New Symmetric Periodic Solutions for the Maxwell-Bloch Differential System

M. R. Cândido· J. Llibre· C. Valls

Received: 4 October 2018 / Accepted: 20 May 2019 / Published online: 3 June 2019
© Springer Nature B.V. 2019

Abstract
We provide sufficient conditions for the existence of a pair of symmetric periodic solutions in the Maxwell-Bloch differential equations modeling laser systems. These periodic solutions come from a zero-Hopf bifurcation studied using recent results in averaging theory.

Keywords Maxwell-Bloch · Averaging theory · Periodic solutions · Zero-Hopf bifurcations

Mathematics Subject Classification (2010) 34C29 · 37C27

1 Introduction and Statement of the Main Result

In nonlinear optics the Maxwell–Bloch equations are used to describe laser systems. These equations were obtained by coupling the Maxwell equations with the Bloch equation (a linear Schrödinger like equation which describes the evolution of atoms
resonantly coupled to the laser field), see [1]. Now in MathSciNet appear 265 articles related with these equations, see for instance [4–6, 9–11].

Recently in [7] it was studied the weak foci and centers of the Maxwell-Bloch system

\[
\begin{align*}
\dot{u} &= -au + v, \\
\dot{v} &= -bv + uw, \\
\dot{w} &= -c(w - \delta) - 4uv.
\end{align*}
\] (1)

For \( c = 0 \) the differential system (1) has a singular line \( \{(u, v, w)|u = 0, v = 0\} \); for \( c \neq 0 \) and \( ac(\delta - ab) < 0 \) the differential system (1) has one equilibrium \( p_0 = (0, 0, \delta) \); and for \( c \neq 0 \) and \( ac(\delta - ab) > 0 \) the differential system (1) has three equilibria \( p_+ = (u^*, v^*, w^*) \), \( p_- = (-u^*, -v^*, w^*) \) and \( p_0 \), where

\[
\begin{align*}
u^* &= \sqrt{\frac{c(\delta - ab)}{4a}}, \\
v^* &= a\sqrt{\frac{c(\delta - ab)}{4a}}, \\
w^* &= ab.
\end{align*}
\]

For \( a = \delta = 0 \) the differential system has the singular line \( L = \{(u, v, w)|v = 0, w = 0\} \). The periodic orbits bifurcating from the equilibrium \( p_0 \) was studied in [2]. Here we complete this study analyzing the periodic orbits which bifurcate form the other two singularities.

We define a zero-Hopf equilibrium of a 3-dimensional autonomous differential system as an equilibrium point having two purely conjugate imaginary eigenvalues and a zero eigenvalue. The next result characterizes the zero-Hopf equilibria of system (1) that lies over the singular line \( L \).

**Proposition 1** Consider \( a = \delta = 0, c = -b, \omega \in (0, \infty) \) and

\[
q_\pm = \left( \pm \frac{1}{2} \sqrt{b^2 + \omega^2}, 0, 0 \right) \in L.
\]

The only zero-Hopf equilibria of system (1) in the singular line \( L \) are \( q_\pm \).

Proposition (1) is proved in Section 2. Perturbing the condition \( a = \delta = 0 \) the line of singularity \( L \) disappears. However the next result shows that there are two equilibrium points in \( L \) which produce an isolated periodic solution due to a zero-Hopf bifurcation.

**Theorem 2** Let \( \omega \in (0, \infty) \),

\[
\begin{align*}
a &= \varepsilon^3 a_3, \\
b &= b_0 - \varepsilon c_1 - \varepsilon^2 c_2 + \varepsilon^3 b_3, \\
c &= -b_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \varepsilon^3 \left( a_3 - b_3 + \frac{2a_3 b_0^2}{\omega^2} \right), \\
\delta &= -\varepsilon^3 a_3 \omega^2 b_0.
\end{align*}
\]
with \((a_3, b_0, b_3, c_1, c_2) \in \mathbb{R}^5\) and \(\varepsilon\) a small parameter. Then for \(|\varepsilon| \neq 0\) sufficiently small the Maxwell-Bloch differential system (1) has two symmetric isolated periodic solutions bifurcating from the equilibrium points \(q_\pm \in L\) when \(\varepsilon = 0\) and \(2a_3c_1(1 + 6b_0^2/(\omega^2 - 5b_0^2)) > 0\).

Theorem 2 is proved in Section 2.

### 2 The Proofs

**Proof of Proposition 1** Consider \(q = (\bar{u}, 0, 0) \in L\). In the following we discuss the conditions for \(q\) being a zero-Hopf equilibrium point of system (1). The characteristic equation at \(q\) is given as

\[
-\lambda^3 + \lambda^2(-b - c) + \lambda(-bc - 4\bar{u}^2) = 0. \tag{2}
\]

It is easy to check that (2) has the pair of pure imaginary roots \(\pm i\omega (\omega > 0)\) if and only if \(\bar{u} = \pm \frac{1}{2}\sqrt{b^2 + \omega^2}\) and \(c = -b\).

Since system (1) is invariant under the transformation \((x, y, z) \rightarrow (-x, -y, z)\) we proceed the proof only for the point \(q_-\).

**Proof of Theorem 2** Assuming the conditions of Theorem 2 and translating \(q_-\) to the origin of coordinates, the differential system (1) writes

\[
\begin{align*}
\dot{u} &= v + \varepsilon \frac{a_3}{2} \left( \sqrt{b_0^2 + \omega^2} - 2u \right), \\
\dot{v} &= -\frac{w}{2} \sqrt{b_0^2 + \omega^2} - b_0 v + uv + \varepsilon c_1 v + \varepsilon^2 c_2 v - \varepsilon^3 b_3 v, \\
\dot{w} &= 2v \sqrt{b_0^2 + \omega^2} + b_0 w - 4uv - \varepsilon c_1 w - \varepsilon^2 c_2 w \\
&\quad + \varepsilon^3 \left( a_3 \left( w \left( -\frac{2b_0^2}{\omega^2} - 1 \right) + \omega^2 \right) + b_3 w \right) \\
&\quad - \varepsilon^4 \frac{a_3 c_1 \omega^2}{b_0} - \varepsilon^5 \frac{a_3 c_2 \omega^2}{b_0} \\
&\quad + \varepsilon^6 \frac{a_3 (\omega^2(b_3 - a_3) - 2a_3 b_0^2)}{b_0}. \tag{3}
\end{align*}
\]

In order to write the linear part of system (3) into its Jordan normal form, we do the linear change of variables \((u, v, w) \rightarrow (x, y, z)\) where

\[
(u, v, w) = \left( z - \frac{x}{\omega}, y, -\frac{2(b_0 y + x \omega)}{\sqrt{b_0^2 + \omega^2}} \right).
\]
The differential system (3) becomes

\[
\begin{align*}
    \dot{x} &= -\omega y + \frac{2(\omega z - x)}{\omega^2 \sqrt{b_0^2 + \omega^2}} \left(2b_0^2 y + b_0 x \omega + y \omega^2\right) - \frac{c_1 \varepsilon (2b_0 y + x \omega)}{\omega} - \frac{c_2 \varepsilon^2 (2b_0 y + x \omega)}{\omega} \\
    + \varepsilon^3 \left(-\frac{a_3 (2b_0^2 + \omega^2) (b_0 y + x \omega)}{\omega^3} - \frac{1}{2} a_3 \omega \sqrt{b_0^2 + \omega^2} + b_3 \left(\frac{2b_0 y}{\omega} + x\right)\right) \\
    &\quad + \frac{a_3 c_1 \omega \epsilon^4}{2b_0} \sqrt{b_0^2 + \omega^2} + \frac{a_3 c_2 \omega \epsilon^5}{2b_0} \sqrt{b_0^2 + \omega^2} \\
    &\quad + \frac{a_3 \epsilon^6}{2b_0 \omega} \sqrt{b_0^2 + \omega^2} (2a_3 b_0^2 + \omega^2 (a_3 - b_3)) \\
    \dot{y} &= \omega x + \frac{2(x - \omega z) (b_0 y + x \omega)}{\omega \sqrt{b_0^2 + \omega^2}} + c_1 y \epsilon + c_2 y \epsilon^2 - b_3 y \epsilon^3, \\
    \dot{z} &= \frac{2(\omega z - x)}{\omega^2 \sqrt{b_0^2 + \omega^2}} \left(2b_0^2 y + b_0 x \omega + y \omega^2\right) - \frac{c_1 \varepsilon (2b_0 y + x \omega)}{\omega^2} \\
    &\quad - \frac{c_2 \varepsilon^2 (2b_0 y + x \omega)}{\omega^2} \\
    &\quad + \varepsilon^3 \left(b_3 \omega^2 (2b_0 y + x \omega) - a_3 \left(2b_0^3 y + 2b_0^2 x \omega + b_0 y \omega^2 + \omega^4 z\right)\right) \\
    &\quad + \frac{a_3 c_1 \epsilon^4}{2b_0} \sqrt{b_0^2 + \omega^2} + \frac{a_3 c_2 \epsilon^5}{2b_0} \sqrt{b_0^2 + \omega^2} \\
    &\quad + \frac{a_3 \epsilon^6}{2b_0 \omega^2} \sqrt{b_0^2 + \omega^2} (2a_3 b_0^2 + \omega^2 (a_3 - b_3)) \\
    \end{align*}
\]

(4)

To study the periodic orbits of system (4) when \(0 < |\varepsilon| \ll 1\), we introduce the cylindrical coordinates \(x = R \cos \theta\), \(y = R \sin \theta\) and \(z = Z\). Doing this transformation system (4) becomes

\[
\frac{dR}{dt} = -\frac{2b_0 R^2 \cos(\theta)(b_0 \sin(2\theta) + \omega \cos(2\theta))}{\omega^2 \sqrt{b_0^2 + \omega^2}} + \frac{2b_0 RZ(b_0 \sin(2\theta) + \omega \cos(2\theta))}{\omega \sqrt{b_0^2 + \omega^2}} + O(\varepsilon)
\]

\[
= R(\theta, R, Z)
\]
\[
\frac{d\theta}{dt} = \omega + \frac{2R \cos(\theta) \left( b_0^2 (-\cos(2\theta)) + b_0^2 + b_0 \omega \sin(2\theta) + \omega^2 \right)}{\omega^2 \sqrt{b_0^2 + \omega^2}} \\
- \frac{2Z \left( b_0^2 (-\cos(2\theta)) + b_0^2 + b_0 \omega \sin(2\theta) + \omega^2 \right)}{\omega \sqrt{b_0^2 + \omega^2}} + O(\epsilon) \\
= O(\theta, R, Z), \\
\frac{dZ}{dt} = -\frac{2R^2 \cos(\theta) \left( (2b_0^2 + \omega^2) \sin(\theta) + b_0 \omega \cos(\theta) \right)}{\omega^3 \sqrt{b_0^2 + \omega^2}} \\
+ \frac{2RZ \left( (2b_0^2 + \omega^2) \sin(\theta) + b_0 \omega \cos(\theta) \right)}{\omega^2 \sqrt{b_0^2 + \omega^2}} + O(\epsilon) \\
= Z(\theta, R, Z).
\]

Rescaling the variables \((R, Z)\) of system \((5)\) as \(R = \epsilon^2 r,\ Z = \epsilon z,\) and taking \(\theta\) as the new independent we obtain the equivalent differential system

\[
\left( \frac{dr}{d\theta}, \frac{dz}{d\theta} \right) = \epsilon F_1(r, z, \theta) + \epsilon^2 F_2(r, z, \theta) + \epsilon^3 F_3(r, z, \theta) + \epsilon^4 F_4(r, z, \theta) + O(\epsilon^5),
\]

where \(F_i(r, z, \theta)\) for \(i = 1, \ldots, 4\) are given in Appendix B.

Applying Theorem 3 from Appendix A to system \((6)\) we calculate the correspondent averaging functions

\[
\begin{align*}
g_1(r, z) &= (0, 0), \\
g_2(r, z) &= (0, 0), \\
g_3(r, z) &= \left( 0, \frac{\pi \left( a_3 (b_0^2 + \omega^2) \left( 2b_0^2 c_1 - 4b_0 z \sqrt{b_0^2 + \omega^2} + c_1 \omega^2 \right) - 2b_0^2 r^2 \right)}{b_0 \omega^3 \sqrt{b_0^2 + \omega^2}} \right), \\
g_4(r, z) &= \left( 4\pi a_3 r z \sqrt{b_0^2 + \omega^2} (5b_0^2 + \omega^2) \right) \omega^5 - \frac{\pi r}{b_0 \omega^5 (b_0^2 + \omega^2)} \left( a_3 c_1 (b_0^2 + \omega^2) \right) \\
&\quad \left( 10b_0^4 + 9b_0^2 \omega^2 + \omega^4 \right) - b_0^2 r^2 \left( 5b_0^2 + 3\omega^2 \right) \right), \\
&\quad - \frac{2\pi z}{b_0 \omega^5 (b_0^2 + \omega^2)} \left( a_3 c_1 \left( b_0^2 + \omega^2 \right) \left( 12b_0^4 + 13b_0^2 \omega^2 + 2\omega^4 \right) - 2b_0^2 r^2 \left( 5b_0^2 + 4\omega^2 \right) \right) \\
&\quad + \frac{\pi}{b_0 \omega^5} \left( 2b_0 c_1 r^2 \left( 5b_0^2 + \omega^2 \right) - a_3 \left( b_0^2 + \omega^2 \right) \right) \\
&\quad \left( 6b_0^3 \left( c_1^2 + 4z^2 \right) + 2b_0 \omega \left( -b_0 c_2 + 2c_1^2 + 6z^2 \right) - c_2 \omega^4 \right) \right).
\end{align*}
\]

Note that the averaged equation \(g_3(r, z)\) vanishes over the graph

\[
\mathcal{Z} = \{ z_{\alpha} = (\alpha, \beta(\alpha)) : \alpha \in \mathbb{R}^+ \} \subset U.
\]
where \( \beta(\alpha) = \frac{a_3 c_1 (b_0^2 + \omega^2) (2b_0^2 + \omega^2) - 2b_0^2 \alpha^2}{4a_3 b_0 (b_0^2 + \omega^2)^{3/2}}. \) Furthermore the Jacobian matrix of \( g_3 \) at \( z_\alpha \) is

\[
Dg_3(z_\alpha) = \begin{pmatrix}
0 & 0 \\
-\frac{4b_0 \pi \alpha}{\omega \sqrt{b_0^2 + \omega^2}} & -\frac{4a_3 \pi (b_0^2 + \omega^2)}{\omega^3}
\end{pmatrix}. \tag{7}
\]

From (7) we have that \( \Delta_\alpha = -4a_3 \pi \left( b_0^2 + \omega^2 \right)/\omega^3 \neq 0. \) This verifies the conditions (i) and (ii) of Theorem 3. Thus we calculate the function \( f(x) \) and we get

\[
f(\alpha) = \frac{\pi b_0 \alpha (\omega^2 (\alpha^2 - 2a_3 b_0^2 c_1) - 2a_3 c_1 \omega^4 - 5b_0^2 \alpha^2)}{\omega^5 (b_0^2 + \omega^2)}.
\]

It is easy to check that \( f(\alpha) \) has the positive zero

\[
\alpha^* = \omega \sqrt{2a_3 c_1 \left( \frac{6b_0^2}{\omega^2 - 5b_0^2} + 1 \right)} \quad \text{if} \quad 2a_3 c_1 \left( \frac{6b_0^2}{\omega^2 - 5b_0^2} + 1 \right) > 0.
\]

Moreover, \( \alpha^* \) is a simple zero because \( f'(\alpha^*) = 4\pi a_3 b_0 c_1/\omega^3 \neq 0. \) Thus the result follows from applying Theorem 3. This conclude the proof. \( \square \)

**Acknowledgments** The first author is supported by FAPESP 2018/07344-0. The second author is partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grant MTM2016-77278-P (FEDER), the Agència de Gestió d’Ajuts Universitaris i de Recerca 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The third author is partially supported by FCT/Portugal through UID/MAT/04459/2013.

**Appendix A: Averaging Theory**

We consider differential systems of the form

\[
\dot{x} = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 F_3(t, x) + \varepsilon^4 F_4(t, x) + \varepsilon^5 \tilde{F}(t, x, \varepsilon), \tag{8}
\]

with \( x \) in some open subset \( \Omega \) of \( \mathbb{R}^n \), \( t \in [0, \infty) \), \( \varepsilon \in [-\varepsilon_0, \varepsilon_0] \). We assume \( F_i \) and \( \tilde{F} \) for all \( i = 1, 2, 3, 4 \) are \( T \)-periodic in the variable \( t \). Let \( x(t, z, 0) \) be the solution of the unperturbed system

\[
\dot{x} = F_0(t, x),
\]

such that \( x(0, z, 0) = z \). We define \( M(t, z) \) the fundamental matrix of the linear differential system

\[
\dot{y} = \frac{\partial F_0(t, x(t, z, 0))}{\partial x} y,
\]

such that \( M(0, z) \) is the \( n \times n \) identity matrix. The *displacement map* of system (8) is defined as

\[
d(z, \varepsilon) = x(T, z, \varepsilon) - z. \tag{9}
\]

In order to have \( d(z, \varepsilon) \) well defined we assume that for \( |\varepsilon| \neq 0 \) sufficiently small the following hypothesis holds:

\[(H) \quad \text{there exists an open set } U \subset \Omega \text{ such that for all } z \in U \text{ the solution } x(t, z, \varepsilon) \text{ is defined on the interval } [0, t(z, \varepsilon)) \text{ with } t(z, \varepsilon) > T. \]
This hypothesis is always satisfied when the unperturbed system has a manifold of $T$-periodic solutions. The standard method of averaging for finding periodic solutions consists in write the displacement map (9) in power series of $\varepsilon$ as follows

$$d(z, \varepsilon) = g_0(z) + \varepsilon g_1(z) + \varepsilon^2 g_2(z) + \varepsilon^3 g_3(z) + \varepsilon^4 g(z, \varepsilon),$$

Where for $i = 0, 1, 2, 3, 4$ we have

$$g_i(z) = M(T, z)^{-1} y_i(T, z) \frac{1}{i!},$$

being

$$y_0(t, z) = x(t, z, 0) - z,$$

$$y_1(t, z) = M(t, z) \int_0^t M(\tau, z)^{-1} F_1(\tau, x(\tau, z, 0)) d\tau,$$

$$y_2(t, z) = M(t, z) \int_0^t M(\tau, z)^{-1} \left[ 2F_2(\tau, x(\tau, z, 0)) + 2 \frac{\partial F_1}{\partial x}(\tau, x(\tau, x, 0)) y_1(\tau, z) \right. \right.$$

$$\frac{\partial^2 F_0}{\partial x^2}(\tau, x(\tau, z, 0)) y_1(\tau, z)^2 \left. \right] d\tau,$$

$$y_3(t, z) = M(t, z) \int_0^t M(\tau, z)^{-1} \left[ 6F_3(\tau, x(\tau, z, 0)) + 6 \frac{\partial F_2}{\partial x}(\tau, x(\tau, z, 0)) y_1(\tau, z)^2 \right. \right.$$

$$\left. \left. + 3 \frac{\partial F_1}{\partial x}(\tau, x(\tau, z, 0)) y_1(\tau, z)^2 \right] + 3 \frac{\partial F_0}{\partial x}(\tau, x(\tau, z, 0)) y_1(\tau, z) + 3 \frac{\partial^2 F_0}{\partial x^2}(\tau, x(\tau, z, 0)) y_1(\tau, z)^2 \right],$$

$$y_4(t, z) = M(t, z) \int_0^t M(\tau, z)^{-1} \left[ 24F_4(\tau, x(\tau, z, 0)) + 24 \frac{\partial F_3}{\partial x}(\tau, x(\tau, x, 0)) y_1(\tau, z)^3 \right.$$

$$+ 12 \frac{\partial^2 F_2}{\partial x^2}(\tau, x(\tau, z, 0)) y_1(\tau, z)^2 + 12 \frac{\partial F_2}{\partial x}(\tau, x(\tau, z, 0)) y_2(\tau, z) \right. \right.$$

$$+ 12 \frac{\partial F_1}{\partial x}(\tau, x(\tau, z, 0)) y_1(\tau, z) \bigcirc y_2(\tau, z) + 4 \frac{\partial^3 F_1}{\partial x^3}(\tau, x(\tau, z, 0)) y_1(\tau, z)^3 \right. \right.$$

$$+ 4 \frac{\partial F_1}{\partial x}(\tau, x(\tau, z, 0)) y_3(\tau, z) + 3 \frac{\partial^2 F_0}{\partial x^2}(\tau, x(\tau, z, 0)) y_2(\tau, z)^2 \right.$$

$$+ 4 \frac{\partial^2 F_0}{\partial x^2}(\tau, x(\tau, z, 0)) y_1(\tau, z) \bigcirc y_2(\tau, z) + 6 \frac{\partial^3 F_0}{\partial x^3}(\tau, x(\tau, z, 0)) y_1(\tau, z)^2 \right.$$

$$\left. \left. \left. \bigcirc y_2(\tau, z) + \frac{\partial^4 F_0}{\partial x^4}(\tau, x(\tau, z, 0)) y_1(\tau, z)^4 \right] d\tau. \right. \right. \right.$$

The functions $g_1, g_2, g_3$ and $g_4$ will be called here the averaged functions of order 1, 2, 3 and 4 respectively of system (8).

We say that system (8) has a periodic solution bifurcating from the point $z_0$ if there exists a branch of solutions $x(t, z(\varepsilon), \varepsilon)$ such that the displacement function satisfies $d(z(\varepsilon), \varepsilon) = 0$ and $z(0) = z_0$. 
Let $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ and $\pi^\perp : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ denote the projections onto the first $m$ coordinates and onto the last $n-m$ coordinates, respectively. For a point $z \in U$ we also consider $z = (a, b) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. Consider the graph
\[ Z = \{ z_\alpha = (\alpha, \beta(\alpha)) : \alpha \in V \} \subset U \]
such that $m < n$, $V$ is an open set of $\mathbb{R}^m$ and $\beta : \mathbb{V} \to \mathbb{R}^{n-m}$ is a $C^4$ function.

The next theorem provides sufficient conditions for the existence of periodic solutions of the differential system (8) when the set $Z$ is a continuum of zeros to the first non vanishing averaged equation.

**Theorem 3** Let $r \in \{0, 1, 2, 3\}$ such that $r$ is the first subindex such that $g_r \not\equiv 0$. In addition to hypothesis (H) assume that
(i) the averaged function $g_r$ vanishes on $Z$. That is $g_r(z_\alpha) = 0$ for all $\alpha \in \mathbb{V}$, and
(ii) the Jacobian matrix
\[ Dg_r(z_\alpha) = \begin{pmatrix} \Lambda_\alpha & \Gamma_\alpha \\ B_\alpha & \Delta_\alpha \end{pmatrix}, \]
where $\Lambda_\alpha = D_a \pi g_r(z_\alpha)$, $\Gamma_\alpha = D_b \pi g_r(z_\alpha)$, $B_\alpha = D_a \pi^\perp g_r(z_\alpha)$ and $\Delta_\alpha = D_b \pi^\perp g_r(z_\alpha)$, satisfies that $\det(\Delta_\alpha) \neq 0$ for all $\alpha \in \mathbb{V}$.

We define the function
\[ f(\alpha) = -\Gamma_\alpha \Delta_\alpha^{-1} \pi^\perp g_{r+1}(z_\alpha) + \pi g_{r+1}(z_\alpha), \]

Then the following statements hold.
(a) If there exists $\alpha^* \in V$ such that $f(\alpha^*) = 0$ and $\det(Df(\alpha^*)) \neq 0$, for $|\varepsilon| \neq 0$ sufficiently small, then there is an initial condition $z(\varepsilon) \in U$ such that $z(0) = z_{\alpha^*}$ and the solution $x(t, z(\varepsilon), \varepsilon)$ of system (8) is $T$-periodic.

For a proof of Theorem 3 see [3]. The ideas of the proof were first presented in [8].

**Appendix B: The Functions $F_i(r, z, \theta)$ for $i = 1, 2, 3, 4$**

In the following functions we take $S = \sin \theta$, $C = \cos \theta$, $S_2 = \sin(2\theta)$, $C_2 = \cos(2\theta)$ and $C_3 = \cos(3\theta)$.

\[ F_1(r, z, \theta) = \begin{pmatrix} \frac{2r}{2\omega^2 \sqrt{b_0^2 + \omega^2}} \left( 2b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) (b_0 S_2 + \omega C_2) \\ -a_3 \omega^2 \left( b_0^2 + \omega^2 \right) C, 0 \end{pmatrix} \]
\[ F_2(r, z, \theta) = \left( \frac{1}{4\omega^4} - \frac{1}{r (b_0^2 + \omega^2)} \right) \left( \omega S \left( a_3 \omega \left( b_0^2 + \omega^2 \right) \right) \\
+ 4r C \left( c_1 \sqrt{b_0^2 + \omega^2} - 2b_0 z \right) \right) + 2b_0 c_1 r \sqrt{b_0^2 + \omega^2} \\
+ 2b_0 r C_2 \left( 2b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) - 4r \left( b_0^2 + \omega^2 \right) \right) \\
\left( a_3 \omega \left( b_0^2 + \omega^2 \right) C - 2r \left( b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) (b_0 S_2 + \omega C_2) \right) \\
+ \frac{2\omega}{b_0 \sqrt{b_0^2 + \omega^2}} \left( C \left( b_0^2 \left( a_3 c_1 \omega^2 - 2r^2 \omega \right) + a_3 c_1 \omega^5 - 4b_0^3 r^2 S_2 \right) \\
- 2b_0 \omega \left( c_2 \sqrt{b_0^2 + \omega^2} \left( b_0^2 S_2 + \omega C_2 \right) + b_0 r C_3 \right) \right), \frac{r}{\omega^3 \sqrt{b_0^2 + \omega^2}} \\
\left( 2S \left( -b_0 c_1 \sqrt{b_0^2 + \omega^2} + 2b_0^2 \omega + 2b_0^2 z + \omega^2 z \right) \\
+ \omega C \left( 2b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) \right) \right), \\
\frac{1}{4\omega^5} \left( \frac{1}{2r^2 \omega \sqrt{b_0^2 + \omega^2}} \left( 2r \left( 2b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) (b_0 S_2 \\
+ \omega C_2) - a_3 \omega \left( b_0^2 + \omega^2 \right) C \right) \left( \frac{1}{b_0^2 + \omega^2} \left( \omega S \left( a_3 \omega \left( b_0^2 + \omega^2 \right) \right) \\
+ 4r C \left( c_1 \sqrt{b_0^2 + \omega^2} - 2b_0 z \right) \right) \right) + 2b_0 c_1 r \sqrt{b_0^2 + \omega^2} + 2b_0 r C_2 \\
\left( 2b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) - 4r \left( b_0^2 + \omega^2 \right) \right)^2 - \frac{2r \omega}{b_0 \sqrt{b_0^2 + \omega^2}} \\
\left( -\omega S \left( a_3 c_1 \omega^2 \left( b_0^2 + \omega^2 \right) - 4b_0 c_2 r \sqrt{b_0^2 + \omega^2} \\
(b_0 S + \omega C) \right) + 4b_0 r^2 \left( 2b_0^2 + \omega^2 \right) S^2 C + 8b_0^2 r^2 \omega S C^2 + 4b_0^2 r^2 \omega^2 C^3 \right) \right) \\
+ \frac{1}{b_0 r \left( b_0^2 + \omega^2 \right)} \left( \omega C \left( 4b_0^2 r^2 C_2 - a_3 c_1 \omega^2 \left( b_0^2 + \omega^2 \right) \right) + 8b_0^3 r^2 S C^2 \\
+ 2b_0 c_2 r \omega \sqrt{b_0^2 + \omega^2} \left( b_0 S_2 + \omega C_2 \right) \right) \left( \omega S \left( a_3 \omega \left( b_0^2 + \omega^2 \right) \right) \\
+ 4r C \left( c_1 \sqrt{b_0^2 + \omega^2} - 2b_0 z \right) \right) + 2b_0 c_1 r \sqrt{b_0^2 + \omega^2} + 2b_0 r C_2 \\
\left( 2b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) - 4r \left( b_0^2 + \omega^2 \right) \right) + \frac{2\omega}{b_0} \right) \right) \right) \]
\[
\left(-4a_3b_0^3r_0\omega C^2 + a_1 C \left( c_2\omega^4 \sqrt{b_0^2 + \omega^2} - 4b_0^4 S \right) - b_0 r\omega^2((a_3 - 2b_3)(b_0 S_2 + \omega C_2 + a_3\omega)) \right),
\]

\[
\frac{1}{2\omega^5} \left( \frac{\omega}{b_0 \sqrt{b_0^2 + \omega^2}} \left( a_3\omega^3 \left( b_0^2 c_1 - 2b_0 z \sqrt{b_0^2 + \omega^2} + c_1\omega^2 \right) - 2b_0 c_2 r\omega^2 \sqrt{b_0^2 + \omega^2} C - 4b_0 r \left( b_0 c_2 \omega \sqrt{b_0^2 + \omega^2} S \right) + r \left( 2b_0^2 + \omega^2 \right) SC + b_0 r \omega C^2 \right) - \frac{1}{b_0^2 + \omega^2} \left( 2S \left( -b_0 c_1 \sqrt{b_0^2 + \omega^2 + 2b_0^2 z + \omega^2} C + \omega C \left( 2b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) \right) \left( \omega S \left( a_3\omega \left( b_0^2 + \omega^2 \right) + 4r C \left( c_1 \sqrt{b_0^2 + \omega^2 - 2b_0 z} \right) \right) + 2b_0 c_1 r \sqrt{b_0^2 + \omega^2} + 2b_0 r C_2 \left( 2b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) ^{-4r z \left( b_0^2 + \omega^2 \right) } \right) \right),
\]

\[
F_4(r, z, \theta) = \left( -4r \omega^2 C^2 - S \left( b_0 \delta_3 \left( b_0^2 + \omega^2 \right) \right) + 2r \omega C \left( 4b_0 z + (b_1 - c_1) \sqrt{b_0^2 + \omega^2} \right) + 2r S \left( 4z b_0^2 + (b_1 - c_1) \sqrt{b_0^2 + \omega^2 b_0 + 2z\omega^2} \right) \right) + \frac{1}{2r^2 \omega^3} \left( -4b_0 r^2 \omega C^3 - 2r \left( 4r S b_0^2 + c_2 \omega^2 \sqrt{b_0^2 + \omega^2} \right) C^2 - \omega \left( -4b_0 r^2 S^2 - 2b_0 (b_2 - c_2) r \sqrt{b_0^2 + \omega^2} S + c_1 \delta_3 \left( b_0^2 + \omega^2 \right) \right) C \right. \\
\left. -2b_2 r \omega^2 \sqrt{b_0^2 + \omega^2} S^2 \right) \left( \frac{1}{b_0^2 + \omega^2} \left( 4r \omega^2 C^2 + S \left( b_0 \delta_3 \left( b_0^2 + \omega^2 \right) \right) + 2r \omega C \left( 4b_0 z + (b_1 - c_1) \sqrt{b_0^2 + \omega^2} \right) + 2r S \left( 4z b_0^2 + (b_1 - c_1) \sqrt{b_0^2 + \omega^2 b_0 + 2z\omega^2} \right) \right) \right)^2
\]
\[
+ \frac{2r \omega}{\sqrt{b_0^2 + \omega^2}} \left( -4r^2 \omega^2 C^3 - 8b_0 r^2 \omega S C^2 - S \left( 2 \left( 2b_0^2 + \omega^2 \right) S_2 r^2 - 2b_0 (b_2 - c_2) \omega \sqrt{b_0^2 + \omega^2} \right) + c_1 \delta_3 \omega \left( b_0^2 + \omega^2 \right) \right) + (b_2 - c_2) r \omega^2 S_2 \sqrt{b_0^2 + \omega^2} \right) \\
- \frac{1}{4r^3 \omega^4} \left( 2r \omega \left( 2b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) \right)
\]

\[C^2 + b_0 \left( \delta_3 \left( b_0^2 + \omega^2 \right) + 2r S \left( 4b_0 z + (b_1 - c_1) \sqrt{b_0^2 + \omega^2} \right) \right) - 2r \omega \left( 2b_0 z + b_1 \sqrt{b_0^2 + \omega^2} \right) S^2 \right) \left( -4r^2 S (2(b_3 - c_3) r \omega C + 2b_0 (b_3 - c_3) r S \\
- c_2 \delta_3 \sqrt{b_0^2 + \omega^2} \right) \omega^4 + \frac{2r \omega}{b_0^2 + \omega^2} \left( 4r z \omega^2 C^2 + S \left( b_0 \delta_3 \left( b_0^2 + \omega^2 \right) \right) \\
+ r \omega C \left( 4b_0 z + (b_1 - c_1) \sqrt{b_0^2 + \omega^2} \right) \\
+ 2r S \left( 4r z \omega^2 C^2 + S \left( b_0 \delta_3 \left( b_0^2 + \omega^2 \right) \right) \right) \right) \left( \frac{1}{b_0^2 + \omega^2} \right) \left( 4r z \omega^2 C^2 + S \left( b_0 \delta_3 \left( b_0^2 + \omega^2 \right) \right) + 2r \omega C \left( 4b_0 z + (b_1 - c_1) \sqrt{b_0^2 + \omega^2} \right) \\
+ 2r S \left( 4r z \omega^2 C^2 + S \left( b_0 \delta_3 \left( b_0^2 + \omega^2 \right) \right) \right) \right) \right) + \frac{1}{2r \omega^2} \left( -4r^2 \omega^2 C^3 - 8b_0 r^2 \omega S C^2 - S \left( 2 \left( 2b_0^2 + \omega^2 \right) S_2 r^2 - 2b_0 (b_2 - c_2) \omega \sqrt{b_0^2 + \omega^2} \right) \\
+ c_1 \delta_3 \omega \left( b_0^2 + \omega^2 \right) \right) + (b_2 - c_2) r \omega^2 S_2 \sqrt{b_0^2 + \omega^2} \right) \right) \right) \right), \\
\]

\[- \frac{1}{2 \omega^2} \left( -2a_3 r \omega C + 2c_3 r \omega C - 2b_0 b_3 r S + 2b_0 c_3 r S + \frac{1}{2r \omega^3} \left( b_0^2 + \omega^2 \right) \right) \left( 4b_0 r^2 \omega C^2 + 2r \left( c_2 - a_2 \right) \sqrt{b_0^2 + \omega^2} \right) \right) \right) C \]
\[
+ \frac{2r \omega}{\sqrt{b_0^2 + \omega^2}} \left( -4r^2 \omega^2 C^3 - 8b_0 r^2 \omega C^2 S - S \left( 2b_0^2 + \omega^2 \right) S_2 r^2 
- 2b_0 (b_2 - c_2 \omega \sqrt{b_0^2 + \omega^2} S + c_1 \delta_3 \omega \left( b_0^2 + \omega^2 \right) ) 
+ (b_2 - c_2 \omega \sqrt{b_0^2 + \omega^2} ) \right) ) 
+ c_2 \delta_3 \left( b_0^2 + \omega^2 \right) \right).
\]

References

1. Arecchi, F.T.: Chaos and generalized multistability in quantum optics. Phys. Scr. 9, 85–92 (1985)
2. Cândido, M.R., Llibre, J., Novaes, D.D.: Persistence of periodic solutions for higher order perturbed differential systems via Lyapunov-Schmidt reduction. Nonlinearity 30, 35–60 (2017)
3. Cândido, M.R., Llibre, J.: Stability of periodic orbits in the averaging theory: Applications to Lorenz and Thomas’ differential systems. Int. J. Bifurcat. Chaos 28, 1830007-14 (2018)
4. Casu, I., Lazareanu, C.: Stability and integrability aspects for the Maxwell-Bloch equations with the rotating wave approximation. Regul. Chaotic Dyn. 22(2), 109–121 (2017)
5. Lazareanu, C.: The real-valued Maxwell-Bloch equations with controls: from a Hamilton-Poisson system to a chaotic one. Int. J. Bifur. Chaos Appl. Sci. Engrg. 27(9), 1750143, 17 (2017)
6. Lazareanu, C.: On a Hamilton-Poisson approach of the Maxwell-Bloch equations with a control. Math. Phys. Anal. Geom. 20(3), Art. 20, 22 (2017)
7. Liu, L., Aybar, O.O., Romanovski, V.G., Zhang, W.: Identifying weak foci and centers in Maxwell–Bloch system. J. Math. Anal. Appl. 430, 549–571 (2015)
8. Llibre, J., Novaes, D.D.: Improving the averaging theory for computing periodic solutions of the differential equations. Z. Angew. Math. Phys. 66(4), 1401–1412. https://doi.org/10.1007/s00033-014-0460-3 (2015)
9. Zuo, D.W.: Modulation instability and breathers synchronization of the nonlinear Schrödinger–Maxwell–Bloch equation. Appl. Math Lett. 79, 182–186 (2018)
10. Wang, L., Wang, Z.Q., Sun, W.R., Shi, Y.Y., Li, M., Xu, M.: Dynamics of Peregrine combs and Peregrine walls in an inhomogeneous Hirota and Maxwell-Bloch system. Commun. Nonlinear Sci. Numer. Simul. 47, 190–199 (2017)
11. Wei, J., Wang, X., Geng, X.: Periodic and rational solutions of the reduced Maxwell-Bloch equations. Commun. Nonlinear Sci. Numer. Simul. 59, 1–14 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.