of the time-fractional differential operator. Finally, in Section 6 we discuss the extension of our analysis to a multi-term time-fractional model.

2. Notation and Preliminaries. We shall first introduce notation and recall some preliminary results. Let \((\cdot, \cdot)\) be the inner product in \(L^2(\Omega)\) and \(\| \cdot \|\) the induced norm. The space \(H^r(\Omega)\) denotes the standard Sobolev space with the usual norm \(\| \cdot \|_{H^r(\Omega)}\). Let \(\{\lambda_j\}_{j=1}^\infty\) and \(\{\phi_j\}_{j=1}^\infty\) denote respectively the Dirichlet eigenvalues and eigenfunctions of the symmetric and uniformly elliptic operator \(L\) on the domain \(\Omega\), with \(\{\phi_j\}_{j=1}^\infty\) being an orthonormal basis in \(L^2(\Omega)\). For \(r \geq 0\), we define the Hilbert space

\[
\hat{H}^r(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{j=1}^\infty \lambda_j^r (v, \phi_j)^2 < \infty \right\}
\]

equipped with the norm

\[
\|v\|_{\hat{H}^r(\Omega)} = \left( \sum_{j=1}^\infty \lambda_j^r (v, \phi_j)^2 \right)^{1/2}.
\]

Note that \(\|v\|_{\hat{H}^r(\Omega)} = (L^r v, v)^{1/2} = \|L^{r/2} v\|\). It is shown, see, for instance, \[36\] Lemma 3.1], that for \(r\) a nonnegative integer, \(\hat{H}^r(\Omega)\) consists of all functions \(v\) in \(H^r(\Omega)\) which satisfy the boundary conditions \(L^j v = 0\) on \(\partial \Omega\) for \(j < r/2\), and that the norm \(\| \cdot \|_{\hat{H}^r(\Omega)}\) is equivalent to the usual norm in \(H^r(\Omega)\). For \(r > 0\) we also define \(\hat{H}^{-r}(\Omega)\) to be the dual space of \(\hat{H}^r(\Omega)\). Since \(\hat{H}^1(\Omega)\) and \(H^1_0(\Omega)\) coincide, so does \(H^{-1}(\Omega)\) and \(\hat{H}^{-1}(\Omega)\), the dual space of \(H^1_0(\Omega)\). Note that \(\{H^r(\Omega)\}, r \geq -1\), form a Hilbert scale of interpolation spaces. Thus, we denote by \(\| \cdot \|_{H^r(\Omega)}\) the norm on the interpolation scale between \(H^1_0(\Omega)\) and \(L^2(\Omega)\) when \(r \in [0, 1]\) and the norm on the interpolation scale between \(L^2(\Omega)\) and \(H^{-1}(\Omega)\) when \(r \in [-1, 0]\). Then, the norms \(\hat{H}^r(\Omega)\) and \(H^r(\Omega)\) are equivalent for any \(r \in [-1, 1]\) by interpolation.

Regularity properties of the solution \(u\) of the time-fractional problem (1.1) play a key role in the error analysis of the finite element method, particularly, since \(u\) has singularity near \(t = 0\), even for smooth given data. From \[35\] and \[24\], we recall the regularity results for the problem (1.1) in terms of the initial data \(u_0\) for the homogeneous problem \((f = 0)\). In particular, for \(t > 0\),

\[
\|\partial_t^\alpha u\| + \|Lu\| \leq C t^{-\alpha}\|u_0\|, \quad (2.1)
\]
and for $r \geq 0$,
\[
t^\ell \|u^{(r)}(t)\|_{H^{\ell+\nu}(\Omega)} \leq Ct^{-\alpha\mu/2}\|u_0\|_{H^r(\Omega)}, \quad \ell = 0, 1,
\]
where $0 \leq \mu \leq 2$ when $\ell = 0$ and $-2 \leq \mu \leq 2$ when $\ell = 1$.

Next, we recall some properties of the fractional operators $\mathcal{I}^\alpha$ and $\partial_t^\alpha$ that will be used in the subsequent sections. For piecewise time continuous functions $\varphi : [0, T] \to X$, where $X$ is a Hilbert space with inner product $(\cdot, \cdot)_X$ and norm $| \cdot |_X$, it is well-known that $\int_0^T (\mathcal{I}^\alpha \varphi, \varphi)_X \, dt \geq 0$. Furthermore, by [26, Lemma A.1], it follows that for $\varphi \in W^{1,1}(0, T; X)$, $\int_0^T (R \partial_t^\alpha \varphi, \varphi)_X \, dt \geq 0$.

Using the result in [32, Lemma 3.1 (iii)] and the inequality $\cos(\alpha \pi/2) \geq 1 - \alpha$, we obtain the following continuity property of $\mathcal{I}^\alpha$: for suitable functions $\varphi$ and $\psi$,
\[
\int_0^t (\mathcal{I}^\alpha \varphi, \psi)_X \, ds \leq \epsilon \int_0^t (\mathcal{I}^\alpha \varphi, \varphi)_X \, ds + \frac{1}{4\epsilon(1-\alpha)} \int_0^t (\mathcal{I}^\alpha \psi, \psi)_X \, ds, \quad \text{for } \epsilon > 0.
\]

In our analysis, we shall also make use of the following inequality which holds by combining Lemmas 2.1 and 2.2 in [23]: if $\varphi(0) = 0$, then
\[
|\varphi(t)|_X^\alpha \leq \frac{t^\alpha}{\alpha^2} \int_0^t (\mathcal{I}^{1-\alpha} \varphi', \varphi')_X \, ds, \quad \text{for } t > 0.
\]

Finally, we recall the following identity which follows from the generalized Leibniz formula:
\[
R \partial_t^\alpha (t \varphi) = t R \partial_t^\alpha \varphi + \alpha \mathcal{I}^{1-\alpha} \varphi.
\]
Since $\partial_t^\alpha \varphi(t) = R \partial_t^\alpha (\varphi(t) - \varphi(0))$, we see that
\[
\partial_t^\alpha (t \varphi) = t R \partial_t^\alpha \varphi + \alpha \mathcal{I}^{1-\alpha} \varphi + t \omega_{1-\alpha}(t) \varphi(0).
\]

For the rest of the paper, $C$ is a generic constant that may depend on $\alpha$ and $T$, but is independent of the spatial mesh size element $h$.

3. General error estimates. Given the elliptic problem
\[
Lu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]
with $f \in L^2(\Omega)$, we now define the solution operator $T : L^2(\Omega) \to H^1_0(\Omega)$ by
\[
a(Tf, v) = (f, v) \quad \forall v \in H^1_0(\Omega),
\]
where $a(u, v) = (A \nabla u, \nabla v) + (cu, v)$. Note that $T : L^2(\Omega) \to L^2(\Omega)$ is compact, selfadjoint and positive definite. In terms of $T$, we may write the initial-boundary value problem (1.1) as
\[
T \partial_t^\alpha u + u = Tf, \quad t > 0, \quad u(0) = u_0.
\]

For the purpose of approximating the solution of this problem, let $V_h \subset L^2(\Omega)$ be a family of finite dimensional spaces that depends on $h$, $0 < h < 1$. We assume that we are given a corresponding family of linear operators $T_h : L^2(\Omega) \to V_h$ which approximate $T$. Then consider the semidiscrete problem: find $u_h(t) \in V_h$ for $t \geq 0$ such that
\[
T_h \partial_t^\alpha u_h + u_h = T_h f, \quad t > 0, \quad u_h(0) = u_{0h} \in V_h.
\]
where $u_{\text{h}}$ is some approximation to $u_0$. We shall make the assumptions that $T_h$ is selfadjoint, positive semidefinite on $L^2(\Omega)$ and positive definite on $V_h$. Let $e(t) = u_h(t) - u(t)$ denote the error at time $t$. Then, by subtracting (3.2) from (3.1), we find that $e$ satisfies

$$T_h \partial_t^\alpha e(t) + e(t) = (T_h - T)(f - \partial_t^\alpha u)(t), \quad t > 0. \tag{3.3}$$

With $\rho(t) = (T_h - T)(f - \partial_t^\alpha u)(t)$, we, thus, obtain

$$T_h \partial_t^\alpha e(t) + e(t) = \rho(t), \quad t > 0, \tag{3.4}$$

with an initial data $e(0) \in L^2(\Omega)$. Then, by using the positive square root of $T_h$, we deduce from the positivity property of $I^\alpha$ that for $t > 0$,

$$\int_0^t (T_h I^\alpha \varphi, \varphi) \, ds \geq 0. \tag{3.5}$$

and, similarly,

$$\int_0^t (T_h R \partial_t^\alpha \varphi, \varphi) \, ds \geq 0. \tag{3.6}$$

Our task now is to derive estimates for $e$ in terms of $\rho$. We begin by proving the following result.

**Lemma 3.1.** Let $e \in C([0, T], L^2(\Omega))$ such that $T_h \partial_t^\alpha e(t) + e(t) = \rho(t)$ for $t > 0$. Then

$$\|e(t)\| \leq \|e(0)\| + 4 \left( \|\rho(0)\| + \int_0^t \|\rho(s)\| \, ds \right). \tag{3.7}$$

In addition, if $T_h e(0) = 0$, then

$$\beta \int_0^t \|T_h \partial_t^\alpha e(s)\|^2 \, ds + (1 - \beta) \int_0^t \|e(s)\|^2 \, ds \leq \int_0^t \|\rho(s)\|^2 \, ds, \quad \beta = 0, 1. \tag{3.8}$$

**Proof.** Form the $L^2(\Omega)$-inner product between (3.4) and $e_t$ to find that

$$(T_h I^{1-\alpha} e_t, e_t) + \frac{1}{2} \frac{d}{dt} \|e\|^2 = (e, \rho_t) = \frac{d}{dt} (\rho, e) - (\rho_t, e).$$

Integrate with respect to time and observe that $\int_0^t (T_h I^{1-\alpha} e_t(s), e_t(s)) \, ds \geq 0$ by (3.5). Then, it follows that

$$\|e(t)\|^2 \leq \|e(0)\|^2 + 2 \left( \|\rho(t)\| \|e(t)\| + \|\rho(0)\| \|e(0)\| + \int_0^t \|\rho_t\| \|e\| \, ds \right) \leq \sup_{s \leq t} \|e(s)\| \left( \|e(0)\| + 2 \|\rho(t)\| + 2 \|\rho(0)\| + 2 \int_0^t \|\rho_t\| \, ds \right) \leq \sup_{s \leq t} \|e(s)\| \left( \|e(0)\| + 4 \|\rho(0)\| + 4 \int_0^t \|\rho_t\| \, ds \right).$$

Here, we have used $\rho(t) = \rho(0) + \int_0^t \rho_t(s) \, ds$. Now, (3.7) follows by replacing $\|e(t)\|^2$ with $\|e(t)\| \sup_{s \leq t} \|e(s)\|$ on the left-hand side. For the second estimate (3.8), we form the $L^2(\Omega)$-inner product between (3.4) and $e$ and obtain

$$(T_h \partial_t^\alpha e, e) + \|e\|^2 = (\rho, e).$$
Then, we integrate with respect to time and note that \( \int_0^t (T_h \partial_t^\alpha e(s), e(s)) \, ds \geq 0 \) since \( T_h e(0) = 0 \) to derive (3.8) for \( \beta = 0 \). The estimate with \( \beta = 1 \) follows analogously after taking the \( L^2(\Omega) \)-inner product between (3.4) and \( T_h \partial_t^\alpha e \) and proceeding similarly. This completes the rest of the proof. \( \square \)

**Remark 3.1.** We integrate (3.4) over \((0, t)\), keeping in mind that \( T_h e(0) = 0 \) and noting that \( T_h \mathcal{I}^{1-\alpha} e = T_h \partial_t^\alpha e \) to find that

\[
T_h \partial_t^\alpha \tilde{e} + \tilde{e} = \tilde{\rho}, \quad \tilde{e}(t) = \int_0^t e(s) \, ds. \tag{3.9}
\]

An application of Lemma 3.1 yields \( \|\tilde{e}\| \leq 4 \int_0^t \|\rho(s)\| \, ds \), and hence,

\[
\|T_h \mathcal{I}^{1-\alpha} e\| \leq \|\tilde{e}\| + \|\tilde{\rho}\| \leq C \int_0^t \|\rho(s)\| \, ds.
\]

This implies \( \|T_h \mathcal{I}^{1-\alpha} e\|^2 \leq Ct \int_0^t \|\rho(s)\|^2 \, ds \). Again, using Lemma 3.1, we deduce

\[
\beta \int_0^t \|T_h \partial_t^\alpha \tilde{e}(s)\|^2 \, ds + (1 - \beta) \int_0^t \|\tilde{e}(s)\|^2 \, ds \leq \int_0^t \|\tilde{\rho}(s)\|^2 \, ds, \quad \beta \in [0, 1]. \tag{3.10}
\]

**Lemma 3.2.** Let \( e \in C([0, T], L^2(\Omega)) \) such that

\[
T_h \partial_t^\alpha e(t) + e(t) = \rho(t), \quad t > 0, \quad T_h e(0) = 0. \tag{3.11}
\]

Then

\[
\int_0^t s^2 \|e(s)\|^2 \, ds \leq 2 \int_0^t (s^2 \|\rho\|^2 + 4\|\tilde{\rho}\|^2) \, ds.
\]

**Proof.** Multiply (3.11) by \( t \) and use the identity (2.6) so that

\[
T_h \partial_t^\alpha (te) + te = t\rho + \alpha T_h \mathcal{I}^{1-\alpha} e. \tag{3.12}
\]

Form the \( L^2(\Omega) \)-inner product between (3.12) and \( te \), and integrate over \((0, t)\). Then, a use of the positivity property (3.6) shows

\[
\int_0^t s^2 \|e(s)\|^2 \, ds \leq 2 \int_0^t (s^2 \|\rho(s)\|^2 + \|T_h \mathcal{I}^{1-\alpha} e(s)\|^2) \, ds.
\]

Now, the result follows by using (3.10) with \( \beta = 1 \). This completes the rest of the proof. \( \square \)

**Lemma 3.3.** Under the assumption of Lemma 3.2 there holds for \( t > 0 \),

\[
\|e(t)\|^2 \leq C \left( \|\rho(t)\|^2 + \frac{1}{t} \int_0^t (\|\rho\|^2 + 3s^2 \|\rho\|^2) \, ds \right).
\]

**Proof.** Take the \( L^2(\Omega) \)-inner product between (3.12) and \( (te)_t \) to find that

\[
(T_h \mathcal{I}^{1-\alpha} (te)_t, (te)_t) + \frac{1}{2} \frac{d}{dt} \|te(t)\|^2 = (t\rho, (te)_t) + \alpha (T_h \mathcal{I}^{1-\alpha} e, (te)_t)
\]

\[
= \frac{d}{dt} (t\rho, te) - ((t\rho)_t, te) + \alpha \frac{d}{dt} (T_h \mathcal{I}^{1-\alpha} e, te) - \alpha (T_h \partial_t^\alpha e, te).
\]
Integrate over $(0, t)$ and use the positivity property \((5.5)\) to obtain
\[
\frac{1}{2}t^2\|e(t)\|^2 \leq \|te\| (\|t\rho\| + \alpha\|T_h\mathcal{L}^{-\alpha}e\|) + \int_0^t s(\|\rho + s\rho_t\| + \alpha\|T_h\partial_t^\alpha e\|)\|e\|ds.
\]
Hence, we derive
\[
t^2\|e(t)\|^2 \leq C \left( t^2\|\rho\|^2 + \|T_h\mathcal{L}^{-\alpha}e\|^2 + t \int_0^t (\|\rho\|^2 + \|s\rho_t\|^2 + \|T_h\partial_t^\alpha e\|^2 + \|e\|^2)ds \right).
\]
Note that, by Lemma \((3.1)\) we arrive at \(
\int_0^t (\|T_h\partial_t^\alpha e\|^2 + \|e\|^2) ds \leq 2 \int_0^t \|\rho\|^2 ds \text{ and also } \|T_h\mathcal{L}^{-\alpha}e\|^2 \leq Ct \int_t^1 \|\rho(s)\|^2 ds. \)
This completes the proof. \(\square\)

We shall now prove the main result of this section.

**Lemma 3.4.** Under the assumption of Lemma \((3.2)\), there holds for \(t > 0\),
\[
t^2\|e(t)\|^2 \leq C \left( t^2\|\rho(t)\|^2 + \|\tilde{\rho}\|^2 + \frac{1}{t} \int_0^t (\|\rho_t\|^2 + \|s\rho\|^2 + \|\tilde{\rho}\|^2) ds \right).
\]

**Proof.** Note that from \((3.12)\),
\[T_h\partial_t^\alpha (te) + te = \eta,
\]
where \(\eta = t\rho + \alpha T_h\partial_t^\alpha \tilde{e}. \) Then, by the estimate in Lemma \((3.3)\)
\[
\|te(t)\|^2 \leq C \left( \|\eta(t)\|^2 + \frac{1}{t} \int_0^t (\|\eta\|^2 + \|s\eta\|^2) ds \right).
\]
Since \(T_h\partial_t^\alpha \tilde{e} = \tilde{\rho} - \tilde{\rho}\), it follows that
\[
\|\eta(t)\|^2 \leq C \left( t^2\|\rho(t)\|^2 + \|\tilde{\rho}(t)\|^2 + \|\tilde{e}(t)\|^2 \right)
\]
\[
\leq C \left( t^2\|\rho(t)\|^2 + \|\tilde{\rho}(t)\|^2 + \frac{1}{t} \int_0^t (\|s\rho\|^2 + \|\tilde{\rho}\|^2) ds \right),
\]
where the last term is obtained by applying Lemma \((3.3)\) to \((3.9)\). For the time derivative in the integral on the right-hand side of \((3.14)\), we note using Lemma \((3.2)\) that
\[
\int_0^t \|s\eta(s)\|^2 ds \leq C \left( \int_0^t (\|s^2\rho_t(s)\|^2 + \|s\rho(s)\|^2 + s^2\|T_h\partial_t^\alpha e\|^2) ds \right)
\]
\[
\leq C \left( \int_0^t (\|s^2\rho_t(s)\|^2 + \|s\rho(s)\|^2 + s^2\|e(s)\|^2) ds \right)
\]
\[
\leq C \left( \int_0^t (\|s^2\rho_t(s)\|^2 + \|s\rho(s)\|^2 + \|\tilde{\rho}(s)\|^2) ds \right).
\]
On substitution of \((3.15)\) and \((3.16)\) in \((3.14)\), we arrive at \((3.17)\) and this completes the lemma. \(\square\)

As an immediate consequence, we obtain the following lemma.

**Lemma 3.5.** Under the assumption of Lemma \((3.2)\), there holds for \(t > 0\),
\[
\|e(t)\| \leq Ct^{-1} \sup_{s \leq t} (\|\tilde{\rho}(s)\| + s\|\rho(s)\| + s^2\|\rho_t(s)\|).
\]
Remark 3.2. Note that the estimate (3.17) is still valid in the limiting case \( \alpha = 1 \), i.e., for the parabolic problem. This estimate is established in [36, Formula (3.16)].

Remark 3.3. In the above analysis, it is possible to replace the \( L^2(\Omega) \)-inner product \((\cdot, \cdot)\) by any inner or semi-inner product \((\cdot, \cdot)\) for which \((T_h w, w)\) is nonnegative. As an example, note that since \( T_h \) is selfadjoint and positive semidefinite on \( L^2(\Omega) \), the followings
\[
(v, w)_{-r,h} = (T_h^r v, w),\quad \|v\|_{-r,h} = (T_h^r v, v)^{1/2}
\]
define a semi-inner product and a semi-norm. Applying these, for instance, in the proof of Lemma 3.1, the estimate (3.7) becomes
\[
\|e(t)\|_{-r,h} \leq \|e(0)\|_{-r,h} + 4 \left( \|\rho(0)\|_{-r,h} + \int_0^t \|\rho_t(s)\|_{-r,h} \, ds \right). \tag{3.18}
\]
This basic estimate has been used in [37] to prove certain superconvergence results.

4. Applications: Galerkin FE methods. In this part, we present some applications of our analysis to approximate the solution of (1.1) by Galerkin FE methods, and derive optimal \( L^2(\Omega) \)-error estimates for problems with smooth and nonsmooth initial data. The Galerkin methods include the standard \( C^0 \)-conforming FE method on both convex and nonconvex domains, and some nonconforming methods. Other Galerkin approximation methods, such as Galerkin spectral methods, are in many ways similar to Galerkin FE methods, as the main difference is in the choice of the finite-dimensional approximating spaces.

4.1. \( C^0 \)-conforming FE method. The weak formulation for problem (1.1) is to seek \( u : (0, T] \to H^1_0(\Omega) \) such that
\[
(\partial_t^\alpha u, v) + a(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega), \quad t > 0, \quad u(0) = u_0, \tag{4.1}
\]
where \( a(\cdot, \cdot) \) is already defined. The approximate solution \( u_h \) will be sought in the finite element space
\[
V_h = \{ v_h \in C^0(\overline{\Omega}) : v_h|_K \text{ is linear for all } K \in T_h \text{ and } v_h|_{\partial \Omega} = 0 \},
\]
where \( T_h \) is a family of shape-regular partitions of the domain \( \overline{\Omega} \) into triangles \( K \), with \( h = \max_{K \in T_h} h_K \), where \( h_K \) denotes the diameter of the element \( K \). The semidiscrete Galerkin FEM for problem for (4.1) is then defined as: find \( u_h : (0, T] \to V_h \) such that
\[
(\partial_t^\alpha u_h, v_h) + a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, \quad t > 0, \quad u_h(0) = u_{0h}, \tag{4.2}
\]
where \( u_{0h} \in V_h \) is a suitable approximation of \( u_0 \).

To derive error estimates, we introduce some more notation. Let problem \( T : L^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega) \) be the solution operator of the elliptic problem corresponding to (1.1), i.e., for \( f \in L^2(\Omega) \), we define \( Tf \) by
\[
a(Tf, v) = (f, v) \quad \forall v \in H^1_0(\Omega). \tag{4.3}
\]
Then, \( T \) is a bounded, selfadjoint and positive definite operator on \( L^2(\Omega) \). Note that from (4.1),
\[
a(u, v) = (f - \partial_t^\alpha u, v) \quad \forall v \in H^1_0(\Omega),
\]
and hence, we have an equivalent formulation as

$$T \partial^\alpha_t u + u = Tf, \quad t > 0, \quad u(0) = u_0. \quad (4.4)$$

Similarly, we let $T_h : L^2(\Omega) \to V_h$ be the solution operator of the corresponding discrete elliptic problem:

$$a(T_h f, \chi) = (f, \chi) \quad \forall \chi \in V_h. \quad (4.5)$$

Then, (4.2) is equivalently rewritten as

$$T_h \partial^\alpha_t u_h + u_h = T_h f, \quad t > 0, \quad u_h(0) = u_{0h}. \quad (4.6)$$

The operator $T_h$ is selfadjoint, positive semidefinite on $L^2(\Omega)$ and positive definite on $V_h$, see [3], and satisfies the following property:

$$\|\nabla^\ell (T_h - T)f\| \leq CH^{2-\ell}\|f\| \quad \forall f \in L^2(\Omega), \quad \ell = 0, 1. \quad (4.7)$$

Furthermore, it is easily verified that

$$T_h = T_h P_h \quad \text{and} \quad T_h = R_h T,$$

where $P_h$ is the orthogonal projection of $L^2(\Omega)$ onto $V_h$ defined by $(P_h v - v, \chi) = 0$ $\forall \chi \in V_h$, and $R_h$ is the Ritz projection $R_h : H^2(\Omega) \to V_h$ defined by the following relation: $a(R_h v - v, \chi) = 0$ $\forall \chi \in V_h$. For $t \in (0, T)$, we define the projection error $\rho(t) = R_h u(t) - u(t)$. Then, $\rho$ satisfies the following estimates [7]: for $\ell = 0, 1$,

$$\|\rho^\ell(t)\|_{H^\ell(\Omega)} \leq C h^{m-j} \|u^\ell(t)\|_{H^m(\Omega)}, \quad j = 0, 1, \quad m = 1, 2. \quad (4.8)$$

Now we prove the following theorem.

**Theorem 4.1.** Let $u$ and $u_h$ be the solutions of (4.1) and (4.2), respectively, with $f = 0$ and $u_h(0) = P_h u_0$. Then, for $u_0 \in H^2(\Omega)$, $0 \leq \delta \leq 2$,

$$\|(u_h - u)(t)\| \leq C h^2 t^{-\alpha(2-\delta)/2} \|u_0\|_{H^\delta(\Omega)}, \quad t > 0. \quad (4.9)$$

**Proof.** Let $e(t) = u_h(t) - u(t)$ be the error of the FE approximation at time $t$. Then, from (4.11) and (4.1), the error $e$ satisfies

$$T_h \partial^\alpha_t e + e = \rho, \quad \rho = (T_h - T)Lu. \quad (4.9)$$

Note that, with $u_{0h} = P_h u_0$, $T_h e(0) = 0$ since $(e(0), \chi) = 0 \forall \chi \in V_h$. Hence, we are now in position to apply Lemma 3.3. By (4.7), we deduce

$$\|e(t)\| \leq C h^2 t^{-1} \sup_{s \leq t} \|L \tilde{u}(s)\| + s \|Lu(s)\| + s^2 \|Lu(s)\|. \quad (4.10)$$

Using the regularity property in (4.7), we obtain for $u_0 \in H^\delta(\Omega)$ with $0 \leq \delta \leq 2$,

$$\|u(s)\|_{H^\delta(\Omega)} \leq C s^{-\alpha(2-\delta)/2} \|u_0\|_{H^\delta(\Omega)},$$

Hence, since $2 - \delta < 2$, we find

$$\|\tilde{u}(s)\|_{H^\delta(\Omega)} \leq \int_0^t \|u(\xi)\|_{H^\delta(\Omega)} d\xi \leq C s^{-\alpha(2-\delta)/2+1} \|u_0\|_{H^\delta(\Omega)}.$$
and, similarly,
\[ s\|u_t(s)\|_{\dot{H}^2(\Omega)} \leq C s^{-\alpha(2-\delta)/2}\|u_0\|_{\dot{H}^1(\Omega)}. \]

Combining these estimates with (4.10) completes the proof. \(\Box\)

**Remark 4.1.** By splitting the error
\[ u_h - u = (u_h - R_hu) + (R_hu - u) =: \theta + \rho, \]
noting that \(\|\theta(t)\| \leq \|(u_h - u)(t)\| + \|\rho(t)\|,\) and using the Ritz projection bound in (4.8) (with \(j = 1\) and \(m = 2\)), we conclude that the estimate in Theorem 4.1 is valid for \(\theta\). Under the quasi-uniformity condition on \(V_h\), the inverse inequality \(\|\nabla\theta(t)\| \leq Ch^{-1}\|\theta(t)\|\), and the estimate \(\|\rho(t)\|_{\dot{H}^1(\Omega)} \leq Ch\|u(t)\|_{H^2(\Omega)} \leq Ct^{-\alpha(2-\delta)/2}\|u_0\|_{\dot{H}^1(\Omega)},\) which follows from (4.8) (with \(j = 1\) and \(m = 2\)) and the regularity property (2.2), we obtain the following optimal error estimate in the \(H^1(\Omega)\)-norm:
\[ \|\nabla(u_h - u)(t)\| \leq C h^{\alpha(2-\delta)/2}\|u_0\|_{\dot{H}^1(\Omega)} \quad \text{for} \quad t \in (0, T] \text{ with } 0 \leq \delta \leq 2. \] (4.11)

**Remark 4.2.** For smooth initial data \(u_0 \in \dot{H}^2(\Omega)\), the estimate in Theorem 4.1 remains valid when \(u_h(0) = R_hu_0\). Indeed, let \(\bar{u}_h\) denote the solution of (1.2) with \(\bar{u}_h(0) = R_hu_0\). Then, \(\xi := u_h - \bar{u}_h\) satisfies
\[ Th\partial_t^\alpha \xi + \xi = 0, \quad t > 0, \quad \xi(0) = P_hu_0 - R_hu_0. \]
Since \(\xi \in \dot{V}_h\), it follows that \(\partial_t^\alpha \xi + \mathcal{L}_h \xi = 0\), where \(\mathcal{L}_h\) is the discrete operator \(\mathcal{L}_h : \dot{V}_h \to \dot{V}_h\) defined by
\[ (\mathcal{L}_h u, v) = a(u, v) \quad \forall u, v \in \dot{V}_h. \]
Then, a regularity result similar to (2.2), yields
\[ \|\xi(t)\| \leq \|(P_h - R_h)u_0\| \leq C\|u_0\|_{\dot{H}^2(\Omega)}. \]
The \(L^2(\Omega)\)-estimate follows then by the triangle inequality.

**Remark 4.3.** Instead of imposing Dirichlet boundary conditions in (1.1) we could have considered, for instance, homogeneous Neumann type boundary conditions. Assuming in such a case that \(c(x) \geq c_0 > 0\) a.e. in \(\Omega\), the operator \(\mathcal{L}\) is again positive definite, so the spaces \(\dot{H}^r(\Omega)\) may be defined in an analogous way. According to (2.2), the smoothing property (2.2) still holds and we may again introduce \(T\) and \(T_h\) and then consider both problems (4.4) and (4.6). The analysis covers this case of boundary conditions.

### 4.2. FE method on nonconvex domain.

Our next target is to study the FE approximation in the the case when the domain \(\Omega\) is a nonconvex polygonal domain in \(\mathbb{R}^2\), with (for simplicity) exactly one reentrant angle \(\omega \in (\pi, 2\pi),\) and set \(\beta = \pi/\omega \in (\frac{1}{2}, 1)\). For the special case of an L-shaped domain, \(\omega = 3\pi/2\) and \(\beta = 2/3\). It is well-known that for such a domain, the regularity of the solution of the elliptic problem \(\mathcal{L}u = f\) in \(\Omega\), \(u = 0\) on \(\partial\Omega\) in limited as a result of the singularity near the reentrant corner. Furthermore, the optimal FE error in \(L^2(\Omega)\)-norm for this problem is reduced from \(O(h^2)\) to \(O(h^{3/2})\). Indeed, we have the following error estimate:
\[ \|T_h f - f\| + h^\beta \|\nabla(T_h f - f)\| \leq C_s h^{2\beta} \|f\|_{\dot{H}^{1+\epsilon}(\Omega)}, \quad (\beta < s \leq 1), \] (4.12)
where $C_s$ depends on $s$, see [5]. We shall now demonstrate that, for the homogeneous problem, an optimal $O(h^{2\beta})$ error estimate holds for the semidiscrete approximation for smooth and nonsmooth initial data.

**Theorem 4.2.** Let $u$ and $u_h$ be the solutions of (1.1) and (4.2), respectively, with $f = 0$ and $u_h(0) = P_h u_0$. Assume that $\Omega$ is nonconvex and has exactly one reentrant angle. Then, we have, for $\beta < s \leq 1$, with $C = C_s$,

$$
\| (u_h - u)(t) \| \leq Ch^{2\beta} t^{-\alpha(1+s-\delta)/2} \| u_0 \|_{H^s(\Omega)}, \quad 0 \leq \delta \leq 1 + s.
$$

**Proof.** It is worth noticing that, while proving the estimate in Lemma 3.5, the convexity of $\Omega$ was not actually required. Using Lemma 3.5 and (4.12), we find that for $\beta < s \leq 1$

$$
\| e(t) \| \leq C_s h^{2\beta} t^{-1} \sup_{s \leq t} \left( \| \tilde{u}(s) \|_{H^{1+s}(\Omega)} + s \| u(s) \|_{H^{1+s}(\Omega)} + s^2 \| u_t(s) \|_{H^{1+s}(\Omega)} \right). \tag{4.13}
$$

Recalling that

$$
\| u(t) \|_{H^{s_1}(\Omega)} \leq C t^{-\alpha(s_1-s_2)/2} \| u(t) \|_{H^{s_2}(\Omega)}, \quad 0 \leq s_2 \leq s_1,
$$

we then complete the proof by following the arguments in the proofs of Theorems 4.1.

**4.3. Nonconforming FE methods.** Now, we come to the error analysis of nonconforming FE methods for problem (1.1). As an example, we consider the method by Crouzeix and Raviart [8], based on the nonconforming FE space

$$
\tilde{V}_h = \{ v \in L^2(\Omega) : u|_K \text{ is linear for all } K \in T_h, \text{ } v \text{ is continuous at the midpoints of the interior edges, } \text{ and } v = 0 \text{ at the midpoints of edges on } \partial \Omega \}.
$$

The discrete problem becomes: find $u_h(t) \in \tilde{V}_h$ such that

$$
(\partial_t^s u_h, v) + a_h(u_h, v) = (f, v) \quad \forall v \in \tilde{V}_h, \tag{4.14}
$$

where the bilinear form $a_h : \tilde{V}_h \times \tilde{V}_h \to \mathbb{R}$ is defined by

$$
a_h(u, v) = \sum_{K \in T_h} \int_K (A \nabla u \cdot \nabla v + cuv) \, dx, \tag{4.15}
$$

with associated broken norm

$$
\| v \|_{h} = \left( \sum_{K \in T_h} \int_K (A \nabla v \cdot \nabla v) \, dx \right)^{1/2}.
$$

Note that $\| \cdot \|_h$ is indeed a norm on $\tilde{V}_h$. Let $T_h : L^2(\Omega) \to \tilde{V}_h$ be the solution operator of the corresponding discrete elliptic problem:

$$
a_h(T_h f, \chi) = (f, \chi) \quad \forall \chi \in \tilde{V}_h.
Then, since \( a_h(\cdot, \cdot) \) is symmetric, the operator \( T_h \) is selfadjoint and positive semidefinite on \( L^2(\Omega) \): for all \( f, g \in L^2(\Omega) \)

\[
(f, T_h g) = a_h(T_h f, T_h g) = (T_h f, g) \quad \text{and} \quad (f, T_h f) = \|T_h f\|_h^2 \geq 0,
\]

and clearly, \( T_h \) is positive definite on \( \tilde{V}_h \). Furthermore, the following well-known estimate holds:

\[
\|T_h f - T f\| + h\|T_h f - T f\|_h \leq C h^2 \|f\|. \tag{4.16}
\]

With \( u_{0h} \) being the the \( L^2 \)-projection of \( u_0 \) on \( \tilde{V}_h \) so that \( T_h(u_{0h} - u_0) = 0 \), we deduce that the error estimate in Theorem 4.1 holds true for the Crouzeix-Raviart nonconforming FE solution \( u_h \).

Our analysis can also be applied to other nonconforming methods, including Nitsche’s method [33] and the Lagrange multiplier method of Babuska [1]. In Nitsche’s method, the bilinear form \( a_h(\cdot, \cdot) \) in (4.15), with \( L = -\Delta \), is given by

\[
a_h(u, \chi) = a(u, \chi) - \left( \frac{\partial u}{\partial n}, \chi \right) - \left( u, \frac{\partial \chi}{\partial n} \right) + \beta h^{-1}(u, \chi),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L^2(\partial \Omega) \), \( \partial u/\partial n \) the conormal derivative on \( \partial \Omega \) and \( \beta \) a positive constant.

Then, since \( \beta h \) is positive definite on \( \tilde{V}_h \), we formally obtain \( \partial_\xi \sigma \) in (5.2). By Green’s formula, we deduce that the error estimate in Theorem 2.1 holds true for the Crouzeix-Raviart nonconforming FE solution \( u_h \).

\[
\tilde{V}_h = \{ \chi \in C^0(\overline{\Omega}) : \chi|_K \in P_1(K) \},
\]

without any boundary conditions imposed on \( \partial \Omega \).

5. Applications: Mixed FE methods. In this section, we consider the mixed form of the problem (1.1) and establish a priori error estimates for smooth and nonsmooth initial data. To simplify the presentation, we choose \( L = -\Delta \). By introducing the new variable \( \sigma = \nabla u \), the problem can be formulated as

\[
\partial_\xi^\alpha u - \nabla \cdot \sigma = f, \quad \sigma = \nabla u, \quad u = 0 \quad \text{on} \partial \Omega,
\]

with \( u(0) = u_0 \). Let \( H(div; \Omega) = \{ \sigma \in (L^2(\Omega))^2 : \nabla \cdot \sigma \in L^2(\Omega) \} \) be a Hilbert space equipped with norm \( \|\sigma\|_{H} = (\|\sigma\|^2 + \|\nabla \cdot \sigma\|^2)^{1/2} \). Then, with \( V = L^2(\Omega) \) and \( W = H(div; \Omega) \), the weak mixed formulation of (1.1) is defined as follows: find \( (u, \sigma) : (0, T) \rightarrow V \times W \) such that

\[
(\partial_\xi^\alpha u, v) - (\nabla \cdot \sigma, v) = (f, v) \quad \forall v \in V; \tag{5.1}
\]

\[
(\sigma, w) + (u, \nabla \cdot w) = 0 \quad \forall w \in W, \tag{5.2}
\]

with \( u(0) = u_0 \). Note that the boundary condition \( u = 0 \) on \( \partial \Omega \) is implicitly contained in (5.2). By Green’s formula, we formally obtain \( \sigma = \nabla u \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \).

Well-posedness of problem (5.1) is established in [35] based on a spectral decomposition approach. In particular, for \( u_0 \in L^2(\Omega) \) and \( f = 0 \), it is shown that the problem (5.1) has a unique weak solution \( u \in C([0, T], L^2(\Omega)) \cap C((0, T], H^2(\Omega)) \) with \( \partial_\xi^\alpha u \in C((0, T], L^2(\Omega)) \), see [35] Theorem 2.1. The regularity results for the inhomogeneous problem with a vanishing initial data are given in [35] Theorem 2.2. The well-posedness of (5.1)-(5.2) can then be established using the equivalence of the weak formulation (1.1) and the mixed formulation (5.1)-(5.2) based on the results in [35].
5.1. Semidiscrete mixed FE problem. For the semidiscrete mixed formulation corresponding to (5.1)-(5.2), let, as before, $T_h$ be a shape-regular partition of the polygonal convex domain $\bar{\Omega}$ into triangles $K$ of diameter $h_K$. Further, let $V_h$ and $W_h$ be appropriate finite element subspaces of $V$ and $W$ satisfying the Ladyzenskaya-Babuska-Brezzi (LBB) condition. For example, let $V_h$ and $W_h$ be the Raviart-Thomas spaces $[34]$ of index $\ell \geq 0$ defined by

$$V_h = \{ v \in L^2(\Omega) : v|_K \in P_\ell(K) \forall K \in T_h \}$$

and

$$W_h = \{ v \in H(div,\Omega) : v|_K \in RT_\ell(K) \forall K \in T_h \}$$

where $RT_\ell(K) = (P_\ell(K))^2 + xP_\ell(K)$. We note that high order Raviart-Thomas elements do not lead to optimal error estimates due to the limited solution regularity. Hence, we shall consider only the case $\ell = 1$. For more examples of these spaces including Brezzi-Douglas-Marini spaces and Brezzi-Douglas-Fortin-Marini spaces, etc., see [4].

The corresponding semidiscrete mixed finite element approximation is to seek a pair $(u_h, \sigma_h) : (0,T] \to V_h \times W_h$ such that

$$(\partial_t^\alpha u_h, v_h) - (\nabla \cdot \sigma_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, \quad (5.3)$$

$$(\sigma_h, w_h) + (u_h, \nabla \cdot w_h) = 0 \quad \forall w_h \in W_h, \quad (5.4)$$

with $u_h(0) = u_{0h}$, where $u_{0h}$ an appropriate approximation of $u_0$ in $V_h$. With bases for $V_h$ and $W_h$, the matrix form of the discrete problem is

$$A\partial_t^\alpha U - B\Sigma = F,$$

$$B^TU + D\Sigma = 0, \quad \text{for } t > 0, \quad U(0) \text{ given,}$$

where $U$ and $\Sigma$ are vectors corresponding to $u_h$ and $\sigma_h$. It is easily seen that the matrices $A$ and $D$ are positive definite. Eliminating $\Sigma$, we have the system of fractional ODEs

$$A\partial_t^\alpha U + BD^{-1}B^TU = F, \quad \text{for } t > 0, \quad U(0) \text{ given,}$$

which by standard results in fractional ODE theory has a unique solution, see [21, Chapter 3].

For $(u, \sigma) \in V \times W$, we define the intermediate mixed projection as the pair $(\tilde{u}_h, \tilde{\sigma}_h) \in V_h \times W_h$ satisfying

$$(\nabla \cdot (\sigma - \tilde{\sigma}_h), v_h) = 0 \quad \forall v_h \in V_h, \quad (5.5)$$

$$((\sigma - \tilde{\sigma}_h), w_h) + (u - \tilde{u}_h, \nabla \cdot w_h) = 0 \quad \forall w_h \in W_h. \quad (5.6)$$

Then, the following estimates hold, see for instance [17, Theorem 1.1],

$$\|u - \tilde{u}_h\| \leq Ch^2\|u\|_{H^2(\Omega)}, \quad \|\sigma - \tilde{\sigma}_h\| \leq Ch^s\|u\|_{H^{s+1}(\Omega)}, \quad s = 1, 2, \quad (5.7)$$

and on quasi-uniform meshes,

$$\|u - \tilde{u}_h\|_{L^\infty(\Omega)} \leq Ch^s |\ln h| \|u\|_{H^{s+1}(\Omega)}, \quad s = 1, 2. \quad (5.8)$$

In our error analysis, we shall use the following result, see [17, Lemma 1.2].
Lemma 5.1. There exists a constant $C$ such that for any pair $(\theta_h, z) \in V_h \times (L^2(\Omega))^2$ satisfying
\[
(z, w_h) + (\theta_h, \nabla \cdot w_h) = 0 \quad \forall w_h \in W_h,
\]
we have
\[
\|\theta_h\|_{L^\infty(\Omega)} \leq C|\ln h| \|z\|.
\]

We now start deriving error estimates for smooth initial data using energy arguments. Since the problem has a limited smoothing property, integration in time with a $t$ type weight is an essential tool to provide optimal error estimates. This idea has been used in [19] and [30] to derive optimal error bounds for problems (1.3) and (1.1), respectively. A similar approach applied to mixed finite element methods for parabolic problems has also been exploited in [12].

5.2. Error estimates with smooth initial data. For the error analysis, define $e_u = u_h - u$ and $e_\sigma = \sigma_h - \sigma$. Then, from (5.1)-(5.2) and (5.3)-(5.4), $e_u$ and $e_\sigma$ satisfy the following equations
\[
(\partial_t^\alpha e_u, v_h) - (\nabla \cdot e_\sigma, v_h) = 0 \quad \forall v_h \in V_h, \quad (5.9)
\]
\[
(e_\sigma, w_h) + (e_u, \nabla \cdot w_h) = 0 \quad \forall w_h \in W_h. \quad (5.10)
\]

To derive a priori error estimates for the semidiscrete FE problem (5.3)-(5.4), we split the errors
\[
e_u := (u_h - \tilde{u}_h) - (\tilde{u}_h - u) =: \theta - \rho,
\]
\[
e_\sigma := (\sigma_h - \tilde{\sigma}_h) - (\tilde{\sigma}_h - \sigma) =: \xi - \zeta.
\]

From (5.9)-(5.10), we note that $\theta$ and $\xi$ satisfy
\[
(\partial_t^\alpha e_u, v_h) - (\nabla \cdot e_\sigma, v_h) = 0 \quad \forall v_h \in V_h, \quad (5.11)
\]
\[
(\xi, w_h) + (\theta, \nabla \cdot w_h) = 0 \quad \forall w_h \in W_h. \quad (5.12)
\]

In the next lemma, we derive preliminary bounds for $e_u$ and $\xi$. To do so, we let $u_h(0) = P_h u_0$, where $P_h$ denotes here the $L^2$-projection of $V$ onto $V_h$.

Lemma 5.2. For $0 < t \leq T$, we have
\[
\int_0^t (I^{1-\alpha} e_u, e_u) \, ds + \|I\xi(t)\|^2 \leq C \int_0^t |(I^{1-\alpha} \rho, \rho)| \, ds.
\]

Proof. Integrate (5.11) over the time interval $(0, t)$ and use the identity $I^{2-\alpha} v'(t) = \int_0^t (I^{1-\alpha} v(t) - \omega_{2-\alpha}(t)v(0))$ to obtain
\[
(I^{1-\alpha} e_u, v_h) - (\nabla \cdot I\xi, v_h) = \omega_{2-\alpha}(t)(e_u(0), v_h) \quad \forall v_h \in V_h. \quad (5.13)
\]

Since $u_h(0) = P_h u_0$, $(e_u(0), \chi) = 0$, and Therefore
\[
(I^{1-\alpha} e_u, v_h) - (\nabla \cdot I\xi, v_h) = 0 \quad \forall v_h \in V_h. \quad (5.14)
\]
Now choose \( v_h = \theta \) in (5.12) and \( w_h = I \xi \) in (5.12), and add the resulting equations to obtain after integration
\[
\int_0^t (I^{1-\alpha} e_u, e_u) \, ds + \int_0^t (\xi, I \xi) \, ds = -\int_0^t (I^{1-\alpha} e_u, \rho) \, ds.
\]
By the continuity of the operator \( I^{1-\alpha} \) in (2.3) with \( \epsilon = 1/2 \), we see that
\[
\left| \int_0^t (I^{1-\alpha} e_u, \rho) \, ds \right| \leq \frac{1}{2} \int_0^t \|I^{1-\alpha} e_u, e_u\| + C(\alpha) \int_0^t \|I^{1-\alpha} \rho\| \, ds.
\]
Noting that \( \langle \xi, I \xi \rangle = \frac{1}{2 \pi} \|I \xi\|^2 \), we deduce
\[
\int_0^t (I^{1-\alpha} e_u, e_u) \, ds + \|I \xi\|^2 \leq C \int_0^t \|I^{1-\alpha} \rho\| \, ds.
\]
This completes the proof.

In the next lemma, we derive an upper bound for \( \theta \) and \( \xi \). This bound leads to optimal convergence rates in the \( L^2(\Omega) \)-norm of \( e_u \) and \( e_\sigma \), and a quasi-optimal convergence rate in \( L^\infty(\Omega) \)-norm for \( e_u \).

**Lemma 5.3.** For \( 0 < t \leq T \), we have
\[
\|\theta(t)\|^2 + \|\xi(t)\|^2 \leq C t^{\alpha-2} \int_0^t \left[ \|I^{1-\alpha} \rho\| + \|I^{1-\alpha} \rho_1\| \|\rho_1\| \right] \, ds.
\]

**Proof.** Multiply both sides of (5.11) by \( t \) and use (2.6) to find with \( \theta_1 = t \theta \) and \( \xi_1 = t \xi \) that
\[
(I^{1-\alpha} \theta_1', v_h) - (\nabla \cdot \xi_1, v_h) = (I^{1-\alpha} \rho_1', \xi_1, v_h) \quad \forall v_h \in V_h.
\]
Next multiply both sides of (5.12) by \( t \) and differentiate with respect to time to arrive at
\[
(\xi_1', w_h) + (\theta_1', \nabla \cdot w_h) = 0 \quad \forall w_h \in W_h.
\]
Choose \( v_h = \theta_1' \) in (5.15) and \( w_h = \xi_1 \) in (5.16), then add the resulting equations to obtain after integration
\[
\int_0^t (I^{1-\alpha} \theta_1', \theta_1') \, ds + \int_0^t (\xi_1', \xi_1) \, ds = -\int_0^t (I^{1-\alpha} \rho_1', \theta_1') \, ds + \alpha \int_0^t (I^{1-\alpha} e_u, \theta_1') \, ds.
\]
Note that \( (\xi_1', \xi_1) = \frac{1}{2 \pi} \|t \xi\|^2 \). Using the continuity of the operator \( I^{1-\alpha} \) and the estimate in Lemma 5.3, we obtain after simplification
\[
\int_0^t (I^{1-\alpha} \theta_1', \theta_1') \, ds + t^2 \|\xi(t)\|^2 \leq C \int_0^t \|I^{1-\alpha} \rho_1', \rho_1'\| \, ds + C \int_0^t \|I^{1-\alpha} \rho, \rho\| \, ds.
\]
Then, the desired estimate follows from (2.4). This concludes the proof.

Using the previous lemmas, we now derive optimal error estimates for the semidiscrete mixed finite element problem with smooth initial data.

**Theorem 5.4.** Let \( (u, \sigma) \) and \( (u_h, \sigma_h) \) be the solutions of (5.1) - (5.2) and (5.3) - (5.4), respectively, with \( f = 0 \) and \( u_{h0} = P_h u_0 \). Then, for \( u_0 \in H^\delta(\Omega) \) with \( \delta \in [1, 2] \), the following error estimates hold:
\[
\| (u_h - u)(t) \| \leq C h^2 t^{-\alpha(2-\delta)/2} \| u_0 \|_{H^\delta(\Omega)}, \quad t > 0,
\]
\[
(5.19)
\]
\[ \| (\sigma_h - \sigma) (t) \| \leq C h^2 t^{-\alpha + (3 - \delta)/2} \| u_0 \|_{H^s(\Omega)}, \quad t > 0, \quad (5.20) \]

and with an additional quasi-uniformity condition on the mesh,

\[ \| (u_h - u)(t) \|_{L^\infty(\Omega)} \leq C h^2 | \ln h | t^{-\alpha + (3 - \delta)/2} \| u_0 \|_{H^s(\Omega)}, \quad t > 0. \quad (5.21) \]

**Proof.** Using the first estimate in (5.7) and (2.2), we find after integration that for \( t \in (0, T] \),

\[ \int_0^t \left[ \| I^{1-\rho} \| \| \rho \| + \| I^{1-\rho} \| \| \rho \| \right] ds \leq C h^4 \int_0^t s^{1+\alpha(\delta-3)} ds \| u_0 \|_{H^s(\Omega)} = C h^4 t^{2+\alpha(\delta-3)} \| u_0 \|_{H^s(\Omega)}, \quad \delta \in [1, 2]. \]

Then, from Lemma 5.3 it follows that

\[ \| \theta(t) \| \leq C h^2 t^{-\alpha(2-\delta)/2} \| u_0 \|_{H^s(\Omega)}, \quad \delta \in [1, 2]. \]

The bound (5.19) follows now from the decomposition \( u_h - u = \theta - \rho \), and the estimate of \( \rho \) in (5.7). To establish (5.20), we first note that Lemma 5.3 and previous estimates yield

\[ \| \psi(t) \| \leq C h^2 t^{-\alpha(3-\delta)/2} \| u_0 \|_{H^s(\Omega)}, \quad \delta \in [1, 2]. \]

From (5.7) and (2.2), we arrive at

\[ \| \zeta(t) \| \leq C h^2 \| u(t) \|_{H^s(\Omega)} \leq C h^2 t^{-\alpha + (3 - \delta)/2} \| u_0 \|_{H^s(\Omega)}, \quad \delta \in [1, 2]. \]

Then, (5.21) follows from the decomposition \( \sigma_h - \sigma = \xi - \zeta \). Finally, in order to show (5.21), we apply Lemma 5.1 to (5.12) and obtain, by the quasi-uniformity of \( \mathcal{T}_h \),

\[ \| \theta(t) \|_{L^\infty(\Omega)} \leq C | \ln h | \| \psi(t) \|. \]

Hence, by Lemma 5.3,

\[ \| \theta(t) \|_{L^\infty(\Omega)} \leq C h^2 | \ln h | t^{-\alpha(3-\delta)/2} \| u_0 \|_{H^s(\Omega)}, \quad \delta \in [1, 2]. \]

Together with the estimate (5.8), this completes the proof of (5.21). \( \Box \)

### 5.3. Error estimates with nonsmooth initial data.

Our next purpose is to derive error estimates for nonsmooth initial data, i.e., for \( u_0 \in L^2(\Omega) \). To this end, we combine our analysis developed in Section 3 with the results of the previous subsection. For a given function \( f \in L^2(\Omega) \), let \( (u_h, \sigma_h) \in V_h \times W_h \) be the unique solution of the mixed elliptic problem

\[ \begin{align*}
- (\nabla \cdot \sigma_h, v_h) &= (f, v_h) \quad \forall v_h \in V_h, \\
(\sigma_h, w_h) + (u_h, \nabla \cdot w_h) &= 0 \quad \forall w_h \in W_h.
\end{align*} \quad (5.22, 5.23) \]

Then, we define a pair of operators \( (T_h, S_h) : L^2(\Omega) \to V_h \times W_h \) as \( T_h f = u_h \) and \( S_h f = \sigma_h \). With \( T : L^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega) \) being the solution operator of the continuous problem (1.1), the following result holds (see [17, Lemma 1.5]):

**Lemma 5.5.** The operator \( T_h : L^2(\Omega) \to V_h \) defined by \( T_h f = u_h \) is selfadjoint, positive semidefinite on \( L^2(\Omega) \) and positive definite on \( V_h \). Further

\[ \| T_h f - T f \| \leq C h^2 \| f \|. \]
We are now ready to prove the following nonsmooth data error estimates. In the proof, we need the following inverse property:

$$\|\nabla \cdot \xi\| \leq Ch^{-1}\|\xi\| \quad \forall \xi \in W_h.$$ (5.24)

**Theorem 5.6.** Let $$(u, \sigma)$$ and $$(u_h, \sigma_h)$$ be the solutions of (5.1) - (5.2) and (5.3) - (5.4), respectively, with $$f = 0$$ and $$u_{0h} = P_h u_0$$. Then

$$\|(u_h - u)(t)\| \leq C h^2 t^{-\alpha} \|u_0\|, \quad t > 0.$$ (5.25)

If the mesh is quasi-uniform, then

$$\|(\sigma_h - \sigma)(t)\| \leq C h t^{-\alpha} \|u_0\|, \quad t > 0,$$ (5.26)

and

$$\|(u_h - u)(t)\|_{L^\infty(\Omega)} \leq C h |\ln h| t^{-\alpha} \|u_0\|, \quad t > 0.$$ (5.27)

**Proof.** From the definition of the operator $$T_h$$ above, the semidiscrete problem may also be written as

$$T_h \partial_t^\alpha u_h + u_h = 0, \quad t > 0, \quad u_h(0) = P_h u_0.$$ 

Recalling the definition of the continuous operator $$T$$, we deduce that

$$T_h \partial_t^\alpha e_u + e_u = -(T_h - T) \Delta u, \quad t > 0, \quad T_h e_u(0) = 0.$$ 

Since $$T_h$$ satisfies the properties in Lemma 5.5, the estimate (5.25) follows immediately from Lemma 5.8 and the regularity result in (2.2). In order to show (5.26), we use (5.12) and the inverse inequality (5.24) to obtain

$$\|\xi(t)\|^2 \leq \|\theta\| \|\nabla \cdot \xi\| \leq C h^{-1}\|\theta\| \|\xi\|.$$ 

Since, by (5.25) and (5.7), $$\|\theta\| \leq \|e_u\| + \|\rho\| \leq C h^2 t^{-\alpha} \|u_0\|$$, it follows that

$$\|\xi(t)\| \leq C h^{-1} t^{-\alpha} \|u_0\|.$$ (5.28)

Together with

$$\|\xi(t)\| \leq C h \|u(t)\|_2 \leq C h t^{-\alpha} \|u_0\|,$$

this establishes (5.25). Finally, we derive (5.27) by using the estimates $$\|\theta(t)\|_{L^\infty(\Omega)} \leq C |\ln h| \|\xi(t)\|$$ and (5.28).

**Remark 5.1.** The results in Theorems 5.4 are optimal with respect to the polynomial degree and data regularity. In the limiting case $$\alpha = 1$$, we find the bounds derived in [17] for the parabolic problem. The nonsmooth data error estimate (5.20) established in Theorem 5.6 is also optimal, whereas the last two error bounds are not. This is due to the limited smoothing property of the time-fractional equation. Note that due to the presence of the limited smoothing property, high order finite elements do not provide better error estimates in the case of nonsmooth initial data. Finally, it is worth to mention that the analysis of mixed methods extends to problems on nonconvex domains.
6. Multi-term time-fractional problem. In this section, we briefly discuss
the extension of our analysis to the following multi-term time-fractional diffusion
problem:
\[ P(\partial_t)u + Lu = f \text{ in } \Omega \times (0, T], \quad u(0) = u_0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \times (0, T], \quad (6.1) \]
where the multi-term differential operator \( P(\partial_t) \) is defined by
\[ P(\partial_t) = \partial_t^\alpha + \sum_{i=1}^m b_i \partial_t^{\alpha_i}, \]
with \( 0 < \alpha_m \leq \cdots \leq \alpha_1 \leq \alpha < 1 \) being the orders of the fractional Caputo derivatives, and \( b_i > 0, i = 1, \ldots, m \). The multi-term differential operator \( P(R\partial_t) \) is defined
analogously. The model (6.1) was developed to improve the modeling accuracy of
the single-term model (1.1) for describing anomalous diffusion. With the notation of
Section 3, we consider the following initial value problem:
\[ T_h P(\partial_t) e(t) + e(t) = \rho(t), \quad t > 0, \quad (6.2) \]
with an initial data in \( L^2(\Omega) \). The operator \( T_h \) satisfies the conditions stated in
Section 3. Then, we have the following result.

**Lemma 6.1.** Let \( e \in C([0, T], L^2(\Omega)) \) satisfy (6.2) with \( T_h e(0) = 0 \). Then, there
holds for \( t > 0 \),
\[ \|e(t)\| \leq Ct^{-1} \sup_{s \leq t} (\|\hat{\rho}(s)\| + s\|\rho(s)\| + s^2\|\rho_t(s)\|). \]

**Proof.** We first introduce the time-fractional integral operator \( Q(I) \) defined by
\[ Q(I) = \alpha I^{1-\alpha} + \sum_{i=1}^m \alpha_i b_i I^{1-\alpha_i}. \]
Then, results similar to (3.5) and (3.6), follow from the following positivity properties:
\[ \int_0^t (Q(I)\varphi, \varphi) \, ds \geq 0 \quad \text{and} \quad \int_0^t (P(R\partial_t)\varphi, \varphi) \, ds \geq 0. \quad (6.3) \]
Furthermore, the generalized Leibniz formula takes the form: with \( \varphi(0) = 0 \),
\[ P(\partial_t)(t\varphi) = t P(\partial_t)\varphi + Q(I)\varphi. \quad (6.4) \]
Using (6.3) and (6.4), we then prove Lemma 6.1 by following line-by-line the proofs
of Lemmas 3.1-3.4 where \( \partial_t^\alpha \) is replaced by \( P(\partial_t) \) and \( \alpha I^{1-\alpha} \) is replaced by \( Q(I) \).

Regularity properties of the solution of problem (6.1) can be found in [15]. For
\( f = 0 \) and \( u_0 \in H^\varsigma(\Omega), \varsigma \in [0, 2] \), it is shown that (see [15] Theorem 2.1)
\[ \|P(\partial_t)^t u(t)\|_{H^\varsigma(\Omega)} \leq Ct^{-\alpha(\ell + (p - q)/2)}\|u_0\|_{H^\varsigma(\Omega)}, \quad t > 0, \]
where for \( \ell = 0, 0 \leq p - q \leq 2 \) and for \( \ell = 1, -2 \leq p - q \leq 0 \). In addition to
these results, one can verify that the solution of (6.1) satisfies the regularity property
stated in (2.2). As an immediate consequence, we conclude that all the the error
estimates achieved in Section 4 for problem (1.1) and in Section 5 for the mixed form
remain valid for the multi-term time-fractional problem (6.1) based on our analysis,
with the only exception that some minor modifications are needed in the proof of
Theorem 5.4. Theorem 4.1 provides, in particular, an improvement of the nonsmooth
data error estimate established in [15] Theorem 3.2] where an additional log factor is
involved.
7. Conclusions. In this paper we provided a unified error analysis for semidiscrete time-fractional parabolic problems and derive optimal error estimates for both smooth and nonsmooth initial data. The analysis depends on known properties of the associated elliptic problems. Examples including spatial approximations by conforming and nonconforming Galerkin FEMs, and by FEM on nonconvex domains have been discussed. Further examples, including space-time fractional parabolic equations can be considered. Particularly interesting in this study, is the mixed form which fits within the framework of the present analysis. Error estimates in gradient and maximum norms deserve further investigation. An interesting future research direction is the analysis of mixed finite element methods applied to the time-fractional Stokes equations.

Acknowledgements. The author thanks Prof. Amiya K. Pani for valuable comments and suggestions.

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