CONVERGENCE AND NON-CONVERGENCE OF SCALED SELF-INTERACTING RANDOM WALKS TO BROWNIAN MOTION PERTURBED AT EXTREMA

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Abstract. We use generalized Ray-Knight theorems introduced by B. Tóth in [Tóth96] together with techniques developed for excited random walks as main tools for establishing positive and negative results concerning convergence of some classes of diffusively scaled self-interacting random walks (SIRWs) to Brownian motions perturbed at extrema (BMPE). The work [Tóth96] studied two classes of SIRWs: asymptotically free and polynomially self-repelling walks. For both classes B. Tóth has shown, in particular, that the distribution function of a scaled SIRW observed at independent geometric times converges to that of a BMPE indicated by the generalized Ray-Knight theorem for this SIRW. The question of weak convergence of one-dimensional distributions of scaled SIRW remained open. In this paper, on the one hand, we prove a full functional limit theorem for a large class of asymptotically free SIRWs which includes asymptotically free walks considered in [Tóth96]. On the other hand, we show that rescaled polynomially self-repelling SIRWs do not converge to the BMPE predicted by the corresponding generalized Ray-Knight theorems and, hence, do not converge to any BMPE.

1. Introduction and main results

This work has as its starting point the paper of B. Tóth [Tóth96], which along with [Tóth94] and [Tóth95] greatly expanded the domain of processes to which the classical Ray–Knight approach could be applied. It introduced two families of nearest neighbor self-interacting random walks: the asymptotically free random walks and the polynomially self-repelling random walks. This approach, relying on tree structures and so inapplicable to higher dimensions, established correct scaling and then convergence in distribution of appropriately scaled hitting times of the random walks. It was also very close to establishing the convergence in law of one point distributions in that it identified the limiting law were such a limit to exist.

B. Tóth also noted that the Ray–Knight theorems for the asymptotically free and polynomially self repelling random walks were the same as analogous results for a previously studied continuous model, in this case the Ray–Knight theorems for Brownian motions perturbed at extrema (BMPE) (or multiples thereof) with parameters determined by the particularities of the original model. This raised the possibility of finding a general result whereby a functional limit theorem for the rescaled walk could simply be proven via establishing the Ray–Knight theorems and minimal technical conditions. A number of results for excited random walks, [DK12, KP16, KMP22], weighed positively on this possibility except that the “technical conditions” remained elusive and the treatment varied significantly from model to model.

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1See [CPY98, Section3].
In this article we show both the usefulness and the limitations of the Ray–Knight approach to establishing scaling limits. On the one hand, we prove that the rescaled asymptotically free random walks do indeed converge to a BMPE. On the other hand, we show that the polynomially self-repelling random walks do not converge to the natural BMPE limit suggested by the Ray–Knight theorems. This nonconvergence result is important as it shows that there is, in fact, no general theorem that permits easy passage from Ray–Knight theorems to functional convergence, and opens up the question of what natural conditions on the original processes ensure functional convergence. It also motivates the study of families of processes having the same Ray–Knight behavior.

1.1. Model description. We consider a discrete time nearest neighbor self-interacting random walk \((X_i)_{i \geq 0}\) on \(\mathbb{Z}\) which starts at \(X_0 = 0\) and at times \(i \in \mathbb{N}\) jumps to one of the two nearest neighbors: if \(X_{i-1} = x\) then \(X_i\) is equal to \(x \pm 1\) with probabilities dependent on the numbers of crossings of undirected bonds \(\{x, x \pm 1\}\) prior to \(i\). More precisely, let \(\Omega = \{\omega = (\omega_i)_{i \geq 0} : \omega_0 = 0, |\omega_i - \omega_{i-1}| = 1 \forall i \in \mathbb{N}\}\) be a set of all nearest neighbor paths originating at 0, \(F, i \geq 0\), be the \(\sigma\)-algebra generated by all subsets of \(\Omega\) of the form \(\{\omega_j = x_j, \forall j \in \{0, 1, \ldots, i\}\}\), and \(F = \sigma(\cup_{i=0}^{\infty} F_i)\). We set \(r^0_x = r^0_x = 0\) for all \(x \in \mathbb{Z}\) and for each \(\omega \in \Omega\) define

\[
(1) \quad r^i_x(\omega) := \sum_{j=1}^{i} 1_{\{x, x+1\}}(\{\omega_{j-1}, \omega_j\}); \quad \ell^i_x := \sum_{j=1}^{i} 1_{\{x-1, x\}}(\{\omega_{j-1}, \omega_j\}), \quad i \in \mathbb{N},
\]

where for two sets \(A\) and \(S\), \(1_A(S) = 1\) if \(S \subseteq A\) and 0 otherwise. We assume that under the probability measure \(P\) on \((\Omega, F)\) the self-interacting random walk (SIRW) \(X = (X_i(\omega))_{i \geq 0}\) has the following dynamics: \(X_0 = 0,\)

\[
(2) \quad P(X_{i+1} = X_i + 1 | F_i) = 1 - P(X_{i+1} = X_i - 1 | F_i) = \frac{w(r^i_{X_i})}{w(r^i_{X_i}) + w(\ell^i_{X_i})}, \quad i \geq 0,
\]

where \(w : \mathbb{N}_0 \to (0, \infty)\) is a given weight function. Properties of \(w\) (in particular, its asymptotics at infinity) determine the type of SIRW and its long term behavior. If the weight function \(w\) is non-increasing then the SIRW is said to be self-repelling. If \(w\) is non-decreasing then the SIRW is said to be self-attracting.

Fix \(\alpha \geq 0, p \in (0, 1], B \in \mathbb{R}, \kappa > 0\) and let \(w : \mathbb{N}_0 \to (0, \infty)\) satisfy

\[
(3) \quad \frac{1}{w(n)} = n^{\alpha} \left(1 + \frac{2pB}{n^p} + O \left(\frac{1}{n^{1+\kappa}}\right)\right) \quad \text{as} \quad n \to \infty.
\]

This model was introduced and studied by B. Tóth in [Tot96]. According to the terminology of [Tot96], setting \(\alpha = 0\) in (3) places the model in the asymptotically free regime: \(w(n) \sim 1\), while \(\alpha > 0\) corresponds to the polynomially self-repelling case: \(w(n) \sim n^{-\alpha}\). In [Tot96], B. Tóth proved functional limit theorems for the local time processes of SIRWs (a.k.a. generalized Ray–Knight Theorems) and local limit theorems for the position of a random walker observed at independent geometric times with linearly growing means ([Tot96 Theorems 2A, 2B]). These local limit theorems imply that if one-dimensional distributions of rescaled SIRWs converge weakly, then the limiting distribution must be that of (a multiple of) a Brownian motion perturbed at its extrema, see Definition [1.1]. Even though the original paper [Tot96] considered only \(p = \kappa = 1\), the relevant results of that work can be extended to \(p \in (0, 1]\) and \(\kappa > 0\). The question of weak convergence (not just along an independent geometric sequence of times) of one-dimensional distributions of the rescaled position of the walk to the conjectured limit remained open and has motivated our work.
1.2. Main results.

**Definition 1.1.** Given \( \theta^+, \theta^- < 1 \), a BMPE \( W^{\theta^+, \theta^-} = (W^{\theta^+, \theta^-}(t), t \geq 0) \) with parameters \((\theta^+, \theta^-)\) is the pathwise unique solution of the equation

\[
W(t) = B(t) + \theta^+ \sup_{s \leq t} W(s) + \theta^- \inf_{s \leq t} W(s), \quad t \geq 0, \quad W(0) = 0,
\]

where \((B(t))_{t \geq 0}\) is a standard Brownian motion.

It is known that this solution is adapted to the filtration of \((B(t))_{t \geq 0}\), has Brownian scaling\(^2\), and the triple \((\inf_{s \leq t} W^{\theta^+, \theta^-}(s), W^{\theta^+, \theta^-}(t), \sup_{s \leq t} W^{\theta^+, \theta^-}(s)), \ t \geq 0\), is a strong Markov process. (See [PW97, CPY98, CD99, Dav99] and references therein.)

B. Davis ([Dav96, Dav99]) has shown that BMPEs arise naturally as diffusive limits of perturbed random walks, so-called pq-walks. More specifically, consider a nearest neighbor random walk which starts with an unbiased jump from the origin and continues to make unbiased jumps to one of its neighbors except when it visits an extremal point of its current range. If at time \( n \in \mathbb{N} \) the walk is at its maximum up to time \( n \) then the probability that it jumps to the right in the next step is \( p := (2 - \theta^+)^{-1} \). Similarly, if at time \( n \in \mathbb{N} \) the walk is at its current minimum then the probability that it jumps to the right in the next step is \( q := 1 - (2 - \theta^-)^{-1} \). ([Dav96, Theorem 1.2] states that linearly interpolated and rescaled perturbed random walks converge weakly to the process \( W^{\theta^+, \theta^-} \).

We note that when \( \theta^+ = \theta^- = \theta \) the perturbed walk is a special case of SIRW with \( w(0) = (2 - \theta)^{-1} \) and \( w(n) = 1 \) for \( n \in \mathbb{N} \). Hence, the idea that a class of asymptotically free SIRWs with much more general weight functions \( w \) might have BMPEs as scaling limits seems plausible but not very intuitive: while it is not difficult to see that the range of a SIRW after \( n \) steps is of order \( \sqrt{n} \), it is not at all clear why after the diffusive scaling and away from the extrema the SIRW should behave essentially as a Brownian motion.

BMPEs also appear as scaling limits of other non-markovian random walks on \( \mathbb{Z} \) such as once reinforced random walks ([Dav90, Dav96]), excited random walks ([Del11, DK12, KP16, KMP22]), and rotor walks with defects ([HLSH18]). A few of the above models are of the type where the (recurrent\(^3\)) random walk after visiting a site “in the bulk” (i.e., away from the boundary of its current range) a certain fixed number of times makes only unbiased steps from that site\(^4\), and for these models the convergence to a BMPE seems very intuitive. On the other hand, for excited random walks with Markovian cookie stacks the work [KMP22] established the convergence to a multiple of BMPE, even though the Brownian behavior “in the bulk” is not seen at the random walk level. Establishing this behavior is non-trivial and requires intermediate “mesoscopic” coarse graining of space and time.

Our first main result is the functional limit theorem for the asymptotically free case \( \alpha = 0 \). Following the notation in [Tóth96], define

\[
U_1(n) := \sum_{j=0}^{n-1} (w(2j))^{-1} \quad \text{and} \quad V_1(n) := \sum_{j=0}^{n-1} (w(2j+1))^{-1}.
\]

Set

\[
\gamma := \lim_{n \to \infty} (V_1(n) - U_1(n)).
\]

\(^2\)(BMPE) has the same distribution as \((W^{\theta^+, \theta^-}(t))_{t \geq 0}\) for all \( c > 0 \).

\(^3\)This result was shown in [Dav96] under the additional assumption \(|\theta^+\theta^-| < (1 - \theta^+)(1 - \theta^-)\) which was removed in [Dav99, Theorem 4.8].

\(^4\)In the sense that the walk visits each \( x \in \mathbb{Z} \) infinitely many times with probability 1.

\(^5\)E.g., this is the case for once reinforced random walks, pq-walks, and excited random walks with bounded cookie stacks ([DK12]).
Note that if $w$ satisfies (3) with $\alpha = 0$ and $p, \varepsilon > 0$ then the above limit always exists. This $\gamma$ is the main parameter which identifies the limiting process in Theorem 1.1. It is related to the parameter $\delta$ in [Tót96] by the equation $\gamma = 1 - \delta/2$.

It is easy to show that if $w$ is monotone, then $\gamma < 1$. More precisely, if $w$ is non-increasing (self-repelling case), then $\gamma \in (0, 1)$, and if $w$ is non-decreasing (self-attracting case), then $\gamma \leq 0$. For reader’s convenience we reproduce the argument given in [Tót96, p. 1340]. If $w$ is non-increasing, then $1/w$ is non-decreasing and

$$\gamma = \lim_{n \to \infty} (V_1(n) - U_1(n)) = \sum_{j=0}^{\infty} \left( \frac{1}{w(2j+1)} - \frac{1}{w(2j)} \right) \geq 0.$$  

On the other hand, $U_1(n+1) - U_1(n) = (w(2n))^{-1} \to 1$ as $n \to \infty$, and we also get

$$\gamma = 1 + \lim_{n \to \infty} (V_1(n) - U_1(n+1)) = 1 - \frac{1}{w(0)} - \lim_{n \to \infty} \sum_{j=0}^{n-1} \left( \frac{1}{w(2j+2)} - \frac{1}{w(2j+1)} \right)$$

$$= 1 - \frac{1}{w(0)} - \sum_{j=1}^{\infty} \left( \frac{1}{w(2j)} - \frac{1}{w(2j-1)} \right) < 1.$$  

If $w$ is non-decreasing, then $V_1(n) - U_1(n) \leq 0$ for all $n$ and $\gamma \leq 0$.

**Theorem 1.1.** Let $w : \mathbb{N}_0 \to (0, \infty)$ be monotone and satisfy (3) with $\alpha = 0$, $p \in (1/2, 1]$, and $\varepsilon > 0$. Consider a SIRW $(X_i)_{i \geq 0}$ defined in (2) with $X_0 = 0$. Then

$$\left( \frac{X_{nt}}{\sqrt{n}} \right)_{t \geq 0} \implies (W^{\gamma}; \gamma)_{t \geq 0} \text{ as } n \to \infty$$

in the standard Skorokhod topology on $D([0, \infty))$.

**Remark 1.2.** Note that the case $p > 1$ is covered by (3) with $B = 0$. The restriction to $p > 1/2$ guarantees that the series representing the “total drift” at a single site (see [11] and Lemma 2.3) converges absolutely. We think that the case $p \in (0, 1/2]$ can be considered as well but will certainly require a different and technically more involved treatment such as coarse graining in the spirit of [KMP22].

**Remark 1.3.** Following [Tót96] we have assumed that $w$ is monotone. If $w$ is not monotone (and $\alpha = 0$), then $\gamma$ can be any real number. Monotonicity assumption on $w$ naturally restricts the parameter $\gamma$ to “recurrent” values. Our arguments, provided that $\gamma < 1$, do not require monotonicity except for the results which we quote from [Tót96]. However, we believe that one could treat non-monotonic $w$ with a slightly different method and that for $\gamma < 1$ the results will remain the same. The removal of the monotonicity assumption would permit arbitrary values of $\gamma$ including “transient” values (i.e., $\gamma > 1$). In this case the approach of [KMP11] would likely give results of the character of those found in [KMP11] and [KZ13, Section 6]. We leave these questions to an interested reader.

Our second result concerns the polynomially self-repelling case $\alpha > 0$. This negative result is rather unexpected and, for this reason, is, in the authors’ opinion, particularly interesting. Let

$$W_\alpha(t) := (2\alpha + 1)^{1/2}W_\alpha^1(t), \quad t \geq 0,$$

where $W_\alpha^{1,1/2}$ is a BMPE with $\theta^+ = \theta^- = 1/2$ (see Definition [1.1]), and recall two facts which make the process $W_\alpha$ the natural candidate for a weak limit of $X^{(n)} := (X^{(n)}(t))_{t \geq 0} := (n^{-1/2}X_{[nt]})_{t \geq 0}$ as $n \to \infty$.

(i) The generalized Ray–Knight theorems for the sequence of processes $(X^{(n)})_{n \in \mathbb{N}}$, [Tót96, Theorem 1B, Corollary 1B], correspond to Ray–Knight theorems for $W_\alpha$ (see [CPY98, Theorems 3.1, 3.4]).
(ii) If for a fixed \( t > 0 \) the sequence \( (X^{(n)}(t))_{n \in \mathbb{N}} \) converges in distribution, then the limiting distribution must be that of \( W_\alpha(t) \) (see \cite{T6996} Theorem 2B, (3.2.9), (3.2.11), and Remark on p. 1334)).

These facts identify the process \( W_\alpha \) as the only possible weak limit of \( (X^{(n)})_{n \in \mathbb{N}} \) in the set \( \{cW^{\theta^+, \theta^-}, \theta^+ \vee \theta^- < 1, c \in \mathbb{R}\} \) of all scalar multiples of BMPEs. Moreover, our first main result, Theorem 1.1, shows that the BMPEs “predicted” by exactly the same information as in (i), (ii) are bona fide weak limits for a class of asymptotically free SIRWs.

**Theorem 1.4.** Let \( \alpha > 0 \), \( (W_\alpha(t))_{t \geq 0} \) be given by (6), and \( w(n) = (n + 1)^{-\alpha} \) for all \( n \in \mathbb{N}_0 \). Consider a SIRW \( (X_i)_{i \geq 0} \) defined in (2) with \( X_0 = 0 \). Then

\[
\left( \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \right)_{t \geq 0} \Rightarrow (W_\alpha(t))_{t \geq 0} \quad \text{as} \quad n \to \infty
\]

in the standard Skorokhod topology on \( D([0, \infty)) \).

**Remark 1.5.** The result presented in Theorem 1.4 and its proof remain valid for a more general weight function \( w \) given by (3) with \( p = \kappa = 1 \) as in \cite{T6996}. We have chosen to write out the arguments for \( w(n) = (n + 1)^{-\alpha}, n \geq 0 \), simply for clarity’s sake and also as our result is a counterexample to the functional convergence of rescaled polynomially self-repelling SIRWs to BMPEs.

Already at this point we would like to say a few words as to why in the polynomially self-repelling case we must reject the only natural candidate \( W_\alpha \) as a possible weak limit of \( X^{(n)}, n \geq 0 \). A more detailed discussion is given right after Proposition 3.2. In Proposition 3.2 we study the behavior of increments of the local time processes associated to \( X^{(n)} \) rather than just the behavior of cumulative local times which appear in generalized Ray–Knight theorems. We discover that these increments scale with a different constant than those of \( W_\alpha \) and this violates the additive property of the limiting local time processes which is inherent to BMPEs. Since our observations are in terms of local time processes which are not continuous path functionals, the proof of this result is rather technical: we have to approximate local times with occupation times over small intervals and then take an appropriate subsequence along which the convergence fails.

**Remark 1.6** (and an open problem). For the polynomially self-repelling case the question of whether \( X^{(n)}(t) \Rightarrow W_\alpha(t) \) for a fixed \( t > 0 \) unfortunately remains open. Provided that one dimensional distributions converge, one would also like to study weak convergence at the process level and find a viable candidate for a weak process limit of \( (X^{(n)})_{n \geq 1} \). The last task is particularly interesting in view of Theorem 1.4.

### 1.3. Organization of the paper

The proof of Theorem 1.1 except for derivations of several technical results, is given Section 2. In Section 3 we prove Theorem 1.4 modulo a key Proposition 3.2 which we state and discuss in Section 3 but prove only at the end of Section 4 after a thorough treatment of the generalized Pólya urn model of left and right jumps from a single site. More specifically, Section 4 studies the process of “discrepancies” between the numbers of left and right jumps from one site and lays a foundation for the technical results of the previous two sections. Appendix A concerns generalized Ray–Knight theorems for SIRWs where we define “branching-like processes” and adapt as necessary some of the results previously obtained in \cite{T6996}. These results are used in Appendix B which contains proofs of Proposition 2.1 and Lemma 2.2 from Section 2. Proofs of two lemmas for the polynomially self-repelling case and an auxiliary Lemma C.1 are given in Appendix C.

### 1.4. Notation

Below we gather some of the notation used throughout the paper.
BESQ$\delta$ - the square of a Bessel process of generalized dimension $\delta \in \mathbb{R}$ ([RY99] Definition (1.1), Chapter XI) and ([GJY03] (28)). For $\alpha \geq 0$ and $\delta \in \mathbb{R}$ we denote by $Z^{(\alpha, \delta)}$ a BESQ$\delta$ process divided by $2(2\alpha + 1)$. When $\alpha = 0$ (asymptotically free case) we shall drop it from the notation and write simply $Z^{(\delta)}$.

BLP - branching-like process (Appendix A).

BMPE - Brownian motion perturbed at its extrema (Definition 1.1).

SIRW - self-interacting random walk.

Weak convergence (denoted by $\Rightarrow$) and tightness of stochastic processes in $D([0, T])$ or $D([0, \infty))$ will always be understood with respect to the standard (i.e., $J_1$) Skorokhod topology.

For a generic stochastic process $(Y_t)_{t \geq 0}$ indexed either by $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ or $t \in [0, \infty)$ and $x \in \mathbb{R}$ we define

$$(7) \quad \tau^Y_x := \inf\{t > 0 : Y_t \geq x\} \quad \text{and} \quad \sigma^Y_x := \inf\{t > 0 : Y_t \leq x\} \quad \text{where} \quad \inf \varnothing := \infty.$$ 

$P^Y_y$ and $E^Y_y$ will be used to indicate that $Y(0) = y$.

We denote by $L(x, n) = \sum_{i=0}^n \mathbb{1}_{\{X_i = x\}}$ the number of visits to $x$ by the random walk by time $n$. The number of upcrossings (resp. downcrossings) by the random walk of the directed bond $(x, x+1)$, $x \in \mathbb{Z}$, (resp. $(x, x-1)$) up to time $i \in \mathbb{N}$ will be denoted by $E^i(x)$ (resp. $D^i(x)$), i.e.,

$$(8) \quad E^i(x) = \sum_{j=0}^{i-1} \mathbb{1}_{\{X_j = x, X_{j+1} = x+1\}}; \quad D^i(x) = \sum_{j=0}^{i-1} \mathbb{1}_{\{X_j = x, X_{j+1} = x-1\}},$$

where $\mathbb{1}_A$ is the indicator function of set $A$.

Given two probability measures $P_1$ and $P_2$ on the Borel sigma-field $\mathcal{B}$ of the Skorokhod metric space $(D([0, 1], d^\alpha))$ ([Bil99] (12.16)), we denote by dist $(P_1, P_2)$ the Prokhorov distance between $P_1$ and $P_2$ ([Bil99] p. 72):

$$\text{dist} (P_1, P_2) = \inf \{\varepsilon > 0 : \forall A \in \mathcal{B}, P_1(A) \leq P_2(A^\varepsilon) + \varepsilon \text{ and } P_2(A) \leq P_1(A^\varepsilon) + \varepsilon\},$$

where $A^\varepsilon = \{\omega \in D([0, 1]) : \exists \tilde{\omega} \in A, d^\alpha(\omega, \tilde{\omega}) < \varepsilon\}$. Recall that if dist $(P_N, P) \to 0$ then $P_N \Rightarrow P$ as $N \to \infty$. In particular, for every $\varepsilon > 0$ and every continuous bounded functional $F$ on $D([0, 1])$ there is an $\eta = \eta(\varepsilon, F, P) > 0$ such that

$$(9) \quad \text{dist} (P_N, P) < \eta \quad \text{implies} \quad \left| \int F dP_N - \int F dP \right| < \varepsilon.$$  

2. Asymptotically free case: proof of Theorem 1.1

For the proof of Theorem 1.1 we will decompose the random walk as $X_n = M_n + \Gamma_n$, where

$$(10) \quad \Gamma_0 = 0, \quad \Gamma_n = \sum_{i=0}^{n-1} E \left[ X_{i+1} - X_i \mid \mathcal{F}_i^X \right], \quad \mathcal{F}_i^X = \sigma(X_j, j \leq i), \quad n \in \mathbb{N}.$$ 

Note that with this choice of $\Gamma_n$, the process $M_n = X_n - \Gamma_n$, $n \in \mathbb{N}_0$, is a martingale with respect to the filtration $\mathcal{F}_n^X$, $n \in \mathbb{N}_0$. The key idea in the proof of Theorem 1.1 is that the martingale term in this decomposition will converge to a Brownian motion (Lemma 2.7) while $\Gamma_n$, which accounts for the accumulated drift experienced by the walk up to time $n$, will be approximated by a linear combination of the running minimum and maximum of the walk.

A similar strategy as the one we employ here has been used previously for other self-interacting random walks (see [Do11], [DK12], [KP16], [HLSH18]). Since the proofs of a few of the technical results that we need are quite similar to those in the existing literature, we will state these technical results here and give their proofs in Appendix B.
Lemma 2.4. Let the weight function

\[ w(x,k) = \text{left/right steps from a fixed site using a generalized Pólya urn.} \]

for any \( w \) converges, (ii) \( 3.3 \] (albeit somewhat simpler) and is given in Appendix B. Let

\[ \text{visited much very few times. In particular, we will need the following lemma.} \]

Theorem 1.1 we will need to show that there are not too many sites in the range which have been

\[ \text{range of the walk up to time } n \] since we expect a limiting distribution with diffusive scalin
g, it is natural that most sites in the

\[ \text{Let } \delta \]

Proposition 2.1. Let the weight function

\[ \gamma \]

where the constant \( \gamma \) is as defined in \( 5 \) and is given in Appendix B. The proofs of Lemmas 2.3 and 2.4 will be given in Section 4.1 where we analyze the sequence of left/right steps from a fixed site using a generalized Pólya urn.

\[ \text{Lemma 2.3. Let the weight function } w \text{ be as in } 3 \text{ with } \alpha = 0, \text{ and let } \gamma_+ = \gamma \lor 0 \text{ where } \gamma \text{ is defined in } 5. \text{ Then for any } M > 0 \text{ and any } b > \frac{\gamma}{2} \text{ we have} \]

\[ \lim_{n \to \infty} P \left( \sup_{k \leq nt} \sum_{x \in [I_{k-1}, I_k]} 1_{\{x \leq nt, s_i \leq M\}} \geq 4n^b \right) = 0, \quad \forall M > 0. \]

The proof of Lemma 2.2 is an adaptation of proofs in \[ \text{KM11, KP16, KMP22} \text{ and is given in Appendix B.} \]

A key point in our approximation of the accumulated drift term by a linear combination of \( S_n \) and \( I_n \) (Lemma 2.8 below) is that for each site \( x \in \mathbb{Z} \), there is a random but finite accumulated drift from all visits to \( x \):

\[ \text{(11) } \delta_x = \sum_{i=0}^{\infty} E \left[ X_{i+1} - X_i \mid F_i \right] 1_{\{X_i = x\}}, \quad x \in \mathbb{Z}. \]

It is not a priori obvious that the sum in (11) converges. However, the following lemmas show that under the proper assumptions on the weight function \( w \) the series not only converges absolutely, a.s., but also has finite moments of all orders. Moreover, \( E[\delta_x] \) can be explicitly calculated.

\[ \text{Lemma 2.3. Let the weight function } w \text{ be as in } 3 \text{ with } \alpha = 0, \text{ and } \kappa > 0. \text{ If} \]

\[ \delta_x = \sum_{i=0}^{\infty} |E \left[ X_{i+1} - X_i \mid F_i \right] 1_{\{X_i = x\}}|, \quad x \in \mathbb{Z}. \]

then \( E[|\delta_x|^M] < \infty \) for all \( M > 0 \) and \( x \in \mathbb{Z} \). In particular, this implies that the sum in (11) converges, \( P \)-a.s., for all \( x \in \mathbb{Z} \), and \( E[|\delta_x|^M] < \infty \) for all \( x \in \mathbb{Z} \) and \( M > 0 \).

\[ \text{Lemma 2.4. Let the weight function } w \text{ be as in } 3 \text{ with } \alpha = 0, \text{ and } \kappa > 0. \text{ Then} \]

\[ E[\delta_x] = \text{sign(x)} \gamma, \quad \text{where the constant } \gamma \text{ is as defined in } 5 \text{ and sign(0) := 0}. \]

The proofs of Lemmas 2.3 and 2.4 will be given in Section 4.1 where we analyze the sequence of left/right steps from a fixed site using a generalized Pólya urn.
Remark 2.5. A crucial observation that will be used in the proof of Theorem 1.1 below, is that the sequences of random variables \((\delta_x)_{x \in \mathbb{Z}}\) and \((\delta_x)_{x \in \mathbb{Z}}\) are both sequences of independent random variables. This is due to the fact that any step of the random walk depends only on the behavior of the walk at previous visits to the current location so that the sequence of left/right steps from each site can be generated by independent realizations of generalized Pólya urns (see Section 4). Moreover, since for any site \(x < 0\) (or for any \(x > 0\) respectively) the process of generating the sequence of left/right steps from \(x\) is the same, it follows that the sequences \((\delta_x)_{x \geq 1}\) and \((\delta_x)_{x \leq -1}\) are both respectively i.i.d.

Remark 2.6. Another fact that we will use is that there is a natural symmetry in the behavior of the walk to the right of the origin and to the left of the origin. In particular, we will repeatedly use that \(S_n^X \overset{\text{Law}}{=} -I_n^X\) for all \(n \geq 0\), \(\delta_x \overset{\text{Law}}{=} -\delta_x\) and \(\bar{\delta}_x \overset{\text{Law}}{=} \bar{\delta}_1\) for all \(x \neq 0\).

Having collected the main tools we are now ready to move on to the proof of Theorem 1.1.

2.1 Proof of Theorem 1.1. Recall that \(M_n = X_n - \Gamma_n\), \(n \in \mathbb{N}_0\), is a martingale. The first step in the proof of Theorem 1.1 is to show that this martingale converges under diffusive scaling to a standard Brownian motion.

Lemma 2.7. The process \(\left(\frac{M_{\lfloor nt\rfloor}}{\sqrt{n}}\right)_{t \geq 0}\) converges in distribution to a standard Brownian motion.

Proof. Since \(M_n\) is a martingale with bounded increments, it is enough (see [Bill99, Theorem 18.2]) to show that \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} E[(M_{i+1} - M_i)^2 | F_i^X] = 1\), in probability. We have

\[
E[(M_{i+1} - M_i)^2 | F_i^X] = E[(X_{i+1} - X_i - X_{i+1} - X_i) | F_i^X]^2 | F_i^X] \leq 1 - E[X_{i+1} - X_i | F_i^X]^2.
\]

Therefore, it is enough to prove that \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} E[X_{i+1} - X_i | F_i^X]^2 = 0\), in probability. The estimate

\[
\sum_{i=0}^{n-1} E[X_{i+1} - X_i | F_i^X]^2 \leq \sum_{x \in \{I_n^X, S_n^X\}} \sum_{i=0}^{\infty} |E[X_{i+1} - X_i | F_i^X]| \mathbb{1}_{\{X_i = x\}} = \sum_{x \in \{I_n^X, S_n^X\}} \bar{\delta}_x
\]

implies that

\[
P\left(\frac{1}{n} \sum_{i=0}^{n-1} E[X_{i+1} - X_i | F_i^X]^2 \geq \varepsilon\right) \leq P(I_n^X \leq -n^{3/4}) + P(S_n^X \geq n^{3/4}) + P\left(\sum_{|x| \leq n^{3/4}} \bar{\delta}_x \geq \varepsilon n\right)
\]

\[
\leq 2P(S_n^X \geq n^{3/4}) + \frac{E[\bar{\delta}_0] + 2n^{3/4}E[\bar{\delta}_1]}{\varepsilon n},
\]

where in the last inequality we used symmetry considerations noted in Remark 2.6. By Proposition 2.1 and Lemma 2.3 the right side vanishes as \(n \to \infty\). This completes the proof. \(\square\)

Lemma 2.8. Let \(w\) be as in (3) with \(\alpha = 0\) and \(p \in (1/2, 1]\). Then,

\[
\lim_{n \to \infty} P\left(\sup_{k \leq nt} |\Gamma_k - \gamma(S_k^X + I_k^X)| > \varepsilon\sqrt{n}\right) = 0, \quad \forall \varepsilon, t > 0.
\]

Proof. For any \(x \in \mathbb{Z}\) and \(m \geq 1\) let \(\delta_{x,m}\) be total drift accumulated in the first \(m\) visits to \(x:\)

\[
(12) \quad \delta_{x,m} = \sum_{i=0}^{\infty} E\left[X_{i+1} - X_i \mid F_i^X\right] \mathbb{1}_{\{X_i = x, \mathcal{L}(x,i) \leq m\}}, \quad \forall x \in \mathbb{Z}, \ m \geq 1.
\]
With this notation we have that $\Gamma_k = \sum_{x \in [t^X_{k-1}, I^X_{k-1}]} \delta_{x, \mathcal{L}(x, k-1)}$. Also, it follows from Lemma 2.4 that $\gamma(S^X_{k-1} + I^X_{k-1}) = \sum_{x \in [S^X_{k}, I^X_{k}]} E[\delta_x]$. Combining these two facts we see that

$$
P\left(\sup_{k \leq nt} |\Gamma_k - \gamma(S^X_{k-1} + I^X_{k-1})| > \varepsilon \sqrt{n}\right)$$

$$\leq P\left(\sup_{k \leq nt} \left| \sum_{x \in [t^X_{k-1}, I^X_{k-1}]} (\delta_{x, \mathcal{L}(x, k-1)} - \delta_x) \right| \geq \frac{\varepsilon \sqrt{n}}{2}\right)$$

$$+ P\left(\sup_{k \leq nt} \left| \sum_{x \in [S^X_{k}, I^X_{k}]} (\delta_x - E[\delta_x]) \right| \geq \frac{\varepsilon \sqrt{n}}{2}\right).$$

For the probability in (14), it follows from the symmetry considerations in Remark 2.6 that for any fixed $K > 0$,

$$P\left(\sup_{k \leq nt} \left| \sum_{x \in [t^X_{k-1}, I^X_{k-1}]} (\delta_x - E[\delta_x]) \right| \geq \frac{\varepsilon \sqrt{n}}{2}\right)$$

$$\leq 2P(S^X_{nt} \geq K \sqrt{n}) + P(\delta_0 \geq \frac{\varepsilon \sqrt{n}}{6}) + 2P\left(\max_{k \leq K \sqrt{n}} \left| \sum_{x=1}^{k} (\delta_x - E[\delta_x]) \right| \geq \frac{\varepsilon \sqrt{n}}{6}\right).$$

For any fixed $\varepsilon, K > 0$, the last two probabilities vanish as $n \to \infty$ by Lemma 2.3 and the strong law of large numbers. Since Proposition 2.1 implies that the first probability can be made arbitrarily small (uniformly in $n$) by taking $K$ sufficiently large, we conclude that the probability in (13) goes to 0 as $n \to \infty$ for any $\varepsilon > 0$.

It remains to estimate the probability in (13). To this end, we will fix a parameter $b \in (\frac{\gamma}{2}, \frac{1}{2})$. Then, since $|\delta_x - m - \delta_x| \leq \delta_x$, we get that for any $M > 0$

$$P\left(\sup_{k \leq nt} \left| \sum_{x \in [t^X_{k-1}, I^X_{k-1}]} (\delta_{x, \mathcal{L}(x, k-1)} - \delta_x) \right| \geq \frac{\varepsilon \sqrt{n}}{2}\right)$$

$$\leq P\left(\sup_{k \leq nt} \sum_{x \in [t^X_{k-1}, I^X_{k-1}]} \delta_x 1_{\{\mathcal{L}(x, k-1) \leq M\}} \geq \frac{\varepsilon \sqrt{n}}{4}\right)$$

$$+ P\left(\sup_{k \leq nt} \sum_{x \in [t^X_{k-1}, I^X_{k-1}]} |\delta_{x, \mathcal{L}(x, k-1)} - \delta_x| 1_{\{\mathcal{L}(x, k-1) > M\}} \geq \frac{\varepsilon \sqrt{n}}{4}\right)$$

$$\leq P\left(\max_{|x| \leq nt} \delta_x \geq \frac{\varepsilon}{16} n^{\frac{1}{2} - b}\right) + P\left(\sup_{k \leq nt} \sum_{x \in [t^X_{k-1}, I^X_{k-1}]} 1_{\{\mathcal{L}(x, k-1) \leq M\}} \geq 4n^b\right)$$

$$+ P\left(\sup_{k \leq nt} \sum_{x \in [t^X_{k-1}, I^X_{k-1}]} |\delta_{x, \mathcal{L}(x, k-1)} - \delta_x| 1_{\{\mathcal{L}(x, k-1) > M\}} \geq \frac{\varepsilon \sqrt{n}}{4}\right).$$

Since $b < \frac{1}{2}$, it follows from Lemma 2.3 and Remark 2.3 that the first probability in (15) vanishes as $n \to \infty$, while, since $b > \frac{\gamma}{2}$, the second probability in (15) vanishes by Lemma 2.2. For the
probability in (16), we have for any $K > 0$ that
\[
P\left( \sup_{k \leq nt} \sum_{x \in [I_{k-1}^n, I_k^n]} |\delta_{x,\mathcal{L}(x,k-1)} - \delta_x| \mathbb{1}\{\mathcal{L}(x,k-1) > M\} \geq \frac{\varepsilon \sqrt{n}}{4} \right)
\leq 2P(S_{[nt]-1}^X \geq K \sqrt{n}) + P\left( \sum_{|x| \leq K \sqrt{n}} \left( \sup_{m>M} |\delta_{x,m} - \delta_x| \right) \geq \frac{\varepsilon \sqrt{n}}{4} \right)
\leq 2P(S_{[nt]-1}^X \geq K \sqrt{n}) + \frac{4}{\varepsilon \sqrt{n}} \sum_{|x| \leq K \sqrt{n}} E \left( \sup_{m>M} |\delta_{x,m} - \delta_x| \right)
\leq 2P(S_{[nt]-1}^X \geq K \sqrt{n}) + \frac{4}{\varepsilon \sqrt{n}} E\delta_0 + \frac{8K}{\varepsilon} E \left( \sup_{m>M} |\delta_{1,m} - \delta_1| \right)
\tag{17}
\]
where we have used the symmetry considerations from Remark 2.6 in the first and last inequalities. Since $|\delta_{1,m} - \delta_1| \leq \delta_1$, it follows from Lemma 2.3 and the dominated convergence theorem that $E\left[ \sup_{m>M} |\delta_{1,m} - \delta_1| \right] \to 0$ as $M \to \infty$. Using this together with Proposition 2.1 we can choose $K, M > 0$ so that the first and third terms in (17) are arbitrarily small (uniformly in $n$). Thus, we can conclude that the probability in (16) vanishes as $n \to \infty$.

**Proof of Theorem 7.1** First, we claim that the sequence of processes $(\frac{X_{[nt]}}{\sqrt{n}}, \frac{M_{[nt]}}{\sqrt{n}}, \frac{\Gamma_{[nt]}}{\sqrt{n}})_{t \geq 0}$ is tight in $D([0, \infty))$ and that any subsequential limit is concentrated on continuous paths. This follows from the decomposition $X_n = M_n + \Gamma_n$, the tightness of the martingale term $M_n$ in Lemma 2.7, the approximation of the accumulated drift term $\Gamma_n$ by $\gamma(I_n^X + S_n^X)$ from Lemma 2.8 and the tightness of the running extrema from Lemma 2.1. The details of this argument are almost identical to the proof of [KP16, Lemma 4.4] and are therefore omitted.

By Proposition 2.1 and Lemmas 2.7 and 2.8 we can then conclude that the process triple $(\frac{1}{\sqrt{n}} \left( X_{[nt]}, M_{[nt]}, \Gamma_{[nt]} \right))_{t \geq 0}$ is a tight sequence in $D([0, \infty))^3$ and that any subsequential limit $(Y_1(t), Y_2(t), Y_3(t))_{t \geq 0}$ is a continuous process such that $Y_2$ is a standard Brownian motion, $Y_3(t) = \gamma(\sup_{s \leq t} Y_1(s) + \inf_{s \leq t} Y_1(s))$ for all $t \geq 0$, $P$-a.s., and $Y_1(t) = Y_2(t) + Y_3(t)$. That is, $Y_1$ is a $(\gamma, \gamma)$-BMPE as claimed. \hfill \Box

**3. Polynomially self-repelling case**

In this section we prove Theorem 1.4 modulo a version of a generalized Ray–Knight theorem, Proposition 3.2. This proposition deals with increments of numbers of upcrossings and is a key fact behind the argument below. Its proof is based, in turn, on a detailed analysis of a generalized Pólya urn model which we carry out later, in Section 4.

Recall the definition of the process $W_\alpha$ in (9) and let $I_\alpha(t) := \inf_{0 \leq s \leq t} W_\alpha(s)$ and $S_\alpha(t) := \sup_{0 \leq s \leq t} W_\alpha(s)$. Then the triple $(I_\alpha(t), W_\alpha(t), S_\alpha(t))$ is a strong Markov process\footnote{relative to any filtration with respect to which $(B(t))_{t \geq 0}$, that appears in the definition $W_\alpha$, is a Brownian motion, see [CPY98, Section 2.3].} which satisfies the equation
\[
W_\alpha(t) = B_\alpha(t) + \frac{1}{2} S_\alpha(t) + \frac{1}{2} I_\alpha(t), \quad W_\alpha(0) = 0, \quad t \geq 0,
\]
where $B_\alpha(t) = \sqrt{2\alpha + 1} B(t), \quad t \geq 0$, and $(B(t))_{t \geq 0}$ is a standard Brownian motion. Recall also that $Z^{(\alpha, \delta)}$ is $(2(2\alpha + 1))^{-1}$ times a BESQ\footnote{relative to any filtration with respect to which $(B(t))_{t \geq 0}$, that appears in the definition $W_\alpha$, is a Brownian motion, see [CPY98, Section 2.3].}. The process $Z^{(\alpha, \delta)}$ has a non-random initial point and solves
\[
dZ^{(\alpha, \delta)}(x) = \frac{\delta}{2(2\alpha + 1)} \, dx + \frac{1}{\sqrt{2\alpha + 1}} \sqrt{2Z^{(\alpha, \delta)}(x)} \, dB(x),
\]
0 ≤ x ≤ \inf\{y > 0 : Z^{(\alpha,\delta)}(y) = 0\}. Throughout this section we also assume that Z^{(\alpha,\delta)} is always absorbed upon hitting 0. Let
\begin{equation}
L_t^{W_\alpha}(x) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{[x,x+\varepsilon]}(W_\alpha(s)) \, ds, \quad x \in \mathbb{R}, \ t ≥ 0,
\end{equation}
be a half of the local time of W_\alpha at x by time t and set \( T_{\ell}^{L_t^{W_\alpha}} := \inf\{t ≥ 0 : L_t^{W_\alpha}(0) > \ell\} \), \( \ell \in [0, \infty) \). We shall write \( L_t^{W_\alpha} \) instead of more cumbersome \( T_{\ell}^{L_t^{W_\alpha}} \).

### Proposition 3.1

For every M > 0, \( \left( L_t^{W_\alpha}(x) \right)_{x ≥ 0} \overset{\text{Law}}{\rightarrow} \left( Z^{(\alpha,1)}(x) \right)_{x ≥ 0} \) with \( Z^{(\alpha,1)}(0) = M \). As \( M \to \infty \),
\begin{equation}
\left( Z^{W_\alpha}(x) \right)_{x ≥ 0} := \left( L_t^{W_\alpha}(x) - L_t^{W_\alpha}(0) \right)_{x ≥ 0} \implies \left( Z^{(\alpha,0)}(x) \right)_{x ≥ 0}, \quad Z^{(\alpha,0)}(0) = 1.
\end{equation}

**Proof.** The first statement follows from [CPY98, Theorem 3.1]. To show (20), we note that \( L_t^{W_\alpha}(x_0) > 0 \) for some \( x_0 > 0 \) implies that \( L_t^{W_\alpha}(x) > 0 \) on \([0, x_0]\) and also that \( S_\alpha(T^{W_\alpha}) > x_0 \). Thus, on the event \( \{L_t^{W_\alpha}(x_0) > 0\} \) the process \( \left( L_t^{W_\alpha}(x) - L_t^{W_\alpha}(0) \right)_{0 ≤ x ≤ x_0} \) has the same distribution as \( \left( L_t^{W_\alpha}(x) \right)_{0 ≤ x ≤ x_0} \). By scaling properties of Brownian motion and the second Ray–Knight theorem for the standard Brownian motion, the process \( L_t^{W_\alpha}(x), \ x ∈ [0, x_0], \) has the same distribution as \( Z^{(\alpha,0)}(x), \ x ∈ [0, x_0] \). Since for every \( x_0 > 0 \) the probability of \( \{L_t^{W_\alpha}(x_0) > 0\} \) goes to 1 as \( M \to \infty \), (20) follows.

\[ \square \]

The key observation we use to prove Theorem 1.4 is that the discrete analog of the left hand side of (20) constructed with \( X^{(n)} \) in place of \( W_\alpha \) is close in distribution (Proposition 3.2 below) to \( Z^{(0,0)} \), which is a half of BESQ\(^0\) and, therefore, is different from \( Z^{(\alpha,0)} \). Before continuing our discussion let us state this result rigorously.

Recall (8) and for \( \ell ≥ 0 \) define \( T_\ell^E := \inf\{k ≥ 0 : E^k(0) > \ell\} \). Just as before, we shall write \( E^T_t(x) \) instead of \( E^T_t^E(x) \).

### Proposition 3.2

For every fixed \( M > 0 \), as \( N \to \infty \), \( (N^{-1}E^{T^N(x)}(\lfloor Nx \rfloor))_{x ≥ 0} \implies (Z^{(\alpha,1)}(x))_{x ≥ 0} \) with \( Z^{(\alpha,1)}(0) = M \). Moreover, for every \( \eta > 0 \) and \( c \in [0, 1/2] \) there exists \( M_0 > 0 \) such that for every \( M ≥ M_0 \) there is an \( N(M) \) so that for all \( N ≥ N(M) \),
\begin{equation}
\text{dist}(P_{Z_{N,0}}^{M,c}, P_{1-2c}^{Z_{0,0}}) < \eta,
\end{equation}
where \( P_{Z_{N,0}}^{M,c} \) denotes the law of the process
\[ Z_{N,0}^{M,c}(x) := \frac{1}{N} (E^{T^N(M+1)}(\lfloor Nx \rfloor) - E^{T^N(M+1)}(\lfloor Nx \rfloor)), \ x ∈ [0, 1], \]
and \( P_{1-2c}^{Z_{0,0}} \) denotes the law of \( (Z^{(0,0)}(x))_{x∈[0,1]} \) with \( Z^{(0,0)}(0) = 1 - 2c \).

### Remark 3.3

We note that the first claim immediately follows from [Tóth96, Theorem 1B]. Here we offer an informal discussion as to why the “limiting” process \( Z^{(0,0)} \) with \( c = 0 \) in (21) is different from the limiting process \( Z^{(\alpha,0)} \) in (20). We have already mentioned that this fact is the key to the proof of Theorem 1.4. The appearance of \( Z^{(0,0)} \) in (21) is not intuitive and is based on a careful analysis of processes of left and right jumps of X from a single site. More precisely, if we look at the rescaled difference between the number of jumps of X to the right between the \( KR\)-th and \((K+1)R\)-th jumps to the left from a single site then, as \( R \) goes to infinity, this rescaled difference

\[ \text{The choice to work with a half of the local time corresponds to the fact that for the walk we consider processes of edge local times.} \]

\[ \text{8} L_{T_1}^{W_\alpha} \text{ is defined analogously to } L_{T_1}^{W_\alpha}. \]

\[ \text{9} \text{i.e., the process } (\mathcal{D}_n - \mathcal{D}_m) \sqrt{n} \text{ with } n = KR \text{ and } m = (K+1)R, \text{ see Proposition 2.}\]
has approximately zero mean and variance \( v(\alpha, K) = 1 + O(K^{-1}) \to 1 \) as \( K \) grows large. On the other hand, for \( K = 0 \) the variance converges to \( v(\alpha, 0) = (2\alpha + 1)^{-1} \) (Proposition 3.3). This is exactly the factor which enters generalized Ray–Knight theorems making the limiting processes in the first claims of Propositions 3.1 and 3.2 to be \( Z^{(\alpha,1)} \) instead of \( Z^{(0,1)} \). The dependence of \( v(\alpha, K) \) on \( K \) reflects the dependence of this rescaled difference on the history of the walk prior to the \( KR \)-th jump to the left. As a consequence of this dependence, the “limiting” process in (21) is \( Z^{(0,0)} \) and not \( Z^{(\alpha,0)} \) as in (20). It is these findings that allow us to rule out BMPE as a possible weak limit. The mentioned above Proposition 3.3 and the proof of Proposition 3.2 are given at the end of Subsection 4.2.

The fact that multiples of the limiting BESQ processes in (20) and (21) are different does not immediately imply the statement of Theorem 1.2, since the difference is expressed in terms of the local time processes, which are not a.s. continuous functionals on the path space. Nevertheless, replacing local times with averages of occupation times of small intervals we shall be able to prove Theorem 1.4.

**Proof of Theorem 1.4** We work on \( D([0, \infty)) \) with the topology generated by one of the equivalent Skorokhod metrics (see, for example, [Bil99, (16.4), (12.16)]). For \( \delta > 0 \) and \( \ell \in [0, \infty) \) define

\[
\mathcal{T}_{\delta,\ell}(\omega) = \inf \left\{ t \geq 0 : \frac{1}{2\ell} \int_0^t \mathbb{1}_{[0,\delta]}(\omega(s)) \, ds > \ell \right\}; \quad G_{\delta,\ell}(\omega) = \frac{1}{2} \int_{\mathcal{T}_{\delta,\ell}} \mathbb{1}_{[0,1]}(\omega(s)) \, ds.
\]

We shall show that, on the one hand,

(UB) for each \( \varepsilon > 0 \) there are \( M_0 = M_0(\varepsilon), \delta_0 = \delta_0(\varepsilon) > 0 \) such that for all \( M \geq M_0, \delta \in (0, \delta_0), \)

\[
(22) \quad E^{W_\alpha}[G^2_{\delta,M}] - E^{\left[ \left( \int_0^1 Z^{(\alpha,0)}(x) \, dx \right)^2 \right]}^{\text{Lem. 3.1}} - \frac{1}{2} \left( 1 + \frac{2}{3(2\alpha + 1)} \right) \leq \varepsilon,
\]

while, on the other hand,

(LB) for each \( \varepsilon > 0 \) there are \( M \geq M_0(\varepsilon), 0 < \delta \leq \delta_0(\varepsilon) \) and \( K_0 = K_0(\varepsilon) \) such that for all \( K \geq K_0 \) and all sufficiently large \( n \) of the form \( n = N^2 \)

\[
(23) \quad E^{X^{(n)}}[G^2_{\delta,M} \wedge K] - E^{\left[ \left( \int_0^1 Z^{(0,0)}(x) \, dx \right)^2 \right]}^{\text{Lem. 3.1}} - \frac{5}{3} \geq -2\varepsilon.
\]

Once this is done, we take an \( \varepsilon \in \left( 0, \frac{\alpha}{3(2\alpha + 1)} \right) \), choose \( M_0, \delta_0 \) as in (UB) and then \( \delta, M, \) and \( K \) as in (LB), and get that for all large \( n \) of the form \( n = N^2 \)

\[
E^{X^{(n)}}[G^2_{\delta,M} \wedge K] \geq E^{W_\alpha}[G^2_{\delta,M}] + \varepsilon \geq E^{W_\alpha}[G^2_{\delta,M} \wedge K] + \varepsilon.
\]

Since \( G^2_{\delta,M} \wedge K \) is a continuous bounded functional on \( D([0, \infty)) \) (see Lemma 3.4 below), the conclusion of Theorem 1.4 follows. \( \square \)

It is left to prove bounds (UB) and (LB). We shall need the following technical lemma. Its proof is given in Appendix C.

**Lemma 3.4.** For all \( \delta > 0, M \geq 0, G_{\delta,M} : \Omega \to \mathbb{R} \) is a \( P^{W_\alpha} \)-a.s. continuous functional. Moreover, there is \( \delta_0 \in (0, 1/2) \) such that for every \( p \geq 1 \), \( \sup_{M \geq 0, \delta \in (0, \delta_0]} E^{W_\alpha}[\left| (G_{\delta,M})^p \right|] < \infty. \)

**Proof of (UB).** Let \( T_{\delta,M} = \inf\{s > T_{\delta,M} : W_\alpha(s) = 0\} \) and \( T_{\delta,M,s} = \inf\{t > T_{\delta,M} : L^W_\alpha(0) - L^W_{T_{\delta,M}}(s) = s\}. \) We fix \( \lambda > 0 \) (eventually we will fix \( \lambda \in (0, 1) \) small) and introduce two “bad”

\[
\text{[11]} L^W_{T_{\delta,M}}(0) \text{ denotes the half of the local time at 0 by time } t.
\]
We first give a bound for $P(A_1 \cup A_2) < \eta$ for all $M \geq M_0, \delta \in (0, \delta_0]$. On $A_1 \cup A_2$ we introduce the process $\tilde{B}_\alpha(s) := B_\alpha(T_{M+1} - B_\alpha(T_{M+1}'))$ for $s \in [0, \nu - T_{M+1}]$ where $\nu = \inf\{s > T_{M+1} : W_\alpha(s) = I_\alpha(T_{M+1}) \}$ or $S_\alpha(T_{M+1})$. Then on $(A_1 \cup A_2)^c$ we have

$$\int_{T_{M+1}}^{T_{M+1}'} \mathbb{1}_{[0,1]}(W_\alpha(s)) ds - 2\lambda \leq \int_{T_{M+1}}^{T_{M+1}'} \mathbb{1}_{[0,1]}(W_\alpha(s)) ds = \int_0^{T_{M+1}'} \mathbb{1}_{[0,1]}(\tilde{B}_\alpha(s)) ds.$$  

We note that on $(A_1 \cup A_2)^c$ the process $(\tilde{B}_\alpha(s))_{s \geq 0}$ has the same distribution as $(B_\alpha(s))_{s \geq 0}$ and that, by the second Ray–Knight theorem for the standard Brownian motion and scaling,

$$\frac{1}{2} \int_0^{T_{M+1}'} \mathbb{1}_{[0,1]}(B_\alpha(s)) ds \overset{\text{law}}{=} \int_0^1 Z^{(\alpha,0)}(x) dx, \text{ where } Z^{(\alpha,0)}(0) = 1.$$  

Hence,

$$E^{W_\alpha}[G^{2}_{M+1} - E^{W_\alpha}[G^{2}_{M+1} \mathbb{1}_{A_1 \cup A_2}]] + E^{W_\alpha}[G^{4}_{M+1} \mathbb{1}_{A_1 \cup A_2}].$$

for a universal constant $C$. Finally, we choose $\lambda$ to be small enough so that $C\lambda \leq \varepsilon/2$ and then use Lemmas 3.3 and 3.5 to conclude that there exist $M_0, \delta_0$ such that the last term in the above formula is less than $\varepsilon/2$ whenever $M \geq M_0$ and $\delta \in (0, \delta_0]$. This completes the proof of (UB).

**Lemma 3.5.** Fix $\lambda > 0$ and define $A_1$ and $A_2$ by \(24\) and \(25\). For each $\eta > 0$ there exist $M_0, \delta_0 > 0$ depending on $\eta$ and $\lambda$ such that $P(A_1 \cup A_2) < \eta$ for all $M \geq M_0, \delta \in (0, \delta_0]$.  

**Proof.** We first give a bound for $P(A_1)$. If $[-\sqrt{M}, \sqrt{M}] \subset [I_\alpha(T_{M+1}), S_\alpha(T_{M+1})]$, then $W_\alpha(T_{M+1} + \cdot)$ has the same law as the scaled Brownian motion $B_\alpha(\cdot)$ started at some point $z \in [0, \delta]$ until exiting $[-\sqrt{M}, \sqrt{M}]$. Also note that on the event $A_1$, at the time when the process $W_\alpha(T_{M+1} + \cdot)$ exits $[I_\alpha(T_{M+1}), S_\alpha(T_{M+1})]$, there must be a point in $(0, \delta]$ such that a half of the local time at this point does not exceed $2$. Therefore,

$$P(A_1) \leq P(T_{M+1} < \tau_{\sqrt{M}} \sqrt{W_\alpha} \vee \sigma_{\sqrt{W_\alpha}}) + \sup_{z \in (0, \delta]} P_z \left( \inf_{x \in (0, \delta]} L_{\tau_{\sqrt{M}} \vee W_\alpha}^{B_\alpha \wedge z} (x) \leq 2 \right) \leq 2P \left( \frac{1}{\delta} \int_0^{\delta} L_{\tau_{\sqrt{M}}} W_\alpha(x) dx > M \right) + \sup_{z \in [-\delta, \delta]} 2P_z \left( \inf_{x \in [-\delta, \delta]} L_{\tau_{\sqrt{M}}}^{B_\alpha}(x) \leq 2 \right).$$

Using the Ray–Knight theorems for BMPE and Brownian motion, respectively, one can show that by choosing $\delta_0$ sufficiently small and $M_0$ sufficiently large we can ensure that $P(A_1) \leq \eta/3$ for all $M \geq M_0$ and $\delta \in (0, \delta_0]$. Our remaining task is to bound $P(A_2 \cap A_1^c)$. We shall start with the set $A_2 := \{T_{M+1}' - T_{M+1} \geq \lambda\} \cap A_1^c$. On $A_1^c$, we are simply estimating the probability that the process $B_\alpha$ which started at
some point in \([0, \delta]\) does not hit 0 by time \(\lambda\). Taking the worst case scenario \(B_\alpha(0) = \delta\) and using the reflection principle for \(B_\alpha\) we get that for all \(\delta \in (0, \delta_0]\)

\[
P(A'_2) \leq 1 - 2P(B_\alpha(\lambda) \geq \delta \mid B_\alpha(0) = 0) \leq \frac{2}{\sqrt{2\pi}} \int_0^\delta \frac{\sqrt{\delta}}{(2\pi)^{3/2}} e^{-x^2/2} dx \leq \frac{\delta_0}{\sqrt{\lambda}} < \eta/3,
\]

if we choose \(\delta_0\) sufficiently small.

We turn to the set \(A'_2 := \{|T_{\delta,M+1} - T_{\delta,M,1}| \geq \lambda\} \cap A'_1\). Letting

\[
\{\{|T_{\delta,M+1} - T_{\delta,M,1}| \geq \lambda\} = I \cup II \cup III := \{T_{\delta,M+1} < T_{\delta,M,1-\delta^{1/3}}\} \cup \{T_{\delta,M+1} > T_{\delta,M,1+\delta^{1/3}}\} \cup \{T_{\delta,M,1+\delta^{1/3}} - T_{\delta,M,1-\delta^{1/3}} \geq \lambda\},
\]

we will bound \(P(I \cap A'_1), P(II \cap A'_1),\) and \(P(III \cap A'_1)\) separately. For \(P(I \cap A'_1)\), taking into account that \(\frac{1}{2\delta} \int_{T_{\delta,M}}^{T_{\delta,M+1}} \mathbf{1}_{[0,\delta]}(W_\alpha(s)) ds = 1\) we get that

\[
P(I \cap A'_1) \leq P\left( I \cap A'_1 \cap \left\{ \frac{1}{2\delta} \int_{T_{\delta,M}}^{T_{\delta,M+1}} \mathbf{1}_{[0,\delta]}(W_\alpha(s)) ds \leq \frac{\delta^{1/3}}{2} \right\} \right) + P\left( I \cap A'_1 \cap \left\{ \frac{1}{2\delta} \int_{T_{\delta,M}}^{T_{\delta,M+1}} \mathbf{1}_{[0,\delta]}(W_\alpha(s)) ds > \frac{\delta^{1/3}}{2} \right\} \right)
\]

The last probability is bounded by the probability that a Brownian motion \(B_\alpha\) started at a point in \([0, \delta]\) doesn’t hit the origin by time \(\delta^{1/3}\), and thus repeating the argument giving the bound for \(P(A'_2)\) we get that the last probability is less than \(\frac{1}{2}\). For the next to last term, since the process \(W_\alpha\) is a Brownian motion on the time interval in the integral, it follows from the second Ray–Knight theorem for \(B_\alpha\) that this term is bounded above by

\[
P_{1-\delta^{1/3}} \left( \frac{1}{\delta} \int_0^\delta Z^{(\alpha,0)}(x) dx > 1 - \frac{\delta^{1/3}}{2} \right)
\]

\[
\leq P_{1-\delta^{1/3}} \left( \left| \frac{1}{\delta} \int_0^\delta Z^{(\alpha,0)}(x) dx - \left( 1 - \frac{\delta^{1/3}}{2} \right) \right| > \frac{\delta^{1/3}}{2} \right) \leq 3\delta^{1/3},
\]

where in the last line we used Chebyshev’s inequality and Lemma \(\text{C.1}\). Hence, \(P(I \cap A'_1) \leq 4\delta^{1/3}\).

The estimate of \(P(II \cap A'_1)\) is very similar, since

\[
P(II \cap A'_1) \leq P\left( A'_1 \cap \left\{ \frac{1}{2\delta} \int_{T_{\delta,M}}^{T_{\delta,M+1}} \mathbf{1}_{[0,\delta]}(W_\alpha(s)) ds \leq 1 \right\} \right)
\]

\[
\leq P_{1+\delta^{1/3}} \left( \frac{1}{\delta} \int_0^\delta Z^{(\alpha,0)}(x) dx \leq 1 \right),
\]

and then using an argument similar to the bound in \(\text{(26)}\) we get that this is at most \(\frac{2}{3}\delta^{1/3}(1 + \delta^{1/3})\). Finally, note that \(P(III \cap A'_1) \leq P\left( L^{B_\alpha}_\lambda(0) \leq 2\delta^{1/3} \right) \to 0\) as \(\delta \to 0\). We conclude that choosing \(\delta_0\) sufficiently small we can ensure that \(P(A'_2) < \eta/3\). \(\square\)
Proof of (LB). Given \( \varepsilon > 0 \), we pick and fix \( c > 0 \) so small and \( K_0 \) so large that

\[
E_{1-2c} \left[ \left( \int_0^1 Z^{(0,0)}(x) \, dx \right)^2 \wedge K_0 \right] \geq \frac{5}{3} - \frac{\varepsilon}{2}.
\]

To justify this, we first take \( c \) sufficiently small to ensure that \( E_{1-2c} \left[ \left( \int_0^1 Z^{(0,0)}(x) \, dx \right)^2 \wedge K_0 \right] \geq \frac{5}{3} - \frac{\varepsilon}{4} \), which can be done via scaling and monotone convergence, and then find \( K_0 \) by applying the monotone convergence theorem.

We then choose \( \eta > 0 \) small so that for this \( K_0 \) the implication (27) holds for the functional \( F(\omega) = \left( \int_0^1 \omega(s) \, ds \right)^2 \wedge K_0 \), \( \omega \in D([0,1]) \). Next, for \( c \) chosen as in (27) we take \( M \geq M_0 \), where \( M_0 \) is as in (UB), and such that (see Proposition 3.2) \( \text{dist}(P^{Z^{M,c}_N}, P^{Z^{(0,0)}_N}) < \eta \) for all \( N \geq N(M) \).

It remains to pick \( \delta \in (0, \delta_0) \), where \( \delta_0 \) is as in (UB). Let

\[
J_n = \left\{ nt_{\delta,M}(X^{(n)}) < T_{(M+1-c)\sqrt{n}}^\varepsilon \right\} \cap \left\{ T_{(M+1-c)\sqrt{n}}^{E_{\varepsilon}} < nt_{\delta,M+1}(X^{(n)}) \right\}.
\]

By Lemma 3.6 below we can choose \( \delta \) so small that for all \( n = N^2 \) large \( P^{X^{(n)}}(J_n) \geq 1 - \varepsilon/(2K_0) \). Finally, note that if \( n = N^2 \) then the definitions of the functional \( G_{\delta,M} \) and the event \( J_n \) imply that

\[
G_{\delta,M}(X^{(n)}) \geq 1_{J_n} \left( \frac{1}{2N^2} \sum_{k=T_{(M+1-c)N}} T_{(M+1-c)N}^{-1} 1_{[0,N]}(X_k) \right) \geq 1_{J_n} \int_0^1 Z^{M,c}_N(x) \, dx,
\]

Putting everything together we get that for our choice of \( c, K_0, \eta, M, \delta \), all \( K \geq K_0 \) and all sufficiently large \( n \) of the form \( n = N^2 \)

\[
E^{X^{(n)}} \left[ G_{\delta,M}^2 \wedge K \right] - \frac{5}{3} \geq E^{X^{(n)}} \left[ G_{\delta,M}^2 \wedge K_0 \right] - E_{1-2c} \left[ \left( \int_0^1 Z^{(0,0)}(x) \, dx \right)^2 \wedge K_0 \right] - \frac{\varepsilon}{2}
\]

\[
\geq E^{X^{(n)}} \left[ G_{\delta,M}^2 \wedge K_0 \right] - E^{X^{(n)}} \left[ \left( \int_0^1 Z^{M,c}_N(x) \, dx \right)^2 \wedge K_0 \right] - \frac{3\varepsilon}{2}
\]

\[
\geq -E^{X^{(n)}} \left[ 1_{J_n} \left( \int_0^1 Z^{M,c}_N(x) \, dx \right)^2 \wedge K_0 \right] - \frac{3\varepsilon}{2} \geq -2\varepsilon.
\]

This completes the proof of (LB). \( \square \)

**Lemma 3.6.** Given \( \varepsilon, c, M > 0 \) there exists \( \delta_0 > 0 \) such that for all \( \delta \in (0, \delta_0) \) and all \( n = N^2 \) large,

\[
P^{X^{(n)}}(nt_{\delta,M}(X^{(n)}) < T_{(M+1-c)\sqrt{n}}^{E_{\varepsilon}} \wedge nT_{\delta,M+1}(X^{(n)})) \geq 1 - \varepsilon.
\]

**Proof.** First of all, note that for all \( M', \delta > 0 \) and integers \( N \)

\[
\sum_{i=0}^{T_{NM'}^{-1}} 1_{[0,N\delta]}(X_i) = \sum_{x=0}^{[N\delta]} \left( T_{NM'}^E(x) + \mathcal{E}_{NM'}^{T_{NM'}}(x) \right)
\]

\[
= 2 \left( \sum_{x=0}^{[N\delta]-1} \mathcal{E}_{NM'}^{T_{NM'}}(x) \right) + \mathcal{E}_{NM'}^{T_{NM'}}(-1) + \mathcal{E}_{NM'}^{T_{NM'}}([N\delta]) - 1,
\]

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where the second equality is due to the fact that at time $T_{N}^{\varepsilon}$ the walk has just completed a jump from 0 to 1 and, thus, $D_{T_{N}^{\varepsilon}}(x) = \mathcal{E}_{N}^{\varepsilon}(x-1) - \mathbb{1}_{\{1\}}(x)$. This together with the bound
\[
\left| \frac{1}{\delta N^2} \sum_{x=0}^{[N\delta]-1} \mathcal{E}_{N}^{\varepsilon}(x) - N M' \right| \leq \sup_{0 \leq x < [N\delta]} \left| \mathcal{E}_{N}^{\varepsilon}(k) - M' \right|
\]
gives that for all large $N$
\[
\left\{ nT_{\delta,M}(X^{(n)}) \geq \mathcal{T}_{(M+c)\sqrt{n}}^{\varepsilon} \right\} \subset \left\{ \frac{1}{2\delta N^2} \sum_{i=0}^{[N\delta]-1} \mathbb{1}_{[0,\delta N]}(X_i) \leq M \right\} \equiv \left\{ \frac{1}{\delta N^2} \sum_{x=0}^{[N\delta]-1} \left| \mathcal{E}_{(M+c)N}^{\varepsilon}(x) - (N(M+c)) \right| \geq \frac{c}{2} \right\}
\]
\[
\subset \left\{ \sup_{0 \leq x < [N\delta]} \left| \mathcal{E}_{(M+c)N}^{\varepsilon}(x) - (N(M+c)) \right| \geq \frac{c}{2} \right\}.
\]
It follows from Proposition 3.2 that for $\delta_0$ chosen sufficiently small the probability of the last event is less than $\varepsilon/2$ for all $N$ large. The event $\left\{ t_{\delta,M+1}^{\varepsilon}(X^{(n)}) \right\}$ can be treated similarly, and we omit the details.

4. Results about generalized Pólya urn model

In this section we shall recall relevant facts about generalized Pólya’s urns and derive several results needed for proofs of Lemma 2.3, Lemma 2.4, and Proposition 3.2 deferred from the previous two sections, thus, completing the proofs of our main results.

The sequence of left/right steps from a fixed site can be thought of as being generated by a generalized Pólya urn process. The urn process is slightly different depending on whether the site is to the left of the origin, the right of the origin, or at the origin and thus, following [Tót96], we first describe the family of generalized Pólya urn processes, and then later we will specialize to the urn processes which correspond to generating steps of the random walk at a fixed site. Given sequences of positive numbers $\{b(i)\}_{i \geq 0}$ and $\{r(i)\}_{i \geq 0}$, the generalized Pólya urn process $\{(\mathcal{B}_n, \mathcal{R}_n)\}_{n \geq 0}$ is a Markov chain on $\mathbb{N}_0^2$ started at $(\mathcal{B}_0, \mathcal{R}_0) = (0, 0)$ with transition probabilities given by
\[
P((\mathcal{B}_{n+1}, \mathcal{R}_{n+1}) = (i+1, j) \mid (\mathcal{B}_n, \mathcal{R}_n) = (i, j)) = \frac{b(i)}{b(i) + r(j)}, \quad \text{and}
\]
\[
P((\mathcal{B}_{n+1}, \mathcal{R}_{n+1}) = (i, j+1) \mid (\mathcal{B}_n, \mathcal{R}_n) = (i, j)) = \frac{r(j)}{b(i) + r(j)}, \quad i, j \geq 0.
\]
If we consider this as being generated by drawing red/blue balls from an urn, then $\mathcal{B}_n$ and $\mathcal{R}_n$ will be the numbers of blue and red balls, respectively, drawn from the urn up to time $n$.

For $* \in \{-, +, 0\}$ we will let $\{(\mathcal{B}_n^*, \mathcal{R}_n^*)\}_{n \geq 0}$ be the generalized Pólya urn process corresponding to the sequences $\{b^*(i)\}_{i \geq 0}$ and $\{r^*(i)\}_{i \geq 0}$
\[
b^-(i) = w(2i), \quad b^+(i) = w(2i + 1), \quad \text{and} \quad b^0(i) = w(2i), \quad \text{for } i \geq 0.
\]
\[
r^-(i) = w(2i + 1), \quad r^+(i) = w(2i), \quad \text{and} \quad r^0(i) = w(2i), \quad \text{for } i \geq 0.
\]
With these choices of parameters, at any site $x$ to the left of the origin the sequence of left/right steps on successive visits to $x$ has the same distribution as the sequence of blue/red draws from the generalized Pólya urn process $\{(\mathcal{B}_n, \mathcal{R}_n)\}_{n \geq 0}$. Similarly, the process $\{(\mathcal{B}_n^+, \mathcal{R}_n^+)\}_{n \geq 0}$ corresponds to left/right steps at sites to the right of the origin and $\{(\mathcal{B}_n^0, \mathcal{R}_n^0)\}_{n \geq 0}$ corresponds to left/right steps at the origin.
When considering any of the urn models described above, we will let \( \tau_k^{3^*} \) (or \( \tau_k^{8^*} \)) denote the number of trials until a blue (or red) ball is selected for the \( k \)-th time. More explicitly, letting \( \tau_0^{3^*} = \tau_0^{8^*} = 0 \) we have for \( k \geq 1 \) that
\[
(29) \quad \tau_k^{3^*} = \inf\{n > \tau_{k-1}^{3^*} : \mathcal{B}_n = \mathcal{B}_{n-1} + 1\} \quad \text{and} \quad \tau_k^{8^*} = \inf\{n > \tau_{k-1}^{8^*} : \mathcal{R}_n = \mathcal{R}_{n-1} + 1\}.
\]
We will also let \( \mathcal{D}_n^* \) be signed difference in the number of red balls and blue balls drawn in the first \( n \) steps of the urn process. That is,
\[
(30) \quad \mathcal{D}_n^* = \mathcal{R}_n^* - \mathcal{B}_n^*.
\]

**Rubin’s construction.** It will be helpful at times to use an equivalent construction of a generalized Pólya urn process (due to Rubin, see [Dav90]) using exponential random variables. Suppose that \( B_1, B_2, \ldots, R_1, R_2, \ldots \) are independent random variables with \( B_i \sim \text{Exp}(b(i-1)) \) and \( R_i \sim \text{Exp}(r(i-1)) \), for \( i \geq 1 \). We then make a red mark on \((0, \infty)\) at \( \sum_{i=1}^k R_i \) for every \( k \geq 1 \) and similarly a blue mark on \((0, \infty)\) at \( \sum_{i=1}^k B_i \) for every \( k \geq 1 \). The urn process can then be constructed by reading off the sequence of red and blue marks in order: every red mark corresponds to drawing a red ball (\( \mathbb{R} \) increases by one) and every blue mark corresponds to drawing a blue ball (\( \mathcal{B} \) increases by one).

**Lemma 4.1.** Let \( w \) be as in (3) with either (1) \( \alpha = 0 \) or (2) \( w(n) = (n+1)^{-\alpha} \) for some \( \alpha > 0 \). There exist constants \( C, c > 0 \) such that for any \( * \in \{-, +, 0\} \)
\[
P(\mathcal{D}_n^* \geq m) \leq Ce^{-cmn^2} \quad \forall n, m \in \mathbb{N}.
\]

**Proof.** Without loss of generality we take \( * = + \) and drop it from the notation. We will only give an upper bound for the right tail probabilities \( P(\mathcal{D}_n^* \geq m) \) since the left tail probabilities \( P(\mathcal{D}_n^* \leq -m) \) can be handled similarly. It follows from Rubin’s construction that
\[
(31) \quad P(\mathcal{D}_n^* \geq m) = P\left( \sum_{i=1}^{n+m} R_i < \sum_{i=1}^{n} B_i \right) = P\left( \sum_{i=1}^{n} B_i - \sum_{i=1}^{n+m} R_i > 0 \right),
\]
where the exponential random variables in Rubin’s construction here have distribution \( R_i \sim \text{Exp}(w(2i - 2)) \) and \( B_i \sim \text{Exp}(w(2i - 1)) \) for all \( i \geq 1 \). We will control the probability above slightly differently in the cases \( \alpha = 0 \) and \( \alpha > 0 \), respectively.

**Case I: \( \alpha = 0 \).** Recalling the definitions of \( U_1 \) and \( V_1 \) in (3), we have that \( E\left(\sum_{i=1}^n B_i - E[\sum_{i=1}^n B_i]\right) = V_1(n) - U_1(n+m) \). Therefore, we can write
\[
(32) \quad P(\mathcal{D}_n^* \geq m) = P\left( \sum_{i=1}^{n} (B_i - E[B_i]) - \sum_{i=1}^{n+m} (R_i - E[R_i]) > U_1(n+m) - V_1(n) \right).
\]
The assumption that \( \alpha = 0 \) implies that the \( w(i) \) are uniformly bounded above, and thus \( U_1(n+m) - U_1(n) \geq \delta m \) for some \( \delta \). Since (5) implies that the difference \( U_1(n) - V_1(n) \) is also bounded for all \( n \), we can conclude that there exist \( m_0 \in \mathbb{N} \) such that \( U_1(n+m) - V_1(n) \geq \delta m + U_1(n) - V_1(n) \geq \delta m/2 \), for all \( m \geq m_0 \). Applying this to (32) we have
\[
P(\mathcal{D}_n^* \geq m) \leq P\left( \sum_{i=1}^{n} (B_i - E[B_i]) + \sum_{i=1}^{n+m} (E[R_i] - R_i) > \frac{\delta m}{2} \right), \quad \forall n \geq 1, m \geq m_0.
\]
Since \( B_i \) and \( R_i \) are exponential random variables whose parameters are uniformly bounded away from 0 and \( \infty \), it follows that there exist constants \( g, t_0 > 0 \) so that we can bound the moment generating functions of the centered random variables by \( \sup_i \left( E[e^{t(B_i - E[B_i])}] \vee E[e^{t(E[R_i] - R_i)]} \right) \leq e^{gt^2/2} \) for all \( |t| \leq t_0 \). Therefore, by [Pet75, Theorem III.15], there exists \( c > 0 \) such that for all
$n \geq 1$ and $m \geq m_0$, $P(\mathcal{D}_{\tau_2^B} \geq m) \leq \exp\left(\frac{-cm^2}{n^2m}\right)$. By choosing a constant $C > 0$ sufficiently large we have that $P(\mathcal{D}_{\tau_2^B} \geq m) \leq C \exp\left(\frac{-cm^2}{n^2m}\right)$ for all $n, m \geq 1$.

**Case II:** $w(n) = (n + 1)^{-\alpha}$ **for some** $\alpha > 0$. In this case, note that the exponential random variables in (31) have means $E[R_i] = (2i - 1)^{\alpha}$ and $E[B_i] = (2i)^{\alpha}$, $i \geq 1$. Therefore, if we let $(R'_i)_{j \geq 1}$ be a sequence of independent exponential random variables which is also independent from $(B_i)_{i \geq 1}$ and such that $R'_i = R_i$ for $i \leq n$ and $R'_i \sim \text{Exp}(1/(2n + 1)^{\alpha})$ for $i > n$, we have from (31) that

$$P(\mathcal{D}_{\tau_2^B} \geq m) \leq P\left(\sum_{i=1}^{n} B_i - \sum_{i=1}^{n+m} R'_i > 0\right)$$

$$= P\left(\sum_{i=1}^{n} (B_i - E[B_i]) - \sum_{i=1}^{n+m} (R'_i - E[R'_i]) > \sum_{i=1}^{n} ((2i - 1)^{\alpha} - (2i)^{\alpha}) + m(2n + 1)^{\alpha}\right).$$

Since $\sum_{i=1}^{n} ((2i - 1)^{\alpha} - (2i)^{\alpha}) \sim -2^{\alpha-1}n^{\alpha}$ as $n \to \infty$, it follows that there exists $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$ and $n \geq 1$ we have

$$P(\mathcal{D}_{\tau_2^B} \geq m) \leq P\left(\sum_{i=1}^{n} (B_i - E[B_i]) - \sum_{i=1}^{n+m} (R'_i - E[R'_i]) > 2^{\alpha-1}mn^{\alpha}\right)$$

$$= P\left(\sum_{i=1}^{n} \frac{B_i - E[B_i]}{n^{\alpha}} + \sum_{i=1}^{n+m} \frac{E[R'_i] - R'_i}{n^{\alpha}} > 2^{\alpha-1}m\right).$$

Since we can again obtain uniform bounds on the moment generating functions of the random variables in the sum above of the form

$$\sup_{i} \left( E\left[e^t \frac{B_i - E[B_i]}{n^{\alpha}}\right] \vee E\left[e^t \frac{E[R'_i] - R'_i}{n^{\alpha}}\right]\right) \leq e^{rt^2}, \quad \forall t \leq |t_0|,$$

for some $g, t_0 > 0$, we can finish the proof just as in the case $\alpha = 0$ above by using the large deviation bounds in [Pet75].

**Remark 4.2.** Since $\mathfrak{R}_n + \mathfrak{B}_n = n$, it follows that $\mathcal{D}_n = n - 2\mathfrak{B}_n$. Replacing $n$ with $\tau_2^B$ we get that $\mathcal{D}_{\tau_2^B} = \tau_2^B - 2n$. Thus, Lemma 4.1 gives concentration bounds on $\tau_2^B$ as well. Moreover, since $|D_{n+1} - D_n| = 1$, we also have that

$$|D_{\tau_2^B} - D_{2n}| \leq |\tau_2^B - 2n| = |D_{\tau_2^B}|,$$

and, thus,

$$|D_{2n}| \leq |D_{2n} - D_{\tau_2^B}| + |D_{\tau_2^B}| \leq |\tau_2^B - 2n| + |D_{\tau_2^B}| = 2|D_{\tau_2^B}|.$$

### 4.1. Asymptotically free case: accumulated drift at a single site.

**Proof of Lemma 2.3** Since the sum in $\delta_x$ depends only on the behavior of the walk on successive visits to the site $x$, we can analyze $\delta_x$ using one of the generalized Pólya urn processes. We shall only treat the case $x > 0$ as the proofs in the other cases are similar. Thus, we will only be using the urn process $(\mathfrak{R}_n^+, \mathfrak{R}_n^+)$. To simplify the notation, we will omit the superscript $+$ throughout the proof.

We re-write $\delta_x$ as follows:

$$\bar{\delta}_x = \sum_{i=0}^{\infty} \frac{|w(r^+_i) - w(r^-_i)|}{w(r^+_i) + w(r^-_i)} \mathbb{I}\{X_i = x\} = \sum_{n=0}^{\infty} \frac{|w(2\mathfrak{R}_n) - w(2\mathfrak{R}_n + 1)|}{w(2\mathfrak{R}_n) + w(2\mathfrak{R}_n + 1)}.$$
Since \( \alpha = 0, \left( \frac{1}{w(2\mathfrak{R}_n)} + \frac{1}{w(2\mathfrak{B}_{n+1})} \right)^{-1} \leq \frac{1}{2} \sup_i w(i) < \infty \). Hence, it is enough to prove that
\[
E \left[ \left( \sum_{n=0}^{\infty} \left| \frac{1}{w(2\mathfrak{R}_n)} - \frac{1}{w(2\mathfrak{B}_n + 1)} \right| \right)^{M} \right] < \infty, \quad \forall M > 0.
\]

To do this, we will show that the sum inside the expectation has tails that decay faster than any polynomial. Letting \( C_0 := \sup_i (1/w(i)) \in [1, \infty) \) we get
\[
P \left( \sum_{n=0}^{\infty} \left| \frac{1}{w(2\mathfrak{R}_n)} - \frac{1}{w(2\mathfrak{B}_n + 1)} \right| \geq 4C_0m \right)
\leq P \left( \tau^\mathfrak{B}_m \geq 3m \right) + P \left( \sum_{n \geq \tau^\mathfrak{B}_m} \left| \frac{1}{w(2\mathfrak{R}_n)} - \frac{1}{w(2\mathfrak{B}_n + 1)} \right| \geq C_0m \right).
\]

For the first term in (36), it follows from Remark 4.2 and Lemma 4.1 that \( P \left( \tau^\mathfrak{B}_m \geq 3m \right) = P \left( \mathfrak{D}^\mathfrak{B}_m \geq m \right) \leq Ce^{-cm} \). Thus, it remains to show that the second term in (36) decreases faster than any polynomial in \( m \). To this end, we first rewrite
\[
\sum_{n \geq \tau^\mathfrak{B}_m} \left| \frac{1}{w(2\mathfrak{R}_n)} - \frac{1}{w(2\mathfrak{B}_n + 1)} \right| = \sum_{n=\tau^\mathfrak{B}_m}^{\infty} \sum_{i=\tau^\mathfrak{B}_n}^{\infty} \left| \frac{1}{w(2\mathfrak{R}_n)} - \frac{1}{w(2\mathfrak{B}_n + 1)} \right| = \sum_{n=\tau^\mathfrak{B}_m}^{\infty} \sum_{i=\tau^\mathfrak{B}_n}^{\infty} \frac{1}{w(2n-2i) - w(2i+1)}.
\]

where the last equality follows from the fact that \( \mathfrak{R}_n = n - \mathfrak{B}_n \) for all \( n \) and \( \mathfrak{B}_n = i \) for every \( n \in [\tau^\mathfrak{B}_i, \tau^\mathfrak{B}_{i+1}) \). Next, we fix a parameter \( 0 < \varepsilon' < \min\{p - \frac{1}{2}, \varepsilon\} \). If \( |\mathfrak{D}^\mathfrak{B}_i| = |\mathfrak{B}_i - 2i| \leq (\log i)^{\frac{1}{2}} \) and \( \tau^\mathfrak{B}_{i+1} - \tau^\mathfrak{B}_i \leq i\varepsilon' \) for all \( i \geq m \) then the last sum above does not exceed
\[
\sum_{i=m}^{\infty} i^{\varepsilon'} \max_{|n-2i| \leq \frac{1}{2}(\log i)^{\frac{1}{2}}} \left| \frac{1}{w(2n-2i)} - \frac{1}{w(2i+1)} \right| \leq \sum_{i=m}^{\infty} i^{\varepsilon'} \max_{|n-2i| \leq \frac{1}{2}(\log i)^{\frac{1}{2}}} \left| \frac{1}{w(2n-2i)} - \frac{1}{w(2i+1)} \right| \leq C \sum_{i=m}^{\infty} \frac{i^{\varepsilon'}}{i^{\varepsilon'}}.
\]

Note that by our choice of \( \varepsilon' \), the sum in the last line is finite and thus can be made arbitrarily small by taking \( m \) sufficiently large. Thus, we conclude that for all sufficiently large \( m \)
\[
P \left( \sum_{n \geq \tau^\mathfrak{B}_m} \left| \frac{1}{w(2\mathfrak{R}_n)} - \frac{1}{w(2\mathfrak{B}_n + 1)} \right| \geq C_0m \right) \leq P \left( |\mathfrak{D}^\mathfrak{B}_i| > (\log i)^{\frac{1}{2}} \right. \text{ or } \tau^\mathfrak{B}_{i+1} - \tau^\mathfrak{B}_i > i\varepsilon', \text{ for some } i \geq m \right) \leq \sum_{i=m}^{\infty} \left\{ P \left( |\mathfrak{D}^\mathfrak{B}_i| > (\log i)^{\frac{1}{2}} \right) + P \left( \tau^\mathfrak{B}_{i+1} - \tau^\mathfrak{B}_i > i\varepsilon' \right) \right\}.
\]

The first probability in the last line is bounded by \( Ce^{-(\log i)^{\frac{1}{2}}} \) by Lemma 4.1. For the last one, the assumption \( \alpha = 0 \) implies that the probability of the next draw in the urn process being a blue
ball is uniformly bounded below by some $q > 0$ so that $\tau_{i+1}^m - \tau_i^m$ is stochastically dominated by a Geo($q$) random variable. In particular, this implies that $P\left(\frac{\tau_{i+1}^m}{\tau_i^m} > i\varepsilon\right) \leq Ce^{-ci\varepsilon}$ for some $C, c > 0$. Thus, we have

$$P\left(\sum_{n \geq \tau_m^m} \frac{1}{w(2R_n)} - \frac{1}{w(2B_n + 1)} \geq C_0\right) \leq C \sum_{i=m}^{\infty} \left(e^{-c \log i^2} + e^{-ci}\right).$$

This completes the proof of the lemma. \hfill \Box

**Proof of Lemma 2.4.** Lemma 2.3 implies that $E[\delta_x]$ is finite for all $x \in \mathbb{Z}$. From this and the symmetry considerations in Remark 2.6 we conclude that $E[\delta_0] = 0$ and $E[\delta_x] = -E[\delta_{-x}]$ for $x \neq 0$. Therefore, we need only to prove that $E[\delta_x] = \gamma$ for all $x > 0$. Using the connection with the generalized Pólya urn processes as in (35)\footnote{Since $x > 0$, we only deal with the Pólya urn process $(\mathcal{B}_j, \mathcal{R}_j)$, and thus we will omit the superscripts $+$.} we see that it is enough to show that

$$E\left[\sum_{j=0}^{\infty} \frac{w(2R_j) - w(2B_j + 1)}{w(2R_j) + w(2B_j + 1)}\right] = \lim_{n \to \infty} E\left[\mathcal{D}_{i-\gamma} - \mathcal{D}_i \mid \mathcal{F}_{i-\gamma}^{\mathcal{B}, \mathcal{R}}\right] = \gamma.\tag{37}$$

To justify the first equality in (37), set $\mathcal{F}_i^{\mathcal{B}, \mathcal{R}} := \sigma((\mathcal{B}_j, \mathcal{R}_j), j \leq i)$ and note that

$$E\left[\mathcal{D}_{i+1} - \mathcal{D}_i \mid \mathcal{F}_i^{\mathcal{B}, \mathcal{R}}\right] = \frac{w(2R_i) - w(2B_i + 1)}{w(2R_i) + w(2B_i + 1)}.\tag{38}$$

Then, by Lemma 2.3 and the dominated convergence theorem we get

$$E\left[\sum_{i=0}^{\infty} \frac{w(2R_i) - w(2B_i + 1)}{w(2R_i) + w(2B_i + 1)}\right] = \lim_{n \to \infty} E\left[\sum_{i=0}^{\tau_n^{-1}} E\left[\mathcal{D}_{i+1} - \mathcal{D}_i \mid \mathcal{F}_i^{\mathcal{B}, \mathcal{R}}\right]\right]$$

$$= \lim_{n \to \infty} E\left[\sum_{i=0}^{\infty} E\left[\mathcal{D}_{i+1} - \mathcal{D}_i \mid \mathcal{F}_i^{\mathcal{B}, \mathcal{R}}\right]\right],\tag{39}$$

where in the last equality we used that $\{i < \tau_n^{-1}\} \in \mathcal{F}_i^{\mathcal{B}, \mathcal{R}}$ for any $i, n \geq 0$. For the expectations in the last line, note that for any fixed $n$,

$$E\left[\sum_{i=0}^{\infty} E\left[\mathcal{D}_{i+1} - \mathcal{D}_i \mathcal{1}_{\{i < \tau_n^{-1}\}} \mid \mathcal{F}_i^{\mathcal{B}, \mathcal{R}}\right]\right] = \sum_{i=0}^{\infty} E\left[\mathcal{D}_{i+1} - \mathcal{D}_i \mathcal{1}_{\{i < \tau_n^{-1}\}} \mid \mathcal{F}_i^{\mathcal{B}, \mathcal{R}}\right]\right]$$

$$= \sum_{i=0}^{\infty} E\left[\mathcal{D}_{i+1} - \mathcal{D}_i \mathcal{1}_{\{i < \tau_n^{-1}\}}\right]$$

$$= E\left[\sum_{i=0}^{\infty} (\mathcal{D}_{i+1} - \mathcal{D}_i) \mathcal{1}_{\{i < \tau_n^{-1}\}}\right] = E\left[\mathcal{D}_{\tau_n^{-1}}\right],\tag{40}$$

where the interchange of the expectation and sum in the first and third equality is justified by the dominated convergence theorem, since $|\mathcal{D}_{i+1} - \mathcal{D}_i| = 1$ and $E[\tau_n^{-1}] < \infty$ for all $n$ (by Remark 2.2). Combining (39) and (40) proves the first equality in (37).

For the second equality in (37), recall the definitions of $U_1(n)$ and $V_1(n)$ in (41), and note that it follows from [16, Lemma 1] that for all $n \geq 1,$

$$E\left[U_1(\mathcal{R}_n)\right] = E\left[U_1(\mathcal{D}_n + n)\right] = E\left[\frac{\mathcal{D}_n + n-1}{w(2i)}\right] = \sum_{i=0}^{n-1} \frac{1}{w(2i + 1)} = V_1(n).$$
Subtracting $U_1(n)$ from both sides we get

\begin{equation}
V_1(n) - U_1(n) = E \left[ \frac{\mathcal{D}_{\tau_{\max}} + n - 1}{w(2i)} - \sum_{i=0}^{n-1} \frac{1}{w(2i)} \right]
= E \left[ \mathcal{D}_{\tau_{\max}} + \text{sign}(\mathcal{D}_{\tau_{\max}}) \max\{n + \mathcal{D}_{\tau_{\max}}, n\} - 1 \right].
\end{equation}

To handle the sum inside the expectation on the right, note that since $\alpha = 0$ all the terms in the sum are uniformly bounded and that if $|\mathcal{D}_{\tau_{\max}}| < (\log n)\sqrt{n}$ then (for $n$ large enough) all of the terms inside the sum are at most $Cn^{-p}$ for some $C > 0$. Thus, we can conclude that

\[ E \left[ \sum_{i=\min\{n + \mathcal{D}_{\tau_{\max}}, n\}}^{\max\{n + \mathcal{D}_{\tau_{\max}}, n\} - 1} \frac{1}{w(2i)} - 1 \right] \leq O \left( \frac{\log n}{n^{p/2}} \right) + CE \left[ |\mathcal{D}_{\tau_{\max}}| \mathbb{1}\{|\mathcal{D}_{\tau_{\max}}|\geq(\log n)\sqrt{n} \} \right], \]

and Lemma 4.1 implies that the last term on the right vanishes as $n \to \infty$. Therefore, if $p > 1/2$ then $V_1(n) - U_1(n) = E|\mathcal{D}_{\tau_{\max}}| + o(1)$. Taking $n \to \infty$ and using the definition of $\gamma$ in (5) completes the proof of the second equality in (37).

4.2. **Polynomially self-repelling case.** Throughout this subsection we will be dealing only with the Pólya urn process $(\mathfrak{B}_j^+, \mathfrak{R}_j^+)$ and so the superscripts $+$ will be suppressed. Recall also that throughout this subsection $w(n) = (n + 1)^{-\alpha}$ where $\alpha > 0$.

Our main goal is to prove Proposition 3.2 using the generalized Pólya urns as a tool. Thus, it is important to first explain the connection between the $\mathcal{E}$-processes in Proposition 3.2 and the generalized Pólya urn processes. It is not hard to see that for any fixed $\ell \geq 0$, the process $(\mathcal{E}^\ell_t(j))_{j \geq 0}$ is a time inhomogeneous Markov chain started from $\mathcal{E}^\ell_t(0) = \ell + 1$ with transition probabilities related to the discrepancy urn process $\mathcal{D}$ as follows. Since at time $T_t$ the walk has just completed a step from 0 to 1, the last visit to site 1 before time $T_t$ resulted in a jump to the left and this was the $\ell$-th jump to the left from 1. Thus, $\mathcal{E}^\ell_t(1)$ is equal to the number of right steps from 1 before the $\ell$-th left jump, and since the sequence of left/right jumps at each site can be generated by a generalized Pólya urn, it follows that the increment $\mathcal{E}^\ell_t(1) - \mathcal{E}^\ell_t(0)$ has the same distribution as $\mathfrak{R}_{\tau^\ell_t} - (\ell + 1) = \mathfrak{R}_{\tau^\ell_t} - 1$. Similarly, the last visit to any site $x \geq 2$ before time $T_t$ was a step to the left and the number of such left steps from $x$ is $\mathcal{D}^\ell_t(x) = \mathcal{E}^\ell_t(x - 1)$. By similar reasoning as above, it follows that conditioned on $\mathcal{D}^\ell_t(j) = n$ the increment $\mathcal{E}^\ell_t(j + 1) - \mathcal{E}^\ell_t(j)$ has the same distribution as $\mathfrak{R}_{\tau^\ell_t}$. That is,

\begin{equation}
P \left( \mathcal{E}^\ell_t(j + 1) - \mathcal{E}^\ell_t(j) = k \mid \mathcal{E}^\ell_t(j) = n \right) = P \left( \mathfrak{R}_{\tau^\ell_t} = k \right), \quad j \geq 1.
\end{equation}

Proposition 3.2 concerns processes of the form $(\mathcal{E}^{(M+1-c)n}(j) - \mathcal{E}^{(M+1-c)n}(j))_{j \geq 0}$. By similar reasoning as above, we see that the conditional distributions of the increments of this process have the same distributions as $\mathfrak{R}_{\tau_t^{M+1-n}} - \mathfrak{R}_{\tau_t^{M+1-n}}$ with $m$ and $n$ determined by the conditioning. Thus, the majority of this subsection will be devoted to obtaining good estimates on the mean and variance of $\mathfrak{R}_{\tau_t^{M+1-n}} - \mathfrak{R}_{\tau_t^{M+1-n}}$ when $m$ and $n$ are both large.

4.2.1. **Variance estimates.** The estimates on $\text{Var}(\mathfrak{R}_{\tau_t^{M+1-n}} - \mathfrak{R}_{\tau_t^{M+1-n}})$ that we will need for the proof of Proposition 3.2 will be given in Proposition 4.8 but since we will take a somewhat winding path to prove this, it is helpful to give the reader an outline of where we are headed. First of all, instead of analyzing differences of the discrepancy process $\mathcal{D}$ at random stopping times, we will use a martingale approximation to obtain estimates on differences of the discrepancy process at
deterministic times. This is accomplished in Lemma 4.3 and Corollaries 4.4 and 4.5. The next task is then in justifying the change from the discrepancy process at the stopping times \( \tau_n \) to the deterministic time \( 2n \). In particular, in Lemma 4.7 we will obtain estimates on \( E[(\mathcal{D}_{\tau_n} - \mathcal{D}_{2n})^2] \), and combining this with the previous results we obtain the asymptotics of \( \text{Var}(\mathcal{D}_{\tau_n} - \mathcal{D}_{\tau_n}) \) in Proposition 4.8.

Obtaining estimates on the variance of \( \mathcal{D}_m - \mathcal{D}_n \) directly is not optimal because the increments \( \mathcal{D}_{i+1} - \mathcal{D}_i \) are too strongly correlated. On the other hand, as can be seen from the proof of the next result, for \( \delta > 0 \) fixed and \( n \) large the process \( \left( \frac{\mathcal{D}_{\tau_n}}{\sqrt{n}} t^\alpha \right)_{t \geq \delta} \) is approximately a martingale, and this observation underlies the following lemma.

**Lemma 4.3.** For \( \delta \in (0, 2) \) fixed, there exists a constant \( C_\delta \) such that for \( n \) sufficiently large and \( \delta n \leq k \leq m \leq 2n \) we have

\[
\left| \text{Var} \left( \mathcal{D}_m \left( \frac{m}{n} \right)^\alpha - \mathcal{D}_k \left( \frac{k}{n} \right)^\alpha \right) - n \int_{k/n}^{m/n} u^{2\alpha} \, du \right| \leq C_\delta \sqrt{n} \log^2 n.
\]

**Proof.** Throughout the proof we will let \( C_\delta \) denote a constant which depends on \( \delta \) but not on \( n \) and which can change from line to line. Let \( \Delta_i = (\mathcal{D}_{i+1} - \mathcal{D}_i) - E \left[ \mathcal{D}_{i+1} - \mathcal{D}_i \mid \mathcal{F}_{i,n} \right] \). Note that by (38) and our choice of \( w \),

\[
E \left[ \mathcal{D}_{i+1} - \mathcal{D}_i \mid \mathcal{F}_{i,n} \right] = \frac{(2\mathcal{B}_i + 2)^\alpha - (2\mathcal{B}_i + 1)^\alpha}{(2\mathcal{B}_i + 2)^\alpha + (2\mathcal{B}_i + 1)^\alpha} = \frac{(i - \mathcal{D}_i + 1)^\alpha - (i + \mathcal{D}_i + 1)^\alpha}{(i - \mathcal{D}_i + 1)^\alpha + (i + \mathcal{D}_i + 1)^\alpha} = \frac{1 - \frac{\mathcal{D}_i}{i} - \frac{1}{i}}{1 + \frac{\mathcal{D}_i}{i} + \frac{1}{i}}.
\]

(43)

Therefore, we have \( \mathcal{D}_{i+1} = \mathcal{D}_i + \Delta_i - \frac{\mathcal{D}_i}{i} + \varepsilon_i^n \) and

\[
\mathcal{D}_{i+1} \left( \frac{i+1}{n} \right)^\alpha - \mathcal{D}_i \left( \frac{i}{n} \right)^\alpha = \left( \mathcal{D}_i + \Delta_i - \frac{\mathcal{D}_i}{i} + \varepsilon_i^n \right) \left( \frac{i+1}{n} \right)^\alpha - \mathcal{D}_i \left( \frac{i}{n} \right)^\alpha = \Delta_i \left( \frac{i+1}{n} \right)^\alpha + \mathcal{D}_i \left( \frac{i+1}{n} \right)^\alpha \left( 1 - \frac{\alpha}{i} \right) - \left( \frac{i}{n} \right)^\alpha \varepsilon_i^n \left( \frac{i+1}{n} \right)^\alpha.
\]

Summing over \( i \in [k, m] \) gives

\[
\mathcal{D}_m \left( \frac{m}{n} \right)^\alpha - \mathcal{D}_k \left( \frac{k}{n} \right)^\alpha = \sum_{i=k}^{m-1} \left( \Delta_i \left( \frac{i+1}{n} \right)^\alpha + \mathcal{D}_i \left( \frac{i+1}{n} \right)^\alpha \left( 1 - \frac{\alpha}{i} \right) - \left( \frac{i}{n} \right)^\alpha \varepsilon_i^n \left( \frac{i+1}{n} \right)^\alpha \right).
\]

To get asymptotics on the variance of this sum, we will first get good bounds on the variances of the sums of the three terms inside the braces separately. That is, we will show that

\[
\max_{\delta n \leq k \leq m \leq 2n} \left| \text{Var} \left( \sum_{i=k}^{m-1} \Delta_i \left( \frac{i+1}{n} \right)^\alpha \right) - n \int_{k/n}^{m/n} u^{2\alpha} \, du \right| \leq C_\delta n^{1/5},
\]

(45)

\[
\max_{\delta n \leq k \leq m \leq 2n} \text{Var} \left( \sum_{i=k}^{m-1} \mathcal{D}_i \left( \frac{i+1}{n} \right)^\alpha \left( 1 - \frac{\alpha}{i} \right) - \left( \frac{i}{n} \right)^\alpha \right) \leq C_\delta (\log^2 n)/n,
\]

(46)

\[
\max_{\delta n \leq k \leq m \leq 2n} \text{Var} \left( \sum_{i=k}^{m-1} \varepsilon_i^n \left( \frac{i+1}{n} \right)^\alpha \right) \leq C_\delta \log^4 n.
\]

(47)
To see that these bounds are enough to finish the proof, first note that by expanding the variance of (44) into variance and covariance terms, bounding the covariance terms by products of square roots of the variances, and using the variance bounds in (46) and (47), one obtains for \( n \) sufficiently large and \( \delta n \leq k \leq m \leq 2n \) that

$$\left| \operatorname{Var} \left( D_m \left( \frac{m}{n} \right)^\alpha - D_k \left( \frac{k}{n} \right)^\alpha \right) - \operatorname{Var} \left( \sum_{i=k}^{m-1} \Delta_i \left( \frac{i+1}{n} \right)^\alpha \right) \right| \leq C_\delta \log^4 n + C_\delta \log^2 n \sqrt{\operatorname{Var} \left( \sum_{i=k}^{m-1} \Delta_i \left( \frac{i+1}{n} \right)^\alpha \right)}.$$  

The proof is then finished by using the variance bounds in (45).

It remains now to prove the variance bounds in (45)–(47).

**Proof of (45):** Since the terms \( \Delta_i \left( \frac{i+1}{n} \right)^\alpha \) form the increments of a martingale we have that

$$\operatorname{Var} \left( \sum_{i=k}^{m-1} \Delta_i \left( \frac{i+1}{n} \right)^\alpha \right) = \sum_{i=k}^{m-1} \left( \frac{i+1}{n} \right)^{2\alpha} E \left[ \Delta_i^2 \right] = \sum_{i=k}^{m-1} \left( \frac{i+1}{n} \right)^{2\alpha} \left( 1 - E \left[ B_{i+1} - D_i \mid \mathcal{F}_i^{\mathcal{B}, \mathcal{R}_i} \right]^2 \right).$$

Note that an integral approximation shows that \( \sum_{i=k}^{m-1} \left( \frac{i+1}{n} \right)^{2\alpha} = n \int_{k/n}^{m/n} u^{2\alpha} \, du + O(1) \), where the \( O(1) \) error term is uniform over all \( \delta n \leq k \leq m \leq 2n \). To finish the proof of (45) we need to get bounds on \( E \left[ B_{i+1} - D_i \mid \mathcal{F}_i^{\mathcal{B}, \mathcal{R}_i} \right]^2 \) that are uniform over \( i \in [\delta n, 2n] \). Recalling (43), we conclude that there exists a constant \( C_\delta \) such that if \( i \in [\delta n, 2n] \) and \( |D_i| \leq n^{3/5} \) then

$$E \left[ B_{i+1} - D_i \mid \mathcal{F}_i^{\mathcal{B}, \mathcal{R}_i} \right] \leq C_\delta n^{-2/5}.$$  

Then, since \( |D_{i+1} - D_i| = 1 \), we get

$$\max_{i \in [\delta n, 2n]} E \left[ B_{i+1} - D_i \mid \mathcal{F}_i^{\mathcal{B}, \mathcal{R}_i} \right]^2 \leq \max_{i \in [\delta n, 2n]} P \left( |D_i| > n^{3/5} \right) + C_\delta^2 n^{-4/5}.$$  

An application of (34) and Lemma 4.1 completes the proof of (45).

**Proof of (46):** Since it is easy to see that \( \max_{[\delta n] \leq i < 2n} \left| \left( \frac{i+1}{n} \right)^\alpha - \left( \frac{i}{n} \right)^\alpha \right| \leq C_\delta \), we have

$$\max_{\delta n \leq k < m \leq 2n} \operatorname{Var} \left( \sum_{i=k}^{m-1} \Delta_i \left( \frac{i+1}{n} \right)^\alpha \left( 1 - \frac{\alpha}{i} \right) - \left( \frac{i}{n} \right)^\alpha \right) \leq \max_{\delta n \leq k < m \leq 2n} E \left[ \sum_{i=k}^{m-1} \left( \frac{i+1}{n} \right)^\alpha \left( 1 - \frac{\alpha}{i} \right) - \left( \frac{i}{n} \right)^\alpha \right]^2 \leq C_\delta n^{-2} E \left[ \max_{i \in [\delta n, 2n]} \mathcal{D}_i^2 \right],$$

and (46) then follows from this, (34), and the tail bounds in Lemma 4.1.

**Proof of (47):** Recalling that \( |\varepsilon_i^n| \leq C(\mathcal{D}_i^2 + i)/2^2 \) for all \( i \) we have that

$$\max_{\delta n \leq k < m \leq 2n} \operatorname{Var} \left( \sum_{i=k}^{m-1} \varepsilon_i^n \left( \frac{i+1}{n} \right)^\alpha \right) \leq \max_{\delta n \leq k < m \leq 2n} E \left[ \sum_{i=k}^{m-1} |\varepsilon_i^n| \left( \frac{i+1}{n} \right)^\alpha \right]^2 \leq C_\delta \left( n^{-2} E \left[ \max_{i \in [\delta n, 2n]} \mathcal{D}_i^4 \right] + 1 \right).$$
and (37) then follows from this, (34), and the tail bounds in Lemma 1.1.

Naively taking $k = 0$ and $m = n$ in Lemma 4.3 suggests the approximation $\text{Var}(\mathcal{D}_n) \approx n \int_0^1 u^{2\alpha} \, du = \frac{n}{2\alpha+1}$. While this does not follow directly from Lemma 4.3 due to the requirement that $k \geq \delta n$, the following corollary shows that this approximation is, in fact, valid.

**Corollary 4.4.** \( \lim_{n \to \infty} \frac{\text{Var}(\mathcal{D}_n)}{n} = \frac{1}{2\alpha+1}. \)

**Proof.** First of all, for any fixed $\delta > 0$ note that

\[
\text{Var}(\mathcal{D}_n) = \text{Var}(\mathcal{D}_n - \mathcal{D}_{[\delta n]}) ([\delta n]/n)^\alpha + ([\delta n]/n)^{2\alpha} \text{Var}(\mathcal{D}_{[\delta n]}) + 2 \text{Cov}(\mathcal{D}_n - \mathcal{D}_{[\delta n]}) ([\delta n]/n)^\alpha, \mathcal{D}_{[\delta n]} ([\delta n]/n)^\alpha).
\]

Note that it follows from (34) and Lemma 4.1 that there is a constant $C > 0$ (not depending on $\delta$ or $n$) such that $\text{Var}(\mathcal{D}_{[\delta n]}) \leq C\delta n$. By Lemma 4.3 there is a constant $C_\delta > 0$ such that for $n$ sufficiently large

\[
\left| \frac{\text{Var}(\mathcal{D}_n)}{n} - \int_0^1 u^{2\alpha} \, du \right| \leq \int_0^1 u^{2\alpha} \, du + \int_0^\delta u^{2\alpha} \, du + C\delta^{2\alpha+1} + 2n^{-1} \sqrt{\text{Var}(\mathcal{D}_n - \mathcal{D}_{[\delta n]}) ([\delta n]/n)^\alpha} \sqrt{\text{Var}(\mathcal{D}_{[\delta n]}) ([\delta n]/n)^\alpha} \leq C\delta n^{-1/2} \log^2 n + C\delta^{2\alpha+1} + 2\sqrt{n} C\delta^{1/2} \log^2 n \sqrt{2\delta^{2\alpha+1}}.
\]

Taking $n \to \infty$ and then $\delta \to 0$ then completes the proof.

Finally, combining Lemma 4.3 and Corollary 4.4 yields the following bounds on $\text{Var}(\mathcal{D}_m - \mathcal{D}_n)$.

**Corollary 4.5.** There exists a constant $C > 0$ such that for all $n$ sufficiently large and $m \in [n, 2n]$, (48)

\[
|\text{Var}(\mathcal{D}_m - \mathcal{D}_n) - (m-n)| \leq C\sqrt{n} \log^2 n + C(m-n)^2/n.
\]

**Proof.** Using the decomposition

\[
\mathcal{D}_m - \mathcal{D}_n = \left(\frac{m}{n}\right)^{-\alpha} (\mathcal{D}_m (\frac{m}{n})^\alpha - \mathcal{D}_n) + \left(\frac{m}{n}\right)^{-\alpha} (1 - \frac{m}{n}) \mathcal{D}_n,
\]

we have that

\[
\text{Var}(\mathcal{D}_m - \mathcal{D}_n) = \left(\frac{m}{n}\right)^{-2\alpha} \text{Var}(\mathcal{D}_m (\frac{m}{n})^\alpha - \mathcal{D}_n) + \left(\frac{m}{n}\right)^{-\alpha} \left(\frac{m}{n}\right)^{-1} \text{Var}(\mathcal{D}_n) + 2 \left(\frac{m}{n}\right)^{-\alpha} \left(\frac{m}{n}\right)^{-\alpha} \text{Cov}(\mathcal{D}_m (\frac{m}{n})^\alpha - \mathcal{D}_n, \mathcal{D}_n).
\]

By Lemma 4.3 and Corollary 4.5 the two variance terms on the right can be approximated by $n \int_0^{m/n} u^{2\alpha} \, du = \frac{n}{2\alpha+1} ((m/n)^{2\alpha+1} - 1)$ and $\frac{n}{2\alpha+1}$, respectively. With these approximations, the sum of the first two terms is approximated by

\[
\left(\frac{m}{n}\right)^{-2\alpha} \frac{n}{2\alpha+1} \left(\frac{m}{n}\right)^{-\alpha} \left(1 - \frac{m}{n}\right)^2 \frac{n}{2\alpha+1} = \frac{n}{2\alpha+1} \left(1 + \frac{m}{n} - 2 \left(\frac{m}{n}\right)^{-\alpha}\right).
\]

Therefore, we can conclude for $n$ large enough and $m \in [n, 2n]$ that

\[
\left| \text{Var}(\mathcal{D}_m - \mathcal{D}_n) - \frac{n}{2\alpha+1} \left(1 + \frac{m}{n} - 2 \left(\frac{m}{n}\right)^{-\alpha}\right) \right| \leq \left(\frac{m}{n}\right)^{-2\alpha} \text{Var}(\mathcal{D}_m (\frac{m}{n})^\alpha - \mathcal{D}_n) - n \int_0^{m/n} u^{2\alpha} \, du + \left(\frac{m}{n}\right)^{-\alpha} \left(\frac{m}{n}\right)^{-1} \text{Var}(\mathcal{D}_n) - \frac{n}{2\alpha+1} \right| + 2 \left(\frac{m}{n}\right)^{-\alpha} \left(\frac{m}{n}\right)^{-\alpha} \text{Cov}(\mathcal{D}_m (\frac{m}{n})^\alpha - \mathcal{D}_n, \mathcal{D}_n) \leq C\sqrt{n} \log^2 n + C(n \left(\frac{m}{n}\right)^{-\alpha} - 1)^2 + C \text{Cov}(\mathcal{D}_m (\frac{m}{n})^\alpha - \mathcal{D}_n, \mathcal{D}_n).
For the covariance term in the last line above, recall the decomposition in (44) and note that \( \text{Cov}(\Delta_i, D_n) = 0 \) for all \( i \geq n \). Thus,

\[
\left| \text{Cov} \left( D_m \left( \frac{m}{n} \right) - D_n, D_n \right) \right| \\
\leq \sqrt{\text{Var} \left( \sum_{i=n}^{m-1} \Delta_i \left( \left( \frac{i+1}{n} \right)^{\alpha} - \left( \frac{i}{n} \right)^{\alpha} \right) \right) \text{Var}(D_n)} \\
+ \sqrt{\text{Var} \left( \sum_{i=n}^{m-1} \varepsilon^n_i \left( \frac{i+1}{n} \right)^{\alpha} \right) \text{Var}(D_n)} \\n\leq C \sqrt{n \log n},
\]

where in the last inequality we used the variance bounds from (46) and (47).

Thus far we have shown that for all \( m \in [n, 2n] \) and \( n \) sufficiently large,

\[
\left| \text{Var}(D_m - D_n) - \frac{n}{2^{2\alpha+1}} \left( 1 + \frac{m}{n} - 2 \left( \frac{m}{n} \right)^{-\alpha} \right) \right| \leq C \sqrt{n \log n} + C n \left( \left( \frac{m}{n} \right)^{-\alpha} - 1 \right)^2.
\]

Finally, we note that there is a \( C > 0 \) such that uniformly over all \( m \geq n \geq 1 \)

\[
\left| \frac{1}{2^{2\alpha+1}} \left( 1 + \frac{m}{n} - 2 \left( \frac{m}{n} \right)^{-\alpha} \right) - \frac{m-n}{m} \right| \leq C \left( \frac{m-n}{m} \right)^2 \text{ and } \left( \left( \frac{m}{n} \right)^{-\alpha} - 1 \right)^2 \leq C \left( \frac{m-n}{m} \right)^2.
\]

From this the inequality in (48) follows easily. \( \square \)

The next step in obtaining bounds on \( \text{Var}(\mathcal{D}_{\tau_{2n}} - \mathcal{D}_{\tau_{n}}) \) is to obtain control on the difference \( \mathcal{D}_{\tau_{n}} - \mathcal{D}_{2n} \). For this, the following lemma will be useful.

**Lemma 4.6.** There exist \( M_0, c_1 \in (0, \infty) \) such that for all \( M \geq M_0, n \in \mathbb{N}, \) and \( y \in [0, \sqrt{n}] \)

\[
P \left( \sup_{\tau_{2n} \leq i \leq \tau_{M+1} n} |D_i - D_{\tau_{2n}}| \geq y \sqrt{n} \right) \leq \frac{1}{c_1} e^{-c_1 y^2}.
\]

The proof of this Lemma is given in Appendix C.

**Lemma 4.7.** There exists a constant \( C > 0 \) such that

\[
E \left[ \left( \mathcal{D}_{\tau_{2n}} - \mathcal{D}_{2n} \right)^2 \right] \leq C \sqrt{n}.
\]

**Proof.** We will show that

\[
E \left[ \left( \mathcal{D}_{\tau_{2n}} - \mathcal{D}_{2n} \right)^2 \right] = \int_0^\infty 2y P \left( |\mathcal{D}_{\tau_{2n}} - \mathcal{D}_{2n}| \geq yn^{1/4} \right) dy
\]

is bounded uniformly in \( n \). By (43) and Lemma 4.1 we get

\[
\int_{n^{3/8}}^\infty 2y P \left( |\mathcal{D}_{\tau_{2n}} - \mathcal{D}_{2n}| \geq yn^{1/4} \right) dy \leq \int_{n^{3/8}}^\infty 2y P \left( |\mathcal{D}_{\tau_{2n}}| \geq yn^{1/4} \right) dy \leq C \sqrt{n} e^{-cn^{1/4}}.
\]
Thus, it remains only to get good bounds on $P\left(|D_{\tau_n^B} - D_{2n}| \geq yn^{1/4}\right)$ for $y \leq n^{3/8}$. To this end, note that

$$P\left(|D_{\tau_n^B} - D_{2n}| \geq yn^{1/4}\right) \leq P\left(|\tau_n^B - 2n| \geq y\sqrt{n}\right) + P\left(|D_i - D_{\tau_n^B}| \geq yn^{1/4}, \text{ for some } |i - \tau_n^B| \leq y\sqrt{n}\right) \leq P\left(|D_{\tau_n^B}| \geq y\sqrt{n}\right) + P\left(\sup_{\tau_n^B - y\sqrt{n} \leq i \leq \tau_n^B + y\sqrt{n}} |D_i - D_{\tau_n^B}| \geq \frac{1}{2} yn^{1/4}\right) \leq Ce^{-cy^2} + Ce^{-cy},$$

where the second inequality follows from the fact that $D_{\tau_n^B} = \tau_n^B - 2n$ and the last inequality follows from Lemmas 4.1 and 4.6 for $n$ sufficiently large and $y \leq n^{3/8}$.

Finally, we obtain asymptotics for $\text{Var}(D_{\tau_n^B})$ and a bound on $\text{Var}(D_{\tau_n^B} - D_{\tau_n^B}) - 2(m - n)$.

**Proposition 4.8.** \(\lim_{n \to \infty} \frac{\text{Var}(D_{\tau_n^B})}{2n} = \frac{1}{2\alpha + 1}\). Moreover, there is a constant $C > 0$ such that for $n$ sufficiently large and $m \in [n, 2n]$,

$$\left|\text{Var}(D_{\tau_n^B} - D_{\tau_n^B}) - 2(m - n)\right| \leq Cn^{3/4} + C\frac{(m - n)^2}{n}.$$

**Proof.** Using the decomposition

$$D_{\tau_n^B} - D_{\tau_n^B} = (D_{2m} - D_{2n}) + (D_{\tau_n^B} - D_{2m}) - (D_{\tau_n^B} - D_{2n}),$$

we can expand the variance into the variance and covariance terms on the right. From this one obtains that for $m \in [n, 2n]$ and $n$ sufficiently large

$$\left|\text{Var}(D_{\tau_n^B} - D_{\tau_n^B}) - 2(m - n)\right| \leq |\text{Var}(D_{2m} - D_{2n}) - 2(m - n)| + \text{Var}(D_{\tau_n^B} - D_{2m}) + \text{Var}(D_{\tau_n^B} - D_{2n}) + 2\sqrt{\text{Var}(D_{2m} - D_{2n})}\left\{\sqrt{\text{Var}(D_{\tau_n^B} - D_{2m})} + \sqrt{\text{Var}(D_{\tau_n^B} - D_{2n})}\right\} + 2\sqrt{\text{Var}(D_{\tau_n^B} - D_{2m})}\sqrt{\text{Var}(D_{\tau_n^B} - D_{2n})} \leq Cn^{3/4} + C\frac{(m - n)^2}{n},$$

where the last inequality follows from the variance estimates in Corollary 4.5 and Lemma 4.7. The proof of the limit of $\frac{\text{Var}(D_{\tau_n^B})}{n}$ follows by a similar argument and using Corollary 4.5 and Lemma 4.7. \(\square\)

**4.2.2. The drift for the urn process.** Our main goal in this subsection is to prove that $E[D_{\tau_n^B}] \to \frac{1}{\sqrt{2(2\alpha + 1)}}$ (Proposition 4.10). We will need the following lemma which slightly improves on the bound $E[D_{\tau_n^B}] = O(\sqrt{n})$ that follows from Lemma 4.4.

**Lemma 4.9.** \(\lim_{n \to \infty} \frac{1}{\sqrt{n}} E[D_{\tau_n^B}] = 0\).

**Proof.** First of all, since Lemma 4.7 implies that $E \left[|D_{\tau_n^B} - D_{2n}|\right] \leq Cn^{1/4}$, it is enough to prove that $\lim_{n \to \infty} \frac{1}{\sqrt{n}} E[D_n] = 0$. To this end, for any fixed $\delta > 0$ we have from (34) and Lemma 4.4 that

$$|E[D_n]| \leq \left|E\left[D_n - D_{[\delta n]}\right]\left([\delta n]/n\right)\right| + C\delta^{\alpha + \frac{1}{2}} \sqrt{n},$$

where the second inequality follows from the fact that $D_{\tau_n^B} = \tau_n^B - 2n$ and the last inequality follows from Lemmas 4.1 and 4.6 for $n$ sufficiently large and $y \leq n^{3/8}$. \(\square\)
For the first term on the right side, we use the decomposition in (44), the fact that $E[\Delta_i] = 0$ for all $i$, and the second moment bounds that were used in the proofs of the variance bounds in (46)–(47) to obtain that

$$
\|E\left[\mathfrak{D}_n - \mathfrak{D}_{[\delta n]} \left(\lfloor \delta n \rfloor/n\right)^\alpha\right]\|
$$

\begin{align*}
\leq & E\left[\sum_{i=[\delta n]}^{n-1} \mathfrak{D}_i \left(\frac{i+1}{n}^\alpha \left(1 - \frac{\alpha}{i}\right) - \left(\frac{i}{n}\right)^\alpha\right)\right] + E\left[\sum_{i=[\delta n]}^{n-1} \varepsilon_i \left(\frac{i+1}{n}^\alpha\right)\right] \\
\leq & C\delta\left(\log n\right)/\sqrt{n} + C\delta\log^2 n.
\end{align*}

Combining (49) and (50) we obtain that $\limsup_{n \to \infty} \frac{1}{\sqrt{n}} E[\mathfrak{D}_n] \leq C\delta^{\alpha + \frac{1}{2}}$, and taking $\delta \to 0$ we get that $\lim_{n \to \infty} \frac{1}{\sqrt{n}} E[\mathfrak{D}_n] = 0$, which completes the proof of the lemma. \hfill \Box

**Proposition 4.10.** $\lim_{n \to \infty} E[\mathfrak{D}_{r_n^\alpha}] = \frac{1}{2(2\alpha + 1)}$.

**Proof.** Using the first equality in (41) and the assumption that $w(i) = (i+1)^{-\alpha}$, we get

$$
\sum_{j=0}^{n-1} (2j + 2)^\alpha - (2j + 1)^\alpha
$$

\begin{align*}
= & E\left[\sum_{i=0}^{2n-1} \left(2i + 1\right)^\alpha - \sum_{i=0}^{2n-1} \left(2i + 1\right)^\alpha\right] \\
= & E\left[\mathfrak{D}_{r_n^\alpha}\right] (2n)^\alpha + E\left[\sum_{i=n}^{n+2n-1} \left\{(2i + 1)^\alpha - (2n)^\alpha\right\}\right],
\end{align*}

where in the last equality and throughout the remainder of the proof we use the convention that $\sum_{i=n}^{m-1} = 0$ if $m = 0$ and $\sum_{i=n}^{m-1} = -\sum_{i=n}^{m-1}$ if $m < 0$. By integral approximations, the sum on the left above is easily seen to equal $\frac{1}{2}(2n)^\alpha + o(n^\alpha)$, and so to finish the proof, we need to obtain good asymptotics for the last expectation on the right. We will analyze the sum inside this expectation differently depending on whether or not $|\mathfrak{D}_{r_n^\alpha}| \leq n^{3/5}$. Since the sum inside the expectation is always bounded by $C(n + |\mathfrak{D}_{r_n^\alpha}|)^{\alpha + 1}$ for a fixed constant $C > 0$, it follows that

$$
E\left[\sum_{i=n}^{n+2n-1} \left\{(2i + 1)^\alpha - (2n)^\alpha\right\} \mathbb{1}_{\{|\mathfrak{D}_{r_n^\alpha}| > n^{3/5}\}}\right] \leq CE\left[\left(n + |\mathfrak{D}_{r_n^\alpha}|\right)^{\alpha + 1}\mathbb{1}_{\{|\mathfrak{D}_{r_n^\alpha}| > n^{3/5}\}}\right],
$$

and this upper bound vanishes as $n \to \infty$ by Lemma 4.1. To control the sum inside the expectation when $|\mathfrak{D}_{r_n^\alpha}| \leq n^{3/5}$, first note that by a Taylor series approximation there is a constant $C > 0$ such that

$$
\max_{i:|i-n|\leq n^{3/5}} \left|(2i + 1)^\alpha - (2n)^\alpha - (2n)^\alpha \frac{i-n}{n}\right| \leq Cn^{\alpha - \frac{4}{5}}.
$$

Using this and the fact that $\sum_{i=n}^{n+m-1}(i-n) = \frac{|m||m-1|}{2}$ we have that on the event $\{|\mathfrak{D}_{r_n^\alpha}| \leq n^{3/5}\}$,

$$
\sum_{i=n}^{n+2n-1} \left\{(2i + 1)^\alpha - (2n)^\alpha\right\} - (2n)^\alpha \mathfrak{D}_{r_n^\alpha} \left|\mathfrak{D}_{r_n^\alpha} - 1\right| \left|\mathfrak{D}_{r_n^\alpha}\right| \leq Cn^{\alpha - \frac{4}{5}} |\mathfrak{D}_{r_n^\alpha}| \leq Cn^{\alpha - \frac{1}{5}}.
$$

Note that it follows from Lemma 4.1, Proposition 4.8 and Lemma 4.9 that

$$
\lim_{n \to \infty} \frac{1}{\sqrt{2n}} E\left[\left(\frac{\mathfrak{D}_{r_n^\alpha}}{\mathfrak{D}_{r_n^\alpha} - 1}\right)^2\mathbb{1}_{\{|\mathfrak{D}_{r_n^\alpha}| \leq n^{3/5}\}}\right] = \lim_{n \to \infty} E\left[\left(\frac{\mathfrak{D}_{r_n^\alpha}}{\sqrt{2n}}\right)^2\right] = \frac{1}{2\alpha + 1}.
$$
Combining the analysis of both the case \(|D_{r,2}^n| > n^{3/5}\) and \(|D_{r,2}^n| \leq n^{3/5}\), we obtain that
\[
E \left[ \sum_{i=n}^{n+D_{r,2}^n-1} (2i+1)^\alpha - (2n)^\alpha \right] = (2n)^\alpha \frac{\alpha}{2\alpha + 1} + o(n^\alpha).
\]
Thus far we have shown that
\[
\frac{1}{2} (2n)^\alpha + o(n^\alpha) = E \left[ \sum_{i=n}^{n+D_{r,2}^n-1} (2i+1)^\alpha - (2n)^\alpha \right] = (2n)^\alpha \frac{\alpha}{2\alpha + 1} + o(n^\alpha).
\]
Dividing both sides by \((2n)^\alpha\) and then taking \(n \to \infty\) finishes the proof.

4.2.3. Proof of Proposition 3.2. As noted in the remark following Proposition 3.2, we need only prove the claim in (21).

For purely notational reasons, we give a proof only for \(c = 0\). We argue by contradiction and suppose that there exists a \(\eta_0 > 0\) for which the conclusion fails. This implies the existence of a sequence of integers \(M_r \to \infty\) as \(r \to \infty\) such that for each \(r\) there exists arbitrarily large \(N\) such that \(\text{dist}(P^{Z_{N_r}^{M_r,0}}, P^{Z(0,0)}) \geq \eta_0\). We note that the space \((D([0,1]), d^o)\) (see [Bil99, (12.16)]) is a complete separable metric space. So by [Bil99, p. 72], the Prokhorov metric on the set of probability measures on \(D([0,1])\) gives the topology of convergence in distribution. Thus for any choice of \(N_r, r \geq 1\), such that
\[
\max \left\{ Z^{(a,1)}(s) - M_r \mid s \leq \frac{1}{rN_r}, \ell \geq 1\right\} = 1,
\]
the processes \(\{Z^{M_r,0}_{N_r}\}_{N_r \geq 1}\) cannot converge to \(Z(0,0)\) in law. The desired contradiction will be arrived at by showing that we can find a sequence \(N_r \to \infty\) satisfying (51) but for which \(\{Z^{M_r,0}_{N_r}\}_{N_r \geq 1}\) does converge to \(Z(0,0)\) in law.

To choose the sequence \(N_r\), we first note that the first conclusion in Proposition 3.2 implies that
\[
\lim_{N \to \infty} P \left( \max_{j \leq N} \left| \mathcal{E}^{T_{M_r,N}}(j) - M_rN \right| \geq \frac{M_rN}{2} \right) = 0,
\]
where the last equality follows from standard rescaling of Bessel squared processes. Since clearly the right side vanishes as \(M_r \to \infty\) we can choose a sequence of integers \(N_r \geq M_r^4\) such that
\[
\lim_{r \to \infty} P \left( \max_{j \leq N_r} \left| \mathcal{E}^{T_{M_r,N}N}(j) - M_rN_r \right| \geq \frac{M_rN_r}{2} \right) = 0.
\]

To prove that \(Z^{M_r,0}_{N_r}\) converges to \(Z(0,0)\) in law, we will apply [EK86, Theorem 4.1, p. 354]. As noted in the paragraph preceding (42), the process \(\mathcal{E}^{T_r}\) is a time inhomogeneous Markov chain, but if we define \(\hat{\mathcal{E}}^{T_r}(j) = \mathcal{E}^{T_r}(j) - 1_{(0)}(j)\) then \(\hat{\mathcal{E}}^{T_r}\) is a time homogeneous Markov chain and so we will work with this process instead. We first consider (and then modify) the two dimensional process \((Y^r_1(s), Y^r_2(s))\) defined by
\[
Y^r_1(s) = \frac{\mathcal{E}^{T_r(1)},N_rN_r((sN_r)) - \mathcal{E}^{T_r,N_rN_r}((sN_r))}{N_r}, \quad \text{and} \quad Y^r_2(s) = \frac{\mathcal{E}^{T_r,N_rN_r}((sN_r))}{M_rN_r}.
\]
We are really only interested in proving that \(Y^r_1\) converges in distribution to \(Z(0,0)\); the second coordinate \(Y^r_2\) is included for convenience of the proof because \(Y^r_1\) is not a Markov chain but the
joint process \((Y_1^r, Y_2^r)\) is. Note that to make the proof simpler we (intentionally) overscaled \(Y_2^r\) so that it will converge to a constant. Define

\[
T_r := \inf \left\{ j \in \mathbb{N} : \left| \hat{\mathcal{E}}^{T_{MrN_r}}(j) - M_rN_r \right| \geq \frac{M_rN_r}{2} \right\},
\]

so that (52) becomes \(\lim_{r \to \infty} P(T_r \leq N_r) = 0\). We then take \(Z_i^r(s) := Y_i^r(s \cap (T_r/N_r))\) so that \(Z_i^r(s) = Y_i^r(s), \forall s \leq 1, i \in \{1, 2\},\) with probability tending to 1 as \(r \to \infty\). As in [EKS6] Theorem 4.1 we choose processes \((B_i^r(s))_{s \in [0,1]}, i \in \{1, 2\},\) so that \(Z_i^r(s) - B_i^r(s), s \in [0,1],\) is a martingale with respect to \(\mathcal{F}_s^r = \sigma(Z_i^r(u), Z_j^r(u), u \leq s)\). Obviously, the processes \(Z_i^r, i = 1, 2,\) are constant on 

\[
[j - 1]/N_r, \frac{j}{N_r},
\]

so we can write

\[
B_i^r(s) = \sum_{j=0}^{[sN_r]-1} b_i^r(j), \quad \text{where} \quad b_i^r(j) = E \left[ Z_i^r \left( \frac{j+1}{N_r} \right) - Z_i^r \left( \frac{j}{N_r} \right) \mid \mathcal{F}_{j/N_r}^r \right].
\]

Similarly, we introduce \((A_{ik}^r(s))_{s \in [0,1], i, k \in \{1, 2\}},\) so that \((Z_i^r(s) - B_i^r(s))(Z_k^r(s) - B_k^r(s)) - A_{ik}^r(s), s \in [0,1],\) is a martingale for all \(i, k \in \{1, 2\}\). Again we can write

\[
A_{ik}^r(s) = \sum_{j=0}^{[sN_r]-1} a_{ik}^r(j), \quad \text{where} \quad a_{ik}^r(j) = \text{Cov} \left( Z_i^r \left( \frac{j+1}{N_r} \right), Z_k^r \left( \frac{j+1}{N_r} \right) \mid \mathcal{F}_{j/N_r}^r \right).
\]

Finishing the proof of Proposition 3.2 amounts to verifying the conditions of [EKS6] Theorem 4.1 with processes \(b_1, b_2, a_{12}, a_{22} \equiv 0,\) and \(a_{11}(x) = 2x.\) That is, letting

\[
\sigma_\ell = \inf\{s \geq 0 : Z_1^r(s) \geq \ell\}, \quad r \in \mathbb{N}, \quad \ell > 1,
\]

it is sufficient to show that for any fixed \(\ell > 1\) we have

\[
limit_{r \to \infty} E \left[ \sup_{s \leq \sigma_\ell^r \wedge 1} |Z_i^r(s) - Z_i^r(s-)|^2 \right] = 0, \quad \text{for} \quad i \in \{1, 2\},
\]

\[
limit_{r \to \infty} E \left[ \sup_{s \leq \sigma_\ell^r \wedge 1} |B_i^r(s) - B_i^r(s-)|^2 \right] = 0, \quad \text{for} \quad i \in \{1, 2\},
\]

\[
limit_{r \to \infty} E \left[ \sup_{s \leq \sigma_\ell^r \wedge 1} |A_{ik}^r(s) - A_{ik}^r(s-)| \right] = 0, \quad \text{for} \quad i, k \in \{1, 2\},
\]

\[
limit_{r \to \infty} \sup_{s \leq \sigma_\ell^r \wedge 1} |B_i^r(s)| = 0, \quad \text{in probability,} \quad \text{for} \quad i \in \{1, 2\},
\]

\[
limit_{r \to \infty} \sup_{s \leq \sigma_\ell^r \wedge 1} |A_{12}^r(s)| = 0, \quad \text{in probability,} \quad \text{for} \quad i \in \{1, 2\}, \quad \text{and}
\]

\[
limit_{r \to \infty} \sup_{s \leq \sigma_\ell^r \wedge 1} \left| A_{11}^r(s) - 2 \int_0^s Z_i^r(s) \, ds \right| = 0, \quad \text{in probability.}
\]

We first address (55) and (58). Recalling the definitions above, and using (42) we have that

\[
b_i^r(j) = \frac{1}{M_rN_r} E \left[ \hat{\mathcal{E}}^{T_{MrN_r}} ((j + 1) \wedge T_r) - \hat{\mathcal{E}}^{T_{MrN_r}} (j \wedge T_r) \mid \hat{\mathcal{E}}^{T_{MrN_r}}(x), x \leq j \right] = \frac{1}{M_rN_r} E \left[ D_{\sigma_{\ell}^r, n} \right] \mathbbm{1}_{\{j < T_r\}} \quad \text{where} \quad n = \hat{\mathcal{E}}^{T_{MrN_r}}(j) \leq \frac{1}{M_rN_r} E[D_{\sigma_{\ell}^r, n}],
\]
It follows from Proposition [1.10] that for $r$ sufficiently large we have $|b_2^r(j)| \leq (M_r N_r)^{-1}$ for all $j$. Therefore, we can conclude that

$$\sup_{s \leq 1} (B_2^r(s) - B_2^r(s-))^2 = \sup_{j \leq N_r} (b_2^r(j))^2 \leq (N_r M_r)^{-2} \rightarrow 0,$$

and also

$$\sup_{s \leq 1} |B_2^r(s)| \leq N_r \sup_{j \leq N_r} |b_2^r(j)| \leq M_r^{-1} \rightarrow 0.$$

A similar computation yields

$$b_1^r(j) = \frac{1}{N_r} E \left[ (\mathcal{D}_{\tau_m^r} - \mathcal{D}_{\tau_n^r})^2 \right] \mathbb{1}_{\{j < T_r\}},$$

where $n = \tilde{\mathcal{E}}^{T_{M_r N_r}}(j)$ and $m = \tilde{\mathcal{E}}^{T_{(M_r+1)N_r}}(j).$

Since Proposition [1.10] implies that $E \left[ (\mathcal{D}_{\tau_m^r} - \mathcal{D}_{\tau_n^r})^2 \right] \rightarrow 0$ as both $m$ and $n$ go to $\infty$ and since $\tilde{\mathcal{E}}^{T_{(M_r+1)N_r}}(j) \geq \tilde{\mathcal{E}}^{T_{M_r N_r}}(j) \geq \frac{M_r N_r}{2}$ on the event $\{j < T_r\}$, it follows that

$$\sup_{s \leq 1} (B_1^r(s) - B_1^r(s-))^2 \rightarrow 0 \quad \text{and} \quad \sup_{s \leq 1} |B_1^r(s)| \rightarrow 0, \quad \text{as } r \rightarrow \infty.$$

We now consider (57), (59), and (60). For this, note that

$$a_{11}^r(j) = \frac{1}{N_r^2} \text{Var} \left( \tilde{\mathcal{E}}^{T_{M_r N_r}} ((j+1) \wedge T_r) \mid \tilde{\mathcal{E}}^{T_{M_r N_r}}(x), x \leq j \right)$$

and similarly

$$a_{21}^r(j) = \frac{1}{N_r^2} \text{Var} \left( \mathcal{D}_{\tau_m^r} - \mathcal{D}_{\tau_n^r} \right) \mathbb{1}_{\{j < T_r\}}, \quad \text{where } n = \tilde{\mathcal{E}}^{T_{M_r N_r}}(j) \text{ and } m = \tilde{\mathcal{E}}^{T_{(M_r+1)N_r}}(j).$$

We remark that in the analysis of the terms involving $a_{11}^r(j)$ we will use the fact that if $j < \sigma^r N_r$ then $\tilde{\mathcal{E}}^{T_{(M_r+1)N_r}}(j) - \tilde{\mathcal{E}}^{T_{M_r N_r}}(j) \leq \ell N_r$. Then to prove (57), note that Proposition [1.8] implies that for $r$ sufficiently large

$$\sup_{s \leq \sigma^r N_r} |A_1^r(s) - A_1^r(s-)| = \max_{j < \sigma^r N_r} |a_1^r(j)|$$

$$\leq \max_{|n-M_r N_r| \leq M_r N_r} \frac{1}{N_r^2} \text{Var} (\mathcal{D}_{\tau_m^r} - \mathcal{D}_{\tau_n^r}) \leq \frac{C}{N_r},$$

and

$$\sup_{s \leq 1} |A_{22}^r(s) - A_{22}^r(s-)| = \max_{j < N_r} |a_{22}^r(j)| \leq \max_{|n-M_r N_r| \leq M_r N_r} \frac{\text{Var} (\mathcal{D}_{\tau_n^r})}{(M_r N_r)^2} \leq \frac{C}{M_r N_r}.$$

(Note that to obtain the last inequality in the second line above we are also using that $N_r \geq M_r^{4/3}$.) This is enough to prove (57). The last two bounds yield (59), since $\sup_{s \leq 1} |A_{12}^r(s)| \leq N_r \max_{j < N_r} |a_{12}^r(j)|$.

For (60), note that Proposition [1.8] together with the definition of the stopping times $T_r$ and $\sigma^r$, implies that for $r$ sufficiently large

$$\sup_{j < T_r \wedge (N_r \sigma^r)} \left| a_{11}^r(j) - \frac{2}{N_r^2} \mathcal{Z}_1^r (j/N_r) \right| \leq \frac{C}{N_r^2} \left( \left( \frac{3 M_r N_r}{2} \right)^{3/4} + \frac{(2 \ell N_r)^2}{M_r N_r} \right) \leq \frac{C M_r^{3/4}}{N_r^{5/4}} + \frac{C}{M_r N_r}.$$
Thus, on the event \( \{ T_r \geq N_r \} \), we have that
\[
\sup_{s \leq \sigma_r^T \wedge 1} \left| A_{11}^r(s) - \int_0^s 2Z_1^r(s) \, ds \right| \leq \sup_{s \leq \sigma_r^T \wedge 1} \left\{ \sum_{j=0}^{\lfloor N_r s \rfloor - 1} \left| a_{1j}^r \right| - \frac{2}{N_r} Z_1^r(j/N_r) \right\} \leq \frac{CM_r^{3/4}}{N_r^{1/4}} + C + \frac{\ell}{N_r}.
\]
Since \( M_r \to \infty \) as \( r \to \infty \) and \( N_r \geq M_r^4 \), this bound vanishes as \( r \to \infty \), and since \( P(T_r \leq N_r) \to 0 \) as \( r \to \infty \), this completes the proof of (59).

Finally, we will prove (55) for the case \( i = 1 \) as a similar (but simpler) proof works for \( i = 2 \) as well. First of all, note that
\[
E \left[ \sup_{s \leq \sigma_r^T \wedge 1} \left| Z_1^r(s) - Z_1^r(s-) \right|^2 \right] = E \left[ \max_{j < \sigma_r^T \wedge N_r \wedge T_r} \left| Z_1^r(j/N_r) - Z_1^r(j/(N_r)^{1/2}) \right|^2 \right] \leq \frac{M_r^2}{N_r} + \int_{M_r/\sqrt{N_r}}^\infty 2xP \left( \max_{j < \sigma_r^T \wedge N_r \wedge T_r} \left| Z_1^r(j/N_r) - Z_1^r(j/(N_r)^{1/2}) \right| > x \right) \, dx.
\]
To bound the probabilities in the integral on the right, first recall that if \( \hat{\xi}_{T_{M_r N_r}}^r(j) = n \) and \( \hat{\xi}_{T_{(M_r+1)N_r}}^r(j) = m \) then \( Z_1^r(j/N_r) - Z_1^r(j/(N_r)^{1/2}) \) has the same distribution as \( \frac{D_{r, n} - D_{r, m}}{N_r} \), and moreover if \( j < (\sigma_r^T N_r \wedge T_r) \) then these values of \( n \) and \( m \) must be such that \( n - M_r N_r \leq \frac{M_r N_r}{2} \) and \( n \leq m \leq n + \ell N_r \). Therefore, conditioning on \( \hat{\xi}_{T_{M_r N_r}}^r(j) \) and \( \hat{\xi}_{T_{(M_r+1)N_r}}^r(j) \) gives
\[
P \left( \left| Z_1^r(j/N_r) - Z_1^r(j/(N_r)^{1/2}) \right| > x, j < (\sigma_r^T N_r \wedge T_r) \right) \leq \sum_{m, n} P \left( D_{r, n} - D_{r, m} \right) \leq \frac{x N_r}{2} \right) \leq C \exp \left\{ -c \left( \frac{x^2 N_r}{2} \right) \right\},
\]
where the last inequality holds by Lemma [41]. Thus, we can conclude that
\[
E \left[ \sup_{s \leq 1} \left| Z_1^r(s) - Z_1^r(s-) \right|^2 \right] \leq \frac{M_r^2}{N_r} + \int_{M_r/\sqrt{N_r}}^\infty 2xC \exp \left\{ -c \left( \frac{x^2 N_r}{2} \right) \right\} \, dx.
\]
Since \( M_r \to \infty \) as \( r \to \infty \) and \( N_r \geq M_r^4 \), this upper bound vanishes as \( r \to \infty \), thus completing the proof of (55) for \( i = 1 \).

\[ \square \]

**Appendix A. Branching-like processes and generalized Ray-Knight theorems**

Tóth’s analysis of the type of SIRWs considered in this paper has been done primarily through the study of the directed edge local times of the walk stopped at certain stopping times. In this appendix we will first recall the definition of these processes and their connection with the directed edge local times of the random walk and then prove a few results which we shall need in Appendix [3].

We begin by defining two homogeneous Markov chains on \( \mathbb{N}_0 \). The transition probabilities will be given in terms of the generalized Pólya urn models from section [4].
• \( \zeta := (\zeta_k)_{k \geq 0} \) is a Markov chain with transition probabilities given by

\[
P(\zeta_{k+1} = j \mid \zeta_k = i) = P\left( R^0_{i+1} = j \right), \quad \forall i, j \geq 0.
\]

(61)

• \( \tilde{\zeta} := (\tilde{\zeta}_k)_{k \geq 0} \) is a Markov chain with transition probabilities given by

\[
P(\tilde{\zeta}_{k+1} = j \mid \tilde{\zeta}_k = i) = P\left( \mathfrak{B}^+_{i+1} = j \right), \quad \forall i, j \geq 0.
\]

(62)

**Remark A.1.** We will refer to \( \zeta \) and \( \tilde{\zeta} \) as BLPs due to the fact that in the case where the weight function \( w(\cdot) \equiv 1 \) we have that \( \zeta \) is a Galton-Watson branching process with \( \text{Geo}(1/2) \) offspring distribution and \( \zeta \) is a Galton-Watson branching process with one immigrant before reproduction and \( \text{Geo}(1/2) \) offspring distribution.

**Remark A.2.** Note that in \( [\text{Tó}t96] \) these Markov chains are defined slightly differently so that

\[
P(\zeta_{k+1} = j \mid \zeta_k = i) = P\left( \mathfrak{B}^+_{i+1} = j \right), \quad \forall i, j \geq 0.
\]

and

\[
P(\tilde{\zeta}_{k+1} = j \mid \tilde{\zeta}_k = i) = P\left( \mathfrak{B}^-_{i+1} = j \right), \quad \forall i, j \geq 0.
\]

However, it is easy to see that the difference in these definitions is simply an interchange of the labels “red” and “blue” and changing the corresponding parameters of the urn model accordingly. Thus, the Markov chains \( \zeta \) and \( \tilde{\zeta} \) defined as in (61) and (62) are equivalent to those in \( [\text{Tó}t96] \).

The BLPs \( \zeta \) and \( \tilde{\zeta} \) are related to the \( \mathcal{E} \) and \( \mathcal{D} \) processes of local times of directed edges as defined in \( [3] \) when the random walk is stopped at certain special stopping times. There are various choices of the stopping times that can be used, but we will discuss this connection here only for the stopping times that will be needed for our purposes.

For any \( z \in \mathbb{Z} \) and \( m \in \mathbb{N} \), let \( \tau_{z,m} = \min\{n \geq 0 : \mathcal{L}(z,n) = m\} \) be the \( m \)-th time the random walk reaches \( z \). Following similar reasoning as in the paragraph above \( [12] \), one can see that the process \( (\mathcal{E}^{\tau_{z,m}}(x))_{x \geq z} \) is a time inhomogeneous Markov chain. Indeed, using the fact that

\[
\mathcal{D}^{\tau_{z,m}}(x) = \mathcal{E}^{\tau_{z,m}}(x-1) + 1 \text{ if } z < x \leq 0, \quad \text{and} \quad \mathcal{D}^{\tau_{z,m}}(x) = \mathcal{E}^{\tau_{z,m}}(x-1) \text{ if } x > z \lor 0,
\]

and the fact that the sequence of left/right steps from \( x \) can be generated by the Pólya urn process \( (\mathfrak{B}_n^-, \mathfrak{R}_n) \) if \( x < 0 \) or \( (\mathfrak{B}_n^+, \mathfrak{R}_n) \) if \( x > 0 \), it follows that

\[
P(\mathcal{E}^{\tau_{z,m}}(x) = j \mid \mathcal{E}^{\tau_{z,m}}(x-1) = i) = P(\zeta_1 = j \mid \zeta_0 = i), \quad \text{if } z < x < 0, \quad \text{and}
\]

(63)

\[
P(\mathcal{E}^{\tau_{z,m}}(x) = j \mid \mathcal{E}^{\tau_{z,m}}(x-1) = i) = P\left( \tilde{\zeta}_1 = j \mid \tilde{\zeta}_0 = i \right), \quad \text{if } x > z \lor 0.
\]

Thus, \( (\mathcal{E}^{\tau_{z,m}}(x))_{x \leq z < 0} \) has the distribution of the BLP \( \zeta \) with a random initial condition given by the distribution of \( \mathcal{E}^{\tau_{z,m}}(z) \), and \( (\mathcal{E}^{\tau_{z,m}}(x))_{x \geq z \lor 0} \) has the distribution of the BLP \( \tilde{\zeta} \) with a random initial condition given by the distribution of \( \mathcal{E}^{\tau_{z,m}}(z \lor 0) \).

The results in this section will only be needed for the proof of our results in the asymptotically free case \( (\alpha = 0) \), and so we will restrict our discussion to this case. The following result due to Tóth \( [\text{Tó}t96] \) shows that when \( \alpha = 0 \) the BLPs \( \zeta \) and \( \tilde{\zeta} \) have scaling limits which are multiples of squared Bessel processes of dimension \( 2 - 2\gamma \) and \( 2\gamma \), respectively.

**Proposition A.3** \( [\text{Tó}t96] \). Assume that \( w \) is as in \( [3] \) with \( \alpha = 0 \).
(1) For $n \geq 1$ let $\zeta^{(n)}_k = (\zeta_k^{(n)})_{k \geq 0}$ have the distribution of the BLP $\zeta$ with initial condition $\zeta_0^{(n)} = \lfloor \gamma n \rfloor$ for some $y \geq 0$, and let $Z_n(t) = \frac{\zeta^{(n)}_k}{n}$ for $n \geq 1$ and $t \geq 0$. Then, on the space $D([0, \infty))$ we have

\[ Z_n(\cdot) \overset{D}{=} Z^{(2\gamma)}(\cdot). \]

(2) For $n \geq 1$ let $\tilde{\zeta}^{(n)}_k = (\tilde{\zeta}^{(n)}_k)_{k \geq 0}$ have the distribution of the BLP $\tilde{\zeta}$ with initial condition $\zeta_0^{(n)} = \lfloor \gamma n \rfloor$ for some $y > 0$, and let $\tilde{Z}_n(t) = \frac{\tilde{\zeta}^{(n)}_k}{n}$ for $n \geq 1$ and $t \geq 0$. Then, on the space $D([0, \infty)) \times [0, \infty)$ we have

\[ \left( \tilde{Z}_n(\cdot), \sigma_0, \tilde{Z}^{(2\gamma)} \right) \overset{D}{=} \left( Z^{(2\gamma)}(\cdot), \sigma_0, Z^{(2\gamma)} \right). \]

Remark A.4. While in [Tot96] it was assumed that the weight function $w$ was as in [3] with $p = \kappa = 1$, a careful reading of the proofs of the diffusion approximations in [Tot96] Section 5] shows that the argument goes through for $p \in (0, 1]$ and $\kappa > 0$ with very few modifications. In particular, the only thing that changes in [Tot96] Lemma 2A is the error terms. For instance, if following Tóth’s notation, we let $F(x) = E[V_1(\zeta_0) | V_1(\zeta_0) = x - x]$, then the asymptotics $F(x) = 1 - \gamma + O(x^{-1})$ in [Tot96] Lemma 2A can be replaced (using the same proof) with $F(x) = 1 - \gamma + O(x^{-(p / \kappa)})$. Similar modifications can be made to the other error terms in [Tot96] Lemma 2A, and from this point on the proof of the diffusion approximations for the BLPs goes through without any changes.

In addition to the diffusion approximation for the BLPs, we will also need a few results which give information on the distributions of hitting probabilities of the BLP $\zeta$.

Lemma A.5. Let $w$ be as in (3) with $\alpha = 0$. If $\gamma \geq 0$, then for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that $P_0^\zeta(\sigma_0^\zeta > n) \geq C_\varepsilon n^{-\gamma - \varepsilon}$ for all $n$ large enough.

Remark A.6. We suspect that the true tail asymptotics are of the form $P_0^\zeta(\sigma_0^\zeta > n) \sim C n^{-\gamma}$ when $\gamma > 0$ and $P_0^\zeta(\sigma_0^\zeta > n) \sim C / \log n$ when $\gamma = 0$, but proving such precise asymptotics would require significant extra work and the asymptotics in Lemma A.5 are sufficient for our purposes in the remainder of the paper.

Proof of Lemma A.5. First of all, note monotonicity of the Markov process $\zeta$ with respect to its initial condition and the Strong Markov property together imply that

\[ P_0^\zeta(\sigma_0^\zeta > n) \geq P_0^\zeta(\sigma_0^\zeta > \tau_0^\zeta) P_0^\zeta(\sigma_0^\zeta > n) \]

For the second probability on the right, the diffusion approximation implies that

\[ \liminf_{n \to \infty} P_0^\zeta(\sigma_0^\zeta > n) \geq \lim_{n \to \infty} P_0^\zeta(\sigma_0^{n/2} > n) = P(\sigma_0^{(2-2\gamma)} > 1 | Z_0^{(2-2\gamma)} = 1) > 0, \]

and thus it is enough to show that $P_0^\zeta(\sigma_0^\zeta > \tau_0^\zeta) \geq C_\varepsilon n^{-\gamma - \varepsilon}$. To this end, first note that the diffusion approximation implies that for any $x \in (0, 1)$,

\[ \lim_{k \to \infty} P_{2k-1}^\zeta(\tau_{2k}^\zeta < \sigma_0^\zeta) \geq \lim_{k \to \infty} P_{2k-1}^\zeta(\tau_{2k}^\zeta < x_{2k-1}^\zeta) \]

\[ = P(\tau_2^{(2n-2\gamma)} < \sigma_0^{(2-2\gamma)} | Z^{(2-2\gamma)}(0) = 1) = \begin{cases} \frac{1-x^\gamma}{2^{\gamma} - x} & \text{if } \gamma > 0 \\ \frac{\log(x)}{\log(x/2)} & \text{if } \gamma = 0 \end{cases} \]

(Note that the last probability can be computed using martingale properties of $(Z_t^{(2-2\gamma)})_t$ when $\gamma > 0$ and of $\log(Z_t^{(2-2\gamma)})$ when $\gamma = 0$.) Taking $x \to 0$ we can conclude that $\liminf_{k \to \infty} P_{2k-1}^\zeta(\tau_{2k}^\zeta < \sigma_0^\zeta) \geq 2^{-\gamma}$, and thus for any $\varepsilon > 0$ there exists a $k_0 = k_0(\varepsilon)$ so that

\[ P_{2k-1}^\zeta(\tau_{2k}^\zeta < \sigma_0^\zeta) \geq 2^{-\gamma - \varepsilon}, \quad \forall k \geq k_0. \]
Then, for $n \geq 2^{k_0}$ we have
\[ P_0^\zeta (\sigma_0^\zeta > \tau_n^\zeta) \geq P_0^\zeta (\sigma_0^\zeta > \tau_{2^{k_0}}^\zeta) = P_0^\zeta (\sigma_0^\zeta > \tau_{2^{k_0}-1}^\zeta) \prod_{k=k_0}^{[\log_2 n]} P_0^\zeta (\tau_{2k}^\zeta < \sigma_0^\zeta | \tau_{2^{k-1}}^\zeta < \sigma_0^\zeta) \]
\[ \geq P_0^\zeta (\sigma_0^\zeta > \tau_{2^{k_0}-1}^\zeta) \prod_{k=k_0}^{[\log_2 n]} P_{2k}^\zeta (\tau_{2k}^\zeta < \sigma_0^\zeta) \]
\[ \geq P_0^\zeta (\sigma_0^\zeta > \tau_{2^{k_0}-1}^\zeta) (2^{-\gamma-\varepsilon})^{[\log_2 n]} \geq P_0^\zeta (\zeta_1 \geq 2^{k_0-1}) 2^{-\gamma-\varepsilon} n^{-\gamma-\varepsilon}. \]
\[ \square \]

In contrast to the previous lemma, since $Z^{(2-2\gamma)}$ is transient when $\gamma < 0$ it is natural to expect that with positive probability the process $\zeta$ never goes too far below where it starts. The following lemma makes precise the sort of statement that we will need later.

**Lemma A.7.** Assume $\gamma < 0$. Then, there exists a constant $c \geq 0$ so that $P_{4n}^\zeta (\sigma_n^\zeta = \infty) \geq c$ for all $n \geq 1$.

**Proof.** If $k = [\log_2 n] + 1$, then $P_{4n}^\zeta (\sigma_n^\zeta = \infty) \geq P_{2k}^\zeta (\sigma_{2k-1}^\zeta = \infty)$. Therefore, it is enough to show that
\[ P_{2k}^\zeta (\sigma_{2k-1}^\zeta = \infty) \geq c > 0, \quad \forall k \geq 0. \]

For notational convenience we will let $n_k = 2^k$ and $m_k = 2^k - \lfloor 2^{k/3} \rfloor$ for $k \geq 0$. We claim that to prove (67) it is enough to show that there exists $q > 1/2$ and $k_1 \in \mathbb{N}$ such that
\[ P_{m_k}^\zeta (\tau_{\ell_{n_k+1}}^\zeta < \sigma_{\ell_{n_k-1}}^\zeta) \geq q, \quad \forall k \geq k_1, \]
and
\[ P_{m_k}^\zeta (\zeta_{\ell_{n_k+1}}^\zeta \wedge \rho_{\ell_{n_k+1}}^\zeta < m_k-1) \leq C n_k e^{-c n_k^{1/3}}, \quad \forall k \geq k_1. \]

To see that (68) and (69) imply (67), first note that it’s enough to prove (67) only for $k \geq k_1$ and secondly that we may assume without loss of generality that $k_1$ is large enough so that $n_{k-1} < m_k$ for all $k \geq k_1$. Now consider a Markov process $\{L_i\}_{i \geq 0}$ on $\{\partial\} \cup \{k_1, k_1+1, k_1+2, \ldots\}$ with $\partial$ being an absorbing state and where at states $k \geq k_1$ the process jumps to the right with probability $q$, to the left (if $k > k_1$) with probability $1 - q - C n_k e^{-c n_k^{1/3}}$, and otherwise jumps to the absorbing state $\partial$. Using the strong Markov property, together with the fact that $\zeta$ is monotone with respect to its starting point, one can then use (65) and a coupling argument to conclude that $P_{2k}^\zeta (\sigma_{2k-1}^\zeta = \infty) \geq P_k^L (\sigma_{\ell_{n_k+1}}^L \wedge \sigma_{\ell_{n_k-1}}^L = \infty) \geq P_{k_1}^L (\sigma_{\ell_{n_k+1}}^L = \infty)$, where $\sigma_{\ell_{n_k+1}}^L$ is the infimum of $\{i \geq 0 : L_i = \partial\}$. Finally, basic simple random walk computations can show that $P_{k_1}^L (\sigma_{\ell_{n_k+1}}^L = \infty) > 0$. Thus, it remains only to show (68) and (69).

For (68), for any $\varepsilon > 0$ we have that for $k$ sufficiently large (depending on $\varepsilon$) we have $\lfloor (2+\varepsilon) m_k \rfloor \geq n_{k+1}$ and $\lfloor (1/2+\varepsilon) m_k \rfloor \geq n_{k-1}$. Therefore,
\[ \liminf_{k \to \infty} P_{m_k}^\zeta (\tau_{\ell_{n_k+1}}^\zeta < \sigma_{\ell_{n_k-1}}^\zeta) \geq \lim_{k \to \infty} P_{m_k}^\zeta (\tau_{\ell_{(2+\varepsilon) m_k}}^\zeta < \sigma_{\ell_{(1/2+\varepsilon) m_k}}^\zeta) \]
\[ = \frac{1 - (\frac{1}{2} + \varepsilon)^\gamma}{(2 + \varepsilon)^\gamma - (\frac{1}{2} + \varepsilon)^\gamma}, \]
where the last equality follows from the diffusion approximation in (65) and an explicit hitting probability computation for the limiting diffusion using the fact that $(Z_t^{(2-2\gamma)})^\gamma$ is a martingale. Taking
$\epsilon \to 0$ we get that $\liminf_{k \to \infty} P_{mk}^\zeta (\tau_{n_{k+1}}^\zeta < \sigma_{n_{k-1}}^\zeta) \geq \frac{1-(1/2)^\gamma}{2\gamma(1/2)},$ where the last inequality follows from the assumption that $\gamma < 0$. This is enough to prove (68).

For (69), first note that

$$P_{mk}^\zeta (\zeta_{n_{k+1}}^\zeta, \tau_{n_{k+1}}^\zeta < m_{k-1})$$

$$= \sum_{i \geq 1} \sum_{z \in (n_{k-1}, n_{k+1})} P_{mk}^\zeta (\sigma_{n_{k-1}}^\zeta, \tau_{n_{k+1}}^\zeta = i, \zeta_{l-1} = z, \zeta_{l} < m_{k-1})$$

$$= \sum_{i \geq 1} \sum_{z \in (n_{k-1}, n_{k+1})} P_{mk}^\zeta (\sigma_{n_{k-1}}^\zeta, \tau_{n_{k+1}}^\zeta \geq i, \zeta_{l-1} = z) P_{z}^\zeta (\zeta_{l} < m_{k-1})$$

(70)

For the first term on the right in (70), note that given $\zeta_0 = n_{k-1}+1$ then $\zeta_l$ has the same distribution as $\mathcal{R}_{n_{k-1}+2} = \mathcal{D}_{n_{k-1}+2} + n_{k-1} + 2$. Therefore, it follows from Lemma 4.1 that for $k$ sufficiently large there are constants $c, C > 0$ such that

$$P_{n_{k-1}+1}^\zeta (\zeta_l < m_{k-1}) = P (\mathcal{D}_{n_{k-1}+2} < -[n_{k-1}/2] - 2) \leq Ce^{-cn_{k-1}^{1/3}}.$$ 

It remains only to prove that the expected value in (70) is bounded above by $Cn_k$. To this end, it is enough to show that there exists a $c \in (0, 1)$ and $k_2 \in \mathbb{N}$ such that

$$P_{n_{k-1}+1}^\zeta (\sigma_{n_{k-1}}^\zeta, \tau_{n_{k+1}}^\zeta > \ell n_{k}) \leq (1 - c)^l,$$ 

for all $k \geq k_2$ and $l \geq 1$.

To see this, note that the Markov property implies that

$$P_{n_{k-1}+1}^\zeta (\sigma_{n_{k-1}}^\zeta, \tau_{n_{k+1}}^\zeta > \ell n_{k}) \leq \left( \max_{z \in (n_{k-1}, n_{k+1})} P_{z}^\zeta (\sigma_{n_{k-1}}^\zeta, \tau_{n_{k+1}}^\zeta > n_{k}) \right)^l.$$ 

(73)

If we let $z_k \in (n_{k-1}, n_{k+1})$ be one of the maximizers in the right side, then by taking subsequences as $k \to \infty$ so the right side achieves the limsup and then taking a further subsequence so that $z_k/n_{k} \to y \in [1/2, 2]$ we can apply the diffusion approximation of the BLP $\zeta$ by $Z^{(2-2\gamma)}$ to obtain that

$$\limsup_{k \to \infty} \max_{z \in (n_{k-1}, n_{k+1})} P_{z}^\zeta (\sigma_{n_{k-1}}^\zeta, \tau_{n_{k+1}}^\zeta > n_{k}) \leq \sup_{y \in [1/2, 2]} P_{y}^{Z^{(2-2\gamma)}} (\sigma_{1/2}^{Z^{(2-2\gamma)}} \wedge \tau_{2}^{Z^{(2-2\gamma)}} > 1) < 1.$$ 

Together with (73) this is enough to prove (72). \hfill \Box

APPENDIX B. PROOFS OF TECHNICAL RESULTS - ASYMPTOTICALLY FREE CASE

(i) Process level tightness of extrema.

Proof of Proposition 27. We will only give the proof for the tightness of the sequence $I_n^X$ since the proof is similar for $S_n^X$. It is enough to show that

$$\lim \limsup_{\delta \to 0} \sup_{n \to \infty} \left( \sup_{k, l \leq nt, |k-l| \leq n\delta} |I_k^X - I_l^X| \geq 2\varepsilon \sqrt{n} \right) = 0, \quad \forall \varepsilon > 0.$$ 

To this end, note that if the running minimum decreases by at least $2\varepsilon \sqrt{n}$ in less than $n\delta$ steps, then there must be some interval $[-m \lceil \varepsilon \sqrt{n} \rceil, -(m-1) \lceil \varepsilon \sqrt{n} \rceil]$ with $m \geq 1$ which the random walk crosses.
from right to left in less than \( n\delta \) steps. Moreover, note that this interval must have \( m \leq \lceil t/\delta \rceil \) since otherwise it will take more than time \( nt \) to cross all these intervals. Therefore,

\[
P \left( \sup_{k, \ell \leq nt \atop \ell \leq nt}\left| I_k^X - I_{\ell}^X \right| \geq 2\varepsilon \sqrt{n} \right) \leq \sum_{m=1}^{\lceil t/\delta \rceil} P \left( \tau_{m|\varepsilon \sqrt{n}|,1} - \tau_{-(m-1)|\varepsilon \sqrt{n}|,1} \leq \delta n \right)
\]

\[
\leq \sum_{m=1}^{\lceil t/\delta \rceil} P \left( 2 \sum_{i=m\lceil \varepsilon \sqrt{n} \rceil+1} P \left( 2 \sum_{i=m\lceil \varepsilon \sqrt{n} \rceil+1} P \left( 2 \sum_{i=m\lceil \varepsilon \sqrt{n} \rceil+1} \delta n \right) = \lceil t/\delta \rceil \right) \right),
\]

where the last equality follows from (13). By Proposition A.3(1), this upper bound converges to \( |t/\delta| P \left( \int_0^1 Z(2^{-2\gamma})(t) dt < \frac{t}{2\varepsilon} \right) Z(2^{-2\gamma})(0) = 0 \) as \( n \to \infty \). To finish we need to prove that this expression vanishes as \( \delta \to 0 \), and this is accomplished by the following lemma.

**Lemma B.1.** There exist constants \( C, c > 0 \) (depending only on \( \gamma \)) such that

\[
P \left( \int_0^1 Z(2^{-2\gamma})(t) dt < x \right) \leq Ce^{-cx^{-1/2}}, \quad \forall x > 0.
\]

**Proof.** By the scaling property of BESQ processes we have for any \( a > 0 \) that

\[
\int_0^1 Z(2^{-2\gamma})(t) dt = a^2 \int_0^{1/a} Z(2^{-2\gamma})(at) dt \overset{\text{law}}{=} a^2 \int_0^{1/a} Z(2^{-2\gamma})(t) dt,
\]

where the last equality in law is when the Bessel processes on both sides are started at \( Z(2^{-2\gamma})(0) = 0 \). Applying this with \( a = \sqrt{x} \) gives

\[
P \left( \int_0^1 Z(2^{-2\gamma})(t) dt < x \right) = P \left( \int_0^{x^{-1/2}} Z(2^{-2\gamma})(t) dt < 1 \right) = P \left( \int_{i=1}^i Z(2^{-2\gamma})(t) dt < 1 \right)
\]

\[
\leq P \left( \int_{i=1}^i Z(2^{-2\gamma})(t) dt < 1 \right), \quad \text{for } 1 \leq i \leq \lfloor x^{-1/2} \rfloor
\]

\[
\leq P \left( \int_0^1 Z(2^{-2\gamma})(t) dt < 1 \right), \quad \text{for } \lfloor x^{-1/2} \rfloor
\]

where the last equality follows from the Markov property and the monotonicity of BESQ processes with respect to their initial condition. The proof of the lemma is then finished by noting that

\[
P \left( \int_0^1 Z(2^{-2\gamma})(t) dt < 1 \right) < 1.
\]

The proof of Proposition 2.1 is now complete. \( \square \)

(ii) **Control of the number of rarely visited sites.** The proof of Lemma 2.2 below follows a strategy that is similar to the proofs of analogous statements in [KM11, KP16, KMP22] but is somewhat simpler. Recall that \( \mathcal{L}(x, n-1) = \mathcal{E}^n(x-1) + \mathcal{E}^n(x) + \mathbb{1}_{\{X_n \leq x \leq 0\}} - \mathbb{1}_{\{0 < x < X_n\}} \) and that the \( \mathcal{E} \) processes are related to the BLPs \( \zeta \) and \( \tilde{\zeta} \) as in (33) and (34). Hence, the control of the number of sites with small local times in Lemma 2.2 can be obtained by first proving similar results for the BLPs \( \zeta \) and \( \tilde{\zeta} \).

Since the process \( \zeta \) is absorbing at zero, we will only need to bound the number of times it goes below \( M \) before hitting zero for the first time. To this end, note that if the event \( \sum_{i=0}^{k-1} \mathbb{1}_{\{\zeta_i \leq M\}} \geq k \) occurs, then the first \( k - 1 \) times the process \( \zeta \) enters \([0, M]\) its next step
must necessarily not be to zero. Using the monotonicity of the process \( \zeta \) with respect to the initial condition, we have that

\[
(74) \quad \sup_{m \geq 1} P_m^\zeta \left( \sum_{i=0}^{\sigma_1^\zeta-1} 1_{\{\zeta_i \leq M\}} \geq k \right) \leq \left( 1 - P_M^\zeta(\zeta_0 = 0) \right)^{k-1},
\]

and note that \( P_M^\zeta(\zeta_0 = 0) > 0 \) so upper bound in (74) decays exponentially in \( k \) for \( M \) fixed.

Since the process \( \zeta \) is not absorbing at zero, we will need to control the time the process \( \zeta \) spends below \( M \) up to a fixed time. The following lemma will be sufficient for our purposes.

**Lemma B.2.** Let \( w(\cdot) \) be as in (3) with \( \alpha = 0 \). Then for any \( M, K > 0 \) and \( b > \frac{\gamma \sqrt{n}}{2} =: \frac{2+\varepsilon}{2} \) there are constants \( C, c, r > 0 \) (depending on \( K, M \) and \( b \)) such that

\[
\sup_{m \geq 0} P_m^\zeta \left( \sum_{i \leq K \sqrt{n}} 1_{\{\zeta_i \leq M\}} > n^b \right) \leq C e^{-cn^r}, \quad \forall n \geq 1.
\]

**Proof.** Since the process \( \zeta \) is monotone with respect to the initial condition, it is enough to prove the upper bound only for the case \( \zeta_0 = 0 \). We begin by fixing some \( d \in (\frac{2+\varepsilon}{2}, b) \). Now, if the event \( \{\sum_{i \leq K \sqrt{n}} 1_{\{\zeta_i \leq M\}} \geq n^b\} \) occurs then either (1) the process \( \zeta \) returns to \( 0 \) at least \( \lfloor n^d \rfloor \) times in the first \( K \sqrt{n} \) steps of the Markov chain or (2) in one of the first \( \lfloor n^d \rfloor \) excursions from \( 0 \) the process spends at least \( n^{b-d} \) steps below \( M \). Therefore, it follows that

\[
P_0^\zeta \left( \sum_{i \leq K \sqrt{n}} 1_{\{\zeta_i \leq M\}} \geq n^b \right)
\]

\[
\leq \sup_{i \leq K \sqrt{n}} P_0^\zeta \left( \sum_{i=0}^{\sigma_1^\zeta-1} 1_{\{\zeta_i \geq 0\}} \geq n^d \right) + P_0^\zeta \left( \sum_{i=0}^{\sigma_1^\zeta-1} 1_{\{\zeta_i \geq M\}} \geq n^{b-d} \right)
\]

\[
\leq \left( 1 - P_0^\zeta(\sigma_0^\zeta \geq K \sqrt{n}) \right)^{\lfloor n^d \rfloor} + n^d \left( 1 - P_M^\zeta(\zeta_1 = 0) \right)^{n^{b-d}-1},
\]

where the second term in the last inequality follows from a similar argument as the one preceding (74). We will handle the first term in the last line differently depending on whether \( \gamma \in [0, 1) \) or \( \gamma < 0 \). If \( \gamma \in [0, 1) \), then choosing an \( \varepsilon > 0 \) small enough so that \( \frac{2+\varepsilon}{2} < d \) it follows from Lemma A.5 that \( \left( 1 - P_0^\zeta(\sigma_0^\zeta \geq K \sqrt{n}) \right)^{\lfloor n^d \rfloor} \leq C' e^{-c'n^{d-2+\varepsilon}} \) for some \( C', c' > 0 \) (depending on \( K \) and \( \varepsilon \)).

On the other hand, if \( \gamma < 0 \) then it follows from Lemma A.7 that \( \left( 1 - P_0^\zeta(\sigma_0^\zeta \geq K \sqrt{n}) \right)^{\lfloor n^d \rfloor} \leq (1 - P_0^\zeta(\sigma_0^\zeta = \infty))^{\lfloor n^d \rfloor} \leq (1 - c)^{\lfloor n^d \rfloor} \) for some \( c > 0 \). In either case, these bounds on the first term are enough to complete the proof.

**Proof of Lemma 2.2.** First of all, note that \( \mathcal{L}(x, k-1) \geq \max\{\mathcal{D}^k(x), \mathcal{E}^k(x)\} \). Therefore,

\[
P \left( \sup_{k \leq nt} \sum_{x \in [I_{k-1}^X, S_{k-1}^N]} 1_{\{\mathcal{L}(x, k-1) \leq M\}} \geq 4n^b \right)
\]

\[
\leq P \left( \sup_{k \leq nt} \sum_{x \in [I_k^X, S_k^N]} 1_{\{\mathcal{E}^k(x) \leq M\}} \geq 2n^b \right) + P \left( \sup_{k \leq nt} \sum_{x \in [I_k^X, S_k^N]} 1_{\{\mathcal{D}^k(x) \leq M\}} \geq 2n^b \right).
\]
Proof of Lemma 3.4. We start with the proof of the statement. It then follows from (74) and Lemma B.2 that this last expression vanishes as \( n \to \infty \) since the second probability can be handled by a symmetric argument.

Due to (i)–(iii), for \( \ell \in \{M, M+1\} \), the process \( \tau_{z,m} \leq K \sqrt{n} \) is tight (see [109] Corollary 1A or Proposition 2.1 above), given any \( \varepsilon > 0 \) we can choose \( K \) large enough so that \( P(\max_{k \leq nt} |X_k| > K \sqrt{n}) < \varepsilon \) and thus

\[
P\left( \sup_{k \leq nt} \sum_{x \in [X_k, S_k^x]} \mathbb{1}_{\{E^k(x) \leq M\}} \geq 2n^b \right)
\]

\[
\leq \varepsilon + \sum_{|z| \leq K \sqrt{n}} \sum_{k \leq nt} \mathbb{P}(\tau_{z,m} = k, \sum_{x \in [z, S_k^x]} \mathbb{1}_{\{E^k(x) \leq M\}} \geq 2n^b)
\]

\[
\leq \varepsilon + \sum_{|z| \leq K \sqrt{n}} \sum_{k \leq nt} \left\{ \mathbb{P}\left( \sum_{x \in [z \land 0, -1]} \mathbb{1}_{\{E^x(x) \leq M\}} \geq n^b \right) + \mathbb{P}\left( \sum_{x \geq z \lor 0} \mathbb{1}_{\{E^x(x) \leq M\}} \geq n^b - 1 \right) \right\}.
\]

Since the process \( \{E^x(x)\}_{x \in [z \land 0, -1]} \) is distributed like the BLP \( \zeta \) and the process \( \{E^x(x)\}_{x \geq z \lor 0} \) is distributed like the BLP \( \zeta \) (both with random initial conditions), we have that this last sum is bounded above by

\[
(K + 1)tn^{5/2} \left\{ \sup_{m \geq 0} P_m^C \left( \sum_{i \leq K \sqrt{n}} \mathbb{1}_{\{\zeta_i \leq M\}} \geq n^b \right) + \sup_{m \geq 1} P_m^C \left( \sum_{i = 0}^{\zeta_i - 1} \mathbb{1}_{\{\zeta_i \leq M\}} \geq n^b - 1 \right) \right\}.
\]

It then follows from (74) and Lemma B.2 that this last expression vanishes as \( n \to \infty \) for any fixed \( K, M, t > 0 \) and \( b > \frac{M}{2} \). Since \( \varepsilon > 0 \) was arbitrary, this completes the proof of the lemma.

**APPENDIX C. PROOFS OF TECHNICAL RESULTS - POLYNOMIALLY SELF-REPELLING CASE**

**Proof of Lemma 3.3** We start with the proof of the \( P^W_{\text{a.s.}} \) continuity of \( G_{\delta,M} \). Fix \( \delta > 0 \) and \( M \geq 0 \). Let \( \mathcal{C} \) be a subset of \( C([0, \infty)) \) which consists of functions \( \omega \) such that

(i) \( \text{meas}\{t \geq 0 : \omega(t) \in [0, \delta]\} = \infty; \)

(ii) for \( \ell \in \{M, M+1\} \), \( T_{\delta,\ell}(\omega) = T_{\delta,\ell}(\omega); \)

(iii) \( \text{meas}\{t \geq 0 : \omega(t) \in [0, \delta, 1]\} = 0, \)

where \( \text{meas}(\cdot) \) denotes Lebesgue measure. Property (i) guarantees that \( T_{\delta,\ell}(\omega) < \infty \) for all \( \ell \geq 0 \). Property (ii) ensures that \( T_{\delta,\ell}(\omega) \) is continuous in \( \ell \) at \( \ell \in \{M, M+1\} \) (it is right-continuous in \( \ell \) by the definition). Observe that \( P^W_{\text{a.s.}}(\mathcal{C}) = 1. \)

**Step 1.** Due to (i)–(iii), for \( \ell \in \{M, M+1\} \), the functional \( T_{\delta,\ell} \) is continuous on \( D([0, \infty)) \) at every \( \omega \in \mathcal{C} \). More precisely, given \( \omega \in \mathcal{C} \) and \( \ell \in \{M, M+1\} \), put \( R(\ell, \omega) = T_{\delta,\ell}(\omega) + 1 \). Then for \( \varepsilon > 0 \) there is a \( \lambda > 0 \) (depending on \( \varepsilon, \delta, \) and \( \omega \)) such that for each \( \omega' \in D([0, \infty)) \) satisfying \( \sup_{t \leq R(\ell, \omega)} |\omega'(t) - \omega(t)| < \lambda \), we have \( |T_{\delta,\ell}(\omega) - T_{\delta,\ell}(\omega')| < \varepsilon. \)
Step 2. Fix $\varepsilon > 0$, $\omega \in \mathcal{C}$, and let $R = R(M + 1, \omega)$. Then for every $\omega' \in D([0, \infty))$ such that $\sup_{t \leq R} |\omega'(t) - \omega(t)| < \lambda$ where $\lambda$ is sufficiently small we get

$$|G_{\delta, M}(\omega) - G_{\delta, M}(\omega')| \leq \int_{T_{\delta, M}(\omega)}^{T_{\delta, M+1}(\omega)} |1_{[0,1]}(\omega(t)) - 1_{[0,1]}(\omega'(t))| \, dt$$

$$+ |T_{\delta, M}(\omega) - T_{\delta, M}(\omega')| + |T_{\delta, M+1}(\omega) - T_{\delta, M+1}(\omega')|$$

We are left to estimate the first term in the right hand side. It is equal to

$$\int_{T_{\delta, M}(\omega)}^{T_{\delta, M+1}(\omega)} |1_{[0,1]}(\omega(t)) - 1_{[0,1]}(\omega'(t))| \, dt$$

$$\leq \int_{T_{\delta, M}(\omega)}^{T_{\delta, M+1}(\omega)} 1_{[\lambda, 1-\lambda, 1+\lambda]}(\omega(t)) \, dt \leq \int_{0}^{R} 1_{[\lambda, 1-\lambda, 1+\lambda]}(\omega(t)) \, dt \leq \varepsilon,$$
We just need to get a bound on \( \max_{x \in [0,1]} E^B_{x^t} \left[ (T_{\delta,1})^p \right] \) uniformly over all small \( \delta > 0 \). Since \( L^{B_\delta}_{t}(0) \to \infty \) as \( t \to \infty \) with probability 1, there is a \( t > 0 \) such that
\[
P_{0}^{B_\delta} (L_{t}(0) > 2) \geq \frac{3}{4},
\]
which implies \( P_{0}^{B_\delta} \left( r_{2} \leq t \right) \geq \frac{3}{4} \).

Since \( T_{\delta,1} \implies r_{1}^{L(0)} \) as \( \delta \to 0 \), we may choose \( \delta_{0} > 0 \) so that \( P_{0}^{B_\delta} \left( T_{\delta,1} \leq r_{2}^{L(0)} \right) \geq 3/4 \) for all \( \delta \in (0, \delta_{0}) \). This will imply that
\[
P_{0}^{B_\delta} (T_{\delta,1} \leq t) \geq P_{0}^{B_\delta} \left( T_{\delta,1} < r_{2}^{L(0)} \right) \geq \frac{1}{2} \text{ for all } \delta \in (0, \delta_{0}].
\]

To move the starting point from 0 to \( x \) we note that by the strong Markov property,
\[
P_{x}^{B_\delta} (T_{\delta,1} \leq t + 1) \geq P_{1}^{B_\delta} (\tau_{0} < 1) P_{0}^{B_\delta} (T_{\delta,1} \leq t) \geq \frac{1}{2} P_{1}^{B_\delta} (\tau_{0} < 1) =: c > 0,
\]
for all \( x \in [0,1] \), and thus again by the strong Markov property,
\[
\max_{x \in [0,1]} P_{x}^{B_\delta} (T_{\delta,1} > n(t+1)) \leq (1 - c)^n, \text{ for all } n \in \mathbb{N}.
\]

This immediately gives the desired upper bound and completes the proof. \( \square \)

**Lemma C.1.** Let \( Z^{(\alpha,0)} \) be the stochastic process defined by (18) which starts at \( s > 0 \) and is absorbed upon hitting 0. Then for all \( y, \alpha \in [0, \infty) \),
\[
E \left( \int_{0}^{y} Z^{(\alpha,0)}(x) \, dx \right) = y s, \quad \text{Var} \left( \int_{0}^{y} Z^{(\alpha,0)}(x) \, dx \right) = \frac{2y^3s}{3(1+2\alpha)}.
\]

**Proof.** The first statement is immediate from (18). Integration by parts gives
\[
\int_{0}^{y} Z^{(\alpha,0)}(x) \, dx - y s = \frac{1}{\sqrt{2\alpha + 1}} \int_{0}^{y} (y - x)^{\alpha} \sqrt{2Z^{(\alpha,0)}(x)} \, dB(x).
\]

By Itô’s isometry and again (18) we get that
\[
\text{Var} \left( \int_{0}^{y} Z^{(\alpha,0)}(x) \, dx \right) = \frac{2}{2\alpha + 1} \int_{0}^{y} (y - x)^{2\alpha} E \left[ Z^{(\alpha,0)}(x) \right] \, dx = \frac{2y^3s}{3(1+2\alpha)}.
\]

**Proof of Lemma 4.6.** Without loss of generality we can assume that \( M \in \mathbb{N} \) and \( y \geq 1 \). Let
\[
A^{+} := \left\{ \sup_{\tau_{Mn}^{\alpha} < i \leq \tau_{(M+1)n}^{\alpha}} (\mathcal{D}_{i} - \mathcal{D}_{x_{Mn}^{\alpha}}) \geq y\sqrt{n} \right\}
\]
and
\[
A^{-} := \left\{ \inf_{\tau_{Mn}^{\alpha} < i \leq \tau_{(M+1)n}^{\alpha}} (\mathcal{D}_{i} - \mathcal{D}_{x_{Mn}^{\alpha}}) \leq -y\sqrt{n} \right\},
\]
and note that we need to bound \( P(A^{+} \cup A^{-}) \). Since
\[
P \left( A^{+} \cup A^{-} \right) \leq P \left( (A^{+} \cup A^{-}) \cap \left\{ |\mathcal{D}_{x_{Mn}^{\alpha}}| \leq y\sqrt{Mn} \right\} \right) + P \left( |\mathcal{D}_{x_{Mn}^{\alpha}}| > y\sqrt{Mn} \right),
\]
and since the last probability is bounded as necessary by Lemma 4.1 we only need to estimate the first term on the right hand side of the above inequality. We have
\[
P \left( (A^{+} \cup A^{-}) \cap \left\{ |\mathcal{D}_{x_{Mn}^{\alpha}}| \leq y\sqrt{Mn} \right\} \right) \leq P \left( A^{+} \left| |\mathcal{D}_{x_{Mn}^{\alpha}}| \leq y\sqrt{Mn} \right. \right) + P \left( A^{-} \left| |\mathcal{D}_{x_{Mn}^{\alpha}}| \leq y\sqrt{Mn} \right. \right).
\]

Due to the self-repelling property of the model, the probability of \( A^{+} \) is a non-increasing function of \( \mathcal{D}_{x_{Mn}^{\alpha}} \) (recall that the discrepancy is the number of red balls drawn minus the number of blue
balls drawn, and the more red balls drawn - the smaller the probability to draw a red ball in the future). Similarly, the probability of \( A^- \) is a non-decreasing function of \( D_{r,M} \). Therefore,

\[
P\left((A^+ \cup A^-) \cap \{ |D_{r,M}| \leq y\sqrt{Mn} \}\right) \leq P\left(A^+ \mid D_{r,M} = -\lfloor y\sqrt{Mn}\rfloor\right) + P\left(A^- \mid D_{r,M} = \lfloor y\sqrt{Mn}\rfloor\right).
\]

The estimates of the last two probabilities are very similar, so we shall only bound the first of them.

We think of \( D = (D_i)_{\tau_{Mn} \leq i \leq \tau_{Mn+1,n}} \) as a nearest neighbor walk which goes up each time a red ball is drawn and down if a blue ball is drawn. The walk starts at \(-\lfloor y\sqrt{Mn}\rfloor\) at time \( \tau_{Mn} \). We note that every \( D \)-walk path in \( A^+ \) must hit \( a := -\lfloor y\sqrt{Mn}\rfloor + \lfloor y\sqrt{n}\rfloor \) in no more than \( 2(n-1) + \lfloor y\sqrt{n}\rfloor \) steps, and thus we shall estimate the probability of the “larger event”

\[
A_1^+ := \left\{ \sup_{\tau_{Mn} \leq i \leq \tau_{Mn} + 2n + \lfloor y\sqrt{n}\rfloor} (D_i - D_{r,M}) \geq y\sqrt{n}\right\} \supset A^+.
\]

To this end, we write

\[
P(A_1^+ \mid \{D_{r,M} = -\lfloor y\sqrt{Mn}\rfloor\})
\]

\[
(79) \quad \leq P\left(A_1^+ \cap (B^-)^c \mid \{D_{r,M} = -\lfloor y\sqrt{Mn}\rfloor\}\right) + P(B^- \mid \{D_{r,M} = -\lfloor y\sqrt{Mn}\rfloor\}),
\]

where

\[
B^- := \left\{ \inf_{\tau_{Mn} \leq i \leq \tau_{Mn} + 2n + \lfloor y\sqrt{n}\rfloor} (D_i - D_{r,M}) \leq -\lfloor y\sqrt{Mn}\rfloor \right\}.
\]

We will bound probabilities in (79) by coupling the \( D \)-walk with appropriate simple random walks, and, thus, in the remainder of the proof we will denote a simple random walk which starts at 0 and steps to the right with probability \( q \in (0,1) \) on each step by \( S_q = \{S_{q,n}\}_{n \geq 0} \).

For the first probability in (79), note that as long as \( \exists D_i \leq 0 \) the probability that the \( (i+1) \)-th step is up is at least \( 1/2 \). Thus, we can couple the shifted \( D \)-walk \((D_{i} + \lfloor y\sqrt{Mn}\rfloor)_{i \geq 1}\) with a simple symmetric random walk \( S_{1/2} \) so that the shifted \( D \)-walk is bounded below by \( S_{1/2} \) until the simple random walk \( S_{1/2} \) goes above \( \lfloor y\sqrt{Mn}\rfloor \). From this we obtain that

\[
P\left(B^- \mid \{D_{r,M} = -\lfloor y\sqrt{Mn}\rfloor\}\right) \leq P\left(\max_{k \leq 2n + \lfloor y\sqrt{n}\rfloor} \left| S_{1/2,k} \right| \geq \lfloor y\sqrt{Mn}\rfloor\right)
\]

\[
\leq 4P\left(S_{1/2,2n} \geq \lfloor y\sqrt{Mn}\rfloor\right)
\]

where the second inequality follows from the reflection principle and the assumption that \( y \leq \sqrt{n} \).

Finally, applying Chebyshev’s inequality to the last probability on the right we obtain that there exist constants \( M_0, c_2 \in (0,\infty) \) such that

\[
P\left(B^- \mid \{D_{r,M} = -\lfloor y\sqrt{Mn}\rfloor\}\right) \leq \frac{1}{c_2} e^{-c_2 y^2} n \quad \text{for all } M \geq M_0, n \geq 1 \text{ and } y \leq \sqrt{n}.
\]

It is left to bound the first term in (79). To this end, we first note that given \( D_i \), the probability that the \( D_{i+1} - D_i \) (i.e., the \( D \)-walk takes a jump up) is equal to

\[
\frac{(2\mathcal{B}_i + 2)^\alpha}{(2\mathcal{B}_i + 2)^\alpha + (2(\mathcal{B}_i + D_i) + 1)^\alpha} = \frac{1}{1 + \left(\frac{D_i - \mathcal{B}_i}{\mathcal{B}_i + 1}\right)^\alpha}.
\]

\[1\] At the hitting time of \( a \) the number of up steps taken is equal to the number of down steps plus \( \lfloor y\sqrt{n}\rfloor \) but due to the restriction imposed by \( A^+ \) the number of down steps taken is at most \( n - 1 \), so the total number of steps taken up to this hitting time does not exceed \( 2(n-1) + \lfloor y\sqrt{n}\rfloor \).
Since for every $D$-walk path in the event $A_1^+ \cap (B^-)^c$ we have $D_i \geq -2y\sqrt{Mn}$ and $\mathfrak{B}_i \geq Mn$, we have that the probability of jumping up can always be bounded above by

$$P \left( A_1^+ \cap (B^-)^c \mid D_{\frac{n}{M}} = -\lfloor y\sqrt{Mn} \rfloor \right) \leq 2P \left( \frac{\max_{k \leq 2n + |y\sqrt{Mn}|} S_{q_y,k} \geq y\sqrt{n}}{\sqrt{Mn}} \right),$$

where the second inequality follows again from a reflection principle and the fact that $\max_{k \leq 2n + |y\sqrt{Mn}|} S_{q_y,k} \geq y\sqrt{n}$, and we can conclude from Hoeffding’s inequality that there is a $c_4 > 0$ such that

$$P \left( A_1^+ \cap (B^-)^c \mid D_{\frac{n}{M}} = -\lfloor y\sqrt{Mn} \rfloor \right) \leq 2P \left( S_{q_y,2n + |y\sqrt{Mn}|} - E[S_{q_y,2n + |y\sqrt{Mn}|}] \geq \frac{y\sqrt{n}}{2} \right) \leq \frac{1}{c_3}e^{-c_3y^2},$$

for all $M \geq (12k_0)^2$, $n \geq 1$ and $y \leq \sqrt{n}$.

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\footnote{Note that the constant $q_y$ depends on the parameters $\alpha$, $M$ and $n$ as well, but we emphasize only the dependence on $y$ because our final bound will be depending only on $y$ and uniform over $M \geq M_0$ and $n \geq 1$.}
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