Green’s functions and method of images: an interdisciplinary topic usually cast aside in physics textbooks

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Abstract

In the present work we discuss how to address the solution of electrostatic problems, in professional cycle, using Green’s functions and the Poisson’s equation. By using this procedure, it was possible to verify its relation with the method of images as an interdisciplinary approach in didactic physics textbooks. For this, it was considered the structural role that mathematics, specially the Green’s function, have in physical thought presented in the method of images.

Keywords: Green’s Functions, Method of Images, Physics Textbooks

1 Introduction

One of the usual problems on electrostatics consists in obtaining the electric field (or electric potential) generated by a charge distribution in certain region of the space. It is possible to solve this problem by direct integration over the charge distribution or

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tackling the Poisson’s equation, subjected to a set of boundary conditions imposed on
the field (or potential).

Solving non-homogeneous differential equations using Green’s Functions is one of the
most powerful forms of describing the solution for a problem of this kind. However, a
great number of classical books on electrodynamics do not explore Poisson’s Equation
solutions using this method. Instead, the common approach uses the method of images, a
very interesting way to solve the problem, once it requires a deep physical interpretation.

The method of images can be used when we are trying to obtain the electrostatic field
generated by charge distribution near a conductive surface. This procedure takes into
account the symmetry of the problem adding a charge image located outside the region
of interest. From this new arrangement, it’s possible to reconstruct the same boundary
conditions of the initial problem without charge image.

However, it seems at first that this procedure is barely or not related to the resolution
of Poisson’s (or Laplace) equation. The method, on the contrary, is completely compatible
with the more general procedure of solving Poisson’s equation via Green’s functions. It is
a practical and conceptually elegant mathematical tool, even though it is not general. In
addition, it assumes the existence of virtual image charges in regions where the solution is
not valid, what seems somewhat artificial for seems introducing in teaching and learning
processes.

One of the authors in [1] developed a classification of tasks on electrostatics in which
are presented four primary classes of situations addressed to: (i) calculation of electrostatic fields; (ii) symbolic representation of the electrostatic field; (iii) analogical representation of the electrostatic field; (iv) description of electrostatic interactions.

All these primary classes include four or five secondary classes of situations whose
classification is based on the objects, variables and unknowns presented by the problem.
According to authors in case (i) there are five classes of situations. Among them the
most complicated/complex is the one related to calculation of electrostatic fields (or potentials) due to unknown charge distributions, which include, for instance, conductors in electrostatic equilibrium.

Their arguments rely upon two main epistemological reasons related to the necessary
thought operations for mastering these problems: advanced mathematical techniques for
problem-solving and conceptual deepness demanded for physical interpretation of charge
redistribution.

The relation among these concepts has been barely explored and, moreover, the cal-
culations by the methods of images are often restricted to the case of point charges, with
exception of Reitz, Milford and Christy [2] that include the problem of linear images, cases
in which it is possible to calculate the potential due to very large, electrically charged,
wires placed in regions containing conductors.

More complicated situations involving well-known and unknown electric charge distributions are rarely discussed using the Green’s functions method. Nevertheless, Machado [3] and Jackson [4], solve the problem of a discharged and grounded sphere in front of a point charge through this method, a situation often solved by using the method of images.

Panofsky and Phillips [5] approach the general problem of Green’s function by discussing general, mathematical and physical features, although they do not elaborate the discussion for specific problems. The present work presents a discussion on how to approach electrostatics in the professional cycle from the point of view of solving Green’s functions for Poisson’s equation, can be articulated to the method of images in an interdisciplinary approach. Our framework takes into account the structural role that Mathematics (Green’s functions) have in Physical thought (method of images).

The structure of this paper is presented as follows. In Section 2 presents a brief discussion about mathematical structures of physical thought. In Section 3 the problem of Green’s functions is presented from a historical point of view until the complete mathematical formulation of the solution of the Poisson’s equation, considering three-dimensional and two-dimensional cases. Section 4 presents a set of electrostatic problems whose solution was obtained by Green’s function to verify the relation with method of image. Final remarks are made in the Section 5.

2 Mathematics structures Physical thought

The relation between Mathematics and Physics is not just historical, but also epistemological. The junction among Physics, Astronomy and Mathematics in the Copernican Revolution fully stresses this fact. Expressing physical ideas in mathematical terms, on the other hand, is much more than a predictive tool, because it involves structuring physical thought in function of mathematic enunciations. It is not necessary to defend the role of Mathematics in Physics, because it is blatantly obvious. However, it is fundamental to discuss which role is developed in teaching and learning these disciplines.

Karam [6] and Rebello et al. [7] state that the results of studies on transference from Mathematics to Physics are strikingly clear about the hindrances faced by the students in this task, once using the first in the second envolves more than a simple correspondence relation between two distinct conceptual domains. In other words, that means this association is very different from the rote use of formulas.

Thus, approaching the role of Mathematics in Physics, requires differentiating its technical role (tool-like) and its structural role (reason-like). The first one can be assumed
when it's used in the second one. In Table 2 some characteristics of the technical dimension concerning the role of mathematics are pointed (extracted from Karam [6]).

Therefore, Karam [6] states that the technical role of Mathematics is associated with calculations developed in a disconnected way from physical problems (e.g., plug-and-chug), while its counterpart, the structural one, is related to the use of Mathematics to reason about the physical world, that is, to establish reference to it. Although the first is important for mastering the second one, the technical domain is not sufficient to lead students to the structural level [6]. The author highlights it is impossible to detach conceptual understanding and mathematical structures use, and points some important characteristics of this feature, which we present in the table 2.

We then seek to discuss the structural role of Green's function in Physics by explaining its relation with the method of images by modeling and comprehending problems containing known and unknown charge distributions.

| Technical | |
| --- | --- |
| Blindly use an equation to solve quantitative problems | |
| Focus on mechanic or algorithmic manipulations | |
| Use arguments of authority | |
| Rote memorization of equations and rules | |
| Fragmented knowledge | |
| Identify superficial similarities between equations | |
| Mathematics seen as calculation tool | |
| Mathematics seen as language used to represent and communicate | |

Table 1: Technical dimension concerning the role of mathematics in physics - author: Karam

| Structural | |
| --- | --- |
| Derive an equation from physical principles using logical reasoning | |
| Focus on physical interpretations or consequences | |
| Justify the use of specific mathematical structures to model physical phenomena | |
| Structured knowledge: connect apparently different physical assumptions through logic | |
| Recognize profound analogies and common mathematical structures behind different physical phenomena | |
| Mathematics seen as reasoning instrument | |
| Mathematics seen as essential to define physical concepts and structure physical thought | |

Table 2: Structural dimension concerning the role of mathematics in physics - author: Karam
3 Green’s Functions and Poisson’s equation

In this section the problem of Green’s function is presented from a historical point of view and it is discussed the apparent contradiction between the fact that differential operators applied in Green’s Functions are expressed in terms of the Dirac Delta “function”¹ initially elaborated by Paul Dirac and, then, formalized by Laurent Schwartz in XX century [8].

How could Green alive between XVIII and XIX centuries, write his formulation in XX century notation? The reason is: he did not do that. In the following section, we discuss how was Green’s approach and how it is different from the version used in this paper.

3.1 A brief history of Green’s Functions

George Green was born on July 14th of 1793, Nottingham, England and died on May 31st of 1841, in the same town. It was one of the biggest exponents in Mathematical-Physics of the region, being the first to introduce the concept of Potential and the method of Green’s functions, largely used until the present days in many fields on Physics. However, it seems he has been forgotten for a while, what would imply posthumous recognition for his work, due to his popularization in works of William Thomson, known as Lord Kelvin [8].

Nonetheless, if his work was so important both for Mathematics and Physics, why it remained obscure in history? Cannell [8] enumerates factors like: his premature death, at the age 47; the fact of going to Cambridge to study lately and then returning to Nottingham, without establishing personally in the former city; his graduation in math in a relatively advanced age; the development of abstract works for the period he lived, without drawing attention of the scientific community, more worried with practical questions at that time; the advanced nature of his work, barely understood for much scientists of the poque.

Electromagnetism was not a commonplace subject at Green’s time, it became so solely after Kelvin and Faraday. Green knew, however, the works of Laplace, Legendre and Lacroix and had access to a translation to English of the *Mécanique Celeste* due to Pierre Laplace, made by John Toplin, his tutor in Nottingham Free Grammar School, a Leibnizian (what explains his preference for ”d-ism” instead of the Newtonian ”dot-ism”). Green also deeply knew the work of Poisson in Magnetism, probably accessed by attending to the *Nottingham Subscription Library*. The mathematician was interested, in his essay on electricity and magnetism (1828), in inverse-type problem related to the

¹Actually, the Dirac Delta function is a limit of a sequence, that is, a distribution for which there is mathematical foundation and formulation.
electric potential (physical quantity named after Green), namely, "knowing the potential how can we determine the electric fluid (electric charge) density in a ground conductor of any form?". The former solution to Poisson’s equation was given by means of an integral (in modern notation)\[^2\]

\[
\nabla^2 \phi = \frac{-\rho}{\varepsilon_0} \\
\phi = \frac{1}{4\pi\varepsilon_0} \int_{V} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'.
\]

In other words, Green was interested in determining the charge distribution from operations on the potential function, whose negative gradient would result in the force on an unit charge exerted on this conductor (nowadays, we interpret it as the electric field). For this, Green developed a work in mathematical analysis and constructed what we know by Green’s theorem. It is derived nowadays using the divergence theorem, due to Gauss-Ostrogradsky, by integrating by parts. After using the theorem, Green investigated what happens in the neighborhood of a point charge located in $\vec{r} = \vec{r}'$ (modern notation), evaluating the limit of the solution for $\vec{r} \to \vec{r}'$, that is, when the calculation of the potential is made near the point which the charge is placed. He then carried out to the following function (modern notation)\[^3\]

\[
G(\vec{r}, \vec{r}') = \frac{1}{4\pi|\vec{r} - \vec{r}'|}.
\]

Green also applied his solution and succeeded in finding a formula relating the unknown surface charge density in a conductor with the known potential in its surface; his solution is likewise discontinuous, what is physically feasible, once electric charges (or fluid, at that time), were known to stay concentrated in the conductor’s surface. The mathematician checked if the function satisfies Laplace’s equation outside the source and considered the Green function as a response to an unitary impulse [8], exactly as is done nowadays.

In 1930, 102 years later, Paul Dirac introduced his famous “delta functions” without proper mathematical rigor, although with a significant practical value. In modern notation the differential equation satisfied by Green’s functions are presented in function of these “improper functions”, as Dirac called them. Nevertheless, the formalization of such mathematical elements just turned possible after the work of Laurent Schwartz, in the

\[^2\] At that time, the Poisson’s equation was written like $\nabla^2 \phi = -4\pi \rho$, what changes the solution to $\phi = \int_{V} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$.

\[^3\] Once we changed the original Poisson’s equation for the international system, the original Green function was $G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}$.
50’s on the theory of distributions, in which he describes the “delta functions” as limits of a sequence, *id est*, a distribution. This is the reason why, in our calculus, we use modern notation to find Greens’ functions. We can clearly see how Green was ahead of his time.

### 3.2 Poisson’s Equation

Electric charges are held stationary by other forces than the ones of electric origin, such as molecular binding forces. Since charges are stationary, no electric currents and, thus, no magnetic fields are presented (*\vec{B} = 0*). For a stationary electric charge distribution, described by \( \rho(\vec{r}) \), the associated electrostatic field satisfies the following set of differential equations,

\[
\begin{align*}
\nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0}, \\
\n\nabla \times \vec{E} &= 0.
\end{align*}
\]

Accordingly to the Helmholtz’s theorem [11], once both divergence and curl of a sufficiently smooth, rapid decaying, vector field are known, the problem can solved. For the electrostatic field, the solution for \( \vec{E} \) can be written as the gradient of a scalar function \( \varphi(\vec{r}) \), since it is irrotational:

\[
\vec{E} = -\nabla \varphi,
\]

where \( \varphi(r) \) is well-known as the electrostatic potential. Replacing (6) in the equation (4) leads to the Poisson’s equation:

\[
\nabla^2 \varphi = -\frac{\rho}{\epsilon_0},
\]

and when regions without electric charge distribution are considered, \( \rho = 0 \), becomes

\[
\nabla^2 \varphi = 0,
\]

equation known as Laplace’s equation.

In electrostatics problems the solution will be unique if the boundary conditions are imposed on the potential \( \varphi(\vec{r}) \) (on the electrostatic field \( \vec{E}(\vec{r}) \)) in some point of the space, accordingly to the Uniqueness theorem [4]. If we impose the boundary conditions on \( \varphi(\vec{r}) \), these are known as Dirichlet boundary conditions. However, when the boundary conditions are applied to \( \vec{E}(\vec{r}) \), they are denoted as Neumann boundary conditions [4]. Another possibility is to apply mixed boundary conditions, both on \( \varphi(\vec{r}) \) and \( \vec{E}(\vec{r}) \). In this case, we call Robbin’s boundary conditions.
3.3 Green’s functions

In general, solving the scalar differential equation (7) for $\varphi$ is easier than solving vector differential equations for $\vec{E}$, (4) and (5) equations. We can apply Green’s function in (7), and from this follows the n-dimensional equation below,

$$\nabla^2 G(\vec{r}, \vec{r'}) = -\delta^{(n)}(\vec{r} - \vec{r'}).$$ (9)

Considering the Green’s identities [9, 10], it is possible to obtain an expression for Electrostatic Potential

$$\nabla \cdot (\varphi \nabla G) = \varphi \nabla \cdot (\nabla G) + \nabla \varphi \cdot \nabla G, \quad (10)$$

$$\nabla \cdot (G \nabla \varphi) = G \nabla \cdot (\nabla \varphi) + \nabla \varphi \cdot \nabla G, \quad (11)$$

what leads to

$$\int (\varphi \nabla^2 G - G \nabla^2 \varphi) \, dV = \int (\varphi \nabla G - G \nabla \varphi) \cdot \hat{n} \, dS. \quad (12)$$

Then, using (7) and Green’s Identity (10) – (11) in equation (12), one can obtain

$$\varphi(\vec{r}) = \frac{1}{\epsilon_0} \int G \rho \, dV' + \oint G \frac{\partial \varphi}{\partial n} \, dS' - \oint \varphi \frac{\partial G}{\partial n} \, dS', \quad (13)$$

which is the general solution for an electrostatic potential and, consequently, for the electric field. The second and third terms in equation (13) are associated with the choice of the boundary conditions to which the electric charge density is subject. Once the boundary conditions are defined, equation (13) will have a unique and well-defined solution according to the uniqueness theorem.

In a great number of physical problems that includes conductors in electrostatic equilibrium and zero potential, it is adequate to apply both on the Potential and on the Green’s Function, the Dirichlet’s boundary conditions. This implies,

$$\varphi(\vec{r}) = \frac{1}{\epsilon_0} \int G \rho \, dV'. \quad (14)$$

For this kind of problem is always possible to add into the Green’s function a solution to Laplace’s equation, denoted by $(G_L(\vec{r}, \vec{r}'))$, which satisfies physical and mathematical boundary conditions. Therefore, the full Green’s function will be written as

$$G(\vec{r}, \vec{r'}) = G_D(\vec{r}, \vec{r'}) + G_L(\vec{r}, \vec{r'}), \quad (15)$$

where $G_D(\vec{r}, \vec{r'})$ depends exclusively the dimensions of Laplacian operator, whereas
$G_L(\vec{r}, \vec{r}')$ depends on the boundary conditions. In the next section, we shall determine the expression for $G_D$ in 3-dimensional and 2-dimensional cases for Laplacian operator.

### 3.3.1 Green’s function for Poisson’s Equation

#### Three-dimensional case

The analytical expression for the Green’s function in three dimensions will be determined. It is necessary to apply a Fourier Transform, leading Green’s function to $k-$space. Then the inverse Fourier transform must be used to find the solution in the coordinates space.

The Fourier transform and its inverse for Green’s function $G_D$ are presented below, respectively

$$G_D^k \equiv G_D(\vec{k}, \vec{r}') = \int_{-\infty}^{\infty} G_D(\vec{r}, \vec{r}') e^{i\vec{k} \cdot \vec{r}} d^3 r, \quad (16)$$

$$G_D(\vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} G_D^k e^{-i\vec{k} \cdot \vec{r}} d^3 k, \quad (17)$$

Applying the Fourier transform on equation (9) with $n = 3$ and integrating by parts we obtain the following expression

$$G_D^k = \frac{e^{i\vec{k} \cdot \vec{r}'}}{(k_x^2 + k_y^2 + k_z^2)}, \quad (18)$$

which represents the Green’s function in $k-$space. Applying inverse Fourier transform in (18), we will recover the expression of $G_D$ in coordinates space,

$$G_D(\vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} G_D^k e^{-i\vec{k} \cdot \vec{R}} d^3 k, \quad (19)$$

with $\vec{R} = \vec{r} - \vec{r}'$. The integral in equation (19) becomes simpler by an adequate variable change. Once the integrand does not depend on variable $\phi$, integration results in numerical factor equals to $2\pi$. Rewriting the exponent in equation (19) as $\vec{k} \cdot \vec{R} = kR\cos\theta$, we integrate in the variable $\theta$ to find the following result

$$G_D(\vec{r}, \vec{r}') = \frac{1}{2\pi^2 R} \int_{0}^{\infty} \frac{\sin(kR)}{k} \frac{dk}{k}. \quad (20)$$

The integral in (20) can be taken to the complex plane, with part of it being an integral along the real axis and the other one along a contour $\Gamma$ extending to infinity. By Jordan’s Lemma the second integral mentioned vanishes in infinity, once this function is obviously analytic.
It is also possible to write the sine function in exponential form, what implies two integrals

$$\int \frac{\sin(z)}{z} \, dz = \int_{-\infty}^{\infty} \frac{\sin(z)}{z} \, dz = \frac{1}{2i} \oint \frac{e^{iz} - e^{-iz}}{z} \, dz,$$  \hspace{1cm} (21)

whose integration on path $\Gamma$ is, by convention, positive (counterclockwise).

We chose the path $\Gamma_1$ for the first integral so that it circles (and excludes) the pole $z = 0$ coming from the left ($-\infty \to +\infty$) along the real axis, oriented counterclockwise (positive). We chose path $\Gamma_2$ for the second integral in a way that the pole is included and singularity removed, also coming from the left along the real axis, but oriented clockwise (negative). Then, we find the following result

$$\int_{-\infty}^{\infty} \frac{\sin(z)}{z} \, dz = \frac{2\pi}{2i} e^{0} = \pi.$$  \hspace{1cm} (22)

Now, we can find the the Green’s function for the Laplace’s equation for the three-dimensional case,

$$G_D(\vec{r}, \vec{r}') = \frac{1}{4\pi |\vec{r} - \vec{r}'|},$$  \hspace{1cm} (23)

We will discuss the two-dimensional Laplacian operator case in sequence.

**Two-dimensional case**

Evoking the the Green Function for the two-dimensional Poisson’s equation with $n = 2$,

$$\nabla^2 G_D(\vec{r}, \vec{r}') = -\delta^2(\vec{r} - \vec{r}'),$$  \hspace{1cm} (24)

enunciating both Fourier direct and inverse transforms

$$G_k^D = G_D(\vec{k}, \vec{r}') = \int_{-\infty}^{\infty} G_D(\vec{r}, \vec{r}') e^{i\vec{k} \cdot \vec{r}} \, d^2 r,$$  \hspace{1cm} (25)

$$G_D(\vec{r}, \vec{r}') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} G_k^D e^{-i\vec{k} \cdot \vec{r}} \, d^2 k,$$  \hspace{1cm} (26)

we can follow the similar procedure in three-dimensional case and obtain

$$G_k^D = \frac{e^{i\vec{k} \cdot \vec{r}'}}{k_x^2 + k_y^2},$$  \hspace{1cm} (27)

$$G_D(\vec{r}, \vec{r}') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{e^{i\vec{k} \cdot \vec{r}'} e^{-i\vec{k} \cdot \vec{r}}}{k_x^2 + k_y^2} \, d^2 k.$$  \hspace{1cm} (28)

To solve the integration in equation (28), we can change variables to polar coordinates.
and apply the scalar product

$$G_D(\vec{r}, \vec{r}') = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} \frac{e^{ikR \cos \theta}}{k} dk d\theta,$$  \hspace{1cm} (29)

recognizing the integral in $\theta$ as $2\pi J_0(kR)$, where $J_0(kR)$ is the zero-order Bessel function

$$G_D(\vec{r}, \vec{r}') = \frac{1}{(2\pi)} \int_0^\infty \frac{J_0(kR)}{k} dk d\theta.$$  \hspace{1cm} (30)

The integration on (30) can be done if we derivate the Green function with respect to variable $R$,

$$\frac{dG_D(\vec{r}, \vec{r}')}{dR} = \frac{1}{(2\pi)} \int_0^\infty \frac{\partial}{\partial R} \left[ \frac{J_0(kR)}{k} \right] dk d\theta$$

$$= \frac{1}{(2\pi)} \left[ \frac{J_0(kR)}{R} \right]_0^\infty = -\frac{1}{(2\pi R)},$$  \hspace{1cm} (31)

and, carrying out the integration with respect to variable $R$, we have

$$G_D(\vec{r} - \vec{r}') = -\frac{1}{2\pi} \ln |\vec{r} - \vec{r}'|,$$  \hspace{1cm} (32)

where the equation (32) represents the Green’s function for two-dimensional Laplace’s case.

## 4 Solving electrostatic problem by Green’s Function

In previous section we argued about the expressions of Green’s functions in three and two-dimensional cases. Now, we will direct our attention to obtain the solution a set of problems using the Green’s function and verifying the relation with the method of images.

### 4.1 Point Charge placed near a Grounded Infinite Plane Conductor

Let’s consider a point electric charge, $q$, placed a distance $d$ along the $z$ axis of an infinite thin grounded plate along the $xy$ plane. What is electrical potential produced in a region $z > 0$ in space?

The Green’s function for the three-dimension problem, in this case, admits a solution for Laplace’s equation adjusted to Dirichlet boundary conditions for both Green Function and Electric Potential.
The ground conductor is mathematically structured as having null electric potential over its surface at \( z = 0 \). Meanwhile, the Green’s Function reduces the problem of a continuous, and in this case unknown, distribution to the one of a point charge, exactly what the method of images proposes.

Considering the equation (23), the full Green’s function to this problem will be

\[
G(\vec{r}, \vec{r}') = \frac{1}{4\pi \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} + G_L(\vec{r}, \vec{r}'),
\]

(33)

On the boundary \( \vec{S} = (x, y, z = 0) \), for every point located on the plate we must have the Green’s function equals zero

\[
G(S, \vec{r}') = \frac{1}{4\pi \sqrt{(x-x')^2 + (y-y')^2 + z'^2}} + G_L(S, \vec{r}') = 0.
\]

(34)

Thus, it is possible to see, by inspection, that the function \( G_L \) must have the following expression

\[
G_L(\vec{r}, \vec{r}') = -\frac{1}{4\pi \sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}},
\]

(35)

to ensure the condition given by equation (34) will be valid. Therefore, insofar the only known electric charge is a point one, it can be modeled by a Dirac Delta function charge density, whose infinity point is located at point \((0, 0, d)\).

In other words, \( \rho(x, y, z) = q \delta(z - d) \delta(y - 0) \delta(x - 0) \) in such a way that integrating over the volume in equation (14) leads to:

\[
\varphi(x, y, z) = \frac{q}{4\pi \varepsilon_0 \sqrt{(x)^2 + (y)^2 + (z-d)^2}} - \frac{q}{4\pi \varepsilon_0 \sqrt{(x)^2 + (y)^2 + (z+d)^2}}.
\]

(36)

We can verify that this expression is a solution for Laplace’s Equation for \( z > 0 \), except for \( z = d \) where it diverges. Besides, the solution satisfy the imposed boundary conditions. The result is the same obtained by the method of images as seen in the figure.

It is important to highlight the symmetry of Green’s Function in the inversion between point and source locations. Structuring Physical thought (withdraw the plane by a point charge) in such a way to make the Electrical Potential to vanish in that surface is a matter related to the mathematical point of view.
Figure 1: Contour lines for the electrostatic potential \(36\). We can verify where we must place an image charge (blue) in a way that maintains a null potential over the grounded plane conductor (black thick line).

**Electric Field**

After determining the electric potential, it is possible to find the associated electric field. Therefore, using the equation \(6\), we find

\[
\vec{E} = \frac{q}{4\pi\epsilon_0} \left( E_x \hat{i} + E_y \hat{j} + E_z \hat{k} \right),
\]

(37)

where the components of electric field are given by

\[
E_x = \frac{x}{((d-z)^2 + r^2)^{3/2}} - \frac{x}{((d+z)^2 + r^2)^{3/2}},
\]

\[
E_y = \frac{y}{((d-z)^2 + r^2)^{3/2}} - \frac{y}{((d+z)^2 + r^2)^{3/2}},
\]

\[
E_z = \frac{d-z}{((d-z)^2 + r^2)^{3/2}} - \frac{d+z}{((d+z)^2 + r^2)^{3/2}},
\]

(38)

with \(r^2 = x^2 + y^2\).
4.1.1 Infinite charged wire placed near a Grounded Infinite Plane Conductor

Consider an infinite charged wire placed at distance \( d \) along the \( x \) axis, near a grounded infinite plane conductor. It is possible to use the Green function in two-dimensional case \((32)\), to adjust a solution to Laplace’s equation \( G_L \). The potential is zero on the charged plane, what leads to the full Green’s function

\[
G(x, x', y, y') = -\frac{1}{2\pi} \ln \sqrt{\frac{(x - x')^2 + (y - y')^2}{(x + x')^2 + (y - y')^2}}, \tag{39}
\]

where we considered the Dirichlet for the determination of the Green’s function \( G_L \). The equation \((39)\) represents the same result obtained by the method of images.

To obtain the electrostatic potential for this case, we must integrate the Green function over the volume in equation \((14)\). Considering the charge density function given by \( \rho = \lambda \delta(x-d)\delta(y-0) \), one obtains

\[
\varphi(x, y) = -\frac{\lambda}{2\pi \varepsilon_0} \ln \frac{(x - d)^2 + (y)^2}{(x + d)^2 + (y)^2}. \tag{40}
\]

From equation \((40)\) it is possible to study the equipotential surfaces, if the argument of the logarithm function is a constant

\[
\frac{(x - d)^2 + y^2}{(x + d)^2 + y^2} = m. \tag{41}
\]

Thus represents a circumference with equation

\[
\left[ x - \left( d \frac{(1 + m^2)}{1 - m^2} \right) \right]^2 + y^2 = \left( \frac{2md}{1 - m^2} \right)^2. \tag{42}
\]

For the case \( m = 1 \) in equation \((41)\), the radius of the circumference will be infinite, which represents a plane. As long as the solution fits \( \varphi(S) = 0 \) and \( x_0 = \infty \), the equipotentials are on the plane and at infinity. Considering the cases with \( m < 1 \), the equipotentials surfaces represent circles of radii \( r = \frac{2md}{1 - m^2} \) centered at point \( x_0 = d \frac{(1 + m^2)}{(1 - m^2)} \), as presented in figure \([2]\).
Electric Field

From the electrostatic potential (40), the resultant electric field can be found using the equation (6), given by the following expression

\[ \vec{E}(\vec{r}) = E_x \hat{i} + E_y \hat{j}, \]  

where

\[ E_x = \frac{\lambda d}{\pi \epsilon_0} \left\{ \frac{d^2 - x^2 + y^2}{[(d - x)^2 + y^2][(d + x)^2 + y^2]} \right\}, \]  

and

\[ E_y = \frac{\lambda d}{\pi \epsilon_0} \left\{ \frac{2xy}{[(d - x)^2 + y^2][(d + x)^2 + y^2]} \right\}, \]

represents the two cartesian coordinates of electric field.

4.2 Point Charge placed near a Grounded Spherical Conductor

We shall solve the classical problem of finding the potential inside a grounded sphere of radius \( R \), centered at the origin, due to a point charge inside the sphere at position \( \vec{r}' \), as showed at figure 3. The full Green Function for this problem is given by
Figure 3: Diagram illustrating the Laplace’s equation for a sphere of radius R, with a point charge located at $\vec{r}'$. 

\[
G(r, \theta, r', \theta') = \frac{1}{4\pi} \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos(\theta - \theta')}} + G_L(r, \theta, r', \theta'). 
\]  
(46)

where we already considered the Green’s function (23) and the spherical symmetry of the problem to write the distance $|\vec{r} - \vec{r}'|$. 

In the similar way, it is necessary to add $G_L$ into the full Green’s function (46). Over the surface of the sphere, for any polar angle, $\theta$, the electrical potential always will be null. This is equivalent to make the Green function (46) vanish for $r = R$, what leads to an expression for $G_L$

\[
G_L(R, \theta, r', \theta') = -\frac{1}{4\pi} \frac{1}{\sqrt{R^2 + r'^2 - 2Rr'\cos(\theta - \theta')}} ,
\]  
(47)

and, by inspection, we verify that in the point $r$ the $G_L$ has the following form,

\[
G_L(r, \theta, r', \theta') = -\frac{1}{4\pi} \frac{1}{\sqrt{r'^2 + R^2 - 2rr'\cos(\theta - \theta')}} .
\]  
(48)

which corresponds to the Green’s function inside the sphere, for a point image charge $q'$ outside it at point $r' = \frac{R^2}{r}$. The equation (48) is the only one that leads to a vanishing Green’s function over the surface of the sphere.

Now, we must find the associated electrostatic potential by integrating over the volume in equation (14), assuming a charge distribution like $\rho(\vec{r}) = \frac{q}{r^2} \delta(r' - d)\delta(\theta' - 0)\delta(\phi' - 0)$. We find,

\[
\varphi(r, \theta) = \frac{1}{4\pi \epsilon_0} \left\{ \frac{q}{\sqrt{r^2 + d^2 - 2rd \cos \theta}} - \frac{(qR/d)}{\sqrt{r^2 + \frac{R^4}{d^2} - 2r \frac{R^2}{d} \cos \theta}} \right\} . 
\]  
(49)
The result obtained in equation (49) can be derived by using the method of images. Considering a negative image charge placed a distance \( r' = \frac{R^2}{d} \), from the centre of the spherical shell and a charge \( q' = -(qR/d) \) produces the same results showed in (49) as represented in the figure 4.

**Electric Field**

From the electrostatic potential (49), it is possible to find the electric field using the equation (7), which is expressed in spherical coordinates as

\[
\vec{E}(r, \theta) = E_r \hat{r} + E_\theta \hat{\theta},
\]

where

\[
E_r = \frac{q}{4\pi\epsilon_0} \frac{(r - d \cos \theta)}{[r^2 - 2rd \cos \theta + d^2]^{3/2}} - \frac{R}{d} \frac{(R - \frac{R^2}{d} \cos \theta)}{[r^2 - 2r\frac{R^2}{d} \cos \theta + \frac{R^4}{d^2}]^{3/2}} \tag{51}
\]

and

\[
E_\theta = \frac{q}{4\pi\epsilon_0} \frac{d \sin \theta}{(d^2 - 2dr \cos \theta + r^2)^{3/2}} - \frac{R}{d} \frac{\frac{R^2}{d} \sin \theta}{d \frac{R^2}{d^2} \sin \theta + r^2} \frac{R^4}{d^2} \tag{52}
\]

and the lines of force are represented in the figure 4.

Considering that the inner charge lies on the z-axis, the induced charge density at surface of the sphere will be described by a function of the polar angle \( \theta \)

\[
\sigma(\theta) = \epsilon_0 \frac{\partial V}{\partial r} \bigg|_{r=R} = -\frac{q}{4\pi R (R^2 + d^2 - 2dR \cos \theta)^{3/2}} \tag{53}
\]

and the total charge on the surface of sphere can be found by integrating over all angles,

\[
Q_t = \int_0^\pi \int_0^{2\pi} \sigma(\theta) d\Omega = -q. \tag{54}
\]

What would happen if the charge \( q \) was outside of the grounded sphere? In this case, this problem can be solved using this procedure in similar way. Assuming the charge \( q \) is located at position \( \vec{r}' = d \) outside of a grounded sphere of radius \( R \), the electrostatic potential outside is given by the sum of the potentials of the charge and its image charge \( q' \) inside the sphere.
Figure 4: Lines of force due to the electrostatic field $\vec{E}(\vec{r})$ and the equipotential surfaces for a positive point charge (red) inside the spherical shell of radius $R$. The blue charge represents the image charge $q'$, and it guarantees the electric field is null over the surface of the spherical shell.

5 Conclusion

It has been showed in this paper that it is possible to establish comparison between Green’s function and the method of images in electrostatic problems. The method of images relies upon a strong sense of physical interpretation, while the technique of Green’s function is a powerful form of solving problems involving differential equations.

The solution attached to the image charge appears as a solution for Laplace’s equation, satisfying the boundary conditions associated. On the other hand, Green’s function method is more general technique than the one due to calculation by image charges.

However, in a physics problem, without the interpretation, connecting these two instances, the mathematical knowledge relates in an non-substantive way and may be anchored to non-relevant prior knowledge. Therefore, it leads to non-elaborated ideas as, for example, “problems involving conductors are solved by Green’s function” or “problems involving conductors are solved by the method of images”, what places this kind of relation closer to the rote learning pole and further from the meaningful learning pole [12].

Nevertheless, in parallel, the methods may be meaningful both in Physics and Math-
ematics, once it is possible to learn about conductors in electrostatic equilibrium while conceptually and operationally tackling only using Green’s functions.

The authors defend, as does Karam[6], that mathematical knowledge structures physical thought and that gives meaning to mathematical knowledge through situations that make the concept of Green’s function useful and meaningful in the field of physics [13], permitting transference to the domain of Mathematics[7].

The value of this article underlies in showing a deep relation between physical thought and mathematical structure in a case of electromagnetism (professional cycle). Offering a wider view on the role of Mathematics in Physics than the common views of Mathematics as tool (operationalistic function) or as a merely language (restricted communicative function).

Another intricate point in the discussion is the fact that this knowledge is necessarily tied to epistemological features that can not be cast aside. Green himself obviously did not knew the Dirac Delta function, neither Dirac himself had a formal proof of its validity, which was developed by Schwartz, but this did not stopped them from doing elaborated Mathematics.

Similar epistemological difference can be found among the works of Newton (or Leibniz) and the ones by Weierstrass [14]. For as much the notion of function due to the latter mathematician approaches the concept of number (static view), the one due to the former in closer to the concept of variable (dynamic view) [14].

Related to this, is the unmentionable wide failure in Calculus teaching in the first year of any course of Exact Sciences [14], whose cause is, partially, associated with the disregarding of this feature into teaching-learning processes: students often study textbooks approaching the concept in a Weierstrassian perspective, which is much further (and much more formal) from students’ prior knowledge than it should be. It is reasonable to be expected for reproduction all over Brazil. In spite of the existence of great teachers and students in these courses, this epistemological features is beyond their will power or applied didactical methodology in the teaching processes.

Returning to the discussion of Green’s function, we advise that its interpretation should be approached to the notion of point source, as Green himself did, because this can provide conditions for comprehension of more modern concepts as, for example, the Dirac Delta Function. Without this epistemological ingredient, the process of interdisciplinary interaction between Mathamatics and Physics in classroom can blatantly fail in reaching its objective of coming up with conditions for meaningful learning [12].

The authors expect to contribute, by means of discussion of these simple examples, to demonstrate the feasibility of discussing in an integrated manner the method of images (with high degree of physical interpretation) and the technique of Green’s function (with
high degree of mathematical power) in classroom.

The authors also look forward to discuss principles related to providing condition not just for comprehension of the secondary class of situations $\Gamma_E$ pointed in [11], but seeking for interdisciplinary integration between Physics and Mathematics in a manner of promoting reasoning founded in the thesis that Mathematics structure Physical thought [6] and that Physics may give sense to concepts of Mathematics [13].

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