PREDICTION IN A NON-HOMOGENEOUS POISSON CLUSTER MODEL

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Abstract. A non-homogeneous Poisson cluster model is studied, motivated by insurance applications. The Poisson center process which expresses arrival times of claims, triggers off cluster member processes which correspond to number or amount of payments. The cluster member process is an additive process. Given the past observations of the process we consider expected values of future increments and their mean squared errors, aiming the application in claims reserving problems. Our proposed process can cope with non-homogeneous observations such as the seasonality of claims arrival or the reducing property of payment processes, which are unavailable in the former models where both center and member processes are time homogeneous. Hence results presented in this paper are significant extensions toward applications. We also give numerical examples to show how non-homogeneity appears in predictions.

1. Introduction

A Cluster point process is one of the most important classes of point processes, which has two driving processes, the process of cluster center and the process of each cluster (see e.g. Daley and Vere-Jones [3] or Westcott [17]). The Poisson cluster process is a version of cluster point processes whose process of center is a Poisson process. The process has been applied to a wide-range of different fields such as earthquake aftershocks [16], motor traffic [4], computer failure times [8] and broadband traffics [3] to name just a few. For more on the history and applications we refer to [3].

Motivated by insurance applications we will investigate the Poisson cluster process of the form,

\[ M(t) = \sum_{j=1}^{N(1)} L_j(t - T_j), \quad t \geq 1, \]

where \(0 < T_1 < T_2 < \cdots\) are points of non-homogeneous Poisson (NP for short) processes \(N(t)\) and \((L_j)\), \(j = 1, 2, \ldots\) are an iid sequence of additive processes with \(L_j(t) = 0, \text{ a.s. for } t \leq 0, \) such that \((T_j)\) and \((L_j)\) are independent. In the insurance context \(T_j \leq 1\) would be the arrival of claims within a year, and \((L_j(t - T_j))_{j: T_j \leq 1}\) are the corresponding payment processes from an insurance company to policyholders. We could also regard the cluster as the counting process of payment number. Hence \(M(t)\) would be the total number or amount of payments for the claims arriving in a year and being paid in the interval \([0, t], \ t \geq 1\). Historically such kind of stochastic process modeling goes back to Lundberg (1903) (see comments in [11, p.224]) who introduced the Poisson process for a simple claim counting process. Norberg [12, 13] has been considered to give publicity to the point process approach in a non-life insurance context.

Our focus in this paper is on the prediction of the future increments

\[ M(t, t+s) = M(t+s) - M(t), \quad t \geq 1, s > 0, \]

for some suitable \(\sigma\)-fields \(\mathcal{F}_t\) i.e. we will calculate \(E[M(t, t+s) \mid \mathcal{F}_t]\) and evaluate the mean squared error of the prediction. These kind of problems are known to be claims reserving problems, which

Key words and phrases. Poisson cluster model, prediction, conditional expectation, claims reserving, insurance, Lévy process, shot noise.

Mathematics Subject Substitution. Primary 60K30, Secondary 60G25 60G55.

This research was supported by JSPS KAKENHI Grant number 25870879 Grant-in-Aid for Young Scientists (B).
have been intensively studied from old times. We refer e.g. to Chapter 11 of Mikosch [11] for the recent development of the topic, where several interesting methods including famous chain ladder method are well explained.

In reference to prediction problems with the model (1.1), Mikosch [11] introduced the model into the claims reserving problems with a simple setting such that both the center process and clusters are homogeneous Poisson processes, where numerically tractable form of predictor $E[M(t, t + s) | M(t)]$ is also obtained. More generally, Matsui and Mikosch [9] consider Lévy or truncated compound Poisson for clusters and obtain analytic forms of both prediction and its mean squared error. Matsui [10] introduced a variation of the model (1.1) which starts randomly given number of cluster processes at each jump point of underlying process $N(t)$ and also obtain predictors and their errors. In a different path Jessen et al. [5] takes simpler but useful point process modeling for the problem. See also Rolski and Tomanek [14] which investigates asymptotics of conditional moments arising from prediction problems. Notice that almost all processes used in the the context are included in the class of Lévy processes, which implies that increments are time-homogeneous.

In this paper we introduce non-homogeneity into both underlying Poisson process $N(t)$ and clusters $L_j$ by using the additive processes such that the processes have independent but not always stationary increments. More precisely, we assume a $NP$ process for $N$, whereas each cluster $L_j$ is assumed to be an additive process which is given by a certain integral of a general Poisson random measure. Our intention here is to model the seasonality of claims arrivals and the curved line of payment numbers or amounts which are naturally observed from data (see e.g. Table 2 of [5]). Again we emphasize that in the former models [11], [9] or [10], they intensively use Lévy clusters which are the processes of stationary independent increments and therefore are time homogeneous.

This paper is organized as follows. In section 2 we consider the model with additive Lévy processes and obtain the conditional characteristic function (ch.f. for abbreviation) $E[e^{ixM(t, t+s)} | M(t)]$. Based on the derived ch.f. we investigate expressions of $E[M(t, t+s) | M(t)]$ where $NP$ clusters and non-homogeneous negative binomial clusters are considered. In both cases, we derive recursive algorithm to calculate exact values of predictors and their conditional mean squared errors. In Section 3 the prediction $E[M(t, t+s) | F_t]$ with different σ-fields $F_t$ is investigated where we notice the delay in reporting times of claims and consider the number of reported claims until time $t$ for $F_t$. Exact analytic forms for both predictors and their mean squared errors are calculated. In the final section, we give numerical examples to see how the non-homogeneity affects the predictors.

Finally, we briefly explain basics of an additive process $\{L(t)\}_{t \geq 0}$ based on Sato [15, p.53]. It is well known that the process is stochastically continuous, and has independent increments with càdlàg path starting at $L(0) = 0$ a.s. The distribution of the process $\{L(t)\}_{t \geq 0}$ at time $t$ is determined by its generating triplet $(A_t, \nu_t, \gamma_t)$ since this determines the corresponding ch.f. Among additive processes we work on the process of so called jump part such that the distribution of time $t$ is given by the inversion of

\begin{equation}
E[e^{ixL(t)}] = \exp \left\{ \int_{(0,t] \times \mathbb{R}} (e^{ixu} - 1)\nu(d(u,v)) \right\},
\end{equation}

where a measure $\nu$ on $(0,\infty) \times \mathbb{R}$ satisfies $\nu((0,\cdot] \times \{0\}) = 0$, $\nu(\{t\} \times \mathbb{R}) = 0$ and

$$
\int_{(0,t] \times \mathbb{R}} (\lfloor u \rfloor \wedge 1)\nu(d(u,v)) < \infty, \quad \text{for } t \geq 0.
$$

In this case the generating triplet is $(0, \nu_t, 0)$ with $\nu_t(B) := \nu(\{0,t\} \times B)$ for any Borel set $B \in B(\mathbb{R})$. The first condition means $\nu_t(\{0\}) = 0$ and the second one implies stochastic continuity, whereas the third controls smoothness of the path. In view of (1.2) one see that an additive process has an integral representation by Poisson random measure on $(0,\infty) \times \mathbb{R}$ with intensity measure $\nu$. We refer to Theorem 19.2 and 19.3 of Sato [15] for the jump part of an additive process. Although we could treat more general additive process by including the continuous part or another version of
jump part, the prediction procedure would be more complicated and we confine the process of the cluster as such.

2. Prediction in Poisson cluster model

In this section firstly we give general prediction results which are valid for all additive Lévy clusters given by (1.1) and then we investigate numerically tractable expressions with examples. More precisely, we study expressions of the conditional expectation of $M(t, t + s)$ given $M(t)$, $t \geq 1, s > 0$ and its mean squared error.

The main difference of our prediction from Matsui and Mikosch [9] is that we can not use the stationary increments of cluster center nor cluster member processes and hence expressions for predictors require some devices and are more complicated. However, by discarding time homogeneity of Lévy processes, we can introduce time dependency into the process of the cluster, which is of critical importance in applications.

The following is basic for the model (1.1).

Lemma 2.1. Assume the model (1.1) with iid additive processes $L_k$, $k = 1, 2, \ldots$ and a NP process $N$ with mean measure $\Lambda$ such that $\Lambda[0, \infty) < \infty$. We write the generic of processes $L_k$ as $L$. Then the ch.f. is given by

$$E[e^{ixM(t)}] = \exp \left\{ \int_{[0,1]} (E[e^{ixL(t-u)}] - 1)\Lambda(du) \right\}$$

for $t \geq 1$ and $x \in \mathbb{R}$. Moreover, assume that $E[L(t)]$ finitely exists for all $t \geq 0$, then

$$E[M(t)] = \int_{[0,1]} E[L(t-u)]\Lambda(du), \quad t \geq 1.$$

Assume that $E[L^2(t)]$ is finite for all $t \geq 0$. Then, for $1 \leq s \leq t$,

$$\text{Cov}(M(s), M(t)) = \int_{[0,1]} (E[L^2(s-u)] + E[L(s-u)]E[L(s-u,t-u)])\Lambda(du).$$

We are starting to observe the conditional ch.f. of $M(t, t + s)$ given $M(t)$.

Lemma 2.2. Assume the model (1.1) with iid additive processes $L_k$, $k = 1, 2, \ldots$ given by (1.2) and a NP process $N$ with the mean value function $\Lambda(\cdot)$. For $m = 1, 2, \ldots, s > 0$, $t \geq 1$ and $x \in \mathbb{R}$, the conditional ch.f. of $M(t, t + s)$ given $\{M(t) \in A\}$ for any Borel set $A$ has the following form

\begin{equation}
\hat{f}_A(x) = E[e^{ixM(t,t+s)} \mid M(t) \in A] = E\left[ \exp \left\{ \sum_{j=1}^{N(1)} \int_{t-V_j,t+s-V_j} \right\} \right] \frac{P\left( \sum_{j=1}^{N(1)} L_j(t-V_j) \in A \mid (V_j) \right)}{P\left( \sum_{j=1}^{N(1)} L_j(t-V_j) \in A \right)}
\end{equation}

for an iid sequence $(V_j)$ with density function

\begin{equation}
F(dx) = \Lambda(dx)/\Lambda(1), \quad 0 \leq x \leq 1
\end{equation}

such that $(V_j)$, $(L_j)$ and $N$ are mutually independent.

Proof. Since $M(t)$ is measurable with respect to $\sigma$-filed by $(T_j)$, $(L_j(t - T_j))$, we use the iteration property of conditional expectation to calculate

$$E[e^{ixM(t,t+s)} \mid M(t)] = E\left[ E\left[ \prod_{j=1}^{\infty} e^{ix(T_j \leq 1)L_j(t-T_j,t+s-T_j)} \mid (T_j), (L_j(t - T_j)) \right] \mid M(t) \right].$$
Accordingly, for any Borel set \( A \), we obtain

\[
E[e^{ixM(t,t+s)} | M(t) \in A] = \frac{E\left[\prod_{j=1}^{N(1)} E[e^{ixL(t-T_j,t+s-T_j)} | T_j] 1\{M(t) \in A\}\right]}{P(M(t) \in A)}
\]

\[
= \frac{E\left[\prod_{j=1}^{N(1)} E[e^{ixL(t-T_j,t+s-T_j)} | T_j] 1\{M(t) \in A\} | (T_j)\right]}{P(M(t) \in A)}
\]

\[
= \frac{E\left[\prod_{j=1}^{N(1)} E[e^{ixL(t-T_j,t+s-T_j)} | T_j] P(M(t) \in A | (T_j))\right]}{P(M(t) \in A)}
\]

Since quantities

\[
\prod_{j=1}^{N(1)} E[e^{ixL(t-T_j,t+s-T_j)} | T_j] \quad \text{and} \quad P\left(\sum_{j=1}^{N(1)} L_j(t-T_j) \in A | (T_j)\right)
\]

do not depend on the order of \((T_j)\), the order statistic property of Poisson yields

\[
E\left[\prod_{j=1}^{N(1)} E[e^{ixL(t-T_j,t+s-T_j)} | T_j] P\left(\sum_{j=1}^{N(1)} L_j(t-T_j) \in A | (T_j)\right) | N(1)\right]
\]

\[
= E\left[\prod_{j=1}^{N(1)} E[e^{ixL(t-V_j,t+s-V_j)} | V_j] P\left(\sum_{j=1}^{N(1)} L_j(t-V_j) \in A | (V_j)\right)\right],
\]

where \((V_j)\) is the iid sequence whose common distribution is by (2.2). Now we insert this and (1.2) into the final expression and we obtain the result.

Based on \( \hat{f}_A(x) \), we see important examples in the following subsections.

2.1. Non-homogeneous Poisson clusters. We consider the model of (1.1) with \( NP \) clusters \( L_j, j = 1, 2, \ldots \) such that the generic cluster process \( L \) at time \( t \) has c.h.f.,

\[
E[e^{iuL(t)}] = e^{\mu(t)e^{iu}-1}, \quad u \in \mathbb{R}.
\]

where \( \mu(t) := \nu((0,t] \times \{1\}) \), i.e. the measure \( \nu \) in (1.2) has the support only on \((0,t] \times \{1\}\). The measure \( \mu(t) \) is generally called the mean value function or intensity measure of the Poisson process (see Sec.19 of Sato [15] or Sec.7.2 of Mikosch [11]). By the condition \( \nu(\{t\} \times \mathbb{R}) = 0 \) (stochastic continuity) before, \( \mu(t) \) is assumed to be continuous in \( t \). Moreover, we assume that \( \mu(0,\infty) < \infty \). Notice that the Poisson process is one of the most important processes among additive processes. Besides, it is a basic process for modeling the claim reserves in the non-life insurance context.
Before constructing prediction we define some notations. Let \( \phi(Y_1, Y_2)(z_1, z_2) \) be the Laplace transform of a bivariate random variable \((Y_1, Y_2)\), \( \phi(Y_1, Y_2)(z_1, z_2) := E[e^{-z_1Y_1-z_2Y_2}] \), \( z_1 \geq 0, z_2 \geq 0 \) and denote its \((m, n)\)th partial derivatives by \( \phi^{(m,n)}(Y_1, Y_2)(z_1, z_2) \), whereas \( \phi_Y^{(m)}(t) \ldots \) denotes simply \( m \)th derivative of \( \phi_Y(z) = E[e^{-zY}] \) with \( z \geq 0 \). Throughout we use a random sum

\[
(2.4) 
R_{N(1)}(t) := \sum_{j=1}^{N(1)} \mu(t - V_j), \quad t \geq 1, 
\]

where \((V_j)\) is an iid random sequence with common density \((2.2)\).

**Lemma 2.3.** Assume the model \((1.1)\) with iid NP processes \(L_k, k = 1, 2, \ldots\) with mean value function \(\mu(\cdot)\) and a NP process \(N\) with the intensity measure \(\Lambda(\cdot)\). Then for \(m = 1, 2, \ldots\) and \(x \in \mathbb{R}\) the conditional ch.f. of \(M(t, t+s), t \geq 1, s > 0\) given \(\{M(t) = m\}\) has the following form

\[
(2.5) 
\hat{f}_m(x) = E[e^{ixM(t,t+s)} \mid M(t) = m] 
= \frac{\phi_{R_{N(1)}(t), R_{N(1)}(t+s)}(e^{ix}, 1 - e^{ix})}{\phi_{R_{N(1)}(1)}^{(m)}(1)}, 
\]

where the random element \(R_{N(1)}(1)\) is given by \((2.4)\).

**Proof.** By inserting the ch.f. \((2.3)\) into the expression \((2.1)\) in Lemma \((2.2)\) we observe

\[
E[e^{ixM(t,t+s)} \mid M(t) = m] 
= E\left[\frac{\sum_{j=1}^{N(1)} \mu(t-T_j, t+s-T_j)(e^{ix}-1)P\left(\sum_{j=1}^{N(1)} L_j(t-V_j) = m \mid (V_j)\right)}{\sum_{j=1}^{N(1)} P(L_j(t-V_j) = m)}\right]. 
\]

The aggregation property of Poisson processes (Prop. 7.3.11 of \([11]\)) yields

\[
P\left(\sum_{j=1}^{N(1)} L_j(t-V_j) = m \mid (V_j)\right) = \left(\sum_{j=1}^{N(1)} \mu(t-V_j)\right)^m \frac{m!}{m!} e^{-\sum_{j=1}^{N(1)} \mu(t-V_j)} \text{ a.s.} 
\]

from which it follows that

\[
e^{\sum_{j=1}^{N(1)} \mu(t-V_j, t+s-V_j)(e^{ix}-1)P\left(\sum_{j=1}^{N(1)} L_j(t-V_j) = m \mid (V_j)\right)} = \frac{(\sum_{j=1}^{N(1)} \mu(t-V_j))^m}{m!} e^{\sum_{j=1}^{N(1)} \mu(t+s-V_j)(e^{ix}-1) - \mu(t-V_j)e^{ix}} 
= \frac{(-1)^m}{m!} \left(\sum_{j=1}^{N(1)} \mu(t+s-V_j)(e^{ix}-1) - \mu(t-V_j)y\right)_y^{(m)} \bigg|_{y=e^{ix}}. 
\]

Now taking expectation for \((V_j)\), we obtain by Fubini’s theorem that

\[
P\left(\sum_{j=1}^{N(1)} L_j(t-T_j) = m\right) = \frac{(-1)^m \phi_{R_{N(1)}(t)}^{(m)}(1)}{m!}, 
\]

\[
E\left[e^{\sum_{j=1}^{N(1)} \mu(t-V_j, t+s-V_j)(e^{ix}-1)P\left(\sum_{j=1}^{N(1)} L_j(t-T_j) = m \mid (V_j)\right)}\right] 
= \frac{(-1)^m}{m!} \left(\phi_{R_{N(1)}(t), R_{N(1)}(t+s)}(y, 1 - e^{ix})\right)_y^{(m)} \bigg|_{y=e^{ix}} 
\]

for \(x \in \mathbb{R}\). Hence we obtain the result. \(\square\)
Now by differentiating (2.5) sufficiently often, we obtain the conditional moments.

**Theorem 2.4.** Assume the model (1.1) with iid NP processes \( L_k, k = 1, 2, \ldots \) with the mean value function \( \mu(\cdot) \) and a NP process \( N \) with the mean value function \( \Lambda(\cdot) \). Then the prediction \( \tilde{M}_m(t, t + s) \) of \( M(t, t + s) \) given \( \{M(t) = m\} \) has the form

\[
\tilde{M}_m(t, t + s) = \frac{\varphi_R^{(m+1,0)}(R_{N(1)}(t), R_{N(1)}(t+s))(1,0) - \varphi_R^{(m,1)}(R_{N(1)}(t), R_{N(1)}(t+s))(1,0)}{\varphi_R^{(m)}(R_{N(1)}(t))(1)}
\]

and the conditional variance of \( M(t, t + s) \) given \( \{M(t) = m\} \) is

\[
\text{Var}(M(t, t + s) \mid M(t) = m) = \frac{\varphi_R^{(m+2,0)}(R_{N(1)}(t), R_{N(1)}(t+s))(1,0) - 2\varphi_R^{(m+1,1)}(R_{N(1)}(t), R_{N(1)}(t+s))(1,0) + \varphi_R^{(m,2)}(R_{N(1)}(t), R_{N(1)}(t+s))(1,0)}{\varphi_R^{(m)}(R_{N(1)}(t))(1)}
\]

where \( R_{N(1)} \) is the random sum (2.4).

It is desirable to obtain an explicit expression for the unconditional mean squared error \( E[(M(t, t + s) - E[M(t, t + s) \mid M(t)])^2] \), since it gives a certain measure for evaluating goodness of predictors. However, in the light of expressions (2.6) and (2.7) it seems intractable (see Remark 2.2 of Matsui and Mikosch [9]). Hence we content with conditional moments which are provided with numerically tractable expressions.

In what follows we investigate further expressions of (2.6) and (2.7). It is convenient to observe the bivariate Laplace transform of \( (R_{N(1)}(t), R_{N(1)}(t+s)) \) i.e.

\[
\phi_{R_{N(1)}(t), R_{N(1)}(t+s)}(y, z) = E[e^{-yR_{N(1)}(t)-zR_{N(1)}(t+s)}] = E[\prod_{j=1}^{N(1)} E[e^{-y(t+V_j)-z(t+s-V_j)}] \mid N(1)] = E\left[ \left( \int_0^1 e^{-y(t-v)-z(t+s-v)} \frac{\Lambda(dv)}{\Lambda(1)} \right)^{N(1)} \right] = \exp \left\{ \int_0^1 (e^{-y(t-v)-z(t+s-v)} - 1) \Lambda(1) \right\} = e^{\Lambda(1)\phi_{R_{N(1)}(t), R_{N(1)}(t+s)}(y, z) - 1}
\]

and derivatives of \( \Lambda(1)\phi_{R_{N(1)}(t), R_{N(1)}(t+s)}(y, z) \) with respect to \( z \) at \( z = 0 \),

\[
\psi_j(y) = \Lambda(1)\phi_{R_{N(1)}(t), R_{N(1)}(t+s)}^{(0,j)}(y, 0) = \int_0^1 (-\mu(t+s-v))^j e^{-\mu(t-v)} \Lambda(1) dv, \quad j = 0, 1, 2,
\]

where we note that \( \Lambda(1)\phi_{R_{N(1)}(t)}(y) = \psi_0(y) \).

**Lemma 2.5.** Let \( \ell = 1, 2, \ldots \) and \( j = 0, 1, 2 \). Let \( \phi_{R_{N(1)}(t), R_{N(1)}(t+s)}^{(\ell,j)}(1,0) \) be the \((\ell, j)\)th partial derivative of \( \phi_{R_{N(1)}(t), R_{N(1)}(t+s)}(y, z) \) at \((y, z) = (1, 0)\) and let \( \psi_j^{(\ell)}(1) \) be the \( \ell \)th derivative of \( \psi_j(y) \) at \( y = 1 \). Then, the following recursive relations hold.

\[
\psi_j^{(\ell)}(1) = \int_0^1 (-\mu(t+s-v))^j (-\mu(t-v))^{\ell} e^{-\mu(t-v)} \Lambda(1) dv, \quad j = 0, 1, 2,
\]
Var(\text{homogeneous}) z with respect to \(1\) independent and follows (see e.g. [15, p.20]). The marginal distribution at time \((2.9)\)

\[
\begin{align*}
\phi^{(\ell,0)}_{RN(1)(t),RN(1)(t+s)}(1,0) &= \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} \phi^{(k,0)}_{RN(1)(t),RN(1)(t+s)}(1,0)\psi_0^{(\ell-k)}(1), \\
\phi^{(\ell,1)}_{RN(1)(t),RN(1)(t+s)}(1,0) &= \sum_{k=0}^{\ell} \binom{\ell}{k} \phi^{(k,0)}_{RN(1)(t),RN(1)(t+s)}(1,0)\psi_1^{(\ell-k)}(1), \\
\phi^{(\ell,2)}_{RN(1)(t),RN(1)(t+s)}(1,0) &= \sum_{k=0}^{\ell} \binom{\ell}{k} \phi^{(k,1)}_{RN(1)(t),RN(1)(t+s)}(1,0)\psi_1^{(\ell-k)}(1) + \sum_{k=0}^{\ell} \binom{\ell}{k} \phi^{(k,0)}_{RN(1)(t),RN(1)(t+s)}(1,0)\psi_2^{(\ell-k)}(1).
\end{align*}
\]

Proof. We differentiate ch.f. (2.8) to see

\[
\begin{align*}
\phi^{(1,0)}_{RN(1)(t),RN(1)(t+s)}(y,0) &= \phi_{RN(1)(t),RN(1)(t+s)}(y,0)\psi_0^{(1)}(y), \\
\phi^{(0,1)}_{RN(1)(t),RN(1)(t+s)}(y,0) &= \phi_{RN(1)(t),RN(1)(t+s)}(y,0)\psi_1(y), \\
\phi^{(0,2)}_{RN(1)(t),RN(1)(t+s)}(y,0) &= \phi^{(0,1)}_{RN(1)(t),RN(1)(t+s)}(y,0)\psi_1(y) + \phi_{RN(1)(t),RN(1)(t+s)}(y,0)\psi_2(y).
\end{align*}
\]

Applications of the Leibniz’s rule to these quantities yield our desired results. \(\Box\)

2.2. Non-homogeneous negative binomial clusters. We consider a negative binomial \(NB\) for short) process for the generic random process \(L\) of clusters \(L_k, k = 1, 2, \ldots\) Let \(\mu(t) > 0\) be a continuous function of time \(t\) with \(\mu(0) = 0\) and let \(p \in (0, 1)\) so that \(q = 1 - p\). \((1.2)\)

is constructed with \(\nu(d(u,v)) = \mu(du) \times \sigma(dv)\) where \(\sigma\) is concentrated on positive integer and \(\sigma(\{k\}) = k^{-1}q^k, k = 1, 2, \ldots\) Accordingly the ch.f. of \(L(t)\) has the form

\[
E[e^{ixL(t)}] = \left(\frac{p}{1-qe^{ix}}\right)^{\mu(t)}, \quad x \in \mathbb{R}
\]

(see e.g. [15] p.20). The marginal distribution at time \(t\) of the process follows \(NB\) with parameters \(\mu(t)\) and \(p\) (we also write \(NB(\mu(t),p)\) for abbreviation) i.e.

\[
P(L(t) = k) = \binom{\mu(t) + k - 1}{k}p^{\mu(t)}q^k \quad k = 1, 2, \ldots
\]

such that the mean and variance of the process are respectively given by \(E[L(t)] = \mu(t)q/p\) and \(\text{Var}(L(t)) = \mu(t)q/p^2\). The distributions of increments \(L(t) - L(s), 0 \leq s \leq t < \infty\) are mutually independent and follows \(NB(\mu(t) - \mu(s), p)\). Although there exist only a few references for non-homogeneous \(NB\) process, e.g. Carrillo [2], for homogeneous \(NB\) process, detailed distributional properties are given in e.g. Kozubowski and Podgorski [7] (see also Johnson et al. [6]).

Throughout we use the bivariate probability generating function \(G_{Y_1,Y_2}(z_1, z_2) := E[z_1^{Y_1}z_2^{Y_2}], |z_1z_2| \leq 1, (z_1, z_2) \in \mathbb{C}^2\) of an integer valued random vector \((Y_1, Y_2)\), and its \((k,\ell)\)th derivatives \(G^{(k,\ell)}_{Y_1,Y_2}(z_1, z_2)\) with respect to \((z_1, z_2), \{k,\ell\} = 1, 2, \ldots\) Moreover, the notation \((\cdot)^{(m)}\) denotes the \(m\)th derivative of the quantity in the brace. We again use the random sum (2.4) where \(\mu\) is replaced by that of \(NB(\mu(t), p)\). We abbreviate \(G_{RN(1)(t),RN(1)(t+s)}(z_1, z_2)\) to \(G_{t,t+s}(z_1, z_2)\) throughout this section.

It is convenient to start with the conditional ch.f. of \(M(t, t+s]\) given \(M(t)\).
Lemma 2.6. Assume the model (1.1) with iid NB(\(\mu(t), p\)) additive clusters \(L_k, k = 1, 2, \ldots\) such that \(\mu(t) > 0\) is continuous and \(p \in (0, 1)\). Then conditional ch.f. of \(M(t, t + s)\) given \(\{N(t) = m\}\) has the form,

\[
\hat{f}_m(x) = \frac{1}{p(p^{m-1}G_{R_N(1)}(p))^m} \sum_{k=0}^{m-1} \binom{m}{k} \frac{(m-1)!}{(m-k-1)!} (1 - qe^{ix})^{m-k}G_{t,s}^{(m-k,0)}(1 - qe^{ix}, p/(1 - qe^{ix})),
\]

where \(R_{N(1)}\) is the random sum of (2.4) and \(q = 1 - p\).

Proof. We apply (2.9) to the expression (2.1) of Lemma 2.2, which yields the conditional ch.f. for NB clusters as

\[
E[e^{ixM(t,t+s)} | M(t) = m] = E\left[\frac{(p/(1 - qe^{ix}))^{\sum_{j=1}^{N(1)}(t-V_j,t+s-V_j)}P\left(\sum_{j=1}^{N(1)}L_j(t - V_j) = m | (V_j)\right)}{P\left(\sum_{j=1}^{N(1)}L_j(t - V_j) = m\right)}\right].
\]

Since \(N(1)\) is measurable with the \(\sigma\)-filed by \((V_j)\) and since \((L_j)\) is independent of \((V_j)\), it follows from (2.10) that

\[
P(\sum_{j=1}^{N(1)}L_j(t - V_j) = m | (V_j)) = \frac{p(1-p)^{m-1}(R_{N(1)} + m - 1)!}{(R_{N(1)} - 1)!} p^{R_{N(1)} - 1}
\]

\[
= \frac{p(1-p)^m}{m!} (p^{R_{N(1)}+m-1})(m).
\]

We apply a similar calculation to the enumerator to obtain

\[
(p/(1 - qe^{ix}))^{\sum_{j=1}^{N(1)}(t-V_j,t+s-V_j)}P\left(\sum_{j=1}^{N(1)}L_j(t - V_j) = m | (V_j)\right)
\]

\[
= (\gamma^{R_{N(1)}(t)+m-1})(\gamma = 1 - qe^{ix} (1 - qe^{ix})(p/(1 - qe^{ix}))^{R_{N(1)}(t+s)}(1 - p)^m)
\]

\[
= \frac{(1-p)^m}{m!}(1 - qe^{ix})^{m-1}G_{t,s}^{(m-k,0)}(\gamma, p/(1 - qe^{ix}))^{(m)} |_{\gamma = 1 - qe^{ix}}.
\]

Now taking expectation for both quantities under notations of differentiation, which is justified by Fubini’s theorem, we conclude the result.

Differentiation of the conditional ch.f. at \(x = 0\) several times yields the following result.

Theorem 2.7. Let \(L\) be NB(\(\mu(t), p\)) process such that \(\mu(t) > 0\) is continuous and \(p \in (0, 1)\). Then the prediction \(M(t, t + s)\) given \(\{M(t) = m\}\) is

\[
\hat{M}_m(t, t + s) = \frac{1}{p(p^{m-1}G_{R_N(1)}(p))^m} \sum_{k=0}^{m-1} \binom{m}{k} \frac{(m-1)!}{(m-k-1)!} (-p^{m-k-1}q)
\]

\[
\times \left\{ (m-k)G_{t,s}^{(m-k,0)}(p, 1) + pG_{t,s}^{(m-k+1,0)}(p, 1) - G_{t,s}^{(m-k,1)}(p, 1) \right\}
\]

and the conditional mean squared error has the form

\[
\text{Var}(M(t, t + s) | M(t) = m)
\]

\[
= -\hat{M}_m^2(t, t + s) + \frac{1}{p(p^{m-1}G_{R_N(1)}(p))^m} \sum_{k=0}^{m-1} \binom{m}{k} \frac{(m-1)!}{(m-k-1)!} p^{m-k-2} \frac{q \times \left[ (m-k)(m-k)q - 1 \right] G_{t,s}^{(m-k,0)}(p, 1)}{(m-k)(m-k-1)}.
\]
Proposition 2.8.\(\) Let \(k, \ell = 1, 2, \ldots\) and \(G^{(k,\ell)}_{t,t+s}(p,1)\) be the \((k,\ell)\)th derivatives of \(G_{t,t+s}(z_1,z_2)\) at \((z_1,z_2) = (p,1)\) and let \(H_j^{(k)}(p)\), \(j = 0, 1, 2\) be the \(k\)th derivative of \(H_j(z_1)\) at \(z_1 = p\). Then the following recursive relations hold.

\[
H_j^{(\ell)}(p) = \int_0^1 \frac{\Gamma(\mu + s - v + 1)}{\Gamma(\mu + t - v + 1 - j) \Gamma(\mu + t - v + 1 - \ell)} z_1^{\mu(t-v)} \Lambda(dv)
\]

and

\[
G^{(\ell,0)}_{t,t+s}(p,1) = \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} G^{(k)}_{R^{(1)}(t)}(p) \cdot H_0^{(\ell-k)}(p)
\]

\[
G^{(\ell,1)}_{t,t+s}(p,1) = \sum_{k=0}^{\ell} \binom{\ell}{k} G^{(k)}_{R^{(1)}(t)}(p) \cdot H_1^{(\ell-k)}(p)
\]

\[
G^{(\ell,2)}_{t,t+s}(p,1) = \sum_{k=0}^{\ell} \binom{\ell}{k} \left\{ G^{(k,1)}_{t,t+s}(p) \cdot H_1^{(\ell-k)}(p) + G^{(k)}_{R^{(1)}(t)}(p) \cdot H_2^{(\ell-k)}(p) \right\}.
\]

Proof.\(\) We differentiate \(G_{t,t+s}(z_1,z_2)\) with respect to \(z_1\) and \(z_2\) proper times at \(z_1 = 1\) and obtain

\[
G^{(1,0)}_{t,t+s}(z_1,1) = G^{(1,0)}_{R^{(1)}(t)}(z_1) \cdot H_0^{(1)}(z_1)
\]

\[
G^{(0,1)}_{t,t+s}(z_1,1) = G^{(0,1)}_{R^{(1)}(t)}(z_1) \cdot H_1(z_1)
\]

\[
G^{(0,2)}_{t,t+s}(z_1,1) = G^{(0,2)}_{R^{(1)}(t)}(z_1,1) \cdot H_1(z_1) + G^{(0,1)}_{R^{(1)}(t)}(z_1) \cdot H_2(z_1)
\]

Applications of Leibniz's rule to these quantities together with (2.11) yield the result. \(\square\)

3. Prediction with delay in reporting

In this section we introduce the time difference \(D_k > 0\) between the arrival time \(T_k\) of \(k\)th claim and its reporting time, i.e. the report of \(k\)th claim is coming at time \(T_k + D_k\), and then we start the
cluster process \(L_k\). Accordingly in the model of \(L(t-T_k)\) are replaced by \(L(t-(T_k+D_k))\) and we will work with model

\[
M(t) = \sum_{j=1}^{\infty} I_{(T_j \leq 1)} L_j (t-(T_j + D_j)), \quad T_j \geq 1, \ D_j \geq 0.
\]

We assume that the generic random element \(D\) of iid sequences \((D_k)\) takes positive values with common distribution \(F_D\) such that \((D_k)\) is independent of \((L_k)\) and \(N\).

Recall that usually the total claims number \(N(1)\) may not be available at time \(t \geq 1\), while we know the reported number of claims,

\[
\hat{N}(t) = \# \{ k \geq 1 : T_k + D_k \leq t, T_k \in [0,1] \}.
\]

In what follows, we will consider the prediction \(M(t,t+s)\) based on \(\hat{N}(t)\), namely we will calculate the conditional expectation

\[
\hat{M}(t,t+s) = E[M(t,t+s) \mid \hat{N}(t) = \ell], \ \ell = 0,1,2,\ldots
\]

First we specify the distribution of \(\hat{N}(t)\). Let \(Q\) be a Poisson random measure on the space \(E = [0, 1]\times[0,\infty)\) with mean measure \(\nu = \Lambda \times F_D\). Then \(N(1)\) and \(\hat{N}(t)\) have the Poisson integral representation (c.f. Ex. 7.3.6 in Mikosch [11]),

\[
(3.1) \quad N(1) = \int_E Q(ds,dy) = \int_{s=0}^{1} \int_{y=0}^{t-s} Q(ds,dy) + \int_{s=0}^{1} \int_{y=t-s}^{\infty} Q(ds,dy)
\]

\[
(3.2) \quad = \hat{N}(t) - \lfloor N(1) - \hat{N}(t) \rfloor,
\]

where random variables \(N(1) - \hat{N}(t)\) and \(\hat{N}(t)\) are independent and Poisson distributed with parameters

\[
E[N(1) - \hat{N}(t)] = \int_{0}^{1} \int_{t-s}^{\infty} \Lambda(ds)F(dy) = \hat{\Lambda}(t) \quad \text{and} \quad E[\hat{N}(t)] = \Lambda(1) - \hat{\Lambda}(t).
\]

It is convenient to start with the conditional ch.f. \(M(t,t+s)\) given \(\hat{N}(t)\).

**Lemma 3.1.** Let \(\ell = 0,1,2,\ldots\), \(t \geq 1\) and \(s > 0\). The conditional ch.f. of \(M(t,t+s)\) given \(\{\hat{N}(t) = \ell\}\) has the form

\[
E[e^{ixM(t,t+s)} \mid \hat{N}(t) = \ell] = (E[e^{ixL(t-s-z-t)}])^\ell
\]

\[
\times \exp \left\{ - \int_{v=0}^{1} \int_{r=t-v}^{t+s-v} (1 - E[e^{ixL(t+s-v-r)}]) \Lambda(dv) F_D(dr) \right\},
\]

where \(Z\) has distribution \(\Lambda \ast F_D / E[\hat{N}(t)]\).

**Proof.** Since \(\hat{N}(t)\) is measurable with respect to \(\sigma\)-field by \((T_j)\) and \((D_j)\), the conditional ch.f. of \(M(t,t+s)\) on \((T_j)\) and \((D_j)\) has the form,

\[
E[e^{ixM(t,t+s)} \mid (T_j), (D_j)]
\]

\[
= E\left[ \prod_{k=1}^{\infty} \exp\{ixL_k(t-T_k-D_k, t+s-T_k-D_k)1_{\{T_k \leq 1\}} \} \mid (T_j), (D_j) \right]
\]

\[
= \prod_{k:T_k \leq 1, T_k + D_k \leq t}^{\infty} E[e^{ixL(t-T_k-D_k, t+s-T_k-D_k)} \mid T_k, D_k]
\]

\[
\times \prod_{k:T_k \leq 1, T_k + D_k > t}^{\infty} E[e^{ixL(t-s-T_k-D_k)} \mid T_k, D_k],
\]
where in the last step we notice \( L(t - T_k - D_k) = 0 \) a.s. for \( k : T_k + D_k \geq t \). We proceed calculation by the chain rule of conditional expectation to obtain
\[
E[e^{ixM(t,t+s)} | \tilde{N}(t)] = E \left[ \exp \left\{ \sum_{k:T_k \leq t, T_k + D_k \leq t} \log E[e^{ixL(t-T_k-D_k,t+s-T_k-D_k)} | T_k, D_k] \right\} \times \exp \left\{ \sum_{k:T_k \leq t, T_k + D_k > t} \log E[e^{ixL(t+s-T_k-D_k)} | T_k, D_k] \right\} | \tilde{N}(t) \right] 
\]
\[
= E \left[ \exp \left\{ \sum_{j=1}^{\tilde{N}(t)} \log E[e^{ixL(t-Z_j,t+s-Z_j)}] \right\} | \tilde{N}(t) \right] \times \exp \left\{ - \int_{v=0}^{1} \int_{r=t}^{t+s-v} (1 - E[e^{ixL(t+s-v-r)}]) \Lambda(dv) F_D(dr) \right\} 
\]
\[
= (E[e^{ixL(t-Z,t+s-Z)}])^{\tilde{N}(t)} \exp \left\{ - \int_{v=0}^{1} \int_{r=t}^{t+s-v} (1 - E[e^{ixL(t+s-v-r)}]) \Lambda(dv) F_D(dr) \right\},
\]
where \((Z_j)\) is the iid random sequence with generic random element \(Z\). The last expression coincides with the result and the proof is over.

Note that due to the convolution \( G := \Lambda * F_D \), the last term in (3.3) has another expression
\[
\exp \left\{ - \int_{v=0}^{1} \int_{0}^{\infty} (1 - E[e^{ixL(t+s-v-r)}] 1_{\{t \leq r + v \leq t + s\}}) \Lambda(dv) F_D(dr) \right\} 
\]
\[
= \exp \left\{ - E[\tilde{N}(t,t+s)] \int_{t}^{t+s} (1 - E[e^{ixL(t+s-w)}]) G(dw) / E[\tilde{N}(t,t+s)] \right\} 
\]
\[
= E \left[ \left( E[e^{ixL(t+s-W)}] \right)^{\tilde{N}(t,t+s)} \right],
\]
where \(W\) is independent of \(L\) and has distribution \(G(dw)/E[\tilde{N}(t,t+s)]\) on \((t,t+s]\).

Now we differentiate the conditional ch.f. at \(x=0\) proper times to obtain the following result.

**Theorem 3.2.** Consider the model (1.1) with iid additive clusters \(L_k, k = 1, 2, \ldots\) given by (1.2) such that cluster processes start at time \((T_k + D_k)\). Let \(\ell = 0, 1, 2, \ldots\), \(t \geq 1\) and \(s > 0\).

(i) The prediction \(\tilde{M}_{\ell}(t,t+s)\) of \(M(t,t+s)\) given \(\{\tilde{N}(t) = \ell\}\) is given by
\[
\tilde{M}_{\ell}(t,t+s) = \ell J_1 + H_1,
\]
where
\[
J_i = \int_{0}^{t} E[L^i(t-u,t+s-u)] \frac{\Lambda * F_D(du)}{E[\tilde{N}(t)]}, \quad i = 1, 2
\]
and
\[ H_i = \int_0^1 \int_{r=t-v}^{t+s-v} E[L(t+s-v-r)] \Lambda(dv) F_D(dr), \quad i = 1, 2. \]

(ii) The conditional variance of \( M(t,t+s) \) given \( \{\tilde{N}(t) = \ell\} \) is
\[ \text{Var}(M(t,t+s) \mid \tilde{N}(t) = \ell) = \ell J_2 - \ell J_1^2 + H_2. \]

**Remark 3.3.** Since \( E[\tilde{N}(t)] = \Lambda(1) - \tilde{N}(t) \), we evaluate the error of prediction \( \tilde{M} \) by
\[ E[(M(t,t+s) - E[M(t,t+s) \mid \tilde{N}(t)])^2] = E[\text{Var}(M(t,t+s) \mid \tilde{N}(t))] = (\Lambda(1) - \tilde{N}(t))(J_2 - J_1^2) + H_2, \]
which we could not do in the prediction by \( \hat{M} \) of Section \ref{sec:3.2}

Applying Theorem \ref{thm:3.2}, we calculate the following examples.

**Lemma 3.4.** Let \( L \) be a NP process with the mean value function \( \mu(t) > 0 \). Then
\[ \tilde{M}_\ell(t,t+s) = \ell \int_0^t \mu(t-u,t+s-u) \frac{\Lambda * F_D(du)}{E[\tilde{N}(t)]} + \int_0^1 \int_{r=-v}^{t+s-v} \mu(t+s-v-r) \Lambda(dv) F_D(dr) \]
and the conditional variance of \( M(t,t+s) \) given \( \tilde{N}(t) = \ell \) is
\[ \text{Var}(M(t,t+s) \mid \tilde{N}(t) = \ell) = \tilde{M}_\ell(t,t+s) - \ell \left( \int_0^t \mu(t-u,t+s-u) \frac{\Lambda * F_D(du)}{E[\tilde{N}(t)]} \right)^2 \]
\[ + \ell \int_0^t \mu^2(t-u,t+s-u) \frac{\Lambda * F_D(du)}{E[\tilde{N}(t)]} \]
\[ + \int_0^1 \int_{r=-v}^{t+s-v} \mu^2(t+s-v-r) \Lambda(dv) F_D(dr). \]

**Lemma 3.5.** Let \( L \) be NB process defined in \ref{sec:2.10}. Then the prediction \( \tilde{M}_\ell(t,t+s) \) of \( M(t,t+s) \) given \( \{\tilde{N}(t) = \ell\} \) is given by
\[ \tilde{M}_\ell(t,t+s) = \ell \frac{q}{p} \int_0^t \mu(t-u,t+s-u) \frac{\Lambda * F_D(du)}{E[\tilde{N}(t)]} + \frac{q}{p} \int_0^1 \int_{r=-v}^{t+s-v} \mu(t+s-v-r) \Lambda(dv) F_D(dr) \]
and the conditional variance of \( M(t,t+s) \) given \( \tilde{N}(t) = \ell \) is
\[ \text{Var}(M(t,t+s) \mid \tilde{N}(t) = \ell) = \frac{\tilde{M}_\ell(t,t+s)}{p} + \ell \frac{q^2}{p^2} \int_0^t \mu^2(t-u,t+s-u) \frac{\Lambda * F_D(du)}{E[\tilde{N}(t)]} \]
\[ - \ell \frac{q^2}{p^2} \left( \int_0^t \mu(t-u,t+s-u) \frac{\Lambda * F_D(du)}{E[\tilde{N}(t)]} \right)^2 \]
\[ + \frac{q^2}{p^2} \int_0^1 \int_{r=-v}^{t+s-v} \mu^2(t+s-v-r) \Lambda(dv) F_D(dr). \]

### 4. Numerical examples and some discussion

In this section we will observe how non-homogeneity affects the predictor with several examples. We consider the predictor \( \tilde{M}_m(t,t+s) \) (see Subsection \ref{sec:2.1}) with NP clusters \( L_j, j = 1, 2, \ldots \) under different mean value functions \( \mu \) where we keep the underlying Poisson processes \( N \) homogeneous. For the mean value function of the process \( N \), two cases \( E[N(x)] = \Lambda_1(x) = 30x, \Lambda_2(x) = 60x \) are examined, whereas we set three mean value functions for the cluster \( L \), which are
\[ \mu_1(x) = 5x, \quad \mu_2(x) = \frac{5x}{1 + x^2}, \quad \text{and} \quad \mu_3(x) = 5x^2. \]
The middle one is a decreasing function while other two are increasing ones. We plot the predictor 
\( \hat{M}_m(1,2) \) as function of \( m \) for \( m = 10 \sim 170 \) in Figure 4. We also make a straight dot line from the
initial value to the end value for comparison. In the light of Figure 4 we see non-linearity of
\( \hat{M}_m(1,2) \) as a function \( m \) in all cases, and sizes of \( \hat{M}_m(1,2) \) properly reflect the strength of intensity
functions.

Finally, we mention how our model (1.1) could be estimated from data. The process \( N(t) \) may
be estimated from the claims arrivals observations, whereas the generic process \( L \) of clusters \( (L_j) \)
would be estimated from observed payment streams. Nowadays statistical estimations of stochastic
processes are well established and since our model uses basic processes which are not restrictive,
we may have no difficulty in estimation. Then once the model (1.1) is estimated, the prediction of
future payment amount would be possible by our proposed method.

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Figure 1. Graphs of conditional expectation of $M_m(1, 2)$ based on the recursive algorithm given in Subsection 2.1. Top left, $(\Lambda_1, \mu_1)$. Top right, $(\Lambda_2, \mu_1)$. Middle left, $(\Lambda_1, \mu_2)$. Middle right, $(\Lambda_2, \mu_2)$. Bottom left, $(\Lambda_1, \mu_3)$. Bottom right, $(\Lambda_2, \mu_3)$. 