Robust Least Squares for Quantized Data
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Abstract

In this paper we formulate and solve a robust least squares problem for a system of linear equations subject to quantization error. Ordinary least squares fails to consider uncertainty in the data matrices, modeling all noise in the observed signal. Total least squares accounts for uncertainty in the data matrix, but necessarily increases the condition number of the system compared to ordinary least squares. Tikhonov regularization or ridge regression, is frequently employed to combat ill-conditioning, but requires heuristic parameter tuning which presents a host of challenges and places strong assumptions on parameter prior distributions. The proposed method also requires selection of a parameter, but it can be chosen in a natural way, e.g., a matrix rounded to the 4th digit uses an uncertainty bounding parameter of $0.5 \times 10^{-4}$. We show here that our robust method is theoretically appropriate, tractable, and performs favorably against ordinary and total least squares for both residual and absolute error reduction.

Index Terms

Least squares approximation, Quantization error, Ridge regression, Robust optimization, Subgradient methods, Tikhonov regularization, Total least squares

I. Introduction

The primary goal of this paper is to recover unknown parameters from a noisy observation and an uncertain linear operator. In particular, our mathematical model for robust least squares is

$$\min_x \left\{ \max_{\Delta \in \mathcal{U}} \| (A + \Delta)x - b \|_2 \right\}.$$  \hspace{1cm} (1)

The above is referred to as our robust optimization (RO) problem. We use $\| \cdot \|$ to denote the Euclidean norm and $\mathcal{U}$ is the uncertainty set from which perturbations in $A$ are drawn.

The above RO formulation is motivated by two situations. In both cases, we let $\bar{A}$ and $\bar{x}$ represent the true and unknown data matrix and parameter vector, respectively. We model $b = \bar{A}\bar{x} + \eta$ with $\eta$ Gaussian and only have knowledge of $A = \bar{A} - \Delta$. We don’t know $\Delta$ explicitly but can make inferences based on the problem. In the first situation, we consider a data matrix subject to quantization or round-off error. Suppose the observed matrix $A$ has elements rounded to the hundredth place. Our uncertainty set can be written as $\mathcal{U} = \{ \Delta \in \mathbb{R}^{m \times n} : \| \Delta \|_\infty \leq \delta \}$ with $\delta = 0.005$. If $A_{i,j} = 0.540$, then we know the true $\bar{A}_{i,j} \in (0.535, 0.545]$, hence $\Delta_{i,j} \in (-0.005, 0.005]$. The norm $\| \cdot \|_\infty$ takes the maximum absolute value of any element in the matrix.

The second problem considers a data matrix with uncertainty proportional to the magnitude of each entry i.e. $\mathcal{U} = \{ \Delta \in \mathbb{R}^{m \times n} : \Delta_{i,j} \in (-p|A_{i,j}|, p|A_{i,j}|) \}$. Here, $p$ denotes a proportionality constant. Data subject to $\pm 1\%$ uncertainty would have $p = 0.01$. The two cases cover the effects of finite-precision in fixed and floating point representations, respectively. In both problems, the uncertainty sets are specified element-wise allowing us to decouple along rows.

Note that this differs from other robust least-squares problems considered in [1], [2] and classic papers from the late 1990’s [3]. Those works usually made special assumptions like the constraint and objective norms match (e.g. minimize $\| x \|_p$ subject to $\| Ax - b \|_p \leq v$) or column-wise separability. The work in [2] is similar regarding row-wise separability, but they impose a hard constraint on the 2-norm of the solution i.e., $\| x \| \leq k$ for $k \in \mathbb{R}$. For an overview of robust optimization, see [4] and [5] with the latter focusing on implementation.

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A. Limitations of Ordinary and Total Least Squares

The ordinary least squares (OLS) problem seeks a vector $x$ to minimize the residual given by

$$
\min_x \|Ax - b\|^2.
$$

(2)

We focus our attention on the over-determined case where $m > n$ and further assume that $A$ is full rank. It is well known that the closed form solution to (2) is given by

$$
\hat{x}_{\text{OLS}} = (A^T A)^{-1} A^T b.
$$

(3)

A key assumption for OLS is that $A$ is known with certainty and $b$ is subject to additive noise. The OLS solution (3) is the maximum likelihood estimator (MLE) for the model of Gaussian noise in $b$. When $A$ is well-conditioned, the solution of (2) is robust to perturbations in both $A$ and $b$ with error analysis found in many introductory numerical analysis texts. As a result, we consider cases of poor conditioning where $x$ exhibits extreme sensitivity to small changes in the operator $A$ and data $b$, making the OLS solution a poor estimator.

In practice, it is uncommon to know $A$ precisely. Typical causes of uncertainty are sampling error, measurement error, human error, modeling error, or rounding error. There were attempts at addressing this model limitation in [6] but these relied on small magnitude errors to use a truncated series approximation for an inverse. Total least square (TLS) was developed in response to this lack of symmetry in uncertainty. Rather than isolating noise to the observed signal or right hand side, the TLS model allows for an uncertain operator and is given by $(A + \Delta)\bar{x} = b + \eta$ where $A$ and $b$ are observed with $\Delta$ and $\eta$ a random matrix and vector, respectively. We assume that $\Delta$ and $\eta$ have been scaled to have the same variance; see [7] for a modern overview of the topic. The solution to the TLS problem is the $x$ that solves

$$
\min_{x, \Delta, \eta} \|\Delta, \eta\|_F
$$

subject to $(A + \Delta)x = b + \eta$

where $[\Delta, \eta] \in \mathbb{R}^{m \times (n+1)}$. $A$ and $b$ are the observed matrix/signal pair, and $\|\cdot\|_F$ is the Frobenius norm. It was shown in [8] that (4) can be solved via a singular value decomposition (SVD) with the closed form solution of

$$
\hat{x}_{\text{TLS}} = (A^T A - \sigma_{n+1}^2 I)^{-1} A^T b
$$

(5)

where $\sigma_{n+1} \in \mathbb{R}$ being the smallest singular value of the augmented matrix $[A, b] \in \mathbb{R}^{m \times (n+1)}$. Similar to OLS, the TLS solution yields a MLE for the model of Gaussian noise in $A$ and $b$. It should be noted that $(A^T A - \sigma_{n+1}^2 I)$ is necessarily worse conditioned than $A^T A$. Since $A^T A$ is positive definite by virtue of $A$ being full rank, all eigenvalues of $A^T A$ are shifted closer to zero by the amount of $\sigma_{n+1}^2$. For a small spectral gap between $\sigma_n$ and $\sigma_{n+1}$, the matrix will be close to singular and solutions will be extremely sensitive to perturbations in $A$.

This can be understood intuitively; uncertainty in $A$ permits additional degrees of freedom allowing the model to “fit” noise. To illustrate, consider the simple linear regression problem. We have access to several regressor/response pairs $(a, b) \in \mathbb{R}^2$. Suppose our data is generated from the model $Ax = \eta \sim \mathcal{N}(0, \sigma^2 I)$. As a result, we consider cases of poor conditioning where $x$ exhibits extreme sensitivity to small changes in the operator $A$ and data $b$, making the OLS solution a poor estimator.

Beyond problems with conditioning, both OLS and TLS implicitly assume normality in $b$ (and $A$ for TLS) since (3) and (5) are the respective MLE’s for Gaussian uncertainty. These estimators are theoretically inappropriate for the case of quantization error in $A$ where the uncertainty is bounded and uniformly distributed around observed values.
Fig. 1. Instance of data sampled from a constant function, $\tilde{b} = 0$ subject to additive noise. That is, $b = \tilde{b} + \delta = \delta$. Solid line indicates ground truth model ($\tilde{b} = 0$). Dashed line shows OLS model ($b = a \cdot \hat{\mathbf{x}}_{\text{OLS}}$). TLS model is dot-dashed line ($b = a \cdot \hat{\mathbf{x}}_{\text{TLS}}$). Since TLS minimizes orthogonal distance, the model over-fits data.

B. The Trouble with Tikhonov Regularization

Poor conditioning as observed in TLS is often combated with Tikhonov regularization or ridge regression (RR) whereby solutions with large norms are penalized via an $\ell_2$ regularization term. The solution to the ridge regression problem solves

$$
\min_{\mathbf{x}} \left\{ \| A\mathbf{x} - \mathbf{b} \|_2^2 + \| \lambda \mathbf{x} \|_2^2 \right\}
$$

with $\lambda \geq 0$. The minimizing $\mathbf{x}$ has a closed form solution of

$$
\hat{\mathbf{x}}_{\text{RR}} = (A^T A + \lambda^2 I)^{-1} A^T \mathbf{b}.
$$

There is a structural connection between the RR and TLS; their solutions are nearly identical with TLS subtracting from the diagonal of $A^T A$ and RR adding to it. This can be understood from a Bayesian statistics point of view. Large values of $\lambda$ correspond to stronger evidence of lower variance in $\mathbf{x}$ (and zero mean). In the case of an uncertain data matrix, we should have less confidence in low variance.

Golub, Hansen, and O’Leary explored the interplay between TLS and RR in [9]. They considered a generalized version of the regularized total least squares (RRTLS) problem

$$
\min_{\mathbf{x}, \Delta, \eta} \left\| \begin{bmatrix} \Delta & \eta \end{bmatrix} \right\|_F
$$

subject to

$$(A + \Delta)\mathbf{x} = \mathbf{b} + \eta$$

$$\| \mathbf{x} \| \leq \gamma. \quad (8)$$

The second constraint is equivalent to imposing a regularization term on the objective of the TLS problem. When the inequality constraint is replaced by an equality, the solution to (8) is

$$
\hat{x}_{\text{RRTLS}} = (A^T A + \alpha I)^{-1} A^T \mathbf{b} \quad \text{with} \quad \alpha = \mu (1 + \| \mathbf{x} \|_2^2) - \frac{\| \mathbf{b} - A\mathbf{x} \|_2^2}{1 + \| \mathbf{x} \|_2^2} \quad (9)
$$

and $\mu$ the corresponding Lagrange multiplier. Note that for all $\gamma > 0$, we have $\alpha \geq -\sigma_{n+1}^2$. Similarly, $\alpha = -\sigma_{n+1}^2$ when $\gamma = \| \hat{x}_{\text{TLS}} \|$ and $\alpha = 0$ when $\gamma = \| \hat{x}_{\text{OLS}} \|$. This suggests that regularized total least squares merely adds a standard regularization term to the poorly conditioned TLS matrix, $A^T A - \sigma_{n+1}^2 I$.

Given the close interplay between TLS, RR, and RRTLS, it is reasonable to solve either TLS or RR since they are effectively the same problem with different parameters; the additional term becomes regularizing when $\alpha = \lambda^2 > 0$. Although TLS is most appropriate for the case of an uncertain matrix, its “deregularizing” effect inflames issues with conditioning making it an unpopular choice for solving typical linear inverse problems.
There is also the tricky consideration of choosing an appropriate regularization parameter, whether in the standard RR form of (6) or $\gamma$ in (3). Although there are many approaches for choosing a parameter such as Morozov’s discrepancy principle (MDP), the unbiased predictive risk estimator method (UPR), the generalized cross validation method (GCV), or the “elbow” or “L” method to name a few, parameter selection is based on heuristic arguments or requires unknown information a priori. A detailed treatment of the above methods and their analysis can be found in [10]. Throughout the remainder of this paper, we drop our discussion of RRTLS, focusing instead on TLS and RR.

C. Robust Least Squares

The central focus of robust optimization (RO) is to find a solution that is feasible over all possible realizations of uncertain variables. In our case, $\Delta \in \mathcal{U}$ is unknown where $\mathcal{U} = \{ M \in \mathbb{R}^{m \times n} : |M| \leq D \}$ and $D$ is an element-wise positive matrix chosen in a principled fashion, i.e., the degree to which matrix entries are quantized. Our robustified version is written as a minimax optimization problem in (1) with the appropriate $\mathcal{U}$. The inner maximization problem guards against over-fitting, effectively regularizing our solution, and reducing sensitivity to perturbations in $A$. Note that RO avoids placing a statistical prior on $\Delta$, which can be a strength or weakness depending on the model.

The outline of the rest of the paper is as follows: in section II we derive a closed form solution to the inner maximization problem thereby showing its tractability, section III discusses computational methods for solving problem (1), and section IV provides the results to numerical experiments followed by a conclusion.

II.CLOSED FORM SOLUTION OF INNER OBJECTIVE AND THEORY

A. Floating Point Uncertainty

We begin by breaking (1) into an inner maximization and outer minimization problem:

$$\min_x \left\{ \max_{|\Delta| \leq D} \| (A + \Delta)x - b \|_2 \right\} \iff \min_x f(x) \text{ subject to } f(x) = \max_{|\Delta| \leq D} \| (A + \Delta)x - b \|_2^2. \tag{10}$$

Here $|\Delta| \leq D$ indicates that the magnitude of elements in $\Delta$ are bound by the corresponding non-negative components of $D$. That is, $D_{i,j}$ constrains element $\Delta_{i,j}$ of the uncertainty matrix. The inner maximization problem will recover $f(x)$. Because the function $x \mapsto \| (A + \Delta)x - b \|^2$ is convex in $x$ and the supremum over an arbitrary family of convex functions is convex, it follows that $f$ is convex making $\min_x f(x)$ an unconstrained convex optimization problem. To evaluate $f$ and find a subgradient, it remains to find the maximizing $\Delta \in \mathbb{R}^{m \times n}$.

**Theorem 1.** The maximizing function $f(x)$ in (10) is given by

$$f(x) = \| Ax - b \|^2 + 2(|Ax - b|, D|x|) + \| D|x| \|^2 \tag{11}$$

where $|x|$ and $|Ax-b|$ denote the vectors of component-wise absolute values.

**Proof.** Since $f(x) = \max_{|\Delta| \leq D} \| (A + \Delta)x - b \|^2$ must be maximized over $\Delta$, we can fix $x$, define $c := Ax - b$, then treat it as constant. Exploiting row-wise separability, we write

$$\max_{|\Delta| \leq D} \| \Delta x + c \|^2 = \sum_{i=1}^m \max_{|\Delta_i| \leq D_i} \| \Delta_i^T x + c_i \|^2 \tag{12}$$

where $\Delta_i^T$ and $D_i^T$ are the $i^{th}$ row of $\Delta$ and $D$, respectively, and $c_i$ is the $i^{th}$ element of vector $c$. We now work row by row. Note that we can switch to absolute values rather than squares when maximizing for each row. Applying the triangle inequality gives an upper bound

$$|\Delta_i^T x + c_i| \leq |c_i| + \sum_{j=1}^n |\Delta_{i,j}| |x_j| \tag{13}$$

$$\leq |c_i| + \sum_{j=1}^n D_{i,j} |x_j| = D_i^T |x| + |c_i|.$$
It is easily verified that the upper bound is achieved when
\[ \Delta_{i,j} = D_{i,j} \text{sign}(x_j \cdot c_i) \] (14)
making it a solution to the inner maximization problem. Recalling that \( c_i = |Ax - b|_i \), we simplify to
\[ f(x) = \|Ax - b\|^2 + 2\langle |Ax - b|, D|x| \rangle + \|D|x|\|^2. \] (15)

Using Theorem 1, the optimization problem in (10) can be rewritten as
\[
\min_x f(x) = \min_x \left\{ \|Ax - b\|^2 + 2\langle |Ax - b|, D|x| \rangle + \|D|x|\|^2 \right\}.
\] (16)
In general, \( f \) is not differentiable (e.g., let \( m = n = 1 \) and \( A = D = b = 1 \), then \( f(x) = (x - 1)^2 + 2|x| \cdot |x - 1| + x^2 \) is not differentiable at \( x \in \{0, 1\} \)), but we are guaranteed a subgradient by virtue of convexity. Furthermore, a generalization of Danskin’s Theorem [11] provides us with a method to find elements of the subgradient for all \( x \). In particular, we have
\[
 f'(x) := 2(A + \Delta_x)^T [(A + \Delta_x)x - b] \in \partial f(x)
\] (17)
where \( \Delta_x \) indicates the optimal \( \Delta \) for a given \( x \) as provided in Eq. (14).

B. Fixed Point Uncertainty

Fixed point uncertainty is a special case of (10) with all elements bound by the same value. We can write this constraint as \( \|\Delta\|_\infty \leq \delta \) with \( \|\Delta\|_\infty \) representing the largest magnitude element of \( \Delta \). Problem (10) becomes
\[
\min_x \left\{ \max_{\|\Delta\|_\infty \leq \delta} \|Ax + \Delta\| - \|b\|^2 \right\}
\] (18)
This instance gives rise to the following corollary.

**Corollary 1.** For fixed point uncertainty, problem (18) reduces to
\[
\min_x \left\{ \|Ax - b\|^2 + 2\delta \|x\|_1 + m\delta^2 \|x\|^2 \right\}. \] (19)

**Proof.** We take \( 1_k \in \mathbb{R}^k \) to be the vector composed of ones. Note that \( \|\Delta\|_\infty \leq \delta \) is equivalent to \( |\Delta| \leq D \) when \( D = \delta 1_m 1_k^T \). By Theorem 1, the result follows easily using properties of inner products. \( \square \)

It is helpful for computation to have the optimal \( \Delta \) given by
\[
\Delta_{i,j} = \delta \cdot \text{sign}(Ax - b_i \cdot x_j). \] (20)
To fix notation, let \( \hat{x}_{\text{RO}} \) denote the argument minimizing (18). We focus on fixed point error for the remainder of the paper.

C. Explicit Regularization of Robust Objective

The robust formulation presented above is intended to address uncertainty in the data matrix from rounding error. Although implicit regularization occurs via the 3rd term in both (16) and (19), we don’t address poor conditioning of \( A \) directly. To illustrate this point, note that \( \hat{x}_{\text{RO}} \rightarrow \hat{x}_{\text{OLS}} \) as \( \delta \rightarrow 0 \). If \( A \) is poorly conditioned, such behavior is undesirable. This can be remedied by adding an \( \ell_2 \) regularization term and solving
\[
\min_x \left\{ f(x) + \lambda^2 \|x\|^2 \right\} = \min_x \left\{ (\|Ax - b\|^2 + 2\delta \|x\|_1 \|Ax - b\|_1 + m\delta^2 \|x\|^2_1) + \lambda^2 \|x\|^2 \right\}. \] (21)
Since the robust objective and regularization term are both convex, so is their sum. Furthermore, the modified objective is \( 2\lambda^2 \) strongly convex yielding a unique minimizer, \( \hat{x}_{\text{RO},\lambda} \). Strong convexity also implies a bounded sequence of iterates, i.e., \( \|x_k - x^*\| \leq M \), and therefore guarantees convergence under the mirror-descent method. In fact, the same technique works with regularizers other than \( \ell_2 \), and can be solved via proximal mirror-descent methods when the regularizer is smooth or has an easy-to-compute proximity operator [12] Ch. 9], though we focus on \( \ell_2 \) regularization in this paper.
III. Algorithms

Equipped with an element of the subdifferential, we can employ a variety of solvers. Bundle methods are a promising choice as they sequentially form an approximation to the objective. This is done by using the location, function value, and subgradient of previous iterates to lower bound the objective with supporting hyperplanes. At each step, a direction finding quadratic program must be solved, but can be dealt with rapidly thanks to software such as CVXGEN [13] and ECOS [14]. A survey on the topic can be found in [15].

Since the problem is unconstrained, another option is to use smooth optimization techniques, and in particular quasi-Newton methods that form low-rank approximations of $\nabla^2 f$. Our objective $f$ is not differentiable, much less twice so, and consequently these methods requiring second derivatives seem theoretically inappropriate. However, there is a growing body of literature going back to Lemaire [10] recognizing the empirical success of these methods. See [17], [18] and references therein. The success of these methods depends on the smoothness near the solution. Using the minFunc MATLAB package [19] with random and zero vector initializations, and using the package’s default solver of limited-memory BFGS [20], we observed fast and accurate convergence for several test problems.

For our numerical experiments, we elected to use subgradient descent for its simplicity and flexibility. Indeed, we can easily handle constraints via projections or a regularization term $h(x)$ with proximal operators. Recent advances in proximal subgradient descent methods and their analysis can be found in [21] and [22]. Convergence may be slow, but is remedied by using a suitable warm start. We observe that several of the RR solutions are good initializations. In fact, we can exploit the computational availability of RR solutions to devise a warm-start strategy. To this end, we $x_\lambda$ solve the RR problem given in (6) with parameter $\lambda$ and define

$$R(\lambda) = \|Ax_\lambda - b\|^2 + 2\delta\|x_\lambda\|_1 + m\delta^2\|x_\lambda\|^2.$$  \hfill (22)

For the cost of a one-time SVD of $A$, we can solve the RR problem for a new choice of $\lambda$ with a matrix-vector multiply. Using a one-dimensional line search, we find the RR parameter whose solution minimizes the robust objective, that is

$$\tau = \arg\min_\lambda R(\lambda),$$ \hfill (23)

and take $x_0 = (A^T A + \tau^2 I)^{-1} A^T b$ as a warm start for the subgradient descent algorithm. This initialization is heuristic, but does a reasonable job in practice, accelerating convergence compared to starting with 0, $\hat{x}_{OLS}$, $\hat{x}_{TLS}$, or $\hat{x}_{RR\lambda}$ (for $\lambda$ specified by UPR, GCV, or MDP).

A drawback of the subgradient descent method is that a step-size scheme must be chosen in advance. As a result, we chose a standard diminishing, non-summable step length given by

$$t_k = \frac{1}{\sqrt{k+1}} \cdot \|f'(x_k)\|.$$ \hfill (24)

A discussion on the choice of step lengths can be found in [23]. Pseudocode is provided in Algorithm 1. The proximal version is the same, except line 8 in Algo. 1 should be replaced with $x_k \leftarrow \text{prox}_{t_k \delta} (x_{k-1} - t_{k-1} g_{k-1})$, and if $g_{k-1} = 0$ (which would indicate optimality if there is no regularizer or constraint), we step a distance of $\frac{1}{\sqrt{k+1}}$ in the direction of the largest gradient observed so far, so that the update is still well-defined.

IV. Numerical Experiments

A. Setup

We use MATLAB’s pseudo-random number generator randn to draw 10,000 standard normal matrices, $A \in \mathbb{R}^{30 \times 15}$. The bar notation in this section indicates true and unobserved values. We ensure ill-conditioning by fixing the smallest singular value of $A$. We do so by taking the SVD such that $\bar{A} = U \Sigma V^T$ where $U$ and $V$ are unitary and $\Sigma$ is a diagonal matrix with singular values along its diagonal in descending order. By resetting $\Sigma_{15, 15} = 10^{-4} \times \Sigma_{1, 1}$ and redefining $\bar{A} := U \Sigma V^T$, we ensure that each $\bar{A}$ is ill-conditioned (with condition number of $10^4$) and will exhibit sensitivity to perturbations in the data.

Our true solution, $\bar{x}$, has the first element fixed at $\bar{x}_1 = \pm 10,000$ with its sign being drawn uniformly at random and the remaining elements drawn from a standard normal distribution, $\bar{x}_{2:15} \sim \mathcal{N}(0, I)$. The decision to use a single large element is intentional since RR biases solutions towards zero resulting in
observed A are approximately 30 (exact for mean). To ensure feasibility, i.e., a real root, we choose since 
η component Gaussian random variable, m ∥ least-squares problem directly using standard software like
since it is computationally cheap and easy to program, though it is also possible to solve a constrained
is the solution to (27) and the parameter
δ in Section I-B, and λ with
in (26) based on the observed accuracy, e.g., if A is rounded to the 2nd decimal, then δ = 0.5 · 10−2. The results for (29) are presented in the next subsection.

The regularization parameters for UPR and GCV are recovered using an approximate line search for
the corresponding objectives provided in Vogel’s text [10]. We also conduct a line search to solve (23) for
x best solution to (6) with parameter
τ. When
A x = b := A R x, and then generate a noisy signal by letting b := b + η where η ∼ N(0, \|b\|2
SNR). In our experiments, we fix SNR = 10. Finally, we quantize A which gives us our observed A, then solve the following problems
\[ \hat{x}_{\text{OLS}} = \arg\min_x \|Ax - b\|^2 \] (25)
\[ \hat{x}_{\text{RO}} = \arg\min_x \left\{ \max_{\|\Delta\|_{\infty} \leq \delta} \| (A + \Delta)x - b\|^2 \right\} \] (26)
\[ \hat{x}_{\text{RR}} = \arg\min_x \left\{ \|Ax - b\|^2 + \lambda^2 \|x\|^2 \right\} \] (27)
\[ \hat{x}_{\text{TLS}} = \left\{ \arg\min_x, \Delta, \eta \right\} \| \Delta, \eta \|_F \] subject to \( (A + \Delta)x = b + \eta \) (28)
\[ \hat{x}_{\text{RRO}} = \arg\min_x \left\{ \max_{\|\Delta\|_{\infty} \leq \delta} \left( \| (A + \Delta)x - b\|^2 + \lambda^2 \|x\|^2 \right) \right\} \] (29)

with λ in (27) chosen according to one of the UPR, GCV or MDP parameter selection strategies discussed in Section I-B and δ in (26) based on the observed accuracy, e.g., if A is rounded to the 2nd decimal, then δ = 0.5 · 10−2. The results for (29) are presented in the next subsection.

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Fig. 2. Mean error $\|e\| = \|\hat{x} - \bar{x}\|$ over 10k simulations as a function of the digit to which matrix entries were rounded to. Zero solution shows error if using $\hat{x} = 0$. RR UPR omitted for clarity since error closely tracks that of RR GCV. Note that $\delta = 0.5 \times 10^{\text{round to digit}}$.

Fig. 3. Histograms of component-wise absolute error, $|\hat{x}_i - \bar{x}_i|$, along horizontal axis for 10k simulations with matrix entries rounded to the thousandth place (round to digit = 3 and $\delta = 0.5 \times 10^{-3}$). Vertical axis is frequency. The axes are fixed for comparison in RO and RR plots; scales differ for OLS and TLS. Left: histograms for solution components drawn from $N(0,1)$. Right: histograms for solution components drawn from $\{\pm 10^4\}$. For big component values, RR estimate error is localized away from zero indicating large bias.

$$P(\|\hat{A}\hat{x} - b\|^2 - \rho \leq 0) = 95\%,$$ which can be easily calculated using $\chi^2$ inverse CDF tables. For our experiments, $\rho \approx \frac{2|\bar{b}|^2}{3\text{SNR}}$. An appropriate $\rho$ is generally unknown a priori.

B. Standard Robust Objective Results

The results are displayed in Figure 2 which shows mean error as a function of rounding digit for $\hat{A}$, and Figure 3 which fixes the rounding digit to the thousands place and reports the entire histogram for component-wise absolute error. The first observation is that Figure 2 shows the robust solution does well at reducing error in the face of heavy quantization. The RO solution converges to the OLS solution, as expected, when $\delta \to 0$, i.e., when the quantization effect is small. This might be undesirable, especially when $A$ is ill-conditioned, and can be addressed via the inclusion of a regularization term as discussed in the next experiment.
A second observation is that RO doesn’t sacrifice accuracy for bias as RR does. This phenomenon can be observed in Figure 3 which depicts histograms of component-wise absolute error for different methods. The plots to the left show error behavior for small components of the solution, that is, for \( \hat{x}_{2:15} \sim \mathcal{N}(0, I) \). Since values are close to zero, RR is expected to perform well at estimating the true solution. Indeed, RR implicitly assumes a prior with mean zero. The histograms on the right show absolute errors for “large” components drawn from \( \{\pm 10^4\} \). Rather than localizing about zero, the absolute error’s mode is observed around 1,000 for GCV and UPR and 4,000 for MDP. Heavier penalty terms (bigger \( \lambda \)) place less weight on the LS term in the objective and bias estimates more aggressively towards zero. Typical \( \lambda \) values for GCV, UPR, and MDP are 1.08, 1.10, and 3.61, respectively. The robust solution accurately estimates the large and small components values without the extreme errors observed with OLS and TLS; that is, RO appears to have a low-variance, comparable to RR, but also low-bias, comparable or better than OLS.

It is worth noting that in our experiment, extreme quantization increases fidelity for the OLS, TLS, and GCV solutions, which is counter-intuitive. This appears to be related to quantization improving conditioning as seen in Table I. Making a precise statement is difficult because one can easily find examples of quantization increasing and decreasing the condition number.

### C. Regularized Robust Objective Results

When conditioning is a concern, convergence of the robust solution to the OLS solution for small \( \delta \) is undesirable. To address this problem, a regularization term may be added to the robust objective. We conducted the same experiment above but solved the regularized robust problem from (21) to get \( \hat{x}_{\text{RRO}} \). As expected, \( \hat{x}_{\text{RRO}} \rightarrow \hat{x}_{\text{RR}} \) when \( \delta \rightarrow 0 \). Figure 5 shows the behavior of the regularized robust solutions for different values of \( \lambda \). The RR errors are included for comparison. Importantly, the regularized robust solution achieves higher accuracy in regimes of heavy quantization and converges to the regularized solution, addressing issues with sensitivity of the OLS solution to noise.

For large regularization parameters like MDP, the \( \ell_2 \) penalty term tends to swamp the robust objective for all but the most extreme quantization levels and provides limited benefit. Under moderate regularization such as GCV, the robust version successfully reduces mean error as seen in Figure 4 and avoids extreme errors. Visually, this can be seen in the bottom panels of Figure 5 where the standard RR solution returns estimates with large errors, i.e., greater than \( 10^4 \) (last bin aggregates extreme error), while the robust version does not.

### V. Conclusion

In this paper, we presented a robust method for addressing fixed and floating point uncertainty in matrices due to quantization error. After reviewing the limitations of standard methods, we formulated a tractable robust objective and presented algorithms to solve it. The only parameter necessary in our formulation can be chosen in a principled fashion, i.e., by observing the degree to which matrix elements are rounded. We provide
Fig. 4. Mean norm of error $\|\epsilon\| = \|\hat{x} - \bar{x}\|$ over 10k simulations of $RR_\lambda$ and the robust counter part, $RRO_\lambda$. Note that $\delta = 0.5 \times 10^3$.

Fig. 5. Histograms of robust regularized component-wise error overlaid with RR error for MDP and GCV. The matrices used were rounded the thousandth place, e.g., $\delta = 0.5 \times 10^{-3}$. Errors of magnitude $> 10^4$ aggregated in last bin. Left: component-wise error for elements with true distribution $\mathcal{N}(0, 1)$. Right: component-wise error for elements drawn from $\{\pm 10^4 \}$. For the large MDP parameter, the $\ell_2$ penalty term overpowers the robust terms and RRO offers little advantage. Modest regularization such as GCV benefits from robust formulation; note large errors for standard RR in GCV plots.

A modified robust objective to use with ill-conditioned systems via the inclusion of an $\ell_2$ penalty term. In this case, a regularization parameter must be chosen, but it is no additional work over Tikhonov regularization. Our numerical experiments show that robust least squares outperforms ordinary least squares, total least squares, and ridge regression, effectively balancing bias and accuracy. The proposed methods provides a tool for dealing with imprecise data in ill-conditioned systems.

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