Jarzynski-like Equality of Nonequilibrium Information Production Based on Quantum Cross Entropy

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The two-time measurement scheme is well studied in the context of quantum fluctuation theorem. However, it becomes infeasible when the random variable determined by a single measurement trajectory is associated with the von-Neumann entropy of the quantum states. We employ the one-time measurement scheme to derive a Jarzynski-like equality of nonequilibrium information production by proposing an information production distribution based on the quantum cross entropy. The derived equality further enables one to explore the roles of the quantum cross entropy in quantum communications, quantum machine learning and quantum thermodynamics.

I. Introduction

Quantum thermodynamics explores the laws of the thermodynamics in the nanoscale from the perspective of the quantum information science [1–10]. On such scales, statistical fluctuations become more significant, and have principally been accounted for by fluctuation theorem [11–13]. The discovery of the fluctuation theorem is one of the most important accomplishments in the thermodynamics to date [16]. The fluctuation theorem can be regarded as a first principle in thermodynamics, from which many fundamental principles of thermodynamic phenomena can be derived, such as arrow of time [17] and response theory [18, 19].

More recently, fluctuation theorem have equally been used to characterize information processing tasks. For example, Sagawa and Ueda [20], and Fujitani and Suzuki [21], related the fluctuation theorem with an efficiency of the feedback control for the manipulation of the total entropy production via measurements. The relation between the fluctuation theorem and the adiabaticity of the process was revealed by considering the state distinguishability [22, 23]. In the context of quantum computing and communications, Gardas and Deffner [24] demonstrated that the fluctuation theorem can be used to determine the dynamics of the quantum systems and the susceptibility to the thermal noise. Also, Kafri and Deffner [25] related the fluctuation theorem and the Holevo information [26–28], which upper bounds the amount of classical information that can be transmitted through the quantum channel.

A standard approach to the fluctuation theorem in the quantum regime is the two-time measurement (TTM) scheme [29–38], in which the distribution of the measurement outcomes is constructed by the projection measurements on the system before and after the process. The first measurement corresponds to the state preparation of the input state, which is an ensemble of the eigenstates of the first measurement weighted by the probabilities of obtaining the corresponding outcomes, while the second measurement can be independent from the output state [25, 39].

While this scheme corresponds to the classical approach in stochastic thermodynamics [40], in the quantum regime, it is considered to be inconsistent because it does not take into account the quantum coherence [41] and the informational contribution of back-action of projection measurements [22]. Particularly, for the information production (namely the von-Neumann entropy gain), one needs to fully obtain the information of the output state because the second measurement is strictly dependent on the principal components of the output state, which requires the quantum state tomography. Therefore, from the practical and conceptual perspective, the TTM scheme is infeasible when we want to deal with information production in the context of fluctuation theorem. Also, while there are other approaches beyond the TTM scheme, such as the Bayesian method [42, 43] and quasiprobability [44, 45], in order to deal with information production, they all strictly require the quantum state tomography; therefore, we need to find an alternative approach to deal with the information production.

To solve this problem, we employ so-called one-time measurement (OTM) scheme, which was proposed by Deffner, Paz and Zurek in Ref. [22]. In this scheme, similar to the TTM scheme, we perform a projection measurement initially, which corresponds to the state preparation of the input state. However, what differs from the TTM scheme is that the second projection measurement is avoided, so that the corresponding distribution of

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the measurement outcomes is determined by the conditional expectation of the observable of interest given the initial measurement outcome. This quantity can be estimated if the post-measurement state of the initial measurement and the dynamics are known. Particularly, for the information production, we do not have to diagonalize the output state in the OTM scheme, so that the OTM scheme is the only option.

This paper is organized as the following. In Sec. II, we first propose an information production distribution for an input state of rank $r$ and a quantum channel. Then, we derive the Jarzynski-like equality and the lower bound on the total information production, which particularly becomes significant when we need to consider the information flow of the system in the quantum processes [46, 47]. We demonstrate that the lower bound is characterized by the quantum cross entropy. While there were less attentions on the quantum cross entropy, recently, the relations of the quantum cross entropy with the maximum likelihood principle in the machine learning [48] and the quantum source coding [49] have been explored. In our paper, we further explore the roles of quantum cross entropy in various protocols. In Sec. III, we discuss the applications of our result to quantum communications, quantum machine learning and quantum thermodynamics by focusing on the quantum autoencoder (QAE) protocol [50, 51], which is a quantum data compression protocol assisted by the variational quantum algorithms (VQAs) [52–57], and the maximum available work theorem [58] in the quantum thermodynamic systems, followed by the conclusion in Sec. IV.

II. Main Results

Let us consider a Hilbert space $\mathcal{H}$ of dimension $d \equiv \dim(\mathcal{H})$. Let $\mathcal{B}(\mathcal{H})$ denote the set of the density matrices acting on $\mathcal{H}$. We initially prepare a quantum state $\rho_0 \in \mathcal{B}(\mathcal{H})$, and perform a measurement with an observable $P \equiv \sum_{i=1}^r a(p_i)\Pi_i$, where $\Pi_i \equiv \ket{p_i}\bra{p_i}$ are the projectors on the eigenbases of $P$. Suppose that the outcome is $a(p_i)$. Then, the post-measurement state is given by $\rho_{in} = \sum_{i=1}^r p_i \ket{p_i}\bra{p_i}$ with $p_i \equiv \text{Tr}[\rho_0\Pi_i]$. Then, in general, the input state is given by $[25, 39]$

$$\rho_{in} = \sum_{i=1}^r p_i \ket{p_i}\bra{p_i}, \quad (1)$$

where $r \equiv \text{rank}(\rho_{in})$ denotes the rank of the input state. The state $\rho_{in}$ is an ensemble of the eigenbases of the initial measurement $P$ weighted by the probabilities of obtaining the outcomes $a(p_i)$; therefore, we can regard the initial measurement as a protocol of the state preparation of $\rho_{in}$. In this case, $p_i$ satisfies the following conditions $0 < p_i \leq 1$ ($1 \leq i \leq r$), $p_i = 0$ ($r + 1 \leq i \leq d$), and $\sum_{i=1}^r p_i = 1$.

Let $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}')$ be a quantum channel, which is a completely positive and trace-preserving (CPTP) map [59]. Through this channel, the output state $\rho_{out} \in \mathcal{B}(\mathcal{H}')$ is given by

$$\rho_{out} \equiv \Phi(\rho_{in}). \quad (2)$$

The total information production is defined as

$$\Delta S \equiv S(\rho_{out}) - S(\rho_{in}), \quad (3)$$

where $S(\rho) \equiv -\text{Tr}[\rho \ln \rho]$ denotes the von-Neumann entropy of the quantum state $\rho$.

Here, we propose the following information production distribution in the OTM scheme [60]

$$\tilde{P}(\sigma) \equiv \sum_{i=1}^r p_i \delta(\sigma - C(\rho_{in})_{\rho_{out}} - \ln p_i), \quad (4)$$

where $C(\rho_1, \rho_2) \equiv -\text{Tr}[\rho_1 \ln \rho_2]$ denotes the quantum cross entropy of $\rho_1$ with respect to $\rho_2$. Let $\text{supp}(\rho)$ denote the support of a quantum state $\rho$. Then, note that $C(\rho_1, \rho_2) < \infty$ (supp$(\rho_1) \subseteq$ supp$(\rho_2)$) and $C(\rho_1, \rho_2) = \infty$ (otherwise). Also, by definition, we have $C(\rho, \rho) = S(\rho)$. In Eq. (4), due to $\rho_{out} = \Phi(\rho_{in}) = \sum_{i=1}^r p_i \Phi(\rho_{in}) |p_i\rangle \langle p_i|$, we have supp$(\Phi(\rho_{in}) |p_i\rangle \langle p_i|) \subseteq$ supp$(\rho_{out})$, so that $C(\Phi(\rho_{in}) |p_i\rangle \langle p_i|), \rho_{out}) < \infty$.

With this distribution, the average of $\sigma$ with respect to $\tilde{P}(\sigma)$ becomes the exact information production

$$\langle \sigma \rangle_{\tilde{P}} = S(\rho_{out}) - S(\rho_{in}) = \Delta S, \quad (5)$$

where we used the linearity on the first argument of the quantum cross entropy [48] and Eq. (2). Let $r' \equiv \text{rank}(\rho_{out})$ be the rank of the output state. Then, we can interpret the random variable $\sigma$ as follows. Let $\{q_j, |q_j\rangle\}_{j=1}^{r'}$ denote an eigensystem of $\rho_{out}$. Let us define the transition probability $P(j|i) \equiv \langle q_j | \Phi(\rho_{in}) |p_i\rangle \langle p_i| q_j\rangle$. Then, in the OTM scheme, $\sigma$ randomly takes $\sum_j (-\ln q_j) P(j|i) + \ln p_i = \sum_j (-\ln q_j + \ln p_i) P(j|i)$, which is the conditional expectation of the information production given the initial measurement outcome.

Here, $\sigma$ can be also identified to be a random variable as a source of the information production $\Delta S$. Therefore, $\tilde{P}(\sigma)$ is a good definition. Averaging the exponentiated information production with respect to the distribution in Eq. (4), we can obtain our main result:

**Theorem 1** (Jarzynski-like equality of nonequilibrium information production). The Jarzynski-like equality of nonequilibrium information production is

$$\langle e^{-\sigma} \rangle_{\tilde{P}} = \sum_{i=1}^r e^{-C(\Phi(\rho_{in}) |p_i\rangle \langle p_i|), \rho_{out})}, \quad (6)$$

which results in

$$\Delta S \geq L_{otm}, \quad (7)$$

where $L_{otm}$ is defined as

$$L_{otm} \equiv -\ln \left( \sum_{i=1}^r e^{-C(\Phi(\rho_{in}) |p_i\rangle \langle p_i|), \rho_{out})} \right). \quad (8)$$
Proof. From Eqs. (1) and (4), we have
\[ \langle e^{-\sigma} \rangle_R = \int d\sigma \bar{P}(\sigma) e^{-\sigma} \]
\[ = \sum_{i=1}^{r} p_i e^{-C(\Phi(|p_i\rangle_1|\rho_{\text{out}}}) e^{-\ln p_i} \]
\[ = \sum_{i=1}^{r} e^{-C(\Phi(|p_i\rangle_1|\rho_{\text{out}}}), \]
which proves Eq. (6). From Jensen’s inequality \( \langle e^{-\sigma} \rangle_R \geq e^{-\langle \sigma \rangle_R} \) and Eq. (5), we obtain Eq. (7) [61]. \[ \square \]

When \( \Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) is particularly a unital map (i.e., \( \Phi(\mathbb{1}) = \mathbb{1} \), where \( \mathbb{1} \) denotes the identity matrix acting on \( \mathcal{H} \)), it well known that we have \( \Delta S \geq 0 \) [62]. However, we can obtain a tighter bound as demonstrated in the following corollary:

**Corollary 1** (Lower bound from OTM scheme under a unital map). When \( \Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) is a unital map, \( L_{\text{otm}} \) is a tighter bound on \( \Delta S \) as
\[ \Delta S \geq L_{\text{otm}} \geq 0. \]  

**Proof.** Given an input state \( \rho_{in} = \sum_{i=1}^{r} p_i |p_i\rangle_1 \langle p_i| \) of rank \( r \), let \( \Pi_{in} \equiv \sum_{i=1}^{r} |p_i\rangle_1 \langle p_i| \) be the projectors onto the support (null space) of \( \rho_{in} \). Let \( \Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) be a unital map, i.e. \( \Phi(\mathbb{1}) = \mathbb{1} \). Because the quantum cross entropy can be lower bounded by using the state overlap [48], we have
\[ C(\Phi(|p_i\rangle_1|\rho_{\text{out}}}) \geq -\ln \text{Tr} [\Phi(|p_i\rangle_1|\rho_{\text{out}})]. \]  

Then, due to the linearity of the CPTP map, we can obtain
\[ \sum_{i=1}^{r} e^{-C(\Phi(|p_i\rangle_1|\rho_{\text{out}}}) \leq \text{Tr} [\Phi(\Pi_{in})\rho_{\text{out}}]. \]  

Because \( \Pi_{in} + \Pi_{in} = \mathbb{1} \), from
\[ \Phi(\mathbb{1}) = \Phi(\Pi_{in}) + \Phi(\Pi_{in}) = \mathbb{1}, \]  
we can obtain
\[ \text{Tr} [\Phi(\Pi_{in})\rho_{\text{out}}] = 1 - \text{Tr} [\Phi(\Pi_{in})\rho_{\text{out}}] \leq 1. \]  

Therefore, from Eqs. (8), (12) and (13), we obtain Eq. (10), which proves Corollary. 1. \[ \square \]

### III. Examples

In this section, we illustrate two applications of our result: quantum autoencoder and quantum thermodynamics.

### A. Quantum Autoencoder

As our first example, we demonstrate the application of our result in the quantum autoencoder (QAE) proposed by Romero, Olson and Aspuru-Guzik in Ref. [50]. The QAE is a quantum analogue of the (classical) variational autoencoder [63]. In the QAE, the encoding and decoding operations are described by a parameterized quantum circuit. The original quantum data is compressed to the latent system by tracing over the other subsystem. Then, one prepares the fresh qubits, and decompresses the quantum data through the decoding operation acting on the fresh-qubit system and the latent system. The goal of the protocol is to recover the quantum data in the output, implying that a low-dimensional feature quantum state is well extracted through the encoding process; thus, we can use the resulting decoding process as a generative model to produce a quantum state outside the training quantum dataset by fluctuating the feature state. The cost function dependent on these tunable parameters, which measures the distance between the output and input state, is constructed by the quantum computer, and the set of the parameters is optimized through training the cost function with the classical computers. Recently, as a practical near-term quantum algorithm, the QAE has been widely explored both theoretically and experimentally [64–75].

Let us describe the setup of the QAE below. We consider a composite Hilbert space \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \), where \( \mathcal{H}_A \) (\( \mathcal{H}_B \)) denotes the Hilbert space of the reduced quantum system \( A \) (\( B \)). For the followings, let us regard \( \mathcal{H}_A \) as the latent Hilbert space, into which we compress our quantum data. Also, let us write \( d_j \) as the dimension of the reduced Hilbert space \( \mathcal{H}_j \), \( d_j \equiv \dim(\mathcal{H}_j) \) (\( j = A, B \)), so that the dimension of the total system is given by \( d = d_A d_B \). Following Ref. [50], we consider the following scenario (see Fig. 1). In this setup, we apply a parameterized unitary \( U \) to the input state \( \rho_{in} \) and perform the partial trace over \( \mathcal{H}_B \) to compress the data by applying the unitary \( U^\dagger \) to generate the output state \( \rho_{out} \).

*FIG. 1. Quantum Autoencoder:* We use \( U \) to compress the input state \( \rho_{in} \) into the reduced Hilbert space \( \mathcal{H}_A \), and use the state \( \rho_B \) of the fresh qubits in \( \mathcal{H}_B \) to decompress the data by applying the unitary \( U^\dagger \) to generate the output state \( \rho_{out} \).
To discuss the Jarzynski-like equality, it is convenient to define the compressed states
\[ \rho_A \equiv \text{Tr}_B \left[ U \rho_{in} U^\dagger \right] \]
\[ \rho_A^{(i)} \equiv \text{Tr}_B \left[ U |\psi_i\rangle \langle \psi_i| U^\dagger \right]. \]
(16) (17)
Therefore, we can write \( \rho_A = \sum_{i=1}^d p_i \rho_A^{(i)} \), so that we have \( \text{supp}(\rho_A^{(i)}) \subseteq \text{supp}(\rho_A) \).

Given this setup above, we can relate \( L_{\text{otm}} \) to the classical information transmission and the cost function in QAE, which demonstrates the roles of the quantum cross entropy in quantum communications and quantum machine learning in the framework of QAE protocol.

Let us first derive the expression of \( L_{\text{otm}} \) in QAE. From Eq. (15), we have
\[ C(\Phi(|p_i\rangle \langle p_i|), \rho_{out}) = S(\rho_B) + C(\rho_A^{(i)}, \rho_A). \]
(18)
Therefore, we can write
\[ \langle e^{-\sigma} \rangle_B = e^{-S(\rho_B)} \sum_{i=1}^r e^{-C(\rho_A^{(i)}, \rho_A)}, \]
so that \( L_{\text{otm}} \) is given by
\[ L_{\text{otm}} = S(\rho_B) - \ln \left( \sum_{i=1}^r e^{-C(\rho_A^{(i)}, \rho_A)} \right). \]
(19) (20)
An important observation is that \( \langle e^{-\sigma} \rangle_B \) includes two terms which characterize the protocols of the QAE. One is the von-Neumann entropy \( S(\rho_B) \), which is the informational contribution from the state preparation protocol in the fresh-qubit system \( \mathcal{H}_B \). The other one is associated with the quantum cross entropy \( C(\rho_A^{(i)}, \rho_A) \) with respect to the latent Hilbert space \( \mathcal{H}_A \). This quantity can be regarded as a term characterizing the compression protocol of the QAE. In the following, we explore the roles of the quantum cross entropy in the quantum communications and quantum machine learning from the relation of the lower bound \( L_{\text{otm}} \) to the loss of Holevo information and the global cost function of the QAE.

1. Relation to the loss of Holevo information in QAE

Here, we explore the relation between \( L_{\text{otm}} \) and the entropic disturbance. Entropic disturbance is the loss of Holevo information through a given quantum channel \( \Phi \) [46, 76]. Hence, it quantifies the loss of the maximum amount of classical information transmittable through the quantum channel. In Ref. [46], a lower bound on \( \Delta\chi \) was derived. Here, for a given quantum channel \( \Phi \), we provide an upper bound on the entropic disturbance by using \( L_{\text{otm}} \) to provide a operational meaning to the quantum cross entropy in terms of classical information transmission.

The entropic disturbance is defined as follows. Let \( \mathcal{L} \equiv \{ |p_i\rangle \langle p_i| \}_{i=1}^r \) denote an ensemble of input state \( \rho_{in} \equiv \sum_{i=1}^r p_i \rho_i \) and \( \Phi(\mathcal{L}) \equiv \{ |p_i\rangle \langle p_i| \}_{i=1}^r \) denote the ensemble of output state \( \rho_{out} \equiv \Phi(\rho_{in}) = \sum_{i=1}^r p_i \Phi(\rho_i) \). Entropic disturbance is defined as \( \Delta\chi \equiv \chi(\mathcal{L}) - \chi(\Phi(\mathcal{L})) \), where \( \chi(\mathcal{L}) \equiv \sum_{i=1}^r p_i S(\rho_i) \) and \( \chi(\Phi(\mathcal{L})) \equiv S(\rho_{out}) - \sum_{i=1}^r p_i S(\Phi(\rho_i)) \) are the Holevo information of \( \rho_{in} \) and \( \rho_{out} \), respectively. In our case, we have \( \rho_i = |p_i\rangle \langle p_i| \), so that \( S(\rho_i) = S(|p_i\rangle \langle p_i|) = 0 \). Therefore, due to \( \Delta S \geq L_{\text{otm}} \) and Eq. (8), we can obtain
\[ \Delta\chi \leq \ln \left( \sum_{i=1}^r e^{-C(\rho(\rho_{in}^{(i)}) \rho_{out}^{(i)}))} + \sum_{i=1}^r p_i S(\Phi(|p_i\rangle \langle p_i|)), \right. \]
(21)
which shows that the upper bound of the entropic disturbance can be characterized by the quantum cross entropy.

Now, let us consider the case of QAE, in which \( \Phi \) satisfies Eq. (15). Due to \( S(\Phi(|p_i\rangle \langle p_i|)) = S(\rho_A^{(i)}) + S(\rho_B), \) we have \( \sum_{i=1}^r p_i S(\Phi(|p_i\rangle \langle p_i|)) = \sum_{i=1}^r p_i S(\rho_A^{(i)}) + S(\rho_B), \) so that the upper bound on \( \Delta\chi \) in QAE is given by
\[ \Delta\chi \leq \sum_{i=1}^r p_i S(\rho_A^{(i)}) + \ln \left( \sum_{i=1}^r e^{-C(\rho_A^{(i)}, \rho_A)} \right). \]
(22)
Therefore, the information and the quantum cross entropy of the compressed states contributes to setting an upper bound on entropic disturbance of in the QAE protocol. Also, note that for the QAE, \( \Delta\chi \) can be explicitly written as
\[ \Delta\chi = S(\rho_{in}) - S(\rho_A) + \sum_{i=1}^r p_i S(\rho_A^{(i)}), \]
(23)
which implies that the loss of the maximum amount of classical information in the QAE protocol is independent of the choice of \( \rho_B \) but strictly dependent on the compressed and input state.

2. Relation to the global cost function of QAE

The lower bound \( L_{\text{otm}} \) can be also related to the performance of the QAE, which can be characterized by its global cost function. The cost function of the QAE is well-defined when the fresh-qubit state \( \rho_B \) is a pure state
\[ \rho_B = |\psi\rangle \langle \psi| . \]
(24)
In this case, from Eq. (20) and \( S(\rho_B) = S(|\psi\rangle \langle \psi|) = 0 \), we have
\[ L_{\text{otm}} = -\ln \left( \sum_{i=1}^r e^{-C(\rho_A^{(i)}, \rho_A)} \right). \]
(25)
Let \( \eta_B \) denote the reduced state \( \eta_B \equiv \text{Tr}_A[U \rho_{in} U^\dagger] \). Then, from Refs. [50, 78], the global cost function \( \mathcal{E} \) can be given by
\[ \mathcal{E} \equiv 1 - \langle \psi | \eta_B | \psi \rangle, \]
(26)
which satisfies $0 \leq \mathcal{C} \leq 1$. The ultimate goal of this protocol is to find an optimal unitary $U_*$ to realize $\rho_{\text{in}} = \rho_{\text{out}}$. In this optimal case, the global cost function is $\mathcal{C} = 0$. Then, by using $d_B$ the dimension of $\mathcal{H}_B$ and the global cost function, we can obtain the following inequality (See Appendix A for the proof)

$$L_{\text{otm}} \leq \Delta S \leq 2 \ln \left( \frac{1}{\sqrt{1-\mathcal{C}}} + \sqrt{(d_B-1)\mathcal{C}} \right). \quad (27)$$

From Eq. (25), we can finally obtain

$$\sum_{i=1}^{r} e^{-C(\rho_A^{(i)},\rho_A)} \geq \left( \frac{1}{\sqrt{1-\mathcal{C}}} + \sqrt{(d_B-1)\mathcal{C}} \right)^2, \quad (28)$$

which shows that the quantum cross entropy plays a role as an informational contribution of the compressed state to the performance of the QAE protocol. Here, note that, due to $0 \leq \mathcal{C} \leq 1$, we have $0 \leq 2 \ln \left( \frac{1}{\sqrt{1-\mathcal{C}}} + \sqrt{(d_B-1)\mathcal{C}} \right) \leq \ln(d_B)$, where $\mathcal{C} = 0$ (i.e., $\eta_B = |\psi\rangle\langle\psi|$) leads to the minimum, and $\mathcal{C} = 1-1/d_B$ (i.e., $\eta_B = \mathbb{I}_B/d_B$) leads to the maximum.

We can also check the consistency of Eq. (27) by considering the optimal case. The optimal unitary $U_*$ is a disentangling gate [75], so that $U_*$ satisfies

$$U_*\rho_{\text{in}}U_\dagger = \rho_A \otimes |\psi\rangle\langle\psi|$$

$$U_*|p_i\rangle\langle p_i|U_\dagger = \rho_A^{(i)} \otimes |\psi\rangle\langle\psi|. \quad (29)$$

Then, by using the unitary invariance of the quantum cross entropy, we obtain

$$C(\rho_A^{(i)},\rho_A) = C(\rho_A^{(i)} \otimes |\psi\rangle\langle\psi|, \rho_A \otimes |\psi\rangle\langle\psi|) = C(|p_i\rangle\langle p_i|, \rho_{\text{in}}) = -\ln p_i. \quad (30)$$

In this way, we have $L_{\text{otm}} = 0$. By definition, in the optimal case, we have $\mathcal{C} = 0$; therefore, $\ln \left( \frac{1}{\sqrt{1-\mathcal{C}}} + \sqrt{(d_B-1)\mathcal{C}} \right) = 0$. From $L_{\text{otm}} \leq \Delta S \leq 2 \ln \left( \frac{1}{\sqrt{1-\mathcal{C}}} + \sqrt{(d_B-1)\mathcal{C}} \right)$, we get the expected result $\Delta S = 0$ for the optimal unitary case.

### B. Maximum Available Work Theorem

For the second example, we explore the role of quantum cross entropy in work extraction from the quantum thermodynamic systems by from the relation between $L_{\text{otm}}$ and the maximum available work theorem [58].

In Ref. [58], a generic quantum thermodynamic system is regarded as a tripartite system composed of the system, work reservoir and heat bath. In this setup, the work $\langle W \rangle$, the internal energy change of the system $\Delta E_s$ and the internal energy change of the heat bath $\Delta E_b$ satisfy the first law of thermodynamics $\Delta E_s + \Delta E_b = \langle W \rangle$.

Also, when $\Delta S$ and $\Delta S_b$ denote the von-Neumann entropy change of the system and heat bath, respectively, the second law of thermodynamics states $\Delta S + \Delta S_b \geq 0$. Because the heat reservoir is so large, which can be regarded as being always in equilibrium at inverse temperature $\beta$, we can write $\Delta S_b = \beta \Delta E_b$. Then, the maximum available work theorem states

$$\langle W \rangle \geq \Delta E_s - \beta^{-1} \Delta S \equiv \Delta \mathcal{E}, \quad (31)$$

where $\mathcal{E}$ is called exergy or availability, which quantifies the maximally available work.

In this setup, from Theorem. 1, we can obtain the upper bound on the exergy $\Delta \mathcal{E}$ as

$$\Delta \mathcal{E} \leq \Delta E_s + \beta^{-1} \ln \left( \frac{1}{\sum_{i=1}^{r} e^{-C(\Phi(|p_i\rangle\langle p_i|),\rho_{\text{out}})}} \right). \quad (32)$$

This demonstrates the informational contribution of the quantum cross entropy in extracting maximally available work in the quantum thermodynamic systems. If there is no work reservoir, i.e., $\langle W \rangle = 0$, the corresponding maximum available work theorem becomes $\Delta S \geq -\beta \Delta E_b$. However, when the system undergoes the energy-emitting process ($\Delta E_b \geq 0$) described by a unital evolution, from Corollary 1, we have a tighter bound as

$$\Delta S \geq L_{\text{otm}} \geq -\beta \Delta E_b. \quad (33)$$

A good example of the energy-emitting unital evolution is the spin-boson model [79]. Let us consider a system $\mathcal{H}_s$ initially prepared in $\rho_{\text{in}}$ coupled to a heat bath $\mathcal{H}_b$, whose initial state is prepared in the Gibbs state

$$\rho_b^{\text{eq}} = \frac{e^{-\beta H_b}}{Z}, \quad (34)$$

where $Z \equiv \text{Tr} \left[ e^{-\beta H_b} \right]$ is the canonical partition function with inverse temperature $\beta$ and $H_b$ the time-independent bare Hamiltonian of the bath. Then, when $\Phi$ is a thermal operation [4–6] from $t = 0$ to $t = \tau$,

$$\rho_{\text{out}} = \Phi(\rho_{\text{in}}) = \text{Tr}_b \left[ U_\tau (\rho_{\text{in}} \otimes \rho_b^{\text{eq}}) U_\tau^\dagger \right]. \quad (35)$$

In the quantum thermodynamic setup of the spin-boson model, $\mathcal{H}_s$ and $\mathcal{H}_b$ usually describes a two-level atomic system and the bosonic heat bath, respectively (See Fig. 2), and the atomic system in Eq. (35) undergoes the dephasing process, which is described by a unitary map. This model can be described by the following time-independent Hamiltonian (we set $\hbar = 1$)

$$H = \frac{\omega_0}{2} \sigma_z + H_b + \sigma_z \otimes \sum_k (g_k a_k + g_k^* a_k^\dagger), \quad (36)$$

where

$$H_b = \sum_k \omega_k a_k^\dagger a_k \quad (37)$$
During the evolution from the initial state to the final state, the system undergoes a unitary evolution. The Hamiltonian of the spin-boson model is described in Eq. (36).

is the bare Hamiltonian of the boson fields, and \( z = \text{diag}(1, -1) \) is the Pauli’s Z operator acting on the atom. \( \omega_k \) are the angular frequencies of the atom and the \( k \)-th boson mode, and \( g_k \) denotes the coupling strength between the atom and the \( k \)-th boson mode. Here, in general \( g_k \) is a complex number, and \( g_k^* \) denotes the complex conjugate of \( g_k \). Also, \( a_k(a_k^\dagger) \) is the annihilation (creation) operator of \( k \)-th mode of the boson fields. The Hamiltonian Eq. (36) describes an interaction between atom and boson fields with multiple modes, which leads to the dephasing process of the atomic system. In interaction picture, we have

\[
H(t) = z \otimes \sum_k (g_k a_k e^{-i\omega_k t} + g_k^* a_k^\dagger e^{i\omega_k t}).
\]

During the evolution from \( t = 0 \) to \( t = \tau \), the internal energy change of the heat bath becomes (See Appendix B for the proof)

\[
\Delta E_b \equiv \text{Tr} \left[ U_\tau (\rho_{in} \otimes \rho_b^{eq}) U_\tau^\dagger H_b \right] - \text{Tr} \left[ \rho_b^{eq} H_b \right]
\]

\[
= \sum_k \omega_k |g_k|^2 \left( \frac{\sin(\omega_k \tau/2)}{\omega_k/2} \right)^2 \geq 0,
\]

which shows that the system undergoes the energy-emitting process, which can be verified from the energy conservation of the total system. Considering the noise spectral density \( J(\omega) = \sum_k |g_k|^2 \omega \delta(\omega - \omega_k) \), we can obtain

\[
\Delta E_b = \int_{-\infty}^{\infty} J(\omega) \left( \frac{\sin(\omega \tau/2)}{\omega^2} \right)^2 d\omega.
\]

Since we have \( \lim_{\tau \to \infty} \frac{\sin(\omega \tau/2)}{\omega^2} = \delta(\omega/2) = 2\delta(\omega) \), where we used the relation \( \lim_{\tau \to \infty} \left( \frac{\tau \sin(\omega \tau)}{\omega \tau} \right) = \delta(\omega) \), we obtain

\[
\lim_{\tau \to \infty} \Delta E_b = \int_{-\infty}^{\infty} 4J(\omega)\delta^2(\omega) d\omega = 4J(0)\delta(0) = 0.
\]

IV. Conclusion

In conclusion, we have proposed a distribution of an information production for a quantum state of arbitrary rank and a quantum channel by adopting the one-time measurement scheme. The derived Jarzynski-like equality and the lower bound on the total information production are characterized by the quantum cross entropy, which further enables one to explore the roles of quantum cross entropy in quantum communications, quantum machine learning and quantum thermodynamics. By focusing on the quantum autoencoder, we have explored the informational contributions of the quantum cross entropy of the compressed states to the loss of the maximum classical information transmittable through the circuit and the performance of the protocol characterized by the global cost function. We have also demonstrated the application of our result in the quantum thermodynamics by exploring relation between the quantum cross entropy and the maximum available work theorem. Our result can provide insights of the quantum cross entropy as a resource to achieve some tasks. As a valuable future direction, we will explore reverse process in the OTM scheme, which still remains open.

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Appendix

A. Proof of Eq. (27)

Here, we provide a detailed proof of Eq. (27). First, by using the cost function $\mathcal{C}$, we can obtain an upper bound on $\Delta S$. From

$$
\rho_{\text{out}} = U^\dagger (\rho_A \otimes |\psi\rangle\langle\psi|) U
$$

(A1)

with $\rho_A \equiv \text{Tr}_B[U\rho_{\text{in}}U^\dagger]$, because the von-Neumann entropy is unitarily invariant, we have

$$
S(\rho_{\text{out}}) = S(\rho_A \otimes |\psi\rangle\langle\psi|) = S(\rho_A) \\
S(\rho_{\text{in}}) = S(U\rho_{\text{in}}U^\dagger),
$$

(A2)

which leads to

$$
\Delta S = S(\rho_A) - S(U\rho_{\text{in}}U^\dagger).
$$

(A3)

Because $\eta_B \equiv \text{Tr}_A[U\rho_{\text{in}}U^\dagger]$, from Araki-Lieb inequality [80]

$$
|S(\rho_A) - S(\eta_B)| \leq S(U\rho_{\text{in}}U^\dagger),
$$

(A4)

we have

$$
\Delta S \leq S(\eta_B).
$$

(A5)

By using $d_B$ the dimension of the Hilbert space $\mathcal{H}_B$, $S(\eta_B)$ can be upper bounded as

$$
S(\eta_B) = \ln(d_B) - S\left(\eta_B \parallel \frac{1}{d_B}\right) \leq \ln(d_B) - S_{\text{min}}\left(\eta_B \parallel \frac{1}{d_B}\right),
$$

(A6)

where

$$
S_{\text{min}}(\rho_1||\rho_2) \equiv -\ln(F[\rho_1, \rho_2])
$$

(A7)

denotes the sandwiched min relative entropy of $\rho_1$ with respect to $\rho_2$ [81–83] with the standard quantum fidelity defined as

$$
F[\rho_1, \rho_2] \equiv \left(\text{Tr}\left[\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}\right]\right)^2.
$$

(A8)

Therefore, we have

$$
S(\eta_B) \leq \ln \left(d_B F\left[\eta_B, \frac{1}{d_B}\right]\right).
$$

(A9)

Here, we consider so-called generalized quantum fidelity [84–87], which is defined as

$$
\tilde{F}[\sigma_1, \sigma_2] \equiv \left(\sqrt{F[\sigma_1, \sigma_2]} + \sqrt{1 - \text{Tr}[\sigma_1](1 - \text{Tr}[\sigma_2])}\right)^2.
$$

(A10)

Here, note that $\sigma_1$ and $\sigma_2$ are the sub-normalized states i.e. $0 \leq \text{Tr}[\sigma_1] \leq 1$ and $0 \leq \text{Tr}[\sigma_2] \leq 1$. Because applying the projection operator $|\psi\rangle\langle\psi|$ is described by the completely positive trace non-increasing (CPTNI) map [87], from the monotonicity of the generalized quantum fidelity under the CPTNI maps [84–87], we have

$$
F\left[\eta_B, \frac{1}{d_B}\right] \leq \tilde{F}\left[|\psi\rangle\langle\psi|\eta_B|\psi\rangle\langle\psi|, \frac{1}{d_B}|\psi\rangle\langle\psi|\right].
$$

(A11)

Since we have

$$
\tilde{F}\left[|\psi\rangle\langle\psi|\eta_B|\psi\rangle\langle\psi|, \frac{1}{d_B}|\psi\rangle\langle\psi|\right] \leq \left(\sqrt{1 - \mathcal{C}} + \sqrt{(d_B - 1)\mathcal{C}}\right)^2,
$$

(A12)

we can obtain

$$
\Delta S \leq S(\eta_B) \leq 2\ln\left(\sqrt{1 - \mathcal{C}} + \sqrt{(d_B - 1)\mathcal{C}}\right),
$$

(A13)

which states that the information production in quantum autoencoder with pure fresh-qubit state can be upper bounded by using $\mathcal{C}$. Therefore, from Theorem. 1, we can finally arrive at Eq. (27)

$$
L_{\text{otm}} \leq \Delta S \leq 2\ln\left(\sqrt{1 - \mathcal{C}} + \sqrt{(d_B - 1)\mathcal{C}}\right).
$$

(A14)

B. Proof of Eq. (39)

We demonstrate the detailed proof of Eq. (39) based on Refs. [88, 89]. The Hamiltonian of the spin-boson model is

$$
H = \frac{\omega_0}{2}\sigma_z + H_b + \sigma_z \otimes \sum_k (g_k a_k + g_k^\dagger a_k^\dagger),
$$

(B1)

where we define $\sigma_z \equiv \left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$ and $a_k(a_k^\dagger)$ as the annihilation (creation) operator of $k$-th mode of the boson heat bath. The annihilation and creation operators satisfy the commutation relation

$$
[a_k, a_{k'}^\dagger] = \delta_{kk'}, [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0.
$$

(B2)
Also, $H_b$ is defined as
\[ H_b = \sum_k \omega_k a_k^\dagger a_k. \]  
(B3)

Then, the Hamiltonian in the interaction picture becomes
\[ H(t) = \sigma_z \otimes \sum_k (g_k a_k e^{-i\omega_k t} + g_k^* a_k^\dagger e^{i\omega_k t}). \]  
(B4)

Using Magnus expansion, the propagator becomes
\[ U_t = \exp \left[ -it(\mathcal{H}_0 + \mathcal{H}_1) \right], \]  
(B5)

where the higher terms are vanishing because $[H(t_1), H(t_2)]$ becomes just a number (See Eq. (B10)). Here, we define
\[ \mathcal{H}_0 \equiv \frac{1}{t} \int_0^t H(t_1) dt_1 \]  
and
\[ \mathcal{H}_1 \equiv -\frac{i}{2t} \int_0^t dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)]. \]  
(B7)

More explicitly, $\mathcal{H}_0$ can be written as
\[ \mathcal{H}_0 = \sigma_z \otimes \sum_k \left( G_k(t) a_k + G^*_k(t) a_k^\dagger \right), \]  
(B8)

where
\[ G_k(t) \equiv g_k \frac{\sin(\omega_k t/2) e^{-i\omega_k t/2}}{\omega_k t/2}. \]  
(B9)

For $\mathcal{H}_1$, because we have
\[ [H(t_1), H(t_2)] = -2it \sum_k |g_k|^2 \sin(\omega_k (t_1 - t_2)) \]  
(B10)

and
\[ \int_0^t dt_1 \int_0^{t_1} dt_2 \sin(\omega_k (t_1 - t_2)) = \frac{1}{\omega_k} \left( t - \frac{1}{\omega_k} \sin(\omega_k t) \right), \]  
(B11)

we can write
\[ \mathcal{H}_1 = -\sum_k \frac{|g_k|^2}{\omega_k} \left( 1 - \frac{\sin(\omega_k t)}{\omega_k t} \right) \in \mathbb{R}, \]  
(B12)

which is just a real number.

Therefore, from Eqs. (B5), (B8), and (B12), the propagator becomes
\[ U_t = \exp \left[ -it \sum_k \left( \sigma_z \otimes (G_k(t) a_k + G^*_k(t) a_k^\dagger) \right) \right] e^{-it\mathcal{H}_1}. \]  
(B13)

With this propagator, due to
\[ \sum_{k'} \sigma_z \otimes (G_{k'}(t)a_{k'} + G^*_{k'}(t)a_{k'}^\dagger), a_k = -G^*_k(t)\sigma_z \]  
(B14)
and
\[ \sum_{k'} \sigma_z \otimes (G_{k'}(t)a_{k'} + G^*_{k'}(t)a_{k'}^\dagger), a_k^\dagger = G_k(t)\sigma_z, \]  
(B15)

from Baker-Hausdorff-Campbell’s formula, we have
\[ U_t^\dagger a_k U_t = a_k - itG^*_k(t)\sigma_z \]  
(B16)

\[ U_t^\dagger a_k^\dagger U_t = a_k^\dagger + itG_k(t)\sigma_z. \]  
(B17)

Therefore, we can write
\[ U_t^\dagger H_b U_t = \sum_k \omega_k \left( U_t^\dagger a_k^\dagger U_t \right) \left( U_t^\dagger a_k U_t \right) \]  
\[ = H_b + it \sum_k \omega_k \sigma_z \otimes (G_k(t) a_k - G^*_k(t) a_k^\dagger) \]  
(B18)

\[ + \sum_k \omega_k |G_k(t)|^2 t^2. \]

Let $\rho_{\text{in}}$ be the input state of the system with a rank $r$, and the initial state of the boson heat bath be the Gibbs state
\[ \rho_b^{\text{eq}} = e^{-\beta H_b} / Z. \]  
(B19)

Note that, with $a_k$ and $a^\dagger_k$, for all $k$, we have
\[ \text{Tr} [\rho_{\text{in}} a_k^\dagger a_k] = \text{Tr} [\rho_b^{\text{eq}} a_k^\dagger a_k] = 0. \]  
(B20)

We assume that the two-level atomic system and bosonic field are initially decoupled. Therefore, the initial state of the total system is $\rho_{\text{in}} \otimes \rho_b^{\text{eq}}$, so that the evolution of the atomic system from $t = 0$ to $t = \tau$ is described by the following thermal operation
\[ \rho_{\text{out}} = \Phi(\rho_{\text{in}}) = \text{Tr}_\tau [U_\tau (\rho_{\text{in}} \otimes \rho_b^{\text{eq}}) U_\tau^\dagger]. \]  
(B21)

The internal energy change of the heat bath during the evolution can be defined as the difference in the average energy of the heat bath at $t = \tau$ and $t = 0$
\[ \Delta E_b \equiv \text{Tr} [U_\tau (\rho_{\text{in}} \otimes \rho_b^{\text{eq}}) U_\tau^\dagger H_b] - \text{Tr} [\rho_b^{\text{eq}} H_b]. \]  
(B22)

From Eqs. (B9), (B16) and (B20), $\Delta E_b$ can be explicitly written as
\[ \Delta E_b = \sum_k \omega_k |g_k|^2 \left( \frac{\sin(\omega_k \tau/2)}{\omega_k^2/2} \right)^2 \geq 0. \]  
(B23)

C. Second-law-like Inequality Involving Guessed Heat

The information production distribution $\overline{P}(\sigma)$ can be related to the distribution of the internal energy difference in OTM scheme $\overline{P}(\Delta E_b)$ in a very special case,
which leads to a second-law-like inequality involving the guessed heat introduced in Ref. [89]. Let $H_s(t)$ be the system’s bare Hamiltonian, which is time-dependent. Also, suppose that the system is initially decoupled from the heat bath, which is initially prepared in a Gibbs state

$$\rho^\text{eq}_b = \frac{e^{-\beta H_b}}{Z_b}, \quad (C1)$$

where $H_b$ is the bath’s bare Hamiltonian, which is time-independent. Here, $Z_b \equiv \text{Tr} \left[ e^{-\beta H_b} \right]$ is the partition function defined by $H_b$. Let $H_{\text{int}}$ be the interaction Hamiltonian. Then, the unitary operator $U_t$ describing the time evolution of the total system follows the Schrödinger’s equation $\partial_t U_t = -i(H_s(t) + H_b + H_{\text{int}})U_t$ with $U_0 \equiv 1$. Evolving the total system from $t = 0$ to $t = \tau$ and focusing on the system alone, we have

$$\rho_{\text{out}} = \Phi(\rho_{\text{in}}) = \text{Tr}_b \left[ U_\tau (\rho_{\text{in}} \otimes \rho^\text{eq}_b) U_\tau^\dagger \right]. \quad (C2)$$

Let $\{E_i, |E_i\rangle\}_{i=1}^d$ be an eigensystem of $H_s(0)$. When we have

$$\rho_{\text{in}} = \rho_{s}^\text{eq}(0) \equiv \frac{e^{-\beta H_s(0)}}{Z_0}, \quad (C3)$$

$$\rho_{\text{out}} = \rho_{s}^\text{eq}(\tau) \equiv \frac{e^{-\beta H_s(\tau)}}{Z_\tau},$$

where $Z_0 \equiv \text{Tr} \left[ e^{-\beta H_s(0)} \right]$ and $Z_\tau \equiv \text{Tr} \left[ e^{-\beta H_s(\tau)} \right]$ are the partition functions defined by $H_s(0)$ and $H_s(\tau)$, respectively, from Eq. (4), the information production distribution in the OTM scheme becomes

$$\tilde{P}(\sigma) = \frac{1}{\beta} \sum_{i=1}^d \frac{e^{-\beta E_i}}{Z_0} \delta \left( \frac{\sigma}{\beta} + \Delta F - \Delta \tilde{E}(E_i) \right), \quad (C4)$$

where

$$\Delta \tilde{E}(E_i) \equiv \text{Tr} \left[ \Phi(|E_i\rangle\langle E_i|) H_s(\tau) \right] - E_i, \quad (C5)$$

and

$$\Delta F \equiv -\beta^{-1} \ln \left( \frac{Z_\tau}{Z_0} \right) \quad (C6)$$

is the equilibrium Helmholtz free energy difference. From Ref. [89], $\tilde{P}(\Delta E_s)$ is given by

$$\tilde{P}(\Delta E_s) = \sum_{i=1}^d \frac{e^{-\beta E_i}}{Z_0} \delta \left( \Delta E_s - \Delta \tilde{E}(E_i) \right). \quad (C7)$$

Therefore, we can write

$$\tilde{P}(\Delta E_s) = \beta \tilde{P}(\sigma) \quad (C8)$$

with the random variable $\sigma$ being

$$\sigma = \beta(\Delta E_s - \Delta F) \quad (C9)$$

Following Ref. [89], we have

$$\langle e^{-\beta \Delta E_s} \rangle_\tilde{P} = e^{-\beta \Delta F} e^{-\beta \langle \tilde{Q}\rangle_b} e^{-S(\Theta_{sb}(\tau))|\rho^\text{eq}_s(\tau)\otimes \rho^\text{eq}_b)} \quad (C10)$$

where

$$\Theta_{sb}(\tau) \equiv \sum_{i=1}^d \frac{e^{-\beta \text{Tr}[\Phi(|E_i\rangle\langle E_i|) H_s(\tau)]}}{Z_\tau} \text{Tr}_b \left[ |E_i\rangle\langle E_i| \otimes \rho^\text{eq}_b \right] U_\tau^\dagger \quad (C11)$$

is called “guessed state”, and $\langle \tilde{Q}\rangle_b$ is a heat-like quantity called “guessed heat” defined as

$$\langle \tilde{Q}\rangle_b \equiv \text{Tr}_b \left[ H_b \rho^\text{eq}_b \right] - \text{Tr}_b \left[ H_b \Theta_{sb}(\tau) \right]. \quad (C12)$$

This heat-like quantity describes an energy dissipation from the heat bath as if its final state is $\text{Tr}_s \left[ \Theta_{sb}(\tau) \right]$ the reduced state of the guessed state. From Eq. (C8) and (C9), we can obtain

$$\langle e^{-\beta \Delta E_s} \rangle_\tilde{P} = e^{-\beta \Delta F} \langle e^{-\sigma} \rangle_\tilde{P}, \quad (C13)$$

From Eq. (C10), we can finally write

$$\langle e^{-\sigma} \rangle_\tilde{P} = e^{-\beta \langle \tilde{Q}\rangle_b} e^{-S(\Theta_{sb}(\tau))|\rho^\text{eq}_s(\tau)\otimes \rho^\text{eq}_b)} \quad (C14)$$

By using Jensen’s inequality and the non-negativity of the quantum relative entropy, we can arrive at

$$\Delta S - \beta \langle \tilde{Q}\rangle_b \geq 0, \quad (C15)$$

which is the second-law-like inequality involving the guessed heat.

[1] S. Deffner and S. Campbell, Quantum Thermodynamics (Morgan and Claypool Publishers, San Rafael, 2019).

[2] F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso, Thermodynamics in the Quantum Regime (Springer, 2019).

[3] S. Vinjanampathy and J. Anders, Quantum thermodynamics, Contemp. Phys. 57, 545 (2016).

[4] T. Sagawa, Thermodynamics of information processing in small systems, Prog. Theor. Phys. 127, 1 (2012).

[5] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, The role of quantum information in thermodynamics—a topical review, J. Phys. A: Math. Theor. 49, 143001 (2016).

[6] N. H. Y. Ng and M. P. Woods, Resource theory of
Quantum computing in the NISQ era and beyond, Quantum Sci. Technol. 2, 045001 (2017).

K. H. Wan, O. Dahlsten, H. Kristjánsson, R. Gardner, and M. S. Kim, Quantum generalisation of feedforward neural networks, npj Quantum Inf. 3, 1 (2017).

J. Preskill, Quantum computing in the NISQ era and beyond, Quantum 2, 79 (2018).

J. R. McClean, J. Romero, R. Babbush, and A. Aspuru-Guzik, The theory of variational hybrid quantum-classical algorithms, New J. Phys. 18, 023023 (2016).

T. Jones, S. Endo, S. McArdle, X. Yuan, and S. C. Benjamin, Variational quantum algorithms for discovering Hamiltonian spectra, Phys. Rev. A 99, 062304 (2019).

K. M. Nakashishi, K. Fujii, and S. Todo, Sequential minimal optimization for quantum-classical hybrid algorithms, Phys. Rev. Research 2, 043158 (2020).

M. Cerezo, A. Andersson, R. Babbush, S. C. Benjamin, S. Endo, K. Fujii, J. R. McClean, K. Mitarai, X. Yuan, L. Cincio, and P. J. Coles, Variational quantum algorithms, Nat. Rev. Phys. 3, 625 (2021).

K. Bharti, A. Cervera-Lierta, T. H. Kyaw, T. Haug, S. Alperin-Lea, A. Anand, M. Degroote, H. Heinonen, J. S. Kottmann, T. Menke, et al., Noisy intermediate-scale quantum algorithms, Rev. Mod. Phys. 94, 015004 (2022).

S. Defnner, Kibble-Zurek scaling of the irreversible entropy production, Phys. Rev. E 96, 052125 (2017).

M. M. Wilde, Quantum information theory (Cambridge University Press, 2013).

The OTM scheme has been utilized to explore work and heat in the open quantum system [89], its classical correspondence [23], heat exchange [90], and work as an external observable [91]. Particularly, a second-law-like inequality involving the guessed heat introduced in Ref. [80] can be derived by using $P(\sigma)$ (See Appendix C).

Note that our main claims are the derivation of the general integrated fluctuation theorems, which hold for any states and quantum channels, and its potential of characterizing the quantum protocol with the quantum cross entropy. We leave the comparison between the tight bound derived in Refs. [46, 47] and our bound derived in the OTM scheme as an open problem.

M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information: 10th Anniversary Edition, 10th ed. (Cambridge University Press, New York, NY, USA, 2010).

D. P. Kingma and M. Welling, An Introduction to Variational Autoencoders, Found. Trends Mach. Learn. 12, 307 (2019).

C. Bravo-Prieto, Quantum autoencoders with enhanced data encoding, Mach. Learn.: Sci. Technol. 2, 035028 (2021).

D. Bondarenko and P. Feldmann, Quantum autoencoders to denoise quantum data, Phys. Rev. Lett. 124, 130502 (2020).

D. F. Locher, L. Cardarelli, and M. Müller, Quantum error correction with quantum autoencoders, arXiv:2202.00555 (2022).

C. Cao and X. Wang, Noise-assisted quantum autoencoder, Phys. Rev. Appl. 15, 054012 (2021).

G. R. Steinbrecher, J. P. Olson, D. Englund, and J. Carolan, Quantum optical neural networks, npj Quantum Inf. 5, 1 (2019).

Y. Du and D. Tao, On exploring practical potentials of quantum auto-encoder with advantages, arXiv:2106.15432 (2021).

A. Pepper, N. Tischler, and G. J. Pryde, Experimental realization of a quantum autoencoder: The compression of qutrits via machine learning, Phys. Rev. Lett. 122, 054127 (2022).

S. Patel, B. Collis, W. Duong, D. Koch, M. Cutugno, L. Wissing, and P. Alsing, Information loss and run time from practical application of quantum data compression, arXiv preprint arXiv:2203.11332 (2022).

V. S. Ngairangbam, M. Spanowsky, and M. Takeuchi, Anomaly detection in high-energy physics using a quantum autoencoder, Phys. Rev. D 105, 055004 (2022).

C.-J. Huang, H. Ma, Q. Yin, J.-F. Tang, D. Dong, C. Chen, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, Realization of a quantum autoencoder for lossless compression of quantum data, Phys. Rev. A 102, 032412 (2020).

H. Ma, C.-J. Huang, C. Chen, D. Dong, Y. Wang, R.-B. Wu, and G.-Y. Xiang, On compression rate of quantum autoencoders: Control design, numerical and experimental realization, arXiv:2005.11149 (2020).

F. Buscemi and M. Horodecki, Towards a unified approach to information-disturbance tradeoffs in quantum measurements, Open Syst. Inf. Dyn. 16, 29 (2009).

When $\Phi$ is a unitary operation, we have $L_{\text{stt}} = 0$ and $S(\Phi(\rho), \rho) = 0$. Because the entropic disturbance is invariant under the unitary operation $\Delta X = 0$, we can say that the upper bound is tight for the unitary operation.

M. Cerezo, A. Sone, T. Volkoff, L. Cincio, and P. J. Coles, Cost-function-dependent barren plateaus in shallow quantum neural networks, Nat. Commun. 12, 1791 (2021).

M. A. Schlosshauer, Decoherence: and the quantum-to-classical transition (Springer Science & Business Media, 2007).

H. Araki and E. H. Lieb, Entropy inequalities, Commun. Math. Phys. 18, 160 (1970).

M. M. Wilde, A. Winter, and D. Yang, Strong converse for the classical capacity of entanglement-breaking and hadamard channels via a sandwiched rényi relative entropy, Commun. Math. Phys. 331, 593 (2014).

S. Beigi, Sandwiched rényi divergence satisfies data processing inequality, J. Math. Phys. 54, 122202 (2013).

T. Sagawa, Entropy, divergence, and majorization in classical and quantum thermodynamics, arXiv: 2007.09974 (2020).

M. Tomamichel, R. Colbeck, and R. Renner, Duality between smooth min-and max-entropies, IEEE Trans. Inf. Theory 56, 4674 (2010).

M. Tomamichel, Quantum Information Processing with Finite Resources: Mathematical Foundations, Vol. 5 (Springer, 2015).

V. Cappellini, H.-J. Sommers, and K. Życzkowski, Subnormalized states and trace-nonincreasing maps, J. Math. Phys. 48, 052110 (2007).
[87] M. Cerezo, A. Poremba, L. Cincio, and P. J. Coles, Variational quantum fidelity estimation, Quantum 4, 248 (2020).

[88] P. Cappellaro, 22.51 Quantum Theory of Radiation Interactions Fall 2012”, Massachusetts Institute of Technology: MIT OpenCourseWare (2012).

[89] A. Sone, Y.-X. Liu, and P. Cappellaro, Quantum Jarzynski equality in open quantum systems from the one-time measurement scheme, Phys. Rev. Lett. 125, 060602 (2020).

[90] A. Sone, D. O. Soares-Pinto, and S. Deffner, Exchange fluctuation theorems for strongly interacting quantum pumps, arXiv preprint arXiv:2209.12927 (2022).

[91] K. Beyer, K. Luoma, and W. T. Strunz, Work as an external quantum observable and an operational quantum work fluctuation theorem, Phys. Rev. Research 2, 033508 (2020).