SYMMETRIC TENSOR NUCLEAR NORMS

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Abstract. This paper studies nuclear norms of symmetric tensors. As recently shown by Friedland and Lim, the nuclear norm of a symmetric tensor can be achieved at a symmetric decomposition. We discuss how to compute symmetric tensor nuclear norms, depending on the tensor order and the ground field. Lasserre relaxations are proposed for the computation. The theoretical properties of the relaxations are studied. For symmetric tensors, we can compute their nuclear norms, as well as the nuclear decompositions. The proposed methods can be extended to nonsymmetric tensors.

1. Introduction

Let $F$ be a field (either the real field $\mathbb{R}$ or the complex one $\mathbb{C}$). Let $F_{n_1 \times \cdots \times n_m}$ be the space of tensors of order $m$ and dimension $(n_1, \ldots, n_m)$. Each tensor in $F_{n_1 \times \cdots \times n_m}$ can be represented by an $m$-dimensional hypermatrix (or array)

$$A = (A_{i_1 \cdots i_m})$$

with each entry $A_{i_1 \cdots i_m} \in F$ and $1 \leq i_1 \leq n_1, \ldots, 1 \leq i_m \leq n_m$. For two tensors $A, B \in F_{n_1 \times \cdots \times n_m}$, their hermitian inner product is defined as

$$(1.1) \quad A \cdot B := \sum_{1 \leq i_j \leq n_j, j=1,\ldots,m} A_{i_1 \ldots i_m} \bar{B}_{i_1 \ldots i_m}. \tag{1.1}$$

(The bar $\bar{}$ denotes the complex conjugate.) This induces the Hilbert-Schmidt norm

$$(1.2) \quad \|A\| := \sqrt{A \cdot A}. \tag{1.2}$$

For vectors $x^{(1)} \in F_{n_1}, \ldots, x^{(m)} \in F_{n_m}$, $x^{(1)} \otimes \cdots \otimes x^{(m)}$ denotes their standard tensor product, i.e.,

$$(x^{(1)} \otimes \cdots \otimes x^{(m)})_{i_1 \ldots i_m} = (x^{(1)})_{i_1} \cdots (x^{(m)})_{i_m}. \tag{1.2}$$

The spectral norm of $A$, depending on the field $F$, is defined as

$$(1.3) \quad \|A\|_{\sigma,F} := \max \{|A \cdot x^{(1)} \otimes \cdots \otimes x^{(m)}| : \|x^{(j)}\| = 1, x^{(j)} \in F_{n_j}\}. \tag{1.3}$$

In the above, $\| \cdot \|$ denotes the standard Euclidean vector norm. The nuclear norm of $A$, also depending on $F$, is defined as

$$(1.4) \quad \|A\|_{*,F} := \min \left\{ \sum_{i=1}^r |\lambda_i| : \left| A = \sum_{i=1}^r \lambda_i v^{(i,1)} \otimes \cdots \otimes v^{(i,m)} \right| \right\}. \tag{1.4}$$

The spectral norm $\| \cdot \|_{\sigma,F}$ is dual to the nuclear norm $\| \cdot \|_{*,F}$ (cf. [6]):

$$\|A\|_{\sigma,F} = \max \{ |A \cdot \lambda' | : \|\lambda'\|_{*,F} = 1 \}, \tag{1.4}$$

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\[ \|A\|_{\star, F} = \max \{|A \bullet Y| : \|Y\|_{\sigma, F} = 1\}. \]

Spectral and nuclear tensor norms have important applications, e.g., signal processing and blind identification ([21, 22]), tensor completion and recovery ([24, 34]), low rank tensor approximations ([8, 29, 35]). When the order \( m > 2 \), the computation of spectral and nuclear norms is NP-hard ([9, 10, 13]). In [5], the nuclear norms of several interesting tensors were studied. We refer to [14, 15] for tensor theory and applications.

This paper focuses on nuclear norms of symmetric tensors. Recall that a tensor \( A \in \mathbb{F}^{n_1 \times \cdots \times n_m} \) is symmetric if \( n_1 = \cdots = n_m \) and
\[
A_{i_1 \ldots i_m} = A_{j_1 \ldots j_m}
\]
whenever \((i_1, \ldots, i_m)\) is a permutation of \((j_1, \ldots, j_m)\). Let \( S_m(\mathbb{F}^n) \) be the space of all \( n \)-dimensional symmetric tensors of order \( m \) and over the field \( \mathbb{F} \). For convenience, denote the symmetric tensor power \( x \otimes^m := x \otimes \cdots \otimes x \) (\( x \) is repeated \( m \) times).

For a symmetric tensor \( A \in S^m(\mathbb{F}^n) \), its spectral and nuclear norms can be simplified as (for \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \))
\[
(1.5) \quad \|A\|_{\sigma, F} = \max \{|A \bullet x^{\otimes m}| : \|x\| = 1, x \in \mathbb{F}^n\},
\]
\[
(1.6) \quad \|A\|_{\star, F} = \min \left\{ \sum_{i=1}^r |\lambda_i| : A = \sum_{i=1}^r \lambda_i (v_i)^{\otimes m}, \|v_i\| = 1, v_i \in \mathbb{F}^n, \lambda_i \in \mathbb{F}\right\}.
\]

The equality (1.5) can be found in Banach [1], Friedland [7], Friedland and Ottaviani [8], and Zhang et al. [35]. The equality (1.6) was recently proved by Friedland and Lim [9]. In (1.6), the decomposition of \( A \), for which the minimum is achieved, is called a nuclear decomposition as in [9]. When \( A \) is a real tensor,
\[
\|A\|_{\sigma, \mathbb{R}} \leq \|A\|_{\sigma, \mathbb{C}}, \quad \|A\|_{\star, \mathbb{R}} \geq \|A\|_{\star, \mathbb{C}}.
\]

The strict inequalities are possible in the above. Explicit examples can be found in [9] and in §6 of this paper.

The computation of tensor nuclear norms can be formulated as a moment optimization problem. When \( A \) is a real cubic symmetric tensor (i.e., \( m = 3 \)), Tang and Shah [33] pointed out that the real nuclear norm \( \|A\|_{\star, \mathbb{R}} \) is equal to the optimal value of the moment optimization problem
\[
(1.7) \quad \min \int_S 1d\mu \quad s.t. \quad A = \int_S x \otimes x \otimes xd\mu
\]
where \( \mu \) is a Borel measure variable whose support is contained in the unit sphere
\[
(1.8) \quad S := \{x \in \mathbb{R}^n | \|x\| = 1\}.
\]

The equality constraint in (1.7) gives cubic moments of \( \mu \), while the objective is the total mass of \( \mu \). Lasserre’s hierarchy of semidefinite relaxations [16, 19] can be applied to solve (1.7), as proposed in [33]. This gives a sequence of lower bounds, say, \( \{\rho_k\} \), for the real nuclear norm \( \|A\|_{\star, \mathbb{R}} \). It can be shown that \( \rho_k \to \|A\|_{\star, \mathbb{R}} \) as \( k \to \infty \). However, in computational practice, it is very difficult to check the convergence, i.e., how do we detect if \( \rho_k \) is equal to, or close to, \( \|A\|_{\star, \mathbb{R}} \)? When the convergence occurs, how can we get a nuclear decomposition? To the best of the author’s knowledge, there was few prior work on these two questions. The major difficulty is that the flat extension condition (cf. [4, 6, 11]), which is often used for
solving moment problems, is usually not satisfied for solving (1.7). This causes the embarrassing fact that the nuclear norm is often not known, although it can be approximated as close as possible in theory. Moreover, when the order \( m \) is even, or the field \( \mathbb{F} = \mathbb{C} \), the nuclear norm \( \|A\|_{*, \mathbb{F}} \) is no longer equal to the optimal value of (1.7).

In this paper, we propose methods for computing nuclear norms of symmetric tensors, for both odd and even orders, over both the real and complex fields. We give detailed theoretical analysis and computational implementation.

- When the order \( m \) is odd and \( \mathbb{F} = \mathbb{R} \), the nuclear norm \( \|A\|_{*, \mathbb{R}} \) equals the optimal value of (1.7), as shown in [33].
- When the order \( m \) is even and \( \mathbb{F} = \mathbb{R} \), the nuclear norm \( \|A\|_{*, \mathbb{R}} \) is no longer equal to the optimal value of (1.7). We construct a new moment optimization problem whose optimal value equals \( \|A\|_{*, \mathbb{R}} \).
- When \( \mathbb{F} = \mathbb{C} \), we construct a new moment optimization problem whose optimal value equals \( \|A\|_{*, \mathbb{C}} \), for both even and odd orders.

Lasserre relaxations in [10, 19] are efficient for solving these moment optimization problems. We can get a sequence of lower bounds for the nuclear norm \( \|A\|_{*, \mathbb{F}} \), which is denoted as \( \{\|A\|_{k, \mathbb{F}}\}_{k=1}^{\infty} \). (The integer \( k \) is called the relaxation order.) We prove the asymptotic convergence \( \|A\|_{k, \mathbb{F}} \to \|A\|_{*, \mathbb{F}} \) as the relaxation order \( k \to \infty \). In computational practice, the finite convergence often occurs, i.e., \( \|A\|_{k, \mathbb{F}} = \|A\|_{*, \mathbb{F}} \) for some \( k \). We show how to detect \( \|A\|_{k, \mathbb{F}} = \|A\|_{*, \mathbb{F}} \) and how to compute nuclear decompositions. This can be done by solving a truncated moment problem. We also prove conditions that guarantee finite convergence.

The paper is organized as follows. Section 3 discusses nuclear norms when the order \( m \) is odd and \( \mathbb{F} = \mathbb{R} \). Section 4 discusses nuclear norms when \( m \) is even and \( \mathbb{F} = \mathbb{R} \). Section 5 discusses nuclear norms when the field \( \mathbb{F} = \mathbb{C} \). The numerical experiments are given in Section 6. Some preliminary results are given in Section 2. The extensions to nonsymmetric tensors are given in Section 7.

2. Preliminaries

**Notation.** The symbol \( \mathbb{N} \) (resp., \( \mathbb{R} \), \( \mathbb{C} \)) denotes the set of nonnegative integers (resp., real, complex numbers). For \( x = (x_1, \ldots, x_n) \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), denote

\[
x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| := |\alpha_1| + \cdots + |\alpha_n|.
\]

For a degree \( d > 0 \), denote the set of monomial powers

\[
\begin{align*}
\mathbb{N}^n_{[0, d]} &:= \{ \alpha \in \mathbb{N}^n : 0 \leq |\alpha| \leq d \}, \\
\mathbb{N}^n_{(d)} &:= \{ \alpha \in \mathbb{N}^n : |\alpha| = d \}, \\
\mathbb{N}^n_{[0, d)} &:= \mathbb{N}^n_{(d)} \cup \{0\}.
\end{align*}
\]

Denote the vector of monomials:

\[
[x]_{0,m} := (x^\alpha)_{|\alpha|=0,m}.
\]

The symbol \( \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n] \) denotes the ring of polynomials in \( x := (x_1, \ldots, x_n) \) and with real coefficients, while \( \mathbb{R}[x]_d \) denotes the set of polynomials in \( \mathbb{R}[x] \) with degrees up to \( d \). We use \( \mathbb{R}[x]_{d}^{hom} \) to denote the set of homogeneous polynomials in \( \mathbb{R}[x] \) with degree \( d \). For the complex field \( \mathbb{C} \), the \( \mathbb{C}[x] \) and \( \mathbb{C}[x]_d \) are similarly defined. The \( deg(p) \) denotes the total degree of a polynomial \( p \). For \( t \in \mathbb{R} \), \( \lceil t \rceil \) (resp., \( \lfloor t \rfloor \)) denotes the smallest integer not smaller (resp., the largest integer not bigger) than \( t \). For a matrix \( A \), \( A^T \) denotes its transpose. For a symmetric matrix
X, X ≥ 0 (resp., X > 0) means X is positive semidefinite (resp., positive definite). The e_i denotes the standard ith unit vector, and e is the vector of all ones.

In the following, we review some basics in polynomial optimization and moment problems. We refer to [17, 18, 20] for details. A polynomial p ∈ ℜ[x] is said to be a sum of squares (SOS) if p = p_1^2 + ⋯ + p_k^2 for some p_1, . . . , p_k ∈ ℜ[x]. The set of all SOS polynomials in x is denoted as Σ[x]. For a degree m, denote the truncation

$$\Sigma[x]_m := \Sigma[x] \cap ℜ[x]_m.$$ 

For a tuple g = (g_1, . . . , g_i) of polynomials, its quadratic module is the set

$$Qmod(g) := \Sigma[x] + g_1 \cdot \Sigma[x] + \cdots + g_i \cdot \Sigma[x].$$

The kth truncation of Qmod(g) is the set

$$Qmod(g)_k := \Sigma[x]_k + g_1 \cdot \Sigma[x]_d_1 + \cdots + g_i \cdot \Sigma[x]_d_i$$

where each d_i = k − deg(g_i). Note that

$$Qmod(g) = \bigcup_{k \in ℤ} Qmod(g)_k.$$ 

For a tuple h = (h_1, . . . , h_s) of polynomials, the ideal it generates is the set

$$Ideal(h) := h_1 \cdot ℜ[x] + \cdots + h_m \cdot ℜ[x].$$

The kth truncation of Ideal(h) is the set

$$Ideal(h)_k := h_1 \cdot ℜ[x]_{k − \deg(h_1)} + \cdots + h_m \cdot ℜ[x]_{k − \deg(h_m)}.$$

Clearly, Ideal(h) = \bigcup_{k \in ℤ} Ideal(h)_k.

Let g, h be as above. Consider the set

$$K = \{x \in ℜ^n : h(x) = 0, \; g(x) ≥ 0\}.$$ 

Clearly, if f ∈ Ideal(h) + Qmod(g), then f ≥ 0 on the set K. The reverse is also true under certain conditions. The set Ideal(h) + Qmod(g) is said to be archimedean if N − \|x\|^2 ∈ I(h) + Q(g) for some scalar N > 0.

**Theorem 2.1.** ([31]) Let h, g, K be as above. Assume Ideal(h) + Qmod(g) is archimedean. If a polynomial f > 0 on K, then f ∈ Ideal(h) + Qmod(g).

The above theorem is called Putinar’s Positivstellensatz in the literature. Interestingly, when f ≥ 0 on K, we also have f ∈ Ideal(h) + Qmod(g), under general optimality conditions (cf. [27]).

Let ℜ_n^N be the space of multi-sequences indexed by α ∈ ℜ_n^N (see the notation [21]). A vector in ℜ_n^N is called a truncated multi-sequence (tms) of degree d. Every z ∈ ℜ_n^N can be labelled as

$$z = (z_\alpha)_{\alpha \in ℜ_n^N}.$$ 

For t ≤ d and z ∈ ℜ_n^N, denote the truncation:

$$z|_{\{0, m\}} := (z_\alpha)_{\alpha \in ℜ_n^{\{0, m\}}}.$$ 

For p = Σ_{α ∈ ℜ_n^N} p_α x^α ∈ ℜ[x]_d and z ∈ ℜ_n^N, define the product

$$\langle p, z \rangle := \sum_{\alpha \in ℜ_n^N} p_\alpha z_\alpha.$$
In the above, each $p_\alpha$ is a coefficient. For a polynomial $q \in \mathbb{R}[x]_{2k}$ and a tms $z \in \mathbb{R}^{[n,2k]}$, the product $\langle qp_1 p_2, z \rangle$ is a quadratic form in the coefficients of $p_1, p_2$. The $k$th localizing matrix of $q$, generated by a tms $z \in \mathbb{R}^{[n,2k]}$, is the symmetric matrix $L_q^{(k)}(z)$ such that

$$\langle qp_1 p_2, z \rangle = \text{vec}(p_1)^T \left( L_q^{(k)}(z) \right) \text{vec}(p_2)$$

for all $p_1, p_2 \in \mathbb{R}[x]$ with $\deg(p_1), \deg(p_2) \leq 2k - \lceil \deg(q)/2 \rceil$. In the above, $\text{vec}(p_i)$ denotes the coefficient vector of $p_i$. When $q = 1$ (the constant one polynomial), $L_q^{(k)}(z)$ is reduced to the so-called moment matrix and is denoted as

$$M_k(z) := L_1^{(k)}(z).$$

We refer to \[4, 11\] for localizing and moment matrices.

3. Odd order tensors with $F = \mathbb{R}$

Assume the field $F = \mathbb{R}$ and the order $m$ is odd. We discuss how to compute the real nuclear norm $\|A\|_{*,\mathbb{R}}$ of a tensor $A \in \mathbb{S}^m(\mathbb{R}^n)$. Note that $\lambda_i(v_i)_{\otimes m} = (-\lambda_i)(-v_i)_{\otimes m}$, when $m$ is odd. In the decomposition of $A$ as in (1.6), one can generally assume $\lambda_i \geq 0$, so

$$\|A\|_{*,\mathbb{R}} = \min \left\{ \sum_{i=1}^r \lambda_i : A = \sum_{i=1}^r \lambda_i(v_i)_{\otimes m}, \lambda_i \geq 0, \|v_i\| = 1, v_i \in \mathbb{R}^n \right\}.$$

Let $\mathcal{B}(S)$ be the set of Borel measures supported on the unit sphere $S$ as in (1.8). As pointed out in \[33\], $\|A\|_{*,\mathbb{R}}$ equals the optimal value of

$$\min \left\{ \int 1 d\mu : \text{s.t. } A = \int x_{\otimes m} d\mu, \mu \in \mathcal{B}(S) \right\}.$$

Let $\mathbf{a} \in \mathbb{R}^{N_{(m)}}$ be the vector of tensor entries of $A$ such that

$$\mathbf{a}_\alpha = A_{i_1 \cdots i_m} \quad \text{if} \quad x^\alpha = x_{i_1} \cdots x_{i_m}.$$

The equality constraint in (3.2) is equivalent to that

$$\mathbf{a}_\alpha = \int x^\alpha d\mu \quad (\alpha \in N_{(m)}).$$

Define the cone of moments

$$\mathcal{R}(0,m) := \left\{ y \in \mathbb{R}^{N_{(0,m)}} \left| \exists \mu \in \mathcal{B}(S) \text{ s.t. } y_\alpha = \int x^\alpha d\mu \forall \alpha \in N_{(0,m)} \right. \right\}.$$ 

The cone $\mathcal{R}(0,m)$ is closed, convex, and has nonempty interior \[26\ Prop. 3.2\]. The optimization problem (3.2) is equivalent to

$$\min \left\{ (y)_0 : \text{s.t. } (y)_\alpha = \mathbf{a}_\alpha \quad (\alpha \in N_{(m)}), \quad y \in \mathcal{R}(0,m) \right\}.$$
3.1. An algorithm. The cone $\mathcal{R}_{0,m}$ can be approximated by semidefinite relaxations. Denote the cones

\begin{equation}
\mathcal{S}^k := \{ z \in \mathbb{R}^{N_{0,2k}} \mid M_k(z) \geq 0, L_1(z) = 0 \},
\end{equation}

\begin{equation}
\mathcal{S}_{0,m}^k := \{ y \in \mathbb{R}^{N_{0,m}} \mid \exists z \in \mathcal{S}^k, y = z|_{\{0,m\}} \}.
\end{equation}

It can be shown that (cf. [25, Prop. 3.3])

\begin{equation}
\mathcal{R}_{0,m} = \bigcap_{k \geq m/2} \mathcal{S}_{0,m}^k.
\end{equation}

This leads to the hierarchy of semidefinite relaxations

\begin{equation}
\begin{cases}
\|A\|_{k*,\mathbb{R}} := \min \langle z \rangle_0 \\
\text{s.t.} \quad \langle z \rangle_\alpha = a_\alpha (\alpha \in N_{0,m}), \\
\quad z \in \mathcal{S}^k,
\end{cases}
\end{equation}

for the relaxation orders $k = m_0, m_0 + 1, \ldots$, where $m_0 := \lfloor m/2 \rfloor$. Since $\mathcal{R}_{0,m} \subseteq \mathcal{S}^{2k+2} \subseteq \mathcal{S}^k$ for all $k$, we have the monotonicity relationship

\begin{equation}
\|A\|_{k*,\mathbb{R}} \leq \cdots \leq \|A\|_{1*,\mathbb{R}} \leq \cdots \leq \|A\|_{\ast,\mathbb{R}}.
\end{equation}

Algorithm 3.1. Given a tensor $A \in \mathbb{R}^m(n)$ with odd $m$, let $k = m_0$ and do:

Step 1 Solve the semidefinite relaxation (3.9), for an optimizer $z^k$.

Step 2 Let $y^k := z^k|_{\{0,m\}}$ (see (2.4) for the truncation). Check whether $y^k \in \mathcal{R}_{0,m}$ or not. If yes, then $\|A\|_{\ast,\mathbb{R}} = \|A\|_{k*,\mathbb{R}}$ and go to Step 3; otherwise, let $k := k + 1$ and go to Step 1.

Step 3 Compute the decomposition of $y^k$ as

\begin{equation}
y^k = \lambda_1[v_1]_{0,m} + \cdots + \lambda_r[v_r]_{0,m}
\end{equation}

with all $\lambda_i > 0, v_i \in S$. This gives the nuclear decomposition

\begin{equation}
A = \lambda_1(v_1) \otimes m + \cdots + \lambda_r(v_r) \otimes m
\end{equation}

such that $\lambda_1 + \cdots + \lambda_r = \|A\|_{\ast,\mathbb{R}}$.

In the above, the method in [25] can be applied to check whether $y^k \in \mathcal{R}_{0,m}$ or not. If yes, a nuclear decomposition can also be obtained. It requires to solve a moment optimization problem whose objective is randomly generated.

3.2. Convergence properties. The dual cone of the set $\mathcal{R}_{0,m}$ is

\begin{equation}
\mathcal{P}(S)_{0,m} := \{ t + q \mid t \in \mathbb{R}, q \in \mathbb{R}_{[x]_m^{\text{hom}}}, t + q \geq 0 \text{ on } S \}.
\end{equation}

So, the dual optimization problem of (3.3) is

\begin{equation}
\max_{p \in \mathbb{R}_{[x]_m^{\text{hom}}}} \langle p, a \rangle \text{ s.t. } 1 - p \in \mathcal{P}(S)_{0,m}.
\end{equation}

Lemma 3.2. Let $a$ be the vector as in (3.3). Then, both (3.3) and (3.3) achieve the same optimal value, which equals the nuclear norm $\|A\|_{\ast,\mathbb{R}}$.

Proof. Clearly, $p = 0$ (the zero polynomial) is an interior point of (3.12). By the linear conic duality theory [2, §2.4], (3.5) and (3.12) have the same optimal value which is $\|A\|_{\ast,\mathbb{R}}$, and (3.3) achieves it. The feasible set of (3.12) is compact. This is because $|p| \leq 1$ on the unit sphere and $p$ is a form of degree $m$. So, (3.12) also achieves its optimal value, which equals $\|A\|_{\ast,\mathbb{R}}$. \qed
Denote the nonnegative polynomial cones:

\[(3.13) \quad Q_k := \text{Ideal}_{2k}(1 - \|x\|^2) + \Sigma[x]_{2k}, \quad Q := \bigcup_{k \geq 1} Q_k.\]

The cones \(Q_k\) and \(\mathcal{S}^{2k}\) are dual to each other (cf. [26]). So, the dual optimization problem of \((3.9)\) is

\[(3.14) \quad \max_{p \in \mathbb{R}[x]_m^{\text{hom}}} \langle p, a \rangle \quad \text{s.t.} \quad 1 - p \in Q_k.
\]

Some properties of Lasserre relaxations were mentioned in [33]. For completeness of the paper, we give the properties with more details and rigorous proofs.

**Lemma 3.3.** Let \(a\) be the vector as in \((3.9)\). Then, both \((3.9)\) and \((3.14)\) achieve the same optimal value, which equals \(\|A\|_{k^*, \mathbb{R}}\). Moreover, for each \(k \geq m_0\), \(\|A\|_{k^*, \mathbb{R}}\) is a norm function in \(A \in \mathbb{S}^n(\mathbb{R}^n)\).

**Proof.**

For each \(k \geq m_0\), \(p = 0\) is an interior point of \((3.14)\). So, \((3.9)\) and \((3.14)\) have the same optimal value \(\|A\|_{k^*, \mathbb{R}}\), and \((3.9)\) achieves it (cf. [2, §2.4]). The set \(Q_k\) is closed, which can be implied by Theorem 3.1 of [26] (also see Theorem 3.35 of [20]). When \(1 - p \in Q_k\), \(|p| \leq 1\) on the unit sphere \(S\). Hence, the feasible set of \((3.14)\) is compact, and it also achieves its optimal value. In the following, we prove that \(\|A\|_{k^*, \mathbb{R}}\) is a norm function in \(A\).

1. Because \(M_k(z) \succeq 0\), \((z)_{0 \geq 0}\). So \(\|A\|_{k^*, \mathbb{R}} \geq 0\) for all \(A\).
2. Let \(z^*\) be an optimizer such that \(\|A\|_{k^*, \mathbb{R}} = (z^*)_0\). If \(\|A\|_{k^*, \mathbb{R}} = 0\), then \((z^*)_0 = 0\) and \(z^* = 0\), because \(M_k(z^*) \succeq 0\) and \(L_{1-\|x\|^2}^{(k)}(z^*) = 0\). So, \(a = 0\) and \(A\) must be the zero tensor.
3. First, we show that \(\| - A\|_{k^*, \mathbb{R}} = \|A\|_{k^*, \mathbb{R}}\). For \(z \in \mathbb{R}^{N_0[0, 2k]}\), define \(s(z) \in \mathbb{R}^{N_0[0, 2k]}\) be such that

\[(s(z))_\alpha = (-1)^{|\alpha|}(z)_\alpha, \quad \forall \alpha \in N_0[0, 2k].\]

One can verify that (1 denotes the vector of all ones)

\[M_k(s(z)) = \text{diag}([-1]_k)M_k(z)\text{diag}([-1]_k),\]

\[L_{1-\|x\|^2}^{(k)}(s(z)) = \text{diag}([-1]_{k-1})L_{1-\|x\|^2}^{(k)}(z)\text{diag}([-1]_{k-1}).\]

Thus, \(z\) is feasible for \((3.9)\) with tensor \(A\) if and only if \(s(z)\) is feasible for \((3.9)\) with tensor \(-A\). Since \((z)_0 = (s(z))_0\), we get \(\| - A\|_{k^*, \mathbb{R}} = \|A\|_{k^*, \mathbb{R}}\).

Second, we show that \(\|tA\|_{k^*, \mathbb{R}} = t\|A\|_{k^*, \mathbb{R}}\) for all \(t > 0\). For \(t > 0\), \(z\) is feasible for \((3.9)\) with tensor \(A\) if and only if \(tz\) is feasible for \((3.9)\) with tensor \(tA\). Note that \((z)_0 = (tz)_0\) for \(t > 0\). So, \(\|tA\|_{k^*, \mathbb{R}} = t\|A\|_{k^*, \mathbb{R}}\) for all \(t > 0\).

The above two cases imply that

\[\|tA\|_{k^*, \mathbb{R}} = |t| \cdot \|A\|_{k^*, \mathbb{R}} \quad \forall A \in \mathbb{S}^n(\mathbb{R}^n), \forall t \in \mathbb{R}.\]

4. The feasible set of \((3.9)\) is a convex set in \((z, A)\). Its objective is a linear function in \(z\). By the result in [2, §3.2.5], \(\|A\|_{k^*, \mathbb{R}}\) is a convex function in \(A\), so \(\|A + B\|_{k^*, \mathbb{R}} \leq \|A\|_{k^*, \mathbb{R}} + \|B\|_{k^*, \mathbb{R}}\) for all \(A, B\).

The convergence of Algorithm 3.1 is summarized as follows.
Algorithm 3.1 has the following properties:

(i) By Lemma 3.2, for every \( p^* \) an optimizer of \((3.12)\), if \( 1 - p^* \in Q \), then \( \|A\|_{k^*, R} = \|A\|_{*, R} \) for all \( k \) sufficiently big.

(ii) If \( y^k \in \mathcal{R}_{(0, m)} \) for some order \( k \), then \( \|A\|_{*, R} = \|A\|_{k, R} \).

(iii) The sequence \( \{y^k\}_{k=m_0}^\infty \) converges to a point in \( \mathcal{R}_{(0, m)} \).

Proof. (i) By Lemma 3.2 for every \( \epsilon > 0 \), there exists \( p_1 \in \mathbb{R}[x]^\text{hom} \) such that

\[ 1 - p_1 > 0 \text{ on } S, \quad \langle p_1, a \rangle \geq \|A\|_{*, R} - \epsilon. \]

By Theorem 2.1 there exists \( k_1 \) such that \( 1 - p_1 \in Q_{k_1} \). By Lemma 3.3 we get

\[ \|A\|_{k_1, R} \geq \|A\|_{*, R} - \epsilon. \]

The monotonicity relation \((5.10)\) and the above imply that

\[ \|A\|_{*, R} \geq \lim_{k \to \infty} \|A\|_{k, R} \geq \|A\|_{*, R} - \epsilon. \]

Since \( \epsilon > 0 \) can be arbitrarily small, the item (i) follows directly.

(ii) If \( 1 - p^* \in Q \), then \( 1 - p^* \in Q_{k_2} \) for some \( k_2 \). By Lemma 3.2 we know

\[ \|A\|_{k^*, R} = \langle p^*, a \rangle \leq \|A\|_{k_2, R}. \]

Then, \((3.10)\) implies that \( \|A\|_{k^*, R} = \|A\|_{*, R} \) for all \( k \geq k_2 \).

(iii) If \( y^k \in \mathcal{R}_{(0, m)} \) for some \( k \), then \( \|A\|_{k^*, R} \geq \|A\|_{*, R} \), by Lemmas 3.2 and 3.3.

Then, the equality \( \|A\|_{k^*, R} = \|A\|_{*, R} \) follows from \((3.10)\).

(iv) Note the relations

\[ (y^k)_0 = \|A\|_{k, R}, \quad (y^k)_a = a, \quad (\forall a \in \mathbb{N}^{\text{hom}}). \]

Since \( \|A\|_{k, R} \to \|A\|_{*, R} \), we know the limit \( y^k \) of the sequence \( \{y^k\} \) must exist. For all \( k \geq m/2 \), we have \( y^k \in \mathcal{R}_{(0, m)} \). The distance between \( \mathcal{R}^k_{(0, m)} \) and \( \mathcal{R}_{(0, m)} \) tends to zero as \( k \to \infty \) (cf. [20 Prop. 3.4]), so \( y^k \in \mathcal{R}_{(0, m)} \). It can also be implied by the equality \((3.3)\). \( \square \)

In Theorem 3.4(iii), we always have \( 1 - p^* \geq 0 \) on \( S \). Under some general conditions, we further have \( 1 - p^* \in Q \), as shown in [27]. Thus, Algorithm 3.1 usually has finite convergence, which is confirmed by numerical experiments in [16].

4. Even order tensors with \( F = \mathbb{R} \)

Assume the order \( m \) is even and the field \( F = \mathbb{R} \). For a symmetric tensor \( A \in \mathbb{S}^m(\mathbb{R}^n) \), the sign of \( \lambda_i \) in \((1.6)\) cannot be generally assumed to be positive. However, we can always decompose \( A \) as \((1.6) \) is the vector of all ones)

\[
\begin{align*}
\mathcal{A} &= \sum_{i=1}^{r_1} \lambda_1^+(v^+_i)^{\otimes m} - \sum_{i=1}^{r_2} \lambda_1^-(v_i^-)^{\otimes m}, \\
\lambda_1^+ &\geq 0, \quad \|v^+_i\| = 1, \quad 1^T v^+_i \geq 0, \quad v^+_i \in \mathbb{R}^n, \\
\lambda_1^- &\geq 0, \quad \|v^-_i\| = 1, \quad 1^T v^-_i \geq 0, \quad v^-_i \in \mathbb{R}^n.
\end{align*}
\]

Let \( \mathcal{B}(S^+) \) be the set of Borel measures supported in the half unit sphere

\[ S^+ := \{x \in \mathbb{R}^n \mid \|x\| = 1, 1^T x \geq 0\}. \]
Clearly, the weighted Dirac measures
\[ \mu^+ := \sum_{i=1}^{r_1} \lambda_i^+ \delta_{v_i^+}, \quad \mu^- := \sum_{i=1}^{r_2} \lambda_i^- \delta_{v_i^-} \]
belong to \( \mathcal{B}(S^+) \). The decomposition (4.11) is equivalent to
\[ (4.3) \quad \mathcal{A} = \int x \otimes m \, d\mu^+ - \int x \otimes m \, d\mu^- . \]

Reversely, if there exist \( \mu^+, \mu^- \in \mathcal{B}(S^+) \) satisfying (4.13), then \( \mathcal{A} \) has a decomposition as in (4.1) (cf. [25, Prop. 3.3]). Therefore, the nuclear norm \( \|A\|_{*, \mathbb{R}} \) equals the optimal value of the problem
\[ (4.4) \quad \begin{cases} \min \{ \int 1 \, d\mu^+ + \int 1 \, d\mu^- \} \\ \text{s.t.} \\ \mathcal{A} = \int x \otimes m \, d\mu^+ - \int x \otimes m \, d\mu^- , \end{cases} \quad \mu^+, \mu^- \in \mathcal{B}(S^+) . \]

Let \( a \in \mathbb{R}^{N_n(m)} \) be the vector such that
\[ (4.5) \quad a_\alpha = \mathcal{A}_{i_1 \cdots i_m} \quad \text{if} \quad x^\alpha = x_{i_1} \cdots x_{i_m} . \]
Denote the cone of moments
\[ (4.6) \quad \mathcal{R}^+_n(0, m) := \left\{ y \in \mathbb{R}^{N_n(m)} \mid \exists \mu \in \mathcal{B}(S^+) \text{ such that } y_\alpha = \int x^\alpha \, d\mu \text{ for } \alpha \in N_n(n, m) \right\} . \]

Then, (4.4) is equivalent to
\[ (4.7) \quad \begin{cases} \min \{ (y^+)_0 + (y^-)_0 \} \\ \text{s.t.} \\ (y^+)_\alpha - (y^-)_\alpha = a_\alpha \, (\alpha \in N_n(n, m)) , \\ y^+, y^- \in \mathcal{R}^+_n(0, m) . \end{cases} \]

4.1. **An algorithm.** The cone \( \mathcal{R}^+_n(0, m) \) can be approximated by semidefinite relaxations. Denote the cones
\[ (4.8) \quad \mathcal{R}^{+, 2k}_n(0, m) := \left\{ z \in \mathbb{R}^{N_n(0, 2k)} \mid M_k(z) \succeq 0 , \, L_1^{(k)}(z) \preceq 0 , \, L_{1-n}^{(k)}(z) = 0 \right\} , \]
\[ (4.9) \quad \mathcal{R}^{+, 2k}_n(0, m) := \left\{ y \in \mathbb{R}^{N_n(0, m)} \mid \exists z \in \mathcal{R}^{+, 2k} , \, y = z |_{(0, m)} \right\} . \]

Note that \( \mathcal{R}^{+, 2k}_n(0, m) \) is a projection of \( \mathcal{R}^{+, 2k}_n \) and \( \mathcal{R}^+_n(0, m) \subseteq \mathcal{R}^{+, 2k}_n(0, m) \) for all \( k \). As shown in [26], it holds that
\[ (4.10) \quad \mathcal{R}^+_n(0, m) = \bigcap_{k \geq m/2} \mathcal{R}^{+, 2k}_n(0, m) . \]

So, we get the hierarchy of semidefinite relaxations for solving (4.7):
\[ (4.11) \quad \begin{cases} \|A\|_{k, *, \mathbb{R}} := \min_{z^+, z^-} (z^+)_0 + (z^-)_0 \\ \text{s.t.} \\ (z^+)_\alpha - (z^-)_\alpha = a_\alpha \, (\alpha \in N_n(n, m)) , \\ z^+, z^- \in \mathcal{R}^{+, 2k} , \end{cases} \]
for \( k = m_0, m_0 + 1, \ldots (m_0 = [m/2]) \). Similar to (4.10), we also have the monotonicity relationship
\[ (4.12) \quad \|A\|_{m_0, *, \mathbb{R}} \leq \cdots \leq \|A\|_{k, *, \mathbb{R}} \leq \cdots \leq \|A\|_{*, \mathbb{R}} . \]

**Algorithm 4.1.** For a given tensor \( A \in S^n(\mathbb{R}^n) \), let \( k = m_0 \) and do:

1. **Step 1** Solve the semidefinite relaxation (4.11), for an optimizer \( (z^+, k, z^-) \).
Step 2 Let \( y^{+,k} := z^{+,k}|_{[0,m]} \), \( y^{-,k} := z^{-,k}|_{[0,m]} \) (see (2.4) for the truncation). Check whether \( y^{+,k}, y^{-,k} \in \mathcal{R}^{+}_{[0,m]} \) or not. If they both belong, then \( \|A\|_{*,R} = \|A\|_{k*,R} \) and go to Step 3; otherwise, let \( k := k + 1 \) and go to Step 1.

Step 3 Compute the decompositions of \( y^{+,k}, y^{-,k} \) as
\[
y^{+,k} = \sum_{i=1}^{r_1} \lambda_i^+ [v_i^+]_{0,m}, \quad y^{-,k} = \sum_{i=1}^{r_2} \lambda_i^- [v_i^-]_{0,m},
\]
with all \( \lambda_i^+ > 0, \lambda_i^- > 0 \) and \( v_i^+, v_i^- \in S^+ \). The above gives the nuclear decomposition:
\[
A = \sum_{i=1}^{r_1} \lambda_i^+ (v_i^+) \otimes m - \sum_{i=1}^{r_2} \lambda_i^- (v_i^-) \otimes m
\]
such that \( \sum_{i=1}^{r_1} \lambda_i^+ + \sum_{i=1}^{r_2} \lambda_i^- = \|A\|_{*,R} \).

In the above, the method in [25] can be applied to check if \( y^{+,k}, y^{-,k} \in \mathcal{R}^{+}_{[0,m]} \) or not. If yes, a nuclear decomposition can also be obtained. In Step 3, it is possible that \( r_1 = 0 \) or \( r_2 = 0 \), for which case the corresponding \( y^{+,k} \) or \( y^{-,k} \) is the vector of all zeros. Note that Algorithm 4.1 can also be applied to compute \( \|A\|_{*,R} \) even if the order \( m \) is odd.

4.2. Convergence properties. The dual cone of the set \( \mathcal{R}^{+}_{[0,m]} \) is
\[
\mathcal{R}(S^+)_0 := \{ t + p \mid t \in \mathbb{R}, p \in \mathbb{R}[x]_{m}^{\text{hom}}, t + p \geq 0 \text{ on } S^+ \}.
\]
So, the dual optimization problem of (4.7) is
\[
(4.13) \quad \max_{p \in \mathbb{R}[x]_{m}^{\text{hom}}} \langle p, a \rangle \quad \text{s.t.} \quad 1 \pm p \in \mathcal{R}(S^+)_0.
\]

Lemma 4.2. Let \( a \) be the vector as in (4.9). Then, both (4.7) and (4.13) achieve the same optimal value which equals \( \|A\|_{*,R} \).

Proof. The feasible set of (4.7) is always nonempty, say, \((\hat{y}^+, \hat{y}^-)\) is a feasible pair. Let \( \xi \) be an interior point of \( \mathcal{R}^{+}_{[0,m]} \). Then \( \hat{y}^+ + \xi, \hat{y}^- + \xi \) are both interior points of \( \mathcal{R}^{+}_{[0,m]} \). The zero polynomial \( p = 0 \) is an interior point of (4.13). By the linear conic duality theory [2, §2.4], the optimal values of (4.7) and (4.13) are equal, and they both achieve it. The optimal value of (4.7) is \( \|A\|_{*,R} \), so it is also the optimal value of (4.13).

Next, we study the properties of the relaxation (4.12). Denote the cones of nonnegative polynomials:
\[
(4.14) \quad Q_k^+ := \text{Ideal}_{2k}(1 - \|x\|^2) + \text{Qmod}_{2k}(1^T x), \quad Q^+ := \bigcup_{k \geq 1} Q_k^+.
\]
The cones \( Q_k^+ \) and \( \mathcal{R}^{+,2k} \) are dual to each other (cf. [26]), so the dual optimization problem of (4.11) is
\[
(4.15) \quad \max_{p \in \mathbb{R}[x]_{m}^{\text{hom}}} \langle p, a \rangle \quad \text{s.t.} \quad 1 \pm p \in Q_k^+.
\]

Lemma 4.3. Let \( a \) be the vector of entries of \( A \) as in (4.9). For each \( k \geq m_0 \), both (4.17) and (4.19) achieve the same optimal value \( \|A\|_{k*,R} \). Moreover, \( \|A\|_{k*,R} \) is a norm function in \( \mathcal{A} \in S^m(\mathbb{R}^n) \).
Algorithm 4.1 has the following properties:

1. Because \((z^+)_0 > 0, (z^-)_0 > 0\), we must have \(\|A\|_{k^*, \mathbb{R}} > 0\) for all \(A\).
2. Let \((z^+, z^-)\) be such that \(\|A\|_{k^*, \mathbb{R}} = (z^+_0 + (z^-)_0)\). If \(\|A\|_{k^*, \mathbb{R}} = 0\), then \((z^+_0 + (z^-)_0) = 0\), and hence and \(z^+ = z^- = 0\). So, \(A\) must be the zero tensor.
3. Let \(s(z)\) be the function as in the proof of Lemma 3.3. One can similarly prove that \((z^+, z^-)\) is feasible for (4.11) with tensor \(A\) if and only if \((s(z^+), s(z^-))\) is feasible (4.11) with tensor \(-A\). This implies that \(\|A\|_{k^*, \mathbb{R}} = \|\|A\|_{k^*, \mathbb{R}}\| \mathbb{R}^n\). Similarly, one can show that \(\|tA\|_{k^*, \mathbb{R}} = t\|A\|_{k^*, \mathbb{R}}\) for \(t > 0\). Therefore, \(\|tA\|_{k^*, \mathbb{R}} = t\|A\|_{k^*, \mathbb{R}}\) for all \(A\) and for all \(t \in \mathbb{R}\).
4. For all tensors \(A, B\), the triangular inequality \(\|A + B\|_{k^*, \mathbb{R}} \leq \|A\|_{k^*, \mathbb{R}} + \|B\|_{k^*, \mathbb{R}}\) follows from the fact that the feasible set of (4.11) is a convex set in \((z, A)\) and its objective is linear in \(z\).

The convergence properties of Algorithm 4.1 are as follows.

**Theorem 4.4.** Let \(\|A\|_{k^*, \mathbb{R}}\) be the optimal value of (4.11). For all \(A \in S^n(\mathbb{R}^n)\), Algorithm 4.1 has the following properties:

1. \(\lim_{k \to \infty} \|A\|_{k^*, \mathbb{R}} = \|A\|_{*^*, \mathbb{R}}\).
2. Let \(p^*\) be an optimizer of (4.11). If \(1 \pm p^* \in Q^+\), then \(\|A\|_{k^*, \mathbb{R}} = \|A\|_{*^*, \mathbb{R}}\)

(i) \(\|A\|_{k^*, \mathbb{R}} \geq \|A\|_{*^*, \mathbb{R}}\) for all \(k\) sufficiently big.

**Proof.** (i) By Lemma 4.2, for every \(\epsilon > 0\), there exists \(p_1 \in \mathbb{R}[x]_{m}^{hom}\) such that

\[1 \pm p_1 > 0 \text{ on } S, \quad \langle p_1, a \rangle \geq \|A\|_{*^*, \mathbb{R}} - \epsilon.\]

By Theorem 2.1, there exists \(k_1\) such that \(1 \pm p_1 \in Q_{k_1}^+\). By Lemma 4.3, we can get

\[\|A\|_{k_1, \mathbb{R}} \geq \|A\|_{*^*, \mathbb{R}} - \epsilon.\]

The relation (4.12) and the above imply that

\[\|A\|_{*^*, \mathbb{R}} \geq \lim_{k \to \infty} \|A\|_{k^*, \mathbb{R}} \geq \|A\|_{*^*, \mathbb{R}} - \epsilon.\]

The item (i) follows from that \(\epsilon > 0\) can be arbitrarily small.

(ii)-(iii): The proof is the same as for Theorem 3.4 (ii)-(iii), by using Lemmas 4.2 and 4.3.
one can see that the sequence of diagonal entries of $M_k(z^{+},k)$, $M_k(z^{-},k)$ are bounded.

Then, we can show that the sequence

\[
L_{k}^{(z^{+},k)} = L_{k}^{(z^{-},k)} = 0,
\]

one can see that the sequence of diagonal entries of $M_k(z^{+},k)$, $M_k(z^{-},k)$ is bounded. Then, we can show that the sequence $\{(z^{+},k), z^{-},k\}$ is bounded. This implies that $\{(y^{+},k), y^{-},k\}_{k=m_0}^{\infty}$ is also bounded. When $\hat{y}^{+}, \hat{y}^{-}$ is one of its accumulation points, we can get $\hat{y}^{+} + \hat{y}^{-} = \|A\|_{*,R}$ by evaluating the limit. Note that $y^{+},k, y^{-},k \in \mathcal{F}_{0,n}^{+}$ for all $k$. The distance between $\mathcal{F}_{0,n}^{+}$ tends to zero as $k \to \infty$ (cf. [26, Prop. 3.4]), so we have $\hat{y}^{+}, \hat{y}^{-} \in \mathcal{F}_{0,n}^{+}$. It can also be implied by [4,10]. Next, write down the decompositions:

\[
\hat{y}^{+} = \sum_{i=1}^{r_1} \lambda_i^{+}[v_i^{+}]_{0,m}, \quad \hat{y}^{-} = \sum_{i=1}^{r_2} \lambda_i^{-}[v_i^{-}]_{0,m},
\]

with all $\lambda_i^{+} \geq 0$, $\lambda_i^{-} \geq 0$, and $v_i^{+}, v_i^{-} \in S^{+}$. They give the real nuclear decomposition $A = A_1 - A_2$, with

\[
A_1 = \sum_{i=1}^{r_1} \lambda_i^{+}[v_i^{+}]^{\circ m}, \quad A_2 = \sum_{i=1}^{r_2} \lambda_i^{-}[v_i^{-}]^{\circ m}.
\]

When the nuclear decomposition of $A$ is unique, the decompositions of $A_1, A_2$ are also unique. So, the accumulation point $(\hat{y}^{+}, \hat{y}^{-})$ is unique and $(y^{+},k, y^{-},k)$ must converge to it as $k \to \infty$. \hfill \Box

In Theorem 4.4(ii), we always have $1 \pm p^{*} \geq 0$ on $S^{+}$. Under some general optimality conditions, it holds that $1 \pm p^{*} \in Q^{+}$. So, Algorithm 4.1 generally has finite convergence. This is confirmed by numerical experiments in [46.

5. Nuclear norms with $F = C$

When the ground field $F = C$, the nuclear norm of $A \in S^{m}(\mathbb{C}^{n})$ is

\[
\|A\|_{*,C} = \min \left\{ \sum_{i=1}^{r} \lambda_i : A = \sum_{i=1}^{r} \lambda_i[w_i]^{\circ m}, \|w_i\| = 1, w_i \in \mathbb{C}^{n} \right\}.
\]

First, we formulate an optimization problem for computing $\|A\|_{*,C}$.

**Lemma 5.1.** For all $A \in S^{m}(\mathbb{C}^{n})$, $\|A\|_{*,C}$ equals the optimal value of

\[
\min \sum_{i=1}^{r} \lambda_i \quad \text{s.t.} \quad A = \sum_{i=1}^{r} \lambda_i(u_i + \sqrt{-1}v_i)^{\circ m},
\]

\[
\lambda_i \geq 0, \|u_i\|^2 + \|v_i\|^2 = 1, u_i, v_i \in \mathbb{R}^{n},
\]

\[
1^T v_i \geq 0, \sin(\frac{2\pi}{m}) u_i - \cos(\frac{2\pi}{m}) 1^T v_i \geq 0.
\]

In the above, $1$ is the vector of all ones.

**Proof.** The decomposition of $A$ as in (5.1) is equivalent to

\[
A = \sum_{i=1}^{r} \lambda_i(\tau_i w_i)_{0,m}^{\circ m},
\]
for all unitary $\tau_i \in \mathbb{C}$ with $\tau_i^m = 1$. Write
\[ w_i = u_i + \sqrt{-1}v_i, \quad u_i, v_i \in \mathbb{R}^n, \]
then
\[ 1^T w_i = (1^T u_i) + \sqrt{-1}(1^T v_i), \]
\[ 1^T (\tau_i w_i) = \tau_i \left( (1^T u_i) + \sqrt{-1}(1^T v_i) \right). \]
Write $1^T w_i = re^{\sqrt{-1}\theta}$ with $r \geq 0$, $0 \leq \theta < 2\pi$. There always exists $k \in \{0, 1, \ldots, m-1\}$ such that
\[ 0 \leq \theta - 2k\pi/m < 2\pi/m. \]
If we choose $\tau_i = e^{-2\pi k\sqrt{-1}/m}$, then
\[ 1^T (\tau_i w_i) = re^{\sqrt{-1}\theta_1}, \quad 0 \leq \theta_1 < 2\pi/m. \]
Therefore, without loss of generality, we can assume $1^T w_i = re^{\sqrt{-1}\theta}$, with $0 \leq \theta < 2\pi/m$, in (5.1). This means that $(\text{Im} \, \| \cdot \|)$ denotes the imaginary part
\[ \text{Im}(1^T w_i) \geq 0, \quad \text{Im}(e^{-2\pi k\sqrt{-1}/m}1^T w_i) \leq 0, \]
which are equivalent to the conditions
\[ 1^T v_1 \geq 0, \quad \sin(2\pi/m)1^T u_i - \cos(2\pi/m)1^T v_i \geq 0. \]
Then, the lemma follows from (5.1).

A complex vector in $\mathbb{C}^n$ can be represented by a $2n$-dimensional real vector. Let $x = (x^{re}, x^{im})$ with
\[ x^{re} = (x_1, \ldots, x_n), \quad x^{im} = (x_{n+1}, \ldots, x_{2n}). \]
Denote the set
\[ S^c := \left\{ x = (x^{re}, x^{im}) \bigg| \|x^{re}\|^2 + \|x^{im}\|^2 = 1, \ x^{re}, x^{im} \in \mathbb{R}^n, \ 1^T x^{im} \geq 0, \ \sin(2\pi/m)1^T x^{re} - \cos(2\pi/m)1^T x^{im} \geq 0 \right\}. \]
For the decomposition of $A$ as in (5.2), the weighted Dirac measure
\[ \mu := \lambda_1 \delta_{(u_1, v_1)} + \cdots + \lambda_r \delta_{(u_r, v_r)} \]
belongs to $\mathcal{B}(S^c)$, the set of Borel measures supported on $S^c$. It satisfies
\[ (5.4) \quad A = \int (x^{re} + \sqrt{-1}x^{im}) \otimes m \, d\mu. \]
Note that $\lambda_1 + \cdots + \lambda_r = \int 1 \, d\mu$. Conversely, for every $\mu \in \mathcal{B}(S^c)$ satisfying (5.4), we can always get a decomposition of $A$ as in (5.2). This can be implied by [25, Prop. 3.3]. By Lemma [5.1], $\|A\|_{s, \mathbb{C}}$ equals the optimal value of
\[ \left\{ \min \left\{ \int 1 \, d\mu \right\} \right. \] s.t. \[ A = \int (x^{re} + \sqrt{-1}x^{im}) \otimes m \, d\mu, \] $\mu \in \mathcal{B}(S^c)$. \]
Note that $S^c = \{ x \in \mathbb{R}^{2n} : h(x) = 0, g_1(x) \geq 0, g_2(x) \geq 0 \}$ where
\[ h := x^T x - 1, \quad g_1 := 1^T x^{im}, \quad g_2 := \sin(2\pi/m)1^T x^{re} - \cos(2\pi/m)1^T x^{im}. \]
Let $a^{re}, a^{im} \in \mathbb{R}^{n(m)}$ be the real vectors such that
\[ a^{re} + \sqrt{-1}a^{im} = A, \quad \text{if} \quad x^\alpha = x_{i_1} \cdots x_{i_m}. \]
Indeed, it holds that (cf. [26, Prop. 3.3])
\[ (x_1 + \sqrt{-1} x_{n+1})^{\alpha_1} \cdots (x_n + \sqrt{-1} x_{2n})^{\alpha_n} = R_\alpha(x) + \sqrt{-1} T_\alpha(x), \]
for real polynomials $R_\alpha, T_\alpha \in \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_{2n}]$. Then,
\[
\int (x_1 + \sqrt{-1} x_{n+1})^{\alpha_1} \cdots (x_n + \sqrt{-1} x_{2n})^{\alpha_n} d\mu = \int R_\alpha(x) d\mu + \sqrt{-1} \int T_\alpha(x) d\mu.
\]
Hence, (5.5) is equivalent to
\[
\begin{align*}
\min & \int 1 d\mu \\
\text{s.t.} & \quad a^e_\alpha = \int R_\alpha(x) d\mu \ (\alpha \in \mathbb{N}^n_{\{m\}}), \\
& \quad a^m_\alpha = \int T_\alpha(x) d\mu \ (\alpha \in \mathbb{N}^n_{\{m\}}), \\
& \quad \mu \in \mathcal{B}(S^c).
\end{align*}
\]
To solve (5.9), we can replace $\mu$ by the vector of its moments. Denote the moment cone
\[
\mathcal{R}^c_{\{0,m\}} := \left\{ y \in \mathbb{R}^{2n}_{\{0,m\}} \mid \exists \mu \in \mathcal{B}(S^c) \text{ such that } y_\beta = \int x^\beta d\mu \text{ for } \beta \in \mathbb{N}^{2n}_{\{0,m\}} \right\}.
\]
So, (5.9) is equivalent to the optimization problem
\[
\begin{align*}
\min & \quad (y)_0 \\
\text{s.t.} & \quad (R_\alpha, y) = a^e_\alpha \ (\alpha \in \mathbb{N}^n_{\{m\}}), \\
& \quad (T_\alpha, y) = a^m_\alpha \ (\alpha \in \mathbb{N}^n_{\{m\}}), \\
& \quad y \in \mathcal{R}^c_{\{0,m\}}.
\end{align*}
\]
5.1. An algorithm. The cone $\mathcal{R}^c_{\{0,m\}}$ can be approximated by semidefinite relaxations. For $h, g_1, g_2$ as in (5.4), denote the cones
\[
\mathcal{R}^{c,2k}_{\{0,m\}} := \left\{ z \in \mathbb{R}^{2n}_{\{0,2k\}} \mid M_k(z) \succeq 0, L_{h(k)}(z) = 0, L_{g_1(k)}(z) \succeq 0, L_{g_2(k)}(z) \succeq 0 \right\},
\]
\[
\mathcal{R}^{c,2k}_{\{0,m\}} := \left\{ y \in \mathbb{R}^{2n}_{\{0,m\}} \mid \exists z \in \mathcal{R}^{c,2k}_{\{0,m\}}, \ y = z|_{\{0,m\}} \right\}.
\]
Clearly, $\mathcal{R}^{c,2k}_{\{0,m\}}$ is a projection of $\mathcal{R}^{c,2k}_{\{0,m\}}$. For all $k \geq m/2$, we have
\[
\mathcal{R}^c_{\{0,m\}} \subseteq \mathcal{R}^{c,2k}_{\{0,m\}} \subseteq \mathcal{R}^{c,2k}_{\{0,m\}}.
\]
Indeed, it holds that (cf. [26 Prop. 3.3])
\[
\mathcal{R}^c_{\{0,m\}} = \bigcap_{k \geq m/2} \mathcal{R}^{c,2k}_{\{0,m\}}.
\]
This produces the hierarchy of semidefinite relaxations
\[
\begin{align*}
\|A\|_{k^*, c} := \min & \quad (z)_0 \\
\text{s.t.} & \quad (R_\alpha, z) = a^e_\alpha \ (\alpha \in \mathbb{N}^n_{\{m\}}), \\
& \quad (T_\alpha, z) = a^m_\alpha \ (\alpha \in \mathbb{N}^n_{\{m\}}), \\
& \quad z \in \mathcal{R}^{c,2k},
\end{align*}
\]
for $k = m_0, m_0 + 1, \ldots, (m_0 = \lfloor m/2 \rfloor)$. Like (4.12), we also have
\[
\|A\|_{m_0, c} \leq \cdots \leq \|A\|_{k^*, c} \leq \cdots \leq \|A\|_{*, c}.
\]
Algorithm 5.2. For a given tensor \( A \in \mathbb{S}^m(\mathbb{C}^n) \), let \( k = m_0 \) and do:

Step 1 Solve the semidefinite relaxation (5.15), for an optimizer \( z^k \).

Step 2 Let \( y^k := z^k|_{\{0, m\}} \) (see (2.3) for the truncation). Check whether or not \( y^k \in \mathcal{R}_{\{0, m\}} \). If yes, then \( \|A\|_{\ast, \mathbb{R}} = \|A\|_{k, \mathbb{C}} \) and go to Step 3; otherwise, let \( k := k + 1 \) and go to Step 1.

Step 3 Compute the decompositions of \( y^k \) as

\[
 y^k = \lambda_1[(u_1, v_1)]_{0, m} + \cdots + \lambda_r[(u_r, v_r)]_{0, m}
\]

with all \( \lambda_i > 0, \quad (u_i, v_i) \in S^r \). This gives the nuclear decomposition

\[
 A = \lambda_1(u_1 + \sqrt{-1}v_1)_{\otimes m} + \cdots + \lambda_r(u_r + \sqrt{-1}v_r)_{\otimes m}
\]

such that \( \sum_{i=1}^{\infty} \lambda_i = \|A\|_{\ast, \mathbb{C}} \).

In the above, the method in [25] can be used to check if \( y^k \in \mathcal{R}_{\{0, m\}} \) or not. If yes, we can also get a nuclear decomposition. It requires to solve a moment optimization problem whose objective is randomly generated.

5.2. Convergence properties. Denote the real polynomial vectors:

\[
 R(x) := (R_\alpha(x))_{\alpha \in \mathbb{N}^n_{\{m\}}}, \quad T(x) := (T_\alpha(x))_{\alpha \in \mathbb{N}^n_{\{m\}}},
\]

where \( R_\alpha, T_\alpha \) are as in (5.8). Their length \( D = \binom{2n+m-1}{m} \). Denote

\[
 \mathcal{P}(S^r)_{0, m} := \{ t + p \mid t \in \mathbb{R}, \: p \in \mathbb{R}[x]_{\text{hom}}^m, \: t + p \geq 0 \text{ on } S^r \}.
\]

The cones \( \mathcal{R}_{\{0, m\}} \) and \( \mathcal{P}(S^r)_{0, m} \) are dual to each other [26], so the dual optimization problem of (5.11) is

\[
 \begin{array}{ll}
 \max & p_1^T a^r + p_2^T a^i \\
 \text{s.t.} & 1 - p_1^T R(x) - p_2^T T(x) \in \mathcal{P}(S^r)_{0, m}.
 \end{array}
\]

Lemma 5.3. Let \( a^r, a^i \) be as in (5.7). Then, both (5.11) and (5.18) achieve the same optimal value which equals \( \|A\|_{\ast, \mathbb{C}} \).

Proof. The origin is an interior point of (5.18). So, (5.11) and (5.18) have the same optimal value, and (5.11) achieves it (cf. [26, §2.4]). In the next, we prove that (5.18) also achieves its optimal value. Let

\[
 w = x^r + \sqrt{-1}x^i, \quad q_\alpha = (p_1)_\alpha - \sqrt{-1}(p_2)_\alpha.
\]

\[
 q(w) = \sum_{|\alpha| = m} q_\alpha(R_\alpha(x) + \sqrt{-1}T_\alpha(x)) = \sum_{|\alpha| = m} q_\alpha w^\alpha.
\]

Clearly, \( q(w) \) is a form of degree \( m \) and in \( w \in \mathbb{C}^n \), and (Re denotes the real part)

\[
 \text{Re } q(w) = p_1^T R(x) + p_2^T T(x).
\]

Let \( B = \{ x^r + \sqrt{-1}x^i : (x^r, x^i) \in S^r \} \), which is a subset of the complex unit sphere \( \|w\| = 1 \), when \( (p_1, p_2) \) is feasible for (5.18), the polynomial

\[
 p(x) := 1 - p_1^T R(x) - p_2^T T(x) \geq 0 \quad \text{on } S^r.
\]

So, \( \text{Re } q(w) \leq 1 \) for all \( w \in B \). For all \( w \in \mathbb{C}^n \) with \( \|w\| = 1 \), there exist \( r^m = 1 \) and \( a \in B \) such that \( w = \tau a \). This is shown in the proof of Lemma 5.1 so

\[
 \text{Re } q(w) = \text{Re } q(\tau a) = \text{Re } q(a) \leq 1.
\]
The above is true for all unit complex vectors \( w \), hence
\[
\text{Re } q(w) \leq 1 \quad \forall w \in \mathbb{C}^n : ||w|| = 1.
\]
Because \( q(w) \) is homogeneous in \( w \), the above implies that
\[
|q(w)| \leq 1 \quad \forall w \in \mathbb{C}^n : ||w|| = 1.
\]
So, there exists \( M > 0 \) such that \( ||\text{vec}(q)|| \leq M \) for all \( q \) satisfying the above. Since \( ||\text{vec}(q)||^2 = ||\text{vec}(p_1)||^2 + ||\text{vec}(p_2)||^2 \), the feasible set of (5.18) is compact. So, (5.18) must achieve its optimal value.

Next, we study the properties of the relaxation (5.15). For \( h, g := (g_1, g_2) \) as in (5.0), denote the cones of polynomials (5.19)
\[
Q_k^c := \text{Ideal}(h)_{2k} + \text{Qmod}_{2k}(g), \quad Q^c := \bigcup_{k \geq 1} Q_k^c.
\]
The cones \( Q_k^c \) and \( \mathcal{P}^{c, 2k} \) are dual to each other [26], so the dual optimization problem of (5.15) is
\[
(5.20) \quad \begin{cases}
\max_{p_1, p_2 \in \mathbb{R}^n} \quad \frac{1}{2} p_1^T a^{re} + p_2^T a^{im} \\
\text{s.t.} \quad 1 - p_1^T R(x) - p_2^T T(x) \in Q_k^c.
\end{cases}
\]

**Lemma 5.4.** Let \( a^{re}, a^{im} \) be the real vectors as in (5.7). Then, for each \( k \geq m_0 \), both (5.13) and (5.20) achieve the same optimal value which equals \( ||A||_{k^*, c} \). Moreover, \( ||A||_{k^*, C} \) is a norm function in \( A \in \mathcal{S}^m(\mathbb{C}^n) \).

**Proof.** The proof is almost the same as for Lemmas 3.3 and 4.3. For each \( k \geq m_0 \), the origin is an interior point of (5.20). The vanishing ideal of \( S^c \) is Ideal(\( h \)), so the set \( Q_k^c \) is closed, implied by Theorem 3.35 of [20] or Theorem 3.1 of [23]. In the proof of Lemma 6.3, we showed that the feasible set of (5.13) is compact. Since \( Q_k^c \subseteq \mathcal{P}(S^c)_{(0, m)} \), the feasible set of (5.20) is also compact. By the linear conic duality theory [23 §2.4], both (5.13) and (5.20) achieve the same optimal value. We can similarly prove that \( ||A||_{k^*, C} \) is a norm function in \( A \). We omit the proof here, since it is almost the same as for Lemmas 6.3 and 4.3.

The convergence properties of Algorithm 5.2 are as follows.

**Theorem 5.5.** Let \( ||A||_{k^*, C} \) be the optimal value of (5.15). For all \( A \in \mathcal{S}^m(\mathbb{C}^n) \), Algorithm 5.2 has the following properties:

(i) \( \lim_{k \to \infty} ||A||_{k^*, C} = ||A||_{*, C} \).

(ii) Let \( (p_1^*, p_2^*) \) be an optimal pair for (5.18). If
\[
1 - (p_1^*)^T R(x) - (p_2^*)^T T(x) \in Q^c,
\]
then \( ||A||_{k^*, C} = ||A||_{*, C} \) for all \( k \) sufficiently big.

(iii) If \( y^k \in \mathcal{P}_{(0, m)} \) for some order \( k \), then \( ||A||_{k^*, C} = ||A||_{*, C} \).

(iv) The sequence \( \{y^k\}_{k=m_0}^\infty \) is bounded, and each of its accumulation points belongs to \( \mathcal{P}_{(0, m)} \). Moreover, if the nuclear decomposition of \( A \) over \( \mathbb{C} \) is unique, then \( y^k \) converges to a point in \( \mathcal{P}_{(0, m)} \) as \( k \to \infty \).

**Proof.** (i) By Lemma 5.3, for every \( \epsilon > 0 \), there exist \( s_1, s_2 \in \mathbb{R}^D \) such that
\[
1 - (s_1)^T R - (s_2)^T T > 0 \quad \text{on} \quad S^c, \quad \langle s_1, a^{re} \rangle + \langle s_2, a^{im} \rangle \geq ||A||_{*, C} - \epsilon.
\]
By Theorem 2.1 there exists $k_1$ such that
\[ 1 - (s_1)^T R - (s_2)^T T \in Q_{k_1}. \]

By Lemma 5.4, we can get $\| A \|_{k_1, \mathbb{C}} \geq \| A \|_{*, \mathbb{C}} - \epsilon$. The monotonicity relation (5.10) and the above imply that
\[ \| A \|_{*, \mathbb{C}} \geq \lim_{k \to \infty} \| A \|_{k_1, \mathbb{C}} \geq \| A \|_{*, \mathbb{C}} - \epsilon. \]

Since $\epsilon > 0$ can be arbitrarily small, the item (i) follows directly.

(ii) If $1 - (p_1)^T R - (p_2)^T T \in Q_{\epsilon}$, then $1 - (p_1)^T R - (p_2)^T T \in Q_{k_2}$ for some $k_2 \in \mathbb{N}$. By Lemma 5.3 we know that
\[ \| A \|_{*, \mathbb{C}} = \| p_1^* a^{r, e} + p_2^* a^{im} \| \leq \| A \|_{k_2, \mathbb{C}}. \]

Then, (5.10) implies that $\| A \|_{k_1, \mathbb{C}} = \| A \|_{*, \mathbb{C}}$ for all $k \geq k_2$.

(iii) If $y^k \in S_{(0, m)}^c$ for some order $k$, then $\| A \|_{k, \mathbb{C}} \geq \| A \|_{*, \mathbb{C}}$, by Lemma 5.3.

The equality $\| A \|_{k, \mathbb{C}} = \| A \|_{*, \mathbb{C}}$ follows from (5.10).

(iv) Note that $(z^k)_0 = (y^k)_0 = \| A \|_{k, \mathbb{C}}$ for all $k$. The condition $L_{1-\|x\|^2}^{(k)}(z^k) = 0$ implies that
\[ (z^k)_{2e_1+2\beta} + \cdots + (z^k)_{2e_2+2\beta} = (z^k)_{2\beta} \]
for all $\beta \in \mathbb{N}_{0, 2k}$. By induction, one can easily show that
\[ (z^k)_{2\beta} \leq (z^k)_0 \quad \forall \beta \in \mathbb{N}_{0, 2k}. \]

Since $y^k$ is a truncation of $z^k$ and $M_k(z^k) \geq 0$, we get
\[ |(y^k)_\beta|^2 = |(z^k)_\beta|^2 \leq (z^k)_{2\beta}(z^k)_0 \leq |(z^k)_0|^2 = |(y^k)_0|^2 = \| A \|_{*, \mathbb{C}}^2. \]

This shows that the sequence $\{y^k\}$ is bounded. For all $k \geq m/2$, it holds that $y^k \in S_{(0, m)}^{c, 2k}$. The distance between $S_{(0, m)}^{c, 2k}$ and $S_{(0, m)}^{c}$ tends to zero as $k \to \infty$ (cf. [26, Prop. 3.4]). Therefore, every accumulation point $\hat{y}$ of the sequence $\{y^k\}$ belongs to $S_{(0, m)}^{c}$. This can also be implied by (5.14). So, $\hat{y} = \sum_{i=1}^{r} \lambda_i (u_i, v_i)_{0, m}$, with $\lambda_i > 0$ and $(u_i, v_i) \in S^c$. The feasibility condition in (5.15) and the relation (5.8) imply that
\[ A = \sum_{i=1}^{r} \lambda_i (u_i + \sqrt{-1}v_i) \otimes m. \]

When the nuclear decomposition of $A$ is unique, $\lambda_i$ and $(u_i, v_i) \in S^c$ are also uniquely determined. So, the accumulation point $\hat{y}$ is unique and $y^k$ converges to a point in $S_{(0, m)}^{c}$ as $k \to \infty$. \hfill \Box

6. Numerical examples

This section presents numerical experiments for nuclear norms of symmetric tensors. The computation is implemented in MATLAB R2012a, on a Lenovo Laptop with CPU at 2.90GHz and RAM 16.0G. Algorithm 5.1 is applied for real nuclear norms of real odd order tensors, Algorithm 5.1 is for real nuclear norms of real even order tensors, while Algorithm 5.2 is for complex nuclear norms of all tensors. These algorithms can be implemented in software Gloptipoly 3 [12] by calling the semidefinite program package SeDuMi [32].

Since our methods are numerical, we display only four decimal digits for the computational results. For a nuclear decomposition $A = (u_1) \otimes m + \cdots + (u_r) \otimes m$, we display it by listing the vectors $u_1, \ldots, u_r$, column by column, from the left to right.
Example 6.1. ([9]) (i) Consider the tensor in $S^4(\mathbb{R}^2)$ such that
\[ A = \frac{1}{\sqrt{3}} (e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1). \]
We got $\|A\| = \|A\|_{2*,R} = 3$. It took about 1 second. The real nuclear decomposition $A = \sum_{i=1}^{3} (u_i)^{\otimes 3}$ is
\[
\begin{array}{ccc}
0.0000 & -0.7937 & 0.7937 \\
0.5774 & 0.4582 & 0.4582
\end{array}
\]
and the complex nuclear decomposition $A = \sum_{i=1}^{3} (w_i)^{\otimes 3}$ is
\[
\begin{array}{ccc}
-0.5873 & 0.27401 & 0.45824 \\
0.2944 & 0.3511 & 0.65821 \\
0.5456 & 0.54561 & 0.65821
\end{array}
\]
The nuclear norms are the same as in [9].

(ii) Consider the tensor in $S^4(\mathbb{R}^2)$ such that
\[ A = \frac{1}{2} (e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 - e_2 \otimes e_2 \otimes e_2). \]
We got $\|A\|_{2*,R} = \|A\|_{2*,C} = 2$ and $\|A\|_{2*,C} = \sqrt{2}$. It took about 1 second. The real nuclear decomposition $A = \sum_{i=1}^{3} (u_i)^{\otimes 3}$ is
\[
\begin{array}{ccc}
0.0000 & -0.7565 & 0.7565 \\
-0.8736 & 0.4368 & 0.4368
\end{array}
\]
while the complex nuclear decomposition $A = \sum_{i=1}^{2} (w_i)^{\otimes 3}$ is
\[
\begin{array}{ccc}
0.5456 & -0.31501 & -0.54561 & + 0.31501 \\
0.3150 & + 0.54561 & 0.3150 & + 0.54561
\end{array}
\]
The nuclear norms are the same as in [9].

Next, we see some tensors of order four.

Example 6.2. (i) Consider the tensor $A \in S^4(\mathbb{R}^3)$ such that
\[ A = e^{\otimes 4} - e_1^{\otimes 4} - e_2^{\otimes 4} - e_3^{\otimes 4}. \]
We got $\|A\|_{2*,R} = 12$ and $\|A\|_{2*,C} = 11.8960$. It took about 15 seconds. The real nuclear decomposition is the same as above. The complex nuclear decomposition $A = \sum_{i=1}^{3} (w_i)^{\otimes 4}$ is
\[
\begin{array}{ccc}
-0.1152 & 0.07141 & 0.0332 \\
0.5316 & 0.56961 & 0.6145 \\
-0.1158 & 0.07521 & 0.0288
\end{array}
\]
\[
\begin{array}{ccc}
0.0997 & 0.09681 & 0.5905 \\
0.0997 & 0.09681 & 0.5905 \\
0.0997 & 0.09681 & 0.5905
\end{array}
\]
\[
\begin{array}{ccc}
0.1285 & 0.012121 & 0.0163 \\
0.1285 & 0.012121 & 0.0163 \\
0.1285 & 0.012121 & 0.0163
\end{array}
\]
(ii) Consider the tensor $A \in S^4(\mathbb{R}^3)$ such that
\[ A = (e_1 + e_2)^{\otimes 4} + (e_1 + e_3)^{\otimes 4} - (e_2 + e_3)^{\otimes 4}. \]
We got $\|A\|_{2*,R} = 12$. It took about 12 seconds. The real and complex nuclear decompositions are the same as above.

(iii) Consider the tensor $A \in S^4(\mathbb{R}^3)$ such that
\[ A = (e_1 + e_2 - e_3)^{\otimes 4} + (e_1 - e_2 + e_3)^{\otimes 4} + (e_1 + e_2 + e_3)^{\otimes 4} - (e_3)^{\otimes 4}. \]
We got $\|A\|_{2*,R} = 36$. It took about 7 seconds. The real and complex nuclear decompositions are the same as above.

The following are some examples of complex-valued tensors.
Example 6.3. (i) Consider the tensor $A \in S^3(\mathbb{C}^3)$ such that

$$A_{i_1i_2i_3} = \sqrt{-1}^{i_1i_2i_3}.$$ We got $\|A\|_{*,\mathbb{C}} = \|A\|_{2*,\mathbb{C}} \approx 8.8759$. It took about 1 second. The nuclear decomposition $A = \sum_{i=1}^5 (w_i) \otimes i^3$ is

$$\begin{align*}
0.6024 \pm 0.3478i & -0.6019 \pm 0.3475i \\
0.0000 - 0.0000i & 0.0000 - 0.0000i \\
-0.6024 - 0.3478i & 0.6019 - 0.3475i \\
0.0000 \pm 0.0000i & 0.0000 \pm 0.0000i \\
\end{align*}$$

(ii) Consider the tensor $A \in S^5(\mathbb{C}^3)$ such that

$$A_{i_1i_2i_3i_4i_5} = (\sqrt{-1})^{i_1} + (-1)^{i_2} + (\sqrt{-1})^{i_3} + (1)^{i_4};$$ We got $\|A\|_{*,\mathbb{C}} = \|A\|_{3*,\mathbb{C}} \approx 26.9569$. It took about 17 seconds. The nuclear decomposition $A = \sum_{i=1}^7 (w_i) \otimes i^4$ is

$$\begin{align*}
0.6274 + 0.5703i & -0.7099 + 0.2840i \\
0.6275 + 0.5699i & 0.5471 + 0.1352i \\
-0.5940 - 0.6576i & 0.3296 + 0.1164i \\
-0.1432 + 0.8472i & 0.9097 + 0.9276i \\
0.1440 - 0.8475i & 0.3732 - 0.7060i \\
0.9490 + 0.2131i & 0.4154 - 0.7460i \\
\end{align*}$$

(iii) Consider the tensor $A \in S^6(\mathbb{C}^3)$ such that

$$A_{i_1i_2i_3i_4i_5i_6} = (\sqrt{-1})^{i_1i_2i_3i_4i_5} + (\sqrt{-1})^{i_1i_2i_3i_4i_5}.$$ We got $\|A\|_{*,\mathbb{C}} = \|A\|_{2*,\mathbb{C}} \approx 49.5626$. It took about 4.7 seconds. The nuclear decomposition $A = \sum_{i=1}^6 (w_i) \otimes i^5$ is

$$\begin{align*}
0.2711 + 0.8336i & 0.8651 + 0.0003i \\
-0.2255 - 0.6963i & -0.7224 - 0.0008i \\
0.2711 + 0.8336i & 0.8651 + 0.0003i \\
0.5542 + 0.0006i & 0.8654 + 0.4276i \\
1.1499 + 0.0041 & -0.3356 + 0.9198i \\
0.5542 + 0.0006i & 0.8654 + 0.4276i \\
\end{align*}$$

(iv) Consider the tensor $A \in S^6(\mathbb{R}^n)$ such that

$$A_{i_1i_2i_3} = i_1 + i_2 + i_3.$$ We got $\|A\|_{*,\mathbb{R}} = \|A\|_{2*,\mathbb{R}} = 686$. It took about 4.8 seconds. The nuclear decomposition is

$$A = \begin{bmatrix} 1 & 1 \\ -2\sqrt{-1} & 2\sqrt{-1} \end{bmatrix} \otimes 6.$$

Example 6.4. Consider the tensor $A \in S^3(\mathbb{R}^n)$ such that

$$A_{i_1i_2i_3} = i_1 + i_2 + i_3.$$ For a range of values of $n$, the real and complex nuclear norms $\|A\|_{*,\mathbb{R}}$ and $\|A\|_{*,\mathbb{C}}$ are reported in Table 1. We list the order $k$ for which $\|A\|_{*,\mathbb{R}} = \|A\|_{k*,\mathbb{R}}$ and the length of the nuclear decomposition, as well as the consumed time (in seconds). For neatness, we only display nuclear decompositions for $n = 3$. The real nuclear decomposition $A = \sum_{i=1}^3 (w_i) \otimes i^3$ is

$$\begin{align*}
0.3699 & -0.8633 \\
-0.1899 & -0.3318 \\
-0.7487 & 0.1996 \\
\end{align*}$$

while the complex nuclear decomposition $A = \sum_{i=1}^3 (w_i) \otimes i^3$ is

$$\begin{align*}
0.3699 & + 0.6690i \\
-0.2054 & + 0.5318i \\
-0.5868 & + 0.3745i \\
\end{align*}$$
Table 1. Nuclear norms of the tensor in Example 6.4

| n | F | ∥A∥_F^* | k | length | time |
|---|---|---------|---|--------|------|
| 2 | R | 13.4164 | 2 | 3 | 0.81 |
| 2 | C | 13.2114 | 2 | 3 | 1.31 |
| 3 | R | 33.6749 | 2 | 3 | 0.90 |
| 3 | C | 32.9505 | 2 | 3 | 1.92 |
| 4 | R | 65.7267 | 2 | 3 | 0.93 |
| 4 | C | 64.0886 | 2 | 3 | 3.73 |
| 5 | R | 111.2430 | 2 | 3 | 0.97 |
| 6 | R | 171.7091 | 2 | 3 | 1.08 |
| 7 | R | 248.4754 | 2 | 3 | 1.23 |
| 8 | R | 342.7886 | 2 | 3 | 1.67 |
| 9 | R | 455.8125 | 2 | 3 | 2.45 |
| 10 | R | 588.6425 | 2 | 3 | 2.66 |

Example 6.5. Consider the tensor $A \in S^4(\mathbb{R}^n)$ such that

$$A_{i_1i_2i_3i_4} = \cos \left( \frac{1}{i_1} + \frac{1}{i_2} + \frac{1}{i_3} + \frac{1}{i_4} \right).$$

The nuclear norms, the order $k$ for which $\|A\|_{F^*} = \|A\|_{k^*F}$, the lengths of the nuclear decompositions, and the consumed time (in seconds) are displayed in Table 2 for a range of values of $n$. For neatness, we only display nuclear decompositions for $n = 3$. The real nuclear decomposition is $A = (u_1)^{\otimes 4} + (u_2)^{\otimes 4} - (u_3)^{\otimes 4} - (u_4)^{\otimes 4}$, where $u_1, u_2, u_3, u_4$ are respectively given as

-0.0261 0.9989 -0.6615 1.0988
0.7131 0.1816 0.2850 0.9044
0.9287 -0.1114 0.5965 0.7863

The complex nuclear decomposition $A = \sum_{i=1}^{4}(w_i)^{\otimes 4}$ is

-0.0001 - 0.1505i 0.2673 0.7394 0.7395 0.6659 + 0.6676i
-0.0010 + 0.3845i 0.5967 + 0.3287i 0.3287 + 0.5965i 0.5021 + 0.5032i
-0.0013 + 0.5481i 0.6771 + 0.1666i 0.1680 + 0.6759i 0.4172 + 0.4181i

□
For a tensor $\mathcal{A} \in \mathcal{S}^m(\mathbb{C}^n)$, define

$$\mathcal{A}(x) := \sum_{i_1, \ldots, i_m = 1}^n A_{i_1 \ldots i_m} x_{i_1} \cdots x_{i_m}. \tag{6.1}$$

Clearly, $\mathcal{A}(x)$ is a homogeneous polynomial in $x := (x_1, \ldots, x_n)$ and of degree $m$. There is a bijection between the symmetric tensor space $\mathcal{S}^m(\mathbb{C}^n)$ and the space of homogeneous polynomials of degree $m$ (cf. [28, 30]). So, we can equivalently display $\mathcal{A}$ by showing the polynomial $\mathcal{A}(x)$. Moreover, the decomposition $\mathcal{A} = \sum_{i=1}^r \pm (u_i)^\odot m$ is equivalent to $\mathcal{A}(x) = \sum_{i=1}^r \pm (u_i^T x)^m$. Thus, we can also display a nuclear decomposition by writing $\mathcal{A}(x)$ as a sum of power of linear forms.

**Example 6.6.** (i) Consider the tensor $\mathcal{A} \in \mathcal{S}^4(\mathbb{R}^3)$ such that

$$\mathcal{A}(x) = x_1 x_2 x_3.$$ 

We got $\|\mathcal{A}\|_{*,\mathbb{R}} = \|\mathcal{A}\|_{2*,\mathbb{R}} = \|\mathcal{A}\|_{*,\mathbb{C}} = \|\mathcal{A}\|_{2*,\mathbb{C}} = \sqrt{3}/2$. The real nuclear decomposition of $\mathcal{A}$ is given as

$$\mathcal{A}(x) = \frac{1}{27} \left( (-x_1 - x_2 + x_3)^3 + (-x_1 + x_2 - x_3)^3 + (x_1 - x_2 - x_3)^3 + (x_1 + x_2 + x_3)^3 \right).$$

The above also serves as a complex nuclear decomposition.

(ii) Consider the tensor $\mathcal{A} \in \mathcal{S}^4(\mathbb{R}^4)$ such that

$$\mathcal{A}(x) = x_1 x_2 x_3 x_4.$$ 

We got $\|\mathcal{A}\|_{*,\mathbb{R}} = \|\mathcal{A}\|_{2*,\mathbb{R}} = \|\mathcal{A}\|_{*,\mathbb{C}} = \|\mathcal{A}\|_{2*,\mathbb{C}} = 2/3$. The real nuclear decomposition is given as

$$\mathcal{A}(x) = \frac{1}{144} \left( (-x_1 - x_2 + x_3 + x_4)^4 + (-x_1 + x_2 - x_3 + x_4)^4 + (-x_1 + x_2 + x_3 - x_4)^4 + (+x_1 + x_2 + x_3 + x_4)^4 - (-x_1 + x_2 + x_3 + x_4)^4 - (x_1 - x_2 + x_3 + x_4)^4 - (x_1 + x_2 - x_3 + x_4)^4 - (x_1 + x_2 + x_3 - x_4)^4 - (x_1 + x_2 + x_3 + x_4)^4 \right).$$

The above also serves as a complex nuclear decomposition.

(iii) Consider the tensor $\mathcal{A} \in \mathcal{S}^4(\mathbb{R}^2)$ such that

$$\mathcal{A}(x) = x_1^2 x_2^2.$$ 

We got $\|\mathcal{A}\|_{*,\mathbb{R}} = \|\mathcal{A}\|_{2*,\mathbb{R}} = 1$. The real nuclear decomposition is

$$\mathcal{A}(x) = \frac{1}{12} (-x_1 + x_2)^4 + \frac{1}{12} (x_1 + x_2)^4 - \frac{1}{6} x_1^4 - \frac{1}{6} x_2^4.$$ 

The complex nuclear norm $\|\mathcal{A}\|_{*,\mathbb{C}} = \|\mathcal{A}\|_{2*,\mathbb{C}} = 2/3$, and the decomposition is

$$\mathcal{A}(x) = \frac{1}{24} \left( (x_1 - x_2)^4 + (x_1 + x_2)^4 - (x_1 + \sqrt{-1} x_2)^4 - (x_1 - \sqrt{-1} x_2)^4 \right).$$

(iv) Consider the tensor $\mathcal{A} \in \mathcal{S}^4(\mathbb{R}^3)$ such that

$$\mathcal{A}(x) = x_1^2 x_2^2 + x_2^2 x_3^2 + x_1^2 x_3^2.$$ 

We got $\|\mathcal{A}\|_{*,\mathbb{R}} = \|\mathcal{A}\|_{2*,\mathbb{R}} = 2$. The real nuclear decomposition is

$$\mathcal{A}(x) = \frac{1}{24} \left( (x_1 - x_2 + x_3)^4 + (x_1 + x_2 - x_3)^4 + (-x_1 + x_2 + x_3)^4 + (x_1 + x_2 + x_3)^4 - \frac{1}{6} \left( x_1^4 + x_2^4 + x_3^4 \right) \right).$$
Interestingly, for all SOEP tensors as in (6.2), we have
\[ A \in S^4(\mathbb{R}^3) \] such that
\[ \|A\|_{*,C} = \|A\|_{2*,C} = 5/3, \] and the decomposition is
\[
A(x) = \frac{1}{36} \left( -x_1 + x_2 + x_3 \right)^4 + \left( x_1 - x_2 + x_3 \right)^4 + \left( x_1 + x_2 - x_3 \right)^4 + \left( x_1 + x_2 + x_3 \right)^4 \\
- \left( x_1 - x_3^2 \right) + \left( x_2 - x_3^2 \right) + \left( x_1 + x_3^2 \right) + \left( x_1 + x_3^2 \right).
\]

(v) Consider the tensor \( A \in S^4(\mathbb{R}^3) \) such that
\[ \|A\|_{*,R} = \|A\|_{2*,R} = \|A\|_{*,C} = \|A\|_{2*,C} = 5, \] with the nuclear decomposition
\[
A(x) = \frac{1}{12} \left( (x_1 - x_2 + x_3)^4 + (x_1 + x_2 - x_3)^4 + (-x_1 + x_2 + x_3)^4 + (x_1 + x_2 + x_3)^4 \right).
\]
The above also serves as a complex nuclear decomposition. \( \square \)

**Example 6.7.** For \( a, b, c \in \mathbb{R}^n \), the symmetrization of the rank-1 nonsymmetric tensor \( a \otimes b \otimes c \) is
\[
sym(a \otimes b \otimes c) := \frac{1}{6} (a \otimes b \otimes c + a \otimes c \otimes b + b \otimes a \otimes c + b \otimes c \otimes a + a \otimes b \otimes c).
\]
One wonders whether \( \|\text{sym}(a \otimes b \otimes c)\|_{*,R} = \|a\| \cdot \|b\| \cdot |c| \) or not. Indeed, this is usually not true. Typically, we have the inequalities
\[
\|\text{sym}(a \otimes b \otimes c)\|_{*,C} < \|\text{sym}(a \otimes b \otimes c)\|_{*,R} < \|a\| \cdot \|b\| \cdot |c|.
\]
For instance, consider the following tensor in \( S^3(\mathbb{R}^3) \)
\[
A = \text{sym}(e_1 \otimes (e_1 + e_2) \otimes (e_1 + e_2 + e_3)).
\]
We can compute that \( \|A\|_{*,C} \approx 2.2276, \|A\|_{*,R} \approx 2.4190, \) but
\[
\|e_1\| \cdot \|e_1 + e_2\| \cdot \|e_1 + e_2 + e_3\| = \sqrt{6} \approx 2.4495.
\]
The computed real nuclear decomposition \( A = \sum_{i=1}^5 (u_i) \otimes 3 \) is
\[
\begin{array}{cccccc}
-0.2896 & 0.0169 & -0.4750 & -0.0947 & 1.0423 \\
0.1803 & -0.5617 & 0.1042 & -0.2944 & 0.1806 \\
-0.2891 & -0.3122 & 0.3114 & 0.1865 & 0.2630
\end{array}
\]
The computed complex nuclear decomposition \( A = \sum_{i=1}^8 (w_i) \otimes 3 \) is
\[
\begin{array}{cccccc}
-0.2795 + 0.2861i & -0.2882 + 0.2878i & -0.4808 + 0.8481i & -0.2489 + 0.0952i \\
0.1094 + 0.2873i & -0.2098 + 0.2076i & -0.2776 + 0.4375i & 0.1376 + 0.3928i \\
-0.2009 + 0.0696i & 0.2301 + 0.1457i & -0.1146 + 0.2252i & 0.1359 + 0.1297i \\
-0.1212 + 0.0018i & 0.3286 - 0.1764i & 0.2244 - 0.1049i & 0.2196 - 0.1357i \\
0.0304 - 0.0450i & 0.2410 + 0.1640i & 0.1103 + 0.2279i & 0.2235 + 0.2601i \\
0.0268 + 0.0974i & -0.0996 + 0.2903i & 0.1464 - 0.1210i & 0.0750 + 0.1106i
\end{array}
\]
However, if \( a = b = c \), then \( \|\text{sym}(a \otimes b \otimes c)\|_{*,F} = \|a\|^3 \) for \( F = \mathbb{R}, \mathbb{C} \). \( \square \)

**Example 6.8.** (Sum of Even Powers) For an even order \( m \), consider the tensors \( A \in S^m(\mathbb{R}^n) \) of the form such that
\[
(6.2) \quad A = (a_1)^\otimes m + \cdots + (a_r)^\otimes m,
\]
with \( a_1, \ldots, a_r \in \mathbb{R}^n \). Such a tensor is called a sum of even powers (SOEP). Interestingly, for all SOEP tensors as in (6.2), we have
\[
(6.3) \quad \|A\|_{*,R} = \|A\|_{*,C} = \|a_1\|^m + \cdots + \|a_r\|^m.
\]
Clearly, $\|A\|_{\star, C} \leq \|A\|_{\star, \mathbb{R}} \leq \sum_{i=1}^r \|a_i\|^m$. The reverse inequalities are actually also true. Let $B$ be the tensor such that $B(x) = (x^T x)^{m/2}$. Then, $\|B\|_{\sigma, C} = 1$. By the duality relation, we get
\[
\|A\|_{\star, C} \geq A \bullet B = \sum_{i=1}^r B(a_i) = \sum_{i=1}^r \|a_i\|^m.
\]
So, (6.3) is true. It can also be proved by applying Lemma 4.1 of [9]. Moreover, a nonsymmetric cubic tensor $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ can be computed as
\[
\text{Algorithm 4.1 and 5.2 confirmed that } \|A\|_{\star} = \|A\|_{\star, C} = 120. \quad \square
\]

### 7. Extensions to nonsymmetric tensors

The methods in this paper can be naturally extended to nonsymmetric tensors. A similar discussion was made in [33]. For convenience, we show how to do this for a nonsymmetric cubic tensor $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$. Clearly, its real nuclear norm can be computed as
\[
\|A\|_{\star, \mathbb{R}} = \min \left\{ \sum_{i=1}^r \lambda_i \right\} \quad \text{such that } A = \sum_{i=1}^r \lambda_i v^{(i,1)} \otimes v^{(i,2)} \otimes v^{(i,3)}, \quad \lambda_i \geq 0, \quad \|v^{(i,j)}\| = 1, \quad v^{(i,j)} \in \mathbb{R}^{n_j}
\]

One can similarly show that $\|A\|_{\star, \mathbb{R}}$ is equal to the minimum value of the optimization problem
\[
\min \int 1 d\mu \quad \text{s.t.} \quad A = \int x^{(1)} \otimes x^{(2)} \otimes x^{(3)} d\mu, \quad \mu \in \mathcal{B}(T).
\]
In the above, the variables $x^{(j)} \in \mathbb{R}^{n_j}$, and $\mathcal{B}(T)$ is the set of Borel measures supported on the set
\[
T := \left\{(x^{(1)}, x^{(2)}, x^{(3)}): \|x^{(1)}\| = \|x^{(2)}\| = \|x^{(3)}\| = 1\right\}.
\]
Similarly, we can define the cone of moments (denote $[n] := \{1, \ldots, n\}$)
\[
\mathcal{R}_{\{0,3\}}^{n_1 \times n_2 \times n_3} := \left\{ y \in \mathbb{R}^{1+n_1n_2n_3} \middle| \exists \mu \in \mathcal{B}(T) \text{ s.t. } (y)_{ijk} = \int (x^{(1)})_i (x^{(2)})_j (x^{(3)})_k d\mu \right\}
\]
One can show that (7.2) is equivalent to
\[
\min \quad (y)_{000} \quad \text{s.t.} \quad (y)_{ijk} = A_{ijk} \quad (\forall i, j, k), \quad y \in \mathcal{R}_{\{0,3\}}^{n_1 \times n_2 \times n_3}.
\]
A similar version of Algorithm 3.1 can be applied to solve (7.4).
Example 7.1. Consider the nonsymmetric tensor $A \in \mathbb{R}^{2 \times 2 \times 2}$ such that
$$ A_{ijk} = i - j - k. $$
By solving (7.4), we get $\|A\|_{*,\mathbb{R}} = 6.0000$. A real nuclear decomposition of $A$ is given as
$$\begin{bmatrix} 1.4363 \\ 0.3140 \end{bmatrix} \otimes \begin{bmatrix} -0.9146 \\ -1.516 \end{bmatrix} \otimes \begin{bmatrix} 0.9296 \\ 1.1394 \end{bmatrix} + \begin{bmatrix} 0.5484 \\ 0.6978 \end{bmatrix} \otimes \begin{bmatrix} 0.8846 \\ 0.0732 \end{bmatrix} \otimes \begin{bmatrix} 0.6501 \\ -0.6040 \end{bmatrix}.$$ 

When $\mathbb{F} = \mathbb{C}$, we can similarly compute the complex nuclear norm $\|A\|_{*,\mathbb{C}}$, by considering each $x^{(j)}$ as a complex variable. For nonsymmetric tensors, it is usually much harder to compute the nuclear norm $\|A\|_{*,\mathbb{R}}$ or $\|A\|_{*,\mathbb{C}}$. This is because the variable $x$ has much higher dimension than for the case of symmetric tensors, which makes the moment optimization problem like (7.4) very difficult to solve.

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