A Riemann theta function formula with its application to double periodic wave solutions of nonlinear equations

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Abstract. Based on a Riemann theta function and Hirota’s bilinear form, a lucid and straightforward way is presented to explicitly construct double periodic wave solutions for both nonlinear differential and difference equations. Once such an equation is written in a bilinear form, its periodic wave solutions can be directly obtained by using an unified theta function formula. The relations between the periodic wave solutions and soliton solutions are rigorously established. The efficiency of our proposed method can be demonstrated on a class variety of nonlinear equations such as those considered in this paper, shall water wave equation, (2+1)-dimensional Bogoyavlenskii-Schiff equation and differential-difference KdV equation.

Keywords: Nonlinear equations; Hirota’s bilinear method; Riemann theta function; double periodic wave solutions; soliton solutions.

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1. Introduction

The bilinear derivative method developed by Hirota is a powerful and direct approach to construct exact solution of nonlinear equations. Once a nonlinear equation is written in bilinear forms by a dependent variable transformation, then multi-soliton solutions are usually obtained \[^{[1]-[6]}\]. It was based on Hirota forms that Nakamura proposed a convenient way to construct a kind of quasi-periodic solutions of nonlinear equations \[^{[7][8]}\], where the periodic wave solutions of the KdV equation and the Boussinesq equation were obtained. Such a method indeed exhibits some advantages. For example, it does not need any Lax pairs and Riemann surface for the considered equation, allows the explicit construction of multi-periodic wave solutions, only relies on the existence of the Hirota’s bilinear form, as well

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as all parameters appearing in Riemann matrix are arbitrary. Recently, further development was made to investigate the discrete Toda lattice, (2+1)-dimensional Kadomtsev-Petviashvili equation and Bogoyavlenskii’s breaking soliton equation [9]-[14]. However, where repetitive recursion and computation must be performed for each equation [7]-[14].

The motivation of this paper is to considerably improves the key steps of the above existing methods. To achieve this aim, we devise a theta function bilinear formula, which actually provides us a lucid and straightforward way for applying in a class of nonlinear equations. Once a nonlinear equation is written in bilinear forms, then the double periodic wave solutions of the nonlinear equation can be obtained directly by using the formula. Moreover, we propose a simple and effective method to analyze asymptotic properties of the periodic solutions. As illustrative examples, we shall construct double periodic wave solutions to the shallow water wave equation, (2+1)-dimensional Bogoyavlenskii-Schiff equation and differential-difference KdV equation.

The organization of this paper is as follows. In section 2, we briefly introduce a Hirota bilinear operator and a Riemann theta function. In particular, we provide a key formula for constructing double periodic wave solutions for both differential and difference equations. As applications of our method, in sections 3-5, we construct double periodic wave solutions to the shallow water wave equation, (2+1)-dimensional Bogoyavlenskii-Schiff equation and differential-difference KdV equation, respectively. In addition, it is rigorously shown that the double periodic wave solutions tend to the soliton solutions under small amplitude limits.

2. Hirota bilinear operator and Riemann theta function

To fix the notations we recall briefly some notions that will be used in this paper. The Hirota bilinear operators $D_x, D_t$ and $D_n$ are defined as follows:

$$D_x^m D_t^k f(x, t) \cdot g(x, t) = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^k f(x, t) g(x', t') |_{x'=x, t'=t}$$

$$e^{\delta D_n} f(n) \cdot g(n) = e^{\delta(\partial_n - \partial_{n'})} f(n) g(n') |_{n'=n} = f(n + \delta) g(n - \delta),$$

$$\cosh(\delta D_n) f(n) \cdot g(n) = \frac{1}{2} (e^{\delta D_n} + e^{-\delta D_n}) f(n) \cdot g(n),$$

$$\sinh(\delta D_n) f(n) \cdot g(n) = \frac{1}{2} (e^{\delta D_n} - e^{-\delta D_n}) f(n) \cdot g(n).$$
Proposition 1. The Hirota bilinear operators $D_x, D_t$ and $D_n$ have properties \[D_x^m D_t^n e^{\xi_1} e^{\xi_2} = (\alpha_1 - \alpha_2)^m (\omega_1 - \omega_2)^n e^{\xi_1 + \xi_2},\]
\[e^{\delta D_n} e^{\xi_1} e^{\xi_2} = e^{\delta(v_1 - v_2)} e^{\xi_1 + \xi_2},\]
\[\cosh(\delta D_n) e^{\xi_1} e^{\xi_2} = \cosh(\delta(v_1 - v_2)) e^{\xi_1 + \xi_2},\]
\[\sinh(\delta D_n) e^{\xi_1} e^{\xi_2} = \sinh(\delta(v_1 - v_2)) e^{\xi_1 + \xi_2},\]
where $\xi_j = \alpha_j x + \omega_j t + \nu_j n + \sigma_j$, and $\alpha_j, \omega_j, \nu_j, \sigma_j, j = 1, 2$ are parameters and $n \in \mathbb{Z}$ is a discrete variable. More generally, we have
\[F(D_x, D_t, D_n) e^{\xi_1} e^{\xi_2} = F(\alpha_1 - \alpha_2, \omega_1 - \omega_2, \exp(\delta(v_1 - v_2))) e^{\xi_1 + \xi_2},\]
where $F(D_x, D_t, D_n)$ is a polynomial about operators $D_x, D_t$ and $D_n$. This properties are useful in deriving Hirota’s bilinear form and constructing periodic wave solutions of nonlinear equations.

In the following, we introduce a general Riemann theta function and discuss its periodicity, which plays a central role in the construction of periodic solutions of nonlinear equations. The Riemann theta function reads
\[\vartheta \left[ \begin{array}{c} \xi \\ \tau \end{array} \right] (\xi, \tau) = \sum_{m \in \mathbb{Z}} \exp \{ 2\pi i (\xi + \varepsilon)(m + s) - \pi \tau (m + s)^2 \}.\]
(2.2)
Here the integer value $m \in \mathbb{Z}$, complex parameter $s, \varepsilon \in \mathbb{C}$, and complex phase variables $\xi \in \mathbb{C}$; The $\tau > 0$ which is called the period matrix of the Riemann theta function.

In the definition of the theta function (2.2), for the case $s = \varepsilon = 0$, hereafter we use $\vartheta(\xi, \tau) = \vartheta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (\xi, \tau)$ for simplicity. Moreover, we have $\vartheta \left[ \begin{array}{c} \xi \\ \varepsilon \end{array} \right] (\xi, \tau) = \vartheta(\xi + \varepsilon, \tau)$.

Definition 1. A function $g(t)$ on $\mathbb{C}$ is said to be quasi-periodic in $t$ with fundamental periods $T_1, \ldots, T_k \in \mathbb{C}$, if $T_1, \ldots, T_k$ are linearly dependent over $\mathbb{Z}$ and there exists a function $G(y_1, \ldots, y_k)$, such that
\[G(y_1, \ldots, y_j + T_j, \ldots, y_k) = G(y_1, \ldots, y_j, \ldots, y_k), \quad \text{for all } y_j \in \mathbb{C}, j = 1, \ldots, k.\]
\[G(t, \ldots, t, \ldots, t) = g(t).\]
In particular, $g(t)$ is called double periodic as $k = 2$, and it becomes periodic with $T$ if and only if $T_j = m_j T$, $j = 1, \ldots, k$. \(\square\)

Let’s first see the periodicity of the theta function $\vartheta(\xi, \tau)$.

Proposition 2. \[\vartheta(\xi + 1 + i\tau, \tau) = \exp(-2\pi i \xi + \pi \tau) \vartheta(\xi, \tau).\] (2.3)
We regard the vectors 1 and \( i\tau \) as periods of the theta function \( \vartheta(\xi, \tau) \) with multipliers 1 and \( \exp(-2\pi i\xi + \pi \tau) \), respectively. Here, \( i\tau \) is not a period of theta function \( \vartheta(\xi, \tau) \), but it is the period of the functions \( \partial_\xi^2 \ln \vartheta(\xi, \tau), \partial_\xi \ln[\vartheta(\xi + e, \tau)/\vartheta(\xi + h, \tau)] \) and \( \vartheta(\xi + e, \tau)\vartheta(\xi - e, \tau)/\vartheta(\xi + h, \tau)^2 \).

**Proposition 3.** The meromorphic functions \( f(\xi) \) on \( \mathbb{C} \) are as follow

(i) \[ f(\xi) = \partial_\xi^2 \ln \vartheta(\xi, \tau), \quad \xi \in \mathbb{C}, \]

(ii) \[ f(\xi) = \partial_\xi \ln \frac{\vartheta(\xi + e, \tau)}{\vartheta(\xi + h, \tau)}, \quad \xi, \ e, \ h \in \mathbb{C}. \]

(iii) \[ f(\xi) = \frac{\vartheta(\xi + e, \tau)\vartheta(\xi - e, \tau)}{\vartheta(\xi, \tau)^2}, \quad \xi, \ e, \ h \in \mathbb{C}. \]

then in all three cases (i)–(iii), it holds that

\[ f(\xi + 1 + i\tau) = f(\xi), \quad \xi \in \mathbb{C}, \]

that is, \( f(\xi) \) is a double periodic function with 1 and \( i\tau \).

**Proof.** By using (2.3), it is easy to see that

\[ \frac{\partial_\xi \vartheta(\xi + 1 + i\tau, \tau)}{\vartheta(\xi + 1 + i\tau, \tau)} = -2\pi i + \frac{\partial_\xi \vartheta(\xi, \tau)}{\vartheta(\xi, \tau)}, \]

or equivalently

\[ \partial_\xi \ln \vartheta(\xi + 1 + i\tau, \tau) = -2\pi i + \partial_\xi \ln \vartheta(\xi, \tau). \] (2.5)

Differentiating (2.5) with respective to \( \xi \) again immediately proves the formula (2.4) for the case (i). The formula (2.4) can be proved for the cases (ii) and (iii) in a similar manner. \( \Box \)

**Theorem 1.** Suppose that \( \vartheta \left[ \begin{array}{c} \varepsilon' \\ 0 \end{array} \right] (\xi, \tau) \) and \( \vartheta \left[ \begin{array}{c} \varepsilon \\ 0 \end{array} \right] (\xi, \tau) \) are two Riemann theta functions, in which \( \xi = \alpha x + \omega t + \nu n + \sigma \). Then Hirota bilinear operators \( D_x, D_t \) and \( D_n \) exhibit the following perfect properties when they act on a pair of theta functions

\[
D_x \vartheta \left[ \begin{array}{c} \varepsilon' \\ 0 \end{array} \right] (\xi, \tau) \cdot \vartheta \left[ \begin{array}{c} \varepsilon \\ 0 \end{array} \right] (\xi, \tau) = \sum_{\mu=0,1} \partial_x \vartheta \left[ \begin{array}{c} \varepsilon' - \varepsilon \\ -\mu/2 \end{array} \right] (2\xi, 2\tau) |_{\xi=0} \vartheta \left[ \begin{array}{c} \varepsilon' + \varepsilon \\ \mu/2 \end{array} \right] (2\xi, 2\tau), \]

\[
\exp(\delta D_n) \vartheta \left[ \begin{array}{c} \varepsilon' \\ 0 \end{array} \right] (\xi, \tau) \cdot \vartheta \left[ \begin{array}{c} \varepsilon \\ 0 \end{array} \right] (\xi, \tau) = \sum_{\mu=0,1} \exp(\delta D_n) \vartheta \left[ \begin{array}{c} \varepsilon' - \varepsilon \\ -\mu/2 \end{array} \right] (2\xi, 2\tau) |_{\xi=0} \vartheta \left[ \begin{array}{c} \varepsilon' + \varepsilon \\ \mu/2 \end{array} \right] (2\xi, 2\tau), \]

where the notation \( \sum_{\mu=0,1} \) represents two different transformations corresponding to \( \mu = 0, 1 \).

The bilinear formula for \( t \) is the same as (2.6) by replacing \( \partial_x \) with \( \partial_t \).
In general, for a polynomial operator $F(D_x, D_t, D_n)$ with respect to $D_x, D_t$ and $D_n$, we have the following useful formula

$$F(D_x, D_t, D_n)\vartheta \left[ \frac{\varepsilon'}{0} \right](\xi, \tau) \cdot \vartheta \left[ \frac{\varepsilon}{0} \right](\xi, \tau) = \left[ \sum_{\mu} C(\varepsilon', \varepsilon, \mu) \right] \vartheta \left[ \frac{\varepsilon' + \varepsilon}{\mu/2} \right](2\xi, 2\tau), \tag{2.8}$$

in which, explicitly

$$C(\varepsilon, \varepsilon', \mu) = \sum_{m \in \mathbb{Z}} F(M) \exp \left[ -2\pi \tau (m - \mu/2)^2 - 2\pi i (m - \mu/2)(\varepsilon' - \varepsilon) \right]. \tag{2.9}$$

where we denote vector $M = (4\pi i (m - \mu/2)\alpha, 4\pi i (m - \mu/2)\omega, \exp[4\pi i (m - \mu/2)\delta \nu])$.

**Proof.** Making use of Proposition 1, we obtain the relation

$$D_x \vartheta \left[ \frac{\varepsilon'}{0} \right](\xi, \tau) \cdot \vartheta \left[ \frac{\varepsilon}{0} \right](\xi, \tau) = \sum_{m', \mu \in \mathbb{Z}} D_x \exp\{2\pi i m' (\xi + \varepsilon') - \pi m'^2 \tau\} \cdot \exp\{2\pi i m (\xi + \varepsilon) - \pi m^2 \tau\},$$

$$= \sum_{m', \mu \in \mathbb{Z}} 2\pi i \alpha (m' - m) \exp\{2\pi i (m' + m)\xi - 2\pi i (m'\varepsilon' + m\varepsilon) - \pi \tau [m'^2 + m^2]\}.$$  

By shifting sum index as $m = l' - m'$, then

$$\Delta = \sum_{l', m' \in \mathbb{Z}} 2\pi i \alpha (2m' - l') \exp\{2\pi i l' \xi - 2\pi i [m'\varepsilon' + (l' - m')\varepsilon] - \pi \tau [m'^2 + (l' - m')^2]\}$$

$$= \sum_{l', m' \in \mathbb{Z}} 2\pi i \alpha (2m' - 2l - \mu) \exp\{4\pi i \xi (l + \mu/2) - 2\pi i l' \xi + (l' - m')\varepsilon - \pi [m'^2 + (m' - 2l - \mu)^2]\tau\}$$

Finally letting $m' = k + l$, we conclude that

$$\Delta = \sum_{\mu = 0, 1} \left[ \sum_{k \in \mathbb{Z}} 4\pi i \alpha [k - \mu/2] \exp\{-2\pi i (k - \mu/2)(\varepsilon' - \varepsilon) - 2\pi \tau (k - \mu/2)^2\} \right]$$

$$\times \left[ \sum_{l \in \mathbb{Z}} \exp\{2\pi i (l + \mu/2) (2\xi + \varepsilon' + \varepsilon) - 2\pi \tau (l + \mu/2)^2\} \right]$$

$$= \sum_{\mu = 0, 1} \partial_x \vartheta \left[ \frac{\varepsilon' - \varepsilon}{-\mu/2} \right](2\xi, 2\tau) |_{\xi = 0} \vartheta \left[ \frac{\varepsilon' + \varepsilon}{\mu/2} \right](2\xi, 2\tau),$$

by using the following relations

$$k + l = (k - \mu/2) + (l + \mu/2), \quad k - l - \mu = (k - \mu/2) - (l + \mu/2).$$

In a similar way, we can prove the formula (2.7). The formula (2.8) follows from (2.6) and (2.7). □

**Remark 1.** The formulae (2.8) and (2.9) show that if the following equations are satisfied

$$C(\varepsilon, \varepsilon', \mu) = 0, \tag{2.10}$$
for \( \mu = 0,1 \), then \( \vartheta \left[ \varepsilon' \right]_0 (\xi,\tau) \) and \( \vartheta \left[ \varepsilon \right]_0 (\xi,\tau) \) are periodic wave solutions of the bilinear equation

\[
F(D_x, D_t, D_n) \vartheta \left[ \varepsilon' \right]_0 (\xi,\tau) \cdot \vartheta \left[ \varepsilon \right]_0 (\xi,\tau) = 0.
\]

The formula (2.10) contains two equations which are called constraint equations. This formula actually provides us an unified approach to construct double periodic wave solutions for both differential and difference equations. Once a equation is written bilinear forms, then its periodic wave solutions can be directly obtained by solving system (2.10).

**Theorem 2.** Let \( C(\varepsilon, \varepsilon', \mu) \) and \( F(D_x, D_t, D_n) \) be given in Theorem 1, and make a choice such that \( \varepsilon' - \varepsilon = \pm 1/2 \). Then

(i) If \( F(D_x, D_t, D_n) \) is an even function in the form

\[
F(-D_x, -D_t, -D_n) = F(D_x, D_t, D_n),
\]

then \( C(\varepsilon, \varepsilon', \mu) \) vanishes automatically for the case \( \mu = 1 \), namely

\[
C(\varepsilon, \varepsilon', \mu) = 0, \text{ for } \mu = 1. \tag{2.11}
\]

(ii) If \( F(D_x, D_t, D_n) \) is an odd function in the form

\[
F(-D_x, -D_t, -D_n) = -F(D_x, D_t, D_n),
\]

then \( C(\varepsilon, \varepsilon', \mu) \) vanishes automatically for the case \( \mu = 0 \), namely

\[
C(\varepsilon, \varepsilon', \mu) = 0, \text{ for } \mu = 0. \tag{2.12}
\]

**Proof.** We are going to consider the case where \( F(D_x, D_t, D_n) \) is an even function and prove the formula (2.11). The formula (2.12) is analogous. Making transformation \( m = -\bar{m} + \mu \), and noting \( F(D_x, D_t, D_n) \) is even, we then deduce that

\[
C(\varepsilon, \varepsilon', \mu) = \sum_{\bar{m} \in \mathbb{Z}} F(-\mathcal{M}) \exp \left[ -2\pi r(\bar{m} - \mu/2)^2 + 2\pi i(\bar{m} - \mu/2)(\varepsilon' - \varepsilon) \right]
= C(\varepsilon, \varepsilon', \mu) \exp \left[ 4\pi i(\bar{m} - \mu/2)(\varepsilon' - \varepsilon) \right]
= C(\varepsilon, \varepsilon', \mu) \exp \left( \pm 2\pi i\bar{m} \right) \exp \left( \pm \pi i\mu \right) = -C(\varepsilon, \varepsilon', \mu),
\]

which proves the formula (2.11). \( \Box \)

**Corollary 1.** Let \( \varepsilon'_j - \varepsilon_j = \pm 1/2, \ j = 1, \cdots, N \). Assume \( F(D_x, D_t, D_n) \) is a linear combination of even and odd functions

\[
F(D_x, D_t, D_n) = F_1(D_x, D_t, D_n) + F_2(D_x, D_t, D_n),
\]
where $F_1(D_x, D_t, D_n)$ is even and $F_2(D_x, D_t, D_n)$ is odd. In addition, $C(\varepsilon, \varepsilon', \mu)$ corresponding (2.9) is given by

$$C(\varepsilon, \varepsilon', \mu) = C_1(\varepsilon, \varepsilon', \mu) + C_2(\varepsilon, \varepsilon', \mu),$$

where

$$C_1(\varepsilon, \varepsilon', \mu) = \sum_{m \in \mathbb{Z}^N} F_1(M) \exp \left[ -2\pi \tau (m - \mu/2)^2 - 2\pi i (m - \mu/2)(\varepsilon' - \varepsilon) \right],$$

$$C_2(\varepsilon, \varepsilon', \mu) = \sum_{m \in \mathbb{Z}^N} F_2(M) \exp \left[ -2\pi \tau (m - \mu/2)^2 - 2\pi i (m - \mu/2)(\varepsilon' - \varepsilon) \right].$$

Then

$$C(\varepsilon, \varepsilon', \mu) = C_2(\varepsilon, \varepsilon', \mu) \quad \text{for} \quad \mu = 1,$$

$$C(\varepsilon, \varepsilon', \mu) = C_1(\varepsilon, \varepsilon', \mu) \quad \text{for} \quad \mu = 0. \quad (2.13)$$

**Proof.** In a similar to the proof of Theorem 2, shifting sum index as $m = -\bar{m} + \mu$, and using $F_1(D_x, D_t, D_n)$ even and $F_2(D_x, D_t, D_n)$ odd, we have

$$C(\varepsilon, \varepsilon', \mu) = C_1(\varepsilon, \varepsilon', \mu) + C_2(\varepsilon, \varepsilon', \mu)$$

$$= [C_1(\varepsilon, \varepsilon', \mu) - C_2(\varepsilon, \varepsilon', \mu)] \exp(\pm \pi i \mu). \quad (2.15)$$

Then for $\mu = 1$, the equation (2.15) gives

$$C_1(\varepsilon, \varepsilon', \mu) = 0,$$

which implies the formula (2.13). The formula (2.14) is is analogous. □

The theorem 2 and corollary 1 are very useful to deal with coupled Hirota’s bilinear equations, which will be seen in the following section 4.

3. **The shallow water wave equation**

The shallow water wave equation takes the form \[16\]

$$u_t - u_{xxt} - 3uu_t + 3u \int_{x}^{\infty} u_t dx + u_x = 0, \quad (3.1)$$

which is like to the KdV equation in the family of shallow water wave equations. Hirota and Satsuma obtained soliton solutions of the equation by means of bilinear method \[17\]. Here we construct its a double periodic wave solution and show that the one-soliton solution can be obtained as limiting case of the double periodic solution.

To apply the Hirota bilinear method for constructing double periodic wave solutions of the equation (3.1), we consider a variable transformation

$$u = 2\partial_x^2 \ln f(x,t). \quad (3.2)$$
Substituting (3.2) into (3.1) and integrating with respect to $x$, we then get the following Hirota’s bilinear form

$$F(D_x, D_t)f \cdot f = (D_x D_t + D_x^2 - D_tD_x^3 + c)f \cdot f = 0,$$

where $c$ is an integration constant. In the special case of $c = 0$, starting from the bilinear equation (3.3), it is easy to find its one-soliton solution

$$u_1 = 2\partial_x^2 \ln(1 + e^\eta),$$

with phase variable $\eta = px + \frac{p^2}{p^2 - 1}t + \gamma$ for every $p$ and $\gamma$.

Next, we turn to see the periodicity of the solution (3.2), the function $f$ is chosen to be a Riemann theta function, namely,

$$f(x, t) = \vartheta(\xi, \tau),$$

where phase variable $\xi = \alpha x + \omega t + \sigma$. With Proposition 3, we refer to

$$u = 2\partial_x^2 \ln \vartheta(\xi, \tau) = 2\alpha^2 \partial_x^2 \ln \vartheta(\xi, \tau),$$

which shows that the solution $u$ is a double periodic function with two fundamental periods 1 and $i\tau$.

We introduce the notations by

$$\lambda = e^{-\pi \tau/2}, \quad \vartheta_1(\xi, \lambda) = \vartheta(2\xi, 2\tau) = \sum_{m \in \mathbb{Z}} \lambda^{4m^2} \exp(4i\pi m \xi),$$

$$\vartheta_2(\xi, \lambda) = \vartheta \left[ \begin{array}{c} 0 \\ -1/2 \end{array} \right] (2\xi, 2\tau) = \sum_{m \in \mathbb{Z}} \lambda^{(2m-1)^2} \exp[2i\pi (2m - 1)\xi],$$

where the phase variable $\xi = \alpha x + \omega t + \sigma$.

Substituting (3.5) into (3.3), using formula (2.10) and (3.7) leads to a linear system (corresponding to $\mu = 0$ and $\mu = 1$, respectively)

$$[\vartheta''_1(0, \lambda)\alpha + \vartheta^{(4)}_1(0, \lambda)\alpha^4]\omega + \vartheta_1(0, \lambda)c + \vartheta''_1(0, \lambda)\alpha^2 = 0,$$

$$[\vartheta''_2(0, \lambda)\alpha + \vartheta^{(4)}_2(0, \lambda)\alpha^4]\omega + \vartheta_2(0, \lambda)c + \vartheta''_2(0, \lambda)\alpha^2 = 0,$$

where we have denoted the derivative of $\vartheta_j(\xi, \lambda)$ at $\xi = 0$ by notations

$$\vartheta_j^{(k)}(0, \lambda) = \frac{d^k \vartheta_j(\xi, \lambda)}{d\xi^k}|_{\xi=0}, \quad j = 1, 2; k = 1, 2, 3, 4.$$ 

This system admits an explicit solution $(\omega, c)$. In this way, we obtain an explicit periodic wave solution (3.6) with parameters $\omega, c$ by (3.8), while other parameters $\alpha, \sigma, \tau, \sigma$ are free.

In summary, double periodic wave (3.6) possesses the following features: (i) It is is one-dimensional, i.e. there is a single phase variable $\xi$. Moreover, it has two fundamental periods
and \(i\tau\) in phase variable \(\xi\), but it need not to be periodic in \(x\) and \(t\). (ii) It can be viewed as a parallel superposition of overlapping one-soliton waves, placed one period apart.

In the following, we further consider asymptotic properties of the periodic wave solution. Interestingly, the relation between the periodic wave solution (3.6) and the one-soliton solution (3.4) can be established as follows.

**Theorem 3.** Suppose that the vector \((\omega, c)\) is a solution of the system (3.8), and for the periodic wave solution (3.6), we let

\[
\alpha = \frac{p}{2\pi i}, \quad \sigma = \frac{\gamma + \pi \tau}{2\pi i},
\]

where the \(p\) and \(\gamma\) are given in (3.4). Then we have the following asymptotic properties

\[
c \to 0, \quad 2\pi i \xi - \pi \tau \to \eta = px + \frac{p}{p^2 - 1} t + \gamma,
\]

\[
\vartheta(\xi, \tau) \to 1 + e^{\eta}, \quad \text{as} \quad \lambda \to 0.
\]

In other words, the double periodic solution (3.10) tends to the soliton solution (3.4) under a small amplitude limit, that is,

\[
u \to u_1, \quad \text{as} \quad \lambda \to 0.
\]

**Proof.** Here we will directly use the system (3.8) to analyze asymptotic properties of the periodic solution (3.6). Since the coefficients of system (3.8) are power series about \(\lambda\), its solution \((\omega, c)\) also should be a series about \(\lambda\). We explicitly expand the coefficients of system (3.8) as follows

\[
\vartheta_1(0, \lambda) = 1 + 2\lambda^4 + \cdots, \quad \vartheta_1''(0, \lambda) = -32\pi^2 \lambda^4 + \cdots, \quad \vartheta_1^{(4)}(0, \lambda) = 512\pi^4 \lambda^4 + \cdots, \quad \vartheta_2(0, \lambda) = 2\lambda + 2\lambda^9 + \cdots
\]

\[
\vartheta_2''(0, \lambda) = -8\pi^2 \lambda + \cdots, \quad \vartheta_2^{(4)}(0, \lambda) = 32\pi^4 \lambda + \cdots.
\]

Let the solution of the system (3.8) be in the form

\[
\omega = \omega_0 + \omega_1 \lambda + \omega_2 \lambda^2 + \cdots = \omega_0 + o(\lambda),
\]

\[
c = c_0 + c_1 \lambda + c_2 \lambda^2 + \cdots = c_0 + o(\lambda).
\]

Substituting the expansions (3.11) and (3.12) into the system (3.8) (the second equation is divided by \(\lambda\)) and letting \(\lambda \to 0\), we immediately obtain the following relations

\[
c_0 = 0, \quad (-8\pi^2 \alpha + 32\pi^4 \alpha^3)\omega_0 - 8\pi^2 \alpha^2 = 0,
\]

which implies

\[
c_0 = 0, \quad \omega_0 = \frac{\alpha}{4\pi^2 \alpha^2 - 1}.
\]
Combining (3.12) and (3.13) then yields
\[ c \to 0, \quad 2\pi i \omega \to \frac{2\pi ic}{(2\pi i \alpha)^2 - 1} = \frac{p}{p^2 - 1}, \quad \text{as} \quad \lambda \to 0. \]
Hence we conclude
\[ \hat{\xi} = 2\pi i \xi - \pi \tau = px + 2\pi i \omega t + \gamma \]
\[ \to px + \frac{p}{p^2 - 1} t + \gamma = \eta, \quad \text{as} \quad \lambda \to 0. \quad (3.14) \]

It remains to consider asymptotic properties of the periodic wave solution (3.6) under the limit \( \lambda \to 0 \). By expanding the Riemann theta function \( \vartheta(\xi, \tau) \) and using (3.14), it follows that
\[ \vartheta(\xi, \tau) = 1 + \lambda^2 (e^{2\pi i \xi} + e^{-2\pi i \xi}) + \lambda^4 (e^{4\pi i \xi} + e^{-4\pi i \xi}) + \ldots \]
\[ = 1 + e^{\hat{\xi}} + \lambda^4 (e^{-\hat{\xi}} + e^{2\hat{\xi}}) + \lambda^{12} (e^{-2\hat{\xi}} + e^{3\hat{\xi}}) + \ldots \]
\[ \to 1 + e^{\hat{\xi}} \to 1 + e^{\eta}, \quad \text{as} \quad \lambda \to 0, \]
which together with (3.6) leads to (3.10). Therefore we conclude that the double periodic solution (3.6) just goes to the one-soliton solution (3.4) as the amplitude \( \lambda \to 0 \). □

4. The modified Bogoyavlenskii-Schiff equation

We consider (2+1)-dimensional modified Bogoyavlenskii-Schiff equation \[18\]
\[ u_t - 4u^2u_z - 2u_x \partial_x^{-1}(u^2)_z + u_{xxx} = 0, \quad (4.1) \]
which was deduced from the Miura transformation \[19\]. Equation (4.1) is reduced to the modified KdV equation in the case of \( x = z \).

We shall construct a double periodic wave solution to the equation (4.1) by using Theorem 1 and 2. The equation (4.1) can be described by a coupled system
\[ u = \psi_x, \]
\[ \rho_{xx} + \psi_x^2 + c = 0, \quad (4.2) \]
\[ \psi_t + 2\psi_x \rho_{xx} + \psi_z (\rho_{xx} + \psi_x^2 + c) + \psi_{xxx} = 0. \quad (4.3) \]

We perform the dependent variable transformations
\[ u = \psi_x = \partial_x \ln \left( \frac{f}{g} \right), \quad \rho = \ln(fg), \]
\[ (4.3) \]
them then equation (4.2) is reduced to the following bilinear form
\[ F(D_x)f \cdot g = (D_x^2 + c)f \cdot g = 0, \quad (4.4) \]
\[ G(D_t, D_x, D_z)f \cdot g = (D_t + D_x^2 D_z + cD_z)f \cdot g = 0, \]
where \( c \) is a constant. The equation (4.4) is a type of coupled bilinear equations, which is more difficult to be dealt with than the single bilinear equation (3.3) due to appearance of
two functions and two equations. We will take full advantages of Theorem 2 to reduce the number of constraint equations.

Now we take into account the periodicity of the solution (4.3), in which we take \( f \) and \( g \) as

\[
    f = \vartheta(\xi + e, \tau), \quad g = \vartheta(\xi + h, \tau), \quad e, h \in \mathbb{C},
\]

where phase variable \( \xi = \alpha x + \beta z + \omega t + \sigma \). By means of Proposition 3, we find that the solution

\[
    u = \alpha \partial_\xi \ln \frac{\vartheta(\xi + e, \tau)}{\vartheta(\xi + h, \tau)}
\]

is a double periodic function with two fundamental periods 1 and \( i \tau \).

In the special case of \( c = 0 \), the equation (4.2) admits one-soliton solution

\[
    u_1 = \partial_x \ln \frac{1 + e^\eta}{1 - e^\eta}, \quad (4.6)
\]

where \( \eta = px + qy - p^2 qt + \gamma \) for every \( p, q \) and \( \gamma \).

We take \( e = 0, h = 1/2 \) in (4.5), and therefore

\[
    f = \vartheta(\xi, \tau) = \sum_{m \in \mathbb{Z}} \exp(2\pi i m \xi - \pi m^2 \tau),
\]

\[
    g = \vartheta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (\xi, \tau) = \sum_{m \in \mathbb{Z}} \exp(2\pi i m (\xi + 1/2) - \pi m^2 \tau) \quad (4.7)
\]

Due to the fact that \( F(D_x) \) is an even function, its constraint equations in the formula (2.10) vanish automatically for \( \mu = 1 \). Similarly the constraint equations associated with \( G(D_t, D_x, D_z) \) also vanish automatically for \( \mu = 0 \). Therefore, the Riemann theta function (4.6) is a solution of the bilinear equation (4.4), provided the following equations

\[
    \vartheta_1''(0, \lambda) \alpha^2 + \vartheta_1(0, \lambda) c = 0,
\]

\[
    \vartheta_2'(0, \lambda) \omega + \vartheta_2'(0, \lambda) \beta c + \vartheta_2''(0, \lambda) \alpha^2 \beta = 0, \quad (4.8)
\]

where we introduce the notations by

\[
    \lambda = e^{-\pi \tau/2}, \quad \vartheta_1(\xi, \lambda) = \vartheta(2\xi, 2\tau) = \sum_{m \in \mathbb{Z}} \lambda^{4mn^2} \exp(4i\pi mn \xi),
\]

\[
    \vartheta_2(\xi, \lambda) = \vartheta \left[ \begin{array}{c} 1/2 \\ -1/2 \end{array} \right] (2\xi, 2\tau) = \sum_{m \in \mathbb{Z}} (-1)^m \lambda^{(2m-1)^2} \exp[2i\pi(2m - 1) \xi].
\]

It is obvious that equation (4.8) admits an explicit solution \( \omega \) and \( c \). In this way, a periodic wave solution reads

\[
    u = \partial_x \ln \frac{\vartheta(\xi, \tau)}{\vartheta(\xi + 1/2, \tau)} \quad (4.9)
\]
where parameters \( \omega \) and \( c \) are given by (4.11), while other parameters \( \alpha, \beta, \tau, \sigma \) are free. In summary, double periodic wave (4.9) has the following features: (i) It is one-dimensional and has two fundamental periods 1 and \( i\tau \) in phase variable \( \xi \). (ii) It can be viewed as a parallel superposition of overlapping one-soliton waves, placed one period apart.

In the following, we further consider asymptotic properties of the double periodic wave solution. The relation between the periodic wave solution (4.9) and the one-soliton solution (4.6) can be established as follows.

**Theorem 4.** Suppose that the vector \( (\omega, c)^T \) is a solution of the system (4.8). In the periodic wave solution (4.9), we choose parameters as

\[
\alpha = \frac{p}{2\pi i}, \quad \beta = \frac{q}{2\pi i}, \quad \sigma = \frac{\gamma + \pi \tau}{2\pi i},
\]

where the \( p, q \) and \( \gamma \) are the same as those in (4.6). Then we have the following asymptotic properties

\[
c \to 0, \quad \xi \to \frac{\eta + \pi \tau}{2\pi i}, \quad f \to 1 + e^{\eta}, \quad g \to 1 - e^{\eta}, \quad \text{as} \quad \lambda \to 0.
\]

In other words, the double periodic solution (4.9) tends to the one-soliton solution (4.6) under a small amplitude limit, that is,

\[
u \to u_1, \quad \text{as} \quad \lambda \to 0.
\]

**Proof.** Here we will directly use the system (4.8) to analyze asymptotic properties of periodic solution (4.9). We explicitly expand the coefficients of system (4.8) as follows

\[
\vartheta_1(0, \lambda) = 1 + 2\lambda^4 + \cdots, \quad \vartheta''_1(0, \lambda) = -32\pi^2 \lambda^4 + \cdots,
\]

\[
\vartheta_2(0, \lambda) = -4\pi i\lambda + 12\pi i\lambda^9 + \cdots, \quad \vartheta''_2(0, \lambda) = 16\pi^2 i\lambda - 48\pi^2 i\lambda^9 + \cdots.
\]

Suppose that the solution of the system (4.8) is of the form

\[
\omega = \omega_0 + \omega_1 \lambda + \omega_2 \lambda^2 + \cdots = \omega_0 + o(\lambda),
\]

\[
c = c_0 + c_1 \lambda + c_2 \lambda^2 + \cdots = c_0 + o(\lambda).
\]

Substituting the expansions (4.12) and (4.13) into the system (4.8) and letting \( \lambda \to 0 \), we immediately obtain the following relations

\[
c_0 = 0, \quad -4\pi i\omega_0 + 16\pi^2 i\alpha^2 \beta = 0,
\]

which has a solution

\[
c_0 = 0, \quad w_0 = 4\pi^2 \alpha^2 \beta.
\]

Combining (4.13) and (4.14) leads to

\[
c \to 0, \quad 2\pi i\omega \to 8\pi^2 i\alpha^2 \beta = -p^2 q, \quad \text{as} \quad \lambda \to 0,
\]
or equivalently
\[ \dot{\xi} = 2\pi i \xi - \pi \tau = px + qy + 2\pi i \omega t + \gamma \]
\[ \rightarrow px + qy - p^2 qt + \gamma = \eta, \quad \text{as} \quad \lambda \to 0. \] (4.15)

It remains to identify that the periodic wave (4.9) possesses the same form with the one-soliton solution (4.6) under the limit \( \lambda \to 0 \). For this purpose, we start to expand the functions \( f \) and \( g \) in the form
\[ f = 1 + \lambda^2 (e^{2\pi i \xi} + e^{-2\pi i \xi}) + \lambda^8 (e^{4\pi i \xi} + e^{-4\pi i \xi}) + \cdots \]
\[ g = 1 - \lambda^2 (e^{2\pi i \xi} + e^{-2\pi i \xi}) + \lambda^8 (e^{4\pi i \xi} + e^{-4\pi i \xi}) + \cdots \]
By using (4.13)-(4.15), it follows that
\[ f = 1 + e^{\hat{\xi}} + \lambda^4 (e^{-\hat{\xi}} + e^{2\hat{\xi}}) + \lambda^{12} (e^{-2\hat{\xi}} + e^{3\hat{\xi}}) + \cdots \]
\[ \rightarrow 1 + e^{\hat{\xi}} \rightarrow 1 + e^\eta, \quad \text{as} \quad \lambda \to 0; \]
\[ g = 1 - e^{\hat{\xi}} + \lambda^4 (e^{2\hat{\xi}} - e^{-\hat{\xi}}) + \lambda^{12} (e^{-2\hat{\xi}} - e^{3\hat{\xi}}) + \cdots \]
\[ \rightarrow 1 - e^{\hat{\xi}} \rightarrow 1 - e^\eta, \quad \text{as} \quad \lambda \to 0. \] (4.16)
The expression (4.11) follows from (4.16), and thus we conclude that the double periodic solution (4.9) just goes to the one-soliton solution (4.6) as the amplitude \( \lambda \to 0 \). □

5. The differential-difference KdV equation

We consider differential-difference KdV equation
\[ \frac{d}{dt} \left( \frac{u(n)}{1 + u(n)} \right) = u(n - 1/2) - u(n + 1/2). \] (5.1)
Hirota and Hu have found its soliton solutions and rational solutions \([20][21]\), among them one-soliton solution reads
\[ u_1(n) = \frac{(1 + e^{\eta + p/2})(1 + e^{\eta - p/2})}{(1 + e^\eta)^2} - 1, \] (5.2)
where \( \eta = pn - \sinh(p/2)t + \gamma \) for every \( p \) and \( \gamma \).

We shall construct a periodic wave solutions to the equation (5.1) by using Theorem 1. By means of a variable transformation
\[ u(n) = \frac{f(n + 1/2) f(n - 1/2)}{f(n)^2} - 1, \] (5.3)
the equation (5.1) is reduced to the bilinear equation
\[ \left[ \sinh(\frac{1}{4}D_n) D_t + 2 \sinh(\frac{1}{4}D_n) \sinh(\frac{1}{2}D_n) + c \right] f(n) \cdot f(n) = 0, \] (5.4)
where \( c \) is a constant.
Now we take into account the periodicity of the solution (5.3), in which we take \( f(n) = \vartheta(\xi, \tau) \), where phase variable \( \xi = \nu n + \omega t + \sigma \). Then solution (5.3) is written as

\[
u(n) = \vartheta(\xi + \frac{1}{2} \nu, \tau) \vartheta(\xi - \frac{1}{2} \nu, \tau) - 1. \tag{5.5}\]

By means of Proposition 2, it is easy to deduce that \( u_n \) is a double periodic function with two fundamental periods 1 and \( i\tau \).

Substituting (5.5) into (5.4) and using formula (2.10) leads to a linear system

\[
\sinh(\frac{1}{4} D_n) \vartheta'_{1}(0, \lambda) \omega + \vartheta_{1}(0, \lambda)c + \sinh(\frac{1}{4} D_n) \sinh(\frac{1}{2} D_n) \vartheta_{1}(0, \lambda) = 0, \\
\sinh(\frac{1}{4} D_n) \vartheta'_{2}(0, \lambda) \omega + \vartheta_{2}(0, \lambda)c + \sinh(\frac{1}{4} D_n) \sinh(\frac{1}{2} D_n) \vartheta_{2}(0, \lambda) = 0, 
\tag{5.6}
\]

where \( \vartheta_{1}(\xi, \lambda) \) and \( \vartheta_{2}(\xi, \lambda) \) are the same as those in (3.7) with \( \xi = \nu n + \omega t + \sigma \). By using the solution \( \omega \) and \( c \) of system (5.6), a periodic wave solution is obtained by (5.5).

In the following, we further consider asymptotic properties of the double periodic wave solution. The relation between the periodic wave solution (5.5) and the one-soliton solution (5.2) can be established as follows.

**Theorem 5.** Suppose that the vector \( (\omega, c)^T \) is a solution of the system (5.6). In the periodic wave solution (5.5), we choose parameters as

\[
\nu = \frac{p}{2\pi i}, \quad \sigma = \frac{\gamma + \pi \tau}{2\pi i}, \tag{5.7}
\]

where the \( p \) and \( \gamma \) are the same as those in (5.2). Then we have the following asymptotic properties

\[
c \to 0, \quad \xi \to \frac{\eta + \pi \tau}{2\pi i}, \quad \vartheta(\xi, \tau) \to 1 + e^{\theta}, \quad \text{as} \quad \lambda \to 0.
\]

In other words, the periodic solution (5.5) tends to the one-soliton solution (5.2) under a small amplitude limit, that is,

\[
u(n) \to u_{1}(n), \quad \text{as} \quad \lambda \to 0. \tag{5.8}
\]

**Proof.** Here we will directly use the system (5.6) to analyze asymptotic properties of periodic solution (5.5). We explicitly expand the coefficients of system (5.6) as follows

\[
\vartheta_{1}(0, \lambda) = 1 + 2\lambda^{3} + \cdots, \quad \sinh(\frac{1}{4} D_n) \vartheta'_{1}(0, \lambda) = 8\pi i \sinh(i\pi \nu) \lambda^{4} + \cdots, \\
\sinh(\frac{1}{4} D_n) \sinh(\frac{1}{2} D_n) \vartheta_{1}(0, \lambda) = 2\sinh(i\pi \nu) \sinh(2i\pi \nu) \lambda^{4} + \cdots, \\
\vartheta_{2}(0, \lambda) = 2\lambda + 2\lambda^{9} + \cdots, \quad \sinh(\frac{1}{4} D_n) \vartheta'_{2}(0, \lambda) = 4\pi i \sinh(i\pi \nu/2) \lambda + \cdots, \\
\sinh(\frac{1}{4} D_n) \sinh(\frac{1}{2} D_n) \vartheta_{2}(0, \lambda) = 2\sinh(i\pi \nu) \sinh(i\pi \nu/2) \lambda + \cdots. \tag{5.9}
\]
Suppose that the solution of the system (5.6) is of the form
\[
\omega = \omega_0 + \omega_1 \lambda + \omega_2 \lambda^2 + \cdots = \omega_0 + o(\lambda),
\]
\[
c = c_0 + c_1 \lambda + c_2 \lambda^2 + \cdots = c_0 + o(\lambda).
\]
(5.10)

Substituting the expansions (5.9) and (5.10) into the system (5.6) and letting \( \lambda \to 0 \), we immediately obtain the following relations
\[
c_0 = 0, \quad 4\pi i \sinh(i\pi \nu/2)\omega_0 + 2 \sinh(i\pi \nu/2) \sinh(i\pi \nu) = 0,
\]
which implies
\[
c_0 = 0, \quad w_0 = -\frac{1}{2\pi i} \sinh(i\pi \nu).
\]
(5.11)

Combining (5.9) and (5.10) leads to
\[
c \to 0, \quad 2\pi i \omega \to - \sinh(i\pi \nu) = - \sinh(p/2), \quad \text{as} \quad \lambda \to 0,
\]
or equivalently
\[
\dot{\xi} = 2\pi i \xi - \pi \tau = pn + 2\pi i \omega t + \gamma
\]
\[
\to pn - \sinh(p/2) t + \gamma = \eta, \quad \text{as} \quad \lambda \to 0.
\]
(5.12)

It remains to consider asymptotic properties of the periodic wave solution (5.5) under the limit \( \lambda \to 0 \). By expanding the Riemann theta function \( \vartheta(\xi, \tau) \), it follows that
\[
\vartheta(\xi, \tau) = 1 + e^{\xi} + \lambda^4 (e^{-\xi} + e^{2\xi}) + \lambda^{12} (e^{-2\xi} + e^{3\xi}) + \cdots
\]
\[
\to 1 + e^{\eta}, \quad \text{as} \quad \lambda \to 0,
\]
which together with (5.5) lead to (5.8). Therefore we conclude that the periodic solution (5.5) just goes to the one-soliton solution (5.2) as the amplitude \( \lambda \to 0 \). □

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