Full two-dimensional numerical study of the quantum-Hall Skyrmion crystal

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(Dated: March 22, 2022)

PACS numbers: 73.43.-f,12.39.Dc,05.10.-a

Spin textures arise in the effective action approach to the quantum Hall effect. Up to now these textures, also called Skyrmions, have been mainly studied using approximations. Studies of its finite density form, the crystal, have been limited to a given symmetry and tend to ignore Skyrmion deformation. Using a simulated annealing technique, adapted to problems with a long-range interaction, we are able to present the first full two-dimensional study of quantum Hall Skyrmions and the corresponding crystals. Our results show that only the Skyrmions with topological charge one and two are bound and that the Skyrmion crystal is probably a triangular lattice made of charge-two Skyrmions.

Since Skyrmions are the basic low-energy excitations of the quantum Hall model, usually called the baby Skyrmion model to distinguish it from the three-dimensional one. A thermodynamic approach was used to find its multi-Skyrmions (using simulated annealing) and to study its thermodynamical properties. We have now successfully extended this promising technique to non-local Coulomb terms.

In this Letter, we are therefore able to numerically discuss the structure of the quantum Hall multi-Skyrmions and Skyrmion crystals without any restricting approximations. We will also use our experience on the thermodynamics of the local baby Skyrme model to extrapolate to the quantum Hall model and discuss future research. We will give more details on numerical techniques and more detailed results in a longer paper.

As mentioned, the quantum Hall ground state is ferromagnetic for filling factors $\nu = 1$ and $\nu = 1/3$. The lowest Landau level projected quantum field theory is quite complicated and has interesting quasi-particle states. One can either try to use quantum-mechanical Hartree-Fock approximations, or use a semiclassical Skyrme model. We use the latter approach, where the Skyrme model arises through the effective action of the long-wavelength limit of the quantum field theory. Its Lagrange density has a time dependent term containing a gauge interaction, a gradient expansion term, a Zeeman energy functional, because it contains a non-local Coulomb interaction.

$$
\mathcal{L}_{\text{eff}} = \frac{1}{2} \rho_A (\partial_t n) - \frac{1}{2} \rho_A (\nabla n)^2 - \frac{1}{2} g \mu_B n \cdot B - \nu_{\text{FM}}^2 \frac{e^2}{2} \frac{1}{4\pi\epsilon} \int d^2 r' \frac{q(r)q'(r')}{|r - r'|},
$$

(1)

Here, $\nu_{\text{FM}}$ is the filling factor of the ferromagnetic component of the QH state. The ferromagnetic charge density is $\tilde{\rho} = \frac{\sqrt{2} e}{2\pi l}$, where the magnetic length $l = \sqrt{\hbar/eB}$. The topological charge density, $q(r) = n \cdot (\partial_x n \times \partial_y n)/4\pi$, describes the deviation of the electron density from the ferromagnetic background, $\rho(r) = \tilde{\rho} = \nu_{\text{FM}} q(r)$. The spin stiffness $\rho_s = \frac{1}{16\sqrt{2\pi}} \nu_{\text{FM}}^2 e^2/4\pi\epsilon$. The relative dielectric constant $\epsilon_r (\epsilon = \epsilon_r \epsilon_0)$ and the $g$ factor are properties of the...
sample used.

In this Letter, we ignore the time-dependent term, because we are looking for static solutions. Briefly, its role is to make the field \( \mathbf{n} \) satisfy the Poisson bracket \( \{ \mathbf{n}_i(r), \mathbf{n}_j(r') \} = \epsilon_{ijk} n_k(r) \delta(r - r') \), and the Hamiltonian is equal to the potential energy.

The Zeeman term is proportional to the unit of length squared and the Coulomb term is proportional to the inverse length \( d \). The gradient term is scale invariant. The size of the static soliton solutions of the Skyrme model is thus determined by the balance between the Coulomb and the Zeeman term. Using a virial argument, one can show that at the minimum energy solution the Coulomb energy must be twice the Zeeman energy. We find it numerically preferable to use a unit of length in which the Zeeman and Coulomb terms balance, rather than use the magnetic length as unit. Equating the coefficients, we get \( g \rho_B B d^2 = \nu_{FM}^2 / (4 \pi c d) \), which leads to

\[
\frac{d}{l} = \left( \frac{e^2 \nu_{FM}}{2 \epsilon_B B l} \right)^{1/3}
\]  

(2)

They are our units of length, and the unit of energy becomes \( e^2 / (4 \pi c d) \). The energy density reads \( E = r/d, B = B e_3 \)

\[
E = \frac{1}{2} \left( \int d^2 x \left[ \hat{\rho}_s (\nabla_x n)^2 + n_3 - 1 \right] + \int d^2 x d^2 x' \frac{q(x) q(x')}{|x - x'|} \right).
\] 

(3)

This is thus effectively a one-parameter problem in terms of \( \hat{\rho}_s = \frac{1}{16 \pi^2} \nu_{FM}^2 d^2 \). Note however that the quality of the gradient expansion depends on the size of the Skyrmions (related to \( d \)) as compared to the magnetic length \( l \), and thus comparison with experiment and HF calculations will only make quantitative sense when \( d \gg l \). For “reasonable” parameters

\[
B = 10 \text{T}, \quad \epsilon_r = 13, \quad \text{and} \quad g = -0.44,
\] 

(4)

these take the values \( l = 8.2 \) nm and \( d/l \approx 7 \).

The Skyrmions up to topological charge three have been studied in Ref. [14], using radially symmetric Ansätze. In this paper it was shown that, for a large range of parameters, the charge-two Skyrmion is bound whereas the charge-three Skyrmion is not. They also made a comparison with more microscopic Hartree-Fock results, and discussed the limitations of the classical approach. Several studies of the local baby Skyrme models show that the non-radially symmetric multi-Skyrmions can have far lower energies than their radially symmetric analogues. However, the exact details are highly dependent on the form of the potential term; see [13]. Based on our past experience, we think it is very important to do a full two-dimensional numerical study. We first study the minimization of the energy functional [3]. We discretize the model on a rectangular spatial grid and impose fixed boundary conditions with the vacuum field \( \mathbf{n} = (0, 0, 1) \) on the boundary. We start out with a field configuration of a given topological charge. Then we minimize the energy functional using simulated annealing [10]. We have extended this minimization technique to handle the long-range interaction terms arising from the Coulomb force.

We have investigated the minimal energy solutions with the topological charge one to four; see Fig. [1] for some results. We found that only the charge-two Skyrme is bound. The simulations for charge three and four seem to reveal a bound charge-three Skyrme and a isosceles-triangular charge-four Skyrmion. However, these are finite box effects. When increasing the box size we are left with individual Skyrmions of charge one and two. As energetically expected, a charge-three field splits into a charge-one and a charge-two Skyrmion, and the charge-four Skyrmion splits into two charge-two

![FIG. 1: The Skyrmions from topological charge one to three, for the parameter \( d/l = 7 \). Lighter colors denote a higher charge density, and lengths are expressed in units of \( d \).](image-url)
Skyrmions. We see clear indication that the charge-two Skyrmion might be the building block of any quantum Hall Skyrmion crystal. It is very likely that a change in the parameters of the Lagrangian, and modification of the potential are able to dramatically change the structure of the multi-Skyrmions and possibly lead to stable Skyrmions above charge two. In fact, any change of the effective action, such as higher-order gradient expansion terms, medium modifications to the Coulomb potential (see below), quantum effects, and inhomogeneities can have an impact on weakly bound multi-Skyrmions.

We now turn our attention to the study of the quantum Hall Skyrmion crystal, and produce a full two-dimensional study. We do not impose any symmetry, apart from that imposed by the periodic boundary conditions of the lattice. We need to be very careful when studying crystals with non-local Coulomb interaction terms. The crystal could be thought of as densely packed multi-Skyrmions, the quasi-particles of the Skyrme model, but the Skyrmions are all charged, and the samples used to study Skyrmions are electrically neutral. This means that Skyrmion crystals exist in a positively charged background. This also renormalizes away the infinite Coulomb energy otherwise carried by a crystal. Since we shall use a finite size box for the simulation, we can use an Ewald sum rule to evaluate the sum over all the image charges generated by the periodic boundary conditions. This method explicitly gives us the zero-wave length component of the Coulomb field which we then renormalize away (a more detailed analysis of the procedure will be given in Ref. [12]). The first two parts of the energy expression (3) are not modified when applying periodic BCs, but the Coulomb part becomes a sum over the vectors $R = (n_1 L_1, n_2 L_2)$, where $L_{1,2}$ are the lengths of the simulation volume,

$$\frac{1}{|x - x'|} \to \sum_R \frac{1}{|x - (x' + R)|}.$$  \hspace{1cm} (5)

Using Ewald’s technique, we split the sum into a sum over the real and the reciprocal lattice (the integers $k$ and $l$ are used in the sum over the real lattice, $k'$ and $l'$ for the reciprocal one)

$$V(s_x, s_y) \sum_R \frac{1}{|s + R|} = \sqrt{1/(L_1 L_2)} \left\{ \sum_{k', l'} e^{i2\pi (k's_x/L_1 + l's_y/L_2)} \right.$$

$$\times \Phi \left( \pi \left[ k^2 L_2/L_1 + l^2 L_1/L_2 \right] \right) +$$

$$+ \sum_{kl} \Phi \left( \pi \left[ (k + s_x L_1)^2 L_1 + (l + s_y L_2)^2 L_2 \right] \right) \right\}, \hspace{1cm} (6)$$

where

$$\Phi(x) = \frac{\sqrt{\pi}}{\sqrt{x}} \operatorname{erfc}(\sqrt{x}).$$

The term $k' = l' = 0$ is infinite. Adding a positive background charge leaves us a constant energy shift [12]. Using (6) minus this shift is simple and straightforward. The sums all converge extremely rapidly.

We have performed a study of these finite density quantum Hall systems, for the same parameters as the multi-Skyrmion simulations. Of course, we are already biased by our study of the multi-Skyrmions and expect a lattice of deformed charge-two Skyrmions. Our best guess is a triangular lattice, because it is the preferred lattice of the

![FIG. 2: Magnetisation in triangular charge-two Skyrmion crystals, for increasing densities. The units of length is given in terms of the parameter $d$; for $d/l = 7$ the change in filling of the quantum-Hall state is (from left to right) $\Delta \nu = 0.024, 0.0544,$ and 0.0967. Light colors denote a spin pointing upwards; the darkest denote a downwards spin. The contour levels are the same across all three graphs. See also [16].](image)
local baby Skyrme model \[11\]. Our goal was two-fold: to find the energetically favored symmetry of the lattice of charge-two Skyrmions and to show that other lattice construction are not stable. We started out with many different configurations, mostly lattices made out of charge-one Skyrmions. Nearly all configuration minimized to a configuration made out of charge-two Skyrmions. Taking finite box effects into account, the lattice seems to be triangular. Indeed we believe that the energetically preferred crystal is the triangular lattice made out of deformed charge-two Skyrmions; see Fig. 2. There appears to be a rather deep connection between these results and the complex (CP1) parametrization of the Skyrme field in terms of Weierstrass functions, which will not be pursued here, see however \[12\].

The thermodynamics of the quantum Hall Skyrme model is another interesting challenge. What are the phases of the model at finite density and temperature? Our simulated annealing already contains the thermodynamic partition function, so it is well suited to tackle this question: see the phase portrait of the baby Skyrme model \[11\]. However, the model differs in several important aspects. The baby Skyrme model has a local fourth-order term rather than the global Coulomb term. The fact that the dynamics of the Skyrme field is very different may have an influence on the thermodynamics as well. There other kinetic terms have also been discussed, see Ref. \[17\]. Nonetheless, we expect some similarities, because the crystals are similar.

The most thorough discussion of the phase diagram of Skyrmion crystals based on a “simple” potential Ansatz is given in Ref. \[12\]. Many interesting phases are discussed, and we have tried to simulate a few of those, by starting a simulation in a periodic box with a Skyrme field of the right symmetry. In all cases we found a rapid decay to crystals of charge-two Skyrmions, or extremely frustrated crystals (due to the small box used). This does not mean that such crystals cannot occur, but it seems to be a generic feature of Skyrme-like theories to have a toroidal charge-two Skyrmion that is bound: It seems to be this feature that is driving our results. Thus we would like to argue that at sufficiently high densities the Skyrme crystal looks like the ones shown in Fig. 2. At lower densities the behavior is different. As long as we stick with the effective action \[1\] we shall see isolated clusters with charge two; if we modify the interaction, we expect connected networks, or even random clusters of multi-Skyrmions, as discussed in Ref. \[11\]. We intend to investigate these questions further.

The occurrence of a fluid at finite temperatures, which might well be the phase transition seen in experiment, is also of relevance. It has been argued that quantum fluctuations play a crucial role here as well \[19\], so the question about the nature of the fluid and the phase transition deserves further study.

The authors would like to thank Dr. O Schwindt, Dr. P. Sutcliffe and Prof. N. Manton for useful discussions. This work was supported by research grants (GR/L22331 and GR/N15672) from the Engineering and Physical Sciences Research Council (EPSRC) of Great Britain. T.W. Would like to thank Richard Battye and Trinity college for support during a stay at DAMTP.