On (3,3)-Gaussian Quadratic Stochastic Operators

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Abstract. In this paper, we define \((n,k)\)-Gaussian quadratic stochastic operators and investigate the trajectory behavior of \((3,3)\)-Gaussian quadratic stochastic operators. It is established the necessary and sufficient conditions for regularity of these operators.

1. Introduction

Let \((X,\mathcal{F})\) be a measurable space, where \(X\) is a state space and \(\mathcal{F}\) is \(\sigma\)-algebra on \(X\), and \(S(X,\mathcal{F})\) be the set of all probability measures on \((X,\mathcal{F})\). Let \(\{P(x,y,A) : x, y \in X, A \in \mathcal{F}\}\) be a family of functions on \(X \times X \times \mathcal{F}\) such that for any fixed \(x, y \in X\) \(P(x,y,\cdot) \in S(X,\mathcal{F})\); \(P(x,y,A)\) regarded as a function of two variables \(x\) and \(y\) with fixed \(A \in \mathcal{F}\) is a measurable function on \((X \times X, \mathcal{F} \otimes \mathcal{F})\), and \(P(x,y,A) = P(y,x,A)\) for any \(x, y \in X\) and \(A \in \mathcal{F}\). We consider a nonlinear transformation called quadratic stochastic operator (qso) \(V : S(X,\mathcal{F}) \rightarrow S(X,\mathcal{F})\) defined by

\[
(V\lambda)(A) = \int_X \int_X P(x,y,A)d\lambda(x)d\lambda(y),
\]

where \(A \in \mathcal{F}\) is an arbitrary measurable set.

Assume \(\{V^n\lambda : n = 0, 1, 2, \cdots\}\) is the trajectory of the initial point \(\lambda \in S(X,\mathcal{F})\), where \(V^{n+1}\lambda = V(V^n\lambda)\) for all \(n = 0, 1, 2, \cdots\). One of the main problem for considered dynamical system (1) is to describe the limit points of \(\{V^n\lambda : n = 0, 1, 2, \cdots\}\) for arbitrary \(\lambda \in S(X,\mathcal{F})\). In measure theory, there are various notions of the convergence of measures: weak convergence, strong convergence, total variation convergence. In this paper we will consider strong convergence, which is defined as follows.

**Definition 1** For \((X,\mathcal{F})\) a measurable space, a sequence \(\{\mu_n\}\) is said to converge strongly to a limit \(\mu\) if

\[
\lim_{n \to \infty} \mu_n(A) = \mu(A)
\]

for every set \(A \in \mathcal{F}\).

If a state space \(X = \{1, 2, \cdots, m\}\) be a finite set and corresponding \(\sigma\)-algebra be a power set \(\mathcal{P}(X)\), i.e., the set of all subsets of \(X\), then the set of all probability measures on \((X,\mathcal{F})\) has the following form:

\[
S^{m-1} = \{x = (x_1, x_2, \cdots, x_m) \in R^m : \text{for any } i x_i \geq 0, \text{ and } \sum_{i=1}^m x_i = 1\}
\]
that is called a \((m - 1)\)-dimensional simplex. In this case for any \(i,j \in X\) a probabilistic measure \(P(i,j,\cdot)\) is a discrete measure with \(\sum_{k=1}^{m} P(ij,\{k\}) = 1\), where \(P(ij,\{k\}) = P_{ij,k}\) and corresponding qso \(V\) has the following form

\[
(Vx)_k = \sum_{i,j=1}^{m} p_{ij,k}x_i x_j
\]

for any \(x \in S^{m-1}\) and for all \(k = 1, \ldots, m\), where

\(a) p_{ij,k} \geq 0, \ b) p_{ij,k} = p_{ji,k} \text{ for all } i,j,k; \ c) \sum_{k=1}^{m} p_{ij,k} = 1.\)

Quadratic stochastic operators (3) defined on finite state space can be reinterpreted in terms of evolutionary operator of free population [2, 5, 6, 9, 10, 12, 13, 14, 18, 19, 20], evolutionary games [1], [3], [8], gene conversion [16], [17], and in those forms that has a fair history. In [7, 15], it was given along self-contained exposition of the recent achievements and open problems in the theory of the QSO.

It is natural to introduce and study qso on infinite state space. In addition to mathematical motivation such operators one can interpret as evolutionary operator of free population with infinite genotypes or evolutionary game with infinitely many strategies or gene conversion model with infinite alleles. Below we will use the following evident statement.

**Proposition 1** Let \(\{\lambda_1, \lambda_2, \ldots, \lambda_m\}\) be a finite family of probability measures on \((X, F)\). If for sequence of \(m\)-tupls \(\{(a_1^{(n)}, a_2^{(n)}, \ldots, a_m^{(n)})\} \in S^{m-1} : n = 0, 1, \ldots\) there exists limit \((a_1, a_2, \ldots, a_m) \in S^{m-1}\), then for the sequence of convex combinations of measures \(\{\lambda_1, \lambda_2, \ldots, \lambda_m\}\)

\[
\mu_n = \sum_{i=1}^{m} a_i^{(n)} \lambda_i, n = 0, 1, \ldots
\]

there exists strong limit

\[
\lim_{n \to \infty} \mu_n = \sum_{i=1}^{m} a_i \lambda_i. (4)
\]

**Definition 2** A qso \(V\) is called strong regular if for any initial point \(\lambda \in S(X, F)\) the strong limit

\[
\lim_{n \to \infty} V^n \lambda (5)
\]

exists.

In this paper, we introduce nonlinear transformations generated by a family of Gaussian measures. The modern theory of Gaussian measures lies at the intersection of the theory of random processes, functional analysis, and mathematical physics and is closely connected with diverse applications in quantum field theory, statistical physics, financial mathematics, and other areas of sciences. The study of Gaussian measures combines ideas and methods from probability theory, nonlinear analysis, geometry, linear operators, and topological vector spaces.

Let \(\mathbb{R}\) be the set of real numbers, \(\mathbb{B}\) be the \(\sigma\)-algebra of Borel subsets of \(\mathbb{R}\) and let \(S(\mathbb{R}, \mathbb{B})\) be the set of all probability measures on \((\mathbb{R}, \mathbb{B})\). Let \(\{P(x, y, A) : x, y \in \mathbb{R}, A \in \mathbb{B}\}\) be a family of functions on \(\mathbb{R} \times \mathbb{R} \times \mathbb{B}\) that satisfy the following conditions:

i) \(P(x, y, \cdot) \in S(\mathbb{R}, \mathbb{B})\), for any fixed \(x, y \in \mathbb{R}\), that is, \(P(x, y, \cdot) : \mathbb{B} \to [0, 1]\) is the probabilisty measure on \(\mathbb{B}\);
ii) $P(x, y, A)$ regarded as a function of two variables $x$ and $y$ with fixed $A \in \mathcal{B}$ is measurable function on $(\mathbb{R} \times \mathbb{R}, \mathcal{B} \otimes \mathcal{B})$;

iii) $P(x, y, A) = P(y, x, A)$ for any $x, y \in \mathbb{R}, A \in \mathcal{B}$.

Now we introduce a Gaussian qso as follows [4]. Let $\{P(x, y, A) : x, y \in \mathbb{R}, A \in \mathcal{B}\}$ be a family of functions such that

$$
P(x, y, A) = \frac{1}{\psi(x, y)\sqrt{2\pi}} \int_A e^{-\frac{(u-x)^2}{2\psi^2(x, y)}} du,
$$

where $\varphi(x, y)$ and $\psi(x, y)$ are symmetric measurable functions with $\psi(x, y) > 0$ for any $x, y \in \mathbb{R}$.

**Definition 3** An operator $V$ generated by family of functions (6) is called Gaussian quadratic stochastic operator.

Thus a Gaussian quadratic stochastic operator is determined by pair of symmetric measurable functions $\{\varphi(x, y); \psi(x, y)\}$, where second function $\psi(x, y)$ is strictly positive.

**Definition 4** A Gaussian qso is called $(n, k)$—Gaussian, if $\{\varphi(x, y) ; \psi(x, y)\}$ are simple symmetric functions with $|\{\varphi(x, y) : x, y \in X\}| = n$ and $|\{\psi(x, y) : x, y \in X\}| = k$.

In [4], it is shown that the $(n, k)$—Gaussian qso is regular transformation for case $n, k = 1, 2$. In this paper, we consider $(3, 3)$—Gaussian qso.

2. (3,3)-Gaussian qso
Let $a, b \in \mathbb{R}$ be a fixed numbers with $a < b$. Assume that both functions $\{\varphi(x, y)\}$ and $\psi(x, y)$ are simple functions and take three different values, namely,

$$
\varphi(x, y) = \begin{cases} 
m_1 & \text{if } x < a, y < a \text{ or } x < a, y \geq b \text{ or } x \geq b, y < a \text{ or } x \geq b, y \geq b \\
m_2 & \text{if } \{x < a\} \cup \{x \geq b\}, a \leq y < b \text{ or } a \leq x < b, \{y < a\} \cup \{y \geq b\} \\
m_3 & \text{if } a \leq x < b; a \leq y < b
\end{cases}
$$

and

$$
\psi(x, y) = \begin{cases} 
\sigma_1 & \text{if } x < a, y < a \text{ or } x < a, y \geq b \text{ or } x \geq b, y < a \text{ or } x \geq b, y \geq b \\
\sigma_2 & \text{if } \{x < a\} \cup \{x \geq b\}, a \leq y < b \text{ or } a \leq x < b, \{y < a\} \cup \{y \geq b\} \\
\sigma_3 & \text{if } a \leq x < b; a \leq y < b
\end{cases}
$$

where $\sigma_i > 0$ for $i = 1, 2, 3$.

Let $V$ be the $(3, 3)$—Gaussian qso generated by these functions. For any initial measure $\lambda \in S(\mathbb{R}, \mathcal{B})$ we let $B_1(\lambda) = \lambda((\infty, a))$, $B_2(\lambda) = \lambda([a, b))$ and $B_3(\lambda) = \lambda([b, \infty))$, and respectively for Gaussian measure $\mu$ with mean $m$ and variance $\sigma^2$ we have

$$
B_1(\mu(m, \sigma)) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2} \frac{(u-m)^2}{\sigma^2}} du,
$$

$$
B_2(\mu(m, \sigma)) = \frac{1}{\sigma\sqrt{2\pi}} \int_{a}^{b} e^{-\frac{1}{2} \frac{(u-m)^2}{\sigma^2}} du,
$$

and

$$
B_3(\mu(m, \sigma)) = \frac{1}{\sigma\sqrt{2\pi}} \int_{b}^{\infty} e^{-\frac{1}{2} \frac{(u-m)^2}{\sigma^2}} du.
$$
Now we consider a trajectory of the $(3, 3)$-Gaussian qso. Let $\mu_i$ is the Gaussian measure with mean $m_i$ and variance $\sigma_i^2$, where $i = 1, 2, 3$. Assume $A_z = (-\infty, z]$. For any measure $\lambda \in S(\mathbb{R}, \mathcal{B})$ and arbitrary $z \in \mathbb{R}$ we have

\[
V\lambda(A_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{a} P(x, y, A_z) d\lambda(x) d\lambda(y) + \int_{-\infty}^{a} \int_{-\infty}^{a} P(x, y, A_z) d\lambda(x) d\lambda(y)
\]

Thus by induction for sequence $V^n(\lambda)$ we produce the following recurrent equation

\[
V^{n+1}(A_z) = [B^n_2(\lambda) + B^n_3(\lambda) + 2B_1(\lambda)B_3(\lambda)]\mu_1(A_z) + [2B_2(\lambda)(B_1(\lambda) + B_3(\lambda))]\mu_2(A_z)
\]

That is

\[
V\lambda(A_z) = [B_2^2(\lambda) + B_3^2(\lambda) + 2B_1(\lambda)B_3(\lambda)]\mu_1(A_z) + [2B_2(\lambda)(B_1(\lambda) + B_3(\lambda))]\mu_2(A_z)
\]

From (10) follows that

\[
B_1(V(\lambda)) = B_1(\mu(m_1, \sigma_1))[B_1^2(\lambda) + B_2^2(\lambda) + 2B_1(\lambda)B_3(\lambda)]
\]

Thus by induction for sequence $V^n(\lambda)$ we produce the following recurrent equation

\[
V^{n+1}(A_z) = [B^n_2(\lambda) + B^n_3(\lambda) + 2B_1(\lambda)B_3(\lambda)]\mu_1(A_z) + [2B_2(\lambda)(B_1(\lambda) + B_3(\lambda))]\mu_2(A_z) + [B_2^2(\lambda)]\mu_3(A_z),
\]

where $n = 0, 1, \cdots$, and for parameters $B_1(V(\lambda)), B_2(V(\lambda)), B_3(V(\lambda))$ we have the following
Assume $x = B_1(V^n \lambda)$, $y = B_2(V^n \lambda)$, and $z = B_3(V^n \lambda)$. The recurrence equations (13) one can rewrite as follows:

\begin{align*}
x' &= B_1(\mu(m_1, \sigma_1))[x^2 + z^2 + 2xz] + B_1(\mu(m_2, \sigma_2))[2y(x + z)] + B_1(\mu(m_3, \sigma_3))[y^2] \\
y' &= B_2(\mu(m_1, \sigma_1))[x^2 + z^2 + 2xz] + B_2(\mu(m_2, \sigma_2))[2y(x + z)] + B_2(\mu(m_3, \sigma_3))[y^2] \\
z' &= B_3(\mu(m_1, \sigma_1))[x^2 + z^2 + 2xz] + B_3(\mu(m_2, \sigma_2))[2y(x + z)] + B_3(\mu(m_3, \sigma_3))[y^2]
\end{align*}

Assume $x + z = u$, $y = v$, and $B_1(\mu(m_1, \sigma_1)) + B_3(\mu(m_1, \sigma_1)) = A$, $B_1(\mu(m_2, \sigma_2)) + B_3(\mu(m_2, \sigma_2)) = B$, and $B_1(\mu(m_3, \sigma_3)) + B_3(\mu(m_3, \sigma_3)) = C$. Then, we can reduce the recurrence equations (14) to the following:

\begin{align*}
u' &= Au^2 + 2Buv + Cv^2 \\
v' &= (1 - A)u^2 + 2(1 - B)uv + (1 - C)v^2.
\end{align*}

It is evident that this system of equations define one-dimensional qso on $S^1$. Let $\Delta = 4(1 - A)C + (1 - 2B)^2$ be the discriminant of the quadratic equation generated by (15). It is evident that $0 < \Delta < 5$ and $\Delta$ takes all values in this interval. This qso have been studied by Lyubich [11] (see [15, Sec. 2.2]) and it was proved the following theorem.

**Theorem 1** If $0 < \Delta < 4$, then a one-dimensional qso (15) is a regular and if $4 < \Delta < 5$, then there exists a cycle of second order and all trajectories tend to this cycle except the stationary trajectory starting with fixed point.

It is easy to prove the following statement that establishes the necessary and sufficient conditions for $4 < \Delta < 5$.

**Theorem 2** $4 < \Delta < 5$ if and only if $A \in \left[0, \frac{1}{4}\right]$, $C > \frac{3}{4(1 - A)}$, and

\begin{equation}
B < \frac{1}{2} - \sqrt{1 - (1 - A)C} \quad \text{or} \quad B > \frac{1}{2} + \sqrt{1 - (1 - A)C}.
\end{equation}

According the Proposition 1 the trajectory behavior of the considered (3,3)–Gaussian qso is fully determined by the trajectory behavior of the one-dimensional qso (15), i.e. if for some parameters $A, B, C$ qso (15) is the regular or there exists a cycle of second order, then for parameters $m_i, \sigma_i > 0, i = 1, 2, 3$ such that $B_1(\mu(m_1, \sigma_1)) + B_3(\mu(m_1, \sigma_1)) = A$, $B_1(\mu(m_2, \sigma_2)) + B_3(\mu(m_2, \sigma_2)) = B$, and $B_1(\mu(m_3, \sigma_3)) + B_3(\mu(m_3, \sigma_3)) = C$ the corresponding (3,3)–Gaussian qso will be regular or respectively there exists a cycle of second order. Thus we have proved the following theorem.
Theorem 3 For $(3, 3)$–Gaussian qso with parameters $m_i, \sigma_i > 0, i = 1, 2, 3$ there exists a cycle of second order if and only if the following conditions hold:

(i) 
\[
\frac{1}{\sigma_1\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\frac{(u-m_1)^2}{\sigma_1^2}} \, du > \frac{3}{4},
\]

(ii) 
\[
\frac{1}{\sigma_3\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\frac{(u-m_3)^2}{\sigma_3^2}} \, du < 1 - \frac{3\sqrt{2\pi}\sigma_1}{4 \int_a^b e^{-\frac{1}{2}\frac{(u-m_1)^2}{\sigma_1^2}} \, du},
\]

(iii) 
\[
\frac{1}{\sigma_2\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\frac{(u-m_2)^2}{\sigma_2^2}} \, du > \frac{1}{2} + \sqrt{1 - \left(1 - \frac{1}{\sigma_3\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\frac{(u-m_3)^2}{\sigma_3^2}} \, du\right) \frac{1}{\sigma_1\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\frac{(u-m_1)^2}{\sigma_1^2}} \, du}
\]

or
\[
\frac{1}{\sigma_2\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\frac{(u-m_2)^2}{\sigma_2^2}} \, du < \frac{1}{2} - \sqrt{1 - \left(1 - \frac{1}{\sigma_3\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\frac{(u-m_3)^2}{\sigma_3^2}} \, du\right) \frac{1}{\sigma_1\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\frac{(u-m_1)^2}{\sigma_1^2}} \, du}.
\]

In opposite case all trajectories converge to a single fixed point, i.e. it is a regular transformation.

It is evident that if $m_1 \leq a$ or $m_1 \geq b$ then $1 - A < \frac{1}{2}$, that is the corresponding $(3, 3)$–Gaussian qso is the regular transformation. From the inequality $C > \frac{3}{4(1-A)}$ follows that $C > \frac{3}{4}$, and if

\[
\frac{1}{\sigma_3\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\frac{(u-m_3)^2}{\sigma_3^2}} \, du > \frac{1}{4}
\]

then the corresponding $(3, 3)$–Gaussian qso is the regular transformation. Thus for $(3, 3)$–Gaussian qso to have a cycle of second order it is necessary $a < m_1 < b$ and

\[
\frac{1}{\sigma_3\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\frac{(u-m_3)^2}{\sigma_3^2}} \, du < \frac{1}{4}.
\]

According the following inequality

\[
\frac{1}{\sigma_1\sqrt{2\pi}} \int_{m_1-1.2\sigma_1}^{m_1+1.2\sigma_1} e^{-\frac{1}{2}\frac{(u-m_1)^2}{\sigma_1^2}} \, du > \frac{3}{4}
\]

one can conclude that the first inequality holds if $(m_1 - 1.2\sigma_1, m_1 + 1.2\sigma_1) \subset (a, b)$. The analysis of the third inequality in Theorem 3 is rather complicated problem. It will be the subject of a future work.

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